The role of phase fluctuations in model of superconductivity with the external pair potential

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(Dated: May 2, 2017)

We investigate the stability of the solutions of the BCS model with the external pair potential formulated in a work K.V. Grigorishin arXiv:1605.07080. It has been shown that the cause of superconductivity in this model is the electron-electron interaction only and the potential cannot impose superconductivity since the superconducting ordering will be destroyed by phase fluctuations in a form of quantum turbulence (vortexes, vortex rings, vortex tangles etc.). The external pair potential renormalizes the order parameter in initial superconducting matter and the bosonic scenario of transition to normal state in a clear 3D superconductor occurs unlike BCS theory where the fermionic scenario takes place.

PACS numbers: 74.20.Fg, 74.20.Mn

Keywords: external pair potential, order parameter, topological defect, quantum turbulence

I. INTRODUCTION

According to BCS theory the transition to superconducting state is accompanied by reconstruction of the electron spectrum with appearance of the energy gap $2|\Delta|$ on Fermi level. The superconducting state is characterized by the complex order parameter

$$\Delta(r) = |\Delta|e^{i\phi(r)}$$

(1)

whose modulus is the binding energy of a Cooper pair, and whose phase $\phi(r)$ characterizes the coherence of ensemble of the Cooper pairs. When there is no current in the superconductor then $\phi(r) = \text{const}$. Two scenarios of superconductor-normal metal transition exist. In BCS theory the module $|\Delta|$ becomes zero in the point of phase transition $|\Delta(T_c)| = 0$, and the phase loses its sense automatically. This is fermionic scenario of the transition. However superconducting state can be destroyed with another way: fluctuations of order parameter turns to zero the correlator:

$$\langle \Delta^+(0)\Delta(r) \rangle \to 0 \text{ for } r \to \infty,$$

(2)

at nonzero module of order parameter $|\Delta| \neq 0$. This is bosonic scenario of the transition. The finite module at transition means presence of finite concentration of bound electron pairs, i.e. concentration of bosons in the transition does not become zero. Such regime can be released in system with low electron density due small "rigidity" with respect to the phase’s changes [1, 2]. The bosonic scenario can occur in granular superconductors where Cooper pairs localized in granules so that superconductivity of macroscopic sample disappear but the granules remain superconducting [3–5]. The bosonic scenario takes place in 2D superconductivity [4]: the two-dimensional superconducting systems are distinguished by the possible existence of a gas of fluctuations in the form of spontaneously generated magnetic vortices below the bulk transition temperature $T_{c0}$. The vortices are created in pairs with opposite directions of the on-axis field (vortex-antivortex pairs). A transit around a vortex changes the phase of the wave function by $2\pi$, so free movement of the vortices leads to fluctuations in the phase. If the amplitude of the fluctuations in the phase is sufficiently high, the coherence of the state is lost $\langle e^{i\phi(r)} \rangle = 0$. However the modulus of the order parameter remains nonzero over a larger part of the volume $|\langle \Delta(r) \rangle| \neq 0$ (it goes to zero only inside vortices, near the vortex axis). Within some range of temperatures $T_{BKT} < T < T_{c0}$ Cooper pairs coexist with vortices. The pairs reduce the dissipation, but do not suppress it completely. For temperatures $T < T_{BKT}$ the correlator (2) tends to zero according to a power law $1/r$, thus quasi-long-range order and almost coherent state takes place. It should be noted the scenario of preformed pairs [2, 6, 7]: superconductivity is destroyed due to phase fluctuations at temperature $T = T_{c}$ but Cooper pairs survive up to temperature $T^*$ called the pseudogap temperature well above $T_{c}$, which manifest as the energy gap in the electronic DOS although superconductivity has been destroyed.

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In a work [3] the model of a hypothetical superconductor has been proposed. In this model an interaction energy between (within) structural elements of condensed matter (molecules, nanoparticles, clusters, layers, wires etc.) depends on state of Cooper pairs: if the pair is broken, then energy of the molecular system is changed by quantity \( E_a - E_b \), where \( E_a \) and \( E_b \) are energies of the system after- and before breaking of the pair accordingly. Thus to break the Cooper pair we must make the work against the effective electron-electron attraction and must change the energy of the structural elements:

\[
2|\Delta| \rightarrow 2|\Delta| + v.
\]

We will consider a case \( v > 0 \) only: the breaking of the pair raises of the molecular structure (creation of the pair lowers the energy). In this case the pairs become more stable. The Hamiltonian corresponding to the transformations (3) is

\[
\hat{H} = \hat{H}_{BCS} + \hat{H}_{\nu} = \sum_{k, \sigma} \varepsilon(k) a_{k, \sigma}^+ a_{k, \sigma} - \frac{\lambda}{V} \sum_{k, p} a_{p, \uparrow}^+ a_{-p, \downarrow}^+ a_{-k, \downarrow} a_{k, \uparrow} - \frac{v}{2} \sum_{k} \left[ \frac{\Delta}{|\Delta|} a_{k, \uparrow}^+ a_{-k, \downarrow} + \frac{\Delta^*}{|\Delta|} a_{-k, \downarrow}^+ a_{k, \uparrow} \right],
\]

where \( \hat{H}_{BCS} \) is BCS Hamiltonian: kinetic energy + pairing interaction (\( \lambda > 0 \)). The combinations \( a_{k, \sigma}^+ a_{-k, \sigma}^+ \) and \( a_{-k, \sigma} a_{k, \sigma} \) are creation and annihilation of Cooper pairs operators. The anomalous averages

\[
\Delta^+ = \frac{\lambda}{V} \sum_{p} \langle a_{p, \uparrow}^+ a_{-p, \downarrow}^+ \rangle, \quad \Delta = \frac{\lambda}{V} \sum_{p} \langle a_{-p, \downarrow} a_{p, \uparrow} \rangle,
\]

are the complex order parameter \( \Delta = |\Delta|e^{i\phi} \) (or two-component order parameter: module \( |\Delta| \) and phase \( \phi \)). Due the multipliers \( \frac{\Delta^+}{|\Delta|} \) and \( \frac{\Delta^*}{|\Delta|} \) in \( \hat{H}_{\nu} \) the energy does not depend on the phase \( \phi \) (\( a \rightarrow a e^{i\phi/2}, a^+ \rightarrow a^+ e^{-i\phi/2} \Rightarrow \Delta \rightarrow \Delta e^{i\phi}, \Delta^* \rightarrow \Delta^* e^{-i\phi} \)). Thus both \( \hat{H}_{BCS} \) and \( \hat{H}_{\nu} \) are invariant under the \( U(1) \) transformation. The term \( \hat{H}_{\nu} \) is similar to "source term" in [9], where it means the injection of Cooper pairs into the system. On the other hand, \( \hat{H}_{\nu} \) has a form of an external field acting on a Cooper pairs only, and \( \nu \) is energy of a Cooper pair in this field. We call the field \( \nu \) as the external pair potential (EPP) since the potential is imposed on the electron subsystem by the structural elements of matter. It has been obtained two type of solutions in this model (at large temperatures \( T \to \infty \)):

\[
|\Delta| = \frac{v}{2} \quad \text{if} \quad g = 0
\]

\[
|\Delta| = \frac{g \hbar \omega v}{4k_B T},
\]

where \( \omega \) is the phonon frequency. We can see that in the first case the energy gap is imposed on the system by EPP and it does not depend on temperature. In the second case the energy gap asymptotically tends to zero when temperature rises and \( \Delta = 0 \) if electron-phonon interaction is absent \( g = 0 \).

In this work we consider a question: what is cause of superconductivity in the model (4) - the electron-electron interaction \( \lambda \) or the potential \( \nu \)? In other words: can we impose the superconducting state on the noninteracting system by EPP? This question is closely related with scenario of the superconductor-normal conductor transition: in BCS theory (when \( \nu = 0 \)) we have the fermionic scenario, however presence of EPP can change situation towards the bosonic scenario due important role of phase fluctuations in such system. In this work we explore the phase fluctuations and topological defects (vortexes, vortex rings, vortex tangles and other form of quantum turbulence) of the order parameter \( \Delta(r) \) and their essential difference from the same in Ginzburg-Landau (GL) theory.

**II. TWO TYPES OF SOLUTIONS**

In order to find equilibrium value of the order parameter \( \Delta \) we should obtain system’s energy. Using the BCS wave function \( |BCS\rangle = \prod_k (u(k) + v(k) a_{k, \uparrow}^+ a_{-k, \downarrow}^+) |0\rangle \), where \( u^2(k) + v^2(k) = 1 \), we obtain the energy in a form:

\[
W_s = \langle \hat{H} \rangle = \sum_k 2\varepsilon(k) v^2(k) - \frac{\lambda}{V} \sum_{k, p} u(k)v(k)u(p)v(p) - \frac{\Delta}{|\Delta|} \sum_k u(k)v(k),
\]
where $\Delta$ can be supposed real in the absence of magnetic field and current. The first term is internal kinetic energy, the second term is energy of the electron-electron interaction, the third term is energy of Cooper pairs’ condensate in the external pair field. The functions $v(k)$ and $u(k)$ have to minimize the energy, that is $\frac{dW}{dv^2(k)} = 0$:

$$2\varepsilon(k) - \frac{1 - 2v^2(k)}{u(k)v(k)} \left[ \frac{\lambda}{V} \sum_p u(p)v(p) + \frac{v}{2} \frac{\Delta}{|\Delta|} \right] = 0. \quad (9)$$

There are two different solutions of this equation, which correspond to two different physical situations:

1. **Regime of withholding of the energy gap**

We can suppose

$$\Delta = \frac{\lambda}{V} \sum_p u(p)v(p) + \frac{v}{2} \frac{\Delta}{|\Delta|}; \quad (10)$$

Then we obtain the $u$, $v$ functions in a form

$$v^2(k) = \frac{1}{2} \left( 1 - \frac{\varepsilon(k)}{E} \right), \quad u^2(k) = \frac{1}{2} \left( 1 + \frac{\varepsilon(k)}{E} \right) \quad (11)$$

as in BCS theory with

$$E^2 = \varepsilon^2 + |\Delta|^2, \quad v(k)u(k) = \frac{\Delta}{2E}; \quad (12)$$

From Eq. (10) we can see that if the coupling constant is $\lambda = 0$ then the energy gap is nonzero $|\Delta| = v/2$ (if $v > 0$). Thus the operator $\hat{H}_v$ describes the external source of Cooper pairs injecting the pairs into the system. If to suppose $v = 0$ then we have ordinary BCS theory.

Superconducting state is energetically favorable if the free energy $(8)$ is less than the free energy of normal state i.e $W_s - W_n < 0$. Using method of a work [8] we can obtain the free energy $F_s - F_n$, where if we suppose the absence of EPP $v = 0$ and $T \to T_c$ then we have GL expansion:

$$F_s - F_n = -a|\Delta|^2 + \frac{b}{2}|\Delta|^4 + cq^2|\Delta|^2, \quad (13)$$

where $a = \alpha(T_c - T)$, $\alpha > 0$, $b$ and $c$ are expansion coefficients. The equilibrium value of the energy gap is

$$|\Delta|^2 = \frac{a - cq^2}{b}. \quad (14)$$

If we suppose the absence of electron-electron interaction $\lambda = 0$ then at $T \to \infty$ it is easy to obtain

$$F_s - F_n = -v \frac{w}{2k_BT}|\Delta| + \frac{w}{2k_BT}|\Delta|^2 + \frac{wk_F^2}{144(k_BT)^2m^2q^2}|\Delta|^2 = -a|\Delta| + \frac{b}{2}|\Delta|^2 + \frac{c}{2q^2}|\Delta|^2; \quad (15)$$

where $w$ is a value of energy dimension, $a$, $b$ and $c$ are corresponding expansion coefficients. The equilibrium value of the energy gap is

$$|\Delta| = \frac{a}{b + cq^2} = \frac{v/2}{1 + \frac{c}{b}q^2}. \quad (16)$$

We can see fundamentally differences of the energy gap $(14)$ from the energy gap $(16)$:

1. the gap $(14)$ depends on temperature and $\Delta = 0$ at $T > T_c$. The gap $(16)$ does not depend on temperature: $|\Delta| = v/2 = \text{const}$

2. for the gap $(14)$ the critical momentum $q_c = a/c$ exists such that $\Delta = 0$ at $q > q_c$. The gap $(16)$ tend to zero asymptotically $\Delta(q \to \infty) \to 0$, that is it does not become zero for any momentum.
Thus EPP imposes superconductivity to the system and holds it hardly. In the same time the coherence length is determined as

$$\xi^2/\hbar^2 = \frac{\xi}{a} \propto \frac{1}{T - T_c} \quad \text{GL theory}$$

The coherence length has physical sense of the spatial scale at which the magnitude of the energy gap $|\Delta|$ varies (below critical temperature) or it is a correlation radius of fluctuations (above the critical temperature). In the GL theory $\xi \propto \frac{1}{a}$, but in the regime of the withholding the coherence length is finite quantity, although $q_c = \infty$ formally. This fact is a consequence of the fact that in the GL theory on a spatial scale less than $\xi$ the energy gap is filled with quasiparticles with energy scale $|\Delta|^2/E_F \ll |\Delta|$ [10], for example, each vortex line is equivalent to a normal region of radius $\xi$. In the regime with free energy [15] the EPP withhold the energy gap on all spatial scales.

2. Regime of renormalization

On the other hand, in order to obtain the functions [11] we can suppose

$$\Delta = \frac{\lambda}{V} \sum_P u(p)v(p), \quad (18)$$

however then we will have a dispersion law of quasiparticles and a condensation amplitude in forms

$$E^2 = \varepsilon^2 + |\Delta|^2 \left(1 + \frac{\nu}{2|\Delta|}\right)^2, \quad v(k)u(k) = \frac{\Delta \left(1 + \frac{\nu}{2|\Delta|}\right)}{2E}.$$  \quad (19)

Even if the ordering $\Delta \sim \langle aa\rangle$, $\Delta^+ \sim \langle a^+a^+\rangle$ is absent then the energy gap in the quasiparticles’ specter is present. Thus the state with $\Delta = 0$ can be interpreted as state with a pseudogap. The integration domain in the self-consistency equation [18] must be cut off at some characteristic phonon energy $\hbar\omega$ ($\lambda > 0$ if $|\varepsilon(k)| < \hbar\omega$, outside $\lambda = 0$), then we have

$$\Delta = g \int_{-\hbar\omega}^{\hbar\omega} d\varepsilon \frac{\Delta \left(1 + \frac{\nu}{2|\Delta|}\right)}{2\sqrt{\varepsilon^2 + |\Delta|^2 \left(1 + \frac{\nu}{2|\Delta|}\right)^2}}.$$  \quad (20)

We can see from Eq. (20), that if the coupling constant is $g \equiv \lambda

nu_F = 0$ then $\Delta = 0$. This means, that only electron-electron coupling is the cause of superconductivity but not the potential $\nu$. In this case the operator $\hat{H}_\nu$ is an external field acting on the Cooper pairs only, and $\nu$ is energy of the pair in this field. As stated above, the potential $\nu$ is called as external pair potential since it is imposed on the electron subsystem by the structural elements of a substance (molecular, clusters etc. if their energy depends on state of Cooper pairs), unlike the pair potential $\Delta$, which is result of electron-electron interaction and determined with the self-consistency equation. If $\nu = 0$ we have usual self-consistency equation $\Delta = I(\Delta)$ in BCS theory. In presence of the external pair potential $\nu \neq 0$ the self-consistency condition has a form $\Delta = I(\Delta, \nu) - Eq. (20)$. Thus the potential $\nu$ renormalizes the order parameter $\Delta$.

It should be noticed that $\Delta = 0$ is not solution of the self-consistency equation (20) if $\nu \neq 0$, $g \neq 0$ (due the term $\Delta^\frac{1}{2}$ an uncertainty $\frac{\nu}{2}$ occurs) unlike BCS theory (when $\nu = 0$). The absence of this solution is consequence of the fact that the derivative $\frac{d\nu}{d\Delta}$ does not exist at $\Delta = 0$ because $W(\Delta \to 0) \sim |\Delta|$ and the function $|\Delta|$ is non-differentiable in this point (the derivative has a jump discontinuity), unlike BCS theory where $W(\Delta \to 0) \sim |\Delta|^2$ - Fig.1. Thus the regions, where Eq. (20) has not solutions, should be considered as the normal state of a superconductor. However we can extend the definition of the normal state by the following way. The order parameter is a complex quantity: $\Delta = |\Delta|e^{i\phi}$. As it will be demonstrated below the ratio $\frac{\Delta}{|\Delta|}$ can be zero due the strong fluctuations of the phase $\phi(r,t)$ so that average value $\langle e^{i\phi(r,t)} \rangle = 0$. In this sense $\Delta = 0$ can be considered as a solution of the self-consistency equation (20) and it means the normal state of a superconductor. Then the energy $W_s$ can be determined as $W_s = W_s(\Delta = 0)$:

$$W_s - W_n = V \nu_F \int_{-\hbar\omega}^{\hbar\omega} 2\varepsilon \nu^2 d\varepsilon - g \left( \int_{-\hbar\omega}^{\hbar\omega} \nu d\varepsilon \right)^2 - v \frac{\Delta}{|\Delta|} \int_{-\hbar\omega}^{\hbar\omega} \nu d\varepsilon$$

$$- V \nu_F \int_{-\hbar\omega}^{\hbar\omega} 2\varepsilon \nu^2_0 d\varepsilon - g \left( \int_{-\hbar\omega}^{\hbar\omega} \nu_0 u_0 d\varepsilon \right)^2 - v \int_{-\hbar\omega}^{\hbar\omega} \nu_0 u_0 d\varepsilon,$$  \quad (21)
Figure 1: the free energy (23) at $q = 0$ - line (1), the Ginzburg-Landau free energy (13) at $q = 0$ - line (2).

where

$$v_0^2(\varepsilon) = \frac{1}{2} \left( 1 - \frac{\varepsilon}{E_0} \right), \quad u_0^2(\varepsilon) = \frac{1}{2} \left( 1 + \frac{\varepsilon}{E_0} \right), \quad u_0(\varepsilon)v_0(\varepsilon) = \frac{v^2}{2E_0}, \quad E_0^2 = \varepsilon^2 + \frac{v^2}{2}. \quad (22)$$

This renormalization of the normal state is consequence of the extrapolation in the point $\Delta = 0$ where the self-consistent equation (20) does not have solutions. The expressions for the energies $W_s$ and $W_n$ have not any sense in themselves, only their difference $W_s - W_n$ is a physical quantity. Interpretation for $W_n$ and $E_0$ is following. Let the electron-electron interaction is switched off: $g = 0$. Then charge is still carried by the pairs of electrons (current carriers are the pairs) because for their breaking it must be made the work $v > 0$. This means that the quasiparticles’ spectrum has a gap $\nu/2$. But this state is not superconducting because the ordering $\langle aa \rangle, (a^+a^+)$ is absent. In our opinion such state can be interpreted as state with a pseudogap.

Using method of a work [8] we can obtain the free energy $F_s - F_n$. If the external pair potential is absent $v = 0$ and temperature is slightly less than the critical temperature $T \lesssim T_c$, then we have GL expansion (13). Let consider the case $v > 0$ and limit $T \to \infty$. Then an expansion in powers of $1/T$ and $|\Delta|$ has a form:

$$F_s - F_n = V\nu_F \left[ \frac{\hbar\omega}{2k_BT}|\Delta|^2 - vg \left( \frac{\hbar\omega}{2k_BT} \right)^2 |\Delta| + v \frac{\hbar\omega k_F^2}{144(k_BT)^3 m^2} q^2 |\Delta| \right]. \quad (23)$$

From condition $\frac{dF}{d|\Delta|} = 0$ we obtain

$$|\Delta| = \frac{g\hbar\nu}{4k_BT} - \frac{vk_F^2}{36(k_BT)^2 m^2} q^2. \quad (24)$$

Thus at large temperatures $T \gg T_c(v = 0)$ the energy gap tends to zero asymptotically as $1/T$ with increasing of temperature. Thus, formally, the critical temperature is equal to infinity. However the critical momentum

$$q_c^2 = \frac{9g\hbar\omega m^2 k_BT}{k_F^2} \quad (25)$$

exists such that $\Delta = 0$ at $q > q_c$. In [8] it has been shown that the free energy (23) is equivalent to GL free energy (13) with corresponding coefficients with $a > 0$ at all temperatures, and the coherence length

$$\xi^2 = \hbar^2 \frac{c}{a} \propto \frac{1}{T}. \quad (26)$$

The energy gap $\nu/2$ in the normal state cannot be destroyed by the condensate’s momentum $q$ as in previous regime of withholding of the energy gap. Thus we can see that in any given variants (regime of withholding of the energy gap or the regime of renormalization) EPP causes the existence of the energy gap $\nu/2$. The questions arise: which state can be realized as superconducting and why the state with the gap $\nu/2$ in the regime of renormalization is normal.
III. PHASE FLUCTUATIONS

In the mean field approximation the solution \(14\) minimizes the GL free energy in the superconducting phase. However the order parameter is a complex function and the phase depends on position:

\[
\Delta = \Delta e^{i\phi(r)}. \tag{27}
\]

Inserting this approximation into the GL free energy we get

\[
F_s - F_n = \int \left[ -\alpha |\Delta(r)|^2 + \frac{b}{2} |\Delta(r)|^4 + c |(-ih\nabla) \Delta(r)|^2 \right] dr
= \text{const} + c|\Delta|^2 h^2 \int |\nabla \phi(r)|^2 = \text{const} + Vc|\Delta|^2 \sum_q q^2 \phi^2(q), \tag{28}
\]

where we neglect the fluctuations of the order parameter’s modulus and the transformations \(\Delta(r) = \sum_q \Delta_q e^{iqr/h}, \Delta_q = \frac{1}{\pi} \int \Delta(r) e^{-iqr/h} dr \) have been used. With the help of the equipartition theorem \(Vc|\Delta|^2 q^2 \langle \phi^2(q) \rangle = \frac{1}{2}k_B T\) we obtain a Fourier component of the mean square of phase fluctuation

\[
\langle \phi^2(q) \rangle = \frac{k_B T}{2c|\Delta|^2 q^2 V}. \tag{29}
\]

Let us consider the correlation function

\[
\langle \Delta^+(0) \Delta(r) \rangle = |\Delta|^2 \langle \exp \left( i\phi(0) - i\phi(r) \right) \rangle
= |\Delta|^2 \left( \exp \left( i \sum_q \left[ 1 - e^{iqr/h} \right] \phi_q \right) \right) \approx |\Delta|^2 \exp \left( \frac{-1}{2} \sum_q \left( 1 - e^{iqr/h} \right)^2 \right) \langle \phi^2_q \rangle
= |\Delta|^2 \exp \left( - \frac{k_B T}{2c|\Delta|^2} \right) \int \frac{d^D q}{q^2} \left( 1 - \cos \frac{qr}{h} \right). \tag{30}
\]

For \(D = 1\) we have

\[
\int_0^\infty dq \frac{d^D q}{q^2} \left( 1 - \cos \frac{qr}{h} \right) = \frac{r}{h} \left( \frac{-1}{x} \frac{\cos x}{x} - 0 \right)_{x=q/r} = \frac{q}{r} e^{-qr/h} \Rightarrow \langle \Delta^+(0) \Delta(r) \rangle = |\Delta|^2 e^{-qr/h} = 0 \tag{31}
\]

This means that despite we have nonzero energy gap in each point \(|\Delta| \neq 0\) the mean over system order parameter is absent \(\langle \Delta \rangle = 0\). In 1D case superconductivity is impossible due an infrared divergence (at the lower limit 0). Physically the mechanism of destruction of phase coherence in 1D systems is spontaneous creation of topological defects as the phase slip \(5\) \(11\) \(12\). It should be noticed that the upper limit of integration over modulus of momentum \(q\) in Eq.(30) is the critical momentum \(q_c\). This is consequence of suppression of the energy gap such that \(|\Delta(q)| = 0\) at \(q > q_c\) - Eqs.(14) \(29\).

For \(D = 2\) it is easy to show

\[
\langle \Delta^+(0) \Delta(r) \rangle = |\Delta|^2 \left( \frac{\hbar}{r q_c} \right)^{\frac{k_B T}{4\pi \hbar^2 |\Delta|^2}} \tag{32}
\]

at \(r \to \infty\). Such behavior of the correlator at large distances is known as quasi-long range order. As well known in the normal state the correlator is \(\langle \Delta^+(0) \Delta(r) \rangle \propto e^{-r/\xi}\). At \(T = T_c\) the \(\xi = \infty\), thus at transition to superconducting state the correlator becomes \(\propto \frac{1}{r}\). Hence 2D system is superconducting if

\[
\frac{k_B T}{4\pi \hbar^2 |\Delta|^2} \leq 1. \tag{33}
\]

Such a transition is called the BKT transition. Physically the mechanism of destruction of phase coherence in 2D system is spontaneous creation of topological defects such as vortexes \(5\) \(11\) \(12\). The phase of order parameter \(\phi(r, t)\) in each point of superconductor depends strongly (in each time moment) on position of all vortexes because the phase changes to \(2\pi\) at going around each vortex. This means, that vortexes, moving like particles of liquid, leads to strong fluctuations of the phase \(\phi(r, t)\) so that the mean value of phase becomes zero \(\langle \phi(r, t) \rangle = 0\). Therefore the order
parameter becomes zero too \( \langle \Delta(\mathbf{r}, t) \rangle = |\Delta| \langle \phi(\mathbf{r}, t) \rangle = 0 \). From the above it leads that the asymptotic (24) at \( T \to \infty \) is impossible in 2D due the phase fluctuations.

For \( D = 3 \) it is easy to show

\[
\langle \Delta^+(0)\Delta(\mathbf{r}) \rangle = |\Delta|^2 \exp \left[ -\frac{k_B T}{4\pi^2 \hbar^3 c |\Delta|^2} \left( q_c - \frac{\hbar}{r} S_1 \left( \frac{r q_c}{\hbar} \right) \right) \right].
\]

Let us consider limit cases:

\[
r = 0 \quad \langle \Delta^+(0)\Delta(\mathbf{r}) \rangle = |\Delta|^2 e^0
\]

\[
r \to \infty \quad \langle \Delta^+(0)\Delta(\mathbf{r}) \rangle = |\Delta|^2 \exp \left[ -\frac{k_B T}{4\pi^2 \hbar^3 c |\Delta|^2} \cdot q_c \right] > 0.
\]

Thus the long-range order occurs in this case. In Section II in the regime of withholding of the energy gap (type II solution) we could see the gap (16) tends to zero asymptotically \( \Delta(q \to \infty) \to 0 \), thus \( q_c = \infty \). In this case the integral \((30)\) over \( q \) has an ultraviolet divergence. We can see from Eqs. (32) (35) that \( \langle \Delta^+(0)\Delta(\mathbf{r}) \rangle = 0 \) at all \( r \neq 0 \), thus strong and quasi-long-range order are absent. In such system despite the fact that the energy gap is nonzero \( |\Delta| \neq 0 \) the mean value of the order parameter is zero \( \langle \Delta \rangle = 0 \), hence superconducting response of the system is absent. This reflects the fact that superconducting order parameter is wave function of the condensate of Cooper pairs \( |\Delta| e^{i\phi(r,t)} \), but due the phase fluctuations we have \( \langle \phi(\mathbf{r}, t) \rangle = 0 \). In the regime of renormalization (type II solution) the state with the functions (22) is normal due the same reasons - the gap \( \nu/2 \) cannot be destroyed by any finite the condensate’s momentum \( q \) that is \( q_c = \infty \). In contrast, the superconducting state (25) has the finite critical momentum \( \langle \Delta^+(0)\Delta(\mathbf{r} \to \infty) \rangle = 0 \). It should be noticed that in the GL theory for \( T \to T_c \) we have \( q_c \to 0 \) and \( |\Delta| \to 0 \) - Eq.(14), so that phase fluctuations evolve due suppression of the gap only (in normal state the phase has not any sense) and we have the fermionic scenario of the transition. On the contrary in the regime of renormalization for \( T \to \infty \) we have \( q_c \to \infty \) and \( |\Delta| \to 0 \) - Eq.(25), so that \( \langle \Delta^+(0)\Delta(\mathbf{r}) \rangle = 0 \) i.e. superconducting state is being suppressed by phase fluctuations (the bosonic scenario).

The ultraviolet limit corresponds to small spatial scales (for example the coherence length (26) tends to zero at \( T \to \infty \)). This means that the scale of spatial variation of the order parameter can be less than interatomic distance. The similar phenomenon takes place in HeII, where vortex rings can have minimal radius \( \sim 5\AA \) (14) with the core radius \( \sim 1\AA \) (15) that is less than average distance between atoms \( \sim 3.6\AA \). As noted in (16) such small sizes of the core radius demonstrate that the vortexes in HeII (and in superconductors) are vortexes in the field of amplitude of probability \( \Psi(\mathbf{r}) \), but not in the atomic environment (or in electron gas), and classic formulas on such scale has effective sense. Moreover in our opinion to cut off the integrals by some atomic size is not quite correct because Hamiltonian (4) is operator in continuous and infinite medium. It should be noticed that the infrared limit \( q \to 0 \) is some abstraction too because it is necessary presence of infinite medium but in reality we have finite samples.

IV. THE ASSEMBLE OF TOPOLOGICAL DEFECTS IN THE ULTRAVIOLET LIMIT \( q \to \infty \)

As noted above due the infrared divergence in Eq.(30) the 1D superconductivity is impossible, in 2D system BKT transition occurs and the asymptotic (24) at \( T \to \infty \) is impossible. Physically this means destruction of phase coherence so that \( \langle \Delta(\mathbf{r}) \rangle = 0 \). Mechanism of the destruction is spontaneous creation of topological defects as the phase slip (1D) and vortexes (2D). However we could see that the ultraviolet divergence in Eq.(30) takes place in the regime of withholding of the energy gap due infinity large critical momentum \( q_c \to \infty \). We should investigate mechanism of the destruction of phase coherence in this case.

Let us calculate the second critical field \( H_{c2} \) for the regime of withholding of the energy gap. To do it a simple method of calculation of the field \( H_{c2} \) can be proposed by the example of the GL theory. Let a superconductor is in magnetic field \( \mathbf{H} \parallel Oz \). It is convenient to choose a calibration \( A_y = \mu_0 H x \). Then GL equation has a form (12):

\[
\xi^2 \left[ -\frac{d^2}{dx^2} + \frac{2\pi i}{\Phi_0} \mu_0 H \frac{d}{dy} + \left( \frac{2\pi \mu_0 H}{\Phi_0} \right)^2 x^2 \right] \varphi - \varphi + |\varphi|^2 \varphi = 0,
\]

where \( \varphi(\mathbf{r}) = \Delta(\mathbf{r})/(a/b) \) is a dimensionless variable. Here, unlike the standard method, we retained a term \( \varphi^3 \). When the field strength is of about \( H_{c2} \) the superconductor are pierced by many vortices, so that the order parameter is strongly nonhomogeneous \( \varphi = \varphi(x,y) \), it varies over distances of the order of the coherence length \( \xi \). We can consider the order parameter is real \( \varphi = \varphi^x \) and average it over the system so that \( \langle \varphi(x,y) \rangle = \varphi = const > 0 \). In addition,
we can suppose $x^2 = \xi^2$. Then Eq. (36) takes the form:

$$
\xi^4 \left( \frac{2\pi \mu_0 H}{\Phi_0} \right)^2 \varphi - \varphi + \varphi^3 = 0,
$$

(37)

where $\Phi_0$ is a magnetic flux quantum. Then the order parameter is

$$
\varphi = \sqrt{1 - \xi^4 \left( \frac{2\pi \mu_0 H}{\Phi_0} \right)^2}.
$$

(38)

We can see $\varphi$ decreases with increasing magnetic field. At the field

$$
H_{c2} = \frac{\Phi_0}{2\pi \mu_0 \xi^2}
$$

(39)

the second order phase transition takes place $\varphi(H_{c2}) = 0$. At $H > H_{c2}$ superconducting phase is absent. Besides the gap $\varphi(r)$ in a vortex can be approximated by the formula $\varphi(r) = \tanh (\nu r / \xi)$ [12], where $\nu$ is a constant or order of one. Thus at small $r$ we have $\varphi \propto r$. However near the vortex line $r < \xi$ the energy gap is filled with quasiparticles with energy scale $|\Delta|^2 / E_F \ll |\Delta|$ [10]. Thus the second critical field is such field at which the average distance between the vortices is $\sim \xi$. In other worlds vortexes cannot be closed to each other by a distance less than $\sim \xi$ because it leads to destruction of superconducting state.

Using the foregoing method and the functional (15) we can write Eq. (37) for regime of withholding of energy gap:

$$
\xi^4 \left( \frac{2\pi \mu_0 H}{\Phi_0} \right)^2 \varphi + \varphi - \frac{\varphi}{|\varphi|} = 0.
$$

(40)

Then the gap is

$$
\varphi = \frac{1}{1 + \xi^4 \left( \frac{2\pi \mu_0 H}{\Phi_0} \right)^2}.
$$

(41)

We can see $\varphi$ decreases with increasing magnetic field, however, unlike GL regime and the regime of renormalization, we have that

$$
H_{c2} = \infty.
$$

(42)

This means that any field cannot destroy the energy gap in the system. In such system the vortex lines $r = 0$ is normal only. The vortexes can be closed to each other arbitrarily close without destruction of state with energy gap. This fact leads to following important physical effect.

Let us consider creation of a pair of vortexes with opposite momentums in a 2D system. The energy of a vortex is

$$
\epsilon = d \left( \frac{\Phi_0}{8\pi \mu_0} B(0) \right) = \frac{\Phi_0}{4\pi \lambda} \left( \frac{d}{\mu_0} \ln \frac{\lambda}{\xi} \right),
$$

(43)

where $B(0)$ is magnetic induction in center of the vortex, $\lambda$ is a magnetic penetration depth, $d$ is layer’s thickness. Interaction between the vortexes is $U(r) = -\frac{\Phi_0}{8\pi \lambda^2} \ln \frac{1}{r}$, thus at distances $r > \lambda$ the vortexes do not interact almost and they behave as free from each other. However in other parts of the system the same process take place. Thus the system can be divided into cells of size $\sim \lambda$ of each - Fig. 2. The number of ways to place a vortex in the cell is $\lambda^2 / \xi^2$. Corresponding entropy is $k_B \ln(\lambda^2 / \xi^2)$. Then free energy per a vortex is

$$
F = \epsilon - TS = \frac{d}{\mu_0} \left( \frac{\Phi_0}{4\pi \lambda} \right)^2 \ln \frac{\lambda}{\xi} - k_B T \ln \frac{\lambda^2}{\xi^2} = \left[ \frac{d}{\mu_0} \left( \frac{\Phi_0}{4\pi \lambda} \right)^2 - 2k_B T \right] \ln \frac{\lambda}{\xi}.
$$

(44)

Free vortexes can appear when $F = 0$, that is at temperature

$$
k_B T_{\text{BKT}} = \frac{d\Phi_0^2}{32\pi^2 \mu_0 \lambda^2}.
$$

(45)
The phase of order parameter $\phi(r,t)$ in each point of superconductor depends strongly (in each time moment) on position all vortexes because the phase changes to $2\pi$ at going around each vortex. Moving of the free vortexes leads to strong fluctuations of the phase $\phi(r,t)$ so that the mean value of phase becomes zero $\langle \phi(r,t) \rangle = 0$, hence the order parameter becomes zero too $\langle \Delta(r,t) \rangle = \langle \Delta(\phi(r,t)) \rangle = 0$.

As we have seen above for regime of withholding of energy gap the system is characterized by the finite coherence length $\xi$ [17], but in a vortex the line $r = 0$ is normal only, unlike the GL theory where normal core has radius $\sim \xi$. Thus the energy of a vortex remains finite value because it is determined by magnetic induction $B(0)$ in center of the vortex, however the number of ways to place a vortex in the cell is $\lambda^2/\xi^2$ since the vortex occupies zero square. Then the free energy per a vortex is

$$ F = \epsilon - TS = \frac{d}{\mu_0} \left( \frac{\Phi_0}{4\pi}\right)^2 \ln \frac{\lambda}{\xi} - k_B T \ln \frac{\lambda^2}{\xi^2} = -\infty. $$

Thus we can make the following conclusion: creation of a topological defect (vortex in 2D system or vortex ring, vortex tangle in 3D or other forms in quantum turbulence) in the regime of withholding of energy gap is accompanied by the unlimited increase of entropy but it's energy remains finite. This means that the superconducting state is unstable and destroyed by phase fluctuations so that the order parameter is zero $\langle \Delta(r,t) \rangle = \langle \Delta(\phi(r,t)) \rangle = 0$ but energy gap can be nonzero $\langle \Delta(r,t) \rangle \neq 0$ like in the pseudogap state.

**V. SUMMARY**

In this article we investigate the stability of superconducting state in the BCS model with the external pair potential. This model has two types of solutions: regime of withholding of the energy gap and regime of renormalization. The first type has property: if even the coupling constant is $\lambda = 0$ then the energy gap is nonzero $\Delta = \nu/2$ and it does not depend on temperature, moreover the gap tends to zero asymptotically with momentum of the condensate $q$ unlike GL theory where the critical momentum $q_c$ exists such that $\Delta(q > q_c) = 0$. Thus EPP imposes superconductivity to the system and holds it hardly at all temperatures and on all spatial scales. The second type solution has property: if the coupling constant is $\lambda = 0$ then $\Delta = 0$. This means, that the potential $\nu$ renormalizes the order parameter $\Delta$ only but it is not the cause of superconductivity. At large temperatures $T \gg T_c(\nu = 0)$ the energy gap tends to zero asymptotically as $1/T$. Thus, formally, the critical temperature is equal to infinity. However the critical momentum $q_c$ exists like in ordinary GL theory. In this regime we suppose that the ordering $\Delta \sim \langle a_a \rangle, \Delta^+ \sim \langle a^+_a a^+_a \rangle$ is absent then the energy gap $\nu/2$ presents in the quasiparticles’ specter. We suppose that the state with $\Delta = 0$ can be interpreted as normal state with a pseudogap, i.e. current carriers are the pairs but the pairs are not correlated. The pseudogap state cannot be destroyed by the finite condensate’s momentum $q$ as in previous regime.

We have shown that if EPP is present $\nu > 0$ but electron-electron interaction is absent $g = 0$ that superconductivity is impossible: $\langle \Delta^+(0)\Delta(r) \rangle_{r \to \infty} = 0$ although $\langle \Delta(r) \rangle \neq 0$. This is result of phase fluctuations such that $\langle \Delta(r,t) \rangle = \langle \Delta(\phi(r,t)) \rangle \neq 0$ i.e. the superconducting ordering is destroyed by these fluctuations. Thus the regime of withholding of the energy gap and the normal state of the regime of renormalization is the pseudogap state; there are stable but noncoherent Cooper pairs in the system. This means that we cannot impose the superconducting
state on the noninteracting system by the external pair potential, only electron-electron interaction is cause of superconductivity. We can impose the superconducting state on the noninteracting system in a case only when the operator $\hat{H}_v$ has $U(1) \times U(1)$ symmetrical form $\sum_k \left[ \psi_{k\uparrow} \psi^+_{-k\uparrow} + \psi^+_{k\downarrow} \psi_{-k\downarrow} \right]$, where $v$ is the order parameter of another superconductor, for example, the boundary (proximity) effect, when a superconductor is placed in contact with a normal metal [10] or interband mixing of two order parameters belonging to different bands in a multi-band superconductor [17] occurs. In our model of superconductivity (second type solution) the EPP can renormalize the order parameter and normal state only: the asymptotic $\Delta \propto \frac{1}{T}$ and energy gap in normal state are effects of this renormalization. Physically these results is consequence of the fact that the order parameter is a wave function $|\Delta(r)| e^{i\phi(r)}$ of the Cooper pairs but not the energy gap $|\Delta(r)|$. This means that in the BCS model with EPP the bosonic scenario of suppression of superconducting state occurs - the normal state is a state with noncorrelated pairs, unlike BCS theory where fermionic scenario (break of the pairs) takes place.

As well known the impossibility of 1D superconductivity and BKT transition in 2D systems is mathematical consequence of the infrared divergence in the integral of the correlator $\langle \sum_p \frac{1}{\nu_p} \rangle$. Physically this means destruction of the superconducting ordering by phase fluctuations. Thus the asymptotic (7) cannot be realized in the 2D system. For the regime of withholding of the energy gap and normal state of the regime of renormalization the ultraviolet divergence occurs as a result of stability of the energy gap with respect to the condensate’s momentum, i.e. $q_c = \infty$ unlike GL theory and the regime of renormalization where the critical momentum is finite. The ultraviolet limit corresponds to small spatial scales, to cut off the integrals by some atomic size is not correct because Hamiltonian (1) is operator in continuous and infinite medium, and, for example, the vortexes is vortexes in the field of amplitude of probability $\Psi(r)$, but not in the electron gas, and classic formulas on such scale has effective sense. We have found mechanism of the destruction of phase coherence in this case: creation of a topological defect (vortex in 2D system or vortex ring, vortex tangle in 3D and other forms of quantum turbulence) in the regime of withholding of energy gap is accompanied by the unlimited increase of entropy but it’s energy remains finite. The infinity entropy of the vortex is a result of the fact that in the vortex the line $r = 0$ is normal only unlike the GL theory where normal core has radius $\sim \xi$. Thus the energy of a vortex remains finite value because it is determined by magnetic induction $B(0)$ in center of the vortex, however the number of ways to place a vortex is infinity since its normal core occupies zero square.

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