$p$-adic limits of renormalized logarithmic Euler characteristics

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1 Introduction

Given a countable residually finite group $\Gamma$, we write $\Gamma \rightarrow e$ if $(\Gamma_n)$ is a sequence of normal subgroups of finite index such that any infinite intersection of $\Gamma_n$’s contains only the unit element $e$ of $\Gamma$. Given a $\Gamma$-module $M$ we are interested in the multiplicative Euler characteristics

$$\chi(\Gamma_n, M) = \prod_i |H_i(\Gamma_n, M)| (-1)^i$$

and the limit in the field $\mathbb{Q}_p$ of $p$-adic numbers

$$h_p := \lim_{n \to \infty} (\Gamma : \Gamma_n)^{-1} \log_p \chi(\Gamma_n, M).$$

(2)

Here $\log_p : \mathbb{Q}_p^\times \to \mathbb{Z}_p$ is the branch of the $p$-adic logarithm with $\log_p(p) = 0$. Of course, neither (1) nor (2) will exist in general. We isolate conditions on $M$, in particular $p$-adic expansiveness which guarantee that the Euler characteristics $\chi(\Gamma_n, M)$ are well defined. That notion is a $p$-adic analogue of expansiveness of the dynamical system given by the $\Gamma$-action on the compact Pontrjagin dual $X = M^*$ of $M$. Under further conditions on $\Gamma$ we also show that the renormalized $p$-adic limit in (2) exists and equals the $p$-adic $R$-torsion $\tau_p^R(M)$ of $M$. The

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latter is a $p$-adic analogue of the Li–Thom $L^2$ $R$-torsion $\tau_{(2)}^\Gamma(M)$ of a $\Gamma$-module $M$ which they related to the entropy $h$ of the $\Gamma$-action on $X$. We view the limit $h_p$ in (2) as a version of entropy which values in the $p$-adic numbers and our formula

$$h_p = \tau_p^\Gamma(M)$$

as an analogue of the Li–Thom formula of [LT14],

$$h = \tau_{(2)}^\Gamma(M)$$

in the expansive case. The reason for this point of view is explained in section 6.

Assuming the limit (2) exists, we get a certain amount of arithmetic information about the sequence of Euler characteristics $\chi(\Gamma_n, M)$. For simplicity, let us assume that $h_p \in p\mathbb{Z}_p$ and $p \neq 2$. Then $\exp_p(h_p)$ is defined, where $\exp_p$ is the $p$-adic exponential function. For a number $\chi \in \mathbb{Q}_p^\times$ let $\chi^{(1)}$ be the component in $1 + p\mathbb{Z}_p$ under the standard decomposition

$$\mathbb{Q}_p^\times = \mu_{p-1} \times (1 + p\mathbb{Z}_p) \times p\mathbb{Z}.$$

Then the limit formula (2) is equivalent to the following assertion: There exists a sequence $(u_n), u_n \in \mathbb{Q}_p^\times$ with $u_n \to 1$ such that for large enough $n$ we have

$$\chi(\Gamma_n, M)^{(1)} = \exp_p(h_p)^{(\Gamma; \Gamma_n)} u_n^{(\Gamma; \Gamma_n)}.$$

In particular

$$\chi(\Gamma_n, M)^{(1)} \sim \exp_p(h_p)^{(\Gamma; \Gamma_n)}$$

in the sense that the quotient of both sides tends to 1 in $\mathbb{Q}_p$. For the general case, use Proposition 4.6.

The theory of algebraic dynamical systems $X$ gives some useful insights to the study of multiplicative Euler characteristics and their renormalized limits: One of the motivations for the present paper was the Li–Thom theorem $h = \tau_{(2)}^\Gamma(M)$ mentioned above. As another example, it is known that for $\Gamma = \mathbb{Z}^N$ only the principal prime ideals in a prime filtration for $M$ contribute to the entropy of $X$. The proof uses a positivity argument which does not transfer to the $p$-adic case. However, in the $p$-adic case the analogous assertion for the quantity $h_p$ defined by formula (2) is still true. This follows from a basic result on Serre’s local intersection numbers.
Here is a short review of the individual sections. In §2 we prove that multiplicative Euler characteristics are well defined under suitable $p$-adic conditions. In §2 we do this for modules over augmented rings $R$ and in §3 we specialize to integral group rings $R = \mathbb{Z} \Gamma$. In §4 we review and extend the theory of $\log_p \det_\Gamma$ which we introduced in [Den09]. It is a $p$-adic analogue of $\log \det_{\mathcal{A} \Gamma}$ where $\det_{\mathcal{A} \Gamma}$ is the Fuglede–Kadison determinant on the von Neumann group algebra $\mathcal{A} \Gamma$. In §5 using $\log_p \det_\Gamma$ we introduce $p$-adic $R$-torsion and prove the limit formula $h_p = \tau_p^\Gamma(M)$. In §6 we review some aspects of the theory of algebraic dynamical systems and rephrase the preceding results in dynamical terms. In the final §7 we specialize to $\Gamma = \mathbb{Z} \mathbb{N}$ and relate $\chi(\Gamma_n, M)$ to Serre’s intersection numbers. This allows us to calculate $h_p$ for $p$-adically expansive modules $M$ explicitly. We end with a remark how some of this connects to Arakelov theory on $\mathbb{P}^\mathbb{N}_\mathbb{Z}$.

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## 2 Multiplicative Euler characteristics

Let $R$ be an (associative unital) ring. A (left) $R$-module $M$ is of type $FP$ (resp. $FL$) if there is an exact sequence

$$0 \rightarrow P_d \xrightarrow{2} \cdots \xrightarrow{2} P_0 \xrightarrow{5} M \rightarrow 0$$

of finitely generated projective (resp. free) $R$-modules $P_\nu$ for $0 \leq \nu \leq d$. We say that $R$ is augmented if we are given a ring homomorphism $\varepsilon : R \rightarrow \mathbb{Z}$. This makes $\mathbb{Z}$ a right $R$-module. The following fact follows from the definitions:

**Proposition 2.1.** For an augmented ring $R$ and an $R$-module $M$ of type $FP$ as in (3) the groups

$$\text{Tor}^R_i(\mathbb{Z}, M) = H_i(\mathbb{Z} \otimes_R P_\ast)$$

are finitely generated $\mathbb{Z}$-modules which vanish for $i > d$.

For a $\mathbb{Z}$-module $A$ and a prime number $p$ let $\hat{A} = \varprojlim A/p^nA$ be its $p$-adic completion. Write $A_{p^\infty} = \text{Ker}(p^n : A \rightarrow A)$ and let $(\hat{A}_{p^i})$ be the projective system with transition maps $p : A_{p^{i+1}} \rightarrow A_{p^i}$ for $i \geq 1$.
Lemma 2.2. Let $R$ be a ring with $R_p = 0$. In the situation above, write $P^\bullet$ for the acyclic complex with $M$ in degree $-1$. Then we have:

a) $H_\nu(P^\bullet/p^iP^\bullet) = 0$ for $\nu \neq 1$ and $(H_1(P^\bullet/p^iP^\bullet)) = (M^{p^i})$ as projective systems

b) $H_\nu(\hat{P}^\bullet) = 0$ for $\nu \neq 0, 1$

Proof. For a projective $R$-module $P$ we have $P_p = 0$ since $P$ is a direct summand of a power of $R$ and we assumed that $R_p = 0$. Hence we have exact sequences of complexes for $i \geq 0$

$$0 \longrightarrow M_p[1] \longrightarrow P^\bullet \overset{p^i}{\longrightarrow} P^\bullet \longrightarrow P^\bullet/p^iP^\bullet \longrightarrow 0.$$ (5)

Here $M_p[1]$ is the complex with $M_p$ in degree $-1$ and which is zero elsewhere. Note the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_p[1] & \longrightarrow & P^\bullet & \overset{p^i}{\longrightarrow} & P^\bullet & \longrightarrow & P^\bullet/p^iP^\bullet & \longrightarrow & 0 \\
& & \downarrow{p} & & \downarrow{p} & & \downarrow{} & & \downarrow{} & & \\
0 & \longrightarrow & M_{p^{i-1}}[1] & \longrightarrow & P^\bullet & \overset{p^{i-1}}{\longrightarrow} & P^\bullet & \longrightarrow & P^\bullet/p^{i-1}P^\bullet & \longrightarrow & 0.
\end{array}$$ (6)

From (5) we get an exact sequence of complexes

$$0 \longrightarrow P^\bullet/M_p[1] \overset{\partial^i}{\longrightarrow} P^\bullet \longrightarrow P^\bullet/p^iP^\bullet \longrightarrow 0$$

and hence a long exact homology sequence

$$\ldots \longrightarrow H_{-1}(P^\bullet/M_p[1]) \overset{\partial^i}{\longrightarrow} H_{-1}(P^\bullet) \longrightarrow H_{-1}(P^\bullet/p^iP^\bullet) \longrightarrow 0.$$ (7)

Since $H_\nu(P^\bullet) = 0$ for all $\nu$, we get

$$H_{-1}(P^\bullet/p^iP^\bullet) = 0$$

$$H_0(P^\bullet/p^iP^\bullet) \sim H_{-1}(P^\bullet/M_p[1]) = 0,$$

noting that the map $P_0 \to P_{-1}/M_p = M/M_p$ is surjective. Moreover we find

$$H_1(P^\bullet/p^iP^\bullet) \sim H_0(P^\bullet/M_p[1]) \overset{\pi}{\longrightarrow} M_p^i,$$ (8)
since the sequence

\[ P_1 \xrightarrow{0} P_0 \xrightarrow{\pi} P_{-1}/M_{p^i} = M/M_{p^i} \]

has homology

\[ \text{Ker } \pi / \text{Im } \partial = \pi^{-1}(M_{p^i})/\text{Im } \partial = \pi^{-1}(M_{p^i})/\text{Ker } \pi \xrightarrow{\pi} M_{p^i}. \]

Because of (6), in the isomorphism (8) the projection from the \( i \)-th to the \((i - 1)\)-th term on the left corresponds to \( p : M_{p^i} \to M_{p^{i-1}} \) on the right. Hence we have

\[ (H_1(P^+_{/p^iP^+})) \cong (M_{p^i}) \]

as projective systems. Finally, the long exact sequence (7) implies that

\[ H_\nu(P^+_{/p^iP^+}) \cong H_{\nu-1}(P^+_P) = 0 \quad \text{for } \nu \geq 2. \]

Thus a) is proved. According to [Wei94, Theorem 3.5.8], for the projective system of chain complexes of abelian groups

\[ \cdots \to P^+_P / p^i P^+_P \to P^+_P / p^{i-1} P^+_P \to \cdots \to P^+_P / p^0 P^+_P = 0 \]

we have exact sequences for \( \nu \in \mathbb{Z} \)

\[ 0 \to \lim_{\leftarrow i} H_{\nu+1}(P^+_{/p^iP^+}) \to H_\nu(\hat{P}^+_P) \to \lim_{\leftarrow i} H_\nu(P^+_{/p^iP^+}) \to 0. \]

Using a), they imply the assertions in b).

Let \( R \) be a ring. For any \( R \)-module \( A \) we have a natural map

\[ \varphi_A : \hat{R} \otimes_R A \to \hat{A}, (e_i) \otimes a \mapsto (e_i a). \]

Let \( P \) be a finitely generated projective \( R \)-module. Then there is an \( R \)-module \( Q \) such that \( P \oplus Q = R^n \) as \( R \)-modules for some \( n \geq 1 \). We have \( \varphi_P \oplus \varphi_Q = \varphi_{R^n} \) and since \( \varphi_{R^n} \) is an isomorphism, \( \varphi_P \) is an isomorphism as well.

**Corollary 2.3.** Let \( R \) be a ring with \( R_p = 0 \) and let \( M \) be an \( R \)-module of type \( FP \) with resolution (3).

a) If \( \varinjlim^1 M_p = 0 \) the map

\[ \varphi_M : \hat{R} \otimes_R M \xrightarrow{\sim} \hat{M} \]
is an isomorphism.
b) If $\varprojlim \nu M_\nu = 0$ for $\nu = 0, 1$, the sequence

$$0 \to \hat{P}_d \to \ldots \to \hat{P}_0 \to \hat{M} \to 0 \quad (9)$$

obtained from $(3)$ by $p$-adic completion and the isomorphic sequence

$$0 \to \hat{R} \otimes_R P_d \to \ldots \to \hat{R} \otimes_R P_0 \to \hat{R} \otimes_R M \to 0$$

are both exact.
c) If $M$ is as in b) and if in addition $R$ is augmented by $\varepsilon : R \to \mathbb{Z}$ so that $\mathbb{Z}_p$ becomes a right $\hat{R}$ module via $\hat{\varepsilon} : \hat{R} \to \hat{\mathbb{Z}} = \mathbb{Z}_p$, we have

$$\text{Tor}_i^R(\mathbb{Z}, M) \otimes_\mathbb{Z} \mathbb{Z}_p = \text{Tor}_i^\hat{R}(\mathbb{Z}_p, \hat{M}) = \text{Tor}_i^\hat{R}(\mathbb{Z}_p, \hat{R} \otimes_R M) \quad (10)$$

Proof. a) Consider the commutative diagram

$$\begin{array}{cccccc}
\hat{R} \otimes_R P_1 & \longrightarrow & \hat{R} \otimes_R P_0 & \longrightarrow & \hat{R} \otimes_R M & \longrightarrow 0 \\
\downarrow^{\psi_{P_1}} & & \downarrow^{\psi_{P_0}} & & \downarrow^{\phi_M} & \\
\hat{P}_1 & \longrightarrow & \hat{P}_0 & \longrightarrow & \hat{M} & \longrightarrow 0
\end{array}$$

The maps $\psi_{P_1}$ and $\psi_{P_0}$ are isomorphisms by the above remark. The upper line is exact since $\hat{R} \otimes_R -$ is right exact. The lower line is exact by Lemma 2.2 since $\lim_{\leftarrow i} M_\nu = 0$. Hence $\phi_M$ is an isomorphism.
b) The exactness of $(9)$ follows with the help of Lemma 2.2. Since $\hat{R} \otimes_R P_i = \hat{P}_i$ and $\hat{R} \otimes_R M = \hat{M}$ by a), the second assertion follows.
c) The ring $\mathbb{Z}_p$ is flat over $\mathbb{Z}$. Hence we find

$$\mathbb{Z}_p \otimes_\mathbb{Z} \text{Tor}_i^R(\mathbb{Z}, M) = \mathbb{Z}_p \otimes_\mathbb{Z} H_i(\mathbb{Z} \otimes_R P_\bullet) = H_i(\mathbb{Z}_p \otimes_\hat{R} (\hat{R} \otimes_R P_\bullet)) = \text{Tor}_i^{\hat{R}}(\mathbb{Z}_p, \hat{M})$$

For the last step note that according to b), $\hat{R} \otimes_R P_\bullet$ is a resolution of $\hat{R} \otimes_R M = \hat{M}$ by projective $\hat{R}$-modules.\[\square\]

We can now prove the main result of this section which gives a $p$-adic criterion for a certain multiplicative Euler-characteristic to be defined.
Theorem 2.4. a) Let $R$ be a ring with $R_p = 0$ and $M$ an $R$-module of type $FP$ such that $p^iM = p^{n_0}M$ for some $n_0 \geq 0$ and all $i \geq n_0$. Then we have \( \lim_{\leftarrow} M_{p^i} = 0 \) and
\[
\hat{R} \otimes_R M = \hat{M} = M/p^{n_0}M.
\]
Now assume that in addition $R$ is augmented and $\lim_{\leftarrow} M_{p^i} = 0$. Then we have:

b) For each $i \geq 0$ the abelian group $\text{Tor}_i^R(\mathbb{Z}, M)$ is finite and its $p$-primary part is annihilated by $p^{n_0}$. For $i > d$ with $d$ as in a resolution (3) the groups $\text{Tor}_i^R(\mathbb{Z}, M)$ vanish. In particular the multiplicative Euler characteristics
\[
\chi_R(M) = \prod_i |\text{Tor}_i^R(\mathbb{Z}, M)|^{-1} \in \mathbb{Q}
\]
is well defined.

c) For any resolution (3) of $M$ the sequence
\[
0 \rightarrow \hat{R}_Q \otimes_R P_d \rightarrow \cdots \rightarrow \hat{R}_Q \otimes_R P_0 \rightarrow 0
\]
is exact, where $\hat{R}_Q = \hat{R} \otimes_{\mathbb{Z}} \mathbb{Q} = \hat{R} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. If $n_0 = 0$ above, i.e. if $p : M \rightarrow M$ is surjective, then even the sequence
\[
0 \rightarrow \hat{R} \otimes_R P_d \rightarrow \cdots \rightarrow \hat{R} \otimes_R P_0 \rightarrow 0
\]
is exact.

Remark. For an $R$-module $M$ of type $FP$ with $p : M \rightarrow M$ an isomorphism, all conditions on $M$ in Theorem 2.4 hold and the finite groups $\text{Tor}_i^R(\mathbb{Z}, M)$ have vanishing $p$-primary part. For example, let $M = R^r/R^r \partial$ where $\partial \in M_r(R)$ defines an injective map $\partial : R^r \rightarrow R^r$ by right multiplication. If $R_p = 0$ then $p : M \rightarrow M$ is an isomorphism precisely if $\overline{\partial} \in \text{GL}_r(\overline{R})$ where $\overline{R} = R \otimes_{\mathbb{Z}} \mathbb{F}_p$ and $\overline{\partial} = \partial \otimes \text{id}$. This follows from the snake lemma which gives an exact sequence:
\[
0 \rightarrow M_p \rightarrow \overline{R}^r \rightarrow \overline{R}^r \rightarrow M/p \rightarrow 0.
\]

Proof. a) The condition $p^iM = p^{n_0}M$ for $i \geq n_0$ implies that $\hat{M} = M/p^{n_0}M$. The commutative diagram for $i \geq j \geq n_0$
\[
\begin{array}{ccc}
0 & \rightarrow & M_{p^i} \\
\downarrow & & \downarrow \\
0 & \rightarrow & M_{p^j}
\end{array}
\begin{array}{ccc}
0 & \rightarrow & M \\
\downarrow & & \downarrow \\
p^i M & \rightarrow & p^j M
\end{array}
\begin{array}{ccc}
p^{n_0} M & \rightarrow & 0 \\
\downarrow & & \downarrow \\
p^{n_0} M & \rightarrow & 0
\end{array}
\]
gives isomorphisms
\[ M_{p^i}/p^{i-j}M_{p^i} \cong M/p^{i-j}M. \]

For \( i \geq j + n_0, j \geq n_0 \) it follows that
\[ M_{p^i}/p^{i-j}M_{p^i} \cong M/p^{n_0}M. \]

Hence for \( j \geq n_0 \) the image of \( p^{i-j}M_{p^i} \) in \( M_{p^i} \) is independent of \( i \) if \( i \geq j + n_0 \). Hence \((M_{p^i}) \) satisfies the Mittag–Leffler condition and by [Wei94, Prop. 3.5.7] we therefore have
\[ \lim_{\longleftarrow i} M_{p^i} = 0. \]

By Corollary 2.3 we therefore have
\[ \hat{R} \otimes_R M = \hat{M} = M/p^{n_0}M. \]

b) According to a) and the assumptions on \( M \) we have \( \lim_{\longleftarrow \nu} M_{p^\nu} = 0 \) for \( \nu = 0, 1 \).

The groups \( \text{Tor}_i^R(Z, M) \) are finitely generated abelian groups by Proposition 2.1 which vanish for \( i > d \). Part c) of Corollary 2.3 implies that
\[ \text{Tor}_i^R(Z, M) \otimes_Z Z_p = \text{Tor}_i^R(Z_p, \hat{M}) = \text{Tor}_i^R(Z_p, M/p^{n_0}M). \]

Hence these finitely generated \( Z_p \)-modules are annihilated by \( p^{n_0} \) and therefore finite. Thus the rank of \( \text{Tor}_i^R(Z, M) \) is zero and \( \text{Tor}_i^R(Z, M) \) is a finite abelian group. Its \( p \)-primary part is \( \text{Tor}_i^R(Z, M) \otimes_Z Z_p \) which is annihilated by \( p^{n_0} \) as we just saw.

Part c) of the theorem follows from a) and Corollary 2.3 b). \( \square \)

We add some facts about the \( p \)-conditions on \( M \) which we encountered.

**Proposition 2.5.** Let \( A \) be an abelian group. The following conditions are equivalent:

1) \( \lim_{\longleftarrow i} A_{p^i} = 0 \)
2) \( \text{Hom}((\mathbb{Q}/\mathbb{Z})(p), A) = 0 \) where \((\mathbb{Q}/\mathbb{Z})(p) = \mathbb{Q}_p/\mathbb{Z}_p\) is the \( p \)-primary part of \( \mathbb{Q}/\mathbb{Z} \).

If \( A \) has bounded \( p \)-torsion, i.e. if there is some \( m_0 \geq 1 \) such that \( A_{p^i} = A_{p^{m_0}} \) for \( i \geq m_0 \), then 1), 2) hold.
Proof. Since

\[ A_{p^i} = \text{Hom}(\mathbb{Z}/p^i, A) = \text{Hom}(p^{-i}\mathbb{Z}/\mathbb{Z}, A) \]

we find that

\[ \lim_{\leftarrow} A_{p^i} = \text{Hom}(\lim_{\to} p^{-i}\mathbb{Z}/\mathbb{Z}, A) = \text{Hom}((\mathbb{Q}/\mathbb{Z})(p), A). \]

Hence assertions 1) and 2) are equivalent. If \( A_{p^i} = A_{p^{m_0}} \) for \( i \geq m_0 \), then for any element \((a_i) \in \lim_{\leftarrow} A_{p^i}\) we have \( a_i = p^{m_0}a_{i+m_0} = 0 \) which implies 1). \( \square \)

**Proposition 2.6.** If \( R \) is a (left) Noetherian ring and \( M \) a finitely generated (left) \( R \)-module, then \( \lim_{\leftarrow} M_{p^i} = 0. \)

**Proof.** The \( R \)-module \( M \) is Noetherian and hence the ascending chain of \( R \)-submodules \( M_p \subset M_{p^2} \subset M_{p^3} \subset \ldots \) is stationary. Now the claim follows from Proposition 2.5. \( \square \)

**Remark.** We will be interested in integral group rings \( R = \mathbb{Z}\Gamma \). By a result of Hall they are left- and right-Noetherian for polycyclic-by-finite groups \( \Gamma \) but not in general.

**Proposition 2.7.** Let \( R \) be a ring with \( R_p = 0 \) and let \( M \) be an \( R \)-module of type \( FP \). There are equivalences:

a) \( p^iM = p^{n_0}M \) for all \( i \geq n_0 \) if and only if \( \hat{R} \otimes_R M \) is annihilated by \( p^{n_0}. \)

b) \( (p^iM) \) is stationary for large enough \( i \) if and only if \( R_q \otimes_R M = 0. \)

**Remark.** For \( R = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}] \) assertion b) was first observed by Bräuer \([Bra10]\).

**Proof.** a) The implication “\( \Rightarrow \)” follows from part a) of Theorem 2.4. The natural map \( \hat{R} \to R/p^iR \) is surjective. The induced surjection \( \hat{R}/p^i\hat{R} \to R/p^iR \) is also injective because \( p \)-multiplication is injective on \( R \): If \( (\overline{x}_\nu) \in \hat{R} \) satisfies \( \overline{x}_i = 0 \) then \( x_\nu = p^iy_\nu \) for all \( \nu > i \) with unique \( y_\nu \)’s in \( R \). Moreover \( x_\nu \equiv x_{\nu-1} \mod p^{\nu-1}R \) implies that \( y_\nu \equiv y_{\nu-1} \mod p^{\nu-1}R \) for \( \nu > i \). Setting \( z_\nu = y_{\nu+i} \) we obtain an element \( (\overline{z}_\nu) \in \hat{R} \) with \( p^i(\overline{z}_\nu) = (\overline{x}_\nu) \). Using the isomorphism \( \hat{R}/p^i\hat{R} = R/p^iR \) we can now prove the converse implication in a). We have:

\[ M/p^iM = (R/p^iR) \otimes_R M = (\hat{R}/p^i\hat{R}) \otimes_R M = (\hat{R} \otimes_R M)/p^i(\hat{R} \otimes_R M). \]
If \(p^{n_0}\) annihilates \(\hat{R} \otimes_R M\) it therefore follows that

\[
M/p^iM = \hat{R} \otimes_R M \quad \text{for } i \geq n_0.
\]

Hence the natural map

\[
M/p^iM \longrightarrow M/p^{n_0}M
\]

is an isomorphism for \(i \geq n_0\) and therefore \(p^iM = p^{n_0}M\).

b) follows from a) since modules of type \(FP\) are finitely generated.

**Proposition 2.8.** a) Let \(R\) be a ring and \(0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0\) a short exact sequence. If \(M'\) and \(M''\) are of type \(FP\) (resp. \(FL\)) with resolutions \(F'_{\bullet} \rightarrow M'\) and \(F''_{\bullet} \rightarrow M''\), then there is a commutative diagram with exact lines and columns where \(F_i = F'_i \oplus F''_i\)

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & F'_{i-1} & \longrightarrow & F'_{i-2} & \longrightarrow & F'_{i-3} & \longrightarrow & 0 \\
0 & \longrightarrow & M'_{i-1} & \longrightarrow & M_{i-2} & \longrightarrow & M_{i-3} & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
\]

Here \(i\) and \(\pi\) are the natural inclusion and projection. In particular \(M\) is of type \(FP\) (resp. \(FL\)) as well.

b) Now assume that in addition \(R_p = 0\) and that \(p^nM' = p^{n_0}M'\) for \(n \geq n'_0\) and \(p^nM'' = p^{n_0}M''\) for \(n \geq n''_0\). Then we have that \(p^nM = p^{n_0+n_0}M\) for \(n \geq n'_0 + n''_0\).

c) For any exact sequence \(0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0\) of \(R\)-modules the conditions \(\lim_{\leftarrow i} M'_{pi} = 0\) and \(\lim_{\leftarrow i} M''_{pi} = 0\) imply that \(\lim_{\leftarrow i} M_{pi} = 0\).

**Proof.** a) is the Horseshoe Lemma 2.2.8 of [Wei94].

b) Note first that \(M\) is of type \(FP\) by a). The remaining assertion follows easily using Proposition 2.7 a) or directly as follows: For \(n \geq n_0 = n'_0 + n''_0\) consider \(x \in p^{n_0}M, x = p^{n_0}x_1\) with \(x_1 \in M\). The image of \(p^{n-n_0}x\) in \(M''\) is of the form \(p^{n-n_0}x''\) for some \(x'' \in M\) since \(n - n'_0 \geq n''_0\). Let \(y\) be a preimage of \(x''\) in \(M\). Then \(p^{n_0}x_1 - p^{n-n_0}y\) lies in \(M'\) and hence \(x = p^{n_0}x_1 - p^{n_0}y \in p^{n_0}M' = p^nM'\). Thus \(x \in p^nM\) as claimed.

Assertion c) follows since both \(\lim_{\leftarrow i} M'_{pi}\) and \(M \mapsto M_{pi}\) are left exact functors. □
3 Homology of discrete groups for $p$-adically expansive modules

In this section we consider the case where $R = \mathbb{Z}\Gamma$ is the integral group ring of a discrete group $\Gamma$. We are particularly interested in the case where $\Gamma$ is residually finite. The group ring $\mathbb{Z}\Gamma$ is augmented by $\varepsilon : \mathbb{Z}\Gamma \to \mathbb{Z}$ defined by $\varepsilon(\gamma) = 1$ for all $\gamma \in \Gamma$. By definition

$$H_i(\Gamma, M) = \text{Tor}^Z_i(\mathbb{Z}, M),$$

for a $\Gamma$- or equivalently $\mathbb{Z}\Gamma$-module $M$. There is a natural isomorphism:

$$\hat{\mathbb{Z}}\Gamma = \left\{ \sum_{\gamma \in \Gamma} x_\gamma \gamma \mid x_\gamma \in \mathbb{Z}_p \text{ with } |x_\gamma| \to 0 \text{ for } \gamma \to \infty \right\}. \quad (11)$$

Here $|x_\gamma|$ is the $p$-adic absolute value and $\gamma \to \infty$ means convergence in the cofinite topology of $\Gamma$: For all $\varepsilon > 0$ there is a finite subset $S \subset \Gamma$ such that $|x_\gamma| < \varepsilon$ for all $\gamma \in \Gamma \setminus S$. The augmentation map $\hat{\varepsilon} : \hat{\mathbb{Z}}\Gamma \to \mathbb{Z}_p$ induced by $\varepsilon : \mathbb{Z}\Gamma \to \mathbb{Z}$ sends $\sum_{\gamma \in \Gamma} x_\gamma \gamma$ to the convergent series $\sum_{\gamma \in \Gamma} x_\gamma$. We set

$$c_0(\Gamma) := \hat{\mathbb{Z}}\Gamma \otimes_{\mathbb{Z}} \mathbb{Q} = \left\{ \sum_{\gamma \in \Gamma} x_\gamma \gamma \mid x_\gamma \in \mathbb{Q}_p \text{ with } |x_\gamma| \to 0 \text{ for } \gamma \to \infty \right\}.$$ We call a $\mathbb{Z}\Gamma$-module $M$ of type FL $p$-adically expansive if $c_0(\Gamma) \otimes_{\mathbb{Z}\Gamma} M = 0$ or equivalently if the flag $(p^iM)$ is stationary, see Proposition 2.7 b). More precisely, we call such a module $M$ $p$-adically expansive of exponent $n_0$ if $p^iM = p^{n_0}M$ holds for all $i \geq n_0$. By Proposition 2.8 given an exact sequence of $\mathbb{Z}\Gamma$-modules

$$0 \to M' \to M \to M'' \to 0$$

with $M'$, $M''$ $p$-adically expansive of exponents $n'_0$ and $n''_0$, the module $M$ is $p$-adically expansive of exponent $n'_0 + n''_0$. The module $M$ is $p$-adically expansive of exponent zero i.e. $pM = M$ if and only if $\hat{\mathbb{Z}}\Gamma \otimes_{\mathbb{Z}\Gamma} M = 0$, cf. Proposition 2.7 a). The reason for the name “$p$-adically expansive” will be explained in §6. In the present context, Theorem 2.4 gives:

**Theorem 3.1.** Let $\Gamma$ be a discrete group and let $\Delta$ be a normal subgroup of finite index. Let $M$ be a $p$-adically expansive $\mathbb{Z}\Gamma$-module of exponent $n_0$ and $FP$-resolution (3) of length $d$. Assume also that $\lim_{\leftarrow i} M_{p^i} = 0$. Then the homology
groups $H_i(\Delta, M)$ are finite abelian groups whose $p$-primary part is annihilated by $p^{n_0}$ and which vanish for $i > d$. Moreover the sequence

$$0 \to c_0(\Delta) \otimes_{\mathbb{Z}\Delta} P_d \to \ldots \to c_0(\Delta) \otimes_{\mathbb{Z}\Delta} P_0 \to 0$$

is exact. If $n_0 = 0$ i.e. if $p : M \to M$ is surjective, then even the sequence

$$0 \to \hat{\mathbb{Z}} \otimes_{\mathbb{Z}\Delta} P_d \to \ldots \to \hat{\mathbb{Z}} \otimes_{\mathbb{Z}\Delta} P_0 \to 0$$

is exact.

**Proof.** For any finitely generated $\mathbb{Z}\Gamma$-module $P$ there is an exact sequence of $\mathbb{Z}\Gamma$-modules for some $n$

$$0 \to Q \to (\mathbb{Z}\Gamma)^n \to P \to 0.$$ 

If $P$ is projective, we have $P \oplus Q \cong (\mathbb{Z}\Gamma)^n$ as $\mathbb{Z}\Gamma$-modules. The $\mathbb{Z}\Gamma$-algebra $\mathbb{Z}\Gamma$ is a free $\mathbb{Z}\Delta$-module with basis $\gamma_1, \ldots, \gamma_r$ a system of representatives of $\Delta \setminus \Gamma$, $r = (\Gamma : \Delta)$. Hence $P \oplus Q$ is a free $\mathbb{Z}\Delta$-module of rank $nr$ and in particular $P$ is finitely generated and projective as a $\mathbb{Z}\Delta$-module. Any $\mathbb{Z}\Gamma$-module $M$ of type $FP$ (or $FL$) i.e. with a resolution (3) for $R = \mathbb{Z}\Gamma$ is therefore also of type $FP$ (resp. $FL$) as a $\mathbb{Z}\Delta$-module with the same resolution (3) but now viewed as $\mathbb{Z}\Delta$-modules. It follows that a $p$-adically expansive $\Gamma$-module $M$ (of exponent $n_0$) is also $p$-adically expansive (of exponent $n_0$) as a $\Delta$-module. The result now follows from Theorem 2.4 applied to $M$ as an $R = \mathbb{Z}\Delta$-module. \qed

For a countable residually finite group $\Gamma$, sequences $\Gamma_n \to e$ as in the introduction always exist. If $M$ is a $p$-adically expansive $\Gamma$-module with $\lim_{\Gamma_n} M_{p^i} = 0$ then by Theorem 3.1 the Euler characteristics

$$\chi(\Gamma_n, M) := \prod_{i} |H_i(\Gamma_n, M)|(-1)^i \in \mathbb{Q}^\times$$

(12)

exist for all $n$. We will later be concerned with their renormalized $p$-adic logarithmic limit as $\Gamma_n \to e$.

We now discuss the assumptions on $M$ in Theorem 3.1 in the case where the $\mathbb{Z}\Gamma$-module $M$ is of type $FL$ with $d = 1$. There is thus a resolution

$$0 \to (\mathbb{Z}\Gamma)^s \xrightarrow{\partial} (\mathbb{Z}\Gamma)^r \to M \to 0$$

(13)
and $M$ is $p$-adically expansive i.e. $c_0(\Gamma) \otimes_{\mathbb{Z}\Gamma} M = 0$ if and only if the induced map

$$c_0(\Gamma)^s \overset{\partial}{\to} c_0(\Gamma)^r$$

is surjective. In this case we have $s \geq r$ as one sees by tensoring with $\mathbb{Q}_p$, viewed as a $c_0(\Gamma)$-module via the augmentation map $\hat{\varepsilon} : c_0(\Gamma) \to \mathbb{Q}_p$. The snake lemma applied to (13) leads to an exact sequence

$$0 \to \lim_{\leftarrow i} M_{p^i} \to (\hat{\mathbb{Z}}\Gamma)^s \overset{\partial}{\to} (\hat{\mathbb{Z}}\Gamma)^r$$

which implies that $\lim_{\leftarrow i} M_{p^i} = 0$ if and only if $\partial : (\hat{\mathbb{Z}}\Gamma)^s \to (\hat{\mathbb{Z}}\Gamma)^r$ or equivalently $\partial : c_0(\Gamma)^s \to c_0(\Gamma)^r$ is injective. Replacing $c_0(\Gamma)$ by $\hat{\mathbb{Z}}\Gamma$ we get corresponding statements for $p$-adically expansive modules of degree zero. This proves the following proposition noting that $\mathbb{Z}\Gamma$ is a subring of $c_0(\Gamma)$.

**Proposition 3.2.** In the situation (13) the $\mathbb{Z}\Gamma$-module $M$ is $p$-adically expansive (of degree zero) and satisfies $\lim_{\leftarrow i} M_{p^i} = 0$ if and only if $r = s$ and $\partial \in M_r(\mathbb{Z}\Gamma) \cap \text{GL}_r(c_0(\Gamma))$ (resp. $\partial \in M_r(\mathbb{Z}\Gamma) \cap \text{GL}_r(\hat{\mathbb{Z}}\Gamma)$).

A ring $R$ is called directly finite if $ab = 1$ in $R$ implies $ba = 1$. If the ring $M_r(c_0(\Gamma))$ is directly finite, any surjective $c_0(\Gamma)$-linear map $c_0(\Gamma)^r \to c_0(\Gamma)^r$ is also injective. In an exact sequence

$$0 \to (\mathbb{Z}\Gamma)^r \overset{\partial}{\to} (\mathbb{Z}\Gamma)^r \to M \to 0$$

with $p$-adically expansive $M$ the condition $\lim_{\leftarrow i} M_{p^i} = 0$ is therefore automatic if $M_r(c_0(\Gamma))$ is directly finite. In the somewhat analogous case of the real or complex $L^1$-group algebra of $\Gamma$ it is known that $M_r(L^1(\Gamma))$ is directly finite. This follows from a result of Kaplansky [Kap69], p. 122. For residually finite groups this is easy to see and the same proof works in the $p$-adic case, see [Bra10, section 7.4]:

**Theorem 3.3** (Bräuer). For a residually finite group $\Gamma$ the ring $M_r(c_0(\Gamma))$ is directly finite for each $r \geq 1$.

**Proof.** For a cofinite normal subgroup $\Delta$ of $\Gamma$ set $\overline{\Gamma} = \Gamma/\Delta$ and consider the natural ring homomorphism $c_0(\Gamma) \to c_0(\overline{\Gamma}) = \mathbb{Q}_p[\overline{\Gamma}]$ sending $x = \sum_{\gamma} x_{\gamma} \gamma$ to
\[ \bar{x} = \sum_{\gamma} x_{\gamma} \gamma \Delta = \sum_{\delta \in \Gamma} y_{\delta} \delta \] where \( y_{\delta} = \sum_{\gamma \in \delta} x_{\gamma} \). We claim that the ring homomorphism:

\[ \alpha : c_0(\Gamma) \to \prod_{\Delta} c_0(\Gamma/\Delta) \]

is injective, where the product runs over all \( \Delta \) as above. Assume \( \ker \alpha \neq 0 \). Since \( \ker \alpha \) is an ideal there is then an element \( x \in \ker \alpha \) with \( x_e \neq 0 \). By definition of \( c_0(\Gamma) \) the set \( S \subset \Gamma \) of \( e \neq \gamma \in \Gamma \) with \( |x_{\gamma}| \geq |x_e| > 0 \) is finite. Since \( \Gamma \) is residually finite there is a cofinite normal subgroup \( \Delta \) of \( \Gamma \) with \( \Delta \cap S = \emptyset \). Hence we have \( |x_{\gamma}| < |x_e| \) for all \( e \neq \gamma \in \Delta \). By the ultrametric inequality it follows that

\[ \left| \sum_{\gamma \in \Delta} x_{\gamma} \right| = |x_e| \neq 0. \]

On the other hand \( \sum_{\gamma \in \Delta} x_{\gamma} = 0 \) since \( x \) is mapped to zero in \( c_0(\Gamma/\Delta) \). This is a contradiction and hence \( \alpha \) is injective. It follows that the natural map

\[ M_r(c_0(\Gamma)) \to \prod_{\Delta} M_r(c_0(\Gamma/\Delta)) \]

is injective as well, [Bra10, Corollary 7.15]. Each of the rings \( M_r(c_0(\Gamma/\Delta)) \) is directly finite since \( c_0(\Gamma/\Delta) \) is finite dimensional as a \( \mathbb{Q}_p \)-vector space and \( M_r(c_0(\Gamma/\Delta)) \) is naturally a subring of \( \text{End}_{\mathbb{Q}_p}(c_0(\Gamma/\Delta)) \). Hence \( M_r(c_0(\Gamma)) \) is directly finite as well. \( \square \)

4 Review of a \( p \)-adic determinant

In this section which is based on [Den09] we review the definition of a \( p \)-adic analogue of the Fuglede–Kadison determinant and its properties.

Let \( B \) be a \( \mathbb{Q}_p \)-Banach algebra whose norm takes values in \( p^\mathbb{Z} \cup \{0\} \). Let \( \text{tr}_B : B \to \mathbb{Q}_p \) be a trace functional i.e. a continuous linear map with \( \text{tr}_B(ab) = \text{tr}_B(ba) \) for all \( a, b \in B \). Consider the \( \mathbb{Z}_p \)-Banach algebra \( A = B^0 := \{ b \in B \mid \|b\| \leq 1 \} \) and let

\[ U^1 = 1 + pA = \{ b \in B \mid \|1-b\| < 1 \} \]
be the normal subgroup of 1-units in $A^\times$. Note here that for $a \in A$ the convergent series

$$(1 + pa)^{-1} := \sum_{\nu=0}^{\infty} (-pa)^{\nu}$$

gives an inverse to $1 + pa \in U^1$. There is an exact sequence of groups

$$1 \longrightarrow U^1 \longrightarrow A^\times \xrightarrow{\pi} (A/pA)^\times \longrightarrow 1. \quad (14)$$

The projection $\pi$ is surjective since for $\overline{a} = a + pA$ in $(A/pA)^\times$ there is some $b \in A$ with $ab = 1 + pc$ and $ba = 1 + bd$ with $c, d \in A$. Since $1 + pc$ and $1 + pd$ are units it follows that $a$ is a unit as well.

The logarithmic series

$$\log : U^1 \longrightarrow A, \quad \log u = -\sum_{\nu=1}^{\infty} \frac{(1 - u)^{\nu}}{\nu}$$

converges and defines a continuous map. In [Den09, Theorem 4.1] it is shown that the map

$$\text{tr}_B \log : U^1 \longrightarrow \mathbb{Q}_p$$

(15) is a homomorphism. This is a consequence of the $p$-adic Campbell–Baker–Hausdorff formula. If $\text{tr}_B(A) \subset \mathbb{Z}_p$, then $\text{tr}_B \log$ takes values in $\mathbb{Z}_p$.

Let $\Gamma$ be a discrete group and recall the algebra

$$c_0(\Gamma) = \widehat{\mathbb{Z}\Gamma} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \left\{ \sum_{\gamma} x_{\gamma} \gamma \mid x_{\gamma} \in \mathbb{Q}_p \text{ with } |x_{\gamma}| \to 0 \text{ for } \gamma \to \infty \right\}.$$

Equipped with the norm

$$\left\| \sum_{\gamma} x_{\gamma} \gamma \right\| = \max_{\gamma} |x_{\gamma}|$$

it becomes a $\mathbb{Q}_p$-Banach algebra and the norm topology induced on $\widehat{\mathbb{Z}\Gamma} = c_0(\Gamma)^0$ is the $p$-adic topology on $\widehat{\mathbb{Z}\Gamma}$. A short calculation shows that the map

$$\text{tr}_\Gamma : c_0(\Gamma) \longrightarrow \mathbb{Q}_p, \quad \sum_{\gamma} x_{\gamma} \gamma \longmapsto x_e$$
defines a trace functional. More generally, \( B = M_r(c_0(\Gamma)) \) with norm \( \|(a_{ij})\| = \max_{i,j} \|a_{ij}\| \) is a \( \mathbb{Q}_p \)-Banach algebra with trace functional

\[
\text{tr}_\Gamma : M_r(c_0(\Gamma)) \xrightarrow{\text{tr}} c_0(\Gamma) \xrightarrow{\text{tr}} \mathbb{Q}_p.
\]

The algebra \( A = B^0 \) is given by \( M_r(\widehat{\mathbb{Z}\Gamma}) \) and we have \( U^1 = 1 + pM_r(\widehat{\mathbb{Z}\Gamma}) \). The exact sequence \((14)\) reads as follows here:

\[
1 \longrightarrow 1 + pM_r(\widehat{\mathbb{Z}\Gamma}) \longrightarrow \text{GL}_r(\widehat{\mathbb{Z}\Gamma}) \longrightarrow \text{GL}_r(F_p\Gamma) \longrightarrow 1.
\]

The map \((15)\) is denoted by

\[
\log_p \det_\Gamma := \text{tr}_\Gamma \log : 1 + pM_r(\widehat{\mathbb{Z}\Gamma}) \longrightarrow \mathbb{Z}_p.
\]

It is a continuous homomorphism of groups. We wish to extend \( \log_p \det_\Gamma \) to \( \text{GL}_r(\widehat{\mathbb{Z}\Gamma}) \). For each of the algebras \( R = \mathbb{Z}\Gamma, \widehat{\mathbb{Z}\Gamma}, c_0(\Gamma) \) and \( F_p\Gamma \) we set \( K_T(R) = K_1(R)/\langle \pm \Gamma \rangle \) where \( \langle \rangle : R^\times \rightarrow K_1(R) \) is the natural map. Note that \( K_T(\mathbb{Z}\Gamma) \) is the Whitehead group \( \text{Wh}(\Gamma) \) of \( \Gamma \) and that \( K_T(F_p\Gamma) \) is torsion if and only if \( \text{Wh}^{F_p}(\Gamma) := K_1(F_p\Gamma)/\langle F_p^\times \cdot \Gamma \rangle \) is torsion. The kernel \( \mathcal{N} \) of the exact sequence induced by \((16)\),

\[
1 \longrightarrow \mathcal{N} \longrightarrow K_T(\widehat{\mathbb{Z}\Gamma}) \longrightarrow K_T(F_p\Gamma) \longrightarrow 1
\]

is a quotient of \( 1 + pM_\infty(\widehat{\mathbb{Z}\Gamma}) \) where \( M_\infty(A) \) is the non-unital algebra of infinite matrices \( (a_{ij})_{i,j \geq 1} \) with only finitely many non-zero entries. For residually finite groups \( \Gamma \) we showed in the proof of \cite[Theorem 5.1]{Den09} that the homomorphism \( \log_p \det_\Gamma \) of \((17)\) factors over \( \mathcal{N} \) and conjectured that this should be true for all groups \( \Gamma \). We get the following result, cf. \cite[Theorem 5.1]{Den09}.

**Theorem 4.1.** Let \( \Gamma \) be a countable residually finite discrete group such that \( \text{Wh}^{F_p}(\Gamma) \) is torsion. Then there is a unique homomorphism

\[
\log_p \det_\Gamma : K_T(\widehat{\mathbb{Z}\Gamma}) \longrightarrow \mathbb{Q}_p
\]

with the following property: For every \( r \geq 1 \) the composition

\[
1 + pM_r(\widehat{\mathbb{Z}\Gamma}) \xrightarrow{\text{tr}} \text{GL}_r(\widehat{\mathbb{Z}\Gamma}) \longrightarrow K_1(\widehat{\mathbb{Z}\Gamma}) \longrightarrow K_T(\widehat{\mathbb{Z}\Gamma}) \xrightarrow{\log_p \det_\Gamma} \mathbb{Q}_p
\]

coincides with the map \((17)\).
Remark. \( \text{Wh}^{F_p}(\Gamma) \) is torsion for torsion free elementary amenable groups by [FL03] and for a class of groups that comprises all word hyperbolic groups by [BLR08]. For \( \Gamma \) as in the theorem we define the homomorphism \( \log_p \det_{\Gamma} \) on \( \text{GL}_r(\hat{\Gamma}) \) to be the composition
\[
\log_p \det_{\Gamma} : \text{GL}_r(\hat{\Gamma}) \to K_T(\hat{\Gamma}) \xrightarrow{\log_p \det_{\Gamma}} \mathbb{Q}_p .
\] (18)

Explicitely it is given as follows: For a matrix \( f \in \text{GL}_r(\hat{\Gamma}) \) there are integers \( N \geq 1 \) and \( s \geq r \) such that in \( \text{GL}_s(\hat{\Gamma}) \) we have
\[
f^N = i(\pm \gamma) \varepsilon g .
\]

Here \( \varepsilon \) is a product of \( s \times s \)-elementary matrices (identity matrices with one off-diagonal entry), \( g \) is in \( 1 + pM_s(\hat{\Gamma}) \) and \( i(\pm \gamma) \) for some \( \gamma \in \Gamma \) is the \( s \times s \)-matrix
\[
\begin{pmatrix}
\pm \gamma & 0 \\
0 & 1_{s-1}
\end{pmatrix}.
\]

Then we have
\[
\log_p \det_{\Gamma} f = \frac{1}{N} \log_p \det_{\Gamma} g = \frac{1}{N} \text{tr}_{\Gamma} \log g .
\]

For a countable residually finite group \( \Gamma \) let \((\Gamma_n)\) be a sequence of cofinite normal subgroups with \( \Gamma_n \to e \). The quotient \( \Gamma^{(n)} = \Gamma / \Gamma_n \) is a finite group. For \( f \in \text{GL}_r(c_0(\Gamma)) \) let \( f^{(n)} \) be its image in
\[
\text{GL}_r(c_0(\Gamma^{(n)})) = \text{GL}_r(\mathbb{Q}_p[\Gamma^{(n)}]) = \text{Aut}_{\mathbb{Q}_p[\Gamma^{(n)}]}(\mathbb{Q}_p[\Gamma^{(n)}]^r)
\]
under the canonical homomorphism. We write \( \det_{\mathbb{Q}_p}(f^{(n)}) \) for the determinant of \( f^{(n)} \) viewed as a \( \mathbb{Q}_p \)-linear automorphism of the \( r(\Gamma : \Gamma_n) \)-dimensional \( \mathbb{Q}_p \)-vector space \( \mathbb{Q}_p[\Gamma^{(n)}]^r \). Let \( \log_p : \mathbb{Q}_p^\times \to \mathbb{Z}_p \) be the \( p \)-adic logarithm normalized by the condition \( \log_p(p) = 0 \).

The logarithmic determinant \( \log_p \det_{\Gamma} \) which is defined for operators on generally infinite dimensional \( p \)-adic Banach spaces can be approximated by renormalized logarithmic determinants of operators on finite dimensional vector spaces if \( \Gamma \) is residually finite. In [Den09, Proposition 5.5] we proved

**Theorem 4.2.** For a countable residually finite discrete group \( \Gamma \) and a sequence \( \Gamma_n \to e \) the following formula holds if \( f \in 1 + pM_r(\hat{\Gamma}) \) or if \( f \in \text{GL}_r(\hat{\Gamma}) \) and \( \text{Wh}^{F_p}(\Gamma) \) is torsion:
\[
\log_p \det_{\Gamma} f = \lim_{n \to \infty} \left( \Gamma : \Gamma_n \right)^{-1} \log_p(\det_{\mathbb{Q}_p}(f^{(n)})) \quad \text{in} \quad \mathbb{Q}_p .
\]
Remark. Note that the nature of both sides in this formula is very different. For example, for \( f \in 1 + p M_r(\overline{Z}\Gamma) \subset \text{GL}_r(\overline{Z}\Gamma) \) the left hand side is defined for all \( \Gamma \) by \((17)\) whereas the right hand side requires \( \Gamma \) to be residually finite. We would also like to define the left hand side for all \( f \in \text{GL}_r(c_0(\Gamma)) \), so that the formula holds in this generality but we do not know how to do this in general.

Corollary 4.3. Let \( \Gamma \) be a countable residually finite group. We have \( \log \det_{\Gamma} f = 0 \) if \( f \in 1 + p M_r(\overline{Z}\Gamma) \) or if \( f \in \text{GL}_r(\overline{Z}\Gamma) \) and \( \text{Wh}^{\mathbb{F}_p}(\Gamma) \) is torsion.

Proof. This follows from Theorem 4.2 because \( f^{(n)} \) respects the \( \mathbb{Z} \)-lattice \( \mathbb{Z}[\Gamma^{(n)}]^r \) in \( \mathbb{Q}_p[\Gamma^{(n)}]^r \) and hence \( \det_{\mathbb{Q}_p}(f^{(n)}) = \pm 1 \).

For finitely generated abelian groups \( \Gamma \), Theorem 4.2 can be generalized to all \( f \in \text{GL}_r(c_0(\Gamma)) \) because in this case there is the usual determinant \( \det_{c_0(\Gamma)} : \text{GL}_r(c_0(\Gamma)) \rightarrow c_0(\Gamma)^\times \).

We explain this in the case where \( \Gamma = \mathbb{Z}^N \), see also [Brä10, §4.3] for a related approach. By a classical result on Tate algebras, see e.g. [Den09, Example 2.3], we have a direct product decomposition:

\[ c_0(\Gamma)^\times = p^Z \mu_{p-1} \Gamma U_1^\Gamma \quad \text{for} \quad \Gamma = \mathbb{Z}^N. \quad (19) \]

Here \( \mu_{p-1} \subset \mathbb{Z}_p^\times \) is the group of \( p-1 \)th roots of unity in \( \mathbb{Q}_p^\times \) and \( U_1^\Gamma = 1 + p\widehat{\mathbb{Z}}\Gamma \) is the subgroup of 1-units in \( c_0(\Gamma) \). For \( f \in \text{GL}_r(c_0(\Gamma)) \), according to \((19)\) we may write

\[ \det_{c_0(\Gamma)}(f) = p^\nu \zeta g \quad \text{in} \quad c_0(\Gamma)^\times \quad \text{with} \quad g \in U_1^\Gamma \quad \text{etc.} \quad (20) \]

We define

\[ \log_p \det_{\Gamma} f := \log_p \det_{\Gamma} g \quad \text{as in} \quad (17). \quad (21) \]

In this way we obtain a homomorphism

\[ \log_p \det_{\Gamma} : K_T(c_0(\Gamma)) \rightarrow \mathbb{Q}_p, \]

which extends the map in Theorem 4.1. This follows from the uniqueness assertion in Theorem 4.1 using the formula

\[ \text{tr} \log f = \log \det_{c_0(\Gamma)} f \quad \text{in} \quad c_0(\Gamma) \quad (22) \]
for $f \in 1 + pM_r(\widehat{\mathbb{Z}}\Gamma)$. Namely, for such $f$ we have $g = \det c_0(\Gamma)(f)$ and hence

$$
\log_p \det \Gamma f \overset{[21]}{=} \log_p \det \Gamma (\det c_0(\Gamma) f) = \log_p \det c_0(\Gamma) f
$$

$$
\overset{[22]}{=} \text{tr}_\Gamma \log \det c_0(\Gamma) f
$$

$$
\overset{[17]}{=} \text{tr}_\Gamma \text{tr} \log f
$$

$$
= \text{tr}_\Gamma \log f \overset{[17]}{=} \log_p \det \Gamma f.
$$

One way to prove equation (22) is by embedding the Tate algebra

$$
c_0(\Gamma) \cong \mathbb{Q}_p\langle t_1^{\pm 1}, \ldots, t_N^{\pm 1} \rangle,
$$

into its quotient field and applying [Har77, Appendix C, Lemma 4.1].

**Corollary 4.4.** For $\Gamma = \mathbb{Z}^N$ the limit formula in Theorem 4.2 extends to all $f \in \text{GL}_r(c_0(\Gamma))$.

**Proof.** Using [Bou70, § 9, No 4, Lemma 1] we find

$$
\det_{\mathbb{Q}_p}(f^{(n)}) = \det_{\mathbb{Q}_p}(\varepsilon^{(n)}) \text{ where } \varepsilon^{(n)} = \det_{c_0(\Gamma^{(n)})}(f^{(n)}).
$$

Equation (20) gives

$$
\varepsilon^{(n)} = (\det_{c_0(\Gamma)}(f))^{(n)} = (p^\nu \zeta^n g^{(n)}).
$$

Since $\log_p$ vanishes on roots of unity and powers of $p$ we obtain

$$
\log_p \det_{\mathbb{Q}_p}(f^{(n)}) = \log_p \det_{\mathbb{Q}_p}(g^{(n)}).
$$

On the other hand, by definition (21) we have

$$
\log_p \det \Gamma f = \log_p \det \Gamma g.
$$

Now the Corollary follows from Theorem 4.2 applied to $g$. \qed

**Remark 4.5.** By Corollary 4.3 we know that $\log_p \det \Gamma f = 0$ for $f \in \text{GL}_r(\mathbb{Z}\Gamma)$ where $\Gamma = \mathbb{Z}^N$. This can also be seen from definition (21) since $\det_{c_0(\Gamma)}(f) \in (\mathbb{Z}^N)^\times = \mu_2 \Gamma$, so that $g = 1$. Note here that $\mathbb{Z}\Gamma \cong \mathbb{Z}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}]$. 

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We end this section with a reformulation of the existence of limits of renormalized $p$-adic logarithms. Consider the direct product decompositions

$$Q_p^\times = \mu_{p-1} \times \left(1 + p\mathbb{Z}_p\right) \times \mathbb{Z}$$  for $p \neq 2$$

and

$$Q_2^\times = \mu_2 \times \left(1 + 4\mathbb{Z}_2\right) \times 2\mathbb{Z}$$  for $p = 2$.

For $\chi \in Q_p^\times$ let $\chi^{(1)}$ be the component in $1 + p\mathbb{Z}_p$ for $p \neq 2$ resp. in $1 + 4\mathbb{Z}_2$ for $p = 2$.

**Proposition 4.6.** Consider numbers $\chi_n \in Q_p^\times$, $a_n \in \mathbb{Z} \setminus 0$ and $h_p \in \mathbb{Q}_p$. Then the following assertions are equivalent:

1) $\lim_{n \to \infty} \frac{1}{a_n} \log_p \chi_n = h_p$ in $\mathbb{Q}_p$.

2) For some (equivalently, all) integer(s) $k \in \mathbb{Z} \setminus 0$ with $|kh_p| < 1$ if $p \neq 2$ resp. $|kh_2| < 1/2$ if $p = 2$, there is a sequence $(u_n)$ in $Q_p^\times$ with $u_n \to 1$ such that

$$\chi_n^{(1)k} = \exp_p(h_p)^{a_n} u_n^{k a_n}$$  for large enough $n$.

In particular

$$\chi_n^{(1)k} \sim \exp_p(h_p)^{a_n} \text{ in } \mathbb{Q}_p \text{ as } n \to \infty .$$

**Proof.** 1) $\Rightarrow$ 2) Writing

$$\frac{1}{a_n} \log_p \chi_n = h_p + \varepsilon_n$$  with $\varepsilon_n \to 0$$

we get

$$\log_p \chi_n^{(1)k} = k \log_p \chi_n^{(1)} = k \log_p \chi_n = a_n kh_p + a_n k\varepsilon_n .$$

By assumption $\chi_n^{(1)k}$ lies in $1 + p\mathbb{Z}_p$ for $p \neq 2$ and in $1 + 4\mathbb{Z}_2$ for $p = 2$. Hence we have

$$\exp_p \log_p \chi_n^{(1)k} = \chi_n^{(1)k} .$$

On the other hand $|kh_p| < (p\sqrt[p]{p})^{-1}$ by assumption and $|\varepsilon_n| < (p\sqrt[p]{p})^{-1}$ for $n$ large enough. Hence we find

$$\exp_p(a_n kh_p + a_n k\varepsilon_n) = \exp_p(h_p)^{a_n} \exp_p(\varepsilon_n)^{k a_n} .$$

Setting $u_n := \exp_p(\varepsilon_n)$ we obtain the formula for $\chi_n^{(1)k}$ claimed in 2).

2) $\Rightarrow$ 1) follows similarly by applying $\log_p$ in 2).
5 \textit{$p$-adic $R$-torsion}

In this section we prove that a renormalized $p$-adic limit of logarithms of multiplicative Euler characteristics exists and equals a suitably defined $p$-adic version of $R$-torsion.

Consider a discrete group $\Gamma$ and a (left) $\mathbb{Z}\Gamma$-module $M$ of type $FL$. Choose a resolution
\[
0 \rightarrow F_d \xrightarrow{\partial} \ldots \xrightarrow{\partial} F_0 \xrightarrow{\pi} M \rightarrow 0
\]
of $M$ by finitely generated free $\mathbb{Z}\Gamma$-modules $F_\nu$ for $0 \leq \nu \leq d$. For each $\nu$ we also fix a $\mathbb{Z}\Gamma$-basis $b_\nu$ of $F_\nu$. Assume that the complex
\[
0 \rightarrow \widehat{\mathbb{Z}} \Gamma \otimes_{\mathbb{Z} \Gamma} F_d \xrightarrow{\partial} \ldots \xrightarrow{\partial} \widehat{\mathbb{Z}} \Gamma \otimes_{\mathbb{Z} \Gamma} F_0 \rightarrow 0
\]
where $\hat{\partial} := \text{id} \otimes \partial$ is acyclic. We write $\tau(F_\bullet, b_\bullet)$ for the torsion in $K_T(\widehat{\mathbb{Z}} \Gamma)$ of this complex equipped with the induced bases $1 \otimes b_\nu$, cf. [Mil66, §3,4]. Consider an exact sequence of $\mathbb{Z}\Gamma$-modules
\[
0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0
\]
of $\mathbb{Z}\Gamma$-modules. Assume that $(F'_\bullet \rightarrow M', b'_\bullet)$ and $(F''_\bullet \rightarrow M'', b''_\bullet)$ are based resolutions as above such that the complexes $\widehat{\mathbb{Z}} \Gamma \otimes_{\mathbb{Z} \Gamma} F'_\bullet$ and $\widehat{\mathbb{Z}} \Gamma \otimes_{\mathbb{Z} \Gamma} F''_\bullet$ are exact. Equipping an $FL$-resolution $F_\bullet \rightarrow M$ as in Proposition 2.8 for $R = \mathbb{Z}\Gamma$ with bases $b_\nu = b'_\nu \cup b''_\nu$ on $F_\nu = F'_\nu \oplus F''_\nu$ we have
\[
\tau(F_\bullet, b_\bullet) = \tau(F'_\bullet, b'_\bullet)\tau(F''_\bullet, b''_\bullet) \text{ in } K_T(\widehat{\mathbb{Z}} \Gamma).
\]
This is a special case of [Mil66, Theorem 3.1, §4]. According to [Coh73, §15,16] there is the following formula for the torsion: Since (24) is an acyclic complex of free and hence projective $\hat{\mathbb{Z}} \Gamma$-modules, it admits a chain contraction i.e. a degree-one $\hat{\mathbb{Z}} \Gamma$-module homomorphism
\[
\delta : \hat{\mathbb{Z}} \Gamma \otimes_{\mathbb{Z} \Gamma} F_\bullet \rightarrow \hat{\mathbb{Z}} \Gamma \otimes_{\mathbb{Z} \Gamma} F_\bullet
\]
such that $\delta \hat{\partial} + \hat{\partial} \delta = \text{id}$. Set
\[
F_{\text{odd}} = F_1 \oplus F_3 \oplus \ldots \quad \text{and} \quad F_{\text{even}} = F_0 \oplus F_2 \oplus \ldots
\]
and
\[
b_{\text{odd}} = b_1 \cup b_3 \cup \ldots \quad \text{and} \quad b_{\text{even}} = b_0 \cup b_2 \cup \ldots
\]
The map

\[(\hat{\partial} + \delta)_{\text{odd}} := (\hat{\partial} + \delta)|_{\hat{\mathbb{Z}}^\Gamma \otimes_{\mathbb{Z}^\Gamma} F_{\text{odd}}} : \hat{\mathbb{Z}}^\Gamma \otimes_{\mathbb{Z}^\Gamma} F_{\text{odd}} \longrightarrow \hat{\mathbb{Z}}^\Gamma \otimes_{\mathbb{Z}^\Gamma} F_{\text{even}}\]

is an isomorphism of finitely generated free \(\hat{\mathbb{Z}}^\Gamma\)-modules. Using the chosen bases \(1 \otimes b_{\text{odd}}\) and \(1 \otimes b_{\text{even}}\) together with some ordering we may view \((\hat{\partial} + \delta)_{\text{odd}}\) as an automorphism \((\hat{\partial} + \delta)_{\text{odd}}^*\) of \((\hat{\mathbb{Z}}^\Gamma)^r\) where

\[r = \sum_{i \text{ odd}} \text{rank}_{\mathbb{Z}^\Gamma} F_i = \sum_{i \text{ even}} \text{rank}_{\mathbb{Z}^\Gamma} F_i. \tag{26}\]

We identify \(\text{Aut}_{\hat{\mathbb{Z}}^\Gamma}(\hat{\mathbb{Z}}^\Gamma)^r\) with \(\text{GL}_r(\hat{\mathbb{Z}}^\Gamma)\) by sending \(f\) to the matrix \(\langle f \rangle = (a_{ij})\) defined by \(f(e_i) = \sum_j a_{ij}e_j\) where \(e_1, \ldots, e_r\) is the standard basis of \((\hat{\mathbb{Z}}^\Gamma)^r\). Let

\[\pi : \text{GL}_r(\hat{\mathbb{Z}}^\Gamma) \longrightarrow K_T(\hat{\mathbb{Z}}^\Gamma)\]

be the canonical homomorphism. Then we have

\[\tau(F_{\ast}, b_{\ast}) = \pi(\langle (\hat{\partial} + \delta)_{\text{odd}}^* \rangle). \tag{27}\]

This is independent of the chosen orderings of the bases. For different bases \(b_{\nu}'\) of \(F_{\nu}\) it follows e.g. from (27) that

\[\tau(F_{\ast}, b_{\ast}') = \tau(F_{\ast}, b_{\ast})\pi(S) \text{ in } K_T(\hat{\mathbb{Z}}^\Gamma) \tag{28}\]

for some matrix \(S \in \text{GL}_r(\mathbb{Z}^\Gamma)\).

Now assume that \(\Gamma\) is countable and residually finite and that \(\text{Wh}^{\mathbb{Z}_p}(\Gamma)\) is a torsion group. Then we can apply the homomorphism

\[\log_p \det_\Gamma : K_T(\hat{\mathbb{Z}}^\Gamma) \longrightarrow \mathbb{Q}_p\]

of Theorem 4.1 to \(\tau(F_{\ast}, b_{\ast})\). We call

\[\tau_p^\Gamma(M) := \log_p \det_\Gamma(\tau(F_{\ast}, b_{\ast})) \in \mathbb{Q}_p \tag{29}\]

the \(p\)-adic \(R\)-torsion of \(M\). Because of formula (28) and Corollary 4.3 it is independent of the choice of bases \(b_{\ast}\). In the situation of (25) we have

\[\tau_p^\Gamma(M) = \tau_p^\Gamma(M') + \tau_p^\Gamma(M''). \tag{30}\]
**Theorem 5.1.** Let $\Gamma$ be countable and residually finite with $\text{Wh}^F C_p(\Gamma)$ torsion. Let $M$ be a $\mathbb{Z}\Gamma$-module of type $FL$ with $pM = M$ and $\lim_{i} M_{p^i} = 0$. Then the following assertions hold:

a) The $p$-adic Reidemeister torsion $\tau_p^\Gamma(M) \in \mathbb{Q}_p$ is defined and independent of the chosen based resolution $(23)$ of $M$.

b) Choose a sequence $\Gamma_n \rightarrow e$ of normal subgroups $\Gamma_n$ of finite index in $\Gamma$ and consider the multiplicative Euler characteristics $\chi(\Gamma_n, M) \in \mathbb{Q}^\times$ from $(12)$. Then we have

$$\tau_p^\Gamma(M) = \lim_{n \rightarrow \infty} (\Gamma : \Gamma_n)^{-1} \log_p \chi(\Gamma_n, M) \text{ in } \mathbb{Q}_p . \quad (31)$$

**Example.** For $d = 1$ i.e. for resolutions of the form

$$0 \longrightarrow (\mathbb{Z}\Gamma)^s \xrightarrow{\partial} (\mathbb{Z}\Gamma)^r \longrightarrow M \longrightarrow 0$$

we showed in Proposition $3.2$ that the conditions on $M$ in Theorem $5.1$ are equivalent to $r = s$ and $\partial \in M_r(\mathbb{Z}\Gamma) \cap \text{GL}_r(\hat{\mathbb{Z}}\Gamma)$. In this case we have $\tau_p^\Gamma(M) = \log_p \det \partial$ and $\chi(\Gamma_n, M) = |\det_{\mathbb{Q}_p}(\partial^{(n)})|$ using $(33)$ and an argument as in the proof of Proposition $5.2$. Theorem $5.1$ reduces to the limit formula in Theorem $4.2$.

**Remark.** Given an exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ of $\mathbb{Z}\Gamma$-modules, where $M'$ and $M''$ satisfy the conditions of Theorem $5.1$, the module $M$ satisfies those conditions as well by Proposition $2.8$. According to $(30)$ we therefore have $\tau_p^\Gamma(M) = \tau_p^\Gamma(M') + \tau_p^\Gamma(M'')$. This also follows from equation $(31)$.

**Proof.** By Theorem $3.1$ we know that for any resolution $(23)$ of $M$ the complex $(24)$ is acyclic. Hence $\tau_p^\Gamma(M)$ is defined. Independence of the resolution will follow from b). Again by Theorem $3.1$ the homology groups $H_i(\Gamma_n, M)$ are finite for all $i$ and vanish for almost all $i$, so that

$$\chi(\Gamma_n, M) = \prod_i |H_i(\Gamma_n, M)|(-1)^i \in \mathbb{Q}^\times$$

is well defined. In order to prove formula $(31)$ we apply Theorem $4.2$ to $f = (\hat{\partial} + \delta)_{\text{odd}} \in \text{GL}_r(\hat{\mathbb{Z}}\Gamma)$. This gives the equality

$$\tau_p^\Gamma(M) = \lim_{n \rightarrow \infty} (\Gamma : \Gamma_n)^{-1} \log_p (\det_{\mathbb{Q}_p}(f^{(n)})) . \quad (32)$$
Let $({F}_n^{(n)}, {\partial}^{(n)})$ be the complex of free $\mathbb{Z}\Gamma^{(n)}$-modules of finite rank obtained by tensoring the complex $(F_\bullet, \partial)$ with $\mathbb{Z}\Gamma^{(n)}$ via the natural map $\mathbb{Z}\Gamma \to \mathbb{Z}\Gamma^{(n)}$. We have
\[ F^{(n)}_\bullet = \mathbb{Z}\Gamma^{(n)} \otimes_{\mathbb{Z}\Gamma} F_\bullet = (\mathbb{Z} \otimes_{\mathbb{Z}\Gamma^{(n)}} \mathbb{Z}\Gamma) \otimes_{\mathbb{Z}\Gamma} F_\bullet = \mathbb{Z} \otimes_{\mathbb{Z}\Gamma^{(n)}} F_\bullet. \] 
(33)

Here $\mathbb{Z}$ is viewed as a $\mathbb{Z}\Gamma^{(n)}$-module via the augmentation and the right $\mathbb{Z}\Gamma$-equivariant isomorphism
\[ \mathbb{Z} \otimes_{\mathbb{Z}\Gamma^{(n)}} \mathbb{Z}\Gamma \xrightarrow{\sim} \mathbb{Z}\Gamma^{(n)} \]
is induced by the map $\mathbb{Z} \times \mathbb{Z}\Gamma \to \mathbb{Z}\Gamma^{(n)}$ which sends $(\nu, \sum n_\gamma \gamma)$ to $\nu \sum n_\gamma \gamma$. We will use the identifications $F^{(n)}_\bullet = \mathbb{Z} \otimes_{\mathbb{Z}\Gamma^{(n)}} F_\bullet$, $\partial^{(n)} = \text{id} \otimes \partial$ and view $(F^{(n)}_\bullet, \partial^{(n)})$ simply as a complex of finitely generated free $\mathbb{Z}$-modules. Tensoring the acyclic complex (34) of $\widetilde{\mathbb{Z}\Gamma}$-modules via the natural map
\[ \widetilde{\mathbb{Z}\Gamma} = \varprojlim_i \mathbb{Z}\Gamma / p^i \mathbb{Z}\Gamma \longrightarrow \varprojlim_i \mathbb{Z}\Gamma^{(n)} / p^i \mathbb{Z}\Gamma^{(n)} = \mathbb{Z}_p \Gamma^{(n)} \]
with $\mathbb{Z}_p \Gamma^{(n)}$ we obtain the complex of free $\mathbb{Z}_p$-modules
\[ \mathbb{Z}_p \Gamma^{(n)} \otimes_{\mathbb{Z}\Gamma} \widetilde{\mathbb{Z}\Gamma} \otimes_{\mathbb{Z}\Gamma} F_\bullet = \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}\Gamma^{(n)} \otimes_{\mathbb{Z}\Gamma} F_\bullet = \mathbb{Z}_p \otimes_{\mathbb{Z}} F^{(n)}_\bullet = \mathbb{Z}_p \otimes_{\mathbb{Z}\Gamma^{(n)}} F_\bullet. \]
(34)

The differential $\hat{\partial}^{(n)} := \text{id} \otimes \widetilde{\mathbb{Z}\Gamma} \hat{\partial}$ on the left corresponds to the differential $\text{id} \otimes_{\mathbb{Z}\Gamma^{(n)}} \partial$ on $\mathbb{Z}_p \otimes_{\mathbb{Z}\Gamma^{(n)}} F_\bullet$. The complex (34) carries the chain contraction $\delta^{(n)} := \text{id} \otimes \widetilde{\mathbb{Z}\Gamma} \delta$ and it is therefore acyclic. The isomorphism
\[ (\hat{\partial} + \delta)^{(n)}_{\text{odd}} := \text{id} \otimes \widetilde{\mathbb{Z}\Gamma} (\hat{\partial} + \delta)_{\text{odd}} \]
can be identified with the isomorphism of $\mathbb{Z}_p$-modules
\[ (\text{id} \otimes_{\mathbb{Z}\Gamma^{(n)}} \partial + \delta)^{(n)}_{\text{odd}} := \text{id} \otimes_{\mathbb{Z}\Gamma^{(n)}} \partial + \delta^{(n)}_{\text{odd}} |_{\mathbb{Z}_p \otimes_{\mathbb{Z}\Gamma^{(n)}} F_{\text{odd}}} : \mathbb{Z}_p \otimes_{\mathbb{Z}\Gamma^{(n)}} F_{\text{odd}} \xrightarrow{\sim} \mathbb{Z}_p \otimes_{\mathbb{Z}\Gamma^{(n)}} F_{\text{even}}. \]
(35)

Fixing $\mathbb{Z}$-bases of the free $\mathbb{Z}$-modules $\mathbb{Z} \otimes_{\mathbb{Z}\Gamma^{(n)}} F_\bullet = F^{(n)}_\bullet$ we may view $(\text{id} \otimes_{\mathbb{Z}\Gamma^{(n)}} \partial + \delta)^{(n)}_{\text{odd}}$ as an element of $\text{GL}_{r(\Gamma;\Gamma^{(n)})}(\mathbb{Z}_p)$ with $r$ defined in (26) above. Its image in $K_T(\mathbb{Q}_p) = K_1(\mathbb{Q}_p)/\langle \pm 1 \rangle$ is the torsion $\tau(\mathbb{Q}_p \otimes_{\mathbb{Z}\Gamma^{(n)}} F_\bullet)$ of the based acyclic complex $(\mathbb{Q}_p \otimes_{\mathbb{Z}\Gamma^{(n)}} F_\bullet, \text{id} \otimes_{\mathbb{Z}\Gamma^{(n)}} \partial)$. It is independent of the chosen $\mathbb{Z}$-bases since $K_1(\mathbb{Z}) = \mathbb{Z}^\times = \{ \pm 1 \}$. In formula (32) the matrix $f^{(n)}$ for $f = (\hat{\partial} + \delta)_{\text{odd}}^{(n)}$ is viewed as an element of $\text{GL}_{r(\Gamma;\Gamma^{(n)})}(\mathbb{Q}_p)$. The image of $f^{(n)}$ in $K_T(\mathbb{Q}_p)$ is equal to $\tau(\mathbb{Q}_p \otimes_{\mathbb{Z}\Gamma^{(n)}} F_\bullet)$ by the above considerations. Consider the composition
\[ \det_{\mathbb{Q}_p} K_T(\mathbb{Q}_p) \xrightarrow{\sim} \mathbb{Q}_p^\times / \{ \pm 1 \}. \]
Since \( \det_{\mathbb{Q}_p} f^{(n)} = \det_{\mathbb{Q}_p} \tau(Q_p \otimes_{\mathbb{Z}_{\Gamma_n}} F_\bullet) \) in (52), it suffices to show that
\[
\det_{\mathbb{Q}_p} \tau(Q_p \otimes_{\mathbb{Z}_{\Gamma_n}} F_\bullet) = \chi(\Gamma_n, M) \quad \text{in } \mathbb{Q}_p^\times/\{\pm 1\}.
\] (36)

By Theorem 3.1 the homology groups \( H_i(\Gamma_n, M) = H_i(\mathbb{Z} \otimes_{\mathbb{Z}_{\Gamma_n}} F_\bullet) \) are finite. Hence the complex \( \mathbb{Q} \otimes_{\mathbb{Z}_{\Gamma_n}} F_\bullet \) is acyclic. Its torsion
\[
\tau(Q \otimes_{\mathbb{Z}_{\Gamma_n}} F_\bullet) \in K_T(Q) \sim \mathbb{Q}_p^\times/\{\pm 1\}
\]
with respect to any \( \mathbb{Z} \)-bases of the groups \( \mathbb{Z} \otimes_{\mathbb{Z}_{\Gamma_n}} F_\bullet \) is independent of the choice of bases since \( K_1(\mathbb{Z}) = \{\pm 1\} \). Since \( \tau(Q \otimes_{\mathbb{Z}_{\Gamma_n}} F_\bullet) \) is mapped to \( \tau(Q_p \otimes_{\mathbb{Z}_{\Gamma_n}} F_\bullet) \) under the natural map \( K_T(Q) \to K_T(Q_p) \), we have
\[
\det_{\mathbb{Q}_p} \tau(Q_p \otimes_{\mathbb{Z}_{\Gamma_n}} F_\bullet) = \det \tau(Q \otimes_{\mathbb{Z}_{\Gamma_n}} F_\bullet) \quad \text{in } \mathbb{Q}_p^\times/\{\pm 1\}.
\]
Thus (36) is a consequence of the formula
\[
\det_{\mathbb{Q}_p} \tau(Q \otimes_{\mathbb{Z}_{\Gamma_n}} F_\bullet) = \chi(\Gamma_n, M) \quad \text{in } \mathbb{Q}_p^\times/\{\pm 1\}
\]
which is a special case of the following proposition applied to \( L_\bullet = \mathbb{Z} \otimes_{\mathbb{Z}_{\Gamma_n}} F_\bullet \) not noting that \( H_i(L_\bullet) = H_i(\Gamma_n, M) \).

**Proposition 5.2.** Let \( 0 \to L_d \xrightarrow{\partial} \ldots \xrightarrow{\partial} L_0 \to 0 \) be a complex of finitely generated free \( \mathbb{Z} \)-modules whose homology groups are finite. The torsion \( \tau(Q \otimes L_\bullet) \in K_T(Q) \) of the acyclic complex \( Q \otimes L_\bullet \) equipped with bases coming from \( \mathbb{Z} \)-bases of the \( L_\nu \) is independent of the bases and we have
\[
\det_{\mathbb{Q}} \tau(Q \otimes L_\bullet) = \prod_i |H^i(L_\bullet)|^{-1} \quad \text{in } \mathbb{Q}_p^\times/\{\pm 1\}.
\]

**Proof.** By induction. For \( d = 0 \) the assertion is trivial. For \( d = 1 \) we have \( 0 \to L_1 \xrightarrow{\partial} L_0 \to 0 \) with \( \partial : Q \otimes L_1 \sim Q \otimes L_0 \) an isomorphism. Then \( \delta = \partial_{\mathbb{Q}}^{-1} \) gives a chain retraction and
\[
\det_{\mathbb{Q}} \tau(Q \otimes L_\bullet) = \det_{\mathbb{Q}}((\partial_{\mathbb{Q}} + \delta)_{\text{odd}}) = \det_{\mathbb{Q}}(\partial_{\mathbb{Q}}) \in \mathbb{Q}_p^\times/\{\pm 1\}.
\]

Here, after identifying \( L_0 \) and \( L_1 \) with \( \mathbb{Z}^N \) for \( N = \text{rk} L_0 = \text{rk} L_1 \) we view \( \partial_{\mathbb{Q}} \) as an automorphism of \( \mathbb{Q}^N \). The map \( \partial \) is injective since \( \partial_{\mathbb{Q}} \) is injective and since \( L_1 \) is free. It is known that
\[
|\det(\partial_{\mathbb{Q}})| = |\text{coker } \partial|,
\]
25
as follows from the theorem on elementary divisors. Since \( H_1(L_\ast) = \ker \partial = 0 \) and \( H_0(L_\ast) = \coker \partial \) it follows that
\[
\det_Q \tau(Q \otimes L_\ast) = |H_0(L_\ast)| |H_1(L_\ast)|^{-1} \text{ in } \mathbb{Q}^\times/\{\pm 1\}.
\]
Now assume that \( d \geq 2 \) and that the proposition holds for \( d \) replaced by \( d - 1 \).
Consider the commutative diagram with exact lines:

It defines an exact sequence of complexes
\[
0 \longrightarrow L_1 \longrightarrow L_2 \longrightarrow \cdots \longrightarrow L_d \longrightarrow 0.
\]
Noting that \( \ker \partial \subset L_1 \) and \( L_1/\ker \partial \cong L_0 \) are injective we see that both \( L_\ast' \) and \( L_\ast'' \) are complexes of finitely generated free \( \mathbb{Z} \)-modules. By [Mil66, Theorem 3.1] we have
\[
\tau(Q \otimes L_\ast) = \tau(Q \otimes L_\ast') \tau(Q \otimes L_\ast'') \text{ in } K_T(Q).
\]
It follows from the definition of torsion in [Mil66, §3.4] that shifting the numbering of a complex by one replaces its torsion by its inverse. Let \( L_\ast'[1] \) be the
complex obtained from $L'$ by putting $L_\nu$ in degree $\nu - 1$ for $\nu \geq 2$ and $\text{Ker} \partial$ in degree zero. Then we have

$$\tau(Q \otimes L'[1]) = \tau(Q \otimes L'[1])^{-1} \text{ in } K_T(Q),$$

and by the induction hypotheses

$$\det Q \tau(Q \otimes L'[1]) = d \prod_{i=1}^d |H_i(L_\bullet)|^{-1} \text{ in } \mathbb{Q}^\times/\{\pm 1\}.$$ 

Together with the equation from the case $d = 1$,

$$\text{det}_Q \tau(Q \otimes L''[1]) = |H_0(L_\bullet)|$$

and (37), (38) we get the assertion for $L_\bullet$:

$$\det Q \tau(Q \otimes L_\bullet) = d \prod_{i=0}^d |H_i(L_\bullet)|^{-1} \text{ in } \mathbb{Q}^\times/\{\pm 1\}.$$ 

In Theorem 5.1 we had to assume that $pM = M$ i.e. that $M$ was $p$-adically expansive of exponent zero because except for $\Gamma = \mathbb{Z}^N$ we only know how to define $\log_p \text{det}_\Gamma$ on $K_T(\hat{\mathbb{Z}}\Gamma)$ and not on $K_T(c_0(\Gamma))$.

Assume that for a $\mathbb{Z}\Gamma$-module of type $FL$ with a based resolution (23) the complex

$$0 \rightarrow c_0(\Gamma) \otimes_{\mathbb{Z}\Gamma} F_d \rightarrow \ldots \rightarrow c_0(\Gamma) \otimes_{\mathbb{Z}\Gamma} F_0 \rightarrow 0$$

is acyclic. Let $\tau(F_\bullet, b_\bullet) \in K_T(c_0(\Gamma))$ be its torsion. It is multiplicative in short exact sequences as in (25). Assume that we are given a “reasonable” definition of a homomorphism

$$\log_p \text{det}_\Gamma : K_T(c_0(\Gamma)) \rightarrow \mathbb{Q}_p.$$ 

Then as in (29) we can define the $p$-adic $R$-torsion

$$\tau_p^\Gamma(M) = \log_p \text{det}_\Gamma(\tau(F_\bullet, b_\bullet)) \in \mathbb{Q}_p.$$ 

It is independent of the choices of bases $b_\nu$ if we have $\log_p \text{det}_\Gamma f = 0$ for $f \in \text{GL}_r(\mathbb{Z}\Gamma)$. If $\Gamma$ is countable and residually finite we define “reasonable” to
mean that \( \log \det_\Gamma \) is compatible with (17) and that the analogue of the limit formula
\[
\log_p \det_\Gamma f = \lim_{n \to \infty} (\Gamma : \Gamma)^{-1} \log_p (\det_{Q_p} f^{(n)}) \quad \text{in} \quad Q_p
\]
should hold for all \( f \in \text{GL}_r(c_0(\Gamma)) \). The limit formula implies the above condition \( \log_p \det_\Gamma f = 0 \) for \( f \in \text{GL}_r(\mathbb{Z}_\Gamma) \). Trivial modifications of the proof of Theorem 5.1 then give the following result which applies in particular to \( \Gamma = \mathbb{Z}^N \).

**Theorem 5.3.** Let \( \Gamma \) be a countable and residually finite group for which there is a “reasonable” definition of \( \log_p \det_\Gamma \) on \( K_T(c_0(\Gamma)) \) in the above sense. Let \( M \) be a \( \mathbb{Z}_\Gamma \)-module of type \( \text{FL} \) which is \( p \)-adically expansive and such that \( \lim_{\rho \to e} M_{\rho} = 0 \). Then \( \tau^\Gamma_p(M) \in Q_p \) is defined and independent of the resolution (23). Moreover for any sequence \( \Gamma_n \to e \) we have
\[
\tau^\Gamma_p(M) = \lim_{n \to \infty} (\Gamma : \Gamma_n)^{-1} \log_p \chi(\Gamma_n, M) \quad \text{in} \quad Q_p .
\]

Note that the multiplicative Euler characteristics \( \chi(\Gamma_n, M) \) exist by Theorem 3.1. It follows from the definition or the limit formula that \( \tau^\Gamma_p \) is additive in short exact sequences \( 0 \to M' \to M \to M'' \to 0 \). If the assumptions of the Theorem are satisfied for \( M' \) and \( M'' \) they hold for \( M \) as well by Proposition 2.8 for \( R = \mathbb{Z}_\Gamma \). As mentioned before, Theorem 5.3 applies to \( \Gamma = \mathbb{Z}^N \) with \( \log_p \det_\Gamma \) defined by formula (21), because in Corollary 4.4 we proved the required limit formula. In this case the conditions on \( M \) in Theorem 5.3 can be simplified:

**Remark 5.4.** For \( \Gamma = \mathbb{Z}^N \) a \( \mathbb{Z}_\Gamma \)-module \( M \) is of type \( \text{FL} \) if and only if it is finitely generated. In this case there exist \( \text{FL} \)-resolutions (23) of length \( d \leq N + 1 \). Moreover we have \( \lim_{\rho \to e} M_{\rho} = 0 \).

**Proof.** Since \( \mathbb{Z}_\Gamma \cong \mathbb{Z}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}] \) is a Noetherian ring every finitely generated module \( M \) has a resolution by finitely generated free \( \mathbb{Z}_\Gamma \)-modules
\[
\ldots \to F_1 \xrightarrow{\partial} F_0 \to M \to 0 .
\]
The global dimension of \( \mathbb{Z}_\Gamma \) is \( N + 1 \). Hence the projective dimension \( d \) of \( M \) is \( d \leq N + 1 \). By [Wei94, Lemma 4.1.6], in the resolution
\[
0 \to F_d/\ker \partial \xrightarrow{\partial} F_{d-1} \to \ldots \to F_0 \to M \to 0
\]
the finitely generated module $F_d/\text{Ker} \, \partial$ is projective. It was pointed out in [Swa78, p. 111] that the Quillen–Suslin theorem is valid for $\mathbb{Z} \Gamma$. Hence every projective module over $\mathbb{Z} \Gamma$ and in particular $F_d/\text{Ker} \, \partial$ is free. The converse implication is clear. The vanishing of $\lim_{\leftarrow i} M_{p^i}$ is a special case of Proposition 2.6.

**Corollary 5.5.** For $\Gamma = \mathbb{Z}^N$ let $M$ be a $p$-adically expansive $\mathbb{Z} \Gamma$-module. Then the $p$-adic $R$-torsion $\tau_p^\Gamma(M) \in \mathbb{Q}_p$ (defined using $\log_p \det_\Gamma$ from (21)) satisfies

$$\tau_p^\Gamma(M) = \lim_{n \to \infty} (\Gamma : \Gamma_n)^{-1} \log_p \chi(\Gamma_n, M) \quad \text{in} \quad \mathbb{Q}_p$$

for a sequence $\Gamma_n \to 0$.

In section 7 we will see that for $\Gamma = \mathbb{Z}^N$ the multiplicative Euler characteristics $\chi(\Gamma_n, M)$ have a very nice interpretation as intersection numbers on arithmetic schemes. We end this section by mentioning a geometric characterization of $p$-adic expansiveness in the case $\Gamma = \mathbb{Z}^N$. Set $R = \mathbb{Z} \Gamma = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}]$. Let

$$T_p^N = \{(z_1, \ldots, z_n) \in \overline{\mathbb{Q}}_p^N \mid |z_i|_p = 1 \text{ for } i = 1, \ldots, N\}$$

be the $p$-adic $N$-torus in $\overline{\mathbb{Q}}_p$. Recall that the support of an $R$-module $M$ is $\text{supp} \, M = \text{spec} \, R/\text{Ann} \,(M)$ where $\text{Ann} \,(M)$ is the annihilator ideal of $M$ in $R$. A prime ideal $\mathfrak{p}$ of $R$ is in $\text{supp} \, M$ if and only if $M_{\mathfrak{p}} \neq 0$. For a finite $R$-module $M$ let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal prime ideals of $\text{supp} \, M$. Then

$$\text{supp} \, M = V(\mathfrak{p}_1) \cup \ldots \cup V(\mathfrak{p}_r)$$

with $V(\mathfrak{p}) = \text{spec} \, R/\mathfrak{p}$ is the decomposition of $\text{supp} \, M$ into irreducible components. The prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are also the minimal associated prime ideals of $M$, see [Mat80, Theorem 6.5]. In [Brä10, Theorem 3.3] the following characterizations of $p$-adic expansiveness are shown:

**Theorem 5.6 (Bräuer).** For $\Gamma = \mathbb{Z}^N$ a finitely generated $\mathbb{Z} \Gamma$-module $M$ is $p$-adically expansive if and only if either of the following equivalent conditions holds:

a)$$T_p^N \cap (\text{supp} \, M)(\overline{\mathbb{Q}}_p) = \emptyset \quad (39)$$

b) The module $M$ is $S_p$-torsion, where $S_p$ is the multiplicative system $S_p = \mathbb{Z} \Gamma \cap c_0(\Gamma)^\times$. 29
formulates condition (39) in terms of associated primes of $M$. This is equivalent to our formulation since $\text{Ass}(M)$ and $\text{supp} M$ have the same minimal primes.

6 Algebraic dynamical systems

Actions of countable discrete groups $\Gamma$ on compact abelian topological groups $X$ by continuous group automorphisms are called “algebraic”. The Pontrjagin dual $M = X^*$ is a discrete abelian group with the induced $\Gamma$-action, and hence a $\mathbb{Z}\Gamma$-module. In this way we obtain a contravariant equivalence of the category of algebraic $\Gamma$-actions and the category of $\mathbb{Z}\Gamma$-modules $M$. For commutative groups $\Gamma$ the latter category is equivalent to the category of quasicoherent sheaves on the scheme $\text{spec} \mathbb{Z}\Gamma$. Algebraic dynamical systems are studied in depth in the book [Sch95] with an emphasis on the case $\Gamma = \mathbb{Z}^N$. Often dynamical properties of the $\Gamma$-action on $X$ have an appealing algebraic or (for $\Gamma = \mathbb{Z}^N$) algebraic-geometric characterization in terms of the $\mathbb{Z}\Gamma$-module $M$ or the corresponding sheaf on $\mathbb{A}^N = \text{spec} \mathbb{Z}\Gamma$. For more studies of algebraic actions of non-commutative groups we refer e.g. to [Den06], [CL15], [LT14] for example.

An algebraic action of $\Gamma$ on $X$ is expansive if and only if there is a neighborhood $U$ of $0 \in X$ such that $\bigcap_{\gamma \in \Gamma} \gamma U = \{0\}$. Expansiveness is equivalent to the condition that the $\mathbb{Z}\Gamma$-module $M$ is finitely generated and $L^1\Gamma \otimes_{\mathbb{Z}\Gamma} M = 0$ cf. [CL15, Theorem 3.1]. For $\Gamma = \mathbb{Z}^N$ it is also equivalent to the condition

$$T^N \cap (\text{supp} M)(\mathbb{C}) = \emptyset. \quad (40)$$

Here

$$T^N = \{(z_1, \ldots, z_N) \in \mathbb{C}^N \mid |z_i| = 1 \text{ for } i = 1, \ldots, N\}$$

is the real $N$-torus in $\mathbb{C}^N$. This follows from [Sch95, Theorem 6.5] because the minimal primes of $\text{Ass}(M)$ and $\text{supp} M$ are the same. For discrete groups $\Gamma$ there is a $p$-adic analogue of $L^1\Gamma$, however in the $p$-adic context, because of the ultrametric inequality it is more natural to work with the algebra $c_0(\Gamma)$. Thus it is natural to call the $\Gamma$-action $p$-adically expansive if $c_0(\Gamma) \otimes_{\mathbb{Z}\Gamma} M = 0$ and — for technical reasons — if the $\mathbb{Z}\Gamma$-module $M$ is of type $FP$. In other words, the $\Gamma$-action on $X$ is $p$-adically expansive if and only if $M$ is a $p$-adically expansive
$\mathbb{Z}\Gamma$-module. Incidentally, for $\Gamma = \mathbb{Z}^N$ a $\mathbb{Z}\Gamma$-module is of type $FP$ if and only if it is finitely generated. The analogy between conditions (39) and (40) in the classical and the $p$-adic case lends further credibility to our definition of $p$-adic expansiveness.

**Proposition 6.1.** Consider an algebraic action of a countable discrete group $\Gamma$ on a compact abelian group $X$ such that $M = X^*$ is a $\mathbb{Z}\Gamma$-module of type $FL$. Then the following conditions are equivalent:

a) The action of $\Gamma$ on $X$ is $p$-adically expansive.

b) There is an integer $n_0 \geq 0$ with $X_{p^{n_0}} = X_{p^n}$ for all $n \geq n_0$ where $X_{p^n} = \ker(p^n : X \to X)$.

**Proof.** We know that a) is equivalent to $M/p^nM = M/p^{n_0}M$ for some $n_0$ and all $n \geq n_0$. Via Pontrjagin duality this is equivalent to b). \qed

It is somewhat odd that condition b) depends only the structure of $X$ as an abelian group and not as in the classical case on the $\Gamma$-action.

For algebraic actions of amenable groups $\Gamma$ the topological entropy $h$ coincides with the measure theoretic entropy with respect to the Haar probability measure on $X$. Both definitions of entropy do not seem to have an analogue with values in the $p$-adic numbers. For an expansive action of $\Gamma$ on $X$ it follows from the definition that for every cofinite normal subgroup $\Delta$ of $\Gamma$ the number of fixed points $X^\Delta$ of $\Delta$ on $X$ is finite. If $\Gamma$ is residually finite one may therefore consider the periodic points entropy

$$h_{\text{per}} := \lim_{n \to \infty} (\Gamma : \Gamma_n)^{-1} \log |X^{\Gamma_n}| \in [0, \infty] \quad \text{for } \Gamma_n \to e \quad (41)$$

if it exists. For $f \in M_N(\mathbb{Z}\Gamma)$ the $\Gamma$-action on $X_f = M_f^*$ with $M_f = (\mathbb{Z}\Gamma)^N/(\mathbb{Z}\Gamma)^N f$ is expansive if and only if $L^1\Gamma \otimes_{\mathbb{Z}\Gamma} M_f = 0$ i.e. if there is some $g \in M_N(L^1\Gamma)$ with $gf = 1$. Since as mentioned above $M_N(L^1\Gamma)$ is directly finite this means that $f \in \text{GL}_N(L^1\Gamma)$. In this case, at least for $N = 1$ it is known by [DS07] that $h_{\text{per}}$ exists and that it can be expressed by the Fuglede–Kadison determinant on $K_1$ of the von Neumann algebra of $\Gamma$

$$h_{\text{per}} = \log \det_{\text{Fug}} f \quad (42)$$

Moreover, if in addition $\Gamma$ is amenable so that the entropy of the $\Gamma$-action on $X_f$ is defined, we have

$$h = h_{\text{per}} \quad (43)$$
In [Den09] we replaced the classical logarithm in formula (41) by the \( p \)-adic logarithm \( \log_p \) and called the resulting quantity
\[
\Psi_p := \lim_{n \to \infty} (\Gamma : \Gamma_n)^{-1} \log_p |X^{\Gamma_n}| \tag{44}
\]
the \( p \)-adic (periodic points) entropy if the limit exists for all \( \Gamma_n \to e \). We also introduced \( \log_p \det_{\Gamma} f \) as an analogue of \( \log \det_{\Lambda_T} f \). For \( X = X_f \) we proved the formula
\[
\Psi_p = \log_p \det_{\Gamma} f \tag{45}
\]
under the additional conditions on \( \Gamma \) and \( f \) of Theorem 4.2. For \( \Gamma = \mathbb{Z}^N \) formula (43) is known to be true for arbitrary expansive algebraic actions by [Sch95, Theorem 21.1].

In the \( p \)-adic case however, it was noted by Bräuer [Brä10, Example 7.1], that even for \( \Gamma = \mathbb{Z} \) there exist \( p \)-adically expansive actions for which the corresponding limit (44) does not exist: Consider
\[
\mathbb{F}_4 = \mathbb{F}_2[t]/(t^2 + t + 1) = \mathbb{Z}[t, t^{-1}]/(2, t^2 + t + 1)
\]
as a \( \mathbb{Z}[t] \)-module. Thus \( t \) acts by multiplication with the generator \( \xi := t \mod (2, t^2 + t + 1) \) on \( \mathbb{F}_4 \). Fix a prime number \( p \neq 2 \). Then multiplication by \( p \) on \( M = \mathbb{F}_4 \) is invertible. Hence \( M \) is \( p \)-adically expansive (of exponent zero) and \( \lim_{n \to \infty} M_{p^n} = 0 \). Consider the sequence \( \Gamma_n = 3n\mathbb{Z} \to 0 \).
We have \( |X^{\Gamma_n}| = |X| = 4 \) and hence
\[
(\Gamma : \Gamma_n)^{-1} \log_p |X^{\Gamma_n}| = (3n)^{-1} \log_p |X^{\Gamma_n}| = (3n)^{-1} \log_p 4.
\]
Since \( \log_p 4 \neq 0 \) for \( p \neq 2 \) this sequence does not converge in \( \mathbb{Q}_p \) for \( n \to \infty \).
Bräuer showed that \( \log_p \det_{\mathbb{Z}^N} f \) only depends on the \( \mathbb{Z}[\mathbb{Z}] \)-module \( M_f \) and gave a natural extension of \( \log_p \det_{\mathbb{Z}^N} \) to all \( p \)-adically expansive \( \mathbb{Z}[\mathbb{Z}] \)-modules \( \mathbb{M} \), see (53) below. For the module \( M = \mathbb{F}_4 \) above he obtained the value \( \log_p \det_{\mathbb{Z}^N} M = 0 \), so that we should have \( \Psi_p = 0 \) in this case. The starting points of the present note were the following observations:

- For residually finite \( \Gamma \) and \( X = X_f \) with \( f \in M_r(\mathbb{Z} \Gamma) \) \( p \)-adically expansive, we have \( H^i(\Gamma, X) = 0 \) for \( i \geq 1 \) and therefore
\[
|X^{\Gamma_n}| = \chi(\Gamma_n, X) := \prod_{i} |H^i(\Gamma_n, X)|^{(-1)^i}. \tag{46}
\]
This is shown after the proof of Proposition 6.2 below.
For $\Gamma \cong \mathbb{Z}$ and any finite $\Gamma$-module $X$ we have $\chi(\Gamma, X) = 1$. This follows from the exact sequence

$$0 \rightarrow H^0(\Gamma, X) \rightarrow X \xrightarrow{f - \text{id}} X \rightarrow H^1(\Gamma, X) \rightarrow 0$$

where $f$ is the automorphism of $X$ corresponding to a generator of $\Gamma \cong \mathbb{Z}$.

Thus, redefining the $p$-adic periodic points entropy as the limit

$$h_{p_{\text{per}}} := \lim_{n \to \infty} (\Gamma : \Gamma_n)^{-1} \log_p \chi(\Gamma_n, X) \quad \text{for } \Gamma_n \to e$$

(47)

if it exists, we get the same quantity as before for principal $p$-adically expansive actions. Moreover, for the non-principal $\Gamma = \mathbb{Z}$-action on $X = \mathbb{F}_4^*$ above we have $\chi(\Gamma_n, X) = 1$ for any non-trivial subgroup $\Gamma_n$ of $\Gamma$ and hence $h_{p_{\text{per}}} = 0$ for all $\Gamma_n \to 0$ as suggested by Bräuer’s considerations.

Even in the classical case it looks natural in relation to entropy to consider the following limit — if it exists

$$\lim_{n \to \infty} (\Gamma : \Gamma_n)^{-1} \log \chi(\Gamma_n, X).$$

(48)

Namely, entropy is known to be additive in short exact sequences and the logarithmic Euler characteristic has the same property. Of course, with (48) there is the additional complication that the groups $H^i(\Gamma_n, X)$ have to be finite and zero for large $i$. This is the case for expansive algebraic actions on $X$ if $M = X^*$ is of type FL and if $L^1 \Gamma$ is a flat $\mathbb{Z}\Gamma$-module. However flatness fails if $\Gamma$ contains a free group in two variables, cf. [CL15, Theorem 3.1.(4) and Remark 3.4]. On the other hand, by Theorem 3.1 and Proposition 6.2 below we know that $p$-adic expansiveness and the condition $\lim_{M_{p'}} = 0$ imply the finiteness of all $H^i(\Gamma_n, X)$. We will now rephrase the main theorem of the previous section as a calculation of the modified $p$-adic periodic points entropy (47).

**Proposition 6.2.** Let $\Gamma$ be a discrete group and $M$ a discrete $\mathbb{Z}\Gamma$-module. Then the cohomology groups $H^i(\Gamma, X)$ of the compact Pontrjagin dual $X = M^*$ are compact and there is a natural topological isomorphism

$$H_i(\Gamma, M)^* = H^i(\Gamma, X)$$

for each $i \geq 0$.

(49)

In particular $H^i(\Gamma, X)$ is finite (resp. zero) if and only if $H_i(\Gamma, M)$ is finite (resp. zero) and in this case $|H^i(\Gamma, X)| = |H_i(\Gamma, M)|$. Thus

$$\chi(\Gamma, X) = \prod_i |H^i(\Gamma, X)|^{(-1)^i} \chi(\Gamma, X) = \prod_i |H^i(\Gamma, X)|^{(-1)^i}.$$
is defined if and only
\[ \chi(\Gamma, M) = \prod_i |H_i(\Gamma, M)|^{(-1)^i} \]
is defined. In this case we have \( \chi(\Gamma, X) = \chi(\Gamma, M) \).

**Proof.** For a free right \( Z\Gamma \)-module \( L \) on a set of generators \( S \) the isomorphism
\[ \text{Hom}_{Z\Gamma}(L, X) = X^S, \; \alpha \mapsto (\alpha(s))_{s \in S} \]
turns \( \text{Hom}_{Z\Gamma}(L, X) \) into a compact abelian group. The topology is independent of the choice of generators \( S \) of \( L \). There is a natural isomorphism of compact groups
\[ (L \otimes_{Z\Gamma} M)^* \sim \rightarrow \text{Hom}_{Z\Gamma}(L, X). \] (50)
It is obtained by sending a bilinear map \( \varphi : L \times M \to S^1 \) with \( \varphi(la, m) = \varphi(l, am) \) for \( a \in Z\Gamma, l \in L, m \in M \) to the \( Z\Gamma \)-equivariant map \( \Phi : L \to \text{Hom}(M, S^1) \) given by \( \Phi(l)(m) = \varphi(l, m) \). Now choose a resolution \( \ldots \to L_1 \to L_0 \to Z \to 0 \) of the \( Z\Gamma \)-module \( Z \) by free right \( Z\Gamma \)-modules \( L_i \). Using (50) we obtain a topological isomorphism of complexes of compact groups
\[ (L_* \otimes_{Z\Gamma} M)^* \sim \rightarrow \text{Hom}_{Z\Gamma}(L_*, X). \]
Taking cohomology and using that Pontrjagin duality is an exact functor we obtain the desired topological isomorphism
\[ H^i(\Gamma, M)^* \sim \rightarrow H^i(\Gamma, X). \]
\[ \square \]

In the situation of formula (46) we have \( c_0(\Gamma) \otimes_{Z\Gamma} M_f = 0 \) and hence \( c_0(\Gamma)^r = c_0(\Gamma)^r f \). Thus \( f \) has a left inverse and using Theorem 3.3 it follows that \( f \in \text{GL}_r(c_0(\Gamma)) \). We get a short exact sequence
\[ 0 \longrightarrow (Z\Gamma)^r \overset{f}{\longrightarrow} (Z\Gamma)^r \longrightarrow M \longrightarrow 0. \]
Using the isomorphism \( Z \otimes_{Z\Gamma^n} Z\Gamma = Z\Gamma^{(n)} \) we find
\[ H_i(\Gamma_n, M) = H_i((Z\Gamma^{(n)})^r \overset{f^{(n)}}{\longrightarrow} (Z\Gamma^{(n)})^r) \]
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and therefore \( H_i(\Gamma_n, M) = 0 \) for \( i \geq 2 \). Since \( f \in \text{GL}_r(c_0(\Gamma)) \) we have \( f^{(n)} \in \text{GL}_r(\mathbb{Q}_p \Gamma^{(n)}) \). Thus the map \( f^{(n)} : \mathbb{Z} \Gamma^{(n)} \to \mathbb{Z} \Gamma^{(n)} \) is injective and hence \( H_1(\Gamma_n, M) = 0 \). Using Proposition \( \ref{prop:injective} \) we conclude that in the situation of (46) we have \( H^i(\Gamma_n, X) = 0 \) for \( i \geq 1 \) as claimed in (46).

**Theorem 6.3.** Under the conditions on \( \Gamma \) and \( M \) in Theorem 5.1 resp. Corollary 5.5 the \( p \)-adic entropy (47) of \( X = M^* \)

\[
h_p^\text{per} := \lim_{n \to \infty} (\Gamma : \Gamma_n)^{-1} \log_p \chi(\Gamma_n, X) \quad \text{for} \quad \Gamma_n \to e
\]

is well defined and equal to the \( p \)-adic \( R \)-torsion of \( M \):

\[
h_p^\text{per} = \tau_p^R(M).
\]

This is a reformulation of Theorem 5.1 and Corollary 5.5 taking into account Proposition \( \ref{prop:injective} \). It was motivated by an analogous (and deeper) result of Li and Thom [LT14, Theorem 1.1], who express classical entropy of algebraic dynamical systems of type \( FL \) for amenable groups as an \( L^2 \)-torsion.

In the rest of this section we give a review of Bräuer’s definition of \( \log_p \det_\Gamma(M) \) for \( \Gamma = \mathbb{Z}^N \) and show that it agrees with \( \tau_p^R(M) \).

Set \( R = \mathbb{Z}[\mathbb{Z}^N] \) and let \( \mathcal{M}_{S_p}(R) \) be the category of finitely generated \( R \)-modules which are \( S_p = R \cap c_0(\mathbb{Z}^N)^\times \)-torsion. According to Theorem 5.6 these are exactly the \( p \)-adically expansive \( R \)-modules. This is an exact subcategory of the category of all \( R \)-modules and one has a localization sequence [Bas68, IX Theorem 6.3 and Corollary 6.4]

\[
R^\times = K_1(R) \longrightarrow K_1(R[S_p^{-1}]) \xrightarrow{\delta} K_0(\mathcal{M}_{S_p}(R)) \longrightarrow K_0(R) \xrightarrow{i} K_0(R[S_p^{-1}]) \longrightarrow 0.
\]

(51)

The map \( i \) is injective since \( K_0(R) = K_0(\mathbb{Z}) = \mathbb{Z} \) and since \( K_0(R[S_p^{-1}]) \) surjects onto \( \mathbb{Z} \) via the rank. Hence \( \delta \) induces an isomorphism

\[
\overline{\delta} : K_1(R[S_p^{-1}])/R^\times \xrightarrow{\sim} K_0(\mathcal{M}_{S_p}(R)),
\]

whose inverse is denoted by \( \text{cl}_p \). Noting that \( R^\times = \pm \mathbb{Z}^N \), we have:

\[
K_1(c_0(\mathbb{Z}^N))/R^\times = K_T(c_0(\mathbb{Z}^N)).
\]

(52)
Following [Brä10 4.3], we may therefore consider the composition
\[
\log_p \det_{\mathbb{Z}^N} : K_0(\mathcal{M}_{S_p}(R)) \xrightarrow{c_1} K_1(R[S_p^{-1}]) / R^\times \longrightarrow K_{-1}(c_0(\mathbb{Z}^N)) \xrightarrow{\log_p \det_{\mathbb{Z}^N}} \mathbb{Q}_p.
\]
(53)
Here the last map is defined by equation (21).

**Proposition 6.4.** Set $\Gamma = \mathbb{Z}^N$. For any $p$-adically expansive $R = \mathbb{Z}\Gamma$-module $M$ we have:
\[
\log_p \det_{\Gamma}[M] = \tau_p^{\Gamma}(M).
\]

**Proof.** We recall the definition of $\delta$ in (51). For $f \in \text{GL}_r(R[S_p^{-1}])$ choose a common denominator $s \in S_p$ for the entries of $f$ and set $g = sf \in M_r(R) \cap \text{GL}_r(R[S_p^{-1}]) \subset M_r(R) \cap \text{GL}_r(c_0(\Gamma))$. Then $R^r / R^r g$ and $R^r / R^r s$ are both $S_p$-torsion. Namely choose $t \in S$ with $tf^{-1} \in M_r(R)$. Then $st \in S$ annihilates $R^r / R^r g$. We have
\[
\delta(f) = [R^r / R^r g] - [R^r / R^r s] \quad \text{in } K_0(\mathcal{M}_{S_p}(R)).
\]
Both $\log_p \det_{\Gamma}$ and $\tau_p^{\Gamma}$ are additive on $K_0(\mathcal{M}_{S_p}(R))$ i.e. on short exact sequences of $p$-adically expansive $R$-modules. Hence it suffices to check equality in Proposition 6.4 for the modules $M_f = R^r / R^r f$ with $f \in M_r(R) \cap \text{GL}_r(R[S_p^{-1}])$. For them we have $c_1[M_f] = [f]$ and hence
\[
\log_p \det_{\Gamma}[M_f] = \log_p \det_{\Gamma} f.
\]
On the other hand it is immediate from the definition of $\tau_p^{\Gamma}$ that we have
\[
\tau_p^{\Gamma}(M_f) = \log_p \det_{\Gamma} f.
\]
\[\square\]

### 7 Euler characteristics and intersection theory

In this section we set $\Gamma = \mathbb{Z}^N$ so that $\mathbb{Z}\Gamma = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}]$ and we denote by $\Delta \subset \Gamma$ a subgroup of finite index with finite quotient group $G$. For suitable $\mathbb{Z}\Gamma$-modules $M$ we will express the multiplicative Euler-characteristic $\chi(\Delta, M)$ in terms of Serre’s intersection numbers \[\text{Ser00}.\]
We call a $\mathbb{Z}\Gamma$-module $M$ expansive if it is finitely generated and if $L^1\Gamma \otimes_{\mathbb{Z}\Gamma} M = 0$. This is equivalent to the algebraic dynamical system $X = M^*$ being expansive. Recall that the $\mathbb{Z}\Gamma$-module $M$ is $p$-adically expansive if and only if it is finitely generated and $c_0(\Gamma) \otimes_{\mathbb{Z}\Gamma} M = 0$.

**Proposition 7.1.** Let $M$ be an expansive or $p$-adically expansive $\mathbb{Z}\Gamma$-module. Then $\mathbb{Z}G \otimes_{\mathbb{Z}\Gamma} M$ is finite.

**Proof.** Since $G$ is finite, the finitely generated $\mathbb{Z}G$-module $A = \mathbb{Z}G \otimes_{\mathbb{Z}\Gamma} M$ is finitely generated as an abelian group. The natural surjective ring homomorphisms

$L^1\Gamma \twoheadrightarrow L^1G = \mathbb{C}G$ and $c_0(\Gamma) \longrightarrow c_0(G) = \mathbb{Q}_pG$

therefore induce surjections

$L^1\Gamma \otimes_{\mathbb{Z}\Gamma} M \twoheadrightarrow \mathbb{C} \otimes_{\mathbb{Z}} A$ and $c_0(\Gamma) \otimes_{\mathbb{Z}\Gamma} M \twoheadrightarrow \mathbb{Q}_p \otimes_{\mathbb{Z}} A$.

Thus (p-adic) expansiveness implies that $A$ is a torsion group, hence finite. \(\square\)

Set $\mathfrak{m} = \text{spec } \mathbb{Z}G$ viewed as a closed subgroup scheme of $\mathfrak{g} = \text{spec } \mathbb{Z}\Gamma = \mathfrak{g}_{m,\mathbb{Z}}$. The subgroup $\mu(\mathbb{Q}) \subset \mathfrak{g}(\mathbb{Q}) = (\mathbb{Q}^\times)^N$ is finite and hence contained in $\mu(\mathbb{Q})^N$ where $\mu(\mathbb{Q})$ is the group of roots of unity in $\mathbb{Q}^\times$. For $\Delta = n\Gamma = (n\mathbb{Z})^N$ for example, we have $G = (\mathbb{Z}/n)^N$ and hence $\mathfrak{m} = \mu_n^{\mathbb{Z}}$ where $\mu_n^{\mathbb{Z}}$ is the group-scheme over $\mathbb{Z}$ of $n$-th roots of unity. Recall the affine scheme $\text{supp } M = \text{spec } (\mathbb{Z}\Gamma/\text{Ann } (M))$.

**Proposition 7.2.** Let $M$ be a finitely generated $\mathbb{Z}\Gamma$-module. The group $\mathbb{Z}G \otimes_{\mathbb{Z}\Gamma} M$ is finite if and only if $\mathfrak{m}(\mathbb{Q}) \cap (\text{supp } M)(\mathbb{Q}) = \emptyset$. In this case the groups $\text{Tor}^Z_{\mathbb{Z}\Gamma} (\mathbb{Z}G, M)$ are finite as well and viewing them as $\mathbb{Z}\Gamma$-modules we have

$$\text{Tor}^Z_{\mathbb{Z}\Gamma} (\mathbb{Z}G, M) = \bigoplus_{m} \text{Tor}^Z_{\mathbb{Z}\Gamma}((\mathbb{Z}G)_m, M_m).$$

Here $m$ runs over the finitely many (closed) points of $\mathfrak{m} \cap \text{supp } M$ in $\mathfrak{g}$.

**Proof.** The group $\mathbb{Z}G \otimes_{\mathbb{Z}\Gamma} M$ is finitely generated as a $\mathbb{Z}G$-module and hence as a $\mathbb{Z}$-module. Hence it is finite if and only if $\mathbb{Q}G \otimes_{\mathbb{Z}\Gamma} M = 0$ or equivalently $S := \text{supp } \mathbb{Q}G \otimes_{\mathbb{Z}\Gamma} M = \emptyset$. We have

$$S = \mathfrak{m}(\mathbb{Q}) \cap (\text{supp } M)(\mathbb{Q}) \quad \text{in } \mathfrak{g}_\mathbb{Q} = \mathfrak{g} \otimes_{\mathbb{Z}} \mathbb{Q},$$

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and this is empty if and only if
\[ S(\overline{Q}) = \mu(\overline{Q}) \cap (\text{supp } M)(\overline{Q}) \]
is empty. Set \( I = \text{Ker } (\mathbb{Z}\Gamma \to \mathbb{Z}G) \). The ring \( \mathbb{Z}\Gamma \) acts on \( \text{Tor}_{i}^{\mathbb{Z}\Gamma}(\mathbb{Z}G, M) \) via its quotient
\[ \overline{R} = \mathbb{Z}G \otimes_{\mathbb{Z}\Gamma} (\mathbb{Z}\Gamma/\text{Ann } (M)) = \mathbb{Z}\Gamma/(I + \text{Ann } (M)) \, . \]
Since \( \text{spec } \overline{R} = \mu \cap \text{supp } M \) is finite over \( \text{spec } \mathbb{Z} \) and its generic fibre is empty by assumption, \( \text{spec } \overline{R} \) is 0-dimensional and consists of finitely many closed points. They correspond to the maximal ideals \( m \) in \( \mathbb{Z}\Gamma \) containing both \( I \) and \( \text{Ann } (M) \). By the structure theorem for Artin algebras, we have
\[ \overline{R} = \prod_{m} R_{m} \, . \]
Since localization is exact, we get
\[ \text{Tor}_{i}^{\mathbb{Z}\Gamma}(\mathbb{Z}G, M) = \text{Tor}_{i}^{\mathbb{Z}\Gamma}(\mathbb{Z}G, M) \otimes_{R} \overline{R} = \bigoplus_{m} \text{Tor}_{i}^{(\mathbb{Z}\Gamma)_{m}}((\mathbb{Z}G)_{m}, M_{m}) \, . \]
The \( \text{Tor} \)-groups are finite since they are finitely generated \( \overline{R} \)-modules and the ring \( \overline{R} \) is finite. \( \square \)

**Remark.** Choosing embeddings \( \overline{Q} \subset \overline{Q}_{p} \) and \( \overline{Q} \subset \mathbb{C} \) we have \( \mu(\overline{Q}) = \mu(\overline{Q}_{p}) \subset T_{p}^{N} \) and \( \mu(\overline{Q}) = \mu(\mathbb{C}) \subset T^{N} \). Using the characterizations (39) resp. (40) of (p-adic) expansiveness, the first part of Proposition 7.2 gives a more geometric proof of Proposition 7.1.

In [Ser00] Serre developed a theory of local intersection numbers using higher Tor’s. We review the main definitions and results.

Let \( A \) be a regular local ring and \( M_{1}, M_{2} \) finite \( A \)-modules such that the length of \( M_{1} \otimes_{A} M_{2} \) is finite. Then the intersection multiplicity
\[ i(M_{1}, M_{2}; A) = \sum_{i=0}^{\dim A} (-1)^{i} l_{A}(\text{Tor}_{i}^{A}(M_{1}, M_{2})) \]
is well defined, and moreover \( \dim M_{1} + \dim M_{2} \leq \dim A \) where \( \dim M = \dim \text{supp } M \). Roberts [Rob85] and Gillet–Soulé [GS85] showed that we have
\[ i_{A}(M_{1}, M_{2}; A) = 0 \quad \text{if } \dim M_{1} + \dim M_{2} < \dim A . \] (54)
Gabber proved non-negativity:

\[ i_A(M_1, M_2; A) \geq 0 . \]  

(55)

Positivity of \( i_A(M_1, M_2; A) \) for \( \dim M_1 + \dim M_2 = \dim A \) is not yet known in general. All these properties were conjectured by Serre and proven by him in several special cases, in particular for equicharacteristic or unramified local rings \( A \).

**Corollary 7.3.** Let \( M \) be a finitely generated \( \mathbb{Z} \Gamma \)-module, \( \Gamma = \mathbb{Z}^N \) and let \( \Delta \subset \Gamma \) be a subgroup of finite index with quotient group \( G \). Assume that \( \mathfrak{m}(\mathfrak{Q}) \cap (\text{supp } M)(\mathfrak{Q}) = \emptyset \) where \( \mathfrak{m} = \text{spec } \mathbb{Z}G \subset \mathfrak{G} \) or equivalently that \( \mathbb{Z}G \otimes_{\mathbb{Z} \Gamma} M \) is finite. Then we have:

\[ \chi(\Delta, M) = \prod_{m} Nm^{i((\mathbb{Z}G)_m, M_m; (\mathbb{Z} \Gamma)_m)} . \]  

(56)

Here \( m \) runs over those maximal ideals in \( \mathfrak{m} \cap \text{supp } M \) for which \( \dim M_m = N \) and we write \( Nm = |\mathbb{Z} \Gamma / m| \). Moreover \( \chi(\Delta, M) \geq 1 \) is an integer.

**Proof.** Let \( F_* \rightarrow M \) be a free resolution of the \( \mathbb{Z} \Gamma \)-module \( M \). It is also a free resolution by \( \mathbb{Z} \Delta \)-modules. We have

\[ \mathbb{Z}G \otimes_{\mathbb{Z} \Gamma} F_* = \mathbb{Z} \otimes_{\mathbb{Z} \Delta} F_* \]

canonically as complexes and therefore

\[ \text{Tor}_i^{\mathbb{Z} \Gamma}(\mathbb{Z}G, M) = \text{Tor}_i^{\mathbb{Z} \Delta}(\mathbb{Z}, M) = H_i(\Delta, M) . \]

Formula (56) now follows from Proposition 7.2 and assertions (54), (55) noting that \( \dim(\mathbb{Z} \Gamma)_m = N + 1 \) for the maximal ideals \( m \) of \( \mathbb{Z} \Gamma \) and \( \dim(\mathbb{Z}G)_m = 1 \) for the closed points \( m \) of \( \mathfrak{m} \). By Hilbert’s weak Nullstellensatz the field \( \mathbb{Z} \Gamma / m \) is finite for every maximal ideal \( m \) and hence \( Nm \) is defined.

**Remark.** Note that the conditions on \( M \) in Corollary 7.3 are satisfied if \( M \) is (\( p \)-adically) expansive.

For a finitely generated \( \mathbb{Z} \Gamma \)-module \( M \) there is a filtration by submodules \( 0 = M_0 \subset M_1 \subset \ldots \subset M_n = M \) such that for \( 1 \leq i \leq n \) we have \( M_i/M_{i-1} \cong \mathbb{Z} \Gamma / p_i \)
with \( p_i \in \text{supp} M \). If \( \mu(\overline{Q}) \cap (\text{supp} M)(\overline{Q}) = \emptyset \) then \( \mu(\overline{Q}) \cap \text{supp} (Z\Gamma/p_i)(\overline{Q}) = \emptyset \) as well, since

\[
\text{supp} \; Z\Gamma/p_i = V(p_i) \subset \text{supp} \; M.
\]

Hence, if \( \chi(\Delta, M) \) is defined, \( \chi(\Delta, Z\Gamma/p_i) \) is defined for all \( i \) as well and we have

\[
\chi(\Delta, M) = \prod_{i=1}^{n} \chi(\Delta, Z\Gamma/p_i) \tag{57}
\]

since \( \chi(\Delta, \cdot) \) is multiplicative in short exact sequences.

**Proposition 7.4.** Let \( p \) be a prime ideal in \( Z\Gamma \) with \( \mu(\overline{Q}) \cap V(p)(\overline{Q}) = \emptyset \). If \( p \) is not principal, then \( \chi(\Delta, Z\Gamma/p) = 1 \).

**Proof.** Since \( Z\Gamma \) is a Noetherian unique factorization domain, the non-principal prime ideals \( p \) have height \( \text{ht}(p) \geq 2 \). The dimension inequality (in fact an equality)

\[
\dim Z\Gamma/p + \text{ht}(p) \leq \dim Z\Gamma = N + 1
\]

therefore gives

\[
\dim(Z\Gamma/p)_m \leq \dim Z\Gamma/p \leq N - 1 \quad \text{for} \; m \in \text{supp} Z\Gamma/p.
\]

Now the assertion follows from formula (56) applied to \( M = Z\Gamma/p \). \( \square \)

**Corollary 7.5.** Let \( M \) be as in Corollary 7.3 and let \( p_i = (f_i) \) for \( i \in I \subset \{1, \ldots, n\} \) be the principal prime ideals in \( Z\Gamma \). Then we have

\[
\chi(\Delta, M) = \prod_{i \in I} \chi(\Delta, Z\Gamma/f_iZ\Gamma) \quad \text{where} \; \chi(\Delta, Z\Gamma/f_iZ\Gamma) = |ZG/f_iZG|.
\]

**Proof.** The condition \( \mu(\overline{Q}) \cap \text{spec} Z\Gamma/p_i = \emptyset \) means that \( ZG \otimes_{Z\Gamma} Z\Gamma/p_i = ZG/p_iZG \) is finite by Proposition 7.2. For \( p_i = (f_i) \) this shows that \( f_i : QG \to QG \) is surjective and hence injective. Thus

\[
H_{\nu}(\Delta, Z\Gamma/f_iZ\Gamma) = H_{\nu}(ZG \xrightarrow{f_i} ZG)
\]

is zero for \( \nu \geq 1 \) and equal to \( ZG/f_iZG \) for \( \nu = 0 \). Using Proposition 7.3 and formula (57), the assertion follows. \( \square \)
We will now apply the preceding results to the calculation of $p$-adic entropy $h_{p}^{\text{per}}$ as defined in formula (47) or equivalently of $p$-adic $R$-torsion.

**Corollary 7.6.** Let $X$ be a $p$-adically expansive $\Gamma = \mathbb{Z}^N$-algebraic dynamical system and $M = X^\ast$ the corresponding $\mathbb{Z}\Gamma$-module. With notations as in Corollary 7.3 we have:

$$h_{p}^{\text{per}} = \tau_{p}^{\Gamma}(M) = \sum_{i \in I} \log_{p} \det_{\Gamma} f_{i}.$$  

In particular $h_{p}^{\text{per}} = 0$ if all the prime ideals $p_i$ occurring in $M$ are non-principal.

**Proof.** This follows from Theorem 6.3, Propositions 7.1, 7.2 and Corollary 7.5.

**Remark.** For classical entropy we have the analogous formula

$$h = \sum_{i \in I} \log \det_{\mathbb{N}\Gamma} f_{i},$$

which holds even without expansiveness assumptions. See [Sch95, Corollary 18.5 and Proposition 18.6] and use that the logarithm of the Fuglede–Kadison determinant $\log \det_{\mathbb{N}\Gamma} f_{i}$ equals the Mahler measure of $f_{i}$

$$m(f_i) = \int_{T^{N}} \log |f_i| \, d\mu.$$  

Here $\mu$ is the Haar measure of $T^{N}$. This is proved using the Yuzvinskii addition formula for entropy and an argument based on the monotonicity of entropy. The latter is used to show that algebraic dynamical systems $X = (\mathbb{Z}\Gamma/p)^\ast$ for non-principal prime ideals have zero entropy. This argument does not transfer to $p$-adic entropy and we replaced it by the vanishing result (54) for local intersection numbers.

We now relate classical entropy with multiplicative Euler characteristics:

**Corollary 7.7.** Let $X$ be an expansive $\Gamma = \mathbb{Z}^N$-algebraic dynamical system. Then we have

$$h = \lim_{n \to \infty} (\Gamma : \Gamma_n)^{-1} \log \chi(\Gamma_n, X) \quad \text{for } \Gamma_n \to 0.$$  

(58)
Remark. In [Sch95, VI 21] the formula
\[ h = \lim_{n \to \infty} (\Gamma : \Gamma_n)^{-1} \log |X^{\Gamma_n}| \] (59)
is proved for expansive \( X \). Thus \( \chi(\Gamma_n, X) \) can be replaced by \( |H^0(\Gamma_n, X)| \) and the limit remains the same. Formula (58) is more natural than formula (59) because both the entropy and the logarithmic Euler characteristics are additive in short exact sequences of algebraic dynamical systems. On the other hand, formula (59) is more explicit.

Proof of Corollary 7.7 By additivity of entropy and its vanishing for non-principal actions, we have
\[ h = \sum_{i \in I} h_{f_i} \text{ where } h_f = \text{ entropy of } X_f := (\Gamma / f \Gamma)^* \]where
\[ h_{f_i} = \lim_{n \to \infty} (\Gamma : \Gamma_n)^{-1} \log |X_{f_i}^{\Gamma_n}| \]
Using Proposition 6.2 and Corollary 7.5 we find
\[ \chi(\Gamma_n, X) = \prod_{i \in I} |X_{f_i}^{\Gamma_n}| . \]
Combining this with formula (59) applied to \( X = X_{f_i} \):
\[ h_{f_i} = \lim_{n \to \infty} (\Gamma : \Gamma_n)^{-1} \log |X_{f_i}^{\Gamma_n}| \]
we obtain (58).

We end with a remark about a connection to Arakelov theory [Sou92]. Embedding \( G = G_{m,Z} \) into \( \mathbb{P} = \mathbb{P}_{\mathbb{Z}}^N \), the closed subscheme \( \mu \) of \( G \) is also closed in \( \mathbb{P} \) since it is finite over \( \text{spec } \mathbb{Z} \). For \( M = \mathbb{Z} \Gamma / \mathfrak{a} \) consider \( V(\mathfrak{a}) = \text{spec } \mathbb{Z} \Gamma / \mathfrak{a} \) and its closure \( \overline{V(\mathfrak{a})} \) in \( \mathbb{P} \). We have \( \mu \cap V(\mathfrak{a}) = \mu \cap \overline{V(\mathfrak{a})} \) since \( \mu \) is closed in \( \mathbb{P} \) and therefore the right side of the formula for log \( \chi(\Delta, M) \) obtained from (56) is the sum of all the non-archimedean local Arakelov intersection numbers of \( \mu \) and \( \overline{V(\mathfrak{a})} \) in \( \mathbb{P}^N \).

For principal \( \mathfrak{a} = (f) \) the global intersection pairing on \( \mathbb{P}^N \) with the Arakelov cycle \( (\text{div } f, -\log |f|^2) \) is zero. Hence log \( \chi(\Delta, M) \) then equals the negative of the sum over the local archimedean intersection numbers. We assume \( M = \mathbb{Z} \Gamma / f \mathbb{Z} \Gamma \) to be expansive here, i.e. \( f(z) = 0 \) for all \( z \in T^N \).
The Archimedean contribution to the global intersection number is \( \sum_{\zeta \in \mu(C)} \log |f(\zeta)| \).

For \( \Delta = \Gamma_n = (n\mathbb{Z})^N \) we therefore find

\[
(\Gamma : \Gamma_n)^{-1} \log \chi(\Gamma_n, Z\Gamma/f\Gamma) = n^{-N} \sum_{\zeta \in \mu_n(C)^N} \log |f(\zeta)|
\]

and this converges for \( n \to \infty \) to

\[
\int_{T^N} \log |f| d\mu = \log \det_{\Lambda T} f = h,
\]

as it has to.

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