Equilibrium problems for vector potentials with semidefinite interaction matrices and constrained masses

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Abstract

We prove existence and uniqueness of a solution to the problem of minimizing the logarithmic energy of vector potentials associated to a $d$-tuple of positive measures supported on closed subsets of the complex plane. The assumptions we make on the interaction matrix are weaker than the usual ones and we also let the masses of the measures vary in a compact subset of $\mathbb{R}^d_+$. The solution is characterized in terms of variational inequalities. Finally, we review a few examples taken from the recent literature that are related to our results.

Key words: weighted energy minimisation problems, vector potentials, external fields, equilibrium conditions, graph theory, Nikishin systems.

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1 Introduction

Vector equilibrium problems in logarithmic potential theory have been studied for a few decades and have shown to be crucial in the investigation of many problems in approximation theory, like those involving multiple orthogonal polynomials (e.g. Hermite-Padé approximants, in particular Angelesco and Nikishin systems). This approach has been very fruitful in the analysis of numerous questions in numerical or applied mathematics, to name only a few, eigenvalue distribution of Toeplitz matrices, models in random matrix theory, determinantal processes, e.g. non-intersecting random paths. Vector equilibrium problems were first considered in [14, 15]. The book [20] contains a nice introduction to the subject. Equilibrium problems on general locally compact spaces are studied in [21, 28, 29].

We first introduce some notations. Let $\mu$ be a (positive) Borel measure with closed support in $\mathbb{C}$ and set

$$U^\mu(z) = \int \log \frac{1}{|z - x|} d\mu(x),$$

for its logarithmic potential. Assume that $\mu$ has not too much mass at infinity (in a sense to be specified later), so that the above integral converges for $|z - x|$ large. Then, the
logarithmic potential is a superharmonic function from $\mathbb{C}$ to $(-\infty, \infty]$, and the energy of $\mu$ is defined as

$$I(\mu) = \iint \log \frac{1}{|x-y|} d\mu(x)d\mu(y) = \int U^\mu(x)d\mu(x) > -\infty.$$ 

For a subset $\Sigma$ of $\mathbb{C}$, let

$$\mathcal{M}(\Sigma) = \{ \mu \text{ Borel measure, of finite mass, supported in } \Sigma, \text{ and } I(\mu) < \infty \}, \quad (1.2)$$

and

$$\mathcal{M}_t(\Sigma) = \{ \mu \in \mathcal{M}(\Sigma), \| \mu \| = t \},$$

where $\| \mu \|$ denotes the total mass of the measure $\mu$. For two measures $\mu, \nu \in \mathcal{M}(\Sigma)$, we define the so-called mutual energy

$$I(\mu, \nu) = \iint \log \frac{1}{|x-y|} d\mu(x)d\nu(y). \quad (1.3)$$

Again, if $\mu$ and $\nu$ do not have too much mass at infinity, this integral converges for $|x-y|$ large, and is well-defined in $(-\infty, +\infty]$.

Throughout, we let

$$\Delta = (\Delta_1, \ldots, \Delta_d), \quad \bigcup_{i=1}^d \Delta_i \subset \mathbb{C}, \quad (1.4)$$

be a $d$-tuple of closed non polar sets of $\mathbb{C}$, i.e. of positive logarithmic capacities

$$\text{cap} (\Delta_i) > 0, \quad i = 1, \ldots, d,$$

and we define the cartesian products

$$\mathcal{M}^d(\Delta) = \mathcal{M}(\Delta_1) \times \cdots \times \mathcal{M}(\Delta_d), \quad \mathcal{M}^d_1(\Delta) = \mathcal{M}_1(\Delta_1) \times \cdots \times \mathcal{M}_1(\Delta_d).$$

Assume for the moment that the $\Delta_i, i = 1, \ldots, d$, are compact sets. For two $d$-tuples of measures

$$\mu = (\mu_1, \ldots, \mu_d)^t \in \mathcal{M}^d(\Delta), \quad \nu = (\nu_1, \ldots, \nu_d)^t \in \mathcal{M}^d(\Delta),$$

we define the mutual energy of $\mu$ and $\nu$ as

$$J(\mu, \nu) = \sum_{j=1}^d I(\mu_j, \nu_j),$$

which is finite. Actually, the compactness of the $\Delta_i$ entails that the mutual energy of two measures of finite energies is also finite.

Let $C = (c_{i,j})$ be a real symmetric positive definite matrix of order $d$, such that

$$\forall (i,j), \text{ if } \Delta_i \cap \Delta_j \neq \emptyset \text{ then } c_{i,j} \geq 0. \quad (1.5)$$

The energy of $\mu$ with respect to the interaction matrix $C$ is defined as

$$J(\mu) = J(C\mu, \mu) = \sum_{i,j=1}^d c_{i,j} I(\mu_i, \mu_j).$$
Note that, because of [1.5], $J(\mu)$ is always well defined (even if some of the components of $\mu$ have infinite energies). Now, the extremal problem is the following:

\[
J^* = \inf \{ J(\mu), \quad \mu \in \mathcal{M}_d^d(\Delta) \},
\]

and characterize the extremal tuple of measures $\mu^*$ in $\mathcal{M}_d^d(\Delta)$, for which the infimum is attained.

As the sets $\Delta_i$ are assumed to be of positive capacity, a solution $\mu^*$ to this problem, with $J^* = J(\mu^*) < \infty$, exists, and it is unique. The proof of existence is based on the fact that the mutual energy (1.3) is lower semi-continuous, which implies together with (1.5), that $\mu \mapsto J(\mu)$ is also lower semi-continuous. Moreover, the map is strictly convex on the set $\mathcal{M}_d^d(\Delta)$, from which uniqueness follows, see [20] Propositions 5.4.1 and 5.4.2.

A characterization of the solution can be given via the so-called equilibrium conditions. For that, we introduce the partial potentials

\[
U^\mu_i(x) = \sum_{j=1}^{d} c_{i,j} U^\mu_j(x), \quad i = 1, \ldots, d,
\]

where the scalar potentials $U^\mu_j(x)$ have been defined in (1.1). Then, $d$-tuple of measures $\mu$ solves the minimization problem if and only if there exist constants $w_i$, such that, for $i = 1, \ldots, d$,

\[
\begin{align*}
U^\mu_i(x) &\geq w_i, \quad \text{quasi-everywhere on } \Delta_i, \quad (1.6) \\
U^\mu_i(x) &\leq w_i, \quad \text{everywhere on } \text{supp}(\mu_i), \quad (1.7)
\end{align*}
\]

where quasi-everywhere means everywhere up to a set of capacity zero. Proofs of these results can be found in [20, Chapter 5].

Remark 1.1. For some $x \in C$ it may happen that $U^\mu_j(x) = +\infty$ for several indices $j$. However, the partial potential $U^\mu_i$ is well defined quasi-everywhere since positive measures of finite mass and compact support have a finite potential quasi-everywhere, see [27, Theorem III.16].

Regarding applications, it is also very useful to consider an additional external field in equilibrium problems. The main reference for the study of equilibrium problems in presence of an external fields is the book [24].

Let $Q = (Q_j)_{j=1, \ldots, d}$ be a vector of lower semi-continuous functions,

\[
Q_j : \Delta_j \to (-\infty, \infty], \quad j = 1, \ldots, d,
\]

and define the weighted energy of a tuple of measures $\mu \in \mathcal{M}^d(\Delta)$ in the presence of the external field $Q$ as

\[
J_Q(\mu) = J(\mu) + 2 \sum_{j=1}^{d} \int Q_j d\mu_j. \quad (1.8)
\]
For $\mu \in M^d(\Delta)$, we have mentioned that $J(\mu) = J(C\mu, \mu)$ is finite. By lower-semicontinuity, each $Q_j$ is bounded from below on $\Delta_j$, $j = 1, \ldots, d$. Hence, the integrals in (1.8) are well-defined and $J_Q(\mu) > -\infty$. It can also be checked that, in $M^d(\Delta)$, there exists at least one measure $\mu$ with $J_Q(\mu) < \infty$, see the proof of Theorem 1.7 (i).

Then, the extremal problem of minimizing the weighted energies

$$\{J_Q(\mu), \; \mu \in M^d(\Delta)\},$$

is solved by a unique $d$-tuple of measures $\mu^* \in M^d_1(\Delta)$, with $J_Q(\mu^*) < \infty$, and it is characterized by the existence of constants $w_i^Q$, such that, for $i = 1, \ldots, d$,

$$U^\mu_i(x) + Q_i(x) \geq w_i^Q, \quad \text{quasi-everywhere on } \Delta_i,$$

$$U^\mu_i(x) + Q_i(x) \leq w_i^Q, \quad \text{everywhere on } \text{supp } (\mu_i).$$

For a proof in the scalar case $d = 1$, we refer to [15] and [24, Theorem I.1.3]. The vector problem with external fields is considered in [15], see also [13].

In the past few years, generalizations of the above vector equilibrium problems have appeared repeatedly in the literature. By generalizations, we mean that the assumptions on the interaction matrix or on the masses were relaxed in various ways. For instance, in [3, 4], one allows for sets which no longer satisfy the compatibility condition (1.5), since some $\Delta_j$ are intervals with a common endpoint. In [1, 2, 3], one considers interaction matrices which are only positive semidefinite. In these papers, the authors also minimize $J$ not over the set $M^d_1(\Delta)$ of tuples of probability measures but over the set

$$M^d_K(\Delta) = \{\mu = (\mu_1, \ldots, \mu_d)^t \in M^d(\Delta), \; ||\mu|| = (||\mu_1||, \ldots, ||\mu_d||)^t \in K\},$$

where $K$ is a non-empty compact subset of the set $\mathbb{R}^d_+$ of $d$-tuples of non negative real numbers. In addition, one considers in [3, 4, 7, 9, 10, 12, 24, 26] extremal problems with not necessarily compact sets $\Delta_j$. In the papers [3, 9, 10, 12, 26], a solution satisfying the extremal properties (1.6)–(1.7) or (1.10)–(1.11) could be exhibited directly through some algebraic equation hence settling the problem of existence of a minimizer.

The goal of this paper is to provide a more systematic approach, by showing existence, uniqueness, and characterization of the extremal solution for a large class of generalized equilibrium problems. At this point, the following simple examples are instructive, since they show that some care has to be taken when weakening the assumptions of the minimization problem.

**Example 1.2.** Consider the data

$$C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \Delta_1 = [-1/2, 0], \quad \Delta_2 = [0, 1/2],$$

where $C$ is positive semidefinite, and the problem of finding the minimum $J^*$ of the corresponding energy

$$J(\mu) = I(\mu_1 - \mu_2) \geq 0, \quad \mu = (\mu_1, \mu_2)^t \in M^2_1(\Delta).$$

It is known that the same problem on the pair of subsets $\Delta_{1,n} = [-1/2, -1/n]$ and $\Delta_{2,n} = [1/n, 1/2]$, $n \geq 1$, admits the minimal energy $J^*_n$ with

$$J^*_n = 1 \quad \text{cap } (\Delta_{1,n}, \Delta_{2,n}) = 2\pi K(2/n) K'(2/n),$$
where \( \text{cap}(\Delta_{1,n}, \Delta_{2,n}) \) denotes the capacity of the condenser with plates \( \Delta_{1,n} \) and \( \Delta_{2,n} \). The explicit value given in the second equality, in terms of the complete and complementary elliptic integrals of the first kind \( K \) and \( K' \), can be found in [19], and may also be derived from Example II.5.14 in [24, pp.133-134]. Since

\[
K(k) = \frac{\pi}{2} + O(k^2), \quad K'(k) = -\log k + O(1), \quad \text{as } k \to 0,
\]

we obtain by letting \( n \) tend to infinity, that \( J^* = 0 \). However, this value cannot be reached by a couple of measures \( (\mu_1, \mu_2) \) of finite energy since \( I(\mu_1 - \mu_2) = 0 \) would imply \( \mu_1 = \mu_2 \), see Lemma 2.1 below.

More generally, for a rank 1 interaction matrix \( C = y y^t \) with \( y \in \{-1, 1\}^d \), our vector equilibrium problem corresponds to the electrostatics of a condenser with external field, see, e.g., [24, Chapter VIII]. Here one usually assumes disjoint \( \Delta_j \) in order to ensure existence and uniqueness of an extremal tuple of measures, though, as we will see below, we may somewhat relax this condition.

Next, we present three simple examples where existence of an extremal tuple of measures holds but not uniqueness.

**Example 1.3.** Consider the data

\[
C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \Delta_1 = \Delta_2 = [-1, 1], \quad K = \{ (x, y) \in \mathbb{R}^2, x + y = 1, x \geq 0, y \geq 0 \},
\]

then \( J(\mu_1, \mu_2) = I(\mu_1 + \mu_2) \) is minimal over \( M^2_K(\Delta) \) for any couples \( (x\omega_{[-1,1]}, y\omega_{[-1,1]}) \), \( x + y = 1 \), where \( \omega_{[-1,1]} \) denotes the equilibrium measure of \([-1, 1]\).

Here, one may show that \( J \) is convex but not strictly convex over \( M^2_K(\Delta) \). Notice also that there is even not a unique minimizer over \( M^2_K(\Delta) \).

**Example 1.4.** Consider the data

\[
C = I_2, \quad \Delta_1 = \Delta_2 = [-1, 1], \quad K = \{ (x, y) \in \mathbb{R}^2, x^2 + y^2 = 1, x \geq 0, y \geq 0 \},
\]

then

\[
J(\mu_1, \mu_2) = I(\mu_1) + I(\mu_2),
\]

which is minimal when both measures \( \mu_1 \) and \( \mu_2 \) are multiples of the equilibrium measure \( \omega_{[-1,1]} \) of \([-1, 1]\). Hence, any couple \( (x\omega_{[-1,1]}, y\omega_{[-1,1]}) \) with \( x^2 + y^2 = 1 \) belongs to \( M^2_K(\Delta) \) and gives the minimum value \( \log 2 \) of the energy \( J \).

For this example, it is not difficult to show that \( J \) is strictly convex over \( M^2(\Delta) \), but the non-uniqueness of the extremal tuple of measures comes from the lack of convexity of \( K \). The next example shows that even convexity of \( K \) does not allow to conclude.

**Example 1.5.** Consider the data

\[
C = I_2, \quad \Delta_1 = \Delta_2 = [-4, 4], \quad K = \{ (x, y) \in \mathbb{R}^2, x + y = 1, x \geq 0, y \geq 0 \},
\]

then \( J(\mu_1, \mu_2) = I(\mu_1) + I(\mu_2) \) is minimal when both measures \( \mu_1 \) and \( \mu_2 \) are multiples of the equilibrium measure \( \omega_{[-4,4]} \) of \([-4, 4]\), and in this case \( J(x\omega_{[-4,4]}, y\omega_{[-4,4]}) = (x^2 + y^2)I(\omega_{[-4,4]}) \). Since \( I(\omega_{[-4,4]}) = -\log(2) < 0 \), we get the minimal value \( -\log 2 \) both for \( (\omega_{[-4,4]}, 0) \) and \( (0, \omega_{[-4,4]}) \) (and \( J \) is no longer convex).
In this work, we want to extend the afore-mentioned results about the minimization of \((1.9)\) to the following situation:

(i) The sets \(\Delta_i\), \(i = 1, \ldots, d\), are closed sets of \(\mathbb{C}\) (instead of compact sets).

(ii) The interaction matrix \(C \in \mathbb{R}^{d \times d}\), of rank \(r\) say, is positive semi-definite (instead of definite).

(iii) The compatibility condition \((1.5)\) is not necessarily satisfied.

(iv) The minimization of \(J_Q\) is performed over \(\mathcal{M}_K^d(\Delta)\) instead of \(\mathcal{M}_1^d(\Delta)\).

To cope with the non-compactness of the sets \(\Delta_i\), we need to add to the defining properties of the set \(\mathcal{M}(\Sigma)\), see \((1.2)\), a growth condition at infinity.

Hence, from now on, the set \(\mathcal{M}(\Sigma)\) will consist of Borel measures \(\mu\) of finite mass, supported on \(\Sigma\), of finite energy, and such that
\[
\int \log(1 + |t|)d\mu(t) < \infty. \tag{1.12}
\]

The set of \(d\)-tuples of measures \(\mathcal{M}_K^d(\Delta)\) is redefined accordingly, i.e. we assume that condition \((1.12)\) is satisfied component-wise.

For a positive measure \(\mu\) of finite mass, satisfying \((1.12)\), we have
\[
U^\mu(z) \geq -\|\mu\| \log(1 + |z|) - \int \log(1 + |t|)d\mu(t) > -\infty, \quad z \in \mathbb{C}. \tag{1.13}
\]

The question raised in Remark \([1.1]\) about the well-definedness of partial potentials can be answered in the same manner since the assertion given there still holds true for measures in \(\mathcal{M}(\Sigma)\), see Lemma \([2.3]\). For two measures \(\mu\) and \(\nu\) of finite masses, satisfying \((1.12)\), we have
\[
I(\mu, \nu) \geq -\|\mu\| \int \log(1 + |t|)d\nu(t) - \|\nu\| \int \log(1 + |t|)d\mu(t) > -\infty,
\]
and in particular \(I(\mu) > -\infty\). Moreover, denoting by \(\tilde{\mu}\) the normalized measure \(\mu/\|\mu\|\) for a non-zero \(\mu \in \mathcal{M}(\Sigma)\), it is known that the inequality
\[
I(\tilde{\mu} - \tilde{\nu}) \geq 0, \quad \mu, \nu \in \mathcal{M}(\Sigma),
\]
holds true, see Lemma \([2.1]\). In particular, we have \(2I(\tilde{\mu}, \tilde{\nu}) \leq I(\tilde{\mu}) + I(\tilde{\nu})\), and since, by definition of \(\mathcal{M}(\Sigma)\), the energies of \(\mu\) and \(\nu\) are finite, it then follows that the mutual energy \(I(\mu, \nu)\) is finite as well. As a consequence, for \(\mu \in \mathcal{M}_K^d(\Delta)\), the energy \(J(\mu)\) is always well defined in \(\mathbb{R}\).

For the external fields, we also need some growth condition at infinity. Throughout, we assume that \(Q = (Q_j)_j\) is a vector of admissible functions, in the sense\(^1\) of \([24\text{, Chapter VIII.1}]\) :
Definition 1.6. Let \( \Sigma \) be a closed subset of \( \mathbb{C} \) of positive capacity. A function \( f : \Sigma \to (-\infty, \infty] \) is said to be \textit{admissible} if it satisfies the following three conditions:

(i) \( f \) is lower semi-continuous,

(ii) \( f \) is finite on a set of positive capacity,

(iii) \( f(x)/\log|x| \to \infty \) as \( |x| \to \infty \) (in case \( \Sigma \) is unbounded).

In view of the preceding examples, we also have to add assumptions linking the matrix of interaction \( C \) to the topology of the sets \( \Delta_j \). For the proof of the existence of an extremal tuple of measures we will assume that

\[
\exists y \in \text{Im}(C), \quad \forall (i, j), \quad \text{if } \text{dist}(\Delta_i, \Delta_j) = 0 \quad \text{then} \quad y_i y_j > 0, \quad (1.14)
\]

whereas, for uniqueness, we will also impose that, for any subset of indices \( I \subset \{1, 2, \ldots, d\} \), different from a singleton,

\[
\text{if the columns } (C_i)_{i \in I} \text{ of } C \text{ are linearly dependent then } \text{cap} \left( \bigcap_{i \in I} \Delta_i \right) = 0. \quad (1.15)
\]

Notice that both conditions (1.14) and (1.15) are trivially true for positive definite interaction matrices \( C \) (for condition (1.14) take \( y = (1, \ldots, 1)^t \)). Such interaction matrices appear e.g. when studying the asymptotic behavior of Angelesco or Nikishin systems in approximation theory.

It is instructive to have a closer look at vector equilibrium problems corresponding to condensers, namely with interaction matrices \( C = yy^t \) of rank 1, \( y \in \{-1, 1\}^d \). In this case, (1.14) is equivalent to (1.5), it tells us that any two plates \( \Delta_j \) with charges of opposite sign have positive distance, and (1.15) requires in addition that any two plates \( \Delta_j \) with charges of the same sign have an intersection of capacity zero. Finally, notice that condition (1.14) fails to hold for Example 1.2, whereas condition (1.15) fails to hold for Example 1.3. For the other two examples, conditions (1.14) and (1.15) hold, indicating that there should be additional restrictions on the set \( K \).

We now state the two main results of our paper. The first result shows, under assumption (1.14), the existence of a solution to our minimization problem.

Theorem 1.7. Consider some nonempty compact set \( K \subset \mathbb{R}_d^+ \), and assume that the positive semidefinite interaction matrix \( C \) satisfies (1.14). Let

\[
J_Q^* := \inf \{ J_Q(\mu), \ \mu \in \mathcal{M}_K^d(\Delta) \}. \quad (1.16)
\]

Then, the following assertions hold.

(a) \( J_Q^* \) is finite.

(b) There exists a \( d \)-tuple of measures \( \mu^* \in \mathcal{M}_K^d(\Delta) \), such that \( J_Q(\mu^*) = J_Q^* \).

Our second result is about uniqueness of a minimizer of the extremal problem (1.16), and about its characterization by equilibrium conditions, the so-called Euler–Lagrange inequalities. Here we restrict ourselves to measures \( \mu \) whose vector of masses \( (||\mu_1||, \ldots, ||\mu_d||) \) lies in a non–empty compact polyhedron \( K \) of \( \mathbb{R}_d^+ \).

\(^2\) In particular, Example 1.3 tells us that the classical condition (1.5) only ensures strict convexity in case of invertible interaction matrices.
Theorem 1.8. Assume that the positive semidefinite interaction matrix $C$ satisfies the assumptions (1.14) and (1.15), and that the set of masses $K$ consists of a non-empty compact polyhedron of the form

$$K = \{ x \in \mathbb{R}^d, Ax = a \},$$

(1.17)

with $A \in \mathbb{R}^{m \times d}$ and $a \in \mathbb{R}^m$, where we suppose in addition that

$$\text{Ker}(A) \subset \text{Ker}(C).$$

(1.18)

Then, the following assertions hold true,

(a) There exists a unique $d$-tuple of measures $\mu^* \in \mathcal{M}_K^d(\Delta)$, of finite energy $J_Q(\mu^*) < \infty$, such that

$$J_Q(\mu^*) = \inf \{ J_Q(\mu), \mu \in \mathcal{M}_K^d(\Delta) \}.$$  

(b) The $d$-tuple of measures

$$\mu = (\mu_1, \ldots, \mu_d) \in \mathcal{M}_K^d(\Delta)$$

is the minimizer of $J_Q$ over $\mathcal{M}_K^d(\Delta)$ if and only if there exists $F \in \mathbb{R}^m$ such that, for $i = 1, \ldots, d$,

$$U_i^\mu(x) + Q_i(x) \geq (A^t F)_i \quad \text{quasi–everywhere on } \Delta_i,$$

(1.19)

$$U_i^\mu(x) + Q_i(x) \leq (A^t F)_i \quad \mu_i–\text{almost everywhere on } \Delta_i.$$  

(1.20)

Remark 1.9. Notice that Theorem 1.8 includes the particular case $A = I_d$ of a singleton $K$, where we prescribe the mass of all components of our tuple of measures. Non-singleton $K$ of the form (1.17) have been considered first in [1, 2, 3], where the authors impose equality in (1.18). From Example 1.5 we learn that in general the condition (1.18) cannot be dropped for establishing uniqueness.

As said before, in case of invertible $C$, all our (somehow technical) assumptions are trivially true for any configuration of sets $\Delta_j$ as in (1.4).

Corollary 1.10. In case of a symmetric positive definite interaction matrix $C$ and a singleton $K = \{ a \}$, there exists one and only one minimizer of $J_Q$ over $\mathcal{M}_K^d(\Delta)$, which is characterized by the equilibrium conditions (1.19) and (1.20) for $A = I_d$.

Example 1.11. Let

$$C = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \Delta_2 \subset \Delta_1 \subset \mathbb{C}, \quad K = \{ (a_1, a_2) \} \subset \mathbb{R}^2_+, \quad a_2 \leq 2 a_1$$

then, according to (1.19) and (1.20), the couple of measures

$$\mu_1 = (a_1 - \frac{a_2}{2}) \omega_{\Delta_1} + \frac{a_2}{2} \omega_{\Delta_2}, \quad \mu_2 = a_2 \omega_{\Delta_2},$$

minimizes $J$ over $\mathcal{M}_K^d(\Delta)$. As in Example 1.2 we can give an electrostatic interpretation in terms of a condenser with two plates $\Delta_1$ and $\Delta_2$ of opposite charge. However, here the Nikishin interaction matrix $C$ translates some di-electric medium where particles of equal charge have stronger interaction than those of opposite sign. We observe the somehow surprising fact that there exists a unique electrostatic equilibrium even if the two plates overlap. Notice that Nikishin systems with touching intervals $\Delta_j$ have been considered before in the literature without addressing this issue.
The remainder of the paper is organized as follows. In Section 2, we gather several preliminary results that are needed in the sequel. In Section 3, we give the proof of Theorem 1.7. We also derive, under an additional condition, that the components of the solution have compact supports. In Section 4, we give the proof of Theorem 1.8. In Section 5, we review a few examples taken from the literature that are related to our results. Some open questions are discussed in Section 6.

2 Preliminary Results

Let us first recall the important fact that the energy of a signed measure of mass 0 is non negative.

**Lemma 2.1.** Let $\mu, \nu \in M(\mathbb{C})$ with $\|\mu\| = \|\nu\|$. Then

$$I(\mu - \nu) \geq 0,$$

and $I(\mu - \nu) = 0$ if and only if $\mu = \nu$.

**Proof.** See [24, Lemma I.1.8] for measures $\mu, \nu$ with compact support, and [7, Theorem 2.5] for the unbounded case, see also [25, Theorem 4.1].

We proceed with a few results which are well-known when the supports of the measures are compact, but for which we have not always found references in the non compact case. We defer the proofs of these results to the Appendix.

**Lemma 2.2.** Let $\mu \in M(\mathbb{C})$. Then $\mu(E) = 0$ for every Borel polar set $E$.

**Proof.** See [24, Remark I.1.7] for supp $(\mu)$ compact, and Appendix A for the general case.

**Lemma 2.3.** Let $\mu$ be a positive measure of finite mass, satisfying (1.12). Then, the potential $U^\mu(z)$ can be $+\infty$ only on a Borel set of capacity 0.

**Proof.** It is well-known that the assertion holds true for any super-harmonic function on $\mathbb{C}$, not identically $+\infty$, see [22, Theorem 3.5.1]. In particular, it holds true for the potential $U^\mu(z)$.

Throughout we will use weak convergence of Borel measures. Let $(\mu_n)_n$ be a bounded sequence of Borel measures on $\mathbb{C}$,

$$\|\mu_n\| \leq c < \infty, \quad n \in \mathbb{N}.$$ 

We recall that the sequence $\mu_n$ tends weakly to a measure $\mu$, as $n \to \infty$, if

$$\int f d\mu_n \to \int f d\mu, \quad (2.1)$$

for every bounded, continuous, real-valued function $f$ on $\mathbb{C}$. In the literature, the notion of vague convergence is also used, where it is assumed that (2.1) holds true only for continuous function $f$ on $\mathbb{C}$ with compact support. Clearly, vague convergence is weaker than the weak convergence. For example, the sequence $\delta_n$ of Dirac measures at $x = n$ converges vaguely to 0, although it does not converge weakly. For some comments on these two different notions of convergence of measures, one may have a look to [8, pp.134–137].
Lemma 2.4. Assume that the bounded sequence $\mu_n$ tends weakly to $\mu$, and let $Q$ be a lower bounded, lower semi-continuous function on $\mathbb{C}$. Then

$$\int Q d\mu \leq \liminf_{n \to \infty} \int Q d\mu_n.$$ 

Proof. See [24, Theorem 0.1.4] for $\mu_n$ all supported in a compact set, and Appendix A for the general case.

Definition 2.5. A bounded sequence of measures $(\mu_n)_{n \geq 0}$ in $\mathcal{M}(\mathbb{C})$ is said to be (i) tight if:

$$\forall \epsilon > 0, \, \exists \text{ compact set } K \subset \mathbb{C}, \, \forall n \in \mathbb{N}, \int_{\mathbb{C} \setminus K} d\mu_n(t) \leq \epsilon,$$

(ii) log-tight if:

$$\forall \epsilon > 0, \, \exists \text{ compact set } K \subset \mathbb{C}, \, \forall n \in \mathbb{N}, \int_{\mathbb{C} \setminus K} \log(1 + |t|) d\mu_n(t) \leq \epsilon.$$  \hspace{1cm} (2.2)

The notion of tightness of a bounded set of measures is classical, see e.g. [5]. The notion of log-tightness is slightly stronger. Note that, from assumption (1.12), each individual measure $\mu \in \mathcal{M}(\mathbb{C})$ satisfies inequality (2.2). Here, for log-tightness of a sequence, we ask this condition to be satisfied uniformly with respect to $n$.

Theorem 2.6 (Prohorov). Let $(\mu_n)_{n \geq 0}$ be a tight sequence of probability measures on $\mathbb{C}$. Then, there is a subsequence of $(\mu_n)_{n \geq 0}$ which is weakly convergent to a probability measure on $\mathbb{C}$.

Proof. See Helly’s selection theorem [24, Theorem 0.1.3] for $\mu_n$ all supported in some compact set, [5, Theorem 5.1] in a general metric space, and [11, Theorem 9.3.3] for the special case of the euclidean space $\mathbb{R}^k$.

Remark 2.7. The Prohorov theorem is actually stronger than Theorem 2.6, in that it also states, in the converse direction, that a weakly convergent sequence of measures is tight.

Lemma 2.8. Let $(\mu_n)_{n \geq 0}$ and $(\nu_n)_{n \geq 0}$ be bounded log-tight sequences of measures in $\mathcal{M}(\mathbb{C})$. Assume $\mu$ and $\nu$ are two Borel measures such that $\mu_n \to \mu$ and $\nu_n \to \nu$ in the weak topology. Then

$$I(\mu, \nu) \leq \liminf_{n \to \infty} I(\mu_n, \nu_n).$$ \hspace{1cm} (2.3)

Proof. See [20, Theorem 5.2.1] for all $\mu_n, \nu_n$ supported in some compact set, and Appendix A for the general case.

Let us proceed with establishing four propositions, among which we prove the positiveness of $J$, the lower semi-continuity of $J_Q$, and an inequality relating the weighted energy $J_Q(\mu)$ with the scalar energies of the components of $\mu$. 

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Throughout, we write the positive semidefinite matrix $C$ of rank $r$ as a full rank factorization of the form

$$ C = B^tB, \quad B \text{ matrix of dimensions } (r, d), \ r \leq d, \text{ of rank } r. \quad (2.4) $$

Such a factorization is obtained, e.g., from the Jordan decomposition of $C$ by recalling that there exists an orthonormal basis of eigenvectors of $C$. First, we generalize Lemma 2.1 to our vector setting.

**Proposition 2.9.** Let $\mu, \nu \in \mathcal{M}_R^d(\Delta)$ with tuples of masses verifying $B\|\mu\| = B\|\nu\|$. Then,

$$ J(\mu - \nu) \geq 0. \quad (2.5) $$

Moreover, if condition (1.15) holds true, then

$$ J(\mu - \nu) = 0 \text{ if and only if } \mu = \nu. \quad (2.6) $$

**Proof.** Let $\lambda = B(\mu - \nu)$. Then, we may write

$$ J(\mu - \nu) = J(\lambda, \lambda) = \sum_{j=1}^r I(\lambda_j). $$

By assumption, each component $\lambda_j$ of $\lambda$, with Hahn decomposition $\lambda_j = \lambda_{j,+} - \lambda_{j,-}$, is a signed measure of mass 0, whose absolute value $\lambda_{j,+} + \lambda_{j,-}$ is of finite energy. Hence Lemma 2.1 applies, showing that each $I(\lambda_j)$ is non negative, so that (2.5) holds true.

We also know from Lemma 2.1 that $J(\mu - \nu) > 0$ if $\lambda_j \neq 0$ for at least one index $j$. Hence, to establish (2.6) it only remains to show that $\mu \neq \nu$ implies $\lambda \neq 0$. This property is trivial for positive definite $C$ and thus invertible $B$. In our setting with semidefinite $C$, we will need assumption (1.15).

Assuming $\mu - \nu \neq 0$, we deduce that there exists an index $i_0$ and a Borel set $N$ such that $(\mu_{i_0} - \nu_{i_0})(N) \neq 0$. Now, we consider the partition

$$ \bigcup_{j=1,\ldots,d} \Delta_j = \bigcup_{I \subset \{1, \ldots, d\}, \ I \neq \emptyset} E_I, \quad E_I = \left( \bigcap_{i \in I} \Delta_i \right) \cap \left( \bigcap_{i \notin I} \Delta_i \right), $$

where some of the $E_I$ may be empty sets. This induces a partition of $N$,

$$ N = \bigcup_{I \subset \{1, \ldots, d\}, \ I \neq \emptyset} N_I, \quad N_I = N \cap E_I, $$

so that $(\mu_{i_0} - \nu_{i_0})(N) = \sum_I (\mu_{i_0} - \nu_{i_0})(N_I)$. Therefore there exists a subset $I \subset \{1, \ldots, d\}$ such that $(\mu_{i_0} - \nu_{i_0})(N_I) \neq 0$, and

$$ \forall i \notin I, \quad (\mu_i - \nu_i)(N_I) = (\mu_i - \nu_i)(N_I \setminus \Delta_i) = 0, \quad (2.7) $$

since $\text{supp}(\mu_i - \nu_i) \subset \Delta_i$. Note also that either $\mu_{i_0}(N_I)$ or $\nu_{i_0}(N_I)$ is nonzero, so that $N_I$ is of positive capacity by Lemma 2.2. Denote by $\tilde{B}$, and $\tilde{C}$, the submatrix of $B$, and of $C$, respectively, obtained from selecting the columns of indices belonging to $I$. Since with $N_I$ also $\bigcap_{i \in I} \Delta_i$ has positive capacity, we obtain from condition (1.15) that $\tilde{C}$ and thus $\tilde{B}$ has full column rank. By (2.7), the relation $B(\mu - \nu)(N_I) = \lambda(N_I)$ simplifies to $\tilde{B}(\mu - \nu)_{i \in I}(N_I) = \lambda(N_I)$, which cannot be zero. Thus $\lambda \neq 0$. \qed
As in the classical case, the main ingredient in the proof of Theorem 1.7 will be the lower semi-continuity of the functional $J_Q$. We note that the proof does not use the compatibility condition (1.5).

**Proposition 2.10.** Let $(\mu^{(n)})_{n \geq 0}$ be a sequence of $d$-tuples of measures in $\mathcal{M}_K^d(\Delta)$ which is log-tight (in the component–wise sense) and assume that $\mu^{(n)}$ tends to a $d$-tuple of measures $\mu \in \mathcal{M}_K^d(\Delta)$, again component–wise, as $n \to \infty$, in the weak topology. Then

$$J_Q(\mu) \leq \liminf_{n \to \infty} J_Q(\mu^{(n)}).$$

**Proof.** We first show the asserted inequality for the map $\mu \mapsto J(\mu)$. For that, we will use convolution of scalar finite Borel measures $\mu$ and $\nu$, which, for a Borel set $B \subset \mathbb{C}$, is defined as follows,

$$(\mu * \nu)(B) = \int \nu(B - t) d\mu(t) = \int \mu(B - t) d\nu(t).$$

The convolution $\mu * \nu$ is a positive measure such that

$$\text{supp} (\mu * \nu) \subset \text{supp} (\mu) + \text{supp} (\nu), \quad (\mu * \nu)(\mathbb{C}) = \mu(\mathbb{C}) \nu(\mathbb{C}).$$

From

$$(\mu * \nu)(B) = (\mu \times \nu)(\{(x, y), \ x + y \in B\}),$$

it is easy to see that convolution is a commutative and associative operation. We will also use the convolution of a function $h$ with a measure $\mu$,

$$h * \mu(z) = \int h(z - t) d\mu(t),$$

so that the potential $U^\mu$ coincides with the convolution $-\log |\cdot| * \mu$.

Let $\lambda_N$ be the equilibrium measure of the circle centered at 0 of radius $e^{-N}$. Its potential is easily computed:

$$U^{\lambda_N}(x) = \min \left(N, \log \frac{1}{|x|}\right),$$

see e.g. [24, Example 0.5.7]. It is a continuous function tending pointwise to $\log(1/|x|)$, $x \neq 0$, as $N$ tends to $\infty$. Then, by associativity and commutativity of the convolution, we get

$$U^{\mu*\lambda_N}(z) = -\log |\cdot| * (\mu * \lambda_N)(z) = (-\log |\cdot| * \lambda_N) * \mu(z) = \int U^{\lambda_N}(z - x) d\mu(x),$$

and for the mutual energies, we have

$$I(\mu * \lambda_N, \nu) = \int U^{\lambda_N}(x - y) d\mu(x) d\nu(y), \quad (2.8)$$

$$I(\mu * \lambda_N, \nu * \lambda_N) = \int U^{\lambda_N}(x - y) d\mu(x) d(\nu * \lambda_N)(y) = \int (U^{\lambda_N} * (\nu * \lambda_N))(x) d\mu(x)$$

$$= \int U^{\lambda_N*\lambda_N}(x - y) d\mu(x) d\nu(y). \quad (2.9)$$
From the definition of $U^{\lambda_N}$ follows that $I(\mu * \lambda_N, \nu * \lambda_N) \leq I(\mu, \nu)$. In particular, $I(\mu * \lambda_N) < \infty$ if $I(\mu) < \infty$. Moreover,

$$\int \log(1 + |x|)d(\mu * \lambda_N) = \iint \log(1 + |x + y|)d\mu(x)d\lambda_N(y) \leq \int \log(1 + e^{-N} + |x|)d\mu(x)$$

$$\leq \log(1 + e^{-N})\|\mu\| + \int \log(1 + |x|)d\mu(x) < \infty.$$  

Hence, for any closed subset $\Sigma$ of $\mathbb{C}$, the measure $\mu * \lambda_N$ lies in $\mathcal{M}(\Sigma + D(0, e^{-N}))$ if $\mu \in \mathcal{M}(\Sigma)$.

Now, consider a log-tight sequence $\mu^{(n)} \in \mathcal{M}^d_\Delta$ such that

$$\mu^{(n)} \rightharpoonup \mu \in \mathcal{M}^d_\Delta,$$

in the weak sense. Let $N$ be given. From the above remarks, the $d$-tuple of measures $\mu^{(n)} * \lambda_N$, where the convolution is taken componentwise, belongs to $\mathcal{M}^d_\Delta(D(0, e^{-N}))$, and the masses of $\mu^{(n)}$ and $\mu^{(n)} * \lambda_N$ are the same. Thus, from (2.5), we get

$$J(\mu^{(n)} - \mu^{(n)} * \lambda_N) \geq 0,$$

or equivalently,

$$J(\mu^{(n)}) \geq \sum_{i,j=1}^d c_{i,j} \left( I(\mu^{(n)}_i, \mu^{(n)}_j * \lambda_N) + I(\mu^{(n)}_i * \lambda_N, \mu^{(n)}_j) - I(\mu^{(n)}_i * \lambda_N, \mu^{(n)}_j * \lambda_N) \right).$$

Let us consider the first energy in the right-hand side of the above inequality. Since $\mu^{(n)}_i * \lambda_N$ is a log-tight family which tends weakly to $\mu_i * \lambda_N$, Lemma 2.8 tells us that,

$$\lim_{n \to \infty} I(\mu^{(n)}_i, \mu^{(n)}_j * \lambda_N) \geq I(\mu_i, \mu_j * \lambda_N). \tag{2.10}$$

Actually, we have more. Indeed, redoing the proof of Lemma 2.8 with the kernel $U^{\lambda_N}(x - y)$ instead of $\log(|x - y|^{-1})$, we now get an integrand in the first integral of (A.1) which is bounded and continuous. Hence, (2.10) can be strengthened to

$$\lim_{n \to \infty} I(\mu^{(n)}_i, \mu^{(n)}_j * \lambda_N) = I(\mu_i, \mu_j * \lambda_N).$$

The limits

$$\lim_{n \to \infty} I(\mu^{(n)}_i * \lambda_N, \mu^{(n)}_j) = I(\mu_i * \lambda_N, \mu_j), \quad \lim_{n \to \infty} I(\mu^{(n)}_i * \lambda_N, \mu^{(n)}_j * \lambda_N) = I(\mu_i * \lambda_N, \mu_j * \lambda_N)$$

are proven in the same way. Consequently, we obtain that

$$\lim_{n \to \infty} J(\mu^{(n)}) \geq \sum_{i,j=1}^d c_{i,j} \left( I(\mu_i, \mu_j * \lambda_N) + I(\mu_i * \lambda_N, \mu_j) - I(\mu_i * \lambda_N, \mu_j * \lambda_N) \right).$$

where the right-hand side has a sense since we assume that the limit measure $\mu \in \mathcal{M}^d_\Delta$ (all its components have finite energy). Finally, both potentials $U^{\lambda_N}$ and $U^{\lambda_N * \lambda_N}$ tend
pointwise to $\log(1/|x|)$ for $x \neq 0$, as $N \to \infty$. They are dominated by $|\log(1/|x|)|$, and moreover,

$$
\int \int \left| \log \frac{1}{|x-y|} \right| d\mu_i(x) d\mu_j(y) \leq I(\mu_i, \mu_j) + 2\|\mu_j\| \int \log(1 + |x|) d\mu_i(x) + 2\|\mu_i\| \int \log(1 + |x|) d\mu_j(x),
$$

which is finite because $I(\mu_i, \mu_j)$ is and we have (1.12). Hence, from the dominated convergence theorem, we get

$$
\lim_{N \to \infty} I(\mu_i, \mu_j \ast \lambda_N) = I(\mu_i, \mu_j), \quad \lim_{N \to \infty} I(\mu_i \ast \lambda_N, \mu_j \ast \lambda_N) = I(\mu_i, \mu_j),
$$

implying that

$$
\liminf_{n \to \infty} J(\mu(n)) \geq J(\mu).
$$

Since the external fields $Q_j$ are lower semi-continuous and lower bounded, the fact that

$$
\liminf_{n \to \infty} \int Q_j d\mu_j(n) \geq \int Q_j d\mu_j, \quad j = 1, \ldots, d,
$$

follows from Lemma 2.4.

The aim of the next proposition is to show an inequality which will be used in the proof of Proposition 2.12. It asserts that the scalar energy of a linear combination $\sum_j y_j \mu_j$ of bounded measures $\mu_j$ in $\mathcal{M}(\mathbb{C})$, with given coefficients $y_j$, is lower bounded, independently of the $\mu_j$, as soon as it is weighted by a multiple $\gamma Q$ of the external field, with $\gamma$ an arbitrary small positive number. Such a result is needed only to cope with unbounded $\Delta_j$ since, for compact $\Delta_j$, it is not difficult to derive a lower bound for the energy of a signed measure which does not involve external fields.

**Proposition 2.11.** Let $Q = (Q_1, \ldots, Q_d)^t$ be an admissible external field and let $y = (y_1, \ldots, y_d)^t$ be a given vector in $\mathbb{R}^d$. Then,

$$
\forall \gamma > 0, \quad \exists \Gamma \in \mathbb{R}, \quad \forall \mu \in \mathcal{M}_K^d(\Delta), \quad \Gamma \leq I(y^t \mu) + \gamma \int Q^t d\mu. \quad (2.11)
$$

**Proof.** Since the union $\Sigma$ of the sets $\Delta_i$, $i = 1, \ldots, d$, is different from $\mathbb{C}$, recall (1.4), there exist some $z_0 \in \mathbb{C}$ and some $r < 1$ say, such that the disk $D(z_0, 2r)$ does not intersect $\Sigma$. Let $\omega_D$ be the equilibrium measure of the disk $D = D(z_0, r)$ and let

$$
\tau = \lambda - \lambda(\mathbb{C})\omega_D,
$$

where $\lambda$ denotes the scalar signed measure $y^t \mu$. Since $I(\lambda)$ is finite, $I(\tau)$ is finite as well and Lemma 2.1 applies : $I(\tau) \geq 0$, or equivalently

$$
I(\lambda) \geq 2\lambda(\mathbb{C})I(\lambda, \omega_D) + \lambda(\mathbb{C})^2 \log(r)
$$

$$
= 2\lambda(\mathbb{C}) \sum_{j=1}^d y_j I(\mu_j, \omega_D) + \lambda(\mathbb{C})^2 \log(r). \quad (2.12)
$$

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All the mutual energies \( I(\mu_j, \omega_D) \) can be bounded above:

\[
I(\mu_j, \omega_D) = \int \int \log \frac{1}{1 - |t|} d\mu_j d\omega_D \leq \log \left( \frac{1}{r} \right) M_j(K), \tag{2.13}
\]

with \( M_j(K) = \sup_{\mu \in \mathcal{M}_K^d(\Delta)} \| \mu_j \| \). Moreover, the \( I(\mu_j, \omega_D) \) can also be lower bounded. First, note that, in view of the third condition of admissibility in Definition 1.6 and the fact that \( Q_j \) is lower bounded on compact sets, we have

\[
\forall \gamma_j > 0, \quad \exists \Gamma_j \in \mathbb{R}, \quad \forall z \in \Delta_j, \quad \log(1 + |z|) \leq \gamma_j Q_j(z) + \Gamma_j.
\]

Then,

\[
-I(\mu_j, \omega_D) \leq \int \int \log(1 + |z|) d\mu_j(z) + \| \mu_j \| \int \int \log(1 + |t|) d\omega_D(t)
\]

\[
\leq \gamma_j \int Q_j d\mu_j + \Gamma_j \| \mu_j \| + \| \mu_j \| \sup_{t \in D} \log(1 + |t|)
\]

\[
\leq \gamma_j \int Q_j d\mu_j + M_j(K) \left( \Gamma_j + \sup_{t \in D} \log(1 + |t|) \right), \tag{2.14}
\]

and the proposition follows from plugging inequalities (2.13) or (2.14) into (2.12), according to the sign of \( \lambda(\mathcal{C})y_j \), and noting that \( \lambda(\mathcal{C}) \) is bounded both above and below independently of \( \mu \).

Next, we show that the weighted energy of a tuple of measures \( \mu \in \mathcal{M}_K^d(\Delta) \) dominates the energies of its components. This result requires the condition (1.14).

**Proposition 2.12.** Assume that the \( d \)-tuple of closed sets \( \Delta \) and the interaction matrix \( C \) satisfy (1.14). Then, there exist positive constants \( a_0 \) and \( a_1 \) such that

\[
\forall \mu = (\mu_1, \ldots, \mu_d) \in \mathcal{M}_K^d(\Delta), \quad \sum_{j=1}^d I(\mu_j) \leq a_1 J_Q(\mu) + a_0. \tag{2.15}
\]

**Proof.** Consider a vector \( y \) in the range of \( C = B^t B \) that satisfies (1.14), and note that, since for all indices \( i \), \( y_i^2 > 0 \), the minimum \( m = \min(y_i^2) \) is positive. Let \( x \) be a non-zero vector in \( \mathbb{R}^r \) such that \( y = B^t x \), and \( Q \) be an orthogonal matrix with \( x/\|x\| \) as its first column. Then, the first row of \( Q^t B \) is \( y^t/\|x\| \) and

\[
J(\mu) = J(C \mu, \mu) = J(Q^t B \mu, Q^t B \mu) = \frac{1}{\|x\|^2} J \left( \sum_{j=1}^d y_j \mu_j \right) + \sum_{k=2}^r I(\lambda_k),
\]

where we have set \( (\lambda_1, \ldots, \lambda_r)^t = Q^t B \mu \). Next, we have the following lower bounds for the energies,

\[
I(\mu_j, \mu_k) \geq -\| \mu_k \| \int \log(1 + |t|) d\mu_j(t) - \| \mu_j \| \int \log(1 + |z|) d\mu_k(z)
\]

\[
\geq -\gamma_{j,k} \int Q_j d\mu_j + \int Q_k d\mu_k - \Gamma_{j,k}, \tag{2.16}
\]
where, as in the proof of Proposition 2.11, the positive real number $\gamma_{j,k}$ can be arbitrarily small and $\Gamma_{j,k}$ is a sufficiently large number. Hence, from (2.16) applied with $j = k$, we deduce
\[
m \sum_j \left( I(\mu_j) + 2\gamma_{j,j} \int Q_j d\mu_j + \Gamma_{j,j} \right) \leq \sum_j y_j^2 \left( I(\mu_j) + 2\gamma_{j,j} \int Q_j d\mu_j + \Gamma_{j,j} \right),
\]
so that
\[
m \sum_j I(\mu_j) \leq \sum_j y_j^2 I(\mu_j) + \sum_j (y_j^2 - m) \left( 2\gamma_{j,j} \int Q_j d\mu_j + \Gamma_{j,j} \right)
= I \left( \sum_j y_j \mu_j \right) - \sum_{j \neq k} y_j y_k I(\mu_j, \mu_k) + \sum_j (y_j^2 - m) \left( 2\gamma_{j,j} \int Q_j d\mu_j + \Gamma_{j,j} \right)
= \|x\|^2 J_Q(\mu) - 2\|x\|^2 \sum_{j=1}^d \int Q_j d\mu_j - \|x\|^2 \sum_{k=2}^r I(\lambda_k) - \sum_{j \neq k} y_j y_k I(\mu_j, \mu_k)
+ \sum_j (y_j^2 - m) \left( 2\gamma_{j,j} \int Q_j d\mu_j + \Gamma_{j,j} \right).
\]
(2.17)

For the signed measures $\lambda_k$, we have lower bounds provided by Proposition 2.11
\[
I(\lambda_k) \geq -\gamma_k \sum_{j=1}^d \int Q_j d\mu_j + \Gamma_k, \quad k = 1, \ldots, r.
\]
(2.18)

Finally, for indices $j, k$ such that $y_j y_k < 0$, we know from (1.14) that $\text{dist}(\Delta_j, \Delta_k)$ is positive so that in this case we also have the upper bound,
\[
I(\mu_j, \mu_k) \leq \log \left( \frac{1}{\text{dist}(\Delta_j, \Delta_k)} \right) \|\mu_j\| \|\mu_k\|.
\]
(2.19)

Making use of (2.16), (2.18) and (2.19) into (2.17) leads to
\[
m \sum_{j=1}^d I(\mu_j) \leq \|x\|^2 J_Q(\mu) - c \sum_{j=1}^d \int Q_j d\mu_j - \Gamma,
\]
where $c$ is a positive real number since the constants $\gamma_k, \gamma_{j,j}$, and $\gamma_{j,k}$ are arbitrarily small. As the external fields $Q_j$ are lower bounded, the above inequality implies (2.15) with two constants $a_0$ and $a_1$ that depend only on the tuple of sets $\Delta$, the interaction matrix $C$ and the compact set of masses $K$.

3  Existence of a solution

In this section, we give the proof of Theorem 1.7. We also prove, under an additional technical assumption, that the components of a solution have compact supports.
Proof of Theorem 1.7. We show that $J^*_Q < +\infty$ as in [23, Theorem I.1.3(a)]. For $\epsilon > 0$, the sets $\Delta_j(\epsilon) = \{ x \in \Delta_j : Q_j(x) \leq 1/\epsilon \}$ are closed and thus compact by assumption on $Q_j$. Since $Q_j$ is finite on a set of positive capacity, $\Delta_j(\epsilon)$ is of positive capacity for sufficiently small $\epsilon > 0$. Denoting by $\omega_{\Delta_j(\epsilon)}$ the equilibrium measure of such a $\Delta_j(\epsilon)$ of positive capacity, $I(\omega_{\Delta_j(\epsilon)}) < \infty$, we find for the $d$-tuple of measures $\mu \in M^d_K(\Delta)$ with $\mu_j = b_j \omega_{\Delta_j(\epsilon)}$, $j = 1, \ldots, d$, and $b = (b_j) \in K$ that $J_Q(\mu) < \infty$. Next, we prove that $J^*_Q > -\infty$. We have

$$J_Q(\mu) = \sum_{k=1}^r I(\lambda_k) + 2 \sum_{j=1}^d \int Q_j d\mu_j,$$

where $(\lambda_1, \ldots, \lambda_r)^t = B\mu$, and the energies $I(\lambda_k)$ satisfy inequalities of the type (2.11) with arbitrarily small positive constants $\gamma_k$, the sum of which can be made less than 1. Hence, there exists a constant $\Gamma$ such that

$$\forall \mu \in M^d_K(\Delta), \quad J_Q(\mu) \geq \sum_{j=1}^d \int Q_j d\mu_j - \Gamma \geq -\sum_{j=1}^d |q_j| M_j(K) - \Gamma, \quad (3.1)$$

with $q_j = \inf_{z \in \mathbb{C}} Q_j(z) > -\infty$, $M_j(K) = \sup_{\mu \in M^d_K(\Delta)} \|\mu_j\| < \infty$, $j = 1, \ldots, d$.

This finishes the proof of assertion (a).

The proof of assertion (b) follows usual lines, see e.g. [20, Chapter 5], by constructing $\mu^*$ as a weak limit of a minimizing sequence of $J_Q$. We first note, in view of (3.1), that for minimizing the energy $J_Q$, it is sufficient to consider the subset $\mathcal{T}$ of $M^d_K(\Delta)$ consisting of $d$-tuples of measures $\mu$ such that

$$\sum_{j=1}^d \int Q_j d\mu_j \leq J^*_Q + \Gamma + 1. \quad (3.2)$$

Let us show that $\mathcal{T}$ is a log-tight family. For $\mu \in \mathcal{T}$, we have

$$\sum_{j=1}^d \int (Q_j - q_j) d\mu_j \leq J^*_Q + \Gamma + 1 + \sum_{j=1}^d |q_j| M_j(K).$$

We simply denote by $\alpha$ the right-hand side of the above inequality. Let $\epsilon > 0$ be given. Since the $Q_j$ are admissible, there exists a compact set $K \subset \mathbb{C}$ such that

$$\sum_j (Q_j(x) - q_j) \geq \frac{\alpha}{\epsilon} \log(1 + |x|), \quad x \in \mathbb{C} \setminus K.$$

Consequently, for any $d$-tuple of measures $\mu$ in $\mathcal{T}$,

$$\sum_j \int_{\mathbb{C} \setminus K} \log(1 + |x|) d\mu_j \leq \frac{\epsilon}{\alpha} \sum_j \int_{\mathbb{C} \setminus K} (Q_j(x) - q_j) d\mu_j \leq \frac{\epsilon}{\alpha} \sum_j \int_{\mathbb{C}} (Q_j(x) - q_j) d\mu_j \leq \epsilon,$$
which shows that the set $\mathcal{T}$ is indeed log-tight. Now, consider a minimizing sequence of $d$-tuples of measures $\mu^{(n)} \in \mathcal{T}$, namely

$$
\lim_{n \to \infty} J_Q(\mu^{(n)}) = J^*_Q.
$$

The family $\mathcal{T}$ being log-tight, it is a fortiori tight, so that by Theorem 2.6 there exists a subsequence, that we still denote by $\mu^{(n)}$, having a weak limit $\mu^*$. Its components $\mu^*_j$ are supported on $\Delta_j$, and its $d$-tuple of masses belongs to $K$. Since $\log(1+|x|)$ is a continuous and lower bounded function, we get from Lemma 2.4 that

$$
\int \log(1 + |x|) d\mu^*_j(x) \leq \liminf_{n \to \infty} \int \log(1 + |x|) d\mu^{(n)}_j, \quad j = 1, \ldots, d.
$$

Moreover, up to an additive constant, $\log(1+|x|)$ is upper bounded by $Q_j(x)$, inequality (3.2) holds true for the sequence $\mu^{(n)}_j$, and

$$
-q_j M_j(K) \leq q_j \|\mu_j^{(n)}\| \leq \int Q_j d\mu^{(n)}_j, \quad j = 1, \ldots, d.
$$

Therefore, we may deduce that

$$
\int \log(1 + |x|) d\mu^*_j(x) < \infty, \quad j = 1, \ldots, d.
$$

Next, we show that each component $\mu^*_j$ is of finite energy. From Lemma 2.8 follows that

$$
I(\mu^*_k) \leq \liminf_{n \to \infty} I(\mu^{(n)}_k), \quad k = 1, \ldots, d.
$$

Adding these inequalities over $k$, and noting that, in view of Proposition 2.12 the sum obtained on the right-hand side is finite, we get

$$
I(\mu^*_j) \leq \liminf_{n \to \infty} \sum_{k=1}^d I(\mu^{(n)}_k) - \sum_{k \neq j} I(\mu^*_k) \leq a_1 \liminf_{n \to \infty} J_Q(\mu^{(n)}) + a_0 - \sum_{k \neq j} I(\mu^*_k)
$$

$$
= a_1 J^*_Q + a_0 - \sum_{k \neq j} I(\mu^*_k) < \infty,
$$

where the last inequality comes from

$$
I(\mu^*_k) \geq -2\|\mu^*_k\| \int \log(1 + |x|) d\mu^*_k(x) > -\infty \quad k = 1, \ldots, d.
$$

Consequently, $\mu^* \in \mathcal{M}_K^d(\Delta)$. From the lower semi–continuity of $J_Q$ established in Proposition 2.10 we conclude that $J^*_Q \geq J_Q(\mu^*)$, and thus $J^*_Q = J_Q(\mu^*)$, showing that $\mu^*$ is a minimizer of the extremal problem (1.16). \qed

We now turn to the question of whether the supports of the components of an extremal tuple of measures as in Theorem 1.7 are compact sets. This property was shown to hold true under more restrictive conditions on the matrix $C$ and the tuple of sets $\Delta$ in [4, 24]. In our generalized setting, we have the following result.
Theorem 3.1. Let \( \mu \in \mathcal{M}_K^d(\Delta) \) be a solution to the minimization problem (1.16). Then, the components \( \mu_i, i = 1, \ldots, d \), of \( \mu \), have compact supports if and only if the following assertion holds true:

there exists a real \( \alpha \) and a number \( M > 0 \) such that, for all pair \((i, j)\) with \( \Delta_i \) and \( \Delta_j \) unbounded and \( c_{i,j} < 0 \), there holds

\[
c_{i,j}U^{\mu_j}(z) + \frac{1}{d}Q_i(z) \geq \alpha, \quad \mu_i-\text{almost everywhere on } \Delta_i \setminus D_M,
\]

where \( D_M \) denotes the closed disk of radius \( M \) centered at zero.

Remark 3.2. The assumption (3.4) bears some similarity with assumption [A3] in [4, Definition 2.1], where it is assumed that the functions

\[
c_{i,j}\log \frac{1}{|z-t|} + \frac{Q_i(z) + Q_j(t)}{d}, \quad i, j = 1, \ldots, d
\]

are uniformly lower bounded on \( \Delta_i \times \Delta_j \).

Remark 3.3. Assumption (3.4) is trivially satisfied if

\[
\forall i, j, \quad \text{if } \Delta_i \text{ and } \Delta_j \text{ are unbounded then } c_{i,j} \geq 0.
\]

This condition can be seen as an analog of (1.5) where we only consider the point at infinity in the intersection of \( \Delta_i \) and \( \Delta_j \) (in the Riemann sphere). Of course, it is more restrictive than the condition (3.4) but it has the advantage that, for a given extremal problem, it can be checked at once from the data if it holds true or not.

Remark 3.4. Condition (3.4) follows from (1.14) in the case of a matrix \( C \) of rank 1, for instance when considering a condenser as in [24, Chapter VIII]. Indeed, here necessarily \( C \) is a positive multiple of \( yy^t \) with the vector \( y \) as in (1.14). Thus \( c_{i,j} < 0 \) implies that \( y_i y_j < 0 \), and hence for all \( z \in \Delta_i \)

\[
U^{\mu_j}(z) \leq \|\mu_j\| \log \left( \frac{1}{\text{dist}(\Delta_i, \Delta_j)} \right).
\]

Consequently, (3.4) follows by recalling that \( Q_i \) is lower bounded. Hence, as in [24, Theorem VIII.1.4], we may conclude that the components of an extremal tuple of measures in (1.16) in the case rank(\( C \)) = 1 have compact support.

Proof. Suppose first that the support of the measures \( \mu_i \) are compact. Then, for \( M \) sufficiently large, the sets \( \text{supp} (\mu_i) \setminus D_M \) are empty sets so that (3.4) is trivially true.

Conversely, let us show that \( \mu_i \) has a compact support if (3.4) holds. We first establish a property of \( \mu_i \) similar to [20, Lemma 5.4.1], namely,

\[
\forall \nu_i \in \mathcal{M}_{\|\mu_i\|}(\Delta_i) : \int (U_i^\mu + Q_i)d(\nu_i - \mu_i) \geq 0. \quad (3.5)
\]
For a proof of (3.5), we define $\nu \in \mathcal{M}_K^d(\Delta)$ by $\nu_j = \mu_j$ for $j \neq i$. Notice that $\mu + t(\nu - \mu) \in \mathcal{M}_K^d(\Delta)$ for any $0 < t < 1$, and hence by definition of $\mu$
\begin{align*}
0 & \leq J_Q(\mu + t(\nu - \mu)) - J_Q(\mu) \\
& = 2t \int (U^\mu_i + Q_i)d(\nu_i - \mu_i) + t^2J(\nu - \mu).
\end{align*}
Dividing by $t$ and letting $t \to 0$ gives the desired inequality (3.5).

For our proof of compactness of $\text{supp} (\mu_i)$, we may suppose without loss of generality that $\Delta_i$ is unbounded, $\|\mu_i\| > 0$, and that $\mu_i(D_M) > 0$, where for the last property we possibly choose a larger $M$. We consider
\begin{equation*}
\nu_i := \frac{\|\mu_i\|}{\mu_i(D_M)}\mu_i|_{D_M}
\end{equation*}
being clearly an element of $\mathcal{M}_{\|\mu_i\|}(\Delta_i)$. Then we may rewrite condition (3.5) as
\begin{align*}
\left(\frac{\|\mu_i\|}{\mu_i(D_M)} - 1\right) \int_{|z| \leq M} (U^\mu_i + Q_i)d\mu_i - \int_{|z| > M} (U^\mu_i + Q_i)d\mu_i \geq 0,
\end{align*}
or
\begin{equation}
(\|\mu_i\| - \mu_i(D_M))\alpha_0 \geq \|\mu_i\| \int_{|z| > M} (U^\mu_i(z) + Q_i(z))d\mu_i(z),
\end{equation}
with the finite constant
\begin{equation*}
\alpha_0 := \int (U^\mu_i(z) + Q_i(z))d\mu_i(z).
\end{equation*}
It remains to show that $U^\mu_i(z) + Q_i(z)$ is sufficiently large $\mu_i$–almost everywhere on $\Delta_i \setminus D_M$.

For this, notice first that, by possibly making $\alpha$ smaller and $M$ larger, (3.4) also holds for all indices $j$ with $c_{i,j} < 0$ and compact $\Delta_j$ since then $\text{supp} (\mu_j)$ is compact. In case $c_{i,j} \geq 0$ we use (1.13) to conclude that, for $\mu_i$–almost all $z \in \Delta_i \setminus D_M$,
\begin{align*}
U^\mu_i(z) + Q_i(z) & \geq \sum_{j, c_{i,j} \geq 0} c_{i,j}U^\mu_j(z) + \sum_{j, c_{i,j} < 0} (\alpha - \frac{1}{d}Q_i(z)) + Q_i(z) \\
& \geq - \sum_{j, c_{i,j} \geq 0} c_{i,j}\|\mu_j\| \log(1 + |z|) + \frac{1}{d}Q_i(z) + \alpha_1
\end{align*}
for some constant $\alpha_1$. Here we have used the fact that $c_{i,i} \geq 0$. According to the third condition of admissibility in Definition 1.6, i.e. the behavior of $Q_i$ at infinity, we may now possibly choose a larger $M$ such that $U^\mu_i(z) + Q_i(z) \geq (\alpha_0 + 1)/\|\mu_i\|$ for $\mu_i$–almost all $z \in \Delta_i \setminus D_M$. Hence inequality (3.6) becomes
\begin{equation*}
(\|\mu_i\| - \mu_i(D_M))\alpha_0 \geq (\|\mu_i\| - \mu_i(D_M))(\alpha_0 + 1),
\end{equation*}
implicating that $\|\mu_i\| = \mu_i(D_M)$, and the fact that $\mu_i$ has compact support. □
4 Uniqueness and equilibrium conditions

Proof of Theorem 1.8(a). Our proof relies on Proposition 2.9, but otherwise the arguments of [20] or [2, Proof of Theorem 1.1] carry over to our more general setting.

It is sufficient to show that the application \( \mu \mapsto J_Q(\mu) \) is strictly convex in the convex subset of \( \mathcal{M}_K(\Delta) \) consisting of \( d \)-tuples of measures \( \mu \) with finite \( J_Q \)-energy. By finiteness of the \( J \)-energy on \( \mathcal{M}_K(\Delta) \), that simply boils down to

\[
\int Q_j d\mu_j < \infty, \quad j = 1, \ldots, d.
\]

For two distinct \( d \)-tuples of measures \( \mu \) and \( \nu \) of finite \( J_Q \)-energies, we have

\[
\frac{1}{2} \left( J_Q(\mu) + J_Q(\nu) \right) - J_Q \left( \frac{\mu + \nu}{2} \right) = \frac{1}{2} \left( J(\mu) + J(\nu) \right) - J \left( \frac{\mu + \nu}{2} \right) = \sum_{i,j=1}^{d} c_{ij} \left( \frac{1}{2} \left( I(\mu_i, \mu_j) + I(\nu_i, \nu_j) \right) - I \left( \frac{\mu_i + \nu_i}{2}, \frac{\mu_j + \nu_j}{2} \right) \right) = J \left( \frac{\mu - \nu}{2} \right),
\]

and it only remains to show that the last term is positive. By the definition (1.17) of \( K \), the vector of masses \( \|\mu\| - \|\nu\| \) is an element of the kernel of the matrix \( A \), which by (2.4) and (1.18) is a subset of the kernel of \( C \) and thus of \( B \). Hence the strict convexity follows from Proposition 2.9.

Remark 4.1. There are other sufficient conditions to ensure strict convexity of the map \( \mu \mapsto J_Q(\mu) \) on \( d \)-tuples of measures of \( \mathcal{M}_K(\Delta) \) of finite \( J_Q \)-energy, for instance we may replace (1.18) by the requirement that the union of the \( \Delta_j \) is compact, with capacity less than 1. Another sufficient condition for strict convexity, namely

\[
\forall i \neq j, \quad \text{if} \quad \Delta_i \cap \Delta_j \neq \emptyset \quad \text{then} \quad c_{ij} = 0, \quad (4.1)
\]

has been considered in [2, 16]. Notice that (4.1) is stronger than (1.5), and that (1.5) alone does not imply strict convexity, see Example 1.3.

Remark 4.2. We claim that if there is equality in assumption (1.18) then (1.14) holds. To see this, notice that from the full rank decomposition \( C = B^tB \) and from the assumption \( \text{Im} (C) = \text{Im} (A) \) we conclude that there exists a matrix \( E \) such that \( A = EB \), implying that we may rewrite the non-empty compact \( K \) as \( K = \{ x \in \mathbb{R}_+^d, Bx = b \} \) for a suitable vector \( b \in \mathbb{R}^r \). Writing \( e = (1, \ldots, 1)^t \in \mathbb{R}^d \), we conclude that the linear optimization problem \( \max \{ e^t x, Bx = b, x \geq 0 \} \) has an optimal solution. In particular (6, Theorem 19.12), there is a Lagrange multiplier \( \lambda \in \mathbb{R}^r \) with \( B^t \lambda \geq e \). Hence \( y := B^t \lambda \) is an element of \( \text{Im} (C) = \text{Im} (B^t) \) with strictly positive components, implying (1.14).

Before entering the details of the proof of assertion (b) of Theorem 1.8, we shortly comment on the equilibrium conditions (1.19) and (1.20). First recall from Lemma 2.3 that the potentials \( U^{(j)} \) for \( j = 1, \ldots, d \) are finite and hence \( U^{(j)} + Q_i \) is well-defined in
\[ \Delta_i \setminus \Delta_{i,\infty} \text{ with } \Delta_{i,\infty} \subset \Delta_i \text{ some polar Borel set. Also, } U_i^\mu + Q_i \text{ as a sum of measurable functions is measurable, and hence both sets } \\
\Delta_{i,+} = \{ z \in \Delta_i \setminus \Delta_{i,\infty} : U_i^\mu(z) + Q_i(z) > (A^iF)_i \}, \\
\Delta_{i,-} = \{ z \in \Delta_i \setminus \Delta_{i,\infty} : U_i^\mu(z) + Q_i(z) < (A^iF)_i \}, \]

are Borel sets. Hence \([1.19]\) means that \(\Delta_{i,\infty} \cup \Delta_{i,+} \) is polar, whereas \([1.20]\) can be equivalently rewritten as \(\mu_i(\Delta_{i,\infty} \cup \Delta_{i,+}) = 0\).

As in [20, Lemma 5.4.2] we have to establish a different characterization of an extremal tuple of measures which generalizes (3.5).

**Lemma 4.3.** The \(d\)-tuple of measures \(\mu = (\mu_1, \ldots, \mu_d) \in \mathcal{M}_K^d(\Delta)\) with \(J_Q(\mu) < \infty\) is extremal for (1.16) if and only if for any \(d\)-tuple of measures \(\nu = (\nu_1, \ldots, \nu_d) \in \mathcal{M}_K^d(\Delta)\) with \(J_Q(\nu) < \infty\) we have

\[ \sum_{i=1}^d \int (U_i^\mu + Q_i) d\nu_i \geq \sum_{i=1}^d \int (U_i^\mu + Q_i) d\mu_i. \] (4.2)

**Proof.** In order to see that (4.2) is necessary for optimality, notice that, for all \(0 < t \leq 1\), we have \(\mu + t(\nu - \mu) \in \mathcal{M}_K^d(\Delta)\), with

\[ J_Q(\mu + t(\nu - \mu)) - J_Q(\mu) = 2t \sum_{i=1}^d \int (U_i^\mu + Q_i) d(\nu_i - \mu_i) + t^2 J(\nu - \mu) \] (4.3)

being nonnegative. Dividing by \(t\) and letting \(t \to 0\) gives (4.2). Conversely, we recall from Proposition 2.9 that \(J(\nu - \mu) \geq 0\). Injecting (4.2) into (4.3) for \(t = 1\), we conclude as required that \(\mu\) is extremal. \(\square\)

**Proof of Theorem 1.8(b).** Suppose first that \(\mu \in \mathcal{M}_K^d(\Delta)\) satisfies (1.19) and (1.20). Then \(\mu_i(\Delta_{i,\infty} \cup \Delta_{i,+}) = 0\), and integrating (1.20) with respect to \(\mu_i\) shows that \(J_Q(\mu) < \infty\). Let now \(\nu \in \mathcal{M}_K^d(\Delta)\) with \(J_Q(\nu) < \infty\). Then \(\nu_i(\Delta_{i,\infty} \cup \Delta_{i,-}) = 0\) by Lemma 2.2. Hence integrating (1.19) with respect to \(\nu_i\) and (1.20) with respect to \(\mu_i\) gives

\[ \sum_{i=1}^d \int (U_i^\mu + Q_i) d\nu_i - \sum_{i=1}^d \int (U_i^\mu + Q_i) d\mu_i \geq \sum_{i=1}^d (\|\nu_i\| - \|\mu_i\|)(A^iF)_i = 0, \]

the last equality following from the definition of the polyhedron of masses \(K\). Hence \(\mu\) is extremal according to Lemma 4.3.

Suppose now that \(\mu \in \mathcal{M}_K^d(\Delta)\) is extremal. Consider the set of indices \(I = \{ i \in \{1, \ldots, d\} : \|\mu_i\| > 0 \}\), set for \(i \in I\)

\[ w_i := \frac{1}{\|\mu_i\|} \int (U_i^\mu + Q_i) d\mu_i, \]

and consider as before the Borel sets \(\Delta_{i,+} = \{ z \in \Delta_i \setminus \Delta_{i,\infty} : U_i^\mu(z) + Q_i(z) > w_i \}\) and \(\Delta_{i,-} = \{ z \in \Delta_i \setminus \Delta_{i,\infty} : U_i^\mu(z) + Q_i(z) < w_i \}\). Following [20, Proposition 5.4.4], we claim that, for \(i \in I\),

\[ U_i^\mu(x) + Q_i(x) \geq w_i, \quad \text{quasi–everywhere on } \Delta_i. \] (4.4)
Suppose the contrary for some \( i \in I \). Since \( \Delta_{i,\infty} \) is polar, we conclude that \( \Delta_{i,-} \) is of positive capacity. Thus there exists a compact set \( E \subset \Delta_i \) with \( U_{i}^\mu \) well-defined and finite on \( E \), \( \text{cap}(E) > 0 \), and \( U_{i}^\mu(x) + Q_i(x) < w_i \) for all \( x \in E \). Taking any \( \nu_i \in \mathcal{M}_{\|\mu_i\|}(E) \), then with \( \nu_j = \mu_j \) for \( j \neq i \) we get \( \nu \in \mathcal{M}_R^\nu(\Delta) \) and, by Lemma 4.3
\[
0 \leq \sum_{t=1}^{d} \int (U_{i}^\mu + Q_i) d(\nu_t - \mu_t) = \int (U_{i}^\mu + Q_i) d\nu_t - \|\nu_t\| w_i,
\]
but the term on the right is negative by construction of \( E \) and \( \nu_t \), a contradiction. Thus also (4.5) is true. Thus we have shown so far that, for indices
\[
\|\mu_i\| w_i, \quad \mu_i \text{-almost everywhere.} \tag{4.5}
\]
Suppose the contrary for some \( i \in I \). Since \( \mu_i(\Delta_{i,\infty}) = 0 \) by Lemma 2.2, we get \( \mu_i(\Delta_{i,+}) > 0 \). Applying, e.g., [23, Theorem 2.18], we conclude that there exists a compact set \( E \subset \Delta_i \) with \( U_{i}^\mu \) well-defined and finite on \( E \), \( \mu_i(E) > 0 \), and \( U_{i}^\mu(x) + Q_i(x) > w_i \) for all \( x \in E \). A combination of Lemma 2.2 with (4.4) tells us that
\[
\int_{\Delta_i \setminus E} (U_{i}^\mu + Q_i) d\mu_i \geq w_i \mu_i(\Delta_i \setminus E),
\]
and thus
\[
\|\mu_i\| w_i \geq \int_{E} (U_{i}^\mu + Q_i) d\mu_i + w_i \mu_i(\Delta_i \setminus E) > w_i \mu_i(E) + w_i \mu_i(\Delta_i \setminus E),
\]
a contradiction. Hence also (4.5) is true. Thus we have shown so far that, for indices \( i \) with \( \|\mu_i\| > 0 \), (1.19) and (1.20) hold true if we replace \( (A^t F)_i \) by a suitable constant \( w_i \in \mathbb{R} \). It remains to relate these constants \( w_i \) with \( A \) and also to discuss the partial potentials \( U_{i}^\mu \) for indices \( i \) such that \( \|\mu_i\| = 0 \). For this purpose, similar to [2, Part 3 of proof of Theorem 1.2], we consider the quadratic optimization problem in \( \mathbb{R}^d \),
\[
\min \{ x^t H x + 2h^t x, \quad x \in K \},
\]
where \( H \in \mathbb{R}^{d \times d} \) and \( h \in \mathbb{R}^d \) with
\[
H_{i,j} = c_{i,j} I(\nu_i, \nu_j), \quad h_i = \int Q_i d\nu_i, \quad i, j = 1, \ldots, d
\]
and the probability measures \( \nu_i \in \mathcal{M}_1(\Delta_i) \) are defined by \( \nu_i = \mu_i/\|\mu_i\| \) if \( \|\mu_i\| \neq 0 \), and else arbitrary but fixed. Then, by Theorem 1.8(a), \( \|\mu\| \in K \) is the unique solution of the above quadratic problem. From [6, Theorem 19.12] we know that there exist Lagrange multipliers \( F \in \mathbb{R}^m \) and \( G \in \mathbb{R}^d \) such that
\[
H\|\mu\| + h = A^t F + G, \quad \forall i, \quad G_i \geq 0, \quad \|\mu_i\| G_i = 0. \tag{4.6}
\]
In case \( \|\mu_i\| \neq 0 \) we find from (4.4) and (4.5) that
\[
(H\|\mu\| + h)_i = \sum_{j=1}^{d} c_{i,j} I\left( \frac{\mu_i}{\|\mu_i\|}, \mu_j \right) + \int Q_i \frac{d\mu_i}{\|\mu_i\|} = \int (U_{i}^\mu + Q_i) \frac{d\mu_i}{\|\mu_i\|} = w_i.
\]
Also, $G_i = 0$, and hence $(H\|\mu\| + h)_i = w_i = (A^tF)_i$. In particular, relations (4.4) and (4.5) imply the desired relations (1.19) and (1.20). In case $\|\mu_i\| = 0$ we learn from (4.6) that

$$\forall \nu_i \in \mathcal{M}_1(\Delta_i), \quad \int (U^\mu_i + Q_i) d\nu_i \geq (A^tF)_i,$$

implying (1.19), and assertion (1.20) is trivially true.

5 A review of some examples

Many recently studied problems, such as e.g. the behavior of Hermite-Padé approximants, the limit eigenvalue distribution of banded Toeplitz matrices, or the limit distribution of non-intersecting brownian paths, translate into vector equilibrium problems with external fields. Existence and uniqueness of the solution were shown under conditions that are actually covered by the results of the previous sections. The above-mentioned equilibrium problems can be stated in terms of graphs. We recall that for a graph $G = (V, E)$, the set of edges $E$ is a subset of the cartesian product $V \times V$, where $V$ denotes the set of vertices. For multigraphs we allow for repeated edges between two given vertices. We also remind the reader that the incidence matrix $A$ is labelled in rows by vertices and in columns by edges, with a column corresponding to an edge from the vertex $u$ to the vertex $v$ has entry $-1$ at row $u$, 1 at row $v$ and 0 elsewhere.

In what follows, we always suppose that a graph or a multigraph $G = (V, E)$ is given. We denote its incidence matrix by $A$ and we consider as interaction matrix the matrix $C = A^tA$, together with the polyhedron of masses $K = \{ x \in \mathbb{R}^d_+, Ax = a \}$. In what follows $K$ is supposed to contain at least one element with strictly positive components. For instance, for the graph of figure 1, we have

$$A = \begin{pmatrix}
-1 & -1 & 0 \\
0 & 1 & 1 \\
1 & 0 & -1
\end{pmatrix}, \quad C = \begin{pmatrix}
2 & 1 & -1 \\
1 & 2 & 1 \\
-1 & 1 & 2
\end{pmatrix}.$$  \hspace{1cm} (5.1)

Figure 1: A graph with undirected cycle but no directed cycle

As a consequence, the interaction matrix $C$ is indexed in rows and columns by the edges and it can be checked that its entries are $-2, -1, 0, 1, 2$ with the following interpretation

$$C_{\alpha,\beta} = \begin{cases}
2 & \text{if } \alpha = \beta \\
1 & \text{if } \alpha \rightarrow \beta \\
-1 & \text{if } \beta \rightarrow \alpha \\
-2 & \text{if } \beta \rightarrow \alpha \\
0 & \text{elsewhere (i.e. } \alpha, \beta \text{ do not have any vertex in common)}
\end{cases}.$$
By construction, the matrix $C$ is always positive semi-definite. To each edge $i$ we associate a closed set $\Delta_i$ and a measure $\mu_i$ supported on $\Delta_i$.

We can interpret the different assumptions we made in the previous sections about the matrix $C$ and the supports $\Delta_i$ in terms of graph theory.

**Proposition 5.1.** The following assertions hold true:

(a) The following three statements are equivalent: (i) matrix $C$ is invertible; (ii) $G$ has no undirected cycle; (iii) the polyhedron of masses $K = \{x \in \mathbb{R}^d_+; Ax = a\}$ is a singleton.

(b) The polyhedron of masses $K$ is compact if and only if $G$ has no directed cycle.

(c) Condition (1.5) is equivalent to the fact that any two edges which follow each other correspond to non intersecting sets $\Delta_i$ and $\Delta_j$.

(d) Condition (4.1) is equivalent to the fact that any two distinct edges corresponding to intersecting sets $\Delta_i$ do not have any vertex in common.

(e) Condition (1.15) is equivalent to:

\[
\forall \text{ set } I \text{ of edges of } \mathcal{E} \text{ forming an undirected cycle in } G, \ \text{cap} (\bigcap_{\alpha \in I} \Delta_\alpha) = 0.
\]

(f) Let $G^*$ be the undirected intersection graph of the sets $\{\Delta_i\}_{i=1}^d$ that is, the vertices of $G^*$ are the edges of $G$ and there is an edge in $G^*$ between $i$ and $j$ if the corresponding sets $\Delta_i$ and $\Delta_j$ are intersecting. Condition (1.14) is equivalent to:

each connected component of $G^*$ corresponds to a subgraph in $G$ that does not contain a directed cycle.

We do not present here complete proofs for these assertions which follow from graph theory. Notice however that (a) is based on the classical fact that the rank of an incidence matrix is given by the number of its columns iff the underlying graph has no undirected cycle. Assertions (c) and (d) immediately follow from the above graph interpretation of the entries of $C$.

Condition (4.1) is obviously stronger than (1.5). From the graph theory interpretation given in assertions (d) and (e) we see that (4.1) implies (1.15). From assertions (b) and (f) we see that the compactness of the polyhedron $K$ implies (1.14), as noticed already in Remark 4.2.

The first vector equilibrium problems using the terminology of graphs were studied in [16], where systems of Markov functions generated by a rooted tree $G = (\mathcal{V}, \mathcal{E})$, the so-called generalized Nikishin systems, were considered. Recall that a tree is a connected graph without undirected cycles. In particular, by properties (a), (e), and (f) of Proposition 5.1, $C$ is invertible and conditions (1.14) and (1.15) are satisfied. So the result [16, Theorem 1] also follows from our work, and we may drop in [16, Theorem 1] any further requirements on the sets $\Delta_j$ like (4.1) or (1.5). The authors associate to each vertex in $\mathcal{V}$ a Markov function, and to each edge $\alpha$ in $\mathcal{E}$ a measure with support in an interval $\Delta_\alpha$. 


This class includes the well-known Nikishin systems, see Figure 2 (a), and the Angelesco systems, see Figure 2 (b), with interaction matrices $C$ respectively given by

$$
\begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{pmatrix},
\begin{pmatrix}
2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{pmatrix}.
$$

The solution of their extremal problem is related to the limit distributions of the zeros of the polynomial denominators of the Hermite-Padé approximants to the generalized Nikishin systems.

In [2], the results of [16] were generalized to rooted multigraphs $G = (V, E, O)$ with a root $O$, that is, multigraphs which have no directed cycles but do have directed paths from $O$ to any other vertex. An example of such a graph with undirected cycles is shown in Figure 3. By generalizing the ideas of [16], the graph is associated to a system of Markov functions with intersecting supports. According to assertion (f) of Proposition 5.1, condition (1.14) holds since there are no directed cycles. Also, as said before, the condition (4.1) imposed in [2] implies (1.15). Thus [2, Theorem 1.1] dealing with $K$ as in (1.17) is covered by our work as well. The Hermite-Padé approximants to specific systems of Markov functions related to graphs with cycles were also investigated in [26] in connection with applications to number theory.

Another vector equilibrium problem appears in [1] and [3] in the study of the asymptotics of diagonal simultaneous Hermite-Padé approximants to two analytic functions with separated pairs of branch points. The authors define the class $H(C \setminus \Gamma)$ of holomorphic functions in $C \setminus \Gamma$, where $\Gamma$ is a piecewise analytic arc joining two points $a$ and $b$ in $C$. A typical example of such a function is

$$
f(z) = \log \left( \frac{z - a}{z - b} \right).
$$

For $f_1 \in H(C \setminus \Gamma_1)$, $f_2 \in H(C \setminus \Gamma_2)$, with

$$
\Delta_1 = \Gamma_1, \quad \Delta_2 = \text{Clos}(\Gamma_2 \setminus \Gamma_1),
$$

and $\Delta_3$ a piecewise analytic arc containing the intersection $\Delta_1 \cap \Delta_2$, they show the existence and uniqueness of a triple of measures

$$\mu = (\mu_1, \mu_2, \mu_3) \text{ with } \text{supp}(\mu_i) \subset \Delta_i, \quad i = 1, 2, 3,$$

minimizing the energy $J(\mu)$, where the interaction matrix $C$ is given in (5.1), corresponding to the graph in Figure 1 and the set of masses is given by

$$K = \left\{ x \in \mathbb{R}^3_+, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} = \left\{ x \in \mathbb{R}^3_+, \ Ax = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Notice that this graph contains an undirected cycle but, since $\text{cap}(\Delta_1 \cap \Delta_2) = 0$, we are again in the settings of our theorems. The measure $\mu_1 + \mu_2$ is the limit distribution of the poles of the diagonal simultaneous Hermite-Padé approximants of the functions $(f_1, f_2)$, and the measure $\mu_3$ describes the limit distribution of the extra interpolation points to $f_1$.

In [9], the limit distribution of non-intersecting one-dimensional Brownian paths with prescribed starting and ending points is also characterized by a vector equilibrium problem. As explained in [9], there is an underlying undirected graph $G_u$ whose edges connect vertices in the set of starting points with vertices in the set of ending points, that is, a bipartite graph. The authors show, in addition, that their graph is a tree, see [9, Proposition 2.1]. In [9, Corollary 2.9.], they establish existence and uniqueness of a solution to an extremal vector equilibrium problem with interaction matrix $C = (B'B)/2$, $B$ being the incidence matrix of $G_u$, with quadratic external fields, fixed masses, and sets $\Delta_j = \mathbb{R}$. The supports of the extremal measures are compact, and describe the limiting behavior of such non-intersecting one-dimensional Brownian paths. In order to relate [9, Corollary 2.9.] to our findings, notice that, by considering the natural orientation of edges from starting points to ending points, we get a directed graph $G$ which is both a tree and a bipartite graph, see the example in Figure 4. Using this last property, it is not difficult to see that $B'B = A'A$ with $A$ the incidence matrix of $G$. Thus, we learn from assertion (a) of Proposition 5.1 that $C$ is invertible, see also [2, Proposition 2.8], and that $K$ is a singleton. In particular, both conditions (1.14) and (1.15) are true and even the condition (1.5) holds. Nevertheless, [9, Corollary 2.9.] is not a consequence of [20, Chapter 5] since the sets $\Delta_j$ are not compact. However, the quadratic external fields of [9] are admissible in the sense of our Definition 1.6 and thus existence, uniqueness and equilibrium conditions for an extremal tuple of measures also follow from our general findings. Note also that the compactness of the supports of these extremal measures follows from Remark 3.3 since all entries of $C$ are non negative.
6 Conclusion

In this paper we have shown existence and uniqueness of an extremal tuple of measures for a vector generalization of a weighted energy problem in logarithmic potential theory with a polyhedron of masses, substantially weakening the assumptions typically assumed in other papers on this subject. We have also derived a characterisation of such an extremal tuple of measure in terms of equilibrium conditions for the vector potentials.

We have not been able to prove in our general setting that the supports of the components of the extremal tuple of measure are always compact. We conjecture that, because of the growth of the external field at infinity and condition (1.14), it should be true. In any case, we note that the variational inequality (1.20) implies that the potentials $U_{\mu_j}$ such that $c_{i,j} < 0$ satisfy $U_{\mu_j}(z)/\log |z| \to \infty$ as $z \in \Delta_i$ tends to infinity (up to a set of $\mu_i$-measure zero). Hence, in view of assertion (ii) of [18, Theorem 5.7.15], we may at least conclude that the support of $\mu_i$ is the union of a set of $\mu_i$-measure zero and a set thin at infinity.

There are also examples of vector–valued extremal problems in logarithmic potential theory where the external fields have a slow increase near $\infty$, or are even not present. For instance, in [12], the authors describe the limiting eigenvalue distribution of banded Toeplitz matrices. It is obtained as a component of the solution of a vector equilibrium problem with a positive definite interaction matrix $C$ (namely the one of a Nikishin system), without any external field at all. Also, in [10], these results have been extended to Toeplitz matrices with rational symbol, and in this case the vector equilibrium problem includes external fields of the form $Q(z) = C \log(|z|)$. In these examples, it may happen that the extremal measures do not have a bounded support. For a general analysis of such examples, one should work on the Riemann sphere instead of the complex plane, see the recent contribution [17] in case of positive definite $C$.

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References

[1] A. I. Aptekarev, Asymptotics of Hermite-Padé Approximants for two functions with branch points, Doklady Mathematics 78, 717-719 (2008).

[2] A.I. Aptekarev, V.G. Lysov, Systems of Markov functions generated by graphs and the asymptotics of their Hermite-Padé approximants, Sbornik Mathematics 201, 183-234 (2010).

[3] A.I. Aptekarev, A.B.J. Kuijlaars, W. Van Assche, Asymptotics of Hermite-Padé rational approximants for two analytic functions with separated pairs of branch points (case of genus 0), Int. Math. Res. Pap. IMRP 2008, Art. ID rpm007, 128pp.

[4] F. Balogh, M. Bertola, Regularity of a vector potential problem and its spectral curve, J. Approx. Theory 161, 353–370 (2009).
[5] P. Billingsley, Convergence of probability measures, 2nd Ed., Wiley, New York, 1999.

[6] J.-F. Bonnans, J.-Ch. Gilbert, Cl. Lemarchal, C.A. Sagastizábal, Numerical Optimization, Universitext, Springer-Verlag, Berlin, 2006.

[7] U. Cegrell, S. Kolodziej, N. Levenberg, Two problems on potential theory for unbounded sets, Math. Scand. 83, 265–276 (1998).

[8] P. Deift, Orthogonal Polynomials and Random Matrices: a Riemann-Hilbert approach. Courant Lecture Notes in Mathematics Vol. 3, Amer. Math. Soc., Providence R.I. 1999.

[9] S. Delvaux, A.B.J. Kuijlaars, A graph-based equilibrium problem for the limiting distribution of non-intersecting Brownian motions at low temperature, Constr. Approx. 32, 467–512 (2010).

[10] S. Delvaux, M. Duits, An equilibrium problem for the limiting eigenvalue distribution of rational Toeplitz matrices, SIAM J. Matrix Anal. 31, 1894–1914 (2010).

[11] R.M. Dudley, Real Analysis and Probability, Cambridge Univ. Press, Cambridge, 2002.

[12] M. Duits, A.B.J. Kuijlaars, An equilibrium problem for the limiting eigenvalue distribution of banded Toeplitz matrices, SIAM J. Matrix Anal. Appl. 30, 173–196 (2008).

[13] U. Fidalgo Prieto, G. Lopez Lagomasino, Rate of convergence of generalized Hermite-Padé approximants of Nikishin systems, Constr. Approx. 23, 165–196 (2006).

[14] A.A. Gonchar, E.A. Rakhmanov, On equilibrium problems for vector potentials, Russian Math. Surveys, 40, 183–184 (1985).

[15] A.A. Gonchar, E.A. Rakhmanov, Equilibrium measure and the distribution of zeros of extremal polynomials, Math USSR Sb. 53, 119–130 (1986).

[16] A.A. Gonchar, E.A. Rakhmanov, V.N. Sorokin, Hermite-Padé approximants for systems of Markov-type functions, Sb. Math, 188, 671–696 (1997).

[17] A. Hardy, A.B.J. Kuijlaars, Weakly admissible vector equilibrium problems, manuscript 2011.

[18] L.L. Helms, Potential Theory, Universitext, Springer-Verlag, Berlin, 2009.

[19] A.L. Levin and D.S. Lubinsky, Green equilibrium measures and representations of an external field, J. of Approx.theory 113, 298-323 (2001).

[20] E. M. Nikishin, V. N. Sorokin, Rational Approximations and Orthogonality, Transl. Amer. Math. Soc., Vol. 92, Providence, R.I., 1991.

[21] M. Ohtsuka, On potentials in locally compact spaces, J. Sci. Hiroshima Univ., ser. A 1, 135-352 (1961).
A Appendix

Proof of Lemma 2.2. Assume E is a Borel set such that \( \mu(E) > 0 \). By regularity of \( \mu \), there exists a compact subset \( K \) of \( E \) with \( \mu(K) > 0 \). Set \( \tilde{\mu} = \mu|_K \). Then,

\[
I(\tilde{\mu}) = I(\mu) + \int_{\mathbb{C} \setminus K} \int_{\mathbb{C}} \log(|z - t|)d\mu(z)d\mu(t) + \int_K \int_{\mathbb{C} \setminus K} \log(|z - t|)d\mu(z)d\mu(t)
\]

\[
\leq I(\mu) + 4 \int_{\mathbb{C}} \int_{\mathbb{C}} \log(1 + |t|)d\mu(t)d\mu(z),
\]

which shows that \( I(\tilde{\mu}) < \infty \) and thus \( \text{cap}(E) > 0 \).

Proof of Lemma 2.4. By [22, Theorem 2.1.3], there exists an increasing sequence of continuous functions \( \tilde{h}_m \) which converges pointwise to \( Q \). Assume \( Q \) is lower bounded by \( c \in \mathbb{R} \). Set \( \tilde{h}_m = \min(c + m, \max(c, h_m)) \).

Then, \( (\tilde{h}_m)_m \) is an increasing sequence of continuous bounded functions that still tends pointwise to \( Q \) and we have

\[
\lim \inf_{n \to \infty} \int Qd\mu_n \geq \lim_{m \to \infty} \lim \inf_{n \to \infty} \int \tilde{h}_m d\mu_n = \lim_{m \to \infty} \int \tilde{h}_m d\mu = \int Qd\mu,
\]

where in the last equality we use the monotone convergence theorem.
Proof of Lemma 2.8. Let $\epsilon > 0$ be given and let $M > 1$ be such that
\[
\forall n \geq 0, \quad \int \int_{|x-y| \geq M} \log(|x - y|) d\mu_n(x) d\nu_n(y) \leq \epsilon.
\]
Note that the existence of $M$ follows from the simple inequalities
\[
0 \leq \log(|x - y|) \leq \log(1 + |x|) + \log(1 + |y|),
\]
satisfied for $|x - y| \geq 1$, the fact that the masses of the measures are uniformly bounded, and the log-tightness of the sequences. We also set $h(t)$ for a continuous function on $\mathbb{R}_+$ such that
\[
0 \leq h(t) \leq 1, \quad h(t) = 1 \text{ for } t \leq M, \quad h(t) = 0 \text{ for } t \geq M + 1.
\]
Then, we have
\[
I(\mu_n, \nu_n) = \int \int \log(|x - y|^{-1}) h(|x - y|) d\mu_n(x) d\nu_n(y) \\
+ \int \int \log(|x - y|^{-1})(1 - h(|x - y|)) d\mu_n(x) d\nu_n(y).
\]
(A.1)

On one hand, the cartesian product measure $\mu_n \times \nu_n$ tends weakly to $\mu \times \nu$, see [3, Theorem 2.8] or \[11\, Theorem 9.5.9\], and the integrand in the first integral is lower semi-continuous and lower bounded on $\mathbb{C}$. Hence, Lemma 2.4 applies (more precisely a version of it on $\mathbb{C}^2$ which holds true as well). On the other hand, the second integral has a modulus less than $\epsilon$ uniformly in $n$. Consequently,
\[
\liminf_{n \to \infty} I(\mu_n, \nu_n) \geq \int \int \log(|x - y|^{-1}) h(|x - y|) d\mu(x) d\nu(y) - \epsilon \\
= I(\mu, \nu) - \int \int \log(|x - y|)(1 - h(|x - y|)) d\mu(x) d\nu(y) - \epsilon.
\]
The integrand in the last integral is continuous and lower bounded on $\mathbb{C}$. Hence, again by Lemma 2.4, this integral is less than
\[
\liminf_{n \to \infty} \int \int \log(|x - y|)(1 - h(|x - y|)) d\mu_n(x) d\nu_n(y) \leq \epsilon,
\]
which implies
\[
\liminf_{n \to \infty} I(\mu_n, \nu_n) \geq I(\mu, \nu) - 2\epsilon.
\]
Since $\epsilon > 0$ is arbitrary, (2.3) follows. \qed

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