Fermi distribution of semiclassical non-equilibrium Fermi states

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When a classical device suddenly perturbs a degenerate Fermi gas a semiclassical non-equilibrium Fermi state arises. Semiclassical Fermi states are characterized by a Fermi energy or Fermi momentum that slowly depends on space or/and time. We show that the Fermi distribution of a semiclassical Fermi state has a universal nature. It is described by Airy functions regardless of the details of the perturbation. In this letter we also give a general discussion of coherent Fermi states.

1. Introduction Among various excitations of a degenerate Fermi gas, coherent Fermi states play a special role. A typical (not coherent) excitation of a Fermi gas consists of a finite number of holes below the Fermi level and a finite number of particles above it. Instead, a coherent Fermi state involves an infinite superposition of particles and holes arranged in such a manner that one can still think in terms of a Fermi sea (with no holes in it), the Fermi level of which depends on time and space.

Coherent states appear in numerous recent proposals about generating coherent quantum states in nanoelectronic devices and fermionic cooled atomic systems. These states can be used to transmit quantum information, test properties of electronic systems and to generate many-particles entangled states.

Coherent Fermi states can be obtained by different means. One is a sudden perturbation of the Fermi gas. For example, a smooth potential well, the spatial extent of which much larger than the Fermi length, is applied to a Fermi gas. Fermions are trapped in the well. Then the well is suddenly removed. An excited state of the Fermi gas obtained in this way is a coherent Fermi state. This kind of perturbation is typical for various manipulations with cooled fermionic atomic gases.

A realistic way to generate coherent Fermi states in electronic systems is by applying a time dependent potential along a point contact, typifying many manipulations with nanoelectronic devices \(^1\)\(^2\).

Although the realization of coherent states in electronic systems experimentally is more challenging than in atomic systems, we will routinely talk about electrons.

From a theoretical standpoint coherent Fermi states are an important concept revealing fundamental properties of Fermi statistics. Coherent states appeared in other disciplines not directly related to electronic physics. Random matrix theory (RMT) \(^3\), non-linear waves \(^4\), crystal growth \(^5\)\(^6\), various determinental stochastic processes \(^7\), asymmetric diffusion processes \(^8\), to name but a few.

Unless a special effort is made (see e.g., \(^2\)) coherent Fermi states involve many electrons and are such that space-time gradients of the electronic density are much smaller than the Fermi scale. These state arise as a result of perturbing the Fermi gas by a classical device. We call them semiclassical Fermi states. They are the main object of this paper. We will show that semiclassical Fermi states show great degree of universality as well as their single electron counterpart studied in \(^1\)\(^2\).

A general coherent Fermi state is a unitary transformation of the ground state \(|0\rangle\) of a Fermi gas: \(|U\rangle = U|0\rangle\), \(U = e^{i\int \Xi(x)\rho(x)dx - i\int \Pi(x)v(x)dx}\), where \(\rho\) and \(v\) are operators of electronic density and velocity \([\rho(x), v(y)] = -\frac{\hbar}{m}\nabla\delta(x-y)\) and \(\Xi(x)\) and \(\Pi(x)\) are two real functions characterizing the state \(^4\)\(^9\).

For the purpose of this paper it is sufficient to assume that the motion of electrons is one dimensional and chiral (electrons move to the right), although most of the results we discuss are not limited to one-dimensional electronic gases. To this end, edge states in the Integer Quantum Hall effect may serve as a prototype. In a chiral sector the operators of density and velocity are identical \(v(\text{right}) = \frac{\hbar}{m}\\rho(\text{right})\). The contribution of the right sector is easy to take into account. In this case the Fermi coherent state
\[
|U\rangle = e^{i\int \Phi(x)\rho(x)dx}|0\rangle
\]

is characterized by a single function \(\Phi\).

The function \(\Phi\) can be understood as the action of an instantaneous perturbation by a potential \(\hbar v_F \nabla \Phi(x)\), or, if we ignore the electronic dispersion the function \(\Phi\) is the action of a time-dependent gate voltage \(eV(t) = -\hbar \frac{\hbar}{m} \Phi(x_0 - v_F t)\) applied through a point contact (located at \(x_0\)) \(^1\)\(^6\).

Fermi coherent states feature inhomogeneous electric density \(\rho(x)\) and current \(I(x)\). Their expectation values give a meaning to the functions \(\Xi\) and \(\Pi\). In the chiral state, where the electric current and the electronic density are proportional, their expectation values are gradients of \(\Phi\)
\[
\hbar \langle U|\rho(x)|U\rangle = \frac{\hbar}{e v_F} \langle U|I(x)|U\rangle = p_F + \hbar \nabla \Phi.
\]

Consequently \(\hbar \nabla \Phi(x)\) plays the role of the space-time modulation of the Fermi point, such that all states with momentum (or energy) less than \(p_F(x) = \hbar \nabla \Phi(x)\) (or \(E_F(x) = v_F \hbar \nabla \Phi\)) are occupied (no holes in the Fermi sea) as shown in Fig. \(^1\). We shall refer to the function \(p_F(x)\) as the Fermi surface, with some abuse of nomenclature. We shall also term the region in phase space around \(p_F(x)\) as the ‘Fermi surf’.
The result for the Fermi number and Wigner function.

Near a maxima, but well above the minima of the surf

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Its typical range is larger than the momentum spacing

states. These states involve many excited electrons.

FIG. 2: Universal asymptotes of the Fermi number. The
graph (blue curve) shows the Fermi number in units of $\hbar/\ell$
for the example $(\rho(x)) = \rho_0 + (n/\ell)\cos(x/\ell)$ computed from
Eq. [17]. Black curves are the asymptotic forms obtained
from the universal Fermi number formula [4]. The universal
asymptotes are magnified.

The question we address in this letter is: what is the
Fermi distribution in the 'Fermi surf' of a modulated
Fermi point, namely around and between extrema of
$P_F(x)$?

We will be especially interested in semiclassical Fermi
states. These states involve many excited electrons.
Its typical range is larger than the momentum spacing
$h\nabla \Phi \gg \Delta$ (but still is smaller than the Fermi momentum).

We show that, quite interestingly, the semiclassical
Fermi surf features a universal Fermi distribution. There
the Wigner function (11) is described by the function
$A_1(s)$ is $f_s^{\infty} A_1(s')ds'$:

$$n_F(x, p + P_F(x)) \approx A_1\left(2^{2/3}\kappa p\right),$$

where the scale $\kappa = [\hbar^2 P_F''(x_\ast)/2]^{-1/3}$, and the offset
$P_F(x)$ are the only information about the state that enters
the formula. This formula holds close to any point
where the Fermi surf is concave $P_F''(x) < 0$. The Fermi
occupation number (12) also displays universal behavior.
Near a maxima, but well above the minima of the surf
the Fermi number reads

$$n_F(P_F(x_\ast) + p) \approx \kappa\Delta \left[\left[A_1'(\kappa p)\right]^2 - (\kappa p)A_1''(\kappa p)\right].$$

If the Fermi surf is convex rather than concave, then par-
ticle hole symmetry $n_F(p, x) \rightarrow 1 - n_F(-p, x)$ provides
the result for the Fermi number and Wigner function.

FIG. 1: Modulated Fermi Edge. The Wigner function and
Fermi number are almost 1 below the Edge (shaded area)
and vanish above the Edge, having universal character in the
'surf'.

FIG. 3: Universal behavior of the Wigner function vs. mo-
mentum for a concave Fermi surface obtained from [3].

Figures [12] illustrate the universal regimes. Here $\Delta$
is the momentum spacing ($2\pi\hbar/\Delta$ is the system volume).

I The goal of the letter is twofold: to emphasize these
simple, albeit universal, distributions, and, also to collect
a few major facts about Fermi coherent states.

2. Coherent Fermi states The formal definition of
a Fermi coherent states starts with the current algebra
(see e.g., [11]). To simplify the discussion and formulas
we consider only one chiral (right) part of the current
algebra.

Current modes are Fourier harmonics of the electronic
density $\rho(x) = \sum_{k>0} e^{+ikx}J_k$. An electronic current
mode $J_k = \sum_p c_k^p c_{p+k}$ (we count electronic momentum
from the Fermi momentum $p_F$), creates a superposition
of particle-hole excitations with momentum $k$. Positive
modes annihilate the ground state, $\{0\}$, a state where
all momenta below $p_F$ are filled: $J_k\{0\} = 0, \ k > 0$.
Negative modes are Hermitian conjugated to the posi-
tive modes $J_{-k} = (J_{+k})^\dagger$.

Chiral currents obey a current (or Tomonaga) algebra:

$$[J_k, J_l] = \frac{k}{\Delta} \delta_{k+l, 0}.$$  (5)

A Fermi coherent state $\{U\}$ is defined as an eigenstate of
positive current modes:

$$J_k\{U\} = p_k\{U\}, \ k > 0.$$  (6)

As follows from (1), $p_k = \Delta \int e^{-i k x} d\Phi(x)/(2\pi)$ are
positive Fourier modes of the function $\Phi(x)$. Assuming that
the total number of particles in the coherent state is the same as in
the ground state, or that the dc component of current is $I_0 = \frac{\pi e}{\hbar}\rho_F$, i.e., $\int d\Phi = 0$, we obtain

$$\{U\} = Z^{-1/2} e^{i\sum_{k>0} \frac{1}{\Delta} J_{-k} \{0\}}, \ Z = e^{\sum_{k>0} \frac{1}{\Delta} |p_k|^2}.$$  (7)

Using normal ordering with respect to the ground state
(where all positive modes of the current are placed to the
right of negative modes) the unitary operator reads [17]:

$$e^{i \int \Phi(x) \rho(x) dx} = Z^{-\frac{1}{2}} e^{i \int \Phi(x) \rho(x) dx} =$$

$$= Z^{-\frac{1}{2}} e^{i \sum_{k>0} \frac{1}{\Delta} p_k J_{-k} e^{-\sum_{k>0} \frac{1}{\Delta} p_k J_k}}.$$  (8)
A coherent state represents an electronic wave-packet which is fully characterized by the electronic density. It is a simple exercise involving the algebra of the current operators to show that the function $\nabla \Phi(x)$ is a non-uniform part of the density as is in $[2]$. Alternatively, one can use $Z$ as a generation function $\langle \rho(x) \rangle \equiv \langle U|\rho(x)|U \rangle = \rho_0 + 2\text{Re}\sum_{k>0} k e^{ikx} \partial_{p_k} \log Z$.

Coherent states obey the Wick theorem. The Wick theorem allows to compute a correlation function of any finite number of electronic operators, as a determinant over the one-fermiionic function $K(x_1, x_2) \equiv \langle U|\psi^\dagger(x_1)\psi(x_2)|U \rangle$, where $\psi(x) = (\hbar/\Delta)^{1/2} \sum_p e^{i\kappa p x} c_p$ is an electronic operator. The one-fermiionic function can be computed with the help of the formula:

$$U\psi(x)U^{-1} = e^{-i\Phi(x)}\psi(x)$$

which leads to the expression:

$$K(x_1, x_2) = \frac{e^{i\Phi(x_1)-\Phi(x_2)} - \frac{i}{\hbar} P_F(x_1-x_2) - 1}{i(x_1-x_2)}$$

valid for $|x_1-x_2| < \hbar$. An equivalent object appears in RMT where it is often called - Dyson’s kernel. We adopt this name. As points merge one recovers the density $[2]$

$$K(x, x) = \langle \rho(x) \rangle = \nabla \Phi.$$

3. Wigner function and Fermi occupation number

The Wigner function is defined as Wigner transform of $P$ where we denoted

$$n_F(x, p) = \frac{1}{2\pi} \int K(x + \frac{y}{2}, x - \frac{y}{2}) e^{-\frac{i}{\hbar} py} dy$$

The meaning of the Wigner function is clarified away from the surf. There it means an occupation of electrons in the phase space $(x, p)$: 1 below a surf, 0 above. On the surf Wigner function is not necessarily positive.

The Fermi number

$$n_F(p) = \langle U|\rho^p c_p|U \rangle = \frac{\Delta}{2\pi \hbar} \int n_F(x, p) dx$$

is the Wigner function averaged over space.

Combining $[10]$ and $[11]$ we write

$$n_F(x, p) = \frac{1}{2\pi i} \int e^{\frac{ix}{\hbar} J} e^{-\frac{i}{\hbar} (P_F(x')-p)x'} \frac{dy}{y-i0}$$

where we denoted $P_F(x) = \hbar \nabla \Phi = \langle \rho(x) \rangle$ as in $[2]$.

Below we evaluate the integral $[13]$ semiclassically bearing in mind that $\Phi$ is of a finite order as $h \to 0$.

A universal regime arises at the Fermi surf, $\kappa |p - P_F(x)| \simeq 1$. In this case it is sufficient to expand $P_F(x)$ in a Taylor series around extremas of $P_F(x)$ to second order $P_F(x) = P_F(x_0) + \frac{1}{2} P_F''(x_0) (x-x_0)^2 + \ldots$. Then the integral $[13]$ becomes the Airy integral given in Eq. $[3]$. Further integration over space yields $[4]$.

In this regime the Dyson kernel in the momentum space $K_{p_1, p_2} \equiv \langle U|\rho^p c_{p_1} c_{p_2}|U \rangle$ reads:

$$K_{p_1, p_2} \simeq \Delta \text{Ai}''(\kappa p_1) \text{Ai}''(\kappa p_2) - \text{Ai}(\kappa p_1) \text{Ai}'(\kappa p_2)$$

which is the celebrated Airy kernel appearing in numerous problems as the limiting shape of crystals $[5]$, asymmetric diffusion $[8]$, edge distribution of eigenvalues of random matrices $[13]$, etc.

This is the Fermi number $[1]$, which can be directly obtained from the kernel by taking a limit $p_1 \to p_2$ in $[14]$. At large positive momenta $(\kappa p \to +\infty)$ the Fermi number behaves as $n_F(p) \sim \frac{\Delta}{2\pi} e^{-\frac{\pi}{4p}}$ and as $\sim \frac{\Delta}{\pi} \left( \kappa \sqrt{-p \kappa} - \frac{1}{4p} \cos \left( \frac{4}{3} (\kappa p) \frac{3}{2} \right) \right)$ for large negative momenta within the surf.

Away from the universal region of the surf the Fermi distribution can be computed within a saddle point approximation. The saddle point of the integral $[13]$ is:

$$P_F(x + \frac{y}{2}) + P_F(x - \frac{y}{2}) = 2p$$

It has pairs of solutions $\pm y_F(x, p)$. Let $P_{\text{max}} = \max(P_F(x))$ and $P_{\text{min}} = \min(P_F(x))$ be adjacent extrema of the surf. Without loss of generality we may assume that $(x, p)$ is outside the Fermi sea $p > P_F(x)$. The particle hole symmetry $n_F \to 1 - n_F$ helps to recover the case when the momentum is inside the sea. If $p$ is in the surf, $p \in (P_{\text{min}}, P_{\text{max}})$, then some saddle point pairs of $[15]$ may be real. Their contribution produces oscillatory features with a suppressed amplitude. If $p$ hovers above the surf, $p > P_{\text{max}}$, then the saddle points are imaginary. Their contributions are exponentially small.

Between two adjacent extrema the Wigner function reads:

$$n_F(x, p) \approx \sqrt{\frac{\hbar}{8\pi |y_F|}} \times \left\{ \begin{array}{ll} 2 \sin \left( \Omega - \frac{\pi}{4} \right), & \text{if } p \in (P_{\text{min}}, P_{\text{max}}) \text{ and } p > P_{\text{max}}, \\
\frac{\hbar}{8\pi |y_F|}, & \text{if } p \in (P_{\text{min}}, P_{\text{max}}) \text{ and } p < P_{\text{max}}, \\
\end{array} \right.$$
This formula allows to compare the asymptotes near the edges to the universal expression above. Using the homogeneous asymptote of Bessel function $J_m(m-(m/2)^{1/3}) \sim (2/m)^{1/3} A_m(\zeta)$ at large $m$, one recovers [14] and [4]. Fig. 2 illustrates the universal asymptote.

4. Holomorphic Fermions as coherent states

To contrast semiclassical coherent Fermi states and quantum coherent Fermi states, we briefly discuss special coherent states known as holomorphic fermions [13].

Holomorphic fermions are defined as a superposition of fermionic modes $\psi(z) = \sum_{p} e^{ipz} c_p$, with a complex “coordinate” $\text{Im} \zeta < 0$.

Holomorphic fermions are coherent states since they can be represented as an exponential of a Bose field - displacement of electrons $\varphi(z) = \sum_{k \neq 0} \frac{1}{ik} e^{ikz} J_k \[11\]$

$$\psi(z) = c_{fp} \cdot e^{i\varphi(z)}; \tag{18}$$

A function $\Phi$ for a string of fermions $\prod_{i=1}^{n} (\psi(z_i)\psi(\zeta_i)) |0\rangle$ is $e^{i\Phi(x)} = \prod_{i=1}^{n} \frac{x - z - \zeta_i}{x - z} \cdot e^{iz \zeta_i}$.

The density (or current) of these states consists of Lorentzian relations to the theory of Random Matrices. Fig. 2 illustrates the universal asymptote.

Fermi coherent states and Random Matrix Ensembles (see e.g., [15] for derivation [18]). Up to a normalization

$$\Psi_p(x) \sim e^{-\frac{1}{2} Y_p(x)}, \tag{20}$$

This formula features the complex curve $Y_p(z) = -ipz + \sum_{k \geq 0} \frac{p_k}{k} e^{ikz}$, a useful characteristic of the coherent state. The function $Y_p(z) + ipz$ is analytic in the upper half-plane.

5. Normalization

Computing the Slater determinant of [20] we obtain

$$\Psi(x_1, \ldots, x_n) = Z_n^{-1/2} e^{-\frac{1}{2} \sum_{i=1}^{n} (V(x_i) - ip_F x_i)} \Delta(x), \tag{22}$$

where $\Delta(x) = \det \left( e^{ip_F x_i} \right)_{p,l \leq n} = \prod_{i>j} \left( e^{ip_F x_i} - e^{ip_F x_j} \right)$ is the VanderMonde determinant.

The normalization factor in [22]

$$Z_n = \int \prod_{n \geq i > j \geq 1} \left| e^{ip_F x_i} - e^{ip_F x_j} \right|^2 \prod_{i=1}^{n} e^{-V(x_i)} dx_i \tag{23}$$

is the the partition function of eigenvalues of a circular unitary $n \times n$-matrix [8]. At the limit of vanishing spacing one replaces $e^{ip_F} \rightarrow 1 + i px$. In this case coherent Fermi state is described by Random Hermitian Matrix ensemble.

If $n$ is large, Eq. [22] can be interpreted as a coordinate representation of the coherent state. A Fermi coherent state may be thought as a Fermi sea filled by particles (without holes) with wave functions [20] and $p = 1, \ldots, n$. The coordinate representation provides another avenue to compute matrix elements discussed in this paper as a limit $n \rightarrow \infty$, $p_F/n \rightarrow \infty$. Some of them have been studied for various reasons in the theory of Random Matrix Ensembles (see e.g., [15] for derivation of the Dyson kernel).
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[17] The formula contains the physics of the Anderson orthogonality catastrophe [2]: the overlap between the ground state of a Fermi gas and a coherent state of a Fermi gas describing a localized potential vanishes with a power of the level spacing. If the energy dependence of the scattering phase $\delta$ of the potential is a smooth at the Fermi energy $p_k \to \delta/\pi$ at small $k$. As a result, the overlap $|\langle 0|U|\rangle|^2 = Z^{-1} = e^{\sum_{k=0}^{k} \frac{1}{2} |p_k|^2} \sim (\Delta/p_F)^{(\delta/\pi)^2}$ vanishes with the spacing.