A simulation based teaching method to explain the concept of probability theory; an alternative learning method

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Abstract. A concept of defining classical probability is often based on the equally likely outcome assumption of an experiment, so the experiment should be based on the equality outcome of media or tool used. If the equality assumption is not met then the alternative for defining probability is done by using the concept of relative frequency. The concept of a relative frequency means that the probability of an event is the proportion of the event occurred in an experiment with infinite trials. Mathematically If \( P(A) \) denotes a probability of event \( A \), \( f(A) \) denotes the frequency of event \( A \) occurred, and \( n \) denotes the number of trials in an experiment, then \( P(A) = \lim_{n \to \infty} \frac{f(A)}{n} \), assumed the limit exists. This paper proves the existence of limit for the relative frequency as the number of trials goes infinitely through simulation. This simulation can finally be used as an alternative learning method in understanding the concept of probability.

1. Introduction

Advances in technology in the field of hardware accompanied by the development of software to make a lot of computing problems that have not been solved before, today can be completed even with a high degree of accuracy. The advancement of computational technology greatly influences the learning model that has not been shown visually. A simulated learning model of mathematics also continues to grow in line with the development of computing technology. Specifically in the learning model of probability theory, students can get a deeper insight of understanding the concept of probability theory through the usage of the computing technology. There are various researchers introduced the usage of computer as a tool to help understanding of the abstract or difficult concepts of mathematics and educations. Stephens et al in [1] is using simulation and resampling methods to teach inferential concepts. A study conducted by Novak show that students rated the simulation used in the study as a more exciting in comparison to lecture, teamwork or reading a textbook [2]. Ferreira et al. in [3] have studied about aspects of high school students’ learning of probability in a context where they are supported by the statistical software R. They point out that “the use of R allowed students to extend their reasoning beyond that developed from paper-and-pencil approaches, since it made it possible for them to work with larger number of simulations, and go beyond the standard equiprobability assumption in coin tosses [3]”. Other topics about using simulation methods can be found in [4].
According to Linde [5], modeling experiments involving randomness, that is, experiments with uncertainty outcome, is basic concern of probability theory. Historically, probability and game have very close relationship. For example, Fermat and Pascal solved the problem of estimating the fair amount to be given to each player if the game is interrupted by proportionally dividing the stakes among each player’s chances [6]. Further, as mentioned by Vasudevan in [7], “Probabilities were thus first introduced as part of a general theory of the conditions of fair play in games of chances”. The instruments such as used in the tossing of coin and the casting of dice are almost perfectly symmetric [7], since this is reasonable in many chance games [6].

In the classical definition, probability is a fraction of the number of favorable cases to a particular event divided by the number of all cases possible [6]. A concept of defining classical probability is often based on the equally likely outcome assumption of an experiment, so the experiment should be based on the equality outcome of media or tool used. Vasudevan remark in his paper, “In what follows, the ‘classical theory of probability’ signifies the assortment of historical approaches to the subject which place an emphasis on combinatorial methods of reasoning, or accept, in some form, the classical definition of probability as the ratio of favorable to total possible outcomes, all such outcomes being equally possible [7]”.

The classical theory of probability that often assumes concepts of equally likely to occur from an experiment, or usually stated in a phrase of “equally likely outcome” often becomes a problem in experimental probability when the tools used does not follow that assumption. This problem is not easy to explain throughout the limited trial of experiments. In the tossing of coin that has one side heavier, e.g. head is twice as heavy as tail, the initial assumption, equally likely outcome, is not met. Then the alternative for defining probability is done by using the concept of relative frequency or frequentist approach. This approach closely related to infinite trial or large number of trials. Von Mises is the founder of frequentist approach in probability theory [8].

Learning probability as frequentist concepts may face some difficulty cause of limited experience. In the probability teaching, a definite thought that there would be a single, absolute and right answer as in mathematics cannot be used are other difficulties [9]. Koparan in [10] showed that probability teaching based on simulation increased the prediction and related inference skills of the prospective teachers and generally influenced the success of the students in a positive way. He also listed other advantages of simulation as follows; simulations can help students to understand the concepts in elementary probability and provide for sensitivity estimation, simulations have positive effects on the inference skills on the experimental probability problems. Additionally, according to Koparan, simulations handle small and large sampling answer “what-if” questions, interaction among variables, time compression, a better understanding of the concepts, and conditions close to reality [10].

In this paper, simulation will be used in learning probability to generate large number of trial and to help understanding on frequentist approach on probability. Through simulation the existence number for probability obtain by frequentist concept will be shown. Finally, it can be used as an alternative learning method in understanding the concept of probability, which is the main purpose of this paper.

2. Reviews of Classical Definition

Classical definition of probability is based on equally likely outcome assumption. Let Ω be the outcome space. Probability defined as a fraction of the number of favorable cases to a particular event divided by the number of all cases possible [6]. Probability of event \( A \) is equal to the number of outcomes in \( A \) divided by the total number of possible outcomes in a random experiment also known as the classical definition of probability [11]. We can write this definition as,

\[
P(A) = \frac{n(A)}{n(\Omega)},
\]

where, \( P(A) \) probability of event \( A \).

In classical definition, the experiment should be based on the equality outcome of media or tool used. When this assumption is not met, then the alternative for defining probability is done by using
the concept of relative frequency or frequentist approach. As frequentist approach, probability is a fraction of number of case among the first \( n \) observation as \( n \) going to infinity \[8\]. Additionally, Grami in \[11\] illustrated, in the frequency interpretation, the probability of an event \( A \) is the expected or estimated relative frequency of \( A \) in a large number of trials or number of trial approaches infinity. Mathematically, if \( P(A) \) denotes a probability of event \( A \), \( f_n \) denotes the frequency of event \( A \) occurred, and \( n \) denotes the number of trials in an experiment, then we can write as

\[
P(A) = \lim_{n \to \infty} \frac{f_n}{n},
\]

assumed the limit existed.

3. Simulation Methods

In this section, we firstly simulate some experiments using statistical software, R, to show the existence of probability measure with the concept of relative frequency. Secondly, two distributions, Binomial and Negative Binomial (NB) distribution are also simulated by the relative frequency concept. Finally, the simulation results are compared to related theoretical results.

3.1. Simulation of Probability

Frequentist approach of probability concept is closely related to large number of two trials or observations. The concept is learned through two types of experiment, tossing a coin and casting a dice. Each type of experiment is simulated in two conditions of the tool, equally likely outcome, and unequally likely outcome. Let \( P(A), 0 < P(A) < 1 \), probability of an event \( A \) occurred and \( P_n(A) = f_n/n \) be the relative frequency of event \( A \) in the first \( n \) observations. Law of Large Number states \( P_n(A) \rightarrow P(A) \) as \( n \to \infty \). In the first case we evaluate experiments of tossing a fair coin, and casting a balance dice. Theoretically based on classical definition, the probability of event \( A \), head shown up, is \( 1/2 \), \( P(A) = 1/2 \). Likewise, the probability of event \( B \), a six spot shown up, in the experiment of casting a dice, is \( 1/6 \). For the second case, we evaluate experiments of tossing an unfair coin, and an unbalanced dice, where for a coin, we assume that the probability of \( A \), head shown up, \( P(A) = 1/3 \), and for the dice, we assume that the probability of \( B \), a six spots shown up is \( 1/2 \), and the probability of the other spots are equal, \( 1/10 \). Further, we simulate the experiments by developing a function using statistical R software. Each experiment repeatedly tossing the coin, say it \( n \) times, for \( n = 1,2,\ldots,10000 \) is depicted in figure 1, and for some values of \( n \) is shown in Table 2 and 3.

Further, let parameters \( \Omega, N, p, n, A, m, r \) to be sample space, number of trials, the probability of each elementary outcome of the experiment, the particular numbers of trials, particular event being ecuated, number of iteration, and target for number of successful (for NB distribution), respectively. The simulation algorithm of the experiment is,

**Step 1.** Generate \( N \) trials of experiments,
**Step 2.** Generate proportion or relative frequency of event \( A \) occurred, and
**Step 3.** Repeat \( N \) trials of experiment \( m \) times to get relative frequency distribution of event occurred.

Parameters used for several cases discuss above is noted in Table 1.

| Case      | \( \Omega \) | \( N \)   | \( p \)  | \( \Delta n \) | \( A \) | \( m \) | \( r \) |
|-----------|-------------|-----------|---------|----------------|-------|-------|-------|
| Fair Case | \((H,T)\)   | 10000     | \(1/2,1/2\) | \(5,10,50,100,500,1000,5000,10000\) | H     | 1000  | 5     |
|           | \((1,2,3,4,5,6)\) | 10000 | \(1/6,1/6,1/6,1/6,1/6,1/6\) | \(5,10,50,100,500,1000,5000,10000\) | H     | 1000  | 5     |
| Unfair Case | \((H,T)\)   | 10000     | \(1/3,2/3\) | \(5,10,50,100,500,1000,5000,10000\) | 6     | 1000  | 5     |
|           | \((1,2,3,4,5,6)\) | 10000 | \(1/10,1/10,1/10,1/10,1/10,1/10,1/2\) | \(5,10,50,100,500,1000,5000,10000\) | 6     | 1000  | 5     |
3.2. Simulation of Distributions
Using relative frequency concept, we simulated two distributions, Binomial and NB distribution. Similar to probability, both distributions learned through simulation of tossing a coin and casting a dice.

Let there are $N$ independent observations (tossing a coin or a dice) with probability of success, $p$, for each observation. Success for tossing a coin or a dice observation is when head shown up or six spots shown up, respectively. Further, this $N$ independent observations is repeated $m$ times.

3.2.1. Binomial Distribution. Let $X$ is number of success in $N$ independent observations. Theoretically, $X$ describes Binomial distribution. Density of $X$ is fraction of frequency relative divided by number of iteration, $m$. Let $P_r(X) = \frac{f_X}{m}$ be the relative frequency of number of success after $m$ repetition. This relative frequency tends to $p$. To simulate Binomial distribution of $X$, we use same function as developing in probability simulation. Distinguishingly, step 1 and step 3 uses to generate Binomial distribution. Several cases simulate in section 3.1 is also simulated for Binomial distribution.

3.2.2. Negative Binomial Distribution. Let $r$, a fixed positive integer, be target for number of successful observations. NB distribution describes the behavior of the number of failures observed before the $r^{th}$ success occurred. Let $Y$ be the number of failures observed before the $r^{th}$ success occurred. To simulate NB distribution of $Y$ we use same function as developing in probability simulation with additional step to calculate $Y$. Further, step 1 is modified to adjust an additional step. Several cases simulate in section 3.1 is also simulated for NB distribution.

4. Result and Conclusions
In this section, we discuss simulation results of both cases of two type experiments in Table 1. We compared relative frequency result for two type simulation of experiments to show the existence of probability to theoretical probability. By the relative frequency concept, Binomial and NB distribution is generated for each case.

4.1. Fair Case
Figure 1 shows relative frequency of event $A$ in tossing a fair coin $n$ times ($n = 1, 2, ..., 10000$), relative frequency, relative frequency distribution, Binomial distribution of $X$ and NB distribution of $Y$, respectively. Similarly Figure 2 shows the results for event $B$, six spots shown up in tossing a balance dice. Its relative frequency value for some $n$ of event $A$ and event $B$, respectively show in Table 2.

Relative frequency of event $A$, $P_r(A)$, for some $n$ tends to 0.5 as $n$ large. After 10000 observations, event $A$ occurred 4963 times and its relative frequency is 0.4934, close enough to theoretical probability 0.5. In its empirical distribution of relative frequency indicate that in 1000 iteration almost of relative frequency numbers distributed around 0.5. Its number of success, $X$, in 1000 iteration has Binomial distribution and number of failures observed before the 5th success occurred, $Y$, describes NB distribution.

| Table 2. Relative frequency of simulation results for fair case. |
|--------------------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $n$ | 5 | 10 | 50 | 100 | 500 | 1000 | 5000 | 10000 |
| $P_r(A)$ | 0.4 | 0.6 | 0.62 | 0.57 | 0.556 | 0.531 | 0.5016 | 0.4963 |
| $P_r(B)$ | 0.0 | 0.2 | 0.2 | 0.18 | 0.192 | 0.184 | 0.167 | 0.1673 |

Relative frequency of event $B, P_r(B)$, for some $n$ tends to 0.167 as $n$ large. After 10000 observations, event occurred 1673 times and its relative frequency is 0.1673, close enough to theoretical probability 1/6. In its empirical distribution of relative frequency indicate that in 1000 iteration almost of relative frequency numbers distributed around 1/6. Further, $X$ and $Y$ have Binomial and NB distribution, respectively.
4.2. Unfair Case

Figure 3 and Figure 4 shows relative frequency, relative frequency distribution, Binomial distribution of \( X \) and NB distribution of \( Y \) in tossing a unfair coin or an unbalanced dice, respectively. Its relative frequency value for some \( n \) for event \( A \) and event \( B \), respectively show in Table 3.

| \( n \) | 5   | 10  | 50  | 100 | 500 | 1000 | 5000 | 10000 |
|--------|-----|-----|-----|-----|-----|------|------|-------|
| \( P_s(A) \) | 0.6 | 0.4 | 0.38 | 0.43 | 0.444 | 0.469 | 0.4984 | 0.5037 |
| \( P_s(B) \) | 0.3 | 0.34 | 0.34 | 0.354 | 0.347 | 0.3326 | 0.3303 |
Relative frequency of event \( A \), \( P_n(A) \) for some \( n \) tends to 0.33 as \( n \) large. After 10000 observations, event \( A \) occurred 3303 times and its relative frequency is 0.3303, close enough to theoretical probability 1/3. Its empirical distribution of relative frequency indicate that in 1000 iteration almost of relative frequency numbers distributed around 0.33. Its number of success, \( X \), in 1000 iteration has Binomial distribution and number of failures observed before the 5th success occurred, \( Y \), describes Negative Binomial distribution.

Relative frequency of event \( B \), \( P_n(B) \) for some \( n \) tends to 0.5 as \( n \) large. After 10000 observations, event \( B \) occurred 5037 times and its relative frequency is 0.5037, close enough to theoretical probability 1/2. \( X \) and \( Y \) are Binomial and NB distribution, respectively.

Figure 3. Simulation results of unfair coin. (a) Relative frequency of even A. (b) Frequency distribution of \( P(A) \). (c) Binomial distribution with \( n = 10,000 \), \( p = 1/3 \). (d) NB distribution with \( r = 5 \), \( p = 1/3 \)

Figure 4. Simulation results of unbalanced dice. (a) Relative frequency of even B (b) Frequency distribution of \( P(B) \). (c) Binomial distribution with \( n = 10,000 \), \( p = 1/2 \). (d) NB distribution with \( r = 5 \), \( p = 1/2 \)
Acknowledgements
In this opportunity, I would like to express my sincere gratitude to, the Research Department of the University of Sumatera Utara that has provided financing aid for me to attend the conference. I would also like to thank the Dean of FMIPA, University of Sumatera Utara, for his given permit to me to carry out this research and to present it in this International seminar.

References
[1] Stephens M, Carver R and McCormack D 2014 Proc. of the 9th Int. Conf. on Teaching Statistics
[2] Novak E 2014 J. Comput. Assist. Lear 30(2), 148
[3] Ferreira R D S, Kataoka V Y and Karrer M 2014 Stat. Educ. Res. J 13(2), 132
[4] Pilz J, Rasch D, Melas V B and Moder K 2015 Statistics and Simulation vol 231 (Cham: Springer) pp 3
[5] Linde W 2016 Probability Theory (Berlin: De Gruyter) p 1
[6] Batanero C, Chernof E J, Engel J, Lee H S and Sanchez E 2016 In Res. Teach. Learn. Probab pp 1-33
[7] Vasudevan A 2018. Stud. Hist. Philos. Sci 67 32
[8] Maistrov L E 2014 Probability theory: Ahistorical sketch (Academic Press) pp 3
[9] Koparan T 2015 Int. J. Math. Educ. Sci. Technol 46(1), 94
[10] Koparan T and Yilmaz G K 2015 Univers. J. Educ. Res 3(11), 775
[11] Grami A 2019 Probability, Random Variables, Statistics, and Random Processes (Hoboken: Wiley) pp 4