Fuzzy chaotic centred quasi-uniform space

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Abstract
In this paper, the concept of fuzzy chaotic centred pre-distinctiveness space, fuzzy chaotic centred distinctiveness space, fuzzy chaotic centred filter base, fuzzy chaotic centred quasi-uniform structure, fuzzy chaotic centred quasi-uniform space, fuzzy chaotic centred tight and fuzzy chaotic centred quasi-uniform subspace are introduced and their interesting properties are discussed.

Keywords
Fuzzy chaotic centred pre-distinctiveness space, fuzzy chaotic centred distinctiveness space, fuzzy chaotic centred filter base, fuzzy chaotic centred quasi-uniform structure, fuzzy chaotic centred quasi-uniform space, fuzzy chaotic centred tight and fuzzy chaotic centred quasi-uniform subspace.

AMS Subject Classification
54A40, 03E72.

1. Introduction
Fuzzy set theory, which was established by Zadeh[12] (1965), has emerged as an incredible method of representing quantitatively and controlling the imprecision in decision-making problems. Fuzzy sets have applications in many fields such as information [7] and control [8]. Chang [4] introduced and developed the concept of fuzzy topological spaces. In 2007, the concept centred systems in fuzzy topological spaces introduced by Uma, Roja and Balasubramanian [10]. The concept of chaotic in general metric space was introduced by R. L. Devaney [5]. The elementary properties of chaos (Devaney definition of chaos) were established in [1] and [2]. Furthermore, the properties of chaos were developed and studied in [11]. In this paper, the concept of fuzzy chaotic centred pre-distinctiveness space, fuzzy chaotic centred distinctiveness space, fuzzy chaotic centred filter base, fuzzy chaotic centred quasi-uniform structure, fuzzy chaotic centred quasi-uniform space, fuzzy chaotic centred tight and fuzzy chaotic centred quasi-uniform subspace are introduced and their interesting properties are discussed.

2. Preliminaries

Definition 2.1. [12] A fuzzy set in X is a function with domain X and values in I, that is an element of \( I^X \).

Definition 2.2. [3] Let S be a set. Then \( S \subseteq P(S) \) is called a texturing of S and \((S, \mathcal{S})\) is called a texture space or simply a texture if

1. \((S, \subseteq)\) is complete lattice containing S and \( \phi \) which has the property that arbitrary meets coincide with intersections, and finite joins coincide with unions. i.e.,
   \[ \bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i, A_i \in \mathcal{S}, i \in I, \text{ for all index sets } I, \]
   \[ \bigvee_{i \in I} A_i = \bigcup_{i \in I} A_i, A_i \in \mathcal{S}, i \in I, \text{ for all index sets } I. \]

2. \( \mathcal{S} \) is completely distributive.

3. \( \mathcal{S} \) separates the points of S. That is, given \( s_1 \neq s_2 \) in S, we have \( A \in \mathcal{S} \) with \( s_2 \in A, s_1 \notin A \).

A surjection \( \sigma : \mathcal{S} \to \mathcal{S} \) is called a complementation if \( \sigma^2(P) = P \) for all \( P \in \mathcal{S} \) and \( P \subseteq Q \in \mathcal{S} \) implies \( \sigma(Q) \subseteq \sigma(P) \). A texture with a complementation is said to be complemented.

Example 2.3. If X is a set and P(X) the powerset of X, then (X, P(X)) is the discrete texture on X. For \( x \in X \), \( P_x = x \) and \( Q_x = X / x \).

Definition 2.4. [6] Let \((X, \tau)\) be a fuzzy topological space. Let \( f : X \to X \) be any mapping. The fuzzy orbit set under the mapping \( f \) which is in fuzzy topology \( \tau \) is called fuzzy...
Definition 2.5. [9] Let \((X, \tau)\) be a fuzzy topological space and \(\lambda \in KF(X)\) (Where \(KF(X)\) is a collection of all nonempty fuzzy compact subsets of \(X\)). Let \(f : X \to X\) be any mapping. Then \(f\) is fuzzy chaotic with respect to \(\lambda\) if

(i) \(\text{cl} \, FO_f(\lambda) = 1\),

(ii) \(P\) is fuzzy dense.

Notation 2.6. (i) \(FC(\lambda) = \{ f : X \to X / f \text{ is fuzzy chaotic with respect to } \lambda \text{ where } \lambda \text{ is a fuzzy set in } X \}\).

(ii) \(FCH(X) = \{ \lambda \in KF(X) / FC(\lambda) \neq \phi \}\).

Definition 2.7. [9] A fuzzy topological space \((X, \tau)\) is called a fuzzy chaos space if \(FCH(X) \neq \phi\). If \((X, \tau)\) is fuzzy chaos space then the elements of the FCH(X) are called chaotic sets in \(X\).

Definition 2.8. [9] Let \((X, \tau)\) be a fuzzy chaos space. Let \(\mathfrak{C}\) be the collection of fuzzy chaotic sets in \(X\) satisfying the following conditions:

(i) \(0, 1 \in \mathfrak{C}\),

(ii) if \(\mu_1, \mu_2 \in \mathfrak{C}\), then \(\mu_1 \land \mu_2 \in \mathfrak{C}\),

(iii) if \(\{ \mu_j : j \in J \} \subset \mathfrak{C}\), then \(\bigvee_{j \in J} \mu_j \in \mathfrak{C}\).

Then \(\mathfrak{C}\) is called the fuzzy chaotic structure in \(X\). The triple \((X, \tau, \mathfrak{C})\) is called fuzzy chaotic structure space. The elements of \(\mathfrak{C}\) are called fuzzy chaotic open sets. The complement of fuzzy chaotic open set is called fuzzy chaotic closed set.

Definition 2.9. [9] A fuzzy chaotic structure space \((X, \tau, \mathfrak{C})\) is called a fuzzy chaotic Hausdorff space if for each pair of non zero fuzzy chaotic sets \(\lambda\) and \(\mu\) such that \(\lambda \neq \mu\), then there exist fuzzy chaotic open sets \(\gamma\) and \(\delta\) such that \(\lambda \leq \gamma\) and \(\mu \leq \delta\) and \(\gamma \land \delta\).

3. Fuzzy Chaotic Centred Quasi-Uniform Space

Definition 3.1. Let \(\mathfrak{X} = \{ \{ p_i \}_{i \in J} \}\) be a nonempty set of fuzzy chaotic centred systems and let \(U, V\) be subsets of the cartesian product of \(\mathfrak{X} \times \mathfrak{X}\). Define certain subsets as follows:

\[
U \circ V \equiv \{ (p_i, q_i) : \exists r_i \in \mathfrak{X} \text{ such that } ((p_i, r_i) \in U \text{ and } (r_i, q_i) \in V) \},
\]

\[
U^1 \equiv U, U^{n+1} \equiv U \circ U^n \text{ (n = 1, 2, ...)},
\]

\[
U^{-1} \equiv \{ (p_i, q_i) : (q_i, p_i) \in U \} \text{ and, for } p_i \in \mathfrak{X},
\]

\[
U \setminus p_i \equiv \{ q_i \in \mathfrak{X} : (p_i, q_i) \in U \}.
\]

\(U\) is called symmetric if \(U = U^{-1}\).

Definition 3.2. Let \(S\) be any set and \(\mathbb{B}\) be a nonempty subset of \(S\). Then \(\mathbb{B}\) is called a fuzzy chaotic centred filter base on \(S\) if the intersection of two sets in \(\mathbb{B}\) contains a set in \(\mathbb{B}\).

Definition 3.3. Let \(S\) be any set and \(\mathbb{B}\) is a fuzzy chaotic centred filter base on \(S\). If \(\mathbb{B}\) satisfies the following conditions:

(i) \(U \in \mathbb{B}\) implies \(FCH(U) \neq \emptyset\).

(ii) For all \((p_i, q_i) \in S, p_i = q_i\) if and only if \((p_i, q_i) \in U\) for each \(U \in \mathbb{B}\).

(iii) For each \(U \in \mathbb{B}\) there exists \(V \in \mathbb{B}\) such that \(V^2 \subset U\).

(iv) For each \(U \in \mathbb{B}\) there exists \(V \in \mathbb{B}\) such that \(\mathfrak{X} \times \mathfrak{X} = U \cup -V\).

Then \(\mathbb{B}\) is called a fuzzy chaotic centred quasi-uniform structure on \(\mathfrak{X}\). The elements of \(\mathbb{B}\) are called the fuzzy chaotic centred entourages of (fuzzy chaotic centred quasi-uniform structure on) \(\mathfrak{X}\), and the pair \((\mathfrak{X}, \mathbb{B})\) itself is called a fuzzy chaotic centred quasi-uniform space.

Note 3.5. The intersection of all the fuzzy chaotic centred entourages in \(\mathbb{B}\) is the diagonal. \(\Delta = \{ (p_i, p_i) : p_i \in \mathfrak{X} \}\) of \(\mathfrak{X} \times \mathfrak{X}\) by (U2).

Note 3.6. The inequality relation is symmetric, (U2) implies that if \(p_i = q_i\), then \((p_i, q_i) \in U\) for all \(U \in \mathbb{B}\).

Definition 3.7. The standard (fuzzy chaotic centred uniform) inequality on a fuzzy chaotic centred quasi-uniform space \((\mathfrak{X}, \mathbb{B})\) is defined by \(p_i \neq q_i\) if there exists \(U \in \mathbb{B}\) such that \((p_i, q_i) \notin U\) or there exists \(U \in \mathbb{B}\) such that \((q_i, p_i) \notin U\).

Definition 3.8. The standard inequality on a fuzzy chaotic centred quasi-uniform space \((\mathfrak{X}, \mathbb{B})\) is said to be fuzzy chaotic centred tight if \(\neg (p_i \neq q_i) \Rightarrow p_i = q_i\).

Definition 3.9. Let \((\mathfrak{X}, \mathbb{B})\) be a fuzzy chaotic centred quasi-uniform space and let \(\mathfrak{Q}\) be a nonempty subset of \(\mathfrak{X}\). A fuzzy chaotic centred quasi-uniform structure \(\mathbb{B}\) on \(\mathfrak{X}\) induces a fuzzy chaotic centred quasi-uniform structure \(\mathbb{B}_{\mathfrak{Q}}\) on a nonempty subset \(\mathfrak{Q}\) of \(\mathfrak{X}\). The fuzzy chaotic centred entourages of \(\mathbb{B}_{\mathfrak{Q}}\) are the sets \(U \cap (\mathfrak{Q} \times \mathfrak{Q})\) with \(U \in \mathbb{B}\). Then \(\mathfrak{Q}\) together with \(\mathbb{B}_{\mathfrak{Q}}\) is called fuzzy chaotic centred quasi-uniform subspace of \(\mathfrak{X}\). The fuzzy chaotic centred quasi-uniform structure \(\mathbb{B}_{\mathfrak{Q}}\) is called the fuzzy chaotic centred subspace of quasi-uniform structure on \(\mathfrak{Q}\).
Proposition 3.10. Let \((X, \mathcal{U})\) be a fuzzy chaotic centred quasi-uniform space and let \(U\) be a fuzzy chaotic centred entourage of \((X, \mathcal{U})\). Then either \(p_i \neq q_i\) or \((p_i, q_i) \in U\), for every \(p_i, q_i \in X\).

Proof. By (U4), there exists \(V \in \mathcal{U}\) such that \(X \times X = U \cup \neg V\). If \((p_i, q_i) \in \neg V\), then \(p_i \neq q_i\) by the definition of the standard inequality on \(X\).

Proposition 3.11. The standard inequality on a fuzzy chaotic centred quasi-uniform space is fuzzy chaotic centred tight.

Proof. Let \((X, \mathcal{U})\) be a fuzzy chaotic centred quasi-uniform space and let \(p_i\) and \(q_i\) be fuzzy chaotic centred systems of \(X\) such that \(\neg (p_i \neq q_i)\). Then by Proposition 3.10 \((p_i, q_i) \in U\) for each \(U \in \mathcal{U}\). Therefore by (U2) \(p_i = q_i\).

Proposition 3.12. Let \((X, \mathcal{U})\) be a fuzzy chaotic centred quasi-uniform space and let \(V\) be a fuzzy chaotic centred entourage of \((X, \mathcal{U})\). Let \(n\) be an integer \(\geq 2\). Then \(V^n\) is a fuzzy chaotic centred entourage, and \(V^{n-1} \subset V^n\).

Proof. Let \((p_i, q_i) \in V^{n-1}\), by (U2) \((p_i, p_i) \in V\). Hence \((p_i, q_i) \in V \circ V^{n-1} = V^n\). Therefore by (U1) and (F2) \(V^n\) is fuzzy chaotic centred entourage.

Proposition 3.13. Let \((X, \mathcal{U})\) be a fuzzy chaotic centred quasi-uniform space and let \(U \in \mathcal{U}\). Then there exists \(V \in \mathcal{U}\) such that \(V^3 \subset U\).

Proof. Let us construct \(W, V \in \mathcal{U}\) such that \(W^2 \subset U\) and \(V^2 \subset W\) by using (U3) twice. By Proposition 3.12 \(V \circ V^2 \subset W\). Hence \(V^3 \subset U\).

Proposition 3.14. Let \((X, \mathcal{U})\) be a fuzzy chaotic centred quasi-uniform space and let \(U\) be a fuzzy chaotic centred entourage of \((X, \mathcal{U})\). Then there exists a fuzzy chaotic centred entourage \(V\) such that \(V^2 \subset U\) and \(\neg U \subset \sim V\).

Proof. By Proposition 3.13, choose a fuzzy chaotic centred entourage \(V\) such that \(V^3 \subset U\). By Proposition 3.12, \(V^2 \subset U\) and there exists a fuzzy chaotic centred entourage \(W\) such that \(X \times X = V \cup \neg W\) by (U4). Consider \((p_i, q_i)\) in \(\neg U\) and \((r_i, s_i)\) in \(V\). If \((p_i, r_i) \in V\) and \((s_i, q_i) \in V\), then \((p_i, q_i) \in V^3 \subset U\), which is a contradiction. Hence either \((p_i, r_i) \in \neg W\) and \(p_i \neq r_i\), or \((s_i, q_i) \in \neg W\) and \(s_i \neq q_i\). Thus \((p_i, q_i) \neq (r_i, s_i)\). It follows that \(\neg U \subset \sim V\).

Proposition 3.15. Let \((X, \mathcal{U})\) be a fuzzy chaotic centred quasi-uniform space. If \(U\) is a fuzzy chaotic centred entourage of \((X, \mathcal{U})\), then for all integers \(n \geq 2\), there exists a fuzzy chaotic centred entourage \(V\) such that \(V^n \subset U\) and \(X \times X = U \cup \sim V\).

Proof. If \(U\) is a fuzzy chaotic centred entourage of \((X, \mathcal{U})\), then there exists a fuzzy chaotic centred entourage \(W \in \mathcal{U}\) such that \(X \times X = U \cup \neg W\). By Proposition 3.14, there exists a fuzzy chaotic centred entourage \(V \in \mathcal{U}\) such that \(\neg W \subset \sim V\). Also by Proposition 3.14, there exists a fuzzy chaotic centred entourage \(V\) such that \(V^2 \subset U\), which gives the proof for \(n = 2\). Since the proof is true for \(n = 2\), by the induction hypothesis it is true for power of 2, say \(2^k\) where \(k\) is a positive integer. Let \(2^k > n\). Hence there exists a fuzzy chaotic centred entourage \(V \in \mathcal{U}\) such that \(V^{2k} \subset U\) and \(X \times X = U \cup \sim V\). By Proposition 3.12, \(V^n \subset V^{2k}\). Thus \(V^n \subset V^{2k} \subset U\), which completes the proof.

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