Exact Conservation of Quantum Numbers in the Statistical Description of High Energy Particle Reactions

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1 Introduction

In high energy particle collisions, the interaction volume is often very small and large deviations can occur from the thermodynamic limit depending on the beam energy. In some cases it may be easy to produce a pion since one needs only 140 MeV for its rest mass, however, to produce an anti-proton one needs at least twice the rest mass, 1.88 GeV and not simply 0.94 GeV, since they can only be produced in pairs. The probability to produce anti-protons therefore cannot simply follow the same law (e.g. a Boltzmann distribution) as the production of pions. Fortunately, statistical mechanics provides us with the tools to take into account constraints like baryon number or strangeness conservation. This will be presented in these lecture notes.

Basically, if the system is large and hot, the corrections are negligible. In \textit{Pb–Pb} collisions at the CERN/SPS collider they are negligible because the system is large enough but for \textit{p–p} collisions at the same energy these corrections are large and must be taken into account because the final system is too small.

The main statistical concepts are collected in Ch. 3. In chapter 3, a statistical method for taking the exact strangeness conservation into account is presented. This is applied in Ch. 3.1 to describe the particle production as observed at the GSI in \textit{Ni+Ni} collisions. The exact treatment of quantum numbers is extended to include the strangeness, baryon number and charge in Ch. 4. A comparison of numerical results with the AGS E802 data is given in Ch. 4.1. Finally in Ch. 5, the generalization of the method to arbitrary internal symmetry is reviewed.

2 Quantum Statistical Concepts in Brief

Throughout this paper we follow the usual convention of natural units, so the speed of light, Planck constant and Boltzmann constant have values $c \equiv 1$, $\hbar \equiv 1$ and $k \equiv 1$, respectively.
All the physical information about a collection of particles is contained in a density operator, $\hat{\rho}$. The average of an observable $A$ in this statistical ensemble is calculated as $\langle A \rangle = \text{tr}(\hat{\rho}\hat{A})$, where $\hat{A}$ is a Hermitian operator corresponding to the observable. Using the density operator, one defines the entropy of the system considered as $S = -\text{tr}(\hat{\rho}\ln \hat{\rho})$.

In any system in nature, the entropy is known to tend to its maximum, so one has to find a representation of the density operator satisfying this condition. In thermodynamical equilibrium, the average occupations of different quantum states do not change in time, so $\partial \hat{\rho}/\partial t = 0$. The density operator satisfies the equation of motion of the form $i\partial \hat{\rho}/\partial t = -[\hat{\rho}, \hat{H}]$, where $\hat{H}$ is the Hamiltonian of the system. Thus, the thermodynamical, stationary density operator is diagonal in the basis formed by Hamiltonian eigenstates.

The choice of constraints used in maximizing the entropy defines the type of the statistical ensemble obtained. The closed system with fixed energy, $E$, volume, $V$, and number of particles $i$, $N_i$, is a microcanonical ensemble. System in heat bath (the ensemble average of energy $\langle E \rangle = \text{tr}(\hat{\rho}\hat{H})$, $V$ and $N_i$ are conserved) corresponds to canonical ensemble. Further, if we let the particle number $N_i$ fluctuate such that the average $\text{tr}(\hat{\rho}\hat{N}_i)$ is conserved, we obtain a grand canonical ensemble.

Maximization of entropy using the canonical boundary conditions leads to the density operator of the form $\hat{\rho} = e^{-\beta \hat{H}}/Z$, where $\beta$ is the inverse of temperature $T$, and $Z$ is the canonical partition function,

$$Z = \text{tr} e^{-\beta \hat{H}} = \sum_i e^{-\beta E_i(N)}.$$  \hspace{1cm} (1)

Here $i$ labels the different quantum states in the system and $E_i(N)$ is the eigenvalue of the $N$ particle Hamiltonian. In the last step the trace is expressed in the basis of Hamiltonian eigenstates. Once knowing the correct partition function, one is able to calculate the thermodynamical quantities describing the system. For example, the average energy is $\langle E \rangle = T^2 \partial \ln Z/\partial T$.

Choosing the grand canonical boundary conditions yields

$$\hat{\rho}_G = e^{-\beta(\hat{H} - \mu_i \hat{N}_i)}/Z_G.$$  

Here we have employed the grand canonical partition function,

$$Z_G(T, \{\lambda_i\}, V) = \text{tr} e^{-\beta(\hat{H} - \mu_i \hat{N}_i)} = \prod_i \sum_{N_i} \lambda_i^{N_i} Z_{N_i},$$  \hspace{1cm} (2)

where $\lambda_i = e^{\beta \mu_i}$ is the fugacity of the particle species $i$ and $Z_{N_i}$ is the $N$ particle canonical partition function of the species $i$. Chemical potentials $\mu_i$ take care of particle number conservation in an average sense. In this work, we are mainly interested in the mean particle numbers in the grand canonical system, so we employ frequently the equation

$$\langle N_i \rangle = \lambda_i \frac{\partial \ln Z_G}{\partial \lambda_i}.$$  \hspace{1cm} (3)
In properly quantized, finite volume system the partition function is often rather difficult to compute. In this work we never know the exact geometry of the system, so we settle for a finite volume, $V$, sample of the infinite volume system. Thus, the summation over discrete quantum states in partition function is changed to simple phase space integration over continuum. The one particle canonical partition function of particle $i$ is now given by

$$Z_i^1 = g_i V \frac{V}{2\pi^2} \int_0^\infty dp \, p^2 e^{-\beta \sqrt{p^2 + m_i^2}},$$

where $g_i$ is the spin degeneration factor, and $m_i$ is the mass of the particle $i$. Using the previous result and taking care of the correct occupation of quantum states, the grand canonical partition function can be written in the form

$$\ln Z_G(T, \{\lambda_i\}, V) = \sum_i g_i V \frac{V}{2\pi^2} \int_0^\infty dp \, p^2 \ln \left[ 1 + \eta_i \lambda_i e^{-\beta \sqrt{p^2 + m_i^2}} \right]^\eta_i,$$

where $\eta$ is the statistics factor: $\eta_i = 1$ for fermions and $\eta_i = -1$ for bosons. Now we can write the mean particle number as

$$\langle N_i \rangle = g_i V \frac{V}{2\pi^2} \int_0^\infty dp \, p^2 \left[ \lambda_i^{-1} e^{\beta \sqrt{p^2 + m_i^2}} + \eta_i \right]^{-1}.$$

In a rare gas (i.e. Boltzmann) limit, which is mostly applied here, we just put the statistics factors $\eta_i = 0$ in particle numbers formula, or let the possible occupation of one particle states be only one to obtain

$$\ln Z_G = \sum_i \lambda_i Z_i^1.$$
3 Exact Strangeness Conservation

Let us consider first a gas composed of neutral kaons and antikaons and request that the overall strangeness be zero. The canonical (with respect to strangeness) partition function is given by

\[ Z_{S=0} = \prod_{i=0}^{\infty} \left( \sum_{n_i=0}^{\infty} \frac{1}{n_i！} e^{-\beta \varepsilon_i n_i} \right) \left( \sum_{\pi_i=0}^{\infty} \frac{1}{\pi_i！} e^{-\beta \bar{\varepsilon}_i \pi_i} \right) \delta_{n_{K^0}, n_{\bar{K}^0}}, \]  

where \( \varepsilon_i \) is the energy and \( n_i \) the number of \( K^0 \)'s in the state denoted by \( i \). The Kronecker delta ensures that the overall strangeness is zero:

\[ n_{K^0} - n_{\bar{K}^0} = \sum_{i=0}^{\infty} (n_i - \pi_i) = 0. \]

By including the \( 1/n_i！ \) and \( 1/\pi_i！ \) one gets the sums over all distinct states. Replacing the sums over single particle levels by the Boltzmannian momentum integrals and using the Dirac representation

\[ \delta(n - m) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha e^{i(n-m)\alpha} \]

of delta function, one obtains the following result

\[ Z_{S=0} = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \exp \left[ \frac{V}{2\pi^2} \int_0^{\infty} dp p^2 e^{-\beta \varepsilon_{K^0} + i\alpha} \right] \times \exp \left[ \frac{V}{2\pi^2} \int_0^{\infty} dp p^2 e^{-\beta \bar{\varepsilon}_{\bar{K}^0} - i\alpha} \right], \]

where \( \varepsilon_{K^0} = \sqrt{p^2 + m_{K^0}^2} \). Applying the notation of single particle partition function [13] gives

\[ Z_{S=0} = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \exp \left( Z_{K^0}^1 e^{i\alpha} + Z_{\bar{K}^0}^1 e^{-i\alpha} \right). \]

By expanding the exponentials in power series it is easy to perform the \( \alpha \)-integration to obtain

\[ Z_{S=0} = \sum_{p=0}^{\infty} \frac{1}{(p!)^2} (Z_{K^0}^1)^p (Z_{\bar{K}^0}^1)^p. \]

This series converges to a modified Bessel function

\[ Z_{S=0} = I_0(2\sqrt{Z_{K^0}^1 Z_{\bar{K}^0}^1}), \]

which is generally defined by

\[ I_\nu(x) = \sum_{l=0}^{\infty} \frac{1}{\pi} \frac{1}{(l + \nu)！} \left( \frac{x}{2} \right)^{2l+\nu}. \]
The generalization of the calculation to a gas containing any particles $i$ carrying strangeness $S_i = 0, \pm 1$ is straightforward. For this case one obtains

$$Z_{S=0} = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \exp\left(\mathcal{N}_1 e^{i\alpha} + \mathcal{N}_{-1} e^{-i\alpha} + \mathcal{N}_0\right),$$

(17)

where $\mathcal{N}_x$ stands for the sum of single particle partition functions of particles having strangeness $S_i = x$:

$$\begin{cases}
\mathcal{N}_1 &= Z^1_{K^+} + Z^1_{K^0} + \ldots + Z^1_{\Lambda^+} + Z^1_{\Sigma^0} + \ldots \\
\mathcal{N}_{-1} &= Z^1_{K^-} + Z^1_{K^0} + \ldots + Z^1_{\Lambda^0} + Z^1_{\Sigma^{-}} + \ldots \\
\mathcal{N}_0 &= Z^1_{\pi^+} + Z^1_{\pi^0} + Z^1_{\pi^-} + Z^1_{\eta} + \ldots + Z^1_{p} + Z^1_{n} + \ldots
\end{cases}$$

(18)

In this case, where the strangeness is being treated canonically and the baryon number and electric charge grand canonically, the Boltzmannian one species partition function is

$$Z^1_{i} = \lambda_B^i \lambda_Q^i g_i \frac{V}{2\pi^2} \int_0^{\infty} dp p^2 e^{\beta \sqrt{p^2 + m^2}}.$$

(19)

Performing the power expansion of the exponentials and the integration in eq. (17) we are left with

$$Z_{S=0} = Z_0 \sum_{p=0}^{\infty} \frac{1}{p!} (\mathcal{N}_1 \mathcal{N}_{-1})^p = Z_0 I_0(2 \sqrt{\mathcal{N}_1 \mathcal{N}_{-1}}),$$

(20)

where $Z_0$ is the grand canonical partition function for particles having strangeness zero. To calculate the average abundancies of particles, we substitute the fictitious strangeness fugacity and derive

$$\langle N_i \rangle = \lambda_i \frac{\partial \ln Z_{S=0}}{\partial \lambda_i} \bigg|_{\lambda_S = 1},$$

(21)

which gives

$$\langle N_i \rangle = Z^1_{i} \left(\frac{\mathcal{N}_1}{\mathcal{N}_{-1}}\right)^{S_i} \frac{I_{S_i}(2 \sqrt{\mathcal{N}_1 \mathcal{N}_{-1}})}{I_0(2 \sqrt{\mathcal{N}_1 \mathcal{N}_{-1}})}.$$  

(22)

Each term in the sum of eq. (20) is the product of a strangeness plus one and a strangeness minus one particle and one sees the exact strangeness conservation explicitly at work. Due to the strict conservation, the number of strange particles increases nonlinearly with volume, which is illustrated in Fig. (1).

The method used above and the expression of the partition function $Z_{S=0}$ indicate that to impose the strict strangeness conservation, one projects the
Fig. 1. Nonlinear volume dependence of strange particle abundance in canonical treatment (see eq. 22) compared to grand canonical case (X ≡ 1).

grand canonical partition function $Z_G(T, \lambda_B, \lambda_Q, \lambda_S)$ onto the state with strangeness $S$,

$$Z_S = \frac{1}{2\pi} \int_0^{2\pi} d\alpha e^{-iS\alpha} Z_G(T, \lambda_B, \lambda_Q, \lambda_S), \quad (23)$$

where the fugacity factor $\lambda_S$ has been replaced by $\lambda_S = e^{i\alpha}$. (24)

The partition function for a gas containing particles with strangeness $S_i = 0, \pm 1, \pm 2$ is given by

$$Z_S = \frac{Z_0}{2\pi} \int_0^{2\pi} d\alpha e^{-iS\alpha} \exp \left\{ N_1 e^{i\alpha} + N_{-1} e^{-i\alpha} \right\}$$

$$+ N_{2\alpha}(\lambda_B e^{-2i\alpha} + \lambda_B^{-1} e^{2i\alpha}) + N_{2\alpha}^{-}(\lambda_B \lambda_Q e^{-i2\alpha} + \lambda_B^{-1} \lambda_Q e^{i2\alpha}) \right\}, \quad (25)$$

where the sums $N_{2\alpha}$ and $N_{2\alpha}^{-}$ include also the heavier resonances carrying the same quantum numbers, as the $N_{\pm 1}$ do. Using the generating function for modified Bessel functions $I_\nu$ defined by

$$\exp \left\{ \frac{\rho}{2} \left( t + \frac{1}{t} \right) \right\} = \sum_{\nu=-\infty}^{\infty} I_\nu(\rho)t^\nu \quad (26)$$

and expanding again the exponentials in power series we obtain

$$Z_S = \frac{Z_0}{2\pi} \int_0^{2\pi} d\alpha e^{-iS\alpha} \sum_{m=-\infty}^{\infty} I_m(2N_{2\alpha}) \lambda_B^m e^{-i2m\alpha} \quad (27)$$
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\[ \times \sum_{n=-\infty}^{\infty} I_n(2N_{\Xi^-}) \lambda_B^{n} \lambda_Q^{-n} e^{-2i\alpha} \sum_{p=0}^{\infty} \frac{1}{p!} N_p^p e^{i\alpha} \sum_{q=0}^{\infty} \frac{1}{q!} N_q^q e^{-i\alpha}. \]

Carrying out the integration and rearranging the summations the result can be expressed in terms of \( I_n \)-functions:

\[ Z_S = Z_0 \sum_{m=-\infty}^{\infty} I_m(2N_{\Xi^-}) \lambda_B^m \sum_{n=-\infty}^{\infty} I_n(2N_{\Xi^-}) \lambda_B^n \lambda_Q^{-n} \]
\[ \times \sum_{p=0}^{\infty} \frac{1}{p!} (-S + p - 2m - 2n)! \]
\[ \times (\sqrt{N_1 N_{-1}})^{-S+2p-2m-2n} \left( \frac{N_1}{N_{-1}} \right)^{S+m+n} \]
\[ = Z_0 \sum_{m=-\infty}^{\infty} I_m(2N_{\Xi^-}) \lambda_B^m \sum_{n=-\infty}^{\infty} I_n(2N_{\Xi^-}) \lambda_B^n \lambda_Q^{-n} \]
\[ \times \left( \sqrt{\frac{N_1}{N_{-1}}} \right)^{S+2m+2n} I_{S+2m+2n}(2\sqrt{N_1 N_{-1}}). \] (28)

Using the same techniques the result can be generalized to the case, where the \( \Omega^- \)-like hadrons (\( S_1 = \pm 3 \)) are included as well:

\[ Z_S = Z_0 \sum_{m=-\infty}^{\infty} I_m(2N_{\Xi^-}) \lambda_B^m \sum_{n=-\infty}^{\infty} I_n(2N_{\Xi^-}) \lambda_B^n \lambda_Q^{-n} \]
\[ \times \sum_{l=-\infty}^{\infty} I_l(2N_{\Omega^-}) \lambda_B^l \lambda_Q^{-l} \]
\[ \times \left( \sqrt{\frac{N_1}{N_{-1}}} \right)^{S+2l+2m+2n} I_{S+2m+2n+3l}(2\sqrt{N_1 N_{-1}}). \] (29)

The mean number of hadrons \( i \) with strangeness \( S_i \) in the system becomes

\[ \langle N_i \rangle = Z_i^{1} \frac{Z_{S-S_i}}{Z_{S}} \left( \sqrt{\frac{N_1}{N_{-1}}} \right)^{-S_i}. \] (30)

The modified Bessel functions decrease quickly with increasing absolute value of their indices, so the numerical evaluation of mean particle numbers is not cumbersome.

3.1 Application of \( Z_S \)

In a recent paper [1] we have analysed the particle production in GSI SIS Ni+Ni experiments. We addressed especially the abundance of kaons who
can not be described by hadronic gas model in its standard form. Although the size of the Ni system is relatively large the corrections due to exact strangeness conservation turned out to be crucial at low temperatures, \( T < 100 \text{ MeV} \), involved. At these temperatures the width of resonances had to be taken into account. A summary of our results for particle ratios is presented in table 1.

**Table 1.** Particle ratios given by present model compared to experimental results. The best fit value, \( \mu_B = 0.72 \text{ GeV} \), for the baryon chemical potential is used.

| Ratio         | Model \( R = 4.2 \text{ fm} \) | Model \( R = 3 \text{ fm} \) | Data \( T = 65 \text{ MeV} \) | Data \( T = 75 \text{ MeV} \) | \( \delta \) |
|--------------|-------------------------------|-------------------------------|--------------------------|--------------------------|------|
| \( K^+/K^- \) | 25.7                         | 22.4                         | 23.9                     | 21.1                     | 21 \pm 9 |
| \( K^+ / \pi^+ \) | 0.0071                       | 0.0339                       | 0.0027                   | 0.0132                   | 0.0074 \pm 0.0021 |
| \( \phi / K^- \) | 0.103                        | 0.082                        | 0.276                    | 0.212                    | 0.1 |
| \( \pi^+ / \pi^- \) | 0.893                        | 0.895                        | 0.894                    | 0.898                    | 0.89 |
| \( \eta / \pi^0 \) | 0.008                        | 0.015                        | 0.008                    | 0.015                    | 0.037 \pm 0.002 |
| \( \pi^+ / p \) | 0.225                        | 0.247                        | 0.224                    | 0.246                    | 0.195 \pm 0.020 |
| \( \pi^0 / B \) | 0.104                        | 0.108                        | 0.104                    | 0.107                    | 0.125 \pm 0.007 |
| \( d / p \) | 0.129                        | 0.188                        | 0.129                    | 0.188                    | 0.26 |

The measured hadronic ratios with corresponding errorbars are described as bands in the \((T, \mu_B)\) plane as shown in Fig. 1. The intervals of temperature and of chemical potential

\[
T = 70 \pm 10 \text{MeV} \\
\mu_B = 720 \pm 30 \text{MeV}
\]

give a good fit to the data. The freeze-out radius of \( R \simeq 4 \text{ fm} \) was extracted from the volume dependence of the ratios \( K^+ / \pi^+ \) and \( \phi / K^- \).

### 4 Exact Baryon, Charge and Strangeness Conservation

In the case of three exactly conserved, additive quantum numbers we start from the single particle partition function of particle \( i \),

\[
Z_i = \sum_j g_j \frac{V}{2\pi^2} \int_0^\infty dp \, p^2 e^{-\beta\varepsilon_i} \delta_{B_i,B} \delta_{Q_i,Q} \delta_{S_i,S}.
\]  

(31)

Making use of the integral representation for \( \delta \)-functions and the overall conservation constraints

\[
\sum B_i = B, \quad \sum Q_i = Q, \quad \sum S_i = S,
\]  

(32)
Fig. 2. Curves in the ($\mu_B, T$) plane corresponding to the hadronic ratios indicated. The interaction volume corresponds to the radius of 4.2 fm, and the isospin asymmetry is $B/2Q = 1.04$.

the resulting integral corresponds to a projection of the grand canonical partition function onto the desired values of $B$, $Q$ and $S$:

$$Z_{B,Q,S} = \frac{1}{(2\pi)^3} \int_0^{2\pi} d\psi e^{-iB\psi} \int_0^{2\pi} d\phi e^{-iQ\phi} \int_0^{2\pi} da e^{-iS\alpha} \times Z_G(T, \lambda_B, \lambda_Q, \lambda_S).$$

(33)

Here the fugacity factors have been replaced by

$$\lambda_B = e^{i\psi}, \quad \lambda_Q = e^{i\phi}, \quad \lambda_S = e^{i\alpha}.$$  

(34)

As the contributions always come pairwise for particles and antiparticles, the fugacity factors will give rise to the cosine of the corresponding angle. It is useful to group the particles according to their quantum numbers. Leaving out charm, bottom and $S_i \geq 2$ particles we are left with ten categories which will be labeled by their particle content. For instance, $N_{K^0}$ stands for the sum of one particle partition functions of neutral strange and antistrange mesons ($K^0$, $\bar{K}^0$, $K^{*0}$, $\bar{K}^{*0}$, ...), $N_{K^\pm}$ stands for charged strange mesons ($K^+$, $K^-$, $K^{*+}$, $K^{*-}$, ...), and $N_A$ stands for all neutral hyperons and antihyperons. With these notations the partition function can be rewritten in the form

$$Z_{B,Q,S} = \frac{1}{(2\pi)^3} \int_0^{2\pi} d\psi e^{-iB\psi} \int_0^{2\pi} d\phi e^{-iQ\phi} \int_0^{2\pi} da e^{-iS\alpha}$$

(35)
\[
\times \exp\left(2N_n \cos \psi + 2N_\pi \cos \phi + 2N_K^0 \cos \alpha + 2N_K^- \cos(\phi + \alpha) + 2N_p \cos(\psi + \phi) + 2N_{\Delta -} \cos(\psi - \phi) + 2N_{\Delta^+} \cos(\psi + 2\phi) + 2N_A \cos(\psi - \alpha) + 2N_{\Sigma^+} \cos(\psi + \alpha) + 2N_{\Sigma^-} \cos(\psi - \phi - \alpha)\right).
\]

The integration above can not be done directly due to cosine terms of multiple angles. To circumvent this difficulty, we introduce a new angle whenever more than one appears. For example, in the term involving \(N_p\) we introduce an intermediate angle \(\xi\) in the following way

\[
1 = \int_0^{2\pi} d\xi \delta(\psi + \phi - \xi) = \sum_{\nu = -\infty}^{\infty} \frac{1}{2\pi} \int_0^{\pi} d\xi e^{i\nu(\psi + \phi - \xi)}. \tag{36}
\]

The application of the integral representation of the modified Bessel function,

\[
I_n(z) = \frac{1}{\pi} \int_0^{\pi} d\omega e^{z \cos \omega} \cos n\omega, \tag{37}
\]

allows one to write the partition function in the form

\[
Z_{B,Q,S}(T,V) = Z_0 \left( \prod_{\nu = 1}^{7} \sum_{n_\nu = -\infty}^{\infty} \right)
\times I_{-B+n_2+n_3+n_4+n_5+n_6+n_7}(2N_n)
\times I_{-Q+n_1+n_2-n_3+n_5-n_6+2n_7}(2N_\pi)
\times I_{-S+n_1-n_4-n_5-n_6}(2N_K^0)
\times I_{n_1}(2N_{K^+}^r)I_{n_2}(2N_p)I_{n_3}(2N_{\Delta^-})
\times I_{n_4}(2N_A)I_{n_5}(2N_{\Sigma^+})I_{n_6}(2N_{\Sigma^-})I_{n_7}(2N_{\Delta^+}). \tag{38}
\]

The differentiation of eq. (33) for particle abundances yields the result

\[
\langle N_i \rangle = \frac{Z_{B-B,Q-Q,S-S,S_i}}{Z_{B,Q,S}}. \tag{39}
\]

The evaluation of the canonical partition function with three simultaneously conserved quantum numbers becomes numerically very time consuming for large values of \(B\). So far, for systems with \(B > 20\) we have been forced to resort to the grand canonical treatment.

### 4.1 Application of \(Z_{B,Q,S}\)

In order to compare our numerical results with the E802 experimental data shown in Table 2, we estimate the baryon number and charge of the experimental system via geometrical considerations. Letting \(R_P\) and \(R_T\) be the radius of a projectile and target nucleus respectively, we assume that the radii are directly proportional to the cubic roots of the mass numbers \(A_P\)
and $A_T$ of the interacting nuclei. In the case of central collisions the interaction region is taken to be a cylinder of radius $R_P$, length $2\sqrt{R_P^2 - R_T^2}$ plus, two remaining spherical segments at the ends of the cylinder with $R_T$ as the radius. We find the total number of participating nucleons (or, baryon number, $B$) and the total charge $Q$ to be

$$B = A_P + A_T \left\{ 1 - \left[ 1 - \left( \frac{A_P}{A_T} \right)^{2/3} \right]^{3/2} \right\}$$

(40)

$$Q = Z_P + Z_T \left\{ 1 - \left[ 1 - \left( \frac{A_P}{A_T} \right)^{2/3} \right]^{3/2} \right\}$$

(41)

respectively.

The comparison of our numerical results with the E802 data (Table 2) of the $K^+/{\pi}^+$ and $K^-/{\pi}^-$ ratios as functions of the size of the system or, equivalently, $B$ are shown in Fig. 3. In the upper figure we observe that the theoretical curve for $B/2Q = 5/6$ approximates the $K^+/{\pi}^+$ data best. The theoretical curves lie systematically above the data but drop closer as $B/2Q$ decreases towards the collision value. The effect of the isospin asymmetry of the system is seen also in the $K^-/{\pi}^-$ data comparison. As the ratio $B/2Q$ approaches the collision value the theoretical curves begin to approximate the data more closely.

**Table 2.** Experimental results reported by the E802 collaboration. $B$ and $Q$ are calculated using equations 40 and 41.

| Collision | $K^+/{\pi}^+$ | Ref. | $K^-/{\pi}^-$ | Ref. | $B$ | $Q$ |
|-----------|---------------|------|---------------|------|-----|-----|
| $p + ^4Be_9$ | 7.8$\pm$0.4% | [11,12] | 2.0$\pm$0.2% | [11] | 3.9 | 2.3 |
| $p + ^{13}Al_{27}$ | 9.9$\pm$0.5% | [12] | 5.4 | 3.1 |
| $p + ^{30}Cu_{64}$ | 10.8$\pm$0.6% | [12] | 6.9 | 3.7 |
| $p + ^{79}Au_{197}$ | 12.5$\pm$0.6% | [11,12] | 2.8$\pm$0.3% | [11] | 9.7 | 4.5 |
| $^{14}Si_{28} + ^{79}Au_{197}$ | 18.2$\pm$0.9% | [12] | 3.2$\pm$0.3% | [11] | 102.7 | 44.0 |
| 19.2$\pm$3% | [13] | 3.6$\pm$0.8% | [13] | |

5 Generalization of the Projection Method

In this section, we review the projection method generalized to arbitrary internal symmetry of the system in addition to $U(1)$ of strangeness and $U(1)\times U(1)\times U(1)$ of baryon number, electric charge and strangeness. For complete derivation, see the original texts of Turko, Redlich and Hagedorn [14].
Fig. 3. Thermal model expectations for the production ratios $K^+/\pi^+$ and $K^-/\pi^-$ at a temperature of 100 MeV and a baryon density of 0.04 fm$^{-3}$ compared to experimental results from the Brookhaven AGS. The experimental ratios from $Si-Au$ collisions ($B \sim 10^3$) is moved to $B = 21$ for the sake of convenience.

A general method is suitable for non-abelian symmetries, such as $SU(2)$ of isospin [13] or angular momentum [14], and $SU(3)$ of color [17] as well.

If the system is exactly symmetric under the operations of internal symmetry group $G$, the corresponding group generators $Q_k$ have the same eigenstates as the Hamiltonian. Thus

$$[\hat{H}, Q_k] = 0, \ k = 1, \ldots, n,$$

(42)
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where \( n \) is the number of parameters in the group. Let us define the generating function \( \hat{Z} \) as \( \hat{Z} = \text{tr}[U(g)e^{-\beta H}] \), where \( U(g) \) is an unitary representation of the group. With the aid of irreducible presentations of \( U \), this decomposes to

\[
\hat{Z} = \sum_{\nu} \frac{\hat{\chi}_\nu(g)}{d(\nu)} Z_\nu. \tag{43}
\]

Here we used a character \( \hat{\chi}_\nu(g) \) and the dimension \( d(\nu) \) of an irreducible presentation \( U_\nu(g) \), and the corresponding canonical partition function \( Z_\nu \).

Using the orthogonality of characters,

\[
\int d\mu(g) \hat{\chi}_\nu(g) \hat{\chi}_{\nu'}(g) = \delta_{\nu\nu'}, \tag{44}
\]

we may compute the canonical partition function once we know the generating function:

\[
Z_\nu = d(\nu) \int d\mu(g) \hat{\chi}_\nu(g) \hat{Z}. \tag{45}
\]

Further investigation of the generating function reveals that

\[
\hat{Z} = \text{tr} \exp \left( -\beta \hat{H} + i \sum_{k=1}^{r} Q_k \gamma_k \right) = \prod_{j=1}^{\infty} \prod_{\rho=1}^{d(\nu)} \sum_{n} \exp \left[ n \left( -\beta E_j + i \sum_{k=1}^{r} q^{(\rho)}_k \gamma_k \right) \right]. \tag{46}
\]

In the last step, we have expressed the trace in the basis of \( n \)-particle Hamiltonian eigenstates. The \( q^{(\rho)}_k \) are the conserved charges, and the \( \gamma_k \) are the variables of the Cartan subgroup of the group \( G \) of rank \( r \). Eq. (46) resembles the grand canonical partition function, and is actually obtained from it by the Wick rotation: \( \beta \mu_i \to -i\gamma_i \).

As an example, let us choose the internal symmetry of the system correspond to \( U(1)_{q_1} \times \cdots \times U(1)_{q_r} \), where the \( q_i \) are the conserved charges. The character of \( U(1)_{q_i} \) is \( e^{i\eta_i \gamma_i} \), so the character of the direct product group is \( \exp(i \sum_{i=1}^{r} q_i \gamma_i) \). The canonical partition function respecting the exact conservation of charges \( q_i \) has now the form

\[
Z_{q_1, \ldots, q_r}(T, V) = \frac{1}{(2\pi)^r} \int_{0}^{2\pi} d\gamma_1 \cdots \int_{0}^{2\pi} d\gamma_r \times \exp \left[ -i \sum_{i=1}^{r} q_i \gamma_i \right] \hat{Z}(T, V, \gamma_1, \cdots, \gamma_r). \tag{47}
\]

The special cases, \( Z_S \) and \( Z_{B,Q,S} \) for a Boltzmannian hadron resonance gas are considered in previous sections.
6 Summary

The particle abundances have been computed in the canonical formalism using the formulation for the exact conservation of baryon number, strangeness and charge in the thermal model of particle production. A good agreement with the experimental data of GSI Ni+Ni collisions and of E802 collaboration in p−A collisions was reported.

The good agreement with chemical equilibrium does not mean that the particle spectra should follow exactly a Boltzmann distribution since the momenta of particles can be severely affected by flow. As an example, a model with Bjorken expansion in the longitudinal direction will still have its particle ratios determined by Boltzmann factors even though the longitudinal distribution is nowhere near a Boltzmann distribution [18].

References

1. J. Cleymans, D. Elliott, A. Keränen and E. Suhonen, Phys. Rev. C57 (1998) 3319
2. KaoS Collaboration, P. Barth et al., Phys. Rev. Lett. 78 (1997) 4007
3. KaoS Collaboration, H. Oeschler, ‘‘Hadrons in Dense Matter and Hadrosynthesis‘‘, these proceedings.
4. FOPI Collaboration, Y. Leifels : ‘‘FOPI Results - Strangeness in 4π.’’ Talk presented at the International Workshop : ‘‘Hadrons in Dense Matter,’’ GSI, Darmstadt, July 2-4, 1997
5. TAPS Collaboration, M. Appenheimer et al., GSI 97-1, page 58; R. Averbeck, ‘‘Hadronische Materie bei SIS-Energien - Eine Thermodynamische Analyse.’’ (unpublished), presented at the DPG Frühjahrstagung, Göttingen, February 1997
6. FOPI Collaboration, D. Best et al., Nucl. Phys. A625 (1997) 367
7. FOPI Collaboration, N. Herrmann, Nucl. Phys. A610 (1996) 49c
8. FOPI Collaboration, D. Pelte et al. , Z. Phys. A359 (1997) 55
9. KaoS Collaboration, C. MüNZ et al. , Z. Phys. A352 (1995) 175; Z. Phys. A357 (1997) 399
10. J. Cleymans, A. Keränen, M. Marais and E. Suhonen, Phys. Rev. C56 (1997) 2474
11. T. Abbot et al (E802 Collaboration), Phys. Rev. Lett. 66 (1991) 1567
12. T. Abbot et al (E802 Collaboration), Phys. Rev. D45 (1992) 3906
13. T. Abbot et al (E802 Collaboration), Phys. Rev. Lett. 64 (1990) 847
14. K. Redlich and L. Turko, Z. Phys. C5 (1980) 201; L. Turko, Phys. Lett. B104 (1981) 153; R. Hagedorn and K. Redlich, Z. Phys. C27 (1985) 541
15. B. Müller and J. Rafelski, Phys. Lett. B116 (1982) 274
16. W. Bliem, P. Koch and U. Heinz, Z. Phys. C63 (1994) 637
17. Th. Elze and W. Greiner, Phys. Rev. A33 (1986) 1879
18. J. Cleymans, 3rd International Conference on “Physics and Astrophysics of Quark-Gluon Plasma”, Jaipur, India, March 17 - 21, 1997 to be published, nucl-th/9704046