Bose-Einstein condensed supermassive black holes: a case of renormalized quantum field theory in curved space-time

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Abstract

This paper investigates the question whether a realistic black hole can be in principal similar to a star, having a large but finite redshift at its horizon. If matter spreads throughout the interior of a supermassive black hole with mass $M \sim 10^9 M_\odot$, it has an average density comparable to air and it may arise from a Bose-Einstein condensate of densely packed H-atoms. Within the Relativistic Theory of Gravitation with a positive cosmological constant, a bosonic quantum field describing H atoms is coupled to the curvature scalar with dimensionless coupling $\xi$. In the Bose-Einstein condensed groundstate an exact, self-consistent solution for the metric occurs for a certain large value of $\xi$, quadratic in the black hole mass. It is put forward that $\xi$ is set by proper choice of the background metric as a first step of a renormalization approach, while otherwise the non-linearities are small. The black hole has a hair, the binding energy. Fluctuations about the ground state are considered.

Key words: supermassive black hole, quantum field theory, Bose Einstein condensation, renormalization

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1. Introduction

The standard knowledge that black holes (BHs) have no hair (i.e., they are determined by mass, charge and spin only) has recently been tested versus observational data from quasars. Schild et al. carefully examine why standard models of quasars with the various structural elements fail to be compatible with the observations of a luminous ring at the innermost edge of the accretion disc of the quasar Q0957+561 A,B. The observations imply that this quasar contains an observable magnetic moment, a “hair”, which represents strong evidence that it does not have an event horizon \cite{Schild2006}. On a different track, one of us presented a class of exactly solvable BHs of with the Schwarzschild on the outside having with one hair, namely the binding energy \cite{Nieuwenhuizen2008}.

A great deal of observational activity is presently underway to measure with 10 micro-arcsec resolution the Sgr A black hole object at the center of our Galaxy, because it is widely claimed that detecting the shadow of the black hole would prove the existence of an event horizon. However the black hole with hair objects either indicated by the Schild et al \cite{Schild2006} observations [1] or by the Nieuwenhuizen \cite{Nieuwenhuizen2008} theory \cite{Nieuwenhuizen2008}, would also be a compact object with a strong gravity shadow.
that is virtually indistinguishable from a black hole event horizon. Indeed, the outer metrics are close to the Schwarzschild metric. The two classes of objects are in fact best distinguished from one another by the existence of a light cylinder effect originating the dusty torus, and the clear demonstration of such light cylinder effects in ordinary quasars by Schilf et al (2009) favors the magnetic over the standard black hole [3].

Quasars are probably supermassive black holes which occur at the center of most galaxies. Even though the one of our Galaxy weighs only about 4 million solar masses, the typical weight is a billion solars, with the present champion at 17 billion. Since the Schwarzschild radius of a black hole scales with its mass, \( r_S = 2GM/c^2 \), its density decays as \( 1/M^2 \), implying that bigger black holes have a lower average density. This motivates to consider a supermassive black hole, say with \( M = 10^9 M_\odot \). Then the average mass density is on the order of grams per liter, comparable to air. The fact that they are enormously heavy only because of their size, suggests that for these objects the physics may be not too difficult or unfamiliar, and it invites to study their internal structure as a standard problem in a standard theory.

These estimates of the average density confront the theoretical descriptions. On the basis of the Schwarzschild, Kerr and Kerr-Newman metrics, black holes (BHs) are described as singular objects with all matter localized in the center or, if rotating, on an infinitely thin ring. Recent approaches challenge this assumption and consider matter just spread throughout the interior [4–6].

Let us point at the following simple connection. Supermassive BHs occur in the center of many galaxies and weigh about \( M_{\text{BH}} = 0.0012 M_{\text{bulge}} \) [7]. Let us assume that they consist of hydrogen atoms and that mass and particle number are related as \( M \equiv \nu N m_N \) with \( m_N \) the nucleon mass and some \( \nu \leq 1 \). Neglecting rotation, we may compare the H number density within the Schwarzschild radius \( R_S = 2GM/c^2 \), i.e. \( n_H = 3N/4\pi r_S^3 \) with the one of densely packed, non-overlapping H-atoms, that is, with the Bohr density \( n_B \equiv 3/4\pi \alpha_0^3 \), with \( \alpha_0 \approx 0.529 \) the Bohr radius. This yields a mass \( c^2(n_B^3/8G^2m_N)^{1/2} = 8.26 \cdot 10^7 M_\odot \) (we take \( \nu = 1 \) here), which indeed lies in the range of observed supermassive black holes. The corresponding mass density is 2.7 kg/liter, but scaling as \( 1/M^2 \) for larger black holes, it is as low as 0.6 g/liter for the \( 17 \cdot 10^9 M_\odot \) black hole, already below the one of air. We shall therefore take as characteristic mass scale \( 10^9 M_\odot = 1.98 \cdot 10^{39} \) kg and write

\[ M \equiv M_9 \cdot 10^9 M_\odot. \] (1)

The assumption of spread-out matter poses the question how matter can have a pressure that allows such a state. It was proposed originally by Sacharov that the vacuum equation of state \( p = -\rho \) could describe matter at superhigh densities [8]. Laughlin and coworkers assume that matter near the horizon could be in its Bose-Einstein condensed (BEC) phase, modeled by the vacuum equation of state [4]. Dymnikova considers BHs obeying it in the interior, which, however, have one or two horizons [5]. Mazur and Mottola take the BEC idea over to the interior, and investigate a “gravostar”, of which the interior obeys the vacuum equation of state, and which is surrounded by a thin shell of normal matter having the stiff equation of state \( p = +\rho \). This solution is regular everywhere [6]. A related subject is the description of boson stars [9,10].

One of us considered a modification of general relativity (GR), the Relativistic Theory of Gravitation (RTG) that differs from GR only at very high redshift, in particular near the horizon of black holes [11]. This modification appears to describe the BH interior like a star, be it with strong but finite redshifts.

We shall study a supermassive BH that exist as a self-gravitating hydrogen cloud, in a Bose-Einstein condensed phase. Hereto we employ the Relativistic Theory of Gravitation (RTG), which reproduces all weak gravitational effects in the solar system [12,13] as well as the \( \Lambda \)-CDM cosmology [11]. We shall extend the recent approach to this line of research started by one of us [2,14,15], where an exact interior solution to the supermassive black hole problem was found, in a theory which is a modification of General Relativity. This Schwarzschild-type black hole has a hair, namely the binding energy of the matter out of which it is composed. It has a very large but finite redshift at the horizon, so there is no sharp horizon, no Hawking radiation and no role for Bekenstein-Hawking entropy. From a principle point of view, this approach, when extended to the rotating situation with an accretion disk, may describe the above discussed quasar observations.

Section 2 discusses aspects of RTG and its generalization of the Schwarzschild metric, together with its deformation near the horizon. Section 3 introduces the
quantum field theory for the H-atoms and their Bose-Einstein condensation. Section 4 presents an exact solution of the interior metric. Section 5 deals with general solution near the horizon. Section 6 discusses excitation about the groundstate and the paper closes with a discussion in section 7.

2. Relativistic Theory of Gravitation

We consider a static metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, where $x^\mu = (ct, r, \theta, \phi)$, with $\mu = 0, 1, 2, 3$. Here and in the sequel, summation over repeated indices is implied. In case of spherical symmetry it has the form

$$ds^2 = U(r) c^2 dt^2 - V(r) dr^2 - W^2(r) d\Omega^2$$

(2)

with $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$. The metric tensor can be read off, $g_{\mu\nu} = \text{diag}(U, -V, -W^2, -W^2 \sin^2 \theta)$, while $g^{\mu\nu} = (g^{-1})_{\mu\nu}$. The gravitational energy density arises from the Landau-Lifshitz pseudo-tensor [16], generalized to become a tensor in Minkowski space-time [17,11]. For the metric (2) it takes the form

$$t^0_0 = \frac{c^4 W^2}{8\pi G r^2} \left( -\frac{r^2 V'W'W}{V} + r^3 V' - 5r^2 W'^2 \right. \left. + \frac{2r^2 VW'}{W} + 8r W W' - 2r^2 V - 3W^2 \right).$$

(3)

Let us start with the General Theory of Relativity (GTR, GR). The Schwarzschild metric reads in the harmonic symmetry it has the form

$$U_S = \frac{1}{V_S} = 1 - \frac{2M}{r + M}, \quad W_S = r + M,$$

(4)

where $M$ is the gravitational length

$$\overline{M} = \frac{GM}{c^2}.$$  

(5)

The metric is singular at the horizon $W_S = 2\overline{M}, r = \overline{M}$ and it involves the gravitational energy density

$$t^0_0 = \frac{c^4 \overline{M}}{4\pi G r^2} \frac{d}{dr} (r + \overline{M})^3 \frac{(r + \overline{M})}{2r^3 (r - \overline{M})}.$$  

(6)

Its quadratic divergence at $\overline{M}$ presents an often overlooked, physical peculiarity, that induces a negative infinite contribution to the total energy. In other gauges the singularity can be moved completely to the origin. All by all, the situation is puzzling. For this reason, we shall switch to RTG with matter not located at the singularity $r = 0$, but just spread out within the horizon.

We focus on RTG, which describes gravitation as a field in Minkowski space [12,13] and possesses the same gravitational energy momentum tensor and thus also the gravitational energy density Eq. (3) [2]. It extends the Hilbert-Einstein action with the cosmological term and a bimetric coupling between the Minkowski $(\gamma)$ and Riemann $(g)$ metrics,

$$L = -\frac{c^4 R}{16\pi G} - \rho_\Lambda + \frac{1}{2} \rho_{bi} \gamma_{\mu\nu} g^{\mu\nu} + L_{\text{mat}},$$

(7)

(For $\rho_{bi} = 0$ it is just a field theoretic description of GTR.) The dimensionless action is $S'/\hbar = (1/\hbar c) \int d^4 x \sqrt{-g} L$, where $x_0 = ct$. One has the Einstein equations

$$C^\mu_\nu \equiv R^\mu_\nu - \frac{1}{2} g^\mu_\nu R = \frac{8\pi G}{c^4} T^\mu_\nu,$$

$$T^\mu_\nu = T^{\mu\nu}_\Lambda + T^{\mu\nu}_{bi},$$

(8)

(9)

where $T^{\mu\nu}$ comes from the matter, $T^{\mu\nu}_\Lambda$ from the cosmological term and $T^{\mu\nu}_{bi}$ from the bimetric term. They have the dimension of energy/volume and elements $(\rho_i, -p_i)$ of the form

$$\rho_{tot} = \rho + \rho_\Lambda + \frac{\rho_{bi}}{2U} - \frac{\rho_{bi}^2}{2V},$$

$$p_i = p_i + \rho_\Lambda - \frac{\rho_{bi}}{2U} - \frac{\rho_{bi}^2}{2V}.$$  

(10)

with $i = r, \theta, \phi$. The mass density is $\rho/c^2$. It is customary to choose the value $\rho_{bi} = \rho_\Lambda$ in order to allow a Minkowski metric $U = V = 1, W = r$ in the absence of matter, because then $\rho_{tot} = p_i^{tot} = 0$. We shall fix $\rho_\Lambda = \rho_{bi}$ to the observed positive cosmological constant [2]. However, historically the opposite choice $\rho_\Lambda < 0$ was considered and the cosmological data were described by an additional inflaton field [13], so the sign of $\rho_{bi}$ is still disputed. We show that the possibility to solve a realistic black hole settles that indeed $\Lambda_{bi} > 0$ is the physically interesting case. The new point of RTG is that $g_{00} = U$ can be very small but still positive. Despite the smallness of $\rho_{bi}$, the $\rho_{bi}/U$ term becomes relevant near the horizon [12,2] and regularizes the singularities of the Schwarzschild metric. So we continue with $\rho_{tot} \approx \rho + \rho_{bi}/2U, p_i^{tot} \approx p_i + \rho_{bi}/2U$. 


The \( \rho_{bi} \) term in (7) violates general coordinate invariance and the consistency requirement \( T_{\mu\nu}^{\mu\nu} = 0 \) imposes the harmonic gauge condition

\[
\frac{U'}{U} - \frac{V'}{V} + \frac{W'}{W} = \frac{4rV}{W^2}.
\]  

(11)
The residual gauge group actually would lead to an infinite gravitational energy, so its physical subgroup is empty, making the solution unique [18].

In post-Newtonian approximations and in applications to e.g., stars, the \( \rho_{\lambda} \) and \( \rho_{bi} \) terms are negligible, which will bring back the general coordinate invariance of GTR. Gravitational radiation in e.g. X-ray binaries is the same as in GTR. But near the horizon of a black hole the large redshift will make the \( \rho_{bi}/U \) term sizeable, and deeply change the theory.

The \( ^0 \), \( ^1 \), and \( ^2 \) components of the Einstein equations read respectively

\[
\frac{1}{W^2} \frac{W''}{V W^2} - \frac{2W'''}{V W} + \frac{V'''}{V^2 W} = \frac{8\pi G}{c^4} \rho_{tot},
\]

\[
\frac{1}{W^2} \frac{W''}{V W^2} \frac{U''}{U W} = - \frac{8\pi G}{c^4} \rho_{tot},
\]

\[
- \frac{U''}{2UV} \frac{W''}{V W} + \frac{U'''}{4U^2 V} + \frac{V'''}{4UV^2} + \frac{V''}{2V W} = - \frac{8\pi G}{c^4} \rho_{tot}.
\]  

(12)
As usual, the last equation is automatically satisfied by energy conservation \( T_{\mu\nu}^{\mu\nu} = 0 \). The Ricci scalar becomes

\[
R = \frac{8\pi G}{c^4} \left( - \rho_{tot} + \rho_{\mu} + 2\rho_{\perp} \right) \equiv R_{bi} + R_{m} \quad \text{(13)}
\]

\[
R_{bi} = \frac{8\pi G \rho_{bi}}{c^4}, \quad R_{m} = \frac{8\pi G}{c^4} \left( - \rho + \rho_{\mu} + 2\rho_{\perp} \right). \quad \text{(14)}
\]

We define the inverse length \( \mu_{bi} \) by

\[
\rho_{bi} = \frac{\mu_{bi}^2}{16\pi G}, \quad \mu_{bi} = \sqrt{2A} = 2.38 \cdot 10^{-23} \frac{c^2}{GM_{\odot}} \quad \text{(15)}
\]

with \( \Lambda \) the cosmological constant. We shall often encounter the combination

\[
\hat{\rho}_{bi} = \frac{\mu_{bi} GM}{c^2} = 2.38 \cdot 10^{-23} \frac{M}{M_{\odot}} = 2.38 \cdot 10^{-14} M \quad \text{(16)}
\]

2.1. Generalization of the Schwarzschild solution

The Schwarzschild metric solves the above equations in the limit \( \rho \rightarrow 0 \), \( \rho_{\lambda} = \rho_{bi} \rightarrow 0 \). A more general solution within RTG is found as follows. Assume that \( U \), \( V \) and \( W \) are still related in the Schwarzschild manner \( U = 1 - 2M/W \), \( V = W^2/U \). This solves the two Einstein equations (12) without matter, \( \rho_{tot} = \rho_{tot} = 0 \). Inserting this in the harmonic constraint we have

\[
\frac{W''}{W^2} W (W - 2M) + \frac{2(W - M)}{W r'} = 2r.
\]  

(17)
Going to the inverse function \( r(W) \) this becomes linear,

\[
W (W - 2M) r'' + 2(W - M) r' = 2r
\]  

(18)
One solution is \( r = A(W - M) \). The method of variation of constants brings

\[
\frac{A''}{A'} = - \frac{1}{W - M} \frac{2}{W - 2M} \quad \text{(19)}
\]

So with integration constant \( C \) one has

\[
A' = \frac{- C M^3}{W(W - M)^2(W - 2M)} \quad \text{(20)}
\]

It brings \( A \) and from this, choosing \( A(\infty) = 1, \)

\[
r = W - M + C \left[ W - M - \frac{W}{2} \log \frac{W}{W - 2M} - M \right].
\]  

(21)
The horizon \( U = 0 \), \( W = 2M \) is located at \( r = \infty \) for \( C > 0 \) and at \( -\infty \) for \( C < 0 \). Clearly, a small \( C \) is expected to avoid such peculiarities.

Near the horizon the bimetric coupling induces a deformation of the Schwarzschild metric, which regularizes its singularity [12,13]. For small values of the dimensionless product \( \mu_{bi} M \), a scaling form was presented by one of us [2],

\[
r = M \frac{1 + \eta (\epsilon^\zeta + \zeta + \log \eta + 2)}{1 - \eta (\epsilon^\zeta + \zeta + \log \eta + 2)} \quad \text{with} \quad U = \eta \epsilon^\zeta, \quad \text{(22)}
\]

\[
V = \frac{\epsilon^\zeta}{\eta(1 + \epsilon^\zeta)^2}, \quad W = \frac{2M}{1 - \eta \epsilon^\zeta - \tilde{\mu}_{bi}^2 (\zeta + \omega)}.
\]

Here \( \zeta \) is the running variable and \( \eta \sim \mu_{bi} M \) a small scale. For \( \mu_{bi} M \ll \eta \epsilon^\zeta \ll 1 \) it coincides with the generalized Schwarzschild solution (21), with \( C = -4\eta \) indeed being small. Thus the singularities of (21), \( U \) and \( V \) are smoothly deformed in (22), due to the bimetric term \( \rho_{bi}/U \).

\(^3\) An exact solution was found before by A. V. Genk and A. A. Tron.
3. Quantum field theory in curved space

Let our H-atoms be described by a scalar, bosonic creation field operator \( \hat{a}_i \psi_i(r) e^{-iE_i t} \hbar \), where \( i = \{ n, \ell, m \} \), \( \{ \hat{a}_i, \hat{a}^\dagger_i \} = \delta_{ij} \) and eigenfunctions factor as \( \psi_i(r) = \phi_n(r) Y_{\ell m}(\theta, \phi) \). The total field \[ \hat{\psi}_i = \hat{\psi} + \hat{\psi}^\dagger \]
is Hermitean. We assume that the two-particle interaction can be replaced by a \( \delta \)-potential. This leads to a quartic Lagrangian density \( L_{\text{mat}} = \frac{1}{2} \partial^\mu \hat{\psi}_i \partial_\mu \hat{\psi}_i - \frac{m^2 c^2}{2 \hbar^2} + \frac{\xi}{2} \hat{\psi}_i^2 - \frac{\lambda}{2 \hbar c} \hat{\psi}_i^4 \), \[ (25) \]
where \( \partial_\mu = \partial / \partial x^\mu \), \( \partial^\mu = \partial / \partial x^\mu \), and \( \lambda \) is dimensionless. With field dimension \( [\psi] = [\sqrt{\hbar c / \ell_P}] \), where \[ \ell_P = \sqrt{\hbar G / c} \], \( m_P = \sqrt{\hbar c G} \), \[ (26) \]
are the Planck length and the Planck mass, respectively, the Lagrangian density has dimension \([L_{\text{mat}}] = [\hbar c / \ell_P^2] \), i.e., energy density, as it should.

For a field in curved space the renormalization group generates the coupling to the Ricci curvature scalar \( R \) entering eq. (25) \[ [9] \]. Its strength \( \xi \) has to be obtained from renormalization arguments. Popular values are \( \xi = 0 \) and \( \xi = \frac{1}{2} \). In our earlier work \[ [2] \] we have treated it as a phenomenological parameter, here we shall argue that its large, mass dependent value is self-generated.

The conjugate momentum field is \[ \hat{\pi}_i^\mu = \frac{\partial L_{\text{mat}}}{\partial (\partial_\mu \psi_i)} = \partial^\mu \hat{\psi}_i \]. \[ (27) \]
The equal-time commutation relation is for the field is \[ n_{\mu}[\hat{\psi}(r, t), \hat{\pi}^\mu(r', t)] = i\hbar c \delta^{(3)}(r - r') \sqrt{-g_3} \], \[ (28) \]
where \( n_{\mu} = \delta_\mu^0 \sqrt{U} \) is a time-like unit vector and \( g_3 = -V W^4 \sin^2 \theta \) the determinant of the spatial part of the metric. The Hamiltonian density is

\[ \hat{H}_t = \sum_i \hat{a}_i^\dagger \hat{a}_i - L_{\text{mat}}^{\text{full}} = \frac{\partial^\mu \hat{\psi}_i \partial_\mu \hat{\psi}_i + \partial^\mu \hat{\psi}_i \partial_\mu \hat{\psi}_i}{2} + \frac{m^2 c^2}{2 \hbar^2} + \frac{\xi}{2} \hat{\psi}_i^2 + \frac{\lambda}{2 \hbar c} \hat{\psi}_i^4 \]
where \( i = 1, 2, 3 \) sums the spatial components.

Bose-Einstein condensation can be described in the rotating wave approximation, yielding, after normal ordering, the Lagrangian density \[ [20] \]
\[ L_{\text{mat}} = g_{\mu\nu} \partial_\mu \hat{\psi}^\dagger \partial_\nu \hat{\psi} - (\bar{m}^2 + \xi R) \hat{\psi}^\dagger \hat{\psi} - \frac{\lambda}{4 \hbar c} \hat{\psi}^4 \]
where we define the inverse Compton length \[ m = \frac{m c}{\hbar} \]. \[ (30) \]

The equal-time commutator reduces to \[ n_0 [\hat{\psi}(r, t), \partial_0 \hat{\psi}^\dagger(r', t)] + n_0 [\hat{\psi}^\dagger(r, t), \partial_0 \hat{\psi}(r', t)] \]
\[ = i\hbar c \delta^{(3)}(r - r') \sqrt{-g_3} \], \[ (32) \]
which reads when written out in eigenfunctions \[ n_0 \int_j 2i E_j \hat{\psi}_j(r) \hat{\psi}^\dagger_j(r') = i\hbar c \delta^{(3)}(r - r') \sqrt{-g_3} \]. \[ (33) \]

Multiplying this with \( \sqrt{-g_3} \psi_j^*(r) d^3 r \) leads to the orthonormality \( (2E_j / \hbar c)^2 \int d^3 r n_0 \sqrt{-g_3} \psi_j^* \psi_j = \delta_{ji} \). The Hamiltonian density becomes
\[ \hat{H} = \partial^\mu \hat{\psi}^\dagger \partial_\mu \hat{\psi} + \partial^\mu \hat{\psi}^\dagger \partial_\mu \hat{\psi} + \bar{m}^2 + \xi R \hat{\psi}^\dagger \hat{\psi} + \frac{\lambda}{4 \hbar c} \hat{\psi}^4 \]. \[ (34) \]

The non-relativistic, flat space Gross-Pitaevskii equation reads \[ [20] \]
\[ i\hbar \partial_t \Psi_0 = -\frac{\hbar^2}{2m} \nabla^2 \Psi_0 + g |\Psi_0| \Psi_0 \], \[ (35) \]
with \( g = 4\pi \hbar^2 a_s / m \) modeling the two particle interaction by the scattering length \( a_s \). For hydrogen in flat space one has \[ [21] \]
\( a_s = 0.32 a_0 \) singlet state, \( a_s = 1.34 a_0 \) triplet.

We shall continue with the singlet value. Our first task is to connect \( \lambda \) to \( g \). The relativistic form of (35) is
\[ \eta^{\mu\nu} \partial_\mu \partial_\nu \psi_0 + \bar{m}^2 \psi_0 + \frac{\lambda N_0}{2\hbar c} |\psi_0|^2 \psi_0 = 0 \]
where \( \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1) \) is the Minkowski metric and \( \psi_0 = \hbar c \Psi / \sqrt{2N_0 m c^2} \), while the coupling,
\[ \lambda = \frac{8m^2c^4}{\hbar^2} = 32\pi a_smc = 32\pi a_sM = 8.10 \cdot 10^6 \]  

(37)

is dimensionless. Eq. (36) derives from the groundstate of a field (23) with \( N_0 \) particles in the groundstate of a field theory (30) with metric \( g^{\mu\nu} = \eta^{\mu\nu} \) and \( \xi = 0 \). Going to curved space is achieved by taking a general \( g^{\mu\nu} \), which yields the bare Lagrangian density, still having \( \xi = 0 \) and \( \lambda \) from (37). Renormalization arguments will add the \( \xi R \) terms, and it may consequently depend on BH parameters such as the mass.

3.1. Self-consistent field theory

The material energy-momentum tensor \( (T_m)_{\mu\nu} \) is derived from the quantum field theory. We do not include the effect of the \( \xi R \)-term in it. The energy density reads

\[ \rho_m = \frac{\langle \partial_\mu \hat{\psi}^\dagger \partial_\nu \hat{\psi} \rangle}{c^2U} + \frac{\langle \hat{\psi} \partial_\mu \partial_\nu \hat{\psi} \rangle}{V} + \frac{\langle \hat{\psi} \partial_\mu \partial_\nu \hat{\psi} \rangle}{W^2} + \lambda \langle \hat{\psi}^\dagger \hat{\psi} \rangle. \]  

(39)

The pressures \( p_r^m, p_\theta^m, p_\phi^m \) have this shape with signature \( (+ + + + + +) \), \( (+ + + + + -) \), and \( (+ + + + - -) \), respectively. Spherical symmetry will imply that \( \rho_\phi^m = p_\phi^m \). For a uniform groundstate \( p_m \) is isotropic,

\[ (\rho_m, p_m) = \frac{E_0}{2U} |\Psi_0^2| \pm \left( \frac{m^2}{2E_0} |\Psi_0^2| + \frac{\lambda |\Psi_0^2|}{16E_0} \right). \]  

(40)

They consist of a vacuum part \( p = -\rho = \text{const.} \) and a stiff part \( p = +\rho \sim 1/U \), the types studied in [6] and in [15]. In the non-relativistic \( (E_0 = mc^2) \) and flat space \( (U = 1) \) limit, they reduce for \( \lambda = 0 \) to the expected results \( \rho_m = mc^2 |\Psi_0^2| \) and \( p_m = 0 \).

The energy momentum tensor in the Einstein equations has two further terms. Because of the \( \xi R \)-term in (30), the Einstein equations embody a direct backreaction \( (B) \) of matter on curvature, \( G_{\mu\nu} = 8\pi Gc^{-4}(T_{\mu\nu}^m + T_{\mu\nu}^\Lambda + T_{\mu\nu}^B) - BG_{\mu\nu} \), with

\[ B = \frac{16\pi G}{c^4} \xi (\hat{\psi}^\dagger \hat{\psi}) \equiv B_0 + B_c, \]  

(41)

\[ B_0 = \frac{16\pi G}{c^4} \xi |\psi_0|^2, \quad B_c = \frac{16\pi G}{c^4} \xi (\hat{\psi}^\dagger \hat{\psi}). \]

If \( B \) depends on \( r \), the derivation of the Einstein equations brings from the term \( \xi R(\hat{\psi}^\dagger \hat{\psi}) \) also derivatives of \( B \), which induces another new term \( T_{\mu\nu}^B \). Let us in general define the dimensionless density and pressures

\[ \bar{\rho} = \frac{8\pi G}{c^4} M^2 \rho, \quad \bar{p}_r = \frac{8\pi G}{c^4} M^2 p_r. \]  

(42)

The elements of \( (T_\mu)^{\nu}_B \equiv \text{diag}(\rho_B, -p_r^B, -p_\theta^B, -p_\phi^B) \) can then be expressed as

\[ \bar{\rho}_B = \bar{M}^2 \left( \frac{B''}{V} + \frac{2rB'}{W^2} - \frac{B'U'}{VW} \right), \]

(43)

\[ \bar{p}_r^B = -\bar{M}^2 \left( \frac{B'U'}{VW} - \frac{2B''}{VW} \right), \]

(44)

\[ \bar{p}_\theta^B = -\bar{M}^2 \left( \frac{B''}{V} - \frac{2rB'}{W^2} - \frac{B'U'}{VW} \right), \]

(45)

where the explicit factors \( r \) arise from inserting the harmonic constraint (11).

In terms of the function \( B_0 \), eq. (39) and the pressures read

\[ \rho_m = \frac{E_0}{U} B_0 \frac{B''}{2} + \frac{B_0^2}{4U} + \frac{m^2}{16\pi \xi} (B_0 + \frac{\lambda}{2} B_0^2) + \rho_c, \]

(46)

\[ p_r^B = \frac{E_0}{U} B_0 \frac{B''}{2} + \frac{B_0^2}{4U} - \frac{m^2}{16\pi \xi} (B_0 + \frac{\lambda}{2} B_0^2) + \rho_c, \]

(47)

where

\[ E_0 \equiv \frac{E_0}{mc^2}, \]

(48)

and where \( \rho_c, p_c \) and \( p_c^\phi \) arise from the excited states. They are derived in (16b). Together with \( B_c \), they represent all fluctuations of the problem about the ground state at temperature \( T > 0 \); in fact, even at \( T = 0 \) these fluctuations exist and they already bring important effects, see [20] in general and the discussion below for our BH. Furthermore, the \( B_0^2 \sim |\psi_0|^2 \) term is of relative order

\[ \tilde{\lambda} \equiv \frac{\lambda m_p^2}{32\pi m^2 \xi}. \]  

(49)

Eq. (8) involves a total energy momentum tensor

\[ T_{\mu\nu} = T_{\mu\nu}^m + T_{\mu\nu}^\Lambda + T_{\mu\nu}^B \equiv T_{\mu\nu} + T_{\mu\nu}^\Lambda + T_{\mu\nu}^B, \]  

(50)
As mentioned, of the last two terms in (50) only the \( \rho_0/U \) contributions are relevant for us. \( T_\rho \equiv \text{diag}(\rho, -p_r, -p_\perp, -p_\perp) \) has dimensionless elements

\[
\bar{p} = \frac{M^2}{1+B} \left[ \begin{array}{c}
\frac{B''}{V} + 2rB'V - B'U' + \frac{B'_0}{8\xi B_0V} \\
+ \frac{m^2}{2\xi} B_0 - \frac{m^2}{4\xi} \lambda B_0^2 + \frac{2E_0}{4\xi} - \frac{\mu_0^2}{M} \left( B + \frac{\bar{p}_e}{M} \right)
\end{array} \right],
\]

\[
\bar{p}_e = \frac{M^2}{1+B} \left[ -\frac{B''}{V} - 2rB'V - B'W' + B''^2 - \frac{B'_0}{8\xi B_0V} \\
- \frac{m^2}{2\xi} B_0 - \frac{m^2}{4\xi} \lambda B_0^2 + \frac{2E_0}{4\xi} - \frac{\mu_0^2}{M} \left( B + \frac{\bar{p}_e}{M} \right)
\end{array} \right],
\]

\[
\bar{p}_\perp = \frac{M^2}{1+B} \left[ \begin{array}{c}
\frac{B''}{V} - 2rB'V - B'W' + B''^2 - \frac{B'_0}{8\xi B_0V} \\
- \frac{m^2}{2\xi} B_0 - \frac{m^2}{4\xi} \lambda B_0^2 + \frac{2E_0}{4\xi} - \frac{\mu_0^2}{M} \left( B + \frac{\bar{p}_e}{M} \right)
\end{array} \right].
\]

The Ricci scalar follows from \( R = \rho_0'/2U + R_m \), where \( R_m = -8\pi GT/c^4 \) reads

\[
R_m = -\frac{1}{\xi(1+B)} \left[ \begin{array}{c}
\frac{3B''}{V} + 6\xi B' + \frac{B'_0^2}{2V} + \frac{m^2}{2\xi} B_0 - \xi \frac{\lambda B_0}{2} + \frac{\xi \bar{T}_e}{2M}
\end{array} \right]
\]

\[
+ 2m^2 B_0 + m^2 \lambda B_0^2 - \frac{2E_0}{2U} - \frac{\mu_0^2}{M} \left( B + \frac{\bar{p}_e}{M} \right).
\]

The last term is proportional to \( T_e = \rho_e - p_\perp - 2p'_\perp \), the trace of the energy momentum tensor from excited states.

The Gross-Pitaevskii equation,

\[
\frac{\partial \psi_0}{\partial U} + \nabla V \psi_0 + \frac{2r}{W} \psi_0 = \left( m + \xi R \right) \psi_0 + \frac{\lambda N}{2\beta c} \psi_0 \psi_0^3 \psi_0^3
\]

may be expressed in terms of \( B_0 \approx \psi_0^2 \) and \( B = B_0 + B_e \),

\[
-6\xi B_0 \frac{B''}{V} + 2rB' - (1 + B) \left( \frac{B'_0}{2V} + \frac{2B'_0}{W^2} \right) + \frac{(1 + B_e)B'_0}{2B_0 V} + 2\frac{m}{B_0}[1 + B_0(1 + B_e) - B_0]
\]

\[
+ \frac{2E_0}{U} - \frac{\mu_0^2}{U} - \frac{\mu_0^2}{U} B_0 B_e = \frac{2\xi B_0 + \bar{T}_e}{M}.
\]

We can now first verify that the total energy momentum tensor is conserved due to the harmonic condition (11). The terms \( B_0^2/U \) are singular because we shall consider \( U(0) = 0 \) and \( B_0(0) > 0 \); they drop out from (51,51,52) and (52,54) for \( T_0 = \mu_0 \sqrt{\xi}/2 \), that is,

\[
E_0^2 = \frac{1}{2} \xi h^2 c^2 \rho_0 = \frac{8\pi G\rho_0^2}{c^2} \xi \rho_0.
\]

We shall verify later that \( B_e(0) = 0 \), so the \( B_0 B_e/U \) term is indeed less singular. We may decompose \( \xi \) as

\[
\xi = \frac{\xi_0}{1 + \xi_e}.
\]

We shall later see that \( \xi_e \) is small. The leading term is

\[
\xi_0 \equiv \frac{2m^2 M^2}{3m_B^2} = 3.28 \cdot 10^{55} M_0^2
\]

In the regime \( B, B_0 \gg 1/\xi \), Eq. (54) simplifies,

\[
- \frac{B''}{V} - 2rB' - \frac{m^2}{2\xi} (1 - B_0 + \lambda B_0) = S_B
\]

\[
S_B = \frac{B'_0}{V} + 2rB'_0 - \frac{m^2}{2\xi} (1 + \lambda B_0) B_e + \frac{\mu_0^2}{6U} + \frac{\bar{T}_e}{3M}
\]

In terms of \( \psi_0 \) the Gross-Pitaevskii equation (58) reads

\[
-\frac{\psi''_0}{V} - \frac{2r\psi'_0}{W} - \frac{B'_0^2}{4m_B^2} + \frac{m_1}{6m_0} \frac{\xi \bar{T}_e}{2M} = \frac{S_B}{2B_0} \psi_0.
\]

Either (58) or (59) determines \( B_0 \) or \( \psi_0 \), once \( B_e \) and \( T_e \) are known. To leading order, they can be omitted. It is simpler, however, to work with \( B = B_0 + B_e \) rather than with \( B_0 \) itself, which satisfies

\[
B'' + 2rB' - \frac{m^2}{3\xi} (1 - B + \lambda B) = \frac{S_B}{2M},
\]

\[
S_B = \frac{2m^2 M^2}{3\xi} [2 - (1 - \lambda B + B_0)] B_e - \frac{\mu_0^2}{3U} - \frac{2\bar{T}_e}{3}
\]

It also holds that

\[
\bar{m}^2 \xi R_m = -\lambda \bar{m}^2 B - \frac{B''}{4B^2V} + \frac{B''}{2B V} + \frac{rB'}{BW^2} \approx -\lambda \bar{m}^2 B - \frac{B''}{4B^2V} + \frac{1 - B + \lambda B}{6B M^2} - \frac{\delta T_m}{6BM^2}
\]

where the first relation is exact and the second holds for \( B \gg 1/\xi \).

4. Exact solution in the interior

Since \( \lambda \) will turn out to be very small, we first take it zero. We also neglect the excitation terms \( B_e \) and \( T_e \)
Solving Eq. (60) for $\bar{\lambda} = 0$ and a constant $B$, we find a relation and, with (55) as consequence,
\begin{equation}
B = 1, \quad N|\psi_0^2| = \frac{E_0}{8\pi^2} = \frac{\mu_{bi}}{E_0} = \frac{\mu_{bi}}{8\pi \sqrt{2\xi E_p}}.
\end{equation}

The value $B = 1$ expresses a 100% direct backreaction of matter on the metric.

Instead of searching a finite $U$, as for boson stars, [9,10] we assume a very small $U$ with $U(0) = 0$, coded by a parameter $u$, of the form
\begin{equation}
U = \frac{1}{2} u \mu_{bi} W^2.
\end{equation}

It is customary to introduce the mass function $M(r)$ and $\overline{M}(r) = GM(r)/c^2$, defined by
\begin{equation}
V = \frac{W^2}{1 - 2\overline{M}/W}.
\end{equation}

The 00 and 11 Einstein equations then take the form
\begin{equation}
\overline{M} = \frac{4\pi G}{c^4} W^2 W'^2 \rho_{tot}, \quad \left(\frac{1}{2} W - \overline{M}\right) \frac{U'}{U} - \frac{\overline{M} W'}{W} = \frac{4\pi G}{c^4} W^2 V^2 \rho_{tot}.
\end{equation}

We may combine (10) and (51) together with $B = 1$ and the Ansatz (63), to obtain
\begin{equation}
\left(\rho_{tot}, \rho_{r}^{tot}\right) = \frac{c^4}{4\pi G W^2} \left(\frac{1}{4u} \pm \frac{\overline{M} W^2}{8\xi}\right).
\end{equation}

In RTG there is a solution of (65) and (66),
\begin{equation}
\overline{M} = \frac{W}{4} + \frac{3\pi W^3}{24\xi}, \quad u = 1.
\end{equation}

It would not exist within GTR, as is seen by taking $\mu_{bi} \to 0$, $u \to \infty$ first. For the Schwarzschild black hole the horizon occurs when $\overline{M} = M$ for $W = 2M$. Concerning the outside metric, we will be close to that situation. This implies that a mass $M$ requires the large, non-constant coupling $\xi = \xi_0 \xi_1$ with
\begin{equation}
\xi_0 \equiv \frac{2m^2 M^2}{3m_p^2} = 3.28 \cdot 10^{55} \left(\frac{M}{10^9 M_{\odot}}\right)^2.
\end{equation}

and $\xi_1 = 1$ up to possible small corrections to be discussed further on. Interestingly, $\xi_0$ is a measure for the area of the black hole $(4\pi \overline{M}^2)$ expressed in units of the square of the hydrogen Compton length $1/\overline{m} = h/mc$, so it is a measure of the surface entropy. Notice that in the Bekenstein-Hawking entropy the area is expressed in squares of the Planck length. In this work we shall consider the $T = 0$ situation, so that the entropy is exactly zero.

We now get
\begin{equation}
E_0 = \mu_{bi} \sqrt{\frac{1}{3} \xi} = \frac{1}{\sqrt{3}} \mu_{bi} GM m = \frac{1}{\sqrt{3}} \bar{u} m c^2.
\end{equation}

and coupling
\begin{equation}
\bar{\lambda} = \frac{\lambda m_p^2}{32\pi m^2\xi} = \frac{3a_s \hbar c^4}{2G^3 m^3 M^2} = 4.166 \cdot 10^{-13} \frac{1}{M_9^2}.
\end{equation}

Though the latter is small, the product $\bar{\lambda} \xi$ is large,
\begin{equation}
\bar{\lambda} \equiv 6\lambda \xi = \frac{6a_s c^2}{GM} = 8.19 \cdot 10^{53}.
\end{equation}

Let us introduce the ‘Riemann’ variables $x$ and $y$ by
\begin{equation}
x = \frac{W}{2M}, \quad y = \sqrt{1 - x^2},
\end{equation}

so that $U = 2\rho_{bi} x^2$. With (63), (64) and $\overline{M} = \frac{1}{2} \overline{M}(x + x^3)$ from (67), the harmonic constraint (11) brings
\begin{equation}
\frac{2x^2}{x} - 2x^2 - \frac{2x^2}{1 - x^2} = \frac{4x}{x} = \frac{8x^2}{x^2(1 - x^2)}.
\end{equation}

Going to the inverse function $r(x)$ makes it linear,
\begin{equation}
x^2(1 - x^2)r_x + x(3 - 4x^2)r_y = 4r.
\end{equation}

In terms of the variable $y$ this transforms into
\begin{equation}
x^2 y_{yy} - 4x^2 y_{y} = 4r.
\end{equation}

The solution is then remarkably simple,
\begin{equation}
r = \frac{r_1}{\sqrt{1 - y^2}} \left(1 + \frac{y}{\sqrt{y}}\right) \left(1 - y\right) \left(1 + y\right) \sqrt{\xi/2}.
\end{equation}

(The second independent solution with $\sqrt{\xi} \to -\sqrt{\xi}$ or $y \to -y$ is singular at $r = 0$, $y = 1$.) It will hold that $r_1 \approx \overline{M}$. This determines the metric functions $W'$ and $V$,
\begin{equation}
W' = \sqrt{\frac{x}{2}}^{2 - \xi} y(1 + y)^{\xi}, \quad V = \frac{2W'^2}{y^2} = \frac{5}{2} y^{4 - 2\xi}(1 + y)^{2\xi}.
\end{equation}

Putting these results together, it now follows that
\[ \rho = \frac{3c^4}{64\pi GM^2}, \quad p_r = p_\perp = p = \frac{3c^4}{64\pi GM^2}. \]  

(76)

So in the interior we reproduce the vacuum equation of state \( \rho = -p = \text{const.} \), that is, \( \bar{\rho} = -\bar{p} = 3/8 \).

To understand the structure of the problem, we again take \( \lambda = 0 \). Then for any \( A \) there is the solution

\[ B(x) = 1 + Ay = 1 + A\sqrt{1 - x^2}. \]  

(77)

Surprisingly, their \( A \)-dependence factors out, keeping a vacuum equation of state \( \bar{\rho} = -\bar{p} = 3/8 \), so (77) is an exact, non-uniform solution of the same metric. It can be verified that \( \psi_0 \sim \sqrt{1 + Ay} \) solves the Gross-Pitaevskii equation (60) at \( \bar{\lambda} = 0 \) and without excited states terms, as it should.

4.1. Normalization

To normalize \( \psi_0 \), we need the 3d volume element in the future time direction, \( d\Sigma^\alpha = drd\theta d\phi r^2 \sqrt{-g_3} \equiv \delta_0^\alpha dV \), set by the timelike unit vector \( n^\alpha = \delta^\alpha_0 / \sqrt{U} \) and \( g_3 = -V W^3 \sin^2 \theta \). This results in \( dV = drd\Omega \sqrt{V/U} W^2 \). It then holds that \( \sqrt{-g}d^4x = U d\Sigma d^3V \).

The general inner product \([19]\) \( (\psi_1, \psi_2) = (-i/\hbar c) \int d\Sigma^\alpha (\psi_1^* \partial_\alpha \psi_2 - \partial_\alpha \psi_1^* \psi_2) \) defines the orthonormality, already noticed below (33),

\[ (\psi_i, \psi_j) = \frac{E_i + E_j}{\hbar^2 c^2} \int dV \psi_i^* \psi_j^* \equiv \delta_{ij}, \]  

(78)

with the volume element

\[ dV = d\Sigma d\Omega \sqrt{g} / \mu_{bi} \]  

(79)

it yields a volume \( V = 32 \pi M^2 / \mu_{bi} = 4\pi(2M^3 / \mu_{bi}) \) and

\[ |\Psi_0^2| = \frac{2E_0}{\hbar^2 c^2} N_0 |\psi_0^2| = \frac{\sqrt{3\mu_{bi}} c^6}{16\pi G^2 m M^2}(1 + Ay). \]  

(80)

4.2. Properties of the solution

The horizon is located at \( r = M, x = 1 \). We may integrate (80) over the BH, which yields \( N_0 \). Alternatively, from the definition (41) of \( B \), we may consider \( \int dV B \), making use of the normalization \( (\psi_0, \psi_0) = 1 \). Either way, this yields

\[ N = N_0 + N_e = 2\sqrt{3} \frac{M}{m} \int_0^1 dy [B_0(y) + B_1(y)]. \]  

(81)

It allows to relate the BH energy to the groundstate occupation,

\[ M^2 = \nu N_0 m c^2, \quad \nu = \frac{1}{(2 + A)\sqrt{3}}. \]  

(82)

Clearly, the energy \( M^2 \) of the BH can be at best 29% of the rest energy \( N_0 m c^2 \) of the constituent hydrogen atoms. In the BH formation, the major part of the energy, \( (1 - \nu)N_0 m c^2 \) minus the potential energy, has to be radiated out. This may explain the large luminosity of quasars.

The leading part of the energy of the quantum field

\[ E^{(0)}_\psi \equiv \int dV \frac{\lambda}{4\hbar c} (\hat{\psi}^\dagger \hat{\psi})^2 = \frac{3\lambda}{2\mu_{bi}} M^2 \int_0^1 dy B^2 \]  

(83)

equals

\[ E^{(0)}_\psi = \frac{9\alpha_i \hbar^2 c^6}{4\mu_{bi} G^2 m^3 M^2}(1 + A + \frac{A^2}{3}). \]  

(84)

With \( \lambda / \mu_{bi} = 17.5/M_0^2 \) it is of the order of the total energy \( M^2 \), and it may exceed it. To obtain the total energy, the gravitational energy density has to be taken into account, which is partly positive and partly negative. Due to a sum rule the total energy is always \( M^2 \) [2].

At the origin the solution exhibits the powerlaw singularities:

\[ U = U_1 r^{\gamma_\nu}, \quad V = \frac{1}{2} \gamma_\nu W_1^2 r^{\gamma_\nu - 2}, \quad W = W_1 r^{\frac{1}{2} \gamma_\nu}. \]  

(85)

where \( \gamma_\nu = \frac{1}{2} (\sqrt{5} + 1) \) is the golden mean. But if we take \( W \) as the coordinate, we have in the interior the shape

\[ ds^2 = \frac{1}{2} \mu_{bi} W^2 c^2 dt^2 - \frac{2dW^2}{1 - W^2/4M^2} - W^2 d\Omega^2, \]  

(86)

which is regular at its origin, with the term \( 2dW^2 \) coding the above powerlaw singularities in \( r \).
5. General solution near the horizon

Near the horizon the exact solution will be deformed. In general we may code the functions $U(r)$ and $V(r)$ in new functions $u(r)$ and $v(r)$, 

$$U = 2\rho_0^2 x^2 u, \quad V = \frac{8M}{y^2} v,$$  \hspace{1cm} (87)

and $W = 2Mx$. With (11) we then have for arbitrary $f$

$$\frac{f_{rr}}{V} + \frac{2rf_r}{W^2} = \frac{x^2 f_{yy}}{8Mv} + \frac{x^2 f_y}{8Mv} + \left[-4y + \frac{x^2 (u_y - v_y)}{v} \right] f_y \frac{v}{8Mv},$$

since the $W''$ terms cancel, as they should, because one can also start with $W$ as variable instead of $r$. The leading shape of the Gross Pitaevskii Eq. (60) can be written as function of $y$, (we neglect $\lambda$, as it will be much smaller than $B_e$ and $T_e$)

$$\frac{x^2}{4v} B_{yy} - \frac{y}{v} B_y + \frac{x^2}{yv} \left[ \frac{u_y}{u} - \frac{v_y}{v} \right] B_y + B = 1 = S_B^r$$

$$S_B^r = 2(1 + \xi_e)B_e - \frac{B}{2} \rho_0^2 = -\frac{2}{3} T_e - \frac{\xi_e}{B_e} - 1. \hspace{1cm} (88)$$

In this equation $B = B_0 + B_e$ consists of contributions of both the groundstate and excited states, while $B_e$ and $T_e$ involve the latter only. While $v = 1$ in the BH interior, it grows as $\sim e^{\xi_e}$ beyond the horizon, possibly enhancing the effect of the fluctuations. In principle this equation may therefore describe a decay of $B_0$ to zero, embedded in excited states that decay slower.

The Einstein equations read in terms of the new functions $u$ and $v$

$$\frac{1}{x^2} \left[ 2 - \frac{y^2}{v^2} \right] + \frac{1}{v} \left[ 2 - \frac{y u_y}{v} \right] = \frac{1}{x^2 u} + 8\rho, \hspace{1cm} (89)$$

$$\frac{2}{x^2} - 3y^2 v \frac{y u_y}{u v} = \frac{1}{x^2 u} - 8\rho.$$  

Expressing the shapes (51) in $y$, we have

$$\rho = \frac{x^2 B_{yy} - (3y + \frac{x^2 u_y}{v}) B_y}{8v(1 + B)} + \frac{y B_0 + 3\lambda B_0^2}{8v(1 + B)} + \frac{\rho_o}{1 + B^2}$$

$$\rho_e = \frac{(3y - \frac{x^2 u_y}{v}) B_y}{8v(1 + B)} - \frac{6B + 3\lambda B_0^2}{8v(1 + B)} + \rho_e,$$  \hspace{1cm} (90)

$$\rho_\perp = \frac{-x^2 B_{yy} + [3y - \frac{x^2 (u_y - u)}{v}]] B_y}{8v(1 + B)} - \frac{6 B_0 + 3\lambda B_0^2}{8v(1 + B)}$$

$$+ \frac{\rho_\parallel}{1 + B^2}.$$  

In the region $B \gg 1/\xi$ we have the simplifications

$$\rho = \frac{(y - \frac{x^2 u_y}{v}) B_y}{8v(1 + B)} + \frac{4 + 2B + \lambda(4B + 3B^2)}{8v(1 + B)} + \rho_0,$$

$$\rho_e = \frac{-y B_y}{8v(1 + B)} + \frac{4 + 2B + \lambda(4B + 3B^2)}{8v(1 + B)}$$

$$\frac{\rho_\parallel}{\rho_\parallel} = \frac{\rho_\parallel}{\rho_\parallel},$$

with the following source terms at $\lambda = 0$

$$\rho_\parallel = \frac{\xi_e(2 + B) + 4\rho_e - 3(1 + \xi_e)B_e + 2S_B^r}{4(1 + B)}$$

$$\rho_e = \frac{6B}{8(1 + B)} + \frac{4\rho_e + 3(1 + \xi_e)B_e}{4(1 + B)}. \hspace{1cm} (91)$$

The equation (72) for the Minkowski coordinate now reads

$$x^2 y^2 r_x + x \left[ \frac{4y^2 - 1 + \frac{x^2 u_y}{u} - v^2}{2} \right] r_x = 4vr. \hspace{1cm} (92)$$

6. Excitations

Let us reformulate our theory on a new basis. We go to a new coordinate $z$,

$$x = \frac{W}{2M} = \frac{1}{\cosh(z/\sqrt{2})}, \quad y = -\tanh\frac{z}{\sqrt{2}},$$  \hspace{1cm} (93)

so that $z = -\infty$ at $r = 0$ and $z = 0$ at $r = 1/M$, a dimensionless time $s$ and a scaled energy,

$$s = \frac{\mu_0 c t}{\sqrt{2}}, \quad \tilde{E} = \sqrt{\frac{\tilde{T}}{\mu_0 c} E}.$$  \hspace{1cm} (94)

The line element then becomes

$$ds^2 = 4M^2 x^2 (udx^2 - v dz^2 - d\Omega^2),$$  \hspace{1cm} (95)

which is Minkovskian in the exact solution where $u = v = 1$. It corresponds to volume elements

$$d^4r\sqrt{-g} = d^4\tilde{r}\sqrt{u v} 16M x^4, \quad d^4\tilde{r} = d\tilde{r} dz d\Omega$$

$$dV = d\nu d\Omega \frac{8M^2}{\mu_0} \sqrt{\frac{\tilde{T}}{u}} = d\tilde{r} dz \Omega \frac{4M^2 x^2}{\mu_0} \sqrt{2v/\tilde{u}}$$

In terms of the new field

$$\delta \psi \equiv \psi - \bar{a}_0 \psi_0 = \sqrt{\frac{\tilde{c}}{2M u^{1/4}}} \tilde{x},$$  \hspace{1cm} (96)

the innerproduct $(\delta \psi, \delta \bar{\psi})$ just becomes

$$d\tilde{r} dz \Omega \frac{4M^2 x^2}{\mu_0} \sqrt{2v/\tilde{u}}$$

$$the \innerproduct (\delta \psi, \delta \bar{\psi}) just becomes
\( (\chi_i, \chi_j) \equiv (\hat{E}_i + \hat{E}_j) \int \frac{dz}{u} \frac{\chi_i}{u} \chi_j = \delta_{ij}. \) (99)

For the kinetic term it holds that
\[
d^4r \sqrt{-g} \frac{\partial^\mu \hat{\psi}^\dagger}{\partial x^\mu} \frac{\partial \hat{\psi}}{\partial \bar{x}} = d^4r V \psi \left( \frac{\partial \hat{\psi} \hat{\psi}}{\partial x} \right)
\]
(100)

where \( \hat{\chi} = (v/u)^{1/2} \bar{\chi}/x \) and \( L = -i[\partial \theta, (1/\sin \theta)\partial \phi] \)
is the angular momentum operator in units of \( \hbar \). Likewise, the \(-1/(m^2 + \xi R)\partial \hat{\psi} \hat{\psi} \) term becomes
\[
(6\tilde{\lambda} \xi_0 B \bar{x} - V_{01} - \frac{\xi}{u} \frac{\hbar c \sqrt{v}}{16\pi^2} \frac{1}{1/4} \chi x)
\]
(101)

with potential
\[
V_{01} = \frac{x^4 B_2^2}{8vB^2} - \left( 1 - \lambda \frac{1}{B} \right) x^2 - 2x^2 T_e \frac{\bar{x}}{3B}.
\]
In vacuum, \( z \to 0^+ \), \( R_m \to 0 \) and \( V_{01} \to 6\xi \), which acts as an infinite barrier. More precisely, it leads to a decay at a scale of the Compton wavelength \( h/mc = 1/m \). Finally the interaction term brings quadratic terms with potential \( V \psi \) \( \hat{\psi} \) up to second order. Noting the time-dependence \( \psi_0 \) \( |\psi_0| e^{-i\mu_s} \) with
\[
\hat{\mu} = \hat{E}_0 = \sqrt{\xi},
\]
this yields in a straightforward manner
\[
- \frac{3\tilde{\lambda} \xi_0 B_0 \hbar c \sqrt{v}}{16\pi^2} \frac{1}{1/4} \chi x + e^{2i\mu_s} \bar{\chi}^2 + e^{-2i\mu_s} \bar{\chi}^2,
\]
(102)

where \( B_0 \) is the groundstate contribution to \( B \). Notice that (101) and (103) involve the same prefactor \( \tilde{\lambda} \xi_0 = \hat{\lambda}/6\xi_1 \). Together this brings an action
\[
S_{\text{mat}} = \hbar \int d\tau dz \frac{df}{\nu} (L_2 + L_{\text{int}})
\]
(104)

with, after a partial integration, a Lagrangian density
\[
L_2 = \frac{\partial \chi^\dagger}{\partial u} \frac{\partial \chi}{\partial x} - \frac{1}{v} \frac{\partial \chi^\dagger}{\partial x} \frac{\partial \chi}{\partial u} + \frac{L}{\chi} \cdot L \chi
\]
(105)

where derivatives are with respect to \( z \). From these equations one can show that
\[
(\omega_i - \omega_j^*) \int dz \frac{V}{\nu} (u_i u_j^* - v_i v_j^*) = 0
\]
(111)

The complex Bogoliubov functions \( u_i \) and \( v_i \) should not be mistaken for the positive metric functions \( u \) and \( v \).
and with real \( \omega \), this confirms the inner product (109).

In the interior one has \( u = v = 1 \). At \( \omega_\ell = \ell \) and \( T_\omega \to 0 \) the above equations then allow the exact solution \( u_0 = -v_0 = x\sqrt{1 + Ay} \), which corresponds to \( \pm \psi_0 \) on this basis, so the solvability stems from the one of the Gross-Pitaevskii equation. This situation is related a gauge transformation that changes the phase of the groundstate \( \psi_0 \) [20].

For excited states \( \omega_\ell \mu \sim \ell^2 \sim V \sim \tilde{\lambda} \) are large, so both \( V_0 \) and \( \omega_\ell^2 \) can be neglected. Then \( \omega \sim \tilde{\lambda}\mu \). In the regime \( z \ll -1 \) one has \( V \ll 1 \), so one expects

\[
u_i = c_ne^{iknz}, \quad v_i = -\frac{c_n}{2k^2}Ve^{iknz}. \tag{112}
\]

We can deal with the boundary conditions of these excited states as with plane waves, e. g. by requiring that \( e^{iknz} = 1 \) at some large \( z = -L \) and take \( L \to \infty \) at the end. The normalization constant is then

\[
c_n = \frac{1}{\sqrt{2\mu L}} \quad \text{(113)}
\]

Another regularization is to assume that \( k = k' - ik'' \) has a small imaginary part; this will keep all integrals starting at \(-\infty \) finite. Then \( c_n \sim \sqrt{k''} \). Below we shall employ sine-modes and impose the hard wall boundary condition at \( z = -L \).

6.1. Between the center and the peak of the potential

In the typical case where \( A > 0 \) the potential \( V = \tilde{\lambda}(1 - y^2)(1 + Ay) \) has a maximum at

\[
z_c = -\sqrt{2}\arctanh y_c, \quad y_c = \frac{\sqrt{1 + 3A^2} - 1}{3A}, \tag{114}
\]

which goes to zero for \( A \to 0 \), but remains finite for \( A \to \infty \). The region \( -\infty < z < z_c \), which covers the whole interior when \( A \to 0 \), is considered first.

Since \( V = \lambda x^2 B \) is large, the excited states can be analyzed with the WKB method. The function

\[
s = u_i + v_i, \quad \text{satisfies with}
\]

\[
\omega_i \equiv \frac{k^2}{2\sqrt{\xi}}, \quad \ell(\ell + 1) \approx \ell^2
\]

the equation

\[
k^4 s = s^{(\ell'' - 2\ell^2)} + \ell^4 s + \ell^2 2Vs - 2(Vs)'' \quad \text{.} \tag{117}
\]

We make the Ansatz

\[
s(z) = \text{const.} \quad e^{iS(z) - \tau(z)} \quad \text{(118)}
\]

where \( S = O(\tilde{\lambda}^{1/2}) \), \( \tau = O(\tilde{\lambda}^0) \) and higher order corrections may be neglected. The role of \( L \) will be discussed below. We have at leading order

\[
S' + 2(V + \ell^2) S'' + \ell^4 + 2V \ell^2 = k^4 \quad \text{.} \tag{119}
\]

with the solution

\[
S'' = \sqrt{k^4 + V^2} - V - \ell^2. \tag{120}
\]

At a given location \( z \) the solution is of plane wave type when \( S' \) is real, which occurs provided \( \ell \) is limited,

\[
\ell^2 \leq \ell^2(z) \equiv \sqrt{k^4 + V^2(z) - V(z)} \tag{121}
\]

We shall not need \( S \) itself. At next order we find

\[
4(S'^3 - S'V - S'\ell^2)\tau'' = 2(S'^3 - S'V - S'\ell^2)' \quad \text{.} \tag{122}
\]

with solution fixed to \( \tau = 0 \) for \( z \to -\infty \) (\( V \to 0 \)),

\[
\tau = \frac{1}{4} \ln \frac{k^4 + V^2}{k^4} + \frac{1}{4} \ln \frac{\sqrt{k^4 + V^2} - V - \ell^2}{k^2 - \ell^2} \quad \text{.} \tag{123}
\]

To get \( u_i \) and \( v_i \) to leading order is now easy. Since

\[
(u_i, v_i) = \frac{1}{\sqrt{\mu L}} \frac{k^2 \pm (V + \sqrt{k^4 + V^2})}{2k(k^4 + V^2)^{1/4}} \times \left( \frac{k^2 - \ell^2}{\sqrt{k^4 + V^2} - V - \ell^2} \right)^{1/4} \sin S \quad \text{.} \tag{125}
\]

For \( z \to -\infty \) (\( V \to 0 \)) one has indeed \( u_i \to \sin k z/\sqrt{\mu L} \), \( v_i \to 0 \). If we impose a hard wall boundary condition \( u_i = 0, v_i = 0 \) at \( z = -L \), the normalization (109) is satisfied, since it is determined by values \( z \ll -1 \). As customary for plane wave problems, \( L \) will be taken to infinity at the end.

For \( \ell > \ell_i(z) \), the action \( S \) becomes imaginary, expressing a damping of the wave that has to penetrate the potential barrier to reach this position \( z \). These states lead to negligible corrections.

We can calculate the Hamiltonian
\[ H = \text{const.} + \sum_i E_i^2 b_i^\dagger b_i, \]  
with
\[ \tilde{E}_i^2 = \int dz [(|u'|^2 + |v'|^2 + c + V)(|u|^2 + |v|^2) + V(u_i u_i^* + u_i v_i^*)]. \]  
(127)

With \(|u'|^2 + |v'|^2 = S^2(|u|^2 + |v|^2)|, this becomes
\[ \tilde{E}_i^2 = \int dz [\xi + \sqrt{k^4 + V^2}(|u|^2 + |v|^2) + V(u_i u_i^* + u_i v_i^*)]. \]  
(128)

As in the normalization, these integrals are dominated by large negative \( z \)-values, where \( V \rightarrow 0 \), so that we simply get
\[ \tilde{E}_i = \sqrt{\xi + k^2} \approx \mu + \frac{k^2}{2\mu}. \]  
(129)

It is degenerate (independent of \( \ell, m \)) as in the quantum Hall effect, though here there are no spectral gaps. At a given location \( z \) the wavefunctions \( u_i, v_i \) are oscillating (not damped) provided \( \ell \leq \ell_{\text{c}}(z) \).  

6.1.1. Contribution to the fraction of excited states

We can now consider \( B_e \), the excited states contribution to the direct back reaction \( B = B_0 + B_e \), that arises due to the interaction term, even at \( T = 0 \). We start from the definition
\[ B_e = \frac{16\pi G}{c^2} \xi \langle \hat{\psi}^\dagger \hat{\psi} \rangle \]  
(130)

and express this as
\[ B_e = \frac{8\pi m^2 \sqrt{\xi}}{3m^2 \sqrt{x^2 \xi}} (\xi^\dagger \xi) \equiv \varepsilon B_e(z), \]  
\[ \varepsilon = \frac{4\pi m^2 \lambda^{3/2}}{3m^2 \sqrt{\xi}}, \quad B_e = \frac{2\sqrt{x^2 \xi}}{\lambda^{3/2} x^2 \sqrt{u}} \sum_i |v_i|^2. \]  
(131)

In the exactly solvable case the sine modes, that exist for \( z \leq z_c \), i.e., to the left of the peak of \( V \), behave as \( \sin kz \) for \( z \rightarrow -\infty \). The hard wall boundary condition \( u_i = 0, v_i = 0 \) at \( z = L \) brings the quantization \( k_n = n\pi/L, n = 1, 2, \ldots \). Replacing \( \sin^2 S \rightarrow \frac{1}{2} \), this yields at a given position \( z \),
\[ B_e = \frac{2\sqrt{\xi} L}{\lambda^{3/2} x^2 \pi} \int_0^\infty dk \int_0^{\ell^2} d\ell^2 |v_i|^2 \]  
\[ = \int_0^\infty \frac{dk}{\pi} \frac{(k^2 + V - k^2)^2}{4\lambda^{3/2} x^2 k^2 K^2} \]  
\[ \times \left[ k\sqrt{K^2 - V} + (k^2 + V - K^2) \arcsin \sqrt{\frac{K^2 - V}{k^2 + V - K^2}} \right]. \]  
(132)

where \( V \equiv V(z) \) and \( K^2 \equiv k^2 + V^2 \). The integral gives
\[ B_e(z) = \frac{0.236792}{x^2 \lambda^{3/2}} V^{3/2} = 0.236792 x B^{3/2}. \]  
(133)

so the result neatly vanishes at the origin \( x \rightarrow 0 \),
\[ B_e = 0.757 B^{3/2} x \frac{M}{10^9 M_\odot}. \]  
(134)

According to (131) is has the characteristic strength
\[ \varepsilon = \frac{4\pi m^2 \lambda^{3/2}}{3m^2 \sqrt{\xi}}, \quad B_e = \frac{2\sqrt{x^2 \xi}}{\lambda^{3/2} x^2 \sqrt{u}} \sum_i |v_i|^2. \]  
(135)

so our variable \( \varepsilon \) is of the same order of magnitude, confirming the expectation of the introduction that the relevant physical parameter is \( na_s^* \), which is small for \( M \gg 10^9 M_\odot \).

6.2. Between the peak of the potential and the horizon

We take \( V = \infty \) beyond the horizon. Then it has quasi-bound states in the region \( z_c < z < 0 \); they are not true bound states because \( V \) drops to zero for \( z \ll z_c \). But since the energy barrier \( \sim \lambda \) will be very large, the tunneling into the interior will be extremely small, and we shall neglect it.

For \( U = V = 1 \) we can now copy previous solution (120), (125). At given value of \( \ell, a \) real valued \( S' \) starts at \( z = z_t \), set by
\[ V(z_t) = \frac{\ell^4 - \ell^4}{2\ell^2}. \]  
(137)

The surface state “lives” in the interval \( z_t \leq z \leq 0 \). The smallest \( z_t \) arises when \( z_t = z_c \) and it has \( \ell^2 = \sqrt{k^4 + V^2} - V_c \), while \( z_t \rightarrow 0 \) for the maximum \( \ell^2 = \sqrt{k^4 + \lambda^2} - \lambda \). The solution may now be written as
\begin{equation}
(u, v) = \frac{1}{\sqrt{\mu L_1}} \frac{k^2 \pm (\sqrt{k^4 + V^2} + V)}{2k(\sqrt{k^4 + V^2})^{1/4}} \sin S(z) \times (\sqrt{k^4 + V^2} - V - \ell^2)^{1/4}.
\end{equation}

(138)

which also depends on \( z \) through \( V(z) \). Since \( \sin^2 S(z) \) oscillates fast, it can be replaced by \( \frac{1}{2} \), so the normalization (109) is achieved by

\begin{equation}
L_i = \int_{z_i}^{0} \frac{dz}{\sqrt{\left[ (k^4 + V^2)(\sqrt{k^4 + V^2} - V - \ell^2) \right]^{1/2}}}. \tag{139}
\end{equation}

Recalling that \( S(z_i) = 0 \), the hard wall boundary condition \( u_i = v_i = s = 0 \) at \( z = 0 \) can be fulfilled provided the phase \( S(0) \) is an integer \( n \) times \( \pi \). At given \( \ell \) this defines the eigenvalue \( k_n \).

6.2.1. Contribution to the fraction of excited states

It is instructive to investigate whether these states cause divergent effects for states localized close to the horizon, those with \( |z_i| \ll 1 \). At fixed \( k \) and \( \ell \), we have

\begin{equation}
S^2 = \hat{k}^2 - \bar{\lambda} + \hat{\lambda} \bar{A} z - \ell^2 = \bar{\lambda} \bar{A} (z - z_i) \tag{140}
\end{equation}

with

\begin{equation}
\hat{k} = (k^4 + \bar{\lambda}^2)^{1/4}, \quad \bar{A} = \frac{A(\hat{k}^2 - \bar{\lambda})}{\sqrt{2k^2}}, \tag{141}
\end{equation}

and

\begin{equation}
z_i = \frac{\bar{\lambda} + \ell^2 - \hat{k}^2}{\lambda \bar{A}}. \tag{142}
\end{equation}

The maximal \( \ell \) at a given \( z \) is

\begin{equation}
\ell_i^2(z) = \hat{k}^2 - \bar{\lambda} + \bar{\lambda} \bar{A} z. \tag{143}
\end{equation}

This brings

\begin{equation}
L_i = \frac{(\hat{k}^2 + \bar{\lambda}) \sqrt{-z_i}}{\hat{k}^2 \sqrt{\lambda \bar{A}}} = \frac{(\hat{k}^2 + \bar{\lambda}) \sqrt{k^2 - \lambda - \ell^2}}{k^2 \lambda \bar{A}}. \tag{144}
\end{equation}

From \( S(z_i) = 0 \) we get

\begin{equation}
S(z) = \frac{2}{3} \sqrt{\lambda \bar{A}} (z - z_i)^{3/2} \tag{145}
\end{equation}

implying

\begin{equation}
S(0) = \frac{2}{3\lambda \bar{A}} (\hat{k}^2 - \bar{\lambda} - \ell^2)^{3/2}. \tag{146}
\end{equation}

So we may set

\begin{equation}
\frac{dn}{dk} = \frac{1}{\pi} \frac{\pi \lambda k^2}{\sqrt{2k^2 - \lambda - \ell^2}}. \tag{147}
\end{equation}

We can now calculate at given small \( z \)

\begin{equation}
\begin{aligned}
\sum_i \epsilon_i^2(z) &= \int \frac{d\ell_i^2}{\ell_i^2} \int \frac{dn}{dk} \epsilon_i^2(z) \\
&= \int \frac{d\ell_i^2}{\ell_i^2} \int \frac{d\ell_i^2}{16\pi \mu \hat{k}^2(\lambda + k^2) \sqrt{\ell_i^2(z) - \ell^2}}
\end{aligned} \tag{148}
\end{equation}

where we replaced \( \sin^2 S(z) \) by \( \frac{1}{2} \). Since the singularity at \( \ell_i(z) \) can be integrated, these states brings no special large contribution near the horizon \( y = 0 \). The outcome is of order \( \tilde{\lambda}^{5/2}/\ell^2 \), as for the modes near the origin, so both type of modes bring comparable excitations, \( B_0 \sim \epsilon \), as one would expect.

6.3. States close to the horizon

States localized close to the horizon can be studied analytically, we define

\begin{equation}
k^4 = \ell^4 + 2\ell^2 \bar{\lambda} (1 - \frac{z_i}{\sqrt{2}}), \quad z_i = \frac{\ell^4 - k^4}{\sqrt{2}\ell^2 \bar{\lambda}} \tag{149}
\end{equation}

\( \bar{z} = C \bar{z} \)

and supposing \( s(z) = f(\bar{z}_i - \bar{z}) \)

\begin{equation}
C^4 f^{\bar{z}_i} - 2(\ell^2 + \bar{\lambda})C^2 f^{\bar{z}_i} + 2\ell^2 \bar{\lambda} (\bar{z}_i - \bar{z}) C \bar{\lambda} = 0 \tag{151}
\end{equation}

so to leading order

\begin{equation}
C = \frac{\tilde{\lambda}^{1/3}}{2^{1/6}} \left( \frac{\ell^2}{\ell^2 + \bar{\lambda}} \right)^{1/3} \approx \frac{\tilde{\lambda}^{1/3}}{2^{1/6}} \left( 1 - \frac{\bar{\lambda}}{\sqrt{k^4 + \lambda^2}} \right)^{1/3} \tag{152}
\end{equation}

The solution is

\begin{equation}
f_i(\bar{z}) = A_i(\bar{z}_i - \bar{z}) \tag{153}
\end{equation}

which oscillates for \( \bar{z}_i < \bar{z} < 0 \) and decays for \( \bar{z} < \bar{z}_i \). The combination

\begin{equation}
f^2(\bar{z}) = \frac{C A_i^2 (\bar{z}_i - \bar{z})}{2\sqrt{\ell^4 \int_{\bar{z}_i}^{\infty} dy A^2(y)}} \approx \frac{\pi C A_i^2 (\bar{z}_i - \bar{z})}{2\sqrt{-\bar{z}_i \bar{z}}} \tag{154}
\end{equation}

is normalized to \( \int d\bar{z} f^2 = 1 \), so this results in
\[(u^2, v^2) = \frac{(w + V \pm k^2)^2}{4k^2(w + V)} f^2 \]

The zeros of Ai(\(\bar{z}_i\)) occur at

\[|\bar{z}_i| = \left(\frac{3\pi n}{2}\right)^{2/3}, \quad n = \frac{2}{3\pi} |z_i|^{3/2}, \]

so this yields

\[dn = \frac{1}{\pi} \sqrt{\bar{z}_i} d\bar{z}_i \]

Putting things together yields

\[\bar{B}_c(z) = \frac{1}{2(2\lambda A^2)^{1/6}} \int_0^\infty d\bar{z}_i \int_0^\infty dx \left(\frac{X - 1}{X}\right)^{1/3}\]

\[\frac{(X + 1 - x)^2}{2X(X + 1)} \frac{1}{A^2} \left[\bar{z}_i - \left(\frac{X - 1}{X}\right)^{1/3} Z\right]. \quad (158)\]

where

\[X = \sqrt{x^2 + 1}, \quad Z = \frac{\bar{\lambda}^{1/3} z}{2^{1/3} A^{1/6}}. \quad (159)\]

In the \(\bar{z}_i\) integral only the small values are reliable and the result is not of order unity but of order \(\bar{\lambda}^{-1/6} = 4.7 \times 10^{-8}\). Still, the result vanishes exactly at \(z = 0\) before we pass from a sum to the integral. The derivative is well defined, however,

\[\bar{B}'_c(z) = -\frac{\bar{\lambda}^{1/6}}{2^{4/3} A^{2/3}} \int_0^\infty dx \left(\frac{X - 1}{X}\right)^{2/3}\]

\[\frac{(X + 1 - x)^2}{2X(X + 1)} \frac{1}{A^2} \left[-\left(\frac{X - 1}{X}\right)^{1/3} Z\right]. \quad (160)\]

At \(z = 0\) it takes the value

\[\bar{B}'_c(0) = -\frac{0.157682 \bar{\lambda}^{1/6}}{A^{2/3}}. \quad (161)\]

while for \(z \to -\infty\) it decays as

\[\bar{B}'_c(z) = \frac{9^{3/4}}{8^{21/4}} \frac{\Gamma(\frac{3}{4})}{\sqrt{\pi}} \frac{1}{A^{1/2} \lambda |z|^{7/6} \bar{\lambda}^{1/2}}. \quad (162)\]

where we used

\[\int_0^\infty dy y^{5/2} A^2(y) = \frac{4^{3/4} \Gamma(\frac{7}{4})}{4\sqrt{\pi}} = 0.471436. \quad (163)\]

6.4. Fluctuation energy of the matter field

Let us calculate the energy density and pressures of the quantum field. We need the following contributions

\[\rho_c^\epsilon = \langle \partial_\xi \partial^\dagger \psi \partial_\xi \partial^\dagger \psi \rangle = \frac{\hbar c}{16M^4 x^4} \langle \partial_\xi \partial^\dagger \psi \partial_\xi \partial^\dagger \psi \rangle,\]

\[\rho_\epsilon^\gamma = \frac{\hbar c}{16M^4} \langle \partial_\xi \partial^\dagger \psi \partial_\xi \partial^\dagger \psi \rangle, \quad (159)\]

\[\rho_\epsilon^\gamma = \frac{\hbar c}{16M^4} \langle \partial_\xi \partial^\dagger \psi \partial_\xi \partial^\dagger \psi \rangle, \quad (164)\]

They determine

\[\rho_c = \rho_c^\epsilon + \rho_\epsilon^\gamma + \rho_\epsilon^\phi + \rho_\epsilon^\mu + \rho_\epsilon^\rho, \quad (155)\]

\[\rho^\epsilon = \rho^\epsilon_\mu + \rho^\epsilon_\rho - \rho^\phi - \rho^\phi, \quad (165)\]

which implies that

\[T_c = -2\rho^\epsilon + 2\rho^\mu + 2\rho^\rho + 4\rho^\phi + 4\rho^\rho. \quad (166)\]

The leading terms at \(T = 0\) are indeed also of order \(\epsilon_c\),

\[\rho^\epsilon_c = \frac{1}{8\pi^2} B_c, \quad \rho^\phi_c = \frac{3}{4} B_c, \quad (167)\]

the other terms are smaller by a factor \(\bar{\lambda}\) at least. So

\[\rho_c = \left(\frac{1}{8\pi^2} + \frac{3}{4}\right) B_c, \quad \rho^\phi_c = \rho^\phi_\mu = \left(\frac{1}{8\pi^2} - \frac{3}{4}\right) B_c, \quad (168)\]

This now implies that the sources of the GP equation, (88), and of the Einstein equations, (90) and (91), are

\[S_B = 0, \quad \bar{\rho}_c - \frac{3}{4} B_c = \bar{\rho}_c + \frac{3}{4} B_c = \frac{1}{8\pi^2} B_c \quad (169)\]

6.5. Reaction of the metric on the fluctuations

In order to investigate whether the matching of interior and exterior solutions can be achieved near the horizon, we investigate the reaction of the metric to
fluctuations of the exact solution caused by the terms \( \rho_e, p'_e \) and \( p'_x \). We set
\[
B = 1 + B_1, \quad u = 1 + u_1, \quad v = 1 + v_1, \quad \xi_1 = 1 + \xi(0)
\]
where \( B_1, u_1, v_1 \) and \( \xi_2 \) are of order \( \varepsilon \).

6.5.1. The solvable case \( A = 0 \)

The \( B \)-equation becomes to linear order in \( \varepsilon \)
\[
x^2 \frac{d^2 B}{dx^2} - y B' + B_1 = -\frac{2}{3} \tilde{T}_e. \tag{171}
\]
In this section, derivatives are with respect to \( y \). The homogeneous solutions are
\[
B^{(1)}_1 = y, \quad B^{(2)}_1 = \frac{1}{2x^2} - \frac{3}{2} + \frac{3y}{4} \ln \frac{1 - y}{1 + y}, \tag{172}
\]
and they have a Wronskian
\[
W = B^{(1)}_1 B^{(2)}_1 - B^{(1)}_1 B^{(2)}_1 = \frac{1}{x^4}. \tag{173}
\]
The solution for \( B_1 \) therefore reads
\[
B_1 = b_1 y + B^{(1)}_1(y) \int_1^y dy \frac{8\tilde{T}_m B^{(2)}_1}{3x^2 W} \]
\[
- B^{(2)}_1(y) \int_1^y dy \frac{8\tilde{T}_m B^{(1)}_1}{3x^2 W} = b_1 y \tag{174}
\]
\[
+ \frac{4}{3} B^{(2)}_1(y) \int_0^y dy \int_0^x dx \frac{x^3 \delta T_m - 4y}{3} \int_0^x dy \frac{x^3}{y} \delta T_m B^{(1)}_1,
\]
with \( b_1 \) an integration constant. The last expression exhibits regularity at the origin \( x = 0 \), even when \( \delta T_m \) has a \( 1/x \) singularity, as we discussed above.

\( v_1 \) may be solved from the \( 1^- \)-Einstein equation,
\[
v_1 = \frac{x^2}{y} \left[ u_1 - x^2 u'_1 - \frac{4}{3} \rho_m + \frac{i}{2} (B_1 - y B'_1) \right], \tag{175}
\]
after which \( u_1 \) satisfies
\[
x^2 u''_1 - 4y u'_1 + \frac{4}{x^2} u_1 = s \tag{176}
\]
with source term
\[
s = -4 \hat{\rho}_1 + 14 \hat{\rho}_4 - \frac{x^2}{2y} (\hat{\rho}'_1 - \hat{\rho}_4) \tag{177}
\]
The homogeneous solutions \( P/x \) and \( Q/x \) involve the associated Legendre functions \( P \equiv P_1^{\sqrt{3}}, Q \equiv Q_1^{\sqrt{3}}, \)
\[
P(y) = \left( \frac{1 + y}{1 - y} \right)^{\frac{i}{2} \sqrt{3}} \left( 1 + \frac{i y}{\sqrt{3}} \right), \quad Q(y) = P^*(y) \tag{178}
\]
The solution then reads
\[
u_1 = \frac{Q}{x} \int_1^y dy \frac{P_s}{x W} - \frac{P}{x} \int_1^y dy \frac{Q_s}{x W} \]
\[
= \frac{P}{x} \int_0^y dy \frac{Q_s}{y W} - \frac{Q}{x} \int_0^y dy \frac{P_s}{y W} \tag{179}
\]
with the Wronskian
\[
W = PQ' - QP' = \frac{8i}{\sqrt{3} x^2}. \tag{180}
\]

It is imaginary, so \( u_1 \) is real. (179) is regular at \( x = 0 \), so no homogeneous solutions can be added. It now follows that \( v_1(y) \) diverges as \( c_{-2}/y^2 \) near the horizon \( y = 0 \). We get
\[
c_{-2} = \frac{u_1(0)}{3} - \frac{4}{3} (\hat{\rho}_1(0) - \hat{\rho}_4(0)) + \frac{i}{4} B_1(0) \tag{181}
\]
with
\[
u_1(0) = \frac{i \sqrt{3}}{8} \int_0^1 dy (Q - P) x s \tag{182}
\]
For \( A = 0 \) one has
\[
\hat{\rho}_1 = \frac{\rho_a}{x}, \quad \hat{\rho}_4 = 6 \rho_a x \tag{183}
\]
with \( \rho_a = 2.80959 \times 10^{-15} M_\odot^3 \) a positive amplitude. This gives \( u_1(0) = 13.6518 \rho_a \), while \( B_1(0) = -496 \rho_a/45 \). Together they yield
\[
c_{-2} = 5.70616 \rho_a, \tag{184}
\]
which is positive, and showing an upturn of \( v \) in the narrow region \( y \sim \sqrt{a} \) near the horizon.

6.5.2. The general case \( A > 0 \)

When \( A > 0 \), the \( B_1 \) equation gets coupled to the \( u_1, v_1 \) equations. No explicit solution of the linearized problem has been found. Inspection of the equations near \( y = 0 \) reveals that now a singularity
\[
v_1 = \frac{c_{-2}}{y^2} + O(y^0), \quad B_1 = -\frac{A c_{-2}}{2y} + O(y^0), \tag{185}
\]
is allowed. From the above case \( A = 0 \) it is to be expected that \( c_{-2} \) remains positive, so at \( A > 0 \) also \( B_1 \) is
singular. The signs are the ones expected for approaching the vacuum; $B$ decays while $v$ increases, towards its high peak slightly beyond the horizon. In retrospect, the induced decay of $B$ also indicates that $A > 0$ is the typical case, rather than the no-hair value $A = 0$. This important fact gives hope that a self-consistent treatment near the horizon achieves to match the exact solution in the interior with the deformed Schwarzschild metric in the exterior. It remains an open problem to consider this behavior in a self-consistent way.

7. Conclusion

We have questioned the general wisdom that static BHs have all their mass in the center and that its interior cannot be described by present theories based on General Theory of Relativity. Estimates show that a picture of closely packed H atoms naturally applies to the supermassive BH’s in the center of galaxies, $M \sim 10^9 M_\odot$. We therefore attempt to describe them as more or less normal objects like stars.

We present within the Relativistic Theory of Gravitation (RTG), an exact solution for a BH, of which the interior is governed by quantum matter in its Bose-Einstein condensed phase. Powerlaw singularities occur at the origin, that get absorbed in the Riemann description of the metric. Elsewhere, the solution is regular. The redshift at the horizon is finite, though of the order $1/\bar{\mu}_{bh} \sim 10^{14}$.

This solution is still to be matched with the Schwarzschild metric, which near the horizon is deformed in RTG. We have carefully derived the complete fluctuation spectrum about the groundstate. The matching of the inner metric with the outer metric at the horizon has been considered in a first order perturbative approach, which shows an enhancement effect near the horizon. The full problem still has to be carried out, and it has to be done self-consistently. It remains as a task for future to show that this indeed leads to a proper decay of matter and behavior of the metric near the horizon.

Our BH is a quantum fluid confined by its own gravitation. It puts forward that a BH is just an intense gas cloud, without an event horizon, as was also deduced from the Schind et al. observations [1,3]. In the interior, time keeps its standard role. No Planckian physics is involved; Hawking radiation is absent and Bekenstein-Hawking entropy plays no role. In our zero-temperature situation the entropy of the quantum field vanishes. Because the Schwarzschild singularity is cut off by the bimetric coupling, there is no connection with any form of quantum gravity, even though the redshift at the horizon is of order $10^{14}$.

Schild et al. [1,3] have explained their observations in term of a magnetic dipole moment of the black hole, a “hair”. Our BH also has one “hair”, the binding energy, expressed as $E_{bind} = Nmc^2 - Mc^2 = (1 - \nu)Nmc^2$. Here $\nu$ can take any value below $1/2\sqrt{3} = 29\%$. We confirmed that the previously derived solution of the Gross-Pitaevskii equation with a free parameter, $A$ or $\nu(A)$, shows up also as a zero mode in the fluctuation spectrum. As one would expect for a classical theory of gravitation, when the quantum matter in the BH has reached a certain groundstate, the classical metric allows the system still to go to a lower energy state. Indeed, the passage of celestial bodie will induce oscillations in the metric and emission of gravitational waves, which, upon re-equilibration, increase the binding energy, finally up to 100% of the rest energy of its constituents, $Nmc^2$. In that final state the mass is completely balanced by the binding energy, making it look like a zero mass object. In its stable state, the BH has a fraction strictly-less-than-one-half of the ground state energy of the constituents, so the major part of the zero-point energy has to be emitted in radiation. This property may explain the enormous luminosity of quasars.

It has been assumed that renormalization couples to the matter field density to the curvature scalar with a strength $\xi$. This parameter is chosen appropriately in a first step of renormalization of the scalar field theory with a large quartic coupling. The value $\xi = 2m^2M^2/3m^2P \sim 3 \cdot 10^{55} (M/10^9 M_\odot)^2$ allows an exact and explicit solution of the interior metric, any other value would be inconsistent. The fluctuation spectrum is well defined, and various very large or very small numbers finally combine into reasonable prefactors. It is noticed that the leading corrections of the matter field are of order $\sqrt{n_a}^2$, as it happens for in Bose-Einstein condensation in a box. The physical reason for this, diluteness of the gas of H-atoms, was put forward in the introduction. For this reason we expect that the field theory for supermassive BHs with $M \gg 10^9 M_\odot$ can be renormalized perturbatively after the first step.
that fixes the leading value of $\xi$.

We have set out the lines for studying the fluctuation spectrum near the horizon. It is left as a task for future to show that they indeed fluently connect the empty space metric of the exterior (i.e. the deformed Schwarzschild metric) with our exact solution for the metric in the interior.

An important question is whether formation of realistic supermassive BHs brings the matter indeed in or near the Bose-Einstein condensed groundstate. This would require the study of the finite temperature situation. Extension to finite temperatures, not presented here, will exhibit a $T^{3/2}$ fraction of thermal atoms. Also the stability of the solution needs to be studied.

Calculation of the normal mode spectrum may lead to predictions that deviate from the ones of GTR; this spectrum may be observed in the foreseeable future.

We failed to apply our approach to GTR, technically because it lacks compensation for the $1/U$ terms, that in this solution are truly singular at the origin. If no other solution exists for the considered physical situation, GTR must be abandoned and replaced by another theory, RTG being the first candidate. In view of its smaller symmetry group, this may have far reaching consequences for singularities in classical gravitation – they would probably be regularized – and for quantum approaches to gravitation, since the primary space-time, Minkowski space-time, needs no quantization.

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