ON THERMODYNAMICALLY CONSISTENT STEFAN PROBLEMS WITH VARIABLE SURFACE ENERGY

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Abstract. A thermodynamically consistent two-phase Stefan problem with temperature-dependent surface tension and with or without kinetic undercooling is studied. It is shown that these problems generate local semiflows in well-defined state manifolds. If a solution does not exhibit singularities, it is proved that it exists globally in time and converges towards an equilibrium of the problem. In addition, stability and instability of equilibria is studied. In particular, it is shown that multiple spheres of the same radius are unstable if surface heat capacity is small; however, if kinetic undercooling is absent, they are stable if surface heat capacity is sufficiently large.

1. Introduction

In the recent paper [18] the authors studied Stefan problems with surface tension and with or without kinetic undercooling which are consistent with thermodynamics, in the sense that the total energy is preserved and the total entropy is strictly increasing along nonconstant smooth solutions.

1. To formulate this problem, let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^2$, $n \geq 2$. $\Omega$ is occupied by a material that can undergo phase changes: at time $t$, phase $i$ occupies the subdomain $\Omega_i(t)$ of $\Omega$, respectively, with $i = 1, 2$. We assume that $\partial \Omega_1(t) \cap \partial \Omega = \emptyset$; this means that no boundary contact can occur. The closed compact hypersurface $\Gamma(t) := \partial \Omega_1(t) \subset \Omega$ forms the interface between the phases.

The problem consists in finding a family of closed compact hypersurfaces $\Gamma(t)$ contained in $\Omega$ and an appropriately smooth function $u : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ such that

$$\begin{cases}
\kappa(u) \partial_t u - \text{div}(d(u) \nabla u) = 0 & \text{in } \Omega \setminus \Gamma(t) \\
\partial_t u = 0 & \text{on } \partial \Omega \\
[u] = 0 & \text{on } \Gamma(t) \\
\left[\psi(u) + \sigma \mathcal{H}ight] = \gamma(u)V & \text{on } \Gamma(t) \\
\left[d(u) \partial_n u\right] = (l(u) - \gamma(u)V)V & \text{on } \Gamma(t) \\
u(0) = u_0 & \text{in } \Omega, \\
\Gamma(0) = \Gamma_0.
\end{cases}$$

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Here \( u(t) \) denotes the (absolute) temperature, \( \nu(t) \) the outer normal field of \( \Omega_1(t) \), \( V(t) \) the normal velocity of \( \Gamma(t) \), \( \mathcal{H}(t) = \mathcal{H}(\Gamma(t)) = -\operatorname{div}_{\Gamma(t)} \nu(t) \) the sum of the principal curvatures, and \([v] = v_2|_{\Gamma(t)} - v_1|_{\Gamma(t)} \) the jump of a function \( v \) across \( \Gamma(t) \). Since \( u \) means absolute temperature we always assume that \( u > 0 \).

Several quantities are derived from the free energies \( \psi_i(u) \) as follows:
- \( \epsilon_i(u) := \psi_i(u) + u\eta_i(u) \) denotes the internal energy in phase \( i \),
- \( \eta_i(u) := -\psi'_i(u) \) the entropy,
- \( \kappa_i(u) := \epsilon'_i(u) = -u\psi''_i(u) \) the heat capacity,
- \( l(u) := u[\psi'(u)] = -u[\eta(u)] \) the latent heat.

Furthermore, \( d_i(u) > 0 \) denotes the coefficient of heat conduction in Fourier’s law, \( \gamma(u) \geq 0 \) the coefficient of kinetic undercooling, and \( \sigma > 0 \) the coefficient of surface tension. In the sequel we drop the index \( i \), as there is no danger of confusion; we just keep in mind that the coefficients in the bulk depend on the phases.

The temperature is assumed to be continuous across the interface. However, the free energy and the conductivities depend on the respective phases, and hence the jumps \( \varphi(u) := [\psi(u)], [\kappa(u)], [\eta(u)], [d(u)] \) are in general non-zero at the interface. Throughout we require that the heat capacities \( \kappa_i(u) \) and diffusivities \( d_i(u) \) are strictly positive over the whole temperature range \( u > 0 \), and that \( \varphi \) has exactly one zero \( u_m > 0 \) called the melting temperature.

If we assume that the coefficient of surface tension \( \sigma \) is constant, then this model is consistent with the laws of thermodynamics. In fact, the total energy of the system is given by
\[
E(u, \Gamma) = \int_{\Omega \setminus \Gamma} \epsilon(u) \, dx + \int_{\Gamma} \sigma \, ds, \tag{1.2}
\]
and by the transport and surface transport theorem we have for smooth solutions
\[
\frac{d}{dt} E(u(t), \Gamma(t)) = -\int_{\Gamma} \{ [d(u) \partial_{\nu} u] + [\epsilon(u)] V + \sigma \mathcal{H} V \} \, ds
\]
\[
= -\int_{\Gamma} \{ [d(u) \partial_{\nu} u] - (l(u) - \gamma(u)V)V \} \, ds = 0,
\]
and thus, energy is conserved. Also the total entropy \( \Phi(u, \Gamma) \) defined by
\[
\Phi(u, \Gamma) = \int_{\Omega \setminus \Gamma} \eta(u) \, dx \tag{1.3}
\]
is nondecreasing along smooth solutions, as
\[
\frac{d}{dt} \Phi(u(t), \Gamma(t)) = \int_{\Omega} \frac{1}{u^2} d(u) |\nabla u|^2 \, dx - \int_{\Gamma} \frac{1}{u} \{ [d(u) \partial_{\nu} u] + u[\eta(u)] V \} \, ds
\]
\[
= \int_{\Omega} \frac{1}{u^2} d(u) |\nabla u|^2 \, dx + \int_{\Gamma} \frac{1}{u} \gamma(u) V^2 \, ds \geq 0.
\]

2. In this paper we consider the physically important case where surface tension \( \sigma = \sigma(u) \) is a function of surface temperature \( u \). Then, following [9] and [11], the surface energy will be \( \int_{\Gamma} \epsilon_\Gamma(u) \, ds \) instead of \( \int_{\Gamma} \sigma \, ds \), where \( \epsilon_\Gamma(u) \) denotes the
density of surface energy. In addition, one has to take into account the total surface entropy \( \int_{\Gamma} \eta_T(u) \, ds \), as well as balance of surface energy. The latter means that the Stefan law has to be replaced by a dynamic equation on the moving interface \( \Gamma(t) \) of the form

\[
\kappa_T(u) \partial_t u - \text{div}_\Gamma (d_T(u) \nabla u) = \left[ d(u) \partial_n u \right] - \left( l(u) - \gamma(u) V + l_T(u) \mathcal{H} \right) V,
\]

where \( \partial_{t,n} \) denotes the time derivative in normal direction, see (1.7). As in the bulk we define on the interface

- \( \epsilon_T(u) := \sigma(u) + u \eta_T(u) \) denotes the surface internal energy,
- \( \eta_T(u) := -\sigma'(u) \) the surface entropy,
- \( \kappa_T(u) := \epsilon_T'(u) = -u \sigma''(u) \) the surface heat capacity,
- \( l_T(u) := u \sigma'(u) = -u \eta_T(u) \) the surface latent heat.

We also employ a Fourier law on the interface to describe surface heat conduction, i.e. we set \( q_T := -d_T(u) \nabla \Gamma u \), which should be present as soon as the interface has heat capacity. Recall that \( u \) is assumed to be continuous across the interface, hence the surface temperature

\[
u_T := u|_{\Gamma},
\]

is well-defined.

Obviously, if \( \sigma \) is constant then \( \epsilon_T = \sigma \), and \( \eta_T = \kappa_T = l_T = 0 \), hence this model reduces to (1.4). On the other hand, if \( \sigma \) is linear in \( u \) we still have \( \kappa_T = 0 \) and then it makes sense to also set \( d_T \equiv 0 \), to obtain the modified Stefan law

\[
\left[ d(u) \partial_n u \right] = \left( l(u) - \gamma(u) V + l_T(u) \mathcal{H} \right) V,
\]

which differs from the Stefan law in (1.1) only by replacing \( l(u) \) by \( l(u) + l_T(u) \mathcal{H} \).

This is just a minor modification of (1.1), and its analysis remains essentially the same as in [15]. The only difference is that the stability condition for the equilibria, and in case \( \gamma \equiv 0 \) also the well-posedness condition, changes. More precisely, the well-posedness condition changes from \( \varphi' \neq 0 \) to \( \lambda' \neq 0 \) where \( \lambda := \varphi(s)/\sigma(s) \), and the stability condition modifies by replacing \( \varphi'/\sigma \) by \( \lambda' \).

Therefore we concentrate here on the case where \( \kappa_T(u), d_T(u) > 0 \), which means that \( \sigma \) is strictly concave. It has been shown experimentally that positive surface heat capacity \( \kappa_T \) is important in certain practical situations; see [2] for recent work in this direction. Experimental evidence also show that \( \sigma \) is strictly decreasing, hence admits exactly one zero \( u_c > 0 \); \( \sigma(u) \) is positive in \( (0,u_c) \) and negative for \( u > u_c \). Physically, it is reasonable to assume \( u_c > u_m \). It turns out that the analysis of the problem with nonlinear surface tension is considerably different from the linear case. In the sequel we always assume that

\[
d_i, \psi_i, d_T, \sigma, \gamma \in C^3(0, u_c), \quad d_i, \kappa_i, d_T, \kappa_T, \sigma > 0 \text{ on } (0, u_c), \quad i = 1, 2, \quad (1.5)
\]

if not stated otherwise. Further, we let \( \gamma \equiv 0 \) if there is no undercooling, or \( \gamma > 0 \) on \( (0, u_c) \) if undercooling is present, and we restrict our attention to the temperature range \( u \in (0, u_c) \).
With these restrictions on the parameter functions, we consider the following problem:

\[
\begin{align*}
\kappa(u) \partial_t u - \text{div}(d(u) \nabla u) &= 0 \quad \text{in } \Omega \setminus \Gamma(t) \\
\partial_n u &= 0 \quad \text{on } \partial \Omega \\
[u] = 0, \quad u_{\Gamma} &= u \quad \text{on } \Gamma(t) \\
\varphi(u_{\Gamma}) + \sigma(u_{\Gamma}) \mathcal{H} &= \gamma(u_{\Gamma}) V \quad \text{on } \Gamma(t) \\
\kappa_{\Gamma}(u_{\Gamma}) \partial_{t,n} u_{\Gamma} - \text{div}_{\Gamma}(d_{\Gamma}(u_{\Gamma}) \nabla_{\Gamma} u_{\Gamma}) &= \\
= [d(u) \partial_n u] - (l(u_{\Gamma}) + l_{\Gamma}(u_{\Gamma}) \mathcal{H} - \gamma(u_{\Gamma}) V) V \quad \text{on } \Gamma(t) \\
u(0) &= u_0 \quad \text{in } \Omega, \\
\Gamma(0) &= \Gamma_0.
\end{align*}
\]

(1.6)

Here \( \varphi(u) = [\psi(u)] \), and \( \partial_{t,n} u_{\Gamma} \) denotes the time derivative of \( u_{\Gamma} \) in normal direction, defined by

\[
\partial_{t,n} u_{\Gamma}(t,p) := \frac{d}{d\tau} u_{\Gamma}(t + \tau, x(t + \tau, p)) |_{\tau=0}, \quad t > 0, \quad p \in \Gamma(t),
\]

(1.7)

with \( \{x(t + \tau, p) \in \mathbb{R}^n : (\tau, p) \in (-\varepsilon, \varepsilon) \times \Gamma(t)\} \) the flow induced by the normal vector field \( (V \nu) \). That is, \([\tau \mapsto x(t + \tau, p)]\) defines for each \( p \in \Gamma(t) \) a flow line through \( p \) with

\[
\frac{d}{d\tau} x(t + \tau, p) = (V \nu)(t + \tau, x(t + \tau, p)), \quad x(t + \tau, p) \in \Gamma(t + \tau), \quad \tau \in (-\varepsilon, \varepsilon),
\]

and \( x(t, p) = p \). The existence of a unique trajectory

\[
\{x(t + \tau, p) \in \mathbb{R}^n : \tau \in (-\varepsilon, \varepsilon)\}, \quad p \in \Gamma(t),
\]

with the above properties is not completely obvious, see for instance [12] for a proof.

Note that the (non-degenerate) equilibria for this problem are the same as those for (1.1): the temperature is constant, and the disperse phase \( \Omega_1 \) consists of finitely many nonintersecting balls of the same radius. We shall prove that such an equilibrium is stable in the state manifold \( \mathcal{S} \mathcal{M} \) defined below if \( \Omega_1 \) is connected and the stability condition introduced in the next section holds. Such an equilibrium will be a local maximum of the total entropy, as we found before in [13] for the case of constant surface tension. To the best of our knowledge, there is no mathematical work on thermodynamically consistent Stefan problems with surface tension depending on the temperature.

3. The case where undercooling is present is the simpler one, as both equations on the interface are dynamic equations. In particular, the Gibbs-Thomson identity

\[
\gamma(u_{\Gamma}) V - \sigma(u_{\Gamma}) \mathcal{H} = \varphi(u_{\Gamma})
\]

can be understood as a mean curvature flow for the evolution of the surface, modified by physics.
If there is no undercooling, it is convenient to eliminate the time derivative of $u_T$ from the energy balance on the interface. In fact, differentiating the Gibbs-Thomson law w.r.t. time $t$ and, with $\lambda(s) = \varphi(s)/\sigma(s)$, we obtain

$$\lambda'(u_T)\partial_{\nu}u_T + H'(\Gamma)\nu = 0 \quad \text{on } \Gamma(t),$$

hence substitution into surface energy balance yields with

$$T(\nu_T) := \omega_T(\nu_T) - H'(\Gamma), \quad \omega_T(\nu_T) := \lambda'(u_T)(l(u_T) - \nu_T\lambda(u_T))/\kappa_T(u_T),$$

the relation

$$T(\nu_T)V = \frac{\lambda'(u_T)}{\kappa_T(u_T)} \left\{ \text{div}_{\Gamma}(d_T(\nu_T)\nabla_{\Gamma}u) + [d(u)\nu_u] \right\}. \quad (1.8)$$

As $V$ should be determined only by the state of the system and should not depend on time derivatives of other variables, this indicates that the problem without undercooling is not well-posed if the operator $T(\nu_T)$ is not invertible in $L_2(\Gamma)$, as $V$ might not be well-defined. On the other hand if $T(\nu_T)$ is invertible, then

$$V = [T(\nu_T)]^{-1} \frac{\lambda'(u_T)}{\kappa_T(u_T)} \left\{ \text{div}_{\Gamma}(d_T(\nu_T)\nabla_{\Gamma}u_T) + [d(u)\nu_u] \right\} \quad (1.9)$$

uniquely determines the interfacial velocity $V$, gaining two derivatives in space, and showing that the right hand side of surface energy balance is of lower order. Note that

$$\omega_T(s) = s\sigma(s)[\lambda'(s)]^2/\kappa_T(s) \geq 0 \quad \text{in } (0, u_c), \quad (1.10)$$

and $\omega_T(s) = 0$ if and only if $\lambda'(s) = 0$. Therefore the well-posedness condition becomes more complex compared to the case $\kappa_T \equiv 0$.

Going one step further, taking the surface gradient of the Gibbs-Thomson relation yields the identity

$$\kappa_T(u_T)V - d_T(u_T)H(\Gamma) = \kappa_T(u_T)\{f_T(u_T) + F_T(u, u_T)\}, \quad (1.11)$$

as will be shown in Section 6. Here the function $f_T$ is the antiderivative of $\lambda(d_T/\kappa_T)'$ vanishing at $s = u_m$, and $F_T$ is nonlocal in space and of lower order. So also in the case where undercooling is absent we obtain a mean curvature flow, modified by physics.

We would like to point out a phenomenon, in absence of kinetic undercooling, which is due to positive surface heat capacity $\kappa_T$. If $\kappa_T$ at an equilibrium is large enough, then such a steady state is stable, even if the interface is disconnected! Hence, this case seems to prevent the onset of Ostwald ripening. However, as we shall see in the next section, such equilibria cannot be maxima of the total entropy.

The plan for this paper is as follows. In Section 2 we discuss some fundamental physical properties of the Stefan problem with variable surface tension. In particular, it is shown that the negative total entropy is a strict Lyapunov functional for the problem, and we characterize and analyze the equilibria of the system. The direct mapping method based on the Hanzawa transform, first introduced in [8], is discussed in Section 3. This way the problem is reduced to a quasilinear parabolic problem. In Section 4 we consider the full linearization of the problem at a given
equilibrium, and we prove that these are normally hyperbolic, generically. The last two sections deal with the analysis of the nonlinear problem with and without kinetic undercooling. The analysis is based on results for abstract quasilinear parabolic problems, in particular on the generalized principle of linearized stability, see [10, 17]. We refer here to [3, 13, 15] for information on maximal regularity in \(L^p\)- and weighted \(L^p\)-spaces, and to [5, 6, 7, 18] for more background information concerning the Stefan problem.

2. Energy, Entropy and Equilibria

(a) The total energy of the system (1.6) is given by

\[
E(u, \Gamma) = \int_{\Omega \setminus \Gamma} \epsilon(u) \, dx + \int_{\Gamma} \epsilon_{\Gamma}(u_{\Gamma}) \, ds,
\]

and by the transport and surface transport theorem we have for smooth solutions

\[
\frac{d}{dt} E(u, \Gamma) = \int_{\Omega} \kappa(u) \partial_t u \, dx - \int_{\Gamma} \{\epsilon(u)V + \text{div}_\Gamma (d_{\Gamma}(u_{\Gamma}) \nabla_{\Gamma} u_{\Gamma}) \}
\]

\[
= \int_{\Omega} \left\{ -[d(u)\partial_n u] - \left[ \epsilon(u) \right] V + \text{div}_\Gamma (d_{\Gamma}(u_{\Gamma}) \nabla_{\Gamma} u_{\Gamma}) \right\} \, dx
\]

\[
= - \int_{\Gamma} \{ [\psi(u)] + \sigma(u_{\Gamma}) \mathcal{H} - \gamma(u_{\Gamma}) V \} V \, ds = 0
\]

by the Gibbs-Thomson law, and thus, energy is conserved.

(b) The total entropy of the system, given by

\[
\Phi(u, \Gamma) = \int_{\Omega \setminus \Gamma} \eta(u) \, dx + \int_{\Gamma} \eta_{\Gamma}(u_{\Gamma}) \, ds,
\]

satisfies

\[
\frac{d}{dt} \Phi(u, \Gamma) = \int_{\Omega} \eta'(u) \partial_t u \, dx + \int_{\Gamma} \left\{ \epsilon_{\Gamma}(u_{\Gamma}) \partial_{\Gamma} u_{\Gamma} - \left[ \eta(u_{\Gamma}) \right] + \eta_{\Gamma}(u_{\Gamma}) \mathcal{H} V \right\} \, ds
\]

\[
= \int_{\Omega} \left\{ \frac{1}{u} \kappa(u) \partial_t u \right\} \, dx + \int_{\Gamma} \left\{ \frac{1}{u_{\Gamma}} \kappa_{\Gamma}(u_{\Gamma}) \partial_{\Gamma} u_{\Gamma} + \left( l(u) + l_{\Gamma}(u_{\Gamma}) \right) \mathcal{H} V \right\} \, ds
\]

\[
= \int_{\Omega} \frac{1}{u^2} d(u) |\nabla u|^2 \, dx
\]

\[
+ \int_{\Gamma} \frac{1}{u_{\Gamma}} \{-[d(u)\partial_n u] + \text{div}_\Gamma (d_{\Gamma}(u_{\Gamma}) \nabla_{\Gamma} u_{\Gamma}) + \left[ d(u)\partial_n u \right] + \left[ \gamma(u_{\Gamma}) V \right] \}
\]

\[
= \int_{\Omega} \frac{1}{u^2} d(u) |\nabla u|^2 \, dx + \int_{\Gamma} \frac{1}{u_{\Gamma}} \left\{ d_{\Gamma}(u_{\Gamma}) |\nabla_{\Gamma} u_{\Gamma}|^2 + u_{\Gamma} \gamma(u_{\Gamma}) V^2 \right\} \, ds \geq 0,
\]

where we employed the transport theorem, the surface transport theorem and (1.6). In particular, the negative total entropy is a Lyapunov functional for problem (1.6).
Then by the method of Lagrange multipliers, there is

\[ \text{varying} \]

The derivatives of the functionals are given by

\[ u \]

Setting first \( u(t_0) = \text{const} = u_\Gamma(t_0) \) in \( \Omega \), and \( \mathcal{H}(t_0) = -[\psi(u(t_0))]/\sigma(u_\Gamma(t_0)) = \text{const} \), provided we have

\[ \|\psi(s)\| = 0 \implies \sigma(s) > 0. \]

Physically, this assumption is plausible, as it means that at melting temperature \( u_m > 0 \) (defined as the unique positive zero of the function \( \varphi(s) := [\psi(s)] \)) the surface tension \( \sigma(u_m) \) is positive. Since \( \Omega \) is bounded, we may conclude that \( \Gamma(t_0) \) is a union of finitely many, say \( m \), disjoint spheres of equal radius, i.e. \( (u(t_0), \Gamma(t_0)) \) is an equilibrium. Therefore, the limit sets of solutions in the state manifold defined below are contained in the \( (mn + 1) \)-dimensional manifold of equilibria

\[ \mathcal{E} = \{ (u_\ast, \bigcup_{1 \leq i \leq m} S_{R_\ast}(x_i)) : 0 < u_\ast < u_c, \ [\psi(u_\ast)] = (n - 1)\sigma(u_\ast)/R_\ast, \]

\[ B_{R_\ast}(x_i) \subset \Omega, B_{R_\ast}(x_i) \cap B_{R_\ast}(x_k) = \emptyset, k \neq i \}, \]

where \( S_{R_\ast}(x_i) \) denotes the sphere with radius \( R_\ast \) and center \( x_i \).

(d) Another interesting observation is the following. Consider the critical points of the functional \( \Phi(u, u_\Gamma, \Gamma) \) with constraint \( \mathcal{E}(u, u_\Gamma, \Gamma) = \mathcal{E}_0 \), say on

\[ U := \{ (u, u_\Gamma, \Gamma) : u \in C(\overline{\Omega} \setminus \Gamma), \ \Gamma \in \mathcal{M}\mathcal{H}^2(\Omega), \ u_\Gamma \in C(\Gamma), \ u, u_\Gamma > 0 \}, \]

see below for the definition of \( \mathcal{M}\mathcal{H}^2(\Omega) \). So here we do not assume from the beginning that \( u \) is continuous across \( \Gamma \), and \( u_\Gamma \) denotes surface temperature. Then by the method of Lagrange multipliers, there is \( \mu \in \mathbb{R} \) such that at a critical point \( (u_\ast, u_{\Gamma_\ast}, \Gamma_\ast) \) we have

\[ \Phi'(u_\ast, u_{\Gamma_\ast}, \Gamma_\ast) + \mu \mathcal{E}'(u_\ast, u_{\Gamma_\ast}, \Gamma_\ast) = 0. \]

The derivatives of the functionals are given by

\[ \langle \Phi'(u, u_\Gamma, \Gamma)(v, v_\Gamma, h) \rangle = (\eta'(u)[v]\Omega + (\eta_\Gamma'(u_\Gamma)[v_\Gamma])_\Gamma - ([\eta(u)] + \eta_\Gamma(u_\Gamma)\mathcal{H}(\Gamma)[h])_\Gamma, \]

and

\[ \langle \mathcal{E}'(u, u_\Gamma, \Gamma)(v, v_\Gamma, h) \rangle = (\epsilon'(u)[v]\Omega + (\epsilon_\Gamma'(u_\Gamma)[v_\Gamma])_\Gamma - ([\epsilon(u)] + \epsilon_\Gamma(u_\Gamma)\mathcal{H}(\Gamma)[h])_\Gamma. \]

Setting first \( v_\Gamma = h = 0 \) and varying \( v \) in \( \Omega \), we obtain

\[ \eta'(u_\ast) + \mu \epsilon'(u_\ast) = 0 \quad \text{in} \ \Omega, \]

varying \( v_\Gamma \) yields

\[ \eta_\Gamma'(u_{\Gamma_\ast}) + \mu \epsilon_\Gamma'(u_{\Gamma_\ast}) = 0 \quad \text{on} \ \Gamma_\ast, \]
and finally varying \( h \) we get
\[
\eta(u) + \eta'(u)\mathcal{H}(\Gamma) + \mu(\varepsilon(u) + \epsilon(\Gamma)) = 0 \text{ on } \Gamma_*.
\]
The relations \( \eta(u) = -\psi'(u) \) and \( \varepsilon(\Gamma) = -\psi'(u) - u\psi''(u) \) imply \( 0 = -\psi''(u)(1 + \mu u) \), and this shows that \( u_* = -1/\mu \) is constant in \( \Omega \), since \( \kappa(u) = -u\psi''(u) > 0 \) for all \( u > 0 \) by assumption. Similarly on \( \Gamma_* \) we obtain \( u_{\Gamma_*} = -1/\mu \) constant as well, provided \( \kappa_{\Gamma}(u_{\Gamma}) > 0 \), hence in particular \( u_* \equiv u_{\Gamma_*} \). This further implies the Gibbs-Thomson relation \( [\psi(u_*)] + \sigma(u_*)\mathcal{H}(\Gamma_*) = 0 \). Since \( u_* \) is constant we see that \( \mathcal{H}(\Gamma_*) \) is constant, by (2.3). Therefore \( \Gamma_* \) is a sphere whenever connected, and a union of finitely many disjoint spheres of equal size otherwise. Thus the critical points of the entropy functional for prescribed energy are precisely the equilibria of problem (1.6).

(e) Going further, suppose we have an equilibrium \( e_* : = (u_*, \Gamma_*, \Gamma_*) \) where the total entropy has a local maximum w.r.t. the constraint \( \mathcal{E} = E_0 \) constant. Then \( \mathcal{D}_* = \{ \Phi + \mu E \}''(e_*) \) is negative semi-definite on the kernel of \( \mathcal{E}'(e_*) \), where \( \mu = -1/\mu_* \) is the fixed Lagrange multiplier found above. The kernel of \( \mathcal{E}'(e) \) is given by the identity
\[
(\kappa(u)v, v)_\Omega + (\kappa_{\Gamma}(u_{\Gamma})v_{\Gamma}, v_{\Gamma}) + u_*(l_*|h)_\Gamma = 0,
\]
which at equilibrium yields
\[
(\kappa_* v, v)_\Omega + (\kappa_{\Gamma_*}v_{\Gamma}, v_{\Gamma}) + u_*(l_*|h)_\Gamma = 0,
\]
(2.6)
where \( \kappa_* := \kappa(u_*) \), \( \kappa_{\Gamma_*} := \kappa_{\Gamma}(u_*) \) and
\[
l_* := \frac{1}{u_*} \left[ (l(u_*))_{\Omega} + l_{\Gamma}(u_*)\mathcal{H}(\Gamma_*) \right] = [\psi'(u_*)] + [\sigma'(u_*)](\mathcal{H}(\Gamma_*)).
\]
(2.7)
On the other hand, a straightforward calculation yields with \( z = (v, v_{\Gamma}, h) \)
\[
-\langle \mathcal{D}_* z | z \rangle = \frac{1}{u_*} \left[ (\kappa_* v, v)_\Omega + (\kappa_{\Gamma_*} v_{\Gamma}, v_{\Gamma}) - \sigma_0 u_* (\mathcal{H}(\Gamma_*) h|h)_\Gamma \right],
\]
where \( \kappa_{\Gamma_*} = \kappa_{\Gamma}(u_* \) and \( \sigma_0 = \sigma(u_* \). As \( \kappa_* \) and \( \kappa_{\Gamma_*} \) are positive, we see that the form \( \langle \mathcal{D}_* z | z \rangle \) is negative semi-definite as soon as \( \mathcal{H}(\Gamma_*) \) is negative semi-definite. We have
\[
\mathcal{H}(\Gamma_*) = (n - 1)/R_*^2 + \Delta_*,
\]
where \( \Delta_* \) denotes the Laplace-Beltrami operator on \( \Gamma_* \) and \( R_* \) means the radius of an equilibrium sphere. To derive necessary conditions for an equilibrium \( e_* \) to be a local maximum of entropy, we consider two cases.

1. Suppose that \( \Gamma_* \) is not connected, i.e. \( \Gamma_* \) is a finite union of spheres \( \Gamma^k_* \). Set \( v = v_{\Gamma} = 0 \), and let \( h = h_k \) be constant on \( \Gamma^k_* \) with \( \sum_k h_k = 0 \). Then the constraint (2.6) holds, and with \( \omega_n \) the area of the unit sphere in \( \mathbb{R}^n \)
\[
\langle \mathcal{D} z | z \rangle = (\sigma_0 u_*)((n - 1)/R_*^2 + \Delta_*)\omega_n R_*^{n-1} \sum h_k^2 > 0,
\]
hence $\mathcal{D}$ cannot be negative semi-definite in this case, as $\sigma_* > 0$ by (2.3). Thus if $e_*$ is an equilibrium with maximal total entropy, then $\Gamma_*$ must be connected, and hence both phases are connected.

2. Assume that $\Gamma_*$ is connected. With $h = -(\kappa_*|1|_\Omega - \kappa_{\Gamma_*}|\Gamma_*|$, $v = v_\Gamma = u_* l_\Gamma|\Gamma_*$ we see that $\mathcal{D}$ negative semi-definite on the kernel of $E'(e_*)$ implies the condition

$$\zeta_* := \zeta(u_*) := \frac{(n-1)\sigma_*[|\kappa_*|_\Omega + \kappa_{\Gamma_*}|\Gamma_*|]}{u_* l_\Gamma^2 R_*^2|\Gamma_*|} \leq 1. \tag{2.9}$$

We will see below that connectedness of $\Gamma_*$ and the strong stability condition $\zeta_* < 1$ are sufficient for stability of the equilibrium $e_*$. We point out below that the quantity $\zeta_*$ defined in (2.9) coincides with the analog quantity in [18, Definition (1.11)] in case $\kappa_{\Gamma_*} = 0$ and $\sigma_\ast$ constant. (Note that $l_* = l(u_*)/u_*$ in this case, which differs from the definition of $l_*$ in [18]).

(f) Summarizing, we have shown

- The total energy is constant along smooth solutions of (1.6).
- The negative total entropy is a strict Lyapunov functional for (1.6).
- The equilibria of (1.6) are precisely the critical points of the entropy functional with prescribed energy.
- If the entropy functional with prescribed energy has a local maximum at $e_* = (u_*,u_{\Gamma*},\Gamma_*)$ then $\Gamma_*$ is connected.
- If $\Gamma_*$ is connected, a necessary condition for a critical point $(u_*,u_{\Gamma*},\Gamma_*)$ to be a local minimum of the entropy functional with prescribed energy is inequality (2.9).

(g) Now let us look at the energy of an equilibrium as a function of temperature. Suppose we have an equilibrium $(u,\Gamma)$ at a given energy level $E_0$, and assume that $\Gamma$ consists of $m$ disjoint spheres of radius $R$ contained in $\Omega$. Then

$$0 < R < R_m := \sup \{ R > 0 : \Omega \text{ contains } m \text{ disjoint ball of radius } R \},$$

and with $\varphi(u) := [\psi(u)]$ we have

$$0 = \varphi(u) + \sigma(u)\mathcal{H}(\Gamma) = \varphi(u) - (n-1)\sigma(u)/R,$$

and hence $R = R(u) = (n-1)\sigma(u)/\varphi(u)$. Further we have

$$E_\varphi(u) := E(u,\Gamma) = \int_\Gamma \epsilon(u) \, dx + \int_\Gamma \epsilon_\Gamma(u) \, ds$$

$$= \epsilon_2(u)|\Omega| - |\Omega|[\epsilon(u)] + \epsilon_\Gamma(u)|\Gamma|$$

$$= \epsilon_2(u)|\Omega| - (m\omega_n/n)R(u)^n[\epsilon(u)] + m\omega_n R(u)^{n-1}\epsilon_\Gamma(u)$$

$$= \epsilon_2(u)|\Omega| + c_{n,m} \left[ \frac{\sigma(u)^n}{\varphi(u)^{n-1}} - u \frac{d}{du} \frac{\sigma(u)^n}{\varphi(u)^{n-1}} \right],$$

where $c_{n,m} = m\omega_n(n-1)^{n-1}/n$. Thus we obtain for the total energy of an equilibrium

$$E_\varphi(u) = \delta(u) - u\delta'(u), \quad \delta(u) = |\Omega|\psi_2(u) + c_{n,m} \frac{\sigma(u)^n}{\varphi(u)^{n-1}}. \tag{2.10}$$
Then we may approximate $\Gamma$ by a real analytic hypersurface $\Sigma$ (or merely $\Sigma$ is as small as we want. More precisely, for each $\eta > 0$ there is a real analytic hypersurface $\Sigma$ that satisfies the following condition: for every $\nu \in \mathbb{R}^n$, the Hausdorff distance between $\Gamma$ and $\Sigma$ for $\nu$ is less than $\eta$. This is true because $\Gamma$ is a bounded domain and $\Sigma$ is a closed hypersurface of class $C^2$, and the curvature of $\Gamma$ is bounded.

Let us look at the derivative of the function $E_\nu(u)$. A simple calculation yields

$$E'_\nu(u) = -u\delta''(u) = -|\Omega|\psi'^2(u) - c_{n,m}u \frac{d}{du} \left[ \frac{\sigma'(\varphi)}{\varphi} \right]_{n-1}(\sigma/\varphi)^{n-1} \sigma' - (n-1)(\sigma/\varphi)^n \varphi'$$

$$= |\Omega|\kappa_2(u) - c_{n,m}u [n(\sigma/\varphi)^{n-1}\sigma'' - (n-1)(\sigma/\varphi)^n \varphi'']$$

$$- c_{n,m}u(n-1)(\sigma/\varphi)^{n-1}u[(\sigma')^2/\sigma - 2\sigma'(\varphi)/\varphi + \sigma(\varphi')^2/\varphi^2]$$

$$= |\Omega|\kappa_2(u) + |\Gamma|\kappa_T(u) - |\Omega_1|\kappa(u) - (R^2|\Gamma|/(n-1)\sigma)u[\varphi' - \sigma'(\varphi/\sigma)]^2$$

$$= [(\kappa(u)|\Omega_1 + |\Gamma|\kappa_T(u)] - (R^2(u)|\Gamma|/(n-1)\sigma(u))u[\psi'(u)] + \sigma'(u)\mathcal{H}(\Gamma)]^2.$$ 

Therefore the stability condition $\kappa(u) \leq 1$ is equivalent to $E'_\nu(u) \leq 0$, an alternative interpretation to the one obtained above.

### 3. Transformation to a Fixed Interface

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial \Omega$ of class $C^2$, and suppose $\Gamma \subset \Omega$ is a closed hypersurface of class $C^2$, i.e. a $C^2$-manifold which is the boundary of a bounded domain $\Omega_1 \subset \Omega$. We then set $\Omega_2 = \Omega \setminus \Omega_1$. Note that while $\Omega_2$ is connected, $\Omega_1$ may be disconnected. However, $\Omega_1$ consists of finitely many components only, as $\partial \Omega_1 = \Gamma$ by assumption is a manifold, at least of class $C^2$.

Recall that the second order bundle of $\Gamma$ is given by

$$\mathcal{N}^2\Gamma := \{(p, \nu_\Gamma(p), L_\Gamma(p)) : p \in \Gamma\}.$$ 

Note that the Weingarten map $L_\Gamma$ (also called the shape operator, or the second fundamental tensor) is defined by

$$L_\Gamma(p) = -\nabla_\Gamma \nu_\Gamma(p), \quad p \in \Gamma,$$

where $\nabla_\Gamma$ denotes the surface gradient on $\Gamma$. The eigenvalues $\kappa_j(p)$ of $L_\Gamma(p)$ are the principal curvatures of $\Gamma$ at $p \in \Gamma$, and we have $|L_\Gamma(p)| = \max_j |\kappa_j(p)|$. The curvature $\mathcal{H}_\Gamma(p)$ is defined by

$$\mathcal{H}_\Gamma(p) = \sum_{j=1}^{n-1} \kappa_j(p) = \text{tr}L_\Gamma(p) = -\text{div}_\Gamma \nu_\Gamma(p),$$

where $\text{div}_\Gamma$ means surface divergence. Recall also that the Hausdorff distance $d_H$ between the two closed subsets $A, B \subset \mathbb{R}^m$ is defined by

$$d_H(A, B) := \max \left\{ \sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A) \right\}.$$ 

Then we may approximate $\Gamma$ by a real analytic hypersurface $\Sigma$ (or merely $\Sigma \in C^3$), in the sense that the Hausdorff distance of the second order bundles of $\Gamma$ and $\Sigma$ is as small as we want. More precisely, for each $\eta > 0$ there is a real analytic...
closed hypersurface such that \( d_H(N^2\Sigma, N^2\Gamma) \leq \eta \). If \( \eta > 0 \) is small enough, then \( \Sigma \) bounds a domain \( \Omega^2_1 \) with \( \Omega^2_1 \subset \Omega \), and we set \( \Omega^2_2 = \Omega \setminus \Omega^2_1 \).

It is well known that such a hypersurface \( \Sigma \) admits a tubular neighborhood, which means that there is a \( \delta > 0 \) such that the map
\[
\Lambda : \Sigma \times (-\delta, \delta) \to \mathbb{R}^n
\]
\[
\Lambda(p, r) := p + r\nu_\Sigma(p)
\]
is a diffeomorphism from \( \Sigma \times (-\delta, \delta) \) onto \( \mathcal{R}(\Lambda) \). The inverse
\[
\Lambda^{-1} : \mathcal{R}(\Lambda) \mapsto \Sigma \times (-\delta, \delta)
\]
of this map is conveniently decomposed as
\[
\Lambda^{-1}(x) = (\Pi_\Sigma(x), d_\Sigma(x)), \quad x \in \mathcal{R}(\Lambda).
\]
Here \( \Pi_\Sigma(x) \) means the nonlinear orthogonal projection of \( x \) to \( \Sigma \) and \( d_\Sigma(x) \) the signed distance from \( x \) to \( \Sigma \); so \( |d_\Sigma(x)| = \text{dist}(x, \Sigma) \) and \( d_\Sigma(x) < 0 \) iff \( x \in \Omega^2_1 \). In particular we have \( \mathcal{R}(\Lambda) = \{ x \in \mathbb{R}^n : \text{dist}(x, \Sigma) < \delta \} \).

On the one hand, \( a \) is determined by the curvatures of \( \Sigma \), i.e. we must have
\[
0 < a < \min \left\{ \left| \frac{1}{\kappa_j(p)} \right| : j = 1, \ldots, n-1, \ p \in \Sigma \right\},
\]
where \( \kappa_j(p) \) mean the principal curvatures of \( \Sigma \) at \( p \in \Sigma \). But on the other hand, \( a \) is also connected to the topology of \( \Sigma \), which can be expressed as follows. Since \( \Sigma \) is a compact (smooth) manifold of dimension \( n - 1 \) it satisfies a (interior and exterior) ball condition, which means that there is a radius \( r_\Sigma \geq 0 \) such that for each point \( p \in \Sigma \) there are \( x_j \in \Omega^2_j \), \( j = 1, 2 \), such that \( B_{r_\Sigma}(x_j) \subset \Omega^2_j \), and \( \bar{B}_{r_\Sigma}(x_j) \cap \Sigma = \{ p \} \). Choosing \( r_\Sigma \) maximal, we then must also have \( a < r_\Sigma \).

In the sequel we fix
\[
a = \frac{1}{2} \min \left\{ r_\Sigma, \frac{1}{|\kappa_j(p)|} : j = 1, \ldots, n-1, \ p \in \Sigma \right\}.
\]

For later use we note that the derivatives of \( \Pi_\Sigma(x) \) and \( d_\Sigma(x) \) are given by
\[
\nabla d_\Sigma(x) = \nu_\Sigma(\Pi_\Sigma(x)), \quad \Pi'_\Sigma(x) = M_0(d_\Sigma(x), \Pi(x)) P_\Sigma(\Pi_\Sigma(x))
\]
for \( |d_\Sigma(x)| < a \), where \( P_\Sigma(p) = I - \nu_\Sigma(p) \otimes \nu_\Sigma(p) \) denotes the orthogonal projection onto the tangent space \( T_p\Sigma \) of \( \Sigma \) at \( p \in \Sigma \), and
\[
M_0(r)(p) = (I - rL_\Sigma(p))^{-1}, \quad (r, p) \in (-a, a) \times \Sigma.
\]

Note that
\[
|M_0(r)(p)| \leq 1/(1 - r|L_\Sigma(p)|) \leq 2, \quad \text{for all} \ (r, p) \in (-a, a) \times \Sigma.
\]

Setting \( \Gamma = \Gamma(t) \), we may use the map \( \Lambda \) to parameterize the unknown free boundary \( \Gamma(t) \) over \( \Sigma \) by means of a height function \( h(t, p) \) via
\[
\Gamma(t) = \{ p + h(t, p)\nu_\Sigma(p) : p \in \Sigma, \ t \geq 0 \},
\]
at least for small \( |h|_{\infty} \). Extend this diffeomorphism to all of \( \bar{\Omega} \) by means of
\[
\Xi_h(t, x) = x + \chi(d_\Sigma(x)/a)h(t, \Pi_\Sigma(x))\nu_\Sigma(\Pi_\Sigma(x)) =: x + \xi_h(t, x).
\]
Here \( \chi \) denotes a suitable cut-off function. More precisely, \( \chi \in \mathcal{D}(\mathbb{R}), \) \( 0 \leq \chi \leq 1, \) \( \chi(r) = 1 \) for \( |r| < 1/3, \) and \( \chi(r) = 0 \) for \( |r| > 2/3. \) Note that \( \Xi_h(t, x) = x \) for \( |d(x)| > 2a/3, \) and

\[
\Xi_h^{-1}(t, x) = x - h(t, x)\nu_\Sigma(x), \quad x \in \Sigma,
\]

for \( |h|_{\infty} \) sufficiently small.

Setting

\[
v(t, x) = u(t, \Xi_h(t, x)) \quad \text{or} \quad u(t, x) = v(t, \Xi_h^{-1}(t, x))
\]

we have this way transformed the time varying regions \( \Omega \setminus \Gamma(t) \) to the fixed domain \( \Omega \setminus \Sigma. \) This is the direct mapping method, also called Hanzawa transformation.

By means of this transformation, we obtain the following transformed problem.

\[
\begin{cases}
\kappa(v)\partial_t v + A(v, \rho)v = \kappa(v)\mathcal{R}(\rho)v & \text{in } \Omega \setminus \Sigma \\
\partial_v v = 0 & \text{on } \partial\Omega \\
[v] = 0, \quad v_T = v & \text{on } \Sigma \\
[\psi(v_T)] + \sigma(v_T)\mathcal{H}(\rho) - \gamma(v_T)\beta(\rho)\partial_t \rho = 0 & \text{on } \Sigma \\
\kappa_T(v_T)\partial_t v_T + C(v_T, \rho)v_T + \mathcal{B}(v, \rho)v_T = -\{l(v_T) + l_T(v_T)\mathcal{H}(\rho) - \gamma(v_T)\beta(\rho)\partial_t \rho\} \partial_t \rho & \text{on } \Sigma \\
v(0) = v_0, \quad \rho(0) = \rho_0.
\end{cases}
\]

(3.2)

Here \( A(v, \rho), \mathcal{B}(v, \rho) \) and \( C(v_T, \rho) \) denote the transformed versions of the operators of \(-\text{div}(d\nabla), \) \([-d\partial_v], \) and \(-\text{div}_T(d_T\nabla_T), \) respectively. Moreover, \( \mathcal{H}(\rho) \) means the mean curvature of \( \Gamma, \beta(\rho) = (\nu_\Sigma|v_T(\rho)), \) the term \( \beta(\rho)\partial_t \rho \) represents the normal velocity \( V, \) and

\[
\mathcal{R}(\rho)v = \partial_v v - \partial_t u \circ \Xi_h.
\]

The system (3.2) is a quasi-linear parabolic problem on the domain \( \Omega \) with fixed interface \( \Sigma \subset \Omega \) with dynamic boundary conditions.

To elaborate on the structure of this problem in more detail, we calculate

\[
D\Xi_h = I + D\xi_h, \quad [D\xi_h]^{-1} = I - [I + D\xi_h]^{-1}D\xi_h =: I - M_1(\rho)^T.
\]

where \( D \) denotes the derivative with respect to the space variables. Hence \( D\xi_h = 0 \) for \( |d_\Sigma(x)| > 2a/3 \) and

\[
D\xi_h(t, x) = \frac{1}{a} \chi'(d_\Sigma(x)/a)\rho(t, \Pi_\Sigma(x))\nu_\Sigma(\Pi_\Sigma(x)) \otimes \nu_\Sigma(\Pi_\Sigma(x))
+ \chi(d_\Sigma(x)/a)\nu_\Sigma(\Pi_\Sigma(x)) \otimes M_0(d_\Sigma(x))\n- \chi(d_\Sigma(x)/a)\rho(t, \Pi_\Sigma(x))L_\Sigma(\Pi_\Sigma(x))M_0(d_\Sigma(x))P_\Sigma(\Pi_\Sigma(x))
\]

for \( 0 \leq |d_\Sigma(x)| \leq 2a/3. \) In particular, for \( x \in \Sigma \) we have

\[
D\xi_h(t, x) = \nu_\Sigma(x) \otimes \nabla_\Sigma \rho(t, x) - \rho(t, x) L_\Sigma(x)P_\Sigma(x),
\]

and

\[
[D\xi_h]^T(t, x) = \nabla_\Sigma \rho(t, x) \otimes \nu_\Sigma(x) - \rho(t, x) L_\Sigma(x),
\]
since $L_\Sigma(x)$ is symmetric and has range in $T_\Sigma \Sigma$. Therefore, $[I + D\xi_\rho]$ is boundedly invertible, if $\rho$ and $\nabla\Sigma\rho$ are sufficiently small, and

$$||[I + D\xi_\rho]^{-1}\|_\infty \leq 2 \text{ for } |\rho|_\infty \leq \frac{1}{4(||\chi'||_\infty/a + 2\max_j |\kappa_j|)}.$$

Employing this notation we obtain

$$\nabla u \circ \Xi_\rho = ([D\Xi_\rho^{-1}]^T \circ \Xi_\rho)\nabla v = [D\Xi_\rho]^{-1,T}\nabla v = (I - M_1(\rho))\nabla v,$$

and for a vector field $q = \bar q \circ \Xi_\rho$

$$(\nabla|\bar q|) \circ \Xi_\rho = ([D\Xi_\rho^{-1}]^T \circ \Xi_\rho)\nabla|q| = ([D\Xi_\rho]^{-1,T}\nabla|q| = ((I - M_1(\rho))\nabla|q|).$$

Further we have

$$\partial_t u \circ \Xi_\rho = \partial_t v - ([D\Xi_\rho^{-1}]^T \circ \Xi_\rho)\nabla|\partial_t \Xi_\rho| = \partial_t v - ([D\Xi_\rho]^{-1,T}\nabla|\partial_t \Xi_\rho| = \partial_t v - (\nabla v)(I - M_1^T(\rho))\partial_t \Xi_\rho),$$

hence

$$R(\rho)v = (\nabla v)(I - M_1^T(\rho))\partial_t \Xi_\rho).$$

The normal time derivative transforms as

$$\partial_{t,n} u_T \circ \Xi_\rho = \partial_t v + (\nabla_T v_T|\nu_\Sigma) V = \partial_t v_T,$$

as $\nabla_T v_T$ is perpendicular to $\nu_\Sigma$.

With the Weingarten tensor $L_\Sigma = -\nabla_\Sigma \nu_\Sigma$ we obtain

$$\nu_T(\rho) = \beta(\rho)(\nu_\Sigma - \alpha(\rho)), \quad \alpha(\rho) = M_0(\rho)\nabla_\Sigma \rho,$$

$$M_0(\rho) = (I - \rho L_\Sigma)^{-1}, \quad \beta(\rho) = (1 + |\alpha(\rho)|^2)^{-1/2},$$

and

$$V = (\partial_t \Xi_\rho|\nu_T) = (\nu_\Sigma|\nu_T(\rho))\partial_t \rho = \beta(\rho)\partial_t \rho.$$

For the mean curvature $H(\rho)$ we have

$$H(\rho) = \beta(\rho)\{\text{tr}[M_0(\rho)(L_\Sigma + \nabla_\Sigma \alpha(\rho)) - \beta^2(\rho)(M_0(\rho)\alpha(\rho))(\nabla_\Sigma \alpha(\rho))\alpha(\rho)]\},$$

an expression involving second order derivatives of $\rho$ only linearly. More precisely,

$$H(\rho) = \beta(\rho)G(\rho) : \nabla_\Sigma \rho + \beta(\rho)F(\rho),$$

$$G(\rho) = M_0^2(\rho) - \beta^2(\rho)M_0(\rho)\nabla_\Sigma \rho \otimes M_0(\rho)\nabla_\Sigma \rho.$$

Note that $\beta$ as well as $F$ and $G$ only depend on $\rho$ and $\nabla_\Sigma \rho$. The linearization of the curvature at $\rho = 0$ is given by

$$H'(0) = \text{tr} L_\Sigma^2 + \Delta_\Sigma.$$
The full linearization at an equilibrium \((u_*, u_{Γ_*}, Γ_*)\) with \(u_{Γ_*} = u_*\), \(Γ_* = \cup_k Σ_k\) a finite union of disjoint spheres contained in \(Ω\) and with radius \(R_* > 0\) given by \(R_* = (n - 1)σ(u_*)/∥ψ(u_*)∥\), reads

\[
\begin{align*}
\kappa_* \partial_t v - d_* \Delta v &= \kappa_* f & \text{in } Ω \setminus Γ_* \\
\partial_v v &= 0 & \text{on } ∂Ω \\
[v] = 0, ν_T = v & \text{on } Γ_* \\
k_{Γ_*} \partial_t v_{Γ_*} - d_{Γ_*} \Delta v_{Γ_*} - [d_* \partial_v v] + l_* u_* \partial_v ρ &= k_{Γ_*} f_{Γ_*} & \text{on } Γ_* \\
l_* ν_T - σ_* A_* ρ - γ_* \partial_v ρ &= g & \text{on } Γ_* \\
v(0) &= ν_0, ρ(0) = ρ_0.
\end{align*}
\]

(4.1)

Here

\[
\kappa_* = κ(u_*) > 0, \quad k_{Γ_*} = k_{Γ}(u_*) > 0, \quad d_* = d(u_*) > 0, \quad d_{Γ_*} = d_{Γ}(u_*) > 0, \quad σ_* = σ(u_*) > 0, \quad γ_* = γ(u_*) ≥ 0,
\]

and as in (2.7)

\[
l_* = [ψ'(u_*)] + σ'(u_*) H(Γ_*) = φ'(u_*) - σ'(u_*) φ(u_*)/σ(u_*) = σ(u_*) Λ'(u_*)
\]

and

\[
A_* = -\left(\frac{n - 1}{R_*^2} + Δ_*, \right)
\]

where \(Δ_*\) denotes the Laplace-Beltrami operator on \(Γ_*\).
4.1. Maximal Regularity. We begin with the case $\gamma_*>0$, which is the simpler one. Define the operator $L$ in

$$X_0 := L_p(\Omega) \times W^r_p(\Gamma_*) \times W^s_p(\Gamma_*)$$

with

$$X_1 := W^2_p(\Omega \setminus \Gamma_*) \times W^{2+r}_p(\Gamma_*) \times W^{2+s}_p(\Gamma_*)$$

by means of

$$D(L) = \{(v, v_T, \rho) \in X_1 : [v] = 0, \ v_T = v \text{ on } \Gamma_*, \ \partial_{\nu} v = 0 \text{ on } \partial \Omega\},$$

$$L = \begin{bmatrix}
\frac{-(d_*/\kappa_*)}{\Delta} & 0 & 0 \\
-([d_*/\kappa_*] \partial_{\nu}) & \frac{(l^2_*/\gamma_*) - d_*/\kappa_*}{\Delta} & -(l_* u_*/\gamma_* \kappa_*) A_* \\
0 & -(l_*/\gamma_*) & (\sigma_*/\gamma_*) A_*
\end{bmatrix}$$

In case $\gamma_*>0$, problem (4.1) is equivalent to the Cauchy problem

$$\dot{z} + Lz = (f, f_T - (l_* u_*/\gamma_* \kappa_*) g, g), \quad z(0) = z_0,$$

where $z = (v, v_T, \rho)$ and $z_0 = (v_0, v_0|_{\Gamma_*}, \rho_0)$. The main result on problem (4.1) for $\gamma_*>0$ is the following.

**Theorem 4.1.** Let $1 < p < \infty$, $\gamma_*>0$, and

$$-1/p \leq r \leq 1 - 1/p, \quad r \leq s \leq r + 2.$$  

Then for each finite interval $J = [0, a]$, there is a unique solution

$$(v, v_T, \rho) \in E(J) := H^1_p(J; X_0) \cap L_p(J; X_1)$$

of (4.1) if and only if the data $(f, f_T, g)$ and $(v_0, v_{T0}, \rho_0)$ satisfy

$$(f, f_T, g) \in F(J) = L_p(J; X_0),$$

$$(v_0, v_{T0}, \rho_0) \in W^{2-2/p}_p(\Omega \setminus \Gamma_*) \times W^{2+r-2/p}_p(\Gamma_*) \times W^{2+s-2/p}_p(\Gamma_*)$$

and the compatibility conditions

$$[v_0] = 0, \quad v_{T0} = v_0 \quad \text{on } \Gamma_*, \quad \partial_{\nu} v = 0 \quad \text{on } \partial \Omega.$$

The operator $-L$ defined above generates an analytic $C_0$-semigroup in $X_0$ with maximal regularity of type $L_p$.

**Proof.** Looking at the entries of $L$ we see that $L : X_1 \to X_0$ is bounded provided $r \leq 1 - 1/p, \ r \leq s$, and $s \leq r + 2$. The compatibility condition $v_T = v|_{\Gamma_*}$ implies $r + 2 \geq 2 - 1/p$. This explains the constraints on the parameters $r$ and $s$. To obtain maximal $L_p$-regularity, we first consider the case $s > r$. Then $L$ is lower triangular up to perturbation. So we may solve the problem for $(v, v_T)$ with maximal $L_p$-regularity (cf. [4] for the one-phase case) first and then that for $\rho$. In the other case we have $r = s$. Then the second term in the third line in the definition of $L$ is of lower order, hence $\rho$ decouples from $(v, v_T)$. This way we also obtain maximal $L_p$-regularity. Since the Cauchy problem for $L$ has maximal $L_p$-regularity, we can now infer from [14] Proposition 1.2 that $-L$ generates an analytic $C_0$-semigroup in $X_0$. \[\square\]
We note that if $l_* = 0$ and $\gamma_* = 0$ then the linear problem is not well-posed. In fact, in this case the linear Gibbs-Thomson relation reads

$$-\sigma_* A_* \rho = g,$$

which is not well-posed as the kernel of $A_*$ is non-trivial and $A_*$ is not surjective.

Now we consider the case $l_* \neq 0$ and $\gamma_* = 0$. For the solution space we fix again $r, s \in \mathbb{R}$ with $r \leq s \leq r + 2$, $-1/p \leq r \leq 1 - 1/p$, and consider

$$(v, v_T, \rho) \in E(J) = H^1_p(J, X_0) \cap L_p(J; X_1).$$

Then by trace theory the space of data becomes

$$(f, f_T, g) \in F_0(J) := L_p(J; L^p(\Omega)) \times L_p(J; W^p_\gamma(\Gamma_*))$$

and the space of initial values will be

$$(v_0, v_{T0}, \rho_0) \in W^{2-2/p}_p(\Omega \setminus \Gamma_*) \times W^{r+2-2/p}(\Gamma_*) \times W^{s+3-2/p}(\Gamma_*),$$

with compatibilities

$$[v_0] = 0, \quad v_{T0} = v_0, \quad l_* v_{T0} - \sigma_* A_\rho = g(0) \quad \text{on} \quad \Gamma_*, \quad \partial_\nu v = 0 \quad \text{on} \quad \partial \Omega.$$ 

To obtain maximal $L^p$-regularity, we replace $v_T$ by the Gibbs-Thomson relation, which for $\gamma_* = 0$ is an elliptic equation. We obtain $v_T = (\sigma_* / l_*) A_\rho + g / l_*$. Inserting this expression into the energy balance on the surface $\Gamma_*$ yields

$$\left(l_* u_* + (\kappa_\gamma \sigma_* / l_*) A_*\right) \partial_t \rho - d_\gamma \Delta_* v_T - [d_\gamma \partial_\nu v] = \kappa_\gamma (f_T - \partial_\nu g / l_*). \quad (4.2)$$

Moreover, we obtain

$$d_\gamma \Delta_* v_T = (l_* u_* + (\kappa_\gamma \sigma_*/ l_*) A_*) (d_\gamma A_* A_\rho - (l_* u_* d_\gamma A_*/ \kappa_\gamma A_*) \Delta_* \rho$$

and

$$- (l_* u_* d_\gamma A_*/ \kappa_\gamma A_*) \Delta_* \rho + (d_\gamma / l_*) \Delta_* g.$$

Now we assume that

$$u_* l_*^2 R_*^2 \neq (n - 1) \kappa_\gamma \sigma_*,$$

which is equivalent to invertibility of the operator $A_0 := l_* u_* + (\kappa_\gamma \sigma_*/ l_*) A_*$. Applying its inverse to (4.2) we arrive at the following equation for $\rho$:

$$\partial_t \rho - (d_\gamma A_*/ \kappa_\gamma A_*) \Delta_* \rho + A_0^{-1} \{ (u_* l_* d_\gamma A_*/ \kappa_\gamma A_*) \Delta_* \rho - [d_\gamma \partial_\nu v] \} = \bar{g}, \quad (4.4)$$

with

$$\bar{g} = A_0^{-1} \{ \kappa_\gamma f_T - ((\kappa_\gamma / l_*) \partial_\nu g - (d_\gamma / l_*) \Delta_* g) \}.$$

Solving equation (4.4) for $\partial_t \rho$ we obtain for $v_T$:

$$\kappa_\gamma \partial_t v_T - d_\gamma \Delta_* v_T - [d_\gamma \partial_\nu v] + l_* u_* A_0^{-1} \{ d_\gamma \Delta_* v_T + [d_\gamma \partial_\nu v] \} = \tilde{f}_T. \quad (4.5)$$

where

$$\tilde{f}_T = \kappa_\gamma \{ f_T - l_* u_* A_0^{-1} (f_T - \partial_\nu g / l_*) \}.$$ 

Then by the regularity of $f_T$ and $g$ and with $r \leq s \leq r + 2$ we see that

$$\tilde{f}_T \in L_p(J; W^r_p(\Gamma_*)), \quad \bar{g} \in L_p(J; W^{s+3-2/p}_p(\Gamma_*)).$$
So the linear problem (4.1) can be recast as an evolution equation in $X$ and $L$

with $L_0 = L_{00} + L_{01}$ defined by

$$D(L_{00}) = \{(v, v_\Gamma, \rho) \in X_1 : [v] = 0, \ v_\Gamma = v \text{ on } \Gamma_*, \ \partial_\nu v = 0 \text{ on } \partial \Omega\},$$

and

$$L_{00} = \begin{bmatrix}
-(d_s/\kappa_s) \Delta & 0 & 0 \\
-(d_s/\kappa_s) \partial_\nu & -d_s^2/\kappa_s & 0 \\
-A_0^{-1}[d_s \partial_\nu] & 0 & -(d_s/\kappa_s) \Delta
\end{bmatrix},$$

and

$$L_{01} = \begin{bmatrix}
0 & 0 & 0 \\
(l_* u_s/\kappa_s) A_0^{-1}[d_s \partial_\nu] & (l_* u_s d_s/\kappa_s) A_0^{-1}\Delta & 0 \\
0 & 0 & (l_* u_s d_s/\kappa_s) A_0^{-1}\Delta
\end{bmatrix}.$$

Looking at $L_0$ we first note that $L_{01}$ is a lower order perturbation of $L_{00}$. The latter is lower triangular, and the problem for $(v, v_\Gamma)$ as above has maximal $L_p$-regularity in $X_0$. As the diagonal entry in the equation for $\rho$ has maximal $L_p$-regularity as well we may conclude that $-L_0$ generates an analytic $C_0$-semigroup with maximal regularity in $X_0$. More precisely, we have the following result.

**Theorem 4.2.** Let $1 < p < \infty$, $\gamma_* = 0$, $-1/p \leq r \leq 1 - 1/p$, $r \leq s \leq r + 2$, $l_* \neq 0$, and assume $u_* l_*^2 R_*^2 \neq \kappa_* \sigma_* (n-1)$.

Then for each interval $J = [0, a]$, there is a unique solution $(v, v_\Gamma, \rho) \in E(J)$ of (4.1) if and only if the data $(f, f_\Gamma, g)$ and $(v_0, v_{\Gamma 0}, \rho_0)$ satisfy

$$(f, f_\Gamma, g) \in F_0(\Omega),$$

$$(v_0, v_{\Gamma 0}, \rho_0) \in W^{r+2-2/p}(\Omega) \times W^{s+2-2/p}(\Gamma_*) \times W^s_{p=2-2/p}(\Gamma_*),$$

and the compatibility conditions

$$[v_0] = 0, \ v_{\Gamma 0} = v_0, \ l_* v_0 - \sigma_* A_0^\gamma = g(0) \text{ on } \Gamma_*, \ \partial_\nu v = 0 \text{ on } \partial \Omega.$$ 

The operator $-L_0$ defined above generates an analytic $C_0$-semigroup in $X_0$ with maximal regularity of type $L_p$.

Note that the compatibility condition $l_* v_0 - \sigma_* A_0^\gamma = g(0)$ allows to recover the Gibbs-Thomson relation from the dynamic equations. Indeed, it follows from (4.2)-(4.3) that the function $w := v_\Gamma - ((\sigma_*/l_*) A_0^\gamma + g/l_*)$ satisfies

$$\kappa_* \partial_\nu w - d_0^\gamma \Delta w = l_* u_* (\partial_\nu \rho - A_0^{-1}[d_0^\gamma \Delta w + [d_0 \partial_\nu v] + \kappa_*(f_\Gamma - \partial_\nu g/l_*))]$$

$$= -l_* u_* A_0^{-1} d_0^\gamma \Delta w,$$

and this shows that $w$ is a solution of the parabolic equation

$$\kappa_* \partial_\nu w - d_0^\gamma \Delta w + l_* u_* d_0^\gamma A_0^{-1} \Delta w = 0, \ w(0) = 0 \text{ on } \Gamma_*.$$ (4.6)

As $w \equiv 0$ is the unique solution of (4.6) we conclude that the Gibbs-Thomson relation is satisfied.
4.2. The Eigenvalue Problem. By compact embedding, the spectrum of $L$ consists only of countably many discrete eigenvalues of finite multiplicity and is independent of $p$. Therefore it is enough to consider the case $p=2$. In the following, we will use the notation

$$
(u|v)_\Omega := (u|v)_{L^2(\Omega)} := \int_\Omega uv \, dx, \quad u, v \in L^2(\Omega),
$$

$$
(g|h)_{\Gamma_*} := (g|h)_{L^2(\Gamma_*)} := \int_{\Gamma_*} gh \, ds, \quad g, h \in L^2(\Gamma_*),
$$

for the $L^2$ inner product in $\Omega$ and $\Gamma_*$, respectively. Moreover, we set $|v|_{\Omega} = (v|v)_\Omega^{1/2}$ and $|g|_{\Gamma_*} = (g|g)_{\Gamma_*}^{1/2}$. The eigenvalue problem reads as follows:

$$
\begin{align*}
\kappa_* \lambda v - d_* \Delta v &= 0 \quad \text{in } \Omega \setminus \Gamma_*, \\
\partial_\nu v &= 0 \quad \text{on } \partial \Omega, \\
|v| &= 0 \quad \text{on } \Gamma_*, \\
\kappa_* \lambda v - d_{\Gamma_*} \Delta_{\Gamma_*} v - [d_\gamma \partial_\nu v] + l_* u_* \lambda \rho &= 0 \quad \text{on } \Gamma_*.
\end{align*}
$$

Let $\lambda \neq 0$ be an eigenvalue with eigenfunction $(v, \rho) \neq 0$. Then (4.7) yields

$$
0 = \lambda|\sqrt{\kappa_*} v|_{\Omega}^2 - (d_* \Delta v|v)_{\Omega} = \lambda|\sqrt{\kappa_*} v|_{\Omega}^2 + |\sqrt{d_*} \nabla v|_{\Omega}^2 + ([d_\gamma \partial_\nu v]|v)_{\Gamma_*}.
$$

On the other hand, we have on the interface

$$
0 = \kappa_* \lambda|v|_{\Gamma_*}^2 - d_{\Gamma_*} (d_\gamma \partial_\nu v)_{\Gamma_*} - ([d_\gamma \partial_\nu v]|v)_{\Gamma_*} + \lambda u_* l_* (\rho|v)_{\Gamma_*},
$$

$$
= \lambda \kappa_* |v|_{\Gamma_*}^2 + d_{\Gamma_*} |\nabla v|_{\Gamma_*}^2 - ([d_\gamma \partial_\nu v]|v)_{\Gamma_*} + \lambda u_* l_* (\rho|v)_{\Gamma_*}.
$$

Adding these identities we obtain

$$
0 = \lambda|\sqrt{\kappa_*} v|_{\Omega}^2 + |\sqrt{d_*} \nabla v|_{\Omega}^2 + \lambda \kappa_* |v|_{\Gamma_*}^2 + d_{\Gamma_*} |\nabla v|_{\Gamma_*}^2 + \lambda u_* l_* (\rho|v)_{\Gamma_*}.
$$

hence employing the Gibbs-Thomson law this results into the relation

$$
\lambda|\sqrt{\kappa_*} v|_{\Omega}^2 + |\sqrt{d_*} \nabla v|_{\Omega}^2 + \lambda \kappa_* |v|_{\Gamma_*}^2 + d_{\Gamma_*} |\nabla v|_{\Gamma_*}^2 + \lambda u_* \sigma_* (A_* \rho)_{\Gamma_*} + \gamma_* u_* |\lambda|^{2} |\rho|_{\Gamma_*}^2 = 0.
$$

Since $A_*$ is selfadjoint in $L^2(\Gamma_*)$, this identity shows that all eigenvalues of $L$ are real. Decomposing $v = v_0 + \tilde{v}$, $\nu_\gamma = \nu_\gamma \rho_0 + \bar{\rho}$, with the normalizations

$$(\kappa_*|v_0)_{\Omega} = (\nu_\gamma|v_0)_{\Gamma_*} = (\rho_0|\bar{\rho})_{\Gamma_*} = 0,$$

this identity can be rewritten as

$$
\lambda \left\{ |\sqrt{\kappa_*} v_0|_{\Omega}^2 + \kappa_* |v_0|_{\Gamma_*}^2 + \sigma_* u_* (A_* \rho_0)_{\Gamma_*} + \lambda u_* \gamma_* |\rho_0|_{\Gamma_*}^2 \right\}
$$

$$
+ |\sqrt{d_*} \nabla v_0|_{\Omega}^2 + d_{\Gamma_*} |\nabla v_0|_{\Gamma_*}^2
$$

$$
+ \lambda \left[ (\kappa_*|1)^2 + \kappa_* |v_0|_{\Gamma_*}^2 - \sigma_* u_* n \frac{1}{R_*^2} [\Gamma_* |\rho|^2 + \gamma_* u_* \lambda |\Gamma_*|_{\rho}^2] \right] = 0.
$$

In case $\Gamma_*$ is connected, $A_*$ is positive semi-definite on functions with mean zero, and hence the bracket determines whether there are positive eigenvalues. Taking
Note that the tangent space of \( E \) at \( \gamma \) has \( \gamma \) as a coefficient of undercooling. Theorem 4.3.

It is shown in \cite{16} that in case \( \gamma = 0 \), the stability condition \( \gamma = 1 \) is satisfied.

In this case \( \gamma = 0 \) on \( \Gamma \) if \( \gamma = 0 \) on \( \Gamma \) is semi-simple if \( \gamma = 1 \) and \( \gamma = 0 \), then all eigenvalues of \( -L \) are negative, except for the eigenvalue 0.

If \( \gamma = 1 \), and \( \gamma = 1 \) in case \( \gamma = 0 \), then there are precisely \( m \) positive eigenvalues of \( -L \), where \( m \) denotes the number of equilibrium spheres.

If \( \gamma = 1 \) and \( \gamma = 1 \) in case \( \gamma = 0 \) then \( -L \) has precisely \( m - 1 \) positive eigenvalues.

If \( \gamma = 1 \) and \( \gamma = 0 \) then \( -L \) has precisely \( m - 1 \) positive eigenvalues.

Theorem 4.3. Let \( \sigma > 0 \), \( \gamma \geq 0 \), \( l \neq 0 \),

\[ \eta = (n-1)\sigma \kappa \Gamma \gamma \beta \]

and assume that the interface \( \Gamma \) consists of \( m \geq 1 \) components. Let

\[ \zeta = (n-1)\sigma \kappa \gamma \beta \Gamma \gamma \]

and let the equilibrium energy \( E \) be defined as in \eqref{2.3}. Then

(i) \( E(u_\gamma) = (\zeta - 1)u_\gamma R_\gamma \Gamma \gamma \beta \][n-1] \sigma \Gamma \gamma \]

(ii) 0 is an eigenvalue of \( L \) with geometric multiplicity \( mn + 1 \).

(iii) 0 is semi-simple if \( \zeta \neq 1 \).

(iv) If \( \gamma > 1 \) and \( \gamma = 0 \), then all eigenvalues of \( -L \) are negative, except for the eigenvalue 0.

(v) If \( \gamma > 1 \), and \( \gamma < 1 \) in case \( \gamma = 0 \), then there are precisely \( m \) positive eigenvalues of \( -L \), where \( m \) denotes the number of equilibrium spheres.

(vi) If \( \gamma \leq 1 \), and \( \gamma < 1 \) in case \( \gamma = 0 \) then \( -L \) has precisely \( m - 1 \) positive eigenvalues.

(vii) \( N(L) \) is isomorphic to the tangent space \( T_{(u, \gamma, \gamma)} \) of \( \gamma \) at \( (u, \gamma, \gamma) \) \( \in \gamma \).

Remarks 4.4. (a) Formally, the result is also true if \( \gamma = 0 \) and \( \gamma > 0 \). In this case \( E(u_\gamma) = (\kappa \gamma \beta \Gamma \gamma \beta \][n-1] \sigma \Gamma \gamma \beta \]

(b) Note that \( \zeta \) does not depend on the diffusivities \( d_\gamma \), \( d_\gamma \), nor on the coefficient of undercooling \( \gamma \).

(c) It is shown in \cite{17} that in case \( \zeta = 1 \) and \( \gamma \) connected, the eigenvalue 0 is no longer semi-simple: its algebraic multiplicity rises by 1 to \( n + 2 \).
(d) It is remarkable that in case kinetic undercooling is absent, large surface heat capacity, i.e. $\eta^* > 1$, stabilizes the system, even in such a way that multiple spheres are stable, in contrast to the case $\eta^* < 1$.

(e) We can show that, in case $\gamma^* = 0$, if $\eta^*$ increases to 1 then all positive eigenvalues go to $\infty$.

We recall a result on the Dirichlet-to-Neumann operator $D\lambda$, $\lambda \geq 0$ which is defined as follows. Let $g \in H^{3/2}(\Gamma^*)$ be given. Solve the elliptic transmission problem

$$
\begin{cases}
\kappa_* \lambda w - d_* \Delta w = 0 \quad \text{in } \Omega \setminus \Gamma^*, \\
\partial_\nu w = 0 \quad \text{on } \partial \Omega, \\
[w] = 0 \quad \text{on } \Gamma^*, \\
w = g \quad \text{on } \Gamma^*,
\end{cases}
$$

(4.9)

and define $D\lambda g = -[d\partial_\nu w] \in H^{1/2}(\Gamma^*)$.

**Lemma 4.5.** The Dirichlet-to-Neumann operator $D\lambda$ has the following well-known properties.

(a) $(D\lambda g)(\Gamma^*) = \lambda |\kappa^{1/2}_* w|_{H^1_0}^2 + |d^{1/2}_* \nabla w|_{H^1_0}^2$, for all $g \in H^{3/2}(\Gamma^*)$;

(b) $|D\lambda g|_{\Gamma^*} \leq C[\lambda^{1/2} |g|_{H^1(\Gamma^*)} + |g|_{H^2(\Gamma^*)}]$, for all $g \in H^{3/2}(\Gamma^*)$ and $\lambda \geq 1$;

(c) $(D\lambda g)(\Gamma^*) \geq c\lambda^{1/2} |g|_{H^1(\Gamma^*)}^2$, for all $g \in H^{3/2}(\Gamma^*)$ and $\lambda \geq 1$.

In particular, $D\lambda$ extends to a self-adjoint positive definite linear operator in $L_2(\Gamma^*)$ with domain $H^{1/2}_0(\Gamma^*)$.

4.3. **Proof of Theorem 4.3** For the case that $\kappa_{\Gamma^*} = d_{\Gamma^*} = 0$ this result is proved in [18]. Assertion (i) follows from the considerations in part (g) of the introduction. Assertions (i), (iii), and (vii) only involve the kernel of $L$ and the manifold of equilibria. Since both are the same as in the case $\kappa_{\Gamma^*} = d_{\Gamma^*} = 0$, the proofs of (i), (iii) and (vii) given in [18] remain valid in the more general situation considered here. The first part of assertion (iv) has been proved above, it thus remains to prove the assertions in (v) and (vi), and the second part of (iv).

If the stability condition $\zeta^* \leq 1$ does not hold or if $\Gamma^*$ is disconnected, then there is always a positive eigenvalue. It is a delicate task to prove this. The principal idea to attack this problem is as follows: suppose $\lambda > 0$ is an eigenvalue, and that $\rho$ is known; solve the resolvent diffusion problem

$$
\begin{cases}
\kappa_* \lambda v - d_* \Delta v = 0 \quad \text{in } \Omega \setminus \Gamma^*, \\
\partial_\nu v = 0 \quad \text{on } \partial \Omega, \\
[v] = 0 \quad \text{on } \Gamma^*, \\
v = v_T \quad \text{on } \Gamma^*,
\end{cases}
$$

(4.10)

to get $-[d_* \partial_\nu v] =: D\lambda v_T$. Next we solve the resolvent surface diffusion problem

$$
\lambda \kappa^* v_T - d^* \Delta v_T + D\lambda v_T = h,
$$
to the result

\[ v_T = T_\lambda h := (\lambda \kappa_{\Gamma_*} - d_{\Gamma_*} \Delta_* + D_\lambda)^{-1} h. \]

Setting \( h = -\lambda u_* \rho \) this implies with the linearized Gibbs-Thomson law the equation

\[ [(l^2 u_*) \lambda_\Delta + \gamma_* \lambda] \rho + \sigma_* A_* \rho = 0. \]  

(4.11)

\( \lambda > 0 \) is an eigenvalue of \( -L \) if and only if (4.11) admits a nontrivial solution. We consider this problem in \( L_2(\Gamma_*) \). Then \( A_* \) is selfadjoint in \( L_2(\Gamma_*) \) and

\[ \sigma_* (A_* \rho) |_{\Gamma_*} \geq -\frac{(n-1)\sigma_*}{R_*^2} |\rho|_{\Gamma_*}^2, \]

for each \( \rho \in D(A_*) = H^2_0(\Gamma_*) \). Moreover, since \( A_* \) has compact resolvent, the operator

\[ B_\lambda := [(l^2 u_*) \lambda_\Delta + \gamma_* \lambda] + \sigma_* A_* \]  

(4.12)

has compact resolvent as well, for each \( \lambda > 0 \). Therefore the spectrum of \( B_\lambda \) consists only of eigenvalues which, in addition, are real. We intend to prove that in case either \( \Gamma_* \) is disconnected or the stability condition does not hold, \( B_{\lambda_0} \) has 0 as an eigenvalue, for some \( \lambda_0 > 0 \). This has been achieved in [15] in the simpler case where \( \kappa_{\Gamma_*} = d_{\Gamma_*} = 0 \), in which case \( T_\lambda \) is the Neumann-to-Dirichlet operator for (4.10). Here we try to use similar ideas as in [15], namely we investigate \( B_\lambda \) for small and for large values of \( \lambda \). However, in the situation of this paper this will be more involved.

For this purpose we need more information about \( T_\lambda \). So we first consider the problem

\[ \begin{align*}
\kappa_* \lambda_\Delta - d_* \Delta v &= 0 \quad \text{in} \quad \Omega \setminus \Gamma_* \\
\partial_\nu v &= 0 \quad \text{on} \quad \partial \Omega \\
[v] &= 0 \quad \text{on} \quad \Gamma_* \\
\lambda \kappa_{\Gamma_*} - d_{\Gamma_*} \Delta_* v - [d_* \partial_\nu v] &= g \quad \text{on} \quad \Gamma_*.
\end{align*} \]  

(4.13)

As we have seen above this problem has a unique solution for each \( \lambda > 0 \), denoted by \( v = S_\lambda g \). Obviously for \( \lambda = 0 \) this problem has a one-dimensional eigenspace spanned by the constant function \( e \equiv 1 \). The problem is solvable if and only if the mean value of \( g \) is zero, i.e. if \( g \in L_{2,0}(\Gamma_*) \). This implies by compactness that \( S_\lambda g \to S_0 g \) as \( T_\lambda \to T_0 \) as \( \lambda \to 0^+ \), whenever \( g \) has mean zero, where \( S_0 g \) means the unique solution of (4.13) for \( \lambda = 0 \) with mean zero.

(a) Suppose that \( \Gamma_* \) is disconnected. If the interface \( \Gamma_* \) consists of \( m \) components \( \Gamma^k, k = 1, \ldots, m \), we set \( e_k = 1 \) on \( \Gamma^k \) and zero elsewhere. Let \( \rho = \sum a_k e_k \neq 0 \) with \( \sum a_k = 0 \), hence \( Q_0 \rho = \rho \), where \( Q_0 \) is the canonical projection onto \( L_{2,0}(\Gamma_*) \) in \( L_2(\Gamma_*), Q_0 \rho := \rho - (\rho|e)_{\Gamma_*}/|\Gamma_*| \). Then

\[ \lim_{\lambda \to 0} \lambda T_\lambda \rho = \lim_{\lambda \to 0} \lambda T_\lambda Q_0 \rho = 0, \]

since \( T_\lambda Q_0 \) is bounded as \( \lambda \to 0 \). This implies

\[ \lim_{\lambda \to 0} (B_\lambda \rho |_{\Gamma_*}) = -\frac{(n-1)\sigma_*}{R_*^2} \sum_k |\Gamma^k| a_k^2 < 0. \]
Therefore \( B_\lambda \) is not positive semi-definite for small \( \lambda \).

(b) Suppose next that \( \Gamma_* \) is connected. Consider \( \rho = e \). Then we have

\[
(B_\lambda e|e)_{\Gamma_*} = u_* \lambda^2 \rho(T_\lambda e|e)_{\Gamma_*} + \lambda \gamma_* e|e_{\Gamma_*}^2 - ((n - 1)\sigma_*/R^2)|e_{\Gamma_*}^2.
\]

We compute the limit \( \lim_{\lambda \to 0} \lambda (T_\lambda e|e)_{\Gamma_*} \) as follows. First solve the problem

\[
\begin{cases}
-\Delta v = -\kappa_* a_0 & \text{in } \Omega \backslash \Gamma_* \\
\partial_\nu v = 0 & \text{on } \partial \Omega \\
\|v\| = 0 & \text{on } \Gamma_*
\end{cases}
\]

(4.14)

where \( a_0 = |\Gamma_*|/((\kappa_*|\Omega + \kappa_*|\Gamma_*]) \), which is solvable since the necessary compatibility condition holds. Let \( v_0 \) denote the solution which satisfies the normalization condition \( v_0 |\Omega + \kappa_* (v_0 |1)_{\Gamma_*} = 0 \). Then \( v_\lambda := \frac{S_\lambda e - v_0 - a_0}{\lambda} \) satisfies the problem

\[
\begin{cases}
\kappa_* \lambda v_\lambda - \Delta v_\lambda = -\kappa_* v_0 & \text{in } \Omega \backslash \Gamma_* \\
\partial_\nu v_\lambda = 0 & \text{on } \partial \Omega \\
v_\lambda = 0 & \text{on } \Gamma_*
\end{cases}
\]

(4.15)

By the normalization \( (\kappa_* |v_0 |\Omega + \kappa_* (v_0 |1)_{\Gamma_*} = 0 \) we see that the compatibility condition for (4.13) holds for each \( \lambda > 0 \), and so we conclude that \( v_\lambda \) is bounded in \( W^2_2(\Omega \backslash \Gamma_*) \) as \( \lambda \to 0 \), it even converges to 0. Hence we have

\[
\lim_{\lambda \to 0} \lambda T_\lambda e = \lim_{\lambda \to 0} \lambda (\lambda v_\lambda + \lambda v_0)|\Gamma_* + a_0| = a_0.
\]

This then implies

\[
\lim_{\lambda \to 0} (B_\lambda e|e)_{\Gamma_*} = \frac{|\Gamma_*|^2}{(\kappa_*|\Omega + \kappa_*|\Gamma_*]} - \frac{(n - 1)\sigma_*/R^2} < 0,
\]

if the stability condition does not hold, i.e. if \( \zeta_* > 1 \). Therefore also in this case \( B_\lambda \) is not positive semi-definite for small \( \lambda > 0 \).

(c) Next we consider the behavior of \( (B_\lambda \rho |\rho)_{\Gamma_*} \) as \( \lambda \to \infty \). We intend to show that \( B_\lambda \) is positive definite for large \( \lambda \). We have

\[
\lambda T_\lambda = \lambda (\kappa_* \lambda - \kappa_\pi \Delta_\pi + D_\lambda)^{-1} \to 1/\kappa_* \quad \text{for } \lambda \to \infty,
\]

as \( D_\lambda \) is of lower order, by (b) of Lemma 4.5. This implies for a given \( g \in D(A_\pi) \)

\[
(B_\lambda g|g)_{\Gamma_*} = \lambda^2 \rho(T_\lambda g|g)_{\Gamma_*} + \sigma_*(A_\rho g|g)_{\Gamma_*} + \gamma_* |g|_{\Gamma_*}^2 \]

\[
\geq (\gamma_* \lambda - \frac{(n - 1)\sigma_*}{R^2})|g|_{\Gamma_*}^2 + \lambda^2 \rho(T_\lambda g|g)_{\Gamma_*} \]

\[
\sim (\gamma_* \lambda - \frac{(n - 1)\sigma_*}{\kappa_*} + \frac{\lambda^2}{\kappa_*})|g|_{\Gamma_*}^2.
\]
as $\lambda \to \infty$. We have thus shown that $B_\lambda$ is positive definite if $\gamma_* > 0$ and

$$\gamma_* = 0 \quad \text{and} \quad \frac{\lambda^2 u_*}{\kappa_* R_*} > \frac{(n-1)\sigma_*}{R_*^2}. \tag{4.16}$$

In particular, for $\gamma_* = 0$ and small $\lambda^2$ the latter condition condition will be violated, in general.

(d) In summary, concentrating on the cases $\gamma_* > 0$ or $\gamma_* = 0$ we have shown that $B_\lambda$ is not positive semi-definite for small $\lambda > 0$ if either $\Gamma_*$ is not connected or the stability condition does not hold, and $B_\lambda$ is always positive definite for large $\lambda$.

Let $\lambda_0 = \sup\{\lambda > 0 : B_\mu \text{ is not positive semi-definite for each } \mu \in (0, \lambda]\}$.

Since $B_\lambda$ has compact resolvent, $B_\lambda$ has a negative eigenvalue for each $\lambda < \lambda_0$. This implies that 0 is an eigenvalue of $B_{\lambda_0}$, thereby proving that $-L$ admits the positive eigenvalue $\lambda_0$.

Moreover, we have also shown that

$$B_0 \rho = \lim_{\lambda \to 0} \left[ \frac{\lambda^2}{\lambda_0} \lambda T_\lambda \rho + \gamma_* \lambda \rho + \sigma_* A_* \rho \right] = \frac{\lambda^2 u_* |\Gamma_*|}{(\kappa_* |\Omega| + \kappa_* |\Gamma_*|)} P_0 \rho + \sigma_* A_* \rho,$$

where $P_0 \rho := (I - Q_0) \rho = (\rho |\epsilon_1|_{\Gamma_*} / |\Gamma_*|)$. Therefore, $B_0$ has the eigenvalue

$$\frac{u_* |\Gamma_*|}{(\kappa_* |\Omega| + \kappa_* |\Gamma_*|)} - \frac{(n-1)\sigma_*}{R_*^2} = \frac{u_* |\Gamma_*|}{(\kappa_* |\Omega| + \kappa_* |\Gamma_*|)} - \frac{(n-1)\sigma_*}{R_*^2} R_*^2 \tag{4.16},$$

with eigenfunction $\epsilon_1$, and in case $m > 1$ it also has the eigenvalue $-(n-1)\sigma_*/R_*^2$ with precisely $m - 1$ linearly independent eigenfunctions of the form $\sum_k a_k \epsilon_k$ with $\sum_k a_k = 0$.

As $\lambda$ varies from 0 to $\lambda_0$, all the negative eigenvalues of $B_0$ identified above will eventually have to cross 0 along the real axis. At each of these occasions, $-L$ will inherit at least one positive eigenvalue, which will then remain positive. This implies that $-L$ has exactly $m$ positive eigenvalues if the stability condition does not hold, and $m - 1$ otherwise. This covers the case $\gamma_* > 0$ as well as $\gamma_* = 0$.

(e) To cover the remaining we assume $\gamma_* = 0$ and $\kappa_* (n-1)/R_* > u_* |\Gamma_*|/\sigma_* = \delta_*$. Suppose $\lambda > 0$ is an eigenvalue of $L_0$, then there is $\rho \neq 0$ such that

$$(\lambda \kappa_* - d_* \Delta_* + D_\lambda) A_* \rho + \lambda \delta_* \rho = 0.$$

Multiplying this equation in $L_2(\Gamma_*)$ by $A_* \rho$ and integrating by parts one obtains the identity

$$\lambda \kappa_* |A_* \rho|_{\Gamma_*}^2 + d_* |\nabla A_* \rho|_{\Gamma_*}^2 + (D_\lambda A_* \rho |A_* \rho)_{\Gamma_*} + \lambda \delta_* (A_* \rho |\rho)_{\Gamma_*} = 0.$$

As $D_\lambda$ is positive definite in $L_2(\Gamma_*)$ this equation implies

$$\lambda \kappa_* |A_* \rho|_{\Gamma_*}^2 + \lambda \delta_* (A_* \rho |\rho)_{\Gamma_*} \leq 0.$$
Let $P$ denote the projection onto the kernel $\mathcal{N}(\Delta_\ast)$ and $Q = I - P$. Since $P,Q$ commute with $A_\ast$ this implies
\[
\lambda \kappa_{\Gamma_\ast} |A_\ast Q \rho|^2_{\Gamma_\ast} + \lambda \kappa_{\Gamma_\ast} |A_\ast P \rho|^2_{\Gamma_\ast} + \lambda \delta_\ast (A_\ast P \rho | P \rho)_{\Gamma_\ast} \leq 0,
\]
as $A_\ast$ is positive semi-definite on $\mathcal{R}(Q) = \mathcal{R}(\Delta_\ast)$. Now $A_\ast = -((n-1)/R_2^2)P$ and
\[
0 \geq \lambda \kappa_{\Gamma_\ast} |A_\ast P \rho|^2_{\Gamma_\ast} + \lambda \delta_\ast (A_\ast P \rho | P \rho)_{\Gamma_\ast} = \lambda \frac{n-1}{R^2} \left( \kappa_{\Gamma_\ast} \frac{n-1}{R^2} - \delta_\ast \right) |P \rho|^2_{\Gamma_\ast} \geq 0,
\]
hence $P \rho = 0$ and $A_\ast Q \rho = 0$. This implies $A_\ast \rho = 0$ and therefore $\rho = 0$ as $\delta_\ast > 0$. This shows that there are no positive eigenvalues of $L_0$ in case $\gamma_\ast = 0$ and $\kappa_{\Gamma_\ast} (n-1)/R_2^2 > u_\ast l_2^2/\sigma_\ast$. This completes the proof.

5. The Semiflow in Presence of Kinetic Undercooling

In this section we assume throughout $\gamma(s) > 0$ for all $0 < s < u_\ast$, i.e. kinetic undercooling is present at the relevant temperature range. In this case we may apply the results in [17] and [10], resulting in a rather complete analysis of the problem.

5.1. Local Well-Posedness. To prove local well-posedness we employ the direct mapping method as introduced in Section 3. As base space we use
\[
X_0 = L_p(\Omega) \times W_p^{1-1/p}(\Sigma) \times W_p^{1-1/p}(\Sigma),
\]
and we set
\[
X_1 = \{(v,v_{\Gamma},\rho) \in H^2_p(\Omega \setminus \Sigma) \times W_p^{2-1/p}(\Sigma) \times W_p^{3-1/p}(\Sigma) : [v] = 0, v_{\Gamma} = v|_{\Sigma}, \partial_{\nu} v|_{\partial \Omega} = 0 \}.
\]
The trace space $X_\gamma$ then becomes for $p > n + 2$
\[
X_\gamma = \{(v,v_{\Gamma},\rho) \in W_p^{2-2/p}(\Omega \setminus \Sigma) \times W_p^{2-3/p}(\Sigma) \times W_p^{3-3/p}(\Sigma) : [v] = 0, v_{\Gamma} = v|_{\Sigma}, \partial_{\nu} v|_{\partial \Omega} = 0 \},
\]
and that with the time weight $t^{1-\mu}$, $1 \geq \mu > 1/p$,
\[
X_{\gamma,\mu} = \{(v,v_{\Gamma},\rho) \in W_p^{2\mu-2/p}(\Omega \setminus \Sigma) \times W_p^{2\mu-3/p}(\Sigma) \times W_p^{2\mu+1-3/p}(\Sigma) : [v] = 0, v_{\Gamma} = v|_{\Sigma}, \partial_{\nu} v|_{\partial \Omega} = 0 \},
\]
Note that
\[
X_{\gamma,\mu} \hookrightarrow BUC^1(\Omega \setminus \Sigma) \times C^1(\Sigma) \times C^2(\Sigma), \tag{5.1}
\]
provided $2\mu > 1 + (n+2)/p$, which is feasible as $p > n + 2$. In the sequel, we only consider this range of $\mu$. We want to rewrite system (3.2) abstractly as the quasilinear problem in $X_0$
\[
\dot{z} + A(z)z = F(z), \quad z(0) = z_0, \tag{5.2}
\]
with \( z = (v, v_T, \rho) \) and \( z_0 = (v_0, v_{T0}, \rho_0) \). Here the quasilinear part \( A(z) \) is the diagonal matrix operator defined by

\[
-A(z) = \text{diag} \left[ \frac{(d(v)/\kappa(v))(\Delta - M_2(\rho) : \nabla^2)}{(d_{T1}(v_T)/\kappa_1(v_T))(P_{T1}(\rho)M_0(\rho))^2 : \nabla^2_{\Sigma}} \right]
\]

with \( M_2(\rho) = M_1(\rho) + M_1^T(\rho) - M_1(\rho)M_1^T(\rho) \). The semilinear part \( F(z) \) is given by

\[
\begin{align*}
\mathcal{R}(\rho)v + \frac{1}{\kappa(v)} \left\{ d'(v) \left| (I - M_1(\rho) \nabla v)^2 - d(v)((I - M_1(\rho) : \nabla M_1(\rho)) \nabla v) \right| \right\} \\
- \frac{1}{\kappa_1(v_T)} \left[ (B(v_T, \rho)v - [(l(v_T) + l_T(v_T))H(\rho) - \gamma(v_T)\beta(\rho)\partial_T \rho]\beta(\rho)\partial_T \rho + m_3) \right] \\
\varphi(v_T)/\beta(\rho)\gamma(v_T) + \sigma(v_T)F(\rho)/\gamma(v_T)
\end{align*}
\]

where \( \varphi(s) = [\psi(s)] \) and

\[ m_3 = -d_{T1}(v_T)(P_{T1}(\rho)M_0(\rho))^2 : \nabla^2_{\Sigma} v_T - C(v_T, \rho)v_T. \]

We note that \( m_3 \) depends on \( v_T, \nabla_{\Sigma} v_T \), and on \( \rho, \nabla_{\Sigma} \rho, \nabla_{\Sigma}^2 \rho \), but not on \( \nabla_{\Sigma}^2 v_T \), hence is of lower order. Apparently, the first two components of \( F(z) \) contain the time derivative \( \partial_T \rho \); we may replace it by

\[ \partial_T \rho = \{ \varphi(v_T) + H(\rho) \}/\beta(\rho)\gamma(v_T), \]

to see that it is of lower order as well.

Now fix a ball \( B := B_{X_{\gamma,\mu}}(z_0, R) \subset X_{\gamma,\mu} \), where \( |\rho_0|_{C^1(\Sigma)} \leq \eta \) for some sufficiently small \( \eta > 0 \). Then it is not difficult to verify that

\( (A, F) \in C^1(B, B(X_1, X_0) \times X_0) \)

provided \( d_i, \psi_i, d_T, \sigma, \gamma \in C^3(0, \infty) \) and \( d_j, \kappa_j, \sigma, \gamma > 0 \) on \( (0, u_c), j = 1, 2, \Gamma \), and provided \( 2 \geq 2\mu > 1 + n + 2/p \) as before. Moreover, as \( A(z) \) is diagonal, well-known results about elliptic differential operators show that \( A(z) \) has the property of maximal regularity of type \( L_p \), and also of type \( L_{p,\mu} \), for each \( z \in B \). In fact, for small \( \eta > 0 \), \( A(z) \) is small perturbation of

\[ A_{\eta}(z) = \text{diag}[ - (d(v)/\kappa(v))\Delta, -(d_{T1}(v_T)/\kappa_1(v_T))\Delta_{\Sigma}, -(\sigma(v_T)/\gamma(v_T))\Delta_{\Sigma} ] \]

Therefore we may apply \[10\] Theorem 2.1 to obtain local well-posedness of \((5.2)\), i.e. a unique local solution

\[ z \in H^1_{p,\mu}((0, a); X_0) \cap L_{p,\mu}((0, a); X_1) \hookrightarrow C([0, a]; X_{\gamma,\mu}) \cap C((0, a]; X_{\gamma}) \]

which depends continuously on the initial value \( z_0 \in B \). The resulting solution map \( [z_0 \mapsto z(t)] \) defines a local semiflow in \( X_{\gamma,\mu} \).
5.2. Nonlinear Stability of Equilibria. Let \( \epsilon_* = (u_*, u_{\Gamma*}, \Gamma_*) \) denote an equilibrium as in Section 4. In this case we choose \( \Sigma = \Gamma_* \) as a reference manifold, and as shown in the previous subsection we obtain the abstract quasilinear parabolic problem

\[
\dot{z} + A(z)z = F(z), \quad z(0) = z_0,
\]

with \( X_0, X_1, X_\gamma \) as above. We set \( z_* = (u_*, u_{\Gamma*}, 0) \). Assuming that \( \zeta_* \neq 0 \) in the stability condition, we have shown in Section 4 that the equilibrium \( z_* \) is normally hyperbolic. Therefore we may apply \cite{17} Theorems 2.1 and 6.1 to obtain the following result.

**Theorem 5.1.** Let \( p > n + 2 \). Suppose \( \gamma > 0 \) on \((0, u_*)\) and the assumptions of \eqref{1.3} hold true. As above \( \mathcal{E} \) denotes the set of equilibria of \eqref{5.3}, and we fix some \( z_* \in \mathcal{E} \). Then we have

(a) If \( \Gamma_* \) is connected and \( \zeta_* < 1 \) then \( z_* \) is stable in \( X_\gamma \), and there exists \( \delta > 0 \) such that the unique solution \( z(t) \) of \eqref{5.3} with initial value \( z_0 \in X_\gamma \) satisfying \( |z_0 - z_*|_\gamma < \delta \) exists on \( \mathbb{R}_+ \) and converges at an exponential rate in \( X_\gamma \) to some \( z_\infty \in \mathcal{E} \) as \( t \to \infty \).

(b) If \( \Gamma_* \) is disconnected or if \( \zeta_* > 1 \) then \( z_* \) is unstable in \( X_\gamma \), and even in \( X_0 \). For each sufficiently small \( \rho > 0 \) there is \( \delta \in (0, \rho] \) such that the solution \( z(t) \) of \eqref{5.3} with initial value \( z_0 \in X_\gamma \) subject to \( |z_0 - z_*|_\gamma < \delta \) either satisfies

(i) \( \text{dist}_{X_\gamma}(z(t_0); \mathcal{E}) > \rho \) for some finite time \( t_0 > 0 \); or

(ii) \( z(t) \) exists on \( \mathbb{R}_+ \) and converges at an exponential rate in \( X_\gamma \) to some \( z_\infty \in \mathcal{E} \).

**Remark 5.2.** The only equilibria which are excluded from our analysis are those with \( \zeta_* = 1 \), which means \( \mathcal{E}'_\epsilon(u_*) = 0 \). These are critical points of the function \( \mathcal{E}_\epsilon(u) \) at which a bifurcation may occur. In fact, if such \( u_* \) is a maximum or a minimum of \( \mathcal{E}_\epsilon \) then two branches of \( \mathcal{E} \) meet at \( u_* \), a stable and an unstable one, which means that \( (u_*, \Gamma_*) \) is a turning point in \( \mathcal{E} \).

5.3. The Local Semiflow on the State Manifold. Here we follow the approach introduced in \cite{11} for the two-phase Navier-Stokes problem and in \cite{18} for the two-phase Stefan problem, see also \cite{10} for the Mullins-Sekerka problem.

We denote by \( \mathcal{MH}^2(\Omega) \) the closed \( C^2 \)-hypersurfaces contained in \( \Omega \). It can be shown that \( \mathcal{MH}^2(\Omega) \) is a \( C^2 \)-manifold: the charts are the parameterizations over a given hypersurface \( \Sigma \) according to Section 3, and the tangent space consists of the normal vector fields on \( \Sigma \). We define a metric on \( \mathcal{MH}^2(\Omega) \) by means of

\[
d_{\mathcal{MH}^2}(\Sigma_1, \Sigma_2) := d_H(N^2\Sigma_1, N^2\Sigma_2),
\]

where \( d_H \) denotes the Hausdorff metric on the compact subsets of \( \mathbb{R}^n \) introduced in Section 2. This way \( \mathcal{MH}^2(\Omega) \) becomes a Banach manifold of class \( C^2 \).

Let \( d_\Sigma(x) \) denote the signed distance for \( \Sigma \) as in Section 2. We may then define the canonical level function \( \varphi_\Sigma \) by means of

\[
\varphi_\Sigma(x) = \phi(d_\Sigma(x)), \quad x \in \mathbb{R}^n,
\]
where
\[ \phi(s) = s \chi(s/a) + (1 - \chi(s/a)) \text{sgn } s, \quad s \in \mathbb{R}. \]

Then it is easy to see that \( \Sigma = \varphi_{\Sigma}^{-1}(0) \), and \( \nabla \varphi_{\Sigma}(x) = \nu_{\Sigma}(x) \), for \( x \in \Sigma \). Moreover, 0 is an eigenvalue of \( \nabla^2 \varphi_{\Sigma}(x) \), and the remaining eigenvalues of \( \nabla^2 \varphi_{\Sigma}(x) \) are the principal curvatures of \( \Sigma \) at \( x \in \Sigma \).

If we consider the subset \( \mathcal{M}^2(\Omega, r) \) of \( \mathcal{M}^2(\Omega) \) which consists of all closed hypersurfaces \( \Gamma \in \mathcal{M}^2(\Omega) \) such that \( \Gamma \subset \Omega \) satisfies a (interior and exterior) ball condition with fixed radius \( r > 0 \), then the map
\[ \Upsilon : \mathcal{M}^2(\Omega, r) \to C^2(\bar{\Omega}), \quad \Upsilon(\Gamma) := \varphi_{\Gamma}, \tag{5.4} \]
is an isomorphism of the metric space \( \mathcal{M}^2(\Omega, r) \) onto \( \Upsilon(\mathcal{M}^2(\Omega, r)) \subset C^2(\bar{\Omega}) \).

Let \( s - (n - 1)/p > 2 \). Then we define
\[ W^s_p(\Omega, r) := \{ \Gamma \in \mathcal{M}^2(\Omega, r) : \varphi_{\Gamma} \in W^s_p(\Omega) \}. \tag{5.5} \]
In this case the local charts for \( \Gamma \) can be chosen of class \( W^s_p \) as well. A subset \( A \subset W^s_p(\Omega, r) \) is said to be (relatively) compact, if \( \Upsilon(A) \subset W^s_p(\Omega) \) is (relatively) compact.

As an ambient space for the state manifold of (1.6) we consider the product space \( C(\bar{G}) \times \mathcal{M}^2 \), due to continuity of temperature and curvature.

We define the state manifold \( \mathcal{SM} \) for (1.6) as follows:
\[ \mathcal{SM} := \{(u, \Gamma) \in C(\bar{\Omega}) \times \mathcal{M}^2 : u \in W^{2-2/p}(\Omega \setminus \Gamma), \Gamma \in W^{3-3/p}, \]
\[ 0 < u < u_c \text{ in } \bar{\Omega}, \quad \partial \nu u = 0 \text{ on } \partial \Omega \}. \tag{5.6} \]
Charts for this manifold are obtained by the charts induced by \( \mathcal{M}^2(\Omega) \) followed by a Hanzawa transformation as in Section 3. Note that there is no need to incorporate the dummy variable \( u_\Gamma \) into the definition of the state manifold, as \( u_\Gamma = u|_{\Gamma} \) whenever \( u_\Gamma \) appears.

Applying the result in subsection 5.1 and re-parameterizing the interface repeatedly, we see that (1.6) yields a local semiflow on \( \mathcal{SM} \).

**Theorem 5.3.** Let \( p > n + 2 \). Suppose \( \gamma > 0 \) on \((0, u_c)\) and the assumptions of (1.3) hold true.

Then problem (1.6) generates a local semiflow on the state manifold \( \mathcal{SM} \). Each solution \((u, \Gamma)\) exists on a maximal time interval \([0, t_*)\), where \( t_* = t_*(u_0, \Gamma_0) \).

### 5.4. Global Existence and Convergence

There are several obstructions to global existence for the Stefan problem with variable surface tension (1.6):

- **regularity:** the norms of \( u(t) \) or \( \Gamma(t) \) become unbounded;
- **well-posedness:** the temperature may reach 0 or \( u_c \);
- **geometry:** the topology of the interface changes;
- or the interface touches the boundary of \( \Omega \);
- or the interface contracts to a point.
Let \((u, \Gamma)\) be a solution in the state manifold \(\mathcal{SM}\). By a uniform ball condition we mean the existence of a radius \(r_0 > 0\) such that for each \(t\), at each point \(x \in \Gamma(t)\) there exist centers \(x_i \in \Omega_i(t)\) such that \(B_{r_0}(x_i) \subset \Omega_i\) and \(\Gamma(t) \cap B_{r_0}(x_i) = \{x\}\), \(i = 1, 2\). Note that this condition bounds the curvature of \(\Gamma(t)\), prevents it from shrinking to a point, from touching the outer boundary \(\partial \Omega\), and from undergoing topological changes.

With this property, combining the semiflow for (1.6) with the Lyapunov functional and compactness we obtain the following result.

**Theorem 5.4.** Let \(p > n + 2\). Suppose \(\gamma > 0\) on \((0, u_c)\) and the assumptions of (1.5) hold true. Suppose that \((u, \Gamma)\) is a solution of (1.6) in the state manifold \(\mathcal{SM}\) on its maximal time interval \([0, t_s]\). Assume the following on \([0, t_s]\): there is a constant \(M > 0\) such that

1. \(|u(t)|_{W_p^{3-2/p}} + |\Gamma(t)|_{W_p^{3-3/p}} \leq M < \infty\);
2. \(0 < 1/M \leq u(t) \leq u_c - 1/M\);
3. \(\Gamma(t)\) satisfies a uniform ball condition.

Then \(t_s = \infty\), i.e. the solution exists globally, and it converges in \(\mathcal{SM}\) to some equilibrium \((u_\infty, \Gamma_\infty)\) in \(\mathcal{E}\). On the contrary, if \((u(t), \Gamma(t))\) is a global solution in \(\mathcal{SM}\) which converges to an equilibrium \((u_s, \Gamma_s)\) in \(\mathcal{SM}\) as \(t \to \infty\), then properties (i)-(iii) are valid.

**Proof.** Assume that assertions (i)-(iii) are valid. Then \(\Gamma([0, t_s]) \subset W_p^{3-3/p}(\Omega, r)\) is bounded, hence relatively compact in \(W_p^{3-3/p-\varepsilon}(\Omega, r)\). Thus we may cover this set by finitely many balls with centers \(\Sigma_k\) real analytic in such a way that \(\text{dist}_{W_p^{3-3/p-\varepsilon}}(\Gamma(t), \Sigma_j) \leq \delta\) for some \(j = j(t), t \in [0, t_s]\). Let \(J_k = \{t \in [0, t_s] : j(t) = k\}\). Using for each \(k\) a Hanzawa-transformation \(\Xi_k\), we see that the pull backs \(\{u(t, \cdot) \circ \Xi_k : t \in J_k\}\) are bounded in \(W_p^{3-2/p}(\Omega, \Sigma_k)\), hence relatively compact in \(W_p^{2-2/p-\varepsilon}(\Omega, \Sigma_k)\). Employing now the results in subsection 5.1 we obtain solutions \((u^1, \Gamma^1)\) with initial configurations \((u(t), \Gamma(t))\) in the state manifold on a common time interval, say \([0, \tau]\), and by uniqueness we have

\[ (u^1(\tau), \Gamma^1(\tau)) = (u(t + \tau), \Gamma(t + \tau)). \]

Continuous dependence implies then relative compactness of \((u(\cdot), \Gamma(\cdot))\) in \(\mathcal{SM}\). In particular, \(t_s = \infty\) and the orbit \((u, \Gamma)(\mathbb{R}_+) \subset \mathcal{SM}\) is relatively compact. The negative total entropy is a strict Lyapunov functional, hence the limit set \(\omega(u, \Gamma) \subset \mathcal{SM}\) of a solution is contained in the set \(\mathcal{E}\) of equilibria. By compactness \(\omega(u, \Gamma) \subset \mathcal{SM}\) is non-empty, hence the solution comes close to \(\mathcal{E}\), and stays there. Then we may apply the convergence result Theorem 5.1. The converse is proved by a compactness argument. \(\square\)

### 6. The Semiflow without Kinetic Undercooling

In this section we assume throughout \(\gamma(s) = 0\) for all \(s > 0\), i.e kinetic undercooling is absent. In this case we may apply the results in [17] and [10] too, but we
have to work harder to apply them. At first we prove (1.11) as follows. According to (1.8) we know that

$$T_\Gamma(u_T)V := (\omega_T(u_T) - \mathcal{H}'(\Gamma))V = \frac{\lambda'(u_T)}{\kappa_T(u_T)} \{ \text{div}_\Gamma(d_T(u_T) \nabla \Gamma u_T) + [d(u) \partial_v u] \}. $$

Next we observe

$$\frac{\lambda'(u_T)}{\kappa_T(u_T)} \text{div}_\Gamma(d_T(u_T) \nabla \Gamma u_T)$$

$$= \frac{1}{\kappa_T(u_T)} \text{div}_\Gamma(d_T(u_T) \nabla \Gamma \lambda(u_T)) - \frac{d_T(u_T)}{\kappa_T(u_T)} \lambda''(u_T) |\nabla \Gamma u_T|^2$$

$$= \text{div}_\Gamma \left( \frac{d_T(u_T)}{\kappa_T(u_T)} \nabla \Gamma \lambda(u_T) \right) - \frac{d_T(u_T)}{\kappa_T(u_T)} \left\{ \lambda''(u_T) - \lambda'(u_T) \frac{\kappa_T'(u_T)}{\kappa_T(u_T)} \right\} |\nabla \Gamma u_T|^2$$

$$= \Delta_T h_T(u_T) \frac{d_T(u_T)}{\kappa_T(u_T)} \left\{ \lambda''(u_T) - \lambda'(u_T) \frac{\kappa_T'(u_T)}{\kappa_T(u_T)} \right\} |\nabla \Gamma u_T|^2$$

where $h_T$ denotes the antiderivative of $d_T \lambda'/\kappa_T$ with $h_T(u_m) = 0$. We note that by a partial integration

$$h_T(s) = \lambda(s) \frac{d\tau(s)}{\kappa_T(s)} - \int_{u_m}^s \lambda(\tau) \frac{d\tau'(\tau)}{\kappa_T(\tau)} d\tau =: \lambda(s) \frac{d\tau(s)}{\kappa_T(s)} - f_T(s).$$

Now employing $\lambda(u_T) = -\mathcal{H}(\Gamma)$ leads to the identity

$$T_\Gamma(u_T)\{V - \frac{d_T(u_T)}{\kappa_T(u_T)} \mathcal{H}(\Gamma) - f_T(u_T)\}$$

$$= \frac{\lambda'(u_T)}{\kappa_T(u_T)} [d(u) \partial_v u] - \frac{d_T(u_T)}{\kappa_T(u_T)} \left\{ \lambda''(u_T) - \lambda'(u_T) \frac{\kappa_T'(u_T)}{\kappa_T(u_T)} \right\} |\nabla \Gamma u_T|^2$$

$$+ [\omega_T(u_T) - \text{tr} L_T^2] h_T(u_T),$$

hence applying the inverse of $T_\Gamma(u_T)$ we arrive at

$$\kappa_T(u_T) V - d_T(u_T) \mathcal{H}(\Gamma) = \kappa_T(u_T) \{ f_T(u_T) + F_T(u, u_T) \},$$

(6.1)

where

$$F_T(u, u_T) = [\kappa_T(u_T) T_\Gamma(u_T)]^{-1} \left\{ \lambda'(u_T) [d(u) \partial_v u] - d_T(u_T) \left\{ \lambda''(u_T) - \lambda'(u_T) \frac{\kappa_T'(u_T)}{\kappa_T(u_T)} \right\} |\nabla \Gamma u_T|^2$$

$$+ \kappa_T(u_T) [\omega_T(u_T) - \text{tr} L_T^2] h_T(u_T) \right\}. $$

In the sequel we will replace the Gibbs-Thomson law by the dynamic equation (6.1) plus the compatibility condition $\varphi(u_{T0}) + \sigma(u_{T0}) \mathcal{H}(\Gamma_0) = 0$ at time $t = 0$. 
6.1. Local Well-Posedness. To prove local well-posedness we employ the direct mapping method as introduced in Section 3. As base space we use as in Section 5
\[ X_0 = L_p(\Omega) \times W_p^{-1/p}(\Sigma) \times W_p^{-1-1/p}(\Sigma), \]
and we let \( X_1, X_\gamma \) and \( X_{\gamma,\mu} \) as defined there.

We rewrite system (6.2) abstractly as the quasilinear problem in \( X_0 \)
\[ \dot{z} + A_0(z)z = F_0(z), \quad z(0) = z_0, \quad (6.2) \]
where \( z = (v, v_\Gamma, \rho) \) and \( z_0 = (v_0, v_\Gamma_0, \rho_0) \). Here the quasilinear part \( A_0(z) \) is the diagonal matrix operator defined by
\[ -A_0(z) = \text{diag} \left[ \frac{(d(v)/\kappa(v))(\Delta - M_2(\rho) : \nabla^2)}{\kappa(v)}, \frac{(d_\Gamma(v_\Gamma)/\kappa_\Gamma(v_\Gamma))(P_\Gamma(\rho) M_0(\rho))^2 : \nabla_\Sigma^2}{\beta(\rho)} \right] \]
with \( M_2(\rho) = M_2(\rho) + M_1^T(\rho) - M_1(\rho) M_1(\rho) \). The semilinear part \( F_0(z) \) is given by
\[ \mathcal{R}(\rho) v + \frac{1}{\kappa(v)} \left( \{d(v)|I - M_2(\rho)\nabla v|^2 - d(v)(I - M_1(\rho)) : \nabla M_1(\rho)|\nabla v| \} \right. \]
\[ - \frac{1}{\kappa_\Gamma(v_\Gamma)} \left( \{B(v_\Gamma, \rho)v - [l(v_\Gamma) + l_\Gamma(v_\Gamma)]H(\rho) - \gamma(v_\Gamma) \beta(\rho) \partial_\rho \beta(\rho) \partial_\rho \rho + m_3 \} \right) \]
where by abuse of notation \( F_\Gamma \) here means the transformed \( F_\Gamma \) introduced previously, and where
\[ m_3 = -d_\Gamma(v_\Gamma)(P_\Gamma(\rho) M_0(\rho))^2 : \nabla_\Sigma^2 v_\Gamma - C(v_\Gamma, \rho)v_\Gamma. \]
Again, the first two components of \( F_0(z) \) contain the time derivative \( \partial_\rho \). We replace it by the transformed version of (6.1)
\[ \partial_\rho = \left\{ f_\Gamma(v_\Gamma) + F_\Gamma(v, v_\Gamma, \rho) + d_\Gamma(v_\Gamma)\kappa_\Gamma(v_\Gamma)H(\rho) \right\}/\beta(\rho), \]
to see that it leads to a lower order term, as in Section 5.

Provided that \( T_{\Gamma_0}(v_{\Gamma_0}) \) is invertible we may proceed as in Section 5, applying Theorem 2.1 in [10], to obtain local well-posedness, i.e. a unique local solution
\[ z \in H_{p,\mu}^1((0, a); X_0) \cap L_{p,\mu}(X_0; X_1) \to C([0, a]; X_{\gamma,\mu}) \cap C([0, a]; X_\gamma) \]
which depends continuously on the initial value \( z_0 \in \mathcal{B} \). The resulting solution map \( [z_0 \mapsto z(t)] \) defines a local semiflow in \( X_{\gamma,\mu} \).

6.2. Nonlinear Stability of Equilibria. Let \( e_\ast = (u_\ast, u_{\Gamma_\ast}, \Gamma_\ast) \) denote an equilibrium as in Section 4. In this case we choose \( \Sigma = \Gamma_\ast \) as a reference manifold, and

as shown in the previous subsection we obtain the abstract quasilinear parabolic problem
\[ \dot{z} + A_0(z)z = F_0(z), \quad z(0) = z_0, \quad (6.3) \]
with \( X_0, X_1, X_\ast \) as above. We set \( z_\ast = (u_\ast, u_{\Gamma\ast}, 0) \). Assuming well-posedness and \( \zeta_\ast \neq 1 \) in stability condition, we have shown in Section 4 that the equilibrium \( e_\ast \) is normally hyperbolic. Therefore we may apply once more [17], Theorems 2.1 and 6.1 to obtain the following result.

**Theorem 6.1.** Let \( p > n + 2 \). Suppose \( \gamma \equiv 0, \sigma \in C^4(0, u_c) \), and the assumptions of (1.3) hold true. As above \( E \) denotes the set of equilibria of (1.3), and we fix some \( z_\ast \in E \). Assume that the well-posedness condition

\[
l_\ast \neq 0 \quad \text{and} \quad u_\ast l_\ast^2/\sigma_\ast \neq \kappa_\ast(n - 1)/R_\ast^2 \tag{6.4}
\]

is satisfied. Then we have

(a) If \( \Gamma_\ast \) is connected and \( \zeta_\ast < 1 \), or if \( \kappa_\ast(n - 1)/R_\ast^2 > u_\ast l_\ast^2/\sigma_\ast \) then \( z_\ast \) is stable in \( X_\gamma \), and there exists \( \delta > 0 \) such that the unique solution \( z(t) \) of (5.3) with initial value \( z_0 \in X_\gamma \) satisfying \( |z_0 - z_\gamma|_\gamma < \delta \) exists on \( \mathbb{R}_+ \) and converges at an exponential rate in \( X_\gamma \) to some \( z_\infty \in E \) as \( t \to \infty \).

(b) If \( \kappa_\ast(n - 1)/R_\ast^2 < u_\ast l_\ast^2/\sigma_\ast \), and if \( \Gamma_\ast \) is disconnected or if \( \zeta_\ast > 1 \) then \( z_\ast \) is unstable in \( X_\gamma \) and even in \( X_0 \). For each sufficiently small \( \rho > 0 \) there is \( \delta \in (0, \rho) \) such that the solution \( z(t) \) of (5.3) with initial value \( z_0 \in X_\gamma \) subject to

\[
|z_0 - z_\gamma|_\gamma < \delta \quad \text{either satisfies}
\]

(i) \( \text{dist}_{X_\gamma}(z(t_0); E) > \rho \) for some finite time \( t_0 > 0 \); or

(ii) \( z(t) \) exists on \( \mathbb{R}_+ \) and converges at exponential rate in \( X_\gamma \) to some \( z_\infty \in E \).

Thus the only cases which are excluded are \( \zeta_\ast = 1 \), and the two values where the well-posedness condition (6.4) is violated.

6.3. The Local Semiflow on the State Manifold. We define the state manifolds \( \mathcal{SM}_0 \) for (1.0) in case \( \gamma \equiv 0 \) as follows.

\[
\mathcal{SM}_0 := \{(u, \Gamma) \in C(\overline{\Omega}) \times \mathcal{MH}^2 : u \in W_p^{2-2/p}(\Omega \setminus \Gamma), \Gamma \in W_p^{3-3/p}, \\
0 < u < u_c \text{ in } \Omega, \partial_\nu u = 0 \text{ on } \partial\Omega, \\
\lambda(u_\Gamma) + \mathcal{H}(\Gamma) = 0 \text{ on } \Gamma, T_\Gamma(u_\Gamma) \text{ is invertible in } L_2(\Gamma) \}.
\tag{6.5}
\]

Charts for this manifold are obtained by the charts induced by \( \mathcal{MH}^2(\Omega) \) followed by a Hanzawa transformation as in Section 3.

Applying the result of subsection 6.1 and re-parameterizing the interface repeatedly, we see that (1.6) with \( \gamma \equiv 0 \) yields a local semiflow on \( \mathcal{SM}_0 \).

**Theorem 6.2.** Let \( p > n + 2 \). Suppose \( \gamma \equiv 0, \sigma \in C^4(0, u_c) \), and the assumptions of (1.3) hold true.

Then problem (1.6) generates a local semiflow on the state manifold \( \mathcal{SM}_0 \). Each solution \( (u, \Gamma) \) exists on a maximal time interval \([0, t_\ast)\), where \( t_\ast = t_\ast(u_0, \Gamma_0) \).
6.4. Global Existence and Convergence. In addition to the obstructions to global existence for the Stefan problem with variable surface tension in the presence of kinetic undercooling there is an additional possibility for loss of well-posedness:

- **regularity**: the norms of $u(t)$ or $\Gamma(t)$ become unbounded;
- **well-posedness**: the temperature may reach 0 or $u_c$; or $T_\Gamma(u_\Gamma)$ may become non-invertible;
- **geometry**: the topology of the interface changes; or the interface touches the boundary of $\Omega$; or the interface contracts to a point.

We set $E_0 = SM_0 \cap E$. As in Section 5, combining the semiflow for \((1.6)\) with the Lyapunov functional and compactness we obtain the following result.

**Theorem 6.3.** Let $p > n + 2$. Suppose $\gamma \equiv 0$, $\sigma \in C^4(0, u_c)$, and the assumptions of \((1.5)\) hold true. Suppose that $(u, \Gamma)$ is a solution of \((1.6)\) in the state manifold $SM_0$ on its maximal time interval $[0, t_\ast)$. Assume the following on $[0, t_\ast)$:

1. $|u(t)|_{W^{2-2/p}_p} + |\Gamma(t)|_{W^{3-3/p}_p} \leq M < \infty$;
2. $0 < 1/M \leq u(t) \leq u_c - 1/M$;
3. $|\mu_j(t)| \geq 1/M$ holds for the eigenvalues of $T_\Gamma(u_\Gamma)$;
4. $\Gamma(t)$ satisfies a uniform ball condition.

Then $t_\ast = \infty$, i.e. the solution exists globally, and it converges in $SM_0$ to an equilibrium $(u_\infty, \Gamma_\infty) \in E_0$. Conversely, if $(u(t), \Gamma(t))$ is a global solution in $SM_0$ which converges to an equilibrium $(u_\infty, \Gamma_\infty) \in E_0$ in $SM_0$ as $t \to \infty$, then the properties (i)-(iv) are valid.

**Proof.** The proof follows the same lines as that of Theorem 5.2. □

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