Schwinger-Dyson approach for a Lifshitz-type Yukawa model

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Abstract

We consider a 3+1 dimensional field theory at a Lifshitz point for a dynamical critical exponent $z=3$, with a scalar and a fermion field coupled via a Yukawa interaction. Using the non-perturbative Schwinger-Dyson approach we calculate quantum corrections to the effective action. We demonstrate that a first order derivative kinetic term as well as a mass term for the fermion arise dynamically. This signals the restoration of Lorentz symmetry in the IR regime of the single fermion model, although for theories with more than one fermionic species such a conclusion will require fine-tuning of couplings. The limitations of the model and our approach are discussed.

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1 Introduction

Quantum field theory models, in which the UV behavior is governed by a Lifshitz-type fixed point have attracted attention recently, as their renormalization properties appear significantly improved, compared to models with a Lorentz symmetric Gaussian fixed point. A novel quantum gravity model, which claims power counting renormalizability, has been formulated recently by Horava in [1, 2]. This scenario is based on an anisotropy between space and time coordinates, which is expressed via the scalings \( t \to b^z t \) and \( x \to bx \), where \( z \) is a dynamical critical exponent. For \( z \neq 1 \) the UV behavior of the model is governed by a nonstandard Lifshitz fixed point, while for \( z = 1 \) we recover the well known Gaussian fixed point. Note that in the Horava model, \( z = 3 \) is chosen.

Horava gravity has stimulated an extended research on cosmology and black hole solutions, see for example [3, 4, 5, 6, 7, 8]. We would like to note that Horava gravity is a non-relativistic theory, however it is expected that general relativity is recovered in the IR limit. Moreover, some possible inconsistencies on Horava gravity have been remarked in [9, 10, 11], but they will not be discussed here.

Independently of general relativity, quantum field theory models in flat space-time with anisotropy have been studied as well. For example, a thorough study on renormalization properties of models with a Lifshitz-type fixed point, is presented in [12, 13, 14, 15], and the Standard Model in this Lorentz violating approach is examined in [16]. Also, the renormalizability of scalar field theory at the Lifshitz point is examined in [17], and in [18] renormalizable models with a Lifshitz fixed point are constructed, whereas a renormalizable asymptotically free Yang Mills theory, in 4+1 dimensions, is given in [19]. As far as dynamical mass generation is concerned, a four-fermion interaction has been studied in the framework of Lifshitz-like theories [20], where the authors find a gap equation for the fermion mass, and the \( CP^{N-1} \) model at the Lifshitz point is discussed in [21]. In addition, [22] shows some perturbative properties of Lifshitz-like theories containing scalars and fermions, where an extension of supersymmetry to a Lorentz non-invariant theory is studied. For a presentation of renormalization group equations in the case of a scalar field, see [23]. Finally, in [24] a U(1) Gauge theory in 2+1 dimensions with \( z = 2 \) is considered.

The literature mainly deals with perturbative studies, and our aim here is to study a simple field theory model, in the framework of the non-perturbative
Schwinger-Dyson approach. In particular, we consider a Lifshitz-type model, in flat space time and in 3+1 dimensions, for a dynamical critical exponent $z=3$, with a scalar and a fermion field interacting via a Yukawa coupling. For the construction of the bare action of the model, we use only the quadratic marginal operators (kinetic terms), with dimension six, plus a Yukawa interaction term with a dimensionful coupling. Note that the construction of more complicated models, including other marginal and relevant operators (for $z=3$) is possible. However, in this work we will restrict our study to a Yukawa interaction only, in order to deal with a tractable system of equations, describing the dynamical generation of mass and Lorentz symmetry for fermions.

We stress that, for the specific model we examine, the generation of a term of the form $\lambda \bar{\psi} \gamma^\mu \partial_\mu \psi$ is enough for the restoration of Lorentz symmetry in the IR limit of the fermionic sector. However, as Ref. [23] points out, the issue of Lorentz symmetry restoration becomes problematic when one considers models with more than one species of particles, and we will discuss this point before the conclusion.

To summarize, this study is based on the Schwinger-Dyson approach, for which we derive in Appendix A the corresponding equation for the fermion self energy. The latter is parametrized by two dressed parameters, $m_f$ and $\lambda$, via the operators $m_f^3 \bar{\psi} \psi$ and $\lambda i \bar{\psi} \gamma^\mu \partial_\mu \psi$, and the corresponding self consistent equations are solved. Note that the parameter $\lambda$ controls the restoration of Lorentz symmetry in the fermionic IR sector. The coupled evolutions of these two parameters with the Yukawa coupling is presented in fig. 2, where it is found that there exists a critical value for the coupling, above which quantum corrections can generate simultaneously a Lorentz invariant kinetic term and a mass for fermions. We also comment on the physical relevance of our model and the limits of our approximations in Appendix B and Appendix C.

2 Free systems

We construct in this section the free scalar and fermion models, and derive the corresponding propagators which will be used for the loop calculations in the next section.
2.1 Scalar field

Here we remind the reader the construction of an anisotropic scalar model, in $D+1$ dimensions, starting with the action

$$S_b = \frac{1}{2} \int dt d^Dx \left( \dot{\phi}^2 - \phi (-\Delta)^z \phi \right),$$  \hspace{1cm} (1)

where a dot over a letter represents a time derivative. The action (1) describes a free scalar theory, with the following mass dimensions

$$[x^k] = -1 \quad [t] = -z \quad [\phi] = \frac{D-z}{2},$$  \hspace{1cm} (2)

and leads to the following equation of motion

$$\ddot{\phi} + (-\Delta)^z \phi = 0.$$  \hspace{1cm} (3)

We look for a solution by assuming the separation of variable

$$\phi(t, x) = \xi(t) \exp \{ i \cdot p \cdot x \},$$  \hspace{1cm} (4)

which leads to

$$\ddot{\xi} + (p^2)^z \xi = 0.$$  \hspace{1cm} (5)

We obtain

$$\xi = \xi_0 \exp (\pm i\omega t), \quad \omega = (p^2)^{\frac{z}{2}},$$  \hspace{1cm} (6)

where $\xi_0$ is a constant, such that the solutions of the form of eq.(4) represent plane waves in D+1 dimensions, and the Feynman propagator for the scalar field, which will be used in order to calculate loop diagrams, is

$$G_b(\omega, p) = \frac{i}{\omega^2 - (p^2)^z + i\epsilon},$$  \hspace{1cm} (7)

where $[\omega] = z$. If we include a mass term $-\frac{1}{2} m_b^2 \phi^2$ in the action of eq.(1) the scalar field propagator is modified as

$$\tilde{G}_b(\omega, p) = \frac{i}{\omega^2 - (p^2)^z - m_b^2 + i\epsilon},$$  \hspace{1cm} (8)

where $[m_b] = 1$. 

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2.2 Fermionic field

The action for the free fermionic model is

$$S_f = \int dtd^Dx \left\{ \bar{\psi}i\gamma^0\dot{\psi} + \bar{\psi}(-\Delta)^{\frac{z-1}{2}}(i\gamma^k\partial_k)\psi \right\}, \quad (9)$$

where we have included only quadratic marginal operators which correspond to a Lifshitz fixed point at the ultraviolet. A dimensional analysis gives

$$[x^k] = -1 \quad [t] = -z \quad [\psi] = \frac{D}{2}, \quad (10)$$

and the equation of motion is:

$$i\gamma^0\dot{\psi} + (-\Delta)^{\frac{z-1}{2}}(i\gamma^k\partial_k)\psi = 0. \quad (11)$$

We make the following ansatz for the solution of the above equation

$$\psi(t, x) = \theta(t)\hat{\psi}_p \exp\{ip\cdot x\}. \quad (12)$$

where the spinor part $\hat{\psi}_p$ is normalized according to the equation $\hat{\psi}_p^\dagger \hat{\psi}_p = 1$. If we multiply with the Hermitian conjugate we obtain

$$\ddot{\psi} + (-\Delta)^{\frac{z-1}{2}} \psi = 0 \quad (13)$$

The solution (12) should satisfy eq. (13), hence we obtain

$$\theta(t) = \theta_0 \exp(\pm it\omega), \quad \omega = (p^2)^{\frac{z}{2}} \quad (14)$$

where $\theta_0$ is a constant, such that the solutions (12) represent plane waves in D+1 dimensions. The Feynman propagator for the fermion field is

$$G_f(\omega, p) = i \frac{\omega\gamma^0 - (p^2)^{\frac{z-1}{2}}(p \cdot \gamma) + i\varepsilon}{\omega^2 - (p^2)^{z} + i\varepsilon} \quad (15)$$

where $[\omega] = z$. We can include the mass term $-m_f^2\bar{\psi}\psi$ in the action (9), where $[m_f] = 1$, as well as an additional quadratic term $\lambda\bar{\psi}(i\gamma^k\partial_k)\psi$, where $[\lambda] = z - 1$, such that the fermion propagator is finally

$$\tilde{G}_f(\omega, p) = i \frac{\omega\gamma^0 - \left[(p^2)^{\frac{z-1}{2}} + \lambda\right](p \cdot \gamma) + m_f^2}{\omega^2 - \left[(p^2)^{\frac{z-1}{2}} + \lambda\right]^2 p^2 - m_f^{2z} + i\varepsilon} \quad (16)$$

5This term is quadratic in the fermion field, but it is not marginal for $z \neq 1$. 


3 Dynamics

3.1 Model and Schwinger Dyson equations

We now consider the simplest interaction between scalars and fermions in the Lifshitz context, through a Yukawa coupling, and start with the following bare action

\[
S = \int dt d^D x \left\{ \bar{\psi} i \gamma^0 \psi + \bar{\psi} (-\Delta)^{\frac{z-1}{2}} (i \gamma^k \partial_k) \psi + \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \phi (-\Delta)^{z} \phi - \frac{1}{2} m_0^2 \phi^2 - g \bar{\psi} \psi \right\},
\]

where the coupling constant has dimension \([g] = \frac{3z-D}{2}\). In the framework of the gradient expansion, we will consider quantum corrections up to the first order in momentum only, such that we will look at the corrections to the scalar mass, and will allow the dynamical generation of a fermion mass term \(-m_f^2 \bar{\psi} \psi\) and of the additional first order fermionic kinetic term \(\lambda \bar{\psi} (i \gamma^k \partial_k) \psi\), in order to study the restoration of Lorentz invariance for fermions.

In the action (17), we start with a bare scalar mass in order to absorb the only UV divergence which will appear, as we will see, in the corrections to the scalar mass. No UV divergence will appear in the fermion self energy, due to the higher order derivatives, and for this reason \(m_f\) and \(\lambda\) can be taken equal to zero in the bare action. We note here that, also because of higher derivatives, the UV divergence we will find in the corrections to the scalar mass is logarithmic for \(D = z = 3\), and not quadratic as it is in a Lorentz-invariant theory.

We will use here the Schwinger Dyson approach to calculate the fermion and scalar self energies, which is non-perturbative and represents a resummation of graphs, avoiding IR divergences, because of the presence of a fermion mass and first order derivative kinetic term, both generated dynamically. Also, studies of dynamical mass generation usually lead to a mass which is non-analytical in the coupling constant, which cannot be found with a naive loop-expansion, and one therefore needs a non-perturbative approach. We show in the Appendix that the corresponding Schwinger-Dyson equation for the fermion self energy \(\Sigma_f = G_f^{-1} - G_f^{-1}\) is

\[
\Sigma_f = ig G_f \Theta G_b,
\]

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\[ \Sigma_f = \Gamma_{\text{bare}} + \Gamma_{\text{dressed}} \]

Figure 1: The fermion self energy given by the Schwinger Dyson equation in the rainbow approximation. A solid thick line represents the dressed fermion propagator, a solid thin line the bare fermion propagator, and a dashed line represents the dressed scalar propagator (which, in our approximation, is like the bare propagator, but with the renormalized mass instead of the bare one). As one can see, the fermion self energy is obtained as a resummation of an infinite number of graphs, which is at the origin of the non-perturbative feature of the results.

where \( G_f, G_b \) and \( \Theta \) are respectively the dressed fermion propagator, the dressed boson propagator and the dressed vertex. The equation (18) is self consistent, since it displays the dressed quantities on both sides, and therefore corresponds to a resummation of all quantum corrections (see fig. (1)).

Using the exact equation (18), we can study the dynamical generation of mass and first order derivative terms for fermions, and we will make the following assumptions:

- We neglect quantum corrections to the vertex, which corresponds to the so-called ladder or rainbow approximation [25], and we therefore consider \( \Theta \simeq g \). The corresponding partial resummation provided by the Schwinger-Dyson equations (18) is the dominant one for the study of dynamical mass generation[6]. We show in Appendix B that this approximation is well controlled in the regime where we observe dynamical mass generation;

- We also neglect the renormalization of the bare fermion kinetic term, which is consistent in the framework of the gradient expansion, if we

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[6] There is in principle an infinite tower of Schwinger-Dyson equations, which are self consistent equations for every \( n \)-point function, each involving the \( n + 1 \)-point function. A given truncation of this tower of coupled equations consists then is a specific resummation of graphs for each correlation function.
take into account first order derivative corrections to the fermion dynamics only;

- Also because of the gradient expansion, we consider a momentum-independent dynamical mass, since the latter would be quadratic in the momentum. In addition, the dominant contribution of the loop integral appearing in the Schwinger-Dyson equation (18) arises from low momentum, since no UV divergence occurs in the calculation of the fermion self energy. This approximation is discussed in Appendix C.

In what follows, we will concentrate on the case \( D = z = 3 \).

### 3.2 Scalar sector

It can be shown, as done in the Appendix for the fermion self energy, that the Schwinger Dyson equation for the scalar self energy reads

\[
\Sigma_b = \text{Tr}\{G_b^{-1} - G_b^{-1}\} = ig\text{Tr}\{G_f \Theta G_f\}.
\]  

(19)

As we will see in the next subsection, the operators \( \bar{\psi}\psi \) and \( \bar{\psi}(i\gamma^k \partial_k)\psi \) will be generated dynamically, such that we assume here that the dressed fermion propagator has the form (16), \( G_f = \tilde{G}_f \), where \( m_f \) and \( \lambda \) are generated dynamically. The scalar mass, after a Wick rotation, is then obtained from eq. (19) for vanishing momentum, which reads

\[
m_b^6 - m_0^6 = 4g^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^3p}{(2\pi)^3} \frac{\omega^2 + (p^2 + \lambda)^2 p^2 - m_f^6}{\omega^2 + (p^2 + \lambda)^2 p^2 + m_f^6}^2,
\]  

(20)

The integration over \( \omega \) leads to a logarithmically-divergent integral over \( p \):

\[
m_b^6 = m_0^6 + \frac{g^2}{\pi^2} \int_0^\Lambda \frac{p^4(p^2 + \lambda)^2 dp}{p^2(p^2 + \lambda)^2 + m_f^6} \]

\[
= m_0^6 + \frac{g^2}{\pi^2} \left( \ln \left( \frac{\Lambda}{m_f} \right) + \frac{2 \ln 2 - 1}{3} \right) + O(\Lambda^{-2}),
\]  

(21)

where \( \Lambda \) is the cut off in the 3-dimensional \( p \) space. Although eq. (21) apparently contains an IR divergence for \( m_f = 0 \), this divergence is actually not present in this case, if \( \lambda \neq 0 \), since we have then

\[
m_b^6 = m_0^6 + \frac{g^2}{2\pi^2} \ln \left( 1 + \frac{\Lambda^2}{\lambda} \right),
\]  

(22)
and \( \lambda \) plays the role of IR cut off.

In what follows, the bare mass \( m_0 \) will be chosen such that the renormalized mass \( m_b \) is finite and fixed. This renormalized mass will play the role of IR cut off for the calculation of the fermion self energy.

### 3.3 Fermion sector and self-consistent equations

The fermion self energy is calculated from the bare propagator (15) and the dressed propagator which is assumed to have the form (16), such that

\[
\Sigma_f(k) = -\lambda(k \cdot \gamma) - m^3_f. \tag{23}
\]

Furthermore, if we assume that the dressed scalar propagator has the form \( \mathcal{G}_b = \tilde{\mathcal{G}}_b \), where \( m_b \) is the renormalized, finite scalar mass (21), the right-hand side of the Schwinger-Dyson equation (18) is (for vanishing frequency and after a Wick rotation)

\[
\frac{1}{\omega^2 + (p - k)^6 + m_b^6}. \tag{24}
\]

This is a convergent integral, and, together with the self energy (23), leads to the self consistent equations which must be satisfied by \( \lambda \) and \( m_f \):

(i) The equation the fermion dynamical mass should satisfy is obtained by taking the trace of the Schwinger-Dyson equation (18), for \( k = 0 \):

\[
m^3_f = \frac{g^2}{(2\pi)^4} \int_{-\infty}^{\infty} d\omega \int \frac{m^3_f \, d^3p}{\omega^2 + p^6 + m_b^6}\frac{\omega \, p^2 \gamma^0 - (p^2 + \lambda) (p \cdot \gamma) + m^3_f}{\omega^2 + (p^2 + \lambda)^2 p^2 + m_f^6}. \tag{25}
\]

If \( m_f \neq 0 \), the integration over \( \omega \) shows that the dynamical mass must satisfy

\[
\frac{4\pi^2}{g^2} = \int_0^\infty \frac{p^2 dp}{A_b A_f(A_b + A_f)}, \tag{26}
\]

where

\[
A_b = \sqrt{p^6 + m_b^6} \quad A_f = \sqrt{p^2(p^2 + \lambda)^2 + m_f^6}. \tag{27}
\]
(ii) The equation for the coefficient $\lambda$ is obtained by expanding the self energy in $k$, and keeping the linear contribution only in order to identify it with the corresponding term in eq. (23). Using the following equality, valid for any function $f$,

$$
\int d^D p (k \cdot p) (p \cdot \gamma) f(p^2) = \frac{\Omega_D}{D} (k \cdot \gamma) \int_0^\infty dp \ p^{D+1} f(p^2),
$$

where $\Omega_D$ is the solid angle in dimension $D$, and identifying the coefficients of $(k \cdot \gamma)$ in the Schwinger Dyson equation, we obtain the following self consistent equation for $\lambda$

$$
\lambda = \frac{g^2}{2\pi^3} \int_0^\infty d\omega \int_0^\infty \frac{p^8 (p^2 + \lambda) dp}{(\omega^2 + A_b^2) [(\omega^2 + A^2_f)]}
= \frac{g^2}{4\pi^2} \int_0^\infty dp \ p^8 (p^2 + \lambda) \frac{2A_b + A_f}{A_b^2 A_f (A_b + A_f)^2},
$$

where $A_f, A_b$ are given in eq. (27). Finally, we are left with the two self-consistent coupled equations (26,29), which have to be solved simultaneously to find the parameters $(m_f, \lambda)$ which can be generated dynamically.

### 3.4 Numerical analysis and discussion

In this section, we present our numerical analysis, we comment on the physical relevance of our model and the limits of our approximation.

In order to solve the system of eqs. (25) and (29) we have to distinguish two cases: a) $m_f = 0$, and b) $m_f \neq 0$.

a) If $m_f = 0$ eq. (25) is satisfied automatically, then we have checked numerically that eq. (29) can be solved with respect to $\lambda$, for all the range of the free parameters $g$ and $m_b$. However, this class of solutions with $m_f = 0, \lambda \neq 0$ is not accepted because of the IR divergences which arise when we go to second order approximation in momentum, as we explain in Appendix C. For this reason this class of solutions is not presented here.

b) If we assume that $m_f \neq 0$ we can divide eq. (25) by the factor $m_f^3$ to obtain eq. (26) in the previous section. Then we can rescale the other parameters of the theory with the renormalized mass of the scalar field $m_b$, to obtain the following dimensionless parameters:

$$
\mu = \frac{m_f}{m_b} \quad l = \frac{\lambda}{m_b^2} \quad \varepsilon = \frac{g}{2\pi m_b^3},
$$

(30)
and the set of coupled equations to solve is, from eqs. (26, 29),

\[
1 = \varepsilon^2 \int_0^\infty \frac{x^2 \, dx}{\tilde{A}_b \tilde{A}_f (\tilde{A}_b + \tilde{A}_f)}
\]

\[
1 = \frac{\varepsilon^2}{l} \int_0^\infty dx \, x^8 (x^2 + l) \frac{2 \tilde{A}_b + \tilde{A}_f}{\tilde{A}_b^3 \tilde{A}_f (\tilde{A}_b + \tilde{A}_f)^2}, \tag{31}
\]

where

\[
\tilde{A}_b = \sqrt{1 + x^6} \quad \tilde{A}_f = \sqrt{\mu^6 + x^2(x^2 + l)^2}. \tag{32}
\]

We solve the above algebraic system of equations numerically, and a unique solution for the pair \( (l, \mu) \) is obtained, if the dimensionless coupling \( \varepsilon \) is larger than the threshold \( \varepsilon_c \simeq 1.3263 \). The results for the parameters \( l^{1/2} \) and \( \mu \) as a function of the dimensionless coupling \( \varepsilon \) are presented in fig. (2). We would like to stress that the singular point \( \mu = 0, \sqrt{7} = 0.529 \) (for \( \varepsilon = \varepsilon_c \)) is not a solution of the system of equations (31) because of the restriction \( \mu \neq 0 \). However the system of equations (31) has solutions arbitrarily close to the singular point if \( \varepsilon > \varepsilon_c \), while for \( \varepsilon < \varepsilon_c \) we have no real solutions.

According to the above results Lorentz symmetry arises for the specific model we examine, in the IR limit when \( p \ll \lambda^{1/2} \). Indeed, from the dispersion relation for the free fermion

\[
\frac{\omega^2}{\lambda^2} = \left( \frac{p^2}{\lambda} + 1 \right)^2 p^2 + \frac{m_f^6}{\lambda^2}, \tag{33}
\]

and for \( p \ll \lambda^{1/2} \), we obtain

\[
E^2 \simeq p^2 + \tilde{m}_f^2, \tag{34}
\]

where we define the rescaled parameters \( E = \omega/\lambda \) and \( \tilde{m}_f = m_f^3/\lambda \) that correspond to the fermion energy and mass with the correct dimensions \([E] = \tilde{m}_f] = 1\).

However, if \( m_f \simeq \lambda^{1/2} \), the limit \( p \ll \lambda^{1/2} \) implies that \( p \ll m_f \), and the behavior of the particle is then nonrelativistic, the kinetic energy of the fermion is given by \( p^2/2m_f \) (note that \( m_f = \tilde{m}_f \) for \( \lambda^{1/2} \simeq m_f \)). We observe in fig. (2) that there is a small region for which we obtain a relativistic fermion, in particular for \( \varepsilon > \varepsilon_c \) when \( \varepsilon \) is close to the critical value \( \varepsilon_c \). \( l^{1/2} \) becomes significantly larger than \( \mu \). For \( \varepsilon \gg \varepsilon_c \), the mass of the fermion increases and becomes comparable to \( \lambda^{1/2} \), this means that the relativistic behavior for fermions is restricted to a narrow set of values for the coupling.
Figure 2: The parameters $\mu = m_f/m_b$, $l^{1/2} = \lambda^{1/2}/m_b$ as a function of $\varepsilon = g/(2\pi m_b^3)$. The system of equations (31) has a unique solution for $\varepsilon > \varepsilon_c$, which for $\varepsilon \to \varepsilon_c$ tends asymptotically to $\mu = 0$ and $l^{1/2} \approx 0.529$. For $\varepsilon \leq \varepsilon_c$ we have checked numerically that the system (31) has no solution.

The non-perturbative analysis with SD equations, in this article, was based on the following two approximations: 1) ladder approximation, 2) first order approximation in momentum.

1) In the case of ladder approximation we make the assumption that the bare coupling $g$ is almost equal to the dressed coupling $\Theta$. It seems that this consideration is consistent with our results, hence we do not expect significant corrections in fig. (2). In particular, in Appendix B we compute the one loop vertex diagram and show that the dressed coupling $\Theta^{(1)}$ receives small corrections and is therefore close to the bare coupling $g$.

2) For $\varepsilon \simeq \varepsilon_c$, our approximation of taking into account only the first order in momentum in the propagators might not be reliable: higher order in momentum terms become significantly strong due to IR divergences for $m_f = 0$, hence different Ansätze for the propagators should be considered, with a larger number of unknown parameters. This would generate a considerably more involved numerical problem to solve and is beyond the scope of this
article. However, as we explain in Appendix C, we believe that the effects of higher orders would smoothen the singular behaviour of the solution when $\varepsilon \to \varepsilon_c$, and that the present singularity is rather an artifact arising from the first order approximation in momentum.

Finally we would like to note that the issue of Lorentz symmetry restoration becomes problematic when we consider more realistic models with different species of fermions. In particular, the authors of Ref. [23] find that, although renormalization group flows reduce the difference in the velocities of different species towards the IR, this is not enough to achieve the experimental precision of the speed of light, unless we assume an unnatural fine-tuning between the Lorentz violating terms of the UV action of the model. These results are demonstrated in the framework of perturbation theory by considering several simple Lifshitz-type models, and we emphasize that the non-perturbative mechanism we propose in this paper suffers from similar problems, as we discuss further in the main part of this paper. For example if we had considered an extension of our model with two species of fermions, with different Yukawa couplings $g_1$ and $g_2$, we would have encountered the generation of two first order kinetic terms with different $\lambda_1$ and $\lambda_2$ in the IR. This would mean that the limiting velocities of fermions would be significantly different in the IR, unless we had chosen $g_1$ extremely close to $g_2$. Such fine-tuning is not consistent with the logic of standard model where the Yukawa couplings are significantly different for different species of fermions.

4 Conclusions

We considered a 3+1 dimensional model with a Lifshitz-type fixed point ($z = 3$) in which a scalar and a fermion field interact via a Yukawa term. The effect of dynamical mass generation, as well as the restoration of Lorentz symmetry in the IR limit was examined in the framework of Schwinger-Dyson equations.

An interesting point in this model is that the interaction is super renormalizable, and the only UV divergence present comes from the scalar self-energy diagram. Note that, in contrast to the standard case ($z = 1$) in which the divergence is quadratic, the divergence in our model is logarithmic, due to the higher powers of momentum in the propagators. In order to absorb the UV divergence in our model, we introduce a bare mass for the scalar such that our effective theory does not depend on the cutoff of the theory.
On the other hand, a bare fermion mass is not necessary in the action since a dynamical mass is generated quantum mechanically and is finite.

Note that the absence of quadratic divergences in the scalar self energy diagram sets the hierarchy problem on a new basis, as the scalar field mass flows logarithmically with the UV cutoff, see eq.(21). However, the absence of quadratic divergences in our model can not be considered as a resolution to the hierarchy problem as we do not consider all the degrees of freedom of the Standard Model (see also [26]).

The ansatz for the scalar and fermion self energies is based on a linear approximation in the external momentum $k$, and we do not discuss here the possibility of generating a Lorentz-invariant kinetic term for the scalar field as this term would be of order $k^2$. The equations arising from the Schwinger-Dyson approach are solved numerically and the results are presented in fig.(2). We find that there is a critical value $g_c$ for the Yukawa coupling, above which Lorentz symmetry is restored and a mass is generated in the low energy limit of the fermionic sector.

Note, that there are other marginal and relevant operators which can be included in the UV action, for example the interaction terms $g_1 \phi^2 \bar{\Psi} \Psi$, $g_2 (\bar{\Psi} \Psi)^4$, $g_3 \phi^4$, etc. However, had we included these terms in the bare action, the corresponding Schwinger-Dyson system of equations would have become exceedingly complicated and would in any case have lead us beyond the scope of this article. As a first step in a non-perturbative approach, we restricted our analysis to the most economical model, namely a Yukawa interaction, which is enough to demonstrate the dynamical generation of a mass and a first order kinetic term for the fermion.

Finally, this non-perturbative mechanism for the restoration of Lorentz symmetry in models defined at a Lifshitz point may be useful for the study of other theories with immediate phenomenological interest, such as QED or Higgs models, which are proposed for future investigation.

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Appendix A: Schwinger-Dyson equation

The partition function of the theory corresponding to the bare action (17) is

\[
Z[j, \bar{\eta}, \eta] = \int D[\phi, \psi, \psi] \exp \left\{ iS + i \int dt d^D x \left( j\phi + \bar{\eta}\psi + \psi\eta \right) \right\} = \exp \{ iW[j, \bar{\eta}, \eta] \}, \tag{35}
\]

where \( j, \bar{\eta}, \eta \) are the sources for \( \phi, \psi, \bar{\psi} \) respectively, and \( W \) is the connected graphs generator functional. The functional derivatives of the latter define the classical fields \( \phi_c, \psi_c, \bar{\psi}_c \)

\[
\frac{\delta W}{\delta j} = \frac{1}{Z} < \phi > \equiv \phi_c \\
\frac{\delta W}{\delta \bar{\eta}} = \frac{1}{Z} < \psi > \equiv \psi_c \\
\frac{\delta W}{\delta \eta} = -\frac{1}{Z} < \bar{\psi} > \equiv -\bar{\psi}_c, \tag{36}
\]

where

\[
< \cdots > = \int D[\phi, \psi, \bar{\psi}] (\cdots) \exp \left\{ iS + i \int dt d^D x \left( j\phi + \bar{\eta}\psi + \psi\eta \right) \right\}. \tag{37}
\]

The proper graphs generator functional \( \Gamma[\phi_c, \psi_c, \bar{\psi}_c] \) is defined as the Legendre transform of \( W \),

\[
\Gamma = W - \int dt d^D x \left( j\phi_c + \bar{\eta}\psi_c + \bar{\psi}_c\eta \right), \tag{38}
\]

where the sources have to be understood as functionals of the classical fields. It is easy to check that

\[
\frac{\delta \Gamma}{\delta \phi_c} = -j \\
\frac{\delta \Gamma}{\delta \psi_c} = \bar{\eta} \\
\frac{\delta \Gamma}{\delta \bar{\psi}_c} = -\eta \\
\frac{\delta^2 \Gamma}{\delta \psi_c \delta \bar{\psi}_c} = - \left( \frac{\delta^2 W}{\delta \eta \delta \bar{\eta}} \right)^{-1}. \tag{39}
\]
The first step for the derivation of a self consistent equation involving the dressed propagators and vertex is to note that the functional integral of a functional derivative vanishes, such that
\[
\left\langle \frac{\delta S}{\delta \psi} + \eta \right\rangle = 0.
\] (40)

Using the different derivatives (39), we obtain then
\[
\frac{\delta \Gamma}{\delta \bar{\psi}_c} = \left( i\gamma^0 \partial_t + (-\Delta) \frac{-1}{2} (i\gamma^k \partial_k) \right) \psi_c - \frac{g}{Z} < \phi \psi >.
\] (41)

The vertex, the bare and dressed fermion propagators are respectively
\[
\Theta = \left( \frac{\delta^3 \Gamma}{\delta \phi_c \delta \psi_c \delta \bar{\psi}_c} \right)_0,
\]
\[
G_f^{-1} = \left( \frac{\delta^2 S}{\delta \psi \delta \bar{\psi}} \right)_0,
\]
\[
G_f^{-1} = \left( \frac{\delta^2 \Gamma}{\delta \bar{\psi}_c \delta \psi_c} \right)_0,
\] (42)

where the index 0 refers to vanishing fields, such that a functional derivative of eq.(41) gives for the fermion self energy
\[
\Sigma_f = G_f^{-1} - G_f^{-1} = -\frac{g}{Z} \left( \frac{\delta}{\delta \psi_c} < \phi \psi > \right)_0.
\] (43)

We then express \( < \phi \psi > \) in terms of derivatives of \( W \):
\[
\frac{\delta^2 W}{\delta j \delta \bar{\eta}} = -i\phi_c \psi_c + \frac{i}{Z} < \phi \psi >,
\] (44)

such that
\[
\left( \frac{\delta}{\delta \bar{\psi}_c} < \phi \psi > \right)_0 = -i \left( \frac{\delta}{\delta \psi_c} \frac{\delta^2 W}{\delta j \delta \bar{\eta}} \right)_0
\]
\[
= -i \left( \frac{\delta^3 W}{\delta \eta \delta j \delta \bar{\eta}} \frac{\delta \eta}{\delta j} \right)_0.
\] (45)
\[
\begin{align*}
&= i \left( \frac{\delta}{\delta j} \left( \frac{\delta^2 \Gamma}{\delta \psi_c \delta \psi_c} \right)^{-1} \frac{\delta \eta}{\delta j} \right)_0 \\
&= i \left( \left( \frac{\delta^2 \Gamma}{\delta \psi_c \delta \psi_c} \right)^{-1} \left( \frac{\delta^3 \Gamma}{\delta \phi_c \delta \psi_c \delta \psi_c} \right) \frac{\delta \phi_c}{\delta j} \left( \frac{\delta^2 \Gamma}{\delta \psi_c \delta \psi_c} \right)^{-1} \frac{\delta^2 \Gamma}{\delta \psi_c \delta \psi_c} \right)_0 \\
&= i \mathcal{G}_f \Theta \left( \frac{\delta j}{\delta \phi_c} \right)_0^{-1} = -i \mathcal{G}_f \Theta \mathcal{G}_b, \quad (46)
\end{align*}
\]

and the Schwinger-Dyson equation for the fermion self energy is finally, from eq. (43),
\[
\Sigma_f = ig \mathcal{G}_f \Theta \mathcal{G}_b. \quad (47)
\]

We stress that this equation is exact, and represents a resummation of all quantum corrections, since it involves the dressed quantities on both sides of the equation.

**Appendix B: Validity of the ladder approximation**

We check here that the ladder approximation is consistent and calculate corrections to the coupling constant. This calculation is one-loop like, but it takes into account the fermion mass generated dynamically: this is a similar approach to Schwinger Dyson equations, and avoids IR divergences.

This correction is given by the following three-point graph, for vanishing incoming momentum:
\[
\begin{align*}
\Theta^{(1)} &= g + ig^3 \text{tr} \int \frac{d\omega}{2\pi} \frac{d^3 p}{(2\pi)^3} \left( \frac{\omega \gamma^0 - p^2 (\mathbf{p} \cdot \gamma) + m_f^3}{\omega^2 - p^6 - m_f^6 + i\epsilon} \right)^2 \frac{1}{\omega^6 - p^6 - m_b^6 + i\epsilon}, \\
\end{align*}
\]

and, after a Wick rotation,
\[
\Theta^{(1)} = g + \frac{g^3}{\pi^3} \int \frac{d\omega}{2\pi} \frac{d^3 p}{(2\pi)^3} \frac{-\omega^2 - p^6 + m_f^6}{(\omega^2 + p^6 + m_b^6)(\omega^2 + p^6 + m_f^6)^2}.
\]

In the phase where the fermion dynamical mass is generated, \(m_f \simeq m_b\) (near the singular point of fig. 2 but not very close to it), such that
\[
\Theta^{(1)} \simeq g + \frac{g^3}{\pi^3} \int \frac{d\omega}{2\pi} \frac{d^3 p}{(2\pi)^3} \frac{-\omega^2 - p^6 + m_b^6}{(\omega^2 + p^6 + m_b^6)^3},
\]

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\[
\begin{align*}
&= g + \frac{g^3}{4\pi^2} \int p^2 dp \left( \frac{3m_b^6}{(p^6 + m_b^6)^{5/2}} - \frac{2}{(p^6 + m_b^6)^{3/2}} \right) \\
&= g + \frac{g^3}{12\pi^2 m_b^6} \int_0^\infty dx \left( \frac{3}{(1 + x^2)^{5/2}} - \frac{2}{(1 + x^2)^{3/2}} \right)
\end{align*}
\]

The last integral vanishes, and \( \Theta^{(1)} \approx g \). As a consequence, corrections to the coupling constant can be neglected, and the ladder approximation that we used is justified.

**Appendix C: Beyond first order approximation in momentum**

In order to go beyond the first order approximation in momentum, we have to expand the self energy diagrams up to second order in external momentum,

\[
\Sigma_f(k) = -m_f^3 - \lambda_f (k \cdot \gamma) + Z_f k^2 \\
\Sigma_b(k) = m_b^6 + \lambda_b k^2 - m_0^6
\]

In this way we introduce two more parameters \( Z_f \) and \( \lambda_b \), and we have to solve numerically a system of four coupled equations with four unknown parameters:

\[
\mu = m_f/m_b \quad l_f = \lambda_f/m_b^2 \quad l_b = \lambda_b/m_b^4 \quad z_f = Z_f/m_b,
\]

which would satisfy equations of the form

\[
\begin{align*}
F_1(\mu, l_f, l_b, z_f, g/m_b^2) &= \mu \\
F_2(\mu, l_f, l_b, z_f, g/m_b^3) &= l_f \\
F_3(\mu, l_f, l_b, z_f, g/m_b^3) &= l_b \\
F_4(\mu, l_f, l_b, z_f, g/m_b^3) &= z_f
\end{align*}
\]

where the functions \( F_1, F_2, F_3, F_4 \) are integrals over the energy \( \omega \) and the momentum \( p \). Note that these functions could be obtained by expanding the fermion and scalar self energies up to second order in momentum \( k \), but the corresponding set of equations would lead to a much more involved numerical problem, and would not change the essential point, i.e. the dynamical generation of Lorentz-invariant terms.
Nevertheless, one can predict the effect these corrections would have, in the vicinity of the critical point of fig. 2. In our first order approximation in momentum, we disregarded the solutions ($\mu = 0, l_f \neq 0$), whereas they satisfy our set of self-consistent equations, since $l_f$ acts as an IR regulator when $m_f = 0$. The reason we disregard these solutions is the following: we checked that at $m_f \to 0$, the derivatives $d^2 \Sigma_f(k)/dk^2$ and $d^2 \Sigma_b(k)/dk^2$ become infinite whereas they should be finite, since they are related to wave function renormalizations. As a consequence, we believe that the existence of this singularity is an artifact of the first order approximation in momentum and we expect that second order corrections would smoothen the singular behaviour of the dynamical mass.

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