Incremental embedding for temporal networks

Tomasz Kajdanowicz¹, Kamil Tagowski¹, Maciej Falkiewicz¹, Piotr Bielak¹, Przemysław Kazienko¹, and Nitesh V. Chawla¹,²

¹ Department of Computational Intelligence, Wrocław University of Science and Technology, Poland
² Department of Computer Science and Engineering, University of Notre Dame, Notre Dame, IN, USA

Abstract. Prediction over edges and nodes in graphs requires appropriate and efficiently achieved data representation. Recent research on representation learning for dynamic networks resulted in a significant progress. However, the more precise and accurate methods, the greater computational and memory complexity. Here, we introduce ICMEN – the first-in-class incremental meta-embedding method that produces vector representations of nodes respecting temporal dependencies in the graph. ICMEN efficiently constructs nodes’ embedding from historical representations by linearly convex combinations making the process less memory demanding than state-of-the-art embedding algorithms. The method is capable of constructing representation for inactive and new nodes without a need to re-embed.

The results of link prediction on several real-world datasets shown that applying ICMEN incremental meta-method to any base embedding approach, we receive similar results and save memory and computational power. Taken together, our work proposes a new way of efficient online representation learning in dynamic complex networks.

Keywords: Representation learning · Temporal graph embedding · Incremental network embedding

1 Introduction

Many vital tasks in network analysis involve prediction over edges and nodes. For instance in node classification, we aim at predicting most likely node’s label. In a social network, this might be an interest or preference of the user, or in citation network, the research area the paper belongs to [7]. In link prediction, we intend to model the existence of a link between pair of nodes. Predicted links in social networks may denote real-life friends and in citation networks related but unmentioned references.

It is recently recognized, that the majority of real-world networks are naturally dynamic. It means they evolve over time and nodes as well as links can appear or disappear. We know so far, that considering temporal information about the network allows their better understanding and modeling [9–10], especially for supervised machine learning tasks.
The classical approach to machine learning requires discriminating and independent features. Vector representations in node classification or link prediction are obtained from some transformation of the graph. Historically typical approaches were using domain-specific expert knowledge to construct hand-made features or just adopted some of the structural properties. However, it does not generalize well across different prediction tasks and usually it strongly depends on the domain. Learning the representation by solving some optimization problem is an alternative.

While learning representation in networks one must find a trade-off in computational and memory complexity as well as generalization ability. This is especially important while dealing with extensively large networks with long history. Due to our best knowledge majority of existing state-of-the-art methods requires access to all edges that appeared in the network to obtain a 'full' vector representation. It causes difficulties in efficient representation learning in terms of memory and computation power requirements. Moreover, existing methods may not be able to provide representation for nodes that appeared after the learning time. Performing incremental learning would be a solution.

Overall, there are two main approaches to how dynamic networks composed of event stream data is processed to keep up-to-date nodes' representation: either (1) computation at each new communication event or (2) processing increments of a given size (number of events, from a period). We will mainly consider the latter setting in this paper.

**Present work.** In this work, we address the problem of learning the network representation in an incremental manner. We introduce ICMEN – an incremental meta-embedding method that produces vector representations of nodes respecting temporal dependencies in the network.

Our key contribution is in providing a meta-learning that can utilize any of existing graph embedding methods. Moreover, it is fully incremental allowing representation learning for much larger dynamic graphs than the previous methods. Our method is capable of working in online manner, where nodes’ representation is obtained from their previous representations updated by the most recent node activities in the network (increments) and calibrated to increase representation stability. We achieve this by developing a linear convex combination method that calculates representation for nodes from their historical representations from previous periods. We propose an algorithm for learning of combination parameters using Dirichlet-Multinomial model. The resulting method is flexible, giving us accurate and fully automatic results of how combined nodes’ representation should look like. The algorithm benefits from important property of embedding methods that are constructed with some notion of random walks and approximate probabilities with the softmax. We have observed that the correlations of pair-wise distances of nodes’ representations are invariant with respect to repeated embedding learning for the same network. Obviously, they are numerically different due to the random nature of walks, initialization and imperfect optimization algorithms. Nevertheless, representations obtained from repeated embeddings are geometrically congruent and scaled, which means the
figures obtained are linear transformations of others (have the same shape and size after scaling, rotation and reflection). We aim at revealing these transformations by providing calibration mechanism. Going further, we hypothesized that for networks, that have gradual changes in connection patterns, representation of networks built from events for the following periods (connection data increments) should hold the same property. This allowed us to propose the idea to combine representations for consecutive increments by linearly convex folding keeping similar generalization ability to represent the whole data. This direction is fully supported by the recent findings about real-world dynamic social networks, which does not show sudden concept drifts in communication patterns and are rather gradual in changes [17].

Our experiments focus on common prediction task in networks - link prediction, where we aim at predicting the existence of an edge given a pair of nodes. We compare the results of ICMEN performance obtained in incremental manner to the results trained on the whole data at once. The procedure was performed for two state-of-the-art base node embedding methods: node2vec [4] and CTDNE [10]. We experiment on several dynamic real-world networks from diverse domains. Evaluations demonstrate that ICMEN does not provide significantly worse results compared to the non-incremental setting. The algorithm shows even competitive performance for datasets that are dynamic and require representation reflecting recent connection patterns. Computationally, some parts of ICMEN are parallelizable, and it can scale to large networks.

Overall, the paper provides the following contributions:

1. We propose a new incremental embedding meta-method (ICMEN) that efficiently constructs nodes’ representation from historical embeddings (provided by any method) by linearly convex combinations.
2. We demonstrate the fully incremental manner of the method that is capable of constructing representation for new nodes appearing in time without a need to re-embed the old nodes as well as provide embeddings for temporally inactive nodes.
3. We show how ICMEN calibration mechanism is in accordance with network dynamics showing its ability to steadily track proper representation of nodes over time without access to historical source data and having negligible computational and memory complexity.
4. The generalized version of ICMEN can aggregate embeddings for multiple increments at once dynamically adjusting their importance.
5. We present the ability of the method to extend from nodes to pairs of nodes for edge-based prediction tasks by means of Hadamard operator.
6. We empirically evaluate ICMEN for link prediction on several real-world datasets showing that making a particular embedding method incremental using ICMEN, we receive similar results.

The rest of the paper is structured as follows. The literature review is presented in Sec. 2. The framework of incremental dynamic graph embedding and technical details are shown in Sec. 3. The empirical evaluation of the ICMEN method for link prediction task on five real-world networks can be found in Sec. 4 along
with execution time analysis. We conclude and discuss the framework as well as highlight some promising directions for future research in Sec. 5. A reference implementation of the ICMEN method are available on the project repository: https://gitlab.com/tgem/incremental

2 Related Work

Machine learning research community has been working extensively on feature engineering for many years. Complex structures like networks, large datasets and their dynamics are among the most vital challenges in this domain. The mainstream research direction is to transfer (embed) all of them into representative node vectors in an efficient way required in real-world applications.

Table 1. Comparison of incremental embedding methods for graphs and their time and memory complexity analysis. STAT, DYN – produces embeddings for static or dynamic graphs. IN-S, IN-N/I – provides incremental learning for the same set of nodes or also for new and inactive nodes. PAR – can be parallelized. META – is meta method, $d$ – embedding dimensionality, $|V|$, $|E|$, $I$ – number of vertices, edges, iterations, respectively, $\gamma$, $\omega$ – number and length of random walks, $L$ – number of negative samples.

| Method          | STAT | DYN | IN-S | IN-N/I | PAR | META | Time complexity | Memory complex |
|-----------------|------|-----|------|--------|-----|------|----------------|----------------|
| DeepWalk’14     | ✓    | ✓   | ✓    | ✓      | ✓   | ✓    | $O(d|V|\log|V|)$ | $O(\gamma\omega|E|)$ |
| LINE’15         | ✓    | ✓   | ✓    | ×      | ✓   | ✓    | $O(d|E|)$      | $O(|E|)$        |
| Node2vec’16     | ✓    | ✓   | ✓    | ✓      | ✓   | ✓    | $O(d|V|)$      | $O(\gamma\omega|E|)$ |
| M-NMF’17        | ◌    | ✓   | ✓    | ✓      | ✓   | ✓    | $O(d|V|^2)$    | $O(|E|)$        |
| CTDNE’18        | ×    | ✓   | ✓    | ✓      | ×   | ✓    | $O(d|V|)$      | $O(\gamma\omega|E|)$ |
| DynGEM’18       | ✓    | ✓   | ✓    | ✓      | ✓   | ✓    | $O(d|V|)$      | $O(\gamma\omega|E|)$ |
| DynGraph2vec’18 | ◌    | ✓   | ✓    | ✓      | ×   | ×    | autoencoder    | autoencoder/lstm |
| Node2vec-dynlink’18 | × | ✓  | ✓    | ✓      | ✓   | ✓    | $O(d|V|)$      | $O(\gamma\omega|E|)$ |
| HTNE’18         | ×    | ✓   | ✓    | ✓      | ×   | ✓    | $O(d|V|)$      | $O(|E|)$        |
| ICMEN (our) ‘19 | ✓    | ✓   | ✓    | ✓      | ✓   | ✓    | $O(d^2|V|)$    | $O(d|V|)$       |

A classical approach to node embedding is DeepWalk [11] that learns node representations using neural networks. It was based on random walks in graphs, which later were fed into the skipgram model. The values from the hidden layers were directly used as embeddings - node vectors. In order to overcome high computational cost of softmax, the authors used hierarchical softmax. The model captures the 2nd order node proximity in the graph. The LINE algorithm [14] addresses the efficiency problem since many solutions are suitable only for small graphs. It is based on an objective function that considers both local (1st and 2nd order node proximity’s) and global network structure. To improve scalability, an edge sampling and asynchronous stochastic gradient descend together with negative sampling were used. Node2vec [11] is an important modification of DeepWalk. The random walks are parametrized here by $p$ and $q$ to control both
DFS- and BFS-like behavior. Also hierarchical softmax is replaced by negative sampling. In M-NMF, matrix factorization was used to reflect both microscopic and mesoscopic structures – communities [15]. All previous methods operate only on static graphs. To respect dynamics of temporal networks, CTDNE – the Continuous-time Dynamic Network Embedding was introduced [10]. It relies on temporal random walks that preserve a time specific order of edges: the timestamp of any edge has to follow the timestamps of the previous edges in the walk sequence. The recent DynGEM model for embedding of temporal graphs [5] was designed for better time complexity than the ones for static graphs. It estimates the node embeddings of the graph snapshot at timestep $t$ based on the embeddings from the preceding step $t-1$. The model is based on a deep autoencoder, whose weights are initialized with its previous weights, so it only works with its own embedding method. It also handles the new nodes using a custom layer expansion algorithm (PropSize). The authors improved their model and proposed dyngraph2vec [6]. Also here, deep learning is used, but instead of relying only on an autoencoder model also LSTM cells and a fusion of both is considered. Additionally, it is possible to consider multiple graph snapshots at once – the input of the models is defined as a list of snapshots: $G_1, ..., G_t$ and the output – the predicted graph snapshot $G_{t+1}$. The hidden layers are the latent representations of the last snapshot $G_{t+1}$. The n2v-dyn-link method is a combination of node2vec and temporal random walks [18]. It considers two approaches. The first is a static embedding: the graph is divided into multiple snapshots and the vanilla node2vec embedding is applied to each. The final representation is a concatenation of all subsequent embeddings. This is hardly scalable: the dimensionality of the target embedding may be too high. The node2vec model is extended with temporal random walks in the second approach. The authors added all nodes from the test to train graph to handle the representations for the nodes that do not exist in the train data. The newest solution HTNE [19] uses the concept of a Hawkes process [6] to model a node’s neighbourhood. The method is an extension of the well-known skip-gram model where the distance between vectors is calculated as intensity of the stochastic process. The background intensity is just the negative squared Euclidean distance and the excitation function is modeled by a combination of softmax and exponential kernel function. Due to the explicit modelling of the neighborhood, the method does not require generating random walks, instead it takes a time-attributed sequence of events for each node.

3 Incremental Embedding Framework

We model the dynamic network $G = \langle G_{0:n} \rangle$ as a sequence of $n$ ($n \geq 2$) discrete graph snapshots $G_{0,1}, G_{1,2}, ..., G_{n-1,n}$ ordered in time, where $G_{t-1,t} = (V_{t-1,t}, E_{t-1,t})$, is a graph increment (snapshot) which contains all the vertices $V_{t-1,t}$ and edges $E_{t-1,t}$ that appeared in the increment source data in the given period $[t-1, t)$. All graphs in the sequence preserve node indexing, i.e. we are able to match nodes from any two increments that correspond to each other, if they occur in these increments. Similarly, we define a sequence of related
node embeddings $F_{0,1}, F_{1,2}, ..., F_{n-1,n}$, where $F_{t-1,t}$ is the embedding calculated independently for $G_{t-1,t}$ using its source data. The node embedding $F_{t-1,t}$ is a matrix of size $|V_{t-1,t}| \times d$, where each row contains a $d$-dimensional embedding for a single network node.

The problem to solve is to find the embedding $F_{0,n}$ of the full graph $G_{0,n}$, without calculating it from all source data from period $[0, n]$, but based on already computed embeddings for the increments. Since nodes appear and disappear in individual increments $G_{t-1,t}$, we want to embed all nodes that have ever occurred in $G_{0,n}$ to have the final matrix of size $|V| \times d$, where $V = \bigcup_{t=1}^{n} V_{t-1,t}$.

More formally, incremental embedding for dynamic networks is formulated as a linearly convex combination of nodes’ vector representations obtained from successive data increments that build up the dynamic network. We assume that $V$ denotes a node set of dynamic graph $G$, $v \in V$ is a single node, $C(v)$ is a set of all contexts of node $v$ in the skip-gram model generated from random walks performed on graph $G$, $c \in C$ is a single context. $D = (V, C)$ is the set of all nodes and their context pairs: $f_{c(v)}$ and $f_{v(c)} \in \mathbb{R}^d$ vector representations, e.g. extracted from random walks for all $c$ and $v$. The parameters $\Psi$ are $f_{c(v)}, f_{v(c)}$ for $v \in V$ (a total of $|C| \times |V| \times d$ parameters). We consider now the node embedding, following the notation of [2], as conditional probabilities $p(c|v)$.

A population of random walks, the goal is to set the parameters $\Psi$ of $p(c|v; \Psi)$ so as to maximize random walks probability, see Eq. 1.

$$\arg\max_{\Psi} \prod_{v \in V} \left[ \prod_{c \in C(v)} p(c|v; \Psi) \right] = \arg\max_{\Psi} \prod_{(v, c) \in D} p(c|v; \Psi)$$ (1)

The conditional probability in neural-network language allow parameterizing the skip-gram model using softmax:

$$p(c|v, \Psi) = \frac{e^{f_{c(v)}^T f_{v(c)}}}{\sum_{c' \in C} e^{f_{c'(v)}^T f_{v(c')}}},$$ (2)

Where $\cdot$ is the dot product of Euclidean space. Instead of maximizing the likelihood in Eq. 1, we switch to log-likelihood:

$$\arg\max_{\Psi} \sum_{(v, c) \in D} \log p(c|v) = \arg\max_{\Psi} \sum_{(v, c) \in D} \left( \log e^{f_{c(v)}^T f_{v(c)}} - \log \sum_{c' \in C} e^{f_{c'(v)}^T f_{v(c')}} \right)$$ (3)

Let’s now consider the incremental setting of node embedding problem. Assuming that node set $V$ remains the same for all increments, applying ICMEN for a pair of $(n-1, n)$ of increments (represented by $f_{n-1}$ and $f_n$ respectively)
we aim at approximating Eq. 3 with

$$\argmax_{\Psi} \alpha \left( \sum_{(v,c) \in D_{n-1}} \left( \log e^{f_{n-1}(c)} - \log \sum_{c' \in C} e^{f_{n-1}(c')} \right) \right) +$$

$$+ (1 - \alpha) \left( \sum_{(v,c) \in D_n} \left( \log e^{f_n(c)} - \log \sum_{c' \in C} e^{f_n(c')} \right) \right),$$

where $\alpha$ is a convex combination parameter which we show how to fit from data in Sec. 3.3.

We assume that pairs $(v, c)_{n-1}$ and $(v, c)_n$ are not significantly different from each other, i.e. for the same set of $v$ we have similarly related $c$. The proposed approach leads straightforwardly to the capability of incremental learning. When a new portion of data for period $[t - 1, t)$ arrives, it might be used to calculate a new increment representation $f_t$. It is combined with the previously computed embedding $f_{t-1}$ representing all previous periods, i.e., for the entire period $[0, t - 1)$, see Fig. 2(a). It results in the compound embedding representing the dynamic graph for the whole period $[0, n)$.

What is more, the proposed approach can be easily fed by any other node embedding technique instead of softmax over some random walks notion as it was already shown as equivalent to multiple approaches in [12]. In other words, the proposed method uses existing embeddings of graph increments and generates the approximate embedding of the graph built of all subsequent increments.

Due to the randomness, commonly included in the training procedure (e.g. random walks, representation initialization, etc.), for a given graph $G_{t-2, t-1}$, we can obtain multiple realizations of embeddings that are equivalent but not identical. If they capture the same knowledge, then one is approximately a linear combination of another (an exact linear combination would be valid if both embeddings reached global optimum). The same holds for two increments representations $f_{t-1}$ and $f_t$ – we should calibrate them first to apply Eq. 4. For that purpose, we introduce the concept of reference nodes which do not evolve between $G_{t-2, t-1}$ and $G_{t-1, t}$, see Sec. 3.1 and Eq. 8 for details.

### 3.1 Basic Incremental ICMEN Model

We propose a new method for incremental node embedding on temporal graphs – ICMEN (Incremental Convex Meta Embedding for Nodes). To start with the definition of the basic ICMEN algorithm:

$$ICMEN(\alpha),$$

where $\alpha \in [0, 1]$ is a real valued combination coefficient (see below). The rule for the estimation of a new embedding $\tilde{F}_{t-2, t}$ based on two previous ones $F_{t-2, t-1}$ and $F_{t-1, t}$ is defined as follows:

$$\tilde{F}_{t-2, t} = \alpha F_{t-2, t-1} + (1 - \alpha) F_{t-1, t}$$

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As we can see the $\alpha$ parameter defines how much (or how less) the first embedding impacts on the resulting estimate. Typically, $\alpha < 0.5$ to make older data influence less than the newer one. Using this rule to derive the aggregated embedding for two periods, we can define the algorithm for finding the full graph embedding $\tilde{F}_{0,n}$, see Fig. 2(a). Having $\tilde{F}_{0,1} = F_{0,1}$, we can iteratively estimate $\tilde{F}_{0,t}$, for $t = 2$ to $t = n$:

$$\tilde{F}_{0,t} = \alpha \tilde{F}_{0,t-1} + (1 - \alpha)F_{t-1,t}$$ \hspace{1cm} (7)

Note that $\alpha$ is constant for all $t$. Eq. 7 holds for representations that are already calibrated, i.e. embeddings correspond to each other in scale, reflection, rotation and translation. However, if they are not geometrically congruent, which is the case in almost all embedding methods, we propose to apply the procedure of increment calibration. It can be formalized as a search for matrix $A^*$ such that:

$$A^* = \arg\min_A \| \tilde{F}_{0,t-1} - F_{t-1,t}A \|,$$ \hspace{1cm} (8)

where $\| \cdot \|$ is any norm valid for the problem and can be solved by Multi-Output Regression methods. To make the calibration matrix $A^*$ capture the appropriate transformations we must obtain it for a subset of nodes, i.e. reference nodes, that are believed to be time-invariant. Ideally, they should be chosen based on expert knowledge or objective selection procedure. We introduce for that purpose the notion of node scoring function with domain of some temporal node centrality measure. This function quantifies to what extent a given node has changed between $G_{0,t-1}$ and $G_{t-1,t}$. Finally, we construct nodes ranking accordingly and select top $N$ vertices as reference. The size of the reference nodes set is crucial. In order to train a good Multi-Output Regressor we need sufficient number of training examples. On the other hand, we do not want to use as reference nodes that have evolved. In practical implementations we can consider many different centrality measures and scoring functions, e.g. temporal degree. Once $A^*$ is found it modifies embeddings by $F_{t-1,t}A^*$. 

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Fig. 1. The idea of the **Generalized ICMEN** method: (1) calculate the embedding $F_{n-1,n}$ for most recent graph snapshot $G_{n-1,n}$ (the embeddings of previous snapshots are computed beforehand); (2) calibrate the embeddings for a given set of reference nodes using Multi-Output Regression; (3) estimate ICMEN’s parameters by (3a) performing link prediction on most recent graph snapshot for each embedding $F_{t-1,t}$ and aggregating the class counts, (3b) feed these counts into the Dirichlet-Multinomial model, (4) combine all embeddings using the Generalized ICMEN method, Eq. 12.
Next step in the method is devoted to initialization of all reference nodes in $F_{t-1,t}$ with their previous representation from $F_{0,1-1}$. It may happen that few selected reference nodes are not active in $(t-1,t)$ period, whereas they are essential for learning of the embedding for $(t-1,t)$. Thus we need to assure they will appear in the training process. This extends $G_{t-1,t}$ with disconnected nodes. For random walks based embedding methods it modifies negative sampling distribution by adding unigrams with $\gamma$ number of occurrence (number random walks started at each node).

Fig. 2. (a) ICMEN method for incremental representation learning with linear convex combination of increments’ embeddings. (b) The generalized ICMEN model combining $K$ embeddings.

3.2 Generalized ICMEN model

Now, we want to propose an extended version of our ICMEN method and express it with two parameters, which enables us to combine more than two embeddings at once, making ICMEN more general, Fig. 2(b):

$$ICMEN(\alpha, K),$$

where $K \in \mathbb{N}$ is the number of last embeddings combined by the algorithm (in Sec. 3.1 we always combined 2 embeddings), and $\alpha \in \mathbb{R}^K$ is now a $K$-dimensional real-valued vector:

$$\alpha = [\alpha_1, ..., \alpha_K]^\top$$

which is constrained to $\{\alpha : 0 \leq \alpha_k \leq 1, \sum_{k=1}^K \alpha_k = 1\}$ namely the $K-1$-dimensional simplex $S_{K-1}$.

Let $F_K$ denote the vector of $K$ embeddings before time $t$ (for the whole graph $t = n$):

$$F_K = [\bar{F}_{0,t-K+1}, F_{t-K+1,t-K+2}, ..., F_{t-1,t}]^\top$$

The embedding rule can be defined as follows:

$$\bar{F}_{t-K,t} = \alpha^\top \cdot F_K$$
which is the dot product of $\alpha$ and the sequence of embeddings $F_K$. Note that we need to perform calibration between a pair of consecutive embeddings iteratively.

The generalized ICMEN version with $K$ combined embeddings allows dealing with the case where one is interested in achieving the recent node’s representation based on some previously obtained representations for $K-1$ previous periods, Fig. 2(b). The overview of processing algorithm is presented in Fig. 1.

### 3.3 Estimating model’s parameters

Basic ICMEN model has only one parameter to estimate: $\alpha$, given a pair of calibrated embeddings. However, there are infinitely many possible solutions (as $\alpha$ is real valued), we can use a set $A$ of sample values and test our algorithm on it. A significant problem arises while using the generalized model with $K$ parameters (Sec. 3.2). Assuming that each parameter has the same number of considered values $|A|$, the model has a search space of size $O(|A|^K)$, i.e., it grows exponentially with each new dimension. Hence, we need to find a way to estimate the model parameters using an algorithm that is cheaper than a full search over the entire parameter space.

We propose an algorithm that uses Bayesian inference with assumption of Dirichlet-Multinomial distribution. Each embedding $F_{t-1,t}$ is a separate class in a multinomial distribution. To estimate the parameters $\hat{\alpha} = \{\hat{\alpha}_1, \ldots, \hat{\alpha}_K\}$, we generate a sample from this multinomial distribution where link prediction task measures class counts (correct predictions). The last graph snapshot $G_{n-1,n}$ is split into train and test sets. Negative edges are sampled in the test set. If a link was correctly predicted with several embeddings, we randomly choose only single representation. If none of them can provide correct classification of the link, such an edge is removed from the sample. The prior for $\alpha$ is the Dirichlet distribution:

$$Dir(\alpha|\beta) = \frac{1}{B(\beta)} \prod_{k=1}^{K} \alpha_k^{\beta_k-1} \mathbf{1}_{(S_{K-1})}(\alpha),$$  \hspace{1cm} (13)

where $B()$ is the beta function used for normalization purposes. $\beta$ reflects the prior knowledge about the distribution. In our case, we will define two settings: uniform: $\beta_1 = \beta_2 = \ldots = \beta_K = 1$, where representations are equally important, and increasing: $\beta_1 < \beta_2 < \ldots < \beta_K$, where more recent embeddings are assumed to be more significant. The likelihood function has the form:

$$p(D|\alpha) = \prod_{k=1}^{K} \alpha_k^{N_k}$$  \hspace{1cm} (14)

where $D = [N_1, N_2, ..., N_K]$ is the vector of class counts (embeddings’ successes) from the experiment described above. $N = \sum_{k=1}^{K} N_k$ is the total size of the sample.

Using the above assumptions we estimate the ICMEN parameters $\hat{\alpha}$ as the maximum a posteriori probability MAP of the Dirichlet-Multinomial model [8]:

$$\hat{\alpha}_i = \frac{N_i + \beta_i - 1}{N + \sum_{k=1}^{K} \beta_k - K}$$  \hspace{1cm} (15)
3.4 Learning edge feature vectors with ICMEN

The ICMEN method provides the ability to represent pairs of nodes instead of individual nodes only. This is especially important in such tasks as link prediction, where we forecast whether a link exists between two nodes in a network or not.

Given two nodes \( u, v \in V \), we utilize a binary operator over the corresponding \( d \)-dimensional feature vectors \( f_u \) and \( f_v \) in order to generate an edge representation \( e_{uv} \) such that \( e : V \times V \rightarrow \mathbb{R}^d \).

According to the previously reported results [4], we agree to use element-wise multiplication, i.e. Hadamard (\( \otimes \)) operator, i.e. for the \( i \)th coordinate of the edge representation \( e_{uv} \), we have: \( e_{uv,i} = < f_u \otimes f_v >_i = f_{u,i} \cdot f_{v,i} \).

3.5 Missing embeddings: new and inactive nodes

A surpassing feature of our method differentiating it from others is its ability to producing up-to-date representation for newly introduced or partially inactive nodes. In the basic ICMEN algorithm, the way we establish the embedding for new or disappearing nodes is trivial. In such case at time \( t \), we only have one out of two embeddings and Eq. [7] cannot be applied. For new nodes, there exists an appropriate embedding \( F_{t-1,t} \) and then the combined embedding \( F_{0,t} = F_{t-1,t} \). If a node suspended its activity in the second period, its embedding \( F_{0,t} \) needs to be updated and then the combined embedding \( F_{0,t} = F_{0,t-1} \).

For the generalized ICMEN method, the problem is more complex. The number of input embeddings \( K' \) varies between 1 and \( K \). If there are all embeddings available, we simply apply Eq. [12] and if \( K' = 1 \) the proceed the same as for the basic ICMEN approach. Otherwise, we take the estimated \( \alpha \) coefficients (see Sec. 3.3) for all \( K' \) available embeddings for a given node \( (1 < K' < K) \). Next, we normalize them, so that they sum up to 1. Then, we apply Eq. [12] assuming that we only have \( K' \) components.

3.6 Complexity Analysis

To estimate the time and memory complexity of our algorithm, we consider the scenario of a single incremental step, i.e. the moment when we have \( n - 1 \) past embeddings \( F_{0,1}, F_{1,2}, ..., F_{n-2,n-1} \) and a new graph snapshot \( G_{n-1,n} \) appears. We use a random walk based embedding algorithm (node2vec) and the generalized version of ICMEN. The steps required to get the new embedding estimate \( \hat{F}_{0,n} \) were presented in Fig. [1]. We denote \( O_T(\cdot) \) and \( O_S(\cdot) \) as the time and memory complexities, respectively. Analyzing each algorithm step separately, we get:

1. first, we run node2vec: \( O_T(node2vec) = O(d|V_{n-1,n}|) \) and \( O_S(node2vec) = O(\gamma \omega |E_{n-1,n}|) \);
2. we gather reference nodes by sorting all nodes from the last snapshot by their centrality measure; it takes \( O_T(sort) = O(|V_{n-2,n-1} \cup V_{n-1,n}|) \), \( O_S(sort) = O(|V_{n-2,n-1} \cup V_{n-1,n}|) \); next, we train and perform prediction with Multi Output Regression on reference nodes: \( O_T(MOR) = O(d^2|V_{n-2,n-1} \cup V_{n-1,n}|) \), \( O_S(MOR) = O(d|V_{n-2,n-1} \cup V_{n-1,n}|) \),
(3a) Link prediction requires sampling of absent edges (in the same number as existing edges; Negative Sampling) takes: $O_T(NS) = O_S(NS) = O(|E_n|)$; we train Logistic Regression using all $n$ embeddings: $O_T(LR) = O(nd|E_n|)$, $O_S(LR) = O(nd|E_n|)$, finally we apply LR and aggregate the Class Counts, which takes $O_T(CC) = O(n * |E_n|)$ and $O_S(CC) = O(n)$.

(3b) the estimation of $\alpha$ using Dirichlet-Multinomial model is done by simple division of $n$ numbers, so it takes: $O_T(dirichlet) = O_S(dirichlet) = O(n)$.

(4) for the combination of all embeddings, we perform a Weighted Matrix Addition, which takes: $O_T(WMA) = O_S(WMA) = O(d|V_0| + d|V_1| + ... + d|V_{n-1}|) = O(d|V|)$, where $|V|$ is the number of vertices in the whole graph.

Therefore, the time and memory complexities of ICMEN equals the sum of all above steps’ complexities. The size of the last snapshot $V_{inc}, E_{inc}$ is much smaller than of all previous ones $V_{non-inc}, E_{non-inc}$: $V_{inc} << V_{non-inc}, E_{inc} << E_{non-inc}$. Hence, we can approximate the total ICMEN complexity to: $O_T(ICMEN) \approx O(d^2|V|)$ and $O_S(ICMEN) \approx O(d|V|)$.

4 Experiments

The performance of both basic and generalized ICMEN method is analyzed in this section. For that purpose, we used five dynamic graph datasets obtained from NetworkRepository [13], see Table 2 for their profiles. We sorted edges in each graph ascending by time, and utilized the first 75% of them of each graph to obtain representations. The remaining 25% is used for evaluation on link prediction task – we consider these edges as existing links and sample an equal number of non-existing links.

Table 2. Graph dataset statistics. $|E|$ - no. of temporal edges (events), $|E|/|V|$ - average no. of events per node, Directed - is the graph directed or not.

| Dataset         | $|V|$ | $|E|$ | $|E|/|V|$ | Timespan | Directed |
|-----------------|------|------|----------|----------|----------|
| FB-forum        | 899  | 33.7K| 37.5     | 164      | $\times$ |
| FB-messages     | 1899 | 61.7K| 32.5     | 216      | $\times$ |
| Enron-employees | 151  | 50.5K| 334.9    | 1138     | $\times$ |
| Hypertext09     | 113  | 20.8K| 184.2    | 2        | $\times$ |
| Radoslaw-email  | 167  | 82.9K| 496.5    | 271      | $\sqrt{}$|

4.1 Experimental Setup

We split the first 75% of edges into various number of snapshots $n \in \{2, 3, \ldots, 10\}$. For each such snapshot $G_{t-1,t}$, we calculate its embedding $F_{t-1,t}$ using separately two different random-walk based embedding methods: Node2vec [4] – parameters
$p, q \in \{0.25, 0.5, 1, 2, 4\}$ obtained from the grid search for each dataset and CTDNE [10] – the initial edge sampling strategy and temporal neighbour selection are based on the uniform distribution.

Calibration was performed with arbitrary chosen node scoring function $\zeta$ and centrality measure $g$ – temporal degree. The time-invariance of each node $v$ between two snapshots $t - 1$ and $t$ was, therefore, estimated by $\zeta(v_{t-1}, v_t) = |g(v_{t-1}) - g(v_t)|(\pi/2 - \arctan(\min(g(v_{t-1}), g(v_t))))$. Nodes with low activity have a much more penalized difference than highly active ones. For each dataset and number of splits individually, we estimated the number of reference nodes to be taken into calibration by greedy search. Calibration matrix $A^*$ was obtained by Ridge Regression at each dimension separately. We investigate several regularization constants and report the best results.

Each embedding is independently computed five times. The same parameters are used for two base embedding methods, namely Node2Vec and CTDNE, i.e.: random walk length – 80 (with min. 10), no. of random walks per node – 10, context window size – 10, embedding dimension $d$ – 128.

We utilize a logistic regression classifier for link prediction and AUC metric to measure the classification performance.

### 4.2 Baseline methods

To get a better overview of how well both, basic and generalized ICMEN variants behave, compared to existing algorithms, we chose the following baseline methods: Node2vec, CTDNE, HTNE and three model variants of dyngraph2vec. The three first of them were trained and evaluated using the whole graph, whereas the dyngraph2vec methods used existing splits (2 up to 10). We retrain each method 5 times, similarly to the ICMEN evaluation.

We also wanted to use the DynGEM algorithm from the implementation provided by its authors [https://github.com/palash1992/DynamicGEM](https://github.com/palash1992/DynamicGEM). Unfortunately, after careful review of this code, we noticed that the implementation ($staticAE$) completely differs from the idea provided in the paper. The weights are not propagated between subsequent timesteps and the proposed PropSize algorithm is not implemented at all. Therefore, we decided not to use this implementation.

### 4.3 Link prediction

We examined ICMEN method using link prediction task and reporting AUC and computation time. The results were compared to state-of-the-art baselines.

We observe in Fig. 3 that the performance of distinct ICMEN configurations differs across datasets and base embedding methods. For instance hypertext09 and radoslaw-email benefits from temporal random walks in CTDNE base embedding. Overall, the ICMEN method tends to preserve similar performance for different number of splits.

While comparing ICMEN to other baseline methods, Fig. 4, we can spot, that there are no leaders that for all datasets provide the highest AUC. We do not
Fig. 3. Performance of link prediction for different number of discrete graph snapshots (X-axis): \textbf{Basic ICMEN}, \textbf{ICMEN} – the generalized ICMEN. Base embeddings: \textbf{N2V/CTDNE} - Node2vec/CTDNE snapshots embeddings. Estimation: \textbf{Best} - parameters found using grid search, \textbf{Dir Uni/Inc} - Dirichlet uniform/increasing strategy.

observe statistically significant difference between the results of the best ICMEN model and the others. Overall, our method provides comparable and even better results than the baselines, but it operates in a fully incremental manner.

An important feature of the incremental approach provided by ICMEN is its faster data processing. Fig. 5. ICMEN commonly outperforms all other methods, especially with the increasing number of increments.

5 Conclusion and Future Work

We introduced ICMEN, a new incremental embedding meta-method that efficiently constructs nodes’ representation by linearly convex combinations of any
already available node embeddings. It allows dealing with large temporal networks using periodical (increment-based) updates folding new activities (arriving increments) and historical representations not accessing the source data. As a result, the method is less memory demanding than any other embedding algorithm. We demonstrated how the method is capable of constructing representation for inactive and new nodes without a need to re-embed the old nodes. Moreover, the introduced calibration mechanism enables us to compare representations in time. We shown that making a particular base embedding method incremental using ICMEN, we receive similar results in link prediction task, saving memory while remaining computationally competitive.

In the future, we would like to develop a mechanism for automatic recognition of the moment in time where the embeddings should be retrained to account drifts in the graph.

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References

1. Borchani, H., Varando, G., Bielza, C., Larrañaga, P.: A survey on multi-output regression. WIREs Data Mining Knowl Discov 5, 216–233 (2015)
2. Goldberg, Y., Levy, O.: word2vec Explained: Deriving Mikolov et al.‘s Negative-Sampling Word-Embedding Method. arXiv:1402.3722 (2014)
3. Goyal, P., Chhetri, S.R., Canedo, A.: dyngraph2vec: Capturing network dynamics using dynamic graph representation learning. CoRR abs/1809.02657 (2018)
4. Grover, A., Leskovec, J.: node2vec: Scalable feature learning for networks. In: KDD’2016. pp. 855–864. ACM (2016)
5. Kamra, N., Goyal, P., He, X., Liu, Y.: Dyngem: Deep embedding method for dynamic graphs. In: Workshop on Represent. Learning for Graphs, ReLiG (2017)
6. Laub, P.J., Taimre, T., Pollett, P.K.: Hawkes Processes. arXiv:1507.02822 (2015)
7. McCallum, A.K., Nigam, K., Rennie, J., Seymore, K.: Automating the construction of internet portals with machine learning. Inf. Retrieval 3(2), 127–163 (2000)
8. Murphy, K.K.: Machine Learning: A Probabilistic Perspective. MIT Press (2012)
9. Newman, M.E.: The structure of scientific collaboration networks. Proceedings of the national academy of sciences 98(2), 404–409 (2001)
10. Nguyen, G.H., Lee, J.B., Rossi, R.A., Ahmed, N.K., Koh, E., Kim, S.: Continuous-time dynamic network embeddings. In: WWW ’18. pp. 969–976 (2018)
11. Perozzi, B., Al-Rfou, R., Skiena, S.: Deepwalk: Online learning of social representations. CoRR abs/1403.6652 (2014)
12. Qiu, J., Dong, Y., Ma, H., Li, J., Wang, K., Tang, J.: Network Embedding as Matrix Factorization. In: WSDM’18. pp. 459–467. ACM Press (2018)
13. Rossi, R.A., Ahmed, N.K.: The network data repository with interactive graph analytics and visualization. In: AAAI. pp. 4292–4293. AAAI Press (2015)
14. Tang, J., Qu, M., Wang, M., Zhang, M., Yan, J., Mei, Q.: LINE: large-scale information network embedding. CoRR abs/1503.03578 (2015)
15. Wang, X., Cui, P., Wang, J., Pei, J., Zhu, W., Yang, S.: Community preserving network embedding. In: AAAI. pp. 203–209. AAAI Press (2017)
16. Watts, D.J., Strogatz, S.H.: Collective dynamics of ‘small-world’ networks. Nature 393(6684), 440 (1998)
17. Wei, W., Carley, K.M.: Measuring temporal patterns in dynamic social networks. ACM Transactions on Knowledge Discovery from Data 10(1), 9 (2015)
18. Winter, S.D., Decuyper, T., Mitrovic, S., Baesens, B., Weerdt, J.D.: Combining temporal aspects of dynamic networks with node2vec for a more efficient dynamic link prediction. In: ASONAM 2018. pp. 1234–1241 (2018)
19. Zuo, Y., Liu, G., Lin, H., Guo, J., Hu, X., Wu, J.: Embedding temporal network via neighborhood formation. In: ACM SIGKDD 2018. pp. 2857–2866. ACM (2018)