Generalized additive bases, König’s lemma, and the Erdős-Turán conjecture*

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Abstract

Let \( A \) be a set of nonnegative integers. For every nonnegative integer \( n \) and positive integer \( h \), let \( r_A(n,h) \) denote the number of representations of \( n \) in the form \( n = a_1 + a_2 + \cdots + a_h \), where \( a_1, a_2, \ldots, a_h \in A \) and \( a_1 \leq a_2 \leq \cdots \leq a_h \). The infinite set \( A \) is called a basis of order \( h \) if \( r_A(n,h) \geq 1 \) for every nonnegative integer \( n \). Erdős and Turán conjectured that \( \limsup_{n \to \infty} r_A(n,2) = \infty \) for every basis \( A \) of order 2. This paper introduces a new class of additive bases and a general additive problem, a special case of which is the Erdős-Turán conjecture. König’s lemma on the existence of infinite paths in certain graphs is used to prove that this general problem is equivalent to a related problem about finite sets of nonnegative integers.

1 Representation functions and the Erdős-Turán conjecture

Let \( \mathbb{N}_0 \) and \( \mathbb{Z} \) denote the nonnegative integers and integers, respectively. Let \( A \) be a finite set of integers. We denote the largest element of \( A \) by \( \max(A) \) and the cardinality of \( A \) by \( \text{card}(A) \). For any real numbers \( a \) and \( b \), we denote by \( [a,b] \) the finite set of integers \( n \) such that \( a \leq n \leq b \).

For any set \( A \) of integers, we denote by \( r_A(n,h) \) the number of representations of \( n \) in the form \( n = a_1 + a_2 + \cdots + a_h \), where \( a_1, a_2, \ldots, a_h \in A \) and \( a_1 \leq a_2 \leq \cdots \leq a_h \). The function \( r_A \) is called the unordered representation function of the set \( A \), or, simply, the representation function of \( A \).

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The set $A$ of nonnegative integers is called a \textit{basis of order $h$} if every nonnegative integer can be represented as the sum of $h$ not necessarily distinct elements of $A$. If $A$ is a basis of order $h$ with representation function $r_A$, then

$$1 \leq r_A(n, h) < \infty$$

for all nonnegative integers $n$. We call $A$ an \textit{asymptotic basis of order $h$} if

$$r_A(n, h) = f(n)$$

for all $n \geq 0$, where $f : \mathbb{N}_0 \to \mathbb{N}_0 \cup \{\infty\}$ is any function such that $\text{card}(f^{-1}(0)) < \infty$.

In the case of additive bases for the set of all integers, Nathanson \cite{5} proved that every function is a representation function, that is, if $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ satisfies the condition $\text{card}(f^{-1}(0)) < \infty$, then for every $h \geq 2$ there exists a set $A$ of integers such that $r_A(n, h) = f(n)$ for every integer $n$.

A special case of the representation function problem for nonnegative integers with $h = 2$ is the conjecture of Erdős and Turán \cite{2} that the representation function $r_A(n, 2)$ of an asymptotic basis $A$ of order 2 must be unbounded, that is,

$$\liminf_{n \to \infty} r_A(n, 2) > 0 \implies \limsup_{n \to \infty} r_A(n, 2) = \infty.$$

This is an important unsolved problem in additive number theory.

Dowd \cite{1} and Grekos, Haddad, Helou, and Pikho \cite{3} have given various equivalent formulations of the Erdős-Turán conjecture. In particular, Dowd proved that there exists a set $A$ of nonnegative integers and a number $c$ such that $r_A(n, 2) \in [1, c]$ for all nonnegative integers $n$ if and only if for every $N$ there exists a finite set $A_N$ of nonnegative integers with $\max(A_N) \geq N$ and $r_A(n, 2) \in [1, c]$ for all $n = 0, 1, \ldots, \max(A_N)$. In this paper we apply Dowd’s method to obtain similar results for a new class of generalized additive bases.

\section{Generalized additive bases}

We extend the idea of an additive basis of order $h$ as follows: Let $\mathcal{H} = \{H_n\}_{n=0}^\infty$ be a sequence of nonempty finite sets of positive integers. For any set $A$ of nonnegative integers, we define the representation function

$$r_A(n, H_n) = \sum_{h_n \in H_n} r_A(n, h_n).$$

The set $A$ of nonnegative integers will be called a \textit{basis of order $\mathcal{H}$} if

$$r_A(n, H_n) \geq 1$$

for all $n \geq 0$, and an \textit{asymptotic basis of order $\mathcal{H}$} if the representation function satisfies \text{\textit{(1)}} for all sufficiently large $n$.

Let $\mathcal{R} = \{R_n\}_{n=0}^\infty$ be a sequence of nonempty finite sets of positive integers. If

$$r_A(n, H_n) \in R_n$$

for all $n \geq 0$, and an \textit{asymptotic basis of order $\mathcal{H}$} if the representation function satisfies \text{\textit{(1)}} for all sufficiently large $n$. In this paper we apply Dowd’s method to obtain similar results for a new class of generalized additive bases.
for every nonnegative integer $n$, then $A$ will be called an $R$-basis of order $\mathcal{H}$. Since each $R_n$ is a nonempty set of positive integers, it follows that every $R$-basis of order $\mathcal{H}$ is a basis of order $\mathcal{H}$. We shall call the set $A$ an asymptotic $R$-basis of order $\mathcal{H}$ if (2) holds for all sufficiently large $n$.

For any sequences $\mathcal{H} = \{H_n\}_{n=0}^\infty$ and $\mathcal{R} = \{R_n\}_{n=0}^\infty$ of nonempty finite sets of positive integers, we can ask if there exists an $R$-basis of order $\mathcal{H}$ or an asymptotic $R$-basis of order $\mathcal{H}$. This is the generalized representation function problem. The original Erdős-Turán conjecture corresponds to the special case $H_n = \{2\}$ and $R_n = [1, c]$ for all $n \geq 0$. It is an open problem to determine the number of distinct $\mathcal{R}$-bases of order $\mathcal{H}$ and asymptotic $\mathcal{R}$-bases of order $\mathcal{H}$ for a given pair of sequences $\mathcal{H}$ and $\mathcal{R}$.

Let $h \geq 2$ and let $f$ be a function such that $f(n)$ is a positive integer for every nonnegative integer $n$. We introduce the sets $H_n = \{h\}$ and $R_n = \{f(n)\}$ for all $n$, and the sequences $\mathcal{H} = \{H_n\}_{n=0}^\infty$ and $\mathcal{R} = \{R_n\}_{n=0}^\infty$. Then an $R$-basis of order $\mathcal{H}$ is a basis $A$ of order $h$ whose representation function satisfies $r_A(n, h) = f(n)$ for all $n \in \mathbb{N}_0$, and so the representation function problem for bases of order $h$ is a special case of the generalized representation function problem.

An $R$-basis of order $\mathcal{H}$ is not necessarily infinite. For example, if $H_0 = R_0 = \{1\}$ and if $H_n = \{n\}$ and $1 \in R_n$ for all $n \geq 1$, then the set $\{0, 1\}$ is an $R$-basis of order $\mathcal{H}$.

**Theorem 1** Let $\mathcal{H} = \{H_n\}_{n=0}^\infty$ be a sequence of nonempty finite sets of positive integers. There exists a finite set $A$ that is a basis of order $\mathcal{H}$ or an asymptotic basis of order $\mathcal{H}$ if and only if

$$
\liminf_{n \to \infty} \frac{\max(H_n)}{n} > 0.
$$

**Proof.** Let $h_n^* = \max(H_n)$. Let $A$ be a finite set of nonnegative integers that is a basis of order $\mathcal{H}$. Then $0, 1 \in A$ and so $\max(A) \geq 1$. Every positive integer $n$ can be represented as the sum of $h_n$ elements of $A$ for some $h_n \in H_n$, and so

$$
n \leq h_n \max(A) \leq h_n^* \max(A).
$$

It follows that

$$
\liminf_{n \to \infty} \frac{h_n^*}{n} \geq \frac{1}{\max(A)} > 0.
$$

Conversely, if $\liminf_{n \to \infty} h_n^*/n > 0$, then there exists a positive integer $m$ such that

$$
h_n^* \geq \frac{n}{m}
$$

for all $n \geq 0$. Consider the finite set $A = [0, m]$. By the division algorithm, every positive integer $n$ can be written in the form $n = qm + r$, where $q$ and $r$ are nonnegative integers and $0 \leq r \leq m - 1$. If $r = 0$, then $q = n/m \leq h_n^*$ and

$$
n = q \cdot m + (h_n^* - q) \cdot 0 \in h_n^* A.
$$
If \( 1 \leq r \leq m - 1 \), then \( q = (n - r)/m < h_n^* \). Since \( h_n^* \) and \( q \) are integers, it follows that \( h_n^* \geq q + 1 \) and

\[
n = q \cdot m + 1 \cdot r + (h_n^* - q - 1) \cdot 0 \in h_n^*A.
\]

In both cases, \( r_A(n, H_n) \geq r_A(n, h_n^*) \geq 1 \), and the finite set \( A \) is a basis of order \( \mathcal{H} \).

If \( A \) is an asymptotic basis of order \( \mathcal{H} \), then \( r_A(n, H_n) = 0 \) for only finitely many \( n \in \mathbb{N}_0 \), and so there is a finite set \( F \) of nonnegative integers such that \( A \cup F \) is a basis of order \( \mathcal{H} \). Therefore, there exists a finite set that is a basis of order \( \mathcal{H} \) if and only if there exists a finite set that is an asymptotic basis of order \( \mathcal{H} \). This completes the proof. \( \square \)

Let \( \mathcal{H} = \{ H_n \}_{n=0}^{\infty} \) and \( \mathcal{R} = \{ R_n \}_{n=0}^{\infty} \) be sequences of nonempty finite sets of positive integers. A nonempty finite set \( A \) of nonnegative integers will be called a finite basis of order \( \mathcal{H} \) if

\[
r_A(n, H_n) \geq 1
\]

for all \( n \in [0, \max(A)] \), and a finite \( \mathcal{R} \)-basis of order \( \mathcal{H} \) if

\[
r_A(n, H_n) \in R_n
\]

for all \( n \in [0, \max(A)] \).

**Theorem 2** Let \( \mathcal{H} = \{ H_n \}_{n=0}^{\infty} \) and \( \mathcal{R} = \{ R_n \}_{n=0}^{\infty} \) be sequences of nonempty finite sets of positive integers.

(i) If \( A \) is a basis of order \( \mathcal{H} \), or if \( A \) is a finite basis of order \( \mathcal{H} \) with \( \max(A) \geq 1 \), then \( 0, 1 \in A \).

(ii) If \( A \) is an \( \mathcal{R} \)-basis of order \( \mathcal{H} \) or if \( A \) is a finite \( \mathcal{R} \)-basis of order \( \mathcal{H} \) with \( \max(A) \geq 1 \), then \( \text{card}(H_0) \in R_0 \) and \( \text{card}(H_1) \in R_1 \).

(iii) If \( A \) is an \( \mathcal{R} \)-basis of order \( \mathcal{H} \), then \( A_N = A \cap [0, N] \) is a finite \( \mathcal{R} \)-basis of order \( \mathcal{H} \) for every \( N \geq 0 \).

(iv) If \( A \neq \{0\} \) is a finite \( \mathcal{R} \)-basis of order \( \mathcal{H} \), then \( F' = F \setminus \{\max(F)\} \) is also a finite \( \mathcal{R} \)-basis of order \( \mathcal{H} \).

**Proof.** To prove (i) and (ii), we observe that if \( r_A(0, H_0) \geq 1 \), then \( 0 \in A \). If \( r_A(1, H_1) \geq 1 \), then \( 1 \in A \). Since, for every \( h \geq 1 \), both 0 and 1 have unique representations as sums of exactly \( h \) nonnegative integers, it follows that if \( r_A(0, H_0) \in R_0 \), then

\[
r_A(0, H_0) = \sum_{h_0 \in H_0} r_A(0, h_0) = \sum_{h_0 \in H_0} 1 = \text{card}(H_0) \in R_0.
\]

Similarly, if \( r_A(1, H_1) \in R_1 \), then

\[
r_A(1, H_1) = \text{card}(H_1) \in R_1.
\]

The statements (iii) and (iv) follow immediately from the definition of a finite basis. \( \square \)
3 König’s lemma

The principal tool in this paper is König’s lemma on the existence of infinite paths in trees. For completeness, we include a short proof below.

A graph $G$ consists of a nonempty set $\{v\}$, whose elements are called vertices, and a set $\{e\}$, whose elements are called edges. Each edge is a set $e = \{v, v'\}$, where $v$ and $v'$ are vertices and $v \neq v'$. Thus, we are considering only graphs without loops or multiple edges.

We use the following terminology. The vertices $v$ and $v'$ are called adjacent if $\{v, v'\}$ is an edge. The degree of a vertex $v$ is the number of edges $e$ with $v \in e$. A path in $G$ from vertex $v$ to vertex $v'$ is a sequence of vertices $v_0, v_1, v_2, \ldots, v_n$ such that $v_0 = v$, $v_n = v'$, and $v_i$ is adjacent to $v_j$ for all $i = 1, \ldots, n$. We define the length of this path by $n$. The graph $G$ is connected if for every two vertices $v$ and $v'$ with $v \neq v'$ there is a path from $v$ to $v'$. A graph is connected if and only if, for some vertex $v_0$, there is a path from $v_0$ to $v$ for every vertex $v \neq v_0$.

A simple path in $G$ is a path whose vertices are pairwise distinct. A simple circuit is a sequence of vertices $v_0, v_1, v_2, \ldots, v_n$ such that $n \geq 3$, $\{v_{i-1}, v_i\}$ is an edge for $i = 1, \ldots, n$, $v_i \neq v_j$ for $0 \leq i < j \leq n - 1$, and $v_0 = v_n$. A graph $G$ has no simple circuits if and only if, for every pair of distinct vertices $v$ and $v'$, there is at most one simple path from $v$ to $v'$. An infinite simple path is an infinite sequence of pairwise distinct vertices $v_0, v_1, v_2, \ldots$ such that $v_{i-1}$ is adjacent to $v_i$ for all $i \geq 1$.

A tree is a connected graph with no simple circuits. A rooted tree is a tree with a distinguished vertex, called the root of the tree. In a rooted tree, for every vertex $v$ different from the root, there is a unique simple path in the tree from the root to $v$.

**Theorem 3 (König’s lemma)** If $T$ is a rooted tree with infinitely many vertices such that every vertex has finite degree, then $T$ contains an infinite simple path beginning at the root.

**Proof.** Let $v_0$ be the root of the tree. We use induction to prove that for every $n$ there is a simple path $v_0, v_1, \ldots, v_n$ such that the tree $T$ contains infinitely many vertices $v$ for which the unique simple path from the root $v_0$ to $v$ begins with the vertices $v_0, v_1, \ldots, v_n$. Since $T$ has infinitely many vertices, the root $v_0$ satisfies this condition.

Let $n \geq 1$, and assume that we have constructed a simple path $v_0, v_1, \ldots, v_{n-1}$ of vertices of the tree $T$ with the property that $T$ contains an infinite set $I_{n-1}$ of vertices such that, for every $v \in I_{n-1}$, the unique simple path from $v_0$ to $v$ passes through vertex $v_{n-1}$. Since the degree of $v_{n-1}$ is finite, the set $F_n$ of vertices $v \neq v_{n-2}$ that are adjacent to $v_{n-1}$ is a finite set. For every vertex $v \in I_{n-1}$, there is a unique simple path in $T$ that begins at $v_0$, passes through $v_{n-1}$ and exactly one of the vertices in $F_n$, and ends at $v$. By the pigeonhole principle, since $I_{n-1}$ is infinite, there is a vertex $v_n \in F_n$ and an infinite set $I_n \subseteq I_{n-1}$ of vertices such that, for every $v \in I_n$, the unique path from $v_0$ to $v$
passes through \( v_n \). This completes the induction. The vertices \( v_0, v_1, v_2, \ldots \) are pairwise distinct, and \( v_0, v_1, v_2, \ldots \) is an infinite simple path in \( T \). \( \square \)

4 The generalized representation function problem

In this section we prove that there exists an infinite \( \mathcal{R} \)-basis of order \( \mathcal{H} \) if and only if there exist arbitrarily large finite \( \mathcal{R} \)-bases of order \( \mathcal{H} \).

**Theorem 4** Let \( \mathcal{R} = \{ R_n \}_{n=0}^{\infty} \) and \( \mathcal{H} = \{ H_n \}_{n=0}^{\infty} \) be sequences of nonempty finite sets of positive integers such that

\[
\lim_{n \to \infty} \frac{\max(H_n)}{n} = 0. 
\]

There exists an \( \mathcal{R} \)-basis \( A \) of order \( \mathcal{H} \) if and only if for every \( N \) there exists a finite \( \mathcal{R} \)-basis \( A_N \) of order \( \mathcal{H} \) with \( \max(A_N) \geq N \).

**Proof.** If \( A \) is a \( \mathcal{R} \)-basis of order \( \mathcal{H} \), then, by (3) and Theorem 1, the set \( A \) is infinite, hence for every \( N \) there is an integer \( a(N) \in A \) with \( a(N) \geq N \). By Theorem 2, the set \( A_N = A \cap [0, a(N)] \) is a finite \( \mathcal{R} \)-basis of order \( \mathcal{H} \) with \( \max(A_N) \geq N \).

Conversely, suppose that for every \( N \) there exists a finite \( \mathcal{R} \)-basis \( A_N \) of order \( \mathcal{H} \) with \( \max(A_N) \geq N \). If \( N \geq 1 \), then \( 0, 1 \in A_N \) and the sets \( \{0\} \) and \( \{0, 1\} \) are finite \( \mathcal{R} \)-bases of order \( \mathcal{H} \).

We construct the graph \( T \) whose vertices are the finite \( \mathcal{R} \)-bases of order \( \mathcal{H} \). This graph has infinitely many vertices, since there are finite \( \mathcal{R} \)-bases of order \( \mathcal{H} \) with arbitrarily large maximum elements.

Vertices \( V \) and \( V' \) will be called adjacent in this graph if \( V' \subseteq V \) and \( V \setminus V' = \{ \max(V) \} \). The sets \( \{0\} \) and \( \{0, 1\} \) are adjacent vertices of this graph, and \( \{0, 1\} \) is the only vertex adjacent to \( \{0\} \). If \( V \) is a vertex and \( \text{card}(V) \geq 2 \), then it follows from Theorem 2 that \( V' = V \setminus \{ \max(V) \} \) is a vertex. Moreover, \( V' \) is the unique vertex adjacent to \( V \) in \( T \) such that \( \text{card}(V') = \text{card}(V) - 1 \). If \( V'' \) is adjacent to \( V \) and \( V'' \neq V' \), then \( V = V'' \setminus \{ \max(V'') \} \) and \( \text{card}(V'') = \text{card}(V) + 1 \).

We shall prove that \( T \) is a rooted tree with root \( V_0 = \{0\} \). Let \( V = \{a_0, a_1, \ldots, a_n\} \) be a vertex, where \( 0 = a_0 < a_1 < \cdots < a_n \). For every \( k = 0, 1, \ldots, n \), the set \( V_k = \{a_0, a_1, \ldots, a_k\} \) is a finite \( \mathcal{R} \)-basis of order \( \mathcal{H} \), hence is a vertex of \( T \). Then \( V_0 = \{0\} \), \( V_n = V \), and \( V_0, V_1, \ldots, V_n \) is a simple path in \( T \) from the root \( V_0 \) to \( V \). It follows that the graph \( T \) is connected.

Suppose that \( n \geq 3 \) and \( V_0, V_1, \ldots, V_{n-1}, V_n \) is a simple circuit in \( T \), where \( V_n = V_0 \). Let \( V_{n+1} = V_1 \). Since each vertex is a finite set of integers, we can choose \( k \in [1, n] \) such that \( V_k \) is a vertex in the circuit of maximum cardinality. Vertices \( V_{k-1} \) and \( V_{k+1} \) are adjacent to \( V_k \), hence \( \text{card}(V_k) - \text{card}(V_{k-1}) = \pm 1 \) and \( \text{card}(V_k) - \text{card}(V_{k+1}) = \pm 1 \). The maximality of \( \text{card}(V_k) \) implies
that \( \text{card}(V_k) - \text{card}(V_{k-1}) = \text{card}(V_k) - \text{card}(V_{k+1}) = 1 \), and so \( V_{k-1} = V_k \setminus \{\max(V_k)\} = V_{k+1} \), which is impossible. Therefore, \( T \) contains no simple circuit, and so \( T \) is a tree.

To apply König’s lemma, we must prove that every vertex of this tree has finite degree. The only vertex adjacent to the root \( \{0\} \) is \( \{0,1\} \), hence \( \{0\} \) has finite degree. Let \( V \neq \{0\} \) be a vertex of \( T \). Then \( 1 \in V \) and so \( \max(V) \geq 1 \). Suppose that the \( V \) is adjacent to infinitely many vertices \( V' \). The only subset of \( V \) that is a vertex adjacent to \( V \) is \( V \setminus \{\max(V)\} \). Every other vertex \( V' \) adjacent to \( V \) is a superset of \( V \) of the form \( V' = V \cup \{\max(V')\} \). For each such \( V' \), the integer \( n = \max(V') - 1 \) must be an element of the sumset \( h \in H \) for some \( h \in H \), and so

\[
\begin{align*}
n &\leq h \max(V) \leq \max(H) \max(V),
\end{align*}
\]

Since \( n \in h \), it follows that

\[
\limsup_{n \to \infty} \frac{\max(H)}{n} \geq \frac{1}{\max(V)} > 0,
\]

which contradicts (3). Thus, every vertex of the infinite tree \( T \) has finite degree. By König’s lemma, the tree must contain an infinite simple path \( \{0\} = V_0, V_1, V_2, \ldots \). For each nonnegative integer \( n \), let \( a_n = \max(V_n) \). Then \( V_n = \{a_0, a_1, \ldots, a_n\} \) for all \( n = 0, 1, 2, \ldots \). Let

\[
A = \{a_n\}_{n=0}^{\infty} = \bigcup_{n=0}^{\infty} V_n.
\]

Since \( a_n \geq n \), it follows that

\[
r_A(n, H_n) = \sum_{h_n \in H_n} r_A(n, h_n) = \sum_{h_n \in H_n} r_{V_n}(n, h_n) = r_{V_n}(n, H_n) \in R_n,
\]

and so \( A \) is a \( R \)-basis of order \( \mathcal{H} \). This completes the proof. \( \square \)

**Theorem 5** Let \( h \geq 2 \) and let \( f \) be a function such that \( f(n) \) is a positive integer for every nonnegative integer \( n \). There exists a basis \( A \) of order \( h \) with representation function \( r_A(n, h) = f(n) \) if and only if for every \( N \) there exists a finite set \( A_N \) of nonnegative integers with \( \max(A_N) \geq N \) and \( r_A(n, h) = f(n) \) for all \( n = 0, 1, \ldots, \max(A_N) \).

**Proof.** This follows immediately from Theorem 4 with \( H_n = \{h\} \) and \( R_n = \{f(n)\} \) for all nonnegative integers \( n \). \( \square \)

Applying Theorem 4 to the classical Erdős-Turán conjecture, we obtain the following result of Dowd [1, Theorem 2.1].

**Theorem 6** Let \( c \geq 1 \) and \( h \geq 2 \). There exists a basis \( A \) of order \( h \) such that

\[
r_A(n, h) \leq c \quad \text{for all } n \geq 0
\]
if and only if, for every $N$, there exists a finite set $A_N$ of nonnegative integers such that $\max(A_N) \geq N$ and

$$1 \leq r_{A_N}(n, h) \leq c$$

for all $n = 0, 1, \ldots, \max(A_N)$.

**Proof.** This follows immediately from Theorem 4 with $H_n = \{h\}$ and $R_n = [1, c]$ for all $n \geq 0$. □

## 5 Ordered representation functions

There are other important representation functions in additive number theory. For example, for any set $A$ of integers, the **ordered representation function** $r_A'(n, h)$ counts the number of $h$-tuples $(a_1, \ldots, a_h) \in A^h$ such that $a_1 + \cdots + a_h = n$. Nathanson [4] proved the following uniqueness theorem for ordered representation functions: For any function $f : \mathbb{N}_0 \to \mathbb{N}_0$ and for any positive integer $h$, there exists at most one set $A$ of nonnegative integers such that $r_A'(n, h) = f(n)$ for all $n \in \mathbb{N}_0$. He also showed that uniqueness does not hold if ordered representation functions only eventually coincide, and he described all pairs of sets $A$ and $B$ of nonnegative integers such that $r_A'(n, 2) = r_B'(n, 2)$ for all sufficiently large integers $n$.

We can define **basis of order $H$** and **$\mathcal{R}$-basis of order $H$** in terms of the ordered representation function. Theorem 4 is also true for ordered representation functions.

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