Abstract: We employ the concept of interval-valued functions to state and prove an existence result for the Henstock-Kurzweil-Stieltjes-◊-double integral on time scales.

Keywords: Existence, Double integral, Henstock-Kurzweil integral, Interval-valued functions, Time scales

1 Introduction

The Henstock-Kurzweil integral is a generalization of Riemann integral that was studied independently by Henstock [5] and Kurzweil [6]. The Henstock-Kurzweil-Stieltjes integral is a generalized Riemann-Stieltjes integral which shares the same properties. The theory of interval analysis can be traced back to the celebrated book of Moore et al. [7]. In 2016, Yoon [11] presented some properties of interval-valued Henstock-Stieltjes integral on time scales. The Henstock-Kurzweil delta integral on time scales was introduced by Peterson and Thompson [9] and Henstock-Kurzweil integrals on time scales was studied by Thompson [10]. We relate the time scales version of integration to the usual form, most of the properties of a time scale integral can be realized by using the methods tailored to the time scale setting (see [2], [3], [4], [8], [9], [10]).

Some basic properties such as uniqueness and Bolzano Cauchy criterion of fuzzy Henstock-Kurzweil-Stieltjes-◊-double integral on time scales are stated and proved by the authors in [1]. In this paper, the authors are concerned with an existence result for Henstock-Kurzweil-Stieltjes-◊-double integral of interval-valued functions on time scales because of its various applications in the theory of integration.

A time scale \( \mathbb{T} \) is any closed non-empty subset of \( \mathbb{R} \), with the topology inherited from the standard topology on the real numbers \( \mathbb{R} \). Let \( a, b \in \mathbb{T}_1, c, d \in \mathbb{T}_2 \), where \( a < d, c < d \), and a rectangle \( \mathcal{R} = [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} = \{(t, s) : t \in [a, b], s \in [c, d], t \in \mathbb{T}_1, s \in \mathbb{T}_2 \} \). Let \( g_1, g_2 : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R} \) be two non-decreasing functions on \( [a, b]_{\mathbb{T}_1} \) and \( [c, d]_{\mathbb{T}_2} \), respectively. Let \( F : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R} \) be bounded on \( \mathcal{R} \). Let \( P_1 \) and \( P_2 \) be two partitions of \( [a, b]_{\mathbb{T}_1} \) and \( [c, d]_{\mathbb{T}_2} \) such that \( P_1 = \{t_0, t_1, ..., t_n\} \subset [a, b]_{\mathbb{T}_1} \) and \( P_2 = \{s_0, s_1, ..., s_n\} \subset [c, d]_{\mathbb{T}_2} \). Let \( \xi_1, \xi_2, ..., \xi_n \) denote an arbitrary selection of points from \( [a, b]_{\mathbb{T}_1} \) with \( \xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}, i = 1, 2, ..., n \). Similarly, let \( \zeta_1, \zeta_2, ..., \zeta_n \) denote an arbitrary selection of points from \( [c, d]_{\mathbb{T}_2} \) with \( \zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}, j = 1, 2, ..., k \).

Definition 1.1. Let \( \mathbb{T}_1, \mathbb{T}_2 \) be two given time scales and let \( \mathbb{T}_1 \times \mathbb{T}_2 = \{(x, y) : x \in \mathbb{T}_1, y \in \mathbb{T}_2 \} \) which is an Hausdorff metric space with the metric (distance) \( d_H \), define \( d_H(A, B) = d_H((a, c), (b, d)) = ((b - a)^2 + (d - c)^2) \) as the distance between \( A \) and \( B \). Then
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\[d_H(A, B) = \max \left( |A^- - B^-|, |A^+ - B^+| \right) \]
\[= \max \left( \left| (a^-, c^-) - (b^-, d^-) \right|, \left| (a^+, c^+) - (b^+, d^+) \right| \right) \]
\[= \max \left( (b^- - a^-)^2 + (d^- - c^-)^2, (b^+ - a^+)^2 + (d^+ - c^+)^2 \right)^{\frac{1}{2}}.\]

We now introduce Henstock-Kurzweil-Stieltjes-\(\bigcirc\)-double integral over versions in \(\mathbb{T}_1 \times \mathbb{T}_2\).

**Definition 1.2.** Let \(F : [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} \to \mathbb{R}\) be a bounded function on \(\mathcal{R}\) and let \(g\) be a non-decreasing function defined on \([a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}\) with partitions \(P_1 = \{t_0, t_1, \ldots, t_n\} \subset [a, b]_{\mathbb{T}_1}\) with tag points \(\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}\) for \(i = 1, 2, \ldots, n\) and \(P_2 = \{s_0, s_1, \ldots, s_k\} \subset [c, d]_{\mathbb{T}_2}\) with tag points \(\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}\) for \(j = 1, 2, \ldots, k\). Then

\[
S(P_1, P_2, F, g_1, g_2) = \sum_{i=1}^{n} \sum_{j=1}^{k} F(\xi_i, \zeta_j)(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1}))
\]

is defined as Henstock-Kurzweil-Stieltjes-\(\bigcirc\)-double sum of \(F\) with respect to functions \(g_1\) and \(g_2\). Let \(P = P_1 \times P_2\), then the Henstock-Kurzweil-Stieltjes-\(\bigcirc\)-double sum of \(F\) with respect to functions \(g_1\) and \(g_2\) is denoted by \(S(P, F, g_1, g_2)\).

**Definition 1.3.** Let \(F : [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} \to I_\mathbb{R}\) be an interval-valued function on \(\mathcal{R} = [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}: t \in [a, b]_{\mathbb{T}_1}, s \in [c, d]_{\mathbb{T}_1}\). We say that \(F\) is Henstock-Kurzweil-Stieltjes-\(\bigcirc\)-double integrable with respect to non-decreasing functions \(g_1, g_2\) defined on \([a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}\) if there is a number \(\alpha\), a member of \(\mathcal{R}\) such that for every \(\varepsilon > 0\), there is a \(\bigcirc\)-gauge \(\delta\) (or \(\gamma\)) such that

\[
d_H(S(P, F, g_1, g_2), I_0) < \varepsilon
\]

provided that \(P_1 = \{t_0, t_1, \ldots, t_n\} \subset [a, b]_{\mathbb{T}_1}\) with tag points \(\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}\) for \(i = 1, \ldots, n\) and \(P_2 = \{s_0, s_1, \ldots, s_k\} \subset [c, d]_{\mathbb{T}_2}\) with tag points \(\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}\), \(j = 1, 2, \ldots, k\) are \(\delta\)-fine (or \(\gamma\)) partitions of \([a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}\).

We say that \(I_0\) is the Henstock-Kurzweil-Stieltjes-\(\bigcirc\)-double integral of \(F\) with respect to \(g_1\) and \(g_2\) defined on \([a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}\), and write

\[
\int \int_\mathcal{R} F(t, s) \bigcirc g_1(t) \bigcirc g_2(s) = I_0.
\]

**2 The Main Results**

We need the following definitions to prove an existence theorem of interval Henstock-Kurzweil-Stieltjes-\(\bigcirc\)-double integral on time scales.

**Definition 2.1.** A function \(F : [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} \to I_\mathbb{R}\) is bounded on \([a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}\) with respect to \(g_1\) and \(g_2\) if there exists \(M \geq 0\) in \(I_\mathbb{R}\) such that

\[
|F(t, s)| \leq M, \text{ for all } t, s \in [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}.
\]

**Definition 2.2.** A function \(F : [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} \to I_\mathbb{R}\) is continuous at \(t_0, s_0 \in [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}\), if for any \(\varepsilon > 0\) there exists a positive \(\delta = \delta(t_0, s_0)\) such that whenever \(t, s \in [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}\),

\[
d_H(F(t, s), F(t_0, s_0)) < \varepsilon
\]
implies
\[ d_H((t, s), (t_0, s_0)) < \delta. \]

If \( F : [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} \to I_{\mathbb{R}} \) is uniformly continuous on \([a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}\), then it is continuous on \([a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}\).

If \( P_1 \) and \( P_2 \) are tagged partitions of \([a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}\), then there exists \( \mathcal{P} \) a collection of all divisions of \([a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}\). The variation of \( F \) over \( \mathcal{P} \) is given by
\[
\text{var}(F, \mathcal{P}) = \sum_{P_1} \sum_{P_2} d_H(F(t, s), F(t_0, s_0)).
\]

Note that for any division \( \mathcal{P} \) of \([a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}\), \( \text{var}(F, \mathcal{P}) \) is a continuous function on \([a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}\).

**Definition 2.3.** A function \( F : [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} \to I_{\mathbb{R}} \) is said to be of bounded variation on \([a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}\) if
\[
\text{BV}_F = \text{BV}(F, [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}) = \sup_{P \in \mathcal{P}([a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2})} \text{var}(F, \mathcal{P})
\]
is continuous on \([a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}\).

**Theorem 2.1.** Let \( F : [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} \to I_{\mathbb{R}} \) be of bounded variation on \([a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}\). Then the variation of \( F \) is additive; that is, if \( a \leq \alpha \leq b \) and \( c \leq \beta \leq d \), then
\[
\text{Var}(F; [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}) = \text{Var}(F; [a, \alpha]_{\mathbb{T}_1} \times [c, \beta]_{\mathbb{T}_2}) + \text{Var}(F; [\alpha, b]_{\mathbb{T}_1} \times [\beta, d]_{\mathbb{T}_2}).
\]

**Proof.** Suppose that \( F : [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} \to I_{\mathbb{R}} \) is of bounded variation. Let \( \alpha, \beta \in [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} \) and let \( P_1 \) and \( P_2 \) be two partitions of \([a, b]_{\mathbb{T}_1}\) and \([c, d]_{\mathbb{T}_2}\) such that \( P_1 = \{t_0, t_1, ..., t_n\} \subset [a, b]_{\mathbb{T}_1}\) and \( P_2 = \{s_0, s_1, ..., s_n\} \subset [c, d]_{\mathbb{T}_2}\). Let \( \xi_1, \xi_2, ..., \xi_n \) denote an arbitrary selection of points from \([a, b]_{\mathbb{T}_1}\) with \( \xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}, i = 1, 2, ..., n \). Similarly, let \( \zeta_1, \zeta_2, ..., \zeta_n \) denote an arbitrary selection of points from \([c, d]_{\mathbb{T}_2}\) with \( \zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}, j = 1, 2, ..., k \). Then \( P'_1 = \{t_0, t_1, ..., t_n\} \subset [a, b]_{\mathbb{T}_1}\) and \( P'_2 = \{s_0, s_1, ..., s_n\} \subset [c, d]_{\mathbb{T}_2}\) are refinements of \( P_1 \) and \( P_2 \) obtained by adjoining \( \alpha \) and \( \beta \) to \( P_1 \) and \( P_2 \) respectively. Thus
\[
\sum_{P_1} \sum_{P_2} d_H(F(t, s), F(t_0, s_0)) \leq \sum_{P'_1} \sum_{P'_2} d_H(F(t, s), F(t_0, s_0))
\]
where \( P_1 \) and \( P_2 \) are partitions of \([a, b]_{\mathbb{T}_1}\) and \([c, d]_{\mathbb{T}_2}\) respectively and \( P'_1 \) and \( P'_2 \) are also partitions of \([a, b]_{\mathbb{T}_1}\) and \([c, d]_{\mathbb{T}_2}\) respectively. Then \( P' = P'_1 \cup P'_2 \) and that
\[
\sum_{P_1} \sum_{P_2} d_H(F(t, s), F(t_0, s_0)) \leq \sup_{P \in \mathcal{P}([a, \alpha]_{\mathbb{T}_1} \times [c, \beta]_{\mathbb{T}_2})} \left( \sum_{P} d_H(F(t, s), F(t_0, s_0)) \right)
\]
and
\[
\sum_{P'_1} \sum_{P'_2} d_H(F(t, s), F(t_0, s_0)) \leq \sup_{P \in \mathcal{P}([a, \alpha]_{\mathbb{T}_1} \times [c, \beta]_{\mathbb{T}_2})} \left( \sum_{P} d_H(F(t, s), F(t_0, s_0)) \right)
\]
Hence,
Hence,

$$\var(F; [a, b]_{T_1} \times [c, d]_{T_2}) = \sup_{P \in \mathcal{P}(F; [a, b]_{T_1} \times [c, d]_{T_2})} \left( \sum_{P} \sum_{P} d_H(F(t, s), F(t_0, s_0)) \right)$$

$$\leq \var(F; [a, b]_{T_1} \times [\beta, d]_{T_2}) + \var(F; [a, \alpha]_{T_1} \times [c, \beta]_{T_2}).$$

On the other hand, for any $P_1' = \{t_0, t_1, \alpha, ..., t_n\} \subset [a, b]_{T_1}$ and $P_2' = \{s_0, s_1, \beta, ..., s_n\} \subset [c, d]_{T_2}$ are refinements of $P_1$ and $P_2$ obtained by adjoining $\alpha$ and $\beta$ to $P_1$ and $P_2$ respectively. Then $P' = P_1' \cup P_2' \in \mathcal{P}_{\alpha, \beta}(a, b]_{T_1} \times [c, d]_{T_2}$ is the set of all divisions of $[a, b]_{T_1} \times [c, d]_{T_2}$ with $\alpha$ and $\beta$ as the division points. Hence,

$$\sup_{P' \in \mathcal{P}_{\alpha, \beta}(F; [a, b]_{T_1} \times [c, d]_{T_2})} \left( \sum_{P'} \sum_{P'} d_H(F(t, s), F(t_0, s_0)) \right)$$

$$\leq \sup_{P \in \mathcal{P}(F; [a, b]_{T_1} \times [c, d]_{T_2})} \left( \sum_{P} \sum_{P} d_H(F(t, s), F(t_0, s_0)) \right)$$

$$= \var(F; [a, b]_{T_1} \times [c, d]_{T_2}).$$

Thus,

$$\var(F; [a, b]_{T_1} \times [\beta, d]_{T_2}) + \var(F; [a, \alpha]_{T_1} \times [c, \beta]_{T_2})$$

$$\leq \sup_{P' \in \mathcal{P}_{\alpha, \beta}(F; [a, b]_{T_1} \times [c, d]_{T_2})} \left( \sum_{P'} \sum_{P'} d_H(F(t, s), F(t_0, s_0)) \right)$$

$$\leq \var(F; [a, b]_{T_1} \times [c, d]_{T_2}).$$

Therefore, combining the two inequalities, we have

$$\var(F; [a, b]_{T_1} \times [\beta, d]_{T_2}) + \var(F; [a, \alpha]_{T_1} \times [c, \beta]_{T_2}) = \var(F; [a, b]_{T_1} \times [c, d]_{T_2}).$$

This completes the proof.

**Theorem 2.2. [Existence Theorem]** Let $F : [a, b]_{T_1} \times [c, d]_{T_2} \rightarrow I_R$ be a continuous function and $g : [a, b]_{T_1} \times [c, d]_{T_2} \rightarrow I_R$ be of bounded variation on $[a, b]_{T_1} \times [c, d]_{T_2}$, then $F$ is Henstock-Kurzweil-Stieltjes-\(\bigcirc\)-double integrable on $[a, b]_{T_1} \times [c, d]_{T_2}$.

**Proof.** Let $\varepsilon > 0$. Since $g$ is of bounded variation, $\var_g \in I_R$ and $g = g_1 \times g_2$. This means that there exists $M > 0$ such that $\var_g(t, s) \leq M$ for all $t, s \in [a, b]_{T_1} \times [c, d]_{T_2}$. Since $F$ is continuous on $[a, b]_{T_1} \times [c, d]_{T_2}$, for all $t_0, s_0 \in [a, b]_{T_1} \times [c, d]_{T_2}$ there exists a positive $\delta_0(t_0, s_0)$ such that whenever $t, s \in [a, b]_{T_1} \times [c, d]_{T_2}$ with

$$d_H(t, s), (t_0, s_0)) < \delta_0,$$

we have

$$d_H(F(t, s), F(t_0, s_0)) < \varepsilon.$$ 

Let a positive gauge $\delta$ be defined on $[a, b]_{T_1} \times [c, d]_{T_2}$ by $\delta = \frac{\delta_0}{2}$, for all $t, s \in [a, b]_{T_1} \times [c, d]_{T_2}$. Let

$$P_1 = \{(a, t_1), (t_1, t_2), \xi_1, ..., (t_{n-1}, b], \xi_n) \} \subset [a, b]_{T_1}$$

and

$$P_2 = \{(c, s_1), (s_1, s_2), \xi_2, ..., ([s_{k-1}, d], \xi_k) \} \subset [c, d]_{T_2}$$

be $\delta$-fine tagged divisions of $[a, b]_{T_1} \times [c, d]_{T_2}$. Then, there exists a tagged division $P_0$ such that $P_1 < P_0$ and $P_2 < P_0$. Now, for every $(t_{i-1}, t_i), (t_{j-1}, t_j) \in d_H$ and $(s_{j-1}, s_j), (s_{k-1}, d) \in d_H$: $i = 1, 2, ..., n; \quad j = 1, 2, ..., k$.

Now we have the difference

$$d_H((t_{i-1}, s_{j-1}), (t_i, s_j)) = F(\xi_i, \xi_j)[(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1}))] - S(F, g_1, P_{i,j})$$
where

\[ P_{i,j} = \left\{ \left[ X_{q-1}^{(i,j)}, X_q^{(i,j)} \right], s_q^{(i,j)} \right\}, X_0^{(i,j)} = (t_{i-1}, s_{j-1}), X_{m_i}^{(i,j)} = (t_i, s_j), q - 1 < m_{i,j} \]

is a refinement of \(((t_{i-1}, s_{j-1}), (t_i, s_j)), (\xi, \zeta)\) in \(P_0\). Then

\[ d_H((t_{i-1}, s_{j-1}), (t_i, s_j)) = \sum_{i=1}^{n} \sum_{j=1}^{k} \left( \sum_{q=1}^{m_{i,j}} F(\xi, \zeta) - F(s_q^{(i,j)}) \right) \left( g(X_{q}^{(i,j)}) - g(X_q^{(i,j)}) \right). \]

Now, \(s_q^{(i,j)}, (\xi, \zeta) \in ((t_{i-1}, s_{j-1}), (t_i, s_j)) \subseteq (\xi, \zeta) - \delta(\xi, \zeta), (\xi, \zeta) + \delta(\xi, \zeta)\) which implies that

\[ |(\xi, \zeta) - s_q^{(i,j)}| \leq d_H((t_{i-1}, s_{j-1}), (t_i, s_j)) < \delta(\xi, \zeta). \]

By continuity of \(F\) at \((\xi, \zeta)\),

\[ |s_q^{(i,j)} - (\xi, \zeta)| < \delta(\xi, \zeta) = \frac{\delta_0(\xi, \zeta)}{2} \leq \delta_0(\xi, \zeta) \]

it implies that

\[ d_H(F(s_q^{(i,j)}) - F(\xi, \zeta)) < \epsilon. \]

So,

\[ d_H((t_{i-1}, s_{j-1}), (t_i, s_j)) = \sum_{i=1}^{n} \sum_{j=1}^{k} \left( \sum_{q=1}^{m_{i,j}} F(\xi, \zeta) - F(s_q^{(i,j)}) \right) \left( g(X_{q}^{(i,j)}) - g(X_q^{(i,j)}) \right). \]

Hence, by Theorem 2.1, we have
\[ d_H(S(P, F, g) - S(P_0, F, g)) \]
\[ = d_H \left( \sum_{i=1}^{n} \sum_{j=1}^{k} F(\xi_i, \zeta_j)\{(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1}))\} \right) \]
\[ = d_H \left( \sum_{i=1}^{n} \sum_{j=1}^{k} \{ F(\xi_i, \zeta_j)\{(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1}))\}, SP_{i,j}, F, g) \} \right) \]
\[ \leq \sum_{i=1}^{n} \sum_{j=1}^{k} \left| d_H((t_{i-1}, s_{j-1}), (t_i, s_j)) \right| \]
\[ = \sum_{i=1}^{n} \sum_{j=1}^{k} \left| \frac{m_{i,j}}{K} \right| \left| (g(X_{i,j}^{s(i,j)} - g(X_{i,j}^{t(i,j)})) \right| \]
\[ \leq \frac{\varepsilon}{K} \sum_{i=1}^{n} \sum_{j=1}^{k} \left| Var[g, (t_{i-1}, s_{j-1}), (t_i, s_j)] \right| \]
\[ = \frac{\varepsilon}{K} Var_{q} < \frac{\varepsilon}{K} K = \varepsilon. \]

Similarly,
\[ d_H(S(Q, F, g), S(P_0, F, g)) < \varepsilon. \]

Thus,
\[ d_H(S(P, F, g), S(Q, F, g)) \leq d_H(S(P, F, g), S(P_0, F, g)) + D(S(P_0, F, g), S(Q, F, g)) \]
\[ < \varepsilon + \varepsilon = 2\varepsilon. \]

By Cauchy criterion theorem in [2], \( F \) is Henstock-Kurzweil-Stieltjes-\( \ast \)-double integrable on \([a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} \).

\[ \square \]

References

[1] Afariogun, D. A., Mogbademu, A. A. and Olaoluwa, H. O., 2021, On fuzzy Henstock-Kurzweil-Stieltjes-\( \ast \)-double integral on time scales, J Math Anal and Model, 2(2), 38 - 49.

[2] Avsec, S., Bannish, B., Johnson, B. and Meckler, S., 2006, The Henstock-Kurzweil delta integral on unbounded time scales, Panamer Math J, 16(3), 77 - 98.
[3] Bartosiewicz, Z. and Pawluzewicz, E., 2006, Realizations of linear control systems on time scales, Control and Cybernetics, 35 (4), 769-786.

[4] Bohner, M. and Peterson, A., 2001, Dynamic equations on time scales, Birkhauser Boston, MA.

[5] Henstock, R., 1961, Definitions of Riemann type of the variational integrals, Proc. London Math. Soc., 11(3), 402 - 418.

[6] Kurzweil, J., 1957, Generalized ordinary differential equation and continuous dependence on a parameter, CMJ, 7(82), 418 - 449.

[7] Moore, R. E., Kearfott, R. B. and Cloud, M. J., 2009, Introduction to interval analysis, Society for Industrial and Applied Mathematics, 3600 University City Science Center Philadelphia, PA, United State.

[8] Mozyrska, D., Pawlusiewicz, E. and Torres, D. F. M., 2009, The Riemann-Stieltjes integral on time scales, Austr. J. Math. Anal. Appl., 7(1), 1 - 14.

[9] Peterson, A. and Thompson, B., 2006, Henstock-Kurzweil delta and nabla integrals, J. Math. Anal. Appl., 323(1), 162-178.

[10] Thompson, B., 2008, Henstock-Kurzweil integrals on time scales, Panamer. Math. J., 18(1), 1 - 19.

[11] Yoon, J. H., 2016, On Henstock-Stieltjes Integrals of Interval-Valued Functions on Time Scales, Journal of the Chungcheong Mathematical Society, 29, 109 - 115.