A Dunkl-Gamma Type Operator in Terms of Generalization of Two-Variable Hermite Polynomials

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Abstract. The goal of this paper is to present a Dunkl-Gamma type operator with the help of generalization of the two-variable Hermite polynomials and to derive its approximating properties via the classical modulus of continuity, second modulus of continuity and Peetre’s $K$-functional.

1. Introduction

By now, several research workers have investigated linear positive operators and their approximation properties, see for instance [1], [2], [3], [4], [5], [6] and references so on. Furthermore, many authors have studied linear positive operators containing generating functions and given some approximation properties of these operators. To see such operators, we give the references such as Altın et. al [7], Doğru et. al [8], Krech [9], Olgun et. al [10], Sucu et. al [11], Taşdelen et. al [12], Varma et. al [13, 14].

Latterly, with the help of Dunkl exponential function, several authors have defined some linear positive operators. First of them is a Dunkl analogue of Szász operators given in [15] as follows:

$$S_n^*(g; x) = \frac{1}{e_\nu(nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_\nu(k)} g \left( \frac{k + 2\nu \theta_k}{n} \right) ; \quad n \in \mathbb{N}, \nu, \ x \in [0, \infty) \quad (1.1)$$

for $g \in C[0, \infty)$. Here the Dunkl exponential function is defined by

$$e_\nu(x) = \sum_{k=0}^{\infty} x^k \frac{\gamma_\nu(k)}{\gamma_\nu(k)} \quad (1.2)$$

for $\nu > -\frac{1}{2}$ and the coefficients $\gamma_\nu$ are given by

$$\gamma_\nu(2k) = \frac{2^{2k} k! \Gamma(k + \nu + 1/2)}{\Gamma(k + \nu + 1/2)} \quad \text{and} \quad \gamma_\nu(2k + 1) = \frac{2^{2k+1} k! \Gamma(k + \nu + 3/2)}{\Gamma(k + \nu + 1/2)} \quad (1.3)$$

Also, for the coefficients $\gamma_\nu$, the following recursion relation holds

$$\frac{\gamma_\nu(k + 1)}{\gamma_\nu(k)} = (2\nu \theta_{k+1} + k + 1), \quad k \in \mathbb{N}_0, \quad (1.4)$$

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where \( \theta_k \) is defined by

\[
\theta_k = \begin{cases} 
0, & \text{if } k = 2p \\
1, & \text{if } k = 2p + 1
\end{cases}
\tag{1.5}
\]

for \( p \in \mathbb{N}_0 \) in [16]. Then, İçöz and Çekim have given a Stancu-type generalization of Szász-Kantorovich operators and \( q \)-Szász operators with the help of the Dunkl exponential function in [17, 18].

Next, Wafi and Rao [19] has introduced Szász–Gamma operators based on Dunkl analogue as

\[
D_n^f(x) = \frac{1}{e_{\mu}(nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_{\mu}(k)} \frac{n^{k+2\mu\theta_k+\lambda+1}}{\Gamma(k+2\mu\theta_k+\lambda+1)} \int_0^\infty t^{k+2\mu\theta_k+\lambda} e^{-nt} f(t) dt,
\tag{1.6}
\]

where \( \lambda \geq 0 \) and \( \Gamma \) is Gamma function defined by

\[
\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \text{ for } x > 0.
\tag{1.7}
\]

Finally, Aktaş et. al [20] has introduced the operator \( T_n(f; x) \) for \( n \in \mathbb{N} \)

\[
T_n(f; x) := \frac{1}{e_{\alpha x^2} e_{\mu}(nx)} \sum_{k=0}^{\infty} \frac{h_{\mu}^n(n, \alpha)}{\gamma_{\mu}(k)} x^k f\left(\frac{k + 2\mu\theta_k}{n}\right),
\]

where \( \alpha \geq 0, \mu \geq 0 \) and \( x \in [0, \infty) \), via the Dunkl generalization of two-variable Hermite polynomials, \( h_{\mu}^n(\xi, \alpha) \) in [21] defined as follows

\[
\sum_{n=0}^{\infty} \frac{h_{\mu}^n(\xi, \alpha)}{\gamma_{\mu}(n)} t^n = e^{\alpha t^2} e_{\mu}(\xi t).
\tag{1.8}
\]

Here

\[
h_{\mu}^n(\xi, \alpha) = \frac{\gamma_{\mu}(n) H_{\mu}^n(\xi, \alpha)}{n!}
\]

and \( H_{\mu}^n(\xi, \alpha) \) has the following explicit representation

\[
H_{\mu}^n(\xi, \alpha) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{\alpha^k \xi^{n-2k}}{k! \gamma_{\mu}(n-2k)}.
\]

We note that \( H_{\mu}^n(\xi, \alpha) \) reduces to the two-variable Hermite polynomials defined by

\[
H_n(\xi, \alpha) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{\alpha^k \xi^{n-2k}}{k!(n-2k)!}
\tag{1.9}
\]

as \( \mu = 0 \), see detail [22]. In the case of \( \mu = 0 \), the operator \( T_n(f; x) \) gives the operator \( G_n^\alpha(f; x) \) defined by Krech [9] as follows

\[
G_n^\alpha(f; x) = e^{-(nx^2+\alpha x^2)} \sum_{k=0}^{\infty} \frac{x^k}{k!} H_k(n, \alpha) f\left(\frac{k}{n}\right), \quad x \in \mathbb{R}^+_0 := [0, \infty),
\]

where \( H_k \) is the two variable Hermite polynomial in [13]. Furthermore, recently, some sequences of Dunkl operators and Dunkl-Gamma type operators in terms of Appell polynomials have been defined and approximation properties of these operators have been investigated [23, 24].

The paper is organized as follows. In the next section, we introduce a Dunkl-Gamma type operator consisting of the generalization of two-variable Hermite polynomials. In the third section, the rates of convergence of the operator are obtained.
by means of the classical modulus of continuity, second modulus of continuity, Peetre’s $K$-functional and the Lipschitz class $\text{Lip}_M(\gamma)$.

2. The Dunkl-Gamma Type Operator

Firstly, before we introduce our operator, let us give some features and results related to $h^\mu_n(\xi, \alpha)$ generated by the Dunkl generalization of two-variable Hermite polynomials in [13].

We first recall the following definition and lemma in [16].

**Definition 1.** [16] Assume that $\mu \in \mathbb{C}, x \in \mathbb{C}$. On all entire functions $\varphi$ on $\mathbb{C}$, Rosenblum defines the linear operator $D_\mu$ as follows:

$$D_{\mu,x}(\varphi(x)) = (D_{\mu}\varphi)(x) = \varphi'(x) + \frac{\mu}{x}(\varphi(x) - \varphi(-x)), \quad x \in \mathbb{C}. \quad (2.1)$$

**Lemma 1.** [16] Assume that $\varphi, \psi$ are entire functions. With the help of the linear operator $D_\mu$, the following relations are satisfied:

i) $D^j_\mu : x^n \to \frac{\gamma(n)}{\gamma(n-j)}x^{n-j}, \quad j = 0, 1, 2, ..., n (n \in \mathbb{N}); \quad D_\mu^j : 1 \to 0$,

ii) $D_\mu(\varphi\psi) = D_\mu(\varphi)\psi + \varphi D_\mu(\psi)$, if $\varphi$ is an even function,

iii) $D_\mu : e^\mu(\lambda x) \to \lambda e^\mu(\lambda x)$.

By using this definition and Lemma[1] the results in the next lemma hold true (see detail [20]).

**Lemma 2.** [20] $h^\mu_n(\xi, \alpha)$ has the following results

(i) $\sum_{n=0}^{\infty} h^\mu_n(\xi, \alpha) x^n = (\xi + 2\alpha t)e^{\alpha t^2}e^\mu(\xi t)$, 

(ii) $\sum_{n=0}^{\infty} h^\mu_n(\xi, \alpha) x^n = (\xi^2 + 4\xi \alpha t + 4\alpha^2 t^2 + 2\alpha)e^{\alpha t^2}e^\mu(\xi t) + 4\alpha e^{\alpha t^2}e^\mu(-\xi t)$.

Now we can define our operator as follows:

**Definition 2.** Via $h^\mu_n(\xi, \alpha)$ given in [13], we consider the operator $S_n(f; x)$, $n \in \mathbb{N}$ given by

$$S_n(f; x) := \frac{1}{e^{\alpha x^2}e^\mu(nx)} \sum_{k=0}^{\infty} h^\mu_k(n, \alpha) x^k \frac{n^{k+2\mu+\lambda+1}}{\Gamma(k+2\mu+\lambda+1)} \int_0^\infty t^{k+2\mu+\lambda} e^{-nt}\,f(t)\,dt \quad (2.2)$$

where $\alpha \geq 0$, $\mu > \frac{1-\lambda}{2}$, $\lambda \geq 0$ and $x \in [0, \infty)$. We note that the operator in (2.2) is positive and linear. For $\alpha = 0$, it reduces to $D^j_n(x)$ given by [16].

**Lemma 3.** The following equations can be derived from the definition of the operator $S_n(f; x)$:

i) $S_n(1; x) = 1$,

ii) $S_n(t; x) = x + 2\alpha x^2 + \frac{\lambda + 1}{n}$,

iii) $S_n(t^2; x) = \left[ \frac{x}{n} \right]^2 \left\{ n^2 + 4\alpha x + 4\alpha^2 x^2 + 2\alpha + 4\alpha e^{(-nx)} \right\} e^\mu(-nx) + 2\mu e^{(-nx)} e^\mu(nx) + \frac{2(\lambda + 2)}{n^2} (n + 2\alpha x)x + \frac{(\lambda + 1)(\lambda + 2)}{n}$.
By using the above equation and the generating function in (1.8), we get the relation (i). Using the definition of Gamma function again, we have

\[ \int_0^\infty t^{k+2\mu\theta_k+\lambda+1} e^{-nt} \, dt = \frac{\Gamma(k+2\mu\theta_k+\lambda+2)}{n^{k+2\mu\theta_k+\lambda+2}}. \]

Thus we get the relations

\[
S_n(t; x) = \frac{1}{e^{\alpha x^2} \mu(n x)} \sum_{k=0}^{\infty} h_k^{\mu}(n, \alpha) \frac{x^k}{\gamma_\mu(k)} \frac{n^{k+2\mu\theta_k+\lambda+1}}{\Gamma(k+2\mu\theta_k+\lambda+1)} \frac{\Gamma(k+2\mu\theta_k+\lambda+2)}{n^{k+2\mu\theta_k+\lambda+2}} \]

The second series in right hand side of the above equation from the generating function in (1.8) is \(\frac{(\lambda+1)}{n}\). Also, if we use the recursion relation in (1.4) for the first term, we get

\[
S_n(t; x) = \frac{1}{n e^{\alpha x^2} \mu(n x)} \sum_{k=1}^{\infty} h_k^{\mu}(n, \alpha) \frac{x^k}{\gamma_\mu(k-1)} + \frac{\lambda+1}{n}.
\]

While we are substituting \(k\) by \(k+1\) and using Lemma 2 (i), we arrive at the relation (ii). From the definition of Gamma function in (1.7) again, the following equality holds

\[ \int_0^\infty t^{k+2\mu\theta_k+\lambda+2} e^{-nt} \, dt = \frac{\Gamma(k+2\mu\theta_k+\lambda+3)}{n^{k+2\mu\theta_k+\lambda+3}}, \]

from which, it follows

\[
S_n(t^2; x) = \frac{1}{e^{\alpha x^2} \mu(n x)} \sum_{k=0}^{\infty} h_k^{\mu}(n, \alpha) \frac{x^k}{\gamma_\mu(k)} \frac{n^{k+2\mu\theta_k+\lambda+1}}{\Gamma(k+2\mu\theta_k+\lambda+1)} \frac{\Gamma(k+2\mu\theta_k+\lambda+3)}{n^{k+2\mu\theta_k+\lambda+3}} \]

The third term in right hand side of the above equation from the generating function in (1.8) is \(\frac{\lambda+2}{n^2}\). Also by taking into account the recursion relation in (1.4) the
for first and second series, we obtain
\[
S_n(t^2; x) = \frac{x}{n^2e^{ax^2}e_{\mu}(nx)} \sum_{k=0}^{\infty} (k + 2\mu \theta_{k+1} + \lambda + 2) \frac{h_{k+1}^\mu(n, \alpha)}{\gamma_{\mu}(k)} x^k \\
+ \frac{(\lambda + 2)x}{n^2e^{ax^2}e_{\mu}(nx)} \sum_{k=0}^{\infty} \frac{h_{k+1}^\mu(n, \alpha)}{\gamma_{\mu}(k)} x^k + \frac{(\lambda + 2)(\lambda + 1)}{n^2}.
\]

Using the equation
\[
\theta_{k+1} = \theta_k + (-1)^k
\]
in [16], it yields
\[
S_n(t^2; x) = \frac{x}{n^2e^{ax^2}e_{\mu}(nx)} \sum_{k=0}^{\infty} (k + 2\mu \theta_{k} + \lambda + 2) \frac{h_{k+1}^\mu(n, \alpha)}{\gamma_{\mu}(k)} x^k \\
+ \frac{2\mu x}{n^2e^{ax^2}e_{\mu}(nx)} \sum_{k=0}^{\infty} \frac{h_{k+1}^\mu(n, \alpha)}{\gamma_{\mu}(k)} (-x)^k \\
+ \frac{2(\lambda + 2)x}{n^2e^{ax^2}e_{\mu}(nx)} \sum_{k=0}^{\infty} \frac{h_{k+1}^\mu(n, \alpha)}{\gamma_{\mu}(k)} x^k + \frac{(\lambda + 2)(\lambda + 1)}{n^2}.
\]

Finally using the recursion relation in (2.3) in the first series, from Lemma 2 (i) for the second and third series and Lemma 4 (ii) for the first series, we complete the proof of (iii).

Remark 1. In case of $\alpha = 0$, the results of Lemma 3 reduce to the results in the paper of Wafi and Rao in [19].

Lemma 4. From the results of Lemma 3 and the linearity of the operator, we can obtain the next results for $S_n$ operator
\[
\Lambda_1 = S_n(t - x; x) = \frac{2\alpha x^2 + \lambda + 1}{n}, \\
\Lambda_2 = S_n((t - x)^2; x) = \frac{1}{n} \left[ \frac{x^2}{n} (4x^2\alpha^2 + 4\lambda\alpha + 10\alpha) + 2x \left( \mu \frac{e_{\mu}(nx)}{e_{\mu}(nx)} + 1 \right) + \frac{(\lambda + 1)(\lambda + 2)}{n} \right].
\]

Taking into account the inequality $\frac{e_{\mu}(nx)}{e_{\mu}(nx)} \leq 1$ for $x \geq 0$ and $\mu > \frac{1}{2}$ and $\frac{e_{\mu}(nx)}{e_{\mu}(nx)} \to 0$ as $x \to \infty$ in [25], we have the following theorem.

Theorem 1. Assume that the function $g$ on the interval $[0, \infty)$ is uniformly continuous bounded function. For each function $g$ on $[0, \infty)$, we can give
\[
S_n(g; x)^{uniformly} \overset{n \to \infty}{\rightarrow} g(x)
\]
on each compact set $A \subset [0, \infty)$ when $n \to \infty$.

Proof. In view of Lemma 3
\[
\lim_{n \to \infty} S_n(t^i; x) = x^i, \ i = 0, 1, 2
\]
is verified where the convergence holds uniformly in each compact subset of $[0, \infty)$. Then, using well known Korovkin Theorem in [26], we give the desired result. □
3. The Convergence Rates of Operator $S_n$

In this part, we obtain some rates of convergence of the operator $S_n$.

**Theorem 2.** If $h \in \text{Lip}_M(\gamma)$, which satisfies the inequality

$$|h(s) - h(t)| \leq M|s - t|^{\gamma}$$

where $s, t \in [0, \infty)$, $0 < \gamma \leq 1$ and $M > 0$, we have

$$|S_n(h; x) - h(x)| \leq M(\Lambda_2)^{\gamma/2}$$

where $\Lambda_2$ is given in Lemma 4.

**Proof.** From the linearity of operator and $h \in \text{Lip}_M(\gamma)$, we get

$$|S_n(h; x) - h(x)| \leq S_n(|h(t) - h(x)|; x) \leq MS_n(|t - x|^{\gamma}; x).$$

Under favour of Hölder's inequality and Lemma 4, we can give the following required inequality

$$|S_n(h; x) - h(x)| \leq M[\Lambda_2]^{2 \gamma}.$$ 

□

**Theorem 3.** The operator $S_n$ in $\mathbb{B}$ satisfies the inequality

$$|S_n(g; x) - g(x)| \leq \left(1 + \sqrt{\frac{\pi}{2}}\right)(4\gamma + 10\lambda + 10\alpha + 2\lambda\frac{\omega(g; \delta)}{\Delta})$$

where $g \in \mathcal{C}[0, \infty)$, which is the space of uniformly continuous functions on $[0, \infty)$, and the modulus of continuity is defined by

$$\omega(g; \delta) := \sup_{s, t \in [0, \infty)} \frac{|g(s) - g(t)|}{|s - t| \leq \delta}$$

for $g \in \mathcal{C}[0, \infty)$.

**Proof.** Firstly we note that the modulus of continuity verifies the following inequality

$$\omega(g; \delta) \leq \omega(g; \delta) \left(\frac{|t - x|}{\delta} + 1\right).$$

Under favour of the linearity of operator, Cauchy-Schwarz’s inequality, and Lemma 4, respectively, it follows

$$|S_n(g; x) - g(x)| \leq S_n(|g(t) - g(x)|; x) \leq \left(1 + \frac{1}{\delta} S_n(|t - x|; x)\right) \omega(g; \delta) \leq \left(1 + \frac{1}{\delta} \sqrt{\Lambda_2}\right) \omega(g; \delta).$$

By choosing $\delta = \frac{1}{\sqrt{n}}$, we complete the proof. □

**Lemma 5.** For $h \in \mathcal{C}_B^2[0, \infty)$, which is denoted by

$$\mathcal{C}_B^2[0, \infty) = \{h \in \mathcal{C}_B[0, \infty) : h', h'' \in \mathcal{C}_B[0, \infty)\}$$

with the norm

$$\|h\|_{\mathcal{C}_B^2[0, \infty)} = \|h\|_{\mathcal{C}_B[0, \infty)} + \|h'\|_{\mathcal{C}_B[0, \infty)} + \|h''\|_{\mathcal{C}_B[0, \infty)}$$
where $C_B(0, \infty)$ is the space of continuous and bounded functions on $[0, \infty)$ with the norm
$$
\|h\|_{C_B(0, \infty)} = \sup_{x \in [0, \infty)} |h(x)|,
$$
the following inequality holds true
$$
|S_n (h; x) - h (x)| \leq \frac{\Lambda^\frac{x}{2}}{2} (2 + \Lambda^\frac{1}{2}) \|h\|_{C_B^2(0, \infty)},
$$
where $\Lambda_2$ is given by in Lemma [4].

**Proof.** With the help of the Taylor’s series of the function $h$, it follows that
$$
h (s) = h (x) + (s - x) h' (x) + \frac{(s - x)^2}{2!} h'' (\zeta)
$$
where $\zeta$ between $x$ and $s$. Then, by applying $S_n$ to this equality and using the linearity of the operator, we get
$$
S_n (h; x) - h (x) = h' (x) S_n (s - x; x) + \frac{h'' (\zeta)}{2} \Lambda_2.
$$
Using Lemma [4] and
$$
S_n (|s - x|; x) \leq \left( S_n \left( (s - x)^2; x \right) \right)^{\frac{x}{2}} = \Lambda_2^\frac{x}{2},
$$
the following inequality is satisfied
$$
|S_n (h; x) - h (x)| \leq \Lambda^\frac{x}{2} \|h'\|_{C_B(0, \infty)} + \frac{\Lambda^\frac{1}{2}}{2} \|h''\|_{C_B(0, \infty)}
$$
$$
\leq \frac{\Lambda^\frac{x}{2}}{2} (2 + \Lambda^\frac{1}{2}) \|h\|_{C_B^2(0, \infty)}.
$$

\[ \square \]

For any $f \in C_B(0, \infty)$ and $\delta > 0$, Peetre’s $K$-functional is given by
$$
K_2 (f; \delta) := \inf \left\{ \|f - g\|_{C_B(0, \infty)} + \delta \|g''\|_{C_B(0, \infty)} : g \in C_B^2(0, \infty) \right\},
$$
where $C_B^2(0, \infty) = \{ g \in C_B(0, \infty) : g', g'' \in C_B(0, \infty) \}$ and the second modulus of continuity $\omega_2 (f; \delta)$ is defined as
$$
\omega_2 \left( f; \sqrt{\delta} \right) := \sup_{0 < s \leq \sqrt{\delta}} \sup_{x \in [0, \infty)} |f (x + 2s) - 2f (x + s) + f (x)|.
$$

(3.5)

Also, the inequality holds
$$
K_2 (f; \delta) \leq c \omega_2 (f; \sqrt{\delta}),
$$
where $c > 0$,

(3.6)

between Peetre’s $K$-functional and second modulus of continuity $\omega_2$ (see [27]).

**Lemma 6.** For $g \in C_B^2(0, \infty)$ and $x \geq 0$, $\alpha, \lambda \geq 0$, we get
$$
|\tilde{S}_n (g; x) - g (x)| \leq \Upsilon (n, x) \|g''\|_{C_B(0, \infty)}
$$
where
$$
\tilde{S}_n (g; x) = S_n (g; x) + g(x) - g \left( x + \frac{2\alpha x^2 + \lambda + 1}{n} \right)
$$
and
$$
\Upsilon (n, x) = \Lambda_1^2 + \Lambda_2.
$$
From (3.7), (3.8) and (3.9), where \( \Upsilon(n; x - \varepsilon) \), for \( t \in [0, \infty) \) we get \( \mathcal{S}_n(t - x; x) = 0 \). For \( g \in C^2_B[0, \infty) \), the Taylor’s expression is
\[
g(t) = g(x) + (t - x)g'(x) + f_x(t - v)g''(v)dv, \quad t \in [0, \infty).
\]
If we apply \( \mathcal{S}_n \) to the last equality and then use \( \mathcal{S}_n(t - x; x) = 0 \), we have
\[
\mathcal{S}_n(g; x) - g(x) = \mathcal{S}_n \left( f_x(t - v)g''(v)dv; x \right)
\]
\[
= \mathcal{S}_n \left( f_x(t - v)g''(v)dv; x \right) - f_x^+ \frac{2ax^2 + \lambda + 1}{n} (x + \frac{2ax^2 + \lambda + 1}{n} - v)g''(v)dv,
\]
from which, it follows
\[
|\mathcal{S}_n(g; x) - g(x)| \leq \mathcal{S}_n \left( f_x(t - v)g''(v)dv; x \right)
\]
\[
+ \int f_x^+ \frac{2ax^2 + \lambda + 1}{n}(x + \frac{2ax^2 + \lambda + 1}{n} - v)g''(v)dv. \tag{3.7}
\]
Since
\[
|f_x(t - v)g''(v)dv| \leq (t - x)^2 \left\| g'' \right\|_{C_B[0, \infty)}, \tag{3.8}
\]
we have
\[
\int f_x^+ \frac{2ax^2 + \lambda + 1}{n}(x + \frac{2ax^2 + \lambda + 1}{n} - v)g''(v)dv \leq \left( \frac{2ax^2 + \lambda + 1}{n} \right)^2 \left\| g'' \right\|_{C_B[0, \infty)}. \tag{3.9}
\]
From (3.7), (3.8) and (3.9),
\[
|\mathcal{S}_n(g; x) - g(x)| \leq \left\{ \mathcal{S}_n \left( (t - x)^2; x \right) + \left( \frac{2ax^2 + \lambda + 1}{n} \right)^2 \right\} \left\| g'' \right\|_{C_B[0, \infty)}
\]
\[
= \Upsilon(n, x) \left\| g'' \right\|_{C_B[0, \infty)},
\]
where \( \Upsilon(n, x) = \Lambda_1^2 + \Lambda_2. \)

**Theorem 4.** Let \( f \in C_B[0, \infty) \) and \( c > 0 \). The following inequality holds
\[
|\mathcal{S}_n(f; x) - f(x)| \leq c \omega_2 \left( f; \frac{1}{2} \sqrt{\Upsilon(n, x)} \right) + \omega(f; \Lambda_1).
\]

**Proof.** For \( f \in C_B[0, \infty) \) and \( g \in C^2_B[0, \infty) \), we get
\[
\mathcal{S}_n(f; g; x) = \mathcal{S}_n(f - g; x) + (f - g)(x) - (f - g) \left( x + \frac{2ax^2 + \lambda + 1}{n} \right)
\]
\[
= \mathcal{S}_n(f; x) + f(x) - f \left( x + \frac{2ax^2 + \lambda + 1}{n} \right) - \mathcal{S}_n(g; x).
\]
On the other hand, we give
\[
\mathcal{S}_n(f; x) - f(x) = \mathcal{S}_n(f - g; x) + \mathcal{S}_n(g; x) - g(x) + f \left( x + \frac{2ax^2 + \lambda + 1}{n} \right) - f(x) - f(x)
\]
\[
= \mathcal{S}_n(f - g; x) - (f(x) - g(x)) + \mathcal{S}_n(g; x) - g(x) + f \left( x + \frac{2ax^2 + \lambda + 1}{n} \right) - f(x).
\]
Using Lemma 6, thus we have
\[
|\mathcal{S}_n(f; x) - f(x)| \leq \left| \mathcal{S}_n(f - g; x) \right| + |f(x) - g(x)| + \left| \mathcal{S}_n(g; x) - g(x) \right| + \left| f \left( x + \frac{2ax^2 + \lambda + 1}{n} \right) - f(x) \right|
\]
\[
\leq 4 \left\| f - g \right\|_{C_B[0, \infty)} + \Upsilon(n, x) \left\| g'' \right\|_{C_B[0, \infty)} + \omega(f; \Lambda_1).
From the inequality (3.6) between Peetre’s $K$-functional and the second modulus of continuity $\omega_2$, we have the desired result. □

4. Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] Atakut, Ç. and Büyükyazıcı, İ., Stancu type generalization of the Favard Szász operators, Appl. Math. Lett., 23 (12) (2010), 1479-1482.
[2] Atakut, Ç. and İspir, N., Approximation by modified Szász–Mirakjan operators on weighted spaces, Proc. Indian Acad. Sci. Math. 112 (2002), 571-578.
[3] Ciupa, A., A class of integral Favard–Szász type operators. Stud. Univ. Babes-Bolyai Math. 40 (1) (1995), 39-47.
[4] Gupta, V., Vasishtha, V. and Gupta, M.K., Rate of convergence of the Szász–Kantorovich–Bezier operators for bounded variation functions, Publ. Inst. Math. (Beograd) (N.S.), 72 (2006), 137–143.
[5] Szász, O., Generalization of S. Bernstein’s polynomials to the infinite interval, J. Res. Nat. Bur. Stand. 45 (1950), 239-245.
[6] Stancu, D.D., Approximation of function by a new class of polynomial operators, Rev. Roum. Math. Pures et Appl., 13 (8) (1968), 1173-1194.
[7] Altın, A., Doğru, O. and Taşdelen, F., The generalization of Meyer-König and Zeller operators by generating functions, J. Math. Anal. Appl., 312 (1) (2005), 181-194.
[8] Doğru, O., Özarslan, M.A. and Taşdelen, F., On positive operators involving a certain class of generating functions, Studia Sci. Math. Hungar., 41 (4) (2004), 415-429.
[9] Krech, G., A note on some positive linear operators associated with the Hermite polynomials, Carpathian J. Math., 32 (1) (2016), 71–77.
[10] Olgun, A., İnce, H. G. and Taşdelen, F., Kantrovich type generalization of Meyer-König and Zeller operators via generating functions, An. Ştiinţ. Univ. ”Ovidius” Constanţa Ser. Mat., 21 (3) (2013), 209–221.
[11] Sucu, S., İçöz, G. and Varma, S., On some extensions of Szász operators including Boas-Buck type polynomials, Abstr. Appl. Anal., Vol. 2012, Article ID 680340, 15 pages.
[12] Taşdelen, F., Aktaş, R., Altın, A., A Kantrovich type of Szász operators including Brenke-type polynomials, Abstract and Applied Analysis, Vol. 2012 (2012), 13 pages.
[13] Varma, S., Sucu, S., İçöz, G., Generalization of Szász operators involving Brenke type polynomials, Comput. Math. Appl., 64 (2) (2012), 121-127.
[14] Varma, S., Taşdelen F., Szász type operators involving Charlier polynomials, Mathematical and Computer Modeling, 56 (2012), 118-122.
[15] Sucu, S., Dunkl analogue of Szász operators, Appl. Math. Comput. 244 (2014), 42-48.
[16] Rosenblum, M., Generalized Hermite polynomials and the Bose-like oscillator calculus, Oper. Theory: Adv. Appl. 73 (1994), 369-396.
[17] İçöz, G., Çekim, B., Dunkl generalization of Szász operators via q-calculus, Journal of Inequalities and Applications, 2015:284 (2015), 11 pages.
[18] İçöz, G., Çekim, B., Stancu-type generalization of Dunkl analogue of Szász–Kantorovich operators, Math. Meth. Appl. Sci., 39 (2016), 1803–1810.
[19] Wafi, A. and Rao, N., Szász–gamma operators based on Dunkl analogue, Iran J Sci Technol Trans Sci., 43(2019),213-223.
[20] Aktaş, R., Çekim, B. and Taşdelen, F., A Dunkl Analogue of Operators Including Two-Variable Hermite polynomials, Bull. Malays. Math. Sci. Soc., 42 (2019), 2795–2805.
[21] Ben Cheikh, Y., Gaied, M. Dunkl–Appell d-orthogonal polynomials. Integral Transforms and Special Functions, 18(8) (2007), 581-597.
[22] Appell, P. and Kampe de Feriet, J., Hypergeometriques et Hyperspheriques: Polynomes d’Hermite, Gauthier-Villars, Paris, 1926.
[23] Sucu, S., Approximation by sequence of operators including Dunkl–Appell polynomials, Bull. Malays. Math. Sci. Soc., 43 (3) (2020), 2455-2464.
[24] Taşdelen, F., Söylemez, D., Aktaş, R., Dunkl-Gamma type operators including Appell polynomials, Complex Analysis and Operator Theory, 13 (2019), 3359–3371.

[25] Milovanović, G. V., Mursaleen, M., & Nasiruzzaman, M. (2018). Modified Stancu type Dunkl generalization of Szász–Kantorovich operators. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 112(1), 135-151.

[26] Korovkin, P.P., On convergence of linear positive operators in the space of continuous functions (Russian). Doklady Akad. Nauk. SSSR (NS) 90 (1953), 961–964.

[27] DeVore, R.A. and Lorentz, G.G., Constructive Approximation, Springer, Berlin, 1993.

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