C²-robust heterodimensional tangencies

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Abstract
In this paper, we give sufficient conditions for the existence of C² robust heterodimensional tangency, and present a non-empty open set in Diff²(M) with dim M ≥ 3 each element of which has a non-degenerate heterodimensional tangency on a C² robust heterodimensional cycle.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The purpose of this paper is to show the existence of a large class of diffeomorphisms on the C² manifold M whose global dynamics is persistently non-dominated on cycles by robust heterodimensional tangencies. To quickly explain our results, let us begin by recalling the key words of non-hyperbolic diffeomorphisms: dominated splittings, homoclinic tangencies, and heterodimensional cycles.

A dominated splitting is perhaps a bottom structure obtained by weakening hyperbolicity, which was first presented by Pliss [19]. Precisely, we say that a compact subset Λ of a smooth closed manifold M which is invariant by f ∈ Diff²(M) has a dominated splitting if TΛM is expressed by the direct sum of Df-invariant subbundles E and F as TΛM = E ⊕ F whose fibres have constant dimensions and there are C > 0 and λ > 1 such that for any integer n > 0, x ∈ Λ and any pair of unitary vectors (u, v) ∈ E(x) × F(x),

\[ \| Df^n(x)u \| \cdot \| Df^n(x)v \|^{-1} < C \lambda^{-n}. \]

If Dfⁿ is uniformly contracting on E and expanding on F, Λ is called hyperbolic.
Let $P$ be a saddle point for $f \in \text{Diff}^r(M)$. A point $Y$ of the intersection between the stable manifold $W^s(P)$ and the unstable manifold $W^u(P)$ is a homoclinic tangency of $P$ if

$P \neq Y, \quad T_Y M \neq T_Y W^s(P) + T_Y W^u(P)$. 

Obviously, if $f$ has a homoclinic tangency, then any $f$-invariant set containing the orbit of the tangency does not have any dominated splitting. Moreover, it is worth noting that Wen [24, theorem A] proved that the inverse is also true in the $C^1$ topology if $f$ is restricted to the preperiodic or prehomoclinic sets, that is, if a dominated splitting cannot be defined on such $f$-invariant sets, then there is a diffeomorphism arbitrarily close to $f$ which has a homoclinic tangency, see also [13]. On the other hand, the presence of dominated splittings is equivalent to the absence of infinitely many periodic sinks or sources in $C^1$ generic diffeomorphisms, see [1].

From the definition, a tangency associated with a periodic point is easily broken by a generic perturbation. However, Newhouse showed in [16] that there is an open set in $\text{Diff}^2(M)$ with $\text{dim}(M) = 2$ arbitrarily close to $f$ in which any diffeomorphism has a robust homoclinic tangency associated with some nontrivial set containing the continuation of the periodic point. On the other hand, Moreira showed in [14] that a similar result does not hold for the generic subset in $\text{Diff}^1(M)$ with $\text{dim}(M) = 2$, see the comments at the end of this section. The Newhouse result was extended to the higher dimensional cases with the $C^2$ topology in [12, 18, 21, 23]. Among others, several results of [18] play important roles in this paper. Thus, we need to work at least in the $C^2$ topology.

Another known cause of non-hyperbolicity is the presence of heterodimensional cycles. For a diffeomorphism $f$ on a smooth manifold $M$ of $\text{dim}(M) \geq 3$, we say that $f$ has a heterodimensional cycle associated with two saddle points $P$ and $Q$ if

$\text{index}(P) \neq \text{index}(Q), \quad W^s(P) \cap W^u(Q) \neq \emptyset, \quad W^s(Q) \cap W^u(P) \neq \emptyset$, 

where index($P$) stands for the dimension of the unstable manifold of $P$. By the first index condition, one of these two intersections, say $W^s(Q) \cap W^u(P)$, satisfies

$\text{dim}(T_X W^u(Q)) + \text{dim}(T_X W^s(P)) < \text{dim}(T_X M)$

for every $X \in W^s(Q) \cap W^u(P)$. On the other hand, the intersection $W^s(Q) \cap W^u(Q)$ satisfies

$\text{dim}(T_Y W^s(P)) + \text{dim}(T_Y W^u(Q)) > \text{dim}(T_Y M)$

for every $Y \in W^s(P) \cap W^u(Q)$. Thus the condition may permit transverse points. The set of such points is denoted by $W^s(P) \pitchfork W^u(Q)$. A non-transverse point

$Y \in (W^s(P) \cap W^u(Q)) \setminus (W^s(P) \pitchfork W^u(Q))$

is called a heterodimensional tangency. We say that such a heterodimensional tangency is strict if

$T_Y W^s(P) = T_Y W^u(Q)$. 

Observe that, when $\text{dim}(M) = 3$, any heterodimensional tangency is of the strict type.

The condition for diffeomorphisms being away from homoclinic tangency leads to a dominated splitting even for a heterodimensional cycle associated with $P$ and $Q$. In fact, Díaz and Rocha showed in [11, theorem A] that if the diffeomorphism is $C^r$ away from diffeomorphisms having homoclinic tangencies associated with continuations of $P$ and $Q$, it can be $C^r$-approximated by one with an invariant set containing the continuations which has a (strong partially hyperbolic) dominated splitting. On the other hand, it is known that, under some volume condition, the absence of dominated splittings brings on heterodimensional cycles in $C^1$ generically [22].
In this paper, we focus on heterodimensional tangencies which interfere with dominated splittings. As for heterodimensional tangencies of three-dimensional diffeomorphisms, several results have been already presented, as follows.

**C^1 Newhouse phenomenon [9].** If \( f \) has a parabolic heterodimensional tangency between \( P \) and \( Q \) which are persistently linked, it can be \( C^1 \)-approximated by a diffeomorphism having either infinitely many sinks or sources. Moreover, if the Jacobian of \( f \) at \( P \) is smaller than 1 and that of \( Q \) is greater than 1, it can be \( C^1 \)-approximated by a diffeomorphism having infinitely many nontrivial minimal Cantor sets.

**Strange attractors, C^2 Newhouse phenomenon [15].** For any generic 2-parameter family \( \{ f_{\mu, \nu} \} \) with \( f_{0,0} = f \) which has a non-degenerate heterodimensional tangency associated with \( P \) and \( Q \), there exists an open set of subfamilies exhibiting infinitely many homoclinic tangencies of the continuation of \( P \) which unfold generically. It follows that one can detect a positive Lebesgue measure set \( \mathcal{A} \) in the \( \mu \nu \)-plane arbitrarily near \( f \) such that for any \( (\mu, \nu) \in \mathcal{A} \), \( f_{\mu, \nu} \) exhibits non-hyperbolic strange attractors. Moreover, one can detect an open set of diffeomorphisms which have \( C^r \)-robust homoclinic tangency for \( r \geq 2 \).

**Renormalization, robust cycle [7].** Under appropriate conditions, one can obtain a renormalization in a neighbourhood of the heterodimensional tangency associated with \( P \) and \( Q \) such that its return maps converge to the Hénon-like family with the centre unstable direction and admit blender-horseshoes. Using this fact, for every \( r \geq 1 \), one can detect \( C^r \)-robust cycles associated with the blender-horseshoe and the continuation of \( P \) arbitrarily \( C^r \)-close to \( f \).

Incidentally, though the above works start from the heterodimensional tangency on a heterodimensional cycle, its robustness is not discussed. However, by a little perturbation, heterodimensional tangencies are broken as well as homoclinic tangencies. Thus, considering Newhouse’s works, it is natural to ask:

**Question I.** Does there exist an open set of diffeomorphisms which have heterodimensional tangencies on heterodimensional cycles?

This paper will be devoted to answering the question and studying related topics. To present our results, we have to introduce several terminologies.

A basic set for \( f \) is a compact hyperbolic transitive locally maximal \( f \)-invariant subset in \( M \). A basic set is nontrivial if it is not an periodic orbit. The dimension of the unstable bundle on a basic set \( \Lambda \) for \( f \) is called the unstable index and denoted by \( \text{index}(\Lambda) \). We say that a diffeomorphism \( f \in \text{Diff}^r(M) \) has a heterodimensional cycle associated with basic sets \( \Lambda_0 \) and \( \Lambda_1 \) if

\[
\text{index}(\Lambda_0) \neq \text{index}(\Lambda_1), \quad W^u(\Lambda_0) \cap W^s(\Lambda_1) \neq \emptyset, \quad W^s(\Lambda_0) \cap W^u(\Lambda_1) \neq \emptyset.
\]

A heterodimensional tangency between \( W^u(q) \) and \( W^s(p) \) with \( q \in \Lambda_0 \) and \( p \in \Lambda_1 \) is defined as in the case of periodic points \( Q, P \). Now let us suppose \( \text{index}(\Lambda_0) > \text{index}(\Lambda_1) \) and define that \( f \) has a \( C' \) robust heterodimensional tangency associated with \( \Lambda_0 \) and \( \Lambda_1 \) if there exists a \( C' \) neighbourhood \( U \) of \( f \) such that, for any \( g \in U \), there exist the continuations \( \Lambda_{0,g} \) of \( \Lambda_0 \) and \( \Lambda_{1,g} \) of \( \Lambda_1 \) which contain points \( \tilde{q} \in \Lambda_{0,g} \) and \( \tilde{p} \in \Lambda_{1,g} \) such that

- \( W^u(\tilde{q}) \) and \( W^s(\tilde{p}) \) contain a heterodimensional tangency.

Moreover, the heterodimensional tangency is in a \( C' \) robust cycle if there exist \( \tilde{q} \in \Lambda_{0,g} \) and \( \tilde{p} \in \Lambda_{1,g} \) such that

- \( W^u(\tilde{p}) \cap W^s(\tilde{q}) \) is non-empty.
In what follows of this paper, we only discuss the case of
\[ \dim M = d \geq 3, \quad \text{index}(\Lambda_0) = d - 1, \quad \text{index}(\Lambda_1) = 1. \]
Under this assumption, a heterodimensional tangency \( Y \in W^u(\tilde{q}) \cap W^s(\tilde{p}) \) can be characterized from a topological viewpoint as follows. Consider a local coordinate \((x_1, \ldots, x_d)\) on a neighbourhood of \( Y \) with \( Y = \emptyset \) such that a small \((d - 1)\)-disc in \( W^s(\tilde{p}) \) containing \( Y \) is given as the graph of a \( C^2 \) function \( u : \mathbb{R}^{d-1} \rightarrow \mathbb{R} \) with \( x_d = u(x_1, \ldots, x_{d-1}) \) and \( \frac{\partial u}{\partial x_i}(0) = 0 \) for \( i = 1, \ldots, d - 1 \). We say that the tangency is non-degenerate if the Hessian matrix \( Hu(\emptyset) = (\frac{\partial^2 u}{\partial x_i \partial x_j}(0)) \) is regular. A non-degenerate tangency is elliptic if all the eigenvalues of the Hessian matrix have the same sign, and otherwise hyperbolic, see [15] for the case of \( d = 3 \).

From now on, we suppose that \( M \) is a closed \( C^2 \) manifold with a Riemannian metric and \( \text{Diff}^2(M) \) is the space of all \( C^2 \) diffeomorphisms on \( M \) endowed with the \( C^2 \) topology. For a given saddle periodic point \( P \) and a basic set \( \Lambda \) of \( f \in \text{Diff}^2(M) \), \( \text{per}(P) \) denotes the minimum period of \( P \), and \( \Lambda_{\epsilon} \) denotes the continuation of \( \Lambda \) for any \( \epsilon \) sufficiently close to \( f \) in \( \text{Diff}^2(M) \).

**Theorem A.** Let \( f \) be an element in \( \text{Diff}^2(M) \) with \( \dim M = d \geq 3 \) which has nontrivial basic sets \( \Lambda_0, \Lambda_1 \) and saddle periodic points \( Q \in \Lambda_0, P \in \Lambda_1 \) such that

1. \( Df_{\text{per}(P)}(P) \) has eigenvalues satisfying
   \[ |\alpha_1| \leq \cdots \leq |\alpha_{d-2}| < |\alpha_c| < 1 < |\alpha_u|, \quad |\alpha_c \alpha_u| < 1, \]
   \( Df_{\text{per}(Q)}(Q) \) has eigenvalues satisfying
   \[ |\beta_1| < 1 < |\beta_c| < |\beta_2| < \cdots < |\beta_{d-2}|. \]
2. \( W^u(Q) \cap W^s(P) \) contains a heterodimensional tangency of elliptic type and \( W^u(Q) \cap W^s(P) \) contains a non-transverse intersection.

Then, there is a non-empty open set \( \mathcal{O} \subset \text{Diff}^2(M) \) whose closure contains \( f \) and that satisfies following condition: for every \( g \in \mathcal{O} \), there exists a nontrivial basic set \( \Lambda_{\epsilon, g} \) of index one such that \( g \) has a heterodimensional tangency of elliptic type associated with \( \Lambda_{\epsilon, g} \) and the continuation \( \Lambda_{\epsilon, g} \) of \( \Lambda_0 \).

The saddle periodic point \( P \) in (1) is called sectionally dissipative in [18], and \( \alpha_c \) and \( \beta_c \) are called the real central contracting and real central expanding eigenvalues, respectively, on the cycle. Though the sectionally dissipative condition in (1) might not be indispensable if one would use [21] instead of [18], it is sufficient to prove theorems B and C. So we will work under the condition to avoid miscellaneous difficulties. Note that the condition (1) implies that the heterodimensional tangency in the condition (2) is of the strict type.

Theorem A says nothing about whether or not the \( C^2 \) robust heterodimensional tangency is in a heterodimensional cycle associated with \( \Lambda_{0, g} \) and \( \Lambda_{2, g} \). However, one can obtain an affirmative answer to question I for \( r \geq 2 \) by supposing that the diffeomorphism in theorem A has a certain basic set called a cu-blender [5, 6, definition 3.1] which will be given in section 7.

**Theorem B.** There exists a non-empty open set in \( \text{Diff}^2(M) \) each element of which has a heterodimensional tangency of elliptic type on a heterodimensional cycle.

Now we consider the relation between heterodimensional tangencies and the absence of dominating splittings. Trivially, if a heterodimensional cycle contains a heterodimensional tangency, then there is no dominating splitting on the cycle. As mentioned above, in the \( C^1 \)
topology, the existence of homoclinic tangencies and the absence of dominating splittings are synonymous in the sense of Wen [24]. In the C^2 topology, the result corresponding to that in [24] is not known yet. Thus, the following question makes sense.

**Question II.** *Can a diffeomorphism with a heterodimensional cycle which does not admit any dominating splitting be C^2-approximated by a diffeomorphism having a heterodimensional tangency on a cycle?*

In the following theorem, we will present open conditions on C^2 diffeomorphisms under which question II is solved affirmatively.

Let \( \Lambda_0 \) and \( \Lambda_1 \) be nontrivial basic sets for \( f \in \text{Diff}^2(M) \) with \( \text{index}(\Lambda_0) = d - 1 \) and \( \text{index}(\Lambda_1) = 1 \), where \( d = \dim(M) \geq 3 \). We suppose furthermore that \( \Lambda_0 \) is a blender horseshoe and \( \Lambda_1 \) is a sectionally dissipative basic set with a real central contracting direction, that is, every periodic point in \( \Lambda_1 \) satisfies the same condition as in (1) of theorem A.

**Theorem C.** Let \( f \) be a diffeomorphism satisfying the above conditions. If \( f \) has a spherical heterodimensional intersection on the heterodimensional cycle associated with \( \Lambda_0 \) and \( \Lambda_1 \), and satisfies the C^2 open conditions given in section 8, then \( f \) is C^2-approximated by a diffeomorphism having a C^2-robust cycle with a robust heterodimensional tangency.

Here we say that \( f \) has a *spherical heterodimensional intersection* associated with \( \Lambda_0 \) and \( \Lambda_1 \) if there are \( q_1 \in \Lambda_0 \) and \( p_1 \in \Lambda_1 \) such that \( W^u(q_1) \cap W^s(p_1) \) contains a \((d - 2)\)-dimensional sphere.

Note that the cycle considered in theorem C is critical in the sense of [10]. Moreover, in the case of \( d = 3 \), Díaz, Nogueira and Pujals proved that the homoclinic classes of \( \Lambda_0 \) and \( \Lambda_1 \) in theorem C do not admit any dominated splitting [9, section 2.3].

We comment here briefly on problems related to our theorems. In the present paper, we have obtained an affirmative answer to question I about the existence of robust heterodimensional tangencies in the C^r topology for any \( r \geq 2 \). Thus, it is natural to ask whether there exist robust heterodimensional tangencies in the C^1 topology. An example is already known, which exhibits a C^1-robust homoclinic tangency in the dimension at least 3, see [2]. However, Moreira [14] showed that any two regular Cantor sets \( K_1 \) and \( K_2 \) have arbitrarily C^1 close to the originals and disjoint to each other. This implies that C^1-robust homoclinic tangencies can not be possible for two-dimensional generic diffeomorphisms. Similarly, we think that C^1-robust heterodimensional tangencies could not be possible for generic diffeomorphisms in any dimension greater than 2.

**Outline of proof of theorem A.** At the end of this section, we will outline the proof of the main theorem. The diffeomorphism \( f \) in theorem A has basic sets \( \Lambda_0, \Lambda_1 \) with \( \text{index}(\Lambda_0) = d - 1 \), \( \text{index}(\Lambda_1) = 1 \) and periodic points \( Q \in \Lambda_0 \) and \( P \in \Lambda_1 \) such that \( W^u(Q, f) \) and \( W^s(P, f) \) have a heterodimensional tangency \( Y \) of elliptic type and \( W^u(P, f) \) and \( W^s(Q, f) \) have a non-transverse intersection \( X \), see figure 1. First we perturb \( f \) near \( X \) slightly so that \( W^u(P, f) \) returns near \( Y \) via a neighbourhood \( U_Q \) of \( Q \). Perturb \( f \) again in a small neighbourhood \( U_Y \) of \( Y \) and get a diffeomorphism \( g_1 \) such that \( W^u(P, g_1) \) and \( W^s(P, g_1) \) have a homoclinic tangency in \( U_Y \). Moreover, the heterodimensional tangency \( Y \in W^u(Q, f) \cap W^s(P, f) \) is broken and becomes a \((d - 2)\)-sphere \( \mathcal{S}_{d-2} \) in \( W^u(Q, g_1) \cap W^s(P, g_1) \) (proposition 2.2). By applying Palis–Viana’s result in [18] which is based on Palis–Takens [17], we have a diffeomorphism \( g_2 \) arbitrarily close to \( g_1 \) which has a new basic set \( \Lambda_2 \) in \( U_Y \) homoclinically related to \( \Lambda_1 \) and such that its stable thickness \( \tau^s(\Lambda_2) \) is very large (proposition 4.1). Let \( A^d_0 \) be a small tubular neighbourhood of \( \mathcal{S}_{d-2} \) in \( W^u(Q, g_2) \). The iterated forward images of \( A^d_0 \) by \( g_2 \) converge to
$W^u(P, g_2)$ and hence return to $U_Y$, see figures 4 and 5. Then we perturb $g_2$ in $U_Y$ and get a diffeomorphism $g_3$ such that $W^u(q, g_3)$ and $W^s(p, g_3)$ have a heterodimensional tangency $r$ for some $q \in \Lambda_0$ and $p \in \Lambda_2$ (proposition 4.3), see figure 5 again. By the gap lemma [16, 17] together with the largeness of $\tau(A_2)$, one can show that, for any diffeomorphism $g$ sufficiently near $g_3$, $W^u(\Lambda_0, g)$ and $W^s(\Lambda_2, g)$ have a heterodimensional tangency in $U_Y$. Theorem A follows directly from this fact.

Note that we start with a heterodimensional tangency associated with $Q \in \Lambda_0$ and $P \in \Lambda_1$. However, our robust heterodimensional tangencies are associated with $\Lambda_{0,g}, \Lambda_{2,g}$ but not with $\Lambda_{0,g}, \Lambda_{1,g}$.

2. Coexistence of spherical intersections and tangencies

Suppose that $M$ is a $C^2$ manifold of dimension $d \geq 3$. The purpose of this section is to find an element of $\text{Diff}^2(M)$ arbitrarily close to the diffeomorphism given in theorem A which has simultaneously a spherical heterodimensional intersection and a homoclinic tangency.

Let $f$ be an element of $\text{Diff}^2(M)$ with periodic points $P$ and $Q$ satisfying (1) and (2) in theorem A. For simplicity, we may suppose that $\text{per}(P) = \text{per}(Q) = 1$ if necessary replacing $f$ by $f^n$ for some $n \in \mathbb{N}$ and $f$ is $C^2$ linearizable in small neighbourhoods $U_P$ of $P$ and $U_Q$ of $Q$. Then, by the condition (1), $f$ can be written in $U_P$ as

$$f(x, y, z) = (A_{1,2}x, \alpha_{1,2}y, \alpha_{1,2}z)$$

(2.1)

where $x = (x_1, \ldots, x_{d-2}) \in \mathbb{R}^{d-2}$, $y \in \mathbb{R}$ and $A_{1,2}$ is a regular $(d-2)$-matrix with eigenvalues $\alpha_1, \ldots, \alpha_{d-2}$ satisfying

$$|\alpha_1| \leq \cdots < |\alpha_{d-2}| < |\alpha_2| < 1 < |\alpha_u|, \quad |\alpha_c\alpha_u| < 1.$$ (2.2)

On the other hand in $U_Q$, $f$ can be written as

$$f(x, y, z) = (B_{1,2}x, \beta_{1,2}y, B_{1,2}z),$$

(2.3)

where $x, y \in \mathbb{R}, z = (z_1, \ldots, z_{d-2}) \in \mathbb{R}^{d-2}$ and $B_{1,2}$ is a regular $(d-2)$-matrix with eigenvalues $\beta_1, \ldots, \beta_{d-2}$ satisfying

$$|\beta_1| < 1 < |\beta_2| < 1 < \cdots < |\beta_{d-2}|.$$ (2.4)

We say that a non-transverse intersection $X \in W^u(P) \cap W^s(Q)$ is quasi-transverse if it satisfies

$$T_X W^u(Q) + T_X W^s(P) = T_X W^s(Q) \oplus T_X W^u(P),$$

see [11]. For the non-transverse intersection $X \in W^u(P) \cap W^s(Q)$ and the heterodimensional tangency $Y \in W^u(P) \cap W^s(Q)$ of elliptic type given in (2) of theorem A, we may furthermore assume that $f$ satisfies the following situations without loss of generality. See figure 1.

Remark 2.1 (on linearizing coordinates).

(i) $X$ is located at $(1, 0, 0)$ with respect to the linearizing coordinate on $U_Q$, where $0 = (0, \ldots, 0) \in \mathbb{R}^{d-2}$. Moreover, $T_X W^u(P)$ and the eigenspace associated with $\beta_c$ are linearly independent.

(ii) $Y$ is located at $(1, 1, 0)$ with respect to the linearizing coordinate on $U_P$ where $1 = (1, \ldots, 1) \in \mathbb{R}^{d-2}$.

(iii) There exist integers $N_1, N_2 > 0$ such that $X = f^{-N_1}(X) \in W^s_{\text{loc}}(P) = (0, 0, 1)$ and $Y = f^{-N_1}(Y) \in W^u_{\text{loc}}(Q) = (0, 1, 1)$. 
We now prove the following proposition under the linearizing coordinates of remark 2.1. Note that, even if $Y \in W_{ss}(P)$ and $\tilde{Y} \in W_{uu}(Q)$, one can show similarly the following proposition by using the linearizing coordinate with $Y = (1, 0, 0)$ and $\tilde{Y} = (0, 0, 1)$, respectively.

**Proposition 2.2.** There exists a $g_1 \in \text{Diff}^2(M)$ arbitrarily $C^2$ close to the above $f \in \text{Diff}^2(M)$ such that $g_1$ has simultaneously the transverse intersection $W^{s}(P, g_1)$ containing a $(d-2)$-sphere in $W^{u}(Q, g_1)$ and a quadratic homoclinic tangency associated with $P$.

Here a $(d-2)$-sphere in $W^{u}(Q, g_1)$ means the boundary of a $C^2$ embedded $(d-1)$-disc in $W^{u}(Q, g_1)$.

For the proof of the proposition, let us prepare a suitable parametrized family in $\text{Diff}^2(M)$ containing $f$. For a sufficiently small $\delta > 0$, let $U_X$ and $U_Y$ be the $2\delta$-neighbourhoods of $X = (1, 0, 0)$ and $Y = (1, 1, 0)$ respectively. To define local perturbations of $g_1$ in $U_X$ and $U_Y$, consider the functions on $U_X$, $U_Y$ defined as

$$H_X(x, y, z) = h(x - 1)h(y) \prod_{i=3}^{n} h(z_i), \quad H_Y(x, y, z) = \prod_{i=1}^{n-2} h(x_i - 1)h(y - 1)h(z),$$

where $h$ is a $C^2$ bump function on $\mathbb{R}$ satisfying

$$h(t) = \begin{cases} 0 & \text{if } 2\delta \leq |t|; \\ 0 < h(t) < 1 & \text{if } \delta < |t| < 2\delta; \\ 1 & \text{if } |t| \leq \delta. \end{cases}$$

Fix $\delta_0 > 0$ which is sufficiently smaller than $\delta$, e.g. $\delta_0 = \delta/100$. Let $\{\varphi_{\mu, \nu}\} (-\delta_0 < \mu, \nu < \delta_0)$ be the family of perturbations in $\text{Diff}^2(M)$ given by

$$\varphi_{\mu, \nu}(x, y, z) = (x, y + \nu H_X(x, y, z), z) \quad \text{if } (x, y, z) \in U_X,$$

$$\varphi_{\mu, \nu}(x, y, z) = (x, y, z + \mu H_Y(x, y, z)) \quad \text{if } (x, y, z) \in U_Y,$$

$$\varphi_{\mu, \nu} = \text{id}_{M \setminus U_X \cup U_Y} \quad \text{otherwise.} \quad (2.5)$$

Using the $\{\varphi_{\mu, \nu}\}$, we define the 2-parameter family $\{f_{\mu, \nu}\}$ by

$$f_{\mu, \nu} = \varphi_{\mu, \nu} \circ f. \quad (2.6)$$

For the definition, it is clear that $f_{\mu, \nu} \to f$ in the $C^2$ topology as $\mu, \nu \to 0$. 

Figure 1. Heterodimensional tangency of elliptic type on a cycle.
Figure 2. Transitions from $U_X$ to $U_Y$ and $U_Y$ to $U_Y$.

Remark 2.3 (about notations).

- Since $P \not\in U_X$ and $Q \not\in U_Y$, the continuations of these points satisfy $P_{\mu,\nu} = P$ and $Q_{\mu,\nu} = Q$ for every $\mu, \nu$ with $-\delta_0 < \mu, \nu < \delta_0$.
- The (global) unstable and stable manifolds for $f_{\mu,\nu}$ of these saddle points are denoted by $W^u(P, f_{\mu,\nu})$, $W^s(P, f_{\mu,\nu})$ and so on. However, since the local unstable and stable manifolds in $U_P$ and $U_Q$ are not affected by the perturbations, these manifolds may be denoted by notations the same as the originals, e.g., $W^u_{\text{loc}}(P)$, $W^s_{\text{loc}}(P)$ and so on.
- For a small $\eta > 0$, the $\eta$-neighbourhoods of $Y$ in $W^u(Q)$ and $\tilde{Y}$ in $W^s(P)$ are denoted by $D^u_{\eta}Y$ and $D^s_{\eta}\tilde{Y}$, respectively.
- Since continuations of $D^u_{\eta}Y$ in $W^u(Q, f_{\mu,\nu})$ vary with respect to $\mu, \nu$, they should be denoted by $D^u_{\eta}(f_{\mu,\nu})$.

Proof of proposition 2.2. Let $\{f_{\mu,\nu}\}$ be the above 2-parameter family with $f_{0,0} = f$. The component $L$ of $W^u(P, f) \cap U_X$ containing $X$ is a segment not parallel to the $y$-axis of the linearizing coordinate of $U_Q$, see in remark 2.1-(i). We move $L$ by the perturbation (2.5) with respect to $\nu$ and have a continuation $L_{\nu}$ of $L$ such that $L_{\nu} \subset W^u(P, f_{0,\nu})$ and $L_{\nu} \to L$ as $\nu \to 0$. Here $L_{\nu} \to L$ means that $L_{\nu}$ $C^2$-converges to $L$.

For an arbitrarily small $\varepsilon > 0$, take an integer $n_0 > 0$ satisfying

$$|\beta_\varepsilon|^{-n_0} < \varepsilon/2 \quad \text{and} \quad K|\beta_\varepsilon|^n_0 < \varepsilon/2,$$

where $K = \|Df_{Nz}(\tilde{Y})\|$. Moreover, one can take the $n_0$ so that, for any $n > n_0$ and $\nu_n := |\beta_\varepsilon|^{-n}$, there exists a segment $\ell_{\nu_n} \subset L_{\nu_n}$ such that $f_{0,\nu_n}^{n}(\ell_{\nu_n})$ is a component of $f_{0,\nu_n}^{n}(L_{\nu_n}) \cap U_Q$ satisfying

$$\text{dist}(f_{0,\nu_n}^{n}(\ell_{\nu_n}), \tilde{Y}) < \varepsilon \quad \text{and} \quad \angle(Tf_{0,\nu_n}^{n}(\ell_{\nu_n}), TW^s_{\text{loc}}(Q)) < \varepsilon,$$

where the distance and angle are defined with respect to the Euclidean metric on $U_Q$ induced by the linearizing coordinate, see figure 2. We may assume that $D^u_{f_{0,\nu_n}^{n}}(f_{0,\nu_n}^{n}) \setminus \{Y\}$ is contained in the region of $U_P \setminus W^s_{\text{loc}}(P)$ with $\varepsilon < 0$.

We set $\hat{\ell}_{\nu_n} := f_{0,\nu_n}^{N_{\varepsilon}} \circ f_{0,\nu_n}^{n}(\ell_{\nu_n})$ and first consider the case that $\hat{\ell}_{\nu_n}$ is disjoint from $W^u_{\text{loc}}(P)$ as in figure 2. Then, we will adjust a value of the other parameter $\mu$ as follows so that the claim of this proposition holds. Let $\hat{\ell}_{\mu,\nu_n}$ be a continuation of $\hat{\ell}_{\nu_n}$ such that $\hat{\ell}_{\mu,\nu_n} \subset W^u(P, f_{\mu,\nu_n})$ and $\hat{\ell}_{\mu,\nu_n} \to \hat{\ell}_{\nu_n}$ as $\mu \to 0$. By the perturbation (2.5) with respect to $\mu$, any point in $U_Y$ close to $Y$ is just moved in parallel with the $z$-axis of $U_P$. Thus, by the intermediate value theorem,
there exists a \( \mu_n \) such that
- \( 0 < |\mu_n| < 2K|\beta^s|\); 
- \( \tilde{\ell}_{\mu_n,\nu_n} \) and \( W^s_{\text{loc}}(P) \) have a quadratic tangency; 
- \( D^s_{Y}(f_{\mu_n,\nu_n}) \) meets \( W^s_{\text{loc}}(P) \) nontrivially and transversely.

Moreover, since \( Y \) is a tangency of elliptic type (remark 2.1-(ii)), \( D^s_{Y}(f_{\mu_n,\nu_n}) \) \( \triangleleft \) \( W^s_{\text{loc}}(P) \) is a \((d-2)\)-sphere in \( W^u(Q, f_{\mu_n,\nu_n}) \cap W^s_{\text{loc}}(P) \).

Next we consider the case that \( \tilde{\ell}_{\nu_n} \) intersects \( W^s_{\text{loc}}(P) \). For \(-1 \leq a < b \leq 1\), let \( A_{a,b} \) be the open subset of \( U_P \) defined as \( A_{a,b} = \{(x, y, z) \in U_P ; a < z < b\} \). Since \( P \) is contained in the basic set \( A_{1,1} \) for any \( 0 < t < 1 \), there exists a continuation of subsurfaces \( H_{\mu,\nu} \) of \( W^s(P, f_{\mu,\nu}) \) and \( W^s_{\text{loc}}(P) \) which are almost horizontal and contained in either \( A_{0,t} \) or \( A_{1-t,0} \) for any \((\mu, \nu)\) sufficiently near \((0, 0)\). Since \( \tilde{\ell}_{\nu_n} \) converges to an arc in \( D^s_{Y} \) as \( n \to \infty \), \( \tilde{\ell}_{\nu_n} \) is contained in \( A_{-1,t} \) and moreover disjoint from \( H_{0,\nu} \) in the case of \( H_{0,\nu} \subset A_{0,t} \) for all sufficiently large \( n \).

When \( H_{0,\nu} \subset A_{0,t} \), there exists \( \mu_n \) with \( 0 < \mu_n < t \) such that \( \tilde{\ell}_{\mu_n,\nu_n} \) and \( H_{\mu_n,\nu_n} \) have a quadratic tangency and \( D^s_{Y}(f_{\mu_n,\nu_n}) \) \( \triangleleft \) \( W^s_{\text{loc}}(P) \) is a \((d-2)\)-sphere, see figure 3(a). When \( H_{0,\nu} \subset A_{-1,0} \), one can choose \( \mu_n \) with \(-t < \mu_n < 0\) so that \( \tilde{\ell}_{\mu_n,\nu_n} \) and \( W^s_{\text{loc}}(P) \) have a quadratic tangency and \( D^s_{Y}(f_{\mu_n,\nu_n}) \) \( \triangleleft \) \( H_{\mu_n,\nu_n} \) is a \((d-2)\)-sphere, see figure 3(b). Since \( t \) can be taken arbitrarily small, we may choose \( \mu_n \) so that \( \lim_{n \to \infty} \mu_n = 0 \).

Thus, in ether case, \( g_1 := f_{\mu_n,\nu_n} \) satisfies our desired conditions. This completes the proof. \( \square \)

3. Stable and unstable thicknesses

In this section, we will recall the definition of the thickness given in Newhouse [16] for a Cantor set \( K \) in \( \mathbb{R} \). Let \( I \) be the minimal interval containing \( K \). A gap of \( K \) is a connected component of \( I \setminus K \). An ordering \( G = \{G_n\} \) of the gaps is called a presentation of \( K \). For any
$x \in \partial G_p$, the $G$-component of $K$ at $x$ is the connected component $C$ of $I \setminus (G_1 \cup \cdots \cup G_n)$ containing $x$. For each such $x$, set $\tau(K, G, x) = \text{Length}(C)/\text{Length}(G)$. Then the thickness of $K$ is given by

$$\tau(K) = \sup_{\psi} \inf_x \tau(K, \psi, x),$$

where the infimum is taken over all boundary points of gaps of $K$. The local thickness of $K$ at $x \in K$ is defined as

$$\tau(K, x) = \limsup_{\varepsilon \to 0} \left\{ \tau(L); L \subset K \cap [x - \varepsilon, x + \varepsilon] \text{ a Cantor set} \right\}.$$ 

The notion of thickness can be extended to that of a nontrivial basic set $\Lambda = \Lambda_\psi$ of index 1 as follows. Let $z$ be a point of $W^s_{\text{loc}}(\Lambda)$ and $\pi : I \to M$ a $C^1$ embedding transverse to $W^s_{\text{loc}}(\Lambda)$ at $z = \pi(0)$, where $I$ is a closed interval containing 0 as an interior point. Actually, $C^1$ projections along leaves of the stable $C^1$ foliation of $\Lambda$ can be used to define $\pi$. The local stable thickness of $\Lambda$ at $z$ is $\tau^s(\Lambda, z) := \tau(\pi^{-1}(W^s_{\text{loc}}(\Lambda)), 0)$. Note that $\tau^s(\Lambda, z)$ is independent of the choice of $\pi$ and has the identical value for every $z \in W^s_{\text{loc}}(\Lambda)$ which is a strictly positive finite number. Thus we may denote it simply by $\tau^s(\Lambda)$ and call the (local) stable thickness of $\Lambda$. On the other hand, it depends continuously on the diffeomorphism, that is, for any $\varphi_1$ sufficiently $C^2$-close to $\varphi_2$, $|\tau^s(\Lambda_{\varphi_1}) - \tau^s(\Lambda_{\varphi_2})|$ is smaller than a given positive constant, see [17, section 4.3]. The local unstable thicknesses $\tau^u(\Lambda')$ of a basic set $\Lambda'$ with index($\Lambda') = d - 1$ is defined similarly.

Let $\Lambda, \Gamma$ be nontrivial basic sets with

$$\text{index}(\Lambda) = d - 1, \quad \text{index}(\Gamma) = 1, \quad (3.1)$$

and let $\gamma : I \to M$ be a $C^1$ embedding transverse to $W^u(p)$ and $W^s(q)$ for some points $p \in \Lambda$ and $q \in \Gamma$. Since $\Lambda$ is a basic set, there exists a Cantor set $K_u$ in some open subinterval $J$ of $I$ and an open segment $\alpha$ in $W^u(p, \Lambda)$ such that $(\gamma(J), \gamma(K_u))$ is $C^1$ diffeomorphic to $(\alpha, \Lambda \cap \gamma)$ along an unstable foliation associated with $W^u(\Lambda)$. A Cantor set $K^s$ in $I$ is defined similarly from $\Gamma$. We say that $K_u$ and $K^s$ are linked if $K^s$ is not contained in a single component of $\mathbb{R} \setminus K_u$ and $K_u$ is not contained in a single component of $\mathbb{R} \setminus K^s$. By the gap lemma ([16, 17]), if $\tau^u(\Lambda) \tau^s(\Gamma) > 1$ and $K_u$ and $K^s$ are linked, then $K_u \cap K^s \neq \emptyset$.

Note that the above explanation could not be applied directly to the basic sets without the condition (3.1). But, under the (codimension-one) sectionally dissipative condition, one can define the stable thickness of $\Lambda$ with the same property as above by using intrinsically $C^1$ projections along the leaves of an intrinsically $C^1$ foliation of $\Lambda$, see [18, section 2-4] for details.

4. Palis–Viana’s setting and heterodimensional tangencies

In what follows, to simplify notations, we denote continuations of the saddles $P, Q$ and the basic sets $\Lambda_0$, $\Lambda_1$ of every perturbed diffeomorphism near $f$ by the same notations as the original ones if it does not cause any confusion.

Let us start with the $C^2$ diffeomorphism $g_1$ which is arbitrarily $C^2$ close to $f$ given in proposition 2.2 and satisfies the setting in Palis–Viana [18]. In fact, $g_1$ has a quadratic homoclinic tangency in $U_f$ associated with the saddle point $P$ satisfying the sectionally dissipative condition (2.2). By Palis–Viana [18], we have the following result.
**Proposition 4.1 (Steps 1–3 in [18, section 7]).** There exists a $g_2 \in \text{Diff}^2(M)$ arbitrarily $C^2$ close to $g_1$ which has a basic set $\Lambda_2$ of index 1 other than $\Lambda_1$ such that

1. $\tau^u(\Lambda_1)\tau^s(\Lambda_2) > 1$;
2. $\Lambda_1$ and $\Lambda_2$ are homoclinically related;
3. There are periodic points $P_1 \in \Lambda_1$, $P_2 \in \Lambda_2$ such that $W^u(P_1, g_2)$ and $W^s(P_2, g_2)$ have a quadratic tangency. □

Actually, the above basic set $\Lambda_2$ is obtained by a renormalization of return maps defined on $U_Y$ which converge to Hénon-like endomorphisms whose stable thickness of the invariant expanding Cantor set is arbitrarily large if $g_2$ is sufficiently close to $g_1 = f_{\mu_n, \nu_n}$, see [18, section 6]. Hence, one can suppose that

\[
\tau^u(\Lambda_1)\tau^s(\Lambda_2) > 1 \quad (4.1)
\]

holds, where $\Lambda_0 = \Lambda_{0, g_2}$ is the continuation of the original basic set $\Lambda_0$ in theorem A.

Now we will show that there exists a diffeomorphism $g_3$ arbitrarily $C^2$ close to $g_2$ and such that $W^u(\Lambda_0, g_2) \cap W^s(\Lambda_2, g_3)$ contains a heterodimensional tangency of elliptic type.

For a given $q \in \Lambda_0$, a compact subset $A$ of $W^u(q, g_2)$ is called an unstable cylinder with foliation $F$ if there exists a $C_1$ diffeomorphism $h : [0, 1] \times S^{d-2} \rightarrow A$ with $F = \{h(t) \times S^{d-2} : 0 \leq t \leq 1\}$. For $p \in \Lambda_1$, we say that a sequence of unstable cylinders $A_n$ in $W^u(q, g_2)$ with foliations $F_n$ $C^1$ converges to an arc $\alpha$ in $W^u(p, g_2)$ if it satisfies the following conditions.

- The diameter of each leaf of $F_n$ is less than a constant $\varepsilon_n > 0$ with $\lim_{n \rightarrow \infty} \varepsilon_n = 0$.
- There exist $C^1$ sections $\sigma_n : [0, 1] \rightarrow A_n \subset M$ meeting all leaves of $F_n$ transversely and $C^1$ converging to an embedding $\sigma_\infty : [0, 1] \rightarrow M$ with $\sigma_\infty([0, 1]) = \alpha$.

**Lemma 4.2.** Let $g_3$ be the diffeomorphism in proposition 4.1. For any periodic points $q \in \Lambda_0$, $p \in \Lambda_1$, there exists a sequence of foliated unstable cylinders $A^u_n$ in $W^u(q, g_2)$ $C^1$ converging to an arc in $W^u_{\text{loc}}(p, g_2)$ containing $p$ as an interior point.

**Proof.** To make the proof clear, we first show the lemma for the fixed points $Q \in \Lambda_0$, $P \in \Lambda_1$. Since $g_2$ is sufficiently $C^2$ close to $g_1$, proposition 2.2 implies that $W^u_{\text{loc}}(P, g_2) \cap W^u(Q, g_2)$ contains a $(d-2)$-sphere $S^{d-2}_0$. Let $A^u_0$ be a closed tubular neighbourhood of $S^{d-2}_0$ in $W^u(Q, g_2)$ with foliation $F_0$, each leaf of which is the intersection of $A^u_0$ and the level surface $z = t$ with...
there exists a sub-cylinder \( A^u_n \) of the unstable cylinder \( g^2_n(A^u_n) \) with \( A^u_n \supset g^2_n(S^{d-2}_0) \) such that the boundary \( \partial A^u_n \) is contained in the union of the level surfaces \( z = \pm 1 \). The foliation \( F_n \) on \( A^u_n \) is defined similarly as \( X_0 \). By (2.1), we conclude that the sequence of \( A^u_n \) converges to the arc \( \alpha = (0, 0) \times [-1, 1] \) in \( W^u_{loc}(P, g_2) \). Note that \( P = (0, 0) \times [0] \) is an interior point of \( \alpha \).

Next, we observe that the assertion of the lemma holds for any periodic points \( q \in A_0 \) and \( p \in \Lambda_1 \). Since \( W^u(q, g_2) \cap W^s(Q, g_2) \neq \emptyset \) and \( W^s(p, g_2) \cap W^u(P, g_2) \neq \emptyset \), by the inclination lemma, there exists a \((d - 1)\)-dimensional disc in \( W^s(q, g_2) \) (respectively \( W^u(p, g_2) \)) which is arbitrarily \( C^1 \)-close to \( W^u_{loc}(Q, g_2) \) (respectively \( W^s_{loc}(P, g_2) \)). Thus, \( W^s(p, g_2) \cap W^u(q, g_2) \) contains a \((d - 2)\)-sphere \( S^{d-2}_0 \) arbitrarily \( C^1 \)-close to \( S^{d-2}_0 \). Moreover, one has a closed tubular neighbourhood \( \tilde{A}^u_0 \) of \( S^{d-2}_0 \) in \( W^u(q, g_2) \) which is arbitrarily \( C^1 \)-close to \( A^u_0 \). Since \( W^s(p, g_2) \cap W^u(P, g_2) \neq \emptyset \), the unstable cylinder \( g^2_n(A^u_0) \) for sufficiently large \( m \) and \( W^u_{loc}(p, g_2) \) has a transverse \((d - 2)\)-sphere intersection. Thus, by a similar argument to the first case, one can obtain sub-cylinders \( \tilde{A}^u_0 \) of \( g^{m+n+1}(\tilde{A}^u_0) \) which \( C^1 \)-converges to the arc \( \tilde{\alpha} \) containing \( p \) in \( W^u_{loc}(p, g_2) \).

\( \square \)

**Proposition 4.3.** There exists a \( g_3 \in \text{Diff}^2(M) \) arbitrarily \( C^2 \) close to \( g_2 \) of proposition 4.1 with the saddle periodic points \( P, Q \) and the continuations of the basic sets \( \Lambda_0, \Lambda_1, \Lambda_2 \) such that

1. \( \Lambda_1 \) and \( \Lambda_2 \) are homoclinically related;
2. \( Q \in A_0, P \in \Lambda_1; \)
3. \( W^u(\Lambda_0, g_3) \cap W^s(\Lambda_2, g_3) \) contains a heterodimensional tangency \( r \) of elliptic type.

**Proof.** The first and second claims are obtained immediately from (2) of proposition 4.1. By (3) of proposition 4.1, there exist periodic points \( P_1 \in \Lambda_1, P_2 \in \Lambda_2 \) and an arc \( L^u \) in \( W^u(P_1, g_2) \) which has a quadratic tangency \( R \) with \( W^s_{loc}(P_2, g_2) \). If necessary replacing \( L^u \) by a shorter arc, we may assume that \( L^u \) is contained entirely in one component of \( U_R \setminus W^s_{loc}(P_2, g_2) \), say in the lower component, where \( U_R \) is the \( \eta \)-neighbourhood of \( R \) in \( M \) with small \( \eta > 0 \). By lemma 4.2, there exists a sequence of unstable cylinders \( A^u_n \) in \( W^u(Q, g_2) \) \( C^1 \) converging to an arc \( \alpha \) in \( W^u_{loc}(P_1, g_2) \) containing \( P_1 \) as an interior point. Then we have a subarc \( \alpha' \) of \( \alpha \) and \( m \in \mathbb{N} \) such that \( g^m_n(\alpha') = L^u \). Let \( A^u_m \) be sub-cylinders of \( A^u \) \( C^1 \) converging to \( \alpha' \). Then \( A^u_m = g^m_n(A^u) \) are unstable cylinders in \( W^u(Q, g_2) \) \( C^1 \) converging to \( L^u \), see figure 5.

Note that the tangency between \( L^u \) and \( W^s(P_2, g_2) \) unfolds generically with respect to the parameter of the Hénon-like family given by the renormalization in proposition 4.1 as

![Figure 5. Creation of heterodimensional tangency.](image-url)
in [18, section 6 and step 3 of section 7]. Thus, by controlling the parameter and applying the intermediate value theorem, one can get a $g_3 \in \text{Diff}^2(M)$ arbitrarily close to $g_3$ such that the continuation $A'^n$ of $A^0$ in $W^u(Q, g_3)$ has a tangency $r$ with $W^s(P_2, g_3)$ and $A'^n \setminus \{r\}$ lies in the lower component of $U_R \setminus W^s_{\text{loc}}(P_2, g_3)$ if we take $n$ sufficiently large. In particular, all the eigenvalues of the Hessian matrix of $A'$ at $r$ relative to $W^s(P_2, g_3)$ are non-positive. Slightly modifying $g_3$ by $C^2$ perturbation if necessary, we may suppose that all these eigenvalues are strictly negative. It follows that the tangency $r$ is of elliptic type. Thus the proof is complete. $$\square$$

The assertion of proposition 4.3 (1) will be used to prove theorem B, see remark 6.1.

Consider the unstable manifold $W^u(q, g_3)$ associated with $q \in \Lambda_0$. Let $\gamma : I = (-\varepsilon, \varepsilon) \to M$ be a short $C^1$ regular curve meeting $W^u(q, g_3)$ transversely at $\gamma(0)$. Since $\Lambda_0$ is a nontrivial basic set of index $d - 1$, there exists a Cantor set $K_{0, g_3}$ in $I$ with $K_{0, g_3} \ni 0$ which is defined as $K^\delta$ in section 3. We say that $W^u(q, g_3)$ is two-sided if $(-\eta, 0) \cap K_{0, g_3} \neq \emptyset$ and $(0, \eta) \cap K_{0, g_3} \neq \emptyset$ for any $0 < \eta < \varepsilon$. The two-sided stable manifold $W^s(p, g_3)$ with $p \in \Lambda_2$ is defined similarly. Since a Cantor set is perfect by definition, there exists $t \in K_{0, g_3}$ arbitrarily close to 0 (possibly $t = 0$) such that the unstable manifold $W^u(q', g_3)$ for a $q' \in \Lambda_0$ with $W^u(q', g_3) \ni \gamma(t)$ is two-sided.

If necessary modifying the diffeomorphism $g_3$ given in proposition 4.3 slightly, one can suppose that $g_3$ satisfies the following.

**Corollary 4.4.** The unstable manifolds $W^u(q, g_3)$ in $W^u(\Lambda_0, g_3)$ and the stable manifold $W^s(p, g_3)$ in $W^s(\Lambda_2, g_3)$ containing the heterodimensional tangency $r$ are two-sided.

**Proof.** Any $C^1$ regular curve in $M$ meeting $W^u(q, g_3)$ with $q \in \Lambda_0$ transversely at $r$ meets also $W^s(p, g_3)$ with $p \in \Lambda_2$ transversely at $r$. By using the curve, one can show that there exist $q' \in \Lambda_0$ and $p' \in \Lambda_2$ such that both $W^u(q', g_3)$ and $W^s(p', g_3)$ are two-sided and intersect arbitrarily small neighbourhood $U$ of $r$ in $M$. Thus, we have a $g_3 \in \text{Diff}^2(M)$ arbitrarily $C^2$ close to $g_3$ such that $W^u(q', g_3)$ and $W^s(p', g_3)$ have a heterodimensional tangency $r'$ of elliptic type in $U$. The proof is completed by setting $p' = p$, $q' = q$, $g_3' = g_3$, $r' = r$. $$\square$$

5. Arc of tangencies of stable and unstable foliations

Recall that $M$ is a closed $C^2$ manifold with dim $M = d \geq 3$. We denote the diffeomorphism $g_3$ obtained in proposition 4.3 by $g$ for simplicity. Thus, $g \in \text{Diff}^2(M)$ has nontrivial basic sets $\Lambda_0$ and $\Lambda_2$ with index($\Lambda_0$) $= d - 1$ and index($\Lambda_2$) $= 1$. Moreover, $W^u(\Lambda_0, g)$ and $W^s(\Lambda_2, g)$ have a heterodimensional tangency. Let $\mathcal{F}$ and $\tilde{\mathcal{F}}$ be stable and unstable foliations associated with $\Lambda_0$ and $\Lambda_2$. Note that $W^u_{\text{loc}}(\Lambda_0)$ and $W^s_{\text{loc}}(\Lambda_2)$ are considered to be sublaminations of $\mathcal{F}$ and $\tilde{\mathcal{F}}$ respectively. Both $\mathcal{F}$, $\tilde{\mathcal{F}}$ are $C^1$ foliations on $M$, but each leaf of $\mathcal{F}$ or $\tilde{\mathcal{F}}$ is a codimension-one $C^2$ submanifold of $M$. The aim of this section is to show that, under suitable conditions, there exists a regular $C^1$ curve $\gamma : (-\varepsilon, \varepsilon) \to M$ which meets leaves of both $\mathcal{F}$ and $\tilde{\mathcal{F}}$ transversely and such that, for any $t \in (-\varepsilon, \varepsilon)$, $\gamma(t)$ is a non-degenerate heterodimensional tangency between leaves of $\mathcal{F}$ and $\tilde{\mathcal{F}}$.

Suppose that there exist leaves $\lambda_0$ of $\mathcal{F}$ and $\tilde{\lambda}_0$ of $\tilde{\mathcal{F}}$ which have a strict tangency at a point $p$ in $M$. A small open neighbourhood $U$ of $p$ in $M$ has a $C^2$ coordinate $(x, z)$ with $p = (0, 0)$ such that the leaf $\lambda_0$ is contained in the level surface $z = 0$, where $x = (x_1, \ldots, x_{d-1})$. Let $\theta = (0, \ldots, 0) \in \mathbb{R}^{d-1}$. This gives the identification of $U$ with an open neighbourhood of $(0, 0)$ in $\mathbb{R}^{d}$. 

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Since $\mathcal{F}$ is a $C^1$ foliation on $M$ with $C^2$ leaves, for a sufficiently small $\delta > 0$, there exists a $C^1$ diffeomorphism

$$\varphi_{\delta} : (-\delta, \delta)^d = (-\delta, \delta)^{d-1} \times (-\delta, \delta) \to U$$

with $\varphi_{\delta}(x, z) = (x, \sigma(x, z))$ and satisfying the following conditions, where we set $\varphi_{\delta} = \varphi$ for short.

- $\sigma(x, z)$ is a $C^2$ map on $x$ and $C^1$ on $z$.
- $\varphi(x, 0) = (x, 0)$ for any $x \in (-\delta, \delta)^{d-1}$.
- For any $z \in (-\delta, \delta)$, $\varphi((x, \sigma(x, z))$ is contained in a leaf of $\mathcal{F}$.

For any $(x, z) \in (-\delta, \delta)^d$, let

$$\tilde{N}(x, z) = (\mu(x, z), v(x, z)) = (\mu_1(x, z), \ldots, \mu_{d-1}(x, z), v(x, z))$$

be the unit tangent vector of $T_{\varphi(x, z)}(U)$ orthogonal to the leaf of $\mathcal{F}$ containing $x, z$ and such that the $d$-th entry $v(x, z)$ of $\tilde{N}(x, z)$ is positive, see figure 6. Consider the $C^1$ map $\psi_{\delta} = \psi : (-\delta, \delta)^d \to \mathbb{R}^{d-1}$ defined as

$$\psi(x, z) = (\tilde{N}(x, z) \cdot \partial_{\alpha_1} \psi(x, z), \ldots, \tilde{N}(x, z) \cdot \partial_{\alpha_d} \psi(x, z)).$$

Then the Jacobian matrix of $\psi(x, z)$ is

$$J \psi(x, z) = \begin{pmatrix} a_{1,1}(x, z) & \cdots & a_{1,d-1}(x, z) & b_1(x, z) \\ \vdots & \ddots & \vdots & \vdots \\ a_{d-1,1}(x, z) & \cdots & a_{d-1,d-1}(x, z) & b_{d-1}(x, z) \end{pmatrix},$$

where

$$a_{i,j}(x, z) = \partial_{x_j} \tilde{N}(x, z) \cdot \partial_{\alpha_i} \psi(x, z) + \tilde{N}(x, z) \cdot \partial_{x_j} \partial_{\alpha_i} \psi(x, z),$$

$$b_j(x, z) = \partial_{z_j} \tilde{N}(x, z) \cdot \partial_{\alpha_j} \psi(x, z) + \tilde{N}(x, z) \cdot \partial_{z_j} \partial_{\alpha_j} \psi(x, z).$$

Since $\tilde{N}(0, 0) = (0, 1)$, $\partial_{x_j} \psi(0, 0) = (0, \ldots, 0, 1, 0, \ldots, 0)$, $\partial_{z_j} \psi(0, 0) = (0, 0, \ldots, 0)$ and $\partial_{x_i} \partial_{x_j} \psi(0, 0) = (0, \partial_{x_i} \partial_{x_j} \psi(0, 0))$, we have

$$J \psi(0) = \begin{pmatrix} \partial_{x_1} \mu_1(0) & \cdots & \partial_{x_{d-1}} \mu_1(0) & \partial_{x_1} \mu_1(0) + \partial_{x_1} x(0) \\ \vdots & \ddots & \vdots & \vdots \\ \partial_{x_1} \mu_{d-1}(0) & \cdots & \partial_{x_1} \mu_{d-1}(0) & \partial_{x_1} \mu_{d-1}(0) + \partial_{x_1} x(0) \end{pmatrix},$$

where $x(0) = (0, 0) \in \mathbb{R}^d$.

Let $h : (-\delta, \delta)^{d-1} \to \mathbb{R}$ be a $C^2$ function such that the graph $\{(x, h(x)) : x \in (-\delta, \delta)^{d-1}\}$ of $h$ is contained in the leaf $\tilde{\lambda}_0$ of $\tilde{F}$. For any $x \in (-\delta, \delta)^{d-1}$ and $i = 1, \ldots, d-1$, the vector $(0, \ldots, 0, 1, 0, \ldots, 0, \partial_{x_i} h(x, 0))$ is tangent to $\tilde{\lambda}_0$ at $(x, h(x))$, see figure 6. It follows that

$$\tilde{N}(x, h(x)) \cdot (0, \ldots, 0, 1, 0, \ldots, 0, \partial_{x_i} h(x)) = \mu_i(x, h(x)) + v(x, h(x)) \partial_{x_i} h(x) = 0.$$
Differentiating the latter equation by $x_j$ ($j = 1, \ldots, d - 1$), we have
\[
\partial_{x_j} \mu_i(x, h(x)) + \partial_{x_i} \mu_j(x, h(x)) \partial_{x_j} h(x) \\
= -\left(\partial_{x_j} v(x, h(x)) + \partial_{x_i} v(x, h(x)) \partial_{x_j} h(x)\right) \partial_{x_j} h(x) \\
- v(x, h(x)) \partial_{x_j} x_j h(x).
\]
Since $h(0) = 0$, $\partial_{x_j} h(0) = 0$ and $v(0, 0) = 1$,
\[
\partial_{x_i} \mu_j(0, 0) = -\partial_{x_j} x_j h(0). \tag{5.3}
\]

**Proposition 5.1.** With the notation as above, the following two conditions are equivalent.

1. The strict tangency of $\lambda_0$ and $\tilde{\lambda}_0$ at $p = (0, 0)$ in $U$ is non-degenerate.
2. There exists a $C^1$ regular curve $\gamma_0 = \gamma : (-\varepsilon, \varepsilon) \rightarrow U$ for a sufficiently small $\varepsilon > 0$ with $\gamma(0) = p$ and such that, for each $t \in (-\varepsilon, \varepsilon)$, the curve $\gamma(t) = (\psi(t), \varepsilon)$ meets leaves $\lambda_t$ of $\mathcal{F}$ and $\tilde{\lambda}_t$ of $\mathcal{F}$ in $\gamma(t)$ transversely. Moreover, $\lambda_t$ and $\tilde{\lambda}_t$ have a non-degenerate strict tangency at $\gamma(t)$.

**Proof.** Since ‘(2) ⇒ (1)’ is obvious, we prove ‘(1) ⇒ (2)’.

Suppose that the tangency of $\lambda_0$ and $\tilde{\lambda}_0$ at $p = (0, 0)$ is non-degenerate, or equivalently that
\[
\det(\partial_{x_j} x_j h(0)) \neq 0. \tag{5.4}
\]
Let $\Psi = \psi : (-\delta, \delta)^d \rightarrow \mathbb{R}^d$ be the $C^1$ map defined as
\[
\Psi(x, z) = (\psi(x, z), z) = (\tilde{\mathcal{N}}(x, z) \cdot \partial_{x_j} \varphi(x, z), \ldots, \tilde{\mathcal{N}}(x, z) \cdot \partial_{x_j} \varphi(x, z), z). \tag{5.5}
\]
By (5.2) and (5.3), the Jacobian matrix of $\Psi$ at $(0, 0)$ has the form
\[
J\Psi(0, 0) = \begin{pmatrix}
-\partial_{x_1} h(0) & \cdots & -\partial_{x_d} h(0) & \partial_{x_1} \mu_1(0, 0) + \partial_{x_1} \alpha(0, 0) \\
\vdots & \ddots & \vdots & \vdots \\
-\partial_{x_d} h(0) & \cdots & -\partial_{x_d} h(0) & \partial_{x_d} \mu_d(0, 0) + \partial_{x_d} \alpha(0, 0) \\
0 & \cdots & 0 & 1
\end{pmatrix}.
\]
Then (5.4) implies that $J\Psi(0, 0)$ is regular. By the inverse function theorem, there exists a $C^1$ local inverse $\Psi^{-1} : (-\delta, \delta)^d \rightarrow (-\delta, \delta)^d$ of $\Psi$ with $\Psi^{-1}(0, 0) = (0, 0)$ for a sufficiently small $\varepsilon > 0$. Consider the $C^1$ map $\gamma : (-\varepsilon, \varepsilon) \rightarrow U$ defined by $\gamma(t) = \varphi^{-1}(0, t)$. Since the $d$-th entry of $\partial_{x_i} \Psi^{-1}(0, t)$ is one, the map $\Psi^{-1}(0, t)$ of $t$ defines a regular curve in $(-\delta, \delta)^d$ passing through the $\varepsilon$-constant level surfaces transversely. This implies that $\gamma(t)$ is a regular curve in $U$ transverse to leaves of $\mathcal{F}$. Let $\lambda_t$ (respectively $\tilde{\lambda}_t$) be the leaf of $\mathcal{F}$ (respectively $\tilde{\mathcal{F}}$) containing $\gamma(t)$. From the definition (5.5) of $\Psi$, we know that the normal vector $\tilde{\mathcal{N}}(\Psi^{-1}(0, t))$ of $\tilde{\lambda}_t$ at $\gamma(t)$ is orthogonal to the tangent vectors $\partial_{x_i} \varphi(\Psi^{-1}(0, t))$ ($i = 1, \ldots, d - 1$) of $\lambda_t$. This shows that $\lambda_t$ is tangent strictly to $\tilde{\lambda}_t$ at $\gamma(t)$. In particular, the curve $\gamma(t)$ meets $\tilde{\lambda}_t$ transversely at $\gamma(t)$ as well as it does $\lambda_t$.

From the continuity of the matrix $J\Psi$, one can assume that the strict tangency of $\lambda_t$ and $\tilde{\lambda}_t$ is non-degenerate for any $t \in (-\varepsilon, \varepsilon)$ if necessary replacing $\varepsilon$ by a smaller positive constant. This completes the proof. \[\square\]

**Corollary 5.2.** If the condition (1) in proposition 5.1 holds, then there is an open neighbourhood $\mathcal{O}_\varepsilon$ of $g$ in $\text{Diff}^\infty(M)$ such that, for any $\tilde{g}$ in $\mathcal{O}_\varepsilon$, there exists a $C^1$ regular curve $\gamma_0 : (-\varepsilon, \varepsilon) \rightarrow M$ satisfying the condition on $\tilde{g}$ corresponding to (2) of proposition 5.1 and depending on $\tilde{g} \in \mathcal{O}_\varepsilon$ continuously.
Proof. According to propositions 1 and 2 in Pollicott [20], there exists a small open neighbourhood \( \mathcal{O}_g \) of \( g \) in \( \text{Diff}^2(M) \) such that, for any \( \tilde{g} \in \mathcal{O}_g \), there are stable and unstable foliations associated, respectively, with \( \Lambda_{0,\tilde{g}} \) and \( \Lambda_{2,\tilde{g}} \). The condition corresponding to (2) of proposition 5.1 and depends continuously on \( \tilde{g} \in \mathcal{O}_g \), where \( \varphi_{\tilde{g}} : (\varepsilon, \varepsilon) \to U \) is a diffeomorphism defined from \( \tilde{g} \) as (5.1).

6. Proof of theorem A

Proof of theorem A. By combining results which have been obtained before this section, we have a sequence \( \{f_n\} \) in \( \text{Diff}^2(M) \) \( C^2 \) converging to \( f \) and satisfying the following conditions (i)–(iv):

(i) By proposition 4.3, each \( f_n \) has saddle periodic points \( Q_{f_n} \), \( P_{f_n} \) and nontrivial basic sets \( \Lambda_{0,f_n}, \Lambda_{1,f_n}, \Lambda_{2,f_n} \) such that \( Q_{f_n} \in \Lambda_{0,f_n}, P_{f_n} \in \Lambda_{1,f_n}, \text{index}(\Lambda_{0,f_n}) = d - 1, \text{index}(\Lambda_{1,f_n}) = 1, \text{and} \Lambda_{1,f_n} \) and \( \Lambda_{2,f_n} \) are homoclinically related, where \( \Lambda_{0,f_n} \) is the continuation of the basic set \( \Lambda_0 = \Lambda \) in theorem A. Moreover, \( W^u(\Lambda_{0,f_n}, f_n) \) and \( W^s(\Lambda_{2,f_n}, f_n) \) have a heterodimensional tangency \( r_n \) of elliptic type.

(ii) By corollary 4.4, we may assume that both the leaves of \( W^u(\Lambda_{0,f_n}, f_n) \) and \( W^s(\Lambda_{2,f_n}, f_n) \) containing \( r_n \) are two-sided.

(iii) By proposition 5.1 and corollary 5.2, there is an open neighbourhood \( \mathcal{O}_n \) of \( f_n \) in \( \text{Diff}^2(M) \) such that, for any \( g \in \mathcal{O}_n \), there exists a \( C^1 \) regular curve \( \gamma_{f_n} : (\varepsilon, \varepsilon) \to M \) which satisfies the condition corresponding to (2) of proposition 5.1 and depends continuously on \( g \in \mathcal{O}_n \). Furthermore, in the case of \( g = f_n \), the curve \( \gamma_{f_n} \) satisfies \( \gamma_{f_n}(0) = r_n \).

(iv) By (4.1) together with the local continuity of thickness in the \( C^2 \) topology,

\[ \tau^u(\Lambda_{0,g}) \tau^s(\Lambda_{2,g}) > 1 \]

for any \( g \in \mathcal{O}_n \) where \( \Lambda_{0,g}, \Lambda_{2,g} \) are the continuations of \( \Lambda_{0,f_n}, \Lambda_{2,f_n} \) respectively.

Let \( K_{0,g} \) and \( K_{2,g} \) be Cantor sets defined as \( K^u \) and \( K^s \) in section 3. By (ii) and (iii), we may assume that \( K_{0,g} \) and \( K_{2,g} \) are linked for any \( g \in \mathcal{O}_n \) if necessary replacing \( \mathcal{O}_n \) by a smaller open neighbourhood of \( f_n \). Since \( \tau(K_{0,g}) \tau(K_{2,g}) > 1 \) by (iv), the gap lemma [16, 17] implies that \( K_{0,g} \cap K_{2,g} \neq \emptyset \) for any \( g \in \mathcal{O}_n \). For any \( t \in K_{0,g} \cap K_{2,g}, \gamma_{g}(t) \) is a heterodimensional tangency of elliptic type associated with \( \Lambda_{0,g} \) and \( \Lambda_{2,g} \). Thus the union \( \mathcal{O} = \bigcup_{n=1}^{\infty} \mathcal{O}_n \) is an open set of \( \text{Diff}^2(M) \) whose closure contains \( f \) and such that each \( g \in \mathcal{O} \) has a heterodimensional tangency of elliptic type associated with the basic sets \( \Lambda_{0,g} \) and \( \Lambda_{2,g} \). This completes the proof of theorem A.

Remark 6.1. By proposition 4.3 (1), one can choose the open set \( \mathcal{O}_n \) in the proof of theorem A so that, for any \( g \in \mathcal{O}_n, \Lambda_{1,g} \) is homoclinically related to \( \Lambda_{2,g} \).

7. Proof of theorem B

To prepare for the proof of theorem B, let us introduce several definitions in the \( C^2 \) topology. Let \( f \in \text{Diff}^2(M) \) with \( \dim M = d \geq 3 \). A nontrivial basic set \( \Lambda_0 \) of index \( \text{index}(\Lambda_0) = d - 1 \) is a cu-blender for \( f \) if there exist a neighbourhood \( \mathcal{U}_f \subset \text{Diff}^2(M) \) of \( f \) and a \( C^1 \) open set \( D \) of \( (d - 2) \)-dimensional discs \( \mathcal{D} \) embedded in \( M \) such that for every \( g \in \mathcal{U}_f \) and every \( D \in \mathcal{D}, \)

\[ W^s_{\text{loc}}(\Lambda_{0,g}, g) \cap D \neq \emptyset. \]
The set $D$ is called the superposition region of the cu-blender. See definition 3.1 in [5, 6].

In fact, the cu-blenders considered in previous works of Bonatti–Díaz belong to a special class of blenders, called blender-horseshoes, see [3, section 1], [5, definition 3.8]. In these works, the blender-horseshoe $\Lambda_0$ is the maximal invariant set of $f$ in a $d$-dimensional ‘cube’ $\Delta$ in $M$ and has a hyperbolic splitting with three nontrivial bundles $T_{\Lambda_0}M = E' \oplus E^{cu} \oplus E^{uu}$ where $E'$ is the stable bundle with $\dim E' = 1$ and $E^{cu} = E^{cu} \oplus E^{uu}$ is the unstable bundle with $\dim E^{cu} = 1$. Moreover, there exists an integer $k > 0$ such that $f^k|\Lambda_0$ is topologically conjugate to the complete shift of two symbols. In practice of [3, section 1], each of the bundles is obtained as a limit of stable, unstable and strong unstable cones $C^s$, $C^{cu}$, $C^{uu}$ defined on $\Delta$, respectively. In particular,

$$E^s \subset C^s, \quad E^{cu} \subset C^{cu} \subset C^u, \quad E^{cu} \oplus E^{uu} \subset C^u.$$ 

Note that the construction of the blender-horseshoe implies that $\Delta \cap f^k(\Delta)$ consists of two components each of which contains a distinguished saddle periodic point in $\Lambda_0$. We may suppose that one of them is a fixed point, say $Q$, while the other is $k$-periodic point, say $Q'$. See figure 7.

We consider vertical discs $D$ through the blender-horseshoe, that is, each $D$ is a $(d - 2)$-dimensional disc tangent to $C^{cu}$ and joining the ‘top’ and the ‘bottom’ of the cube $\Delta$. Then there are two isotopy classes of vertical discs that do not intersect $W^s_{loc}(Q, f)$ (respectively $W^s_{loc}(Q', f^k)$), called discs at the right and at the left of $W^s_{loc}(Q, f)$ (respectively $W^s_{loc}(Q', f^k)$). For example, $W^s_{loc}(Q, f)$ (that is a vertical disc) is at the left of $W^s_{loc}(Q', f^k)$ as in figure 7. Similarly, $W^{uu}_{loc}(Q', f^k)$ is at the right of $W^s_{loc}(Q, f)$. Note that the superposition region $D$ of the blender-horseshoe consists of the vertical discs in between $W^s_{loc}(Q, f)$ and $W^s_{loc}(Q', f^k)$. See for more details in [3, 5, 6, 8].

**Proof of theorem B.** Let $f$ be a $d$-dimensional $C^2$ diffeomorphism with the nontrivial basic sets $\Lambda_0$, $\Lambda_1$ satisfying the same conditions as in (1), (2) of theorem A. Moreover, we suppose that the $\Lambda_0$ is a blender-horseshoe of index$(\Lambda_0) = d - 1$, which is the maximal invariant set of a $d$-dimensional cube $\Delta$. From (1), $P \in \Lambda_1$ is of index one. As to the unstable manifold $W^u(P, f)$ of $P$, we furthermore suppose that there exist a segment $L^u_0$ of $W^u(P, f)$, a constant $\delta_0 > 0$ and a $C^2$ embedding $D : [-1, 1] \times [-\delta_0, \delta_0]^{d-3} \rightarrow \Delta$ such that

- $D([-1, 1] \times \{0^{d-3}\}) = L^u_0$;
- for any $0 < \delta < \delta_0$, $D_\delta(L^u_0) : = D([-1, 1] \times [-\delta, \delta]^{d-3})$ is a vertical disc through the blender-horseshoe $\Lambda_0$ and contained in the superposition region $D$.

Let $\mathcal{O}$ be the open set obtained in theorem A. Then the closure of $\mathcal{O} \cap \mathcal{U}_f$ contains $f$. By theorem A, for every $g \in \mathcal{O} \cap \mathcal{U}_f$, $W^u(\Lambda_{0,f}, g)$ and $W^s(\Lambda_{2,f}, g)$ have a heterodimensional tangency of elliptic type. Note that the blender-horseshoe is robust for any small $C^\prime$.
perturbation with $r \geq 1$, see [6, remark 3.2]. Hence, one has a disc $D_0(L^u_0)$ in $\Delta$ with $L^u_0 \subset W^u(P, g)$ such that $D_0(L^u_0) \to D_0(L^u_1)$ as $g \to f$. Thus, we may assume that $D_0(L^u_0)$ still belongs to $D$. Hence, in particular,

$$W^{u}_{bc}(\Lambda_{0,g}, g) \cap D_0(L^u_0) \neq \emptyset.$$  

By theorem A and remark 6.1, the basic set $\Lambda_{2,g}$ is nontrivial and homoclinically related to $\Lambda_{1,g}$. Therefore, there exists a segment $\tilde{L}^u_g$ in $W^u(\tilde{P}, g)$ for some $\tilde{P} \in \Lambda_{2,g}$ such that $\tilde{L}^u_g$ is arbitrarily $C^1$ close to $L^u_0$ and $D_0(\tilde{L}^u_g)$ is also contained in $D$. Thus,

$$W^{u}_{bc}(\Lambda_{0,g}, g) \cap D_0(\tilde{L}^u_g) \neq \emptyset.$$  

Since $W^{u}_{bc}(\Lambda_{0,g}, g)$ is closed subset of $M$, $W^{u}_{bc}(\Lambda_{0,g}, g)$ and $\tilde{L}^u_g$ have non-empty intersection. It follows that

$$W^u(\Lambda_{0,g}, g) \cap W^u(\Lambda_{2,g}, g) \neq \emptyset.$$  

This completes the proof of theorem B. \hfill \Box

8. Proof of theorem C

Finally, let us prove theorem C. For that purpose, we first consider a $C^2$ diffeomorphism $f$ having nontrivial basic sets $\Lambda_0, \Lambda_1$ with the following $C^2$ open conditions.

(i) Each periodic point in $\Lambda_0, \Lambda_1$ satisfies the same condition as in (1) of theorem A.

(ii) $f$ has a spherical heterodimensional intersection on the heterodimensional cycle associated with $\Lambda_0$ and $\Lambda_1$.

(iii) $\Lambda_0$ is a blender-horseshoe.

Let $D$ be the superposition region of $\Lambda_0$. We denote the union $\cup_{D \in D} D$ by $|D|$. As was mentioned above, $\Delta \cap f^k(\Delta)$ consists of two components. Note that $|D| := |D| \setminus (\Delta \cap f^k(\Delta))$ is disjoint from $\Lambda_0$. We say that a segment $L$ is in superposition in $|D|$ if

- $L \subset [D]$ and $TL \subset C^{au}$;
- there exists a constant $\delta_0 > 0$ and a $C^2$ embedding $D : [-1, 1] \times [-\delta_0, \delta_0]^{d-3} \to \Delta$ for any $0 < \delta < \delta_0$ which satisfies

$$D([-1, 1] \times \{0^{d-3}\}) = L, \quad D([-1, 1] \times [-\delta, \delta]^{d-3}) \subset D.$$  

In addition to (i)-(iii), we suppose the following condition:

(iv) There exists a saddle periodic point $P$ in $\Lambda_1$ such that the unstable manifold $W^u(P, f)$ contains segments $\ell_v^u, L^u_f$ in superposition in $|D|$ whose orbits $O(\ell_v^u, f) = \bigcup_{n \in \mathbb{Z}} f^n(\ell_v^u)$ and $O(L^u_f, f) = \bigcup_{n \in \mathbb{Z}} f^n(L^u_f)$ are disjoint.

Note that there exists an open subset of $\text{Diff}^2(M)$ whose element satisfies the condition (iv). In fact, one can construct an open set of examples from a certain diffeomorphism $f_0 \in \text{Diff}^2(M)$ with a partially hyperbolic saddle-node periodic point $S$ with at least three disjoint orbits of strong homoclinic intersections, i.e.

- $Df_0^{\text{per}}(S)$ has eigenvalues $\alpha, \gamma_1, \ldots, \gamma_{d-2}, \beta$ with $|\alpha| < 1 < |\beta|$, $|\gamma_1| = \cdots = |\gamma_{d-2}| = 1$ and such that $\alpha, \beta$ are real and at least one of $\{\gamma_i\}$ is 1.
- Consider the strong stable and unstable invariant directions $E^{ss}, E^{uu}$ respectively corresponding to the eigenvalues $\alpha, \beta$ of $Df_0^{\text{per}}(S)$. The strong unstable manifold $W^{uu}(S, f_0)$ of $S$ is the unique $f_0$-invariant manifold tangent to $E^{uu}$ of the same dimension as $E^{uu}$. The strong stable manifold $W^{ss}(S, f_0)$ of $S$ is defined similarly by using $E^{ss}$ instead of $E^{uu}$.
\textbf{Figure 8.} (a) Strong homoclinic intersections associated with $S$. (b) $\ell_f^u$ and $L_f^u$ in $\tilde{D}$.

- $W^{ss}(S, f_0) \cap W^{uu}(S, f_0)$ contains at least three different orbits which do not belong to the orbit of $S$. See figure 8(a).

Let us explain how such examples are constructed. For simplicity, we assume that $d = 3$, $S$ is the saddle-node fixed point and $W^{ss}(S, f_0) \cap W^{uu}(S, f_0) \setminus \{S\}$ contains three points $X$, $Y$, $Z$ any one of which is not contained in the orbit of the other ones. Suppose moreover that these points are quasi-transverse intersections associated with $W^{ss}(S, f_0)$ and $W^{uu}(S, f_0)$. After a small $C^2$ perturbation of $f_0$, we can have a diffeomorphism $f$ such that the saddle-node fixed point $S$ splits into two hyperbolic fixed points $Q$ (expanding in the central direction) and $P$ (contracting in the central direction). The saddle points $Q$ and $P$ have different indices and $W^s(P)$ and $W^u(Q)$ have a transverse intersection that contains the interior of a ‘central’ curve jointing $Q$ and $P$. Moreover, from [6, section 3.3], $f$ has the following properties.

- There exists a three-dimensional cube $\Delta$ and an integer $k > 0$ such that $\Delta \cap f^k(\Delta)$ consists of two disjoint components which respectively contain $Q$ and a point $Z_f \in W^u(P, f)$ converging to $Z$ as $f \to f_0$.

- The maximal invariant set in $\Delta$ is a blender-horseshoe $\Lambda_f$ with distinguished fixed point $Q$ and $k$-periodic point $Q'$.

Note that it has the superposition region $D$ between $W^u_{loc}(Q, f)$ and $W^u_{loc}(Q', f^k)$.

Since the orbits $O(X, f_0)$ of $X$ and $O(Y, f_0)$ of $Y$ are disjoint, one has small segments $\sigma_X, \sigma_Y \subset W^{uu}(S, f_0)$ with $X \in \text{Int}\sigma_X, Y \in \text{Int}\sigma_Y$ such that

$$O(\sigma_X, f_0) \cap O(\sigma_Y, f_0) = \emptyset.$$  

Thus, for any $f$ sufficiently $C^2$-close to $f_0$,

$$O(\sigma_X, f) \cap O(\sigma_Y, f) = \emptyset$$  \hspace{1cm} (8.1)

where $\sigma_X, \sigma_Y \subset W^{uu}(S, f)$ are segments such that $\sigma_X \to \sigma_X$ and $\sigma_Y \to \sigma_Y$ as $f \to f_0$. We may assume that $T_X\sigma_X$ and $T_Y\sigma_Y$ are not equal to the central direction of $S$ if necessary slightly $C^2$-perturbing $f$. Hence, for sufficiently large integer $m > 0$, we obtain segments $\ell_f^u \subset f^m(\sigma_X), L_f^u \subset f^m(\sigma_Y)$ which are in superposition in $[\tilde{D}]$. Moreover, by (8.1), we have $O(\ell_f^u, f) \cap O(L_f^u, f) = \emptyset$. Observe that the above property is open and corresponds to (iv).

**Proof of theorem C.** To prove this theorem, we have only to show that an arbitrarily small $C^2$ neighbourhood of the above $f$ with (i)–(iv) contains a diffeomorphism $g$ having a heterodimensional tangency on a heterodimensional cycle associated with saddle periodic points which satisfy (1) and (2) of theorem A.

By the conditions (i)–(ii), we have points $q_1 \in \Lambda_{0,f}$ and $p_1 \in \Lambda_{1,f}$ such that $W^u(q_1, f) \cap W^s(p_1, f)$ contains a $(d-2)$-dimensional sphere $S_f^{d-2}$. See figure 9. Moreover, by (iii)–(iv),
The first perturbation from \( f \) times. There exist points \( q_2 \in \Lambda_0, p_2 \in \Lambda_1 \) and disjoint segments \( \tilde{e}_f^u, L_f^u \in W^u(p_2, f) \) such that \( \tilde{e}_f^u, L_f^u \) are in superposition \([\bar{D}]\) and \( W^s_{\text{loc}}(q_2, f) \cap \tilde{e}_f^u \neq \emptyset \). Note that, in general, \( q_1, p_1, q_2 \) are not periodic points, or there exist points \( q \) such that periodic points are dense in the basic set \( f \times f \). Let us define a periodic point \( \tilde{q}_2 \in \Lambda_0 \) arbitrarily near \( q_2 \), a periodic point \( \tilde{p}_2 \in \Lambda_1 \) arbitrarily near \( p_2 \) and segments \( \tilde{e}_f^u, L_f^u \) of \( W^u(\tilde{p}_2, \tilde{f}) \) with \( \tilde{e}_f^u \rightarrow e_f^u, L_f^u \rightarrow L_f^u \) as \( \tilde{f} \rightarrow f \) such that \( \tilde{e}_f^u, L_f^u \) are in \([\bar{D}]; \tilde{e}_f^u \) and \( W^s_{\text{loc}}(q_2, \tilde{f}) \) have a quasi-transverse intersection \( \tilde{X} \). See figure 10.

(II) The second perturbation from \( \tilde{f} \) to \( \tilde{g} \). On the other hand, since the \((d - 2)\)-dimensional sphere \( S_{d-2}^f \) is contained in the transverse intersection \( W^u(q_1, f) \cap W^s(p_1, f) \) and periodic...
points are dense in the basic sets $\Lambda_0$, $\Lambda_1$, one has the following property:

- for every $\hat{f}$ sufficiently $C^2$-close to $f$, there exists a periodic point $\tilde{q}_1 \in \Lambda_0$ arbitrarily near $q_1$ and a periodic point $\tilde{p}_1 \in \Lambda_1$ arbitrarily near $p_1$ such that $W^u(\tilde{q}_1, \hat{f}) \cap W^s(\tilde{p}_1, \hat{f})$ contains a $(d-2)$-sphere $\tilde{S}_{d-2}^{\hat{f}}$ which has at least two tangencies with leaves of $\mathcal{F}^{ss}(\tilde{p}_1, \hat{f})$. See figure 10.

Since $f|_{\Lambda_1} = \tilde{f}|_{\Lambda_1}$ and $\Lambda_1$ is sectionally dissipative by (i), $\tilde{p}_1$ is a sectionally dissipative periodic point of $\tilde{f}$. Observe that there exists a foliated unstable cylinder $A_u^0$ in $W^u(\tilde{q}_1, \tilde{f})$ and a foliated stable cylinder $A_s^0$ in $W^s(\tilde{p}_1, \tilde{f})$ such that $A_u^0 \cap W^s(\tilde{p}_1, \tilde{f}) = \tilde{f}^N (A_u^0 \cap W^u(\tilde{q}_1, \tilde{f})) = \tilde{S}_{d-2}^{\tilde{f}}$ for some integer $N > 0$. As in the proofs of lemma 4.2 and proposition 4.3, we have sub-cylinders $A_u^N$ of $\tilde{f}^N(A_u^0)$ $C^1$ converging to $\ell^u_{\tilde{f}}$ as $n \to \infty$. See figure 10. Similarly, we have sub-cylinders $A_s^N$ of $\tilde{f}^{-n}(A_s^0)$ $C^1$ converging to a segment in $W^{ss}_c(\tilde{q}_2, \tilde{f})$ centred at $\tilde{X}$ as $n \to \infty$. Hence, by making a $C^2$ perturbation of $\tilde{f}$ similar to the above one in a small neighbourhood of $\tilde{X}$, we can obtain a diffeomorphism $\tilde{g}$ arbitrarily $C^2$ close to $\tilde{f}$ such that $W^u(\tilde{q}_1, \tilde{g})$ and $W^s(\tilde{p}_1, \tilde{g})$ have a heterodimensional tangency of elliptic type $r$, while $L^u_g$ is still contained in $|\tilde{D}|$. See figure 11(a).

**III The third perturbation from $\tilde{g}$ to $g$.** Since $\Lambda_1$ is transitive and contains $\tilde{p}_1, \tilde{p}_2$, there exists a segment $\tilde{L}_g^u$ in $W^u(\tilde{p}_1, \tilde{g})$ arbitrarily $C^2$ close to $L^u_g$ and hence it is also contained in $|\tilde{D}|$. That is, there exists a constant $\delta_0 > 0$ and a $C^2$ embedding $D : [-1, 1] \times [-\delta, \delta]^{d-3} \to \Delta$ such that

$$D([-1, 1] \times \{0^{d-3}\}) = \tilde{L}_g^u, \quad D(\tilde{L}_g^u) := D([-1, 1] \times [-\delta, \delta]^{d-3}) \subset \mathcal{D}$$
for every $0 < \delta < \delta_0$. Therefore,

$$W^s_{\text{loc}}(\Lambda_0, \tilde{g}) \cap D_\delta(\hat{L}_{\delta}^u) \neq \emptyset.$$ 

This implies that there exists a point $\tilde{q}'_1 \in \Lambda_0$ such that

$$W^s_{\text{loc}}(\tilde{q}'_1, \tilde{g}) \cap \hat{L}_{\delta}^u \neq \emptyset.$$ 

Note that, although $\tilde{q}'_1$ may not be a periodic point of $\tilde{g}$, it is an accumulation point of periodic points of $\Lambda_0$. Observe that $W^s_{\text{loc}}(\tilde{q}'_1, \tilde{g}) \cap \hat{L}_{\delta}^u$ consists of a quasi-transverse intersection $Y'$. See figure 11(a). Thus, once again by adding a perturbation similar to the above one in a small neighbourhood of $Y'$, we have a diffeomorphism $g$ arbitrarily $C^2$ close to $\tilde{g}$ such that $W^u(\tilde{p}_1, g)$ and $W^s(\tilde{q}_1, g)$ have a quasi-transverse intersection $Y$. By the condition (iv), it is possible to choose the perturbation which does not break the heterodimensional tangency $r$. See figure 11(b). Since $g|_{\Lambda_1} = \tilde{f}|_{\Lambda_1}$, $\tilde{p}_1$ is sectionally dissipative with respect to $g$. It follows that $g$ is a diffeomorphism satisfying all the conditions in theorem A. Thus, the claim of theorem C follows directly from theorem A. 

\[\square\]

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