1. Introduction

Let $M$ be an $n$-dimensional projective algebraic manifold in certain projective space $\mathbb{CP}^N$. The hyperplane line bundle of $\mathbb{CP}^N$ restricts to an ample line bundle $L$ on $M$, which is called a polarization of $M$. A Kähler metric $g$ is called a polarized metric, if the corresponding Kähler form represents the first Chern class $c_1(L)$ of $L$ in $H^2(M, \mathbb{Z})$. Given any polarized Kähler metric $g$, there is a Hermitian metric $h$ on $L$ whose Ricci form is equal to $\omega_g$. For each positive integer $m > 0$, the Hermitian metric $h_L$ induces the Hermitian metric $h_L^m$ on $L^m$. Let $(E, h_E)$ be a Hermitian vector bundle of rank $r$ with a Hermitian metric $h_E$. Consider the space $\Gamma(M, L^m \otimes E)$ of all holomorphic sections for large $m$. For $U, V \in \Gamma(M, L^m \otimes E)$, the pointwise and the $L^2$ inner products are defined as

$$\langle U(x), V(x) \rangle_{h_L^m \otimes h_E}$$

and

$$(U, V) = \int_M \langle U(x), V(x) \rangle_{h_L^m \otimes h_E} dV_g,$$  \hspace{1cm} (1.1)

respectively, where $dV_g = \frac{\omega_g^n}{n!}$ is the volume form of $g$. Let $\{S_1, \ldots, S_d\}$ be an orthonormal basis of $\Gamma(M, L^m \otimes E)$ with respect to (1.1), where $d = d(m) = \dim \Gamma(M, L^m \otimes E)$. For any $x \in M$, define a matrix $S = S(x)$ by

$$S = \left( \langle S_i, S_j \rangle_{h_L^m \otimes h_E} \right).$$

For any positive integer $b$, define

$$\sigma_b \equiv \text{tr}(S^b).$$  \hspace{1cm} (1.2)
The value $\sigma_b$ is independent of the choice of the orthonormal basis because under different basis of $\Gamma(M, L^m \otimes E)$, the matrices $S$ are similar. Moreover, $S$ is diagonalizable. Since $E$ is of rank $r$, there exists a unitary matrix $Q$ and a diagonal matrix $D$ such that $Q^*SQ = D$, where

$$D_{ij} = \begin{cases} \lambda_i \delta_{ij}, & \text{if } 1 \leq i, j \leq r; \\ 0, & \text{otherwise.} \end{cases}$$ (1.3)

Furthermore, there exists an orthonormal basis $\{T_i\}_{i=1}^d$ such that

$$\sigma_b = \text{tr}(S^b) = \text{tr}(D^b) = \sum_{i=1}^d \|T_i(x)\|_{h_m^b}^{2b}.$$ (1.4)

From (1.3), we have

$$\sigma_b = \sum_{i=1}^r \lambda_i^b,$$

where $\lambda_i$ are the nonzero eigenvalues of $S$. For $b > r$, $\sigma_b$ can be written as a polynomial of $\sigma_1, \cdots, \sigma_r$. Hence we only need to compute $\sigma_1, \cdots, \sigma_r$.

The asymptotic behavior of $\sigma_b$ plays a very important rule in Kähler-Einstein geometry. In the case of $b = 1$, Zelditch [7] and Catlin [1] independently proved the existence of an asymptotic expansion (Tian-Yau-Zeldtich expansion) of the Szegö kernel. In the break through paper of Donaldson [2], using the expansion, he was able to prove the stability for the manifold admitting constant scalar curvature.

The result of Zeldtich and Catlin is stated as follows:

**Theorem** (Zelditch, Catlin). Let $M$ be a compact complex manifold of dimension $n$ (over $\mathbb{C}$) and let $(L, h) \to M$ be a positive Hermitian holomorphic line bundle. Let $x$ be a point of $M$. Let $g$ be the Kähler metric on $M$ corresponding to the Kähler form $\omega_g = \text{Ric}(h)$. For each $m \in \mathbb{N}$, $h$ induces a Hermitian metric $h_m$ on $L^m$. Let $\{s_1^m, \cdots, s_{d_m}^m\}$ be any orthonormal basis of $H^0(M, L^m)$, $d_m = \dim H^0(M, L^m)$, with respect to the inner product (1.1). Then there is an asymptotic expansion:

$$\sigma_1 = \sum_{i=1}^{d_m} \|s_i^m(x)\|_{h_m}^2 \sim a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + \cdots$$ (1.5)

for certain smooth coefficients $a_j(x)$ with $a_0 = 1$. More precisely, for any $k$:

$$\| \sum_{i=1}^{d_m} \|s_i^m(x)\|_{h_m}^2 - \sum_{k=0}^{N} a_j(x)m^{n-k}\|c^\mu \|_C \leq C_{N, \mu} m^{n-N-1},$$

where $C_{N, \mu}$ depends on $N, \mu$ and the manifold $M$. 
In [4], Lu proved that each coefficient $a_j(x)$ is a polynomial of the curvature and its covariant derivatives. In particular, $a_1(x) = \frac{1}{2}\rho(x)$ is half of the scalar curvature of the Kähler manifold. All polynomials $a_j(x)$ can be represented by a polynomial of the curvature and its derivatives. Moreover, Lu and Tian [5, Theorem 3.1] proved that the leading term of $a_j$ is $C\Delta^{j-1}\rho$, where $\rho$ is the scalar curvature and $C = C(j, n)$ is a constant.

In this paper, we establish an asymptotic expansion for $\sigma_b$. Note that both Zelditch and Catlin used Szegö kernel or Bergman kernel in their proofs. Their methods, however, do not apply to the case $b > 1$. To establish the expansion, we go back to peak section estimates. Using the peak section method in [6], we are able to get the expansion of $\sigma_b$, which generalizes the result of Zelditch and Catlin. Our result is:

**Theorem 1.1.** Let $M$ be a compact complex manifold of dimension $n$, $(L, h_L) \to M$ a positive Hermitian holomorphic line bundle and $(E, h_E)$ a Hermitian vector bundle of rank $r$. Let $g$ be the Kähler metric on $M$ corresponding to the Kähler form $\omega_g = \text{Ric}(h_L)$. Let $\Gamma(M, L^m \otimes E)$ be the space of all holomorphic global sections of $L^m \otimes E$, and let $\{T_1, \cdots, T_d\}$ be an orthonormal basis of $\Gamma(M, L^m \otimes E)$. Let

$$\sigma_b = \sum_{i=1}^{d} \|T_i(x)\|_{h^m_L \otimes h_E}^{2b}.$$  
(1.6)

Then for $m$ big enough, there exists an asymptotic expansion

$$\sigma_b(x) \sim a_0(x)m^{bn} + a_1(x)m^{bn-1} + \cdots$$  
(1.7)

for certain smooth coefficients $a_j(x)$. The expansion is in the sense that

$$\|\sigma_b - \sum_{k=0}^{N} a_k m^{bn-k}\|_{C^\mu} \leq C(\mu, N, M)m^{bn-N-1}$$

for positive integers $N, \mu$ and a constant $C(N, \mu, M)$ depending only on $N, \mu$ and the manifold.

The second main result of this paper focuses on compact complex manifolds with analytic Kähler metrics. It is well-known that the Tian-Yau-Zeldtich expansion does not converge in general. Even if it is convergent, it may not converge to $\sigma_1$. We proved that in the case when the metric is analytic, the optimal result may achieve: The asymptotic expansion is convergent and the limit approaches $\sigma_b$ faster than any other polynomials.
Theorem 1.2. With the notations as in the above theorem, suppose that the Hermitian metrics $h_L$ and $h_E$ are real analytic at a fixed point $x$. Then for $m$ big enough, the expansion
\[
\sigma_b(x) \sim a_0(x)m^{bn} + a_1(x)m^{bn-1} + \cdots
\] (1.8)
is convergent for certain smooth coefficients $a_j(x)$. There is a $\delta > 0$ such that the coefficient $a_j(x)$ satisfies
\[
|a_j(x)| < \frac{C}{\delta^j}
\]
for some constant $C$. Moreover, the expansion is convergent in the sense
\[
\|\sigma_b - \sum_{k=0}^N a_km^{bn-k}\|_{C^\mu} \leq Cm^{bn}(\delta m)^{-N-1}
\]
for a constant $C(\mu, \delta)$ which only depends on $\mu$.

Theorem 1.2 gives that

Corollary 1.1. With the notations as in the above theorem, the limit of the series
\[
\lim_{N \to \infty} \sum_{k=0}^N a_km^{bn-k}
\]
exists.

In fact, we prove a little bit more in Theorem 1.2.

Corollary 1.2. With the notations as in the above theorem, we have
\[
\|\sigma_b - \sum_{k=0}^N a_km^{bn-k}\|_{C^\mu} \leq Ce^{-(\log m)^2}.
\]

Proof. Choose $N = \lfloor \log m \rfloor$ to be the integer part of $\log m$. Then
\[
\|\sigma_b - \sum_{k=0}^N a_km^{bn-k}\|_{C^\mu} \leq Cm^{bn-\log m-1}.
\]

On the other hand,
\[
\| \sum_{k=N+1}^\infty a_km^{bn-k}\|_{C^\mu} \leq Cm^{bn-N-1}.
\]

Thus
\[
\|\sigma_b - \sum_{k=0}^\infty a_km^{bn-k}\|_{C^\mu} \leq Cm^{bn-N} \leq Cm^{bn}e^{-(\log m)^2}.
\]
\[\square\]
More precisely, we have the following result

\[ \| \sigma_b - \sum_{k=0}^{\infty} a_k m^{bn-k} \|_{C^\mu} \leq C e^{-\varepsilon (\log m)^2}. \]  

(1.9)

The above result \((1.9)\) was only known in very special cases before. In Liu \[3\], she proved the case for \(b = 1\) on a smooth Riemann surface with constant curvature. On a planar domain with Poincaré metric, Engliš proved the same result.

We have multiple definitions for \(O(\frac{1}{m^{k+1}})\) through this paper. In the case that \(h_L\) and \(h_E\) are \(C^\infty\) as the assumption in Theorem 1.1, it denotes a quantity dominated by \(C/m^{k+1}\) with the constant \(C\) depending only on \(k\) and the geometry of \(M\). In the case that \(h_L\) and \(h_E\) are analytic as the assumption in Theorem 1.2, it denotes a pure constant.

The last part of this paper, we compute the coefficient of the new expansion.

**Theorem 1.3.** With the same notation as in Theorem 1.1, each coefficient \(a_j(x)\) is a homogeneous polynomial of the curvature and its derivatives at \(x\). In particular,

\[ a_0 = r \]

\[ a_1 = \frac{1}{2} br \rho + \rho_E, \]

and the leading term for \(a_k\) for \(k \geq 2\) is

\[ \frac{br k}{(k+1)!} \rho^{k-1} \rho_E, \]

where \(\rho\) is the scalar curvature of \(M\), \(\rho_E\) is the scalar curvature of \(E\), and \(\Delta\) is the Laplace operator of \(M\).

**Remark 1.1.** Apart from the generality of the above results, the \(C^0\)-estimate of the Tian-Yau-Zelditch expansion, when \(b \neq 1\), was essentially known to \[4\]. Thus, technically the proofs of the above theorems are on the \(C^\mu\) estimates. We developed a theorem on the variation of \(K\)-coordinates to achieve this goal. This is a research announcement. Details of the proofs will follow.

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