Abstract. If the Lie group of a non-Abelian theory is replaced by the corresponding q-group, one is led to replace the Lie algebra by two dual algebras. The first of these lies close to the Lie algebra that it is replacing while the second introduces new degrees of freedom. We interpret the theory based on the first algebra as a modification of standard field theory while we propose that the new degrees of freedom introduced by the second algebra describe solitonic rather than point particle sources. We have earlier found that the modified q-electroweak theory differs very little from the standard theory. Here we find a similar result for q-gravity. Both of the modified theories are incomplete, however, and must be completed by the solitonic sector. We propose that the solitonic sector of both q-electroweak and q-gravity have the symmetry of knots associated with $SU_q(2)$. Since the Lorentz group is here deformed, there is no longer the standard classification of particles described by mass and spin. There is instead a classification of irreducible structures determined by $SU_q(2)$. 
1 Introduction.

Since the $q$- and Lie algebras are closely related, it has been natural to study the $q$-theories obtained by replacing the Lie algebras in our current theories and in particular in our description of elementary particles. In doing this a distinction should be made between the Lorentz algebra and the algebras of the standard model since the latter are phenomenological and less solidly based than the former. For example, a $q$-electroweak theory may be obtained by replacing $SU(2)$ electroweak by $SU_q(2)$ electroweak and at the same time retaining the Lorentz group.\(^1\) In general in going from $SU(N)$ to $SU_q(N)$ the original algebra gets replaced by two algebras, the first lying close to the Lie algebra that is being replaced. The second algebra and its attached state space introduces new degrees of freedom that can naturally be associated with non-locality.

We are here using the language of Lie groups rather than Hopf algebras since we want to emphasize a correspondence limit with standard theory in which “internal algebra” describes a deformation of the usual Lie group and “external algebra” describes a deformation of the usual Lie algebra. Since the standard Lie group may be obtained by integrating its Lie algebra, all degrees of freedom of the standard theory are already exposed in the Lie algebra. That is not true in the $q$-theory, and is the reason for discussing both algebras here.

The part of the program associated with the external algebra has been carried out for $q$-electroweak and leads to a modified Weinberg-Salam theory that is not reducible to the standard Weinberg-Salam theory and therefore has slightly different experimental consequences. This so modified Weinberg-Salam theory is not renormalizable and needs to be completed by taking the internal algebra into account.

Here we also discuss the more speculative $q$-Lorentz or $q$-gravity theories. Both $q$-electroweak and $q$-gravity are based on the algebras defined by

$$T^t \epsilon T = \epsilon$$ \hspace{1cm} (1.1)

where

$$\epsilon = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix}$$ \hspace{1cm} (1.2)

For $q$-electroweak $T \epsilon SU_q(2)$ and for $q$-gravity $T \epsilon SL_q(2)$. 

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In our view the really interesting feature of the $q$-theories is the appearance of the two
algebras. It is natural to regard the external algebra, differing little from the parent Lie
algebra, as underlying a modified standard field theory with point particle sources. Then
the internal algebra may be interpreted as underlying a dual description of the same field
but with solitonic sources. It is natural to expect solitons here, since gauge theories are non-
linear theories with attractive self-interactions. There are numerous examples of solitons in
non-linear theories, including spinor solitons, 't Hooft-Polyakov solitons, and Nambu strings
and other stringlike structures formed by attractive self-interactions. In our picture one
would expect the external theory to represent a perturbative description of the full theory,
and the internal theory to provide a non-perturbative description of the full theory as well
as a classification of the solitonic sources. In the simplest model one may assume that the
two algebras implement the same Lagrangian.

Sections 2, 3, and 4 summarize familiar facts about the spin representation of the $q$-
Lorentz group, $q$-spinors, and $\sigma_q$ matrices. Section 5 describes the higher dimensional repre-
sentations of $SU_q(2)$. In Section 6 the curvature of standard Euclidean gravity is expressed
in terms of the spinor connection of $SU(2) \times SU(2)$ and in Section 7 the relation between
external $q$-gravity and standard gravity is examined. It is shown there that the external $q$-
gravity is very close to the standard gravity theory just as external $q$-electroweak is close to
the standard electroweak. Both $q$-electroweak and $q$-gravity are therefore approximately cor-
rect physical theories; but since neither is tree unitary, there must be some missing physics.
In the present situation it is natural to try to identify the missing physics with the internal
theory. Section 10 conjectures that the internal theory may describe closed and knotted flux
tubes that play the role of solitonic sources.

2 The $q$-Lorentz Group.

The formalism of $q$-gravity resembles $q$-electroweak theory in that the affine connection $\Gamma_\mu$
of $q$-gravity lies in $SL_q(2)$ while the corresponding vector potential $A_\mu$ of $q$-electroweak lies
in $SU_q(2)$, a subgroup of $SL_q(2)$.
Let us recall the $q = 1$ limit of $q$-Lorentz by introducing

$$X = x_k \sigma_k \quad k = 0, 1, 2, 3$$

(2.1)

where the $x_k$ are real and $\sigma_k = (1, \vec{\sigma})$ so that

$$X = \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix}$$

(2.2)

Then

$$X^+ = X$$

(2.3)

and

$$\det X = t^2 - x^2 - y^2 - z^2$$

(2.4)

Now introduce the 2-dimensional representation of the Lorentz transformation ($L$) by setting

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(2.5)

where the matrix elements of $L$ are complex and restricted by

$$\det L = 1$$

(2.6)

By (2.6) the number of independent real parameters of $L$ is reduced to six, the number needed to characterize a Lorentz transformation.

Now transform $X$ by

$$X' = L^+ XL$$

(2.7)

where $L^+$ is $L$ adjoint. Then

$$(X')^+ = X'$$

(2.8)

$$\det X' = \det X$$

(2.9)

$$(x_o^2 - \vec{x}^2)' = (x_o^2 - \vec{x}^2)$$

(2.10)

Hence the 6 independent parameters of $L$ may be identified with the parameters of a Lorentz transformation.
The unimodular restriction on $L$ may be expressed as follows:

$$
\epsilon_{ij}L_{im}L_{jn} = \epsilon_{mn}\det L = \epsilon_{mn}
$$  \hfill (2.11)

or

$$
L^t\epsilon L = \epsilon
$$  \hfill (2.12)

where

$$
\epsilon_{mn} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
$$  \hfill (2.13)

One may pass to the $q$-theory by replacing $\epsilon_{mn}$ by

$$
\epsilon_{mn}(q) = \begin{pmatrix} 0 & q_1^{1/2} \\ -q_1^{1/2} & 0 \end{pmatrix}
$$  \hfill (2.14)

and requiring the analog of (2.12)

$$
L^t_q\epsilon(q)L_q = L_q\epsilon_qL^t_q = \epsilon_q
$$  \hfill (2.15)

The matrix elements of $L_q$ must now satisfy the following algebra:

$$
ab = qba \quad ac = qca \quad bc = cb \quad ad - qbc = 1
$$

$$
bd = qdb \quad cd = qdc \quad da - q_1bc = 1
$$  \hfill (2.16)

The $q$-determinant is

$$
ad - qbc
$$  \hfill (2.17)

the natural generalization of (2.11) and (2.14).

### 3 $q$-Spinors.

We shall next drop the subscripts on $L_q$ and $\epsilon_q$ and understand the symbols $L$ and $\epsilon$ to represent the $q$-deformed matrices.

Let $\psi^A$ be a contravariant 2-rowed basis and $\chi^{\dot{A}}$ a contravariant basis for the conjugate representation:

$$
\psi^{\dot{A}'} = L^A_B\psi^B
$$  \hfill (3.1)

$$
\chi^{\dot{A}'} = (L^*)^A_B\chi^B
$$  \hfill (3.2)
Associated with $\psi^A$ and $\chi^A$ are corresponding covariant spinors

$$\psi'_A = \psi_B (L^{-1})_B^A$$  \hfill (3.3)

$$\chi'_A = \chi_B ((L^*)^{-1})_B^A$$  \hfill (3.4)

By (2.15) $\epsilon_{AB}$ is an invariant second rank tensor since

$$\epsilon_{AB} = L_A^C L_B^D \epsilon_{CD}$$  \hfill (3.5)

Then $\psi^A \epsilon_{AB} \chi^B$ is an invariant form:

$$\psi^A' \epsilon_{AB} \chi^B' = (L^A_B \psi^C) \epsilon_{AB} (L^B_D \chi^D)$$  \hfill (3.6)

$$= \psi^C ((L^A_B \epsilon_{AB} L^B_D) \chi^D$$

$$= \psi^C \epsilon_{CD} \chi^D$$  \hfill (3.7)

while

$$(\psi^A \chi^{\dot{X}})' = (L^A_B \psi^B) (L^*)_{\dot{Y}}^{\dot{X}} \chi^{\dot{Y}}$$

$$= L^A_B (\psi^B \chi^{\dot{Y}}) L^+_{\dot{Y}} \dot{X}$$

Then

$$M^{A \dot{X}'} = L^A_B M^B_{\dot{Y}} (L^+)_{\dot{Y}} \dot{X}$$

or

$$M' = L M L^+$$  \hfill (3.8a)

where

$$M^{A \dot{X}} = \psi^A \chi^{\dot{X}}$$  \hfill (3.8b)

Now the remarks following (2.7) do not hold for (3.8a) since the matrix elements of $L$ no longer commute. In particular, $\det M$ and $x_\alpha^2 - \vec{x}^2$ are no longer invariant.

The general $q$-spin tensor may again be defined as

$$u(n, m) = \psi_{k_1...k_n} \chi_{\dot{i}_1...\dot{i}_m}$$  \hfill (3.9)
that transforms like the product of $n$ $q$-spinors and $m$ complex conjugate $q$-spinors. Then $u(n, m)$ is the basis of an irreducible representation of the $q$-Lorentz group.

We define a contravariant $\epsilon$ symbol by

$$\epsilon^{AB}(q) = \epsilon_{AB}(q^{-1})$$

and $\epsilon$ may be used to raise or lower indices to obtain contra- or covariant spin tensors.

## 4 The $\sigma_q$ Matrices

Let

$$(\sigma^m_q)_{BY} = (1, \bar{\sigma})$$

be the usual Pauli matrices. Introduce the matrices contravariant to $(\sigma^m_q)_{BY}$ with respect to the metric $\epsilon_q$:

$$(\sigma^m_q)^{\dot{X}A} = \epsilon^{\dot{X}Y} \epsilon_q \sigma^m_q (\sigma^m_q)_{BY}$$

Then

$$(\sigma^n_q)^{\dot{X}A} = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} -q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

by (3.10). These matrices satisfy the following relations

$$(\sigma^n_q)^{\dot{X}A} (\sigma^n_q)^{Y\dot{X}} = 2\eta^{mn}$$

$$(\sigma^n_q)^{\dot{X}A} (\bar{\sigma}_q)^{Y\dot{B}} = 2\delta^{\dot{X}}_A \delta^B$$

where

$$\eta^{nm} = \begin{pmatrix} \frac{1}{2}(q + q^{-1}) & 0 & 0 & \frac{1}{2}(q - q^{-1}) \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\frac{1}{2}(q - q^{-1}) & 0 & 0 & -\frac{1}{2}(q + q^{-1}) \end{pmatrix}$$

and

$$\det \eta = -\cosh^2 \theta + \sinh^2 \theta = -1$$

where

$$q = e^\theta$$
The equation of the light cone is then

\[ \eta_{nm} x^n x^m = \eta_{00} c^2 t^2 + \eta_{33} z^2 - x^2 - y^2 = 0 \]

or

\[ (\cosh \theta)(c^2 t^2 - z^2) - x^2 - y^2 = 0 \] (4.7)

after the rescaling where

\[ \zeta = (\cosh \theta)^{1/2} z \]
\[ \tau = (\cosh \theta)^{1/2} t \] (4.8b)

The light cone is then rescaled in the \( q \)-theory, but it is not invariant under the deformed Lorentz transformation.

By (3.7) the spinor bilinear \( \psi^t \epsilon \psi \) is invariant under \( q \)-Lorentz transformations satisfying (2.15).

The basic invariant is therefore no longer the interval or the light cone but is instead the spinor cone or the associated \( q \)-commutor

\[ \psi^t \epsilon \psi = 0 \] (4.9a)

or

\[ (\psi^1, \psi^2)_q \equiv \psi^1 \psi^2 - q \psi^2 \psi^1 = 0 \] (4.9b)

Since Einstein-Minkowski spacetime is based on the metrical light cone and since that is not invariant under \( q \)-Lorentz transformations, it must be replaced here by a \( q \)-spacetime that is based on the spinor cone that is invariant under these transformations. Hence if \( q \neq 1 \), the light cone and an associated Einstein-Minkowski space are replaced by a non-commuting algebra and associated non-commuting analogue of group space, as discussed by many authors.

At the same time the elementary particle states, labeled by mass and spin, and defined by the Lorentz and Poincaré algebras, disappear, and one is left with only structures and states defined by the \( SL_q(2) \) algebra.
5 The Irreducible Representation of $SU_q(2)$.

Instead of deforming the spin representation of the Lorentz group one may deform its $SO(4)$ representation. Since $SO(4) = SU(2) \times SU(2)$ we may therefore make use of $SU_q(2)$. The irreducible representations of $SU_q(2)$ are as follows:

\[
D^j_{mm'}(a, \bar{a}, b, \bar{b}) = \Delta^j_{mm'} \sum_{s,t} \left( \frac{n_+}{s} \right)_1 \left( \frac{n_-}{t} \right)_1 q^{(n_+ + 1 - s)}(-1)^t \delta(s + t, n'_+) \\
\times a^s b^{n_+ - s} \bar{b}^t \bar{a}^{n_- - t} \tag{5.1}
\]

where

\[
n_\pm = j \pm m
\]

\[
n'_\pm = j \pm m'
\]

\[
\left( \frac{n}{s} \right)_1 = \frac{\langle n \rangle_1!}{\langle s \rangle_1! (n - s)_1!} \quad \text{and} \quad \langle n \rangle_1 = \frac{q^{2n} - 1}{q^2 - 1}
\]

\[
\Delta^j_{mm'} = \left[ \frac{\langle n'_+ \rangle_1! (n'_-)_1!}{\langle n'_+ \rangle_1! (n'_-)_1!} \right]^{1/2} q_1 = q^{-1}
\]

and the arguments of (5.1) satisfy the following relations

\[
ab = qba \quad a\bar{a} + b\bar{b} = 1 \quad b\bar{b} = \bar{b}b \tag{5.2}
\]

\[
a\bar{b} = q\bar{b}a \quad \bar{a}a + q^2\bar{b}b = 1
\]

if $q$ is real. These relations are obtained from the corresponding relations for $SL_q(2)$ by setting

\[
c = -q_1 \bar{b}
\]

\[
a = \bar{d} \tag{5.3}
\]

in (2.16). Now set

\[
D^{1/2}(a, \bar{a}, b, \bar{b}) = e^{B\sigma_+} e^{\lambda\sigma_3} e^{C\sigma_-} \tag{5.4a}
\]

and expand to terms linear in $(B, C, \theta)$. Here

\[
q = e^\lambda \quad b = Bq^{-\theta} \quad q_1 \bar{b} = -q^{-\theta}C \tag{5.4b}
\]

Then
\[ D_{mm'}^j(B, C, \theta) = D_{mm'}^j(0, 0, 0) + B(J_B^j)_{mm'} + C(J_C^j)_{mm'} + 2\lambda \theta (J_\theta^j)_{mm'} + \ldots \] (5.5)

The non-vanishing matrix coefficients \((J_B^j)_{mm'}, (J_C^j)_{mm'},\) and \((J_\theta^j)_{mm'}\) are by (5.1)

\[
(m - 1|J_B^j|m) = [\langle j + m \rangle q_i^2 \langle j - m + 1 \rangle q_i^2]^{1/2}
\]

\[
(m + 1|J_C^j|m) = [\langle j - m \rangle q_i^2 \langle j + m + 1 \rangle q_i^2]^{1/2} \quad q_1 = q^{-1}
\]

\[
(m|J_\theta^j|m) = m
\]

Then \((B, C, \theta)\) and \((J_B, J_C, J_\theta)\) are generators of two dual algebras satisfying the following commutation rules

\[
(J_B, J_\theta) = -J_B \quad (J_C, J_\theta) = J_C \quad (J_B, J_C) = q_1^{2j - 1}[2J_\theta]
\]

\[
(B, C) = 0 \quad (\theta, B) = B \quad (\theta, C) = C
\]

(5.7)

(5.8)

where

\[
[x] = \frac{q^x - q_1^x}{q - q_1} \quad \langle x \rangle = \frac{q^x - 1}{q - 1}
\]

Here the internal algebra is described by (5.2), (5.4) and (5.8) and the external algebra by (5.7). These are the two algebras previously introduced.

The following commutation relations are implied by (5.6)

\[
J = \frac{1}{2} \text{ (fundamental)}
\]

\[
(J_B, J_\theta) = -J_B \quad (J_C, J_\theta) = J_C \quad (J_B, J_C) = 2J_\theta
\]

(5.10)

\[
J = 1 \text{ (adjoint)}
\]

\[
(J_B, J_\theta) = -J_B \quad (J_C, J_\theta) = J_C \quad (J_B, J_C) = \langle 2 \rangle q_i^2 J_\theta
\]

(5.11)

If \(J > 1\), the right-hand side of (5.7) is not linear in the generators and in that case one cannot speak of structure constants or a deformed Lie algebra.

In the Weinberg-Salam electroweak theory the vector potential lies in the Lie algebra of \(SU(2)\). In the \(q\)-electroweak theory it lies in the external algebra of \(SU_q(2)\). Since only the
fundamental and adjoint representations of the external algebra are needed, the \(q\)-electroweak
theory differs from standard electroweak theory only in the adjoint representation, and there
only slightly. We regard this theory, based on the external algebra, as a perturbative version
of the full \(q\)-electroweak theory that is based on the internal algebra.

In the corresponding gravitational case, the affine connection must lie in the Lie algebra
of \(SL(2)\) or in the Euclidean version in \(O(4)\), or \(SU(2) \times SU(2)\) and in the \(q\)-gravitational
theory we shall consider only the \(SU_q(2) \times SU_q(2)\) representation.

6 Euclidean Gravity.

The \(SO(4)\) group may be factored in the well known way, namely:

\[
SO(4) = SU(2) \times SU(2)
\]

or

\[
e^{i\theta_{\mu\nu}M_{\mu\nu}} = e^{i\theta_{ij}L_i} e^{i\theta_{ki}J_k} \quad i, j, k = 1, 2, 3
\]

\[
e^{i\theta_{0\mu\nu}M_{\mu\nu}} = e^{i\theta_{0ij}L_{ij}^{ij}} e^{i\theta_{0ki}J_{ki}} \quad \mu\nu = 0, 1, 2, 3
\]

where

\[
(L_i, L_j) = i\epsilon_{ijk}L_k
\]

\[
(J_i, J_j) = i\epsilon_{ijk}J_k
\]

\[
(J_i, L_k) = 0
\]

and

\[
J_k = \frac{1}{2}\epsilon_{kij}J_{ij} \quad i, j, k = 1, 2, 3
\]

Here \(L_k\) and \(J_k\) are both generators of \(SU(2)\) and the \(\epsilon_{k\ell s}\) are the structure constants of
\(SU(2)\).

Let us express the \(SU(2) \times SU(2)\) spin connection as follows:

\[
\omega_\mu = \omega^i_\mu s_i + \omega^j_\mu s_j \quad i, j, k = 1, 2, 3
\]
where \( s_i \) and \( s_{jk} \) are two-dimensional representations of \( L_i \) and \( J_{jk} \) respectively. Then the curvature is

\[
R_{\mu\lambda} = \partial_\mu \omega_{\lambda} - \partial_\lambda \omega_\mu + (\omega_\mu, \omega_\lambda) \quad (6.8)
\]

\[
= R^{ok}_{\mu\lambda} s_k + R^{jk}_{\mu\lambda} s_{jk} \quad (6.9)
\]

where

\[
R^{ok}_{\mu\lambda} = \partial_\mu \omega^{ok}_\lambda - \partial_\lambda \omega^{ok}_\mu + \epsilon^{ok}_{\alpha i, oj} \omega^{oi}_\alpha \omega^{oj}_\mu \quad (6.10)
\]

\[
R^{jk}_{\mu\lambda} = \partial_\mu \omega^{jk}_\lambda - \partial_\lambda \omega^{jk}_\mu + i \epsilon^{jk}_{\alpha rs, mn} \omega^{rs}_\alpha \omega^{mn}_\mu \quad (6.11)
\]

Here \( \epsilon^{ok}_{\alpha i, oj} \) and \( \epsilon^{jk}_{\alpha rs, mn} \) restate \( \epsilon_{ijk} \) according to (6.3) and (6.4). Define the antisymmetric matrix \( R^{ab}_{\mu\lambda} \) as follows:

\[
[R^{ab}_{\mu\lambda}] = \begin{bmatrix}
0 & R^{01}_{\mu\lambda} & R^{02}_{\mu\lambda} & R^{03}_{\mu\lambda} \\
0 & R^{12}_{\mu\lambda} & R^{13}_{\mu\lambda} \\
0 & R^{23}_{\mu\lambda} \\
0 & 0
\end{bmatrix} \quad a, b = 0, 1, 2, 3 \quad (6.12)
\]

where

\[
R^{ab}_{\mu\lambda} = -R^{ba}_{\mu\lambda} \quad (6.13)
\]

Then the action for Euclidean gravity is

\[
S = \int V^a \wedge V^b \wedge \epsilon_{abcd} R^{cd} \quad (6.14)
\]

\[
= \int R \sqrt{-g} \ d^4 x \quad \text{if} \quad \det \eta = -1 \quad (6.15)
\]

where \( V^a \) and \( R^{cd} \) are tetrad and curvature forms.

### 7 \( q \)-Gravity.

The \( q \)-gravitational action is

\[
S = \int R \sqrt{-g} \ d^4 x \quad (7.1)
\]

where the metric \( g_{\alpha\beta}(q) \) may be written in terms of the tetrad \( V^a_\alpha \):

\[
g_{\alpha\beta}(q) = V^a_\alpha \eta_{ab}(q) V^b_\beta \quad (7.2)
\]
and $\eta_{ab}(q)$ is given by (4.6). Since
\[ \det \eta(q) = -1 \] (7.3)
we have by (7.2)
\[ \det g(q) = -(\det V)^2 \] (7.4)
The Riemann tensor is
\[ R_{\mu\lambda}(q) = \partial_\mu \Gamma_\lambda(q) - \partial_\lambda \Gamma_\mu(q) + [\Gamma_\mu(q), \Gamma_\lambda(q)] \] (7.5)

where
\[ \Gamma_{\alpha\beta}^\mu(q) = \frac{1}{2} g^{\mu\sigma}(q)(\partial_\alpha g_{\beta\sigma}(q) + \partial_\beta g_{\alpha\sigma}(q) - \partial_\sigma g_{\alpha\beta}(q)) \] (7.6)

By (7.2) the field equations obtained from (7.1) will depend on $q$, if written in terms of $V_\alpha^a$. If they are written in terms of the traditional $g_{\alpha\beta}$ they will be unchanged. It is then only the relation between $g_{\alpha\beta}$ and $V_\alpha^a$ that depends on $q$ as determined by (7.2).

To bring the presentation of $q$-gravity closer to $q$-electroweak, express the curvature in terms of the spin connection (6.7) rather than the Christoffel connection (7.6).

To make this transition to the $q$-theory rewrite (6.7), (6.10), and (6.11) as
\[ \omega_{\mu}(q) = \omega_{\mu}^{oi}(q)s_i(q) + \omega_{\mu}^{jk}(q)s_{jk}(q) \] (7.7)
\[ R_{\mu\lambda}^{ok}(q) = \partial_\mu \omega_{\lambda}^{ok}(q) - \partial_\lambda \omega_{\mu}^{ok}(q) + i \epsilon_{\alpha,\mu,\nu}^{ok}(q)\omega_{\alpha}^{oi}(q)\omega_{\nu}^{oj}(q) \] (7.8)
\[ R_{\mu\lambda}^{jk}(q) = \partial_\mu \omega_{\lambda}^{jk}(q) - \partial_\lambda \omega_{\mu}^{jk}(q) + i \epsilon_{\tau,s,\mu,\nu}^{jk}(q)\omega_{\tau}^{rs}(q)\omega_{\nu}^{mn}(q) \] (7.9)

But
\[ s_{jk}(q) = s_{jk} \quad s_k(q) = s_k \] (7.10)
Then
\[ \epsilon_{m\ell s}(q) = \epsilon_{m\ell s} \] (7.11)
since the fundamental representation is not changed in passing to the $q$-algebra (see (5.10)). Therefore $\epsilon_{\alpha,\mu,\nu}^{ok}(q)$ and $\epsilon_{\tau,s,\mu,\nu}^{jk}(q)$ are also independent of $q$.

Hence this version of $q$-gravity also agrees with Einstein gravity. Therefore whether one attempts to $q$-deform either the Christoffel connection ($\Gamma$) or the spin connection ($\omega$) the equations of the free gravitational field are unchanged.
8 Interacting Fields.

If one considers the total $q$-field characterized by an action describing interacting gravitational, electroweak, and spinor fields, then it would be natural to assume that all groups, including the Lorentz group, are $q$-deformed. In this action the $q$-deformation will appear in terms describing the free electroweak field, in interactions of the electroweak with the gravitational field via the electroweak energy momentum tensor, and in interactions of both electroweak and gravity with the spinor fields if the spinor interactions are described by

$$\psi^t \epsilon_q D_i \psi \quad i = 1, 2$$

(8.1)

where $\epsilon_q$ plays the role of the usual charge conjugation matrix and

$$D_1 = \gamma^\mu (\partial_\mu + eA_\mu)$$

$$D_2 = \gamma^\mu (\partial_\mu + g\omega_\mu)$$

(8.2)

where $D_1$ and $D_2$ are electroweak and gravitational covariant derivatives. Then

$$(\psi^t \epsilon_q D_i \psi)' = (\psi^t L_q^t \epsilon_q (L_q D_i L_q^{-1}) L_q \psi)
= \psi^t \epsilon_q D_i \psi$$

(8.3)

since

$$L_q^t \epsilon_q L_q = \epsilon_q$$

(8.4)

$$D_i' = L_q D_i L_q^{-1}$$

(8.5)

In the $q$-electroweak case, $\epsilon_q$ is two-dimensional, while in the $q$-gravity case, it is four-dimensional to agree with $\omega_\mu$.

9 The Internal Algebra.

One sees that $q$-gravity like $q$-electroweak differs little or not at all from the standard theories in the sector exclusively dependent on the external algebra. One may next speculate about the theory based on the internal algebra.
If one postulates $q$-Lorentz, there is no longer a local Poincaré group permitting the definition of elementary particles in terms of mass and spin. Instead the irreducible structures should be defined by the irreducible representations of the $q$-algebra. Alternatively the elementary structures may be described as knotted loops defined by the $q$-algebra. These knots may be labeled by their Jones polynomials and these polynomials may be generated directly by a recipe based on the $q$-algebra.

The 3-dimensional knots may be characterized by their projections on a plane, where they appear as four-valent plane graphs with extra structure at the vertices in the form of the two types of crossings, shown in figure 1.

The broken arc pair at a crossing indicates the arc that passes underneath the other arc in space. If (a) and (b) in Fig. 1 are rotated counterclockwise and clockwise respectively, so that the overcrossing line lies along the x-axis, then the (−−) channel is composed of the conventional first and third quadrants.

L. Kauffman has shown how to encode a program for generating the Jones polynomial with the matrix $\epsilon_q$ defined in (1.2). He associates a well-defined polynomial $\langle K \rangle$ with an unoriented link $K$. This polynomial is defined recursively in Eqs. (9.1), (9.2), and (9.3):\(^5\)

$$\langle K \rangle = \epsilon \left[ q^{-\frac{1}{2}} \langle K_- \rangle - q^{\frac{1}{2}} \langle K_+ \rangle \right] \quad (9.1)$$

where $K, K_-, \text{ and } K_+$ are shown in figure 2.

$$\langle 0 \ K \rangle = (q + q^{-1}) \langle K \rangle \quad (9.2)$$
In the first formula (9.1) brackets like
\[ \langle \cdots \rangle_{X} \]
(9.4) refer to graphs with one crossing highlighted.

Formula (9.1) asserts that the polynomial for a given diagram is obtained by an additive combination of the polynomials for the diagrams obtained by splicing away the given crossing in the two possible ways, i.e. one may open up either the \( q^{-1/2} \) channel or the \( q^{1/2} \) channel. The small diagrams indicate larger diagrams that differ only as indicated. Formulas (9.2) and (9.3) state that the value of a loop (simple closed curve in the plane) is \( (q + q^{-1}) \) and that if the loop occurs (isolated) inside a large diagram then the value of the polynomial acquires a factor \( (q + q^{-1}) \) from the loop.

Repeated applications of these rules to a graph \( K \) with multiple crossings reduce \( \langle K \rangle \) to a Laurent polynomial in \( q \).

If \( K \) is oriented, then one may form the following invariant of ambient isotopy:
\[
f_{k}(iq^{-1/2}) = (iq^{-1/2})^{-W(K)}\langle K \rangle
\] (9.5)
where \( W(K) \), the sum of the crossing signs, is the writhe of \( K \). For any oriented link \( V_{k}(t) = f_{k}(t^{-1/4}) \) is the one variable Jones polynomial.\(^{5}\)

Written entirely in terms of \( \epsilon_{q} \) the Kauffman rules (9.1), (9.2), and (9.3) read as follows:
\[
\langle K \rangle = \text{Tr} \epsilon_{q}^{\beta} [\sigma_{-}\langle K_{-} \rangle + \sigma_{+}\langle K_{+} \rangle] \quad \sigma_{\pm} = \frac{1}{2}(\sigma_{1} \pm i \sigma_{2})
\]
\[
\langle OK \rangle = \text{Tr} \epsilon_{q}^{\beta} \epsilon_{q} \langle K \rangle
\]
\[
\langle K \rangle = \text{Tr} \epsilon_{q}^{\beta} \epsilon_{q}
\]
10 Summary and Remarks.

As we have seen in Section 2, $\epsilon_q$ is the basic invariant of $SL_q(2)$ and hence of the $q$-Lorentz group. Now we see that it also encodes the Kauffman bracket. If the bracket is physically interpreted as a vacuum expectation value, then $q + q^{-1}$ also acquires a physical meaning as the vacuum expectation value of a loop.

One may establish a correspondence between the irreducible representations of the $q$-algebra, given by (5.1) and labeled by $(j, m, m')$, and the characterization of oriented knots by $(N, w, r)$ where $N$ is the number of crossings, $w$ is the writhe, and $r$ is the Whitney degree or the rotation number of the underlying plane curve. Then we may characterize the knot $(N, w, r)$ by $D_{wr}^N$, also given by (5.1), as well as by the Jones polynomial. The Jones polynomial is numerically valued, while $D_{wr}^N$ is an operator that may be evaluated on the state space attached to the $q$-algebra.

Here $N$, $w$, and $r$ are all integers. In order to include the half-integer representations one writes

$$D_{\frac{N}{2}+\frac{1}{2}}$$

(10.1)

where $N$, $w$, and $r$ are all integers.

Then one may satisfy the knot constraint:

$$r - w = \text{odd}$$

(10.2)

Since the external algebra differs little from the parent Lie algebra it may be chosen as the basis of a modified standard theory that describes point particles. It is then natural to associate the additional degrees of freedom of the internal algebra with the soliton that replaces the point particle, since there is no other place for them in a particle-like description. Indeed if we exclude point particles from the theory, the fields must be their own sources. This point of view is supported by the existence of solitons in various classical non-linear field theories, including in particular gauge theories that exhibit 't Hooft-Polyakov solitons or alternatively cohesive flux tubes or strings. The solitons usually considered are globular or unknotted, but knotted flux tubes are obvious possibilities for representing solitonic sources that are localized at some scale and we see from the present work that they are natural
in $q$-gauge theories. One may also entertain the possibility that the external theory is a perturbative version of the internal theory.

One may then conjecture a quantum theory in which the states are states of knots rather than states of Lorentz particles. We may describe the knots as $q$-Lorentz particles characterized by Jones polynomials or by $D_{N/2}^{1/2}$.

We note that knot states have also emerged from attempts to quantize general relativity. These results have been summarized as follows:

In the canonical formulation of Einstein gravity one may take the dynamical variables to be the spatial components of the Ashtekar connection, $A_i$, and the corresponding components of the conjugate momentum, $E^i$, to be the densitized triad. Then in the absence of cosmological and energy-momentum terms, the constraints are all first class and take the form:

$$D_i E^i = 0 \quad \text{(gauge invariance)} \quad (10.3)$$
$$\text{Tr} \, F_{ij} E^i = 0 \quad \text{(3D-reparametrization)} \quad (10.4)$$
$$\text{Tr} \, F_{ij} E^i E^j = 0 \quad \text{(time reparametrization)} \quad (10.5)$$

where $F_{ij}$ is the curvature computed from $A_i$.

The quantum state satisfying all these quantum constraints may be exhibited as the following integral transform

$$\psi(\gamma) = \int DA \, W(\gamma, A) \psi(A) \quad (10.6a)$$

where the kernel is the Wilson loop

$$W(\gamma, A) = \text{Tr} \, P \, e^{\oint_\gamma A} \quad (10.6b)$$

and $\gamma$ is a non-self-intersecting smooth closed curve while $A$ is the Ashtekar connection. Eq. (10.6) transforms functionals of $A$ into functionals of loops. Since the diffeomorphism class of a smooth non-self-intersecting loop is called a knot, the functions of knot classes satisfy all the constraints of uncoupled quantum general relativity.
From the viewpoint of this paper, however, knotted solitons should appear in any $q$-gauge theory with attractive self-interactions. Here the $\epsilon_q$ symmetry appears as input, but in the quantum gravity work the knot group appears as output of the quantization.

Because external $q$-gravity agrees with standard gravity (as described in Sections 6 and 7), knot solutions to the quantum constraints should appear in external $q$-gravity as well. In addition, as shown in Sections 9 and 10, knots also appear in the internal $q$-theory via $\epsilon_q$, the fundamental invariant of $q$-Lorentz or $SU_q(2)$. Knots therefore appear in both the internal and external sectors of $q$-gravity. Since all fields are coupled to the gravitational field, knots may on these grounds be expected quite generally, and one may conjecture that $SU_q(2)$ plays the role of a universal hidden symmetry.

I thank A. C. Cadavid and C. Fronsdal for useful comments.

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