OPTIMAL REBATE STRATEGIES IN A TWO-ECHELON SUPPLY CHAIN WITH NONLINEAR AND LINEAR MULTIPLICATIVE DEMANDS

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Abstract. We examine the pure rebate strategies in a two-echelon supply chain under stochastic demand with multiplicative error. Given exogenous wholesale price and retail price, we characterize the unique Nash equilibrium when both manufacturer and retailer provide rebate policy to consumers under nonlinear and linear price-dependent demand functions, including iso-elastic multiplicative demand function (EMDF) and linear multiplicative demand function (LMDF). Based on a game theoretical framework, we prove that there still exists a unique equilibrium when the price elasticity is rather small with constraint conditions in the former case. We also find that in this case the retailer (manufacturer) may increase its rebate value in reaction to the manufacturer’s (retailer’s) rebate value in order to stimulate sales, which is contrary to the conventional wisdom that the retailer (manufacturer) will shrink its rebate value to gain an “extra advantage” unfairly. As a result, both parties share the same profit at equilibrium. Further, we compare the expected profit outcomes at equilibrium among joint-rebate game, single-party rebate game and no-rebate game by using numerical examples. It is shown that the joint-rebate policy is not always dominates the others unless the price elasticity is sufficiently flexible.

1. Introduction. Rebates have become one of the most crucial promotional tools by manufacturers or/and retailers due to the incentives to increase their profit in the last decade. A large number of retail stores in the United States offer rebate programs. Millman [27] reports that $10 billion rebates are offered to consumer in 2002, and Grow [19] estimates that even more than 400 million rebate programs are provided in United States with a total value of $6 billion each year. According to the report of the leading E-commerce store Mr.rebates(\texttt{http://www.mrrebates.com/}), over 2000 stores provide rebates ranging from 1% to 20% or certain cash coupons, including Dell, JCPenney, Overstock.com, etc. According to the report

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of Parago, the national largest rebate provider (http://www.parago.com/), more than 50 million rebates is delivered in 2009, and its retail rebating business grows by 33% in 2010.

There are two main benefits of rebates. First, rebates can increase product demand while maintaining consumers' future price expectations of the product ([22]; [23]). Customers can save purchase cost and benefit from the rebate of manufacturer or retailer with a temporary discount on per-unit, allowing the store to maintain the product at its current price. Second, rebates can be more cost effective than price markdowns because of the low claim rate ([13]). Consumers may not redeem cash due to a variety of reasons, such as low-sensitivity to rebate, forgetting to redeem, exceeding deadlines or making mistakes in requisite documentation, and long redemption time. Therefore, the firms who offered the rebates may obtain a great deal of additional profit.

Rebates can appear in various forms such as mail-in rebate (MIR), coupons, and instant rebate. The most common rebate is MIR, which usually requires consumers to follow certain rules to redeem cash, such as collecting the paperwork, filling out forms, and sending out the rebate requests within the correct time frame. The manufacturer and retailer may both offer rebate programs in a supply chain. A survey\(^1\) claims that 50% of retailers and almost half (48%) of manufacturers use the programs as part of their customer loyalty and promotions mix between December 2010 and March 2011 among 75 retail and manufacturing organizations.

There is a stream of literature about rebate issues in supply chains. However, most of the previous work considers MIR offered by the manufacturer while the retailer will not offer MIR. Recently, Cho et al. [13] consider the optimal pricing and rebate strategy under the case where either the manufacturer or the retailer (or both) can offer an MIR. They model the consumer demand based on a simple deterministic model. Geng and Mallik [17] further provide a generalization of the work of [13] by considering stochastic demand and simultaneous and endogenous decision making under the case where both the manufacturer and retailer offer MIRs. However, they focus discussions on an iso-elastic multiplicative demand function (EMDF) with limitation to the scenario when customers earn a much higher price elasticity \(b \geq 2\). Here the price elasticity \(b\), defined as the ratio of the percentage increase of demand relative to 1% increase of the effective-price (the selling price minus the offered rebate), measures the sensitivity of consumers to the product. If \(b > 1\), we say a product is defined as price elastic to demand, and working with this assumption in the demand function is common in the literature (see [8], [31]). When the price elasticity is high, say \(b \geq 2\), it is intuitively reasonable to offer more rebates as it is beneficial to both parties by creating additional demand. However, in realistic business activity, there exist various product categories with relatively small price elasticity of demand (see [38]), say, \(1 < b < 2\). As illustrated in Table 1, we can see that many products such as beer, electricity, mass transit and bus, have price elasticity between 1 and 2. Some questions also naturally arise under this scenario, is there still a unique equilibrium? What kind of best response rebate strategies will the game players take? How does the PE over different intervals affect the rebate strategies on the responses of the manufacturer/retailer? Moreover, the EMDF function is based on a power structure (Nash game) ([33]). What happens if the stochastic demand function has a linear structure? The demand based on a

\(^1\)The results of primary research are performed by Aberdeem Group(2011), readers can get more information to visit http://www.aberdeen.com.
linear multiplicative demand function (LMDF) has also been widely investigated in the literature. See, for example, [2], [18], [30], and [32].

Table 1. Price Elasticity of Demand for Some Products

| Product              | Price Elasticity | Product          | Price Elasticity |
|----------------------|------------------|------------------|------------------|
| Beer                 | 1.13             | Electricity      | 1.14             |
| Mass transit, Bus    | 1.20             | New cars         | 1.20             |
| Charitable giving    | 1.29             | Marijuana        | 1.50             |
| Air travel           | 2.40             | Toilet articles, reparations | 3.04 |

To address the gap in part, this paper extends the work of [17] in two important respects. First, we consider a more general scenario with price elasticity varying on the interval $b > 1$ instead of $b \geq 2$. Second, to make the analysis more complete, we study another structural form of demand based on LMDF, in addition to the EMDF function. Our results show that there still exists an equilibrium when the price elasticity (PE) is rather small with certain conditions. When the PE is below the threshold 2, which means that the consumers are not sensitive to rebates, then increasing the amount of rebate cannot contribute too much additional sales in the market. As a result, the retailer does not have much motivation to offer more rebates to consumers. However, as PE becomes larger ($b \geq 2$), rebate can further stimulate more sales when facing rebate-sensitive consumers. Therefore, it is profitable for the retailer to provide more rebates to consumers as PE increases. Surprisingly, in equilibrium, the manufacturer may also increase its rebate value in reaction to the retailer’s rebate policy by stimulating additional sales, which is contrary to the conventional wisdom that the manufacturer will shrink its rebate value to gain an “extra advantage” unfairly. Interestingly, both parties share the same profit at equilibrium. We also show from numerical examples that as the price elasticity increases over the range $1 < b < 2$, both parties will simultaneously reduce their rebate value whereas they would like to increase the amount of rebate when $b \geq 2$. Furthermore, we compare the expected profit at equilibrium under the three different scenarios, including no-rebate policy and the two one-party-rebate policies. However, we find that the expected profit of both the manufacturer and the retailer is not always higher in the joint rebate game than that of the exclusive game and no-rebate game, which is different from the study of [17]. In other words, the joint-rebate policy at least dominates the others only when the demand elasticity is sufficiently flexible.

2. Literature review. There is a substantial marketing, operations management and economics literature that investigates the rebate issues in supply chains. Taylor [35] sets up a supply chain coordination model under channel rebates with sales effort effects, indicating that when demand is influenced by retailer sales effort, a properly designed target rebate and return contract achieves coordination and a win-win outcome. Lu and Moorthy [26] examine the key differences between coupons and rebates and show the conditions under which one is optimal than the other one. Taylor and Xiao [36] further extend these work to two forms of contracts about incentives for retailer forecasting: a returns contract and a rebates contract. The authors characterize the optimal rebates contract which compensates the retailer for the units she sells to end consumers. It is shown that under rebate the
retailer, manufacturer and total system may benefit from the retailer having inferior forecasting technology. Chiu et al. [12] also discuss the rebate as well as price and returns supply contracts for coordinating supply chains with price-dependent demands, showing that multiple equilibrium policies for channel coordination exist. Chen et al. [11] focus on price promotions provided by manufactures through manufacturer rebates, and analyze the impact on a manufacturer and a retailer. They reveal that unless all of the customers claim the rebate, the rebate always benefits the manufacturer. Chen et al. [10] focus on a different property of rebates with their ability to price discriminate with a consumer among her post purchase states, and argue that the consumer redeems the rebate only in post-purchase states in which her marginal utility of income is high. Arya and Mittendorf [1] also consider the manufacturer-to-consumer rebates and view it as price discrimination. Their results demonstrate that consumer rebates can be particularly useful when a supply chain encounters inefficiencies stemming from strategic inventory buildup by retailers. Khouja [22] formulates and solves models by jointly determining the optimal price, rebate face value, and the optimal order quantity for a price and rebate sensitive deterministic demand. Under realistic conditions, offering rebate can have significant pricing and inventory policy implications and can lead to a significant increase in profit. Arcelus and his partners have published a stream of extended papers on pricing, rebate or ordering policies in a supply chain under the newsvendor framework (Arcelus et al. [2], [3], [4], [5], [6]). For example, Arcelus et al. [5] states that when considering the uncertainty of a stochastic redemption rate, it is contrary to what happens when moving from a deterministic to a stochastic demand, which leads the rebate policy to dominate its price-discount counterpart. Arcelus et al. [6] then study the effectiveness of end consumer rebates from both channels. They find that all the three scenarios are equally profitable and the retailer-only rebate policy is dominant. Liang et al. [25] use “Present-biased preference” theory to characterize consumers’s demand and study the impact of manufacturer rebates on the competition between a national brand (NB) and a private label (PL). They find out that a positive slippage is not always profitable to the manufacturer if the rebate fails to expand the demand of the NB.

There is also some recent work in the context of operations management that considers a game theoretic tool on rebate policy in supply chains. Nagarajan and Sosic [29] survey some applications of cooperative game theory to supply chain management placed in two aspects: profit allocation and stability. Cachon and Netessine [9] discuss both non-cooperative and cooperative game theory in static and dynamic settings, and take careful attention to techniques for demonstrating the existence and uniqueness of equilibrium in non-cooperative games. As a matter of fact, the most common theory used in practice is non-cooperative game theory, in which all information is symmetrical, i.e., the manufacturer or retailer can acquire knowledge about all parameters and distributions in the model setting. Khouja and Zhou [23] examine the use of consumer cash MIR offered by a manufacturer in a Stackelberg game. They show that rebates are profitable to manufacturers if consumers are inconsistent in the sense that the rebate valuation when they make purchase decisions is independent of their redemption probabilities. Aydin et al. [7] build a single-retailer, single-manufacturer supply chain with endogenous manufacturer rebates and retail pricing and also demonstrate a pure strategy Nash equilibrium under two types of manufacturer’s rebates respectively: manufacturer-to-retailer rebate, and manufacturer-to-consumer rebate. Demirag et al. [15] study a game theoretical
model to examine the impact of “retailer incentive” and “customer rebate” promotions on the manufacturer’s pricing and the retailer’s ordering/sale decisions. The main tradeoff is that customer rebates are given to every customer, while the use of retailer incentives is controlled by the retailer.

The current work of us also involves a game theoretical framework in a two-level supply chain. Most of the work cited above consider manufacturer-to-consumer rebate policy, and the study on retailer-to-consumer rebate policy seems to be limited. Some exceptions include [13] and [17]. Cho et al. [13] use a Stackelberg game to analyze the rebate strategies in a two-level supply chain based on a deterministic demand model. Geng and Mallik [17] study the joint decisions of offering MIRs in a single-manufacturer-single-retailer supply chain using a game theoretic framework under demand uncertainty, with primary focus on the additive demand function. They simply discuss the situation of multiplicative demand function only when consumers are highly price-sensitive (b ≥ 2). To the knowledge of us, our paper may be the first to consider the joint-rebate strategies of the manufacturer and retailer when facing a low price-sensitive price elasticity (1 < b < 2) in the multiplicative form. Moreover, we discuss the rebate strategies in supply chains based on another general demand model LMDF, in addition to the EMDF. The random term in both EMDF and LMDF belongs to IFR (increasing failure rate). Table 2 summarizes the major differences of demand models between our work and the relevant literature. The remainder of this paper is organized as follows. In Section 3, we describe

\[ \text{Table 2. Difference of multiplicative demand model between our paper and the literature} \]

| Paper                  | EMDF          | LMDF          | Property of the Random Factor |
|------------------------|---------------|---------------|------------------------------|
|                        | 1 < b < 2     | b ≥ 2         |                             |
| Nevins [30]            | –             | –             | Normal                       |
| Polatoglu [32]         | –             | –             | Exponential                  |
| Petruzzi and Dada [31] | –             | √             | IFR                         |
| Granot and Yin [18]    | –             | –             | Uniform                      |
| Arcelus et al. [5]     | –             | √             | Uniform                      |
| Geng and Mallik [17]   | –             | √             | IFR                         |
| Hu and Li [20]         | –             | √             | IFR                         |
| Our paper              | √             | √             | IFR                         |

our model setting and assumptions. In Sections 4 and Section 5, we discuss the optimal rebate strategies at equilibrium and managerial insights under EMDF and LMDF, respectively. Some concluding remarks are given in Section 6. All proof are provided in the Appendix.

3. The model. Consider a two-level supply chain consisting of one manufacturer and one retailer, the manufacturer produces a single product with unit cost c and sells it at a constant wholesale price w to the single retailer, who in turn sells the product to consumers at a retail price p. In our model, both parties can offer rebates to end consumers in their best interests. Let \( r_R \geq 0 \) and \( r_M \geq 0 \) denote the rebate provided by the retailer and the manufacturer to consumers, respectively. The total market demand is stochastic and effective-price-dependent. The unit effective-price that consumers purchase is \( p - \alpha r_R - \alpha r_M \), where \( \alpha \) represents rebate effectiveness (0 ≤ \( \alpha \) ≤ 1), which is a proxy that measures the effective percentage of one dollar
increase in rebate value relative to one dollar drop in price ([22]). The rebate policy
will be more attractive with a higher rebate effectiveness, and \( \alpha = 1 \) indicates that
the effect of one dollar rebate is equivalent to one dollar of direct price reduction.

In our model setting, let \( a \) be the potential market size and \( b \) be price elasticity
of demand. Here we describe the demand model as

\[
D_i(r_R, r_M, \epsilon) = y_i(r_R, r_M)\epsilon, \ i \in \{e, l\},
\]

\[
y_e(r_R, r_M) = a(p - \alpha r_R - \alpha r_M)^{-b},
\]

\[
y_l(r_R, r_M) = a - b(p - \alpha r_R - \alpha r_M),
\]

where \( y_i(r_R, r_M) \) represents the deterministic component of demand, which is strictly
decreasing with retail price and increasing with rebates and is assumed to be at
least twice differentiable. For simplicity, denote \( D_e(r_R, r_M, \epsilon) \) and \( D_l(r_R, r_M, \epsilon) \) as
EMDF and LMDF, respectively. Note that \( a > 0 \) and \( b > 1 \) EMDF while \( a > 0 \) and \( b > 0 \) in LMDF. In particular, in the case of IMDF, \( b \) is the effective-price
elasticity of demand (or \( DE = -(\hat{p}/y_i)(dy_i/d\hat{p}) \)), it pictures the percentage change
in quantity demanded in response to one percent change in effective-price. Figure
1 gives the curve with different elasticity parameters. The random variable \( \epsilon \) in
the demand function is defined over \([0, +\infty)\), which has a mean of \( \mu \) and a standard
deviation of \( \sigma \). Denote \( f(\cdot) \) and \( F(\cdot) \) as the probability density function(PDF) and
cumulative distribution function (CDF) of \( \epsilon \), respectively. To ensure the solution of
the traditional newsvendor problem(NVP) is unique, we assume \( F(\cdot) \) also enjoys a
unique reverse function \( F^{-1}(\cdot) \). Moreover, define an increasing failure rate(IFR) as
\( r(\cdot) = f(\cdot)/(1 - F(\cdot)) \), which has the derivative \( r'(\cdot) > 0 \). The most commonly used
distributions such as the normal, uniform, exponential, beta, and gamma exhibit
this assumption ([24], [31]).

The sequence of decisions is characterized in Figure 1. At the beginning of the
selling season, the manufacturer determines the amount of rebate \( r_M \) to consumers,
and the retailer chooses her own rebate value \( r_R \) to consumers simultaneously. Once
the rebate strategies are both announced, the retailer will forecast the future market
demand \( D_i \) (For simplicity, we write \( D_i(r_R, r_M, \epsilon) \) as \( D_i \) and make a decision about
order quantity \( Q_i \) and places an order from the manufacturer. After receiving the
order, the manufacturer produces the products and delivers them to the retailer in
time. After the selling season, the uncertain-demand of customers is indeed real-
ized. During the sequence of decision making, all information is symmetrical, which
implies that both parties share all the same parameters and distributions. Here
we assume that the proportion of all consumers redeem the rebate successfully is
\( \beta (0 \leq \beta \leq 1) \), i.e., if \( n \) consumers buy the product, then the total number of redemption
is \( \beta n \), and the rebate cost for each unit product offered by the manufacturer
(retailer) will be \( \beta r_M (\beta r_R) \) ([7]). For expositional simplicity, we do not consider
the unit salvage value due to overstocking and unit shortage penalty incurred from
understocking associated with the on-sale products.

For any given rebate \( r_M \) from the manufacturer, the retailer’s objective is to
maximize the expected profit:

\[
\Pi_R(r_R, Q_i; r_M) = \max_{r_R, Q_i} \mathbb{E}[(p - \beta r_R)(D_i \wedge Q_i) - wQ_i], \ i \in \{e, l\}.
\]

\[2\text{The larger the value of } b, \text{ the more sensitive one percentage demand is to one percentage change in the effective-price } (p - \alpha r_R - \alpha r_M). \text{ If } b > 1, \text{ we say a product is defined as effective-price elastic to demand. If } b = 1, \text{ we say the demand is unit elastic. Otherwise, it is inelastic. Without loss of generality, we focus on the elastic product by assuming } b > 1.\]
Notice that \((p - \beta r_R)(D_i \wedge Q_i)\) represents the retailer’s revenue from the sale of \(\min(D_i, Q_i)\) units minus the redemption cost per unit by selling the product to the customers, while \(wQ_i\) represents the procurement cost of the retailer. Given the manufacturer’s rebate value \(r_M\), the retailer’s aim is to seek optimal rebate value \(r_R\) and order quantity \(Q_i\) to maximize his expected profit.

Similarly, for any given set of retailer’s rebate \(r_R\) and order quantity \(Q_i\), the manufacturer’s expected profit function is

\[
\Pi_M(r_M; r_R, Q_i) = \max_{r_M} E[(w - c)Q_i - \beta r_M(D_i \wedge Q_i)], \ i \in \{e, l\}.
\]

The term \((w - c)Q_i\) represents the revenue from wholesaling that the retailer has procured minus the marginal cost. The second part \(\beta r_M(D_i \wedge Q_i)\) denotes the total rebate payment incurred from manufacturer’s rebate promotion.

In the demand models, we assume the wholesale price \(w\) and the retail price \(p\) are both exogenous. This assumption is reasonable for the reason that the rebate promotion (such as online rebate or MIR) is usually implemented during a short selling period, and the manufacturer and retailer may maintain the list price. On the other hand, the price commitment between buyers and sellers is also widely observed in marketing practice, which guarantees the appeal of rebate promotion. The assumption is also consistent with [11], [14] and [25].

4. Optimal strategy under EMDF. In this Section, we first investigate the rebate strategies of the retailer and the manufacturer under EMDF. In this case, the retailer and manufacturer move simultaneously in offering the rebate to consumers. After both firms’ rebate strategies \((r_R, r_M)\) have been determined, the retailer decides the order quantity to maximize his expected profits. Here we use the standard technique of backward induction to solve the game, and first we derive the retailer’s order quantity \(Q_i\).

4.1. The decision of the retailer. For convenience, we define \(z_c = Q_c/y_c(r_R, r_M)\) as stocking factor, \(\Lambda(z_c) = \int_0^{z_c} (z_c - x)f(x)dx\) as expected leftover factor and \(\Theta(z_c) = \int_{z_c}^{\infty} (x - z_c)f(x)dx\) as expected shortage factor, respectively. This technique is common in the literature (e.g., Arcelus et al. [2], Petruzzi and Dada [31]). The stocking factor \(z_c\) represents the retailer’s service level to consumers in case of stocking out affected by the risk of demand uncertainty. Using this transformation, the order quantity \(Q_c\) can be substituted by \(z_c\). In the following analysis, we also denote \(y_c = y_c(r_R, r_M)\) for simplicity. Then we can rewrite the retailer’s expected profit function as

\[
\hat{\Pi}_R(r_R, z_c; r_M) = (p - \beta r_R - w)y_c\mu - wy_c\Lambda(z_c) - (p - \beta r_R - w)y_c\Theta(z_c).
\]

The term \((p - \beta r_R - w)y_c\mu\) is the expected benefits by satisfying the certainty-order quantity \(Q_c(= y_c\mu)\), which can be interpreted as the riskless profit. And
The retailer’s expected profit function

\[ \Pi_r(r, z; R) = \alpha(p - \beta r_R - w) + \Theta(z) \]

is the profit resulting from the risk of demand-uncertainty.

The retailer’s profit maximization problem with respect to \( z \) is a traditional newsvendor problem. After facilitating the first order condition, we obtain:

\[ \frac{\partial \Pi_r(r, z; R)}{\partial z} = a(p - \alpha r_R - \alpha r_M)(p - \beta r_R)(1 - F(z)) - w. \]  

(7)

By setting Equation (7) to zero, we derive the optimal response \( z_e \) determined by the following Lemma.

Equation (8) implies that the best response stocking factor is directly affected by the retailer’s rebate value. Obviously, the retailer prefers to reduce the risk of revenue loss by improving the service level (avoiding stocking out).

Taking the first derivative of Equation (6) with respect to the retailer’s rebate \( r_R \), then the retailer’s best response rebate can be determined by the following Lemma.

**Lemma 1.** Given the manufacturer \( R_M \) and the retailer’s stocking factor \( z_e \), the retailer’s optimal response rebate is uniquely determined by

\[ R_e(r_M, z_e) = \frac{(b \alpha - \beta) p - b \alpha w z_e / (z_e - \Lambda(z_e))}{(b - 1) \alpha \beta} + \frac{r_M}{b - 1}. \]  

(9)

The expression of \( R_e(r_M, z_e) \) includes two terms. Note that the first one consists of the stocking factor including the expected leftover \( \Lambda(z_e) \) associated with demand uncertainty, and the second one is a positive constant \( \frac{r_M}{b - 1} \) multiplying manufacturer’s rebate. From this equation we can explicitly observe the relationship among the optimal rebate of retailer, stocking factor and manufacturer’s rebate.

Next, we try to establish the existence of a pure-strategy Nash equilibrium by demonstrating that the retailer’s expected profit function (6) is quasi-concave in retailer’s rebate \( r_R \) for a given manufacturer’s rebate \( R_M \). Since \( r_R = [p - w / (1 - F(z_e))] / \beta \) from Equation (8), it follows that \( r_R \) is monotone in \( z_e \). To show the unimodality of \( \Pi_R(r, z_e; R_M), z_e \) in \( r_R \), it suffices to show that \( \Pi_R(r, z_e; R_M), z_e \) as a function of \( z_e \) is quasi-concave for a given \( R_M \). According to the procedure, we have the following results.

**Lemma 2.** Given the manufacturer’s rebate \( R_M \), then we have

(i) The retailer’s expected profit function \( \Pi_R(r, z_e; R_M), z_e \) is quasi-concave in the stocking factor \( z_e \) when \( F(\cdot) \) is a distribution function satisfying the following condition:

\[ \frac{b - 1}{b} \frac{r(z_e)}{1 - F(z_e)} \geq \frac{z_e F(z_e) - \Lambda(z_e)}{[z_e - \Lambda(z_e)]^2}, \]  

(10)

where most common distributions such as uniform, normal, exponential, weibull and gamma follow its validity.

(ii) If the condition for (i) is met and \( \alpha(w - \beta r_M) - (\alpha - \beta)p > 0 \), then there is a unique \( z_e^* \) to maximize the retailer’s expected profit; otherwise, a unique maximizer \( z_e^*(r_M) \) may not exist.

Under some certain conditions, Lemma 2 indicates that the retailer’s response stocking factor \( z_e^*(r_M) \) can be solved uniquely for any given manufacturer’s rebate.
value \( r_M \). From Equation \((8)\), one can conclude that the upper bound of \( z_e \) is \( F^{-1}(1-w/p) \) if the retailer does not offer a rebate, so the optimal response \( z^*_e(r_M) \) we yield in Lemma 2 must satisfy the constraint. This analysis is also similar to the additive demand case in Geng and Mallik [17]. Once the \( z^*_e(r_M) \) is determined for a fixed \( r_M \), then the retailer’s best response rebate value can be inferred by \( r^*(r_M) = \left[ p - w/(1 - F(z^*(r_M))) \right] \). The following results describe the game behavior of the retailer in response to the manufacturer.

**Corollary 1.** The retailer’s best response rebate \( \hat{r}_R(r_M, z_e) \) increases in the manufacturer’s rebate \( r_M \), while the retailer’s best response stocking factor \( \hat{z}_e \) decreases in the manufacturer’s rebate \( r_M \).

It implies that the retailer will increase the rebate value and reduce the service level as the manufacturer’s rebate increases. Geng and Mallik [17] study the similar rebate value for a fixed \( r_M \), then the retailer’s best response rebate value can be inferred by \( r^*(r_M) = \left[ p - w/(1 - F(z^*(r_M))) \right] \). The following results describe the game behavior of the retailer in response to the manufacturer.

**4.2. The Decision of the manufacturer.** Given a response rebate \( r_R \) from the retailer, the manufacturer simultaneously decide on the amount of the best response rebate value \( r_M \) to maximize his expected profit. Due to our assumption that all information is symmetric and complete, the manufacturer can infer the retailer’s stocking factor \( z_e(r_R) \) from Equation \((8)\). The manufacturer’s expected profit function can be simplified from Equation \((5)\) as:

\[
\hat{\Pi}_M(r_M; r_R, z_e(r_R)) = (w - \beta r_M - c) y_e \mu - (w - c) y_e \Lambda(z_e(r_R)) - (w - \beta r_M - c) y_e \Theta(z_e(r_R)).
\]

Similar to the retailer’s expected profit function, the structure of manufacturer’s expected profit function also consists of two terms. The term \((w - \beta r_M - c) y_e \mu\) denotes the riskless revenue by ordering deterministic quantity \( y_e \mu \) at the net price of \( w - \beta r_M - c \). \((w - c) y_e \Lambda(z_e(r_R))\) denotes the loss profit due to demand-uncertainty, and analogously the term \((w - \beta r_M - c) y \Theta(z_e(r_R))\) expresses the expected profit loss of shortages. Taking the first derivative of \(\hat{\Pi}_M(r_M; r_R, z_e(r_R))\) with respect to \( r_M \) and setting it to zero, we have the following results.

**Lemma 3.** (i) Given the retailer’s rebate value \( r_R \) and stocking factor \( z_e \), the manufacturer’s best response rebate \( \hat{r}_M(r_R, z_e) \) can be determined by

\[
\hat{r}_M(r_R) = \frac{bo(w - c)z_e(r_R)/[z_e(r_R) - \Lambda(z_e(r_R))] - \beta p}{(b - 1)\alpha \beta} + \frac{r_R}{b - 1}. \tag{11}
\]

(ii) Given the retailer’s stocking factor \( z_e \), then the manufacturer’s expected function can be rewritten as \(\hat{\Pi}_M(r_M; r_R(z_e), z_e)\), which is strictly concave in \( r_M(z_e) \). Furthermore, the unique \( \hat{r}_M \) is

\[
\hat{r}_M(z_e) = \frac{bo(w - c)z_e((z_e - \Lambda(z_e)) - \omega w/(1 - F(z_e)) + (\alpha - \beta)p}{(b - 1)\alpha \beta}. \tag{12}
\]
Lemma 3 gives the response of the manufacturer’s rebate strategy. In Lemma 3(ii), the rebate value is given in terms of the retailer’s rebate \( r_{R} \). However, because of the technical difficulty to solve \( \hat{r}_{M} \) in terms of \( r_{R} \), here we also invert \( z_{e} = z_{e}(r_{R}) \) to \( r_{R} = r_{R}(z_{e}) \) from Equation (8), i.e., the manufacturer’s best response rebate \( \hat{r}_{M}(z_{e}) \) can be rearranged by the stocking factor \( z_{e} \) in Lemma 3(ii).

**Corollary 2.** The manufacturer’s optimal response rebate \( \hat{r}_{M} \) decreases in the retailer’s stocking factor \( z_{e} \) and increases in the retailer’s rebate \( r_{R} \), respectively.

The above Corollary 2 states that, when the retailer decreases the service level or increases the magnitude of rebate, the manufacturer will increase his rebate value to consumers. As the retailer increases the stocking factor, which means that the retailer need to order more products to meet the customer’s uncertain demand as well as helping to earn higher market reputation and stimulate more demand, the manufacturer would like to shrink the size of rebate in order to reduce the rebate cost. On the other hand, if the retailer increases the rebate value, the selling price that the consumers paid goes down, which in return increases the product demand. As a result, the manufacturer will take a cooperative action by offering more rebates.

### 4.3. Equilibrium analysis.

According to the analysis of decisions from retailer and manufacturer, we can yield a Nash equilibrium by solving the best response equation group from Equation (8), (9), and (12) simultaneously. Thereby we transform the equilibrium of optimization problem over a single variable \( (z_{e}) \):

\[
- [(b - 2)w + c] \frac{z_{e}}{z_{e} - \Lambda(z_{e})} + \frac{(b - 2)w}{1 - F(z_{e})} + (1 - \beta/\alpha)p = 0. \tag{13}
\]

**Theorem 1.** For the optimal joint-rebate strategies under EMD case, the manufacturer will set a rebate value \( \hat{r}_{M}(z_{e}^{*}) \) in Lemma 3(ii), while the retailer will offer a rebate value \( \hat{r}_{R}(\hat{r}_{M}(z_{e}^{*}), z_{e}^{*}) \) in Lemma 1 and order \( Q_{e}^{*} = y(\hat{r}_{R}(z_{e}^{*}), z_{e}^{*}), \hat{r}_{M}(z_{e}^{*}))z_{e}^{*} \) units, where the \( z_{e}^{*} \) is determined by Equation (13). We have

(i) Suppose \( b \geq 2 \), if \( [(1 - \beta/\alpha)p]^{+} \leq c \leq \frac{b}{b - 2}w \), then there exists a unique Nash equilibrium.

(ii) Suppose \( 1 < b \leq 2 \), if \( (2 - b)w \leq c \leq [(1 - \beta/\alpha)p]^{+} \), then there exists a unique Nash equilibrium.

As mentioned above, Theorem 1 completely provides sufficient technical characterizations of the equilibrium in terms of \( z_{e} \) for the simultaneous game. The unique Nash equilibrium is represented by \( z_{e} \) in Equation (13). More specially, we give the equilibrium conditions in the whole feasible region as the price elasticity parameter \( b \) is highly sensitive \( (b \geq 2) \) or not \( (1 < b < 2) \). However, if the corresponding conditions are relaxed, there may not exist an equilibrium. On the one hand, Theorem 1(i) indicates that, the unique Nash equilibrium exists when the manufacturer’s unit production cost is binding with constraints in terms of retail price and wholesale price, i.e., the wholesale price is high and the retailer’s selling price is very small for any given production cost. Intuitively, when the price elasticity is sufficiently high, the condition gives the manufacturer more incentive to offer rebates, and it stimulates the demand and benefits both firms. However, it is may also increase the burden of rebate cost. Theorem 1(ii) characterizes that when the rebate elasticity is rather small, there also exists a unique Nash Equilibrium. The condition yields a higher retail price and a lower wholesale price, which gives more incentive for retailer to offer rebates. As a result, both firms have make a trade-off decision between the rebate-promotion and rebate-payment. Moreover, to insure the term
Furthermore, the retailer’s best response order quantity is $Q^*$. The reason is that both firms prefer $\alpha$ to be high and $\beta$ to be low, resulting in a higher marginal profit for each firm. This is also identical to the study of Aydin et al. [7], in which they find that the total supply chain profit always improves under consumer rebate when the sensitive parameter ($\alpha$) is larger than the redemption rate ($\beta$), and the total supply chain profit may suffer if $\alpha \leq \beta$.

**Corollary 3.** Under the game of EMDF, the retailer and manufacturer share the same expected profit at equilibrium.

Interestingly, we find that the retailer and manufacturer share the same profit at equilibrium under our model assumption. This counter-intuitive finding implies that when the two firms are involved in the simultaneous game, each firm will adjust the rebate strategy to the other’s response. Consequently, they bear the burden of rebate promotion cost and finally divide the total supply chain profit equally. We further illustrate this result in Section 4.4 with numerical studies.

Note that we have discussed the joint optimal rebate strategies offered by the retailer and manufacturer in the game, one interesting question is that whether a firm should launch rebate promotion and how to make decisions for the rebate value. we will further shed light on the equilibrium outcome under the exclusive game framework. First we characterize the results if only the retailer offers rebate to consumers.

**Lemma 4.** (i) If $(b\alpha - \beta)(z^*_e - \Lambda(z^*_e))p - b\alpha w z^*_e > 0$ and $[(b - 2)w - (b - 1)c]z^*_e - \Lambda(z^*_e) + (1 - \hat{\frac{\beta}{\alpha}})p \leq 0$, then only the retailer launch a rebate program to consumers, where $z^*_e$ is uniquely determined by the following equation

$$\frac{b}{b - 1 - \frac{z_e}{\Lambda(z_e)} - 1 - F(z_e)} - \frac{(\alpha - \beta)p}{(b - 1)\alpha \beta} = 0.$$  

(ii) The best response rebate $\hat{r}_e$ that the retailer sets is

$$\hat{r}_e = \frac{(b\alpha - \beta)p - b\alpha w z^*_e/(z^*_e - \Lambda(z^*_e))}{(b - 1)\alpha \beta}.$$  

Furthermore, the retailer’s best response order quantity is $Q^*_e = ye(\hat{r}_e, 0)z^*_e$.

In this special case, the manufacturer will not provide any rebates to consumers such that $\hat{r}_m = 0$. From the certain conditions, an intuitive finding is that the retailer is more willing to use rebate promotion if the retail price is higher and the wholesale price is lower. The following lemma summarizes the results if only the manufacturer offers rebate to consumers.

**Lemma 5.** (i) If $b\alpha(w - c) - \beta p > 0$ and $[(2 - b)w - c]z^*_e - \Lambda(z^*_e) - \frac{\beta}{\alpha} + b - 1 \leq 0$, then only the manufacturer offers a rebate program to end consumers, where $z^*_e = F^{-1}(1 - w/p)$.

(ii) The best response rebate $\hat{r}_m$ that the manufacturer sets at equilibrium is

$$\hat{r}_m = \frac{b\alpha(w - c)z^*_e/[z^*_e - \Lambda(z^*_e)] - \beta p}{(b - 1)\alpha \beta},$$  

furthermore, the retailer’s best response order quantity is $Q^*_e = ye(0, \hat{r}_m)z^*_e$.

Lemma 5 implies that the manufacturer will offer consumer rebate if the margin $w - c$ and rebate effectiveness $\alpha$ is relatively high, but the retail price $p$ and redemption rate $\beta$ is relatively low. Because the higher rebate effectiveness $\alpha$ allows
the manufacturer to offer more rebates, which in turn results in more consumer demand. Further, a lower redemption rate $\beta$ brings more additional profit for the manufacturer. In the timing of this event, the retailer’s strategy is only to make a decision for order quantity $Q^*_e$ in response to the manufacturer’s rebate policy, and the manufacturer’s strategy is to choose the rebate $\hat{r}_M$ in response to retailer’s order quantity.

4.4. Numerical examples. To gain further insights, we study the impact of price elasticity (PE) on the optimal rebate strategy and expected profit outcomes for both firms. Based on the analytical results in Theorem 1, we use the following data: (i) when $1 < b < 2$, $a = 10, c = 3.5, w = 4, p = 8, \epsilon \sim U[0, 50], \alpha = 0.9, \beta = 0.5$; (ii) when $b \geq 2$, $a = 10, c = 3, w = 10, p = 15, \epsilon \sim U[0, 50], \alpha = 0.9, \beta = 0.8$.

From Figure 2, it indicates that the retailer and manufacturer take a similar rebate strategy when the price elasticity changes. First, the optimal rebate values decrease as PE increase if $b < 2$, whereas they will simultaneous increase the rebate value if $b \geq 2$. When PE is very small, the consumers are not sensitive to the rebate, i.e., increasing the amount of rebate cannot contribute much sales to the demand. In this case, it is beneficial to reduce the rebates. Furthermore, as PE becomes larger($b \geq 2$), the consumers are highly sensitive to the rebate promotion, and increasing the rebate value can further stimulate the much demand. As a result, both firms have the incentive to offer more rebates. However, the strategies under the two scenarios result in different impact on the expected profit. The associated profit first increases in PE if $1 < b < 2$, then decreases in PE if $b \geq 2$. The underlying reason is that, as PE changes, rebate promotion is more effective to increase profit in the former case. However, if $b \geq 2$, the profit loss resulting from the rebate cost dominates the rebate promotion, and the profit loss increases as PE gets bigger.
Is the rebate policy always favorable? To answer this question, we also give a basic model if there is not any rebate promotion in the supply chain. The corresponding data is similar to the numerical experiment in Figure 2. Figure 3 depicts the profit outcomes at equilibrium under the models with or without rebate policy when the PE changes. For notational convenience, we denote the retailer’s (manufacturer’s) profit under the case that no parties offer any rebates by R (M), while the retailer’s (manufacturer’s) profit under joint-rebate is R,M. Interestingly, our examples show that the rebate policy is not always profitable for the manufacturer or retailer. When the PE is rather small, the basic model may dominate the model of rebate policy, while the latter will dominate the former as the PE gets larger. The result is intuitive. Lower PE means that sales cannot be effectively stimulated by the rebates, it is profitable for sellers to shrink the rebate scale. As a matter of fact, the best choice is not to launch the rebate promotion. However, when the PE gets bigger, rebates take a significant and positive effect on the demand expansion, which means that the profit improved by additional sales will dominate the profit loss incurred by rebate cost. As a result, the expected profit with joint rebate policy for both parties is higher than that of without rebate policy.

Figure 4 compares the equilibrium profits of the only-retailer-rebate game and simultaneous rebate game. Similarly, here the retailer’s (manufacturer’s) profit under the only-retailer-rebate case is denoted by R (M), and the profit under the joint-rebate policy for both firms is M,R. It illustrates that the simultaneous rebate game results in a higher expected profit than the only-retailer-rebate game when PE is sufficiently large.

Figure 5 compares the equilibrium profits under only-manufacturer-rebate policy and the simultaneous rebate policy with different values of PE, where the capital R (M) denotes the retailer’s (manufacturer’s) profit under only-manufacturer-rebate policy, and R,M denotes the their profits under joint-rebate policy, respectively. we can see that firms’ profit goes up as PE increases in the region (1, 2). As the PE gets bigger, the simultaneous rebate game always dominates the only-manufacturer-rebate game. This phenomenon indicates that in the game of simultaneous rebate policy, the two firms earn a higher profit margin by sharing responsibility for the rebate cost in stimulating sales. Further, if only the manufacturer offers rebate, the retailer’s expected profit is much more than that of the manufacturer.
5. Optimal strategy under LMDF. In this section, we investigate the optimal rebate strategies under the LMDF. Similarly, we define a stocking factor $z_l = Q_l/y_l$ by substituting $z_l$ with $Q_l$. Here we use the same modeling framework to analyze some useful properties and results under LMDF.

Lemma 6. Under the scenario of LMDF, then we have

(i) Given the manufacturer’s rebate value $r_M$, if $b(\alpha - \beta)p - b\alpha(w + \beta r_M) + \beta a > 0$, then the retailer has a unique response $z_l^*(r_M)$ to maximize the expected profit, which is determined by the following equation:

$$
\frac{z_l}{2(z_l - \Lambda(z_l))} - \frac{1}{1 - F(z_l)} + \frac{b(\alpha - \beta)p - b\alpha r_M + \beta a}{2b\alpha w} = 0,
$$

(14)

(ii) Given the retailer’s stocking factor $z_l$, the manufacturer has a unique best response $\tilde{r}_M$ to maximize his expected profit as follows:

$$
\tilde{r}_M(z_l) = \frac{b\alpha(w - c)z_l/ (z_l - \Lambda(z_l)) - b\alpha(p - w/(1 - F(z))) - \beta(a - bp)}{2b\alpha\beta}.
$$

(15)
Lemma 6 gives the outcomes of the game under the scenario of LMDF. We find the results described above have a similar structure with EMDF.

**Corollary 4.** Under the scenario of LMDF,

(i) The retailer’s best response rebate $\tilde{r}_R(r_M, z_l)$ decreases in the manufacturer’s rebate $r_M$ and stocking factor $z_l$ increases in the manufacturer’s rebate $r_M$, respectively.

(ii) The manufacturer’s best response rebate $\tilde{r}_M(r_R, z_l)$ increases in the retailer’s stocking factor $z_l$ decreases in the retailer’s rebate $r_R$, respectively.

(iii) The retailer and manufacturer share the same expected profit at equilibrium.

The above corollary characterizes the best response for the retailer and manufacturer. In contrast to EMDF, it shows different behaviors. When the manufacturer increases his rebate value to end consumers, it stimulates the additional sales via a perceived lower price than the list price. In this case, the retailer increases his stocking factor to improve the service level in the presence of demand uncertainty. The best response of the manufacturer is the same as the retailer. Similar to the case of EMDF, the two firms may offer different rebate value but both earn the same expected profit at equilibrium.

**Theorem 2.** Given the manufacturer’s rebate value $r_R$ and retailer’s rebate value $r_M$, then there is a unique Nash equilibrium in terms of $z_l$ that determined by the following equation:

$$
\frac{3w - c}{3w - z_l - \Lambda(z_l)} = \frac{1}{1 - F(z_l)} + \frac{b_\alpha p + \beta(a - bp)}{3bw} = 0.
$$

Similar to Theorem 1, we can also derive an equilibrium by solving the above equation. Our proof demonstrates that there still exists a unique Nash equilibrium under LMDF. Compared to EMDF, it does not require any additional constraints. Moreover, we also characterize the outcomes when only one firm offers rebate policy.

**Lemma 7.** (i) If $b_\alpha(w - c) - \beta(a - bp) > 0$ and $2b_\alpha p - \beta(a - bp) - b_\alpha(3w - c) \frac{z_l}{z_l - \Lambda(z_l)} \leq 0$, then only the manufacturer offers a rebate program to end consumers, where $z_l^* = F^{-1}(1 - w/p)$.

(ii) The best response rebate $\tilde{r}_M$ that the manufacturer sets at equilibrium is

$$
\tilde{r}_M = \frac{b_\alpha(w - c)z_l^*/(z_l - \Lambda(z_l^*)) - \beta(a - bp)}{2b_\alpha \beta},
$$

and the retailer’s best response order quantity is $Q_1^* = y_l(0, \tilde{r}_M)z_l^*$.

**Lemma 8.** (i) If $(b_\alpha p + b_\beta p - \beta a)(z_l^* - \Lambda(z_l^*)) - b_\alpha wz_l^* > 0$ and $2b_\alpha(2w - c)z_l^* - (b_\alpha p + b_\beta p + \beta a)(z_l^* - \Lambda(z_l^*)) \leq 0$, then only the retailer offers a rebate program to end consumers, where $z_l^*$ is uniquely determined by the following equation

$$
\frac{z_l}{2[z_l - \Lambda(z_l) - 1]} = \frac{1}{1 - F(z_l)} + \frac{b_\alpha p + \beta(a - bp)}{2bw} = 0.
$$

(ii) The best response rebate $\tilde{r}_R$ that the retailer sets is

$$
\tilde{r}_R = \frac{b_\alpha(p - w z_l^*/(z_l^* - \Lambda(z_l^*))) - \beta(a - bp)}{2b_\alpha \beta},
$$

and the retailer’s best response order quantity is $Q_1^* = y(\tilde{r}_R, 0)z_l^*$.
From Lemma 7 and Lemma 8, one of the intuitive findings is that when the wholesale price is high and retail price is relatively low, the manufacturer is more likely to offer rebate while the retailer would not. This is because when the wholesale price is high, the manufacturer earns a higher profit margin so that he is willing to offer rebate for the purpose of stimulating more demand. Through our numerical examples, we also compare the equilibrium outcomes between joint-rebate policy and single-party rebate policy. The results are similar to the case under EMDF. In particular, the former at least dominates the latter. For brevity, here we omit the detail analysis and discussion.

6. Conclusions. Rebate is one of the most effective strategies to generate additional demand in operations management and marketing management. In this paper, following Geng and Mallik [17], we consider a two-echelon supply chain consisting a single manufacturer and retailer and both firms may laugh rebate promotion to stimulate sales in their best interests. First, their work considers the rebate strategies when facing market demand with high price-sensitive consumers, i.e., the price elasticity $b \geq 2$. But they do not address the rebate strategies when consumers is not very sensitive to the price, namely, $1 < b < 2$. Second, Geng and Mallik [17] restrict discussions to a stochastic nonlinear demand function based on EMDF only. However, the linear demand function with multiplicative error has also been widely investigated in the literature. To address the gap, we extend it from the case with high price elasticity $b \geq 2$ to a more general case with $b > 1$. Moreover, in addition to the stochastic nonlinear demand function based on EMDF, we also include the discussion based on a linear price-dependent demand function, LMDF. In our model setting, we shed light on the rebate response behaviors of the retailer and manufacturer using a game framework.

Our analysis shows that there still exists a unique equilibrium when the price elasticity is rather small ($1 < b < 2$), similar to the case with $b \geq 2$. We compare the optimal rebate strategies both with high price elasticity and relatively small price elasticity. We further prove that the retailer and manufacturer share the same expected profit at equilibrium. We characterize some structural properties and also prove the uniqueness and existence of the Nash equilibrium under the exclusive game scenarios, including only-manufacturer-rebate policy and only-retailer rebate-policy. Our numerical examples find that the joint-rebate policy results in both higher expected profit comparing to the single party rebate policy when $b \geq 2$, but the result is not always hold when $1 < b < 2$. The reason is that the rebate provided by the manufacturer and the retailer cannot increase the “perfect” demand which generates more profit(profit loss due to rebate cost) when the consumers are not highly sensitive to the rebates. In addition, when the demand elasticity increases on the interval $b < 2$, the manufacturer and retailer will simultaneously reduce their rebate value, whereas they would like to reduce the amount of rebate as it increases on the interval $b \geq 2$. To gain more insights, we further compare the expected profit at equilibrium among the different scenarios, and results show that the simultaneous rebate policy dominates the others when the price elasticity is sufficiently flexible.

In our paper, we study the pure joint-rebate strategies between a single manufacturer and a single retailer under a decentralized supply chain. How to coordinate the supply chain between the manufacturer and retailer using the rebate promotion programs is probably another interesting research direction. Further, we currently
tackle our model setting in the context of a supply chain involving a single-product-single-period framework. Extending it to a multi-product-multi-period one may become another future challenge research.

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Appendix. Proof of Lemma 1. Recall that the retailer’s expected profit function in Equation (7), we have \( \Pi_R(r, z_R; r_M) = (p - \alpha r_R - \alpha r_M)^{-b}[(p - \beta r_R)(z_e - \Lambda(z_e)) - w z_e] \). Taking the first order derivative of \( \Pi_R(r, z_R; r_M) \) with respect to \( r_R \) and setting it to zero, we have the result of (9). The second order derivatives of \( \Pi_R \) with respect to \( r_R \) when \( \frac{\partial \Pi_R}{\partial r_R} = 0 \) is

\[
\frac{\partial^2 \Pi_R}{\partial r_R^2} \bigg|_{\partial r_R = 0} = ab(b + 1)a^2(p - \alpha r_R - \alpha r_M)^{-b-2}[(p - \beta r_R)(z_e - \Lambda(z_e)) - w z_e] \\
+ 2ab(a - \alpha r_R - \alpha r_M)^{-b-1}[-\beta(z_e - \Lambda(z_e))] \\
= ab(b + 1)
\]

If \( b > 1 \), from the above inequality, then \( \Pi_R(r_R, z; r_M) \) is concave with respect to \( r_R \). Therefore, for any given \( z_e \) and \( r_M \), the result that we obtain in (9) is the retailer’s best response rebate.

\[
\frac{\partial \Pi_R}{\partial z_e} (r, r_R, z_e; r_M) = aw(1 - F(z))(p - \alpha r_R - \alpha r_M)^{-b}L(z_e),
\]

where \( L(z_e) = \frac{b}{b - 1} \frac{z_e}{1 - \Lambda(z_e)} - \frac{1}{1 - F(z)} - \frac{(a - \beta)p + \beta r_M}{(b - 1)aw} \). Due to the left part \( aw(1 - F(z))(p - \alpha r_R - \alpha r_M)^{-b} > 0 \), so if \( \frac{\partial \Pi_R}{\partial z_e}(r_R, z_e, r_M) / \partial z_e = 0 \), then we have \( L(z) = 0 \). The second order derivatives of \( \Pi_R \) with respect to \( z_e \) when \( \frac{\partial \Pi_R}{\partial z_e} = 0 \) is

\[
\frac{\partial^2 \Pi_R}{\partial z_e^2} \bigg|_{\partial z_e = 0} = aw(1 - F(z))(p - \alpha r_R - \alpha r_M)^{-b}L(z_e),
\]

If \( b - 1 > \frac{r(z_e)}{1 - F(z)} - \frac{zF(z_e) - \Lambda(z_e)}{(z_e - \Lambda(z_e))^2} \) > 0, then the above equation is strictly negative, therefore the best response \( z^*_e \) satisfies \( \frac{\partial \Pi_R}{\partial z_e}(r_R, z_e, r_M) / \partial z_e = 0 \). To prove the condition in Lemma 2(ii) is tenable for most common distributions, here we construct a new function

\[
R(z_e) = \frac{b}{b - 1} \frac{z_e}{1 - \Lambda(z_e)} - \frac{1}{1 - F(z_e)}.
\]

Geng and Mallik [17] has proved that \( R(z_e) \) decreases with \( z_e \) when \( b \geq 2 \). More generally, \( R(z_e) \) decreases in \( z_e \) when \( 1 < b < 2 \), and many commonly stochastic distributions including uniform, normal, exponential, gamma, weibull exhibit this property. To overcome the analytical difficulty, similar to Khouja [22], we test
probability density function is

\[ f(z_e) = \frac{1}{\lambda}, \]

and the related cumulative distribution function is

\[ F(z_e) = \frac{z_e}{\lambda}, \]

Thus, the hazard rate \( r(z_e) = f(z_e)/(1 - F(z_e)) = \frac{1}{\lambda - z_e}. \)

Under this case, \( R(z_e) \) is defined by

\[ R(z_e) = \frac{b}{b - 1} \frac{z_e}{z_e - \frac{z_e^2}{2\lambda}} - \frac{1}{1 - \frac{z_e}{\lambda}} = \frac{b}{b - 1} \frac{2\lambda - z_e}{2\lambda - z_e - \lambda - z_e}. \]

Differentiating \( R(z_e) \) with respect to stocking factor \( z_e \) yields

\[ \frac{dR(z_e)}{dz_e} = \frac{2\lambda}{b - 1} \frac{(2\lambda - z_e)^2 - (\lambda - z_e)^2}{(\lambda - z_e)^2}. \]

Then, it follows that

\[ \frac{d^2R(z_e)}{dz_e^2} \bigg|_{dz_e=0} = \frac{b}{b - 1} \frac{4\lambda}{(2\lambda - z_e)^3} - \frac{2\lambda}{(\lambda - z_e)^2} = \frac{2b}{b - 1} \frac{2\lambda - z_e}{b - 1} \frac{dR(z_e)}{dz_e} + \frac{\lambda}{(\lambda - z_e)^2} - \frac{2\lambda}{(\lambda - z_e)^3} = \frac{2\lambda}{(\lambda - z_e)^2} \frac{(2\lambda - z_e)(\lambda - z_e)^2 - 2\lambda - z_e}{2\lambda - z_e} dR(z_e) = \frac{2\lambda}{(\lambda - z_e)^2} (2\lambda - z_e)(\lambda - z_e) < 0. \]

So \( R(z_e) \) is quasi-concave in the stocking factor \( z_e \). Specially, \( \frac{dR(z_e)}{dz_e} \bigg|_{z_e=0} = -1 < 0 \), thus \( R'(z_e) < 0 \), i.e., \( R(z_e) \) decreases in terms of \( z_e \), hence,

\[ \frac{dR(z_e)}{dz_e} = \frac{b}{b - 1} \frac{zF(z_e) - \Lambda(z_e)}{[z_e - \Lambda(z_e)]^2} - \frac{r(z_e)}{1 - F(z_e)} < 0, \]

which can be simplified as

\[ \frac{b - 1}{b - 1} \frac{r(z_e)}{1 - F(z_e)} > z_eF(z_e) - \Lambda(z_e) \frac{z_e - \Lambda(z_e)}{[z_e - \Lambda(z_e)]^2}. \]

Hence, \( \Pi_R(\alpha_R, \beta, \gamma) \) is quasi-concave in \( z_e \). Note that if \( \alpha(w - \beta r_M) - (\alpha - \beta)p > 0 \), \( L(0^+) = \frac{1}{b - 1} - \frac{(\alpha - \beta)p + \alpha \beta r_M}{(b - 1)\alpha w} > 0 \), together with that \( L(+\infty) < 0 \), then there is a unique maximizer \( z_e^* \) that satisfies \( \partial\Pi_R(\alpha_R, \beta, \gamma, z_e, \gamma)/\partial z_e = 0. \)

**Proof of Corollary 1.** (i) According to Lemma 2, recall that

\[ L(z_e) = \frac{b}{b - 1} \frac{z_e}{z_e - \Lambda(z_e)} - \frac{1}{1 - F(z)} - \frac{(\alpha - \beta)p + \alpha \beta r_M}{(b - 1)\alpha w}. \]

Notice that

\[ \frac{\partial L(z_e^*)}{\partial z_e^*} = \frac{\partial [\frac{b}{b - 1} \frac{z_e^*}{z_e^* - \Lambda(z_e^*)} - \frac{1}{1 - F(z_e^*)}]}{\partial z_e^*} < 0, \]

since \( \frac{\partial L(z_e^*)}{\partial r_M} = -\frac{\beta}{(b - 1)\alpha w} < 0 \), it follows that \( \frac{dz_e^*}{dr_M} = \frac{\partial L(z_e^*)}{\partial z_e^*} \frac{\partial z_e^*}{\partial r_M} < 0 \), thus the retailer’s best response stocking factor decreases in the manufacturer’s rebate. Furthermore, note that \( \frac{dr_R}{dr_M} < 0 \) and we have \( \frac{dr_R}{dr_M} = \frac{dz_e^*}{dz_e^*} \frac{dz_e^*}{dr_M} > 0 \), which indicates that the retailer’s optimal rebate increases in manufacturer’s given rebate. \( \square \)
Figure 6. The monotone decreasing trend of $R(z_e)$ with different distributions ($1 < b < 2$)

Proof of Lemma 3. Similar to retailer’s decision, we have the formula-transform of manufacturer’s expected profit,

$$\hat{\Pi}_M(r_R, z_e(r_R); r_M) = a(p - \alpha r_R - \alpha r_M)^{-b}[(w - c)z_e(r_R) - \beta r_M(z_e(r_R) - \Lambda(z_e(r_R))$$]

Taking the first-order derivative of $\hat{\Pi}_M$ with respect to $r_M$ and Setting it to zero yields

$$\hat{r}_M(r_R) = \frac{b\alpha(w - c)z_e(r_R) / [z_e(r_R) - \Lambda(z_e(r_R))] - \beta p}{b - 1} + \frac{r_R}{b - 1}.$$  

Rewriting $r_R$ in terms of $z_e$ from Equation (8), we obtain

$$\hat{r}_M(z_e) = \frac{b\alpha(w - c)z_e / (z_e - \Lambda(z_e)) - \alpha / (1 - F(z_e)) + (\alpha - \beta)p}{(b - 1)\alpha\beta}.$$  

Now we prove that $\hat{\Pi}_M$ is quasi-concave in $r_M$. For ease of calculation, we transform the manufacturer’s profit function as

$$\hat{\Pi}_M(r_M; r_R, z_e) = y(r_M; r_R, z_e)L(r_M; r_R, z_e),$$  

where

$$y(r_M; r_R, z_e) = a(p - \alpha r_R - \alpha r_M)^{-b}, \quad L(r_M; r_R, z_e) = (w - c)z_e - \beta r_M(z_e - \Lambda(z_e)).$$
Taking the first order derivative of $\hat{\Pi}_M(r_M; r_R, z_e(r_R))$ with respect to manufacturer’s rebate value $r_M$ yields

$$\frac{\partial \hat{\Pi}_M(r_M; r_R, z_e(r_R))}{\partial r_M} = \frac{\partial y(r_M; r_R, z_e(r_R))}{\partial r_M} L(r_M; r_R, z_e(r_R)) + y(r_M; r_R, z_e(r_R)) \frac{\partial L(r_M)}{\partial r_M}$$

$$= ab(p - \alpha r_R - \alpha r_M)^{-b\beta}[(w - c)z_e(r_R) - \beta r_M(z_e(r_R) - \Lambda(z_e(r_R)))]$$

$$+ a(p - \alpha r_R - \alpha r_M)^{-b\beta}[-\beta(z_e(r_R) - \Lambda(z_e(r_R)))]$$

Taking the second order partial derivative of $\hat{\Pi}_M(r_M; r_R, z_e(r_R))$ with respect to $r_M$ yields

$$\frac{\partial^2 \hat{\Pi}_M}{\partial r_M^2} = \frac{\partial^2 y}{\partial r_M^2} L(r_M; r_R, z_e) + 2 \frac{\partial y(r_M; r_R, z_e, z_e)}{\partial r_M} \frac{\partial L(r_M; r_R, z_e,r_e)}{\partial r_M} + y(r_M; r_R, z_e) \frac{\partial^2 L}{\partial r_M^2}$$

$$= ab(b+1)\alpha^2(p - \alpha r_R - \alpha r_M)^{-b\beta}[(w - c)z_e(r_R) - \beta r_M(z_e(r_R) - \Lambda(z_e(r_R)))]$$

$$+ 2\alpha^2(b+1)b^{-1}[(w - c)z_e(r_R) - \beta r_M(z_e(r_R) - \Lambda(z_e(r_R)))]$$

$$+ 2\alpha^2(b+1)b^{-1}[(w - c)z_e(r_R) - \beta r_M(z_e(r_R) - \Lambda(z_e(r_R)))]$$

$$= (1-b)\alpha^2(p - \alpha r_R - \alpha r_M)^{-b\beta} \hat{\Pi}_M(r_M; r_R, z_e(r_R)) + 2\alpha^2(b+1)b^{-1} \frac{\partial \hat{\Pi}_M}{\partial r_M}$$

$$\Rightarrow \frac{\partial^2 \hat{\Pi}_M(r_M; r_R, z_e(r_R))}{\partial r_M^2} \bigg|_{b_M^*} = 0 \text{ when } b > 1.$$ 

So the manufacturer’s expected profit function $\hat{\Pi}_M(r_M; r_R, z_e(r_R))$ is quasi-concave in $r_M$. The critical fraction solution $r_M^*(z_e)$ obtained by setting the first order condition to zero is a unique one that maximizes the manufacturer’s expected profit. \(\square\)

**Proof of Corollary 3.** We only need to prove the case of $1 < b < 2$. From Lemma 3, we have

$$\hat{r}_M(z_e) = \frac{1}{(b-1)\beta} \left( \frac{w - c}{w} - \frac{bz}{(b-1)(z_e - \Lambda(z_e))} \right) - \frac{1}{1 - F(z_e)} + \frac{(\alpha - \beta)p}{(b-1)\alpha \beta},$$

when $1 < b < 2$, leading to

$$\frac{d}{dz_e} \left[ \frac{w - c}{w} - \frac{bz}{(b-1)(z_e - \Lambda(z_e))} \right] - \frac{1}{1 - F(z_e)} \right| < \frac{d}{dz_e} \left[ \frac{b}{b-1} \frac{z_e}{z_e - \Lambda(z_e)} - \frac{1}{1 - F(z_e)} \right] < 0.$$ 

Thus, $d^r_M/dz_e < 0$, from Equation (8), $dz_e/dr_R < 0$, thus $\frac{dr_M}{dz_e} > 0$. \(\square\)

**Proof of Theorem 1.** When $b \geq 2$, it has been proved by Geng and Mallik(2011). (ii) When $1 < b < 2$, let

$$R(z_e) = (2 - b)w[(1 + \frac{c}{(b-2)w}) \frac{z_e}{z_e - \Lambda(z_e)} - \frac{1}{1 - F(z_e)}] + (1 - \beta / \alpha)p$$

We can derive that $R(0^+) = (2 - b)w \frac{z_e}{(b-2)w} + (1 - \beta / \alpha)p = (1 - \beta / \alpha)p - c$, $R(+\infty) = -c(0^+)$. If $(1 - \beta / \alpha)p - c > 0$, $R(0^+) > 0$, the equation of Theorem 1 has at least one solution.
Rearranging the equation of $R(z)$, we obtain
\[
\phi(z_e) = \left(1 + \frac{c}{(b-2)w} \right) \frac{z_e}{z_e - \Lambda(z_e)} - \frac{1}{1 - F(z_e)}.
\]
Define:
\[
g(z_e) = \frac{z_e}{z_e - \Lambda(z_e)}, \quad h(z_e) = \frac{1}{1 - F(z_e)},
\]
\[
r(z_e) = \frac{f(z_e)}{1 - F(z_e)}, \quad \Phi(z_e) = z_e - \Lambda(z_e).
\]
Since $\Phi'(z_e) = 1 - F(z_e) \geq 0$, $\Phi(A) = A \geq 0$, so $\Phi(z_e) \geq 0$. Also, we have $\frac{dr(z_e)}{dz_e} > 0$ due to IFR. Taking the first and second derivatives of $\phi(z_e)$ with respect to $z_e$, respectively, we obtain
\[
\frac{d\phi(z_e)}{dz_e} = (1 + \frac{c}{(b-2)w}) \frac{dg(z_e)}{dz_e} - \frac{r(z_e)}{1 - F(z_e)},
\]
\[
\frac{d^2\phi(z_e)}{dz_e^2} = (1 + \frac{c}{(b-2)w}) \left( \frac{z_ef(z_e)}{(z_e - \Lambda(z_e))^2} - \frac{2(1 - F(z_e))}{z_e - \Lambda(z_e)} \right) - \frac{dr(z_e)/dz_e + r^2(z_e)}{1 - F(z_e)}.
\]
Let $\frac{d\phi}{dz_e} = 0$, we have
\[
\frac{d^2\phi(z_e)}{dz_e^2} = \frac{(b-2)w + c}{(b-2)w + c} \frac{r(z_e)}{1 - F(z_e)},
\]
Then we can get
\[
\frac{d^2\phi(z_e)}{dz_e^2} = \frac{(b-2)w + c}{(b-2)w} \frac{r(z_e)}{1 - F(z_e)} - \frac{dr(z_e)/dz_e + r^2(z_e)}{1 - F(z_e)}.
\]
If $(b-2)w + c < 0$, and $1 < b < 2$, it must satisfy $(b-2)w > 0$, i.e., $c \geq (2-b)w$. It means that $\Phi(z_e)$ is quasi-concave in $z_e$ as the above conditions are met. Note that $R(+\infty) < 0$, thus a sufficient condition of $R(z_e)$ satisfying the unimodality is $R(0^+) > 0$. Therefore, we can make a conclusion that the root $z_e^*$ of equilibrium equation is both existent and unique (when $1 < b < 2$), depending the following condition $(2-b)w \leq c < [(1-\beta/\alpha)p]_+^+$. Particularly, if $b = 2$, $\phi(z_e)$ can be rewritten as
\[
\phi(z_e) = -\frac{cz}{z_e - \Lambda}\left(1 - \frac{1}{\beta/\alpha}\right)p = 0, \text{ and } \frac{d\phi(z_e)}{dz_e} = \frac{z_e F(z_e) - \Lambda(z_e)}{(z_e - \Lambda(z_e))^2}.
\]
It is easy to prove $\frac{d\phi(z_e)}{dz_e} < 0$ and $\phi(+\infty) < 0$. Thus, $\phi(z_e)$ strictly decreases in $z_e$. To ensure the unique solution $z_e^*$ exists, it requires that $\phi(0^+) > 0$, i.e., $c < [(1-\beta/\alpha)p]_+^+$, thus $b = 2$ only satisfies the case of $1 < b < 2$. □

**Proof of Corollary 4.** To show $\hat{\Pi}_H(r_R^*; z_e^*; r_M^*) = \hat{\Pi}_M(r_M^*; r_R^*; z_e^*)$, we need to prove
\[
\hat{\Pi}_R(r_R^*(z_e^*), z_e^*; r_M^*(z_e^*)) = \hat{\Pi}_M(r_M^*(z_e^*); r_R^*(z_e^*), z_e^*),
\]
which is equivalent to $(p - \beta r_R^*(z_e))(z_e - \Lambda(z_e)) - wz = (w-c)z_e - \beta r_M^*(z_e)(z_e - \Lambda(z_e))$, or equivalent to
\[
p - \beta r_R^*(z_e) + \beta r_M^*(z_e) = \frac{(2w - c)z_e}{z_e - \Lambda(z_e)}.
\]
Substituting the response function $\hat{r}_R(z_e)$ and $\hat{r}_M(z_e)$ into the above equation, we obtain
\[
-[(b-2)w + c] \frac{z_e}{z_e - \Lambda(z_e)} + \frac{(b-2)w}{1 - F(z_e)} + (1 - \beta/\alpha)p = 0,
\]
which is true due to the equilibrium from Equation (13).

Proof of Lemma 4. If the manufacturer does not offer a rebate program, i.e., \( r_M = 0 \), substitute it into the retailer’s rebate response function (Lemma 1), we obtain the optimal rebate value \( \hat{r}_R(z_e) = \frac{(b\alpha - \beta)p - \beta dwz_e}{(b-1)\alpha \beta} \). To insure the retailer offer a positive rebate, it needs \( r^*_{R}(z_e) \), i.e., \( (b\alpha - \beta)(z_e^* - \Lambda(z_e^*))p - \beta dwz_e^* > 0 \). Substituting the value of \( \hat{r}_R(z_e) \) into the manufacturer’s rebate response function (Lemma 3) and let \( \hat{r}_M(r_M, z_e) \leq 0 \), which yields the other constraint \( [(b-2)w - (b-1)c]z_e^2 - \Lambda(z_e) + (1-\frac{\beta}{\alpha})p \leq 0 \). To get the solution of \( z_e^* \), substituting the value of \( \hat{r}_R(z_e) \) into the NVP solution, then we can derive the unique optimal stocking factor from the equation

\[
\frac{b}{w-1} z_e - \Lambda(z_e) - \frac{(a-\beta)p}{w(1-c)} = 0.
\]

Proof of Lemma 5. If only the manufacturer offers consumer rebate, let \( r_R = 0 \) and substitute it into the NVP solution, we can yields the unique optimal stocking factor (equilibrium) \( z_e^* = F^{-1}(1-w/p) \), then according to the manufacturer’s rebate response function (Lemma 3), we obtain \( \hat{r}_M(z_e) = \frac{b\alpha(w-c)z_e}{z_e - \Lambda(z_e)} - \frac{\alpha w}{w(1-c)} \). To insure \( \hat{r}_M(z_e) > 0 \), we can prove that \( \hat{r}_M(z_e) \) strictly decreases in \( z_e \), if \( \hat{r}_M(0^+) = \frac{b\alpha(w-c)z_e}{z_e - \Lambda(z_e)} - \frac{\alpha w}{w(1-c)} > 0 \), then \( \hat{r}_M(z_e) > 0 \) holds, i.e., the manufacturer will offer a positive rebate value to consumers. Meanwhile, if the retailer offer no rebates, his best response function satisfies \( \hat{r}_R(\hat{r}_M(z_e^*), z_e^*) \leq 0 \), thus we can derive another sufficient condition \( [(2-b)w - c]z_e^* - \frac{b}{\alpha} + b - 1 \leq 0 \).

Proof of Lemma 6. (i) Rewrite the manufacturer’s expected profit function as:

\[
\Pi_R(r_R, z_l; r_M) = [a - b(p - \alpha r_R - \alpha r_M)](p - \beta r_R)(z_l - \Lambda(z_l)) - w z_l.
\]

The first and second order conditions of \( \Pi_R(r_R, z_l; r_M) \) with respect to stocking factor \( z_l \) are given by

\[
\begin{align*}
\frac{\partial \Pi_R(r_R, z_l; r_M)}{\partial z_l} &= [a - b(p - \alpha r_R - \alpha r_M)](1 - F(z_l))(p - \beta r_R) - w, \tag{17} \\
\frac{\partial^2 \Pi_R(r_R, z_l; r_M)}{\partial z_l^2} &= [a - b(p - \alpha r_R - \alpha r_M)](p - \beta r_R)(-f(z_l)), \tag{18} \\
&= -[a - b(p - \alpha r_R - \alpha r_M)]w r(z_l) < 0. \tag{19}
\end{align*}
\]

Thus, \( \Pi_R(r_R, z_l; r_M) \) is strictly concave in \( z_l \). Setting Equation (17) to zero, we obtain:

\[
(1 - F(z_l))(p - \beta r_R) - w = 0. \tag{20}
\]

For convenience, we let \( L(z_l) = (1 - F(z_l))(p - \beta r_R) - w \). Taking the first order derivative of \( \Pi_R(r_R, z_l; r_M) \) with respect to \( r_R \) yields:

\[
\hat{r}_R(z_l, r_M) = \frac{b\alpha[p - w z_l/(z_l - \Lambda(z_l)) - \beta(a - bp) - r_M]}{2b\alpha \beta} - \frac{r_M}{2}. \tag{21}
\]

Similarly, note that

\[
\frac{\partial^2 \Pi_R(r_R, z_l; r_M)}{\partial r_R^2} = -\frac{2b^2 a^2 w}{a - b(p - \alpha r_R - \alpha r_M)} \frac{z_l F(z_l) - \Lambda(z_l)}{1 - F(z_l)} < 0,
\]

thus \( \Pi_R(r_R, z_l; r_M) \) is concave in \( r_R \). Substituting (21) into \( L(z_l) \), we have

\[
L(z_l) = w(1 - F(z_l)) \frac{a z_l}{2(z_l - \Lambda(z_l))} - \frac{1}{1 - F(z_l)} + \frac{b(\alpha - \beta)p - \beta \alpha w}{2b\alpha w} \frac{z_l F(z_l) - \Lambda(z_l)}{1 - F(z_l)}. \tag{22}
\]

Combining Lemma 2 and Equation (22), we can derive:

\[
\frac{d}{dz_l} \left[ \frac{1}{2} \frac{b z_l}{(b-1)(z_l - \Lambda(z_l))} \right] - \frac{1}{1 - F(z_l)} \frac{1}{2} \frac{b z_l}{(b-1)(z_l - \Lambda(z_l))} \frac{1}{1 - F(z_l)} < 0.
\]
Therefore, \( \partial L(z_t)/\partial z_t < 0 \). Together with that if \( L(0^+) = -\frac{1}{2} \left\{ b(\alpha - \beta)p - b\alpha \beta r_M + \beta a \right\} < 0 \), e.g., \( b(\alpha - \beta)p - b\alpha w + \beta r_M + \beta a > 0 \), and \( L(+\infty) < 0 \), then there exist a unique maximizer \( \hat{z}_R(r_M, z_t) \) that satisfies \( \partial \Pi_R(r_M(z_t), z_t(r_M))/\partial z_t = 0 \).

**Proof of Corollary 4.** (i) Similar to Corollary 1 and 2, we have
\[
\frac{\partial L(z_t^*)}{\partial z_t^*} = \left[ \frac{2}{2z_t^* - \Lambda(z_t^*)} - \frac{1}{1 - F(z_t^*)} \right] < 0,
\]
Since \( \frac{\partial L(z_t^*)}{\partial z_t^*} = -\frac{\beta}{2w} < 0 \), if follows that \( \frac{d^2L(z_t^*)}{d^2z_t^*} = -\frac{dL(z_t^*)}{dz_t^*}/dL(z_t^*) < 0 \), thus \( z_t^* \) decreases in \( r_M \). On the other hand, \( \frac{d^2L(z_t^*)}{d^2z_t^*} = \frac{dL(z_t^*)}{dz_t^*} > 0 \), thus \( \hat{r}_R \) decreases in \( r_M \).

(ii) From Equation (15), we have
\[
\hat{r}_M(z_t) = \frac{1}{2\beta} \left\{ \frac{w - c}{w} \frac{z_t}{z_t - \Lambda(z_t)} + \frac{1}{1 - F(z_t)} \right\} + \frac{b\alpha p - \beta(a - bp)}{2\alpha \beta}.
\]
It is easy to verify that
\[
\frac{d^2r_t(z_t)}{dz_t} = \frac{1}{2\beta^2} \left\{ \frac{w - c}{w} \left( \frac{z_t}{z_t - \Lambda(z_t)} \right)^2 + \frac{dr_t(z_t)/dz_t + r_t^2(z_t)}{1 - F(z_t)} \right\} > 0,
\]
implies that the manufacturer’s optimal rebate increases in retailer’s stocking factor. On the other hand, \( \frac{d^3r_t(z_t)}{dz_t^3} = \frac{d^3r_t(z_t)}{dz_t^3} < 0 \), so the manufacturer’s best response decreases in the retailer’s rebate.

**Proof of Theorem 2.** We define a function \( R(z_t) \) as
\[
R(z_t) = \frac{3w - c}{3w} \frac{z_t}{z_t - \Lambda(z_t)} - \frac{1}{1 - F(z_t)} + \frac{b\alpha p + \beta(a - bp)}{3b\alpha w}.
\]
Note that \( R(0^+) = (b\alpha(p - c) + \beta(a - bp))/(3b\alpha w) > 0 \), and \( R(+\infty) \to -\infty(<0) \), thus there exist at least one solution to \( R(z_t) = 0 \).

Furthermore, we also have:
\[
\frac{d}{dz_t} \left[ \frac{3w - c}{3w} \left( \frac{b\alpha}{b - 1}(z_t - \Lambda(z_t)) + \frac{1}{1 - F(z_t)} \right) \right] < \frac{d}{dz_t} \left[ \frac{b\alpha}{b - 1}(z_t - \Lambda(z_t)) + \frac{1}{1 - F(z_t)} \right] < 0.
\]
Therefore, \( R(z_t) \) is monotone in \( z_t \), then the solution \( z_t \) to \( R(z_t) = 0 \) in equilibrium is unique.

**Proof of Lemma 7 and 8.** Proofs of 7 and 8 are similar to 4 and 5, here we omit the details.

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