SCHRODINGER EQUATION APPROACH TO NON-LINEAR
σ-MODELS IN THE LARGE N-LIMIT

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Abstract

Non-linear $d$-dimensional vector $\sigma$-models are studied in the large $N$-limit. It is found that a two-point correlation function obeys a standard Schrödinger equation for a free quantum particle moving in the $\delta$-function quantum well. The threshold problem for bound states in this equation is shown to be equivalent to a critical behavior of these models above and below the Curie point.

The $SU(N)$-symmetric Ginzburg-Landau (GL) $\sigma$-model subject to a uniform magnetic field $H$ is considered in the large-$N$ limit within the Schrödinger equation approach. A upper critical magnetic field line $H_{c2}(T)$ of type-II superconductors for an arbitrary external $H$ is obtained without exploiting the lowest Landau level (LLL) approximation. Both low-$H$ perturbation expansion terms and exponentially small corrections to the LLL approximation are calculated.

Correspondences between the one-particle quantum mechanics and critical phenomena as well as some applications of the above method to other models of statistical mechanics are also discussed.

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I. INTRODUCTION

Non-linear vector chiral models such as the $CP^N$ and $O(N)$, $SU(N)$-symmetric ones with a global non-abelian symmetry are known to possess some remarkable features, in particular, asymptotic freedom and dynamical mass generation. These models are an ideal theoretical laboratory for researchers because they capture all the essentials of critical phenomena (see [1–6] and references therein).

The standard dynamical approach for treating the models mentioned in the large-$N$ limit is based on the non-linear Dyson-Schwinger (DS) equations [7]. In this limit the Hartree-Fock approximation is known to be exact and can be readily solved. That also holds, for instance, for the $0(N)$-symmetric $\Phi^4$-theory and the Gross-Neveu model at $N = \infty$ [4,5,7].

In this paper we will show that in contrast to the DS approach the two-point correlation function $\langle S^*_a(\vec{x}) S_b(0) \rangle$ of the $O(N)$, $SU(N)$, $CP^N$ models in the large-$N$-limit and in the symmetric phase obeys a conventional Schrodinger equation for a free quantum particle moving in a potential $(-T)\delta(\vec{x})$ with $T$ being a temperature.

All the well-known results can be obtained in this way. In other words, statistical mechanics and the elementary quantum mechanics are shown to be somewhat closely related subjects, and the relationship will be developed here via the Schrodinger equation approach.

It is curious that the Schrodinger equation under discussion is just the very same toy model discovered long ago for describing dimensional transmutation phenomena in the non-relativistic quantum mechanics (see the textbook [8]). As a result, one obtains an amusing physical interpretation of dynamical mass generation, critical phenomena, etc. which take place in non-linear vector $\sigma$-models in terms of quantum mechanics.

Being almost evident from the physical standpoint, nevertheless, it is somewhat curious to give a precise formulation of correspondences between quantum field theory and quantum mechanics.

It should be noted that some aspects of this topic have been already discussed in literature. In the recent paper [9] (see also references therein) the critical behavior of the binding energy and of a radius of the bound state $\xi$ near the threshold were considered. It was also found that the critical exponent $\nu$ of the correlation length $\xi$ is coincided to that of the spherical model. The present paper is aimed to develop the approach presented in [9] and to provide a deeper insight into correspondences between one-particle quantum mechanics and critical phenomena.

There is a wide held opinion among experts in the field that a renormalization-group (RG) approach can be applied only for systems with an infinite number of degrees of freedom, which lead to ultraviolet singularities. Since quantum mechanics for a free particle doesn’t exhibit ultraviolet singularities, hence the RG method does not work.

That is wrong because there is a good number of singular potentials in quantum mechanics which require a short-distance regularization, say, the $\delta$-function potential [8]. Moreover, as we will see, just this potential leads to the $\beta$-function coinciding to those of the non-linear $\sigma$-models in the large-$N$ limit.

Next we study the $SU(N)$-symmetric Ginzburg-Landau (GL) model subject to a uniform magnetic field. This system is of great interest in both statistical and condensed matter physics because it is related to the long standing and challenging problem of the critical
behavior of a type-II superconductor near the upper critical magnetic field $H_{c2}(T)$. This problem having a rich and long history has been much discussed in literature (10).

In brief, it is as follows (11). It has been recognized for some time that an external magnetic field drastically changes the critical properties of superconductors. The magnetic field hinders the growth of the thermal fluctuations in the plane perpendicular to $H$, since the growth of their correlation length is restricted by the magnetic length scale $\ell_H = \sqrt{hc/eH}$ which is much shorter than the coherence length $\xi$. This effect of dimensional reduction results in an enhancement of the longitudinal fluctuations leading, in particular, to the increase of the lower critical dimension from 2 to 4.

If critical fluctuations are ignored, the uniform frustration (in the language of spin models of the vortex lattice) eventually leads to a continuous phase transition into the Abrikosov flux lattice state. On the one hand, in contrast to mean-field theory, the standard renormalization group (RG) approach in $6-\epsilon$ dimensions, in fact, failed so far to yield insight on the nature of the phase transition due to the appearance of an infinite number of invariant charges (relevant scaling variables) inherent to the non-renormalizable scalar $\phi^4$ field theory in a field (12,13).

Physical arguments (presented e.g. in [10]) support the existence of a first-order melting transition for the flux lattice. The conventional $1/N$-expansion, when applied to the $SU(N)$-symmetric Ginzburg-Landau (GL) model with an $N$-component order parameter, also gives a first-order phase transition above four dimensions (14).

On the other hand, however, the results obtained within the framework of the $1/N$-expansion are somewhat controversial because in more recent studies a continuous phase transition has been obtained (11,13,16).

The well-known peculiarity of the $1/N$-expansion is the symmetry breaking between particle-hole and particle-particle channels leading to suppressing the p-p channel and to an unstable spectrum of fluctuations around the Abrikosov flux solution (17).

In order to avoid these difficulties and to restore the superconducting phase transition it was recently proposed to modify the quartic term in the Ginzburg-Landau Hamiltonian. Namely, according to (17) the conventional $SU(N)$-symmetric interaction $(\phi^*_a \phi_a)^2$ should be replaced by $2(\phi^*_a \phi_a)^2 - \phi^*_a \phi_b \phi^*_b \phi_a$ being $O(N) \times U(1)$-symmetric.

The question of the nature of the phase transition from the normal into the mixed state of a type-II superconductor not discussing here remains therefore an open problem (for further discussion, see [17]), even before the effects of impurities and the topical question of the vortex glass are to be considered.

It should be also mentioned that in lattice models of superconductors the interaction between thermal fluctuations and the underlying crystal lattice, i.e. when a periodic lattice potential is coupled to the superconducting order parameter, can restore the phase transition into the mixed state (13,16,18–20).

The essential ingredient of all the above papers is making use of the lowest level Landau (LLL) approximation. From a formal point of view this approach describes a phase diagram of a type-II superconductor near the upper critical magnetic field $H_{c2}(T)$, at the best, only for an astronomically large magnetic field as in a vicinity of the neutron star $H \sim H_0 = \Phi_0/\alpha \sim 10^5 T$ with $\Phi_0 = hc/2e$ being the magnetic flux quantum and $\alpha$ being a lattice spacing. In this unphysical (so-called "ultraquantum") limit one has $\ell_H \ll a \ll \xi$. The problem is
whether results obtained within the LLL approximation hold for any $H$ or not.

In this paper we wish to obtain the equation of the phase transition line $H_{c2}(T)$ in the $SU(N)$-symmetric model in the large-$N$ limit for an arbitrary magnetic field without using the LLL approximation. The Schrodinger equation approach developed below will be exploited.

This paper is organized as follows. In Section II we consider the $d$-dimensional $SU(N)$-symmetric non-linear $\sigma$-model in the large-$N$-limit and the conventional Schrodinger equation for the two-point spin correlation functions is derived. Section III develops a RG approach for a free quantum particle moving in the $\delta$-function quantum well. Correspondences between statistical mechanics and quantum mechanics are discussed. Section IV contains a treatment of the GL model subject to a uniform magnetic field. The discussion of some other models and of related issues as well as some concluding remarks are contained in Sec.V.

II. SCHRODINGER EQUATION FOR THE TWO-POINT CORRELATION FUNCTION

We begin with the $d$-dimensional lattice $SU(N)$-symmetric spin model, described by the Hamiltonian

$$H = -J \sum_{<i,j>} [\vec{S}_i \vec{S}_j^* + h.c.]$$

(2.1)

with $\vec{S} = (S_1,...S_N)$ being a $N$-component complex unit vector with the fixed-length constraint

$$\vec{S}(\vec{x})\vec{S}(\vec{x})^* = 1$$

(2.2)

imposed on spins. Here $<i,j>$ indicates that the summation is over all nearest-neighboring sites; $J$ being a spin coupling constant.

The standard GL model is a continuous version of the lattice model Eq.(2.1) obtained after taking an appropriate continuum limit

$$H = \frac{J}{2} \int d^d x |\partial_{\mu} S_a|^2$$

(2.3)

where the summation over $\mu = 1,...,d$ is understood. The partition function associated with eq.(2.3) reads

$$Z = \int \prod_{a=1}^N DS_a DS_a^* \exp(-\frac{H}{T})\delta(|\vec{S}|^2 - 1)$$

(2.4)

The temperature $T$ has been rescaled so as to include $J$ into the coupling constant $T$. Local fields $S_a(\vec{x}), a = 1,...N$ are not free, because of the constraint Eq.(2.2).

Now I am concerned with an equation for the two-point correlation function in the disordered phase $T > T_c$ with the $SU(N)$-symmetry being unbroken.

$$G_{ab}(\vec{x}) = <S_a(\vec{x})S_b^*(0)>$$

(2.5)
Following the standard textbook [5], one makes use of the well-known representation for Eq.(2.5) in terms of path integral

\[ G_{ab}(\vec{x}) = \frac{1}{Z} \int \prod_{a=1}^{N} DS_{a}D\lambda S_{a}(\vec{x})S_{b}^{*}(0) \exp[-A_{1}(\vec{S}(\vec{x}), \lambda(\vec{x}))] \]

\[ Z = \int \prod_{a=1}^{N} DS_{a}D\lambda \exp[-A_{1}(\vec{S}(\vec{x}), \lambda(\vec{x}))] \]

\[ A_{1}(\vec{S}(\vec{x}), \lambda(\vec{x})) \equiv \frac{1}{2T} \int d^{d}x [\partial_{\mu}S_{a}]^{2} + \lambda(|\vec{S}|^{2} - 1)] \quad (2.6) \]

with \( A_{1}(\vec{S}(\vec{x}), \lambda(\vec{x})) \) being an effective action and \( \lambda(\vec{x}) - \) a Lagrange multiplier serving to enforce the constraint Eq.(2.2).

Integrating out fields \( \vec{S}(\vec{x}) \), one arrives at the equation [6]

\[ G_{ab}(\vec{x}) = \frac{1}{Z} \int D\lambda G(\vec{x}, \vec{y}; \lambda) \exp[-A_{2}(\lambda(\vec{x}))] \]

\[ Z = \int D\lambda \exp[-A_{2}(\lambda(\vec{x}))] \]

\[ A_{2}(\lambda(\vec{x})) = \frac{1}{2T} \int d^{d}x [\lambda(\vec{x}) - \frac{N}{2} tr \log(-\Delta + \lambda(\vec{x}))] \]

\[ \Delta \equiv \partial_{\mu}^{2} \quad (2.7) \]

where

\[ G(\vec{x}, \vec{y}; \lambda) = \langle \vec{y}| \frac{1}{-\Delta + \lambda} |\vec{x} \rangle \quad (2.8) \]

If a product \( TN \) remains finite in the large-\( N \) limit \( N \to \infty \), the path integrals in Eq.(2.7) can be easily evaluated by means of the steepest descend method [5,6]

\[ \langle S_{a}(\vec{x})S_{b}^{*}(0) \rangle = T\delta_{ab}G(\vec{x}, \vec{y}; \lambda_{0}) \quad (2.9) \]

with \( \lambda_{0} \) being a saddle point. The equation for the two-point Green’s function under discussion acquires a simple form [5,6,8].

\[ [-\Delta + m^{2}]G_{ab}(\vec{x}) = T\delta_{ab}\delta(\vec{x}) \quad (2.10) \]

Here \( m^{2} \equiv \lambda_{0} \). The inhomogeneous equation Eq.(2.10) can be readily transformed into a homogeneous one by making use of the constraint Eq.(2.2). Bearing in mind that \( G_{cc}(0) = 1 \) the r.h.s of Eq.(2.10) can be rewritten as \( T\delta_{ab}\delta(\vec{x})G_{cc}(\vec{x}) \). As a result, this procedure yields the following equation

\[ [-\Delta + m^{2}]G_{ab}(\vec{x}) = T\delta_{ab}\delta(\vec{x})G_{cc}(\vec{x}) \quad (2.11) \]

Here the Einstein summation convention is assumed.

In the symmetric phase for \( T > T_{c} \) it is possible to introduce an "effective wave function" \( G_{ab}(\vec{x}) \equiv \delta_{ab}\Psi(\vec{x}) \). Substituting this in Eq.(2.11) we obtain the equation
\[ \hat{H} \Psi(\vec{x}) = -|E| \Psi(\vec{x}) \]  
(2.12)

where the Hamiltonian \( \hat{H} \) is given by

\[ \hat{H} = -\Delta - TN\delta(\vec{x}); \quad |E| = m^2 \]  
(2.13)

Eq. (2.12) is the standard Schrödinger equation for a free quantum particle moving in the \( \delta \)-function quantum well, moreover the one-particle Green function under discussion corresponds to the lowest level wave function of the Hamiltonian Eq. (2.13).

It is worth noting that Eq. (2.13) results from a double scaling limit procedure applied to the model under discussion Eq. (2.1), namely, the continuum limit and the large-\( N \) limit.

The constraint Eq. (2.2) giving rise to the potential energy in Eq. (2.13) may be regarded as a boundary condition on \( \Psi(\vec{x}) \).

It is worth noting that the sign of \( T \) in the Hamiltonian Eq. (2.13) corresponding to the attraction, results from a compactness of the underlying manifold (the sphere \( S^{2N-1} = SU(N)/SU(N-1) \) in our case). In turn, it implies that the quantum well has the bound state.

To avoid confusion let’s notice that the Schrödinger equation Eq. (2.12) doesn’t describe a bound state of two particles from the fundamental \( N \)-plet of our model, as it could be seemed at the first sight, because we consider only the one-particle Green function Eq. (2.5).

III. ONE-PARTICLE QUANTUM MECHANICS AND CRITICAL PHENOMENA

In this Section we will consider the RG approach in the context of the Schrödinger equation with the \( \delta \)-function potential and will give precise correspondences between phase transitions in statistical mechanics and the threshold phenomena in quantum mechanics in spirit of [8].

At the outset let’s find eigenfunctions and eigenvalues of the Hamiltonian \( \hat{H} \) Eq. (2.13) belonging to the discrete spectrum.

Suppose for concreteness that \( d = 2 \). The above Hamiltonian \( \hat{H} \) being scale invariant, doesn’t contain a mass parameter. Despite this, the only bound state appears, its wave function and energy value \( |E_{bs}| \equiv m^2 \) can be found by applying the Fourier transform to Eq. (2.12)

\[ [k^2 + m^2]\psi(\vec{k}) = TN\Psi(0) \]  
(4.1)

where

\[ \psi(\vec{k}) = \int d^d x \Psi(\vec{x}) \exp(-i\vec{k}\cdot\vec{x}) \]

\[ \Psi(\vec{x}) = \int \frac{d^d x}{(2\pi)^d} \psi(\vec{k}) \exp(i\vec{k}\cdot\vec{x}) \]  
(4.2)

From Eq. (4.1) it follows that

\[ \Psi(0) = TN\Psi(0) \int \frac{d^2 k}{(2\pi)^2(k^2 + m^2)} \]  
(4.3)
This relation links the energy of the bound state $m^2$ and the coupling constant $T$.

$$1 = TN \int \frac{d^2k}{(2\pi)^2(k^2 + m^2)}$$ (4.4)

The integral in the r.h.s. of Eq.(4.4) is logarithmically divergent. This reflects the fact that $\delta(\vec{x})$ is the so-called ”singular” potential. Likewise quantum field theory it requires introducing a ultraviolet cutoff $\Lambda = a^{-1}$ with $a$ being a width of the quantum well.

One is led to the equation

$$1 = \frac{TN}{2\pi} \log \frac{\Lambda}{m}$$ (4.5)

The energy of the bound state is a physical observable quantity and shouldn’t depend on the cutoff. It implies that the coupling constant acquires the cutoff dependence $T(\Lambda)$ in order to keep $E$ $\Lambda$-independent.

In fact, we can treat Eq.(4.3) as a simplest example of the isospectral deformation, i.e. the transform of a potential energy which doesn’t change the energy spectrum.

It is well known from the theory of solitons and of the inverse scattering transform that in the case of 1D Schrodinger equation there is an infinite dimensional group of isospectral deformations, being isomorphic to a symmetry group of the Korteweg-de-Vries equation (21).

Thus we arrive at the one-loop equation for $T$ being an isospectral deformation in two dimensions

$$\Lambda \frac{dT}{d\Lambda} = -T^2$$ (4.6)

In the $d$-dimensional space one obtains

$$1 = TN\left(\frac{S_d}{(2\pi)^d} \frac{\Lambda^{d-2}}{d-2} - K_d m^{d-2}\right)$$ (4.7)

where

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

$$K_d = \frac{1}{2^d \sin(\pi d/2) \Gamma(d/2) \pi^{(d-2)/2}}$$ (4.8)

where $S_d$ is the area of the unit $d$-dimensional sphere.

It is convenient to introduce a dimensionless coupling constant $t = T\Lambda^{2-d}$. The Gell-Mann-Low equation for $t$ is easily seen to be

$$\Lambda \frac{dt}{d\Lambda} = (d - 2)t - t^2$$ (4.9)

Eq.(4.3) shows that there exists the critical value of coupling constant (the fixed point)
\[ T_c = \frac{(d-2)(2\pi)^d}{NS_d^d} \Lambda^{2-d} \quad (4.10) \]

Now let’s discuss correspondences between the quantum mechanical approach and the statistical physics description. On the one hand, we are dealing with the conventional Schrodinger equation with the singular quantum well. Its solution is well known from the elementary quantum mechanics.

If a depth of this well is small enough: \( T < T_c \), the bound state cannot appear at all. For a deep well \( T > T_c \) there exists the only bound state with \( E_{bs} = -m^2 \). In two dimensions the bound state is known to exist irrespective to a magnitude of the coupling constant \( T > 0 \).

On the other hand, in the language of statistical mechanics and quantum field theory, Eq.(4.6) indicates on asymptotic freedom or dynamical mass generation at \( d = 2 \) and on a continuous phase transition at \( d > 2 \). The ultraviolet stable fixed point of the spherical model \( T_c \) corresponds to a threshold value of the coupling constant in quantum mechanics.

Above \( T_c \) we have the \( N \)-plet of massive scalar bosons, whilst below \( T_c \) the Goldstone massless particles appear, as expected.

Interpreting some features of critical phenomena in terms of quantum mechanics one may go even further by considering the critical correlation length \( \xi \). It is obvious that \( \xi^{-1} \sim |E_{bs}| = m \). As can be seen from Eq.(4.7) in a close vicinity of the threshold \( T_c \) the discrete eigenvalue behaves like \( |E_{bs}| \sim |T - T_c|^{\frac{1}{d-2}} \). In the context of statistical mechanics it means that \( \xi \sim \tau^{-\nu} \) where \( \nu = \frac{1}{d-2} \).

The energy gap \( E_{bs}(T) \) is a continuous function vanishing at \( T = T_c \). It means that the model under discussion undergoes a second-order phase transition. The first-order phase transition would correspond to a jump of the gap at a threshold which seems to be impossible in quantum mechanics.

Above the threshold \( T > T_c \) the asymptotic behavior of the wave function is as follows. The large distance asymptotic \( \xi \ll x \) is given by

\[ \Psi(\vec{\xi}) \sim \frac{exp(-\frac{x}{\xi})}{x^{\frac{d-1}{2}}} \quad (4.11) \]

If the gap vanishes: \( m = 0 \) at \( T = T_c \), then a wave function of the ground state exhibits a scale-invariant behavior

\[ \Psi(\vec{\xi}) \sim x^{2-d} \quad (4.12) \]

Thus, at the Curie point \( T = T_c \) the wave function changes its behavior from exponential at high \( T \) to the “power law” at low \( T \). Below the threshold the lowest eigenvalue of \( \hat{H} \) equals zero, hence we arrive again at Eq.(4.12) corresponding to the correlation function of massless Goldstone excitations \( \langle \pi_a(\vec{x})\pi_a(0) \rangle \) with \( \pi_a(\vec{x}) \) being transverse modes.

All the above correspondences between quantum and statistical mechanics are summarized at the Table 1.

Before moving to the next topic, we make an observation concerning critical exponents. It is somewhat surprising that the \( d \)-dimensional chiral models in the large \( N \)-limit taken at special values of \( d \) have some critical exponents coinciding to those of the 2D Ising model and of the 3-state Potts model on dynamical planar lattices (DPL) (see Table II) [30].
avoid misunderstanding stress that the models under discussion have quite different values of critical exponents $\nu$ and $\eta$ (Indeed, two any statistical models, having different spatial dimensions $d$, cannot have numerically identical critical exponents, because in that case some hyperscaling relations containing $d$ would be broken).

### IV. SU(N)-SYMMETRIC GINZBURG-LANDAU MODEL IN A UNIFORM MAGNETIC FIELD

Here we shall consider the more interesting and nontrivial example of the $SU(N)$-symmetric $d$-dimensional Ginzburg-Landau model Eq.(2.3) subject to a uniform magnetic field $B_{11,22}$.

This model is described by the Hamiltonian

$$H = \frac{1}{2} \int d^d x |(\partial_\mu + i \frac{2\pi}{\Phi_0} A_\mu) S_a|^2$$

where the vector potential $A_\mu$ is taken within the symmetric gauge

$$A = \frac{1}{2} [B, r]$$

where $B$ is taken along the $z$ axis. The partition function associated with Eq.(5.1) reads

$$Z = \int \prod_{a=1}^N DS_a DS^*_a \exp(-\frac{H}{T}) \delta(|S|^2 - 1)$$

The non gauge-invariant two-point correlation function of $\vec{S}(\vec{r})$ is given by

$$G_{ab}(\vec{r}, \vec{r}') = <S_a(\vec{r}) S_b^*(\vec{r}'>)$$

with $G_{ab}(\vec{r}, \vec{r}')$ being a Green function of the $d$-dimensional Schrodinger operator

$$[-i\partial_\mu - \frac{2\pi}{\Phi_0} A_\mu]^2 + m_0^2]G_{ab}(\vec{r}, \vec{r}') = T\delta_{ab}\delta(\vec{r} - \vec{r}')$$

The exact solution of Eq.(5.3) in an arbitrary gauge reads [23]

$$G_{ab}(\vec{r}, \vec{r}') = T\delta_{ab} \exp\left(-i \frac{2\pi}{\Phi_0} \int_{\vec{r}}^{\vec{r}'} dx_\mu A_\mu \right) \left(4\pi\right)^{\frac{d-4}{2}} \int_{0}^{\infty} du \frac{\omega u^{\frac{d-4}{2}}}{2\sinh(u\omega/2)}$$

$$\times \exp\{-m_0^2 u - \frac{(z-z')^2}{4u} - \frac{\omega}{8} \coth\left(\frac{1}{2} u\omega\right)[(x-x')^2 + (y-y')^2]\}$$

where $\omega = \frac{2eB}{\hbar}$ is the cyclotron frequency (here we have set $\hbar = 1$ and $2m = 1$) and $z$ and $z'$ are $(d-2)$-dimensional longitudinal coordinates. The integral in Eq.(5.6) is taken over the straight line connecting the points $\vec{r}$ and $\vec{r}'$.

Taking into account the constraint $|\vec{S}(\vec{x})|^2 = 1$ we arrive at the Schrodinger equation for an effective wave function
\[ \left( -i \partial_\mu - \frac{2\pi}{\Phi_0} A_\mu \right)^2 + m_0^2 \right] \Psi(\mathbf{r}, \mathbf{r}') = T N \delta(\mathbf{r} - \mathbf{r}') \Psi(\mathbf{r}, \mathbf{r}') \]  \hspace{1cm} (5.7)

where \( \delta_{ab} \Psi(\mathbf{r}, \mathbf{r}') \equiv G_{ab}(\mathbf{r}, \mathbf{r}') \).

From the field-theoretical point of view we are dealing with the Abelian Higgs model defined by the Euclidean Lagrangian, Eq.(5.1) (or, equivalently, the \( N \)-component scalar QED), in a large external magnetic field.

It is important for the problem under consideration that an external magnetic field effectively reduces the spatial dimension of the system from \( d \) to \( d - 2 \). Thus, if the spatial dimension of the model Eq.(5.1) is \( 4 + \epsilon \), then due to the dimensional reduction effect it actually equals \( 2 + \epsilon \).

At the first glance it looks like the dimensional reduction in a \( O(N) \)-symmetric Heisenberg ferromagnet subject to a random magnetic field. As a matter of fact, the model Eq.(5.1) differs vastly from a random-field system because of the lack both of quenched disorder and of the hidden supersymmetry [24]. Moreover, it will be seen below that the dimensional reduction is not exact in our case.

In the context of quantum mechanics it is known that an arbitrary weak uniform magnetic field gives rise to the bound state for any 3D quantum well [25]. In other words, it amounts to the dimensional reduction.

Now let’s exploit the boundary condition for the solution Eq.(5.6) at the coinciding points \( \vec{x} = \vec{x}' \):

\[ \Psi(\vec{x}, \vec{x}) = \frac{T N \omega}{(4\pi)^{d/2}} \int_0^\infty \frac{du}{a^d} \frac{\exp(u\omega/2 - m^2 u)}{2u^{(d-2)/2} \sinh(u\omega/2)} = 1 \]  \hspace{1cm} (5.8)

Notice that in order to obtain Eq.(5.8) we have used the mass renormalization \( m_0^2 + \omega/2 = m^2 \). The integral in the r.h.s. of Eq.(5.8) is ultraviolet divergent, a short-distance cutoff to be used. As a result one should deal with the cutoff-dependent coupling constant \( T(a) \) so as to keep the mass independent on \( a \) 

At criticality the mass term vanishes \( m = 0 \), the equation determining the upper critical magnetic field \( H_{c2}(T) \) reads

\[ \frac{1}{T(H)} = \frac{N \omega}{(4\pi)^{d/2}} \int_0^\infty \frac{du}{a^d} \frac{\exp(u\omega/2)}{2u^{(d-2)/2} \sinh(u\omega/2)} \]  \hspace{1cm} (5.9)

Provided \( d < 4 \), the integral at the r.h.s. of Eq.(5.8) diverges at large \( u \) at the critical point. It implies that in this case the \( H_{c2}(T) \)-line doesn’t exist at all.

A straightforward algebra yields the phase boundary \( H_{c2}(T) \) in \( d > 4 \) dimensions:

\[ \frac{T(0)}{T(H)} = \frac{d - 2}{2} \left( \frac{H}{H_0} \right)^{d/2} \int_0^\infty \frac{dt}{H_0} t^{(2-d)/2} \exp t \sinh t \]  \hspace{1cm} (5.10)

where

\[ T(0) = \frac{2(4\pi)^{d/2} a^{d-2}}{N} ; \quad H_0 = \frac{\Phi_0}{a^2} \]  \hspace{1cm} (5.11)

The properties of the upper critical magnetic field \( H_{c2}(T) \) Eq.(5.10) are as follows.
First, the phase transition line exists only above 4 dimensions due to the dimensional reduction.

Second, the characteristic scale of the magnetic field is given by $H_0$. The asymptotic behavior of the r.h.s. of Eq.(5.10) at the low-$H$ and at the high-$H$ regions for $4 < d < 6$ is given by:

1. at low-$H$ when $s \equiv \frac{H}{H_0} \ll 1$

$$\frac{T(0)}{T(H)} = 1 + \frac{d-2}{d-4} s + A_d s^{d/2-2} - \frac{d-2}{3(d-6)} s^2 + \frac{d-2}{45(d-10)} s^4 + 0(s^6) \quad (5.12)$$

where constant $A_d$ is given by

$$A_d = -2 \frac{d-3}{d-4} + \frac{d-2}{2} \int_1^\infty \frac{dt \, t^{d-2} \exp t}{\sinh t}$$

$$+ \frac{d-2}{2} \int_0^1 \frac{dt \, t^{d-2} \exp t - \sinh t - t \sinh t}{t \sinh t} \quad (5.13)$$

In fact, Eq.(5.12) describes a vicinity of the critical end point $T = T_c, H = 0$ where several phases coexist and the second derivative of $T(H)$ with respect to $H$ diverges $T''(H) \sim H^{(d-6)/2}, H \to 0$.

2. at high-$H$ for $1 \ll s$ it is possible to calculate explicitly corrections to the LLL-approximation

$$\frac{T(0)}{T(H)} = 2 \frac{d-2}{d-4} s + \frac{d-2}{2} \exp(-2s) - \frac{(d-2)^2}{8s} \exp(-2s) + 0(s^{-2} \exp(-2s)) \quad (5.14)$$

It is small wonder that the corrections to the LLL approximation are exponentially small, because non LLL levels are massive. In contrast to the first case, provided $s \to \infty$ the $H_{c2}(T)$ line goes away to infinity.

Finally, the critical exponent of the correlation length found from Eq.(5.8) equals $\nu = \frac{1}{\alpha-1}$.

**V. CONCLUDING REMARKS**

Thus, it has been demonstrated that the Schrodinger equation for a free quantum particle moving in a $\delta$-function well truly captures essentials of the critical behavior both above and below $T_c$ for some non-linear vector $\sigma$-models in the large-$N$ limit with different global non-abelian symmetries, namely, $SU(N), 0(N)$ and $CP^N$. All these models in large-$N$ limit are non-trivial systems of interacting Goldstone particles, undergoing a continuous phase transition at a finite temperature. All of them are described by the beta-function Eq.(4.9) being to some extent "universal".

Notice that the above approach cannot be extended to both gauge and matrix models. The reason for that is that in the large-$N$ limit vector non-linear $\sigma$-models are equivalent to a theory of free massless particles. This is not the case for gauge and matrix models.

It is interesting that the results obtained can be applied to some other models of statistical mechanics. For instance, within the framework of the 2D supersymmetric non-linear $\sigma$-model
suggested by E.Witten in [26] it is easy to show that in the large-$N$ limit the two-point correlator of the $\vec{n}$ field

$$<\vec{n}(x)\vec{n}(0)>$$  \hspace{1cm} (6.1)

also obeys Eq.(2.12). It is most remarkable that if $d = 2, N = 3$, the one-loop beta-function of this supersymmetric model is exact! ( [28]).

Another model deserved to be mentioned is the popular Kardar-Parisi-Zhang model. Without going into details, mention that considering the KPZ-equation in the 2-particle sector, one just arrives at the effective Schrodinger equation Eq.(2.12) with the "universal" beta-function Eq.(4.9) [29].

Concerning the continuous GL model in an external magnetic field Eq.(2.12) discussed above, it should be noted that the above approach doesn’t use the standard LLL approximation, results obtained hold for an arbitrary magnetic field.

Corrections to this approximations are found to be exponentially small confirming the conjecture advanced in [13] that the LLL approximation provides a correct description of critical properties at the close vicinity of $H_{c2}(T)$.

This model doesn’t exhibit a phase transition below 4 dimensions. In order to suppress the dimensional reduction one might consider the same model on a lattice in spirit of the approach developed in [11,15,18–20]. Within the Schrodinger equation approach, one gets a remarkable correspondence between a free Bloch electron moving in the crystall lattice in a uniform magnetic field with the impurity ($-T$)$\delta$-function well and the phase-boundary line $H_{c2}(T)$ for type-II superconductors.

More precisely, it is obvious from the physical standpoint that that electron can be captured by this well to create a bound state around the impurity atom. The threshold for the bound state level $|E_{bs}|$ is just the upper-critical magnetic field for a uniformly frustrated model.

In fact, we have got a well known and understood problem of solid state physics where the electron is described by the Azbel-Harper-Hofstadter Hamiltonian with the delta-function quantum well [31,33]. This problem can be solved, at least, numerically in the commensurate case for rational values of frustration, i.e. when the magnetic flux through the plaquette is given by $\frac{\Phi}{\Phi_0} = 2\pi\frac{p}{q}$ where $p, q$ are mutually prime integers. Such investigations would allow one to determine the dependence $T_c(q); q = 1, 2, ...$. This model as well as a model with weak quenched disorder are left for future studies.

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**TABLE I.** Correspondences between quantum mechanics and statistical mechanics

| QUANTUM MECHANICS | STATISTICAL MECHANICS |
|-------------------|------------------------|
| wave function $\psi(\vec{x})$ | correlation function $<S(\vec{x})S(0)^*>$ |
| quantum well $-T\delta(\vec{x})$ | coupling constant $T$ |
| bound state | dynamical mass generation |
| energy gap $E_{bs}$ | reciprocal of the correlation length $\xi^{-1}$ |
| generator of isospectral deformations | beta-function $\beta(T)$ |
| threshold $T_c$ | fixed (or Curie) point $T_c$ |

- $E_{bs} \sim (T - T_c)\nu; \nu = \frac{d-2}{d-2}$
- $\xi \sim (T - T_c)^{-\nu}; \nu = \frac{1}{d-2}$
- $T > T_c; \psi(\vec{x}) \sim \exp(-\frac{\vec{x}}{\xi})x^{1-d}$
- $T > T_c; <\vec{S}(\vec{x})\vec{S}(0)^* > \sim \exp(-\frac{\vec{x}}{\xi})x^{1-d}$
- $T \leq T_c; \psi(\vec{x}) \sim x^{2-d}$
- $T \leq T_c; \text{Goldstone modes} <\vec{S}(\vec{x})\vec{S}(0)^* > \sim x^{2-d}$
- no analogy

**TABLE II.** Critical exponents of the $d$-dimensional $\sigma$-model with $N = \infty$ ($d = 3; 8/3$), and those of the Ising model and of the 3-state Potts model on a dynamical planar lattice

| Critical index | $\sigma$ - model | $\sigma$ - model ($d = 3$) | $\sigma$ - model ($d = 8/3$) |
|----------------|------------------|-----------------------------|-----------------------------|
| $\alpha$       | $\frac{d-4}{d-2}$ | -1                          | -2                          |
| $\beta$        | $\frac{1}{2}$    | $\frac{1}{2}$               | $\frac{1}{2}$               |
| $\gamma$       | $\frac{d+2}{d-2}$| 2                           | 3                           |
| $\delta$       | $\frac{d}{d-2}$  | 5                           | 7                           |
| $d\nu$         | $\frac{d}{d-2}$  | 3                           | 4                           |

**TABLE II.** Critical exponents of the $d$-dimensional $\sigma$-model with $N = \infty$ ($d = 3; 8/3$), and those of the Ising model and of the 3-state Potts model on a dynamical planar lattice