Remarks on the $K$-theory of $C^*$-algebras of products of odometers

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Abstract
We pose a conjecture on the $K$-theory of the self-similar $k$-graph $C^*$-algebra of a standard product of odometers. We generalize the $C^*$-algebra $Q_S$ to any subset of $\mathbb{N}^\times \setminus \{1\}$ and then realize it as the self-similar $k$-graph $C^*$-algebra of a standard product of odometers.

Keywords $C^*$-algebra · $K$-theory · Self-similar $k$-graph · Products of odometers

Mathematics Subject Classification 46L05

1 Introduction
Motivated by the work of Bost and Connes in Ref. [3], Cuntz in Ref. [5] constructed a $C^*$-algebra $Q_\mathbb{N}$ which is strongly related to the $ax + b$-semigroup over $\mathbb{N}$. Later Li in Ref. [15] defined the notion of semigroup $C^*$-algebras and Brownlowe, Ramagge, Robertson, and Whittaker in Ref. [4] defined the boundary quotients of semigroup $C^*$-algebras. Their work branches out to many interesting mathematical areas and, hence, generates a very popular area of $C^*$-algebras.

For any nonempty subset $S \subset \mathbb{N}^\times \setminus \{1\}$ consisting of mutually coprime numbers, Barlak, Omland, and Stammeier in Ref. [2] defined a $C^*$-algebra $Q_S$ which is a direct generalization of the Cuntz algebra $Q_\mathbb{N}$ (by letting $S$ be the set of all prime numbers). Barlak, Omland, and Stammeier decomposed the $K$-theory of $Q_S$ into a free abelian part which they solved in Ref. [2] and a highly nontrivial torsion part. At the end of [2], Barlak, Omland, and Stammeier made a conjecture to the torsion part which is equivalent to a conjecture of $k$-graph $C^*$-algebras about whether the

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\(K\)-theory of the single-vertex \(k\)-graph \(C^*\)-algebra is independent of the factorization rule. Very recently, many authors also found the \(k\)-graph algebra conjecture is highly connected with the famous Yang–Baxter equation (see for example [8, 27]). Therefore, the conjecture of Barlak, Omland, and Stammeier, and the conjecture about the \(k\)-graph \(C^*\)-algebra are extremely important in many ways.

Self-similar actions appear naturally in geometric group theory. Nekrashevych was the first who systematically built a bridge between the \(C^*\)-algebra theory and the self-similar actions (on rooted trees, see [20, 21]). Nekrashevych’s work was recently generalized to the self-similar directed graphs, self-similar directed graph \(C^*\)-algebras by Exel and Pardo in Ref. [7], and to the self-similar \(k\)-graphs, self-similar \(k\)-graph \(C^*\)-algebras by Li and Yang in Ref. [17].

The purpose of this paper is twofold. First, we pose a conjecture (see Conjecture 4.3) on the \(K\)-theory of the self-similar \(k\)-graph \(C^*\)-algebra of a standard product of odometers (see Definition 2.9). Second, we generalize the \(C^*\)-algebra \(Q\) to any subset of \(\mathbb{N}^k \setminus \{1\}\) and then realize it as the self-similar \(k\)-graph \(C^*\)-algebra of \(\Lambda\). Therefore, Conjecture 4.3 naturally includes the conjecture of Barlak, Omland, and Stammeier and the conjecture about the \(k\)-graph \(C^*\)-algebra. We hope the self-similar \(k\)-graph \(C^*\)-algebra setting will provide insight on how to solve these conjectures in the future.

Our paper is organized as follows. In Sect. 2, we provide the background material about self-similar \(k\)-graph \(C^*\)-algebras. In Sect. 3, we use the skew product approach to stabilize every self-similar \(k\)-graph \(C^*\)-algebra. In Sect. 4, we pose a conjecture (see Conjecture 4.3) on the \(K\)-theory of the self-similar \(k\)-graph \(C^*\)-algebra of a standard product of odometers and then we discuss the relationship between Conjecture 4.3 and the conjecture of Barlak, Omland, and Stammeier.

2 Preliminaries

In this section, we recap the background of \(k\)-graphs, \(k\)-graph \(C^*\)-algebras, self-similar \(k\)-graph, self-similar \(k\)-graph \(C^*\)-algebras from [13, 17, 18].

**Definition 2.1** Let \(k\) be a positive integer which is allowed to be infinity. A countable small category \(\Lambda\) is called a \(k\)-graph if there exists a functor \(d: \Lambda \rightarrow \mathbb{N}^k\) satisfying that for \(\gamma \in \Lambda\), \(p, q \in \mathbb{N}^k\) with \(d(\gamma) = p + q\), there exist unique \(\gamma' \in d^{-1}(p)\) and \(\gamma'' \in d^{-1}(q)\) with \(s(\gamma') = r(\gamma'')\) such that \(\gamma = \gamma' / \gamma''\).

**Definition 2.2** Let \(\Lambda\) be a \(k\)-graph. For \(p, q \in \mathbb{N}^k, \lambda \in \mathbb{T}^k\), denote by \(p \vee q := (\max\{p_i, q_i\})_{i=1}^k\), denote by \(\Lambda^p := d^{-1}(p)\), denote by \(\lambda^p := \prod_{i=1}^k \lambda_i^{p_i}\). For \(\mu, \nu \in \Lambda\), define \(\Lambda^{\min}(\mu, \nu) := \{ (\alpha, \beta) \in \Lambda \times \Lambda : \mu \alpha = \nu \beta, d(\mu \alpha) = d(\mu) \vee d(\nu) \}\). For \(A, B \subset \Lambda\), denote by \(AB := \{ \mu \nu : \mu \in A, \nu \in B, s(\mu) = r(\nu) \}\). Denote by \(\{e_i\}_{i=1}^k\) the standard basis of \(\mathbb{N}^k\). For \(1 \leq n \leq k\), denote by \(1_n := \sum_{i=1}^n e_i\).
Definition 2.3 Let $\Lambda$ be a $k$-graph. Then, $\Lambda$ is said to be row-finite if $|v\Lambda^n| < \infty$ for all $v \in \Lambda^0$ and $p \in \mathbb{N}^k$. $\Lambda$ is said to be source-free if $v\Lambda^n \neq \emptyset$ for all $v \in \Lambda^0$ and $p \in \mathbb{N}^k$. $\Lambda$ is said to be finite if $|\Lambda^n| < \infty$ for all $n \in \mathbb{N}^k$.

Throughout the rest of this paper, all $k$-graphs are assumed to be row finite and source free.

Definition 2.4 Let $\Lambda$ be a $k$-graph. Then, the $k$-graph $C^*$-algebra $O_\Lambda$ is defined to be the universal $C^*$-algebra generated by a family of partial isometries $\{s_\lambda : \lambda \in \Lambda\}$ (Cuntz-Krieger $\Lambda$-family) satisfying

1. $\{s_v\}_{v \in \Lambda^0}$ is a family of mutually orthogonal projections;
2. $s_{\mu v} = s_\mu s_v$ if $s(\mu) = r(v)$;
3. $s_\mu s_\nu = s_{s(\mu)}$ for all $\mu \in \Lambda$; and
4. $s_v = \sum_{\mu \in \Lambda^0} s_\mu s_\nu^*$ for all $v \in \Lambda^0, p \in \mathbb{N}^k$.

Definition 2.5 Let $G$ be a countable discrete group, let $\Lambda$ be a $k$-graph, let $\cdot : G \times \Lambda \to \Lambda$, $(g, \mu) \mapsto g \cdot \mu$ be a map, and let $| : G \times \Lambda \to G$, $(g, \mu) \mapsto g|_\mu$ be a map. Then, the pair $(G, \Lambda)$ is called a self-similar $k$-graph if

1. $G \cdot \Lambda^p \subseteq \Lambda^p$ for all $p \in \mathbb{N}^k$;
2. $s(g \cdot \mu) = g \cdot s(\mu)$ and $r(g \cdot \mu) = g \cdot r(\mu)$ for all $g \in G, \mu \in \Lambda$;
3. $g \cdot (\mu \nu) = (g \cdot \mu)(g|_\mu \cdot v)$ for all $g \in G, \mu, v \in \Lambda$ with $s(\mu) = r(v)$;
4. $g|_v = g$ for all $g \in G, v \in \Lambda^0$;
5. $g|_\nu = g|_\mu |_v$ for all $g \in G, \mu, v \in \Lambda$ with $s(\mu) = r(v)$;
6. $1_G|_\mu = 1_G$ for all $\mu \in \Lambda$;
7. $(gh)|_\mu = g|_h \cdot h|_\mu$ for all $g, h \in G, \mu \in \Lambda$.

Remark 2.6 Conditions 3 and 7 of Definition 2.5 define the interaction between the map $\cdot$ and the map $|$. This interaction is the defining characteristic of a self-similar $k$-graph.

Definition 2.7 Let $(G, \Lambda)$ be a self-similar $k$-graph. Define $O_{G,\Lambda}^+$ to be the universal unital $C^*$-algebra generated by a Cuntz–Krieger $\Lambda$-family $\{s_\mu : \mu \in \Lambda\}$ and a family of unitaries $\{u_g : g \in G\}$ satisfying

1. $u_g h = u_g u_h$ for all $g, h \in G$;
2. $u_g s_\mu = s_{g \cdot \mu} u_g|_\mu$ for all $g \in G, \mu \in \Lambda$.

Define $O_{G,\Lambda} := \overline{\operatorname{span}} \{s_\mu u_g s_\nu^* : s(\mu) = g \cdot s(\nu)\}$, which is called the self-similar $k$-graph $C^*$-algebra of $(G, \Lambda)$.

Remark 2.8

1. To see the spanning elements of $O_{G,\Lambda}$ are closed under the multiplication, we calculate that...
\[ S_{\mu} u_{\mu^{i}} s_{\rho} u_{\rho^{j}} s_{\eta} = S_{\mu} u_{\mu^{i}} \left( \sum_{(\xi,\eta) \in \Lambda^{\min}(v,a)} s_{\xi} s_{\eta} \right) u_{\rho^{j}} s_{\rho} \]

\[ = \sum_{(\xi,\eta) \in \Lambda^{\min}(v,a)} s_{\mu(g,\xi)} u_{\mu^{i}} u_{(h(\eta^{-1})_{v})} s_{\rho} s_{\rho^{-1}} \]

2. If \( \Lambda \) is finite, then \( O_{G,\Lambda} \) is a unital \( C^* \)-algebra with the unit \( \sum_{v \in \Lambda} s_{v} \).

3. There exists a strongly continuous homomorphism \( \gamma : \mathbb{T}^k \to \text{Aut}(O_{G,\Lambda}) \), which is called the gauge action, such that \( \gamma_{x}(s_{\mu} u_{\mu^{i}} s_{\nu}) = \lambda^{d(\mu)-d(v)} s_{\mu} u_{\mu^{i}} s_{\nu} \) for all \( \mu, \nu \in \Lambda, g \in G \) with \( s(\mu) = g \cdot s(v) \). By the gauge-invariant uniqueness theorem (see [13, Theorem 3.4]), \( O_{\Lambda} \) embeds in \( O_{G,\Lambda} \) naturally.

**Definition 2.9** [16, Definition 4.6] Let \((G, \Lambda)\) be a self-similar \( k \)-graph. Then, \((G, \Lambda)\) is called a product of odometers if

1. \( G = \mathbb{Z} \);
2. \( \Lambda^{0} = \{v\} \);
3. \( \Lambda^{1} := \{x_{\xi}^{\prime} \}_{\xi \in \mathbb{N}, \xi > 1} \) for all \( 1 \leq i \leq k \);
4. \( 1 \cdot x_{\xi}^{\prime} = x_{\xi(\xi+1) \mod n_{i}}^{\prime} \) for all \( 1 \leq i \leq k, 0 \leq \xi \leq n_{i} - 1 \);
5. \( 1|_{x_{\xi}^{\prime}} = \begin{cases} 0 & \text{if } 0 \leq \xi < n_{i} - 1 \\ 1 & \text{if } \xi = n_{i} - 1 \end{cases} \) for all \( 1 \leq i \leq k, 0 \leq \xi \leq n_{i} - 1 \).

In particular, if \( k = 1 \), then \((G, \Lambda)\) is called an odometer. Moreover, if \( x_{\xi} \cdot x_{\eta} = x_{\xi+\eta} \) for all \( 1 \leq i < j \leq k, 0 \leq \xi, \xi' \leq n_{i} - 1, 0 \leq t, t' \leq n_{j} - 1 \) with \( \xi + t \xi = t' + \xi' n_{j} \), then \((G, \Lambda)\) is called a standard product of odometers. Denote by \( g_{\Lambda} := \gcd\{n_{i} - 1 : 1 \leq i \leq k\} \).

### 3 Self-similar skew products

Kumjian and Pask in Ref. [13, Definition 5.1] defined the notion of skew products of \( k \)-graphs. That is, given a \( k \)-graph \( \Lambda \) and a functor from \( \Lambda \) into a group \( K \), they endowed the Cartesian product \( \Lambda \times K \) with a \( k \)-graph structure and it is called the skew product of \( k \) and \( K \). Since any \( k \)-graph \( \Lambda \) carries a functor which is the degree map \( d : \Lambda \to \mathbb{Z}^{k} \), there is a natural skew product \( \Lambda \) and \( \mathbb{Z}^{k} \), denoted by \( \Lambda \star \mathbb{Z}^{k} \), induced from the degree map (see Definition 3.1).

In this section, for any self-similar \( k \)-graph \((G, \Lambda)\), we construct a self-similar \( k \)-graph \((G, \Lambda \star \mathbb{Z}^{k})\) and we show that \( O_{G,\Lambda \star \mathbb{Z}^{k}} \cong O_{G,\Lambda} \star_{v} \mathbb{T}^{k} \), where \( \gamma : \mathbb{T}^{k} \to \text{Aut}(O_{G,\Lambda}) \) is the gauge action.

**Definition 3.1** (cf. [13, Definition 5.1]) Let \( \Lambda \) be a \( k \)-graph. Define \( \Lambda \star \mathbb{Z}^{k} := \Lambda \times \mathbb{Z}^{k} \); define \((\Lambda \star \mathbb{Z}^{k})^{0} := \Lambda^{0} \times \mathbb{Z}^{k} \); for \((\mu, z) \in \Lambda \star \mathbb{Z}^{k} \), define \( s(\mu, z) := (s(\mu), d(\mu) + z) \); \( r(\mu, z) := (r(\mu), z) \); for \((\mu, z), (\nu, d(\mu) + z) \in \Lambda \star \mathbb{Z}^{k} \) with \( s(\mu) = r(\nu) \), define \((\mu, z) \cdot (\nu, d(\mu) + z) := (\mu \nu, z) \); for \((\mu, z) \in \Lambda \star \mathbb{Z}^{k} \), define \( d(\mu, z) := d(\mu) \). Then, \( \Lambda \star \mathbb{Z}^{k} \) is a \( k \)-graph.
Definition 3.2 Let \((G, \Lambda)\) be a self-similar \(k\)-graph. For \(g \in G, (\mu, z) \in \Lambda \rtimes \mathbb{Z}^k\), define \(g \cdot (\mu, z) := (g \cdot \mu, z)\) and \(g|_{(\mu, z)} := g|_\mu\). Then, \((G, \Lambda \rtimes \mathbb{Z}^k)\) is a self-similar \(k\)-graph.

Definition 3.3 Let \((G, \Lambda)\) be a self-similar \(k\)-graph. Denote by \(\{s_y, u_g\}\) the generators of \(O_{G, \Lambda}^\text{tf}\) and by \(\{t(\mu, z), v_g\}\) the generators of \(O_{G, \Lambda \rtimes \mathbb{Z}^k}^\text{tf}\). We have \(O_{G, \Lambda \rtimes \mathbb{Z}^k} = \overline{\text{span}}\{t(\mu, z-d(\mu))v_g t(\nu, z-d(\nu)) : \mu, \nu \in \Lambda, g \in G, z, \nu \in \mathbb{Z}^k, s(\mu) = g \cdot s(\nu)\} \).

Lemma 3.4 Let \(A\) be a \(C^*\)-algebra, let \(S\) be a set of generators of \(A\), and let \((T_i)_{i \in I}\) be a net of operators in \(M(A)\) such that \((T_i)_{i \in I}\) is uniformly bounded. Suppose that for any \(a \in S \cup S^*\), the nets \((T_ia)_{i \in I}\) and \((T^*a)_{i \in I}\) converge. Then \((T_i)_{i \in I}, (T^*_i)_{i \in I}\) converge strictly and \((\lim_{i \in I} T_i^*) = \lim_{i \in I} T_i^*\).

Theorem 3.5 (cf. [13, Corollary 5.3]) Let \((G, \Lambda)\) be a self-similar \(k\)-graph. Then \(O_{G, \Lambda} \rtimes \gamma \mathbb{T}^k\) is isomorphic to \(O_{G, \Lambda \rtimes \mathbb{Z}^k}\).

Proof Let \((i_A, i_G)\) be the universal covariant homomorphism of \((O_{G, \Lambda} \rtimes \mathbb{T}^k, \gamma)\) in \(M(O_{G, \Lambda} \rtimes \gamma \mathbb{T}^k)\) (cf. [26, Theorem 2.61]).

For \(\mu \in \Lambda, \gamma \in \mathbb{Z}^k\), define \(\pi(s_\mu) := \sum_z t(\mu, z), \pi(u_g) := \sum_z t(\mu, z) v_g\), \(U_\lambda := \sum_{(\gamma, z)} \lambda^{-z} t(\mu, z), (\pi(s_\mu), \pi(u_g)), U_\lambda\) lie in \(M(O_{G, \Lambda \rtimes \mathbb{Z}^k})\) due to Lemma 3.4. It is straightforward to check that there exists a homomorphism \(\pi : O_{G, \Lambda}^\text{tf} \to M(O_{G, \Lambda \rtimes \mathbb{Z}^k})\) which restricts to a nondegenerate homomorphism of \(O_{G, \Lambda}^\text{tf}\). Since \(\pi(s_\mu u_g s^*_\nu) = U_\lambda \pi(s_\mu u_g s^*_\nu) U_\lambda^*\) for all \(\mu, \nu \in \Lambda, g \in G, \lambda \in \mathbb{T}^k\), we get a nondegenerate covariant homomorphism \((\pi, U)\) of \((O_{G, \Lambda} \rtimes \mathbb{T}^k, \gamma)\) in \(M(O_{G, \Lambda \rtimes \mathbb{Z}^k})\). By the universal property of \(O_{G, \Lambda} \rtimes \gamma \mathbb{T}^k\), there is a nondegenerate homomorphism \(\pi \rtimes U : O_{G, \Lambda} \rtimes \gamma \mathbb{T}^k \to M(O_{G, \Lambda \rtimes \mathbb{Z}^k})\) (denote by \(\pi \rtimes U : M(O_{G, \Lambda} \rtimes \gamma \mathbb{T}^k) \to M(O_{G, \Lambda \rtimes \mathbb{Z}^k})\) the unique extension of \(\pi \rtimes U\) such that \(\pi \rtimes U o i_A = \pi\mathbb{T} \rtimes U o i_G = U\). Notice that the image of \(\pi \rtimes U\) actually lies in \(O_{G, \Lambda \rtimes \mathbb{Z}^k}\).

Conversely, for \(\mu \in \Lambda, z \in \mathbb{Z}^k, g \in G\), define \(\rho(t(\mu, z)) := \int \lambda^{d(\mu) + z} i_A(s_\mu) i_G(\lambda) d\lambda, \rho(v_g) := \sum_z i_A(s_\mu u_g)\). It is straightforward to see that there exists a homomorphism \(\rho : O_{G, \Lambda \rtimes \mathbb{Z}^k} \to M(O_{G, \Lambda} \rtimes \gamma \mathbb{T}^k)\). Observe that \(\rho(O_{G, \Lambda \rtimes \mathbb{Z}^k}) \subset O_{G, \Lambda} \rtimes \gamma \mathbb{T}^k\).

It is easy to see that \(\rho \circ (\pi \rtimes U) = \text{id}|_{O_{G, \Lambda} \rtimes \gamma \mathbb{T}^k}\) and \((\pi \rtimes U) \circ \rho|_{O_{G, \Lambda \rtimes \mathbb{Z}^k}} = \text{id}|_{O_{G, \Lambda \rtimes \mathbb{Z}^k}}\).

Hence, we are done.

Corollary 3.6 Let \((G, \Lambda)\) be a self-similar \(k\)-graph. Then there exists a group homomorphism \(\hat{\gamma} : \mathbb{Z}^k \to \text{Aut}(O_{G, \Lambda \rtimes \mathbb{Z}^k})\) such that

1. \(\hat{\gamma}(t(\mu, w-d(\mu))v_g t(\nu, w-d(\nu))) = t(\mu, w-d(\mu))v_g t(\nu, w-d(\nu))\) for all \(\mu, \nu \in \Lambda, g \in G, z, w \in \mathbb{Z}^k\) with \(s(\mu) = g \cdot s(\nu)\);
2. \(O_{G, \Lambda \rtimes \mathbb{Z}^k} \rtimes \hat{\gamma} \mathbb{Z}^k\) is Morita equivalent to \(O_{G, \Lambda}\).

Proof The first statement follows immediately from Theorem 3.5. The second statement follows due to the Takai duality theorem (see [24]).

\[ \text{Birkhäuser} \]
4 K-theory of $C^*$-algebras of products of odometers

4.1 A conjecture on the $K$-theory of $C^*$-algebras of products of odometers

In this subsection, we pose a conjecture on the $K$-theory of the self-similar $k$-graph $C^*$-algebra of a standard product of odometers.

**Proposition 4.1** Let $(\mathbb{Z}, \Lambda)$ be a product of odometers. For $n \geq 1$, define

$$B_n := \begin{cases} \text{span}\{t_{(\mu,n_1_\cdots-d(\mu))}y_{g} t_{(\nu,n_1_\cdots-d(\nu))}^*\}, & \text{if } k \neq 1 \\ \text{span}\{t_{(\mu,n_1_\cdots-d(\mu))}y_{g} t_{(\nu,n_1_\cdots-d(\nu))}^*\}, & \text{if } k = \infty. \end{cases}$$

1. For $n \geq 1$, $B_n$ is a $C^*$-subalgebra of $O_{\mathbb{Z}, \Lambda_*, \mathbb{Z}^i}$.
2. For $n \geq 1$, $B_n \cong C(\mathbb{T}) \otimes K(L^2(\Lambda))$.
3. $(B_n)_{n=1}^\infty$ is an increasing sequence and $O_{\mathbb{Z}, \Lambda_*, \mathbb{Z}^i} = \bigcup_{n=1}^\infty B_n$.
4. $K_1(O_{\mathbb{Z}, \Lambda_*, \mathbb{Z}^i}) \cong \mathbb{Z}$ and

$$K_0(O_{\mathbb{Z}, \Lambda_*, \mathbb{Z}^i}) \cong \left\{ \frac{z}{\prod_{i=1}^k p_i} : z \in \mathbb{Z}, p_1, \ldots, p_k \in \mathbb{N}, \sum_{i=1}^k p_i < \infty \right\}.$$

5. The homomorphism $\tilde{\gamma}$ in Corollary 3.6 induces an action of $\mathbb{Z}^k$ on $K_0(O_{\mathbb{Z}, \Lambda_*, \mathbb{Z}^i})$ such that $z - w/(n_1 \cdots n_k) = w/((n_1 \cdots n_k)z_1 \cdots z_k)$ for all $z \in \mathbb{Z}^k$, $w \in \mathbb{Z}$, and $n \geq 1$, and induces a trivial action $\mathbb{Z}^k$ of on $K_1(O_{\mathbb{Z}, \Lambda_*, \mathbb{Z}^i})$.

**Proof** The proof of this proposition is similar for both the cases $k < \infty$ and $k = \infty$. So, we only prove the proposition for $k < \infty$ and omit the proof of the other case. We assume that $k < \infty$.

For $n \geq 1$, $\mu, \nu, \alpha, \beta \in \Lambda, g, h \in \mathbb{Z}$, we have

$$t_{(\mu,n_1_\cdots-d(\mu))}y_{g} t_{(\nu,n_1_\cdots-d(\nu))}^* t_{(\alpha,n_1_\cdots-d(\alpha))} h_{K_1(\mathbb{Z}, \Lambda_*, \mathbb{Z}^i)} = \delta_{\nu,\alpha} t_{(\mu,n_1_\cdots-d(\mu))}y_{g} t_{(\nu,n_1_\cdots-d(\nu))}^* t_{(\alpha,n_1_\cdots-d(\alpha))} h_{K_1(\mathbb{Z}, \Lambda_*, \mathbb{Z}^i)}.$$

So $B_n$ is a $C^*$-subalgebra of $O_{\mathbb{Z}, \Lambda_*, \mathbb{Z}^i}$.

For $n \geq 1$, by Lemma 3.4, $\sum_{\mu \in \Lambda} t_{(\mu,n_1_\cdots-d(\mu))}y_{g} t_{(\mu,n_1_\cdots-d(\mu))}^*$ is a unitary of $M(B_n)$ for all $g \in \mathbb{Z}$. Denote by $V : \mathbb{T} \to \mathbb{C}$, $\lambda \mapsto \lambda$ the generating unitary of $C(\mathbb{T})$. Then, there exists a homomorphism $\varphi : C(\mathbb{T}) \to M(B_n)$ such that $\varphi(V_g) = \sum_{\mu \in \Lambda} t_{(\mu,n_1_\cdots-d(\mu))}y_{g} t_{(\mu,n_1_\cdots-d(\mu))}^*$ for all $g \in \mathbb{Z}$. Denote by $\{e_{\mu,\nu}\}_{\mu,\nu \in \Lambda}$ the generators of $K(L^2(\Lambda))$. For $\mu, \nu \in \Lambda$, define $E_{\mu,\nu} := t_{(\mu,n_1_\cdots-d(\mu))}y_{g} t_{(\nu,n_1_\cdots-d(\nu))}^*$. For $\mu, \nu, \alpha, \beta \in \Lambda$, we have $E_{\mu,\nu} E_{\alpha,\beta} = \delta_{\nu,\alpha} E_{\mu,\beta}$. So there exists a homomorphism $\psi : K(L^2(\Lambda)) \to B_n$ such that $\psi(e_{\mu,\nu}) = E_{\mu,\nu}$ for all $\mu, \nu \in \Lambda$. Since the images of $\varphi, \psi$ commute, by [19, Theorem 6.3.7] there exists a surjective homomorphism $h : C(\mathbb{T}) \otimes K(L^2(\Lambda)) \to B_n$ such that $h(V_g \otimes e_{\mu,\nu}) = t_{(\mu,n_1_\cdots-d(\mu))}y_{g} t_{(\nu,n_1_\cdots-d(\nu))}^*$ for all $g \in \mathbb{Z}$, $\mu, \nu \in \Lambda$. Notice that there exist two faithful expectations...
\[ E : C(\mathbb{T}) \otimes K(\ell^2(\Lambda)) \to C(\mathbb{T}) \otimes K(\ell^2(\Lambda)), F : B_\eta \to B_\eta \] (the existence of \( F \) follows from \([18, \text{Theorem 3.20}]\)) such that
\[ E(V_g \otimes e_{\mu,k}) = \delta_{g,0} \delta_{\mu,1} C(\mathbb{T}) \otimes e_{\mu,k}, F(t_{(\mu,n_1-\mu,\ldots,\mu,n_{1-k-1,1})}) = \delta_{g,0} \delta_{\mu,1} t_{(\mu,n_1-\mu,\ldots,\mu,n_{1-k-1,1})}. \] It is easy to check that \( h \circ E = F \circ h \) and \( h \) is injective on the image of \( E \). Hence, \( h \) is an isomorphism by \([11, \text{Proposition 3.11}]\). Therefore, \( B_\eta \cong C(\mathbb{T}) \otimes K(\ell^2(\Lambda)) \).

For \( n \geq 1, \mu, \nu \in \Lambda, g \in \mathbb{Z} \), we have
\[
\begin{align*}
I_{(\mu, n_1-\mu)}(V_g t_{(\nu, n_1-\mu)}) &= \sum_{a \in \Lambda^k} t_{(\mu, n_1-\mu)}(V_g t_{(\nu, n_1-\mu)}) t_{(a, n_1)} t_{(\nu, n_1-\mu)} \\
&= \sum_{a \in \Lambda^k} t_{(\mu, n_1-\mu)}(V_g t_{(a, n_1)}) t_{(\nu, n_1-\mu)} \\
&= \sum_{a \in \Lambda^k} t_{(\mu, n_1-\mu, (a, n_1))}(V_g t_{(\nu, n_1-\mu)}).
\end{align*}
\]
So \( B_\eta \subset B_{\eta+1} \).

For \( \mu, \nu \in \Lambda, w \in \mathbb{Z}^k, g \in \mathbb{Z} \), pick up an arbitrary \( n \geq 1 \) such that \( w \leq n_1 \), we have
\[
\begin{align*}
I_{(\mu, w-\mu)}(V_g t_{(w, w-\mu)}) &= \sum_{a \in \Lambda^{k-w}} t_{(\mu, w-\mu)}(V_g t_{(w, w-\mu)}) t_{(a, w)} t_{(w, w-\mu)} \\
&= \sum_{a \in \Lambda^{k-w}} t_{(\mu, w-\mu)}(V_g t_{(a, w)}) t_{(w, w-\mu)} \\
&= \sum_{a \in \Lambda^{k-w}} t_{(\mu, w-\mu, (a, w))}(V_g t_{(w, w-\mu)}).
\end{align*}
\]
Hence, \( O_{\mathbb{Z}^k \Lambda^k} = \bigcup_{n \geq 1} B_n \).

For \( n \geq 1 \), denote by \( t_n : B_{\eta} \to B_{\eta+1} \) the inclusion map. We compute that \( I_{(\nu, n_1-\nu)} = \sum_{\mu \in \Lambda^k} I_{(\mu, (n+1)_1-\nu)} I_{(\nu, (n+1)_1-\nu)} \), so \( K_0(t_n)(1) = n_1 \cdots n_k \). Hence, \( K_0(O_{\mathbb{Z}^k \Lambda^k}) \cong \{ t \in \mathbb{Z} : t \in \mathbb{Z}, p_1, \ldots, p_k \in \mathbb{N} \} \). We calculate that
\[
\begin{align*}
I_{(\nu, n_1-\nu)}(t^*_{(\nu, n_1-\nu)}) &= \sum_{a \in \Lambda^k} t_{(1-a, (n+1)_1-\nu)} V_{1-a}^* t_{(\nu, n_1-\nu)} \\
&= \sum_{a \in \Lambda^k} t_{(1-a, (n+1)_1-\nu)} V_{1-a}^* t_{(\nu, n_1-\nu)}.
\end{align*}
\]
So \( I_{(\nu, n_1-\nu)}(t^*_{(\nu, n_1-\nu)}) = \sum_{a \in \Lambda^k} t_{(1-a, (n+1)_1-\nu)} V_{1-a}^* t_{(\nu, n_1-\nu)} \) in \( K_1(M_{\Lambda^k} \otimes C(\mathbb{T})) \). Since \( 1|_{\delta_{\hat{a}_1, n_1-1}} \cdots \delta_{\hat{a}_k, n_{k,1}-1} = K_1(t_n) = \text{id} \). Hence \( K_1(O_{\mathbb{Z}^k \Lambda^k}) \cong \mathbb{Z} \).
Since \( \widehat{\gamma} \) is a group homomorphism from \( \mathbb{Z}^k \) to \( \text{Aut}(O_{G, \Lambda} \ltimes \mathbb{Z}^k) \), \( \widehat{\gamma} \) induces actions of \( \mathbb{Z}^k \) on \( K_0(O_{G, \Lambda} \ltimes \mathbb{Z}^k) \) and \( K_1(O_{G, \Lambda} \ltimes \mathbb{Z}^k) \). For \( z \in \mathbb{Z}^k, n \geq 1 \), there exists \( m \geq 1 \) such that \( z \leq m1_k \). Then, \( \widehat{\gamma}(v_{(n,1)}) = v_{(n,1+z)} = \sum_{a \in A^{m1_k-z}} t_{a, (m+1)}(a) t_{a, (m+1)}(a) \). So \( K_1(\widehat{\gamma}) = 1/(n_1 \cdots n_k) \). Moreover, we calculate that

\[
\widehat{\gamma}(v_{(n,1)}) = v_{(n,1+z)} = \sum_{a \in A^{m1_k-z}} t_{a, (m+1)}(a) t_{a, (m+1)}(a)
\]

So \( K_1(\widehat{\gamma}) \) is the identity map.

**Remark 4.2** When \( k = 1 \), the \( K \)-theory of the \( C^* \)-algebra \( O_{Z, \Lambda} \) was studied in numerous papers such as [6, 9, 12, 14].

When \( k \geq 2 \), by Corollary 3.6, to determine the \( K \)-theory of \( O_{Z, \Lambda} \), it is equivalent to determine the \( K \)-theory of \( O_{Z, \Lambda} \ltimes \mathbb{Z}^k \). Based on the combination of previous profound work ([1, Corollary 2.5], [10, 6.10], [22, Theorem 2], [23]), it gives rise to a cohomology spectral sequence (see [25]) \( E_r^{p,q} \) for \( p, q \in \mathbb{Z} \), where for \( p, q \in \mathbb{Z} \), we have \( E_r^{p,q} = K_r(O_{Z, \Lambda} \ltimes \mathbb{Z}^k) \ltimes \mathbb{Z}^k \). The differential map is

\[
d_1^{p,q} : E_1^{p,q} \rightarrow E_1^{p+1,q}, \quad g \otimes e \mapsto \sum_{e_i \in \mathbb{Z}^k} (e_i \cdot g - g) \otimes (e \otimes e_i)
\]

for all \( g \in K_q(O_{Z, \Lambda} \ltimes \mathbb{Z}^k), e \in \mathbb{Z}^k \). By the definition of the cohomology spectral sequence, \( E_2^{p,q} := \ker(d_1^{p,q})/\text{Im}(d_1^{p-1,q}) \) for all \( p, q \in \mathbb{Z} \). By Proposition 4.1, each \( d_1^{p,q} \) and each \( d_1^{p,q} \) are clear, so we can determine \( E_2^{p,q} \). When \( q \) is even, following the algorithm from [2, Proposition 6.12], \( E_2^{p,q} = \begin{cases} \mathbb{Z}^k & \text{if } 1 \leq p \leq k \\ 0 & \text{if } p < 1 \text{ or } p > k \end{cases} \). On the other hand, when \( q \) is odd, since the action on \( K_1(O_{Z, \Lambda} \ltimes \mathbb{Z}^k) \) is trivial due to Proposition 4.1, we can easily deduce that the connecting maps \( d_1^{p,q} \) are all zero, hence \( E_2^{p,q} = \begin{cases} \mathbb{Z}^k & \text{if } 0 \leq p \leq k \\ 0 & \text{if } p < 0 \text{ or } p > k \end{cases} \). Since the determination of the \( k + 1 \)-th page of the spectral sequence and further the \( K \)-theory of \( O_{Z, \Lambda} \ltimes \mathbb{Z}^k \) is obstructed by extension issues, we are not able to characterize the \( K \)-theory of \( O_{Z, \Lambda} \ltimes \mathbb{Z}^k \). However, we can make the following conjecture.

**Conjecture 4.3** Let \( (Z, \Lambda) \) be a product of odometers with \( k \geq 2 \). Denote by \( e := (\delta_{i,j})_{i,j=1}^{2^{k-2}} \in (\mathbb{Z}/g\Lambda\mathbb{Z})^{2^{k-2}} \). Then

\[
(K_0(O_{Z,\Lambda}), [1_{O_{Z,\Lambda}}], K_1(O_{Z,\Lambda})) \cong (\mathbb{Z}^{2^{k-1}} \oplus (\mathbb{Z}/g\Lambda\mathbb{Z})^{2^{k-2}}, (0, e), \mathbb{Z}^{2^{k-1}} \oplus (\mathbb{Z}/g\Lambda\mathbb{Z})^{2^{k-2}}).
\]

**4.2 Generalized \( \mathcal{O}_5 \) and a conjecture of Barlak, Omland, and Stammeier**

For any nonempty subset \( S \subset \mathbb{N}^\times \setminus \{1\} \) consisting of mutually coprime numbers, Barlak, Omland, and Stammeier in Ref. [2] defined a unital \( C^* \)-algebra \( \mathcal{O}_S \) and they made a conjecture about the \( K \)-theory of this \( C^* \)-algebra.
Conjecture 4.4 [2, Conjecture 6.5]

\( (K_0(Q_S), [1_{Q_S}], K_1(Q_S)) \cong (\mathbb{Z}^{2^n-1} \oplus (\mathbb{Z}/g_S\mathbb{Z})^{2^{|S|-2}}, (0,e), \mathbb{Z}^{2^n-1} \oplus (\mathbb{Z}/g_S\mathbb{Z})^{2^{|S|-2}}), \)

where \( g_S = \gcd\{n - 1 : n \in S\} \), \( e = (\delta_{1,i} + \mathbb{Z}/g\Lambda\mathbb{Z})_{1 \leq i \leq k} \in (\mathbb{Z}/g\Lambda\mathbb{Z})^{2^{|S|-2}} \).

In the final subsection, we first generalize the construction of \( Q_S \) to an arbitrary nonempty subset of \( \mathbb{N}^\infty \setminus \{1\} \), and we show that \( Q_S \) is indeed a self-similar \(|S|\)-graph\( C^*\)-algebra of a standard product of odometers, finally we connect Conjecture 4.3 with Conjecture 4.4.

Definition 4.5 (cf. [2, Definition 2.1]) Let \( S \) be a nonempty subset of \( \mathbb{N}^\infty \setminus \{1\} \). Define \( Q_S \) to be the universal unital \( C^*\)-algebra generated by a family of isometries \( \{s_n\}_{n \in S} \) and a unitary \( u \) satisfying for any \( n, m \in S \),

1. \( s_n s_m = s_{nm} \);  
2. \( s_n u = u^n s_n \);  
3. \( \sum_{i=0}^{n-1} u^i s_n u^{-i} = 1_{Q_S} \).

Remark 4.6 It is very natural to extend Conjecture 4.4 to the case that \( S \) is any nonempty subset of \( \mathbb{N}^\infty \setminus \{1\} \).

Lemma 4.7 Let \( S \) be a nonempty subset of \( \mathbb{N}^\infty \setminus \{1\} \), and let \( B \) be a unital \( C^*\)-algebra generated by a family of isometries \( \{S_n\}_{n \in S} \) and a unitary \( U \) satisfying Conditions (2), (3) of Definition 4.5. Then for \( n \in S, z \in \mathbb{Z} \setminus n\mathbb{Z} \), we have \( S_n^* U z S_n = 0 \).

Proof We may assume that \( z > 0 \). Write \( z = wn + l \), for some \( w \geq 0, 1 \leq l \leq n - 1 \).

We calculate that

\[
(S_n^* U z S_n)(S_n^* U z S_n)^* = S_n^* U^l U^{wn} S_n S_n^* U^{-wn} U^{-l} S_n
\]

\[
= S_n^* U^l (U^{wn} S_n) (U^{wn} S_n)^* U^{-l} S_n
\]

\[
= S_n^* U^l (S_n U^w) (S_n U^w)^* U^{-l} S_n
\]

\[
= S_n^* U^l S_n S_n^* U^{-l} S_n
\]

\[
= S_n^* U^l \left( 1_B - \sum_{i=1}^{n-1} U^i S_n S_n^* U^{-i} \right) U^{-l} S_n
\]

\[
= 1_B - S_n^* U^w S_n S_n^* U^{-w} S_n - \sum_{1 \leq i \leq n-1, i \neq n-l} S_n^* U^{l+i} S_n S_n^* U^{-l-i} S_n
\]

\[
= - \sum_{1 \leq i \leq n-1, i \neq n-l} S_n^* U^{l+i} S_n S_n^* U^{-l-i} S_n.
\]

So \( (S_n^* U z S_n)(S_n^* U z S_n)^* = 0 \). Hence \( S_n^* U z S_n = 0 \). \( \square \)

Proposition 4.8 Let \( S \) be a nonempty subset of \( \mathbb{N}^\infty \setminus \{1\} \), and let \( B \) be a unital \( C^*\)-algebra generated by a family of isometries \( \{S_n\}_{n \in S} \) and a unitary \( U \) satisfying
Conditions (2), (3) of Definition 4.5. Then \( \{S_n\}_{n \in S} \) and \( U \) satisfy Condition (1) of Definition 4.5 if and only if for any \( n, m \in S \),

\[
S^*_n S_m = \sum_{i=0}^{n-1} S^*_n U^i S_n^* S_n^* U^{-i} S_m
\]

\[
= \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} S^*_n U^i S_n U^j S_m S_n^* U^{-j} S_n^* U^{-i} S_m
\]

\[
= \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} S^*_n U^{i+j} S_n S_m S_n S_n^* U^{-i-j} S_m
\]

\[
= \sum_{i=0}^{nm-1} S^*_n U^i S_n S_m S_n^* U^{-i} S_m
\]

\[
= \sum_{0 \leq i \leq nm-1, 1 \in \mathbb{Z}} U^{i/n} S_m S_n^* U^{-i/m} \text{ (by Lemma 4.7).}
\]

Conversely, suppose that for any \( n, m \in S \),

\[
S^*_n S_m = \sum_{0 \leq i \leq nm-1, 1 \in \mathbb{Z}} U^{i/n} S_m S_n^* U^{-i/m}.
\]

For \( n, m \in S \), we calculate that

\[
S^*_m S_n S_n^* S_n = \sum_{0 \leq i \leq nm-1, 1 \in \mathbb{Z}} (S^*_m U^{i/n} S_m)(S^*_n U^{-i/m} S_n)
\]

\[
= 1_B + \sum_{1 \leq i \leq nm-1, 1 \in \mathbb{Z}} (S^*_m U^{i/n} S_m)(S^*_n U^{-i/m} S_n)
\]

\[
= 1_B \text{ (by Lemma 4.7).}
\]

So

\[
(S_n S_m - S_m S_n)^* (S_n S_m - S_m S_n) = 2 \cdot 1_B - S^*_m S_n S_m S_n - S^*_n S_m S_n S_m = 0.
\]

Hence \( S_n S_m = S_m S_n \). \( \Box \)

**Remark 4.9** Let \((\mathbb{Z}, \Lambda)\) be a standard product of odometers. Then \(O_{\mathbb{Z}, \Lambda}\) is a universal unital \(C^*\)-algebra generated by a family of isometries \(\{s_{x_i}\}_{1 \leq i \leq k, 0 \leq \theta \leq n_i - 1}\) and a unitary \(u\) satisfying that

1. \( \sum_{\theta=0}^{n_i-1} s_{x_i} s_{x_i}^* = 1_{O_{\mathbb{Z}, \Lambda}} \) for all \(1 \leq i \leq k\);
2. \( s_{x_i} s_{x_j} = s_{x_j} s_{x_i} \) for all \(1 \leq i < j \leq k, 0 \leq \theta, \theta' \leq n_i - 1, 0 \leq t, t' \leq n_j - 1\) with \( \theta + tn_i = t' + \theta'n_j \);
3. \[ us_{x_i} = \begin{cases} 
 s_{x_{i+1}} & \text{if } 0 \leq \delta < n_i - 1 \\
 s_{x_0} u & \text{if } \delta = n_i - 1 
 \end{cases} \text{, for all } 1 \leq i \leq k, 0 \leq \delta \leq n_i - 1. \]

**Theorem 4.10** Let \( S \) be a nonempty subset of \( \mathbb{N}^\times \setminus \{1\} \). We enumerate \( S = \{ 1 < n_1 < \cdots < n_k \} \). Denote by \((\mathbb{Z}, \Lambda_S)\) the standard product of odometers such that \( |\Lambda_S^x| = n_i \) for all \( 1 \leq i \leq k \). Then \( Q_S \cong \mathcal{O}_{\mathbb{Z}, \Lambda_S} \).

**Proof** Denote by \( \{s_{n_i}\}_{1 \leq i \leq k} \) and \( u \) the generators of \( Q_S \), and denote by \( \{t_{x_i} : 1 \leq i \leq k, 0 \leq \delta \leq n_i - 1\} \) and \( v \) the generators of \( \mathcal{O}_{\mathbb{Z}, \Lambda_S} \) as discussed in Remark 4.9.

Define \( V := u, T_{x_i} := u^\delta s_{n_i} \) for all \( 1 \leq i \leq k, 0 \leq \delta \leq n_i - 1 \). Then \( \{T_{x_i} : 1 \leq i \leq k, 0 \leq \delta \leq n_i - 1\} \) and \( V \) satisfy Conditions (1)–(3) of Remark 4.9. So there exists a homomorphism \( \pi : \mathcal{O}_{\mathbb{Z}, \Lambda_S} \rightarrow Q_S \) such that \( \pi(t_{x_i}) = u^\delta s_{n_i}, \pi(v) = u \) for all \( 1 \leq i \leq k, 0 \leq \delta \leq n_i - 1 \).

Conversely, define \( U := v, S_{n_i} := t_{x_i} \) for all \( 1 \leq i \leq k \). Then \( \{S_{n_i} : 1 \leq i \leq k\} \) and \( U \) satisfy Conditions (1)–(3) of Definition 4.5. So, there exists a homomorphism \( \rho : Q_S \rightarrow \mathcal{O}_{\mathbb{Z}, \Lambda_S} \) such that \( \rho(u) = v, \rho(s_{n_i}) = s_{x_i} \) for all \( 1 \leq i \leq k \).

Since \( \rho \circ \pi = \text{id}, \pi \circ \rho = \text{id} \), we deduce that \( Q_S \cong \mathcal{O}_{\mathbb{Z}, \Lambda_S} \).

**Remark 4.11** For any nonempty subset \( S \subset \mathbb{N}^\times \setminus \{1\} \), by the above theorem, we can easily deduce that Conjecture 4.4 is contained in Conjecture 4.3.

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