WEYL ASYMPTOTICS FOR PERTURBATIONS OF MORSE POTENTIAL AND CONNECTIONS TO THE RIEMANN ZETA FUNCTION

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Abstract. Let \( N(T; V) \) denote the number of eigenvalues of the Schrödinger operator \(-y'' + V y\) with absolute value less than \( T \). This paper studies the Weyl asymptotics of perturbations of the Schrödinger operator \(-y'' + \frac{1}{4} e^{2t} y\) on \([x_0, \infty)\). In particular, we show that perturbations by functions \( \varepsilon(t) \) that satisfy \( |\varepsilon(t)| \lesssim e^t \) do not change the Weyl asymptotics very much. Special emphasis is placed on connections to the asymptotics of the zeros of the Riemann zeta function.

1. Introduction

It is known that if the Riemann hypothesis is true, then there is a positive semi–definite matrix
\[ H(x) = \begin{pmatrix} h_1(x) & h_2(x) \\ h_2(x) & h_3(x) \end{pmatrix} \]
such that the spectrum of:
\[ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = zH(x) \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} \]
(with self–adjoint boundary conditions on an interval \([a, b]\); here \( b \leq \infty \), is the same as the imaginary parts of the zeros of the Riemann \( \zeta \) function on the line \( \frac{1}{2} + it \). If the coefficients are smooth enough, then this can be transformed to a Schrödinger operator on \([a, b]\) that squares the eigenvalues (see for example [2, 3]).

The purpose of this paper is to indicate some properties that this potential – if it exists – must have. It is well–known that if \( Z(T) \) represents the number of zeros of magnitude less than \( |T| \) of the \( \zeta \) function, then:
\[ Z(T) = \frac{1}{\pi} T \log T + \frac{1}{\pi} (-2 \log 2\pi - 1) T + O(\log T). \]

If a potential, \( V \), exists whose eigenvalues are the squares of the imaginary parts of the zeros of \( \zeta \), and if \( N(T, L_V) \) represents the number of eigenvalues less than \( T \), then the Weyl–asymptotics would satisfy:
\[ Z(T) = \frac{1}{\pi} \sqrt{T} \log \sqrt{T} + \frac{1}{\pi} (-2 \log 2\pi - 1) \sqrt{T} + O(\log \sqrt{T}) \]

We use this to give some properties that such a potential must have to match the asymptotics in this way.

Much of the work in this paper builds off of work by Lagarias in papers such as [2,3]. In [3], it is shown that if \( V(t) = \frac{1}{4} e^{2t} + ke^t \) and if \( L_V := L_{V, x_0, \alpha} f = -f'' + Vf \) is the associated Schrödinger operator on the interval \([x_0, \infty)\), and if \( N(T; V, x_0, \alpha) \) is the number of eigenvalues of \( L_V \) less than \(|T|\), then

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Theorem 1.1. For $L_{Q_k,x_0,\alpha}$ the Weyl Asymptotics satisfy:

$$N(T; V, x_0) = \frac{1}{\pi} \sqrt{T} \log \sqrt{T} + \frac{1}{\pi} (2 \log 2 - 1 - x_0) \sqrt{T} + O(1), \quad (1.1)$$

as $T \to \infty$. The constant in the $O(1)$ depends on $k$ and $x_0$.

There is an $O(1)$ error term here, whereas in the Weyl–asymptotics for a potential that encodes the zeros of $\zeta$ should include an $O(\log \sqrt{T})$ term. We show in this paper that if $V$ is "close" to the potential $\frac{1}{4} e^{2t} + ke^t$, then there is no $O(\log \sqrt{T})$ in the Weyl–asymptotics. On the other hand, we show that if $V$ is "too far" from $V$, then there is an error term that is bigger than $O(\log \sqrt{T})$. These vague categories of "too close" and "too far" (which are defined below) do not form a dichotomy, and so a potential – if it exists – that matches the asymptotics appropriately must be neither "too close" or "too far".

Our first theorem shows that any perturbation by a function $\varepsilon(t)$ that satisfies $|\varepsilon(t)| \leq e^t$ will not really change the asymptotics (below, $Q_0(t) = \frac{1}{2} e^{2t}$):

Theorem 1.2. Let $\varepsilon(t)$ be a function that satisfies $|\varepsilon(t)| \leq e^t$. Then there holds:

$$N(T; Q_0 + \varepsilon, x_0, \alpha) = \frac{1}{\pi} \sqrt{T} \log \sqrt{T} + \frac{1}{\pi} (2 \log 2 - 1 - x_0) \sqrt{T} + O(1),$$

where the constant in the $O(1)$ depends on $x_0$ and $\varepsilon$. In particular:

$$|N(T; Q_0, x_0, \alpha) - N(T; Q_0 + \varepsilon, x_0, \alpha)| < O(1),$$

and so no $\log \sqrt{T}$ is introduced.

On the other hand, we prove that perturbations of $Q_0$ by functions that are bigger (resp.
smaller) than $e^{[1+\delta]x}$ (resp. $-e^{[1+\delta]x}$) for some $\delta > 0$ introduce a term that is on the order of (at least) $T^{\frac{1}{2}} \sqrt{\log T}$:

Theorem 1.3. Let $V$ be a function that satisfies $V(t) \geq \frac{1}{4} e^{2t} + \frac{1}{4} e^{(1+\varepsilon) t}$ (or $V(t) \leq \frac{1}{4} e^{2t} + \frac{1}{4} e^{(1-\varepsilon) t}$) for some $\varepsilon > 0$, then we have:

$$|N(T; V, x_0, \alpha) - N(T; Q_0, x_0, \alpha)| \geq T^{\frac{1}{2}} \sqrt{\log T}.$$

Finally, we prove that if $V(t)$ is sub-exponential on a fixed percentage of every interval $[x_0, R]$, then $N(T; V, x_0)$ does not even match the asymptotics in the first order:

Theorem 1.4. Let $V$ be a real potential and suppose there is a sub–exponential function $W(t)$ (by sub–exponential we mean $\frac{\log W(t)}{t} \to 0$ as $t \to \infty$) such that there is a positive number, $\delta$ such that for all sufficiently large $R$ there holds $|[t : V(t) < W(t)] \cap [x_0, R]| > \delta R$. Then:

$$\frac{N(T; V, x_0)}{\sqrt{T} \log T} \to \infty \text{ as } T \to \infty.$$

Thus, if our goal is to find a potential $V$ such that $N(T; V, x_0) = \frac{1}{\pi} \sqrt{T} \log \sqrt{T} + \frac{1}{\pi} (2 \log 2 - 1 - x_0) \sqrt{T} + O(\log T)$, we can summarize our findings as follows:

(a) Theorem 1.4 gives the heuristic that if $N(T; V, x_0) = O(\sqrt{T} \log \sqrt{T})$, then $V(t)$ must be at least on the order of $e^{\alpha t}$ for some $\alpha > 0$. (More precisely, it can’t be dominated by a sub–exponential function on sets of fixed percentages of intervals $[0, R]$).
(b) Theorem 1.3 says that for potentials of the form $\frac{1}{4}e^{2t} + \varepsilon(t)$ where $\varepsilon(t) > e^{(1+\varepsilon)t}$ (or $\varepsilon(t) < e^{(1+\varepsilon)t}$), $N(T; V, x_0)$ will not have the desired asymptotics. In particular, if $V(t) \simeq e^{at}$ and $\alpha \neq 2$, then $N(T; V, x_0)$ does not have the desired asymptotics. Thus, the heuristic is that if $V$ is a potential with the desired spectral asymptotics, then $V(t)$ must be a small perturbation of $\frac{1}{4}e^{2t}$.

(c) Finally, Theorem 1.2 shows that small perturbations of $\frac{1}{4}e^{2t}$ will not produce the desired asymptotics. The theme of this paper is then that a potential that gives the desired Weyl asymptotics is not just a “small perturbation” of a well–established potential (like the Morse potential). It seems that any potential that gives the desired Weyl asymptotics is going to have to oscillate wildly between sub and super exponential functions, will have singularities, and will probably not be a function (that is, it is a distribution).

Additionally, to make much more progress in this area, a version of Theorem 3.1 that puts less restrictions on $V$ is needed. Theorem 1.2 is a step in this direction and hopefully can be extended to other non–exponential potentials.

For the rest of the paper, we set $\alpha = 0$ and we don’t write the “$\alpha$” in $N(T; V, x_0, \alpha)$. The proofs of Theorems 1.2, 1.4, and 1.3 are in the following sections. Section 2 also contains some material that is used throughout the paper.

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2. Proof of Theorem 1.2

The proof of Theorem 1.2 is based on a very simple observation along with the integral estimate in [3]. We give some background information on which the simple observation is based. This observation is also used in other parts of this paper.

We have this basic theorem due to Sturm; see for example [4, 6]:

**Theorem 2.1.** Let $u$ be a (any) solution to:

$$u'' + (\lambda - g)u = 0$$

and let $v$ be a (any) solution to:

$$v'' + (\lambda - h)u = 0,$$

where $h(x) < g(x)$ (so that $\lambda - g(x) < \lambda - h(x)$). Between any two zeros of $u$, there is a zero of $v$. In particular:

$$\#\{\text{zeros of } v\} \geq \#\{\text{zeros of } u\} - 1$$

We have the following corollary (see also, for example, [5]):

**Corollary 2.2.** Let $g, h$ be as above. Consider the same equations as above but consider only solutions that satisfy a boundary condition at 0 and are in $L^2$ (that is, we have an eigenvalue problem; we consider only the "limit point" case). Then for every $a > 0$ we have:

$$\#\{\text{eigenvalues of second problem in } [0, a]\} \geq \#\{\text{eigenvalues of first problem in } [0, a]\} + O(1).$$
\[ y'' + (\lambda - g)y = 0 \quad (2.1) \]

and
\[ y'' + (\lambda - h)y = 0. \quad (2.2) \]

We want to consider solutions to these equations that satisfy \( y(0) = 0 \) and \( y'(0) = 1 \). Let \( y(x, \lambda; g) \) and \( y(x, \lambda; h) \) denote solutions to the respective problems. We consider the limit point case and we have that \( y(x, \lambda; g) \in L^2 \) if and only if \( \lambda \) is an eigenvalue of \( (2.1) \); and similarly for \( y(x, \lambda; h) \) and \( (2.2) \).

For a fixed \( \lambda \), we know from Theorem 2.1 that \( y(x, \lambda; h) \) has at least as many zeros as \( y(x, \lambda; g) \) (up to an \( O(1) \) error). Furthermore – also by Theorem 2.1 – if \( \lambda_k(g) \) and \( \lambda_{k+1}(g) \) are the \( k^{\text{th}} \) and \((k+1)^{\text{th}}\) eigenvalues for \( (2.1) \), then \( y(x, \lambda; g) \) has \( k \) zeros for \( \lambda_k(g) < \lambda < \lambda_{k+1}(g) \); that is, \( y(x, \lambda; g) \) gains a zero only when \( \lambda \) is an eigenvalue (at which point it gains exactly one zero). Of course, similar statements are true for \( y(x, \lambda; h) \).

Now, to prove the Corollary 2.2 we reason as follows. Starting with \( \lambda = 0 \), increase \( \lambda \) in a continuous manner. If \( \lambda \) passes through two eigenvalues of problem \((2.1)\) with out passing through an eigenvalue of \((2.2)\), then we have a contradiction to Theorem 2.1. Indeed, if \( \lambda = \lambda_2(g) \), and \( \lambda_0(h) > \lambda_2(g) \) then we have that \( y(x, \lambda; g) \) has 2 zeros while \( y(x, \lambda; h) \) has no zeros (Theorem 2.1 says that \( y(x, \lambda; h) \) should have at least one zero). Continue increasing \( \lambda \) this way, noting that whenever it passes through an eigenvalue of problem \((2.1)\), it must pass through at least one eigenvalue of problem \((2.2)\).

We now prove Theorem 1.2.

**Proof of Theorem 1.2.** By assumption, there is a \( C > 0 \) such that \( \frac{1}{4}e^{2t} - Ce^t < e(t) < \frac{1}{4}e^{2t} + Ce^t \). By the theorem of Lagarias \( N(T; \frac{1}{4}e^{2t} \pm Ce^t, x_0) = \frac{1}{\pi} \sqrt{T} \log \sqrt{T} + \frac{1}{2\pi} \log 2 - 1 - x_0) \sqrt{T} + O(1) \). By Corollary 2.2
\[
N(T; \frac{1}{4}e^{2t} + Ce^t) \leq N(T; V, x_0) \leq N(T; \frac{1}{4}e^{2t} - Ce^t).
\]

Since the outer two terms are equal, up to an \( O(1) \) error, this implies that \( N(T, V, x_0) = \frac{1}{\pi} \sqrt{T} \log \sqrt{T} + \frac{1}{2\pi} \log 2 - 1 - x_0) \sqrt{T} + O(1) \). \( \square \)

### 3. Proof of Theorem 1.3

To prove Theorem 1.3 we use Weyl’s law. This is a well-known theorem with several variants. We quote the one from [3]:

**Theorem 3.1.** Let \( V(t) = Q_k(t) \), or \( \frac{1}{4}e^{2t} + \frac{1}{4}e^{(1+\varepsilon)t} \). Then there holds:
\[
N(T; V, x_0) = \frac{1}{\pi} \int_{x_0}^{V^{-1}(T)} \sqrt{T - V(t)} dt + O(1). \quad (3.1)
\]

To prove Theorem 1.3 we use Weyl’s law above and the following two lemmas.

**Lemma 3.2.** Let \( V \) be a potential such that there is an \( \varepsilon > 0 \) such that:
\[
V(t) > \frac{1}{4}e^{2t} + \frac{1}{4}e^{(1+\varepsilon)t}.
\]
Then
\[ |N(T; Q_0, x_0) - N(T; V, x_0)| \geq T^\frac{1}{2} \sqrt{\log T}. \]

**Proof.** First, by corollary 2.2, \( N(T; Q_0, x_0) > N(T; V, x_0) \) (since \( Q_0 < V \)). Also, since \( V(t) \geq \frac{1}{4} e^{2t} + \frac{1}{4} e^{(1+\epsilon)t} \), by Corollary 2.2 it follows that \( N(T; \frac{1}{4} e^{2t} + \frac{1}{4} e^{(1+\epsilon)t}, x_0) > N(T; V, x_0) \) and so:
\[
|N(T; Q_0, x_0) - N(T; V, x_0)| = N(T; Q_0, x_0) - N(T; V, x_0) > N(T; Q_0, x_0) - N(T; \frac{1}{4} e^{2t} + \frac{1}{4} e^{(1+\epsilon)t}, x_0),
\]
and so we get a lower bound on this last term. Letting \( W(t) = \frac{1}{4} (e^{2t} + e^{(1+\epsilon)t}) \), by (3.1), this is equal to:
\[
\int_{t=0}^{Q_0^{-1}(T)} \sqrt{T - Q_0(t)} dt - \int_{t=0}^{W^{-1}(T)} \sqrt{T - W(t)} dt.
\]

For the rest of the proof, let \( T = \frac{1}{4} e^{2u} \) so that \( Q_0^{-1}(T) = u \); also let \( p = W^{-1}(T) \). Thus, we can write the integral above as:
\[
\frac{1}{2} \int_{t=p}^{u} \left( \sqrt{e^{2u} - e^{2t}} - \sqrt{e^{2u} - e^{2t} - e^{(1+\epsilon)t}} \right) dt + \frac{1}{2} \int_{t=p}^{u} \sqrt{e^{2u} - e^{2t}} dt. \tag{3.2}
\]

We estimate the first integral in (3.2). (As a side note, we mention the second integral is “small” and does not contribute much). We first estimate \( u - p \). Since \( p \) satisfies \( e^{2p} + e^{(1+\epsilon)p} = e^{2u} \), by taking \( \log \) on both sides, we find:
\[
2p + \left( \log e^{2p} + e^{(1+\epsilon)p} - \log e^{2p} \right) = 2u,
\]
so that \( 2(u - p) = \left( \log (e^{2p} + e^{(1+\epsilon)p}) - \log e^{2p} \right) \). The quantity in parentheses is estimated as:
\[
\log (e^{2p} + e^{(1+\epsilon)p}) - \log e^{2p} = \log (1 + e^{(\epsilon-1)p}) \simeq e^{(\epsilon-1)p}.
\]

Thus, \( u - p \simeq e^{(\epsilon-1)p} \). Let \( p_\delta = p - \delta p \), where \( \delta > 0 \) is a (small) number that will be chosen later that depends only on \( \epsilon \). We will show that:
\[
\int_{p_\delta}^{p} \sqrt{e^{2u} - e^{2t}} - \sqrt{e^{2u} - e^{2t} - e^{(1+\epsilon)t}} dt \geq \sqrt{pe^{2\epsilon p}}. \tag{3.3}
\]

Note that \( u \simeq \log T \) and \( e^{\frac{1}{2}u} \simeq T^{\frac{1}{2}} \). Thus, this estimate implies the claim since \( \sqrt{pe^{2\epsilon p}} = \sqrt{ue^{2\epsilon u}} \to 0 \) as \( T \to \infty \).

The integrand is estimated as:
\[
\sqrt{e^{2u} - e^{2t}} - \sqrt{e^{2u} - e^{2t} - e^{(1+\epsilon)t}} = \frac{2^{(1+\epsilon)t}}{\sqrt{e^{2u} - e^{2t}} + \sqrt{e^{2u} - e^{2t} - e^{(1+\epsilon)t}}} \geq \frac{2^{(1+\epsilon)t}}{\sqrt{e^{2u} - e^{2p_\delta}}}. \]

Additionally, we have:
\[
\int_{p_\delta}^{p} e^{(1+\epsilon)t} dt \geq \frac{1}{1 + \epsilon} \left( e^{(1+\epsilon)p} - e^{(1+\epsilon)p_\delta} \right) \geq \frac{1}{1 + \epsilon} e^{(1+\epsilon)p_\delta} (p - p_\delta) = \frac{\delta}{1 + \epsilon} e^{(1+\epsilon)(1-\delta)p_\delta} p.
\]

The “\( \geq \)” above follows from the mean value theorem.

Furthermore using the mean value theorem again, we have
\[
e^{2u} - e^{2p_\delta} \leq e^{2u} (u - p_\delta) = e^{2u} ((u - p) + \delta p) = e^{2u} (u - p (1 - \delta)).
\]
Putting together these estimates, we find:

\[ \int_{p_0}^{p} \sqrt{\text{e}^{2u(t)} - \text{e}^{2t}} - \sqrt{\text{e}^{2u(t)} - \text{e}^{(1+\varepsilon)t}} \, dt \geq \frac{\delta}{1 + \varepsilon} \frac{\text{e}^{(1+\varepsilon)(1-\delta)p}}{\text{e}^{2u(t) - \text{p}(1-\delta)}} \geq \frac{\sqrt{\delta p}}{1 + \varepsilon} \text{e}^{p-u} \text{e}^{(1+\varepsilon)p}. \]

Now, choose \( \delta \) so small that \((1 + \varepsilon)\delta < \varepsilon/2\) so that last quantity above is bigger than \( \frac{\sqrt{\delta p}}{1 + \varepsilon} \text{e}^{p-u} \text{e}^{\frac{\varepsilon}{2}} \) (since \( p-u \to 0 \) so \( e^{p-u} \to 1 \)). This completes the proof. \( \square \)

**Remark 3.3.** Note that if \( \varepsilon = 0 \), the estimate above is:

\[ \frac{\sqrt{\delta p}}{1 + \varepsilon} \text{e}^{p-u} \text{e}^{(1+\varepsilon)p} = \frac{\sqrt{\delta p}}{1 + \varepsilon} \text{e}^{p-u} \text{e}^{\delta p} \to 0. \]

**Lemma 3.4.** Let \( V \) be a potential such that there is an \( \varepsilon > 0 \) such that:

\[ V(t) < \frac{1}{4} \text{e}^{2t} - \frac{1}{4} \text{e}^{(1+\varepsilon)t}. \]

Then

\[ |N(T; V, x_0) - N(T; Q_0, x_0)| \geq T^{\varepsilon} \sqrt{\log T}. \]

**Proof.** Similar reasoning above leads us to find a lower bound on:

\[ \int_{t=\mu}^{\tau} \sqrt{\text{e}^{2u(t)} - \text{e}^{(1+\varepsilon)t}} - \sqrt{\text{e}^{2u(t)} - \text{e}^{2t}} \, dt, \]

where \( u_\delta = u - \delta u \) and \( \delta \) depends on \( \varepsilon \) and will be chosen later. We find a lower bound on the integrand as:

\[ \sqrt{\text{e}^{2u(t)} - \text{e}^{(1+\varepsilon)t}} - \sqrt{\text{e}^{2u(t)} - \text{e}^{2t}} > \frac{\text{e}^{(1+\varepsilon)t}}{\sqrt{\text{e}^{2u(t)} - \text{e}^{2t} + \text{e}^{(1+\varepsilon)t}}} > \frac{\text{e}^{(1+\varepsilon)t}}{\sqrt{\text{e}^{2u(t)} + \text{e}^{(1+\varepsilon)t}}} \approx \frac{\text{e}^{(1+\varepsilon)t}}{\text{e}^{u(t)}}. \]

And so:

\[ \int_{t=\mu}^{\tau} \sqrt{\text{e}^{2u(t)} - \text{e}^{(1+\varepsilon)t}} - \sqrt{\text{e}^{2u(t)} - \text{e}^{2t}} \, dt > e^{-u} \int_{t=\mu}^{\tau} \text{e}^{(1+\varepsilon)t} \, dt \geq \frac{\delta u}{1 + \varepsilon} \text{e}^{e^{u(t)}-(1+\varepsilon)\delta u}. \]

As above, choose \( \delta \) small enough so that \((1 + \varepsilon)\delta < \frac{1}{2} \varepsilon \). \( \square \)

## 4. Proof of Theorem 1.4

To prove Theorem 1.4 we use the following well-known theorem:

**Theorem 4.1.** Let \( V \) be a positive potential. Then there holds:

\[ N(T; V, x_0) \simeq \left| \{(t, \xi) : |\xi|^2 + V(t) < T \} \cap \{(t, \xi) : t \geq x_0 \} \right|. \]

We also make the following observation. If \( W(t) \) is sub–exponential (by which we mean that \( \frac{\log W(t)}{t} \to 0 \) as \( t \to \infty \)) then there is a function \( \varepsilon(t) \) with \( \varepsilon(t) \to 0 \), \( t \varepsilon(t) \to \infty \) and \( W(t) = e^{t \varepsilon(t)} \). Indeed, we easily compute \( \varepsilon(t) \) by noting that \( W(t) = e^{t \log W(t)} \) and so \( \varepsilon(t) = \frac{\log W(t)}{t} \). Since \( W(t) \to \infty \) we observe that \( t \varepsilon(t) = \log W(t) \to \infty \). Furthermore, since \( W(t) \) is sub–exponential, we conclude that \( \frac{\log W(t)}{t} \to 0 \) as \( t \to \infty \).

The proof of Theorem 1.4 will follow from this observation and the following lemma.
**Lemma 4.2.** Let $V$ and $W$ be as in Theorem 1.4. Let $\varepsilon(t)$ be a function that satisfies $\varepsilon(t) \to 0$ and $t\varepsilon(t) \to \infty$ as $t \to \infty$. Further assume that there is a positive number $\delta < 1$ such that for all sufficiently large $R$ there holds $\left| \{ t : V(t) < Ce^{tx(t)} \} \cap [x_0, R] \right| > \delta R$. Then:

$$\frac{N(T; V, x_0)}{\sqrt{T \log T}} \to \infty \quad \text{as} \quad T \to \infty.$$ 

**Proof.** Let $\psi(t) := t\varepsilon(t)$ and note that by the properties of $\psi(t)$ we have that $\frac{\psi^{-1}(t)}{t} \to \infty$ as $t \to \infty$. By Theorem 4.1 we have:

$$N(T; V, x_0) \simeq \left| \{ (t, \xi) : \xi^2 + V(t) < T \} \right| \geq \left| \{ (t, \xi) : \xi^2 + V(t) < \frac{T}{2} \} \cap \{ (t, \xi) : V(t) < Ce^{tx(t)} \} \right|$$

$$= \int_{t=x_0}^{\psi^{-1}(\log \frac{T}{2C})} \left( T - Ce^{tx(t)} \right)^{\frac{1}{2}} \mathbb{1}_{\{ V(t) < Ce^{tx(t)} \}}(t) dt.$$

Now, on the set over which the integral above is taken, we have that $T - Ce^{tx(t)} \simeq T$. Furthermore, the measure of the set over which the integral is being taken is at least $\delta \psi^{-1}\left( \log \frac{T}{2C} \right)$. Thus we have that:

$$N(T; V, x_0) \geq \delta \psi^{-1}\left( \log \frac{T}{2C} \right) \sqrt{T}.$$  (4.1)

To prove the desired claim, we need to show that $\frac{\psi^{-1}(\log \frac{T}{2C})}{\log T} \to \infty$. This is equivalent to showing that $\frac{\psi^{-1}(T)}{T} \to \infty$ as $T \to \infty$. But this is true because $\frac{\psi(t)}{t} \to 0$ as $T \to \infty$. Thus, this completes the proof. \hfill \Box

## 5. Remarks and Complements

In this section, we make some concluding remarks and extend some of the results above.

**Proposition 5.1.** If $V(t)$ is “exponential order” (by which we mean there are positive constants $C, a, b$ such that $\frac{1}{b} e^{at} < V(t) < Ce^{bt}$) then $N(T; V) \simeq \sqrt{T \log T}$.

**Proof.** First, we note that by Corollary 2.2, we only need to show that $N(T; e^{kt}) \simeq \sqrt{T \log T}$ for all $k > 0$. To do this, we use Theorem 3.1. Thus:

$$N(T; e^{kt}) \simeq \int_{x_0}^{\frac{1}{k} \log T} \left( T - e^{kt} \right)^{\frac{1}{2}} dt \geq \int_{x_0}^{\frac{1}{k} \log T} \sqrt{T} dt \simeq \sqrt{T \log T}.$$ 

It is even easier to show that $N(T; e^{kt}) \leq \sqrt{T \log T}$ \hfill \Box

Here is a proposition that says that if $V$ is super–exponential, then the Weyl Asymptotics are too small:

**Proposition 5.2.** Let $V(t)$ be a super–exponential potential (by which we mean $\frac{\log V(t)}{t} \to \infty$ as $t \to \infty$) then:

$$\frac{N(T; V)}{\sqrt{T \log T}} \to 0 \quad \text{as} \quad T \to \infty.$$
Proof. Similar to above, we assume that $V(t) > e^{t\varepsilon(t)}$ where $\varepsilon(t) \to \infty$; let $\psi(t) = t\varepsilon(t)$. By Corollary 2.2, we get an upper bound on $N(T; e^{t\varepsilon(t)})$:

$$\int_{x_0}^{\psi^{-1}(\log T)} \left( T - e^{t\varepsilon(t)} \right)^{\frac{1}{2}} dt \leq \sqrt{T} \psi^{-1}(\log T).$$

Now, $\frac{\psi^{-1}(t)}{t} \to 0$ as $t \to \infty$ since $\frac{\psi(t)}{t} \to \infty$ as $t \to \infty$. Thus,

$$\frac{\sqrt{T} \psi^{-1}(\log T)}{\sqrt{T} \log T} = \frac{\psi^{-1}(\log T)}{\log T} \to 0.$$

Finally, we briefly discuss an extension to Theorem 1.4. Recall that in the proof of Theorem 1.4 we had estimate (4.1):

$$N(T; V, x_0) \gtrsim \delta \psi^{-1} \left( \log \frac{T}{2C} \right) \sqrt{T}.$$

Now, let’s make $\delta$ be a function that depends on $R$. That is, we know that $V$ is sub-exponential on sets of size $\delta(R)R$ on the intervals $[0, R]$. Then the above estimate is:

$$N(T; V, x_0) \gtrsim \delta(\log \frac{T}{2C}) \psi^{-1} \left( \log \frac{T}{2C} \right) \sqrt{T}.$$

Thus, $\delta$ can be a decreasing function, so long as:

$$\frac{\delta(\log \frac{T}{2C}) \psi^{-1} \left( \log \frac{T}{2C} \right)}{\log T} \to \infty.$$

So, for example, if $\varepsilon(t) = t^{-\gamma}$, then $\psi(t) = t^{1-\gamma}$ and $\psi^{-1}(t) = t^{1+\gamma}$ for some $\gamma > 0$. Then the estimate above is:

$$\frac{\delta(\log \frac{T}{2C}) \psi^{-1} \left( \log \frac{T}{2C} \right)}{\log T} = \delta(\log \frac{T}{2C}) \left( \log \frac{T}{2C} \right)^{1+\gamma}.$$

So, this still goes to $\infty$ if, for example, $\delta(t) > t^{-\frac{1}{2}}$.

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