THE RIESZ TRANSFORM FOR THE HARMONIC OSCILLATOR
IN SPHERICAL COORDINATES

ÓSCAR CIAURRI AND LUZ RONCAL

Abstract. In this paper we show weighted estimates in mixed norm spaces for the Riesz transform associated with the harmonic oscillator in spherical coordinates. In order to prove the result we need a weighted inequality for a vector-valued extension of the Riesz transform related to the Laguerre expansions which is of independent interest. The main tools to obtain such extension are a weighted inequality for the Riesz transform independent of the order of the involved Laguerre functions and an appropriate adaptation of Rubio de Francia’s extrapolation theorem.

1. Introduction

Let \( H := -\Delta + |\cdot|^2 \) be the harmonic oscillator in \( \mathbb{R}^n \). The eigenfunctions of this operator in \( \mathbb{R}^n \) verify \( H\phi = E\phi \), where \( E \) is the corresponding eigenvalue. There are two complete sets of eigenfunctions for \( H \). By using cartesian coordinates, one obtains the functions

\[
\phi_k(x) = \prod_{i=1}^{n} h_{k_i}(x_i), \quad k = (k_1, \ldots, k_n) \in \mathbb{N}^n,
\]

where \( h_{k_i}(x_i) = (\sqrt{\pi}2^k k_i!)^{-1/2} H_{k_i}(x_i) e^{-x_i^2/2} \), and \( H_j \) denotes the Hermite polynomial of degree \( j \in \mathbb{N} \) (see [13, p. 60]). The system of functions \( \{\phi_k\}_{k \in \mathbb{N}^n} \) is orthonormal and complete in \( L^2(\mathbb{R}^n, dx) \).

But the situation is completely different if we analyze the eigenfunctions of the harmonic oscillator by using spherical coordinates (see (2.1) below). Let \( \mathbb{B}^n \) be the unit ball in \( \mathbb{R}^n \) and \( S^{n-1} = \partial \mathbb{B}^n \). Let \( \mathcal{H}_j \) be the space of spherical harmonics of degree \( j \) in \( n \) variables, and let \( \{Y_{m,\ell}\}_{\ell=1,\ldots,\dim\mathcal{H}_m} \) be an orthonormal basis for \( \mathcal{H}_j \) in \( L^2(S^{n-1}, d\sigma) \), where \( \sigma \) is the surface area measure in \( S^{n-1} \). Then the eigenfunctions of the harmonic oscillator, see [8], are given by

\[
\tilde{\phi}_{m,k,\ell}(x) = \left( \frac{2\Gamma(k+1)}{\Gamma(m-k+n/2)} \right)^{1/2} L_k^{n/2-1+m-2k}(r^2)^{m-2k} Y_{m-2k,\ell}(x') e^{-r^2/2},
\]

where \( x \in \mathbb{B}^d \), \( r = |x| \), \( x' \in S^{n-1} \), \( m \geq 0 \), \( k = 0, \ldots, \lfloor m/2 \rfloor \), \( \ell = 1, \ldots, \dim \mathcal{H}_{m-2k} \), and \( L_k^b \) are Laguerre polynomials of order \( b \) and degree \( k \in \mathbb{N} \), see [13, p. 76].
This system is orthonormal and complete in $L^2(\mathbb{R}^n, dx)$ and the eigenvalues are $E_{m,k,\ell} = (n + 2m)$. Moreover

$$L^2(\mathbb{R}^n, dx) = \bigoplus_{m=0}^{\infty} \mathcal{J}_m$$

with

$$\mathcal{J}_m = \{ f \in C^\infty(\mathbb{R}^n) : Hf = (n + 2m)f \}.$$

One of the main targets of this paper will be the analysis of the Riesz transform related to the system of eigenfunctions of the harmonic oscillator in spherical coordinates.

It could be said that the investigation of conjugacy operators related to discrete and continuous non-trigonometric orthogonal expansions was initiated in the seminal article by B. Muckenhoupt and E. M. Stein [15]. They analyze a substitute of classical conjugacy function in the context of ultraspherical polynomials expansions, Hankel transforms and Fourier-Bessel expansions. Later, the book by Stein [23] propelled the research in Fourier Analysis of general laplacians. It is noteworthy to observe that in the classical one-dimensional case the “continuous” counterpart of the conjugacy is the Hilbert transform, and we have the equivalence between Hilbert transform and the so called Riesz transform. So, abusing of the language, the wording conjugacy and Riesz transform are used as the same thing. For the last forty years, the research developed is huge, and the list of references could be endless. Concerning examples close to our context, the study of Riesz transforms in the setting of multi-dimensional Hermite functions was initiated by S. Thangavelu [25, 26, 27] and continued in [24, 12, 14]. On the other hand, Riesz transforms associated with expansions based on different multi-dimensional Laguerre functions have been investigated by A. Nowak and K. Stempak in [17, 18].

We will analyze the Riesz transform associated with the system given in (1.1) in the so called mixed norm spaces $L^{p,2}(\mathbb{R}^n, r^{n-1} dr d\sigma)$ (see Section 2 for definition). These spaces were first systematically studied by A. Benedek and R. Panzone in [3]. They arise frequently in harmonic analysis when the spherical harmonics are involved. The papers [22, 7, 5, 2] contain good examples of their use. In [6], the authors considered this kind of spaces to study fractional integrals related to the functions $\tilde{\phi}_{m,k,\ell}$.

The boundedness properties of the Riesz transform related to $\tilde{\phi}_{m,k,\ell}$ in the mixed norm spaces will be reduced to two inequalities due to the decomposition of the harmonic oscillator in spherical coordinates. The first one will be a vector-valued inequality for a sequence of Riesz transforms for Laguerre expansions of convolution type and order $(n - 2)/2 + k$ (note that $\tilde{\phi}_{m,k,\ell}(x) = \ell_k^{n/2-1+m-2k}(r)Y_{m-2k,\ell}(x')$, where $\ell_k^\alpha$ denotes the Laguerre functions which are defined in (3.2) below). This is the main point in the proof of our result and it requires very precise estimates of the kernel of the Riesz transform in the Laguerre setting in terms of the order. With these estimates we will be able to apply the Calderón-Zygmund theory. Then a suitable version of the extrapolation theorem of Rubio de Francia will do the rest to deduce the vector-valued extension. The second inequality appearing from the angular part of the harmonic oscillator, will be deduced from the Calderón-Zygmund theory as well.
2. The Riesz Transform for $H$ in Spherical Coordinates

The harmonic oscillator in spherical coordinates can be written as

$$H = -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} + r^2 - \frac{1}{r^2}\Delta_0,$$

where $\Delta_0$ is the spherical part of the Laplacian. It can be checked that

$$H = \delta^* \delta + n,$$

with

$$\delta = x' \left( \frac{\partial}{\partial r} + r \right) + \frac{1}{r} \nabla \delta, \quad \delta^* = -x' \left( \frac{\partial}{\partial r} - r \right) - \frac{1}{r} \nabla \delta,$$

where $\nabla \delta$ is the spherical gradient, which is the spherical part of $\nabla$ and it involves only derivatives in $x'$. Moreover $\Delta_0 = \nabla \delta \cdot \nabla \delta$.

For each $\sigma > 0$, we define the fractional integrals for the harmonic oscillator as

$$H^{-\sigma} f = \sum_{m=0}^{\infty} \frac{1}{(n+2m)^\sigma} \text{Proj}_{J_m} f,$$

where

$$\text{Proj}_{J_m} f = \sum_{k=0}^{[\frac{n}{2}]} \sum_{\ell=1}^{\dim H_{m-2k}} c_{m,k,\ell}(f) \varphi_{m,k,\ell}, \quad c_{m,k,\ell}(f) = \int_{\mathbb{R}^n} f(y) \varphi_{m,k,\ell}(y) \, dy.$$

With the previous definitions the Riesz transform is given as

$$Rf := |\delta H^{-1/2} f|.$$

Observe that (2.2) is $\nabla + x$ in spherical coordinates, being $\nabla$ the usual gradient. Hence $Rf$ coincides with operator $|\nabla + x H^{-1/2}|$.

In order to analyze this kind of operators we introduce the mixed norm spaces, defined as

$$L^{p,2}(\mathbb{R}^n, r^{n-1} \, dr \, d\sigma) = \{ f(x) : \|f\|_{L^{p,2}(\mathbb{R}^n, r^{n-1} \, dr \, d\sigma)} < \infty \},$$

where

$$\|f\|_{L^{p,2}(\mathbb{R}^n, r^{n-1} \, dr \, d\sigma)} = \left( \int_0^{\infty} \left( \int_{S^{n-1}} |f(rx')|^2 \, d\sigma(x') \right)^{p/2} r^{n-1} \, dr \right)^{1/p},$$

with the obvious modification in the case $p = \infty$. The main feature of these spaces is that we consider the $L^2$-norm in the angular part and the $L^p$-norm in the radial one. They are very different from $L^p(\mathbb{R}^n, dx)$; in fact $L^p(\mathbb{R}^n, dx) \subset L^{p,2}(\mathbb{R}^n, r^{n-1} \, dr \, d\sigma)$ for $p > 2$, $L^2(\mathbb{R}^n, dx) = L^{2,2}(\mathbb{R}^n, r^{n-1} \, dr \, d\sigma)$, and $L^{p,2}(\mathbb{R}^n, r^{n-1} \, dr \, d\sigma) \subset L^p(\mathbb{R}^n, dx)$ for $p < 2$. These spaces are the most suitable when spherical harmonics are involved due to the orthogonality of the system in the sphere $S^{n-1}$. Indeed, if a function $f$ on $\mathbb{R}^n$ is expanded in spherical harmonics,

$$f(x) = \sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim H_j} f_{j,\ell}(r) \mathcal{Y}_{j,\ell}(x'),$$

where

$$f_{j,\ell}(r) = \int_{S^{d-1}} f(rx') \mathcal{Y}_{j,\ell}(x') \, d\sigma(x'),$$

with

$$\mathcal{Y}_{j,\ell}(x') = \frac{1}{2} \frac{\partial}{\partial n} \mathcal{Y}_{j,\ell}(x').$$
The functions \( \ell \) denote by \( A \) have \( L \) if \( p \) theorem 2.1. Let we have to define the class of weights involved in them. For \( 1 \leq p < \infty \), we have (\( L \) basis of \( \mathbb{R} \) which is symmetric on \( \mathbb{R} \). Let \( f \), \( w \), and \( \mu \) are defined by \( A \), \( A \), and \( \mu \), respectively. To establish our result related to weighted inequalities for the Riesz transform, we have to define the class of weights involved in them. For \( 1 \leq p < \infty \), we have to define the class of weights involved in them. For \( 1 \leq p < \infty \), we have (\( L \) basis of \( \mathbb{R} \) which is symmetric on \( \mathbb{R} \). Let \( f \), \( w \), and \( \mu \) are defined by \( A \), \( A \), and \( \mu \), respectively. One of the main results of the paper is stated below.

Theorem 2.1. Let \( n \geq 2 \), \( 1 < p < \infty \), and \( w \in A_p^{n/2 - 1} \). Then

\[
\| Rf \|_{L^p(\mathbb{R}^n, w(x) \, dx)} \leq C \| f \|_{L^p(\mathbb{R}^n, w(x) \, dx)},
\]

for each \( f \in L^p(\mathbb{R}^n, w(x) \, dx) \), and with a constant \( C \) depending on \( n \) and \( w \) only.

The proof of Theorem 2.1 will be given in Section 5. The main estimates will be developed in Section 3 and Section 4.

3. Vector-valued inequalities for the Riesz transform for Laguerre expansions of convolution type

Let \( \alpha > -1 \), consider the differential operator given by

\[
L_\alpha = -\frac{d^2}{dx^2} + x^2 - \frac{2\alpha + 1}{x} \frac{d}{dx},
\]

which is symmetric on \( \mathbb{R}^+ \) equipped with the measure \( d\mu_\alpha \). The Laguerre functions \( \ell_k^\alpha \) are defined by

\[
\ell_k^\alpha(x) = \left( \frac{2\Gamma(k+1)}{\Gamma(k+\alpha+1)} \right)^{1/2} L_k^\alpha(x^2)e^{-x^2/2}, \quad x > 0, \quad \alpha > -1.
\]

The functions \( \ell_k^\alpha \) are eigenfunctions of the differential operator (3.1). Indeed, we have \( L_\alpha \ell_k^\alpha = (4k + 2\alpha + 2) \ell_k^\alpha \). Furthermore, the system \( \{ \ell_k^\alpha \}_{k \in \mathbb{N}} \) is an orthonormal basis of \( L^2(\mathbb{R}^+, d\mu_\alpha) \). We will refer to the functions \( \ell_k^\alpha \) as Laguerre functions of convolution type.

It is easily seen that \( L_\alpha \) can be decomposed as

\[
L_\alpha = \delta^*_\alpha \delta_\alpha + 2(\alpha + 1),
\]

where

\[
\delta_\alpha = \frac{d}{dx} + x,
\]
and
\[ \delta^\sigma_\alpha = -\frac{d}{dx} + x - \frac{2\alpha + 1}{x}. \]

We provide now the definition of the Riesz transforms. Since the spectrum of \( L_\alpha \) is separated from zero, we can define the fractional integrals of order \( \sigma \), for each \( \sigma > 0 \), as

\[ (L_\alpha)^{-\sigma} f = \sum_{k=0}^{\infty} \frac{1}{(4k + 2\alpha + 2)\sigma} P^\sigma_k f, \]

with \( P^\sigma_k f = \langle f, \ell^{\sigma}_k \rangle d\mu_\alpha \ell^{\sigma}_k \), where \( \langle f, g \rangle d\mu_\alpha \) means \( \int_{\mathbb{R}_+} f(x)g(x) d\mu_\alpha(x) \). Now, by using \( \frac{d}{dx}L^\sigma_k = -L^\sigma_{k-1} \), \( \alpha > -1 \), \( k \in \mathbb{N} \), see [13, (4.18.6)], we obtain
\[ \delta^\sigma_\alpha \ell^\sigma_k = -2\sqrt{k}x^{\sigma+1}. \]

Therefore, for \( f \in L^2(\mathbb{R}_+, d\mu_\alpha) \) with the expansion \( f = \sum_k \langle f, \ell^\sigma_k \rangle d\mu_\alpha x^{\sigma+1} \), we define the Riesz transform for the expansions of Laguerre functions of convolution type as

\[ R^\sigma f = \delta_\alpha(L_\alpha)^{-1/2} f = -2 \sum_{k=0}^{\infty} \left( \frac{k}{4k + 2\alpha + 2} \right)^{1/2} \langle f, \ell^\sigma_k \rangle d\mu_\alpha x^{\sigma+1}. \]

The system \( \{ x^{\sigma+1} \ell^\sigma_{k-1} \}_{k \in \mathbb{N}} \) is an orthonormal basis in \( L^2(\mathbb{R}_+, d\mu_\alpha) \), see [18, Proposition 4.1]. Therefore, the series above converges in \( L^2(\mathbb{R}_+, d\mu_\alpha) \) and defines a bounded operator therein.

Our result about the Riesz transform for the Laguerre expansions is the following.

**Theorem 3.1.** Let \( \alpha \geq -1/2 \), \( \alpha \geq 1 \), and \( 1 < p, r < \infty \). Define \( u_j(x) = x^{\alpha j} \), \( x \in \mathbb{R}_+, j = 0, 1, \ldots \). Then there exists a constant \( C \) such that
\[ \left\| \left( \sum_{j=0}^{\infty} |u_j R^{\alpha+j}(u_j^{-1}f_j)|^r \right)^{1/r} \right\|_{L^p(\mathbb{R}_+, d\mu_\alpha)} \leq C \left\| \left( \sum_{j=0}^{\infty} |f_j|^r \right)^{1/r} \right\|_{L^p(\mathbb{R}_+, d\mu_\alpha)}, \]
for all \( w \in A^\alpha_p \). Moreover the constant \( C \) depends on \( \alpha \) and \( w \) only.

In order to prove Theorem 3.1, we need two ingredients.

**Proposition 3.2.** Let \( \alpha \geq -1/2 \), \( \alpha \geq 1 \), and \( 1 < r < \infty \). Define \( u_j(x) = x^{\alpha j} \), \( x \in \mathbb{R}_+, j = 0, 1, \ldots \). Then,
\[ \int_0^\infty |u_j R^{\alpha+j}(u_j^{-1}f)(x)|^r w(x) d\mu_\alpha(x) \leq C \int_0^\infty |f(x)|^r w(x) d\mu_\alpha(x), \]
for every weight \( w \in A^\alpha_p \) and with \( C \) independent of \( j \) and depending on \( \alpha \) and \( w \).

**Proposition 3.3.** Let \( \{ T_j \} \) be a sequence of operators and suppose that, for some fixed \( r > 1 \), these operators are uniformly bounded in \( L^r(\mathbb{R}_+, d\mu_\alpha) \) for every weight \( w \in A^\alpha_p \), i. e.
\[ \int_0^\infty |T_j f(x)|^r w(x) d\mu_\alpha(x) \leq C \int_0^\infty |f(x)|^r w(x) d\mu_\alpha(x), \]
with \( C \) independent of \( j \). Then the vector valued inequality
\[ \left\| \left( \sum_{j=0}^{\infty} |T_j f_j|^r \right)^{1/r} \right\|_{L^p(\mathbb{R}_+, d\mu_\alpha)} \leq C \left\| \left( \sum_{j=0}^{\infty} |f_j|^r \right)^{1/r} \right\|_{L^p(\mathbb{R}_+, d\mu_\alpha)} \]
holds for all \( 1 < r, p < \infty \) and \( w \in A^\alpha_p \).
Theorem 3.1 can be deduced easily by combining Proposition 3.2 and Proposition 3.3. Indeed, by taking $T_j = u_j R^{\alpha + aj}(u_{j}^{-1} f)$ we have that $T_j$ is bounded on $L^r(\mathbb{R}_+, w \, d\mu_\alpha)$ for $1 < r < \infty$ and all $w \in A_p^\alpha$, uniformly in $j \geq 0$, by Proposition 3.2. Then, Proposition 3.3 does the rest.

In Subsection 3.1 we will prove Proposition 3.2 and the proof of Proposition 3.3 will be given in Subsection 3.2.

Notation. The constants that do not depend on relevant quantities will be denoted by $C$ and can change from one line to another without further comment. We only note that a constant denoted by $C_\alpha$ depends on $\alpha$ but not on $j$.

3.1. **Proof of Proposition 3.2.** The proof of Proposition 3.2 is based on the theory of Calderón-Zygmund operators defined on spaces of homogeneous type, where the classical weighted Calderón-Zygmund theory is still valid with proper adjustments. Indeed, we can adapt the proof in the classical case for the Lebesgue measure, see [10]. In order to do this it is enough to have at our disposal the weighted boundedness of an appropriate maximal operator. In our case we should consider the boundedness on $L^p(\mathbb{R}_+, w \, d\mu_\alpha)$, with $w \in A_p^\alpha$, for $1 < p < \infty$ of the maximal Hardy-Littlewood operator defined as

\[
M_\alpha f(x) = \sup_{x \in I} \frac{1}{\mu_\alpha(I)} \int_I |f(y)| \, d\mu_\alpha(y),
\]

which follows from a more general result by A. Calderón [4].

We will write the operator $u_j R^{\alpha + aj}(u_{j}^{-1} f)$ as an integral operator in the Calderón-Zygmund sense with kernel as follows

\[
u_j R^{\alpha + aj}(u_{j}^{-1} f)(x) = \int_0^\infty (xy)^{aj} R^{\alpha + aj}(x, y) f(y) \, d\mu_\alpha(y),
\]

where $R^{\alpha + aj}$ is the kernel of the Riesz transform associated with the orthonormal system $\{e_k^{aj}\}_{k \geq 0}$. Then, we will prove growth and smoothness estimates for the kernel. These estimates are contained in the proposition below, that is the heart of the matter.

**Proposition 3.4.** Let $\alpha \geq -1/2$, $a \geq 1$, and $j \geq 0$. Then

\[
|\langle xy \rangle^{aj} R^{\alpha + aj}(x, y) \rangle | \leq \frac{C_1}{\mu_\alpha(B(x, |x - y|))}, \quad x \neq y,
\]

\[
|\nabla_{x,y} |\langle xy \rangle^{aj} R^{\alpha + aj}(x, y) \rangle | \leq \frac{C_2}{|x - y| \mu_\alpha(B(x, |x - y|))}, \quad x \neq y,
\]

with $C_1$ and $C_2$ independent of $j$, and where $\mu_\alpha(B(x, |x - y|)) = \int_{B(x, |x - y|)} d\mu_\alpha$ and $B(x, |x - y|)$ is the ball of center $x$ and radius $|x - y|$.

The case $j = 0$ of the previous proposition is contained in [18], so we will focus on the proof of (3.6) and (3.7) for $j \geq 1$ only.

The heat semigroup related to $L_\alpha$ is initially defined in $L^2(\mathbb{R}_+, d\mu_\alpha)$ as

\[
T_{\alpha, t} f = \sum_{k=0}^\infty e^{-t(4k+2\alpha+2)} \langle f, e_k^\alpha \rangle d\mu_\alpha e_k^\alpha, \quad t > 0.
\]

We can write the heat semigroup $\{T_{\alpha, t}\}_{t > 0}$ as an integral operator

\[
T_{\alpha, t} f(x) = \int_0^\infty G_{\alpha, t}(x, y) f(y) \, d\mu_\alpha(y).
\]
The Laguerre heat kernel is given by
\[ G_{\alpha,t}(x,y) = \sum_{k=0}^{\infty} e^{-(4k+2\alpha+2)t} \ell_k(x)\ell_k(y). \]

The explicit expression for the Laguerre heat kernel is known and it can be found in [13, (4.17.6)]:
\[ G_{\alpha,t}(x,y) = \sinh(2t)^{-1} \exp \left( -\frac{1}{2} \coth(2t)(x^2 + y^2) \right) (xy)^{-\alpha} I_\alpha \left( \frac{xy}{\sinh 2t} \right), \]
with \( I_\alpha \) denoting the modified Bessel function of the first kind and order \( \alpha \), see [13, Chapter 5].

It can be seen in [18, Section 3] that the Riesz transform (3.4) is an operator associated, in the Calderón-Zygmund sense, with the kernel
\[ R^\alpha(x,y) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \delta_\alpha G_{\alpha,t}(x,y)t^{-1/2} \, dt. \]

Let us see now that the operators \( u_j R^{\alpha+aj}(u_j^{-1} f) \) are associated, in the Calderón-Zygmund sense, with the kernel given by
(3.8) \[ (xy)^{aj} R^{\alpha+aj}(x,y) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \delta_\alpha G_{\alpha+aj,t}(x,y)t^{-1/2} \, dt. \]

**Proposition 3.5.** Let \( \alpha \geq -1/2, \ a \geq 1, \) and \( u_j(x) = x^{aj}, \ x \in \mathbb{R}_+, \ j = 0, 1, \ldots \) Take \( f, g \in C_c(\mathbb{R}_+) \) having disjoint supports. Let \( u_j R^{\alpha+aj}(u_j^{-1} f) \) be defined by (3.4). Then
\[ \langle u_j R^{\alpha+aj}(u_j^{-1} f), g \rangle_{du_\alpha} = \int_0^{\infty} \int_0^{\infty} (xy)^{aj} R^{\alpha+aj}(x,y) f(y) g(x) \, d\mu_\alpha(y) \, d\mu_\alpha(x). \]

**Proof.** Let \( u_j^{-1} f = h_1 \) and \( u_j^{-1} g = h_2 \). Then,
\[ \langle u_j R^{\alpha+aj}(u_j^{-1} f), g \rangle_{du_\alpha} = \langle R^{\alpha+aj} h_1, h_2 \rangle_{d\mu_{\alpha+aj}} \]
\[ = \int_0^{\infty} \int_0^{\infty} R^{\alpha+aj}(x,y) h_1(y) h_2(x) \, d\mu_{\alpha+aj}(y) \, d\mu_{\alpha+aj}(x) \]
\[ = \int_0^{\infty} \int_0^{\infty} (xy)^{aj} R^{\alpha+aj}(x,y) f(y) g(x) \, d\mu_\alpha(y) \, d\mu_\alpha(x), \]
since the second identity above was proven in [18, Proposition 3.3]. \qed

Observe that the operator \( u_j R^{\alpha+aj}(u_j^{-1} f) \) is bounded on \( L^2(\mathbb{R}_+, d\mu_\alpha) \). Indeed, its norm is the same as the norm of \( R^{\alpha+aj} \) on \( L^2(\mathbb{R}_+, d\mu_{\alpha+aj}) \) (this follows from the proof of Proposition 3.5), which is proved to be finite after (3.4).

We will find a suitable expression for the kernel (3.8), and this task boils down to expressing the corresponding heat kernel in an appropriate way. We use Schläfli’s integral representation of Poisson’s type for modified Bessel function, see [13, (5.10.22)],
\[ I_\alpha(z) = z^\alpha \int_{-1}^{1} \exp(-zs) \Pi_\alpha(ds), \quad \text{arg } z < \pi, \ \alpha > -\frac{1}{2}, \]
where the measure \( \Pi_\alpha(du) \) is given by
\[ \Pi_\alpha(du) = \frac{(1 - u^2)^{-\alpha-1/2} \, du}{\sqrt{\pi} 2^\alpha \Gamma(\alpha + 1/2)}, \quad \alpha > -1/2. \]
In the limit case $\alpha = -1/2$, we put $\pi_{-1/2} = \frac{1}{2}(\delta_{-1} + \delta_{1})$. Consequently, for $\alpha \geq -1/2$, the kernel $G_{\alpha,t}(x,y)$ can be expressed as

$$G_{\alpha,t}(x,y) = \left(\sinh(2t)\right)^{-1-\alpha} \int_{-1}^{1} \exp \left( -\frac{1}{2} \coth(2t)(x^2 + y^2) - \frac{xy}{\sinh(2t)} \right) \Pi_{\alpha}(ds).$$

Let

$$q_\pm = q_\pm(x,y,s) = x^2 + y^2 \pm 2xys.$$

Meda’s change of variable

$$t = \frac{1}{2} \log \frac{1 + \xi}{1 - \xi}, \quad \xi \in (0,1),$$

leads to

$$G_{\alpha,t}(x,y) = \left(\frac{1 - \xi^2}{2\xi}\right)^{1+\alpha} \int_{-1}^{1} \exp \left( -\frac{1}{4\xi} q_+(x,y,s) - \frac{\xi}{4} q_-(x,y,s) \right) \Pi_{\alpha}(ds).$$

Let

$$\beta_\alpha(\xi) = \sqrt{\frac{2}{\pi}} \left(\frac{1 - \xi^2}{2\xi}\right)^{-1+\alpha} \left(\frac{1 + \xi}{1 - \xi}\right)^{-1/2}.$$

In this way, by (3.8) we get

$$R^\alpha(x,y) = \int_{0}^{1} \beta_\alpha(\xi) \delta_\alpha \int_{-1}^{1} \exp \left( -\frac{q_+}{4\xi} - \frac{\xi q_-}{4} \right) \Pi_{\alpha}(ds) d\xi$$

(3.10)

$$= \int_{-1}^{1} \int_{0}^{1} \beta_\alpha(\xi) \left( x - \frac{1}{2\xi}(x + ys) - \frac{\xi}{2}(x - ys) \right) \exp \left( -\frac{q_+}{4\xi} - \frac{\xi q_-}{4} \right) d\xi \Pi_{\alpha}(ds).$$

The application of Fubini’s theorem above can be justified, see [18, Proposition 5.6].

Throughout the proofs in this section, we will use several elementary facts, that are listed below. First, observe that

(3.11) $\beta_\alpha(\xi) \leq C2^{-\alpha} \begin{cases} \xi^{-\alpha - 3/2}, & 0 < \xi \leq 1/2, \\ \xi^{-\alpha - 1}(1 - \xi^2)^\alpha (-\log(1 - \xi^2))^{-1/2}, & 1/2 < \xi < 1, \end{cases}$

(3.12) $|x - ys| \leq \sqrt{q_-}$

and

(3.13) $|x - ys|^\theta \exp \left( -\frac{\xi q_-}{4} \right) \leq C\xi^{-\theta/2}, \quad \theta > 0.$

Inequality (3.12) is immediate, and (3.13) follows from (3.12) and the inequality

$$x^\gamma e^{-x} \leq \gamma^\gamma e^{-\gamma}, \quad x \in \mathbb{R}_+, \quad \gamma \in \mathbb{R}_+.$$

Let $b > 0$. Define $h(u) := (1 - u)^b u^{v - 1/2}$, for $u \in (0,1)$. Then, for $v \geq 1/2$

(3.15) $h(u) \leq \left( \frac{b}{b + v - 1/2} \right)^b.$

We will also use frequently the following fact without further mention

$$\frac{\Gamma(z + r)}{\Gamma(z + t)} \approx z^{r-t}, \quad z > 0, \quad r, t \in \mathbb{R}.$$
Apart from this, we need several technical lemmas that provide the tools to prove the main estimates. First we show the estimates for the measure of the balls $B(x, |x-y|)$ in the space $(\mathbb{R}_+, d\mu_\alpha)$. The result presented here is a one-dimensional version of [18, Proposition 3.2]. It will be used tacitly throughout the proofs.

**Lemma 3.6.** Let $\alpha \geq -1/2$. Then, for all $x, y \in \mathbb{R}_+$,
\[
\mu_\alpha(B(x, |x-y|)) \simeq |x-y|(x+y+|x-y|)^{2\alpha+1} \simeq |x-y|(x+y)^{2\alpha+1}.
\]

We will also use the following.

**Lemma 3.7.** Let $k, m \in \mathbb{R}$ be such that $m \geq -1/2$ and $k + m > -1/2$. Then
\[
\int_0^1 \xi^{-k} \beta_m(\xi) \exp\left(-\frac{q+}{4\xi}\right) d\xi \leq C_k \frac{2^m \Gamma(m+k+1/2)}{q_+^{m+k+1/2}}.
\]

**Proof.** Split the integral into two parts, $\int_0^{1/2} + \int_{1/2}^1$. For the first integral, the result follows from (3.11) and the following estimate
\[
\int_0^1 \xi^{-a-1} e^{-T/\xi} d\xi \leq T^{-a} \Gamma(a), \quad a > 0,
\]
which, in turn, is a slight modification of [16, Lemma 2.1]. For the second integral, from (3.11), the task is reduced to estimating
\[
2^{-m} \int_{1/2}^1 \xi^{-m-k-1}(1-\xi^2)^m(-\log(1-\xi^2))^{-1/2} \exp\left(-\frac{q+}{4\xi}\right) d\xi.
\]
Now, since $\xi \in (1/2, 1)$, by (3.14) and Stirling’s formula we get
\[
\xi^{-m-k-1} \exp\left(-\frac{q+}{4\xi}\right) \leq C \xi \cdot \xi^{-m-k-1/2} \exp\left(-\frac{q+}{4\xi}\right)
\]
\[
\leq C \xi \left(\frac{4}{q_+}\right)^{m+k+1/2} (m+k+1/2)^{m+k+1/2} e^{-(m+k+1/2)}
\]
\[
\simeq C \xi \left(\frac{4}{q_+}\right)^{m+k+1/2} \Gamma(m+k+3/2)(m+k+1/2)^{-1/2}
\]
\[
= C \xi \left(\frac{4}{q_+}\right)^{m+k+1/2} \Gamma(m+k+1/2)(m+k+1/2)^{1/2}.
\]
With this, we get
\[
2^{-m} \int_{1/2}^1 \xi^{-m-k-1}(1-\xi^2)^m(-\log(1-\xi^2))^{-1/2} \exp\left(-\frac{q+}{4\xi}\right) d\xi
\]
\[
\leq C \left(\frac{4}{q_+}\right)^{m+k+1/2} \Gamma(m+k+1/2)(m+k+1/2)^{1/2}
\]
\[
\times 2^{-m} \int_{1/2}^1 \xi(1-\xi^2)^m(-\log(1-\xi^2))^{-1/2} d\xi
\]
\[
\leq C_k \frac{2^m}{q_+^{m+k+1/2}} \Gamma(m+k+1/2),
\]
where in the last step we made the change of variable $-\log(1-\xi^2) = w$ and noticed that the resulting integral is bounded by $\int_0^\infty e^{-(m+1)w} w^{-1/2} dw = \frac{\Gamma(1/2)}{(m+1)^{1/2}}$. □
The lemma below is Lemma 5.3 in [6].

**Lemma 3.8.** Let $c \geq -1/2$, $0 < B < A$, $\lambda > 0$ and $d \geq 0$. Then
\[
\int_0^1 \frac{(1 - s)^{c+d-1/2}}{(A - Bs)^{c+d+\lambda+1/2}} ds \leq \frac{C(d)}{A^{c+1/2}B^d(A - B)^\lambda},
\]
where
\[
C(d) = \begin{cases} \frac{\Gamma(d)\Gamma(\lambda)}{\Gamma(d+\lambda)}, & d > 0, \\ 1, & d = 0. \end{cases}
\]

The following Lemma will also be helpful.

**Lemma 3.9.** Let $\alpha \geq -1/2$, $a \geq 1$, $j = 1, 2, \ldots, k > 0$ and
\[
I_k = \int_{-1}^1 \int_0^1 \xi^{-k} \beta_\alpha(a_j(\xi)) \exp \left(-\frac{q+4}{4\xi}\right) d\xi \Pi_{\alpha+a_j}(ds).
\]
Then
\[
I_k \leq \frac{C_{\alpha,k}}{(x + y)^{2\alpha+1}(xy)^{a_j}|x - y|^{2k}}.
\]

**Proof.** By applying Lemma 3.7 with $m = \alpha + a_j$ and the change of variable $s = 1 - 2u$ we have
\[
I_k \leq C_{\alpha,k}4^{\alpha+a_j} \frac{\Gamma(\alpha + a_j + 1/2)}{\Gamma(\alpha + a_j + 1/2)} \int_0^1 \frac{u^{\alpha+a_j-1/2}(1 - u)^{\alpha+a_j-1/2}}{((x + y)^2 - 4xyu)^{\alpha+a_j+k+1/2}} du.
\]
Then, by Lemma 3.8 with $c = \alpha$, $d = a_j$ and $\lambda = k$ we conclude that
\[
I_k \leq \frac{C_{\alpha,k}}{(x + y)^{2\alpha+1}(xy)^{a_j}|x - y|^{2k}}.
\]

Now we pass to the proof of Proposition 3.4. Remember that we will prove (3.6) an (3.7) for $j \geq 1$.

3.1.1. **Growth estimates: proof of (3.6).** Let the kernel $R^{\alpha+a_j}(x, y)$ be as in (3.10). We write
\[
R^{\alpha+a_j}(x, y) = J_1 - \frac{1}{2}J_2 - \frac{1}{2}J_3,
\]
where
\[
J_1 := x \int_{-1}^1 \int_0^1 \beta_\alpha(a_j(\xi)) \exp \left(-\frac{q+4}{4\xi}\right) d\xi \Pi_{\alpha+a_j}(ds),
\]
\[
J_2 := \int_{-1}^1 (x + ys) \int_0^1 \xi^{-1} \beta_\alpha(a_j(\xi)) \exp \left(-\frac{q+4}{4\xi}\right) d\xi \Pi_{\alpha+a_j}(ds),
\]
and
\[
J_3 := \int_{-1}^1 (x - ys) \int_0^1 \xi \beta_\alpha(a_j(\xi)) \exp \left(-\frac{q+4}{4\xi}\right) d\xi \Pi_{\alpha+a_j}(ds).
\]

Note that each of the three integrands of the inner integrals above are positive.

Let us begin with the study of $J_2$. First, note that
\[
|J_2| \leq |x - y| \int_{-1}^1 \int_0^1 \xi^{-1} \beta_\alpha(a_j(\xi)) \exp \left(-\frac{q+4}{4\xi}\right) d\xi \Pi_{\alpha+a_j}(ds)
\]
so this case is reduced to that one of
and we can apply again Lemma 3.9 with $k = 1$. Concerning $J_{22}$, we
can consider two cases. First, the case $y > 2x$ is immediate, because it can be easily
seen that $J_{22} \leq C J_{21}$. For the other case, when $y \leq 2x$, we use Lemma 3.7, the
change of variable $s = 1 - 2u$ and (3.15) with $v = a + aj$ and $b = 1/2$, to get

$$J_{22} \leq C \sqrt{xJ} \frac{\Gamma(a + aj + 3/2)}{\Gamma(a + aj + 1/2)} \int_{-1}^{1} \frac{(1 + s)(1 - s^2)^{a+aj-1/2}}{q^{a+aj+3/2}} ds$$

$$= C \sqrt{xJ} \frac{\Gamma(a + aj + 3/2)4^{a+aj}}{\Gamma(a + aj + 1/2)} \int_{0}^{1} \frac{u^{a+aj-1/2}(1 - u)^{a+aj+1/2}}{((x + y)^2 - 4xyu)^{a+aj+3/2}} du$$

$$\leq C \sqrt{xJ} \frac{\Gamma(a + aj + 1/2)\sqrt{a + aj}}{\Gamma(a + aj + 1/2)^{2a+aj}} \frac{1}{\Gamma(aj + 1)} (x + y)^{2a+aj+1/2}|x - y|$$

$$\leq \frac{C}{(xy)^a \mu (B(x,|x-y|))}$$

where, in the last step, we applied Lemma 3.8 with $c = a$, $d = aj+1/2$ and $\lambda = 1/2$.

We continue with $J_3$. It follows from (3.13) that

$$|J_3| \leq C \int_{-1}^{1} \int_{0}^{1} \xi^{1/2} \beta_{a+aj}(\xi) \exp \left(-\frac{q+4\xi}{4\xi}\right) d\xi \Pi_{\alpha+aj}(ds)$$

$$\leq C \int_{-1}^{1} \int_{0}^{1} \xi^{-1/2} \beta_{a+aj}(\xi) \exp \left(-\frac{q+4\xi}{4\xi}\right) d\xi \Pi_{\alpha+aj}(ds)$$

and the required inequality is obtained by using Lemma 3.9 with $k = 1/2$.

Finally, we study $J_1$. We split the outer integral into two parts, $J_1 = x \int_{0}^{1} + x \int_{0}^{1} =: xJ_{11} + xJ_{12}$. For the first case, observe that $x < x - ys$. This and (3.13) imply

$$|J_{11}| \leq \int_{-1}^{0} |x - ys| \int_{0}^{1} \beta_{a+aj}(\xi) \exp \left(-\frac{q+4\xi-\xi q+4}{4\xi}\right) d\xi \Pi_{\alpha+aj}(ds)$$

$$\leq C \int_{-1}^{1} \int_{0}^{1} \xi^{-1/2} \beta_{a+aj}(\xi) \exp \left(-\frac{q+4\xi}{4\xi}\right) d\xi \Pi_{\alpha+aj}(ds)$$

and we can apply again Lemma 3.9 with $k = 1/2$. Concerning $xJ_{12}$, we have

$$|x|J_{12}| \leq \int_{-1}^{0} |x + ys| \int_{0}^{1} \xi^{-1/2} \beta_{a+aj}(\xi) \exp \left(-\frac{q+4\xi}{4\xi}\right) d\xi \Pi_{\alpha+aj}(ds),$$

so this case is reduced to that one of $J_2$, and we are done.

3.1.2. Smoothness estimates: proof of (3.7). Observe that

$$\frac{d}{dx} [(xy)^aj R^{a+aj}(x, y)] = (aj)x^{aj-1}y^aj R^{a+aj}(x, y) + (xy)^aj \frac{d}{dx} (R^{a+aj}(x, y)).$$

Therefore, our first aim is to get the estimate

$$\frac{aj}{x} R^{a+aj}(x, y) \leq \frac{C}{(xy)^a |x - y|^{\mu (B(x,|x-y|))}}.$$
Recall from the previous subsection that \( R^{\alpha+aj}(x, y) \) can be written in terms of three expressions \( J_1, J_2 \) and \( J_3 \). We will prove the estimate above for each one of the corresponding expressions. The proof is systematic and follows similar reasonings as in the growth estimates, so we sketch the hints.

Let us begin with \( J_2 \). As in the previous subsubsection, observe that

\[
\frac{aj}{x} |J_2| \leq C \frac{aj}{x} \frac{\Gamma(\alpha + aj + 3/2)4^{\alpha+aj}}{\Gamma(\alpha + aj + 1/2)} (|x-y| \int_0^1 u^{\alpha+aj-1/2}(1-u)^{\alpha+aj-1/2} \frac{du}{(x+y)^2 - 4xyu}^{\alpha+aj+3/2}) + y \int_0^1 u^{\alpha+aj-1/2}(1-u)^{\alpha+aj+1/2} \frac{du}{(x+y)^2 - 4xyu}^{\alpha+aj+3/2} =: I_{21} + I_{22}.
\]

Consider now two cases. First, if \( 2x \geq y \). For \( I_{21} \), by (3.15) with \( v = \alpha + aj \) and \( b = 1/2 \), and Lemma 3.8 with \( c = \alpha \), \( d = aj - 1/2 \) and \( \lambda = 3/2 \), we have

\[
I_{21} \leq C \frac{|x-y|}{x} \frac{\Gamma(\alpha + aj + 3/2)4^{\alpha+aj}}{\Gamma(\alpha + aj + 1/2)} \frac{aj}{\sqrt{\alpha + aj}} \int_0^1 \frac{(1-u)^{\alpha+aj-1}}{(x+y)^2 - 4xyu}^{\alpha+aj+3/2} du
\]

\[
\leq C_\alpha \frac{|x-y|}{x} \frac{\Gamma(\alpha + aj + 3/2)4^{\alpha+aj}}{\Gamma(\alpha + aj + 1/2)} \frac{aj}{\sqrt{\alpha + aj}} \frac{\Gamma(aj-1/2)}{\Gamma(aj+1)} \times \frac{1}{(x+y)^{2\alpha+1}(4xy)^{aj-1/2}|x-y|^3}
\]

\[
\leq \frac{C_\alpha}{(xy)^{aj}|x-y|\mu_\alpha(B(x, |x-y|))}.
\]

Similarly, for \( I_{22} \), we use (3.15) with \( v = \alpha + aj \) and \( b = 1 \), and Lemma 3.8 with \( c = \alpha \), \( d = aj \) and \( \lambda = 1 \) to get

\[
I_{22} \leq \frac{C_\alpha}{(xy)^{aj}|x-y|\mu_\alpha(B(x, |x-y|))}.
\]

Secondly, if \( 2x < y \). For \( I_{21} \), observe that \( |x-y| \sim y \), use that \( u^{\alpha+aj-1/2}(1-u) \leq C \) and apply Lemma 3.8 with \( c = \alpha \), \( d = aj - 1 \) and \( \lambda = 2 \), thus

\[
I_{21} \leq C a_j \frac{|x-y|}{x} \frac{\Gamma(\alpha + aj + 3/2)4^{\alpha+aj}}{\Gamma(\alpha + aj + 1/2)} \int_0^1 \frac{(1-u)^{\alpha+aj-3/2}}{(x+y)^2 - 4xyu}^{\alpha+aj+3/2} du
\]

\[
\leq C a_j \frac{|x-y|^2}{xy} \frac{\Gamma(\alpha + aj + 3/2)4^{\alpha+aj}}{\Gamma(\alpha + aj + 1/2)} \frac{\Gamma(aj-1)}{\Gamma(aj+1)} \frac{1}{(x+y)^{2\alpha+1}(4xy)^{aj-1}|x-y|^4}
\]

\[
\leq \frac{C_\alpha}{(xy)^{aj}|x-y|\mu_\alpha(B(x, |x-y|))}.
\]

For \( I_{22} \), using that \( (1-u) \leq C \) and \( y \leq 2(y-x) \), it is clear that \( I_{22} \leq CI_{21} \) and we are done.

Now we pass to \( \frac{aj}{x} J_3 \). Applying Lemma 3.7 and reasoning as in the previous subsubsection for \( J_3 \), we have

\[
\frac{aj}{x} |J_3| \leq C \frac{aj}{x} \frac{4^{\alpha+aj}\Gamma(\alpha + aj + 1)}{\Gamma(\alpha + aj + 1/2)} \int_0^1 \frac{u^{\alpha+aj-1/2}(1-u)^{\alpha+aj-1/2}}{(x+y)^2 - 4xyu}^{\alpha+aj+1} du.
\]

We distinguish two cases. If \( 2x \geq y \), we use (3.15) with \( v = \alpha + aj \) and \( b = 1/2 \), and Lemma 3.8 with \( c = \alpha \), \( d = aj - 1/2 \) and \( \lambda = 1 \) in order to get the result. If \( 2x < y \), the fact that \( u^{\alpha+aj-1/2}(1-u) \leq C \) and Lemma 3.8 with \( c = \alpha \), \( d = aj - 1 \) and \( \lambda = 3/2 \) yield the desired estimate.
Finally, with regard to \( \frac{a_j}{x} J_1 \), we split the outer integral into two parts, \( \frac{a_j}{x} J_1 = \frac{a_j}{x} \int_{-1}^{0} + \frac{a_j}{x} \int_{0}^{1} =: \frac{a_j}{x} I_{11} + \frac{a_j}{x} I_{12} \). As for the first summand, analogously to the treatment of \( x J_1 \) in the previous subsection, by (3.13) we have
\[
\frac{a_j}{x} I_{11} \leq C \frac{a_j}{x} \int_{-1}^{1} \int_{0}^{1} \xi^{-1/2} \beta_{\alpha+a_j}(\xi) \exp \left( -\frac{q+}{4\xi} \right) d\xi \Pi_{\alpha+a_j}(ds),
\]
and this case is reduced to the study of \( \frac{a_j}{x} |J_3| \) above. Concerning \( I_{12} \), observe that, with analogous reasonings as in the previous subsection,
\[
\frac{a_j}{x} I_{12} \leq C \frac{a_j}{x} \int_{-1}^{1} |x + ys| \int_{0}^{1} \xi^{-1} \beta_{\alpha+a_j}(\xi) \exp \left( -\frac{q+}{4\xi} \right) d\xi \Pi_{\alpha+a_j}(ds).
\]
and we treat the last expression as we did with \( \frac{a_j}{x} |J_2| \).

Taking into account (3.16), we proceed now with \((xy)^{a_j} d\frac{d}{dx}(R^{\alpha+a_j}(x, y))\). From the expression for the kernel in (3.10), we get
\[
(xy)^{a_j} \frac{d}{dx}(R^{\alpha+a_j}(x, y)) = (xy)^{a_j} \int_{-1}^{1} \int_{0}^{1} \left(1 - \frac{1}{2\xi} - \frac{\xi-2}{2}\right) \beta_{\alpha+a_j}(\xi) \exp \left( -\frac{q+}{4\xi} - \frac{q+\xi q-}{4\xi} \right) d\xi \Pi_{\alpha+a_j}(ds)
\]
\[
+ (xy)^{a_j} \int_{-1}^{1} \int_{0}^{1} \left(x - \frac{3x+ys}{2\xi} - \frac{(x-ys)\xi}{2}\right) \exp \left( -\frac{q+}{4\xi} - \frac{q+\xi q-}{4\xi} \right) d\xi \Pi_{\alpha+a_j}(ds).
\]

Concerning \( S_1 \), we get
\[
|S_1| \leq C \int_{-1}^{1} \int_{0}^{1} \xi^{-1} \beta_{\alpha+a_j}(\xi) \exp \left( -\frac{q+}{4\xi} \right) d\xi \Pi_{\alpha+a_j}(ds)
\]
and we use Lemma 3.9 with \( k = 1 \) to obtain the result.

The study of \( S_2 \) is more involved. We write
\[
S_2 = \sum_{m=1}^{5} S_{2m} = \sum_{m=1}^{5} \int_{-1}^{1} \int_{0}^{1} z_m(x, y, \xi) \beta_{\alpha+a_j}(\xi) \exp \left( -\frac{q+}{4\xi} - \frac{q+\xi q-}{4\xi} \right) d\xi \Pi_{\alpha+a_j}(ds),
\]
where
\[
z_1(x, y, \xi) = -x \left(\frac{x+ys}{2\xi}\right) \xi; \quad z_2(x, y, \xi) = -x \left(\frac{x-ys}{2\xi}\right) \xi; \quad z_3(x, y, \xi) = \left(\frac{x+ys}{2\xi}\right)^2;
\]
\[
z_4(x, y, \xi) = \left(\frac{x^2-y^2 s^2}{2\xi}\right) \xi; \quad \text{and} \quad z_5(x, y, \xi) = \left(\frac{x-ys}{2\xi}\right)^2.\]

Observe that, if \( s < 0 \) then \( |z_1| \leq \frac{|x^2-y^2 s^2|}{2\xi} \), otherwise, if \( s > 0 \) then \( |z_1| \leq \left(\frac{x+ys}{\xi}\right)^2 = |z_3| \). Note also that
\[
|z_4| \leq C \frac{x^2-y^2 s^2}{\xi}.
\]

Therefore, \( |S_{21}| \leq |S_{23}| + Q \), where
\[
Q := \int_{-1}^{1} \int_{0}^{1} \left|\frac{x^2-y^2 s^2}{\xi}\right| \beta_{\alpha+a_j}(\xi) \exp \left( -\frac{q+}{4\xi} - \frac{q+\xi q-}{4\xi} \right) d\xi \Pi_{\alpha+a_j}(ds).
\]

In this way, if we get the desired estimates for \( |S_{23}| \) and \( Q \), then we immediately obtain the same estimates for \( |S_{21}| \) and \( |S_{24}| \). Concerning \( Q \), by (3.13), we get
\[
Q \leq C \int_{-1}^{1} \int_{0}^{1} \left(\frac{x+ys}{\xi^{3/2}}\right) \beta_{\alpha+a_j}(\xi) \exp \left( -\frac{q+}{4\xi} \right) d\xi \Pi_{\alpha+a_j}(ds)
\]
\[
\begin{align*}
&\leq C|x-y| \int_{-1}^{1} \int_{0}^{1} \xi^{-3/2} \beta_{\alpha+aj}(\xi) \exp \left( - \frac{q_+}{4\xi} \right) \, d\xi \, \Pi_{\alpha+aj}(ds) \\
&+ Cy \int_{-1}^{1} (1+s) \int_{0}^{1} \xi^{-3/2} \beta_{\alpha+aj}(\xi) \exp \left( - \frac{q_+}{4\xi} \right) \, d\xi \, \Pi_{\alpha+aj}(ds) =: Q_1 + Q_2.
\end{align*}
\]

The estimate for \(Q_1\) follows immediately from Lemma 3.9 with \(k = 3/2\). As for \(Q_2\), we use Lemma 3.7 with \(k = 3/2\) and \(m = \alpha + aj\), (3.15) with \(v = \alpha + aj\) and \(b = 1/2\), and Lemma 3.8 with \(c = \alpha + 1/2\), \(d = aj\) and \(\lambda = 1\), so that

\[
Q_2 \leq C_\alpha (x+y) \frac{\Gamma(\alpha + aj + 1/2)}{\Gamma(\alpha + aj + 1/2) \sqrt{\alpha + aj}} \frac{1}{\Gamma(aj + 1)} \frac{(x+y)^{2\alpha + 2(4xy)^{aj}|x-y|^2}}{2C_{x-y}^2|\mu_\alpha(B(x, |x-y|))|}.
\]

Now, for \(|S_{23}|\) we can write

\[
|S_{23}| \leq C|x-y|^2 \int_{-1}^{1} \int_{0}^{1} \xi^{-2} \beta_{\alpha+aj}(\xi) \exp \left( - \frac{q_+}{4\xi} - \frac{\xi q_-}{4} \right) \, d\xi \, \Pi_{\alpha+aj}(ds) \\
+ Cy^2 \int_{-1}^{1} (1+s)^2 \int_{0}^{1} \xi^{-2} \beta_{\alpha+aj}(\xi) \exp \left( - \frac{q_+}{4\xi} - \frac{\xi q_-}{4} \right) \, d\xi \, \Pi_{\alpha+aj}(ds) \\
=: T_1 + T_2.
\]

The estimate for \(T_1\) follows immediately by Lemma 3.9 with \(k = 2\). Concerning \(T_2\) we distinguish two cases. If \(y > 2x\), then \(y^2 \simeq |x-y|^2\) and this case is reduced to the study of \(T_1\). If \(y \leq 2x\), then we use Lemma 3.7 with \(k = 2\) and \(m = \alpha + aj\), (3.15) with \(v = \alpha + aj\) and \(b = 1\), and Lemma 3.8 with \(c = \alpha\), \(d = aj + 1\) and \(\lambda = 1\), to obtain

\[
T_2 \leq \frac{C_\alpha}{(xy)^{aj}|x-y|\mu_\alpha(B(x, |x-y|))}.
\]

For \(S_{22}\), observe that \(|x| \leq C(|x-ys| + |x+ys|)|\), hence

\[
|S_{22}| \leq C \int_{-1}^{1} \int_{0}^{1} \frac{|x-ys|}{2} \beta_{\alpha+aj}(\xi) \exp \left( - \frac{q_+}{4\xi} - \frac{\xi q_-}{4} \right) \, d\xi \, \Pi_{\alpha+aj}(ds) \\
+ C \int_{-1}^{1} \int_{0}^{1} \frac{|x-ys|}{2} \beta_{\alpha+aj}(\xi) \exp \left( - \frac{q_+}{4\xi} - \frac{\xi q_-}{4} \right) \, d\xi \, \Pi_{\alpha+aj}(ds) \\
\leq S_{24} + P,
\]

where \(P\) is the second integral. The factor \(S_{24}\) was already studied above, since that case was reduced to the study of \(Q\). On the other hand, we use (3.13) with \(\theta = 2\), and we get

\[
P \leq C \int_{-1}^{1} \int_{0}^{1} \beta_{\alpha+aj}(\xi) \exp \left( - \frac{q_+}{4\xi} \right) \, d\xi \, \Pi_{\alpha+aj}(ds) \\
\leq C \int_{-1}^{1} \int_{0}^{1} \xi^{-1} \beta_{\alpha+aj}(\xi) \exp \left( - \frac{q_+}{4\xi} \right) \, d\xi \, \Pi_{\alpha+aj}(ds)
\]

and the estimate follows from Lemma 3.9 with \(k = 1\). Finally, note that \(S_{25} \leq P\) and we are done.
3.2. Proof of Proposition 3.3. The proof of Proposition 3.3 is a consequence of the extrapolation theorem of Rubio de Francia, see [20, 21], adapted to our setting. Such adaptation is the following.

**Theorem 3.10.** Let \( \alpha > -1 \). Suppose that for some pair of nonnegative functions \((f, g)\), for some fixed \( 1 \leq r < \infty \) and for all \( w \in A^\alpha_r \) we have

\[
\int_0^\infty g(x)^r w(x) \, d\mu_\alpha(x) \leq C \int_0^\infty f(x)^r w(x) \, d\mu_\alpha(x),
\]

with \( C \) depending only on \( w \). Then, for all \( 1 < p < \infty \) and all \( w \in A^\alpha_p \) we have

\[
\int_0^\infty g(x)^p w(x) \, d\mu_\alpha(x) \leq C \int_0^\infty f(x)^p w(x) \, d\mu_\alpha(x).
\]

**Proof.** We follow the proof given by J. Duoandikoetxea in [11, Theorem 3.1]. The main ingredients are the factorization theorem (see [11, Lemma 2.1]) and the construction of the Rubio de Francia weights \( R_f \) and \( RH \) (see [11, Lemma 2.2]) in our context. The first ingredient is available here because of the general factorization theorem proved by Rubio de Francia [21, Section 3]. For the second ingredient we use the maximal Hardy-Littlewood operator given in (3.5) to construct the weights \( R_f \) and \( RH \) as in Lemma 2.1 of [11]. Then the proof follows the same lines as in [11]. □

**Remark 3.11.** Originally the extrapolation theorem was given for sublinear operators, but it turns out that sublinearity is not necessary. Actually even the operator itself does not play any role and all the statements can be given in terms of pairs of nonnegative measurable functions. This observation was made by D. Cruz-Uribe and C. Pérez in [9, Remark 1.11] and then J. Duoandikoetxea adopted this setting in [11].

To deduce Proposition 3.3 from Theorem 3.10 we proceed in the following way. Let \( \{T_j\}_{j \geq 0} \) be a sequence of operators such that each one is bounded in \( L^s(\mathbb{R}^+, w \, d\mu_\alpha) \), for some \( 1 < s < \infty \) and for all \( w \in A^\alpha_s \), uniformly in \( j \geq 0 \). Then for \( 1 < r < \infty \) and for any sequence of functions \( f_j \) in \( L^r(\mathbb{R}^+, w \, d\mu_\alpha) \), we have

\[
\int_0^\infty \sum_{j=0}^\infty |T_j f_j(x)|^r w(x) \, d\mu_\alpha(x) \leq C \int_0^\infty \sum_{j=0}^\infty |f_j(x)|^r w(x) \, d\mu_\alpha(x),
\]

for all \( w \in A^\alpha_r \). Now, in the extrapolation theorem above we make the following choices:

\[
g = \left( \sum_{j=0}^\infty |T_j f_j|^r \right)^{1/r}, \quad f = \left( \sum_{j=0}^\infty |f_j|^r \right)^{1/r}.
\]

With this pair \( (f, g) \), the inequality (3.17) is just the hypothesis of Theorem 3.10. Therefore, for any \( 1 < p < \infty \) and all \( w \in A^\alpha_p \),

\[
\int_0^\infty \left( \sum_{j=0}^\infty |T_j f_j|^r \right)^{p/r} w(x) \, d\mu_\alpha(x) \leq C \int_0^\infty \left( \sum_{j=0}^\infty |f_j|^r \right)^{p/r} w(x) \, d\mu_\alpha(x),
\]

and the proof is completed.
4. Vector-valued inequalities for the fractional integrals for the Laguerre expansions of convolution type

This section is devoted to the analysis of a vector-valued inequality for the fractional integrals of the Laguerre expansions of convolution type. It arises associated to the angular part, $\frac{1}{2} \nabla_0$, of the operator $\delta$ defined in (2.2) and in the definition of the Riesz transform for the harmonic oscillator.

Let $(L_\alpha)^{-1/2}$ be the fractional integral of order $1/2$ for the Laguerre expansions as given in (3.3). Then, we define the operator

$$T^\alpha f(x) = \frac{1}{x}(L_\alpha)^{-1/2} f(x), \quad x \in \mathbb{R}^+.$$ 

The boundedness properties of this operator are contained in the following theorem.

**Theorem 4.1.** Let $\alpha \geq -1/2$, $\alpha \geq 1$, and $1 < p, r < \infty$. Define $u_j(x) = x^{\alpha j}$, $j = 1, 2, \ldots$. Then there exists a constant $C$ such that

$$\left\| \left( \sum_{j=1}^{\infty} |u_j T^{\alpha+j}(u_j^{-1} f_j)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^+, w \, d\mu_\alpha)} \leq C \left\| \left( \sum_{j=0}^{\infty} |f_j|^r \right)^{1/2} \right\|_{L^p(\mathbb{R}^+, w \, d\mu_\alpha)}$$

for all $w \in A^p_\alpha$. Moreover the constant $C$ depends only on $\alpha$ and $w$.

We will deduce the result by using Proposition 3.3 and the following proposition.

**Proposition 4.2.** Let $\alpha \geq -1/2$, $\alpha \geq 1$, and $1 < p < \infty$. Define $u_j(x) = x^{\alpha j}$, $j = 1, 2, \ldots$. Then

$$\int_0^\infty |u_j T^{\alpha+j}(u_j^{-1} f_j)(x)|^p w(x) \, d\mu_\alpha(x) \leq C \int_0^\infty |f(x)|^p w(x) \, d\mu_\alpha(x),$$

for every weight $w \in A^p_\alpha$, where $C$ is a constant independent of $j$ and depending on $\alpha$ and $w$.

The previous proposition follows from the weighted Calderón-Zygmund theory.

**Proposition 4.3.** Let $\alpha \geq -1/2$ and $\alpha \geq 1$. Define $u_j(x) = x^{\alpha j}$, $j = 1, 2, \ldots$. Then

$$\int_0^\infty |u_j T^{\alpha+j}(u_j^{-1} f_j)(x)|^2 \, d\mu_\alpha(x) \leq C \int_0^\infty |f(x)|^2 \, d\mu_\alpha(x),$$

for every $f \in L^2(\mathbb{R}^+, d\mu_\alpha)$, with $C$ independent of $j$.

**Proof.** Take $g = u_j^{-1} f$. Then, inequality (4.1) is implied by

$$\int_0^\infty |(\alpha + aj)T^{\alpha+j}g(x)|^2 \, d\mu_{\alpha + aj}(x) \leq C \int_0^\infty |g(x)|^2 \, d\mu_{\alpha + aj}(x).$$

The identity for the Laguerre polynomials [1, p. 783, 22.7.29]

$$L_{k+1}^{\beta+1}(x) = \frac{1}{x}(x - k - 1)L_k^{\beta+1}(x) + (\beta + k + 1)L_k^{\beta+1}(x)$$

can be rewritten as

$$\frac{\beta}{x} L_k^{\beta}(x) = L_{k+1}^{\beta+1}(x) - L_k^{\beta+1}(x) + \frac{k + 1}{x}(L_k^{\beta+1}(x) - L_k^{\beta}(x)).$$

Now we use the following [1, p. 783, 22.7.30]

$$L_k^{\beta-1}(x) = L_k^{\beta}(x) - L_k^{\beta}(x),$$

where $\lambda$ is a constant independent of $j$ and depending on $\alpha$ and $w$. Theorem 4.1 is deduced from Proposition 4.3.
applied to the differences \( L_k^β(x^2) - L_{k+1}^β(x^2) \) and \( L_k^β(x) - L_{k+1}^β(x) \) and we change \( x \mapsto x^2 \) to deduce that
\[
\frac{\beta}{x} L_k^β(x^2) = \frac{k+1}{x} L_{k+1}^{β-1}(x^2) + xL_k^{β+1}(x^2).
\]

Now from (4.3) and the definition of the functions \( \ell^α_k \) given in (3.2), we conclude
\[
\frac{\beta}{x} \ell^β_k(x) = \frac{k+1}{x} \ell^{α-1}_{k+1}(x) + x\sqrt{k+\beta+1} \ell^{β+1}_k(x).
\]

In this way, the left side of (4.2) is bounded by the sum of
\[
\int_0^\infty \left| \sum_{k=0}^\infty \frac{k+1}{4k+2α+2α_j+2} \ell^{α+j-1}_{k+1}(x) \right| \left| \left( g, \ell^{α+j}_k \right) \right| \left( g, \ell^{α+j}_k \right) \left( g, \ell^{α+j}_k \right) dμ_{α+j}(x),
\]
and
\[
\int_0^\infty \left| \sum_{k=0}^\infty \sqrt{k+α+α_j+1} \ell^{α+j+1}_k(x) \right| \left| \left( g, \ell^{α+j}_k \right) \right| \left( g, \ell^{α+j}_k \right) \left( g, \ell^{α+j}_k \right) dμ_{α+j}(x).
\]

But the two previous summands can be controlled by the right side of (4.2) with a constant \( C \) independent of \( j \).

The operator \( T^α f \) can be written as
\[
T^α f(x) = \int_0^\infty T^α(x,y) f(y) dμ_α(y)
\]
where
\[
T^α(x,y) = \frac{1}{x} \frac{1}{\sqrt{π}} \int_0^∞ G_{α,t}(x,y)t^{-1/2} dt.
\]

So, proceeding as in Proposition 3.5, we can see that the operators \( juj T^{α+α_j}(u_j^{-1} f) \) can be associated, in the Calderón-Zygmund sense, with the kernel
\[
j(xy)^{α+j} T^{α+α_j}(x,y) = \frac{j(xy)^{α_j}}{x} \frac{1}{\sqrt{π}} \int_0^∞ G_{α,t}(x,y)t^{-1/2} dt.
\]

For this kernel, the following estimates of Calderón-Zygmund type are verified.

**Proposition 4.4.** Let \( α \geq -1/2 \), \( α_j \geq 1 \), and \( j \geq 1 \). Then
\[
|j(xy)^{α_j} T^{α+α_j}(x,y)| \leq \frac{C_1}{μ_α(B(x,|x-y|))}, \quad x \neq y,
\]
\[
|j\nabla_{x,y}(xy)^{α_j} T^{α+α_j}(x,y)| \leq \frac{C_2}{|x-y|μ_α(B(x,|x-y|))}, \quad x \neq y,
\]
with \( C_1 \) and \( C_2 \) independent of \( j \), and where \( μ_α(B(x,|x-y|)) = \int_{B(x,|x-y|)} dμ_α \) and \( B(x,|x-y|) \) is the ball of center \( x \) and radius \( |x-y| \).

The proof of this proposition follows the lines of Proposition 3.4. In this case the kernel is written as
\[
T^α(x,y) = \frac{1}{x} \frac{1}{\sqrt{π}} \int_0^1 β_α(ξ) \int_0^1 \exp \left( -\frac{q+ξq - ξ}{4ξ/4} \right) Π_α(ds) dξ,
\]
where \( β_α \) is the function in (3.9). Then, to obtain the estimates in Proposition 4.4 we have to use Lemma 3.7, Lemma 3.8, and Lemma 3.9 as it was done in the proof of Proposition 3.4. The details are omitted.
5. Proof of Theorem 2.1

With the change $j = m - 2k$, we have

$$|Rf(x)|^2 = |\delta H^{-1/2} f(x)|^2 = \sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim \mathcal{H}_j} \sum_{k=0}^{\infty} \frac{c_{2k+j,k,\ell}(f)}{\sqrt{n + 2j + 4k}} \delta \phi_{2k+j,k,\ell}(x) \right|^2$$

$$= \left| x' \sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim \mathcal{H}_j} \sum_{k=0}^{\infty} \frac{c_{2k+j,k,\ell}(f)}{\sqrt{n + 2j + 4k}} (\frac{\partial}{\partial r} + r) (r^j \ell_{n/2-1+j}(r)) \mathcal{Y}_{j,\ell}(x') \right|^2$$

$$+ \frac{1}{r} \sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim \mathcal{H}_j} \sum_{k=0}^{\infty} \frac{c_{2k+j,k,\ell}(f)}{\sqrt{n + 2j + 4k}} r^j \ell_{n/2-1+j}(r) \mathcal{Y}_{j,\ell}(x') \right|^2$$

$$+ \left| x' \sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim \mathcal{H}_j} \sum_{k=0}^{\infty} \frac{c_{2k+j,k,\ell}(f)}{\sqrt{n + 2j + 4k}} r^j \ell_{n/2-1+j}(r) \mathcal{Y}_{j,\ell}(x') \right|^2,$$

where in the last step we used that $\langle x', \nabla_0 \mathcal{Y}_{j,\ell}(x') \rangle = 0$. Now, from the identity (see [19, Lemma 2.2])

$$\int_{\mathbb{R}^{n-1}} \langle \nabla_0 \mathcal{Y}_{j,\ell}(x'), \nabla_0 \mathcal{Y}_{j',\ell'}(x') \rangle \, d\sigma(x') = j(2j + n - 2) \delta_{j,j'} \delta_{\ell,\ell'}$$

and by using that

$$c_{2k+j,k,\ell}(f) = \int_0^\infty (r^{-j} f_{j,\ell}(r)) \ell_{n/2-1+j}(r) r^{n-1+2j} \, dr,$$

where $f_{j,\ell}$ is given by (2.3), it can be easily checked that

$$\int_{\mathbb{R}^{n-1}} |Rf(x)|^2 \, d\sigma(x') \leq \sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim \mathcal{H}_j} \left| r^j \mathcal{R}^{n/2-1+j}((\cdot)^{-j} f_{j,\ell})(r) \right|^2$$

$$+ C \sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim \mathcal{H}_j} \frac{2j + n - 2}{j} \left| j r^j \mathcal{T}^{n/2-1+j}((\cdot)^{-j} f_{j,\ell})(r) \right|^2.$$

Then Theorem 2.1 is an immediate consequence of Theorem 3.1 and Theorem 4.1 because the double sum $\sum_{j=0}^{\infty} \sum_{\ell=1}^{\dim \mathcal{H}_j}$ can be rearranged to become a single sum $\sum_{j=0}^{\infty}$.

**Acknowledgement.** We are very thankful to the two referees of this paper for helpful and valuable remarks. We are also greatly indebted to Gustavo Garrigós for some fruitful discussions on the main topic of this paper.

**References**

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover Publications, New York, 1972.
[2] P. Balodis and A. Córdoba, The convergence of multidimensional Fourier-Bessel series, *J. Anal. Math.* 77 (1999), 269-286.

[3] A. Benedek and R. Panzone, The space $L^p$ with mixed norm, *Duke Math. J.* 28 (1961), 301-324.

[4] A. P. Calderón, Inequalities for the maximal function relative to a metric, *Studia Math.* 57 (1976), 297-306.

[5] A. Carbery, E. Romera, and F. Soria, Radial weights and mixed norm inequalities for the disc multiplier, *J. Funct. Anal.* 109 (1992), 52-75.

[6] Ó. Ciaurri and L. Roncal, Vector-valued extensions for fractional integrals of Laguerre expansions, preprint 2012, arXiv:1212.4715.

[7] A. Córdoba, The disc multiplier, *Duke Math. J.* 58 (1989), 21-29.

[8] K. Coulembier, H. De Bie, and F. Sommen, Orthogonality of Hermite polynomials in superspace and Mehler type formulae, *Proc. Lond. Math. Soc.* 103 (2011), 786-825.

[9] D. Cruz-Uribe and C. Pérez, Two weight extrapolation via the maximal operator, *J. Funct. Anal.* 174 (2000), 1-17.

[10] J. Duoandikoetxea, *Fourier analysis*, American Mathematical Society, Providence, 2001.

[11] J. Duoandikoetxea, Extrapolation of weights revisited: new proofs and sharp bounds, *J. Funct. Anal.* 260 (2011), 1886-1901.

[12] E. Harboure, L. de Rosa, C. Segovia, and J. L. Torrea, $L^p$-dimension free boundedness for Riesz transforms associated to Hermite functions, *Math. Ann.* 328 (2004), 653–682.

[13] N. N. Lebedev, *Special functions and its applications*, Dover, New York, 1972.

[14] F. Lust-Piquard, Dimension free estimates for Riesz transforms associated to the harmonic oscillator on $\mathbb{R}^n$, *Potential Anal.* 24 (2006), 47–62.

[15] B. Muckenhoupt and E. M. Stein, Classical expansions and their relation to conjugate harmonic functions, *Trans. Amer. Math. Soc.* 118 (1965), 17–92.

[16] A. Nowak and K. Stempak, Negative powers of Laguerre operators, *Canad. J. Math.* 64 (2012), 183–216.

[17] A. Nowak and K. Stempak, Riesz transforms and conjugacy for Laguerre function expansions of Hermite type, *J. Funct. Anal.* 244 (2007), 399-443.

[18] A. Nowak and K. Stempak, Riesz transforms for multi-dimensional Laguerre function expansions, *Adv. Math.* 215 (2007), 642-678.

[19] T. E. Pérez, M. A. Piñar, and Y. Xu, Weighted Sobolev orthogonal polynomials on the unit ball, *J. Approx. Theory* 171 (2013), 84-104.

[20] J. L. Rubio de Francia, Factorization and extrapolation of weights, *Bull. Amer. Math. Soc. (N.S.)* 7 (1982), 393-395.

[21] J. L. Rubio de Francia, Factorization theory and $A_p$ weights, *Amer. J. Math.* 106 (1984), 533-547.

[22] J. L. Rubio de Francia, Transference principles for radial multipliers, *Duke Math. J.* 58 (1989), 1-19.

[23] E. M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Annals of Mathematics Studies 63, Princeton Univ. Press, Princeton, NJ, 1970.

[24] K. Stempak and J. L. Torrea, Poisson integrals and Riesz transforms for Hermite function expansions with weights, *J. Funct. Anal.* 202 (2003), 443-472.

[25] S. Thangavelu, Riesz transforms and the wave equation for the Hermite operator, *Comm. Partial Differential Equations* 15 (1990), 1199-1215.

[26] S. Thangavelu, On conjugate Poisson integrals and Riesz transforms for the Hermite function expansions, *Colloq. Math.* 64 (1993), 103–113.

[27] S. Thangavelu, *Lecture notes on Hermite and Laguerre expansions*, Princeton Univ. Press., Princeton, NJ, 1993.

Departamento de Matemáticas y Computación, Universidad de La Rioja, 26004 Logroño, Spain

E-mail address: oscar.ciaurri@unirioja.es, luz.roncal@unirioja.es