Weak-disorder limit at criticality for directed polymers on hierarchical graphs

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Abstract

We prove a distributional limit theorem conjectured in [Journal of Statistical Physics 174, No. 6, 1372-1403 (2019)] for partition functions defining models of directed polymers on diamond hierarchical graphs with disorder variables placed at the graphical edges. The limiting regime involves a joint scaling in which the number of hierarchical layers, \( n \in \mathbb{N} \), of the graphs grows as the inverse temperature, \( \beta \equiv \beta(n) \), vanishes with a fine-tuned dependence on \( n \). The conjecture pertains to the marginally relevant disorder case of the model wherein the branching parameter \( b \in \{2, 3, \ldots\} \) and the segmenting parameter \( s \in \{2, 3, \ldots\} \) determining the hierarchical graphs are equal, which coincides with the diamond fractal embedding the graphs having Hausdorff dimension two. Unlike the analogous weak-disorder scaling limit for random polymer models on hierarchical graphs in the disorder relevant \( b < s \) case (or for the (1+1)-dimensional polymer on the rectangular lattice), the distributional convergence of the partition function when \( b = s \) cannot be approached through a term-by-term convergence to a Wiener chaos expansion, which does not exist for the continuum model emerging in the limit. The analysis proceeds by controlling the distributional convergence of the partition functions in terms of the Wasserstein distance through a perturbative generalization of Stein’s method at a critical step.

1 Introduction

In probabilistic frameworks, a disordered system usually refers to a relatively simple and familiar random object whose “pure” probabilistic law is distorted through its coupling to a random “environment” formed by an array of random variables (local impurities) or a random field. If the size of the model depends on a parameter \( L \in \mathbb{N} \), a central question for these disordered systems is whether typical realizations of the random environment create either a qualitative or only a quantitative change in the law of the random object as \( L \nearrow \infty \). For a given coupling strength \( \beta \in [0, \infty) \) of the system to the environment, these large-scale behaviors are respectively referred to as strongly disordered or weakly disordered. A disordered system is further classified as disorder relevant if it exhibits strong disorder for any fixed \( \beta \) as the system size grows or as disorder irrelevant otherwise. Finally, models at the border between the disorder relevant and disorder irrelevant regimes are referred to as marginally relevant or marginally irrelevant, and these boundary models manifest anomalous finer scaling behavior as the coupling strength vanishes.

One of the most closely studied disorder models is the directed polymer in a random environment, which is a \( d \)-dimensional simple symmetric random walk (SSRW) whose trajectories are reweighed within a Gibbsian formalism that depends on an inverse temperature parameter, \( \beta \), and an array of centered i.i.d. random variables labeled by the time-space lattice \( \{1, \ldots, L\} \times \mathbb{Z}^d \) for a polymer length \( L \in \mathbb{N} \). The parameter \( \beta \) effectively controls the strength of the polymer’s coupling to the environment, and \( \beta = 0 \) corresponds to a pure SSRW. Established results in this field imply that the

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A (d + 1)-polymer model is disorder relevant when d = 1, marginally relevant when d = 2, and disorder irrelevant in all higher dimensions; see Comets’s recent book \[14\]. The focus of this article is on a hierarchical diamond graph analog of the rectangular lattice polymers, but to give a context for our work we will discuss some related continuum limit results for the disorder relevant and marginally relevant rectangular models.

In principle, disorder relevance opens up the possibility that there exists a continuum disorder model that emerges in a joint limit in which the polymer length, L, grows as the inverse temperature \( \beta \equiv \beta(L) \) vanishes with an appropriate dependence on \( L \). A rigorous mathematical result in this direction was developed by Alberts, Khanin, and Quastel in the article \[2\], which proved that the partition function for (1+1)-dimensional directed polymers converges in law to a nontrivial distributional limit, \( Z_{\beta} \), as \( L \to \infty \) and the inverse temperature has the asymptotic form \( \beta = (\hat{\beta} + o(1))L^{-1/4} \) for a fixed parameter value \( \hat{\beta} \in \mathbb{R}_+ \). This scaling limit is referred to as the intermediate disorder regime since it magnifies a parameter region between the weak (\( \beta = 0 \)) and the strong (\( \beta > 0 \)) domains of disorder behavior for the (1 + 1)-dimensional polymer, and it amounts to a continuum/weak-disorder limiting regime in which the polymers are diffusively rescaled towards Brownian motion trajectories while the environmental disorder variables are renormalized towards a white noise field \( W \equiv W(t, x) \) on \([0, 1] \times \mathbb{R} \).

The authors construct the limiting partition functions \( Z_{\beta} \) in terms of Wiener chaos expansions of the field \( W(t, x) \) involving the one-dimensional heat kernel \( g(t', x'; t, x) = \frac{1}{\sqrt{2\pi(t-t')}} \exp\left\{ -\frac{(x-x')^2}{2(t-t')} \right\} \).

A model of continuum directed polymers corresponding to the limiting partition functions laws \( Z_{\beta} \) in \[2\] was discussed more explicitly in \[3\], where \( Z_{\beta} \) is equal in distribution to the total mass of a random measure on \( C([0, 1]) \), i.e., the space of Brownian trajectories. Moreover, the authors use the point-to-point form, \( Z_{\beta} \equiv Z_{\beta}(t', x'; t, x) \), of these limiting partition function laws to construct a solution to the one-dimensional stochastic heat equation (SHE):

\[
\partial_t Z_{\beta} = \frac{1}{2} \partial_x^2 Z_{\beta} + \beta W Z_{\beta}, \quad Z_{\beta}(t, x', t, x) = \delta_0(x' - x).
\]

In the case where \( Z_{\beta} \equiv Z_{\beta}(0, 0; 1, *) \) corresponds to the limit of point-to-line partition functions for polymers starting at the origin, \( Z_{\beta} \) is equal in law to the total mass of a random measure \( M_{\beta} \) on \( C([0, 1]) \) that can be formally expressed as

\[
M_{\beta}(dp) = e^{\beta \int W(p) - \frac{x^2}{2} \mathbb{E}[W(p)]} P(dp) \quad \text{for} \quad p \in C([0, 1]), \quad (1.1)
\]

where \( P \) is the Wiener measure on \( C([0, 1]) \) for a standard Brownian motion and \( \hat{W}(p) := \int_0^1 W(t, p_t) dt \) defines a Gaussian field over \( C([0, 1]) \) with correlation kernel given by the intersection time between paths: \( T(p, q) = \mathbb{E}[\hat{W}(p) \hat{W}(q)] = \int_0^1 \delta(p_t - q_t) dt \). Random measures formally expressed in terms of exponentials of Gaussian fields as in \[14\] are the focus of the theory of Gaussian multiplicative chaos (GMC), and \( M_{\beta} \) is a subcritical GMC for any \( \beta \in \mathbb{R}_+ \) that can be understood through the general approach to GMC theory in \[25\]. The random measures \( M_{\beta} \) are a.s. mutually singular to \( P \) and satisfy

\[
\mathbb{E}[M_{\beta}(dp)] = P(dp) \quad \text{and} \quad \mathbb{E}[M_{\beta}(dp)M_{\beta}(dq)] = e^{\beta^2 T(p, q)} P(dp) P(dq), \quad (1.2)
\]

and in particular \( \mathbb{E}[M_{\beta} \times M_{\beta}] \) is absolutely continuous with respect to \( P \times P \), which is a necessary feature of subcritical GMCs.\(^2\)

Weak-disorder limits analogous to \[2\] for the marginally relevant (2 + 1)-dimensional polymer involve fundamental new mathematical difficulties and are not as well understood as the weak-disorder

\(^1\)The general relationship between disorder relevance and continuum limits is argued for in \[8\].
\(^2\)The field \( \hat{W}(p) \) yields a Gaussian random variable when integrated against a test function \( \psi \in L^2(C([0, 1]), P) \).
\(^3\)See Lemma 34 of \[25\].
regime for the \((1 + 1)\)-polymer despite significant progress in a series of articles \([5, 6, 7, 8, 9]\) by Caravenna, Sun, and Zygouras. In \([4]\) the authors proved that the partition function \(Z_{L, \hat{\beta}}\) for \((2+1)\)-dimensional polymers has the following distributional limit behavior as \(L \to \infty\) when the inverse temperature tends to zero as \(\beta \equiv \beta_L = (\hat{\beta} + o(1))(\pi \log L)^{-1/2}\) for fixed \(\hat{\beta} \in \mathbb{R}_+\):

\[
Z_{L, \beta_L} \xrightarrow{L \to \infty} \mathcal{Z}_\beta := \begin{cases} \exp\{\sigma_\beta \chi - \frac{1}{2} \sigma_\beta^2\} & \hat{\beta} < 1, \\ 0 & \hat{\beta} \geq 1, \end{cases}
\]

(1.3)

where \(\chi\) is a standard normal random variable and \(\sigma^2_\beta := \log\left(\frac{1}{1-\beta}\right)\). In other terms, for \(\hat{\beta} < 1\) the limit law, \(\mathcal{Z}_\beta\), is a mean-one lognormal that converges in probability to zero (while having exploding variance) as \(\hat{\beta} \nearrow 1\). Thus a phase transition from weak disorder to strong disorder occurs at \(\hat{\beta} = 1\) within this weak-coupling limit regime.

A further study of the \((2+1)\)-dimensional directed polymer around the critical point \(\hat{\beta} = 1\) within the weak-disorder limit is undertaken in \([9]\) by choosing a more refined inverse temperature \(\beta \equiv \beta_{L,r}\) that depends on a fixed parameter \(r \in \mathbb{R}\) for which the variance of \(\exp\{\beta_{L,r} \omega - \frac{1}{2} \beta^2_{L,r}\}\) has the form \(\frac{\pi}{\log L}(1 + \frac{r}{\log L} + o\left(\frac{1}{\log L}\right))\) for \(L \gg 1\), where \(\omega\) is a disorder variable. This scaling satisfies \(\beta_{L,r} = (1 + o(1))(\pi \log L)^{-1/2}\), i.e., falls within the critical window of the phase transition (1.3) and the parameter \(r\) comes into play at order \((\log L)^{-3/2}\). For a time parameter \(t \geq 0\), the authors define the following random measures \(\mathcal{Z}_{Lt,\beta_{L,r}}\) on \(\mathbb{R}^2\):

\[
\mathcal{Z}_{Lt,\beta_{L,r}}(dx) := \frac{1}{L} \sum_{y \in \frac{1}{L^2} \mathbb{Z}^2} Z_{Lt,\beta_{L,r}}(y\sqrt{L}) \delta_y(x),
\]

(1.4)

where \(Z_{L, \beta}(x)\) is the partition function for length \(L\) polymers starting from position \(x \in \mathbb{Z}^2\). Using a tightness argument involving bounds for the third moments of the variables \(Z_{Lt,\beta_{L,r}}(\phi) := \int_{\mathbb{R}^2} \phi(x) \mathcal{Z}_{Lt,\beta_{L,r}}(dx)\) for \(\phi \in C_c(\mathbb{R}^2)\), the authors prove the existence of subsequential limits \(L \to \infty\) such that \(\mathcal{Z}_{Lt,\beta_{L,r}}\) converges in law to a random measure \((\mathcal{Z}_{t,r}, \mathbb{R}^2)\) satisfying

\[
\mathbb{E} \left[ \left( \int_{\mathbb{R}^2} \phi(x) \mathcal{Z}_{t,r}(dx) \right)^2 \right] = \int_{\mathbb{R}^2 \times \mathbb{R}^2} \phi(z) \phi(z') K_{t,r}(z - z') dz dz',
\]

(1.5)

where \(K_{t,r}(z - z)\) is a correlation kernel with logarithmic blow up around its diagonal from Bertini and Cancrini’s article \([4]\) on the two-dimensional SHE. The above is related to a recent breakthrough on the two-dimensional SHE at criticality by Gu, Quastel, and Tsai \([20]\). The form (1.5) is consistent with the existence of a \((2+1)\)-dimensional continuum polymer measure \(M^\phi(d\eta)\) on \(C([0, 1], \mathbb{R}^2)\) analogous to the \((1+1)\)-dimensional case in \([3]\) when the starting point of the polymer has an appropriate probability density \(\phi : \mathbb{R}^2 \to [0, \infty)\) (i.e., diffuse initial position). If \(P^\phi\) denotes Wiener motion on \(C([0, 1], \mathbb{R}^2)\) for trajectories starting with initial position \(\phi\), then two independently chosen trajectories will a.s. not intersect. Thus if a continuum disordered polymer measure \(M^\phi\) exists, \(\mathbb{E}[M^\phi \times M^\phi]\) would not be absolutely continuous with respect to the product Wiener measure \(P^\phi \times P^\phi\), unlike the continuum \((1 + 1)\)-dimensional polymer case \([12]\).

In this article, we study a similar limiting regime to that in \([9]\) for the marginally relevant disorder case of directed polymers crossing diamond hierarchical graphs with disorder variables on the graphical edges. More precisely, we prove a distributional limit theorem for the partition function of our polymer model within a window around a critical point that emerges in the weak-disorder regime. The limiting law, \(W_r\), depends on a parameter \(r \in \mathbb{R}\) and is used in \([12]\) to define a one-parameter family of continuum directed polymer models summarized below. Of course, the hierarchical symmetry of the graphs makes a detailed limit analysis within the weak-disorder regime less difficult than for the model of marginally relevant polymers on the rectangular lattice discussed above. The hierarchical setting,
however, provides some insights that are likely general for continuum polymers arising at criticality in systems with marginally relevant disorder.

Hierarchical graphs (“lattices”) are a frequent setting for statistical mechanical toy models because they may retain key characteristics of interest from their non-hierarchical analogs while providing an exact solubility in terms of renormalization transformations; see for instance [17, 18, 21, 22, 23, 24, 27] for recent mathematical work. By the nature of their recursive construction, hierarchical models embed copies of themselves after a change in the controlling parameters for the embedded copies. The articles [15, 16] were the first to study models of directed polymers in a random environment on hierarchical graphs, and, in particular, on diamond graphs, which are a subfamily depending on a pair \( b, s \in \{2, 3, \ldots \} \) of parameters determining the geometrical branching and segmenting structure within a network of pathways. In [23] Lacoin and Moreno analyzed the phase diagram of polymers on diamond graphs when the disorder variables are placed on the vertices, showing that

- strong disorder holds for any \( \beta > 0 \) when \( b \leq s \), and
- when \( b > s \) there is a critical cut-off \( \beta_c > 0 \) for which weak disorder holds when \( \beta \leq \beta_c \) and strong disorder holds for \( \beta \) above \( \beta_c \).

In terms of their disorder relevance, the cases \( b < s \), \( b = s \), and \( b > s \) are analogous respectively to the \( d = 1 \), \( d = 2 \), and \( d \geq 3 \) cases of \((d+1)\)-dimensional polymers on the rectangular lattice. In the disorder relevant \( b < s \) case, [1] proves a limit theorem for the partition functions in an intermediate disorder regime analogous to [2], and [11] defines a continuum polymer model similar to [3], although using GMC for the construction rather than Wiener chaos.

When the model is altered by placing disorder variables on the edges of the graphs rather than the vertices (as in this article), the analysis in [23] goes through essentially unchanged when \( b < s \) or \( b > s \), but for the marginal case of \( b = s \) there is a basic combinatorial difference: for two directed polymers \( p \) and \( q \) chosen independently and uniformly at random,

- the expected number of vertices shared by \( p \) and \( q \) has order \( \log L \) for \( L \gg 1 \), where \( L \) is the length of the polymers, and
- the expected number of edges shared by \( p \) and \( q \) is exactly 1, independent of \( L \). A closer look shows that when \( L \gg 1 \) the polymers will share no edges at all with a probability \( 1 - O(1/\log L) \), and that the expected number of common edges will be of order \( \log L \) in the complementary event.

Thus switching from vertex disorder to edge disorder for the \( b = s \) case of the diamond graph polymer has a similar effect in the weak-disorder limit as the mollifications in [1.4] because two independent two-dimensional SSRW trajectories of length \( L \) and with initial spatial probability densities spread out on the order of \( \sqrt{L} \) have a probability of intersecting that vanishes with order \( 1/\log L \) and, when conditioned on the event that the paths do intersect, an expected number of intersections on the order \( \log L \).

Before moving on to the details of the discrete model in this article, we will briefly summarize the continuum polymer model defined in [12] and its conditional Gaussian multiplicative chaos structure [13]. The limiting partition function law, \( W_r \), derived in later sections is equal in distribution to the total mass of a random measure \( M_r \) on the space \( \Gamma \) of directed paths crossing a compact diamond fractal, \( D \), having Hausdorff dimension two. Each directed path \( p \in \Gamma \) is an isometric embedding of the unit interval \( [0, 1] \) into the fractal, and there is a natural “uniform” probability measure \( \mu \) on \( \Gamma \) (serving as the analog of Wiener measure for the continuum \((1+1)\)-dimensional polymer) for which \( \mathbb{E}[M_r] = \mu \). For directed paths \( p, q \in \Gamma \), the set of intersection times is \( I_{pq} := \{ t \in [0, 1] \mid p(t) = q(t) \} \),

\(^4\)This assertion about the history of directed polymers on the diamond lattice is from [11, Page 73].

\(^5\)In terms of the parameter \( s \), the polymer length has the form \( L = s^n \).
and two paths chosen uniformly at random, i.e., according to the product measure $\mu \times \mu$, have a finite (trivial) number of intersections with probability one. In contrast, the random product measures $M_r \times M_r$ almost surely assign positive weight to the set of pairs $(p,q) \in \Gamma \times \Gamma$ for which $I_{p,q}$ is uncountable, albeit of Hausdorff dimension zero. The size of typical $I_{p,q}$ can be characterized through the exponent $h = 1$ case of the generalized Hausdorff measure $\mathcal{H}_h^\log$ on $[0,1]$ of the form

$$
\mathcal{H}_h^\log(S) := \lim_{\delta \searrow 0} \mathcal{H}_{h,\delta}^\log(S) \quad \text{for} \quad \mathcal{H}_{h,\delta}^\log(S) := \inf_{S \subseteq \cup_{k=1}^\infty} \frac{1}{\delta^h} \sum_{k} \frac{1}{|\log((1/\delta_k)|^h),}
$$

where $S \subset [0,1]$, and the infimum is over all coverings of $S$ by intervals $I$ of length $|I|$ less than $\delta > 0$.

The qualitative difference (trivial to nontrivial) between the typical behavior of the intersection-times set $I_{p,q}$ under the pure measure $\mu \times \mu$ and realizations of the disordered product measure $M_r \times M_r$ is a strong localization property that is not present in the subcritical continuum models [3, 11]. To compare with the $(1+1)$-dimensional continuum polymer measures $M_\beta$ discussed above, the set of intersection times $I_{p,q}$ is appropriately measured by $T(p,q) = \int_0^1 \delta_0(p_t - q_t)dt$ (which is closely related to the dimension-$1/2$ Hausdorff measure of $I_{p,q}$) for both the product Wiener measure $P \times P$ and realizations of $M_\beta \times M_\beta$. Secondly, in contrast with (1.2), the expectation of $M_r \times M_r$ has Lebesgue decomposition with respect to $\mu \times \mu$ given by

$$
\mathbb{E}[M_r \times M_r] = \mu \times \mu + \omega_r,
$$

where the measure $(\Gamma \times \Gamma, \omega_r)$ assigns full weight to the set of pairs $(p,q)$ such that $\mathcal{H}_h^\log(I_{p,q}) = \infty$ for all $h < 1$ and $\mathcal{H}_h^\log(I_{p,q}) = 0$ for all $h > 1$, in other terms, for which $I_{p,q}$ has log-Hausdorff exponent one. The fact that $\mathbb{E}[M_r \times M_r] \neq 0$ implies that $M_r$ is not a ‘critical’ GMC when the expectation $\mathbb{E}[M_r] = \mu$ is a probability measure and thus $\sigma$-finite. The family of random measures $(M_r)_{r \in \mathbb{R}_+}$, however, has a conditional interrelational GMC structure wherein for any $a \in \mathbb{R}_+$ the law of the random measure $M_{r+a}$ can be constructed from $M_r$ as

$$
M_{r+a}(dp) \overset{\mathcal{L}}{=} e^{\sqrt{\pi}W_{M_r}(p)} e^{-\frac{\pi}{4} \mathbb{E}[W_{M_r}^2,(p)]} M_r(dp), \quad p \in \Gamma,
$$

where $\hat{W}_{M_r}(p)$ is a field over $(\Gamma, M_r)$ that is Gaussian when conditioned on $M_r$ and has a correlation kernel $T(p,q) = \mathbb{E}[\hat{W}_{M_r}(p) \hat{W}_{M_r}(q) | M_r]$ roughly equivalent to the generalized Hausdorff measure with exponent $h = 1$, $\mathcal{H}_1^\log(I_{p,q})$, of the set of intersection times. Because the random measures $M_r$ converge in law to the pure measure $\mu$ as $r \searrow -\infty$, the above formally implies that an infinite field strength is required to generate $M_r$ as a GMC on $\mu$.

To be clear, the mathematical content of this article is entirely focused on the distributional convergence of a sequence of partition functions (random variables) and work on the limiting continuum disordered polymer model is in the articles cited above.

2 The set-up and main result

This section begins by defining a family of random measures on directed paths crossing diamond hierarchical graphs and concludes with the statement of Theorem 2.7, which was conjectured in [10] and is the main result of this article.
2.1 Construction of the diamond hierarchical graphs

Hierarchical diamond graphs $D_{b,s}^n$, $n \in \mathbb{N}_0$ are recursively defined through a construction determined by a branching number $b \in \{2, 3, \ldots\}$ and a segmenting number $s \in \{2, 3, \ldots\}$. The zeroth graph, $D_{b,s}^0$, is simply two root vertices, $A$ and $B$, with an edge between them. The first-generation graph, $D_{b,s}^1$, is formed by $b$ parallel branches connecting $A$ and $B$, wherein each branch has $s$ edges running in sequence. For $n \geq 2$ the graph $D_{b,s}^n$ is defined recursively from $D_{b,s}^{n-1}$ by embedding a copy of $D_{b,s}^1$ in place of each edge on $D_{b,s}^{n-1}$. The set of edges, $E_{b,s}^n$, on $D_{b,s}^n$ thus contains $(bs)^n$ elements.

The first three recursively-defined diamond graphs with $b = 3$ and $s = 3$.

A directed path on $D_{b,s}^n$ is a function $p : \{1, \ldots, s^n\} \to E_{b,s}^n$ for which $p(1)$ is incident to $A$, $p(s^n)$ is incident to $B$, and successive edges $p(k)$, $p(k+1)$ share a common vertex for $1 \leq k < s^n$. In other terms, the path moves monotonically from $A$ up to $B$, as seen in the figure. We denote the set of directed paths on $D_{b,s}^n$ by $\Gamma_{b,s}^n$.

2.2 Random Gibbsian measure on directed paths

Next we define a random Gibbs measure on the space $\Gamma_{b,s}^n$ of directed paths. Let $\omega_h$ be an i.i.d. family of random variables labeled by $h \in E_{b,s}^n$ and having mean zero, variance one, and finite exponential moments, $\mathbb{E} \left[ \exp \{ \beta \omega_h \} \right]$ for $\beta \geq 0$. Given an inverse temperature value $\beta \in [0, \infty)$, we define a random path measure on directed paths such that the weight assigned to $p \in \Gamma_{b,s}^n$ is given by

$$M_{\beta,n}^\omega (p) = \frac{1}{|\Gamma_{b,s}^n|} \mathbb{E} \left[ e^{\beta H_{\omega}^n (p)} \right]$$

for path energy

$$H_{\omega}^n (p) := \sum_{a \prec p} \omega_a ,$$

where $a \prec p$ means that the edge $a \in E_{b,s}^n$ lies along the path $p$. At infinite temperature ($\beta = 0$), $M_{\beta,n}^\omega$ is a uniform probability measure on $\Gamma_{b,s}^n$. We denote the total mass of $M_{\beta,n}^\omega$ by

$$W_{\omega}^n (\beta) := M_{\beta,n}^\omega (\Gamma_{b,s}^n) ,$$

which can be written as

$$\frac{1}{|\Gamma_{b,s}^n|} \sum_{p \in \Gamma_{b,s}^n} \prod_{a \prec p} e^{\beta \omega_a}$$

in terms of the disorder variables $\omega_a$. The recursive construction of the diamond graphs implies the following distributional recursive relation for the partition functions $W_{\omega}^n (\beta)$:

$$W_{\omega}^{n+1} (\beta) = \frac{1}{b} \sum_{i=1}^{b} \prod_{j=1}^{s} W_{\omega}^{n,i,j} (\beta) ,$$

in terms of the disorder variables $\omega_a$. The recursive construction of the diamond graphs implies the following distributional recursive relation for the partition functions $W_{\omega}^{n+1} (\beta)$:
where the $W_n^{r(i,j)}(β)$’s are independent copies of the random variable $W_n^ω(β)$. The variances $g_n(β) := \text{Var}(W_n^ω(β))$ are recursively related as $g_{n+1}(β) = M_{b,s}(g_n(β))$ with $M_{b,s} : (0,∞) \to [0,∞)$ defined as

$$M_{b,s}(x) := \frac{1}{b} \left[ (1 + x)^s - 1 \right].$$

Notice that the map $M_{b,s}$ has a fixed point at $x = 0$ and for $0 < x \ll 1$

$$= \begin{cases} \frac{s}{b} x + O(x^2) & s \neq b, \\ x + \frac{b-1}{2} x^2 + O(x^3) & s = b. \end{cases} \quad (2.1)$$

Thus the fixed point is linearly attractive when $b > s$, linearly repelling when $b < s$, and marginally repelling when $b = s$.

### 2.3 High-temperature scaling limits for the Gibbs measure

Our focus is on high-temperature (i.e., weak-disorder) scaling limits in which the hierarchical level parameter, $n$, grows as the inverse temperature $β = β(n)$ decays under an appropriate tuning in $n$ such that the random path measures $M^ω_{β,n}$ converge in distribution to a limiting random measure on paths. This article focuses only on the total mass of the measures while [12] extends this limit analysis to the full measures and discusses some delicate properties of the limiting path measures.

High-temperature scaling limits are only viable in the cases $b < s$ and $b = s$ for which $x = 0$ is a repelling fixed point of the variance map $M_{b,s}$. The article [1] contains a limit theorem for $W_n^ω(β)$ in the case $b < s$, where for a fixed parameter value $r \in \mathbb{R}_+$ the inverse temperature $β \equiv β_{n,r}$ has the large $n$ asymptotic form

$$β_{n,r} = \sqrt{r} \left( \frac{b}{s} \right)^{n/2} + o \left( \left( \frac{b}{s} \right)^{n/2} \right). \quad (2.2)$$

The sequences of random variables $\{W_n^ω(β_{n,r})\}_{n \in \mathbb{N}}$ converge in distribution as $n \to \infty$ to a family of limit laws $W_r$ supported on $(0,∞)$ that satisfy the distributional recursion relation

$$W_{r} = \frac{d}{b} \sum_{i=1}^{b} \prod_{j=1}^{s} W_r^{(i,j)},$$

where $W_r^{(i,j)}$ are i.i.d. copies of $W_r$. The variance, $R_{b,s}(r)$, of $W_r$ satisfies $M_{b,s}(R_{b,s}(r)) = R_{b,s}(\frac{r}{b})$.

Of course, the exponential form of the inverse temperature scaling (2.2) corresponds to the linear repelling (2.1) of the map $M_{b,s}$ from $x = 0$ that occurs in the $b < s$ case.

The main result of the current article is a proof of an analogous limit theorem for $W_n^ω(β)$ in the $b = s$ case. The correct choice of inverse temperature scaling—see below in (2.3)—was introduced in [10] although the results therein were confined to proving convergence of the positive integer moments. Convergence of the integer moments does not imply convergence in law because the higher limiting moments increase super-factorially; see (III) of Theorem [2.4] below. For fixed $b \in \{2,3,4,\ldots\}$ and $r \in \mathbb{R}$, let the sequence $(β_{n,r})_{n \in \mathbb{N}}$ have the large $n$ asymptotics

$$β_{n,r} := \frac{\kappa_b}{\sqrt{n}} - \frac{\tau \kappa_b^2}{2n} + \frac{\kappa_b \eta_b \log n}{n^2} + \frac{\kappa_b r}{n^2} + o \left( \frac{1}{n^2} \right), \quad (2.3)$$

where $τ := E[ω_ω^3]$ is the skew of the disorder variables, $ω_ω$, and the constants $κ_b, η_b > 0$ are defined as

$$κ_b := \sqrt{\frac{2}{b - 1}} \quad \text{and} \quad η_b := \frac{b+1}{3(b-1)}. \quad (2.4)$$
If we let $M_{b,b}^n$ denote the $n$-fold composition of $M_{b,b}$, the variance, $\varphi_n(\beta_{n,r}^{(b)})$, of $W_n^\omega(\beta_{n,r}^{(b)})$ can be written explicitly as

$$\varphi_n(\beta_{n,r}^{(b)}) = M_{b,b}^n(\varphi_0(\beta_{n,r}^{(b)})),$$ where $\varphi_0(\beta_{n,r}^{(b)})$ has the large $n$ asymptotics

$$\varphi_0(\beta_{n,r}^{(b)}) := \text{Var}\left(\frac{e^{\beta_{b,n}^{(b)} \omega}}{E[e^{\beta_{b,n}^{(b)} \omega}]}\right) = \kappa^2 \left(\frac{1}{n} + \frac{\eta_b \log n}{n^2} + \frac{r}{n^2} + o\left(\frac{1}{n^2}\right)\right).$$

The basic observations above combined with Lemma 2.3 below, imply that $\varphi_n(\beta_{n,r}^{(b)})$ converges as $n \to \infty$ to a limit $R_b(r)$ for any $r \in \mathbb{R}$.

**Remark 2.1.** Let us set the skew, $\tau$, of the disorder variables to zero here for simplicity. Theorem 7.1 of [1] states that if $\beta$ falls within a critical window around $n$, and $\beta_{n,r}^{(b)}$ is a critical point for the parameter $\hat{\beta} \in \mathbb{R}^+$, then $W_n^\omega(\hat{\beta}/\sqrt{n})$ has the distributional behaviors listed below depending on $\hat{\beta}$ as $n \to \infty$.

$$W_n^\omega(\hat{\beta}/\sqrt{n}) \overset{d}{\approx} 1 + \frac{1}{\sqrt{n}} \cdot N\left(0, \frac{1}{1/\hat{\beta}^2 - 1/\kappa_b^2}\right) \quad \hat{\beta} < \kappa_b$$

$$W_n^\omega(\hat{\beta}/\sqrt{n}) \overset{d}{\approx} 1 + \frac{1}{\sqrt{\log n}} \cdot N\left(0, \frac{6}{b+1}\right) \quad \hat{\beta} = \kappa_b$$

The variance of $W_n^\omega(\hat{\beta}/\sqrt{n})$ blows up. $\hat{\beta} > \kappa_b$

Thus $\kappa_b$ is a critical point for the parameter $\hat{\beta}$ in the moment behavior of $W_n^\omega(\hat{\beta}/\sqrt{n})$ when $n \to \infty$, and $\beta_{n,r}^{(b)}$ falls within a critical window around $\kappa_b$. The variance blow-up at $\kappa_b$ coincides with the transition to strong disorder as can be seen in the limit model emerging under the scaling (2.3) as $n \to \infty$; see Remark 2.9.

**Remark 2.2.** In terms of the length $L = b^n$ of the directed polymers, the asymptotic form (2.3) implies that $(\beta_{n,r}^{(b)})^2 = \frac{A_b}{\log L} - \frac{B_b}{(\log L)^{3/2}} + \frac{C_b \log \log L}{(\log L)^2} + \frac{D_b r}{(\log L)^3} + o\left(\frac{1}{\log L}\right)$ for some constants $A_b, B_b, C_b, D_b > 0$. This inverse temperature scaling in the critical window is similar to the form [8 Remark 1.1] for (2+1)-dimensional directed polymers on the rectangular lattice except for the inclusion of the term $\log \log L$ and the constants do not depend on the fourth cumulant of the disorder variables here.

### 2.4 Previous results on the centered moments

The lemma and theorem below are results from [10].

**Lemma 2.3** (Variance function). For $b \in \{2, 3, \ldots\}$, there exits a unique continuously differentiable increasing function $R_b : \mathbb{R} \to \mathbb{R}^+$ satisfying the properties (I)-(III) below.

**I** Composition of $R_b(r)$ with the map $M_{b,b}$ translates the parameter $r$: $M_{b,b}(R_b(r)) = R_b(r+1)$.

**II** As $r \to \infty$, $R_b(r)$ grows without bound. As $r \to -\infty$, $R_b(r)$ has the vanishing asymptotics

$$R_b(r) = -\frac{\kappa_b^2}{r} + O\left(\frac{\log^2(-r)}{r^3}\right).$$

**III** The derivative $R_b'(r)$ admits the limiting form

$$R_b'(r) = \lim_{n \to \infty} \frac{\kappa_b^2}{n^2} \prod_{k=1}^n (1 + R_b(r - k))^{b-1}.$$
Moreover, if for some $r \in \mathbb{R}$ the sequence of positive real numbers $(x^{n,r})_{n \in \mathbb{N}}$ has the large $n$ asymptotics
\[
x^{n,r} = \kappa^r_0 \left( \frac{1}{n} + \frac{\eta_r \log n}{n^2} + \frac{r}{n^2} \right) + o\left( \frac{1}{n^2} \right),
\]
then $M^{n}_{b,b}(x^{n,r})$ converges as $n \to \infty$ to $R_b(r)$.

Appendix A contains an elementary but instructive calculation showing the consistency between properties (I) and (II) above. The higher centered moments of $W_n^\omega(\beta_{n,r}^{(b)})$ converge to limits $R_b^{(m)}(r)$ characterized as follows.

**Theorem 2.4** (Limiting higher moments). Fix $b \in \{2, 3, \ldots\}$ and let $s = b$. For each $m \in \{2, 3, \ldots\}$ there is a continuous, increasing function $R_b^{(m)} : [0, \infty) \to [0, \infty)$ such that for any $r \in \mathbb{R}$
\[
\mathbb{E}\left[ \left( W_n^\omega(\beta_{n,r}^{(b)}) - 1 \right)^m \right] \xrightarrow{n \to \infty} R_b^{(m)}(r).
\]
The limit functions $R_b^{(m)}$ satisfy properties (I)-(III) below.

(I) There are multivariate polynomials $P_m : \mathbb{R}^{m-1} \to \mathbb{R}$ with nonnegative coefficients such that for all $r \in \mathbb{R}$
\[
R_b^{(m)}(r+1) = P_m(R_b^{(2)}(r), R_b^{(3)}(r), \ldots, R_b^{(m)}(r)).
\]

(II) $R_b^{(m)}(r)$ grows without bound as $r \to \infty$ and vanishes as $r \to -\infty$ with the asymptotics
\[
R_b^{(m)}(r) \sim \kappa^m_b \frac{m!}{2^{m/2}(m/2)!} |r|^{-m/2} \quad \text{for } m \text{ even and } R_b^{(m)}(r) = O(|r|^{-(m+1)/2}) \quad \text{for } m \text{ odd.}
\]

(III) There is a $c > 0$ such that $\frac{\log \log(R_b^{(m)}(r))}{m} > c$ holds for any fixed $r \in \mathbb{R}$ and large enough $m \in \mathbb{N}$.

**Remark 2.5.** The function $R_b(r)$ in the statement of Lemma 2.3 is equal to $R_b^{(2)}(r)$ in the statement of Theorem 2.4.

**Remark 2.6.** The quantity $\kappa^m_b \frac{m!}{2^{m/2}(m/2)!} |r|^{-m/2}$ in (II) agrees with the $m^{th}$ moment of a centered normal random variable with variance $\kappa^2_b/|r|$.

### 2.5 Main result

As mentioned above, Theorem 2.4 does not imply that $W_n^\omega(\beta_{n,r}^{(b)})$ converges in law as $n \to \infty$ since $R_b^{(m)}(r)$ grows super-factorially with $m \in \mathbb{N}$. Thus the following theorem was left as a conjecture in [10].

**Theorem 2.7.** Fix $b \in \{2, 3, 4, \ldots\}$ and $r \in \mathbb{R}$, and let the sequence $(\beta_{n,r}^{(b)})_{n \in \mathbb{N}}$ have the form (2.3). When $s = b$ there is convergence in distribution as $n \to \infty$
\[
W_n^\omega(\beta_{n,r}^{(b)}) \quad \Rightarrow \quad L_r^{(b)}
\]
to a family of limit laws $\{L_r^{(b)}\}_{r \in \mathbb{R}}$ uniquely determined by (I)-(IV) below.

(I) $L_r^{(b)}$ has mean 1 and variance $R_b(r)$.

(II) For $m \in \{3, 4, \ldots\}$, the $m^{th}$ centered moment of $L_r^{(b)}$ is equal to $R_b^{(m)}(r)$.

(III) Let $W_r$ be a random variable with distribution $L_r^{(b)}$. The centered variables $\sqrt{-r}(W_r - 1)$ converge in law as $r \to -\infty$ to a centered normal with variance $\kappa^2_b$. 

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(IV) If $W_r^{(i,j)}$ are independent variables with distribution $L_r^{(b)}$, then there is equality in distribution

$$W_{r+1} = \frac{1}{b} \sum_{1 \leq i \leq b} \prod_{1 \leq j \leq b} W_r^{(i,j)}.$$ 

**Remark 2.8.** The convergence in distribution of $\sqrt{-r}(W_r - 1)$ to $N(0, \kappa_b^2)$ as $r \to -\infty$ follows from the asymptotics for the centered moments $R_b^{(m)}(r)$ in (II) of Theorem 2.4.

**Remark 2.9.** The family of limit laws in Theorem 2.7 exhibits a transition to strong disorder as $r \to \infty$ in the sense that the random variables $W_r$ converge in probability to zero. This is proved in [13, Section 5] using the GMC structure discussed in the introduction.

### 2.6 The rest of this article

**Notation:** In the remainder of the article, we refer exclusively to the case when the branching parameter and the segmenting parameter of the diamond graphs are equal ($b = s$). The dependence of all previously defined expressions on the parameter $b \in \{2, 3, \ldots\}$ will be suppressed as in the following list of notational identifications:

$$D_n^{(b)} \equiv D_n, \quad \Gamma_n^{(b)} \equiv \Gamma_n, \quad \beta_n^{(b)} \equiv \beta_n, \quad M_{b,b}(x) \equiv M(x), \quad R_b^{(m)}(r) \equiv R^{(m)}(r), \quad \kappa_b \equiv \kappa, \quad \eta_b \equiv \eta.$$ 

$\mathbb{N}$ denotes the positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

**Article organization:**

- Section 3 states Theorem 3.14, which is a slightly strengthened version of Theorem 2.7 that is couched in the language used in the proofs.
- Taken together, Sections 4 & 5 complete the proof of Theorem 3.14 after stating the key technical results in Proposition 5.1 and Lemmas 5.7-5.9 that support the proof.
- Sections 6 & 7 contain the proofs of Proposition 5.1 & Lemmas 5.7-5.9 with some of the relatively routine elements delayed to Section 8.
- Appendix A carries through an instructive consistency check between (I) and (II) of Lemma 2.3 and Appendix B provides some background on the zero bias approach [19] to Stein’s method.

### 3 Reformulation in terms of arrays and Wasserstein distance

This section defines notation and terminology to reformulate Theorem 2.7 as Theorem 3.14, which is stated in the language that we use throughout the remainder of the article.

#### 3.1 Edge-labeled array notation

**Notation 3.1 (Arrays).** Let $x_a$ be real numbers labeled by $E_k$ for some $k \in \mathbb{N}_0$.

- The notation $\{x_a\}_{a \in E_k}$ denotes an element of $\mathbb{R}^{b^{2k}}$, which we refer to as an **array**.
- More generally, if $a \in E_j$ for some $j \in \mathbb{N}$ with $j \leq k$, then $\{x_a\}_{a \in a \cap E_k}$ denotes an element in $\mathbb{R}^{b^{2(k-j)}}$, where we have identified $a$ with its canonically corresponding subset of $E_k$.

**Definition 3.2 (Array maps).** For $k \in \mathbb{N}_0$ and $a \in E_k$, define $a \times (i,j)$ for $i,j \in \{1, \ldots, b\}$ as the element in $E_{k+1}$ corresponding to the $j^{th}$ segment along the $i^{th}$ branch of the embedded copy of $D_1$ in $D_{n+1}$ identified with $a$\(^6\)

\(^6\)This is to be understood in the context of the recursive construction of $D_{n+1}$ from $D_n$ in Section 2.1.
• We define $Q$ as the map that sends an array of real numbers $\{x_{a}\}_{a \in E_k}$ to the contracted array
\[
\{w_{a}\}_{a \in E_{k-1}} := Q\{x_{a}\}_{a \in E_{k}} \quad \text{for} \quad w_{a} := \frac{1}{b} \sum_{i=1}^{b} \left( \prod_{j=1}^{b} \left( 1 + x_{a \times (i,j)} \right) - 1 \right).
\]

• We define $L$ to be the linearization of $Q$ around the zero array:
\[
\{y_{a}\}_{a \in E_{k-1}} := L\{x_{a}\}_{a \in E_{k}} \quad \text{for} \quad y_{a} := \frac{1}{b} \sum_{1 \leq i,j \leq b} x_{a \times (i,j)}.
\]

• We define $E := Q - L$, i.e., the “error” of the linearization.

• For $N \in \mathbb{N}_0$, $Q^N$ and $L^N$ refer to the $N$-fold composition of the maps $Q$ and $L$, respectively.

**Remark 3.3.** We can write $W_n^{\omega}(\beta_{n,r})$ in terms of the operation $Q$ as
\[
W_n^{\omega}(\beta_{n,r}) = 1 + Q^n \{X_h^{(n)}\}_{h \in E_n} \quad \text{for} \quad X_h^{(n)} := \frac{e^{\beta_{n,r} \omega_h}}{E[e^{\beta_{n,r} \omega_h}]} - 1. \quad (3.1)
\]

**Remark 3.4.** Let $\{x_{a}\}_{a \in E_k}$ be an array of i.i.d. centered random variables with variance $\sigma^2$.

(i) $Q\{x_{a}\}_{a \in E_k}$ and $L\{x_{a}\}_{a \in E_k}$ are i.i.d. arrays of centered random variables with variance $M(\sigma^2)$ and $\sigma^2$, respectively. In particular, the operation $L$ preserves the variance of the array variables.

(ii) For $\{y_{a}\}_{a \in E_{k-1}} := L\{x_{a}\}_{a \in E_k}$ and $\{z_{a}\}_{a \in E_{k-1}} := E\{x_{a}\}_{a \in E_k}$, the random variables $y_{a}$ and $z_{a}$ are uncorrelated. Thus the variables in the array $E\{x_{a}\}_{a \in E_k}$ have variance $M(\sigma^2) - \sigma^2$.

(iii) Moreover, the random variable $Q^k\{x_{a}\}_{a \in E_k}$ can be written as the following sum of uncorrelated terms: $Q^k\{x_{a}\}_{a \in E_k} = L^k\{x_{a}\}_{a \in E_k} + \sum_{l=1}^{k} L^{l-1} E Q^{k-l}\{x_{a}\}_{a \in E_k}$.

The lemma below generalizes (iii) in Remark 3.4 and identifies the main source of uncorrelated terms found in this article. The proof follows easily from the multilinear polynomial forms of the maps $Q$, $E$, $L$.

**Lemma 3.5.** Let $\{x_{a}\}_{a \in E_k}$ be an array of independent centered random variables with finite second moments. If $A_l, B_l \in \{Q, E, L\}$ for $l \in \{1, \ldots, k\}$, then the random variables $A_1 \cdots A_k\{x_{a}\}_{a \in E_k}$ and $B_1 \cdots B_k\{x_{a}\}_{a \in E_k}$ are uncorrelated when one of the following sets is nonempty:
\[
S_A := \{l \mid A_l = E \land B_l = L\} \quad \text{and} \quad S_B := \{l \mid B_l = E \land A_l = L\}.
\]

**Proof.** Suppose that $l \in S_A$. The multilinear polynomial $A_1 \cdots A_k\{x_{a}\}_{a \in E_k}$ is a linear combination of monomials $\prod_{a \in U} x_{a}$ for which the set $U \subset E_k$ must contain a pair $a_1, a_2 \in U$ satisfying the following: there exist $f_1, f_2 \in E_l$ and $e \in E_{l-1}$ such that $a_1 \in f_1, a_2 \in f_2, f_1 \neq f_2$, and $f_1, f_2 \in e$. On the other hand, the multilinear polynomial $B_1 \cdots B_k\{x_{a}\}_{a \in E_k}$ does not contain any monomials of this type, so $A_1 \cdots A_k\{x_{a}\}_{a \in E_k}$ and $B_1 \cdots B_k\{x_{a}\}_{a \in E_k}$ are uncorrelated.

**Remark 3.6.** Note that if $\{x_{h}\}_{h \in E_n}$ is an array of i.i.d. centered random variables with variance $\sigma^2$, then $L^n\{x_{h}\}_{h \in E_n} = \frac{1}{n!} \sum_{h \in E_n} x_{h}$ has the form of a central limit-type renormalized sum since $b^n = |E_n|^{1/2}$. Thus if $n \gg k$, then $\{z_{a}\}_{a \in E_k}$ are uncorrelated.

**Definition 3.7.** Let $Q$ be defined as in Definition 3.2 and $n \in \mathbb{N}_0$. 

A Q-pyramidic array is a finite sequence $k = 0, 1, \ldots, n$ of arrays of real numbers $\{x^{(k,n)}_a\}_{a \in E_k}$ satisfying $\{x^{(k-1,n)}_a\}_{a \in E_{k-1}} = Q^k \{x^{(n)}_a\}_{a \in E_k}$ for all $k \neq 0$.

When $k = n$ we condense the superscript: $x^{(n,n)}_h \equiv x^{(n)}_h$ for $h \in E_n$. Moreover, $\{x^{(k,n)}_a\}_{a \in E_k} = Q^k \{x^{(n)}_h\}_{h \in E_n}$ is referred to as the Q-pyramidic array generated from $\{x^{(n)}_h\}_{h \in E_n}$.

**Remark 3.8.** To distinguish the entire Q-pyramidic array from one of its subarrays, $\{x^{(k,n)}_a\}_{a \in E_k}$, we will sometimes write $\{x^{(s,n)}_a\}_{a \in E_s}$.

**Lemma 3.9.** For $n \in \mathbb{N}_0$ and $h \in E_n$, let the random variables $X^{(n)}_h$ be defined as in (3.1) for some fixed parameter value $r \in \mathbb{R}$. For $n \in \mathbb{N}_0$, let $\{X^{(k,n)}_a\}_{a \in E_k} := Q^{n-k} \{X^{(n)}_h\}_{h \in E_n}$ be the Q-pyramidic array generated from $\{X^{(n)}_h\}_{h \in E_n}$.

(I) For each $k$, the variables in the array $\{X^{(k,n)}_a\}_{a \in E_k}$ are i.i.d.

(II) For each $k \geq 1$, the array $Q \{X^{(k,n)}_a\}_{a \in E_k}$ is equal to $\{X^{(k-1,n)}_a\}_{a \in E_{k-1}}$.

(III) For each $k$, the variables in the array $\{X^{(k,n)}_a\}_{a \in E_k}$ are centered and have finite $m^{th}$ moments that converge to $R^{(m)}(r - k)$ as $n \to \infty$ for every $m \in \{2, 3, \ldots\}$.

**Remark 3.10.** Parts (I) and (II) of Lemma 3.9 are immediate consequences of the definition of the variable arrays $\{X^{(k,n)}_a\}_{a \in E_k}$ and part (III) follows by applying Theorem 2.4 to the embedded copy of $D_{n-k}$ within $D_n$ corresponding to the edge $a \in E_k$.

**Definition 3.11.** A sequence $\{\{X^{(s,n)}_a\}_{a \in E_a}\}_{n \in \mathbb{N}}$ of Q-pyramidic arrays of random variables is said to be regular with parameter $r \in \mathbb{R}$ if it satisfies the conclusions (I)-(III) of Lemma 3.9.

### 3.2 A limit theorem for hierarchical arrays

**Theorem 3.12 (Limit law).** For any $r \in \mathbb{R}$, there exists a unique law on sequences in $k \in \mathbb{N}_0$ of edge-labeled arrays of random variables, $\{X^{(k)}_a\}_{a \in E_k}$, taking values in $[-1, \infty)$ and holding the properties (I)-(III) below.

(I) For each $k \in \mathbb{N}_0$, the variables in the array $\{X^{(k)}_a\}_{a \in E_k}$ are i.i.d.

(II) For each $k \in \mathbb{N}$, the array $\{X^{(k-1)}_a\}_{a \in E_{k-1}}$ is equal to $Q \{X^{(k)}_a\}_{a \in E_k}$.

(III) For each $k \in \mathbb{N}_0$, the variables in the array $\{X^{(k)}_a\}_{a \in E_k}$ are centered and have $m^{th}$ moment equal to $R^{(m)}(r - k)$ for all $m \in \{2, 3, \ldots\}$.

**Definition 3.13** (Wasserstein distance). For two probability measures $\mu$ and $\nu$ on $\mathbb{R}$, let $\mathcal{M}_{\mu,\nu}$ be the set of joint measures $J(dx, dy)$ on $\mathbb{R}^2$ with marginals $\mu$ and $\nu$. For $p \geq 1$ assume that $\mu$ and $\nu$ satisfy $\int_{\mathbb{R}} |x|^p \mu(dx) < \infty$ and $\int_{\mathbb{R}} |x|^p \nu(dx) < \infty$. We define the Wasserstein-$p$ distance between $\mu$ and $\nu$ as

$$
\rho_p(\mu, \nu) := \inf_{J \in \mathcal{M}_{\mu,\nu}} \left( \int_{\mathbb{R}^2} |x - y|^p J(dx, dy) \right)^{1/p}.
$$

If $X$ and $Y$ are random variables with distributional measures $\mu$ and $\nu$, respectively, then we extend our notation through the interpretation $\rho_p(X, Y) \equiv \rho_p(\mu, \nu)$. 

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Theorem 3.14. Let \( \{X_a^{(x,n)}\}_{a \in E_k} \) be a regular sequence of \( \mathcal{Q} \)-pyramidal arrays of random variables with parameter \( r \in \mathbb{R} \). For any \( a \in E_k \) the Wasserstein-2 distance between \( X_a^{(k,n)} \) and \( X_a^{(k)} \) vanishes as \( n \to \infty \), and, in particular, the i.i.d. array \( \{X_a^{(k,n)}\}_{a \in E_k} \) (viewed as an element in \( \mathbb{R}^{b_{2k}} \)) converges in law to \( \{X_a^{(k)}\}_{a \in E_k} \) for each \( k \in \mathbb{N}_0 \).

Remark 3.15. The hierarchical symmetry of the model implies that it is sufficient to prove Theorem 3.14 for the case \( k = 0 \) in which the arrays \( \{X_a^{(k,n)}\}_{a \in E_k} \) and \( \{X_a^{(k)}\}_{a \in E_k} \) are single random variables \( X^{(0,n)} \) and \( X^{(0)} \), respectively (on which we have dropped the subscripts). The proof of Theorem 3.14 involves writing \( X^{(0,n)} = \mathcal{Q}N\{X_{\xi}^{(N,N)}\}_{\xi \in E_N} \) and \( X^{(0)} = \mathcal{Q}N\{X_{\xi}^{(N)}\}_{\xi \in E_N} \) for a generation \( N \in \mathbb{N} \) with \( 1 \ll N \ll n \) and introducing arrays of random variables \( \{\widehat{X}_{\xi}^{(N)}\}_{\xi \in E_N} \) (Definition 5.4) for which we show that \( X_{\xi}^{(N,n)} \overset{d}{\approx} \widehat{X}_{\xi}^{(N)} \) and \( X_{\xi}^{(N)} \overset{d}{\approx} \widehat{X}_{\xi}^{(N)} \) in an appropriately strong sense that is characterized in Proposition 5.1.

4 Existence of a limiting hierarchical array

In this section we prove the existence of the hierarchical array of random variables described in Theorem 3.12. The proof is based on a routine tightness argument involving nested subsequences.

Proof of Theorem 3.12 (existence part). For \( k, n \in \mathbb{N}_0 \) with \( 0 < k < n \), define the i.i.d. arrays of random variables \( \{X_a^{(k,n)}\}_{a \in E_k} \) as in Lemma 3.9. By Remark 3.10 for any \( a \in E_k \) the variance of \( X_a^{(k,n)} \) converges to \( R(r-k) \) as \( n \to \infty \). In particular, for any fixed \( k \) the sequence \( \{X_a^{(k,n)}\}_{a \in E_k} \) of variable arrays indexed by \( n \in \mathbb{N} \), viewed as a random vector in \( \mathbb{R}^{b_{2k}} \), is tight. We define \( \xi_{n}^{(k)} \in \mathbb{N} \) inductively in \( k \in \mathbb{N}_0 \) as a nested sequence of subsequences as follows:

- Let \( (\xi_{n}^{(0)})_{n \in \mathbb{N}} \) be a subsequence of \( n = 1, 2, 3, \ldots \) such that the single-element array \( \{X_a^{(0,\xi_{n}^{(0)})}\}_{a \in E_0} \) converges in law as \( n \to \infty \) to a limit \( \{X_a^{(0)}\}_{a \in E_0} \).

- If for \( k \in \mathbb{N}_0 \) the sequence \( (\xi_{n}^{(k)})_{n \in \mathbb{N}} \) has been chosen so that the array \( \{X_a^{(k,\xi_{n}^{(k)})}\}_{a \in E_k} \) converges in law as \( n \to \infty \) to a limiting array \( \{X_a^{(k)}\}_{a \in E_k} \), then we choose \( (\xi_{n}^{(k+1)})_{n \in \mathbb{N}} \) to be a subsequence of \( (\xi_{n}^{(k)})_{n \in \mathbb{N}} \) such that \( \{X_a^{(k+1,\xi_{n}^{(k+1)})}\}_{a \in E_{k+1}} \) converges in law to some limit \( \{X_a^{(k+1)}\}_{a \in E_{k+1}} \).

With the sequence in \( k \in \mathbb{N}_0 \) of limiting arrays \( \{X_a^{(k)}\}_{a \in E_k} \) constructed above, let us consider properties (I)-(III). Property (I) holds immediately from the construction since all of the arrays, \( \{X_a^{(k,n)}\}_{a \in E_k} \), used in the construction are i.i.d. For property (II) notice that for any \( k \in \mathbb{N} \)

\[
\{X_a^{(k-1)}\}_{a \in E_{k-1}} \overset{d}{=} \lim_{n \to \infty} \{X_a^{(k-1,\xi_{n}^{(k-1)})}\}_{a \in E_{k-1}} = \lim_{n \to \infty} \mathcal{Q}\{X_a^{(k,\xi_{n}^{(k)})}\}_{a \in E_k} \overset{d}{=} \mathcal{Q}\{X_a^{(k)}\}_{a \in E_k},
\]

where the second equality follows from part (II) of Lemma 3.9 and the third holds by the continuity of the map \( \mathcal{Q} \). It follows that the sequence in \( k \in \mathbb{N}_0 \) of arrays of random variables \( \{X_a^{(k)}\}_{a \in E_k} \) can be defined on a single probability space such that \( \{X_a^{(k)}\}_{a \in E_k} \) is a.s. equal to \( \mathcal{Q}\{X_a^{(k-1)}\}_{a \in E_{k-1}} \). For property (III), Lemma 3.9 implies that the \( m^{th} \) moment of \( X_a^{(k,n)} \) converges to the limit \( R^{(m)}(r-k) \) for \( a \in E_k \) and \( m \in \{2,3,\ldots\} \). Since this holds for all \( m \), we have that \( \mathbb{E}[\{X_a^{(k)}\}^m] = R^{(m)}(r-k) \) for all \( m \).
The limiting random variables \( \{X_a^{(k)}\}_{a \in E_k} \) take values in \([-1, \infty)\) since the random variables \( \{1 + X_h^{(n)}\}_{h \in E_n} \) are nonnegative by their definition \([3.1]\), and the form of the map \( \mathcal{Q} \) implies that the arrays \( \{1 + X_a^{(k,n)}\}_{a \in E_k} \) for \( \{X_a^{(k,n)}\}_{a \in E_k} := \mathcal{Q}^k\{X_h^{(n)}\}_{h \in E_n} \) must also be nonnegative.

\[
\square
\]

5 Uniqueness of the limiting hierarchical array and universality

The goal of this section is to prove Theorem \(3.14\) and, simultaneously, the uniqueness part of Theorem \(3.12\) after stating the key propositions that enter into the proof.

5.1 \(L^2\)-bound for a contractive dynamics on arrays of random variables

The following proposition provides a condition template by which we can show that the random variables \( \mathcal{Q}^N\{U_e^{(N)}\}_{e \in E_N} \) and \( \mathcal{Q}^N\{V_e^{(N)}\}_{e \in E_N} \) are close together under the \(L^2\)-metric on random variables provided that \( \{(U_e^{(N)}, V_e^{(N)})\}_{e \in E_N} \) is an i.i.d. array of \((\mathbb{R}^2\text{-valued})\) random variables and the variables \( U_e^{(N)} \) and \( V_e^{(N)} \) are close together in \(L^2\). In loose terms, we are bounding the sensitivity of the “dynamics” on arrays generated by the map \( \mathcal{Q} \) to the initial conditions.

**Proposition 5.1.** There exist \( \delta > 0 \) and \( C > 0 \) depending only on \( s \in \mathbb{R} \) such that the statements (i)-(ii) below hold for any i.i.d. array \( \{(U_e^{(N)}, V_e^{(N)})\}_{e \in E_N} \) of centered \(\mathbb{R}^2\text{-valued}\) random variables for which \( U_e^{(N)} \) has the variance bound

\[
\mathbb{E}[(U_e^{(N)})^2] < R(s - N). \tag{5.1}
\]

(i) If \( \mathbb{E}[(V_e^{(N)} - U_e^{(N)})^2] < \delta/N^4 \), then

\[
\mathbb{E}\left[\left(\mathcal{Q}^N\{V_e^{(N)}\}_{e \in E_N} - \mathcal{Q}^N\{U_e^{(N)}\}_{e \in E_N}\right)^2\right]^{\frac{1}{2}} \leq CN^2\mathbb{E}\left[(V_e^{(N)} - U_e^{(N)})^2\right]^{\frac{1}{2}}.
\]

(ii) If \( \mathbb{E}[(V_e^{(N)} - U_e^{(N)})^2] < \delta/N^2 \) and the variables \( U_e^{(N)} \) and \( V_e^{(n)} - U_e^{(N)} \) are uncorrelated, then

\[
\mathbb{E}\left[\left(\mathcal{Q}^N\{V_e^{(N)}\}_{e \in E_N} - \mathcal{Q}^N\{U_e^{(N)}\}_{e \in E_N}\right)^2\right]^{\frac{1}{2}} \leq C\mathbb{E}\left[(V_e^{(N)} - U_e^{(N)})^2\right]^{\frac{1}{2}}.
\]

**Remark 5.2.** In particular, if \( \{(U_e^{(N)}, V_e^{(N)})\}_{e \in E_N} \) is a sequence in \( N \in \mathbb{N} \) of arrays of random variables satisfying the conditions of Proposition \(5.1\) and \( \mathbb{E}[(V_e^{(N)} - U_e^{(N)})^2] = o(1/N^4) \), then the \(L^2\)-distance between \( \mathcal{Q}^N\{V_e^{(N)}\}_{e \in E_N} \) and \( \mathcal{Q}^N\{U_e^{(N)}\}_{e \in E_N} \) vanishes with large \( N \).

5.2 Defining intermediary distributional approximations

Let \( \{X^{(n)}\}_{a \in E_n} \) be a sequence in \( n \in \mathbb{N} \) of \( \mathcal{Q}\)-pyramidal arrays satisfying properties (I)-(III) in the statement of Lemma \(3.9\). Proposition \(5.1\) combined with Remark \(3.15\) suggests a path for proving Theorem \(3.14\) by showing that for \( 1 \leq N \leq n \) and \( e \in E_N \) the \(L^2\)-distance between the random variables \( X_e^{(N,n)} \) and \( X_e^{(N)} \) is small for some coupling of the variables. We will attempt to further orient the reader towards the framework of the analysis in coming sections by heuristically motivating the definitions of three distributional approximations for the random variable \( X_e^{(N,n)} \) that have roles in the proof of Theorem \(3.14\) see Definition \(5.4\). The analysis will be founded on the introduction
of intermediary generational scales \( n(N), \tilde{n}(N) \in \mathbb{N} \) between \( N \) and \( n \) that allow us to identify two sources of central limit-type renormalized sums (in (I) and (II) below) within an approximation for \( X^{(N,n)}_e \). For some fixed \( \epsilon \in (0,1/2) \), it suffices for us to take

\[
\tilde{n}(N) := N + \lfloor N^{\epsilon}/2 \rfloor \quad \text{and} \quad n(N) := N + \lfloor N^{\epsilon} \rfloor ,
\]

so that, in particular, when \( 1 \ll N \ll n \)

\[ N < \tilde{n}(N) < n(N) \ll n , \quad 1 \ll n(N) - \tilde{n}(N) , \quad \text{and} \quad 1 \ll \tilde{n}(N) - N. \]

For notational neatness, we will suppress the dependence of these generational parameters on \( N \): \( \tilde{n}(N) \equiv \tilde{n} \) and \( n(N) \equiv n \).

**Remark 5.3.** To enable the reader distinguish at a glance between arrays having the four distinct generational parameters \( N < \tilde{n} < n \ll n \), we will maintain a rigid indexing convention in which the arrays with generation numbers \( N, \tilde{n}, n, \) are respectively dummy indexed by the letters \( e, f, g, h \):

\[
\{ x_e \}_{e \in E_N}, \quad \{ x_f \}_{f \in E_{\tilde{n}}}, \quad \{ x_g \}_{g \in E_n}, \quad \{ x_h \}_{h \in E_n}.
\]

Recall that if \( \{ x_a \}_{a \in E_k} \) is an array and \( a \in E_{\ell} \) for some \( 0 \leq \ell \leq k \), then \( \{ x_a \}_{a \in E_{\ell} \cap E_k} \) refers to the subarray labeled by all \( a \in E_k \) canonically embedded in \( a \). From Definition 3.6, we can write

\[ X^{(N,n)}_e = Q^{n-N} \{ X^{(n)}_h \}_{h \in \ell \cap E_n} . \]

For any \( n \) between \( N \) and \( n \), this equality can be rewritten using the identity \( Q = L + E \) as

\[ X^{(N,n)}_e = L^{n-N} Q^{n-n} \{ X^{(n)}_h \}_{h \in \ell \cap E_n} + \sum_{k=1}^{n-N} L^{k-1} E Q^{n-N-k} \{ X^{(n)}_h \}_{h \in \ell \cap E_n} . \]

In Section 7.1 we show that replacing \( Q^{n-N-k} \) by the partial linearization \( L^{n-N-k} Q^{n-n} \) in the sum above yields negligible errors, and thus the above is approximately

\[ \approx L^{n-N} Q^{n-n} \{ X^{(n)}_h \}_{h \in \ell \cap E_n} + \sum_{k=1}^{n-N} L^{k-1} E L^{n-N-k} Q^{n-n} \{ X^{(n)}_h \}_{h \in \ell \cap E_n} . \]

For any \( \tilde{n} \) between \( N \) and \( n \), we can rearrange the above as

\[ = \left( L^{\tilde{n}-N} + \sum_{k=1}^{\tilde{n}-N} L^{k-1} E L^{\tilde{n}-N-k} \right) \left( L^{n-\tilde{n}} Q^{n-n} \{ X^{(n)}_h \}_{h \in \ell \cap E_n} \right) \]

\[ + L^{\tilde{n}-N} \left( \sum_{k=1}^{n-\tilde{n}} L^{k-1} E L^{n-\tilde{n}-k} \{ X^{(n)}_h \}_{h \in \ell \cap E_n} \right) . \]

The underbraced expressions above are central limit-type renormalized sums (recall Remark 3.6), and thus admit Gaussian approximations when \( n - \tilde{n} \gg 1 \) and \( \tilde{n} - N \gg 1 \):

(I) The variables in the array \( \{ Y^{(n)}_f \}_{f \in E_{\tilde{n}}} := L^{n-\tilde{n}} Q^{n-n} \{ X^{(n)}_h \}_{h \in \ell \cap E_n} \) are approximately distributed as

\[ Y^{(n)}_f \overset{d}{\approx} Y^{(n)}_f \sim \mathcal{N}(0,R(r-n)) \]

because the variables in the array \( Q^{n-n} \{ X^{(n)}_h \}_{h \in \ell \cap E_n} \) have variance approximately equal to \( R(r-n) \) by Lemma 3.9.
For $Z_{f}^{N,n} := \sum_{k=1}^{n-\tilde{n}} \mathcal{L}^{k-1} L \mathcal{L}^{n-k} \{X_{h}^{(n)}\}_{h \in f \cap E_{a}}$, the variable $Z_{e}^{N,n} := \mathcal{L}^{\tilde{n}-N} \{Z_{f}^{N,n}\}_{f \in \cap E_{a}}$ has approximate distribution

$$Z_{e}^{N,n} \overset{d}{\approx} Z_{e}^{(N)} \sim \mathcal{N}(0, \varsigma_{N}^{2}) \quad \text{where} \quad \varsigma_{N}^{2} := (n - \tilde{n})(R(r - n + 1) - R(r - n)) \quad \text{(5.6)}$$

The variance $\varsigma_{N}^{2}$ arises through approximations involving Remark 3.4, property (III) of Lemma 3.9 and the identity $M(R(r)) = R(r + 1)$ from Lemma 2.3.

The above line of heuristic reasoning suggests that variables in the array $\{X_{e}^{(N,n)}\}_{e \in E_{N}}$ are close in distribution to the variables in the array $\{\hat{X}_{e}^{(N)}\}_{e \in E_{N}}$ defined in (iii) of Definition 5.4 below.

**Definition 5.4.** For $\epsilon \in (0, 1/2)$ and $\hat{n}, n \in \mathbb{N}$ defined as in (5.2), let the i.i.d. arrays of random variables $\{Y_{f}^{N,n}\}_{f \in E_{a}}, \{Z_{e}^{N,n}\}_{e \in E_{N}}, \{Y_{f}^{(N)}\}_{f \in E_{a}}$ and $\{Z_{e}^{(N)}\}_{e \in E_{N}}$ be defined as in (I) and (II) above.

(i) We define variables in the array $\{\hat{X}_{e}^{N,n}\}_{e \in E_{N}}$ as

$$\hat{X}_{e}^{N,n} := \mathcal{L}^{\tilde{n}-N} \{Y_{f}^{N,n}\}_{f \in \cap E_{a}} + \sum_{k=1}^{\tilde{n}-N} \mathcal{L}^{k-1} L \mathcal{L}^{n-k} \{Y_{f}^{N,n}\}_{f \in \cap E_{a}} + Z_{e}^{N,n}.$$  

(ii) For $\{Y_{f}^{N,n}\}_{f \in E_{a}}$ and $\{Z_{e}^{N,n}\}_{e \in E_{N}}$ independent, we define the i.i.d. array $\{\hat{X}_{e}^{N,n}\}_{e \in E_{N}}$ to have variables with distribution

$$\hat{X}_{e}^{N,n} \overset{d}{=} \mathcal{L}^{\tilde{n}-N} \{Y_{f}^{N,n}\}_{f \in \cap E_{a}} + \sum_{k=1}^{\tilde{n}-N} \mathcal{L}^{k-1} L \mathcal{L}^{n-k} \{Y_{f}^{N,n}\}_{f \in \cap E_{a}} + Z_{e}^{(N)}.$$

(iii) For $\{Y_{f}^{(N)}\}_{f \in E_{a}}$ and $\{Z_{e}^{(N)}\}_{e \in E_{N}}$ independent, we define the i.i.d. array $\{\hat{X}_{e}^{(N)}\}_{e \in E_{N}}$ to have variables with distribution

$$\hat{X}_{e}^{(N)} := \mathcal{L}^{\tilde{n}-N} \{Y_{f}^{(N)}\}_{f \in \cap E_{a}} + \sum_{k=1}^{\tilde{n}-N} \mathcal{L}^{k-1} L \mathcal{L}^{n-k} \{Y_{f}^{(N)}\}_{f \in \cap E_{a}} + Z_{e}^{(N)}.$$

**Remark 5.5.** The superscripts of the variables $\hat{X}_{e}^{N,n}, \hat{X}_{e}^{N,n}, \hat{X}_{e}^{(N)}, Y_{f}^{N,n}, Y_{f}^{(N)}, Z_{f}^{N,n},$ and $Z_{e}^{(N)}$ refer to their dependence on the underlying generational parameters $N, n \in \mathbb{N}$ with $N < n$, whereas the superscript of $X_{e}^{(N,n)}$ (with the round brackets) denotes more specifically that the random variable $X_{e}^{(N,n)}$ is an element of a generation-$N$ layer of a $\mathcal{Q}$-pyramidal array generated from a generation-$n$ array, $\{X_{h}^{(n)}\}_{h \in E_{n}}$.

**Remark 5.6.** Note that the variable $\hat{X}_{e}^{N,n}$ is equal to (5.3). In the proof of Theorem 2.7 in the next section, $\hat{X}_{e}^{N,n}$ and $\hat{X}_{e}^{N,n}$ have the role of distributional intermediaries between $X_{e}^{(N,n)}$ and $\hat{X}_{e}^{(N)}$.

### 5.3 Proof of Theorem 3.14

For $N < n$ and $e \in E_{N}$, let the random variables $\hat{X}_{e}^{N,n}, \hat{X}_{e}^{N,n}$, and $\hat{X}_{e}^{(N)}$ be defined as in Section 5.2 for a regular sequence, $\{(X_{a}^{(n)})_{a \in E_{a}}\}_{n \in \mathbb{N}}$, of $\mathcal{Q}$-pyramidal arrays with parameter $r \in \mathbb{R}$ and some choice of $\epsilon \in (0, 1/2)$. The following lemmas imply that the pairs $(X_{e}^{(N,n)}, \hat{X}_{e}^{N,n})$, $(\hat{X}_{e}^{N,n}, \hat{X}_{e}^{N,n})$, and $(\hat{X}_{e}^{N,n}, \hat{X}_{e}^{(N)})$ satisfy the conditions (i) or (ii) of Proposition 5.1 after appropriate couplings of the variables for the latter two pairs.
Lemma 5.7. The random variables \( X_e^{(N,n)} - \tilde{X}_e^{N,n} \) and \( \tilde{X}_e^{N,n} \) are uncorrelated. Moreover, there is a sequence \( \{a_N\}_{N \in \mathbb{N}} \) of positive constants satisfying \( a_N = o(N^{-1}) \) with \( N \gg 1 \) such that for all large enough \( n \in \mathbb{N} \)

\[
\mathbb{E}\left[ (X_e^{(N,n)} - \tilde{X}_e^{N,n})^2 \right] < a_N.
\]

Lemma 5.8. There is a sequence of positive constants \( \{b_N\}_{N \in \mathbb{N}} \) that vanish super-polynomially with \( N \gg 1 \) such that for all large enough \( n \in \mathbb{N} \)

\[
\rho_2(\tilde{X}_e^{N,n}, \tilde{X}_e^{(N)}) < b_N.
\]

Lemma 5.9. There is a sequence of positive constants \( \{b_N\}_{N \in \mathbb{N}} \) that vanish super-polynomially with \( N \gg 1 \) such that for all large enough \( n \in \mathbb{N} \)

\[
\rho_2(\tilde{X}_e^{N,n}, \tilde{X}_e^{(N)}) < b_N.
\]

Remark 5.10. By definition of the metric \( \rho_2 \), Lemmas 5.8 & 5.9 imply that there are couplings \( (\tilde{X}_e^{N,n}, \tilde{X}_e^{(N)}) \) and \( (\tilde{X}_e^{N,n}, \tilde{X}_e^{(N)}) \) such that \( \mathbb{E}[\tilde{X}_e^{N,n} - \tilde{X}_e^{(N)}]^2 \) and \( \mathbb{E}[\tilde{X}_e^{N,n} - \tilde{X}_e^{(N)}]^2 \) are less than \( b_N^2 \).

The following easy corollary verifies the condition (5.1) in the statement of Proposition 5.1 for the pairs of random variables discussed above, and its proof is in Section 8.1

Corollary 5.11. For any \( s \in (r, \infty) \), the inequality \( \mathbb{E}[(U_e^{(N)})^2] < R(s - N) \) holds for \( U_e^{(N)} \) equal to \( X_e^{N,n}, \tilde{X}_e^{N,n}, \) and \( \tilde{X}_e^{(N)} \) for large enough \( N \) and \( n > N \).

Remark 5.12. The relevant sense of a given statement holding “for large enough \( N \) and \( n > N \)” will always be that there exists a constant \( \lambda > 0 \) and a function \( \Lambda : \mathbb{N} \to (0, \infty) \) such that the statement is true whenever \( N > \lambda \) and \( n > \Lambda(N) \).

Let us temporarily assume Proposition 5.1, Lemmas 5.7-5.9 and Corollary 5.11 to complete the remainder of the proof of Theorem 3.14.

Proof of Theorem 3.14. Let \( \{X_a^{(s,n)}\}_{a \in E_k} \) be a regular sequence of \( Q \)-pyramidal arrays of random variables with parameter \( r \in \mathbb{R} \). By Remark 3.15 it suffices for us to focus on distributional convergence in the case \( k = 0 \), in which the array \( \{X_a^{(0,n)}\}_{a \in E_k} \) consists of a single random variable, \( X^{(0,n)} \). We have divided the analysis below into parts (a)-(d).

(a) Setting Up: For \( N \leq n \) let the arrays of random variables \( \{\tilde{X}_e^{N,n}\}_{e \in E_N}, \{\tilde{X}_e^{N,n}\}_{e \in E_N}, \) and \( \{\tilde{X}_e^{(N)}\}_{e \in E_N} \) be defined as in Definition 5.4. We will show that the \( \rho_2 \) distance between \( X^{(0,n)} \) and \( Q^N \{\tilde{X}_e^{(N)}\}_{e \in E_N} \) converges to zero as \( N \) and \( n \) grow. Writing \( X^{(0,n)} = Q^N \{X_e^{(N,n)}\}_{e \in E_N} \) and applying the triangle inequality yields

\[
\rho_2\left( X^{(0,n)}, Q^N \{\tilde{X}_e^{(N)}\}_{e \in E_N} \right) \leq \rho_2\left( Q^N \{X_e^{(N,n)}\}_{e \in E_N}, Q^N \{\tilde{X}_e^{N,n}\}_{e \in E_N} \right) + \rho_2\left( Q^N \{\tilde{X}_e^{N,n}\}_{e \in E_N}, Q^N \{\tilde{X}_e^{N,n}\}_{e \in E_N} \right) + \rho_2\left( Q^N \{\tilde{X}_e^{N,n}\}_{e \in E_N}, Q^N \{\tilde{X}_e^{(N)}\}_{e \in E_N} \right).
\]
For any particular couplings of the above three pairs of random variables, we have
\[
\begin{align*}
&\leq \mathbb{E} \left[ \left( Q^N \{ X_e^{(N,n)} \}_{e \in E_N} - Q^N \{ \tilde{X}_e^{(N,n)} \}_{e \in E_N} \right)^2 \right]^{\frac{1}{2}} \\
&\quad + \mathbb{E} \left[ \left( Q^N \{ \tilde{X}_e^{(N,n)} \}_{e \in E_N} - Q^N \{ \tilde{X}_e^{(N)} \}_{e \in E_N} \right)^2 \right]^{\frac{1}{2}} \\
&\quad + \mathbb{E} \left[ \left( Q^N \{ \tilde{X}_e^{(N)} \}_{e \in E_N} - Q^N \{ \tilde{X}_e^{(N)} \}_{e \in E_N} \right)^2 \right]^{\frac{1}{2}}. \tag{5.7}
\end{align*}
\]

The random variables $Q^N \{ \tilde{X}_e^{(N,n)} \}_{e \in E_N}$ and $Q^N \{ X_e^{(N,n)} \}_{e \in E_N}$ are already defined in the same probability space, and we will not require any special coupling between them. Notice that the expressions on the right side above have the form of those expressions bounded in Proposition 5.1.

(b) Verifying the conditions of Proposition 5.1: By Lemma 5.7 the variables $X_e^{(N,n)} - \tilde{X}_e^{(N,n)}$ and $\tilde{X}_e^{(N)}$ are uncorrelated, and there is a positive sequence $\{ a_N \}_{N \in \mathbb{N}}$ with $a_N = o(N^{-1})$ such that
\[
\mathbb{E} \left[ (X_e^{(N,n)} - \tilde{X}_e^{(N,n)})^2 \right] < a_N^2
\]
for large enough $n$. By Lemmas 5.8 & 5.9 there is a positive sequence $\{ b_N \}_{N \in \mathbb{N}}$ with $b_N = o(N^{-2})$ and i.i.d. couplings $\{ (\tilde{X}_e^{(N,n)}, \tilde{X}_e^{(N,n)}) \}_{e \in E_N}$ and $\{ (\tilde{X}_e^{(N)}, \tilde{X}_e^{(N)}) \}_{e \in E_N}$ such that
\[
\mathbb{E} \left[ (\tilde{X}_e^{(N,n)} - \tilde{X}_e^{(N,n)})^2 \right] < b_N^2 \quad \text{and} \quad \mathbb{E} \left[ (\tilde{X}_e^{(N,n)} - \tilde{X}_e^{(N)})^2 \right] < b_N^2
\]
for large enough $n$. Corollary 5.11 implies that the arrays $\{ X_e^{(N,n)} \}_{e \in E_N}$, $\{ \tilde{X}_e^{(N,n)} \}_{e \in E_N}$, $\{ \tilde{X}_e^{(N)} \}_{e \in E_N}$ satisfy condition (5.1) of Proposition 5.1 for any $s \in (r, \infty)$ and large $N, n \in \mathbb{N}$. Moreover, the above considerations imply that for large enough $N$ and $n > N$ we have the following:

- the array $\{ (\tilde{X}_e^{(N,n)}, X_e^{(N,n)}) \}_{e \in E_N}$ satisfies the conditions for part (ii) of Proposition 5.1 with $(\tilde{X}_e^{(N,n)}, X_e^{(N,n)}) = (T_e^{(N)}, V_e^{(N)})$,
- the arrays $\{ (\tilde{X}_e^{(N,n)}, \tilde{X}_e^{(N)}) \}_{e \in E_N}$ satisfy the conditions for part (i) of Proposition 5.1 and
- the arrays $\{ (\tilde{X}_e^{(N,n)}, \tilde{X}_e^{(N)}) \}_{e \in E_N}$ satisfy the conditions for part (i) of Proposition 5.1

(c) Returning to (5.7): Therefore with three applications of Proposition 5.1 to the right side of (5.7) there is a $C > 0$ such that for large enough $N, n \in \mathbb{N}$ we have the first inequality below.
\[
\begin{align*}
\rho_2 \left( X^{(0,n)}, Q^N \{ \tilde{X}_e^{(N)} \}_{e \in E_N} \right) \\
\leq CN \mathbb{E} \left[ (X_e^{(N,n)} - \tilde{X}_e^{(N,n)})^2 \right]^{\frac{1}{2}} + CN^2 \mathbb{E} \left[ (\tilde{X}_e^{(N,n)} - \tilde{X}_e^{(N,n)})^2 \right]^{\frac{1}{2}} + CN^2 \mathbb{E} \left[ (\tilde{X}_e^{(N,n)} - \tilde{X}_e^{(N)})^2 \right]^{\frac{1}{2}} \\
\leq CN a_N + CN^2 b_N + CN^2 b_N.
\end{align*}
\]

The second inequality holds by Lemmas 5.7–5.9 As $N \to \infty$ the above goes to zero by the asymptotic properties of $a_N$ and $b_N$.

(d) Connecting with the random array constructed in Section 4: We have established that the $\rho_2$ distance between $X^{(0,n)}$ and $Q^N \{ \tilde{X}_e^{(N)} \}_{e \in E_N}$ vanishes as $n$ and $N$ grow. Let $\{ X_a^{(k)} \}_{a \in E_k}$ be the sequence in $k \in \mathbb{N}_0$ of arrays of random variables constructed in Section 4 for parameter $r \in \mathbb{R}$. Note that the arrays $\{ X_a^{(k)} \}_{a \in E_k}$ form a regular sequence of $Q$-pyramidical arrays of random variables.
with no \( n \in \mathbb{N} \) dependence. Thus we can apply our above result with \( \{ X^{(k,n)}_{a} \}_{a \in E_{k}} := \{ X^{(k)}_{a} \}_{a \in E_{k}} \) to get that the \( \rho_2 \)-distance between \( X^{(0)} \) and \( Q^{N} \{ \tilde{X}_{e}^{(N)} \} \in E_{N} \) converges to zero as \( N \to \infty \). Therefore, \( \rho_2( X^{(0,n)}, X^{(0)} ) \) vanishes with large \( n \) and the law of \( X^{(0)} \) must be unique.

\[ \square \]

6 Proof of Proposition 5.1

Proof of Proposition 5.1. For \( 0 \leq k \leq N \), define the i.i.d. arrays of random variables

\[
\{ U_{a}^{(k,N)} \}_{a \in E_{k}} := Q^{N-k} \{ U_{e}^{(N)} \}_{e \in E_{N}} \quad \text{and} \quad \{ V_{a}^{(k,N)} \}_{a \in E_{k}} := Q^{N-k} \{ V_{e}^{(N)} \}_{e \in E_{N}}
\]

and \( W_{a}^{(k,N)} := V_{a}^{(k,N)} - U_{a}^{(k,N)} \). The variables \( U_{a}^{(k,N)} \), \( V_{a}^{(k,N)} \), and \( W_{a}^{(k,N)} \) have mean zero, and \( U_{a}^{(k,N)} \) has variance

\[
\left( \sigma_{k}^{(N)} \right)^{2} := \text{Var}(U_{a}^{(k,N)}) = M^{N-k}(\left( \sigma_{k}^{(N)} \right)^{2}) \leq M^{N-k}(R(s - N)) = R(s - k) \quad (6.1)
\]

for \( (\sigma^{(N)})^{2} := \mathbb{E}[(U_{e}^{(N)})^{2}] \) by Remark 3.4. The inequality uses our assumption that the variance of \( U_{e}^{(N)} \) is smaller than \( R(s - N) \) and the last equality is property (I) of Lemma 2.3.

We have the following recursive relation for the variables \( W_{a}^{(k,N)} \)

\[
W_{a}^{(k,N)} = \frac{1}{b} \sum_{i=1}^{b} \prod_{j=1}^{b} \left( 1 + U_{a \times (i,j)}^{(k+1,N)} + W_{a \times (i,j)}^{(k+1,N)} \right) - \frac{1}{b} \sum_{i=1}^{b} \prod_{j=1}^{b} \left( 1 + U_{a \times (i,j)}^{(k+1,N)} \right).
\]

Expanding the products on the left and cancelling yields

\[
= \frac{1}{b} \sum_{i=1}^{b} \left( \sum_{j=1}^{b} W_{a \times (i,j)}^{(k+1,N)} + \sum_{1 \leq j, J \leq b} W_{a \times (i,j)}^{(k+1,N)} U_{a \times (i,J)}^{(k+1,N)} \right) + \sum_{A,B \subseteq \{1,...,b\}} A \cap B = \emptyset \text{ and } |A| \geq 1 \text{ or } |B| \geq 2 \left( \prod_{j \in A} W_{a \times (i,j)}^{(k+1,N)} \prod_{j \in B} U_{a \times (i,j)}^{(k+1,N)} \right). \tag{6.2}
\]

Since the arrays are i.i.d. and centered, the recursive formula above shows, by induction, that if \( W_{a}^{(N)} := V_{a}^{(N)} - U_{a}^{(N)} \) is uncorrelated with \( U_{e}^{(N)} \) for all \( e \in E_{N} \) then \( W_{a}^{(k,N)} \) is uncorrelated with \( U_{a}^{(k,N)} \) for all \( 0 \leq k < N \) and \( a \in E_{k} \). In particular if \( U_{e}^{(N)} \) and \( V_{e}^{(N)} - U_{e}^{(N)} \) are uncorrelated, then \( Q^{N} \{ U_{e}^{(N)} \} \in E_{N} \) and \( Q^{N} \{ V_{e}^{(N)} \} \in E_{N} \) are uncorrelated.

Define the multivariate polynomial

\[
P(x, y, z) := \sum_{A,B \subseteq \{1,...,b\}} A \cap B = \emptyset \text{ and } |A| \geq 1 \text{ or } |B| \geq 2 \left( \min(|A|, |B|) \sum_{u=0}^{\min(|A|, |B|)} \binom{|A|}{u} \binom{|B|}{u} x^{|A| - u} y^{|B| - u} z^{|u|} \right).
\]

Let \( \left( \phi_{k}^{(N)} \right)^{2} \) denote the second moment of \( W_{a}^{(k,N)} \) and define \( u_{k}^{(N)} := \mathbb{E}[U_{a}^{(k,N)} W_{a}^{(k,N)}] \). Taking the second moment of (6.2) yields

\[
\left( \phi_{k}^{(N)} \right)^{2} = \left( \phi_{k+1}^{(N)} \right)^{2} + (b - 1) \left( \phi_{k+1}^{(N)} \right)^{2} \left( \sigma_{k+1}^{(N)} \right)^{2} + (b - 1) \left( u_{k+1}^{(N)} \right)^{2} + P \left( \left( \phi_{k+1}^{(N)} \right)^{2}, \left( \sigma_{k+1}^{(N)} \right)^{2}, u_{k+1}^{(N)} \right).
\]
Define $\epsilon \in \{0, 2\}$ as $\epsilon = 0$ when $U_e^{(N)}$ and $W_e^{(N)} = V_e^{(N)} - U_e^{(N)}$ are uncorrelated and as $\epsilon = 2$ otherwise. Using that $\kappa^2 := \frac{2}{\kappa - 1}$ and applying Cauchy-Schwarz to bound $u_{k+1}^{(n)}$ by $\sigma_{k+1}^{(N)} u_{k+1}^{(n)}$ yields

$$
\leq \left( \vartheta_{k+1}^{(N)} \right)^2 + \frac{2 + \epsilon}{\kappa^2} \left( \vartheta_{k+1}^{(N)} \right)^2 \sigma_{k+1}^{(N)^2} + P \left( \left( \vartheta_{k+1}^{(N)} \right)^2, \sigma_{k+1}^{(N)}, u_{k+1}^{(n)} \right).
$$

There exists a $c > 0$ such that $P(x, y, z) \leq cx(x+y^2)$ for all $(x, y, z)$ with $0 \leq x, y \leq 1$ and $|z| \leq \sqrt{x+y}$, so thus we have

$$
\leq \left( \vartheta_{k+1}^{(N)} \right)^2 + \frac{2 + \epsilon}{\kappa^2} \left( \vartheta_{k+1}^{(N)} \right)^2 \sigma_{k+1}^{(N)^2} + c \left( \vartheta_{k+1}^{(N)} \right)^2 \left( \vartheta_{k+1}^{(N)} \right)^2 (R(s-k)-1) + c \left( \vartheta_{k+1}^{(N)} \right)^2 \left( \vartheta_{k+1}^{(N)} \right)^2 (R(s-k-1))^2,
$$

where the last inequality follows from (6.1).

We will assume that $s < -1$ in the analysis below so that the terms $\ell - s$ in the sums over $\ell \in \mathbb{N}$ below are positive and bounded away from 0. The general case uses the same ideas but with somewhat messier expressions that involve separating out the finite part of the sums over $\ell \in \mathbb{N}$ with $\ell - s < 2$.

Recall that $\left( \vartheta_{k}^{(N)} \right)^2 := \mathbb{E} \left[ (V_e^{(N)} - U_e^{(N)})^2 \right]$ for $e \in E_N$. Suppose that $\left( \vartheta_{k}^{(N)} \right)^2 < \delta/N^{2+\epsilon}$, where

$$
\delta := \left( \inf_{N \in \mathbb{N}} \frac{(R(s-k))^{2+\epsilon} N^{2+\epsilon}}{(N-s)^{2+\epsilon}} \right)^2 \exp \left\{ -\frac{2 + \epsilon}{\kappa^2} \sum_{\ell=1}^{\infty} \left( \frac{R(s-\ell)}{ \ell - s} \right) - \frac{\kappa^2}{\ell - s} \left( \frac{R(s-\ell)}{\ell - s} \right)^2 \right\}.
$$

Note that $\delta > 0$ since property (II) in Lemma 2.3 implies that the series $\sum_{\ell=1}^{\infty} (R(s-\ell) - \frac{\kappa^2}{\ell - s})$ and $\sum_{\ell=1}^{\infty} (R(s-\ell))^2$ are summable and the asymptotics $R(-r) \sim \frac{\kappa^2}{r}$ with $r \gg 1$ implies that the infimum is nonzero.

Let $k^{(N)}$ be the smallest $k \in \mathbb{N}$ such that $\left( \vartheta_{k}^{(N)} \right)^2 \leq \left( R(s-k) \right)^2$. For $k \in \mathbb{N}$ with $k+1 \in [k^{(N)}, N]$, we have the inequality

$$
\left( \vartheta_{k}^{(N)} \right)^2 \leq \left( \vartheta_{k+1}^{(N)} \right)^2 + \frac{2 + \epsilon}{\kappa^2} \left( \vartheta_{k+1}^{(N)} \right)^2 R(s-k-1) + 2c \left( \vartheta_{k+1}^{(N)} \right)^2 \left( R(s-k-1) \right)^2
\leq \left( \vartheta_{k+1}^{(N)} \right)^2 \exp \left\{ \frac{2 + \epsilon}{\kappa^2} R(s-k-1) + 2c \left( R(s-k-1) \right)^2 \right\}.
$$

Applying the above recursively and rearranging yields

$$
\leq \left( \vartheta_{N}^{(N)} \right)^2 \exp \left\{ \frac{2 + \epsilon}{\kappa^2} \sum_{\ell=k+1}^{N} \left( R(s-\ell) \right)^2 \right\}
= \left( \vartheta_{N}^{(N)} \right)^2 \exp \left\{ \left( 2 + \epsilon \right) \sum_{\ell=k+1}^{N} \frac{1}{\ell - s} + \frac{2 + \epsilon}{\kappa^2} \sum_{\ell=k+1}^{N} \left( R(s-\ell) - \frac{\kappa^2}{\ell - s} \right)
+ 2c \sum_{\ell=k+1}^{N} \left( R(s-\ell) \right)^2 \right\}.
$$

The sum $\sum_{\ell=k+1}^{N} \frac{1}{\ell - s}$ is a Riemann lower bound for $\int_{k}^{N} \frac{1}{t-s}dt = \log \left( \frac{N-s}{k-s} \right)$, so the above is smaller than

$$
\leq \left( \vartheta_{N}^{(N)} \right)^2 \left( \frac{N-s}{k-s} \right)^{2+\epsilon} \exp \left\{ \frac{2 + \epsilon}{\kappa^2} \sum_{\ell=k+1}^{N} \left( R(s-\ell) - \frac{\kappa^2}{\ell - s} \right) + 2c \sum_{\ell=k+1}^{N} \left( R(s-\ell) \right)^2 \right\}.
$$
By definition of $\delta > 0$

$$\leq \frac{(R(s - k))^2}{\delta} N^{2+\epsilon} \left( \varrho_N^{(N)} \right)^2.$$ 

Notice that $(\varrho_N^{(N)})^2$ is smaller than $(R(s - k))^2$ because $(\varrho_N^{(N)})^2 < \delta / N^{2+\epsilon}$. Hence, $k \geq k^{(N)}$ and by induction on $k$ we can deduce that $k^{(N)} = 0$. Therefore we can apply the above inequality with $k = 0$ to get

$$\mathbb{E} \left[ (Q^N \{ V_e^{(N)} \}_{e \in E_N} - Q^N \{ U_e^{(N)} \}_{e \in E_N} )^2 \right] = (\varrho_0^{(N)})^2 \leq C^2 N^{2+\epsilon} \left( \varrho_N^{(N)} \right)^2,$$

where $C := R(s)/\delta^2$. Since $(\varrho_N^{(N)})^2 := \mathbb{E}[(V_e^{(N)} - U_e^{(N)})^2]$, the proof is complete. 

7 The three approximation lemmas

In this section, we will prove Lemmas 5.7 & 5.9. Recall from Sections 5.2 & 5.3 that Lemma 5.7 involves bounding the error of a partial linearization of the map $Q^{n-N}$ when acting on the array $\{X_h^{(n)}\}_{h \in E_n}$ and Lemmas 5.8 & 5.9 are both Gaussian approximations driven by central limit-type renormalization that occurs at different generational scales.

7.1 Proof of Lemma 5.7

Proof of Lemma 5.7: The variables $X_e^{(N,n)}$ and $\hat{X}_e^{N,n}$ are uncorrelated by Lemma 3.5 and have mean zero, so the $L^2$-distance between $X_e^{(N,n)}$ and $\hat{X}_e^{N,n}$ can be written as

$$\mathbb{E} \left[ (X_e^{(N,n)} - \hat{X}_e^{N,n})^2 \right] = \mathbb{E} \left[ (X_e^{(N,n)})^2 \right] - \mathbb{E} \left[ (\hat{X}_e^{N,n})^2 \right],$$

and by definition of $\hat{X}_e^{N,n}$ the above is equal to

$$= \mathbb{E} \left[ (X_e^{(N,n)})^2 \right] - \mathbb{E} \left[ \left( \mathcal{L}^{n-N} \{ X_g^{(n,n)} \}_{g \in e \cap E_n} + \sum_{k=1}^{n-N} \mathcal{L}^{k-1} \mathcal{E} \mathcal{L}^{n-N-k} \{ X_g^{(n,n)} \}_{g \in e \cap E_n} \right)^2 \right]$$

$$= \mathbb{E} \left[ (X_e^{(N,n)})^2 \right] - \left( \mathbb{E} \left[ \left( \mathcal{L}^{n-N} \{ X_g^{(n,n)} \}_{g \in e \cap E_n} \right)^2 \right] + \sum_{k=1}^{n-N} \mathbb{E} \left[ \left( \mathcal{L}^{k-1} \mathcal{E} \mathcal{L}^{n-N-k} \{ X_g^{(n,n)} \}_{g \in e \cap E_n} \right)^2 \right] \right).$$

For $a \in E_k$ with $0 \leq k \leq n$, the random variable $X_{a}^{(n,k)} := Q^{n-k} \{ X_h^{(n)} \}_{h \in a \cap E_n}$ has variance $\sigma_{k,n}^2 := M^{n-k}(\sigma_n^2)$, where $\sigma_n^2 := \text{Var}(X_n^{(n)})$. By Remark 3.4, we can write the above as

$$= \sigma_{N,n}^2 - \sigma_{n,n}^2 - (\mathbf{n} - N) \left( M(\sigma_{n,n}^2) - \sigma_{n,n}^2 \right).$$

For any fixed $k \in \mathbb{N}_0$ the sequence $\sigma_{k,n}^2$ converges as $n \to \infty$ to $R(r-k)$ since a regular sequence of $Q$-pyramidal arrays of random variables, $\{ X_{a}^{(n)} \}_{a \in E_n}$ with parameter $r$ by definition satisfies property (III) in the statement of Lemma 3.9. It follows that there is a sequence $\{ \xi_N(n) \}_{n \in \mathbb{N}}$ such that $\xi_N(n) \to 0$ as $n \to \infty$ for each $N$, and the above is equal to

$$= R(r-N) - R(r-n) - (\mathbf{n} - N) \left( M(R(r-n)) - R(r-n) \right) + \xi_N(n)$$

$$= R(r-N) - R(r-n) - (\mathbf{n} - N) \left( R(r-n+1) - R(r-n) \right) + \xi_N(n).$$

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By (II) of Lemma 2.3, expanding $R(r - N)$ and $R(r - n)$ with large $N$ yields
\[
= \left( \frac{\kappa^2}{N - r} + \frac{\eta \kappa^2 \log N}{N^2} + O \left( \frac{\log^2 N}{N^3} \right) \right) - \left( \frac{\kappa^2}{n - r} + \frac{\eta \kappa^2 \log n}{n^2} + O \left( \frac{\log^2 n}{n^3} \right) \right) \\
- (n - N) \left( \frac{\kappa^2}{n - r - 1} + \frac{\eta \kappa^2 \log n}{n^2} + O \left( \frac{\log^2 n}{n^3} \right) \right) - \left( \frac{\kappa^2}{n - r} + \frac{\eta \kappa^2 \log n}{n^2} + O \left( \frac{\log^2 n}{n^3} \right) \right) \\
+ \xi_N(n) .
\]
Since $n - N = O(N^\epsilon)$ with $N \gg 1$, we have that
\[
= (n - N) \left( \frac{\kappa^2}{(N - r)^2} + O \left( \frac{1}{N^{3 - \epsilon}} \right) \right) - (n - N) \left( \frac{\kappa^2}{(n - r)^2} + O \left( \frac{\log n}{n^3} \right) \right) + \xi_N(n) .
\]
The above is $O(1/N^{3 - 2\epsilon}) + \xi_N(n)$, which gives us the result by the restriction $\epsilon \in (0, 1/2)$.

\[7.2\] A generalization of Stein’s auxiliary functions

Before moving to the proof of Lemma 5.8 we will discuss a generalized version of the auxiliary functions used in Stein’s method [26], which is a general strategy for proving the central limit theorem under the Wasserstein-1 metric. For random variables $X$ and $Y$ with $\mathbb{E}[|X|], \mathbb{E}[|Y|] < \infty$, the Wasserstein-1 distance has the dual form
\[
\rho_1(X, Y) = \sup_{H \in \text{Lip}_1} \left( \mathbb{E}[H(X)] - \mathbb{E}[H(Y)] \right) ,
\]
where $\text{Lip}_1$ is the collection of all Lipshitz functions on $\mathbb{R}$ with Lipshitz constant $\leq 1$. Given $H \in \text{Lip}_1$ define the auxiliary function $f : \mathbb{R} \to \mathbb{R}$
\[
f(x) := e^{\frac{x^2}{2}} \int_{-\infty}^{x} \left( H(t) - \hat{H} \right) e^{-\frac{t^2}{2}} dt , \quad \text{where} \quad \hat{H} := \int_{-\infty}^{\infty} H(r) e^{-\frac{r^2}{2}} \sqrt{2\pi} dr . \quad (7.1)
\]
The function $f$ solves the differential equation
\[
H(x) - \hat{H} := f'(x) - xf(x) \quad (7.2)
\]
and has the following convenient uniform bounds on its first two derivatives:
\[
\sup_{x \in \mathbb{R}} |f'(x)| \leq 1 \quad \text{and} \quad \sup_{x \in \mathbb{R}} |f''(x)| \leq 2 . \quad (7.3)
\]
Thus if $X$ is a random variable with finite variance and $X \sim \mathcal{N}(0, 1)$ then
\[
\mathbb{E}[H(X)] - \mathbb{E}[H(X)] = \mathbb{E}[f'(X)] - \mathbb{E}[Xf(X)] . \quad (7.4)
\]
The usefulness of the auxiliary function, $f$, is that the Wasserstein-1 distance between the distributions of $X$ and $X$ can be reduced to a quantity only involving $X$.

We will require a perturbative generalization of Stein’s method that bounds the Wasserstein-1 distance between random variables of the form $X := Y + Z$ and $X := Y + Z$ for variables $Y, Z, Z$ satisfying that $Z$ is centered with $\text{Var}(Z) = 1$ and $Z \sim \mathcal{N}(0, 1)$ is independent of $Y$. In other words, we would like to show how to bound the error of replacing the random variable $Z$ with a standard normal $Z$ independent of $Y$. In this case we will define an auxiliary function $F : \mathbb{R}^2 \to \mathbb{R}$ for a given $H \in \text{Lip}_1$ that satisfies the following partial differential equation analogous to (7.2):
\[
H(y + z) - \int_{\mathbb{R}} H(y + r) e^{-\frac{r^2}{2}} \sqrt{2\pi} dr := \partial_z F(y, z) - zF(y, z) . \quad (7.5)
\]
The following proposition, whose proof is in Section 8.2, provides bounds for the first- and second-order partial derivatives of $F$ in analogy to (7.3).
Proposition 7.1. Define $F : \mathbb{R}^2 \to \mathbb{R}$ for $H \in \text{Lip}_1$ through the formula

$$F(y, z) := e^{\frac{y^2}{2}} \int_{-\infty}^{z} \left( H(y + t) - \int_{\mathbb{R}} H(y + r) e^{-\frac{r^2}{2\pi}} dr \right) e^{-\frac{r^2}{2\pi}} dt .$$

For all $(y, z) \in \mathbb{R}^2$,

$$|\partial_y F(y, z)| \leq \sqrt{\pi/2} , \quad |\partial_z F(y, z)| \leq 1 , \quad \text{and} \quad |\partial_y^2 F(y, z)|, |\partial_y \partial_z F(y, z)|, |\partial_z^2 F(y, z)| \leq 2 .$$

Corollary 7.2. Define $F_\sigma : \mathbb{R}^2 \to \mathbb{R}$ for $H \in \text{Lip}_1$ through the formula

$$F_\sigma(y, z) := \frac{1}{\sigma} e^{\frac{y^2}{2\sigma^2}} \int_{-\infty}^{z} \left( H(y + t) - \int_{\mathbb{R}} H(y + r) e^{-\frac{r^2}{2\sigma^2}} dr \right) e^{-\frac{r^2}{2\sigma^2}} dt .$$

The function $F_\sigma(y, z)$ solves the partial differential equation

$$H(y + z) - \int_{\mathbb{R}} H(y + r) e^{-\frac{r^2}{2\sigma^2}} dr = \sigma \frac{\partial}{\partial z} F_\sigma(y, z) - \frac{z}{\sigma} F_\sigma(y, z) ,$$

and for all $(y, z) \in \mathbb{R}^2$,

$$|\partial_y F_\sigma(y, z)| \leq \sqrt{\pi/2} , \quad |\partial_z F_\sigma(y, z)| \leq 1 , \quad \text{and} \quad |\partial_y^2 F_\sigma(y, z)|, |\partial_y \partial_z F_\sigma(y, z)|, |\partial_z^2 F_\sigma(y, z)| \leq 2 / \sigma .$$

Proof. Define $\tilde{F}_\sigma(y, z) := \frac{1}{\sigma} F_\sigma(\sigma y, \sigma z)$ and $\tilde{H}_\sigma(z) := \frac{1}{\sigma} H(\sigma z)$. Then

$$\tilde{F}_\sigma(y, z) := e^{\frac{y^2}{2\sigma^2}} \int_{-\infty}^{z} \left( \tilde{H}_\sigma(y + t) - \int_{\mathbb{R}} \tilde{H}_\sigma(y + r) e^{-\frac{r^2}{2\sigma^2}} dr \right) e^{-\frac{r^2}{2\sigma^2}} dt .$$

Since $\tilde{H}_\sigma(z) \in \text{Lip}_1$, it follows from Proposition 7.1 that the first- and second-order derivatives of $\tilde{F}_\sigma$ have the bounds therein. From the equation $F_\sigma(y, z) = \sigma \hat{F}_\sigma(y / \sigma, z / \sigma)$ we see that the derivatives of $F_\sigma$ have the desired bounds. \qed

7.3 Proof of Lemma 5.8

Recall that $Y_f^{N, n}$ is defined as in (5.5), $Z_f^{N, n}$ is defined above (5.6), and $\hat{X}_e^{N, n}$ and $\hat{X}_f^{N, n}$ are defined as in Definition 5.4. We will need the following lemma, which collects some statements about the second and fourth moments of these variables.

Lemma 7.3. For $a \in E_k$ define $\sigma_k^{2, n} := \text{Var}(X_a^{(k, n)})$. The statements below hold for $e \in E_N$ and $f \in E_n$.

(i) The variance of $Y_f^{N, n} := \mathcal{L} \cap \hat{n} \left\{ X_g^{(n, n)} \right\}_{g \in f \cap E_n}$ is $\sigma^{2, n}_{n, n}$, and $\lim_{n \to \infty} \sigma_{n, n}^{2, n} = R(r - n)$.

(ii) The variance of $Z_f^{N, n}$ has the large $n$ convergence

$$\lim_{n \to \infty} \sigma_{n, n}^{2, n} = R(r - n).$$

In particular $\sigma_{n, n}^{2, n}$ is bounded by a constant multiple of $(n - \hat{n})(r - n + 1) - R(r - n)$.

(iii) There is a $C > 0$ such that for any $N \in \mathbb{N}$

$$\limsup_{n \to \infty} \mathbb{E} \left[ |Y_f^{N, n}|^4 \right] \leq \frac{C}{N^2} \quad \text{and} \quad \limsup_{n \to \infty} \mathbb{E} \left[ |Z_f^{N, n}|^4 \right] \leq \frac{C(n - \hat{n})^2}{N^4} .$$
(iv) There is a $C > 0$ such that for any $N \in \mathbb{N}$

$$
\limsup_{n \to \infty} \mathbb{E}\left[|\hat{X}^{N,n}_e|^4\right] < \frac{C}{N^2} \quad \text{and} \quad \limsup_{n \to \infty} \mathbb{E}\left[|\hat{X}^{N,n}_e|^4\right] < \frac{C}{N^2}.
$$

The lemma below follows easily from Holder’s inequality and the definition of Wasserstein-$p$ distance.

**Lemma 7.4.** Let $X$ and $Y$ be random variables with finite fourth moments. We have the following bound on the Wasserstein-2 distance between $X$ and $Y$ using the Wasserstein-1 distance:

$$
\rho_2(X, Y) \leq (\rho_1(X, Y))^\frac{1}{2} \left(\mathbb{E}[|X|^4]^{\frac{1}{2}} + \mathbb{E}[|Y|^4]^{\frac{1}{2}}\right).
$$

**Proof of Lemma 7.4.** This proof is divided into parts (a)-(g).

(a) **Notation:** For $e \in E_N$ we can write $\hat{X}^{N,n}_e$ and $\hat{X}^{N,n}_e$ in the forms

$$
\hat{X}^{N,n}_e = \overline{X}^{N,n}_e + \overline{Y}^{N,n}_e + \overline{Z}^{N,n}_e \quad \text{and} \quad \hat{X}^{N,n}_e = \overline{X}^{N,n}_e + \overline{Y}^{N,n}_e + \overline{Z}^{(N)}_e,
$$

where the random variables $\overline{X}^{N,n}_e$, $\overline{Y}^{N,n}_e$, $\overline{Z}^{N,n}_e$ are defined as

$$
\overline{X}^{N,n}_e := \mathcal{L}^{\hat{n}-N}\{Y^{N,n}_f\}_{f \in e \cap E_\hat{n}},
$$

$$
\overline{Y}^{N,n}_e := \sum_{k=1}^N \mathcal{L}^{\hat{n}-N-k}\{Y^{N,n}_f\}_{f \in e \cap E_\hat{n}},
$$

$$
\overline{Z}^{N,n}_e := \mathcal{L}^{\hat{n}-N}\{Z^{(N)}_f\}_{f \in e \cap E_\hat{n}},
$$

and recall that $\overline{Z}^{(N)}_e$ is the normal random variable (independent of $\overline{X}^{N,n}_e$ and $\overline{Y}^{N,n}_e$) defined in (5.6).

(b) **Stein’s method:** Next we will use Stein’s method to bound the Wasserstein-1 distance between $\hat{X}^{N,n}_e$ and $\hat{X}^{N,n}_e$. By definition,

$$
\rho_1(\hat{X}^{N,n}_e, \hat{X}^{N,n}_e) = \sup_{H \in \text{Lip}_1} \left| \mathbb{E}[H(\hat{X}^{N,n}_e)] - \mathbb{E}[H(\hat{X}^{N,n}_e)] \right|.
$$

For a given $H : \mathbb{R} \to \mathbb{R}$ with Lipschitz constant less than 1, define $F : \mathbb{R}^2 \to \mathbb{R}$ as in Corollary 7.2 with $\sigma := \varsigma_N$. Then $F$ is a solution to the partial differential equation

$$
H(x + z) - \mathbb{E}[H(x + Z^{(N)}_e)] = \varsigma_N \partial_2 F(x, z) - \frac{z}{\varsigma_N} F(x, z),
$$

where the expectation is w.r.t. $Z^{(N)}_e \sim \mathcal{N}(0, \varsigma^2_N)$. By Corollary 7.2, the first-order partial derivatives of $F$ are bounded by $\sqrt{\pi/2}$ and the second-order partial derivatives are bounded by $2/\varsigma_N$.

To bound the expression in the supremum of (7.6), we must bound the absolute value of

$$
\mathbb{E}\left[\varsigma_N \partial_2 F(\overline{X}^{N,n}_e + \overline{Y}^{N,n}_e, \overline{Z}^{N,n}_e) - \frac{Z^{N,n}_e}{\varsigma_N} F(\overline{X}^{N,n}_e + \overline{Y}^{N,n}_e, \overline{Z}^{N,n}_e)\right].
$$

Since $\overline{Z}^{N,n}_e$ is a sum over $f \in e \cap E_\hat{n}$ of terms $\frac{1}{\hat{n}-N} Z^{N,n}_f$, the above can be written as

$$
= \varsigma_N \mathbb{E}\left[\partial_2 F(\overline{X}^{N,n}_e + \overline{Y}^{N,n}_e, \overline{Z}^{N,n}_e)\right] - \frac{\varsigma^{-1}}{\hat{n}-N} \sum_{f \in e \cap E_\hat{n}} \mathbb{E}\left[Z^{N,n}_f F(\overline{X}^{N,n}_e + \overline{Y}^{N,n}_e, \overline{Z}^{N,n}_e)\right],
$$

(II)
and with $V^{N,n}_e := (X^{N,n}_e + Y^{N,n}_e, Z^{N,n}_e)$ we have the compact form

$$
= \varsigma_N \mathbb{E}\left[\partial_2 F\left(V^{N,n}_e\right)\right] - \frac{\varsigma^{-1}_N}{bn-N} \sum_{f \in e \cap E_{\tilde{n}}} \mathbb{E}\left[Z^{N,n}_{f} F\left(V^{N,n}_e\right)\right].
$$

(7.7)

As in the usual implementation of Stein’s method, we would like to tease out cancellations between (I) and (II) by writing the random variable $Z^{N,n}_f$ in (II) as a sum of a large term, $Z^{N,n}_f - \frac{1}{b_{n-N}} Z^{N,n}_f$, and a small term, $\frac{1}{b_{n-N}} Z^{N,n}_f$, and then Taylor expanding (II). The complicating feature here is that $X^{N,n}_e + Y^{N,n}_e$ is not independent of $Z^{N,n}_f$.

(c) Identifying the dependent factors: Next we seek to separate out the dependence of the random variables $X^{N,n}_e$ and $Y^{N,n}_e$ on the random variable $Z^{N,n}_f$ for a given $f \in e \cap E_{\tilde{n}}$. More precisely, we can define a term $B^{N,n}_f$ such that the statements (i)-(ii) below hold for the $\mathbb{R}^2$-valued random variable $\Delta^{N,n}_f := \frac{1}{b_{n-N}} (Y^{N,n}_f + Z^{N,n}_f B^{N,n}_f, Z^{N,n}_f)$.

(i) The random variables $Z^{N,n}_f$ and $Y^{N,n}_f$ are independent of $\Delta^{N,n}_f$ and $B^{N,n}_f$.

(ii) The random variables $Z^{N,n}_f$, $Y^{N,n}_f$, and $B^{N,n}_f$ have mean zero and are pairwise uncorrelated.

For $f \in e \cap E_{\tilde{n}}$ the definition of $B^{N,n}_f$ is as follows:

$$
B^{N,n}_f := b_{\tilde{n}-N} \frac{\partial \mathcal{F}}{\partial y_f} \left\{ Y^{N,n}_f \right\} f \in e \cap E_{\tilde{n}},
$$

where the function $\mathcal{F}$, which maps arrays $\{y_a\}_{a \in E_{\tilde{n}-N}}$ into $\mathbb{R}$, is defined below. The variable $Y^{N,n}_e$ is a multilinear function, $\mathcal{F}$, of the array $\{Y^{N,n}_f\}_{f \in e \cap E_{\tilde{n}}}$, where

$$
\mathcal{F}\left\{y_a\right\}_{a \in E_{\tilde{n}-N}} := \sum_{k=1}^{\tilde{n}-N} \mathcal{L}^{k-1} \mathcal{L}^{\tilde{n}-N-k} \left\{y_a\right\}_{a \in E_{\tilde{n}-N}}
$$

$$
= \sum_{k=1}^{\tilde{n}-N} \frac{1}{b_{\tilde{n}-N-k}} \left( 1 + \frac{1}{b_{\tilde{n}-N-k}} \right) \left( 1 + \frac{1}{b_{\tilde{n}-N-k}} \right) \sum_{a \in \alpha \cap E_{\tilde{n}-N}} y_a.
$$

Note that the multilinearity of $\mathcal{F}$ implies that the function $\mathcal{F}\left\{y_a\right\}_{a \in E_{\tilde{n}-N}} - y_a \frac{\partial \mathcal{F}}{\partial y_a} \left\{y_a\right\}_{a \in E_{\tilde{n}-N}}$ is independent of the variable $y_a$, where $\alpha \in E_{\tilde{n}-N}$. Moreover, the partial derivative of $\mathcal{F}$ with respect to $y_a$ has the form

$$
\frac{\partial \mathcal{F}}{\partial y_a} \left\{y_a\right\}_{a \in E_{\tilde{n}-N}} = \frac{1}{b_{\tilde{n}-N}} \sum_{k=1}^{\tilde{n}-N} \left( \prod_{\tilde{a} \in E_k^{(a)}} y_{\tilde{a}} \left( 1 + \frac{1}{b_{\tilde{n}-N-k}} \sum_{a \in \tilde{a} \cap E_{\tilde{n}-N}} y_a \right) - 1 \right),
$$

where $E_k^{(a)}$ is the $(b-1)$-element subset of $E_k$ consisting of elements $\tilde{a}$ with the following three restrictions: (i) $\alpha \not\subset \tilde{a}$, (ii) there is path in $\Gamma_k$ that passes over both $\alpha$ and $\tilde{a}$, (iii) there is an element in $E_{k-1}$ that contains both $\alpha$ and $\tilde{a}$.

By Remark 3.4 and (i) of Lemma 7.3, the second moment of $B^{N,n}_f$ is equal to

$$
\mathbb{E}\left[\left|B^{N,n}_f\right|^2\right] = \left(\tilde{n} - N\right)^2 \left( 1 + \alpha_{\mathbb{N},n}^2 \right)^{b-1} - 1 \leq \frac{C}{N^{1-2e}},
$$

(7.8)
where the inequality holds for some $C > 0$ and all $n$ larger than $N$.

**d) Stein analysis:** Now we are ready to begin an analysis of the expression (7.7). By Taylor’s theorem to second-order, the expression inside the expectation in (II) has the form

\[
Z_f^{N,n} F(\overline{V}_e^{N,n}) = Z_f^{N,n} F(\overline{V}_e^{N,n} - \Delta_f^{N,n}) + \frac{1}{2} Z_f^{N,n} \Delta_f^{N,n} \cdot \nabla F(\overline{V}_e^{N,n} - \Delta_f^{N,n}) + \frac{1}{2} Z_f^{N,n} (\Delta_f^{N,n})^2 \cdot (D_2 F)(\overline{V}_e^{N,n} - r_f \Delta_f^{N,n}),
\]

(7.9)

where $D_2$ is the 2-tensor of second-order derivatives and $r_f$ is some value between 0 and 1 depending on $\overline{V}_e^{N,n}$ and $\Delta_f^{N,n}$. The expectation of the first expression on the right side of (7.9) is zero by observations (i)-(ii) in part (c) above. By definition of $\Delta_f^{N,n}$, the second term on the right side of (7.9) can be written as

\[
Z_f^{N,n} \Delta_f^{N,n} \cdot \nabla F(\overline{V}_e^{N,n} - \Delta_f^{N,n}) = \frac{1}{b^{n-N}} Z_f^{N,n} \left( Y_f^{N,n} + Y_f^{N,n} D_f^{N,n} \right) (\partial_1 F)(\overline{V}_e^{N,n} - \Delta_f^{N,n}) + \frac{1}{b^{n-N}} \left( Z_f^{N,n} \right)^2 (\partial_2 F)(\overline{V}_e^{N,n} - \Delta_f^{N,n}).
\]

(7.10)

Again by observations (i)-(ii) in part (c), the expectation of the first expression on the right side of (7.10) is zero.

As a consequence of the above remarks, taking the expectation of (7.9) leaves us with

\[
E \left[ Z_f^{N,n} F(\overline{V}_e^{N,n}) \right] = \frac{1}{b^{n-N}} E \left[ (Z_f^{N,n})^2 \right] E \left[ (\partial_2 F)(\overline{V}_e^{N,n} - \Delta_f^{N,n}) \right] + \frac{1}{2} Z_f^{N,n} (\Delta_f^{N,n})^2 \cdot (D_2 F)(\overline{V}_e^{N,n} - r_f \Delta_f^{N,n}),
\]

(7.11)

where we have used that $Z_f^{N,n}$ is independent of $\overline{V}_e^{N,n} - \Delta_f^{N,n}$ to factor the first expectation on the right. The right-most expectation on the top line of (7.11) is equal to

\[
E \left[ (\partial_2 F)(\overline{V}_e^{N,n} - \Delta_f^{N,n}) \right] = E \left[ (\partial_2 F)(\overline{V}_e^{N,n}) \right] - \underbrace{E \left[ \int_0^1 \Delta_f^{N,n} \cdot (\nabla \partial_2 F)(\overline{V}_e^{N,n} - r \Delta_f^{N,n}) dr \right]}_{(IV)}.
\]

For $\varsigma_{N,n} := E \left[ (Z_f^{N,n})^2 \right]^{1/2}$, combining (7.11) with (7.7) yields the equality

\[
E \left[ \varsigma_N (\partial_2 F)(\overline{V}_e^{N,n}) \right] = \frac{\overline{V}_e^{N,n}}{\varsigma_N} F(\overline{V}_e^{N,n}) - \frac{\varsigma_N^2}{\varsigma_N} E \left[ (\partial_2 F)(\overline{V}_e^{N,n}) \right] - \varsigma_N \frac{1}{b^{n-N}} + \frac{\varsigma_N^2}{\varsigma_N} \cdot (III) + \frac{\varsigma_N^2}{\varsigma_N} \cdot (IV).
\]

(7.12)

In the above we have used that the expressions (III) and (IV) do not depend on the choice of $f \in e \cap E_0$, and that there are $b^{2(n-N)}$ elements in $e \cap E_0$. The first term on the right side of (7.12) vanishes as $n \to \infty$ because $\partial_2 F$ is bounded by $\sqrt{\pi/2}$ and $\varsigma_{N,n} \to \varsigma_N$ by part (ii) of Lemma 7.3. We will bound the last two terms on the right side of (7.12) in (e) and (f) below.
(e) **Second term on the right side of (7.12):** For any \((x, z) \in \mathbb{R}^2\), the norm of the 2-tensor \(D_2^N F(x, z)\) is bounded by \(4/\zeta_N\) since its components are smaller than \(2/\zeta_N\) as a consequence of Corollary 7.2. Thus we have the second inequality below.

\[
\begin{align*}
&b^{-n} \frac{1}{\zeta_N} \cdot |(III)| \leq \frac{1}{2\zeta_N} b^{-n} \mathbb{E} \left[ \left| Z_f^{n,N} (\Delta_f^{N,n})^2 \cdot (D_2^N F) \left( \sqrt{\mathbb{E}_e^{N,n}} - r_f \right) \right| \right] \\
&\leq \frac{2}{\zeta_N} b^{-n} \mathbb{E} \left[ |Z_f^{n,N}| \left| \Delta_f^{N,n} \right|^2 \right]
\end{align*}
\]

By definition of \(\Delta_f^{N,n}\), the above is equal to

\[
= \frac{2}{\zeta_N} b^{-n} \mathbb{E} \left[ |Z_f^{n,N}| \left( \left| Y_f^{N,n} \right|^2 + Y_f^{N,n} B_f^{N,n} \left| B_f^{N,n} \right|^2 + \left| Z_f^{n,N} \right|^2 \right) \right].
\]

Foil the products and using that \(B_f^{N,n}\) has mean zero and is independent of the variables \(Y_f^{N,n}\) and \(Z_f^{n,N}\), we get

\[
= \frac{2}{\zeta_N} b^{-n} \mathbb{E} \left[ |Z_f^{n,N}| \left( \left| Y_f^{N,n} \right|^2 + \mathbb{E} \left[ |B_f^{N,n}|^2 \right] \right) \left( 1 + \mathbb{E} \left[ |B_f^{N,n}|^2 \right] \right) \right].
\]

Applying the Cauchy-Schwarz inequality to each term above yields that

\[
\leq \frac{2}{\zeta_N} b^{-n} \mathbb{E} \left[ |Z_f^{n,N}| \right] \left( \left( \mathbb{E} \left[ |Y_f^{N,n}|^2 \right] \right)^{\frac{1}{2}} \left( 1 + \mathbb{E} \left[ |B_f^{N,n}|^2 \right] \right) \right) \left( \mathbb{E} \left[ |Z_f^{n,N}|^2 \right] \right)^{\frac{1}{2}}.
\]

By (7.8) and Lemma 7.3, the expression above is bounded for all \(n \in \mathbb{N}\) by

\[
\frac{2N^2}{(n - \mathbf{n}) b^{-n}} \left( \frac{C(n - \mathbf{n})}{N^2} \right)^{\frac{1}{2}} \left( \frac{\sqrt{C}}{N} \left( 1 + \frac{C}{N^{1-2\epsilon}} \right) + (n - \mathbf{n}) \frac{\sqrt{C}}{N^2} \right) \propto \frac{N^{-\frac{1}{2}}}{b^{N^2}}.
\]

(f) **Third term on the right side of (7.12):** To bound the third term on the right side of (7.12), we can use that the vector \((\nabla \partial_2 F)(x, z)\) has norm less \(\sqrt{2}\) times \(2/\zeta_N\), i.e., the bound for the second-order partial derivatives of \(F\), to get

\[
\frac{S_{2,N}}{\zeta_N} \cdot |(IV)| := \frac{S_{2,N}}{\zeta_N} \mathbb{E} \left[ \int_0^1 \Delta_f^{N,n} \cdot (\nabla \partial_2 F)(\sqrt{\mathbb{E}_e^{N,n}} - r \Delta_f^{N,n}) \, dr \right] \leq \frac{2^3 S_{2,N}}{\zeta_N} \mathbb{E} \left[ \left| \Delta_f^{N,n} \right|^2 \right].
\]

By Jensen’s inequality the above is smaller than,

\[
\leq 2^3 \frac{S_{2,N}}{\zeta_N} \mathbb{E} \left[ \left| \Delta_f^{N,n} \right|^2 \right] \frac{1}{2} = \frac{2^3 S_{2,N}}{\zeta_N} \mathbb{E} \left[ \left( \sqrt{\mathbb{E}_e^{N,n}} + Y_f^{N,n} B_f^{N,n} \right)^2 + \left( Z_f^{n,N} \right)^2 \right]^{\frac{1}{2}}.
\]

Since \(Y_f^{N,n}\) and \(B_f^{N,n}\) are independent and \(B_f^{N,n}\) has mean zero,

\[
= \frac{2^3 S_{2,N}}{\zeta_N} \mathbb{E} \left[ \left( \sqrt{\mathbb{E}_e^{N,n}} \right)^2 \right] + \mathbb{E} \left[ |Y_f^{N,n}|^2 \right] \mathbb{E} \left[ |B_f^{N,n}|^2 \right] + \mathbb{E} \left[ |Z_f^{n,N}|^2 \right] \mathbb{E} \left[ |B_f^{N,n}|^2 \right] \mathbb{E} \left[ |Z_f^{n,N}|^2 \right] \left( \mathbb{E} \left[ |Z_f^{n,N}|^2 \right] \right)^{\frac{1}{2}}.
\]

By (7.8) and Lemma 7.3 the above is bounded for all \(n \in \mathbb{N}\) by

\[
\frac{2^3}{b^{-n}} \left( \frac{C}{N} + \frac{C^2}{N^{2-2\epsilon}} + \frac{C(n - \mathbf{n})}{N^2} \right) \propto \frac{N^{-\frac{1}{2}}}{b^{N^2}}.
\]
(g) Extension to Wasserstein-2 distance: Our results above can be summarized by stating that there is a $c > 0$ and a sequence $\{\xi_N(n)\}_{n \in \mathbb{N}}$ such that $\xi_N(n) \to 0$ as $n \to \infty$ for each $N$ and

$$
\rho_1(\widehat{X}_{e,n}^N, \widehat{X}_{e}^N) \leq \xi_N(n) + \frac{c^{N-\frac{a}{2}}}{b^{\frac{N}{2}}}.
$$

(7.13)

By Lemma [7.4] we have that

$$
\rho_2(\widehat{X}_{e,n}^N, \widehat{X}_{e}^N) \leq \left( \rho_1(\widehat{X}_{e,n}^N, \widehat{X}_{e}^N) \right)^\frac{1}{2} \left( \mathbb{E} \left[ |\widehat{X}_{e,n}^N|^4 \right]^\frac{1}{2} + \mathbb{E} \left[ |\widehat{X}_{e}^N|^4 \right]^\frac{1}{2} \right).
$$

The limit supremum of the above vanishes super-polynomially as $n \to \infty$ by (7.13) and part (iv) of Lemma [7.3].

\[ \square \]

### 7.4 Proof of Lemma 5.9

The following lemma is a central limit theorem in which the distance between a normalized sum of i.i.d. random variables and a centered normal random variable of the same variance is measured in terms of the Wasserstein-1 distance. We include a proof using the zero bias transformation of Goldstein and Reinert [19] in Appendix B.

**Lemma 7.5.** Let $X_1, \ldots, X_n$ be i.i.d. centered random variables with variance $\sigma^2$ and finite third absolute moment. Then for $\overline{X}_n := \frac{X_1 + \ldots + X_n}{\sqrt{n}}$ and $X \sim N(0, \sigma^2)$

$$
\rho_1(\overline{X}_n, X) \leq \frac{3}{\sigma^2 \sqrt{n}} \mathbb{E}[|X|^3].
$$

**Corollary 7.6.** Let us take the conditions of Lemma 7.5 and assume in addition that the fourth moment is finite. Then for any $n \in \mathbb{N}$

$$
\rho_2(\overline{X}_n, X) \leq \frac{4}{\sigma^2 \sqrt{n}^\frac{5}{2}} \mathbb{E}[|X|^4]^\frac{5}{12}.
$$

**Proof of Lemma 5.9.** For $e \in E_N$ the variables $\widehat{X}_{e,n}^N$ and $\overline{X}_e^N$ have the form

$$
\mathcal{L}^\hat{N}(\{Y_f\}_{f \in e \cap E_\hat{n}}) + \sum_{k=1}^{\hat{n}-N} \mathcal{L}^{k-1} \mathcal{E} \mathcal{L}^{\hat{n}-N-k}(\{Y_f\}_{f \in e \cap E_\hat{n}}) + \mathcal{Z}_e^N
$$

for $Y_f := Y_f^N$ and $Y_f := Y_f^N$, respectively, where $Y_f^N$ and $Y_f^N$ are defined as in (5.5) and independent of $\mathcal{Z}_e^N$.

For each $f \in e \cap E_\hat{n}$, let $(Y_f^N, Y_f^N)$ be a coupling of the variables $Y_f^N$ and $Y_f^N$ such that

$$
\rho_2(Y_f^N, Y_f^N) = \mathbb{E}\left[ (Y_f^N - Y_f^N)^2 \right]^\frac{1}{2}.
$$

(7.14)

With this coupling, we can bound the $\rho_2$-distance between $\widehat{X}_{e,n}^N$ and $\overline{X}_e^N$ as follows:

$$
\left( \rho_2(\widehat{X}_{e,n}^N, \overline{X}_e^N) \right)^2 \\
\leq \mathbb{E}\left[ \mathcal{L}^{\hat{n}-N}(\{Y_f^N\}_{f \in e \cap E_\hat{n}} - \{Y_f^N\}_{f \in e \cap E_\hat{n}}) \right. \\
\left. + \sum_{k=1}^{\hat{n}-N} \mathcal{L}^{k-1}(\mathcal{E} \mathcal{L}^{\hat{n}-N-k}(\{Y_f^N\}_{f \in e \cap E_\hat{n}} - \mathcal{E} \mathcal{L}^{\hat{n}-N-k}(\{Y_f^N\}_{f \in e \cap E_\hat{n}})) \right]^2.
$$

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Since the terms summed above are uncorrelated and the operation $\mathcal{L}^{k-1}$ acts on i.i.d. arrays of mean zero random variables, we can write the above as

$$
= \sum_{f \in E_n} \frac{1}{b^{2n}} \mathbb{E} \left[ |Y_{f}^{N,n} - Y_{f}^{(N)}|^2 \right] \\
+ \sum_{k=1}^{\hat{n} - N} \frac{1}{b^{2(k-1)}} \sum_{e \in \mathcal{E}} \mathbb{E} \left[ E \{ \tilde{Y}_{e \times (i,j)}^{N,n} \} |1 \leq i, j \leq b| - E \{ \tilde{Y}_{e \times (i,j)}^{(N)} \} |1 \leq i, j \leq b|^2 \right],
$$

(7.15)

where for $e \in E_{N+k-1}$ the arrays within the expectations above are defined as

$$
\{ \tilde{Y}_{e \times (i,j)}^{N,n} \} |1 \leq i, j \leq b| := \mathcal{L}^{\hat{n} - N - k} \{ Y_{f}^{N,n} \} |f \in e \cap E_n \} and \{ \tilde{Y}_{e \times (i,j)}^{(N)} \} |1 \leq i, j \leq b| := \mathcal{L}^{\hat{n} - N - k} \{ Y_{f}^{(N)} \} |f \in e \cap E_n \} .
$$

The elements in the above arrays have variances $\text{Var}(\tilde{Y}_{f}^{N,n}) = \sigma^2_{n,n}$ and $\text{Var}(\tilde{Y}_{f}^{(N)}) = R(r - n)$ since $\mathcal{L}$ preserves the variance of the array variables. For any $1 \leq k \leq \hat{n} - N$, we can write the summand in (7.15) in the form

$$
\mathbb{E} \left[ E \{ \tilde{Y}_{e \times (i,j)}^{N,n} \} |1 \leq i, j \leq b| - E \{ \tilde{Y}_{e \times (i,j)}^{(N)} \} |1 \leq i, j \leq b|^2 \right] = \sum_{i=1}^{b} \frac{1}{b^2} \sum_{|A| \geq 2} \mathbb{E} \left[ \prod_{j \in A} \tilde{Y}_{e \times (i,j)}^{N,n} - \prod_{j \in A} \tilde{Y}_{e \times (i,j)}^{(N)} \right]^2
$$

because the operation $E = Q - \mathcal{L}$ returns $\frac{1}{b} \sum_i (\prod_j (1 + a_{i,j}) - 1 - \sum_j a_{i,j})$ when it acts on an array $\{a_{i,j}\} |1 \leq i, j \leq b|$. By writing $\tilde{Y}_{f}^{N,n} = \tilde{Y}_{f}^{(N)} + (\tilde{Y}_{f}^{N,n} - \tilde{Y}_{f}^{(N)})$ for each $f = e \times (i,j) in the products above and foiling, we get

$$
= \mathbb{E} \left[ \tilde{Y}_{f}^{N,n} - \tilde{Y}_{f}^{(N)} \right]^2 U(\sigma^2_{n,n}, R(r - n)),
$$

where $U(y_1, y_2)$ is a degree $b$ polynomial with nonnegative coefficients and no constant term. The equality $\tilde{Y}_{f}^{N,n} - \tilde{Y}_{f}^{(N)} = \mathcal{L}^{\hat{n} - N - k} \{ Y_{f}^{N,n} - Y_{f}^{(N)} \} |f \in e \cap E_n \}$ implies that the $L^2$-distance between $\tilde{Y}_{f}^{N,n}$ and $\tilde{Y}_{f}^{(N)}$ is equal to the $L^2$-distance between $Y_{f}^{N,n}$ and $Y_{f}^{(N)}$, so by (7.14) the above can be written as

$$
= (\rho_2(Y_{f}^{N,n}, Y_{f}^{(N)}))^2 U(\sigma^2_{n,n}, R(r - n)).
$$

(7.16)

Putting together our results from (7.15) and (7.16), we have

$$
\left( \rho_2(\tilde{X}_{e}^{N,n}, \tilde{X}_{e}^{(N)}) \right)^2 \leq (\rho_2(Y_{f}^{N,n}, Y_{f}^{(N)}))^2 \left( 1 + (\hat{n} - N)U(\sigma^2_{n,n}, R(r - n)) \right).
$$

(7.17)

By Lemma 7.3, $\sigma^2_{n,n}$ converges to $R(r - n)$ as $n \to \infty$. Since $R(r - n) \propto \frac{1}{n}$ for $n \gg 1$ and the polynomial $U$ has no constant or linear terms, there is a $c > 0$ such that for all large enough $N$ and $n$

$$
\left( \rho_2(Y_{f}^{N,n}, Y_{f}^{(N)}) \right)^2 \leq \left( \rho_2(Y_{f}^{N,n}, Y_{f}^{(N)}) \right)^2 \left( 1 + \frac{c \hat{n} - N}{n^2} \right) \leq c \left( \rho_2(Y_{f}^{N,n}, Y_{f}^{(N)}) \right)^2.
$$

(7.18)
The second term on the right side converges to zero as \( n \to \infty \) since \( \sigma_{n,n}^2 \to R(r - n) \). By definition, \( Y_{f}^{N,n} \) is a sum of the i.i.d. random variables \( \frac{1}{b^{n-\bar{n}}} X_{g}^{(N,n)} \) over \( g \in f \cap E_n \) that contains \( b^{2(n-\bar{n})} \) elements. Hence, by Corollary 7.6, we have the inequality below.

\[
\rho^2 \left( Y_{f}^{N,n}, \frac{\sigma_{n,n}}{\sqrt{R(r - n)}} \right) = \rho^2 \left( \frac{1}{b^{n-\bar{n}}} \sum_{g \in f \cap E_n} X_{g}^{(N,n)}, \frac{\sigma_{n,n}}{\sqrt{R(r - n)}} \right) \leq \frac{4}{b^{2(n-\bar{n})}} \mathbb{E} \left[ |X_{g}^{(N,n)}|^4 \right]^{\frac{1}{2}} \xrightarrow{n \to \infty} \frac{4}{b^{2(n-\bar{n})}} \left( R(4)(r - n) \right)^{\frac{1}{2}}.
\]

The convergence above holds by property (III) of Lemma 3.9. The above converges to zero super-polynomially with \( N \gg 1 \) since \( n - \bar{n} \propto N^\epsilon/2 \) and \( R(2m)(r) \propto (-1/r)^m \) for \( -r \gg 1 \).

\[
\square
\]

8 Miscellaneous proofs

8.1 Proofs from Section 5

Proof of Corollary 5.11. The random variables \( X_{e}^{(N,n)} - \hat{X}_{e}^{N,n} \) and \( \hat{X}_{e}^{N,n} \) are uncorrelated, and thus

\[
\mathbb{E} \left[ (\hat{X}_{e}^{N,n})^2 \right] \leq \mathbb{E} \left[ (X_{e}^{(N,n)})^2 \right] \xrightarrow{n \to \infty} R(r - N) < R(s - N),
\]

where the convergence holds since \( \{X_{a}^{(s,n)}\}_{a \in E_n} \) is a regular sequence of \( \mathcal{Q} \)-pyramidal arrays of random variables with parameter \( r \). Thus we have verified the desired condition in the case \( U_{e}^{(N)} := \hat{X}_{e}^{N,n} \) for any \( s \in (r, \infty) \) and large enough \( N, n \in \mathbb{N} \). To extend our result to the case \( U_{e}^{(N)} := \hat{X}_{e}^{N,n} \), first notice that by Lemma 5.8, the limit supremum as \( n \to \infty \) of \( \mathbb{E} \left[ (\hat{X}_{e}^{N,n} - X_{e}^{N,n})^2 \right] \) vanishes super-polynomially as \( N \) grows. Hence foiling this expression yields

\[
\mathbb{E} \left[ (\hat{X}_{e}^{N,n})^2 \right] - \mathbb{E} \left[ (X_{e}^{N,n})^2 \right] = 2 \mathbb{E} \left[ \hat{X}_{e}^{N,n} (\hat{X}_{e}^{N,n} - X_{e}^{N,n}) \right] + \mathbb{E} \left[ (\hat{X}_{e}^{N,n} - X_{e}^{N,n})^2 \right] =: \xi_{N}(n),
\]

and by Cauchy-Schwarz \( \limsup_{n \to \infty} \xi_{N}(n) \) also vanishes super-polynomially with \( N \gg 1 \). Next note that for large \( N \)

\[
R(s - N) - R(r - N) = (s - r) \frac{N^2}{N^2} + o\left( N^{-2} \right),
\]

where the asymptotics hold as a consequence of (II) in Lemma 2.3. Thus, the difference \( R(s - N) - R(r - N) \) is an asymptotic multiple of \( N^{-2} \) that, in particular, is larger than \( \limsup_{n \to \infty} \xi_{N}(n) \) for large \( N \). Therefore the result holds for \( U_{e}^{(N)} := \hat{X}_{e}^{N,n} \). The same reasoning applies to the case \( U_{e}^{(N)} := \hat{X}_{e}^{N,n} \).

\[
\square
\]

8.2 Proofs from Section 7

Proof of Proposition 7.2. The bounds \( \sup_{y,z \in \mathbb{R}} |\partial_{y} F(y, z)| \leq 1 \) and \( \sup_{y,z \in \mathbb{R}} |\partial_{y} \partial_{z} F(y, z)| \leq 2 \) are equivalent to (7.3), so we can focus on the partial derivatives \( \partial_{y}, \partial_{y}^2 \), and \( \partial_{y} \partial_{z} \). Define \( \phi_{-}(t) := \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \) and \( \phi_{+}(t) := 1 - \phi_{-}(t) \). We can rewrite \( H \) in terms of \( H' \) as

\[
H(z) - \int_{\mathbb{R}} H'(r) \frac{e^{-r^2/2}}{\sqrt{2\pi}} dr = \int_{-\infty}^{z} H'(t) \phi_{-}(t) dt - \int_{z}^{\infty} H'(t) \phi_{+}(t) dt.
\]

(8.1)
Moreover, we can rewrite $F$ in the form

$$
F(y, z) = \frac{1}{2} e^{\frac{y^2}{4}} \int_{-\infty}^{\infty} \left( H(y + t) - \int_\mathbb{R} H(y + r) \frac{e^{-\frac{r^2}{2}}}{\sqrt{2\pi}} dr \right) e^{-\frac{t^2}{2}} dt \\
- \frac{1}{2} e^{\frac{y^2}{4}} \int_{z}^{\infty} \left( H(y + t) - \int_\mathbb{R} H(y + r) \frac{e^{-\frac{r^2}{2}}}{\sqrt{2\pi}} dr \right) e^{-\frac{t^2}{2}} dt,
$$

and using the identity (8.1) we have

$$
= e^{\frac{y^2}{4}} \int_{-\infty}^{z} \left( \int_{-\infty}^{t} H'(y + r)\phi_-(r)dr - \int_t^{\infty} H'(y + r)\phi_+(r)dr \right) e^{-\frac{t^2}{2}} dt \\
- e^{\frac{y^2}{4}} \int_{z}^{\infty} \left( \int_{-\infty}^{t} H'(y + r)\phi_-(r)dr - \int_t^{\infty} H'(y + r)\phi_+(r)dr \right) e^{-\frac{t^2}{2}} dt.
$$

Swapping the order of integration yields

$$
= \int_\mathbb{R} G(z, r) H'(y + r)dr = \int_\mathbb{R} G(z, r - y) H'(r)dr,
$$

where $G : \mathbb{R}^2 \to \mathbb{R}$ is the kernel

$$
G(z, r) := \begin{cases} 
-\sqrt{2\pi} e^{\frac{z^2}{4}} \phi_-(z)\phi_+(r) & z < r, \\
-\sqrt{2\pi} e^{\frac{z^2}{4}} \phi_+(z)\phi_-(r) & z \geq r.
\end{cases}
$$

The results will follow by bounding $\sup_{z \in \mathbb{R}} \int_\mathbb{R} |(dG)(z, r)|dr$ for the derivatives $d \in \{ \partial_r, \partial_r^2, \partial_z \partial_r \}$.

The first partial derivative with respect to $r$ has the form

$$
\partial_r G(z, r) = \begin{cases} 
\sqrt{2\pi} e^{\frac{z^2}{4}} \phi_-(z) e^{-\frac{r^2}{2}} & z < r, \\
\sqrt{2\pi} e^{\frac{z^2}{4}} \phi_+(z) e^{-\frac{r^2}{2}} & z \geq r.
\end{cases}
$$

For any $z \in \mathbb{R}$, the equality $\int_\mathbb{R} |\partial_r G(z, r)|dr = 2\sqrt{2\pi} e^{\frac{z^2}{4}} \phi_-(z)\phi_+(z)$ holds, and the right side attains its maximum value, $\sqrt{\pi/2}$, when $z = 0$.

The second-order partial derivatives involving $r$ have the forms $\partial_r^2 G(z, r) = -\delta(z - r) + A_1(z, r)$ and $\partial_z \partial_r G(z, r) = -\delta(z - r) + A_2(z, r)$, where

$$
A_1(z, r) = \begin{cases} 
-\sqrt{2\pi} e^{\frac{z^2}{4}} \phi_-(z) e^{-\frac{r^2}{2}} & z < r, \\
\sqrt{2\pi} e^{\frac{z^2}{4}} \phi_+(z) e^{-\frac{r^2}{2}} & z \geq r,
\end{cases} \quad
A_2(z, r) = \begin{cases} 
(\sqrt{2\pi} e^{\frac{z^2}{4}} \phi_-(z) + 1) e^{-\frac{r^2}{2}} & z < r, \\
(\sqrt{2\pi} e^{\frac{z^2}{4}} \phi_+(z) - 1) e^{-\frac{r^2}{2}} & z \geq r.
\end{cases}
$$

Notice that $1 + \sqrt{2\pi} e^{\frac{z^2}{4}} \phi_-(z)$ and $1 - \sqrt{2\pi} e^{\frac{z^2}{4}} \phi_+(z)$ are nonnegative for all $z \in \mathbb{R}$, and thus we simply have

$$
\int_\mathbb{R} |A_1(z, r)|dr = \phi_-(z) + \phi_+(z) = 1,
$$

and

$$
\int_\mathbb{R} |A_2(z, r)|dr = \left(1 + \sqrt{2\pi} e^{\frac{z^2}{4}} \phi_-(z)\right)\phi_+(z) + \left(1 - \sqrt{2\pi} e^{\frac{z^2}{4}} \phi_+(z)\right)\phi_-(z) = 1.
$$

Therefore $\sup_{z \in \mathbb{R}} \int_\mathbb{R} |(dG)(z, r)|dr \leq 2$ for $d = \partial_r \partial_z$ and $d = \partial_r^2$. 

\[ \square \]
Lemma 3.5. Part (i): For $f \in E_n$ the variance of $Y_f^{N,n} = \mathcal{L}^{n-n}(X_g^{(n,n)})_{g \in f \cap E_n}$ is $\sigma_{n,n}^{(2)} = \text{Var}(X_g^{(n,n)})$ since the operation $\mathcal{L}$ preserves the variance of variables in the array (Remark 3.4). The convergence of $\sigma_{n,n}^{(2)}$ to $R(r-n)$ as $n \to \infty$ holds since $(\{X_a^{(n,n)}\}_{a \in E_n})_{n \in \mathbb{N}}$ is a regular sequence of $\mathcal{Q}$-pyramidal arrays of random variables with parameter $r$ (and thus satisfies property (III) of Lemma 3.9).

Part (ii): Since terms in the sum $Z_f^{N,n} = \sum_{k=\hat{n}+1}^{n} \mathcal{L}^{k-1} \mathcal{E} \mathcal{L}^{n-k}(X_g^{(n,n)})_{g \in f \cap E_n}$ are uncorrelated by Lemma 3.5 we have

$$\varsigma_{N,n}^2 := \text{Var}(Z_f^{N,n}) = \sum_{k=\hat{n}+1}^{n} \text{Var} \left( \mathcal{L}^{k-1} \mathcal{E} \mathcal{L}^{n-k}(X_g^{(n,n)})_{g \in f \cap E_n} \right).$$

By part (ii) of Remark 3.4

$$= (n - \hat{n}) M(x) - x \bigg|_{x = \hat{M}^{n-n}(\sigma_{n}^2)} = (n - \hat{n}) \left( \sigma_{n-1,n}^{(2)} - \sigma_{n,n}^{(2)} \right).$$

Since $\sigma_{k,n}^{(2)}$ converges to $R(r-k)$ with large $n$, there is a sequence $\{\epsilon_N(n)\}_{n \in \mathbb{N}}$ that vanishes as $n \to \infty$ such that the above is equal to

$$= (n - \hat{n}) (R(r-n + 1) - R(r-n)) + \epsilon_N(n),$$

which has the form $\varsigma_{N}^2 + \epsilon_N(n)$.

Part (iii): For $g \in E_n$ define $\sigma_{n,n}^{(m)} := \mathbb{E} \left[ (X_g^{(n,n)})^m \right]$. Also, for $a \in E_k$ with $k \in \{0, \ldots, n\}$ define

$$\tilde{\sigma}_{k,n}^{(m)} := \mathbb{E} \left[ \left( \mathcal{L}^{n-k}(X_g^{(n,n)})_{g \in a \cap E_n} \right)^m \right] = \mathbb{E} \left[ \left( \frac{1}{b^{n-k}} \sum_{g \in a \cap E_n} X_g^{(n,n)} \right)^m \right].$$

For $m = 4$ we have

$$\tilde{\sigma}_{k,n}^{(4)} = \frac{1}{b^{2(n-k)}} \sigma_{n,n}^{(4)} + 3 \frac{b^{2(n-k)} - 1}{b^{2(n-k)}} (\sigma_{n,n}^2)^2.$$

Since $(\{X_a^{(n,n)}\}_{a \in E_n})_{n \in \mathbb{N}}$ is a regular sequence of $\mathcal{Q}$-pyramidal arrays with parameter $r$, the sequence $\sigma_{k,n}^{(m)}$ converges to $R^{(m)}(r-k)$ with large $n$, so we have

$$\tilde{\sigma}_{k,n}^{(4)} \xrightarrow{n \to \infty} \frac{1}{b^{2(n-k)}} R^{(4)}(r-n) + 3 \frac{b^{2(n-k)} - 1}{b^{2(n-k)}} (R(r-n))^2.$$

Applying the above with $k = \hat{n}$ yields that $\mathbb{E}[(Y_f^{N,n})^4] = \tilde{\sigma}_{\hat{n},n}^{(4)} \sim 3\epsilon^4 / N^2$ when $N \gg 1$ since $n = N + \lfloor N\epsilon \rfloor$ and $R(r) \sim \frac{\epsilon^4}{r}$ for $-r \gg 1$ by Lemma 2.3.

For $f \in E_n$ and $\hat{n} \leq k \leq n$, define $\tilde{X}_a^{N,n} := \mathcal{L}^{n-k}(X_g^{(n,n)})_{g \in a \cap E_n}$. The fourth moment of $Z_f^{N,n}$ can be written as

$$\mathbb{E}[(Z_f^{N,n})^4] = \mathbb{E} \left[ \left( \sum_{k=\hat{n}+1}^{n} \mathcal{L}^{k-\hat{n}-1} \mathcal{E} \left( \tilde{X}_a^{N,n} \right) \right)^4 \right].$$

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Since the terms \( \mathcal{L}^{k-\tilde{n}-1} \mathcal{E} \{ \tilde{X}^{N,n}_a \}_{a \in f \cap E_k} \) have mean zero and are uncorrelated by Lemma 3.5, the above has the bound

\[
\mathbb{E} \left( \left( \sum_{k=\tilde{n}+1}^{n} \left( \mathcal{L}^{k-\tilde{n}-1} \mathcal{E} \{ \tilde{X}^{N,n}_a \}_{a \in f \cap E_k} \right)^2 \right)^2 \right) \\
\leq (n - \tilde{n}) \sum_{k=\tilde{n}+1}^{n} \mathbb{E} \left( \mathcal{L}^{k-\tilde{n}-1} \mathcal{E} \{ \tilde{X}^{N,n}_a \}_{a \in f \cap E_k} \right)^4.
\]

(8.2)

A single term from the sum in (8.2) can be written as

\[
\mathbb{E} \left( \mathcal{L}^{k-\tilde{n}-1} \mathcal{E} \{ \tilde{X}^{N,n}_a \}_{a \in f \cap E_k} \right)^4
\]

\[
= \frac{1}{b^{2(k-\tilde{n})}} \mathbb{E} \left( \prod_{j=1}^{b} \left( 1 + \tilde{X}^{N,n}_{a \times (i,j)} \right) - 1 - \sum_{j=1}^{b} \tilde{X}^{N,n}_{a \times (i,j)} \right)^4
\]

\[
+ 3 \frac{b^{2(k-\tilde{n})} - 1}{b^{2(k-\tilde{n})}} \mathbb{E} \left( \prod_{j=1}^{b} \left( 1 + \tilde{X}^{N,n}_{a \times (i,j)} \right) - 1 - \sum_{j=1}^{b} \tilde{X}^{N,n}_{a \times (i,j)} \right)^2
\]

for representatives \( i \in \{1, \ldots, b\} \) and \( a \in f \cap E_{k-1} \). In the above we have used the definition of \( \mathcal{E} \) and that \( \mathcal{L}^{k-\tilde{n}-1} \mathcal{E} \{ \tilde{X}^{N,n}_a \}_{a \in f \cap E_k} \) is a sum of \( b^{2(k-\tilde{n})} \) independent mean zero random variables. There is a degree \( b \) polynomial \( P(x, y) \) of the form \( a_1 x^2 + a_2 xy + a_3 y^4 \) plus higher-order terms for some constants \( a_1, a_2, a_3 > 0 \) such that the above is equal to

\[
P \left( \tilde{\sigma}^{(4)}_{k,n}, \tilde{\sigma}^{(2)}_{k,n} \right) \xrightarrow{n \to \infty} P \left( \frac{1}{b^{2(k-n)}} R^{(4)}(r-n) + 3 \frac{b^{2(n-k)} - 1}{b^{2(n-k)}} \left( R(r-n) \right)^2, R(r-n) \right).
\]

Thus (8.2) is bounded by a constant multiple of \( (n - \tilde{n})^2 / N^4 \) since \( R^{(2m)}(r-n) = O \left( \frac{1}{N^m} \right) \).

Part (iv): These fourth moment bounds are similar, but simpler, to the bound for \( \mathbb{E} \left[ (Y_f^{N,n})^4 \right] \) above. \( \square \)

**Proof of Lemma 7.4.** Let \((X, Y)\) be a coupling such that the \( L^1 \)-distance between the variables \( X \) and \( Y \) is equal to \( \rho_1(X, Y) \). Since \( \rho_2(X, Y) \) is an infimum of the \( L^2 \)-distance over couplings,

\[
\rho_2(X, Y) \leq \mathbb{E} \left[ |X - Y|^2 \right]^{\frac{1}{2}} = \mathbb{E} \left[ |X - Y| \right]^{\frac{1}{2}} \mathbb{E} \left[ |X - Y|^2 \right]^{\frac{1}{2}} \leq \mathbb{E} \left[ |X - Y| \right]^{\frac{1}{2}} \mathbb{E} \left[ |X - Y|^4 \right]^{\frac{1}{4}} \leq \left( \rho_1(X, Y) \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ |X|^4 \right]^{\frac{1}{4}} + \mathbb{E} \left[ |Y|^4 \right]^{\frac{1}{4}} \right),
\]

where the second inequality is Holder’s. The last inequality is the triangle inequality and \((x + y)^p \leq x^p + y^p \) for \( x, y \geq 0 \) for \( p = 2/3 \). \( \square \)

### A Variance function consistency check

There is instructional value in implementing a consistency check between properties (I) and (II) in the statement of Lemma 2.3, i.e., between the claim that \( M(R(r)) = R(r+1) \) and the 

\[
R(r) = \frac{\kappa^2}{-r} + \frac{\kappa^2 \eta \log(-r)}{r^2} + O \left( \frac{\log^2(-r)}{r^3} \right), \quad \text{(A.1)}
\]
where $\kappa^2 := \frac{2}{b-1}$ and $\eta := \frac{b+1}{3(b-1)}$. Fix some $r$ with $-r \gg 1$ and define $V_n = R(r - n)$ for $n \in \mathbb{N}_0$. We begin by writing $R(r)$ as a telescoping sum

$$R(r) = \sum_{k=1}^{\infty} (V_{k+1} - V_k) = \sum_{k=1}^{\infty} (M(V_k) - V_k). \tag{A.2}$$

Since $R(r)$ has the asymptotics (A.1) and the map $M(x) = \frac{1}{b} [(1 + x)^b - 1]$ has the $0 < x \ll 1$ asymptotics $M(x) = x + \frac{b-1}{2} x^2 + \frac{(b-1)(b-2)}{6} x^3 + O(x^4)$, the equality (A.2) can be written as

$$= \frac{b-1}{2} \sum_{k=1}^{\infty} V_k^2 + \frac{(b-1)(b-2)}{6} \sum_{k=1}^{\infty} V_k^3 + \sum_{k=1}^{\infty} O(V_k^4). \tag{b}

We will analyze the expressions (a), (b), and (c) to verify that the right side of (A.2) has the asymptotics (A.1). The expression (c) is $O(1/r^3)$ since the terms $V_k$ are bounded by a constant multiple of $(k - r)^{-1}$.

The expression (a) has the asymptotics

$$(a) = \frac{b-1}{2} \sum_{k=1}^{\infty} \left( \frac{\kappa^2}{k-r} + \frac{\eta \kappa^2 \log(k-r)}{(k-r)^2} + O\left( \frac{\log^2(k-r)}{(k-r)^3} \right) \right)^2.$$

Foil the square and using that $\kappa^{-2} = (b-1)/2$ we get

$$= \kappa^2 \sum_{k=1}^{\infty} \frac{1}{(k-r)^2} + 2\eta \kappa^2 \sum_{k=1}^{\infty} \frac{\log(k-r)}{(k-r)^3} + \sum_{k=1}^{\infty} O\left( \frac{\log^2(k-r)}{(k-r)^4} \right)$$

$$= \frac{\kappa^2}{-r} - \frac{1}{(b-1)r^2} + \eta \kappa^2 \frac{\log(-r)}{r^2} + \frac{\eta \kappa^2}{2r^2} + O\left( \frac{\log^2(-r)}{r^3} \right),$$

where we have used a trapezoidal Riemann approximation to get

$$\sum_{k=1}^{\infty} \frac{1}{(k-r)^2} = -\frac{1}{2r^2} + \frac{1}{2} \sum_{k=1}^{\infty} \left( \frac{1}{(k-r)^2} + \frac{1}{(k-1-r)^2} \right) + O\left( \frac{1}{r^3} \right)$$

$$= -\frac{1}{2r^2} + \frac{1}{-r} \int_{0}^{\infty} \frac{1}{(1+x)^2} dx + O\left( \frac{1}{r^3} \right)$$

$$= -\frac{1}{2r^2} + \frac{1}{-r} + O\left( \frac{1}{r^3} \right),$$

and right-hand Riemann approximations to get

$$\sum_{k=1}^{\infty} \frac{\log(k-r)}{(k-r)^3} = \frac{\log(-r)}{r^2} \int_{0}^{\infty} \frac{1}{(1+x)^3} dx + \frac{1}{r^2} \int_{0}^{\infty} \frac{\log(1+x)}{(1+x)^3} dx + O\left( \frac{\log(-r)}{r^3} \right)$$

$$= \frac{1}{2} \log(-r) + \frac{1}{4r^2} + O\left( \frac{\log(-r)}{r^3} \right).$$
The expression (b) can be written as

\[
(b) = \frac{(b - 1)(b - 2)}{6} \sum_{k=1}^{\infty} \left( \frac{\kappa^2}{k - r} + \frac{\eta \kappa^2 \log(k - r)}{(k - r)^2} + O\left(\frac{\log^2(k - r)}{(k - r)^3}\right) \right)^3
\]

\[
= \frac{b - 2}{3} \sum_{k=1}^{\infty} \left( \frac{\kappa^4}{(k - r)^3} + O\left(\frac{\log(k - r)}{(k - r)^4}\right) \right)
\]

\[
= \frac{b - 2}{3} \kappa^4 \frac{2}{2r^2} + O\left(\frac{\log(-r)}{r^3}\right),
\]

where we have used the Riemann approximation

\[
\sum_{k=1}^{\infty} \frac{1}{(k - r)^3} = \frac{1}{r^2} \int_{0}^{\infty} \frac{1}{1 + x^3} dx + O\left(\frac{1}{r^3}\right) = \frac{1}{2r^2} + O\left(\frac{1}{r^3}\right).
\]

Summing up (a), (b), and (c) gives the desired asymptotics (A.1) as a result of the cancellation

\[
- \frac{1}{(b - 1)r^2} + \frac{\eta \kappa^2}{2r^2} + \frac{b - 2}{3} \kappa^4 \frac{2}{2r^2} = 0
\]

between the underbracketed terms above.

B The zero bias approach to Stein’s method

We will discuss the zero bias variation on Stein’s method introduced in [19], which provides an easy proof of Lemma 7.5 (restated in Lemma B.4).

B.1 Zero bias transformation

Let \(X\) be a centered random variable with variance \(\sigma^2\). The zero bias transformation, \(X^\ast\), of \(X\) is the distribution satisfying

\[
\mathbb{E}[f'(X^\ast)] = \frac{1}{\sigma^2} \mathbb{E}[Xf(X)]
\]

for all absolutely continuous functions \(f\) on \(\mathbb{R}\). The right side above can be written as

\[
\frac{1}{\sigma^2} \mathbb{E}[Xf(X)] = \mathbb{E}\left[\frac{X^2}{\sigma^2} \int_{0}^{X} f'(r) dr \right].
\]

Thus if \(X\) has distribution measure \(\mu\), then \(X^\ast\) is constructed by choosing a number \(x\) using the measure \(\nu(dx) = \frac{x^2}{\sigma^2}\mu(dx)\) and then picking a number uniformly at random from the interval between 0 and \(x\). The normal distribution is the unique fixed point for the zero bias transformation:

Lemma B.1. Let \(X\) be centered random variable with variance \(\sigma^2\). Then \(X \overset{d}{=} X^\ast\) iff \(X \sim \mathcal{N}(0, \sigma^2)\).

Lemma B.2. Let \(X\) be a centered random variable with variance \(\sigma^2\) and finite absolute moment \(\varsigma_n := \mathbb{E}[|X|^n]\) for some \(n \geq 3\). The absolute moment, \(\varsigma^\ast_{n-2}\), of \(X^\ast\) is finite and equal to \(\varsigma^\ast_{n-2} = \frac{\varsigma_n}{\sigma^2(n-1)}\).

Proof. This follows immediately from the definition of \(X^\ast\) since

\[
\varsigma^\ast_{n-2} = \mathbb{E}[|X^\ast|^{n-2}] = \mathbb{E}\left[\frac{X^2}{\sigma^2} \int_{0}^{X} |r|^{n-2} dr \right] = \frac{\mathbb{E}[|X|^n]}{\sigma^2(n-1)} = \frac{\varsigma_n}{\sigma^2(n-1)}.
\]

The following lemma gives a key distributional identity for the zero bias transformation of a finite sum of independent random variables; see, for instance, Lemma 2.2 of [18] for the proof.
Lemma B.3. Let $X_1, \ldots, X_n$ be independent centered random variables with $\text{Var}(X_k) = \sigma_k^2$. Let $i$ be a variable taking values in $\{1, 2, \ldots, n\}$ with probability $P[i = k] = \frac{\sigma_i^2}{\sigma_1^2 + \cdots + \sigma_n^2}$. The distribution of $(X_1 + \cdots + X_n)^*$ has the form

$$(X_1 + \cdots + X_n)^* \overset{d}{=} X_1 + \cdots + X_n + (X_i^* - X_i),$$

where $i$ is independent of the random variables $X_k$ and $X_k^*$. In other terms, the $k$th variable $X_k$ in the sum is replaced by $X_k^*$ with probability $\frac{\sigma_k^2}{\sigma_1^2 + \cdots + \sigma_n^2}$.

B.2 Relation to Stein’s method

Recall that $\rho_1(X, Y) := \sup_{h \in \text{Lip}_1} E[h(X) - h(Y)]$ for two random variables $X$ and $Y$ with finite first absolute moments. Also, recall that the auxiliary function $f$ for a given $h \in \text{Lip}_1$ in Stein’s method satisfies the differential equation

$$f'(x) - \frac{x}{\sigma^2} f(x) = h(x) - \int_{\mathbb{R}} h(r) e^{-r^2/2\sigma^2} dr$$

and that the first- and second-order derivatives have the bounds $\sup_d |f'(x)| \leq 1$ and $\sup_d |f''(x)| \leq 2$. In particular $f'$ is absolutely continuous with Lipschitz constant $\leq 2$. If $X$ is a centered random variable with variance $\sigma^2$ and $X \sim \mathcal{N}(0, \sigma^2)$, then by definition of $X^*$ we have

$$E[h(X) - h(X)] = E[f'(X) - \frac{X}{\sigma^2} f(X)] = E[f'(X) - f'(X^*)]$$

Thus, by supemizing over $h \in \text{Lip}_1$ above, we have the bound $\rho(X, X) \leq 2 \rho(X, X^*)$ since $|f''| \leq 2$. Therefore the Wasserstein-1 norm between $X$ and the normal random variable $X$ is smaller than two times the Wasserstein-1 norm between $X$ and its zero bias transformation.

Lemma B.4. Let $X_1, \ldots, X_n$ be i.i.d. variables with mean 0 and variance $\sigma^2$. For $Y_n := \frac{X_1 + \cdots + X_n}{\sqrt{n}}$, we have the inequality

$$\rho_1(Y_n, Y_n^*) \leq \frac{1}{\sqrt{n}} \rho(X_1, X_1^*)$$

for $Y_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}$. Moreover, if $E[|X_1|^3] < \infty$ and $Y \sim \mathcal{N}(0, \sigma^2)$, then

$$\rho_1(Y_n, Y) \leq \frac{3}{\sqrt{n}} \frac{E[|X_1|^3]}{\sigma^2}.$$

Proof. Let the pairs $(X_k, X_k^*)$ be i.i.d. couplings of the variables $X_k$ and $X_k^*$ such that

$$\rho_1(X_k, X_k^*) = E[|X_k - X_k^*|].$$

Then $\rho_1(Y_n, Y_n^*)$ is bounded as follows:

$$\rho_1(Y_n, Y_n^*) = \sup_{||h||_{\text{Lip}}} E[h(Y_n) - h(Y_n^*)] \leq E[|Y_n - Y_n^*|] = \frac{1}{\sqrt{n}} E[|X_1 - X_1^*|] = \frac{1}{\sqrt{n}} E[|X_1 - X_1^*|],$$

and the last term is equal to $\frac{1}{\sqrt{n}} \rho_1(X_1, X_1^*)$ by assumption. Next we simply observe that

$$\rho_1(X_1, X_1^*) = E[|X_1 - X_1^*|] \leq E[|X_1|] + E[|X_1^*|] \leq E[|X_1|] + \frac{1}{2\sigma^2} E[|X_1|^3] \leq \frac{3}{2\sigma^2} E[|X_1|^3],$$

where the second inequality is by Lemma B.2. The result then holds because $\rho_1(Y_n, Y) \leq 2 \rho(Y_n, Y_n^*)$. 

\[\square\]
Proof of Lemma 7.6. Lemma 7.4 gives us the inequality
\[ \rho_2(X_n, X) \leq (\rho_1(X_n, X))^{\frac{2}{3}} \left( \mathbb{E}[X_n^4]^{\frac{1}{6}} + \mathbb{E}[X]^4 \right). \]

Since \( \mathbb{E}[X^2] = \sigma^2 \) and \( X \sim \mathcal{N}(0, \sigma^2) \), we have \( \mathbb{E}[X^4] = 3\sigma^4 \leq 3\mathbb{E}[X_n^4] \). Thus with Lemma 7.5,
\[ \leq \left( \frac{3}{\sqrt{n}} \mathbb{E}[|X_1|^3] \right)^{\frac{1}{6}} (1 + 3\frac{1}{n}) \mathbb{E}[X_n^4]^{\frac{1}{6}} \leq 4 \left( \frac{1}{\sqrt{n}} \mathbb{E}[X_1^4] \right)^{\frac{1}{6}} \mathbb{E}[X_n^4]^{\frac{1}{6}} \leq 4n^{-\frac{1}{6}} \mathbb{E}[X_1^4]^{\frac{5}{6}}. \]

The second inequality above uses that \( \mathbb{E}[X_n^4] = 3\sigma^4 \left( 1 - \frac{1}{n} \right) + \frac{1}{n} \mathbb{E}[X_1^4] \) is smaller than \( 3\mathbb{E}[X_1^4] \). \( \square \)

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