HARMONICITY OF VECTOR FIELDS ON FOUR-DIMENSIONAL LORENTZIAN LIE GROUPS

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Abstract. We consider four-dimensional Lie groups equipped with left-invariant Lorentzian Einstein metrics, and determine the harmonicity properties of vector fields on these spaces. In some cases, all these vector fields are critical points for the energy functional restricted to vector fields. We also classify vector fields defining harmonic maps, and calculate explicitly the energy of these vector fields.

1. Introduction

In [3] it has been proved that a (simply connected) four-dimensional homogeneous Riemannian manifold is either symmetric, or isometric to a Lie group equipped with a left-invariant Riemannian metric. Following [2], in a sense, by Proposition 2.2 in [6] the classification of four-dimensional Lorentzian Lie groups coincide with the Riemannian ones which on that base, four-dimensional Einstein Lorentzian lie groups were classified. Also investigating critical points of the energy associated to vector fields is an interesting purpose under different points of view. As an example by the Reeb vector field $\xi$ of a contact metric manifold, somebody can see how the criticality of such a vector field is related to the geometry of the manifold ([16], [17]). Recently, it has been [11] proved that critical points of $E : \mathfrak{X}(M) \to \mathbb{R}$, that is, the energy functional restricted to vector fields, are again parallel vector fields. Moreover, in the same paper it also has been determined the tension field associated to a unit vector field $V$, and investigated the problem of determining when $V$ defines a harmonic map.

A Riemannian manifold admitting a parallel vector field is locally reducible, and the same is true for a pseudo-Riemannian manifold admitting an either space-like or time-like parallel vector field. This leads us to consider different situations, where some interesting types of non-parallel vector fields can be characterized in terms of harmonicity properties. We may refer to the recent monograph [9] and some references [14], [15] for an overview on harmonic vector fields.

Let $(M, g)$ be a compact pseudo-Riemannian manifold and $g_s$ be the Sasaki metric on the tangent bundle $TM$, then the energy of a smooth vector field $V : (M, g) \to (TM, g^s)$ on $M$ is;

$$E(V) = \frac{n}{2} \text{vol}(M, g) + \frac{1}{2} \int_M \|\nabla V\|^2 dv$$

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(assuming M compact; in the non-compact case, one works over relatively compact domains see [4]). If \( V : (M, g) \to (TM, g^s) \) be a critical point for the energy functional, then \( V \) is said to define a harmonic map. The Euler-Lagrange equations characterize vector fields \( V \) defining harmonic maps as the ones whose tension field \( \theta(V) = tr(\nabla^2 V) \) vanishes. Consequently, \( V \) defines a harmonic map from \((M, g)\) to \((TM, g^s)\) if and only if

\[
tr[R(\nabla V, V)] = 0, \quad \nabla^* \nabla V = 0,
\]

where with respect to a pseudo-orthonormal local frame \( \{e_1, ..., e_n\} \) on \((M, g)\), with \( \varepsilon_i = g(e_i, e_i) = \pm 1 \) for all indices \( i \), one has

\[
\nabla^* \nabla V = \sum_i \varepsilon_i (\nabla_{e_i} \nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V).
\]

A smooth vector field \( V \) is said to be a harmonic section if and only if it is a critical point of \( E^v(V) = (1/2) \int_M ||\nabla V||^2 dv \) where \( E^v \) is the vertical energy. The corresponding Euler-Lagrange equations are given by

\[
\nabla^* \nabla V = 0,
\]

Let \( \mathfrak{X}^\rho(M) = \{ V \in \mathfrak{X}(M) : ||V||^2 = \rho^2 \} \) and \( \rho \neq 0 \). Then, one can consider vector fields \( V \in \mathfrak{X}(M) \) which are critical points for the energy functional \( E|_{\mathfrak{X}^\rho(M)} \), restricted to vector fields of the same constant length. The Euler-Lagrange equations of this variational condition are given by

\[
\nabla^* \nabla V \text{ is collinear to } V.
\]

As usual, for \( \rho \neq 0 \) condition (1.4) is taken as a definition of critical points for the energy functional restricted to vector fields of the same length in the non-compact case. If \( V \) is a light-like vector field then (1.4) is still a sufficient condition so that \( V \) is a critical point for the energy functional \( E|_{\mathfrak{X}^\rho(M)} \), restricted to light-like vector fields ([4], Theorem 26). In the present paper we shall provide a complete investigation of harmonicity of vector fields on four-dimensional Einstein Lorentzian Lie groups. In [4], four-dimensional Einstein Lorentzian Lie groups were classified into two types, denoted by 16 cases. Let \( G \) be a four-dimensional Einstein Lorentzian Lie group, \( \mathfrak{g} \) the corresponding Lie algebra of \( G \). Once applied to vector fields belonging to \( \mathfrak{g} \), conditions (1.1), (1.2) translate into some systems of algebraic equations for the components of these vector fields. The paper is organized in the following way. In Section 2, we shall recall the definition and basic properties of Einstein Lorentzian Lie algebra, as described in [4]. Harmonicity properties of vector fields of four-dimensional Einstein Lorentzian Lie group of type (a) and (c) will be investigated in Sections 3 and 4, respectively. Finally, the energy of all these vector fields is explicitly calculated in Section 5.

2. Preliminaries

A Lie group \( G \) together with a left-invariant pseudo-Riemannian metric is called a pseudo-Riemannian Lie group. The left-invariant pseudo-Riemannian metric defines an inner product \( g \) on the Lie algebra \( \mathfrak{g} \) of \( G \), and conversely, any inner product on \( \mathfrak{g} \) gives rise to an unique left-invariant metric on \( \mathfrak{g} \). The couple \((\mathfrak{g}, g)\) is called pseudo-Riemannian Lie algebra.
Let \((g, g)\) be a pseudo-Riemannian Lie algebra of dimension \(n\). The Levi-Civita connection defines a product \((u, v) \rightarrow uv\) on \(g\) called Levi-Civita product given by the Koszul formula
\[
2g(uv, w) = g([u, v], w) + g([w, u], v) + g([v, w], u).
\]
We denote by \(\nabla\) the Levi-Civita connection of \((g, g)\) and by \(R\) its curvature tensor, taken with the sign convention
\[
(2.6) \quad R(X, Y) = \nabla_{[X,Y]} - [\nabla X, \nabla Y],
\]
for all smooth vector fields \(X, Y\). In the next section we shall present harmonicity of vector fields on four-dimensional Einstein Lorentzian Lie groups. To this purpose we first describe the Lie brackets as the following theorem.

**Theorem 2.1.** [6] Let \(G\) be a four-dimensional simply connected Lie group. If \(g\) is a left-invariant Lorentzian Einstein metric on \(G\), then the Lie algebra \(g\) of \(G\) is isometric to \(g = \mathfrak{r} \ltimes g_3\), where \(g_3 = \text{span}\{e_1, e_2, e_3\}\) and \(\mathfrak{r} = \text{span}\{e_4\}\), and one of the following cases occurs.

(a) \(\{e_i\}_{i=1}^4\) is a pseudo-orthonormal basis, with \(e_3\) time-like. In this case, \(G\) is isometric to one of the following semi-direct products \(\mathbb{R} \ltimes g_3\):

1. \(\mathbb{R} \ltimes H\), where \(H\) is the Heisenberg group and \(g\) is described by one of the following sets of conditions:
   \[
   \begin{align*}
   (1) & \quad [e_1, e_2] = eAe_1, [e_1, e_3] = Ae_1, [e_1, e_4] = \delta Ae_1, [e_3, e_4] = -2A\delta(e_2 - e_3), \\
   (2) & \quad [e_1, e_2] = e\sqrt{A^2 - B^2}e_1, [e_1, e_3] = -\delta e\sqrt{A^2 - B^2}e_1, [e_1, e_4] = \frac{\delta A + B}{2}e_1, [e_2, e_4] = B(e_2 + \delta e_3), [e_3, e_4] = A(e_2 + \delta e_3), \\
   (3) & \quad [e_1, e_2] = e\sqrt{A^2 - B^2}e_1, [e_1, e_3] = e\sqrt{A^2 - B^2}e_1, [e_2, e_4] = Be_2 - Ae_3, [e_3, e_4] = Ae_2 - \frac{A^2}{B}e_3, \\
   (4) & \quad [e_1, e_2] = e\sqrt{A^2 - B^2}e_1, [e_2, e_3] = A(e_2 + \delta e_3), [e_2, e_4] = Be_2 - Ae_3, [e_3, e_4] = Ae_3, \\
   (5) & \quad [e_1, e_4] = -A + Be_1, [e_2, e_4] = Be_2 - \epsilon\sqrt{A^2 + AB + B^2}e_3, [e_3, e_4] = \epsilon\sqrt{A^2 + AB + B^2}e_3, \\
   (6) & \quad [e_1, e_4] = -2Ae_1, [e_2, e_4] = -5Ae_2 - 6eAe_3, [e_3, e_4] = Ae_3, \\
   (7) & \quad [e_1, e_4] = Ae_1, [e_2, e_4] = -2Ae_2 + Be_3, [e_3, e_4] = Be_2 + Ae_3, \\
   (8) & \quad [e_1, e_4] = e\sqrt{A^2 - B^2}e_1, [e_2, e_4] = e\sqrt{A^2 - B^2}e_1, [e_3, e_4] = Ae_2 + \epsilon\sqrt{A^2 - B^2}e_3, \\
   (9) & \quad [e_1, e_4] = e\sqrt{A^2 - B^2}e_1, [e_2, e_4] = e\sqrt{A^2 - B^2}e_1, [e_3, e_4] = Ae_2 + \epsilon\sqrt{A^2 - B^2}e_3, \\
   (10) & \quad [e_1, e_4] = Ae_1 + e\sqrt{B^2 - A^2 - C^2 - AC}e_2, [e_2, e_4] = e\sqrt{B^2 - A^2 - C^2 - AC}e_1 - (A + C)e_2 - Be_3, [e_3, e_4] = Be_2 + Ce_3, \\
   (11) & \quad [e_1, e_4] = e\sqrt{A^2 - B^2}e_1 + \delta Ae_3, [e_2, e_4] = e\sqrt{A^2 - B^2}e_1 + \delta Ae_3, [e_3, e_4] = Ae_2 + \epsilon\sqrt{A^2 - B^2}e_3, \\
   (c) & \quad \{e_i\}_{i=1}^4\) is a basis, with the inner product \(g\) on \(g\) completely determined by \(g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = 1\) and \(g(e_i, e_j) = 0\) otherwise. In this case, \(G\) is isometric to one of the following semi-direct products \(\mathbb{R} \ltimes g_3\):
   \end{align*}
\]
2. \(\mathbb{R} \ltimes H\), where \(g\) is described by one of the following sets of conditions:

12. \(\mathbb{R} \ltimes H\), where \(g\) is described by one of the following sets of conditions:
(14) \[ [e_1, e_2] = \epsilon \sqrt{((A + D)^2 + 4B^2)}e_3, [e_1, e_4] = -Be_1 + De_2 + Ee_3, [e_2, e_4] = Ae_1 + Be_2 + Ce_3, \]

(c) \( \mathbb{R} \times \mathbb{R}^3 \), where \( g \) is described by one of the following sets of conditions:

(15) \[ [e_1, e_2] = Ae_2 + Be_3, [e_2, e_4] = -Ae_1 + Ce_3, \]

(16) \[ [e_1, e_4] = Ae_1 + Be_2 + Ce_3, [e_2, e_4] = De_1 + Ee_2 + Fe_3, [e_3, e_4] = \frac{(B+D)^2 + 2(A^2 + E^2)}{2(E + A)}e_3 \]

In all the cases listed above, \( \epsilon = \pm 1 \) and \( \delta = \pm 1 \).

Following [6] for an arbitrary four-dimensional Lorentzian Lie algebra \( (g, g) \) and a basis \( \{e_1, e_2, e_3, e_4\} \) of \( g \), the Lorentzian inner product takes one of the following forms:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix},
\]

which each of Einstein examples in case (b) is isometric to one of cases listed in case (a). We treat type (a) and type (c) separately.

3. Harmonicity of vector fields: type (a)

Consider a four-dimensional simply connected Lie group \( G \) of type (a), a pseudo-orthonormal basis \( \{X_i\}_{i=1}^4 \), with \( X_3 \) time-like. Although \( \{X_i\}_{i=1}^4 \) is a pseudo-orthonormal basis, for some purposes we are going to construct the following pseudo-orthonormal frame

\[ e_1 = -X_1, \quad e_2 = X_2, \quad e_3 = -X_3, \quad e_4 = X_4. \]

Using this frame we give the following result.

**Theorem 3.1.** Let \( G \) be a four-dimensional simply connected Lie group of type (a) and \( V = ae_1 + be_2 + ce_3 + de_4 \in g \) be a left-invariant vector field on \( G \) for some real constants \( a, b, c, d \). If \( g \) be a left-invariant Lorentzian Einstein metric on \( G \), then for the different cases of theorem [2.1] type (a), (which \( V \) is restricted to vector fields of the same length), we have:

1. \( V \) is a critical point for the energy functional if and only if \( V = b(e_2 + e_3) \), that is, \( b = c \). In this case \( \epsilon = 1 \), \( \nabla^* \nabla V = -3A^2 V \).
2. \( V \) is a critical point for the energy functional if and only if \( V = b(e_2 + e_3) + de_4 \), that is, \( b = c \). In this case \( \delta = -1 \), \( \nabla^* \nabla V = -\frac{3}{2}(-B + A)^2 V \).
3. \( V \) is a critical point for the energy functional, in this case, \( \nabla^* \nabla V = -\frac{(A^2 - B^2)^2}{B^2} V \).
4. \( V \) is a critical point for the energy functional if and only if \( V = ae_1 + be_2 \). In this case \( \nabla^* \nabla V = (B^2 - A^2) V \).
5. \( V \) is a critical point for the energy functional if and only if \( A = -B \). In this case \( \nabla^* \nabla V = 0 \).
6. \( V \) is a critical point for the energy functional if and only if \( V = b(e_2 + e_3) \), that is, \( b = c \). In this case \( \epsilon = 1 \), \( \nabla^* \nabla V = -13A^2 V \).
7. \( V \) is a critical point for the energy functional if and only if \( V = b(e_2 + e_3) \), that is, \( b = c \). In this case \( \nabla^* \nabla V = (B^2 - A^2 + 3AB) V \).
8. \( V \) is a critical point for the energy functional if and only if \( V = b(e_2 + e_3) \), that is, \( b = c \). In this case \( \epsilon = -1 \), \( \nabla^* \nabla V = -\frac{13}{36}(A + B)^2 V \).
(9) : V is a critical point for the energy functional if and only if \( V = a(e_1 + e_3) \), that is, \( a = c \). In this case \( \epsilon = 1 \), \( \nabla^* \nabla V = -\frac{13}{4} A^2 \).

Proof. The above statement is obtained from a case-by-case argument. As an example, we report the details for case (4) here. Let \( V \) be a critical point for the energy functional restricted to vector fields of the same length, as follows:

\[
\begin{align*}
\nabla e_1 e_1 &= -\epsilon \sqrt{A^2 - B^2} e_2, & \nabla e_1 e_2 &= \epsilon \sqrt{A^2 - B^2} e_1, \\
\nabla e_3 e_3 &= A e_4, & \nabla e_3 e_4 &= A e_3,
\end{align*}
\]

while \( \nabla e_i e_j = 0 \) in the remaining cases.

We can now use (3.8) to calculate \( \nabla e_i V \) for all indices \( i \). We get

\[
\begin{align*}
\nabla e_1 V &= \epsilon \sqrt{A^2 - B^2} (be_1 - ae_2), & \nabla e_2 V &= 0, \\
\nabla e_3 V &= A (de_3 + ce_4), & \nabla e_4 V &= 0,
\end{align*}
\]

From (3.9) it follows at once that there are no parallel vector fields \( V \neq 0 \) belonging to \( g \).

We can now calculate \( \nabla e_i \nabla e_i V \) and \( \nabla \nabla e_i e_i V \) for all indices \( i \). We obtain

\[
\begin{align*}
\nabla e_1 \nabla e_1 V &= -(A^2 - B^2)(ae_1 + be_2), & \nabla e_2 \nabla e_2 V &= 0, \\
\nabla e_3 \nabla e_3 V &= A^2 (ce_3 + de_4), & \nabla e_4 \nabla e_4 V &= 0,
\end{align*}
\]

Thus, we find

\[
\nabla^* \nabla V = \sum_i \epsilon_i (\nabla e_i \nabla e_i V - \nabla \nabla e_i e_i V) = -(A^2 - B^2)(ae_1 + be_2) - A^2(ce_3 + de_4).
\]

Since \( \nabla^* \nabla V = -(A^2 - B^2) V - B^2 (ce_3 + de_4) \), condition (4) results that \( c = d = 0 \).

In the other direction, let \( V = (B^2 - A^2)(ae_1 + be_2) \). A direct calculation yields that \( \nabla^* \nabla V = -(A^2 - B^2) V \). \( \square \)

In this case let \( V \) is a critical point for the energy functional vector field. Clearly, \( \nabla^* \nabla V = 0 \) if and only if \( -(A^2 - B^2) V = 0 \), which means that \( A = B \). Next, using (3.9) and (2.6) we find

\[
\begin{align*}
R(\nabla e_1 V, V) e_1 &= \epsilon^3 \sqrt{(A^2 - B^2)^3 (a^2 + b^2)} e_2, & R(\nabla e_2 V, V) e_2 &= 0, \\
R(\nabla e_3 V, V) e_3 &= 0, & R(\nabla e_4 V, V) e_4 &= 0.
\end{align*}
\]

and so,

\[
tr[R(\nabla V, V)] = \epsilon^3 \sqrt{(A^2 - B^2)^3 (a^2 + b^2)} e_2.
\]

Hence, \( tr[R(\nabla V, V)] = 0 \) if and only if \( A = B \). Applying this argument for other cases of type (a) proves the following classification result.

**Theorem 3.2.** Let \( G \) be a four-dimensional simply connected Lie group of type (a) and \( V \) be a critical point for the energy functional restricted to vector fields of the same length, described by conditions (2) – (9) of theorem 3.1 then for the different cases of theorem 3.1 we have:

(2), (3) and (4): \( V \) defines harmonic map if and only if, \( A = B \).
(5): \( V \) defines harmonic map if and only if, \( A = -B \). In this case \( \epsilon = 1 \).
(7): \( V \) defines harmonic map if and only if, \( A = ((3 - \sqrt{13})/2) B \).
(8): \( V \) defines harmonic map if and only if, \( A = -B \). In this case \( \epsilon = -1 \).
In all cases of theorem 3.1 a straightforward calculation shows that $\nabla_{\bar{e}} u = 0$ for all indices $i$. Therefore, $u$ is a parallel light-like vector field. The existence of a light-like parallel vector field is an interesting phenomenon which has no Riemannian counterpart, and characterizes a class of pseudo-Riemannian manifolds which illustrate many of differences between Riemannian and pseudo-Riemannian settings (see for example [7],[8])

Remark 1. A vector field $V$ is geodesic if $\nabla V V = 0$, and is Killing if $\mathcal{L}_V g = 0$, where $\mathcal{L}$ denotes the Lie derivative. Parallel vector fields are both geodesic and Killing, and vector fields with these special geometric features often have particular harmonicity properties [1,10,12,13]. By standard calculations we obtain the following result.

**Proposition 3.3.** Let $G$ be a four-dimensional simply connected Lie group of type $(a)$ and $V \in \mathfrak{g}$ be a left-invariant vector field on $G$. If $g$ be a left-invariant Lorentzian Einstein metric on $G$, then we have the following classification.

| $(G, g)$ | Geodesic vector fields | Killing vector fields |
|----------|------------------------|-----------------------|
| (1)      | $V = be_2$ or $V = b(e_2 + e_3)$ |✗| |
| (2)      | $V = c(\frac{A}{B} e_2 + e_3) + de_4$ | $V = c(e_3 - e_2)$ is Killing $\iff A = B$ |
| (3)      | $V = c(\frac{A}{B} e_2 + e_3) + de_4$ | $V = ae_1 + c(e_3 - e_2)$ is Killing $\iff A = B$ |
| (4)      | $V = be_2 + de_4$ | $V = ae_1 + be_2$ is Killing $\iff A = B$ |
| (5)      | $V = de_4$ | $V = ae_1$ is Killing $\iff A = -B$ |
| (6)      | $V = de_4$ or $V = b(e_2 + e_3)$ |✗| |
| (7)      | $V = de_4$ or $V = b(e_2 + e_3)$ | $V = b(e_2 - e_3)$ is Killing $\iff A = B$ |
| (8)      | $V = de_4$ or $V = b(e_2 + e_3)$ | $V = ae_1$ is Killing $\iff A = -B$ |
| (9)      | $V = de_4$ or $V = a(e_1 + e_3)$ |✗| |
| (10)     | $V = ce_3$ is geodesic $\iff C = 0$ | $V = ae_1$ is Killing $\iff A = 0$ |
| (11)     | $V = de_4$ |✗| |

In particular, using Proposition 3.3 and Theorems 3.1, 3.2 a straightforward calculation proves the following main classification result.

**Corollary 3.4.** Let $V \in \mathfrak{g}$ be a left-invariant vector field on four-dimensional simply connected Lie group $G$ of type $(a)$. If $g$ be a left-invariant Lorentzian Einstein metric on $G$, then for the different cases of theorem 3.1 the following properties are equivalent (which $V$ is restricted to vector fields of the same length):
The following properties are equivalent

| (G, g) | The following properties are equivalent |
|--------|----------------------------------------|
| (1)    | V is geodesic; \(\equiv V\) is a critical point for the energy functional; \(\equiv\) none of these vector fields is harmonic (in particular, defines a harmonic map); \(\equiv V = b(e_2 + e_3),\) |
| (2)    | V is geodesic; \(\equiv V\) is harmonic if and only if \(A = B; \equiv V\) is a critical point for the energy functional; \(\equiv V\) defines harmonic map if and only if \(A = B; \equiv V\) is Killing if and only if \(A = -B\) and \(d = 0; \equiv V = b(e_2 + e_3) + de_4,\) |
| (3)    | V is harmonic if and only if \(A = B; \equiv V\) is a critical point for the energy functional; \(\equiv V\) defines harmonic map if and only if \(A = B; \equiv V\) is Killing if and only if \(A = B; \equiv V = ae_1 + be_2,\) |
| (4)    | V is harmonic if and only if \(A = B; \equiv V\) is a critical point for the energy functional; \(\equiv V\) defines harmonic map if and only if \(A = B; \equiv V\) is Killing if and only if \(A = B; \equiv V = (ae_1 + be_2,\) |
| (5)    | V is harmonic if and only if \(A = -B; \equiv V\) is a critical point for the energy functional if and only if \(A = -B; \equiv V\) defines harmonic map if and only if \(A = -B; \equiv V\) is Killing if and only if \(A = -B\) and \(b = c = d = 0,\) |
| (6)    | V is geodesic; \(\equiv V\) is a critical point for the energy functional \(\equiv\) none of these vector fields is harmonic (in particular, defines a harmonic map); \(\equiv V = b(e_2 + e_3),\) |
| (7)    | V is geodesic; \(\equiv V\) is harmonic if and only if \(A = ((3 - \sqrt{13})/2)B; \equiv V\) is a critical point for the energy functional; \(\equiv V\) defines harmonic map if and only if \(A = ((3 - \sqrt{13})/2)B; \equiv V\) is Killing if and only if \(A = -B; \equiv V = b(e_2 + e_3),\) |
| (8)    | V is geodesic; \(\equiv V\) is harmonic if and only if \(A = -B; \equiv V\) is a critical point for the energy functional; \(\equiv V\) defines harmonic map if and only if \(A = -B; \equiv V = (e_2 + e_3),\) |
| (9)    | V is geodesic; \(\equiv V\) is a critical point for the energy functional \(\equiv\) none of these vector fields is harmonic (in particular, defines a harmonic map); \(\equiv V = a(e_1 + e_3).\) |

4. Harmonicity of vector fields: type (c)

Consider a four-dimensional simply connected Lie group \(G\) of type (c), a basis \(\{X_i\}_{i=1}^{4}\) with non-zero inner product \(g\) determined by \(g(X_1, X_1) = g(X_2, X_2) = g(X_3, X_4) = g(X_4, X_3) = 1.\) For type (c) we can construct a pseudo-orthonormal frame field \(\{e_1, e_2, e_3, e_4\},\) putting

\[
e_1 = X_1, \quad e_2 = X_2, \quad e_3 = -(1/2)X_3 + X_4, \quad e_4 = (1/2)X_3 + X_4.
\]

Clearly, \(e_3\) is time-like. A vector field \(V \in \mathfrak{g}\) is uniquely determined by its components with respect to the pseudo-orthonormal basis \(e_i.\) Hence \(V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{g}\) be a left-invariant vector field on \(G\) for some real constants \(a, b, c, d.\) Notice that the (constant) norm of \(V\) is given by \(||V||^2 = a^2 + b^2 - c^2 + d^2.\)

For example in case (14) of theorem 2.1, let \(V\) be harmonic. Equations (2.5) and (4.11)
now give
\[
\begin{align*}
\nabla_{e_1} e_1 &= -B(e_3 - e_4), \quad \nabla_{e_1} e_2 = \frac{1}{2}(A + D - \epsilon\alpha)(e_3 - e_4), \\
\nabla_{e_1} e_3 &= \nabla_{e_1} e_4 = -B e_1 + \frac{1}{2}(A + D - \epsilon\alpha) e_2, \\
\nabla_{e_2} e_1 &= \frac{1}{2}(A + D + \epsilon\alpha)(e_3 - e_4), \quad \nabla_{e_2} e_2 = B(e_3 - e_4), \\
\nabla_{e_2} e_3 &= \nabla_{e_2} e_4 = \frac{1}{2}(A + D + \epsilon\alpha) e_1 + B e_2, \\
\nabla_{e_3} e_1 &= \nabla_{e_3} e_2 = \frac{1}{2}(A - D - \epsilon\alpha) e_1 + C(e_3 - e_4), \\
\nabla_{e_3} e_3 &= \nabla_{e_3} e_4 = \nabla_{e_3} e_3 = \nabla_{e_3} e_4 = \nabla_{e_3} e_3 = \nabla_{e_3} e_4 = E e_1 + C e_2,
\end{align*}
\]
where \( \alpha = \sqrt{(A + D)^2 + 4B^2} \). Set \( u = e_3 - e_4 \). Then, from (4.12) we get \( \nabla_{e_i} u = 0 \) for all indices \( i \). Therefore, \( u \) is a parallel light-like vector field. The existence of a light-like parallel vector field is an interesting phenomenon which has no Riemannian counterpart, and characterizes a class of pseudo-Riemannian manifolds which illustrate many of differences between Riemannian and pseudo-Riemannian settings (see for example [7, 8]). For an arbitrary left-invariant vector field \( V = a e_1 + b e_2 + c e_3 + d e_4 \in \mathfrak{g} \) we can now use (4.12) to calculate \( \nabla_{e_i} V \) for all indices \( i \). We get
\[
\begin{align*}
\nabla_{e_1} V &= (c + d)(-B e_1 + \frac{1}{2}(D + A - \epsilon\alpha) e_2) \\
&\quad + \frac{1}{2}(A + D - \epsilon\alpha) b - Ba)u, \\
\nabla_{e_2} V &= (c + d)(\frac{1}{2}(D + A + \epsilon\alpha) e_1 + Be_2) \\
&\quad + \frac{1}{2}(A + D + \epsilon\alpha) a - Bb)u, \\
\nabla_{e_3} V &= \nabla_{e_4} V = (E(c + d) + (\frac{1}{2}(D - A + \epsilon\alpha) b) e_1 \\
&\quad + (C(c + d) - \frac{1}{2}(D - A + \epsilon\alpha) a) e_2 + (Cb + Ea)u,
\end{align*}
\]
where \( \alpha = \sqrt{(A + D)^2 + 4B^2} \). We now calculate
\[
\nabla^* \nabla V = \sum_i \varepsilon_i (\nabla_{e_i} \nabla_{e_i} V - \nabla_{e_i} e_i V) = ((A + D)^2 + 4B^2)(c + d)u.
\]
that is, \( \nabla^* \nabla V \) identically vanishes if and only if \( c = -d \). Using some similar argument for other cases lead to the following results.

**Theorem 4.1.** Let \( G \) be a four-dimensional simply connected Lie group of type (c) and \( V = a e_1 + b e_2 + c e_3 + d e_4 \in \mathfrak{g} \) be a left-invariant vector field on \( G \) for some real constants \( a, b, c, d \). If \( g \) be a left-invariant Lorentzian Einstein metric on \( G \), then for the different cases of theorem 2.1, type (c) we have:

(12), (13), (14) and (16): \( V \) is harmonic if and only if \( V = a e_1 + b e_2 + c u \), that is, \( c = -d \).

(15) : \( V \) is harmonic.

Next, again for this case, using (16) and (11.13), we find
$R(\nabla_{e_1}V, V)e_1 = -R(\nabla_{e_2}V, V)e_2 = \frac{1}{2}B(c + d)^2((A + D)^2 + 4B^2 + (D - A)e\alpha)u,$

$R(\nabla_{e_3}V, V)e_3 = R(\nabla_{e_4}V, V)e_4 = \frac{1}{4}B(c + d)((A + D)^2 + 4B^2 + (D - A)e\alpha)$

$(-2E(c + d) - b(D - A + e\alpha))e_1 - (2C(c + d) - a(D - A + e\alpha))e_2),$

where $\alpha = \sqrt{(A + D)^2 + 4B^2}$. Therefore

$tr[R(\nabla_VV)] = \sum_i \varepsilon_i R(\nabla_{e_i}V, V)e_i = 0.$

We also proved the following.

**Theorem 4.2.** Let $G$ be a four-dimensional simply connected Lie group of type (c) and $V$ be a left-invariant harmonic vector field on $G$, described by conditions (12) – (16) of theorem [4.1] then for the different cases of theorem [4.1] we have:

1. (12), (13), (14) and (16): $V$ defines harmonic map if and only if $c = -d$.
2. (15): $V$ defines harmonic map.

Therefore, for type (c) in all cases, left-invariant harmonic vector fields define harmonic maps, which in cases (12), (13), (14) and (16), form three-parameter families.

Also, with regard to harmonicity properties of invariant vector fields, four-dimensional simply connected Lie groups of type (c) display some particular features. The main geometrical reasons for the special behaviour of these groups are the existence of a parallel light-like vector field. Using remark 4.2 we can easily prove the following classification result.

**Proposition 4.3.** Let $G$ be a four-dimensional simply connected Lie group of type (c) and $V \in \mathfrak{g}$ be a left-invariant vector field on $G$. If $g$ be a left-invariant Lorentzian Einstein metric on $G$, then we have the following classification.

| $(G, g)$ | Geodesic vector fields | Killing vector fields | Parallel vector fields |
|----------|------------------------|----------------------|-----------------------|
| (12)     | $V = be_2 + cu$        | $\times$             | $V = cu$              |
| (13)     | $V = a(e_1 - ((B + C + D)/2A)e_2) + cu$ | $\times$             | $V = cu$              |
| (14)     | $V = cu$               | $V = ce_3$           | $V = cu$              |
| (15)     | $V = b(-e_1 + e_2) + cu$ | $V = b(-(C/B)e_1 + e_2 - (A/B)e_4) + ce_3$ | $V = cu$ |
| (16)     | $V = b(-e_1 + e_2) + cu$ or $V = b(-(B/A)e_1 + e_2) + cu,$ | $V = a(e_1 - e_2)$ | $V = cu$ |

Comparing Proposition 4.3 and Theorem 4.1 with Theorem 4.2 respectively, one sees the following main result which emphasizes once again the special role played by the parallel vector field $u$.

**Corollary 4.4.** Let $V \in \mathfrak{g}$ be a left-invariant vector field on four-dimensional simply connected Lie group $G$ of type (c). If $g$ be a left-invariant Lorentzian Einstein metric on $G$, then for the different cases of theorem 4.1 the following properties are equivalent (which $V$ is restricted to vector fields of the same length):
The following properties are equivalent

(12) \( V \) is geodesic if and only if \( a = 0; \equiv V \) is harmonic; \( \equiv V \) is a critical point for the energy functional \( \equiv V \) defines harmonic map; \( \equiv V \) is parallel if and only if \( a = b = 0 \), that is, \( V \) is collinear to \( u; \equiv V = ae_1 + be_2 + cu \),

(13) \( V \) is geodesic if and only if \( b = -a(B + C + D)/2A; \equiv V \) is harmonic; \( \equiv V \) is a critical point for the energy functional \( \equiv V \) defines harmonic map; \( \equiv V \) is parallel if and only if \( a = b = 0; \equiv V = ae_1 + be_2 + cu \),

(14) \( V \) is geodesic if and only if \( a = b = 0; \equiv V \) is harmonic; \( \equiv V \) is a critical point for the energy functional \( \equiv V \) defines harmonic map; \( \equiv V \) is parallel if and only if \( a = b = 0; \equiv V = ae_1 + be_2 + cu \),

(15) \( V \) is geodesic if and only if \( a = b, c = -d; \equiv V \) is harmonic; \( \equiv V \) is a critical point for the energy functional \( \equiv V \) defines harmonic map; \( \equiv V \) is parallel if and only if \( a = b = 0; \equiv V = ae_1 + be_2 + ce_3 + de_4 \),

(16) \( V \) is geodesic if and only if \( a = -b; \equiv V \) is harmonic; \( \equiv V \) is a critical point for the energy functional \( \equiv V \) defines harmonic map; \( \equiv V \) is Killing if and only if \( a = -b \) and \( c = 0; \equiv V \) is parallel if and only if \( a = b = 0; \equiv V = ae_1 + be_2 + cu \),

5. The energy of vector fields

We calculate explicitly the energy of a vector field \( V \in g \) of a four-dimensional Einstein Lorentzian Lie group. This gives us the opportunity to determine some critical values of the energy functional on four-dimensional Einstein Lorentzian Lie group. We shall first discuss geometric properties of the map \( V \) defined by a vector field \( V \in g \).

**Type(a)** Let \( (G, g) \) be a four-dimensional Einstein Lorentzian Lie group of type \( (a) \), \( \{e_i\}_{i=1}^4 \) a pseudo-orthonormal basis with \( e_3 \) time-like. We now prove the following.

**Proposition 5.1.** Let \( G \) be a four-dimensional simply connected Lie group of type \( (a) \), \( V = ae_1 + be_2 + ce_3 + de_4 \in g \) be a vector field on \( G \) and \( D \) be its relatively compact domain. Denote by \( E_D(V) \) the energy of \( V|_D \). For the different cases of theorem 2.7, type \( (a) \) we have:

\[
(G, g) \quad E_D(V)
\]

(1) \( (2 + A^2(a^2 + 3d^2)/2)volD \)

(2) \( (2 + (-B + A)^2(a^2 + 3d^2)/8)volD \)

(3) \( (2 + (A^2 - B^2)^2||V||^2)volD \)

(4) \( (2 + (A^2 - B^2)(a^2 + b^2)/2))volD \)

(5) \( (2 + (A^2(-4a^2 - 12d^2 - 17c^2 + 24bc - 7b^2)/2))volD \)

(6) \( (2 + A^2(||V||^2 + 2d)/2 - B^2(b^2 - c^2)/2))volD \)

(7) \( (2 + (1/72)(A + B)(A(-4a^2 + 17b^2 + 7c^2 - 12d^2 - 24bc) - B(4a^2 + 7b^2 + 17c^2 + 12d^2 - 24bc)))volD \)

(8) \( (2 + (1/2)A^2(b^2 + 3d^2))volD \). In this case \( \epsilon = 1 \)
We already know from Theorems 3.2 and 4.2 which vector fields in a relatively compact domain of \( G \).

Then, locally,

\[
||\nabla V||^2 = \sum_{i=1}^{n} \varepsilon_i g(\nabla e_i V, \nabla e_i V).
\]

These conclusions are obtained from a case-by-case argument. As an example, If \( V \in g \) is a vector field of a four-dimensional Einstein Lorentzian Lie group of type (a), case (4), then (3.9) easily yield

\[
||\nabla V||^2 = (A^2 - B^2)(a^2 + b^2).
\]

Therefore, \( ||\nabla V|| = 0 \) if and only if \( A = B \). Thus, if \( A = B \) vector fields of the same length, will minimize the energy.

Type(c) Let \( (G, g) \) be a four-dimensional Einstein Lorentzian Lie group of type (c), \( \{e_1, ..., e_4\} \) a local pseudo-orthonormal basis of vector fields described in (3.11) then, from equation (4.13) for case (14), we deduce

\[
||\nabla V||^2 = ((A + D)^2 + 4B^2)(c + d)^2,
\]

which leads us to the following result.

**Proposition 5.2.** Let \( G \) be a four-dimensional simply connected Lie group of type (c), \( V \) be a vector field on \( G \) and \( D \) be its relatively compact domain. Denote by \( E_D(V) \) the energy of \( V \mid_D \). For the different cases of theorem 2.1, type (c) we have:

| \((G, g)\) | \(E_D(V)\) |
|---|---|
| (12) | \((2 + ((A + B)^2 + C^2)(c + d)^2/2))volD\) |
| (13) | \((2 + (B^2 + 4A^2 - 2CB - 2BD + (C + D)^2)(B^2 + 4A^2 + 2CB + 2BD + (C + D)^2)(c + d)^2/32A^2))volD\) |
| (14) | \((2 + ((A + D)^2 + 4B^2)(c + d)^2/2))volD\) |
| (15) | \(2volD\) |
| (16) | \((2 + ((A + B)^2 + A^2 + B^2)(c + d)^2/2))volD\) |

We already know from Theorems 5.2 and 4.2 which vector fields in \( g \) of Einstein Lorentzian Lie group are critical points for the energy functional. Taking into account Propositions (5.1) and (5.2), we then have the following.

**Theorem 5.3.** Let \( (G, g) \) be a four-dimensional Einstein Lorentzian Lie group and \( D \) a relatively compact domain of \( G \), then one of the following cases occurs.

1. \( 2volD \) is the absolute minimum for the energy functional. This minimum is attained by smooth maps defined by invariant vector fields of the form \( V = b(e_2 + e_3). \)
2. \( (2 + 3(A - B)/8)d^2\) is the minimum value of the energy functional \( E_D \). This minimum is attained by smooth maps defined by invariant vector fields of the form \( V = b(e_2 + e_3) + de_4. \)
3. \( (2 + (A^2 - B^2)/2\rho^2)volD \) is the minimum value of the energy functional \( E_D \) restricted to vector fields of constant length \( \rho \). Such a minimum is attained by all vector fields \( V = ae_1 + be_2) \in g \) of length \( ||V|| = \rho = \sqrt{a^2 + b^2}. \)
(4) \( (2 + (A^2 - B^2)\rho^2)\text{vol}D \) is the minimum value of the energy functional \( E_D \) restricted to vector fields of constant length \( \rho \). Such a minimum is attained by all vector fields \( V \in \mathfrak{g} \) of length \( ||V|| = \rho \).

(5) \( E_D = 2\text{vol}D \) is the absolute minimum for the energy functional. This minimum is attained when \( A = -B \). In this case \( \epsilon = 1 \).

(6) \( 2\text{vol}D \) is the absolute minimum for the energy functional. This minimum is attained by smooth maps defined by invariant vector fields of the form \( V = b(\mathbf{e}_2 + \mathbf{e}_3) \).

(7) \( 2\text{vol}D \) is the absolute minimum for the energy functional. This minimum is attained by smooth maps defined by invariant vector fields of the form \( V = b(\mathbf{e}_2 + \mathbf{e}_3) \). Clearly in this case \( ||V||^2 = b^2 - b^2 = 0 \).

(8) \( 2\text{vol}D \) is the absolute minimum for the energy functional. This minimum is attained by smooth maps defined by invariant vector fields of the form \( V = b(\mathbf{e}_2 + \mathbf{e}_3) \).

(9) \( 2\text{vol}D \) is the absolute minimum for the energy functional. This minimum is attained by smooth maps defined by invariant vector fields of the form \( V = b(\mathbf{e}_1 + \mathbf{e}_3) \).

(12, (13), (14) and (16): \( 2\text{vol}D \) is the absolute minimum for the energy functional. This minimum is attained by smooth maps defined by invariant vector fields of the form \( V = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3 \), where \( \mathbf{e}_i \) is the base described in (4).

(15) \( 2\text{vol}D \) is the absolute minimum for the energy functional. This minimum is attained by smooth maps defined by every invariant vector fields \( V \).

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