On optimal control problems with impulsive commutative dynamics

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Abstract—We consider control systems governed by nonlinear O.D.E.’s that are affine in the time-derivative du/dt of the control u. The latter is allowed to be an integrable, possibly of unbounded variation function, which gives the system an impulsive character. As is well-known, the corresponding Cauchy problem cannot be interpreted in terms of Schwartz distributions, even in the commutative case. A robust notion of solution already proposed in the literature is here adopted and slightly generalized to the case where an ordinary, bounded, control is present in the dynamics as well. For a problem in the Mayer form we then investigate the question whether this notion of solution provides a “proper extension” of the standard problem with absolutely continuous controls u. Furthermore, we show that this impulsive problem is a variational limit of problems corresponding to controls u with bounded variation.

I. INTRODUCTION AND BASIC NOTATION

Consider the control system
\[
\dot{x} = f(x, u, v) + \sum_{\alpha=1}^{m} \tilde{g}_\alpha(x, u)\dot{u}^\alpha,
\]
\[
x(a) = \bar{x}, \quad u(a) = \bar{u},
\]
where \( v : [a, b] \to V \subset \mathbb{R}^l \) is a standard bounded control while \( u : [a, b] \to U \subset \mathbb{R}^m \) is an \( L^1 \)-function, which we refer to as the impulsive control. The presence of the derivative \( \dot{u} \) on the right hand-side raises the issue of the definition of a (possibly discontinuous) solution \( x : [a, b] \to \mathbb{R}^n \). Several applications of this type of system are known, e.g. in mechanics, biology and economics. In optimal control theory impulses arise as soon as the control is unbounded and the cost lacks coercivity properties. It is well-known that theory impulses arise as soon as the control is unbounded.

The latter is allowed to be an integrable, possibly of unbounded variation function, which gives the system an impulsive character. As is well-known, the corresponding Cauchy problem cannot be interpreted in terms of Schwartz distributions, even in the commutative case. A robust notion of solution already proposed in the literature is here adopted and slightly generalized to the case where an ordinary, bounded, control is present in the dynamics as well. For a problem in the Mayer form we then investigate the question whether this notion of solution provides a “proper extension” of the standard problem with absolutely continuous controls u. Furthermore, we show that this impulsive problem is a variational limit of problems corresponding to controls u with bounded variation.

A. Notation and preliminaries

Let \( [a, b] \) be a real interval and \( E \subseteq \mathbb{R}^d \). \( L^1([a, b]; E) \) will denote the space of Lebesgue integrable functions defined on \([a, b]\) and having values in \( E \). We shall use \( L^1([a, b]; E) \) to denote the corresponding set of equivalence classes, and whether a Mayer type optimal control problem on the interval \([a, b]\),
\[
\inf_{(u, v) \in L^1 \times L^1} \psi(x(b), u(b)), \quad \text{(3)}
\]
is in fact a proper extension of the standard problem
\[
\inf_{(u, v) \in AC \times L^1} \psi(x(b), u(b)), \quad \text{(4)}
\]
where \( L^1 \) and \( L^1 \) stand for the “set of Lebesgue integrable functions” (on \([a,b]\)) and its quotient set, respectively; while \( AC \) means “absolutely continuous”.

Loosely speaking, a proper extension of a minimum problem is a new problem in which the old one is embedded, in such a way that the domain of the original problem is (somehow) dense in the new domain and the two problems have the same infimum value.

Our motivation to study proper extensions of (4) comes mainly from the need of giving a physically acceptable meaning to typical investigations for optimal control problems involving p.d. solutions. An instance is represented by necessary conditions for optimality. Indeed, in order that such necessary conditions are of practical use one should rule out the occurrence of Lavrentiev-like phenomena, namely the fact that the infimum value of the extended problem is strictly less than that of the original system. Another instance that makes the search for proper extensions reasonable is dynamic programming and its PDE expression, the Hamilton-Jacobi equation (see Section V). Of course, the case where terminal constraints are imposed on the trajectories is of great interest both for necessary conditions and dynamic programming. This case, which poses non-trivial additional difficulties, is investigated in [1].
There exists replaced by other conditions guaranteeing existence of the applications are covered, and 2) we can ensure uniqueness of a generic hypothesis, we impose it here motivated by the functions defined on (Notice, in particular, that the last components of are zero.)

Remark 1.1: While this commutativity assumption is not a generic hypothesis, we impose it here motivated by the following reasons: 1) the scalar case and some mechanical applications are covered, and 2) we can ensure uniqueness of the solution of the impulsive Cauchy problem.

Besides Hypothesis I we shall assume the following:

Hypothesis 2: (i) is compact.
(ii) For every \( v \in V \), \( f(\cdot, v) : R^{n+m} \to R^{n+m} \) is locally Lipschitz continuous and, for every \( (x, u) \in R^{n+m} \) one has that \( f(x, u, \cdot) : V \to R^{n+m} \) is continuous.
(iii) There exists \( M > 0 \) such that \( |f(x, u, v)| \leq M(1 + |(x, u)|) \), for every \( (x, u) \in R^{n+m} \), uniformly in \( v \in V \).
(iv) The vector fields \( g_\alpha : R^{n+m} \to R^{n+m} \) are of class \( C^1 \) and there exists \( N > 0 \) such that \( |g_\alpha(x, u)| \leq N(1 + |(x, u)|) \), for every \( (x, u) \in R^{n+m} \).

We observe that the sublinearity in (iii) and (iv) can be replaced by other conditions guaranteeing existence of the integral trajectories.

Let \( h \) be a locally Lipschitz vector field on a \( C^1 \)-manifold \( M \), and let \( m \in M \). Whenever the solution to

\[
\frac{d}{dt} x(s) = h(x(s)), \quad h(0) = m
\]

is defined on a interval \( I \) containing 0, we use \( \exp(th)(m) \) to denote the value of this solution at time \( t \), for every \( t \in I \). We remark that the identification \( \exp(h) = \exp(1h) \) is consistent with this definition.

II. THE CAUCHY PROBLEM

Let us introduce a change of coordinates \( \phi : R^{n+m} \to R^{n+m} \) defined by

\[
\phi(x, z) := \text{Pr}\left( \exp\left(-z_\alpha g_\alpha\right)(x, z) \right),
\]

where \( \text{Pr} : R^n \times R^m \to R^n \) denote the canonical projection on the first factor, \( \text{Pr}(x, z) := x \), and let the function \( \varphi : R^{n+m} \to R^n \) be defined by

\[
\varphi(x, z) := \text{Pr}\left( \exp\left(-z_\alpha g_\alpha\right)(x, z) \right).
\]

Let us consider the map \( \phi : R^{n+m} \to R^{n+m} \) defined by

\[
\phi(x, z) := (\varphi(x, z), z).
\]

It is straightforward to prove the following result:

Lemma 2.1: Assume that the vector fields \( g_1, \ldots, g_m \) are of class \( C^r \), with \( r \geq 1 \). Then \( \phi \) is a \( C^r \)-diffeomorphism of \( R^{n+m} \) onto itself and, for every \( (\xi, \zeta) \in R^{n+m} \), one has

\[
\phi^{-1}(\xi, \zeta) = (\varphi(\xi, -\zeta), \zeta).
\]

The \( C^r \)-diffeomorphism \( \phi \) induces a \( C^{r-1} \)-diffeomorphism \( D\phi \) on the tangent bundle. For each \( \alpha = 1, \ldots, m \), let us set

\[
F(\xi, \zeta, v) := D\phi(x, z) f(x, z, v),
\]

\[
G_\alpha(\xi, \zeta) := D\phi(x, z) g_\alpha(x, z).
\]

Lemma 2.2: For every \( i = 1, \ldots, n, \alpha = 1, \ldots, m \),

\[
F = \left( \frac{\partial \phi^i}{\partial x^j} \right) \frac{\partial}{\partial x^j}, \quad G_\alpha = \frac{\partial}{\partial z^\alpha},
\]

where we have set \( \varphi = (\varphi^1, \ldots, \varphi^m) \).

Remark 2.1: The proof of Lemma 2.2 (see [6, Lemma 2.1] for details) is in fact a direct consequence of the Simultaneous Flow-Box Theorem (see e.g. [9]).

Notice that the last components of \( F \) are zero. Therefore, in the new coordinates \( (\xi, \zeta) \), the control system (1) turns into the simpler form

\[
\dot{\xi}(t) = \mathcal{F}(\xi(t), u(t), v(t)).
\]

From now on we assume that the data are such that the Cauchy problem for (11) has a unique solution defined on \([a, b]\), for each \( u \in AC([a, b]; R^m), v \in L^1([a, b]; V) \). For instance, one can verify that this property holds true as soon as condition (iv) in Hypothesis 2 is replaced by (iv') below, which implies that \( D\phi \) is globally bounded.

(iv') \( g_\alpha \) and \( C^1 \) are globally Lipschitz.

Lemma 2.3 below concerns relations between the solutions of the control systems in both systems of coordinates.

Since we are going to exploit the diffeomorphism \( \phi : R^{n+m} \to R^{n+m} \) it is convenient to embed (11)-(12) in the \( n + m \)-dimensional Cauchy problem

\[
\begin{cases}
\dot{\xi} = f(x, z, v) + g_\alpha(x, z) \hat{u}^\alpha, \\
\dot{\zeta} = (x, z) (a) = \left( \begin{array}{c} \bar{x} \\ \bar{z} \end{array} \right).
\end{cases}
\]

Recall that the vector fields \( f \) and \( g_\alpha \), are defined in \( R^{n+m} \times V \) and \( R^{n+m} \), respectively. When \( u \in AC([a, b]; R^m) \), for every \( (\bar{x}, \bar{z}) \in R^{m+n} \) and \( v \in L^1([a, b]; V) \), there exists a unique solution to (12) in the interval \([a, b]\). We let \((x, z)(\bar{x}, \bar{z}, u, v)(\cdot)\) denote this solution.

We shall also consider the Cauchy problem

\[
\begin{cases}
\dot{\xi} = F(\xi, \zeta, v) + G_\alpha \hat{u}^\alpha, \\
\dot{\zeta} = (\xi, \zeta) (a) = \left( \begin{array}{c} \xi \\ \zeta \end{array} \right).
\end{cases}
\]

When \( u \in AC([a, b]; R^m) \), there exists a unique solution to (13) in \([a, b]\).

We let \((\xi, \zeta)(\xi, \zeta, u, v)(\cdot)\) denote this solution.
The essential difference between the two systems relies on the fact that the vector fields $G_\alpha$ are constant. This allows us to give a notion of solution for (13) also for merely integrable controls $u$. Indeed, it is natural to set

$$\zeta(t) := \zeta + u(t) - u(a),$$

for all $t \in [a, b]$ and to let $\zeta$ be the Carathéodory solution of the Cauchy problem $\xi = F(\xi, \zeta, v)$, $\xi(a) = \xi_0$.

When $u \in AC([a, b]; \mathbb{R}^m)$ the relation between the two systems is described in Lemma 2.3 below. Let $(\xi, \zeta) (\xi, \zeta, u, v)$ denote the unique solution of (13) associated with $(\xi, \zeta) \in \mathbb{R}^{n+m}$ and $(u, v) \in AC([a, b]; \mathbb{R}^m) \times L^1([a, b]; V)$.

**Lemma 2.3:** Let us consider $(\bar{x}, \bar{z}) \in \mathbb{R}^{n+m}$ and controls $u \in AC([a, b]; \mathbb{R}^m)$, $v \in L^1([a, b]; V)$. Then,

$$\langle \xi, \zeta \rangle (\bar{x}, \bar{z}, u, v)(t) = \phi \left( (x, z)(\bar{x}, \bar{z}, u, v)(t) \right),$$

for all $t \in [a, b]$, where $(\bar{\xi}, \bar{\zeta}) := \phi(\bar{x}, \bar{z})$.

The latter result is a straightforward consequence of the definition of $F$ and $G_\alpha$.

**B. Poincare defined solutions**

Throughout the paper we shall assume that $U$ is an impulse domain:

**Definition 2.1:** Let $U \subseteq \mathbb{R}^m$. $U$ is called an impulse domain if, for every bounded interval $I \subseteq \mathbb{R}$, for each function $u \in L^1(I; U)$ and for every $t \in I$, there exists a sequence $\{u_k\} \subset AC(I; U)$ such that $\|u_k - u\|_1 \to 0$ and $u_k(t) \to u(t)$, when $n \to \infty$.

Examples of impulse domains are:

- $U = \Omega$, with $\Omega$ a bounded, open, connected subset with Lipschitz boundary;
- an embedded differentiable submanifold of $\mathbb{R}^m$;
- a convex subset $U \subseteq \mathbb{R}^m$.

**Definition 2.2:** Consider an initial data $\bar{x} \in \mathbb{R}^n$ and let $(u, v) \in L^1([a, b]; U) \times L^1([a, b]; V)$. We say that a map $x : [a, b] \to \mathbb{R}^n$ is a pointwise defined solution (shortly p.d. solution) of the Cauchy problem (1)-(2) if, for every $t \in [a, b]$, the following conditions are met:

(i) there exists a sequence $\{u_k\} \subset AC([a, b]; U)$ such that $u_k \to u$ in $L^1([a, b]; U)$, $u_k(a) \to u(a)$, $u_k(t) \to u(t)$, when $k \to \infty$, and;

(ii) for each $k \in \mathbb{N}$, there exists a Carathéodory solution $x_k : [a, b] \to \mathbb{R}^n$ of (1)-(2) corresponding to the control $(u_k, v)$, and the initial condition $(\bar{x}, \bar{u} := u(a))$;

(iii) the sequence $\{x_k\}$ has uniformly bounded values and converges to $x$ in $L^1([a, b]; \mathbb{R}^n)$ and, moreover, $\lim_{k \to \infty} x_k(t) = x(t)$.

**Remark 2.2:** When $u$ is absolutely continuous, the notion of e.d. solution is equivalent to the standard concept of Carathéodory solution. Moreover, in [2], we show that the notion of p.d. solution is quite general and when controls $u \in BV$ it coincides with the most known concepts of solution, in the generic case when the Lie brackets do not vanish.

**Theorem 2.4 (Existence, uniqueness, representation):**

For every $\bar{x} \in \mathbb{R}^n$, and every control pair $(u, v) \in L^1([a, b]; U) \times L^1([a, b]; V)$, there exists a unique p.d. solution of the Cauchy problem (1)-(2) defined on $[a, b]$, where we have set $\bar{u} := u(a)$. We shall use $x(\bar{x}, u, v)(\cdot)$ to denote this solution. Moreover, setting $\xi := \varphi(\bar{x}, u(a))$, one has

$$x(\bar{x}, u, v)(t) = \varphi(\xi(t), -u(t)), \quad (15)$$

for all $t \in [a, b]$, where $\xi(\cdot) := \xi(\xi, u, v)(\cdot)$ is the Carathéodory solution of the Cauchy problem

$$\dot{\xi} = F(\xi, u, v), \quad \xi(a) = \xi_0. \quad (16)$$

To prove this theorem, which extends an analogous result in [6] where $f$ did not depend on the standard control $v$, we shall make use of the following result.

**Lemma 2.5 (see [3]):** The following assertions hold true:

(i) For $r > 0$ and $K \subseteq U$ compact, there exists a compact subset $K' \subseteq \mathbb{R}^n$, such that the trajectories $x(\bar{x}, u, v)(\cdot)$ have values in $K'$, whenever we consider $\bar{x} \in B_r(0)$, $u \in AC([a, b]; K)$ and $v \in L^1([a, b]; V)$.

(ii) For each $r$ and $K$ as in (ii), there exists a constant $M > 0$ such that, for every $t \in [a, b]$, for all $x_1, x_2 \in B_r(0)$, for all $u_1, u_2 \in AC([a, b]; K)$ and for every $v \in L^1([a, b]; V)$, one has

$$|x_1(t) - x_2(t)| + |x_1 - x_2|_1 \leq M \left[ |\bar{x}_1 - \bar{x}_2| + |u_1(a) - u_2(a)| + |u_1(t) - u_2(t)| + |u_1 - u_2|_1 \right],$$

where $x_1 := x(\bar{x}_1, u_1, v)$, $x_2 := x(\bar{x}_2, u_2, v)$.

Proof: (of Theorem 2.4) Set $\bar{u} := (u(a), (\bar{\xi}, \bar{\zeta}) := \phi(\bar{x}, \bar{u})$, $\xi(\cdot) := \bar{\xi} + u(\cdot) - \bar{u}$, and let $\xi$ be the solution of the differential equation (11) with initial condition $\xi(a) = \xi_0$.

Observe that $\zeta := \bar{u}$ and hence, $\zeta \equiv u$.

Define $(x, z) := \phi^{-1}(\xi, \zeta)$. Let us show that $(x, z)$ is a p.d. solution of (12). Choose $t \in [a, b]$, and a sequence of absolutely continuous controls $u_k : [a, b] \to U$ converging to $u$ in the $L^1$ topology and verifying $u_k(a) \to \bar{u}$, $u_k(t) \to u(t)$ when $k \to \infty$. Since $u$ is bounded it is not restrictive to assume that the functions $\{u_k\}$ have equibounded values. Let $(\xi_k, \zeta_k)$ be the corresponding solutions to (13) and set

$$x_k(z_k) := \phi^{-1}(\xi_k, \zeta_k). \quad (17)$$

Note, in particular, that the paths $(\xi_k, \zeta_k)$ and $(x_k, z_k)$ are equibounded. Then

$$\| (x, z) - (x_k, z_k) \|_1 = \| \phi^{-1}(\xi, \zeta) - \phi^{-1}(\xi_k, \zeta_k) \|_1 \to k \to \infty 0,$$

as the map $\phi^{-1}$ is Lipschitz continuous on compact sets. Moreover, since $\zeta_k(t) = \bar{\xi} + u_k(t) - u(a)$, one has $\zeta_k(t) \to \xi(t)$. Therefore, in view of (17) and since $\xi_k \to \xi$ uniformly, $(x_k(t), z_k(t)) \to (x(t), z(t))$. This concludes the part concerning existence and representation of a solution.

In order to prove uniqueness, let $x^1(\cdot)$ and $x^2(\cdot)$ be solutions of (1)-(2), both associated with the same data $\bar{x} \in \mathbb{R}^n$, $(u, v) \in L^1 \times L^1$ and where $\bar{u} := u(a)$. Assume by contradiction that there exists $t \in [a, b]$ such that $x^1(t) = x^2(t)$. According to the definition of p.d. solution there exist
Therefore, \( x \) where solution which is a contradiction. The proof is concluded.

\[ \text{right limits of } t \]

\[ t \]

\[ (i) \text{ For each } u \in AC([0,1]; \mathbb{R}) \text{, then, for any } [a, b] \subseteq [1/2, 1], \text{ the associated Carathéodory solution of } (19) \text{ verifies } \]

\[ x(t) = x(a)e^{u(t)} - u(a). \]  

Consider now the \( \mathcal{L}^1 \)-control

\[ u(t) := \begin{cases} 
1, & \text{for } t \in [0, 1/2], \\
0, & \text{for } t \in [1/2, 1].
\end{cases} \]

Observe that, if \( u \in AC([0,1]; \mathbb{R}) \), then, for any \([a, b] \subseteq [1/2, 1] \), the associated Carathéodory solution of (19) verifies

\[ x(t) = x(a)e^{u(t)} - u(a). \]  

(iii) For each \( r \) and \( K \) as in (ii), there exists a constant \( M > 0 \) such that, for every \( t \in [a, b] \), for all \( \bar{x}_1, \bar{x}_2 \in B_r(0) \), for all \( u_1, u_2 \in L^1([a, b]; K) \) and for every \( v \in L^1([a, b]; V) \), one has

\[ |x_1(t) - x_2(t)| + ||x_1 - x_2||_1 \]

\[ \leq M \left(|x_1 - x_2| + |u_1(a) - u_2(a)|ight) \]

\[ + |u_1(t) - u_2(t)| + ||u_1 - u_2||_1 \]

where \( x_1 := x(\bar{x}_1, u_1, v_1), x_2 := x(\bar{x}_2, u_2, v_2). \)

A detailed proof of this result is available in [3].

III. PROPER EXTENSION OF A STANDARD MINIMUM PROBLEM

Let us consider the (standard) optimal control problem

\[ \inf_{(u, v) \in AC \times L^1} \psi(x(b), u(b)), \]  

where it is assumed that:

(i) the cost map \( \psi : \mathbb{R}^{n+m} \to \mathbb{R} \) is continuous;

(ii) \( AC \times L^1 \) stands for \( AC([a, b]; U) \times L^1([a, b]; V) \);

(iii) \( x(\cdot) = x(\bar{x}, u, v)(\cdot) \), i.e. \( x(\cdot) \) is the p.d. solution of the Cauchy problem (1)-(2) where \( \bar{u} := u(a) \).

Our main concern here is to define a proper extension of the minimum problem (25).

Let us give a formal notion of proper extension:

**Definition 3.1:** Let \( E \) be a set and let \( \mathcal{F} : E \to \mathbb{R} \) be a function. A proper extension of a minimum problem

\[ \inf_{e \in E} \mathcal{F}(e) \]  

is a new minimum problem

\[ \inf_{e \in E} \hat{\mathcal{F}}(e) \]  

on a set \( \hat{E} \) endowed with a limit notion and such that there exists an injective map \( i : E \to \hat{E} \) verifying the following properties:

(i) \( \hat{\mathcal{F}}(i(e)) = \mathcal{F}(e) \) for all \( e \in E \) and, moreover, for every \( \hat{e} \in \hat{E} \) there exists a sequence \( (e_k) \) in \( E \) such that, setting \( \hat{e}_k := i(e_k) \), one has

\[ \lim_{k \to \infty} (\hat{e}_k, \hat{\mathcal{F}}(\hat{e}_k)) = (\hat{e}, \hat{\mathcal{F}}(\hat{e})). \]

(ii) \( \inf_{e \in E} \mathcal{F}(e) = \inf_{e \in E} \hat{\mathcal{F}}(e) \).

After identifying \( E \) and \( \hat{E} \) with the set of pairs \((x(\cdot), u(\cdot))\) corresponding to controls in \( AC \times L^1 \) and \( L^1 \times L^1 \), respectively, we wish to investigate the question whether the optimal control problem

\[ \inf_{(u, v) \in L^1 \times L^1} \psi(x(b), u(b)), \]  

is a proper extension (with \( i \) equal to the identity map) of the problem

\[ \inf_{(u, v) \in AC \times L^1} \psi(x(b), u(b)). \]

**Remark 3.1:** Notice that, in view of the definition of p.d. solution, the density property (i) is automatically satisfied.
To investigate the validity of (ii), let us consider the reachable sets (at time $b$ for a fixed initial values $\bar{x}$ and $\bar{u}$):

$$\mathcal{R} := \{(x, u)(b): (u, v) \in L^1 \times L^1, \quad u(a) = \bar{u}, \quad x = x(\bar{x}, u, v)\},$$

$$\mathcal{R}^+ := \{(x, u)(b): (u, v) \in AC \times L^1, \quad u(a) = \bar{u}, \quad x = x(\bar{x}, u, v)\}.$$  \hfill (27) 

Since the (Carathéodory) solution corresponding to an absolutely continuous $u$ is also a p.d. solution, one has

$$\mathcal{R}^+ \subset \mathcal{R}.$$  \hfill (29) 

The inclusion is in general strict. However, the closure of the two sets always coincide.

**Theorem 3.1:**

$$\overline{\mathcal{R}} = \overline{\mathcal{R}}^+.$$  \hfill (30) 

**Proof:** In view of (29) it suffices to prove that $\mathcal{R} \subseteq \mathcal{R}^+$. Assume by contradiction that there exists $y \in \mathcal{R}$ such that

$$d(y, \mathcal{R}^+) = \eta > 0,$$

and let $\{(u_k, v_k)\} \subset L^1([a, b]; U) \times L^1([a, b]; V)$ be a sequence of controls with $u_k(a) = \bar{u}$ and such that the final points $y_k := (x(\bar{x}, u_k, v_k), u_k)$ verify $d(y_k, y) \leq \eta/3$, for all $k \in \mathbb{N}$. Because of the definition of p.d. solution, for every $k \in \mathbb{N}$ there exists $(\bar{u}_k, v_k) \in AC([a, b]; U) \times L^1([a, b]; V)$ such that, setting $\tilde{y}_k := (x(\bar{x}, \bar{u}_k, v_k)(b), \bar{u}_k(b))$, one has $d(\tilde{y}_k, y_k) \leq \eta/3$, so that

$$d(y, y_k) \leq d(y, y_k) + d(y_k, \tilde{y}_k) \leq 2\eta/3,$$

which contradicts (31), as $\tilde{y}_k \in \mathcal{R}^+$. \hfill \blacksquare

We let define the value functions

$$V_{AC}(\bar{x}, \bar{u}) := \inf_{(u, v) \in AC \times L^1} \psi(x(b), u(b)) = \inf_{\mathcal{R}^+} \psi(x, u),$$

$$V_{L^1}(\bar{x}, \bar{u}) := \inf_{(u, v) \in L^1 \times L^1} \psi(x(b), u(b)) = \inf_{\mathcal{R}} \psi(x, u),$$

where it has been made explicit that these values depend on the initial data $(\bar{x}, \bar{u})$.

**Corollary 3.2:** For every $(\bar{x}, \bar{u}) \in \mathbb{R}^{n+m}$, one has

$$V_{AC}(\bar{x}, \bar{u}) = V_{L^1}(\bar{x}, \bar{u}).$$

Hence, also in view of Remark 3.1 we can conclude that problem (25) is a proper extension of problem (26).

**IV. LIMITS OF MINIMUM PROBLEMS WITH BOUNDED VARIATION**

Let us assume that $U$ is a convex set.

When the impulsive (possibly discontinuous) control $u$ has bounded total variation one can give a notion of solution based on the concept of graph completion (see e.g. [5], [10], [12]). This approach differs from the one above and can, in fact, be applied also to systems with no commutativity assumptions. However, if the commutativity hypothesis is standing, one can establish a one-to-one correspondence between the two concepts, as shown in Proposition 4.1 below.

For every $K \geq 0$, let us consider the original system, supplemented with the variable $x_0 = t$,

$$\begin{cases}
    \dot{x}_0 = 1, \\
    \dot{x} = f(x, u, v) + \sum_{\alpha=0}^{m} g_\alpha(x, u)u^\alpha, \\
    (x_0, x)(a) = (a, \bar{x}, \bar{u}),
\end{cases}$$

(33)

where the impulsive controls $u$ belong to the set

$$BV_K([a, b]; U) := \left\{ u: [a, b] \to U, \text{Var}[u] \leq K \right\},$$

where $\text{Var}[u]$ denotes the total variation of $u$. We also consider the subset

$$AC_K([a, b]; U) := AC([a, b]; U) \cap BV_K([a, b]; U).$$

**Definition 4.1:** We shall use $U_K$ to denote the set of maps $(u_0, u) \in Lip([0, 1]; [a, b] \times U)$ such that, for a.a. $s \in [0, 1]$, $u'_b(s) \geq 0$, $\text{Var}[u'] \leq b - a + K$, and, moreover, $u_0(0, 1] = [a, b]$. These maps will be called space-time controls with variation not larger than $K$. Furthermore, $U_K^K \subset U_K$ will denote the subset made of those space-time controls $(u_0, u)$ such that $u'_b > 0$ for a.a. $s \in [0, 1]$.

Let us consider the space-time control system in the interval $[0, 1]$ given by

$$\begin{cases}
    y'_0 = u'_0, \\
    \frac{dy}{ds} = u'_0 f(y, u, v) + \sum_{\alpha=1}^{m} g_\alpha(y, u)u'^\alpha, \\
    (y_0, y, u)(0) = (a, \bar{x}, \bar{u}),
\end{cases}$$

(34)

where the apex denotes differentiation with respect to the pseudo-time $s$, $(u_0, u) \in U_K$, and $v \in L^1([0, 1]; V)$. If $u$ is absolutely continuous, (34) can be regarded as an ad hoc Lipschitz continuous time-reparameterization of (33), as it is made precise in the following statement (whose proof merely relies on the chain rule for derivatives).

**Proposition 4.1:** Let us consider controls $(u, v) \in AC_K([a, b]; U) \times L^1([a, b]; V)$ and an initial data $\bar{x} \in \mathbb{R}^n$. Let us set

$$s(t) := \frac{\int_0^t \int_0^{[\hat{u}]\,dr}}{\int_0^t 1 + [\hat{u}]\,dr}, \quad t(\cdot) = y_0(\cdot) := s^{-1}(\cdot),$$

and $u_0(s) := t(s), \quad u(s) = u \circ t(s), \quad v(s) := v \circ t(s)$.

Then, $(u_0, u) \in U_K$, $v \in L^1([0, 1]; V)$ and, setting $x(\cdot) := x(\bar{x}, u, v)(\cdot), \quad y(\cdot) := y(\bar{x}, u, v)(\cdot)$, one has

$$x \circ t(s) = y(s), \quad \text{for all } s \in [0, 1].$$

Conversely, if $(u_0, u) \in U_K$, $v \in L^1([0, 1]; V)$, setting $s(\cdot) := u_0^{-1}(\cdot)$ and $u(t) = u \circ s(t), \quad v(t) := v \circ s(t)$, one has that $(u, v) \in AC_K([a, b]; U) \times L^1([a, b]; V)$ and

$$x(t) = y \circ s(t), \quad \text{for all } t \in [a, b],$$

where $x(\cdot) := x(\bar{x}, u, v)(\cdot), \quad y(\cdot) := y(\bar{x}, u, v)(\cdot)$.

On the other hand, the space-time control system makes sense also when we allow $u'_b(s) = 0$ on some interval $[s_1, s_2] \subseteq [0, 1]$. This accounts for a trajectory’s jump at $t = u_0(s_1) = u_0(s_2)$}. Notice that the trajectory ‘during’

2 Here $Lip([0, 1]; [a, b] \times U)$ denotes the space of Lipschitz continuous function defined in $[0, 1]$ and with values in $[a, b] \times U$.
the jump is governed by the dynamics $\sum_{i=0}^{m} g_i(y, u)u_i'$. The commutativity hypothesis is here crucial, for it implies that the magnitude $y(s_2) - y(s_1)$ of the jump is independent of the path $[s_1, s_2] \to u(s)$.

Consider now the reachable sets (at time $b$):

$$\mathcal{R}_K := \{(x, u)(b) : (u, v) \in BV_K \times L^1\},$$

$$\mathcal{R}_K^+ := \{(x, u)(b) : (u, v) \in AC_K \times L^1\},$$

$$\mathcal{R}_K^{BV} := \{(y(1), u(1)) : ((u_0, u), v) \in \mathcal{U}_K \times L^1\},$$

$$\mathcal{R}_K^{BV+} := \{(y(1), u(1)) : ((u_0, u), v) \in \mathcal{U}_K \times L^1\},$$

where it is meant that the involved trajectories are the solutions of the corresponding Cauchy problems with given initial point $(\bar{x}, \bar{u})$.

It follows easily that

$$\mathcal{R}_K^{BV+} \subset \mathcal{R}_K^{BV} \quad \text{for all } K > 0,$$

$$0 \leq K_1 < K_2 \Rightarrow \mathcal{R}_K^{BV+} \subset \mathcal{R}_K^{BV+}, \quad \mathcal{R}_K^{BV} \subset \mathcal{R}_{K_2}^{BV}.$$ Moreover, in view of Proposition 4.1, absolutely continuous solutions of (33) coincide with solutions of (34) corresponding to a control $(\bar{x}, \bar{u})$.

It follows easily that

$$\mathcal{R}_K^{+} = \mathcal{R}_K^{BV+}, \quad \text{for all } K \geq 0.$$

Furthermore,

$$\mathcal{R}_K = \mathcal{R}_K^{BV}.$$ (42)

This identity can be verified by exploiting the commutativity assumption (Hypothesis 1), which makes all the graph completions equivalent, and then by associating to each control $u \in BV$ its rectilinear graph completion. The latter is a Lipschitz continuous path in space-time obtained by bridging the discontinuities of $u$ by means of rectilinear segments.

One can also prove (see [11]) the following statement.

**Proposition 4.2:** For every solution $y(\bar{x}, u_0, u, v)$ corresponding to a control $(u_0, u) \in \mathcal{U}_K$ there exists a sequence $\{(u_{0h}, u_h)\}_{h \in \mathbb{N}}$ in $\mathcal{U}_K$ such that

$$(u_{0h}, u_h) \to (u_0, u), \quad y(\bar{x}, u_{0h}, u_h, v) \to y(\bar{x}, u_0, u, v),$$

uniformly on $[0, 1]$. In particular, one gets,

$$\mathcal{R}_K^{BV+} = \mathcal{R}_K^{BV}.$$ (43)

**Remark 4.1:** (see [5]) If the vector field $\bar{f}$ is independent of the ordinary control $v$, then the set of solutions to (34) corresponding to controls in $\mathcal{U}_K$ is closed in the $C^0$-topology. In particular, the reachable set $\mathcal{R}_K^{BV}$ is compact, so that

$$\mathcal{R}_K^{BV} = \mathcal{R}_K^{BV+}. $$ (44)

**Remark 4.2:** Let us point out that relations (40), (41), Proposition 4.2 and Remark 4.1 are valid also in the case when the commutativity in Hypothesis 1 is not imposed.

**Theorem 4.3:**

$$\mathcal{R} = \bigcup_{K \geq 0} \mathcal{R}_K^{BV} = \bigcup_{K \geq 0} \mathcal{R}_K^{BV+}. $$ (44)

We refer to [3] for a proof of the latter result.

Let us to consider the value functions corresponding to problems with bounded variation:

$$V_{AC_K}(\bar{x}, \bar{u}) = \inf_{(u,v) \in AC_K \times L^1} \psi(x(b), u(b)),$$

$$V_{BV_K}^{+}(\bar{x}, \bar{u}) = \inf_{(u_0, u, v) \in \mathcal{U}_K \times L^1} \psi(y(1), u(1)),$$

$$V_{BV_K}(\bar{x}, \bar{u}) = \inf_{(u_0, u, v) \in \mathcal{U}_K \times L^1} \psi(y(1), u(1)).$$

**Corollary 4.4:** For every $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times U$, one has

$$\lim_{K \to \infty} V_{BV_K} (\bar{x}, \bar{u}) = V(\bar{x}, \bar{u}).$$ (45)

V. CONSIDERATIONS ON DYNAMIC PROGRAMMING

For every $K \geq 0$, let us consider the map $W_K : [a, b] \times \mathbb{R}^M \times [0, K] \to \mathbb{R}$ defined by letting $W_K(t, x, u, k)$ be the value function of the (impulsive) minimum problem in $[t, b[$ with $u$-variation less than or equal to $K - k$. By a reparameterization approach akin to the one in [11] one might prove that $W$ is continuous and is the unique solution of a boundary value problem for a Hamilton-Jacobi equation involving the compactified Hamiltonian

$$H(t, x, u, k, p_t, p_x, p_u, p_k) := \sup_{w_0 \in [0, 1]} H(t, x, u, k, p_t, p_x, p_u, p_k; w_0, w, v),$$

where the $H$ is defined by

$$H(t, x, u, k, p_t, p_x, p_u, p_k; w_0, w, v) := (p_t + p_x \cdot f(x, u, v))w_0 + (p_x \cdot g_a + p_u \cdot u_a)w_a + p_k |w_a|.$$ Notice that $W_K(a, x, u, 0) = V_{BV_K}(x, u)$, for all $(x, u) \in \mathbb{R}^n \times U$. In particular the Hamilton-Jacobi equation

$$H(t, x, u, k, \nabla W_K) = 0$$

may be utilized for both sufficient conditions of optimality and numerical analysis of the problem with $\text{Var}(u) \leq K$.

Via Corollary 4.4 one can then address the general problem.

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