BRST symmetry for a torus knot

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received 26 June 2017; accepted in final form 21 September 2017
published online 24 October 2017

PACS 11.10.Ef – Lagrangian and Hamiltonian approach
PACS 11.30.-j – Symmetry and conservation laws

Abstract – We develop BRST symmetry for the first time for a particle on the surface of a torus knot by analyzing the constraints of the system. The theory contains 2nd-class constraints and has been extended by introducing the Wess-Zumino term to convert it into a theory with first-class constraints. BFV analysis of the extended theory is performed to construct BRST/anti-BRST symmetries for the particle on a torus knot. The nilpotent BRST/anti-BRST charges which generate such symmetries are constructed explicitly. The states annihilated by these nilpotent charges consist of the physical Hilbert space. We indicate how various effective theories on the surface of the torus knot are related through the generalized version of the BRST transformation with finite-field–dependent parameters.

Introduction. – Knot \(^{[1,2]}\) theory, based on mathematical concepts has found immense applications in various branches of frontier physics. Knot invariants in physical systems were introduced long ago and has got considerable impact during the last one and half decades \(^{[1–9]}\), especially when interpreted as Wilson loop observable in the Chern-Simons (CS) theory \(^{[7]}\). The discussion on topological string approach to the torus knot invariants are presented in ref. \(^{[7]}\). In the context of gauge theory, knot invariant theories relate 3d symmetry to the CS sub-manifolds and 3d SUSY gauge theory. It also plays an important role in various other problems like, the inequivalent quantization problem \(^{[4]}\), in the role of topology in defining the vacuum state in gauge theories \(^{[5]}\), in understanding band theory in solids \(^{[6]}\). Various longstanding problems related with the connection between knot theory and quantum field theory were discussed in \(^{[7]}\). Dynamics and symmetries of the particles constraints to move on the surface of a torus knot have recently been addressed through an Hamiltonian analysis \(^{[9]}\).

BRST quantization \(^{[10]}\) is an important and powerful technique to deal with a system with constraints \(^{[11]}\). It enlarges the phase space of a gauge theory and restores the symmetry of the gauge fixed action in the extended phase space keeping the physical contents of the theory unchanged. BRST symmetry plays a very important role in renormalizing spontaneously broken theories, like standard model and hence it is extremely important to investigate it for different systems. To the best of our knowledge BRST formulation for a particle on a torus knot has not been developed yet. This motivates us in the study of BRST symmetry for a particle on the surface of a torus knot. In the present work we make an important step forward in formulating BRST symmetry for a particle constrained to move on a torus knot. We study the particle on a torus knot following the technique of Dirac’s constraints analysis \(^{[11]}\). The system is shown to contain 2nd-class constraints. We introduce the Wess-Zumino term to recast the system in a gauge invariant fashion in the extended Hilbert space. We further develop the BFV (Batalin-Fradkin-Vilkovisky) formulation of this extended theory using the constraints in the theory \(^{[12–15]}\). The nilpotent BRST and anti-BRST charges are constructed which generate the transformations using the constraints in the theory. These nilpotent BRST charges annihilate the states in the physical Hilbert space which is shown to be consistent with the constraints present in the theory. We indicate how various BRST invariant effective theories on the surface of a torus knot are interlinked by considering the finite-field–dependent version of the BRST (FFBRST) transformation, introduced by Joglekar and Mandal \(^{[16]}\) about 22 years ago. FFBRST transformations are the generalization of the usual BRST transformation where the usual infinitesimal, anti-commuting constant transformation parameter is replaced by a field-dependent but global and anti-commuting parameter. Such generalized transformation protects the nilpotency and retains
the symmetry of the gauge fixed effective actions. The remarkable property of such transformations are that they relate the generating functional corresponding to different effective actions. The non-trivial Jacobian of the path integral measure under such a finite transformation is responsible for all the new results. In virtue of this remarkable property, FFBRST transformations have been investigated extensively and have found many applications in various gauge field theoretic systems [17–31]. A similar generalization of the BRST transformation with the same remarkable property, FFBRST transformations have been responsible for all the new results. In virtue of this remarkable property of such transformations are that they relate the generating functional corresponding to different effective actions. The non-trivial Jacobian of the path integral measure under such a finite transformation is responsible for all the new results. 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which is invariant under the following time-dependent gauge transformations:

\[ \delta \lambda = \dot{f}(t), \quad \delta \theta = -\frac{f(t)}{2p}, \quad \delta \phi = \frac{f(t)}{2q}, \quad \delta \alpha = -f(t), \]
\[ \delta p_\eta = \delta p_\theta = \delta p_\phi = 0, \quad \delta b = \delta \Pi_\alpha = \delta \eta = \delta \Pi_\lambda = 0, \quad (11) \]

where \( f(t) \) is an arbitrary function of time. To construct the Hamiltonian for this gauge invariant theory we construct the canonical momenta corresponding to this modified Lagrangian and they are written as

\[ p_\eta = \frac{ma^2}{(\cosh \eta - \cos(\theta - \frac{\alpha}{2p}))^2} \dot{\eta}, \quad \Pi_\lambda = 0, \]
\[ p_\theta = \frac{ma^2}{(\cosh \eta - \cos(\theta - \frac{\alpha}{2p}))^2} \left( \dot{\theta} - \frac{\dot{\alpha}}{2p} \right), \]
\[ p_\phi = \frac{ma^2 \sinh^2 \eta}{(\cosh \eta - \cos(\theta - \frac{\alpha}{2p}))^2} \left( \dot{\phi} - \frac{\dot{\alpha}}{2q} \right), \]
\[ \Pi_\alpha = \frac{ma^2}{2(\cosh \eta - \cos(\theta - \frac{\alpha}{2p}))^2} \left( \frac{\dot{\alpha}}{2} \left( \frac{1}{p^2} + \sinh^2 \eta \right) \right. \]
\[ + \left. \left( \frac{\dot{\theta}}{p} + \sinh \eta \frac{\dot{\phi}}{q} \right) \right) - \left( p \theta + q \phi - \alpha \right). \quad (12) \]

The only primary constraint for this extended theory is

\[ \Psi_1 \equiv \Pi_\lambda \approx 0. \quad (13) \]

The Hamiltonian corresponding to the Lagrangian \( L' \) is written as

\[ H' = p_\eta \dot{\eta} + p_\theta \dot{\theta} + p_\phi \dot{\phi} + \Pi_\alpha \dot{\alpha} - L'. \quad (14) \]

The total Hamiltonian after using the Lagrange multiplier \( \beta \) corresponding to the primary constraint \( \Pi_\lambda \) is obtained as

\[ H_T = \frac{(\cosh \eta - \cos(\theta - \frac{\alpha}{2p}))^2}{2ma^2} \left[ p_\eta^2 + p_\theta^2 + \frac{p_\phi^2}{\sinh^2 \eta} \right] \]
\[ - \lambda \left( \Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q} \right) + \beta \Pi_\lambda. \quad (15) \]

Using Dirac’s method of constraint analysis [11], we obtain the secondary constraint

\[ \Psi_2 \equiv \left( \Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q} \right) \approx 0. \quad (16) \]

There is no tertiary constraint corresponding to this total Hamiltonian as

\[ \Psi_3 = \dot{\Psi}_2 = \left[ H_T', \left( \Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q} \right) \right] = 0. \quad (17) \]

This extended theory thus has only first-class constraints.

**BFV formalism for a torus knot.** – To discuss all possible nilpotent symmetries we further extend the theory using BFV formalism [12–15]. In the BFV formulation associated with this system, we introduce a pair of canonically conjugate ghost fields \((c, \bar{c})\) with ghost number 1 and \(-1\), respectively, for the primary constraint \( \Pi_\lambda \approx 0 \) and another pair of ghost fields \((\bar{c}, \bar{p})\) with ghost number \(-1 \) and \(1\), respectively, for the secondary constraint, \((\Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q}) \approx 0 \). The effective action for a particle on the surface of the torus knot in extended phase space is then written as

\[ S_{\text{eff}} = \int dt \left[ p_\eta \dot{\eta} + p_\theta \dot{\theta} + p_\phi \dot{\phi} + \Pi_\alpha \dot{\alpha} - \Pi_\lambda \dot{\lambda} \right. \]
\[ - \frac{(\cosh \eta - \cos(\theta - \frac{\alpha}{2p}))^2}{2ma^2} \left\{ p_\eta^2 + p_\theta^2 \right. \]
\[ + \left. \frac{p_\phi^2}{\sinh^2 \eta} \right\} + \dot{c}P + \dot{\bar{c}}\bar{P} - \{Q_\bar{b}, \psi \}, \quad (18) \]

where \(Q_\bar{b}\) is the BRST charge and has been constructed using the constraints of the system as

\[ Q_\bar{b} = ic \left( \Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q} \right) - i\bar{P}\Pi_\lambda. \quad (19) \]

The canonical brackets for all the dynamical variables are written as

\[ [\eta, p_\eta] = [\theta, p_\theta] = [\phi, p_\phi] = [\alpha, \Pi_\alpha] = [\lambda, \Pi_\lambda] = [\bar{c}, \bar{\psi}] = i; \quad [c, \bar{\psi}] = -i. \quad (20) \]

The nilpotent BRST transformation corresponding to this action is constructed using the relation \( s_\delta \Phi = -[Q_\bar{b}, \Phi] \) which is related to the infinitesimal BRST transformation as \( \delta_\Phi = s_\delta \Phi \lambda \). Here \( \lambda \) is the infinitesimal BRST parameter. Here the minus sign is for the bosonic variable and plus is for the fermionic variable. The BRST transformation for the particle on a torus knot is then written as

\[ s_b \lambda = \bar{P}, \quad s_b \theta = -\frac{c}{2p}, \quad s_b \phi = -\frac{c}{2q}, \quad s_b \alpha = -c, \]
\[ s_b p_\eta = s_b p_\theta = s_b p_\phi = 0, \quad s_b \bar{P} = \left( \Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q} \right), \]
\[ s_b \bar{c} = \Pi_\lambda = b, \quad s_b c = s_b b = s_b \Pi_\alpha = s_b \eta = s_b \Pi_\lambda = 0. \quad (21) \]

One can check that these transformations are nilpotent.

In BFV formulation the generating functional is independent of the gauge fixing fermion [12–14], hence we have the liberty to choose it in the convenient form as

\[ \Psi = p \lambda + \bar{c} \left( p \theta + q \phi + \alpha + \frac{\Pi_\lambda}{2} \right). \quad (22) \]
Using the expressions for $Q_b$ and $\Psi$, the effective action (18) is written as

$$S_{\text{eff}} = \int dt \left[ p_\eta \dot{\eta} + p_\theta \dot{\theta} + p_\phi \dot{\phi} + \Pi_\alpha \dot{\alpha} - \Pi_\lambda \dot{\lambda} - \frac{(\cosh \eta - \cos(\theta - \frac{2p}{2q}))^2}{2ma^2} \left( p_\eta^2 + p_\theta^2 + p_\phi^2 \right) \sinh^2 \eta \right] + \dot{\epsilon} P - \ddot{\epsilon} P - \frac{2\epsilon \eta}{2q} \left[ \Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q} \right] + 2\epsilon \Pi_\lambda \left( \dot{\theta} + q\phi + \alpha + \frac{\Pi_\lambda}{2} \right)$$

and the generating functional for this effective theory is represented as

$$Z_\psi = \int D\phi \, \exp \left[ i S_{\text{eff}} \right]$$

The measure $D\phi = \prod \, d\xi$, where $\xi$ are all the dynamical variables of the theory. Now integrating this generating functional over $P$ and $\dot{P}$, we get

$$Z_\psi = \int D\phi' \exp \left[ i \int dt \left[ p_\eta \dot{\eta} + p_\theta \dot{\theta} + p_\phi \dot{\phi} + \Pi_\alpha \dot{\alpha} - \Pi_\lambda \dot{\lambda} \right] - \frac{(\cosh \eta - \cos(\theta - \frac{2p}{2q}))^2}{2ma^2} \left( p_\eta^2 + p_\theta^2 + p_\phi^2 \right) \sinh^2 \eta \right] + \dot{\epsilon} c + \lambda \left( \Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q} \right) - 2\epsilon c + \Pi_\lambda \left( \dot{\theta} + q\phi + \alpha + \frac{\Pi_\lambda}{2} \right) \right]$$

where $D\phi'$ is the path integral measure for the effective theory when integrations over fields $P$ and $\dot{P}$ are carried out. Further integrating over $\Pi_\lambda$ we obtain an effective generating functional as

$$Z_\psi = \int D\phi'' \exp \left[ i \int dt \left[ p_\eta \dot{\eta} + p_\theta \dot{\theta} + p_\phi \dot{\phi} + \Pi_\alpha \dot{\alpha} \right] - \frac{(\cosh \eta - \cos(\theta - \frac{2p}{2q}))^2}{2ma^2} \left( p_\eta^2 + p_\theta^2 + p_\phi^2 \right) \sinh^2 \eta \right] + \dot{\epsilon} c + \lambda \left( \Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q} \right) - 2\epsilon c - \frac{(\lambda - (p\theta + q\phi + \alpha))^2}{2} \right]$$

where $D\phi''$ is the path integral measure corresponding to all the dynamical variables involved in the effective action. The BRST symmetry transformation for this effective theory is written as

$$s_b \theta = -\frac{\epsilon}{2p}, \quad s_b \phi = -\frac{\epsilon}{2q}, \quad s_b \alpha = -\epsilon,$n

$$s_b \eta = s_b \theta = s_b \phi = 0, \quad s_b c = -\left( \lambda - (p\theta + q\phi + \alpha) \right), \quad s_b \lambda = \dot{\epsilon}, \quad s_b c = s_b \Pi_\alpha = s_b \eta = s_b \Pi_\lambda = 0.$$

It is straightforward to check the nilpotency ($s_b^2 = 0$) of these transformations. Equations of motion for $\dot{c}$ are used to show the nilpotency of the $c$ field.

**BRST and anti-BRST charge.** – In this section we show that the physical subspace of the system is consistent with the constraints of the system. The physical states are annihilated by the BRST charge in eq. (19)

$$Q_b |\psi\rangle = 0 = \left\{ i c \left( \Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q} \right) - i c \Pi_\lambda \right\} |\psi\rangle =$$

$$\left\{ i c \left( \Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q} \right) |\psi\rangle - i c \Pi_\lambda |\psi\rangle \right\}. \quad (28)$$

This implies that

$$\left( \Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q} \right) |\psi\rangle = 0, \quad \Pi_\lambda |\psi\rangle = 0. \quad (29)$$

The Hamiltonian (15) is also invariant under anti-BRST transformation in which the role of $c$ and $\dot{c}$ is interchanged. Anti-BRST transformations for this theory are written as

$$s_{ab} \theta = \frac{\epsilon}{2p}, \quad s_{ab} \phi = \frac{\epsilon}{2q}, \quad s_{ab} \alpha = \dot{\epsilon} \frac{\epsilon}{2q}, \quad s_{ab} \lambda = -\dot{\epsilon}, \quad s_{ab} c = s_{ab} b = s_{ab} \Pi_\alpha = s_{ab} \eta = s_{ab} \Pi_\lambda = 0.$$

It is straightforward to check the nilpotency ($s_{ab}^2 = 0$) of these transformations. Equations of motion for $c$ are used to show the nilpotency of the $\dot{c}$ field.

The nilpotent charge for the anti-BRST symmetry in (30) is constructed as

$$Q_{ab} = -i c \left( \Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q} \right) + i c \Pi_\lambda. \quad (31)$$

Like BRST charge, anti-BRST charges $Q_{ab}$ also generate the anti-BRST transformations in (30) through the following commutation and anti-commutation relations:

$$s_{ab} \theta = -[Q_{ab}, \theta] = \frac{\epsilon}{2p}, \quad s_{ab} \phi = -[Q_{ab}, \phi] = \frac{\epsilon}{2q}, \quad s_{ab} \alpha = -[Q_{ab}, \alpha] = \dot{\epsilon}, \quad s_{ab} \lambda = -[Q_{ab}, \lambda] = -\dot{\epsilon}, \quad s_{ab} c = -[Q_{ab}, c] = -\Pi_\lambda. \quad (32)$$

Anti-BRST charge too annihilates the states of the physical Hilbert space:

$$Q_{ab} |\psi\rangle = 0,$n

$$-i c \left( \Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q} \right) + i c \Pi_\lambda |\psi\rangle = 0, \quad (33)$$

or

$$\left( \Pi_\alpha + \frac{p_\theta}{2p} + \frac{p_\phi}{2q} \right) |\psi\rangle = 0, \quad \Pi_\lambda |\psi\rangle = 0. \quad (34)$$
The anti-BRST charge too projects on the physical subspace of the total Hilbert space. Thus the anti-BRST charge plays exactly the same role as the BRST charge. It is straightforward to check that these charges are nilpotent, i.e., \( Q_b^2 = 0 \) and satisfy
\[
\{ Q_b, Q_{ab} \} = 0.
\]

**FFBRST for a torus knot.** – In this section we show that these nilpotent symmetries can be generalized by making the parameter finite and field dependent following the work of Joglekar and Mandal [17]. The BRST transformations are generated from the BRST charge using the relation \( \delta_b \phi = -[Q_b, \phi] \delta \Lambda \), where \( \delta \Lambda \) is an infinitesimal anti-commuting global parameter. Following their technique, the anti-commuting BRST parameter \( \delta \Lambda \) is generalized to be finite field dependent instead of using the infinitesimal but space-time-independent parameter \( \Theta \). Since the parameter is finite in nature unlike in the usual case, the path integral measure is not invariant under such finite transformation. The Jacobian for these transformations for a certain \( \Theta[\phi] \) can be calculated in the following way:
\[
D \phi = J(k) D \phi'(k) = J(k + dk) D \phi'(k + dk),
\]
where a numerical parameter \( k \) (\( 0 \leq k \leq 1 \)) has been introduced to execute the finite transformation in a mathematically convenient way. All the fields are taken to be \( k \)-dependent in such a fashion that \( \phi(x, 0) = \phi(x) \) and \( \phi(x, k = 1) = \phi'(x) \). \( S_{eff} \) is invariant under FFBRST which is constructed by considering successive infinitesimal BRST transformations \( (\phi(k) \rightarrow \phi(k + dk)) \). The non-trivial Jacoiban \( J(k) \) can be written as local functional of fields and will be replaced as \( e^{i S_1(\phi(k), k)} \) if the condition [17]
\[
\int D\phi(k) \left[ \frac{1}{J(k)} \frac{dJ(k)}{dk} - i \frac{dS_1}{dk} \right] e^{i(S_1 + S_{\phi})} = 0 \quad (37)
\]
holds, where \( \frac{dS_1}{dk} \) is a total derivative of \( S_1 \) with respect to \( k \) in which the dependence on \( \phi(k) \) is also differentiated. The change in the Jacobian is calculated as
\[
\frac{J(k)}{J(k + dk)} = \Sigma_{\phi} \pm \frac{\delta \phi(x, k + dk)}{\delta \phi(x, k)} = 1 + \frac{1}{J(k)} \left( \frac{dJ(k)}{dk} \right) dk,
\]
where \( \Sigma_{\phi} \pm = \) is for bosonic and fermionic fields (\( \phi \)), respectively. We know that the effective action for a particle on the surface of a torus knot using BFV formulation is written in (18) and the BRST transformation is given by (21). The finite version of this BRST is then written as
\[
\delta_b \lambda = \hat{P} \Theta, \quad \delta_b \theta = -\frac{c}{2p} \Theta, \quad \delta_b \phi = -\frac{c}{2q} \Theta, \quad \delta_b \alpha = -c \Theta,
\]
\[
\delta_b p_\theta = \delta_b p_\phi = \delta_b p_\phi = 0, \quad \delta_b P = \left( \Pi_{\alpha} + \frac{p_\theta}{2p} + \frac{p_\phi}{2q} \right) \Theta,
\]
\[
\delta_b \bar{c} = \Pi_\lambda \Theta = b \Theta, \quad \delta_b b = \delta_b \Pi_{\alpha} = \delta_b \eta = \delta \Pi_{\lambda} = 0,
\]
where \( \Theta \) is a finite-field-dependent, global and anti-commuting parameter. It is straightforward to check that under this transformation too, the effective action in (23) is invariant. The generating functional for this effective theory is then written as
\[
Z_\psi = \int D\Phi \exp \left[ i \int dt \left( p_\eta \dot{\eta} + p_\phi \dot{\phi} + p_\phi \dot{\phi} + \Pi_{\alpha} \dot{\alpha} - \Pi_{\lambda} \lambda \right)
\]
\[
+ \left( \frac{\cosh \eta - \cos(\theta - \frac{\pi}{n})}{2 \alpha^2} \right)^2 \left( p_\phi^2 + p_\theta^2 + \frac{p_\phi^2}{\sinh^2 \eta} \right)
\]
\[
+ \dot{c} P + \dot{c} \dot{P} - \dot{P} P + \lambda \left( \Pi_{\alpha} + \frac{p_\theta}{2p} + \frac{p_\phi}{2q} \right)
\]
\[
- 2\alpha c + \Pi_\lambda \left( \frac{\theta + g \phi + \alpha + \frac{\Pi_\lambda}{2} \right),
\]
where
\[
\Phi = \prod d\eta dp_\theta dp_\phi d\phi d\Pi_{\alpha} d\Pi_\lambda dP d\dot{P} d\dot{c} \quad (41)
\]
is the path integral measure in the total phase space. This path integral measure is not invariant under such FFBRST transformation as already mentioned. It gives rise to a Jacobian in the extended phase space which is calculated using (38). Using the condition in (37), one can calculate the extra part in the action \( S_1 \) for some specific choices of the finite parameter \( \Theta \).

Now we consider a simple example of FFBRST to show the connection between two effective theories explicitly. For that we choose the finite BRST parameter \( \Theta = \int dk \Theta(k) \) where \( \Theta \) is given as
\[
\Theta = i \gamma \int dt' c(y, k) \Pi_\lambda(y, k).
\]
The change in the Jacobian is calculated for this particular parameter as
\[
\frac{1}{J(k)} \frac{dJ(k)}{dk} = -i \gamma \int dt' \Pi_\lambda^2(y, k).
\]
We make an ansatz for \( S_1 \) as
\[
S_1 = i \int dt \xi_1(k) \Pi_\lambda^2,
\]
where \( \xi_1(k) \) is a \( k \)-dependent arbitrary parameter. Now,
\[
\frac{dS_1}{dk} = i \int dt \xi'_1(k) \Pi_\lambda^2,
\]
for the non-trivial Jacobian \( J(k) \) and the transformation is given by (21). The finite
By satisfying the condition in (37) we find $\xi_1 = -\gamma k$. The FFBRST with finite parameter $\Theta$ as given in (42) changes this generating functional as

$$Z = \int D\phi(k)e^{i(S_1 + S_{1ff})}$$

$$= \int D\Phi \exp \left[ i \int dt \left( p_\theta \dot{\phi} + p_\phi \dot{\theta} + \Pi_\theta \dot{\phi} + \Pi_\phi \dot{\theta} - \Pi_\lambda \lambda - \Pi_{\lambda'} \lambda' - \Pi_{\lambda''} \lambda'' \right) + \frac{(\cosh \eta - \cos(\theta - \frac{\pi k}{2}))^2 }{2ma^2} \left( \frac{p_\theta^2 + p_\phi^2 + \gamma^2}{\sinh^2 \eta} \right) + \left( \left( \frac{\lambda'}{2} - \gamma k \right) \right) \right]. \quad (46)$$

Here the generating functional at $k = 0$ will give a pure theory for a free particle on the surface of a torus knot with a gauge parameter $\lambda'$ and at $k = 1$, the generating functional for same theory with a different gauge parameter $\lambda'' = \lambda' - 2\gamma$. Even though we have considered a very simple example, our formulation is valid to connect any two generating functionals corresponding to different effective theories on the surface of the torus knot. We have further extended the BRST formulation by considering the trans-connection between any two effective theories can be made in a straightforward manner following the prescriptions outlined in this work.

**Conclusion.** – The mathematical concept of the knot theory is very useful in describing various physical systems and it has been extensively used to study many different phenomena in physics. However, there was no BRST formulation for a particle on the surface of a torus knot. In this work we systematically developed the BRST/anti-BRST formulation for the first time for a particle moving on a torus knot. Using Dirac’s constraint analysis we found all the constraints of this system. Further, we have extended this theory to include the Wess-Zumino term to recast this theory as gauge theory. Using BFV formulation, the BRST/anti-BRST invariant effective action for a particle moving on a torus knot has been developed. Nilpotent charges which generate these symmetries have been calculated explicitly. The physical states which are annihilated by these nilpotent charges are consistent with the constraints of the system. Our formulation is independent of a particular choice of a torus knot. We have further extended the BRST formulation by considering the transformation parameter finite and field dependent. We indicate how different effective theories on the surface of the torus knot are related through such a finite transformation through the non-trivial Jacobian factor. In support of our result we explicitly relate the generating functionals of two effective theories with different gauge fixing parameters. Using FFBRST with a suitable finite parameter the connection between any two effective theories can be made in a straightforward manner following the prescriptions outlined in this work.

One of us (VKP) acknowledges the University Grant Commission (UGC), India, for its financial assistance under the CSIR-UGC JRF/SRF scheme.

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