Determinantal quartic surfaces
with a definite Hermitian representation

Martin Helsø

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Abstract

We give a bound on the number of isolated, essential singularities of
determinantal quartic surfaces in 3-space. We also provide examples
of different configurations of real singularities on quartic surfaces with
a definite Hermitian determinantal representation, and conjecture an
extension of a theorem by Degtyarev and Itenberg.

1 Introduction

Representing a polynomial as the determinant of a linear matrix is a problem
dating back at least to Hesse [Hes44]. Determinantal representations have
applications in areas such as linear algebra, operator theory, convex optimisation
and algebraic geometry.

A homogenous polynomial \( f \in \mathbb{C}[x_0, \ldots, x_n]_d \) of degree \( d \) has a determinantal representation if \( f(x) = \det(M_d(x)) \), where

\[
M_d(x) := M_d(x_0, \ldots, x_n) := M_{d,0}x_0 + \cdots + M_{d,n}x_n
\]  

(1.1)

for some \((d \times d)\)-matrices \( M_{d,0}, \ldots, M_{d,n} \). We say that the representation is Hermitian if \( M_{d,0}, \ldots, M_{d,n} \) are Hermitian matrices, and it is symmetric if \( M_{d,0}, \ldots, M_{d,n} \) are symmetric matrices. The hypersurface \( \mathcal{V}(f) \subset \mathbb{C}P^n \) is called determinantal if \( f \) possesses a determinantal representation; \( \mathcal{V}(f) \) is called a symmetroid if the representation is symmetric.

Let \( f \in \mathbb{R}[x_0, \ldots, x_n]_d \) be a real polynomial with a Hermitian determinantal representation (1.1). The representation is definite if the matrix \( M_d(e) \) is positive definite for some point \( e \in \mathbb{R}P^n \). The eigenvalues of a Hermitian matrix are real. It follows that every real line through \( e \) only meets the hypersurface \( \mathcal{V}(f) \subset \mathbb{C}P^n \) in real points. A polynomial with this property is called hyperbolic with respect to \( e \). The connected component of \( e \) in \( \mathbb{R}P^n \setminus \mathcal{V}_R(f) \) is called the hyperbolicity cone of \( f \) with respect to \( e \).

If \( M_{d,0}, \ldots, M_{d,n} \) in (1.1) are real, symmetric matrices, then the set

\[
\{ x \in \mathbb{R}P^n \mid M_d(x) \text{ is semidefinite} \}
\]

is called a spectrahedron. It is easy to see that all spectrahedra are hyperbolicity cones. The converse statement, that all hyperbolicity cones are spectrahedra, is called the generalised Lax conjecture and is an object of much interest. A partial
result is the Helton–Vinnikov theorem, which implies that all hyperbolicity cones in \( \mathbb{R}^{n^2} \) are spectrahedra [HV07; LPR05]. Let \( f \in \mathbb{R}[x_0, \ldots, x_n] \) be polynomial with a definite Hermitian representation (1.1), and suppose that \( e \in \mathbb{R}^{n^2} \) is such that \( M_d(e) \) is positive definite. We note that the hyperbolicity cone of \( f \) with respect to \( e \) is a spectrahedron, by following [PV13, Corollary 5.1]: We can write \( M_d(x) = A_d(x) + iB_d(x) \), where \( A_d(x) \) is real symmetric and \( B_d(x) \) is real antisymmetric. We define the real symmetric \((2d \times 2d)\)-matrix

\[
A_{2d}(x) := \begin{bmatrix}
A_d(x) & B_d^T(x) \\
B_d(x) & A_d(x)
\end{bmatrix} = \begin{bmatrix}
A_d(x) & -B_d(x) \\
B_d(x) & A_d(x)
\end{bmatrix}. \tag{1.2}
\]

Let

\[
U := \frac{\sqrt{2}}{2} \begin{bmatrix} I_d & iI_d \\
iI_d & I_d \end{bmatrix}
\]

where \( I_d \) is the identity matrix of size \( d \). After the change of coordinates

\[
\overline{U} A_{2d} U^T = \begin{bmatrix} A + iB & 0 \\ 0 & A - iB \end{bmatrix} = \begin{bmatrix} M_d & 0 \\ 0 & \overline{M}_d \end{bmatrix},
\]

we see that

\[
\det(A_{2d}(x)) = \det(M_d(x)) \det(\overline{M}_d(x)) = f^2.
\]

The polynomials \( f \) and \( f^2 \) have the same hyperbolicity cone with respect to \( e \), which is the spectrahedron defined by \( A_{2d}(x) \).

The rank and corank of a point \( x \in \mathbb{C}P^n \) are defined as rank \( M_d(x) \) and corank \( M_d(x) \), respectively. The rank-\( k \) locus of \( M_d(x) \) is the set of points in \( \mathbb{C}P^n \) with rank less than or equal to \( k \). The hypersurface \( V(f) \) is equal to the rank-(\( d - 1 \)) locus. The points \( x \in V(f) \) with corank \( x \geq 2 \) with respect to \( M_d(x) \) are always singular on \( V(f) \), and they are called essential singularities. A point \( x \) with corank \( x = 1 \) is generally not singular, but if \( x \in \text{Sing } V(f) \), then \( x \) is called an accidental singularity. The multiplicity of a point \( x \in V(f) \) is greater than or equal to its corank. Since the rank-(\( d - 2 \)) locus is given by the vanishing of the \((d - 1) \times (d - 1)\)-minors of \( M_d(x) \), the singular locus of \( V(f) \) is at least \((n - 4)\)-dimensional, and it is at least \((n - 3)\)-dimensional if \( V(f) \) is a symmetroid. Moreover, if \( V(f) \) is a generic symmetroid of degree \( d \), then \( \text{Sing } V(f) \) has degree \((d^2 - 1)\) and contains no accidental singularities [Pio06, Proposition 1.5; Sal65, p. 420].

We restrict the attention to quartic determinantal surfaces in \( \mathbb{P}^3 \). A generic determinantal surface is smooth, while a generic quartic symmetroid has ten rank-2 points that are nodes, that is, isolated quadratic singularities. A nodal quartic symmetroid is called transversal if it has ten rank-2 nodes and no further singularities. The study of quartic symmetroids originated with Cayley [Cay69a]. Recently, real quartic symmetroids with a nonempty spectrahedron have gained attention. Using the global Torelli theorem for K3-surfaces, Degtyarev and Itenberg proved the following:

**Theorem 1.1** ([DI11, Theorem 1.1]). There exists a real transversal quartic symmetroid with a nonempty spectrahedron, having \( \rho \) real nodes, of which \( \sigma \) nodes lie on the boundary of the spectrahedron, if and only if \( 0 \leq \sigma \leq \rho \), both even, and \( 2 \leq \rho \leq 10 \).
Ottem et al. presented an algorithmic proof of Theorem 1.1, and for each pair \((\rho, \sigma)\) satisfying the inequalities in the theorem, they gave an example of a symmetroid with the corresponding configuration of nodes [Ott+15]. An analogue to Theorem 1.1 for rational quartic symmetroids is proven in [HR18, Theorem 1.7].

We prove the following bound on the number of isolated rank-2 points of a determinantal quartic surface in Section 2:

**Theorem 1.2.** Let \(S_4 \subset \mathbb{P}^3\) be a determinantal quartic surface with only isolated, simple singularities. Let \(\eta\) be the number of essential singularities of \(S_4\). Then \(\eta \leq 8\), unless \(S_4\) is a symmetroid, in which case \(\eta = 10\).

We expect that Theorem 1.2 is well-known, but we have not been able to find a reference in the literature. As remarked above, a definite Hermitian determinantal representation gives rise to a spectrahedron. The real singularities may lie on or off the spectrahedron. We surmise a generalisation of Theorem 1.1 to Hermitian representations:

**Conjecture 1.3.** Suppose that \(\mathcal{V}(f) \subset \mathbb{R}^3\) is a real quartic surface, where \(f\) admits a definite Hermitian determinantal representation \(M_4(x)\). Assume that the complex surface \(\mathcal{V}_C(f) \subset \mathbb{C}^3\) has \(\eta\) isolated nodes, all of which are essential nodes with respect to \(M_4(x)\), and that \(\mathcal{V}(f)\) has \(\rho\) real nodes, of which \(\sigma\) real nodes lie on the spectrahedron defined by \(M_4(x)\). Then \(f\) exists if and only if \(0 \leq \sigma \leq \rho \leq \eta \leq 8\) and \(\rho \equiv \eta \pmod{2}\), or \(0 \leq \sigma \leq \rho \leq \eta = 10\), \(\rho \geq 2\) and \(\sigma \equiv \rho \equiv \eta \pmod{2}\).

In Section 3, we consider the real singularities of quartic surfaces with a Hermitian determinantal representation. In particular, Section 3.1 explains why we expect fewer restrictions on \(\rho\) and \(\sigma\) for \(0 \leq \eta \leq 8\) than for \(\eta = 10\) in Conjecture 1.3. After that, we describe our strategy for finding examples of surfaces with a given triple \((\eta, \rho, \sigma)\). Table 1 shows the progress towards proving Conjecture 1.3. The existence of all cases for \(\eta = 10\) is given by Theorem 1.1, and explicit examples are given in [Ott+15]. The known examples for \(0 \leq \eta \leq 8\) are listed in Section 4.

| \(\eta\) | \((\rho, \sigma)\) |
|---|---|
| 0 | Known examples: \((0, 0)\) |
| 1 | Known examples: \((1, 1), (1, 0)\) |
| 2 | Known examples: \((2, 2), (2, 1), (2, 0), (0, 0)\) |
| 3 | Known examples: \((3, 3), (3, 2), (3, 1), (3, 0), (1, 1), (1, 0)\) |
| 4 | Known examples: \((4, 4), (4, 3), (4, 2), (4, 1), (4, 0), (2, 2), (2, 1), (0, 0)\) |

Continued on the next page.
2 Essential singularities on determinantal quartic surfaces

A quartic surface \( V(f) \subset \mathbb{P}^3 \) with only isolated singularities, can have zero to sixteen nodes. If \( f \) has a symmetric determinantal representation, exactly ten of the nodes — counted with multiplicity — are essential singularities. For each of the sixteen nodes on a Kummer surface \( V(f) \), there exists a symmetric determinantal representation of \( f \) such that the node is essential in that representation [Ott+15, p. 597]. On the other hand, [Jes16, Article 9] describes a quartic symmetroid with eleven nodes, where one of the nodes is an accidental singularity in every symmetric determinantal representation. It is natural to ask how many nodes can be essential singularities when we consider nonsymmetric determinantal representations. We show that the maximum number of isolated, essential singularities is obtained precisely with a symmetric determinantal representation.

Determinantal quartic surfaces are characterised by containing a projectively normal sextic curve with genus 3 [Sch81]. Coble noted that this follows because the Picard group of a general quartic surface is generated by a plane section [Cob82, p. 39]. Given a determinantal representation

\[
M_4(x) := \begin{bmatrix}
m_{00} & m_{10} & m_{20} & m_{30} \\
m_{01} & m_{11} & m_{21} & m_{31} \\
m_{02} & m_{12} & m_{22} & m_{32} \\
m_{03} & m_{13} & m_{23} & m_{33}
\end{bmatrix}
\]

of a quartic surface \( S_4 := V(\det(M_4(x))) \), two families of genus 3 sextics on \( S_4 \) can be described as follows: The \((3 \times 3)\)-minors of a \((4 \times 3)\)-submatrix of
\[ M_4(x) \] define a sextic curve of genus 3; the four curves obtained by the different \((4 \times 3)\)-submatrices span the first family \(C_1\). The second family \(C_2\) is defined similarly by replacing the \((4 \times 3)\)-submatrices with \((3 \times 4)\)-submatrices. If \(M_4(x)\) is symmetric, \(C_1\) and \(C_2\) coincide; if the representation is Hermitian, the two families are complex conjugates. Because these curves are given by the vanishing of some \((3 \times 3)\)-minors, they contain the rank-2 locus of \(S_4\). That is, they contain the set of essential singularities on \(S_4\). Moreover, for each curve \(C_1 \subset C_2\) there is a curve \(C_2 \subset C_2\) such that \(C_1 \cup C_2\) is the complete intersection of \(S_4\) and a cubic surface \(S_3\). In particular, the union of the curves defined by the \((3 \times 3)\)-minors of

\[
A_1 := \begin{bmatrix}
m_{00} & m_{10} & m_{20} \\
m_{01} & m_{11} & m_{21} \\
m_{02} & m_{12} & m_{22} \\
m_{03} & m_{13} & m_{23}
\end{bmatrix} \quad \text{and} \quad A_2 := \begin{bmatrix}
m_{00} & m_{10} & m_{20} & m_{30} \\
m_{01} & m_{11} & m_{21} & m_{31} \\
m_{02} & m_{12} & m_{22} & m_{32}
\end{bmatrix}
\]

is the intersection of \(S_4\) and the surface \(S_3\) defined by their common \((3 \times 3)\)-minor

\[
\begin{vmatrix}
m_{00} & m_{10} & m_{20} \\
m_{01} & m_{11} & m_{21} \\
m_{02} & m_{12} & m_{22}
\end{vmatrix} = 0.
\]

The curves in \(C_1\) and \(C_2\) are nonhyperelliptic. Indeed, a sextic curve of genus 3 in \(\mathbb{P}^3\) is nonhyperelliptic if and only if it is projectively normal [Dol12, Exercise 4.10]. We give an elementary argument showing that the curve \(C\) defined by \(A_1\), is nonhyperelliptic: Suppose that the entries \(m_{ij}\) in \(A_1\) are linear forms in the variables \(x_0, x_1, x_2\) and \(x_3\). Note that \(C\) is the solution set to the equation

\[
A_1 \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (2.1)
\]

We can rewrite (2.1) as

\[
A'_1 \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (2.2)
\]

where \(A'_1\) is a \((4 \times 4)\)-matrix with linear entries in \(y_0, y_1, y_2\). Both (2.1) and (2.2) define the same curve \(K\) in \(\mathbb{P}^2 \times \mathbb{P}^3\). Then \(C\) is the projection of \(K\) to \(\mathbb{P}^3\), and the curve \(C'\) given by \(\text{det}(A'_1) = 0\) is the projection to \(\mathbb{P}^2\). The curve \(C'\) is a smooth planar quartic curve, hence nonhyperelliptic [Har77, Example IV.5.2.1]. It follows that \(C\) is nonhyperelliptic as well.

We are now ready to prove the bound on the possible number of essential singularities on a quartic surface.

**Proof of Theorem 1.2.** Without loss of generality, we may assume that \(S_4\) has only essential singularities, \(P_1, \ldots, P_n\). Since \(S_4\) is determinantal, there are smooth, sextic curves \(C_1, C_2 \subset S_4\) of genus 3 passing through \(P_1, \ldots, P_n\). Moreover, we may assume that \(C_1 \cup C_2 = S_4 \cap S_3\) for some cubic surface \(S_3\).

Let \(\pi: S_4 \to S_3\) be the blow-up of \(S_4\) at \(P_1, \ldots, P_n\). Then \(S_4\) is a smooth K3-surface. The exceptional divisor \(E_i\) over \(P_i\) is a \((-2)\)-curve satisfying \(h \cdot E_i = 0\),
where \( h \) is the class of the preimage of a plane section of \( S_4 \). Because \( C_1 \) passes through \( P_1 \), the strict transform \( \tilde{C}_1 \) satisfies \( \tilde{C}_1 \cdot E_i = 1 \). Furthermore, the adjunction formula gives

\[
\tilde{C}_1 \cdot \tilde{C}_1 = 2g_{\tilde{C}_1} - 2 = 2 \cdot 3 - 2 = 4,
\]

since the canonical divisor on \( \tilde{S}_4 \) is trivial. It follows from \( C_1 \cup C_2 = S_4 \cap S_3 \) that the total transform \( \pi^{-1}(C_1 \cup C_2) = 3h \). The curve \( C_1 \cup C_2 \) is double at \( P_1, \ldots, P_\eta \), so \( \pi^{-1}(C_1 \cup C_2) = \tilde{C}_1 + \tilde{C}_2 = \sum_{i=1}^\eta E_i \). Thus

\[
\tilde{C}_1 + \tilde{C}_2 = \pi^{-1}(C_1 \cup C_2) - \sum_{i=1}^\eta E_i = 3h - \sum_{i=1}^\eta E_i.
\]  

(2.3)

We intersect both sides of (2.3) with \( \tilde{C}_1 \):

\[
\tilde{C}_1 \cdot (\tilde{C}_1 + \tilde{C}_2) = \tilde{C}_1 \cdot \left(3h - \sum_{i=1}^\eta E_i\right).
\]  

(2.4)

By using the facts above and that \( \tilde{C}_1 \cdot h = \deg(\tilde{C}_1) = 6 \), Equation (2.4) yields

\[
\tilde{C}_1 \cdot \tilde{C}_2 = 14 - \eta.
\]

The linear system \( |\tilde{C}_1| \) gives rise to a morphism \( \varphi : \tilde{S}_4 \to \mathbb{P}^3 \). Because \( \tilde{C}_1 \) is a nonhyperelliptic curve of genus 3 with \( \tilde{C}_1 \cdot \tilde{C}_1 = 4 \), it is mapped to a plane curve \( C'_1 := \varphi(\tilde{C}_1) \) of degree 4. Thus the image \( S'_4 := \varphi(\tilde{S}_4) \) is a quartic surface. Assume first that \( S_4 \) is not a symmetroid, so \( C_1 \neq C_2 \). Then \( C'_2 := \varphi(\tilde{C}_2) \) is not a plane section. Because \( C'_2 \) is nonhyperelliptic and spans \( \mathbb{P}^3 \), we have \( \deg(C'_2) \geq 6 \). Since \( C'_1 \) is a plane section, we have \( \deg(C'_2) = \tilde{C}_1 \cdot \tilde{C}_2 \). Therefore

\[
14 - \eta = \tilde{C}_1 \cdot \tilde{C}_2 \geq 6,
\]

(2.5)

so we get \( \eta \leq 8 \).

Assume now that \( S_4 \) is a symmetroid, so \( C_1 = C_2 \). Then \( \tilde{C}_1 \cdot \tilde{C}_2 = \tilde{C}_1 \cdot \tilde{C}_1 = 4 \). Adjusting for this in the right-hand side of (2.5) gives \( \eta = 10 \). Hence we have recovered the well-known fact that a quartic symmetroid with only isolated singularities has ten rank-2 points. \( \blacksquare \)

3 Real singularities on quartics with a Hermitian determinantal representation

Consider \( \mathbb{CP}^{15} \) as the projectivisation of the vector space over \( \mathbb{C} \) spanned by Hermitian \((4 \times 4)\)-matrices. Then the rank-2 locus \( X_2 \) of \( \mathbb{CP}^{15} \) is given by the vanishing of the \((3 \times 3)\)-minors of a general Hermitian \((4 \times 4)\)-matrix; it is an elevenfold of degree 20. A quartic surface \( S_4 \subset \mathbb{P}^3 \) with a Hermitian determinantal representation corresponds to a linear 3-space \( H \subset \mathbb{CP}^{15} \) and essential singularities on \( S_4 \) corresponds to the intersection of \( H \) with \( X_2 \). This is helpful for finding examples of Hermitian determinantal representations with a specific singular locus.

In [Hel19; Hel20; HR18], symmetroids are studied via the associated quadratic form. We wish to use this technique to determine the real part of \( X_2 \). Let
We summarise the correspondence between\(A\) and \(H\). After a \([\text{Macaulay2}]\) calculation, we find that the \(\text{baselocus}\) follows that \(\text{Sing}(\text{W})\) has \(2\) \(\text{rank}\) \(A\) \(\times\) \(\text{rank}(A)\) \(\times\) \(\text{rank}(A)\) \(\times\) \(\text{rank}(A)\) \(\times\) \(\text{rank}(A)\).\n
In addition, if \(A\) is \(\text{Hermitian}\), it is \(\text{unitarily diagonalisable}\). If \(A\) is \(\text{diagonal}\) then it is \(\text{unitarily diagonalisable}\). Hence \(\text{rank}(A) = 2\). In addition, if \(M\) is \(\text{diagonal}\) then \(\text{rank}(\text{W})\), then we can define its \(\text{spectrahedron}\) in terms of \(M\) only, because

\[
\{ x \in \mathbb{R}^{15} \mid M_4(x) \text{ is semidefinite} \} = \{ x \in \mathbb{R}^{15} \mid A_8(x) \text{ is semidefinite} \}.
\]

The construction of \(A_8\) from \(M_4\) allows us to view the point \(x := [x_{00} : x_{01} : x_{02} : x_{03} : x_{11} : x_{12} : x_{13} : x_{22} : x_{23} : x_{33} : y_{01} : y_{02} : y_{03} : y_{12} : y_{13} : y_{23}]\) in \(\mathbb{C}^{15}\) as the symmetric \((8 \times 8)\)-matrix

\[
A_8(x) := \begin{bmatrix}
    x_{00} & x_{01} & x_{02} & x_{03} & 0 & -y_{01} & -y_{02} & -y_{03} \\
    x_{01} & x_{11} & x_{12} & x_{13} & y_{01} & 0 & -y_{12} & -y_{13} \\
    x_{02} & x_{12} & x_{22} & x_{23} & y_{02} & y_{12} & 0 & -y_{23} \\
    x_{03} & x_{13} & x_{23} & x_{33} & y_{03} & y_{13} & y_{23} & 0 \\
    0 & y_{01} & y_{02} & y_{03} & x_{00} & x_{01} & x_{02} & x_{03} \\
    -y_{01} & 0 & y_{12} & y_{13} & x_{01} & x_{11} & x_{12} & x_{13} \\
    -y_{02} & -y_{12} & 0 & y_{23} & x_{02} & x_{12} & x_{22} & x_{23} \\
    -y_{03} & -y_{13} & -y_{23} & 0 & x_{03} & x_{13} & x_{23} & x_{33}
\end{bmatrix}.
\]

Let \(y := [y_0, \ldots, y_7]\). Denote by \(Q_{A_8}(x)\) the quadric \(V(yA_8(x)y^T) \subset \mathbb{C}^{15}\) associated to \(A_8(x)\), and let

\[
W := \left\{ Q_{A_8}(x) \mid x \in \mathbb{C}^{15} \right\}.
\]

After a \([\text{Macaulay2}]\) calculation, we find that the \(\text{baselocus}\) \(\text{Bl}(W) \subset \mathbb{C}^{15}\) of \(W\) consists of two disjoint, complex conjugate \(3\)\(-\text{spaces}\) \(H_\text{Bl} := V(l_0, l_1, l_2, l_3)\) and \(\overline{H_\text{Bl}} := V(l_0^\ast, l_1^\ast, l_2^\ast, l_3^\ast)\), where \(l_j\) is the linear form \(y_j + iy_{j+4}\) for \(j = 0, 1, 2, 3\). We summarise the correspondence between \(M_4\) and \(A_8\):

**Lemma 3.1.** There is an isomorphism between the projectivisation of the vector space over \(\mathbb{C}\) spanned by Hermitian \((4 \times 4)\)-matrices of rank \(k\), and the space of rank-\(2k\) quadrics in \(\mathbb{C}^{15}\) passing through the two given disjoint \(3\)-spaces.

A point in \(X_2\) corresponds to a rank-4 quadric in \(W\). We will now count the rank-4 quadrics in \(W\). Let \(Q\) be a rank-4 quadric in \(W\). By the rank-nullity theorem, \(\text{Sing}(Q)\) is a \(3\)-space. Since \(Q\) contains \(\text{Bl}(W)\) and is irreducible, it follows that \(\text{Sing}(Q)\) intersects \(H_\text{Bl}\) and \(\overline{H_\text{Bl}}\) in a line each. A \(3\)-space \(H\) which intersects \(H_\text{Bl}\) and \(\overline{H_\text{Bl}}\) in a line each, is spanned by those lines, since \(H_\text{Bl}\) and \(\overline{H_\text{Bl}}\) are disjoint. The Grassmannian \(\mathbb{G}(1, 3)\) of lines in a \(3\)-space is \(4\)-dimensional.
Hence there is an 8-dimensional space of 3-spaces that are spanned by a line in $H_{43}$ and a line in $\overline{H}_{43}$. The 3-space $H$ is the singular locus of a web $W_H \subset W$ of quadrics. This can be seen by projecting $\mathbb{P}^7 \to \mathbb{P}^3$ with $H$ as projection centre. The 3-spaces $H_{43}$ and $\overline{H}_{43}$ are projected onto two skew lines $L_{43}, \overline{L}_{43} \subset \mathbb{P}^3$. A rank-4 quadric $Q \subset \mathbb{P}^7$ singular at $H$ and containing $H_{43}$ and $\overline{H}_{43}$ is projected onto a quadric in $\mathbb{P}^3$ containing $L_{43}$ and $\overline{L}_{43}$. There is a web of quadrics in $\mathbb{P}^3$ containing $L_{43}$ and $\overline{L}_{43}$. In total, we get that the space of rank-4 quadrics in $W$ has dimension $8 + 3 = 11$. Since there is a bijection between this space and $X_2$, this is as expected.

A real point in $X_2$ corresponds to a real rank-4 quadric $Q$ in $W$. Then $H := \text{Sing}(Q)$ is real. Thus $H$ intersects $H_{43}$ and $\overline{H}_{43}$ in two complex conjugate lines, $L, \overline{L}$. Hence $H_L := \langle L, \overline{L} \rangle$ is the unique real 3-space which contains $L$ and is the singular locus of a rank-4 quadric in $W$. As noted above, there is a web of quadrics singular at $H_L$. Since the Grassmannian $G(1, 3)$ of lines in $H_{43}$ is 4-dimensional, we get that the locus of real rank-4 quadrics in $W$ has dimension $3 + 4 = 7$. Hence the real part of the rank-2 locus $X_2$ is 7-dimensional.

**Proposition 3.2.** The rank-2 locus $X_2$ of $\mathbb{CP}^{15}$, the projectivisation of the vector space over $\mathbb{C}$ spanned by Hermitian $(4 \times 4)$-matrices, is 11-dimensional. The real part of $X_2$ is 7-dimensional.

We derive a parameterisation of the real rank-4 quadrics in $W$, which in turn corresponds to a parameterisation of the real part of $X_2$. A line $L$ in $H_{43}$ is given by

$$V \left( y_0 + iy_4, y_1 + iy_5, y_2 + iy_6, y_3 + iy_7, \sum_{j=0}^{7} (a_j + ib_j)y_j, \sum_{j=0}^{7} (c_j + id_j)y_j \right)$$

for $a_j, b_j, c_j, d_j \in \mathbb{R}$. Since $H_L$ is the only real 3-space containing $L$, we deduce that $H_L = V(\ell_0, \ell_1, \ell_2, \ell_3)$, where

$$\ell_0 := \sum_{j=0}^{3} \left( (a_j - b_{j+4})y_j + (a_{j+4} + b_j)y_{j+4} \right),$$

$$\ell_1 := \sum_{j=0}^{3} \left( (a_j - b_{j+4})y_{j+4} - (a_{j+4} + b_j)y_j \right),$$

$$\ell_2 := \sum_{j=0}^{3} \left( (c_j - d_{j+4})y_j + (c_{j+4} + d_j)y_{j+4} \right),$$

$$\ell_3 := \sum_{j=0}^{3} \left( (c_j - d_{j+4})y_{j+4} - (c_{j+4} + d_j)y_j \right).$$

The quadrics that are singular at $H_L$ are given by quadratic polynomials in $\ell_0, \ell_1, \ell_2, \ell_3$. Moreover, to be contained in $W$, the quadrics must contain $H_{43}$ and $\overline{H}_{43}$. We compute that the quadrics in $W$ that are singular at $H_L$, generate the ideal

$$I := (\ell_0^2 + \ell_1^2, \ell_0\ell_2 + \ell_1\ell_3, \ell_0\ell_3 - \ell_1\ell_2, \ell_2^2 + \ell_3^2).$$

Hence a real rank-4 quadric in $W$ is on the form

$$V \left( a(\ell_0^2 + \ell_1^2) + b(\ell_0\ell_2 + \ell_1\ell_3) + c(\ell_0\ell_3 - \ell_1\ell_2) + d(\ell_2^2 + \ell_3^2) \right) \quad (3.1)$$
for $a, b, c, d \in \mathbb{R}$. It would be interesting to have equations for the real part of $X_2$, but deriving them from (3.1) has been too computationally demanding.

### 3.1 Definite representations

Let $\mathcal{V}(f) \subset \mathbb{RP}^3$ be a quartic surface with a definite Hermitian determinantal representation $M_4(x)$. Then $M_4(x)$ gives rise to the spectrahedron

$$\{ x \in \mathbb{RP}^3 \mid M_4(x) \text{ is semidefinite} \}.$$  

If $M_4(x)$ is semidefinite but not definite at a point $x$, then $x \in \mathcal{V}(f)$, and in particular, $x$ lies on the boundary of the spectrahedron. Since $\mathcal{V}(f)$ is real, singularities are either real or occur in complex conjugate pairs. Hence we have $\rho \equiv \eta \pmod{2}$ in Conjecture 1.3. For $\eta < 10$, we do not expect any further restrictions on $\rho$ and $\sigma$, as we explain below.

Hermitian determinantal representations are subtly different from symmetric representations, which is why we do not have that $\sigma$ is even and $\rho \geq 2$ for all Hermitian determinantal representations with $\eta < 10$. For instance, the defining characteristic of a transversal symmetroid $\mathcal{V}(f)$ is that the projection from one of the nodes $P$ is ramified along the union of two cubic curves $R_1, R_2 \subset \mathbb{P}^2$ [Cay69b, p. 200]. It is showed in [Ott+15, Proof of Theorem 1.1, p. 600] that if $\mathcal{V}(f)$ has a nonempty spectrahedron $S$ and $P \notin S$, then the nodes on $S$ are projected onto the intersection points of the ovals of $R_1$ and $R_2$. These intersect in an even number of points by the Jordan curve theorem, hence $\sigma$ is even. 

A priori, the ramification curve for the projection from $P$ does not impose restrictions on $\sigma$ if $\mathcal{V}(f)$ is not a symmetroid.

In the special case of a transversal quartic symmetroid which contains a line, a simple reason for $\rho \neq 0$, is that the line passes through an odd number of rank-2 points, at least one of which must be real. A general proof of $\rho \neq 0$ for a transversal quartic symmetroid with a nonempty spectrahedron can be found in [Ott+15, Lemma 4.2]. Not all lines on a quartic surface with a Hermitian determinantal representation contain an odd number of rank-2 points:

**Example 3.3.** The pencil

$$\begin{pmatrix}
0 & x_0 - ix_0 & x_1 & 0 \\
x_0 + ix_0 & x_0 & x_1 & x_0 + ix_0 \\
x_1 & x_0 & x_0 & x_0 \\
0 & -ix_0 & x_0 & x_0 \\
\end{pmatrix}$$

has rank 3 at all points, except at $[1 : 0]$ and $[0 : 1]$, where the rank is 2.

### 3.1.1 Finding examples

In Section 4, we list examples of definite Hermitian determinantal representations with different values of $(\eta, \rho, \sigma)$. The list is missing the sixty-four triples in Table 1 predicted by Conjecture 1.3. Both [HR18; Ott+15] present examples of symmetroids with a nonempty spectrahedron, that are found using random searches. Our examples are not found in this way. The problem with drawing random Hermitian $(4 \times 4)$-matrices $M_4,i$ in search of a definite determinantal representation (1.1) with a given triple $(\eta, \rho, \sigma)$, is that we do not know of a matrix form for $M_4,i$ that will guarantee at least $\eta \geq 4$ essential nodes. For
\( \eta = 0, 1, 2, 3 \) it straightforward to achieve at least \( \eta \) nodes: Let \( M_{4,0} \) be a definite matrix, and let \( \eta \) of \( M_{4,1}, M_{4,2}, M_{4,3} \) be rank-2 matrices. Moreover, we can control \( \sigma \) by choosing the definiteness of the rank-2 matrices. This will in general yield a representation with \((\eta, \eta, \sigma)\). Note that \( M_{4,i} \) corresponds to a real point by construction, hence \( \rho = \eta \). To get \( \rho < \eta \), we let \( M_{4,0} \) and \( M_{4,1} \) be two rank-4 matrices that span a pencil that contains two complex conjugate rank-2 points.

For \( 4 \leq \eta \leq 8 \), our strategy has been the following: A transversal quartic symmetroid with a nonempty spectrahedron corresponds to a 3-space \( H \subset \mathbb{C}P^3 \) that intersects the rank-2 locus \( X_2 \) in ten points. We try to deform \( H \) whilst keeping a definite point and some nodes. More precisely, we choose four points \( P_1, P_2, P_3, P_4 \) in \( H \cap X_2 \) that span \( H \). Let \( Q \subset \mathbb{P}^3 \) be the associated quadric at \( P_1 \). There is a web of rank-4 quadrics singular at \( \text{Sing}(Q) \), which corresponds to a 3-space \( Y_2 \subset X_2 \). We replace \( P_1 \) with a point \( P'_1 \) in \( Y_2 \). The 3-space \( H' \) spanned by \( P'_1, P_2, P_3, P_4 \) intersects \( X_2 \) in \( \eta \geq 4 \) points. Then \( H' \) corresponds to a Hermitian representation with \( \eta \) essential singularities. There is a complete list of examples of all possible values of \( \rho \) and \( \sigma \) for \( \eta = 10 \) in [Ott+15]. We used these as a starting point to get different values of \( \rho \) and \( \sigma \) for \( 4 \leq \eta \leq 8 \).

### 4 Examples of quartics with a definite Hermitian determinantal representation

Below is a list of definite Hermitian determinantal representations that define quartic surfaces with only essential singularities. For each matrix \( M_4(x) \), we specify the configuration \((\eta, \rho, \sigma)\) of essential singularities and a point \( e \in \mathbb{R}P^3 \) such that \( M_4(e) \) is definite. We omit examples with \( \eta = 10 \). A list of examples for \( \eta = 10 \) with all possible values for \( \rho \) and \( \sigma \) is found in [Ott+15].

#### 4.1 Zero rank-2 points

\((0, 0, 0)\):

\[
\begin{bmatrix}
  x_3 & x_0 & x_2 + ix_1 & 0 \\
  x_0 & x_3 & x_1 & x_1 - ix_2 \\
  x_2 - ix_1 & x_1 & 2x_0 + x_3 & x_1 + x_2 \\
  0 & x_1 + ix_2 & x_1 + x_2 & x_3
\end{bmatrix}
\]

\( e := [0 : 0 : 0 : 1] \)

![Figure 4.1: A real, quartic surface with a definite Hermitian determinantal representation and \((\eta, \rho, \sigma) = (0, 0, 0)\).](image)
4.2 One rank-2 point

$(1, 1, 1)$:

\[
\begin{bmatrix}
  x_0 + x_3 & 0 & x_2 + ix_1 & 0 \\
  0 & 2x_0 + x_3 & x_1 & x_1 - ix_2 \\
  x_2 - ix_1 & x_1 & x_2 + x_3 & x_1 + x_2 \\
  0 & x_1 + ix_2 & x_1 + x_2 & x_3 
\end{bmatrix}
\]

\[e := [0 : 0 : 0 : 1]\]

$(1, 1, 0)$:

\[
\begin{bmatrix}
  x_3 & x_0 & x_2 + ix_1 & 0 \\
  x_0 & x_3 & x_1 & x_1 - ix_2 \\
  x_2 - ix_1 & x_1 & x_2 + x_3 & x_1 + x_2 \\
  0 & x_1 + ix_2 & x_1 + x_2 & x_3 
\end{bmatrix}
\]

\[e := [0 : 0 : 0 : 1]\]

Figure 4.2: A real, quartic surface with a definite Hermitian determinantal representation and $(\eta, \rho, \sigma) = (1, 1, 0)$.

4.3 Two rank-2 points

$(2, 2, 2)$:

\[
\begin{bmatrix}
  x_0 + x_1 + x_3 & 0 & x_2 + ix_1 & 0 \\
  0 & 2x_0 + x_1 + x_3 & x_1 & x_1 - ix_2 \\
  x_2 - ix_1 & x_1 & 2x_1 + x_3 & x_1 + x_2 \\
  0 & x_1 + ix_2 & x_1 + x_2 & x_1 + x_3 
\end{bmatrix}
\]

\[e := [0 : 0 : 0 : 1]\]

$(2, 2, 1)$:

\[
\begin{bmatrix}
  x_1 + x_3 & x_0 & x_2 + ix_1 & 0 \\
  x_0 & x_1 + x_3 & x_1 & x_1 - ix_2 \\
  x_2 - ix_1 & x_1 & 2x_1 + x_3 & x_1 + x_2 \\
  0 & x_1 + ix_2 & x_1 + x_2 & x_1 + x_3 
\end{bmatrix}
\]

\[e := [0 : 0 : 0 : 1]\]
(2, 2, 0):
\[
\begin{bmatrix}
 x_3 & x_0 & x_2 & x_1 \\
 x_0 & x_3 & 0 & -ix_2 \\
 x_2 & 0 & x_3 & x_2 + ix_1 \\
 x_1 & ix_2 & x_2 - ix_1 & x_3
\end{bmatrix}
\]
\[e := [0 : 0 : 0 : 1]\]

(2, 0, 0):
\[
\begin{bmatrix}
 2x_3 & x_2 & x_2 + x_3 & x_0 + ix_1 \\
 x_2 & x_3 & x_0 + ix_1 & 0 \\
 x_2 + x_3 & x_0 - ix_1 & x_2 + x_3 & x_2 \\
 x_0 - ix_1 & 0 & x_2 & x_2 + x_3
\end{bmatrix}
\]
\[e := [0 : 0 : 0 : 1]\]

Figure 4.3: A real, quartic surface with a definite Hermitian determinantal representation and \((\eta, \rho, \sigma) = (2, 2, 1)\).

4.4 Three rank-2 points

(3, 3, 3):
\[
\begin{bmatrix}
 x_0 + x_1 + x_2 + x_3 & x_1 + x_2 & -x_1 + x_2 + ix_0 & ix_2 \\
 x_1 + x_2 & x_0 + 2x_1 + 2x_2 + x_3 & x_0 - x_1 + x_2 & x_0 + x_1 + x_2 \\
 -x_1 + x_2 - ix_0 & x_0 - x_1 + x_2 & 2x_0 + x_1 + x_2 + x_3 & x_0 + ix_2 \\
 -ix_2 & x_0 + x_1 + x_2 & x_0 - ix_2 & x_0 + x_1 + 3x_2 + x_3
\end{bmatrix}
\]
\[e := [0 : 0 : 0 : 1]\]

(3, 3, 2):
\[
\begin{bmatrix}
 x_0 + x_2 + x_3 & x_0 & x_0 & 0 \\
 x_0 & 2x_0 + 2x_2 + x_3 & x_0 & x_0 - ix_1 \\
 x_0 & x_0 & x_0 + x_3 & x_1 \\
 0 & x_0 + ix_1 & x_1 & x_0 + x_3
\end{bmatrix}
\]
\[e := [0 : 0 : 0 : 1]\]
(3, 3, 1):
\[
\begin{bmatrix}
  x_2 + x_3 & x_0 & ix_2 & 0 \\
  -ix_2 & x_2 + x_3 & 2x_2 + x_3 & x_1 + x_2 \\
  0 & x_2 + ix_1 & x_1 + x_2 & x_2 + x_3 \\
\end{bmatrix}
\]

\( \mathbf{e} := [0 : 0 : 0 : 1] \)

(3, 3, 0):
\[
\begin{bmatrix}
  x_3 & x_0 & x_2 & 0 \\
  x_0 & x_3 & ix_2 & -ix_1 \\
  x_2 -ix_2 & x_2 + x_3 & x_1 \\
  0 & ix_1 & x_1 & x_3 \\
\end{bmatrix}
\]

\( \mathbf{e} := [0 : 0 : 0 : 1] \)

(3, 1, 1):
\[
\begin{bmatrix}
  x_2 + 2x_3 & 0 & x_3 + ix_2 & x_0 + ix_1 \\
  0 & x_2 + x_3 & x_0 + x_2 + ix_1 & x_2 \\
  x_3 - ix_2 & x_0 + x_2 - ix_1 & 2x_2 + x_3 & x_2 \\
  x_0 - ix_1 & x_2 & x_2 & x_2 + x_3 \\
\end{bmatrix}
\]

\( \mathbf{e} := [0 : 0 : 0 : 1] \)

(3, 1, 0):
\[
\begin{bmatrix}
  2x_3 & 0 & x_3 & x_0 + x_2 + ix_1 \\
  0 & x_3 & x_0 + ix_1 & x_2 \\
  x_3 & x_0 - ix_1 & x_3 & x_2 \\
  x_0 + x_2 - ix_1 & x_2 & x_2 & x_3 \\
\end{bmatrix}
\]

\( \mathbf{e} := [0 : 0 : 0 : 1] \)

Figure 4.4: A real, quartic surface with a definite Hermitian determinantal representation and \((\eta, \rho, \sigma) = (3, 3, 1)\).
4.5 Four rank-2 points

(4, 4, 4): \( M_4(x) := M_{4,0}x_0 + M_{4,1}x_1 + M_{4,2}x_2 + M_{4,3}x_3 \), where

\[
M_{4,0} := \begin{bmatrix} 1 & 9 & 24 & 18 \\ 9 & -23 & 0 & -30 \\ 24 & 0 & -36 & 36 \\ 18 & -30 & 36 & 0 \end{bmatrix}, \quad M_{4,1} := \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 4 & 0 & 6 \\ -3 & 0 & 9 & 0 \\ 0 & 6 & 0 & 9 \end{bmatrix},
\]

\[
M_{4,2} := \begin{bmatrix} 31 & -9 & -12 + 36i & -18 \\ -9 & 43 & 36 & 42 \\ -12 - 36i & 36 & 72 & 0 \\ -18 & 42 & 0 & 0 \end{bmatrix}, \quad M_{4,3} := \begin{bmatrix} 2 & 3 & 6 & 3i \\ 3 & 2 & 3 & -3 \\ 6 & 3 & -6 & 3i \\ -3i & -3 & -3i & 0 \end{bmatrix},
\]

\( e := [0 : 12 : 0 : 1] \)

(4, 4, 3): \( M_4(x) := M_{4,0}x_0 + M_{4,1}x_1 + M_{4,2}x_2 + M_{4,3}x_3 \), where

\[
M_{4,0} := \begin{bmatrix} 26 & 0 & 12 & 18 \\ 0 & -4 & 0 & -6 \\ 12 & 0 & -9 & 9 \\ 18 & -6 & 9 & 0 \end{bmatrix}, \quad M_{4,1} := \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 4 & 0 & 6 \\ -3 & 0 & 9 & 0 \\ 0 & 6 & 0 & 9 \end{bmatrix},
\]

\[
M_{4,2} := \begin{bmatrix} -2 & 0 & -1 + i & -2 \\ 0 & 1 & 1 & 1 \\ -1 - i & 1 & 2 & 0 \\ -2 & 1 & 0 & 0 \end{bmatrix}, \quad M_{4,3} := \begin{bmatrix} 2 & 3 & 6 & 3i \\ 3 & 2 & 3 & -3 \\ 6 & 3 & -6 & 3i \\ -3i & -3 & -3i & 0 \end{bmatrix},
\]

\( e := [0 : 4 : 0 : 3] \)

(4, 4, 2):

\[
\begin{bmatrix} x_0 + x_3 & x_0 & x_2 & 0 \\ x_0 & 2x_0 + x_3 & ix_2 & -ix_1 \\ x_2 & -ix_2 & x_2 + x_3 & x_1 \\ 0 & ix_1 & x_1 & x_3 \end{bmatrix}
\]

\( e := [0 : 0 : 0 : 1] \)

(4, 4, 1):

\[
\begin{bmatrix} x_0 + 2x_2 - 5x_3 & -x_0 + x_2 + 2x_3 & x_0 + x_3 & 2x_0 + ix_3 \\ -x_0 + x_2 - 2x_3 & x_0 - 2x_2 + 8x_3 & 2x_0 - 2x_2 + 7x_3 & x_1 + x_2 + 4x_3 \\ x_0 + x_3 & 2x_1 - 2x_2 + 7x_3 & 2x_1 - 2x_2 + 7x_3 & x_1 + ix_3 \\ 2x_0 - ix_3 & x_1 - x_2 + 4x_3 & x_1 - ix_3 & x_2 \end{bmatrix}
\]

\( e := [0 : 5 : 32 : 10] \)

(4, 4, 0):

\[
\begin{bmatrix} x_3 & x_0 & x_1 + ix_2 & 0 \\ x_0 & x_3 & 0 & -ix_1 \\ x_1 - ix_2 & 0 & x_3 & x_1 + ix_2 \\ 0 & ix_1 & x_1 - ix_2 & x_3 \end{bmatrix}
\]

\( e := [0 : 0 : 0 : 1] \)
Figure 4.5: A real, quartic surface with a definite Hermitian determinantal representation and \((\eta, \rho, \sigma) = (4, 4, 2)\).

4.6 Five rank-2 points

\((5, 5, 3)\):

\[
\begin{bmatrix}
x_0 + x_3 & ix_0 & x_2 & 0 \\
-ix_0 & x_0 + x_3 & ix_2 & -ix_1 \\
x_2 & -ix_2 & x_2 + x_3 & x_1 \\
0 & ix_1 & x_1 & x_3
\end{bmatrix}
\]

e := [0 : 0 : 0 : 1]
(5, 5, 2):
$$\begin{bmatrix}
x_0 + x_2 + x_3 & x_0 + x_2 & x_0 - x_2 & 0 \\
x_0 + x_2 & 2x_0 + 2x_2 + x_3 & x_0 - x_2 & x_0 + x_2 - ix_1 \\
x_0 - x_2 & x_0 - x_2 & x_0 + x_2 + x_3 & x_1 \\
0 & x_0 + x_2 + ix_1 & x_1 & x_0 + x_2 + x_3
\end{bmatrix}$$

\(e := [0 : 0 : 0 : 1]\)

(5, 5, 1):
$$\begin{bmatrix}
x_0 + 9x_1 - x_2 - x_3 & -x_0 + 3x_1 + x_2 - x_3 & x_0 + ix_3 & 2x_0 \\
x_0 + 3x_1 + x_2 - x_3 & x_3 & x_3 & x_3 \\
x_0 - ix_3 & x_3 & x_1 - x_2 + 2x_3 & 4x_1 \\
2x_0 & x_3 & 4x_1 & 4x_2
\end{bmatrix}$$

\(e := [0 : 21 : 22 : 80]\)

Figure 4.6: A real, quartic surface with a definite Hermitian determinantal representation and \((\eta, \rho, \sigma) = (5, 5, 2)\).

4.7 Six rank-2 points

(6, 6, 4):
$$\begin{bmatrix}
x_3 & x_0 & x_1 & 0 \\
x_0 & x_3 & 0 & -ix_1 \\
x_1 & 0 & x_2 + x_3 & x_1 + ix_2 \\
0 & ix_1 & x_1 - ix_2 & x_3
\end{bmatrix}$$

\(e := [0 : 0 : 0 : 1]\)

(6, 6, 3):
$$\begin{bmatrix}
x_0 + x_2 + x_3 & x_0 & x_1 & ix_2 \\
x_0 & x_0 - x_2 + x_3 & x_1 & x_0 \\
x_1 & x_1 & -x_2 + x_3 & ix_2 \\
-ix_2 & x_0 & -ix_2 & x_3
\end{bmatrix}$$

\(e := [0 : 0 : 1 : 3]\)
Figure 4.7: A real, quartic surface with a definite Hermitian determinantal representation and \((\eta, \rho, \sigma) = (6, 6, 3)\). Not all nodes are visible from this angle.

4.8 Seven rank-2 points

\((7, 7, 5)\):

\[
\begin{bmatrix}
 x_0 + x_3 & 0 & x_2 & 0 \\
 0 & x_0 + x_3 & ix_2 & -ix_1 \\
 x_2 & -ix_2 & x_2 + x_3 & x_1 \\
 0 & ix_1 & x_1 & x_3
\end{bmatrix}
\]

\(e := [0 : 0 : 0 : 1]\)

\((7, 7, 4)\):

\[
\begin{bmatrix}
 -x_2 + x_3 & x_0 - ix_0i + x_2 - x_3 & x_1 & -x_0 + x_3 \\
 x_0 + ix_0 + x_2 - x_3 & -x_0 - x_2 + 2x_3 & x_1 & x_0 + ix_0 + x_2 - x_3 \\
 x_1 & x_1 & x_0 & x_2 \\
 -x_0 + x_3 & x_0 - ix_0 + x_2 - x_3 & x_2 & x_3
\end{bmatrix}
\]

\(e := [1 : 0 : 0 : 2]\)

Figure 4.8: A real, quartic surface with a definite Hermitian determinantal representation and \((\eta, \rho, \sigma) = (7, 7, 4)\). Not all nodes are visible from this angle.
4.9 Eight rank-2 points

\( (8, 8, 5) : \begin{bmatrix} -x_2 + x_3 & x_0 - i(x_0 + x_2 - x_3) & x_1 & -x_0 + x_3 \\ x_0 + i(x_0 + x_2 - x_3) & 2x_2 & x_1 & x_0 + i(x_0 + x_2 - x_3) \\ x_1 & x_1 & x_0 & x_2 \\ -x_0 + x_3 & x_0 - i(x_0 + x_2 - x_3) & x_2 & x_3 \end{bmatrix} \)

e := [1 : 0 : 0 : 2]

\( (8, 8, 4) : \begin{bmatrix} 2x_0 + x_3 & ix_0 & x_2 & 0 \\ -ix_0 & x_0 + x_3 & ix_2 & -ix_1 \\ x_2 & -ix_2 & x_2 + x_3 & x_1 \\ 0 & ix_1 & x_1 & x_3 \end{bmatrix} \)

e := [0 : 0 : 0 : 1]

\( (8, 6, 4) : \begin{bmatrix} x_0 + x_3 & 0 & x_2 & 0 \\ 0 & 2x_0 + x_3 & ix_2 & -ix_1 \\ x_2 & -ix_2 & x_2 + x_3 & x_1 \\ 0 & ix_1 & x_1 & x_3 \end{bmatrix} \)

e := [0 : 0 : 0 : 1]

Figure 4.9: A real, quartic surface with a definite Hermitian determinantal representation and \((\eta, \rho, \sigma) = (8, 8, 5)\). Not all nodes are visible from this angle.

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**Author’s address:** Martin Helsø, University of Oslo, Postboks 1053 Blindern, 0316 Oslo, Norway, martibhe@math.uio.no

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