Rigid equivalences of 5-dimensional 2-nondegenerate rigid real hypersurfaces $M^5 \subset \mathbb{C}^3$ of constant Levi rank 1

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ABSTRACT. We study the local equivalence problem for real-analytic ($\mathcal{C}^\infty$) hypersurfaces $M^5 \subset \mathbb{C}^3$ which, in some holomorphic coordinates $(z_1, z_2, w) \in \mathbb{C}^3$ with $w = u + iv$, are rigid in the sense that their graphing functions:

$$u = F(z_1, z_2, \bar{z}_1, \bar{z}_2)$$

are independent of $v$. Specifically, we study the group $\text{Hol}_{\text{rigid}}(M)$ of rigid local biholomorphic transformations of the form:

$$\begin{pmatrix} z_1, z_2, w \end{pmatrix} \mapsto \begin{pmatrix} f_1(z_1, z_2), f_2(z_1, z_2), aw + g(z_1, z_2) \end{pmatrix},$$

where $a \in \mathbb{R}\{0\}$ and $D(f_1, f_2) \neq 0$, which preserve rigidity of hypersurfaces.

After performing a Cartan-type reduction to an appropriate $\{e\}$-structure, we find exactly two primary invariants $I_0$ and $V_0$, which we express explicitly in terms of the 5-jet of the graphing function $F$ of $M$. The identical vanishing $0 \equiv I_0(J^5F) \equiv V_0(J^5F)$ then provides a necessary and sufficient condition for $M$ to be locally rigidly-biholomorphic to the known model hypersurface:

$$M_{LC} : u = \frac{z_1 \bar{z}_1 + \frac{1}{2}z_2^2 \bar{z}_2 + \frac{1}{4}z_1^2 \bar{z}_2}{1 - z_2 \bar{z}_2}.$$ 

We establish that $\dim \text{Hol}_{\text{rigid}}(M) \leq 7 = \dim \text{Hol}_{\text{rigid}}(M_{LC})$ always.

If one of these two primary invariants $I_0 \neq 0$ or $V_0 \neq 0$ does not vanish identically, then on either of the two Zariski-open sets $\{ p \in M : I_0(p) \neq 0 \}$ or $\{ p \in M : V_0(p) \neq 0 \}$, we show that this rigid equivalence problem between rigid hypersurfaces reduces to an equivalence problem for a certain 5-dimensional $\{e\}$-structure on $M$, that is, we get an invariant absolute parallelism on $M^5$. Hence $\dim \text{Hol}_{\text{rigid}}(M)$ drops from 7 to 5, illustrating the gap phenomenon.

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1. Introduction

In appropriate affine coordinates \((z_1, z_2, w) \in \mathbb{C}^3\) with \(w = u + iv\), a real-analytic \((\mathcal{C}^\omega)\) real hypersurface \(M^5 \subset \mathbb{C}^3\) may locally be represented as the graph of a \(\mathcal{C}^\omega\) function \(F\) over the 5-dimensional real hyperplane \(\mathbb{C}z_1 \times \mathbb{C}z_2 \times \mathbb{R}\). When \(F\) is independent of \(v\):

\[
M : \quad u = F(z_1, z_2, \bar{z}_1, \bar{z}_2),
\]

the hypersurface is called rigid.

Its fundamental CR-bundle:

\[
T^{1,0}M := (\mathbb{C} \otimes_{\mathbb{R}} TM) \cap T^{1,0}\mathbb{C}^3
\]

is of complex rank 2 = \(\text{CRdim} M\), as well as its conjugate \(T^{0,1}M = \overline{T^{1,0}M}\).

Relevant foundational material for CR geometry focused on the local biholomorphic equivalence problem of \(\mathcal{C}^\omega\) CR submanifolds \(M \subset \mathbb{C}^N\) has been set up in the memoir [11], to which readers will be referred for details.

The Levi forms at various points \(p \in M\) are maps measuring Lie bracket non-involutivity [11, p. 45]:

\[
T^{1,0}_p M \times T^{1,0}_p M \rightarrow \mathbb{C} \otimes_{\mathbb{R}} T_p M \quad \mod (T^{1,0}_p M \oplus T^{0,1}_p M),
\]

where \(\mathcal{M}\) and \(\mathcal{N}\) are any two local sections of \(T^{1,0}M\) defined near \(p\) which extend \(\mathcal{M}_p = \mathcal{M}|_p \) and \(\mathcal{N}_p = \mathcal{N}|_p\), the result being independent of extensions.

Levi forms are known to be biholomorphically invariant. In terms of two natural intrinsic generators for \(T^{1,0}M\):

\[
\mathcal{L}_1 := \frac{\partial}{\partial z_1} - iFz_1 \frac{\partial}{\partial v} \quad \text{and} \quad \mathcal{L}_2 := \frac{\partial}{\partial z_2} - iFz_2 \frac{\partial}{\partial v},
\]

the Levi forms at all points \(p \in M\) identify with the matrix-valued map:

\[
\mathcal{L}_F^M(p) := 2 \begin{pmatrix} F_{z_1\bar{z}_1} & F_{z_2\bar{z}_1} \\ F_{z_1\bar{z}_2} & F_{z_2\bar{z}_2} \end{pmatrix}(p).
\]

Throughout this article, we will make two main (invariant) assumptions. The first one is that the rank of \(\mathcal{L}_F^M(p)\) be constant equal to 1 at every point \(p \in M\).

Since \(2 = \text{rank} T^{1,0}M\), this implies that there is a rank 1 Levi kernel subbundle:

\[
K^{1,0}M \subset T^{1,0}M,
\]

which is generated by the vector field:

\[
\mathcal{K} := k \mathcal{L}_1 + \mathcal{L}_2,
\]

incorporating the slant function:

\[
k := -\frac{F_{z_2\bar{z}_1}}{F_{z_1\bar{z}_1}}.
\]

Indeed, a direct check convinces that both \([\mathcal{K}, \overline{\mathcal{L}}_1]\) and \([\mathcal{K}, \overline{\mathcal{L}}_2]\) vanish modulo \(T^{1,0}M \oplus T^{0,1}M\). The known involutivity properties of the Levi kernel subbundle \(K^{1,0}M \subset T^{1,0}M\)
Its Lie algebra, obtained by differentiating the holomorphisms, is:

\[ [K^{1,0}M, K^{1,0}M] \subset K^{1,0}M, \]
\[ [K^{0,1}M, K^{0,1}M] \subset K^{0,1}M, \]
\[ [K^{1,0}M, K^{0,1}M] \subset K^{1,0}M \oplus K^{0,1}M. \]

Another fundamental function will also be needed in a while:

\[ P := \frac{F_{z_1 \overline{z}_1}}{F_{z_2 \overline{z}_2}}. \]

All this justifies the introduction of the so-called Freeman form ([11, p. 89]):

\[ K_p^{1,0}M \times (T_p^{1,0}M \mod K_p^{1,0}M) \longrightarrow T_p^{1,0}M \oplus T_p^{0,1}M \mod (K_p^{1,0}M \oplus T_p^{0,1}M), \]
\[ (\mathcal{K}_p, \mathcal{L}_p) \longmapsto [\mathcal{K}, \overline{\mathcal{L}}]_p \mod (K_p^{1,0}M \oplus T_p^{0,1}M), \]

where \( \mathcal{K} \) and \( \mathcal{L} \) are any two local sections of \( K^{1,0}M \) and of \( T^{1,0}M \) defined near \( p \) which extend \( \mathcal{K}_p = \mathcal{K}|_p \) and \( \mathcal{L}_p = \mathcal{L}|_p \), the result being independent of extensions. In bases, these Freeman forms at various points \( p \in M \) are simply maps \( \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C} \). They are known to be biholomorphically invariant.

Our second main (invariant) assumption will be that the rank of the Freeman form be maximal equal to 1 at every point \( p \in M \). Such \( M \) are called 2-nondegenerate at \( p \).

A computation:

\[ [\mathcal{K}, \overline{\mathcal{L}}_1] = [k \mathcal{L}_1 + \mathcal{L}_2, \overline{\mathcal{L}}_1] = -\overline{\mathcal{L}}_1(k) \mathcal{L}_1 + k [\mathcal{L}_1, \overline{\mathcal{L}}_1] + [\mathcal{L}_2, \overline{\mathcal{L}}_1], \]

shows that \( M \) is 2-nondegenerate at \( p \in M \) \iff \( \overline{\mathcal{L}}_1(k)(p) \neq 0 \).

Next, for a \( C^\omega \) hypersurface \( M^5 \subset \mathbb{C}^3 \), define the Lie pseudogroup:

\[ \text{Hol}_{\text{rigid}}(M) := \left\{ h : M \longrightarrow M \text{ local rigid biholomorphism} \right\}. \]

Its Lie algebra, obtained by differentiating 1-parameter local groups of rigid biholomorphisms, is:

\[ \text{Lie}(\text{Hol}_{\text{rigid}}(M)) = \mathfrak{hol}_{\text{rigid}}(M) \]
\[ := \left\{ X = A_1(z_1, z_2) \frac{\partial}{\partial z_1} + A_2(z_1, z_2) \frac{\partial}{\partial z_2} + (\alpha w + B(z_1, z_2)) \frac{\partial}{\partial w} : \right. \]
\[ \left. (X + \overline{X})|_M \text{ is tangent to } M \right\}, \]

where \( A_1, A_2, B \) are holomorphic functions of only \( (z_1, z_2) \), and where \( \alpha \in \mathbb{R} \).

Our first result is the elementary

**Proposition 1.1.** For the model hypersurface:

\[ M_{\text{LC}} : \quad u = \frac{z_1 \overline{z}_1 + \frac{1}{2} z_1^2 \overline{z}_2 + \frac{1}{2} z_1^2 z_2}{1 - z_2 \overline{z}_2}, \]
the Lie algebra $\mathfrak{hol}_{\text{rigid}}(M_{\text{LC}})$ of infinitesimal biholomorphisms is 7-dimensional, generated by:

\[
\begin{align*}
X^1 &= i\partial_w, \\
X^2 &= z_1\partial_{z_1} + 2w\partial_w, \\
X^3 &= iz_1\partial_{z_1} + 2iz_2\partial_{z_2}, \\
X^4 &= (z_2 - 1)\partial_{z_1} - 2z_1\partial_w, \\
X^5 &= (i + iz_2)\partial_{z_1} - 2iz_1\partial_w, \\
X^6 &= z_1z_2\partial_{z_1} + (z_2^2 - 1)\partial_{z_2} - z_1^2\partial_w, \\
X^7 &= iz_1z_2\partial_{z_1} + (iz_2^2 + i)\partial_{z_2} - iz_1^2\partial_w.
\end{align*}
\]

Next, we conduct the Cartan process for rigid biholomorphic equivalences to this model $M_{\text{LC}}$, reaching a representation of a Lie algebra isomorphic to the dual of the one generated by $X^1, \ldots, X^7$.

**Theorem 1.2.** A basis for the Maurer-Cartan forms on the local Lie group $\text{Hol}_{\text{rigid}}(M_{\text{LC}})$ is provided by 7-differential 1-forms:

\[
\{\rho, \kappa, \zeta, \bar{\rho}, \bar{\kappa}, \bar{\zeta}, \alpha, \bar{\alpha}\},
\]

where $\bar{\rho} = \rho$ is real, which enjoys the 7 structure equations with constant coefficients:

\[
\begin{align*}
\frac{\partial \rho}{\partial z_1} &= (\alpha + \bar{\alpha}) \wedge \rho + i\kappa \wedge \bar{\rho}, \\
\frac{\partial \kappa}{\partial z_1} &= \alpha \wedge \kappa + \zeta \wedge \bar{\rho}, \\
\frac{\partial \zeta}{\partial z_1} &= (\alpha - \bar{\alpha}) \wedge \zeta, \\
\frac{\partial \alpha}{\partial z_1} &= \zeta \wedge \bar{\zeta}.
\end{align*}
\]

This preliminary study of the model $M_{\text{LC}}$ then constitutes our guiding map within the general problem. Recall that two fundamental functions expressed in terms of $F$ are:

\[
k := -\frac{F_{z_2\overline{z}_1}}{F_{z_1\overline{z}_1}} \quad \text{and} \quad P := \frac{F_{z_1\overline{z}_1\overline{z}_1}}{F_{z_1\overline{z}_1}}.
\]

**Theorem 1.3.** The equivalence problem under local rigid biholomorphisms of $C^\omega$ rigid real hypersurfaces $\{u = F(z_1, z_2, \overline{z}_1, \overline{z}_2)\}$ in $\mathbb{C}^3$ whose Levi form has constant rank 1 and which are everywhere 2-nondegenerate reduces to classifying $\{e\}$-structures on the 7-dimensional bundle $M^5 \times \mathbb{C}$ equipped with coordinates $(z_1, z_2, \overline{z}_1, \overline{z}_2, v, c, \bar{c})$ together with a coframe of 7 differential 1-forms:

\[
\{\rho, \kappa, \zeta, \bar{\rho}, \bar{\kappa}, \bar{\zeta}, \alpha, \bar{\alpha}\},
\]

which satisfy invariant structure equations of the shape:

\[
\begin{align*}
\frac{\partial \rho}{\partial z_1} &= (\alpha + \bar{\alpha}) \wedge \rho + i\kappa \wedge \bar{\rho}, \\
\frac{\partial \kappa}{\partial z_1} &= \alpha \wedge \kappa + \zeta \wedge \bar{\rho}, \\
\frac{\partial \zeta}{\partial z_1} &= (\alpha - \bar{\alpha}) \wedge \zeta + \frac{1}{c}I_0 \kappa \wedge \bar{\rho} + \frac{1}{cc}V_0 \kappa \wedge \bar{\rho}, \\
\frac{\partial \alpha}{\partial z_1} &= \zeta \wedge \bar{\zeta} - \frac{1}{c}I_0 \zeta \wedge \bar{\rho} + \frac{1}{cc}Q_0 \kappa \wedge \bar{\rho} + \frac{1}{cc}T_0 \zeta \wedge \kappa,
\end{align*}
\]

conjugate equations for $\frac{\partial \bar{\rho}}{\partial \overline{z}_1}$, $\frac{\partial \bar{\kappa}}{\partial \overline{z}_1}$, $\frac{\partial \bar{\alpha}}{\partial \overline{z}_1}$ being understood.
Theorem 1.4. A 2-nondegenerate Cartan equations of a uniquely defined Lie group. Hence as a corollary, we obtain the Cartan theory, the identical vanishing of all invariants provide constant coefficients Maurer-Cartan equations of a uniquely defined Lie group. Hence as a corollary, we obtain the

Corollary 1.6. All such rigid structure.

Because of the size of computations, we will not attempt to set up such an explicit proof is completed if one does not require to make explicit the

Theorem 1.5. Let \( M^5 \subset \mathbb{C}^3 \) be a local rigid 2-nondegenerate \( \mathcal{C}^\omega \) constant Levi rank 1 hypersurface. If either \( I_0 \neq 0 \) or \( V_0 \neq 0 \) everywhere on \( M \), the local rigid-biholomorphic equivalence problem reduces to an invariant 5-dimensional \( \{e\} \)-structure on \( M \).

In fact, once the last remaining group parameter \( c \in \mathbb{C}^* \) is seen to be normalizable from either:

\[
\frac{1}{c} J_0 = 1 \quad \text{or} \quad \frac{1}{c^2} V_0 = 1,
\]

the proof is completed if one does not require to make explicit the \( \{e\} \)-structure on \( M \). Because of the size of computations, we will not attempt to set up such an explicit \( \{e\} \)-structure.

Lastly, from general Cartan theory, we deduce the

Corollary 1.6. All such rigid \( M^5 \subset \mathbb{C}^3 \) that are not rigidly-biholomorphic to the model \( M_{\text{LC}} \) satisfy

\[
\dim \text{Hol}_{\text{rigid}}(M) \leq 5.
\]

In continuation with these results, a further task appears: to classify up to rigid biholomorphisms the ‘submaximal’ hypersurfaces with \( \dim \text{Hol}_{\text{rigid}}(M) = 5 \) whose rigid biholomorphic group is locally transitive. Another question would be to classify under rigid biholomorphisms those rigid \( M^5 \subset \mathbb{C}^3 \) that have identically vanishing Pocchiola invariants \( 0 \equiv W_0 = J_0 \), hence which are equivalent to \( M_{\text{LC}} \), but under a general biholomorphism, not necessarily rigid. Upcoming publications will be devoted to advances in these directions.
2. Recall on the geometry of CR real hypersurfaces

Let \((z_1, z_2, w)\) be holomorphic coordinates in \(\mathbb{C}^3\) with \(w = u + iv\), and let \(M^5 \subset \mathbb{C}^3\) be a real-analytic, real hypersurface passing through the origin. Assuming that the real hypersurface is smooth at the origin, and that the vector
\[
\left. \frac{\partial}{\partial u} \right|_0 \notin T_0 M
\]
does not lie in the vector subspace \(T_0 M \subset T_0 \mathbb{C}^3\). The implicit function theorem therefore implies the existence of a real analytic (denoted by \(C^\omega\)) graphing function such that \(M^5\) is represented near the origin by
\[
u = F(z_1, z_2, \bar{z}_1, \bar{z}_2, v).
\]

Définition 2.1. The smooth real hypersurface \(M^5 \subset \mathbb{C}^3\) is rigid at the origin if \(M^5\) may be represented by a graphing function \(u = F(z_1, z_2, \bar{z}_1, \bar{z}_2)\), where the function \(F\) is independent of \(v\).

Hypothesis 2.2. In the rest of the article, we will assume that \(M^5\) is rigid.

The complexified tangent bundle \(\mathbb{C}TM = TM \otimes \mathbb{R} \mathbb{C}\) inherits from \(\mathbb{C}T\mathbb{C}^3 = T\mathbb{C}^3 \otimes \mathbb{R} \mathbb{C}\) two biholomorphically invariant complex vector bundles
\[
T^{1,0}M := \mathbb{C}TM \cap T^{1,0}\mathbb{C}^3, \quad T^{0,1}M := \mathbb{C}TM \cap T^{0,1}\mathbb{C}^3 = \overline{T^{1,0}M}.
\]
The two vector fields
\[
\mathcal{L}_1 := \frac{\partial}{\partial z_1} + A^1 \frac{\partial}{\partial v} \quad \text{and} \quad \mathcal{L}_2 := \frac{\partial}{\partial z_2} + A^2 \frac{\partial}{\partial v},
\]
with
\[
A^1 := -iF_{z_1}, \quad \text{and} \quad A^2 := -iF_{z_2},
\]
then form a \(T^{1,0}M\) frame. The differential 1-form
\[
\rho_0 = dv + iF_{z_1} dz^1 + iF_{z_2} dz^2 - i\overline{F}_{\bar{z}_1} d\bar{z}^1 - i\overline{F}_{\bar{z}_2} d\bar{z}^2
\]
has the kernel
\[
\ker \rho_0 = \{ \rho_0 = 0 \} = T^{1,0}M \oplus T^{0,1}M.
\]
By a formula in Merker-Pocchiola-Sabzevari [11], page 82, the Levi matrix is shown to be
\[
\text{Levi}(M) = \begin{pmatrix}
\rho_0(i[\mathcal{L}_1, \overline{\mathcal{L}_1}]) & \rho_0(i[\mathcal{L}_2, \overline{\mathcal{L}_1}]) \\
\rho_0(i[\mathcal{L}_1, \overline{\mathcal{L}_2}]) & \rho_0(i[\mathcal{L}_2, \overline{\mathcal{L}_2}])
\end{pmatrix}
= 2 \begin{pmatrix}
F_{z_1 \bar{z}_1} & F_{z_2 \bar{z}_1} \\
F_{z_1 \bar{z}_2} & F_{z_2 \bar{z}_2}
\end{pmatrix},
\]
which is not identically zero if \(M\) is further assumed to be not Levi-flat. After a change of coordinates in the \((z_1, z_2)\) space, without loss of generality,
\[
\rho_0(i[\mathcal{L}_1, \overline{\mathcal{L}_2}]) = 2F_{z_1 \bar{z}_1} \neq 0
\]
everywhere on \(M\), and hence the vector field
\[
\mathcal{F} := i[\mathcal{L}_1, \overline{\mathcal{L}_1}] = 2F_{z_1 \bar{z}_1} \frac{\partial}{\partial v} := \ell \frac{\partial}{\partial v}
\]
vanishes nowhere on \(M\).
2.1. **The rank 1 hypothesis.** We will also make a further

**Hypothesis 2.4.** The smooth real-analytic (rigid) real-hypersurface $M$ is of constant Levi rank 1.

With this hypothesis, the collection of 1-dimensional kernels $K_{1,p}^M$ of the Levi form at all points $p \in M$ spans a real-analytic sub-distribution of the $T^{1,0}M$ bundle $K_{1,0}^M \subset T^{1,0}M$, satisfying the following inclusions

\[
[K_{1,0}^M, K_{1,0}^M] \subset K_{1,0}^M,
\]
\[
[K_{0,1}^M, K_{0,1}^M] \subset K_{0,1}^M,
\]
\[
[K_{1,0}^M, K_{0,1}^M] \subset K_{1,0}^M \oplus K_{0,1}^M.
\]

To construct a generator $\mathcal{K}$ of the Levi kernel, introduce a slant function $k$ satisfying

\[
\begin{pmatrix}
F_{z_1 \bar{z}_1} \\
F_{z_2 \bar{z}_2} \\
F_{z_1 \bar{z}_2}
\end{pmatrix}
\begin{pmatrix}
k_1 \\
1
\end{pmatrix}
= 0.
\]

The first equation then implies that $k = -\frac{F_{z_2 \bar{z}_1}}{F_{z_1 \bar{z}_1}}$ while the same $k$ satisfies the second equation $k F_{z_1 \bar{z}_2} + F_{z_2 \bar{z}_2} = 0$ trivially by using the vanishing determinant of the matrix. Then the Levi kernel sub-bundle $\mathcal{K} \subset T^{1,0}M$ is of complex rank 1 and is generated by the vector field $\mathcal{K} = k \mathcal{L}_1 + \mathcal{L}_2$.

The slant function enjoys the following property

**Proposition 2.5** (See Merker-Pocchiola-Sabzevari [11]). The smooth real-analytic (rigid) real hypersurface $M$ is 2-nondegenerate in the sense of Freeman if and only if $\overline{\mathcal{L}}_1(k) \neq 0$ everywhere on $M$.

In the rigid case, a direct calculation shows that

\[
\mathcal{L}_1(k) = -\frac{F_{z_2 \bar{z}_1} F_{z_2 \bar{z}_2} + F_{z_2 \bar{z}_1} F_{z_1 \bar{z}_2}}{(F_{z_1 \bar{z}_1})^2},
\]
\[
\overline{\mathcal{L}}_1(k) = -\frac{F_{z_1 \bar{z}_1} F_{z_2 \bar{z}_1} + F_{z_2 \bar{z}_1} F_{z_1 \bar{z}_2}}{(F_{z_1 \bar{z}_1})^2},
\]
\[
\mathcal{F}(k) = 0.
\]

Moreover, introduce the next fundamental function

\[
P = \frac{\ell_{z_1}}{\ell} = \frac{F_{z_1 \bar{z}_1 z_2}}{F_{z_1 \bar{z}_1}}.
\]
Lemma 2.7 (See Pocchiola [12] or Foo-Merker [3]). The following 3 functional identities hold on $M$:

\[ \mathcal{K}(\bar{k}) \equiv 0, \]
\[ \mathcal{K}(P) \equiv -P\mathcal{L}_1(k) - \mathcal{L}_1(\mathcal{L}_1(k)), \]
\[ \mathcal{K}(-P) \equiv -P\mathcal{L}_1^*(k) - \overline{\mathcal{L}_1}^*(\mathcal{L}_1(k)). \]

(2.8)

According to Pocchiola [12] page 37, there are 10 Lie bracket identities

\[ [\mathcal{T}, \mathcal{L}_1] \equiv -P\mathcal{T}, \]
\[ [\mathcal{T}, \mathcal{K}] \equiv \mathcal{L}_1(k)\mathcal{T} + 0, \]
\[ [\mathcal{T}, \overline{\mathcal{L}_1}] \equiv -\mathcal{P}\mathcal{T}, \]
\[ [\mathcal{L}_1, \mathcal{K}] \equiv \mathcal{L}_1(\bar{k})\mathcal{L}_1, \]
\[ [\mathcal{L}_1, \mathcal{K}] \equiv \mathcal{L}_1(\bar{k})\mathcal{L}_1, \]
\[ [\mathcal{K}, \mathcal{K}] \equiv 0, \]
\[ [\mathcal{K}, \mathcal{L}_1] \equiv \mathcal{L}_1(k)\mathcal{L}_1, \]
\[ [\mathcal{K}, \mathcal{L}_1] \equiv \mathcal{L}_1^*(k)\mathcal{L}_1, \]
\[ [\mathcal{L}_1, \mathcal{K}] \equiv \mathcal{L}_1^*(\bar{k})\mathcal{L}_1. \]

(2.9)

where the "+0" is deliberately added to show the difference from the general case. The following 1-forms

\[ \rho_0 = \frac{1}{\ell} (dv - A^1dz_1 - A^2dz_2 - \bar{A}^1d\bar{z}_1 - \bar{A}^2d\bar{z}_2), \]
\[ \kappa_0 = dz_1 - \bar{k}dz_2, \]
\[ \zeta_0 = dz_2, \]
\[ \bar{\kappa}_0 = d\bar{z}_1 - \bar{k}d\bar{z}_2, \]
\[ \bar{\zeta}_0 = d\bar{z}_2, \]

are, by a simple computation, dual to the corresponding vector fields $\mathcal{T}, \mathcal{L}_1, \mathcal{K}, \overline{\mathcal{L}_1}, \overline{\mathcal{K}}$. Using the Cartan-Lie formula which states that for any smooth vector fields $X, Y$ and any smooth 1-form $\omega$, one has

\[ d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]), \]

the initial Darboux-Cartan structure is therefore obtained:

\[ d\rho_0 = P\rho_0 \wedge \kappa_0 - \mathcal{L}_1(k) \rho_0 \wedge \zeta_0 + P\rho_0 \wedge \bar{\kappa}_0 - \overline{\mathcal{L}_1}^*(\bar{k}) \rho_0 \wedge \bar{\zeta}_0 + i\kappa_0 \wedge \bar{\kappa}_0, \]
\[ d\kappa_0 = -\mathcal{L}_1(k) \kappa_0 \wedge \zeta_0 + \overline{\mathcal{L}_1}^*(k) \zeta_0 \wedge \bar{\kappa}_0, \]
\[ d\zeta_0 = 0. \]

(2.11)

Here, conjugate equations for $d\overline{\kappa}_0$ and for $d\overline{\zeta}_0$ are not written, as they can be immediately deduced.

3. Initial $G$-structure for rigid equivalences

of rigid real hypersurfaces

Our objective is to study absolute parallelism of rigid equivalences of rigid real hypersurfaces using Cartan method. We introduce the

Définition 3.1. Two local rigid real hypersurfaces at the origin are rigidly equivalent if there exists a biholomorphic map of the form

\[ \varphi : (z_1, z_2, w) \mapsto (z'_1, z'_2, w') := (f(z_1, z_2), g(z_1, z_2), aw + h(z_1, z_2)), \]
for some $a \in \mathbb{R}^\times$, and local holomorphic functions $f$, $g$, $h$, transforming one hypersurface into the other.

To make sure that the definition makes sense, let $M'$ be another rigid real hypersurface in the target space of the form

$$w' + \bar{w}' = F'(z'_1, z'_2, \bar{z}'_1, \bar{z}'_2) = 0.$$ 

Then the pullback by $\varphi$ of the defining function is

$$0 = a\frac{w + \bar{w}}{2} + \left( \frac{1}{2} h(z_1, z_2) + \frac{1}{2} \bar{h}(\bar{z}_1, \bar{z}_2) - F'(f(z_1, z_2), g(z_1, z_2), \bar{f}(\bar{z}_1, \bar{z}_2), \bar{g}(\bar{z}_1 \bar{z}_2)) \right)$$

which is again a defining function of a rigid real hypersurface.

Since $\varphi$ is holomorphic, its differential $\varphi^*: \mathbb{C}T\mathbb{C}^3 \to \mathbb{C}T\mathbb{C}^3$ stabilises the holomorphic $(1, 0)$ and the anti-holomorphic $(0, 1)$ vector fields:

$$\varphi^* T^{1,0} M \subseteq T^{1,0} M,$$
$$\varphi^* T^{0,1} M \subseteq T^{0,1} M. \quad (3.2)$$

Furthermore, by the invariance of the Freeman forms, the pushforward maps $\varphi^*$ also respects the Levi kernel distributions

$$\varphi^* K^{1,0} M \subseteq K^{1,0} M.$$

Consequently, there exist functions $f'$, $c'$ and $e'$ on $M'$ such that

$$\varphi^*(\mathcal{K}) = f'\mathcal{K}',$$
$$\varphi^*(\mathcal{L}_1) = c'\mathcal{L}_1' + e'\mathcal{K}'. \quad (3.3)$$

The difference with the articles of Pocchiola [12], Merker-Pocchiola [10] and Foo-Merker [3] is that the rigid equivalence assumption made on the map $\varphi: M \to M'$ between two rigid real hypersurfaces greatly simplifies the initial $G$-structure, especially because $\varphi^* T$ is a multiple of $T'$ by a function that vanishes nowhere on $M'$. In fact, if $R(z'_1, z'_2, \bar{z}'_1, \bar{z}'_2, v')$ is any $\mathcal{C}^\infty$ function on $M'$, then by definition of the pushforward of a vector field,

$$\left( \varphi^* \mathcal{T} \right) \left( R(z'_1, z'_2, \bar{z}'_1, \bar{z}'_2, v') \right) = \mathcal{T}(R \circ \varphi)$$
$$= \ell \frac{\partial}{\partial v}(R(f(z_1, z_2), g(z_1, z_2), \bar{f}(\bar{z}_1, \bar{z}_2), \bar{g}(\bar{z}_1, \bar{z}_2), av + \Im(h(z_1, z_2))))$$
$$= a\ell \frac{\partial R}{\partial v'} \circ \varphi$$
$$= a \ell \frac{\partial}{\partial v'} \left( \frac{\partial}{\partial v'} \circ \varphi \right)_{(\mathcal{T}' R) \circ \varphi}$$
$$= a \ell \frac{\partial}{\partial v'} (\mathcal{T}' R) \circ \varphi,$$

whence

$$\varphi^* \mathcal{T} = a\ell \mathcal{T}'. \quad (3.4)$$
Hence, there exists a real-valued function $a'$ nowhere vanishing on $M'$ such that

$$\varphi_\ast \mathcal{T} = a' \mathcal{T}' .$$

In fact, this function is determined since

$$a' \mathcal{T}' = \varphi_\ast \mathcal{T} = \varphi_\ast (i_\mathcal{L}_1, \mathcal{I}_1)$$

$$= i_\varphi \mathcal{T}_1, \mathcal{T}_1 = i_\varphi \mathcal{T}_1, \mathcal{T}_1 = c' c' i_\varphi \mathcal{T}_1, \mathcal{T}_1 .$$

This implies that

$$a' = c' c' .$$

Summarising, we therefore have the following matrix

$$\varphi_\ast \begin{pmatrix} \mathcal{T} \\ \mathcal{L}_1 \\ \mathcal{K} \\ \mathcal{T}_1 \\ \mathcal{K}_1 \end{pmatrix} = \begin{pmatrix} c' c' & 0 & 0 & 0 & 0 \\ 0 & c' & e' & 0 & 0 \\ 0 & 0 & f' & 0 & 0 \\ 0 & 0 & 0 & c' & e' \\ 0 & 0 & 0 & 0 & \bar{f}' \end{pmatrix} \begin{pmatrix} \mathcal{T}' \\ \mathcal{L}'_1 \\ \mathcal{K}' \\ \mathcal{T}'_1 \\ \mathcal{K}'_1 \end{pmatrix} .$$

Taking transposition of the matrix, one obtains the pullback formula for the two coframes

$$\varphi_\ast \begin{pmatrix} \rho' \\ \kappa' \\ \zeta' \\ \kappa_0' \\ \zeta_0' \end{pmatrix} = \begin{pmatrix} c' c' & 0 & 0 & 0 & 0 \\ 0 & c' & 0 & 0 & 0 \\ 0 & e' & f' & 0 & 0 \\ 0 & 0 & 0 & c' & 0 \\ 0 & 0 & 0 & 0 & \bar{e}' \bar{f}' \end{pmatrix} \begin{pmatrix} \rho \\ \kappa_0 \\ \zeta_0 \\ \kappa_0' \\ \zeta_0' \end{pmatrix} .$$

In conclusion, for the rigid CR transformation between rigid CR real hypersurfaces, the initial $G$-structure is constituted by the following 5 by 5 matrices

$$\begin{pmatrix} c\bar{c} & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 \\ 0 & e & f & 0 & 0 \\ 0 & 0 & 0 & \bar{c} & 0 \\ 0 & 0 & 0 & \bar{e} & \bar{f} \end{pmatrix}$$

with the free complex variables

$$c, f \in \mathbb{C} - \{0\}, \quad \text{and} \quad e \in \mathbb{C} .$$

4. Cartan equivalence method for the model case

Before starting the Cartan equivalence method for rigid equivalences of $C^\omega$ smooth rigid real hypersurfaces, a study of the equivalence method for the model case is necessary to obtain a model $\{e\}$-structure, which will serve as a reference for the general case. Recall that the model case is the tube over the future light cone, denoted by Pocchiola’s notation as MLC, is locally defined by the following rigid equation

$$u = \frac{z_1 \bar{z}_1 + \frac{1}{2} z_1^2 \bar{z}_2 + \frac{1}{2} \bar{z}_1^2 z_2}{1 - z_2 \bar{z}_2} .$$
The vector fields $\mathcal{L}_1$, $\mathcal{K}$, $\mathcal{L}_2$, $\mathcal{K}$, $\mathcal{T}$, which constitute a frame for the complexified tangent bundle of $M_{1C}$, thus have the following expressions

\[
\mathcal{L}_1 = \frac{\partial}{\partial z_1} - i \frac{\bar{z}_1 + z_1 \bar{z}_2}{1 - z_2 \bar{z}_2} \frac{\partial}{\partial v}, \\
\mathcal{K} = \frac{\bar{z}_1 + z_1 \bar{z}_2}{1 - z_2 \bar{z}_2} \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \frac{i}{2} \frac{\bar{z}_1^2 + 2z_1 \bar{z}_1 \bar{z}_2 + z_1^2 \bar{z}_2^2}{(1 - z_2 \bar{z}_2)^2} \frac{\partial}{\partial v}, \\
\mathcal{T} = -\frac{2}{1 - z_2 \bar{z}_2} \frac{\partial}{\partial v},
\]  

(4.1)

and the slant function is given by

\[
k = -\frac{\bar{z}_1 + z_1 \bar{z}_2}{1 - z_2 \bar{z}_2}.
\]

The initial coframe according to Pocchiola (model case) \[13\] has the form

\[
\rho_0 = -\frac{i}{2} (\bar{z}_1 + z_1 \bar{z}_2) \, dz_1 - \frac{i}{4} \frac{\bar{z}_1^2 + 2z_1 \bar{z}_1 \bar{z}_2 + z_1^2 \bar{z}_2^2}{1 - z_2 \bar{z}_2} \, dz_2 \\
+ \frac{i}{2} (z_1 + \bar{z}_1 \bar{z}_2) \, dz_2 + \frac{i}{4} \frac{z_1^2 + 2z_1 \bar{z}_1 \bar{z}_2 + \bar{z}_1^2 \bar{z}_2^2}{1 - z_2 \bar{z}_2} \, dz_2 + \frac{1}{2} (-1 + z_2 \bar{z}_2) \, dv, \\
\kappa_0 = dz_1 + \frac{\bar{z}_1 + z_1 \bar{z}_2}{1 - z_2 \bar{z}_2} \, dz_2, \\
\zeta_0 = dz_2,
\]  

(4.2)

which then satisfy the following structure equations

\[
d\rho_0 = \frac{\bar{z}_2}{1 - z_2 \bar{z}_2} \, \rho_0 \wedge \zeta_0 + \frac{z_2}{1 - z_2 \bar{z}_2} \, \rho_0 \wedge \bar{\zeta}_0 + i \kappa_0 \wedge \bar{\kappa}_0, \\
d\kappa_0 = \frac{\bar{z}_2}{1 - z_2 \bar{z}_2} \, \kappa_0 \wedge \zeta_0 - \frac{1}{1 - z_2 \bar{z}_2} \, \zeta_0 \wedge \bar{\kappa}_0, \\
d\zeta_0 = 0.
\]  

(4.3)

In the case of rigid biholomorphisms as previously explained, the transformation group, denoted by $g$, acts on the coframe $(\rho_0, \kappa_0, \zeta_0)$ by the matrix

\[
g = \begin{pmatrix} c \bar{c} & 0 & 0 \\ 0 & c & 0 \\ 0 & e & f \end{pmatrix}
\]

while ignoring the $T^{0,1} M$ counterpart. Its inverse

\[
g^{-1} = \begin{pmatrix} \frac{1}{cc} & 0 & 0 \\ 0 & \frac{1}{c} & 0 \\ 0 & -\frac{e}{cf} & \frac{1}{f} \end{pmatrix}
\]

provides the following Maurer-Cartan matrix of 1-forms

\[
dg \cdot g^{-1} = \begin{pmatrix} \alpha + \bar{\alpha} & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & \delta & \varepsilon \end{pmatrix},
\]
where the 1-forms \( \alpha, \delta, \) and \( \varepsilon \) take on the following expressions

\[
\alpha = \frac{dc}{c}, \\
\delta = \frac{de}{c} - e \frac{df}{f}, \\
\varepsilon = \frac{df}{f}.
\]

(4.4)

Hence after some computation

\[
d\rho = (\alpha + \tilde{\alpha}) \wedge \rho - \frac{e \bar{z}_2}{cf(1 - z_2 \bar{z}_2)} \rho \wedge \kappa + \frac{\bar{z}_2}{f(1 - z_2 \bar{z}_2)} \rho \wedge \zeta \\
- \frac{\bar{e}_2}{cf(1 - z_2 \bar{z}_2)} \rho \wedge \bar{\kappa} + \frac{z_2}{f(1 - z_2 \bar{z}_2)} \rho \wedge \bar{\zeta} + i \kappa \wedge \bar{\kappa}, \\
d\kappa = \alpha \wedge \kappa + \frac{\bar{z}_2}{f(1 - z_2 \bar{z}_2)} \kappa \wedge \zeta + \frac{ce}{ccf(1 - z_2 \bar{z}_2)} \kappa \wedge \bar{\kappa} - \frac{c}{ccf(1 - z_2 \bar{z}_2)} \zeta \wedge \bar{\kappa}, \\
d\zeta = \delta \wedge \kappa + \varepsilon \wedge \zeta + \frac{e \bar{z}_2}{cf(1 - z_2 \bar{z}_2)} \kappa \wedge \zeta + \frac{e^2}{ccf(1 - z_2 \bar{z}_2)} \kappa \wedge \bar{\kappa} \\
+ \frac{e}{ef(1 - z_2 \bar{z}_2)} \zeta \wedge \bar{\kappa}.
\]

(4.5)

In the rest of the article, we will adopt the following order for the coefficients appearing in front of the 2-forms:

\[
\begin{array}{cccccccc}
\rho_0 \wedge \kappa_0 & \rho_0 \wedge \zeta_0 & \rho_0 \wedge \bar{\kappa}_0 & \rho_0 \wedge \bar{\zeta}_0 \\
\kappa_0 \wedge \zeta_0 & \kappa_0 \wedge \bar{\kappa}_0 & \kappa_0 \wedge \bar{\zeta}_0 \\
\zeta_0 \wedge \bar{\kappa}_0 & \zeta_0 \wedge \bar{\zeta}_0 & \bar{\kappa}_0 \wedge \bar{\zeta}_0
\end{array}
\]

(4.6)

Therefore, the 2-forms may be abbreviated as

\[
d\rho = (\alpha + \tilde{\alpha}) \wedge \rho + R_1 \rho \wedge \kappa + R_2 \rho \wedge \zeta + R_3 \rho \wedge \bar{\kappa} + R_4 \rho \wedge \bar{\zeta} \\
+ i \kappa \wedge \bar{\kappa}, \\
d\kappa = \alpha \wedge \kappa + K_5 \kappa \wedge \zeta + K_6 \kappa \wedge \bar{\kappa} + K_7 \zeta \wedge \bar{\kappa}, \\
d\zeta = \delta \wedge \kappa + \varepsilon \wedge \zeta + Z_5 \kappa \wedge \zeta + Z_6 \kappa \wedge \bar{\kappa} + Z_8 \zeta \wedge \bar{\kappa}.
\]

(4.7)

Observe that \( R_3 = \overline{R}_1 \) and \( R_4 = \overline{R}_2 \). We will then proceed with the absorption, which can be done by replacing \( \alpha, \delta \) and \( \varepsilon \) with the new Maurer-Cartan 1-forms

\[
\alpha = \hat{\alpha} - x_\rho \rho - x_\kappa \kappa - x_\zeta \zeta - x_{\bar{\kappa}} \bar{\kappa} - x_{\bar{\zeta}} \bar{\zeta}, \\
\delta = \hat{\delta} - y_\rho \rho - y_\kappa \kappa - y_\zeta \zeta - y_{\bar{\kappa}} \bar{\kappa} - y_{\bar{\zeta}} \bar{\zeta}, \\
\varepsilon = \hat{\varepsilon} - z_\rho \rho - z_\kappa \kappa - z_\zeta \zeta - z_{\bar{\kappa}} \bar{\kappa} - z_{\bar{\zeta}} \bar{\zeta}.
\]

(4.8)
for certain unknowns \( x_\bullet, y_\bullet \) and \( z_\bullet \). Therefore the 2-forms may be re-written as
\[
d\rho = (\dot{\alpha} + \bar{\alpha}) \land \rho + (R_1 + x_\kappa + \bar{x}_\kappa) \rho \land \kappa + (R_2 + x_\zeta + \bar{x}_\zeta) \rho \land \zeta + (R_3 + x_\kappa + \bar{x}_\kappa) \rho \land \bar{\kappa} + (R_4 + x_\zeta + \bar{x}_\zeta) \rho \land \bar{\zeta},
\]
\[
d\kappa = \dot{\alpha} \land \kappa + (K_5 + x_\zeta) \kappa \land \zeta + (K_6 + x_\bar{\kappa}) \kappa \land \bar{\kappa} - x_\rho \rho \land \kappa + i\kappa \land \bar{\kappa},
\]
\[
d\zeta = \dot{\delta} \land \zeta + \bar{\delta} \land \bar{\zeta} - y_\rho \rho \land \zeta + z_\rho \rho \land \zeta + (Z_5 + y_\zeta - z_\bar{\kappa}) \kappa \land \zeta + (Z_6 + y_\bar{\kappa}) \kappa \land \bar{\kappa} + (Z_8 + z_\bar{\kappa}) \zeta \land \bar{\kappa} + y_\zeta \kappa \land \bar{\zeta} + z_\zeta \zeta \land \bar{\zeta}.
\]

This therefore leads to the following set of equations
\[
x_\kappa + \bar{x}_\kappa = -\frac{e\bar{z}_2}{c(1 - z_2\bar{z}_2)}, \quad x_\zeta + \bar{x}_\zeta = \frac{\bar{z}_2}{f(1 - z_2\bar{z}_2)}, \quad x_\rho = 0, \quad y_{\zeta} = 0, \quad z_{\rho} = 0, \quad z_{\zeta} = 0
\]
\[
y_{\kappa} = -\frac{e\bar{z}_2}{c(1 - z_2\bar{z}_2)}, \quad y_{\bar{\kappa}} = -\frac{e\bar{z}_2}{c(1 - z_2\bar{z}_2)}, \quad y_{\bar{\zeta}} = 0, \quad z_{\bar{\zeta}} = 0
\]
\[
x_\kappa = -\frac{e\bar{z}_2}{c(1 - z_2\bar{z}_2)}, \quad x_\bar{\kappa} = -\frac{e\bar{z}_2}{c(1 - z_2\bar{z}_2)}, \quad x_\bar{\zeta} = 0, \quad y_{\zeta} = 0, \quad z_{\rho} = 0
\]
\[
(4.10)
\]

These equations have solutions which result in the absorption of all the torsions except \( K_7 \), and hence
\[
d\rho = (\dot{\alpha} + \bar{\alpha}) \land \rho + i\kappa \land \bar{\kappa},
\]
\[
d\kappa = \dot{\alpha} \land \kappa - \frac{c}{c(1 - z_2\bar{z}_2)} \kappa \land \bar{\kappa},
\]
\[
d\zeta = \dot{\delta} \land \zeta + \bar{\delta} \land \bar{\zeta} \land \kappa.
\]

As in Pocchiola (model case) \[13\], the essential torsion
\[
\frac{c}{c(1 - z_2\bar{z}_2)}
\]
may be normalised to 1 by making the following choice
\[
f = -\frac{c}{c(1 - z_2\bar{z}_2)}.
\]

With this normalisation being made, we proceed with the second loop of the Cartan’s equivalence method. The new transformation group then becomes
\[
\begin{pmatrix}
\rho \\
\kappa \\
\zeta
\end{pmatrix} = \begin{pmatrix}
c\bar{c} & 0 & 0 \\
0 & c & 0 \\
0 & e & \xi
\end{pmatrix} \begin{pmatrix}
\rho_0 \\
\kappa_0 \\
\zeta_0
\end{pmatrix} \quad \text{with}
\begin{pmatrix}
\rho_0 \\
\kappa_0 \\
\zeta_0
\end{pmatrix} = \begin{pmatrix}
c\bar{c} & 0 & 0 \\
0 & c & 0 \\
0 & e & \xi
\end{pmatrix} \begin{pmatrix}
\rho \\
\kappa \\
\zeta
\end{pmatrix}
\]
\[
(4.12)
\]
with a change of the base coframe \((\rho_0, \kappa_0, \zeta_0) \mapsto (\rho_0, \kappa_0, \hat{\zeta}_0)\) via
\[
\hat{\zeta}_0 := -\frac{1}{1 - z_2\bar{\zeta}_0} \zeta_0.
\]
According to Pocchiola (model case) [13], the 2-forms become
\[
d\rho_0 = -\bar{z}_2 \rho_0 \wedge \hat{\zeta}_0 - z_2 \rho_0 \wedge \bar{\zeta}_0 + i\kappa_0 \wedge \bar{\kappa}_0,
\]
\[
d\kappa_0 = -\bar{z}_2 \kappa_0 \wedge \hat{\zeta}_0 + \hat{\zeta}_0 \wedge \bar{\kappa}_0,
\]
\[
d\hat{\z}_0 = z_2 \hat{\z}_0 \wedge \bar{\z}_0.
\]
Moreover, one has the following Maurer-Cartan matrix of 1-forms
\[
\begin{pmatrix}
\alpha + \bar{\alpha} & 0 & 0 \\
0 & \alpha & 0 \\
0 & \delta & \alpha - \bar{\alpha}
\end{pmatrix},
\]
where
\[
\alpha = \frac{dc}{c}, \quad \text{and} \quad \delta = \frac{de}{c} - \frac{e}{c} \left(\frac{dc}{c} - \frac{dc}{c}\right).
\]
A computation by hand gives
\[
d\rho = (\alpha + \bar{\alpha}) \wedge \rho + \frac{e\bar{c}}{c^2} \rho \wedge \kappa - \bar{z}_2 \frac{\bar{c}}{c} \rho \wedge \zeta + \frac{e\bar{c}}{c^2} \rho \wedge \bar{\kappa} - z_2 \frac{c}{\bar{c}} \rho \wedge \bar{\zeta} + i \kappa \wedge \bar{\kappa},
\]
\[
= (\alpha + \bar{\alpha}) \wedge \rho + R_1 \rho \wedge \kappa + R_2 \rho \wedge \bar{\zeta} + \bar{R}_1 \rho \wedge \bar{\kappa} + R_2 \rho \wedge \bar{\zeta} + i \kappa \wedge \bar{\kappa},
\]
\[
d\kappa = \alpha \wedge \kappa - \bar{z}_2 \frac{\bar{c}}{c} \kappa \wedge \zeta - \frac{e}{c} \kappa \wedge \bar{\kappa} + \zeta \wedge \bar{\kappa}
\]
\[
= \alpha \wedge \kappa + K_5 \kappa \wedge \zeta + K_6 \kappa \wedge \bar{\kappa} + \zeta \wedge \bar{\kappa},
\]
\[
d\zeta = \delta \wedge \kappa + (\alpha - \bar{\alpha}) \wedge \zeta - \bar{z}_2 \frac{\bar{c}}{c^2} \kappa \wedge \zeta + \left(-\frac{e^2}{c^2} + z_2 \frac{\bar{e}c}{c^2}\right) \kappa \wedge \bar{\kappa}
\]
\[
+ \left(\frac{e}{c} - z_2 \frac{\bar{e}c}{c^2}\right) \zeta \wedge \bar{\kappa} - z_2 \frac{c}{\bar{c}} \kappa \wedge \bar{\zeta} + \frac{z_2c}{\bar{c}} \zeta \wedge \bar{\zeta}
\]
\[
= \delta \wedge \kappa + (\alpha - \bar{\alpha}) \wedge \zeta + Z_5 \kappa \wedge \zeta + Z_6 \kappa \wedge \bar{\kappa} + Z_8 \zeta \wedge \bar{\kappa} + Z_7 \kappa \wedge \bar{\zeta} + Z_9 \zeta \wedge \bar{\zeta}.
\]
Then we proceed with the absorption by setting
\[
\alpha = \hat{\alpha} - x_\rho \rho - x_\kappa \kappa - x_\zeta \zeta - x_{\bar{\kappa}} \bar{\kappa} - x_{\bar{\zeta}} \bar{\zeta},
\]
\[
\delta = \hat{\delta} - y_\rho \rho - y_\kappa \kappa - y_\zeta \zeta - y_{\bar{\kappa}} \bar{\kappa} - y_{\bar{\zeta}} \bar{\zeta},
\]
and we obtain
\[
d\rho = (\hat{\alpha} + \bar{\hat{\alpha}}) \wedge \rho + (R_1 + x_\kappa + x_{\bar{\kappa}}) \rho \wedge \kappa + (R_2 + x_\zeta + x_{-\bar{\zeta}}) \rho \wedge \zeta
\]
\[
+ \left(\bar{R}_1 + x_\kappa + x_{\bar{\kappa}}\right) \rho \wedge \bar{\kappa} + \left(\bar{R}_2 + x_\zeta + x_{-\bar{\zeta}}\right) \rho \wedge \bar{\zeta} + i \kappa \wedge \bar{\kappa},
\]
\[
d\kappa = \hat{\alpha} \wedge \kappa - x_\rho \rho \wedge \kappa + (K_5 + x_\zeta) \kappa \wedge \zeta + (K_6 + x_\kappa) \kappa \wedge \bar{\kappa} + x_\zeta \kappa \wedge \bar{\kappa} + \zeta \wedge \bar{\kappa},
\]
\[
d\zeta = \hat{\delta} \wedge \kappa + (\hat{\alpha} - \bar{\hat{\alpha}}) \wedge \zeta - y_\rho \rho \wedge \kappa + (x_{\bar{\rho}} - x_\rho) \rho \wedge \zeta + (Z_5 - x_\kappa + y_\zeta + x_{\bar{\kappa}}) \kappa \wedge \zeta
\]
\[
+ (Z_6 + y_\kappa) \kappa \wedge \bar{\kappa} + Z_7 \kappa \wedge \bar{\zeta} + (Z_8 - x_{\bar{\zeta}} + x_\kappa) \zeta \wedge \bar{\kappa} + (Z_9 - x_{\bar{\zeta}} + x_\kappa) \zeta \wedge \bar{\zeta}.
This leads to another set of absorption equations

\[
\begin{align*}
    x_\kappa + \overline{x_\kappa} &= -\overline{z_2 c^2} e c, \\
    x_\zeta + \overline{x_\zeta} &= \overline{z_2 c} \zeta, \\
    x_\kappa + \overline{x_\kappa} &= \overline{e c} \overline{c^2}, \\
    x_\zeta + \overline{x_\zeta} &= z_2 \overline{c} c, \\
    x_\rho &= 0, \\
    x_\kappa &= -\overline{e c}, \\
    x_\zeta &= -\overline{e c}.
\end{align*}
\]  

(4.15)

The following equations

\[
\begin{align*}
    x_\kappa + \overline{x_\kappa} &= -\overline{z_2 c^2} e c, \\
    x_\kappa &= -\overline{e c}, \\
    x_\kappa - \overline{x_\kappa} &= -\overline{c^2} e c.
\end{align*}
\]  

(4.16)

force us to conclude that \( e = 0 \), which is consistent with Pocchiola (model case) [13], page 146, where he sets

\[
d = -i \frac{e^2 c}{2 c}.
\]

In our case, \( d = 0 \) due to our rigidity assumption, and thus we are led to \( e = 0 \).

This new normalisation gives rise to the new transformation group,

\[
\begin{pmatrix}
    \rho \\
    \kappa \\
    \zeta
\end{pmatrix} =
\begin{pmatrix}
    c c & 0 & 0 \\
    0 & c & 0 \\
    0 & 0 & \overline{c}
\end{pmatrix}
\begin{pmatrix}
    \rho_0 \\
    \kappa_0 \\
    \zeta_0
\end{pmatrix}
\]

with the new Maurer-Cartan matrix

\[
dg \cdot g^{-1} =
\begin{pmatrix}
    \alpha + \tilde{\alpha} & 0 & 0 \\
    0 & \alpha & 0 \\
    0 & 0 & \alpha - \tilde{\alpha}
\end{pmatrix}
\]

and the following 2-forms

\[
\begin{align*}
    d\rho &= (\alpha + \tilde{\alpha}) \wedge \rho - \overline{z_2 c} \rho \wedge \zeta - z_2 \overline{c} \rho \wedge \overline{\zeta} + i \kappa \wedge \overline{\kappa}, \\
    d\kappa &= \alpha \wedge \kappa - \overline{z_2 c} \kappa \wedge \zeta + \zeta \wedge \overline{\kappa}, \\
    d\zeta &= (\alpha - \tilde{\alpha}) \wedge \zeta + z_2 \overline{c} \zeta \wedge \overline{\zeta}. \\
\end{align*}
\]  

(4.17)

We will proceed with the absorption process by setting

\[
\alpha = \tilde{\alpha} - x_\rho \rho - x_\kappa \kappa - x_\zeta \zeta - x_\kappa \overline{\kappa} - x_\zeta \overline{\zeta},
\]
which leads to
\[
    d\rho = (\hat{\alpha} + \bar{\hat{\alpha}}) \wedge \rho + (x\zeta + \bar{x}\zeta - \bar{z}_2\bar{c}) \rho \wedge \zeta + (x\bar{\zeta} + \bar{x}\bar{\zeta} - \bar{z}_2\bar{c}) \rho \wedge \bar{\zeta} \\
    + (x\kappa + \bar{x}\kappa) \rho \wedge \kappa + (x\bar{\kappa} + \bar{x}\bar{\kappa}) \rho \wedge \bar{\kappa} + i\kappa \wedge \bar{\kappa},
\]
\[
    d\kappa = \hat{\alpha} \wedge \kappa + \zeta \wedge \bar{\kappa},
\]
\[
    d\zeta = (\hat{\alpha} - \bar{\hat{\alpha}}) \wedge \zeta.
\]

To remove all the torsions, one has to solve for \(x_\bullet\) the following system of linear equations
\[
x_\zeta + \bar{x}_\zeta = \bar{z}_2 \bar{c},
\]
\[
x_\bar{\zeta} + x\zeta = \bar{z}_2 \bar{c},
\]
\[
x_\kappa + \bar{x}_\kappa = 0,
\]
\[
x_\rho = 0,
\]
\[
x_\bar{\kappa} = 0,
\]
\[
x_\bar{\zeta} = 0,
\]
\[
x_\zeta = \bar{z}_2 \bar{c},
\]
\[
- x_\rho + \bar{x}_\rho = 0,
\]
\[
- x_\bar{\kappa} + \bar{x}_\bar{\kappa} = 0,
\]
\[
x_\bar{\kappa} - x_\kappa = 0,
\]
\[
x_\zeta - \bar{x}_\zeta + \frac{c}{\bar{c}} = 0.
\]

This time the solution set is unambiguous:
\[
x_\rho = 0, \quad x_\kappa = 0, \quad x_\zeta = \bar{z}_2 \frac{c}{\bar{c}}, \quad x_\bar{\kappa} = 0, \quad x_\bar{\zeta} = 0,
\]
and since the degree of indeterminacy is zero, Cartan’s test tells us that there is no need for prolongation. The absorption takes place and we get
\[
    d\rho = (\hat{\alpha} + \bar{\hat{\alpha}}) \wedge \rho + i\kappa \wedge \bar{\kappa},
\]
\[
    d\kappa = \hat{\alpha} \wedge \kappa + \zeta \wedge \bar{\kappa},
\]
\[
    d\zeta = (\hat{\alpha} - \bar{\hat{\alpha}}) \wedge \zeta.
\]

The \(\{e\}\)-structure is then completed by the following

**Proposition 4.21.** One has \(d\hat{\alpha} = \zeta \wedge \bar{\zeta}\).

**Proof.** Applying the Poincaré derivative on both sides of the three equations above, we get
\[
    (d\hat{\alpha} + d\bar{\hat{\alpha}}) \wedge \rho = 0, \quad (d\hat{\alpha} - \bar{\zeta} \wedge \bar{\zeta}) \wedge \kappa = 0, \quad (d\hat{\alpha} - \bar{\hat{\alpha}}) \wedge \zeta = 0.
\]
By applying complex conjugation to both sides of the third equation, one has an additional relation

\[(d\hat{\alpha} - d\tilde{\alpha}) \wedge \tilde{\zeta} = 0.\]  

(4.25)

In the second equation \((4.23)\), Cartan’s lemma provides a 1-form \(A\) so that

\[d\hat{\alpha} = \zeta \wedge \tilde{\zeta} + A \wedge \kappa.\]

Hence in \((4.22), (4.24)\) and \((4.25)\),

\[
\begin{align*}
(d\hat{\alpha} + d\tilde{\alpha}) \wedge \rho &= A \wedge \kappa \wedge \rho + \bar{A} \wedge \bar{\kappa} \wedge \rho = 0; \\
(d\hat{\alpha} - d\tilde{\alpha}) \wedge \zeta &= A \wedge \kappa \wedge \zeta - \bar{A} \wedge \bar{\kappa} \wedge \zeta = 0; \\
(d\hat{\alpha} - d\tilde{\alpha}) \wedge \bar{\zeta} &= A \wedge \kappa \wedge \bar{\zeta} - \bar{A} \wedge \bar{\kappa} \wedge \bar{\zeta} = 0.
\end{align*}
\]

Wedging with \(\zeta\) on both sides of the first equation, and by \(\rho\) on the second, we get

\[
\begin{align*}
A \wedge \kappa \wedge \rho \wedge \zeta + \bar{A} \wedge \bar{\kappa} \wedge \rho \wedge \zeta &= 0, \\
A \wedge \kappa \wedge \rho \wedge \zeta - \bar{A} \wedge \bar{\kappa} \wedge \rho \wedge \zeta &= 0.
\end{align*}
\]

Similarly, wedging with \(\bar{\zeta}\) on both sides of the first equation, and by \(\rho\) on the third, it comes

\[
\begin{align*}
A \wedge \kappa \wedge \rho \wedge \bar{\zeta} + \bar{A} \wedge \bar{\kappa} \wedge \rho \wedge \bar{\zeta} &= 0, \\
A \wedge \kappa \wedge \rho \wedge \bar{\zeta} - \bar{A} \wedge \bar{\kappa} \wedge \rho \wedge \bar{\zeta} &= 0.
\end{align*}
\]

Therefore,

\[
\begin{align*}
A \wedge \kappa \wedge \rho \wedge \zeta &= 0, \\
A \wedge \kappa \wedge \rho \wedge \bar{\zeta} &= 0.
\end{align*}
\]

(4.26)

This implies the existence of functions \(f\) and \(g\) with

\[A = f\rho + g\kappa.\]

Hence

\[d\hat{\alpha} = \zeta \wedge \bar{\zeta} + f\rho \wedge \kappa.\]

Substituting this into \((4.24)\),

\[0 = (\zeta \wedge \bar{\zeta} + f\rho \wedge \kappa - \bar{\zeta} \wedge \zeta - \bar{f}\rho \wedge \bar{\kappa}) \wedge \zeta = f\rho \wedge \kappa \wedge \zeta - \bar{f}\rho \wedge \bar{\kappa} \wedge \zeta,
\]

we conclude by linear independence of these 3-forms that \(f = 0\). \(\square\)

4.1. **Summary.** For the model case, there exists a coframe \((\rho, \kappa, \zeta, \alpha, \bar{\kappa}, \bar{\zeta}, \bar{\alpha})\) satisfying the following structure equations

\[
\begin{align*}
\text{d}\rho &= (\alpha + \bar{\alpha}) \wedge \rho + i\kappa \wedge \bar{\kappa}, \\
\text{d}\kappa &= \alpha \wedge \kappa + \zeta \wedge \bar{\kappa}, \\
\text{d}\zeta &= (\alpha - \bar{\alpha}) \wedge \zeta, \\
\text{d}\hat{\alpha} &= \zeta \wedge \bar{\zeta},
\end{align*}
\]

(4.27)

along with the conjugates \(d\rho, d\bar{\kappa}, d\bar{\zeta}\) and \(d\hat{\alpha}\). Observe that \(\alpha\) cannot be purely imaginary as seen during the absorption of the final Cartan process. This therefore constitutes the Maurer-Cartan constant coefficients equations for the 7 dimensional complex Lie algebra of automorphisms of the model light cone \(M_{LC}\). We will confirm that this is \(\text{aut}_{\text{CR}}(M_{LC})\), arguing by means of vector fields.
5. Representation by vector fields

By a result of Gaussier-Merker [4, 5], it is known that the Lie algebra of infinitesimal CR automorphisms of the tube over future light cone $M_{LC}$ is generated by the following 10 holomorphic vector fields

\[ X^1 = i\partial_w, \]
\[ X^2 = z_1\partial_{z_1} + 2w\partial_w, \]
\[ X^3 = iz_1\partial_{z_1} + 2iz_2\partial_{z_2}, \]
\[ X^4 = (z_2 - 1)\partial_{z_1} - 2z_1\partial_w, \]
\[ X^5 = (i + iz_2)\partial_{z_1} - 2iz_1\partial_w, \]
\[ X^6 = z_1z_2\partial_{z_1} + (z_2^2 - 1)\partial_{z_2} - z_1^2\partial_w, \]
\[ X^7 = iz_1z_2\partial_{z_1} + (iz_2^2 + i)\partial_{z_2} - iz_1^2\partial_w, \]
\[ X^8 = iwz_1\partial_{z_1} - iz_2^2\partial_{z_2} + iw^2\partial_w, \]
\[ X^9 = (iz_1^2 - iz_2^2 + iw)\partial_{z_1} + (2z_1z_2 + 2z_1)\partial_{z_2} + 2wz_1\partial_w, \]
\[ X^{10} = (-iz_1^2 + iwz_2 - iw)\partial_{z_1} + (-2iz_1z_2 + 2z_1)\partial_{z_2} - 2iwz_1\partial_w. \]

(5.1)

It can be shown that for each $1 \leq i \leq 10$, the vector field $X^i + X^i$ is tangent to $M_{LC}$. The commutator table of these 10 vector fields is as follows.

|    | $X^1$ | $X^2$ | $X^3$ | $X^4$ | $X^5$ | $X^6$ | $X^7$ | $X^8$ | $X^9$ | $X^{10}$ |
|----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-----------|
| $X^1$ | 0     | 2$X^1$| 0     | 0     | 0     | 0     | 0     | $-X^2$| $-X^5$| $-X^4$    |
| $X^2$ | 0     | 0     | $-X^4$| $-X^8$| 0     | 0     | $2X^8$| $X^9$ | $X^{10}$| $X^3$     |
| $X^3$ | 0     | $X^6$ | $X^6$ | $-X^4$| 0     | $2X^8$| $X^9$ | $2X^6 - 2X^2$| $-2X^7 + 2X^3$| $2X^6 + 2X^2$|
| $X^4$ | 0     | $X^8$ | $-X^4$| $X^9$ | $2X^7 + 2X^{10}$| $X^{10}$| $X^9$ | 0     | 0     | 0         |
| $X^5$ | 0     | 0     | $-2X^3$| 0     | $-X^9$| $X^{10}$| $X^9$ | 0     | 0     | 0         |
| $X^6$ | 0     | 0     | 0     | $4X^6$| 0     | 0     | $4X^6$| 0     | 0     | 0         |
| $X^7$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0         |
| $X^8$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0         |
| $X^{10}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0         |

It is therefore clear from the table above that the vector fields $X^1, \ldots, X^7$ generate a Lie sub-algebra, which we will denote by $\mathfrak{h}$. Next, we are going to find out which among these 10 vector fields have integral curves that define local rigid automorphisms of $\mathbb{C}^3$ (in the sense of Definition 3.1).

Recall that an integral curve of a vector field $X$ on $\mathbb{C}^3$ is the map

$$\gamma : \mathbb{R} \to \mathbb{C}^3$$

satisfying the following differential equation with initial condition:

$$\frac{d\gamma}{dt}\bigg|_{\gamma(t)} = X|_{\gamma(t)},$$

$$\gamma(0) = p.$$  

(5.2)
Usually such an integral curve at \( p \) is denoted by

\[
\exp(tX)(p) := \gamma(t) \quad (\gamma(0) = p).
\]

Due to the following identity

\[
\exp(-tX) \exp(tX)(p) = p,
\]

an integral curve therefore defines an automorphism of \( \mathbb{C}^3 \) for each fixed \( t \):

\[
\exp(tX) : \mathbb{C}^3 \to \mathbb{C}^3 \\
p \mapsto \exp(tX)(p).
\]

For notational ease, we will let \( p_1, p_2 \) and \( p_3 \) denote the coordinates of \( \gamma(0) = (\gamma_1(0), \gamma_2(0), \gamma_3(0)) = (p_1, p_2, p_3) \).

5.1. **Vector field** \( X^1 \). Integral curve:

\[
(\gamma_1(t), \gamma_2(t), \gamma_3(t)) = (p_1, p_2, p_3 + it).
\]

Therefore for each fixed \( t \), the holomorphic map

\[
(z_1, z_2, w) \mapsto (z_1, z_2, w + it)
\]

is rigid (we see \( it \) as a constant holomorphic function).

5.2. **Vector field** \( X^2 \). Integral curve:

\[
(\gamma_1(t), \gamma_2(t), \gamma_3(t)) = (e^tp_1, p_2, e^{2it}p_3).
\]

Then for each fixed \( t \), the holomorphic map

\[
(z_1, z_2, w) \mapsto (e^t z_1, z_2, e^{2it}w)
\]

is rigid.

5.3. **Vector field** \( X^3 \). Integral curve:

\[
(\gamma_1(t), \gamma_2(t), \gamma_3(t)) = (e^{it}p_1, e^{2it}p_2, p_3).
\]

Therefore for each fixed \( t \), the holomorphic map

\[
(z_1, z_2, w) \mapsto (e^{it} z_1, e^{2it} z_2, w)
\]

is rigid.

5.4. **Vector field** \( X^4 \). Integral curve:

\[
(\gamma_1(t), \gamma_2(t), \gamma_3(t)) = ((p_2 - 1)t + p_1, p_2, -(p_2 - 1)t^2 - 2p_1t + p_3).
\]

For each fixed \( t \), the holomorphic map

\[
(z_1, z_2, w) \mapsto (z_2 - 1)t + z_1, z_2, w - ((z_2 - 1)t^2 + 2z_1t))
\]

is rigid.

5.5. **Vector field** \( X^5 \). Integral curve:

\[
(\gamma_1(t), \gamma_2(t), \gamma_3(t)) = (p_1 + i(p_2 + 1)t, p_2, p_3 - 2ip_1t + (p_2 + 1)t^2).
\]

For each fixed \( t \), the holomorphic map

\[
(z_1, z_2, w) \mapsto (z_1 + i(z_2 + 1)t), z_2, w - 2iz_1t + (z_2 + 1)t^2)
\]

is therefore rigid.
5.6. Vector field $X^6$. The integral curve $(\gamma_1(t), \gamma_2(t), \gamma_3(t))$ is given by the following equations

\[
\begin{align*}
\gamma_1(t) &= \frac{2p_1(1 + p_2)e^t}{(1 + p_2)(1 + p_2 + e^{2t}(1 - p_2))}, \\
\gamma_2(t) &= \frac{(1 + p_2) - e^{2t}(1 - p_2)}{(1 + p_2) + e^{2t}(1 - p_2)}, \\
\gamma_3(t) &= p_3 + \frac{p_1^2}{1 - p_2} - \frac{2p_1^2}{1 - p_2 (1 + p_2) + (1 - p_2)e^{2t}}.
\end{align*}
\]

(5.4)

For each fixed $t$, the holomorphic map

\[
(z_1, z_2, w) \mapsto \left( \frac{2z_1(1 + z_2)e^t}{(1 + z_2)(1 + z_2 + e^{2t}(1 - z_2))}, \frac{(1 + z_2) - e^{2t}(1 - z_2)}{(1 + z_2) + e^{2t}(1 - z_2)}, \right.
\]

\[
\left. w + \frac{z_1^2}{1 - z_2} - \frac{2z_1^2}{1 - z_2 (1 + z_2) + e^{2t}(1 - z_2)} \right)
\]

is therefore rigid.

5.7. Vector field $X^7$. The integral curve $(\gamma_1(t), \gamma_2(t), \gamma_3(t))$ is given by

\[
\begin{align*}
\gamma_1(t) &= \frac{ip_1}{p_2 \sinh(t) + i \cosh(t)}, \\
\gamma_2(t) &= \frac{-p_2 - it \tanh(t)}{i + p_2 \tanh(t)}, \\
\gamma_3(t) &= p_3 + \frac{p_1^2 \sinh(t)}{p_2 \sinh(t) + i \cosh(t)}.
\end{align*}
\]

Hence for each fixed $t$, the holomorphic map

\[
(z_1, z_2, w) \mapsto \left( \frac{iz_1}{z_2 \sinh(t) + i \cosh(t)}, \frac{-z_2 - it \tanh(t)}{i + z_2 \tanh(t)}, w + \frac{z_1^2 \sinh(t)}{z_2 \sinh(t) + i \cosh(t)} \right)
\]

is rigid.

One can deduce directly from the table that the Lie algebra $\mathfrak{h}$ is neither semi-simple nor reductive. Indeed, the Killing form applied to the first vector field vanishes

\[
\text{trace}(\text{ad}(X^1)\text{ad}(X^j)) = 0, \quad (j = 1, \ldots, 7)
\]

and hence $\mathfrak{h}$ is not semi-simple by Cartan’s criterion. Moreover, suppose by means of \textit{reductio ad absurdum} that $\mathfrak{h}$ is reductive, then it has a decomposition

\[
\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{z}(\mathfrak{h}),
\]

where $\mathfrak{s}$ is a semi-simple Lie sub-algebra and $\mathfrak{z}(\mathfrak{h})$ is the centre of $\mathfrak{h}$. But it is clear from the table that $\mathfrak{h}$ has no element in the centre except the zero vector field, and hence

\[
\mathfrak{h} = \mathfrak{s}
\]

so that $\mathfrak{h}$ is semi-simple, a contradiction.

We will now proceed to establish a link between the Maurer-Cartan coframe

\[
(\rho, \kappa, \zeta, \alpha, \bar{\kappa}, \bar{\zeta}, \bar{\alpha})
\]

(5.5)
appearing in the structure equations in the previous sections, and the vector fields $X^1, \ldots, X^7$. In fact, let

$$\partial \rho, \partial \kappa, \partial \zeta, \partial \alpha, \partial \bar{\kappa}, \partial \bar{\zeta}, \partial \bar{\alpha}$$

be the right-invariant vector fields that are respective duals to the 1-forms in equation (5.5), and let $\mathfrak{h}'$ be the Lie algebra generated by these vector fields. In what follows, the link will be established by seeking a Lie algebra isomorphism

$$\tau : \mathfrak{h} \rightarrow \mathfrak{h}'$$

between $\mathfrak{h}$ and $\mathfrak{h}'$.

We make the following recall which can be found in Olver [14], page 257. Consider a set of 1-forms $\theta = \{\theta^1, \ldots, \theta^m\}$ on a manifold $M$ producing the fundamental structure equations

$$d\theta^i = \sum_{1 \leq j < k \leq m} T^i_{jk} \theta^j \wedge \theta^k \quad (i = 1, \ldots, m).$$

If $\partial \theta^i$ are the vector fields dual to $\theta^i$, one has the following commutation relations

$$[\partial \theta^i, \partial \theta^k] = -\sum_{i=1}^m T^i_{jk} \partial \theta^j \quad (1 \leq i < j \leq m).$$

Following this formula, and if we adopt the order of indices

$$\rho < \kappa < \zeta < \alpha < \bar{\kappa} < \bar{\zeta} < \bar{\alpha},$$

the Maurer-Cartan structure equations in equation (4.27) therefore provide the following commutator table of the vector fields:

|       | $\partial \rho$ | $\partial \kappa$ | $\partial \zeta$ | $\partial \alpha$ | $\partial \bar{\kappa}$ | $\partial \bar{\zeta}$ | $\partial \bar{\alpha}$ |
|-------|-----------------|-------------------|-----------------|-------------------|-------------------------|-------------------------|-------------------------|
| $\partial \rho$ | 0                | 0                 | 0               | $\partial \rho$  | 0                       | 0                       | $\partial \bar{\rho}$  |
| $\partial \kappa$ | 0                | 0                 | 0               | $\partial \kappa$ | $-i\partial \rho$         | $\partial \kappa$       | 0                       |
| $\partial \zeta$ | 0                | 0                 | 0               | $\partial \zeta$ | $-\partial \rho$          | $\partial \kappa$       | $-\partial \alpha + \partial \bar{\alpha} - \partial \zeta$ |
| $\partial \alpha$ | $-\partial \rho$ | $-\partial \kappa$ | $-\partial \zeta$ | 0                 | 0                       | $\partial \zeta$        | 0                       |
| $\partial \bar{\kappa}$ | 0               | $i\partial \rho$  | $\partial \kappa$ | 0                 | 0                       | 0                       | $\partial \bar{\kappa}$ |
| $\partial \bar{\zeta}$ | 0               | $-\partial \kappa$ | $-\partial \alpha + \partial \bar{\alpha}$ | 0                 | 0                       | $\partial \zeta$        | 0                       |
| $\partial \bar{\alpha}$ | $-\partial \rho$ | 0                 | $\partial \zeta$ | 0                 | $-\partial \kappa$        | $-\partial \zeta$       | 0                       |

Let $W^1, \ldots, W^7$ be the vector fields defined by

$$W^1 := -\frac{i}{2} \partial \rho, \quad W^4 := \partial \kappa - \partial \bar{\kappa},$$
$$W^2 := \partial \alpha + \partial \bar{\alpha}, \quad W^5 := \partial \kappa + \partial \bar{\kappa},$$
$$W^3 := \partial \zeta - \partial \bar{\zeta}, \quad W^6 := \partial \zeta + \partial \bar{\zeta},$$
$$W^7 := -\partial \alpha + \partial \bar{\alpha}. \quad (5.7)$$

Using the commutator table above, one has the following table of Lie brackets of various vector fields $W^i$:
which is the same as the commutator table of the vector fields $X^1, \ldots, X^7$. Therefore the map which sends for each $i = 1, \ldots, 7$:

\[
\tau : \mathfrak{h} \longrightarrow \mathfrak{h}' \\
X^i \mapsto \tau(X^i) := W^i
\]

(5.8)
defines a Lie algebra isomorphism. The following theorem summarises what has been done so far for the rigid automorphisms of the model case:

**Theorem 5.9.** The set of infinitesimal rigid CR-automorphisms of the tube over the future light cone

\[
\text{MLC} : (\text{Re}z_1)^2 - (\text{Re}z_2)^2 - (\text{Re}z_3)^2 = 0, \quad \text{Re}z_1 > 0,
\]

is a 7-dimensional Lie sub-algebra of the set of all of its infinitesimal CR-automorphisms. A basis for the Maurer-Cartan forms of the infinitesimal rigid CR-automorphisms is provided by the 7 differential 1-forms $\rho, \kappa, \zeta, \alpha, \bar{\kappa}, \bar{\zeta}, \bar{\alpha}$ on $\text{MLC} \times \mathbb{C}$ which satisfy the following Maurer-Cartan equations:

\[
\begin{align*}
d\rho &= (\alpha + \bar{\alpha}) \wedge \rho + i\kappa \wedge \bar{\kappa}, \\
d\kappa &= \alpha \wedge \kappa + \zeta \wedge \bar{\kappa}, \\
d\zeta &= (\alpha - \bar{\alpha}) \wedge \zeta, \\
d\alpha &= \zeta \wedge \bar{\zeta}, \\
d\bar{\kappa} &= \bar{\alpha} \wedge \bar{\kappa} + \bar{\zeta} \wedge \kappa, \\
d\bar{\zeta} &= -(\alpha - \bar{\alpha}) \wedge \bar{\zeta}, \\
d\bar{\alpha} &= -\zeta \wedge \bar{\zeta}.
\end{align*}
\]

(5.10)

Moreover, if $\{\partial_\rho, \partial_\kappa, \partial_\zeta, \partial_\alpha, \partial_{\bar{\kappa}}, \partial_{\bar{\zeta}}, \partial_{\bar{\alpha}}\}$ is a set of right-invariant vector fields that are dual to the respective coframe 1-forms $\{\rho, \kappa, \zeta, \alpha, \bar{\kappa}, \bar{\zeta}, \bar{\alpha}\}$, then there is an isomorphism of Lie algebras between the Lie algebra $\mathfrak{h}'$ generated by these vector fields, and the Lie algebra of infinitesimal rigid automorphisms of the tube over the future light cone. \(\square\)

6. The general case

The previous theorem shows that the Maurer-Cartan form that we have obtained, together with the structure equations, give a good setup for the equivalence problem. Recall from equations (2.10) and (2.11) that the Darboux-Cartan structure equations are given by
the 1-forms \( \{\rho_0, \kappa_0, \zeta_0\} \) with

\[
d\rho_0 = P \rho_0 \land \kappa_0 - \mathcal{L}_1(k) \rho_0 \land \zeta_0 + \overline{P} \rho_0 \land \overline{\kappa_0} - \overline{\mathcal{F}_1(k)} \rho_0 \land \overline{\zeta_0} + i \kappa_0 \land \overline{\kappa_0},
\]

\[
d\kappa_0 = -\mathcal{L}_1(k) \kappa_0 \land \zeta_0 + \overline{\mathcal{F}_1(k)} \zeta_0 \land \overline{\kappa_0},
\]

\[
d\zeta_0 = 0.
\]

In equation (3.7), the group transformation of the \((1,0)\) coframe is determined by the matrix

\[
\omega = \begin{pmatrix} \rho \\ \kappa \\ \zeta \end{pmatrix} = \begin{pmatrix} c \alpha & 0 & 0 \\ 0 & c & 0 \\ 0 & e & f \end{pmatrix} \begin{pmatrix} \rho_0 \\ \kappa_0 \\ \zeta_0 \end{pmatrix} := g \omega_0.
\]

We will also continue to adopt the order of coefficients as stated in equation (4.6)

7. Cartan process: first loop

Using the formula

\[
d\omega = (dg)^{-1} \omega + g d\omega_0,
\]

the Maurer-Cartan form is

\[
(dg)^{-1} = \begin{pmatrix} \alpha + \bar{\alpha} & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & \delta & \varepsilon \end{pmatrix},
\]

where \(\alpha, \delta\) and \(\varepsilon\) are given by those in equation (4.4). A direct computation shows that

\[
d\rho = \alpha \land \rho + \bar{\alpha} \land \rho + \left(\frac{P}{c} + \frac{e \mathcal{L}_1(k)}{cf}\right) \rho \land \kappa + \left(\frac{\overline{P}}{c} + \frac{e \overline{\mathcal{F}_1(k)}}{cf}\right) \rho \land \overline{\kappa}
\]

\[
+ \left(\frac{-\mathcal{L}_1(k)}{f}\right) \rho \land \zeta + \left(\frac{-\overline{\mathcal{F}_1(k)}}{f}\right) \rho \land \overline{\zeta} + i \kappa \land \overline{\kappa},
\]

\[
d\kappa = \alpha \land \kappa + \left(\frac{-\mathcal{L}_1(k)}{f}\right) \kappa \land \zeta + \left(\frac{-e \mathcal{L}_1(k)}{cf}\right) \kappa \land \overline{\kappa} + \left(\frac{e \overline{\mathcal{F}_1(k)}}{cf}\right) \kappa \land \overline{\kappa},
\]

\[
d\zeta = \delta \land \kappa + \varepsilon \land \zeta + \left(\frac{-e \mathcal{L}_1(k)}{cf}\right) \kappa \land \zeta + \left(\frac{-e^2 \overline{\mathcal{F}_1(k)}}{ccf}\right) \kappa \land \overline{\kappa}
\]

\[
+ \left(\frac{e \mathcal{L}_1(k)}{cf}\right) \zeta \land \overline{\kappa}.
\]

We proceed with the absorption by setting

\[
\alpha = \hat{\alpha} - x_\rho \rho - x_\kappa \kappa - x_\zeta \zeta - x_{\overline{\kappa}} \overline{\kappa} - x_{\overline{\zeta}} \overline{\zeta},
\]

\[
\delta = \hat{\delta} - y_\rho \rho - y_\kappa \kappa - y_\zeta \zeta - y_{\overline{\kappa}} \overline{\kappa} - y_{\overline{\zeta}} \overline{\zeta},
\]

\[
\varepsilon = \hat{\varepsilon} - z_\rho \rho - z_\kappa \kappa - z_\zeta \zeta - z_{\overline{\kappa}} \overline{\kappa} - z_{\overline{\zeta}} \overline{\zeta}.
\]

Solving a system of linear equations to eliminate as many torsions as possible, one obtains

\[
d\rho = (\hat{\alpha} + \overline{\alpha}) \land \rho + i \kappa \land \overline{\kappa},
\]

\[
d\kappa = \hat{\alpha} \land \kappa + \frac{e \mathcal{L}_1(k)}{cf} \zeta \land \overline{\kappa},
\]

\[
d\zeta = \hat{\delta} \land \kappa + \hat{\varepsilon} \land \zeta.
\]
Notice that the function
\[ \frac{c L_1(k)}{\bar{c}} \]
is nowhere vanishing, and hence the torsion that appears in \( d\kappa \) may be normalised to 1 by setting
\[ f = \frac{c L_1(k)}{\bar{c}}. \]

8. Cartan process: second loop

With this normalisation, we proceed with a change of the base coframe
\[ \hat{\zeta}_0 := \underline{L}_1(\kappa)\zeta_0, \]
so that the new transformation group becomes
\[
\begin{pmatrix}
\rho \\
\kappa \\
\zeta
\end{pmatrix} =
\begin{pmatrix}
\bar{c} c & 0 & 0 \\
0 & c & 0 \\
0 & e & \bar{c}
\end{pmatrix}
\begin{pmatrix}
\rho_0 \\
\kappa_0 \\
\zeta_0
\end{pmatrix}.
\]

Observe that both functions vanish identically
\[ \mathcal{F}(k) \equiv 0, \quad \mathcal{F}(L_1(k)) \equiv 0, \]
since both \( k \) and \( L_1(k) \) are independent of \( v \). Using equation (5.5) of Foo-Merker [3], the new Darboux-Cartan structure equations become
\[
\begin{align*}
    d\rho_0 &= P \rho \wedge \kappa_0 - \frac{L_1(k)}{L_1(k)} \rho \wedge \hat{\zeta}_0 + \bar{P} \rho_0 \wedge \bar{\kappa}_0 - \frac{\overline{L}_1(k)}{\overline{L}_1(k)} \rho_0 \wedge \overline{\zeta}_0 + i\kappa_0 \wedge \bar{\kappa}_0, \\
    d\kappa_0 &= -\frac{L_1(k)}{L_1(k)} \kappa_0 \wedge \hat{\zeta}_0 + \bar{\kappa}_0 \wedge \bar{\kappa}_0, \\
    d\hat{\zeta}_0 &= \frac{\mathcal{L}_1(\mathcal{L}_1(k))}{\mathcal{L}_1(k)} \kappa_0 \wedge \hat{\zeta}_0 - \frac{\overline{\mathcal{L}_1(k)}}{\overline{\mathcal{L}_1(k)}} \hat{\zeta}_0 \wedge \bar{\pi}_0 + \frac{\overline{\mathcal{L}_1(k)}}{\overline{\mathcal{L}_1(k)}} \hat{\zeta}_0 \wedge \overline{\zeta}_0.
\end{align*}
\]

Moreover, one has the following Maurer-Cartan matrix
\[
(\mathbf{d}g)g^{-1} = \begin{pmatrix}
\alpha + \bar{\alpha} & 0 & 0 \\
0 & \alpha & 0 \\
0 & \delta & \alpha - \bar{\alpha}
\end{pmatrix},
\]
with the 1-forms
\[
\alpha = \frac{dc}{\bar{c}}, \quad \delta = \frac{de}{\bar{c}} - \frac{e}{\bar{c}} \left( \frac{dc}{\bar{c}} - \frac{d\bar{c}}{\bar{c}} \right).
\]
One obtains therefore
\[ d\rho = (\alpha + \bar{\alpha}) \wedge \rho + \left( \frac{P}{c} + \frac{\mathcal{L}_1(k) \bar{e}c}{\mathcal{L}_1(k)c^2} \right) \rho \wedge \kappa + \left( -\frac{\mathcal{L}_1(k) \bar{c}}{\mathcal{L}_1(k)c} \right) \rho \wedge \zeta \]
\[ + \left( \frac{\bar{P}}{c} + \frac{\mathcal{L}_1(k) ec}{\mathcal{L}_1(k)c^2} \right) \rho \wedge \bar{\kappa} + \left( -\frac{\mathcal{L}_1(k) c}{\mathcal{L}_1(k)c} \right) \rho \wedge \bar{\zeta} + i\kappa \wedge \bar{\kappa}, \]
\[ d\kappa = \alpha \wedge \kappa + \left( -\frac{\mathcal{L}_1(k) \bar{c}}{\mathcal{L}_1(k)c} \right) \kappa \wedge \zeta - \frac{e}{c} \kappa \wedge \bar{\kappa} + \zeta \wedge \bar{\kappa}, \]
\[ d\zeta = \delta \wedge \kappa + (\alpha - \bar{\alpha}) \wedge \zeta + \left( -\frac{\mathcal{L}_1(k) e\bar{c}}{\mathcal{L}_1(k)c^2} + \frac{\mathcal{L}_1(\mathcal{L}_1(k))}{\mathcal{L}_1(k)} \frac{1}{c} \right) \kappa \wedge \zeta \]
\[ + \left( -\frac{e^2}{c^2} + \frac{\mathcal{L}(k) e\bar{e}}{\mathcal{L}_1(k)c^2} + \frac{\mathcal{L}_1(\mathcal{L}_1(k))e}{\mathcal{L}_1(k)c} \right) \kappa \wedge \pi + \left( \frac{e}{c} - \frac{\mathcal{L}_1(k) e\bar{c}}{\mathcal{L}_1(k)c^2} - \frac{\mathcal{L}_1(\mathcal{L}_1(k))}{\mathcal{L}_1(k)c} \frac{1}{c} \right) \zeta \wedge \bar{\kappa} \]
\[ - \frac{\mathcal{L}_1(\bar{k}) e}{\mathcal{L}_1(k)c} \kappa \wedge \bar{\zeta} + \frac{c\mathcal{L}_1(k)}{\mathcal{L}_1(k)} \zeta \wedge \bar{\zeta}. \]

As before, we proceed with the absorption by setting
\[ \alpha = \hat{\alpha} - x_\rho \rho - x_\kappa \kappa - x_\zeta \zeta - x_{\bar{\kappa}} \bar{\kappa} - x_{\bar{\zeta}} \bar{\zeta}, \]
\[ \delta = \hat{\delta} - y_\rho \rho - y_\kappa \kappa - y_\zeta \zeta - y_{\bar{\kappa}} \bar{\kappa} - y_{\bar{\zeta}} \bar{\zeta}. \]

The equations that need attention are
\[ x_{\bar{\kappa}} + x_\kappa = -\frac{\bar{P}}{c} - \frac{\mathcal{L}_1(k) \bar{e}c}{\mathcal{L}_1(k)c^2}, \]
\[ x_{\bar{\kappa}} = \frac{e}{c}, \]
\[ x_\kappa - x_{\bar{\kappa}} = \frac{e}{c} + \frac{\mathcal{L}_1(k) \bar{e}c}{\mathcal{L}_1(k)c^2} + \frac{\mathcal{L}_1(\mathcal{L}_1(k))}{\mathcal{L}_1(k)c} \frac{1}{c}. \]

For the linear equations to have solutions, one therefore has to make the following choice for \( e \):
\[ e = \frac{c}{\bar{c}} \left( -\frac{1}{3} \bar{P} + \frac{1}{3} \frac{\mathcal{L}_1(\mathcal{L}_1(k))}{\mathcal{L}_1(k)} \right). \]

We remark as well that in [3], a similar normalisation is done during second loop of the Cartan process where the following choice for \( b \) is made:
\[ b = -i\bar{e}c + \frac{i}{3} \left( \frac{\mathcal{L}_1 \mathcal{L}_1(k)}{\mathcal{L}_1(k)} - \bar{P} \right), \]
so that when \( b = 0 \) due to rigidity assumption, the same expression for \( e \) is also obtained.
At this stage, we set
\[ B := -\frac{1}{3} \bar{P} + \frac{1}{3} \frac{\mathcal{L}_1(\mathcal{L}_1(k))}{\mathcal{L}_1(k)}. \]
9. Final loop

We make another change of base coframe by setting

\[ \zeta'_0 = \zeta_0 + B \kappa_0. \]

The new transformation group becomes

\[
\begin{pmatrix} \rho \\ \kappa \\ \zeta \end{pmatrix} = \begin{pmatrix} c \xi & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} \rho_0 \\ \kappa_0 \\ \zeta'_0 \end{pmatrix},
\]

with the new Darboux-Cartan structure:

\[
d\rho_0 = \left( P + B \frac{\mathcal{L}_1(k)}{\mathcal{L}_1(k)} \right) \rho_0 \wedge \kappa_0 - \frac{\mathcal{L}_1(k)}{\mathcal{L}_1(k)} \rho_0 \wedge \zeta'_0
\]
\[
+ \left( P + B \frac{\mathcal{L}_1(k)}{\mathcal{L}_1(k)} \right) \rho_0 \wedge \bar{\kappa}_0 - \frac{\mathcal{L}_1(k)}{\mathcal{L}_1(k)} \rho_0 \wedge \bar{\zeta}'_0
\]
\[
+ i \kappa_0 \wedge \bar{\kappa}_0
\]
\[
= \left( P - \frac{\mathcal{L}_1(k)}{3 \mathcal{L}_1(k)} \right) \kappa_0 \wedge \zeta'_0 - \frac{\mathcal{L}_1(k)}{3 \mathcal{L}_1(k)^2} \rho_0 \wedge \zeta'_0
\]
\[
+ \left( P - \frac{\mathcal{L}_1(k)}{3 \mathcal{L}_1(k)} \right) \rho_0 \wedge \bar{\kappa}_0
\]
\[
= \left( P - \frac{\mathcal{L}_1(k)}{3 \mathcal{L}_1(k)} \right) \kappa_0 \wedge \zeta'_0
\]
\[
= \left( P - \frac{\mathcal{L}_1(k)}{3 \mathcal{L}_1(k)} \right) \kappa_0 \wedge \bar{\kappa}_0
\]
\[
=: R_1 \rho_0 \wedge \kappa + R_2 \rho_0 \wedge \zeta'_0 + \bar{R}_1 \rho_0 \wedge \bar{\kappa}_0 + \bar{R}_2 \rho_0 \wedge \bar{\zeta}'_0 + i \kappa_0 \wedge \bar{\kappa}_0,
\]

\[
d\kappa_0 = -\frac{\mathcal{L}_1(k)}{\mathcal{L}_1(k)} \kappa_0 \wedge \zeta'_0
\]
\[
= -\frac{\mathcal{L}_1(k)}{\mathcal{L}_1(k)} \kappa_0 \wedge \bar{\kappa}_0
\]
\[
=: K_5 \kappa_0 \wedge \zeta'_0 + K_6 \kappa_0 \wedge \bar{\kappa}_0
\]

The 2-form \(d\zeta'_0\) requires a bit of computation, as will be seen in the proof of the following

**Proposition 9.2.** One has

\[
d\zeta'_0 = \left( -B \frac{\mathcal{L}_1(k)}{\mathcal{L}_1(k)} + \frac{\mathcal{L}_1(\mathcal{L}_1(k))}{\mathcal{L}_1(k)} - \frac{\mathcal{K}(B)}{\mathcal{L}_1(k)} \right) \kappa_0 \wedge \zeta'_0
\]
\[
+ \left( B - \frac{\mathcal{L}_1(\mathcal{L}_1(k))}{\mathcal{L}_1(k)} - \frac{B \mathcal{L}_1(\mathcal{L}_1(k))}{\mathcal{L}_1(k)} - \mathcal{L}_1(B) \right) \rho_0 \wedge \zeta'_0
\]
\[
=: Z_5 \kappa_0 \wedge \zeta'_0 + Z_6 \kappa_0 \wedge \bar{\kappa}_0 + Z_8 \rho_0 \wedge \bar{\kappa}_0 + Z_9 \rho_0 \wedge \bar{\zeta}'_0.
Proof. Using the transformation $\zeta_0' = \hat{\zeta}_0 + B\kappa_0$, the 2-forms $d\hat{\zeta}_0$ and $d\kappa_0$ are expressed in terms of the new coframe $(\rho, \kappa_0, \zeta_0')$ as

\[
d\hat{\zeta}_0 = \frac{L_1(k)}{\mathcal{L}_1(k)} \kappa_0 \wedge \zeta'_0 - B \frac{\mathcal{L}_1(k)}{\mathcal{L}_1(k)} \kappa_0 \wedge \zeta_0 - \left( \frac{\mathcal{L}_1(k)}{\mathcal{L}_1(k)} + B \frac{\mathcal{L}_1(k)}{\mathcal{L}_1(k)} \right) \zeta'_0 \wedge \pi_0
\]

\[
+ \left( B \frac{\mathcal{L}_1(k)}{\mathcal{L}_1(k)} + B \frac{\mathcal{L}_1(k)}{\mathcal{L}_1(k)} \right) \kappa_0 \wedge \pi_0 + \frac{\mathcal{L}_1(k)}{\mathcal{L}_1(k)} \zeta'_0 \wedge \pi_0,
\]

as well as

\[
d\kappa_0 = -\frac{L_1(k)}{\mathcal{L}_1(k)} \kappa_0 \wedge \zeta'_0 - B \kappa_0 \wedge \pi_0 + \zeta'_0 \wedge \pi_0.
\]

Moreover one has for the 1-form $dB$ the following expansion

\[
dB = \mathcal{T}(B) \rho_0 + L_1(B) \kappa_0 + \mathcal{K}(B) \zeta_0 + \mathcal{L}_1(B) \pi_0 + \mathcal{K}(B) \zeta_0.
\]

By rigidity assumption, $\mathcal{T}(B) \equiv 0$; and by using the Assertion 7.4 on page 26 of Foo-Merker [3],

\[
\mathcal{K}(B) = -B \mathcal{L}_1(\kappa).
\]

Using these two observations, the 1-form $dB$ is therefore

\[
dB = \left( L_1(B) - B \frac{\mathcal{K}(B)}{\mathcal{L}_1(k)} \right) \kappa_0 + \frac{\mathcal{K}(B)}{\mathcal{L}_1(k)} \zeta_0 + \left( \mathcal{L}_1(B) + B \frac{\mathcal{L}_1(k)}{\mathcal{L}_1(k)} \right) \pi_0 - B \frac{\mathcal{L}_1(k)}{\mathcal{L}_1(k)} \zeta_0.
\]

Substituting $d\hat{\zeta}_0$, $d\kappa_0$ and $dB$ in the following identity

\[
d\zeta'_0 = d\hat{\zeta}_0 + dB \wedge \kappa_0 + B \, d\kappa_0
\]

by the expressions computed above finishes the proof of the proposition. \hfill \Box

Explicitly,

\[
d\zeta'_0 = \left( \frac{PL_1(k)}{3L_1(k)} - \frac{L_1(k)\mathcal{L}_1(k)}{3L_1(k)^2} + \frac{L_1(k)}{3L_1(k)} \right) \kappa_0 \wedge \zeta'_0
\]

\[
+ \left( \frac{-P^2}{9} - \frac{P\mathcal{L}_1(k)}{9\mathcal{L}_1(k)} + \frac{5\mathcal{L}_1(k)}{9\mathcal{L}_1(k)^2} \right) \kappa_0 \wedge \pi_0
\]

\[
+ \left( \frac{-P}{3} - \frac{2\mathcal{L}_1(k)}{3\mathcal{L}_1(k)} + \frac{P\mathcal{L}_1(k)}{3\mathcal{L}_1(k)} - \frac{\mathcal{L}_1(k)}{3\mathcal{L}_1(k)^2} \right) \zeta'_0 \wedge \pi_0
\]

\[
+ \frac{\mathcal{L}_1(k)}{\mathcal{L}_1(k)} \zeta'_0 \wedge \pi_0.
\]
After transformation, the new 2-forms $d\rho$, $d\kappa$ and $d\zeta$ become

\[
d\rho = (\alpha + \alpha) \wedge \rho + \frac{1}{c} R_1 \rho \wedge \kappa + \frac{\bar{c}}{c} R_2 \rho \wedge \zeta + \frac{1}{\bar{c}} R_1 \rho \wedge \bar{\kappa} + \frac{c}{\bar{c}} R_2 \rho \wedge \bar{\zeta} + i\kappa \wedge \bar{\kappa},
\]

\[
d\kappa = \alpha \wedge \kappa + \frac{\bar{c}}{c} K_5 \kappa \wedge \zeta + \frac{1}{c} K_6 \kappa \wedge \bar{\kappa} + \zeta \wedge \bar{\kappa},
\]

\[
d\zeta = (\alpha - \bar{\alpha}) \wedge \zeta + \frac{1}{c} Z_5 \kappa \wedge \zeta + \frac{1}{\bar{c}} Z_6 \kappa \wedge \bar{\kappa} + \frac{1}{\bar{c}} Z_8 \zeta \wedge \bar{\kappa} + \frac{c}{\bar{c}} Z_9 \zeta \wedge \bar{\zeta}.
\]

By setting the new Maurer-Cartan 1-form as

\[
\alpha := \hat{\alpha} - x_\rho \rho - x_\kappa \kappa - x_\zeta \zeta - x_{\bar{\kappa}} \bar{\kappa} - x_{\bar{\zeta}} \bar{\zeta},
\]

with

\[
x_\rho = 0, \quad x_\kappa = -\frac{1}{c} R_1 + \frac{1}{c} K_6, \quad x_\bar{\kappa} = \frac{1}{c} B, \quad x_\zeta = \frac{\bar{c} \mathcal{L}_1(k)}{\mathcal{L}_1(k)}, \quad x_{\bar{\zeta}} = 0,
\]

the final absorbed equations become:

\[
d\rho = (\hat{\alpha} + \bar{\alpha}) \wedge \rho + i\kappa \wedge \bar{\kappa}, \quad (9.5)
\]

\[
d\kappa = \hat{\alpha} \wedge \kappa + \zeta \wedge \bar{\kappa}, \quad (9.6)
\]

\[
d\zeta = (\hat{\alpha} - \bar{\alpha}) \wedge \zeta + \frac{1}{c} (Z_5 - \bar{Z}_8) \kappa \wedge \zeta + \frac{1}{\bar{c}} Z_6 \kappa \wedge \bar{\kappa}.
\]

\[
10. \text{ The } \{e\}-\text{structure.}
\]

This time, for ease of notation, we write

\[
S_5 = \frac{1}{c} (Z_5 - \bar{Z}_8) := \frac{1}{c} I_0, \quad S_6 = \frac{1}{\bar{c}^2} Z_6 := \frac{1}{\bar{c}^2} V_0.
\]

If we write

\[
\psi := -S_5 \zeta - S_6 \bar{\kappa},
\]

equation (9.7) may be written otherwise as

\[
d\zeta = (\hat{\alpha} - \bar{\alpha}) \wedge \zeta + \psi \wedge \kappa.
\]

Based on the model case in Section 4, one should obtain for $d\hat{\alpha}$ the following:

\[
d\hat{\alpha} = \zeta \wedge \bar{\zeta} + \cdots,
\]

where the remaining terms are 2-forms that vanish in the model case. Taking exterior derivatives of both sides of equations (9.5), (9.6) and (9.7):

\[
0 = (d\hat{\alpha} + d\bar{\alpha}) \wedge \rho,
\]

\[
0 = (d\hat{\alpha} - \zeta \wedge \bar{\zeta} + S_5 \zeta \wedge \bar{\kappa}) \wedge \kappa
\]

\[
0 = (d\hat{\alpha} - d\bar{\alpha}) \wedge \zeta - (\hat{\alpha} - \bar{\alpha}) \wedge d\zeta + d\psi \wedge \kappa - \psi \wedge \alpha \wedge \kappa.
\]

In the second equation of (10.1), Cartan’s lemma provides a 1-form $A$ with

\[
d\hat{\alpha} = \zeta \wedge \bar{\zeta} - S_5 \zeta \wedge \bar{\kappa} + A \wedge \kappa.
\]

To study $A$, write it as a formal linear combination of the 1-forms with unknown coefficients:

\[
A = A_\rho \rho + A_\kappa \kappa + A_\zeta \zeta + A_{\bar{\kappa}} \bar{\kappa} + A_{\bar{\zeta}} \bar{\zeta} + A_{\bar{\alpha}} \bar{\alpha}.
\]
From the first equation of (10.1), one obtains
\[ A_\zeta = S_5, \quad A_\bar{\zeta} = 0, \quad A_\bar{\kappa} \text{ is real,} \quad A_\bar{\alpha} = A_\bar{\bar{\alpha}} = 0, \]
and so
\[ d\bar{\alpha} = \zeta \wedge \bar{\zeta} - S_5 \zeta \wedge \bar{\kappa} + A_{\rho \bar{\kappa}} \wedge \kappa + A_{\bar{\kappa} \bar{\bar{\kappa}}} \wedge \kappa + S_5 \bar{\zeta} \wedge \kappa. \]

Using this expression of \( d\bar{\alpha} \) in the third equation of (10.1), the remaining coefficients of \( A \) are therefore obtained:
\[ A_\rho = 0, \quad 0 = 2A_{\bar{\kappa}} \bar{\kappa} \wedge \kappa \wedge \zeta + \bar{\alpha} \wedge \psi \wedge \kappa \wedge \bar{\zeta} + d\psi \wedge \kappa \wedge \bar{\zeta}. \]

We expand \( d\psi \) so that
\[ d\psi \wedge \kappa \wedge \bar{\zeta} = (-dS_5 \zeta \wedge \bar{\kappa} - dS_5 \bar{\zeta} \wedge \kappa - dS_6 \bar{\kappa} \wedge \bar{\zeta} \wedge \kappa + d\psi \wedge \kappa \wedge \bar{\zeta}) \]

which is a secondary invariant.

To make sure that the equation does make sense, the term on the right needs to be verified that it is real-valued. This requires some computation. First we need a lemma:

**Lemma 10.3.** On the \( G \)-structure \( M \times G^2 \) with coordinates \((z_1, z_2, \bar{z}_1, \bar{z}_2, v, c, \bar{c})\), let \( F : M \times G^2 \to \mathbb{C} \) be a function. Then
\[
\begin{align*}
\frac{dF}{c} &= c \partial_k F \, \bar{\alpha} + \bar{c} \partial_k F \, \bar{\bar{\alpha}} + \left( \frac{1}{cc} \partial_k (F) - c x_\kappa \partial_k F - c \bar{c} \bar{x}_\kappa \partial_k F \right) \rho \\
&\quad + \left( \frac{1}{c} \left( \frac{L_1 (F) - B \mathcal{H} (F)}{L_1 (k)} \right) - c x_\kappa \partial_k F - c \bar{c} \bar{x}_\kappa \partial_k F \right) \kappa \\
&\quad + \left( \frac{\bar{c}}{c} \mathcal{H} (F) \right) \zeta \\
&\quad + \left( \frac{1}{c} \left( \frac{L_1 (F) - B \mathcal{H} (F)}{L_1 (k)} \right) - c x_\bar{\kappa} \partial_k F - c \bar{c} \bar{x}_\bar{\kappa} \partial_k F \right) \bar{\kappa} \\
&\quad + \left( \frac{c \mathcal{H} (F)}{c} \frac{L_1 (F)}{L_1 (k)} - c x_\zeta \partial_k F - c \bar{c} \bar{x}_\zeta \partial_k F \right) \bar{\zeta} \\
&:= \partial_\alpha (F) \, \alpha + \partial_\kappa (F) \, \kappa + \partial_\rho (F) \, \rho + \partial_\kappa (F) \, \kappa + \partial_\zeta (F) \, \zeta \\
&\quad + \partial_{\bar{\kappa}} (F) \, \bar{\kappa} + \partial_{\bar{\zeta}} (F) \, \bar{\zeta},
\end{align*}
\]

The proof of the lemma is done by straightforward computation which will be skipped. With the solution to the absorption equations (9.4), we therefore have the following vector fields:
\[ \partial_\alpha := c \partial_c, \]
\[ \partial_\rho := \frac{1}{cc} \mathcal{T}, \]
\[ \partial_\kappa := \frac{1}{c} \left( \mathcal{L}_1 - \frac{B \mathcal{K}}{\mathcal{L}_1(k)} \right) - c \left( -\frac{1}{c} R_1 + \frac{1}{c^2} \mathcal{K}_6 \right) \partial_c + \frac{cc}{c} \mathcal{B} \partial_c, \]
\[ \partial_\zeta = \frac{c}{cc} \mathcal{K} \mathcal{F}_1(k) + \frac{cc}{c} \mathcal{F}_1(k) \partial_c, \]

while the vector fields \( \partial_\kappa, \partial_\rho, \partial_\zeta \) are respective complex conjugates of \( \partial_\alpha, \partial_\kappa, \partial_\zeta \). As a result:

\[ (S_5)_{\bar{\kappa}} - (S_6)_{\bar{\zeta}} = \frac{1}{cc} \left( \mathcal{F}_1(I_0) - \frac{B \mathcal{K}}{\mathcal{L}_1(k)} + BI_0 - \frac{\mathcal{K}(V_0)}{\mathcal{L}_1(k)} \right) := \frac{1}{cc} Q_0 \]
\[ = \frac{1}{cc} \left( \mathcal{F}_1(Z_5) - \mathcal{F}_1(Z_8) - B \frac{\mathcal{K}(Z_5)}{\mathcal{L}_1(k)} + B \frac{\mathcal{K}(Z_8)}{\mathcal{L}_1(k)} + BZ_5 - BZ_8 - \frac{\mathcal{K}(Z_6)}{\mathcal{L}_1(k)} \right). \]

We will also need the following

**Lemma 10.6.** *One has the following identity*

\[ \mathcal{F}_1(Z_5) - \frac{\mathcal{K}(Z_5)}{\mathcal{L}_1(k)} = B \frac{\mathcal{K}(Z_5)}{\mathcal{L}_1(k)} + Z_5 K_6 - Z_5 K_5 - \mathcal{L}_1(Z_8) + B \frac{\mathcal{K}(Z_8)}{\mathcal{L}_1(k)} + Z_8 K_6 + Z_9 Z_6. \]

**Proof.** We will compute the terms on the left-hand side by applying \( d^2 \equiv 0 \) to the third equation of equation (9.1). Doing so, while wedging on both sides of \( d^2 \zeta' = 0 \) with \( \rho \wedge \bar{\zeta}' \), one should get

\[ 0 = (Z_5)_{\kappa_0} - Z_5 K_5 - Z_5 Z_8 - (Z_6)_{\zeta_0} + Z_6 K_5 + (Z_8)_{\kappa_0} + Z_8 Z_5 - Z_8 K_6 - Z_8 Z_6 \rho_0 \wedge \kappa_0 \wedge \bar{\kappa}_0 \wedge \zeta_0 \wedge \bar{\zeta}_0. \]

Finally, for any function \( G \) independent of \( c \), one uses the following formula

\[ dG = T(G) \rho + \left( \mathcal{L}_1(G) - \frac{B \mathcal{K}(G)}{\mathcal{L}_1(k)} \right) \kappa_0 + \frac{\mathcal{K}(G)}{\mathcal{L}_1(k)} \zeta_0 \]
\[ + \left( \mathcal{F}_1(G) - \frac{B \mathcal{K}(G)}{\mathcal{L}_1(k)} \right) \bar{\kappa}_0 + \frac{\mathcal{K}(G)}{\mathcal{L}_1(k)} \bar{\zeta}_0. \]

The proof is therefore complete by applying this to \((Z_5)_{\kappa_0}, (Z_6)_{\zeta_0}\) and \((Z_8)_{\kappa_0}\). \( \square \)

Substituting the identity into \( A_{\bar{k}} \), one has therefore

\[ -2 A_{\bar{k}} = \frac{1}{cc} \left( -Z_6 K_5 + Z_9 Z_6 \right) - \mathcal{L}_1(Z_8) - \mathcal{F}_1(Z_8) + B \frac{\mathcal{K}(Z_8)}{\mathcal{L}_1(k)} + B \frac{\mathcal{K}(Z_6)}{\mathcal{L}_1(k)} - Z_8 B - Z_8 B, \]

and observing that \( Z_9 = -\bar{K}_3 \), the coefficient \( A_{\bar{k}} \) is thus real-valued, and the \{e\}-structure is finally complete.

We have therefore proved Theorem 1.3.

In the interest of computations, the secondary invariant

\[ Q_0 := \frac{1}{2} \left( \mathcal{F}_1(I_0) - \frac{B \mathcal{K}(I_0)}{\mathcal{L}_1(k)} + BI_0 - \frac{\mathcal{K}(V_0)}{\mathcal{L}_1(k)} \right) \]

may further be simplified using the following:
Proposition 10.8. Under the Levi degeneracy assumption, one has:

$$\frac{\mathcal{F}(I_0)}{\mathcal{L}_1(k)} = -2I_0.$$ 

Proof. We remark here that the Levi-degeneracy condition is necessary to normalise the expression and thus it cannot be dropped. It is implicitly used in $d\rho$ the first equation of the following $\{e\}$-structure:

$$d\rho = (\alpha + \bar{\alpha}) \wedge \rho + i\kappa \wedge \bar{\pi},$$

$$d\kappa = \alpha \wedge \kappa + \zeta \wedge \pi,$$

$$d\zeta = (\alpha - \bar{\alpha}) \wedge \zeta + \frac{1}{c}\bar{I}_0 \kappa \wedge \zeta + \frac{1}{\bar{c}c}V_0 \kappa \wedge \pi,$$

$$d\alpha = \zeta \wedge \alpha - \frac{1}{c}\bar{I}_0 \zeta \wedge \alpha + \frac{1}{\bar{c}c}Q_0 \kappa \wedge \alpha + \frac{1}{c}I_0 \bar{\pi} \wedge \kappa.$$

Applying Poincaré derivative to the third equation $d\zeta$ and using $d^2 \equiv 0$, while wedging on both sides with $\alpha \wedge \pi \wedge \rho \wedge \bar{\pi}$, we obtain

$$0 = d\alpha \wedge \zeta \wedge \alpha \wedge \pi \wedge \rho \wedge \bar{\pi} - d\bar{\alpha} \wedge \zeta \wedge \alpha \wedge \pi \wedge \rho \wedge \bar{\pi}$$

$$+ \partial_\zeta \left( \frac{1}{c}I_0 \right) \bar{\zeta} \wedge \kappa \wedge \zeta \wedge \alpha \wedge \pi \wedge \rho \wedge \bar{\pi},$$

where $\partial_\zeta$ is the following vector field coming from equation (10.5):

$$\partial_\zeta = \frac{c}{\bar{c}} \frac{\mathcal{F}}{\mathcal{L}_1(k)} - c \frac{\mathcal{L}_1(k)}{\mathcal{L}_1(k)} \partial_c.$$ 

Then using $d\alpha$ and $d\bar{\alpha}$ from the $\{e\}$-structure, we obtain the desired identity. □

Thus we recover the expression of $Q_0$ as appeared in the introduction.

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