A comparison of the Almgren-Pitts and the Allen-Cahn min-max theory

Akashdeep Dey *

Abstract

Min-max theory for the Allen-Cahn equation was developed by Guaraco [Gua18] and Gaspar-Guaraco [GG18]. They showed that the Allen-Cahn widths are greater than or equal to the Almgren-Pitts widths. In this article we will prove that the reverse inequalities also hold i.e. the Allen-Cahn widths are less than or equal to the Almgren-Pitts widths. Hence, the Almgren-Pitts widths and the Allen-Cahn widths coincide. We will also show that all the closed minimal hypersurfaces (with optimal regularity) which are obtained from the Allen-Cahn min-max theory are also produced by the Almgren-Pitts min-max theory. As a consequence, we will point out that the index upper bound in the Almgren-Pitts setting, proved by Marques-Neves [MN16] and Li [Li19a], can also be obtained from the index upper bound in the Allen-Cahn setting, proved by Gaspar [Gas20] and Hiesmayr [Hie18].

1 Introduction

Minimal submanifolds are defined by the condition that they are the critical points of the area functional. In [Alm62, Alm65], Almgren studied the topology of the space of cycles and developed a min-max theory for the area functional. He proved that any closed, Riemannian manifold $(M^{n+1}, g)$ contains a minimal variety of dimension $l$ for every $1 \leq l \leq n$. The regularity theory in the co-dimension 1 case was further developed by Pitts [Pit81] and Schoen-Simon [SS81]. They proved that in a closed, Riemannian manifold $(M^{n+1}, g)$, $n+1 \geq 3$, there exists a closed, minimal hypersurface which is smooth and embedded outside a singular set of Hausdorff dimension $\leq n - 7$.

In recent years, there have been a lot of research activities in the Almgren-Pitts min-max theory. By the work of Marques-Neves [MN17] and Song [Son18], every closed Riemannian manifold $(M^{n+1}, g)$, $3 \leq n+1 \leq 7$, contains infinitely many closed, minimal hypersurfaces. This was conjectured by Yau [Yau82]. In [IMN18], Irie, Marques and Neves proved that for a generic metric $g$ on $M$, the union of all closed, minimal hypersurfaces is dense in $(M, g)$. This theorem was later quantified by Marques, Neves and Song in [MNS19] where they proved that for a generic metric there exists an equidistributed sequence of closed, minimal hypersurfaces in $(M, g)$. In higher dimensions, Li [Li19b] has proved that every closed Riemannian manifold equipped with a generic metric contains infinitely many closed minimal hypersurfaces with

*Email: adey@math.princeton.edu, dey.akash01@gmail.com
optimal regularity. The Weyl law for the volume spectrum \( \{ \omega_k \}_{k=1}^\infty \), proved by Liokumovich, Marques and Neves [LMN18] played a major role in the arguments of [IMN18, MNS19, Li19b].

The Morse index of the minimal hypersurfaces produced by the Almgren-Pitts min-max theory has been obtained by Marques and Neves when the ambient dimension \( 3 \leq n + 1 \leq 7 \). In [MN16], Marques and Neves showed that the index of the min-max minimal hypersurface is bounded from above by the dimension of the parameter space. Zhou [Zho19] has proved that for a generic (bumpy) metric, the min-max minimal hypersurfaces have multiplicity one which was conjectured by Marques and Neves. Using the Morse index upper bound [MN16] and multiplicity one theorem [Zho19], Marques and Neves [MN18] have proved the following theorem. For a generic (bumpy) metric there exists a sequence of closed, embedded, two-sided minimal hypersurfaces \( \{ \Sigma_k \}_{k=1}^\infty \) in \((M^{n+1},g)\) such that Ind(\(\Sigma_k\)) = \(k\) and \(H^n(\Sigma_k) = \omega_k \sim k^{\frac{1}{n+1}}\).

This theorem has been generalized by Marques, Montezuma and Neves in [MMN20] where they have proved the strong Morse inequalities for the area functional. In higher dimensions, Morse index upper bound has been proved by Li [Li19a].

In [Gua18], Guaraco introduced a new approach for the min-max construction of minimal hypersurfaces which was further developed by Gaspar and Guaraco in [GG18]. This approach is based on the study of the limiting behaviour of solutions to the Allen-Cahn equation. The Allen-Cahn equation (with parameter \( \varepsilon > 0 \)) is the following semi-linear, elliptic PDE

\[
AC_\varepsilon(u) := \varepsilon \Delta u - \varepsilon^{-1} W'(u) = 0
\]

where \( W : \mathbb{R} \to \mathbb{R} \) is a double well potential e.g. \( W(t) = \frac{1}{4}(1-t^2)^2 \). The solutions of this equation are precisely the critical points of the energy functional

\[
E_\varepsilon(u) = \int_M \frac{\varepsilon |\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}.
\]

Building on the work of Hutchinson-Tonegawa [HT00], Tonegawa [Ton05] and Tonegawa-Wickramasekera [TW12], Guaraco [Gua18] proved that if \( \{u_i\}_{i=1}^\infty \) is a sequence of solutions to the Allen-Cahn equation \( AC_\varepsilon_i(u_i) = 0 \), \( \varepsilon_i \to 0 \) with \( E_\varepsilon_i(u_i) \) and Ind(\(u_i\)) are uniformly bounded, then, possibly after passing to a subsequence, the level sets of \( u_i \) accumulate around a closed, minimal hypersurface with optimal regularity. (Such a minimal hypersurface is called a limit-interface.) Moreover, by a mountain-pass argument he proved the existence of critical points of \( E_\varepsilon \) with uniformly bounded energy and Morse index. In this way he obtained a new proof of the previously mentioned theorem of Almgren-Pitts-Schoen-Simon. The index of the limit-interface is bounded by the index of the solutions. This was proved by Hiesmayr [Hie18] assuming the limit-interface is two-sided and by Gaspar [Gas20] in the general case.

In [GG18, GG19], Gaspar and Guaraco studied the phase transition spectrum which is the Allen-Cahn analogue of the volume spectrum. They proved that the phase transition spectrum satisfies a Weyl law similar to the volume spectrum and gave alternative proofs of the density [IMN18] and the equidistribution [MNS19] theorems. In [CM20], Chodosh and Mantoulidis proved the multiplicity one conjecture in the Allen-Cahn setting in dimension 3 and the upper semicontinuity of the Morse index when the limit-interface has multiplicity one. As a consequence, they proved that for a generic (bumpy) metric \( g \) on a closed manifold \( M^3 \), there
exists a sequence of closed, embedded, two-sided minimal surfaces \( \{ \Sigma_k \}_{k=1}^{\infty} \) in \((M^3, g)\) such that \( \text{Ind}(\Sigma_k) = k \) and area\( (\Sigma_k) \sim k^{1/3} \).

If \( \Sigma \) is a non-degenerate, separating, closed, embedded minimal hypersurface in a closed Riemannian manifold, Pacard and Ritoré [PR03] constructed solutions of the Allen-Cahn equation \( AC_\varepsilon(u) = 0 \) for sufficiently small \( \varepsilon > 0 \) whose level sets converge to \( \Sigma \). The uniqueness of these solutions has been proved by Guaraco, Marques and Neves [GMN19]. The construction of Pacard and Ritoré has been extended by Caju and Gaspar [CG19] in the case when all the Jacobi fields of \( \Sigma \) are induced by the ambient isometries.

In the present article we will be interested in the question to what extent the Almgren-Pitts min-max theory and the Allen-Cahn min-max theory agree. Part of this question has been answered by Guaraco [Gua18] and Gaspar-Guaraco [GG18]; they proved that the Almgren-Pitts widths are less than or equal to the Allen-Cahn widths. The aim of this article is to prove the reverse inequality i.e the Allen-Cahn widths are less than or equal to the Almgren-Pitts widths.

To precisely state our main result, we need some facts about the universal \( G \)-principal bundle. We will follow the book by Dieck [Die08, Chapter 14.4] and the paper by Gaspar and Guaraco [GG18, Appendix B] where further details can be found. Let \( G \) be a topological group and \( p_G : EG \to BG \) be a universal \( G \)-principal bundle (which is unique up to isomorphism). Given a topological space \( B \), there exists a one-to-one correspondence between the set of homotopy classes of maps \( B \to BG \) and the set of isomorphism classes of numerable \( G \)-principal bundles over \( B \). If \( f_1, f_2 : B \to BG \) are homotopic, \( f_1^*EG \) and \( f_2^*EG \) are isomorphic numerable principal \( G \)-bundles over \( B \). Conversely, if \( E \) is a numerable free \( G \)-space, there exists a \( G \)-map from \( E \) to \( EG \) which is unique up to \( G \)-homotopy. Denoting \( B = E/G \), if \( F_1, F_2 : E \to EG \) are \( G \)-maps, they descend to homotopic maps \( f_1, f_2 : B \to BG \). We also note the following facts: a numerable \( G \)-principal bundle \( p : \mathcal{E} \to \mathcal{B} \) is universal if \( \mathcal{E} \) is a contractible topological space [Die08, 14.4.12]; each open covering of a paracompact space is numerable [Die08, 13.1.3].

We refer to Section 2 for the definitions and notations used in the rest of this section. Let \((M^{n+1}, g)\) be a closed Riemannian manifold, \( n + 1 \geq 3 \). Let \( X \) be a cubical complex and we fix a double cover \( \pi : \tilde{X} \to X \). Since the space \( I_{n+1}(M^{n+1}; \mathbb{F}; \mathbb{Z}_2) \) is contractible [MN18] and every metric space is paracompact, \( \partial : I_{n+1}(M^{n+1}; \mathbb{F}; \mathbb{Z}_2) \to \mathbb{Z}_n(M^{n+1}; \mathbb{F}; \mathbb{Z}_2) \) is a universal \( \mathbb{Z}_2 \)-principal bundle. We denote by \( \Pi \) the homotopy class of maps \( X \to \mathbb{Z}_n(M^{n+1}; \mathbb{F}; \mathbb{Z}_2) \) corresponding to the double cover \( \pi : \tilde{X} \to X \). More concretely, \( \Pi \) is the set of all maps \( \Phi : X \to \mathbb{Z}_n(M^{n+1}; \mathbb{F}; \mathbb{Z}_2) \) such that \( \ker(\Phi_s) = \text{im}(\pi_s) \) where

\[
\Phi_s : \pi_1(X) \to \pi_1(\mathbb{Z}_n(M^{n+1}; \mathbb{F}; \mathbb{Z}_2)) \quad \text{and} \quad \pi_s : \pi_1(\tilde{X}) \to \pi_1(X)
\]

are the maps induced by \( \Phi, \pi \).

Similarly, \( H^1(M) \setminus \{0\} \) is contractible and there is a free \( \mathbb{Z}_2 \) action on this space given by \( u \mapsto -u \). Therefore, \( H^1(M) \setminus \{0\} \) (equipped with the \( \mathbb{Z}_2 \) action) is the total space of a universal \( \mathbb{Z}_2 \)-principal bundle. Let \( \bar{\Pi} \) denote the set of all \( \mathbb{Z}_2 \)-equivariant maps \( h : \tilde{X} \to H^1(M) \setminus \{0\} \) i.e. if \( T : \tilde{X} \to \tilde{X} \) is the deck transformation, \( h(T(x)) = -h(x) \) for all \( x \in \tilde{X} \).
The following theorem follows from the work of Guaraco [Gua18] and Gaspar-Guaraco [GG18].

**Theorem 1.1** ([Gua18, GG18]). Let $L_{AP}(\Pi)$ be the Almgren-Pitts width of $\Pi$ ((2.1)) and $L_\varepsilon(\tilde{\Pi})$ be the $\varepsilon$-Allen-Cahn width of $\tilde{\Pi}$ ((2.2)). Then the following inequality holds.

$$L_{AP}(\Pi) \leq \frac{1}{2\sigma} \liminf_{\varepsilon \to 0^+} L_\varepsilon(\tilde{\Pi}). \quad (1.1)$$

As a consequence, the following inequality holds between the volume spectrum and the phase transition spectrum.

$$\omega_p \leq \frac{1}{2\sigma} \liminf_{\varepsilon \to 0^+} c_\varepsilon(p) \forall p \in \mathbb{N}. \quad (1.2)$$

In the present article we will show that the reverse inequality also holds. More precisely, we will prove the following theorem.

**Theorem 1.2.** We have the following inequality between the Almgren-Pitts width and the $\varepsilon$-Allen-Cahn width.

$$\frac{1}{2\sigma} \limsup_{\varepsilon \to 0^+} L_\varepsilon(\tilde{\Pi}) \leq L_{AP}(\Pi). \quad (1.3)$$

As a consequence we have,

$$\frac{1}{2\sigma} \limsup_{\varepsilon \to 0^+} c_\varepsilon(p) \leq \omega_p \forall p \in \mathbb{N}. \quad (1.4)$$

Hence, combining (1.1) and (1.3) we conclude that $\frac{1}{2\sigma} \lim_{\varepsilon \to 0^+} L_\varepsilon(\tilde{\Pi})$ exists and is equal to $L_{AP}(\Pi)$. Similarly, (1.2) and (1.4) together imply that $\frac{1}{2\sigma} \lim_{\varepsilon \to 0^+} c_\varepsilon(p)$ exists and is equal to $\omega_p$ for all $p \in \mathbb{N}$. When the ambient dimension $3 \leq n + 1 \leq 7$, it was proved by Gaspar and Guaraco [GG19] that $\lim_{\varepsilon \to 0^+} c_\varepsilon(p)$ exists.

The next theorem essentially follows from the work of Hutchinson-Tonegawa [HT00], Guaraco [Gua18] and Gaspar-Guaraco [GG18]. Informally speaking, it says that all the minimal hypersurfaces obtained from the Allen-Cahn min-max theory are also produced by the Almgren-Pitts min-max theory.

**Theorem 1.3.** Let $C_{AC}(\tilde{\Pi})$ be as defined at the end of Section 2.5 and $C_{AP}(\Pi)$ be as defined at the end of Section 2.3. If $V \in C_{AC}(\tilde{\Pi})$, then $V \in C_{AP}(\Pi)$ as well.

Combining the index estimate of Gaspar [Gas20] (Theorem 2.2) and the above Theorem 1.3, one can obtain an alternative proof of the following Morse index upper bound in the Almgren-Pitts min-max theory proved by Marques-Neves [MN16] and Li [Li19a].

**Theorem 1.4** ([MN16, Li19a]). Let $\dim(X) = \dim(\tilde{X}) = k$. There exists $V \in C_{AP}(\Pi)$ such that $\text{Ind}(\text{spt}(V))$ is less than or equal to $k$.

Indeed, by the min-max theory for the Allen-Cahn functional (see Section 2.4), for all sufficiently small $\varepsilon > 0$ there exists a min-max critical point $u_\varepsilon$ of $E_\varepsilon$ (corresponding to the homotopy class $\tilde{\Pi}$) such that $\text{Ind}(u_\varepsilon) \leq k$. Hence, by Theorem 2.1, 2.2 and 1.3, there exists

$$V \in C_{AC}(\tilde{\Pi}) \subset C_{AP}(\Pi)$$

such that $\text{Ind}(\text{spt}(V)) \leq k$. 

4
Acknowledgements. I am very grateful to my advisor Prof. Fernando Codá Marques for many helpful discussions and for his support and guidance. The author is partially supported by NSF grant DMS-1811840.

2 Notations and Preliminaries

2.1 Notations

Here we summarize the notations which will be frequently used later.

$[m]$ the set $\{1, 2, \ldots, m\}$

$\mathcal{H}^s$ the Hausdorff measure of dimension $s$

$A \cup B$ the disjoint union of $A$ and $B$

$\text{int}(A), \overline{A}$ the interior of $A$, the closure of $A$ (in a topological space)

$\mathcal{C}(M)$ the space of Caccioppoli sets in $M$

$\partial A$ the topological boundary of $A$ (in a topological space) $= \overline{A} \setminus \text{int}(A)$; $\partial$ will also denote the boundary of a current or the boundary of a cell in a cell-complex.

$\partial^* E$ the reduced boundary of a Caccioppoli set $E$

$[\Sigma]$ the current associated to the rectifiable set $\Sigma$

$\|V\|\Sigma$ the Radon measure associated to the varifold $V$

$B^c$ the complement of $B$ in $M$ i.e. $M \setminus B$

$B(p, r)$ the geodesic ball centered at $p$ with radius $r$

$A(p, r, R)$ the annulus centered at $p$ with radii $r < R$

$d(-, S)$ distance from a set $S \subset (M, g)$

$N_\rho(S)$ the set of points $x \in (M, g)$ such that $d(x, S) \leq \rho$

$T_\rho(S)$ the set of points $x \in (M, g)$ such that $d(x, S) = \rho$

$H^1(M)$ the Sobolev space $\{f \in L^2(M) : \text{distributional derivative } \nabla f \in L^2(M)\}$

2.2 Varifolds

Here we will briefly discuss the notion of varifold; further details can be found in Simon’s book [Sim83]. Given a Riemannian manifold $(M^{n+1}, g)$, let $G_n M$ denote the Grassmanian bundle of $n$-dimensional hyperplanes over $M$. An $n$-varifold in $M$ is a positive Radon measure on $G_n M$. If $V$ is an $n$-varifold and $p : G_n M \to M$ is the canonical projection map, $\|V\| = p_* V$ is a Radon measure on $M$; $\|V\|(A) = V(p^{-1}(A))$. The topology on the space of $n$-varifolds is given by the weak* topology i.e. a net $\{V_i\}_{i \in I}$ converges to $V$ if and only if

$$\int_{G_n M} f(x, \omega) dV_i(x, \omega) \to \int_{G_n M} f(x, \omega) dV(x, \omega)$$

for all $f \in C_c(G_n M)$. If $\Sigma \subset M$ is $n$-rectifiable and $\theta : \Sigma \to [0, \infty)$ is in $L^1_{\text{loc}}(\Sigma, \mathcal{H}^n)$, the $n$-varifold $\nu(\Sigma, \theta)$ is defined by

$$\nu(\Sigma, \theta)(f) = \int_{\Sigma} f(x, T_x \Sigma) \theta(x) \, d\mathcal{H}^n(x)$$
where \( T_x \Sigma \) is the approximate tangent space of \( \Sigma \) at \( x \) which exists \( \mathcal{H}^n|\Sigma \) a.e. Such a varifold is called a **rectifiable n-varifold**. When \( \theta \) is the constant function 1, \( \nu(\Sigma, \theta) \) is denoted by \( |\Sigma| \); it is called the **varifold associated to \( \Sigma \)**.

If \( \varphi : M \to M \) is a \( C^1 \) map and \( V \) is an \( n \)-varifold in \( M \), the push-forward varifold \( \varphi_#V \) is defined as follows.

\[
(\varphi_#V)(f) = \int_{G^+_nM} f(\varphi(x), D\varphi|_x(\omega)) J\varphi(x,\omega) dV(x,\omega)
\]

where

\[
J\varphi(x,\omega) = \left( \det \left( (D\varphi(x)|_\omega)^t \circ (D\varphi(x)|_\omega) \right) \right)^{1/2}
\]

is the Jacobian factor and

\[
G^+_nM = \{(x, \omega) \in G_nM : J\varphi(x,\omega) \neq 0 \}.
\]

If \( V = \nu(\Sigma, \theta) \) is a rectifiable \( n \)-varifold, \( \varphi_#V = \nu(\varphi(\Sigma), \tilde{\theta}); \tilde{\theta} : \varphi(\Sigma) \to \mathbb{R} \) is defined by

\[
\tilde{\theta}(y) = \sum_{x \in \varphi^{-1}(y) \cap \Sigma} \theta(x).
\]

We denote by \( \mathcal{V}_n(M) \) the closure of the space of rectifiable \( n \)-varifolds in \( M \) with respect to the above varifold weak topology. The \( F \) metric on \( \mathcal{V}_n(M) \) is defined as follows \([\text{Pit81}, \text{page 66}]\).

\[
F(V,W) = \sup \{ |V(f) - W(f)| : f \in C^0_c(G_nM), |f| \leq 1, \text{Lip}(f) \leq 1 \}.
\]

For every \( a > 0 \), the \( F \)-metric topology and the varifold weak topology coincide on the set \( \{ V \in \mathcal{V}_n(M) : \|V\|(M) \leq a \} \).

### 2.3 Almgren-Pitts min-max theory

In this subsection, we will recall some of the definitions in the Almgren-Pitts min-max theory; we refer to the papers by Marques and Neves \([\text{MN14, MN17, MN16, MN18}], \text{Schoen and Simon [SS81]}\) and the book by Pitts \([\text{Pit81}]\) for more details. To discuss the Almgren-Pitts min-max theory we need to introduce the following spaces of currents. Let \( (M^{n+1}, g) \) be a closed Riemannian manifold. \( \mathbf{I}(M^{n+1}; \mathbb{Z}_2) \) is the space of \( l \)-dimensional mod 2 flat chains in \( M \); we only need to consider \( l = n, n+1 \). \( \mathcal{Z}_n(M^{n+1}; \mathbb{Z}_2) \) denotes the space of flat chains \( T \in \mathbf{I}_n(M; \mathbb{Z}_2) \) such that \( T = \partial U \) for some \( U \in \mathbf{I}_{n+1}(M; \mathbb{Z}_2) \). For \( T \in \mathcal{Z}_n(M; \mathbb{Z}_2) \), \( |T| \) stands for the varifold associated to \( T \) and \( \|T\| \) is the radon measure associated to \( |T| \). \( F \) and \( M \) denote the flat and mass norm on \( \mathbf{I}(M; \mathbb{Z}_2) \). When \( l = n+1 \), these two norms coincide. The \( F \) metric on the space of currents is defined as follows.

\[
F(U_1, U_2) = F(U_1, U_2) + F(|\partial U_1|, |\partial U_2|) \text{ if } U_1, U_2 \in \mathbf{I}_{n+1}(M; \mathbb{Z}_2);
\]

\[
F(T_1, T_2) = F(T_1, T_2) + F(|T_1|, |T_2|) \text{ if } T_1, T_2 \in \mathbf{I}_n(M; \mathbb{Z}_2).
\]

It is proved in \([\text{MN18}]\) that the space \( \mathbf{I}_{n+1}(M; F; \mathbb{Z}_2) \) is contractible and the boundary map \( \partial : \mathbf{I}_{n+1}(M; F; \mathbb{Z}_2) \to \mathcal{Z}_n(M; F; \mathbb{Z}_2) \) is a two sheeted covering map. By the constancy theorem, if \( U_1, U_2 \in \mathbf{I}_{n+1}(M; F; \mathbb{Z}_2) \) such that \( \partial U_1 = \partial U_2 \), either \( U_1 = U_2 \) or \( U_1 + U_2 = [M] \).
Let \( X, \Pi \) be as in Section 1. The Almgren-Pitts width of the homotopy class \( \Pi \) is defined by

\[
L_{AP}(\Pi) = \inf_{\Phi \in \Pi} \sup_{x \in X} \{M(\Phi(x))\}. \tag{2.1}
\]

A sequence of maps \( \Phi_i : X \to Z_n(M;F;\mathbb{Z}_2) \) in \( \Pi \) is called a minimizing sequence if

\[
\limsup_{i \to \infty} \sup_{x \in X} \{M(\Phi_i(x))\} = L_{AP}(\Pi).
\]

The critical set of a minimizing sequence \( \{\Phi_i\} \), denoted by \( C(\{\Phi_i\}) \), is the set of all varifolds \( V \in V_n(M) \) such that \( \|V\|(M) = L_{AP}(\Pi) \) and there exist sequences \( \{i_j\} \subset \{i\} \) and \( \{x_j\} \subset X \) such that

\[
\lim_{j \to \infty} F(V, |\Phi_{i_j}(x_j)|) = 0.
\]

We define \( C_{AP}(\Pi) \) to be the set of all varifolds \( V \in V_n(M) \) such that \( V \in C(\{\Phi_i\}) \) for some minimizing sequence \( \{\Phi_i\} \subset \Pi \), \( V \) is a stationary, integral varifold and \( \text{spt}(V) \) is a closed, minimal hypersurface with optimal regularity (i.e. smooth and embedded outside a singular set of Hausdorff dimension \( \leq n - 7 \)). The theorem of Almgren-Pitts-Schoen-Simon guarantees that \( C_{AP}(\Pi) \) is non-empty.

### 2.4 Allen-Cahn min-max theory

We now briefly discuss the min-max theory for the Allen-Cahn functional following the papers by Guaraco [Gua18] and Gaspar-Guaraco [GG18] where further details can be found. Let \( W : \mathbb{R} \to \mathbb{R} \) be a smooth, symmetric, double well potential. More precisely, \( W \) has the following properties.

\( W \geq 0; W(-t) = W(t) \) for all \( t \in \mathbb{R}; W \) has exactly three critical points \( 0, \pm 1; W(\pm 1) = 0 \) and \( W''(\pm 1) > 0 \) i.e. \( \pm 1 \) are non-degenerate minima; \( 0 \) is a local maximum.

The Allen-Cahn energy (with parameter \( \varepsilon > 0 \)) is given by

\[
E_\varepsilon(u) = \int_M \varepsilon \frac{|
abla u|^2}{2} + \frac{W(u)}{\varepsilon}.
\]

As mentioned earlier,

\[
AC_\varepsilon(u) := \varepsilon \Delta u - \varepsilon^{-1}W'(u) = 0
\]

if and only if \( u \) is a critical point of \( E_\varepsilon \).

Let \( \tilde{X}, \tilde{\Pi} \) be as in Section 1. The \( \varepsilon \)-Allen-Cahn width of the homotopy class \( \tilde{\Pi} \) is defined by

\[
L_\varepsilon(\tilde{\Pi}) = \inf_{h \in \tilde{\Pi}} \sup_{x \in \tilde{X}} E_\varepsilon(h(x)). \tag{2.2}
\]

A sequence of maps \( h_i : \tilde{X} \to H^1(M) \setminus \{0\} \) in \( \tilde{\Pi} \) is called a minimizing sequence for \( E_\varepsilon \) if

\[
\limsup_{i \to \infty} \sup_{x \in \tilde{X}} E_\varepsilon(h_i(x)) = L_\varepsilon(\tilde{\Pi})
\]

\( u \) is called a min-max critical point of \( E_\varepsilon \) (corresponding to the homotopy class \( \tilde{\Pi} \)) if \( u \) is a critical point of \( E_\varepsilon \) with \( E_\varepsilon(u) = L_\varepsilon(\tilde{\Pi}) \) and

\[
\lim_{i \to \infty} d_{H^1(M)}(u, h_i(\tilde{X})) = 0
\]
where \( \{h_i\} \) is a minimizing sequence for \( E_\varepsilon \) in \( \tilde{\Pi} \).

As \( W \) is an even function, \( E_\varepsilon \) is invariant under the \( \mathbb{Z}_2 \) action on \( H^1(M) \) given by \( u \mapsto -u \). Moreover, as proved in [Gua18, Proposition 4.4], \( E_\varepsilon \) satisfies the Palais-Smale condition for bounded sequences. Hence, as explained in [GG18], if \( \varepsilon > 0 \) satisfies

\[
L_\varepsilon(\tilde{\Pi}) < E_\varepsilon(0) = \frac{W(0)}{\varepsilon} \text{Vol}(M,g) \tag{2.3}
\]

(which holds for \( \varepsilon \) sufficiently small by (1.3)), one can apply Corollary 10.5 of [Gho93] to the \( \mathbb{Z}_2 \)-homotopic family \( \tilde{\Pi} \) to conclude that there exists a min-max critical point \( u_\varepsilon \) of \( E_\varepsilon \) (corresponding to the homotopy class \( \tilde{\Pi} \)) such that \( \text{Ind}(u_\varepsilon) \leq k \). (Here \( k \) is the dimension of the parameter space \( \tilde{X} \).) The restriction on \( \varepsilon \) given by (2.3) is due to the fact that the space \( H^1(M) \setminus \{0\} \) is not complete; (2.3) ensures that a minimizing sequence for \( E_\varepsilon \) is bounded away from 0.

### 2.5 Convergence of the phase interfaces

Let us define \( F : \mathbb{R} \to \mathbb{R} \) and the energy constant \( \sigma \) as follows.

\[
F(a) = \int_0^a \sqrt{W(s)/2} \, ds; \quad \sigma = \int_{-1}^1 \sqrt{W(s)/2} \, ds \quad \text{so that} \quad F(\pm1) = \pm \frac{\sigma}{2}. \tag{2.4}
\]

Let \( u \in C^1(M), w = F \circ u \). The \( n \)-varifold associated to \( u \) is defined by

\[
V[u](A) = \int_{-\infty}^{\infty} |\{w = s\}|(A) \, ds
\]

for every Borel set \( A \subset G_n M \). On a closed manifold, if \( AC_\varepsilon(u) = 0 \) and \( u \) is not identically equal to \( \pm1, |u| < 1 \) [GG18, Lemma 2.2]; in that case, in the definition of \( V[u] \) the integral can be taken over the interval \((-\sigma/2, \sigma/2)\).

Building on the work of Hutchinson-Tonegawa [HT00], Tonegawa [Ton05] and Tonegawa-Wickramasekera [TW12], Guaraco [Gua18] has proved the following theorem.

**Theorem 2.1** ([HT00, Ton05, TW12, Gua18]). Let \( \{u_i : M \to (-1,1)\}_{i=1}^\infty \) be a sequence of smooth functions such that

(i) \( AC_\varepsilon(u_i) = 0 \) with \( \varepsilon_i \to 0 \) as \( i \to \infty \);

(ii) There exists \( E_0 > 0 \) and \( I_0 \in \mathbb{N}_0 \) such that \( E_\varepsilon(u_i) \leq E_0 \) and \( \text{Ind}(u_i) \leq I_0 \) for all \( i \in \mathbb{N} \).

Then, there exists a stationary, integral varifold \( V \) such that possibly after passing to a subsequence, \( V[u_i] \to V \) in the sense of varifolds. Moreover,

\[
\|V\|(M) = \frac{1}{2\sigma} \lim_{i \to \infty} E_\varepsilon(u_i)
\]

and \( \text{spt}(V) \) is a closed, minimal hypersurface with optimal regularity.
The proof of the regularity of the limit-interface depends on the regularity theory of stable, minimal hypersurfaces developed by Wickramasekera [Wic14].

The upper bound for the Morse index of the limit-interface in the above theorem was proved by Gaspar [Gas20] and Hiesmayr [Hie18] (when the limit-interface is two sided).

**Theorem 2.2** ([Gas20, Hie18]). The Morse index of the limit-interface $\text{spt}(V)$ in the above Theorem 2.1 is less than or equal to $I_0$.

Lastly, we introduce the following definition. $C_{AC}(\tilde{\Pi})$ is the set of all stationary, integral $n$-varifolds $V$ such that $\text{spt}(V)$ is a closed, minimal hypersurface with optimal regularity and $V$ is the varifold limit of $V[u_i]$ for some sequence $\{u_i\}_{i=1}^{\infty}$ such that $u_i$ is a min-max critical point of $E_{\varepsilon_i}$ (with $\varepsilon_i \to 0$) corresponding to the homotopy class $\tilde{\Pi}$. By the discussion of Section 2.4 and Theorem 2.1, $C_{AC}(\tilde{\Pi})$ is non-empty.

### 3 Proof of the width inequality

In this section we will prove our main Theorem 1.2. Let us fix $\eta > 0$. Let $L = L_{AP}(\Pi)$. By the interpolation theorems of Pitts and Marques-Neves [MN17, MN18] there exists $\Phi : X \to \mathbb{Z}_n(M^{n+1}; M; \mathbb{Z}_2)$ such that

$$\sup_{x \in X} \{ M(\Phi(x)) \} < L + \eta.$$  \hspace{1cm} (3.1)

We choose $\tilde{\Phi} : \tilde{X} \to \mathcal{C}(M)$ which is a lift of $\Phi$ i.e. for all $x \in \tilde{X}$,

$$\lbrack \partial^* \tilde{\Phi}(x) \rbrack = \partial \lbrack \tilde{\Phi}(x) \rbrack = \Phi(\pi(x)).$$

$\tilde{\Phi}$ is $\mathbb{Z}_2$-equivariant i.e. if $T : \tilde{X} \to \tilde{X}$ is the deck transformation, $\lbrack \tilde{\Phi}(x) \rbrack + \lbrack \tilde{\Phi}(T(x)) \rbrack = [M]$ for all $x \in \tilde{X}$.

### 3.1 Approximation of a Caccioppoli set by open sets with smooth boundary

In this subsection, following the book by Giusti [Giu84] and the paper by Miranda-Pallara-Paronetto-Preunkert [MPPP07], we briefly discuss the fact that a Caccioppoli set can be approximated by open sets with smooth boundary. We begin with the following theorem.

**Theorem 3.1** ([Giu84, Theorem 1.17], [MPPP07, Proposition 1.4]). Let $E \in \mathcal{C}(M)$. There exists a sequence of smooth functions $\{f_j : M \to \mathbb{R}\}_{j=1}^{\infty}$ such that $0 \leq f_j \leq 1$ for all $j$ and

$$\lim_{j \to \infty} \int_M |f_j - \chi_E| \, d\mathcal{H}^{n+1} = 0 \quad \text{and} \quad \int_M |D\chi_E| = \lim_{j \to \infty} \int_M |Df_j|.$$  

Following [Giu84, Proof of Theorem 1.24], for $t \in (0,1)$, let us define $E_{j,t} = \{f_j > t\}$. Then,

$$|f_j - \chi_E| > t \text{ on } E_{j,t} \setminus E \quad \text{and} \quad |f_j - \chi_E| \geq 1 - t \text{ on } E \setminus E_{j,t} \hspace{1cm} (3.2)$$

which implies

$$\int_M |f_j - \chi_E| \, d\mathcal{H}^{n+1} \geq \min\{t, 1-t\} \int_M |\chi_E - \chi_E| \, d\mathcal{H}^{n+1}. \hspace{1cm} (3.3)$$
Hence, for all \( t \in (0, 1) \),

\[
\lim_{j \to \infty} \int_M |\chi_{E_j,t} - \chi_E| \, d\mathcal{H}^{n+1} = 0
\]

(3.4)

\[
\implies \int_M |D\chi_E| \leq \liminf_{j \to \infty} \int_M |D\chi_{E_j,t}|.
\]

(3.5)

Therefore, using Theorem 3.1, the co-area formula for the BV function and (3.5) we obtain the following inequalities.

\[
\int_M |D\chi_E| = \lim_{j \to \infty} \int_M |Df_j| \geq \int_0^1 \left( \liminf_{j \to \infty} \int_M |D\chi_{E_j,t}| \right) \, dt \geq \int_M |D\chi_E|.
\]

This implies

\[
\liminf_{j \to \infty} \int_M |D\chi_{E_j,t}| = \int_M |D\chi_E| \text{ for a.e. } t \in (0, 1).
\]

(3.6)

We choose \( t_0 \in (0, 1) \) such that \( t_0 \) is a regular value of \( f_j \) for all \( j \) and (3.6) holds for \( t = t_0 \).

Further, possibly after passing to a subsequence, we can assume that

\[
\lim_{j \to \infty} \int_M |D\chi_{E_j,t_0}| = \int_M |D\chi_E|.
\]

(3.7)

Let us define \( E_j = \overline{E_{j,t_0}} \). Since

\[
E_{j,t_0} \subset \overline{E_{j,t_0}} \subset E_j \cup \{ f_j = t_0 \},
\]

we have \( \mathcal{H}^{n+1}(\overline{E_{j,t_0}} \setminus E_j) \). From (3.4) and (3.7) we conclude that

\[
\chi_{E_j} \to \chi_E \text{ in } L^1(M) \text{ and } \lim_{j \to \infty} \int_M |D\chi_E| = \int_M |D\chi_E|.
\]

(3.8)

By [Pit81, 2.1(18)(f), page-63], (3.8) implies that \([\partial^* E_j]\) converges to \([\partial^* E]\) in \( F \).

Let us fix \( p \in M \) and \( R > 0 \). Using (3.8),

\[
\lim_{j \to \infty} \int_0^R \left( \int_{\partial B(p,t)} |\chi_{E_j} - \chi_E| \, d\mathcal{H}^n \right) \, dt = \lim_{j \to \infty} \int_{B(p,R)} |\chi_{E_j} - \chi_E| \, d\mathcal{H}^{n+1} = 0.
\]

Hence, there exists a subsequence \( \{\chi_{E_{js}}\} \subset \{\chi_{E_j}\} \) such that

\[
\lim_{s \to \infty} \int_{\partial B(p,t)} |\chi_{E_{js}} - \chi_E| \, d\mathcal{H}^n = 0 \text{ for a.e. } t \in (0, R).
\]

Next, we define \( E_j = \overline{\{f_j < t_0\}} \) and \( F = M \setminus E \). Then, \( E_j \cap F_j \subset \{f_j = t_0\} \) and \( E_j \cup F_j = M \).

Therefore, \([E_j] + [F_j] = [M] \), \([\partial^* E_j] = [\partial^* F_j] \) and \( \chi_{E_j} \to \chi_F \) in \( L^1(M) \). We note that \( \partial E_j \subset \{f_j = t_0\} \). As the reduced boundary is a subset of the topological boundary, we also have \( \partial^* E_j = \partial^* F_j \subset \{f_j = t_0\} \). Since \( \{f_j = t_0\} \) is a smooth, closed hypersurface in \( M \),
for each \(a \in \{f_j = t_0\}\) there exist \(\rho > 0\) and co-ordinates \(\{x_1, x_2, \ldots, x_{n+1}\}\) on \(B(a, \rho)\) such that \(x_i(a) = 0\) for all \(i\) and

\[
(B(a, \rho) \cap \{f_j = t_0\}) = \{x \in B(a, \rho) : x_{n+1} = 0\}.
\]

Let \(G_1 = \{x \in B(a, \rho) : x_{n+1} < 0\}\) and \(G_2 = \{x \in B(a, \rho) : x_{n+1} > 0\}\) so that

\[
(B(a, \rho) \setminus \{f_j = t_0\}) = G_1 \cup G_2.
\]

\(G_1\) and \(G_2\) are connected open sets. We have the following three mutually exclusive cases.

1. \(f_j > t_0\) on both \(G_1\) and \(G_2\); this implies \(a \in \text{int}(E_j) \setminus F_j\);
2. \(f_j < t_0\) on both \(G_1\) and \(G_2\); this implies \(a \in \text{int}(F_j) \setminus E_j\);
3. \(f_j > t_0\) on one of \(G_1\) and \(G_2\), and \(f_j < t_0\) on the other; this implies \(a \in E_j \cap F_j\).

From the above three cases we can conclude that \(\partial^* E_j = \partial^* F_j = E_j \cap F_j = \partial E_j = \partial F_j\) which is a smooth, closed, embedded hypersurface in \(M\). Indeed, the set of points \(a \in \{f_j = t_0\}\) for which the above item (3) holds is both open and closed in \(\{f_j = t_0\}\); hence it is the union of certain connected components of \(\{f_j = t_0\}\).

From the above discussion we arrive at the following proposition.

**Proposition 3.2.** Let \(E \in C(M)\) and \(F = M \setminus E\). Then, for each \(j \in \mathbb{N}\) there exist closed sets \(E_j, F_j \in C(M)\) such that the followings hold.

1. \([E_j] + [F_j] = [M]\) and \(M = E_j \cup F_j\).
2. \(\chi_{E_j} \to \chi_E\) and \(\chi_{F_j} \to \chi_F\) in \(L^1(M)\).
3. \(\partial^* E_j = \partial^* F_j = E_j \cap F_j = \partial E_j = \partial F_j\) is a smooth, closed, embedded hypersurface in \(M\).
4. \([\partial^* E_j] = [\partial^* F_j]\) converges to \([\partial^* E]\) in \(\mathbf{F}\).
5. For every \(p \in M\) and \(R > 0\) there exist subsequences \(\{\chi_{E_{js}}\} \subset \{\chi_{E_j}\}\) and \(\{\chi_{F_{js}}\} \subset \{\chi_{F_j}\}\) such that

\[
\lim_{s \to \infty} \int_{\partial B(p,t)} |\chi_{E_{js}} - \chi_E| d\mathcal{H}^n = 0 \quad \text{and} \quad \lim_{s \to \infty} \int_{\partial B(p,t)} |\chi_{F_{js}} - \chi_F| d\mathcal{H}^n = 0
\]

for a.e. \(t \in (0, R)\).

### 3.2 Preliminary constructions

Let \(D\) be a countable, dense subset of \(M\) and \(\text{inj}(M)\) be the injectivity radius of \((M, g)\). We consider

\[\mathcal{B} = \{B(p,t) : p \in D, t \in (0, \text{inj}(M)) \cap \mathbb{Q}\}\]

which is also a countable set. Let us assume that \(M\) is isometrically embedded in some Euclidean space \(\mathbb{R}^m\). We have the following theorem which is a consequence of Sard’s theorem.
**Theorem 3.3.** ([Nic11, Corollary 1.25]) Let \( \Sigma \) be a closed submanifold of \( \mathbb{R}^m \). For \( v \in \mathbb{R}^m \) we define \( f_v : \mathbb{R}^m \to \mathbb{R} \) by \( f_v(x) = \langle x, v \rangle \). Then, there exists a generic set \( V \subset \mathbb{R}^m \) (depending on \( \Sigma \)) such that for all \( v \in V \), \( f_v|\Sigma \) is a Morse function on \( \Sigma \).

By the above theorem, there exists \( \omega \in \mathbb{R}^m \) such that \( f_\omega|_M \) and \( f_\omega|_{\partial B} \) are Morse functions for all \( B \in \mathcal{B} \). By composing scaling and translation with \( f_\omega \), we can assume that \( f_\omega(M) = [1/3, 2/3] \). From now on, whenever we will consider \( f_\omega \), it will be assumed that \( f_\omega : M \to [1/3, 2/3] \).

Let us choose \( r_0 \in (0, \text{inj}(M)) \) such that

- \( \mathcal{H}^n(f_\omega^{-1}(t) \cap \overline{B(p, r_0)}) < \eta/2 \) for all \( t \in [1/3, 2/3], p \in M \);
- \( \mathcal{M}(\Phi(x) \mathbb{L} \overline{B}(p, r_0)) < \eta \) for all \( x \in X, p \in M \);

where \( \Phi \) is as chosen at the beginning of Section 3. Such a choice of \( r_0 \) is possible because of the ‘no concentration of mass property’ ([MN17]). We choose \( p_i \in D \) such that

\[
M = \bigcup_{i=1}^{I} B(p_i, r_0/4); \quad B_i^0 := B(p_i, r_0).
\]

Hence, in particular, we have

\[
\mathcal{H}^n(f_\omega^{-1}(t) \cap \overline{B}(p_i, r_0)) < \eta/2 \quad \forall t \in [1/3, 2/3], i \in [I];
\]

\[
\mathcal{M}(\Phi(x) \mathbb{L} \overline{B}(p, r_0)) < \eta \quad \forall x \in X, i \in [I].
\]

**Lemma 3.4.** There exists \( r_1 \in (r_0/2, 3r_0/4) \cap \mathbb{Q}, \delta \in (0, r_0/8) \) such that

\[
\mathcal{M}(\Phi(x) \mathbb{L} \overline{A}(p_i, r_1 - 2\delta, r_1 + \delta)) < \frac{\eta}{I}
\]

for all \( x \in X \) and \( i \in [I] \).

**Proof.** By the compactness of \( X \), there exists \( \{x_j\}_{j=1}^{J} \subset X \) such that

\[
\forall x \in X, \exists j \in [J] \text{ such that } \mathcal{M}(\Phi(x) - \Phi(x_j)) < \frac{\eta}{2I}.
\]

For \( i \in [I], j \in [J] \) we define

\[
\mathcal{R}_{ij} = \left\{ r' \in (r_0/2, 3r_0/4) : \mathcal{M}(\Phi(x_j) \mathbb{L} \partial B(p_i, r')) < \frac{\eta}{2I} \right\}.
\]

We note that

\[
(r_0/2, 3r_0/4) \setminus \mathcal{R}_{ij} \text{ is finite } \forall i, j.
\]

Hence, we can choose

\[
r_1 \in \left( \bigcap_{i \in [I], j \in [J]} \mathcal{R}_{ij} \right) \cap (r_0/2, 3r_0/4) \cap \mathbb{Q}.
\]

Therefore, we have

\[
\mathcal{M}(\Phi(x_j) \mathbb{L} \partial B(p_i, r_1)) < \frac{\eta}{2I} \quad \forall i, j.
\]
Hence, we can choose $\delta \in (0, r_0/8)$ such that
\[
M(\Phi(x_j) \setminus \overline{A}(p_i, r_1 - 2\delta, r_1 + \delta)) < \frac{\eta}{2I} \quad \forall \ i, j;
\]
which implies (by (3.11))
\[
M(\Phi(x) \setminus \overline{A}(p_i, r_1 - 2\delta, r_1 + \delta)) < \frac{\eta}{I} \quad \forall \ x \in X i \in [I].
\]

**Lemma 3.5.** There exists $\delta \in (0, r_0/8)$ such that
\[
H^n(f^{-1}_\omega(t) \cap \overline{A}(p_i, r_1 - 2\delta, r_1 + \delta)) < \frac{\eta}{2I}
\]
for all $t \in [1/3, 2/3]$, $i \in [I]$.

**Proof.** We assume by contradiction that there exist $\{t_j\}_{j=1}^\infty \subset [1/3, 2/3]$ and $\{d_j\}_{j=1}^\infty \subset \mathbb{R}^+$ such that $d_j \to 0$ and for some $i \in [I]$
\[
H^n(f^{-1}_\omega(t_j) \cap \overline{A}(p_i, r_1 - 2d_j, r_1 + d_j)) \geq \frac{\eta}{2I}
\]
holds for all $j \in \mathbb{N}$. Without loss of generality we can assume that $t_j \to t$. Denoting
\[
\mu_j = H^\mathbb{L}(f^{-1}_\omega(t_j)) \quad \text{and} \quad \mu = H^\mathbb{L}(f^{-1}_\omega(t)),
\]
we have that $\mu_j$ weakly converges to $\mu$ (in the sense of radon measure). Let us fix $l \in \mathbb{N}$. There exists $j_0 \in \mathbb{N}$ such that $d_j \leq l^{-1}$ for all $j \geq j_0$. Hence,
\[
\mu_j(\overline{A}(p_i, r_1 - 2l^{-1}, r_1 + l^{-1})) \geq \frac{\eta}{2I}
\]
for all $j \geq j_0$. Therefore,
\[
\mu(\overline{A}(p_i, r_1 - 2l^{-1}, r_1 + l^{-1})) \geq \limsup_{j \to \infty} \mu_j(\overline{A}(p_i, r_1 - 2l^{-1}, r_1 + l^{-1})) \geq \frac{\eta}{2I}.
\]
This holds for all $l \in \mathbb{N}$. Denoting $A_l = \overline{A}(p_i, r_1 - 2l^{-1}, r_1 + l^{-1})$ we have
\[
A_{l+1} \subset A_l \quad \text{and} \quad \bigcap_{l=1}^\infty A_l = \partial B(p_i, r_1).
\]
This implies
\[
H^n(f^{-1}_\omega(t) \cap \partial B(p_i, r_1)) = \mu(\partial B(p_i, r_1)) = \lim_{l \to \infty} \mu(A_l) \geq \frac{\eta}{2I}.
\]
However, this is not possible. Indeed, $r_1 \in \mathbb{Q}$ (Lemma 3.4) and by the choice of $\omega$, $f_\omega|\partial B(p_i, r_1)$ is a Morse function on $\partial B(p_i, r_1)$; hence $H^n(f^{-1}_\omega(t) \cap \partial B(p_i, r_1))$ must be 0. \qed
We can assume that the two δ’ s appearing in Lemma 3.4 and Lemma 3.5 are the same. Next, we modify \( f_\omega \) near the points where it achieves local maxima or local minima (which are not global maxima or minima) to get another Morse function \( f : M \to [1/3, 2/3] \) such that \( f \) has no non-global local maxima or local minima and for all \( t \in [1/3, 2/3], i \in [l] \)

\[
\mathcal{H}^n (f^{-1}(t) \cap B_i^\delta) < \eta \quad \text{and} \quad \mathcal{H}^n (f^{-1}(t) \cap \bigcup (p_i, r_1 - 2\delta, r_1 + \delta)) < \frac{\eta}{I}.
\]

Let us introduce the following notation which will be used later.

\[
\mathcal{B}_i^* = B(p_i, r_1); \quad \mathcal{A}_1 = \bigcup_{i=1}^l A(p_i, r_1 - 2\delta, r_1 + \delta);
\]

\[
\mathcal{A}_2 = \bigcup_{i=1}^l A(p_i, r_1 - 3\delta/2, r_1 + \delta/2); \quad \mathcal{A} = \bigcup_{i=1}^l A(p_i, r_1 - \delta, r_1).
\]

### 3.3 Cell complex structure on the parameter space

For \( l \in \mathbb{N}, \mathcal{I}[l] \) is the cell complex on \( \mathcal{I} = [0, 1] \) whose 0-cells are

\[
[0], \ [l^{-1}], \ldots, [1 - l^{-1}], \ [1]
\]

and 1-cells are

\[
[0, l^{-1}], \ [l^{-1}, 2l^{-1}], \ldots, [1 - l^{-1}, 1].
\]

\( \mathcal{I}[l] \) denotes the cell complex on \( \mathcal{I} = [0, 1] \) whose cells are \( \alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_m \) where each \( \alpha_j \in \mathcal{I}[l] \). We note that \( \mathcal{I}[1] \) is the standard cell complex on \( \mathcal{I} \). Similarly, if \( Y \) is a subcomplex of \( \mathcal{I}[1] \), \( Y[l] \) is the union of all the cells in \( \mathcal{I}[l] \) whose support is contained in \( Y \). If \( Y \) is a cell complex, \( Y_p \) will denote the set of all \( p \)-cells in \( Y \); if \( \alpha, \beta \in \mathcal{Y} \) such that \( \beta \) is a face of \( \alpha \) (in the definition of the face, we do not insist that \( \dim(\beta) < \dim(\alpha) \) or \( \dim(\beta) = \dim(\alpha) - 1 \)), we use the notation \( \beta < \alpha \).

If \( \lambda = [il^{-1}, (i + 1)l^{-1}] \in \mathcal{I}[l] \), there exists a canonical map

\[
\Delta_\lambda : \mathcal{I} \to \mathcal{I}; \quad \Delta_\lambda(t) = (i + t)l^{-1}
\]

such that \( \Delta_\lambda : \mathcal{I} \to \lambda \) is a homeomorphism. Similarly, if \( \alpha = \alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_m \in \mathcal{I}[l]_p \), there exists a canonical map \( \Delta_\alpha : \mathcal{I}^p \to \mathcal{I}^m \) defined as follows. There exist precisely \( p \) indices

\[
j_1 < j_2 < \cdots < j_p \quad \text{such that} \quad \dim(\alpha_{j_s}) = 1 \ \forall s = 1, 2, \ldots, p.
\]

We define

\[
(\Delta_\alpha(t_1, t_2, \ldots, t_p))_j = \begin{cases} \Delta_{\alpha_{j_s}}(t_s) & \text{if } j = j_s, \\ \alpha_j & \text{if } j \notin \{j_1, j_2, \ldots, j_p\} \iff \dim(\alpha_j) = 0. \end{cases}
\]

\( \Delta_\alpha : \mathcal{I}^p \to \alpha \) is a homeomorphism. Let \( D_\alpha : \alpha \to \mathcal{I}^p \) be the inverse of \( \Delta_\alpha \).
If \( \beta < \alpha \) and \( D_\alpha(\beta) = q \prec \mathcal{J}^p \), then the following compatibility relation holds.
\[
\Delta_\alpha \circ \Delta_q = \Delta_\beta.
\] (3.17)

We recall from Section 1 that \( X \) is a subcomplex of \( \mathcal{J}^N \) for some \( N \) and \( \pi : \tilde{X} \to X \) is a double cover. For \( l \in \mathbb{N} \), \( X[l] \) is the cell complex on \( X \) as defined above. \( \tilde{X}[l] \) denotes the cell complex on \( \tilde{X} \) whose cells are pre-images of the cells of \( X[l] \) via the map \( \pi \). We choose \( K \in \mathbb{N} \) such that the followings hold. (Here \( \Phi \) and \( \tilde{\Phi} \) are as chosen at the beginning of Section 3.)

If \( x_1, x_2 \in X[K]_0 \) belong to a common cell in \( X[K] \), \( M(\tilde{\Phi}(x_1) - \Phi(x_2)) < \eta \). (3.18)

If \( y_1, y_2 \in \tilde{X}[K]_0 \) belong to a common cell in \( \tilde{X}[K] \), \( \mathcal{H}^{n+1}(\tilde{\Phi}(y_1)\Delta \tilde{\Phi}(y_2)) < \frac{\eta \delta}{T} \). (3.19)

where \( \delta \) is as in Section 3.2 (Lemma 3.4, (3.13)).

Let \( \{ q : q \in [Q]\} \) be all the cells of \( X[K] \) indexed in such a way that \( \dim(c_{q_1}) \leq \dim(c_{q_2}) \) if \( q_1 \leq q_2 \). Let \( \{ q, f_q : q \in [Q]\} \) be the cells of \( \tilde{X}[K] \) so that \( \pi^{-1}(c_q) = e_q \cup f_q \). Since \( c_q \) is contractible, \( \pi|_{e_q} : e_q \to c_q \) and \( \pi|_{f_q} : f_q \to c_q \) are homeomorphisms. Let us introduce the following notation \((d = \dim(c_q))\).

\[
\Delta_{c_q} = \Delta_q : \mathcal{J}^d \to c_q ; \quad D_{c_q} = D_q : c_q \to \mathcal{J}^d ;
\]
\[
(\pi|_{e_q})^{-1} \circ \Delta_q = \Delta^1_q : \mathcal{J}^d \to e_q ; \quad D_q \circ (\pi|_{e_q}) = D^1_q : e_q \to \mathcal{J}^d ;
\]
\[
(\pi|_{f_q})^{-1} \circ \Delta_q = \Delta^2_q : \mathcal{J}^d \to f_q ; \quad D_q \circ (\pi|_{f_q}) = D^2_q : f_q \to \mathcal{J}^d .
\] (3.20)

### 3.4 Construction of an almost smooth sweepout

The next proposition follows from Proposition 3.2.

**Proposition 3.6.** There exists a sequence \( \{ \tilde{\Phi}_j : \tilde{X}[K]_0 \to \mathcal{C}(M) \}_{j=1}^\infty \) such that the followings hold for all \( j \in \mathbb{N} \) and \( x \in \tilde{X}[K]_0 \).

(i) \( \tilde{\Phi}_j(x) \) is a closed subset of \( M \).

(ii) \( [\tilde{\Phi}_j(x)] + [\tilde{\Phi}_j(T(x))] = [M] \) and \( M = \tilde{\Phi}_j(x) \cup \tilde{\Phi}_j(T(x)) \).

(iii) As \( j \to \infty \), \( \chi_{\tilde{\Phi}_j(x)} \to \chi_{\Phi(x)} \) in \( L^1(M) \).

(iv) \( \partial \tilde{\Phi}_j(x) = \partial \tilde{\Phi}_j(T(x)) = \tilde{\Phi}_j(x) \cap \tilde{\Phi}_j(T(x)) = \partial^+ \tilde{\Phi}_j(x) = \partial^+ \tilde{\Phi}_j(T(x)) \) is a smooth, closed hypersurface in \( M \).

(v) Defining \( \Phi_j(x) = \partial^+ \tilde{\Phi}_j(x) \), as \( j \to \infty \), \( [\Phi_j(x)] \to \Phi(\pi(x)) \) in \( \mathcal{F} \).

(vi) For all \( i \in [I] \),
\[
\lim_{j \to \infty} \int_{\partial B(p_i, t)} |\chi_{\tilde{\Phi}_j(x)} - \chi_{\tilde{\Phi}(x)}| \, d\mathcal{H}^n = 0
\]
for a.e. \( t \in (0, r_1) \).
We will approximate \( \tilde{\Phi} \) by a discrete and ‘almost smooth’ sweepout \( \tilde{\Psi} : \tilde{X}[K]_0 \to C(M) \) which will be constructed using the \( \tilde{\Phi}_j \)'s. The construction is motivated by the interpolation theorems of Almgren [Alm62], Pitts [Pit81, 4.5], Marques-Neves [MN14, Theorem 14.1] and Chambers-Liokumovich [CL20, Lemma 6.2]. The construction is divided into three parts.

**Part 1.** We recall that \( \{e_q, f_q : q \in [Q]\} \) are the cells of \( \tilde{X}[K] \). For \( q \in [Q] \) we define the following collection of balls

\[
\mathcal{B}(q) = \{B(p_i, r_i(q)) : i \in [I]\}
\]

(3.21)

where \( r_i(q) \in (r_1 - \delta, r_1) \) (\( \delta \) is as in Section 3.2) and \( r_i(q) \) is chosen inductively so that the following conditions are satisfied.

(i) \( \|\Phi(\pi(x))\| (\partial B(p_i, r_i(q))) = 0 \) \( \forall x \in (e_q)_0 \).

(ii) \( \partial B(p_i, r_i(q)) \) is transverse to \( \tilde{\Phi}_j(x) \) for all \( x \in (e_q)_0 \) and \( j \in \mathbb{N} \).

(iii) \( \partial B(p_i, r_i(q)) \) is transverse to \( \partial B(p_s, r_s(q)) \) for all \( s < i \).

(iv) \( \partial B(p_i, r_i(q)) \) is transverse to \( \partial B(p_{j'}, r_{j'}(q')) \) for all \( q' < q \) and \( j \in [I] \).

(v) \( \chi_{\tilde{\Phi}_j(x)} \) converges to \( \chi_{\tilde{\Phi}(x)} \) in \( L^1(M, \mathcal{H}^n \mathcal{L}, \partial B(p_i, r_i(q))) \) for all \( x \in (e_q)_0 \cup (f_q)_0 \).

(vi) If \( m = \dim(e_q) = \dim(f_q) \),

\[
\int_{\partial B(p_i, r_i(q))} |\chi_{\tilde{\Phi}(x)} - \chi_{\tilde{\Phi}(x')}| \, d\mathcal{H}^n < \frac{2^{2m} \eta}{I}
\]

for all \( x, x' \in (e_q)_0 \) and for all \( x, x' \in (f_q)_0 \).

Next we choose

\[ r_1 > r > \max\{r_i(q) : i \in [I], q \in [Q]\} \]

such that the followings hold.

\[
\|\Phi(\pi(x))\| (\partial B(p_i, r)) = 0 \quad \forall x \in \tilde{X}[K]_0, \ i \in [I];
\]

(3.22)

\[
\|\tilde{\Phi}_j(x)\| (\partial B(p_i, r)) = 0 \quad \forall j \in \mathbb{N}, \ x \in \tilde{X}[K]_0, \ i \in [I].
\]

(3.23)

We introduce the notation

\[
B_i(q) = B(p_i, r_i(q)) \quad ; \quad B_i = B(p_i, r).
\]

(3.24)

We note that by (3.9), for all \( q \in [Q] \),

\[
\bigcup_{i \in [I]} B_i(q) = M = \bigcup_{i \in [I]} B_i.
\]

**Part 2.** Let us introduce the following definitions.

\[
\mathcal{R}_1 = \{B_i : i \in [I]\} \cup \{M \setminus B_i : i \in [I]\}, \quad \mathcal{R}_2 = \{U_1 \cap U_2 \cap \cdots \cap U_s : s \in \mathbb{N} \text{ and each } U_j \in \mathcal{R}_1\},
\]

\(^{1}\)For this item we need to use item (vi) of Proposition 3.6.

\(^{2}\)For this item we need to use (3.19).
\[\mathcal{R} = \{V_1 \cup V_2 \cup \cdots \cup V_t : t \in \mathbb{N} \text{ and each } V_j \in \mathcal{R}_2\}\]

\(\mathcal{R}_1, \mathcal{R}_2\) and \(\mathcal{R}\) are finite sets. In a topological space for any two sets \(A\) and \(B\),
\[
\partial(A \cap B), \partial(A \cup B) \subset \partial A \cup \partial B.
\]  
\tag{3.25}

Hence, for any \(R \in \mathcal{R}\),
\[
\partial R \subset \bigcup_{i \in [I]} \partial B_i \implies \|\Phi(x)\| (\partial R) = 0 \forall x \in \tilde{X}[K]_0.
\]

Moreover, \(R\) is an open subset of \(M\). Therefore, by Proposition 3.6 item (v), as \(j \to \infty\)
\[
\|\Phi_j(x)\| (R) \to \|\Phi(x)\| (R) \forall x \in \tilde{X}[K]_0.
\]  
\tag{3.26}

We also note that \(M \in \mathcal{R}\) as \(\cup_{i \in [I]} B_i = M\).

**Proposition 3.7.** There exists \(\gamma \in \mathbb{N}\) such that the followings hold.

(i) \(|\|\Phi(x)\| (R) - \|\Phi(x)\| (R)| < \eta\) for all \(x \in \tilde{X}[K]_0\) and \(R \in \mathcal{R}\).

(ii) \(H^n(\Phi(x)) < L + 2\eta \forall x \in \tilde{X}[K]_0\).

(iii) \(H^n(\Phi \cap \overline{B_i}) < \eta \forall x \in \tilde{X}[K]_0\).

(iv) \(H^n(\Phi \cap \bar{A}) < \eta \forall x \in \tilde{X}[K]_0\).

(v) For \(q \in [Q]\) and \(i \in [I]\), if \(\text{dim}(e_q) = \text{dim}(f_q) = m\),
\[
\int_{\partial B_i(q)} |\chi \Phi_{\gamma}(x) - \chi \Phi_{\gamma}(x')| dH^n \leq \frac{2^{2m} \eta}{I}
\]  
\tag{3.27}

if \(x, x' \in (e_q)\) or if \(x, x' \in (f_q)\).

**Proof.** (i) follows from (3.26). Since, \(M \in \mathcal{R}\), (ii) follows from (3.1) and (i). To obtain (iii) we note that Proposition 3.6 item (v) and (3.10) imply for each \(x \in \tilde{X}[K]_0\),
\[
\lim_{j \to \infty} \|\Phi_j(x)\| (\overline{B_i}) \leq \|\Phi(x)\| (\overline{B_i}) < \eta.
\]

Similarly, (iv) follows from Proposition 3.6 item (v) and Lemma 3.4. Finally, item (v) follows from items (v) and (vi) of the definition of \(r_i(q)\). \(\square\)

Next we define
\[
\mathcal{S} = \left\{ \Phi_{\gamma}(x) \cap \partial B_i(q) : x \in \tilde{X}[K]_0, i \in [I], q \in [Q] \right\} \bigcup \left\{ \partial B_i(q) \cap \partial B_j(q') : i, j \in [I]; q, q' \in [Q] \right\}
\]

where \(\gamma\) is as in Proposition 3.7. Let \(\mathcal{S}\) be the closed subset of \(M\) which is the union of all the elements of \(\mathcal{S}\). By the transversality assumptions in the definition of \(r_i(q)\), each non-empty element of \(\mathcal{S}\) is a smooth, closed, co-dimension 2 submanifold of \(M\). Hence, for all \(\Sigma \in \mathcal{S}\)
\[
H^n(T_p(\Sigma)) = O(\rho)
\]

17
where the constants in \(O(\rho)\) depend on \(\Sigma\). Therefore, there exist constants \(C, \rho_0\) depending on the submanifolds contained in the set \(\mathcal{S}\) such that
\[
\mathcal{H}^n(\mathcal{T}_\rho(S)) \leq C \rho \quad \forall \rho \leq \rho_0.
\]

(3.28)

**Part 3.** Let \(\mathcal{K}(M)\) be the set of all closed subsets of \(M\). We are now going to prove that there exists a discrete, ‘almost smooth’ sweepout \(\tilde{\Psi} : \tilde{X}[KI]_0 \to \mathcal{K}(M)\) which approximates \(\tilde{\Phi} : \tilde{X} \to \mathcal{C}(M)\). With some additional work it is possible to ensure that for all \(v \in \tilde{X}[KI]_0\), \(\tilde{\Psi}(v) \in \mathcal{C}(M)\); however we do not need this fact to prove Theorem 1.2. Let us introduce the following notation. Let \(\mathcal{K}, \mathcal{K}'\) be closed subsets of \(M\). \(B\) is a normal geodesic ball and \(A = M \setminus \overline{B}\). We define
\[
\Omega(\mathcal{K}, \mathcal{K}'; B) = (\mathcal{K} \cap \overline{A}) \cup (\mathcal{K}' \cap \overline{B})
\]
which is also a closed subset of \(M\). This definition is motivated by the construction of “nested sweepouts” by Chambers and Liokumovich [CL20, Section 6]. In the following proposition and in its proof, we will use various notations which were introduced previously.

**Proposition 3.8.** There exists a map \(\tilde{\Psi} : \tilde{X}[KI]_0 \to \mathcal{K}(M)\) such that \(\tilde{\Psi}\) has the property \(\tilde{\Psi}(x) = \tilde{\Phi}_\gamma(x)\) for all \(x \in \tilde{X}[K]_0\).

Moreover, denoting
\[
\Psi(v) = \Psi(T(v)) = \tilde{\Psi}(v) \cap \tilde{\Psi}(T(v)) \text{ for } v \in \tilde{X}[KI]_0;
\]
\[
\tilde{\Psi} \circ \Delta^1_q = \tilde{\Xi}_q^1, \quad \tilde{\Psi} \circ \Delta^2_q = \tilde{\Xi}_q^2,
\]
for \(q \in [Q]\) and \(m = \dim(e_q) = \dim(f_q), \tilde{\Psi}|_{e_q[I]}_0, \tilde{\Psi}|_{f_q[I]}_0\) have the following properties.

\((P0)_{m,q}\) For \(s = 1, 2\),
\[
\tilde{\Xi}_q^s(\xi', i^{1-1}) = \Omega \left(\tilde{\Xi}_q^s(\xi', (i-1)^{1-1}), \tilde{\Xi}_q^s(\xi', 1); B_i(q)\right)
\]
for all \(\xi' \in \mathcal{S}^{m-1}[I]_0 \setminus (\partial \mathcal{S}^{m-1}) [I]_0\) and \(i \in [I]\).

\((P1)_{m,q}\) For all \(v \in e_q[I]_0, M = \tilde{\Psi}(v) \cup \tilde{\Psi}(T(v))\).

\((P2)_{m,q}\) For all \(v \in e_q[I]_0,
\[
\tilde{\Psi}(v) \subseteq \bigcup_{x \in (e_q)_0} \tilde{\Phi}_\gamma(x).
\]
Similarly, for all \(v' \in f_q[I]_0,
\[
\tilde{\Psi}(v') \subseteq \bigcup_{x' \in (f_q)_0} \tilde{\Phi}_\gamma(x').
\]

\((P3)_{m,q}\) Let
\[
\mathcal{F}_q = \{s \in [Q] : e_s < e_q \text{ or } f_s < e_q\}.
\]
For all \(v \in e_q[I]_0,
\[
\partial \tilde{\Psi}(v), \partial \tilde{\Psi}(T(v)) \subseteq \left\{ \bigcup_{x \in (e_q)_0} \Phi_\gamma(x) \right\} \cup \left\{ \bigcup_{i \in [I], s \in \mathcal{F}_q} \partial B_i(s) \right\}.
\]

18
(P4)_{m,q} For all \( v \in e_q[I_0] \), \( \Psi(v) \setminus \mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \ldots \cup \mathcal{S}_l \) where each \( \mathcal{S}_j \) is an open subset of a hypersurface belonging the set

\[
\left\{ \Phi_i(x) : x \in (e_q)_0 \right\} \cup \left\{ \partial B_i(s) : i \in I, \ s \in \mathcal{F}_q \right\}.
\]

Here \( \mathcal{F}_q \) is as in (P3)_{m,q}.

(P5)_{m,q} Suppose \( v, v' \in e_q[I_0] \) or \( v, v' \in f_q[I_0] \); \( e \in e_q[I_1] \cup f_q[I_1] \) such that \( v, v' \prec e \). Then, there exists \( B = B_i(e) \) for some \( i(e) \in I \) such that

\[
\begin{align*}
\Psi(v) \cap (M \setminus (A \cup B)) &= \Psi(v') \cap (M \setminus (A \cup B)) ; \\
\tilde{\Psi}(v) \cap (M \setminus (A \cup B)) &= \tilde{\Psi}(v') \cap (M \setminus (A \cup B)) ; \\
\tilde{\Psi}(T(v)) \cap (M \setminus (A \cup B)) &= \tilde{\Psi}(T(v')) \cap (M \setminus (A \cup B)) .
\end{align*}
\]

Moreover, \( B \) can be characterized as follows. Without loss of generality, let \( v, v' \in e_q[I_0] \). If \( D_q^1(v) = (\xi_1, \xi_2, \ldots, \xi_m) \) and \( D_q^1(v') = (\xi'_1, \xi'_2, \ldots, \xi'_m) \), there exists a unique index \( j \in [m] \) such that \( \xi_j \neq \xi'_j \). Let \( i \in I \) be such that \( \{\xi_j, \xi'_j\} = \{(i-1)I^{-1}, iI^{-1}\} \). Then \( B = B_i \).

(P6)_{m,q} For all \( i \in I \) and \( v \in e_q[I_0] \),

\[
\mathcal{H}^n \left( \Psi(v) \cap \overline{B_i} \right) < 2^{4m+2} \eta.
\]

(P7)_{m,q} For all \( v \in e_q[I_0] \),

\[
\mathcal{H}^n \left( \Psi(v) \cap \overline{B_1} \right) < 2^{4m+2} \eta.
\]

(P8)_{m,q} For all \( v \in e_q[I_0] \), \( x \in (e_q)_0 \) and \( R \in \mathcal{R} \),

\[
\left| \mathcal{H}^n (\Psi(v) \cap R) - \mathcal{H}^n (\Phi_i(x) \cap R) \right| < 2^{4m+2} \eta.
\]

In particular,

\[
\mathcal{H}^n (\Psi(v)) < L + (2^{4m+2} + 2 \eta).
\]

Proof. We will inductively construct \( \tilde{\Psi} : e_q[I_0], f_q[I_0] \to \mathcal{K}(M) \). If \( \dim(e_q) = \dim(f_q) = 0 \), then \( e_q[I] = e_q \) and \( f_q[I] = f_q \). In that case we define

\[
\tilde{\Psi}(e_q) = \tilde{\Phi}_i(e_q) \quad \text{and} \quad \tilde{\Psi}(f_q) = \tilde{\Phi}_i(f_q)
\]

which is precisely the property (P). Moreover, for this definition of \( \tilde{\Psi} \) on the 0-cells \( e_q, f_q \), properties (P0)_{0,q} - (P8)_{0,q} are also satisfied. (We need to use Proposition 3.7.)

Let us assume that \( d \geq 1 \) and \( \tilde{\Psi} \) is defined on

\[
\bigcup_{\dim(e_q) \leq d-1} \left( e_q[I_0] \cup f_q[I_0] \right)
\]

and if \( \dim(e) = \dim(f) = (d-1), \ \tilde{\Psi}|_{e_q[I_0]}, \tilde{\Psi}|_{f_q[I_0]} \) have the properties \((P0)_{(d-1),s} - (P8)_{(d-1),s} \).
Let $\nu \in [Q]$ be such that $\dim(e_\nu) = \dim(f_\nu) = d$. We define $\tilde{\Xi}_\nu^1, \tilde{\Xi}_\nu^2$ on $\mathcal{I}^d[I_0]$ and $\tilde{\Psi}$ on $e_\nu[I_0], f_\nu[I_0]$ as follows. By our assumption, $\tilde{\Psi}$ is defined on $(\partial e_\nu)[I_0], (\partial f_\nu)[I_0]$. For $z \in (\partial \mathcal{I}^d)[I_0], \Delta^1_\nu(z) \in (\partial e_\nu)[I_0]$ and $\Delta^2_\nu(z) \in (\partial f_\nu)[I_0]$; for $s = 1, 2$ we define

$$
\tilde{\Xi}_\nu^s(z) = \left(\tilde{\Psi} \circ \Delta^s_\nu\right)(z).
$$

(3.31)

If $z \in \mathcal{I}^d[I_0] \setminus (\partial \mathcal{I}^d)[I_0]$, we write $z = (z', iI^{-1})$ with $z' \in \mathcal{I}^{d-1}[I_0]$ and $1 \leq i \leq 1 - I^{-1}$; for $s = 1, 2$, we define

$$
\tilde{\Xi}_\nu^s(z', iI^{-1}) = \Omega \left(\tilde{\Xi}_\nu^s(z', (i-1)I^{-1}), \tilde{\Xi}_\nu^s(z', 1); B_i(\nu)\right).
$$

(3.32)

For $v \in e_\nu[I_0] \cup f_\nu[I_0]$, we define

$$
\tilde{\Psi}(v) = \begin{cases} 
\left(\tilde{\Xi}_\nu^1 \circ D^1_\nu\right)(v) & \text{if } v \in e_\nu[I_0]; \\
\left(\tilde{\Xi}_\nu^2 \circ D^2_\nu\right)(v) & \text{if } v \in f_\nu[I_0].
\end{cases}
$$

(3.33)

We note that by (3.31), $\tilde{\Psi}$ is well-defined on $(\partial e_\nu)[I_0], (\partial f_\nu)[I_0]$. Let us also set

$$
\Xi_\nu(z) = \tilde{\Xi}_\nu^1(z) \cap \tilde{\Xi}_\nu^2(z), \quad z \in \mathcal{I}^d[I_0]
$$

so that

$$
\Psi(v) = \begin{cases} 
(\Xi_\nu \circ D^1_\nu)(v) & \text{if } v \in e_\nu[I_0]; \\
(\Xi_\nu \circ D^2_\nu)(v) & \text{if } v \in f_\nu[I_0].
\end{cases}
$$

(3.35)

We need to show that $\tilde{\Psi}$ defined in this way on $e_\nu[I_0], f_\nu[I_0]$ have the properties $(P0)_{d,\nu} - (P8)_{d,\nu}$.

Before we start, for convenience let us introduce the following notation.

$$
B_i = B_i(\nu), \quad A_i = M \setminus \overline{B_i}, \quad S_i = \partial B_i = \partial A_i = \overline{A_i} \cap \overline{B_i}.
$$

We fix $z' \in \mathcal{I}^{d-1}[I_0] \setminus (\partial \mathcal{I}^{d-1})[I_0]$ and define

$$
K_i = \tilde{\Xi}_\nu^1(z', iI^{-1}), \quad L_i = \tilde{\Xi}_\nu^2(z', iI^{-1}), \quad \Sigma_i = \Xi_\nu(z', iI^{-1}) = K_i \cap L_i.
$$

From (3.32), for $i = 1, 2, \ldots, (I-1)$ we obtain,

$$
K_i = (K_{i-1} \cap \overline{A_i}) \cup (K_i \cap \overline{B_i}); \quad L_i = (L_{i-1} \cap \overline{A_i}) \cup (L_i \cap \overline{B_i}); \quad \Sigma_i = (\Sigma_{i-1} \cap \overline{A_i}) \cup (\Sigma_i \cap \overline{B_i}) \cup (K_{i-1} \cap L_i \cap S_i) \cup (K_i \cap L_{i-1} \cap S_i).
$$

(3.36)

Using induction one can prove the following.

$$
K_i = \left[K_0 \cap (\bigcup_{s=1}^i B_s)^c\right] \bigcup \left[K_i \cap (\bigcup_{s=1}^i \overline{B_s})\right]; \\
L_i = \left[L_0 \cap (\bigcup_{s=1}^i B_s)^c\right] \bigcup \left[L_i \cap (\bigcup_{s=1}^i \overline{B_s})\right]; \\
\Sigma_i = \left[\Sigma_0 \cap (\bigcup_{s=1}^i B_s)^c\right] \bigcup \left[\Sigma_i \cap (\bigcup_{s=1}^i \overline{B_s})\right] \bigcup \Lambda_i \quad \text{where} \quad \Lambda_i \subset \bigcup_{t=1}^i S_t.
$$

(3.37)
For \( i > 1 \) we have,
\[
\Lambda_i = (\Lambda_{i-1} \cap \overline{A}_i) \cup (K_{i-1} \cap L_I) \cup (K_I \cap L_I - L_i).
\] (3.38)

From the equations in (3.37), we conclude that the equations in (3.36) hold for \( i = I \) as well. We want to denote the top and bottom faces of \( e_\nu \) and \( f_\nu \) by the indices \( \tau \) and \( \beta \in [Q] \) i.e.
\[
\left\{ \Delta^1_{\nu}((\mathcal{S}^{d-1} \times \{1\})), \Delta^2_{\nu}((\mathcal{S}^{d-1} \times \{1\}) \right\} = \{ e_\tau, f_\tau \};
\]
\[
\left\{ \Delta^1_{\nu}((\mathcal{S}^{d-1} \times \{0\})), \Delta^2_{\nu}((\mathcal{S}^{d-1} \times \{0\}) \right\} = \{ e_\beta, f_\beta \}.
\]

We note that the top face of \( e_\nu \) could be either \( e_\tau \) (in that case the top face of \( f_\nu \) is \( f_\tau \)) or \( f_\tau \) (in that case the top face of \( f_\nu \) is \( e_\tau \)); similar remark applies for the bottom face. For the ease of the presentation, we will assume that \( e_\tau \) is the top face of \( e_\nu \) and \( e_\beta \) is the bottom face of \( e_\nu \). The other cases will be entirely analogous.

We now proceed in steps to check that \( \tilde{\Psi}_{\nu[I_0]} \), \( \tilde{\Psi}_{\nu[I_0]} \) have the properties \( (P0)_{d,\nu} - (P8)_{d,\nu} \).

**Step 0.** \( (P0)_{d,\nu} \) is satisfied because of our definition of \( \tilde{\Xi}_1^{\nu} \) and \( \tilde{\Xi}_2^{\nu} \) i.e. equation (3.32).

**Step 1.** By the induction hypothesis, \( \tilde{\Psi} \) restricted to \( \nu[I_0], f_\beta[I_0] \) (and \( e_\tau[I_0], f_\tau[I_0] \)) satisfy \( (P1)_{(d-1),\beta} \) (and \( (P1)_{d-1}, \tau \)). Hence,
\[ K_0 \cup L_0 = M = K_I \cup L_I \]
which together with (3.37) imply
\[ M = K_i \cup L_i. \]

**Step 2.** By our assumption, \( \tilde{\Psi} \) restricted to \( e_\beta[I_0], f_\beta[I_0] \) (and \( e_\tau[I_0], f_\tau[I_0] \)) satisfy \( (P2)_{(d-1),\beta} \) (and \( (P2)_{d-1}, \tau \)). Hence,
\[ K_i \subset \bigcup_{x \in (e_\beta)_{\nu}} \Phi_\gamma(x); \quad K_I \subset \bigcup_{x \in (e_\tau)_{\nu}} \Phi_\gamma(x). \]
Therefore, using (3.37),
\[ K_i \subset \bigcup_{x \in (e_\nu)_{\nu}} \Phi_\gamma(x) \quad \forall i \in [I]. \]
Similarly, we can show that
\[ L_i \subset \bigcup_{x \in (f_\nu)_{\nu}} \Phi_\gamma(x) \quad \forall i \in [I]. \]

**Step 3.** By the induction hypothesis,
\[
\partial K_0 \subset \left\{ \bigcup_{x \in (e_\beta)_{\nu}} \Phi_\gamma(x) \bigcup \bigcup_{i \in [I], s \in \mathcal{S}_\beta} \partial B_i(s) \right\};
\] (3.39)
\[
\partial K_I \subset \left\{ \bigcup_{x \in (e_\tau)_{\nu}} \Phi_\gamma(x) \bigcup \bigcup_{i \in [I], s \in \mathcal{S}_\tau} \partial B_i(s) \right\}.
\] (3.40)
Moreover, by (3.25) and (3.36)
\[ \partial K_i \subset \partial K_{i-1} \cup \partial K_I \cup S_i \implies \partial K_i \subset \partial K_0 \cup \partial K_I \cup \left( \bigcup_{j=1}^{i} S_j \right). \] (3.41)

Combining (3.39), (3.40) and (3.41), we obtain
\[ \partial K_i \subset \left\{ \bigcup_{x \in (e_\nu)_0} \Phi_\gamma(x) \right\} \bigcup \left\{ \bigcup_{i \in [I], s \in \mathcal{F}_\beta \cup \mathcal{F}_\tau} \partial B_i(s) \right\} \bigcup \left\{ \bigcup_{t=1}^i \partial B_t \right\}. \] (3.42)

We can arrive at a similar conclusion for \( \partial L_i \) as well.

**Step 4.** For \( j = 0, 1, \ldots, I \), let \( \hat{\Sigma}_j := \Sigma_j \setminus S \). By the induction hypothesis
\[ \hat{\Sigma}_0 = S^0_1 \cup S^0_2 \cup \ldots \cup S^0_0 \]
where each \( S^0_0 \) is an open subset of a hypersurface belonging to the set
\[ \left\{ \Phi_\gamma(x) : x \in (e_\beta)_0 \right\} \bigcup \left\{ \partial B_i(s) : i \in [I], s \in \mathcal{F}_\beta \right\}; \]
and
\[ \hat{\Sigma}_I = S^I_1 \cup S^I_2 \cup \ldots \cup S^I_I \]
where each \( S^I_0 \) is an open subset of a hypersurface belonging to the set
\[ \left\{ \Phi_\gamma(x) : x \in (e_\tau)_0 \right\} \bigcup \left\{ \partial B_i(s) : i \in [I], s \in \mathcal{F}_\tau \right\}. \] (3.43)

By induction let us assume that
\[ \hat{\Sigma}_{i-1} = S^{i-1}_1 \cup S^{i-1}_2 \cup \ldots \cup S^{i-1}_I \]
where each \( S^{i-1}_0 \) is an open subset of a hypersurface belonging to the set
\[ \left\{ \Phi_\gamma(x) : x \in (e_\nu)_0 \right\} \bigcup \left\{ \partial B_i(s) : i \in [I], s \in \mathcal{F}_\beta \cup \mathcal{F}_\tau \right\} \bigcup \left\{ \partial B_t : t \in [i-1] \right\}. \] (3.44)

Then,
\[ (\Sigma_{i-1} \cap \overline{A}_i) \setminus S = (\hat{\Sigma}_{i-1} \cap \overline{A}_i) \setminus S = (\hat{\Sigma}_{i-1} \cap \overline{A}_i) \setminus S \] (3.45)
as by (3.44) \( \hat{\Sigma}_{i-1} \cap \partial B_i \subset S \). Further, \( (\hat{\Sigma}_{i-1} \cap A_i) \setminus S \) is an open subset of \( \hat{\Sigma}_{i-1} \) as \( A_i \subset M \) is open and \( S \subset M \) is closed. Similarly, using (3.43),
\[ (\Sigma_I \cap \overline{B}_i) \setminus S = (\hat{\Sigma}_I \cap B_i) \setminus S \] (3.46)
which is an open subset of \( \hat{\Sigma}_I \).
\[ (K_{i-1} \cap L_I \cap S_i) \setminus S = (int(K_{i-1} \cap L_I) \cap S_i) \setminus S \] (3.47)
as by (3.25), \( \partial(K_{i-1} \cap L_I) \subset \partial K_{i-1} \cup \partial L_I \); by (3.42) and by the induction hypothesis \((P3)_{(d-1),\tau}\),
\[ (\partial K_{i-1} \cup \partial L_I) \cap S_i \subset S. \]
Similarly,
\[(K_i \cap L_{i-1} \cap S_i) \setminus S = (\text{int}(K_i \cap L_{i-1}) \cap S_i) \setminus S.\] (3.48)

Moreover,
\[M = B_i \cup S_i \cup A_i.\]

Hence, using (3.36) and equations (3.43) – (3.48), we obtain
\[\hat{\Sigma}_i = S_i^1 \cup S_i^2 \cup \ldots \cup S_i^l_i\]

where each \(S_j^i\) is an open subset of a hypersurface belonging to the set
\[\{\Phi_\gamma(x) : x \in (e_\nu)_0\} \cup \{\partial B_i(s) : i \in [I], s \in \mathcal{F}_\beta \cup \mathcal{F}_\tau\} \cup \{\partial B_t : t \in [I]\}.\] (3.49)

**Step 5.** Suppose \(v, v' \in (\partial e_\nu)[I]_0\). Let \(\rho \in [Q]\) such that \(\lambda_\rho < e_\nu\) (\(\lambda_\rho\) could be \(e_\rho\) or \(f_\rho\)) with \(\dim(\lambda_\rho) = d - 1\) and \(v, v' \in \lambda_\rho[I]_0\). Then, by the inductive hypothesis (P5)\(_{(d-1),\rho}\) and by the compatibility relation (3.17), we get (P5)\(_{(d,\nu)}\) for this case.

Now, we assume that \(v \notin (\partial e_\nu)[I]_0\); \(D_\rho^1(v) = (z_1, z_2, \ldots, z_d) = (z', z_d), D_\rho^1(v') = (z'_1, z'_2, \ldots, z'_d) = (z'', z'_d)\). If \(z'_d \neq z''_d\), using (3.36) we get (P5)\(_{(d,\nu)}\). Otherwise, \(z'_d = z_d\) and there are two possibilities: either \(v' \notin (\partial e_\nu)[I]_0\) or \(v' \in (\partial e_\nu)[I]_0\). Let us assume the second possibility (the other case is similar), i.e. there exists \(\rho \in [Q]\) such that \(\lambda_\rho < e_\nu\) (\(\lambda_\rho = e_\rho\) or \(f_\rho\)) with \(\dim(\lambda_\rho) = d - 1\) and \(v' \in \lambda_\rho[I]_0\). We define,
\[B_i' = B_i(\rho), \quad A_i' = M \setminus \overline{B_i'}, \quad S_i' = \partial B_i' = \partial A_i';\]
\[K_i' = \hat{\Xi}_\nu^1(z'', iI^{-1}), \quad L_i' = \hat{\Xi}_\nu^2(z'', iI^{-1}), \quad \Sigma_i' = \Xi_\nu(z'', iI^{-1}) = K_i' \cap L_i'.\]

Using (P0)\(_{(d-1),\rho}\) and the compatibility relation (3.17), we can deduce the following relations similar to those in (3.37).
\[K_i' = \left[K_0' \cap (\bigcup_{s=1}^l B_s')^c\right] \cup \left[K_i' \cap (\bigcup_{s=1}^l \overline{B_s'})\right];\]
\[L_i' = \left[L_0' \cap (\bigcup_{s=1}^l B_s')^c\right] \cup \left[L_i' \cap (\bigcup_{s=1}^l \overline{B_s'})\right];\] (3.50)
\[\Sigma_i' = \left[\Sigma_0' \cap (\bigcup_{s=1}^l B_s')^c\right] \cup \left[\Sigma_i' \cap (\bigcup_{s=1}^l \overline{B_s'})\right] \cup A_i' \quad \text{where} \quad A_i' \subset \bigcup_{t=1}^l S_i'.\]

(P5)\(_{(d-1),\beta}\) and (P5)\(_{(d-1),\tau}\) imply
\[\Sigma_0 \cap (M \setminus (A \cup B)) = \Sigma_0' \cap (M \setminus (A \cup B))\];
\[\Sigma_I \cap (M \setminus (A \cup B)) = \Sigma_I' \cap (M \setminus (A \cup B));\] (3.51)

where \(B = B_j; j\) is such that if \(z_I \neq z'_I, \{z_I, z'_I\} = \{(j - 1)I^{-1}, jI^{-1}\}\). We note that for all \(s \in [I], \quad B_s \Delta B_s', \partial B_s, \partial B_s' \subset A.\)

23
Hence,

\[(\bigcup_{i=1}^{j} B_s)^c \Delta (\bigcup_{i=1}^{j} B'_s)^c \subset A \implies (\bigcup_{i=1}^{j} B_s)^c \cap (A \cup B)^c = (\bigcup_{i=1}^{j} B'_s)^c \cap (A \cup B)^c; \]

\[(\bigcup_{i=1}^{j} B_s)^c \Delta (\bigcup_{i=1}^{j} B'_s)^c \subset A \implies (\bigcup_{i=1}^{j} B_s)^c \cap (A \cup B)^c = (\bigcup_{i=1}^{j} B'_s)^c \cap (A \cup B)^c; \]

\[\Lambda_i \cap (A \cup B)^c = \emptyset = \Lambda'_i \cap (A \cup B)^c \]

(3.52)

Using the equations (3.37), (3.50), (3.51) and (3.52), we obtain

\[\Sigma_i \cap (M \setminus (A \cup B)) = \Sigma'_i \cap (M \setminus (A \cup B)).\]

for all \(i \in [I]\). By a similar argument, one can also show that

\[K_i \cap (M \setminus (A \cup B)) = K'_i \cap (M \setminus (A \cup B));\]

and

\[L_i \cap (M \setminus (A \cup B)) = L'_i \cap (M \setminus (A \cup B)).\]

**Step 6.** Using (3.38), for \(i > 1\),

\[\mathcal{H}^n(\Lambda_i) \leq \mathcal{H}^n(\Lambda_{i-1}) + \mathcal{H}^n(K_{i-1} \cap L_I \cap S_i) + \mathcal{H}^n(K_I \cap L_{i-1} \cap S_i). \]

(3.53)

By \((P2)_{d,\nu}\),

\[K_{i-1} \subset \bigcup_{x \in (e_\nu)_0} \tilde{\Phi}_\gamma(x), \quad L_I \subset \bigcup_{x \in (f_\nu)_0} \tilde{\Phi}_\gamma(x).\]

Hence,

\[\mathcal{H}^n(S_i \cap K_{i-1} \cap L_I) \leq \sum_{x_1 \in (e_\nu)_0, x_2 \in (f_\nu)_0} \mathcal{H}^n \left( S_i \cap \tilde{\Phi}_\gamma(x_1) \cap \tilde{\Phi}_\gamma(x_2) \right). \]

(3.54)

For \(x_1 \in (e_\nu)_0, x_2 \in (f_\nu)_0,\)

\[\mathcal{H}^n \left( S_i \cap \tilde{\Phi}_\gamma(x_1) \cap \tilde{\Phi}_\gamma(x_2) \right) = \mathcal{H}^n \left( S_i \cap \tilde{\Phi}_\gamma(x_1) \cap \tilde{\Phi}_\gamma(x_2) \cap \tilde{\Phi}_\gamma(T(x_2)) \right) \]

\[+ \mathcal{H}^n \left( S_i \cap \tilde{\Phi}_\gamma(x_1) \cap \tilde{\Phi}_\gamma(x_2) \cap \left( M - \tilde{\Phi}_\gamma(T(x_2)) \right) \right) \]

\[= 0 + \mathcal{H}^n \left( S_i \cap \left( \tilde{\Phi}_\gamma(x_1) - \tilde{\Phi}_\gamma(T(x_2)) \right) \right) \]

\[< \frac{2^{2d}\eta}{I}. \]

(3.55)

In the second equality we have used the fact that \(S_i\) is transverse to \(\Phi_\gamma(x_2) = \tilde{\Phi}_\gamma(x_2) \cap \tilde{\Phi}_\gamma(T(x_2))\) (which was assumed in the definition of \(r_i(q)\)) and \((P1)_{d,\nu}\). In the last inequality we have used Proposition 3.7, item (v). Therefore, combining (3.53) – (3.55), we obtain

\[\mathcal{H}^n(\Lambda_i) < \mathcal{H}^n(\Lambda_{i-1}) + \frac{2^{2d+1}\eta}{I} \text{ for } i > 1.\]

Moreover, by the above argument,

\[\mathcal{H}^n(\Lambda_1) < \frac{2^{2d+1}\eta}{I}.\]
Hence, for all $i \in [I]$,
\[
H^n(\Lambda_i) < \frac{2^{4d+1} \eta}{I} \leq 2^{4d+1} \eta. \tag{3.56}
\]
Using this inequality along with (3.37) and (P6)$_{(d-1), \beta}$, (P6)$_{(d-1), \tau}$, we conclude that for all $j \in [I]$,
\[
H^n \left( \Sigma_i \cap \overline{B^c_j} \right) \leq H^n \left( \Sigma_0 \cap \overline{B^c_j} \right) + H^n \left( \Sigma_I \cap \overline{B^c_j} \right) + H^n(\Lambda_i) \\
< (2^{4d-2} + 2^{4d-2} + 2^{4d+1}) \eta \\
< 2^{4d+2} \eta. \tag{3.57}
\]

**Step 7.** Using (3.56), (3.37) and (P7)$_{(d-1), \beta}$, (P7)$_{(d-1), \tau}$, we obtain as in Step 6,
\[
H^n(\Sigma_i \cap \overline{A}_1) \leq H^n(\Sigma_0 \cap \overline{A}_1) + H^n(\Sigma_I \cap \overline{A}_1) + H^n(\Lambda_i) \\
< (2^{4d-2} + 2^{4d-2} + 2^{4d+1}) \eta \\
< 2^{4d+2} \eta. \tag{3.58}
\]

**Step 8.** We note that for all $s \in [I]$,
\[
\overline{B}^c_s \subset B^c_s \subset \overline{B}^c_s \cup \overline{A}; \quad \overline{B}^c_s \subset B_s \subset \overline{B}^c_s \cup \overline{A}.
\]
Therefore, for all $i \in [I]$,
\[
\left( \bigcup_{s=1}^i B_s \right)^c \subset \left( \bigcup_{s=1}^i B_s \right)^c \subset \left( \bigcup_{s=1}^i \overline{B}_s \right)^c \cup \overline{A}; \\
\bigcup_{s=1}^i \overline{B}_s \subset \bigcup_{s=1}^i B_s \subset \left( \bigcup_{s=1}^i \overline{B}_s \right) \cup \overline{A}.
\]
Hence, using (3.37), we deduce the following inequalities.
\[
H^n(\Sigma_i \cap R) \leq H^n \left( \Sigma_0 \cap \left( \bigcup_{s=1}^i \overline{B}_s \right)^c \cap R \right) + H^n(\Sigma_0 \cap \overline{A}) + H^n(\Sigma_I \cap \left( \bigcup_{s=1}^i B_s \right) \cap R) + H^n(\Lambda_i); \tag{3.59}
\]
\[
H^n(\Sigma_i \cap R) \geq H^n \left( \Sigma_0 \cap \left( \bigcup_{s=1}^i \overline{B}_s \right)^c \cap R \right) + H^n(\Sigma_I \cap \left( \bigcup_{s=1}^i B_s \right) \cap R) - H^n(\Sigma_I \cap \overline{A}). \tag{3.60}
\]
Denoting
\[
R_1 = \left( \bigcup_{s=1}^i \overline{B}_s \right)^c \cap R, \quad R_2 = \left( \bigcup_{s=1}^i B_s \right) \cap R, \tag{3.61}
\]
using (3.59), (3.60), (P7)$_{(d-1), \beta}$, (P7)$_{(d-1), \tau}$ and (3.56) we obtain,
\[
\left| H^n(\Sigma_i \cap R) - H^n(\Sigma_0 \cap R_1) - H^n(\Sigma_I \cap R_2) \right| \\
\leq H^n(\Sigma_0 \cap \overline{A}) + H^n(\Sigma_I \cap \overline{A}) + H^n(\Lambda_i) \\
\leq (2.2^{4d-2} + 2^{4d+1}) \eta. \tag{3.62}
\]
Let $x_0 \in (e_\nu)_0$; without loss of generality we can assume that $x_0 \in (e_\beta)_0$ i.e. $D^{1}_{\nu}(x_0) = (\xi, 0)$ for some $\xi \in \mathcal{I}^{d-1}$. Let $x_1 \in (e_\tau)_0$ such that $x_1 = \Delta^{1}_{\nu}(\xi, 1)$. We note that $R \in \mathcal{R}$ implies $R_1,R_2 \in \mathcal{R}$ as well. Further, by the definition of $B_{s}, \|\Phi_{\gamma}(x_0)\| (\partial B_{s}) = 0$; hence,

$$\mathcal{H}^{n}(\Phi_{\gamma}(x_0) \cap R) = \mathcal{H}^{n}(\Phi_{\gamma}(x_0) \cap R_1) + \mathcal{H}^{n}(\Phi_{\gamma}(x_0) \cap R_2).$$

Therefore, using (3.62), (P8)$_{(d-1),\beta}$, (P8)$_{(d-1),\tau}$, Proposition 3.7 (i) and (3.18) we get the following estimate.

$$\begin{align*}
|\mathcal{H}^{n}(\Sigma_i \cap R) - \mathcal{H}^{n}(\Phi_{\gamma}(x_0) \cap R)| \\
\leq |\mathcal{H}^{n}(\Sigma_i \cap R) - \mathcal{H}^{n}(\Sigma_0 \cap R_1)| + |\mathcal{H}^{n}(\Sigma_0 \cap R_1) - \mathcal{H}^{n}(\Phi_{\gamma}(x_0) \cap R_1)| \\
+ |\mathcal{H}^{n}(\Phi_{\gamma}(x_0) \cap R_1) - \mathcal{H}^{n}(\Phi_{\gamma}(x_0) \cap R_2)| \\
< (2^{4d-1} + 2^{4d+1} + 2^{4d-2} + 2^{4d-2} + 3)\eta \\
< 2^{4d+2}\eta. \quad (3.63)
\end{align*}$$

Since $M \in \mathcal{R}$, (3.63) and Proposition 3.7, (ii) imply

$$\mathcal{H}^{n}(\Sigma_i) \leq L + (2^{4d+2} + 2)\eta. \quad \square$$

### 3.5 Approximate solution of the Allen-Cahn equation

Here we will briefly discuss an approximate solution of the Allen-Cahn equation whose energy is concentrated in a tubular neighbourhood of a closed, two-sided hypersurface (with mild singularities). We will follow the paper by Guaraco [Gua18]; further details can be found there.

Let $h: \mathbb{R} \to \mathbb{R}$ be the unique solution of the following ODE.

$$\dot{\varphi}(t) = \sqrt{2W(\varphi(t))}; \quad \varphi(0) = 0.$$

For all $t \in \mathbb{R}$, $-1 < h(t) < 1$ and as $t \to \pm\infty$, $(h(t) \mp 1)$ converges to zero exponentially fast.

$h_\varepsilon(t) = h(t/\varepsilon)$ is a solution of the one dimensional Allen-Cahn equation

$$\varepsilon^2 \ddot{\varphi}(t) = W'(\varphi(t))$$

with finite total energy:

$$\int_{-\infty}^{\infty} \left( \frac{\varepsilon}{2} h_\varepsilon(t)^2 + W(h_\varepsilon(t)) \right) dt = 2\sigma.$$

For $\varepsilon > 0$, we define Lipschitz continuous function

$$g_\varepsilon(t) = \begin{cases} 
 h_\varepsilon(t) & \text{if } |t| \leq \sqrt{\varepsilon}; \\
 h_\varepsilon(\sqrt{\varepsilon}) + \left( \frac{1}{\sqrt{\varepsilon}} - 1 \right) (1 - h_\varepsilon(\sqrt{\varepsilon})) & \text{if } \sqrt{\varepsilon} \leq t \leq 2\sqrt{\varepsilon}; \\
 1 & \text{if } t \geq 2\sqrt{\varepsilon}; \\
 h_\varepsilon(-\sqrt{\varepsilon}) + \left( \frac{1}{\sqrt{\varepsilon}} + 1 \right) (1 + h_\varepsilon(-\sqrt{\varepsilon})) & \text{if } -2\sqrt{\varepsilon} \leq t \leq -\sqrt{\varepsilon}; \\
 -1 & \text{if } t \leq -2\sqrt{\varepsilon}.
\end{cases}$$
Suppose \( d : M \to \mathbb{R} \) is a Lipschitz continuous function such that \( \| \nabla d \| = 1 \) a.e. Let \( u_\varepsilon = g_\varepsilon \circ d \), \( U \subset M \). Using the notation

\[
E_\varepsilon(u,U) = \int_U \frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon},
\]

we compute

\[
E_\varepsilon(u_\varepsilon,U) = \int_{U \cap \{ |d| \leq 2\varepsilon \}} \left[ \frac{\varepsilon}{2} g_\varepsilon(d(y))^2 + \frac{1}{\varepsilon} W(g_\varepsilon(d(y))) \right] \, \mathcal{H}^{n+1}(y)
\]

\[
= \int_{|\tau| \leq 2\varepsilon} \left[ \frac{\varepsilon}{2} \dot{g}_\varepsilon(\tau)^2 + \frac{1}{\varepsilon} W(\dot{g}_\varepsilon(\tau)) \right] \mathcal{H}^n(U \cap \{ d = \tau \}) \, d\tau
\]

\[
\leq \int_{|\tau| \leq \varepsilon} \left[ \frac{\varepsilon}{2} \dot{h}_\varepsilon(\tau)^2 + \frac{1}{\varepsilon} W(\dot{h}_\varepsilon(\tau)) \right] \mathcal{H}^n(U \cap \{ d = \tau \}) \, d\tau
\]

(3.64)

\[
+ \left( \frac{1}{2} (1 - h_\varepsilon(\sqrt{\varepsilon}))^2 + \frac{1}{\varepsilon} W(h_\varepsilon(\sqrt{\varepsilon})) \right) \, \text{Vol}(M, g)
\]

(3.65)

Let

\[
\Lambda_\varepsilon = \sup_{|\tau| \leq \sqrt{\varepsilon}} \mathcal{H}^n(U \cap \{ d = \tau \})
\]

Then, the integral in (3.64) is bounded by \( 2\sigma \Lambda_\varepsilon \). Further, there exists \( \varepsilon_0 = \varepsilon_0(W, g, \eta) \) such that for all \( \varepsilon \leq \varepsilon_0 \) the expression in (3.65) is bounded by \( 2\sigma \eta \). Hence,

\[
E_\varepsilon(u_\varepsilon,U) \leq 2\sigma (\Lambda_\varepsilon + \eta) \forall \varepsilon \leq \varepsilon_0.
\]

(3.66)

For \( v \in \tilde{X}[K]_0 \), we define \( d_v : M \to \mathbb{R} \) as follows.

\[
d_v(p) = \begin{cases} -d(p, \Psi(v)) & \text{if } p \in \tilde{\Psi}(v); \\ d(p, \Psi(v)) & \text{if } p \in \tilde{\Psi}(T(v)). \end{cases}
\]

(3.67)

By the definition of \( \Psi(v) \) and (P1) of Proposition 3.8, \( d_v \) is a well-defined continuous function on \( M \). Moreover, by [Gua18, Proposition 9.1], \( d_v \) is Lipschitz continuous and \( \| \nabla d_v \| = 1 \) a.e.

Let \( p \in U \cap \{ d_v = \tau \} \). Then, either \( d(p, S) = |\tau| \) or there exists \( z \in (\Psi(v) \setminus S) \) with \( d(p, z) = |\tau| \) such that \( p = \exp_z(\tau \mathbf{n}(z)) \) where \( \mathbf{n}(z) \) is the unit normal to \( \Psi(v) \setminus S \) at \( z \) pointing inside \( \tilde{\Psi}(T(v)) \). We must have

\[
z \in N_{|\tau|}(U) \cap (\Psi(v) \setminus S).
\]

Hence, we can write

\[
\mathcal{H}^n(U \cap \{ d_v = \tau \}) \leq \mathcal{H}^n(T_{|\tau|}(S)) + \int_{N_{|\tau|}(U) \cap (\Psi(v) \setminus S)} |J \exp_z(\tau \mathbf{n}(z))| \, d\mathcal{H}^n(z)
\]

(3.68)

where \( J \exp_z(\tau \mathbf{n}(z)) \) is the Jacobian factor of the map \( z \mapsto \exp_z(\tau \mathbf{n}(z)) \) which can be estimated as follows.

Let

\[
\mathcal{M} = \{ \Phi_\gamma(x) : x \in \tilde{X}[K]_0 \} \bigcup \{ \partial B_i(q) : i \in [I], q \in [Q] \}.
\]
Using (P4) of Proposition 3.8 and [War66, Corollary 4.2, Theorem 4.3], [Gua18, Proposition 9.4], one can deduce the following estimate. Let \( \lambda > 0 \) be such that

\[
H_\Sigma(v, v) \leq \lambda \langle v, v \rangle \quad \forall \Sigma \in \mathcal{S}' , \quad v \in T\Sigma.
\]

Here \( H_\Sigma \) denotes the second fundamental form of \( \Sigma \). Then, there exist \( \tau_0 \) and \( C_1 \) depending only on \( \lambda \), the ambient dimension \( n + 1 \) and \( g \) such that

\[
|J \exp_\varepsilon(\tau \mathbf{n}(z))| \leq (1 + \lambda \varepsilon) \quad \forall \varepsilon \in \Psi(v) \setminus S.
\]

Hence, using (3.28) and (3.68), there exists \( \tau_1 = \tau_1(\mathcal{S}, \mathcal{S}', n, g) \) such that for all \( |\tau| \leq \tau_1 \)

\[
\mathcal{H}^n(\mathcal{U} \cap \{d_v = \tau\}) \leq C|\tau| + (1 + C_1|\tau|)\mathcal{H}^n(\mathcal{N}_1(\mathcal{U}) \cap \Psi(v)).
\]

This estimate along with (P6) and (P7) of Proposition 3.8, (3.30) and (3.66) gives the following proposition. We recall that \( k \) is the dimension of the parameter spaces \( X \) and \( \bar{X} \).

**Proposition 3.9.** There exists \( \varepsilon_1 = \varepsilon_1(\eta, \tau_1, \varepsilon_0, \delta, r_0 - r_1) \) such that for all \( v \in \bar{X}[KT]_0, \varepsilon \leq \varepsilon_1 \) and \( i \in [I] \), denoting \( \vartheta^v_\varepsilon = g_\varepsilon \circ d_v \), we have

\[
E_\varepsilon(\vartheta^v_\varepsilon, \mathbf{B}_1^1) \leq 2\sigma(2^{4k+2} + 2)\eta; \quad E_\varepsilon(\vartheta^v_\varepsilon, \mathbf{A}_2) \leq 2\sigma(2^{4k+2} + 2)\eta; \quad E_\varepsilon(\vartheta^v_\varepsilon, M) \leq 2\sigma(L + (2^{4k+2} + 4)\eta).
\]

We recall from Section 3.2 that \( f : M \to [1/3, 2/3] \) is a Morse function with no non-global local maxima or minima; hence the map \( t \mapsto f^{-1}(t) \) is continuous in the Hausdorff topology. Following [Gua18, Section 7], we define the following functions. For \( t \in [1/3, 2/3] \), let

\[
d^{(t)}(p) = \begin{cases} -d(p, f^{-1}(t)) & \text{if } f(p) \leq t; \\ d(p, f^{-1}(t)) & \text{if } f(p) \geq t. \end{cases}
\]

We define \( w_\varepsilon : [0, 1] \to H^1(M) \) as follows.

\[
w_\varepsilon(t) = \begin{cases} g_\varepsilon \circ d^{(t)} & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}; \\ 1 - 3t(1 - w_\varepsilon(1/3)) & \text{if } 0 \leq t \leq \frac{1}{3}; \\ -1 + 3(1 - t)(1 + w_\varepsilon(2/3)) & \text{if } \frac{2}{3} \leq t \leq 1. \end{cases} \tag{3.69}
\]

Since \( f^{-1}(t) \) varies continuously in the Hausdorff topology, \( w_\varepsilon : [0, 1] \to H^1(M) \) is continuous [Gua18, Proposition 9.2]. The following proposition follows from [Gua18, Section 9.6] and the estimates (3.13), (3.66).

**Proposition 3.10.** There exists \( \varepsilon_2 = \varepsilon_2(\eta, f, \varepsilon_0, \delta, r_0 - r_1) \) such that for all \( t \in [0, 1], \varepsilon \leq \varepsilon_2 \) and \( i \in [I] \), we have

\[
E_\varepsilon(w_\varepsilon(t), \mathbf{B}_1^1) \leq 6\sigma\eta; \quad E_\varepsilon(w_\varepsilon(t), \mathbf{A}_2) \leq 6\sigma\eta.
\]

### 3.6 Construction of sweepout in \( H^1(M) \setminus \{0\} \) and proof of Theorem 1.2

The next proposition essentially follows from the argument in [GT01, Proof of Lemma 7.5].

**Proposition 3.11.** \( u \mapsto |u| \) is a continuous map from \( H^1(M) \) to itself.
Proof. We define \( \kappa : \mathbb{R} \to \mathbb{R} \) as follows

\[
\kappa(t) = \begin{cases} 
1 & \text{if } t > 0; \\
0 & \text{if } t = 0; \\
-1 & \text{if } t < 0.
\end{cases}
\]

By [GT01, Lemma 7.6] \( v \in H^1(M) \) implies \( |v| \in H^1(M) \) with \( D|v| = \kappa(v)Dv \).

We need to show that if \( u_n \to u \) in \( H^1(M) \), \( |u_n| \to |u| \) in \( H^1(M) \). We will use the following fact. Let \( \{x_n\} \) be a sequence in a topological space \( X \) and \( x \in X \) such that every subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) has a further subsequence \( \{x_{n_{k_i}}\} \) which converges to \( x \). Then \( \{x_n\} \) converges to \( x \). Therefore, without loss of generality we can assume that \( u_n \) converges to \( u \) pointwise a.e.

\[
|u_n| \to |u| \text{ in } L^2(M) \text{ because of the inequality } ||u_n| - |u|| \leq |u_n - u|.
\]

Using the fact that \( Du = 0 \text{ a.e. on } \{u = 0\} \), we compute

\[
\int_M |\kappa(u_n)Du_n - \kappa(u)Du|^2 \leq \int_M 2|\kappa(u_n)|^2|Du_n - Du|^2 + 2|\kappa(u_n) - \kappa(u)|^2|Du|^2 \\
\leq 2 \int_M |Du_n - Du|^2 + 2 \int_{\{u \neq 0\}} |\kappa(u_n) - \kappa(u)|^2|Du|^2.
\]

The first integral converges to 0. If \( u(y) \neq 0 \) and \( u_n(y) \) converges to \( u(y) \), \( \kappa(u_n(y)) \) converges to \( \kappa(u(y)) \) as well. Hence, by the dominated convergence theorem, the second integral also converges to 0. \( \square \)

For \( x \in \mathbb{R} \), let \( x^+ = \max\{x, 0\} \); \( x^- = \min\{x, 0\} \). We define the maps \( \phi, \psi, \theta : H^1(M) \times H^1(M) \times H^1(M) \to H^1(M) \) as follows.

\[
\phi(u_0, u_1, w) = \min\{\max\{u_0, -w\}, \max\{u_1, w\}\}; \\
\psi(u_0, u_1, w) = \max\{\min\{u_0, w\}, \min\{u_1, -w\}\}; \\
\theta(u_0, u_1, w) = \phi(u_0, u_1, w)^+ + \psi(u_0, u_1, w)^-.
\] (3.70)

**Proposition 3.12.** The map \( \theta \) defined above has the following properties.

(i) \( \theta \) is a continuous map.

(ii) \( \theta(-u_0, -u_1, w) = -\theta(u_0, u_1, w) \).

(iii) If \( |u_0|, |u_1| \leq 1 \), then \( \theta(u_0, u_1, 1) = u_0, \theta(u_0, u_1, -1) = u_1 \) where \( 1 \) denotes the constant function 1.

(iv) If \( u_0(p) = u_1(p) \), then \( \theta(u_0, u_1, w)(p) = u_0(p) = u_1(p) \).

(v) For every \( U \subset M \), \( E_\varepsilon(\theta(u_0, u_1, w), U) \leq E_\varepsilon(u_0, U) + E_\varepsilon(u_1, U) + E_\varepsilon(w, U) \).

(vi) \( \theta(u_0, u_1, w)(p) = 0 \) implies \( p \in \{u_0 = 0\} \cup \{u_1 = 0\} \cup \{w = 0\} \).
Proof. We have the following identities.
\[
\max\{a, b\} = \frac{a + b + |a - b|}{2}; \quad \min\{a, b\} = \frac{a + b - |a - b|}{2}.
\]
Hence, the continuity of φ, ψ and θ follows from Proposition 3.11. We note that φ(−u₀, −u₁, w) = −ψ(u₀, u₁, w) which implies (ii). (iii) also follows from direct computation. Indeed, if |u₀|, |u₁| ≤ 1, φ(u₀, u₁, 1) = u₀ = ψ(u₀, u₁, 1) and φ(u₀, u₁, −1) = u₁ = ψ(u₀, u₁, −1). To prove (iv), we need the following identity. If a ≥ 0,
\[
\min\{\max\{a, −b\}, \max\{a, b\}\} = a.
\]
Indeed, the identity holds if \( \max\{a, b\} = a = \max\{a, −b\} \). It also holds if \( \max\{a, −b\} = a \) and \( \max\{a, b\} = b > a \). It is not possible for both \( \max\{a, −b\} \) and \( \max\{a, b\} \) to be different from a as this will imply \( b > a, −b > a \implies 0 > a \). Moreover, if \( a ≥ 0 \), either \( \min\{a, b\} ≥ 0 \) (if \( b ≥ 0 \)) or \( \min\{a, −b\} ≥ 0 \) (if \( b ≤ 0 \)); hence
\[
\max\{\min\{a, b\}, \min\{a, −b\}\} ≥ 0.
\]
(3.71) and (3.72) give (iv) if \( u₀(p) = u₁(p) ≥ 0 \). The other case is similar.

For \( i = 0, 1 \), let \( Uᵢ = \{uᵢ < 0\} \) and \( Vᵢ = \{uᵢ > 0\} \). We also set \( G = \{w < 0\} \), \( H = \{w > 0\} \), \( Z = \{w = 0\} \). Then we have (suppressing the dependence of φ, ψ, θ on \( u₀, u₁, w \)),
\[
\begin{align*}
\phi(y) < 0 &\iff y ∈ (U₀ ∩ H)∪(U₁ ∩ G); \\
\phi(y) > 0 &\iff y ∈ (V₀ ∩ H)∪(V₁ ∩ G)∪(V₀ ∩ V₁ ∩ Z); \\
ψ(y) < 0 &\iff y ∈ (U₀ ∩ H)∪(U₁ ∩ G)∪(U₀ ∩ U₁ ∩ Z); \\
ψ(y) > 0 &\iff y ∈ (V₀ ∩ H)∪(V₁ ∩ G).
\end{align*}
\]
Further,
\[
\theta(y) = \begin{cases} 
\phi(y) &\text{if } y ∈ \{φ ≥ 0\} ∩ \{ψ ≥ 0\}; \\
ψ(y) &\text{if } y ∈ \{φ ≤ 0\} ∩ \{ψ ≤ 0\}.
\end{cases}
\]
(3.74)

It follows from (3.73) that
\[
M = (\{φ ≥ 0\} ∩ \{ψ ≥ 0\}) ∪ (\{φ ≤ 0\} ∩ \{ψ ≤ 0\}).
\]
Hence, using the definition of φ and ψ, (3.74) implies that
\[
\theta(y) ∈ \{u₀(y), u₁(y), w(y), −w(y)\}; \quad \nabla \theta(y) ∈ \{\nabla u₀(y), \nabla u₁(y), \nabla w(y), −\nabla w(y)\}.
\]
(3.75)
Therefore, we have the following point-wise estimates.
\[
|\nabla \theta(y)|^2 ≤ |\nabla u₀(y)|^2 + |\nabla u₁(y)|^2 + |\nabla w(y)|^2; \quad W(θ(y)) ≤ W(u₀(y)) + W(u₁(y)) + W(w(y))
\]
which give item (v) of the proposition. (3.75) also implies item (vi). □

Let \( α ∈ \tilde{X}[KI]_j \) so that \( μ = π(α) ∈ X[KI]_j \). We define the following maps.
\[
(π|_α)^{-1} \circ Δ_{μ} = Δ^*_α : \mathcal{I}^j → α; \quad Δ^{*}_{μ} \circ (π|_α) = D^*_α : α → \mathcal{I}^j.
\]
Proposition 3.13. Let $\epsilon^* = \min\{\epsilon_1, \epsilon_2, (\delta/4)^2, 4^{-1}(r_1 - r)^2\}$. There exists a continuous, $\mathbb{Z}_2$-equivariant map $\zeta_\epsilon : \tilde{X} \to H^1(M) \setminus \{0\}$ with the property

\[(\mathcal{P})_0 \text{ For all } v \in \tilde{X}[KI]_0, \zeta_\epsilon(v) = \hat{\vartheta}_\epsilon^v;\]

where $\hat{\vartheta}_\epsilon^v$ is as in Proposition 3.9. Moreover, if $\alpha \in \tilde{X}[KI]_j$, $\zeta_\epsilon|_\alpha$ has the following properties.

\[(\mathcal{P}0)_{j,\alpha} \text{ Let } \hat{\zeta}_\epsilon^\alpha = \zeta_\epsilon \circ \Delta^\bullet. \text{ Then, for all } z = (z', z_j) \in \mathcal{J}^j \text{ (with } z' \in \mathcal{J}^{j-1}), \]

\[\hat{\zeta}_\epsilon^\alpha(z', z_j) = \theta \left( \zeta_\epsilon(z', 0), \hat{\zeta}_\epsilon(z', 1), w_\epsilon(z_j) \right);\]

where $w_\epsilon$ is as defined in (3.69).

\[(\mathcal{P}1)_{j,\alpha} \text{ We define } \mathcal{I}_\alpha = \{i(e) : e \in \alpha_1\} \text{ where } i(e) \text{ is as in Proposition 3.8, (P5). For all } \epsilon \leq \epsilon^*, \]

\[\forall t \in \alpha \text{ and } v \in \alpha_0, \]

\[y \notin A_2 \bigcup \left( \bigcup_{i \in \mathcal{I}_\alpha} \mathcal{B}_i^1 \right) \implies \zeta_\epsilon(t)(y) = \zeta_\epsilon(v)(y).\]

\[(\mathcal{P}2)_{j,\alpha} \text{ For all } \epsilon \leq \epsilon^*, i \in [I] \text{ and } t \in \alpha, \]

\[
\frac{1}{2\sigma} E_\epsilon(\zeta_\epsilon(t), \mathcal{B}_i^1) \leq 3^j(2^{4k+2} + 2) \eta; \quad \frac{1}{2\sigma} E_\epsilon(\zeta_\epsilon(t), A_2) \leq 3^j(2^{4k+2} + 2) \eta.
\]

As a consequence, for all $\epsilon \leq \epsilon^*$ and $t \in \alpha$,

\[
\frac{1}{2\sigma} E_\epsilon(\zeta_\epsilon(t), M) \leq L + (2^{4k+2} + 4) \eta \epsilon^* + (j + 1)3^j(2^{4k+2} + 2) \eta. \quad (3.76)
\]

Proof. The map $\zeta_\epsilon$ will be defined inductively on the cells of $\tilde{X}[KI]$. The equivariance of $\zeta_\epsilon$ and the fact that $\zeta_\epsilon(x)$ is not identically 0 for all $x \in \tilde{X}$ will be proved at the end (after proving the other properties of $\zeta_\epsilon$). If $v \in \tilde{X}[KI]_0$, we define $\zeta_\epsilon(v) = \hat{\vartheta}_\epsilon^v$ which is precisely the property $(\mathcal{P})_0$. The properties $(\mathcal{P}0)_{0,v}$ and $(\mathcal{P}1)_{0,v}$ are vacuous; $(\mathcal{P}2)_{0,v}$ is satisfied because of Proposition 3.9. Let us assume that for some $p \geq 1$, $\zeta_\epsilon$ has been defined on

\[\bigcup_{j \leq p-1} \tilde{X}[KI]_j\]

and if $\varrho \in \tilde{X}[KI]_{p-1}$, $\zeta_\epsilon|_{\varrho}$ has the properties $(\mathcal{P}0)_{(p-1),\varrho}$ - $(\mathcal{P}2)_{(p-1),\varrho}$.

Let $\lambda \in \tilde{X}[KI]_p$. By our assumption, $\zeta_\epsilon$ is already defined on $\partial \lambda$. For $z \in \partial \mathcal{T}^p$, we define

\[\hat{\zeta}_\epsilon^\lambda(z) = (\zeta_\epsilon \circ \Delta^\bullet)(z).\]

For arbitrary $z = (z', z_p) \in \mathcal{J}^p$ with $z' \in \mathcal{J}^{p-1}$, we define

\[
\hat{\zeta}_\epsilon^\lambda(z', z_p) = \theta \left( \zeta_\epsilon^\lambda(z', 0), \hat{\zeta}_\epsilon^\lambda(z', 1), w_\epsilon(z_p) \right) \quad (3.77)
\]

where $w_\epsilon$ is as defined in (3.69). By Proposition 3.12 (iii), this equation indeed holds for $z_p = 0, 1$. Further, (3.77) is also valid if $z \in \partial \mathcal{T}^p$ because of our induction hypothesis that $\zeta_\epsilon$
restricted $\partial \lambda$ satisfies $(P0)$ and the compatibility relation (3.17). For $t \in \lambda$, $\zeta_\epsilon(t)$ is defined by $\zeta_\epsilon(t) = (\hat{\zeta}_\epsilon \circ D^*_\lambda)(t)$. $\zeta_\epsilon$ is continuous on $\lambda$ as $\zeta_\epsilon|\partial \lambda$ is continuous and the maps $\theta$, $w_\epsilon$ are continuous.

It remains to verify that $\zeta_\epsilon|\lambda$ satisfies the properties $(P0)_{p,\lambda} - (P2)_{p,\lambda}$. $(P0)_{p,\lambda}$ holds by the definition (3.77). To prove $(P1)_{p,\lambda}$, we first show that if $v, v' \in \tilde{X}[KI]_0$ and $e \in \tilde{X}[KI]_1$ such that $v, v' < e$,

$$y \notin A_2 \cup B^1_{i(e)} \implies \zeta_\epsilon(v)(y) = \zeta_\epsilon(v')(y).$$

(3.78)

We set $B^1 = B^1_{i(e)}$ and $B = B_{i(e)}$. Either $y \in \tilde{\Psi}(v)$ or $y \in \tilde{\Psi}(T(v))$. We assume that $y \in \tilde{\Psi}(T(v))$ (the other case is analogous); hence, (using Proposition 3.8 (P5))

$$y \in \tilde{\Psi}(T(v)) \cap (A_2 \cup B^1)^c = \tilde{\Psi}(T(v')) \cap (A_2 \cup B^1)^c$$

Thus, $d_v(y) = d(y, \Psi(v)), d_{v'}(y) = d(y, \Psi(v'))$. If both $d_v(y)$ and $d_{v'}(y)$ are $\geq 2\sqrt{\epsilon}$, $\zeta_\epsilon(v)(y) = 1 = \zeta_\epsilon(v')(y)$.

Otherwise, let us say $d_v(y) = d(y, \Psi(v)) < 2\sqrt{\epsilon}$. We will show that $d_v(y) = d_{v'}(y)$. Let $y' \in \Psi(v)$ such that $d(y, y') = d(y, \Psi(v))$. Therefore,

$$d(y, y') = d(y, \Psi(v)) < 2\sqrt{\epsilon} \leq \min\{\delta/2, r_1 - r\}.$$

(3.79)

Hence, $y \notin A_2 \cup B^1$ implies $y' \notin A \cup B$. Thus, (again using Proposition 3.8 (P5))

$$y' \in \Psi(v) \cap (A \cup B)^c = \Psi(v') \cap (A \cup B)^c.$$  

This gives

$$d(y, \Psi(v)) = d(y, \Psi(v) \cap (A \cup B)^c);$$

and $d(y, \Psi(v')) \leq d(y, y') < 2\sqrt{\epsilon}$ which also implies (by the above reasoning)

$$d(y, \Psi(v')) = d(y, \Psi(v') \cap (A \cup B)^c).$$

Since $\Psi(v) \cap (A \cup B)^c = \Psi(v') \cap (A \cup B)^c$, we get $d_v(y) = d_{v'}(y)$.

Thus we have proved (3.78) which together with (3.77), induction hypothesis $(P1)_{(p-1), \partial \lambda}$ and Proposition 3.12 (iv) gives $(P1)_{p,\lambda}$.

Finally we prove $(P2)_{p,\lambda}$. Using (3.77) together with Proposition 3.12 (v), the induction hypothesis $(P2)_{(p-1), \partial \lambda}$ and Proposition 3.10 we obtain the following estimate ($\mathcal{U}$ stands for $B^1_i$ for some $i \in [I]$ or $A_2$.)

$$\frac{1}{2\sigma}E_\epsilon \left( \hat{\zeta}_\epsilon(z', z_p), \mathcal{U} \right) \leq \frac{1}{2\sigma}E_\epsilon \left( \hat{\zeta}_\epsilon(z', 0), \mathcal{U} \right) + \frac{1}{2\sigma}E_\epsilon \left( \hat{\zeta}_\epsilon(z', 1), \mathcal{U} \right) + \frac{1}{2\sigma}E_\epsilon \left( w_\epsilon(z_p), \mathcal{U} \right)$$

$$\leq 2.3^{p-1}(2^{4k+2} + 2)\eta + 3\eta$$

$$\leq 3^p(2^{4k+2} + 2)\eta.$$
This estimate along with \((P1)_{p, \lambda}, (P)_{0}\) and Proposition 3.9 gives for all \(\varepsilon \leq \varepsilon^*\), \(t \in \alpha\) and \(v \in \alpha_0\),
\[
\frac{1}{2\sigma} E_{\varepsilon}(\zeta_\varepsilon(t), M) \leq \frac{1}{2\sigma} E_{\varepsilon}(\zeta_\varepsilon(v), M) + \sum_{i \in I_\lambda} \frac{1}{2\sigma} E_{\varepsilon}(\zeta_\varepsilon(t), B^i_1) + \frac{1}{2\sigma} E_{\varepsilon}(\zeta_\varepsilon(t), A_2) \leq L + (2^{4k+2} + 4)\eta + (p + 1)3^p(2^{4k+2} + 2)\eta.
\]
In the last line we have used the fact that \(|I_\lambda| \leq p\) which follows from the characterization of \(i(\varepsilon)\) in Proposition 3.8 (P5).

Now we show that the map \(\zeta_\varepsilon\) constructed above is \(\mathbb{Z}_2\)-equivariant. From the definition of \(d_v\) ((3.67)), it follows that \(d_{T(v)} = -d_v\) for all \(v \in \tilde{X}[KI]_0\). Hence, by \((P)_{0}\), \(\zeta_\varepsilon(T(v)) = -\zeta_\varepsilon(v)\) for all \(v \in \tilde{X}[KI]_0\). Now we can use induction along with \((P0)\) and Proposition 3.12 (ii) to conclude that \(\zeta_\varepsilon(T(x)) = -\zeta_\varepsilon(x)\) for all \(x \in \tilde{X}\).

Lastly, we prove that for all \(x \in \tilde{X}\), \(\mathcal{H}^{n+1}(\{\zeta_\varepsilon(x) = 0\}) = 0\). This will in particular imply that for all \(x \in \tilde{X}\), \(\zeta_\varepsilon(x)\) is not identically equal to 0. If \(v \in \tilde{X}[KI]_0\), by (3.67) and \((P)_{0}\), \(\{\zeta_\varepsilon(v) = 0\} = \Psi(v)\) and \(\mathcal{H}^{n+1}(\Psi(v)) = 0\) by Proposition 3.8 (P4). Hence, \(\mathcal{H}^{n+1}(\{\zeta_\varepsilon(v) = 0\}) = 0\) for all \(v \in \tilde{X}[KI]_0\). Now, as before we can use induction along with \((P0)\) and Proposition 3.12 (vi) to conclude that \(\mathcal{H}^{n+1}(\{\zeta_\varepsilon(x) = 0\}) = 0\) for all \(x \in \tilde{X}\).

**Proof of Theorem 1.2.** By Proposition 3.13, for every \(\eta > 0\) there exists \(\varepsilon^* > 0\) such that for all \(\varepsilon \leq \varepsilon^*\) there exists \(\zeta_\varepsilon \in \bar{\Pi}\) such that
\[
\frac{1}{2\sigma} \sup_{x \in \tilde{X}} E_{\varepsilon}(\zeta_\varepsilon(x)) \leq L + \alpha(k)\eta \quad (by \ (3.76))
\]
where \(\alpha(k)\) is a constant which depends only on \(k = \dim(\tilde{X})\). Hence,
\[
\frac{1}{2\sigma} L_{\varepsilon}(\bar{\Pi}) \leq L + \alpha(k)\eta \quad \forall \varepsilon \leq \varepsilon^* \implies \frac{1}{2\sigma} \limsup_{\varepsilon \to 0^+} L_{\varepsilon}(\bar{\Pi}) \leq L + \alpha(k)\eta.
\]
Since \(\eta > 0\) is arbitrary, this implies (1.3).

The following facts follow from [GG18, Section 6]. \(\text{Ind}_{\mathbb{Z}_2}(\tilde{X}) \geq p + 1\) if and only if each map \(\Phi \in \Pi\) is a \(p\)-sweepout. Moreover,
\[
c_\varepsilon(p) = \inf_{\Pi} L_{\varepsilon}(\bar{\Pi})
\]
where the infimum is taken over all \(\bar{\Pi}\) such that \(\text{Ind}_{\mathbb{Z}_2}(\tilde{X}) \geq p + 1\). Similarly,
\[
\omega_p = \inf_{\Pi} L_{AP}(\Pi)
\]
where the infimum is taken over all \(\Pi\) such that each map in the homotopy class \(\Pi\) is a \(p\)-sweepout.

33
We fix \( p \in \mathbb{N} \). For each \( j \in \mathbb{N} \), there exist double cover \( \tilde{X}_j \to X_j \) and the corresponding homotopy classes \( \Pi_j, \tilde{\Pi}_j \) (as discussed in Section 1) such that \( \text{Ind}_{\mathbb{Z}_2}(\tilde{X}_j) \geq p + 1 \) and

\[
L_{AP}(\Pi_j) < \omega_p + \frac{1}{j}.
\]

By (1.3),

\[
\frac{1}{2\sigma} \limsup_{\varepsilon \to 0^+} L_{\varepsilon}(\tilde{\Pi}_j) \leq L_{AP}(\Pi_j) < \omega_p + \frac{1}{j}.
\]

Hence, there exists \( \tilde{\varepsilon} > 0 \) such that for all \( \varepsilon \leq \tilde{\varepsilon} \)

\[
\frac{1}{2\sigma} c_\varepsilon(p) \leq \frac{1}{2\sigma} L_{\varepsilon}(\tilde{\Pi}_j) < \omega_p + \frac{1}{j}
\]

which implies

\[
\frac{1}{2\sigma} \limsup_{\varepsilon \to 0^+} c_\varepsilon(p) \leq \omega_p + \frac{1}{j}.
\]

Since this holds for all \( j \in \mathbb{N} \), we obtain (1.4). \( \square \)

4 Proof of Theorem 1.3

In this section we will discuss how Theorem 1.3 can be proved using the results contained in the papers [HT00, Gua18, GG18]. We recall the function \( F \) defined in (2.4). For \( \beta \in (0, 1) \), we set \( \sigma_\beta = F^{-1}(1 - \beta) \).

**Proposition 4.1.** Let \( \{u_i : M \to (-1, 1)\}_{i=1}^{\infty} \) be a sequence smooth functions such that the followings hold.

(i) \( AC_{\varepsilon_i}(u_i) = 0 \) with \( \varepsilon_i \to 0 \) as \( i \to \infty \).

(ii) There exists \( E_0 \) such that \( E_{\varepsilon_i}(u_i) \leq E_0 \) for all \( i \in \mathbb{N} \).

(iii) \( V \) is a stationary, integral varifold such that \( V_i = V[u_i] \to V \) and \( \text{spt}(V) \) has optimal regularity.

Then, for all \( s > 0 \) there exists \( b_0 \in (0, 1/2] \) such that the following holds. Denoting \( w_i = F \circ u_i \), for all \( b \in (0, b_0] \) there exists \( i_0 \in \mathbb{N} \) depending on \( s \) and \( b \) such that for all \( i \geq i_0 \) there exists \( t_i \in [-\sigma_b/2, \sigma_b/2] \) for which \( \{w_i > t_i\}, \{w_i < t_i\} \in C(M), \{\{w_i > t_i\}\} + \{\{w_i < t_i\}\} = [M], \partial\{\{w_i = t_i\}\} = \{\{w_i = t_i\}\} \) and \( F(V_i|\{w_i = t_i\}) \leq s \).

**Proof.** Throughout the proof, we will use the notation of [HT00]. Let \( p \in (M, g) \) and \( 0 < r < \text{inj}(M)/4 \). Identifying \( \mathbb{R}^{n+1} \) with \( T_p M \), let us define \( f_{p,r} : B^{n+1}(0, 4) \to M \),

\[
f_{p,r}(v) = \exp_p(rv) \quad \text{and} \quad g_{p,r} = r^{-2}f_{p,r}^*g.
\]

As explained in [Gua18], there exists \( 0 < r_0 < \text{inj}(M)/4 \), depending on \( (M^{n+1}, g) \), such that if \( p \in M, 0 < r \leq r_0 \) and \( \{u_i\}_{i=1}^{\infty} \) are solutions of \( AC_{\varepsilon_i}(u_i) = 0 \) on \( (B^{n+1}(0, 4), g_{p,r}) \) with \( \varepsilon_i \to 0 \), all the results of [HT00] continue to hold.
For simplicity, let us assume that $V = N|\Sigma|$ where $\Sigma$ is a closed, minimal hypersurface with optimal regularity. The general case will be similar. Let

$$s_0 = 2\sigma(N + 4 + 2\sigma N)^{-1}. \quad (4.1)$$

and we fix $0 < s \leq s_0$. (The assumption $s \leq s_0$ will be useful later.) By [HT00, Proposition 5.1] there exists $b_0 \in (0, 1/2]$ such that

$$\limsup_{i \to \infty} \int_{\{|u_i| \geq 1 - b_0\}} \frac{W(u_i)}{\varepsilon_i} \leq s.$$

Let us fix $b \in (0, b_0]$. Using the above equation and [HT00, Proposition 4.3],

$$\limsup_{i \to \infty} \int_{\{|u_i| \geq 1 - b\}} |\nabla w_i| \leq s. \quad (4.2)$$

There exists an open set $\Omega$ containing $\text{reg}(\Sigma)$ such that $d\Sigma = d(-, \Sigma)$ is smooth on $\Omega$ and the nearest point projection map $P : \Omega \to \Sigma$ is also smooth on $\Omega$. By [MN16, Proof of Theorem 5.2], $\text{im}(P) = \text{reg}(\Sigma)$. We choose $U_1 \subset \subset U_2 \subset \subset \text{reg}(\Sigma)$ such that

$$N\mathcal{H}^n(\Sigma \setminus U_1) < s. \quad (4.3)$$

For $U \subset \subset \text{reg}(\Sigma)$, let

$$\mathcal{R}_r U = \{v : v \in T^\perp \Sigma \text{ with } \|v\| < r, p \in U\}.$$ 

There exists $\rho > 0$ such that $\exp : \mathcal{R}_\rho U_2 \to \Omega$ is a diffeomorphism onto its image. For $j = 1, 2$, let $U_j = \exp(\mathcal{R}_\rho U_j)$ so that

$$P(U_j) = U_j = \Sigma \cap U_j \quad \text{and} \quad U_1 \subset \subset U_2 \subset \subset \Omega.$$ 

We can assume that (choosing a smaller $\rho$ if necessary), the Jacobian factor

$$JP(y, S) \leq 2 \forall (y, S) \in G_n U_2. \quad (4.4)$$

On $\Omega$, following [HT00, Section 5], we define

$$v_i = \begin{cases} \frac{\langle \nabla u_i, \nabla d \Sigma \rangle}{|\nabla u_i|} & \text{if } |\nabla u_i| \neq 0; \\ 0 & \text{if } |\nabla u_i| = 0. \end{cases}$$

Let $\tau_i = (1 - v_i^2)\varepsilon_i|\nabla u_i|^2$. We choose a compactly supported function $0 \leq \chi_1 \leq 1$ such that $\text{spt}(\chi_1) \subset \Omega$ and $\chi_1 \equiv 1$ on $U_2$. Since, $V_i \to V$ is the varifold sense, using [HT00, Proposition 4.3] we obtain ([HT00, (5.7)])

$$\lim_{i \to \infty} \int_\Omega \chi_1 \tau_i = \lim_{i \to \infty} \int \chi_1(1 - v_i^2)|\nabla w_i| = 0 \implies \lim_{i \to \infty} \int_{U_2} \tau_i = 0. \quad (4.5)$$

Let us use the notation

$$\xi_i = \frac{\varepsilon_i|\nabla u_i|^2}{2} - \frac{W(u_i)}{\varepsilon_i}.$$ 

Since the level sets $u_i^{-1}(t)$ converge to $\Sigma$ in the Hausdorff sense, we can choose a sequence $r_i$ such that
• $r_i \to 0$;
• $\{|u_i| \leq 1 - b\} \subset N_{r_i}(\Sigma)$;
• $\varepsilon_i/r_i \to 0$.

By [HT00, Proposition 4.3] and (4.5),
\[ \lim_{i \to \infty} \int_{U_2} |\xi_i| + \tau_i = 0. \]

Let us choose a sequence $\eta_i \to 0$ such that
\[ \lim_{i \to \infty} \eta_i^{-1} \int_{U_2} |\xi_i| + \tau_i = 0. \]

There exists $i_1 \in \mathbb{N}$ such that for all $i \geq i_1$ we have
• $r_i \leq r_0$ where $r_0$ is as defined at the beginning of the proof;
• $r_i < d(\overline{U_1}, \partial U_2)$.

As we discussed at the beginning of the proof, for any $p \in M$ and $0 < r \leq r_0$, if $u_\varepsilon$ is a solution of $AC_\varepsilon(u_\varepsilon) = 0$ on $(B^{n+1}(0,4), g_{p,r})$, Proposition 5.5 and 5.6 of [HT00] hold (with the constants depending additionally on $r_0$). For our fixed choice of $s$ and $b$, we choose $L$ via Proposition 5.5 and 5.6. For $i \geq i_1$, let
\[ G_i = U_1 \cap \left\{ y : \int_{B(y,r)} |\xi_i| + \tau_i \leq \eta_i r^n \text{ if } 4 \varepsilon_i L \leq r \leq r_i \right\}. \]

$G_i$ may not be $\mathcal{H}^n$-measurable; for our later purpose we choose an $\mathcal{H}^n$-measurable set $\mathcal{G}_i \subset G_i$ as follows. By the Besicovitch covering theorem, there exist $\{B_j\}_{j=1}^\ell$ such that each $B_j$ is a collection of at-most countably many mutually disjoint open balls and
\[ \overline{U_1} \setminus G_i \subset \bigcup_{j=1}^\ell \bigcup_{B \in B_j} B. \]

Let
\[ \mathcal{G}_i = \left( \bigcup_{j=1}^\ell \bigcup_{B \in B_j} B \right)^c \cap U_1 \subset G_i \]
which is a compact set. Using the monotonicity formula for the scaled energy one can show the following ([HT00, (5.9)]).
\[ \|V_i\| (\overline{U_1} \setminus \mathcal{G}_i) + \mathcal{H}^n (P(\overline{U_1} \setminus \mathcal{G}_i)) \leq c(s,W,g)\eta_i^{-1} \int_{U_2} |\xi_i| + \tau_i \quad (4.6) \]
which converges to 0 as $i \to \infty$ by our choice of the sequence $\{\eta_i\}$.

We define
\[ S_i^t = \{ w_i = t\}. \]
For a.e. $t \in [-\sigma_b/2, \sigma_b/2]$, $\{w_i > t\}, \{w_i < t\} \in \mathcal{C}(M)$; \(\|[w_i > t]\| + \|[w_i < t]\| = |M|\); and \(\partial([w_i > t]) = ([w_i = t])\). For such $t \in [-\sigma_b/2, \sigma_b/2]$, $i \geq i_1$ and Lipschitz continuous function $\varphi : G_n \mathcal{M}^{-1} \rightarrow \mathbb{R}$ with $|\varphi| \leq 1$, \(\text{Lip}(\varphi) \leq 1\), we obtain the following estimates.

\[
\left|S_i^t(\varphi) - N|\Sigma| \varphi\right| \leq \left|S_i^t \cap \overline{U}_1(\varphi) - N|\Sigma \cap \overline{U}_1| \varphi\right| + \|S_i^t\|(U_1^c) + N\|\Sigma\|(U_1^c); \tag{4.7}
\]
\[
\left|S_i^t \cap \overline{U}_1(\varphi) - N|\Sigma \cap \overline{U}_1| \varphi\right| \leq \left|S_i^t \cap G_i(\varphi) - N|P(G_i)| \varphi\right| + \|S_i^t\|(U_1 \setminus G_i) + N\mathcal{H}^n(P(U_1 \setminus G_i)); \tag{4.8}
\]
\[
\left|S_i^t \cap G_i(\varphi) - N|P(G_i)| \varphi\right| \leq \left|S_i^t \cap G_i(\varphi) - P_\#|S_i^t \cap G_i(\varphi)\right| + P_\#|S_i^t \cap G_i| \varphi - N|P(G_i)| \varphi\right|; \tag{4.9}
\]
\[
\left|S_i^t \cap G_i(\varphi) - P_\#|S_i^t \cap G_i| \varphi\right| \leq \int_{S_i^t \cap G_i} \varphi(y, T_y S_i^t) - \varphi(P(y, DP|_y(T_y S_i^t))) JP(y, T_y S_i^t) d\mathcal{H}^n(y). \tag{4.10}
\]

Using [HT00, Proposition 5.6] and a scaling argument as in the Proof of Theorem 1 of [HT00], as $i \rightarrow \infty$

\[
d_{G_nM}((y, T_y S_i^t), (P(y, DP|_y(T_y S_i^t)))) \rightarrow 0 \text{ and } JP(y, T_y S_i^t) \rightarrow 1
\]
uniformly for $|t| \leq \sigma_b/2$ and $y \in S_i^t \cap G_i$. Here $d_{G_nM}$ denotes the distance in $G_nM$. Hence, there exists a sequence of positive real numbers $\{\theta_i\}_{i=1}^\infty$ (which does not depend on $t$) such that

\[
\lim_{i \rightarrow \infty} \theta_i = 0 \text{ and } \sup_{\varphi} \left|S_i^t \cap G_i(\varphi) - P_\#|S_i^t \cap G_i| \varphi\right| \leq \theta_i \mathcal{H}^n(S_i^t) \tag{4.11}
\]
for all $t \in [-\sigma_b/2, \sigma_b/2]$. Let us choose a cut-off function $0 \leq \chi_2 \leq 1$ with \text{spt}(\chi_2) \subset U_2$ and $\chi_2 \equiv 1$ on $\overline{U}_1$. If $p : G_nM \rightarrow M$ is the canonical projection map, $\chi_2 = \chi_2 \circ p$.

For $y \in P(G_i)$, let $m_i^t(y)$ be the cardinality of the set $P^{-1}(y) \cap S_i^t \cap G_i$. Using [HT00, Proposition 5.5, 5.6] and a scaling argument as in the Proof of Theorem 1 of [HT00], there exists $i_2 \in \mathbb{N}$ such that for all $i \geq i_2$, $y \in P(G_i)$ and $|t| \leq \sigma_b/2$, $m_i^t(y) \leq N$. (Here we need to use $s \leq s_0$.) Hence,

\[
\left|P_\#|S_i^t \cap G_i| \varphi - N|P(G_i)| \varphi\right|
\leq \int_{P(G_i)} (N - m_i^t(y)) d\mathcal{H}^n(y)
= N|P(G_i)|(\chi_2) - P_\#|S_i^t \cap G_i|(\chi_2)
\leq N|\Sigma|(\chi_2) - P_\#|S_i^t|(\chi_2) + 2\|S_i^t\|(\overline{U}_1 \setminus G_i) + 2\|S_i^t\|(U_1^c) \text{ (by (4.4))}. \tag{4.12}
\]
We will now integrate the various error terms obtained above with respect to \( t \) in the interval \([-\sigma_b/2, \sigma_b/2]\) and let \( i \to \infty \).

\[
\limsup_{i \to \infty} \int_{-\sigma_b/2}^{\sigma_b/2} \| S_i^t \| (U_i^t) \, dt \leq \limsup_{i \to \infty} \sigma \| V_i \| (U_i^t) \leq s \quad \text{(by (4.3))};
\]

\[
\int_{-\sigma_b/2}^{\sigma_b/2} N \| \Sigma \| (U_i^t) \, dt \leq s \quad \text{(by (4.3))};
\]

\[
\lim_{i \to \infty} \int_{-\sigma_b/2}^{\sigma_b/2} \| S_i^t \| (\overline{U}_1 \setminus \mathcal{F}_i) + N\mathcal{H}^{n}(P(\overline{U}_1 \setminus \mathcal{F}_i)) \, dt = 0 \quad \text{(by (4.6))};
\]

\[
\limsup_{i \to \infty} \int_{-\sigma_b/2}^{\sigma_b/2} \| S_i^t \| (M) \, dt \leq \limsup_{i \to \infty} \sigma \| V_i \| (M) = \sigma N\mathcal{H}^{n}(\Sigma). \quad \text{(4.16)}
\]

We note that \( V_i \to N|\Sigma| \) in the varifold sense implies \( P_{\#}V_i(\chi_2) \to N|\Sigma|(\chi_2) \). \( (P_{\#}V_i(\chi_2) \) is well-defined as \( \text{spt}(\chi_2) \subset U_2 \). Hence, using (4.4) and (4.2),

\[
\limsup_{i \to \infty} \int_{-\sigma_b/2}^{\sigma_b/2} (N|\Sigma|(\chi_2) - P_{\#}|S_i^t|(\chi_2)) \, dt \\
\leq \lim_{i \to \infty} \sigma (N|\Sigma|(\chi_2) - P_{\#}V_i(\chi_2)) + \limsup_{i \to \infty} \int_{\{|u_i| \geq 1-b\}} |\nabla u_i| \\
\leq 2s. \quad \text{(4.17)}
\]

Hence, using the equations (4.7)-(4.17), we conclude that for all \( i \geq i_2 \), there exists a measurable function \( \Theta_i : [-\sigma_b/2, \sigma_b/2] \to \mathbb{R} \) such that

\[
\mathbf{F}(N|\Sigma|, |S_i^t|) \leq \Theta_i(t) \quad \forall t \in [-\sigma_b/2, \sigma_b/2] \quad \text{and} \quad \limsup_{i \to \infty} \int_{-\sigma_b/2}^{\sigma_b/2} \Theta_i(t) \, dt \leq (2 + 4\sigma)s.
\]

Hence, there exists \( i_3 \in \mathbb{N} \) such that for all \( i \geq i_3 \),

\[
\int_{-\sigma_b/2}^{\sigma_b/2} \Theta_i(t) \, dt \leq (3 + 4\sigma)s.
\]

Therefore, for \( i \geq i_3 \), there exists \( t_i \in [-\sigma_b/2, \sigma_b/2] \) such that

\[
\mathbf{F}(N|\Sigma|, |S_i^t|) \leq \Theta_i(t_i) \leq (3 + 4\sigma)s^{-1}. \]

Since \( s \in (0, s_0) \) is arbitrary and \( b \leq b_0 \leq 1/2 \), this finishes the proof of the proposition. \( \square \)

**Proposition 4.2.** Let \( \{u_i : M \to (-1, 1)\}_{i=1}^{\infty} \) be a sequence of smooth functions such that items (i) – (iii) of Proposition 4.1 are satisfied. Additionally, we assume that \( u_i \) is a min-max critical point of \( E_{\varepsilon_i} \) corresponding to the homotopy class \( \tilde{\Pi} \). Let us set \( L_{\varepsilon_i} = L_{\varepsilon_i}(\tilde{\Pi}) \) so that

\[
L = L_{AP}(\Pi) = \frac{1}{2\sigma} \lim_{i \to \infty} L_{\varepsilon_i} = \| V \| (M).
\]

Then, (using the notation from Proposition 4.1) for every \( s > 0 \) and \( b \in (0, b_0(s)] \), there exists \( i_0^* \geq i_0 \) such that the following holds. For all \( i \geq i_0^* \), there exists \( \Phi_i : X \to \mathbb{Z}_n(M^{n+1}; \mathcal{M}; \mathbb{Z}_2) \), \( x_i^* \in X \) and \( \delta_i > 0 \) such that \( \Phi_i \in \Pi, \delta_i \to 0 \),

\[
\sup_{x \in X} \mathbf{M}(\Phi_i(x)) \leq \max \left\{ \frac{1}{2\sigma_b}(L_{\varepsilon_i} + \varepsilon_i), L + s \right\} + \delta_i \quad \text{and} \quad \mathbf{F}(V, |\Phi_i(x_i^*)|) \leq s. \quad \text{(4.18)}
\]

38
Proof. Since $u_i$ is a min-max critical point of $E_{\varepsilon_i}$ corresponding to the homotopy class $\tilde{H}$, for each $i$, there exists a sequence of continuous, $\mathbb{Z}_2$-equivariant maps $\{h_j^i : \tilde{X} \to H^1(M) \setminus \{0\}\}_{j=1}^{\infty}$ such that
\[
\sup_{x \in \tilde{X}, j \in \mathbb{N}} E_{\varepsilon_i}(h_j^i(x)) \leq L_{\varepsilon_i} + \varepsilon_i \quad \text{and} \quad \lim_{j \to \infty} d_{H^1(M)}(u_i, h_j^i(\tilde{X})) = 0.
\]
The next Lemma is a restatement of Lemma 8.10 and 8.11 of [Gua18].

Lemma 4.3 ([Gua18, Lemma 8.10, 8.11]). Let $h_1, h_2 \in H^1(M)$. For $\delta \in (0, 1)$, we set $C_\delta = W(1 - \delta) > 0$. Then, for all $\varepsilon > 0$,
\[
H^{n+1}(\{|h_1| \leq 1 - \delta\}) \leq \varepsilon C_\delta^{-1} E_{\varepsilon}(h_1).
\]

Let $\alpha_0 \in (-1 + \delta, 1 - \delta)$ be such that for $j = 1, 2$, $\Omega_j = \{h_j > \alpha_0\} \in C(M)$. Then, for all $\varepsilon > 0$,
\[
H^{n+1}(\Omega_1 \setminus \Omega_2) \leq \varepsilon C_\delta^{-1} E_{\varepsilon}(h_2) + (\alpha_0 + 1 - \delta)^2 \|h_1 - h_2\|_{H^1(M)}^2.
\]
As a consequence, for $j = 1, 2$, if $\lambda_j \in (-1 + \delta, 1 - \delta)$ such that $T_j = \partial\{[h_j > \lambda_j]\} \in \mathbb{Z}_n(M^{n+1}; \mathbb{Z}_2)$,
\[
\mathcal{F}(T_1, T_2) \leq 2\varepsilon C_\delta^{-1}(E_{\varepsilon}(h_1) + E_{\varepsilon}(h_2)) + 2(\alpha_0 + 1 - \delta)^2 \|h_1 - h_2\|_{H^1(M)}^2 \forall \varepsilon > 0.
\]

We recall that $X$ is a subcomplex of $\mathcal{J}^N[1]$ for some $N \in \mathbb{N}$. There exists $\alpha \in (-1 + b, 1 - b)$ such that for all $i, j \in \mathbb{N}$ and $x \in \pi^{-1}(X \cap Q^N)$, $\{h^i_j(x) > \alpha\} \in C(M)$. Let us fix $i \geq i_0$ ($i_0$ is as in Proposition 4.1); let $j_0 \in \mathbb{N}$ such that
\[
d_{H^1(M)}(u_i, h^i_{j_0}(\tilde{X})) \leq \varepsilon_i / 2.
\]

For simplicity, let us denote the map $h^i_{j_0}$ by $h$. We choose $l_i \in \mathbb{N}$ such that if $x, x'$ belong to a common cell in $\tilde{X}[3^{-l_i}]$, $\|h(x) - h(x')\|_{H^1(M)} \leq \varepsilon_i / 2$. Let $x^*_i \in \tilde{X}[3^{-l_i}]$ such that
\[
d_{H^1(M)}(u_i, h(x^*_i)) = d_{H^1(M)}(u_i, h\left(\tilde{X}[3^{-l_i}]\right))
\]
which is bounded by $\varepsilon_i$ by our choice of $j_0$ and $l_i$. Following the argument in [Gua18,GG18], there exists a function $\lambda : \tilde{X} \to (-1 + b, 1 + b)$ such that for all $x \in \tilde{X}$ the following conditions are satisfied.

- $\lambda(T(x)) = -\lambda(x)$;
- $\{h(x) > \lambda(x), h(x) < \lambda(x)\} \in C(M)$ with $\|[h(x) > \lambda(x)]\| + |\{h(x) < \lambda(x)\}| = |M|$;
- Denoting $\hat{h} = F \circ h$ and $\hat{\lambda} = F \circ \lambda$; $2\sigma b_d(M(\partial\{[\hat{h}(x) > \hat{\lambda}(x)]\}) \leq E_{\varepsilon_i}(h(x))$.

One can define a discrete, $\mathbb{Z}_2$-equivariant map $\tilde{\varphi}_i : \tilde{X}[3^{-l_i}] \to \mathbf{I}_{n+1}(M^{n+1}; \mathbb{Z}_2)$ which is fine in the flat norm as follows ($w_i, t_i$ are as in Proposition 4.1).

\[
\tilde{\varphi}_i(x) = \begin{cases} 
\{[\hat{h}(x) > \hat{\lambda}(x)]\} & \text{if } x \notin \{x^*_i, T(x^*_i)\}; \\
\{[w_i > t_i]\} & \text{if } x = x^*_i; \\
\{[w_i < t_i]\} & \text{if } x = T(x^*_i).
\end{cases}
\]
If \( \varphi_i = \partial \circ \hat{\varphi}_i \), using Lemma 4.3, fineness of \( \varphi_i \) with respect to the flat norm

\[
F^\ast(\varphi_i) \leq 4\varepsilon_i C^{-1}_b (L_{\hat{\varepsilon}_i} + \varepsilon_i) + 2(1 - b + \alpha)^{-2}\varepsilon_i^2
\]

which converges to 0 as \( i \to \infty \). Moreover, \( F(V, \{\{w_i = t_i\}\}) \leq s \) implies that \( M(\{\{w_i = t_i\}\}) \leq L + s \). Hence,

\[
\sup_{x \in X[3^{-i}]_0} M(\varphi_i(x)) \leq \max \left\{ \frac{1}{2\sigma_b} (L_{\hat{\varepsilon}_i} + \varepsilon_i), L + s \right\} + \delta_i.
\]

As argued in [Gua18,GG18], one can apply the interpolation theorem of Zhou [Zho17, Proposition 5.8] to produce a sequence of discrete maps whose fineness with respect to the mass norm converges to 0 and then, using the interpolation theorem of Marques and Neves [MN14, Theorem 14.1], one can find a sequence of maps continuous in the mass norm. More precisely, there exists \( i_0^* \geq i_0 \) such that for all \( i \geq i_0^* \) there exist \( \Phi_i : X \to Z_n(M^{n+1};\mathbb{M};\mathbb{Z}_2) \) and \( \delta_i > 0 \) such that \( \Phi_i \in \Pi, \delta_i \to 0 \) and

\[
\sup_{x \in X} M(\Phi_i(x)) \leq \max \left\{ \frac{1}{2\sigma_b} (L_{\hat{\varepsilon}_i} + \varepsilon_i), L + s \right\} + \delta_i.
\]

Moreover, \( \Phi_i(x) = \varphi_i(x) \) for all \( x \in X[3^{-i}]_0 \). In particular, \( \Phi_i(x_i^*) = \varphi_i(x_i^*) = \{\{w_i = t_i\}\} \); hence,

\[
F(V, \{\Phi_i(x_i^*)\}) \leq s.
\]

**Proof of Theorem 1.3.** By letting \( s \to 0, b \to 0 \) and \( i \to \infty \) in the above Proposition 4.2, we obtain Theorem 1.3. More precisely, let \( \{s_m\}_{m=1}^{\infty} \) be a sequence such that \( s_m \to 0 \). We choose \( b_m \in (0, b_0(s_m)] \) such that \( b_m \to 0 \). By Proposition 4.2, for every \( m \in \mathbb{N} \), there exist \( \Psi_m : X \to Z_n(M^{n+1};\mathbb{M};\mathbb{Z}_2) \), \( i(m) \in \mathbb{N} \) and \( x_m^* \in X \) such that \( \Psi_m \in \Pi, i(m) > i(m - 1) \),

\[
\sup_{x \in X} M(\Psi_m(x)) \leq \max \left\{ \frac{1}{2\sigma_{b_m}} (L_{\hat{\varepsilon}_{i(m)}} + \varepsilon_{i(m)}), L + s_m \right\} + s_m \quad \text{and} \quad F(V, \{\Psi_m(x_m^*)\}) \leq s_m.
\]

This implies \( \{\Psi_m\}_{m=1}^{\infty} \) is a minimizing sequence in \( \Pi \) and \( V \in C(\{\Psi_m\}) \).

**References**

[Alm62] F. Almgren, *The homotopy groups of the integral cycle groups*, Topology (1962), 257–299.

[Alm65] ———, *The theory of varifolds*, Mimeographed notes, Princeton, 1965.

[CG19] R. Caju and P. Gaspar, *Solutions of the Allen-Cahn equation on closed manifolds in the presence of symmetry*, arXiv:1906.05938 [math.DG] (2019).

[CL20] G. R. Chambers and Y. Liokumovich, *Existence of minimal hypersurfaces in complete manifolds of finite volume*, Invent. Math. 219 (2020), 179–217.

[CM20] O. Chodosh and C. Mantoulidis, *Minimal surfaces and the Allen-Cahn equation on 3-manifolds: index, multiplicity, and curvature estimates*, Ann. of Math. 191 (2020), no. 1, 213–328.

[Die08] T. tom Dieck, *Algebraic Topology*, European Mathematical Society, Paris, 2008.
[Gas20] Pedro Gaspar, *The second inner variation of energy and the Morse index of limit interfaces*, J. Geom. Anal. 30 (2020), 69–85.

[GG18] P. Gaspar and M. A. M. Guaraco, *The Allen-Cahn equation on closed manifolds*, Calc. Var. Partial Differ. Equ. 57 (2018), 101.

[GG19] ———, *The Weyl law for the phase transition spectrum and the density of minimal hypersurfaces*, Geom. Funct. Anal. 29 (2019), no. 2, 382–410.

[Gho93] N. Ghoussoub, *Duality and perturbation methods in critical point theory*, Cambridge Tracts in Mathematics, vol. 107, Cambridge University Press, Cambridge, 1993. With appendices by David Robinson.

[Giu84] E. Giusti, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, Birkhäuser Verlag, Basel, 1984.

[GMN19] M. A. M. Guaraco, F. C. Marques, and A. Neves, *Multiplicity one and strictly stable Allen-Cahn minimal hypersurfaces*, arXiv:1912.08997 [math.DG] (2019).

[GT01] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of second order*, Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[Hie18] F. Hiesmayr, *Spectrum and index of two-sided Allen-Cahn minimal hypersurfaces*, Communications in Partial Differential Equations 43 (2018), no. 11, 1541–1565.

[HT00] J. E. Hutchinson and Y. Tonegawa, *Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory*, Calc. Var. Partial Differ. Equ. 10(1) (2000), 49–84.

[MN14] F. C. Marques and A. Neves, *Min–max theory and the Willmore conjecture*, Ann. Math. 179 (2014), 683–782.

[MN16] ———, *Topology of the space of cycles and existence of minimal varieties*, Surveys in Differential Geometry 21(1) (2016), 165–177.

[MN18] ———, *Morse index of multiplicity one min-max minimal hypersurfaces*, arXiv:1803.04273 [math.DG] (2018).

[MN17] ———, *Existence of infinitely many minimal hypersurfaces in positive Ricci curvature*, Invent. Math. 209 (2017), no. 2, 577–616.

[MN16] ———, *Morse index and multiplicity of min-max minimal hypersurfaces*, Cambridge J. Math. 4 (2016), no. 4, 463–511.

[MNS19] F. C. Marques, A. Neves, and A. Song, *Equidistribution of minimal hypersurfaces in generic metrics*, Invent. Math. 216 (2019), no. 2, 421–443.

[MPPP07] M. Miranda, Jr, D. Pallara, F Paronetto, and M. Preunkert, *Heat semigroup and functions of bounded variation on Riemannian manifolds*, J. Reine Angew. Math. 613 (2007), 99–119.

[Nic11] L. Nicolaescu, *An invitation to Morse theory*, Universitext, Springer, 2011.

[Pit81] J. Pitts, *Existence and regularity of minimal surfaces on Riemannian manifolds*, Mathematical Notes 27, Princeton University Press, Princeton, 1981.
[Sim83] L. Simon, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, Canberra, 1983.

[Son18] A. Song, *Existence of infinitely many minimal hypersurfaces in closed manifolds*, arXiv:1806.08816 [math.DG] (2018).

[SS81] R. Schoen and L. Simon, *Regularity of stable minimal hypersurfaces*, Comm. Pure Appl. Math. **34** (1981), 741–797.

[Ton05] Y. Tonegawa, *On stable critical points for a singular perturbation problem*, Comm. Anal. Geom. **13**(2) (2005), 439–459.

[TW12] Y. Tonegawa and N. Wickramasekera, *Stable phase interfaces in the van der Waals–Cahn–Hilliard theory*, J. Reine Angew. Math. **668** (2012), 191–210.

[War66] F. W. Warner, *Extension of the Rauch comparison theorem to submanifolds*, Trans. Amer. Math. Soc. **122** (1966), 341–356.

[Wic14] N. Wickramasekera, *A general regularity theory for stable codimension 1 integral varifolds*, Ann. of Math. **179**(3) (2014), 843–1007.

[Yau82] S.-T. Yau, *Seminar on Differential Geometry, Problem section*, Ann. of Math. Stud., 102, Princeton Univ. Press, Princeton, N.J., 1982.

[Zho17] X. Zhou, *Min-max hypersurface in manifold of positive Ricci curvature*, J. Differential Geom. **105** (2017), no. 2, 291–343.

[Zho19] ________, *On the multiplicity one conjecture in min-max theory*, arXiv:1901.01173 [math.DG] (2019).