Transport in boundary-driven quantum spin systems: one-way street for the energy current

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Abstract
We study transport properties in boundary-driven asymmetric quantum spin chains given by XXZ and XXX Heisenberg models. Our approach exploits symmetry transformations in the Lindblad master equation associated to the dynamics of the systems. We describe the mathematical steps to build the unitary transformations related to the symmetry properties. For general target polarizations, we show the occurrence of the one-way street phenomenon for the energy current, namely, the energy current does not change in magnitude and direction as we invert the baths at the boundaries. We also analyze the spin current in some situations, and we prove the uniqueness of the steady state for all investigated cases. Our results, involving nontrivial properties of the energy flow, shall interest researchers working on the control and manipulation of quantum transport.

Keywords: energy transport, boundary-driven quantum systems, symmetries in the Lindblad equation

1. Introduction
A bedrock of nonequilibrium statistical physics is the understanding of the transport laws [1–3]. In particular, the study of the energy flow properties is of theoretical and experimental interest: a good example is the investigation of thermal rectification. Motivated by the amazing progress of modern electronics due to the invention of transistor, electric diode and other nonlinear solid state devices, several works are devoted to the investigations of asymmetries in

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the energy current in order to build thermal diodes [4, 5], a device in which the magnitude of the energy current changes as we invert the system between two baths.

A subject of increasing attention nowadays is the study of such transport laws in the quantum regime. Stimulated by the emerging field of quantum thermodynamics, by the development of nanotechnology and the possibility of experimental manipulation of small quantum systems, the study of quantum models becomes mandatory.

Quantum spin chains, in specific, are exhaustively investigated. They are the archetypal models of open quantum systems, and are related to problems in several different areas: condensed matter, cold atoms, optics, quantum information, etc. Their boundary-driven versions, i.e., systems with target polarization at the boundaries are recurrently studied [6–10]. The energy current of these boundary-driven systems, differently of the weakly coupled models, usually involves heat and work [11–13]. It is an important information: when we ignore the work component, incorrect conclusions may be obtained [13, 14].

The present article addresses the investigation of some (a)symmetries in the energy current of boundary-driven Heisenberg (XXX) and XXZ models. In specific, we show the occurrence of the one-way street phenomenon for the energy current to several asymmetric Heisenberg and XXZ with general cases of different boundary polarizations. Such a phenomenon means that the energy current is the same as we invert the baths at the boundaries, that is, it does not change in magnitude and direction. Thus, the phenomenon is, in some way, related to (but stronger than) rectification.

It is important to emphasize that, as said, the energy current is not only heat, and so, there is no thermodynamic inconsistency in the occurrence of the one-way street effect. For more details, see references [11–13].

The dynamics associated to the models, as usual, is given by a Lindblad master equation (LME) [15]. To establish our results we exploit symmetries of the density matrix and of the LME. And these results are independent of the system size and of the transport regime.

The existence of the one-way street phenomenon was established in reference [16] by a direct computation of the steady density matrix and the energy current for a XXZ model with $\sigma^z$ polarization at the boundaries. The argument of symmetries appeared in reference [17] for the same case, and in a recent letter [18] we stated, without presenting a mathematical proof, the possibility of a ubiquitous occurrence of such phenomenon for systems with general target spin polarization at the boundaries. In the present paper, we give the mathematical proofs for the energy current property; we also show that, in some cases, the spin current changes the sign as we invert the baths, differently of the energy flow. And, an important mathematical point, we prove the uniqueness of the steady distributions for the cases treated here.

The rest of the paper is organized as follows. In section 2, we introduce the model and describe the approach. In section 3, we analyze several cases of different target polarizations and present the mathematical steps. In section 4, we prove the uniqueness of the steady states. Section 5 is devoted to the final remarks.

2. Models and approach

Now we introduce the models to be treated here, the LME, the approach to be used and some previous results.

We consider here standard quantum spin models, namely, the XXZ and Heisenberg (XXX) chains. For the Hamiltonian of the asymmetric version of the spin 1/2 XXZ chain, we take

$$
\mathcal{H} = \sum_{i=1}^{N-1} \left\{ \alpha \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y \right) + \Delta \sigma_i^z \sigma_{i+1}^z \right\},
$$

(1)
where $\sigma^\beta_i$ ($\beta = x, y, z$) are the Pauli matrices. We are interested in cases involving asymmetric distributions for the anisotropy parameter $\Delta_\alpha$, for example, a graded distribution: $\Delta_1 < \Delta_2 < \cdots < \Delta_{N-1}$.

For the Heisenberg model, we take the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N-1} \alpha_i \left( \sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1} + \sigma^z_i \sigma^z_{i+1} \right),$$

(2)

where $\alpha_i$ is asymmetrically distributed.

The open quantum systems to be analyzed are given by the steady states of the LME

$$\frac{d\rho}{dt} = i[\rho, \mathcal{H}] + \mathcal{L}(\rho),$$

(3)

where we assume $\hbar = 1$, $\rho$ is the density matrix, the dissipator $\mathcal{L}(\rho)$ describes the coupling with the baths and it is given by

$$\mathcal{L}(\rho) = \mathcal{L}_L(\rho) + \mathcal{L}_R(\rho),$$

$$\mathcal{L}_{L,R}(\rho) = \sum_{i=\pm} L_i \rho L_i^\dagger - \frac{1}{2} \{ L_i^L, \rho \},$$

(4)

$\{.,.\}$ above describes the anti-commutator; different $L_s$ will be specified later.

The spin and the energy current are derived from the LME and continuity equations, see, e.g., reference [19] for details. For the XXZ chain, the spin current is

$$\langle J^M_j \rangle = 2\alpha \langle \sigma^y_j \sigma^y_{j+1} - \sigma^z_j \sigma^z_{j+1} \rangle.$$

(5)

Adding in the Hamiltonian the interaction with an external magnetic field $\sum_{j=1}^N B_j \sigma^z_j$, the energy current becomes

$$\langle J^E_j \rangle = \langle J^{\text{XXZ}}_j \rangle + \langle J^M_j \rangle,$$

$$\langle J^{\text{XXZ}}_j \rangle = 2\alpha \left( \alpha \left( \sigma^y_{j-1} \sigma^y_{j+1} - \sigma^z_{j-1} \sigma^z_{j+1} \right) + \Delta_{j-1,j} \left( \sigma^z_{j-1} \sigma^y_{j+1} - \sigma^z_{j-1} \sigma^y_{j+1} \right) + \Delta_{j,j+1} \left( \sigma^z_{j-1} \sigma^y_{j+1} - \sigma^z_{j-1} \sigma^y_{j+1} \right) \right),$$

(6)

$$\langle J^M_j \rangle = \frac{1}{2} B_j (J_{j-1} + J_j).$$

It is important to recall that there is a remarkable difference between symmetric and asymmetric XXZ chains. For the symmetric case we have $\langle J^{\text{XXZ}}_j \rangle = 0$ [19]. And so, the energy current becomes proportional to the spin current, and vanishes as $B = 0$. But it does not follow in the asymmetric case as shown, by a direct computation, in reference [16] for a system with $\sigma^z$ polarization at the boundaries.

Turning to the Heisenberg Hamiltonian, the expressions for the currents become

$$\langle J^M_j \rangle = 2\alpha \left( \sigma^y_j \sigma^y_{j+1} - \sigma^z_j \sigma^z_{j+1} \right),$$

(7)

$$\langle J^{\text{XXZ}}_j \rangle = 2\alpha \left( \alpha \left( \sigma^y_{j-1} \sigma^y_{j+1} - \sigma^z_{j-1} \sigma^z_{j+1} \right) + \left( \sigma^z_{j-1} \sigma^y_{j+1} - \sigma^z_{j-1} \sigma^y_{j+1} \right) + \left( \sigma^z_{j-1} \sigma^y_{j+1} - \sigma^z_{j-1} \sigma^y_{j+1} \right) \right).$$

(8)
We now describe our strategy to prove the current properties, in particular, the one-way street phenomenon. In some way, we follow Popkov and Livi [20]. We exploit symmetries in the LME to show that, if $\rho$ is a steady state solution of the LME, then there is a unitary transformation $U$ (to be built) such that $U\rho U^\dagger$ is a solution of the LME with inverted baths. By uniqueness (to be proved), it is the steady state with inverted baths. Then we turn to the energy current and show that the average with the new steady state is the same of that with the initial steady state. That is, the energy current does not change as we invert the baths: this is the one-way street phenomenon.

To be precise, in the steady state the LME becomes

$$0 = -i[H, \rho] + \mathcal{L}(\rho).$$

It means that, in order to perform our analysis, we must find a unitary transformation $U$ such that

$$H = UH_0 U^\dagger, \quad \mathcal{L}_{\text{inv. baths}}(U\rho U^\dagger) = U\mathcal{L} U^\dagger.$$ (9)

Moreover, to show the one-way street phenomenon we need to prove that $UJU^\dagger = J$.

In the next section, we find $U$ for several different dissipators, i.e., several different boundary polarizations, such that these relations are satisfied.

### 3. Unitary transformations and symmetry results

Now we will build the unitary transformations in order to exploit the symmetries of the Lindblad equations and prove some current properties, in particular, the one-way street phenomenon for the energy current.

We begin by noting that any unitary matrix can be written as (the reader can prove it)

$$U = \begin{pmatrix} a & b \\ -e^{i\varphi}b^* & e^{i\varphi}a^* \end{pmatrix},$$ (10)

where $a, b \in \mathbb{C}, \varphi \in \mathbb{R}$ and $|a|^2 + |b|^2 = 1$.

Then, we analyze several cases involving different boundary polarizations. We also investigate two different graded systems: the two first cases are related to the XXZ chain, and the other ones to Heisenberg model. First we take the case in which the polarization is in the $x$ direction in one boundary, and in some generic angle in the plane $xy$ in the other boundary. Precisely, we consider the Lindblad operators

$$K_+^L = \sqrt{\gamma(1 + f)} \left( \frac{\sigma_x^+ + i\sigma_y^+}{2} \right),$$

$$K_-^L = \sqrt{\gamma(1 - f)} \left( \frac{\sigma_x^- - i\sigma_y^-}{2} \right),$$

$$K_+^R = \sqrt{\gamma(1 - f)} \left( \frac{\cos \theta \sigma_x^+ + \sin \theta \sigma_y^+ + i\sigma_z^+}{2} \right),$$

$$K_-^R = \sqrt{\gamma(1 + f)} \left( \frac{\cos \theta \sigma_x^- + \sin \theta \sigma_y^- - i\sigma_z^-}{2} \right),$$ (11)

where $\gamma$ is the coupling constant and $f$ is the driving strength (we take $f_L = -f_R = f$).
To perform the change between the baths, we need to find a unitary operator such that

\[
U(\sigma^y + i\sigma^z)U^\dagger = \cos\theta \sigma^x + \sin\theta \sigma^y - i\sigma^z, \\
U(\sigma^y - i\sigma^z)U^\dagger = \cos\theta \sigma^x + \sin\theta \sigma^y + i\sigma^z, \\
U(\cos\theta \sigma^x + \sin\theta \sigma^y + i\sigma^z)U^\dagger = \sigma^y - i\sigma^z, \\
U(\cos\theta \sigma^x + \sin\theta \sigma^y - i\sigma^z)U^\dagger = \sigma^y + i\sigma^z.
\]

We may still have factors such as \(-1, i\) or \(-i\) on the right hand side without any further problem.

Given such conditions, we see that it is enough to find a unitary matrix \(A\) such that the operation \(A(\cdot)A^\dagger\) transforms as:

\[
\begin{align*}
\sigma^y &\rightarrow \cos\theta \sigma^x + \sin\theta \sigma^y \rightarrow (1) \\
\sigma^z &\rightarrow (2) \\
\sigma^z &\rightarrow (3) -\sigma^z
\end{align*}
\]

And so, \(U\) will be given by

\[
U = A \otimes A \otimes \ldots \otimes A.
\]

Carrying out the computation:

\[
A\sigma^zA^\dagger = \begin{pmatrix} a & b \\ -e^{i\varphi}b^* & e^{i\varphi}a^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A^\dagger = \begin{pmatrix} |a|^2 - |b|^2 & -e^{-i\varphi}ab - e^{i\varphi}ab \\ -e^{i\varphi}b^*a^* - e^{-i\varphi}b^*a^* & |b|^2 - |a|^2 \end{pmatrix}.
\]

But we want

\[
A\sigma^zA^\dagger = -\sigma^z = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

It implies that \(|b|^2 - |a|^2 = 1\). As we already have \(|b|^2 + |a|^2 = 1\), then \(|a|^2 = 0\), and so, \(a = 0\), \(|b|^2 = 1\).

Now the matrix \(A\) is given by

\[
A = \begin{pmatrix} 0 & b \\ -e^{i\varphi}b^* & 0 \end{pmatrix}.
\]

To find \(b\) we perform the computation

\[
A\sigma^yA^\dagger = \begin{pmatrix} 0 & b^* \\ -e^{i\varphi}b^* & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} A^\dagger = \begin{pmatrix} 0 & -ie^{-i\varphi}b^2 \\ ie^{i\varphi}b^2 & 0 \end{pmatrix}.
\]

We want

\[
A\sigma^yA^\dagger = \cos\theta \sigma^x + \sin\theta \sigma^y = \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix},
\]

\[
A\sigma^yA^\dagger = \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}.
\]
we take $\varphi = \theta$, and it leads us to $-ib^2 = 1$. Then, it is enough to take $b = \frac{1}{\sqrt{2}}$. Hence,

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 + i \\ -e^{i\varphi}(1 - i) & 0 \end{pmatrix}$$

(19)

is the desired unitary matrix.

Carrying out some computation we find that

$$A(\cos \theta \sigma^x + \sin \theta \sigma^y)A^\dagger = \sigma^x.$$  

Now we analyze the XXZ Hamiltonian

$$H = \sum_{i=1}^{N-1} \alpha(\sigma_i^x \sigma_{i+1}^x + \sigma_i^z \sigma_{i+1}^z) + \Delta_i \sigma_i^z \sigma_{i+1}^z$$

First, we note that

$$A \sigma^z A^\dagger = -\sin \theta \sigma^x + \cos \theta \sigma^y.$$

Then, it follows that

$$UHU^\dagger = \sum_{i=1}^{N-1} \alpha(A \sigma_i^z A^\dagger A \sigma_i^z A^\dagger + A \sigma_i^z A^\dagger A \sigma_i^z A^\dagger) + \Delta_i A \sigma_i^z A^\dagger A \sigma_i^z A^\dagger$$

$$= \ldots$$

$$= \sum_{i=1}^{N-1} \alpha(\sigma_i^x \sigma_{i+1}^x + \sigma_i^z \sigma_{i+1}^z) + \Delta_i \sigma_i^z \sigma_{i+1}^z.$$  

(20)

That is, $UHU^\dagger = H$.

Before investigating the effect of $U$ on the currents, we note that

$$A^\dagger \sigma^x A = -\sin \theta \sigma^x + \cos \theta \sigma^y,$$

$$A^\dagger \sigma^y A = \cos \theta \sigma^x + \sin \theta \sigma^y,$$

$$A^\dagger \sigma^z A = -\sigma^z.$$  

The energy current is

$$\hat{J}_E = 2\alpha \left[ \alpha \left( \sigma_{i-1}^z \sigma_i^z \sigma_{i+1}^z - \sigma_{i-1}^z \sigma_i^z \sigma_{i+1}^z \right) + \Delta_{i-1} \left( \sigma_{i-1}^z \sigma_i^z \sigma_{i+1}^z - \sigma_{i-1}^z \sigma_i^z \sigma_{i+1}^z \right) \right]$$

$$+ \Delta_i \left( \sigma_{i-1}^z \sigma_i^z \sigma_{i+1}^z - \sigma_{i-1}^z \sigma_i^z \sigma_{i+1}^z \right),$$  

(22)

and so the effect of $U$ is

$$U^\dagger \hat{J}_E U = 2\alpha \left\{ \alpha \left[ \cos \theta \sigma_{i-1}^z + \sin \theta \sigma_{i-1}^z \right] \left( -\sigma_i^z \right) \left( -\sin \theta \sigma_{i+1}^z + \cos \theta \sigma_{i+1}^z \right) \right.$$

$$- \left( -\sin \theta \sigma_{i-1}^z + \cos \theta \sigma_{i-1}^z \right) \left( -\sigma_i^z \right) \left( \cos \theta \sigma_{i+1}^z + \sin \theta \sigma_{i+1}^z \right) \right.$$  

$$+ \Delta_{i-1} \left( -\sigma_{i-1}^z \right) \left( -\sin \theta \sigma_i^z + \cos \theta \sigma_i^z \right) \left( \cos \theta \sigma_{i+1}^z + \sin \theta \sigma_{i+1}^z \right)$$

$$- \left( -\sigma_{i-1}^z \right) \left( \cos \theta \sigma_i^z + \sin \theta \sigma_i^z \right) \left( \cos \theta \sigma_{i+1}^z + \sin \theta \sigma_{i+1}^z \right) \right\]$$
It shows the occurrence of the one-way street phenomenon: the energy current is the same, it keeps the same value and direction as we invert the reservoirs at the boundaries.

Taking the spin current

$$J^M = 2\alpha(\sigma^x_{t+1} - \sigma^y_{t+1})$$

the effect of $U$ is

$$U^\dagger J^M U = 2\alpha \left[A^1\sigma^x_{t+1}A - A^1\sigma^y_{t+1}A^1\right]$$

$$= 2\alpha \left[\left(-\sin \theta \sigma^x_t + \cos \theta \sigma^y_t\right)\left(\cos \theta \sigma^x_{t+1} + \sin \theta \sigma^y_{t+1}\right)
- \left(\cos \theta \sigma^x_t + \sin \theta \sigma^y_t\right)\left(-\sin \theta \sigma^x_{t+1} + \cos \theta \sigma^y_{t+1}\right)\right]$$

$$= 2\alpha \left[-\sin \theta \cos \theta \sigma^x_t \sigma^y_{t+1} - \sin^2 \theta \sigma^y_t \sigma^y_{t+1} + \cos^2 \theta \sigma^y_t \sigma^y_{t+1}
+ \sin \theta \cos \theta \sigma^x_t \sigma^x_{t+1} + \sin \theta \cos \theta \sigma^y_t \sigma^x_{t+1} - \cos^2 \theta \sigma^y_t \sigma^x_{t+1}
+ \sin^2 \theta \sigma^y_t \sigma^x_{t+1} - \sin \theta \cos \theta \sigma^y_t \sigma^y_{t+1}\right]$$

$$= 2\alpha \left(\sigma^x_t \sigma^y_{t+1} - \sigma^y_t \sigma^x_{t+1}\right).$$

That is,

$$U^\dagger J^M U = -J^M,$$  \hspace{1cm} (26)

in other words, the spin current keeps the value and inverts the direction as we invert the reservoirs at the boundaries; there is no spin rectification, no further effect.

X–Y orthogonal polarization. Let us consider the set of Lindblad operators for one boundary

$$L_1 = \alpha(\sigma^x_t + i\sigma^y_t), \hspace{0.5cm} L_2 = \beta(\sigma^x_t - i\sigma^y_t),$$
$$V_1 = p(\sigma^x_t + i\sigma^y_t), \hspace{0.5cm} V_2 = q(\sigma^x_t - i\sigma^y_t),$$
$$W_1 = u(\sigma^x_t + i\sigma^y_t), \hspace{0.5cm} W_2 = v(\sigma^x_t - i\sigma^y_t).$$

And, for the other boundary,

$$L_3 = \beta(\sigma^x_N + i\sigma^y_N), \hspace{0.5cm} L_4 = \alpha(\sigma^x_N - i\sigma^y_N),$$
$$V_3 = v(\sigma^x_N + i\sigma^y_N), \hspace{0.5cm} V_4 = u(\sigma^x_N - i\sigma^y_N),$$
$$W_3 = q(\sigma^x_N + i\sigma^y_N), \hspace{0.5cm} W_4 = p(\sigma^x_N - i\sigma^y_N).$$

For this case, the inversion of the baths can be given by a unitary operator $U = A \otimes A \otimes \ldots \otimes A$ such that

$$A\sigma^x A^\dagger = -\sigma^y, \hspace{0.5cm} A\sigma^y A^\dagger = -\sigma^x, \hspace{0.5cm} A\sigma^z A^\dagger = -\sigma^z.$$  \hspace{1cm} (29)
Indeed, with such an operator we have the transformations

\[
L_1 \rightarrow -iL_4, \quad L_2 \rightarrow iL_3, \quad V_1 \rightarrow -iV_4, \quad V_2 \rightarrow iV_1,
\]
\[
W_1 \rightarrow -iW_4, \quad W_2 \rightarrow iW_3, \quad L_3 \rightarrow -iL_2, \quad L_4 \rightarrow iL_1,
\]
\[
V_3 \rightarrow -iV_2, \quad V_4 \rightarrow iV_3, \quad L_1 \rightarrow -iL_4, \quad L_2 \rightarrow iL_3,
\]
\[
V_1 \rightarrow -iV_4, \quad V_2 \rightarrow iV_1, \quad W_1 \rightarrow -iW_4, \quad W_2 \rightarrow iW_3.
\]

(30)

We will use the general representation for a matrix \( A \in SU(2) \):

\[
A = \begin{pmatrix} a_r + ia_i & b_r + ib_i \\ -b_r + ib_i & a_r - ia_i \end{pmatrix},
\]

(31)

where \( a_r, a_i, b_r, \) and \( b_i \in \mathbb{R} \) satisfy \( a_r^2 + a_i^2 + b_r^2 + b_i^2 = 1 \).

Turning to the computations,

\[
A_{\sigma}^z A^\dagger = \begin{pmatrix} a_r + ia_i & b_r + ib_i \\ -b_r + ib_i & a_r - ia_i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A^\dagger
\]

\[
= \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},
\]

(32)

where

\[
c_{11} = 2(a_r b_r + a_i b_i),
\]

\[
c_{12} = a_r^2 + b_i^2 - a_i^2 - b_r^2 + 2(a_r a_i - b_r b_i)i,
\]

\[
c_{21} = a_r^2 + b_i^2 - a_i^2 - b_r^2 + 2(b_r b_r - a_r a_i)i,
\]

\[
c_{22} = -2(a_r b_r + a_i b_i).
\]

(33)

As we want

\[
A_{\sigma}^z A^\dagger = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},
\]

(34)

we must have

\[
a_r b_r + a_i b_i = 0,
\]

\[
a_r^2 - a_i^2 + b_r^2 - b_i^2 = 0,
\]

\[
a_r a_i - b_r b_i = \frac{1}{2}
\]

(35)

For the \( \sigma^z \) transformation

\[
A_{\sigma}^z A^\dagger = \begin{pmatrix} a_r + ia_i & b_r + ib_i \\ -b_r + ib_i & a_r - ia_i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A^\dagger
\]

\[
= \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix},
\]

(36)

where

\[
d_{11} = a_r^2 + a_i^2 - b_r^2 - b_i^2,
\]

\[
d_{12} = -a_r b_r - ia_i b_i - ia_r b_r + a_r b_i - a_i b_r - ia_r b_i - ib_r a_i - b_r a_i,
\]

\[
d_{21} = -b_r a_i + ib_i a_i + ib_r a_i - a_r b_r + ia_r b_i + ia_r b_i + a_r b_i,
\]

\[
d_{22} = b_r^2 + b_i^2 - a_r^2 - a_i^2.
\]

(37)
As we want
\[ A\sigma^x A^\dagger = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \]  
(38)
we must have
\[ a_r^2 + a_r^2 - b_r^2 - b_r^2 = -1, \]
\[ b_1a_r - a_r b_r = 0, \]  
(39)
\[ b_2a_r + a_r b_r = 0. \]

It is easy to see that a solution is
\[ a_r = 0 = a_i, \]
\[ b_r = -\frac{1}{\sqrt{2}} = -b_i. \]  
(40)
And so,
\[ A = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 + i \\ 1 + i & 0 \end{pmatrix} = \frac{i}{\sqrt{2}} (\sigma^x - \sigma^y) \]  
(41)
is the searched matrix.

We note that we have (as expected)
\[ U\sigma^x U^\dagger = \frac{1}{2} \begin{pmatrix} 0 & -1 + i \\ 1 + i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} -1 - i & 0 \\ 0 & 1 + i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} = -\sigma^z. \]  
(42)
The energy current for the graded XXZ chain is
\[ \tilde{j}_{\text{XXZ}} = 2\alpha \left[ \alpha (\sigma^x_{i-1} \sigma^y_{i+1} - \sigma^x_{i-1} \sigma^x_{i+1}) + \Delta_i (\sigma^y_{i-1} \sigma^y_{i+1} - \sigma^y_{i-1} \sigma^y_{i+1}) \right. \]
\[ + \left. \Delta_i (\sigma^z_{i-1} \sigma^z_{i+1} - \sigma^z_{i-1} \sigma^z_{i+1}) \right]. \]  
(43)
For the action of \( U \), noting that \( A^\dagger \sigma^x A = -\sigma^y \), \( A^\dagger \sigma^y A = -\sigma^x \), \( A^\dagger \sigma^z A = -\sigma^z \), we have
\[ U^\dagger \tilde{j}_{\text{XXZ}} U = -2\alpha \left[ \alpha (\sigma^x_{i-1} \sigma^y_{i+1} - \sigma^x_{i-1} \sigma^x_{i+1}) + \Delta_i (\sigma^y_{i-1} \sigma^y_{i+1} - \sigma^y_{i-1} \sigma^y_{i+1}) \right. \]
\[ + \left. \Delta_i (\sigma^z_{i-1} \sigma^z_{i+1} - \sigma^z_{i-1} \sigma^z_{i+1}) \right] \]
\[ = 2\alpha \left[ \alpha (\sigma^x_{i-1} \sigma^y_{i+1} - \sigma^x_{i-1} \sigma^y_{i+1}) + \Delta_i (\sigma^y_{i-1} \sigma^y_{i+1} - \sigma^y_{i-1} \sigma^y_{i+1}) \right. \]
\[ + \left. \Delta_i (\sigma^z_{i-1} \sigma^z_{i+1} - \sigma^z_{i-1} \sigma^z_{i+1}) \right] \]
\[ = \tilde{j}_{\text{XXZ}}. \]  
(44)
that is, the one-way street phenomenon holds.

For the spin current
\[ \tilde{j}^M = 2\alpha (\sigma^x_{i} \sigma^y_{i+1} - \sigma^y_{i} \sigma^x_{i+1}). \]  
(45)
it follows
\[ U^\dagger J^{\dagger} U = 2\alpha (\sigma^y_i \sigma^x_{i+1} - \sigma^x_i \sigma^y_{i+1}) \]
\[ = -2\alpha (\sigma^x_i \sigma^y_{i+1} - \sigma^y_i \sigma^x_{i+1}) \]
\[ = -\hat{J}^{\dagger}, \] (46)

that is, the current is inverted without rectification or any other effect.

**Y–YZ polarization.** We now investigate the chain in which the first spin is target on the **Y** direction and the last one target on some direction on **YZ** plane. We also turn to the Heisenberg models.

Precisely, we consider the Lindblad operators
\[ K^L_+ = \sqrt{\gamma(1+f)} \left( \frac{\sigma^x_i + i\sigma^y_i}{2} \right), \]
\[ K^L_- = \sqrt{\gamma(1-f)} \left( \frac{\sigma^y_i - i\sigma^x_i}{2} \right), \]
\[ K^R_+ = \sqrt{\gamma(1-f)} \left( \frac{\cos \theta \sigma^y_N + \sin \theta \sigma^z_N + i\sigma^x_N}{2} \right), \]
\[ K^R_- = \sqrt{\gamma(1+f)} \left( \frac{\cos \theta \sigma^y_N + \sin \theta \sigma^z_N - i\sigma^x_N}{2} \right). \] (47)

To perform the baths inversion, it is enough to find \( A \) such that \( A(\cdot)A^\dagger \) makes the transformations
\[ \sigma^z \longrightarrow (1) \cos \theta \sigma^y + \sin \theta \sigma^z \longrightarrow (2) \sigma^z, \]
\[ \sigma^x \longrightarrow (3) -\sigma^x. \]

We will use the representation of a unitary matrix in **SU(2):**
\[ A = \begin{pmatrix} a_r + ia_i & b_r + ib_i \\ -b_r + ib_i & a_r - ia_i \end{pmatrix}, \] (48)

where \( a_r^2 + a_i^2 + b_r^2 + b_i^2 = 1 \).

We begin studying the condition (3) above:
\[ A\sigma^x A^\dagger = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \] (49)

where we want
\[ = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \] (50)

It leads to
\[ c_{11} = 2(a_r b_r + a_i b_i), \]
\[ c_{12} = a_r^2 + b_r^2 - a_i^2 - b_i^2 + 2(a_r a_i - b_r b_i)i, \]
\[ c_{21} = a_r^2 + b_r^2 - a_i^2 - b_i^2 + 2(b_r a_i - a_r b_i)i, \]
\[ c_{22} = -2(a_r b_r + a_i b_i), \] (51)
and we must have
\[ a_r a_r = -a_i b_r, \]
\[ a_i a_i = b_r b_r, \]
\[ a_r^2 + b_r^2 - a_i^2 - b_i^2 = -1, \] (52)
\[ a_r^2 + b_r^2 + a_i^2 + b_i^2 = 1. \]
We can take \( a_r = b_i = 0 \), and so, we stay with
\[ A = \left( \begin{array}{cc} a_i & b_r \\ -b_r & -ia_i \end{array} \right), \] (53)
where \( a_i^2 + b_r^2 = 1 \).

Let us satisfy condition (1). We have
\[ A \sigma^z A^\dagger = \left( \begin{array}{cc} d_{11} & d_{12} \\ d_{21} & d_{22} \end{array} \right), \] (54)
and we want
\[ = \left( \begin{array}{cc} \sin \theta & -i \cos \theta \\ i \cos \theta & -\sin \theta \end{array} \right). \] (55)
That is
\[ d_{11} = a_i^2 - b_r^2, \]
\[ d_{12} = -2ia_i b_r, \]
\[ d_{21} = 2ia_i b_r, \]
\[ d_{22} = b_r^2 - a_i^2, \] (56)
and so
\[ a_i^2 - b_r^2 = \sin \theta, \]
\[ a_i b_r = \frac{\cos \theta}{2}. \] (57)

Consequently,
\[ a_i b_r = \frac{\cos \theta}{2} = \frac{\left| \cos \theta \right|}{2}, \] (58)
for \( 0 \leq \theta \leq \pi/2 \) which are our angles of interest.

We take
\[ a_i = \frac{\sqrt{1 + \sin \theta}}{\sqrt{2}}, \]
\[ b_r = \frac{\sqrt{1 - \sin \theta}}{\sqrt{2}}. \] (59)

Moreover
\[ a_i^2 + b_r^2 = \frac{1 + \sin \theta + 1 - \sin \theta}{2} = 1, \]
\[ a_i^2 - b_r^2 = \frac{1 + \sin \theta - 1 + \sin \theta}{2} = \sin \theta. \] (60)
Then, the final form of $A$ is

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} i \sqrt{1 + \sin \theta} & \sqrt{1 - \sin \theta} \\ -i \sqrt{1 + \sin \theta} & -\sqrt{1 - \sin \theta} \end{pmatrix}. \tag{61}$$

We know that

$$A \sigma^i A^\dagger = \cos \theta \sigma^i + \sin \theta \sigma^i.$$

A simple computation shows that

$$A \sigma^i A^\dagger = \cos \theta \sigma^i - \sin \theta \sigma^i.$$

Noting that $A^\dagger = -A$, it follows

$$A \sigma^i A^\dagger = (-A)^i \sigma^i (-A) = A^j \sigma^j A,$$

and the same for $\sigma^x$ and $\sigma^z$.

For the graded Heisenberg Hamiltonian we have

$$UHU^\dagger = \sum_{i=1}^{N-1} \alpha_i \left\{ -\sigma^i_{i+1}(\sigma^x_{i+1}) + (\cos \theta \sigma^i_{i+1} - \sin \theta \sigma^i_{i+1}) \right\} \left\{ \cos \theta \sigma^i_{i+1} - \sin \theta \sigma^i_{i+1} - \sigma^y_{i+1} \right\}$$

$$= \sum_{i=1}^{N-1} \alpha_i \left\{ \sigma^i_{i+1} + \cos^2 \theta \sigma^i_{i+1} - \cos \theta \sin \theta^i_{i+1} \right\}$$

$$- \sin \theta \cos \theta^i_{i+1} + \sin^2 \theta \sigma^i_{i+1} + \cos^2 \theta \sigma^i_{i+1}$$

$$+ \cos \theta \sin \theta \sigma^i_{i+1} + \sin \theta \cos \theta \sigma^i_{i+1}$$

$$= \sum_{i=1}^{N-1} \alpha_i \left\{ \sigma^i_{i+1} + \sigma^i_{i+1} \sigma^y_{i+1} \right\} = H. \tag{62}$$

For the energy current

$$UJ^xU^\dagger = U \left\{ \left( \sigma^x_{i-1} \sigma^i_{i+1} + \sigma^y_{i-1} \sigma^z_{i+1} + \sigma^z_{i-1} \sigma^y_{i+1} \right) - \sigma^x_{i-1} \sigma^i_{i+1} - \sigma^y_{i-1} \sigma^z_{i+1} - \sigma^z_{i-1} \sigma^y_{i+1} \right\} U^\dagger, \tag{63}$$

we have

$$UJ^xU^\dagger = 2\alpha_{i-1} \left\{ \left( \sigma^x_{i-1} \sigma^i_{i+1} + \sigma^y_{i-1} \sigma^z_{i+1} + \sigma^z_{i-1} \sigma^y_{i+1} \right) \right\} \left( \cos \theta \sigma^i_{i+1} + \sin \theta \sigma^i_{i+1} \right)$$

$$+ \left\{ \cos \theta \sigma^i_{i+1} - \sin \theta \sigma^i_{i+1} \right\} \left( \cos \theta \sigma^i_{i+1} + \sin \theta \sigma^i_{i+1} \right)$$

$$+ \left\{ \cos \theta \sigma^i_{i+1} + \sin \theta \sigma^i_{i+1} \right\} \left( \cos \theta \sigma^i_{i+1} - \sin \theta \sigma^i_{i+1} \right)$$

$$- \left\{ \cos \theta \sigma^i_{i+1} + \sin \theta \sigma^i_{i+1} \right\} \left( \cos \theta \sigma^i_{i+1} - \sin \theta \sigma^i_{i+1} \right)$$

$$- \left\{ \cos \theta \sigma^i_{i+1} - \sin \theta \sigma^i_{i+1} \right\} \left( \cos \theta \sigma^i_{i+1} - \sin \theta \sigma^i_{i+1} \right)$$

$$- \left\{ \cos \theta \sigma^i_{i+1} + \sin \theta \sigma^i_{i+1} \right\} \left( \cos \theta \sigma^i_{i+1} + \sin \theta \sigma^i_{i+1} \right), \tag{64}$$

$$12$$
and so

$$UJ^EU^\dagger = 2\alpha_{i-1}\alpha_i \left\{ -\cos^2 \theta \sigma_{i-1}^x \sigma_{i+1}^y - \cos \theta \sin \theta \sigma_{i-1}^z \sigma_{i+1}^z + \sin \theta \cos \theta \sigma_{i-1}^x \sigma_{i+1}^z + \sin^2 \theta \sigma_{i-1}^x \sigma_{i+1}^z - \cos^2 \theta \sigma_{i-1}^x \sigma_{i+1}^z \right\}$$

that shows the occurrence of the one-way street phenomenon.

**Y–Z orthogonal polarizations.** Let us consider the set of Lindblad operators at one boundary, say the left one, given by

- \( L_1 = \alpha(\sigma_i^+ + i\sigma_i^-) \)
- \( L_2 = \beta(\sigma_i^+ - i\sigma_i^-) \)
- \( V_1 = p(\sigma_i^+ + i\sigma_i^-) \)
- \( V_2 = q(\sigma_i^+ - i\sigma_i^-) \)

and for the right boundary

- \( L_3 = \nu(\sigma_N^+ + i\sigma_N^-) \)
- \( L_4 = \lambda(\sigma_N^+ - i\sigma_N^-) \)
- \( V_3 = \rho(\sigma_N^+ + i\sigma_N^-) \)
- \( V_4 = \sigma(\sigma_N^+ - i\sigma_N^-) \)

To invert the baths it is enough to find an operator \( A \) such that

(I) \( A \sigma^x A^\dagger = -\sigma^x \Leftrightarrow -\sigma^x = A^\dagger \sigma^x A \)

(II) \( A \sigma^y A^\dagger = -\sigma^y \Leftrightarrow -\sigma^y = A^\dagger \sigma^y A \)

(III) \( A \sigma^z A^\dagger = -\sigma^z \Leftrightarrow -\sigma^z = A^\dagger \sigma^z A \)

Indeed, in such case, we will have the transformations

- \( L_1 \to -iV_4 \)
- \( L_2 \to iV_3 \)
- \( V_1 \to -iV_4 \)
- \( V_2 \to iV_3 \)

and also

- \( L_3 \to -iV_2 \)
- \( L_4 \to iV_1 \)
- \( V_3 \to -iV_2 \)
- \( V_4 \to iV_1 \)
Consequently, the dissipator transforms as
\[ U \mathcal{L}(\rho) U^\dagger = \mathcal{L}(\rho, \text{invertedbaths}). \] (71)

As we show below, it is enough to use a representation for \( A \) in \( SU(2) \)
\[ A = \begin{pmatrix} a_r + ia_i & b_r + ib_i \\ -b_r + ib_i & a_r - ia_i \end{pmatrix}, \]
where \( a_r, a_i, b_r, b_i \in \mathbb{R} \) and \( a_r^2 + a_i^2 + b_r^2 + b_i^2 = 1 \).
Computing (I):
\[ A \sigma^x A^\dagger = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}, \] (72)
where
\[ z_{11} = b_r a_r - ib_r a_i + ib_r a_i + b_r a_i + b_r a_r - ia_r b_i + ib_r a_i + a_r b_i, \]
\[ z_{12} = -b_r^2 + ib_r b_i - ib_r b_i - b_r^2 + a_r^2 + a_r a_r - a_r^2, \]
\[ z_{21} = a_r^2 - ia_r a_i - ia_r a_r - a_r^2 - b_r^2 + ib_r b_i + ib_r b_i + b_r^2, \]
\[ z_{22} = -a_r b_r - ib_r a_r + ib_r a_r - a_r b_r - b_r a_r - ia_r b_i + ib_r a_r - a_r b_i. \]

We want
\[ A \sigma^x A^\dagger = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \]
And so,
\[ a_i b_i + a_r b_r = 0, \]
\[ a_r^2 - a_i^2 + b_r^2 - b_i^2 = -1, \]
\[ b_r b_i = a_r a_r. \]

We still want from (II)
\[ A \sigma^y A^\dagger = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \] (74)
that leads us to
\[ a_i b_r - a_r b_i = -\frac{1}{2}, \]
\[ a_r a_r = -b_r b_r, \]
\[ a_r^2 - a_i^2 + b_r^2 - b_i^2 = 0. \]

From the equations above we have \( a_r a_r = 0 = b_r b_r \). We choose \( a_r = b_i = 0 \) and, consequently,
\[ a_i^2 = b_r^2 = -(1 + b_r^2) \Rightarrow -2a_i^2 = -1 \Rightarrow a_i = \frac{1}{\sqrt{2}}. \]
Hence,
\[ b_r = -\frac{1}{\sqrt{2}}. \] (75)
And we obtain for $A$ the final form

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -1 \\ 1 & -i \end{pmatrix} = \frac{i}{\sqrt{2}} (-\sigma^z + \sigma^y).$$

(76)

A short computation shows us that (III) follows:

$$A\sigma^y A^\dagger = -\sigma^y,$$

as expected.

It is easy to see that the transformations keep the Hamiltonian of the graded Heisenberg model unchanged, i.e.,

$$UHU^\dagger = \frac{1}{2} \sum_{i=1}^{N-1} \alpha_i (\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+ + \sigma_i^y \sigma_{i+1}^y) = H.$$  

(77)

For the energy current, we have

$$U^\dagger \hat{J}^E U = 2 \alpha_{i-1} \alpha_i (\sigma_{i-1}^- \sigma_i^+ \sigma_{i+1}^- - \sigma_{i-1}^- \sigma_i^+ \sigma_{i+1}^- - \sigma_{i-1}^- \sigma_i^+ \sigma_{i+1}^- + \sigma_{i-1}^- \sigma_i^+ \sigma_{i+1}^-)$$

$$= \hat{J}^E,$$

(78)

that is, the one-way street phenomenon holds.

**Z–XZ polarization.** We now consider the case involving a $\sigma^z$ target polarization at one side, and on a rotated axis on plane $XZ$ for the other side. That is, we consider the Lindblad operators as

$$K^L_+ = \sqrt{\gamma (1 + f)} \left( \frac{\sigma_i^+ + i \sigma_i^-}{2} \right), \quad K^L_- = \sqrt{\gamma (1 - f)} \left( \frac{\sigma_i^- - i \sigma_i^+}{2} \right),$$

$$K^R_+ = \sqrt{\gamma (1 - f)} \left( \frac{\cos \theta \sigma_i^x + \sin \theta \sigma_i^y + i \sigma_i^z}{2} \right),$$

$$K^R_- = \sqrt{\gamma (1 + f)} \left( \frac{\cos \theta \sigma_i^x + \sin \theta \sigma_i^y - i \sigma_i^z}{2} \right).$$

Again, we search for one operator related to baths inversion. We use the general representation of $SU(2)$. After manipulations similar to those previously described, we find

$$A = \frac{i}{\sqrt{2}} \begin{pmatrix} \sqrt{1 - \cos \theta} & \sqrt{1 + \cos \theta} \\ \sqrt{1 + \cos \theta} & -\sqrt{1 - \cos \theta} \end{pmatrix}.$$  

(79)

Then we study the effect of $U = A \otimes A \otimes \cdots \otimes A$ on the Heisenberg Hamiltonian and on the energy current. We have

$$UHU^\dagger = \cdots$$

$$= \frac{1}{2} \sum_{i=1}^{N-1} \alpha_i \left[ (\cos \theta \sigma_i^x + \sin \theta \sigma_i^y) (\cos \theta \sigma_{i+1}^x + \sin \theta \sigma_{i+1}^y) 

+ (-\sigma_i^y)(-\sigma_{i+1}^y) + (\sin \theta \sigma_i^x - \cos \theta \sigma_i^y) (\sin \theta \sigma_{i+1}^x - \cos \theta \sigma_{i+1}^y) \right]$$

$$= \frac{1}{2} \sum_{i=1}^{N-1} \alpha_i \left[ \cos^2 \theta \sigma_i^x \sigma_{i+1}^x + \cos \theta \sin \theta \sigma_i^x \sigma_{i+1}^x + \sin \theta \cos \theta \sigma_i^y \sigma_{i+1}^y \right]$$

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\[ + \sin^2 \theta \sigma^z_{i+1} + \sigma^y_{i+1} + \sin^2 \theta \sigma^z_{i+1} - \sin \cos \theta \sigma^z_{i+1} \\
- \cos \theta \sin \theta \sigma^z_{i+1} + \cos \theta \sigma^z_{i+1} \]
\[ = \frac{1}{2} \sum_{j=1}^{N-1} \alpha_i (\sigma^x_{i+1} + \sigma^y_{i+1} + \sigma^z_{i+1}) \]
\[ = H, \] (80)

as expected.

For the energy current of the Heisenberg model we have
\[ U^j \mathcal{F} U = 2\alpha_{i-1} \alpha_i \left[ (-\sigma^y_i \sigma^x_{i+1} + \cos \theta \sigma^y_i) (\cos \theta \sigma^z_{i+1} + \sin \theta \sigma^z_{i+1}) \right. \]
\[ - (\cos \theta \sigma^z_{i+1} + \sin \theta \sigma^z_{i-1}) (\sin \theta \sigma^z_{i+1} - \cos \theta \sigma^z_{i-1}) \]
\[ + (\sin \theta \sigma^z_{i+1} - \cos \theta \sigma^z_{i-1}) (\cos \sigma^z_{i+1} + \sin \sigma^z_{i-1}) \]
\[ - (-\sigma^y_i) (\cos \theta \sigma^z_{i+1} + \sin \theta \sigma^z_{i-1}) (\sin \theta \sigma^z_{i+1} - \cos \theta \sigma^z_{i-1}) \]
\[ + (\cos \theta \sigma^z_{i+1} + \sin \theta \sigma^z_{i-1}) (-\sigma^y_i) (\sin \theta \sigma^z_{i+1} - \cos \theta \sigma^z_{i-1}) \]
\[ - (\sin \theta \sigma^z_{i+1} - \cos \theta \sigma^z_{i-1}) (-\sigma^y_i) (\cos \theta \sigma^z_{i+1} + \sin \theta \sigma^z_{i-1}) \]
\[ = 2\alpha_{i-1} \alpha_i \left[ (-\sin \theta \sigma^y_i \sigma^z_{i+1} + \cos \theta \sigma^z_{i+1} + \sin \theta \sigma^z_{i-1}) \right. \]
\[ - (\cos \theta \sigma^z_{i+1} + \sin \theta \sigma^z_{i-1}) (\sin \theta \sigma^z_{i+1} - \cos \theta \sigma^z_{i-1}) \]
\[ + (\sin \theta \sigma^z_{i+1} - \cos \theta \sigma^z_{i-1}) (\cos \theta \sigma^z_{i+1} + \sin \theta \sigma^z_{i-1}) \]
\[ + (\cos \theta \sigma^z_{i+1} \sigma^z_{i+1} + \sin \theta \sigma^z_{i+1} \sigma^z_{i+1}) (\sin \theta \sigma^z_{i+1} - \cos \theta \sigma^z_{i+1}) \]
\[ - (\cos \theta \sigma^z_{i+1} + \sin \theta \sigma^z_{i-1}) (-\sin \theta \sigma^z_{i+1} + \cos \theta \sigma^z_{i-1}) \]
\[ + (\sin \theta \sigma^z_{i+1} - \cos \theta \sigma^z_{i-1}) (-\sin \theta \sigma^z_{i+1} + \cos \theta \sigma^z_{i-1}) \]
\[ = 2\alpha_{i-1} \alpha_i \left[ (-\sin \theta \cos \sigma^y_i \sigma^z_{i+1} \sigma^z_{i+1} - \sin \theta \sigma^y_i \sigma^z_{i+1} \sigma^z_{i+1} \right. \]
\[ + \cos^2 \theta \sigma^y_i \sigma^z_{i+1} + \sin \theta \cos \sigma^y_i \sigma^z_{i+1} + \cos \theta \sin \theta \sigma^y_i \sigma^z_{i+1} \]
\[ - \sin \theta \cos \sigma^y_i \sigma^z_{i+1} \sigma^z_{i+1} - \sin \theta \cos \sigma^y_i \sigma^z_{i+1} \sigma^z_{i+1} \]
\[ + \cos \theta \sin \theta \sigma^z_{i+1} \sigma^z_{i+1} + \cos \theta \sin \theta \sigma^z_{i+1} \sigma^z_{i+1} \]
\[ + \sin \theta \cos \theta \sigma^z_{i+1} \sigma^z_{i+1} + \sin \theta \cos \theta \sigma^z_{i+1} \sigma^z_{i+1} \]
\[ - \cos^2 \theta \sigma^z_{i+1} \sigma^z_{i+1} - \cos \theta \sin \theta \sigma^z_{i+1} \sigma^z_{i+1} \right], \] (81)

and using \( \cos^2 x + \sin^2 x = 1 \), we obtain
\[ U^j \mathcal{F} U = 2\alpha_{i-1} \alpha_i \left[ -\sigma^y_{i-1} \sigma^x_{i+1} + \sigma^y_{i-1} \sigma^x_{i+1} - \sigma^y_{i-1} \sigma^x_{i+1} \right. \]
\[ + \sigma^y_{i-1} \sigma^x_{i+1} + \sigma^y_{i-1} \sigma^x_{i+1} - \sigma^y_{i-1} \sigma^x_{i+1} \]
\[ = 2 \alpha_{i-1} \alpha_i \left[ \sigma_{i-1}^y \sigma_{i+1}^x - \sigma_{i+1}^x \sigma_{i-1}^y + \sigma_{i-1}^x \sigma_{i+1}^y + \sigma_{i+1}^y \sigma_{i-1}^x \right] \]
\[ - \sigma_{i-1}^y \sigma_{i+1}^x + \sigma_{i+1}^x \sigma_{i-1}^y - \sigma_{i-1}^x \sigma_{i+1}^y \]
\[ = \hat{J}_E, \quad (82) \]

that is, one-way street phenomenon.

**X–Z orthogonal polarization.** Now, for one boundary we take the Lindblad operators

\[ L_1 = \alpha (\sigma_1^x + i \sigma_1^y), \quad L_2 = \beta (\sigma_1^x - i \sigma_1^y), \]
\[ V_1 = p (\sigma_1^y + i \sigma_1^z), \quad V_2 = q (\sigma_1^y - i \sigma_1^z), \]
\[ W_1 = u (\sigma_1^z + i \sigma_1^x), \quad W_2 = v (\sigma_1^z - i \sigma_1^x), \]

and, for the opposite boundary,

\[ L_3 = q (\sigma_N^x + i \sigma_N^y), \quad L_4 = p (\sigma_N^x - i \sigma_N^y), \]
\[ V_3 = \beta (\sigma_N^y + i \sigma_N^z), \quad V_4 = \alpha (\sigma_N^y - i \sigma_N^z), \]
\[ W_3 = v (\sigma_N^z + i \sigma_N^x), \quad W_4 = u (\sigma_N^z - i \sigma_N^x). \]

To implement the bath inversion, it is enough to find a unitary operator \( A \) such that

\[ (I) A \sigma^y A^\dagger = - \sigma^y \Leftrightarrow - \sigma^y = A^\dagger \sigma^y A, \]
\[ (II) A \sigma^z A^\dagger = - \sigma^z \Leftrightarrow - \sigma^z = A^\dagger \sigma^z A, \]
\[ (III) A \sigma^x A^\dagger = - \sigma^x \Leftrightarrow - \sigma^x = A^\dagger \sigma^x A, \]

since its action will perform the transformations

\[ L_1 \rightarrow -i V_4, \quad L_2 \rightarrow i V_3, \]
\[ V_1 \rightarrow -i L_4, \quad V_2 \rightarrow i L_3, \]
\[ W_1 \rightarrow -i W_4, \quad W_2 \rightarrow i W_3, \quad (85) \]

and also

\[ L_3 \rightarrow -i V_2, \quad L_4 \rightarrow i V_1, \]
\[ V_3 \rightarrow -i L_3, \quad V_4 \rightarrow i L_4, \]
\[ W_3 \rightarrow -i W_2, \quad W_2 \rightarrow i W_1. \]

After some algebraic manipulations, we find

\[ A = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ i & i \end{pmatrix} = \frac{i}{\sqrt{2}} (\sigma^x - \sigma^z). \]

And everything follows: the Heisenberg Hamiltonian is preserved under the transformations, as well as the energy current, i.e., the one-way street phenomenon holds.

### 4. Steady state uniqueness

Now we prove the uniqueness of the steady state for all the cases previously analyzed here.
As well known, the steady state is unique if the set of Lindblad operators together with the Hamiltonian are enough to generate the whole Pauli algebra \[21\] involving all sites 1, 2, \ldots, N. Here, in our prove, we follow Prosen \[21\].

In any of the previous analyzed cases, the Lindblad operators are given in terms of \(\sigma^+\) and \(\sigma^-\), or \(\Gamma^+ = \sigma^z + \frac{\sigma^x}{2}\) and \(\Gamma^- = \sigma^z - \frac{\sigma^x}{2}\) in one of the sides of the system (1 or \(N\)), or in terms of \(\Pi^+ = \sigma^y + i\sigma^z\) and \(\Pi^- = \sigma^y - i\sigma^z\). But this last case is reduced to the first one by the relations

\[
[\Pi^+, \Pi^-] = \sigma^x,
\]

\[
\Pi^+ + \Pi^- = \sigma^y,
\]

and the other reduces to the first one due

\[
-i [\Gamma^+, \Gamma^-] = \sigma^x,
\]

\[
\Gamma^+ + \Gamma^- = \sigma^y,
\]

(88)

Thus, let us show that having \(\sigma^+\) and \(\sigma^-\) in one of the sides is enough to generate the whole algebra (of course, with the Hamiltonian). To prove it, we will show that the following relations are valid

\[
\sigma^+_2 = \frac{1}{4} \sigma^+_1 [\sigma^+_1, [H, \sigma^+_1]],
\]

\[
\sigma^+_j = -\sigma^+_j - \frac{1}{2} \sigma^+_{j-1} [\sigma^+_{j-1}, \sigma^+_j H \sigma^+_{j-1}],
\]

for \(j = 3, 4, \ldots, n\) and the conjugate

\[
\sigma^-_2 = \frac{1}{4} [\sigma^-_1, [H, \sigma^-_1]] \sigma^-_1,
\]

\[
\sigma^-_j = -\sigma^-_j + \frac{1}{2} [\sigma^-_{j-1}, \sigma^-_{j-1} H \sigma^-_{j-1}] \sigma^-_{j-1},
\]

(90)

we recall that \([\sigma^+, \sigma^-] = 2\sigma^z\). With the previous relations, we get the set \(\{\sigma^+_j, \sigma^-_j; j = 1, \ldots, n\}\) that generates the whole Pauli algebra.

First, we rewrite the XXZ Hamiltonian as

\[
H = \sum_{j=1}^{n-1} (2\sigma^+_j \sigma^+_{j+1} + 2\sigma^-_j \sigma^-_{j+1} + \Delta \sigma^+_j \sigma^-_{j+1}).
\]

(92)

Talking about algebraic properties, the constants \(\alpha\) and \(\Delta\) are not important (as well as the difference between \(\Delta_j\) and \(\Delta_{j+1}\)). And so, our computation follows also for the Heisenberg model.

We have

\[
[H, \sigma_i] = 2 \left[ \sigma_i^+ \sigma^-_2, \sigma^-_1 \right] + 2 \left[ \sigma^-_i \sigma^+_2, \sigma^-_1 \right]
\]

\[
= 2 \left[ \sigma^-_i, \sigma^-_1 \right] \sigma^-_2 + 2 \left[ \sigma_i, \sigma_i \right] \sigma^+_2
\]

\[
= -4 \sigma^-_i \sigma^-_2 + 4 \sigma^+_i \sigma^+_2,
\]

(93)
and so

\[
[\sigma^+_{j-1}, [H, \sigma^+_{j}]] = -4 \left[ \sigma^+_{j-1}, \sigma^+_1 \sigma^+_2 \right] + 4 \left[ \sigma^+_{j-1}, \sigma^+_1 \sigma^+_2 \right] \\
= 4 \left[ \sigma^+_1, \sigma^+_2 \right] \sigma^+_2 \\
= 4 \sigma^+_1 \sigma^+_2.
\]

Consequently

\[
\frac{1}{4} \sigma^+_1 [\sigma^+_1, [H, \sigma^+_1]] = \frac{1}{4} \sigma^+_1 4 \sigma^+_1 \sigma^+_2 = \sigma^+_2.
\]

as we wanted.

Carrying out the computation

\[
\sigma^+_{j-1} H \sigma^+_{j-1} = \sigma^+_{j-1} \left( \sum_{k=1}^{N-1} 2 \sigma^+_k \sigma^-_{k+1} + 2 \sigma^-_k \sigma^+_{k+1} + \Delta \sigma^+_k \sigma^-_{k+1} \right) \sigma^+_{j-1}
\]

\[
= \left( 0 + 2 \frac{I + \sigma^-_{j-1}}{2} \sigma^+_{j-1} + \Delta (-\sigma^-_{j-1}) \sigma^+_{j-1} \right) k=1 \sigma^+_{j-1}
\]

\[
+ \sigma^+_{j-1} \left( 2 \sigma^+_{j-2} \frac{I - \sigma^-_{j-1}}{2} + 0 + \Delta \sigma^+_2 \sigma^-_{j-1} \right) \sigma^+_{j-1}
\]

\[
= (I + \sigma^-_{j-1}) \sigma^+_{j-1} \sigma^+_{j-1} - \Delta \sigma^+_1 \sigma^-_{j-1} + \sigma^+_{j-1} \sigma^-_{j-2} (I - \sigma^-_{j-1}) + \Delta \sigma^+_1 \sigma^-_{j-2} \sigma^+_{j-1}
\]

\[
= (I + \sigma^-_{j-1}) \sigma^+_{j-1} \sigma^+_{j-1} + \sigma^-_{j-1} \sigma^+_{j-1} (I - \sigma^-_{j-1})
\]

\[
= \sigma^+_{j-1} \sigma^+_{j-1} + \sigma^-_{j-1} \sigma^+_{j-1} - \sigma^+_{j-1} \sigma^-_{j-1} + \sigma^+_{j-1} \sigma^-_{j-1}
\]

\[
= 2 \sigma^+_{j-2} \sigma^+_{j-1} + 2 \sigma^-_{j-1} \sigma^+_{j-1}.
\]

hence,

\[
\left[ \sigma_{j-1}, \sigma^+_{j-1} H \sigma^+_{j-1} \right] = 2 \left[ \sigma_{j-1}, \sigma^+_{j-2} \sigma^-_{j-1} \right] + 2 \left[ \sigma_{j-1}, \sigma^+_{j-1} \sigma^-_{j-1} \right]
\]

\[
= 2 \sigma^+_{j-2} (-\sigma^-_{j-1}) + 2 (-\sigma^+_{j-1} \sigma^+_{j-1})
\]

\[
= -2 \sigma^+_{j-2} \sigma^-_{j-1} + \sigma^+_{j-1} \sigma^-_{j-1}.
\]

and so,

\[
\frac{1}{2} \sigma^-_{j-1} [\sigma^+_{j-1}, \sigma^+_{j-1} H \sigma^+_{j-1}] = -\frac{1}{2} \sigma^+_{j-1} (-2)(\sigma^+_{j-2} \sigma^-_{j-1} + \sigma^+_{j-1} \sigma^-_{j-1})
\]

\[
= \sigma^+_{j-2} + \sigma^+_{j-1}.
\]

For the adjoint, we have

\[
\sigma^-_2 = (\sigma^+_2) = (\frac{1}{4} \sigma^+_1 [\sigma^+_1, [H, \sigma^+_1]])
\]

\[
= \frac{1}{4} [\sigma^+_1, [H, \sigma^+_1]] \sigma^+_1 = \frac{1}{4} (-)[\sigma^+_1, [H, \sigma^+_1]] \sigma^+_1
\]

\[
= \frac{1}{4} (-)[\sigma^+_1, (-)[H, \sigma^+_1]] \sigma^+_1 = \frac{1}{4} [\sigma^+_1, [H, \sigma^+_1]] \sigma^+_1.
\]
where we used the identity
\[
[A, B]^\dagger = (AB - BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = [B^\dagger, A^\dagger] = -[A^\dagger, B^\dagger].
\]
We also have
\[
\sigma^-_j = (\sigma^+_j)^{\dagger} = (-\sigma^+_j)^{\dagger} = -\frac{1}{2}\sigma^+_j\sigma^-_{j-1}\sigma^+_{j-1}H\sigma^-_{j-1}\sigma^+_{j-1})^{\dagger}
= -\sigma^-_{j-2} + \frac{1}{2}[\sigma^+_j, \sigma^-_{j-1}]H\sigma_{j-1}^\dagger\sigma^-_{j-1},
\]
and with these results we can conclude the proof.

5. Final remarks

We believe that our results showing the general occurrence of a nontrivial property of energy transport in quantum spin systems will enhance the interest of researchers in quantum transport. It is worth to emphasize that the one-way street phenomenon shown here is an effect stronger than rectification, even a perfect rectification.

These boundary-driven quantum spin systems are the archetypal models of nonequilibrium statistical physics, and the asymmetric versions proposed here are not only theoretical proposals. Graded materials, for example, i.e., asymmetric systems with structure changing gradually in space, are abundant in nature and can be also built. They are recurrently studied in different areas: material science, optics, etc. An example of graded thermal diode has been already experimentally constructed [22]: a carbon and boron nitride nanotube, externally coated with heavy molecules.

It is important to stress that, in particular, asymmetric versions of XXZ and Heisenberg chains seem to be realizable. In references [23, 24], it is shown the possibility to engineer these quantum spin Hamiltonians with different values for the structural parameters $\alpha$ and $\Delta$.

Finally, still concerning the realizability of such systems, recent experimental works with Rydberg atoms in optical traps [25, 26] appear associated to Heisenberg and XXZ models.

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