Outerstring graphs are $\chi$-bounded

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ABSTRACT
An outerstring graph is an intersection graph of curves lying in a halfplane with one endpoint on the boundary of the halfplane. It is proved that the outerstring graphs are $\chi$-bounded, that is, their chromatic number is bounded by a function of their clique number. This generalizes a series of previous results on $\chi$-boundedness of outerstring graphs with various restrictions of the shape of the curves or the number of times the pairs of curves can intersect. This also implies that the intersection graphs of $x$-monotone curves with bounded clique number have chromatic number $O(\log n)$, improving the previous polylogarithmic upper bound. The assumption that each curve has an endpoint on the boundary of the halfplane is justified by the known fact that triangle-free intersection graphs of straight-line segments can have arbitrarily large chromatic number.

Keywords
Geometric intersection graphs, outerstring graphs, chromatic number, $\chi$-boundedness

1. INTRODUCTION
The intersection graph of a finite family of sets $\mathcal{F}$ is the graph with vertex set $\mathcal{F}$ such that two members of $\mathcal{F}$ are connected by an edge if and only if they intersect. In this work, we consider families of curves $\mathcal{F}$ lying in a closed halfplane such that each curve $c \in \mathcal{F}$ has exactly one point on the boundary of the halfplane which is an endpoint of $c$. Such families of curves are called grounded, and their intersection graphs are called outerstring graphs.

The chromatic number of a graph $G$, denoted by $\chi(G)$, is the minimum number of colors sufficient to color the vertices of $G$ properly, that is, so that no two adjacent vertices obtain the same color. The clique number of a graph $G$, denoted by $\omega(G)$, is the maximum size of a clique in $G$, that is, a set of pairwise adjacent vertices in $G$. A class of graphs is $\chi$-bounded if there is a function $f: \mathbb{N} \to \mathbb{N}$ such that every graph $G$ in the class satisfies $\chi(G) \leq f(\omega(G))$.

In this paper, we establish the following result.

THEOREM. The class of outerstring graphs is $\chi$-bounded.

Outerstring graphs are a subclass of string graphs, which are intersection graphs of arbitrary curves in the plane. It is known, however, that the class of string graphs is not $\chi$-bounded [21].

Related work. The study of the chromatic number of intersection graphs of geometric objects in the plane has been initiated in 1960 in a seminal paper of Asplund and Grünbaum [3], who proved that intersection graphs of axis-parallel rectangles are $\chi$-bounded with the bound $\chi = O(\omega^2)$. It started to develop in the 1980s, partly stimulated by practical applications in channel assignment, map labeling, and VLSI design. Gyárfás [9, 10] proved $\chi$-boundedness of intersection graphs of chords of a circle. This was generalized by Kostochka and Kratochvíl [13] to polygons inscribed in a circle. Malešińska, Piskorz and Weissenfels [17] showed that intersection graphs of discs satisfy $\chi \leq 6\omega - 6$, while Peeters [23] proved $\chi \leq 3\omega - 2$ for unit discs. Both results on discs were generalized by Kim, Kostochka and Nakprasit [11], who showed that intersection graphs of families of homothets (uniformly scaled and translated copies) or translates of a fixed convex compact set in the plane satisfy $\chi \leq 6\omega - 6$ or $\chi \leq 3\omega - 2$, respectively. These results actually show a property stronger than $\chi$-boundedness, namely, that the average degree is bounded by a function of $\omega$. The strongest such result on geometric intersection graphs, due to Fox and Pach [6], asserts that string graphs excluding a fixed bipartite subgraph have bounded average degree. See also [12] for a survey of similar results.
McGuinness [18] proved that intersection graphs of L-shapes (shapes consisting of a horizontal and a vertical segment joined to form the letter ‘L’) intersecting a common horizontal line are $\chi$-bounded. Later [19], he proved that triangle-free intersection graphs of simple grounded families of compact arc-connected sets have bounded chromatic number. Suk [24] proved that intersection graphs of simple families of $x$-monotone curves grounded on a vertical line are $\chi$-bounded. Lasoni et al. [16] generalized both these results, showing that intersection graphs of simple grounded families of compact arc-connected sets are $\chi$-bounded. Here grounded means that the sets are contained in a halfplane and the intersection of any set with the boundary of the halfplane is a non-empty segment, and simple means that the intersection of any tuple of sets is arc-connected (possibly empty). Our present result generalizes all the ones in this paragraph, removing the restriction on the number of intersection points of pairs of curves.

Several results are known showing that some classes of geometric intersection graphs are not $\chi$-bounded. Burling [5] constructed triangle-free intersection graphs of axis-parallel boxes in $\mathbb{R}^3$ with arbitrarily large chromatic number. Using essentially the same construction, Pawlik et al. [21, 22] showed the existence of triangle-free intersection graphs of line segments and many other kinds of geometric shapes in the plane with arbitrarily large chromatic number. These constructions show that some restriction on the geometric layout of the families of objects considered, like the one that the family is grounded, is indeed necessary to guarantee $\chi$-boundedness.

The previous best upper bound on the chromatic number of outerstring graphs was of order $(\log n)^{O((\log \omega)^{\omega^{-1}})}$, where $n$ denotes the number of vertices, due to Fox and Pach [7], who established this bound for the chromatic number of arbitrary string graphs. The above-mentioned result of Suk [24] implies that intersection graphs of simple families of $x$-monotone curves with bounded clique number have chromatic number $O(\log n)$. On the other hand, the above-mentioned construction of Pawlik et al. [21] produces triangle-free segment intersection graphs with chromatic number $\Omega(\log \log n)$. Very recently, Krawczyk and Walczak [15] found a construction of string graphs with chromatic number $\Omega((\log \log n)^{\omega^{-1}})$. It is possible that all string graphs have chromatic number of order $(\log \log n)^{f(\omega)}$ for some function $f$. So far, bounds of this kind have been proved only for very special (but still not $\chi$-bounded) families of curves [14, 15].

Bounds on the chromatic number of intersection graphs of curves come useful in the study of so-called quasi-planar geometric and topological graphs. A topological graph is a graph drawn in the plane so that the curves representing edges do not pass through vertices other than their endpoints. Such a graph is $k$-quasi-planar if it does not have $k$ pairwise crossing edges (two curves that intersect only in a common endpoint are not considered as crossing by this definition). Hence 2-quasi-planar graphs are just planar graphs. A well-known conjecture [4, Problem 9.6.1] asserts that $k$-quasi-planar topological graphs on $n$ vertices have $O(n)$ edges for every fixed $k$. Given a $k$-quasi-planar topological graph $G$, if we shorten the edges a little at their endpoints so as to keep all proper crossings, then we obtain a family of curves whose intersection graph has clique number at most $k - 1$. Its proper coloring with $c$ colors yields an edge-decomposition of $G$ into $c$ planar graphs, hence the bound of $O(cn)$ on the number of edges of $G$ follows.

The quasi-planar graph conjecture has been proved for 3-quasi-planar simple topological graphs (that is, such that any two edges intersect in at most one point) by Agarwal et al. [2], then for all 3-quasi-planar topological graphs by Pach, Radoičić and Tóth [20], and for 4-quasi-planar topological graphs by Ackerman [1]. Valtr [26, 27] proved the bound of $O(n \log n)$ on the number of edges in $k$-quasi-planar simple topological graphs with edges drawn as $x$-monotone curves. An improvement of this result due to Fox, Pach and Suk [8] concludes the same without the simplicity condition. They also proved the bound of $2^{\alpha(n)} n \log n$ on the number of edges in $k$-quasi-planar simple topological graphs, where $\alpha(n)$ denotes the inverse Ackermann function and $c$ depends only on $k$. Suk and Walczak [25] showed the same bound for $k$-quasi-planar topological graphs in which any two edges cross a bounded number of times, and improved the bound for simple topological graphs to $O(n \log n)$.

**Corollaries.** One immediate consequence of our present result is that intersection graphs of grounded families of compact arc-connected sets in the plane are $\chi$-bounded, because each such set can be approximated by a grounded curve. This improves the result of Lasoni et al. [16] for simple grounded families of such sets. Another consequence is that intersection graphs of families of curves with the property that there is a straight-line intersecting every curve in the family in exactly one point are $\chi$-bounded. Indeed, we can independently color the parts of curves lying on each side of the line, and then, for the entire curves, we can use the pairs of colors thus obtained. This together with a standard divide-and-conquer argument implies that intersection graphs of $x$-monotone curves with bounded clique number have chromatic number $O(\log n)$, which yields an alternative proof of the result of Fox, Pach and Suk [8] that $k$-quasi-planar simple topological graphs with edges drawn as $x$-monotone curves have $O(n \log n)$ edges.

By the same argument as it is used in [25] for simple families of curves, we obtain the following: intersection graphs of families of curves with the property that one of the curves intersects every other curve in the family in exactly one point are $\chi$-bounded. If we were able to prove the same statement but with a relaxed condition—that one of the curves intersects every other in a bounded number of points, then this would imply, by the argument in [25], the bound of $O(n \log n)$ on the number of edges of $k$-quasi-planar topological graphs in which any two edges cross a bounded number of times.

**Conjecture.** For every $t$, the class of intersection graphs of families of curves with the property that one of the curves intersects every other in at least one and at most $t$ points is $\chi$-bounded.

2. **Preliminaries**

We fix the underlying halfplane of outerstring graphs to be
the closed halfplane above the horizontal axis. We call the horizontal axis bounding this halfplane the baseline. Therefore, we call a family of curves grounded if each curve in the family is contained in this halfplane, has one endpoint on the baseline, and does not intersect the baseline in any other point. We call the endpoint of a curve $c$ that lies on the baseline the basepoint of $c$. We can assume without loss of generality that the basepoints of all curves in a grounded family are pairwise distinct. Therefore, there is a natural left-to-right order of the curves in a grounded family corresponding to the order of their basepoints on the baseline. We denote this order by $\prec$, that is, $c_1 \prec c_2$ means that the basepoint of $c_1$ lies to the left of the basepoint of $c_2$. This notation naturally extends to families of curves: $F_1 \prec F_2$ denotes that $c_1 \prec c_2$ for any $c_1 \in F_1$ and $c_2 \in F_2$. If $F$ is a grounded family of curves and $c_1, c_2 \in F$ are two curves with $c_1 \prec c_2$, then we define $F(c_1, c_2) = \{ c \in F : c_1 \prec c \prec c_2 \}$.

For convenience, we denote the chromatic number and the clique number of the intersection graph of a grounded family of curves $F$ by $\chi(F)$ and $\omega(F)$, respectively.

The following lemma is essentially due to McGuinness [18, Lemma 2.1]. In its formulation below (adapted to our setting), it is proved in [16].

**Lemma 2.1.** If $F$ is a grounded family of curves with $\chi(F) > 2\alpha(\beta + 1)$, where $\alpha, \beta \geq 0$, then there is a subfamily $H \subset F$ such that $\chi(H) > \alpha$ and $\chi(F(u, v)) > \beta$ for any intersecting curves $u, v \in H$.

A special case with $\alpha = 1$ is the following corollary.

**Corollary 2.2.** If $F$ is a grounded family of curves with $\chi(F) > 2(\beta + 1)$, where $\beta \geq 0$, then there are two intersecting curves $u, v \in F$ such that $\chi(F(u, v)) > \beta$.

The exterior of a grounded family of curves $F$, denoted by $\text{ext}(F)$, is the unique unbounded arc-connected component of the closed halfplane above the baseline with the set $\bigcup F$ removed. If $F$ is a grounded family of curves and $u, v \in F$ are two intersecting curves, then it is an immediate consequence of the Jordan curve theorem that $\text{ext}(F)$ is disjoint from the part of the baseline between the basepoints of $u$ and $v$. Therefore, every curve intersecting $\text{ext}(F)$ whose basepoint lies between the basepoints of $u$ and $v$ must intersect $u$ or $v$.

A family $G \subset F$ is externally supported in $F$ if for any curve $p \in G$ there is a curve $s \in F$ that intersects $p$ and $\text{ext}(G)$. See Figure 1 for an illustration. The following lemma applies ideas of Gyárfás [9], which were also subsequently used in [16, 18, 19, 24].

**Lemma 2.3.** Every grounded family $F$ with $\omega(F) \geq 2$ has a subfamily $G$ that is externally supported in $F$ and satisfies $\chi(G) \geq \chi(F)/2$.

**Proof.** We can assume without loss of generality that the intersection graph of $F$ is connected, as otherwise we can take the component with maximum chromatic number. Let $c_0$ be the curve in $F$ with leftmost basepoint. For $i \geq 0$, let $F_i$ denote the family of curves in $F$ that are at distance $i$ from $c_0$ in the intersection graph of $F$. It follows that $F_0 = \{ c_0 \}$ and, for $|i - j| > 1$, each curve in $F_i$ is disjoint from each curve in $F_j$.

From each curve in $F_j$, clearly, we have $\chi(\bigcup F_i) \geq \chi(F)/2$ or $\chi(\bigcup F_{k+1}) \geq \chi(F)/2$, and therefore there is $d \geq 1$ with $\chi(F_d) \geq \chi(F)/2$.

We claim that $F_d$ is externally supported in $F$. Fix $c_d \in F_d$, and let $c_0 \ldots c_d$ be a shortest path from $c_0$ to $c_d$ in the intersection graph of $F$. Since $c_0$ is the curve in $F$ with leftmost basepoint, it intersects $\text{ext}(F_d)$. Moreover, $c_0 \ldots c_{d-2}$ are disjoint from $\bigcup F_d$, as otherwise there would be a curve in $F_d$ at distance less than $d$ to $c_0$. Therefore, all $c_0 \ldots c_{d-2}$ are entirely contained in $\text{ext}(F_d)$. This implies that $c_{d-1}$ intersects $\text{ext}(F_d)$, and by definition it also intersects $c_d$. □

### 3. PROOF SETUP

Here is our main theorem in the form that we are going to prove.

**Theorem (rephrased).** For every $k \geq 1$, the chromatic number of grounded families of curves $F$ with $\omega(F) \leq k$ is bounded by a constant depending only on $k$.

The proof proceeds by induction on $k$. The base case of $k = 1$ is trivial. Therefore, for the induction step, we assume that $k \geq 2$ and every grounded family of curves $F$ with $\omega(F) \leq k - 1$ satisfies $\chi(F) \leq \xi$ for some constant $\xi$. This context of an induction step and the meanings of $k$ and $\xi$ are maintained throughout the rest of the paper. The following observation, used without explicit reference everywhere from now on, explains how the induction hypothesis is applied.

**Observation.** Let $F$ be a grounded family of curves with $\omega(F) \leq k$, let $c_1, \ldots, c_n \in F$, and let $G \subset F \setminus \{ c_1, \ldots, c_n \}$. If each curve in $G$ intersects at least one of $c_1, \ldots, c_n$, then $\chi(G) \leq n\xi$.

**Proof.** For each $i$, the family of curves in $G$ intersecting $c_i$ has clique number at most $k - 1$, so by the induction hypothesis, it has chromatic number at most $\xi$. Summing up over all $i$, we obtain $\chi(G) \leq n\xi$. □

Most of the induction step goes by contraposition. We assume given a grounded family of curves $F$ with $\omega(F) \leq k$ and $\chi(F)$ large, and we show that some specific structure can be found in $F$. The goal is to find a $(k+1)$-clique, as this...
will contradict the assumption that $\omega(\mathcal{F}) \leq k$, thus showing that $\chi(\mathcal{F})$ cannot be too large.

Our proof consists of two parts. In the first part, we show that if $\omega(\mathcal{F}) \leq k$ and $\chi(\mathcal{F})$ is large enough, then $\mathcal{F}$ contains a subfamily with large chromatic number and a specific property, which we call supported by a skeleton (defined in the next section). This is again achieved by contraposition: we show that if every skeleton-supported subfamily of such a family $\mathcal{F}$ has small chromatic number, then $\mathcal{F}$ contains a $(k+1)$-clique. In the second part, we use the existence of skeleton-supported subfamilies recursively, finally also finding a $(k+1)$-clique in $\mathcal{F}$.

4. FIRST PART OF THE PROOF: FINDING A SKELETON-SUPPORTED FAMILY

Let $\mathcal{F}$ be a grounded family of curves. A skeleton in $\mathcal{F}$ is a triple $(u, v, \mathcal{S})$ consisting of a pair of intersecting curves $u$ and $v$ and a family $\mathcal{S} \subset \mathcal{F}(u, v)$ of pairwise disjoint curves. The curves in $\mathcal{S}$ are called the supports of the skeleton. We say that a family of curves $\mathcal{P} \subset \mathcal{F}(u, v)$ is supported by a skeleton $(u, v, \mathcal{S})$ if the following conditions are satisfied:

- no curve in $\mathcal{P}$ intersects $u$ or $v$,
- every curve in $\mathcal{P}$ intersects some curve $s \in \mathcal{S}$ in its part between the basepoint of $s$ and the first intersection point of $s$ with $u \cup v$.

See Figure 2 for an illustration. In this section, we show that if the chromatic number of $\mathcal{F}$ is large enough, then we can find a subfamily of $\mathcal{F}$ with large chromatic number supported by a skeleton in $\mathcal{F}$. That is, we are going to prove the following.

**Lemma 4.1.** There is a function $f : \mathbb{N} \to \mathbb{N}$ such that the following holds: for every $\alpha \in \mathbb{N}$, if $\mathcal{F}$ is a grounded family of curves with $\omega(\mathcal{F}) \leq k$ and $\chi(\mathcal{F}) > f(\alpha)$, then there is a subfamily $\mathcal{P} \subset \mathcal{F}$ with $\chi(\mathcal{P}) > \alpha$ supported by a skeleton in $\mathcal{F}$.

For the rest of this section, we assume that $\mathcal{F}$ is a grounded family of curves with $\omega(\mathcal{F}) \leq k$. In order to find a subfamily of $\mathcal{F}$ with large chromatic number supported by a skeleton in $\mathcal{F}$, we are going to show that if $\mathcal{F}$ has large chromatic number and no such subfamily exists, then $\mathcal{F}$ contains a $(k+1)$-clique.

A bracket in $\mathcal{F}$ is a pair $(\mathcal{P}, \mathcal{S})$ of subfamilies of $\mathcal{F}$ with the following properties:

- $\mathcal{P} \prec \mathcal{S}$ or $\mathcal{S} \prec \mathcal{P}$,
- every curve in $\mathcal{P}$ intersects some curve in $\mathcal{S}$,
- for every curve $s \in \mathcal{S}$, there is a curve $p \in \mathcal{P}$ such that $s$ is the first curve in $\mathcal{S}$ intersected by $p$ as going from the basepoint of $p$.

See Figure 3 for an illustration. For such a bracket and a curve $p \in \mathcal{P}$, we define

- $s(p)$ to be the first curve in $\mathcal{S}$ intersected by $p$ as going from the basepoint,
- $s'(p)$ to be the part of $p$ between its basepoint and its first intersection point with $s(p)$ (and thus with any member of $\mathcal{S}$) as going from the basepoint, excluding the intersection point with $s(p)$,
- $I(p)$ to be the closed region bounded by $s'(p)$, $s(p)$ and the part of the baseline between the basepoints of $p$ and $s(p)$.

Note that every $s'(p)$ is disjoint from every curve in $\mathcal{S}$. We also define $I = \bigcap_{p \in \mathcal{P}} I(p)$ and $E = \text{ext}(\mathcal{P} \cup \mathcal{S})$. We call $I$ and $E$ the interior and the exterior of the bracket, respectively.

**Lemma 4.2.** In the above setting, if $c$ is a curve intersecting both $I$ and $E$, then $c$ intersects $p$ or $s(p)$ for every $p \in \mathcal{P}$.

**Proof.** This is a direct consequence of the definitions of $I$ and $E$ and the Jordan curve theorem.

**Lemma 4.3.** In the above setting, if $\mathcal{H}$ is a family of curves in $\mathcal{F}$ intersecting the boundary of $I$ outside the baseline, then $\chi(\mathcal{H}) \leq 2k\xi$.

**Proof.** Assume without loss of generality that $\mathcal{S} \prec \mathcal{P}$. We are going to prove that the families of curves $\mathcal{P}_I = \{p \in \mathcal{P} : p' \cap I \neq \emptyset\}$ and $\mathcal{S}_I = \{s \in \mathcal{S} : s \cap I \neq \emptyset\}$ are cliques.

Suppose there are two curves $p_1, p_2 \in \mathcal{P}_I$ with $p_1 \prec p_2$ that do not intersect. It follows that $p_2'$ is disjoint from $p_1'$, $s(p_1)$ and the part of the baseline between the basepoints
of $p_1$ and $s(p_1)$. Hence it is disjoint from $f(p_1)$. This and $I \subset I(p_1)$ contradict the assumption that $p_0 \cap I \neq \emptyset$.

Now, suppose there are two curves $s_1, s_2 \in S_i$ with $s_1 \prec s_2$ that do not intersect. There are curves $p_1, p_2 \in P$ such that $s_1 = s(p_1)$ and $s_2 = s(p_2)$. It follows that $s_1$ is disjoint from $p_2$, $s(p_2)$ and the part of the baseline between the basepoints of $p_2$ and $s(p_2)$. Hence it is disjoint from $I(p_2)$. As before, this and $I \subset I(p_2)$ contradict the assumption that $s_1 \cap I \neq \emptyset$.

Since $P_1$ and $S_i$ are cliques, we have $|P_1| \leq k$ and $|S_i| \leq k$. Every point of the boundary of $I$ outside the baseline belongs to some curve in $P_1 \cup S_i$. Hence every curve in $H$ intersects at least one of the curves in $P_1 \cup S_i$. For every $c \in P_1 \cup S_i$, the curves in $H$ intersecting $c$ have clique number at most $k-1$ and thus chromatic number at most $\chi$. This and $|P_1 \cup S_i| \leq 2k$ yield $\chi(H) \leq 2k\chi$. □

A sequence of brackets $((P_1, S_1), \ldots, (P_n, S_n))$ with interiors $I_1, \ldots, I_n$ and exteriors $E_1, \ldots, E_n$, respectively, is an $n$-bracket system if it has the following properties:

- every curve in $P_i$ is entirely contained in $I_{i+1} \cap \cdots \cap I_n$;
- every curve in $S_i$ intersects $E_{i+1} \cap \cdots \cap E_n$.

**Lemma 4.4.** Let $((P_1, S_1), \ldots, (P_n, S_n))$ be an $n$-bracket system in $F$. If every $P_i$ has chromatic number greater than $(n-1)\chi$, then there are curves $s_i \in S_i$ such that $\{s_1, \ldots, s_n\}$ is a clique.

**Proof.** We proceed by induction on $n$. For $n = 1$, by the assumption that $\chi(P_1) > 0$, we have $P_1 \neq \emptyset$, so $S_1 \neq \emptyset$ and we can choose any $s_1 \in S_1$. Now, suppose $n \geq 2$. Choose any $s_1 \in S_1$. Since $s_1$ intersects a curve in $P_1$ and by the properties of a bracket system, $s_1$ intersects $I_2 \cap \cdots \cap I_n$ and $E_2 \cap \cdots \cap E_n$. For $2 \leq i \leq n$, the chromatic number of the curves in $P_i$ that intersect $s_1$ is at most $\chi$. Let $P_i'$ be the curves in $P_i$ that do not intersect $s_1$, and let $S_1' = \{s(p) : p \in P_i'\}$, for $2 \leq i \leq n$. It follows that $\chi(P_i') \geq \chi(P_i) - \chi > (n-2)\chi$. Therefore, we can apply the induction hypothesis to the $(n-1)$-bracket system $((P_2, S_2), \ldots, (P_n, S_n))$ to find curves $s_i \in S_i$ such that $\{s_2, \ldots, s_n\}$ is a clique. By Lemma 4.2, $s_i$ intersects every curve in $S_2, \ldots, S_n$. In particular, $s_1$ intersects $s_2, \ldots, s_n$, so $\{s_1, s_2, \ldots, s_n\}$ is a clique. □

If there is a $(k+1)$-bracket system $((P_0, S_0), \ldots, (P_k, S_k))$ in $F$ such that every $P_i$ has chromatic number greater than $k\chi$, then Lemma 4.4 gives us a $(k+1)$-clique in $F$, which contradicts the assumption that $\omega(F) \leq k$. Therefore, the following completes the proof of Lemma 4.1.

**Lemma 4.5.** There is a function $f : \mathbb{N} \to \mathbb{N}$ such that the following holds: for every $a \in \mathbb{N}$, if $\chi(F) > f(a)$ and every subfamily $P \subset F$ supported by a skeleton in $F$ satisfies $\chi(P) \leq a$, then there is a $(k+1)$-bracket system in $F$.

**Proof.** We fix $a \in \mathbb{N}$ and define $\beta_0 = 0$, $\beta_{i+1} = 2\beta_i + (2a + 6k)\chi + 2$ for $0 \leq i \leq k$, $\gamma = 2^{k+2}(\beta_k + 2\chi + 3)$.

We prove that it is enough to set $f(a) = \gamma$. Thus assume $\chi(F) > \gamma$.

First, by repeated application of Lemma 2.3, we find families $F_0, \ldots, F_{k+1}$ with the following properties:

- $F = F_0 \supset F_1 \supset \cdots \supset F_{k+1}$;
- $F_{k+1}$ is externally supported in $F_i$, for $0 \leq i \leq k$;
- $\chi(F_i) > \gamma/2^i$ for $0 \leq i \leq k+1$.

In particular, $\chi(F_{k+1}) > \gamma/2^{k+1} = 2(\beta_k + 1 + 2\chi + 1)$. By Corollary 2.2, there is a pair of intersecting curves $u, v \in F_{k+1}$ such that $\chi(F_k(u, v)) > \beta_k + 2\chi$. The curves in $F_{k+1}(u, v)$ that intersect $u$ or $v$ have chromatic number at most $2\chi$. Let $G$ be the family of curves in $F_{k+1}(u, v)$ that do not intersect any of $u, v$. It follows that $\chi(G) > \beta_k + 1$.

Now, we are going to find families $G_0, \ldots, G_k$ and bracket systems $(P_0, S_0), \ldots, (P_k, S_k)$ with interiors $I_0, \ldots, I_k$, respectively, so as to satisfy the following:

- $G_0 \subset \cdots \subset G_k \subset G_{k+1} = G$;
- $\chi(G_i) > \beta_i$ for $0 \leq i \leq k+1$;
- $P_i \subset G_{i+1}$ and $\chi(P_i) = k\chi + 1$, for $0 \leq i \leq k$;
- $S_i \subset F_i$ and every curve in $S_i$ intersects $\text{ext}(F_{i+1})$, for $0 \leq i \leq k$;
- every curve in $G_i$ is entirely contained in $I_i \cap \cdots \cap I_k$, for $0 \leq i \leq k$.

This is enough to prove the lemma, because this implies that every curve in $P_i$ is entirely contained in $I_i \cap \cdots \cap I_k$ and every curve in $S_i$ intersects $\text{ext}(F_{i+1}) \subset E_i \cap \cdots \cap E_k$, so $((P_0, S_0), \ldots, (P_k, S_k))$ is a $(k+1)$-bracket system. We start by setting $G_{k+1} = G$. Now, we find the families $G_i, P_i$, and $S_i$ by reverse induction on $i$ (from $k$ to $0$). Thus we suppose that we already have $G_{i+1}$ and show how to find $G_i, P_i$, and $S_i$.

Let $Q_i$ be the family of curves in $F_i(u, v)$ that intersect $u$ or $v$. We have $\chi(Q_0) \leq 2k\chi$. Therefore, $Q_0$ can be partitioned into $2k$ subfamilies $Q_0^1, \ldots, Q_0^{2k}$ so that each $Q_i^j$ consists of pairwise disjoint curves. Hence each $Q_i^j$ is the family of supports of skeleton $(u, v, Q_i^j)$ by our assumption, the chromatic number of the curves supported by each of these skeletons is at most $\alpha$. Let $H_i$ be the family of those curves in $G_{i+1}$ that are supported by none of these skeletons. It follows that $\chi(H_i) > \chi(H_{i+1}) - 2\alpha k > \beta_{i+1} - 2\alpha k = 2\beta_i + 6k\chi + 2$.

Since $H_i \subset F_{i+1}$ and $F_{i+1}$ is externally supported in $F_i$, for every $p \in H_i$, there is a curve $s \in F_i$ that intersects $p$ and $\text{ext}(F_{i+1})$. Let $s(p)$ denote the first curve $s$ with this property intersected by $p$ as going from the basepoint. Let $H_i^{\alpha}$ be the family of those curves $p \in H_i$ for which the basepoint of $s(p)$ lies to the left of the basepoint of $p$, and let $H_i^{\beta}$ be the family of those curves $p \in H_i$ for which the basepoint of $s(p)$ lies to the right of the basepoint of $p$. We have $H_i = H_i^\alpha \cup H_i^\beta$, so $\chi(H_i^{\alpha}) \geq \chi(H_i)/2$ or $\chi(H_i^{\beta}) \geq \chi(H_i)/2$.

Suppose $\chi(H_i^{\alpha}) \geq \chi(H_i)/2$. It follows that $\chi(H_i^{\alpha}) \geq \beta_i + 3k\chi + 1$. Choose $C_i \subset H_i^{\alpha}$ so that the intersection graph of $C_i$ is connected and $\chi(C_i) = \chi(H_i^{\alpha})$. We claim that for every $p \in C_i$, the basepoint of $s(p)$ lies to the left of the entire $C_i$. Suppose this is not the case, that is, there exists $q \in C_i$ such that the basepoint of $s(p)$ lies between the basepoints of $q$ and $p$. Since the intersection graph of $C_i$ is connected, it contains a path $p_0 \cdots p_t$ with $p_0 = p$ and $p_t = q$. By the Jordan curve theorem, one of the curves $p_0, \ldots, p_t$ must intersect the point of $s(p)$ from its basepoint to its first intersection point with $u \cup v$. Therefore, one of the curves $p_0, \ldots, p_t$ is supported by the skeleton $(u, v, Q_i^j)$, where $s(p) \in Q_i^j$. This contradicts the fact that $p_0, \ldots, p_t \in H_i$. 140
Choose $P_i \subset C$, so that $C_i \setminus P_i \prec P_i$ and $\chi(P) = k\xi + 1$. This can be done by processing the curves in $C_i$ from right to left in the order of their basepoints and adding them to $P_i$ until its chromatic number reaches $k\xi + 1$. Let $S_i = \{s(p) : p \in P_i\}$. It follows that $(P_i, S_i)$ is a bracket and $S_i \subset C_i \setminus P_i \prec P_i$. Let $I_i$ be the interior of $(P_i, S_i)$. By Lemma 4.3, the curves in $C_i \setminus P_i$ intersecting the boundary of $I_i$ have chromatic number at most $2k\xi$. Let $G_i$ consist of the curves in $C_i \setminus P_i$ that lie entirely in $I_i$. It follows that $\chi(G_i) \geq \chi(C_i) - 2k\xi = \chi(H_i^\ell) - 3k\xi - 1 > \beta_i$. We have thus found families $G_i, P_i$ and $S_i$ with all the requested properties.

The case $\chi(H_i^\ell) > \chi(H_i^\ell)/2$ is analogous. \qed

5. SECOND PART OF THE PROOF

Let $F$ be a grounded family of curves. For a clique $K \subset F$, let $\ell(K)$ denote the curve in $K$ with leftmost basepoint, and let $r(K)$ denote the curve in $K$ whose basepoint is second from the left. Let $\ell'(K)$ be the part of $\ell(K)$ between its basepoint and its first intersection point with $r(K)$ as going from the basepoint. Let $r'(K)$ be the part of $r(K)$ between its basepoint and its intersection point with $\ell'(K)$, excluding this intersection point. It follows that $\ell'(K) \cup r'(K)$ is a curve connecting the basepoint of $\ell(K)$ with the basepoint of $r(K)$. We call a curve $s \in F(\ell(K), r(K))$ left for $K$ if $s$ intersects $\ell'(K)$ and the part of $s$ between its basepoint and its first intersection point with $\ell'(K)$ as going from the basepoint does not intersect $r'(K)$. Similarly, we call a curve $s \in F(\ell(K), r(K))$ right for $K$ if $s$ intersects $r'(K)$ and the part of its basepoint and its first intersection point with $r'(K)$ as going from the basepoint does not intersect $\ell'(K)$. Clearly, every curve $p \in F(\ell(K), r(K))$ intersecting $\ell'(K) \cup r'(K)$ is either left or right for $K$.

A sequence $(K_1, \ldots, K_n)$ of cliques in $F$ of sizes $k_1, \ldots, k_n \geq 2$, respectively, is a $(k_1, \ldots, k_n)$-clique system if it satisfies the following conditions:

- $K_j \subset F(\ell(K_j), r(K_j))$ for $1 \leq i < j \leq n$,
- all curves in $K_j$ intersect $\ell'(K_i) \cup r'(K_i)$ and are all left for $K_i$ or all right for $K_i$.

See Figure 4 for an illustration. The first condition implies $\ell(K_1) < \ldots < \ell(K_n) < r(K_n) < \ldots < r(K_1)$. A clique system can be empty, that is, we allow $n = 0$. We say that a curve $s \in F(\ell(K_n), r(K_n))$ crosses the clique system $(K_1, \ldots, K_n)$ if it intersects $(\ell(K_1) \cup \ldots \cup K_n)$. A curve in $F(\ell(K_j), r(K_j))$ that crosses $(K_1, \ldots, K_n)$ must intersect $\ell'(K_j) \cup r'(K_j)$ for $1 \leq j \leq n$. The signature in $(K_1, \ldots, K_n)$ of such a curve $s$ is the sequence $\Sigma(s) = (\sigma_1(s), \ldots, \sigma_n(s))$ of 0s and 1s defined as follows:

$$
\sigma_j(s) = \begin{cases} 
0 & \text{if } s \text{ is left for } K_j, \\
1 & \text{if } s \text{ is right for } K_j.
\end{cases}
$$

The second condition on the clique system above means that the signatures in $(K_1, \ldots, K_{n-1})$ of all curves in $K_i$ are the same, for $1 \leq i \leq n$.

**Lemma 5.1.** In the setting above, if $s_1, s_2, s_3$ are pairwise disjoint curves in $F(\ell(K_n), r(K_n))$ crossing the clique system $(K_1, \ldots, K_n)$, $s_1 \prec s_2 \prec s_3$, and $\Sigma(s_1) = \Sigma(s_2) = \Sigma(s_3)$, then $\Sigma(s_1) = \Sigma(s_2) = \Sigma(s_3)$.

**Figure 4:** A $(3, 2)$-clique system $(K_1, K_2)$, where $K_1 = \{p_1, p_2, p_3\}$ and $K_2 = \{q_1, q_2\}$, crossed by a curve $s$; $\ell(K_1) = p_1$, $r(K_1) = p_2$, $\ell(K_2) = q_1$, $r(K_2) = q_2$; the curves $q_1$ and $q_2$ are left for $K_1$; the curve $s$ is right for $K_3$ (because it crosses $r'(K_3)$ before $\ell'(K_1)$ as going from the basepoint) and $K_2$.

**Proof.** Fix $j$ with $1 \leq j \leq n$. Since $\Sigma(s_1) = \Sigma(s_3)$, the curves $s_1$ and $s_3$ are both left or both right for $K_j$. Suppose they are both left for $K_j$. Since $s_2$ is disjoint from $s_1$ and $s_3$, it follows from the Jordan curve theorem that $s_2$ must intersect $\ell'(K_j)$ before it can intersect $r'(K_j)$ as going from the basepoint of $s_2$. Therefore, $s_2$ is also left for $K_j$. If $s_1$ and $s_3$ are right for $K_j$, then the same argument shows that $s_2$ is also right for $K_j$. \qed

We are going to prove the following.

**Lemma 5.2.** For $t \geq 2$, there is a function $g_t : \mathbb{N}^2 \to \mathbb{N}$ with the following property: for any $\alpha, \beta \in \mathbb{N}$, if every grounded family of curves $F$ with $\omega(F) \leq k$ and $\chi(F) > \alpha$ contains a $(k_1, \ldots, k_\alpha, t)$-clique system, then every grounded family of curves $F$ with $\omega(F) \leq k$ and $\chi(F) > g_t(\alpha, \beta)$ contains a $(k_1, \ldots, k_\alpha, t)$-clique system.

Once we prove Lemma 5.2, the induction step of our main theorem will be complete. Indeed, it will follow that every family of curves $F$ with $\omega(F) \leq k$ and $\chi(F) > g_{t+1}(0, 0)$ contains a $(k+1)$-clique, which it cannot contain if $\omega(F) \leq k$. Therefore, it will follow that $g_{t+1}(0, 0)$ is the upper bound on the chromatic number of any grounded family of curves $F$ with $\omega(F) \leq k$. The most involved case in the proof of Lemma 5.2 is when $t = 2$, which we are going to settle now. Then, the general case will follow by an easy induction.

**Lemma 5.3.** There is a function $g_2 : \mathbb{N}^2 \to \mathbb{N}$ with the following property: for any $\alpha, \beta \in \mathbb{N}$, if every grounded family of curves $F$ with $\omega(F) \leq k$ and $\chi(F) > \alpha$ contains a $(k_1, \ldots, k_\alpha, 2)$-clique system, then every grounded family of curves $F$ with $\omega(F) \leq k$ and $\chi(F) > g_2(\alpha, \beta)$ contains a $(k_1, \ldots, k_\alpha, 2)$-clique system.

**Proof.** Let $f$ be the function claimed by Lemma 4.1. We fix $\alpha, \beta \in \mathbb{N}$ and define

$$
m = 2^{\alpha} + 1, \quad \beta = 2\alpha((2^{2\alpha+2} + 2m)\xi + 1).
$$

We prove that it is enough to set $g_2(\alpha, \beta) = f(m)(\beta) + 1$, where $f(m)$ denotes the $m$-fold composition of $f$. Thus we assume that $F$ is a grounded family of curves with $\omega(F) \leq k$. 

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and $\chi(F) > f'(m) + 1$, and that every subfamily $\mathcal{H} \subset F$ with $\chi(\mathcal{H}) > \alpha$ contains a $(k_1, \ldots, k_n)$-clique system. If $n = 0$, then all we need to conclude that $F$ contains a $(2)$-clique system are two intersecting curves in $F$, which exist as $\chi(F) > 1$. So we assume $n \geq 1$.

First, by repeated application of Lemma 4.1, we find families $F_0, \ldots, F_{m-1}$ and skeletons $(u_1, v_1, S_1), \ldots, (u_m, v_m, S_m)$ with the following properties:

- $F = F_0 \supset F_1 \supset \ldots \supset F_m$.
- $(u_i, v_i, S_i)$ is a skeleton in $F_{i-1}$, for $1 \leq i \leq m$.
- $F_i$ is supported by the skeleton $(u_i, v_i, S_i)$, for $1 \leq i \leq m$.
- $\chi(F_i) > f'(m-i)(\beta) \text{ for } 0 < i \leq m$.

In particular, $\chi(F_m) > \beta = 2(2^{m+2} + 2m)\xi + 1)$. By Lemma 2.1, there is a subfamily $H \subset F_m$ such that $\chi(H) > \alpha$ and $\chi(F(p, q)) > 2^{m+1} + 2m\xi$ for any intersecting $p, q \in H$. Since $\chi(H) > \alpha$, there is a $(k_1, \ldots, k_n)$-clique system $(K_1, \ldots, K_m)$ in $H$. Let $\ell = \ell(K_m)$ and $r = r(K_m)$. It follows that $\chi(F_m(\ell, r)) > 2^{m+1} + 2m\xi$.

For every $p \in F_m$ and $1 \leq i \leq m$, since $p$ is supported by the skeleton $(u_i, v_i, S_i)$, there is a curve $s_i \in S_i$ with the property that $p$ intersects $s_i$ in the part between the basepoint of $s_i$ and the first intersection point of $s_i$ with $u_i \cup v_i$ as going from the basepoint. Choose any such curve $s_i$ and denote it by $s_i(p)$.

Let $\mathcal{G} = \{p \in F_m(\ell, r) : s_0(p), \ldots, s_m(p) \in F(\ell, r)\}$. If $p \in F_m(\ell, r)$ and $s_i(p) \notin F(\ell, r)$, then, by the Jordan curve theorem, $p$ must intersect the curve in $S_i$ whose basepoint is rightmost to the left of the basepoint of $\ell$ or the curve in $S_i$ whose basepoint is leftmost to the right of the basepoint of $r$. Therefore, the curves $p \in F_m(\ell, r)$ with $s_i(p) \notin F(\ell, r)$ have chromatic number at most $2\xi$. It follows that $\chi(\mathcal{G}) \geq \chi(F_m(\ell, r)) - 2m\xi > 2^{m+1}\xi$.

Every curve $s \in \bigcup_{i \leq n} S_n(\ell)$ intersects $\text{ext}(F_m)$, so it must intersect every $\ell(K_j) \cup r(K_j)$ with $1 \leq j \leq n$. For such a curve $s$, let $\Sigma(s)$ denote the signature of $s$ with respect to the clique system $(K_1, \ldots, K_n)$. There are at most $2^n$ different signatures $\Sigma(s)$. There are at most $2^n$ different sequences of signatures $\Sigma(s_1(p)), \ldots, \Sigma(s_m(p))$ for $p \in \mathcal{G}$. Therefore, there is a subfamily $\mathcal{P} \subset \mathcal{G}$ with $\chi(\mathcal{P}) \geq \chi(\mathcal{G})/2m > \xi$ such that all the sequences of signatures $\Sigma(s_1(p)), \ldots, \Sigma(s_m(p))$ for $p \in \mathcal{P}$ are equal to some $(\Sigma_1, \ldots, \Sigma_n)$. Since $m = 2^n + 1$, there are two indices $i$ and $j$ with $i < j$ such that $\Sigma_i = \Sigma_j$. That is, all the signatures $\Sigma(s_i(p))$ and $\Sigma(s_j(p))$ for $p \in \mathcal{P}$ are equal to some $\Sigma = \Sigma_i = \Sigma_j$.

Let $s_i^0$ and $s_i^1$ be the curves of the form $s_i(p)$ with $p \in \mathcal{P}$ whose basepoints are leftmost and rightmost, respectively. If there is a curve of the form $s_i^0(p)$ with $p \in \mathcal{P}$ and with basepoint to the left of $s_i^0$, then let $s_i^0$ denote the curve with this property whose basepoint is leftmost. Similarly, if there is a curve of the form $s_i^1(p)$ with $p \in \mathcal{P}$ and with basepoint to the right of $s_i^0$, then let $s_i^1$ denote the curve with this property whose basepoint is rightmost. Since every curve $p \in F(\ell, s_i^0)$ intersects $s_i(p)$, it must intersect $s_i^0$, by the Jordan curve theorem. Hence $\chi(F(\ell, s_i^0)) \leq \xi$. Similarly, we have $\chi(F(\ell, s_i^1)) \leq \xi$. Therefore, $\chi(\mathcal{P}) = \chi(F(\ell, s_i^0)) + \chi(F(\ell, s_i^1)) = \chi(\mathcal{P}) - \chi(F(\ell, s_i^0)) - \chi(F(\ell, s_i^1)) > 2\xi$. Again, by the Jordan curve theorem, every curve $p \in \mathcal{P}(s_i^1, s_i^0)$ with $s_i(p) \notin F(s_i^1, s_i^0)$ must intersect $s_i^0$ or $s_i^1$, so the chromatic number of these curves is at most $2\xi$. Therefore, there is a curve $p \in \mathcal{P}(s_i^1, s_i^0)$ with $s_i(p) \in F(s_i^1, s_i^0)$.

Since $s_j(p) \in F_j$ and $i < j$, the curve $s_j(p)$ is supported by the skeleton $(u_i, v_i, S_i)$. This means that there is a support $s \in S_i$ such that $s_j(p)$ intersects the part of $s$ between its basepoint and its first intersection point with $u_i \cup v_i$ as going from the basepoint. Choose $s$ to be the first support with this property that $s_j(p)$ intersects as going from the basepoint of $s_j(p)$. Since $s_i^0 < s_j(p) < s_i^1$ and the members of $S_i$ are pairwise disjoint, it follows from the Jordan curve theorem that either $s$ is one of $s_i^0$, $s_i^1$ or we have $s_i^0 < s < s_i^1$. Now, since $\Sigma(s_j(p)) = \Sigma(s_i^0) = \Sigma$, Lemma 5.1 yields $\Sigma(s) = \Sigma$. Hence $\Sigma(s) = \Sigma(s_i(p_j))$. This shows that $K_1, \ldots, K_m, \{s, s_j(p_j)\}$ is a $(k_1, \ldots, k_m, 2)$-clique system in $F$. □

PROOF OF Lemma 5.2. We proceed by induction on $t$. The case $t = 2$ has been proved as Lemma 5.3. Thus suppose $t > 3$. Let $g_{t-1}$ be the function claimed by the induction hypothesis. We fix $\alpha, n \in \mathbb{N}$ and define

$$m = 2^n + 1, \quad \beta_0 = \alpha, \quad \beta_{t+1} = g_{t-1}(\beta_t + i) \text{ for } 0 \leq i \leq m - 1.$$ We prove that it is enough to set $g_t(\alpha, n) = \beta_m$.

We assume in the lemma that every grounded family of curves $F$ with $\omega(F) \leq k$ and $\chi(F) > \alpha$ contains a $(k_1, \ldots, k_n)$-clique system. Repeated application of the inductive hypothesis yields the following: every grounded family of curves $F$ with $\omega(F) \leq k$ and $\chi(F) > \beta_m$ contains a $(k_1, \ldots, k_m, t-1, \ldots, t-1)$-clique system. Let $F$ be such a family and $(K_1, \ldots, K_m, L_1, \ldots, L_m)$ be such a clique system in $F$.

For a curve $s \in L_1 \cup \ldots \cup L_m$, let $\Sigma(s)$ denote the signature of $s$ with respect to the clique system $K_1, \ldots, K_n$. Every $\Sigma(s)$ has length $n$, so there are at most $2^n$ different signatures of the form $\Sigma(s)$. Since $m = 2^n + 1$ and all curves $s \in L_i$ have the same signature $\Sigma(s)$ for $1 \leq i \leq m$, there are two indices $i$ and $j$ with $i < j$ such that all curves $s \in L_i \cup L_j$ have the same signature $\Sigma(s)$. Moreover, since $L_i$ and $L_j$ are part of the clique system $(K_1, \ldots, K_n, L_1, \ldots, L_m)$, all curves in $L_j$ are either left or right for $L_i$. If they are left for $L_i$, then $L_i = L_j \cup \{i(L_i)\}$ is a $t$-clique. If they are right for $L_i$, then $L_i = L_j \cup \{r(L_i)\}$ is a $t$-clique. In both cases we conclude that $(K_1, \ldots, K_n, L)$ is a $(k_1, \ldots, k_n, t)$-clique system in $F$. □

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