Silt motion simulation using finite volume particle method

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Abstract. In this paper, we present a 3-D FVPM which features rectangular top-hat kernels. With this method, interaction vectors are computed exactly and efficiently. We introduce a new method to enforce the no-slip boundary condition. With this boundary enforcement, the interaction forces between fluid and wall are computed accurately. We employ the boundary force to predict the motion of rigid spherical silt particles inside the fluid. To validate the model, we simulate the 2-D sedimentation of a single particle in viscous fluid tank and compare results with benchmark data. The particle resolution is verified by convergence study. We also simulate the sedimentation of two particles exhibiting drafting, kissing and tumbling phenomena in 2-D and 3-D. We compare the results with other numerical solutions.

1. Introduction

Fluid flow advecting silts originating from snow and glaciers melting or monsoons can cause severe erosion when passing through the turbines. Erosive wear of hydraulic turbine components is a complex phenomenon, which depends upon different parameters such as silt size, hardness and concentration, velocity of silt and water, angle of impact, and base material properties. Parameters like concentration, velocity and angle of impact are linked to the hydrodynamics of the flow, which reveals the importance of fluid flow in this phenomenon. The purpose of this paper is to present a finite volume particle method which is capable to simulate the water flow and silts motion accurately and efficiently.

The finite volume particle method (FVPM) is a particle-based method introduced by Hietel [1]. This method includes many of the desirable features of mesh-based finite volume methods. FVPM profits from the interaction vectors to weight conservative fluxes exchanged between particles. In this methods, computational nodes are usually moving with material velocity which is compatible with the Lagrangian form of the motion equations. FVPM has some features of SPH but unlike SPH, it is locally conservative regardless of any variation in particle smoothing length. This enables the users to refine the solution by splitting the particles into smaller ones in the region of interest and perform the simulation efficiently. Recently, Quinlan and Nestor [2] proposed a method to compute the interaction vectors for 2-D cases exactly. Following their work, Jahanbakhsh [3] developed an exact FVPM for 3-D cases. This method features the rectangular top-hat kernel and is implemented in SPHEROS software [4] and [5].

In this study, the fluid flow considered as weakly-compressible and the governing equations are discretized by FVPM. Boundary conditions are enforced precisely by placing the particles on the boundary and updating their pressure by the fluids equation of state. To achieve accurate
results, solution is refined near the silt particles. The silt particles are considered as non-resolved rigid particles with constant mass and volume. Using the Lagrangian description of motion, acceleration of the mass center for each particle is calculated from the Newton’s second law. The hydrodynamic forces are computed from boundary particles and the contact forces are modeled between the silt particles according to the Hertz contact theory. This formulation is validated for 2-D and 3-D computations by comparing the results with the benchmark data given for single and pair of particles sedimentation. For two particles sedimentation, we succeed to simulate drafting, kissing and tumbling (DKT) phenomenon. The DKT may be observed when two particles sediment under the action of gravity in a viscous Newtonian fluid and shows the effect of particles wakes on the motion of each other.

The structure of the paper is as following. In the next section, we present the governing equations including fluid and silt equations of motion with the contact modeling. In the discretization section, the FVPM, rectangular top-hat kernel, boundary conditions and solution algorithm are explained. At the end, we present our results.

2. Governing equations

2.1. Fluid flow

Fluid flow is assumed as isothermal and weakly-compressible. The equations of motion are derived from the mass and linear momentum conservation laws

\[
\frac{dp}{dt} = -\rho \nabla \cdot C, \quad \text{and} \quad \frac{d(\rho C)}{dt} = \nabla \cdot \sigma + \rho g
\]  

(1)

where \( \frac{d}{dt} \) denotes the substantial derivative, \( \rho \) is the density, \( C \) is the velocity vector, \( g \) is the gravitational acceleration and \( \sigma \) represents the stress tensor defined as

\[
\sigma = s - p I
\]  

(2)

In eq. (2), \( p \) denotes the static pressure and \( s \) denotes the deviatoric stress tensor. For Newtonian fluid, \( s \) is computed as

\[
s = 2\mu \left( D - \frac{1}{3} \text{tr}(D) I \right)
\]  

(3)

where \( \mu \) is the dynamic viscosity and \( D = \nabla C + (\nabla C)^T \) is the deformation rate tensor. To close the system of equations (1), an equation of state is required. The following equation of state is usually considered for water [6]

\[
p = \rho_o a^2 \left( \left( \frac{\rho}{\rho_o} \right)^{7/6} - 1 \right)
\]  

(4)

where \( a \) is the speed of sound and \( \rho_o \) is the reference density.

2.2. Silt motion

Silt particles are assumed rigid and spherical. Their mass and volume remains constant and their acceleration is found from Newton’s second law

\[
m \frac{dC}{dt} = f^h + f^c + mg, \quad \text{and} \quad I \frac{d\omega}{dt} = T^h
\]  

(5)

where \( m \) and \( I \) are the mass and moment of inertia respectively. \( f^h \) and \( T^h \) are the hydrodynamic force and moment exerted by fluid flow. \( f^c \) is the contact force exerted by the other silt as well.
as wall boundary. The contact force is modeled according to Hertz contact theory. Due to the computational limits, we assume a finite value for the silts elastic modulus $E$, despite the fact that they are rigid. The contact force between two spheres of radii $R_1$ and $R_2$ is given by

$$f_c = \left( \frac{4E}{3} \right) \frac{R_1^2 d^3}{R_1 + R_2} e$$

where, $e$ denotes the unit vector passing through the spheres centers and $d$ is the penetration depth. $R$ is the effective radius and defined as $R = \frac{R_1 R_2}{R_1 + R_2}$

3. Discretization

3.1. Finite Volume Particle Method

In FVPM, the Sheppard function is used as the interpolating or shape function. Sheppard interpolation is equivalent to Moving Least Squares (MLS) interpolating function with zero-order basis [7]. Shepard function $\psi_i(x)$ for a given point $i$ reads

$$\psi_i(x) = \frac{W_i(x)}{\sigma(x)}$$

where $W_i(x) = W_i(x - x_i, h_i)$ is the kernel function and $\sigma$ is the kernels summation. $h_i$ is the smoothing length at the given point and represents the spatial resolution of the interpolation. Sheppard interpolation is zero-order consistent which means that a constant function could be reproduced exactly. These shape functions are used in discretization of the PDE arising from the conservation law

$$\frac{\partial U}{\partial t} + \nabla \cdot F(U) = 0$$

where $U = \{\rho, \rho C\}$ represents the conserved variables and $F$ represents the flux functions. We decompose $F$ into inviscid, $Q$, pressure, $P$, and deviatoric stress, $G$, fluxes as

$$F = Q + P - G$$

where, $Q = \{\rho C, \rho CC\}$, $P = \{0, p I\}$ and $G = \{0, s\}$. According to [3], we write the discretized equations in locally conservative form

$$\frac{d}{dt} (U_i V_i) + \sum_j \left( Q_{ij} - (U_j \dot{x})_{ij} + P_{ij} \right) \cdot \Delta_{ij} - \sum_j G_{ij} \cdot \Delta_{ij} = 0$$

where $V_i$ is particle volume and $\dot{x}_i$ denotes to the particle velocity which is equal to $C_i$ modified by a correction term introduced by Jahanbakhsh [3]. The correction is applied to avoid particle clustering issue. Following Nestor et al. [8], we use the AUSM$^+$ of Liou [9] to discretize the inviscid and pressure fluxes. The term $(U_j \dot{x})_{ij}$ represents the flux change due to the particle motion which is included in inviscid flux according to ALE-type extension of the scheme, as presented by Luo et al. [10]. The interaction vector $\Gamma_{ij}$ and $\Delta_{ij}$ are defined as

$$\Gamma_{ij} = \int_{\Omega} \frac{W_i \nabla W_j}{\sigma^2} dV$$

$$\Delta_{ij} = \Gamma_{ij} - \Gamma_{ji}$$

Deviatoric stress flux $G_{ij}$ are found from velocity gradients. These gradients are computed using Weighted Least Squares approach [3].
The particle volume is evolved by

$$\frac{dV_i}{dt} = \sum_j (\dot{x}_j \cdot \Gamma_{ij} - \dot{x}_i \cdot \Gamma_{ji}) + \dot{x}_i \cdot S_i \quad (13)$$

where \(S_i = \sum_j \Delta S_{ij}\). The term \(\dot{x}_i \cdot S_i\) in eq. (13) is included to satisfy the free surface boundary condition [3]. As discussed by Quinlan and Nestor [2], considering the exact integration of interaction vectors, the \(S_i\) vanishes for the interior particles.

3.2. Rectangular Top-hat Kernel

We use rectangular kernels to compute the particle interaction vectors, \(\Gamma_{ij}\). To compute these vectors, the integral defined in eq. (11) has to be evaluated. For conventional bell-shaped kernels, this integral is difficult or impossible to be evaluated exactly. The alternative approach is to use the Quadrature rules which become very costly in 3-D. Recently, Quinlan and Nestor [2] used 2-D top-hat kernel with circular support, which enables them to compute the integrals exactly and efficiently. Jahanbakhsh [3] use top-hat kernel with rectangular support to compute interaction vectors efficiently and exactly for 3-D case. Rectangular top-hat kernel is defined as

$$W_i(x) = \begin{cases} 1 & \|x - x_i\|_\infty \leq h_i \\ 0 & \|x - x_i\|_\infty > h_i \end{cases} \quad (14)$$

According to eq. (14), \(\nabla W\) is zero everywhere and is not defined at the particle smoothing border. However, the integral corresponding to the interaction vector \(\Gamma_{ij}\) is defined. After some mathematical operations [3], the integral in eq. (11) is simplified as

$$\Gamma_{ij} = -\sum_l \left( \frac{\Delta S_l}{\sigma_l^+ \sigma_l^-}\right) \quad (15)$$

According to eq. (15), the surface corresponding to the intersection of particles \(i\) and \(j\) is partitioned by other particles to \(m\) rectangles. In eq. (15), \(\Delta S\) denotes to the surface vector of partitions, \(\sigma^+\) is the summation of the kernels outside of the intersection zone and \(\sigma^-\) is the summation of the kernels which appear at the surface. A 2-D representation of the partitioning is shown in figure 1a. The rectangular partitions are simplified to line segments for 2-D case.
3.3. Boundary conditions
The no-slip wall boundary condition appears in fluid-silt or fluid-wall interactions. For fluid-silt interaction, we seed a layer of boundary particles in a way that they fit to the silt interface as show in figure (1b). The same approach is used for the wall. Boundary particles behave like fluid and their density is initially set to the fluid reference density \( \rho^\circ \). They are moving with the same velocity as silt or wall known from previous time step. The volume and the mass of boundary particles are evolved according to eq. (13) and (9) respectively. Knowing the density, the pressure is computed from the fluid equation of states (4).

The fluid particles are exerting hydrodynamic force to the boundary particles. This force is computed as

\[
 f_i = \sum_{j \in \text{fluid}} (-p_{ij} I + s_{ij}) \cdot \Delta_{ij} \tag{16}
\]

where \( p_{ij} \) and \( s_{ij} \) are found according to section 3.1. Finally, the total hydrodynamic force and moment exerted to the silt particle found as

\[
 f^h = \sum_{i \in \text{boundary}} f_i \tag{17}
\]

\[
 T^h = \sum_{i \in \text{boundary}} R_i \times f_i \tag{18}
\]

where \( R_i \) is the vector connecting particle \( i \) to the center of silt. See figure (1b).

3.4. Smoothing mass flux
As the pressure and velocity is computed at the same computational node, checker-board oscillations is occurring. Fatehi and Manzari [11] proposed a correction term for SPH method which is added to mass equation and filter out the oscillations. They followed Rhie and Chow [12] approach designed for FVM. We adapt this term to our formulation as

\[
 R_{ij} = \left( \nabla p_i + \nabla p_j - \bar{\nabla} p_{ij} \right) \Delta t \tag{19}
\]

In eq. (19), \( \nabla p_i = \frac{1}{V_i} \sum_j \frac{1}{2} (p_i + p_j) \Delta_{ij} \) and \( \bar{\nabla} p \) is computed using Weighted Least Squares approach [3].

3.5. Time integration
We used second-order explicit Runge-Kutta scheme for the time integration. At first, the conserved variables are updated for the half of the time step using the current time step values

\[
 U^{(t+\frac{\Delta t}{2})} = U^{(t)} - \nabla \cdot F \left( U^{(t)} \right) \frac{\Delta t}{2} \tag{20}
\]

Then, the intermediate fluxes are computed using the half time step values. Finally, the conserved variables for new time step are found by

\[
 U^{(t+\Delta t)} = U^{(t)} - \nabla \cdot F \left( U^{(t+\frac{\Delta t}{2})} \right) \Delta t \tag{21}
\]

For a given CFL number, the \( \Delta t \) is adapted for each step as

\[
 \Delta t = \text{CFL} \times \min \left( \frac{h_i}{a + |C_i|} \right) \tag{22}
\]
4. Results and discussion

4.1. Single particles sedimentation

In this test case we simulate the sedimentation of a 2-D circular silt particle in a tank. The tank is filled with a viscous fluid and the silt particle moves from the rest due to gravity acceleration \( g = (0, -9.81) \text{ m s}^{-2} \). A schematic geometry of the problem is shown in figure 2a. The problem dimensions and fluid and silt properties are summarized as

- Domain dimensions are \( H = 0.06 \text{ m} \) and \( L = 0.02 \text{ m} \).
- Diameter of silt particle is \( d = 0.0025 \text{ m} \).
- Initial position of silt is \((0.01, 0.04) \text{ m}\).
- Silt density is \( \rho_s = 1500 \text{ kg m}^{-3} \).
- Fluid density is \( \rho_f = 1000 \text{ kg m}^{-3} \).
- Fluid viscosity is \( \mu = 0.1 \text{ Pa s} \).

![Schematic Diagram](image1)

![Time History](image2)

Figure 2: (a) Schematic diagrams for the sedimentation of a circular cylinder in a tank. (b) The time history of vertical velocity component for silt sedimentation. The results for three different particle spacing and compared with Glowinski et al. [13] results

Here, we set the speed of sound as \( a = 2 \text{ m s}^{-1} \) and the CFL = 1.0. Following Hashemi et al. [14], we replace the gravity force by buoyancy force \( b \) in the silt motion equation and omit the gravity term in fluid flow equations for computational efficiency. Doing so, following force is added to the r.h.s. of eq. (5).

\[
b = m \left( 1 - \frac{\rho_f}{\rho_s} \right) g
\]  

(23)

Figure 2b depicts the time history of the silt vertical velocity. In this figure, three different particle spacings of \( \delta = \frac{d}{4} \), \( \delta = \frac{d}{8} \) and \( \delta = \frac{d}{16} \) are compared with the FEM solution of Glowinski et al. [13]. As it is visible, resolution of \( \delta = \frac{d}{4} \) is not sufficient to predict the terminal velocity accurately. While, by reducing the particle spacing, the solution is converged to the FEM solution. As Hashemi et al. [14] reported, the small fluctuations in the velocity appears when the silt particle reaches its terminal velocity. These fluctuations are due to the small pressure waves produced in a weakly compressible fluid. However, the smaller particle spacings, the smaller fluctuation observed.
4.2. Drafting, kissing and tumbling
In this test case, two particles with identical density and diameter are accelerating from rest due to the action of gravity. They are placed inside a tank filled with a viscous fluid. At the beginning, the particles have the same horizontal position, but some vertical offset. The trailing particle catches up with the leading one due to the reduced drag in the former particle’s wake. This phenomenon is called drafting. In the next stage, two the particles become close and touch each other in a point called kissing. This stage, is not a stable state in Newtonian fluids and eventually, the trailing particle passes the leading one and separates from it. This stage is called tumbling. Drafting, kissing and tumbling phenomenon experimentally observed for falling solid spheres [15] and numerically simulated for 2-D ([16],[14]) and 3-D ([13],[17],[18]) cases. Here, we simulated this phenomenon in 2-D and 3-D for circular and spherical silt particles respectively.

4.2.1. 2-D case  A schematic geometry of the problem is shown in figure 3a. The problem dimensions and material properties are summarized as

- Domain dimensions are \( H = 0.06 \) m and \( L = 0.02 \) m.
- Diameter of silt particles are \( d = 0.0025 \) m.
- Initial position of trailing particle is \((0, 0.05)\) m.
- Initial position of leading particle is \((0, 0.045)\) m.
- Silt density is \( \rho_s = 1500 \) kg m\(^{-3}\).
- Fluid density is \( \rho_f = 1000 \) kg m\(^{-3}\).
- Fluid viscosity is \( \mu = 0.1 \) Pa s.

The sound speed and CFL are set the same as single particle sedimentation case. We study the resolution convergence by performing the simulation for three different particle spacing of \( \delta = \frac{d}{8}, \delta = \frac{d}{16} \) and \( \delta = \frac{d}{32} \). Figures 3b depicts the time history of the particles’ vertical velocity. It is visible that the results are converging to the finest solution. Figures 4a and 4b depicts the vertical and horizontal velocity results corresponding to \( \delta = \frac{d}{32} \) which are compared with the results reported by Uhlammm [17] and Glowinski et al. [13]. The FVPM result is matched to the Uhlammm [17] solution over drafting and kissing stages. But it shows better agreement with Glowinski et al. [13] result during tumbling.

![Figure 3](image_url)  
Figure 3: (a) Schematic diagrams for the sedimentation of two circular cylinder in a tank. (b) Time history of vertical velocity of the trailing and leading silt particles for three particle spacings.
Figure 4: Time history of (a) vertical and (b) horizontal velocities of the trailing and leading silt particles for particle spacing $\delta = \frac{d}{32}$ compared with Uhlmann [17] and Glowinski et al. [13] results.

Figure 5 shows snapshots of the vorticity for five time steps. Figures 5a and 5b represent the drafting phenomenon and figure 5c represents the kissing stage. In figure 5d, particles are tumbling under the influence of a moment which tries to bring them in more stable state and eventually separate from each other in figure 5e.

Figure 5: Snapshots for circular silt particles sedimentation for different time steps colored by vorticity

4.2.2. 3-D case

For this case, tank is squared section with the side length of $L$ and height of $H$. The problem dimensions and properties are summarized as

- Domain dimensions are $H = 0.04$ m and $L = 0.01$ m.
- Diameter of silt particles are $d = 0.0016$ m.
• Initial position of trailing particle is (0.005, 0.005, 0.0316) m.
• Initial position of leading particle is (0.005, 0.005, 0.035) m.
• Silt density is $\rho_s = 1140$ kg m$^{-3}$.
• Fluid density is $\rho_f = 1000$ kg m$^{-3}$.
• Fluid viscosity is $\mu = 0.1$ Pa s.

We refine the solution by splitting the fluid particles close to the silts according to the strategy presented by Jahanbakhsh [3]. The sound speed and CFL are set the same as 2-D cases. The simulation is performed with the smallest particle spacing of $\delta = \frac{d}{16}$. The vertical velocity of the silts are compared with results from Glowinski et al.[13] and Sharma and Patankar [18] in figure 6. The FVPM results are in good agreement with the other solutions. Figure 7 depicts the snapshots taken in different times with iso-surface of vorticity $|\omega| = 15$ s$^{-1}$. It can be seen that the spheres exhibit the drafting, kissing and tumbling behavior. The tumbling occurs in XZ plane and the two spheres move on either side of the center.

![Figure 6](image_url)

Figure 6: Time history of vertical velocity of the trailing and leading spherical silt particles. The FVPM results corresponding to $\delta = \frac{d}{16}$ is compared with Glowinski et al.[13] and Sharma and Patankar [18] results.

![Figure 7](image_url)

Figure 7: Snapshots for spherical silts particle sedimentation for different time steps with iso-surface of vorticity $|\omega| = 15$ s$^{-1}$
5. Conclusion

In this paper, we presented a 3-D FVPM which features rectangular top-hat kernels. We developed a new method to enforce the no-slip boundary condition and to compute boundary forces. We used the boundary forces to solve the rigid body motion for spherical silt particles. We validated the method by 2-D sedimentation of a single particle in viscous fluid tank. The particle resolution is verified by convergence study. We simulated the drafting, kissing and tumbling phenomena in 2-D and 3-D. The achieved results show a good agreement compared with the other numerical solutions.

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