Bott–Samelson atlases, total positivity, and Poisson structures on some homogeneous spaces

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Abstract
Let $G$ be a connected and simply connected complex semisimple Lie group. For a collection of homogeneous $G$-spaces $G/Q$, we construct a finite atlas $A_{BS}(G/Q)$ on $G/Q$, called the Bott–Samelson atlas, and we prove that all of its coordinate functions are positive with respect to the Lusztig positive structure on $G/Q$. We also show that the standard Poisson structure $\pi_{G/Q}$ on $G/Q$ is presented, in each of the coordinate charts of $A_{BS}(G/Q)$, as a symmetric Poisson CGL extension (or a certain localization thereof) in the sense of Goodearl–Yakimov, making $(G/Q, \pi_{G/Q}, A_{BS}(G/Q))$ into a Poisson–Ore variety. In addition, all coordinate functions in the Bott–Samelson atlas are shown to have complete Hamiltonian flows with respect to the Poisson structure $\pi_{G/Q}$. Examples of $G/Q$ include $G$ itself, $G/T$, $G/B$, and $G/N$, where $T \subset G$ is a maximal torus, $B \subset G$ a Borel subgroup, and $N$ the uniradical of $B$.

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1 Introduction and the main results

1.1 Introduction

Let $G$ be a connected and simply connected complex semi-simple Lie group with Lie algebra $\mathfrak{g}$. Certain geometric aspects of $G$ were revealed after the discovery of quantum groups, among them the notion of total positivity on $G$ and the standard multiplicative Poisson structure $\pi_{st}$ on $G$. Indeed, Lusztig introduced [43–45] the totally positive part $G_{>0}$ of $G$ via his work on representations of the quantized universal enveloping algebra $U_q(\mathfrak{g})$, while Drinfeld introduced [12,13] the standard multiplicative Poisson structure $\pi_{st}$ on $G$ as the semi-classical limit of the (standard) quantum coordinate ring $\mathbb{C}[G]$ of $G$. The pair $(G, \pi_{st})$ is the prototypical example of a complex Poisson Lie group (see, for example, [12,16] and Sect. 4.4).

Both the totally positive part $G_{>0}$ of $G$ defined by Lusztig and the standard Poisson structure on $G$ can be extended to certain homogeneous spaces of $G$. In this paper, for a special collection of homogeneous spaces $G/Q$, we construct a finite atlas on $G/Q$, called the Bott–Samelson atlas, and we prove some remarkable properties of both the Lusztig total positivity and the standard Poisson structure on $G/Q$, expressed through the Bott–Samelson atlas. The homogeneous spaces considered in this paper are precisely all the diagonal $G$-orbits in the double flag variety $(G/B) \times (G/B)$ and in $(G/B) \times (G/N)$, where $B$ is a Borel subgroup of $G$ and $N$ the unipotent radical of $B$. Particular examples include $G$ itself, $G/T$, $G/B$ and $G/N$, where $T \subset B$ is a maximal torus of $G$. See (1.3) for the precise descriptions of all the subgroups $Q$.

To give the precise statements of the main results of the paper, we first briefly review some background on total positivity and Poisson structures.

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1 The word “standard” here refers to the fact that the Poisson structure $\pi_{st}$ is defined using the standard classical $r$-matrix on the Lie algebra of $G$, as opposed to the more general Belavin–Drinfeld ones [4].
The geometric structures underlying total positivity are the so-called positive structures on varieties, as defined in [7,17]. Here we briefly recall (see Sect. 3.1 for details) that a positive structure on an irreducible rational complex variety \(X\) is a positive equivalence class \(\mathcal{P}^X\) of toric charts on \(X\) (see Definition 3.2). A positive structure \(\mathcal{P}^X\) on \(X\) gives rise to a well-defined totally positive part \(X_{>0}\) of \(X\), and thus the notion of total positivity, by setting all the coordinates in one (equivalently, any) toric chart in \(\mathcal{P}^X\) to be positive. A positive structure \(\mathcal{P}^X\) also gives rise to the semi-field \(\text{Pos}(X, \mathcal{P}^X)\) of positive functions, which, by definition, are non-zero rational functions on \(X\) that have subtraction-free expressions in the coordinates of one (equivalently, any) toric chart in \(\mathcal{P}^X\).

For a connected and simply connected complex semi-simple Lie group \(G\), positively equivalent toric charts on \(G\) giving rise to the totally positive part \(G_{>0}\) of Lusztig were described in [43] (although the term “positive structure” was not used). By [21, Theorem 3.1] (see also [19, Theorems 1.11 and 1.12]), all generalized minors on \(G\) are positive functions, and that an element \(g \in G\) lies in \(G_{>0}\) if and only if \(\Delta(g) > 0\) for every generalized minor \(\Delta\). Positive structures on double Bruhat cells and Schubert varieties induced by the Lusztig positive structure on \(G\) have been considered in [1,2,5,18–20]. See also [17] for related positive structures in higher Teichmüller theory.

Extending the case for \(G\) (and some other cases already in the literature), we define in Sect. 3.2 the Lusztig positive structure \(\mathcal{P}_{\text{Lusztig}}^{G/Q}\) on \(G/Q\) for each \(Q\) in (1.3).

Turning to Poisson structures, it is easy to prove (see Sect. 5.1) that for each of the homogeneous spaces \(G/Q\) from the list in (1.3), the standard Poisson structure \(\pi_{\text{std}}\) on \(G\) projects to a well-defined Poisson structure on \(G/Q\), which we will denote by \(\pi_{G/Q}\) and refer to as the standard Poisson structure on \(G/Q\). The pair \((G/Q, \pi_{G/Q})\) is then a prototypical example of a Poisson homogeneous space [14] of the Poisson Lie group \((G, \pi_{\text{std}})\).

It has been noticed, see, for example [25,27], that the quantum coordinate rings of many spaces related to the complex semi-simple Lie group \(G\) can be presented as iterated Ore extensions, a notion from the theory of non-commutative rings [23]. Correspondingly, their semi-classical limits, which are now the (classical) coordinate rings with Poisson brackets, are iterated Poisson–Ore extensions. Here recall [46] that for a Poisson algebra \((A, \{,\})\) over \(\mathbb{C}\), an Ore extension of \((A, \{,\})\) is the \(\mathbb{C}\)-algebra \(A[z]\) with a Poisson bracket \(\{,\}\) which extends the Poisson bracket on \(A\) and satisfies \(\{z, A\} \subset zA + A\). A polynomial Poisson algebra \(A = (\mathbb{C}[z_1, \ldots, z_n], \{,\})\) is called an iterated Poisson–Ore extension if

\[
\{\mathbb{C}[z_1, \ldots, z_{i-1}], z_i\} \subset z_i\mathbb{C}[z_1, \ldots, z_{i-1}] + \mathbb{C}[z_1, \ldots, z_{i-1}], \quad 2 \leq i \leq n. \tag{1.1}
\]

Such an iterated Poisson–Ore extension is said to be symmetric if it also satisfies

\[
\{z_i, \mathbb{C}[z_{i+1}, \ldots, z_n]\} \subset z_i\mathbb{C}[z_{i+1}, \ldots, z_n] + \mathbb{C}[z_{i+1}, \ldots, z_n], \quad 1 \leq i \leq n-1. \tag{1.2}
\]

Iterated Poisson–Ore extensions are Poisson analogs of iterated Ore extensions.

With additional assumptions on the existence of compatible rational actions by algebraic \(\mathbb{C}\)-tori, Goodearl and Yakimov introduced in [25,28] a special class of sym-
metric iterated Poisson–Ore extensions, called symmetric Poisson CGL extensions (named after G. Cauchon, K. Goodearl, and E. Letzter) and developed an extensive theory on such extensions in connection with cluster algebras. In particular, one of the main results of [25,28] says that a presentation of a Poisson algebra \((A, \{, \})\) as a symmetric Poisson CGL extension naturally gives rise to a cluster algebra structure on \(A\) compatible with the Poisson bracket \(\{, \}\) in the sense of [22], i.e., all the extended cluster variables from the same cluster have log-canonical Poisson brackets. See [25–27,29] for applications of the Goodearl–Yakimov theory to classical and quantum cluster structures on the coordinate rings of double Bruhat cells and Schubert cells for symmetrizable Kac–Moody groups.

In this paper, for an irreducible rational \(T\)-Poisson variety \((X, \pi_X)\), where \(T\) is an algebraic \(C\)-torus, we define a \(T\)-Poisson–Ore atlas \(A_X\) for \((X, \pi_X)\) to be an atlas \(A_X\) on \(X\) consisting of \(T\)-invariant coordinate charts, in each one of which \(\pi_X\) is presented as a symmetric Poisson CGL extension or a localization thereof by homogeneous Poisson prime elements (Definition 4.1). By a \(T\)-Poisson–Ore variety, we mean a triple \((X, \pi_X, A_X)\), where \((X, \pi_X)\) is an irreducible rational \(T\)-Poisson variety, and \(A_X\) is a \(T\)-Poisson–Ore atlas for \((X, \pi_X)\). A \(T\)-Poisson–Ore variety is also simply called a Poisson–Ore variety. See Sect. 4.1 for detail.

By [9], an \(n\)-dimensional irreducible rational complex variety \(X\) is said to be uniformly rational if it admits a cover by Zariski open subsets of \(C^n\). A Poisson–Ore variety is thus uniformly rational. Given that the requirements on a Poisson algebra to be an iterated Poisson–Ore extension are very restrictive, and that the changes of coordinates between different coordinate charts are in general highly non-trivial birational transformations, it is a remarkable feature of any Poisson variety if it admits a Poisson–Ore atlas. We prove that this is the case for all the homogeneous spaces \((G/Q, \pi_{G/Q})\) considered in this paper.

What serves as a Poisson–Ore atlas on each \((G/Q, \pi_{G/Q})\) is a finite atlas \(A_{BS}(G/Q)\) on \(G/Q\) constructed in this paper, which we call the Bott–Samelson atlas on \(G/Q\). The Bott–Samelson atlas is canonical in the sense that its construction depends only on the choice of a pinning \(\{T \subset B, \{e_\alpha\}_{\alpha \in \Gamma}\}\) for \(G\) (see [43] and Notation 1.1). In this paper we set up the Bott–Samelson atlas as a bridge connecting the Lusztig positive structure on \(G/Q\) and the standard Poisson structure \(\pi_{G/Q}\).

In a subsequent paper [37], it will be shown, again for \(v \in W\) and \(Q = B(v)\) or \(N(v)\), that the Bott–Samelson atlas on \(G/Q\) gives rise to a cluster open cover of \(G/Q\) compatible with the Lusztig positive structure, i.e., the cluster structures on all the shifted big cells arising from the symmetric Poisson CGL presentations of the Poisson structure \(\pi_{G/Q}\) via the Goodearl–Yakimov theory [25,28], while not in general mutation equivalent, are all compatible with the Lusztig positive structure in the sense that every one of their clusters defines a toric chart in \(\mathcal{P}_{Lusztig}^G\).

While the present paper and the subsequent [37] explore connections between Poisson structures, total positivity, and cluster algebras via the notion of Poisson–Ore varieties, we believe that the latter will also find applications to other areas such as integrable systems (see Theorem C in Sect. 1.2) and tropicalization of positive varieties [1,2,7]. Such topics will be investigated elsewhere.
1.2 Statements of main results and organization of the paper

Notation 1.1 Throughout the paper, we fix a connected and simply-connected complex semisimple Lie group $G$. For any closed subgroup $P$ of $G$, denote the image of $g \in G$ in $G/P$ by $g_P$. The identity element of a group will always be denoted as $e$.

We fix a pinning [43] of $G$, i.e., a collection $\{T \subset B, \{e_\alpha\}_{\alpha \in \Gamma}\}$, where $T$ is a maximal torus of $G$, $B$ is a Borel subgroup of $G$ containing $T$, $\Gamma$ is the set of simple roots corresponding to the pair $(T, B)$, and $e_\alpha$ is a root vector for $\alpha \in \Gamma$. Let $W = N_G(T)$ be the Weyl group, where $N_G(T)$ is the normalizer subgroup of $T$ in $G$. Let $l : W \to \mathbb{N}$ be the length function on $W$. Let $w_0 \in W$ be the longest element, and let $l_0 = l(w_0)$.

Let $N$ be the unipotent radical of $B$. For $v \in W$, let

$$B(v) \overset{\text{def}}{=} B \cap \dot{v}B\dot{v}^{-1} \text{ and } N(v) \overset{\text{def}}{=} N \cap \dot{v}N\dot{v}^{-1} \subset B(v),$$

where for $v \in W$, $\dot{v}$ is any representative of $v$ in $N_G(T)$. \hfill \Box

In this paper, we consider the homogeneous spaces $G/Q$ of $G$, where

$$Q = B(v) \text{ or } N(v), \quad v \in W. \quad (1.3)$$

Consider the diagonal $G$-action on $(G/B) \times (G/B)$ and on $(G/B) \times (G/N)$. It is well-known that the sets of $G$-orbits in both $(G/B) \times (G/B)$ and $(G/B) \times (G/N)$ are indexed by $W$ via $v \mapsto G(e_B, \dot{v}B)$ and $v \mapsto G(e_B, \dot{v}N)$ for $v \in W$. As the stabilizer subgroup of $G$ at $(e_B, \dot{v}B)$ and at $(e_B, \dot{v}N)$ are respectively $B(v)$ and $N(v)$, we can identify the collections $\{G/B(v) : v \in W\}$ and $\{G/N(v) : v \in W\}$ with that of the diagonal $G$-orbits in $(G/B) \times (G/B)$ and in $(G/B) \times (G/N)$ respectively. In particular, since

$$B(w_0) = T, \quad N(w_0) = \{e\}, \quad B(e) = B, \quad N(e) = N,$$

one sees that $G/T$ and $G/B$ are isomorphic to the respective open and closed $G$-orbits in $(G/B) \times (G/B)$, and $G$ and $G/N$ are isomorphic to the respective open and closed $G$-orbits in $(G/B) \times (G/N)$.

The paper consists of three parts.

In the first part of this paper, for each $Q$ in (1.3), we construct the Bott–Samelson atlas $\mathcal{A}_{BS}(G/Q)$ on $G/Q$. More precisely, for each $Q$ in (1.3), we consider the open cover of $G/Q$ by shifted big cells, i.e.,

$$G/Q = \bigcup_{w \in W} wB^-B/Q, \quad (1.4)$$

where $B^-$ is the Borel subgroup of $G$ such that $B^- \cap B = T$. For $u \in W$, let $\mathcal{R}(u)$ be the set of all reduced words of $u$. For $v \in W$, let

$$\mathcal{R}_v = \bigcup_{w \in W} \mathcal{R}(w_0w^{-1}) \times \mathcal{R}(w) \times \mathcal{R}(v).$$
Using the fixed pinning for $G$, we construct, for each $r \in \mathcal{R}(w_0 w^{-1}) \times \mathcal{R}(w) \times \mathcal{R}(v)$, the Bott–Samelson parametrizations

$$\sigma^r_{B(v)} : \mathbb{C}^l \xrightarrow{\sim} wBvB/v \quad \text{and} \quad \sigma^r_{N(v)} : \mathbb{C}^l \times (\mathbb{C}^\times)^d \xrightarrow{\sim} wBvB/N(v),$$

where $l = l(w_0) + l(v) = \dim G/B(v)$, and $d = \dim T$ [see (2.29) and (2.32)]. The Bott–Samelson atlas $\mathcal{A}_{BS}(G/Q)$, for $Q = B(v)$ or $N(v)$, is then defined to be

$$\mathcal{A}_{BS}(G/Q) = \{\sigma^r : r \in \mathcal{R}_v\}.$$  

We refer to any coordinate chart in $\mathcal{A}_{BS}(G/Q)$ as a Bott–Samelson coordinate chart on $G/Q$, and the resulting coordinates (on a shifted big cell) Bott–Samelson coordinates. A Bott–Samelson coordinate, being a regular function on some shifted big cell in $G/Q$, can thus also be regarded as a rational function on $G/Q$.

An outline of the construction of the Bott–Samelson atlas is given in Sect. 1.3 and details, including explicit formulas for all the Bott–Samelson coordinates, are given in Sect. 2.

In the second part of the paper, extending the Lusztig positive structure on $G$, we first define in Sect. 3.2 the Lusztig positive structure $\mathcal{P}^{G/Q}_{\text{Lusztig}}$ on each $G/Q$, and we prove the following Theorem A, presented in more detail as Theorem 3.17 in Sect. 3.4.

**Theorem A** For any $v \in W$ and $Q = B(v)$ or $N(v)$, all the Bott–Samelson coordinates on $G/Q$, when regarded as rational functions on $G/Q$, are positive with respect to $\mathcal{P}^{G/Q}_{\text{Lusztig}}$.

As a consequence of Theorem A, all Bott–Samelson coordinates on $G/Q$ take positive values at every point in the totally positive part of $G/Q$, as defined by $\mathcal{P}^{G/Q}_{\text{Lusztig}}$.

We remark that the Bott–Samelson coordinate charts are not to be confused with toric charts in $\mathcal{P}^{G/Q}_{\text{Lusztig}}$. In particular, given any toric chart in $\mathcal{P}^{G/Q}_{\text{Lusztig}}$ with coordinates $(c_1, \ldots, c_{d(Q)})$ and any Bott–Samelson chart with coordinates $(z_1, \ldots, z_{d(Q)})$, where $d(Q) = \dim(G/Q)$, while Theorem A implies that each $z_j$ has a subtraction-free expression in $(c_1, \ldots, c_{d(Q)})$, the $c_j$’s, in general, do not have subtraction-free expressions in $(z_1, \ldots, z_{d(Q)})$. This already happens for $G/Q = SL(2, \mathbb{C})$ (see Remark 3.18). Furthermore, the changes of coordinates between two Bott–Samelson coordinate charts are in general not positive. See Example 2.13 for the case of $G/Q = G = SL(3, \mathbb{C})$, where two out of the 16 Bott–Samelson coordinate charts on $G$, as well as the coordinate transformations between them, are given.

The third part of the paper concerns the standard Poisson structure $\pi_{G/Q}$ on $G/Q$. The following Theorem B summarizes Theorem 5.6 (for $Q = B(v)$) and Theorem 5.9 (for $Q = N(v)$). See Example 5.10 for an illustration of Theorem B for $SL(4, \mathbb{C})/B$.

**Theorem B** For any $v \in W$ and $Q = B(v)$ or $N(v)$, the Bott–Samelson atlas $\mathcal{A}_{BS}(G/Q)$ is a $T$-Poisson–Ore atlas for the Poisson structure $\pi_{G/Q}$, making $(G/Q, \pi_{G/Q}, \mathcal{A}_{BS}(G/Q))$ into a $T$-Poisson–Ore variety.

For another property of the Bott–Samelson coordinates with respect to the Poisson structure $\pi_{G/Q}$, recall that for a smooth affine complex Poisson variety $(X, \pi_X)$, each...
1.3 Outlines of the construction of the Bott–Samelson atlas and proofs of main

The key ingredients in our construction of the Bott–Samelson atlas

A

Schubert cells in the literature) and denote as

O

It easy to see that dim \( O = l(u) := l(u_1) + \cdots + l(u_r) \). Using the fixed pinning for

G, we will see in Sect. 2.2 that any choice of \( \tilde{u} \in R(u_1) \times \cdots \times R(u_r) \) naturally gives rise to a Bott–Samelson parametrization \( \beta^{\tilde{u}} : C^{l(u)} \to O^u \), and we call the resulting coordinates on \( O^u \) the Bott–Samelson coordinates on \( O^u \) defined by \( \tilde{u} \).
The core of our construction of the Bott–Samelson atlases on \( G/B(v) \) and \( G/N(v) \), where \( v \in W \), consists of two explicit isomorphisms

\[
J^w_{R(v)} : \ wB^- B/B(v) \longrightarrow \mathcal{O}^{w_0 w^{-1}} \times \mathcal{O}^{(w,v)},
\]

\[
J^w_{N(v)} : \ wB^- B/N(v) \longrightarrow \mathcal{O}^{w_0 w^{-1}} \times \mathcal{O}^{(w,v)} \times T,
\]

given in (2.23)–(2.26), where \( w \in W \) is arbitrary. Given any \( r = (w^0, w, v) \in \mathcal{R}(w_0 w^{-1}) \times \mathcal{R}(w) \times \mathcal{R}(v) \), by composing \( J^w_{R(v)} \) and \( J^w_{N(v)} \) with the Bott–Samelson parametrizations of \( \mathcal{O}^{w_0 w^{-1}} \) and \( \mathcal{O}^{(w,v)} \), respectively, we obtain the desired Bott–Samelson parametrizations (see (2.29) and (2.32))

\[
\sigma^r_{R(v)} : \mathbb{C}^l \longrightarrow wB^- B/B(v) \quad \text{and} \quad \sigma^r_{N(v)} : \mathbb{C}^l \times (\mathbb{C}^\times)^d \longrightarrow wB^- B/N(v),
\]

where again \( l = l(w_0) + l(v) = \dim G/B(v) \) and \( d = \dim T \). Note that each shifted big cell \( wB^- B/B(v) \) or \( wB^- B/N(v) \) may have more than one Bott–Samelson parametrization, the precise number being \( |\mathcal{R}(w_0 w^{-1})| + |\mathcal{R}(w)| + |\mathcal{R}(v)| \).

Remark 1.2 When \( Q = B \) (and thus \( v = e \)) and \( w \in W \), the isomorphism

\[
J^w_B : \ wB^- B/B \overset{\sim}{\longrightarrow} \mathcal{O}^{w_0 w^{-1}} \times \mathcal{O}^w
\]

was stated in [35, Lemma A.4], and the explicit formula was given in [34]. Kazhdan and Lusztig used the isomorphism \( J^w_B \) in the proof of [35, Theorem A2], which expresses singularities of Schubert varieties (the closures of Bruhat cells) in terms of Kazhdan–Lusztig polynomials, while A. Knutson, A. Woo, and A. Yong [34] used \( J^w_B \) to show that singularities (and some other invariants) of Richardson varieties in \( G/B \) are determined by that of Schubert varieties. Here recall that a Richardson variety in \( G/B \) is the intersection of a Schubert variety with an opposite Schubert variety (the closure of a \( B^- \)-orbit). The covering of \( G/B \) by the collection \( \{ \mathcal{O}^{w_0 w^{-1}} \times \mathcal{O}^w : w \in W \} \) via the isomorphisms \( J^w_B \) is called a Bruhat atlas on \( G/B \) in [31].

To prove Theorem A, we use the explicit formulas for the Bott–Samelson coordinates given in Propositions 2.10 and 2.11, but in a crucial way we also use [19, Theorem 2.12], which describes certain collections of generalized minors as positive transcendental bases for the fields of rational functions on double Bruhat cells. See Sect. 3.4.

To prove Theorem B, we first give a Poisson geometrical interpretation of the isomorphisms \( J^w_{R(v)} \) and \( J^w_{N(v)} \) and then apply symmetric Poisson CGL extensions associated to generalized Bruhat cells established in [15].

More specifically, it is shown in [40] that for every \( u \in W' \), the generalized Bruhat cell \( \mathcal{O}^u \) carries a so-called standard Poisson structures \( \pi_r \), also defined using the
Poisson structure $\pi_{st}$ on $G$, and it is proved in [15] that the Poisson algebra $(\mathbb{C}[O^u], \pi_r)$ is a symmetric Poisson CGL extension in any of the Bott–Samelson parametrizations of $O^u$ (see Sect. 4.4). For any $v, w \in W$, let $\pi_{1, 2}$ be the unique Poisson structure on $O^{u_0w^{-1}} \times O^{(w, v)}$ and $\pi_{1, 2, 0}$ the unique Poisson structure on $O^{u_0w^{-1}} \times O^{(w, v)} \times T$ such that

$$J^w_{B(v)} : (wB^-B/B(v), \pi_{G/B(v)}) \rightarrow \left(O^{u_0w^{-1}} \times O^{(w, v)}, \pi_{1, 2}\right) \quad \text{and}$$

$$J^w_{N(v)} : (wB^-B/N(v), \pi_{G/N(v)}) \rightarrow \left(O^{u_0w^{-1}} \times O^{(w, v)} \times T, \pi_{1, 2, 0}\right)$$

are Poisson isomorphisms. Our Theorem 5.3, proved in Appendix A, says that $\pi_{1, 2}$ is a mixed product of $\pi_1$ and $\pi_2$, i.e.,

$$\pi_{1, 2} = (\pi_1, 0) + (0, \pi_2) + \mu,$$

where $\mu$ is a certain mixed term expressed using the $T$-actions on $O^{u_0w^{-1}} \times O^{(w, v)}$. A similar statement holds for $\pi_{1, 2, 0}$. Applying a general construction (see Lemma 4.7) on mixed products of symmetric Poisson CGL extensions, we immediately prove that the Poisson structure $\pi_{G/Q}$ is presented as a symmetric Poisson CGL extension in every Bott–Samelson parametrization of $wB^-B/Q$. Details of the presentations are given in Theorem 5.6 for $Q = B(v)$ and in Theorem 5.9 for $Q = N(v)$.

Theorem C is a direct consequence of Theorem B and a general fact on Poisson–Ore varieties. More precisely, in Sect. 4.3, we prove a general fact (see Proposition 4.11) on the completeness of the Hamiltonian flows of all the CGL generators for any symmetric Poisson CGL extension. Consequently (see Theorem 4.12), for any Poisson–Ore variety $(X, \pi_X, A_X)$, all the coordinate functions in any coordinate chart in $A_X$ have complete Hamiltonian flows in that coordinate chart. Theorem C then follows as a special case.

We finish this section by setting up more notation for the rest of the paper.

**Notation 1.3** Continuing with Notation 1.1, let $g$ and $\mathfrak{h}$ be the respective Lie algebras of $G$ and $T$. Recall that $\Gamma \subset \mathfrak{h}^*$ is the set of all simple roots, and for each $\alpha \in \Gamma$, we have fixed a root vector $e_\alpha$ of $\alpha$ as part of the pinning. For $\alpha \in \Gamma$, let $e_{-\alpha}$ be the unique root vector for $-\alpha$ such that $h_\alpha := [e_\alpha, e_{-\alpha}] \subset \mathfrak{h}$ satisfies $\alpha(h_\alpha) = 2$, and let $x_\alpha : \mathbb{C} \to G$ and $x_{-\alpha} : \mathbb{C} \to G$ be the one-parameter subgroups given by

$$x_\alpha(c) = \exp(c e_\alpha), \quad x_{-\alpha}(c) = \exp(c e_{-\alpha}), \quad c \in \mathbb{C}.$$ 

Correspondingly, one also has the co-character $\alpha^\vee : \mathbb{C}^\times \rightarrow T$ such that $\frac{d}{dc}|_{c=1} \alpha^\vee(c) = h_\alpha$.

For $\alpha \in \Gamma$, let $s_\alpha \in W$ be the corresponding simple reflection, and choose the representative $\overline{s_\alpha}$ of $s_\alpha \in W$ in $N_{\alpha}(T)$ by [19, (1.8)]

$$\overline{s_\alpha} = x_\alpha(-1) x_{-\alpha}(1) x_\alpha(-1).$$
Recall that $l : W \to \mathbb{N}$ is the length function on $W$, and that for $w \in W$,
\[ \mathcal{R}(w) = \{ w = (s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}) : \ \alpha_j \in \Gamma \text{ and } s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_l} = w \text{ is reduced} \} \]

Denote $\mathcal{R}(e) = \emptyset$. By [19, Sect. 1.4], for any $w = (s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}) \in \mathcal{R}(w)$,
\[ w \overset{\text{def}}{=} s_{\alpha_1}^{-1}s_{\alpha_2}^{-1}\cdots s_{\alpha_l}^{-1} \]

is a representative of $w$ in $N_G(T)$ independent of the choice of $w \in \mathcal{R}(w)$. Moreover, recall the weak order on $W$ defined as $w_1 \preceq w_2$ if $w = w_1w_2$ and $l(w) = l(w_1)+l(w_2)$, and in such a case $\overline{w} = \overline{w_1}\overline{w_2}$. Denote the (right) action of $W$ on $T$ by
\[ t^w = \overline{w}^{-1}t\overline{w}, \quad t \in T, \ w \in W. \]

For a character $\lambda$ on $T$, the evaluation of $\lambda$ at $t \in T$ is denoted as $t^\lambda \in \mathbb{C}^\times$.

Let $N^-$ be the unipotent radical of $B^-$. For $g \in B^-B = N^-TN$, let $[g]_- \in N^-$, $[g]_0 \in T$, and $[g]_+ \in N$ be the unique elements such that $g = [g]_-[g]_0[g]_+$. For $\alpha \in \Gamma$, let $\omega_{\alpha}$ be the corresponding fundamental weight and let $\Delta^{\omega_{\alpha}}$ be the principal minor on $G$, so that the restriction of $\Delta^{\omega_{\alpha}}$ to $B^-B$ is given by
\[ \Delta^{\omega_{\alpha}}(g) = [g]_0^{\omega_{\alpha}}, \quad g \in B^-B. \]

For $u, v \in W$ and $\alpha \in \Gamma$, let $\Delta_{u\omega_{\alpha},v\omega_{\alpha}}$ be the regular function on $G$ defined by
\[ \Delta_{u\omega_{\alpha},v\omega_{\alpha}}(g) = \Delta^{\omega_{\alpha}}(\overline{u}^{-1}g\overline{v}), \quad g \in G. \]

The functions $\Delta_{u\omega_{\alpha},v\omega_{\alpha}}$ are called generalized minors on $G$ (see [19]). We will need the following property of generalized minors (see the proof of [19, Proposition 2.7]):
\[ \Delta^{\omega_{\alpha}}(w_0^{-1}g^{-1}w_0) = \Delta^{\omega_{\alpha}}(w_0g^{-1}w_0^{-1}) = \Delta^{\omega_{\alpha}^*}(g), \quad g \in G, \quad (1.7) \]

where for $\alpha \in \Gamma$, $\alpha^* = -w_0(\alpha)$. Note that as $s_{\alpha^*}w_0 = w_0s_{\alpha}$, one has
\[ \overline{w_0} = \overline{s_{\alpha^*}}s_{\alpha^*}\overline{w_0} = \overline{s_{\alpha^*}}s_{\alpha^*}\overline{w_0} = \overline{s_{\alpha^*}}w_0s_{\alpha} = \overline{s_{\alpha^*}}w_0s_{\alpha}, \quad (1.8) \]

and thus $\overline{s_{\alpha^*}} = \overline{w_0s_{\alpha}}\overline{w_0^{-1}} = \overline{w_0^{-1}s_{\alpha}}\overline{w_0}$. We also recall from [19, Sect. 2.1] the involutive anti-automorphisms $\tau$ (denoted as $T$ in [19, Sect. 2.1]) and $\iota$ of $G$ given by
\[ t^\tau = t, \quad x_{\alpha}(c)^\tau = x_{-\alpha}(c), \quad t^\iota = t^{-1}, \quad x_{\alpha}(c)^\iota = x_{\alpha}(c), \quad x_{-\alpha}(c)^\iota = x_{-\alpha}(c), \quad (1.8) \]

where $t \in T, \ \alpha \in \Gamma$, and $c \in \mathbb{C}$. By [19, (2.15)], one has
\[ \Delta^{\omega_{\alpha}}((g^{-1})^\iota) = \Delta^{\omega_{\alpha}}(g^\tau) = \Delta^{\omega_{\alpha}}(g), \quad \alpha \in \Gamma, \ g \in G. \quad (1.9) \]
The following facts are from [19, Proposition 2.1]: for any $w \in W$,

$$\overline{w}^r = \overline{w}^{-1} \quad \text{and} \quad \overline{w}^l = \overline{w}^{-1} \quad \text{(1.10)}$$

For an integer $n \geq 1$, let $[1, n] = \{1, 2, \ldots, n\}$. All varieties in this paper are assumed to be smooth. If an algebraic $\mathbb{C}$-torus $\mathbb{T}$ acts an affine variety $X$, define the induced $\mathbb{T}$-action on $\mathbb{C}[X]$ by

$$(t \cdot f)(x) = f(t \cdot x), \quad t \in \mathbb{T}, \ f \in \mathbb{C}[X], \ x \in X.$$  

2 Construction of the Bott–Samelson atlas

2.1 Bott–Samelson coordinates on Bruhat cells

We continue the set-up in Notations 1.1 and 1.3. Recall that for $u \in W$, the $B$-orbit $O^u = BuB/B$ and $B^-$-orbit $B^-uB/B$ in $G/B$ are respectively called the Bruhat (or Schubert) cell and opposite Bruhat (or Schubert) cell corresponding to $u$. Set

$$N_u = N \cap \overline{u}N^\perp \overline{u}^{-1} \quad \text{and} \quad N_u^- = N^- \cap \overline{u}N^\perp \overline{u}^{-1}. \quad \text{(2.1)}$$

It follows from the unique decompositions $BuB = Nu\overline{u}B$ and $B^-uB/B$ that

$$N_u \longrightarrow BuB/B, \ n \mapsto n\overline{u}.B, \quad n \in N_u, \quad \text{(2.2)}$$

$$N_u^- \longrightarrow B^-uB/B, \ m \mapsto m\overline{u}.B, \quad m \in N_u^-, \quad \text{(2.3)}$$

are isomorphisms. Note that if $u = u_1u_2$ and $l(u) = l(u_1) + l(u_2)$, then

$$N_u\overline{u} = (N_{u_1}\overline{u_1})(N_{u_2}\overline{u_2}) \quad \text{(2.4)}$$

is direct product decomposition, from which it follows that

$$N_{u_1} \subset N_u, \quad \overline{u_1}N_{u_2}\overline{u_1}^{-1} \subset N_u, \quad \text{and} \quad \overline{u_1}^{-1}N^-_{u_2} \overline{u_1} \subset N^- \quad \text{(2.5)}$$

Let $u \in W$ and $u = (s_{\alpha_1}, \ldots, s_{\alpha_k}) \in \mathcal{R}(u)$. Recall that for each $\alpha \in \Gamma$ we have the one-parameter subgroup $x_\alpha : \mathbb{C} \rightarrow N$ and $\overline{s_\alpha} \in N_{G}(T)$. For $z = (z_1, \ldots, z_k) \in \mathbb{C}^k$, set

$$g_u(z) := x_{\alpha_1}(z_1)\overline{s_{\alpha_1}} \cdots x_{\alpha_k}(z_k)\overline{s_{\alpha_k}}. \quad \text{(2.6)}$$

By (2.4), $g_u(z) \in N_u\overline{u}$ for every $z \in \mathbb{C}^k$ and that

$$\mathbb{C}^k \ni z = (z_1, \ldots, z_k) \longmapsto g_u(z) \in N_u\overline{u} = N\overline{u} \cap \overline{u}N^- \quad \text{(2.7)}$$
is an isomorphism from $\mathbb{C}^k$ to $N_u\bar{u}$. One thus has the Bott–Samelson parametrization

$$\mathbb{C}^k \ni z = (z_1, \ldots, z_k) \mapsto g_u(z), B \in \mathcal{O}^u. \quad (2.8)$$

The coordinates $(z_1, \ldots, z_k)$ on $\mathcal{O}^u$ via (2.8) are called Bott–Samelson coordinates on $\mathcal{O}^u$ defined by the reduced word $u = (s_{\alpha_1}, \ldots, s_{\alpha_k})$ of $u$.

**Lemma 2.1** Let $u \in W$, $u = (s_{\alpha_1}, \ldots, s_{\alpha_k}) \in \mathcal{R}(u)$, and set $n_u(z) = g_u(z)\bar{u}^{-1} \in N_u$ for $z = (z_1, \ldots, z_k) \in \mathbb{C}^k$. Then

$$z_j = \Delta_{s_{\alpha_1}\cdots s_{\alpha_{j-1}}s_{\alpha_j}, s_{\alpha_1}\cdots s_{\alpha_{j-1}}s_{\alpha_j}}(n_u(z)), \quad j \in [1, k].$$

**Proof** For $i \in [1, k]$, let $g_{\alpha_i}(z_i) = x_{\alpha_i}(z_i)\bar{\alpha_i}$. Let $j \in [1, k]$. By (2.7),

$$s_{\alpha_1} \cdots s_{\alpha_{j-1}}^{-1}g_{\alpha_1}(z_1) \cdots g_{\alpha_{j-1}}(z_{j-1}) \in N^-,$$

$$g_{\alpha_{j+1}}(z_{j+1}) \cdots g_{\alpha_k}(z_k)s_{\alpha_{j+1}}^{-1} \cdots s_{\alpha_k}^{-1} \in N.$$

It follows that

$$\Delta_{s_{\alpha_1} \cdots s_{\alpha_{j-1}}s_{\alpha_j}, s_{\alpha_1} \cdots s_{\alpha_{j-1}}s_{\alpha_j}}(n_u(z)) = \Delta_{s_{\alpha_1} \cdots s_{\alpha_{j-1}}^{-1}g_u(z)s_{\alpha_{j+1}} \cdots s_{\alpha_k}^{-1}}(g_{\alpha_j}(z_j)) = z_j.$$

$\square$

### 2.2 Bott–Samelson coordinates on generalized Bruhat cells

For an integer $r \geq 1$, recall from (1.5) the quotient variety $F_r$ of $G'$ by $B'$, and recall that associated to each $u = (u_1, \ldots, u_r) \in W^r$ one has the generalized Bruhat cell $\mathcal{O}^u \subset F_r$ given in (1.6). For $(g_1, g_2, \ldots, g_r) \in G'$, write $[g_1, g_2, \ldots, g_r]_{F_r} = \omega_r(g_1, \ldots, g_r) \in F_r$, where $\omega_r : G' \to F_r$ is again the projection. One then has the disjoint union

$$F_r = \bigsqcup_{u \in W^r} \mathcal{O}^u, \quad (2.9)$$

generalizing the decomposition $G/B = \bigsqcup_{u \in W} \mathcal{O}^u$. Equip $F_r$ with the $T$-action by

$$t \cdot [g_1, g_2, \ldots, g_r]_{F_r} = [tg_1, g_2, \ldots, g_r]_{F_r}, \quad t \in T, g_1, g_2, \ldots, g_r \in G. \quad (2.10)$$

It is clear that each generalized Bruhat cell $\mathcal{O}^u \subset F_r$ is $T$-invariant.

**Fix** $u = (u_1, \ldots, u_r) \in W^r$. By (2.2), one has the isomorphism

$$N_{u_1} \times N_{u_2} \times \cdots \times N_{u_r} \longrightarrow \mathcal{O}^u,$$

$$(n_1, n_2, \ldots, n_r) \mapsto [n_1\bar{u}_1, n_2\bar{u}_2, \ldots, n_r\bar{u}_r]_{F_r}, \quad (2.11)$$
where \( n_i \in N_{u_i} \) for \( i \in [1, r] \). In particular, \( \dim O^u = l(u_1) + \cdots + l(u_r) \). Let

\[
\tilde{u} = (u_1, u_2, \ldots, u_r) \in \mathcal{R}(u_1) \times \cdots \times \mathcal{R}(u_r).
\]

Let \( l_i = l(u_1) + \cdots + l(u_i) \) for \( i \in [1, r] \), and let \( l = l_r = \dim O^u \). Also write

\[
\tilde{u} = (u_1, u_2, \ldots, u_r) = (s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_r}) \in W^l,
\]

By (2.7), one then has the Bott–Samelson parametrization \( \beta^\tilde{u} : \mathbb{C}^l \to O^u \) given by

\[
\beta^\tilde{u}(z_1, \ldots, z_l) = [g_{u_1}(z_1, \ldots, z_l_1), g_{u_2}(z_{l_1+1}, \ldots, z_l_2), \ldots, g_{u_r}(z_{l_r-1+1}, \ldots, z_l)]_{F_r}.
\]

**Definition 2.2** [15] The coordinates \((z_1, \ldots, z_l)\) via the Bott–Samelson parametrization \( \beta^\tilde{u} : \mathbb{C}^l \to O^u \) are called the Bott–Samelson coordinates on \( O^u \) defined by \( \tilde{u} \).

The following lemma follows immediately from Lemma 2.1.

**Lemma 2.3** For \( u = (u_1, \ldots, u_r) \in W^r \) and \( \tilde{u} = (s_{\alpha_1}, \ldots, s_{\alpha_r}) \in \mathcal{R}(u_1) \times \cdots \times \mathcal{R}(u_r) \) as in (2.12), the Bott–Samelson coordinates \((z_1, \ldots, z_l)\) on \( O^u \) defined by \( \tilde{u} \) are given as follows: for \( i \in [1, r], j \in [l_i-1 + 1, l_i] \) and \( n_i \in N_{u_i} \),

\[
z_j([n_1 u_1, n_2 u_2, \ldots, n_r u_r]_{F_r}) = \Delta_{s_{\alpha_{l_i-1+1}} \cdots s_{\alpha_j-1} \alpha_j}(\omega_{\alpha_j}(n_i)),
\]

where \( s_p = s_{\alpha_p} \) for \( p \in [1, l] \). Furthermore, with respect to the \( T \)-action on \( \mathbb{C}[O^u] \) induced by the \( T \)-action on \( O^u \) in (2.10) (see end of Notation 1.3), one has

\[
t \cdot z_j = t^{s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_j-1}(\alpha_j)} z_j, \quad t \in T, \quad j \in [1, l].
\]

**2.3 Construction of the Bott–Samelson atlas**

Throughout Sect. 2.3, we fix \( v, w \in W \) and let \( Q = B(v) \) or \( N(v) \). We will first construct a decomposition of the shifted big cell \( wB^{-} B/Q \subset G/Q \) using generalized Bruhat cells. We will then introduce Bott–Samelson coordinates on \( wB^{-} B/Q \) using those on generalized Bruhat cells.

Using first the decomposition

\[
wB^{-} B = (\bar{w}N^\bar{w}^{-1}) \bar{w}B,
\]

we write every element in \( wB^{-} B \) uniquely as \( a\bar{w}b \), where \( a \in \bar{w}N^\bar{w}^{-1} \) and \( b \in B \). Using the direct product decompositions

\[
\bar{w}N^\bar{w}^{-1} = N_w^\bar{w} N_w = N_w N_w^\bar{w},
\]

\( \bar{w}N^\bar{w}^{-1} \) is a product of two conjugate Bruhat cells.
where recall that \( N_w = N \cap wN^{-w} \) and \( N^{-w} = N^{-} \cap wN^{-w} \), we further decompose every \( a \in \overline{wN^{-w}} \) uniquely as

\[
a = a_+ a_- = a'_- a'_+ \quad \text{with} \quad a_+, a'_- \in N_w, \quad a_-, a'_+ \in N^{-w}. \tag{2.16}
\]

Since \( a'_- a'^{-1}_+ = (a'_-)^{-1} a_+ \), one has (see Notation 1.3)

\[
a = [a'_- a'^{-1}_+] a_- = [a'_+ a'^{-1}_-] a'_+ . \tag{2.17}
\]

It follows that the map

\[
\overline{wN^{-w}} \longrightarrow N^{-w} \times N_w, \quad a \longmapsto (a_-, a'_+), \tag{2.18}
\]

is an isomorphism, where again \( a \in \overline{wN^{-w}} \) is decomposed as in (2.16). Consequently, for any closed subgroup \( Q \) of \( B \), one has the isomorphism

\[
I^w_Q : \quad wB^{-w} B / Q \longrightarrow (B^{-w}B/B) \times (BwB/Q),
\]

\[
a \bar{w} b. Q \longmapsto (a_- wB, \quad a'_+ \bar{w} b. Q), \tag{2.19}
\]

where again \( a \in \overline{wN^{-w}} \), decomposed as in (2.16), and \( b \in B \).

For \( Q = B(v) \) or \( N(v) \), using the unique decomposition

\[
B = N_v N(v) T,
\]

we can write elements in \( BwB/B(v) \) and in \( BwB/N(v) \) uniquely as

\[
n_1 \bar{w} n_2 B(v) \in BwB/B(v), \quad n_1 \bar{w} n_2 t. N(v) \in BwB/N(v),
\]

where \( n_1 \in N_w, \quad n_2 \in N_v \) and \( t \in T \). It follows that one has the isomorphisms

\[
\xi^{(w,v)}_{B(v)} : \quad BwB/B(v) \longrightarrow O^{(w,v)}, \quad n_1 \bar{w} n_2 B(v) \longmapsto [n_1 \bar{w}, n_2 \bar{v}]_{F_2}, \tag{2.20}
\]

\[
\xi^{(w,v)}_{N(v)} : \quad BwB/N(v) \longrightarrow O^{(w,v)} \times T, \quad n_1 \bar{w} n_2 t. N(v) \longmapsto ([n_1 \bar{w}, n_2 \bar{v}]_{F_2}, \quad t), \tag{2.21}
\]

where again \( n_1 \in N_w, \quad n_2 \in N_v \), and \( t \in T \). On the other hand, the identity

\[
\overline{w_0 w^{-1}} N_w = N_{w_0 w^{-1}} \overline{w_0 w^{-1}},
\]

gives rise to the isomorphism

\[
\xi^w : \quad B^{-w} B / B \longrightarrow O^{w_0 w^{-1}}, \quad m \bar{w}. B \longmapsto \overline{w_0 w^{-1} m^{-1}}. B, \quad m \in N_w. \tag{2.22}
\]
Combining the isomorphism $J^w_\mathcal{O}$ in (2.19) with the isomorphisms in (2.20), (2.21), and (2.22), we get our desired decompositions

$$J^w_{B(v)} : wB^-B/B(v) \xrightarrow{\sim} \mathcal{O}^{u_0w^{-1}} \times \mathcal{O}^{(w,v)}, \quad \text{(2.23)}$$

$$J^w_{N(v)} : wB^-B/N(v) \xrightarrow{\sim} \mathcal{O}^{u_0w^{-1}} \times \mathcal{O}^{(w,v)} \times T, \quad \text{(2.24)}$$

explicitly given as

$$J^w_{B(v)}(a\overline{w}n.B(v)) = \left(\overline{u} a_{-1}^{-1}.B, [a_+^{'} \overline{w}, n\overline{v}]_{F_2}\right), \quad \text{(2.25)}$$

$$J^w_{N(v)}(a\overline{w}nt.N(v)) = \left(\overline{u} a_{-1}^{-1}.B, [a_+^{'} \overline{w}, n\overline{v}]_{F_2}, t\right), \quad \text{(2.26)}$$

where $a \in \overline{w}N\overline{w}^{-1}$, decomposed as in (2.16), $n \in N_v, t \in T$, and $u = u_0w^{-1}$. It is straightforward to prove the following $T$-equivariance of $J^w_{B(v)}$ and $J^w_{N(v)}$.

**Lemma 2.4** The isomorphisms $J^w_{B(v)}$ and $J^w_{N(v)}$ are $T$-equivariant, where $t_1 \in T$ acts on $wB^-B/B(v)$ and $wB^-B/N(v)$ by left translation by $t_1$ and on $\mathcal{O}^{u_0w^{-1}} \times \mathcal{O}^{(w,v)}$ and $\mathcal{O}^{u_0w^{-1}} \times \mathcal{O}^{(w,v)} \times T$ respectively by

$$t_1 \cdot (n_1\overline{u}.B, [n_2\overline{w}, n_3\overline{v}]_{F_2}) = \left(t_1^{u_0^{-1}}n_1\overline{u}.B, [t_1n_2\overline{w}, n_3\overline{v}]_{F_2}\right),$$

$$t_1 \cdot (n_1\overline{u}.B, [n_2\overline{w}, n_3\overline{v}]_{F_2}, t) = \left(t_1^{u_0^{-1}}n_1\overline{u}.B, [t_1n_2\overline{w}, n_3\overline{v}]_{F_2}, t_1^{w}.t\right),$$

where $u = u_0w^{-1} \in W, n_1 \in N_u, n_2 \in N_w, n_3 \in N_v$ and $t \in T$.

It is also straightforward to prove the following for the inverses of $J^w_{B(v)}$ and $J^w_{N(v)}$.

**Lemma 2.5** With $u = u_0w^{-1}$, the inverses of $J^w_{B(v)}$ and $J^w_{N(v)}$ are respectively given by

$$\left(J^w_{B(v)}\right)^{-1}(n_1\overline{u}.B, [n_2\overline{w}, n_3\overline{v}]_{F_2}) = [n_2(\overline{u}^{-1}n_1\overline{u})]^{-1}n_2\overline{w}n_3.B(v) = [n_2(\overline{u}^{-1}n_1\overline{u})]_{+}(\overline{u}^{-1}n_1\overline{u})\overline{w}n_3.B(v),$$

$$\left(J^w_{N(v)}\right)^{-1}(n_1\overline{u}.B, [n_2\overline{w}, n_3\overline{v}]_{F_2}, t) = [n_2(\overline{u}^{-1}n_1\overline{u})]^{-1}n_2\overline{w}n_3t.N(v) = [n_2(\overline{u}^{-1}n_1\overline{u})]_{+}(\overline{u}^{-1}n_1\overline{u})\overline{w}n_3t.N(v),$$

where $n_1 \in N_u, n_2 \in N_w, n_3 \in N_v$, and $t \in T$. 
We can now use the isomorphisms $J^u$ in (2.23) and (2.24) to introduce Bott–Samelson coordinates on $wB^-B/Q$. Let first $Q = B(v)$.

**Notation 2.6** Let again $l_0 = l(w_0)$. For $v, w \in W$, let $k = l(w_0 w^{-1}) = l_0 - l(w)$ and $l = l_0 + l(v)$. Write an element $r \in R(w_0 w^{-1}) \times R(w) \times R(v)$ as $r = (w^0, w, v)$ with

$$w^0 = (s_{\alpha_1}, \ldots, s_{\alpha_k}), \quad w = (s_{\alpha_{k+1}}, \ldots, s_{\alpha_0}), \quad v = (s_{\alpha_{0+1}}, \ldots, s_{\alpha_l}), \quad (2.27)$$

where $\alpha_j \in \Gamma$ for each $j \in [1, l]$.

Recall from (2.13) that associated to $w^0 \in R(w_0 w^{-1})$ and $(w, v) \in R(w) \times R(v)$ we have the parametrizations $\beta^{w^0} : C^k \rightarrow O_{w_0 w^{-1}}$ and $\beta^{(w, v)} : C^{l-k} \rightarrow O_{w, v}$ given by

$$\begin{align*}
\beta^{w^0}(z_1, \ldots, z_k) &= g_{w^0}(z_1, \ldots, z_k)B, \\
\beta^{(w, v)}(z_{k+1}, \ldots, z_l) &= (\{g_{w}(z_{k+1}, \ldots, z_0), \ g_{v}(z_0+1, \ldots, z_l)\}r_2). 
\end{align*}$$

Consequently, one has the parametrization

$$\sigma^r = \beta^{w^0} \times \beta^{(w, v)} : C^l \rightarrow O_{w_0 w^{-1}} \times O_{w, v},$$

$$\sigma^r(z) = (g_{w^0}(z_1, \ldots, z_k)B, \ [g_{w}(z_{k+1}, \ldots, z_0), \ g_{v}(z_0+1, \ldots, z_l)]r_2). \quad (2.28)$$

Combining with $(J^w)_{B(v)}^{-1} : O_{w_0 w^{-1}} \times O_{w, v} \rightarrow wB^-B/B(v)$, we have the isomorphism

$$\sigma^r_{B(v)} \overset{\text{def}}{=} (J^w)_{B(v)}^{-1} \circ \sigma^r : C^l \rightarrow wB^-B/B(v). \quad (2.29)$$

**Definition 2.7** For $w \in W$ and $r = (w^0, w, v) \in R(w_0 w^{-1}) \times R(w) \times R(v)$, the map $\sigma^r_{B(v)} : C^l \rightarrow wB^-B/B(v)$ in (2.29) is called the Bott–Samelson parametrization of $wB^-B/B(v)$ defined by $r$, and the induced coordinates $(z_1, z_2, \ldots, z_l)$ are called the Bott–Samelson coordinates on $wB^-B/B(v)$ defined by $r$. The collection

$$\mathcal{A}_{BS}(G/B(v)) = \{\sigma^r_{B(v)} : w \in W, \ r \in R(w_0 w^{-1}) \times R(w) \times R(v)\}$$

is called the Bott–Samelson atlas on $G/B(v)$, and each $\sigma^r_{B(v)}$ in $\mathcal{A}_{BS}(G/B(v))$ is called a Bott–Samelson coordinate chart on $G/B(v)$. \hfill \Box

Turning to $Q = N(v)$, fix any listing $\omega_1, \ldots, \omega_d$ of all the fundamental weights, and let

$$\sigma : (C^x)^d \rightarrow T, \quad (2.30)$$
be the inverse of the isomorphism $T \to (\mathbb{C}^\times)^d, t \mapsto (t^{\alpha_1}, \ldots, t^{\alpha_d})$. One then has the parametrization
\[
\sigma^r \times \sigma : \mathbb{C}^d \times (\mathbb{C}^\times)^d \to \mathcal{O}^{\mathfrak{w}_0 \mathfrak{w}^{-1}} \times \mathcal{O}^{(w, v)} \times T.
\] (2.31)
Combining with $(J^w_{N(v)})^{-1} : \mathcal{O}^{\mathfrak{w}_0 \mathfrak{w}^{-1}} \times \mathcal{O}^{(w, v)} \times T \to wB^{-}B/N(v)$, one has the isomorphism
\[
\sigma^r_{N(v)} = (J^w_{N(v)})^{-1} \circ (\sigma^r \times \sigma) : \mathbb{C}^d \times (\mathbb{C}^\times)^d \to wB^{-}B/N(v).
\] (2.32)

**Definition 2.8** For $w \in W$ and $r = (w^0, w, v) \in \mathcal{R}(\mathfrak{w}_0 \mathfrak{w}^{-1}) \times \mathcal{R}(w) \times \mathcal{R}(v)$, the map $\sigma^r_{N(v)} : \mathbb{C}^d \times (\mathbb{C}^\times)^d \to wB^{-}B/N(v)$ in (2.32) is called the Bott–Samelson parametrization of $wB^{-}B/N(v)$ defined by $r$, and the induced coordinates $(z_1, z_2, \ldots, z_{l+d})$ are called the Bott–Samelson coordinates on $wB^{-}B/N(v)$ defined by $r$. The collection
\[
\mathcal{A}_{BS}(G/N(v)) = \{\sigma^r_{N(v)} : w \in W, r \in \mathcal{R}(\mathfrak{w}_0 \mathfrak{w}^{-1}) \times \mathcal{R}(w) \times \mathcal{R}(v)\}
\]
is called the Bott–Samelson atlas on $G/N(v)$, and each $\sigma^r_{N(v)}$ in $\mathcal{A}_{BS}(G/N(v))$ is called a Bott–Samelson coordinate chart on $G/N(v)$.

**Example 2.9** Let $v = w_0$ so $G/N(v) = G$. For $w = e$, the Bott–Samelson parametrization $\sigma^r_G : \mathbb{C}^{2l_0} \times (\mathbb{C}^\times)^d \sim B^{-}B$ for $r = (w^0, \emptyset, w_0) \in \mathcal{R}(w_0) \times \mathcal{R}(e) \times \mathcal{R}(w_0)$ is given by
\[
\sigma^r_G(z) = \left(\frac{1}{w_0}g_{w_0}(z_1, \ldots, z_{l_0})^{-1}g_{w_0}(z_{l_0+1}, \ldots, z_{l_0+d})\right) \sigma(z_{l_0+1}, \ldots, z_{l_0+d})
\] for $z = (z_1, \ldots, z_{l_0+d}) \in \mathbb{C}^{2l_0} \times (\mathbb{C}^\times)^d$. Similarly, for $w = w_0$ and for each $r = (\emptyset, w_0, w_0^0) \in \mathcal{R}(e) \times \mathcal{R}(w_0) \times \mathcal{R}(w_0)$, we have the Bott–Samelson parametrization $\sigma^r_G : \mathbb{C}^{2l_0} \times (\mathbb{C}^\times)^d \sim Bw_0B$ given by
\[
\sigma^r_G(z_1, \ldots, z_{l_0+d}) = g_{w_0}(z_1, \ldots, z_{l_0})g_{w_0^0}(z_{l_0+1}, \ldots, z_{l_0+d})\frac{1}{w_0}^{-1} \sigma(z_{l_0+1}, \ldots, z_{l_0+d}).
\]

In the remainder of Sect. 2.3, we express the Bott–Samelson coordinates on $G/B(v)$ and $G/N(v)$ using generalized minors on $G$.

Consider again the case of $Q = B(v)$ first. Fix $w \in W$ and let again $u = w_0 w^{-1}$. Let $r = (w^0, w, v) \in \mathcal{R}(w_0 w^{-1}) \times \mathcal{R}(w) \times \mathcal{R}(v)$ as in (2.27), i.e.,
\[
w^0 = (s_{\alpha_1}, \ldots, s_{\alpha_k}), \quad w = (s_{\alpha_{k+1}}, \ldots, s_{\alpha_0}), \quad v = (s_{\alpha_{l_0+1}}, \ldots, s_{\alpha_l}).
\]
Note then that $(s_{\alpha_1}, s_{\alpha_{k+1}}, \ldots, s_{\alpha_0}) \in \mathcal{R}(w_0)$ and that
\[
l(s_{\alpha_1} \cdots s_{\alpha_k} w) = l(s_{\alpha_i} \cdots s_{\alpha_k}) + l(w), \quad i \in [1, k].
\] (2.33)
Recall from Notation 1.3 that for $\alpha \in \Gamma$ we have $\alpha^* = -w_0(\alpha)$ and that $\bar{s}_\alpha^* = \bar{w}_0^{-1} \bar{s}_\alpha \bar{w}_0$.

**Proposition 2.10** For $w \in W$, write an element in $wB^-B/B(v)$ as $\bar{w}mn.B(v)$ for unique $m \in N^-$ and $n \in N_v$. Then for $r \in \mathcal{R}(w_0 w^{-1}) \times \mathcal{R}(w) \times \mathcal{R}(v)$ as in (2.27), the Bott–Samelson coordinates $(z_1, \ldots, z_j)$ on $wB^-B/B(v)$ defined by $r$ are given by

$$z_j(\bar{w}mn.B(v)) = \begin{cases} 
\Delta_{\bar{s}_{\alpha_j} \cdots \bar{s}_{\alpha_{j-1}} \bar{w} \bar{m}^{-1}} (m), & j \in [1, k], \\
\Delta_{\bar{s}_{\alpha_{j-1}} \cdots \bar{s}_{\alpha_{j-2}} \bar{s}_{\alpha_{j-1}} \alpha^j (\bar{m} \bar{m}^{-1}), & j \in [k + 1, l_0], (2.34) \\
\Delta_{\bar{s}_{\alpha_{l_0+1}} \cdots \bar{s}_{\alpha_{l_1}} \cdots \bar{s}_{\alpha_{l_1-1}} \alpha^j (n), & j \in [l_0 + 1, l]. 
\end{cases}$$

Furthermore, with respect to the $T$-action on $\mathbb{C}[wB^-B/B(v)]$ induced from the $T$-action on $wB^-B/B(v)$ by left translation, each $z_j$ is a $T$-weight vector, with

$$t \cdot z_j = \begin{cases} 
t_{\bar{s}_{\alpha_j} \cdots \bar{s}_{\alpha_{j-1}} \bar{w} \bar{m}^{-1}} (z_j), & j \in [1, k], \\
t_{\bar{s}_{\alpha_{j-1}} \cdots \bar{s}_{\alpha_{j-2}} \bar{s}_{\alpha_{j-1}} \alpha^j (\bar{m} \bar{m}^{-1}), & j \in [k + 1, l]. 
\end{cases} \quad t \in T.$$

**Proof** Fix $m \in N^-$ and $n \in N_v$ and let $a = \bar{w}m^{-1} \in \bar{w}N^-w^{-1}$. Decompose $a$ again as in (2.16), i.e., $a = a_+ a_- = a'_+ a'_-$, where $a_+, a'_+ \in N_w$ and $a_-, a'_- \in N_w$. Let $n_1 = \bar{a}_-^{-1} \bar{u}^{-1} \in N_u$ and $n_2 = a'_+ \in N_w$. By the definition of $J^w_{B(v)}$,

$$J^w_{B(v)}(\bar{w}mn.B(v)) = J^w_{B(v)}(a \bar{w}n.B(v)) = (n_1 \bar{u} B, [n_2 \bar{w}, n \bar{v}]_{F_2}).$$

Write $z_j = z_j(\bar{w}mn.B(v))$ for $j \in [1, l]$. Let first $j \in [1, k]$. By Lemma 2.1,

$$z_j = \Delta^{\omega_{\alpha_j}} (s_{\alpha_1} \cdots s_{\alpha_{j-1}}^{-1} n_1 s_{\alpha_1} \cdots s_{\alpha_j}) = \Delta^{\omega_{\alpha_j}} (s_{\alpha_1} \cdots s_{\alpha_{j-1}}^{-1} (\bar{a}^{-1} \bar{u}^{-1}) s_{\alpha_1} \cdots s_{\alpha_j})$$

$$= \Delta^{\omega_{\alpha_j}} (s_{\alpha_1} \cdots s_{\alpha_k} a_+^{-1} s_{\alpha_{j+1}} \cdots s_{\alpha_k}^{-1}).$$

By (2.33) and (2.5), $s_{\alpha_{j+1}} \cdots s_{\alpha_k} a_+ s_{\alpha_{j+1}} \cdots s_{\alpha_k}^{-1} \in N$. Thus

$$z_j = \Delta^{\omega_{\alpha_j}} (s_{\alpha_1} \cdots s_{\alpha_k} a_+ s_{\alpha_{j+1}} \cdots s_{\alpha_k}^{-1}) = \Delta^{\omega_{\alpha_j}} (s_{\alpha_1} \cdots s_{\alpha_k} a_+ s_{\alpha_{j+1}} \cdots s_{\alpha_k}^{-1})$$

$$= \Delta^{\omega_{\alpha_j}} (s_{\alpha_1} \cdots s_{\alpha_k} a_+^{-1} s_{\alpha_{j+1}} \cdots s_{\alpha_k}^{-1}) = \Delta^{\omega_{\alpha_j}} (s_{\alpha_1} \cdots s_{\alpha_k} a_+ s_{\alpha_{j+1}} \cdots s_{\alpha_k}^{-1}).$$

where in the last step we used again (2.33) for $i = j$ and $j + 1$. By (1.7), one has

$$z_j = \Delta^{\omega_{\alpha_j}^*} (\bar{w}_0^{-1} s_{\alpha_{j+1}} \cdots s_{\alpha_k} \bar{w} m \bar{s}_{\alpha_j} \cdots s_{\alpha_k} \bar{w}^{-1} \bar{w}_0).$$

Note that for $i = j$ or $j + 1$, one has $\bar{s}_{\alpha_i} \cdots s_{\alpha_k} \bar{w} = \bar{s}_{\alpha_i} \cdots s_{\alpha_{i-1}}^{-1} \bar{w}_0$. It follows that

$$z_j = \Delta^{\omega_{\alpha_j}^*} (\bar{w}_0^{-1} s_{\alpha_1} \cdots s_{\alpha_j}^{-1} \bar{w}_0 m \bar{w}_0^{-1} s_{\alpha_1} \cdots s_{\alpha_{j-1}} \bar{w}_0).$$
Proposition 2.11 \[ \text{Assume now that } j \in [k + 1, l_0]. \text{ By Lemma 2.1,} \]
\[
z_j = \Delta^{a_{k+1}} (s_{k+1} \cdots s_{j-1} a_{k+1} \cdots a_j n_2 s_{k+1} \cdots s_{j-1}) = \Delta^{a_{k+1}} (s_{k+1} \cdots s_{j-1} a_{k+1} \cdots a_j).
\]
By (2.5), \( s_{k+1} \cdots s_{j-1} a_{k+1} \cdots a_j \) is an element in \( \mathbb{Z}^+ \). It follows that
\[
z_j = \Delta^{a_{k+1}} (s_{k+1} \cdots s_{j-1} a_{k+1} \cdots a_j) = \Delta^{a_{k+1}} \cdots s_{j-1} a_{k+1} \cdots s_{j-1} a_{k+1} \cdots a_j (w m w^{-1}).
\]
Finally, assume that \( j \in [l_0 + 1, l] \). By Lemma 2.1,
\[
z_j = \Delta^{a_{l_0+1}} (s_{l_0+1} \cdots s_{j-1} a_{l_0+1} \cdots a_j n_2 s_{l_0+1} \cdots s_{j-1}) = \Delta^{a_{l_0+1}} \cdots s_{j-1} a_{l_0+1} \cdots s_{j-1} a_{l_0+1} a_{l_0+1} \cdots a_j (n).
\]
The statement on the \( T \)-weight for each \( z_j \) follows either from a direct calculation using (2.34) or from the \( T \)-equivariance of the isomorphism \( J^w_B(v) \) and Lemma 2.3. □

Turning to \( Q = N(v) \), recall that we have fixed a listing \( \omega_1, \ldots, \omega_d \) of all the fundamental weights to define the isomorphism \( \sigma : (\mathbb{C}^d) \rightarrow T \) in (2.30), which is in turn used in defining the Bott–Samelson coordinate charts on \( G/N(v) \). The following proposition follows directly from Proposition 2.10.

Proposition 2.11 For \( w \in W \), write an element in \( wB^-B/N(v) \) as \( \bar{w}mnt.N(v) \) for unique \( m \in \mathbb{N}^+ \), \( n \in N \), and \( t \in T \). Then for \( r \in R(w_0 w^{-1}) \times R(w) \times R(w) \) as in (2.27), the Bott–Samelson coordinates \((z_1, \ldots, z_{l+d})\) on \( wB^-B/N(v) \) defined by \( r \) are given by
\[
z_{j}(\bar{w}mnt.N(v)) = \begin{cases}
\Delta^{s_{a_j} \cdots s_{a_j} \cdots s_{a_j} \cdots s_{a_j} \cdots s_{a_j}} (m), & j \in [1, k], \\
\Delta^{s_{a_k} \cdots s_{a_k} \cdots s_{a_k} \cdots s_{a_k} \cdots s_{a_k}} (\bar{w} m \bar{w}^{-1}), & j \in [k + 1, l], \\
\Delta^{s_{a_{l_0+1}} \cdots s_{a_{l_0+1}} \cdots s_{a_{l_0+1}} \cdots s_{a_{l_0+1}}} (n), & j \in [l + 1, l + d].
\end{cases}
\]
Furthermore, with respect to the \( T \)-action on \( \mathbb{C}[wB^-B/N(v)] \) induced by the \( T \)-action on \( wB^-B/N(v) \) by left translation, each \( z_j \) is a \( T \)-weight vector, with
\[
t \cdot z_j = \begin{cases}
t^{s_{a_k} \cdots s_{a_k} \cdots s_{a_k} \cdots s_{a_k} \cdots s_{a_k}} (a_j) z_j, & j \in [1, k], \\
t^{s_{a_k} \cdots s_{a_k} \cdots s_{a_k} \cdots s_{a_k} \cdots s_{a_k}} (a_j) z_j, & j \in [k + 1, l], \\
t^{s_{a_k} \cdots s_{a_k} \cdots s_{a_k} \cdots s_{a_k} \cdots s_{a_k}} (a_j) z_j, & j \in [l + 1, l + d],
\end{cases} t \in T.
\]
**Example 2.12** Consider the case of $Q = B$ so that $v = e$. For any $w \in W$ and $r = (w^0, w, \emptyset) \in \mathcal{R}(w_0w^{-1}) \times \mathcal{R}(w) \times \mathcal{R}(e)$ with
\[
 w^0 = (s_{\alpha_1}, \ldots, s_{\alpha_k}) \quad \text{and} \quad w = (s_{\alpha_{k+1}}, \ldots, s_{\alpha_l}),
\]
the Bott–Samelson coordinates on $wB^{-}B/B$ defined by $r$ are given by
\[
 z_j(\overline{w}m.B) = \begin{cases} 
 \Delta_{s_{\alpha_1} \cdots s_{\alpha_j} \cdots s_{\alpha_{k-1}}} \omega_{\alpha_j} (m), & j \in [1, k], \\
 \Delta_{s_{\alpha_{k+1}} \cdots s_{\alpha_{k-1}}} \omega_{\alpha_j} (\overline{w}m\overline{w}^{-1}), & j \in [k+1, l_0].
\end{cases}
\]
(2.35)

**Example 2.13** Consider $G = SL(3, \mathbb{C})$ and $Q = \{e\}$ (so $v = w_0$), with $B$ and $B^-$ consisting, respectively, all the upper-triangular and lower triangular elements in $G$, and denote $s_1 = s_{\alpha_1}$ and $s_2 = s_{\alpha_2}$ for the standard choices of $\alpha_1$ and $\alpha_2$. There are a total of 16 Bott–Samelson coordinate charts on $G$, corresponding to the 16 elements in the set $\bigcup_{w \in W} \mathcal{R}(w_0w^{-1}) \times \mathcal{R}(w) \times \mathcal{R}(w_0)$. Write $g \in G$ as $g = \left(\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33} \end{array} \right)$ and let $\Delta_{i,j,k,l} = \det \left(\begin{array}{cc} a_{ik} & a_{il} \\
 a_{jk} & a_{jl} \end{array} \right)$ for $i < j$ and $k < l$. As the first example, let $w = e$ and choose $r_1 = ((s_1, s_2, s_1), \emptyset, (s_1, s_2, s_1))$. Then the Bott–Samelson parametrization $\sigma^{r_1} = \sigma^{r_1}_G$ of $B^{-}B$ by $\mathbb{C}^6 \times (\mathbb{C}^\times)^2$ maps $\xi = (\xi_1, \ldots, \xi_8) \in \mathbb{C}^6 \times (\mathbb{C}^\times)^2$ to
\[
 \sigma^{r_1}(\xi) = \begin{pmatrix} 
 1 & 0 & 0 \\
 \xi_3 & 1 & 0 \\
 \xi_2 & \xi_1 & 1 \\
 \xi_7 & \xi_4\xi_6 - \xi_5 & \xi_6 \\
 0 & 1 & \xi_6 \\
 0 & 0 & 1 \\
 \xi_3\xi_7 & (\xi_3\xi_4 - \xi_5)/\xi_8 & (\xi_3\xi_4\xi_6 - \xi_3\xi_5 + \xi_6)/\xi_8 \\
 \xi_2\xi_7 & (\xi_2\xi_4 + \xi_1)\xi_8/\xi_7 & (\xi_2\xi_4\xi_6 - \xi_2\xi_5 + \xi_1\xi_6 + 1)/\xi_8 
\end{pmatrix},
\]
and the corresponding Bott–Samelson coordinates $(\xi_1, \ldots, \xi_8)$ on $B^{-}B$ are
\[
\begin{align*}
 \xi_1 &= \frac{\Delta_{13,12}}{\Delta_{12,12}}, \quad \xi_2 = \frac{a_{31}}{a_{11}}, \quad \xi_3 = \frac{a_{21}}{a_{11}}, \quad \xi_4 = \frac{a_{11}a_{12}}{\Delta_{12,12}}, \quad \xi_5 = a_{11}\Delta_{12,23}, \\
 \xi_6 &= \frac{\Delta_{12,12}\Delta_{12,13}}{a_{11}}, \quad \xi_7 = a_{11}, \quad \xi_8 &= \Delta_{12,12}.
\end{align*}
\]
As the second example, let $w = w_0 = s_1s_2s_1$ and $r_2 = (\emptyset, (s_2, s_1, s_2), (s_1, s_2, s_1))$. The Bott–Samelson parametrization $\sigma^{r_2} = \sigma^{r_2}_G$ of $w_0B^{-}B = Bw_0B$ by $\mathbb{C}^6 \times (\mathbb{C}^\times)^2$ maps $z = (z_1, \ldots, z_8) \in \mathbb{C}^6 \times (\mathbb{C}^\times)^2$ to
\[
 \sigma^{r_2}(z) = \overline{w}_0 \begin{pmatrix} 
 1 & 0 & 0 \\
 -z_1 & 1 & 0 \\
 z_2 & -z_3 & 1 \\
 z_4 & z_4z_6 - z_5 & 0 \\
 0 & 1 & z_6 \\
 0 & 0 & 1 \\
 0 & z_8/z_7 & 0 \\
 0 & 0 & 1/z_8 
\end{pmatrix}.
\]
1. For an integer $m \geq 1$, let $\text{Poly}^>^0$ be the set of all non-zero polynomials in $m$ variables with non-negative integer coefficients. Elements in $\text{Poly}^>^0$ will also be called positive integral polynomials in $m$ variables.

3.1 Positive varieties

We first recall the notion of positive varieties.

Notation 3.1 1. For an integer $m \geq 1$, let $\text{Poly}^>^0_m$ be the set of all non-zero polynomials in $m$ variables with non-negative integer coefficients. Elements in $\text{Poly}^>^0_m$ will also be called positive integral polynomials in $m$ variables.

2. For an irreducible rational complex variety $X$ with the field $\mathbb{C}(X)$ of rational functions, and for a subset $\Phi = \{\phi_1, \ldots, \phi_m\}$ of algebraically independent ele-

and the Bott–Samelson coordinates $(\xi_1, \ldots, \xi_8)$ on $w_0B^-B$ are

$$
\begin{align*}
\xi_1 &= z_3/(z_1z_3 - z_2), \quad \xi_2 = 1/z_2, \quad \xi_3 = z_1/z_2, \\
\xi_4 &= \frac{z_2(z_2z_4 - z_3)}{z_1z_3 - z_2}, \quad \xi_5 = z_2(1 - z_1z_4 + (z_1z_3 - z_2)z_5), \\
\xi_6 &= \frac{1}{z_2}(z_1z_3 - z_2)(z_1z_3z_6 - z_2z_6 - z_1), \quad \xi_7 = z_2z_7, \quad \xi_8 = z_8(z_1z_3 - z_2), \\
\xi_9 &= \frac{\xi_2(\xi_2\xi_4 + \xi_1)}{\xi_1\xi_3 - \xi_2}, \quad \xi_5 = \xi_2(\xi_1\xi_3\xi_5 - \xi_2\xi_5 + \xi_3\xi_4 + 1), \\
\xi_6 &= \frac{1}{\xi_2}(\xi_1\xi_3 - \xi_2)(\xi_1\xi_3\xi_6 - \xi_2\xi_6 + \xi_3), \quad \xi_7 = \xi_2\xi_7, \quad \xi_8 = \xi_8(\xi_1\xi_3 - \xi_2).
\end{align*}
$$

3 Positivity of the Bott–Samelson coordinates

In Sect. 3.1 we recall from [1,7,17] the notion of positive structures on complex varieties. In Sect. 3.2 we first recall the Lusztig positive structure on $G$ and then extend it to $G/Q$ for $Q = B(v)$ or $N(v)$ for all $v \in W$. Some results from [8,19] on generalized minors and double Bruhat cells are reviewed in Sect. 3.3, which are then used in Sect. 3.4 to prove positivity of all Bott–Samelson coordinates on $G/Q$ with respect to the Lusztig positive structure.
ments in $\mathbb{C}(X)$, denote by Pos($\phi$) the set of elements $f \in \mathbb{C}(X)$ such that $f = p(\phi_1, \ldots, \phi_m)/q(\phi_1, \ldots, \phi_m)$ for some $p, q \in \text{Poly}_m^{>0}$. Elements in Pos($\phi$) are also said to have subtraction-free expressions in $\phi$.

3. A rational map $F$ from $(\mathbb{C}^\times)^k$ to $(\mathbb{C}^\times)^l$ is said to be positive if the components of $F$ are in $\text{Pos}(c_1, \ldots, c_k)$, where $(c_1, \ldots, c_k)$ are the coordinates on $(\mathbb{C}^\times)^k$.

**Definition 3.2** [7,17] Let $X$ be an $n$-dimensional irreducible rational complex variety.

1. A *toric chart* on $X$ is an open embedding $\rho : (\mathbb{C}^\times)^n \rightarrow X$. Two toric charts

$$\rho_1 : (\mathbb{C}^\times)^n \rightarrow X \text{ and } \rho_2 : (\mathbb{C}^\times)^n \rightarrow X$$

are said to be *positively equivalent* if both $\rho_2^{-1} \circ \rho_1$ and $\rho_1^{-1} \circ \rho_2$ are positive rational maps from $(\mathbb{C}^\times)^n$ to $(\mathbb{C}^\times)^n$. The collection of all toric charts positively equivalent to a given toric chart $\rho$ is called the *positive equivalence class* of $\rho$ and is denoted as $[\rho]$.

2. A *positive structure* on $X$ is a positive equivalence class $\mathcal{P}^X$ of toric charts on $X$.

**Definition 3.3** [7,17]

1. A *positive variety* is a pair $(X, \mathcal{P}^X)$, where $X$ is an irreducible rational complex variety and $\mathcal{P}^X$ a positive structure on $X$. A toric chart $\rho$ on $X$ such that $[\rho] = \mathcal{P}^X$ is also called a *toric chart in* $\mathcal{P}^X$.

2. Given a positive variety $(X, \mathcal{P}^X)$ and a toric chart $\rho : (\mathbb{C}^\times)^n \rightarrow X$ in $\mathcal{P}^X$, the subset

$$(X, \mathcal{P}^X)_{>0} \overset{\text{def}}{=} \rho((\mathbb{R}_{>0})^n)$$

of $X$, which is independent of the choice of $\rho$ in $\mathcal{P}^X$, is called the *totally positive part* of $(X, \mathcal{P}^X)$. Here $\mathbb{R}_{>0}$ is the set of all positive real numbers. When the positive structure $\mathcal{P}^X$ is clearly indicated, we denote $(X, \mathcal{P}^X)_{>0}$ simply by $X_{>0}$ and call it the totally positive part of $X$.

3. Given a positive variety $(X, \mathcal{P}^X)$, a rational function $f \in \mathbb{C}(X)$ is said to be *positive with respect to* $\mathcal{P}^X$ if there exists a toric chart $\rho : (\mathbb{C}^\times)^n \rightarrow X$ in $\mathcal{P}^X$ such that $f \in \text{Pos}(c_1, \ldots, c_n)$, where $(c_1, \ldots, c_n)$ are the local coordinates on $X$ defined by the toric chart $\rho$. Note that the definition of $f$ being positive is independent on the choice of the toric chart $\rho$ in $\mathcal{P}^X$. Denote by Pos($X, \mathcal{P}^X$), or simply by Pos($X$) when $\mathcal{P}^X$ is clearly understood, the set of all rational functions on $X$ that are positive with respect to $\mathcal{P}^X$.

\[\square\]

**Remark 3.4** It is clear that for any integer $m \geq 1$, the set of Poly$^{>0}_m$ of positive integral polynomials in $m$ variables is closed under addition and multiplication. Consequently, for any positive variety $(X, \mathcal{P}^X)$, Pos($X, \mathcal{P}^X$) is a semi-field in the sense that it is closed under addition, multiplication, and division but with no zero element. In particular, for any finite subset $\{f_1, \ldots, f_m\}$ of Pos($X, \mathcal{P}^X$), not necessarily algebraically independent, one has $p(f_1, \ldots, f_m) \in \text{Pos}(X, \mathcal{P}^X)$ for any $p \in \text{Poly}^{>0}_m$. \[\square\]
3.2 The Lusztig positive structure on $G/Q$

Returning to the connected and simply connected complex semisimple Lie group $G$, we first recall the Lusztig positive structure on $G$ and then extend it to $G/B(v)$ and $G/N(v)$ for all $v \in W$.

For $w \in W$, $w = (s_{\alpha_1}, \ldots, s_{\alpha_k}) \in \mathcal{R}(w)$, and $c = (c_1, \ldots, c_k) \in \mathbb{C}^k$, define

$$x_w^-(c) = x_{-\alpha_1}(c_1)x_{-\alpha_2}(c_2) \cdots x_{-\alpha_k}(c_k), \quad x_w^+(c) = x_{\alpha_1}(c_1)x_{\alpha_2}(c_2) \cdots x_{\alpha_k}(c_k). \quad (3.1)$$

As $w$ is a reduced word for $w$, one has (see, for example, [43, (d) of Proposition 2.7])

$$x_w^-((\mathbb{C}^\times)^k) \subset N^- \cap BwB \quad \text{and} \quad x_w^+((\mathbb{C}^\times)^k) \subset N \cap B^-wB^-. \quad (3.2)$$

The following facts are proved in [43, Proposition 2.7] and [5, Theorem 3.1]).

**Lemma 3.5** For any $w \in W$ and $w = (s_{\alpha_1}, \ldots, s_{\alpha_k}) \in \mathcal{R}(w)$,

$$x_w^- : (\mathbb{C}^\times)^k \rightarrow N^- \cap BwB, \quad (c_1, \ldots, c_k) \mapsto x_w^-(c_1, \ldots, c_k), \quad (3.3)$$

$$x_w^+ : (\mathbb{C}^\times)^k \rightarrow N \cap B^-wB^-, \quad (c_1, \ldots, c_k) \mapsto x_w^+(c_1, \ldots, c_k), \quad (3.4)$$

are toric charts, respectively on $N^- \cap BwB$ and $N \cap B^-wB^-$, and their positive equivalence classes are independent of the choice of $w \in \mathcal{R}(w)$.

Let again $\sigma : (\mathbb{C}^\times)^d \rightarrow T$ be the isomorphism in (2.30). For $w_0, w'_0 \in \mathcal{R}(w_0)$, define $\rho_{(w_0, w'_0)} : (\mathbb{C}^\times)^{2l_0+d} \rightarrow G$ by

$$\rho_{(w_0, w'_0)}(c_1, \ldots, c_{2l_0+d}) = x_{w_0}^-(c_1, \ldots, c_{l_0})x_{w'_0}^+(c_{l_0+1}, \ldots, c_{2l_0})\sigma(c_{2l_0+1}, \ldots, c_{2l_0+d}). \quad (3.5)$$

By the unique decomposition $B^-B = N^-NT$ and by Lemma 3.5, $\rho_{(w_0, w'_0)}$ is a toric chart on $G$, and the positive equivalence class $[\rho_{(w_0, w'_0)}]$ of toric charts in $G$ is independent of the choices of $(w_0, w'_0) \in \mathcal{R}(w_0) \times \mathcal{R}(w_0)$ and of $\sigma$.

**Definition 3.6** We will set

$$\mathcal{P}_{Lusztig}^G = [\rho_{(w_0, w'_0)}]$$

for any $(w_0, w'_0) \in \mathcal{R}(w_0) \times \mathcal{R}(w_0)$ and call it the Lusztig positive structure on $G$. □

Denote by $G_{>0}$ the totally positive part of $G$ defined by $\mathcal{P}_{Lusztig}^G$. Then, using any reduced words $w_0, w'_0 \in \mathcal{R}(w_0)$, one has

$$G_{>0} = \{x_{w_0}^-(c_1, \ldots, c_{l_0})x_{w'_0}^+(c_{l_0+1}, \ldots, c_{2l_0})r : c_1, \ldots, c_{2l_0} \in \mathbb{R}_{>0}, r^{\alpha} > 0, \forall \alpha \in \Gamma\},$$
which coincides with the totally positive part of $G$ defined by Lusztig [43].

To define the Lusztig positive structures on $G/B(v)$ and $G/N(v)$ for $v \in W$, we first prove the following lemma.

**Lemma 3.7** For any $v \in W$, the following maps are all open embeddings:

$$
\delta_v : B^-v^{-1}B^- \longrightarrow G/N(v), \quad g \longmapsto g.N(v), \\
\epsilon_v : (N^- \cap Bw_0B) \times (N \cap B^{-v^{-1}}B^-) \times T \longrightarrow G/N(v), \quad (m, n, t) \longmapsto mnt.N(v), \\
\epsilon'_v : (N^- \cap Bw_0B) \times (N \cap B^{-v^{-1}}B^-) \longrightarrow G/B(v), \quad (m, n) \longmapsto mn.B(v).
$$

**Proof** Consider the Zariski open subset $B^-B\overline{v}^{-1}$ of $G$ and its unique decomposition

$$
B^-B\overline{v}^{-1} = B^-\overline{v}^{-1}(N^- \cap \overline{v}N\overline{v}^{-1}).
$$

In view of the unique decompositions

$$
\overline{v}N\overline{v}^{-1} = (N^- \cap \overline{v}N\overline{v}^{-1})N(v) \quad \text{and} \quad B^-v^{-1}B^- = B^-\overline{v}^{-1}(N^- \cap \overline{v}N\overline{v}^{-1}),
$$

one has the unique decomposition $B^-B\overline{v}^{-1} = (B^-v^{-1}B^-)N(v)$, so

$$
B^-v^{-1}B^- \times N(v) \longrightarrow B^-B\overline{v}^{-1}, \quad (g, n) \longmapsto gn, \quad g \in B^-v^{-1}B^-, \quad n \in N(v), \quad (3.6)
$$

is an isomorphism. It follows that $\delta_v$ is an open embedding.

As both $(N^- \cap Bw_0B) \times (N \cap B^{-v^{-1}}B^-) \times T$ and $G/N(v)$ are smooth, irreducible, and of the same dimension, by the Grothendieck–Zariski factorization theorem [30, Theorem 8.12.6], to show that $\epsilon_v$ is an open embedding, it is enough to show that it is injective (see also proof of [19, Theorem 1.2]).

Suppose that $m, m' \in N^- \cap Bw_0B$, $n, n' \in N \cap B^{-v^{-1}}B^-$, and $t, t' \in T$ are such that $mnt.N(v) = m'n't'.N(v)$. Since $mnt, m'n't' \in B^-v^{-1}B^-$, the injectivity of $\delta_v$ implies that $mnt = m'n't'$, and thus $m = m'$, $n = n'$ and $t = t'$. This shows that $\epsilon_v$ is injective and thus an embedding.

Similarly one shows that $\epsilon'_v$ is an open embedding. \qed

Let $v \in W$ and $l = l_0 + l(v) = \dim G/B(v)$, where $l_0 = l(w_0)$. For

$$(w_0, v) = (s_{\alpha_1}, \ldots, s_{\alpha_{l_0}+1}, s_{\alpha_{l_0}+1}, \ldots, s_{\alpha_l}) \in \mathcal{R}(w_0) \times \mathcal{R}(v),$$

and for $c = (c_1, \ldots, c_{l+d}) \in (\mathbb{C}^\times)^{l+d}$, write

$$c(1) = (c_1, \ldots, c_{l_0}) \in (\mathbb{C}^\times)^{l_0}, \quad c(2) = (c_{l_0+1}, \ldots, c_l) \in (\mathbb{C}^\times)^{l(v)},$$

and $c(3) = (c_{l+1}, \ldots, c_{l+d}) \in (\mathbb{C}^\times)^d$, and recall that

$$x_{w_0}^{-}(c(1)) = x_{-\alpha_1}(c_1)x_{-\alpha_2}(c_2) \cdots x_{-\alpha_{l_0}}(c_{l_0}) \in N^- \cap Bw_0B, \quad (3.7)$$
\[ x_{v^{-1}}^+(c(2)) = x_{a_1}(c_1)x_{a_{l_1}^{-1}}(c_{l_1}) \cdots x_{a_{l_0}^{-1}}(c_{l_0}^{-1}) \in N \cap B^{-v^{-1}}B^{-}, \]  
(3.8)

and \( \sigma(c(3)) \in T \), where \( v^{-1} = (s_{a_1}, \ldots, s_{a_{l_0}}) \in \mathcal{R}(v^{-1}) \). Introduce

\[
\rho_{(w_0,v)}^{G/B(v)} : (\mathbb{C}^\times)^l \longrightarrow G/B(v), \ c \longmapsto x_{w_0}^{-}(c(1))x_{v^{-1}}^+(c(2)).B(v), \tag{3.9}
\]

\[
\rho_{(w_0,v)}^{G/N(v)} : (\mathbb{C}^\times)^{l+d} \longrightarrow G/N(v), \ c \longmapsto x_{w_0}^{-}(c(1))x_{v^{-1}}^+(c(2))\sigma(c(3)).N(v). \tag{3.10}
\]

**Lemma 3.8** For \( Q = B(v) \) or \( N(v) \), and for any \((w_0, v) \in \mathcal{R}(w_0) \times \mathcal{R}(v)\), \( \rho_{(w_0,v)}^{G/Q} \) is a toric chart on \( G/Q \), and the positive structure it defines on \( G/Q \) is independent of the choice of \((w_0, v)\) (and of \( \sigma \) when \( Q = N(v) \)).

**Proof** By Lemmas 3.5 and 3.7, \( \rho_{(w_0,v)}^{G/Q} \) is an open embedding and thus a toric chart on \( G/Q \).

For any other choice \((w_0', v') \in \mathcal{R}(w_0) \times \mathcal{R}(v)\), it follows from Lemma 3.7 that

\[
\rho_{(w_0,v)}^{G/B(v)}(c(1), c(2)) = \rho_{(w_0,v')}^{G/B(v)}(c'(1), c'(2))
\]

implies that \( x_{w_0}^{-}(c(1)) = x_{w_0}^{-}(c'(1)) \) and \( x_{v^{-1}}^+(c(2)) = x_{v^{-1}}^+(c'(2)) \). By Lemma 3.5 again, \( Pos(c(1)) = Pos(c'(1)) \) and \( Pos(c(2)) = Pos(c'(2)) \). Thus the two toric charts \( \rho_{(w_0,v)}^{G/B(v)} \) and \( \rho_{(w_0',v')}^{G/B(v)} \) on \( G/B(v) \) are positively equivalent. The case for \( Q = N(v) \) is proved similarly. \( \square \)

**Definition 3.9** For \( v \in W \) and \( Q = B(v) \) or \( N(v) \), define

\[
\mathcal{P}_{Lusztig}^{G/Q} = \left[ \rho_{(w_0,v)}^{G/Q} \right]
\]

for any \((w_0, v) \in \mathcal{R}(w_0) \times \mathcal{R}(v)\) and call it the *Lusztig positive structure* on \( G/Q \). Denote by \( Pos(G/Q) \) the set of all rational functions on \( G/Q \) that are positive with respect to \( \mathcal{P}_{Lusztig}^{G/Q} \). Denote by \((G/Q)_{>0}\) the totally positive part of \( G/Q \) defined by \( \mathcal{P}_{Lusztig}^{G/Q} \). \( \square \)

**Example 3.10** For \( G/B \) so \( v = e \), it follows from the definition of \( \mathcal{P}_{Lusztig}^{G/B} \) that for any \( w_0 \in \mathcal{R}(w_0) \), one has

\[
(G/B)_{>0} = \{ x_{w_0}^{-}(c_1, \ldots, c_{l_0}).B : c_1, \ldots, c_{l_0} \in \mathbb{R}_{>0} \},
\]

which coincides with the totally positive part of \( G/B \) defined by Lusztig in [43, Sect. 8]. \( \square \)

**Remark 3.11** Let \( v \in W \) and recall that double Bruhat cell \( G_{w_0,v^{-1}} \) is defined as

\[
G_{w_0,v^{-1}} = Bw_0B \cap B^{-v^{-1}}B^{-} \subset G.
\]
Consider the map \( \rho_{(w_0,v)} : (\mathbb{C}^\times)^{l+d} \to G^{w_0,v^{-1}} \) given by
\[
\rho_{(w_0,v)}(c_1, \ldots, c_{l+d}) = x_{w_0}^{-}(c_1, \ldots, c_{l_0})x_{v^{-1}}^{+}(c_{l_0+1}, \ldots, c_{l})\sigma(c_{l+1}, \ldots, c_{l+d}).
\]

By [19, Theorem 1.2 and (2.5)], \( \rho_{(w_0,v)} \) is a toric chart on \( G^{w_0,v^{-1}} \). We will refer to the positive equivalence class \( [\rho_{(w_0,v)}] \) as the Lusztig positive structure on \( G^{w_0,v^{-1}} \).

On the other hand, as \( G^{w_0,v^{-1}} \) is a Zariski open subset of \( B^{-v^{-1}}B^- \), the open embedding \( \delta_v : B^{-v^{-1}}B^- \to G/N(v) \) restricts to an open embedding,
\[
\delta_{w_0,v} : G^{w_0,v^{-1}} \to G/N(v), \quad g \mapsto g.N(v).
\]

By construction, \( \delta_{w_0,v} : G^{w_0,v^{-1}} \to G/N(v) \) is a positive open embedding in the sense that for any toric chart \( \rho \) on \( G^{w_0,v^{-1}} \), \( \rho \) is in the Lusztig positive structure on \( G^{w_0,v^{-1}} \) if and only if \( \delta_{w_0,v} \circ \rho \) is a toric chart in Lusztig positive structure on \( G/N(v) \).

3.3 Some auxiliary facts on generalized minors

In this section, we first recall some facts from [19] on flag minors. We then give examples of regular functions on \( G/N(v) \) that are in \( \text{Pos}(G/N(v)) \) for all \( v \in W \).

**Definition 3.12** A generalized minor of the form \( \Delta_{w_0\alpha,\omega_\alpha} \) or \( \Delta_{\alpha_\omega, w_0\alpha} \), where \( w \in W \) and \( \alpha \in \Gamma \), is called a flag minor. For \( w \in W \) and \( w = (s_{\alpha_1}, \ldots, s_{\alpha_k}) \in \mathcal{R}(w) \), set
\[
\Delta_{w,j} = \Delta_{s_{\alpha_1} \cdots s_{\alpha_j} \omega_{\alpha_j} \omega_{\alpha_j}}, \quad \text{and} \quad \Delta_{j,w} = \Delta_{\alpha_{\omega_j} s_{\alpha_1} \cdots s_{\alpha_j} \omega_{\alpha_j}}, \quad j \in [1, k].
\]

A flag minor of the form \( \Delta_{w_0\alpha_\omega, \omega_\omega} \) is invariant under the right translation by elements in \( N \). Similarly, a flag minor of the form \( \Delta_{\alpha_{\omega_\omega}, w_0\alpha_\omega} \) is invariant under the left translation by elements in \( N^{-} \). Flag minors of \( g \in SL(n, \mathbb{C}) \) of size \( i \in [1, n] \) are the determinants of the submatrices of \( g \) formed by the first \( i \) columns and any \( i \) rows, or the first \( i \) rows and any columns (see (6, p. 76)).

Recall from Notation 1.3 the weak order \( \leq \) on \( W \): \( w_1 \leq w \) if \( l(w) = l(w_1) + l(w_1^{-1}w) \). For \( w \in W \), let \( N_w^w = N^- \cap \overline{w}N\overline{w}^- \), and recall that \( N_w = N \cap \overline{w}N\overline{w}^- \). Let
\[
F(w) = \{ \Delta_{w_1\alpha_\omega, w_2\omega_\alpha} | N_w^w : \alpha \in \Gamma, w_2 \leq w_1 \leq w \} \subset \mathbb{C}[N_w^w],
\]
\[
F'(w) = \{ \Delta_{w_1\alpha_\omega, w_2\omega_\alpha} | N_w^w : \alpha \in \Gamma, w_1 \leq w_2 \leq w \} \subset \mathbb{C}[N_w],
\]
and for \( w = (s_{\alpha_1}, \ldots, s_{\alpha_k}) \in \mathcal{R}(w) \), let
\[
F_i(w) = \{ \Delta_{w,j} | N_w^w : j \in [1, k] \} \subset F(w),
\]
\[
F'_i(w) = \{ \Delta_{j,w} | N_w^w : j \in [1, k] \} \subset F'(w).
\]
Lemma 3.13 [19, Theorem 2.22] For any \( w \in W \) and \( w \in R(w) \),

1. \( F_1(w) \) is a transcendental basis of \( \mathbb{C}(N^w) \), and \( F(w) \subset \text{Pos}(F_1(w)) \subset \mathbb{C}(N^w) \);
2. \( F'_1(w) \) is a transcendental basis of \( \mathbb{C}(N_w) \), and \( F'(w) \subset \text{Pos}(F'_1(w)) \subset \mathbb{C}(N_w) \).

Proof (1) is part of [19, Theorem 2.22]. (2) follows from (1), the fact that \( (N_w)^\tau = N_w^- \), and the following identity from [19, Proposition 2.7]:

\[
\Delta_{\alpha_0, \alpha_0}(g) = \Delta_{\alpha_0, \alpha_0}(g^\tau), \quad g \in G, \tag{3.13}
\]

where \( \tau \) is the involutive anti-automorphism of \( G \) given in (1.8). \( \Box \)

Lemma 3.13 can be extended as follows.

Lemma 3.14 For any \( w \in W \) and \( w = (s_{i_1}, \ldots, s_{i_k}) \in R(w) \), \( \{ \Delta_{w, j} \}_{j \in [1, k]} \) is a set of algebraically independent regular functions on \( N^- \), and

\[
\{ \Delta_{w_1, \alpha}, w_2 \alpha \}_{\alpha \in \Gamma} \subset \text{Pos}(\Delta_{w, 1} \alpha, \ldots, \Delta_{w, k} \alpha) \subset \mathbb{C}[N].
\]

Proof Let \( m \in N^- \) and write \( m \) uniquely as \( m = m_1 m_2 \), where \( m_1 \in N^- \setminus \overline{w} N^- \overline{w}^{-1} \) and \( m_2 \in N^- \cap \overline{w} N \overline{w}^{-1} \). Then, for any \( w_2 \leq w_1 \leq w \) and \( \alpha \in \Gamma \), since \( \overline{w}_1^{-1} m_1 \overline{w}_1 \in N^- \) by (2.5), one has

\[
\Delta_{w_1, \alpha}, w_2 \alpha (m) = \Delta_{\alpha, \alpha} (\overline{w}_1^{-1} m_1 \overline{w}_1 \overline{w}_2^{-1} m_2 \overline{w}_2) = \Delta_{w_1, \alpha}, w_2 \alpha (m_2). \tag{3.14}
\]

The statement on \( \{ \Delta_{w, j} \}_{j \in [1, k]} \) now follows from (3.14) and 1) of Lemma 3.13. The statement on \( \{ \Delta_{w, \alpha} \}_{\alpha \in \Gamma} \) is similarly proved using (3.13). \( \Box \)

We now turn to examples of regular functions on \( G/N(v) \), for \( v \in W \), that are positive with respect to \( \rho_{\text{Lusztig}}^{G/N(v)} \). We identify \( \mathbb{C}[G/N(v)] \) with \( \mathbb{C}[G]^{N(v)} \subset \mathbb{C}[G] \), the algebra of right \( N(v) \)-invariant regular functions on \( G \).

Proposition 3.15 For any \( w, v_1 \in W \) such that \( v_1 \leq v \) and for all \( \alpha \in \Gamma \), \( \Delta_{w_0, \alpha}, v_1 \alpha \in \mathbb{C}[G]^{N(v)} \cong \mathbb{C}[G/N(v)] \) and lies in \( \text{Pos}(G/N(v)) \).

Proof By (2.5), \( \overline{w}_1^{-1} N(v) \overline{w}_1 \subset N \). It follows that \( \Delta_{w_0, \alpha}, v_1 \alpha \in \mathbb{C}[G]^{N(v)} \) for any \( w \in W \) and \( \alpha \in \Gamma \).

Choose any \( w_0 = (s_{i_1}, \ldots, s_{i_{l_0}}) \in R(w_0) \) and \( v = (s_{i_{l_0+1}}, \ldots, s_{i_l}) \in R(v) \). For a rational function \( f \) on \( G/N(v) \), define \( \tilde{f} \in \mathbb{C}(c_1, \ldots, c_{l+d}) \) by

\[
\tilde{f} := f \circ \rho_{(w_0, v)}^{G/N(v)} = f \left( x_{w_0}^-(c_1, \ldots, c_{l_0}) x_1^+(c_{l_0+1}, \ldots, c_l) \sigma(c_{l+1}, \ldots, c_{l+d}) \right).
\]
Note that as $\rho_{G/N(v)}^{G/N(v)}$ is an open embedding, $\tilde{f} \neq 0$ if $f \neq 0$. By the definition of $P^{G/N(v)}_{\text{Lusztig}}$, $f \in \text{Pos}(G/N(v))$ if and only if $\tilde{f} \in \text{Pos}(c_1, \ldots, c_{l+d})$. Write $t = \tilde{\sigma}(c_{l+1}, \ldots, c_{l+d})$, and note that for any $\alpha \in \Gamma$, $t^{\omega_{\alpha}}$ is one of the coordinates in $(c_{l+1}, \ldots, c_{l+d})$.

**Special case 1.** Assume first that $v_1 = e$. Then for any $w \in W$ and $\alpha \in \Gamma$, one has

$$\tilde{\Delta}_{w^{\omega_{\alpha}}, \omega_{\alpha}} = t^{\omega_{\alpha}} \Delta_{w^{\omega_{\alpha}}, \omega_{\alpha}}(x_{w_0}^- (c_1, \ldots, c_l)).$$

By [8, Theorem 5.8], $\Delta_{w^{\omega_{\alpha}}, \omega_{\alpha}}(x_{w_0}^- (c_1, \ldots, c_l))$ is a non-zero polynomial in $(c_1, \ldots, c_l)$ with non-negative integer coefficients. Thus $\tilde{\Delta}_{w^{\omega_{\alpha}}, \omega_{\alpha}} \in \text{Pos}(c_1, \ldots, c_{l+d})$.

**Special case 2.** Suppose now that $w = e$. Then for any $v_1 \leq v$ and $\alpha \in \Gamma$, one has

$$\tilde{\Delta}_{\omega_{\alpha}, v_1^{\omega_{\alpha}}} = t^{v_1^{\omega_{\alpha}}} \Delta^{\omega_{\alpha}, v_1^{\omega_{\alpha}}}(x_{v_1^{-1}}^+ (c_{l_0+1}+1, \ldots, c_l)),$$

which, by [8, Theorem 5.8] again, is a non-zero polynomial in $(c_{l_0+1}, \ldots, c_l)$ with non-negative integer coefficients. Thus $\tilde{\Delta}_{\omega_{\alpha}, v_1^{\omega_{\alpha}}} \in \text{Pos}(c_1, \ldots, c_{l+d})$.

Turning to the general case of arbitrary $w$ and $v_1 \leq v$, consider

$$i^* = (w_0, v) = (s_{\alpha_1}, \ldots, s_{\alpha_{l_0}}, s_{\alpha_{l_0+1}}, \ldots, s_{\alpha_l})$$

and let $i$ be the sequence formed by reading $i^*$ backwards. In the notation of [19], $i$ is double reduced word of $(w_0, v^{-1})$, and it gives rise, via [19, (1.22)], to the collection $F(i) = \{ \Delta_{k,i^*} : k \in [1, l + d] \}$ of generalized minors on $G$, where

$$\Delta_{k,i^*} = \begin{cases} 
\Delta_{s_{\alpha_1} \cdots s_{\alpha_{l_0}}^{\omega_{\alpha_k}}, \omega_{\alpha_k}} & k \in [1, l_0], \\
\Delta_{\omega_{\alpha_k}, s_{\alpha_{l_0+1}} \cdots s_{\alpha_{l-1}}^{\omega_{\alpha_k}}} & k \in [l_0 + 1, l], \\
\Delta_{\omega_{\alpha_{l-1}}, v_0^{\omega_{\alpha_{l-1}}}} & k \in [l + 1, l + d].
\end{cases}$$

Note that each $\Delta_{k,i^*}$ in $F(i)$ is a minor of the two special cases discussed above, so $\Delta_{k,i^*} \in \text{Pos}(c_1, \ldots, c_{l+d})$. By [19, Theorem 1.12], every element in

$$F(w_0, v^{-1})|_{G^{w_0,v^{-1}}} := \{ \Delta_{w^{\omega_{\alpha}}, v_1^{\omega_{\alpha}}} |_{G^{w_0,v^{-1}}} : \alpha \in \Gamma, w \in W, v_1 \leq v \}$$

has a subtraction-free expression in the elements in $F(i)|_{G^{w_0,v^{-1}}}$. Thus, $\tilde{\Delta}_{w^{\omega_{\alpha}}, v_1^{\omega_{\alpha}}} \in \text{Pos}(c_1, \ldots, c_{l+d})$ for all $w, v_1 \in W$ with $v_1 \leq v$ and $\alpha \in \Gamma$. □

**Remark 3.16** We remark that for any $v \in W$ and $Q = B(v)$ or $N(v)$, one has

$$(G/Q)_{>0} \subset \bigcap_{w \in W} wB^{-}B/Q.$$
Indeed, for any \( w \in W \) and \( g \in G, g.Q \in wB^−B/Q \) if and only if \( \Delta_{w/\omega_\alpha, \omega_\alpha}(g) \neq 0 \) for all \( \alpha \in \Gamma \). Take any \( (w_0, v) \in \mathcal{R}(w_0) \times \mathcal{R}(v), w \in W, \alpha \in \Gamma, \) and \( c_j \in \mathbb{R}_{>0} \) for \( j \in [1, l + d] \). Then by [8, Theorem 5.8],

\[
\Delta_{w/\omega_\alpha, \omega_\alpha} \left( \rho_{(w_0, v)}(c_1, \ldots, c_l) \right) = \Delta_{w/\omega_\alpha, \omega_\alpha}(x_{w_0}^{-}(c_1, \ldots, c_l)) > 0,
\]

\[
\Delta_{w/\omega_\alpha, \omega_\alpha} \left( \rho_{(w_0, v)}(c_1, \ldots, c_{l+d}) \right) = \sigma(c_{l+1}, \ldots, c_{l+d}) \Delta_{w/\omega_\alpha, \omega_\alpha}(x_{w_0}^{-}(c_1, \ldots, c_l)) > 0.
\]

Thus \((G/Q)_{>0} \subset wB^−B/Q\) for both \( Q = B(v) \) and \( Q = N(v) \).

\[\square\]

### 3.4 Positivity of the Bott–Samelson coordinates on \( G/Q \)

Let again \( v \in W \) and \( Q = B(v) \) or \( N(v) \). We now prove Theorem A stated in the Introduction, namely that all the Bott–Samelson coordinates on \( G/Q \) are positive (rational) functions with respect to the Lusztig positive structure. We prove the following more detailed restatement of Theorem A. For \( v \in W \) and \( Q = B(v) \) or \( N(v) \), let

\[
d(Q) = \dim(G/Q) = \begin{cases} l = l_0 + l(v), & Q = B(v), \\ l + d, & Q = N(v). \end{cases}
\]

**Theorem 3.17** Let \( v \in W \) and \( Q = B(v) \) or \( N(v) \). For any \( w \in W \) and any \( r \in \mathcal{R}(w_0w^{-1}) \times \mathcal{R}(w) \times \mathcal{R}(v) \), all the Bott–Samelson coordinates \((z_1, \ldots, z_{d(Q)})\) on \( wB^−B/Q \) defined by \( r \), when regarded as rational functions on \( G/Q \), are in \( \text{Pos}(G/Q) \).

**Proof** Fix \( w \in W \) and \( r = (w^0, w, v) \in \mathcal{R}(w_0w^{-1}) \times \mathcal{R}(w) \times \mathcal{R}(v) \), where

\[
w^0 = (s_{\alpha_1}, \ldots, s_{\alpha_k}), \quad w = (s_{\alpha_{k+1}}, \ldots, s_{\alpha_0}), \quad \text{and} \quad v = (s_{\alpha_{l_0+1}}, \ldots, s_{\alpha_l}).
\]

We first consider the case of \( Q = N(v) \).

Write an element in \( wB^−B/N(v) \) uniquely as \( g.N(v) \), where \( g = \overline{w}mnt \) for unique \( m \in N^- \), \( n \in N_v \), and \( t \in T \). The values of the Bott–Samelson coordinates \( z_1, \ldots, z_{l+d} \) at \( g.N(v) \in wB^−B/N(v) \) are given in Proposition 2.11.

Assume first that \( j \in [1, k] \). By Proposition 2.11,

\[
z_j(g.N(v)) = \Delta_{s_{\alpha_1}^{-} \cdots s_{\alpha_j}^{-} \omega_{\alpha_j} \cdots s_{\alpha_l}^{-} \omega_{\alpha_l}}(m).
\]

Let \( u^* = (s_{\alpha_1}^*, \ldots, s_{\alpha_k}^*) \in \mathcal{R}(w^{-1}w_0) \) and \( w_0' = (s_{\alpha_{k+1}}, \ldots, s_{\alpha_0}, s_{\alpha_1}^*, \ldots, s_{\alpha_k}^*) \in \mathcal{R}(w_0) \). By Lemma 3.14, there exist \( p, q \in \mathbb{Z}_{>0}^* \) (see Notation 3.1) such that

\[
q(\Delta_{u^*, 1}, \ldots, \Delta_{u^*, j})(m) = z_j(g.N(v)) p(\Delta_{u^*, 1}, \ldots, \Delta_{u^*, j})(m).
\]

(3.15)

On the other hand, for each \( i \in [1, j] \),

\[
\Delta_{u^*, i}(m) = \Delta_{u^*, i}(w_0^{-1}gt^{-1}n^{-1}) = t^{-\omega_{\alpha_i}^*} \Delta_{w_{s_{\alpha_i}^{-} \cdots s_{\alpha_j}^{-} \omega_{\alpha_j} \cdots s_{\alpha_l}^{-} \omega_{\alpha_l}}}(g)
\]
For \( i \in \{1, j\} \), set \( f_i \in \mathbb{C}[wB^{-}B/N(v)] \) by

\[
f_i(g.N(v)) = \frac{\Delta_{w_0, l(w)+i}(g)}{\Delta_{w_0a_{i'}^*, w_0a_j^*}(g)}, \quad g \in \overline{w}N^{-}N_0T.
\]

By (3.15) and (3.16), \( q(f_1, \ldots, f_j) = z_j p(f_1, \ldots, f_j) \in \mathbb{C}[wB^{-}B/N(v)] \). By Proposition 3.15, \( f_i \in \text{Pos}(G/N(v)) \) for each \( i \in \{1, j\} \). Thus \( z_j \in \text{Pos}(G/N(v)) \) by Remark 3.4.

Suppose now that \( j \in [k + 1, l_0] \). By Proposition 2.11 and using the involutive automorphism \( \iota \) of \( G \) given in (1.8) and the identities in (1.9) and (1.10), one has

\[
z_j(g.N(v)) = \Delta_{w^{-1}, 1, \ldots, \Delta_{w^{-1}, l_0-j+1}}(m^{-1})^i
\]

\[
= z_j(g.N(v)) p'(\Delta_{w^{-1}, 1, \ldots, \Delta_{w^{-1}, l_0-j+1}}(m^{-1})^i).
\]

On the other hand, for \( i \in \{1, l_0 - j + 1\} \) and \( i' := l_0 - i + 1 \in \{j, l_0\} \),

\[
\Delta_{w^{-1}, i}(m^{-1})^i = \Delta_{w_0a_{i'}, w_0a_{i'}}(m^{-1})^i = \Delta_{w_0a_{i'}, w_0a_{i'}}(m^{-1})^i.
\]

\[
= \frac{\Delta_{s_{a_{i'+1}} \ldots s_{a_j}, \omega_{a_j}, \ldots, \omega_{a_{i'}}}(g)}{\Delta_{w_0a_{i'}, \omega_{a_j}}(g)}, \quad g \in \overline{w}N^{-}N_0T.
\]

For \( i \in \{1, l_0 - j + 1\} \), set \( f_i' \in \mathbb{C}[wB^{-}B/N(v)] \) by

\[
f_i'(g.N(v)) = \Delta_{s_{a_{i'+1}} \ldots s_{a_j}, \omega_{a_j}, \ldots, \omega_{a_{i'}}}(g)/\Delta_{w_0a_{i'}, \omega_{a_j}}(g), \quad g \in \overline{w}N^{-}N_0T.
\]

One then has \( q'(f_1', \ldots, f_{l_0-j+1}') = z_j p'(f_1', \ldots, f_{l_0-j+1}') \in \mathbb{C}[wB^{-}B/N(v)] \), which, by Proposition 3.15 and Remark 3.4, implies that \( z_j \in \text{Pos}(G/N(v)) \).

Assume now that \( j \in [l_0 + 1, l] \). By Proposition 2.11,

\[
z_j(g.N(v)) = \Delta_{s_{a_{l_0+1}} \ldots s_{a_j}, s_{a_{l_0+1}} \ldots s_{a_j}}(n).
\]
By Lemma 3.14, there exist non-zero \( p'' \), \( q'' \) \in \text{Poly}_{j-l_0}^0 \) such that

\[
q''(\Delta_{1,v}, \ldots, \Delta_{j-l_0,v})(n) = z_j(g.N(v))p''(\Delta_{1,v}, \ldots, \Delta_{j-l_0,v})(n).
\]

On the other hand, for \( i \in [1, j-l_0] \),

\[
\Delta_{i,v}(n) = \Delta^{a_{l_0+i}}(n) s_{a_{l_0+1}} \cdots s_{a_{l_0+i}} = \Delta^{a_{l_0+i}}(m^{-1} n^{-1} s_{a_{l_0+1}} \cdots s_{a_{l_0+i}}) =
\]

\[
t^{-s_{a_{l_0+1}} \cdots s_{a_{l_0+i}} a_{l_0+i}} \Delta w a_{l_0+i} s_{a_{l_0+1}} \cdots s_{a_{l_0+i}} a_{l_0+i}(g).
\]

Writing \( s_{a_{l_0+1}} \cdots s_{a_{l_0+i}} a_{l_0+i} = \sum_{a \in \Gamma} k_a a_a \), one has

\[
t^{s_{a_{l_0+1}} \cdots s_{a_{l_0+i}} a_{l_0+i}} = \prod_{a \in \Gamma} (\Delta w a_{l_0+i})(g)^{k_a}.
\]

For \( i \in [1, j-l_0] \), set \( f_i'' = \mathbb{C}[wB^-B/N(v)] \) by

\[
f_i''(g.N(v)) = \frac{\Delta w a_{l_0+i}, s_{a_{l_0+1}} \cdots s_{a_{l_0+i}} a_{l_0+i}(g)}{\prod_{a \in \Gamma} (\Delta w a_{l_0+i})(g)^{k_a}}, \quad g \in \mathbb{W} N^-N T.
\]

One then has \( q''(f_1'', \ldots, f_{j-l_0}'') = z_j p''(f_1'', \ldots, f_{j-l_0}'') \in \mathbb{C}[wB^-B/N(v)] \), which, again by Proposition 3.15 and Remark 3.4, implies that \( z_j \in \text{Pos}(G/N(v)) \).

For \( j \in [l+1, l+d] \), since \( z_j(g.N(v)) = \Delta w a_{j-l} \omega a_{j-l}(g) \), again \( z_j \in \text{Pos}(G/N(v)) \).

Let now \( Q = B(v) \). Let \( \sigma : G/N(v) \rightarrow G/B(v) \) be the natural projection. Then \( (\sigma^*(z_1), \ldots, \sigma^*(z_l)) \) coincides with the first \( l \) Bott–Samelson coordinates on \( wB^-B/N(v) \) defined by the same \( r \). On the other hand, for any toric chart \( \rho_{(w_0,v)}^{G/B(v)} : (\mathbb{C}^\times)^l \rightarrow G/B(v) \) in (3.9), and for any \( j \in [1, l] \), one has

\[
z_j(\rho_{(w_0,v)}^{G/B(v)}(c_1, \ldots, c_l)) = \sigma^*(z_j)(\rho_{(w_0,v)}^{G/N(v)}(c_1, \ldots, c_l, 1, \ldots, 1)) \in \text{Pos}(c_1, \ldots, c_l).
\]

Thus \( z_j \in \text{Pos}(G/B(v)) \) for each \( j \in [1, l] \).

\( \square \)

**Remark 3.18** As we remarked in the paragraph after the statement of Theorem A in Sect. 1.2, Bott–Samelson charts on \( G/Q \) are not to be confused with toric charts in \( \mathcal{P}_{\text{Lusztig}}^{G/Q} \). Consider the case of \( G/Q = G = SL(2, \mathbb{C}) \): there are two Bott–Samelson charts on \( G \), corresponding to \( w = e \) and \( w = s_1 = s \), respectively given by

\[
\sigma^{(s, \emptyset, s)} : \mathbb{C}^2 \times \mathbb{C}^\times \rightarrow B^-B, \quad (\xi_1, \xi_2, \xi_3) \mapsto \begin{pmatrix} 1 & 0 & 1 \\ \xi_1 & 1 & 0 \\ 0 & \xi_3 & \xi_3^{-1} \end{pmatrix}
\]

\[
= \begin{pmatrix} \xi_3 & \xi_2 \xi_3^{-1} \\ \xi_1 & \xi_3 \\ (\xi_1 \xi_2 + 1) \xi_3^{-1} \end{pmatrix},
\]

\[
\sigma^{(\emptyset, s, s)} : \mathbb{C}^2 \times \mathbb{C}^\times \rightarrow sB^-B, \quad (z_1, z_2, z_3) \mapsto \begin{pmatrix} z_1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & z_3 \end{pmatrix}
\]

\[
= \begin{pmatrix} z_3 & 0 & 0 \\ 0 & z_2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]
\[
\begin{pmatrix}
    z_1 z_3 & (z_1 z_2 - 1) z_3^{-1} \\
    z_3 & z_2 z_3^{-1}
\end{pmatrix}.
\]

The two sets of coordinates are related by
\[
\begin{align*}
    \xi_1 &= z_1^{-1}, & \xi_2 &= z_1 (z_1 z_2 - 1), & \xi_3 &= z_1 z_3, \\
    z_1 &= \xi_1^{-1}, & z_2 &= \xi_1 (\xi_1 \xi_2 + 1), & z_3 &= \xi_1 \xi_3.
\end{align*}
\]

It is a coincidence that in this case the toric chart \( \rho_{(s,s)}^G : (\mathbb{C}^\times)^3 \to SL(2, \mathbb{C}) \) in the Lusztig positive structure is equal to \( \sigma_{(s,s)}^{(s)} |_{(\mathbb{C}^\times)^3} \). Note that while each \( z_j \) has a subtraction-free expression in \( (\xi_1, \xi_2, \xi_3) \), \( \xi_2 \) does not have a subtraction-free expression in \( (z_1, z_2, z_3) \); indeed, otherwise the elements \( g \in sB^{-}B \) with \( (z_1, z_2, z_3) \)-coordinates satisfying \( z_1 = z_2 = 1 \) and \( z_3 > 0 \) would lie in \( G_{>0} \) which is not the case.

**Example 3.19** In this example we take \( G/Q = G = \text{Sp}(2, \mathbb{C}) = \{g \in \text{GL}(4, \mathbb{C}) : g^t J_2 g = J_2 \} \) where \( J_2 = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \) and \( I_2 \) is the identity matrix of size 2. Choose the Cartan subalgebra \( \mathfrak{h} \) of the Lie algebra \( \mathfrak{sp}(2, \mathbb{C}) \) to be the set of all diagonal elements in \( \mathfrak{sp}(2, \mathbb{C}) \) and write elements in \( \mathfrak{h} \) as \( x = \text{diag}(x_1, x_2, -x_1, -x_2) \). Choose the simple roots as \( \alpha_1 = x_1 - x_2 \) and \( \alpha_2 = 2x_2 \), let \( s_1 = s_{\alpha_1} \) and \( s_2 = s_{\alpha_2} \), and choose root vectors
\[
e_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad e_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Then the fundamental weights are \( \omega_{\alpha_1} = x_1 \) and \( \omega_{\alpha_2} = x_1 + x_2 \). Write \( g \in \text{Sp}(2, \mathbb{C}) \) as \( g = (a_{ij})_{i,j=1,...,4} \), and set \( \Delta_{ij,kj} = \det \begin{pmatrix} a_{ik} & a_{jl} \\ a_{jk} & a_{il} \end{pmatrix} \) for \( i < j \) and \( k < l \). Let
\[
\begin{align*}
    \Delta^{(1)} &= \begin{pmatrix}
    \Delta_{\omega\alpha_1,\alpha_1} & \Delta_{\omega\alpha_1,\alpha_2} & \Delta_{\omega\alpha_1,\alpha_3} & \Delta_{\omega\alpha_1,\alpha_4} \\
    \Delta_{\omega\alpha_1,\alpha_2} & \Delta_{\omega\alpha_1,\alpha_2} & \Delta_{\omega\alpha_1,\alpha_3} & \Delta_{\omega\alpha_1,\alpha_4} \\
    \Delta_{\omega\alpha_1,\alpha_3} & \Delta_{\omega\alpha_1,\alpha_3} & \Delta_{\omega\alpha_1,\alpha_3} & \Delta_{\omega\alpha_1,\alpha_4} \\
    \Delta_{\omega\alpha_1,\alpha_4} & \Delta_{\omega\alpha_1,\alpha_4} & \Delta_{\omega\alpha_1,\alpha_4} & \Delta_{\omega\alpha_1,\alpha_4}
\end{pmatrix}, \\
    \Delta^{(2)} &= \begin{pmatrix}
    \Delta_{\omega\alpha_2,\alpha_1} & \Delta_{\omega\alpha_2,\alpha_2} & \Delta_{\omega\alpha_2,\alpha_3} & \Delta_{\omega\alpha_2,\alpha_4} \\
    \Delta_{\omega\alpha_2,\alpha_2} & \Delta_{\omega\alpha_2,\alpha_2} & \Delta_{\omega\alpha_2,\alpha_3} & \Delta_{\omega\alpha_2,\alpha_4} \\
    \Delta_{\omega\alpha_2,\alpha_3} & \Delta_{\omega\alpha_2,\alpha_3} & \Delta_{\omega\alpha_2,\alpha_3} & \Delta_{\omega\alpha_2,\alpha_4} \\
    \Delta_{\omega\alpha_2,\alpha_4} & \Delta_{\omega\alpha_2,\alpha_4} & \Delta_{\omega\alpha_2,\alpha_4} & \Delta_{\omega\alpha_2,\alpha_4}
\end{pmatrix}.
\end{align*}
\]

Then it is straightforward to check that
\[
\Delta^{(1)} = \begin{pmatrix}
    a_{11} & a_{12} & -a_{13} & a_{14} \\
    a_{21} & a_{22} & -a_{23} & a_{24} \\
    -a_{31} & -a_{32} & a_{33} & -a_{34} \\
    a_{41} & a_{42} & -a_{43} & a_{44}
\end{pmatrix},
\]
We review from \([25,28]\) the definitions on symmetric Poisson CGL extensions over \(\mathbb{C}\). Consider now the Bott–Samelson chart on \(G/Q = \text{SP}(2, \mathbb{C})\) for \(w = e\) and \(r = ((s_1, s_2, s_1, s_2), \emptyset, (s_2, s_1, s_2, s_1) ) \in R(w_0) \times R(e) \times R(w_0)\).

A direct calculation gives the Bott–Samelson coordinates \((z_1, \ldots, z_{10})\) on \(B^{-}B\) to be

\[
\begin{align*}
z_1 &= \frac{a_{21}}{a_{11}}, \quad z_2 = \frac{-\Delta_{23,12}}{\Delta_{12,12}} \quad z_3 = \frac{-\Delta_{13,12}}{\Delta_{12,12}} = \frac{(-\Delta_{23,12})a_{11} + (-a_{31})\Delta_{12,12}}{a_{21}\Delta_{12,12}}, \\
z_4 &= \frac{\Delta_{14,12}}{\Delta_{12,12}}, \quad z_5 = \frac{\Delta_{12,12}\Delta_{12,14}}{a_{11}^2}, \quad z_6 = \frac{a_{14}\Delta_{12,12}}{a_{11}}, \\
z_7 &= a_{11}a_{13} + a_{12}a_{14} = \frac{a_{11}^2\Delta_{12,14} + a_{14}^2\Delta_{12,12}}{\Delta_{12,14}}, \\
z_8 &= \frac{a_{11}a_{12}}{\Delta_{12,12}}, \quad z_9 = a_{11}, \quad z_{10} = \Delta_{12,12}.
\end{align*}
\]

Note that the above formulas express the Bott–Samelson coordinates \((z_1, \ldots, z_{10})\) both as regular functions on \(B^{-}B\), given by \(a_{11}\Delta_{12,12} \neq 0\), and as rational functions in the generalized minors of \(\text{SP}(2, \mathbb{C})\) in a subtraction-free manner. In the case of \(z_3\), note that \(-\Delta_{13,12}\) is not a generalized minor for \(\text{SP}(2, \mathbb{C})\), while in the case of \(z_7\), \(a_{13} = -\Delta_{a_1, s_1s_2s_1a_1}\). The second identity for \(z_3\) comes from the Plücker relation for \(\text{GL}(4, \mathbb{C})\):

\[a_{21}\Delta_{13,12} = a_{11}\Delta_{23,12} + a_{31}\Delta_{12,12},\]

and the second identity for \(z_7\) is the result of the following Plücker relation for \(\text{SP}(2, \mathbb{C})\) (the second identity of \([19, (2)\text{ of Theorem }1.16]\) applied to \(u = v = e, i = 1\) and \(j = 2\)):

\[a_{12}a_{14}\Delta_{12,14} = a_{14}^2\Delta_{12,12} + a_{11}(a_{11}\Delta_{12,34} + (-a_{13})\Delta_{12,14}).\]

\(\square\)

### 4 Symmetric Poisson CGL extensions and Poisson–Ore varieties

#### 4.1 Definitions

We review from \([25,28]\) the definitions on symmetric Poisson CGL extensions over \(\mathbb{C}\), although the general theory of \([25,28]\) works for any field of characteristic 0.
In what follows, let $\mathbb{T}$ be an algebraic $\mathbb{C}$-torus. The (additive) group of algebraic characters of $\mathbb{T}$ is denoted by $X(\mathbb{T})$, and we also regard an element in $X(\mathbb{T})$ as in $t^*$ through its differential at the identity element of $\mathbb{T}$, where $t$ is the Lie algebra of $\mathbb{T}$.

Recall that a $\mathbb{T}$-Poisson algebra is a Poisson algebra $R$ over $\mathbb{C}$ with a rational $\mathbb{T}$-action by Poisson isomorphisms. For a $\mathbb{T}$-Poisson algebra $R$, denote the induced action of $h \in t$ on $R$ by $h(r)$ for $r \in R$. More specifically, if $r \in R$ is a $\mathbb{T}$-weight vector with $t \cdot r = t^x r$ for all $t \in \mathbb{T}$, then $h(r) = \chi(h)r$ for all $h \in t$. For a $\mathbb{T}$-Poisson $\mathbb{C}$-algebra $(R, \{,\})$ that is also an integral domain, define a $\mathbb{T}$-homogeneous Poisson prime element to be a prime element $r \in R$ that is a $\mathbb{T}$-weight vector and is such that $\{a, R\} \subseteq aR$.

The following definition of symmetric Poisson CGL extensions is a paraphrase of [25,28].

**Definition 4.1** [25,28]

1. A symmetric $\mathbb{T}$-Poisson CGL extension of length $n$ is a pair $\mathcal{E} = (R, \mathcal{D})$, where $R = (\mathbb{C}[z_1, \ldots, z_n], \{,\})$ is a polynomial Poisson algebra and

$$\mathcal{D} = (\chi_1, \ldots, \chi_n, h_1, \ldots, h_n, h'_1, \ldots, h'_n),$$

(4.1)

with $\chi_j \in X(\mathbb{T})$ and $h_j, h'_j \in t$ for $j \in [1, n]$, satisfying

$$\chi_j(h_j) \neq 0 \quad \text{and} \quad \chi_j(h'_j) \neq 0, \quad \forall \ j \in [1, n],$$

(4.2)

and such that the Poisson bracket $\{,\}$ for $\mathbb{C}[z_1, \ldots, z_n]$ takes the special form

$$\{z_i, z_j\} = \chi_i(h_j)z_i z_j - fi, j = \chi_j(h'_i)z_i z_j - fi, j \quad \text{for some} \quad f_{i,j} \in \mathbb{C}[z_{i+1}, \ldots, z_{j-1}]$$

(4.3)

for all $1 \leq i < j \leq n$, and that the $\mathbb{T}$-action on $\mathbb{C}[z_1, \ldots, z_n]$ by associative algebra isomorphisms via $t \cdot z_j = t^x_jz_j$, where $t \in T$ and $j \in [1, n]$, preserves the Poisson bracket $\{,\}$. For $1 \leq i < j \leq n$,

$$\{z_i, z_j\}_{\log-can} := -\chi_i(h_j)z_i z_j$$

(4.4)

is called the log-canonical term of $\{z_i, z_j\}$. We will also refer to the collection $\mathcal{D}$ in (4.1) as the $\mathbb{T}$-action data on $R$, and to the $z_j$’s as the CGL generators of $\mathcal{E} = (R, \mathcal{D})$.

2. A localized symmetric $\mathbb{T}$-Poisson CGL extension is a pair $\mathcal{E} = (R, \mathcal{D})$, where

$$R = (\mathbb{C}[z_1, \ldots, z_n][y_1^{-1}, \ldots, y_k^{-1}], \{,\}),$$

$$((\mathbb{C}[z_1, \ldots, z_n], \{,\}), \mathcal{D})$$

is a symmetric $\mathbb{T}$-Poisson CGL extension, and $y_1, \ldots, y_k$ are $\mathbb{T}$-homogeneous Poisson prime elements in $\mathbb{C}[z_1, \ldots, z_n].$
Remark 4.2 Given a symmetric $\mathbb{T}$-Poisson CGL extension $\mathcal{E} = (R, D)$ as in Definition 4.1, for any $1 \leq i < j \leq n$, the pair $\mathcal{E}_{[i,j]} = (R_{[i,j]}, D_{[i,j]})$ is also a symmetric $\mathbb{T}$-Poisson CGL extension, where $R_{[i,j]} = \mathbb{C}[x_i, x_{i+1}, \ldots, x_j]$ is a Poisson subalgebra of $R$, and

$$D_{[i,j]} = (\chi_i, \ldots, \chi_j, h_i, \ldots, h_j, h'_i, \ldots, h'_j).$$

In particular, the Poisson algebra $R$ is a symmetric iterated Poisson–Ore extension in the sense explained in Sect. 1.1.

Definition 4.3 Given a $\mathbb{T}$-Poisson algebra $P$ and an algebraic $\mathbb{C}$-torus $\mathbb{T}'$, by a presentation of $P$ as a localized symmetric $\mathbb{T}'$-Poisson CGL extension we mean a triple

$$\mathcal{P} = (\mathcal{E}, E, I),$$

where $\mathcal{E} = (R, D)$ is a localized symmetric $\mathbb{T}'$-Poisson CGL extension, $E : \mathbb{T} \to \mathbb{T}'$ is an embedding of algebraic $\mathbb{C}$-tori, and $I : P \to R$ is an isomorphism of $\mathbb{T}$-Poisson algebras, where $\mathbb{T}$ acts on $R$ through the embedding $E$ and the $\mathbb{T}'$-action on $R$. We drop the adjective “localized” when $(R, D)$ is a symmetric $\mathbb{T}'$-Poisson CGL extension, and we often suppress the mention of $\mathbb{T}$ and $\mathbb{T}'$ and speak of symmetric Poisson CGL extensions and presentations.

We remark that (see, for example, Remark 4.16) that a given $\mathbb{T}$-Poisson algebra may have many different presentations as Poisson CGL extensions.

Example 4.4 Consider the $\mathbb{T}$-Poisson algebra $(\mathbb{C}[\mathbb{T}], 0)$, where 0 stands for the zero Poisson bracket and $\mathbb{T}$ acts on $\mathbb{C}[\mathbb{T}]$ by translation. Let $(\omega_1, \ldots, \omega_d)$ be any basis of the character group $X(\mathbb{T})$, and let $(\omega^*_1, \ldots, \omega^*_d)$ be the basis of $\mathfrak{t}$ dual to $(\omega_1, \ldots, \omega_d)$. Then $((\mathbb{C}[\omega_1, \ldots, \omega_d], 0), D)$ is a symmetric $\mathbb{T}$-Poisson CGL extension, where

$$D = (\omega_1, \ldots, \omega_d, \omega^*_1, \ldots, \omega^*_d, -\omega^*_1, \ldots, -\omega^*_d),$$

and $(\mathcal{E}, \text{Id}_\mathbb{T}, I)$ is a presentation of $(\mathbb{C}[\mathbb{T}], 0)$ as a localized symmetric $\mathbb{T}$-Poisson CGL extension, where $\mathcal{E} = ((\mathbb{C}[\omega_1^\pm, \ldots, \omega_d^\pm], 0), D)$ and $I : \mathbb{C}[\mathbb{T}] \cong \mathbb{C}[\omega_1^\pm, \ldots, \omega_d^\pm]$.

Recall that an affine $\mathbb{T}$-Poisson variety is a Poisson variety $(X, \pi_X)$ together with an algebraic action by $\mathbb{T}$ preserving $\pi_X$. In such a case, $(\mathbb{C}[X], \pi_X)$ becomes a $\mathbb{T}$-Poisson algebra with the induced $\mathbb{T}$-action defined at the end of Notation 1.3.

Definition 4.5 Let $(X, \pi_X)$ be an $n$-dimensional irreducible $\mathbb{T}$-Poisson variety.

1. If $X' \subset X$ is a $\mathbb{T}$-invariant Zariski open subset and $\rho : Z \to X'$ is a parametrization of $X'$ by a Zariski open subset $Z$ of $\mathbb{C}^n$, we say that $\pi_X$ is presented as a localized symmetrical $\mathbb{T}'$-Poisson CGL extension in the coordinate chart $\rho$ (or via the parametrization $\rho : Z \to X'$) if there exists an algebraic $\mathbb{C}$-torus $\mathbb{T}'$, an embedding $E : \mathbb{T} \to \mathbb{T}'$, and $\mathbb{T}'$-action data $D$, such that $((\mathbb{C}[Z], D), E, (\rho^* : \mathbb{C}[X'] \to \mathbb{C}[Z]))$ is a presentation of the $\mathbb{T}$-Poisson algebra $(\mathbb{C}[X'], \pi_X)$ as a localized symmetric $\mathbb{T}'$-Poisson CGL extension.
2. By a $\mathbb{T}$-Poisson–Ore atlas for $(X, \pi_X)$ we mean an atlas $\mathcal{A}_X$ on $X$, consisting of $\mathbb{T}$-invariant coordinate charts parametrized by Zariski open subsets of $\mathbb{C}^n$, such that $\pi_X$ is presented as a localized symmetric Poisson CGL extension in each of the coordinate charts of $\mathcal{A}_X$.

**Definition 4.6** By a $\mathbb{T}$-Poisson–Ore variety, we mean a triple $(X, \pi_X, \mathcal{A}_X)$, where $(X, \pi_X)$ is an irreducible rational $\mathbb{T}$-Poisson variety, and $\mathcal{A}_X$ is a $\mathbb{T}$-Poisson–Ore atlas for $(X, \pi_X)$. A $\mathbb{T}$-Poisson–Ore variety for some algebraic $\mathbb{C}$-torus $\mathbb{T}$ is also simply referred to as a Poisson–Ore variety. \hfill $\square$

### 4.2 Mixed products of symmetric Poisson CGL extensions

Given $\mathbb{T}_i$-Poisson algebras $(R_i, \{,\}_i)$ for $i = 1, 2$, and given any

$$v = \sum_q a_q \otimes b_q \in t_1 \otimes t_2,$$

where $t_1$ and $t_2$ are the respective Lie algebras of $\mathbb{T}_1$ and $\mathbb{T}_2$, define the Poisson bracket $\{,\}_v$ on the tensor product algebra $R_1 \otimes_{\mathbb{C}} R_2$ via

$$\{r_1, r'_1\}_v = \{r_1, r'_1\}_1, \quad \{r_2, r'_2\}_v = \{r_2, r'_2\}_2, \quad r_1, r'_1 \in R_1, \quad r_2, r'_2 \in R_2,$$

$$\{r_1, r_2\}_v = -\sum_q a_q(r_1)b_q(r_2), \quad r_1 \in R_1, \quad r_2 \in R_2. \quad (4.5)$$

Then $(R_1 \otimes_{\mathbb{C}} R_2, \{,\}_v)$ is a $(\mathbb{T}_1 \times \mathbb{T}_2)$-Poisson algebra with the product $(\mathbb{T}_1 \times \mathbb{T}_2)$-action. Suppose now that

$$\mathcal{E}_1 = ((\mathbb{C}[z_1, \ldots, z_n], \{,\}_1), \mathcal{D}_1) \quad \text{and} \quad \mathcal{E}_2 = ((\mathbb{C}[z_{n+1}, \ldots, z_{n+m}], \{,\}_2), \mathcal{D}_2)$$

are symmetric $\mathbb{T}_1$- and $\mathbb{T}_2$-Poisson CGL extensions with respective action data

$$\mathcal{D}_1 = (\chi_1, \ldots, \chi_n, \ h_1, \ldots, h_n, \ h'_1, \ldots, h'_n),$$
$$\mathcal{D}_2 = (\chi_{n+1}, \ldots, \chi_{n+m}, h_{n+1}, \ldots, h_{n+m}, \ h'_{n+1}, \ldots, h'_{n+m}).$$

Identify $\mathbb{C}[z_1, \ldots, z_n] \otimes_{\mathbb{C}} \mathbb{C}[z_{n+1}, \ldots, z_{n+m}] \cong \mathbb{C}[z_1, \ldots, z_{n+m}]$, and for $j \in [1, n + m]$, define $\hat{\chi}_j \in X(\mathbb{T}_1 \times \mathbb{T}_2)$ and $\hat{h}_j, \hat{h}'_j \in t_1 \oplus t_2$ by

$$\hat{\chi}_j = (\chi_j, \ 0) \quad \text{for} \quad j \in [1, n] \quad \text{and} \quad \hat{\chi}_j = (0, \ \chi_j) \quad \text{for} \quad j \in [n + 1, n + m],$$
$$\hat{h}_j = (h_j, \ v^#(\chi_j)), \quad \hat{h}'_j = (h'_j, \ -v^#(\chi_j)), \quad j \in [1, n],$$
$$\hat{h}_j = ((v^{21})^#(\chi_j), \ h_j), \quad \hat{h}'_j = -(v^{21})^#(\chi_j), \ h'_j), \quad j \in [n + 1, n + m],$$

where $v^# : t^*_1 \rightarrow t_2$ and $(v^{21})^# : t^*_2 \rightarrow t_1$ are respectively defined by

$$v^#(\xi_1) = \sum_q a_q \otimes b_q, \quad (v^{21})^#(\xi_2) = \sum_q a_q \otimes b_q, \quad \xi_1 \in t^*_1, \ \xi_2 \in t^*_2.$$
The proof of the following proposition is straightforward and is omitted.

**Lemma 4.7** The pair \( \mathcal{E}_1 \otimes_v \mathcal{E}_2 := (\mathbb{C}[z_1, \ldots, z_{n+m}], \{, \}_v, \mathcal{D}) \), where

\[
\mathcal{D} = (\hat{x}_1, \ldots, \hat{x}_{n+m}, \hat{h}_1, \ldots, \hat{h}_{n+m}, \hat{h}'_1, \ldots, \hat{h}'_{n+m})
\]

is a symmetric \((\mathbb{T}_1 \times \mathbb{T}_2)\)-Poisson CGL extension. We call \( \mathcal{E}_1 \otimes_v \mathcal{E}_2 \) the mixed product of \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) defined by \( v \).

Rephrasing in geometrical terms, if for \( i = 1, 2 \), \((X_i, \pi_i)\) is a \( \mathbb{T}_i \)-Poisson variety, then each \( v = \sum q a_q \otimes b_q \in t_1 \otimes t_2 \) gives rise to the Poisson bi-vector field

\[
\pi_v = (\pi_1, 0) + (0, \pi_2) - \sum q (\rho_1(a_q), 0) \wedge (0, \rho_2(b_q))
\]

on \( X_1 \times X_2 \), where for \( a \in t_i \), \( \rho_i(a) \) is the vector field on \( X \) given by \( \rho_i(a)(x) = \frac{d}{d\lambda}|_{\lambda=0} \exp(\lambda a)x \) for \( x \in X \).

**Definition 4.8** The Poisson structure \( \pi_v \) on \( X_1 \times X_2 \) will be called the **mixed product** of the Poisson structures \( \pi_1 \) and \( \pi_2 \) defined by \( v \).

See [39] for a more general definition and construction of mixed product Poisson structures associated to Poisson Lie groups.

### 4.3 Completeness of Hamiltonian flows of symmetric CGL generators

We start by recalling a definition from [38]. Let \( \mathcal{Q} \) be the algebra of all quasi-polynomials in one complex variable [3, Sect. 26], i.e., all holomorphic functions on \( \mathbb{C} \) of the form

\[
\gamma(c) = \sum_{k=1}^{N} q_k(c)e^{a_k c}, \quad c \in \mathbb{C},
\]

where each \( q_k(c) \in \mathbb{C}[c] \) and the \( a_k \)'s are pairwise distinct complex numbers. A holomorphic map \( \gamma : \mathbb{C} \to \mathbb{C}^n \) is said to have **Property Q** if each of its components is in \( \mathcal{Q} \).

For a smooth affine complex variety \( X \) and a holomorphic curve \( \gamma : \mathbb{C} \to X \) in \( X \), we say that \( \gamma \) has **Property Q** if there exists an embedding \( I : X \hookrightarrow \mathbb{C}^n \) of \( X \) as an affine subvariety of \( \mathbb{C}^n \) such that \( I \circ \gamma : \mathbb{C} \to \mathbb{C}^n \) has Property Q. It is proved in [38, Lemma 1.5] that if a holomorphic curve \( \gamma \) in \( X \) has Property Q, then \( I' \circ \gamma : \mathbb{C} \to \mathbb{C}^{n'} \) has Property Q for all affine embeddings \( I' : X \hookrightarrow \mathbb{C}^{n'} \).

**Definition 4.9** [38] For a smooth affine complex Poisson variety \( (X, \pi_X) \) and \( f \in \mathbb{C}[X] \), we say that \( f \) has **complete Hamiltonian flow with Property Q** if all the integral curves of the Hamiltonian vector field of \( f \) are defined over \( \mathbb{C} \) and have Property Q.
Using the elementary fact that all solution curves of the ODE \( \frac{dx}{dt} = a_0x + b(t) \), where \( a_0 \in \mathbb{C} \) and \( b(t) \in \mathcal{Q} \), are in \( \mathcal{Q} \), the following facts are proved in [38].

Lemma 4.10 [38, Lemmas 1.2 and 1.4] Let \( \pi \) be an algebraic Poisson structure on \( \mathbb{C}^n \) with standard coordinate functions \( x_1, \ldots, x_n \), and let \( f \in \mathbb{C}[\mathbb{C}^n] \). If for each \( j \in [1, n] \) there are \( a_j \in \mathbb{C} \) and \( b_j \in \mathbb{C}[x_1, \ldots, x_{j-1}] \) such that

\[
\{ f, x_j \} = a_jx_jf + b_j,
\]

then \( f \) has complete Hamiltonian flows in \( \mathbb{C}^n \) with Property \( Q \). Furthermore, if a non-zero \( g \in \mathbb{C}[\mathbb{C}^n] \) is such that \( \{ f, g \} = afg \) for some \( a \in \mathbb{C} \), then \( f \) has complete Hamiltonian flows in \( X_g := \{ x \in \mathbb{C}^n : g(x) \neq 0 \} \).

Proposition 4.11 Let \( \mathcal{E} = (R, \mathcal{D}) \) be a localized symmetric Poisson CGL extension as in Definition 4.1, where \( R = (\mathbb{C}[z_1, \ldots, z_n][y_1^{-1}, \ldots, y_k^{-1}], \{ \} \) \). Let \( X \subset \mathbb{C}^n \) be defined by \( y_1y_2 \cdots y_k \neq 0 \). Then for each \( j \in [1, n] \), \( z_j \) has complete Hamiltonian flow with Property \( Q \) in \( X \).

Proof Fix \( j \in [1, n] \). With \( f = z_j \) and \( (x_1, \ldots, x_n) = (z_{j-1}, \ldots, z_1, z_{j+1}, z_{j+2}, \ldots, z_n, z_j) \) as the new set of coordinates for \( \mathbb{C}^n \), it follows from (4.3) that the assumptions in Lemma 4.10 on \( f \) and on \( (x_1, \ldots, x_n) \) are satisfied. Moreover, by [28, Corollary 5.10], for each \( i \in [1, k] \) one has \( \{ z_j, y_i \} = a_{ij}z_jy_i \) for some \( a_{ij} \in \mathbb{C} \). Let \( g = y_1y_2 \cdots y_k \). Then \( \{ z_j, g \} = az_jg \) for some \( a \in \mathbb{C} \). By Lemma 4.10, \( z_j \) has complete Hamiltonian flow with Property \( Q \) in \( X \).

The following theorem is now an immediate consequence of Proposition 4.11 and the definition of Poisson–Ore varieties.

Theorem 4.12 For any Poisson–Ore variety \((X, \pi_X, \mathcal{A}_X)\), all the coordinate functions in any coordinate chart \( \rho : Z \to X \) in \( \mathcal{A}_X \) have complete Hamiltonian flows with Property \( Q \) in \( \rho(Z) \).

4.4 Symmetric Poisson CGL extensions from generalized Bruhat cells

In this section, we review the standard multiplicative Poisson structure \( \pi_{st} \) on \( G \), the standard Poisson structures on generalized Bruhat cells, and the associated symmetric Poisson CGL extensions.

Notation 4.13 Recall that \( g \) and \( \mathfrak{h} \) are the respective Lie algebras of \( G \) and \( T \). We fix a symmetric and non-degenerate invariant bilinear form \( \langle \cdot, \cdot \rangle_g \) on \( g \), and denote by \( \langle \cdot, \cdot \rangle \) both the restriction of \( \langle \cdot, \cdot \rangle_g \) to \( \mathfrak{h} \) and the induced bilinear form on \( \mathfrak{h}^* \). Let again \( d = \dim \mathfrak{h} \), and let \( \{ H_q \}_{q=1}^d \) be any basis of \( \mathfrak{h} \) that is orthonormal with respect to \( \langle \cdot, \cdot \rangle \). Let \( \Delta^+ \subset \mathfrak{h}^* \) be the set of all positive roots.

Recall from Notation 1.3 that for each simple root \( \alpha \), we have fixed root vectors \( e_\alpha \) for \( \alpha \) and \( e_{-\alpha} \) for \( -\alpha \) such that \( \alpha([e_\alpha, e_{-\alpha}]) = 2 \). Choose also such root vectors \( e_\alpha \) and \( e_{-\alpha} \) for every positive root \( \alpha \) that is not simple. It is then easy to see that...
\[ \langle e_\alpha, e_{-\alpha} \rangle \| = \frac{2}{\langle \alpha, \alpha \rangle} \text{ for } \alpha \in \Delta^+. \] The standard quasi-triangular r-matrix on \( g \) is given by [13]

\[ r_{st} = \sum_{q=1}^{d} H_q \otimes H_q + \sum_{\alpha \in \Delta^+} \langle \alpha, \alpha \rangle (e_{-\alpha} \otimes e_{\alpha}) \in g \otimes g, \tag{4.8} \]

which depends only on the choice of the triple \((B, T, \langle , \rangle_g)\) and not on that of the root vectors. Let

\[ \Lambda_{st} = \sum_{\alpha \in \Delta^+} \frac{\langle \alpha, \alpha \rangle}{2} (e_{-\alpha} \otimes e_{\alpha} - e_{\alpha} \otimes e_{-\alpha}) \in \wedge^2 g \]

be the skew-symmetric part of \( r_{st} \). Then the standard multiplicative Poisson structure \( \pi_{st} \) on \( G \) is defined to be the Poisson bi-vector field on \( G \) given by

\[ \pi_{st} = \Lambda_{st}^L - \Lambda_{st}^R, \tag{4.9} \]

where \( \Lambda_{st}^L \) and \( \Lambda_{st}^R \) respectively denote the left and right invariant bi-vector fields on \( G \) with value \( \Lambda_{st} \) at the identity element of \( G \). The Poisson Lie group \((G, \pi_{st})\) is the semi-classical limit of the quantum group \( \mathbb{C}q[G] \) (see [11,16]).

It follows from the definition that \( \pi_{st} \) is invariant under both left and right translations by elements in \( T \), and it is well-known (see, for example, [32,33,36]) that the \( T \)-orbits of symplectic leaves of \( \pi_{st} \) are precisely the double Bruhat cells

\[ G^u,v = BuB \cap B^{-}vB^{-}, \quad u, v \in W. \]

It follows that every \( BwB \), where \( w \in W \), is a Poisson submanifold of \((G, \pi_{st})\).

For any integer \( r \geq 1 \), recall now from (1.5) the quotient variety \( F_r \) of \( G^r \) by \( B^r \). Let again \( \varpi_r : G^r \to F_r \) be the projection. It is shown in [40, Sect. 1.3] and [39, Sect. 7.1] that

\[ \pi_r \triangleq \varpi_r(\pi_{st}^r) \tag{4.10} \]

is a well-defined Poisson structure on \( F_r \), and that for any \( u = (u_1, \ldots, u_r) \in W^r \), the generalized Bruhat cell \( O^u \subset F_r \) is a Poisson submanifold with respect to the Poisson structure \( \pi_r \). The restriction of \( \pi_r \) to \( O^u \) will still be denoted as \( \pi_r \) and is called [15] the standard Poisson structure on \( O^u \). It is also clear that the \( T \)-action on \( F_r \) defined in (2.10) preserves the Poisson structure \( \pi_r \).

Let \( u = (u_1, \ldots, u_r) \in W^r \), and let \( l = l(u_1) + \cdots + l(u_r) \). Recall that associated to \( \tilde{u} = (u_1, \ldots, u_r) \in R(u_1) \times \cdots \times R(u_r) \) one has the Bott–Samelson parametrization \( \beta_{\tilde{u}} : \mathbb{C}^l \to O^u \) in (2.13). Let \( \{ , \}_{\tilde{u}} \) be the Poisson bracket on \( \mathbb{C}[z_1, \ldots, z_l] \) such that

\[ (\beta_{\tilde{u}})^* : (\mathbb{C}[O^u], \pi_r) \to (\mathbb{C}[z_1, \ldots, z_l], \{ , \}_{\tilde{u}}) \tag{4.11} \]
is an isomorphism of Poisson algebras. Regard \( (\mathbb{C}[O^{u}], \pi_r) \) as a \( T \)-Poisson algebra, where \( T \) acts on \( \mathbb{C}[O^{u}] \) through the \( T \)-action on \( O^{u} \) in (2.10), i.e.

\[
 t \cdot z_j = t^{s_{a_1} s_{a_2} \cdots s_{a_{j-1}}(\alpha_j)} z_j, \quad t \in T, \; j \in [1, l],
\]
as in (2.14). For any \( \chi \in \mathfrak{h}^{*} \), let \( \chi^\# \) be the unique element in \( \mathfrak{h} \) satisfying

\[
\chi' (\chi^\#) = \langle \chi, \chi' \rangle, \quad \chi' \in \mathfrak{h}^{*}.
\]

Set \( D_\mathfrak{u} = (\chi_1, \ldots, \chi_l, h_1, \ldots, h_l, h_1', \ldots, h_l') \), where for \( j \in [1, l] \),

\[
\chi_j = s_{a_1} s_{a_2} \cdots s_{a_{j-1}}(\alpha_j) \in \mathfrak{h}^{*}, \quad \text{and} \quad h_j = -h_j' = s_{a_1} s_{a_2} \cdots s_{a_{j-1}}(\alpha_j^\#) \in \mathfrak{h}.
\]

**Theorem 4.14** [15] For any \( u = (u_1, \ldots, u_r) \in W^r \) and \( \mathfrak{u} \in \mathcal{R}(u_1) \times \cdots \times \mathcal{R}(u_r) \),

\[
E_\mathfrak{u} := \left( (\mathbb{C}[z_1, \ldots, z_l], \{ , \}^\#_\mathfrak{u}), D_\mathfrak{u} \right)
\]

is a symmetric \( T \)-Poisson CGL extension. In particular, for \( 1 \leq i < j \leq l \), the log-canonical term \( \{ z_i, z_j \}^\#_\mathfrak{u}, \text{log–can} \) of \( \{ z_i, z_j \}^\#_\mathfrak{u} \) is given by (see (4.4))

\[
\{ z_i, z_j \}^\#_\mathfrak{u}, \text{log–can} = -\chi_i (h_j) z_i z_j = -(s_{a_1} \cdots s_{a_{i-1}}(\alpha_i), s_{a_1} \cdots s_{a_{j-1}}(\alpha_j)) z_i z_j.
\]

**Definition 4.15** For \( u = (u_1, \ldots, u_r) \in W^r \) and \( \mathfrak{u} \in \mathcal{R}(u_1) \times \cdots \times \mathcal{R}(u_r) \), set

\[
\mathcal{P}_\mathfrak{u} = (E_\mathfrak{u}, \text{Id}_r, (\beta^\mathfrak{u}^*)^*).
\]

We call \( \mathcal{P}_\mathfrak{u} \) the symmetric \( T \)-Poisson CGL presentation of \( (\mathbb{C}[O^{u}], \pi_r) \) (in the Bott–Samelson coordinates or via the Bott–Samelson parametrization) defined by \( \mathfrak{u} \).

**Remark 4.16** As an element in \( W \) may have more than one reduced expression, the \( T \)-Poisson algebra \( (\mathbb{C}[O^{u}], \pi_r) \) for \( u \in W^r \) in general has more than one presentation as a symmetric Poisson CGL extension.

**Remark 4.17** By Theorem 4.14, the Poisson bracket \( \{ , \}^\#_\mathfrak{u} \) has the form

\[
\{ z_i, z_j \}^\#_\mathfrak{u} = -\chi_i (h_j) z_i z_j - f_{i,j}, \quad 1 \leq i < j \leq l,
\]

where \( f_{i,j} \in \mathbb{C}[z_{i+1}, \ldots, z_{j-1}] \) for all \( 1 \leq i < j \leq l \). Explicit formulas for the polynomials \( f_{i,j} \) are given in [15] in terms of root strings and the structure constants of the Lie algebra \( \mathfrak{g} \) in any Chevalley basis. We refer to [15] for detail. When \( r = 1 \), Theorem 4.14 is the Poisson analog of the Levendorskii–Soibelman straightening law for the quantum Schubert cell corresponding to \( u_1 \in W \) (see [27, Sect. 9.2] and [10, I.6.10]).
5 The Bott–Samelson atlas is a Poisson–Ore atlas

Continuing with the setting from Sect. 4.4, we first review in Sect. 5.1 the definition of the Poisson structure $\pi_{G/Q}$ for $Q = B(v)$ or $N(v)$ with $v \in W$. We then prove in Sect. 5.2 that for each $w \in W$, the decomposition $J^w_Q$ of $wB^−B/Q$, given in (2.23) and (2.24), identifies the restriction of $\pi_{G/Q}$ to $wB^−B/Q$ with a mixed product of $\pi_1$ on $O^{w_0w_1}$ and $\pi_2$ on $O^{(w,v)}$ (and the zero Poisson structure on $T$ when $Q = N(v)$). Using the presentations of $\pi_1$ and $\pi_2$ as symmetric Poisson CGL extensions via Bott–Samelson parametrizations and the mixed product construction in Lemma 4.7, we obtain presentations of $\pi_{G/Q}$ as symmetric (or localized symmetric) Poisson CGL extensions in every Bott–Samelson coordinate chart on $G/Q$, thereby proving the Theorem B stated in Sect. 1.1. Details on the (localized) symmetric Poisson CGL presentations of $\pi_{G/Q}$ in the Bott–Samelson coordinate charts are given in Theorem 5.6 for $Q = B(v)$ and in Theorem 5.9 for $Q = N(v)$.

5.1 The Poisson structure $\pi_{G/Q}$

Given a Poisson Lie group $(L, \pi)$, recall that a Lie subgroup $M$ of $L$ is called a coisotropic subgroup of $(L, \pi)$ if $M$ is also a coisotropic submanifold of $L$ with respect to $\pi$, i.e., if $\pi(x) \in T_xL \otimes T_xM + T_xM \otimes T_xL$ for all $x \in M$. It is easy to see from the definition that when $M$ is a closed coisotropic subgroup of a Poisson Lie group $(L, \pi)$, the Poisson structure $\pi$ on $L$ projects to a well-defined Poisson structure on $L/M$, called the quotient Poisson structure of $\pi$.

Returning to the Poisson Lie group $(G, \pi_{st})$ with $\pi_{st}$ defined in (4.9), an argument similar to that used in the proof of [41, Lemma 10] shows that for any $v \in W$, $N(v)$ is a coisotropic subgroup of $(G, \pi_{st})$. Since the Poisson structure $\pi_{st}$ is invariant under translation by elements in $T$, $B(v)$ is also a coisotropic subgroup of $(G, \pi_{st})$.

Notation 5.1 For $v \in W$ and $Q = B(v)$ or $N(v)$, denote by $\pi_{G/Q}$ the Poisson structure on $G/Q$ which is the projection to $G/Q$ of the Poisson structure $\pi_{st}$ on $G$. The restriction of $\pi_{G/Q}$ to each shifted big cell $wB^−B/Q$ is also denoted by $\pi_{G/Q}$. Note that $\pi_{G/B} = \pi_1$ in the notation of (4.10).

5.2 The decompositions $J^w_Q$ as Poisson maps

For $v, w \in W$, recall from Sect. 2.3 the isomorphisms

$$J^w_B(v) : wB^−B/B(v) \xrightarrow{\sim} O^{w_0w_1} \times O^{(w,v)},$$

$$J^w_N(v) : wB^−B/N(v) \xrightarrow{\sim} O^{w_0w_1} \times O^{(w,v)} \times T.$$ 

We now identify the respective Poisson structures $J^w_B(v)(\pi_{G/B}(v))$ and $J^w_N(v)(\pi_{G/N(v)})$ on $O^{w_0w_1} \times O^{(w,v)}$ and on $O^{w_0w_1} \times O^{(w,v)} \times T$ as mixed products of the standard Poisson structures $\pi_1$ on $O^{w_0w_1}$ and $\pi_2$ on $O^{(w,v)}$ (and the zero Poisson structure on $T$ when $Q = N(v)$). To this end, for $x \in \mathfrak{h}$, denote by $\rho_r(x)$ the vector field on $F_r$.
generated by the $T$-action on $F_r$ in (2.10) in the direction of $x$, i.e.,

$$\rho_r(x)(p) = \frac{d}{d\lambda}|_{\lambda=0} \exp(\lambda x) \cdot p, \quad p \in F_r. \quad (5.1)$$

The restriction of $\rho_r(x)$ to $O^u$, for $x \in \mathfrak{h}$, will also be denoted by $\rho_r(x)$. For $x \in \mathfrak{h}$, let $x^R$ be the right-invariant (also left-invariant) vector field on $T$ defined by $x$.

**Notation 5.2** Let again $d = \dim_\mathbb{C} T$ and recall that $\{H_q : q \in [1, d]\}$ is a basis of $\mathfrak{h}$ orthonormal with respect to the bilinear form $\langle \cdot, \cdot \rangle$. Define the bi-vector fields

$$\pi_{1,2} = (\pi_1, 0) + (0, \pi_2) + \mu \quad \text{on} \quad O^{w_0w^{-1}} \times O^{(w,v)}, \quad \text{and} \quad (5.2)$$

$$\pi_{1,2,0} = (\pi_1, 0, 0) + (0, \pi_2, 0) + \mu_{12} + \mu_{13} + \mu_{23} \quad \text{on} \quad O^{w_0w^{-1}} \times O^{(w,v)} \times T, \quad (5.3)$$

where the *mixed terms* $\mu$ and $\mu_{12}, \mu_{13}, \mu_{23}$ are respectively given by

$$\mu = \sum_{q=1}^{d} (\rho_1(w_0w^{-1}(H_q)), 0) \wedge (0, \rho_2(H_q)), \quad (5.4)$$

$$\mu_{12} = \sum_{q=1}^{d} (\rho_1(w_0w^{-1}(H_q)), 0, 0) \wedge (0, \rho_2(H_q), 0), \quad (5.5)$$

$$\mu_{13} = \sum_{q=1}^{d} (\rho_1(w_0H_q)), 0, 0) \wedge (0, 0, H_q^R), \quad (5.6)$$

$$\mu_{23} = -\sum_{q=1}^{d} (0, \rho_2(w(H_q)), 0) \wedge (0, 0, H_q^R). \quad (5.7)$$

**Theorem 5.3** For any $w, v \in W$, the maps

$$J^w_{B(v)} : (wB^{-1}B, \pi_{G/B(v)}) \longrightarrow \left( O^{w_0w^{-1}} \times O^{(w,v)}, \pi_{1,2} \right),$$

$$J^w_{N(v)} : (wB^{-1}B/N, \pi_{G/N(v)}) \longrightarrow \left( O^{w_0w^{-1}} \times O^{(w,v)} \times T, \pi_{1,2,0} \right),$$

are Poisson isomorphisms.

Since the proof of Theorem 5.3 is self-contained and can be read independently of the rest of the paper, we present the proof in the Appendix in order not to disrupt the flow of the paper.

**Remark 5.4** Regard $(O^{w_0w^{-1}}, \pi_1)$ and $(O^{(w,v)}, \pi_2)$ as $T$-Poisson varieties, where again $T$ acts on $O^{w_0w^{-1}} \subset F_1$ and on $O^{(w,v)} \subset F_2$ via (2.10). Theorem 5.3 then
says that $\pi_{1,2}$ is the mixed product (in the sense of Definition 4.8) of $\pi_1$ and $\pi_2$ defined by

$$\nu = -\sum_{q=1}^{d} w_0^{-1}(H_q) \otimes H_q \in \mathfrak{h} \otimes \mathfrak{h}.$$ 

Similarly, regard $(\mathcal{O}^{w_0^{-1}} \times \mathcal{O}^{(w, v)}, \pi_{1,2})$ as a $(T \times T)$-Poisson variety with the product $(T \times T)$-action, and regard $(T, 0)$ as a $T$-variety with the left $T$-action by translation. Then $\pi_{1,2,0}$ is the mixed product of $\pi_{1,2}$ and the zero Poisson structure on $T$ defined by

$$\nu' = -\sum_{q=1}^{d} (w_0(H_q), -w(H_q)) \otimes H_q \in (\mathfrak{h} \oplus \mathfrak{h}) \otimes \mathfrak{h}.$$ 

\[\Box\]

### 5.3 Symmetric Poisson CGL presentations of $\pi_{G/Q}$ in Bott–Samelson coordinate charts

Let $v \in W$ and $Q = B(v)$ or $N(v)$. For each $w \in W$, we regard $(\mathbb{C}[wB^{-1}B/Q], \pi_{G/Q})$ as a $T$-Poisson algebra with the $T$-action given by

$$(t \cdot \phi)(g.Q) = \phi(tg.Q), \quad t \in T, \quad \phi \in \mathbb{C}[wB^{-1}B/Q], \quad g \in wB^{-1}B. \quad (5.8)$$

Using Theorems 4.14, 5.3, and the construction of product symmetric Poisson CGL extensions in Lemma 4.7, we obtain symmetric Poisson CGL presentations of $(\mathbb{C}[wB^{-1}B/Q], \pi_{G/Q})$ in the Bott–Samelson coordinates on $wB^{-1}B/Q$. In this section, we give the details of these presentations, which contain, in particular, the action data by the relevant tori on the symmetric Poisson CGL extensions.

For the convenience of the reader, we recall part of Notation 2.6.

**Notation 5.5** Let $l_0 = l(w_0)$, and for $v, w \in W$, let $k = l_0 - l(w)$ and $l = l_0 + l(v)$. Write an element $r \in R(w_0) \times R(w) \times R(v)$ as $r = (w^0, w, v)$ with

$$w^0 = (s_{\alpha_1}, \ldots, s_{\alpha_k}), \quad w = (s_{\alpha_{k+1}}, \ldots, s_{\alpha_{l_0}}), \quad v = (s_{\alpha_{l_0+1}}, \ldots, s_{\alpha_l}).$$

For each $j \in [1, l]$, set

$$\chi_j = s_{\alpha_1} \cdots s_{\alpha_{j-1}}(\alpha_j), \quad h_j = -h'_j = s_{\alpha_1} \cdots s_{\alpha_{j-1}}(\alpha_j^\#), \quad \text{if} \quad j \in [1, k], \quad (5.9)$$

$$\chi_j = s_{\alpha_{k+1}} \cdots s_{\alpha_{j-1}}(\alpha_j), \quad h_j = -h'_j = s_{\alpha_{k+1}} \cdots s_{\alpha_{j-1}}(\alpha_j^\#), \quad \text{if} \quad j \in [k+1, l], \quad (5.10)$$

where for $\chi \in \mathfrak{h}^*$, $\chi^\# \in \mathfrak{h}$ is defined in (4.12). \[\Box\]
The case of $B(v)$. Let $w, v \in W$ and let $r = (w^0, w, v) \in \mathcal{R}(w_0 w^{-1}) \times \mathcal{R}(w) \times \mathcal{R}(v)$ be as in Notation 5.5. Recall from (2.28) and (2.29) the parametrizations

$$
\sigma^r : \mathbb{C}^l \longrightarrow \mathcal{O}^{w_0 w^{-1}} \times \mathcal{O}(w, v),
$$

$$
\sigma^r_{B(v)} = (J^w_{B(v)})^{-1} \circ \sigma^r : \mathbb{C}^l \longrightarrow w B^{-1} B / B(v).
$$

Let $\{,\}_{(w^0,w,v)}$ be the Poisson bracket on $\mathbb{C}[z_1, \ldots, z_l]$ such that

$$(\sigma^r)^* : \left( \mathbb{C}[\mathcal{O}^{w_0 w^{-1}} \times \mathcal{O}(w, v)], \mathcal{O}_{1,2} \right) \longrightarrow \left( \mathbb{C}[z_1, \ldots, z_l], \{,\}_{(w^0,w,v)} \right)$$

is a Poisson isomorphism. By Theorem 5.3, we have the Poisson isomorphism

$$(\sigma^r_{B(v)})^* : \left( \mathbb{C}[w B^{-1} B / B(v)], \pi_{G/B(v)} \right) \longrightarrow \left( \mathbb{C}[z_1, \ldots, z_l], \{,\}_{(w^0,w,v)} \right).$$

Set $\mathcal{D}_{(w^0,w,v)} = (\hat{x}_1, \ldots, \hat{x}_l, \hat{h}_1, \ldots, \hat{h}_l, \hat{h}_1', \ldots, \hat{h}_l')$, where

$$
\hat{x}_j = \begin{cases} (\chi_j, 0), & j \in [1, k], \\
(0, \chi_j), & j \in [k + 1, l], \end{cases}
$$

$$
\hat{h}_j = -\hat{h}'_j = \begin{cases} (h_j, -w_0 w_0(h_j)), & j \in [1, k], \\
(-w_0 w^{-1}(h_j), h_j), & j \in [k + 1, l], \end{cases}
$$

and $\chi_j \in \mathfrak{h}^*$ and $h_j \in \mathfrak{h}$ for $j \in [1, l]$ are given in (5.9) and (5.10), and let

$$
\mathcal{E}_{(w^0,w,v)} := \left( \left( \mathbb{C}[z_1, \ldots, z_l], \{,\}_{(w^0,w,v)} \right), \mathcal{D}_{(w^0,w,v)} \right).
$$

Recall from Theorem 4.14 that one has the symmetric $T$-Poisson CGL extensions

$$
\mathcal{E}_{w^0} = \left( \left( \mathbb{C}[z_1, \ldots, z_k], \{,\}_{w^0} \right), \mathcal{D}_{w^0} \right) \quad \text{and} \quad \mathcal{E}_{(w,v)} = \left( \left( \mathbb{C}[z_{k+1}, \ldots, z_l], \{,\}_{(w,v)} \right), \mathcal{D}_{(w,v)} \right).
$$

One checks directly that $\mathcal{E}_{(w^0,w,v)}$ is the mixed product (see Lemma 4.7) of the symmetric $T$-Poisson CGL extensions $\mathcal{E}_{w^0}$ and $\mathcal{E}_{(w,v)}$ defined by

$$
\nu = -\sum_{q=1}^{d} w_0 w^{-1}(H_q) \otimes H_q \in \mathfrak{h} \otimes \mathfrak{h}.
$$

**Theorem 5.6** For any $w, v \in W$ and $r = (w^0, w, v) \in \mathcal{R}(w_0 w^{-1}) \times \mathcal{R}(w) \times \mathcal{R}(v)$ as in Notation 5.5, the triple

$$
\mathcal{D}^r_{B(v)} := \left( \mathcal{E}_{(w^0,w,v)}, \mathcal{D}_{E_{w^0}}, (\sigma^r_{B(v)})^* \right)
$$

is a presentation of the $T$-Poisson algebra $\left( \mathbb{C}[w B^{-1} B / B(v)], \pi_{G/B(v)} \right)$ as a symmetric $(T \times T)$-Poisson CGL extension, where $E_{w^0} : T \to T \times T, t \to (w_0 w_0^*, t)$ for $t \in T$. 

**Proof** Equip $\mathcal{O}^{w_0w^{-1}} \times \mathcal{O}^{(w,v)}$ with the product $(T \times T)$-action, where $T$ acts on $\mathcal{O}^{w_0w^{-1}} \subset F_1$ and on $\mathcal{O}^{(w,v)} \subset F_2$ via (2.10), and equip $\mathbb{C}[\mathcal{O}^{w_0w^{-1}} \times \mathcal{O}^{(w,v)}]$ with the induced $(T \times T)$-action (see end of Notation 1.3). Let $T \times T$ act on $\mathbb{C}[z_1, \ldots, z_l]$ such that for $j \in [1, l]$, $z_j$ has $(T \times T)$-weight $\hat{\chi}_j$ as given in (5.13), and let $T$ act on $\mathbb{C}[z_1, \ldots, z_l]$ through the embedding $E_w : T \to T \times T$. By (2.14), $(\sigma^r)^*$ in (5.11) is an isomorphism of $(T \times T)$-Poisson algebras. By Lemma 2.4, $(\sigma^r_{B(v)})^*$ in (5.12) is an isomorphism of $T$-Poisson algebras. As

$$(\sigma^r_{B(v)})^* = (\sigma^r)^* \circ \left((J^w_{B(v)})^{-1}\right)^*,$$

the assertion on $\mathcal{P}^r_{B(v)}$ now follows Theorem 5.3. □

**Remark 5.7** For each $r = (w^0, w, v) \in \mathcal{R}(w_0w^{-1}) \times \mathcal{R}(w) \times \mathcal{R}(v)$, recall from (4.11) the Poisson bracket $\{,\}_{(w^0, w, v)}$ on $\mathbb{C}[z_1, \ldots, z_l] \cong \mathbb{C}[\mathcal{O}^{w_0w^{-1}, w, v}]$. Using the definition of the Poisson structure $\pi_{1,2}$ and Theorem 4.14, it is easy to see that

$$\{z_i, z_j\}_{(w^0, w, v)} = \{z_i, z_j\}_{(w^0, w, v)} \quad \text{if} \quad i, j \in [1, k] \quad \text{or} \quad i, j \in [k+1, l],$$

but for $i \in [1, k]$ and $j \in [k+1, l]$, $\{z_i, z_j\}_{(w^0, w, v)}$ is the negative of the log-canonical term of $\{z_i, z_j\}_{(w^0, w, v)}$. For this reason, we may call $\{,\}_{(w^0, w, v)}$ the log-canonical cut of $\{,\}_{(w^0, w, v)}$ at $k$’s place with coefficient $-1$.

**The case of $N(v)$**. We now turn to the $T$-Poisson algebra $\left(\mathbb{C}[wB^{-}B/N(v)], \pi_{G/N(v)}\right)$. Recall again that $\omega_1, \ldots, \omega_d$ is a listing of the set of all fundamental weights and that $z_j = \omega_{j-l} \in \mathbb{C}[T]$ for $j \in [l+1, l+d]$.

Let $\{,\}_{\{w^0, w, v\}}$ be the Poisson bracket on $\mathbb{C}[z_1, \ldots, z_l, z_{l+1}^{\pm 1}, \ldots, z_{l+d}^{\pm 1}]$ such that

$$(\sigma^r \times \sigma)^* : \left(\mathbb{C}[\mathcal{O}^{w_0w^{-1}} \times \mathcal{O}^{(w,v)} \times T], \pi_{1,2,0}\right) \to \left(\mathbb{C}[z_1, \ldots, z_l, z_{l+1}^{\pm 1}, \ldots, z_{l+d}^{\pm 1}], \{,\}_{\{w^0, w, v\}}\right)$$

is a Poisson algebra isomorphism. We first express $\{,\}_{\{w^0, w, v\}}$ in terms of $\{,\}_{\{w^0, w, v\}}$.

**Lemma 5.8** One has $\{z_i, z_j\}_{\{w^0, w, v\}} = \{z_i, z_j\}_{\{w^0, w, v\}}$ for all $i, j \in [1, l]$, and

$$\{z_i, z_j\}_{\{w^0, w, v\}} = \begin{cases} (w_0\omega_{j-l})(hi)z_i z_j, & i \in [1, k], j \in [l+1, l+d], \\ -(w_0\omega_{j-l})(hi)z_i z_j, & i \in [k+1, l], j \in [l+1, l+d], \\ 0, & i, j \in [l+1, l+d]. \end{cases}$$

**Proof** The statements follow directly from the definition of $\pi_{1,2,0}$ given in Theorem 5.3. □
In particular, $\{.,.\}^{\alpha\beta}_{(w^0|w,v)}$ restricts to a Poisson bracket on $\mathbb{C}[z_1, \ldots, z_l, z_{l+1}, \ldots, z_{l+d}]$. By Theorem 5.3, one has an isomorphism of Poisson algebras

$$(\sigma^r_{N(v)})^* : (\mathbb{C}[wB^{-}\mathcal{B}/N(v)], \pi_{G/N(v)}) \rightarrow (\mathbb{C}[z_1, \ldots, z_l, z_{l+1}, \ldots, z_{l+d}], \{.,.\}^{\alpha\beta}_{(w^0|w,v)}).$$

Let $\{\omega_1^+, \ldots, \omega_d^+\}$ be the basis of $\mathfrak{h}$ dual to $\{\omega_1, \ldots, \omega_d\}$. With $\chi_j \in \mathfrak{h}^*$ and $h_j \in \mathfrak{h}$ for $j \in [1, l]$ given in (5.9) and (5.10), let

$$\tilde{\chi}_j = \begin{cases} (\chi_j, 0, 0), & j \in [1, k], \\ (0, \chi_j, 0), & j \in [k+1, l], \\ (0, 0, \omega_{j-l}), & j \in [l+1, l+d], \end{cases}$$

(5.14)

$$\tilde{h}_j = -\tilde{h}_j' = \begin{cases} (h_j, -ww_0(h_j), -w_0(h_j)), & j \in [1, k], \\ (-w_0w^{-1}(h_j), h_j, w(h_j)), & j \in [k+1, l], \\ (-w_0(\omega_j^+), w(\omega_j^+), \omega_j^+), & j \in [l+1, l+d], \end{cases}$$

(5.15)

and let $\mathcal{D}((w^0|w,v), \mathcal{T}) = (\tilde{\chi}_1, \ldots, \tilde{\chi}_{l+d}, \tilde{h}_1, \ldots, \tilde{h}_{l+d}, \tilde{h}_1', \ldots, \tilde{h}_{l+d}')$.

**Theorem 5.9** For any $w, v \in W$ and $r = (w^0, w, v) \in \mathcal{R}(w_0w^{-1}) \times \mathcal{R}(w) \times \mathcal{R}(v)$,

$$\mathcal{E}((w^0|w,v), \mathcal{T}) := \left(\left(\mathbb{C}[z_1, \ldots, z_l, z_{l+1}, \ldots, z_{l+d}], \{.,.\}^{\alpha\beta}_{(w^0|w,v)}\right), \mathcal{D}((w^0|w,v), \mathcal{T})\right)$$

is a symmetric $(T \times T \times T)$-Poisson CGL extension, and

$$\mathcal{P}^r_{N(v)} := \left(\mathcal{E}((w^0|w,v), \mathcal{T}), E'_w, (\sigma^r_{N(v)})^*\right)$$

is a presentation of the $T$-Poisson algebra $(\mathbb{C}[wB^{-}\mathcal{B}/N(v)], \pi_{G/N(v)})$ as a localized symmetric $(T \times T \times T)$-Poisson CGL extension, where

$$\mathcal{E}'((w^0|w,v), \mathcal{T}) := \left(\left(\mathbb{C}[z_1, \ldots, z_l, z_{l+1}, \ldots, z_{l+d}], \{.,.\}^{\alpha\beta}_{(w^0|w,v)}\right), \mathcal{D}((w^0|w,v), \mathcal{T})\right)$$

and $E'_w$ is the embedding $T \rightarrow T \times T \times T$ given by $t \rightarrow (t^w, t, t^w)$ for $t \in T$.

**Proof** Let $T \times T \times T$ act on $\mathbb{C}[z_1, \ldots, z_l, z_{l+1}, \ldots, z_{l+d}]$ such that $z_j$ has $(T \times T \times T)$-weight $\tilde{\chi}_j$ given in (5.14). By the $T$-equivariance of the isomorphism $J^w_{N(v)}$ given in Lemma 2.4 and by (2.14), the isomorphism $(\sigma^r_{N(v)})^*$ is $T$-equivariant, where $t \in T$ acts on $\mathbb{C}[z_1, \ldots, z_l, z_{l+1}, \ldots, z_{l+d}]$ via the embedding $E'_w : T \rightarrow T \times T \times T$. One checks directly that $\mathcal{E}'((w^0|w,v), \mathcal{T})$ is the mixed product (see Definition 4.8) defined by

$$v' = -\sum_{q=1}^d (w_0(H_q), -w(H_q)) \otimes H_q \in (\mathfrak{h} \oplus \mathfrak{h}) \otimes \mathfrak{h}$$
of the symmetric \((T \times T)\)-Poisson CGL extension \(E_{(w_0,w,w)}\) and the symmetric \(T\)-Poisson CGL extension \(((C[z_{l+1}, \ldots, z_{l+d}], 0), D)\) in Example 4.4. The assertion on \(\mathcal{P}_N^{r,v}\) now follows from Theorem 5.3.

\[\Box\]

**Example 5.10** Let \(G = SL(4, \mathbb{C})\) and choose again the Borel subgroups \(B^-\) and \(B\) to consist of lower triangular and upper triangular matrices respectively. Let \(\alpha_1, \alpha_2, \alpha_3\), be the standard listing of the simple roots and set \(s_j = s_{\alpha_i}, i = 1, 2, 3\). We consider two Bott–Samelson coordinate charts on \(G/B\). The Poisson brackets in the coordinates are computed using the computer program in GAP language written by Balazs Elek.

Let first \(w = e\) and \(r_1 = ((s_3, s_2, s_1, s_2, s_3, 0, 0, 0), 0)\). The Bott–Samelson parametrization \(\sigma^{r_1} = \sigma^{r_1}_{G/B} : \mathbb{C}^6 \rightarrow B^-B/B\) is given by

\[
\sigma^{r_1}(\xi_1, \ldots, \xi_6) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-\xi_1 & 1 & 0 & 0 \\
\xi_1 \xi_2 - \xi_2 & \xi_2 & 1 & 0 \\
\Delta_1 & \xi_4 \xi_5 - \xi_5 & \xi_5 & 1
\end{pmatrix} \cdot B \in B^-B/B,
\]

where \(\Delta_1 = \xi_1 \xi_4 \xi_6 - \xi_1 \xi_5 - \xi_2 \xi_6 + \xi_3\). In the \((\xi_1, \ldots, \xi_6)\) coordinates, the Poisson structure \(\pi_{G/B}\) is given by

\[
\begin{align*}
\{\xi_1, \xi_2\} &= -\xi_1 \xi_2, & \{\xi_1, \xi_3\} &= -\xi_1 \xi_3, & \{\xi_1, \xi_4\} &= \xi_1 \xi_4 - 2 \xi_2, \\
\{\xi_1, \xi_5\} &= \xi_1 \xi_5 - 2 \xi_3, & \{\xi_1, \xi_6\} &= 0, & \{\xi_2, \xi_3\} &= -\xi_2 \xi_3, \\
\{\xi_2, \xi_4\} &= -\xi_2 \xi_4, & \{\xi_2, \xi_5\} &= 2 \xi_2 \xi_4, & \{\xi_2, \xi_6\} &= \xi_2 \xi_6 - 2 \xi_3, \\
\{\xi_3, \xi_4\} &= 0, & \{\xi_3, \xi_5\} &= -\xi_3 \xi_5, & \{\xi_3, \xi_6\} &= -\xi_3 \xi_6, \\
\{\xi_4, \xi_5\} &= -\xi_4 \xi_5, & \{\xi_4, \xi_6\} &= \xi_4 \xi_5 - 2 \xi_5, & \{\xi_5, \xi_6\} &= -2 \xi_5 \xi_6. \\
\end{align*}
\]

For another example, take \(w = w_0 = s_1 s_2 s_3 s_1 s_2 s_1\) and \(r_2 = (0, (s_1, s_3, s_2, s_3, s_1, s_2), 0)\). The Bott–Samelson parametrization \(\sigma^{r_2} = \sigma^{r_2}_{G/B} : \mathbb{C}^6 \rightarrow w_0 B^-B/B = B w_0 B/B\) is given by

\[
\sigma^{r_2}(z_1, \ldots, z_6) = \begin{pmatrix}
z_1 z_5 - z_3 & z_4 - z_1 z_6 & z_1 & -1 \\
z_5 & -z_6 & 1 & 0 \\
z_2 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix} \cdot B \in w_0 B^-B/B.
\]

In the \((z_1, \ldots, z_6)\) coordinates, the Poisson structure \(\pi_{G/B}\) is given by

\[
\begin{align*}
\{z_1, z_2\} &= 0, & \{z_1, z_3\} &= -z_1 z_3, & \{z_1, z_4\} &= -z_1 z_4, \\
\{z_1, z_5\} &= z_1 z_5 - 2 z_3, & \{z_1, z_6\} &= z_1 z_6 - 2 z_4, & \{z_2, z_3\} &= -z_2 z_3, \\
\{z_2, z_4\} &= z_2 z_4 - 2 z_3, & \{z_2, z_5\} &= -z_2 z_5, & \{z_2, z_6\} &= z_2 z_6 - 2 z_5, \\
\{z_3, z_4\} &= -z_3 z_4, & \{z_3, z_5\} &= -z_3 z_5, & \{z_3, z_6\} &= -z_3 z_5, \\
\{z_4, z_5\} &= 0, & \{z_4, z_6\} &= -z_4 z_6, & \{z_5, z_6\} &= -z_5 z_6.
\end{align*}
\]
Theorem B stated in Sect. 1.1 is the combination of Theorems 5.6 and 5.9.

In this appendix, we first prove Theorem 5.3 which says that for any $w$, $v \in W$,

$$J^w_{B(v)} : (wB^-B/B(v), \pi_{G/B(v)}) \longrightarrow \left( \mathcal{O}^{w_0w^{-1}} \times \mathcal{O}^{(w,v)}, \pi_{1,2} \right), \quad (A.1)$$

$$J^w_{N(v)} : (wB^-B/N(v), \pi_{G/N(v)}) \longrightarrow \left( \mathcal{O}^{w_0w^{-1}} \times \mathcal{O}^{(w,v)} \times T, \pi_{1,2,0} \right) \quad (A.2)$$

are Poisson isomorphisms, where $J^w_{B(v)}$ and $J^w_{N(v)}$ are given in (2.25) and (2.26), and the Poisson structures $\pi_{1,2}$ and $\pi_{1,2,0}$ are given in (5.2) and (5.3). Here recall that for $Q = B(v)$ or $N(v)$, $\pi_{G/Q}$ is the projection to $G/Q$ of the standard Poisson structure $\pi_{st}$ on $G$. In Sect. A.1, we review some facts on the Poisson Lie group $(G, \pi_{st})$. In Sect. A.2, we prove certain maps involved in the definitions of $J^w_{B(v)}$ and $J^w_{N(v)}$ are Poisson, and we use these facts to prove Theorem 5.3 in Sect. A.3. In Sect. A.4, we determine the $T$-leaves of $(G/Q, \pi_{G/Q})$. 

It is remarkable that such non-trivial changes of coordinates transform one symmetric Poisson CGL extension to another. 

5.4 Proofs of Theorems B and C

Theorem B stated in Sect. 1.1 is the combination of Theorems 5.6 and 5.9.

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Appendix A. Proof of Theorem 5.3 and $T$-leaves of $(G/Q, \pi_{G/Q})$

In this appendix, we first prove Theorem 5.3 which says that for any $v, w \in W$, 

$$J^w_{B(v)} : (wB^-B/B(v), \pi_{G/B(v)}) \longrightarrow \left( \mathcal{O}^{w_0w^{-1}} \times \mathcal{O}^{(w,v)}, \pi_{1,2} \right), \quad (A.1)$$

$$J^w_{N(v)} : (wB^-B/N(v), \pi_{G/N(v)}) \longrightarrow \left( \mathcal{O}^{w_0w^{-1}} \times \mathcal{O}^{(w,v)} \times T, \pi_{1,2,0} \right) \quad (A.2)$$

are Poisson isomorphisms, where $J^w_{B(v)}$ and $J^w_{N(v)}$ are given in (2.25) and (2.26), and the Poisson structures $\pi_{1,2}$ and $\pi_{1,2,0}$ are given in (5.2) and (5.3). Here recall that for $Q = B(v)$ or $N(v)$, $\pi_{G/Q}$ is the projection to $G/Q$ of the standard Poisson structure $\pi_{st}$ on $G$. In Sect. A.1, we review some facts on the Poisson Lie group $(G, \pi_{st})$. In Sect. A.2, we prove certain maps involved in the definitions of $J^w_{B(v)}$ and $J^w_{N(v)}$ are Poisson, and we use these facts to prove Theorem 5.3 in Sect. A.3. In Sect. A.4, we determine the $T$-leaves of $(G/Q, \pi_{G/Q})$. 

5.4 Proofs of Theorems B and C

Theorem B stated in Sect. 1.1 is the combination of Theorems 5.6 and 5.9.

Theorem C stated in Sect. 1.1 follows immediately from Theorems B and 4.12.
A.1 Some facts on the Poisson Lie group \((G, \pi_{\text{st}})\)

We first recall (see, for example, [16, 39]) that given a Poisson Lie group \((L, \pi)\) and a Poisson manifold \((X, \pi_X)\), a left action of \(L\) on \(X\) is said to be Poisson if the action map \((L, \pi) \times (X, \pi_X) \rightarrow (X, \pi_X)\) is Poisson. A Poisson manifold \((X, \pi_X)\) is called a Poisson homogeneous space [14] of a Poisson Lie group \((L, \pi)\) if \((X, \pi_X)\) has a Poisson action by \((L, \pi)\) which is also transitive.

**Example A.1** If \(M\) is a closed coisotropic subgroup (see Sect. 5.1) of a Poisson Lie group \((L, \pi)\), the action of \(L\) on \(L/M\) by left translation makes \(L/M, \pi_{L/M}\) a Poisson homogeneous space of \((L, \pi)\), where \(\pi_{L/M}\) is the projection to \(L/M\) of the Poisson structure \(\pi\) on \(L\). Note that as \(\pi(e) = 0\), \(\pi_{L/M}\) vanishes at \(e.M \in L/M\). In general, it is easy to see from the definitions that if \((X, \pi_X)\) is a Poisson homogeneous space of \((L, \pi)\) and if \(x \in X\) is such that \(\pi_X(x) = 0\), then the stabilizer subgroup \(L_x\) of \(L\) at \(x\) is a coisotropic subgroup of \((L, \pi)\), and the map

\[
(L/L_x, \pi_{L/L_x}) \longrightarrow (X, \pi_X) \quad l \longmapsto lx, \quad l \in L,
\]

is a Poisson isomorphism. \(\square\)

Returning to the Poisson Lie group \((G, \pi_{\text{st}})\), where \(\pi_{\text{st}}\) is given in (4.9), for any \(v \in W\) and \(Q = B(v)\) or \(N(v)\), the Poisson manifold \((G/Q, \pi_G/Q)\) is then a Poisson homogeneous space of \((G, \pi_{\text{st}})\).

We now recall a Drinfeld double of the Poisson Lie group \((G, \pi_{\text{st}})\): associated to the standard quasi-triangular \(r\)-matrix \(r_{\text{st}} \in g \otimes g\) in (4.8), one has the quasi-triangular \(r\)-matrix \(r_{\text{st}}^{(2)} \in (g \oplus g)^{\otimes 2}\) for the direct product Lie algebra \(g \oplus g\), given [39, Sect. 6.1] as (see Notation 4.13)

\[
r_{\text{st}}^{(2)} = (r_{\text{st}}, 0) - (0, r_{\text{st}}^{21}) - \sum_{q=1}^{d} (H_q, 0) \wedge (0, H_q) - \sum_{\alpha \in \Delta^+} (\alpha, \alpha)(e_\alpha, 0) \wedge (0, e_{-\alpha}), \quad (A.3)
\]

where \(r_{\text{st}}^{21} = \tau(r_{\text{st}})\) with \(\tau(x \otimes y) = y \otimes x\) for \(x, y \in g\), and for any vector space \(V\) and \(u, v \in V\), we use the convention that

\[
u \wedge v = u \otimes v - v \otimes u \in V \otimes V.
\]

Let \(\Lambda_{\text{st}}^{(2)} \in \wedge^2(g \oplus g)\) be the skew-symmetric part of \(r_{\text{st}}^{(2)}\), and let \(\Pi_{\text{st}}\) be the multiplicative Poisson structure on the product group \(G \times G\) given by

\[
\Pi_{\text{st}} = \left( r_{\text{st}}^{(2)} \right)^L \left( r_{\text{st}}^{(2)} \right)^R = \left( \Lambda_{\text{st}}^{(2)} \right)^L - \left( \Lambda_{\text{st}}^{(2)} \right)^R. \quad (A.4)
\]

Here, for \(A \in g^{\otimes k}\), \(A^L\) and \(A^R\) respectively denote the left and right invariant tensor fields on \(G\) with value \(A\) at the identity of \(G\). It follows from the definitions that

\[
\Pi_{\text{st}} = (\pi_{\text{st}}, 0) + (0, \pi_{\text{st}}) + \mu_1 + \mu_2, \quad (A.5)
\]
where

\[
\mu_1 = \sum_{q=1}^{d} (H^R_q, 0) \wedge (0, H^R_q) + \sum_{\alpha \in \Delta^+} \langle \alpha, \alpha \rangle (e^R_{\alpha}, 0) \wedge (0, e^R_{-\alpha}). \tag{A.6}
\]

\[
\mu_2 = - \sum_{q=1}^{d} (H^L_q, 0) \wedge (0, H^L_q) - \sum_{\alpha \in \Delta^+} \langle \alpha, \alpha \rangle (e^L_{\alpha}, 0) \wedge (0, e^L_{-\alpha}). \tag{A.7}
\]

The Poisson Lie group \((G \times G, \Pi_{st})\) is a Drinfeld double of the Poisson Lie group \((G, \pi_{st})\) (see, for example, [39, paragraph after Example 6.11]). In particular, the embedding

\[(G, \pi_{st}) \hookrightarrow (G \times G, \Pi_{st}), \quad g \mapsto (g, g), \quad g \in G,
\]

is Poisson, and the projections \((G \times G, \Pi_{st}) \rightarrow (G, \pi_{st})\) to both factors are Poisson.

It follows from (A.5), (A.6) and (A.7) that \(B \times B\) is a coisotropic subgroup of the Poisson Lie group \((G \times G, \Pi_{st})\). Let \(\varpi\) be the projection

\[
\varpi : G \times G \longrightarrow (G \times G)/(B \times B) \cong G/B \times G/B,
\]

and let \(\Pi = \varpi(\Pi_{st})\). It follows from (A.4) and (A.5) that

\[
\Pi = - \varpi \left( \left( \Lambda_{st}^{(2)} \right)^R \right) = (\pi_{G/B}, 0) + (0, \pi_{G/B}) + \varpi(\mu_1).
\]

Let \(G_{\text{diag}} = \{(g, g) : g \in G\} \subset G \times G\) and consider the \(G_{\text{diag}}\)-orbits in \(G/B \times G/B\), which are precisely of the form

\[
G_{\text{diag}}(v) \overset{\text{def}}{=} G_{\text{diag}}(e, B, \overline{v}, B) \subset G/B \times G/B, \quad v \in W.
\]

Note that for \(v \in W\), the stabilizer subgroup of \(G \cong G_{\text{diag}}\) at \((e, B, \overline{v}, B) \in G/B \times G/B\) is precisely \(B(v) = B \cap \overline{v}B\overline{v}^{-1}\).

**Lemma A.2** For each \(v \in W\), \(G_{\text{diag}}(v)\) is a Poisson submanifold of \(G/B \times G/B\) with respect to \(\Pi\), and the \(G\)-equivariant map

\[
(G/B(v), \pi_{G/B(v)}) \longrightarrow (G_{\text{diag}}(v), \Pi), \quad g.B(v) \mapsto (g.B, g\overline{v}.B), \quad g \in G. \tag{A.8}
\]

is a Poisson isomorphism.

**Proof** Let \(v \in W\). By [42, Theorem 2.3], \(G_{\text{diag}}(v)\) is a Poisson submanifold of \(G/B \times G/B\) with respect to \(\Pi\), and, as a \(G \cong G_{\text{diag}}\)-orbit, \((G_{\text{diag}}(v), \Pi)\) is a Poisson homogeneous space of \((G, \pi_{st})\). It is also easy to see that \(\pi_{G/B}(\overline{v}.B) = 0\) and \(\varpi(\mu_1)(e, B, \overline{v}, B) = 0\). It follows that \(\Pi(e, B, \overline{v}, B) = 0\). By Example A.1, the map in (A.8) is an isomorphism of Poisson homogeneous spaces of \((G, \pi_{st})\). \(\square\)
Recall the Poisson manifold \((F_2, \pi_2)\) from Sect. 4.4. By [39, Theorem 7.8] (see also [40, Proposition 5.6]), the map

\[
J_2 : (F_2, \pi_2) \longrightarrow (G/B \times G/B, \, \Pi), \; [g_1, g_2]_{F_2} \longmapsto (g_1B, g_1g_2B),
\]

(A.9)

is a Poisson isomorphism, with

\[
J_2^{-1} : (G/B \times G/B, \, \Pi) \longrightarrow (F_2, \pi_2), \; (h_1B, h_2B) \longmapsto [h_1, h_1^{-1}h_2]_{F_2}.
\]

(A.10)

Note that \(J_2\) is \(G\)-equivariant if \(F_2\) is given the \(G\)-action by

\[
g \cdot [g_1, g_2]_{F_2} = [gg_1, g_2]_{F_2}, \quad g, g_1, g_2 \in G.
\]

(A.11)

By Lemma A.2, we have the following interpretation of the Poisson homogeneous space \((G/B(v), \pi_{G/B(v)})\) as a Poisson submanifold in \((F_2, \pi_2)\).

**Lemma A.3** For any \(v \in W\), the map

\[
E_v : (G/B(v), \, \pi_{G/B(v)}) \longrightarrow (F_2, \pi_2), \; gB(v) \longmapsto [g, \, \overline{v}]_{F_2}, \quad g \in G,
\]

is a Poisson embedding.

### A.2 Some auxiliary Poisson morphisms

Recall that for any \(w \in W\), \(B^{-w}B/B\) is a Poisson submanifold of \((G/B, \pi_{G/B})\) (see [24, Theorem 1.5]), and recall that \(BwB\) is a Poisson submanifold of \((G, \pi_{st})\). Recall also from Sect. 2.3 that using the product decomposition \(\overline{w}N^\perp \overline{w}^{-1} = N_wN_w^{-} = N_w^N\overline{w}\), where again

\[
N_w = N \cap (\overline{w}N^\perp \overline{w}^{-1}) \quad \text{and} \quad N_w^\perp = N^- \cap (\overline{w}N^\perp \overline{w}^{-1}),
\]

every \(a \in \overline{w}N^\perp \overline{w}^{-1}\) can be uniquely written as

\[
a = a_+a_- = a'_+a'_- \quad \text{with} \quad a_+, a'_+ \in N_w, \quad a_-, a'_- \in N_w^-.
\]

(A.12)

**Lemma A.4** For any \(w \in W\), both maps

\[
p_w^1 : (wB^{-w}B, \, \pi_{st}) \longrightarrow (B^{-w}B/B, \, \pi_{G/B}), \; a\overline{wb} \longmapsto a_-\overline{w}B, \quad a \in \overline{w}N^\perp \overline{w}^{-1}, \; b \in B,
\]

\[
p_w^2 : (wB^{-w}B, \, \pi_{st}) \longrightarrow (BwB, \, \pi_{st}), \; a\overline{wb} \longmapsto a'_+\overline{w}b, \quad a \in \overline{w}N^\perp \overline{w}^{-1}, \; b \in B,
\]

are both Poisson, where \(a \in \overline{w}N^\perp \overline{w}^{-1}\) is decomposed as in (A.12). Furthermore, equip \(T\) with the zero Poisson structure. Then the map

\[
j_w \quad (wB^{-w}B, \, \pi_{st}) \longrightarrow (T, 0), \; a\overline{nt} \longmapsto t, \quad a \in \overline{w}N^\perp \overline{w}^{-1}, \; n \in N, \; t \in N,
\]
is also Poisson.

**Proof** Since the projection \((G, \pi_{st})\) to \((G/B, \pi_{G/B})\) is Poisson, to prove \(p_1^w\) is Poisson, it suffices to show that

\[
\tilde{p}_1^w : (wB^-B/B, \pi_{G/B}) \longrightarrow (B^-wB/B, \pi_{G/B}), \quad a\bar{w}.B \mapsto a_-.\bar{w}.B,
\]
is Poisson, where again \(a \in \overline{w}N^-\overline{w}^{-1}\) is decomposed as in (A.12). For \(g, h \in G\), let

\[
\sigma_g : G/B \longrightarrow G/B, \quad g'B \mapsto gg'B, \quad g' \in G,
\]

\[
\sigma_{h,B} : G \longrightarrow G/B, \quad h' \mapsto h'h.B, \quad h' \in G.
\]

Since the left action of \((G, \pi_{st})\) on \((G/B, \pi_{G/B})\) is Poisson, one has

\[
\tilde{p}_1^w(a\bar{w}.B) = \tilde{p}_1^w(a+a_-\bar{w}.B) = (\tilde{p}_1^w \circ \sigma_{a_+})(\pi_{G/B}(a_-\bar{w}.B)) + (\tilde{p}_1^w \circ \sigma_{a_-\bar{w}.B})(\pi_{st}(a_+)).
\]

Since \(B^-wB/B\) is a Poisson submanifold of \((G/B, \pi_{G/B})\), one has

\[
(\tilde{p}_1^w \circ \sigma_{a_+})(\pi_{G/B}(a_-\bar{w}.B)) = \pi_{G/B}(a_-\bar{w}.B).
\]

Since \(N_w\) is a coisotropic subgroup of \((G, \pi_{st})\), \((\tilde{p}_1^w \circ \sigma_{a_-\bar{w}.B})(\pi_{st}(a_+)) = 0\). Thus

\[
\tilde{p}_1^w(a\bar{w}.B) = \pi_{G/B}(a_-\bar{w}.B).
\]

This shows that \(\tilde{p}_1^w\) is Poisson. Similarly, using the multiplicativity of \(\pi_{st}\) and the fact that \(N_w^-\) is a coisotropic subgroup of \((G, \pi_{st})\) (which can be proved using a similar argument as that in the proof of [41, Lemma 10]), one shows that \(p_2^w\) is Poisson.

To show that \(j_w\) is Poisson, since that \(p_2^w\) is Poisson, it suffices to show that \(j_w' : (BwB, \pi_{st}) \longrightarrow (T, 0)\), \(j_w'(g't) = t\), \(g' \in N_w\bar{w}N, t \in T\), is Poisson. By [41, Lemma 10], both \(N_w\bar{w}\) and \(N\) are coisotropic submanifolds of \((G, \pi_{st})\). By the multiplicativity of \(\pi_{st}\), \(N_w\bar{w}N\) is also coisotropic with respect to \(\pi_{st}\). Writing \(g \in BwB\) uniquely as \(g = g't\), where \(g' \in N_w\bar{w}N\) and \(t \in T\), one has \(\pi_{st}(g) = r_t\pi_{st}(g')\). Hence \(j_w'((\pi_{st}(g))) = 0\). \(\square\)

For \(w \in W\) and \(Q = B(v)\) or \(N(v)\), recall from (2.19) the isomorphism

\[
I_Q^w : wB^-B/Q \longrightarrow (B^-wB/B) \times (BwB/Q) \subset (G/B) \times (G/Q), \quad a\bar{w}b.Q \mapsto (a_-\bar{w}.B, a_+\bar{w}b.Q), \quad a \in \overline{w}N^-\overline{w}^{-1}, b \in B,
\]

where, again, \(a \in \overline{w}N^-\overline{w}^{-1}\) is decomposed as in (A.12). Note that \(I_Q^w\) is \(T\)-equivariant, where \(T\) acts on both \(G/B\) and \(G/Q\) by left translation and on \(G/B \times G/Q\) diagonally. For \(\xi \in \mathfrak{h} = \text{Lie}(T)\), let \(\rho_{G/Q}(\xi)\) be the vector field on \(G/Q\) given by

\[
\rho_{G/Q}(\xi)(g.Q) = \frac{d}{dt}|_{t=0} \exp(t\xi)g.Q, \quad g \in G.
\]

(A.13)
Let again \( \{ H_q \}_{q=1}^d \) be a basis of \( \mathfrak{h} \) that is orthonormal with respect to \( \langle \cdot, \cdot \rangle \). Introduce the mixed product Poisson structure \( \pi_{G/B} \triangleright \mu_0 \pi_{G/Q} \) (see Sect. 4.2) on \( (G/B) \times (G/Q) \) by

\[
\pi_{G/B} \triangleright \mu_0 \pi_{G/Q} = (\pi_{G/B}, 0) + (0, \pi_{G/Q}) + \mu_0, \quad \text{where}
\]

\[
\mu_0 = \sum_{q=1}^d (\rho_{G/B}(H_q), 0) \wedge (0, \rho_{G/Q}(H_q)).
\]

It follows from the definition of \( \pi_{G/B} \triangleright \mu_0 \pi_{G/Q} \) that \( (B^{-w}B/B) \times (BwB/Q) \) is a Poisson submanifold of \( ((G/B) \times (G/Q), \pi_{G/B} \triangleright \mu_0 \pi_{G/Q}) \).

**Lemma A.5** For every \( w \in W \), the map

\[
I^w_Q : (wB^{-B}/Q, \pi_{G/Q}) \longrightarrow ((B^{-w}B/B) \times (BwB/Q), \pi_{G/B} \triangleright \mu_0 \pi_{G/Q}) \quad (A.14)
\]

is a Poisson isomorphism.

**Proof** Since \( \pi_{G/Q} \) is a quotient Poisson structure, \( I^w_Q \) is Poisson as long as \( I^w_{[e]} \) is Poisson. Assume thus \( Q = [e] \) and note that in this case \( G/Q = G \) so \( \pi_{G/Q} = \pi_{\text{st}} \). Consider the open submanifold \( (wB^{-B}) \times (wB^{-B}) \) of \( G \times G \) and the map

\[
D_w : (wB^{-B}) \times (wB^{-B}) \longrightarrow (B^{-w}B/B) \times (BwB),
\]

\[
(a\overline{w}b_1, c\overline{w}b_2) \longmapsto (a_{-}\overline{w}B, c'_{+}\overline{w}b_2),
\]

where \( a, c \in \overline{w}N^{-1}, b_1, b_2 \in B \), and \( a = a_+a_- \) and \( c = c'_-c'_+ \) with \( a_+, c'_+ \in N_w \) and \( a_-, c'_- \in N_{\overline{w}} \). We first prove that

\[
D_w : \left( (wB^{-B}) \times (wB^{-B}), \pi_{\text{st}} \right) \longrightarrow \left( (B^{-w}B/B) \times (BwB), \pi_{G/B} \triangleright \mu_0 \pi_{\text{st}} \right) \quad (A.15)
\]

is Poisson. Indeed, recall that \( \pi_{\text{st}} = (\pi_{\text{st}}, 0) + (0, \pi_{\text{st}}) + \mu_1 + \mu_2 \), where \( \mu_1 \) and \( \mu_2 \) are respectively given in (A.6) and (A.7). By Lemma A.4, one has

\[
D_w((\pi_{\text{st}}, 0) + (0, \pi_{\text{st}})) = (\pi_{G/B}, 0) + (0, \pi_{\text{st}}).
\]

By the definition of \( D_w \), \( D_w(x^L, 0) = 0 \) for all \( x \in \mathfrak{b} = \text{Lie}(B) \). Thus \( D_w(\mu_2) = 0 \). It also follows from the definition of \( D_w \) that for \( \alpha \in \Delta^+ \), if \( w^{-1}(\alpha) \in -\Delta^+ \), then \( D_w((e^R_{\alpha}, 0)) = 0 \), and if \( w^{-1}(\alpha) \in \Delta^+ \), then \( D_w((0, e^R_{-\alpha})) = 0 \). Consequently,

\[
D_w(\mu_1) = D_w \left( \sum_{q=1}^d (H^R_q, 0) \wedge (0, H^R_q) \right) = \mu_0.
\]

This shows that \( D_w \) in (A.15) is Poisson. As the diagonal embedding \( (wB^{-B}, \pi_{\text{st}}) \to (B^{-w}B/B) \times (BwB), \pi_{\text{st}} \) is Poisson, we see that \( I^w_{[e]} \) is Poisson. \( \square \)
We now turn to the isomorphism $\zeta^w : B^{-w}B/B \to O^{w_0w^{-1}}$, for $w \in W$, given in (2.22). Recall also that $t^u = \overline{u}^{-1}\overline{t}\overline{u} \in T$ for $t \in T$ and $u \in W$.

**Lemma A.6** For any $w \in W$, the map

$$\zeta^w : (B^{-w}B/B, \pi_{G/B}) \to (O^{w_0w^{-1}}, \pi_{G/B}), \quad m\overline{w}.B \mapsto (\overline{w_0w^{-1}}m^{-1}).B, \quad m \in N_w^-,$$

is a $T$-equivariant Poisson isomorphism, where $t \in T$ acts on $B^{-w}B/B$ by left translation by $t$ and on $O^{w_0w^{-1}}$ by left translation by $t^{uw_0}$.

**Proof** Let $u = w_0w^{-1}$ so that $\overline{u} = \overline{w_0}\overline{w}^{-1}$. It follows from $N_u = \overline{u}N_w^-\overline{u}^{-1}$ that $\zeta^w$ is a well-defined $T$-equivariant isomorphism with the $T$-actions on both sides as described. To show that $\zeta^w$ is a Poisson isomorphism, consider the two Poisson isomorphisms

$$\begin{align*}
\zeta_1 & : (G/B, \pi_{G/B}) \to (G/B^-, \pi_{G/B^-}), \quad \zeta_1(g.B) = g(\overline{w_0})^{-1}.B^-, \quad g \in G, \\
\zeta_2 & : (G/B^-, \pi_{G/B^-}) \to (B^-\backslash G, -\pi_{B^\cdot \backslash G}), \quad \zeta_2(g.B^-) = B^-g^{-1}, \quad g \in G,
\end{align*}$$

where $\pi_{G/B^-}$ and $\pi_{B^\cdot \backslash G}$ respectively denote the Poisson structures on $G/B^-$ and $B^\cdot \backslash G$ that are projections of $\pi_{st}$ on $G$. The restriction of the composition $\zeta_2 \circ \zeta_1$ to $(B^{-w}B/B, \pi_{G/B}) \subset (G/B, \pi_{G/B})$ gives the Poisson isomorphism

$$\zeta_3 : (B^{-w}B/B, \pi_{G/B}) \to (B^-\backslash B^-uB^-, -\pi_{B^\cdot \backslash G}), \quad \zeta_3(m\overline{w}.B) = B^-\overline{w_0}(m\overline{w})^{-1} = B^-\overline{w}m^{-1}, \quad m \in N_w^-.$$

Note that $\overline{u}N_w^- = N\overline{u} \cap \overline{u}N_w^-$. By [41, Lemma 14], the map

$$\zeta_\overline{w} : (B^-\backslash B^-uB^-, -\pi_{B^\cdot \backslash G}) \to (BuB/B, \pi_{G/B}), \quad \zeta_\overline{w}(B^-x) = x.B, \quad x \in N\overline{u} \cap \overline{u}N_w^-,$$

is a Poisson isomorphism. As $\zeta^w = \zeta_\overline{w} \circ \zeta_3$, we see that $\zeta^w$ is a Poisson isomorphism. □

For $u, w \in W$, we now relate $(BwB/N(v), \pi_{G/N(v)})$ and $(BwB/B(v), \pi_{G/B(v)})$. Let

$$K^u_v : BwB/N(v) \to (BwB/B(v)) \times T, \quad n_1\overline{w}n_2t.N(v) \mapsto (n_1\overline{w}n_2.B(v), t), \quad (A.16)$$

where $n_1 \in N_w, n_2 \in N_v$, and $t \in T$. It is clear that $K^u_v$ is a $T$-equivariant isomorphism, where $t_1 \in T$ acts on $BwB/N(v)$ by left translation by $t_1$ and on $(BwB/B(v)) \times T$ by

$$t_1 \cdot (g.B(v), t) = (t_1g.B(v), t_1^w), \quad t_1, t \in T, \quad g \in G.$$
Define the bi-vector field $\pi_{G/B(v)} \triangleright \mu'$ 0 on $(BwB/B(v)) \times T$ by

$$\pi_{G/B(v)} \triangleright \mu' 0 = (\pi_{G/B(v)}, 0) + \mu', \quad \text{where}$$

$$\mu' = -\sum_{q=1}^{d} (\rho_{G/B(v)}(w(H_q)), 0) \wedge (0, H_q^R).$$  \hspace{1cm} (A.17)

**Lemma A.7** For any $w, v \in W$,

$$K^w_v : (BwB/N(v), \pi_{G/N(v)}) \longrightarrow ((BwB/B(v)) \times T, \pi_{G/B(v)} \triangleright \mu' 0)$$

is a Poisson isomorphism.

**Proof** Since the projections

$$\begin{align*}
(BwB, \pi_{st}) &\longrightarrow (BwB/N(v), \pi_{G/N(v)}), \\
(BwB/T, \pi_{G/T}) &\longrightarrow (BwB/B(v), \pi_{G/B(v)})
\end{align*}$$

are $T$-equivariant and Poisson, it is enough to prove Lemma A.7 for $v = w_0$, i.e., to prove that

$$K := K^w_{w_0} : (BwB, \pi_{st}) \longrightarrow ((BwB/T) \times T, \pi_{G/T} \triangleright \mu' 0)$$

is Poisson. Consider again $G \times G$ with the multiplicative Poisson structure $\Pi_{st}$ from (A.4). By (A.5), $(BwB) \times G$ is a Poisson submanifold of $(G \times G, \Pi_{st})$. Define

$$K' : BwB \times G \longrightarrow (G/T) \times T, \quad (g't, g) \longmapsto (gT, t),$$

$$g' \in N\overline{w}N, \quad g \in G, \quad t \in T.$$  

Since $K$ is the composition of $K'$ with the diagonal Poisson embedding $(BwB, \pi_{st}) \hookrightarrow (BwB \times G, \Pi_{st})$, $K$ is Poisson once we prove that

$$K' : (BwB \times G, \Pi_{st}) \longrightarrow ((G/T) \times T, (\pi_{G/T}, 0) + \mu')$$

is Poisson. Recall again that $\Pi_{st} = (\pi_{st}, 0) + (0, \pi_{st}) + \mu_1 + \mu_2$, with $\mu_1$ and $\mu_2$ respectively given in (A.6) and (A.7). By Lemma A.4, $K'(\pi_{st}, 0) = 0$. By the definition of $\pi_{G/T}$, $K'(0, \pi_{st}) = (\pi_{G/T}, 0)$. Thus

$$K'(\pi_{st}, 0) + (0, \pi_{st}) = (\pi_{G/T}, 0).$$

It follows from the definitions that $K'(e^L_0, 0) = K'(e^R_0, 0) = 0$ for all $\alpha \in \Delta^+$ and that $K'(0, x^L_0) = 0$ for $x \in \mathfrak{h}$. Furthermore, it follows from the definition of $K'$ that

$$K'(x^R, 0) = (0, (w^{-1}(x))^R) \quad \text{and} \quad K'(0, x^R) = (\rho_{G/T}(x), 0), \quad x \in \mathfrak{h}.$$
The fact that $K'$ is Poisson now follows from

\[
K'(\mu_1 + \mu_2) = K' \left( \sum_{q=1}^{d} (H_q^R, 0) \right) = \sum_{q=1}^{d} (0, (w^{-1}(H_q))^R) \wedge (\rho_{G/T}(H_q), 0) = \mu'.
\]

For $v, w \in W$, recall from (2.20) and (2.21) the isomorphisms

\[
\zeta^{(w,v)}_{B(v)} : BwB/B(v) \longrightarrow \mathcal{O}^{(w,v)}, \quad n_1 \overline{w}n_2.B(v) \longmapsto [n_1 \overline{w}, n_2 \overline{v}]_{F_2},
\]

\[
\zeta^{(w,v)}_{N(v)} : BwB/N(v) \longrightarrow \mathcal{O}^{(w,v)} \times T, \quad n_1 \overline{w}n_2.T(N(v)) \longmapsto ([n_1 \overline{w}, n_2 \overline{v}]_{F_2}, t),
\]

where $n_1 \in N_w, n_2 \in N_v$ and $t \in T$. Note that $\zeta^{(w,v)}_{B(v)}$ and $\zeta^{(w,v)}_{N(v)}$ are $T$-equivariant, where $t \in T$ acts on $BwB/B(v)$ and $BwB/N(v)$ by left translation by $t_1$ and on $\mathcal{O}^{(w,v)} \subset F_2$ and on $\mathcal{O}^{(w,v)} \times T$ respectively by (see (2.10))

\[
t_1 \cdot [n_1 \overline{w}, n_2 \overline{v}]_{F_2} = [t_1 n_1 \overline{w}, n_2 \overline{v}]_{F_2},
\]

\[
t_1 \cdot ([n_1 \overline{w}, n_2 \overline{v}]_{F_2}, t) = ([t_1 n_1 \overline{w}, n_2 \overline{v}]_{F_2}, t^w_1 t), \quad n_1 \in N_w, n_2 \in N_v, t \in T.
\]

Equip $\mathcal{O}^{(w,v)}$ with the standard Poisson structure $\pi_2$ given in (4.10), and let $\mu''$ be the bi-vector field on $\mathcal{O}^{(w,v)} \times T$ by

\[
\mu'' = - \sum_{q=1}^{d} (\rho_2(w(H_q)), 0) \wedge (0, H_q^R), \tag{A.18}
\]

where $\rho_2$ is defined in (5.1).

**Lemma A.8** The maps

\[
\zeta^{(w,v)}_{B(v)} : (BwB/B(v), \pi_{G/B(v)}) \longrightarrow (\mathcal{O}^{(w,v)}, \pi_2),
\]

\[
\zeta^{(w,v)}_{N(v)} : (BwB/N(v), \pi_{G/N(v)}) \longrightarrow (\mathcal{O}^{(w,v)} \times T, (\pi_2, 0) + \mu'')
\]

are Poisson isomorphisms.

**Proof** As $BwB/B(v)$ is a Poisson submanifold of $(G/B(v), \pi_{G/B(v)})$, by Lemma A.3, one has the $T$-equivariant Poisson embedding

\[
(BwB/B(v), \pi_{G/B(v)}) \longrightarrow (F_2, \pi_2), \quad n \overline{w}b.B(v) \longmapsto [n \overline{w}, \overline{v}]_{F_2} = [n \overline{w}, b \overline{v}]_{F_2},
\]

where $n \in N_w$ and $b \in B$. Since the image of the above embedding is precisely $\mathcal{O}^{(w,v)}$, we see that the $\zeta^{(w,v)}_{B(v)}$ is a Poisson isomorphism. The fact that $\zeta^{(w,v)}_{N(v)} = (\zeta^{(w,v)}_{B(v)} \times \text{Id}) \circ K^{(w)}_v$ is a Poisson isomorphism follows directly from Lemma A.7. \qed
A.3 Proof of Theorem 5.3

Let again $w, v \in W$, and let the notation be as in the statement of Theorem 5.3. First let $Q = B(v)$. By Lemmas A.5, A.6, and A.8, one has the Poisson isomorphisms

$$
(wB^−B/B(v), \pi_{G/B(v)}) \xrightarrow{\xi_w^B(B(v))} ((B^−wB/B) \times (BwB/B(v)), \pi_{G/B} \Join \mu_0 \pi_{G/B(v)})
$$

and

$$
\xi_w^B(B(v)) \quad \xrightarrow{\circ \mu_0 \pi_{G/B(v)}} \quad (\mathcal{O}w_0w^{-1} \times \mathcal{O}^{(w,v)}, \pi_{1,2}).
$$

(A.19)

Since $J^w_B(v) = (\xi^w \times \xi^{(w,v)}_{B(v)}) \circ I^w_B(v)$, one sees that

$$
J^w_{N(v)} : (wB^−B/B(v), \pi_{G/B(v)}) \longrightarrow (\mathcal{O}w_0w^{-1} \times \mathcal{O}^{(w,v)}, \pi_{1,2})
$$

is Poisson. To see that $J^w_{N(v)}$ is Poisson, note that

$$
J^w_{N(v)} = (\Id \times \xi^{(w,v)}_{N(v)}) \circ (\xi^w \times \Id) \circ I^w_{N(v)}.
$$

By Lemmas A.5 and A.6,

$$
((\xi^w \times \Id) \circ I^w_{N(v)}) (\pi_{G/N(v)}) = (\pi_1, 0) + (0, \pi_{G/N(v)}) + \mu'_0,
$$

where $\mu'_0 = \sum_{q=1}^d (\rho_1(w_0w^{-1}(H_q)), 0) \wedge (0, \rho_{G/N(v)}(H_q))$. By Lemma A.8 and the $T$-equivariance of $\xi^{(w,v)}_{N(v)}$, one has

$$
(\Id \times \xi^{(w,v)}_{N(v)})(\pi_1, 0) + (0, \pi_{G/N(v)}) + \mu'_0 = (\pi_1, 0, 0) + (0, \pi_2, 0) + (0, \mu'')
$$

$$
+ \sum_{q=1}^d ((\rho_1(w_0w^{-1}(H_q)), 0, 0) \wedge ((0, \rho_2(H_q), 0) + (0, 0, (w^{-1}(H_q))^R)))
$$

$$
= (\pi_1, 0, 0) + (0, \pi_2, 0) + \mu_{23} + \mu_{12} + \mu_{13} = \pi_{1,2,0}.
$$

It follows that $J^w_{N(v)} : (wB^−B/N(v), \pi_{G/N(v)}) \rightarrow (\mathcal{O}w_0w^{-1} \times \mathcal{O}^{(w,v)} \times T, \pi_{1,2,0})$ is Poisson. This finishes the proof of Theorem 5.3.

A.4 T-leaves of $(G/Q, \pi_{G/Q})$

For any Poisson variety $(X, \pi_X)$ with an action by an algebraic $\mathbb{C}$-torus $\mathbb{T}$ via Poisson isomorphisms, define (see [40]) the $\mathbb{T}$-leaf of $\pi_X$ through $x \in X$ to be $\mathbb{T}\Sigma_x = \bigcup_{t \in \mathbb{T}} t\Sigma_x$, where $\Sigma_x$ is the symplectic leaf of $\pi_X$ through $x$. For the $T$-Poisson variety $(G/Q, \pi_{G/Q})$, where $Q = B(v)$ or $N(v)$ for $v \in W$ and $T$ acts on $G/Q$ by left translation, we now determine the $T$-leaves of $(G/Q, \pi_{G/Q})$. 
Let \( \sigma_{G/Q} : G \to G/Q \) be the projection. Recall the monoidal product \( * \) on \( W \) determined by \( w*s_\alpha = w_\alpha \) if \( l(w_\alpha) = l(w)+1 \) and \( w*s_\alpha = w \) if \( l(w_\alpha) = l(w)-1 \).

**Theorem A.9** For \( v \in W \) and \( Q = B(v) \) or \( N(v) \), the \( T \)-leaves of \((G/Q, \pi_{G/Q})\) are precisely the subvarieties

\[
L_{w,y}^{G/Q} \overset{\text{def}}{=} \sigma_{G/Q} \left( (BwB) \cap B^{-1}yB^{-1} \right) \subset G/Q,
\]

where \( y, w \in W \) and \( y \leq w * v \).

**Proof** Consider first the case of \( Q = B(v) \) and recall from Lemma A.3 the \( T \)-equivariant Poisson embedding

\[
E_v : (G/B(v), \pi_{G/B(v)}) \longrightarrow (F_2, \pi_2), \quad gB(v) \longmapsto [g, \nu]_{F_2}, \quad g \in G,
\]

where \( T \) acts on \( F_2 \) by (2.10). It follows from the Bruhat decomposition \( G = \bigsqcup_{w \in W} BwB \) that the image of \( E_v \) is given by

\[
E_v(G/B(v)) = \bigsqcup_{w \in W} O^{(w,v)} \subset F_2.
\]

The \( T \)-leaves of \((F_r, \pi_r)\), for any \( r \geq 1 \), are determined in [40, Theorem 1.1]. For the case of \( r = 2 \) at hand, let

\[
\mu_{F_2} : F_2 \longrightarrow G/B, \quad [g_1, g_2]_{F_2} \longrightarrow g_{1}g_{2}B.
\]

By [40, Theorem 1.1], the \( T \)-leaves of \((F_2, \pi_2)\) are precisely the intersections

\[
R_{y}^{(w,x)} \overset{\text{def}}{=} O^{(w,x)} \cap \mu_{F_2}^{-1}(B^{-1}yB/B),
\]

where \( w, x, y \in W \) and \( y \leq w * x \). Thus, for each \( w \in W \), \( O^{(w,v)} \subset F_2 \) is a union of all the \( T \)-leaves \( R_{y}^{(w,x)} \) with \( y \leq w * v \). It is easy to see that \( L_{w,y}^{G/B(v)} = E_v^{-1}(R_{y}^{(w,v)}) \) for all \( w, y \in W \) with \( y \leq w * v \) Thus the \( L_{w,y}^{G/B(v)} \)'s are precisely all the \( T \)-leaves of \((G/B(v), \pi_{G/B(v)})\).

Consider now \( Q = N(v) \) and the decomposition \( G/N(v) = \bigsqcup_{w \in W} BwB/N(v) \), where each \( BwB/N(v) \) is a \( T \)-invariant Poisson submanifold with respect to \( \pi_{G/N(v)} \). Let \( w \in W \) and recall from Lemma A.7 the \( T \)-equivariant Poisson isomorphism

\[
K_{v}^{w} : (BwB/N(v), \pi_{G/N(v)}) \longrightarrow ((BwB/B(v)) \times T, \pi_{G/B(v)} \triangleright \mu', 0),
\]

where \( t_1 \in T \) acts on \((BwB/B(v)) \times T \) by

\[
t_1 \cdot (gB(v), t) = (t_1gB(v), t_1^w t), \quad t_1, t \in T, \quad g \in G.
\]
By [38, Lemma 2.23], the \( T \)-leaves of \( ((BwB/B(v)) \times T, \pi_{G/B(v)} \triangleright_{\mu', 0}) \) are precisely of the form \( L \times T \), where \( L \) is a \( T \)-leaf of \( (BwB/B(v), \pi_{G/B(v)}) \). Applying the \( T \)-equivariant Poisson isomorphism \( K^w_v \), one sees that the \( T \)-leaves of \( (BwB/N(v), \pi_{G/N(v)}) \) are precisely \( L_{w,y}^{G/N(v)} \), where \( y \in W \) and \( y \leq w \ast v \). It follows that \( L_{w,y}^{G/N(v)} \), where \( w, y \in W \) and \( y \leq w \ast v \), are all the \( T \)-leaves of \( (G/N(v), \pi_{G/N(v)}) \).

\[ \square \]

**Example A.10** When \( v = w_0 \), so that \( G/N(v) = G \), one has \( w \ast w_0 = w_0 \) for every \( w \in W \), so the condition \( y \leq w \ast w_0 \) is satisfied for every \( y \in W \), and one has \( B^{-y}Bw_0^{-1} = B^{-y}w_0B^{-} \). In this case, Theorem A.9 recovers the well-known result [32,33,36] that the \( T \)-leaves (for the \( T \)-action on \( G \) by left translation) of \( (G, \pi_{st}) \) are precisely all the double Bruhat cells \( G^w_u = BwB \cap B^{-u}B^{-} \), where \( w, u \in W \).

When \( v = e \), so that \( G/B(v) = G/B \), Theorem A.9 recovers the well-known result from [24] that the \( T \)-leaves of \( (G/B, \pi_{G/B}) \) are precisely the open Richardson varieties \( (BwB/B) \cap (B^{-y}B/B) \), where \( w, y \in W \) and \( y \leq w \).

The next examples shows that for any \( w, v \in W \), the double Bruhat cell \( G^{w,v^{-1}} \), as a \( T \)-leaf of \( (G, \pi_{st}) \), can also be embedded into \( G/N(v) \) as a \( T \)-leaf of \( (G/N(v), \pi_{G/N(v)}) \).

**Example A.11** For an arbitrary \( v \in W \), consider the \( T \)-leaf

\[ L_{w,e}^{G/N(v)} = \sigma_{G/N(v)}((BwB) \cap B^{-}Bv^{-1}) \]

of \( (G/N(v), \pi_{G/N(v)}) \). Recall from Lemma 3.7 the embedding

\[ \delta_v : B^{-v^{-1}}B^{-} \longrightarrow G/N(v), \ g \longmapsto gN(v). \]

For \( w \in W \), denote the restriction of \( \delta_v \) to \( G^{w,v^{-1}} = BwB \cap B^{-v^{-1}}B^{-} \) by

\[ \delta_{w,v} = \delta_v|_{G^{w,v^{-1}}} : G^{w,v^{-1}} \longrightarrow G/N(v). \] (A.20)

It then follows that the image of \( \delta_{w,v} \) is precisely the \( T \)-leaf \( L_{w,e}^{G/N(v)} \) of \( (G/N(v), \pi_{G/N(v)}) \). As \( G^{w,v^{-1}} \) is a \( T \)-leaf of \( (G, \pi_{st}) \), we conclude that

\[ \delta_{w,v} : (G^{w,v^{-1}}, \pi_{st}) \sim (L_{w,e}^{G/N(v)}, \pi_{G/N(v)}) \]

is a Poisson isomorphism of \( T \)-leaves.

\[ \square \]

**References**

1. Alekseev, A., Berenstein, A., Hoffman, B., Li, Y.: Poisson structures and potentials. In: Kac, V., Papov, V. (eds.) Lie Groups, Geometry, and Representation Theory, Progress in Mathematics, vol. 326, pp. 1–40. Springer, Berlin (2018)
2. Alekseev, A., Berenstein, A., Hoffman, B., Li, Y.: Langlands duality and Poisson–Lie duality via cluster theory and tropicalization. arXiv:1806.04104
3. Arnold, V.: Ordinary Differential Equations, translation from Russian by Roger Cooke. Springer, Berlin (1992)
4. Belavin, A., Drinfeld, V.: Triangle equations and simple Lie algebras. Soviet Sci. Rev. Sect. C: Math. Phys. Rev. 4, 93–165 (1984)
5. Berenstein, A., Zelevinsky, A.: Total positivity in Schubert varieties. Comment. Math. Helv. 72, 128–166 (1997)
6. Berenstein, A., Fomin, S., Zelevinsky, A.: Parametrizations of canonical bases and totally positive matrices. Adv. Math. 122, 49–149 (1996)
7. Berenstein, A., Kazhdan, D.: Geometric and unipotent crystals II: from unipotent bicrystals to crystal bases. In: Quantum Groups, Contemp. Math., vol. 433, pp. 13–88. Amer. Math. Soc., Providence (2007)
8. Berenstein, A., Zelevinsky, A.: Tensor product multiplicities, canonical bases and totally positive varieties. Invent. Math. 143(1), 77–128 (2001)
9. Bogomolov, F., Böhning, C.: On uniformly rational varieties. In: Topology, Geometry, Integrable Systems, and Mathematical Physics. Amer. Math. Soc. Transl., vol. 234 (2), pp. 33–48. Amer. Math. Soc., Providence (2014)
10. Brown, K., Goodearl, K.: Lectures on Algebraic Quantum Groups. Advanced Courses in Mathematics CRM Barcelona. Birkhäuser, Basel (2002)
11. Chari, V., Pressley, A.: A Guide to Quantum Groups. Cambridge University Press, Cambridge (1994)
12. Drinfeld, V.: Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang–Baxter equations. Sov. Math. Dokl. 27(1), 68–71 (1982)
13. Drinfeld, V.: Quantum groups. Proceedings of the International Congress of Mathematicians, 1 (2) (Berkeley, Calif., 1986), pp. 798–820. Amer. Math. Soc., Providence (1987)
14. Drinfeld, V.: On Poisson homogeneous spaces of Poisson–Lie groups. Theo. Math. Phys. 95, 226–227 (1993)
15. Elek, B., Lu, J.-H.: Bott–Samelson varieties and Poisson–Ore extensions. Int. Math. Res. Not. (2019). https://doi.org/10.1093/imrn/rnz127
16. Etingof, P., Shiffmann, O.: Lectures on Quantum Groups, 2nd edn. International Press, New York (2002)
17. Fock, V., Goncharov, A.: Moduli spaces of local systems and higher Teichmüller theory. Publ. Math. IHES 103, 1–211 (2006)
18. Fomin, S.: Total positivity and cluster algebras. Proceedings of the ICM. vol. II, pp. 125–145, Hindustan Book Agency, New Delhi (2010)
19. Fomin, S., Zelevinsky, A.: Double Bruhat cells and total positivity. J. Am. Math. Soc. 12, 335–380 (1999)
20. Fomin, S., Zelevinsky, A.: Total positivity: tests and parametrizations. Math. Intell. 22, 23–33 (2000)
21. Fomin, S., Zelevinsky, A.: Totally nonnegative and oscillatory elements in semisimple groups. Proc. AMS 128(12), 3749–3759 (2000)
22. Gekhtman, M., Shapiro, M., Vainshtein, A.: Cluster algebras and Poisson geometry. In: Mathematical Surveys and Monographs. The AMS, 167 (2010)
23. Goodearl, K., Warfield, R.: An Introduction to Noncommutative Noetherian Rings. London Mathematical Society Student Text, vol. 61, 2nd edn. Cambridge University Press, Cambridge (2004)
24. Goodearl, K., Yakimov, M.: Poisson structures on affine spaces and flag varieties, II, the general case. Trans. Am. Math. Soc. 361(11), 5753–5780 (2009)
25. Goodearl, K., Yakimov, M.: Quantum cluster algebras and quantum nilpotent algebras. Proc. Natl. Acad. Sci. USA 111(27), 9696–9703 (2014)
26. Goodearl, K., Yakimov, M.: The Berenstein-Zelevinsky quantum cluster algebra conjecture. J. Eur. Math. Soc. (JEMS) 22(8), 2453–2509 (2020). https://doi.org/10.4171/JEMS/969
27. Goodearl, K., Yakimov, M.: Quantum cluster algebra structures on quantum nilpotent algebras. Mem. Am. Math. Soc. 247(1169), 5753–5780 (2017)
28. Goodearl, K., Yakimov, M.: Cluster algebras on nilpotent Poisson algebras. arXiv:1801.01963v2
29. Goodearl, K., Yakimov, M.: Cluster structures on double Bruhat cells (in preparation)
30. Grothendieck, A.: EGA, IV.Publ. Math. IHES, 32 (1967)
31. He, X., Knutson, A., Lu, J.-H.: Bruhat atlases (in preparation)
32. Hodges, T., Levasseur, T.: Primitive ideals of $C_q[SL(3)]$. Commun. Math. Phys. 156(3), 581–605 (1993)
33. Hoffmann, T., Kellendonk, J., Kutz, N., Reshetikhin, N.: Factorization dynamics and Coxeter–Toda lattices. Commun. Math. Phys. 212(2), 297–321 (2000)
34. Knutson, A., Woo, A., Yong, A.: Singularities of Richardson varieties. Math. Res. Lett. 20(02), 391–400 (2013)
35. Kazhdan, D., Lusztig, G.: Representations of Coxeter groups and Hecke algebras. Invent. Math. 53, 165–184 (1979)
36. Kogan, M., Zelevinsky, A.: On symplectic leaves and integrable systems in standard complex semisimple Poisson Lie groups. Int. Math. Res. Not. 32, 1685–1703 (2002)
37. Lu, J.-H.: Cluster open covers of some homogeneous spaces (in preparation)
38. Lu, J.-H., Mi, Y.: Generalized Bruhat cells and completeness of Hamiltonian flows of Kogan–Zelevinsky integrable systems. In: Kac, V., Papov, V. (eds.) Lie Groups, Geometry, and Representation Theory. Progress in Mathematics, vol. 326, pp. 315–365. Springer, Berlin (2018)
39. Lu, J.-H., Mouquin, V.: Mixed product Poisson structures associated to Poisson Lie groups and Lie bialgebras. Int. Math. Res. Not. 19, 5919–5976 (2017)
40. Lu, J.-H., Mouquin, V.: On the $T$-leaves of some Poisson structures related to products of flag varieties. Adv. Math. 306, 1209–1261 (2017)
41. Lu, J.-H., Mouquin, V.: Double Bruhat cells and symplectic groupoids. Trans. Groups 23(3), 765–800 (2018)
42. Lu, J.-H., Yakimov, M.: Group orbits and regular partitions of Poisson manifolds. Commun. Math. Phys. 283(3), 729–748 (2008)
43. Lusztig, G.: Total positivity in reductive groups. In: Lie Theory and Geometry: In Honor of Bertram Kostant, Progr. in Math., vol. 123, pp. 531–568. Birkhäuser, Boston (1994)
44. Lusztig, G.: Introduction to total positivity. In: Positivity in Lie Theory: Open Problems. de Gruyter Expositions in Mathematics, vol. 26, pp. 133–145. de Gruyter, Berlin (1998)
45. Lusztig, G.: A survey of total positivity. Milan J. Math. 76, 125–134 (2008)
46. Oh, S.: Poisson polynomial rings. Commun. Algebra 34(4), 1265–1277 (2006)
47. Yu, S.: On the Knutson–Woo–Yong maps and some Poisson homogeneous spaces. Ph.D. Thesis, the University of Hong Kong (2018)

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