1. Introduction

Marden conjectured that a hyperbolic 3-manifold $M$ with finitely generated fundamental group is tame, i.e. it is homeomorphic to the interior of a compact manifold with boundary [42]. Since then, many consequences of this conjecture have been developed by Kleinian group theorists and 3-manifold topologists. We prove this conjecture in theorem 10.2, actually in slightly more generality for PNC manifolds with hyperbolic cusps (the cusped case is reduced to the non-cusped case in section 6, see the next section for definitions of this terminology).

Many special cases of Marden’s conjecture have been resolved and various criteria for tameness have been developed, see [42, 55, 9, 19, 21, 27, 54, 12, 13, 38]. Bonahon resolved the case where $\pi_1(M)$ is indecomposable, that is $\pi_1(M) \neq A \ast B$. This was generalized by Canary [17, 18] to the case of PNC manifolds with indecomposable fundamental group. Various other cases of limits of tame manifolds being tame have been resolved, culminating in the proof by Brock and Souto that algebraic limits of tame manifolds are tame [55, 21, 12, 13]. In a certain sense, Canary’s covering theorem provides other examples of tame manifolds [20], whereby tameness of a cover implies tameness of a quotient under certain circumstances. Conversely, Canary and Thurston showed that covers (with finitely generated fundamental group) of tame hyperbolic manifolds of infinite volume are tame (this is a purely topological result) [19].

Bonahon in fact proved that hyperbolic 3-manifolds with indecomposable fundamental group are geometrically tame [9]. Geometric tameness was a condition formulated by Thurston (and proved in some special cases by him) which implies that the ends are either geometrically finite or simply degenerate [55]. Canary showed that tame manifolds are geometrically tame [18]. He also showed (generalizing an argument of Thurston [55]) that geometric tameness implies the Ahlfors measure conjecture [3]. Canary’s arguments were generalized for PNC manifolds by Yong Hou [34]. In fact, these arguments give a geometric proof of the Ahlfors finiteness theorem [3]. Other corollaries for Kleinian groups are noted in [19].

Assuming the geometric tameness conjecture, Thurston conjectured that a hyperbolic 3-manifold $N$ is determined by its end invariants: the topological type of $N$, the conformal structure of the domain of discontinuity of each geometrically finite end, and the ending lamination of each simply degenerate end. This was resolved

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last year for tame manifolds by Brock, Canary and Minsky (with major contributions from Masur-Minsky) [50, 47, 49, 43, 44, 48, 16]. This then gives a complete parameterization of isometry types of Kleinian groups. A corollary of the ending lamination conjecture is the density conjecture: any Kleinian group is an algebraic limit of geometrically finite Kleinian groups (shown to be a corollary of the ending lamination conjecture by Kleineidam-Souto [37]).

An application of the covering theorem and geometric tameness (which, as noted, follows from tameness) implies that a cover with finitely generated fundamental group of a finite volume hyperbolic 3-manifold is either geometrically finite or is associated to the fiber of a fibration over $S^1$ (Conjecture C [19]). This has some corollaries. The Simon conjecture [53] says that irreducible (finitely generated) covering spaces of compact 3-manifolds are tame. It follows from Simon’s work and tameness of hyperbolic 3-manifolds that Simon’s conjecture holds for manifolds satisfying the geometrization conjecture.

Another application is that the fundamental group of the figure 8 knot complement is LERF (as well as for the fundamental groups of several other compact 3-manifold groups) [2, 30, 56, 41, 40].

It is shown that the minimal volume orientable hyperbolic 3-manifold has a finite index subgroup which is generated by two elements in [25]. It is conjectured that the Weeks manifold (which has volume $= .9427...$) is the minimal volume orientable hyperbolic 3-manifold. It is known to be minimal volume among arithmetic manifolds [23]. Thus, the commensurator of the fundamental group of the minimal volume orientable hyperbolic 3-manifold must have torsion, so it must cover a non-trivial orbifold with torsion (since a 2-generator group has a hyperelliptic involution which must lie in the commensurator).

Now we outline the argument. The cusped case is reduced to the cusp-free case in section 6. In section 9, it is shown that a PNC manifold with finitely generated fundamental group, satisfying an additional technical condition, has a sequence of homotopy equivalent tame PNC branched covers which limit to it geometrically. In section 13, it is shown how to conclude in this case that the manifold is tame by generalizing an argument of Canary-Minsky [21]. The general case is reduced to this case via section 10 and the PNC covering theorem 14.2. We add an appendix which gives details of an argument of Kleiner [39] which we adapt to show that the boundary of the convex hull of a PNC manifold has area bounded solely in terms of its Euler characteristic and the pinching constants (this is used in the limit argument in section 13).

In several theorems, questions about hyperbolic manifolds have been reduced to questions about Haken manifolds, by “drilling” (or branching) along an appropriately chosen geodesic (see [1],[18],[28]). A reminiscent procedure of grafting allows one to construct geometrically finite manifolds from geometrically infinite ones [15]. The arguments in this paper are inspired by these constructions, and are direct generalizations of Canary’s trick [18] (see also [54]).
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2. Definitions

Let $H^3_b$ be hyperbolic space endowed with a Riemannian metric of constant sectional curvature $b < 0$.

If $C \subset M$ is a subset of a metric space, then we’ll use $N_R(C)$ to denote the set of points of distance $< R$ to $C$.

A PNC (pinched negatively curved) manifold is a Riemannian manifold $(M, g)$, where $g$ is a complete Riemannian metric, and there exist constants $b < a < 0$ such that $b < K(P) < a$, where $P$ is any 2-plane in $G_2(TM)$ and $K(P)$ is the sectional curvature of the metric along $P$ (see [5] for more about PNC manifolds).

Let $\epsilon$ be a Margulis constant for $(M, g)$, $M_{\text{thin}}(\epsilon)$ the set of points at which the injectivity radius is $< \epsilon$ (see 8.3, 10.3 [5] ). A cusp $Q$ of $M$ is a component of $M_{\text{thin}}(\epsilon)$ such that elements of $\pi_1(Q)$ have arbitrarily short loop representatives (equivalently, a cusp is a non-compact component of $M_{\text{thin}}(\epsilon)$). We will denote $M_{\text{thick}}(\epsilon) = M - \text{int} \ M_{\text{thin}}(\epsilon)$.

A PNC manifold $(M^3, g)$ has hyperbolic cusps if for $\epsilon$ small enough, every cusp $Q$ of $M$ is locally isometric to $H^3_b$, for some $b < 0$. Equivalently $Q$ is isometric to a quotient of a horoball in $H^3_b$ by a discrete group of parabolic isometries. We will also consider PNC orbifolds, which have charts locally modelled on a ball in a PNC manifold quotient a finite group of isometries.

A subset $U \subset M$ is convex if for every geodesic $p$ with endpoints $x, y \in U$, $p \subset U$. This is not the usual definition, since we don’t require that $p$ be length-minimizing. Clearly, intersections of convex subsets are convex, so there exists a smallest convex subset. If $M$ is PNC, then let $CH(M)$ denote the convex core, that is the smallest closed convex subset of $M$. $CH(M)$ is homotopy equivalent to $M$, and is generally a codimension zero submanifold, unless $M$ has a totally geodesic invariant subspace.

Let $S$ be a closed surface, with triangulation $\tau$. Let $N$ be a PNC manifold, and $f : S \to N$ a map. Then $f$ is a simplicial ruled surface if $f$ takes each edge of $\tau$ to a geodesic segment, each face of $\tau$ has a foliation by geodesic segments (in fact, this foliation is irrelevant for the arguments, since any such triangular disk will lie in a $\delta$ neighborhood of the edges of the triangle), and the cone angle around every vertex of $\tau$ is $\geq 2\pi$ (see [18, 54]).

A collection of pairwise non-conjugate primitive elements $\alpha \subset \pi_1(M) - \{1\}$ is algebraically diskbusting if for any free product representation $\pi_1(M) = A \ast B$, $A \neq 1 \neq B$, $\alpha \not\subset A \cup B$. This terminology is used, since if $C \subset M$ is a compact core, and
\( \alpha^* \subset C \) is a link whose components represent the conjugacy classes of \( \alpha \), then \( \partial C \) is incompressible in \( M - \alpha^* \), that is \( \alpha^* \) is diskbusting in \( C \).

Let \( M \) be an irreducible, connected, orientable 3-manifold. A 3-manifold with boundary \( J \subset M \) is regular if \( J \) is compact and connected, and \( M - \text{int} \, J \) is irreducible and has no compact component.

A regular exhaustion \( \{ C_n \} \) of \( M \) is a sequence of regular 3-manifolds such that \( C_n \subset \text{int} \, C_{n+1} \) and \( M = \cup_n C_n \).

An open submanifold (but not necessarily properly embedded) \( V \supset J \) is an end reduction of \( M \) at \( J \) if there exists a regular exhaustion \( \{ M_n \} \) with \( V = \cup_n M_n \), such that \( \partial M_n \) is incompressible in \( M - J \), \( V - J \) is irreducible, \( M - V \) has no compact components, and any submanifold \( N \subset M \) with \( J \subset \text{int} \, N \), \( \partial N \) incompressible in \( M - J \) has the property that \( N \) may be isotoped fixing \( J \) so that \( N \subset V \) (this is called the engulfing property).

We need the following generalization of strong convergence of Kleinian groups, which is a type of algebraic and geometric limit (the notion of geometric convergence which we use is much stronger than usual, but suffices for our purposes).

**Definition 2.1.** Let \( N_i \) be a sequence of uniformly PNC manifolds (with finitely generated fundamental group). \( N_i \) converges super strongly to \( M \) if there is a compact core \( C \subset M \), and for all compact connected submanifolds \( K \subset C \subset M \), there is an \( I \) such that for \( i > I \), there is a Riemannian isometry \( \eta_{i,K} : K \to N_i \), with \( \eta_{i,K}|_C : C \to N_i \) is a homotopy equivalence.

**Definition 2.2.** Let \( L \subset M \) be a link in \( M \). Then \( M_L(2,0) \) is the meridional \( \pi \)-orbifold drilling on \( L \), which may be defined in the following way. Take a regular neighborhood \( \mathcal{N}(L) \), then each component is a solid torus (assuming \( M \) is orientable).

If one has a solid torus \( \Delta = D^2 \times S^1 \), let \( K = 0 \times S^1 \), the core of \( \Delta \). There is an involution \( \iota : D^2 \times S^1 \to D^2 \times S^1 \) given by \( \iota(x,y) = (-x, y) \), which is \( \pi \) rotation of \( D^2 \subset \mathbb{C} \) and fixes \( K \). Then \( \Delta/\iota \) has a natural orbifold structure, called \( \Delta_K(2,0) \).

The underlying space is still a solid torus, and we may remove each component of \( \mathcal{N}(L) \), and glue in a copy of \( \Delta_K(2,0) \) so that the meridian orbidisk of \( \Delta_K(2,0) \) is glued to the boundary of the meridian disk of \( \mathcal{N}(L) \). This corresponds to gluing each copy of \( \partial \Delta_K(2,0) \) to \( \partial M - \text{int} \, \mathcal{N}(L) \) by \( (x,y) \equiv (x^2, y) \).

3. PNC Orbifolds

We record the following lemma, whose proof is exactly like that of the manifold case, since the exponential map is an orbifold covering map.

**Lemma 3.1.** (Ghys-Haefliger [29]) If \( \mathcal{O}^n \) is a PNC orbifold, then \( \tilde{\mathcal{O}} \) exists and is a homeomorphic to \( \mathbb{R}^n \) with an induced PNC metric.

We need to use the following branched cover trick, which will occur several times in the argument.

**Lemma 3.2.** Let \( M^3 \) be a PNC manifold, with geodesic link \( \alpha^* \), which has a neighborhood \( \mathcal{N}_R(\alpha^*) \) locally isometric to \( \mathbb{H}^3_b \), \( b < 0 \). Let \( M_{\alpha^*}(2,0) \) be the PNC \( \pi \)-orbifold
obtained by modifying the metric in $\mathcal{N}_R(\alpha^*)$ to get a $\pi$-orbifold along the link $\alpha^*$ [32]. Let $\mathcal{N}_R(\alpha^*) \subset U \subset M$ be a submanifold, and let $V \to M_{\alpha^*}(2,0)$ be an orbifold cover of $M_{\alpha^*}(2,0)$. Suppose that $U_{\alpha^*}(2,0)$ lifts to $V$. Let $V' = V - \mathcal{N}_R(\alpha^*)(2,0) \cup \mathcal{N}_R(\alpha^*)$ be the orbifold obtained by removing the orbifold locus and replacing the original metric on the lift of $\alpha^*$ to $V$. Then $U$ lifts to $V'$.

Proof. It's clear that when we remove $\mathcal{N}_R(\alpha^*)(2,0)$ from $V$, and glue back in $\mathcal{N}_R(\alpha^*)$ to get $V'$, the same operation occurs on the suborbifold $U_{\alpha^*}(2,0)$ to obtain an embedding $U \subset V'$. □

4. END REDUCTIONS

**Theorem 4.1.** (Brin-Thickstun, 2.1-2.3 [11]) Given $J$ a regular submanifold of $M$, an end reduction $V$ at $J$ exists and is unique up to non-ambient isotopy fixing $J$.

Let $\alpha^* \subset M$ be an algebraically diskbusting link, and $V \subset M$ an end reduction of $M$ at $N(\alpha^*)$.

**Theorem 4.2.** (Myers, 9.2 [52]) If $\alpha^* \subset M$ is an algebraically diskbusting link, and $V$ is an end reduction at $N(\alpha^*)$ with $\iota : V \to M$ the inclusion map, then $\iota_* : \pi_1(V) \to \pi_1(M)$ is an isomorphism. It is also clear that $\pi_1(V - \alpha^*)$ injects into $\pi_1(M - \alpha^*)$.

5. PARED MANIFOLD COMPRESSION BODIES

In [8], Bonahon shows that a compact, irreducible 3-manifold can be decomposed along incompressible surfaces into finitely many compression bodies and manifolds with incompressible boundary, which are unique up to isotopy. Thus, if one has a PNC 3-manifold $M$ with finitely generated fundamental group and core $C$, the ends of $C$ will correspond to boundary components of $M$, which are either incompressible or compressible. The ends corresponding to incompressible boundary will be tame, by Canary [17]. Each end corresponding to a compressible boundary component will be associated to a compression body $B$ in Bonahon’s decomposition. Passing to the cover $N$ of $M$ corresponding to the fundamental group of this compression body $B$, $B$ lifts to a core of $N$, and thus the end will be associated to the unique compressible boundary component of $B$. Again, by Bonahon’s result [9], the other ends of $N$ corresponding to incompressible boundary components of $B$ will be tame. Thus, to prove that $M$ is tame, it suffices to prove that the end of $N$ corresponding to the compressible boundary component of $B$ is tame.

We assume that $M$ has a complete PNC Riemannian metric with hyperbolic cusps, whose union we will call $H$. If we choose a small enough Margulis constant $\epsilon$, then we may assume that $H$ is the union of non-compact components of $M_{\text{thin}(\epsilon)}$. Then we will denote $M - \text{int } H = M_H$, the neutered manifold. $\partial M_H$ consists of flat open annuli or tori locally isometric to a horosphere in $H^3_{-\epsilon}$. By the relative core theorem [45], there is a core $(C_H, P) \subset (M_H, \partial M_H)$ which is a homotopy equivalence of pairs, such that $C_H \cap \partial M_H = P$. The manifold $(C_H, P)$ is a pared manifold: $P$ consists
of incompressible annuli and tori in $\partial C_H$ such that no two annuli are parallel in $C_H$ and every torus component of $\partial C_H$ is $\subset P$ (see Def. 4.8, [51]).

6. Reduction of the cusped case to the case with no cusps

As in the previous section, we assume that the compact core is a compression body. We’ll also assume that the manifold is orientable, by passing to a 2-fold orientable cover. For each rank 2 cusp, we perform a high Dehn filling to obtain a PNC manifold with the same (non-cusp) ends, using the method of the Gromov-Thurston $2\pi$ theorem ([32, 6]). The Scott core will remain a compression body.

So we may assume that all of the cusps are rank 1. For each rank one cusp $Q$, $\partial Q$ is an infinite annulus. Fill along the core curve of this annulus by a $2\pi/n$ orbifold, where $n$ is large enough to obtain a PNC orbifold $M'$, again using the method of the $2\pi$ theorem ([32, 6]). The Scott core $C'$ of $M'$ is obtained by adding orbifold 2-handles to the components of the pared locus $P \subset C_H$. Since $M'$ is PNC, $C'$ will be an atoroidal, irreducible orbifold. By the orbifold theorem [7, 24], int $C'$ has a hyperbolic structure. Thus, we may pass to a finite sheeted manifold cover of $M'$ by Selberg’s lemma. If we can show that this cover is tame, then $M'$ will be tame by the finite covering theorem 14.3, and thus our original manifold $M$ is tame.

7. Metric modulo $\epsilon$-thin part

Assume that $M^3$ is a PNC manifold with no cusps. We may choose $\epsilon$ small enough such that each pair of components of $M_{\text{thick}}(\epsilon)$ has distance $> \epsilon$. Define a distance function $d_\epsilon$ on $M_\epsilon = M_{\text{thick}}(\epsilon)$ by $d_\epsilon(x, y) = \inf_{p} \text{length}(p \cap M_\epsilon)$, where $p$ is a path connecting $x$ and $y$. $d_\epsilon$ is a metric on int $M_\epsilon$, whose completion is obtained from $M$ by crushing components of $M_{\text{thick}}(\epsilon)$ to points. Let $N^\epsilon_R(C)$ denote the $R$ neighborhood of $C$ in the metric $d_\epsilon$.

**Lemma 7.1.** If $S \subset M$, and $\text{diam}(S) = \infty$, then $\text{diam}_{d_\epsilon}(S \cap M_\epsilon) = \infty$.

**Proof.** Fix $x \in S \cap M_\epsilon$. Then if $d_\epsilon(x, y) \leq D$, and $p$ is a path in $M$ joining $x$ and $y$ such that $\text{length}(p \cap M_\epsilon) = d_\epsilon(x, y)$, then $p$ may intersect at most $D/\epsilon$ components of $M_{\text{thick}}(\epsilon)$, since the distance between components is at least $\epsilon$. Also, it is clear that each segment of $p \cap M_\epsilon$ has length $\leq D$. Let $N = \lfloor D/\epsilon \rfloor$, and inductively define for $0 \leq k \leq N$, $B_0 = \{x\}$, and $B_k = \mathcal{N}_D(B_{k-1} \cup U_k)$, where $U_k$ is the collection of components of $M_{\text{thick}}(\epsilon)$ meeting $B_{k-1}$. If $B_{k-1}$ is compact, then $U_k$ has finitely many components, so $B_k$ is also compact. Thus, $B_N$ is compact, therefore has bounded diameter, and clearly $\mathcal{N}^\epsilon_{d_\epsilon(x, D)}(x) \subset B_N$, so we see that $\mathcal{N}^\epsilon_{d_\epsilon(x, D)}(x)$ has bounded diameter. Thus, $S \cap M_\epsilon \not\subset \mathcal{N}^\epsilon_{d_\epsilon(x, D)}(x)$ for any $D$, so $\text{diam}_{d_\epsilon}(S \cap M_\epsilon) = \infty$. $\Box$

8. Bounded diameter lemma

The following lemma in various forms is due to Thurston (8.8.5 [55]), Bonahon (1.10 [9]), Canary (3.2.6 [17]), and Canary-Minsky (5.1 [21]). We state it for PNC orbifolds $M$ with no cusps. The proof is essentially the same as the previous proofs.
Lemma 8.1. Given $A \in \mathbb{N}$, and a Margulis constant $\epsilon$, there are constants $D$ and $\mu$ satisfying the following: if $f: \Sigma \to M$ is a simplicial ruled surface with $-\chi(\Sigma) \leq A$, such that every compressible or accidental elliptic curve has length $> \mu$, then $\text{diam}_d(f(\Sigma) \cap M_i) \leq D$.

9. Drilling and geometric limits

Let $(M, \nu)$ be a connected orientable 3-manifold with a PNC metric $\nu$ with no cusps such that $\pi_1(M)$ is finitely generated. We may assume that $M$ has a compact core $C$ which is a compression body. Let $\alpha \subset \pi_1(M)$ be a collection of algebraically diskbusting elements. Let $\alpha^*$ be a union of geodesic representatives of (the conjugacy classes of) $\alpha$ in $M$ (we know that each curve will have distinct representatives, since the elements of $\alpha$ are primitive and non-conjugate). Perturb the metric on $M$ near $\alpha^*$ to a PNC metric $\nu'$ such that $\alpha^*$ is homotopic to an embedded geodesic $\alpha'$, by the method of Lemma 5.5, [18]. Then we may deform the metric near $\alpha'$ to a metric $\nu''$ so that the metric near $\alpha''$ (the geodesic representative of $\alpha$ in $\nu''$) is locally isometric to $\mathbb{H}_b^3$, $b < 0$, by Lemma 3.2 and 4.2, [34]. To simplify notation, we will denote this new metric by $\nu$, and we will assume that the geodesic representative of $\alpha$ is a geodesic link $\alpha^*$ with neighborhood locally isometric to $\mathbb{H}_b^3$.

Let $V$ be an end reduction of $M$ at $\mathcal{N}(\alpha^*)$, let $\{M_i\}$ be a regular exhaustion of $V$, $\alpha^* \subset M_0$, so that $\partial M_i$ is incompressible in $M - \alpha^*$. We may choose a compact core $C \subset V$, and assume that $C \subset M_0$.

Definition 9.1. Let $V \subset M$ be an open irreducible submanifold such that $V = \bigcup_i M_i$ is an exhaustion by regular submanifolds containing $\mathcal{N}(\alpha^*)$ and so that inclusion $V \hookrightarrow M$ induces an isomorphism $\pi_1 V \to \pi_1 M$, and $\partial M_i$ is incompressible in $\pi_1(M - \alpha^*)$. We call such a submanifold an almost end reduction.

These conditions follow if $V$ is an end reduction of $M$ at $\alpha^*$ by 4.2, since $\alpha^*$ is algebraically diskbusting, thus an end reduction is an almost end reduction in this case.

Theorem 9.2. Assume that $\pi_1(V_{\alpha^*}(2,0)) \to \pi_1(M_{\alpha^*}(2,0))$ is an isomorphism, where $V$ is an almost end-reduction (recall def. 2.2 for the notation). Under the above hypotheses, $M$ is a super strong limit of PNC tame $N_i$.

Proof. For simplicity, let’s first consider the case that $M$ can be exhausted by compact cores $M = \bigcup_i C_i$ (in which case, we may assume that $V = M$). This case was shown by Souto to be tame [54], but we will outline how our approach works in this case, since several aspects of the argument are simplified (this argument is actually closely related to Souto’s).

The orbifold $M_{\alpha^*}(2,0)$ is exhausted by suborbifolds $C_{i,\alpha^*}(2,0)$ (for $i \gg 0$) such that $\pi_1(C_{i,\alpha^*}(2,0)) \hookrightarrow \pi_1(M_{\alpha^*}(2,0))$. Since $\pi_1(C_i) \simeq \pi_1(M)$, $\partial C_i$ is incompressible in $M - \text{int} (C_i)$. If $\pi_1(\partial C_i)$ does not inject into $C_{i,\alpha^*}(2,0)$, then by the equivariant Dehn’s lemma, there is an orbifold disk $(D, \partial D) \subset (C_{i,\alpha^*}(2,0), \partial C_i)$. $D$ cannot be disjoint from $\alpha^*$, since $\alpha$ is algebraically diskbusting. So $D$ must be an orbifold disk,
which meets \(\alpha^*\) exactly once. But this means that \(\alpha^*\) represents a generator of \(\pi_1(C_i)\), which means that it is not algebraically diskbusting, a contradiction.

Now, let \(C_i' \to M_{i,\alpha^*}(2,0)\) be the orbifold cover such that \(\pi_1(C_i') = \pi_1(C_{i,\alpha^*}(2,0))\). Then \(\pi_1(C_i')\) is indecomposable, so by Canary’s theorem [18], \(C_i'\) is tame. We have a lift \(C_{i,\alpha^*}(2,0) \hookrightarrow C_i'\), which is a homotopy equivalence of orbifolds. We may create a branched cover of \(M\) by taking \(N_i = C_i' - C_{i,\alpha^*}(2,0) \cup C_i\), where we glue along \(\partial C_i\) (this corresponds to “erasing” the orbifold locus of \(C_i'\), as in lemma 3.2). \(N_i\) has a PNC metric, since we have really only inserted the original metric in \(N(\alpha^*)\) back into \(C_i'\).

The claim is that the manifolds \(N_i\) limit super strongly to \(M\) (recall def. 2.1). To see this, let \(K \subset M\) be a compact subset. Then we may choose \(i\) large enough that \(K \subset C_i\). Then \(C_i\) lifts isometrically up to \(N_i\) by lemma 3.2, and therefore so does \(K\). If \(C \subset K\) is a compact core, then the lift will induce a homotopy equivalence \(\pi_1(C) \simeq \pi_1(C_i') \simeq \pi_1(N_i)\).

In general \(M\) will not be exhausted by compact cores. To try to mimic the above proof, we use the notion of the almost end reduction.

**Lemma 9.3.** \(\pi_1(M_{i,\alpha^*}(2,0)) \to \pi_1(M_{i,\alpha^*}(2,0))\) is an injection for large enough \(i\).

**Proof.** By the definition of almost end reduction, \(\partial M_i\) is incompressible in \(M - \text{int} (M_i)\) and in \(M - \alpha^*\). If \(\pi_1(\partial M_i)\) does not inject into \(\pi_1(M_{i,\alpha^*}(2,0))\), then by the equivariant Dehn’s lemma [46], there exists an orbifold disk \(D_i\) which meets \(\alpha^*\) exactly once (since \(M_i\) is a compression body, we may take the 2-fold branched cover over \(\alpha^*\). Since \(\alpha^*\) does not meet \(\partial M_i\), the only possibility is for the 2-fold action to fix a disk by rotation about a point. One may also prove this using the orbifold theorem [7, 24]). In \(M_i' = M_i - \text{int} N(\alpha^*)\), we get an annulus \(A_i = D_i \cap M_i'\) such that one boundary component lies on \(\partial M_i\) and one lies on a meridian of \(\partial N(\alpha^*)\). Then for \(j > i\), \(A_j\) may be isotoped so that \(A_j \cap M_i'\) is a collection of essential annuli, and has exactly one annulus with a boundary component a meridian of \(\partial N(\alpha^*)\). Let \(H_i\) be the characteristic product region of \(M_i'\) [35, 36], then we may assume that \(A_i \subset H_i\).

There are finitely many components of \(H_i\) of the form \(S^1 \times [0, 1]^2\) which meet \(\partial N(\alpha^*)\) (exercise), and any annulus with one boundary on a meridian of \(\partial N(\alpha^*)\) and the other on \(\partial M_i\) must intersect the core of one of these annuli. We may take a subset \(W \subset \mathbb{N}\) such that for \(i \in W\), for all \(j \in W, j > i, A_j \cap M_i'\) has the same annulus meeting \(\partial N(\alpha^*)\) up to isotopy (which we relabel to be \(A_i\), and relabel \(W\) to be \(\mathbb{N}\)), and such that \((|A_j \cap M_j'|, ..., |A_j \cap M_1'|, |A_j \cap M_0'|)\) is minimal with respect to lexicographic order.

By Haken finiteness, there are only finitely many disjoint non-parallel annuli in \(A_j \cap M_i'\). We may assume that no two are parallel, otherwise we could cut and paste to get an annulus \(A_j'\) in \(M_j'\) meeting \(M_i'\) in fewer components, which decreases \((|A_j \cap M_j'|, ..., |A_j \cap M_1'|, |A_j \cap M_0'|)\) with respect to lexicographic order. Since for \(k > j > i\), \(A_k \cap M_j' \supset A_j\), we see that \(A_j \cap M_i'\) is isotopic to a subset of \(A_k \cap M_i'\). Thus, \(A_j \cap M_i'\) stabilizes. Then we may form the union \(\cup_i A_i\) to get a properly embedded punctured disk in \(V - \text{int} N(\alpha^*)\), which capping off with a meridian disk of \(N(\alpha^*)\),
gives a properly embedded plane $P \subset V$ meeting $\alpha^*$ exactly once. But this implies that the component $\alpha'$ of $\alpha^*$ meeting $P$ is a free generator of $\pi_1(V)$, and therefore $\alpha$ is not a collection of algebraically disk-busting elements, since $\pi_1(V) = \langle \alpha' \rangle \ast \pi_1(V - P)$, where $\alpha - \alpha' \subset \pi_1(V - P)$.

Given this lemma, we may assume that $C \subset M_0$ and $\pi_1(M_{i,\alpha^*}(2,0)) \hookrightarrow \pi_1(M_{\alpha^*}(2,0))$. Let $M'_i \to M_{\alpha^*}(2,0)$ be the orbifold covering such that $\pi_1(M'_i) = \pi_1(M_{i,\alpha^*}(2,0))$, with corresponding lift $M_{i,\alpha^*}(2,0) \hookrightarrow M'_i$. Let $M''_i$ be the manifold which is obtained from $M'_i$ by deleting the orbifold locus. That is, we replace $\mathcal{N}(\alpha^*)(2,0) \subset M'_i$ by $\mathcal{N}(\alpha^*)$ with its original metric, to get the PNC manifold $M''_i$. Then $M''_i$ is a branched cover of $M$, branched over $\alpha^*$. We have a lift $M_i \hookrightarrow M''_i$ obtained by extending $M_i - \mathcal{N}(\alpha^*) \hookrightarrow M'_i$ over $\mathcal{N}(\alpha^*)$, by lemma 3.2. Since $C \subset M_0 \subset M_i$, we have a lift $C \hookrightarrow M''_i$, so let $N_i$ be the cover of $M''_i$ corresponding to $im\{\pi_1(C) \to \pi_1(M''_i)\}$. Note that this construction will be trivial if $V$ is itself tame, and corresponds to the same construction as described above when $M$ is exhausted by compact cores.

Let $K \subset M$ be a compact connected submanifold, and assume $\mathcal{N}(\alpha^*) \subset \text{int } K$. Since $C$ is a compact core for $M$, we may homotope $K$ into $C$, such that the homotopy lies in a connected compact submanifold $K' \subset M$ (where we assume $C \subset K'$). Thus, $im\{\pi_1(K) \to \pi_1(K')\} \subset im\{\pi_1(C) \to \pi_1(K')\}$. Since we are assuming that $\pi_1(M_{\alpha^*}(2,0)) = \pi_1(V_{\alpha^*}(2,0)) = \cup_i \pi_1(M_{i,\alpha^*}(2,0))$, there exists $i$ such that $\pi_1(K_{\alpha^*}(2,0)) \subset \pi_1(M_{i,\alpha^*}(2,0))$. Then there is a lifting $K_{\alpha^*}(2,0) \to M'_i$, by the covering lifting theorem, and therefore we have a lifting $K' \to M''_i$ by lemma 3.2. Because $im\{\pi_1(K) \to \pi_1(M''_i)\} \subset im\{\pi_1(C) \to \pi_1(M''_i)\}$, we may then lift $K \hookrightarrow N_i$. Thus, $M$ is a geometric limit of $N_i$. By our construction, $N_i \simeq M_i$, and $C \subset K \subset M$ a compact core, then $K \hookrightarrow N_i$ restricts to $C \hookrightarrow N_i$ so that $C \simeq N_i$. Thus, $M$ is a super strong limit of $N_i$.

Notice that $M'_i$ has indecomposable fundamental group, and therefore is tame by Bonahon/Canary [9, 17, 18]. Thus, $M''_i$ is tame, and therefore $N_i$ is tame by Cor. 3.2, [19].

10. DRILLING AND TAMENESS

Recall that we have a PNC manifold $M$ and an algebraically diskbusting link $\alpha^* \subset M$. We would like to have $\pi_1(V_{\alpha^*}(2,0)) \cong \pi_1(M_{\alpha^*}(2,0))$, but this might not hold in general.

Lemma 10.1. $M$ is a PNC manifold with $\pi_1(M)$ finitely generated. Let $\alpha^*$ be an algebraically diskbusting geodesic link in $M$, $V$ an almost end-reduction of $M$ at $\alpha^*$. Then $V$ is tame, and may be isotoped so that $V = \text{int } C$, where $C$ is a compact core of $M$.

Proof. Take the orbifold cover $N \to M_{\alpha^*}(2,0)$ such that $\pi_1(N) = \pi_1(V_{\alpha^*}(2,0))$. Then $V_{\alpha^*}(2,0)$ lifts isometrically to $N$. Thus we may fill $\mathcal{N}(\alpha^*)$ back in to get a
negatively curved manifold $N'$ which is a branched cover of $M$ and an isometric embedding $\iota: V \hookrightarrow N'$ by lemma 3.2. Clearly $\iota(V)$ is an almost end-reduction at $\iota(N(\alpha^*)) \subset N'$. By construction, $\pi_1 V_{\alpha^*}(2,0) = \pi_1 N$, so by theorem 9.2, $N'$ is tame. By Haken finiteness for $N' - N(\alpha^*)$, there exists $I$ such that $\partial M_i$ is parallel to $\partial M_j$ in $N' - N(\alpha^*)$ for $i, j > I$, so that they also must be parallel in $V - N(\alpha^*)$. This means that $V$ is tame and may be taken to be the interior of a compact core $C \subset M$, $V = \text{int } C$, by the uniqueness up to isotopy (theorem 4.1).

What we’ve just shown is that every algebraically diskbusting geodesic link $\alpha^*$ is contained in a compact core $C$ of $M$.

**Theorem 10.2.** Let $M$ be PNC (without cusps), and $\pi_1(M)$ finitely generated. Then $M$ is tame.

**Proof.** We may assume that if $C$ is a compact core for $M$, then $C$ is a compression body. Thus all but one of the ends of $M$ will correspond to incompressible surfaces in $\partial C$, so by Canary’s theorem [18] these ends will be tame. Thus, there is a single end $E$ corresponding to the compressible end of $\partial C$ which we need to show is tame. There is a sequence of geodesics $\beta_i^*$ exiting $E$, by the argument of Theorem 3.10, [15]. Now, if $\alpha \in \pi_1 M$ is an algebraically diskbusting element, then $\alpha^* \cup \beta_i^*$ is an algebraically diskbusting link (possibly after perturbing the metric on $M$ slightly). By lemma 4.1, there exists an end reduction $V_i$ at $\alpha^* \cup \beta_i^*$. By lemma 4.2 $V_i$ is an almost end reduction at $\alpha^* \cup \beta_i^*$. Thus, there is a compact core $C$ of $M$ such that $\alpha^* \cup \beta_i^* \subset C$ by lemma 10.1. Since $\alpha$ is algebraically diskbusting, $\partial C$ is incompressible in $M - \alpha^*$. But this implies that we may assume that $C \subset V$, where $V$ is the end-reduction of $M$ at $\alpha^*$, by the engulfing property. Thus, $\beta_i^* \subset V_{\alpha^*}(2,0)$ lifts to $V_{\alpha^*}(2,0) \subset N$, where $N$ is the orbifold cover of $M_{\alpha^*}(2,0)$ corresponding to $\pi_1(V_{\alpha^*}(2,0))$, as in the proof of lemma 10.1. Then the sequence of lifts of geodesics $\beta_i^*$ must exit the end $F$ of $N'$ corresponding to the compressible component of $\partial C$. Thus, the compressible end $F$ of $N'$ is geometrically infinite, and is therefore simply degenerate by [18, 34]. By the PNC covering theorem 14.2, the end $F$ of $N$ must cover finite-to-one an end $E$ of $M_{\alpha^*}(2,0)$. But the incompressible ends of $N'$ cover one-to-one the incompressible ends of $M$, which means that $F$ must cover the compressible end $E$ finite-to-one, and therefore $E$ is tame by 14.3. □

11. **Isoperimetric inequality**

Let $\tilde{\mathcal{CH}}(M) \subset \tilde{M}$ be the universal cover of $\mathcal{CH}(M)$, where $(M, g)$ is a PNC manifold. Let $s \subset \partial \tilde{\mathcal{CH}}(M)$ be a simple closed curve which is contractible in $\partial \tilde{\mathcal{CH}}(M)$. Then $s = \partial D$, $D \subset \tilde{\mathcal{CH}}(M)$.

**Lemma 11.1.** There exists a constant $r$ depending only on the pinching constants, such that $D \subset N_{r + \text{diam}(s)/2}(s)$. 
Proof. Consider $\mathcal{CH}(s) \subset \widetilde{\mathcal{CH}(M)}$. We claim that $D \subset \partial \mathcal{CH}(s)$. Since $\mathcal{CH}(s)$ is a ball, and $s \subset \partial \mathcal{CH}(s)$ (since $s \subset \partial \mathcal{CH}(M)$), then $s$ separates $\partial \mathcal{CH}(s)$ into two disks $\partial \mathcal{CH}(s) = E_1 \cup E_2$, $\partial E_1 = s$. Let $E_1$ be the disk which is closer to $D$. Then $E_1 \cup D$ is the frontier of a closed contractible subset $B \subset \mathcal{CH}(M)$. Then $U = \mathcal{CH}(M) - B \cup E_1$ is a closed convex set. If not, then there is a pair of points $x, y \in U$ and a geodesic $[x, y]$ connecting $x$ and $y$, such that there is a subinterval $[x_1, y_1] \subset M - \text{int } U$ (since $E_1$ is separating in $\mathcal{CH}(M)$). Then $x_1, y_1 \subset \partial U$, so they must lie in $E_1$, otherwise we would violate the convexity of $\mathcal{CH}(M)$. But $E_1 \subset \mathcal{CH}(s)$ which is convex, so $[x_1, y_1] \subset \mathcal{CH}(s) \subset U$, a contradiction. Thus, $\mathcal{CH}(M) \subset U$, so $D = E_1$, as claimed.

For a set $Q$, let $\text{join}(Q)$ be the union of all geodesics connecting pairs of points in $Q$. Clearly $\text{join}(Q) \subset \mathcal{N}_{\text{diam}(Q)/2}(Q)$. Now, by the arguments on pp. 241-243 of [10], $\mathcal{CH}(s) \subset \mathcal{N}_r(\text{join}(s))$, for some $r$ which depends only on the pinching constants of $(M, g)$. Thus, we have $D \subset \mathcal{CH}(s) \subset \mathcal{N}_{r + \text{diam}(s)/2}(s)$. \hfill $\square$

12. Interpolation of simplicial ruled surfaces

The following arguments are modifications of work of Brock in [14], as a way of generalizing the simplicial ruled surface interpolation techniques of Thurston [55] and Canary [19]. We were unable to generalize Canary’s method directly from the hyperbolic case to the PNC case. Let $\Sigma$ be a closed surface of negative Euler characteristic. A pants decomposition $P$ of $\Sigma$ is a maximal collection of isotopy classes of disjoint non-parallel essential simple closed curves on $\Sigma$. Two distinct pants decompositions $P$ and $P'$ are related by an elementary move if $P'$ can be obtained from $P$ by replacing a curve $\alpha \in P$ by a curve $\beta \in P'$ intersecting $\alpha$ minimally (and disjoint from the other curves of $P$, i.e. $\alpha \cap P' = \beta \cap P = \alpha \cap \beta$, and $|\alpha \cap \beta| = 1$ if $\alpha$ is non-separating in the component of $\Sigma - (P - \alpha)$ containing $\alpha$, and $|\alpha \cap \beta| = 2$ if $\alpha$ is separating in the component of $\Sigma - (P - \alpha)$ containing $\alpha$). The pants graph $P(\Sigma)$ is the graph with pants decompositions of $\Sigma$ as vertices, and edges joining pants decompositions which differ by an elementary move.

Given a pants decomposition $P$ of $\Sigma$, we may extend the collection of curves to a standard triangulation. These triangulations are obtained by adding a pair of vertices to each curve of $P$, then extending to a triangulation of each pants region by adding a maximal collection of non-parallel edges (we allow any extension, unlike Brock).

Given a pants decomposition $P$ of $\Sigma$, a standard triangulation $T$ extending $P$, and a $\pi_1$-injective map $f : \Sigma \to N$, where $N$ is a PNC orbifold, we may homotope the map $f : \Sigma \to N$ to a simplicial ruled surface. First, we homotope the loops of $P$ to be geodesic, then we homotope the other edges of $T$ to be geodesic, then we simplicially interpolate the triangles of $T$. This gives a simplicial ruled surface $f^* : \Sigma \to N$, by prop. 3.2.7 [17].

Given a fixed pants decomposition $P$, two standard triangulations $T_1, T_2$ associated to $P$, and simplicial ruled representatives $f_1^* : \Sigma \to N$, $f_2^* : \Sigma \to N$ in a homotopy
class \( f : \Sigma \to N \), we may interpolate continuously between these simplicial ruled representatives. This is done as in Claim 3 of the proof of theorem 2.3, [4]. The moves consist of changing the ruling on a triangle or doing a flip move between triangles which are adjacent in \( \Sigma - P \). We also need a move which moves a vertex along a geodesic representative of a curve of \( P \), so that we can get the vertices to line up. All of these moves preserve the simplicial ruled conditions.

We also need moves to get from \( P \) to \( P' \), where \( P \) and \( P' \) are adjacent pants decompositions. We may choose simplicial ruled representatives \( f_1^* : \Sigma \to N, f_2^* : \Sigma \to N \), where \( f_1^* \) has a standard triangulation with respect to \( P \), and \( f_2^* \) has a standard triangulation with respect to \( P' \).

Claim: We may continuously move \( f_1^* \) and \( f_2^* \) through simplicial ruled surfaces to \( f_1' \) and \( f_2' \), so that \( f_1' \) is homotopic to \( f_2' \) in \( \mathcal{N}_R(f_1'(\Sigma) \cup f_2'(\Sigma)) \), where \( R \) only depends on the pinching constants.

We need only show how the triangulations behave on the 4-punctured sphere and punctured torus regions; we may assume that the standard triangulations agree on the rest of the pair of surfaces, by using the previously described moves to make the standard triangulations line up on this subsurface. For the 4-punctured sphere, insert arcs \( \alpha_1, \alpha_2 \) disjoint from \( \alpha \) and \( \beta_1, \beta_2 \) disjoint from \( \beta \) into both 4-punctured spheres in \( \Sigma \) so that the corresponding edges in \( f_1'(\Sigma) \) and \( f_2'(\Sigma) \) are homotopic in \( M \) rel endpoints, then complete this to a triangulation. \( \alpha \cup \alpha_1 \cup \alpha_2 \) will be geodesic in \( f_1' \), so \( \beta_1 \cup \beta_2 \) will be composed of two geodesics (since these meet \( \alpha \) once). In the other surface \( f_2' \), \( \beta_1 \cup \beta_2 \) will be geodesic. Together, \( f_1'((\beta_i) \cup f_2'((\beta_i)) \) will form a geodesic triangle, and similarly for \( f_1'((\alpha_i) \cup f_2'((\alpha_i)) \), \( i = 1, 2 \) (see figure 12).

There are four such triangles, which we may assume are simplicial ruled. The triangles will lie in a uniform \( \delta \) neighborhood of the edges, by \( \delta \)-hyperbolicity (where \( \delta \) only depends on the pinching constants). If we take four triangles, and one half of each 4-punctured sphere, we get a map \( f : S^2 \to M \) of a sphere of bounded area (depending only on the pinching constants). Let \( \tilde{f} : S^2 \to \tilde{M} \) denote a lift of \( f \) to \( \tilde{M} \).

**Lemma 12.1.** Let \( \tilde{M} \) be a PNC simply connected manifold. Let \( \tilde{f} : S^2 \to \tilde{M} \), then \( \tilde{f} \) is homotopically trivial in \( \mathcal{N}_R(\tilde{f}(S^2)) \), where \( R \) depends only on the area of \( \tilde{f} \).

**Proof.** If we add the bounded complementary regions of \( \tilde{f}(S^2) \), and take a small regular neighborhood to get a submanifold \( U \subset \tilde{M} \), so that \( \tilde{M} - U \) has no bounded components, then \( \pi_2(U) = 0 \) by standard arguments, so \( \tilde{f} \) is homotopically trivial in \( U \). Thus, if \( \tilde{f} \) is not homotopically trivial in \( \mathcal{N}_R(\tilde{f}(S^2)) \), this means that \( \tilde{M} - \mathcal{N}_R(\tilde{f}(S^2)) \) must have a bounded component \( V \) which is contained in a bounded component \( V' \) in the complement of \( \tilde{f}(S^2) \). Choose \( x \in V \), then \( d(x, \tilde{f}(S^2)) > R \). But \( \partial V' \subset \tilde{f}(S^2) \) has \( \text{Area}(\partial V') < \text{Area}(\tilde{f}(S^2)) \). Since \( \text{Vol}(V') > \text{Vol}(B_R(x)) \), by the isoperimetric inequality for \( \tilde{M} \) (e.g. [39]), this implies that \( \text{Area}(\tilde{f}(S^2)) > \text{Area}(\partial V') \) > \( \text{Area}(\partial B_R(x)) \). Thus, choosing \( R \) large enough that \( \text{Area}(\partial B_R(x)) > \text{Area}(\tilde{f}(S^2)) \), we get a contradiction. \( \square \)
By this lemma, \( f \) is homotopically trivial in \( \mathcal{N}_R(f(S^2)) \), by projecting \( \mathcal{N}_R(\tilde{f}(S^2)) \) to \( M \). Thus \( f'_1 \) and \( f'_2 \) will be homotopic in \( \mathcal{N}_R(f'_1(\Sigma) \cup f'_2(\Sigma)) \). A similar construction works for elementary moves involving punctured tori. Since in \( \mathcal{N}_R(f'_1(\Sigma) \cup f'_2(\Sigma)) \), there is a homotopy between the two surfaces, every point in the support of this homotopy will be within a distance \( R \) of a simplicial ruled surface. A theorem of Hatcher, Lochak, and Schneps [33] implies that the pants complex is connected. This implies that we may interpolate between any two simplicial ruled surfaces realizing pants decompositions by surfaces which are bounded distance from simplicial ruled surfaces.

13. TAMENESS OF LIMITS

The following is a generalization of the main theorem of [21].

Lemma 13.1. Suppose \( N_i \) are tame PNC manifolds (with no cusps) which super strongly converge to \( M \). Then \( M \) is tame.

Proof. We may restrict to the case that \( M \) has a compression body core \( C \), since this is the case that we use in the application to tameness. We may assume that the compressible end of \( M \) is not geometrically finite, otherwise we are done.

The first step of the argument is to show that \( \partial \mathcal{C}H(N_i) \to \infty \), in the appropriate sense (Dick Canary suggested this line of argument, which gives a simplification over our previous approach). For simplicity, we will assume that \( M \) has only one end. To modify the argument appropriately, replace \( \mathcal{C}H(N_i) \) with the convex submanifold...
$\mathcal{C}H(N_i) \cup E$, where $E$ consists of the ends of $M - \mathcal{C}H(N_i)$ corresponding to the incompressible components of $\partial C$.

Claim: for all compact connected $K \subset M$, there exists $I$ such that for $i > I$, $\eta_{i,K}(K) \subset \mathcal{C}H(N_i)$ (where the maps $\eta_{i,K}$ come from the definition 2.1 of super strong convergence).

Fix a compact core $C$ for $M$. Suppose not, then there is a connected $K \subset M$ and $J \subset \mathbb{N}$ with $|J| = \infty$, such that $\eta_{i,K}(K) \notin \mathcal{C}H(N_j)$, for all $j \in J$. We may choose an algebraically disk-busting geodesic $\alpha^* \in M$, and assume $C \cup N(\alpha^*) \subset K$ by enlarging $K$. Then there is an $I$ such that for $i \geq I$, we have isometric maps $\eta_{i,\alpha^*} : \alpha^* \to N_i$.

Then there is a $\delta$ depending on $\epsilon$, the Margulis constant, such that $diam_{*}(\partial \mathcal{C}H(N_j)) \leq A/\epsilon + \delta$, for $j \in J$.

The region $N_R^t(\eta_{j,K}(\alpha^*))$ is a finite union of tubular neighborhoods of Margulis tubes and geodesics. Thus, we may perturb $\partial \mathcal{C}H(N_j)$ slightly to meet $\partial N_R^t(\eta_{j,K}(\alpha^*))$ transversely for generic $R$ in a collection of simple curves. Consider the interval $[R_1, R_2]$ such that $L(R) = \text{length}(\partial N_R^t(\eta_{j,K}(\alpha^*)) \cap \partial \mathcal{C}H(N_j)) > \epsilon$, for generic $R \in [R_1, R_2]$, and some component of $\partial N_R^t(\eta_{j,K}(\alpha^*)) \cap \partial \mathcal{C}H(N_j)$ is homotopically non-trivial in $N_j$.

The components of $\partial N_R^t(\eta_{j,K}(\alpha^*)) \cap \partial \mathcal{C}H(N_j)$ which are trivial in $N_j$ but not in $\partial \mathcal{C}H(N_j)$ have length $> f(R_1)$, where $f$ is a monotonic function depending on the pinching constants. This follows since any disk they bound must intersect $\alpha^*$, so by the isoperimetric inequality for a PNC manifold, the length must be greater than that of the boundary of a disk of radius $R_1$ (the argument here is the same as in Lemma 4.1 [21]). So by declaring that $R_1 > f^{-1}(\epsilon)$, we may assume that all components of $\partial N_R^t(\eta_{j,K}(\alpha^*)) \cap \partial \mathcal{C}H(N_j)$ of length $< \epsilon$ which are trivial in $N_j$ are actually trivial in $\partial \mathcal{C}H(N_j)$ for $R \in [R_1, R_2]$.

The reason the interval $[R_1, R_2]$ exists is that if for some $R$, $L(R) \leq \epsilon$, then either a component of $\partial N_R^t(\eta_{j,K}(\alpha^*)) \cap \partial \mathcal{C}H(N_j)$ is homotopically non-trivial in $N_j$ and has length $< \epsilon$, in which case it lies in $M_{\text{th}(\epsilon)}$, and therefore cannot lie in $\partial N_R^t(\eta_{j,K}(\alpha^*))$ for generic $R$, or every component has length $< \epsilon$ and is homotopically trivial in $\mathcal{C}H(N_j)$, in which case either every loop before or every loop after is homotopically trivial (this follows from an outermost loop argument). Then $\epsilon(R_2 - R_1) \leq \int_{R_1}^{R_2} L(r)dr \leq A$ (by the coarea formula). Thus, $R_2 - R_1 \leq A/\epsilon$.

We know that $R_1$ cannot be too large. Every loop in $\partial N_{R_1}^t(\eta_{j,K}(\alpha^*)) \cap \partial \mathcal{C}H(N_j)$ is trivial in $\partial \mathcal{C}H(N_j)$, and therefore bounds a disk of diameter $< r + \epsilon$ by lemma 11.1. Therefore $\partial \mathcal{C}H(N_j)$ lies outside of $N_{R_1-r-\epsilon}(\eta_{j,K}(\alpha^*))$, which would contradict the fact that $\partial \mathcal{C}H(N_j)$ meets $K$ if $R_1$ were too large. Similarly, $R_2$ cannot be too large, since $R_2 \leq R_1 + A/\epsilon$. We also know that all of the loops of $\partial N_{R_2}^t(\eta_{j,K}(\alpha^*)) \cap \partial \mathcal{C}H(N_j)$ bound disks in $\partial \mathcal{C}H(N_j)$ of diameter $< r + \epsilon$. Thus, $\partial \mathcal{C}H(N_j) \subset N_{R(K)}^t(\eta_{j,K}(\alpha^*))$, where $R(K)$ depends on our set $K$.

Thus, if $K \cap \partial \mathcal{C}H(N_j) \neq \emptyset$ for all $j \in J$, then $\partial \mathcal{C}H(N_j) \subset N_{R(K)}^t(\eta_{j,K}(\alpha^*))$. Now, we may take $Q = N_{R(K)}^t(\eta_{j,K}(\alpha^*)) \subset M$, and for $j$ large enough, this maps isometrically to $N_j$ by $\eta_{j,Q}$, and therefore $\eta_{j,Q}(Q) = N_{R(K)}^t(\eta_{j,Q}(\alpha^*))$ in $N_j$. Therefore, $\mathcal{C}H(N_j) \subset$
η_{j,Q}(Q). Pushing back to \( M \), we see that \( M \) is geometrically finite, contradicting our assumption. This finishes the proof of the claim.

The argument of Canary-Minsky \cite{21} proceeds by finding simplicial hyperbolic surfaces which are arbitrarily close to \( \partial \text{CH}(N_i) \). We will work a little more coarsely, by finding a simplicial ruled surface which has points mod \( \epsilon \)-close to \( \partial \text{CH}(N_i) \) (this line of argument was suggested by Minsky).

Take a systole \( g \subset \partial_0 \text{CH}(N_i) = S \) (the compressible component of \( \partial \text{CH}(N_i) \)). By theorems A.1 and A.3, there is an \( l \) depending only on the pinching constants and \( \chi(M) \) such that length \( (g) \leq l \). Then we may homotope \( g \) to a geodesic \( g^* \) in \( \mathcal{N}_R(\partial_0 \text{CH}(N_i)) \), for some \( R \) depending only on \( \chi(M) \) and the pinching constants. If \( R \) is large enough, then \( g \) will be homotopic to \( g^* \) in the complement of \( \mathcal{N}(\eta_{i,\alpha^*}(\alpha^*)) \). We may complete \( g \) to a pants decomposition \( P \), and a standard triangulation of \( S \), to get an incompressible simplicial ruled surface \( f: S \to N_{i,\alpha^*}(2,0) \) realizing \( P \) as geodesics. We may choose a minimal path from \( P \) to a pants decomposition \( P' \) so that \( P' \) has a compressible curve. Then we may use the moves described in section 12 to interpolate between simplicial ruled surface representatives in \( N_{i,\alpha^*}(2,0) \) (where \( \alpha^* \) is a fixed algebraically diskbusting knot in \( M \), which we may map isometrically into \( N_i \) for \( i \) large enough) with standard triangulations associated to \( P \) and \( P' \). The geodesics representative of \( P' \) must meet \( \mathcal{N}(\alpha^*) \), since otherwise the compressible curve of \( P' \) would have a geodesic representative in \( N_i \).

One surface in this homotopy will meet \( \mathcal{N}_R^2(\alpha^*) \), in which case there is a simplicial ruled surface within distance \( R \). We can push these simplicial ruled surfaces to \( M_{\alpha^*}(2,0) \), to get longer and longer sequences of simplicial ruled surfaces exiting the (distinguished) end of \( M_{\alpha^*}(2,0) \) (if the compressible end of \( N_i \) is simply degenerate, then we get these surfaces from the filling theorem 14.1). We may homotope these surfaces in the complement of \( \alpha^* \) by section 12 to simplicial ruled surfaces lying in a compact subset \( K \) of \( M_{\alpha^*}(2,0) \). By Souto’s finiteness theorem (prop. 2, section 2.1, \cite{54}), these surfaces are eventually homotopic (in \( M_{\alpha^*}(2,0) \)), and so we see that we have a homotopy of a surface properly exiting the compressible end of \( M_{\alpha^*}(2,0) \), such that every point in the image of this homotopy is within distance \( R \) of a simplicial ruled surface. Then we may finish off using the argument of the main theorem section 8 \cite{21} or theorem 1, section 3 of \cite{54} to show that \( M \) is tame. \qed

14. PNC covering theorem

Canary proved a covering theorem for simply degenerate ends of hyperbolic manifolds \cite{19}. The main element of Canary’s argument that needs to be generalized to the PNC category is the interpolation of pleated surfaces, which is described in section 12. In fact, for our application, we only need to apply the covering theorem for incompressible ends (this was proven by Thurston \cite{55} in the hyperbolic category).

Theorem 14.1. (the Filling theorem) Let \( N \) be a PNC tame 2-orbifold without cusps, and \( E \) a simply degenerate incompressible (manifold) end of \( N \). There exists a constant \( R \) such that \( E \) has a neighborhood \( U \) homeomorphic to \( S \times [0, \infty) \) such that every
point in some subneighborhood $\hat{U} = S \times [k, \infty)$ is distance $\leq R$ from a simplicial ruled surface $f : S \to \hat{U}$.

Proof. (Sketch) The idea is straightforward here in the incompressible case. Because the end is simply degenerate, we may realize one simplicial ruled surface using a standard triangulation associated to a pants decomposition. Since the end is simply degenerate, we may find simple closed curves on $S$ with geodesic representatives exiting the end. We extend these curves to pants decompositions, and standard triangulations, and then use section 12 to homotope between these and the first surface with surfaces of bounded mod $\epsilon$ diameter, so that every point in the support of the homotopy is bounded distance from a simplicial ruled surface. Then the filling theorem follows from standard topological arguments and the fact that the end has $\infty \mod \epsilon$ diameter by lemma 7.1. □

Theorem 14.2. (the PNC Covering theorem) Let $p : M \to N$ be an orbifold cover, where $N$ is PNC and $M$ is tame. If $E$ is a simply degenerate incompressible manifold end of $M$ (i.e. $E \cong S \times [0, \infty)$ for some surface $S$), then either

1. $E$ has a neighborhood $U$ such that $p$ is finite-to-one on $U$, or
2. $N$ has finite volume and has a finite cover $N'$ which fibers over the circle, such that if $N_S$ denotes the cover of $N'$ associated to the fiber subgroup, then $M$ is finitely covered by $N_S$. Moreover, if $M \neq N_S$, then $M$ is homeomorphic to the interior of a twisted 1-bundle which is double covered by $N_S$.

Proof. (sketch) This theorem follows Canary’s argument closely. If the map of $p|_E$ is not finite-to-one, then by the filling theorem 14.1, there is a point $x \in N$, and infinitely many simplicial ruled surfaces $p \circ f_i : S \to N$ meeting $B_R(x)$, where $f_i : S \to E$ are simplicial ruled surfaces exiting $E$. By the finiteness theorem (see Prop. 2 [54] for a PNC version), infinitely many of these surfaces must be homotopic. By theorem 2.5 [21], we may find embedded surfaces $S_i \subset N(f_i(S)) \subset E$, so that $(S_i \leftarrow N) \simeq f_i$. But if $p \circ S_i$ and $p \circ S_j$ are homotopic in a bounded neighborhood of $x \in N$, for $j \gg i$, then this means we have an embedded lift $f'_i : S_i \to E$ so that $f'_i(S_i)$ is near to $S_j$, and $p(S_i) = p(f'_i(S_i))$. But then the product region between $S_i$ and $f'_i(S_i)$ gives a mapping torus $S_i \times [0,1]/\{(s,0) \sim (f'_i(s),1)\}$, which is a finite sheeted fibered cover of $N$. □

We need a slightly stronger result than that provided by Canary’s theorem:

Lemma 14.3. (finite covering theorem) Let $M \to N$ be an (orbifold) cover, and let $E \subset M$, $F \subset N$ be ends, such that $E \to F$ is finite-to-one. Suppose that $E$ is a tame manifold end, so there is a neighborhood $E' \subset E$ so that $E' \cong S \times [0, \infty)$. Then $F$ is tame as well, so there is a 2-orbifold $O$ and a neighborhood $F' \subset F$ such that $F' \cong O \times [0, \infty)$.

Proof. By replacing $E$ with a finite sheeted cover of $E$, we may assume that the cover $E \to F$ is regular. We may also assume that there is a product neighborhood $E_0$ such that $E \subset E_0$. Let $E_2 \subset E = E_1$ be a product neighborhood, and let $F_3 \subset F_1 = F$ be
a closed end neighborhood so that $\partial F_3$ is a suborbifold, and $F_3$ is covered by $E_3 \subset E_2$. Then $\partial E_3$ is a surface covering $\partial F_3$. Suppose that $\partial E_3$ is compressible in $E_2$. By the equivariant Dehn’s lemma, we may find a set of compressing disks for $\partial E_3$ in $E_1$ which are invariant with respect to the covering translations of $E_1 \rightarrow F_1$, and thus passes to an orbifold compression of $\partial F_3$. Do a maximal set of such compressions to get a closed neighborhood $F'_3$ covered by $E'_3 \subset E_1 \subset E_0$, so that $\partial E'_3$ is incompressible in $E_0$. Since $E_1$ is a product, $\partial E'_3$ must be a surface parallel to $\partial E_1$, since the only incompressible surface in a product is boundary parallel. Continuing in this manner, we get a sequence of equivariant product neighborhoods $E_4, E_5, \ldots$ exiting the end. In $E_i - \text{int} \ E_{i+1}$, the action is standard, so the covering action on $E_i$ must be standard, and we see that $F$ is also a product.

\[ \square \]

Appendix A. Boundary of the convex core

The goal of this appendix is to give details of Kleiner’s argument on pp. 42-43 of [39], which we use to bound the area of the boundary of the convex core. Let $M^3$ be a tame PNC manifold, without cusps, and pinching constants $a < K(P) < b < 0, \forall P \in G_2(TM)$. Assume that $\text{dim}(\mathcal{CH}(M)) = 3$.

**Theorem A.1.** There is a constant $A < 0$ which depends only on the pinching constants such that $\text{Area}(\partial \mathcal{CH}(M)) < A \chi(\partial \mathcal{CH}(M))$.

**Proof.** By geometric tameness, $\partial \mathcal{CH}(M)$ will be a compact surface [17, 34]. The philosophy of the argument is straightforward. If $\partial \mathcal{CH}(M)$ were smooth, then at each point of $\partial \mathcal{CH}(M)$, the Gauss-Kronecker curvature would be zero, otherwise we could push in slightly and get a smaller convex submanifold. Then the Gauss-Bonnet theorem would bound the area of $\partial \mathcal{CH}(M)$. Since $\partial \mathcal{CH}(M)$ is not necessarily smooth, we perform this argument for nearby surfaces which are smooth enough, then take a limit, showing that the average Gauss-Kronecker curvature approaches zero.

We follow the argument of Kleiner [39], including some more details. Kleiner was taking the convex hull of a compact set, but his argument is local, so it generalizes to the convex hull of the limit set of $\pi_1 M$.

Let $\delta : M - \mathcal{CH}(M) \rightarrow (0, \infty)$ be the function such that $\delta(x) = d(x, \mathcal{CH}(M))$. Since balls in a negatively curved manifold are strictly convex (e.g. see section 7.6 [22]), for any $x \in M - \mathcal{CH}(M)$, $B_{\delta(x)}(x) \cap \mathcal{CH}(M) = \xi_0(x)$, for a unique point $\xi_0(x) \in \partial \mathcal{CH}(M)$ (see 1.6 [5], working equivariantly with $\bar{M} - \mathcal{CH}(M)$). Let $E_s$ be the points in $M$ distance $\leq s$ away from $\mathcal{CH}(M)$, so $E_s = \delta^{-1}(0, s] \cup \mathcal{CH}(M)$. Let $C_s = \partial E_s = \delta^{-1}(s)$. We see that $E_s$ is convex, since $\delta$ is a convex function by 1.6(iii) [5]. There is also a nearest point retraction $\xi_s : M - E_s \rightarrow C_s$. These functions are 1-Lipschitz, as shown in 1.6(i) [5].

**Lemma A.2.** The vector field $\nabla \delta$ is locally Lipschitz on $M - E_s$, for any $s > 0$.

**Proof.** This argument follows Federer (pp. 433-440, 4.7 & 4.8 [26]), which proves a corresponding result for convex hulls in Euclidean spaces. We can work in $\bar{M}$ since
we need only a local result, but for simplicity we will abuse notation and use the same notation as for \( M \). The function \( \delta \) is clearly 1-Lipschitz, by the triangle inequality. In fact, \( \delta \) is differentiable, which follows essentially from the continuity of \( \xi_0(x) \) (this follows from the argument below, and see Lemma 4.7 [26]). \( \nabla \delta \) is a unit vector field on \( M - \mathcal{C}H(M) \), orthogonal to the foliation \( \{ C_s \} \). \( \nabla \delta \) is \( C^\infty \) in the direction \( \nabla \delta \), since the flow lines of \( \nabla \delta \) are length-minimizing geodesics. Intuitively, since the surfaces \( C_s \) are supported by balls of the form \( B_s(\xi_0(x)), x \in C_s \), then \( \nabla \delta_{\partial C_s} \) should have derivative bounded by the curvature of the boundary of \( \partial B_s(\xi_0(x)) \).

To make this intuition precise, we compare the vector field \( \nabla \delta \) with radial vector fields. Fix \( s_0 > 0 \), then for any point \( x \in M - E_{s_0} \), \( V_x = \nabla d(\cdot, \xi_0(x)) \) is a \( C^\infty \) (unit) vector field with \( \nabla V_x \) bounded independent of \( x \) (but dependent on \( s_0 \)). This follows since \( \nabla V_x|_{\partial B_x(\xi_0(x))} \) is the shape operator for \( \partial B_x(\xi_0(x)) \), which has bounded norm (dependent on \( s_0 \)) following from theorem 3.14 [22]. We need to show that \( \nabla \delta \) is locally Lipschitz. Fix some bounded open neighborhood \( U \subset M - E_{s_0} \) such that \( \text{diam}(U) \leq \epsilon \), and a smooth chart \( U \leftrightarrow \mathbb{R}^3 \). The Euclidean metric induced on \( U \) is bi-Lipschitz to the metric on \( U \subset M \). Then there is a constant \( C \), such that for any \( x, y \in U \), \( |V_x(x) - V_y(y)| < C|x - y| \). Now, consider distances \( a_1 = d(y, \xi_0(x)), a_2 = d(y, \xi_0(y)), a_3 = d(\xi_0(x), \xi_0(y)) \). Then we have \( a_1 \leq d(x, y) \leq a_3 \). By the Toponogov comparison theorem (see 1.5 [5]), we have \( \langle V_x(y), \nabla \delta(y) \rangle \geq \frac{a_1^2 - a_2^2}{2a_1a_3} = \frac{1}{2} (a_1/a_2 + a_2/a_1 - a_3^2/a_1a_2) \geq 1 - \epsilon^2/2s_0^2 \). Thus, \( |V_x(y) - \nabla \delta(y)| \leq \epsilon/s_0 \). Since \( \nabla \delta(x) = V_x(x) \), then we obtain \( |\nabla \delta(x) - \nabla \delta(y)| \leq |V_x(x) - V_x(y)| + |V_x(y) - \nabla \delta(y)| = O(|x - y|) + O(d(x, y)) = O(|x - y|) \). Thus, \( \nabla \delta \) is locally Lipschitz. Moreover, if \( \nabla \delta \) is \( C^2 \) at \( x \), then \( \nabla \delta \) is \( C^2 \) along the ray \( \xi_0(x)x \), since solutions to the Jacobi equation will be \( C^2 \). By Rademacher’s theorem, \( \nabla \delta \) is \( C^2 \) a.e. Also, \( \nabla \delta_{\partial C_s} \) is Lipschitz, and therefore \( C^2 \) a.e.

Let \( GK_{C_s} \) denote the Gauss-Kronecker curvature of \( C_s \), the product of the principal curvatures, that is the determinant of \( A_{C_s} \), the shape operator. Since \( \nabla \delta_{\partial C_s} \) is \( C^2 \) a.e., this implies that \( GK_{C_s} \) is a well-defined element of \( L^\infty(C_s) \).

Claim: \( \int_{C_s} GK_{C_s} \, d\text{Area}_{C_s} \to 0 \) as \( s \to 0^+ \).

For \( 0 \leq s \leq s_0 \), let \( r_{s_0s} : C_{s_0} \to C_s \) be the Lipschitz nearest point projection (so \( r_{s_0s} = \xi_s|_{C_{s_0}} \)). Assume that \( \nabla \delta \) is \( C^2 \) at \( p \in C_{s_0} \), and \( \kappa = \inf\{ K(P) \} \) where \( P \) runs over all 2-planes. We’ll define \( c_\kappa(t) = \cosh(\sqrt{-\kappa}t), s_\kappa(t) = (-\kappa)^{-\frac{1}{2}} \sinh(\sqrt{-\kappa}t) \). To prove the claim, we need to show that

1. \( |GK_{C_s}(r_{s_0s}(p)) \text{Jac}(r_{s_0s})(p)| \leq F(s_0, \kappa) \), where \( F \) is some function,
2. \( \lim_{s \to 0}(GK_{C_s}(r_{s_0s}(p)) \text{Jac}(r_{s_0s})(p) = 0 \).

Then

\[
\int_{C_s} GK_{C_s} \, d\text{Area}_{C_s} = \int_{C_{s_0}} (GK_{C_s} \circ r_{s_0s}) \text{Jac}(r_{s_0s}) \, d\text{Area}_{C_{s_0}} \to 0
\]
as \( s \to 0 \). Let \( \nu = -\nabla \delta \). Pick \( p \in C_{s_0} \) at which \( \nu \) is differentiable. Let \( \gamma : [0, s_0] \to M \) be the geodesic segment \( \gamma(t) = \exp t\nu(p) \), and for every \( e \in T_p C_{s_0} \), let \( Y \) be the Jacobi field along \( \gamma \) given by \( Y(t) = (\exp \circ (t \cdot \nu))_* e \). Thus, \( Y(0) = e \).

For \( s \in (0, s_0] \) consider maps \( W_{s_0 s} : T_p C_{s_0} \to T_{r_{s_0 s}(p)} C_s \) given by \( e \to \nabla_{\gamma(s_0-s)} Y \). We claim that the maps \( W_{s_0 s} \) are bounded above uniformly in terms of \( s_0 \) and the geometry of \( M \), while the lower bound on \( W_{s_0 s} \), i.e. \( \inf\{|W_{s_0 s} e| | e \in T_p C_{s_0}, |e| = 1\} \), goes to zero as \( s \to 0 \). To see the former, note that the second fundamental form of \( C_{s_0} \) is bounded above uniformly in terms of \( s_0 \) and the geometry of \( M^3 \) since \( C_{s_0} \) is convex and supported from the inside by a ball of radius \( s_0 \); consequently the maps \( W_{s_0 s} \) are bounded above uniformly because they are obtained by solving the Jacobi equation with initial conditions determined by the second fundamental form of \( C_{s_0} \).

Assume that \( |Y(0)| = 1 \). Then

\[
\frac{d}{dt} \langle \nabla Y, \nabla_t Y \rangle = 2 \langle Y', Y'' \rangle - 2 \langle Y', R(\gamma', Y) \gamma' \rangle = -2|Y'||Y R(\gamma', Y)/|Y|, \gamma', Y''/|Y'| \rangle \leq -2\kappa|Y'||Y|.
\]

Assuming \( \lambda \) is the minimum eigenvalue of \( A'' \), we have

\[
\frac{d}{dt} |Y'| = \frac{1}{2|Y'|} \frac{d}{dt} \langle Y', Y'' \rangle \leq -\kappa |Y| \leq -\kappa (c_\kappa - \lambda s_\kappa)(t)|Y(0)| \leq -\kappa c_\kappa(t),
\]

since \( \lambda \geq 0 \) and \( |Y(0)| = 1 \). This follows from theorem 7.4, [22]. Integrating, we obtain

\[
|Y'(t)| \leq \int_0^{s_0-t} -\kappa c_\kappa(u) du + |Y'(0)| = -\kappa s_\kappa(s_0-t) + |Y'(0)|.
\]

Thus, we need to bound \( |Y'(0)| \). We have the formula \( Y'(0) = -A''(Y(0)) \) (p. 321 [22]). Intuitively, \( |A''| \) is bounded, since the surface \( C_{s_0} \) is supported by the ball \( B_{s_0}(\xi_0(p)) \), which has principle curvatures controlled purely by \( \kappa \) and \( s_0 \). By theorem 3.14, [22], if \( \text{tr} A'' \geq (n-1)\lambda \), (in our case, \( n-1 = 2 \)), then the map \( \exp \) has a focal point at distance the smallest zero of the function \( (c_\kappa - \lambda s_\kappa)(t) \). But \( \exp \) has no focal point at distance \( s_0 \), so this means that \( \text{tr} A'' \leq 2c_\kappa(s_0)/s_\kappa(s_0) \). Since \( C_{s_0} \) is convex, the principle curvatures \( \lambda_1, \lambda_2 \geq 0 \), so \( \text{tr} A'' = \lambda_1 + \lambda_2 \geq \lambda_1 \). Thus, \( \lambda_1 \leq 2c_\kappa(s_0)/s_\kappa(s_0) \). Since \( C_{s_0} \) is assumed \( |Y(0)| = 1 \), and \( |A''| = \max(\lambda_1, \lambda_2) \), we have \( |Y'(0)| \leq 2c_\kappa(s_0)/s_\kappa(s_0) \), which clearly only depends on \( \kappa \) and \( s_0 \).

We also need to show that the lower bound \( \inf\{|W_{s_0 s} e| | e \in T_p C_{s_0}, |e| = 1\} \) goes to zero as \( s \to 0 \). Note that we have the factorization \( W_{s_0 s} = A''_{C_s} \circ r_{s_0 s}^* \), where \( A''_{C_s} : T_{\gamma(s_0-s)} C_s \to T_{\gamma(s_0-s)} C_s \) is the Weingarten map for the inward normal to \( C_s \) (given by \( A' u = -(\nabla u \nu)^T \), where \( u \in T_p C_{s_0} \)), and the fact that the lower bound on \( A'' \) goes to zero with \( s \).

We prove this by contradiction. Suppose \( A'' \) does not approach 0. Then there exists a sequence \( s_i \to 0 \) such that \( \inf\{|A''_{C_{s_i}} e| | e \in T_p C_{s_i}, |e| = 1\} \geq \epsilon \). This is also equal to \( \inf(\lambda_1, \lambda_2) \), where \( \lambda_i \) are the eigenvalues for \( A'' \). In \( T_{\gamma(s_0-s)} M \), take a little spherical cap \( C_t \) with radius of curvature \( \epsilon/4 \) and tangent to \( C_{s_i} \). At \( \gamma(s_0 - s_i) \), \( \lambda_i(C_t) = \epsilon/4 \), since the connections agree to first order at the origin. Since \( \lambda_i(C_{s_i}) \geq \epsilon \), then we have
$\lambda_i(C_{s_i}) \geq \epsilon/2$ in a small neighborhood of $\gamma(s - s_i)$ in $C_{s_i}$. Assume that the spherical cap $C'$ is small enough so that $0 \leq \lambda_j(\exp C') \leq \epsilon/2$ for all points in $\exp (C_i)$. Then it follows that $\exp ((C_i) \cap E_{s_i}) = \gamma(s_0 - s_i)$. To see this, note that since the second fundamental forms of a surface $\Sigma \subset M$ and $\exp^{-1}(\Sigma)$ agree at the origin, we may reduce to the Euclidean case. For two surfaces $\Sigma_1, \Sigma_2 \subset \mathbb{R}^3$, which are tangent at 0 to the horizontal plane, and such that $\inf_{|e|=1}(A^2(e, e)) > \sup_{|e|=1}(A^p(e, e))$, where $\nu_j$ is the unit tangent to $\Sigma_j$. Then locally near 0, $\Sigma_1$ lies above $\Sigma_2$. To see this, we note that in a neighborhood of 0, $\Sigma_j$ is a graph over the horizontal plane. The Hessian of $\Sigma_j$ agrees with the second fundamental form at the origin. Thus, the difference of the two graphs $\Sigma_1 - \Sigma_2$ has Hessian $> 0$ in a neighborhood of the origin, and therefore has a local max at the origin, which means that $\Sigma_1$ lies above $\Sigma_2$ (thanks to Ken Bromberg for help with this argument).

Now, we take the cap $\exp (C_i)$, and flow it along normal geodesics distance $s_i$ to a surface $C_i'$. Since $\exp (C_i') \cap E_{s_i} = \gamma(s_0 - s_i)$, we see that $C_i' \cap E_0 = \gamma(s_0)$. By the equation $|Y'(s_i)| \leq \kappa s_i + |Y'(0)|$ (where $Y(t)$ is now the Jacobi field associated to the exponential map from $\exp (C_i)$), we see that if $s_i$ is small enough, then $C_i'$ will remain convex, since $Y'(0)$ cannot be zero. In fact, flowing a bit further, we get a convex cap intersecting $E_0$ in a compact set. We may then cut off a bit of $E_0$ by this cap to get a smaller convex core, a contradiction to the minimality of $CH(M)$ (the argument here is actually very similar to that of lemma 11.1). Thus, $\inf\{|W_{sos} e| \mid e \in T_pC_{s_0}, |e| = 1\}$ goes to zero as $s \to 0$. Since $GK_{C_s}(r_{sos}(p))Jac(r_{sos})(p) = Jac(-W_{sos})$, claims 1 and 2 follow from the bounds on $W_{sos}$.

One can thus make sense of the equation $K = K_{sec} + GK_s$ at points where $C_s$ is $C^2$, where $K$ is the intrinsic sectional curvature of $C_s$, and $K_{sec}$ is the sectional curvature of the plane tangent to $C_s$. Since $K_{sec} \leq b < 0$, we have

$$2\pi \chi(\partial E_s) = \lim_{s \to 0^+} \int_{C_s} Kda = \lim_{s \to 0^+} \int_{C_s} K_{sec} + GK_s da$$

$$\leq \lim_{s \to 0^+} \int_{C_s} bda = b \lim_{s \to 0^+} \text{Area}(C_s),$$

thus $\text{Area}(C_0) \leq \frac{2\pi}{b} \chi(\partial CH(M))$.

$\Box$

**Theorem A.3.** (4.5.4+ [31]) Every closed, orientable surface $V$ of genus $g \geq 1$ with a Riemannian metric admits a closed curve of length \( \leq l \) which is not null-homologous and satisfies $\frac{1}{2}l^2 \leq \text{Area}(V)$.

Now, we apply theorem A.3 to conclude that each component of $\partial CH(M)$ has an embedded curve of bounded length, depending only on the topological type of $\partial CH(M)$.

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