Simulation of Wave Equation on Manifold using DEC

Zheng Xie\textsuperscript{1*} Yujie Ma\textsuperscript{2†}

1. Center of Mathematical Sciences, Zhejiang University (310027), China
2. Key Laboratory of Mathematics Mechanization, Chinese Academy of Sciences, (100090), China

Abstract

The classical numerical methods play important roles in solving wave equation, e.g. finite difference time domain method. However, their computational domain are limited to flat space and the time. This paper deals with the description of discrete exterior calculus method for numerical simulation of wave equation. The advantage of this method is that it can be used to compute equation on the space manifold and the time. The analysis of its stable condition and error is also accomplished.

Keywords: Discrete exterior calculus, Manifold, Wave equation, Laplace operator, Numerical simulation.

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1 Introduction

The wave equation is the prototypical example of a hyperbolic partial differential equation of waves, such as sound waves, light waves and water waves. It arises in fields such as acoustics, electromagnetism, and fluid dynamics \cite{1,2}. To investigate the predictions of wave equation of such phenomena it is often necessary to approximate its solution numerically. A technique suitable for providing numerical solutions to the wave propagation problem

\textsuperscript{*}E-mail: lenozhengxie@yahoo.com.cn Tel./fax: +86 0739 5316081

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is the finite difference time domain (FDTD) method. This is normally defined by looking for an approximate solution on a uniform mesh of points and by replacing the derivatives in the differential equation by difference quotients at points of mesh. The computational domain of this algorithm is limited to flat space and the time [3–7].

Discrete exterior calculus (DEC) constitutes a discrete realization of the exterior differential forms, and therefore, the right framework in which to develop a discretization for differential equations not just on flat space but on manifold [8–16]. The differential operators such as gradient, divergence, and Laplace operator on manifold can also be naturally discretized using DEC. The numerical solution of wave equation on space manifold and the time by the methods of DEC is obtained in this paper. For this equation, an explicit scheme is derived. The analysis of this scheme’s stability shows that the numerical solution becomes unstable unless the time step is restricted.

2 DEC method for wave equation

Wave equation
The wave equation is the prototypical example of a hyperbolic partial differential equation. In its simplest form, the wave equation refers to a scalar function $u$ that satisfies:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u,$$

(1)

where $\Delta$ is the Laplace operator and $c$ is the propagation speed of the wave. More realistic differential equations for waves allow the speed of wave propagation to vary with the frequency of the wave, a phenomenon known as dispersion. In this case, $c$ is replaced by the phase velocity:

$$\frac{\partial^2 u}{\partial t^2} = (\frac{\omega}{k})^2 \Delta u.$$

Another common correction in realistic systems is that the speed is depend on the amplitude of the wave, leading to a nonlinear wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c(u)^2 \Delta u.$$

DEC scheme for wave equation
A discrete differential $k$-form, $k \in \mathbb{Z}$, is the evaluation of the differential $k$-form on all $k$-simplices. A dual form is evaluated on the dual cell. Suppose
each simplex contains its circumcenter. The circumcentric dual cell $D(\sigma_0)$
of simplex $\sigma_0$ is

$$D(\sigma_0) := \bigcup_{\sigma_0 \in \sigma_1 \in \cdots \in \sigma_r} \text{Int}(c(\sigma_0)c(\sigma_1) \cdots c(\sigma_r)),$$

where $\sigma_i$ is all the simplices which contains $\sigma_0, \ldots, \sigma_{i-1}$, $c(\sigma_i)$ is the
circumcenter of $\sigma_i$. In DEC, the basic operators in differential geometry are
approximated as follows:

1. Discrete exterior derivative $d$, this operator is the transpose of the
   incidence matrix of $k$-cells on $k + 1$-cells.

2. Discrete Hodge Star $*$, the operator scales the cells by the volumes of
   the corresponding dual and primal cells.

3. Discrete Laplace operator is $*^{-1}d^T * + d^T * d$.

For some situations, a source having azimuthal symmetry about its axis
is considered. In this case, the 2D triangular discrete manifold as the space
is only need to be considered. Now, we show how to derive an explicit DEC
scheme for Eq.(1) in 2D space manifold and the time. The wave equation
in 3D space and the time can also be computed by a similar approach.

Take Fig.1 as an example for a part of 2D mesh, in which $0, \ldots, F$
are vertices, $1, \ldots, 6$ are the circumcenters of triangles, $a, \ldots, f$
are the circumcenters of edges. Denote $l_{ij}$ as the length of line segment $(i, j)$ and $A_{ijkl}$ as the
area of quadrangle $(i, j, k, l)$.

![Fig.1. A part of 2D mesh](image)

Define

$$l_{12} := l_{1f} + l_{2f}, \quad l_{23} := l_{2a} + l_{3a}, \ldots, \quad l_{61} := l_{6e} + l_{1e},$$
and
\[ P_{123456} := A_{01fe} + A_{02fa} + \cdots + A_{06de}. \]
The diffusion term \( \Delta u \) at vertice 0 is approximated using discrete Laplace operator as follows:
\[
\Delta u_0 \approx \frac{1}{P_{123456}} \left( \frac{l_{23}}{l_{A0}} (u_A - u_0) + \frac{l_{34}}{l_{B0}} (u_B - u_0) + \frac{l_{45}}{l_{C0}} (u_C - u_0) \right. \\
+ \left. \frac{l_{56}}{l_{D0}} (u_D - u_0) + \frac{l_{16}}{l_{E0}} (u_E - u_0) + \frac{l_{12}}{l_{F0}} (u_F - u_0) \right). \tag{2}
\]
The temporal derivative is approximated by middle time differences as follows:
\[
\frac{\partial^2 u^n}{\partial t^2} \approx \frac{1}{(\Delta t)^2} (u_0^{n+1} - 2u^n + u_0^{n-1}), \tag{3}
\]
where \( \Delta t \) is uniform spacing, and \( n\Delta t \) is the coordinate of time. The approximation of Eq.(1) generated by substituting the left-hand sides of (2) and (3) into (1), thus satisfies
\[
Right(2)^{n-1} = \frac{1}{(c\Delta t)^2} \left( u_0^n - 2u_0^{n-1} + u_0^{n-2} \right). \tag{4}
\]

3 Stability, convergence and accuracy

Stability
The Courant-Friedrichs-Lewy condition is a necessary condition for stability while solving certain partial differential equations numerically. Now, this condition is derived for scheme (4). First, this DEC scheme is decomposed into temporal and spacial eigenvalue problems.

The temporal eigenvalue problem:
\[
\frac{\partial^2 u^n}{\partial t^2} = \Lambda u^n
\]
It can be approximated by difference equation
\[
\frac{u_0^{n+1} - 2u_0^n + u_0^{n-1}}{(\Delta t)^2} = \Lambda u_0^n. \tag{5}
\]
Supposing
\[
u_0^{n+1} = u_0^n \cos(\Delta t) \quad u_0^{n-1} = u_0^n \cos(-\Delta t)
\]
and substituting those into Eq.(5), we obtain

\[
\frac{\cos(\Delta t) + \cos(-\Delta t) - 2}{(\Delta t)^2} = \Lambda,
\]

therefore

\[
-\frac{4}{(\Delta t)^2} \leq \Lambda \leq 0.
\]

This is the stable condition for the temporal eigenvalue problem.

The spacial eigenvalue problem:

\[
c^2 \Delta u = \Lambda u
\]

It can be approximated by difference equation (6) based on Fig.1.

\[
\frac{P_{123456}}{c^2} \Lambda u_0 = \frac{l_{23}}{l_{A0}}(u_A - u_0) + \frac{l_{34}}{l_{B0}}(u_B - u_0) + \frac{l_{45}}{l_{C0}}(u_C - u_0) + \frac{l_{56}}{l_{D0}}(u_D - u_0) + \frac{l_{16}}{l_{E0}}(u_E - u_0) + \frac{l_{12}}{l_{F0}}(u_F - u_0)
\]

(6)

Let \( u_i = u_0 \cos(c l_{0i}) \) and substitute into Eq.(6) to obtain

\[
\frac{P_{123456}}{c^2} \Lambda = \frac{l_{23}}{l_{A0}}(\cos(c l_{0A}) - 1) + \frac{l_{34}}{l_{B0}}(\cos(c l_{0B}) - 1) + \frac{l_{45}}{l_{C0}}(\cos(c l_{0C}) - 1) + \frac{l_{56}}{l_{D0}}(\cos(c l_{0D}) - 1) + \frac{l_{16}}{l_{E0}}(\cos(c l_{0E}) - 1) + \frac{l_{12}}{l_{F0}}(\cos(c l_{0F}) - 1)
\]

So we have

\[
-\frac{2c^2}{P_{123456}} \left( \frac{l_{23}}{l_{A0}} + \frac{l_{34}}{l_{B0}} + \frac{l_{45}}{l_{C0}} + \frac{l_{56}}{l_{D0}} + \frac{l_{16}}{l_{E0}} + \frac{l_{12}}{l_{F0}} \right) \leq \Lambda \leq 0.
\]

In order to keep the stability of scheme (4), we need

\[
-\frac{4}{(\Delta t)^2} \leq -\frac{2c^2}{P_{123456}} \left( \frac{l_{23}}{l_{A0}} + \frac{l_{34}}{l_{B0}} + \frac{l_{45}}{l_{C0}} + \frac{l_{56}}{l_{D0}} + \frac{l_{16}}{l_{E0}} + \frac{l_{12}}{l_{F0}} \right)
\]

(7)

for all vertices, namely

\[
c \Delta t \leq \min_{0 \in V} \sqrt{\frac{2P_{123456}}{\frac{l_{23}}{l_{A0}} + \frac{l_{34}}{l_{B0}} + \frac{l_{45}}{l_{C0}} + \frac{l_{56}}{l_{D0}} + \frac{l_{16}}{l_{E0}} + \frac{l_{12}}{l_{F0}}}}.
\]
Convergence

By the definition of truncation error, the solution $\tilde{u}$ of the Eq.(1) satisfies the same relation as scheme (4) except for an additional term $O(\Delta t)^2$ on the right hand side. Thus the error $X_i^n = \tilde{u}_i^n - u_i^n$ is determined from the relation

$$X_i^n = 2X_{i-1}^{n-1} - X_i^{n-2} + \frac{(c\Delta t)^2}{P_{123456}} \left( \frac{l_{23}}{l_{A0}} (X_{A}^{n-1} - X_i^{n-1}) + \frac{l_{34}}{l_{B0}} (X_{B}^{n-1} - X_i^{n-1}) + \frac{l_{45}}{l_{C0}} (X_{C}^{n-1} - X_i^{n-1}) + \frac{l_{56}}{l_{D0}} (X_{D}^{n-1} - X_i^{n-1}) + \frac{l_{16}}{l_{E0}} (X_{E}^{n-1} - X_i^{n-1}) + \frac{l_{12}}{l_{F0}} (X_{F}^{n-1} - X_i^{n-1}) \right) + O(\Delta t)^2.$$  

(8)

Define

$$|X_i^n| = \max_{i \in V} |X_i^n|.$$  

From condition (7), we have

$$\frac{(c\Delta t)^2}{P_{123456}} \left( \frac{l_{23}}{l_{A0}} + \frac{l_{34}}{l_{B0}} + \frac{l_{45}}{l_{C0}} + \frac{l_{56}}{l_{D0}} + \frac{l_{16}}{l_{E0}} + \frac{l_{12}}{l_{F0}} \right) < 2.$$

Hence the coefficient of $X_i^{n-1}$ in Eq.(8) is nonnegative. It follows that

$$|X_i^n| \leq 2|X_i^{n-1}| + |X_i^{n-2}| + O(\Delta t)^2,$$

and hence that

$$|X_i^n| \leq 2|X_i^{n-1}| + |X_i^{n-2}| + O(\Delta t)^2.$$  

Iterating $n$, we obtain

$$|X_i^n| < M_1|X_i^1| + M_0|X_i^0| + O(\Delta t)^2,$$

where $M_1$, $M_0$ are finite value defined on $n$. Since the initial conditions ensure $X^0 = 0$ and $X^1 = 0$, we have

$$\lim_{\Delta t \to 0} |X_i^n| = 0.$$

That is to say the numerical solution approaches the exact solution as the step size goes to 0, and scheme (4) is convergent.
Accuracy

In scheme (4), the space derivative of is approximated by first order difference. Equivalently, $u$ is approximated by linear interpolation functions. Consulting the definition about accuracy of finite volume method, we can say that scheme (4) has first order spatial accuracy. Scheme (4) has second order temporal accuracy, and second order spacial accuracy on rectangular grid with uniform spacing.

4 Algorithm Implementation

The implementation of DEC scheme (4) for wave equation consists of the following steps:

1. Set the simulation parameters. These are the dimensions of the computational mesh and the size of the time step, etc.;

2. Initialize the mesh indexes.

3. Assign current transmitted signal.

4. Compute the value of all spatial nodes and temporarily store the result in the circular buffer for further computation.

5. Visualize the currently computed grid of spatial nodes.

6. Repeat the process from the step 3, until reach the desired total number of iterations.
The flowchart of the scheme (4) can be seen in Fig.2.

Fig.2. The flowchart of scheme (4)

The Fig.3 and Fig.4 show the numerical simulation of Gaussian pulse propagating on the sphere and rabbit by scheme (4) on C# platform.

Fig.3. The propagation of Gaussian pulse on a rabbit
5 Discussion

The DEC scheme for Laplace operator here can also be used to simulate the heat equation, Laplace equation and Poisson equation on manifold.

Discrete Laplace equation

The discrete Laplace equation on surface of regular tetrahedron (Fig.(5)) is

\[
\begin{pmatrix}
1 & 1 & 1 & -3 \\
1 & 1 & -3 & 1 \\
1 & -3 & 1 & 1 \\
-3 & 1 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
u_A \\
u_B \\
u_C \\
u_D \\
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}
\] (9)

The solution of Eq.(9) is

\[u_A = u_B = u_C = u_D = C,\]

where C is arbitrary constant. Eq.(9) is an imprecise approximation of Laplace’s equations on a sphere. Obviously, this equation has constant solution.

Fig.5. The surface of regular tetrahedron
Discrete Poisson equation

Consider a discrete Poisson equation on surface of regular tetrahedron. Suppose the boundary condition is 

\[ u_A = H, \]

then discrete Poisson equation on Fig.(5) is

\[
\begin{pmatrix}
3 & -1 & -1 \\
-1 & 3 & -1 \\
-1 & -1 & 3
\end{pmatrix}
\begin{pmatrix}
u_B \\
u_C \\
u_D
\end{pmatrix} = \begin{pmatrix}
H \\
H \\
H
\end{pmatrix}
\]

(10)

The solution of Eqs.(10) is 

\[ u_B = u_C = u_D = H. \]

Discrete heat equation

The heat equation of temperature \( u \) is

\[
\frac{\partial u}{\partial t} = c \Delta u,
\]

which can be approximated as

\[ u^n_0 = u^{n-1}_0 + c \Delta t \cdot \text{Right}(2)^{n-1}. \]

(11)

The Fig.6 shows the heat diffusion of a constant heat source on the sphere simulated by scheme (11).

![Fig.6. The heat diffusion on a sphere](image)

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