A Simplified Characterisation of Provably Computable Functions of the System ID₁ of Inductive Definitions (Technical Report)

Naohi Eguchi¹ and Andreas Weiermann²

¹ Mathematical Institute, Tohoku University, Japan
eguchi@math.tohoku.ac.jp
² Department of Mathematics, Ghent University, Belgium
weiermann@cage.ugent.be

Abstract. We present a simplified and streamlined characterisation of provably total computable functions of the theory ID₁ of non-iterated inductive definitions. The idea of the simplification is to employ the method of operator-controlled derivations that was originally introduced by Wilfried Buchholz and afterwards applied by the second author to a characterisation of provably total computable functions of Peano arithmetic PA.

Keywords: Provably Computable Functions; System of Inductive Definitions; Ordinal Notation Systems; Operator Controlled Derivations.

1 Introduction

As stated by Gödel’s second incompleteness theorem, any reasonable consistent formal system has an unprovable \( \Pi^0_2 \)-sentence that is true in the standard model of arithmetic. This means that the total (computable) functions whose totality is provable in a consistent system, which are known as provably computable functions or provably total functions, form a proper subclass of total computable functions. It is natural to ask how we can describe the provably total functions of a given system. Not surprisingly provably (total) computable functions are closely related to provable well-ordering, i.e., ordinal analysis. Up to date ordinal analysis for quite strong systems has been accomplished by M. Rathjen [13,14] or T. Arai [12]. On the other hand several successful applications of techniques from ordinal analysis to characterisations of provably computable functions have been provided by B. Blankertz and A. Weiermann [4], W. Buchholz [7], Buchholz, E. A. Cichon and Weiermann [8], M. Michelbrink [10], or G. Takeuti [16]. Surveys on characterisations of provably computable functions of fragments of Peano arithmetic PA contain the monograph [9] by M. Fairtlough and S. S. Wainer.

* The first author is supported by the research project Philosophical Frontiers in Reverse Mathematics sponsored by the John Templeton Foundation.
Modern ordinal analysis is based on the method of local predicativity, that was first introduced by W. Pohlers, c.f. [11,12]. Successful applications of local predicativity to provably computable functions contain works by Blankertz and Weiermann [18] and by Weiermann [5]. However, to the authors’ knowledge, the most successful way in ordinal analysis is based on the method of operator-controlled derivations, an essential simplification of local predicativity, that was introduced by Buchholz [6]. In [19] the second author successfully applied the method of operator-controlled derivations to a streamlined characterisation of provably computable functions of PA. (See also [12] Section 2.1.5.) Technically this work aims to lift up the characterisation in [19] to an impredicative system ID$_1$ of non-iterated inductive definitions. We introduce an ordinal notation system O($\Omega$) and define a computable function $f^\alpha$ for a starting number-theoretic function $f : \mathbb{N} \to \mathbb{N}$ by transfinite recursion on $\alpha \in O(\Omega)$. The ordinal notation system O($\Omega$) comes from a draft [20] of the second author and the transfinite definition of $f^\alpha$ comes from [19]. We show that a function is provably computable in ID$_1$ if and only if it is a Kalmar elementary function in $\{s^\alpha \mid \alpha \in O(\Omega) \text{ and } \alpha < \Omega\}$, where $s$ denotes the successor function $m \mapsto m + 1$ and $\Omega$ denotes the least non-recursive ordinal. (Corollary 12)

2 Preliminaries

In order to make our contribution precise, in this preliminary section we collect the central notions. We write $\mathcal{L}_{PA}$ to denote the standard language of first order theories of arithmetic. In particular we suppose that the constant 0 and the successor function symbol $S$ are included in $\mathcal{L}_{PA}$. For each natural $m$ we use the notation $m$ to denote the corresponding numeral built from 0 and $S$. Let a set variable $X$ denote a subset of $\mathbb{N}$. We write $X(t)$ instead of $t \in X$ for $\mathcal{L}_{PA} \cup \{X\}$. Let $FV_1(A)$ denote the set of free number variables appearing in a formula $A$ and $FV_2(A)$ the set of free set variables in $A$. And then let $\text{FV}(A) := FV_1(A) \cup FV_2(A)$. For a fresh set variable $X$ we call an an $\mathcal{L}_{PA}(X)$-formula $A(x)$ a positive operator form if $FV_1(A(x)) \subseteq \{x\}$, $FV_2(A(x)) = \{X\}$, and $X$ occurs only positively in $A$.

Let $FV_1(A(x)) = \{x\}$. For a formula $F(x)$ such that $x \in FV_1(F(x))$ we write $A(F, t)$ to denote the result of replacing in $A(t)$ every subformula $X(s)$ by $F(s)$. The language $\mathcal{L}_{ID_1}$ of the theory ID$_1$ of non-iterated inductive definitions is defined by $\mathcal{L}_{ID_1} := \mathcal{L}_{PA} \cup \{P_A \mid A \text{ is a positive operator form}\}$ where for each positive operator form $A$, $P_A$ denotes a new unary predicate symbol. We write $\mathcal{T}(\mathcal{L}_{ID_1}, \forall)$ to denote the set of $\mathcal{L}_{ID_1}$-terms and $\mathcal{T}(\mathcal{L}_{ID_1})$ to denote the set of closed $\mathcal{L}_{ID_1}$-terms. The axioms of ID$_1$ consist of the axioms of Peano arithmetic PA in the language $\mathcal{L}_{ID_1}$ and the following new axiom schemata (ID$_1$) and (ID$_2$):

(ID$_1$) $\forall x(A(P_A, x) \rightarrow P_A(x))$.

(ID$_2$) (The universal closure of) $\forall x(A(F, x) \rightarrow F(x)) \rightarrow \forall x(P_A(x) \rightarrow F(x))$, where $F$ is an $\mathcal{L}_{ID_1}$-formula.
For each $n \in \mathbb{N}$ we write $\Sigma_n$ to denote the fragment of Peano arithmetic $\text{PA}$ with induction restricted to $\Sigma_n^0$-formulas. Let $k$ be a natural number and $f : \mathbb{N}^k \to \mathbb{N}$ a number-theoretic function and $T$ be a theory of arithmetic containing $\Sigma_1$. Then we say $f$ is provably computable in $T$ or provably total in $T$ if there exists a $\Sigma_1^0$-formula $A_f(x_1, \ldots, x_k, y)$ such that the following hold:

1. $\text{FV}(A_f) = \text{FV}_1(A_f) = \{x_1, \ldots, x_k, y\}$.
2. For all $m, n \in \mathbb{N}$, $f(m) = n$ holds if and only if $A_f(m, n)$ is true in the standard model $\mathbb{N}$ of $\text{PA}$.
3. $\forall x \exists y A_f(x, y)$ is a theorem in $T$.

It is well known that the provably computable functions of the theory $\Sigma_1$ coincide with the primitive recursive functions. It is also known that the provably computable functions of the theory $\Sigma_2$ coincide with the P¨eter’s multiply recursive functions.

### 3 A non-recursive ordinal notation system $\mathcal{OT}(\mathcal{F})$

In this section we introduce a non-recursive ordinal notation system $\mathcal{OT}(\mathcal{F}) = (\mathcal{OT}(\mathcal{F}), <)$. This new ordinal notation system is employed in the next section. For an element $\alpha \in \mathcal{OT}(\mathcal{F})$ let $\mathcal{OT}(\mathcal{F}) \upharpoonright \alpha$ denote the set $\{\beta \in \mathcal{OT}(\mathcal{F}) \mid \beta < \alpha\}$.

**Definition 1.** We define three sets $\mathcal{SC} \subseteq \mathbb{H} \subseteq \mathcal{OT}(\mathcal{F})$ of ordinal terms and a set $\mathcal{F}$ of unary function symbols simultaneously. Let $0, \varphi, \Omega, S, E$ and $+$ be distinct symbols.

1. $0 \in \mathcal{OT}(\mathcal{F})$ and $\Omega \in \mathcal{SC}$.
2. $\{S, E\} \subseteq \mathcal{F}$.
3. If $\alpha \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, then $S(\alpha) \in \mathcal{OT}(\mathcal{F})$ and $E(\alpha) \in \mathbb{H}$.
4. If $\{\alpha_1, \ldots, \alpha_l\} \subseteq \mathbb{H}$ and $\alpha_1 \geq \cdots \geq \alpha_l$, then $\alpha_1 + \cdots + \alpha_l \in \mathcal{OT}(\mathcal{F})$.
5. If $\{\alpha, \beta\} \subseteq \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, then $\varphi(\alpha, \beta) \in \mathbb{H}$.
6. If $\alpha \in \mathcal{OT}(\mathcal{F})$ and $\xi \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, then $\Omega^\alpha \cdot \xi \in \mathbb{H}$.
7. If $F \in \mathcal{F}$, $\alpha \in \mathcal{OT}(\mathcal{F})$ and $\xi \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, then $F^\alpha(\xi) \in \mathcal{SC}$.
8. If $F \in \mathcal{F}$ and $\alpha \in \mathcal{OT}(\mathcal{F})$, then $F^\alpha \in \mathcal{F}$.

By definition $F(\xi) \in \mathcal{OT}(\mathcal{F})$ holds if $F^\alpha(\xi) \in \mathcal{OT}(\mathcal{F})$ for some $\alpha \in \mathcal{OT}(\mathcal{F})$. We write $\omega^m$ to denote $\varphi^{m_a}$ and $m$ to denote $\omega^0 : m = \sum_{m}^{\omega^0}$.

Let $\text{Ord}$ denote the class of ordinals and $\text{Lim}$ the class of limit ones. We define a semantic $[\cdot]$ for $\mathcal{OT}(\mathcal{F})$, i.e., $[\cdot] : \mathcal{OT}(\mathcal{F}) \to \text{Ord}$. The well ordering $<$ on $\mathcal{OT}(\mathcal{F})$ is defined by $\alpha < \beta \iff [\alpha] < [\beta]$. Let $\Omega_1$ denote the least non-recursive ordinal $\omega_1^{CK}$. For an ordinal $\alpha$ we write $\alpha = N_F \cdot \Omega_1^{\alpha_1} \cdot \beta_1 + \cdots + \Omega_1^{\alpha_l} \cdot \beta_l$ if $\alpha > \alpha_1 > \cdots > \alpha_l$, $\{\beta_1, \ldots, \beta_l\} \subseteq \Omega_1$, and $\alpha = \Omega_1^{\alpha_1} \cdot \beta_1 + \cdots + \Omega_1^{\alpha_l} \cdot \beta_l$. Let $\varepsilon_\alpha$ denote the $\alpha$th epsilon number. One can observe that for each ordinal $\alpha < \varepsilon_{\Omega_1+1}$ there uniquely exists a set $\{\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l\}$ of ordinals such that $\alpha = N_F \cdot \Omega_1^{\alpha_1} \cdot \beta_1 + \cdots + \Omega_1^{\alpha_l} \cdot \beta_l$. For a set $K \subseteq \text{Ord}$ and for an ordinal $\alpha$ we will write $K < \alpha$ to abbreviate $\forall \xi \in K \xi < \alpha$, and dually $\alpha \leq K$ to abbreviate $(\exists \xi \in K) \xi \leq \alpha$.
Definition 2 (Collapsing operators).

1. Let \( \alpha \) be an ordinal such that \( \alpha = \sum_{i=0}^{\alpha_1} \Omega^{\alpha_i}_i \cdot \beta_i + \cdots + \Omega_1^{\alpha_1} \cdot \beta_1 < \varepsilon_{\Omega_1+1} \). The set \( K_\Omega \alpha \) of coefficients of \( \alpha \) is defined by

\[
K_\Omega \alpha = \{ \beta_1, \ldots, \beta_l \} \cup K_\Omega \alpha_1 \cup \cdots \cup K_\Omega \alpha_l.
\]

2. Let \( F : \text{Ord} \to \text{Ord} \) be an ordinal function. Then a function \( F^\alpha : \text{Ord} \to \text{Ord} \) is defined by transfinite recursion on \( \alpha \in \text{Ord} \) by

\[
\begin{cases}
F^0(\xi) = F(\xi), \\
F^\alpha(\xi) = \min \{ \gamma \in \text{Ord} \mid \omega^\gamma = \gamma, \ K_\Omega \alpha \cup \{ \xi \} < \gamma \text{ and } \\
(\forall \eta < \gamma)(\forall \beta < \alpha)(K_\Omega \beta < \gamma \Rightarrow F^\beta(\eta) < \gamma) \}.
\end{cases}
\]

Corollary 3. Let \( F : \text{Ord} \to \text{Ord} \) be an ordinal function. Then \( F^\beta(\eta) < F^\alpha(\xi) \) holds if one of the following holds.

1. \( \beta < \alpha \) and \( K_\Omega \beta \cup \{ \eta \} < F^\alpha(\xi) \).
2. \( \alpha \leq \beta \) and \( F^\beta(\eta) \leq K_\Omega \alpha \).

Proposition 4. Suppose that \( \alpha < \varepsilon_{\Omega_1+1} \), a function \( F : \text{Ord} \to \text{Ord} \) has a \( \Sigma_1 \)-definition in the \( \Omega_1 \)-th stage \( L_{\Omega_1} \) of the constructible hierarchy \( (L_\alpha)_{\alpha \in \text{Ord}} \) and that \( F(\xi) < \Omega_1 \) for all \( \xi < \Omega_1 \). Then \( F^\alpha \) also has a \( \Sigma_1 \)-definition in \( L_{\Omega_1} \) and \( F^\alpha(\xi) < \Omega_1 \) holds for all \( \xi < \Omega_1 \).

Proof. By induction on \( \alpha < \varepsilon_{\Omega_1+1} \). If \( \alpha = 0 \), then \( F^0 \) a \( \Sigma_1 \)-function since so is \( F \), and \( F^0(\xi) = F(\xi) < \Omega_1 \) for all \( \xi < \Omega_1 \). Suppose \( \alpha > 0 \). From elementary facts in generalised recursion theory, c.f. Barwise’s book [3], careful readers will observe that \( F^\alpha \) has a \( \Sigma_1 \)-definition in \( L_{\Omega_1} \), since "\( \xi \in K_\Omega \alpha \)" can be expressed by a \( \Delta_0 \)-formula. To see that \( F^\alpha(\xi) \) for all \( \xi < \Omega_1 \) let us define a function \( \psi : \omega \to \varepsilon_{\Omega_1} \) by

\[
\begin{align*}
\psi(0) &= \min \{ \gamma < \varepsilon_{\Omega_1+1} \mid \omega^\gamma = \gamma \text{ and } K_\Omega \alpha \cup \{ \xi \} < \gamma \}, \\
\psi(m+1) &= \min \{ \gamma < \varepsilon_{\Omega_1+1} \mid \omega^\gamma = \gamma \text{ and } K_\Omega \alpha \cup \{ \xi \} < \gamma \text{ and } \\
&\quad (\forall \eta < \psi(m))(\forall \beta < \alpha)(K_\Omega \beta < \psi(m) \Rightarrow F^\beta(\eta) < \gamma) \}.
\end{align*}
\]

We can see that \( \psi \) is a \( \Sigma_1 \)-function in the same way as we see that \( F^\alpha \) is so.

Claim. \( \psi(m) < \Omega_1 \) for all \( m \in \omega \).

We show that \( \psi(m) < \Omega_1 \) holds by (side) induction on \( m \). In the base case, \( \psi(0) < \Omega_1 \) holds since \( K_\Omega \alpha \cup \{ \xi \} < \Omega_1 \) and \( \Omega_1 \) is closed under the function [E]. Consider the induction step. Let \( \eta < \psi(m) \). Then Side Induction Hypothesis implies \( \eta < \psi(m) < \Omega_1 \). Hence (Main) Induction Hypothesis enables us to deduce \( F^\beta(\eta) < \Omega_1 \) for all \( \beta < \alpha \). Let us define a function \( G : \{ \beta < \alpha \mid K_\Omega \beta < \psi(m) \} \to \Omega_1 \) by \( \beta \mapsto F^\beta(\eta) \). One can see that \( G \) is a \( \Sigma_1 \)-function. On the other hand \( \#\{ \beta < \alpha \mid K_\Omega \beta < \psi(m) \} \leq \omega \) since \( \psi(m) < \Omega_1 \). Here we recall that \( \Omega_1 \)
denotes the least recursively regular ordinal $\omega_1^{CK}$ and hence $L_{\Omega_1}$ is closed under functions whose graphs are of $\Sigma_1$ in $L_{\Omega_1}$. From these we have inequality

$$\psi(m + 1) \leq \sup \{ G(\beta) \mid \beta < \alpha \text{ and } K_{\Omega} \beta < \psi(m) \} < \Omega_1,$$

concluding the claim.

By the claim $\psi$ is a $\Sigma_1$-function in $L_{\Omega_1}$ from $\omega$ to $\Omega_1$. Hence $\sup_{m \in \omega} \psi(m) < \Omega_1$. Define an ordinal $\gamma$ by $\gamma = \sup_{m \in \omega} \psi(m)$. Then $\omega^\gamma = \gamma$, $K_{\Omega} \alpha \cup \{ \xi \} < \gamma$ and $K_{\Omega} \beta < \gamma \Rightarrow F^\beta(\eta) < \gamma$ for all $\eta < \xi$ and for all $\beta < \alpha$. This implies $F^\alpha(\xi) \leq \gamma < \Omega_1$. \hfill \Box

**Proposition 5.** For any $\alpha \in \text{Ord}$, for any $\eta, \xi < \Omega_1$ and for any ordinal function $F : \Omega_1 \rightarrow \Omega_1$, if $\eta < F^\alpha(\xi)$, then $F^\alpha(\eta) \leq F^\alpha(\xi)$.

**Proof.** If $\eta \leq \xi$, then $F^\alpha(\eta) \leq F^\alpha(\xi)$ by the definition of $F^\alpha(\eta)$. Let us consider the case $\xi < \eta < F^\alpha(\xi)$. In this case $K_{\Omega} \alpha \cup \{ \eta \} < F^\alpha(\xi)$ by the definition of $F^\alpha(\xi)$. Suppose that $\beta < \alpha, \gamma < F^\alpha(\xi)$ and $K_{\Omega} \beta < F^\alpha(\xi)$. Then $F^\beta(\gamma) < F^\alpha(\xi)$ again by the definition of $F^\alpha(\xi)$. By the minimality of $F^\alpha(\eta)$ we can conclude $F^\alpha(\eta) \leq F^\alpha(\xi)$. \hfill \Box

**Definition 6.** We define the value $[\alpha] \in \text{Ord}$ of an ordinal term $\alpha \in OT(F)$ by recursion on the length of $\alpha$.

1. $[0] = 0$ and $[\Omega] = \Omega_1$.
2. $[\alpha + \beta] = [\alpha] + [\beta]$.
3. $[\varphi \alpha \beta] = [\varphi][\alpha][\beta]$, where $[\varphi]$ is the standard Veblen function, i.e.,

$$\begin{align*}
[\varphi]0\beta &= \omega^\beta, \\
[\varphi](\alpha + 1)0 &= \sup \{ ([\varphi] \alpha)^n \mid n \in \omega \}, \\
[\varphi] \gamma 0 &= \sup \{ [\varphi] \alpha 0 \mid \alpha < \gamma \} & \text{if } \gamma \in \text{Lim}, \\
[\varphi] (\alpha + 1)(\beta + 1) &= \sup \{ ([\varphi] \alpha)^n ([\varphi] \alpha + 1) \beta + 1 \mid n \in \omega \}, \\
[\varphi] \gamma (\beta + 1) &= \sup \{ [\varphi] \alpha ([\varphi] \beta + 1) \mid \alpha < \gamma \} & \text{if } \gamma \in \text{Lim}, \\
[\varphi] \alpha \gamma &= \sup \{ [\varphi] \alpha \beta \mid \beta < \gamma \} & \text{if } \gamma \in \text{Lim}.
\end{align*}$$

4. $[\Omega^\alpha \cdot \xi] = \Omega_1^{[\alpha]} \cdot [\xi]$.
5. $[S(\alpha)] = [S][\alpha]$, where $S$ denotes the ordinal successor $\alpha \mapsto \alpha + 1$. Clearly $\{ [S] \xi \mid \xi \in \Omega_1 \} \subseteq \Omega_1$.
6. $[E(\alpha)] = [E][\alpha]$, where the function $[E] : \text{Ord} \rightarrow \text{Ord}$ is defined by $[E](\alpha) = \min \{ \xi \in \text{Ord} \mid \omega^\xi = \xi \text{ and } \alpha < \xi \}$. It is also clear that $\{ [E] \xi \mid \xi \in \Omega_1 \} \subseteq \Omega_1$ holds.
7. $[F^\alpha(\xi)] = [F][\alpha][\xi]$.

**Definition 7.** For all $\alpha, \beta \in OT(F)$, $\alpha < \beta$ if $[\alpha] < [\beta]$, and $\alpha = \beta$ if $[\alpha] = [\beta]$.

We will identify each element $\alpha \in OT(F)$ with its value $[\alpha] \in \text{Ord}$. Accordingly we will write $K_{\Omega} \alpha$ instead of $K_{\Omega} [\alpha]$ for $\alpha \in OT(F)$. Further for a finite set $K \subseteq \text{Ord}$ we write $K_{\Omega} \cdot K$ to denote the finite set $\bigcup_{\xi \in K} K_{\Omega} \xi$. By this identification, $\mathbb{I}$ is the set of additively indecomposable ordinals and $\mathbb{SC}$ is the set of strongly critical ordinals, i.e, $\mathbb{SC} \subseteq \mathbb{I} \subseteq \text{Lim} \cup \{ 1 \} \subseteq \text{Ord}$.
Corollary 8. \( F^\alpha(\xi) < \Omega \) for any \( F \in \mathcal{F} \) and \( \xi < \Omega \).

**Proof.** Proof by induction over the build-up of \( F \in \mathcal{F} \).

**Corollary 9.**

1. \( K_\Omega 0 = K_\Omega \Omega = \emptyset \).
2. If \( K_\Omega \alpha < \xi \) and \( \xi \in \text{SC} \), then \( K_\Omega S(\alpha) < \xi \).
3. \( K_\Omega E(\alpha) = \{ E(\alpha) \} \) (since \( \alpha < \Omega \)).
4. If \( K_\Omega \alpha \cup K_\Omega \beta < \xi \) and \( \xi \in \text{SC} \), then \( K_\Omega(\alpha + \beta) < \xi \).
5. \( K_\Omega \varphi \alpha \beta = \{ \varphi \alpha \beta \} \) (since \( \alpha, \beta < \Omega \)). Further, if \( \alpha, \beta < \xi \) and \( \xi \in \text{SC} \), then \( \varphi \alpha \beta < \xi \).
6. \( K_\Omega F^\alpha(\xi) = \{ F^\alpha(\xi) \} \) (since \( \xi < \Omega \)).

By Corollary 8, each function symbol from \( \mathcal{F} \) defines a weakly increasing function \( F : \Omega \to \Omega \) such that \( \xi < F(\xi) \) holds for all \( \xi \in \Omega \). In the rest of this section let \( F \) denote such a function. For a finite set \( K \subseteq \text{Ord} \) we will use the notation \( F[K](\xi) \) to abbreviate \( F(\max(K \cup \{ \xi \})) \).

**Lemma 10.** Let \( K \subseteq \text{Ord} \) be a finite set such that \( K < \Omega \). Then \( (F[K])^\alpha(\xi) \leq F^\alpha[K](\xi) \) for all \( \xi < \Omega \).

**Proof.** By induction on \( \alpha \). For the base case \( (F[K])^0(\xi) = F[K](\xi) = F^0[K](\xi) \). Suppose \( \alpha > 0 \). Then

\[
K_\Omega \alpha \cup \{ \xi \} < F^\alpha(\xi) \leq F^\alpha[K](\xi). \tag{1}
\]

Assume that \( \eta < F^\alpha[K](\xi) \), \( \beta < \alpha \) and \( K_\Omega \beta < F^\alpha[K](\xi) \). Then \( \eta < \Omega \), and hence \( F[K]^{\beta}(\eta) \leq F^{\beta}[K](\eta) \) by IH. Hence

\[
(F[K])^{\beta}(\eta) < F^\alpha[K](\eta) \quad \text{since } K_\Omega K < F^\alpha[K](\eta). \tag{2}
\]

By conditions (1) and (2) we conclude \( (F[K])^\alpha(\xi) \leq F^\alpha[K](\xi) \).

**Lemma 11.** \( (F^\alpha)^{\beta}(\xi) \leq F^{\alpha+\beta}(\xi) \) for all \( \xi < \Omega \).

**Proof.** By induction on \( \beta \). For the base case \( (F^\alpha)^0(\xi) = F^\alpha(\xi) = F^{\alpha+0}(\xi) \). Suppose \( \beta > 0 \). Then

\[
K_\Omega \beta \cup \{ \xi \} < F^{\beta}(\xi) \leq F^{\alpha+\beta}(\xi). \tag{3}
\]

Assume that \( \eta < F^{\alpha+\beta}(\xi) \), \( \beta' < \beta \) and \( K_\Omega \beta' < F^{\alpha+\beta}(\xi) \). Then \( \eta < \Omega \), and hence \( (F^\alpha)^{\beta'}(\eta) \leq F^{\alpha+\beta}(\eta) \) by IH. Hence

\[
(F^\alpha)^{\beta'}(\eta) \leq F^{\alpha+\beta}(\eta) < F^{\alpha+\beta}(\xi). \tag{4}
\]

By conditions (3) and (4) we can conclude \( (F^\alpha)^{\beta}(\xi) \leq F^{\alpha+\beta}(\xi) \).
4 An infinitary proof system ID₁\(^\infty\)

This section introduces the main definition of this paper. We introduce a new infinitary proof system ID₁\(^\infty\) to which the new ordinal notation system is connected and into which every (finite) proof in ID₁ is embedded. For each positive operator form \(A\) and for each ordinal term \(\alpha \in (OT(\mathcal{F}) \cup \Omega)\cup \{\Omega\}\) let \(P_A^<\alpha\) be a new unary predicate symbol. Let us define an infinitary language \(L^*\) of ID₁\(^\infty\) by \(L^* = L_{PA} \cup \{\neq, \notin\} \cup \{P_A^<\alpha, \neg P_A^<\alpha \mid \alpha \in (OT(\mathcal{F}) \cup \Omega)\cup \{\Omega\}\text{ and }A\text{ is a positive operator form}\}\). Let us write \(P_A^<\Omega\) to denote \(P_A\) to have the inclusion \(L_{ID_1} \subseteq L^*\). We write \(T(L^*)\) to denote the set of closed \(L^*\)-terms. Specifically, the language \(L^*\) contains complementary predicate symbol \(\neg P\) for each predicate symbol \(P \in L^*\). We note that the negation \(\neg\) nor the implication \(\rightarrow\) is not included as a logical symbol. The implication \(\neg A\) is defined via de Morgan’s law by \(\neg(P(t)) \equiv P(\bar{t})\) for an atomic formula \(P(t)\), \(\neg(A \land B) \equiv \neg A \lor \neg B, \neg(A \lor B) \equiv \neg A \land \neg B, \neg \forall x A \equiv \exists x \neg A\) and \(\neg \exists x A \equiv \forall x \neg A\). The implication \(A \rightarrow B\) is defined by \(\neg A \lor B\). We start with technical definitions. We will write \(P_A^{<\alpha} t\) and \(\neg P_A^{<\alpha} t\) respectively for \(P_A^{<\alpha}(t)\) and \(\neg P_A^{<\alpha}(t)\).

**Definition 12 (Complexity measures of \(L^*\)-formulas).**

1. The length \(\text{lh}(A)\) of an \(L^*\)-formula \(A\) is the number of the symbols \(P_A^{<\alpha}, \neg P_A^{<\alpha}, \lor, \land, \exists\) and \(\forall\) occurring in \(A\).
2. The rank \(\text{rk}(A)\) of an \(L^*\)-formula \(A\).
   (a) \(\text{rk}(P_A^{<\alpha} t) := \text{rk}(\neg P_A^{<\alpha} t) := \omega \cdot \alpha\).
   (b) \(\text{rk}(A) := 0 \text{ if } A \text{ is an } L_{ID_1}\)-literal.
   (c) \(\text{rk}(A \land B) := \text{rk}(A \lor B) := \max\{\text{rk}(A), \text{rk}(B)\} + 1\).
   (d) \(\text{rk}(\forall x A) := \text{rk}(\exists x A) := \text{rk}(A) + 1\).
3. The set \(\text{k}^{II}(A)\) of \(\Pi\)-coefficients of an \(L^*\)-formula \(A\).
   (a) \(\text{k}^{II}(P_A^{<\alpha} t) := \{0\}, \text{k}^{II}(\neg P_A^{<\alpha} t) := \{0, \alpha\}\).
   (b) \(\text{k}^{II}(A) := \{0\} \text{ if } A \text{ is an } L_{ID_1}\)-literal.
   (c) \(\text{k}^{II}(A \land B) := \text{k}^{II}(A \lor B) := \text{k}^{II}(A) \cup \text{k}^{II}(B)\).
   (d) \(\text{k}^{II}(\forall x A) := \text{k}^{II}(\exists x A) := \text{k}^{II}(A)\).
4. The set \(\text{k}^{\Sigma}(A)\) of \(\Sigma\)-coefficients of an \(L^*\)-formula \(A\).
   \(\text{k}^{\Sigma}(A) := \text{k}^{II}(\neg A)\).
5. The set \(\text{k}(A)\) of all the coefficients of an \(L^*\)-formula \(A\).
   \(\text{k}(A) := \text{k}^{II}(A) \cup \text{k}^{\Sigma}(A)\).
6. The set \(\text{k}^{II}_<(A)\) of \(\Pi\)-coefficients of an \(L^*\)-formula \(A\) less than \(\Omega\).
   \(\text{k}^{II}_<(A) := \text{k}^{II}(A) \mid \Omega\).
   
   The set \(\text{k}^{II}_>(A)\) and \(\kappa(A)\) are defined accordingly.

By definition \(\text{rk}(A) = \text{rk}(\neg A), \text{k}(A) = \text{k}(\neg A)\) and \(\kappa(A) = \kappa(\neg A)\).

**Definition 13 (Complexity measures of \(L^*\)-terms).**

1. The value \(\text{val}(t)\) of a term \(t \in T(L_{ID_1}) = T(L_{PA})\) is the value of the closed term \(t\) in the standard model \(\mathbb{N}\) of the Peano arithmetic PA.
2. A complexity measure \( \text{ord} : \mathcal{T}(\mathcal{L}^*) \to (\mathcal{OT}(\mathcal{F}) \uparrow \Omega) \cup \{ \Omega \} \) is defined by
\[
\begin{align*}
\text{ord}(t) &:= 0 & &\text{if } t \in \mathcal{T}(\mathcal{L}_{\text{ID}}), \\
\text{ord}(\alpha) &:= \xi & &\text{if } \alpha \in \mathcal{OT}(\mathcal{F}).
\end{align*}
\]
3. The norm \( N(\alpha) \) of \( \alpha \in \mathcal{OT}(\mathcal{F}) \).
   (a) \( N(0) = 0 \) and \( N(\Omega) = 1 \).
   (b) \( N(S(\alpha)) = N(\alpha) + 1 \).
   (c) \( N(E(\alpha)) = N(\alpha) + 1 \).
   (d) \( N(\alpha + \beta) = N(\alpha) + N(\beta) \).
   (e) \( N(\varphi \alpha \beta) = N(\alpha) + N(\beta) + 1 \).
   (f) \( N(\Omega^\alpha \cdot \xi) = N(\alpha) + N(\xi) + 1 \).
   (g) \( N(F^\alpha(\xi)) = N(F(\xi)) + N(\alpha) \).

The norm is extended to a complexity measure \( N : \mathcal{T}(\mathcal{L}^*) \to \mathbb{N} \) by
\[
\begin{align*}
N(t) &:= \text{val}(t) & &\text{if } t \in \mathcal{T}(\mathcal{L}_{\text{ID}}), \\
N(\alpha) &:= N(\alpha) & &\text{if } \alpha \in \mathcal{OT}(\mathcal{F}).
\end{align*}
\]

By definition \( N(\omega^\alpha) = N(\varphi 0 \alpha) = N(\alpha) + 1 \) and \( N(m) = N(\omega^0 \cdot m) = m \)
for any \( m < \omega \). This seems to be a good point to explain why we contain the constant \( \Omega \) in \( \mathcal{OT}(\mathcal{F}) \). Having that \( N(\Omega) = 1 \) makes some technicality easier.

**Definition 14.** We define a relation \( \simeq \) between \( \mathcal{L}^* \)-sentences and (infinitary) propositional \( \mathcal{L}^* \)-sentences.

1. \( \neg P_A^\xi <\alpha \equiv \bigwedge_{t \in \mathcal{T}(\mathcal{F}) \mid \alpha} \neg A(P_A^\xi, t) \) and \( P_A^\xi <\alpha \equiv \bigvee_{t \in \mathcal{T}(\mathcal{F}) \mid \alpha} A(P_A^\xi, t) \).
2. \( \forall A(x) \equiv \bigwedge_{t \in \mathcal{T}(\mathcal{L}_{\text{ID}})} A(t) \) and \( \exists x A(x) \equiv \bigvee_{t \in \mathcal{T}(\mathcal{L}_{\text{ID}})} A(t) \).
3. \( \forall x A(x) \equiv \bigwedge_{t \in T(\mathcal{L}_{\text{ID}})} A(t) \) and \( \exists x A(x) \equiv \bigvee_{t \in T(\mathcal{L}_{\text{ID}})} A(t) \).

We call an \( \mathcal{L}^* \)-sentence \( A \) a \( \land \)-type (conjunctive type) if \( A \simeq \bigwedge_{t \in J} A_t \) for some \( A_t \), and a \( \lor \)-type (disjunctive type) if \( A \simeq \bigvee_{t \in J} A_t \) for some \( A_t \). For the sake of simplicity we will write \( \bigwedge_{t \in J} A_t \) instead of \( \bigwedge_{\xi \in \mathcal{T}(\mathcal{F}) \mid \alpha} A_t \) and write \( \bigvee_{\xi \in \mathcal{T}(\mathcal{F}) \mid \alpha} A_t \) accordingly.

**Lemma 15.** 1. If either \( A \simeq \bigwedge_{t \in J} A_t \) or \( A \simeq \bigvee_{t \in J} A_t \), then for all \( i \in J \), \( k^A(A_i) \subseteq k^A(A) \cup \{ \text{ord}(i) \} \) and \( k^\xi(A_i) \subseteq k^\xi(A) \cup \{ \text{ord}(i) \} \).
2. For any \( \alpha \in \mathcal{OT}(\mathcal{F}) \), if \( A \simeq \bigwedge_{\xi \in \alpha} A_{\xi} \), then \( (\exists \sigma \in \mathcal{K}(A))(\forall \xi < \alpha)[\xi \leq \sigma] \).
3. For any \( \mathcal{L}^* \)-sentence \( A \), \( \text{rk}(A) = \omega \cdot \max k(A) + n \) for some \( n \leq \text{lh}(A) \).
4. If \( \text{rk}(A) = \Omega \), then either \( A \equiv P_A^\xi \Omega t \) or \( A \equiv \neg P_A^\xi \Omega t \).
5. If either \( A \simeq \bigwedge_{i \in J} A_i \) or \( A \simeq \bigvee_{i \in J} A_i \), then for all \( i \in J \), \( N(\text{rk}(A_i)) \leq \max\{ N(\text{rk}(A)), \omega \cdot N(\text{ord}) \} \).

Throughout this section we use the symbol \( F \) to denote a weakly increasing ordinal function \( \Omega \to \Omega \) and the symbol \( f \) to denote a number-theoretic function \( \mathbb{N} \to \mathbb{N} \) that enjoys the following conditions.

\((f.1)\) \( f \) is a strictly increasing function such that \( 2m + 1 \leq f(m) \) for all \( m \).

Hence, in particular, \( n + f(m) \leq f(n + m) \) for all \( m \) and \( n \).

\((f.2)\) \( 2 \cdot f(m) \leq f(f(m)) \) for all \( m \).
We will use the notation \( f[n](m) \) to abbreviate \( f(n + m) \). It is easy to see that if the conditions (1) and (2) hold, then for a fixed \( n \) the conditions (3) and (4) also hold.

**Definition 16.** Let \( f : \mathbb{N} \rightarrow \mathbb{N} \) be a number-theoretic function. Then a function \( f^\alpha : \mathbb{N} \rightarrow \mathbb{N} \) is defined by transfinite recursion on \( \alpha \in \mathcal{OT}(\mathcal{F}) \) by

\[
\begin{align*}
f^0(m) &= f(m), \\
f^\alpha(m) &= \max\{f^\beta(f^\beta(m)) \mid \beta < \alpha \text{ and } N(\beta) \leq f[N(\alpha)](m)\} \quad \text{if } 0 < \alpha.
\end{align*}
\]

**Corollary 17.** 1. If \( f \) is strictly increasing, then so is \( f^\alpha \) for any \( \alpha \in \mathcal{OT}(\mathcal{F}) \).
2. If \( \beta < \alpha \) and \( N(\beta) \leq f[N(\alpha)](m) \), then \( f^\beta(m) < f^\alpha(m) \).
3. \( f^\alpha(f^\alpha(m)) \leq f^{\alpha+1}(m) \).

We note that the function \( f^\alpha \) is not a recursive function in general even if \( f \) is recursive since the ordinal notation system \( \langle \mathcal{OT}(\mathcal{F}), \prec \rangle \) is not a recursive system.

**Example 18.** The following are examples of \( f^\alpha \) in case that \( \alpha \leq \omega \) and \( f \) is the successor function \( s : m \mapsto m + 1 \). Let us recall that \( N(n) = N(\omega^0 \cdot n) = n \) for all \( n < \omega \).

1. \( s^1(m) = s^0(s^0(m)) = m + 2 \).
2. \( s^2(m) = s^1(s^1(m)) = m + 4 \).
3. \( s^n(m) = m + 2^n \). (\( n < \omega \))
4. \( s^\omega(m) = m + 2^{m+3} \).

Let us see that \( N(\omega) = 1 \) and hence \( s[N(\omega)](m) = s(1 + m) = m + 2 \). Hence \( s^\omega(m) = f_{\omega+2}(f_{\omega+2}(m)) = m + 2^{m+2} + 2^{m+2} = m + 2^{m+3} \).

**Lemma 19.** Let \( \alpha \in \mathcal{OT}(\mathcal{F}) \) and \( F \in \mathcal{F} \). Then \( N(\alpha) \leq f^{F(\xi)}(0) \).

**Proof.** By induction over the term-construction of \( \alpha \in \mathcal{OT}(\mathcal{F}) \). For the base case \( N(0) = 0 \leq f(0) \leq f^{F\alpha(0)}(0) \) and \( N(\omega) = 1 \leq f(0) \leq f^{F\alpha(0)}(0) \). For the induction step, we only consider the case that \( \alpha = F^{\alpha_0}(\xi) \) for some \( \alpha_0 \neq 0 \) and for some \( \xi < \Omega \). The remaining cases can be treated in similar ways. In this case \( F^{\alpha_0}(0) < F^\alpha(0) \) holds since \( F^{\alpha_0}(0) \leq \{F^{\alpha_0}(\xi)\} = K_\alpha F^{\alpha_0}(\xi) < F^{F^{\alpha_0}(\xi)}(0) = F^\alpha(0) \). It is easy to see that \( F^{F(\xi)}(0) < F^\alpha(0) \) holds. By definition \( N(\alpha) = N(F(\xi)) + N(\alpha_0) \). By IH \( N(F(\xi)) \leq f^{F(\xi)}(0) \) and \( N(\alpha_0) \leq f^{F^{\alpha_0}(0)}(0) \). Hence

\[
N(\alpha) \leq f^{F(\xi)}(0) + f^{F^{\alpha_0}(0)},
\]

\[
\leq f^{F^{\alpha_0}(0)}(f^{F(\xi)}(0)) \text{ since } m + f^{\alpha_0}(0) \leq f^{\omega_0}(m) \text{ for all } m,
\]

\[
\leq f^{F^{\alpha_0}(0)+F(\xi)}(f^{F^{\alpha_0}(0)+F(\xi)}(0))
\]

\[
\leq f^{F^\alpha(0)}(0).
\]

To see that the last inequality is true, we can check \( F^{\alpha_0}(0) + F^{F(\xi)}(0) < F^\alpha(0) \) and \( N(F^{\alpha_0}(0) + F^{F(\xi)}(0)) \leq 2 \cdot N(F^\alpha(0)) \leq f[N(F^\alpha(0))](0) \) from an assumption that \( 2m \leq f(m) \). \( \square \)
**Lemma 20.** Let \( \{\alpha, \beta\} \subseteq OT(\mathcal{F}) \mid \Omega \) and \( F \in \mathcal{F} \). Then, for all \( m \), \((f^\alpha)^\beta(m) \leq f^{F^{\Omega,\alpha+\beta}}(m)\).

**Proof.** If \( \alpha = 0 \), then \((f^\alpha)^\beta(m) = f^\beta(m) \leq f^{F^{\Omega,0+\beta}}(m)\). Suppose \( \alpha \neq 0 \). Then we show the assertion by induction on \( \beta \). If \( \beta = 0 \), then \((f^\alpha)^\beta(m) = f^\alpha(m) \leq f^{F^{\Omega,\alpha}}(m)\). Suppose \( \beta > 0 \). Then there exists \( \gamma < \beta \) such that \( N(\gamma) \leq f^\alpha[N(\beta)](m) \) and \((f^\alpha)^\beta(m) = (f^\alpha)^\gamma((f^\alpha)^\gamma(m))\). By IH

\[
(f^\alpha)^\gamma((f^\alpha)^\gamma(m)) \leq f^{F^{\Omega,\alpha+\gamma}}(m)\]

On the other hand \( N(\beta) \leq f^{F^\beta}(0) \) by Lemma 19. Hence

\[
N(\gamma) \leq f^\alpha(f^{F^\beta}(m)) \quad \text{since } m \leq f(m) \leq f^{F^\beta}(0),
\]

\[
\leq f^{F^{\Omega,\alpha}}+F^\beta(f^{F^{\Omega,\alpha}+F^\beta}(m))
\]

\[
\leq f^{F^{\Omega,\alpha+\beta}}(m).
\]

The second inequality holds since \( \{\alpha, F^\beta(0)\} = K_\Omega \alpha \cup \{F^\beta(0)\} < F^{\Omega,\alpha}(0) + F^\beta(0) \). This implies that

\[
N(F^{\Omega,\alpha+\gamma}(0)) \leq N(F(0)) + N(\alpha) + 1 + f^{F^{\Omega,\alpha+\beta}(0)}(m)
\]

\[
\leq f[N(F^{\Omega,\alpha+\beta}(0))](m).
\]

Further \( F^{\Omega,\alpha+\gamma}(0) < F^{\Omega,\alpha+\beta}(0) \) holds since \( K_\Omega \gamma = \{\gamma\} < \beta \). This together with the inequality 1 yields that

\[
(f^\alpha)^\beta(m) \leq f^{F^{\Omega,\alpha+\gamma}}(m) \leq f^{F^{\Omega,\alpha+\beta}}(m).
\]

\[\square\]

**Lemma 21.**

1. \( f^\alpha[n](m) \leq (f[n])^\alpha(m) \).
2. If \( n \leq m \), then \( (f[n])^\alpha(m) \leq f^\alpha[f^\alpha(f(m))](f(m)) \).

We write \( f[n][m] \) to abbreviate \((f[n])(m)\) and \( f[n]^\alpha \) to abbreviate \((f[n])^\alpha\).

**Proof.** By induction on \( \alpha \). For the base case \( f^0[n](m) = f[n](m) = f[n]^\alpha(m) \). For the induction step, assume \( \alpha > 0 \). Then there exists \( \beta < \alpha \) such that \( N(\beta) \leq f[N(\alpha)](m) \) and \( f^\alpha[n](m) = f^\beta(f^\beta[n](m)) \). Hence

\[
f^\alpha[n](m) \leq f^\beta(f[n]^\beta(m)) \quad \text{by } \text{IH},
\]

\[
\leq f[n]^\beta(f[n]^\beta(m))
\]

\[
\leq f[n]^\alpha(m).
\]

The last inequality holds since \( N(\beta) \leq f[N(\alpha)](m) = f[n][N(\alpha)](m) \).

**Property 1** We show that \( f[n]^\alpha(f(m)) \leq f^\alpha[f^\alpha(f(m))](f(m)) \) holds for all \( m \geq n \) by induction on \( \alpha \). Let \( n \leq m \). For the base case \( f[n]^0(m) \leq f[n](m) \leq f[n]^\alpha(m) \)
\[ f(m+n) \leq f^0(f(m)) + f(m) = f^0[f^0(f(m))](f(m)) \] For the induction step, assume \( \alpha > 0 \). Then there exists \( \beta < \alpha \) such that \( N(\beta) \leq f^0[\Omega]\alpha(\alpha)(m) \) and \( f^{\alpha}(m) = f^{\beta}[\Omega]m(\beta) \). Let us observe that

\[
N(\beta) = f(n + N(\alpha) + m) \leq f(N(\alpha) + m) \quad \text{since} \ n \leq m, \\
\leq f(N(\alpha) + f(m)) \quad \text{from (7)}.
\]

We can see that \( f^{\alpha}(m) \leq f^0[f^0(f(m))](f(m)) \) holds as follows.

\[
f^{\alpha}(m) \leq f^0(f^0(f(m)) + f(m)) \quad \text{by IH}, \\
\leq f^0(f^0(2 \cdot f^0(f(m)) + f(m))) \quad \text{by (f.1)}, \\
\leq f^0(f^0(f^0(f(m))) + f(m)) \quad \text{by (f.2)}, \\
\leq f^0(f^0(f^0(f(m))) + f(m)) \quad \text{by (7)}, \\
\leq f^0(f^0(2 \cdot f^0(f(m)) + f(m))) \quad \text{by (f.1)}, \\
\leq f^0(f^0(f^0(f(m))) + f(m)) \quad \text{by (7)}, \\
\leq f^0(f^0(f^0(f(m))) + f(m)) = f^0[f^0(f(m))](f(m)).
\]

The last inequality holds since \( N(\beta) \leq f(N(\alpha) + f^0(f(m)) + f(m)) \). \( \square \)

**Corollary 22.** If \( n \leq m \), then \( f^{\alpha}(m) \leq f^{\alpha+2}(m) \).

**Proof.** By Lemma \([21][2] \), \( f^{\alpha}(m) \leq f^{\alpha}(f^0(f(m)) + f(m)) \leq f^{\alpha}(f^0(2 \cdot f(m))) \leq f^{\alpha+1}(f^0(f^0(m))) \leq f^{\alpha+1}(f^{\alpha+1}(m)) \leq f^{\alpha+2}(m) \). \( \square \)

We define a relation \( f, F \vdash_{\rho}^\alpha \Gamma \) for a quintuple \( (f, F, \alpha, \rho, \Gamma) \) where \( \alpha < \varepsilon_{\Omega+1}, \rho < \Omega, \omega \) and \( \Gamma \) is a sequent of \( \mathcal{L}^\ast \)-sentences. In this paper a “sequent” means a finite set of formulas. We write \( \Gamma, A \) or \( A, \Gamma \) to denote \( \Gamma \cup \{A\} \). Let us recall that for a finite set \( K \subseteq \text{Ord} \), \( F[K](\xi) \) denotes \( F(\max(K \cup \{\xi\})) \). We will write \( F[\mu](\xi) \) to denote \( F[\{\mu\}](\xi) \). We write \( \text{TRUE}_0 \) to denote the set \( \{A | A \in \mathcal{L}_{PA}\text{-literal true in the standard model} N \text{ of PA}\} \).

**Definition 23.** \( f, F \vdash^\alpha_{\rho} \Gamma \) if

\[
\max\{N(F(0)), N(\alpha)\} \leq f(0), \quad K_{\Omega} \alpha < F(0), \quad \text{(HYP}(f; F; \alpha)) \]

and one of the following holds.

(Ax1) \( \exists x(A(x) \land \mathcal{L}_{ID_1} \text{-literal}, \exists s, t \in T(\mathcal{L}_{ID_1}) \text{ s.t. } FV(A) = \{x\}, \text{val}(s) = \text{val}(t) \) and \( \{\neg A(s), A(t)\} \subseteq \Gamma \).

(Ax2) \( \Gamma \cap \text{TRUE}_0 \neq \emptyset \).

(V) \( \exists A \simeq \bigvee_{\varepsilon \in J} A_{\varepsilon} \in \Gamma, \exists \alpha_0 < \alpha, \exists \varepsilon_0 \in J \text{ s.t. } N(\varepsilon_0) \leq f(0) \text{ ord}(\varepsilon_0) \leq \min\{\alpha, F(0)\}, \text{ and } f, F \vdash^\alpha_{\rho} \Gamma, A_{\varepsilon_0} \).

(A) \( \exists A \simeq \bigwedge_{\varepsilon \in J} A_{\varepsilon} \in \Gamma \text{ s.t. } N(\alpha_0, k_\Omega(A)) \leq f(0), k_\Omega(A) < F(0) \text{ and } (\forall \varepsilon \in J) (\exists \alpha_\varepsilon < \alpha) [f[N(\varepsilon)\varepsilon], F[\text{ord}(\varepsilon)] \vdash^\alpha_{\rho} \Gamma, A_{\varepsilon}]\).

(Cl) \( \exists \varepsilon \in T(\mathcal{L}_{ID_1}), \exists \alpha_0 < \alpha \text{ s.t. } P^\varepsilon_{\varepsilon} \varepsilon \in \Gamma, \Omega < \alpha \text{ and } f, F \vdash^\alpha_{\rho} \Gamma, A(P^\varepsilon_{\varepsilon} \varepsilon, t) \).

(Cut) \( \exists C \text{ an } \mathcal{L}^\ast \text{-sentence of } \land \text{-type, } \exists \alpha_0 < \alpha \text{ s.t. } \max\{h(C), N(\alpha_0), N(\alpha_0, k_\Omega(C))\} \leq f(0), k_\Omega(C) < F(0), rk(C) < \rho, f, F \vdash^\alpha_{\rho} \Gamma, C, \text{ and } f, F \vdash^\alpha_{\rho} \Gamma, \neg C \).
We will call the pair \((f,F)\) operators controlling the derivation that forms \(f,F \vdash^\alpha \Gamma\).

In the sequel we always assume that the operator \(F\) enjoys the following condition \((\text{HYP}(F))\):

\[
\eta < f(\xi) \Rightarrow F(\eta) \leq F(\xi) \quad \text{for any ordinals } \xi, \eta < \Omega. \tag{\text{HYP}(F)}
\]

We note that the hypothesis \((\text{HYP}(F))\) reflects the fact stated in Proposition 5. It is not difficult to see that if the condition \((\text{HYP}(F))\) holds, then the condition \((\text{HYP}(F[\kappa]))\) also holds for any finite set \(\kappa < \Omega\).

**Lemma 24 (Inversion).** Assume that \(A \simeq \bigwedge_{\iota \in J} A_\iota\). If \(f,F \vdash^\alpha \Gamma,A\), then \(f[N(\iota)],F[\text{ord}(\iota)] \vdash^\alpha \Gamma,A_\iota \) for all \(\iota \in J\).

**Proof.** By induction on \(\alpha\). Let \(\iota \in J\). Then we can check that the condition \(\text{HYP}(f[N(\iota)];F[\text{ord}(\iota)];\alpha)\) holds. In particular, by the hypothesis \(\text{HYP}(f;F;\alpha)\) we have \(N(F[\text{ord}(\iota)]) = N(\iota) + N(F(0)) \leq N(\iota) + f(0) \leq f[N(\iota)](0)\). Now the assertion is a straightforward consequence of IH. \(\Box\)

We write \(f \circ g\) to denote the result \(m \mapsto f(g(m))\) of composing \(f\) and \(g\).

**Lemma 25 (Cut-reduction).** Assume that \(C \simeq \bigvee_{\iota \in J} C_\iota\), \(\text{rk}(C) = \rho \neq \Omega\), \(\max\{\lambda(C),N(\max \kappa_0^\iota(C)),N(\max \kappa_1^\iota(C))\} \leq f(g(0))\), and that \(\kappa_0(C) < F(0)\). If \(f,F \vdash^\alpha \Gamma,C\) and \(g,F \vdash^\beta \Gamma,C\), then \(f \circ g,F \vdash^{\alpha+\beta} \Gamma\).

**Proof.** By induction on \(\beta\).

**Case.** \(C\) is not the principal formula of the last rule \((\mathcal{J})\) that forms \(g,F \vdash^\beta \Gamma,C\): We only consider the case that \((\mathcal{J})\) is \((\bigwedge)\). The other cases can be treated similarly. Let us suppose that the sequent \(\Gamma\) contains a formula \(\bigwedge_{\iota \in J} A_\iota\) and the inference rule \((\mathcal{J})\) has the premises \(g[N(\iota)],F[\text{ord}(\iota)] \vdash^{\beta_0} \Gamma,A_\iota,C\) \((\iota \in J)\) for some \(\beta_0 < \beta\). Then, since \(f \circ g[N(\iota)] = (f \circ g)[N(\iota)](0)\) and \(F(0) \leq F[\text{ord}(\iota)](0)\) for all \(\iota \in J\), IH yields the sequent \((f \circ g)[N(\iota)],F[\text{ord}(\iota)] \vdash^{\alpha+\beta_0} \Gamma,A_\iota\) for all \(\iota \in J\). Hence another application of \((\bigwedge)\) yields the sequent \(f \circ g,F \vdash^{\alpha+\beta} \Gamma\).

**Case.** \(C\) is the principal formula of the last rule \((\mathcal{J})\): In this case \((\mathcal{J})\) should be \((\bigvee)\) since \(\text{rk}(C) \neq \Omega\). Let the premise be of the form \(g,F \vdash^\beta \Gamma,C_{\iota_0},C\) for some \(\beta_0 < \beta\) and \(\iota_0 \in J\) such that \(N(\iota_0) \leq g(0)\) and \(\text{ord}(\iota_0) < \min\{\beta,F(0)\}\). IH yields the sequent \(f \circ g,F \vdash^\beta \Gamma,C_{\iota_0}\). \(\quad (8)\)

On the other hand, Inversion lemma yields the sequent \(f[N(\iota_0)],F[\text{ord}(\iota_0)] \vdash^\rho \Gamma,\neg C_{\iota_0}\). Let us observe the following. First, \(f[N(\iota_0)](0) = f(N(\iota_0)) \leq f(g(0)) = (f \circ g)(0)\) since \(N(\iota_0) \leq g(0)\). Secondly, \(F[\text{ord}(\iota_0)](0) \leq F(0)\) by the hypothesis \(\text{HYP}(F)\) since \(\text{ord}(\iota_0) < F(0)\). Hence

\[
(f \circ g,F \vdash^\alpha \Gamma,\neg C_{\iota_0}. \tag{9}
\]
We also observe that \( N(\alpha + \beta) \leq N(\alpha) + N(\beta) \leq f(0) + g(0) \leq (f \circ g)(0) \).

Further \( K_\Omega(\alpha + \beta) < F(0) \) since \( K_\Omega \alpha \cup K_\Omega \beta < F(0) \). Now by an application of (Cut) to the two sequents \( \square \) and \( \square \) we obtain \( f \circ g, F \vdash_\rho^{\alpha + \beta} \Gamma \).

The other cases are similar. \( \square \)

For a sequent \( \Gamma \) we write \( k_\Omega^n(\Gamma) \) to denote the set \( \bigcup_{B \in F} k_\Omega^n(B) \).

**Lemma 26.** Let \( k < \omega \). If \( f, F \vdash_\omega^{\alpha_0+k+2} \Gamma, C \) and \( f, F \vdash_\omega^{\alpha_0+k+2} \Gamma, \neg C \) with a cut formula \( C \) for some \( \alpha_0 < \alpha \) such that \( \rho_k(C) < \Omega + k + 2, \max \{|h(C), N(\max k_\Omega^n(C)), N(\max k_\Omega^n(C))\} \leq f(0) \) and \( k_\Omega(C) < F(0) \). Let \( K_0 \) denote the set \( k_\Omega^n(\Gamma, \neg C) \). Then IH yields the two sequents

\[
\begin{align*}
f^{F_\alpha[K_0][0]+1}, F \vdash_\omega^{\alpha_0+k+1} \Gamma, C, & \quad f^{F_\alpha[K_0][0]+1}, F \vdash_\omega^{\alpha_0+k+1} \Gamma, \neg C.
\end{align*}
\]

Hence Cut-reduction lemma yields the sequent

\[
f^{F_\alpha[K_0][0]+1} \circ f^{F_\alpha[K_0][0]+1}, F \vdash_\omega^{\alpha_0+k+1} \Gamma.
\]

Clearly \( \Omega^\alpha_0 + \Omega^\alpha_0 < \Omega^\alpha \). Further \( N(\Omega^\alpha) = N(\alpha) + 1 \leq f^{F_\alpha[K][0]+1}(0) \) since \( N(\alpha) \leq f(0) = f^0(0) < f^{F_\alpha[0]+1}(0) \). It remains to show that

\[
(f^{F_\alpha[K_0][0]+1} \circ f^{F_\alpha[K_0][0]+1}(0)) \leq f^{F_\alpha[K][0]+1}(0).
\]

Let us see that \( K_0 \subseteq K \cup k_\Omega(C) < F^{\alpha}[K][0] \) since \( k_\Omega(C) < F(0) \). This implies \( F^{\alpha_0}[K_0][0] < F^{\alpha}[K][0] \), and hence \( F^{\alpha_0}[K_0][0] + 1 < F^{\alpha}[K][0] \). We can also see that

\[
N(\max K_0) \leq \max\{N(\max K), N(\max k_\Omega^n(C))\} \leq \max\{N(\max K), f(0)\}.
\]

From this and the inequality \( N(\alpha_0) \leq f(0) \) one can see that

\[
N(F^{\alpha_0}[K_0][0]+1) \leq N(F[K][0]) + N(\alpha_0) + 1
\]

\[
\leq N(F[K][0]) + f(0) + f(0) + 1
\]

\[
\leq f(N(F^{\alpha}[K][0]) + f^{F_\alpha[K][0]+1}(0)).
\]

This allows us to conclude as follows.

\[
(f^{F_\alpha[K_0][0]+1} \circ f^{F_\alpha[K_0][0]+1}(0))
\]

\[
\leq (f^{F_\alpha[K_0][0]+1} \circ f^{F_\alpha[K_0][0]+1}(0))(f^{F_\alpha[K][0]+1}(0))
\]

\[
\leq f^{F_\alpha[K][0]+1}(0).
\]

**Case.** The last rule is (\( \land \)): In this case there exists a formula \( A \simeq \bigwedge_{i \in J} A_i \in \Gamma \) such that \( N(\max k_\Omega^n(A)) \leq f(0), k_\Omega^n(A) < F(0) \) and \( \forall \alpha_i \in J, \exists \alpha_i < \alpha \) s.t. \( f[N(\alpha_i), F[\text{ord}(\alpha_i)] \vdash_\omega^{\alpha_i} \Omega + k + 2, \Gamma, A_i \). By IH \( (f[N(\alpha_i)]F[\text{ord}(\alpha_i)][0]+1, F[\text{ord}(\alpha_i)] \vdash_\omega^{\alpha_i} \Omega + k + 1, \Gamma, A_i) \) for all \( i \in J \).
Claim. \( (f[N(\iota)]) F[\text{ord}(\iota)]^{\alpha}(0) + 1(\iota) \leq f^{F^{\alpha}(0) + 1}[N(\iota)](0) \) for all \( \iota \in J \).

Assuming the claim, \( f^{F^{\alpha}(0) + 1}[N(\iota)], F[\text{ord}(\iota)] \vdash_{\Delta+\kappa+1}^{\omega} \Gamma, A_\iota \) for all \( \iota \in J \) and hence an application of \( (\wedge) \) yields \( f^{F^{\alpha}(0) + 1}, F \vdash_{\Delta+\kappa+1}^{\omega} \Gamma \). To show the claim fix \( \iota \in J \) arbitrarily and let \( n := N(\iota) \). Then Corollary 22 yields
\[
f[n] F[\text{ord}(\iota)]^{\alpha}(0) + 1(0) \leq f^{F[\text{ord}(\iota)]^{\alpha}(0) + 3}(n).
\]
By Lemma 15.2 \( \text{ord}(\iota) \leq k^H_0(A) \) since \( \text{ord}(\iota) \leq \Omega \). Hence \( \text{ord}(\iota) < F(0) \) since \( k^H_0(A) < F(0) \). This together with the hypothesis (HYP(\( F \))) yields \( K_\omega \alpha, \kappa < F[\text{ord}(\iota)] \leq F(0) \leq F^{\alpha}(0) \). Further \( F[\text{ord}(\iota)]^{\alpha}(0) \leq F^{\alpha}(\text{ord}(\iota)) \) by Lemma 10 Hence \( F[\text{ord}(\iota)]^{\alpha}(0) = F^{\alpha}(\text{ord}(\iota)) < F^{\alpha}(0) \) since \( \text{ord}(\iota) < F(0) \leq F^{\alpha}(0) \). And hence
\[
F[\text{ord}(\iota)]^{\alpha}(0) + 3 < F^{\alpha}(0).
\]
As in Example 18 we can see that \( 2n + 3 \leq f^{\alpha}(n) \leq f^{F^{\alpha}(0)}(n) \). Hence
\[
\begin{align*}
N(F[\text{ord}(\iota)]^{\alpha}(0) + 3) &= N(F(0)) + N(\text{ord}(\iota)) + N(\alpha) + 3 \\
&\leq N(F^{\alpha}(0)) + n + f(n) + 3 \quad \text{since } N(\alpha) \leq f[N(\iota)](0) = f(n), \\
&\leq f(N(F^{\alpha}(0)) + 2n + 3) \quad \text{from the condition } (f[1], \\
&\quad \leq f(N(F^{\alpha}(0)) + f^{F^{\alpha}(0)}(n)).
\end{align*}
\]
The two conditions \( (11) \) and \( (12) \) allows us to deduce that
\[
\begin{align*}
f^{F[\text{ord}(\iota)]^{\alpha}(0) + 3}(n) &\leq f^{F[\text{ord}(\iota)]^{\alpha}(0) + 3}(f^{F^{\alpha}(0)}(n)) \\
&\leq f^{F^{\alpha}(0)}(f^{F^{\alpha}(0)}(n)) \\
&\leq f^{F^{\alpha+1}}(n) = f^{F^{\alpha+1}}[n](0).
\end{align*}
\]
Combining the two inequality \( (10) \) and \( (13) \) enables us to conclude the claim, and hence completes this case.

\[ \square \]

Lemma 27 (Predicative Cut-elimination). Assume \( \{\alpha, \beta, \gamma\} < \Omega, N(\alpha) \leq f^{\gamma}(0) \) and \( K_{\Omega} \alpha < F(0) \). If \( f^{\gamma}, F \vdash_{\rho + \omega}^{\beta} \Gamma \), then \( f^{F^{\alpha} + \alpha + \beta}(0) + 1, F \vdash_{\rho + \omega}^{\varphi \alpha \beta} \Gamma \).

Proof. By main induction on \( \alpha \) and side induction on \( \beta \). Let us start with observing the following. First \( N(\varphi \alpha \beta) = N(\alpha) + N(\beta) + 1 \leq f^{\gamma}(0) + f^{\gamma}(0) + 1 \leq f^{\gamma}(f^{\gamma}(0)) + 1 \leq f^{F^{\alpha} + \alpha + \beta}(0) + 1 \). Secondly \( K_{\Omega} \varphi \alpha \beta = \{\varphi \alpha \beta\} < F(0) \) since \( K_{\Omega} \varphi < F(0) \).

Case. The last rule is \( (\wedge) \): In this case there exists a formula \( A \simeq \bigwedge_{\iota \in J} A_{\iota} \in \Gamma \) and for all \( \iota \in J \) there exists \( \beta, \gamma \) such that \( f^{\gamma}[N(\iota)], F[\text{ord}(\iota)] \vdash_{\rho + \omega}^{\beta + \omega} \Gamma, A_{\iota} \). We observe that \( N(\alpha) \leq f^{\gamma}(0) \leq f[N(\iota)]^{\gamma}(0) \) and \( K_{\Omega} \alpha < F(0) \leq F[\text{ord}(\iota)](0) \). Hence Side Induction Hypothesis yields that for all \( \iota \in J \)
\[
f[N(\iota)]^{F^{\alpha} + \alpha + \beta}(0) + 1, F[\text{ord}(\iota)] \vdash_{\rho}^{\varphi \alpha \beta} \Gamma, A_{\iota}.
\]
Let $m := N(\iota)$. Then $f[m]_{\Pi^\alpha + \gamma + \beta}(0)+1(0) \leq f_{\Pi^\alpha + \gamma + \beta}(0)+3[m](0)$ from Corollary 22. Also it holds that $F_{\Pi^\alpha + \gamma + \beta}(0) < F_{\Pi^\alpha + \gamma + \beta}(0)$ for all $\iota \in J$ since $K_{\Omega \beta_{\iota}} < F[\text{ord}(\iota)](0) \leq F(0)$. Further

$$N(F_{\Pi^\alpha + \gamma + \beta}(0)+3) = N(F(0)) + N(\alpha) + N(\gamma) + N(\beta) + 4$$

\[ \leq 2 \cdot f^\gamma(0) + f^{F(0)}(0) + f^m(0) + 4 \] by Lemma 19

\[ \leq f^{F(0)}(f^\gamma(f^\gamma(m))) + 4 \]

\[ \leq f^{F(0)}(f^{\gamma+2}(m)) + 4 \]

\[ \leq f^{F(0)}(f^{\gamma+2}(m)) + f^{\gamma+2}(0) \]

\[ \leq f^{F(0)}(f^{\gamma+2}(m) + f^{\gamma+2}(0)) \]

\[ \leq f^{F(0)}(f^{\gamma+3}(m)) \]

\[ \leq f^{F(0)}(f^{\gamma+2}(m)) \]

\[ \leq f^{F(0)}(f^{\gamma+3}(m)) \]

\[ \leq f^{F(0)}(f^{\gamma+2}(m)) \]

\[ \leq f^{F(0)}(f^{\gamma+3}(m)) \]

(15)

The last inequality holds since $N(F^\gamma(0)+2) = N(F(0)) + N(\gamma) + 2$ is bounded by $f[N(F_{\Pi^\alpha + \gamma + \beta}(0))](m)$. Hence

$$f^{\Pi^\alpha + \gamma + \beta}(0)+3(m) \leq f^{\Pi^\alpha + \gamma + \beta}(0)+3(f^{\Pi^\alpha + \gamma + \beta}(0)(m))$$

\[ \leq f^{\Pi^\alpha + \gamma + \beta}(0)(f^{\Pi^\alpha + \gamma + \beta}(0)(m)) \] by 14,

\[ \leq f^{\Pi^\alpha + \gamma + \beta}(0)(m). \]

This together with 14 allows us to derive the sequent

$$f^{\Pi^\alpha + \gamma + \beta}(0)+1[N(\iota), F[\text{ord}(\iota)] \vdash \phi_{\alpha\beta}, \Gamma, A].$$

An application of ($\land$) yields $f^{\Pi^\alpha + \gamma + \beta}(0)+1, F \vdash \phi_{\alpha\beta}, \Gamma, A$.

**Case.** The last rule is (Cut): In this case there exist a formula $C$ and an ordinal $\beta_0 < \beta$ such that $\text{rk}(C) < \rho + \omega^\alpha$, max\{lh(C), $N(\max k^0_J(C)), N(\max k^0_J(C))\} \leq f^\gamma(0), k_{\Omega}(C) < F(0)$,

$$f^\gamma, F \vdash \beta_0^\rho + \omega^\alpha, C \text{ and } f^\gamma, F \vdash \beta_0^\rho + \omega^\alpha, \Gamma, \neg C.$$ SIH yields $f^{\Pi^\alpha + \gamma + \beta_0}(0)+1, C \vdash \phi_{\alpha\beta_0} \Gamma, C$ and $f^{\Pi^\alpha + \gamma + \beta_0}(0)+1, F \vdash \phi_{\alpha\beta_0} \Gamma, \neg C$. If $\text{rk}(C) < \rho$, then we can apply (Cut), having the conclusion. Suppose that $\rho \leq \text{rk}(C) < \rho + \omega^\alpha$. Then there exist $l < \omega$ and $\alpha_1, \ldots, \alpha_l$ such that $\alpha_i \leq \alpha$ and $\text{rk}(C) = \rho + \omega^{\alpha_1} + \cdots + \omega^{\alpha_l}$. Let $\gamma' := F^{\Pi^\alpha + \gamma + \beta_0}(0)+2$. Then it is easy to observe that $f^{\Pi^\alpha + \gamma + \beta_0}(0)+1 \leq f^\gamma(0) + 4$. Then it is easy to observe that $f^{\Pi^\alpha + \gamma + \beta_0}(0)+1 \leq f^\gamma(m)$ for all $m$. This together with Cut-reduction lemma (Lemma 25) yields

$$f^{\gamma'}, F \vdash \phi_{\alpha\beta_0}^\rho + \omega^\alpha \Gamma.$$

Let us define ordinals $\xi_n$ and $\gamma_n$ by

$$\begin{cases} \xi_0 = \phi_{\alpha\beta_0} + \omega, \\
\xi_{n+1} = \phi_{\alpha_1} \xi_n, \\
\gamma_0 = \gamma' = F^{\Pi^\alpha + \gamma + \beta_0}(0)+2, \\
\gamma_{n+1} = F^{\Pi^\alpha + \gamma + \beta_0}(0)+1. \end{cases}$$

15
Claim. \( f^{\gamma \alpha}, F \vdash \xi_{\rho + \omega^\alpha(2 - n)} G \). (0 \leq n \leq l)

We show the claim by subsidiary induction on \( n \leq l \). The base case follows immediately from \([16]\). For the inductions step suppose \( n < l \). Then by IH we have \( f^{\gamma \alpha}, F \vdash \xi_{\rho + \omega^\alpha(l-(n+1)) + \omega^\alpha} G \). It is easy to see that \( \{ \alpha_1, \xi_n, \gamma_n \} \subseteq \Omega \) and that \( \gamma \leq \gamma_m \) and \( N(\gamma) \leq N(\gamma_m) \) for all \( m \leq l \). Hence

\[
\begin{aligned}
N(\alpha_1) &\leq N(rk(C)) \leq f^\gamma(0), \\
K_{\Omega} \alpha_1 &\leq K_\alpha < F(0).
\end{aligned}
\]

Thus MIH of the lemma yields \( f^{\gamma_{n+1}}, F \vdash \xi_{\rho + \omega^\alpha(l-(n+1))} G \).\( \square \)

By the claim with \( n = l \) we have \( f^{\gamma_n}, F \vdash \xi_{\rho} G \). One can show \( \xi_n < \varphi \alpha \beta \) by a straightforward induction on \( n \). Hence \( \xi_l < \varphi \alpha \beta \). It remains to show that \( f^{\gamma_l}(0) \leq f^{\rho + \omega \alpha + \varphi \alpha \beta}(0+1)(0) \). It is not difficult to check \( \gamma_l < f^{\rho + \omega \alpha + \varphi \alpha \beta}(0+1) \). By simultaneous induction on \( n \) we show the following \([17]\) and \([18]\):

\[
\begin{aligned}
N(\xi_n) &\leq nN(\alpha_1) + 2N(\alpha) + 2N(\beta_0) + 2 + n, \\
N(\gamma_n) &\leq (n + 1)N(F(0)) + \frac{1}{2}n(n + 1)N(\alpha_1) \\
&\quad + (2n + 1)N(\alpha) + N(\gamma) + (2n + 1)N(\beta_0) + 4(n + 1).
\end{aligned}
\]

For the base case

\[
\begin{aligned}
N(\xi_0) &\leq 2(N(\alpha) + N(\beta_0) + 1) \leq 2N(\alpha) + 2N(\beta_0) + 2, \\
N(\gamma_0) &\leq N(F(0)) + N(\alpha) + N(\gamma) + N(\beta_0) + 4.
\end{aligned}
\]

Let us consider the induction step. Assuming \([17]\),

\[
N(\xi_{n+1}) = N(\alpha_1) + N(\xi_n) + 1
\]

\[
\leq (n + 1)N(\alpha_1) + 2N(\alpha) + 2N(\beta_0) + 2 + n + 1.
\]

Assuming both \([17]\) and \([18]\),

\[
\begin{aligned}
N(\gamma_{n+1}) &\leq N(F(0)) + N(\alpha_1) + N(\gamma_n) + N(\xi_n) + 4 \\
&\leq (n + 2)N(F(0)) + \frac{1}{2}n(n + 1) + n + 1)N(\alpha_1) \\
&\quad + (2n + 3)N(\alpha) + N(\gamma) + (2n + 3)N(\beta_0) + 4(n + 1) + 4 \\
&\leq (n + 2)N(F(0)) + \frac{1}{2}(n + 1)(n + 2)N(\alpha_1) \\
&\quad + (2n + 3)N(\alpha) + N(\gamma) + (2n + 3)N(\beta_0) + 4(n + 2).
\end{aligned}
\]

Let us observe that

\[
N(rk(C)) \leq N(\max k(C)) + lh(C) \text{ by Lemma } [15,18]
\]

\[
\leq f^\gamma(0) + f^\gamma(0) \text{ since } lh(C) \leq f^\gamma(0),
\]

\[
\leq f^\gamma(f^\gamma(0)).
\]

16
Hence \( l \leq \text{rk}(C) \leq f^\gamma(f^\gamma(0)) \leq f^{F^\gamma(0)}(0) \) since \( \gamma < F^\gamma(0) \) and \( N(\gamma) \leq N(F^\gamma(0)) \). Further \( \max\{F(0), N(\alpha), N(\beta_0)\} \leq f^\gamma(0) \leq f^{F^\gamma(0)}(0) \) by assumption and \( N(\gamma) \leq f^{F^\gamma(0)}(0) \) by Lemma \[19\]. From these and \[18\],

\[
N(\gamma) \leq (f^{F^\gamma(0)}(0))^3 + 6(f^{F^\gamma(0)}(0))^2 + 8 \cdot f^{F^\gamma(0)}(0) + 4. \tag{20}
\]

On the other hand, from Example \[18\] one can see that \( m^3 + 6m^2 + 8m + 4 \leq f^{F^\gamma(0)}(m) \) holds. Hence by \[20\],

\[
N(\gamma) \leq f^{F^\gamma(0)}(f^{F^\gamma(0)}(0)) \leq f^{F^\Omega^{\alpha+\gamma+\beta}(0)}(0) \leq f(f^{F^\Omega^{\alpha+\gamma+\beta}(0)}(0)). \tag{21}
\]

Hence

\[
f^{\gamma}(0) \leq f^{\gamma}(f^{F^{\Omega^{\alpha+\gamma+\beta}(0)}(0)}) \leq f^{F^{\Omega^{\alpha+\gamma+\beta}(0)}(0)}(0) \leq f^{F^{\Omega^{\alpha+\gamma+\beta}(0)}(0)+1}(0).
\]

This allows us to conclude \( f^{F^{\Omega^{\alpha+\gamma+\beta}(0)}(0)+1}, F \vdash_{\rho}^{\alpha+\beta} \Gamma \).

\[\square\]

**Definition 28.** For each \( L^\ast \)-formula \( B \) let \( B^\alpha \) be the result of replacing in \( B \) every occurrence of \( P^\alpha_{A^\Omega} \) by \( P^\alpha_{A^\xi} \).

**Lemma 29 (Boundedness).** Assume that \( f, F \vdash_{\rho}^{\alpha} \Gamma, A \). Then for all \( \xi \) if \( \alpha \leq \xi \leq F(0) \), \( N(\xi) \leq f(0) \) and \( K_{\xi} \xi < F(0) \), then \( f, F \vdash_{\rho}^{\alpha} \Gamma, A^\xi \).

**Proof.** The claim is trivial if \( F(0) < \alpha \). Assume that \( \alpha \leq F(0) \) and \( f, F \vdash_{\rho}^{\alpha} \Gamma, A \).

By induction on \( \alpha \) we show that for all \( \xi \) if \( \alpha \leq \xi \leq F(0) \), then \( f, F \vdash_{\rho}^{\alpha} \Gamma, A^\xi \).

**Case.** The last rule is \( \langle V \rangle \): If \( A \) is not the principal formula of last rule \( \langle V \rangle \), then the claim follows immediately from IH. Suppose that \( A \simeq \bigvee_{i \in J} A_i \) is the principal formula of \( \langle V \rangle \). Then there exist \( \alpha_0 < \alpha \) and \( \xi_0 \in J \) such that \( \text{ord}(\xi_0) < \alpha \) and \( f, F \vdash_{\rho}^{\alpha_0} \Gamma, A, A_{\xi_0} \). Let \( \alpha \leq \xi \leq F(0) \). Then IH yields \( f, F \vdash_{\rho}^{\alpha_0} \Gamma, A^\xi \). If \( A \not= P^\alpha_{A^\xi} \), then another application of IH and an application of \( \langle V \rangle \) yield \( f, F \vdash_{\rho}^{\alpha_0} \Gamma, A^\xi \). Consider the case that \( A \equiv P^\alpha_{A^\xi} \simeq \bigvee_{\mu < \Omega} A(P^\mu_{A^\xi}, t) \). In this subcase \( A_{\mu_0} \simeq A(P^\mu_{A^\xi}, t) \). Since \( \mu_0 = \text{ord}(\mu_0) < \alpha \leq \xi \), we can apply \( \langle V \rangle \) and then obtain \( f, F \vdash_{\rho}^{\alpha_0} \Gamma, A^\xi \).

**Case.** The last rule is \( \langle A \rangle \): In this case for all \( \xi \in J \) there exists \( \alpha_0 < \alpha \) such that \( f[N(\xi)], F[\text{ord}(\xi)] \vdash_{\rho}^{\alpha_0} \Gamma' \) for a certain \( \Gamma' \). Let us observe that \( f(0) \leq f[\text{ord}(\xi)](0) \). Hence, if \( A \) is not the principal formula of \( \langle A \rangle \), then the claim follows immediately from IH. Suppose that \( A \) is the principal formula of \( \langle A \rangle \). Then \( A \simeq \bigwedge_{\mu \in J} A_\xi \in \Gamma \) and \( \Gamma' \equiv \Gamma, A, A_\xi \). Let \( \alpha \leq \xi \leq F(0) \). Then IH yields \( f[N(\xi)], F[\text{ord}(\xi)] \vdash_{\rho}^{\alpha_0} \Gamma, A^\xi, A_{\xi} \). If \( A \not= \neg P^\alpha_{A^\xi} \), then another application of IH and an application of \( \langle A \rangle \) yield \( f, F \vdash_{\rho}^{\alpha_0} \Gamma, A^\xi \). If \( A \equiv \neg P^\xi_{A^\xi} \), then an application of \( \langle A \rangle \) with \( \mu \leq \xi \leq F(0) < \Omega \) yields \( f, F \vdash_{\rho}^{\alpha_0} \Gamma, \neg P^\xi_{A^\xi} \).

**Case.** The last rule is \( \langle C_{1\Omega} \rangle \): If \( A \) is not the principal formula, then the claim again follows from IH. Let us consider the case that \( A \) is the principal
formula of the last rule ($\text{Cl}_\Omega$) with a premise $f, F \vdash_{\rho} \alpha, \Gamma, P_{\rho, \Delta}^g, t, A(P_{\rho, \Delta}^g, t)$ for some $\alpha_0 < \alpha$ where $A \equiv P_{\rho, \Delta}^g t$. Let $\alpha \leq \xi \leq F(0)$. An application of IH yields $f, F \vdash_{\rho} \alpha, \Gamma, P_{\rho, \Delta}^{<\xi}, A(P_{\rho, \Delta}^{<\xi}, t)$. Another application of IH yields $f, F \vdash_{\rho} \alpha, \Gamma, P_{\rho, \Delta}^{<\alpha_0}, A(P_{\rho, \Delta}^{<\alpha_0}, t)$. Let us observe that $\text{ord}(\alpha_0) = \alpha_0 < \alpha$, $N(\alpha_0) \leq f(0)$, and $\text{ord}(\alpha_0) = \alpha_0 < \alpha \leq F(0)$.

Hence we can apply (\text{Cut}) with $\alpha_0 < \alpha \leq \xi$, concluding $f, F \vdash_{\rho} \alpha, \Gamma, P_{\rho, \Delta}^{<\xi}$.

\hfill \Box

We will write $f, F \vdash_{\Omega} \alpha, \Gamma$ instead of $f, F \vdash_{\rho} \alpha, \Gamma$.

**Lemma 30 (Impredicative Cut-elimination).**

If $f, F \vdash_{\Omega + 1} \alpha, \Gamma$, then $f, F^{\alpha + 1} \vdash_{\Omega} \alpha, \Gamma$.

**Proof.** By induction on $\alpha$. It is easy to check that $f(0) \leq f^{\alpha + 1}(0)$ and $F(0) \leq F^{\alpha}(0)$. It also holds that $K \Omega F^{\alpha}(0) = \{F^{\alpha}(0)\} < F^{\alpha + 1}(0)$. Further,

$$N(F^{\alpha + 1}(0)) = N(F(0)) + N(\alpha) + 1 \leq f(0) + f(0) + 1 \leq f(f(0)) + 1 \leq f^{\alpha + 1}(0).$$

And hence $N(F^{\alpha}(0)) < N(F^{\alpha + 1}(0)) < f^{\alpha + 1}(0)$ in particular. Let $(J)$ denote the last rule that forms $f, F \vdash_{\Omega + 1} \alpha, \Gamma$.

**Case.** $(J)$ is (Cut) with a cut formula $C$: In this case $(J)$ has two premises $f, F \vdash_{\Omega + 1} \alpha, \Gamma, C$ and $f, F \vdash_{\Omega + 1} \alpha, \Gamma, \neg C$ for some $\alpha_0 < \alpha$. IH yields that

$$f^{\alpha_0}(0), F^{\alpha_0 + 1} \vdash_{\Omega} \alpha_0, \Gamma, C,$$

$$f^{\alpha_0}(0), F^{\alpha_0 + 1} \vdash_{\Omega} \alpha_0, \Gamma, \neg C. \tag{22}$$

Let us observe that $F^{\alpha_0}(0) < F^{\alpha}(0)$ since $K \Omega \alpha_0 < F(0) \leq F^{\alpha}(0)$. Similarly $F^{\alpha_0 + 1}(0) \leq F^{\alpha + 1}(0)$ holds. Further

$$N(F^{\alpha_0}(0) + 1) = N(F(0)) + N(\alpha_0) + 1 \leq f(N(F^{\alpha_0}(0) + 1)) = f(N(F^{\alpha_0}(0) + 1)(0)).$$

Hence $f^{\alpha_0 + 1}(0) < f^{\alpha + 1}(0)$.

**Subcase.** $\text{rk}(C) < \Omega$. By Lemma [15] $\text{rk}(C) = \text{rk}(\neg C) \leq \omega \cdot (\max k^{1}_{\Omega}(\neg C)) + \text{lh}(\neg C) < F(0)$ since $k^{1}_{\Omega}(\neg C) \leq k_{\Omega}(C) < F(0)$. Hence $\text{rk}(C) < F(0) \leq F^{\alpha}(0)$. This together with the two sequents (22) and (23) allows us to deduce other two sequents $f^{\alpha_0}(0), F^{\alpha + 1} \vdash_{\Omega} \alpha_0, \Gamma, C$ and $f^{\alpha_0}(0), F^{\alpha + 1} \vdash_{\Omega} \alpha_0, \Gamma, \neg C$. We can apply (Cut) to these two sequents, concluding $f^{\alpha_0}(0), F^{\alpha + 1} \vdash_{\Omega} \alpha_0, \Gamma$.

**Subcase.** $\text{rk}(C) = \Omega$. In this case $C \equiv P_{\Delta}^{<\Omega} t$ by Lemma [14]. Let us observe the following.

1. $N(F^{\alpha_0}(0)) = N(F(0)) + N(\alpha_0) \leq f(0) + f(0) \leq f(f(0)) \leq f^{\alpha_0}(0)$.
2. $K \Omega F^{\alpha}(0) = \{F^{\alpha_0}(0)\} < F^{\alpha + 1}(0)$.  

18
Applying Boundedness lemma (Lemma 24) to the sequent yields the sequent 
\[ f^{\alpha_0 (0)}, F^{\alpha_0 (0)} \vdash \Gamma, F_A^{< \alpha_0 (0)}. \]
As in the previous subcase this induces the sequent
\[ f^{\alpha (0) + 1}, F^{\alpha + 1} \vdash \Gamma, F_A^{< \alpha (0)}. \]  
(24)

On the other hand applying Inversion lemma (Lemma 24) to the sequent yields the sequent
\[ f^{\alpha (0) + 1}[N(F^{\alpha_0 (0)})], F^{\alpha_0 (0)} \vdash \Gamma, \neg P_A^{< \alpha (0)}. \]

By Property 1 we can see that
\[ f^{\alpha (0) + 1}[N(F^{\alpha_0 (0)})](0) \leq f^{\alpha (0) + 1}(f^{\alpha_0 (0)}(0)) \]
\[ \leq f^{\alpha_0 (0) + 1}(0) \text{ and } f^{\alpha_0 + 1}(F^{\alpha_0 })(0) \leq F^{\alpha_0 + 1}(0). \]
These observations induce the sequent
\[ f^{\alpha (0) + 1}, F^{\alpha + 1} \vdash \Gamma, \neg P_A^{< \alpha (0)}. \]  
(25)

Case. \( (J) \) is \( (\land) \) with a principal formula \( A \models \land_{i \in J} A_i \in \Gamma \); In this case \( \forall i \in J, \exists \alpha_i < \alpha \text{ s.t. } f[N(i)], F[\text{ord}(i)] \vdash_{\alpha_i + 1} \Gamma, A_i \). By IH yields the sequent
\[ f[N(i)]F[\text{ord}(i)]^{\alpha_i (0) + 1}, F[\text{ord}(i)]^{\alpha_i + 1} \vdash F[\text{ord}(i)]^{\alpha_i (0)} \Gamma, A_i \]
for all \( i \in J \). In the same way as we showed the claim in the proof of Lemma 26 (p. 14), one can show that for all \( i \in J \)
\[ f[N(i)]F[\text{ord}(i)]^{\alpha_i (0) + 1}(0) \leq f^{\alpha_i (0) + 1}[N(i)(0)], \]
\[ F[\text{ord}(i)]^{\alpha_i + 1}(0) \leq F^{\alpha_i + 1}[\text{ord}(i)](0). \]

These enable us to deduce the sequent
\[ f^{\alpha (0) + 1}[N(i)], F^{\alpha + 1}[\text{ord}(i)] \vdash F[\text{ord}(i)]^{\alpha (0)} \Gamma, A_i \]
for all \( i \in J \). Since \( F[\text{ord}(i)]^{\alpha_i (0)}, \neg F^{\alpha_0 (0)} \) for all \( i \in J \), we can apply \( (\land) \) to this sequent, concluding \( f^{\alpha (0) + 1}, F^{\alpha + 1} \vdash F^{\alpha_0 (0)} \Gamma \).

**Lemma 31 (Witnessing).** For each \( j < l \) let \( B_j(x) \) be a \( \Delta_0^0 \text{-} L_{PA} \)-formula such that \( \forall \text{V} (B_j(x)) = \{ x \} \). Let \( \Gamma \models \exists x_0 B_0(x_0), \ldots, \exists x_{l-1} B_{l-1}(x_{l-1}) \). If \( f, F \vdash_0 \Gamma \) for some \( \alpha \in \mathcal{OT}(\mathcal{F}) \), then there exists a sequence \( \langle m_0, \ldots, m_{l-1} \rangle \) of naturals such that \( \max \{ m_j \mid j < l \} \leq f(0) \) and \( B_0(m_0) \lor \cdots \lor B_{l-1}(m_{l-1}) \) is true in the standard model \( \mathbb{N} \) of PA.

**Proof.** By induction on \( \alpha \). The derivation forming \( f, F \vdash_0 \Gamma \) contains no \( (\text{Cut}) \) rules. Hence the last inference rule should be \( (\forall) \). Thus there exist an ordinal \( \alpha_0 < \alpha \) and a (closed) term \( t \in \mathcal{T}(\mathcal{L}_{\text{ID}_\alpha}) \) such that \( N(t) \leq f(0) \) and \( f, F \vdash_{\alpha_0} \Gamma, B_{l-1}(t) \). By IH there exists a sequence \( \langle m_0, \ldots, m_{l-1} \rangle \) of naturals such that \( \max \{ m_j \mid j < l \} \leq f(0) \) and \( B_0(m_0) \lor \cdots \lor B_{l-1}(m_{l-1}) \lor B_{l-1}(t) \) is true in
\( B_0(m_0) \lor \cdots \lor B_{l-1}(m_{l-1}) \) is already true in \( \mathbb{N} \), then \( \langle m_0, \ldots, m_{l-1} \rangle \) is the desired sequence. Suppose that \( B_0(m_0) \lor \cdots \lor B_{l-1}(m_{l-1}) \) is not true in \( \mathbb{N} \). Then \( B_{l-1}(t) \) must be true. Hence \( B_{l-1}(\text{val}(t)) \) is also true. By definition, \( \text{val}(t) = N(t) \leq f(0) \), and hence \( \langle m_0, \ldots, m_{l-2}, \text{val}(t) \rangle \) is the desired sequence.

\[\square\]

5 Embedding ID\(_1^1\) into ID\(_1^\infty\)

In this section we embed the theory ID\(_1^1\) into the infinitary system ID\(_1^\infty\). Following conventions in the previous section we use the symbol \( f \) to denote a strict increasing function \( f : \mathbb{N} \to \mathbb{N} \) that enjoys the conditions (f\(\text{I}^1\)) and (f\(\text{I}^2\)) (p. 8).

15 We can see that the condition (HYP(\(E\)) holds since \( E(\xi) = \varepsilon_0 \leq E(0) \) for all \( \xi < E(0) = \varepsilon_0 \).

Lemma 32 (Tautology lemma). Let \( s, t \in \mathcal{T}(L_{\text{ID}_1^1}) \), \( \Gamma \) be a sequent of \( \mathcal{L}^*\)-sentences, and \( A(x) \) be an \( \mathcal{L}^\alpha\)-formula such that \( \text{FV}(A) = \{x\} \). If \( \text{val}(s) = \text{val}(t) \), then

\[ f[n], E[k_{\Omega}(A)] \vdash_{0}^{rk(A) \cdot 2} \Gamma, \neg A(s), A(t), \tag{26} \]

where \( n := \max\{N(rk(A)), N(\max k_{\Omega}^{\alpha}(A)), N(\max k_{\Omega}^{\gamma}(A))\} \).

**Proof.** By induction on \( rk(A) \). Let \( n \) denote the maximal among \( N(rk(A)), N(\max k_{\Omega}^{\alpha}(A)) \) and \( N(\max k_{\Omega}^{\gamma}(A)) \). From Lemma 15 one can check that the condition HYP(\( f[n]; E(k_{\Omega}(A)); rk(A) \cdot 2 \)) holds. If \( rk(A) = 0 \), then \( A \) is an \( \mathcal{L}_{\text{ID}_1^1} \)-literal, and hence (26) is an instance of (Ax1). Suppose that \( rk(A) > 0 \). Without loss of generality we can assume that \( A \simeq \bigvee_{i \in I} A_i \). Let \( i \in J \). By Lemma 14 let us observe that \( N(rk(A_i) \cdot 2) < 2\{N(rk(A)), N(i)\} \leq f[N(rk(A))][N(i)](0) \leq f[n](0) \) since \( 2m + 1 \leq f(m) \) for all \( m \) by the condition (f\(\text{I}^1\)). Further by Lemma 13 \( k_{\Omega}(rk(A_i) \cdot 2) \leq k_{\Omega}(A_i) \cup \{ord(i)\} \leq E[k_{\Omega}(A)][ord(i)]. \) Summing up, we have the condition

\[ \text{HYP}(f[n][N(i)]; E[k_{\Omega}(A)][ord(i)]; rk(A_i) \cdot 2). \]

Hence by IH we can obtain the sequent

\[ f[n][N(i)], E[k_{\Omega}(A)][ord(i)] \vdash_{0}^{rk(A_i) \cdot 2} \Gamma, \neg A_i(s), A_i(t). \tag{27} \]

It is not difficult to see \( ord(i) \leq rk(A_i) \leq rk(A \cdot 2 + 1) \) and \( N(rk(A) \cdot 2 + 1) = N(rk(A) \cdot 2 + 1) \leq f[N(rk(A))][N(i)](0) \leq f[n](0) \). This allows us to apply (\( \lor \)) to the sequent (27) yielding

\[ f[n][N(i)], E[k_{\Omega}(A)][ord(i)] \vdash_{0}^{rk(A) \cdot 2 + 1} \Gamma, \neg A_i(s), A(t). \]

We can see that \( rk(A_i) \cdot 2 + 1 < rk(A \cdot 2), N(\max k_{\Omega}^{\alpha}(A)) \leq f[n](0) \) and \( k_{\Omega}^{\alpha}(A) < E[k_{\Omega}(A)] \). Hence we can apply (\( \land \)) concluding (20). \( \square \)
Lemma 33. Let $B_j$ be an $\mathcal{L}_{ID_1}$-sentence for each $j = 0, \ldots, l$. Suppose that $\neg B_0 \lor \cdots \lor (\neg B_{l-1}) \lor B_l$ is a logical consequence in the first order predicate logic with equality. Then there exists a natural $k < \omega$ such that $f[m+k], E \vdash \bigwedge_{0}^{l+2+k} \{B_j \mid 0 \leq j \leq l\}$, where $m = \max\{N(rk(B_j)) \mid j = 0, 1, \ldots, l\}$.

Proof. Let $B_j$ be an $\mathcal{L}_{ID_1}$-sentence for each $j = 0, \ldots, l-1$ and suppose that $B_0 \lor \cdots \lor B_{l-1}$ is a logical consequence in the first order predicate logic with equality. Then we can find a cut-free proof of the sequent $\{B_j \mid 0 \leq j \leq l-1\}$ in an $\text{LK}$-style sequent calculus. More precisely we can find a cut-free proof $P$ of $\{B_j \mid 0 \leq j \leq l-1\}$ in the sequent calculus $G3_m$. (See the book [17] of Troelstra and Schwichtenberg for the definition.) Let $h$ denote the tree height of the cut-free proof $P$. Then by induction on $h$ one can find a witnessing natural $k$ such that $f[m+k], F \vdash \{B_j \mid 0 \leq j \leq l-1\}$ for all $\alpha \geq \Omega + k$. In case $h = 0$ Tautology lemma (Lemma 32) can be applied since for any $\mathcal{L}_{ID_1}$-sentence $A$, $\text{rk}(A) \in \omega \cup \{\omega \cup k \mid k < \omega\}$ and $k(h(A)) \cup k(\Sigma(A)) = k(A) \subseteq \{0, \Omega\}$, and hence $k(h(A)) = \{0\}$ and $\max\{N(\max k(h(A))), N(\max k(\Sigma(A)))\} = 0$. □

Lemma 34. Let $m \in \mathbb{N}$ and $A(x)$ be an $\mathcal{L}_{ID_1}$-formula such that $\text{FV}(A(x)) = \{x\}$. Then for any $t \in T(\mathcal{L}_{ID_1})$ and for any sequent $\Gamma$ of $\mathcal{L}_{ID_1}$-sentences, if $\text{val}(t) = m$, then

$$f[N(\text{rk}(A)) + m], E \vdash \bigwedge_{0}^{(\text{rk}(A)+m)-2} \Gamma, \neg A(0), \forall x(A(x) \rightarrow A(S(x))), A(t). \quad (28)$$

Proof. By induction on $m$. The base case $\text{val}(t) = m = 0$ follows from Tautology lemma (Lemma 32). For the induction step suppose $\text{val}(t) = m + 1$. Fix a sequent $\Gamma$ of $\mathcal{L}_{ID_1}$-sentences. Then (28) holds by IH. On the other hand again by Tautology lemma,

$$f[N(\text{rk}(A))], E \vdash \bigwedge_{0}^{(\text{rk}(A)+m)-2} \Gamma, \neg A(0), \exists x(A(x) \land \neg A(S(x))), A(t), \text{val}(m) \land \neg A(m). \quad (29)$$

An application of (1) to the two sequents (28) and (29) yields

$$f[N(\alpha_m)], E \vdash \bigwedge_{0}^{\alpha_m-2+1} \Gamma, \neg A(0), \exists x(A(x) \land \neg A(S(x))), A(t), \text{val}(m) \land \neg A(m),$$

where $\alpha_m := \text{rk}(A) + m$. The final application of (1) yields

$$f[N(\text{rk}(A)) + m + 1], F \vdash \bigwedge_{0}^{(\text{rk}(A)+m+1)-2} \Gamma, \neg A(0), \exists x(A(x) \land \neg A(S(x))), A(t).$$

□

Lemma 35. Let $\xi \leq \Omega$, $F(x)$ be an $\mathcal{L}_{ID_1}$-formula such that $\text{FV}(F(x)) = \{x\}$ and $B(X)$ be an $X$-positive $\mathcal{L}_{PA}(X)$-formula such that $\text{FV}(B) = \emptyset$. Then

$$f[N(\sigma + \alpha + 1)], E[K_\Omega \xi] \vdash (\sigma + \alpha + 1)-2 \Gamma, \neg \forall x(A(F, x) \rightarrow F(x)), \neg B(Q^\xi, B(F),$$

where $\sigma := \text{rk}(F)$ and $\alpha := \text{rk}(B(P^\xi_A))$. 21
Proof. By main induction on $\xi$ and side induction on $\text{rk}(B(P_A^{<\xi}))$. Let $\text{Cl}_A(F) \equiv \neg\forall x(\mathcal{A}(F,x) \rightarrow F(x)) \equiv \exists x(\mathcal{A}(F,x) \land \neg F(x))$. The argument splits into several cases depending on the shape of the formula $B(X)$.

CASE. $B(X)$ is an $\mathcal{L}_{PA}$-literal: In this case $B$ does not contain the set free variable $X$, and hence Tautology lemma (Lemma 32) can be applied.

CASE. $B \equiv X(t)$ for some $t \in \mathcal{T}(\mathcal{L}_{ID_1})$: In this case $\neg B(P_A^{<\xi}) \equiv \neg P_A^{<\xi}t \equiv \bigwedge_{\eta<\xi} \neg\mathcal{A}(P_A^{<\eta},t)$. Let $\eta < \xi$. Then by MIH

$$f[N(\sigma + \alpha_0 + 1)]_0, E[K_B\eta] \vdash (\sigma + \alpha_0 + 1)^{-2} \Gamma, \text{Cl}_A(F), \neg A(P_A^{<\eta},t), A(F,t), F(t)$$

where $\alpha_0 := \text{rk}(A(P_A^{<\eta},t))$. We note that $\eta < \xi \leq \Omega$ and hence $K_B\eta = \{\eta\} = \{\text{ord}(\eta)\}$. Hence this yields the sequent

$$f[N(\sigma + \alpha)]_0, E[\text{ord}(\eta)] \vdash (\sigma + \alpha_0 + 1)^{-2} \Gamma, \text{Cl}_A(F), \neg A(P_A^{<\eta},t), A(F,t), F(t).$$

An application of $(\land)$ yields the sequent

$$f[N(\sigma + \alpha)]_0, E[K_B\xi] \vdash (\sigma + \alpha)^{-2} \Gamma, \text{Cl}_A(F), \neg P_A^{<\xi}t, A(F,t), F(t). \quad (30)$$

On the other hand by Tautology lemma (Lemma 32),

$$f[N(\sigma + \alpha)]_0, E[K_B\xi] \vdash \text{rk}(F)^{-2} \Gamma, \text{Cl}_A(F), \neg P_A^{<\xi}t, \neg F(t), F(t). \quad (31)$$

Another application of $(\land)$ to the two sequents $(30)$ and $(31)$ yields the sequent

$$f[N(\sigma + \alpha + 1)]_0, E[K_B\xi] \vdash (\sigma + \alpha + 1)^{-2} \Gamma, \text{Cl}_A(F), \neg P_A^{<\xi}t, A(F,t) \land \neg F(t), F(t).$$

An application of $(\lor)$ allows us to conclude

$$f[N(\sigma + \alpha + 1)]_0, E[K_B\xi] \vdash (\sigma + \alpha + 1)^{-2} \Gamma, \text{Cl}_A(F), \neg P_A^{<\xi}t, F(t).$$

CASE. $B(X) \equiv \forall y B_0(X,y)$ for some $\mathcal{L}_{PA}$-formula $B_0(X,y)$: Let $\alpha_0$ denote the ordinal $\text{rk}(B_0(P_A^{<\xi},y))$. Then $\alpha = \alpha_0 + 1$. By the definition of the rank function $\text{rk}$, $\alpha_0 = \text{rk}(B_0(P_A^{<\xi},t))$ for all $t \in \mathcal{T}(\mathcal{L}_{ID_1})$. Fix a closed term $t \in \mathcal{T}(\mathcal{L}_{ID_1})$. Then from SHI we have the sequent

$$f[N(\sigma + \alpha + 1)]_0, E[K_B\xi] \vdash (\sigma + \alpha)^{-2} \Gamma, \text{Cl}_A(F), \neg B_0(P_A^{<\xi},t), B_0(P_A^{<\xi},t).$$

An application of $(\lor)$ yields the sequent

$$f[N(\sigma + \alpha + 1)]_0, E[K_B\xi] \vdash (\sigma + \alpha + 1)^{-2} \Gamma, \text{Cl}_A(F), \neg \forall y B_0(P_A^{<\xi},y), B_0(P_A^{<\xi},t).$$

And an application of $(\land)$ allows us to conclude.

The other cases can be treated in similar ways. \hfill \square

Lemma 36. 1. $f[N(\text{rk}(A(P_A^{<\xi},y))) + 1], E_0^{\Omega^2+\omega} \forall x(\mathcal{A}(P_A^{<\xi}x, F(x,y)) \rightarrow P_A^{<\xi}x).$

2. $f[N(3+t)], E_0^{\Omega^2+\omega} \forall \forall x \{\mathcal{A}(F(x,y), F(x,y)) \rightarrow F(x,y)\} \rightarrow \forall x \{P_A^{<\xi}x \rightarrow F(x,y)\},$

where $y = y_0, \ldots, y_{i-1}$. 

22
Proof. Property 1. Let \( \alpha = \text{rk}(A(P^<_\Omega A), t) \) and \( t \in T(L_{\Omega}^A) \). By the definition of \( \text{rk} \) we can find a natural \( k < \omega \) such that \( \alpha = \text{rk}(A(P^<_\Omega A), t) = \Omega + k \). This implies \( k(A(P^<_\Omega A), t) = \{0, \Omega\} \) and hence \( k_0(A(P^<_\Omega A), t) = \{0\} < E(0) \). By Tautology lemma (Lemma 32),

\[
f[N(\alpha)], E \vdash 2 \cdot P^<_\Omega A t, \neg A(P^<_\Omega A), t, \neg A(P^<_\Omega A), t.
\]

Since \( \Omega < \Omega \cdot 2 + k + 1 = \alpha \cdot 2 + 1 \), we can apply the closure rule \((C\Omega)\) obtaining the sequent

\[
f[N(\alpha)], E \vdash 2 \cdot 2^{+k+1} \cdot \neg A(P^<_\Omega A), t, P^<_\Omega A t.
\]

An application of \((\land)\) followed by an application of \((\lor)\) enables us to conclude

\[
f[N(\alpha) + 1], E \vdash 2^{+k+1} \vdash \forall x(A(P^<_\Omega A), x) \rightarrow P^<_\Omega A x).
\]

Property 2. By definition \( \text{rk}(P^<_\Omega A) = \omega \cdot \Omega = \Omega \). On the other hand \( \text{rk}(F) < \omega \) and hence \( \text{rk}(F) + \text{rk}(P^<_\Omega A) + 1 \cdot 2 = \Omega \cdot 2 + 2 \). Let \( s, t, s_0, \ldots, t_{-1} \in T(L_{\Omega}^A) \). Then by the previous lemma (Lemma 35)

\[
f[2], E \vdash 2^{+k+1} \vdash \forall x(A(F(\cdot, t), x) \rightarrow F(x, t)), \neg P^<_\Omega A t, F(s, t)
\]

since \( N(\Omega + 1) = 2 \). It is not difficult to see that applications of \((\lor), (\land)\) and \((\forall)\) in this order yield the sequent

\[
f[3], E \vdash 2^{+k+5} \forall x(A(F(\cdot, t), x) \rightarrow F(x, t)) \rightarrow \forall x(P^<_\Omega A x \rightarrow F(x, t))
\]

Finally, \( \Gamma \)-fold application of \((\land)\) allows us to conclude. \( \square \)

Let us recall that \( s \) denotes the numerical successor \( m \mapsto m + 1 \).

Theorem 37. Let \( A = \forall x \exists y B(x, y) \) be a \( \Pi^0_2 \)-sentence for a \( \Delta^0_\Omega \)-formula \( B(x, y) \) such that \( \text{FV}(B(x, y)) = \{x, y\} \). If \( L_{\Omega} \vdash A \), then we can an ordinal term \( \alpha < \omega \) built up without the Veblen function symbol \( \varphi \) such that for all \( m = m_0, \ldots, m_{-1} \in \mathbb{N} \) there exists \( n \leq s^\alpha(m_0 + \cdots + m_{-1}) \) such that \( B(m, n) \) is true in the standard model \( \mathbb{N} \) of \( \text{PA} \).

Proof. Assume \( L_{\Omega} \vdash A \). Then there exist \( L_{\Omega} \)-axioms \( A_1, \ldots, A_k \) such that \( (\neg A_1) \lor \cdots (\neg A_k) \lor A \) is a logical consequence in the first order predicate logic with equality. Hence by Lemma 33

\[
f[c_0], E \vdash 2^{+k+5} \forall x(A(F(\cdot, t), x) \rightarrow F(x, t)) \rightarrow \forall x(P^<_\Omega A x \rightarrow F(x, t))
\]

for some constant \( c_0 < \omega \) depending on \( N(\text{rk}(A_1)), \ldots, N(\text{rk}(A_k)), N(\text{rk}(A)) \) and depending also on the tree height of a cut-free \( \text{LK} \)-derivation of the sequent \( \neg A_1, \ldots, \neg A_k, A \). By Lemma 34 and 36 for each \( j = 1, \ldots, k \), there exists a constant \( c_j \) depending on \( \text{rk}(A_j) \) such that \( f[c_j], E \vdash 2^{+k+5} A_j \). Hence \( k \)-fold application of \((\text{Cut})\) yields \( f[c], E \vdash 2^{+k+5} A \), where \( c := \max \{k \} \cup \{c_j \mid j \leq k\} \) and \( d := \max \{\Omega, \text{rk}(A_1), \ldots, \text{rk}(A_k)\} \).
For each \( n \in \mathbb{N} \) and \( \alpha \in \mathcal{O}(\mathcal{F}) \) let us define ordinal \( \Omega_n(\alpha) \) and \( \gamma_n \) by

\[
\Omega_0(\alpha) = \alpha, \quad \gamma_0 = \Omega \cdot 3, \\
\Omega_{n+1}(\alpha) = \Omega^{\Omega_n(\alpha)}, \quad \gamma_{n+1} = \mathcal{E}^{\gamma_n}(0) + 1.
\]

Then \( d \)-fold iteration of Cut-reduction lemma (Lemma 25) yields the sequent 
\[ f[c]^{\gamma_d}, \mathcal{E}^\Omega_{\Omega_d(\Omega \cdot 3)} A. \] 
Hence Impredicative cut-elimination lemma (Lemma 20) yields 
\[ (f[c]^{\gamma_d})^{E^{\Omega_d(\Omega \cdot 3)}(0)}, \mathcal{E}^\Omega_{\Omega_d(\Omega \cdot 3)+1} F \vdash \mathcal{E}^{\Omega_d(\Omega \cdot 3)} A. \]

Let \( F := \mathcal{E}^{\Omega_d(\Omega \cdot 3)+1} \) and \( \beta := \mathcal{E}^{\Omega_d(\Omega \cdot 3)}(0) \). Then \((f[c]^{\gamma_d})^\beta, F \vdash_0^\beta A\) holds. It is not difficult to check that \( \beta < \Omega, N(\beta) \leq (f[c]^{\gamma_d})^\beta \) and \( K_{\Omega} \beta < F(0) \). Hence Predicative cut-elimination lemma (Lemma 27) yields the sequent 
\[ (f[c]^{\gamma_d})^{E^{\Omega \cdot \beta + \beta^2}(0)+1} F \vdash_0^{\beta^2} A. \]

Now let \( f \) denote \( s^\omega \). By Example 18 one can check that the conditions \((s^\omega \underline{1})\) and \((s^\omega \underline{2})\) hold. From Example 18 one will also see that \( s^\omega[c](m) \leq s^\omega(s^\omega(m)) \leq s^{\omega + c + 1}(m) \) for all \( m \). By these we have the inequality 
\[
(s[c]^{\gamma_d})^{E^{\Omega \cdot \beta + \beta^2}(0)+1}(0) \leq ((s^{\omega + c + 1})^{\gamma_d})^{E^{\Omega \cdot \beta + \beta^2}(0)+1}(0).
\]

Thanks to Lemma 20 we can find an ordinal \( \alpha \in \mathcal{O}(\mathcal{F}) \upharpoonright \Omega \) built up without the Veblen function symbol \( \varphi \) such that 
\[
((s^{\omega + c + 1})^{\gamma_d})^{E^{\Omega \cdot \beta + \beta^2}(0)+1}(0) \leq s^\alpha(0).
\]

This together with \((l\)-fold application of\) Inversion lemma (Lemma 24) yields the sequent 
\[
(s^\alpha[m_0] \cdot \cdots [m_{l-1}], F \vdash_0^{\varphi^\beta} \exists y B(m, y),
\]
where \( m = m_0, \ldots, m_{l-1} \). By Witnessing lemma (Lemma 31) we can find a natural \( n \leq s^\alpha[m_0] \cdot \cdots [m_{l-1}](0) = s^\alpha(m_0 + \cdots + m_{l-1}) \) such that \( B(m, n) \) is true in the standard model \( \mathbb{N} \) of \( \text{PA} \).

We say a function \( f \) is elementary (in another function \( g \)) if \( f \) is definable explicitly from the successor \( s \), projection, zero \( 0 \), addition \( + \), multiplication \( \cdot \), cut-off subtraction \( \div (\text{and } g) \), using composition, bounded sums and bounded products, c.f. Rose [15, page 3].

**Corollary 38.** Every function provably computable in \( \text{ID}_1 \) is elementary in \( \{s^\alpha \mid \alpha \in \mathcal{O}(\mathcal{F}) \upharpoonright \Omega \} \).

### 6 A recursive ordinal notation system \( \mathcal{O}(\Omega) \)

In order to obtain a precise characterisation of the provably computable functions of \( \text{ID}_1 \), we introduce a recursive ordinal notation system \( (\mathcal{O}(\Omega), <) \). Essentially \( \mathcal{O}(\Omega) \) is a subsystem of \( \mathcal{O}(\mathcal{F}) \).
Definition 39. We define three sets \( SC \subseteq H \subseteq \mathcal{O}(\Omega) \) of ordinal terms simultaneously. Let \( 0, \Omega, S, + \) be distinct symbols.

1. \( 0 \in \mathcal{O}(\Omega) \) and \( \Omega \in SC \).
2. If \( \alpha \in OT(F) \upharpoonright \Omega \), then \( S(\alpha) \in \mathcal{O}(\Omega) \).
3. If \( \{ \alpha_1, \ldots, \alpha_l \} \subseteq H \) and \( \alpha_1 \geq \cdots \geq \alpha_l \), then \( \alpha_1 + \cdots + \alpha_l \in \mathcal{O}(\Omega) \).
4. If \( \alpha \in \mathcal{O}(\Omega) \), then \( \omega^\alpha \in H \).
5. If \( \alpha \in \mathcal{O}(\Omega) \) and \( \xi \in \mathcal{O}(\Omega) \upharpoonright \Omega \), then \( \Omega^\alpha \cdot \xi \in H \).
6. If \( \alpha \in \mathcal{O}(\Omega) \) and \( \xi \in \mathcal{O}(\Omega) \upharpoonright \Omega \), then \( S^\alpha(\xi) \in SC \).

The relation \( \prec \) on \( \mathcal{O}(\Omega) \) is defined in the obvious way. One will see that \( \mathcal{O}(\Omega) \) is indeed a recursive ordinal notation system. Let us define the norm \( N(\omega^\alpha) \) of \( \omega^\alpha \) in the most natural way, i.e., \( N(\omega^\alpha) = N(\alpha) + 1 \).

Lemma 40. Let \( \alpha \) denote an ordinal term built up in \( OT(F) \) without the Veblen function symbol \( \varphi \). Then there exists an ordinal term \( \alpha' \in \mathcal{O}(\Omega) \) such that \( \alpha \leq \alpha' \) and \( N(\alpha) \leq N(\alpha') \).

Proof. By induction over the term construction of \( \alpha \in OT(F) \). In the base case let us observe that \( E(\alpha) \leq S^1(\alpha) \) for all \( \alpha \prec \Omega \) and that \( N(E(\alpha)) = N(\alpha) + 1 < N(S(\alpha)) + 1 = N(S^1(\alpha)) \). In the induction case we employ Lemma 11.

Lemma 41. For any ordinal term \( \alpha \in OT(F) \) built up without the Veblen function symbol \( \varphi \) there exists an ordinal term \( \alpha' \in \mathcal{O}(\Omega) \) such that \( s^\alpha(m) \leq s^{\alpha'}(m) \) for all \( m \).

Corollary 42. A function is provably computable in \( ID_1 \) if and only if it is elementary in \( \{ s^\alpha \mid \alpha \in \mathcal{O}(\Omega) \upharpoonright \Omega \} \).

The “only if” direction follows from Corollary 38 and Lemma 41. The “if” direction can be seen as follows. One can show that for each \( \alpha \in \mathcal{O}(\Omega) \upharpoonright \Omega \), the system \( ID_1 \) proves that the initial segment \( \langle \mathcal{O}(\Omega) \upharpoonright \Omega, \prec \rangle \) of \( \langle \mathcal{O}(\Omega), \prec \rangle \) is a well-ordering. For the full proof, we kindly refer the readers to, e.g., Pohlers [12] §29. From this one can show that for each \( \alpha \in \mathcal{O}(\Omega) \upharpoonright \Omega \) the function \( s^\alpha \) is provably computable in \( ID_1 \), and hence the assertion.

7 Conclusion

In this technical report we introduce a new approach to provably computable functions, providing a simplified characterisation of those of the system \( ID_1 \) of non-iterated inductive definitions. The simplification is made possible due to the method of operator-controlled derivations that was originally introduced by Wilfried Buchholz [6]. An new idea in this report is to combine the ordinal operators from [6] with the number-theoretic operators from [14], c.f. Definition 28. Ordinal operators contain information much enough to analyse \( \Pi^1_1 \)-consequences of the controlled derivations. In contrast, number-theoretic operators contain information much enough to analyse those \( \Pi^2_0 \)-consequences. It is not difficult to
generalise this approach to the system \( \text{ID}_n \) of \( n \)-fold iterated inductive definitions. Then it is natural to ask whether this approach can be extended to stronger systems like fragments of Kripke-Platek set theories. Extension to strong fragments, e.g., the fragment KPM for recursively Mahlo universes or the fragment KP\( \Pi \) for \( \Pi_3 \)-reflecting universes, is still a challenge.

References

1. T. Arai. Proof Theory for Theories of Ordinals – I: Recursively Mahlo Ordinals. *Annals of Pure and Applied Logic*, 122(1–3):1–85, 2003.
2. T. Arai. Proof Theory for Theories of Ordinals – II: \( \Pi_3 \)-reflection. *Annals of Pure and Applied Logic*, 129(1–3):39–92, 2004.
3. J. Barwise. *Admissible Sets and Structures. An Approach to Definability Theory*, Perspectives in Mathematical Logic. Springer-Verlag, Berlin-New York, 1975.
4. B. Blankertz and A. Weiermann. How to Characterize Provably Total Functions by the Buchholz Operator Method. *Lecture Notes in Logic*, 6:205–213, 1996.
5. B. Blankertz and A. Weiermann. A Uniform Approach for Characterizing the Provably Total Number-Theoretic Functions of KPM and (Some of) its Subsystems. *Studia Logica*, 62:399–427, 1999.
6. W. Buchholz. A Simplified Version of Local Predicativity. In P. Aczel, H. Simmons, and S. Wainer, editors, *Proof Theory*, pages 115–148. Cambridge University Press, Cambridge, 1992.
7. W. Buchholz. Finitary Treatment of Operator Controlled Derivations. *Mathematical Logic Quarterly*, 47(3):363–396, 2001.
8. W. Buchholz, E. A. Cichon, and A. Weiermann. A Uniform Approach to Fundamental Sequences and Hierarchies. *Mathematical Logic Quarterly*, 40(2):273–286, 1994.
9. M. Fairtlough and S. S. Wainer. Hierarchy of Provably Recursive Functions. In S. R. Buss, editor, *Handbook of Proof Theory*, pages 149–207. North Holland, Amsterdam, 1998.
10. M. Michelbrink. A Buchholz Derivation System for the Ordinal Analysis of KP + \( \Pi_3 \)-reflection. *Journal of Symbolic Logic*, 71(4):1237–1283, 2006.
11. W. Pohlers. *Proof Theory. An Introduction*, volume 1407 of *Lecture Notes in Mathematics*. Springer, 1989.
12. W. Pohlers. Subsystems of Set Theory and Second Order Number Theory. In S. R. Buss, editor, *Handbook of Proof Theory*, pages 210–335. North Holland, Amsterdam, 1998.
13. M. Rathjen. Proof-theoretic Analysis of KPM. *Archive for Mathematical Logic*, 30(5–6):377–403, 1991.
14. M. Rathjen. Proof Theory of Reflection. *Annals of Pure and Applied Logic*, 68(2):181–224, 1994.
15. H. E. Rose. *Subrecursion: Functions and Hierarchies*. Clarendon Press, Oxford, 1984.
16. G. Takeuti. *Proof Theory*. North-Holland, Amsterdam, 2nd edition, 1987.
17. A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory*. Cambridge University Press, Cambridge, 2nd edition, 2000.
18. A. Weiermann. How to Characterize Provably Total Functions by Local Predicativity. *Journal of Symbolic Logic*, 61(1):52–69, 1996.
19. A. Weiermann. Classifying the Provably Total Functions of PA. *Bulletin of Symbolic Logic*, 12(2):177–190, 2006.

20. A. Weiermann. A Quick Proof-theoretic Analysis of $ID_1$. 2011. Draft, 7 pages.