Fundamental groups of Galois covers as tools to study non-planar degenerations

Meirav Amram

Department of Mathematics, Shamoon College of Engineering, Ashdod, Israel

September 28, 2021

Abstract

This study establishes a preliminary investigation of geometric objects that degenerate to non-planar shapes, along with their Galois covers and groups.

The study aims to determine the fundamental groups and signatures of the Galois covers of algebraic surfaces in general, as they are invariants of the classification of surfaces in the moduli space. The study investigates the tetrahedron and the double tetrahedron as first examples.

The study’s findings can advance the classification of surfaces and provide further links between algebraic geometry, group theory, and the topology of degenerative processes and their properties.

The resulting groups indicate that the tetrahedron and the double tetrahedron are in different components in the moduli space.

1 Introduction

Classifying algebraic surfaces and studying their moduli space is one of the most thoroughly investigated subjects in algebraic geometry and topology (one can find examples in the works of Catanese [15, 16]). Algebraic surfaces, as geometric objects, can be investigated via their convexity and curvature [11, 21], their degenerations [13, 14], their related moduli properties [17, 25], Chern classes [28], fundamental groups [21, 27], and many other geometrical and topological aspects.

One of the known invariants of classification is the fundamental group $\pi_1(X_{Gal})$ of the Galois cover $X_{Gal}$ of an algebraic surface $X$, with respect to a generic projection $f$ to the projective plane.
$\mathbb{CP}^2$. The group $\pi_1(X_{Gal})$ has a geometric significance in the classification of surfaces because it is equal for all the surfaces in the same connected component in the moduli space.

To determine the group $\pi_1(X_{Gal})$, we provide a beautiful algorithm that involves degeneration of $X$ to a union of planes called $X_0$, with a reverse process of regeneration. We use generic projections of $X$ and $X_0$ onto $\mathbb{CP}^2$ to get their branch curves $S$ and $S_0$, respectively. Then we can see the remarkable correspondence between both curves and apply the braid monodromy technique [31]. We can then use the van Kampen Theorem [36] to get a presentation of the fundamental group $\pi_1(\mathbb{CP}^2 - S)$ of the complement of $S$. In [19] there is an interesting explanation about the correspondence of plane curves and groups. The primary advantage of this algorithm is that we use it when the branch curve $S$ and the fundamental group $\pi_1(\mathbb{CP}^2 - S)$ are difficult to describe. This algorithm lets us recover the curve $S$ and a significant amount of geometric and algebraic information, giving us the group $\pi_1(X_{Gal})$. We can now construct the components of surfaces with non-planar degenerations and with the same (or different) $\pi_1(X_{Gal})$.

We establish the study of non-planar geometric objects that proceeds from the lowest degree to the higher degrees. We are motivated by the generalization of groups in [4] and the generalization of braids in [4]. Past works deal primarily with planar degenerations, which are degenerations that can be depicted on a piece of paper. For example, in [5], we give a full classification of 29 cases of degree 6 planar degenerations. Moreover, previous results regarding non-planar degenerations are quite sporadic. The surfaces we studied are $\mathbb{CP}^1 \times T$ in [4], in which we used the Reide-Meister Schreier method to find $\pi_1(X_{Gal})$, and the pillow degenerations of $K3$ surfaces in [4], in which we presented only the list of braids but no fundamental groups. Here, we suggest the direction of new research that will address even non-planar degenerations from the simplest to the most complex, including determining the fundamental group in the extensive algebraic use of group theory and computational methods. Moreover, [29], which focuses on Chern numbers of Galois covers, motivates us to associate to each object the signature $\chi(X_{Gal})$ of its Galois cover, as a classifying invariant (depending on its sign).

This work is an innovation of two points of view. One can look at the older famous question about the classification of surfaces with heavy group theoretic calculations, and as an originality of a research project about non-planar degenerations, starting with the lowest degrees while using some inductive way to check higher degrees to get a generalization on $\pi_1(X_{Gal})$. In this initiative, we determine $\pi_1(X_{Gal})$ for the objects $T_{(4)}$ and $D(T_{(4)})$, as described in Theorems 3.1 and 3.2. The results will tell us whether these two objects are in different connected components of the moduli space. It might be very interesting to investigate the groups related to these objects, because of other local behavior; they both have singular 3-points, but $D(T_{(4)})$ also has singular 4-points. In the degenerations we consider, only inner singularities can occur, which poses additional challenges because the relations arising from such points are more complicated than in the outer points, see Definition 2.2.
Future studies will continue with more advanced examples of non-planar degenerations containing higher degrees of singularity in the content. In this way, we hope to combine our future results for $\pi_1(X_{\text{Gal}})$ and $\chi(X_{\text{Gal}})$ with the results obtained in [3] and look for their generalizations. This paper aims to position both objects in the moduli space as a foundation stone in the classification of surfaces with non-planar degenerations.

This paper is organized as follows: In Section 2, we present the algorithm and give notations to the braids and elements in the fundamental group. We also explain the degenerations of interest and provide the formulas for Chern numbers and the signatures of the Galois covers. In Subsections 3.1 and 3.2, we calculate the fundamental groups of the Galois covers related to $T(4)$ and $D(T(4))$, respectively. In both subsections, we also give calculations of the Chern numbers and signatures of the Galois covers.

Acknowledgments: We thank Uriel Sinichkin for efficient discussions.

2 Preliminaries and details of the algorithm

In this section, we provide some basic information and outline process steps. We recommend [35], [25], and [20] as sources for relevant information about types of singularities in real algebraic geometry and monodromy and degenerations of projective structures, respectively.

2.1 Preliminaries and algorithm

In this subsection, we explain the algorithm of degeneration and regeneration that provides, for some $n$, a fundamental group of the Galois cover of an algebraic surface embedded in projective space $\mathbb{CP}^n$.

We take a generic projection of the surface $X$ onto the projective plane $\mathbb{CP}^2$. We get the branch curve $S$ in $\mathbb{CP}^2$. It is possible to get a substantial amount of information about $X$. Still, because $S$ is very difficult to describe, and information about braids and groups is not easy to get, we will use the process of degeneration, as described in Definition 2.1.

**Definition 2.1.** Let $\Delta$ be the unit disc, and let $X, Y$ be projective algebraic surfaces. Let $p : Y \to \mathbb{CP}^2$ and $p' : X \to \mathbb{CP}^2$ be projective embeddings. We say that $p'$ is a projective degeneration of $p$ if there exists a flat family $\pi : V \to \Delta$ and an embedding $F : V \to \Delta \times \mathbb{CP}^2$, such that $F$ composed with the first projection is $\pi$, and:

(a) $\pi^{-1}(0) \simeq X$;  
(b) there is a $t_0 \neq 0$ in $\Delta$ such that $\pi^{-1}(t_0) \simeq Y$;  
(c) the family $V - \pi^{-1}(0) \to \Delta - 0$ is smooth;  
(d) restricting to $\pi^{-1}(0)$, $F = 0 \times p'$ under the identification of $\pi^{-1}(0)$ with $X$;
(c) restricting to $\pi^{-1}(t_0)$, $F = t_0 \times p$ under the identification of $\pi^{-1}(t_0)$ with $Y$.

The first step of the algorithm is to construct a flat degeneration of $X$ into a union of planes, denoted as $X_0$. We consider degenerations with only two planes intersecting at a line (because this is the generic case), with each plane homeomorphic to $\mathbb{C}P^2$. The branch curve of $X_0$ is a union of lines that we denote as $S_0$. These lines are the projections of the intersection edges of the planes in $X_0$. In $S_0$ we have singularities that are called $k$-points (i.e., singularities of multiplicity $k$).

In this paper, we consider inner $k$-points; the definition follows.

**Definition 2.2.** We call a $k$-point that is the intersection of $k$ planes $P_1, \ldots, P_k$, s.t. $P_i$ and $P_j$ intersect in a line if $|i - j| = 1$ and additionally, that $P_1$ intersects $P_k$, an inner $k$-point.

Such singularities are considered in [13] and denoted as $E_k$ singularities. In this paper, we have inner 3-points and inner 4-points.

One of the main tools that we use, embedded in the algorithm, is a reverse process of degeneration, which we call regeneration. Using regeneration lemmas from [31], we can recover the original branch curve $S$ from the line arrangement $S_0$.

To make it easier to understand, we illustrate this process in the following diagram.

$$
\begin{array}{ccc}
X \subseteq \mathbb{C}P^n & \overset{\text{degeneration}}{\longrightarrow} & X_0 \subseteq \mathbb{C}P^n \\
\text{generic projection} & & \text{generic projection} \\
S \subseteq \mathbb{C}P^2 & \overset{\text{regeneration}}{\longleftarrow} & S_0 \subseteq \mathbb{C}P^2
\end{array}
$$

So, the regeneration process is the second step of the algorithm; a line in $S_0$ regenerates either to a conic or a double line; tangency points we get during this process will regenerate later to three cusps (see [32] for more details). Therefore, the regenerated branch curve $S$ is a cuspidal curve, and its degree is double that of $S_0$.

To get the list of braids that are associated with $S$, we introduce three types of braids that are related to the singularities in $S$:

1. for a branch point, $Z_{j,j'}$ is a counterclockwise half-twist of $j$ and $j'$ along a path below the real axis,
2. for nodes, $Z^2_{i,j,j'} = Z^2_{i,j} \cdot Z^2_{i,j'}$ and $Z^2_{i',j,j'} = Z^2_{i,j} \cdot Z^2_{i,j'} \cdot Z^2_{i',j} \cdot Z^2_{i',j}$,
3. for cusps, $Z^3_{i,j,j'} = Z^3_{i,j} \cdot (Z^3_{i,j})^2 Z^3_{i,j'} \cdot (Z^3_{i,j}) Z^{-1}_{j,j'}$.

Some of the braids we present in this paper will be conjugated as performed in the formula $a^b = b^{-1} a b$.

Denote $G := \pi_1(\mathbb{C}P^2 - S)$ and its standard generators as $\Gamma_1, \Gamma_1', \ldots, \Gamma_m, \Gamma_m'$. To get a presentation of $G$ through generators $\{\Gamma_j, \Gamma_{j'}\}$ and relations, we need the van Kampen Theorem [36]:

1. for a branch point, $Z_{j,j'}$ corresponds to the relation $\Gamma_j = \Gamma_{j'}$,
2. for nodes, $Z^2_{i,j}$ corresponds to $[\Gamma_i, \Gamma_j] = \Gamma_i \Gamma_j \Gamma_i^{-1} \Gamma_j^{-1} = e$,
(3) for cusps, $Z^3_{ij}$ corresponds to $\langle \Gamma_i, \Gamma_j \rangle = \Gamma_i \Gamma_j \Gamma_i^{-1} \Gamma_j^{-1} \Gamma_i^{-1} = e$.

To each list of relations we add the projective relation $\prod_{j \in m} \Gamma_j = e$ and the parasitic intersections that induce commutative relations. These intersections come from lines in $X_0$ that do not ordinarily intersect, but when projecting $X_0$ onto $\mathbb{CP}^2$, they will. Further details about the fundamental group and parasitic intersections are provided in [30].

Some works on $\pi_1(\mathbb{CP}^2 - S)$ have been done; the reader can find them in Auroux-Donaldson-Katzarkov-Yotov [10] and in A.-Friedman-Teicher [2, 3], Calabri-Ciliberto-Flamini-Miranda [13] and Ciliberto-Lopez-Miranda [18] studied degenerations.

Our algorithm also enables us to compute fundamental groups of Galois covers. We recall from [30] that if $f : X \to \mathbb{CP}^2$ is a generic projection of degree $n$, then $X_{Gal}$, the Galois cover, is defined as follows:

$$X_{Gal} = (X \times_{\mathbb{CP}^2} \ldots \times_{\mathbb{CP}^2} X) - \Delta,$$

where the product is taken $n$ times and $\Delta$ is the diagonal. To apply a theorem of Moishezon-Teicher [30], we define $\tilde{G} := G/\langle \Gamma_j^2, \Gamma_j' \rangle$. Then, there is an exact sequence

$$0 \to \pi_1(X_{Gal}) \to \tilde{G} \to S_n \to 0,$$

where the second map takes the generators $\Gamma_j$ and $\Gamma_j'$ of $G$ to the transposition of the two planes that intersect at line $j$. We thus obtain a presentation of the fundamental group $\pi_1(X_{Gal})$ of the Galois cover.

Galois covers of algebraic surfaces were investigated by Liedtke [27], Moishezon-Teicher [30], and Gieseker [23]. Fundamental groups of Galois covers were studied for $\mathbb{CP}^1 \times T$ ($T$ is a complex torus) in A.-Goldberg [4] and A.-Tan-Xu-Yoshpe [8], for toric varieties in A.-Ogata [7], for surfaces with Zappatic singularity of type $E_k$ in A.-Gong-Tan-Teicher-Xu [6], and for surfaces with degenerations of degree 6 in A.-Gong-Sinichkin-Tan-Xu-Yoshpe [5]. One can get a great deal of information about these groups and their associated Coxeter groups, that we use in the paper, from [12, 24]. Moreover, in [22, 26], there are explanations about the correspondences between graphs and fundamental groups and between graphs and related braids, respectively. We use these correspondences in Section 3.

### 2.2 The degenerations of interest

The two degenerations we consider in this paper are $T_{(4)}$ and $D(T_{(4)})$. Both degenerations are non-planar, and we get them by identifications of common edges. We pay attention that the lowest degree possible for non-planar degeneration $X_0$ of this kind is 4, for which we have the tetrahedron $T_{(4)}$. This tetrahedron is a union of four planes, see Figure 4. Then $D(T_{(4)})$ is the second non-planar degeneration $X_0$, see Figure 5. This degeneration appears as two tetrahedrons without identified bases, each glued to the other along the edges of the missing bases. In Figures 1 and 2 we can see the simplicial complexes of $T_{(4)}$ and $D(T_{(4)})$ respectively.
Before we define the class of degenerations to which those examples correspond, we recommend [34] as a source for relevant information about some combinatorial degeneration data and related Galois covers.

We recall the definition of a planar degeneration:

**Definition 2.3.** A degeneration of smooth toric surface $X$ into a union of planes $X_0$ is said to have a planar representation if:

1. No three planes in $X_0$ intersect in a line.
2. There exists a simplicial complex with connected interior embedded in $\mathbb{R}^2$, s.t. its 2-dimensional cells correspond bijectively to irreducible components of $X_0$; this bijection preserves an incidence relation.

While both $T(4)$ and $D(T(4))$ are not of this type, property (1) in Definition 2.3 holds. We thus can define a more general set of examples:

**Definition 2.4.** Let $\Omega$ be a connected topological manifold of dimension $m$. A degeneration of smooth $m$-dimensional toric variety $X$ into a union of projective spaces $X_0$ is said to be combinatorially homeomorphic to $\Omega$ if:

1. No three irreducible components of $X_0$ intersect in a co-dimension 1 set.
2. There exists a simplicial complex of pure dimension $m$ with a connected interior that is homeomorphic to $\Omega$, s.t. its $m$-dimensional cells correspond bijectively to irreducible components of $X_0$ and this bijection preserves an incidence relation.

This definition generalizes Definition 2.3 because planar degenerations are precisely those that are combinatorially homeomorphic to a disc. Both $T(4)$ and $D(T(4))$ are combinatorially homeomorphic to a sphere, which is a natural topological surface with which to begin a study. Finally, we see that the pillow degeneration considered in [1] is combinatorially homeomorphic to a sphere. In contrast, the degenerations of $\mathbb{CP}^1 \times T$ as in [4] and of $T \times T$ as in [9] are combinatorially homeomorphic to a cylinder and a topological torus, respectively.

We stress that degenerations with planar representations of a bounded degree can be enumerated recursively because the planar simplicial complexes can be enumerated. This work is unlikely to obtain such easy numerations, and much work and thought should be invested to fit the numbers in the picture and produce a geometric object that matches the properties of the algorithm.

Here we provide the details on the construction of the simplicial complexes of both objects.

**Construction 1 (T(4)).** The simplicial complex corresponding to the degeneration of $T(4)$ appears in Figure 1.

- Four triangles correspond to four planes $P_1, P_2, P_3, P_4$.
- Edges 1, \ldots, 6 are the intersections between the planes.
We identify planes $P_1$ and $P_2$ along edge 2, $P_1$ and $P_3$ along edge 1, $P_1$ and $P_4$ along edge 6, $P_2$ and $P_3$ along edge 4, $P_2$ and $P_4$ along edge 3, and $P_3$ and $P_4$ along edge 5.

- $V_1, V_2, V_3, V_4$ are four vertices. Planes $P_1, P_2, P_3$ meet in vertex $V_1$; planes $P_1, P_3, P_4$ meet in vertex $V_2$; planes $P_1, P_2, P_4$ meet in vertex $V_3$; and planes $P_2, P_3, P_4$ meet in vertex $V_4$.

We denote $X_0 = \bigcup_{i=1}^{6} P_i$.

*Figure 1: The simplicial complex of $T_{(4)}$*

**Construction 2** ($D(T_{(4)})$). The simplicial complex corresponding to the degeneration of $D(T_{(4)})$ appears in Figure [2]

- Six triangles correspond to six planes $P_1, \ldots, P_6$.
- Edges 1, \ldots, 9 are the intersections between the planes.
- Plane $P_1$ shares edges 1, 7, and 8 with planes $P_2$, $P_3$, and $P_4$, respectively. Plane $P_2$ shares edges 4 and 6 with planes $P_3$ and $P_5$, respectively. Plane $P_6$ shares with $P_3$, $P_4$, and $P_5$ common edges 2, 3, and 5, respectively. Planes $P_4$ and $P_5$ share common edge 9.
- $V_1, \ldots, V_5$ are five vertices. Planes $P_4, P_5, P_6$ meet in vertex $V_1$; planes $P_1, P_2, P_3$ meet in vertex $V_2$; planes $P_1, P_2, P_4, P_5$ meet in vertex $V_3$; planes $P_1, P_3, P_4, P_6$ meet in vertex $V_4$, and planes $P_2, P_3, P_5, P_6$ meet in vertex $V_5$.

We denote $X_0 = \bigcup_{i=1}^{6} P_i$. 

7
To compute the fundamental groups of the Galois covers of our two surfaces, we work with a dual graph $T$ of each $X_0$, which is defined as follows: The vertices of $T$ are in bijection with the planes in $X_0$, and the vertices corresponding to the planes $P_i$ and $P_j$ are connected by an edge if $P_i$ and $P_j$ intersect in an edge.

In $T_{(4)}$ and $D(T_{(4)})$ we have inner 3-points. Lemma 2.5 about 3-points assists us in the calculations of $\tilde{G}$.

**Lemma 2.5** ([5]). Let $p$ be an inner 3-point in $X_0$ with lines $i < j < k$, see Figure 3. If either $\Gamma_j = \Gamma_j'$ or $\Gamma_k = \Gamma_k'$ holds in $\tilde{G}$ then $\Gamma_l = \Gamma_{l'}$ for all $l \in \{i, j, k\}$. Moreover, $\Gamma_i = \Gamma_{i'}$ always in this case.

![Figure 3: Vertex p in Lemma 2.5](image)

### 2.3 Signatures

Signatures of Galois covers are additional important topological invariants in the classification of algebraic surfaces in the moduli space. We can determine the position of the surface in the moduli space based on the sign of the signature.

Chern numbers, from which we get signatures, were formulated in [30].

**Proposition 2.6.** ([30] Proposition 0.2) The Chern numbers of $X_{Gal}$ are given in the following formulas:

\[ c_1^2(X_{Gal}) = \frac{n!}{4}(m-6)^2, \]
\[ c_2(X_{\text{Gal}}) = n!(3 - m + \frac{d}{4} + \frac{\mu}{2} + \frac{\rho}{6}), \]

where

\[ n = \text{deg } f, \quad m = \text{deg } S, \quad \mu = \text{number of branch points in } S, \]
\[ d = \text{number of nodes in } S, \quad \rho = \text{number of cusps in } S. \]

Then we can calculate in Sections 3.1 and 3.2 the signature \( \chi := \frac{1}{3}(c_1^2 - 2c_2) \) of \( X_{\text{Gal}} \) for both cases.

## 3 Results

In this section, we determine the group \( \pi_1(X_{\text{Gal}}) \) and the signature \( \chi(X_{\text{Gal}}) \), for both \( T(4) \) and \( D(T(4)) \). We apply the algorithm and methods described in detail in Section 2.

### 3.1 The tetrahedron \( T(4) \)

In this section, we consider the tetrahedron \( T(4) \). The tetrahedron presents a non-planar degeneration after the identification along edges 3, 4, and 5 from Figure 1; see Figure 4.

![Figure 4: The tetrahedron \( T(4) \)](image)

In Theorem 3.1, we list the braids related to the branch curve \( S \) of a surface \( X \) with degeneration \( T(4) \), together with the relations of the group \( G \). We then calculate the fundamental group \( \pi_1(X_{\text{Gal}}) \) of the Galois cover of \( X \) and its signature.

**Theorem 3.1.**

1. If \( X \) has degeneration \( T(4) \), the group \( \pi_1(X_{\text{Gal}}) \) is trivial.
2. The signature \( \chi(X_{\text{Gal}}) \) is negative.
Proof. (1) The branch curve $S_0$ in $\mathbb{C}P^2$ is an arrangement of six lines; thus, $S$ has a degree 12.

Vertex $V_1$ gives the following braids:

$$\Delta_1 = Z^3_{1,2} \cdot Z^3_{1,2} \cdot (Z_{2,4}) Z^2_{Z,2} \cdot (Z_{2,4}) Z^2_{Z,2} \cdot (Z_{2,4}) Z^2_{Z,2} \cdot (Z_{2,4}) Z^2_{Z,2},$$

and braids induce the following relations in $G$:

$$\langle \Gamma_1, \Gamma_2 \rangle = (\Gamma_1, \Gamma_2) = (\Gamma_1, \Gamma_2^{-1} \Gamma_2 \Gamma_2) = e \quad (2)$$

$$\langle \Gamma_1, \Gamma_2 \rangle = (\Gamma_1, \Gamma_2) = (\Gamma_1, \Gamma_2^{-1} \Gamma_2 \Gamma_2) = e \quad (3)$$

$$\Gamma_2 \Gamma_2 \Gamma_1 \Gamma_2^{-1} \Gamma_2^{-1} = \Gamma_4 \quad (4)$$

$$\Gamma_2 \Gamma_2 \Gamma_1 \Gamma_2^{-1} \Gamma_2^{-1} = \Gamma_4 \Gamma_4 \Gamma_4^{-1} \quad (5)$$

$$\Gamma_2 \Gamma_2 \Gamma_1 \Gamma_2^{-1} \Gamma_2^{-1} = \Gamma_4 \Gamma_4 \Gamma_4^{-1} \quad (6)$$

$$\Gamma_2 \Gamma_2 \Gamma_1 \Gamma_2^{-1} \Gamma_2^{-1} = \Gamma_4 \Gamma_4 \Gamma_4^{-1} \quad (7)$$

Vertex $V_2$ gives the following braids:

$$\Delta_2 = Z^3_{1,5,5} \cdot Z^3_{1,5,5} \cdot (Z_{5,6}) Z^2_{Z,5,5} \cdot (Z_{5,6}) Z^2_{Z,5,5} \cdot (Z_{5,6}) Z^2_{Z,5,5} \cdot (Z_{5,6}) Z^2_{Z,5,5},$$

and braids induce the following relations in $G$:

$$\langle \Gamma_1, \Gamma_5 \rangle = (\Gamma_1, \Gamma_5) = (\Gamma_1, \Gamma_5^{-1} \Gamma_5 \Gamma_5) = e \quad (8)$$

$$\langle \Gamma_1, \Gamma_5 \rangle = (\Gamma_1, \Gamma_5) = (\Gamma_1, \Gamma_5^{-1} \Gamma_5 \Gamma_5) = e \quad (9)$$

$$\Gamma_5 \Gamma_5 \Gamma_1 \Gamma_5^{-1} \Gamma_5^{-1} = \Gamma_6 \quad (10)$$

$$\Gamma_5 \Gamma_5 \Gamma_1 \Gamma_5^{-1} \Gamma_5^{-1} = \Gamma_6 \Gamma_6 \Gamma_6^{-1} \quad (11)$$

$$\Gamma_5 \Gamma_5 \Gamma_1 \Gamma_5^{-1} \Gamma_5^{-1} = \Gamma_6 \Gamma_6 \Gamma_6^{-1} \quad (12)$$

$$\Gamma_5 \Gamma_5 \Gamma_1 \Gamma_5^{-1} \Gamma_5^{-1} = \Gamma_6 \Gamma_6 \Gamma_6^{-1} \quad (13)$$

Vertex $V_3$ gives the following braids:

$$\Delta_3 = Z^3_{2,3,3} \cdot Z^3_{2,3,3} \cdot (Z_{3,6}) Z^2_{Z,3,3} \cdot (Z_{3,6}) Z^2_{Z,3,3} \cdot (Z_{3,6}) Z^2_{Z,3,3} \cdot (Z_{3,6}) Z^2_{Z,3,3},$$

and braids induce the following relations in $G$:

$$\langle \Gamma_2, \Gamma_3 \rangle = (\Gamma_2, \Gamma_3) = (\Gamma_2, \Gamma_3^{-1} \Gamma_3 \Gamma_3) = e \quad (14)$$

$$\langle \Gamma_2, \Gamma_3 \rangle = (\Gamma_2, \Gamma_3) = (\Gamma_2, \Gamma_3^{-1} \Gamma_3 \Gamma_3) = e \quad (15)$$

$$\Gamma_3 \Gamma_3 \Gamma_2 \Gamma_3^{-1} \Gamma_3^{-1} \Gamma_3^{-1} = \Gamma_6 \quad (16)$$
\[\Gamma_3 \Gamma_3 \Gamma_3 \Gamma_3 \Gamma_3^{-1} \Gamma_3^{-1} = \Gamma_0 \Gamma_3 \Gamma_3^{-1}\]  
(17)

\[\Gamma_3 \Gamma_3 \Gamma_3 \Gamma_3 \Gamma_2 \Gamma_3^{-1} \Gamma_3^{-1} = \Gamma_0'\]  
(18)

\[\Gamma_3 \Gamma_3 \Gamma_3 \Gamma_2 \Gamma_3^{-1} \Gamma_3^{-1} = \Gamma_0 \Gamma_0 \Gamma_0^{-1} .\]  
(19)

Vertex \(V_4\) gives the following braids:

\[\Delta_4 = Z_{\gamma_3, \gamma_4} \cdot Z_{\gamma_3, \gamma_4} \cdot (Z_{\gamma_3, \gamma_4})^{Z_2} \cdot (Z_{\gamma_3, \gamma_4})^{Z_2} \cdot (Z_{\gamma_3, \gamma_4})^{Z_2} \cdot (Z_{\gamma_3, \gamma_4})^{Z_2} .\]  
and braids induce the following relations in \(G\):

\[\langle \Gamma_3, \Gamma_4 \rangle = \langle \Gamma_3, \Gamma_4 \rangle = \langle \Gamma_3, \Gamma_4 \rangle = e\]  
(20)

\[\langle \Gamma_3, \Gamma_4 \rangle = \langle \Gamma_3, \Gamma_4 \rangle = \langle \Gamma_3, \Gamma_4 \rangle = e\]  
(21)

\[\Gamma_4 \Gamma_4 \Gamma_3 \Gamma_3 \Gamma_4^{-1} \Gamma_4^{-1} = \Gamma_0'\]  
(22)

\[\Gamma_4 \Gamma_4 \Gamma_3 \Gamma_3 \Gamma_4^{-1} \Gamma_4^{-1} = \Gamma_0 \Gamma_0 \Gamma_0^{-1} .\]  
(23)

\[\Gamma_4 \Gamma_4 \Gamma_3 \Gamma_3 \Gamma_4^{-1} \Gamma_4^{-1} = \Gamma_0 \Gamma_0 \Gamma_0^{-1} .\]  
(24)

\[\Gamma_4 \Gamma_4 \Gamma_3 \Gamma_3 \Gamma_4^{-1} \Gamma_4^{-1} = \Gamma_0 \Gamma_0 \Gamma_0^{-1} .\]  
(25)

We also have the following parasitic relations and the projective relation:

\[\left[\Gamma_1, \Gamma_3\right] = \left[\Gamma_1, \Gamma_3\right] = \left[\Gamma_1, \Gamma_3\right] = \left[\Gamma_1, \Gamma_3\right] = e\]  
(26)

\[\left[\Gamma_2, \Gamma_5\right] = \left[\Gamma_2, \Gamma_5\right] = \left[\Gamma_2, \Gamma_5\right] = \left[\Gamma_2, \Gamma_5\right] = e\]  
(27)

\[\left[\Gamma_4, \Gamma_6\right] = \left[\Gamma_4, \Gamma_6\right] = \left[\Gamma_4, \Gamma_6\right] = \left[\Gamma_4, \Gamma_6\right] = e\]  
(28)

\[\Gamma_0' \Gamma_0' \Gamma_5 \Gamma_5 \Gamma_4 \Gamma_4 \Gamma_3 \Gamma_3 \Gamma_2 \Gamma_2 \Gamma_1 \Gamma_1 = e .\]  
(29)

Now we calculate \(\tilde{G}\). By Lemma 2.5, \(\Gamma_1 = \Gamma_1\) in vertices \(V_1\) and \(V_2\), \(\Gamma_2 = \Gamma_2\) in vertex \(V_3\), and \(\Gamma_3 = \Gamma_3\) in vertex \(V_4\). It follows again by Lemma 2.5 that \(\Gamma_1 = \Gamma_i\) for \(i = 1, \ldots, 6\). These results simplify the triple relations to

\[\langle \Gamma_1, \Gamma_2 \rangle = \langle \Gamma_1, \Gamma_5 \rangle = \langle \Gamma_2, \Gamma_3 \rangle = \langle \Gamma_3, \Gamma_4 \rangle = e ,\]  
the branch relations to

\[\Gamma_4 = \Gamma_1 \Gamma_2 \Gamma_1, \quad \Gamma_5 = \Gamma_3 \Gamma_4 \Gamma_3, \quad \Gamma_6 = \Gamma_2 \Gamma_3 \Gamma_2, \quad \Gamma_6 = \Gamma_1 \Gamma_4 \Gamma_1 ,\]  
and the commutations to

\[\left[\Gamma_1, \Gamma_3\right] = \left[\Gamma_2, \Gamma_5\right] = \left[\Gamma_4, \Gamma_6\right] = e .\]  

Substituting, for example, \(\Gamma_5 = \Gamma_1 \Gamma_4 \Gamma_1\) in \(\langle \Gamma_1, \Gamma_5 \rangle = e\) gives \(\langle \Gamma_1, \Gamma_6 \rangle = e\), substituting \(\Gamma_2 = \Gamma_1 \Gamma_4 \Gamma_1\) in \(\langle \Gamma_1, \Gamma_2 \rangle = e\) gives \(\langle \Gamma_1, \Gamma_4 \rangle = e\), and so on. In this way, we can complete the list of all triple relations needed in \(\tilde{G}\).
$\tilde{G}$ is thus generated by $\{\Gamma_i | i = 1, \ldots, 6\}$ with the following relations:

$$\Gamma_1^2 = \Gamma_2^2 = \Gamma_3^2 = \Gamma_4^2 = \Gamma_5^2 = \Gamma_6^2 = e \quad (30)$$

$$\langle \Gamma_1, \Gamma_2 \rangle = \langle \Gamma_1, \Gamma_4 \rangle = \langle \Gamma_1, \Gamma_6 \rangle = \langle \Gamma_2, \Gamma_3 \rangle = \langle \Gamma_2, \Gamma_4 \rangle = \langle \Gamma_2, \Gamma_6 \rangle = \langle \Gamma_3, \Gamma_4 \rangle = \langle \Gamma_3, \Gamma_5 \rangle = \langle \Gamma_4, \Gamma_5 \rangle = \langle \Gamma_4, \Gamma_6 \rangle = \langle \Gamma_5, \Gamma_6 \rangle = e \quad (31)$$

$$[\Gamma_1, \Gamma_3] = [\Gamma_2, \Gamma_5] = [\Gamma_4, \Gamma_6] = e \quad (32)$$

$$\Gamma_4 = \Gamma_1 \Gamma_2 \Gamma_1, \quad \Gamma_5 = \Gamma_3 \Gamma_4 \Gamma_3, \quad \Gamma_6 = \Gamma_2 \Gamma_3 \Gamma_2, \quad \Gamma_0 = \Gamma_1 \Gamma_5 \Gamma_1. \quad (33)$$

By substituting $\Gamma_3 = \Gamma_4 \Gamma_5 \Gamma_4$ in $[\Gamma_1, \Gamma_3] = e$, we get $[\Gamma_1, \Gamma_4 \Gamma_5 \Gamma_4] = e$. In a similar way, we can get $[\Gamma_2, \Gamma_1 \Gamma_5 \Gamma_1] = e$, $[\Gamma_3, \Gamma_3 \Gamma_6 \Gamma_3] = e$, and $[\Gamma_4, \Gamma_2 \Gamma_3 \Gamma_2] = e$.

By Theorem 2.3 in [33] we have $\tilde{G} \cong S_4$. Therefore $\pi_1(X_{Gal})$ is trivial.

(2) We write the parameters related to the degeneration $T_{(4)}$:

$$n = 4, \quad m = 12, \quad \mu = 16, \quad d = 12, \quad \rho = 24.$$

We get

$$c_1^2(X_{Gal}) = \frac{4!}{4} (12 - 6)^2 = 9 \cdot 4!,$$

$$c_2(X_{Gal}) = 4! (3 - 12 + \frac{12}{4} + \frac{16}{2} + \frac{24}{6}) = 6 \cdot 4!.$$

Therefore,

$$\chi(X_{Gal}) = \frac{1}{3} (9 \cdot 4! - 12 \cdot 4!) = -4!.$$

\[\square\]

3.2 The double tetrahedron $D(T_{(4)})$

In this section, we take a union of two tetrahedrons of type $T_{(4)}$ each; we omit their bases and identify them along the common edges surrounding the bases. We call it a double tetrahedron and denote it as $D(T_{(4)})$. We obtain the double tetrahedron after identifying the two pieces from Figure 2; see Figure 3.
Figure 5: The double tetrahedron $D(T_{(4)})$
Vertices $V_1$ and $V_2$ are inner 3-points. The identification along edges 2, 6, and 8 creates three vertices $V_3$, $V_4$, and $V_5$, which are inner 4-points, see Figure 6.

![Figure 6: The 4-points in $D(T_{(4)}$)](image)

In Theorem 3.2 we list the braids related to the branch curve $S$ of a surface $X$ with degeneration $D(T_{(4)})$. The braids will induce specific relations in group $G$. We then calculate the fundamental group $\pi_1(X_{Gal})$ and the signature of $X_{Gal}$.

**Theorem 3.2.** (1) Let $X$ be a surface with degeneration $D(T_{(4)})$. Then the group $\pi_1(X_{Gal})$ related to the double tetrahedron in Figure 5 is $\mathbb{Z}_2^4$.

(2) The signature $\chi(X_{Gal})$ is zero.

**Proof.** (1) The branch curve $S_0$ in $\mathbb{C}P^2$ is an arrangement of nine lines; thus, $S$ is a degree 18 curve. Vertex $V_1$ gives the following braids:

\[
\hat{\Delta}_1 = \langle \Gamma_3, \Gamma_5 \rangle = \langle \Gamma_3, \Gamma_5^{-1} \Gamma_3 \Gamma_5 \rangle = e
\]

and braids induce the following relations in $G$:

\[
\langle \Gamma_3, \Gamma_5 \rangle = \langle \Gamma_3, \Gamma_5^{-1} \Gamma_3 \Gamma_5 \rangle = e
\]

(34)

\[
\Gamma_5 \Gamma_3 \Gamma_5^{-1} \Gamma_3 \Gamma_5^{-1} = \Gamma_3^9
\]

(35)

(36)

(37)

(38)

(39)

Vertex $V_2$ gives the following braids:

\[
\hat{\Delta}_2 = \langle \Gamma_1, \Gamma_4 \rangle = \langle \Gamma_1, \Gamma_4^{-1} \Gamma_4 \Gamma_4 \rangle = e
\]

and braids induce the following relations in $G$:

\[
\langle \Gamma_1, \Gamma_4 \rangle = \langle \Gamma_1, \Gamma_4^{-1} \Gamma_4 \Gamma_4 \rangle = e
\]

(40)
\[ \langle \Gamma_1, \Gamma_4 \rangle = \langle \Gamma_1, \Gamma_4' \rangle = \langle \Gamma_1, \Gamma_4^{-1} \Gamma_4' \Gamma_4 \rangle = e \] (41)
\[ \Gamma_4 \Gamma_3 \Gamma_7 \Gamma_5 \Gamma_4^{-1} \Gamma_4'^{-1} \Gamma_4'^{-1} = \Gamma_7 \] (42)
\[ \Gamma_4 \Gamma_3 \Gamma_7 \Gamma_5 \Gamma_4^{-1} \Gamma_4'^{-1} \Gamma_4'^{-1} = \Gamma_7 \Gamma_7 \] (43)
\[ \Gamma_3 \Gamma_4 \Gamma_3 \Gamma_7 \Gamma_5 \Gamma_3^{-1} \Gamma_4^{-1} \Gamma_4'^{-1} = \Gamma_7 \] (44)
\[ \Gamma_3 \Gamma_4 \Gamma_3 \Gamma_7 \Gamma_5 \Gamma_3^{-1} \Gamma_4^{-1} \Gamma_4'^{-1} = \Gamma_7 \Gamma_7 \] (45)

Vertex \( V_3 \) is an inner 4-point. The corresponding relations in \( G \) are:

\[ \langle \Gamma_1, \Gamma_6 \rangle = \langle \Gamma_1, \Gamma_{6'} \rangle = \langle \Gamma_1, \Gamma_6^{-1} \Gamma_{60} \Gamma_6 \rangle = e \] (46)
\[ \langle \Gamma_4, \Gamma_9 \rangle = \langle \Gamma_4', \Gamma_9 \rangle = \langle \Gamma_4^{-1} \Gamma_9 \Gamma_8 \Gamma_9 \rangle = e \] (47)
\[ [\Gamma_6 \Gamma_6 \Gamma_7 \Gamma_6^{-1} \Gamma_9^{-1}, \Gamma_9] = e \] (48)
\[ [\Gamma_6 \Gamma_6 \Gamma_7 \Gamma_6^{-1} \Gamma_9^{-1}, \Gamma_9^{-1} \Gamma_9 \Gamma_9 \Gamma_8 \Gamma_8] = e \] (49)
\[ \langle \Gamma_1, \Gamma_6 \rangle = \langle \Gamma_1, \Gamma_{6'} \rangle = \langle \Gamma_1, \Gamma_6^{-1} \Gamma_{60} \Gamma_6 \rangle = e \] (50)
\[ \langle \Gamma_4, \Gamma_9 \rangle = \langle \Gamma_4', \Gamma_9 \rangle = \langle \Gamma_4^{-1} \Gamma_9 \Gamma_8 \Gamma_9 \rangle = e \] (51)
\[ [\Gamma_{6'} \Gamma_6 \Gamma_7 \Gamma_6^{-1} \Gamma_9^{-1}, \Gamma_9^{-1} \Gamma_9 \Gamma_9 \Gamma_8 \Gamma_8] = e \] (52)
\[ [\Gamma_{6'} \Gamma_6 \Gamma_7 \Gamma_6^{-1} \Gamma_9^{-1}, \Gamma_9^{-1} \Gamma_9 \Gamma_9 \Gamma_8 \Gamma_8] = e \] (53)
\[ \Gamma_{6'} \Gamma_6 \Gamma_7 \Gamma_6^{-1} \Gamma_9^{-1} \Gamma_{60}^{-1} = \Gamma_{90} \Gamma_{90}^{-1} \] (54)
\[ \Gamma_{6'} \Gamma_6 \Gamma_7 \Gamma_6^{-1} \Gamma_9^{-1} \Gamma_{60}^{-1} = \Gamma_{90} \Gamma_{90}^{-1} \Gamma_9^{-1} \] (55)
\[ \Gamma_{6'} \Gamma_6 \Gamma_7 \Gamma_6^{-1} \Gamma_9^{-1} \Gamma_{60}^{-1} = \Gamma_{90} \Gamma_{90}^{-1} \Gamma_9^{-1} \Gamma_9 \] (56)
\[ \Gamma_{6'} \Gamma_6 \Gamma_7 \Gamma_6^{-1} \Gamma_9^{-1} \Gamma_{60}^{-1} = \Gamma_{90} \Gamma_{90}^{-1} \Gamma_9^{-1} \Gamma_9 \] (57)

Vertex \( V_4 \) is an inner 4-point. The corresponding relations in \( G \) are:

\[ \langle \Gamma_2, \Gamma_3 \rangle = \langle \Gamma_2', \Gamma_3' \rangle = \langle \Gamma_2, \Gamma_3^{-1} \Gamma_3' \Gamma_3 \rangle = e \] (58)
\[ \langle \Gamma_7, \Gamma_8 \rangle = \langle \Gamma_7', \Gamma_8 \rangle = \langle \Gamma_7^{-1} \Gamma_7 \Gamma_7, \Gamma_8 \rangle = e \] (59)
\[ [\Gamma_9 \Gamma_9 \Gamma_2 \Gamma_3^{-1} \Gamma_3 \Gamma_3', \Gamma_8] = e \] (60)
\[ [\Gamma_9 \Gamma_9 \Gamma_2 \Gamma_3^{-1} \Gamma_3 \Gamma_3', \Gamma_7 \Gamma_7 \Gamma_7^{-1} \Gamma_8^{-1} \Gamma_8 \Gamma_8 \Gamma_7] = e \] (61)
\[ \langle \Gamma_2, \Gamma_3 \rangle = \langle \Gamma_2, \Gamma_3' \rangle = \langle \Gamma_2, \Gamma_3^{-1} \Gamma_3' \Gamma_3 \rangle = e \] (62)
\[ [\Gamma_9 \Gamma_9 \Gamma_2 \Gamma_3^{-1} \Gamma_3 \Gamma_3', \Gamma_8] = e \] (63)
\[ [\Gamma_9 \Gamma_9 \Gamma_2 \Gamma_3^{-1} \Gamma_3 \Gamma_3', \Gamma_8] = e \] (64)
\[ [\Gamma_9 \Gamma_9 \Gamma_2 \Gamma_3^{-1} \Gamma_3 \Gamma_3', \Gamma_8] = e \] (65)
\[ \Gamma_9 \Gamma_9 \Gamma_2 \Gamma_3^{-1} \Gamma_3 \Gamma_3' \Gamma_3^{-1} \Gamma_3' \Gamma_3 = \Gamma_9 \Gamma_9 \Gamma_8^{-1} \] (66)

15
\[\Gamma_7\Gamma_3\Gamma_2\Gamma_3^{-1}\Gamma_3^{-1}\Gamma_3^{-1} = \Gamma_8\Gamma_7\Gamma_7^{-1}\Gamma_8^{-1}\]

(67)

\[\Gamma_7\Gamma_3\Gamma_2\Gamma_3^{-1}\Gamma_3^{-1}\Gamma_3^{-1} = \Gamma_8\Gamma_7\Gamma_7^{-1}\Gamma_8^{-1}\Gamma_8\]

(68)

\[\Gamma_7\Gamma_3\Gamma_2\Gamma_3^{-1}\Gamma_3^{-1}\Gamma_3^{-1} = \Gamma_8\Gamma_7\Gamma_7^{-1}\Gamma_8^{-1}\Gamma_8^{-1}\Gamma_8.\]

(69)

Vertex \(V_5\) is an inner 4-point. The corresponding relations in \(G\) are:

\[(\Gamma_2, \Gamma_4') = (\Gamma_2', \Gamma_4) = (\Gamma_2', \Gamma_4^{-1}\Gamma_4\Gamma_4') = e\]

(70)

\[(\Gamma_5, \Gamma_6') = (\Gamma_5', \Gamma_6) = (\Gamma_5^{-1}\Gamma_5\Gamma_5, \Gamma_6' = e\]

(71)

\[[\Gamma_4\Gamma_4\Gamma_4^-1\Gamma_4\Gamma_4^-1, \Gamma_4] = e\]

(72)

\[(\Gamma_4\Gamma_4\Gamma_4^-1\Gamma_4\Gamma_4^-1, \Gamma_5^{-1}\Gamma_5\Gamma_5, \Gamma_4' = e\]

(73)

\[(\Gamma_2, \Gamma_4') = (\Gamma_2', \Gamma_4) = (\Gamma_2, \Gamma_4^{-1}\Gamma_4\Gamma_4') = e\]

(74)

\[(\Gamma_5, \Gamma_6^{-1}\Gamma_6\Gamma_6') = (\Gamma_5', \Gamma_6^{-1}\Gamma_6\Gamma_6') = (\Gamma_5^{-1}\Gamma_5\Gamma_5, \Gamma_6^{-1}\Gamma_6\Gamma_6') = e\]

(75)

\[[\Gamma_4\Gamma_4\Gamma_4^{-1}\Gamma_4\Gamma_4^{-1}, \Gamma_6^{-1}\Gamma_6\Gamma_6] = e\]

(76)

\[[\Gamma_4\Gamma_4\Gamma_4^{-1}\Gamma_4\Gamma_4^{-1}, \Gamma_5^{-1}\Gamma_5\Gamma_5, \Gamma_6^{-1}\Gamma_6\Gamma_6\Gamma_6'] = e\]

(77)

\[\Gamma_4\Gamma_4\Gamma_4^{-1}\Gamma_4\Gamma_4^{-1} = \Gamma_6\Gamma_5\Gamma_5^{-1}\]

(78)

\[\Gamma_4\Gamma_4\Gamma_4^{-1}\Gamma_4\Gamma_4^{-1} = \Gamma_6\Gamma_5\Gamma_5^{-1}\Gamma_6^{-1}\]

(79)

\[\Gamma_4\Gamma_4\Gamma_4\Gamma_4^{-1}\Gamma_4\Gamma_4^{-1} = \Gamma_6^{-1}\Gamma_6\Gamma_5\Gamma_5^{-1}\Gamma_6^{-1}\Gamma_6\]

(80)

We also have the following parasitic relations and the projective relation:

\[[\Gamma_1, \Gamma_2] = [\Gamma_1', \Gamma_2] = [\Gamma_1, \Gamma_2] = [\Gamma_1', \Gamma_2] = e\]

(82)

\[[\Gamma_1, \Gamma_3] = [\Gamma_1', \Gamma_3] = [\Gamma_1, \Gamma_3'] = [\Gamma_1', \Gamma_3'] = e\]

(83)

\[[\Gamma_1, \Gamma_5] = [\Gamma_1', \Gamma_5] = [\Gamma_1', \Gamma_5] = e\]

(84)

\[[\Gamma_1, \Gamma_9] = [\Gamma_2, \Gamma_9] = [\Gamma_2', \Gamma_9] = [\Gamma_2', \Gamma_9'] = e\]

(85)

\[[\Gamma_3, \Gamma_4] = [\Gamma_3', \Gamma_4] = [\Gamma_3, \Gamma_4'] = [\Gamma_3', \Gamma_4] = e\]

(86)

\[[\Gamma_3, \Gamma_6] = [\Gamma_3', \Gamma_6] = [\Gamma_3, \Gamma_6'] = [\Gamma_3', \Gamma_6'] = e\]

(87)

\[[\Gamma_4, \Gamma_8] = [\Gamma_4', \Gamma_8] = [\Gamma_4, \Gamma_8'] = [\Gamma_4', \Gamma_8] = e\]

(88)

\[[\Gamma_4, \Gamma_9] = [\Gamma_4', \Gamma_9] = [\Gamma_4, \Gamma_9'] = [\Gamma_4', \Gamma_9'] = e\]

(89)

\[[\Gamma_5, \Gamma_7] = [\Gamma_5', \Gamma_7] = [\Gamma_5, \Gamma_7] = [\Gamma_5', \Gamma_7] = e\]

(90)

\[[\Gamma_5, \Gamma_8] = [\Gamma_5', \Gamma_8] = [\Gamma_5, \Gamma_8] = [\Gamma_5', \Gamma_8'] = e\]

(91)

\[[\Gamma_6, \Gamma_7] = [\Gamma_6', \Gamma_7] = [\Gamma_6, \Gamma_7] = [\Gamma_6', \Gamma_7] = e\]

(92)
\[ [\Gamma_7, \Gamma_9] = [\Gamma_7, \Gamma_9] = [\Gamma_7, \Gamma_9] = e \]  
(93)

\[ \Gamma_9 \Gamma_9 \Gamma_9 \Gamma_9 \Gamma_9 \Gamma_9 \Gamma_9 \gamma = [\Gamma_9, \Gamma_9] = [\Gamma_9, \Gamma_9] = e. \]  
(94)

Now we calculate \( \tilde{G} \). By Lemma 2.5, we have \( \Gamma_1 = \Gamma_1 \) and \( \Gamma_3 = \Gamma_3 \). We substitute \( \Gamma_3 = \Gamma_3 \) in (78) and (79), and get the following two relations:

\[ \Gamma_2 \Gamma_3 \Gamma_2' = \Gamma_8 \Gamma_7 \Gamma_8 \]  
\[ \Gamma_2 \Gamma_3 \Gamma_2' = \Gamma_8 \Gamma_7 \Gamma_7 \Gamma_8. \]

We equate them

\[ \Gamma_8 \Gamma_7 \Gamma_8 = \Gamma_8 \Gamma_7 \Gamma_7 \Gamma_8 \]

to get

\[ \Gamma_7' = \Gamma_7. \]

This result gives \( \Gamma_4 = \Gamma_4 \), using Lemma 2.5. We now substitute \( \Gamma_4 = \Gamma_4 \) in (78) and (79) and use the same technique as above, to get \( \Gamma_5 = \Gamma_5' \). Now by Lemma 2.5, we have \( \Gamma_9 = \Gamma_9' \).

Using the above-resulting equalities and some triple relations, we can simplify (66), (68), (78), and (80) to the following forms:

\[ \Gamma_2' = \Gamma_3 \Gamma_5 \Gamma_7 \Gamma_8 \Gamma_3 \]  
(95)

\[ \Gamma_5' = \Gamma_8 \Gamma_7 \Gamma_9 \Gamma_8 \Gamma_8 \]  
(96)

\[ \Gamma_2' = \Gamma_4 \Gamma_6 \Gamma_3 \Gamma_5 \Gamma_4 \]  
(97)

\[ \Gamma_6' = \Gamma_6 \Gamma_3 \Gamma_4 \Gamma_7 \Gamma_7 \Gamma_6. \]  
(98)

Now we use \( \Gamma_i = \Gamma_i' \) (\( i = 1, 3, 4, 5, 7, 9 \)), and also (95), (96), (97), and (98), to eliminate the generators \( \Gamma_i' \), \( i = 1, \ldots, 9 \). This elimination is not simple and is accompanied by the placement of the expressions (95)-(98) in all the above relations. But there are also many group reductions, which give us more new relations, such as \( \langle \Gamma_1, \Gamma_7 \rangle = e \).

We get a simplified presentation for group \( \tilde{G} \) with the generators \( \Gamma_i ', \ i = 1, \ldots, 9 \) and the following relations:

\[ \Gamma_1' = \Gamma_2' = \Gamma_3' = \Gamma_4' = \Gamma_5' = \Gamma_6' = \Gamma_7' = \Gamma_8' = \Gamma_9' = e \]  
(99)

\[ \langle \Gamma_1, \Gamma_6 \rangle = \langle \Gamma_1, \Gamma_7 \rangle = \langle \Gamma_2, \Gamma_3 \rangle = \langle \Gamma_2, \Gamma_7 \rangle = \langle \Gamma_3, \Gamma_5 \rangle = \langle \Gamma_3, \Gamma_8 \rangle = \langle \Gamma_5, \Gamma_6 \rangle = \langle \Gamma_5, \Gamma_7 \rangle = \langle \Gamma_6, \Gamma_9 \rangle = \langle \Gamma_7, \Gamma_8 \rangle \]  
(100)

\[ [\Gamma_1, \Gamma_3] = [\Gamma_2, \Gamma_6] = [\Gamma_3, \Gamma_4] = [\Gamma_3, \Gamma_6] = [\Gamma_3, \Gamma_7] = [\Gamma_5, \Gamma_2] = [\Gamma_6, \Gamma_7] = [\Gamma_6, \Gamma_8] = [\Gamma_7, \Gamma_9] = e \]  
(101)

\[ [\Gamma_1, \Gamma_8 \Gamma_7 \Gamma_8] = [\Gamma_2, \Gamma_4 \Gamma_7 \Gamma_4] = [\Gamma_2, \Gamma_5 \Gamma_3 \Gamma_5] = [\Gamma_6, \Gamma_5 \Gamma_3 \Gamma_5] = [\Gamma_6, \Gamma_4 \Gamma_4 \Gamma_1] = [\Gamma_8, \Gamma_3 \Gamma_9 \Gamma_3] = e \]  
(102)
\[ \Gamma_7 = \Gamma_1 \Gamma_4 \Gamma_1, \quad \Gamma_3 = \Gamma_3 \Gamma_9 \Gamma_5, \quad \Gamma_9 \Gamma_8 \Gamma_7 \Gamma_9 \Gamma_3 = \Gamma_4 \Gamma_9 \Gamma_5 \Gamma_9 \Gamma_4 \]

and

\[ (\Gamma_1, \Gamma_4) = (\Gamma_1, \Gamma_8) = (\Gamma_2, \Gamma_4) = (\Gamma_2, \Gamma_9) = (\Gamma_8, \Gamma_9) = e \]

\[ [\Gamma_1, \Gamma_2] = [\Gamma_1, \Gamma_5] = [\Gamma_1, \Gamma_9] = [\Gamma_2, \Gamma_8] = [\Gamma_2, \Gamma_9] = [\Gamma_4, \Gamma_8] = [\Gamma_4, \Gamma_9] = [\Gamma_5, \Gamma_8] = e. \]

We still have the simplified projective relation,

\[ \Gamma_8 \Gamma_8 \Gamma_6 \Gamma_2 \Gamma_2 = e, \]

and we will deal with it further in the following paragraphs.

Now we continue to eliminate the generators \( \Gamma_7, \Gamma_3, \) and \( \Gamma_6, \) using (103). We start with the eliminations of \( \Gamma_7 = \Gamma_1 \Gamma_4 \Gamma_1 \) and \( \Gamma_3 = \Gamma_3 \Gamma_9 \Gamma_5. \) After this, we substitute \( \Gamma_7 = \Gamma_1 \Gamma_4 \Gamma_1 \) and \( \Gamma_3 = \Gamma_3 \Gamma_9 \Gamma_5 \) in \( \Gamma_3 \Gamma_8 \Gamma_7 \Gamma_8 \Gamma_3 = \Gamma_4 \Gamma_6 \Gamma_5 \Gamma_6 \Gamma_4 \) to get \( \Gamma_6 = \Gamma_9 \Gamma_8 \Gamma_1 \Gamma_8 \Gamma_9. \) Then we also eliminate \( \Gamma_6. \) The relations that appear in (100)-(103) become redundant.

We get the simplified presentation of \( \tilde{G}, \) with generators \( \Gamma_1, \Gamma_2, \Gamma_4, \Gamma_5, \Gamma_8, \Gamma_9, \) the relations that appear in (104) and (105), the relation

\[ \Gamma_2^2 = \Gamma_4^2 = \Gamma_5^2 = \Gamma_8^2 = \Gamma_9^2 = e \]

, and the relation we will obtain from (106) after performing the substitutions mentioned above. Without (106), this presentation is exactly \( C_Y(T), \) where \( T \) is the graph depicted in Figure 7.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{graph.png}
\caption{Graph T such that \( \tilde{G} \) is a quotient of \( C_Y(T) \)}
\end{figure}

Denote the simplified projective relation (106) as \( \text{Proj}. \) We deal with it using Theorem 6.1 from [33]. According to that theorem, the quotient Coxeter group \( C_Y(T) \) of a graph \( T \) is isomorphic to \( S_n \ltimes A_{t,n}, \) where \( t \) is the number of cycles in \( T \) (i.e., its first Betti number), and \( n \) is the number of vertices it contains. When \( t = 1, \) group \( A_{1,n} \) is the Abelian group generated by \( \{ u_{i,j} \mid 1 \leq i, j \leq n \} \) with the relations \( u_{i,i} = e, \) \( u_{i,j} u_{j,i} = u_{i,j}^{-1}, \) \( u_{i,k} u_{k,j} = u_{i,j}; \) it is obviously isomorphic to \( \mathbb{Z}^{n-1}. \) The action of \( S_n \) on \( A_{t,n} \) is defined by \( \sigma u_{i,j} = u_{\sigma(i), \sigma(j)}. \) In our case, \( t = 1, n = 6, \) and

\[ C_Y(T) \cong S_6 \ltimes A_{1,6} = S_6 \ltimes \mathbb{Z}^5. \]

We can see that

\[ \tilde{G} = \frac{C_Y(T)}{[\text{Proj}]} \]
is a quotient of $C_Y(T)$ by the normal subgroup generated by $Proj$.

Now we assign to each element $\Gamma_i$ ($i = 1, 2, 4, 5, 8, 9$) in $C_Y(T)$ an element in $S_6 \ltimes \mathbb{Z}^5$, see Figure 8. Here, we chose the spanning tree $T_0$ in the proof of [33, Theorem 6.1] to be all the edges but $\Gamma_5$:

$$
\Gamma_1 = (3 \ 4), \quad \Gamma_2 = (5 \ 6), \quad \Gamma_4 = (4 \ 5), \quad \Gamma_5 = (1 \ 6)u_{1,6}, \quad \Gamma_8 = (2 \ 3), \quad \Gamma_9 = (1 \ 2).
$$

![Figure 8: The graph with assignments of elements](image)

We also determine the assigned elements of $\Gamma_3$, $\Gamma_6$, $\Gamma_7$, $\Gamma_2'$, $\Gamma_6'$, and $\Gamma_8'$:

$$
\begin{align*}
\Gamma_3: & \quad \Gamma_3 = \Gamma_0 \Gamma_\bar{2} \Gamma_9 = (1 \ 2)(1 \ 6)u_{1,6}(1 \ 2) = (2 \ 6)u_{2,6} \\
\Gamma_6: & \quad \Gamma_6 = \Gamma_0 \Gamma_\bar{2} \Gamma_1 \Gamma_8 \Gamma_9 = (1 \ 2)(2 \ 3)(3 \ 4)(2 \ 3)(1 \ 2) = (1 \ 4) \\
\Gamma_7: & \quad \Gamma_7 = \Gamma_1 \Gamma_4 \Gamma_1 = (3 \ 4)(4 \ 5)(3 \ 4) = (3 \ 5) \\
\Gamma_8: & \quad \Gamma_2 \Gamma_3 \Gamma_2 = (5 \ 6)(2 \ 6)u_{2,6}(5 \ 6) = (2 \ 5)u_{2,5} \implies \\
& \quad \Gamma_2 \Gamma_3 \Gamma_2 \Gamma_3 \Gamma_6 = (3 \ 5)(2 \ 5)u_{2,5}(3 \ 5) = (2 \ 3)u_{2,3} \implies \\
& \quad \Gamma_8 = \Gamma_8 \Gamma_7 \Gamma_5 \Gamma_2 \Gamma_3 \Gamma_7 \Gamma_5 \Gamma_8 = (2 \ 3)(2 \ 3)u_{2,3}(2 \ 3) = (2 \ 3)u_{3,2} = (2 \ 3)u_{2,3}^{-1} \\
\Gamma_6: & \quad \Gamma_2 \Gamma_4 \Gamma_2 = (5 \ 6)(4 \ 5)(5 \ 6) = (4 \ 6) \implies \\
& \quad \Gamma_3 \Gamma_2 \Gamma_4 \Gamma_5 \Gamma_5 = (1 \ 6)u_{1,6}(4 \ 6)(1 \ 6)u_{1,6} = (1 \ 6)(4 \ 1 \ 6)u_{6,4}u_{1,6} = (1 \ 4)u_{1,4} \implies \\
& \quad \Gamma_6 = \Gamma_6 \Gamma_5 \Gamma_2 \Gamma_4 \Gamma_5 \Gamma_6 \Gamma_8 = (1 \ 4)(1 \ 4)u_{1,4}(1 \ 4) = (1 \ 4)u_{4,1} = (1 \ 4)u_{1,4}^{-1} \\
\Gamma_2': & \quad \Gamma_8 \Gamma_7 \Gamma_8 = (2 \ 3)(3 \ 5)(2 \ 3) = (2 \ 5) \implies \\
& \quad \Gamma_2' = \Gamma_3 \Gamma_8 \Gamma_7 \Gamma_8 \Gamma_3 = (2 \ 6)u_{2,6}(2 \ 5)(2 \ 6)u_{2,6} = (2 \ 6)(2 \ 5)(2 \ 6)u_{5,2}u_{2,6} = (5 \ 6)u_{5,6}.
\end{align*}
$$

Next, we calculate the value of $Proj$ in $S_6 \ltimes \mathbb{Z}^5$:

$$
e = \Gamma_8 \Gamma_3 \Gamma_8 \Gamma_2 \Gamma_2 \Gamma_2 = (2 \ 3)u_{2,3}^{-1}(2 \ 3)(1 \ 4)u_{1,4}^{-1}(1 \ 4)(5 \ 6)u_{5,6}(5 \ 6) = u_{3,2}^{-1}u_{4,1}^{-1}u_{6,5}^{-1} = u_{2,3}u_{1,4}u_{5,6}^{-1} = u_{1,4}u_{2,3}u_{5,6}^{-1} = u_{1,2}u_{2,3}u_{3,4}u_{5,6}^{-1}.
$$

For simplicity, we also call it $Proj$. Denote by $N$ the normal subgroup of $S_6 \ltimes \mathbb{Z}^5$ generated by $Proj$. We then have:

$$
u_{5,6}^2 = u_{5,6}u_{5,6} = (u_{1,2}u_{2,3}u_{3,4}u_{5,6})u_{1,2}^{-1}u_{2,3}^{-1}u_{3,4}^{-1}u_{5,6} = ((5 \ 6)Proj(5 \ 6))Proj^{-1} \in N \implies
$$

$$
u_{4,5}^2 = (1 \ 2 \ 3 \ 4 \ 5 \ 6)u_{5,6}^2(6 \ 5 \ 4 \ 3 \ 2 \ 1) \in N \implies
$$

$$
u_{5,6}^2 - k, 6 = (1 \ 2 \ 3 \ 4 \ 5 \ 6)^k u_{5,6}^2(6 \ 5 \ 4 \ 3 \ 2 \ 1)^k \in N.
$$
In Subsection 3.1, we determined that the double tetrahedron is simply-connected. In Subsection 3.2, we determined that Z_2^5 is normal (and thus equal to N). It is enough to show that \( \sigma^{-1}u_{1,2}u_{3,4}u_{5,6}\sigma \in (u_{1,2}u_{3,4}u_{5,6}) \), for the standard generators \( \sigma = (i \ i + 1) \) of \( S_6 \).

**Conjugation 1:** \((1 \ 2)\Proj(1 \ 2) = (1 \ 2)u_{1,2}u_{3,4}u_{5,6}(1 \ 2) = u_{2,1}u_{3,4}u_{5,6} = u_{1,2}u_{3,4}u_{5,6} = \Proj\)

**Conjugation 2:** \((2 \ 3)\Proj(2 \ 3) = (2 \ 3)u_{1,2}u_{3,4}u_{5,6}(2 \ 3) = u_{1,3}u_{2,4}u_{5,6} = (u_{1,2}u_{2,3})(u_{2,3}u_{3,4})u_{5,6} = u_{1,2}u_{3,4}u_{5,6} = \Proj\)

**Conjugation 3:** \((3 \ 4)\Proj(3 \ 4) = (3 \ 4)u_{1,2}u_{3,4}u_{5,6}(3 \ 4) = u_{1,2}u_{4,3}u_{5,6} = u_{1,2}u_{3,4}u_{5,6} = \Proj\)

**Conjugation 4:** \((4 \ 5)\Proj(4 \ 5) = (4 \ 5)u_{1,2}u_{3,4}u_{5,6}(4 \ 5) = u_{1,2}u_{3,5}u_{4,6} = u_{1,2}(u_{3,4}u_{4,5})(u_{4,5}u_{5,6}) = u_{1,2}u_{3,4}u_{5,6} = \Proj\)

**Conjugation 5:** \((5 \ 6)\Proj(5 \ 6) = (5 \ 6)u_{1,2}u_{3,4}u_{5,6}(5 \ 6) = u_{1,2}u_{3,4}u_{6,5} = u_{1,2}u_{3,4}u_{5,6} = \Proj.\)

Therefore,

\[
\tilde{G} = \frac{S_6 \times \mathbb{Z}_2^5}{\langle \Proj \rangle} = \frac{S_6 \times \mathbb{Z}_2^5}{\langle \Proj \rangle}
\]

and

\[
\pi_1(X_{Gal}) = \mathbb{Z}_2^4.
\]

(2) Now we write the parameters related to the degeneration \( D(T_{(4)}) \):

\[
n = 6, \ m = 18, \ \mu = 20, \ d = 60, \ \rho = 48.
\]

We get

\[
c_1^2(X_{Gal}) = \frac{6!}{4}(18 - 6)^2 = 36 \cdot 6!,
\]

\[
c_2(X_{Gal}) = 6!(3 - 18 + \frac{60}{4} + \frac{20}{2} + \frac{48}{6}) = 18 \cdot 6!.
\]

And therefore,

\[
\chi(X_{Gal}) = \frac{1}{3}(36 \cdot 6! - 36 \cdot 6!) = 0.
\]

\[\square\]

### 3.3 Conclusion

In Subsection 3.1 we determined that \( \pi_1(X_{Gal}) \) is trivial, which means that the Galois cover of the tetrahedron is simply-connected. In Subsection 3.2 we determined that \( \mathbb{Z}_2^4 \) is the group related to the double tetrahedron.
Corollary 3.3. Both geometric objects are positioned in different connected components in the moduli spaces of surfaces.

We are now motivated to continue the inductive investigation of surfaces with non-planar degeneration in the moduli space. We will probably have complicated algebraic computations in groups and daunting complications while resolving high-multiplicity singular points (such as the $k$-points presented in this work). Lemma 2.5 gave us a rule about 3-points that eases group calculations, compared to 4-points that appear in gluing two pieces (for example, as happens for $D(T_{(4)})$ or any two Zappatic degenerations) and for which we still have no similar lemma. We could determine the group very precisely in both cases and get the above corollary, along with the fact that both signatures show that it is true. We will need to develop strategies for attacking various $k$-points and try to determine the general forms of groups.

References

[1] M. Amram, C. Ciliberto, R. Miranda, Mina Teicher, Braid monodromy factorization for a non-prime K3 surface branch curve, *Israel J. Math.* 170 (2009), 61–93. Zbl 1205.14050

[2] M. Amram, M. Friedman, M. Teicher, The fundamental group of the complement of the branch curve of the second Hirzebruch surface, *Topology* 48 (2009), 23–40. Zbl 1207.14017

[3] M. Amram, M. Friedman, M. Teicher, The fundamental group of the branch curve of the complement of the surface $\mathbb{CP}^1 \times T$, *Acta Math. Sin.* 25(9) (2009), 1443–1458. Zbl 1178.14018

[4] M. Amram, D. Goldberg, Higher degree Galois covers of $\mathbb{CP}^1 \times T$, *Algebr. Geom. Topol.* 4 (2004), 841–859. Zbl 1069.14065

[5] M. Amram, C. Gong, U. Sinichkin, et al., Fundamental group of Galois covers of degree 6 surfaces, (2021), [https://doi.org/10.1142/S1793525321500412](https://doi.org/10.1142/S1793525321500412)

[6] M. Amram, C. Gong, S. L. Tan, et al., On the fundamental groups of Galois covers of planar Zappatic deformations of type $E_k$, *Internat. J. Algebra Comput.* 29 (2019), 905–925. Zbl 1423.14307

[7] M. Amram, S. Ogata, Toric varieties–degenerations and fundamental groups, *Michigan Math. J.* 54 (2006), 587–610. Zbl 1148.14302

[8] M. Amram, S.-L. Tan, W.-Y. Xu, et al., Fundamental group of Galois cover of the (2,3)-embedding of $\mathbb{CP}^1 \times T$, *Acta Math. Sin.* 36 (2020), 273–291. Zbl 7189516

[9] M. Amram, M. Teicher, The fundamental group of the complement of the branch curve of $T \times T$ in $\mathbb{C}^2$, *Osaka J. Math.* 40(4) (2003), 857–893. Zbl 1080.14516
[10] D. Auroux, S. Donaldson, L. Katzarkov, et al., Fundamental groups of complements of plane curves and symplectic invariants, *Topology* 43 (2004), 1285–1318. Zbl 1067.53069

[11] R. G. Bettiol, M. Kummer, R. A. E. Mendes, Convex algebraic geometry of curvature operators, *SIAM J. Appl. Algebra Geometry* 5(2) (2021), 200–228. Zbl 7353933

[12] V. Bugaenkoa, Y. Cherniavsky, T. Nagnibedab, R. Shwartz, Weighted Coxeter graphs and generalized geometric representations of Coxeter groups, *Discrete Appl. Math.* 192 (2015), 17–27. Zbl 1319.05064

[13] A. Calabri, C. Ciliberto, F. Flamini, et al., On the $K^2$ of degenerations of surfaces and the multiple point formula, *Ann. of Math.* 165 (2007), 335–395. Zbl 1441.14119

[14] A. Calabri, C. Ciliberto, F. Flamini, et al., On degenerations of surfaces, (2008), arXiv: math/0310009v2.

[15] F. Catanese, On the moduli spaces of surfaces of general type, *J. Differential Geom.* 19 (1984), 483–515. Zbl 0549.14012

[16] F. Catanese, (Some) old and new results on algebraic surfaces, First European Congress of Mathematics, Birkhauser Basel, 1994, 445–490. Zbl 0830.14010

[17] X. Chen, F. Gounelas, C. Liedtke, Curves on K3 surfaces, (2020), https://arxiv.org/abs/1907.01207.

[18] C. Ciliberto, A. Lopez, R. Miranda, Projective degenerations of K3 surfaces, Gaussian maps, and Fano threefolds, *Invent. Math.* 114 (1993), 641–667. Zbl 0807.14028

[19] M. Dettweiler, Plane curve complements and curves on Hurwitz spaces, *J. Reine Angew. Math.* 573 (2004), 19–43. Zbl 1074.14026

[20] K. Fukaya, Y.G. Oh, H. Ohta, K. Ono, Toric degeneration and non-displaceable Lagrangian tori in $S^2 \times S^2$, *Int. Math. Res. Not.* 13 (2012), 2942–2993. Zbl 1250.53077

[21] K. Fukaya, T. Yamaguchi, The fundamental groups of almost nonnegatively curved manifolds, *Ann. of Math.* 136 (1992), 253–333. https://doi.org/10.2307/2946606

[22] G. Gandini, S. Meinert, H. Rüping, The Farrell–Jones conjecture for fundamental groups of graphs of abelian groups, *Groups Geom. Dyn.* 9 (2015), 783–792. Zbl 1325.18004

[23] D. Gieseker, Global moduli for surfaces of general type, *Invent. Math.* 43 (1977), 233–282.

[24] R. Grigorchuk, On a question of Wiegold and torsion images of Coxeter groups, *Algebra Discrete Math.* 4 (2009), 78–96. Zbl 1199.20073

[25] M. Kapovich, On monodromy of complex projective structures, *Invent. math.* 119 (1995), 243–265. Zbl 0839.57011

[26] N. Koban, J. McCammond, J. Meier, The BNS-invariant for the pure braid groups, *Groups Geom. Dyn.* 9 (2015), 665–682. Zbl 1326.20045
[27] C. Liedtke, Fundamental groups of Galois closures of generic projections, *Trans. Amer. Math. Soc.* 362 (2010), 2167–2188. Zbl 1198.14017

[28] M. Manetti, On some components of moduli space of surfaces of general type, *Compos. Math.* 92 (1994), 285–297. Zbl 0849.14016

[29] Y. Miyaoka, On the Chern numbers of surfaces of general type, *Invent. Math.* 42 (1977), 225–237. Zbl 0374.14007

[30] B. Moishezon, M. Teicher, Simply connected algebraic surfaces of positive index, *Invent. Math.* 89 (1987), 601–643. Zbl 0627.14019

[31] B. Moishezon, M. Teicher, Braid group technique in complex geometry II, From arrangements of lines and conics to cuspidal curves, *Algebraic Geometry, Lect. Notes Math.* 1479 (1991), 131–180. Zbl 0764.14014

[32] B. Moishezon, M. Teicher, Braid group technique in complex geometry IV: Braid monodromy of the branch curve $S_3$ of $V_3 \to \mathbb{CP}^2$ and application to $\pi_1(\mathbb{C}^2 - S_3, *)$, *Contemp. Math.* 162 (1994), 332–358.

[33] L. Rowen, M. Teicher, U. Vishne, Coxeter covers of the symmetric groups, *J. Group Theory* 8 (2005), 139–169. Zbl 1120.20040

[34] M. Saiidi, Galois Covers of Degree $p$ and Semi-Stable Reduction of Curves in Mixed Characteristics, *Publ. RIMS, Kyoto Univ.* 43(3) (2007), 661–684. Zbl 1137.14019

[35] F. Sottile, *Real algebraic geometry for geometric constraints*, CRC Handbook on Geometric Constraint Systems Principles, M. Sitharam, A. St. John, and J. Sidman, eds. CRC Press, 2019, 273–285. Zbl 1404.14068

[36] E. R. van Kampen, On the fundamental group of an algebraic curve, *Amer. J. Math.* 55 (1933), 255–260.