The Hilbert Space Representations for SO$_q$(3)-symmetric Quantum Mechanics

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Abstract: The observable algebra $\mathcal{O}$ of SO$_q$(3)-symmetric quantum mechanics is generated by the coordinates $P_i$ and $X_i$ of momentum and position spaces (which are both isomorphic to the SO$_q$(3)-covariant real quantum space $\mathbb{R}^3_q$). Their interrelations are determined with the quantum group covariant differential calculus. For a quantum mechanical representation of $\mathcal{O}$ on a Hilbert space essential self-adjointness of specified observables and compatibility of the covariance of the observable algebra with the action of the unitary continuous corepresentation operator of the compact quantum matrix group SO$_q$(3) are required. It is shown that each such quantum mechanical representation extends uniquely to a self-adjoint representation of $\mathcal{O}$. All these self-adjoint representations are constructed. As an example an SO$_q$(3)-invariant Coulomb potential is introduced, the corresponding Hamiltonian proved to be essentially self-adjoint and its negative eigenvalues calculated with the help of a $q$-deformed Lenz vector.
1. Introduction

Symmetries play an important role in physics. Quantum groups as symmetries of physics had been discovered in the context of integrable systems. They are however expected to play a more general role \[1\] and are investigated as symmetries of some non-commutative spacetime \[2, 3, 4\]. It is hoped that there will be a quantum group symmetric formulation of four-dimensional quantum field theory. On the way to quantum field theory with \(q\)-deformed symmetry first one-particle quantum mechanics has to be understood. In this paper the quantum mechanics of one particle, moving in some \(q\)-deformed \(IR_q(3)\)-space with the configuration space symmetry \(SO_q(3)\), is formulated and investigated.

The outline of this paper is as follows.

In section 2 the compact matrix quantum group \(SO_q(3)\) is introduced. To fix the notation also the notions of quantum group covariance, corepresentation and covariant algebras are recalled. The quantum space \(IR_q^3\) is described by the algebra generated by its three coordinates \(X_i\).

In the next section the observable algebra is introduced as covariant differential operators acting on the functions over the quantum space \(IR_q^3\). While \(X_i\) act by multiplication the momenta \(P_i\) are constructed from the \(SO_q(3)\)-covariant partial derivative operators and shown to describe also a real quantum space \(IR_q^3\) which is henceforth understood as momentum space. The relations between position and momentum operators involve a scaling operator \(\mu\) and the quantized universal enveloping algebra dual to \(SO_q(3)\) represented as angular momentum algebra. Its operators can be characterized as those commuting with both scalar observables \(X^2\) and \(P^2\). The observable algebra exhibits a manifest symmetry between position and momentum coordinates.

The observable algebra has to be represented on a Hilbert space. And the symmetry has to be given by a unitary operator. It is also required that the topological structure of the compact quantum matrix group \(SO_q(3)\) shows up in this representation. Therefore some facts about unitary continuous corepresentations of compact quantum matrix groups are recalled in this section \[4\].

In section 5 some basic notions on Hilbert space representations of unbounded operator algebras are recalled. Then the quantum mechanical representation of \(O\) is defined and motivated. This definition guarantees the compatibility between the topological structure of the compact matrix quantum group and the covariance of the observable algebra. The scalar operators \(P^2\) and \(X^2\) are required to be observable, i. e. represented by essentially self-adjoint operators. In the rest of this section
the consequences of this definition are studied. It comes out that quantum mecha-
nical representation can always be uniquely extended to a self-adjoint representation.
And all these self-adjoint representations can be constructed.

Finally in section 6 a basic result on the self-adjointness of operators of the form $H = P^2 + V$ — to be understood as Hamilton operators of $SO_q(3)$-quantum mechanics — is derived and applied as $SO_q(3)$-symmetric Coulomb potential problem.

2. The $SO_q(3)$-symmetry

Geometrical manifolds can be characterized by algebras of complex functions over
these manifolds which are (in usual geometry) commutative. Quantum space alge-
bras are obtained as deformations of polynomial algebras over manifolds where the
parameter $q$ governs the non-commutativity of the quantum space algebras. The
reality of the manifold leads to a *-algebra structure for the quantum space algebra.
Specializing the manifolds to algebraic Lie groups the polynomial algebra generated
by the Matrix entries of the fundamental representation are $q$-deformed. The group
structure becomes encoded in an additional Hopf algebra structure realized on these
polynomials. In this way the quantum groups can be defined. This will be reviewed
here for the Lie group $SO(3)$ which has a standard $q$-deformation [5, 6]. Consid-
ering $q$-deformations of polynomials of representation spaces of Lie groups one obtains
comodule algebras on which the quantum group algebras coact.

For the definition of the quantum group $SO_q(3)$ one chooses usually the irreducible
representation of $SO(3)$ generated by the orthonormal vectors $rac{1}{\sqrt{2}}(\vec{e}_x + i\vec{e}_y)$, $\vec{e}_z$ and
$rac{1}{\sqrt{2}}(\vec{e}_x - i\vec{e}_y)$ in $\mathbb{C}^3$.

The definition of the quantum group $SO_q(3)$ is based on the fundamental invariant
tensors which are $q$-deformations of the corresponding invariant tensors of the Lie
group $SO(3)$.

2.1. Fundamental tensors of $SO_q(3)$. The deformation parameter $q$ is restricted
by $q \geq 1$. The $q$-antisymmetric tensor $\{\epsilon_{ijk}\}_{ijk=1,2,3}$ and the metric tensor $\{\gamma_{ij}\}_{ij=1,2,3}$
are defined by

$$\epsilon_{1jk} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & \frac{1}{q} \\
0 & -1 & 0
\end{pmatrix}_{jk},$$

$$\epsilon_{2jk} = \begin{pmatrix}
0 & 0 & -1 \\
0 & \sqrt{q} & 0 \\
1 & 0 & 0
\end{pmatrix}_{jk},$$

$$\gamma_{ij} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.$$
\[ \epsilon_{3jk} = \begin{pmatrix} 0 & 1 & 0 \\ -q & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{jk} \quad \text{and} \quad \gamma_{jk} = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{q}} \\ 0 & 1 & 0 \\ \sqrt{q} & 0 & 0 \end{pmatrix}_{jk}. \]

Further the tensors with upper indices \( \{ \gamma^{ij} \}_{ij=1,2,3} \) and \( \{ \epsilon^{ijk} \}_{ijk=1,2,3} \) are defined by

\[ \gamma^{ij} := \gamma_{ij} \quad \text{and} \quad \epsilon^{ijk} := \epsilon_{ijk}. \]

For \( q = 1 \) the above defined tensors coincide with their usual counterparts and are \( q \)-deformations of the completely antisymmetric tensor (however multiplied by \( i \)) and the canonical metric on \( \mathbb{C}^3 \) (in the above chosen basis).

If not otherwise stated the sum convention for repeated indices (one lower, one upper) is always implied. It can be easily checked that \( \epsilon^{ijk} = \epsilon_{abc} \gamma^{cia} \gamma^{bjk} \gamma^{ak} \) and \( \epsilon_{abc} \gamma^{bc} = \epsilon_{abc} \gamma^{ab} = 0 \) hold. The obvious abbreviations \( \epsilon^{ab} := \epsilon_{nm} e^{mb} = \epsilon_{nm} \gamma^{pa} \gamma_{mb} \) and \( \epsilon_{ab}^k := \epsilon_{abc} \gamma^{lk} \) are introduced. They fulfill \( \epsilon_n^{ab} \epsilon_{ab}^m = (q + \frac{1}{q}) \delta_n^m \) and \( \epsilon^{lkn} \epsilon_{ab}^m \epsilon_{ka}^m = (q - 1 + \frac{1}{q}) \epsilon^{lmn} \).

These equations imply that the matrices \( \left( \frac{1}{q+\frac{1}{q}} \epsilon^{ija} \epsilon_{akl} \right) \) and \( \left( \frac{q-1}{q+1+\frac{1}{q}} \gamma^{ij} \gamma_{kl} \right) \) are orthogonal projectors on \( \mathbb{C} \otimes \mathbb{C} \) with three- and one-dimensional ranges respectively. They are understood as the \( q \)-deformed antisymmetrizer and metric projector.

The \( \hat{R} \)-matrix

\[ \hat{R}^{ij}_{kl} := q \delta^i_k \delta^j_l - \epsilon^{ija} \epsilon_{akl} + \left( \frac{1}{q} - 1 \right) \gamma^{ij} \gamma_{kl} \]

with inverse

\[ \hat{R}^{-1ij}_{kl} = \frac{1}{q} \delta^i_k \delta^j_l - \epsilon^{ija} \epsilon_{akl} + (q - 1) \gamma^{ij} \gamma_{kl} \]

obeys the Yang-Baxter-equation \( \hat{R}^{ij}_{ab} \hat{R}^{kl}_{mn} = \hat{R}^{ik}_{de} \hat{R}^{jd}_{lf} \hat{R}^{fe}_{mn} \) which states that this \( \hat{R} \)-matrix provides a representation of the braid group on tensor products of \( \mathbb{C}^3 \) \([3, 4]\) and should be understood as the deformed permutation action on these tensor products.

This preparation suggests the definition of a \( q \)-deformation of the polynomial algebra over the Lie group \( SO(3) \):

**2.2. Definition:** The (unital, associative) \( \mathbb{C} \)-algebra \( \mathcal{A}_{SO_q(3)} \) is generated by the matrix elements \( \{ M^i_j \}_{i,j=1,2,3} \) of the Matrix \( M \) with relations

\[ \epsilon_{ijk} \ M^i_a M^j_b M^k_c = \epsilon_{abc}, \]
\[ \gamma_{ij} \ M^i_a M^j_b = \gamma_{ab}, \]
\[ M^i_a M^j_b \gamma^{ab} = \gamma^{ij}. \]


On these generators the $\mathcal{C}$-antilinear and antimultilicative involution $\ast$ is defined by
\[
(M^i_j)^\ast := \gamma_j^b M^a_b \gamma_{ai}
\]
making $\mathcal{A}_{SO_q(3)}$ a $\ast$-algebra. The tensor product $\mathcal{A}_{SO_q(3)} \otimes \mathcal{A}_{SO_q(3)}$ becomes also a $\ast$-algebra with $(a \otimes b)(c \otimes d) = ac \otimes bd$ and $(a \otimes b)^\ast = a^\ast \otimes b^\ast$ and the unit $1 \otimes 1$.

On $\mathcal{A}_{SO_q(3)}$ one defines the comultiplication $\Delta : \mathcal{A}_{SO_q(3)} \to \mathcal{A}_{SO_q(3)} \otimes \mathcal{A}_{SO_q(3)}$ and the counit $e : \mathcal{A}_{SO_q(3)} \to \mathcal{C}$ as $\ast$-algebra homomorphisms by
\[
\Delta(M^i_j) = M^i_k \otimes M^k_j,
\]
\[
e(M^i_j) = \delta^i_j
\]
and the $\mathcal{C}$-linear antimultiplicative antipode $S : \mathcal{A}_{SO_q(3)} \to \mathcal{A}_{SO_q(3)}$ by
\[
S(M^i_j) = (M^i_j)^\ast.
\]
Thus $\mathcal{A}_{SO_q(3)}$ becomes a $\ast$-Hopf algebra \[7, \] the Hopf algebra $\mathcal{A}_{SO_q(3)}$ of the quantum group $SO_q(3)$.

For $q = 1$, in the case of the undeformed group, one finds immediately that in this definition $\mathcal{A}_{SO(3)}$ coincides with the polynomials of the matrix elements of the fundamental representation of $SO(3)$. For the Hopf algebra mappings one gets for $g, h \in SO(3)$ and a polynomial $F$: $\Delta(F)(g \otimes h) = F(gh)$, $e(F) = F(e)$ and $S(F)(g) = F(g^{-1})$. In the case $q = 1$ one can also see that all the structure of the algebraic group $SO(3)$ is completely contained in the $\ast$-Hopf algebra $\mathcal{A}_{SO_q(3)}$ as defined in \[22\].

2.3. The compact matrix quantum group $(\mathcal{A}_{SO_q(3)}, M)$. There exists a $C^*$-algebra $\mathcal{A}_{SO_q(3)}$ in which the $\ast$-algebra $\mathcal{A}_{SO_q(3)}$ is densely imbedded and the comultiplication $\Delta$ can be extended on $\mathcal{A}_{SO_q(3)}$ as a $C^*$-homomorphism. Then $(\mathcal{A}_{SO_q(3)}, M)$ is a compact matrix quantum group \[8\].

**Proof:** $\mathcal{A}_{SO_q(3)}$ is the sub-$\ast$-Hopf algebra of even degree polynomials of the $\ast$-Hopf algebra of $SU_{\sqrt{q}}(2)$ which has been proved to belong to a compact matrix quantum group \[11, 8\].

2.4. Corepresentations and covariant algebras. The usual notion of representation can now be formulated for quantum groups. For a Hopf algebra $\mathcal{A}$ an $\mathcal{A}$-corepresentation $(V, \Delta_R)$ consists of a $\mathcal{C}$-vector space $V$ with a coaction $\Delta_R : V \to V \otimes \mathcal{A}$ obeying $(\Delta_R \otimes \text{id}) \circ \Delta_R = (\text{id} \otimes \Delta) \circ \Delta_R$ (coassociativity) and $(\text{id} \otimes e) \circ \Delta_R = \text{id}$ (counit). If $V$ is a Hilbert space with $v^i := \langle v | i \rangle$ compatible with the coaction, $(\dagger \otimes \ast) \circ \Delta_R = \Delta_R \circ \dagger$, then $(V, \Delta_R)$ is called unitary. $W \subset V$ is an $\mathcal{A}$-invariant subspace if the restriction of $\Delta_R$ to $W$ defines an $\mathcal{A}$-corepresentation $(W, \Delta_R|_W)$. The
A-corepresentation \((V, \Delta_R)\) is called \(A\)-irreducible if the only \(A\)-invariant subspaces are \(V\) and \(\{0\}\). If the \(A\)-corepresentation space \(V\) is an algebra and the coaction \(\Delta_R\) is an algebra homomorphism \((V, \Delta_R)\) is called an \(A\)-covariant algebra. If \(V\) is in addition a \(*\)-algebra and \((\star \otimes \star) \circ \Delta_R = \Delta_R \circ \star\), then \((V, \Delta_R)\) is called an \(A\)-covariant \(*\)-algebra.

For \(A_{SO_q(3)}\) one has special corepresentations: The scalar corepresentation is given by a one-dimensional vector space and the coaction \(\Delta_R(A) = 1.0\). A vector corepresentation consists of a 3-dimensional \(C\)-vector space \(V\) with basis \(\{A_i\}_{i=1,2,3}\) and the coaction \(\Delta_R(A_i) = A_j \otimes M^j_i\) on them. This basis \(\{A_i\}\) is called an \(SO_q(3)\)-triplet. If the triplet \(\{A_i\}\) is contained in an \(A_{SO_q(3)}\)-covariant \(*\)-algebra and fulfills \(A_i^* = \gamma^{ik} A_k\) then \(\{A_i\}\) is called a real triplet. The real linear combinations will also be used later, they are denoted \(X_i R^{\alpha}_\alpha =: X^R_{\alpha}\) for \(\alpha = 1,2,3\) with the matrix

\[
R^i_{\alpha} := \begin{pmatrix}
\sqrt{\frac{q+1}{2(q^2+1)}} & 0 & \sqrt{\frac{1}{2(q^2+1)}} \\
0 & 1 & 0 \\
i \sqrt{\frac{q+1}{2(q^2+1)}} & 0 & -i \sqrt{\frac{1}{2(q^2+1)}},
\end{pmatrix}
\]

Two (real) \(SO_q(3)\)-triplets contained in a \(A_{SO_q(3)}\)-covariant \(*\)-algebra give rise to the (real) \(SO_q(3)\)-scalar \(A \cdot B := A_i B_j \gamma^{ij}\) and the (real) \(SO_q(3)\)-triplet \([A \times B]_k := \epsilon^{ij} A_i B_j\).

Having now defined the symmetry and its corepresentation one needs still the notion of quantum space, the space on which these symmetries act. It will be constructed from the fundamental corepresentation of the quantum group. Again it is described by the polynomials of its coordinates.

2.5. The quantum space algebra \(A^X_{\mathbb{R}^3_q}\). Let \(\{X_i\}\) be a real triplet. Then the \(A_{SO_q(3)}\)-covariant \(*\)-algebra \(A^X_{\mathbb{R}^3_q}\) is generated by \(\{X_i\}_{i=1,2,3}\) with relations

\[
[X \times X]_k = 0 \quad \text{and} \quad X_i^* = \gamma^{ij} X_j.
\]

In \(A^X_{\mathbb{R}^3_q}\) the scalar \(X^2 := X \cdot X\) is central and real, i.e. \((X^2)^* = X^2\). In the real basis it is \(X^2 = \frac{q+1}{2} \sum_{\alpha=1}(X^R_\alpha)^2\).

3. Differential calculus and Observable algebra

In non-commutative geometry differential calculus is defined purely algebraically. For a covariant differential calculus on a quantum space \([10,2]\) one requires the exis-
tence of covariant operators as deformations of partial derivatives. For the quantum space $\mathcal{F}_q^3$ this had been worked out in \[4\].

### 3.1. Partial derivatives and scaling operator.

The partial derivatives \{\partial_i\}_{i=1,2,3}, linear operators acting on $\mathcal{A}^X_{\mathbb{R}_q^3}$, are defined by $\partial_i(1) = 0$ and the generalized Leibniz rule $\partial_i(X f) = \gamma_{ij} f + X_a \partial_j \left( \frac{1}{q} \hat{R}_{ij}^a b f \right)$ for each $f \in \mathcal{A}^X_{\mathbb{R}_q^3}$. The scaling operator $\mu : \mathcal{A}^X_{\mathbb{R}_q^3} \rightarrow \mathcal{A}^X_{\mathbb{R}_q^3}$ is the linear operator defined by $\mu(1) = 1$ and $\mu(X f) = q X f \Lambda(f)$ for each $f \in \mathcal{A}^X_{\mathbb{R}_q^3}$. The partial derivatives are covariant operators, $\Delta_R \left( \partial_k (f) \right) = (\partial_m \otimes M^m_k) (\Delta_R(f))$, the scaling operator acts as a scalar operator, $\Delta_R (\mu(f)) = (\mu \otimes \text{id}) (\Delta_R(f))$. The elements $X_i, \partial_j$ and $\mu$ act as linear operators on $\mathcal{A}^X_{\mathbb{R}_q^3}$ by left multiplication and left action respectively. They generate an algebra of operators on $\mathcal{A}^X_{\mathbb{R}_q^3}$ which is denoted by $\mathcal{D}^X_{\mathbb{R}_q^3}$. The partial derivatives fulfill $[\partial_k \partial_l]_k = 0$, the relations with the scaling operator are given by $\mu \partial_i = \frac{1}{q} \partial_i \mu$. The scaling operator is invertible in $\mathcal{D}^X_{\mathbb{R}_q^3}$ with $\mu^{-1} := \mu \left( 1 + q^{-2}(1 - q^2)X \cdot \partial + q^{-3}(1 - q^2)X \cdot X \cdot \partial \cdot \partial \right)$. Defining $\partial_i := \mu^2 \left( \partial_i + (q^{-2} - q^{-1})X_i \cdot \partial \cdot \partial \right)$ one gets a second triplet \{\partial_i\} of partial derivatives \[11\]. It obeys $\partial_i X_j = \gamma_{ij} + X_a \partial_b q \hat{R}^{-1ab}_{ij}$. Introducing on $\mathcal{D}^X_{\mathbb{R}_q^3}$ the $*$-involution defined on the generators by $(X_i)^* = \gamma^{ij} X_j$, $(\partial_i)^* = -q^3 \gamma^{ij} \partial_j$ and $\mu^* = q^{-3} \mu^{-1}$, the differential operator algebra $\mathcal{D}^X_{\mathbb{R}_q^3}$ becomes an $\mathcal{A}_{SO(3)}$-covariant $*$-algebra \[11\] \[12\].

### 3.2. The momentum coordinates.

In $\mathcal{D}^X_{\mathbb{R}_q^3}$ one finds the real triplet \{\Pi_i\} with

$$\Pi_i := \frac{1}{i (1 + q^{-3})} \left( \partial_i + q^{-3} \partial_i \right).$$

which fulfills

$$[\Pi \times \Pi]_k = 0 \quad \text{and} \quad \Pi_i^* = \gamma^{ij} \Pi_j$$

and generates in $\mathcal{D}^X_{\mathbb{R}_q^3}$ a covariant $*$-subalgebra which is isomorphic to $\mathcal{A}^X_{\mathbb{R}_q^3}$. (At this stage one finds only that the $\Pi_i$ fulfill all the defining relations for $\mathcal{A}^X_{\mathbb{R}_q^3}$. The isomorphy can be read off from the symmetry \[3\] \[4\] or the representations of this algebra discussed later.) This new quantum space will be understood as the quantum mechanical momentum space. This leads to the definition of the observable algebra below in \[4\].

The relations between position and momentum coordinates, i.e. the $q$-Heisenberg relations, still have to be investigated:

\begin{align}
\left( P_a X_b - X_c P_d q \hat{R}^{-1cd}_{ab} \right) &=: \mu^{-1} \left( \gamma_{ab} W + \epsilon_{ab}^m \frac{q-1}{q^2-q+1} L_m \right), \\
-\left( X_a P_b - P_c X_d q \hat{R}^{-1cd}_{ab} \right) &=: \mu^*^{-1} \left( \gamma_{ab} W + \epsilon_{ab}^m \frac{q-1}{q^2-q+1} L_m \right)
\end{align} (3)
with the real vector triplet \( \{L_i\} \) and the real scalar \( W \) in \( D_{\mathbb{R}^3_q}^X \). The commutation relations with the coordinates and momenta are

\[
L_i Z_j = -\epsilon_i^{cd} \epsilon_{dj}^e Z_c L_e + \epsilon_{ij}^a Z_a W, \quad L \cdot Z = Z \cdot L = 0,
\]

\[
W Z_j = (q - 1 + \frac{1}{q}) Z_j W - (\sqrt{q} - \frac{1}{\sqrt{q}}) \epsilon_{i}^{rs} Z_r L_s
\]

with \( Z_i = X_i, P_i \). In particular it holds \( L_i Z^2 = Z^2 L_i \) and \( W Z^2 = Z^2 W \). This suggests already the name \textit{angular momentum algebra} for the algebra generated by \( \{L_i, W\} \) with the relations

\[
L \cdot L = \frac{W^2 - 1}{(\sqrt{q} - \frac{1}{\sqrt{q}})^2}, \quad [L \times L]_k = L_k W, \quad L_k W = W L_k
\]

and becomes justified by the following result. The relations involving \( \mu = q^{-3} \mu^{*-1} \) are

\[
\mu P_i = \frac{1}{q} P_i \mu, \quad \mu X_i = q X_i \mu, \quad \mu L_i = L_i \mu, \quad \mu W = W \mu.
\]

For the representation theory the following relations are crucial.

3.3. Considering the equation \( [X \times [X \times \partial]]_k = -X_k X \cdot \partial X + X \cdot X \partial_k \) and the \(*\)-conjugated equation one derives in \( D_{\mathbb{R}^3_q}^X \)

\[
X^2 P_i = \frac{C_q}{i (q - 1)} \left[ q^{-1} X_i W \mu^{-1} - q W \mu^{-1} X_i + q^2 X_i W \mu - W \mu X_i \right]
\]

resulting in

\[
X^2 P^2 = C_q^2 \left[ \frac{(1 + q)^2}{q} W^2 - q \mu^2 - 2 - q^{-1} \mu^{-2} \right]
\]

with the constant \( C_q := q^2 (1 + q^3)^{-1} (1 - q)^{-1} \).

3.4. The dual algebra \( \mathcal{A}'_{SO_q(3)} \) of \( \mathcal{A}_{SO_q(3)} \). The space \( \mathcal{A}'_{SO_q(3)} \) of linear functionals on \( \mathcal{A}_{SO_q(3)} \) becomes a \(*\)-algebra with multiplication \( \tilde{\rho}, \tilde{\zeta} \mapsto (\tilde{\rho} \otimes \tilde{\zeta}) \circ \Delta \), unit \( \tilde{1} := e \) and involution \( \tilde{\rho}^*(.) = \tilde{\rho}((S(.)^*)^*). \) The \textit{quantized universal enveloping algebra of SO(3)}. \[ \mathfrak{g}, \mathfrak{h}, \mathfrak{h} \] is the \(*\)-subalgebra \( \mathcal{U}_{SO_q(3)} \) of \( \mathcal{A}'_{SO_q(3)} \) generated by \( \{\tilde{K}^\pm, \tilde{S}^\pm\} \). These linear functionals on \( \mathcal{A}_{SO_q(3)} \) are defined on the generators of \( \mathcal{A}_{SO_q(3)} \) by

\[
\tilde{K}^\pm(M^i) = \begin{pmatrix} q^{\mp 1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{\mp 1} \end{pmatrix}_{ij},
\]

\[
\tilde{L}^+(M^i) = \begin{pmatrix} 0 & \frac{1}{\sqrt{q}} & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}_{ij}, \quad \tilde{L}^-(M^i) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -\sqrt{q} & 0 \end{pmatrix}_{ij}
\]
while their action on elements of higher order of \( \mathcal{A}_{SO_q(3)} \) is defined inductively by

\[
\tilde{K}^{\pm 1}(ab) = \tilde{K}^{\pm 1}(a) \tilde{K}^{\pm 1}(b), \\
\tilde{L}^{\pm}(ab) = e(a)\tilde{L}^{\pm}(b) + \tilde{L}^{\pm}(a)\tilde{K}(b).
\]

They fulfill \( \tilde{L}^{\pm} \tilde{K} = q^{\pm 1} \tilde{K} \tilde{L}^{\pm}, \frac{1}{\sqrt{q}} \tilde{S}^{+} \tilde{S}^{-} = -\sqrt{q} \tilde{S}^{-} \tilde{S}^{+} = e^{-\tilde{K}^2} q^{-1}, \tilde{K}^* = \tilde{K} \) and \( (\tilde{S}^{-})^* = \sqrt{q} \tilde{S}^{+} \). This algebra possesses the real central element

\[
\tilde{W} := \frac{q}{1+q} \tilde{K}^{-1} + \frac{1}{1+q} \tilde{K} + q^{-\frac{1}{2}}(q-1)^{\frac{1}{2}} \tilde{K}^{-1} \tilde{L}^{-} \tilde{L}^{+}.
\]

Each (unitary) corepresentation \((V; \Delta_R)\) of \( \mathcal{A}_{SO_q(3)} \) is canonically a \((\ast,\text{-})\)-representation of \( \mathcal{A}'_{SO_q(3)} \) with \( \mathcal{A}'_{SO_q(3)} \ni \tilde{\rho} \mapsto (\text{id} \otimes \tilde{\rho}) \circ \Delta_R \).

It is clear from this construction that operators \( \tilde{\rho} \) thus lead always to operators on \( \mathcal{A}_{SO_q(3)}^{X} \) which commute with the action of the scalar operators, in particular \( X^2 \) and \( P^2 \). As examples one identifies as operators on \( \mathcal{A}_{\text{Rq}}^{X} \):

\[
(W, L_1, L_2, L_3) = \left( \text{id} \otimes \left( \tilde{W}, \tilde{L}^{+}, \frac{1}{\sqrt{q}}(\tilde{K} - \tilde{W}), \tilde{L}^{-} \right) \right) \circ \Delta_R
\]

preserving the \( \ast \)-structure.

The converse is also true, the property of commuting with \( P^2 \) and \( X^2 \) characterizes the elements of the symmetry algebra completely:

### 3.5. Proposition

The algebra of linear operators on \( \mathcal{A}_{\text{Rq}}^{X} \) commuting with the action of the operators \( X^2, P^2 \in \mathcal{D}_{\text{Rq}}^{X} \) is as an algebra isomorphic to the \( \ast \)-algebra \( \mathcal{A}'_{SO_q(3)} \) of linear functionals on \( \mathcal{A}_{SO_q(3)}^{X} \). For each operator \( \rho : \mathcal{A}_{\text{Rq}}^{X} \rightarrow \mathcal{A}_{\text{Rq}}^{X} \) with \( [\rho, X^2] = [\rho, P^2] = 0 \) there exists a unique \( \tilde{\rho} \in \mathcal{A}'_{SO_q(3)} \) such that \( \rho = (\text{id} \otimes \tilde{\rho}) \circ \Delta_R \).

**Proof:**  The vector space \( \mathcal{A}_{\text{Rq}}^{X} \) can be decomposed as \( \mathcal{A}_{\text{Rq}}^{X} = \bigoplus_{n,l \in \mathbb{N}_0} (X^2)^n Z_l \) where \( Z_l \) consists of homogeneous harmonic polynomials of degree \( l \) in the kernel of \( P^2 \). From [3,13] one reads off that \( Z_l \) is eigenspace of \( W \) with eigenvalue \( c_l = \frac{q^{2l+1} + q^{-l}}{q+1} \). Since the homogeneous polynomials of \( \mathcal{A}_{\text{Rq}}^{X} \) carry a corepresentation of \( \mathcal{A}_{SO_q(3)}^{X} \) and \( X^2 \) transforms trivially it follows that \( Z_l \) is an invariant space in \( \mathcal{A}_{\text{Rq}}^{X} \), therefore also a representation space of \( \mathcal{U}_{SO_q(3)} \). One proves at once that an irreducible representation of \( \mathcal{U}_{SO_q(3)} \) with \( \tilde{W} \mapsto c_l \) is \( 2l+1 \)-dimensional like \( Z_l \). Choosing an orthonormal basis \( \{|lm\} \) according to [1,1,3] this representation becomes a \( \ast \)-representation of \( \mathcal{U}_{SO_q(3)} \). That means that \( (Z_l, \Delta_R|Z_l) \) becomes an irreducible unitary corepresentation of \( \mathcal{A}_{SO_q(3)}^{X} \) with \( \Delta_R : |lm\rangle \mapsto |lm'\rangle \otimes t_{m'm}^{l} \) with \( t_{m'm}^{l} \in \mathcal{A}_{SO_q(3)} \) homogeneous of degree \( l \). [1,8] states that all the set \( \{t_{m'm}^{l} \}_{l,m,m'} \) is
linearly independent. From dimension counting one finds also that this set spans already $\mathcal{A}_{SO_q(3)}$. \{ $X^{2\alpha} |lm\rangle$ \} is a basis of $\mathcal{A}_{IR^3}^X$. From the action of $X^2$ and $P^2$ (using the relation in \ref{3.3}) on this basis one derives for $\rho$ with the properties required that

$$\rho(X^{2\alpha} |lm\rangle) = \sum_{m'} X^{2\alpha} |lm\rangle R^l_{m'm'}$$

with some matrix coefficients $R^l_{m'm'}$. The linear operator $\tilde{\rho} \in \mathcal{A}_{SO_q(3)}$: $t^l_{m'm'} \mapsto R^l_{m'm'}$ is then the solution. $\square$

Summarizing the above the algebra of observables of quantum mechanics on $IR^3_q$ should be defined as follows.

\textbf{3.6. Definition:} The observable algebra $\mathcal{O}$ of $SO_q(3)$-symmetric quantum mechanics is defined to be the $\mathcal{A}_{SO_q(3)}$-covariant $\ast$-algebra which is generated by the phase space coordinates $X_i$ and $P_i$.

The position coordinates $X_i$ and the momentum coordinates $P_i$ generate momentum and position quantum spaces isomorphic to $\mathcal{A}_{IR^3}^X$ (equations \ref{1,2}). Their $\mathcal{A}_{SO_q(3)}$-covariant $q$-commutator closes up to the scaling operator $\mu$ into the angular momentum algebra, generated by the real triplet \{ $L_i$ \} and the real scalar $W$, a subalgebra of the quantized universal enveloping algebra $\mathcal{U}_{SO_q(3)}$, according to equations (3–6).

\textbf{3.7. A $\ast$-automorphism of $\mathcal{O}$}. The symmetry between momentum and position coordinates appears as the $\ast$-algebra automorphism compatible with the quantum group covariance defined by

$$P_i \leftrightarrow -X_i, \quad X_i \leftrightarrow P_i$$

which lets the angular momentum algebra invariant, $S_i \leftrightarrow S_i$, $W \leftrightarrow W$, and maps the scaling operator on its $\ast$-conjugate, $\mu \mapsto \mu^\ast$.

It should be noticed that $\mu^{\pm 2} \in \mathcal{O}$ but $\mu \not\in \mathcal{O}$. Accordingly the operators $\mu^{\pm 1}W$, $\mu^{\pm 1}S_i$, $W^2$, $WS_i$, $WK$, ... $\in \mathcal{O}$ but $W, S_i \not\in \mathcal{O}$. However it will turn out that in each representation in a Hilbert space to be considered later a bounded Hilbert space operator $\mu$ can be uniquely defined which fulfills all the algebraic relations above.

\textbf{4. $\mathcal{A}_{SO_q(3)}$-covariant Hilbert space}

In quantum mechanics a symmetry is given by unitary transformations on the Hilbert space of states and a covariant transformation law for the observables \ref{5}. A continuous symmetry is described by a topological (more special a Lie) group $G$. This topological group is then represented on the Hilbert space $H$ by a continuous mapping $U : G \times H \rightarrow H$ which is also a group homomorphism.
Let us denote with $B(H)$ the bounded operators on the Hilbert space $H$, let $CB(H)$ be the compact operators on $H$. Further let $M(A)$ be the multiplier algebra of the $C^*$-algebra $A$.

### 4.1. Unitary continuous corepresentation.

A unitary continuous corepresentation $(U,H)$ of the compact quantum matrix group $(A,u)$ on the Hilbert space $H$ is given by a unitary operator $U \in M(CB(H) \otimes A) \subset B(H \otimes \ell)$ (where $\ell$ is a Hilbert space on which $A$ is faithfully represented) fulfilling the coaction property $U_{12}U_{13} = (id \otimes \Delta)(U)$. ($U_{12} := U \otimes 1$, $U_{13} := (id \otimes \sigma)(U_{12})$ with the linear transposition operator $\sigma : A \otimes B \mapsto B \otimes A$.) [17].

For a usual compact group $G$ a unitary continuous representation would be a continuous mapping $u : G \times H \rightarrow H$, $g,v \mapsto u(g)v$ or $U : G \rightarrow B(H)$, $g \mapsto U(g)$, i.e. an operator valued continuous (in the strong operator topology) function on $G$, and one could prove that this definition reproduces the above definition.

#### 4.2. Proposition

Let $(U,H)$ be a unitary continuous corepresentation of $(A,u)$. Then it decomposes in finite dimensional irreducible corepresentations $(U_\rho,H_\rho)$, i.e. $H = \bigoplus_{\rho \in \mathcal{I}} K_\rho$ with finite dimensional subspaces of $H$ for a certain index set $\mathcal{I}$. If $E_\rho$ is the orthogonal projector on $K_\rho$ then the restriction $U_\rho := U(E_\rho \otimes 1)$ belongs to $B(K_\rho) \otimes A$ where $A$ is the $*$-algebra generated by the matrix entries of $u$.

**Proof:** This was proved first in [17]. A derivation going along the proof of the classical result for usual compact groups [18] has been given in [19].

This can now be applied to the compact matrix quantum group $SO_q(3)$.

#### 4.3. Hilbert space $*$-representations of $U_{SO_q}(3)$.

Let $(U,H)$ be a unitary continuous corepresentation of $(A_{SO_q(3)},M)$. Let $(U_\rho,H_\rho)$ be as in [17]. Then one defines on the domain $\text{Span}(\bigcup_{\rho \in \mathcal{I}} H_\rho)$ a *-representation of $A'_{SO_q(3)}$ by $\tilde{\chi} \mapsto \sum_\rho (id \otimes \tilde{\chi})(U_\rho)$ by (possibly unbounded) closable operators on this domain. Let their closures be denoted by

$$\tilde{\chi}^U := \sum_\rho (id \otimes \tilde{\chi})(U_\rho).$$

The attention is restricted now to $U_{SO_q}(3)$. The operators $\tilde{K}^U$ and $\tilde{W}^U$ restricted to some $H_\rho$ are self-adjoint and commuting, hence $\tilde{K}^U$ and $\tilde{W}^U$ are self-adjoint and simultaneously diagonalizable on the whole Hilbert space $H$. From these *-representation properties one derives: The Hilbert space decomposes in irreducible representations of $U_{SO_q(3)}$ which are classified by $l \in \frac{1}{2} \mathbb{N}_0$ and $\sigma = \pm 1$: $H_{l,\sigma} =$
Span\{ |l, m\rangle | m = -l, -l + 1, ..., l \} is the finite-dimensional Hilbert space with orthonormal basis vectors \(|l, m\rangle\) [20],

\[
\tilde{K}^U |l, m\rangle = \sigma q^{-l} |l, m\rangle,
\]

\[
\tilde{W}^U |l, m\rangle = \sigma c_l |l, m\rangle
\]

with \(c_l = \frac{q^{l+1} + q^{-l}}{q+1} = 1 + \frac{[l+1]\sqrt{q}[l]\sqrt{q} - \frac{1}{\sqrt{q}}}{q}\),

\[
(\tilde{L}^\pm)^U |l, m\rangle = \alpha_{l,m\pm1} q^{\mp \frac{1}{2}} |l, m \pm 1\rangle
\]

with \(\alpha_{l,m\pm1} = \frac{q^{l+1}}{[2]\sqrt{q}} \sqrt{[l]\sqrt{q}[l+1]\sqrt{q} - [m]\sqrt{q}[m \pm 1]\sqrt{q}}\).

Here the symmetric \(q\)-numbers had been introduced

\[
[x]_p := \frac{p^x - p^{-x}}{p - p^{-1}}.
\]

After this preparation the operator \(U\) can be given explicitly. From the proof of [3,4] the inequivalent unitary corepresentations of \(A_{SO_q(3)}\) are known. They correspond to the integer values for \(l\). Furthermore this fixes \(\sigma = +1\). Thus there exists an orthonormal Hilbert space basis of \(H\) consisting of simultaneous eigenvectors of \(\tilde{W}^U\) and \(\tilde{K}^U\): \{ |\alpha, l, m\rangle : \alpha \in A_l; |m|, l \in \mathbb{N}_0; |m| \leq l \} \} with an \(l\)-dependent index set \(A_l\). The unitary operator \(U\) becomes

\[
U = \sum_{\alpha, l, m, m'} |\alpha, l, m') \langle \alpha, l, m | \otimes t^l_{m'm'}
\]

\[
= \sum_{l, m, m'} \prod_{\mu=0}^{l-m'} \frac{(\tilde{L}^-)^U}{\alpha_{l,l-\mu}} \tilde{P}^U_{l'\mu} \prod_{\mu=0}^{l-m} \frac{(\tilde{L}^-)^U}{\alpha_{l,l-\mu}} \otimes t^l_{m'm'}
\]

with the orthogonal Hilbert space projector \(\tilde{P}^U_{l'\mu}\) projecting on the simultaneous eigenspaces of \(\tilde{W}^U\) and \(\tilde{K}^U\) with eigenvalues \(c_l\) and \(q^{-l}\). The sum has to be understood in the strong operator topology. The matrices \((t^l_{m'm'})\) belong to the \(2l+1\)-dimensional corepresentation of \(A_{SO_q(3)}\), the unitarity is reflected by \(S(t^l_{m'm'}) = t^l_{m'm'}\).

5. Hilbert space of \(SO_q(3)\)-symmetric quantum mechanics

To formulate the definition of \(SO_q(3)\)-quantum mechanics some further points need to be mentioned. The operators in \(\mathcal{O}\) are like in the undeformed case \((q = 1)\) not
bounded. Since they are to be represented on a Hilbert space their domain can not be the full Hilbert space. Instead a domain is to be defined for each operator, and symmetric operators (i.e. those operators $O$ which coincide with their Hilbert space adjoints restricted to the domain of $O$) need not be diagonalizable, i.e. have a projector valued spectral measure.

On the other hand the most fundamental property of an observable is its measurability. That means that a measurement projects any state vector on an eigenspace (or approximate eigenspace in the case of an observable with continuous spectrum) of the measured observable, and the spectrum is real. This property of diagonalizability is fulfilled by self-adjoint operators, i.e. those which coincide with their Hilbert space adjoints.

To represent the observable algebra there must be a common dense domain $D$ of all the operators where the algebra relations of $O$ are fulfilled \cite{21}, the involution $*$ becomes represented by the Hilbert space adjoint. The observables are essentially self-adjoint on this domain.

**5.1. Hilbert space representations.** Let $D$ be a pre-Hilbert space with Hilbert space closure $H = \overline{D}$. Let $\pi$ be an algebra homomorphism of the $*$-algebra $A$ into the linear operators on $D$, $\pi : A \ni a \mapsto \pi(a)$ such that $1 \mapsto 1|_D = \text{id}_D$. Then $(\pi, D)$ will be called a *Hilbert space representation* of $A$. If the algebra involution $*$ becomes represented by the Hilbert space adjoint (the Hilbert space adjoint of the operator $A$ will always be denoted by $A^\dagger$), $\pi(a^*) = \pi(a)^\dagger|_D$, then $(\pi, D)$ will be called a *-representation of the $*$-algebra $A$ in the Hilbert space $H$ \cite{21}.

For each *-representation $(\pi, D)$ of a $*$-algebra $A$ in a Hilbert space $H$ one defines its *adjoint representation* $(\pi^\dagger, D^\dagger)$ with

$$D^\dagger = D(\pi^\dagger) := \bigcap_{a \in A} D \left( \pi(a)^\dagger \right) \quad \text{and} \quad \pi^\dagger(a) := \pi(a^*)^\dagger|_{D^\dagger}$$

which is a representation of $A$ but not necessary a $*$-representation. If a $*$-representation $\pi$ fulfills $\pi = \pi^\dagger$ then it is called a *self-adjoint representation* \cite{22, 21}.

**5.2. Definition.** The state space of $SO_q(3)$-symmetric quantum mechanics is a separable Hilbert space $\mathcal{H}$ with the following additional structure:

(i) $\mathcal{H}$ carries a unitary continuous corepresentation $(U, \mathcal{H})$ of the compact matrix quantum group $(A_{SO_q(3)}, M)$.

(ii) $\mathcal{H}$ carries a $*$-representation $(\pi, D)$ of the observable algebra $O$ with $D \subset \mathcal{H}$.
(iii) The scalars $X^2 := X \cdot X$ and $P^2 := P \cdot P$ in $O$ become represented by observables:
\[ \pi(X^2), \pi(P^2) \] are essentially self-adjoint operators in $\mathcal{H}$.

(iv) The elements of $O$ are represented by covariant operators:
\[ U \left( \pi(a) \otimes 1_{\mathcal{A}'_{SO_q(3)}} \right) U^* = \Delta_R(a) \]
for each element $a \in O$.

(v) The $SO_q(3)$-symmetry manifests itself completely in the observable algebra:
All elements $a \in O$ in the angular momentum algebra, i.e. $[a, P^2] = [a, X^2] = 0$, belong to the $\ast$-representation of $\mathcal{A}'_{SO_q(3)}$ induced by $(\mathcal{H}, U)$:
\[ \tilde{a}_{\mid D} = \pi(a). \]

If all these requirements are fulfilled, then $(\pi, D)$ is called a quantum mechanical representation of $O$.

5.3. Spin degrees of freedom. For the vector operators $\{X_i, P_i\}$ the covariance means certainly $U(\pi(X_i) \otimes 1_{\mathcal{A}'_{SO_q(3)}})U^* = \pi(X_j) \otimes M^j_i$ and $U(\pi(P_i) \otimes 1_{\mathcal{A}'_{SO_q(3)}})U^* = \pi(P_j) \otimes M^j_i$. This implies that the operators $WS_i$ and $W^2$ can be identified to belong to the representation of $\mathcal{U}_{SO_q(3)}$ on the observable algebra. But (v) is not implied by (iv) which only states that the $SO_q(3)$-covariance manifests itself on the observable algebra by the angular momentum algebra as orbital angular momentum. There could be hidden, i.e. internal, degrees of freedom also transforming under the symmetry perhaps even under the quantum group $SU_{\sqrt{q}}$. Let $(u, h)$ be some finite-dimensional irreducible continuous corepresentation of $(\mathcal{A}_{SU_{\sqrt{q}}(2)}, m)$ ($m$ being the matrix of the fundamental representation). Let $\tilde{H} := \mathcal{H} \otimes h$ and define the unitary operator $\tilde{U} := (id \otimes id \otimes m) \circ \sigma_{23}(U \otimes u)$. Let the observable algebra $\tilde{O} := O \otimes \mathcal{U}'_{SO_q(3)}$ be extended by the representation of $\mathcal{U}_{SO_q(3)}$ on $\mathcal{H} \otimes h$. Then also half-integer values for $l$ arise. In this way spin degrees of freedom have to be introduced in the observable algebra to characterize a quantum mechanical state completely. However these possibility is from now on excluded by 5.2 (v).

Because of 5.3 (iv) the $\pi$-representatives of the scalar operators $P^2$ and $X^2$ of $O$ commute on $D$ with the unitary corepresentation operator $U$ and hence also, because of (v), with operators $\tilde{\chi}^U$ derived from $\tilde{\chi} \in \mathcal{A}'_{SO_q(3)}$. Then one may hope that the operators $\pi(X^2)$ (or $\pi(P^2)$ respectively), $\tilde{W}^U$ and $\tilde{K}^U$ can be simultaneously diagonalized on $\mathcal{H}$. For such problems the following theorem will be used.

5.4. Proposition: Let $D_{\text{dense}} \subset \mathcal{H}$ be a common domain for the symmetric Hilbert space operators $a, c$ and $c'$. Let all three operators mutually commute on $D$. Let
$a$ be essentially self-adjoint, and $\pi \geq 1$. Let $c$ and $c'$ be $a$-bounded (i.e. $\exists \lambda > 0 : \|c\phi\| \leq \lambda(\|\phi\| + \|a\phi\|)$ for all $\phi \in D$).

Then $c$ and $c'$ are essentially self-adjoint on $D$. Any other core for $\pi$ is a core for $\pi$ and $\pi'$. All three operators mutually commute strongly.

A core of a closed Hilbert space operator $A$ with domain $D_A$ is a linear subspace $D' \subset D_A$ such that $A|_{D'} = A$. The notion of strong commuting of operators used here is borrowed from [21] and coincides with the notion of commuting in [23].

Proof: That $c$ and $c'$ are self-adjoint on any core for $a$ is a direct consequence of [24] with $N = a$ and $A = c, c'$. That $c$ and $c'$ commute strongly is contents of [25]. That $a$ and $c$ or $c'$ respectively commute strongly is contents of [26].

5.5. Proposition: In each quantum mechanical representation $(\pi, D)$ the operators $\pi(X^2)$, $\tilde{K}^U$ and $\tilde{W}^U$ commute strongly and can be simultaneously diagonalized. This holds also for the operators $\pi(P^2)$, $\tilde{K}^U$ and $\tilde{W}^U$.

Proof: The algebra $\mathcal{A}'_{SO_q(3)}$ acts on $\mathcal{A}_{SO_q(3)}$ by $\tilde{\rho} \mapsto (\text{id} \otimes \tilde{\rho}) \circ \Delta$ and the space $\mathcal{A}_{SO_q(3)}$ decomposes in eigenspaces $\mathcal{A}_{SO_q(3)}(l, m) := \text{Span}\{t_{n,m}^l : n = -l, -l+1, \ldots, l\}$ of $\tilde{K}$ and $\tilde{W}$ with eigenvalues $q^{-m}, c_l > 0$. One defines the linear functionals $\tilde{P}_{lm} \in \mathcal{A}'_{SO_q(3)}$ by

$$\tilde{P}_{lm}|_{\mathcal{A}_{SO_q(3)}(l', m')} = \delta_{l'l} \delta_{mm'}.$$

Then the operators $\tilde{P}_{lm}$ constructed according to [13] are the simultaneous spectral projectors of $\tilde{K}^U$ and $\tilde{W}^U$. They are bounded operators on $\mathcal{H}$ which commute with $\pi(X^2) = \sum_i \pi(X_i)\pi(X_i^*)$ on $D$. Then [14] applies with $c = \tilde{P}_{lm}|D$ and $a = 1|_D + \pi(X^2)$. It states that $\pi(X^2)$, $\tilde{K}^U$ and $\tilde{W}^U$ commute strongly. Using the symmetry $P^2 \leftrightarrow X^2$ one proves the statement for $P^2$. \hfill \Box

From the above proof the definition of the simultaneous spectral projectors $\tilde{P}_{lm}$ of $\tilde{K}^U$ and $\tilde{W}^U$ derived from the corepresentation operator $U$ will be used also in the sequel.

5.6. Proposition: In each quantum mechanical representation $(\pi, D)$ the representatives of the real generators $X_\alpha^R$ and $P_\alpha^R$ of $\mathcal{O}$ are essentially self-adjoint. The adjoint representation $(\pi^1, D^1)$ is self-adjoint.

Proof: Proposition [3,4] with $c = \pi(X_\alpha^R)$ and $a = 1|_D + \pi(X^2)$ with $\lambda = \frac{2}{q+\frac{a}{\pi}}$ applies. It states that $\pi(X_\alpha^R)$ is essentially self-adjoint for any core of $\pi(X^2)$. The symmetry
X ↔ P proves the same for the momenta. This implies the self-adjointness of π† as proved in [27].

That means that every quantum mechanical representation can be uniquely extended to a self-adjoint representation by simply taking the adjoint representation. In all the following only the self-adjoint quantum mechanical representations will be considered.

5.7. In a quantum mechanical representation (π, D) the self-adjoint operators π(X²) and π(P²) are positive, and their kernels are trivial.

Proof: Suppose 0 ≠ |ψ⟩ ∈ H is in the kernel of π(X²). Without restriction |ψ⟩ = ˆP lm |ψ⟩. Let |f⟩ ∈ D. Then it is 0 = ⟨ψ|π(X²)π(P²)|f⟩ = −C_q² ⟨ψ|(q²l² − π(µ²))(q²l² + 2 − π(µ²))π(µ² − 2)|f⟩. For all l ≥ 0 this is in contradiction with µ² = q². The positivity follows from X² = X_i X_i.

5.8. Spectral measures. Since π(X²) and π(P²) are self-adjoint in a quantum mechanical representation (π, D) the spectral theorem (e.g. [28]) applies and one can introduce the projection-valued measures \( \{ P_X \} \) and \( \{ P_P \} \) of π(X²) and π(P²) which fulfill

\[
\begin{align*}
\pi(X²) &= \int_{x \in \mathbb{R}_+} x² dP^X_x \\
\pi(P²) &= \int_{p \in \mathbb{R}_+} p² dP^P_p
\end{align*}
\]

with \( P^X_\Omega = \int_\Omega dP^X_x \), \( P^P_\Omega = \int_\Omega dP^P_p \).

Because of 5.4 the spectral projectors of either of the sets \( \{ π(X²), \hat{W}U, \hat{K}U \} \) or \( \{ π(P²), \hat{W}'U, \hat{K}'U \} \) commute and the spaces

\[
\mathcal{H}_{Ω,l,m}^X := P^X_Ω \hat{P}^U lm \mathcal{H} \quad \text{and} \quad \mathcal{H}_{Ω,l,m}^P := P^P_Ω \hat{P}^U lm \mathcal{H}
\]

(to be thought of as approximate simultaneous eigenspaces) span dense subspaces

\[
\hat{\mathcal{E}}^X := \sum_{0 \leq a < b < \infty \atop |m| \leq l \in \mathbb{N}_0} \mathcal{H}_{[a,b],l,m}^X \quad \text{and} \quad \hat{\mathcal{E}}^P := \sum_{0 \leq a < b < \infty \atop |m| \leq l \in \mathbb{N}_0} \mathcal{H}_{[a,b],l,m}^P
\]

in the Hilbert space \( \mathcal{H} \).

5.9. Let (π, H) be a self-adjoint quantum mechanical representation of \( \mathcal{O} \) on \( H = \mathcal{D} \). Then the following statements hold:

(i) \( \hat{\mathcal{E}}^X \subset \mathcal{D} \) is invariant under \( \pi \). I.e. the restricted representation \( (\pi|\hat{\mathcal{E}}^X, \hat{\mathcal{E}}^X) \) is a *-representation of \( \mathcal{O} \).

(ii) \( \hat{\mathcal{D}} := \hat{\mathcal{E}}^X + \hat{\mathcal{E}}^P \) is invariant under \( \pi \). The *-representation \( (\hat{\pi}, \hat{\mathcal{D}}) := (\pi|\hat{\mathcal{D}}, \hat{\mathcal{D}}) \) determines the representation \( \pi \) completely with \( (\pi, \mathcal{D}) = (\hat{\pi}^!, \hat{\mathcal{D}}^!) \).
Let the conditions and definitions be as in 5.9. Let \( \pi \) be a core of \( \pi(X^X) \), \([5,4]\) can be applied with \( c = \pi(X^X) \), and one concludes that \( \hat{\pi} X \subset D(\pi(X^X)^\dagger) \) is a core for \( \pi(X^X)^\dagger \) and that \( \hat{\pi} X \) is invariant, since the eigenvalues of the operators \( K^U \) and \( \hat{W}^U \) are changed by at most one unit. The action of the operators \( \pi(\mu^{\pm 1}L_i)^\dagger \) and \( \pi(\mu^{\pm 1}W)^\dagger \) can be derived from \([4,3]\) and \( \pi(\mu W)^H(a,b)l,m = H(a^\pm 1,q^\pm 1)l,m \) proving the invariance of \( \hat{\pi} X \) under these operators. Finally, observing \( \pi(X^X)^\dagger \hat{\pi} X = \pi X \), the same conclusions can be drawn for \( \pi(P_i) \) when \([3,5]\) is taken into account. This proves \( \hat{\pi} X \subset D^\dagger = D \). The symmetry \( P \leftrightarrow X \) assures the same conclusions for \( \hat{\pi} P \). Thus \( D \subset D \) is an invariant subset of the domain of \( \pi \). Since the real generators \( \{X^R_\alpha, P^R_\alpha\} \) of \( O \) are represented by essentially self-adjoint operators on \( D \) the last assertion follows \([29]\). □

**5.10. Proposition:** Let the conditions and definitions be as in \([3,3]\). Let \( \hat{\pi}_l \subset \hat{\pi} X \subset \hat{\pi} U \). Then the restriction \( \pi(P^2)|_{\hat{\pi} X} \) is an essentially self-adjoint operator in the Hilbert space \( \hat{\pi} U \). Its spectral measure is given by

\[
dP^P \hat{\pi} U = \sum_{k,k' \in \mathbb{Z}} q^{6(k+k')} N_l^{-2} F_l(q^{2}l+2k) F_l(q^{2}l+2k') \times \pi(\mu^{2k}) dP^X \frac{\mu q^k}{q^{2l+2k'} (1-q)}
\]

with the \( q \)-deformed Bessel functions

\[
F_l(y) = y^l \sum_{k=0}^{\infty} q^{2k} (-1)^k \frac{q^{-2k(l-1)q^{-k(2l+5)}}}{(q^{-4}; q^{-2(2l+3)})_k (q^{-4}; q^{-4})_k} ;
\]

the definitions \( (a;p)_k := \prod_{j=0}^{k-1} (1 - ap^j) \) and \( N_l := q^{-l(l+1)} \frac{\mu q^{2}l-2(2l+3)+\infty}{(q^{-4}; q^{-4})_l} \) and the constant \( C_q = \frac{q^2}{(1+q^2)(1-q)} \) introduced in \([3,3]\).

The function \( F_l \) can also be expressed with the help of \( q \)-hypergeometric functions discussed in \([3,0]\): \( F_l(y) = y^l \Phi_1(0; q^{-2(3+2l)}; q^{-4}, y^2q^{-2l-5}) \). \([3,1]\) states then

\[
\delta_{nm} = Q^n q^{3(m+n)} \sum_{k \in \mathbb{Z}} q^{6k} F_l(q^{2}l+2(k+m)) F_l(q^{2}l+2(k+n))
\]

when \( l \geq -1 \). Introducing the operators \( (K_l f)(y) := q^{-l} f(qy) - q^l f(q^{-1}y) \) and \( \Delta_l f(y) := \frac{C_q}{y^l} K_{l-1} K_l f(y) \) the functions \( F_l \) fulfill the following difference equations (\( l \in \mathbb{N}_0 \)):

\[
\frac{1}{y} (K_l F_l)(y) = \frac{q^{2l-3}}{1-q^{2l+3}} F_{l+1}(y); \quad C_q^2 \Delta_l F_l(y) = \frac{1}{y} K_{l-1} K_l F_l(y) = F_l(y)
\]

which should be compared with \([3,3]\).
Proof: To prove the essential self-adjointness of $\pi(P_2^2)|\hat{\mathcal{E}}_l^X$ its defect indices are investigated. Let $\Psi$ be a $\pm i$-eigenvector of $\pi(P_2^2)|\hat{\mathcal{E}}_l^X$. Then $0 = (\pi(P_2^2) \pm i)P_\Omega^X|\Psi|$. Thus, using the formulas $[33]$, almost everywhere $0 = (-\Delta_l \pm i)d\Psi_y$ with $d\Psi_y := y^{-\frac{2}{\gamma}}dP_\gamma^X\Psi$. This difference equation gives a dependence between $d\Psi_{y_q}^n$ for $y \in [1, q^2]$, but does not correlate different $y \in [1, q^2]$. Hence for each $y \in [1, q^2]$ it leads to a recursion relations which has a two-parametric solution space. The general solution is thus $d\Psi_{y_q}^n = (A_yF_l(\pm i\frac{1}{2}yq^{2n}) + B_yF_{l-1}(\pm i\frac{1}{2}yq^{2n})) d\Psi_y$ with measurable $A_y$ and $B_y$ for $y \in [1, q^2)$. However there is still the norm squared of $\Psi$ given by $\langle \Psi|\Psi \rangle = \int_{y \in [1, q^2)} \langle \Psi|y^{-3}dP_\gamma^X|\Psi \rangle \sum_{n \in \mathbb{Z}} \chi_{q^2k} \chi_{(1-\pm i\frac{1}{2}) \alpha l}^{2k} G_{l,m} := \left( (q^{-4} ; q^{-4(l+m)+\pm i\frac{1}{2}})_{(k-m+1)} (q^{-4} ; q^{-4m+4})_{(k-m+1)} \right)^{-1}$ and $r = \frac{\alpha}{\alpha + 1} \frac{1}{2}$. To investigate the limit $r \to \infty$ one chooses $r = q^{2(l+\alpha)}$ with $t \in 4\mathbb{N}_0$, $\alpha \in \mathbb{R}$. Substituting into $f$ this gives $f(r) = \Phi_1 + \Phi_2$ with $\Phi_2 = q^{2(l+\alpha)} \sum_{k = \frac{1}{4} + 1}^{0} G_{l,m} q^{-2(k-1+32t-2\alpha)} (\pm i)^k =: G_{l,m} q^{-2(l+\alpha) + 2(\alpha+1)+\frac{1}{2}} \psi_l^{(i)}(\alpha)$. Using the limit $G_{l,m} \to 1$ for $t \to \infty$ one has $\lim_{t \to \infty} \psi_l^{(i)}(\alpha) = \psi_l^{(i)}(\alpha) := \sum_{k = -\infty}^{\infty} (\pm i)^k q^{-2(k+\alpha)^2}$. For large $t$, the absolute terms of the individual terms of the alternating sums of real and imaginary part of $\Phi_l$ grow monotonously with $k$ and the sum can be estimated using the last term: $|\Phi_l| < 2G_{l,m} q^{2(l+\alpha) + 2(\alpha+1)+\frac{1}{2}}$. Then one arrives at

$$\lim_{t \to 4\mathbb{N}_0} \frac{\left| f(q^{2(l+\alpha)}) \right|}{\phi(t, \alpha)} = \psi_l^{(i)}(\alpha)$$

with the monotonously growing function $\phi(t, \alpha) = q^{2(l+\alpha) + 2(\alpha+1)+\frac{1}{2}}$. The function $\psi_l^{(i)}$ is continuous and periodic with period 4. Let $\psi^\prime$ be its real part, then $\psi_{l}^{(i)}(\alpha) = \psi^\prime(\alpha) \pm i\psi''(\alpha - 1)$. In $[4]$ it is proved that $\psi''(\alpha) = 0$ implies $\alpha = 1 + 4\mathbb{Z}$. Thus $|\psi_l^{(i)}|$ is strictly positive. Hence the sum above would diverge for $n \to \infty$. Therefore $A_y$ has to vanish almost everywhere.

Doing the same analysis as in $[4]$ or using the results of $[33]$ cited above gives the explicit spectral representation.

Since the image of $dP_{\gamma}^X \hat{\mathcal{D}}_{im}$ belongs to the domain $\hat{\mathcal{D}} \subset \hat{\mathcal{D}}$ the operators $\pi(P_1)$ can be applied. Therefore the term $\hat{\mathcal{D}}_{im} \pi(P_1) \hat{\mathcal{D}}_{im} dP_{\gamma}^X \hat{\mathcal{D}}_{im} \pi(P_1) \hat{\mathcal{D}}_{im}$ leads to the still missing spectral representation of $\pi(P_2^2)$ for $l = 0$, and one gets the following:

5.11. Corollary: The spaces $\hat{\mathcal{E}}^P$ and hence $\hat{\mathcal{D}}$ are uniquely determined by $\hat{\mathcal{E}}^X$. 

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since the equation
\[
d\mathbf{P}_p^* \mathbf{P}_m^U = \sum_{k,k' \in \mathbb{Z}} q^{6(k+k')} N_l^{-2} F_j(q^{\frac{l}{2}+l+2k}) F_j(q^{\frac{l}{2}+l+2k'}) \times \\
\times \pi(\mu^{*2k})d\mathbf{P}_n^X[q^{\frac{l}{2}+l}C_{q^{-1}}] \mathbf{P}_m^U \pi(\mu^{2k'})
\]
is valid for all \( l \in \mathbb{N}_0 \).

Thus to classify all the self-adjoint quantum mechanical representations it is sufficient to construct the spaces \( \mathcal{E}_X \) representing \( \mathcal{O} \).

5.12. Each nontrivial self-adjoint representation of \( \mathcal{O} \) contains some vector \( 0 \neq |\Psi\rangle \in \mathcal{H}_{[1,q],0,0} \).

**Proof:** There exists \( 0 \neq |\Phi\rangle \in \mathcal{H}_{[a,b],l,l} \). One observes that the reducible multiplet \( \pi(X_3)|\Phi\rangle \in \mathcal{H}_{[a,b],l+1,l-1} + \mathcal{H}_{[a,b],l-1,l-1} \) with \( |\Phi\rangle \in \mathcal{H}_{[a,b],l,l} \) because of \( 0 = \mathbf{P}_m^U \pi(X) \mathbf{P}_m^* \). Then one compares the vectors constructed as \( \pi(X_3)|r,l,l\rangle \) and \( \pi(L_3 X_2)|r,l,l\rangle \).

\[
\left\{ \frac{\|\pi(X_3)|\Phi\rangle\|}{\|\pi(L_3 X_2)|\Phi\rangle\|} \geq q^{l[2l+1]} (1 + q^{-2l(2l+1)} \pi^{-1} \pi^{-2}) \right. \\
\left. \geq \frac{1}{q^{l[2l+1]} (1 + q^{-2l(2l+1)} \pi^{-1} \pi^{-2})} \right.
\]

One concludes that they are linearly independent for \( l > 0 \), and \( \mathcal{H}_{[a,b],l-1,l-1} \) is nontrivial if \( l > 0 \). Since the operator \( u := q^{\frac{l}{2}}(W U)^{-1} \pi(\mu W) \) provides an isometry between the spaces \( \mathcal{H}_{[q^{n} a,b],n} \) for \( n \in \mathbb{Z} \) the lemma is proved. \( \square \)

5.13. Irreducible representations. Defining \( \mathcal{E}_\psi := \pi(\mathcal{O})|\psi\rangle \), \( \mathcal{H}_\psi := \overline{\mathcal{E}_\psi} \), \( \pi_\psi := \pi|\mathcal{H}_\psi \) one gets now a self-adjoint quantum mechanical representation of \( \mathcal{O} \) on the Hilbert space \( \mathcal{H}_\psi \). Then its complement is also a self-adjoint quantum mechanical representation \( \mathcal{E}_X \). One observes also at once that this subrepresentation is irreducible if, and only if, \( |\Psi\rangle \in \mathcal{P}_{[1,q]}^X \) is an eigenvector of \( \pi(X_2) \).

Let us suppose now that \( (\pi, \mathcal{D}) \) is irreducible. Take \( |\Psi\rangle \) as before and suppose that it is normalized, \( \langle \Psi|\Psi\rangle = 1 \), and obeys \( \pi(X_2)|\Psi\rangle = x_0^2|\Psi\rangle \) with \( x_0 \in [1,q] \). This irreducible representation will be called from now on \( (\pi_{x_0}, \mathcal{D}_{x_0}) \). Then one defines the following orthonormal basis of \( \mathcal{E}_{x_0}^X \)
\[
\left\{ |x_n, l, m\rangle \, |n \in \mathbb{Z}, l, |m| \in \mathbb{N}_0, |m| \leq l \right\}
\]
of simultaneous eigenvectors of the commuting observable \( \pi_{x_0}(X^2) \) with eigenvalues \( x_n^2 = x_0^2 q^{2n} \) and \( \pi_{x_0}(W^2) \) and \( \pi_{x_0}(L_2 W) \) by
\[
|x_n, 0, 0\rangle = u^n |\Psi\rangle := \pi(q^{-\frac{3}{2}} \mu^{-1} W)^n |\Psi\rangle
\]
\[ |x_n, l, l⟩ := \frac{[2\sqrt{q}[2l + 3]\sqrt{q}]}{q^{n}x_0 [2l + 2]\sqrt{q}} π_{x_0}(X_1) |x_n, l - 1, l - 1⟩ \]
\[ |x_n, l, m⟩ := q^{-\frac{1}{2}} α_{l, m - 1}^{-1} |x_n, l, m + 1⟩ . \]

In this irreducible representation also \( π_{x_0}(P^2) \) has exactly one eigenvalue in \([1, q^2)\). To complement \( \hat{E}^X_{x_0} \) one has still to determine all the eigenvectors of \( π_{x_0}(P^2) \). Defining
\[
p_0 := \frac{\sqrt{q} C_q}{x_0} \]
the simultaneous eigenvectors \( |p_n, l, m⟩ \) of \( π_{x_0}(P^2) \) with eigenvalues \( p_0^2 q^{2n} \) and \( π_{x_0}(W^2) \) and \( π_{x_0}(W L_2) \) in the same manner which span \( \hat{E}^P_{x_0} \) one has still only to give the matrix elements between these both bases. But this can be read off from 5.10:
\[
⟨x_n, l, m|p_{n'}, l', m'⟩ = \delta_{ll'} \delta_{mm'} \frac{1}{N_l} F_l(q^{n+n'+l+\frac{1}{2}}) δ^{(2)}_{n+n'+l} \]
with \( 2δ^{(2)}_{a} := (-1)^a + 1. \)

Now the results 5.13, 5.14 and 5.13 can be summarized as a theorem:

**5.14. Theorem:** Each quantum mechanical representation of \( \mathcal{O} \) can be uniquely extended to a self-adjoint quantum mechanical representation by taking the adjoint representation. Each self-adjoint quantum mechanical representation \((\pi, D)\) is a direct integral sum of the irreducible quantum mechanical representations \((\pi_{x_0}, D_{x_0})\) parametrized by \( x_0 ∈ [1, q) \), and it holds upto unitary equivalence
\[
D \sim \int_{x_0 ∈ [1, q)} D_{x_0} ⊗ dP^X_{x_0} \tilde{P}_{00}^U H, \quad \pi(a) \sim \int_{x_0 ∈ [1, q)} π_{x_0}(a) ⊗ id_{dP^X_{x_0} \tilde{P}_{00}^U H} \]
for \( a ∈ \mathcal{O} \). The operator sets \{\( π(X^2), π(W^2), π(L_2) \)\} and \{\( π(P^2), π(W^2), π(L_2) \)\} respectively are complete sets of commuting observables.

Thus after having characterized a quantum mechanical representation by very natural requirements on the observability of certain operators and the covariance under the quantum group all representations are found. In opposite to the usual quantum mechanical situation where von Neumann’s arguments show the uniqueness of the Schrödinger representation for the Heisenberg algebra [33], in the \( q \)-deformed case a one-parametric set of inequivalent irreducible quantum mechanical representations arise which however have the property that eigenvectors of position and momentum exist.

These results can in the same way be obtained also for \( SO_q(N) \)-symmetric quantum mechanics. The special case \( N = 1 \) is treated in [34] where the fact that \( π(P^2) \) is not essentially self-adjoint on \( E^X \) leads to a lot of ambiguities.
6. Application: Potential Problems

In analogy to usual quantum mechanics one can now consider properties of operators of the form $H = \pi(P^2) + V$ and solve the corresponding eigenvalue equations.

6.1. Let $(\pi_{x_0}, \mathcal{D}_{x_0})$ be the irreducible self-adjoint quantum mechanical representation of $\mathcal{O}$ with $\pi_{x_0}(X^2)$-eigenvalues $x_0^2q^{2n} = x_n^2$. For sequences $f : \mathbb{Z} \to \mathcal{C}$ the mappings $f \mapsto \|f\|_{\infty}$, $f \mapsto \|f\|_2$, $f \mapsto \|f\|_1$ are defined by $\|f\|_{\infty} := \sup_{n \in \mathbb{Z}} |f_n|$, $\|f\|_1 := \sum_{n \in \mathbb{Z}} q^{3n} |f_n|$, $\|f\|_2 := \sqrt{\|f^2\|_1}$. These sequences can be understood as multiplication operators in the $x_0$-representation with the definition $\pi_{x_0}(f) |x_n, l, m\rangle := f_n |x_n, l, m\rangle$ with domain to be specified. Certainly the sequences $f$ with $\|f\|_{\infty} < \infty$ act as bounded operators. Then in the spirit of [36] one can prove the following result.

6.2. Proposition: Let $(\pi_{x_0}, \mathcal{D}_{x_0})$ be the irreducible quantum mechanical representation of $\mathcal{O}$ as before. Let $v$ and $w$ be sequences $\mathbb{Z} \to \mathcal{C}$, $\|v\|_{\infty} < \infty$ and $\|w\|_2 < \infty$ acting as multiplication operators in the $x_0$-representation. In addition let $b$ be a bounded operator on $\mathcal{H}_{x_0}$ which commutes with the corepresentation operator $U$ such that $b\pi_{x_0}(v + w)$ is a symmetric operator on $\mathcal{D}_{x_0}$ (which commutes then strongly with the operator $K^U$ and $\hat{W}^U$). Then $H := \pi_{x_0}(P^2) + b\pi_{x_0}(v + w)$ is essentially self-adjoint on $\mathcal{D}_{x_0}$.

Proof: Let $\mathcal{H} := \mathcal{D}_{x_0}$. That $H$ is essentially self-adjoint will be proven separately for the subspaces $\hat{P}^U_{lm}\mathcal{H}$. The vectors in $|\phi\rangle \in \hat{P}^U_{lm}\mathcal{H}$ define sequences $\phi_n := q^{-\frac{3}{2}n} \langle x_n, l, m|\phi\rangle$ and $\hat{\phi}_n := q^{-\frac{3}{2}n} \langle p_n, l, m|\phi\rangle$ with $\|\phi\|_2 = \|\hat{\phi}\|_2 < \infty$. Then $\|\phi\|_{\infty} \leq \sum_{k \in \mathbb{Z}} \sup_{n+k} \left| q^{\frac{3}{2}(n+k)} \langle x_{n+k}, l, m|p_0, l, m\rangle \right| q^{-\frac{3}{2}k} \|\hat{\phi}_k\| \leq d_l \|\hat{\phi}\|_1$ with the positive constant $d_l$. On the other side the sequence $\Pi : n \mapsto \frac{1}{p_{n+1}^2}$ has finite norm $\|\Pi\|_2$. Let $|\phi\rangle \in \hat{P}^U_{lm}\mathcal{D}_{\pi_{x_0}(P^2)}$ be in the domain where $\pi_{x_0}(P^2)^{\dagger}$ restricted to $\hat{P}^U_{lm}\mathcal{H}$ is self-adjoint. Then $(1 + \pi_{x_0}(P^2))|\phi\rangle \in \mathcal{H}$ lets define the sequence $\Pi^{-1}\hat{\phi}$. With Schwarz’s inequality $\|\hat{\phi}\|_1 \leq \|\Pi\|_2 \|\Pi^{-1}\hat{\phi}\|_2 \leq \|\Pi\|_2 (\|\pi_{x_0}(P^2) |\phi\rangle\| + \|\phi\|)$ applying these inequalities to the vectors $\mu^r |\phi\rangle$ results in $\|\hat{\phi}\|_1 \leq q^{\frac{3}{2}n}\|\Pi\|_2 \|\phi\| + q^{-\frac{3}{2}n}\|\Pi\|_2 \|\pi_{x_0}(P^2) |\phi\rangle\|$. This formula implies that for each $\phi \in \mathcal{H} \subset \hat{P}^U_{lm}$ and each $e > 0$ one can find $g_l > 0$ (depending on $l$ and $e$ but not on the vector $|\phi\rangle$) such that $\|\hat{\phi}\|_1 \leq e\|\pi_{x_0}(P^2) |\phi\rangle\| + g_l \|\phi\|$. Let the operator norm of $b$ be $\|b\|$. Then one proves straight forwardly that choosing $e := \pi\|w\|_2^{-1} \|b\|^{-1}$ and $\gamma := g_l\|w\|_2 \|b\| + \|v\|_{\infty} \|b\|$ with $0 < \tau < 1$ leads to the
inequality
\[ ||b(v + w)|\phi|| \leq \tau||\pi_{x_0}(P^2)|\phi|| + \gamma|||\phi||.\]
Then the Kato-Rellich Theorem \[37\] proves that \( H|\tilde{P}_{lm}^\dagger H \) is essentially self-adjoint on any core of \( \pi_{x_0}(P^2)|\tilde{P}_{lm}^\dagger H \). Thus \( H \) is essentially self-adjoint. \( \Box \)

6.3. Example: The \( SO_q(3) \)-symmetric Coulomb problem. Let \((\pi_{x_0}, D_{x_0})\) be as before. Identify \( a \in \mathcal{O} \) with \( \pi_{x_0}(a) \). Define the bounded operator \( b \) by \( b := -\mu - q^{-1}\mu^* \) where the bounded linear operator \( \mu \) is defined by \( \mu |x_n, l, m\rangle = q^{-\frac{3}{4}}|x_{n-1}, l, m\rangle \) (i.e. this \( \mu \) represents \( \mu \in \mathcal{D}_X^{Y_3} \)). The operator \( \frac{1}{R} = \pi_{x_0}(v + w) \) is defined by the sequences \( v \) and \( w \) acting in the \( X \)-representation and \( (v + w)_n := \frac{1}{x_n} \) and \( v_m = 0 \) for \( m < 0 \). (The such defined operator is uniquely defined by the equation \((\frac{1}{R})^2 X^2 = 1 \) and its positivity.) Then
\[
H := P^2 + b \frac{1}{R} = P^2 - (\mu \frac{1}{R} + \frac{1}{R}\mu^*)
\]
is essentially self-adjoint on \( D_{x_0} \). (It is obvious that the potential \( -\mu \frac{1}{R} - \frac{1}{R}\mu^* \) is a \( q \)-deformation of the usual Coulomb potential.)

Like in the undeformed case one can find the Lenz-vector consisting of the real triplet \( \{A_i\} \) with
\[
A_i := -iq^2C_q [P_i, \tilde{W}^U] + \frac{X_i}{R}
\]
which commutes with the Hamiltonian, \([H, A_i] = 0\), and closes on constant energy spaces together with the angular momentum algebra in a \( q \)-deformed \( SO(4) \)-angular momentum algebra:
\[
[A \times A]_k = -L_k W H, \\
A \cdot A = \frac{q - 1 + \frac{1}{q}}{q - 2 + \frac{2}{q}} W^2 H - \frac{1}{q - 2 + \frac{1}{q}} H + 1, \\
A \cdot L = L \cdot A = 0, \\
[L \times A]_k + [A \times L]_k = WA_k + A_k W, \\
[L \times A]_k - [A \times L]_k = \frac{[2]q}{(\sqrt{q} - \frac{1}{\sqrt{q}})^2} [W, A_k].
\]

The negative spectrum is given by
\[
\left\{ E_n \mid n \in \mathbb{N} , \exists m \in \mathbb{Z} : q^2(q^{-n} - q^n) E_n = x_0 q^m \right\}
\]
with
\[
E_n = -\left(1 + q^{-1}\right)^2 \frac{1}{[n]_q^2}
\]
where the angular momentum is restricted like in the undeformed case by $l + 1 \leq n$. Negative energies arise only for special representations $(\pi_{x_0}, D_{x_0})$. This comes from the fact that the corresponding wave functions are of the form

$$x^{n-1} \sum_k \frac{a^{-k(k+1)}}{(q-x^{-2})^k} (-q^2(q^n - q^{-n})E_n x)^k$$

which are only normalizable for special values of $x_0$.

In this paper a complete framework for $SO_q(3)$-symmetric one-particle quantum mechanics is provided and the representations studied. The observable algebra is a deformation of the usual Heisenberg algebra of coordinates and momenta covariant under the Lie group $SO(3)$ which is obtained by canonical quantization. However if quantization also changed the symmetry as a second order effect but conserved the corepresentation properties then the canonical quantization would have to be modified. Considering $q = 1 + O(h^2)$ the above found algebra would be nothing else than a noncanonical $h$-deformation covariant under the $h^2$-quantized group.

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References

[1] G. Mack, V. Schomerus, *Nucl. Phys. B* 370, 185 (1992).
[2] J. Wess, B. Zumino, *Nucl. Phys. B (Proc. Suppl.)* 18, 302 (1990).
[3] W. Weich, PhD-Thesis (Karlsruhe 1990).
[4] U. Carow-Watamura, M. Schlieker, S. Watamura, Z. Phys. C 49, 439 (1991).
[5] V. G. Drinfeld, *Proceedings of the International Congress of Mathematicians Berkeley* 1, 798 (1986).
[6] L. D. Faddeev, N. Yu. Reshetikhin, L. A. Takhtajan, *Algebra and Analysis* 1, 178 (1989).
[7] E. Abe, *Hopf algebras* (Cambridge U. P., Cambridge 1980).
[8] S. L. Woronowicz, *Comm. Math. Phys.* 111, 139 (1987).
[9] S. L. Woronowicz, *Publ. RIMS, Kyoto Univ.* 23, 117 (1987).
[10] W. Pusz, S. L. Woronowicz, *Rep. Math. Phys.* 27, 232 (1989).
[11] O. Ogievetsky, Lett. Math. Phys. 24, 245 (1992).
[12] O. Ogievetsky, B. Zumino, Lett. Math. Phys 25, 121 (1992).
[13] M. Jimbo, *Lett. Math. Phys.* 10, 63 (1986).
[14] A. Hebecker, W. Weich, *Lett. Math. Phys* 26, 245 (1992).
[15] T. H. Koornwinder, *Lectures at the European School of Group Theory* (Trento 1993).
[16] See §, Theorem 5.7.
[17] P. Podles, S. L. Woronowicz, Comm. Math. Phys. 130, 381 (1990).
[18] A. W. Knapp, Lie groups, Lie algebras, and cohomology (Princeton U. P., Princeton 1988).
[19] B. Drabant, W. Weich, Preprint LMU-TPW 1993-23 (1993).
[20] M. Rosso, Comm. Math. Phys. 117, 581 (1988).
[21] K. Schmüdgen, Unbounded Operator Algebras and Representation Theory (Akademie-Verlag Berlin and Birkhäuser-Verlag Basel 1990).
[22] R. T. Powers, Comm. Math. Phys. 21, 85 (1971).
[23] M. Reed, B. Simon, Methods of Modern Mathematical Physics I,II (Academic Press, New York, 1979).
[24] See [23], Theorem X.37.
[25] See [21], Proposition 7.1.4.
[26] See [21], Lemma 7.1.5.
[27] See [21], Proposition 8.1.12 (v).
[28] See [23], Theorem VIII.6.
[29] See [21], Proposition 8.1.17.
[30] T. H. Koornwinder, R. F. Swarttouw, Trans. Am. Math. Soc. 333, 445 (1992).
[31] See [30], Proposition 2.6, eq. (2.11).
[32] See [14], proof of Proposition 4.
[33] See [21], Proposition 8.3.10.
[34] A. Hebecker, S. Schreckenberg, J. Schwenk, W. Weich, J. Wess, Preprint LMU-TPW 1993-17 (1993).
[35] J. v. Neumann, Math. Ann. 104, 570 (1931).
[36] See [23], Theorem X.15.
[37] See [23], Theorem X.12.