New Sufficient Conditions for Lower Bounding the Optimal Policy of a POMDP using Lehmann Precision

Vikram Krishnamurthy,
Electrical & Computer Engineering,
Cornell Tech
Cornell University, USA.
vikramk@cornell.edu

October 23, 2018

Abstract

This paper provides new sufficient conditions so that the optimal policy of a partially observed Markov decision process (POMDP) can be lower bounded by a myopic policy. The two new proposed conditions, namely, Lehmann precision and copositive dominance, completely fix the problems with two crucial assumptions in the well known papers [8, 12]. For controlled sensing POMDPs, Lehmann precision exploits both convexity and monotonicity of the value function, whereas the classical Blackwell dominance only exploits convexity. Numerical examples are presented where Lehmann precision holds but Blackwell dominance does not hold, thereby illustrating the usefulness of the main result in controlled sensing applications.
1 Introduction

This paper provides sufficient conditions so that the optimal policy of a POMDP is provably lower bounded by a myopic policy. From a practical point of view, this structural result is useful since myopic policies are trivial to compute/implement in large scale POMDPs and also provide a useful initialization for more sophisticated sub-optimal solutions. Structural results are important since in general solving POMDPs is PSPACE-complete; see [11].

The seminal papers [8, 12, 13] give sufficient conditions for two very useful results: (i) the value function of a POMDP to be monotone in the belief state (with respect to the likelihood ratio order and multivariate generalizations) and (ii) for the optimal policy of a POMDP to be lower bounded by a myopic policy. Monotonicity of the value function is crucially important and will be used in our main results below. Regarding lower bounding the optimal policy by a myopic policy, unfortunately, despite the enormous usefulness of such a result, the sufficient conditions given in [8] and [12] are not useful - it is impossible to generate non-trivial examples that satisfy the conditions (c), (e), (f) of [8, Proposition 2] and condition (i) of [12, Theorem 5.6]. Our recent works [5, 6] provided a fix for the conditions on the transition probabilities by using copositive dominance. In this paper, motivated by controlled sensing applications, we provide a complete fix to the conditions on the controlled observation probabilities of the POMDP so that the results of [8, 12] hold for constructing a myopic policy that lower bounds the optimal policy.

This paper is motivated by controlled sensing POMDPs where the observation probabilities (which model an adaptive sensor) are controlled whereas the transition probabilities (which model the Markov chain signal being observed by the sensor) are not controlled. Controlled sensing arises in a variety of applications in reconfigurable sensing (how can a sensor reconfigure its behavior in real time), cognitive radio, adaptive radars, optimal search problems for a Markovian target, and active hypothesis testing. Providing useful sufficient conditions so that the optimal policy for a controlled sensing POMDP is lower bounded by a myopic policy is surprisingly nontrivial. The main new assumption that we will use is the Lehmann precision condition – this single crossing condition proposed in [9] has recently been used extensively in the economics literature, see [3, 2]. Thus far, there has been no way of obtaining structural results for controlled sensing POMDPs that exploit both monotonicity and convexity of the value function. The papers [8, 12] used only monotonicity of the value function (wrt monotone likelihood ratio stochastic order) and the resulting assumptions were not useful (as mentioned above). On the other hand, [15, 12, 13] used only convexity of the value function with Blackwell dominance to construct a lower bound to a controlled sensing POMDP. In this paper, Lehmann precision allows us to use both convexity and monotonicity of the value function to construct the lower bound. Indeed, the Lehmann precision condition on the observation probabilities together with copositive dominance of controlled transition matrices, gives a useful set of conditions for POMDPs which completely fix the problems with the key assumptions in [8] and also [12]. Theorem 3.2 is our main POMDP structural result.

In proving our main result, as an aside we also establish two minor results. First, Theorem 3.4 compares the optimal cumulative rewards of two different POMDPs when the parameters of one dominate the other with respect to Lehmann precision; the result is more useful than the Blackwell dominance case in controlled sensing POMDPs. Second, Theorem 4.3 cleans up the assumption made in [1] which results in the piecewise linear segments of the POMDP value function being monotone vectors. The assumption in [1] is implicit and not easily verifiable. Our proof uses stochastic dominance restricted to certain line segments to show that the conditions in [8] actually do result in monotone...
vectors for the value function for the case of 3 or fewer underlying states.

2 The Partially Observed Markov Decision Process

Consider a discrete time, infinite horizon discounted reward POMDP. A discrete time Markov chain evolves on the state space \( X = \{1, 2, \ldots, X\} \). Denote the action space as \( U = \{1, 2, \ldots, U\} \) and observation space as \( Y \). We consider either \( Y = \{1, 2, \ldots, Y\} \) (finite set) or \( Y = \mathbb{R} \) or \( Y \) is the closed interval \([1, Y]\). Let \( \Pi(X) = \{\pi: \pi(i) \in [0, 1], \sum_{i=1}^{X} \pi(i) = 1\} \) denote the belief space of \( X \)-dimensional probability vectors. For stationary policy \( \mu: \Pi(X) \rightarrow U \), initial belief \( \pi_0 \in \Pi(X) \), discount factor \( \rho \in [0, 1) \), define the discounted cumulative reward:

\[
J_\mu(\pi_0) = \mathbb{E}_\mu \left\{ \sum_{k=0}^{\infty} \rho^k r^*_{\mu(\pi_k)} \pi_k \right\}.
\]  

(1)

Here \( r_u = [r(1, u), \ldots, r(X, u)] \), \( u \in U \) is the reward vector for each sensing action, and the belief state evolves according to Bayes formula as \( \pi_k = T(\pi_{k-1}, y_k, u_k) \) where

\[
T(\pi, y, u) = \frac{B_y(u) P'(u) \pi}{\sigma(\pi, y, u)}, \quad \sigma(\pi, y, u) = 1_X B_y(u) P'(u) \pi, \quad B_y(u) = \text{diag}\{B_{1,y}(u), \ldots, B_{X,y}(u)\}.
\]  

(2)

Here \( 1_X \) represents a \( X \)-dimensional vector of ones, \( P(u) = [P_{ij}]_{X \times X} P_{ij}(u) = \mathbb{P}(x_{k+1} = j | x_k = i, u_k = u) \) denote the controlled transition probabilities. When \( Y \) is a finite set, \( B_{xy}(u) = \mathbb{P}(y_{k+1} = y | x_{k+1} = x, u_k = u) \) denotes the controlled observation probabilities; for \( Y \) continuum, we assume that the conditional distribution \( \mathbb{P}(y_k \leq y | x_k) \) is absolutely continuous wrt the Lebesgue measure and so the controlled conditional probability density function \( B_{xy}(u) = p(y_{k+1} = y | x_{k+1} = x, u_k = u) \) exists.

The aim is to compute the optimal stationary policy \( \mu^*: \Pi(X) \rightarrow U \) such that \( J_{\mu^*}(\pi_0) \leq J_{\mu}(\pi_0) \) for all \( \pi_0 \in \Pi(X) \). Obtaining the optimal policy \( \mu^* \) is equivalent to solving Bellman’s dynamic programming equation: \( \mu^*(\pi) = \arg \max_{u \in U} Q(\pi, u), J_{\mu^*}(\pi_0) = V(\pi_0), \) where

\[
V(\pi) = \max_{u \in U} Q(\pi, u), \quad Q(\pi, u) = r'_u \pi + \rho \sum_{y \in Y} V(T(\pi, y, u)) \sigma(\pi, y, u).
\]  

(3)

Note that for continuum \( Y \), the notation \( \sum_{y \in Y} \) denotes integration wrt \( y \). Also, \( V(\pi) \) is the fixed point of the following value iteration algorithm: Initialize \( V_0(\pi) = 0 \) for \( \pi \in \Pi(X) \). Then

\[
V_{k+1}(\pi) = \max_{u \in U} Q_{k+1}(\pi, u), \quad \mu_k = \arg \max_{u \in U} Q_k(\pi, u),
\]

\[
Q_{k+1}(\pi, u) = r'_u \pi + \rho \sum_{y \in Y} V_k(T(\pi, y, u)) \sigma(\pi, y, u), \quad k = 0, 1, \ldots,
\]  

(4)

Indeed, the sequence \( \{V_k(\pi), k = 0, 1, \ldots\} \) converges uniformly to \( V(\pi) \) on \( \Pi(X) \) geometrically fast. Since \( \Pi(X) \) is continuum, Bellman’s equation (3) and the value iteration algorithm (4) do not directly translate into practical solution methodologies since they need to be evaluated at each \( \pi \in \Pi(X) \).
Almost 50 years ago, [14] showed that when $Y$ is finite, then for any $k$, $V_k(\pi)$ has a finite dimensional piecewise linear and convex characterization. Unfortunately, the number of piecewise linear segments can increase exponentially with the action space dimension $U$ and double exponentially with time $k$. Thus there is strong motivation for structural results: to construct useful myopic lower bounds $\mu(\pi)$ for the optimal policy $\mu^*(\pi)$.

**Remark. Controlled Sensing:** In controlled sensing, the aim is to dynamically decide which sensor (or sensing mode) $u_k$ to choose at each time $k$ to optimize the objective (1). For such POMDPs, the transition matrix $P$, which characterizes the dynamics of the signal being sensed, is functionally independent of the action $u$. Only $r_u$, which models the information acquisition reward of the sensor, and observation probabilities $B(u)$, which models the sensor’s accuracy when it operates in mode $u$, are action dependent.

### 3 Main Structural Result

Although our main motivation stems from controlled sensing (where only the reward and observation matrix are action dependent), we state our main result for general POMDPs where the reward, transition and observation matrices are action dependent; so that the results provide a complete fix to the conditions in [8, 12]. In particular, Assumptions A4 and A6, A7 below provide a complete fix to the problems inherent in conditions (c) and (f) of [8].

**Definition 3.1** (Copositive Ordering of Transition Matrices [5]). Given transition matrices $P(u)$ and $P(u+1)$, we say that $P(u) \preceq P(u+1)$ if the sequence of $X \times X$ matrices $\Gamma^{j,u}$, $j = 1 \ldots, X - 1$ are copositive, i.e.,

$$
\pi' \Gamma^{j,u} \pi \geq 0, \quad \forall \pi \in \Pi(X), \quad \text{for each } j,
$$

where

$$
\Gamma^{j,u} = \frac{1}{2} \left[ \gamma_{m,n}^{j,u} + \gamma_{n,m}^{j,u} \right]_{X \times X}, \quad \gamma_{m,n}^{j,u} = P_{m,j}(u)P_{n,j+1}(u + 1) - P_{m,j+1}(u)P_{n,j}(u + 1).
$$

Our main assumptions are the following:

(A1) [Monotone reward] $r(i,u)$ is increasing in $i$ for each $u \in U$.

(A2) [TP2 transition] $P(u)$ is totally positive of order 2 (TP2): all second-order minors are nonnegative.

(A3) [TP2 observation] $B(u)$, $u \in U$ is TP2.

(A4) [Copositive dominance] $P(u) \preceq P(u+1)$

(A5) [Stochastic dominance of observations] $\sum_{y<j} B_{iy}(u) \leq \sum_{y<j} B_{iy}(u+1)$ for all $i \in X$ and $j \in Y$. Equivalently, $B_i(u) \leq_s B_i(u+1)$ where $B_i(u)$ denotes the $i$-th row of observation matrix $B(u)$ and $\leq_s$ denotes first order stochastic dominance.

---

1Throughout this paper, by increasing, we mean non-decreasing.
2Equivalently, the $i$-th row is monotone likelihood ratio (MLR) dominated by the $(i+1)$-th row for $i = 1, 2, \ldots, X - 1$; MLR dominance is defined in Section 4.
(A6) [Lehmann precision] $\sum_{y \leq j} B_{iy}(u) - \sum_{y \leq l} B_{iy}(u+1)$ changes sign at most once from negative to positive as $i$ increases for all $j, l \in \mathcal{Y}$. We denote this as $B(u+1) >_L B(u)$.

(A7) If $\mathcal{Y} = \mathbb{R}$, then $B_{iy}(u+1)/B_{iy}(u) < \infty$ for $i = 1, \ldots, X$, i.e., absolute continuity holds. If $\mathcal{Y} = \{1, \ldots, \mathcal{Y}\}$ (finite set) then for the boundary values 1 and $\mathcal{Y}$ and $i = 1, \ldots, X$:

$$B_{i1}(u) B_{X1}(u+1) \leq B_{i1}(u+1) B_{X1}(u), \quad B_{iY}(u) B_{XY}(u+1) \geq B_{iY}(u+1) B_{XY}(u). \quad (6)$$

If $\mathcal{Y} = [a, b]$ then (6) holds with 1 and $\mathcal{Y}$ replaced by $a$ and $b$.

The single crossing property A6 is called “Lehmann precision” in [4] and integral precision in [3]; see also [7].

Theorem 3.2 (Main Structural using Lehmann Precision). 1. Controlled Sensing POMDP: Suppose the transition probabilities $P(u)$ are functionally independent of the action, but the observation probabilities $B_i(u)$ are action dependent. Assume A2, A3, A6 (Lehmann precision), A7 hold. Then $Q(\pi, u) - r'_{\pi} \uparrow u$. Therefore, the myopic policy $\mu(\pi) = \arg \max_u r'_{\pi}$ forms a lower bound to the optimal policy in the sense that $\mu^*(\pi) \geq \mu(\pi)$ for all $\pi \in \Pi(X)$.

2. General POMDP: Suppose both the transition probabilities $P(u)$ and observation probabilities $B(u)$ are action dependent. Then under A2, A3, A4 (copositive dominance), A5, A6 (Lehmann precision), A7 the above result holds.

The proof is in Section 4. Theorem 3.2 also holds for any finite horizon (with non-stationary policy).

Discussion

From a practical point of view, Theorem 3.2 is useful since the myopic policy $\mu$ is trivial to compute and implement and gives a guaranteed lower bound to the optimal policy. Also, for beliefs $\pi$ where $\mu(\pi) = U$, the optimal policy $\mu^*(\pi)$ coincides with the myopic policy $\mu(\pi)$.

The rest of this section discusses several implications of Theorem 3.2 and its assumptions.

1. Assumptions: Assumptions A1 to A7 along with Theorem 3.2 completely fixes the problems with the assumptions in [8] and [12].

Assumptions A1, A2, A3 and A5 correspond to conditions (a), (c), (d), (e) in [8, Proposition 1, Proposition 2]. Indeed, [8] proves that A1, A2, A3 are sufficient for $V(\pi)$ to increase with respect to $\pi$ (w.r.t. monotone likelihood ratio order).

(i) Assumption A1. In Theorem 3.2, A1 (monotone rewards) is only required for general POMDPs; it is not required for controlled sensing POMDPs. Moreover, for general POMDPs, A1 can be replaced by the following condition which depends only on the transition probabilities:

(A1') There exists $f \in \mathbb{R}^X$ such that $\Delta_u \overset{\text{defn}}{=} (I - \rho P(u)) f$ is a strictly increasing vector for each action $u \in \mathcal{U}$.

A1' implies that there exists a POMDP with monotone increasing reward vectors $r_u + \Delta_u$ that has the same optimal policy as the original POMDP. To explain A1', suppose the reward vectors $r_u, u \in \mathcal{U}$ are arbitrary; not necessarily monotone. For $f \in \mathbb{R}^X$, define $W(\pi) = V(\pi) + f' \pi$. Then it is easily seen that $W(\pi)$ satisfies Bellman’s equation (3) with reward vector $r_u + \Delta_u$, and the optimal
policy remains unchanged. Thus under A1’ one can choose f so that \( r_u + \Delta_u \) is increasing, while the optimal policy remains unchanged.

For controlled sensing POMDPs, A1’ always holds; hence Statement 1 of Theorem 3.2 does not require A1. Since \( P \) and \( \Delta \) in A1’ do not depend on \( u \), choose \( \tilde{r} = \max_{i,u,j,u'} r(i,u) - r(j,u') \) and select \( \Delta \) with elements \( \Delta(i) = i\tilde{r} \). Clearly, \( r_u + \Delta \) is an increasing vector, and \( f = (I - \rho P)^{-1} \Delta \) explicitly satisfies A1’.

(ii) A2, A3, and A5. A2 and A3 are standard TP2 assumptions [8]; see [5] for several controlled sensing examples. A5 is also used in [8]; but is not required for the controlled sensing result (statement 1 of Theorem 3.2).

(iii) Key new assumptions. Let us focus on A4, A6 and A7 which are the key new assumptions that replace Assumption (c) and (f) in [8] Proposition 2. Assumptions (c) and (f) in [8] are sufficient for \( \sigma(\pi, u) \leq \sigma(\pi, u + 1) \) and \( T(\pi, y, u) \leq T(\pi, y, u + 1) \) for all \( \pi \in \Pi(X) \). Unfortunately, Assumptions (c) and (f) in [8] are mutually exclusive apart from trivial cases.

The copositive condition A4 on the transition probabilities presented in our recent work [6, 5] fixes Assumption (c) in [8] that \( P(1) \leq TP2 P(2) \); such TP2 dominance only holds if \( P(1) = P(2) \) or rank 1, and so is not useful.

Our main new assumption is the Lehmann precision condition A6 on the observation probabilities. This fixes the condition (f) in [8] that \( B_{iy}(2)B_{i+1,y} \leq B_{i+1,y}(2)B_{iy}(1) \). Apart from the trivial case \( B(1) = B(2) \), it is impossible for two stochastic matrices \( B(1), B(2) \) to satisfy condition (f) and A5 (condition (d) in [8]) simultaneously. In comparison, there is a continuum of useful examples that satisfy the conditions A5 and A6 (Lehmann precision) in Theorem 3.2 see examples below.

Finally, A7 is an absolute continuity condition. When the observation space is finite or has finite support, A7 puts conditions on the observation probabilities at the boundary values \( y = 1 \) and \( y = Y \), and is therefore not restrictive. A7 is a sufficient condition for the range of the final component of the updated belief for action \( u \) to be a subset of that for action \( u + 1 \), i.e., \( \{ e^\pi T(\pi, u, y), y \in \mathcal{Y} \} \subseteq \{ e^\pi T(\pi, u + 1, y), y \in \mathcal{Y} \} \).

2. Continuous observations POMDPs: One specific case where A6 holds is the additive noise sensing case where \( y_k = x_k + w_k \) where the additive noise \( w_k \) is an independent and identically distributed sequence of random variables with density \( p_u(\cdot | u) \). Then \( B_{iy} = p_u(y - i|u) \). Then it can be shown [9] that A6 holds iff \( B_{iy}(u) \) is larger than \( B_{iy}(u+1) \) with respect to the dispersive stochastic order.

3. Blackwell dominance vs Lehmann Precision: As mentioned in Section 1 thus far the only known cases of structural results for controlled sensing POMDPs involves Blackwell dominance [12, 13]. Since Theorem 3.2 uses Lehmann precision to give a new set of conditions for controlled sensing compared to Blackwell dominance, it is worthwhile comparing Blackwell dominance with Lehmann precision.

Suppose \( B(1) = B(2) \times L \) where \( L \) is a stochastic matrix. Then \( B(2) \) is said to Blackwell dominate \( B(1) \); denoted as \( B(2) \succ_B B(1) \). Intuitively \( B(1) \) is noisier than \( B(2) \). It is well known using a straightforward Jensen’s inequality argument that the following result holds:

**Theorem 3.3** (Blackwell dominance, [15, 12]).

1. **Controlled Sensing POMDP:** Suppose \( P \) is functionally independent of the action. Then \( B(u + 1) \succ_B B(u), u = 1, \ldots, U - 1 \) is a sufficient condition for the conclusion of Theorem 3.2 to hold.

2. **General POMDP:** Suppose A7, A2, A3, A4 hold. Then \( B(u + 1) \succ_B B(u) \) is a sufficient condition for the conclusion of Theorem 3.2 to hold.
Blackwell dominance exploits only the convexity of the value function. In comparison, Lehman precision in Theorem 3.2 exploits both the monotonicity and convexity of the value function. Below we discuss several examples where Blackwell dominance does not hold, but Lehman precision holds.

**Examples.** (i) Here are two examples of the observation matrices that satisfy assumptions A3, A6, A7 implying that the assumptions of statement 1 of Theorem 3.2 hold: \( X = 3, Y = 3, U = 2 \),

\[
\begin{align*}
\text{Ex1. } B(1) &= \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{bmatrix}, & B(2) &= \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.2 & 0.7 & 0.1 \\ 0 & 0.2 & 0.8 \end{bmatrix} \\
\text{Ex2. } B(1) &= \begin{bmatrix} 0.44847 & 0.30706 & 0.24447 \\ 0.33443 & 0.28762 & 0.37795 \\ 0.32463 & 0.28971 & 0.38565 \end{bmatrix}, & B(2) &= \begin{bmatrix} 0.170021 & 0.410485 & 0.419494 \\ 0.106500 & 0.433559 & 0.459941 \\ 0.020739 & 0.263223 & 0.716038 \end{bmatrix}
\end{align*}
\]

Actually for the second example above, A5 also holds implying that statement 1 and statement 2 of Theorem 3.2 hold. Interestingly, in both examples above, \( B(2) \) does not Blackwell dominate \( B(1) \); this illustrates the usefulness of Theorem 3.2 compared to Theorem 3.3.

(ii) Consider a controlled sensing problem with \( X = Y \) arbitrary positive integers, and \( U = 2 \) sensors; choosing either sensor 1 or sensor 2 yields a noisy observation at most one unit different from the Markov state, i.e., \( B(1) \) and \( B(2) \) are tridiagonal matrices. Sensor 1 is more accurate for states 2, \ldots, \( X - 1 \), while sensor 2 is more accurate for states 1 and \( X \). That is, \( B_{ii}(1) = p, B_{i,i+1}(1) = B_{i,i-1}(1) = (1-p)/2, B_{ii}(2) = q, B_{i,i+1}(2) = (1-p)/2 - q \) with the first and last rows as \( B_{11}(1) = B_{XX}(1) = p, B_{12}(1) = B_{XX-1}(1) = 1 - p, B_{11}(2) = B_{XX}(2) > p, B_{12}(2) = B_{XX-1}(2) < 1 - p \). Then A3, A6, A7 hold and so Part 1 of Theorem 3.2 holds. Blackwell dominance does not hold for this example.

(iii) A consequence of [1] is that for symmetric \( 2 \times 2 \) matrices \( B(1), B(2) \), if \( B_{11}(1) \leq B_{11}(2) \), then Blackwell dominance is equivalent to Lehman precision A6. Also A7 automatically holds. This is easy to show, see [3]: \( B(2) >_B B(1) \) since \( L = B^{-1}(2)B(1) \) is a valid stochastic matrix as can be verified by explicit symbolic computation.

4. **Blackwell dominance vs Lehman Precision in Hierarchical Sensing:** A quirk with Blackwell dominance is that the multiplication order matters. If the multiplication order is reversed, i.e., suppose \( B(1) = M \times B(2) \) where \( M \) is a stochastic matrix, then even though \( B(1) \) is still more “noisy” than \( B(2) \), Blackwell dominance (i.e., \( B(1) = B(2) \times L \) where \( L \) is a stochastic matrix) does not necessarily hold. As an example consider

\[
X = 3, Y = 3, U = 2, \quad B(1) = \begin{bmatrix} 0.3229 & 0.4703 & 0.2068 \\ 0.2237 & 0.4902 & 0.2861 \\ 0.1587 & 0.4620 & 0.3793 \end{bmatrix}, \quad B(2) = \begin{bmatrix} 0.4387 & 0.5190 & 0.0423 \\ 0.2455 & 0.6625 & 0.0920 \\ 0.0615 & 0.2829 & 0.6556 \end{bmatrix}
\]

Then there exists a stochastic matrix \( M \) such that \( B(1) = M \times B(2) \) but Blackwell dominance does not hold since \( B(1) \neq B(2) \times L \) for stochastic matrix \( L \). But A3, A6 (Lehmann precision) and A7 hold for this example and therefore statement 1 of Theorem 3.2 holds.

**Controlled Hierarchical Sensing:** In controlled sensing involving hierarchical sensors (such as hierarchical social networks), level \( l \) of the network receives signal \( x_k \) distorted by the confusion matrix \( M^l \) (\( l \)-th power of stochastic matrix \( M \)), where \( l \in \{0, 1, \ldots, U - 1\} \). That is, each level of the network observes a noisy version of the previous level. Observing (polling) level \( l \) of the network has observation probabilities \( B \) conditional on the noisy message at level \( l \). Therefore the conditional
Subject: POMDP Structural Results for Controlled Sensing using Lehmann Precision

Vikram Krishnamurthy

\[ x_k \sim P \]

\[ M \]

\[ B \]

\[ B(3) \]

\[ B \]

\[ B(2) \]

\[ B \]

\[ B(1) \]

Figure 1: Controlled Hierarchical Sensing where Blackwell dominance does not necessarily hold. Level \( l \) of the network receives the Markovian signal \( x_k \) distorted by the confusion matrix \( M^l \). Polling any specific level has observation probabilities \( B \); so the conditional probabilities of \( y \) at level \( l \) given \( x \) is specified by stochastic matrix \( M^l B \).

Even though \( B(u) \) is more noisy than \( B(u + 1) \), Blackwell dominance does not hold (due to the reverse multiplication order). Yet using Lehmann precision, Theorem 3.2 holds (under the stated assumptions).

5. **How does the optimal cumulative reward depend on Lehmann precision?** Consider two controlled sensing POMDPs with model parameters \( \theta = (P, B(1), \ldots, B(U)) \) and \( \bar{\theta} = (P, \bar{B}(1), \ldots, \bar{B}(U)) \) and identical rewards. Let \( \mu^*(\theta) \) and \( \bar{\mu}^*(\bar{\theta}) \) denote the corresponding optimal policies and let \( J_{\mu^*(\theta)}(\pi) \) and \( J_{\bar{\mu}^*(\bar{\theta})}(\pi) \) defined in (1) denote the respective discounted cumulative rewards when using the optimal policies.

**Theorem 3.4.**

1. (Lehmann precision) Suppose \( B(u) >_L \bar{B}(u) \) for \( u \in \{1, \ldots, U\} \) (see A6 for notation) and A1, A2, A3, A7 hold. Then \( J_{\mu^*(\theta)}(\pi) \geq J_{\bar{\mu}^*(\bar{\theta})}(\pi) \).

2. (Blackwell dominance) Suppose \( B(u) >_B \bar{B}(u) \) for \( u \in \{1, \ldots, U\} \). Then \( J_{\mu^*(\theta)}(\pi) \geq J_{\bar{\mu}^*(\bar{\theta})}(\pi) \).

The proof is similar to that of Statement 2 in Section 4.3 and thus omitted. Even though computing the optimal policy of a POMDP is intractable, Theorem 3.4 facilitates comparing the optimal rewards of two different POMDPs with different observation probabilities. Statement (2) deals with the Blackwell dominance case; see [5, Theorem 14.8.1]. It says that in controlled sensing, the optimal reward of a POMDP \( \bar{\theta} \) with nosier observations is smaller than that of the POMDP \( \theta \); this is intuitively obvious.

Statement 1 is more useful than Statement 2 in controlled sensing applications, since Lehmann precision does not necessarily require that \( \bar{\theta} \) has more noisy observations than \( \theta \). In controlled hierarchical sensing discussed above, Statement 1 says that certain networks intrinsically yield lower optimal cumulative reward than others. For example, consider two networks where network 1 has intrinsic confusion matrix \( M \) and network 2 has intrinsic confusion matrix \( M = ML \) for some stochastic matrix \( L \). Then although Blackwell dominance does not hold (due to the reverse multiplication order), Statement 1 says that controlled sensing with network 1 yields a larger cumulative reward (assuming the conditions of Theorem 3.4 hold).
6. **Monotone vectors in value function for** $X \leq 3$. It is well known since [14] that the value function $V_k(\pi) = \arg\max_{\gamma_i^j} \gamma_i^j \pi$ in (4) is piecewise linear and convex in $\pi$ for any finite $k$. Almost 40 years ago, [1] gave conditions under which the elements of each vector $\gamma_i$ are increasing. Unfortunately the conditions in [1] were implicit and not easily verifiable. As an aside, Theorem 4.3 in Sec 4.2 shows that under A1, A2, A3 Albright’s result is true for $X \leq 3$.

4 Proof of Main Result

**Theorem 3.2**

Here is some intuition. Classical convex dominance is defined for scalar convex functions $\phi : \mathbb{R} \to \mathbb{R}$. In a POMDP the value function $V : \Pi(X) \to \mathbb{R}$ and so at first sight is incompatible with convex dominance. So the proof proceeds in two steps. First we work with the value function on certain line segments in the unit simplex (belief space); see Figure 2 for a visual illustration. On each such line segment monotone likelihood ratio dominance becomes a total order and so the value function is convex and increasing. Because of this scalar representation of the belief on each such line, one can use the classical representation of the convex value function as the sum of one-dimensional wedge functions. We then prove convex dominance of the value function in terms of such wedge functions - the key sufficient condition involves the Lehmann precision condition A6. Finally, since any belief (point) in the belief space (unit simplex) lies on one such line, the proof holds for any belief in the simplex.

4.1 Notation and Definitions

**Monotone likelihood ratio dominance and first order dominance** Below $\pi(i)$ denotes the $i$-th element of belief $\pi \in \Pi(X)$. Let $\pi_1, \pi_2 \in \Pi(X)$ denote two beliefs. $\pi_1$ dominates $\pi_2$ with respect to the MLR order, denoted as $\pi_1 \succeq_r \pi_2$, if $\pi_1(i) \pi_2(j) \leq \pi_2(i) \pi_1(j)$ $i < j, i, j \in \{1, \ldots, X\}$. $\pi_1$ dominates $\pi_2$ with respect to first order dominance, denoted as $\pi_1 \succeq_s \pi_2$ if $\sum_{i > j} \pi_1(i) \geq \sum_{i \geq j} \pi_2(i)$ for $j \in \{1, \ldots, X\}$. A function $\phi : \Pi(X) \to \mathbb{R}$ is said to be MLR (resp. first order) increasing if $\pi_1 \succeq_r \pi_2$ (resp. $\pi_1 \succeq_s \pi_2$) implies $\phi(\pi_1) \geq \phi(\pi_2)$.

For state-space dimension $X = 2$, MLR is a complete order and coincides with first order stochastic dominance. For state-space dimension $X > 2$, MLR dominance implies first order dominance. MLR is a partial order, i.e., $[\Pi(X), \succeq_r]$ is a partially ordered set (poset) since it is not always possible to order any two belief states $\pi \in \Pi(X)$. However, on line segments in the simplex defined below (see also Figure 2), MLR is a total ordering; this property is crucial for our proofs below.

Let $e_i, i \in \{1, 2, \ldots, X\}$ denote the unit $X$-dimensional vector with 1 in the $i$-th position. For $i = 1$ and $i = X$, define the sub simplex $\mathcal{H}_i \subset \Pi(X)$ as

$$\mathcal{H}_i = \{ \pi \in \Pi(X) : \pi(i) = 0 \}. \quad (7)$$

Denote belief states that lie in $\mathcal{H}_i$ by $\bar{\pi}$. For each $\bar{\pi} \in \mathcal{H}_i$, construct the line segment $l(e_i, \bar{\pi})$ that connects $\bar{\pi}$ to $e_i$. Thus $l(e_i, \bar{\pi})$ comprises of belief states $\pi$ of the form:

$$l(e_i, \bar{\pi}) = \{ \pi \in \Pi(X) : \pi = (1 - \epsilon) \bar{\pi} + \epsilon e_i, 0 \leq \epsilon \leq 1 \}, \bar{\pi} \in \mathcal{H}_i. \quad (8)$$

This is why structural results which exploit convexity in POMDPs dating back to [1] work with two state POMDPs.
Figure 2: Illustration of line segments \( l(e_X, \bar{\pi}) \) when \( X = 3 \). The belief space \( \Pi(3) \) lies in an equilateral triangle (2-dimensional unit simplex) with vertices \( e_1 = [1, 0, 0]' \), \( e_2 = [0, 1, 0]' \) and \( e_3 = [0, 0, 1]' \). Any belief \( \pi \in \Pi(3) \) lies on one such dotted line \( l(e_3, \bar{\pi}) \) where belief \( \bar{\pi} = [\pi(1)/(1 - \pi(3)), \pi(2)/(1 - \pi(3))], 0]' \) lies on the hyperplane \( \mathcal{H}_3 \) opposite \( e_3 \). On each line segment \( l(e_3, \bar{\pi}) \) MLR dominance is a total order. Theorem 4.3 shows that the value function is convex and increasing on each such line segment. Theorem 4.5 shows convex dominance on each such line segment; thereby establishing the main result Theorem 3.2.

**Definition 4.1** (MLR ordering \( \succeq_{L_i} \) on lines). \( \pi_1 \) is greater than \( \pi_2 \) with respect to the MLR ordering on the line \( l(e_i, \bar{\pi}) \) — denoted as \( \pi_1 \succeq_{L_i} \pi_2 \), if \( \pi_1, \pi_2 \in l(e_i, \bar{\pi}) \) for some \( \bar{\pi} \in \mathcal{H}_i \), i.e., \( \pi_1, \pi_2 \) are on the same line connected to vertex \( e_i \) of simplex \( \Pi(X) \), and \( \pi_1 \geq_{r} \pi_2 \).

Note that \( [\Pi(X), \succeq_{L_X}] \) and \( [\Pi(X), \succeq_{L_1}] \) are chains, i.e., all elements \( \pi, \pi_2 \in l(e_X, \bar{\pi}) \) are comparable, i.e., either \( \pi \succeq_{L_X} \pi_2 \) or \( \pi_2 \succeq_{L_X} \pi \) (and similarly for \( l(e_1, \bar{\pi}) \)). Figure 2 illustrates this. In Lemma 4.2, we summarize useful properties of \( [\Pi(X), \succeq_{L_1}] \) that will be used in our proofs.

**Lemma 4.2.** The following properties hold on \( [\Pi(X), \succeq_{r}], [l(e_X, \bar{\pi}), \succeq_{L_X}] \).

(i) On \( [\Pi(X), \succeq_{r}] \), \( e_1 \) is the least and \( e_X \) is the greatest element. On \( [l(e_X, \bar{\pi}), \succeq_{L_1}] \), \( \bar{\pi} \) is the least and \( e_X \) is the greatest element.

(ii) Convex combinations of MLR comparable belief states form a chain. For any \( \gamma \in [0, 1] \), \( \pi \preceq_{r} \pi_2 \implies \pi \preceq_{r} \gamma \pi + (1 - \gamma) \pi_2 \preceq_{r} \pi_2 \).

(iii) All points on a line \( l(e_X, \bar{\pi}) \) are MLR comparable. Consider any two points \( \pi^{\gamma_1}, \pi^{\gamma_2} \in l(e_X, \bar{\pi}) \). Then \( \gamma_1 \geq \gamma_2 \), implies \( \pi^{\gamma_1} \succeq_{L_i} \pi^{\gamma_2} \).

### 4.2 Three key results

**Theorem 4.3** (Monotone value function). Under A1, A2 and A3

1. The value functions \( V_k(\pi) \) in (4) and \( V(\pi) \) in (3) are MLR increasing and convex on \( \Pi(X) \). Therefore \( V_k(\pi) \) and \( V(\pi) \) are increasing and convex on each line \( l(e_X, \bar{\pi}) \).

2. (a) For any finite \( k \), the value function \( V_k(\pi) = \max_{i \in L_k} \gamma_i^L_k \pi \) in (4) is piecewise linear and convex.

   (b) The vector \( \gamma_{ik} = [\gamma_{ik}(1), \ldots, \gamma_{ik}(X)]' \) satisfies: \( \gamma_{ik}(1) \leq \gamma_{ik}(j), j \in \{2, \ldots, X - 1\} \leq \gamma_{ik}(X) \). Therefore, for \( X \leq 3 \), each vector \( \gamma_{ik} \) has increasing elements.

3. On any line \( l(\bar{\pi}, e_X) \) the value function is of the form

\[
V_k(\pi) = \sum_{i=1}^{n} \max(\alpha_i e_X' \pi - f_i, 0), \quad \pi \in l(e_X, \bar{\pi}) \tag{9}
\]

\(^4\)A chain is totally ordered subset of a partially ordered set.
where \( \alpha_i \geq 0, e_X \) is the unit vector with 1 in the \( X \)-th element, and \( f_i \in \mathbb{R} \).

**Proof.** Proof of Theorem 4.3 Regarding Statement 1, [8] proved that the value function is MLR monotone on \( \Pi(X) \). Convexity of the value function on the belief space goes back to [14]. Therefore, the value function is monotone and convex on each line segment \( l(e_X, \bar{\pi}) \). Statement 2(a) is in [14]. The proof of Statement 2(b) follows from the fact that \( V(\pi) \) is increasing on lines towards \( e_1 \) which implies \( \gamma_{ik}(1) \leq \gamma_{ik}(j), j = 2, \ldots, X \) and also increasing on lines towards \( e_X \) which implies \( \gamma_{ik}(X) \geq \gamma_{ik}(j), j = 1, X - 1 \). For \( X = 3 \) this implies \( \gamma_{ik}(1) \leq \gamma_{ij}(2) \leq \gamma_{ik}(3) \).

The proof of Statement 3 is as follows: Start with Statement 2(a), namely, \( V_k(\pi) = \max_{\pi \in \Pi_k} \gamma_{ik} \bar{\pi} \). Obviously, all beliefs \( \pi \in \Pi(X) \) that lie on each line segment \( l(e_X, \bar{\pi}) \) satisfy the straight line equation

\[
\pi = \pi(X) e_X + (1 - \pi(X)) \bar{\pi}, \quad \pi \in l(e_X, \bar{\pi})
\]

Therefore each piecewise linear segment \( \gamma_{i}' \bar{\pi} \) of the value function on the line \( l(e_X, \bar{\pi}) \) has the form

\[
\gamma_{i}' \bar{\pi} = \gamma_{i} \bar{\pi} + \pi(X) \left( \gamma_{i} - \gamma_{i}' \bar{\pi} \right)
\]

implying that for \( \pi \in l(e_X, \bar{\pi}) \), the value function \( V_k(\pi) \) has the explicit representation

\[
V_k(\pi) = \max_{i \in I_k} \gamma_{i}' \bar{\pi} + \pi(X) \left( \gamma_{i} - \gamma_{i}' \bar{\pi} \right), \quad (10)
\]

in terms of the scalar variable \( \pi(X) \in [0, 1] \). Statement 1 showed that \( V_k(\pi) \) on each such line \( l(e_X, \bar{\pi}) \) is increasing and convex. Next, any increasing convex function on a line (i.e., a convex function that maps \( \mathbb{R} \) to \( \mathbb{R} \)) is the maximum of a countable set of increasing linear (wedge) functions; see [10, Theorem 1.5.7]. Therefore, given the explicit representation (10) of \( V_k(\pi) \) in terms of the scalar variable \( \pi(X) \) for \( \pi \in l(e_X, \bar{\pi}) \), it follows that for sufficiently large \( n \),

\[
V_k(\pi) = \sum_{i=1}^{n} \max(\alpha_i \pi(X) - f_i, 0), \quad \pi(X) \in [0, 1],
\]

for some constants \( \alpha_i \geq 0, f_i \in \mathbb{R} \). Equivalently,

\[
V_k(\pi) = \sum_{i=1}^{n} \max(\alpha_i e_X' \pi - f_i, 0), \quad \pi \in l(e_X, \bar{\pi}).
\]

The following result is required for establishing our main result when \( \mathcal{Y} \) is either finite or has finite support. \( \text{A7} \) is the crucial assumption here.

**Theorem 4.4** (Finite support observation distributions). Suppose \( \mathcal{Y} = [a, b] \). Assume \( \text{A2} \text{ A3} \text{ A7} \)
Then \( \{e_X' T(\pi, y, u), y \in \mathcal{Y}\} \subseteq \{e_X' T(\pi, y, u + 1), y \in \mathcal{Y}\} \).

**Proof.** Proof of Theorem 4.4 Since \( T(\pi, y, u) \uparrow y \) under \( \text{A3} \) and \( \uparrow \pi \) under \( \text{A2} \) it suffices to show that

\[
e_X' T(\pi, a, u + 1) \leq e_X' T(\pi, a, u), \quad \text{and} \quad e_X' T(\pi, b, u + 1) \geq e_X' T(\pi, b, u) \quad (11)
\]

The first inequality in (11) is equivalent to \( \frac{1}{B_{X,a}(u)} P_{X}^{\pi} \leq \frac{1}{B_{X,a}(u+1)} P_{X}^{\pi} \). Since the numerators are convex combinations of \( B_{ia}(u) \) and \( B_{ia}(u+1), i = 1, \ldots, X \), respectively, \( \text{A7} \) is a sufficient condition for the inequality to hold. A similar proof holds for the second inequality in (11).
\textbf{Theorem 4.5} (Convex dominance for controlled sensing POMDP). Suppose $P(u)$ is functionally independent of $u$. Assume A$\Box$ A$\Box$ A$\Box$ Then the following convex dominance holds for $\alpha > 0$:

$$
\sum_{y \in \mathcal{Y}} (\alpha e'_{X} T(\pi, y, u)) - f(\pi, y, u) \uparrow u
$$

\textit{Proof.} Proof of Theorem 4.5 For notational convenience assume the actions are $u = 1, 2$. Also since\footnote{If $\alpha = 0$, the result holds trivially and there is nothing to prove.} $\alpha > 0$, dividing through by $\alpha$, we need to prove that for $\lambda \in \mathbb{R}$,

$$
\psi(\lambda) \overset{\text{defn}}{=} \sum_{y \in \mathcal{Y}^\lambda} [e'_{X} T(\pi, y, 2) - \lambda] \sigma(\pi, y, 2) - \sum_{y \in \mathcal{Y}^\lambda} [e'_{X} T(\pi, y, 1) - \lambda] \sigma(\pi, y, 1)
$$

where $\mathcal{Y}^\lambda = \{ y : e'_{X} T(\pi, y, u) > \lambda \}$, $u = 1, 2$. Note for $\lambda > 1$ clearly $\mathcal{Y}^\lambda = \emptyset$ since $e'_{X} T(\pi, y, u)$ is the last component of the updated belief; and therefore $\psi(\lambda) = 0$ for $\lambda \geq 1$. Also, for $\lambda \leq 0$, $\mathcal{Y}^\lambda = \mathcal{Y}$ and so $\psi(\lambda) = 0$ for $\lambda < 0$. So we only need to prove $\psi(\lambda) \geq 0$ for $\lambda \in (0, 1)$.

\textbf{Case 1.} $\mathcal{Y} = \mathbb{R}$: Denote $\mathcal{Y}^\lambda_u = \mathcal{Y} - \mathcal{Y}^\lambda$ for $u = 1, 2$. By A$\Box$ $T(\pi, y, u) \uparrow y$ wrt MLR order. So $e'_{X} T(\pi, y, u)$ is a non-empty.

Define $y^*_u = \inf \{ y : e'_{X} T(\pi, y, u) = \lambda \}$. Therefore $\mathcal{Y}^\lambda_u = (y^*_u, \infty)$ for some $y^*_u \in \mathbb{R}$ and the complement set $\mathcal{Y}^\lambda_u = (-\infty, y^*_u]$. By absolute continuity condition A$\Box$ for $\lambda \in (0, 1)$, $\mathcal{Y}^\lambda_u$ is non-empty.

We establish (13) for $\lambda \in (0, 1)$ by showing\footnote{If $B_{iy}(u)$ is discontinuous in $y$ then choose $y^*_u = \sup \{ y : e'_{X} T(\pi, y, u) \leq \lambda \}$ and assign $e'_{X} T(\pi, y^*_u, u) = \lambda$; since $y^*_u$ has measure zero it does not affect the optimal policy.} that $\psi(\lambda^*) \geq 0$ at all stationary points $\lambda^*$ such that $d\psi(\lambda)/d\lambda = 0$. Note that

$$
\psi(\lambda) = \sum_{y \in \mathcal{Y}^\lambda} \left[ e'_{X} B_y(2) P' \pi - \lambda 1' B_y(2) P' \pi \right] - \sum_{y \in \mathcal{Y}^\lambda} \left[ e'_{X} B_y(1) P' \pi - \lambda 1' B_y(1) P' \pi \right]
$$

$$
= (e'_{X} - \lambda 1)' \left[ \sum_{y \in \mathcal{Y}^\lambda} B_y(2) - \sum_{y \in \mathcal{Y}^\lambda} B_y(1) \right] P' \pi
$$

$$
= (e'_{X} - \lambda 1)' \left[ \sum_{y \in \mathcal{Y}^\lambda} B_y(1) - \sum_{y \in \mathcal{Y}^\lambda} B_y(2) \right] P' \pi
$$

$$
= \sum_{i=1}^{X} \frac{(e'_{X}(i) - \lambda)}{\alpha_i} \sgn \left[ \sum_{y \in \mathcal{Y}^\lambda_1} B_{iy}(1) - \sum_{y \in \mathcal{Y}^\lambda_2} B_{iy}(2) \right] \sum_{y \in \mathcal{Y}^\lambda_1} B_{iy}(1) - \sum_{y \in \mathcal{Y}^\lambda_2} B_{iy}(2) \right] (P' \pi)_i
$$

(14)

Let us next evaluate the stationary points of $\psi(\lambda)$ for $\lambda \in (0, 1)$.

\footnote{Since $\psi(0) = \psi(1) = 0$, clearly if $\psi(\lambda) \geq 0$ at its stationary points (minima), then $\psi(\lambda) \geq 0$ for all $\lambda \in [0, 1]$.}
Lemma 4.6. For $\psi(\lambda)$ defined in (13), the gradient wrt $\lambda \in (0, 1)$ is
\[
\frac{d\psi(\lambda)}{d\lambda} = -1' \left[ \sum_{y \in \mathcal{Y}_1^\lambda} B_y(1) - \sum_{y \in \mathcal{Y}_2^\lambda} B_y(2) \right] P' \pi
\]  
(15)

(Proof at the end of this subsection).

Thus the stationary points of $\psi(\lambda)$ satisfy
\[
\frac{d\psi(\lambda)}{d\lambda} = 1' \left[ \sum_{y \in \mathcal{Y}_1^\lambda} B_y(1) - \sum_{y \in \mathcal{Y}_2^\lambda} B_y(2) \right] P' \pi = \sum_i \beta_i p_i = 0.
\]  
(16)

So it only remains to show that $\psi(\lambda)$ is non-negative at these stationary points. To establish this we use the FKG (Fortuin-Kasteleyn-Ginibre) inequality on (14). In our framework the FKG inequality reads: If $\alpha$, $\beta$ are generic increasing vectors and $p$ a generic probability mass function, then
\[
\sum_i \alpha_i \beta_i p_i \geq \sum_i \alpha_i p_i \sum_i \beta_i p_i.
\]

Clearly in (14), $\alpha_i$ is increasing since the elements $(\epsilon_X - \lambda 1)$ are increasing; $\beta_i$ is increasing by A6. $p_i$ is non-negative and thus proportional to a probability mass function. Also from (15), $\sum_i \beta_i p_i = 0$. So, applying FKG inequality to (14) yields $\psi(\lambda) = \sum_i \alpha_i \beta_i p_i \geq 0$. Thus we have established (12) for $\mathcal{Y} = \mathbb{R}$.

Case 2. $\mathcal{Y} = [a, b]$: Next we prove (12) for the finite support case where $\mathcal{Y}$ is the interval $[a, b]$. The key difference compared to the case $\mathcal{Y} = \mathbb{R}$ is that it is possible (if appropriate assumptions are not made) in (13) that $\mathcal{Y}_2^\lambda = \emptyset$ and $\mathcal{Y}_1^\lambda$ is non-empty which would make $\psi(\lambda)$ defined in (13) negative. Assumption A7 along with Theorem 4.4 prevents this from happening. Indeed, from Theorem 4.4 A2 A3 A7 imply that there are three possibilities: (i) $\mathcal{Y}_2^\lambda = \emptyset$ and $\mathcal{Y}_1^\lambda = \emptyset$: clearly $\psi(\lambda) = 0$. (ii) $\mathcal{Y}_2^\lambda \neq \emptyset$ and $\mathcal{Y}_1^\lambda = \emptyset$: clearly from (13), $\psi(\lambda) \geq 0$. (iii) $\mathcal{Y}_1^\lambda$ and $\mathcal{Y}_2^\lambda$ are both non-empty. The proof for this case follows exactly as in the proof for $\mathcal{Y} = \mathbb{R}$ above. (Theorem 4.4 implies $\mathcal{Y}_2^\lambda = \emptyset$ and $\mathcal{Y}_1^\lambda \neq \emptyset$ is impossible.)

Case 3. $\mathcal{Y}$ is finite: Finally, we prove (12) for the case $\mathcal{Y} = \{1, 2, \ldots, Y\}$. Construct the piecewise constant probability density function $O_{io} = B_{iy}$ for $o \in [y, y + 1)$ and $y \in \{1, 2, \ldots, Y\}$. It is easily seen that $T(\pi, o, u) = T(\pi, y, u)$ and the value function and optimal policy remain unchanged. Then the above proof for the finite support case applies. \hfill \Box

Proof. Proof of Lemma 4.6 Here we prove Lemma 4.6 that was used to evaluate the gradient of $\psi(\lambda)$ in the proof above. For $t \in \mathbb{R}$, define $\mathcal{Y}_u^t = \{ y : e'_X T(\pi, y, u) > t \}$, $u = 1, 2$. Start with (13), and noting that $\sum_y |e'_X T(\pi, y, u) - \lambda|^+ \sigma(\pi, y, u) = \int_{-\infty}^{\lambda} |t-\lambda|^+ \sum_y I(e'_X T(\pi, y, u) \geq t) dt$, we have
\[
\psi(\lambda) = \int_{-\infty}^{\lambda} |t-\lambda|^+ \left[ \sum_{y \in \mathcal{Y}_2^\lambda} \sigma(\pi, y, 2) - \sum_{y \in \mathcal{Y}_1^\lambda} \sigma(\pi, y, 1) \right] dt = \int_{-\infty}^{\lambda} 1' \left[ \sum_{y \in \mathcal{Y}_2^\lambda} B_y(1) - \sum_{y \in \mathcal{Y}_1^\lambda} B_y(2) \right] P' \pi dt
\]
where the second equality follows since $\int_{-\infty}^{\lambda} f(t)g(t) dt = f(\infty)g(\infty) - f(\lambda)g(\lambda) - \int_{\lambda}^{\infty} g(x) df(x)$ for generic $f, g$. Then evaluating $d\psi(\lambda)/d\lambda$ completes the proof.
A more intuitive proof involving Dirac delta (generalized) functions is as follows: From (14),

\[
\frac{d\psi(\lambda)}{d\lambda} = -1' \left[ \sum_{y \in Y_1^l} B_y(1) - \sum_{y \in Y_2^l} B_y(2) \right] P' \pi
\]

\[+ (e_X - \lambda 1')' \left[ \sum_{y \in Y} \delta(\lambda - e'_X T(\pi, y^*_\lambda, 1)) B_y(1) - \sum_{y \in Y} \delta(\lambda - e'_X T(\pi, y^*_\lambda, 2)) B_y(2) \right] P' \pi\]

where \(\delta(\lambda - e'_X T(\pi, y^*_\lambda, u))\) denotes the Dirac delta function centered at \(e'_X T(\pi, y^*_\lambda, u)\). Next note that

\[(e_X - \lambda 1')' \sum_{y \in Y} \delta(\lambda - e'_X T(\pi, y^*_\lambda, u)) B_y(u) P' \pi = (e_X - e'_X T(\pi, y^*_\lambda, u) 1') B_y^*(u) P' \pi = 0\]

so that the second line of (17) vanishes.

### 4.3 Proof of Theorem 3.2

With Theorems 4.3 and 4.5 we can now complete the proof\(^8\).

**Statement 1 (Controlled Sensing).** Assuming A1, A2 and A3, the result (9) yields for all \(\pi \in l(e_X, \bar{\pi})\),

\[
\sum_{y \in Y} V_k(T(\pi, y, u)) \sigma(\pi, y, u) = \sum_{i=1}^{n} \sum_{y \in Y} \max(\alpha_i e'_X T(\pi, y, u) - f_i, 0) \sigma(\pi, y, u)
\]

Assuming A3, A6, A7 it follows from Theorem 4.5 that each term \(\sum_{y \in Y} \max(\alpha_i e'_X T(\pi, y, u) - f_i, 0) \sigma(\pi, y, u) \uparrow u\). This implies \(\sum_{i=1}^{n} \sum_{y \in Y} \max(\alpha_i e'_X T(\pi, y, u) - f_i, 0) \sigma(\pi, y, u) \uparrow u\). We have thus proved that

\[
\sum_{y} V_k(T(\pi, y, u + 1)) \sigma(\pi, y, u + 1) \geq \sum_{y} V_k(T(\pi, y, u)) \sigma(\pi, y, u)
\]

or equivalently, in terms of the notation in (4), \(Q_k(\pi, u + 1) - Q_k(\pi, u) \geq r'_{u+1} \mu^* - r'_u \mu^*\). Therefore \(r'_{u+1} \mu^* \geq r'_u \mu^* \implies \mu^*(\pi) = u + 1\), i.e., \(\mu^*_k(\pi) \geq \mu^*_k(\pi)\) for all \(\pi \in l(e_X, \bar{\pi})\). Finally, any belief \(\pi \in \Pi(X)\) lies on one such line segment \(l(e_X, \bar{\pi}) = \{\pi : \pi = (1 - \epsilon) \bar{\pi} + \epsilon e_X\}\) where explicitly, \(\epsilon = \pi(X)\) and \(\pi(i) = \pi(i)/(1 - \pi(X))\), \(i = 1, \ldots, X - 1\). Therefore, \(\mu^*_k(\pi) \geq \mu^*_k(\pi)\) for each \(\pi \in \Pi(X)\). Finally, for the infinite horizon discounted case, the value iteration algorithm (4) converges uniformly; that is, \(V_k(\pi)\) converges uniformly to \(V(\pi)\) on \(\Pi(X)\), so the results hold for \(V(\pi)\).

**Statement 2 (General POMDP).** To simplify notation, assume \(u \in U = \{1, 2\}\). With \(V(\pi)\) denoting the value function of the POMDP, recall that for action \(u = 1\), the POMDP parameters are

\(^8\)Recall A1 is not required for controlled sensing since A1' automatically holds; we mention it here for the general POMDP proof.
$P(1), B(1)$ and for action $u = 2$, the parameters are $P(2), B(2)$. Define the fictitious action $u = a$ with parameters $P(1), B(2)$. Then Statement 1 implies that under $A_1, A_2, A_3, A_6, A_7$ that

$$
\sum_y V(T(\pi, y, 1)) \sigma(\pi, y, 1) \leq \sum_y V(T(\pi, y, a)) \sigma(\pi, y, a)
$$

(18)

since actions 2 and $a$ have the same transition matrix. Also under copositive dominance $A_4$ $T(\pi, y, a) \leq_r T(\pi, y, 2)$. From Theorem 4.3 $V(\pi)$ is MLR increasing implying that $V(T(\pi, y, a)) \leq V(T(\pi, y, 2))$. Finally, $A_2, A_5$ imply that $\sigma(\pi, \cdot, a) \leq_s \sigma(\pi, \cdot, 2)$. Therefore,

$$
\sum_y V(T(\pi, y, a)) \sigma(\pi, y, a) \leq \sum_y V(T(\pi, y, 2)) \sigma(\pi, y, a) \leq \sum_y V(T(\pi, y, 2)) \sigma(\pi, y, 2)
$$

Combining this with (18) proves the result.

References

[1] S. Albright. Structural results for partially observed Markov decision processes. *Operations Research*, 27(5):1041–1053, Sept.-Oct. 1979.

[2] S. Athey and J. Levin. The value of information in monotone decision problems. *Research in Economics*, 72:101–116, 2018.

[3] J.-J. Ganuza and J. S. Penalva. Signal orderings based on dispersion and the supply of private information in auctions. *Econometrica*, 78(3):1007–1030, 2010.

[4] I. Jewitt. Information order in decision and agency problems. *Nuffield College*, 2007.

[5] V. Krishnamurthy. *Partially Observed Markov Decision Processes. From Filtering to Controlled Sensing*. Cambridge University Press, 2016.

[6] V. Krishnamurthy and U. Pareek. Myopic bounds for optimal policy of POMDPs: An extension of Lovejoy’s structural results. *Operations Research*, 62(2):428–434, 2015.

[7] E. L. Lehmann. Comparing location experiments. *Annals of Statistics*, 16(2):521–533, 1988.

[8] W. S. Lovejoy. Some monotonicity results for partially observed Markov decision processes. *Operations Research*, 35(5):736–743, Sept.-Oct. 1987.

[9] T. Mizuno. A relation between positive dependence of signal and the variability of conditional expectation given signal. *Journal of applied probability*, 43(4):1181–1185, 2006.

[10] A. Muller and D. Stoyan. *Comparison Methods for Stochastic Models and Risk*. Wiley, 2002.

[11] C. H. Papadimitriou and J. Tsitsiklis. The complexity of Markov decision processes. *Mathematics of Operations Research*, 12(3):441–450, 1987.

[12] U. Rieder. Structural results for partially observed control models. *Methods and Models of Operations Research*, 35(6):473–490, 1991.
[13] U. Rieder and R. Zagst. Monotonicity and bounds for convex stochastic control models. *Mathematical Methods of Operations Research*, 39(2):187–207, June 1994.

[14] E. J. Sondik. *The optimal control of partially observed Markov processes*. PhD thesis, Electrical Engineering, Stanford University, 1971.

[15] C. C. White and D. P. Harrington. Application of Jensen’s inequality to adaptive suboptimal design. *Journal of Optimization Theory and Applications*, 32(1):89–99, 1980.