Double Explore-then-Commit: Asymptotic Optimality and Beyond

Tianyuan Jin\textsuperscript{*} and Pan Xu\textsuperscript{†} and Xiaokui Xiao\textsuperscript{‡} and Quanquan Gu\textsuperscript{§}

Abstract

We study the two-armed bandit problem with subGaussian rewards. The explore-then-commit (ETC) strategy, which consists of an exploration phase followed by an exploitation phase, is one of the most widely used algorithms in a variety of online decision applications. Nevertheless, it has been shown in Garivier et al. (2016) that ETC is suboptimal in the asymptotic sense as the horizon grows, and thus, is worse than fully sequential strategies such as Upper Confidence Bound (UCB). In this paper, we argue that a variant of ETC algorithm can actually achieve the asymptotically optimal regret bounds for multi-armed bandit problems as UCB-type algorithms do. Specifically, we propose a double explore-then-commit (DETC) algorithm that has two exploration and exploitation phases. We prove that DETC achieves the asymptotically optimal regret bound as the time horizon goes to infinity. To our knowledge, DETC is the first non-fully-sequential algorithm that achieves such asymptotic optimality. In addition, we extend DETC to batched bandit problems, where (i) the exploration process is split into a small number of batches and (ii) the round complexity is of central interest. We prove that a batched version of DETC can achieve the asymptotic optimality with only constant round complexity. This is the first batched bandit algorithm that can attain asymptotic optimality in terms of both regret and round complexity.

1 Introduction

We study how to conduct efficient exploration in the two-armed bandits problem. Our analysis focuses on a simple sequential decision problem played on \( T \) time steps. The decision maker chooses an arm \( A_t \in \{1, 2\} \) at time \( t \) and observes a reward \( r_t \) from a 1-subGaussian distribution with mean value \( \mu_{A_t} \), where \( \mu_1, \mu_2 \in \mathbb{R} \) are unknown. The performance of any strategy for the bandit problem is measured by its expected cumulative regret at time \( T \), i.e., \( R_\mu(T) \), which is defined as

\[
R_\mu(T) = T \max\{\mu_1, \mu_2\} - \mathbb{E}_\mu[\sum_{t=1}^T r_t].
\] (1.1)
For the two-armed bandit problem, Lai and Robbins (1985); Katehakis and Robbins (1995) showed that the regret of any strategy could not be smaller than $2 \log T / \Delta$ when $T$ approaches infinity, i.e.,

$$\lim \inf_{T \to \infty} R_\mu(T) / \log T \geq 2 / \Delta,$$  

(1.2)

where $\Delta = |\mu_1 - \mu_2|$ is the suboptimal gap between the mean rewards of the two arms. When $\Delta$ is a known parameter, the asymptotic lower bound (Garivier et al., 2016) can be improved to

$$\lim \inf_{T \to \infty} R_\mu(T) / \log T \geq 1 / (2\Delta).$$  

(1.3)

We denote $\lim_{T \to \infty} R_\mu(T) / \log T$ as the asymptotic regret rate. A strategy with an asymptotic regret rate no larger than $2 / \Delta$ for unknown gap or $1 / (2\Delta)$ for known gap is called asymptotically optimal.

The most natural approach for solving the above bandit problem is to first pull two arms alternately for a fixed number of times (referred to as the exploration stage), and then pull the arm with the larger average reward repeatedly (referred to as the exploitation stage). The length of the exploration stage can also be a data-dependent stopping time. Such strategies fall into the class of approaches named explore-then-commit (ETC) (Perchet et al., 2016; Garivier et al., 2016), which are simple and widely implemented in various online applications, such as clinical trials, crowdsourcing and marketing (Agarwal et al., 2017; Perchet et al., 2016; Gao et al., 2019). In addition, ETC-type strategies have been widely used in more complicated problems such as the batched bandit problem (Perchet et al., 2016; Agarwal et al., 2017; Gao et al., 2019; Jin et al., 2019). In the batched model, one is allowed to adaptively draw samples and adjust sampling strategy in rounds. In each round, one can query any number of arms, but the outcomes are only revealed at the end of the round. The goal in batched models is to minimize the regret as well as the number of rounds simultaneously. Regarding the regret analysis, Garivier and Kaufmann (2016) suggested that carefully-tuned variants of such two-phrase strategies might be near-optimal. Yet Garivier et al. (2016) later proved that such strategies are actually suboptimal in the sense that they cannot achieve the asymptotic optimal lower bounds as shown in (1.2) or (1.3).

In the current literature of multi-armed bandits algorithms, all existing asymptotically optimal strategies (such as UCB (Garivier and Cappé, 2011), Thompson Sampling (Agrawal and Goyal, 2017), Bayes UCB (Kaufmann, 2016)) are fully-sequential, which means they need to adjust the strategy adaptively based on the outcome at each step. However, this can be rather time consuming or even infeasible in real applications such as clinic trials, where it is impossible to measure the outcome (i.e., reward) for each patient before the treatment proceeds. In contrast, ETC-type strategies only consist of distinct exploration and exploitation stages, where outcomes are only needed at the stage switching time. Thus, a natural and open question is:

**Can non-fully-sequential strategies such as ETC achieve the asymptotic optimal regret?**

In this paper, we answer the above question affirmatively by proposing a double explore-then-commit (DETC) algorithm that consists of two exploration and two exploitation stages, which directly improves the ETC-type algorithms proposed in Garivier et al. (2016). The key idea of DETC is that after the first explore-then-commit phase, with high probability, we will commit the best arm whose average reward concentrates on its mean. In the second explore-then-commit phase, since one arm’s mean reward is already well estimated, we only need to explore the other arm. In this way, the sampling error in the final exploitation stage only comes from the other arm. In
contrast, in single ETC-type algorithms, the sampling error comes from both arms, which incurs suboptimal regret. Compared with UCB-type algorithms, our result suggests that it is not necessary to use the outcome at each time step to achieve the asymptotic optimality.

**Main Contributions.** We first study the case when the suboptimal gap $\Delta = |\mu_1 - \mu_2|$ is known. In this case, we prove that our proposed DETC algorithm achieves the asymptotically optimal regret rate $1/(2\Delta)$, the instance-dependent optimal regret $O(\log(T\Delta^2)/\Delta)$ and the minimax optimal regret $O(\sqrt{T})$. This result significantly improves the $4/\Delta$ asymptotic regret rate of ETC with fixed length and the $1/\Delta$ asymptotic regret rate of SPRT-ETC with data-dependent stopping time for the exploration stage proposed in Garivier et al. (2016). For the case $\Delta$ is unknown, we prove that the DETC strategy achieves the asymptotically optimal regret rate $2/\Delta$, the instance-dependent regret $O(\log(T\Delta^2)/\Delta)$ and the minimax optimal regret $O(\sqrt{T})$. This again improves the $4/\Delta$ asymptotic regret rate of the BAI-ETC algorithm proposed in Garivier et al. (2016). In both cases, this is the first time that the regrets of ETC-type algorithms have been proved to match the asymptotic lower bounds and therefore are asymptotically optimal. Moreover, Garivier et al. (2016) proved that the $1/\Delta$ asymptotic regret rate for the known gap case and the $4/\Delta$ asymptotic regret rate for the unknown gap case are not improvable in ‘single’ explore-then-commit algorithms, which justifies the essence of the double exploration technique in DETC in order to achieve the asymptotic regret.

We also study the batched bandits problem (Perchet et al., 2016) where the round complexity is of central interest. We prove that a simple variant of the proposed DETC algorithm can achieve $O(1)$ round complexity while maintaining the asymptotically optimal regret for two-armed bandits. This is the first batched bandit algorithm that achieves the asymptotic optimality in regret and the optimal round complexity.

**Notation** We denote $\log^+(x) = \max\{0, \log x\}$. We use notations $\lfloor x \rfloor$ (or $\lceil x \rceil$) to denote the largest integer that is no larger (or no smaller) than $x$. We use $O(T)$ to hide constants that are independent of $T$. A random variable $X$ is said to follow 1-subGaussian distribution, if it holds that $E_X[\exp(\lambda X - \lambda E_X[X])] \leq \exp(\lambda^2/2)$ for all $\lambda \in \mathbb{R}$.

## 2 Related Work

For regret minimization in stochastic bandit problems, Lai and Robbins (1985) proved the first asymptotically lower bound that any strategy must have at least $C(\mu) \log(T)(1 - o(1))$ regret when the horizon $T$ approaches infinity, where $C(\mu)$ is a constant. Later, strategies such as UCB (Garivier and Cappé, 2011), Thompson Sampling (Agrawal and Goyal, 2017) and Bayes UCB (Kaufmann, 2016) are all shown to be asymptotically optimal for unknown suboptimal gap. For the known gap setting, Garivier et al. (2016) developed the $\Delta$-UCB algorithm that matches the lower bound. For $K$-armed bandits with finite horizon $T$, the problem-independent lower bound (Auer et al., 2002) states that any strategy has at least a regret in the order of $O(\sqrt{KT})$, which is called the minimax-optimal regret. MOSS proposed by Audibert and Bubeck (2009) is the first strategy that is minimax optimal for $K$-armed bandits. Furthermore, Ada-UCB (Lattimore, 2018) algorithm is proved to be asymptotically optimal and almost problem-dependent optimal. For two-armed bandits, an algorithm with regret in the order of $O(\log(T\Delta^2)/\Delta)$ is called instance dependent optimal (Lattimore and Szepesvári).
Perchet et al. (2016) studied the two-armed batched bandit problem. They developed two policies that are minimax optimal and instance-dependent optimal respectively, and proved that their round cost is near optimal. Recently, Gao et al. (2019) used similar policies for $K$-armed batched bandits and proved that their batch cost is also near optimal. The batch setting is also studied in the best arm identification problem (Agarwal et al., 2017; Jin et al., 2019).

3 Double Explore-then-Commit Strategies

The vanilla ETC strategy (Perchet et al., 2016; Garivier et al., 2016) consists of two stages: in stage one, the agent pulls all arms for the same number of times, which can be a fixed integer or a data-dependent stopping time, leading to the FB-ETC and SPRT-ETC (or BAI-ETC) algorithms in (Garivier et al., 2016); in stage two, the agent pulls the arm that achieves the best average reward according to the outcome of stage one. As we mentioned before, none of these algorithms can achieve the asymptotic optimality. To address this, we propose the following double explore-then-commit strategy.

3.1 Double Explore-then-Commit Algorithm for Known Gaps

We assume w.l.o.g. that arm 1 is optimal. We first consider the case where the suboptimal gap $\Delta = \mu_1 - \mu_2$ is known. We propose a double explore-then-commit (DETC) algorithm, which consists of four stages. The details are displayed in Algorithm 1.

At the initialization step, we pull both arms once, after which we set the current time step $t = 2$. In Stage I, DETC plays both arms for $\tau_1 = 4\lceil \log(T_1 \Delta^2) / \Delta^2 \rceil$ times respectively, where $T_1 \in \mathbb{N}^+$ is a predefined parameter. For any time step $t$, we define $T_k(t)$ to be the total number of times that arm $k (k = 1, 2)$ has been played, i.e., $T_k(t) = \sum_{i=1}^{t} I\{A_i = k\}$, where $A_i$ is the arm played at time step $i$. Then we can define the average reward of arm $k$ at time step $t$ as $\hat{\mu}_k(t) := \sum_{i=1}^{t} I\{A_i = k\} r_i / T_k(t)$, where $r_i$ is the reward received by the algorithm at time $i$.

In Stage II, DETC repeatedly pulls arm $i'$ with the largest average reward at the end of Stage I, namely, $i' = \arg \max_{k=1,2} \hat{\mu}_{k, \tau_1}$, where $\hat{\mu}_{k, \tau_1}$ is the average reward of arm $k$ after its $\tau_1$-th pull. Note that before Stage II, arm $1'$ has been played for $\tau_1$ times. We will terminate Stage II after the total number of time steps for playing arm $1'$ reaches $T_1$. It is worth noting that Stage I and Stage II are similar to existing ETC algorithms (Perchet et al., 2016; Garivier et al., 2016), where these two stages are referred to as the Explore and the Commit stages respectively.

The key difference here is that instead of playing arm $1'$ till the end of the horizon ($T$), our Algorithm 1 sets a check point $T_1 < T$. After arm $1'$ has been played for $T_1$ times, we stop and check the average reward of the arm that is not chosen in Stage II, which is denoted by $2'$. The motivation for this halting follows from a natural question: What if we have chosen the wrong arm to commit? Even though arm $2'$ is not chosen based on the outcome of Stage I, it can still be optimal due to random sampling errors. To avoid such a case, we pull arm $2'$ for more steps such that the average rewards of both arms can be more distinguished from each other. Specifically, in Stage III of Algorithm 1, arm $2'$ is repeatedly played until

$$2(1 - \epsilon_T) t_2 \Delta |\mu' - \theta_{2', t_2}| \geq \log(T \Delta^2),$$

(3.1)
Algorithm 1 Double Exploration-then-Commit (DETC) for Known Gaps

**input** $T$, $\epsilon_T$ and $\Delta$

1. **Initialization:** $A_1 = 1$, $A_2 = 2$, $T_1 = \lfloor 2 \log(T\Delta^2)/(\epsilon_T^2 \cdot \Delta^2) \rfloor$, $t \leftarrow 2$

   **Stage I: Explore all arms uniformly**
   2. **while** $t \leq 8 \lfloor \log(T\Delta^2)/\Delta^2 \rfloor$ **do**
   3. Choose $A_{t+1} = 1$ and $A_{t+2} = 2$, $t \leftarrow t + 2$
   4. **end while**

   **Stage II: Commit the arm with the largest average reward**
   5. $1' \leftarrow \arg\max_i \hat{\mu}_i(t)$
   6. **while** $T_{1'}(t) \leq T_1$ **do**
   7. Choose $A_{t+1} = 1'$, $t \leftarrow t + 1$
   8. **end while**

   **Stage III: Explore the unchosen arm in Stage II**
   9. $\mu' \leftarrow \hat{\mu}_{1'}(t)$, $t_2 \leftarrow 0$, $\theta_{2',s}$ is the recalculated average reward of arm $2'$ after its $s$-th pull in Stage III and $\theta_{2',s} = 0$, for $s = 0$;
   10. **while** $2(1 - \epsilon_T)t_2 \Delta \mid \mu' - \theta_{2',t_2} \mid \leq \log(T\Delta^2)$ **do**
   11. $A_{t+1} = 2'$, $t \leftarrow t + 1$, $t_2 = t_2 + 1$
   12. **end while**

   **Stage IV: Commit the arm with the largest average reward after double exploration**
   13. $a = 1' \{ \hat{\mu}_{1'}(t) \geq \theta_{2',t_2} \} + 2' \{ \hat{\mu}_{1'}(t) < \theta_{2',t_2} \}$
   14. **while** $t \leq T$ **do**
   15. Play arm $a$, $t \leftarrow t + 1$
   16. **end while**

where $\epsilon_T > 0$ is a parameter, $t_2$ is the total number of steps arm $2'$ has been pulled in Stage III, $\theta_{2',t_2}$ is the average reward of arm $2'$ in Stage III and $\mu'$ is the average reward of arm $1'$ at the end of Stage II. Note that $\mu' = \hat{\mu}_{1'}(t)$ throughout Stage III since arm $1'$ is not pulled in this stage.

At the end of Stage II, the average reward $\mu'$ for arm $1'$ already concentrates on its expected reward. Therefore, in Stage III of DETC, the sampling error only comes from pulling arm $2'$. Hence, our DETC algorithm offsets the drawback ETC algorithms where the sampling error comes from both arms. In the remainder of the algorithm (Stage IV), we just pull the arm that achieves the largest empirical reward from the previous three stages.

Now, we show that Algorithm 1 achieves the asymptotic optimal regret. Note that if $T\Delta^2 < 1$, the regret bound is $T\Delta < \sqrt{T}$. Hence, in the following theorem, we assume $T\Delta^2 \geq 1$.

**Theorem 3.1.** Suppose $T\Delta^2 \geq 1$. If $T_1 \Delta^2 \geq 1$, the regret of Algorithm 1 is upper bounded as

$$R_\mu(T) \leq 6\Delta + \frac{4}{\Delta} \log(T_1 \Delta^2) + \frac{2}{T} \log(T\Delta^2) + 2 \sqrt{T\Delta^2} + 2 \sqrt{T\Delta^2} \Delta + \frac{\sqrt{\log(T\Delta^2)} + 2}{(1 - \epsilon_T)^2 \Delta}.$$  

Furthermore, let $\epsilon_T = \min\{\sqrt{\log(T\Delta^2)/(\Delta^2 \log^2 T)}, 1/2\}$, then $\limsup_{T \to \infty} R_\mu(T)/\log T \leq 1/(2\Delta)$.
and \( R_\mu(T) = O(\Delta + \log(T\Delta^2)/\Delta) = O(\sqrt{T} + \Delta) \).

The above theorem states that Algorithm 1 simultaneously achieves the asymptotically optimal regret rate \( 1/(2\Delta) \), instance dependent optimal regret bound \( O(\Delta + \log(T\Delta^2)/\Delta) \) and minimax optimal regret bound \( O(\Delta + \sqrt{T}) \), when parameters \( \epsilon_T \) and \( T_1 \) are properly chosen. In comparison, the ETC algorithm in Garivier et al. (2016) can only achieve \( 1/\Delta \) asymptotic regret rate, which is suboptimal for multi-armed bandit problems (Lai and Robbins, 1985). It is important to note that, Garivier et al. (2016) also proved a lower bound for asymptotic optimality of ETC and showed that the \( 1/\Delta \) asymptotic regret rate of ‘single’ explore-then-commit algorithms cannot be improved. Therefore, the double exploration techniques in our DETC is indeed essential for breaking the \( 1/\Delta \) barrier in the asymptotic regret rate.

It is worth noting that the asymptotic optimality is also achieved by the \( \Delta \)-UCB algorithm in Garivier et al. (2016), which is a fully sequential strategy. Nevertheless, our DETC is the first explore-then-commit (non-fully sequential) algorithm that can achieve the asymptotically optimal regret for multi-armed bandit problems. Compared with \( \Delta \)-UCB, DETC has distinct phases of exploration and exploitation which makes the implementation simple and more practical. A more important feature of DETC for batched bandit models will be illustrated in Section 4.

### 3.1.1 Proof of Theorem 3.1

Now we are going to prove Theorem 3.1. Let \( \tau_2 \) be the total number of times arm 2 is played in Stage III of Algorithm 1. We know that \( \tau_2 \) is a random variable. Recall that \( \mu_1 > \mu_2 \) and \( \Delta = \mu_1 - \mu_2 \). Let \( N_2(T) \) denote the total number of times Algorithm 1 plays arm 2, which is calculated as

\[
N_2(T) = \tau_1 + (T_1 - \tau_1) \mathbb{1}\{\hat{\mu}_1(\tau_1) < \hat{\mu}_2(\tau_1)\} + \tau_2 \mathbb{1}\{\hat{\mu}_1(\tau_1) \geq \hat{\mu}_2(\tau_1)\} + (T - T_1 - \tau_1 - \tau_2) \mathbb{1}\{a = 2\}. \tag{3.2}
\]

Then the regret of Algorithm 1 \( R_\mu(T) = \mathbb{E}[\Delta N_2(T)] \) can be decomposed as follows

\[
R_\mu(T) \leq \mathbb{E}[\Delta \tau_1 + \Delta(T_1 - \tau_1) \mathbb{1}\{\hat{\mu}_1(\tau_1) < \hat{\mu}_2(\tau_1)\} + \Delta \tau_2 \mathbb{1}\{\hat{\mu}_1(\tau_1) \geq \hat{\mu}_2(\tau_1)\} + \Delta T \mathbb{1}\{a = 2\}] \\
\leq \mathbb{E}[\Delta \tau_1 + \Delta T_1 \mathbb{P}(\hat{\mu}_1(\tau_1) < \hat{\mu}_2(\tau_1)) + \Delta \tau_2 \mathbb{P}(\hat{\mu}_1(\tau_1) \geq \hat{\mu}_2(\tau_1)) + \Delta T \mathbb{P}(a = 2)] \\
\leq \Delta \tau_1 + \underbrace{\Delta T_1 \mathbb{P}(\tau_1 < T_1, 1' = 2)}_{I_1} + \underbrace{\Delta \mathbb{E}[\tau_2]}_{I_2} + \underbrace{\Delta T \mathbb{P}(\tau_2 < T, a = 2)}_{I_3}. \tag{3.3}
\]

In what follows, we will bound these terms separately.

**Bounding term \( I_1 \):** Let \( X_i \) and \( Y_i \) be the rewards from playing arm 1 and arm 2 for the \( i \)-th time respectively. Thus \( X_i - \mu_1 \) and \( Y_i - \mu_2 \) are 1-subGaussian random variables. Let \( S_0 = 0 \) and \( S_n = (X_1 - Y_1) + \cdots + (X_n - Y_n) \) for every \( n \geq 1 \). Then \( X_i - Y_i - \Delta \) is a \( \sqrt{2} \)-subGaussian random variable. Applying Lemma A.1 with any \( \epsilon > 0 \), we get

\[
\mathbb{P}(S_{\tau_1}/\tau_1 \leq \Delta - \epsilon) \leq \exp(-\tau_1 \epsilon^2/4) \leq \exp(-\epsilon^2 \log(T_1 \Delta^2)/\Delta^2), \tag{3.4}
\]

where in the last inequality we plugged in the fact that \( \tau_1 \geq 4 \log(T_1 \Delta^2)/\Delta^2 \). By setting \( \epsilon = \Delta \) in
the above inequality, we further obtain $\mathbb{P}(\tau_1 < T_1, 1' = 2) = \mathbb{P}(S_{\tau_1}/\tau_1 \leq 0) \leq 1/(T_1 \Delta^2)$. Hence
\begin{equation}
T_1 \Delta \mathbb{P}(\tau_1 < T_1, 1' = 2) \leq 1/\Delta. \tag{3.5}
\end{equation}
Recall that $T_1 \geq 2 \log (T \Delta^2) / (\epsilon_T^2 \Delta^2)$. Applying Lemma A.1 and union bound, $\mathbb{P}(\mu_1' - \epsilon_T \Delta \leq \mu' \leq \mu_1' + \epsilon_T \Delta) \geq 1 - 2/(T \Delta^2)$. Thus the expected total regret for the case that $\mu' \notin [\mu_1' - \epsilon_T \Delta, \mu_1' + \epsilon_T \Delta]$ is bounded by $2/\Delta$. Therefore, in the remaining proof of terms $I_2$ and $I_3$, we prove the results conditional on the event that $\mu' \in [\mu_1' - \epsilon_T \Delta, \mu_1' + \epsilon_T \Delta]$.

**Bounding term $I_2$:** We first assume that the chosen arm $1'$ is the best arm, i.e., $1' = 1$. In this case, we know that arm $2' = 2$ is played in *Stage III* of Algorithm 1. Let us first define the following notations for the simplicity of the presentation:
\begin{align}
Z_0 = 0, \quad Z_i = \mu' - Y_{i+\tau_1}, \quad S_0' = 0, \quad S_n' = Z_1 + \cdots + Z_n, \tag{3.6}
\end{align}
where $Y_{i+\tau_1}$ is the reward from playing arm 2 for the $i$-th time in *Stage III*. For any $x > 0$, we define $n_x = (\log(T \Delta^2) + x)/(2(1 - \epsilon_T)^2 \Delta^2)$. We also define a check point parameter $x_0 = 2 \sqrt{\log(T \Delta^2)}$.

Note that in *Stage III* of Algorithm 1 (Line 10), it holds that
\begin{equation}
2(1 - \epsilon_T) \Delta |S_{t_2}'| = 2(1 - \epsilon_T) t_2 \Delta |\mu' - \theta_{2', t_2}| < \log(T \Delta^2),
\end{equation}
for $t_2 \leq \tau_2 - 1$. We have
\begin{equation}
\left\{ \tau_2 - 1 \geq \left[ \frac{\log(T \Delta^2) + x}{2(1 - \epsilon_T)^2 \Delta^2} \right] \right\} = \left\{ \tau_2 - 1 \geq \left[ n_x \right] \right\} \subseteq \left\{ S_{[n_x]}' \leq \frac{\log(T \Delta^2)}{2(1 - \epsilon_T) \Delta} \right\}. \tag{3.7}
\end{equation}
Let us denote $\Delta' = \mu' - \mathbb{E}[Y_{i+\tau_1}]$. Thus $Z_i - \Delta'$ is 1-subGaussian and
\begin{equation}
\Delta' = \mu' - \mathbb{E}[Y_{i+\tau_1}] = \mu' - \mu_2 \geq \mu_1 - \epsilon_T \Delta - \mu_2 = (1 - \epsilon_T) \Delta. \tag{3.8}
\end{equation}
Applying Lemma A.1, for any $\epsilon > 0$ we have
\begin{equation}
\mathbb{P}(S_{[n_x]}' / [n_x] \leq \Delta' - \epsilon) \leq \exp(-[n_x] \epsilon^2 / 2). \tag{3.9}
\end{equation}
Let $\epsilon = (1 - \epsilon_T) \Delta x / (\log(T \Delta^2) + x)$, then
\begin{align}
[n_x] (\Delta' - \epsilon) & \geq [n_x] ((1 - \epsilon_T) \Delta - \epsilon) \geq \log(T \Delta^2) / (2(1 - \epsilon_T) \Delta).
\end{align}
Plugging this relationship into (3.9) yields
\begin{equation}
\mathbb{P} \left( S_{[n_x]}' \leq \frac{\log(T \Delta^2)}{2(1 - \epsilon_T) \Delta} \right) \leq \mathbb{P} \left( S_{[n_x]}' \leq [n_x] (\Delta' - \epsilon) \right) \leq \exp\left( -\frac{x^2}{4(\log(T \Delta^2) + x)} \right), \tag{3.10}
\end{equation}
which combined with (3.7) further implies
\begin{equation}
\mathbb{P} \left( \tau_2 - 1 \geq \left[ \frac{\log(T \Delta^2) + x}{2(1 - \epsilon_T)^2 \Delta^2} \right] \right) \leq \exp\left( -\frac{x^2}{4(\log(T \Delta^2) + x)} \right).\end{equation}
Recall that we have \( x_0 = 2\sqrt{\log(T\Delta^2)} \). Then for any \( x \geq x_0 \) it is clear that \( x\sqrt{\log(T\Delta^2)/2} \geq \log(T\Delta^2) \). We have

\[
\int_{n_0}^{\infty} \mathbb{P}(\tau_2 - 2 \geq v)dv = \int_{n_0}^{\infty} \mathbb{P}_\mu(\tau_2 - 2 \geq \frac{\log(T\Delta^2) + x}{2(1 - \epsilon_T)^2\Delta^2}) \frac{dx}{2(1 - \epsilon_T)^2\Delta^2} \\
\leq \int_{n_0}^{\infty} \mathbb{P}_\mu(\tau_2 - 1 \geq \frac{\log(T\Delta^2) + x}{2(1 - \epsilon_T)^2\Delta^2}) \frac{dx}{2(1 - \epsilon_T)^2\Delta^2} \\
\leq \frac{1}{2(1 - \epsilon_T)^2\Delta^2} \int_{n_0}^{\infty} \exp\left(-\frac{x^2}{4(\log(T\Delta^2) + x)}\right)dx \\
\leq \frac{1}{2(1 - \epsilon_T)^2\Delta^2} \int_{n_0}^{\infty} \exp\left(-\frac{x}{2\sqrt{\log(T\Delta^2) + 4}}\right)dx \\
\leq \frac{1}{2(1 - \epsilon_T)^2\Delta^2} \int_{0}^{\infty} \exp\left(-\frac{x}{2\sqrt{\log(T\Delta^2) + 4}}\right)dx \\
= \frac{\sqrt{\log(T\Delta^2) + 2}}{(1 - \epsilon_T)^2\Delta^2}.
\]

The expectation of \( \Delta \tau_2 \) can be calculated as follows

\[
\Delta \mathbb{E}[\tau_2] = \Delta \int_{0}^{\infty} \mathbb{P}(\tau_2 > v)dv \\
= \Delta \int_{0}^{n_0 + x_0^2} \mathbb{P}(\tau_2 > v)dv + \Delta \int_{n_0 + x_0^2}^{\infty} \mathbb{P}_\mu(\tau_2 - 2 \geq v)dv \\
\leq 2\Delta + \frac{\log(T\Delta^2) + 2\sqrt{\log(T\Delta^2)}}{2(1 - \epsilon_T)^2\Delta} + \frac{\sqrt{\log(T\Delta^2) + 2}}{(1 - \epsilon_T)^2\Delta}.
\]

Finally, if \( 1' \) is the suboptimal arm, i.e., \( 1' = 2 \). The proof is similar to the above one and we only need to change the definitions in (3.6) to \( Z_i = X_{i+\tau_1} - \mu' \) and \( \Delta' = \mathbb{E}[X_{i+\tau_1}] - \mu' \). It is easy to verify that the regret bound still satisfies (3.12). This completes the bound for term \( I_2 \).

**Bounding term \( I_3 \):** We again start with assuming that \( 1' = 1 \). We prove that \( \mathbb{P}(\tau_2 < T, a = 2) \leq 1/(T\Delta^2) \). Recall the definition that \( S'_n = \sum_{i=1}^{n} Z_i \) and \( Z_i = \mu' - Y_{i+\tau_1} \). Note that \( Z_i - \Delta' \) is 1-subGaussian and \( \Delta' \geq (1 - \epsilon_T)\Delta \). Then from definition of 1-subGaussian, we have

\[
\mathbb{E}[\exp(-2\Delta(1 - \epsilon_T)Z_1)] = \mathbb{E}[\exp(-2\Delta(1 - \epsilon_T)Z_1 + 2\Delta\Delta'(1 - \epsilon_T) - 2\Delta\Delta'(1 - \epsilon_T))] \\
= \mathbb{E}[\exp((-2\Delta(1 - \epsilon_T)(Z_1 - \Delta') - 2\Delta\Delta'(1 - \epsilon_T))] \\
\leq \exp((-2(1 - \epsilon_T)^2/2 - 2(1 - \epsilon_T)\Delta\Delta')) \\
\leq \exp(2(1 - \epsilon_T)\Delta((1 - \epsilon_T)\Delta - \Delta')) \\
\leq 1,
\]

where the first inequality comes from the definition of subGaussian random variables. Therefore \( M_n = \exp(-2\Delta(1 - \epsilon_T)S'_n) \) is a supermartingale. Let \( \tau' = T \wedge \inf\{n > 1 : S'_n \leq -\log(T\Delta^2)/(2\Delta(1 - \epsilon_T))\} \)
be a stopping time. Observe that
\[
\{\tau_2 < T, a = 2\} \subseteq \{\exists 1 < n < T : S'_n \leq -\log(T\Delta^2)/(2\Delta(1-\epsilon_T))\} = \{\tau' < T\}. \tag{3.14}
\]
Applying Doob’s optional stopping theorem (Durrett, 2019) yields \(E[\tau'] = \tau\) where the last equation is due to the fact that
\[
\frac{1}{\epsilon_T}(1-\epsilon_T)\log(T\Delta^2) = T\Delta^2 \quad \text{on the event \{\tau_2 < T\}},
\]
it holds
\[
\mathbb{P}(\tau_2 < T, a = 2) \leq \mathbb{P}(\tau' < T) = \mathbb{P}(\tau' < T, a = 2) \leq \mathbb{E}[M_{\tau'}]/(T\Delta^2) \leq 1/(T\Delta^2). \tag{3.15}
\]
where the second inequality comes from Markov’s inequality. Similarly, if \(1' = 2\), \(\mathbb{P}(\tau_2 < T, a = 2) \leq 1/(T\Delta^2)\) still holds. And thus term \(I_3\) can be upper bounded by \(1/\Delta\).

**Completing the proof:** The expected total regret for the case that \(\mu' \notin [\mu_2 - \epsilon_T \Delta, \mu_2 + \epsilon_T \Delta]\) is bounded by \(2/\Delta\). Substituting (3.5), (3.12) and (3.15) into (3.3) yields the total regret as follows.
\[
R_\mu(T) \leq 6\Delta + \frac{4}{\Delta} + \frac{4\log(T\Delta^2)}{\Delta} + \frac{\log(T\Delta^2)}{2(1-\epsilon_T)^2\Delta} + \frac{\sqrt{\log(T\Delta^2)} + 2}{(1-\epsilon_T)^2\Delta}.
\]
Recall the choice of \(\epsilon_T\) in Theorem 3.1. By our choice that \(T_1 = [2\log(T\Delta^2)/(\epsilon_T^2\Delta^2)]\), we have
\[
T_1 \leq 1 + \max\{\log^2 T, 8\log(T\Delta^2)/\Delta^2\}, \tag{3.16}
\]
which immediately implies, \(\lim_{T \to \infty} 4\log(T_1\Delta^2)/(\Delta \log T) = 0\). Also note that \(\lim_{T \to \infty} \epsilon_T = 0\). Thus, we have \(\lim_{T \to \infty} R_\mu(T)/\log T = 1/(2\Delta)\).

By (3.16), we know that \(T_1\Delta^2 = O(\log T)\), which results in the worst case regret bound as
\[
R_\mu(T) = O\left(\Delta + \frac{1}{\Delta} + \frac{\log(T\Delta^2)}{\Delta} + \frac{\log(T\Delta^2)}{\Delta}\right) = O\left(\Delta + \frac{\log(T\Delta^2)}{\Delta}\right) = O(\Delta + \sqrt{T}),
\]
where the last equation is due to the fact that \(T\Delta^2 > 1\) and \(\log x \leq 2\sqrt{x}\) for \(x > 1\).

### 3.2 Double Explore-then-Commit Algorithm for Unknown Gaps

In real world applications, the suboptimal gap \(\Delta\) is often unknown. Thus, we should design an algorithm without the knowledge of \(\Delta\) in contrast to Algorithm 1. To this end, we propose a new double explore-then-commit (DETC) algorithm where the gap \(\Delta\) is unknown, which is displayed in Algorithm 2.

Similar to Algorithm 1, Algorithm 2 also consists of four stages, where *Stage I* and *Stage III* are double exploration stages that ensure we have chosen the right arm to pull in the subsequent stages. Since we do not have access to \(\Delta\), we derive the stopping rule for *Stage I* by comparing the empirical average rewards of both arms. Once we have obtained empirical estimates of the mean rewards that are able to distinguish two arms in the sense that \(|\hat{\mu}_1(t) - \hat{\mu}_2(t)| \geq \sqrt{16\log^2(T_1/t)/t}\), we terminate *Stage I*. Here \(t\) is the current time step of the algorithm and \(T_1\) is a predefined parameter. Similar to Algorithm 1, based on the outcomes of *Stage I*, we choose arm \(1' = \arg\max_{i=1,2} \hat{\mu}_i(t)\) at the end of *Stage I* and pull this arm repeatedly throughout *Stage II*. In *Stage III*, we turn to pull arm \(2'\).
Algorithm 2 Double Exploration-then-Commit (DETC) for Unknown Gaps

input $T, T_1$

1: Initialization: $A_1 = 1, A_2 = 2, t \leftarrow 2$

Stage I: Explore all arms uniformly

2: while $|\hat{\mu}_1(t) - \hat{\mu}_2(t)| < \sqrt{\frac{16}{T} \log^+(T_1/t)}$ do
3: Choose $A_{t+1} = 1$ and $A_{t+2} = 2, t \leftarrow t + 2$
4: end while

Stage II: Commit the arm with the largest average reward

5: $1' \leftarrow \arg \max_i \hat{\mu}_i(t)$;
6: while $t \leq T_1$ do
7: Choose $A_{t+1} = 1'$, $t \leftarrow t + 1$
8: end while

Stage III: Explore the unchosen arm in Stage II

9: $\mu' \leftarrow \hat{\mu}_1(t), t_2 \leftarrow 0, \theta_{2,s}$ is the recalculated average reward of arm $2'$ after its $s$-th pull in Stage III and $\theta_{2,s} = 0, s = 0$;
10: while $|\mu' - \theta_{2,t_2}| < \sqrt{\frac{2}{T_2} \log \left( \frac{T_2}{T_1} \left( \log^2 \frac{T_2}{T_1} + 1 \right) \right)}$ do
11: $A_{t+1} = 2', t \leftarrow t + 1, t_2 \leftarrow t_2 + 1$
12: end while

Stage IV: Commit the arm with the largest average reward after double exploration

13: $a = 1' \mathbb{1}\{\hat{\mu}_1(t) \geq \theta_{2,t_2}\} + 2' \mathbb{1}\{\hat{\mu}_1(t) < \theta_{2,t_2}\}$;
14: while $t \leq T$ do
15: Play arm $a$, $t \leftarrow t + 1$
16: end while

that is not chosen in Stage II until the average reward of arm $2'$ is significantly larger or smaller than that of arm $1'$ chosen in Stage II. Note that in both exploration stages, we do not need the information of the suboptimal gap $\Delta$.

In the following theorem, we present the regret bound of Algorithm 2 and show that this regret is also asymptotically optimal and minimax optimal in this setting.

Theorem 3.2. Let $\epsilon_T = \sqrt{2 \log(T \Delta^2)/(T_1 \Delta^2)}$. Suppose that $\epsilon_T \in (0, 1/2)$ and $T \Delta^2 \geq 16e^3$, then

$$R_\mu(T) \leq 2\Delta + \frac{36 + 8 \log^+(T_1 \Delta^2/4) + 2\sqrt{8\pi \log^+(T_1 \Delta^2)}}{\Delta} \frac{482 + 2 \log(4T \Delta^2(\log^2(4T \Delta^2) + 1)) + \sqrt{4\pi \log(4T \Delta^2(\log^2(4T \Delta^2) + 1))}}{(1 - \epsilon_T)^2 \Delta}.$$  

Moreover, if we choose $T_1 = \log^2 T$, then $\lim_{T \to \infty} R_\mu(T)/\log T = 2/\Delta$. If we choose $T_1 = T$, then $R_\mu(T) = O(\Delta + \log(T \Delta^2)/\Delta) = O(\sqrt{T} + \Delta)$. 

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The proof of Theorem 3.2 resembles that of Theorem 3.1. Due to the space limit, we defer it to the appendix. Here we provide some comparison between existing algorithms and Algorithm 2. For two-arm bandits, Lai and Robbins (1985) proved that the asymptotically optimal regret rate is $2/\Delta$. This optimal bound has been achieved by a series of fully sequential bandits algorithms such as UCB (Garivier and Cappé, 2011; Lattimore, 2018), Thompson sampling (Agrawal and Goyal, 2017), Ada-UCB (Kaufmann et al., 2018), etc. All these algorithms are fully sequential, which means they have to examine the outcome from current pull before it can decide which arm to pull in the next time step. In the unknown gap setting, Garivier et al. (2016) proved a $4/\Delta$ lower bound for ‘single’ explore-then-commit algorithms. Therefore, in order to break the $4/\Delta$ barrier in the asymptotic regret rate, our double exploration technique in Algorithm 2 is crucial. In addition, by setting $T_1 = T$, Algorithm 2 is also instance-dependent optimal and minimax optimal. Thus, it is still an interesting open problem for ETC to achieve all three optimalities with the same parameter $T_1$. We will discuss it in more detail in Section 5.

4 Asymptotically Optimal DETC in Batched Bandit Models

The proposed DETC algorithms in this paper can be easily extended to batched bandit problems (Perchet et al., 2016; Gao et al., 2019). In this section, we present simple modifications to Algorithms 1 and 2 and prove that they not only achieve the asymptotically optimal regret bounds but also enjoy $O(1)$ round complexities.

4.1 Batched DETC for Known Gaps

We use the same notations that are used in Section 3.1. The batched DETC algorithm is identical to Algorithm 1 except the stopping rule of Stage III. More specifically, let $\tau_0 = 2 + \log(T\Delta^2)/(2(1 - \epsilon_T)^2\Delta^2)$. In Stage III of Algorithm 2 (Lines 10-12), instead of testing every $t_2$, we only test it at the following time grid:

$$\tau_0 + \frac{2\sqrt{\log(T\Delta^2) + 4}}{2(1 - \epsilon_T)^2\Delta^2}, \quad \tau_0 + \frac{2(2\sqrt{\log(T\Delta^2) + 4})}{2(1 - \epsilon_T)^2\Delta^2}, \quad \tau_0 + \frac{3(2\sqrt{\log(T\Delta^2) + 4})}{2(1 - \epsilon_T)^2\Delta^2}, \ldots \quad (4.1)$$

At each time point in the grid listed above, we query the results of the bandits pulled since the last time point. Between two time points, we play the arm $2'$ without accessing the results. The period between two time points is also referred to as a round (Perchet et al., 2016). Reducing the total number of queries, namely, the round complexity, is an important research topic in the batched bandit problem.

Now, we present the round complexity of the batched version of Algorithm 1.

**Theorem 4.1.** In Stage III of Algorithm 1, if we test the stopping condition at time grid in (4.1), then the expected number of rounds used in Algorithm 1 is $O(1)$. The regret is asymptotically optimal and for $T\Delta^2 \geq 1$, the regret is also minimax optimal.

**Remark 4.2.** Compared with the fully sequentially adaptive bandit algorithms such as UCB, which needs $O(T)$ rounds of queries, our DETC algorithm only needs $O(1)$ rounds of queries and achieves the asymptotic optimality and minimax optimality for two-armed bandit problems. Compared with
the constant round algorithm FB-ETC Garivier et al. (2016), our DETC algorithm improves the asymptotic regret rate of FB-ETC (i.e., $4/\Delta$) by a factor of 8.

**Proof.** The analysis is very similar to that of Theorem 3.1 and thus we will use the same notations therein. Note that Stage I requires 1 round of queries since $\tau_1$ is fixed. In addition, Stage II and Stage IV need 1 query at the beginning of stages respectively. Now it remains to calculate the total rounds for Stage III.

Without loss of generality, we assume that $1' = 1$. We first deal with the case $\mu' \in [\mu_1' - \epsilon_T \Delta, \mu_1' + \epsilon_T \Delta]$. Let $x_i = i(2\sqrt{\log(T\Delta^2)} + 4)$ and $n_{x_i} = \tau_0 + x_i/(2(1 - \epsilon_T)^2\Delta^2)$. For simplicity, assume $x_i, n_{x_i} \in \mathbb{N}^+$. From (3.10), we have

$$P(\tau_2 > n_{x_i}) \leq P(S_{n_{x_i}} \leq \log(T\Delta^2)/(2(1 - \epsilon_T)\Delta)) \leq \exp\left(-\frac{x_i^2}{4(\log(T\Delta^2) + x_i)}\right) \leq \exp\left(-\frac{x_i}{2\sqrt{\log(T\Delta^2)} + 4}\right) \leq 2^{-i}. \quad (4.2)$$

Thus, the expected number of rounds of queries needed in Stage III of Algorithm 1 is upper bounded by $\sum_{i=1}^{\infty} i/2^i = 2$. For the case that $\mu' \notin [\mu_1' - \epsilon_T \Delta, \mu_1' + \epsilon_T \Delta]$, we have $P(\mu' \notin [\mu_1' - \epsilon_T \Delta, \mu_1' + \epsilon_T \Delta]) \leq 2/(T\Delta^2)$. Note that the increment between consecutive test time points is $(2\sqrt{\log(T\Delta^2)} + 4)/(2(1 - \epsilon_T)^2\Delta^2)$, thus the expected number of test time points is at most $T(1 - \epsilon_T)^2\Delta^2/(\sqrt{\log(T\Delta^2)})$. Then the expected number of rounds for this case is bounded by $2(1 - \epsilon_T)^2/(\sqrt{\log(T\Delta^2)})$. For $T \to \infty$, the expected number of rounds cost for this case is 0. To summarize, the round complexity of Algorithm 2 is $O(1)$.

Following the same proof in (3.11) and (3.12), it is easy to verify that $E[\tau_2] \leq \tau_0 + (2\sqrt{\log(T\Delta^2)} + 4)/(2(1 - \epsilon_T)^2\Delta^2)$, which is no larger than the bound in (3.12). The bounds for other terms remain the same. Therefore, the batched version of Algorithm 1 is still asymptotically optimal, instance-dependent optimal and minimax optimal.

### 4.2 Batched DETC for Unknown Gaps

In the unknown gap case, both the stopping rules of Stage I and Stage III in Algorithm 2 need to be modified. In what follows, we describe a variant of Algorithm 2 that only needs to check the results of pulls at certain time points in Stage I and Stage III. In particular, let $T_1 = \log^2 T$. In Stage I, we query the results and test the condition in Line 2 at the following time grid:

$$t = 2\sqrt{\log T}, 4\sqrt{\log T}, 6\sqrt{\log T}, \cdots \quad (4.3)$$

In Stage III, we query the results and test the condition in Line 10. The first test happens at time point $t_2 = N_1$, where

$$N_1 = (2 \log T)/\log \log T. \quad (4.4)$$
Based on the test result in the first round, we use \( \hat{\Delta} = |\mu' - \theta_{2',N_1}| \) as an estimate of \( \Delta' \). Then the subsequent test in *Stage III* happens at the following time grid

\[
2/\hat{\Delta}^2 N_2 \log(T \log^3 T) + 1/\hat{\Delta}^2 N_2 (\log T)^{\frac{3}{2}}, \\
2/\hat{\Delta}^2 N_2 \log(T \log^3 T) + 2/\hat{\Delta}^2 N_2 (\log T)^{\frac{3}{2}}, \\
2/\hat{\Delta}^2 N_2 \log(T \log^3 T) + 3/\hat{\Delta}^2 N_2 (\log T)^{\frac{3}{2}}, \ldots, \ldots, \log^2 T,
\]

where \( N_2 = (1 + (\log T)^{-\frac{1}{4}})^2 \) is a notation for simplicity. Another difference from Algorithm 2 is that we require \( t_2 \leq T_1 \). Thus we will terminate *Stage III* after at most \( T_1 = \log^2 T \) pulls of arm \( 2' \).

**Theorem 4.3.** Let \( T_1 = \log^2 T \) in Algorithm 2. If we test the stopping condition of *Stage I* at time grid in (4.3) and test the stopping condition of *Stage III* at time grid in (4.4) and (4.5), then the expected number of rounds used in Algorithm 2 is \( O(1) \). Moreover, the regret of Algorithm 2 is asymptotically optimal, i.e., \( \lim_{T \to \infty} R_{\mu}(T) / \log T = 2/\Delta \).

Perchet et al. (2016) proved that any algorithm achieving the minimax optimality or instance dependent optimality will cost at least \( \Omega(\log \log T) \) or \( \Omega(\log T / \log \log T) \) rounds respectively. Therefore, we only focus on deriving the asymptotic optimality along with a constant round complexity in the batched bandits setting. It remains an interesting problem to achieve instance-dependent optimality or minimax optimality with optimal round cost together with the asymptotic optimality. We leave it for future work.

5 Conclusion and Future Work

In this paper, we close the gap between the suboptimality of ETC algorithms and the optimality of fully sequential strategies. We propose a double explore-then-commit (DETC) strategy and prove that DETC is asymptotically optimal for subGaussian rewards. We also extend our DETC algorithm to the batched bandit problem (Perchet et al., 2016) and prove that DETC enjoys a constant round complexity while achieving the asymptotic optimality. For unknown gap, for the sake of simplicity, our algorithms cannot achieve the asymptotic optimality, minimax optimality and instance-dependent optimality at the same time. Nevertheless, we believe obtaining such an optimal algorithm is possible by incorporating the Exponential-Gap-Elimination (EGE) technique, which is widely used in best arm identification problems (Karnin et al., 2013; Chen et al., 2017) and also regret minimization problems (Auer and Ortner, 2010). We will leave it as a future work.

Another possible future direction is to extend our algorithm from two-armed bandits to multi-armed bandits problems. We believe the core ideas and intuitions have been well captured by our algorithm and the extension to \( K \)-armed bandits are mostly technical (see, for instance, Gao et al. (2019)).

A Proof of the Regret Bound of Algorithm 2

Now we provide the proof for Theorem 3.2. We first present the some technical lemmas that characterizes the concentration properties of subGaussian random variables.
Lemma A.1 (Corollary 5.5 in Lattimore and Szepesvári). Assume that \( X_1, \ldots, X_n \) are independent, \( \sigma \)-subGaussian random variables centered around \( \mu \). Then for any \( \epsilon > 0 \)
\[
\mathbb{P}(\hat{\mu} \geq \mu + \epsilon) \leq \exp\left(-\frac{n\epsilon^2}{2\sigma^2}\right) \quad \text{and} \quad \mathbb{P}(\hat{\mu} \leq \mu - \epsilon) \leq \exp\left(-\frac{n\epsilon^2}{2\sigma^2}\right),
\]
where \( \hat{\mu} = \frac{1}{n} \sum_{t=1}^{n} X_t \).

Note that the stopping time of Stage I in Algorithm 2 adaptively depends on the samples. Therefore, the Hoeffding’s inequality in Lemma A.1 is not directly applicable. To address this issue, we provide the following variant of the famous maximal inequality.

Lemma A.2 (Maximal Inequality). Let \( N \) and \( M \) be extended real numbers in \( \mathbb{R}^+ \) and \( \mathbb{R}^+ \cup \{+\infty\} \). Let \( \gamma \) be a real number in \( \mathbb{R}^+ \), and let \( \hat{\mu}_n = \sum_{s=1}^{n} X_s / n \) be the empirical mean of \( n \) random variables identically independently distributed according to 1-subGaussian distribution. Then
\[
\mathbb{P}(\exists N \leq n \leq M, \hat{\mu}_n + \gamma \leq 0) \leq \exp\left(-\frac{N\gamma^2}{2}\right). \tag{A.2}
\]

Similar inequality has also been proved in Ménard and Garivier (2017) for bounding the KL divergence between two exponential family distributions for different arms.

Lemma A.3. Let \( \delta > 0 \) be a constant and \( M_1, M_2, \ldots, M_n \) be 1-subGaussian random variables with zero means. Denote \( \hat{\mu}_n = \sum_{s=1}^{n} M_s / n \). Then the following statements hold:

1. for any \( T_1 \leq T \),
\[
\sum_{n=1}^{T} \mathbb{P}\left( \hat{\mu}_n + \sqrt{\frac{4}{n} \log^+(\frac{T_1}{n})} \geq \delta \right) \leq 1 + \frac{4 \log^+(T_1\delta^2)}{\delta^2} + \frac{2}{\delta^2} + \frac{\sqrt{8\pi \log^+(T_1\delta^2)}}{\delta^2}; \tag{A.3}
\]

2. if \( T\delta^2 \geq \epsilon^2 \), then
\[
\sum_{n=1}^{T} \mathbb{P}\left( \hat{\mu}_n + \sqrt{\frac{2}{n} \log \left( \frac{T}{n} \left( \log^2 \frac{T}{n} + 1 \right) \right)} \geq \delta \right)
\leq 1 + \frac{2 \log(T\delta^2(\log^2(T\delta^2) + 1))}{\delta^2} + \frac{2}{\delta^2} + \frac{\sqrt{4\pi \log(T\delta^2(\log^2(T\delta^2) + 1))}}{\delta^2}; \tag{A.4}
\]

3. if \( T\delta^2 \geq 4\epsilon^3 \), then
\[
\mathbb{P}\left( \exists s \leq T : \hat{\mu}_s + \sqrt{\frac{2}{s} \log \left( \frac{T}{s} \left( \log^2 \frac{T}{s} + 1 \right) \right)} + \delta \leq 0 \right) \leq \frac{4(16\epsilon^2 + 1)}{T\delta^2}. \tag{A.5}
\]

Proof of Theorem 3.2. Let \( \tau_1 \) be the number of times each arm is played in Stage I of Algorithm 2 and \( \tau_2 \) be the total number of times arm 2’ is played in Stage III of Algorithm 2. Similar to (3.3),
the regret of Algorithm 2 can be decomposed as follows

\[ R_i(T) \leq \underbrace{DT_1 \mathbb{P} (\tau_1 < T, 1' = 2)}_{I_1} + \underbrace{\Delta \mathbb{E}[\tau_1] + \Delta \mathbb{E}[\tau_2]}_{I_2} + \underbrace{\Delta T \mathbb{P} (\tau_2 < T, a = 2)}_{I_3}. \tag{A.6} \]

**Bounding term I_1:** Let \( X_s \) and \( Y_s \) be the reward of arm 1 and 2 when they are pulled for the \( s \)-th time respectively, \( s = 1, 2, \ldots \). Let \( Z_s = (X_s - Y_s - \Delta) / \sqrt{2} \). Then \( Z_s \) is a 1-subGaussian random variable with zero mean. Let \( S_s = \sum_{i=1}^{s} Z_i \). Recall that \( \hat{\mu}_{k,s} \) is the average reward for arm \( k \) after its \( s \)-th pull. Applying the standard peeling technique, we have

\[
\mathbb{P}(\tau_1 < T, 1' = 2) \leq \mathbb{P} \left( \exists s \in \mathbb{N} : 2s \leq T, \hat{\mu}_{1,s} - \hat{\mu}_{2,s} \leq -\sqrt{8 \log^+(T_1/(2s))} \right) \\
\quad \leq \mathbb{P} \left( \exists s \geq 1 : \frac{\sum_{i=1}^{s} Z_i}{s} \leq -\sqrt{4 \log^+(T_1/(2s)) - \frac{\Delta}{\sqrt{2}}} \right) \\
\quad \leq \sum_{j=0}^{\infty} \mathbb{P} \left( \exists s \in [2^j, 2^{j+1}] : \frac{\sum_{i=1}^{s} Z_i}{s} \leq -\sqrt{4 \log^+(T_1/(2s)) + \frac{\Delta}{\sqrt{2}}} \right) \\
\quad \leq \sum_{j=0}^{\infty} \exp \left( -2^{j-1} \left( \sqrt{\frac{\log^+(T_1/2^{j+2})}{2} + \frac{\Delta}{\sqrt{2}}} \right)^2 \right),
\]

where the last inequality comes from Lemma A.2. Therefore, we have

\[
\mathbb{P}(\tau_1 < T, 1' = 2) \leq \sum_{j=0}^{\infty} \exp \left( -\log^+ \left( \frac{T_1}{2^{j+2}} \right) - 2^{j-2} \Delta^2 \right) \\
\quad = \frac{1}{T_1} \sum_{j=0}^{\infty} 2^{j+2} \exp(-2^{j-2} \Delta^2) \\
\quad \leq \frac{16}{e T_1 \Delta^2} + \frac{1}{T_1} \int_{0}^{\infty} 2^{j+2} \exp(-2^{j-2} \Delta^2) dj \\
\quad \leq \frac{16}{e T_1 \Delta^2} + \frac{-16}{\log 2 \cdot T_1 \Delta^2} \exp(-2^{j-2} \Delta^2) \bigg|_{0}^{\infty} \\
\quad \leq \frac{30}{T_1 \Delta^2}, \tag{A.7}
\]

where the first inequality we used the factor that \((x + y)^2 \geq x^2 + y^2\), \( x, y \geq 0 \), the third inequality follows the fact that the integral function has a maximum value \( 16/(e T_1 \Delta^2) \) and for such function we have \( \sum_{j=0}^{\infty} f(j) \leq \max_{j \in [0, \infty)} f(j) + \int_{0}^{\infty} f(j) dj \). Thus, we have proved that \( DT_1 \mathbb{P}(\tau_1 < T, 1' = 2) \leq 30/\Delta \).

Recall that \( \epsilon_T = \sqrt{2 \log(T \Delta^2) / (T_1 \Delta^2)} \) and \( \epsilon_T \in (0, 1/2) \). Using the same argument in the proof of Theorem 3.1, we can show that the expected total regret for the case that \( \mu' \notin [\mu_{1'} - \epsilon_T \Delta, \mu_{1'} + \epsilon_T \Delta] \) is bounded by \( 2/\Delta \). Moreover, it is easy to see that we also have \( \mathbb{P}(\mu_{1'} - \epsilon_T \Delta \leq \mu' \leq \mu_{1'} + \epsilon_T \Delta) \geq \epsilon_T \).
$1 - 2/(T\Delta^2)$. Therefore, in what follows, we bound terms $I_2$ and $I_3$ conditional on the event that $\mu' \in [\mu_1' - \epsilon_T\Delta, \mu_1' + \epsilon_T\Delta]$.

**Bounding term $I_2$:** By the definition of $\tau_1$ and the stopping rule of Stage I in Algorithm 2, we have

$$
\mathbb{E}[\tau_1] = \sum_{s=1}^{T} \mathbb{P}(\tau_1 \geq s) \leq \sum_{s=1}^{T/2} \mathbb{P}(\hat{\mu}_{1,s} - \hat{\mu}_{2,s} \leq \sqrt{\frac{8\log^+ (T_1/(2s))}{s}}) \\
= \sum_{s=1}^{T/2} \mathbb{P}\left( \frac{\sum_{i=1}^{s} Z_i}{s} \leq \sqrt{\frac{4}{s} \log^+ \left( \frac{T_1}{2s} \right) - \frac{\Delta}{\sqrt{2}}} \right) \\
\leq \sum_{s=1}^{T} \mathbb{P}\left( - \frac{\sum_{i=1}^{s} Z_i}{s} + \sqrt{\frac{4}{s} \log^+ \left( \frac{T_1/2}{s} \right)} \geq \frac{\Delta}{\sqrt{2}} \right) \\
\leq 1 + \frac{8\log^+ (T_1\Delta^2/4)}{\Delta^2} + \frac{4}{\Delta^2} + \frac{2\sqrt{8\pi} \log^+ (T_1\Delta^2/4)}{\Delta^2}, \quad (A.8)
$$

where the equality is by the definition of $\sum_{i=1}^{s} Z_i/s = \sum_{i=1}^{s} (X_i - Y_i - \Delta)/(\sqrt{2}s) = (\hat{\mu}_{1,s} - \hat{\mu}_{2,s} - \Delta)/\sqrt{2}$, and the last inequality is due to the first statement of Lemma A.3 since $Z_i$ are 1-subGaussian variables as well.

To bound $\mathbb{E}[\tau_2]$, we first assume that the chosen arm $1'$ at the end of Stage I of Algorithm 2 is the best arm, i.e., $1' = 1$. Let $\Delta' = \mu' - \mathbb{E}[Y_{i+\tau_1}]$. Then $\Delta' \in [(1 - \epsilon_T)\Delta, (1 + \epsilon_T)\Delta]$. Since $\epsilon_T \in (0, 1/2)$ and $T\Delta^2 \geq 16e^3$, we have $T(\Delta')^2 \geq (1 - \epsilon_T)^2 T\Delta^2 \geq 4e^3$. Let $W_i = \mu' - Y_{i+\tau_1} - \Delta'$. Then $-W_i$ is 1-subGaussian random variable. By the stopping rule of Stage III in Algorithm 2, it holds that

$$
\mathbb{E}[\tau_2] \leq \sum_{t_2=1}^{T} \mathbb{P}(\tau_2 \geq t_2) \\
= \sum_{t_2=1}^{T} \mathbb{P}(\mu' - \theta_{2',t_2} \leq \sqrt{\frac{2}{t_2} \log \left( \frac{T}{t_2} \left( \log^2 \frac{T}{t_2} + 1 \right) \right)}) \\
= \sum_{t_2=1}^{T} \mathbb{P}\left( - \frac{\sum_{i=1}^{t_2} W_i}{t_2} + \sqrt{\frac{2}{t_2} \log \left( \frac{T}{t_2} \left( \log^2 \frac{T}{t_2} + 1 \right) \right)} \geq \Delta' \right) \\
\leq 1 + \frac{2\log(T(\Delta')^2(\log^2(T(\Delta')^2) + 1))}{(\Delta')^2} + \frac{2}{(\Delta')^2} \\
+ \frac{\sqrt{4\pi \log(T(\Delta')^2(\log^2(T(\Delta')^2) + 1))}}{(\Delta')^2}. \quad (A.9)
$$

where the last inequality is due to the second statement of Lemma A.3 and $-W_i$ are 1-subGaussian. When $1' = 2$ is the sub-optimal arm, using the same argument, we can derive same bound as in (A.9) for $\mathbb{E}[\tau_2]$.

**Bounding term $I_3$:** Again, we first assume that the chosen arm $1'$ is the best arm, i.e., $1' = 1$. By
definition, we have $\sum_i W_i/s = \mu' - \theta_{2',s} - \Delta'$ and $W_i$ is 1-subGaussian with zero mean. Recall that we have $T(\Delta')^2 \geq 4\epsilon^3$. By the third statement of Lemma A.3, we have

$$\mathbb{P}(\tau_2 < T, a = 2) \leq \mathbb{P}\left(\exists t_2 \geq 1, \mu' - \theta_{2',t_2} + \sqrt{\frac{2}{t_2} \log \left(\frac{T}{t_2} \left(\log^2 \frac{T}{t_2} + 1\right)\right)} \leq 0\right)$$

$$\leq \mathbb{P}\left(\exists t_2 \geq 1, \mu' - \theta_{2',t_2} - \Delta' + \Delta' + \sqrt{\frac{2}{t_2} \log \left(\frac{T}{t_2} \left(\log^2 \frac{T}{t_2} + 1\right)\right)} \leq 0\right)$$

$$\leq \frac{4(16\epsilon^2 + 1)}{T(\Delta')^2}. \quad (A.10)$$

When $1' = 2$ is sub-optimal. The proof is similar to the previous one. In particular, we only need to change the notations to $\Delta' = \mathbb{E}[X_{i+T_1}] - \mu'$, which also satisfies $\Delta' \in [(1 - \epsilon_T)\Delta, (1 + \epsilon_T)\Delta]$. Hence, we still have (A.10) hold.

**Completing the proof:** Note that the expected total regret for the case that $\mu' \notin [\mu_{1'} - \epsilon_T\Delta, \mu_{1'} + \epsilon_T\Delta]$ is bounded by $2/\Delta$. Therefore, substituting (A.7), (A.8), (A.9) and (A.10) into (A.6), we have

$$R_{\mu}(T) \leq 2\Delta + \frac{36 + 8\log^+(T_1\Delta^2/4) + 2\sqrt{8\pi \log^+(T_1\Delta^2)}}{\Delta}$$

$$+ \frac{482 + 2\log(4T\Delta^2(\log^2(4T\Delta^2) + 1)) + \sqrt{4\pi \log(4T\Delta^2(\log^2(4T\Delta^2) + 1))}}{(1 - \epsilon_T)^2\Delta}.$$

Recall that $\epsilon_T^2 = 2 \log(T\Delta^2)/(T_1\Delta^2)$. Let $T_1 = \log^2 T$. When $T \to \infty$, we have $\epsilon_T \to 0$, and hence $\lim_{T \to \infty} R_{\mu}(T)/T = 2/\Delta$. On the other hand, if we choose $T_1 = T$, since $T\Delta^2 \geq 4\epsilon^3$ and $\epsilon_T \leq 1/2$, the total regret should be $R_{\mu}(T) = O(\Delta + \log(T\Delta^2)/\Delta) = O(\sqrt{T} + \Delta)$.

**B Proof of the Concentration Lemmas**

In this section, we provide the proof of the concentration lemma and the maximal inequality for subGaussian random variables.

**B.1 Proof of Lemma A.2**

*Proof.* From definition of 1-subGaussian random variables, it holds that

$$\mathbb{E}\left[\exp\left(\lambda \sum_{s=1}^{n} - X_s\right)\right] \leq \exp\left(\frac{n\lambda^2}{2}\right), \quad (B.1)$$

for all $\lambda > 0$ and $n \in \mathbb{N}^+$. By definition, we have

$$\mathbb{P}(\exists N \leq n \leq M, \hat{\mu}_n + \gamma \leq 0) = \mathbb{P}\left(\exists N \leq n \leq M, \sum_{s=1}^{n} X_s + n\gamma \leq 0\right)$$

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\[ \leq \mathbb{P} \left( \min_{N \leq n \leq M} \exp \left( \lambda \sum_{s=1}^{n} X_s \right) \leq \exp(\lambda N \gamma) \right) \]

\[ = \mathbb{P} \left( \max_{N \leq n \leq M} \exp \left( \lambda \sum_{s=1}^{n} -X_s \right) \geq \exp(\lambda N \gamma) \right) \]

\[ \leq \frac{\mathbb{E}[\exp(\lambda \sum_{s=1}^{M} -X_s)]}{\exp(\lambda N \gamma)} \]

\[ \leq \exp \left( \frac{M \lambda^2}{2} - \lambda N \gamma \right) \]

\[ \leq \exp \left( -\frac{N \gamma^2}{2} \right), \]

where the second inequality is from Doob’s submartingale inequality (Revuz and Yor, 2013) and the fact that that \( \exp(\lambda \sum_{s=1}^{n} -X_s) \) is a submartingale, the third inequality is due to (B.1) since \( N \) is fixed, and the last inequality holds if we choose \( \lambda = \gamma \).

\[ \square \]

### B.2 Proof of Lemma A.3

To prove Lemma A.3, we also need the following technical lemma from Ménard and Garivier (2017).

**Lemma B.1.** For all \( \beta > 1 \) we have

\[ \frac{1}{e^{\log(\beta)/\beta} - 1} \leq 2 \max\{\beta, \beta/(\beta - 1)\}. \]  

(B.2)

**Proof of Lemma A.3.** For the first statement, let \( \gamma' = 4 \log^+(T_1 \delta^2)/\delta^2 \). Note that for \( n \geq \gamma' \), it holds that \( n \delta^2 \geq 4 \) and

\[ \delta \sqrt{\frac{\gamma'}{n}} = \sqrt{\frac{4}{n} \log^+(T_1 \delta^2)} \geq \sqrt{\frac{4}{n} \log^+(T_1 / n)}. \]  

(B.3)

Therefore, we have

\[ \sum_{n=1}^{T} \mathbb{P} \left( \tilde{\mu}_n + \sqrt{\frac{4}{n} \log^+(\frac{T_1}{n})} \geq \delta \right) \leq \gamma' + \sum_{n=\lceil \gamma' \rceil}^{T} \mathbb{P} \left( \tilde{\mu}_n \geq \delta \left( 1 - \sqrt{\frac{\gamma'}{n}} \right) \right) \]

\[ \leq \gamma' + \sum_{n=\lceil \gamma' \rceil}^{\infty} \exp \left( - \frac{\delta^2 (\sqrt{n} - \sqrt{\gamma'})^2}{2} \right) \]  

(B.4)

\[ \leq \gamma' + 1 + \int_{\gamma'}^{\infty} \exp \left( - \frac{\delta^2 (\sqrt{x} - \sqrt{\gamma'})^2}{2} \right) dx \]

\[ \leq \gamma' + 1 + \frac{2}{\delta^2} \int_{0}^{\infty} \left( \frac{y}{\delta} + \sqrt{\gamma'} \right) \exp(-y^2/2)dy \]

\[ \leq \gamma' + 1 + \frac{2}{\delta^2} + \frac{2 \sqrt{2 \pi \gamma'}}{\delta}, \]  

(B.5)

where (B.4) is the result of Lemma A.1 and (B.5) is due to the fact that \( \int_{0}^{\infty} y \exp(-y^2/2)dy = 1 \).
and \( \int_0^\infty \exp(-y^2/2) dy = \sqrt{2\pi}/2 \). (B.5) immediately implies the claim in the first statement:

\[
\sum_{n=1}^T \mathbb{P}\left( \hat{\mu}_n + \sqrt{\frac{4}{n} \log^+(T_1/n)} \geq \delta \right) \leq \gamma' + \sum_{n=[\sqrt{\gamma}]}^T \mathbb{P}\left( \hat{\mu}_n \geq \delta \left(1 - \sqrt{\frac{\gamma}{n}}\right)\right)
\]

\[
\leq \gamma' + 1 + \frac{2}{\delta^2} + \frac{\sqrt{2\pi\gamma}}{\delta}.
\]  

(B.6)

Plugging \( \gamma' = 4 \log^+(T_1\delta^2)/\delta^2 \) to above equation, we obtain

\[
\sum_{n=1}^T \mathbb{P}\left( \hat{\mu}_n + \sqrt{\frac{4}{n} \log^+(T_1/n)} \geq \delta \right) \leq 1 + \frac{4\log^+(T_1\delta^2)}{\delta^2} + \frac{2}{\delta^2} + \frac{\sqrt{8\pi\log^+(T_1\delta^2)}}{\delta^2}.
\]  

(B.7)

For the second statement, its proof is similar to that of the first one. Let us define the following quantity:

\[
\gamma = \frac{2 \log(T\delta^2(\log^2(T\delta^2) + 1))}{\delta^2}.
\]  

(B.8)

Note that for all \( n \geq \gamma \), it holds that

\[
\delta \sqrt{\frac{\gamma}{n}} = \sqrt{\frac{2 \log(T\delta^2(\log^2(T\delta^2) + 1))}{n}} \geq \sqrt{\frac{2 \log \left( \frac{T}{n} \left( \log^2 \frac{T}{n} + 1 \right) \right)}{n}},
\]  

(B.9)

where we used the fact that \( T\delta^2 \geq e^2 \) and hence \( \delta^2 = 2 \log(T\delta^2(\log^2(T\delta^2) + 1))/\gamma \geq 1/\gamma \geq 1/n \). Therefore, using the same argument in (B.5) we can show that

\[
\sum_{n=1}^T \mathbb{P}\left( \hat{\mu}_n + \sqrt{\frac{2}{n} \log \left( \frac{T}{n} \left( \log^2 \frac{T}{n} + 1 \right) \right)} \geq \delta \right) \leq 1 + \frac{2 \log(T\delta^2(\log^2(T\delta^2) + 1))}{\delta^2} + \frac{2}{\delta^2} + \frac{\sqrt{4\pi\log(T\delta^2(\log^2(T\delta^2) + 1))}}{\delta^2}.
\]

To prove the last statement, we borrow the idea from Ménard and Garivier (2017) for proving the regret of kl-UCB++. Define \( f(\delta) = 2/\delta^2 \log(T\delta^2/4) \). Then we can decompose the event \( \{\exists s : s \leq T\} \) into two cases: \( \{\exists s : s \leq f(\delta)\} \) and \( \{\exists s : f(\delta) \leq s \leq T\} \).

\[
\mathbb{P}\left( \exists s \leq T : \hat{\mu}_s + \sqrt{\frac{2}{s} \log \left( \frac{T}{s} \left( \log^2 \frac{T}{s} + 1 \right) \right)} + \delta \leq 0 \right)
\]

\[
\leq \mathbb{P}\left( \exists s \leq f(\delta) : \hat{\mu}_s \leq -\sqrt{\frac{2}{s} \log \left( \frac{T}{s} \left( \log^2 \frac{T}{s} + 1 \right) \right)} \right) + \mathbb{P}(\exists s, f(\delta) \leq s \leq T : \hat{\mu}_s \leq -\delta).
\]  

(B.10)

Note that when \( T\delta^2 \geq 4e^3 \), \( f(\delta) \geq 0 \). Let \( \beta > 1 \) be a parameter that will be chosen later. Applying
the peeling technique, we can bound term $A_1$ as follows.

\[ A_1 \leq \sum_{\ell=0}^{\infty} \mathbb{P} \left( \exists s, \frac{f(\delta)}{\beta^{\ell+1}} \leq s \leq \frac{f(\delta)}{\beta^{\ell}} : \hat{\mu}_s + \sqrt{\frac{2}{s} \log \left( \frac{T}{s} \left( \log^2 \frac{T}{s} + 1 \right) \right)} \leq 0 \right). \quad (B.11) \]

For each $\ell = 0, 1, \ldots$, define $\gamma_\ell$ to be

\[ \gamma_\ell = \frac{\beta^\ell}{f(\delta)} \log \left( \frac{T \beta^\ell}{2f(\delta)} \left( 1 + \log^2 \frac{T}{2f(\delta)} \right) \right), \quad (B.12) \]

which by definition immediately implies

\[ \sqrt{2\gamma_\ell} = \sqrt{\frac{2\beta^\ell}{f(\delta)} \log \left( \frac{T \beta^\ell}{2f(\delta)} \left( 1 + \log^2 \frac{T}{2f(\delta)} \right) \right)} \leq \sqrt{\frac{2}{s} \log \left( \frac{T}{2s} \left( \log^2 \frac{T}{s} + 1 \right) \right)}, \]

where in the above inequality we used the fact that $s \leq f(\delta)/\beta^\ell$ and that $f(\delta) \geq s/2$ since $\beta > 1$. Therefore, we have

\[ \mathbb{P} \left( \exists s, \frac{f(\delta)}{\beta^{\ell+1}} \leq s \leq \frac{f(\delta)}{\beta^{\ell}} : \hat{\mu}_s + \sqrt{2\gamma_\ell} \leq 0 \right) \leq \mathbb{P} \left( \exists \frac{f(\delta)}{\beta^{\ell+1}} \leq s \leq \frac{f(\delta)}{\beta^{\ell}} : \hat{\mu}_s + \sqrt{2\gamma_\ell} \leq 0 \right) \leq \exp \left( -\frac{f(\delta)}{\beta^{\ell+1} \gamma_\ell} \right) = e^{-\ell \log(\beta)/\beta - C/\beta}, \quad (B.13) \]

where the second inequality is by Doob’s maximal inequality (Lemma A.2), the last equation is due to the definition of $\gamma_\ell$, and the parameter $C$ is defined to be

\[ C := \log \left( \frac{T}{2f(\delta)} \left( 1 + \log^2 \frac{T}{2f(\delta)} \right) \right). \quad (B.14) \]

Substituting (B.13) back into (B.11), we get

\[ A_1 \leq \sum_{\ell=0}^{\infty} e^{-\ell \log(\beta)/\beta - C/\beta} = \frac{e^{-C/\beta}}{1 - e^{-\log(\beta)/\beta}} \leq \frac{e^{1-C/\beta}}{e^{\log(\beta)/\beta} - 1} \leq 2e \max(\beta, \beta/(\beta - 1)) e^{-C/\beta}, \]

where the second inequality is due to $\log \beta \leq \beta$ and thus $e^{\log(\beta)/\beta} \leq e$, and the last inequality comes from Lemma B.1. Since $T \delta^2 \geq 4e^3$, we have $T/(2f(\delta)) = T \delta^2/(4 \log(T \delta^2/4)) \geq \sqrt{T \delta^2/4} \geq e^{3/2}$, which further implies

\[ C = \log \left( \frac{T}{2f(\delta)} \left( 1 + \log^2 \frac{T}{2f(\delta)} \right) \right) \geq \log \left( \frac{T}{2f(\delta)} \right) = \log \left( \frac{T \delta^2}{4 \log(T \delta^2/4)} \right) \geq \frac{3}{2}. \quad (B.15) \]

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Now we choose \( \beta := C/(C - 1) \), so that \( 1 < \beta \leq 2C \) and \( \beta/(\beta - 1) = C \). Together with the definition of \( f \), this choice immediately yields \( A_1 \leq 4eCe^{-C/\beta} = 4e^2Ce^{-C} \). Note that

\[
Ce^{-C} = \left( \frac{T}{2f(\delta)} \left( 1 + \log^2 \frac{T}{2f(\delta)} \right) \right)^{-1} \log \left( \frac{T}{2f(\delta)} \left( 1 + \log^2 \frac{T}{2f(\delta)} \right) \right) \\
\leq \frac{2f(\delta)}{T \log^2(T/(2f(\delta)))} \log \left( \frac{T}{2f(\delta)} \left( 1 + \log^2 \frac{T}{2f(\delta)} \right) \right) \\
\leq \frac{4f(\delta)}{T \log(T/(2f(\delta)))} \\
= \frac{8 \log(T\delta^2/4)}{T\delta^2 \log([T\delta^2/4] / \log(T\delta^2/4))} \\
\leq \frac{16}{T\delta^2},
\]

where in the second and the third inequalities, we used the fact that for all \( x \geq e^{3/2} \),

\[
\frac{\log(x(1 + \log^2 x))}{\log x} \leq 2 \quad \text{and} \quad \frac{\log x}{\log(x/\log x)} \leq 2.
\]

Therefore, we have proved so far \( A_1 \leq 64e^2/(T\delta^2) \). For term \( A_2 \) in (B.10), we can again apply the maximal inequality in Lemma A.2 and obtain

\[
A_2 = \mathbb{P}(\exists s, f(\delta) \leq s \leq T : \hat{\mu}_s \leq -\delta) \leq e^{-\delta^2 f(\delta)/2} = \frac{4}{T\delta^2}.
\]

Finally, combining the above results, we get

\[
\mathbb{P} \left( \exists s \leq f(\delta), \hat{\mu}_s + \sqrt{\frac{2}{s} \log \left( \frac{T}{s} \left( \log^2 \frac{T}{s} + 1 \right) \right)} + \delta \leq 0 \right) \leq \frac{4(16e^2 + 1)}{T\delta^2}.
\]

This completes the proof.

\[\square\]

C Round Complexity of DETC for Batched Bandits with Unknown Gaps

In this section, we derive the round complexity of Algorithm 2 for batched bandits and prove that it still enjoys the asymptotic optimality. Note that in this setting, our focus is on the asymptotic regret bound and thus we assume that \( T \) is sufficiently large in the following proof to simplify the presentation.

Proof of Theorem 4.3. For the sake of simplicity, we use the same notations that are used in Theorem 3.2 and its proof. To compute the round complexity and regret of Stage I, we first compute the probability that \( \tau_1 > 2t\sqrt{\log T} \). We assume \( T \) is large enough such that it satisfies

\[
\sqrt{\log T} \geq 16 \log^+(T_1 \Delta^2/2)/\Delta^2,
\]

\[\text{(C.1)}\]
where we recall that \( T_1 = \log^2 T \). Let \( s_i = 2i\sqrt{\log T} \) for \( i = 1, 2, \ldots \) and \( \gamma = 4\log^+(T_1 \Delta^2/2)/\Delta^2 \).

From (C.1), it is easy to verify that \( s_i \geq 32i/\Delta^2 \), \( \gamma/s_i \leq 1/8 \) and \( \sqrt{4\log^+(T_1/2s_i)/s_i} \leq \Delta \sqrt{\gamma/s_i} \).

The stopping rule in \textit{Stage I} implies

\[
\mathbb{P}(\tau_1 \geq s_i) \leq \mathbb{P}\left( \hat{\mu}_{1,s_i} - \hat{\mu}_{2,s_i} \leq \sqrt{\frac{8}{s_i} \log^+ \left( \frac{T_1}{2s_i} \right)} \right)
\]

\[
= \mathbb{P}\left( \sum_{i=1}^{s_i} \frac{Z_i}{s_i} \leq \sqrt{\frac{4}{s_i} \log^+ \left( \frac{T_1}{2s_i} \right) - \frac{\Delta}{\sqrt{2}}} \right)
\]

\[
\leq \mathbb{P}\left( \sum_{i=1}^{s_i} \frac{Z_i}{s_i} \leq \Delta \sqrt{\frac{\gamma}{s_i} - \frac{\Delta}{\sqrt{2}}} \right)
\]

\[
\leq \exp\left( - \frac{s_i \Delta^2}{2} \left( \frac{1}{\sqrt{2}} - \sqrt{\frac{\gamma}{s_i}} \right)^2 \right)
\]

\[
\leq \exp(-i) \leq 2^{-i},
\]

where the third inequality follows from Lemma A.1 and the fourth inequality is due to the fact that \( s_i \geq 32i/\Delta^2 \) and \( \gamma/s_i \leq 1/8 \). Hence by the choice of testing points in (4.3), the expected number of rounds needed in \textit{Stage I} of Algorithm 2 is upper bounded by \( \sum_{i=1}^{\infty} i/2^i \leq 2 \). The expectation of \( \tau_1 \) is upper bounded by \( \mathbb{E}[\tau_1] \leq \sum_{i=1}^{\infty} 2i \sqrt{\log T}/2^i \leq 4\sqrt{\log T} \), which matches the bound derived in (A.8).

Now we focus on bounding term \( \Delta \mathbb{E}[\tau_2] \) and the round complexity in \textit{Stage III}. Without loss of generality, we assume \( \mu' = 1 \). Similar to the argument in the proof of Theorem 4.1, we can show that for the case \( \mu' \notin [\mu_T - \epsilon_T^2 \Delta, \mu_T + \epsilon_T^2 \Delta] \), here \( \epsilon_T = \sqrt{2} \epsilon_T \). Applying Lemma A.1 and note that \( \epsilon_T = \sqrt{4\log(T \Delta^2)/(T_1 \Delta^2)} \), we have \( \mathbb{P}(\mu' \notin [\mu_T - \epsilon_T^2 \Delta, \mu_T + \epsilon_T^2 \Delta]) \leq 2/(T^2 \Delta^4) \), the expected number of test time points in \textit{Stage III} is \( O(1/(T \Delta^4)) \) which goes to zero when \( T \to \infty \). Therefore, in the rest of the proof, we derive the results conditional on the event that \( \mu' \in [\mu_T - \epsilon_T^2 \Delta, \mu_T + \epsilon_T^2 \Delta] \).

Recall that this condition also implies \( \Delta' \in [(1 - \epsilon_T^2)\Delta, (1 + \epsilon_T^2)\Delta] \), where \( \epsilon_T = \sqrt{\log(T \Delta^2)/(T_1 \Delta^2)} \) and \( T_1 = \log^2 T \). When \( T \) is large enough such that it satisfies

\[
\sqrt{\frac{4\log(T \Delta^2)}{\Delta^2 \log^2 T}} \leq \frac{1}{(\log T)^{1/3}},
\]

we have \( \epsilon_T \leq 1/(\log T)^{1/3} \). Furthermore, we can also choose a large \( T \) such that

\[
\sqrt{\log T (\Delta')^2} \geq 2(\log \log T)^2.
\]

Applying Lemma A.1, we have

\[
\mathbb{P}\left( \mu_T - \Delta' (\log T)^{-1/4} \leq \theta_{x',N_1} \leq \mu_T + \Delta' (\log T)^{-1/4} \right) \geq 1 - 2 \exp\left( - \frac{2 \log T (\Delta')^2}{2 \sqrt{\log T \log \log T}} \right)
\]

\[
\geq 1 - \frac{2}{\log^2 T},
\]

(C.5)
where the last inequality follows by (C.4). This means that after the first round of Stage III in Algorithm 2, the average reward for arm 2' concentrates around the true value $\mu_{2'}$ with a high probability.

We first consider the case that $\mu_{2'} - \Delta' \sqrt{\log T} \leq \theta_{2', N_1} \leq \mu_{2'} + \Delta' \sqrt{\log T}$. Define

$$s'_i = \frac{2(1 + 1/\sqrt{T})^2 \log(T \log^3 T)}{\Delta^2} + i(1 + 1/\sqrt{T})^2 (\log T)^{3/2},$$

$$\gamma' = \frac{2 \log (T(\Delta')^2[\log^2 (T(\Delta')^2) + 1])}{(\Delta')^2},$$

for $i = 1, 2, \ldots$. Recall the definition of test time points in (4.5), we know that the $(i + 1)$-th test in Stage III happens at time step $t_2 = s'_i$. We choose a large enough $T$ such that

$$\log^3 T \geq (\Delta')^2 (\log^2 (T(\Delta')^2) + 1). \quad (C.6)$$

Recall that $\mu' - \mu_{2'} = \Delta'$. Hence $\hat{\Delta} = \mu' - \theta_{2', N_1} \in [(1 - 1/\sqrt{T})\Delta', (1 + 1/\sqrt{T})\Delta']$. Then we have

$$2(1 + 1/\sqrt{T})^2 \log(T \log^3 T) \geq \frac{2 \log(T \log^3 T)}{(\Delta')^2} \geq \gamma', \quad (C.7)$$

where the last inequality is due to (C.6). On the other hand, we also have

$$s'_i \geq \frac{2(1 + 1/\sqrt{T})^2 \log(T \log^3 T)}{\Delta^2} \geq \frac{2}{(\Delta')^2}. \quad (C.8)$$

Therefore, by the definition of $\gamma'$, it holds that

$$\Delta' \sqrt{\frac{\gamma'}{s'_i}} = \sqrt{\frac{2}{s'_i} \log(T(\Delta')^2[\log^2 (T(\Delta')^2) + 1])} \geq \sqrt{\frac{2}{s'_i} \log \left( \frac{T}{s'_i} \left( \log^2 \left( \frac{T}{s'_i} \right) + 1 \right) \right)}.$$

Recall the definition $W_i = \mu' - Y_{i+\tau_1} - \Delta'$ used in (A.9). From the stopping rule of Stage III in Algorithm 2, we obtain

$$\mathbb{P}(\tau_2 \geq s'_i) \leq \mathbb{P} \left( \mu' - \theta_{2', s'_i} \leq \sqrt{\frac{2}{s'_i} \log \left( \frac{T}{s'_i} \left( \log^2 \left( \frac{T}{s'_i} \right) + 1 \right) \right)} \right)$$

$$= \mathbb{P} \left( \sum_{i=1}^{s'_i} W_i + \Delta' \leq \sqrt{\frac{2}{s'_i} \log \left( \frac{T}{s'_i} \left( \log^2 \left( \frac{T}{s'_i} \right) + 1 \right) \right)} \right)$$

$$\leq \exp \left( - \frac{(s'_i \Delta')^2}{2} \left( 1 - \sqrt{\frac{\gamma'}{s'_i}} \right)^2 \right)$$

$$= \exp \left( - \frac{(\Delta')^2}{2} \left( \sqrt{s'_i} - \sqrt{\gamma'} \right)^2 \right)$$

$$= \exp \left( - \frac{(\Delta')^2}{2} \left( \frac{s'_i - \gamma'}{\sqrt{s'_i} + \sqrt{\gamma'}} \right)^2 \right)$$

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The expectation of $T$ where the second inequality from Lemma A.1 and in the last inequality we used the fact that $s'_i - \gamma' \geq i(1 + \sqrt[3]{\log T})^2(\log T)^{\frac{2}{3}}/(\Delta^2) \geq i(\log T)^{\frac{2}{3}}/(\Delta')^2$ by (C.7). Choose sufficiently large $T$ to ensure

$$(\log T)^{\frac{2}{3}} \geq 8s'_i(\Delta')^2.$$  

(C.10)

Substituting (C.10) back into (C.9) yields $P(\tau_2 \geq s'_i) \leq 1/2^i$. Then the expected rounds used in Stage III of Algorithm 2 is upper bounded by $\sum_{i=1}^{\infty} i/2^i \leq 2$. Recall that from (C.3), $\epsilon'_p \leq 1/(\log T)^{\frac{3}{4}}$. The expectation of $\tau_2$ is upper bounded by

$$\mathbb{E}[\tau_2] \leq s'_1 + \sum_{i=2} (s'_i - s'_1)P(\tau_2 \geq s'_i)$$

$$\leq \frac{2(1 + 1/(\log T)^{\frac{1}{2}})^2\log(T\log T)}{\Delta^2} + \frac{2(1 + 1/\sqrt{\log T})^2(\log T)^{\frac{2}{3}}}{\Delta^2}$$

$$\leq \frac{2(1 + 1/(\log T)^{\frac{1}{2}})^2\log(T\log T) + 2(1 + 1/(\log T)^{\frac{1}{2}})^2(\log T)^{\frac{2}{3}}}{(1 - 1/(\log T)^{\frac{1}{2}})^2(1 - 1/(\log T)^{\frac{1}{2}})^2\Delta^2},$$

(C.11)

where the last inequality is due to $\Delta' \in [(1 - \epsilon'_p)\Delta, (1 + \epsilon'_p)\Delta]$.

For the case $\theta_{2, N_1} \notin [\mu_{2'} - \Delta/(\log T)^{\frac{1}{2}}, \mu_{2'} + \Delta/(\log T)^{\frac{1}{2}}]$. Note that $\tau_2 \leq \log^2 T$ and we have $P(\theta_{2, N_1} \notin [\mu_{2'} - \Delta/(\log T)^{\frac{1}{2}}, \mu_{2'} + \Delta/(\log T)^{\frac{1}{2}}]) \leq 2/\log^2 T$ by (C.5). Therefore $\mathbb{E}[\tau_2]$ can be upper bounded by 2, which is dominated by (C.11). By (4.5), the gap of neighboring test time points is at least $(1 + 1/\sqrt{\log T})^2(\log T)^{\frac{2}{3}}/(\Delta^2)$ and $\tau_2 \leq \log^2 T$. The expected rounds is upper bounded by

$$P(\theta_{2, N_1} \notin [\mu_{2'} - \Delta/(\sqrt{\log T}), \mu_{2'} + \Delta/(\sqrt{\log T})]) \cdot \frac{\tau_2^2\Delta^2}{(1 + 1/\sqrt{\log T})^2(\log T)^{\frac{2}{3}}}$$

$$\leq \frac{2\Delta^2}{(1 + 1/\sqrt{\log T})^2(\log T)^{\frac{2}{3}}}$$

$$\leq \frac{2(1 + (\log T)^{\frac{1}{2}})^2\Delta^2}{(\log T)^{\frac{2}{3}}},$$

(C.12)

which is less than 1 if $T$ is chosen large enough. Hence, the expected round cost is at most 1.

Note that the above analysis does not change the regret incurred in Stage III. A slight difference of this proof from that of Theorem 3.2 arises when we terminate Stage III with $t_2 = \log^2 T$. In this case, we have tested more than $\log^2 T$ samples for both arm 1 and 2. Let $G_0 = 0$ and $G_n = (X_1 - Y_{1+n}) + \cdots + (X_n - Y_{n+n})$ for every $n \geq 1$. Then $X_i - Y_{i+n} - \Delta$ is a $\sqrt{2}$-subGaussian random variable. Applying Lemma A.1 with $\epsilon = \Delta$ yields

$$P\left(\frac{G_{\tau_2}}{\tau_2} \leq 0\right) \leq \exp\left(-\frac{\tau_2 \Delta^2}{4}\right).$$

Note that $\tau_2 = \log^2 T$, we further obtain $P(a = 2) = P(G_{\tau_2} \leq 0) \leq \exp(-\Delta^2 \log^2 T/4) \leq 1/T$, where
in the last inequality we again choose large enough $T$ to ensure
\[
\exp(-\Delta^2 \log^2 T/4) \leq \frac{1}{T}.
\]  
(C.13)

Therefore, we have proved that
\[
P(a = 2) \leq \frac{1}{T}.
\]  
(C.14)

To summarize, we can choose a sufficiently large $T$ such that all the conditions (C.1), (C.3), (C.4), (C.6), (C.10) and (C.13) are satisfied simultaneously. Then the round complexity of Algorithm 2 is $O(1)$. Since the only difference of our batched algorithm from Algorithm 2 is the stopping rules of Stage I and Stage III, we only need to combine the regret for terms (C.11) and (C.14) and note that $\Delta E[\tau_1] \leq 4\Delta \sqrt{\log T}$ to obtain the total regret. We can get that for $T \to \infty$, $\lim_{T \to \infty} R(T)/\log T = 2/\Delta$. \qed

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