Beta Laguerre processes in a high temperature regime

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Abstract

Beta Laguerre processes which are generalizations of the eigenvalue process of Wishart/Laguerre processes can be defined as the square of radial Dunkl processes of type B. In this paper, we study the limiting behavior of their empirical measure processes. By the moment method, we show the convergence to a limit in a high temperature regime, a regime where $\beta N \to \text{const} \in (0, \infty)$, where $\beta$ is the inverse temperature parameter and $N$ is the system size. This is a dynamic version of a recent result on the convergence of the empirical measures of beta Laguerre ensembles in the same regime.

Keywords: beta Laguerre processes ; radial Dunkl processes ; beta Laguerre ensembles ; high temperature regime ; the moment method ;

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1 Introduction

Start from an $M \times N$ Brownian matrix $G_t, (M \geq N)$, which consists of independent Brownian motions, we form the matrix process $S_t = G_t^T G_t$, called a Wishart process, where $A^T$ denotes the transpose of a matrix $A$. In case $G_0 = 0$, for any fixed $t > 0$, the matrix $S_t$ is a sample covariance matrix, called a Wishart matrix in honor of John Wishart who first introduced the random matrix model. The process $S_t$ then satisfies the following stochastic differential equation (SDE) in the space of non-negative definite matrices of order $N$,

$$dS_t = \sqrt{S_t} dB_t + dB_t^T \sqrt{S_t} + M I_N dt.$$  (1)

Here $(B_t)_{t \geq 0}$ is an $N \times N$ Brownian matrix, and $I_N$ denotes the identity matrix of order $N$. The SDEs for the eigenvalues $0 \leq \lambda_1(t) \leq \cdots \leq \lambda_N(t)$ of $S_t$ can also be derived

$$d\lambda_i = 2\sqrt{\lambda_i} dB_i + \left( M - N + 1 + \sum_{j \neq i} \frac{2\lambda_i}{\lambda_i - \lambda_j} \right) dt, \quad 1 \leq i \leq N,$$

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with \( \{b_i(t)\}_{i=1,...,N} \) independent standard Brownian motions \([4, 5]\). We also call the solution \( S_t \) of the matrix-valued SDE \((1)\), which uniquely exists when \( M > N - 1 \) is not necessary an integer number, a Wishart process.

For \( M \geq N \), the eigenvalues \( 0 \leq \lambda_1 \leq \cdots \leq \lambda_N \) of the Wishart matrix \( S_t \) (in case \( G_0 = 0 \)) have the following joint density

\[
\frac{1}{Z_{M,N}} \prod_{i<j} |\lambda_j - \lambda_i| \prod_{i=1}^{N} \lambda_i^{\frac{1}{2}(M-N+1)-1} e^{-\lambda_i},
\]

with \( Z_{M,N} \) the normalizing constant. Note that the joint density of the eigenvalues of \( S_t \) can be derived immediately for any \( t > 0 \) because \( S_t \) has the same distribution with \( 2tS_{\frac{1}{2}} \). Laguerre matrices are the complex version of Wishart matrices whose eigenvalues have a similar form of joint density. Readers who are interested in those two random matrix models are referred to a monograph \([19]\). As generalizations of Wishart/Laguerre matrices, beta Laguerre ensembles (\( \beta \mathrm{LE} \) for short) are defined to be the ensembles of \( N \) non-negative points \( 0 \leq \lambda_1 \leq \cdots \leq \lambda_N \) with joint density

\[
\frac{1}{Z_{N,\alpha,\beta}} \prod_{i<j} |\lambda_j - \lambda_i|^\beta \prod_{i=1}^{N} \lambda_i^{\alpha-1} e^{-\lambda_i}, \tag{2}
\]

where \( Z_{N,\alpha,\beta} \) is the normalizing constant and \( \alpha, \beta > 0 \). The ensemble \((2)\) is now realized as the eigenvalues of a random tridiagonal matrix \([10]\).

Beta Laguerre processes, viewed as generalizations of the eigenvalue process of Wishart processes, or a dynamic version of beta Laguerre ensembles, have the following SDE form

\[
d\lambda_i = 2\sqrt{\lambda_i} db_i + \left(2\alpha + \beta \sum_{j \neq i} \frac{2\lambda_i}{\lambda_i - \lambda_j}\right) dt, \quad 1 \leq i \leq N. \tag{3}
\]

In case \( \beta = 2 \), they are eigenvalues of complex Wishart processes or Laguerre processes \([8, 16, 17]\). There is no matrix model yet for general \( \beta \not\in \{1, 2\} \). However, it was observed in \([9]\) that beta Laguerre processes are the square of radial Dunkl processes of type B, and hence, the above SDEs are meaningful for \( \beta > 0 \) and \( \alpha > 1/2 \).

In the study of beta ensembles, the parameter \( \beta \) is regarded as the inverse temperature and is usually fixed when considering the limiting behavior. For fixed \( \beta \), the empirical distribution of the beta Laguerre ensemble \((2)\) under a suitable scaling converges to the Marchenko–Pastur distribution as \( N \to \infty \). Here the parameter \( \alpha \) varies and determines the parameter of the Marchenko–Pastur distribution. A dynamic version of the Marchenko–Pastur law was studied in \([6]\).

The paper, however, aims to investigate the limiting behavior of beta Laguerre processes in a high temperature regime, a regime where \( \beta N \to const \in (0, \infty) \). Beta Laguerre ensembles in this regime have been studied \([1, 23]\). It turns out that the limiting measure of the empirical distributions is a family of probability measures of associated Laguerre polynomials. Our main result is a dynamic version of that static result.

Let us explain our problem in more details. We deal with the following beta
Laguerre processes of Ornstein–Uhlenbeck type,
\[
\begin{aligned}
    d\lambda_i &= \sqrt{2\lambda_i} db_i - \lambda_i dt + \frac{\beta}{2} \sum_{j:j\neq i} \frac{2\lambda_i}{\lambda_i - \lambda_j} dt, \\
    \lambda_i(0) &= \lambda^{(N,i)}_0, \\
\end{aligned}
\]
where \(0 \leq \lambda^{(N,1)}_0 \leq \cdots \leq \lambda^{(N,N)}_0\) are initial data. Here for convenience, the system of equations is scaled so that its stationary distribution coincides with the beta Laguerre ensemble (2). The processes \(\{\lambda_i(t)\}\) can be defined as the square of the Ornstein–Uhlenbeck type of radial Dunkl processes of type B, and hence, their SDEs are well defined for \(\beta > 0\) and \(\alpha > 1/2\). We are interested in investigating the limiting behavior of the empirical measure process
\[
\mu^{(N)}_t = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i(t)}
\]
in the regime where \(\beta N \to 2c \in (0, \infty)\). Here \(c > 0\) and \(\alpha > 1/2\) are assumed to be fixed. A method to deal with this kind of problems has been well developed [6, 7, 20]. By imitating arguments from those works, we can immediately derive the following result.

**Theorem 1.1.** Assume that the initial measure \(\mu^{(N)}_0\) converges weakly to a probability measure \(\mu_0\) and satisfies
\[
\sup_N \int \log(1 + x) d\mu^{(N)}_0 < \infty.
\]
Then for any \(T > 0\), the sequence \((\mu^{(N)}_t)_{0 \leq t \leq T}\) is tight in the space \(C([0, T], \mathcal{P}(\mathbb{R}_{\geq 0}))\) and any limit is supported on the set of continuous probability measure-valued processes \((\mu_t)_{0 \leq t \leq T}\) satisfying the integro-differential equation
\[
\langle \mu_t, f \rangle = \langle \mu_0, f \rangle + \int_0^t \langle \mu_s, \alpha f' - xf' + xf'' \rangle ds \\
+ c \int_0^t \left( \int \int \frac{xf'(x) - yf'(y)}{x-y} d\mu_s(x) d\mu_s(y) \right) ds, \quad t \in [0, T],
\]
for all \(f \in C^2_0 = \{ f : [0, \infty) \to \mathbb{R} : f, f', f'' \text{ bounded} \}\) with \(xf', xf''\) bounded. Here \(C([0, T], \mathcal{P}(\mathbb{R}_{\geq 0}))\) is the space of continuous mappings from \([0, T]\) to the space \(\mathcal{P}(\mathbb{R}_{\geq 0})\) of probability measures on \([0, \infty)\) endowed with the uniform topology, and \(\langle \mu, f \rangle = \int f d\mu\) for a measure \(\mu\) and an integrable function \(f\).

By this theorem, the sequence \((\mu^{(N)}_t)_{0 \leq t \leq T}\) will converge in distribution to a deterministic limit once the integro-differential equation (5) is shown to have a unique solution. This paper is not devoted to study the integro-differential equation in more details. Instead, we are going to use the moment method to establish the convergence of \((\mu^{(N)}_t)_{0 \leq t \leq T}\). By the moment method, we simply mean that the limiting behavior of the empirical measure processes can be derived by studying their moment processes. Under some moments assumptions (H1 and H2 in Sect. 3), we will show by induction that the \(k\)th moment process of \(\mu^{(N)}_t\) converges in probability (as random elements
in the space $C([0,T],\mathbb{R})$ of continuous functions on $[0,T]$ endowed with the uniform norm) to a deterministic limit $m_k(t)$. Here the limit $m_k(t)$ is defined inductively as the solution to the following initial value ordinary differential equation (ODE)

$$
\begin{aligned}
& m'_k(t) = -k \left( m_k(t) + (\alpha + k - 1)m_{k-1}(t) + c \sum_{i=0}^{k-1} m_i(t)m_{k-i-1}(t) \right), \\
& m_k(0) = \lim_{N \to \infty} \langle \mu_0^{(N)}, x^k \rangle,
\end{aligned}
$$

where $m_0 \equiv 1$. Let $\mu_t$ be the unique probability measure-valued process with moments $\{m_k(t)\}$. (It is unique under our moments assumptions.) Then the convergence of every moment process implies that the sequence $(\mu_t^{(N)})_{1 \leq t \leq T}$ converges in probability to $(\mu_t)_{0 \leq t \leq T}$ as $N \to \infty$ (as random elements in $C([0,T],\mathcal{P}(\mathbb{R}_{\geq 0}))$). Namely, we obtain the following result.

**Theorem 1.2.** Assume that Conditions $H1$ and $H2$ are satisfied. Then for any $T > 0$, the sequence of empirical measure processes $\mu_t^{(N)}$ converges in probability in $C([0,T],\mathcal{P}(\mathbb{R}_{\geq 0}))$ to a continuous probability measure-valued process $\mu_t$ as $N \to \infty$.

In addition, we will also show that as $t \to \infty$,

$$
m_k(t) \to \int x^k d\nu_{\alpha,c},
$$

where $\nu_{\alpha,c}$ is the limiting measure of the beta Laguerre ensemble (2) as $\beta N \to 2c$ [23]. Thus, by using the moment method, we are able to complete the following diagram in a high temperature regime where $\beta N \to 2c$

$$
\begin{array}{ccc}
\beta LE(N) & \xrightarrow{N \to \infty} & \nu_{\alpha,c} \\
\downarrow t \to \infty & & \downarrow t \to \infty \\
\mu_t^{(N)} & \xrightarrow{N \to \infty} & \mu_t
\end{array}
$$

We note here that the moment method also works for the following models: Dyson’s Brownian motion models which were already studied in [7, 20], beta Laguerre processes (3) (the usual type) and beta Laguerre processes in a regime where $\beta N \to \infty$.

The paper is organized as follows. In Sect. 2, we shortly introduce the type-B radial Dunkl process of Ornstein–Uhlenbeck type, and then define beta Laguerre processes. The limiting behavior of the empirical measure processes is studied in Sect. 3.

## 2 Beta Laguerre processes

### 2.1 The B-type radial Dunkl process of Ornstein–Uhlenbeck type

Consider the closed subset of $\mathbb{R}^N$ given by

$$
\mathbb{W}_B := \{ x \in \mathbb{R}^N : 0 \leq x_1 \leq \cdots \leq x_N \}.
$$

(6)

The B-type radial Dunkl process of Ornstein–Uhlenbeck type is defined as the Markov process with infinitesimal generator

$$
L_k[f](x) := \frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} f(x) + \sum_{i=1}^{N} \left( \frac{k_1}{x_1} + k_2 \sum_{j:j \neq i} \frac{2x_i}{x_i^2 - x_j^2} - \frac{x_i}{2} \right) \frac{\partial}{\partial x_i} f(x)
$$

(7)
for suitable $f \in C^2(\mathbb{W}_B)$. The two parameters $k_1, k_2 > 0$ are the *multiplicities* of the root system of type B, which is expressed in terms of the canonical basis vectors $\{e_i\}_{i=1}^N$ as

$$B_N := \{e_i - e_j, 1 \leq i \neq j \leq N\} \cup \{\pm (e_i + e_j), 1 \leq i < j \leq N\} \cup \{\pm e\}_{i=1}^N. \quad (8)$$

The transition density of the Markov process was found in [22]. Let $\hat{p}(t, y|x)$ be the transition density (the density of arriving at $y$ after a time $t > 0$ having started from $x$) of the process without confinement (that is, without the restoring drift term $-x_i/2$). Then the transition density of the Ornstein–Uhlenbeck type process is given by

$$p(t, y|x) = \hat{p}(1 - e^{-t}, y|x e^{-t/2}) \quad \text{(Sect. 10 in [22])}$$

$$p(t, y|x) = \frac{1}{c_k(1 - e^{-t})^{N/2}} \prod_{i=1}^N \frac{y_{i1}^{2k_1}}{(1 - e^{-t})^{k_1}} \prod_{1 \leq m < n \leq N} \left(\frac{y_{mn}^2 - y_{mn}^2}{1 - e^{-t}}\right)^{2k_2} \times \exp\left(-\frac{\|y\|^2 + \|x\|^2 - \frac{t}{2}}{2(1 - e^{-t})}\right) \sum_{\sigma \in W_B} E_k \left(\frac{x e^{-t/2}}{\sqrt{1 - e^{-t}}}, \frac{\sigma y}{\sqrt{1 - e^{-t}}}, \sqrt{1 - e^{-t}}\right). \quad (9)$$

Let us now explain notations in the above formula. The reflection operators along the root system generate the Weyl group $W_B$ of all permutations and component-wise sign changes of vectors in $\mathbb{R}^N$. The function $E_k$ is the *Dunkl kernel*, the joint eigenfunction of Dunkl operators [11] of type B, and the explicit form of the sum over $\sigma \in W_B$ is given by a multivariate hypergeometric function [2], though we do not require its explicit form here. Finally, the normalization constant $c_k$ is given by the Selberg integral

$$c_k := 2^N N! \int_{\mathbb{W}_B} e^{-\|x\|^2/2} \prod_{i=1}^N \frac{x_i^{2k_1}}{\prod_{1 \leq i < j \leq N} (x_j^2 - x_i^2)^{2k_2}} d^N x. \quad (10)$$

We have used $\|\cdot\|$ to denote the Euclidean norm in $\mathbb{R}^N$.

The process can be expressed in SDE form by reading off the infinitesimal generator: if we denote the process by $X(t)$ with $X(0) = x$, then each component of its SDEs reads

$$dX_i(t) = db_i(t) + \left(\frac{k_1}{X_i(t)} + k_2 \sum_{j \neq i} \frac{2X_i(t)}{X_j^2(t) - X_j^2(t)} - \frac{X_i(t)}{2}\right) dt, \quad i = 1, \ldots, N, \quad (11)$$

with $\{b_i(t)\}_{i=1}^N$ standard Brownian motions. The above SDEs can also be treated via an approach in [7] (see also [9]).

Because the law $p(t, y|x)$ is controlled by Gaussian functions, we can use an inequality ([21])

$$\sum_{\sigma \in W_B} E_k(x, \sigma y) \leq 2^N N! \exp(\|x\|\|y\|), \quad (12)$$

to show that $\mathbb{E}[\|X_i\|^{2m}]$ is uniformly bounded in $t$, for each $m \in \{1, 2, \ldots\}$. This is a crucial property we need when using the moment method.
2.2 Beta Laguerre processes

Let $\lambda_i = X_i^2/2, i = 1, \ldots, N$, with $\{X_i\}$ the solution of the SDEs (11). Then $\{\lambda_i\}$, called beta Laguerre processes, satisfy the following SDEs

$$d\lambda_i = \sqrt{2\lambda_i} db_i - \lambda_i dt + \left( k_1 + \frac{1}{2} \right) dt + \sum_{j:j\neq i} \frac{2\lambda_i}{\lambda_i - \lambda_j} dt$$

$$= \sqrt{2\lambda_i} db_i - \lambda_i dt + \alpha dt + \sum_{j:j\neq i} \frac{2\lambda_i}{\lambda_i - \lambda_j} dt, \quad i = 1, \ldots, N,$$

(13)

where $\alpha = k_1 + 1/2 > 1/2$ and $\beta = 2k_2 > 0$. For $\beta \in \{1, 2\}$, they are realized as the eigenvalue process of Wishart/Laguerre processes where the above SDEs are defined in the usual sense and $\{\lambda_i\}$ never collide [4, 5, 16, 17].

It is clear from the explicit expression for the joint density of $\{X_i(t)\}_{i=1,\ldots,N}$ in (9) that the distribution of $\{\lambda_i(t)\}_{i=1,\ldots,N}$, staring from any initial point, converges weakly to the beta Laguerre ensemble (2) as $t \to \infty$.

3 Convergence of the empirical measure process

3.1 Assumptions

We study the limiting behavior of the empirical measure process

$$\mu^{(N)}_t = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(t)}$$

in the regime where $\beta N \rightarrow 2c \in (0, \infty)$. For simplicity, let $c \in (0, \infty)$ be fixed and $\beta = 2c/N$. The SDEs can be rewritten as

$$\left\{\begin{array}{l}
d\lambda_i = \sqrt{2\lambda_i} db_i - \lambda_i dt + \alpha dt + \frac{2\lambda_i}{N} \sum_{j:j\neq i} \frac{2\lambda_i}{\lambda_i - \lambda_j} dt, \\
\lambda_i(0) = \lambda^{(N,i)}_0,
\end{array}\right. \quad i = 1, \ldots, N. \quad (14)$$

Here the initial data satisfy $0 \leq \lambda^{(N,1)}_0 \leq \lambda^{(N,2)}_0 \leq \cdots \leq \lambda^{(N,N)}_0$. We make the following assumptions.

**H0.** The initial probability measure $\mu^{(N)}_0 = N^{-1} \sum_{i=1}^{N} \delta_{\lambda^{(N,i)}_0}$ converges weakly to a probability measure $\mu_0$ as $N \to \infty$, and satisfies

$$\sup_N \int \log(1 + x) d\mu^{(N)}_0 < \infty. \quad (15)$$

**H1.** Each moment of $\mu^{(N)}_0$ converges, that is, for each $k = 1, 2, \ldots$,

$$\langle \mu^{(N)}_0, x^k \rangle = \frac{1}{N} \sum_{i=1}^{N} (\lambda^{(N,i)}_0)^k \rightarrow a_k \quad \text{as} \quad N \to \infty.$$ 

**H2.** The sequence of initial moments $\{a_k\}$ does not grow too fast in the sense that

$$\sum_{k=1}^{\infty} \lambda_k^{-\frac{1}{3k}} = \infty,$$
where \( \{ \Lambda_k \} \) is defined recursively as

\[
\Lambda_1 = (\alpha + c) \vee a_1, \quad \Lambda_k = (\alpha + k - 1 + ck) \Lambda_{k-1} \vee a_k, \quad k = 2, 3, \ldots.
\]

Note that Conditions \( \mathbf{H1} \) and \( \mathbf{H2} \) together imply Condition \( \mathbf{H0} \). Indeed, under Condition \( \mathbf{H2} \), the sequence of moments \( \{a_k\} \) satisfies

\[
\sum_{k=1}^{\infty} a_k^{-\frac{s}{n}} \geq \sum_{k=1}^{\infty} \Lambda_k^{-\frac{s}{n}} = \infty.
\]

This is Carleman’s sufficient condition under which a probability measure \( \mu_0 \) on \([0, \infty)\) whose moments match the sequence \( \{f_i\} \) is unique. Together with Condition \( \mathbf{H1} \), it follows that the sequence of probability measures \( \mu^{(N)}_0 \) converges weakly to \( \mu_0 \) (see [3, Theorem 30.2] or [12, §3.3.5], for example). Since \( \log(1 + x) \leq x \), for \( x \geq 0 \), the condition (15) is clear.

### 3.2 A standard method

Let \( T > 0 \) be fixed. Denote by \( \mathcal{C}([0, T], \mathcal{P}(\mathbb{R}_{\geq 0})) \) the space of continuous mappings \( \mu: [0, T] \to \mathcal{P}(\mathbb{R}_{\geq 0}) \) endowed with the uniform topology. Here \( \mathcal{P}(\mathbb{R}_{\geq 0}) \) is the space of probability measure measures on \( \mathbb{R}_{\geq 0} = [0, \infty) \) endowed with the weak topology. Then the empirical probability measure \( (\mu^{(N)}_t)_{0 \leq t \leq T} \) becomes a random element on \( \mathcal{C}([0, T], \mathcal{P}(\mathbb{R}_{\geq 0})) \).

Imitate arguments used in [7, 20], we can immediately obtain the following result.

**Theorem 3.1.** Assume that Condition \( \mathbf{H0} \) is satisfied. Then the sequence \( \mu^{(N)}_t \) is tight in \( \mathcal{C}([0, T], \mathcal{P}(\mathbb{R}_{\geq 0})) \) and any limit is supported on the set of continuous probability measure-valued processes \( \{\mu_t\}_{0 \leq t \leq T} \) satisfying the integro-differential equation

\[
\langle \mu_t, f \rangle = \langle \mu_0, f \rangle + \int_0^t \langle \mu_s, \alpha f' - xf'' + xf'' \rangle \, ds + c \int_0^t \left( \int \frac{xf'(x) - yf'(y)}{x-y} \, d\mu_s(x)d\mu_s(y) \right) \, ds, \quad t \in [0, T],
\]

for all \( f \in C^2_b = \{ f: [0, \infty) \to \mathbb{R}, f, f', f'' \text{ bounded} \} \) with \( xf', xf'' \) bounded.

The proof of this theorem relies on the following formula which is a direct application of Itô’s formula

\[
d\langle \mu^{(N)}_t, f(x) \rangle = \sum_{i=1}^{N} \frac{1}{N} \sqrt{2 \lambda_i} f'(\lambda_i) \, dB_i + \langle \mu^{(N)}_t, \alpha f'(x) - xf''(x) + xf''(x) \rangle \, dt
\]

\[
+ c \int \int \frac{xf'(x) - yf'(y)}{x-y} \, d\mu^{(N)}_t(x) \, d\mu^{(N)}_t(y) \, dt
\]

\[
- \frac{c}{N} \langle \mu^{(N)}_t, xf''(x) + f'(x) \rangle \, dt,
\]

for \( f \in C^2(\mathbb{R}_{\geq 0}) \). Then the arguments can run in exactly the same way as those used in [7, 20], and hence we omit the details here.
Remark 3.2. Assume that $\mu_t$ is a probability measure-valued process satisfying the equation (16). Let

$$S = S(t, z) = \langle \mu_t, (\cdot - z)^{-1} \rangle = \int \frac{d\mu_t(x)}{x-z}, \quad (t \geq 0, z \in \mathbb{C} \setminus \mathbb{R}),$$

be the Stieltjes transform of $\mu_t$. Then the equation (16) with $f = 1/(x-z)$ yields the following partial differential equation for $S$,

$$\frac{\partial S}{\partial t} = S + (2 + z - \alpha) \frac{\partial S}{\partial z} + z \frac{\partial^2 S}{\partial z^2} + c \left( S^2 + 2zS \frac{\partial S}{\partial z} \right).$$

If the above equation admits a unique solution, then so is the equation (16). At present, we do not know how to deal with these equations.

3.3 The moment method

In this section, we introduce the moment method to study the limiting behavior of $\mu_t^{(N)}$. We first show the following result.

Theorem 3.3. Assume that Condition $H1$ is satisfied. Then for any $k = 1, 2, \ldots$, the $k$th moment process

$$S_k^{(N)}(t) = \frac{1}{N} \sum_{i=1}^{N} \lambda_i(t)^k$$

converges in probability to a deterministic differentiable function $m_k(t)$ which is defined inductively as the solution to the following initial value ODE

$$\begin{cases}
m'_k(t) = -k \left( m_k(t) + (\alpha + k - 1) m_{k-1}(t) + c \sum_{i=0}^{k-1} m_i(t) m_{k-i-1}(t) \right), \\
m_k(0) = a_k,
\end{cases}$$

(18)

where $m_0 \equiv 1$. To be more precise, this means that for any $T > 0$, as random elements in the space of continuous functions $C([0, T], \mathbb{R})$ endowed with the uniform norm, the sequence $\{S_k^{(N)}(t)\}$ converges in probability to $m_k(t)$.

We need some preparations to prove this theorem. To begin with, let us express the equation (17) with $f = x^k$ in the following form

$$dS_k^{(N)}(t) = \sum_{i=1}^{N} k \frac{\lambda_i}{N} \sqrt{2\lambda_i \lambda_{i-1}^k} db_i - kS_k^{(N)}(t) dt$$

$$+ \alpha kS_{k-1}^{(N)}(t) dt + ck \sum_{i=0}^{k-1} S_i^{(N)}(t) S_{k-i-1}^{(N)}(t) dt$$

$$+ k(k-1)S_{k-1}^{(N)}(t) dt - \frac{ck^2}{N} S_{k-1}^{(N)}(t) dt$$

$$=: dM_k^{(N)}(t) - kS_k^{(N)}(t) dt + F_k^{(N)}(t) dt.$$ 

Here $M_k^{(N)}(t)$ is a martingale, because of the uniform boundedness of $E[\|X_t\|^{2m}]$ (a statement following the equation (12)), with the quadratic variation

$$\langle M_k^{(N)} \rangle_t = \frac{2k^2}{N} \int_0^t \sum_{i=1}^{N} \lambda_i(s)^{2k-1} ds.$$

(19)
Now we write $S_k^{(N)}(t)$ in the integral form

$$S_k^{(N)}(t) = \langle \mu_0^{(N)}, x^k \rangle - k \int_0^t S_k^{(N)}(s) ds + M_k^{(N)}(t) + \int_0^t F_k^{(N)}(s) ds$$

$$= : \langle \mu_0^{(N)}, x^k \rangle - k \int_0^t S_k^{(N)}(s) ds + \Phi_k^{(N)}(t). \quad (20)$$

Here note that $\Phi_k^{(N)}(t)$ is a continuous function with $\Phi_k^{(N)}(0) = 0$. In addition, observe that the above is an ODE for $\Psi(t) = \int_0^t S_k^{(N)}(s) ds$. Thus $S_k^{(N)}(t)$ has the following explicit expression

$$S_k^{(N)}(t) = \langle \mu_0^{(N)}, x^k \rangle e^{-kt} + \Phi_k^{(N)}(t) - k \left( \int_0^t \Phi_k^{(N)}(s) e^{ks} ds \right) e^{-kt}. \quad (21)$$

Let $T$ be fixed. Let $\mathbb{X} = (\mathcal{C}([0,T], \mathbb{R}), \| \cdot \|)$ be the space of continuous functions on $[0,T]$ endowed with the supremum norm. Then $\mathbb{X}$ is a complete separable metric space. We consider $S_k^{(N)}, M_k^{(N)}$ and $F_k^{(N)}$ as random elements on $\mathbb{X}$.

**Definition 3.4.** Let $X^{(N)}$ and $X$ be $\mathbb{X}$-valued random elements defined on the same probability space. The sequence $X^{(N)}$ is said to converge in probability to $X$ if $\|X^{(N)} - X\|$ converges in probability to 0, that is, for any $\varepsilon > 0$,

$$\lim_{N \to \infty} \mathbb{P}(\|X^{(N)} - X\| \geq \varepsilon) = 0.$$ 

Note that when $X$ is deterministic, then the condition that $X^{(N)}$ is defined on the same probability space is not necessary.

The addition and multiplication operators on $\mathbb{X}$ are defined pointwisely as usual. Based on the estimates that

$$\|x + y\| \leq \|x\| + \|y\|, \quad \|xy\| \leq \|x\| \|y\|, \quad x, y \in \mathbb{X},$$

we can easily show the following. Assume that $X^{(N)}$ (resp. $Y^{(N)}$) converges to $X$ (resp. $Y$) in probability (as random elements on $\mathbb{X}$). Then the following hold.

(i) $X^{(N)} + Y^{(N)}$ (resp. $X^{(N)}Y^{(N)}$) converges to $X + Y$ (resp. $XY$) in probability.

(ii) $\int_0^t X^{(N)}(s) ds$ converges to $\int_0^t X(s) ds$ in probability.

Back to our problem, we now show that the martingale part $M_k^{(N)}$ converges in probability to zero.

**Lemma 3.5.** $M_k^{(N)}$ converges in probability to 0 (in $\mathbb{X}$) as $N \to \infty$.

**Proof.** By using Doob's martingale inequality, we first estimate

$$\mathbb{P}(\|M_k^{(N)}\| \geq \varepsilon) = \mathbb{P}\left( \sup_{0 \leq t \leq T} |M_k^{(N)}(t)| \geq \varepsilon \right) \leq \frac{\mathbb{E}[M_k^{(N)}(T)^2]}{\varepsilon^2}.$$ 

From this, it suffices to show that $\mathbb{E}[M_k^{(N)}(T)^2] \to 0$ as $N \to \infty$. Note from the quadratic formula (19) that

$$\mathbb{E}[M_k^{(N)}(T)^2] = \frac{2k^2}{N} \mathbb{E}\left[ \int_0^T \frac{\sum_{i=1}^N \lambda_i(s) 2^{k-1}}{N} ds \right] = \frac{2k^2}{N} \int_0^T \mathbb{E}[S_{2^{k-1}}(s)] ds.$$
Therefore, it now suffices to show that for each fixed \( k \), there is a constant \( D_k \) such that
\[
S_k^{(N)}(t) := \mathbb{E}[S_k^{(N)}(t)] \leq D_k,
\]
for all \( t \in [0, T] \) and all \( N \).

Take the expectation in both sides of the identity (20), we get that
\[
S_k^{(N)}(t) = \langle \mu_0^{(N)}, x^k \rangle - k \int_0^t S_k^{(N)}(s) ds + \int_0^t \mathbb{E}[F_k^{(N)}(s)] ds.
\]

Note that Condition \( \textbf{H1} \) implies that the initial moment \( \langle \mu_0^{(N)}, x^k \rangle \) is uniformly bounded. Since \( S_i^{(N)}(t)S_j^{(N)}(t) \leq S_i^{(N)}(t) \), if follows that
\[
F_k^{(N)}(t) \leq C_k S_k^{(N)}(t), \quad \mathbb{E}[F_k^{(N)}(t)] \leq C_k \mathbb{E}[S_k^{(N)}(t)],
\]
and hence,
\[
\mathbb{E}[F_k^{(N)}(t)] \leq C_k \mathbb{E}[S_k^{(N)}(t)] \leq C_k D_{k-1}, \quad t \in [0, T],
\]
for some constant \( C_k \) not depending on \( N \). Then the desired uniform boundedness follows immediately by induction. The proof is complete. □

**Proof of Theorem 3.3.** Based on the formula (21), we prove this theorem by induction. The case \( k = 0 \) is trivial. Assume for now that for \( l = 0, 1, \ldots, k - 1 \), the sequence \( S_l^{(N)} \) converges in probability to a differentiable function \( m_l \) (as random elements in \( X \)). We need to show that the sequence \( S_k^{(N)} \) converges in probability to \( m_k \) which satisfies the ODE (18).

By the induction hypothesis, it is clear that
\[
F_k^{(N)}(t) \rightarrow k(\alpha + k - 1)m_k(t) + ck \sum_{i=0}^{k-1} m_i(t)m_{k-i-1}(t) =: F(t) \quad \text{in probability}.
\]
Together with Lemma 3.5, it follows that the function \( \Phi_k^{(N)} \) (defined in (20)) converges in probability to \( \int_0^t F(s) ds \). Therefore, \( S_k^{(N)} \) converges in probability to the limit \( m_k \) given by
\[
m_k(t) = a_k e^{-kt} + F(t) - k \left( \int_0^t F(s) e^{ks} ds \right) e^{-kt}.
\]
Since the function \( F(t) \) is differentiable by the induction hypothesis, we conclude that \( m_k(t) \) is also differentiable, and thus, it satisfies the ODE (18). The proof is complete. □

Next, we study the ODE (18) in more details.

**Lemma 3.6.** Define a sequence \( \{C_{k,0}\}_{k \geq 0} \) as follows
\[
\begin{cases}
C_{0,0} = 1, \\
C_{k,0} = (\alpha + k - 1)C_{k-1,0} + c \sum_{i=0}^{k-1} C_{i,0}C_{k-i-1,0}, \quad k \geq 1,
\end{cases}
\]
Then for each \( k \), the \( k \)th moment process \( m_k(t) \) has the form
\[
m_k(t) = C_{k,0} + \sum_{i=1}^{k} C_{k,i} e^{-it},
\]

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where \( C_{k,i} \) are constants. In particular, \( \lim_{t \to \infty} m_k(t) = C_{k,0} \). In addition, it holds that

\[
\sup_{t \geq 0} m_k(t) \leq \Lambda_k,
\]

where \( \{\Lambda_k\} \) is the sequence in Condition H2.

Proof. Again, we prove this lemma by induction.

The case \( k = 1 \). The first moment process \( m_1(t) \) satisfies the following ODE

\[
\begin{align*}
    m_1'(t) &= (\alpha + c) - m_1(t), \\
    m_1(0) &= (\mu_0, x) = a_1.
\end{align*}
\]

Solving the equation gives the explicit formula

\[
m_1(t) = (\alpha + c)(1 - e^{-t}) + a_1 e^{-t} = C_{1,0} + C_{1,1} e^{-t}.
\]

In particular,

\[
m_1(t) \leq (\alpha + c) \vee a_1 = \Lambda_1.
\]

The case \( k \geq 2 \). By induction, the ODE for \( m_k(t) \) can be written as

\[
m_k'(t) = -km_k(t) + kC_{k,0} + \sum_{i=1}^{k-1} D_{k,i} e^{-it},
\]

where

\[
C_{k,0} = (\alpha + k - 1)C_{k-1,0} + c \sum_{i=0}^{k-1} C_{i,0} C_{k-i-1,0}.
\]

This implies an explicit formula for \( m_k(t) \). For the upper bound, since \( m_i(t)m_j(t) \leq m_{i+j}(t) \), it follows that

\[
m_k'(t) \leq -km_k(t) + k(\alpha + k - 1)m_{k-1}(t) + k^2 c m_{k-1}(t),
\]

from which we deduce that

\[
m_k(t) \leq (\alpha + k - 1 + ck)\Lambda_{k-1} \vee a_k = \Lambda_k.
\]

Lemma 3.7. The solution to the initial value ODE

\[
\phi'(t) = -k\phi(t) + F(t), \quad \phi(0) = \phi_0,
\]

is of the form

\[
\phi(t) = \left( \phi_0 + \int_0^t F(s) e^{ks} ds \right) e^{-kt}.
\]

Consequently, if \( F(t) \leq G(t), t \geq 0 \), then

\[
\phi(t) \leq \psi(t), \quad (t \geq 0),
\]

where \( \psi(t) \) is the solution to the equation

\[
\psi'(t) = -k\psi(t) + G(t), \quad \psi(t) = \phi_0.
\]
Lemma 3.8. The solution to the initial value ODE

\[ \phi'(t) = k(-\phi + D), \quad \phi(0) = C, \]

where \( k, C, D > 0 \) are constants, is given by

\[ \phi(t) = D(1 - e^{-kt}) + Ce^{-kt}. \]

Consequently,

\[ \sup_{t \geq 0} \phi(t) \leq C \lor D. \]

We are now ready to state the main result of this paper.

Theorem 3.9. Assume that Conditions \( H1 \) and \( H2 \) are satisfied. Then for any \( T > 0 \), the sequence of empirical measure processes \( \mu^{(N)}_t \) converges in probability in \( C([0, T], \mathcal{P}(\mathbb{R}_{\geq 0})) \) to a continuous probability measure-valued process \( \mu_t \) as \( N \to \infty \). Here \( \mu_t \) is the unique measure whose moments are given by \( \{m_k(t)\}_{k=1}^\infty \).

Proof. Under Conditions \( H1 \) and \( H2 \), Theorem 3.3 and Lemma 3.6 implies that for each \( t \), the sequence of limit moments \( \{m_k(t)\} \) satisfies

\[ \sum_{k=1}^{\infty} m_k(t)^{-\frac{1}{2k}} \geq \sum_{k=1}^{\infty} \Lambda_k^{-\frac{1}{2k}} = \infty. \]

Therefore, there is a unique probability measure \( \mu_t \) on \([0, \infty)\) whose moments are \( \{m_k(t)\} \). The process \( (\mu_t)_{t \geq 0} \) is continuous because \( \mu_t \) is determined by moments. It follows from Theorem A.1 below that the sequence \( \mu^{(N)}_t \) converges in probability to \( \mu_t \) in \( C([0, T], \mathcal{P}(\mathbb{R}_{\geq 0})) \), for each \( T > 0 \). The proof is complete. \( \square \)

3.4 Beta Laguerre ensembles at high temperature

Let us recall the beta Laguerre ensemble from the equation (2)

\[ \frac{1}{Z_{N,\alpha,\beta}} \times \prod_{i<j} |\lambda_j - \lambda_i|^{\beta} \prod_{l=1}^{N} \lambda_l^{\alpha-1}e^{-\lambda_l}, \quad 0 \leq \lambda_1 \leq \cdots \leq \lambda_N. \]

In the regime where \( \beta N \to 2c \in (0, \infty) \) and \( \alpha \) is fixed, the limiting behavior of the empirical distributions has been studied in [1, 23]. It was shown that as \( N \to \infty \), the empirical distribution

\[ L_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i} \]

converges weakly to the probability measure \( \nu_{\alpha,c} \) which is the probability measure of associated Laguerre orthogonal polynomials (model II) [23]. It is the spectral measure of the following Jacobi matrix

\[ J_{\alpha,c} = \begin{pmatrix} \sqrt{\alpha + c} & \sqrt{\alpha + c + 1} & \ldots \\ \sqrt{c + 1} & \sqrt{\alpha + c + 1} & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}. \]
that is, the measure $\nu_{\alpha,c}$ is determined by moments with moments given by

$$\langle \nu_{\alpha,c}, x^k \rangle = (J_{\alpha,c})^k(1, 1) =: u_k, \quad k = 0, 1, 2, \ldots.$$ 

The density and the Stieltjes transform of $\nu_{\alpha,c}$ were calculated in [15]

$$\nu_{\alpha,c}(x) = \frac{1}{\Gamma(c+1)\Gamma(c+\alpha)} \frac{x^{\alpha-1}e^{-x}}{\Psi(c, 1-\alpha; xe^{-it})^2}, \quad x \geq 0,$$

$$S_{\nu_{\alpha,c}}(z) = \int_0^\infty \frac{\nu_{\alpha,c}(x)dx}{x-z} = \frac{\Psi(c+1, 2-\alpha; z)}{\Psi(c, 1-\alpha; z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$ 

Here $\Psi(a, b; z)$ is Tricomi’s confluent hypergeometric function.

On the other hand, using ideas in [14], we can show that the sequence of moments $\{u_k\}$ satisfies the self-convolutive equation

$$u_k = (\alpha + k - 1)u_{k-1} + c \sum_{i=0}^{k-1} u_i u_{k-i-1}, \quad k = 1, 2, \ldots \quad (22)$$

Therefore the sequence $\{C_{k,0}\}$ in Lemma 3.6 coincides with the sequence of moments $\{u_k\}$ of $\nu_{\alpha,c}$. Note that from the self-convolutive equation, we can also calculate explicitly the density of $\nu_{\alpha,c}$ by using the result in [18]. Thus, Theorem 3.9 and Lemma 3.6 imply that the limit process $\mu_t$ satisfies

$$\lim_{t \to \infty} \mu_t = \nu_{\alpha,c}.$$ 

To summary, in a high temperature regime where $\beta N \to 2c \in (0, \infty0$, with fixed $\alpha$, we have finished showing the following diagram

$$\begin{array}{ccc}
\mu_t^{(N)} & \xrightarrow{N \to \infty} & \mu_t \\
\downarrow t \to \infty && \downarrow t \to \infty \\
\beta LE(N) & \xrightarrow{N \to \infty} & \nu_{\alpha,c}
\end{array}$$

A Convergence of probability measure-valued processes

Let $\mathcal{Y} = \mathcal{C}([0, T], \mathcal{P}(\mathbb{R}))$ be the space of continuous mappings $\mu: [0, T] \to \mathcal{P}(\mathbb{R})$ endowed with the topology of uniform convergence, where $\mathcal{P}(\mathbb{R})$ is the space of probability measures on $\mathbb{R}$ endowed with the weak topology. For definiteness, we consider the Lévy–Prokhorov metric on $\mathcal{P}(\mathbb{R})$ which makes it a complete and separable metric space. Then $\mathcal{Y}$ can be metrizable to become a complete separable metric space. Recall that $\mathcal{X} = \mathcal{C}([0, T], \mathbb{R})$ is the space of continuous functions on $[0, T]$ endowed with the uniform norm. We are going to show the following result which can be roughly stated as the convergence of moments implies the convergence of measures at the process level.

**Theorem A.1.** Let $\mu^{(N)}$ be a sequence of random elements on $\mathcal{Y}$. Assume that for each $k$, the $k$th moment process $\langle \mu^{(N)}(t), x^k \rangle$ is an $\mathcal{X}$-valued random element converging in probability to a non-random limit $m_k(t)$. For each $t \in [0, T]$, let $\mu(t)$ be a probability measure having moments $\{m_k(t)\}_{k \geq 1}$. Assume further that the measure $\mu(t), t \in [0, T]$, is determined by moments. Then $\mu = (\mu(t))_{0 \leq t \leq T}$ is an element in $\mathcal{Y}$, and the sequence $\mu^{(N)}$ converges in probability to $\mu$ as $N \to \infty$ as $\mathcal{Y}$-valued random elements.
Analogous to the case of random probability measures case ([13, Lemma 2.2]), the above theorem follows directly from the following deterministic result.

**Lemma A.2.** Let \( \{\mu^{(N)}(t)\} \) be a sequence in \( Y \) such that for each \( k = 1, 2, \ldots \), the sequence \( \{\mu^{(N)}(t), x^k\} \subset X \) converges uniformly to a limit \( m_k(t) \). Assume that for each \( t \in [0, T] \), the sequence of moments \( \{m_k(t)\} \) uniquely determines the probability measure \( \mu(t) \). Then \( \mu = (\mu(t))_{0 \leq t \leq T} \in Y \) and the sequence \( \{\mu^{(N)}\} \) converges to \( \mu \).

**Proof.** Since the functions \( m_k(t) \) are continuous and for each \( t \) the measure \( \mu(t) \) is determined by moments, it is clear that \( \mu \) is an element of \( Y \). We will show that \( \{\mu^{(N)}\} \) converges to \( \mu \) by contradiction. Indeed, assume for contradiction that the sequence \( \{\mu^{(N)}\} \) does not converge to \( \mu \). Then we can find a subsequence \( \{N_l\} \subset \mathbb{N} \), a sequence \( \{t_l\} \subset [0, T] \) converging to \( t \) such that \( \mu^{N_l}(t_l) \) does not converge to \( \mu(t) \). However, each moment of \( \mu^{N_l}(t_l) \) converges to that of \( \mu(t) \) by the uniform convergence assumption, implying the weak convergence of probability measure, which is a contradiction. The lemma is proved. \( \square \)

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