On the regularity of static axially symmetric solutions in $SU(2)$ Yang-Mills-dilaton theory

Burkhard Kleihaus

NUIM, Department of Mathematical Physics, Maynooth, Co. Kildare, Ireland
(September 19, 2018)

The regularity of static axially symmetric solutions in $SU(2)$ Yang-Mills-dilaton theory is examined. We show that the solutions obtained previously within a singular Ansatz for the non-abelian gauge field can be gauge transformed into a regular form. The local form of the gauge transformation is given on the singular axis and at the origin.

PACS number(s): 11.15.Kc
I. INTRODUCTION

Static spherically symmetric solutions in non-abelian gauge theories have been investigated for a long time. The $SU(2)$ monopoles in the Georgi-Glashow model \cite{1} and in extended models \cite{2}, as well as the sphalerons in the Weinberg-Salam model at vanishing mixing angle \cite{3}, in Yang-Mills-dilaton \cite{4}, Einstein-Yang-Mills \cite{5} and Einstein-Yang-Mills-dilaton \cite{6} theory are some examples. All these solutions are localized in space and possess finite energy and finite energy density.

Besides the static spherically symmetric solutions there exist solutions with axial symmetry only, which also are localized in space and possess finite energy and finite energy density. In the Georgi-Glashow model these are $SU(2)$ multi-monopole solutions investigated numerically in the self-dual limit by Rebbi and Rossi \cite{7} and later analytically by Forgacs, Horvath and Palla \cite{8} and others \cite{9}. A numerical study of the axially symmetric $SU(2)$ multi-monopole solutions in the non self-dual case has been done recently in Ref. \cite{10} for the Georgi-Glashow model and in Ref. \cite{11} for the extended monopole model. In the Weinberg-Salam model the sphaleron at finite mixing angle \cite{12}, the sphaleron anti-sphaleron solution \cite{13} and the multi-sphalerons \cite{14} possess axial symmetry only. The same holds for the $SU(2)$ multi-sphalerons in Yang-Mills-dilaton \cite{15}, Einstein-Yang-Mills and Einstein-Yang-Mills-dilaton \cite{16} \cite{17} theories.

For the construction of static axially symmetric solutions it is convenient to use appropriate Ansätze for the gauge potential and the Higgs field, if present. For the multimonomopes, Forgacs et al. \cite{8} use the static axially symmetric $SU(2)$ Ansatz of Manton \cite{20}, which is inspired by an Ansatz of Witten \cite{21}. The Ansatz of Rebbi and Rossi \cite{7} generalizes the Ansatz of Manton \cite{21} for winding number $|n| > 1$, corresponding to the magnetic charge in the Georgi-Glashow model. As the winding number characterizes non-trivial maps between two two-dimensional spheres, it also characterizes topologically different sectors in the configuration space of the theory. This Ansatz is parameterized by four functions for the gauge potential and two functions for the Higgs field. This Ansatz possesses a residual abelian gauge degree of freedom, which has to be fixed by a gauge constraint. The Ansatz is singular in the sense that it is a priori not well defined on the symmetry axis and at the origin. Rebbi and Rossi discuss the corresponding regularity conditions to be imposed on the functions parameterizing the Ansatz \cite{7}.

The Ansätze used in \cite{10,11,14–19} derive from the Ansatz of Rebbi and Rossi \cite{7}. Boundary conditions have been imposed on the functions to ensure a finite energy density, but regularity conditions on the symmetry axis and at the origin have not been imposed, to ensure a well defined gauge potential. Thus, the solutions are given in a singular form and one may question whether the solutions themselves are regular for winding number $|n| > 1$. On the other hand, one should keep in mind, that the gauge potential is a gauge variant quantity and all physical conclusions concern only gauge invariant quantities. Any regular gauge potential can be gauge transformed into a singular gauge potential. For example, the gauge potential of the ’t Hooft-Polyakov monopole becomes singular in the unitary gauge, see e. g. \cite{22}. However, it is not true that any singular gauge potential can be gauge transformed into a regular one. If the gauge potential is somewhere singular the field strength tensor must be calculated carefully in order not to loose contributions from the singular part of the gauge potential, which are overlooked easily by naive calculation. Once it is shown, that the naive calculation of the field strength tensor gives the correct result, one can find the equation of motion from the action principle. There is no need for a globally regular gauge potential, as long as for any point there exist a neighborhood on which a regular gauge potential can be defined and gauge potentials on intersections of neighborhoods can be transformed into each other by regular gauge transformations. The Dirac monopole e. g., possesses singularities along the negative $z$-axis, which can be removed locally by a gauge transformation, but the singularity at the origin persists - it represents a physical quantity, the magnetic point charge.

Turning back to the non-abelian gauge fields, we notice, that in case no point-like charges are expected to occur, gauge transformations may exist, which transform locally the singular gauge potentials of the solutions constructed in \cite{10,11,14–19} into regular gauge potentials. In general such gauge transformations are singular themselves. The important point is that these singularities of the gauge transformations should not be too strong to produce additional contributions to the field strength tensor or gauge invariant quantities.

In this paper we will consider the multisphalerons in Yang-Mills-dilaton theory \cite{15} and give the local gauge transformations leading to a regular gauge potential.

The paper is organized as follows. In Section I we will present the static axially symmetric Ansatz of the gauge potential, derive the field strength tensor and the Lagrange density of the Yang-Mills-dilaton theory as well as the differential equations for the gauge field functions and the dilaton function. The regularity conditions are also discussed. In Section II the local form of the gauge transformation is derived along the symmetry axis and at the origin. The discussion and conclusions are given in Section III.
II. STATIC AXIALLY SYMMETRIC ANSATZ

The static axially symmetric Ansatz \( A_0, A_r, A_\theta, A_\varphi \) for the \( su(2) \) valued gauge potential \( A = A_\mu dx^\mu \) is given in spherical coordinates \( r, \theta, \varphi \) by

\[
\begin{align*}
A_0 &= 0, \\
A_r &= \frac{1}{2gr} H_1 \tau_\varphi^n, \\
A_\theta &= \frac{1}{2g} (1 - H_2) \tau_\varphi^n, \\
A_\varphi &= -\frac{n}{2g} \sin \theta (H_3 \tau_\rho^n + (1 - H_4) \tau_\theta^n) \\
&= -\frac{n}{2g} \sin \theta (F_3 \tau_\rho^n + F_4 \tau_3),
\end{align*}
\]

(1)

where the \( su(2) \) matrices \( \tau_\lambda^n, \lambda = \rho, \varphi, r, \theta \) are defined in terms of the Pauli matrices \( \tau_1, \tau_2, \tau_3 \) by

\[
\begin{align*}
\tau_\rho^n &= \cos(n\varphi) \tau_1 + \sin(n\varphi) \tau_2, \\
\tau_\varphi^n &= -\sin(n\varphi) \tau_1 + \cos(n\varphi) \tau_2, \\
\tau_r^n &= \sin \theta \tau_\rho^n + \cos \theta \tau_3, \\
\tau_\theta^n &= \cos \theta \tau_\rho^n - \sin \theta \tau_3.
\end{align*}
\]

Here the integer \( n \) denotes the winding number. As symmetry axis we have chosen the \( z \)-axis. The functions \( H_i, i = 1, \ldots, 4 \) depend on the variables \( r \) and \( \theta \) only, \( H_i(r, \theta) = H_i(r) \). The functions

\[
F_3 = \sin \theta H_3 + \cos \theta (1 - H_4), \\
F_4 = \cos \theta H_3 - \sin \theta (1 - H_4)
\]

have been defined for later convenience. In the following \( A_0 \) will be zero in all gauges and we will consider the spatial components of the gauge potential only. We fix the gauge coupling constant to \( g = 1 \).

From the requirement of finite energy density, see later Eq. \( \ref{energy} \), we find that on the \( z \)-axis the functions \( H_1 \) and \( H_3 \) have to vanish, while the functions \( H_2 \) and \( H_4 \) have to be equal to each other, \( H_2(r, \theta = 0) = H_4(r, \theta = 0) = f(r) \), whereas at the origin the functions \( H_1 \) and \( H_3 \) take the value zero, while the functions \( H_2 \) and \( H_4 \) take the value 1.

The Ansatz Eq. \( \ref{ansatz} \) is singular in the sense that it is not well defined on the \( z \)-axis and the origin. These singularities originate from the functions \( \sin(n\varphi) \) and \( \cos(n\varphi) \), which are not well defined at \( \rho = 0, \rho = \sqrt{x^2 + y^2} \). In order to exhibit the singularities more clearly we turn to Cartesian coordinates \( (x, y, z) = (r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta) = (\rho \cos \varphi, \rho \sin \varphi, z) \). In these coordinates the components of the gauge potential become

\[
\begin{align*}
A_x &= \frac{x}{2r \rho} \left( \frac{\rho}{r} H_1 + \frac{z}{r} (1 - H_2) \right) \tau_\rho^n + \frac{ny}{2r \rho} \left( \frac{\rho}{r} H_3 + \frac{z}{r} (1 - H_4) \right) \tau_\theta^n + \frac{n x}{2r \rho} \left( \frac{\rho}{r} H_3 - \frac{z}{r} (1 - H_4) \right) \tau_3, \\
A_y &= \frac{y}{2r \rho} \left( \frac{\rho}{r} H_1 + \frac{z}{r} (1 - H_2) \right) \tau_\rho^n - \frac{nx}{2r \rho} \left( \frac{\rho}{r} H_3 + \frac{z}{r} (1 - H_4) \right) \tau_\theta^n + \frac{n z}{2r \rho} \left( \frac{\rho}{r} H_3 - \frac{z}{r} (1 - H_4) \right) \tau_3, \\
A_z &= \frac{1}{2r} \left( \frac{z}{r} H_1 - \frac{\rho}{r} (1 - H_2) \right) \tau_\varphi^n,
\end{align*}
\]

(2)

where the matrices \( \tau_\lambda^n, \lambda = \varphi, \rho \) contain powers of \( \frac{z}{r} \) and \( \frac{\rho}{r} \) up to order \(|n| \) which are not well defined on the \( z \)-axis and at the origin. Note, that in spite of being not well defined on the \( z \)-axis and at the origin the gauge potential is sufficiently regular to be locally integrable.

Turning to the calculation of the field strength tensor

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i \left[ A_\mu, A_\nu \right],
\]

partial differentiation has to be carried out carefully. However, taking into account the behavior of the functions \( H_i \) on the \( z \)-axis, we see, that in the gauge potential Eq. \( \ref{ansatz} \) terms like \( \frac{\rho}{r} \), \( \frac{z}{r} \) do not arise, which could lead to \( \delta \)-functions
along the z-axis, i. e. \( \partial_x(\phi) + \partial_y(\phi) = -2\pi\delta(x)\delta(y) \). This is in contrast to the Dirac monopole, where the gauge field blows up near the negative z-axis and the field strength tensor does pick up \( \delta \)-functions, but this singular part is an artifact.

Similarly, at the origin no terms like \( \frac{1}{r^2} \) arise in the gauge potential Eq. (3), which could lead to \( \delta \)-functions.

Consequently, it is straightforward to calculate the field strength tensor with the Ansatz Eq. (3). The components of the field strength tensor become

\[
F_{r\theta} = -\frac{1}{2r} (\partial_\theta H_1 + r\partial_r H_2) \tau^\mu_\nu^n,
\]

\[
F_{r\varphi} = -n\sin\theta \frac{1}{2r} \left[ (r\partial_r H_3 - H_1 H_4) \tau^\mu_\nu^n - (r\partial_r H_4 + H_1 H_3 + \cot \theta H_4) \tau^\mu_\nu^n \right],
\]

\[
F_{\theta\varphi} = -n\sin\theta \frac{1}{2} \left[ (\partial_\theta H_3 - 1 + H_2 H_4 + \cot \theta H_3) \tau^\mu_\nu^n - (\partial_\theta H_4 - H_2 H_3 - \cot \theta (H_2 - H_4)) \tau^\mu_\nu^n \right].
\]

Like the gauge potential, the field strength tensor is not well defined on the z-axis and at the origin. However, the Lagrange density \( \mathcal{L}_F = -\frac{1}{2} Tr (F_{\mu\nu}F^{\mu\nu}) \) is well defined due to the normalization of the \( \tau^\mu_\nu^n \) matrices, \( Tr(\tau^A_\mu \tau^A_\nu) = 2\delta_{\mu\nu} \).

In this article we consider Yang-Mills-dilaton theory [4,15], where the dilaton is a scalar field \( \phi(r) \) which couples to the Yang-Mills field. The Lagrange density \( \mathcal{L} \) is given by

\[
-\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + e^{-2\kappa \phi} \frac{1}{2} Tr (F_{\mu\nu}F^{\mu\nu}),
\]

where \( \kappa \) is the dilaton coupling constant.

For a static axially symmetric Ansatz for the gauge field, it is consistent to assume that the dilaton field possesses axial symmetry, too. Thus we can consider the dilaton function \( \phi(r) \) as function of \( r \) and \( \theta \) only, \( \phi(r) = \phi(r, \theta) \).

Next we change to dimensionless variables, \( r \rightarrow \frac{r}{\sqrt{\kappa}} \), \( \phi \rightarrow \frac{\phi}{\kappa} \), and insert the Ansatz into the Lagrange density to obtain the dimensionless Lagrange density

\[
-\mathcal{L}[H, \phi] = \frac{8}{\kappa^4} \mathcal{L}[H, \phi] = \frac{1}{4\kappa^4} e^{2\phi} \left\{ [r\partial_r H_2 + \partial_\theta H_1]^2 + \xi [r\partial_r H_1 - \partial_\theta H_2]^2 \right. \\
+ n^2 \left[ (r\partial_r H_3 - H_1 H_4)^2 + [r\partial_r H_4 + H_1 (H_3 + \cot \theta)]^2 \right] \\
+ [\partial_\theta H_3 + H_2 H_4 - 1]^2 + [\partial_\theta H_4 + (H_2 - H_4) \cot \theta - H_2 H_3]^2 \left\} \\
+ \frac{1}{2r^2} (\partial_\mu \phi)^2 + (\partial_\phi \phi)^2,
\]

where the gauge constraint \( [r\partial_r H_1 - \partial_\theta H_2]^2 = 0 \) has been added with Lagrange multiplier \( \xi \).

The Euler-Lagrange equations which extremize \( \int L(H, \phi)d^4x \) are given by the following system of second order non-linear partial differential equations for the gauge field functions \( H_i(r, \theta) \) and the dilaton function \( \phi(r, \theta) \).

\[
0 = n^2 \left\{ \sin^2 \theta \left[ H_1 (1 - H_3^2 - H_4^2) - r\partial_r H_4 H_3 + r\partial_r H_3 H_4 \right] - \sin \theta \cos \theta [r\partial_r H_4 + 2H_1 H_3] - H_1 \right\} \\
+ \sin^2 \theta \left[ 2\partial_\theta \phi (r\partial_r H_2 + \partial_\theta H_1) + \partial_\theta^2 H_1 + 2\partial_\theta r\partial_r H_2 \right] + \sin \theta \cos \theta \left[ \partial_\theta H_1 + r\partial_r H_2 \right] \\
+ \xi \sin^2 \theta \left[ 2r\partial_\phi (r\partial_r H_1 - \partial_\theta H_2) + r^2 \partial_\mu \phi \partial_\mu H_1 - \partial_\theta r\partial_r H_2 + \partial_\theta H_2 \right],
\]

(4)

\[
0 = n^2 \left\{ \sin^2 \theta \left[ H_2 (1 - H_3^2 - H_4^2) + \partial_\theta H_4 H_3 - \partial_\theta H_3 H_4 \right] + \sin \theta \cos \theta [\partial_\theta H_4 - 2H_2 H_3] - (H_2 - H_4) \right\} \\
+ \sin^2 \theta \left[ 2r\partial_\theta \phi (r\partial_r H_2 + \partial_\theta H_1) + r^2 \partial_\mu \phi \partial_\mu H_2 + r\partial_\theta r\partial_r H_1 - \partial_\theta H_2 \right] \\
- \xi \left\{ 2r\partial_\phi (r\partial_r H_1 - \partial_\theta H_2) + \partial_\theta r\partial_r H_1 + \sin \theta \cos \theta [r\partial_r H_1 - \theta_\theta H_2] \right\},
\]

(5)

\[
0 = \sin^2 \theta \left[ r^2 \partial_\mu \phi \partial_\mu H_3 + \partial_\theta^2 H_3 - H_1^2 H_5 + H_1 H_4 - 2r\partial_r H_4 H_1 - r\partial_r H_1 H_4 - H_2 H_3 + 2\partial_\theta H_4 H_2 + \partial_\theta H_2 H_4 \\
- 2 (r\partial_\phi (H_1 H_4 - r\partial_r H_3) + \partial_\theta \phi (1 - H_2 H_3 - \partial_\theta H_3))] \\
+ \sin \theta \cos \theta \left[ \partial_\theta H_3 - H_1^2 - H_2^2 + H_2 H_4 + 2H_3 \partial_\theta \phi \right] - H_3,
\]

(6)

\[
0 = \sin^2 \theta \left[ r^2 \partial_\mu \phi \partial_\mu H_4 + 2r\partial_r H_3 H_1 + r\partial_r H_1 H_3 - 2\partial_\theta H_3 H_2 - \partial_\theta H_2 H_3 - H_4 (H_2^2 + H_1^2) + H_2 - H_1 H_3 \\
+ 2 (\partial_\theta \phi (\partial_\theta H_4 - H_2 H_3) + r\partial_r \phi (r\partial_r H_4 + H_1 H_3)) \right]
\]
\[ + \sin \theta \cos \theta \left[ r \partial_r H_1 - H_1 - H_2 H_3 - \partial_{\theta} (H_2 - H_4) + 2 (r \partial_r \phi H_1 - \partial_{\theta} \phi (H_2 - H_4)) \right] + (H_2 - H_4) , \]

\[ 0 = \sin \theta \left( r^2 \partial_{\theta}^2 \phi + \partial_{\theta}^2 \phi \right) + (2 \sin \theta r \partial_r \phi + \cos \theta \partial_{\theta} \phi) \]

\[ - \frac{\sin \theta}{2r^2} e^{2\phi} \left\{ \left[ r \partial_r H_2 + \partial_\theta H_1 \right]^2 + \xi \left[ r \partial_r H_1 - \partial_\theta H_2 \right]^2 \right. \]

\[ + n^2 \left( \left[ r \partial_r H_3 - H_1 H_4 \right]^2 + \left[ r \partial_r H_4 + H_3 (\cot \theta) \right]^2 \right) \]

\[ + \left[ \partial_\theta H_3 + H_3 \cot \theta + H_2 H_4 - 1 \right]^2 + \left[ \partial_\theta H_4 + (H_4 - H_2) \cot \theta - H_2 H_3 \right]^2 \right\} \]

This system has to be solved subject to the boundary conditions,

| Condition                                      | At the Origin                                                                 | At Infinity                                                                 |
|-----------------------------------------------|-------------------------------------------------------------------------------|-------------------------------------------------------------------------------|
| \( H_1(0, \theta) = H_3(0, \theta) = 0 \)   | \( H_1(\infty, \theta) = H_3(\infty, \theta) = 0 \)                        | \( H_2(\infty, \theta) = H_4(\infty, \theta) = \pm 1 \)                      |
| \( \partial_\tau \phi(r, \theta) = 0 \)       | \( \phi(\infty, \theta) = 1 \)                                              |                                                                              |
| \( \partial_\theta H_2(r, 0) = \partial_\theta H_4(r, 0) = 0 \) | \( \partial_\theta H_2(r, \frac{\pi}{2}) = \partial_\theta H_4(r, \frac{\pi}{2}) = 0 \) | \( \partial_\theta \phi(r, \frac{\pi}{2}) = 0 \)                           |

These conditions follow from the requirement of finite energy density and symmetry considerations, see [17]. No non-trivial explicit solutions of this boundary value problem are known. However, for winding number \( n \leq 4 \) axially symmetric solutions have been constructed numerically in [13] using a different parameterization of the Ansatz.

For solutions of the differential equations (4)-(8) the gauge potential Eq. (2) is not well defined on the \( z \)-axis and at the origin. Regularity requires that in the gauge potential Eq. (2) the coefficients multiplying the matrices \( \tau^\rho \cdot \tau^n \) contain at least a power \( \rho^{\mid n \mid} \). Let us denote the gauge field functions of a regular gauge potential \( \hat{A} \) by \( \hat{H}_i \), etc. Then near the \( z \)-axis the following behavior for the functions \( \hat{H}_i \) is required

\[ \hat{H}_1 \sim \rho^{\mid n \mid}, \quad 1 - \hat{H}_2 \sim \rho^{\mid n \mid + 1}, \]

\[ \hat{F}_3 = \left[ \frac{\partial}{\partial \theta} \hat{H}_3 + \frac{\partial}{\partial \phi} (1 - \hat{H}_4) \right] \sim \rho^{\mid n \mid + 1}, \quad \hat{F}_4 = \left[ \frac{\partial}{\partial \theta} \hat{H}_3 - \frac{\partial}{\partial \phi} (1 - \hat{H}_4) \right] \sim \rho. \]

In the following section we will find the local form of gauge transformations which lead exactly to the desired behavior of the gauge field functions, Eqs. (10).

**III. GAUGE TRANSFORMATIONS**

The gauge potential is a gauge variant quantity. Let \( U(r) \) be a \( SU(2) \) valued function and \( A = A_\mu dx^\mu \) a \( su(2) \) valued gauge potential. The gauge transformed gauge potential \( \hat{A} = A_\mu dx^\mu \) is defined by

\[ \hat{A} = U \hat{A} U^\dagger + i d U U^\dagger. \]

In this section the gauge degree of freedom will be used to transform the singular gauge potential into a regular one.

First assume there is a regular gauge potential \( \hat{A} = \hat{A}_r \rho^{\mid n \mid} \tau^\rho \, d\theta + \hat{A}_0 \rho^{\mid n \mid} \tau^n \, d\theta + (\hat{A}_\phi \rho^{\mid n \mid} \tau^\phi + \hat{A}_\phi \rho^{\mid n \mid} \tau^\phi + \hat{A}_3 \rho^{\mid n \mid} \tau^3) d\phi \) and that it is related by a gauge transformation matrix \( U(r) \) to the singular gauge potential \( A = A_r \rho^{\mid n \mid} \tau^\rho \, d\theta + A_0 \rho^{\mid n \mid} \tau^n \, d\theta + (A_\phi \rho^{\mid n \mid} \tau^\phi + A_3 \rho^{\mid n \mid} \tau^3) d\phi \) of the same form. We find for the difference of the gauge potentials

\[ A_\mu - \hat{A}_\mu = i \partial_\mu U^\dagger U + U^\dagger [\hat{A}_\mu, U]. \]

For small \( \rho \) the regular gauge potential behaves like \( \hat{A} = \hat{A}_3 \tau_3 d\phi + O(\rho^{\mid n \mid}) \). Consequently we find from Eq. (11)
\begin{align*}
A^{(\varphi)} & = i \left( \partial_\varphi U^\dagger U \right)^{(\varphi)} + O(\rho^n), \\
A^{(\rho)} & = i \left( \partial_\rho U^\dagger U \right)^{(\rho)} + O(\rho^n), \\
0 & = i \left( \partial_\rho U^\dagger U \right)^{(\lambda)}, \quad \lambda = \rho, \ z \\
0 & = i \left( \partial_\rho U^\dagger U \right)^{(\lambda)}, \quad \lambda = \rho, \ z
\end{align*}

where we have defined the expansion $\partial_\mu U^\dagger U = (\partial_\mu U^\dagger U)^{(\rho)} \tau^n + (\partial_\mu U^\dagger U)^{(\varphi)} \tau^\varphi + (\partial_\mu U^\dagger U)^{(z)} \tau_3$. Under the assumption stated above, we find from the first two equations that the $A_r$ and $A_\theta$ components of the singular gauge potential behave like a pure gauge for small $\rho$, whereas the last two equations put constraints on the gauge transformation matrix $U(\vec{r})$. Exploiting the residual abelian gauge degree of freedom of the Ansatz, these constraints can be solved by choosing a special form for the matrix $U(\vec{r})$,

$$U(\vec{r}) = \exp \left( \frac{1}{2} \Gamma \tau^n \right),$$

where the gauge transformation function $\Gamma$ depends on $r$ and $\theta$ only. Indeed, this gauge transformation leaves the form of the Ansatz invariant. The functions $H_i$ transform as

\begin{align*}
H_1 & \rightarrow \hat{H}_1 = H_1 - r \partial_\rho \Gamma, \\
H_2 & \rightarrow \hat{H}_2 = H_2 + \partial_\rho \Gamma, \\
H_3 & \rightarrow \hat{H}_3 = \cos \Gamma (H_3 + \cot \theta) - \sin \Gamma H_4 - \cot \theta, \\
H_4 & \rightarrow \hat{H}_4 = \sin \Gamma (H_3 + \cot \theta) + \cos \Gamma H_4.
\end{align*}

The assumption, that there exist a gauge transformation which transforms a singular gauge potential into a regular one of the same form, is very strong. However, with the differential equations at hand it can be checked straightforwardly whether the $A_r$ and $A_\theta$ components of the gauge potential form a pure gauge for small $\rho$. If this is the case, then the gauge transformation function $\Gamma(r, \theta)$ can be found for small $\rho$. In addition it has to be checked that the same gauge transformation applied to the $A_r$ component of the gauge potential also leads to a regular form.

We will proceed in the following way. At the $z$-axis we will expand the functions $H_i(r, \theta)$ and $\phi(r, \theta)$ in powers of $\sin \theta$. The coefficients will be functions of $r$ only. We insert the expansion of the functions $H_i(r, \theta)$ and $\phi(r, \theta)$ into the partial differential equations and expand the right hand side of Eqs. (4)-(6) in powers of $\sin \theta$. For a solution of the partial differential equations the coefficients in the expansion of the partial differential equations have to vanish. Setting these coefficients equal to zero results in a set of relations for the coefficient functions of the expansion of the functions $H_i(r, \theta)$ and $\phi(r, \theta)$. This does not determine the functions completely. However, it enables us to find the local form of gauge transformations which lead to a gauge transformed potential which is well defined on the $z$-axis.

In a similar way we will expand the functions $H_i(r, \theta)$ and $\phi(r, \theta)$ at the origin in powers of $r \sin \theta$ and $r \cos \theta$. The coefficients in the expansion of the functions are constants then. Relations between these constants will be obtained, again, by inserting the expansion of the functions into the differential equations, expanding the expressions Eqs. (4)-(6) in powers of $r \sin \theta$ and $r \cos \theta$ and setting the coefficients in this expansion equal to zero. From this analysis the local form of a gauge transformation at the origin can be found, which again leads to a gauge transformed potential which is well defined at the origin.

Finally we will compare the gauge transformation at the origin with the gauge transformations near the $z$-axis for small values of $r$. As a result we will find, that the both local forms coincide in the region where both expansions should hold.

Because our main interest is the behavior of the gauge potential at the $z$-axis and at the origin, we will not exhibit the expansion of the dilaton function in the following.

In this paper we will restrict to the case with winding number $n = 2, 3, 4$. The single steps in the calculations are elementary but tedious and will be omitted. For winding number $n = 3, 4$ large expressions arise in the expansion at the origin. These expressions are exhibited in the Appendix.

**A. Gauge Transformation at the $z$-axis**

We start by examining the behavior of the functions $H_i$ near the positive $z$-axis. We will expand these functions $H_i$ in terms of $\sin \theta$. From the equations of motion we will find relations between the coefficients which depend on $r$. 

6
The expansions up to third order in $\theta$ was obtained before in ref. [7] for the more general case of Einstein-Yang-Mills-dilaton theory, i.e. including the coupling of axially symmetric Yang-Mills and dilaton fields to gravity [10,17,19].

$$n = 2$$

The expansions in $\sin \theta$ to third order, which are consistent with the boundary conditions Eqs. (3), are given by

$$H_1(r, \theta) = \tilde{H}_{11}(r) \sin \theta + \frac{\tilde{H}_{12}(r)}{2} \sin^2 \theta + \frac{\tilde{H}_{13}(r)}{6} \sin^3 \theta + O(\sin^4 \theta) ,$$

$$H_2(r, \theta) = f(r) + \frac{\tilde{H}_{22}(r)}{2} \sin^2 \theta + \frac{\tilde{H}_{23}(r)}{6} \sin^3 \theta + O(\sin^4 \theta) ,$$

$$H_3(r, \theta) = g(r) \sin \theta + \frac{\tilde{H}_{32}(r)}{2} \sin^2 \theta + \frac{\tilde{H}_{33}(r)}{6} \sin^3 \theta + O(\sin^4 \theta) ,$$

$$H_4(r, \theta) = f(r) + \frac{\tilde{H}_{42}(r)}{2} \sin^2 \theta + \frac{\tilde{H}_{43}(r)}{6} \sin^3 \theta + O(\sin^4 \theta) .$$

(17)

Inserting (17) into the differential equations (14)-(18) and expanding in terms of $\sin \theta$ the following relations between the coefficient functions $\tilde{H}_{ij}(r)$ are found,

$$\tilde{H}_{11} = -r \partial_r f ,$$

$$\tilde{H}_{22} = -r \partial_r (r \partial_r f) ,$$

$$\tilde{H}_{23} = r \partial_r \tilde{H}_{12} ,$$

$$\tilde{H}_{32} = 0 ,$$

$$\tilde{H}_{42} = \frac{1}{3} \left[ 2 f (f^2 + 3 g - 1) - [r \partial_r]^2 f \right] .$$

The functions $\tilde{H}_{52}(r)$ and $\tilde{H}_{43}(r)$ may also be expressed in terms of $f(r)$, $g(r)$, $\tilde{H}_{12}(r)$, $\tilde{H}_{13}(r)$ and other functions. However these expressions look very complicated and their detailed structure is not needed in the following.

Expanding further $\sin \theta$ in terms of $\theta$ the functions $H_1$ and $H_2$ become

$$H_1 = -r \partial_r f \theta + \tilde{H}_{12} \frac{\theta^2}{2} + O(\theta^3)$$

$$H_2 = f - [r \partial_r]^2 f \frac{\theta^2}{2} + r \partial_r \tilde{H}_{12} \frac{\theta^3}{6} + O(\theta^4)$$

$$= \partial_\theta \left\{ f \theta - [r \partial_r]^2 f \frac{\theta^3}{6} \right\} + r \partial_r \tilde{H}_{12} \frac{\theta^3}{6} + O(\theta^4)$$

Comparing with Eqs. (14) and (15) we see that a gauge transformation (14) with

$$\Gamma = \Gamma^{(2)}(r, \theta) \equiv - \left\{ (f - 1) \theta - [r \partial_r]^2 f \frac{\theta^3}{6} \right\}$$

(18)

yields gauge transformed functions

$$\tilde{H}_1 = \tilde{H}_{12} \frac{\theta^2}{2} + O(\theta^3) = \tilde{H}_{12} \frac{\rho^2}{2} + O(\rho^3) ,$$

$$1 - \tilde{H}_2 = -r \partial_r \tilde{H}_{12} \frac{\theta^3}{6} + O(\theta^4) = -r \partial_r \tilde{H}_{12} \frac{\rho^3}{6} + O(\rho^4) ,$$

which have the desired behavior on the positive z-axis, Eqs. (11).

Performing the same gauge transformation to the functions $H_3$ and $H_4$ and calculating the gauge transformed functions $\tilde{F}_3$ and $\tilde{F}_4$, we find

7
which also have the desired behavior on the positive z-axis, Eqs. (11).

Turning to Cartesian coordinates we find for the non-vanishing components of the gauge potential, Eqs. (2), to lowest order in the expansion

\[
\hat{A}_x = -\frac{x}{12r^3} \left[ r \partial_r \hat{H}_{12} - 3 \hat{H}_{12} \right] \rho^2 \tau_r^2 - \frac{y}{6r^4} \hat{H}_{43} \rho^2 \tau_{\rho}^2 + \frac{y}{2r^2} \left[ f^2 + 2g - 1 \right] \tau_3 ,
\]

\[
\hat{A}_y = -\frac{y}{12r^3} \left[ r \partial_r \hat{H}_{12} - 3 \hat{H}_{12} \right] \rho^2 \tau_r^2 + \frac{x}{6r^4} \hat{H}_{43} \rho^2 \tau_{\rho}^2 - \frac{x}{2r^2} \left[ f^2 + 2g - 1 \right] \tau_3 ,
\]

\[
\hat{A}_z = \frac{1}{4r^3} \hat{H}_{12} \rho^2 \tau_r^2 .
\]

\[n = 3\]

For winding number \(n = 3\) we consider the expansions in \(\sin \theta\) to fourth order. They are given by

\[
H_1(r, \theta) = \hat{H}_{11}(r) \sin \theta + \frac{\hat{H}_{13}(r)}{6} \sin^3 \theta + O(\sin^5 \theta) ,
\]

\[
H_2(r, \theta) = f(r) + \frac{\hat{H}_{22}(r)}{2} \sin^2 \theta + \frac{\hat{H}_{24}}{24} \sin^4 \theta + O(\sin^5 \theta) ,
\]

\[
H_3(r, \theta) = g(r) \sin \theta + \frac{\hat{H}_{33}(r)}{6} \sin^3 \theta + O(\sin^5 \theta) ,
\]

\[
H_4(r, \theta) = f(r) + \frac{\hat{H}_{42}(r)}{2} \sin^2 \theta + \frac{\hat{H}_{44}(r)}{24} \sin^4 \theta + O(\sin^5 \theta) ,
\]

where we have omitted the vanishing terms. The following relations between the coefficient functions are found from the expansion of the differential equations (3)-(8),

\[
\hat{H}_{11} = -r \partial_r f ,
\]

\[
\hat{H}_{22} = -r \partial_r (r \partial_r f) ,
\]

\[
\hat{H}_{24} = 3 \hat{H}_{22} + (r \partial_r \hat{H}_{13} ) ,
\]

\[
\hat{H}_{42} = \frac{1}{3} \left[ 2f(f^2 + 3g - 1) - [r \partial_r]^2 f \right] .
\]

The functions \(\hat{H}_{33}(r)\) and \(\hat{H}_{44}(r)\) may also be expressed in terms of \(f(r)\), \(g(r)\), \(\hat{H}_{13}(r)\), and other functions. Expanding \(\sin \theta\) for small \(\theta\) the functions \(H_1\) and \(H_2\) become

\[
H_1 = -r \partial_r f \theta + (\hat{H}_{13} + r \partial_r f) \frac{\theta^3}{6} + O(\theta^5) ,
\]

\[
H_2 = f - [r \partial_r]^2 f \frac{\theta^2}{2} + (r \partial_r \hat{H}_{13} + [r \partial_r]^2 f) \frac{\theta^4}{24} + O(\theta^5)
\]

\[
\partial_\theta \left( f \theta - [r \partial_r]^2 f \frac{\theta^3}{6} \right) + (r \partial_r \hat{H}_{13} + [r \partial_r]^2 f) \frac{\theta^4}{24} + O(\theta^5) .
\]

Comparing with Eqs. (13) and (14) we see that for winding number \(n = 3\) a gauge transformation

\[
\Gamma = \Gamma^{(3)}(r, \theta) \equiv - \left\{ (f - 1) \theta - [r \partial_r]^2 f \frac{\theta^3}{6} \right\}
\]

yields gauge transformed functions
\[ \dot{H}_1 = (\dot{H}_{13} + r \dot{\partial}_r f - [r \partial_r f]^3) \theta_2^3 + O(\theta^5) = \frac{\dot{H}_{13} + r \dot{\partial}_r f - [r \partial_r f]^3 f}{6r^3} \rho^3 + O(\rho^5), \]
\[ 1 - \dot{H}_2 = -(r \dot{\partial}_r \dot{H}_{13} + [r \partial_r f]^2) \theta_2^4 + O(\theta^5) = \frac{-r \dot{\partial}_r \dot{H}_{13} + [r \partial_r f]^2 f}{24r^4} \rho^4 + O(\rho^5), \]

which have the desired behavior on the positive \( z \)-axis, Eqs. (11).

Performing the same gauge transformation to the functions \( \dot{H}_3 \) and \( \dot{H}_4 \) and calculating the gauge transformed functions \( \hat{F}_3 \) and \( \hat{F}_4 \) we find
\[
\hat{F}_3 = \sin \theta \dot{H}_3 + \cos \theta (1 - \dot{H}_4) = F_{33} \sin^4 \theta + O(\theta^5) = \frac{F_{33}}{r^4} \rho^4 + O(\rho^5),
\]
\[
\hat{F}_4 = \cos \theta \dot{H}_3 - \sin \theta (1 - \dot{H}_4) = \left( f^2 + 2g - 1 \right) \frac{\theta^5}{2} + O(\theta^5) = \frac{f^2 + 2g - 1}{2r} \rho + O(\rho^3),
\]

with
\[
F_{33} = \frac{1}{120} \left( 20g(2f^3 + f - r \partial_r^2 f) + 20\dot{H}_{33} f - 5\dot{H}_{44} + 4f(4f^4 - 4 - 5f [r \partial_r]^2 f) \right).
\]

Again, these functions have the desired behavior on the positive \( z \)-axis, Eqs. (11).

For winding number \( n = 3 \) we find for the non vanishing components of the gauge potential in Cartesian coordinates
\[
\dot{A}_x = -\frac{x}{48r^5} \left( r \partial_r \dot{H}_{13} + [r \partial_r f]^2 f - 4(\dot{H}_{13} + r \partial_r f - [r \partial_r f]^3 f) \right) \rho^3 \tau_\phi^3 + \frac{3y}{2r^5} F_{33} \rho^3 \tau_\phi^3 + \frac{3y}{4r^2} [f^2 + 2g - 1] \tau_3,
\]
\[
\dot{A}_y = -\frac{y}{48r^5} \left( r \partial_r \dot{H}_{13} + [r \partial_r f]^2 f - 4(\dot{H}_{13} + r \partial_r f - [r \partial_r f]^3 f) \right) \rho^3 \tau_\phi^3 - \frac{3x}{2r^5} F_{33} \rho^3 \tau_\rho^3 - \frac{3x}{4r^2} [f^2 + 2g - 1] \tau_3,
\]
\[
\dot{A}_z = \frac{1}{12r^4} \left( \dot{H}_{13} + r \partial_r f - [r \partial_r f]^3 f \right) \rho^3 \tau_\phi^3,
\]
to lowest order in the expansion
\[ n = 4 \]

Repeating the calculations of the last two subsections we expand the gauge field functions in \( \sin \theta \) to fifth order,

\[
H_1(r, \theta) = \dot{H}_{11}(r) \sin \theta + \frac{\dot{H}_{13}(r)}{6} \sin^3 \theta + \frac{\dot{H}_{14}(r)}{24} \sin^5 \theta + \frac{\dot{H}_{15}(r)}{120} \sin^7 \theta + O(\sin^8 \theta),
\]
\[
H_2(r, \theta) = f(r) + \frac{\dot{H}_{22}(r)}{2} \sin^2 \theta + \frac{\dot{H}_{24}(r)}{24} \sin^4 \theta + \frac{\dot{H}_{25}(r)}{120} \sin^6 \theta + O(\sin^6 \theta),
\]
\[
H_3(r, \theta) = g(r) \sin \theta + \frac{\dot{H}_{33}(r)}{6} \sin^3 \theta + \frac{\dot{H}_{34}(r)}{120} \sin^5 \theta + O(\sin^6 \theta),
\]
\[
H_4(r, \theta) = f(r) + \frac{\dot{H}_{42}(r)}{2} \sin^2 \theta + \frac{\dot{H}_{44}(r)}{24} \sin^4 \theta + \frac{\dot{H}_{45}(r)}{120} \sin^6 \theta + O(\sin^6 \theta),
\]

where again vanishing terms have been omitted.

From the expansion of the differential equations (4)–(8) the following relations between the coefficient functions are found,

\[
\dot{H}_{11} = -r \partial_r f,
\]
\[
\dot{H}_{13} = -r \partial_r (f + \dot{H}_{22})
\]
\[
= - \left\{ r \partial_r f - [r \partial_r f]^2 f \right\},
\]
\[
\dot{H}_{22} = -r \partial_r (r \partial_r f),
\]
\[
\dot{H}_{24} = 4 \dot{H}_{22} - r \partial_r (r \partial_r \dot{H}_{22})
\]
\[
= -4 [r \partial_r f]^2 f - [r \partial_r f]^4 f,
\]
\[
\dot{H}_{25} = r \partial_r \dot{H}_{14},
\]
\[ \tilde{H}_{42} = \frac{1}{3} \left[ 2f(f^2 + 3g - 1) - |r\partial_r|^2 f \right], \]
\[ \tilde{H}_{44} = -\frac{1}{5} \left\{ |r\partial_r|^4 \tilde{H}_{22} - 4f^2(4f^4 + 10fg + 5\tilde{H}_{22}) - 4f(5\tilde{H}_{33} - 4 + 5g) - 20\tilde{H}_{22g} \right\} = \frac{1}{5} \left\{ |r\partial_r|^4 f + 4f^2(4f^4 + 10fg - 5|r\partial_r|^2 f) + 4f(5\tilde{H}_{33} - 4 + 5g) - 20g|r\partial_r|^2 f \right\}. \]

The functions \( \tilde{H}_{15}(r) \), \( \tilde{H}_{35}(r) \), \( \tilde{H}_{14}(r) \) may also be expressed in terms of \( f(r) \), \( g(r) \), \( \tilde{H}_{14}(r) \) and other functions. Expanding \( \sin \theta \) in terms of \( \theta \) the functions \( H_1 \) and \( H_2 \) become

\[ H_1 = -r\partial_r f\theta + [r\partial_r]^3 f \frac{\theta^4}{6} + \tilde{H}_{14} \frac{\theta^4}{24} + O(\theta^5), \]
\[ H_2 = f - [r\partial_r]^2 f \frac{\theta^2}{2} + [r\partial_r]^4 f \frac{\theta^4}{24} + r\partial_r\tilde{H}_{14} \frac{\theta^5}{120} + O(\theta^6), \]
\[ \partial_\theta \left\{ f\theta - [r\partial_r]^2 f \frac{\theta^3}{6} + [r\partial_r]^4 f \frac{\theta^5}{120} \right\} + r\partial_r\tilde{H}_{14} \frac{\theta^5}{120} + O(\theta^6). \]

As anticipated, we find that for winding number \( n = 4 \) a gauge transformation with

\[ \Gamma = \Gamma^{(4)}(r, \theta) = -\left\{ (f - 1)\theta - [r\partial_r]^2 f \frac{\theta^3}{6} + [r\partial_r]^4 f \frac{\theta^5}{120} \right\} \]

yields gauge transformed functions

\[ \tilde{H}_1 = \tilde{H}_{14} \frac{\theta^4}{24} + O(\theta^5) = \frac{\tilde{H}_{14}}{24r^2}\rho^4 + O(\rho^5), \]
\[ 1 - \tilde{H}_2 = -r\partial_r\tilde{H}_{14} \frac{\theta^5}{120} + O(\theta^6) = -\frac{r\partial_r\tilde{H}_{14}}{120r^3}\rho^5 + O(\rho^6), \]

which have the desired behavior on the positive \( z \)-axis, Eqs. \( (10) \).

Performing the same gauge transformation to the functions \( H_3 \) and \( H_4 \) and calculating the gauge transformed functions \( \tilde{F}_3 \) and \( \tilde{F}_4 \) we find

\[ \tilde{F}_3 = \sin \theta \tilde{H}_3 + \cos \theta (1 - \tilde{H}_4) = -\tilde{H}_{45} \frac{\theta^5}{120} + O(\theta^6) = -\frac{\tilde{H}_{45}}{120r^5}\rho^5 + O(\rho^6), \]
\[ \tilde{F}_4 = \cos \theta \tilde{H}_4 - \sin \theta (1 - \tilde{H}_4) = (f^2 + 2g - 1) \frac{\theta}{2} + O(\theta^3) = \frac{f^2 + 2g - 1}{2r}\rho + O(\rho^3), \]

which also have the desired behavior on the positive \( z \)-axis, Eqs. \( (10) \).

Turning to Cartesian coordinates we find for the non vanishing components of the gauge potential to lowest order in the expansion

\[ \tilde{A}_x = -\frac{x}{240r^6} \left[ r\partial_r \tilde{H}_{14} - 5\tilde{H}_{14} \right] \rho^4 \phi^4 - \frac{y}{60r^6} \tilde{H}_{45} \rho^4 \phi^4 + \frac{y}{r^2} \left[ f^2 + 2g - 1 \right] \gamma_3, \]
\[ \tilde{A}_y = -\frac{y}{240r^6} \left[ r\partial_r \tilde{H}_{14} - 5\tilde{H}_{14} \right] \rho^4 \phi^4 + \frac{x}{60r^6} \tilde{H}_{45} \rho^4 \phi^4 - \frac{x}{r^2} \left[ f^2 + 2g - 1 \right] \gamma_3, \]
\[ \tilde{A}_z = \frac{1}{48r^5} \tilde{H}_{14} \rho^4 \phi^4. \]

The gauge transformation on the negative \( z \)-axis can be found in analogy to the analysis on the positive \( z \)-axis, taking into account the invariance of the differential equations \( (3)-(8) \) under the transformation

\[ \theta \to \pi - \theta, \quad \partial_\theta \to -\partial_\theta, \quad H_1 \to -H_1, \quad H_3 \to -H_3, \quad H_2 \to H_2, \quad H_4 \to H_4. \]

\[ \text{(19)} \]

B. Gauge Transformation at the origin

The same strategy for finding a suitable gauge transformation can be applied at the origin. It turns out, that for winding number \( n \) an expansion to the order \( n + 3 \) is required.
The analysis of the differential equations at the origin gives the following expressions for the expansion of the gauge field functions,

\[ H_1(r, \theta) = r^2 \sin \theta \cos \theta h_{12} + r^3 \sin^2 \theta \cos \theta h_{13} + r^4 \sin \theta \cos \theta \left( 1 - 2 \sin^2 \theta \right) h_{14} + r^5 \sin^2 \theta \cos \theta \left( h_{15} + H_1^{(54)} \sin^2 \theta \right) + O(r^6) \]

\[ = \partial_r \left\{ \frac{1}{4} r^2 \sin(2\theta) h_{12} + \frac{1}{16} r^4 \sin(4\theta) h_{14} \right\} + r^3 \sin^2 \theta \cos \theta h_{13} + r^5 \sin^2 \theta \cos \theta \left( h_{15} + H_1^{(54)} \sin^2 \theta \right) + O(r^6) , \]

\[ H_2(r, \theta) = 1 - \frac{1}{2} r^2 (1 - 2 \sin^2 \theta) h_{12} + r^3 \sin^3 \theta h_{13} - \frac{1}{4} r^4 h_{14} (1 - 8 \sin^2 \theta + 8 \sin^4 \theta) + \frac{1}{3} r^5 \sin^3 \theta \left( 5 h_{15} + 3 H_1^{(54)} \sin^2 \theta \right) + O(r^6) \]

\[ = - \partial_r \left\{ \frac{1}{4} r^2 \sin(2\theta) h_{12} + \frac{1}{16} r^4 \sin(4\theta) h_{14} \right\} + 1 + r^3 \sin^3 \theta h_{13} + \frac{1}{3} r^5 \sin^3 \theta \left( 5 h_{15} + 3 H_1^{(54)} \sin^2 \theta \right) + O(r^6) , \]

\[ H_3(r, \theta) = r^2 \sin \theta \cos \theta h_{32} + r^4 \sin \theta \cos \theta \left( h_{34c} + h_{34s} \sin^2 \theta \right) + r^5 \sin^4 \cos \theta H_3^{(54)} + O(r^6) , \]

\[ H_4(r, \theta) = 1 - \frac{1}{2} r^2 (h_{12} - 2 h_{32} \sin^2 \theta) + r^4 \left( - \frac{1}{4} h_{14} + H_4^{(42)} \sin^2 \theta + h_{34s} \sin^4 \theta \right) - r^5 \sin^3 \theta \cos^2 \theta H_3^{(54)} + O(r^6) , \]

with

\[ H_3^{(54)} = \frac{1}{2} e^{2\phi_0} h_{13} (h_{12} - 2 h_{32})^2 + h_{13s} \phi^{(2)} + \frac{1}{12} h_{15} , \]

\[ H_4^{(42)} = \frac{1}{4} \left( h_{12}^2 - 2 h_{12} h_{32} + 2 h_{14} + 4 h_{34c} \right) , \]

\[ H_1^{(54)} = - \frac{1}{6} e^{2\phi_0} h_{13} (h_{12} - 2 h_{32})^2 + \frac{1}{3} h_{13s} \phi^{(2)} - \frac{1}{12} \left( 23 h_{15} + 4 h_{13} (h_{12} - 2 h_{32}) \right) , \]

\[ \phi^{(2)} = \frac{2}{(h_{12} - 2 h_{32})} \left( - \frac{9 h_{12}^2}{32} + 16 h_{12} h_{32} - 6 h_{14} - 40 h_{34c} - h_{34s} \right) \]

and where \( \phi_0 \) and \( h_{1j} \) are constants.

It can be seen easily, that a gauge transformation with

\[ \Gamma = \Gamma^{(2)}_{(o)}(r, \theta) = \left\{ \frac{1}{4} r^2 \sin(2\theta) h_{12} + \frac{1}{16} r^4 \sin(4\theta) h_{14} \right\} + O(r^6) \]

removes the low order terms in \( H_1 \) and \( 1 - H_2 \). The gauge transformed functions become

\[ \hat{H}_1 = r^2 \sin^2 \theta \cos \theta h_{13} + r^5 \sin^2 \theta \cos \theta \left( h_{15} + H_1^{(54)} \sin^2 \theta \right) + O(r^6) \]

\[ = z \rho^2 \left[ h_{13} + z^2 h_{15} + \rho^2 (H_1^{(54)} + h_{15}) \right] + O(r^6) , \]

\[ 1 - \hat{H}_2 = -r^3 \sin^3 \theta h_{13} - \frac{1}{3} r^5 \sin^3 \theta \left( 5 h_{15} + 3 H_1^{(54)} \sin^2 \theta \right) + O(r^6) \]

\[ = -\rho^3 \left[ h_{13} + \frac{1}{3} (5 z^2 h_{15} + \rho^2 (3 H_1^{(54)} + 5 h_{15})) \right] + O(r^6) . \]

Performing the same gauge transformation to the functions \( H_3 \) and \( H_4 \) we find for the functions \( \hat{F}_3 \) and \( \hat{F}_4 \)
\[ \hat{F}_3 = \frac{1}{12} z r^3 F_{33o} + O(r^6), \]
\[ \hat{F}_4 = -\frac{1}{2} r \rho (h_{12} - 2h_{32}) + O(r^4), \]

with
\[ F_{33o} = \frac{1}{12} \left[ h_{15} + 12h_{13} \phi^{(2)} + 6e^{2\phi_0} h_{13}(h_{12} - 2h_{32})^2 \right]. \]

Thus the gauge transformed functions fulfill the requirement, Eqs. (10) up to order \( r^n \).

For the gauge transformed functions we find
\[ \hat{A}_r = -\frac{1}{3} z x h_{13} \rho^2 r^2 + z y F_{33o} \rho^2 r^2 - \frac{1}{2} y (h_{12} - 2h_{32}) r_3, \]
\[ \hat{A}_\theta = -\frac{1}{3} z y h_{13} \rho^2 r^2 - z x F_{33o} \rho^2 r^2 + \frac{1}{2} x (h_{12} - 2h_{32}) r_3, \]
\[ \hat{A}_\varphi = \frac{1}{2} h_{13} \rho^2 r^2, \]
to leading order in the expansion Eq. (20).

\[ n = 3 \text{ and } n = 4 \]

The same analysis can be performed for higher winding numbers. In this section we give the results for winding number \( n = 3 \) and \( n = 4 \). For winding number \( n = 3 \) the expansion to sixth order is found to be

\[ H_1(r, \theta) = r^2 \sin \theta \cos \theta h_{12} + r^4 \sin \theta \cos \theta (h_{14s} \sin^2 \theta + h_{14c} \cos^2 \theta) + r^6 \sin \theta \cos \theta \left( a_{11} + a_{13} \sin^2 \theta + H_1^{(65)} \sin^4 \theta \right), \]
\[ H_2(r, \theta) = 1 - \frac{1}{2} r^2 (1 - 2 \sin^2 \theta) h_{12} - \frac{1}{4} r^4 (h_{14c} - 8h_{14s} \sin^2 \theta + 4(h_{14c} - h_{14s}) \sin^4 \theta) - \frac{1}{6} r^6 \left( a_{11} - 18a_{11} \sin^2 \theta - 9a_{13} \sin^4 \theta - 6H_1^{(65)} \sin^6 \theta \right), \]
\[ H_3(r, \theta) = r^2 \sin \theta \cos \theta h_{32} + r^4 \cos \theta \left( H_3^{(41)} \sin \theta + H_3^{(43)} \sin^3 \theta \right) + r^6 \cos \theta \left( H_3^{(61)} \sin \theta + H_3^{(63)} \sin^3 \theta + H_3^{(65)} \sin^5 \theta \right), \]
\[ H_4(r, \theta) = 1 - \frac{1}{2} r^2 (h_{12} - 2h_{32} \sin^2 \theta) + r^4 \left( \frac{1}{4} h_{14c} + H_4^{(42)} \sin^2 \theta + H_4^{(43)} \sin^4 \theta \right) + r^6 \left( -\frac{1}{6} a_{11} + H_4^{(62)} \sin^2 \theta + H_4^{(64)} \sin^4 \theta + H_4^{(66)} \sin^6 \theta \right), \]

where \( h_{ij} \), \( a_{ij} \), and \( H_i^{(jk)} \) are constants. The details of the expressions \( H_i^{(jk)} \) are given in the Appendix.

For \( n = 3 \) the gauge transformation function becomes
\[ \Gamma = \Gamma^{(3)}(r, \theta) = \left\{ \frac{1}{4} r^2 \sin(2\theta) h_{12} + \frac{1}{16} r^4 \sin(4\theta) h_{14c} + \frac{1}{36} r^6 \sin(6\theta) a_{11} \right\} + O(r^7). \]

For the gauge transformed functions we find
\[ \hat{H}_1 = z \rho^3 (h_{14s} + h_{14c}) + \frac{1}{3} z \rho^3 \left( 3(a_{13} + H_1^{(65)}) \rho^2 + (16a_{11} + 3a_{13}) z^2 \right) + O(r^7), \]
\[ 1 - \hat{H}_2 = \rho^4 (h_{14s} + h_{14c}) - \rho^4 \frac{1}{6} \left( (16a_{11} + 3(3a_{13} + 2H_1^{(65)})) \rho^2 + 3(16a_{11} + 3a_{13}) z^2 \right) + O(r^7), \]
\[ \hat{F}_3 = z r \rho^4 F_{33o} + O(r^7), \]
\[ \hat{F}_4 = -\frac{1}{2} r \rho (h_{12} - 2h_{32}) + O(r^4), \]
where $F_{34a}$ is a constant given by

$$F_{34a} = \frac{1}{90} \left[ 16a_{11} + 3a_{13} + 48(h_{14x} + h_{14c})(h_{12} - 2h_{32})^2 \right].$$

Near the origin the components of the transformed gauge potential in Cartesian coordinates become

$$\hat{A}_x = \frac{1}{12} z^2 (16a_{11} + 3a_{13}) r^3 \tau^3_x + \frac{3}{2} z y F_{34o} \rho^3 \tau^3_o - \frac{3}{4} y (h_{12} - 2h_{32}) \tau_3,$$
$$\hat{A}_y = -\frac{1}{12} z y (16a_{11} + 3a_{13}) \rho^3 \tau^3_o - \frac{3}{2} z x F_{34o} \rho^3 \tau^3_o + \frac{3}{4} x (h_{12} - 2h_{32}) \tau_3,$$
$$\hat{A}_z = \frac{1}{2} (h_{14x} + h_{14c}) \rho^3 \tau^3_o,$$

to leading order in the expansion Eq. (21).

For winding number $n = 4$ the expansion to seventh order is found to be

$$H_1(r, \theta) = r^2 \sin \theta \cos \theta h_{12} - r^4 \sin \theta \cos \theta h_{14}(1 - 2 \sin^2 \theta) + \ldots$$
$$H_2(r, \theta) = 1 - \frac{1}{2} r^2 (1 - 2 \sin^2 \theta) h_{12} + \frac{1}{4} r^4 h_{14}(1 - 8 \sin^2 \theta + 8 \sin^4 \theta) + \ldots$$
$$H_3(r, \theta) = r^2 \sin \theta \cos \theta h_{32} + r^4 \cos \theta \left( H_3^{(41)} \sin \theta + H_3^{(43)} \sin^3 \theta \right) + \ldots$$
$$H_4(r, \theta) = 1 - \frac{1}{2} r^2 (h_{12} - 2h_{32} \sin^2 \theta) + r^4 \left( \frac{1}{4} h_{14} + H_4^{(42)} \sin^2 \theta + H_3^{(43)} \sin^4 \theta \right) + \ldots$$

with

$$H_1^{(7)} = -e^{2\phi_0} h_{15} \cos^2 \theta (h_{12} - 2h_{32})^2 + h_{15} \phi^{(2)} \cos^2 \theta + \frac{1}{8} (23d_{12} \sin^2 \theta - 15d_{12} - 12h_{15}(h_{12} - 2h_{32}) \cos^2 \theta),$$
$$H_2^{(7)} = -\frac{1}{5} e^{2\phi_0} h_{15} (7 - 5 \sin^2 \theta)(h_{12} - 2h_{32})^2 + \frac{1}{5} h_{15} \phi^{(2)} (7 - 5 \sin^2 \theta) - \frac{3}{10} h_{15}(h_{12} - 2h_{32})(7 - 5 \sin^2 \theta) - \frac{1}{8} d_{12}(21 - 23 \sin^2 \theta),$$
$$H_3^{(7)} = \frac{13}{20} e^{2\phi_0} h_{15}(h_{12} - 2h_{32})^2 + \frac{7}{20} h_{15} \phi^{(2)} - \frac{1}{160} (5d_{12} + 4h_{15}(h_{12} - 2h_{32})),$$

where $h_{ij}$, $a_{11}$, $d_{12}$, $\phi^{(2)}$, $\phi_0$ and the $H_i^{(jk)}$ are constants. The details of the expressions $H_i^{(jk)}$ are given in the Appendix.

For $n = 4$ the gauge transformation function becomes

$$\Gamma = \Gamma^{(4)}(r, \theta) = \left\{ \frac{1}{4} r^2 \sin(2\theta) h_{12} - \frac{1}{16} r^4 \sin(4\theta) h_{14} + \frac{1}{36} r^6 \sin(6\theta) a_{11} \right\} + O(r^7).$$

Near the origin the transformed gauge potential in Cartesian coordinates becomes

$$\hat{A}_x = z x F_{15o} \rho^4 \tau^4_x - z y F_{35o} \rho^4 \tau^4_o - y (h_{12} - 2h_{32}) \tau_3,$$
$$\hat{A}_y = z y F_{15o} \rho^4 \tau^4_o - z x F_{35o} \rho^4 \tau^4_o + x (h_{12} - 2h_{32}) \tau_3,$$
$$\hat{A}_z = \frac{1}{2} h_{15} \rho^3 \tau^3_o,$$
with
\[ F_{150} = \frac{1}{40} \left( 15d_{12} + 4h_{15} \left[ 3(h_{12} - 2h_{32}) - 2\phi^{(2)} + 2e^{2\phi_0}(h_{12} - 2h_{32})^2 \right] \right) , \]
\[ F_{350} = -\frac{1}{80} \left( 5d_{12} + 4h_{15} \left[ h_{12} - 2h_{32} - 14\phi^{(2)} - 26e^{2\phi_0}(h_{12} - 2h_{32})^2 \right] \right) , \]
to leading order in the expansion Eq. (22).

As for winding number \( n = 2 \) the gauge transformed functions for \( n = 3 \) and \( n = 4 \) fulfill the requirement, Eqs. (10) up to order \( r^5 \) and \( r^7 \), respectively.

C. Gauge Transformation on the overlap

Next we compare the gauge transformations \( \Gamma^{(n)}_{(z)} \) on the positive \( z \)-axis and \( \Gamma^{(n)}_{(o)} \) at the origin. By comparison of the expansions of the functions \( H_i(r, \theta) \) near the \( z \)-axis and near the origin we can determine the expansion of the function \( f(r) \) at the origin and calculate the gauge transformation function \( \Gamma_{(z)}(r, \theta) \) for small values of \( r, \theta \). For winding number \( n = 4 \) for example, we find
\[
\Gamma_{(z)}^{(4)}(r, \theta) = \left( -\frac{1}{2}r^2 h_{12} + \frac{1}{4}r^4 h_{14} - \frac{1}{6}r^6 a_{11} \right) \theta \\
- (-2r^2 h_{12} + 4r^4 h_{14} - 6r^6 a_{11}) \frac{1}{6} \theta^3 \\
+ (8r^2 h_{12} + 64r^4 h_{14} - 216r^6 a_{11}) \frac{1}{120} \theta^5 \\
= -\frac{1}{4}r^2 h_{12}(2\theta - \frac{1}{6}(2\theta)^3 + \frac{1}{120}(2\theta)^5) \\
+ \frac{1}{16}r^4 h_{14}(4\theta - \frac{1}{6}(4\theta)^3 + \frac{1}{120}(4\theta)^5) \\
- \frac{1}{36}r^6 a_{11}(6\theta - \frac{1}{6}(6\theta)^3 + \frac{1}{120}(6\theta)^5) .
\]
The same expression can be obtained by expanding \( \Gamma^{(4)}_{(o)}(r, \theta) \), Eq. (23), for small values of \( \theta \). Thus both gauge transformations coincide on the neighborhood where the expansion for small \( \theta \) and small \( r \) are comparable. The same holds for winding number \( n = 2, 3 \).

Similarly, the gauge transformation function on the negative \( z \)-axis can be compared with \( \Gamma^{(n)}_{(o)}(r, \theta) \). Again the both gauge transformations coincide where the expansions are comparable.

IV. DISCUSSION AND CONCLUSIONS

In this paper we have examined the question as to how static axially symmetric solutions of the Yang-Mills-dilaton theory, given within a singular Ansatz of the gauge potential, can be gauge transformed into regular form. We have shown that the field strength tensor can be calculated straightforwardly from the singular Ansatz, if we impose the condition of finite energy density on the gauge field functions. Although the field strength tensor itself is not well defined on the singular axis and the origin, the Lagrange density is well defined in terms of the gauge field functions. Consequently the equations of motion reduce to a boundary value problem for the gauge field functions and the dilaton function. From the analysis of this boundary value problem we found that the singular parts of the \( A_r \) and \( A_\theta \) components of the gauge potential behave like a pure gauge along the singular axis and at the origin. With suitable local gauge transformations the non-regular parts have been gauged away. Applying the same gauge transformation to the \( A_z \) component of the singular gauge potential leads to a regular form, too. Thus in total the gauge transformed gauge potential is regular.

The gauge transformation itself is not well defined. However, it does not introduce new terms in the field strength tensor. It can be checked, that along the singular axis and at the origin the field strength tensor \( \tilde{F}_{\mu\nu}[A] \), calculated from the regular gauge transformed potential \( \tilde{A} \), coincides with the gauge rotated field strength tensor \( U F_{\mu\nu}[A] U^\dagger \) in terms of the singular gauge potential \( A \), provided the gauge field functions are used in their local expansions, i.e. for solutions of the boundary value problem.
The gauge transformation functions $\Gamma^{(n)}_{i \alpha}(r)$ are determined only in the vicinity of the $z$-axis or the origin, respectively, and have to be considered as local gauge transformations. Away from the $z$-axis or the origin the gauge transformation matrices $U_{(z)}$, $U_{(0)}$ might approach unity sufficiently fast. Then the gauge transformed functions $\tilde{H}_i$ behave like the original functions $H_i$ except in a region near the $z$-axis or the origin.

The gauge transformation derives from the behavior of the gauge field functions on the singular axis and at the origin which in turn depend on the theory under consideration. Thus, the gauge transformations given in this paper for Yang-Mills-dilaton theory do not lead in general to regular gauge potentials in different theories involving the static axially symmetric Ansatz, Eq. (1). However, we have outlined a possible way how one can proceed to find suitable gauge transformations in theories, where multi-monopoles [10,11] or multi-sphalerons [14,16–19] have been constructed within the Ansatz Eq. (1). For example, for the multi-sphalerons in Einstein-Yang-Mills(-dilaton) theory the expansion on the positive $z$-axis up to third order in $\theta$ is given in ref. [1]. From their result it can be seen easily, that for winding number $n = 2$ the gauge transformation leading to a regular gauge potential along the positive $z$-axis coincides with the corresponding gauge transformation given in this paper, Eq. (18). We conjecture, that the same result will hold for the static axially symmetric black hole solutions in Einstein-Yang-Mills(-dilaton) theory constructed in ref. [19]. More work has to be done for the sphaleron and black hole solutions carrying larger winding numbers [26].

Finally let us compare with the spherically symmetric case. The Ansatz for the gauge potential

$$A_i(\vec{r}) = \frac{a(r) - 1}{2r} \epsilon_{ij\alpha} \frac{r^j}{r} \tau_{\alpha},$$

is not well defined at the origin, $r = 0$, even when the boundary condition $a(0) = 1$ is imposed. Regularity requires $a(r) = 1 - b r^2 + O(r^3)$, where $b$ is a constant. This condition is not guaranteed a priori. Instead, for a given model it has to be checked whether the regularity condition follows from the analysis of the solution at the origin, i.e. at the singular point. The similarity to the axially symmetric case is striking. In that case we have also performed the analysis at the singularities, the $z$-axis and the origin, and found that the gauge potential is regular, if we apply suitable gauge transformations.

**Acknowledgments** The author thanks Y. Brihaye, P. Kosinski and J. Kunz for helpful discussions. This work was carried out under Basic Science Research project SC/97/636 of FORBAIRT.
E. E. Donets and D. V. Gal’tsov, Stringy sphalerons and non-abelian black holes, Phys. Lett. B302 (1993) 411;
G. Lavrelashvili and D. Maison, Regular and black hole solutions of Einstein-Yang-Mills-dilaton theory, Nucl. Phys. B410 (1993) 407;
T. Torii and K. Maeda, Black holes with non-Abelian hair and their thermodynamical properties, Phys. Rev. D 48 (1993) 1643;
P. Bizon, Saddle points of stringy action, Acta Phys. Pol. B 24 (1993) 1209;
C. M. O’Neill, Einstein-Yang-Mills theory with a massive dilaton and axion: String-inspired regular and black hole solutions, Phys. Rev. D 50 (1994) 865;
B. Kleihaus, J. Kunz and A. Sood, SU(3) Einstein-Yang-Mills-dilaton sphalerons and black holes, Phys. Lett. B372 (1996) 204;
B. Kleihaus, J. Kunz and A. Sood, Sequences of Einstein-Yang-Mills-dilaton black holes, Phys. Rev. D 54 (1996) 5070.

C. Rebbi and P. Rossi, Multimonopole solutions in the Prasad-Sommerfield limit, Phys. Rev. D 22 (1980) 2010.
P. Forgacs, Z. Horváth and L. Palla, Non-linear superposition of monopoles, Nucl. Phys. B192 (1981) 141.
R. S. Ward, A Yang-Mills-Higgs monopole of charge 2, Commun. Math. Phys. 79 (1981) 317;
M. K. Prasad, Exact Yang-Mills Higgs monopole solutions of arbitrary charge, Commun. Math. Phys. 80 (1981) 137.

B. Kleihaus, J. Kunz and D. H. Tchrakian, Interaction energy of ’t Hooft-Polyakov monopoles, Mod. Phys. Lett. A13 (1998) 2523.

B. Kleihaus, D. O’Keeffe and D. H. Tchrakian, Interaction energies of generalised monopoles, Nucl. Phys. B 536 (1998) 381.

B. Kleihaus, J. Kunz and Y. Brihaye, The electroweak sphaleron at physical mixing angle, Phys. Lett. B273 (1991) 100;
J. Kunz, B. Kleihaus and Y. Brihaye, Sphalerons at finite mixing angle, Phys. Rev. D 46 (1992) 3587;
Y. Brihaye, B. Kleihaus and J. Kunz, Sphalerons at finite mixing angle and singular gauges, Phys. Rev. D 47 (1993) 1664.

F. R. Klinkhamer, Construction of a new electroweak sphaleron, Nucl. Phys. B410 (1993) 343.

B. Kleihaus and J. Kunz, Multisphalerons in the weak interactions, Phys. Lett. B329 (1994) 61;
B. Kleihaus and J. Kunz, Multisphalerons in the Weinberg-Salam theory, Phys. Rev. D 50 (1994) 5343;
Y. Brihaye and J. Kunz, On axially symmetric solutions in the electroweak theory, Phys. Rev. D 50 (1994) 1051.

B. Kleihaus and J. Kunz, Axially symmetric multisphalerons in Yang-Mills-dilaton theory, Phys. Lett. B392 (1997) 135.

B. Kleihaus and J. Kunz, Static axially symmetric solutions of Einstein-Yang-Mills-dilaton theory, Phys. Rev. Lett. 78 (1997) 2527.

B. Kleihaus and J. Kunz, Static axially symmetric Einstein-Yang-Mills-dilaton solutions: Regular solutions, Phys. Rev. D 57 (1998) 834.

H. W. Braden and V. Varela, Bogomol’nyi equations and solutions for Einstein-Yang-Mills-dilaton-sigma models, Phys. Rev. D 58, 124020 (1998).

B. Kleihaus and J. Kunz, Static black hole solutions with axial symmetry, Phys. Rev. Lett. 79 (1997) 1595;
B. Kleihaus and J. Kunz, Static axially symmetric Einstein-Yang-Mills-dilaton solutions. II. Black hole solutions, Phys. Rev. D 57 (1998) 6138.

N. S. Manton, Complex structure of monopoles, Nucl. Phys. B135 (1978) 319.

E. Witten, Phys. Rev. Lett. 38 (1977) 121

J. Arafune, P. G. O. Freund and C. J. Goebel, Topology of Higgs fields, J. Math. Phys. 16 (1975) 433.

P. Goddard and D. I. Olive, Magnetic monopoles in gauge field theories, Rep. Prog. Phys. 41 (1978) 1357.

G. Wentzel, Comments on Dirac’s Theory of Magnetic Monopoles, Progr. Theor. Phys.(Kyoto) Suppl. 37 and 38 (1966) 163.

T. T. Wu and C. N. Yang, Concept of nonintegrable phase factors and global formulation of gauge fields, Phys. Rev. D 12 (1975) 3857.

J. Kunz and B. Kleihaus, in preparation.
APPENDIX

In this section the expressions $H_i^{(jk)}$ used in section III for winding number $n = 3$ and $n = 4$ are given.

**$n = 3$**

\[ H_1^{(65)} = \frac{1}{80} \left( 9(h_{14c} + h_{14s}) \left( 4\phi^{(2)} - 5(h_{12} - 2h_{32}) - 3e^{2\phi_0} (h_{12} - 2h_{32})^2 \right) - 2(144a_{11} + 67a_{13}) \right) \]

\[ H_3^{(43)} = 2\phi^{(2)} (h_{12} - 2h_{32}) + \frac{1}{8} (9h_{12}^2 - 16h_{12}h_{32} + 6h_{14c} + 40h_{34}) \]

\[ H_3^{(41)} = -2\phi^{(2)} (h_{12} - 2h_{32}) - \frac{1}{8} (9h_{12}^2 - 16h_{12}h_{32} + 6h_{14c} + 32h_{34}) \]

\[ H_3^{(65)} = \frac{3}{2} e^{2\phi_0} \phi^{(2)} (h_{12} - 2h_{32})^3 - 12 e^{2\phi_0} (h_{12} - 2h_{32})^2 (h_{14c} + h_{14s}) + \frac{4}{3} (\phi^{(2)})^2 (h_{12} - 2h_{32}) \]

\[ H_3^{(63)} = -3e^{2\phi_0} \phi^{(2)} (h_{12} - 2h_{32})^3 - \frac{84}{5} e^{2\phi_0} (h_{12} - 2h_{32})^2 (h_{14c} + h_{14s}) - \frac{8}{3} (\phi^{(2)})^2 (h_{12} - 2h_{32}) \]

\[ H_3^{(61)} = \frac{3}{2} e^{2\phi_0} \phi^{(2)} (h_{12} - 2h_{32})^3 - \frac{24}{5} e^{2\phi_0} (h_{12} - 2h_{32})^2 (h_{14c} + h_{14s}) + \frac{4}{3} (\phi^{(2)})^2 (h_{12} - 2h_{32}) \]

\[ H_4^{(42)} = -2\phi^{(2)} (h_{12} - 2h_{32}) - \frac{1}{8} (7h_{12}^2 - 12h_{12}h_{32} + 2h_{14c} + 32h_{34}) \]

\[ H_4^{(66)} = \frac{3}{2} e^{2\phi_0} \phi^{(2)} (h_{12} - 2h_{32})^3 - 12 e^{2\phi_0} (h_{12} - 2h_{32})^2 (h_{14c} + h_{14s}) + \frac{4}{3} (\phi^{(2)})^2 (h_{12} - 2h_{32}) \]

\[ H_4^{(64)} = -3e^{2\phi_0} \phi^{(2)} (h_{12} - 2h_{32})^3 + \frac{81}{5} e^{2\phi_0} (h_{12} - 2h_{32})^2 (h_{14c} + h_{14s}) - \frac{8}{3} (\phi^{(2)})^2 (h_{12} - 2h_{32}) \]

\[ H_4^{(62)} = \frac{3}{2} e^{2\phi_0} \phi^{(2)} (h_{12} - 2h_{32})^3 - \frac{24}{5} e^{2\phi_0} (h_{12} - 2h_{32})^2 (h_{14c} + h_{14s}) + \frac{4}{3} (\phi^{(2)})^2 (h_{12} - 2h_{32}) \]

\[ H_4^{(61)} = \frac{3}{2} e^{2\phi_0} \phi^{(2)} (h_{12} - 2h_{32})^3 - \frac{24}{5} e^{2\phi_0} (h_{12} - 2h_{32})^2 (h_{14c} + h_{14s}) + \frac{4}{3} (\phi^{(2)})^2 (h_{12} - 2h_{32}) \]
\[ n = 4 \]

\[ H_3^{(43)} = 2\phi^{(2)}(h_{12} - 2h_{32}) + \frac{1}{8}(9h_{12}^2 - 16h_{12}h_{32} - 6h_{14} + 40h_{34}) \]

\[ H_3^{(41)} = -2\phi^{(2)}(h_{12} - 2h_{32}) - \frac{1}{8}(9h_{12}^2 - 16h_{12}h_{32} - 6h_{14} + 32h_{34}) \]

\[ H_3^{(65)} = \frac{8}{3}e^{2\phi_0}\phi^{(2)}(h_{12} - 2h_{32})^3 + \frac{4}{3}(\phi^{(2)})^2(h_{12} - 2h_{32}) \]
\[ - \frac{1}{3}\left( (5\phi^{(2)}h_{12} + 28\phi_{41})(h_{12} - 2h_{32}) - 8\phi^{(2)}(h_{14} - 4h_{34}) \right) \]
\[ + \frac{1}{12}(-188a_{12} + 1512a_{32} - 147h_{12}^3 + 297h_{12}^2h_{32} + 333h_{12}h_{14} - 576h_{12}h_{34} - 342h_{14}h_{32}) \]

\[ H_3^{(63)} = -\frac{16}{3}e^{2\phi_0}\phi^{(2)}(h_{12} - 2h_{32})^3 - \frac{8}{3}(\phi^{(2)})^2(h_{12} - 2h_{32}) \]
\[ + \frac{1}{3}\left( (7\phi^{(2)}h_{12} + 44\phi_{41})(h_{12} - 2h_{32}) - 10(h_{14} - 4h_{34})\phi^{(2)} \right) \]
\[ + \frac{1}{12}(272a_{12} - 2016a_{32} + 219h_{12}^3 - 441h_{12}^2h_{32} - 486h_{12}h_{14} + 864h_{12}h_{34} + 486h_{14}h_{32}) \]

\[ H_3^{(61)} = \frac{8}{3}e^{2\phi_0}\phi^{(2)}(h_{12} - 2h_{32})^3 + \frac{4}{3}(\phi^{(2)})^2(h_{12} - 2h_{32}) \]
\[ - \frac{2}{3}\left( (\phi^{(2)}h_{12} + 2\phi_{41})(h_{12} - 2h_{32}) - (h_{14} - 4h_{34})\phi^{(2)} \right) \]
\[ + \frac{1}{24}(-28a_{12} + 192a_{32} - 24h_{12}^3 + 48h_{12}^2h_{32} + 51h_{12}h_{14} - 96h_{12}h_{34} - 48h_{14}h_{32}) \]

\[ H_4^{(42)} = -2\phi^{(2)}(h_{12} - 2h_{32}) - \frac{1}{8}(7h_{12}^2 - 12h_{12}h_{32} - 2h_{14} + 32h_{34}) \]

\[ H_4^{(64)} = -\frac{16}{3}e^{2\phi_0}\phi^{(2)}(h_{12} - 2h_{32})^3 - \frac{8}{3}(\phi^{(2)})^2(h_{12} - 2h_{32}) \]
\[ + \frac{2}{3}\left( (2\phi^{(2)}h_{12} + 22\phi_{41})(h_{12} - 2h_{32}) - 5(h_{14} - 4h_{34})\phi^{(2)} \right) \]
\[ + \frac{1}{144}(416a_{12} - 4032a_{32} + 345h_{12}^3 - 702h_{12}^2h_{32} - 846h_{12}h_{14} + 1368h_{12}h_{34} + 900h_{14}h_{32}) \]

\[ H_4^{(62)} = \frac{8}{3}e^{2\phi_0}\phi^{(2)}(h_{12} - 2h_{32})^3 + \frac{4}{3}(\phi^{(2)})^2(h_{12} - 2h_{32}) \]
\[ + \frac{1}{3}\left( (\phi^{(2)}h_{12} - 16\phi_{41})(h_{12} - 2h_{32}) + 2(h_{14} - 4h_{34})\phi^{(2)} \right) \]
\[ + \frac{1}{144}(-40a_{12} + 1152a_{32} - 69h_{12}^3 + 144h_{12}^2h_{32} + 216h_{12}h_{14} - 288h_{12}h_{34} - 252h_{14}h_{32}) \]