On the contact interaction of two identical stringers with an elastic semi-infinite continuous or vertically cracked plate

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Abstract. This paper considers two connected contact problems on the interaction of stringers with an elastic semi-infinite plate. In the first problem, an elastic half-infinite continuous plate is reinforced on its boundary by two identical stringers exposed to a tensile external force. In the second problem, in the presence of the same stringers, the plate contains a collinear system of cracks on its vertical axis. The solution of both problems is reduced to the solution of singular integral equations (SIE) that are solved by a known numerical-analytical method.

1. Introduction
The problems of contact interaction of thin-walled elements such as stringers with massive deformable bodies generalize and develop the classical contact problems of the theory of elasticity. Due to their theoretical and practical significance, these problems are the subject of many studies [1–6]. A brief review of the main works and results is given in [7]. The problems of interaction between stringers and massive elastic bodies with cracks [8–12] are of significant interest.

The present paper considers two contact problems related to each other. In the first problem, an elastic half-infinite continuous plate which, in the right-hand rectangular coordinate system, occupies the lower half-plane is reinforced on its boundary by two identical stringers located symmetrically with respect to the origin of the coordinate system. At the far ends of the stringers, horizontal tensional concentrated forces, equal in value but opposite in direction, are applied. In addition, tensional distributed tangential forces of given intensities act on the upper faces of the stringers along their middle lines. The plate itself is exposed to the action of uniformly distributed horizontal forces tensile in the horizontal direction at infinity. This problem is mathematical formulated as the Prandtl integro-differential equation for stringers in the well-known Melan physical model [1–2]. The procedure of solving the latter by the method of Chebyshev polynomials [6] is reduced to solving a regular infinite system of linear algebraic equations (SLAE). Here the solution of the governing SIE of this problem is constructed by a method which is simpler from the standpoint of computational mathematics. Namely, the solution is represented in the form of a product of a weight function that takes into account the characteristic singularities of the contact stresses under the stringers and some regular function.
of the H"older class. In a particular case where the stringers are not loaded or are under the influence of a self-balanced system of external forces, we also use the Multopp method [13].

Further, we use the Kolosov–Muskhelishvili complex potentials [14] to determine the normal breaking stresses on the vertical axis of the plate at the origin of the coordinate system. An analysis of the formula for these stresses shows that, as the near ends of the stringers tend to zero, i.e., when they infinitely approach each other, these stresses increase infinitely attaining the value of the brittle fracture limit for a given material. As a result, the vertical cracking of the elastic plate occurs.

Proceeding from this fact, the second problem closely related to the first problem is considered. We have the first problem configuration, but there is a crack of finite length on the vertical axis and the crack edges can be loaded with vertical forces. The following two cases of the crack location are discussed:

(a) when it is located inside the plate at some distance from its boundary;
(b) when the crack tip vertically reaches the boundary of the plate.

In both cases, the discussed problem is described by a system of two SIE.

In the case (a), the dislocation densities on the crack edges are unbounded at its tips, and they have ordinary root singularities. In the case (b), the dislocation densities are unbounded at the tip of the lower internal crack and, in general, are bounded at the upper tip. The solution of the governing system of the SIE is constructed by a known numerical-analytical method [15–17].

2. Solution of the first boundary-value problem for an elastic quarter-plane

Let the elastic semi-infinite plate with a rectangular coordinate system $Oxy$, which occupies the lower half-plane $y \leq 0$, have modulus of elasticity $E$, Poisson’s ratio $\nu$, and width $d_1$. Further, suppose that such a plate, on its boundary $y = 0$, on the segments $[-a, -b]$ and $[b, a]$ symmetrically located about the origin, is reinforced by two identical elastic stringers of modulus of elasticity $E_s$; $d_2$ is the width and $h$ is the height. We assume that the stringers are also loaded symmetrically, and horizontal concentrated forces $P$ equal in values and opposite in direction act at the centers of their end-sections $y = \pm a$, while the sections $y = \pm b$ are free from external forces and tensile tangential forces of intensity $\tau_+(y)$ ($y \in (-a, -b) \cup (b, a); \tau_+(y) = -\tau_-(y)$) act along the middle lines of the upper edges of the stringers. At infinity, the plate is subject to the action of uniformly distributed tensile horizontal forces of intensity $P_0$.

Under these assumptions, within the physical model of the one-dimensional elastic continuum of stringers [1–2], it is required to determine the distribution of tangential contact stresses $\tau_-(y)$ referred to the unit width $d_1$ of the plate, and in addition $\tau_-(y) = -\tau_+(y)$.

We introduce the governing SIE of the problem. Following [6], we consider the equilibrium of the part $[b, r]$ of the right stringer. We obtain

$$\sigma_s(r) = \frac{1}{A_s} \left[ d \int_b^r \tau_-(s) \, ds - \int_b^{r} \tau_+(s) \, ds \right] \quad (r = y \in [b, a]). \quad (1)$$

Here $\sigma_s(r)$ ($r = y$) is the axial stress in the section $r$ of the stringer, $d = \min(d_1, d_2)$ is the effective width of the contact zone, and $A_s = d_2 h$ is the cross-section area of the stringer. According to Hooke’s law, from (1), for the axial deformation $\varepsilon_s(r)$ of the stringer, we obtain

$$\varepsilon_s(r) = \frac{1}{A_s E_s} \left[ d \int_b^r \tau_-(s) \, ds - \int_b^{r} \tau_+(s) \, ds \right] \quad (y = r \in (b, a)). \quad (2)$$

Similarly considering the same equilibrium of the part $[r, a]$ of the right stringer, we obtain

$$\varepsilon_s(r) = \frac{1}{A_s E_s} \left[ -d \int_r^a \tau_-(s) \, ds + \int_r^{a} \tau_+(s) \, ds \right] \quad (y = r \in (b, a)). \quad (3)$$
Now, combining (2) and (3), we find
\[ \varepsilon_s(r) = \frac{1}{2A_0E_A} \left[ \int_b^a \frac{d}{r} \frac{\left( r - s \right)}{\tau_-(s)} ds - \int_a^b \frac{d}{r} \frac{\left( r - s \right)}{\tau_+(s)} ds \right] \quad (y = r \in (b, a)). \] (4)

Note that setting \( r = a \) in (1), we obtain the equilibrium condition for the right strirger:
\[ d \int_b^a \tau_-(s) ds = Q; \quad Q = P + T_+ = P + \int_b^a \tau_+(s) ds. \] (5)

On the other hand, for the derivative of the horizontal displacements \( u_y(y, 0) \) of points of the elastic semi-infinite plate, we have [6]
\[ u_y'(y, 0) = \frac{4}{\pi E} d \int_b^a \frac{s \tau_-(s) ds}{s^2 - y^2} + \frac{P_0}{E} (-\infty < y < \infty). \] (6)

Next, we write the condition of contact between the right strirger and the plate
\[ u_y'(y, 0) = \varepsilon_s(y) \quad (-\infty < y < \infty). \] (7)

Substituting (4) and (6) into (7), we obtain the following governing SIE of the problem:
\[ \frac{4}{\pi E} d \int_b^a \frac{s \tau_-(s) ds}{s^2 - y^2} + \frac{P_0}{E} = \frac{1}{2A_0E_A} \left[ d \int_b^a \frac{\left( r - s \right)}{\tau_-(s)} ds - \int_a^b \frac{d}{r} \frac{\left( r - s \right)}{\tau_+(s)} ds \right] \] (5)
\[ (b < y = r < a). \]

Now we introduce dimensionless quantities in (8) and in (5):
\[ \xi = \frac{r}{a}, \quad \eta = \frac{s}{a}, \quad \tau(\xi) = \frac{d\tau_-(a\xi)}{d_1E}, \quad \tau_0(\xi) = \frac{\tau_+(a\xi)}{d_1E}, \quad p_0 = \frac{P_0}{E}, \quad \rho = \frac{b}{a}, \]
\[ \alpha_k = \frac{a k}{a}, \quad \beta_k = \frac{b k}{a} \quad (k = 1, n), \quad L_0 = \bigcup_{k=1}^n (\alpha_k, \beta_k), \quad \Lambda = \frac{ad_1E}{2A_0E_A}. \]

As a result, SIE (8) becomes
\[ \frac{2}{\pi} \int_{\rho}^1 \frac{2\pi \tau(\eta) d\eta}{\eta^2 - \xi^2} + p_0 = \Lambda \left[ \int_{\rho}^1 \frac{\left( \xi - \eta \right)}{\tau(\eta)} d\eta \right] - \int_{\rho}^1 \frac{\left( \xi - \eta \right)\tau_0(\eta)}{d_1E} d\eta \quad (\rho < \xi < 1), \] (9)
and condition (5) takes the form
\[ \int_{\rho}^1 \tau(\eta) d\eta = Q_0, \quad Q_0 = \frac{Q}{ad_1E}. \] (10)

Further, in (9) and in (10), the interval \((\rho, 1)\) is transformed into the interval \((-1, 1)\) [6] for which we set
\[ \xi = \sqrt{c}t + d, \quad \eta = \sqrt{cu} + d \quad \left( c = \frac{1 - \rho^2}{2}, \quad d = \frac{1 + \rho^2}{2}, \quad -1 < t, u < 1 \right). \]

As a result, SIE (9) is transformed to the form
\[ \frac{2}{\pi} \int_{-1}^1 \left[ \frac{1}{u - t} + \frac{\pi \Lambda c}{4\sqrt{cu} + d} \right] \frac{\left( u - t \right)}{\tau(\sqrt{cu} + d)} du = f_0(t) \quad (-1 < t < 1), \]
\[ \chi(u) = \tau(\sqrt{cu} + d), \quad f_0(t) = f_0(\sqrt{ct} + d), \quad f(\xi) = -p_0 + \Lambda \int_{\rho}^1 \frac{\left( \eta - \xi \right)}{\tau_0(\eta)} d\eta, \] (11)
and condition (10), to the form

$$\int_{-1}^{1} \frac{\chi(u) \, du}{\sqrt{cu + d}} = \frac{Q_0}{c}.$$  \hfill (12)

Now let us solve SIE (11)–(12). For this purpose, following the well-known procedure [15–17], we set

$$\chi(u) = \frac{X(u)}{\sqrt{1 - u^2}} \quad (-1 < u < 1),$$

where $X(u)$ is the Hölder function on the interval $[-1, 1]$. As a result, the process of solving SIE (11)–(12) reduces to solving the SLAE

$$\sum_{n=1}^{M} L_{m,n} X_n = c_m \quad (m = 1, M),$$  \hfill (13)

$$L_{m,n} = \begin{cases} 
\frac{2}{M} \left[ \frac{1}{u_n - t_m} + \frac{\pi \Delta c}{4\sqrt{cu_n + d}} \text{sign} (u_n - t_m) \right], & n = 1, M, \quad m = 1, M - 1, \\
\frac{\pi}{M} \sqrt{cu_n + d}, & n = 1, M, \quad m = M, \\
f_0(t_m), & m = 1, M - 1, \\
\frac{Q_0}{c}, & m = M, \\
\cos \left( \frac{2n - 1}{2M} \pi \right), & n = 1, M, \\
\cos \frac{\pi m}{M}, & m = 1, M - 1.
\end{cases}$$

Here $u_n$ and $t_m$ are Chebyshev nodes, i.e., the roots of the Chebyshev polynomials of the first kind $T_m(u)$ and of the second kind $U_{M-1}(t)$, respectively.

After solving SLAE (13), the reduced dimensionless tangential stresses is calculated by the formula

$$\chi(u_n) = \frac{X(u_n)}{\sqrt{1 - u_n^2}} \quad (n = 1, M, \quad u_n = \cos \left( \frac{(2n - 1)\pi}{2M} \right)).$$

Writing formula (1) for the axial stresses in the form

$$\sigma_s(r) = \frac{1}{2A_s} \left[ d \int_{a}^{b} \text{sign} (r - s) \tau_- (s) \, ds - \int_{b}^{a} \text{sign} (r - s) \tau_+ (s) \, ds \right] \quad (r = y \in [b, a]),$$

turning to the interval $(-1, 1)$, and using the formulas given above, we can obtain the dimensionless axial stresses by solving SLAE (13).

### 3. Stress state of an elastic half-plane with a collinear system of vertical cracks

The problem considered in the previous section can be solved exactly if the stringers are absolutely rigid in tension and compression and absolutely flexible in bending. On the basis of this solution, using the normal Kolosov–Muskhelishvili complex potentials [14], one can determine the normal breaking stresses on the vertical axis of the elastic semi-infinite plate. An analysis of the formula for these stresses shows that they increase infinitely as the near ends of the stringers approach each other ($b \to +0$) and, consequently, attain the value of the brittle fracture limit for a given material. As a result, the elastic semi-infinite plate cracks, and a vertical crack occurs on the edge. Therefore, to understand the further development of a tensile elastic semi-infinite plate with two identical symmetrically disposed stringers, it is necessary to study the contact interaction between the stringers and the elastic semi-infinite plate with a vertical edge crack or, in general, with a collinear system of vertical cracks.
Proceeding from this fact, in the contact problem on two identical stringers on an elastic semi-infinite plate described in the previous section, we assume that the plate contains a collinear system of cracks \( L = \bigcup_{k=1}^{\infty} [a_k, b_k] \) on its vertical axis \( Ox \) directed inwards. In addition, we assume that, on the edges of the system of cracks \( L \), forces of intensity \( p(x) \) normal to the edge are distributed which leads to normal separation, but there are no tangential forces.

Under these assumptions and again within the physical model of the one-dimensional elastic set of stringers \([1–2]\), it is required to determine the distribution of tangential contact stresses \( \tau - \Omega \) referred to unit width \( d_1 \) of the plate and the following characteristics of the cracks: the breaking stresses outside the cracks and along the line of their location, the SIF, and the crack openings.

We introduce the governing SIE of the problem. For this purpose, we first construct the solution of the first boundary-value problem for the right elastic quarter-plane-wedge \( \Omega_+ = \{0 < r < \infty, 0 < \vartheta \leq \pi/2\} \) with the right angle in the polar coordinates \( r, \vartheta \):

\[
\begin{align*}
\tau_r \big|_{\vartheta=+0} &= 0 \quad (0 < r < \infty), \quad \sigma_\vartheta \big|_{\vartheta=+0} = -q_0^+ (r) = \begin{cases} -p(r), & r \in L, \\ -\sigma(r), & r \in [0, \infty) \setminus L, \end{cases} \\
\tau_r \big|_{\vartheta=\pi/2-0} &= \tau_- (r) \quad (0 < r < \infty), \quad \sigma_\vartheta \big|_{\vartheta=\pi/2-0} = 0 \quad (0 < r < \infty),
\end{align*}
\]

where \( \sigma_\vartheta \) and \( \tau_r \) are, respectively, the components of the normal and tangential stresses in the polar coordinates. In this coordinate system, we also write the equations without volume forces \([18]\):

\[
\begin{align*}
(\lambda + 2\mu) \frac{\partial e}{\partial r} - \frac{\mu}{r} \frac{\partial}{\partial \vartheta} (\chi) = 0, \quad (\lambda + 2\mu) \frac{1}{r} \frac{\partial e}{\partial \vartheta} + \frac{\partial}{\partial r} \left( \frac{\chi}{r} \right) = 0, \\
e = \frac{\partial u_r}{\partial r} + \frac{1}{r} \left( u_r + \frac{\partial u_\vartheta}{\partial \vartheta} \right), \quad \chi = r \frac{\partial u_\vartheta}{\partial r} + u_\vartheta - \frac{\partial u_r}{\partial \vartheta} = 2r \omega_z,
\end{align*}
\]

and Hooke’s law

\[
\sigma_r = 2\mu \frac{\partial u_r}{\partial r} + \lambda e, \quad \sigma_\vartheta = \frac{2\mu}{r} \left( \frac{\partial u_\vartheta}{\partial r} + u_r \right) + \lambda e, \quad \tau_r \vartheta = \mu \left( \frac{\partial u_\vartheta}{\partial r} - \frac{u_\vartheta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \vartheta} \right).
\]

Here \( u_r \) and \( u_\vartheta \) are the respective components of displacements of the elastic body point in the radial and circular directions, \( e \) is the relative volume expansion, \( \omega_z \) is the angle of rotation about the axis \( Oz \) perpendicular to the plane of the wedge \( \Omega_+ \), \( \lambda \) and \( \mu \) are the Lame constants of the elastic body expressed in terms of Young’s modulus and Poisson’s ratio:

\[
\lambda = \frac{E\nu}{(1 + \nu)(1 + 2\nu)}, \quad \mu = G = \frac{E}{2(1 + \nu)}.
\]

The solution of boundary-value problem (14)–(16) is constructed using the Mellin integral transformation with respect to the variable \( r \). As a result, we have

\[
\begin{align*}
\frac{du_r(r, \pi/2)}{dr} &= \frac{\lambda + 2\mu}{\pi \mu (\lambda + \mu)} \int_0^\infty \frac{s \tau_- (s) ds}{s^2 - r^2} + \frac{2}{\pi} \int_0^\infty \frac{sr^2}{(s^2 + r^2)^2} \varphi^2 (s) ds \quad (0 < r < \infty), \\
q_0^+ (r) &= -\frac{4}{\pi} \int_0^\infty \frac{s^3 \tau_- (s) ds}{(s^2 + r^2)^2} - \frac{\mu (\lambda + \mu)}{\pi (\lambda + 2\mu)} \int_0^\infty \left[ \frac{1}{s - r} - \frac{1}{s + r} + \frac{6r}{(s + r)^2} - \frac{4r^2}{(s + r)^3} \right] \varphi^2 (s) ds.
\end{align*}
\]

Here we introduced the dislocation density on the crack edges

\[
\varphi (r) = \frac{2}{\pi} \frac{du_\vartheta(r, +0)}{dr} \quad (r = x),
\]
and \( \varphi(r) = 0 \) outside the cracks on \( L' = [0, \infty) \setminus L \).

Now we can derive the system of the governing SIE of the posed problem. For this, we write the condition of contact between the right stringer and the plate

\[
\frac{du_r(r, \pi/2)}{dr} = \varepsilon_s(r) \quad (b < r < a)
\]  

(19)

and the condition which follows from the second equality in (14)

\[
q_0^+(r) = p(r) \quad (r \in L).
\]  

(20)

In addition, according to (6), the term \( P_0/E \), which takes into account the longitudinal deformation of the plate due to tensile forces at infinity, must be added to the right-hand side of (17) and the function \( p(r) \) must be replaced by a function \( p(r) + P_0 \) in (20). Then we use (17)–(18) and (19)–(20) to arrive at the following governing system of SIE of the posed problem:

\[
\frac{4}{\pi E \, d_1} \int_b^a \frac{sr_-(s) \, ds}{s^2 - r^2} + \frac{2}{\pi} \int_L \frac{sr^2 \varphi(s) \, ds}{(s^2 + r^2)^2} + \frac{P_0}{E} = \frac{1}{2A_k E_s} \left[ d \int_b^a \tau_-(s) \, ds - \int_b^a \tau_+(s) \, ds \right] \quad (b < r < a).
\]

\[
\frac{1}{\pi} \int_L \left[ \frac{2r}{s^2 - r^2} + \frac{6r^2}{(s + r)^2} - \frac{4r^2}{(s + r)^3} \right] \varphi(s) \, ds + \frac{16}{\pi E \, d_1} \int_b^a \frac{s^3 \varphi(s) \, ds}{(s^2 + r^2)^2} = -\frac{4}{E} q(r)
\]

(21)

For \( r \in L; \ q(r) = p(r) + P_0 \).

The solution of the SIE system of two equations (21) must satisfy the equilibrium condition of the right stringer (5) and the continuity conditions for the displacements at the tips of the cracks:

\[
\int_{\alpha_k}^{b_k} \varphi(s) \, ds = 0 \quad (k = 1, n).
\]  

(22)

We introduce the above-mentioned dimensionless variables in (21), (5) and (22), supplementing them with the following ones:

\[
p_0(\xi) = \frac{p(a \xi)}{E}, \quad q_0(\xi) = \frac{q(a \xi)}{E} = p_0 + p_0(\xi), \quad \psi(\xi) = \varphi(a \xi).
\]

As a result, system (21) becomes

\[
\frac{2}{\pi} \int_{\rho}^{1} \frac{2\eta \tau(\eta) \, d\eta}{\eta^2 - \xi^2} + \frac{2}{\pi} \int_{L_0} \frac{\eta \xi^2 \psi(\eta) \, d\eta}{(\xi^2 + \eta^2)^2} + p_0
\]

\[
= \Lambda \left[ \int_{\rho}^{1} \text{sign} (\xi - \eta) \tau(\eta) \, d\eta - \int_{\rho}^{1} \text{sign} (\xi - \eta) \tau_0(\eta) \, d\eta \right] \quad (\rho < \xi < 1),
\]

\[
\frac{1}{\pi} \int_{L_0} \left[ \frac{2\xi}{\eta^2 - \xi^2} + \frac{6\xi}{(\eta + \xi)^2} - \frac{4\xi^2}{(\eta + \xi)^3} \right] \psi(\eta) \, d\eta + \frac{16}{\pi} \int_0^{1} \frac{\eta^3 \tau(\eta) \, d\eta}{(\eta^2 + \xi^2)^2} = -4q_0(\xi) \quad (\xi \in L_0),
\]

(23)

condition (5) takes the form (10), and conditions (22) become

\[
\int_{\alpha_k}^{b_k} \psi(\eta) \, d\eta = 0 \quad (k = 1, n).
\]

(24)
Then, turning again to (23), we consider this equation on \(L' = [0, \infty) \setminus L\) and take into account the second condition in (14). After passing to the above-mentioned dimensionless variables, we have

\[
\sigma_0(\xi) = \frac{\sigma(a\xi)}{E} = -\frac{4}{\pi} \int_\rho^1 \frac{\eta^2 \tau(\eta)}{(\eta^2 + \xi^2)^2} - \frac{1}{4\pi} \int_{L_0} \left[ \frac{2\xi}{\eta^2 - \xi^2} + \frac{6\xi}{(\eta + \xi)^2} - \frac{4\xi^2}{(\eta + \xi)^3} \right] \psi(\eta) \, d\eta \tag{25}
\]

\((\xi \in L'_0 = [0, \infty) \setminus L_0)\).

Further, for simplicity, we can confine to one crack and replace the interval \(L_0 = (\alpha_1, \beta_1)\) by an interval \((\alpha, \beta)\). Then system (23) takes the form

\[
\frac{2}{\pi} \int_\rho^1 \left[ \frac{2\eta}{\eta^2 - \xi^2} + \frac{\pi \Lambda}{2} \text{sign}(\eta - \xi) \right] \tau(\eta) \, d\eta + \frac{2}{\pi} \int_\alpha^\beta \frac{\eta^2 \psi(\eta) \, d\eta}{(\xi^2 + \eta^2)^2} = f_0(\xi) \quad (\xi \in (\rho, 1)),
\]

\[
\frac{1}{\pi} \int_\alpha^\beta \left[ \frac{2\xi}{\eta^2 - \xi^2} + \frac{6\xi}{(\eta + \xi)^2} - \frac{4\xi^2}{(\eta + \xi)^3} \right] \psi(\eta) \, d\eta + \frac{16}{\pi} \int_\rho^1 \frac{\eta^2 \tau(\eta) \, d\eta}{(\eta^2 + \xi^2)^2} = -4q_0(\xi) \quad (\xi \in (\alpha, \beta)),
\]

\[f_0(\xi) = -p_0 + \Lambda \int_\rho^1 \text{sign}(\eta - \xi) \tau_0(\eta) \, d\eta.\]

Condition (10) remains unchanged, and conditions (24) are reduced to the condition

\[
\int_\alpha^\beta \psi(\eta) \, d\eta = 0. \tag{27}
\]

Further in (26), as well as in (10) and (27), we transform all intervals into an interval \((-1, 1)\) for which:

1. in the first integral (26), we set

\[
\xi = \sqrt{ct + d}, \quad \eta = \sqrt{cu + d} \quad \left(c = \frac{1 - \rho^2}{2}, \quad d = \frac{1 + \rho^2}{2}, \quad -1 < t, u < 1\right);
\]

2. in the second integral (26), we set

\[
\xi = \sqrt{ct + d}, \quad \eta = \sqrt{\gamma u + \delta}, \quad \gamma = \frac{\beta^2 - \alpha^2}{2}, \quad \delta = \frac{\beta^2 + \alpha^2}{2} \quad (-1 < t, u < 1);
\]

3. in the third integral, we set

\[
\xi = \sqrt{\gamma t + \delta}, \quad \eta = \sqrt{\gamma u + \delta} \quad (-1 < t, u < 1);
\]

4. in the fourth integral (25), we set

\[
\eta = \sqrt{cu + d}, \quad \xi = \sqrt{\gamma t + \delta} \quad (-1 < t, u < 1);
\]

5. in condition (10), we set \(\eta = \sqrt{cu + d}\);

6. in condition (26), we set \(\eta = \sqrt{\gamma u + \delta}\).

As a result, SIE system (26) is transformed to the form

\[
\frac{2}{\pi} \int_{-1}^1 \left[ \frac{1}{u - t} + \frac{\pi \Lambda c}{4\sqrt{cu + d}} \text{sign}(u - t) \right] \chi(u) \, du + \frac{\gamma}{\pi} \int_{-1}^1 \frac{(ct + d)\omega(u) \, du}{(ct + cu + d + \gamma t + \delta)^2} = h_0(t),
\]

\[
\frac{1}{\pi} \int_{-1}^1 \left[ \frac{1}{u - t} + \frac{3\gamma}{(\gamma u + \delta + \sqrt{\gamma t + \delta})^2} - \frac{2\gamma \sqrt{\gamma t + \delta}}{(\gamma u + \delta + \sqrt{\gamma t + \delta})^3} \right] \sqrt{\frac{\gamma t + \delta}{\gamma u + \delta}} \omega(u) \, du
\]

\[
+ \frac{8c}{\pi} \int_{-1}^1 \frac{(cu + d)\chi(u) \, du}{(cu + \gamma t + d + \gamma u)^2} = -4q_0(t) \quad (-1 < t < 1),
\]

\[\chi(u) = \tau(\sqrt{cu + d}), \quad \omega(u) = \psi(\sqrt{\gamma u + \delta}), \quad h_0(t) = f_0(\sqrt{ct + d}), \quad g_0(t) = g_0(\sqrt{\gamma t + \delta});\]
condition (10) is transformed to the form (12), and condition (27) is transformed to the form
\[ \int_{-1}^{1} \frac{\omega(u) \, du}{\sqrt{\gamma u + \delta}} = 0. \]  \hspace{1cm} (29)

In (25), where \( L_0 = (\alpha, \beta) \), we also move to the interval \((-1, 1)\). As a result,
\[ \Sigma_0(t) = \sigma_0(\sqrt{\gamma t + \delta}) = -\frac{2c}{\pi} \int_{-1}^{1} (cu + d) \chi(u) \, du - \frac{1}{4\pi} \int_{-1}^{1} \left[ \frac{1}{u - t} + \frac{3\gamma}{(\sqrt{\gamma u + \delta} + \sqrt{\gamma t + \delta})^2} \right] \sqrt{\gamma u + \delta} \omega(u) \, du \quad (t \in (-\delta/\gamma, -1) \cup (1, \infty)). \]  \hspace{1cm} (30)

4. On the stress state of an elastic half-plane with a collinear system of absolutely rigid thin inclusions on the vertical axis

As was mentioned above, to solve this SIE system, we apply the numerical-analytic method for solving SIE on the basis of the Gauss quadrature formulas for Chebyshev nodes for calculating Cauchy-type integrals and ordinary integrals in [15–17]:
\[ \chi(u) = \frac{X(u)}{\sqrt{1 - u^2}}, \quad \omega(u) = \frac{\Omega(u)}{\sqrt{1 - u^2}} \quad (-1 < u < 1), \]  \hspace{1cm} (31)

where \( X(u) \) and \( \Omega(u) \) are the Hölder functions on the interval \([-1, 1]\), and using the procedure of this method, we reduce SIE system (28), (12), (29) to the finite system of linear algebraic equations
\[ \sum_{n=1}^{2M} K_{mn} Z_n = a_m \quad (m = \overline{1, 2M}) \]  \hspace{1cm} (32)
\[ Z_n = \begin{cases} X_n, & n = \overline{1, M}, \\ Y_{n-M}, & n = \overline{M+1, 2M}, \end{cases} \quad X_n = X(u_n), \quad Y_n = \Omega(u_n) \quad (n = \overline{1, M}), \]
\[ u_n = \cos \left( \frac{(2n - 1)\pi}{2M} \right) \quad (n = \overline{1, M}), \quad t_m = \cos \frac{\pi m}{M} \quad (m = \overline{1, M-1}), \]
\[ a_m = \begin{cases} f_0(t_m), & m = \overline{1, M-1}, \\ Q_0, & m = M, \\ -4q_0(t_{m-M}), & m = \overline{M+1, 2M-1}, \\ 0, & m = 2M, \end{cases} \]
of the solution of the system of equations (32):
\[ x_n \text{ is unbounded at the point } x = a. \]

As a result, a crack spreads out onto the plate boundary. To prevent the crack growth, it is possible to connect crack edges near the point \( x = a \). We use (31), (32) to transform formula (33) and take into account the expression of the well-known integral from [19] (p. 175, formula (21)). After simple transformations, we have
\[ K_{mn} = \begin{cases} 
\frac{2}{\pi} \left[ \frac{1}{M} \frac{u_n - t_m}{4\sqrt{cu_n + d}} \text{sign}(u_n - t_m) \right], & \text{for } m = 1, M - 1, \ n = 1, M, \\
\frac{M}{\sqrt{cu_n + d}} c_{t_m} + d & \text{for } m = M, \ n = 1, M, \\
\frac{M}{\sqrt{cu_n + d}} \left( c_{t_m} + \gamma u_n - M + d + \delta \right)^2 & \text{for } m = 1, M - 1, \ n = M + 1, 2M, \\
\frac{1}{M} \left[ \frac{u_n - M - t_M - M}{(\sqrt{\gamma u_n - M + \delta} + \sqrt{\gamma t_M - M + \delta})^2} \right] & \text{for } m = M + 1, 2M - 1, \ n = 1, M, \\
\frac{1}{\pi} \frac{\gamma t_{n-M} + \delta}{\gamma u_n - M + \delta} & \text{for } m = 2M, \ n = M + 1, 2M.
\end{cases} \]

Here \( M \) is any natural number, and \( u_n \) and \( t_m \) are Chebyshev nodes, i.e., the roots of the Chebyshev polynomials of the first and second kinds.

Let us calculate the characteristics of the problem under study. For the SIF at the upper end point \( x = a_1 \) of the crack, we have
\[ K_I = \lim_{x \to a_1-0} \left[ \sqrt{2\pi(a_1 - r)} \sigma_0 \right] = -\sqrt{aE} \lim_{\xi \to a-0} \left[ \sqrt{2\pi(\alpha - \xi)} \sigma_0(\xi) \right], \]

from which, for the dimensionless SIF, we obtain
\[ K_I^{(0)} = \frac{K_I}{\sqrt{aE}} = \lim_{\xi \to a-0} \left[ \sqrt{2\pi(\alpha - \xi)} \sigma_0(\xi) \right]. \tag{33} \]

We use (31), (32) to transform formula (33) and take into account the expression of the well-known integral from [19] (p. 175, formula (21)). After simple transformations, we have
\[ K_I^{(0)} = \frac{\sqrt{\pi}}{4\sqrt{2}} \sqrt{\frac{\gamma}{\alpha}} \Omega(-1) \tag{34} \]

or, substituting the value \( \Omega(-1) \) for the Lagrange interpolation polynomial and using the solution of the system of equations (32), we obtain
\[ K_I^{(0)} = \frac{\sqrt{\pi}}{4\sqrt{2}} \sqrt{\frac{\gamma}{\alpha}} \sum_{n=M+1}^{2M} (-1)^{M+n} Z_n \tan \left( \frac{2n-1}{4}\right). \tag{35} \]

It follows from (34) or (35) that, as \( \alpha \to +0 \) (\( a_1 \to +0 \)), the SIF \( K_I^{(0)} \) infinitely increases and, as a result, a crack spreads out onto the plate boundary. To prevent the crack growth, it is possible to connect crack edges near the point \( x = a_1 \) by an inclusion, which is modelled by continuously distributed linear springs [20].

We note that, in the case of an edge crack \( (a_1 = 0) \), the solution of SIE system (26) or (28) must be constructed in the class of functions bounded at the point \( x = a_1 \) (\( \xi = \alpha \)) and unbounded at the point \( x = b_1 \) (\( \xi = \beta \)).

We also note that the dimensionless breaking stresses \( \Sigma_0(t) \) from (30) are expressed in terms of the solution of the system of equations (32):
\[ \Sigma_0(t) = -\frac{2\xi}{M} \sum_{n=1}^{M} \frac{(cu_n + d)Z_n}{(cu_n + \gamma t + d + \delta)^2} - \frac{\pi}{4M} \sum_{n=M+1}^{2M} \frac{1}{u_n - M - t} + \frac{3\gamma}{\sqrt{\gamma u_n - M + \delta} + \sqrt{\gamma t + \delta}^2} \]
\[ - \sqrt{\frac{\gamma t + \delta}{\gamma u_n - M + \delta} + \sqrt{\gamma t + \delta}^2} Z_n \quad (t \in (-\delta/\gamma, -1) \cup (1, \infty)). \]
The crack opening is determined by the formula

$$
\Phi(r) = \int_{a_1}^{b_1} \text{sign}(r - s)\varphi(s) \, ds \quad (a_1 \leq r \leq b_1).
$$

Transforming this integral to the interval $(-1, 1)$, we obtain

$$
\Phi_0(t) = \Phi(a\sqrt{ct + d}) = \frac{\gamma}{4} \int_{-1}^{1} \text{sign}(t - u) \frac{\omega(u) \, du}{\sqrt{\gamma u + \delta}}
= \frac{\pi \gamma}{4M} \sum_{n=M+1}^{2M} \frac{Z_n}{\sqrt{\gamma u_{n-M} + \delta}} \text{sign}(t - u_{n-M}) \quad (-1 \leq t \leq 1).
$$

The dimensionless tangential contact stresses under the right stringer, according to the first formula in (31) through the solution of the system (32), are determined by the formula

$$
\chi(u_n) = \frac{Z_n}{\sqrt{1 - u_n^2}} \quad \left( n = 1, M, \quad u_n = \cos \left( \frac{(2n - 1)\pi}{2M} \right) \right).
$$

Let us consider the axial directions in the right stringer. By formula (1) and by the formula related to it, which is obtained from (1) by replacing the interval $(b, r)$ with the interval $(r, a)$, we have

$$
\sigma_s(r) = \frac{1}{2A_s} \left[ d \int_b^a \text{sign}(r - s)\tau-(s) \, ds - \int_b^a \text{sign}(r - s)\tau+(s) \, ds \right] \quad (r \in [b, a]).
$$

Hence, after passing to the interval $(-1, 1)$ and to dimensionless variables, we obtain

$$
\Sigma_s^{(0)}(t) = \frac{\Lambda_0 c}{2} \int_{-1}^{1} \text{sign}(t - u) \frac{[X(u) - X_0(u)] \, du}{\sqrt{cu + d}}
= \frac{\pi \Lambda_0 c}{2M} \sum_{n=1}^{M} \text{sign}(t - u_n) \frac{Z_n}{\sqrt{1 - u_n^2}X_0(u)} \quad (-1 \leq t \leq 1),
$$

$$
\Sigma_s^{(0)}(t) = \sigma_s^{(0)}(\sqrt{ct + d}), \quad \sigma_s^{(0)}(\xi) = \frac{\sigma_s(a\xi)}{E}, \quad \Lambda_0 = \frac{ad_1}{2A_s}, \quad X(u) = \tau(\sqrt{cu + d}), \quad X_0(u) = \tau_0(\sqrt{cu + d}).
$$

In special cases when there are only stringers or only cracks, system of SIE (21), (26) or (28) splits into separate SIEs. We note that the problem of the stress state of an elastic semi-infinite plate containing only internal cracks on its vertical axis in the absence of stringer can be constructed by the Multopp method [16].

In the general case, different combinations of mutual locations of stringers and cracks are possible, which leads to various problems.

**Conclusions**

The above-considered problems directly related to the interaction of stress concentrators (such as stringers and cracks) with massive deformable bodies, which are often encountered in applications, are of great theoretical and practical interest. Different combinations of mutual locations of stringers and cracks lead to various problems which can be studied by using the results and approaches proposed in this paper.
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