On Ideals of $L^1$–algebras of Compact Quantum Groups

Benjamin Anderson-Sackaney *

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Abstract

We develop a notion of a non-commutative hull for a left ideal of the $L^1$-algebra of a compact quantum group $G$. A notion of non-commutative spectral synthesis for compact quantum groups is achieved as well. It is shown that a certain Ditkin’s property at infinity (which includes those $G$ where $\hat{G}$ has the approximation property) is equivalent to every hull having synthesis. We use this work to extend recent work of White that characterizes the weak∗ closed ideals of a measure algebra of a compact group to those of the measure algebra of a coamenable compact quantum group. In the sequel, we use this work to study bounded right approximate identities of certain left ideals of $L^1(G)$ in relation to coamenability of $G$.

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1 Introduction

Let us fix a compact group $G$. Recall that for a unitary representation $\pi : G \to \mathcal{B}(\mathcal{H}_\pi)$, the corresponding $L^1$ representation is given by $\pi(f) = \int_G f(t)\pi(t)\,dt$ for $f \in L^1(G)$. Then $L^1(G)$ embeds contractively into $C^*(G)$, with dense range, via the map $f \mapsto \oplus_{\pi \in \text{Irr}(G)} \pi(f)$. Through representation theory, we can describe the closed left ideals of $L^1(G)$.

Theorem 1.1. [19] The closed left ideals of $L^1(G)$ are of the form

$I(E) = \{f \in L^1(G) : \pi(f)(E_\pi) = 0, \pi \in \text{Irr}(G)\}$

where $E = (E_\pi)_{\pi \in \text{Irr}(G)}$ for subspaces $E_\pi \subseteq \mathcal{H}_\pi$.

Now let $G$ be locally compact. The Fourier algebra $A(G)$ is commutative, with Gelfand spectrum $\hat{G}$, so the irreducible corepresentations of $\hat{G}$ are identified with $G$ as the point evaluations $e_t : A(G) \to \mathbb{C}$, $t \in G$. Recall, $A(G)$ embeds contractively into $C_0(G)$ with

*Department of Pure Mathematics, University of Waterloo, Waterloo. Email: b8anders@uwaterloo.ca
dense range as the coefficient functions of the left regular representation. For a closed subset $E \subseteq G$, we will write $I(E) = \{ u \in A(G) : u|_E = 0 \} = \{ u \in A(G) : e_t(u) = 0, \; t \in E \subseteq G = \text{Irr}(\hat{G}) \}$ and $j(E) = \{ u \in A(G) : u \text{ has compact support disjoint from } E \}$. The ideal $I(E)$ is always closed. Recall, since $A(G)$ is Tauberian \[12\], for any ideal $I \subseteq A(G)$ we have $j(E) \subseteq I \subseteq I(E)$ for some closed subset $E \subseteq G$ \[14\] Chap. X Section 1. The closed subset $E$ is said to be a set of synthesis if $j(E) = I(E)$. Our ability to describe the closed ideals of $A(G)$ is limited to the following.

**Theorem 1.2.** \[26\] Let $G$ be a locally compact group. Then every closed subset of $G$ is a set of synthesis if and only if $G$ is discrete and $u \in uA(G)$ for all $u \in A(G)$.

In the general scheme of locally compact quantum groups (LCQGs), the compact and discrete quantum groups are dual to one another (cf. \[55\] and \[45\]). So, it is reasonable to attempt to unify Theorems 1.1 and 1.4 at the level of compact quantum groups (CQGs). Indeed, for a compact group we refer to the collection $(E_x)_{x \in \text{Irr}(G)}$ as the hull of the $I(E)$, and adopting the language used for ideals in Fourier algebras of locally compact groups, Theorem 1.1 says every hull of a closed left ideal of the $L^1$-algebra of a compact group is a “set of synthesis.” Then, since $L^1(G)$ always has a bounded approximate identity (bai), we see Theorem 1.1 is dual to Theorem 1.4 reduced to discrete groups. Using the analogies found between the representation theory of compact quantum groups (CQGs) in general and compact groups (cf. \[55\]), we formulate notions of hull and synthesis, and prove a generalization of Theorems 1.1 and 1.4 at the level of CQGs (Theorem 3.10).

Recently White \[54\] characterized the weak$^*$ closed left ideals of measure algebras of compact groups using the description of closed left ideals of their $L^1$-algebras achieved with Theorem 1.1. We show White’s techniques extend directly to coamenable CQGs in order to achieve a similar description (Theorem 3.10). Leptin’s theorem states that $A(G)$ has a bai if and only if $G$ is amenable, which is to say in the language of quantum groups, $G$ is amenable if and only if $\hat{G}$ is coamenable. Recall that $G$ is always coamenable and $G$ is always amenable. It is a classical result that the augmentation ideal $L^0_\beta(G) = \{ f \in L^1(G) : \int_G f = 0 \}$ admits a bai if and only if a locally compact group $G$ is amenable. More generally, the following is a result of Lau \[35\] (shown at the level of $F$-algebras).

**Theorem 1.3.** \[44\] A LCQG $G$ is coamenable and amenable if and only if the augmentation ideal $L^0_\beta(G)$ has a bai.

More generally, we wish to know when closed left ideals of $L^1$-algebras of LCQGs admit bounded right approximate identities (brais). For amenable and SIN $G$, Forrest \[14\] showed the closed ideal $I(H)$ of functions in $A(G)$ vanishing on a closed subgroup $H$ always admits a bai. This was extended to general amenable groups in \[15\] and conversely it was shown that if $I(H)$ admits a bai for some closed subgroup $H$, then $G$ is amenable in \[14\], thus providing us the complete theorem:

**Theorem 1.4.** \[14, 15\] Let $G$ be a locally compact group and $H$ a closed subgroup. Then $G$ is amenable if and only if $I(H)$ has a bai.
Now let $J^1(G, H)$ be the closed left ideal of functions in $L^1(G)$ which average over the right action of $H$ on $G$ to zero. Caprace and Monod showed the following in [7]:

**Theorem 1.5.** [7] Let $G$ be a locally compact group and $H$ a closed subgroup. Then $H$ is amenable if and only if $J^1(G, H)$ has a brai.

In particular, when $G$ is compact, Theorem 1.5 simply says $J^1(G, H)$ always has a brai.

We can view $J^1(G, H)$ and $I(H)$ as being generalizations of the augmentation ideal (in their respective settings) by the fact $J^1(G, G) = L^1_0(G) = I(\{e\}) = A_0(G)$. In this work we reduce to the setting of CQGs. For discrete $G$, the von Neumann subalgebras $VN(H) \subseteq VN(G)$ are exactly the compact quasi-subgroups and so are the subalgebras $L^\infty(G/H) \subseteq L^\infty(G)$ for compact $G$ (cf. [28] and [30]). Note also that $VN(H)_\perp = I(H)$ and $L^\infty(G/H)_\perp = J^1(G, H)$. So, in this work, we study the preannihilators of compact quasi-subgroups and obtain partial progress towards the classification of those with brais (Corollary 4.16).

Section 2 will comprise the preliminaries for locally compact quantum groups where we will in particular recall the theory behind closed quantum subgroups and more generally, invariant subspaces.

In Section 3, we will develop the notion of a hull $E$ of a closed left ideal $I \subseteq L^1(G)$ and then will classify the compact quantum groups such that $\overline{J(E)} = I(E)$, for each hull $E$, in terms of Ditkin’s property at infinity (or property left $D_\infty$, a property which has recently achieved a new characterization [1], (Theorem 3.10). In particular, we can describe the closed left ideals of compact quantum groups whose dual has the approximation property. Then we will show White’s techniques [54] for classifying the weak* closed left ideals of the measure algebra of a compact group extend to the setting of coamenable compact quantum groups (Theorem 3.16). We will conclude the section with a brief discussion of property left $D_\infty$ and provide examples of CQGs which are weakly amenable and consequently have property left $D_\infty$.

Finally, in Section 4 we study the closed left ideals of $L^1(G)$ which admit a brai, with special emphasis on the preannihilator space $J^0(N)$ of a compact quasi–subgroup $N$ (the natural quantum analogue of a closed subgroup of a compact group). We also study the associated weak* closed left ideal $J^0(N)$ in $M^\infty(G)$ and in a certain case, show $J^0(N) = \overline{J^1(N)}^{wk*}$ if and only if $G$ is coamenable if and only if $J^1(N)$ admits a bounded right approximate identity (Theorems 4.14 and 4.15 and Corollary 4.16). We conclude the section by showing whenever $N \neq X = \langle X \rangle$ for $x \in Gr(G)$, that $\lambda_\perp$ possesses a bounded approximate identity if and only if $G$ is coamenable (Theorem 4.25). In this context, we think of $X$ as being a “quantum coset” of the compact quasi–subgroup $N$. We end by illustrating these last results on discrete crossed products equipped with the structure of a compact quantum group.

## 2 Preliminaries

### 2.1 Locally Compact Quantum Groups

The notion of a quantum group we will be using in this paper is the one developed by Kusterman and Vaes [35]. We use the references [34] and [48]. A **locally compact quantum group** (LCQG) $G$ is a quadruple $(L^\infty(G), \Delta_G, \psi_L, \psi_R)$ where $L^\infty(G)$ is a von Neumann algebra; $\Delta_G : L^\infty(G) \to L^\infty(G) \otimes L^\infty(G)$ a normal unital $*$-homomorphism satisfying $(\Delta_G \otimes \text{id})\Delta_G = (\text{id} \otimes \Delta_G)\Delta_G$ (coassociativity); and $\psi_L$ and $\psi_R$ are normal semifinite faithful
weights on $L^\infty(\mathbb{G})$ satisfying
\[
\psi_L(f \otimes \text{id})\Delta_\mathbb{G}(x) = f(1)\psi_L(x), \quad f \in L^\infty(\mathbb{G}), x \in L^\infty(\mathbb{G})\psi_L \quad \text{(left invariance)}
\]
and
\[
\psi_R(\text{id} \otimes f)\Delta_\mathbb{G}(x) = f(1)\psi_R(x), \quad f \in L^\infty(\mathbb{G}), x \in L^\infty(\mathbb{G})\psi_R \quad \text{(right invariance)},
\]
where $L^\infty(\mathbb{G})\psi_L$ and $L^\infty(\mathbb{G})\psi_R$ are the set of integrable elements of $L^\infty(\mathbb{G})$ with respect to $\psi_L$ and $\psi_R$ respectively. We call $\Delta_\mathbb{G}$ the co–product and $\psi_L$ and $\psi_R$ the left and right Haar weights respectively, of $\mathbb{G}$.

Using $\psi_L$, we can build a GNS Hilbert space $L^2(\mathbb{G})$ representing $L^\infty(\mathbb{G})$. There exists a unitary $W_\mathbb{G} \in L^\infty(\mathbb{G})\overline{\otimes} B(L^2(\mathbb{G}))$ such that $\Delta_\mathbb{G}(x) = W_\mathbb{G}(1 \otimes x)W_\mathbb{G}^*$. The unitaries $W_\mathbb{G}$ is known as the left fundamental unitaries of $\mathbb{G}$. The predual $L^1(\mathbb{G}) := L^\infty(\mathbb{G})_*$ is a Banach algebra with respect to the product $f \star g \mapsto (f \otimes g)\Delta_\mathbb{G}$ known as convolution. This naturally provides us with left and right dual actions on $L^\infty(\mathbb{G})$, realized by the equations
\[
f \star x = (\text{id} \otimes f)\Delta_\mathbb{G}(x) \quad \text{and} \quad x \star f = (f \otimes \text{id})\Delta_\mathbb{G}(x).
\]
Unfortunately, $L^1(\mathbb{G})$ is not generally a $*$–algebra. There is, however, a dense $*$–subalgebra we can build. The antipode $S_\mathbb{G} : D(S) \to L^\infty(\mathbb{G})$ is an unbounded linear anti–automorphism, (where $D(S)$ denotes the weak* dense domain of $S$), satisfying the identity $(\text{id} \otimes S)(W_\mathbb{G}) = W_\mathbb{G}^*$. Then
\[
L^1_\\#(\mathbb{G}) = \{f \in L^1(\mathbb{G}) : \text{there exists } g \in L^1(\mathbb{G}) \text{ such that } g(x) = \overline{(S_\mathbb{G}(x))}, \ x \in D(S)\}
\]
is an involutive algebra that is dense in $L^1(\mathbb{G})$ with involution $f \mapsto \overline{f \circ S}$, (see [32]). Note that the unitary antipode $R_\mathbb{G}$ is the unitary in the polar decomposition of $S_\mathbb{G}$.

A co–representation operator is an element $U \in L^\infty(\mathbb{G})\overline{\otimes} B(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, such that $(\text{id} \otimes \Delta_\mathbb{G})(U) = U_{13}U_{23}$ where $U_{23} = 1 \otimes U$ and $U_{13} = (\Sigma \otimes \text{id})U_{23}(\Sigma \otimes \text{id})$, and $\Sigma$ is the flip map. Co–representation operators $U^\pi$ are in one–to–one correspondence with representations $\pi : L^1(\mathbb{G}) \to B(\mathcal{H}_x)$ with which the relationship is realized by setting $\pi(f) = (f \otimes \text{id})(U^\pi)$. The unitary co–representation operators correspond to representations that restrict to a $*$–representation on $L^1_\\#(\mathbb{G})$. We will simply refer to such representations as $*$–representations.

Representations $\pi$ and $\rho$ are unitarily equivalent if there exists a unitary $U \in B(\mathcal{H}_x)$ such that
\[
(1 \otimes U^\ast)U^\pi(1 \otimes U) = U^\rho.
\]
Whenever we have a representation $\pi$, we are choosing a representative from the equivalence class of unitarily equivalence representations. We call a co–representation operator $U^\pi$ irreducible if the corresponding representation $\pi$ is irreducible. We will use the notation $\text{Irr}(\mathbb{G})$ to denote a family of irreducible $*$–representations on $L^1(\mathbb{G})$, each chosen from distinct equivalence classes of $*$–representations with respect to unitary equivalence.

The fundamental unitary $W_\mathbb{G}$ is a unitary co–representation operator and the corresponding $*$–representation $\lambda_\mathbb{G}$ is called the left regular representation of $\mathbb{G}$.

The von Neumann algebra $\lambda_\mathbb{G}(L^1_\\#(\mathbb{G}))^{\text{weak}}$ has the structure of an LCQG, which in particular, has a co–product unitarily implemented by $W_\mathbb{G}^* = \Sigma(W_\mathbb{G}^*)$. The underlying LCQG, $\hat{\mathbb{G}}$, is called the dual LCQG of $\mathbb{G}$. It turns out that $L^\infty(\mathbb{G}) = \lambda_\mathbb{G}(L^1_\\#(\mathbb{G}))^{\text{weak}}$, which we view as being Pontryagin duality at the level of LCQGs.
There is also $C^*$-algebraic framework underlying the theory of LCQGs. The reduced $C^*$-algebra is $C_r^0(G) := \overline{\lambda_H(L^1(\hat{G}))}$. The map $a \mapsto \Delta^r(a) := W_G(1 \otimes a)W_G^*$ defines a non–degenerate $*$--homomorphism $C_r^0(G) \to M(C^0_G)\otimes_{min} C^0_G$ satisfying coassociativity and is known as the co–product of $C_r^0(G)$. By building a universal $*$–representation $\varpi_G$, we can build the universal $C^*$–algebra $C_u^0(G) := \overline{\varpi_G(L^1(\hat{G}))}^{||\cdot||_u}$ which also comes equipped with a co–product $\Delta_u^G$. The counit is a character $\epsilon_G^u : C_u^0(G) \to \mathbb{C}$ satisfying $(\epsilon_G^u \otimes \epsilon_G^u)\Delta_u^G = \text{id} = (\epsilon_G^u \otimes \epsilon_G^u)\Delta_u^G$. There is a surjective $*$–homomorphism $\Gamma_G : C_u^0(G) \to C_r^0(G)$ such that $(\Gamma_G \otimes \Gamma_G)\Delta_u^G = \Delta_r^G \circ \Gamma_G$ which is known as the reducing morphism.

The dual $M^u(G) := C_u^0(G)^*$ is a unital Banach algebra with respect to the product $\mu \times \nu \mapsto \mu \ast \nu := \langle \mu \otimes \nu \rangle\Delta_u$, which again, is called convolution, and unit $\epsilon_G^u$. We call $M^u(G)$ the universal measure algebra of $G$. Similarly $M^r(G) := C_r^0(G)^*$ is Banach algebra known as the reduced measure algebra. We remark that $M^r(G) = L^1(G)$ and the adjoint $\Gamma_G^* : M^u(G) \to M^r(G)$ gives us a completely isometric embedding, and furthermore, $M^u(G)$ contains $L^1(G)$ as an ideal through this embedding. Whenever $M^r(G)$ turns out to be unital (so that $C_r^0(G)$ admits a “reduced” counit), we denote this unit by $\epsilon_G^u$.

**Remark 2.1.** The examples where $L^\infty(G)$ is commutative are the locally compact groups. We call the dual of a locally compact group a locally compact co–group and we use the notation $C_u^0(G) = C^*_u(G)$, $C_r^0(G) = C^*_r(G)$, $L^1(\hat{G}) = \mathcal{A}(G)$, $C_0^u(\hat{G})^* = \mathcal{B}(G)$, and $L^\infty(\hat{G}) = VN(G)$, i.e., $\hat{G} = \langle VN(G), \Delta, \varphi \rangle$ where $\Delta_G^r(\lambda_G(s)) = \lambda_G(s) \otimes \lambda_G(s)$ and $\varphi = \psi_L = \psi_R$ is the Plancherel weight [27 Chapter IV, Section 3]. The Banach algebra $\mathcal{A}(G)$ is called the Fourier algebra and $\mathcal{B}(G)$ the Fourier–Stieletts algebra. The locally compact co–groups comprise the examples where $L^1(G)$ is commutative, i.e., $L^\infty(G)$ is cocommutative.

### 2.2 Compact Quantum Groups and Fourier Analysis

The compact quantum groups (CQGs) are the LCQGs $G$ such that $L^1(\hat{G})$ is unital, meaning that $C_r^0(G)$ and $C_u^0(G)$ are both unital (cf. [55] and [43]). In this case, we follow the custom of writing $C^*(G)$ and $C_u^*(G)$ for the universal and reduced $C^*$–algebras respectively instead. It follows that $\psi_L = \psi_R = \hbar_G \in L^1(G)$ is a state called the Haar state of $G$, the irreducible representations are finite dimensional, and every $*$–representation decomposes into a direct sum of irreducibles. Given an irreducible representation $\pi$, we will write $U^\pi = \{u_{i,j}^\pi\}$ with respect to an orthonormal basis (ONB) $\{e_i^\pi\}$ of $H_\pi$. Then, $$\text{Pol}(G) := \text{span}\{u_{i,j}^\pi : 1 \leq i, j \leq n_\pi, \pi \in \text{Irr}(G)\}$$ is a $*$–algebra that identifies with a norm dense $*$–subalgebra of $C_u^0(G)$ and $C_r^1(G)$, and a weak* dense $*$–subalgebra of $L^\infty(G)$. It is actually a Hopf $*$–algebra with co–product $\Delta_G := \Delta_G|_{\text{Pol}(G)} \to \text{Pol}(G) \otimes \text{Pol}(G)$, counit $\epsilon_G := \epsilon_G|_{\text{Pol}(G)}$, and antipode $S_G|_{L^\infty(G)}$, which satisfy

$$\Delta_G(u_{i,j}^\pi) = \sum_{t=1}^{n_\pi} u_{i,t}^\pi \otimes u_{t,j}^\pi, \quad \epsilon_G(u_{i,j}^\pi) = \delta_{i,j}, \quad S_G(u_{i,j}^\pi) = (u_{j,i}^\pi)^*$$

Vital to the theory of CQGs is Fourier analysis, which is essentially analysis of Fourier transform, which is the contractive embedding $L^1(\hat{G}) \to C_r^0(\hat{G})$ induced by the left regular
representation. So, a discussion about Fourier analysis involves a discussion about the duals of CQGs. We recommend [53, Section 2] as reference on the following discussion.

A **discrete quantum group (DQG)** is the Pontryagin dual of a CQG. We briefly outline how this works here. It turns out we have the decomposition \( \lambda_G = \bigoplus_{\pi \in \text{Irr}(G)} \pi = \mathcal{O}_G \) (cf. [55]). From this, we have the algebra decompositions

\[
c_{0}(\hat{G}) := \bigoplus_{\pi \in \text{Irr}(G)} M_{n_\pi} = C^0_0(\hat{G}) = C^u(\hat{G})
\]

\[
\ell^\infty(\hat{G}) := \bigoplus_{\pi \in \text{Irr}(G)} M_{n_\pi} = L^\infty(\hat{G}).
\]

In particular, the Fourier transform is the map

\[
f \mapsto \lambda_G(f) = \bigoplus_{\pi \in \text{Irr}(G)} \pi(f) \in c_{0}(\hat{G})
\]

contractively embedding \( L^1(G) \) into \( c_{0}(\hat{G}) \) as a dense subspace. The left and right Haar weights on \( \hat{G} \) are realized as the direct sums:

\[
\psi_L = \bigoplus_{\pi \in \text{Irr}(G)} \text{tr}(F_\pi) \text{tr}(F_\pi^*) \, , \quad \psi_R = \bigoplus_{\pi \in \text{Irr}(G)} \text{tr}(F_\pi) \text{tr}(F_\pi^{-1})
\]

for some positive, invertible matrices \( F_\pi \in M_{n_\pi} \), uniquely determined with the normalization \( \text{tr}(F_\pi) = \text{tr}(F_\pi^{-1}) > 0 \). For each \( \pi \in \text{Irr}(G) \), an ONB may be chosen so that \( F_\pi \) is diagonal (see [48] for general information and [10] for why they can be taken to be diagonal).

We also have a linear contraction \( L^\infty(\hat{G}) \to L^1(\hat{G}) \) given by the map \( x \mapsto \tilde{x} := h_G \cdot x \), where \((h_G \cdot x)(y) = h(xy)\). Then, if we let \( e_{i,j}^\pi \in M_{n_\pi} \) be the matrix unit with 1 in the \((i,j)\) entry,

\[
e_{i,j}^\pi = \sum_{k=1}^{n_\pi} \frac{1}{\text{tr}(F_\pi)} (F_\pi)_{i,k}^{-1} (u_{k,j}^\pi)^*.
\]

In particular, \( \lambda_G(\text{Pol}(G)) = \bigoplus_{\pi \in \text{Irr}(G)} M_{n_\pi} =: c_{00}(\hat{G}) \).

We have the following convolution formula that will be of use to us: for \( f \in L^1(G) \), \( x \in L^\infty(G) \), and \( \pi \in \text{Irr}(G) \),

\[
\pi(f \ast x) = \pi(\hat{x}) \pi(f \circ S_G^{-1}).
\]

See Remark 3.7 for the apparent domain issue in the above equation.

### 2.3 Invariant Subspaces, Ideals, and Quotients

Given a locally compact group \( G \), the von Neumann subalgebras of \( L^\infty(G) \) which are right invariant with respect to the natural action of \( L^1(G) \) on \( L^\infty(G) \) are exactly those of the form \( L^\infty(G/H) \) for a closed subgroup \( H \). Analogously, the von Neuman subalgebras of \( VN(G) \) which invariant with respect to the natural action of \( A(G) \) on \( VN(G) \) are of the form \( VN(H) \) for a closed subgroup \( H \) (cf. [25, Chapter 3.4]). Thus we define the following.

**Definition 2.2.** For a LCQG \( G \), we say a subset \( E \subseteq L^\infty(G) \) is **right invariant** if \( E \ast f \subseteq E \) for all \( f \in L^1(G) \). Likewise, we say a \( E \subseteq C^u(G) \) (or \( C^r(G) \)) is **right invariant** if \( E \ast \mu \subseteq E \) for all \( \mu \in C^u(G) \).
CQGs are amenable. Likewise, DQGs have unital Lits dual, the converse remains an open problem. For LCQGs it is not too difficult to show coamenability of a LCQG implies amenability of generalizing Leptin’s theorem for locally compact groups. On the other hand, while generally invariant state, that is, a state \( m \in L^\infty(G) \) for all \( f \in L^1(G) \).

\[ \Delta_G(N) \subseteq L^\infty(G) \overline{\otimes} N. \]

From the bipolar theorem, it is easy to see that if \( X \subseteq L^\infty(G) \) is a right invariant subspace, then \( X_\perp \) is a closed left ideal in \( L^1(G) \), and whenever \( I \) is a left ideal in \( L^1(G) \), \( I_\perp \) is a weak* closed right invariant subspace of \( L^\infty(G) \). We package this observation and similar ones into the following.

**Proposition 2.4.** The norm closed right ideals of \( L^1(G) \) are in one to one correspondence with the weak* closed right invariant subspaces of \( L^\infty(G) \) via

\[ L^1(G) \ni I \iff I_\perp \subseteq \sigma, L^\infty(G), \]

and the weak* closed right ideals of \( M^*(G) \) and \( M^u(G) \) are in one to one correspondence with the the norm closed left invariance subspaces of \( C^*_0(G) \) and \( C^u_0(G) \) via

\[ C^*_0(G), C^u_0(G) \ni X \iff X_\perp \subseteq \sigma, M^*(G), M^u(G). \]

**Remark 2.3.** It is straightforward seeing that a von Neumann subalgebra \( N \subseteq L^\infty(G) \) is right invariant if and only if

\[ \Delta_G(N) \subseteq L^\infty(G) \overline{\otimes} N. \]

Of special interest are the invariant subalgebras. If \( N \) is a right invariant von Neumann subalgebra of \( L^\infty(G) \), then we call \( N \) a right coideal. In this case we will denote \( J^1(N) = N_\perp \). Notice that \( N_* \cong L^1(G)/J^1(N) \) as Banach spaces. Here is an alternative view on the relationship between \( N \) and \( J^1(N) \) afforded by weak* closures. Using weak*-weak* continuity of the inclusion \( N \subseteq L^\infty(G) \), find the preadjoint

\[ T_N : L^1(G) \to N_* \]

Then \( J^1(N) = \text{ker}(T_N) \). Notice that whenever \( N \) is invariant, \( J^1(N) \) is a two sided ideal, and so \( T_N \) is an algebraic homomorphism, meaning \( N_* \) has a Banach algebra structure inherited from the quotient algebra \( L^1(G)/J^1(N) \).

### 2.4 Multipliers and Approximation Properties

We say a LCQG \( G \) is coamenable if \( L^1(G) \) bai and is amenable if \( L^\infty(G) \) possesses a left invariant state, that is, a state \( m \in L^\infty(G) \) such that \( f(\text{id} \otimes m) \Delta_G(x) = f(1)m(x) \) for all \( f \in L^1(G) \).

**Remark 2.5.** The Haar state for a CQG is, in particular, a left invariant state, meaning CQGs are amenable. Likewise, DQGs have unital \( L^1 \)-algebras and hence are coamenable. It also turns out that a CQG is coamenable if and only if its dual is amenable \( \mathbb{R} \), generalizing Leptin’s theorem for locally compact groups. On the other hand, while generally for LCQGs it is not too difficult to show coamenability of a LCQG implies amenability of its dual, the converse remains an open problem.
Remark 2.6. We have that $G$ is coamenable if and only if $C^0(G) \cong C^0_0(G)$ and so in this case, we will simply write $C(G)$ and similarly $M(G)$ for the (in this case) distinguished measure algebra.

We will also be interested in weakened versions of amenability. In lieu of the duality between coamenability and amenability (for CQG/DQGs), the natural choice is to weaken boundedness of a left or right bai (blai or brai) in $L^1(G)$. To discuss relevant versions of this in the literature, we must first discuss multipliers and completely bounded multipliers. We recommend [6,20] as references for the following discussions.

Definition 2.7. A left multiplier of a Banach algebra $A$ is a bounded linear right $A$-module map $m : A \to A$. We denote the left multipliers on $A$ by $M^l(A)$. We denote the completely bounded left multipliers by $M^l_{cb}(A) := M^l(A) \cap CB(A)$.

Remark 2.8. Note that the elements $m \in M^l_{cb}(A)$ are exactly those such that $m^*$ is completely bounded.

We note $M^l(A)$ is a Banach algebra, viewed as a subalgebra of $B(A)$, which has $A$ embedded contractively as an ideal via the map $a \mapsto m_a$ where $m_a(b) = ab$ for $b \in A$, and we will denote the adjoint $M^*_a := m_a^*$. Similarly, $M^l_{cb}(A)$ is a c.c. Banach algebra, viewed as an operator subspace of $CB(A)$, which has $A$ embedded into $M^l_{cb}(A)$ completely contractively. For a LCQG $G$, because $M^u(G)$ contains $L^1(G)$ as an ideal, we get that $M^u(G)$ embeds completely contractively into $M^l_{cb}(L^1(G))$ via the map $\mu \mapsto m_\mu$ where $m_\mu(f) = \mu \ast f$ for $f \in L^1(G)$ and again we denote the adjoint by $M^*_\mu$. Note also that $M^l_{cb}(G) = id$.

The double centralizers of a Banach algebra $A$ are pairs $(L, R)$ of left and right multipliers $L \in M^l(A)$ and $R \in M^r(A)$ satisfying $aL(b) = R(a)b$. We define the double centralizers by $M(A)$, which also turns out to be a Banach algebra, and has $A$ contractively embedded as an ideal via the map $a \mapsto (l_a, r_a)$ where $l_a(b) = ab$ and $r_a(b) = ba$. There is also a contractive embedding $M(A) \subseteq M^l(A)$. Similarly, we define completely bounded double centralizers of a c.c. Banach algebra $A$, which are double centralizers whose associated bounded linear maps and completely bounded. We denote the completely bounded double centralizers by $M^l_{cb}(A)$. Similarly, we have completely contractive embeddings $A \subseteq M^l_{cb}(A) \subseteq M^l_{cb}(A)$. For LCQGs, we have $M^u(G) \subseteq M^l_{cb}(L^1(G))$.

Remark 2.9. Whenever $G$ is coamenable, it is the case that $M(L^1(G)) = M^l(L^1(G)) = M(G)$. For locally compact co-groups, this property characterizes amenability. That $M(A(G)) = B(G)$ implies amenability for discrete $G$ is due to [36] (and generally is due to Losert [36]), and Losert extended this to the case of $M^l_{cb}(A(G))$ in an unpublished manuscript. For discrete $G$, however, see [5]. For a LCQG $G$ in general, we also have that the completely isometric equalities $M^l_{cb}(L^1(G)) = M^r(G) = M^r(G)$ characterizes coamenability (cf. [20]).

A first weakening, then, would be to loosen the boundedness criterion of the bai.

Definition 2.10. We say a LCQG $G$ is weakly amenable if there exists a net $(f_i) \subseteq L^1(G)$ such that $f_i \ast f \to f$ for all $f \in L^1(G)$ and $\sup_i \|f_i\|_{M^l_{cb}} < \infty$.

There is another relevant, even weaker version of amenability.

Definition 2.11. We say a LCQG $G$ has the approximation property (AP) if there exists a net $(f_i) \subseteq L^1(\hat{G})$ such that $M_{f_i} \to id_{L^\infty(\hat{G})}$ in the stable point weak* topology of $CB(L^\infty(\hat{G}))$, by which we mean, for a separable Hilbert space $\mathcal{H}$, we have $$\varphi(M_{f_i} \otimes id)(x) \to \varphi(x)$$
for all $a \in L^\infty(\hat{G}) \bar{\otimes} \mathcal{B}(\mathcal{H})$ and $\varphi \in L^1(\hat{G}) \bar{\otimes} \mathcal{B}(\mathcal{H})_*$.

We have the following description of the AP, which we point out implies $L^1(\mathcal{G})$ has a left approximate identity when we let $H = \{e\}$, where $H$ is as denoted in the following proposition. Note that when $H$ is compact, $1_H$ is the identity element in $A(H)$.

**Proposition 2.12.** For a LCQG $\mathcal{G}$, TFAE:

1. $\hat{\mathcal{G}}$ has the AP;
2. for every compact group $H$, there is a net $(f_i) \subseteq L^1(\mathcal{G})$ such that
   $$||(f_i \otimes 1_H) \ast g - g||_1 \to 0$$
   for all $g \in L^1(\hat{\mathcal{G}}) \bar{\otimes} A(H)$;
3. and there is a net $(f_i) \subseteq L^1(\mathcal{G})$ such that
   $$||(f_i \otimes 1_{SU(2)}) \ast g - g||_1 \to 0$$
   for all $g \in L^1(\hat{\mathcal{G}}) \bar{\otimes} A(SU(2))$.

**Proof.** The proof follows verbatim of the proof of [18, Theorem 1.11]: the techniques are entirely functional analytic and pass directly to LCQGs.

Note that the map $M_{cb}(L^1(\mathcal{G})) \ni m \mapsto m^* \in CB^*_l(L^\infty(\mathcal{G}))$ is a completely isometric isomorphism, where $CB^*_l(L^\infty(\mathcal{G}))$ denotes the normal completely bounded right $L^1(\mathcal{G})$-module maps on $L^\infty(\mathcal{G})$. Crann pointed out in [8, Proposition 3.2] that the work of Kraus and Ruan [31, Theorem 2.2] extends directly to CQGs to give us the following.

**Proposition 2.13.** Let $\mathcal{G}$ be a CQG. Then $\omega_{T,\varphi} \in Q^l_{cb}(L^1(\mathcal{G}))$ for all $T \in L^\infty(\mathcal{G}) \bar{\otimes} \mathcal{B}(\mathcal{H})$ and $\varphi \in L^1(\mathcal{G}) \bar{\otimes} T(\mathcal{H})$.

With Proposition 2.13 in hand, a proof verbatim to the proof in [31, Theorem 5.4] shows the following, which allows us to view the AP of the duals of CQGs within our framework of weakening coamenability. For convenience, we will supply a proof.

**Proposition 2.14.** We have that a DQG $\mathcal{G}$ has the AP if and only if there exists a net $(e_i) \subseteq L^1(\hat{\mathcal{G}})$ such that $e_i \to e^e_{\mathcal{G}}$ in the $\sigma(M^1_{cb}(L^1(\hat{\mathcal{G}})), Q_{cb}(L^1(\hat{\mathcal{G}})))$ topology on $M^1_{cb}(L^1(\hat{\mathcal{G}}))$.

**Proof.** Suppose $\mathcal{G}$ has the AP. Let $(e_i) \subseteq L^1(\hat{\mathcal{G}})$ be a net such that $M^1_{e_i}$ converges in the stable point weak* topology to id. In particular, for $A \in C^\sigma(\mathcal{G}) \otimes_{min} K(\mathcal{H})$ and $\varphi \in A(\mathcal{G}) \bar{\otimes} T(\mathcal{H})$

$$\omega_{A,\varphi}(M_{e_i}) = \varphi(M_{e_i} \otimes id)(A) \to \varphi(A) = \omega_{A,\varphi}(M_{e^e_{\mathcal{G}}})$$
which says exactly that \( e_i \to e_{i}^{\#} \) in the weak* topology on \( M_{cb}^1(L^1(\hat{G})) \).

Conversely, suppose \( e_i \to e_{i}^{\#} \) in the weak* topology on \( M_{cb}^1(L^1(\hat{G})) \). Then, for \( T \in VN(\mathcal{G})\overline{\otimes} \mathcal{B}(\mathcal{H}) \) and \( \varphi \in A(\mathcal{G})\otimes T(\mathcal{H}) \), using Proposition \ref{prop:vanishing} we have

\[
\varphi(M_{e_i} \otimes \text{id})(T) = \omega_{T,\varphi}(M_{e_i}) \to \omega_{T,\varphi}(M_{e_{i}^{\#}}) = \varphi(T)
\]

which says exactly that \( M_{e_i} \to \text{id} \) in the stable point weak* topology.

**Remark 2.15.** We are unaware of a version of Proposition \ref{prop:vanishing} for general LCQGs. To prove the result, we would require a general version of Proposition \ref{prop:vanishing} however, their proof makes essential use of the underlying Hopf algebras of CQGs and does not clearly extend to general LCQGs.

An immediate observation from Proposition \ref{prop:vanishing} is the following.

**Corollary 2.16.** A weakly amenable DQG has the AP.

3 Structure of Left Ideals

3.1 Left Ideals of \( L^1 \)-algebras

For the rest of this paper \( \mathcal{G} \) will always denote a CQG unless otherwise specified. Recall that for \( x \in \text{Pol}(\mathcal{G}) \), \( \hat{x} = h \cdot x \). So, fix a CQG \( \mathcal{G} \) and let

\[
E = \left\{ E_{\pi} \in \text{Irr}(\mathcal{G}) \right\}
\]

where each \( E_{\pi} \subseteq \mathcal{H}_{\pi} \) is a subspace (possibly trivial or all of \( \mathcal{H}_{\pi} \)). We will write

\[
I(E) = \left\{ f \in L^1(\mathcal{G}) : \pi(f)(E_{\pi}) = 0, \ E_{\pi} \in E \right\}
\]

and

\[
j(E) = I(E) \cap \text{Pol}(\mathcal{G}) = I(E) \cap \lambda_{\mathcal{G}}^{-1}(c_{00}(\hat{\mathcal{G}})),
\]

It is easy to check \( j(E) \) is a left ideal and \( I(E) \) a closed left ideal in \( L^1(\mathcal{G}) \). Then we will refer to such a set \( E \) as the **hull** of any ideal \( I \) containing \( j(E) \) and contained in \( I(E) \).

**Remark 3.1.** It should be addressed that \( I(E) \), and hence \( j(E) \), is independent of the choice of representatives in \( \text{Irr}(\mathcal{G}) \) (where the subspaces \( E_{\pi} \) are chosen up to isomorphism). Indeed, suppose \( \pi \) is unitarily equivalent to \( \rho \), and write \( (1 \otimes U^*)U(1 \otimes U) = U^{\rho} \). Note that the unitary \( U \) is a Hilbert space isomorphism \( U : \mathcal{H}_{\pi} \to \mathcal{H}_{\rho} \). Then \( \pi(f) \xi = U \rho(f)U^* \xi \) shows \( \pi(f)\xi = 0 \) if and only if \( \rho(f)U^* \xi = 0 \). In particular, we have \( UE_{\rho} = E_{\pi} \).

**Definition 3.2.** We say \( E \) is a set of synthesis if \( I(E) = j(E) \).

Before proceeding, we recall the following well-known fact.

**Lemma 3.3.** Let \( I \) be a left ideal in some matrix algebra \( M_n \). Then \( I = \left\{ A \in M_n : A(E) = 0 \right\} \) for some subspace \( E \subseteq \mathbb{C}^n \).

**Proof.** The proof can be found, for example, in \cite{19} Lemma 38.11. We will note, however, that \( E \) is the kernel of some element of \( I \). Actually, we can write \( I = \left\{ A \in M_n : \ker(A) \supseteq \ker(A_0) \right\} \) for some \( A_0 \in I \).

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Proposition 3.4. Let $\mathbb{G}$ be a CQG and $I \leq L^1(\mathbb{G})$ a left ideal. Then there is a hull $E$ such that $j(E) \subseteq I \subseteq I(E)$.

Proof. We follow the methods used for compact groups in [19]. Let $1_\pi$ be the projection onto $M_n$. Then $\pi(f) = 1_\pi \lambda_\mathbb{G}(f)$, and so from density, combined with the fact $M_n$ is finite dimensional, $\pi(L^1(\mathbb{G})) = M_n$. Consequently, $\pi(I)$ is a left ideal in $M_n$. Then, using Lemma 3.3, we can write $\pi(I) = \{\pi(f) \in M_n : f \in L^1(\mathbb{G}), \pi(f)(E_\pi) = 0\}$ for some subspace $E_\pi \subseteq \mathcal{H}_\pi$. Let $E = (E_\pi)_{\pi \in \text{Irr}(\mathbb{G})}$, where each $E_\pi \subseteq \mathcal{H}_\pi$ is the aforementioned subspace for each $\pi$. From here, it is easy to see that $I \subseteq I(E)$.

Now take $f \in j(E)$, so $\lambda_\mathbb{G}(f) = \oplus_{i=1}^n \pi_i(f)$ for some $\pi_1, \ldots, \pi_n \in \text{Irr}(\mathbb{G})$. Since $\oplus_{i=1}^n \pi_i(j(E)) = \oplus_{i=1}^n \pi_i(I)$, we can find $g \in I$ so that $\oplus_{i=1}^n \pi_i(g) = \lambda_\mathbb{G}(f)$. Set $P = \oplus_{i=1}^n I_{\pi_i}$ and let $e \in L^1(\mathbb{G})$ be such that $\lambda_\mathbb{G}(e) = P$. Then

$$\lambda_\mathbb{G}(I) \ni \lambda_\mathbb{G}(e * g) = P \lambda_\mathbb{G}(g) = \bigoplus_{i=1}^n \pi_i(g) = \lambda_\mathbb{G}(f).$$

The two-sided case is as follows.

Corollary 3.5. Let $\mathbb{G}$ be a CQG and $I \leq L^1(\mathbb{G})$ an ideal. Then there exists a hull of $\mathbb{G}$, say $E$, such that $j(E) \subseteq I \subseteq I(E)$ where each $E_\pi \in E$ is either $\mathcal{H}_\pi$ or $\{0\}$.

Proof. Following Proposition 3.4 what is left is noticing that each $E_\pi \in E$ must satisfy either $E_\pi = \mathcal{H}_\pi$ or $E_\pi = \{0\}$. Inspecting the proof of Proposition 3.4 the result follows because $\pi(I)$ is a two sided ideal $M_n$: we either have $\pi(I) = M_n$ or $\pi(I) = \{0\}$.

Given a hull $E$, we will denote

$$\text{Pol}(\mathbb{G}, E) = \{u_{\xi,\eta} : \xi \in E_\pi, \eta \in \mathcal{H}_\pi, \pi \in \text{Irr}(\mathbb{G})\},$$

where $u_{\xi,\eta} = (\text{id} \otimes w_{\eta,\xi})(U^\pi)$ and $w_{\eta,\xi}(T) = \langle T \eta, \xi \rangle$ for $T \in M_n$. Then, we will denote

$$C^r(\mathbb{G}, E) = \text{Pol}(\mathbb{G}, E)^{||\cdot||}, \quad C^0(\mathbb{G}, E) = \text{Pol}(\mathbb{G}, E)^{||\cdot||_w}, \quad \text{and } L^\infty(\mathbb{G}, E) = C^r(\mathbb{G}, E)^{wk*}.$$

Proposition 3.6. Let $\mathbb{G}$ be a CQG and $E$ a hull. Then

1. $j(E) = \left\{x \in L^\infty(\mathbb{G}) : \text{im}(\pi(f \circ S_{\mathbb{G}^{-1}})) \subseteq \ker((\text{id} \otimes f)(\Delta_\mathbb{G})), \ f \in I(E), \ \pi \in \text{Irr}(\mathbb{G})\right\}$

2. and $I(E)^{wk*} = \left\{u_{\xi,\eta} \in \text{Pol}(\mathbb{G}) : \pi(f)(\xi) \neq 0, \ f \in I(E), \ \pi \in \text{Irr}(\mathbb{G}), \xi \in E_\pi\right\}^{wk*}$.

Remark 3.7. Because the symbol $f \circ S_{\mathbb{G}^{-1}}$ is defined only for $f \in L^1(\mathbb{G})$, an explanation of the notation in Proposition 3.6 is in order. We set

$$\pi(f \circ S_{\mathbb{G}^{-1}}) := (f \otimes \text{id})(S_{\mathbb{G}^{-1}} \otimes \text{id})U^\pi$$

which is defined because $S_{\mathbb{G}^{-1}}|_{\text{Pol}(\mathbb{G})} : \text{Pol}(\mathbb{G}) \to \text{Pol}(\mathbb{G})$ is a bijection.
Proof. 1. We will first notice

\[
\{ x \in L^\infty(\mathbb{G}) : \text{im}(\pi(f \circ S^{-1}_G)) \subseteq \ker(\pi(\hat{x})), \ f \in I(E), \pi \in Irr(\mathbb{G}) \}^{wk*}
\]

\[
= \{ x \in L^\infty(\mathbb{G}) : \text{im}(\pi(f \circ S^{-1}_G)) \subseteq \ker(\pi(\hat{x})), \ f \in j(E), \pi \in Irr(\mathbb{G}) \}^{wk*}
\]

and then will show

\[
j(E)^\perp = \{ x \in L^\infty(\mathbb{G}) : \text{im}(\pi(f \circ S^{-1}_G)) \subseteq \ker(\pi(\hat{x})), \ f \in j(E), \pi \in Irr(\mathbb{G}) \}^{wk*}
\]

Accordingly, suppose \( x \in L^\infty(\mathbb{G}) \) satisfies \( \text{im}(\pi(h \circ S^{-1}_G)) \subseteq \ker(\pi(\hat{x})) \) for all \( h \in j(E) \). For \( f \in I(E) \), find \( g \in j(E) \) such that \( \pi(f \circ S^{-1}_G) = \pi(g \circ S^{-1}_G) \). Then

\[
\pi(\hat{x}) \pi(f \circ S^{-1}_G) = \pi(\hat{x}) \pi(g \circ S^{-1}_G) = 0,
\]

as desired. The reverse containment is obvious.

Moving on to the main proof, let \( x \in j(E)^\perp \). Then, for \( f \in j(E) \),

\[
f \ast x = (\text{id} \otimes f)\Delta_G(x) = 0
\]

thanks to right invariance of \( j(E)^\perp \). Therefore,

\[
0 = \pi(f \hat{\ast} x) = \pi(\hat{x}) \pi(f \circ S^{-1}_G),
\]

(1)

in other words, \( \text{im}(\pi(f \circ S^{-1}_G)) \subseteq \ker(\pi(\hat{x})) \).

Conversely, take \( x \in L^\infty(\mathbb{G}) \) such that \( \text{im}(\pi(f \circ S^{-1}_G)) \subseteq \ker(\pi(\hat{x})) \) for all \( f \in j(E) \).

Following equation (1) in reverse tells us \( f \ast x = 0 \). Since \( f \in \lambda_G^{-1}(c_0(\hat{\mathbb{G}})) \), we can find \( \pi_1, \ldots, \pi_n \in Irr(\mathbb{G}) \) so that \( \pi(f) = 0 \) if \( \pi_i \neq \pi \in Irr(\mathbb{G}) \) for all \( i \). Let \( \epsilon_{\oplus_{i=1}^n \pi_i} = \lambda_G^{-1}(1_{\oplus_{i=1}^n M_{\pi_i}}) \) via the identification \( \oplus_{i=1}^n \pi_i \subseteq c_0(\hat{\mathbb{G}}) \). Then

\[
\lambda_G(\epsilon_{\oplus_{i=1}^n \pi_i} \ast f) = (\oplus_{i=1}^n \pi_i)(\epsilon_{\oplus_{i=1}^n \pi_i})(\oplus_{i=1}^n \pi_i)(f) = \lambda_G(f),
\]

so \( \epsilon_{\oplus_{i=1}^n \pi_i} \ast f = f \). Similarly, \( f \ast \epsilon_{\oplus_{i=1}^n \pi_i} = f \). Therefore,

\[
0 = \epsilon_{\oplus_{i=1}^n \pi_i}(f \ast x) = f(x),
\]

as desired.

Now we justify

\[
j(E)^\perp = \bigcap_{f \in I(E)} \ker((\text{id} \otimes f)\Delta_G).
\]

All of the work for this part of the claim has already been done. For \( x \in j(E)^\perp \) and \( f \in I(E) \), notice that we can repeat the steps of the converse above to get \( f \ast x = 0 \).

Conversely, notice the forward implication above actually depended on having \( f \ast x = 0 \).

2. For \( \pi \in Irr(\mathbb{G}) \), pick an ONB \( \{ e^*_j \} \) by choosing an ONB for \( E_\pi \) and then extending it to \( H_\pi \). Then for \( f \in I(E) \) we have

\[
0 = \pi(f)(E_\pi) = [f(u_{i,j}^E)](E_\pi)
\]

if and only if \( f(u_{i,j}^E) = 0 \) for every \( e^*_j \in E_{\pi} \). The rest is clear. \( \square \)
Corollary 3.8. Let $\mathbb{G}$ be a CQG. If $X \leq_{\ell} L^\infty(\mathbb{G})$ is a right invariant subspace, then there exists a hull $E$ such that

$$L^\infty(\mathbb{G}, E) \subseteq X \subseteq \bigcap_{j \in I(E)} \ker((\id \otimes f)\Delta_G).$$

Proof. This follows immediately from Proposition 3.6 and Proposition 3.4.

Before getting to the main theorem, we still need to think about the singly generated ideals.

Lemma 3.9. Let $\mathbb{G}$ be a CQG. Fix $f \in L^1(\mathbb{G})$ and let $E$ be the hull of $\mathbb{G}$ associated with the closed principal left ideal $L^1(\mathbb{G}) \ast f$. The following hold:

1. we have $E_\pi = \ker(\pi(f))$ for each $E_\pi \in E$;
2. $f \in I(E)$;
3. if $E$ is a set of synthesis, then $f \in L^1(\mathbb{G}) \ast f$;
4. $L^1(\mathbb{G}) \ast f = \ker((\id \otimes f)\Delta_G)$.

Proof. 1. This follows easily from the fact $\pi(g \ast f)(E_\pi) = \pi(g)\pi(f)(E_\pi) = 0$ for each $g \in L^1(\mathbb{G})$.

2. This follows immediately by definition of $I(E)$ and from 1.

3. If $E$ is a set of synthesis, then from 1., 2., and Proposition 3.4

$$j(E) = L^1(\mathbb{G}) \ast f = I(E) \ni f.$$

4. If $f \ast x = 0$, then $g \ast f(x) = g(f \ast x) = 0$ for each $f \in L^1(\mathbb{G})$, that is, $x \in (L^1(\mathbb{G}) \ast f)_\perp$. Conversely, if $x \in (L^1(\mathbb{G}) \ast f)_\perp$, then $0 = g \ast f(x) = g(f \ast x)$ for all $g \in L^1(\mathbb{G})$, which implies $f \ast x = 0$.

Theorem 3.10. Let $\mathbb{G}$ be a CQG. Then every hull is a set of synthesis if and only if $f \in L^1(\mathbb{G}) \ast f$ for all $f \in L^1(\mathbb{G})$.

Proof. If we assume every hull is a set of synthesis, then in particular, from Lemma 3.9 we have $f \in L^1(\mathbb{G}) \ast f$ for every $f \in L^1(\mathbb{G})$. Conversely, because of Proposition 3.6 all we need to show is $I(E) \subseteq \bigcap_{f \in I(E)} \ker((\id \otimes f)\Delta_G)_\perp$. So, take $f \in I(E)$ and let $x \in L^\infty(\mathbb{G})$ satisfy $f \ast x = 0$. Find a net $(g_i) \subseteq L^1(\mathbb{G})$ such that $g_i \ast f \to f$. Then

$$0 = g_i(f \ast x) = g_i \ast f(x) \to f(x).$$

From Lemma 3.4 and Theorem 3.10 we immediately conclude the following.

Corollary 3.11. Let $\mathbb{G}$ be a CQG such that $f \in L^1(\mathbb{G}) \ast f$ for all $f \in L^1(\mathbb{G})$. The closed left ideals of $L^1(\mathbb{G})$ are of the form $I(E)$ for some hull $E$.

In light of Theorem 3.10 we make the following definition.

Definition 3.12. We say a LCQG $\mathbb{G}$ has Ditkin’s left property at infinity (or property left $D_\infty$), if $f \in L^1(\mathbb{G}) \ast f$ for every $f \in L^1(\mathbb{G})$. 

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3.2 weak∗ Closed Left Ideals of Measure Algebras

The main result of this subsection is that we achieve a characterization of the weak∗ closed ideals of the measure algebra of a coamenable CQG. Essentially, we will show White’s techniques [54] generalize to the setting of CQGs. Before getting to this, however, we will begin by discussing some more general things about ideals of measure algebras of (not necessarily coamenable) CQGs.

First note, from the hulls of a CQG $G$, we identify the closed left ideals $I(E)$ of $L^1(G)$. Then, using that $L^1(G) \subseteq M^r(G), M^u(G)$ isometrically as an ideal, we identify the weak∗ closed left ideals $I(E)^{wke} \subseteq M^r(G)$ and $I(E)^{wke} \subseteq M^u(G)$. As we will see shortly, for coamenable $G$, this process identifies all weak∗ closed left ideals of $M(G)$. We are interested in another process to find weak∗ closed left ideals in $M^u(G)$ and $M^r(G)$.

Because we have the embedding $\text{Pol}(G) \subseteq C^u(G)$, we can immediately extend $\pi \in \text{Irr}(G)$ to a representation $\pi : M^u(G) \to M_{n_\pi}$ by setting

$$\pi(\mu) = (\mu \otimes \text{id})(U^\pi) = [\mu(u^\pi_{i,j})].$$

With this in hand, given a hull $E$, we will define

$$I^u(E) = \{\mu \in M^u(G) : \pi(\mu)(E_\pi) = 0, \ E_\pi \in E\}$$

and

$$I^r(E) = \{\mu \in M^r(G) : \pi(\mu)(E_\pi) = 0, \ E_\pi \in E\},$$

which are both easily checked to be weak∗ left closed ideals in $M^u(G)$ and $M^r(G)$ respectively. We also have the following.

**Proposition 3.13.** Let $G$ be a CQG. Then $I^u(E) = C^u(G, E)^\perp$ and $I^r(E) = C^r(G, E)^\perp$.

**Proof.** The proof follows similarly to the analogous result in Proposition 3.6.

Now we will work towards the main result. The techniques involve exploiting the following sort of objects.

**Definition 3.14.** A Banach algebra $A$ is **compliant** if $M(A)$ is a dual space and the maps $M(A) \to A$, $\mu \mapsto \mu a$ and $\mu \mapsto a\mu$, for $a \in A$, are weak∗-weakly continuous.

Recall, for coamenable $G$ we have $M(G) = M^l(L^1(G))$ (cf. [20]).

**Proposition 3.15.** Let $G$ be a coamenable LCQG. Then $L^1(G)$ is compliant if and only if $G$ is compact.

**Proof.** According to [53, Proposition 5.8 (i)], compliance of $L^1(G)$ implies it is an ideal in $L^1(G)^{**}$. Then [46, Theorem 3.8] implies $G$ is compact. Conversely, if $G$ is compact then, thanks to [43, Theorem 2.3],

$$C(G) = L^1(G) * L^\infty(G) = L^\infty(G) * L^1(G)$$

where we have used Cohen’s factorization theorem. From here, [54, Lemma 5.7] says $L^1(G)$ is compliant.

**Theorem 3.16.** Let $G$ be a coamenable CQG. The weak∗ closed left ideals of $M(G)$ are of the form $I^u(E)$ for some hull $E$ of $G$. 

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Proof. Since $L^1(\mathbb{G})$ is compliant, because of \cite[Theorem 5.10]{F}, the weak* closed left ideals of $M(\mathbb{G})$ are of the form $T^{w^*}$ for a closed left ideal $I$ of $L^1(\mathbb{G})$. Now apply Corollary \ref{3.11} to get that $I = I(E)$ for some hull $E$. By definition $L^1(\mathbb{G}) \cap I^u(E) = I(E)$, and so using \cite[Theorem 5.10]{F}, we get $I^u(E) = T^{w^*}$ as desired.

Corollary 3.17. Let $\mathbb{G}$ be a coamenable CQG. The closed right invariant subspaces of $C(\mathbb{G})$ are of the form $C(\mathbb{G}, E)$ for some hull $E$.

Proof. This follows from Theorem \ref{3.10} and Proposition \ref{3.6}.

3.3 Ditkin’s Property at Infinity and Examples

Even for locally compact groups, property left $D_\infty$ is a rather opaque condition and, to our knowledge, there are no known examples of locally compact groups with property left $D_\infty$ \cite[Section 6.7]{F}. Recently a characterization of property left $D_\infty$ for locally compact co–groups has been obtained by Andreou \cite{A}. Using the techniques developed there, Andreou obtained a new proof that AP implies property left $D_\infty$ using techniques based around Fubini tensor products (a result which may also be read from \cite[Theorem 1.11]{E}). For this section, we will write down some basic equivalent formulations of property left $D_\infty$ (which we were recorded by Andreou for locally compact groups). Then we will provide examples of CQGs with property left $D_\infty$.

We will say $x \in L^\infty(\mathbb{G})$ satisfies condition $(H)$ if $x \in L^1(\mathbb{G}) \ast x^{w^*} \subseteq L^\infty(\mathbb{G})$.

Proposition 3.18. Let $\mathbb{G}$ be a CQG. TFAE:

1. $\hat{\mathbb{G}}$ has property left $D_\infty$;
2. every $x \in L^\infty(\mathbb{G})$ satisfies condition $(H)$;
3. for $x \in L^\infty(\mathbb{G})$ and $f \in L^1(\mathbb{G})$, $f \ast x = 0$ implies $f(x) = 0$;
4. and for all $X \preceq_r L^\infty(\mathbb{G})$ we have $x \ast f \in X$ for all $f \in L^1(\mathbb{G})$ if and only if $x \in X$.

Proof. First, (4 $\implies$ 2) follows verbatim to the corresponding statement in \cite[Proposition 6.7]{A}. Now we note that commutativity of the Fourier Algebra appears in the proof of the corresponding statement in \cite[Proposition 6.7]{A} of (1 $\implies$ 3), so we must supply our own proof in the CQG setting here to obtain (1 $\iff$ 2 $\iff$ 3). With that said, (2 $\implies$ 1) does follow verbatim from \cite[Proposition 6.7]{A} and the converse follows from a similar Hahn–Banach argument. Then (1 $\iff$ 3) follows from the observation

$$f \in L^1(\mathbb{G}) \ast f \iff \ker(f \otimes \text{id}) \Delta_\mathbb{G} = L^1(\mathbb{G}) \ast f^\perp \subseteq \{f\}^\perp.$$ 

To reiterate, we have (4 $\implies$ 3 $\iff$ 2 $\iff$ 1).

For (3 $\implies$ 4), take $f \in L^1(\mathbb{G})$, so $x \ast f \in X$, which means for $g \in X_\perp$ that

$$0 = g(x \ast f) = f(g \ast x).$$

Since $f \in L^1(\mathbb{G})$ was arbitrary, we deduce that $g \ast x = 0$, which means $g(x) = 0$, that is $x \in X$ as desired.

The following is an immediate consequence of Proposition \ref{2.12}.

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Proposition 3.19. If a DQG $\hat{G}$ has the AP, then it has property left $D_{\infty}$.

Now we point out weakly amenable examples in the literature.

Example 3.20. The duals of the free unitary and orthogonal compact quantum groups $U_k^+$ and $O_k^+$, quantum permutation groups $S_n^+$, and quantum reflection groups $H_n^{+,(s)}$ are all weakly amenable [6].

4 Coamenability and Ideals

4.1 Compact Quasi–Subgroups

In this section, our main goal is to understand the left ideals of the $L^1$–algebra of a CQG which admit a braï, paying special attention to those coming from the compact quasi–subgroups, the quantum analogue of a compact subgroup. We will also consider briefly related left ideals in the measure algebras. Note that $G$ we will always denote a CQG unless otherwise specified.

We begin by recalling the following result of [37].

Theorem 4.1. [37, Theorem 3.1] Let $G$ be a LCQG and $I \triangleleft L^1(G)$ be a closed left ideal. Then $I$ has a braï only if there exists a right $L^1(G)$–module projection $L^\infty(G) \to I^\perp$.

Proof. See the corresponding reference for a proof. We will point out, however, that the projection onto $I^\perp$ is of the form $x \mapsto e^* x$, where $e \in L^\infty(G)$ is a weak $^*$ cluster point of the given braï, and $e^* x$ denotes the natural action of $L^\infty(G)^*$ on $L^\infty(G)$.

Remark 4.2. According to [21, Theorem 3.1], (recall, $G$ is compact) the bounded right $L^1(G)$–module maps $B_R L^1(G)(L^\infty(G))$ are normal, i.e., we have

$$B_R L^1(G)(L^\infty(G)) = \{ (m)^* : m \in M^1(L^1(G)) \}.$$ 

Now, if $I$ is a closed left that has a braï with weak$^*$ cluster point $e \in L^\infty(G)^*$ (afforded by Banach–Alaoglu), then we get $I = L^1(G)^* e$, where $f * e$ denotes the natural action of $f$ on $L^\infty(G)^*$ (see also the proof of [37, Theorem 2.2]).

In light of Theorem 4.1 and Remark 4.2, we will be focusing on those left ideals admitting braïs with which their corresponding right $L^1(G)$–module projections are positive.

Definition 4.3. A compact quasi–subgroup of $G$ is a right coideal $N$ of $L^\infty(G)$ such that there exists a normal right $L^1(G)$–module conditional expectation $E^R_N : L^\infty(G) \to N$.

Equivalently we can find such a $^*$–homomorphism $\pi^u_G : C^u(G) \to C^u(H)$ satisfying

$$\Delta^u_G \circ \pi^u_H = (\pi^u_G \otimes \pi^u_H) \Delta^u_G.$$

Equivalently we can find such a $^*$–homomorphism $\pi^u_H : \operatorname{Pol}(G) \to \operatorname{Pol}(H)$. The quotient space $G/H$ is defined by setting

$$\operatorname{Pol}(G/H) = \{ a \in \operatorname{Pol}(G/H) : (\operatorname{id} \otimes \pi^u_G) \Delta_G(a) = a \otimes 1 \}.$$

We denote $L^\infty(G/H) = \overline{\operatorname{Pol}(G/H)}^{\text{wk}^*}$ etc.
The quotients associated with closed quantum subgroups of a CQG fall under the umbrella of compact quasi-subgroups. Using the embedding $(\pi_{\mathbb{H}})^* : M^u(\mathbb{H}) \to M^u(\mathbb{G})$ we find that $\omega_{\mathbb{G}/\mathbb{H}} = h_{\mathbb{H}} \circ \pi_{\mathbb{H}} \in M^u(\mathbb{G})$ is an idempotent state. Then the adjoint of the map $f \mapsto f * \omega_{\mathbb{G}/\mathbb{H}}$ is a normal right $L^1(\mathbb{G})$-module conditional expectation $L^\infty(\mathbb{G}/\mathbb{H})$. 

**Remark 4.4.** 1. It turns out that the compact quasi–subgroups are in 1–1 correspondence with the idempotent states in $M^u(\mathbb{G})$ (cf. [28]). The relationship is as follows: the compact quasi–subgroups take the form $M^l_{\omega}(L^\infty(\mathbb{G}))$ where $\omega \in M^u(\mathbb{G})$ is an idempotent state and we recall that $M^l_{\omega}$ is the adjoint of the left multiplier $m^l_{\omega} \in M^l(L^1(\mathbb{G}))$ associated with $\omega$ (cf. Section 2.6).

Note in particular that if $\mathbb{H}$ is a closed quantum subgroup of $\mathbb{G}$, then $L^\infty(\mathbb{G}/\mathbb{H}) = M^l_{\mathbb{H} \circ \pi_{\mathbb{H}}}(L^\infty(\mathbb{G}))$, so $L^\infty(\mathbb{G}/\mathbb{H})$ is a compact quasi-subgroup of $\mathbb{G}$. We also point out that

$$M^l_{\omega_N}(L^\infty(\mathbb{G}))(1) = \{ f \in L^1(\mathbb{G}) : ||f||_1 \leq 1 \}$$

where $\omega_N$ is the idempotent state associated with $N$ (cf. [11]).

2. It is a theorem of Kawada and Itô [29] that the closed subgroups of a compact group $G$ are in 1–1 correspondence with the idempotent states in $M^u(\mathbb{G})$, where $L^\infty(G/H) = M^l_{\omega}(L^\infty(\mathbb{G}))$. Likewise, for a discrete group $\Gamma$, we have a 1–1 correspondence between subgroups and idempotent states in $B(\Gamma)$ via $\Lambda \leq \Gamma \iff 1_\Lambda \in B(\Gamma)$, and we have $VN(\Gamma) \cdot 1_\Lambda = VN(\Lambda)$ (see [22]). So, we can view the notion of a compact quasi–subgroup as the quantum analogue of the quotient algebra of a closed subgroup of a compact group / group von Neumann algebra of a subgroup of a discrete group. Note that, in general, not all compact quasi–subgroups arise from closed quantum subgroups. We can see this even for discrete co–groups as any non–normal subgroup (of the underlying discrete group) gives rise to a compact quasi–subgroup but not a closed quantum subgroup. A non-cocommutative example can be found in [42].

3. Not every right coideal of a CQG is a compact quasi-subgroup. The Podleś spheres of $SU_q(2)$, where $q \neq 2$, give uncountably many examples for a fixed $q$ (see [13]).

Recall, for an invariant subalgebra $N_1$, $N_*$ has a Banach algebra structure inherited directly from the quotient $L^1(\mathbb{G})/J^1(N) \cong N_*$. If we assume $N_*$ has a bai (so in the case $N = L^\infty(\mathbb{G}/\mathbb{H})$ for some normal closed quantum subgroup $\mathbb{H}$, $\mathbb{G}/\mathbb{H}$ is coamenable), then we can easily transfer bais between $L^1(\mathbb{G})$ and $J^1(N)$ from the results found in [11] and [13].

**Proposition 4.5.** Let $I \subseteq L^1(\mathbb{G})$ be a closed two–sided ideal and suppose $L^1(\mathbb{G})/I$ has a bai. Then $\mathbb{G}$ is coamenable if and only if $I$ has a bai.

**Proof.** If $L^1(\mathbb{G})/I$ and $I$ both have bais, then we can build a bai for $L^1(\mathbb{G})$ [11 Pg. 43]. The converse is covered by the more general fact that given a Banach algebra $A$ which has a bai and closed left ideal $J$, $J$ has a bai if and only if $J^\perp$ is right invariantly complemented in $A^*$, i.e., there is a right $A$–module projection $P : A^* \to J^\perp$ (cf. [13] 4.1.4 Pg. 42). 

Now, given a compact quasi–subgroup $N$, we will denote the corresponding idempotent state by $\omega_N$. Then, for compact $\mathbb{G}$, we have a projection

$$\bar{E}_N^R := (E_N^R)|_{Pol(\mathbb{G})} = (\id \otimes \omega_N)\Delta_{\mathbb{G}} : Pol(\mathbb{G}) \to Pol(N) := \bar{E}_N^R(Pol(\mathbb{G}))$$
onto a right invariant subalgebra of $\text{Pol}(G)$ satisfying

$\overline{\text{Pol}(N)^+} = N$.

See also [10] Section 2 for a discussion in the case of CQGs.

**Remark 4.6.** Wang [51] showed that normality is equivalent to having $[\omega_{L^\infty(G/H)}(u_{i,j}^\pi)] = I_{n_\pi}$ or 0 for all $\pi \in \text{Irr}(G)$, from which it was also shown for normal $H$,

$$\text{Pol}(G/H) = \text{Pol}(G, E_H)$$

where $E_H = (E_\pi)_{\pi \in \text{Irr}(G)}$ is the hull such that $E_\pi = \mathcal{H}_\pi$ if $[\omega_{L^\infty(G/H)}(u_{i,j}^\pi)] = I_{n_\pi}$ and $E_\pi = \{0\}$ otherwise. In particular, $L^\infty(G, E_H) = L^\infty(G/H)$.

The above remark generalizes to the following for compact quasi–subgroups (and uses the same techniques as Wang).

**Lemma 4.7.** Let $N$ be a compact quasi–subgroup. Then there exists an orthonormal basis $\{e_i^\pi\}$ of $\mathcal{H}_\pi$ so that $u_{i,j}^\pi \in N$ if and only if $\omega_N(u_{i,j}^\pi) = 1$, and

$$\text{Pol}(N) = \text{span}\{u_{i,j}^\pi : 1 \leq i \leq n_\pi, e_j^\pi \in E_\pi\}.$$  

**Proof.** Fix $\pi \in \text{Irr}(G)$. Since $\omega_N$ is an idempotent state, $\pi(\omega_N)$ is an orthogonal projection. Choose an ONB $\{e_i^\pi\}$ so that $\pi(\omega_N)$ is diagonal, so, $\omega_N(u_{i,j}^\pi) = \delta_{i,j}$ or 0. If $u_{i,j}^\pi \in N$, then

$$u_{i,j}^\pi = E_N^\pi(u_{i,j}^\pi) = (\text{id} \otimes \omega_N)\Delta_N(u_{i,j}^\pi) = \omega_N(u_{i,j}^\pi)u_{i,j}^\pi$$

implies $\omega_N(u_{i,j}^\pi) = 1$ and otherwise, $u_{i,j}^\pi \neq \omega_N(u_{i,j}^\pi)u_{i,j}^\pi$, which means $\omega_N(u_{i,j}^\pi) = 0$.

Notice that we have shown $E_N^R(u_{i,j}^\pi) = u_{i,j}^\pi$ or 0. The second claim follows.

**Corollary 4.8.** Let $G$ be a CQG and $N$ a compact quasi–subgroup with hull $E_N$. Then $\text{Pol}(G, E_N) = \text{Pol}(N)$, and furthermore, $L^\infty(G, E_N) = N$.

Now fix a compact quasi–subgroup $N$. We will build from it canonical “continuous function spaces” and “measure spaces”. Accordingly, we will define

$$C^u(N) := \overline{\text{Pol}(N)^+}$$

and

$$C^r(N) := \Gamma_G(C^u(N)),$$

where we recall that $\Gamma_G : C^u(G) \rightarrow C^r(G)$ is the reducing morphism, and so, by definition, we have a surjective $\ast$–homomorphism $\Gamma_G|_{C^u(N)} : C^u(N) \rightarrow C^r(N)$. Note that since $\Gamma_G(\text{Pol}(N)) \subseteq N$, we have $C^r(N) \subseteq N$ and by weak density of $\text{Pol}(N)$ in $N$, we have $C^u(N)^+ = N$. We also have the right $M^u(G)$–module conditional expectation

$$E_{C^u(N)}^R := (\text{id} \otimes \omega_N)\Delta_N^u : C^u(G) \rightarrow C^u(N).$$

Then we will set

$$M^u(N) := C^u(N)^+$$

and

$$M^r(N)^+ := C^r(N).$$

Then, by definition, the adjoint is a completely isometric embedding:

$$(\Gamma_G|_{C^u(N)})^* : M^r(N) \rightarrow M^u(N).$$

Now, by taking the adjoint of the inclusion $C^u(N) \subseteq C^u(G)$, we obtain a surjective weak$^*$–weak$^*$ continuous linear map

$$T_N^u : M^u(G) \rightarrow M^u(N)$$

whose kernel we denote by $J^u(N)$, which of course satisfies $J^u(N) = C^u(N)^\perp$. 

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Remark 4.9. Note that in the case of a quotient $G/H$, it is not hard to show that

$$C^u(G/H) := \{ a \in C^u(G) : (\id \otimes \pi^u_H) \Delta^u_G(a) = a \otimes 1 \} = C^u(L^\infty(G/H)).$$

Furthermore, we will denote $C^r(G/H) = C^r(L^\infty(G/H))$ etc.

For the moment we will consider quotients of closed quantum subgroups. The following notion was formulated in [23].

Definition 4.10. For a CQG $G$, we say a quotient $G/H$ is coamenable if $\pi^u_H : C^u(G) \to C^u(H)$ admits a reduced version, that is, there exists $\pi^r_H : C^r(G) \to C^r(H)$ such that $\Gamma_H \circ \pi^u_H = \pi^r_H \circ \Gamma_G$ (where $\Gamma_G : C^u(G) \to C^r(G)$ is the reducing morphism).

Now we state a useful necessary condition for coamenability of a quotient motivating the condition $\omega_N \in M^r(G) = L^1(G) - \ker$, which we will be using for the main results of this subsection.

Proposition 4.11. Let $G$ be a CQG and $H$ a closed quantum subgroup. If $G/H$ is coamenable, then $\omega_{L^\infty(G/H)} \in M^r(G) \subseteq M^u(G)$.

Proof. Recall that $\omega_{L^\infty(G/H)} = h^u_H \circ \pi^u_H$ and the completely isometric embedding $M^r(G) \subseteq M^u(G)$ is given by the adjoint of $\Gamma_G$. Recall also that we can factorize $h^u_H = h^u_{GH} \circ \Gamma_H$. Then

$$M^r(G) \ni h^u_H \circ \pi^u_H \circ \Gamma_G = h^u_{GH} \circ \Gamma_H \circ \pi^u_H = h^u_{GH} \circ \pi^u_H.$$

So, by assuming $\omega_N \in M^r(G)$, we know that this condition holds at least for coamenable quotients (compare Corollary 4.10 with Proposition 4.9). Next we take a look at the associated left ideals in $M^u(G)$.

Proposition 4.12. Let $G$ be a CQG and $N$ a compact quasi-subgroup. Then $J^u(N)$ has a right unit.

Proof. Let $G$ be a CQG and $N$ a compact quasi-subgroup. First notice that for $\mu \in J^u(N)$ and $a \in C^u(G)$,

$$0 = \mu(E^R_{C^u(N)}(a)) = \mu((\id \otimes \omega_N) \Delta^u_G(a)) = \mu \omega_N(a)$$

Therefore, $\mu * (\epsilon^u_G - \omega_N) = \mu$ for all $\mu \in J^u(N)$. Finally, by choosing an ONB as in Lemma 4.7

$$(\epsilon^u_G - \omega_N)(u^\pi_{i,j}) = \delta_{i,j} - \delta_{i,j} = 0$$

for all $u^\pi_{i,j} \in \Pol(N)$. Then, from density of $\Pol(N)$ in $C^u(N)$, we have $\epsilon^u_G - \omega_N = 0$, that is, $\epsilon^u_G - \omega_N \in J^u(N)$.

Corollary 4.13. Let $G$ be a CQG and $N$ an invariant subalgebra. Then $J^u(N)$ has an identity element.

Proof. A similarly proof to Proposition 4.12 shows $\epsilon^u_G - \omega_N$ is also a left identity.

Accordingly, we will denote the right (or two-sided when appropriate) identity of $J^u(N)$ by $e^u$. Notice then that

$$J^u(N) = J^u(N) * e^u \subseteq M^u(G) * e^u \subseteq J^u(N),$$
So
\[ J^u(N) = M^u(\mathcal{G}) \ast e^u. \]
A natural question to ask is, when can we approximate \( e^u \) from \( J^1(N) \)? In particular, how does this relate to the existence of a brai in \( J^1(N) \)? The answer is as follows.

**Theorem 4.14.** Let \( N \) be a compact quasi–subgroup. If \( J^1(N) \) has a brai then \( \overline{J^1(N)}^{wk^*} = J^u(N) \), where the weak* topology is the one induced by \( C^u(\mathcal{G}) \).

**Proof.** Assume \( J^1(N) \) has a brai \( (e_j) \) and pass to a weak* convergent subnet with limit point \( e \in \overline{J^1(N)}^{wk^*} \subseteq M^u(\mathcal{G}) \). Before proceeding with the proof, we point out some intermediate facts. We will first show \( \overline{J^1(N)}^{wk^*} = M^u(\mathcal{G}) \ast e \). Since \( L^1(\mathcal{G}) \) is an ideal in \( M^u(\mathcal{G}) \), for \( \mu \in M^u(\mathcal{G}) \) and \( f \in J^1(N) \) we have \( \mu \ast f \ast e_j \in J^1(N) \) for all \( j \in J^1(N) \), from which we conclude that \( \mu \ast f \in J^1(N) \). In particular, we have \( \mu \ast e_j \in J^1(N) \) for all \( j \) and so by taking limits, \( \mu \ast e \in \overline{J^1(N)}^{wk^*} \).

Next we will show
\[ J^u(N) \ast e = \overline{J^1(N)}^{wk^*}. \tag{2} \]
Clearly \( J^1(N) \subseteq J^u(N) \), from which we immediately deduce \( e \ast e^u = e \). Then,
\[ J^1(N) \subseteq J^u(N) \ast e = J^u(N) \ast e \ast e \subseteq M^u(\mathcal{G}) \ast e = \overline{J^1(N)}^{wk^*}, \]
using that \( M^u(\mathcal{G}) \) is an ideal in \( M^u(\mathcal{G}) \), and so \( J^u(N) \ast e = \overline{J^1(N)}^{wk^*} \) as desired.

Set \( \omega_N^r = e^u_G - e \). For \( f \in L^1(\mathcal{G}) \), we have
\[ f \circ M^l_{\omega_N^r} = 0 \iff f \ast \omega_N^r = 0 \iff f \ast e = f. \]
So, \( f \circ M^l_{\omega_N^r} = 0 \) for all \( f \in J^1(N) \), which implies \( M^l_{\omega_N^r} (L^\infty(\mathcal{G})) \subseteq N \). Then, since \( M^l_{\omega_N^r} |_N = \text{id}_N \),
\[ M^l_{\omega_N^r} \ast \omega_N = M^l_{\omega_N^r} \circ M^l_{\omega_N^r} = M^l_{\omega_N^r}. \]
Recall from the proof of Proposition 4.12 that \( \omega_N = e^u_G - e^u \). Then, by injectivity of \( \mu \mapsto M^l_{\mu} \), we get
\[ (e^u_G - e) \ast (e^u_G - e^u) = \omega_N \ast \omega_N = \omega_N^r = e^u_G - e, \]
from which we have \( e \ast e^u = e^u \). Therefore, using (2),
\[ \overline{J^1(N)}^{wk^*} = \overline{J^1(N)}^{wk^*} \ast e^u = J^u(N) \ast e \ast e^u = J^u(N) \ast e^u = J^u(N). \]
\[ \square \]

Our question of weakly approximating elements of \( J^u(N) \) by elements of \( J^1(N) \) turns out to relate to coamenability of \( \mathcal{G} \).

**Theorem 4.15.** Let \( \mathcal{G} \) be a CQG and \( N \) a compact quasi–subgroup. If \( \mathcal{G} \) is coamenable then \( \overline{J^1(N)}^{wk^*} = J^u(N) \). Conversely, if \( \omega_N \in M^u(\mathcal{G}) \) and \( \overline{J^1(N)}^{wk^*} = J^u(N) \), then \( \mathcal{G} \) is coamenable.
Proof. Assume $\mathbb{G}$ is coamenable. We first note that $C^u(N) = C^r(N)$, so we will simply write $C(N)$. Because of [54, Theorem 5.10] (cf. Theorem 3.16), it suffices to show $J^u(N) \cap L^1(\mathbb{G}) = J^1(N)$. First, clearly $J^1(N) \subseteq J^u(N)$. For the reverse containment, take $a \in C(\mathbb{G})$. Then for $a \in C(N)$ and $f \in J^u(N) \cap L^1(\mathbb{G})$,

$$0 = T_N^u(f)(a) = f(a) = T_N(f)(a)$$

which implies $f(N) = 0$ by weak* density of $C(N)$ in $N$ and normality of $f$.

Conversely, $M^r(\mathbb{G}) = L^1(\mathbb{G})^{wk*}$ contains $\omega_N + \epsilon^u = \epsilon_G^u$, where the equality was noted in the proof of Proposition 4.12. This implies coamenability of $\mathbb{G}$. \qed

A coamenability result we are looking for presents itself as follows.

Corollary 4.16. Let $\mathbb{G}$ be a CQG and $N$ an compact quasi–subgroup such that $\omega_N \in M^r(\mathbb{G})$. Then $J^1(N)$ has a brai if and only if $\mathbb{G}$ is coamenable.

Proof. If $J^1(N)$ has a brai, then apply Theorems 4.14 and 4.15 to get coamenability of $\mathbb{G}$. The converse is a special case of the following more general fact: if $A$ is a Banach algebra with a bai, then a closed left ideal $J$ has a brai if and only if there is a right $A$–module projection $A^* \rightarrow J^\perp$ (cf. [13, 4.1.4 Pg. 42]). \qed

From Proposition 4.11 and Corollary 4.16, we also deduce the following.

Corollary 4.17. Let $\mathbb{G}$ be a CQG and $\mathbb{H}$ a closed quantum subgroup such that $\mathbb{G}/\mathbb{H}$ is a coamenable quotient. Then $\mathbb{G}$ is coamenable if and only if $J^1(\mathbb{G},\mathbb{H})$ has a brai.

A compact quasi–subgroup $N$ is open if $\omega_N \in L^1(\mathbb{G})$. It was shown in [28] that the open quasi–subgroups of a CQG are the finite dimensional right coideals. Using Corollary 4.16 we obtain the following.

Corollary 4.18. Let $\mathbb{G}$ be a CQG and $N$ an open compact quasi–subgroup. Then $\mathbb{G}$ is coamenable if and only if $J^1(N)$ has a brai.

4.2 Quantum Cosets of Compact Quasi–Subgroups

For this subsection, we take another approach towards studying left ideals admitting brais. We will exploit the intrinsic group in order to lift Forrest’s techniques to the CQG setting. We also analyze the associated left ideals of a class of objects slightly more general than the compact quasi–subgroups, namely, we consider also the “quantum cosets” of compact quasi–subgroups.

Definition 4.19. The group

$$Gr(\mathbb{G}) = \{ x \in L^\infty(\mathbb{G})^{-1} : \Delta_G(x) = x \otimes x \}$$

is called the intrinsic group of $\hat{\mathbb{G}}$. 

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Remark 4.20. Our reference for the following discussion is [24]. We actually have that each element of \( Gr(G) \) is unitary and is a locally compact group when equipped with the weak* topology. It is straightforward seeing that \( Gr(G) = sp(L^1(G)) \). Alternatively, one can identify \( Gr(G) \subseteq Irr(G) \) as the 1–dimensional unitary representations, so in particular, we have \( Gr(G) \subseteq Pol(G) \). Note that whenever \( G \) is compact, the von Neumann algebra generated by \( Gr(G) \) is of the form \( VN(\Gamma) \) for some discrete group \( \Gamma \). We will abuse notation and simply write \( Gr(G) = \Gamma \).

We will make use of the following.

Lemma 4.21. Let \( N \) be a compact quasi–subgroup. Then for \( x \in Gr(G) \),

\[
    xN \cap N = \begin{cases} N & \text{if } x \in N \\ \{0\} & \text{otherwise} \end{cases}
\]

Proof. From Lemma 4.7 we know that \( \omega_N(x) = 1 \) if \( x \in N \) and \( \omega_N(x) = 0 \) if \( x \notin N \). Then the equation

\[
    (id \otimes \omega_N)\Delta_G(x) = \omega_N(x)x
\]

tell us \( E^R_N(x) = x \) if \( x \in N \) and \( E^R_N(x) = 0 \) otherwise. Then for \( y \in N \), using that \( E^R_N \) is a conditional expectation, we have

\[
    xy = E^R_N(xy) = E^R_N(x)y
\]

if and only if \( E^R_N(x) = x \).

Given \( x \in L^\infty(G) \), we denote \( x \cdot f \in L^1(G) \) as the action such that \((f \cdot x)(y) = f(xy)\) for all \( y \in L^\infty(G) \). If \( x \in Gr(G) \), then, since \( x \) is a unitary, \( \cdot : L^1(G) \to L^1(G) \) is an isometric algebra automorphism.

Lemma 4.22. Let \( X \subseteq L^\infty(G) \) be a right invariant weak* closed subspace. For \( x \in Gr(G) \),

\[
    X_\perp \cdot x = (xX)_\perp
\]

is a closed left ideal. If \( X_\perp \) is two–sided, then \( X_\perp \cdot x \) is two–sided.

Proof. Since \((X_\perp)^\perp = X\), it is clear that \( X_\perp \cdot x \subseteq (xX)_\perp \). For \( f \in (xX)_\perp \), it can be shown using a Hahn–Banach argument that \( f \cdot x^{-1} \in X_\perp \). Then \( f = (f \cdot x^{-1}) \cdot x \in X_\perp \cdot x \). For the remaining claim, it is easy to see that \( X_\perp \cdot x \) is closed. Then for \( f \in L^1(G) \) and \( y \in X \),

\[
    (yx) \cdot f = (f \otimes id)(x \otimes x)\Delta_G(y) \in xX
\]

because \((f \cdot x) \otimes id)\Delta_G(y) \in X \). So \( xX \) is right invariant, meaning \((xX)_\perp = X_\perp \cdot x \) is a left ideal. If \( X \) is also left invariant, then left invariance of \( xX \) follows similarly.

Remark 4.23. 1. In the context of a discrete co–group \( \hat{\Gamma} \), the spaces \( Nx \) are the weak* closed subspaces generated by a coset. Indeed, if \( \Lambda \) is a proper subgroup and we take \( s \in Gr(\hat{\Gamma}) \setminus VN(\Lambda) = \Gamma \setminus \Lambda \), then \( s \cdot I(\Lambda) = I(As) \). In light of this, we call \( Nx \) a quantum coset of \( N \).
2. Let $N$ be a compact quasi–subgroup. Note that for $x \in Gr(\mathbb{G}) \setminus N$, $xN$ does not contain $1$ and so cannot be a von Neumann algebra and is not a compact quasi–subgroup. We did see, however, in the above lemma that $xN$ is a weak* closed right invariant subspace of $L^\infty(\mathbb{G})$. Next we will note $M_{I^+}^{(2)} : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ is a projection onto $xN$. To see this, first notice $x\operatorname{Pol}(N)$ is weak* dense in $xN$ because $L^\infty(\mathbb{G}) \ni y \mapsto x\overline{y} \in L^\infty(\mathbb{G})$ is a weak*–weak* homeomorphic linear bijection. Therefore it suffices to check $(\operatorname{id} \otimes \omega_N : x^{-1}) \Delta_G$ is a projection onto $x\operatorname{Pol}(N)$. For this, if we take $y \in \operatorname{Pol}(N)$,

$$(\operatorname{id} \otimes \omega_N : x^{-1}) \Delta_G(xy) = x(\operatorname{id} \otimes \omega_N) \Delta_G(y) = xy$$

and if we take $y \in \operatorname{Pol}(\mathbb{G})$,

$$(\operatorname{id} \otimes \omega_N : x^{-1}) \Delta_G(y) = x(\operatorname{id} \otimes \omega_N) \Delta_G(x^{-1}y) \in x\operatorname{Pol}(N).$$

We point out that the idempotent functional $\omega_N : x^{-1}$ is easily seen to be a contractive idempotent. Contractive idempotents and their associated weak* closed right invariant subspaces were studied in [27,10] (at the level LCQGs). While given a contractive idempotent $\omega \in M^u(\mathbb{G})$, $M^u(\mathbb{G})$ is not an algebra, it is a ternary ring of operators (TRO), i.e., whenever $x,y,z \in M^u(\mathbb{G})$, $xy^*z \in M^u(\mathbb{G})$.

**Lemma 4.24.** Let $N$ be a compact quasi–subgroup. For $x \in Gr(\mathbb{G}) \cap (L^\infty(\mathbb{G}) \setminus N)$, $T_N(J^1(N) \cdot x) = N_*$.

**Proof.** For each $y \in N$, using $xN \cap N = \{0\}$ from Lemma [1.21] find $f \in J^1(N) \cdot x$ so that $f(y) \neq 0$. Then $T_N(f)(y) = f(y) \neq 0$, from which, using a straightforward Hahn–Banach argument and that $T_N$ is open (open mapping theorem) and hence closed, we see that $T_N(J^1(N) \cdot x) = N_*$ as desired. \qed

The following theorem is the statement that $\mathbb{G}$ is coamenable if and only if the preannihilator of an invariant quantum coset has a bai.

**Theorem 4.25.** Let $\mathbb{G}$ be a CQG and $X$ a weak* closed invariant subspace of $L^\infty(\mathbb{G})$. Suppose $\{sX : s \in Gr(\mathbb{G})\}$ has a compact quasi–subgroup and at least two elements. Then $\mathbb{G}$ is coamenable if and only if $X_\perp$ has a bai.

**Proof.** Let $N \subseteq \{sX : s \in Gr(\mathbb{G})\}$ denote the compact quasi–subgroup. As discussed in Remark [1.23] we know $N = x_0X$ for some $x_0 \in Gr(\mathbb{G})$ and from Lemma [1.22] $J^1(N) = X_\perp \cdot x_0$ is a two–sided ideal (and so $N$ is actually an invariant compact quasi–subgroup).

The proof is a generalization of the argument employed by Forrest [14]. Suppose $X_\perp$ has a bai. Now, for $f \in X_\perp$, $y \in L^\infty(\mathbb{G})$, and $x \in Gr(\mathbb{G})$

$$||e_j \cdot x \ast (f \cdot x) - f \cdot x||_1 = \sup_{y \in B_1(L^\infty(\mathbb{G}))} |(e_j \otimes f) \Delta_G(xy) - f(xy)|$$

$$= \sup_{y \in B_1(L^\infty(\mathbb{G}))} |(e_j \otimes f) \Delta_G(y) - f(y)|$$

$$= \sup_{y \in B_1(L^\infty(\mathbb{G}))} |e_j \cdot f(y) - f(y)|$$

$$= ||e_j \cdot f - f||_1 \to 0$$

where in the second last equality, we used the fact $x$ is a unitary. A similar proof shows $f \ast (e_j \cdot x) \to f$, so $e_j \cdot x$ is a bai on $X_\perp \cdot x$. Now, we know $X \cdot Gr(\mathbb{G})$ has two elements, one
of which is $N$. Without loss of generalization, we will suppose the other element is $X$. So, we have that $J^1(N)$ and $X_\perp = J^1(N) \cdot x_0^{-1}$ both have bais. Then from invariance of $N$, we know $T_N$ is an algebraic homomorphism and coupling this fact with Lemma 4.24 finds us a bai on $N = T_N(J^1(N) \cdot x_0^{-1})$. Then we apply Proposition 1.3.

For the converse, from the discussion in Remark 1.23 we have a right $L^1(G)$–module projection $L^\infty(G) \to X$ induced by the idempotent functional $\omega_N \cdot x_0^{-1}$. The rest is identical to the proof of the converse of Proposition 1.24.

**Corollary 4.26.** Let $G$ be a CQG and $X$ a weak$^*$ closed right invariant subspace of $L^\infty(G)$. Suppose $\{sX : s \in Gr(G)\}$ has a compact quasi-subgroup $N$ such that $\omega_N \in M^r(G)$ and at least two elements. Then $G$ is coamenable if and only if $X_\perp$ has a brai.

**Proof.** Let $N \subseteq \{sX : s \in Gr(G)\}$ be the given compact quasi–subgroup. In the proof of Theorem 1.25 we showed $J^1(N)$ also has a brai. Then from Corollary 4.16 we know $G$ is coamenable. The proof of the converse is identical to the proof of the converse in Theorem 1.25.

**Remark 4.27.** In particular, if we have a compact quasi–subgroup $N \subseteq L^\infty(G)$ such that $Gr(G) \setminus N$ is non–trivial, then we are in a situation satisfying hypothesis of Theorem 1.25.

### 4.3 Examples: Discrete Crossed Products

With what follows, we use [0] as a reference.

**Definition 4.28.** A discrete C$^*$–dynamical system is a triple $(A, \Gamma, \alpha)$ where $A$ is a unital C$^*$–algebra, $\Gamma$ is a discrete group, and $\alpha : \Gamma \to Aut(A)$ is a continuous homomorphism.

Given a discrete C$^*$–dynamical system, we denote the finitely supported $A$–valued functions on $\Gamma$

$$A[\Gamma] = \text{span}\{as : a \in A, s \in \Gamma\}.$$  

We view the symbols $a \in A$ and $s \in \Gamma$ as the elements $a = ae$ and $1s = s$ in $A[\Gamma]$, which we assert to satisfy $sas^{-1} = \alpha(s)(a)$ for all $s \in \Gamma$ and $a \in A$, and has the following $*$-algebraic structure

$$(as)(bt) = aa(s)(b)s^{-1}t$$

for $t \in \Gamma$ and $b \in A$ (note that we only needed $A$ to be a unital $*$-algebra). In other words, $A[\Gamma]$ is a $*$–algebra that contains a copy of $A$ and a copy of $\Gamma$ as unitaries such that $\alpha$ is inner.

A covariant representation of $(A, \Gamma, \alpha)$ is a pair $(\pi_A, \pi_\Gamma)$ such that $\pi_A : A \to \mathcal{B}(\mathcal{H})$ is a $*$-representation and $\pi : \Gamma \to \mathcal{B}(\mathcal{H})$ a unitary representation satisfying the covariance equation

$$\pi_A(\alpha(s)(a)) = \pi_\Gamma(s)\pi_A(a)\pi_\Gamma(s)^*.$$  

A covariant representation gives rise to a representation $\pi_A \rtimes_\alpha \pi_\Gamma : A[\Gamma] \to \mathcal{B}(\mathcal{H})$ by setting $\pi_A \rtimes_\alpha \pi_\Gamma(s) = \pi_A(a)\pi(s)$. If we let $\theta : A \to \mathcal{B}(\mathcal{H}_\theta)$ be a faithful $*$-representation, then we can define a canonical covariant representation $(\pi^\theta, \lambda^\theta)$ of $(A, \Gamma, \alpha)$ by defining

$$\pi^\theta : A \to \mathcal{B}(L^2(\Gamma, \mathcal{H}_\theta)), \quad \pi^\theta(a)\xi(s) = \theta(\alpha(s^{-1})(a))\xi(s)$$

and

$$\lambda^\theta : \Gamma \to \mathcal{B}(L^2(\Gamma, \mathcal{H}_A)), \quad \lambda^\theta(t)\xi(s) = \xi(t^{-1}s)$$

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for $a \in A$, $s, t \in \Gamma$, and $\xi \in L^2(\Gamma, \mathcal{H}_\theta)$. Then the reduced crossed product is the $C^*$-algebra

$$A \rtimes_{r\alpha} \Gamma := \pi^{\theta} \rtimes_{\alpha} \lambda^{\theta}(A[\Gamma])^{\| \cdot \|_r},$$

which we note is independent of the choice of faithful representation on $A$ [9]. We also obtain the universal crossed product by setting

$$A \rtimes_{u\alpha} \Gamma := \pi(A[\Gamma])^{\| \cdot \|_u},$$

where

$$\|sa\|_u = \sup \{ \|\pi_A \rtimes_{\alpha} \pi_\Gamma (sa)\| : (\pi_A, \pi_\Gamma) \text{ is a covariant representation of } (A, \Gamma, \alpha) \}.$$

Note that $A \rtimes_{r\alpha} \Gamma$ and $A \rtimes_{u\alpha} \Gamma$ contain copies isometric of $A$ and $C^*_\theta(\Gamma)$ and $C^*(\Gamma)$ respectively, and in each case, we will abuse notation and denote the copy of each element $a \in A$ and $s \in \Gamma$ by $a$ and $s$ respectively.

We also have a von Neumann algebraic version of the crossed product. Our main reference will be [47], Chapter X.

**Definition 4.29.** A discrete $W^*$-dynamical system is a triple $(M, \Gamma, \beta)$ where $M$ is a von Neumann algebra, $\Gamma$ is a discrete group, and $\alpha : \Gamma \to Aut(M)$ is a weak* continuous homomorphism.

In this case, if $\theta : M \to \mathcal{B}(\mathcal{H}_\theta)$ is a normal $*$-representation, using similar definitions as in the $C^*$-algebra case, we can build a canonical pair of representations $(\pi^{\theta}, \lambda^{\theta})$. Then

$$M \rtimes_{\beta} \Gamma := (\pi^{\theta}(M) \lambda^{\theta}(\Gamma))''$$

is the discrete von Neumann crossed product of $(M, \Gamma, \beta)$. We note that $M \rtimes_{\beta} \Gamma$ contains an isometric copy of $M$ and $VN(\Gamma)$ and, as before, we will abuse notation and denote the copy of each $x \in M$ and $s \in \Gamma$ by $x$ and $s$ respectively.

If $G$ is a CQG and $\alpha$ intertwines $\Delta_\delta^G$, then we call $(C^u(G), \Gamma, \alpha)$ a Woronowicz $C^*$-dynamical system. A discrete crossed product of a Woronowicz $C^*$-dynamical system has an underlying CQG whose structure is described in the following.

**Theorem 4.30.** [50, 52] Let $(C^u(G), \Gamma, \alpha)$ be a Woronowicz $C^*$-dynamical system. Then $\alpha$ induces an action of $\Gamma$ on $C^u(G)$ and $L^\infty(G)$ (which intertwines their respect coproducts and we again denote by $\alpha$), and there exists a CQG (denoted $G \rtimes_{\alpha} \hat{\Gamma}$) such that:

1. $\text{Irr}(G \rtimes_{\alpha} \hat{\Gamma}) = \{ su_{i,j}^\pi : \pi \in \text{Irr}(G), \ s \in \Gamma \}$;
2. $\text{Pol}(G)[\Gamma] = \text{Pol}(G \rtimes_{\alpha} \hat{\Gamma})$;
3. $C^u(G) \rtimes_{u\alpha} \Gamma = C^u(G \rtimes_{\alpha} \hat{\Gamma})$;
4. $C^*(G) \rtimes_{r\alpha} \Gamma = C^*(G \rtimes_{\alpha} \hat{\Gamma})$;
5. $L^\infty(G) \rtimes_{\beta} \Gamma = L^\infty(G \rtimes_{\alpha} \hat{\Gamma})$;
6. $h_{G \rtimes_{\alpha} \hat{\Gamma}} = h_G \rtimes_{\alpha} 1_{(\subset)}$;
7. $\Delta_{G \rtimes_{\alpha} \hat{\Gamma}}|_{L^\infty(G)} = \Delta_G$ and $\Delta_{G \rtimes_{\alpha} \hat{\Gamma}}|_{VN(\Gamma)} = \Delta_{\Gamma}$.
Remark 4.31.

1. Note for Theorem 4.33. mend the reference [38].

product constructions. Their form can be gleaned from [17] where the structure of “Fourier
For elements of $C_\alpha \hat{\otimes} \Gamma$ ∈ $C_\alpha \hat{\otimes} \Gamma$ by setting $\mu(s) = \epsilon_G(a)s$

Proposition 4.32. Let $G \times_\alpha \hat{\otimes} \Gamma \times_\alpha \hat{\otimes} \Gamma$ be a crossed product. For $\mu, \nu \in M^u(G \times_\alpha \hat{\otimes} \Gamma)$, we have $\mu * \nu(s) = \mu(s) * \nu$ for all $s \in \Gamma$. In particular, for $u, \nu \in B(\Gamma)$ and $\varphi, \psi \in M^u(G)$, $u \times_\alpha \varphi * (\nu \times_\alpha \psi) = (uv) \times_\alpha (\varphi * \psi)$.

Proof. For $s \in \Gamma$ and $a \in C^u(G)$, we compute:

$\mu \times_\alpha \nu \Delta^u_{G \times_\alpha \hat{\otimes} \Gamma}(as) = (\mu \times_\alpha \nu)(s \times_\alpha \psi) \Delta^u_{G \times_\alpha \hat{\otimes} \Gamma}(a) = (\mu(s) \times_\alpha \nu(s))(a)$.

8. and $\hat{\Gamma}$ is a normal closed quantum subgroup of $G \times_\alpha \hat{\otimes} \Gamma$ and $G = (G \times_\alpha \hat{\otimes} \Gamma)/\hat{\Gamma}$ via the Hopf *-homomorphism

$Pol(G \times_\alpha \hat{\otimes} \Gamma) \rightarrow C[\Gamma], \pi_\hat{\Gamma}(sa) = \epsilon_G(a)s$

for $s \in \Gamma$ and $a \in Pol(G)$.

Remark 4.31.

1. Note for $u \in B(\Gamma) = C^*(\Gamma)^*$ and $\varphi \in M^u(G)$, $u \times_\alpha \varphi \in M^u(G \times_\alpha \hat{\otimes} \Gamma)$ denotes the functional such that

$u \times_\alpha \varphi(ta) = u(t)\varphi(a), \ t \in \Gamma, a \in C^u(G)$.

2. Following Theorem 4.30 8., clearly every closed quantum subgroup of $G \times_\alpha \hat{\otimes} \Gamma$. More generally, we see from 7. that every right invariant subspace of $L^\infty(G)$ and $VN(\Gamma)$ is also a right invariant subspace of $L^\infty(G \times_\alpha \hat{\otimes} \Gamma)$.

3. We obtain many examples from the pair of any CQG $G$ and discrete group $\Gamma$ via the trivial action $id : \Gamma \rightarrow Aut(L^\infty(G))$, which is defined by $id(s)(x) = x$ for all $s \in \Gamma$ and $x \in L^\infty(G)$. In this case, we get $L^\infty(G \times_\alpha \hat{\otimes} \Gamma) = L^\infty(G) \otimes VN(\Gamma)$ as von Neumann algebras.

Maintaining the same notation as Theorem 4.30, we will call $G \times_\alpha \hat{\otimes} \Gamma$ the crossed product of $G$ and $\hat{\Gamma}$ by $\alpha$.

We can use the ideas from [17] to describe the universal and reduced measure algebras of a crossed product. Indeed, we identify any $\mu \in M^u(G \times_\alpha \hat{\otimes} \Gamma)$ with an element of $C_\alpha(\Gamma, M^u(G))$ by setting $\mu(s) = \mu(sa)$ for $s \in \Gamma$ and $a \in C^u(G)$. Then, by definition,

$M^u(G \times_\alpha \hat{\otimes} \Gamma) = \left\{ \mu \in C_\alpha(\Gamma, M^u(G)) : \sup_{s \in \Gamma} \sum_{s \in \Gamma} \mu(sa) < \infty \right\}$.

Convolution can be realized as follows, which is a result we believe is well-known.

Proposition 4.32. Let $G \times_\alpha \hat{\otimes} \Gamma$ be a crossed product. For $\mu, \nu \in M^u(G \times_\alpha \hat{\otimes} \Gamma)$, when viewed as elements of $C_\alpha(\Gamma, M^u(G))$, we have $\mu * \nu(s) = \mu(s) * \nu(s)$ for all $s \in \Gamma$. In particular, for $u, \nu \in B(\Gamma)$ and $\varphi, \psi \in M^u(G)$, $(u \times_\alpha \varphi) * (\nu \times_\alpha \psi) = (uv) \times_\alpha (\varphi * \psi)$.

Proof. For $s \in \Gamma$ and $a \in C^u(G)$, we compute:

$\mu \times_\alpha \nu \Delta^u_{G \times_\alpha \hat{\otimes} \Gamma}(as) = (\mu \times_\alpha \nu)(s \times_\alpha \psi) \Delta^u_{G \times_\alpha \hat{\otimes} \Gamma}(a) = (\mu(s) \times_\alpha \nu(s))(a)$.

We will, in particular, care about the $L^1$-algebras of CQGs arising from the crossed product constructions. Their form can be gleaned from [17] where the structure of “Fourier spaces” of crossed products has been described. For more on the general theory, we recommend the reference [38].

Theorem 4.33. [25] Definition 3.1] Let $G \times_\alpha \hat{\otimes} \Gamma$ be a crossed product. The elements of $L^1(G \times_\alpha \hat{\otimes} \Gamma)$ identify with continuous functions $f : \Gamma \rightarrow L^1(G)$ such that $f(t)(x) = \hat{f}(tx)$ for $t \in \Gamma$ and $x \in L^\infty(G)$, where $\hat{f} \in L^\infty(G \times_\alpha \hat{\otimes} \Gamma)^*$ is of the form

$\hat{f}(T) = \sum_{n \geq 1} \langle T \xi_n, \eta_n \rangle$

for some $(\xi_n), (\eta_n) \subseteq L^2(\Gamma, L^2(G))$ with $\sum_{n \geq 1} ||\xi_n||, \sum_{n \geq 1} ||\eta_n|| < \infty$ and $T \in L^\infty(G \times_\alpha \hat{\otimes} \Gamma)$. 

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Now let us apply our results to shed some light on some of the closed ideals of $L^1(G \rtimes \alpha \Gamma)$ which admit bai.

**Proposition 4.34.** Let $(G, \Gamma, \alpha)$ be a Woronowicz dynamical system. Then the following hold:

1. if we assume $\text{Gr}(G) \neq \{1\}$, $G \rtimes \Gamma$ is coamenable if and only if $J^1(VN(\Gamma)) \cdot x$ has a bai, where $x \in \text{Gr}(G)$;

2. $G \rtimes \Gamma$ is coamenable if and only if any $J^1(N) \cdot s$ has a bai, where $s \in \Gamma$ and $Ns = VN(\Lambda s)$ for some proper subgroup $\Lambda$ of $\Gamma$ or $Ns$ is an (invariant) quantum coset of $L^\infty(\Gamma)$.

**Proof.** First note that $\Gamma, \text{Gr}(G) \subseteq \text{Gr}(G \rtimes \Gamma)$ and $\text{Gr}(G) \cap \Gamma = \{1\}$. For 1, because $\text{Gr}(G) \neq \{1\}$, we can find $x \in \text{Gr}(G) \subseteq \text{Gr}(G \rtimes \Gamma) \setminus \Gamma$ and we apply Theorem 4.25. Likewise, for 2, we can find $x \in \Gamma \subseteq \text{Gr}(G \rtimes \Gamma) \setminus \text{Gr}(G)$ or non–trivial $x \in \Lambda \setminus \Gamma \subseteq \text{Gr}(G \rtimes \Gamma)$, and then we apply Theorem 4.25

5 Open Problems

We will present problems left over from our investigations.

We have characterized the CQGs where every hull is a set of synthesis (Theorem 3.10) as the CQGs with property left $D_\infty$. This means the closed left ideals (and consequently the weak* closed right invariant subspaces of $L^\infty(G)$) are classified for the CQGs satisfying property left $D_\infty$. This leaves us with the following very open ended question.

**Question 5.1.** Which hulls of a CQG are always sets of synthesis?

For example, the closed subgroups of a locally compact groups are always sets of synthesis (cf. [25]). So, we ask the following more specific question.

**Question 5.2.** Are the hulls of right coideals sets of synthesis?

We have made partial progress towards identifying when the left ideals $J^1(N)$ associated with a compact quasi–subgroup admit a brai. While we have a complete characterization in terms of the condition $J^u(N) = J^1(N)^{\omega_{k^+}}$ (Theorem 4.14), our characterization in terms of coamenability of $G$ (Corollary 4.10) requires what is essentially a coamenability type condition on $N$. This leaves us with the following question.

**Question 5.3.** Given a CQG and compact quasi–subgroup $N$, if $J^1(N)$ has brai, then do we have $\omega_N \in M^r(G)$?

Successfully answering the above question means we can say $G$ is coamenable if and only if $J^1(N)$ admits a brai.

We have also characterized coamenability of $G$ in terms of the existence of brais on the associated left ideals of a very small class of TROs associated with a contractive idempotent (Theorem 4.25 and Corollary 4.26). Namely, if we set $X = M^u_L(L^\infty(G))$ where $\omega \in M^u(G)$ is a contractive idempotent, we require $Gr(G) \cap (L^\infty(G) \setminus X) \neq \emptyset$ and one of two things: either $X$ is invariant or $\omega \in M^r(G)$. Therefore we ask the following general question.

**Question 5.4.** Let $G$ be a CQG and $\omega \in M^u(G)$ a contractive idempotent. Do we have that $M^u_L(L^\infty(G))_{\perp}$ admits a brai if and only if $G$ is coamenable?
References

[1] D. Andreou, Crossed Products of Dual Operator Spaces and a Characterization of Groups With the Approximation Property, arXiv:2004.07169 (2020).
[2] E. Bédos, G. Murphy, and L. Tuset, Co-amenability of Compact Quantum Groups, Journal of Geometry and Physics 40 (2001), no. 2, 129–153.
[3] ____, Amenability and Co–amenability of Algebraic Quantum Groups, International Journal of Mathematics and Mathematical Sciences 31 (2002), no. 10.
[4] ____, Amenability and Co–amenability of Algebraic Quantum Groups II, Journal of Functional Analysis 201 (2003), no. 2, 303–340.
[5] M. Bożejko and G. Fendler, Herz-Schur Multipliers and Completely Bounded Multipliers of the Fourier Algebra of Locally Compact Groups, Boll. Un. Mat. Ital. A (6) 3 (1984), no. 2.
[6] M. Brannan, Approximation Properties for Locally Compact Quantum Groups, Banach Center Publications 111 (2017).
[7] P. Caprace and N. Monod, Relative Amenability, Groups, Geometry, and Dynamics 8 (2013), 747–774.
[8] J. Crann, Amenability and Covariant Injectivity of Locally Compact Quantum Groups II, Canadian Journal of Mathematics 65 (2017), no. 5.
[9] K. Davidson, C*-algebras by Example, Fields Institute Monographs (1996).
[10] M. Daws, Operator Biprojectivity of Compact Quantum Groups, Proceedings of the American Mathematical Society 138 (2010), no. 4.
[11] R. Doran and J. Wichman, Approximate Identities and Factorization in Banach Modules, Springer Lecture Notes in Mathematics, Springer-Verlag, Berlin 768 (1979).
[12] P. Eymard, L’algèbre de Fourier d’un groupe localement compact, Bull. Soc. Math. France 92 (1964).
[13] B. Forrest, Amenability and Ideals in the Fourier Algebra of Locally Compact Groups, Ph.D Thesis, University of Alberta (1987).
[14] ____, Amenability and Ideals in A(G), J. Austral. Math. Soc. 53 (1992), no. Series A, 143–155.
[15] B. Forrest, E. Kaniuth, T. Lau, and N. Spronk, Ideals with Bounded Approximate Identities in Fourier Algebras, Journal of Functional Analysis 203 (2003), no. 1.
[16] U. Franz, H.H. Lee, and A. Skalski, Integration Over the Quantum Diagonal Subgroup and Associated Fourier–like Algebras, International Journal of Mathematics 27 (2016), no. 9.
[17] M. Fujita, Banach Algebra Structure in Fourier Spaces and Generalization of Harmonic Analysis on Locally Compact Groups, J. Math. Soc. Japan 31 (1979), no. 1.
[18] U. Haagerup and J. Kraus, Approximation Properties for Group C*-Algebras and Group Von Neumann Algebras, Transactions of the American Mathematical Society 344 (1994), no. 2.
[19] E. Hewitt and K. A. Ross, Abstract Harmonic Analysis. Vol II: Structure and Analysis for Compact Groups. Analysis on Locally Compact Abelian Groups, (1970).
[20] Z. Hu, M. Neufang, and Z.-J. Ruan, Completely Bounded Multipliers Over Locally Compact Quantum Groups, Proceedings of the London Mathematical Society 103 (2011), no. 1.
[21] ____, Module Maps Over Locally Compact Quantum Groups, Studia Mathematica 211 (2012).
[22] M. Ilie and N. Spronk, Completely Bounded Homomorphisms of the Fourier Algebras, J. Func. Anal. 225 (2005), no. 5.
[23] M. Kalantar, P. Kasprzak, A. Skalski, and R. Vergnioux, Noncommutative Furstenberg Boundary, arXiv:2002.00657v1 (2020).
[24] M. Kalantar and M. Neufang, From Quantum Groups to Groups, Canadian Journal of Mathematics 65 (2013), no. 5, 1073–1095.
[25] E. Kaniuth and A. Lau, Fourier and Fourier–Stieltjes Algebras on Locally Compact Groups, AMS Mathematical Surveys and Monographs 231 (2018).
[26] E. Kaniuth and Anthony T. Lau, Spectral Synthesis for $A(G)$ and Subspaces of $V.N(G)$, Proceedings of the American Mathematical Society 129 (2001), no. 11, 3253–3263.
[27] P. Kasprzak, Shifts of Group–Like Projections and Contractive Idempotent Functionals for Locally Compact Quantum Groups, International Journal of Mathematics 29 (2018), no. 2.
[28] P. Kasprzak and P. Sołtan, The Lattice of Idempotent States of a Locally Compact Quantum Group, Publications of the Research Institute for Mathematical Sciences 56 (2018), no. 1.
[29] Y. Kawada and K. Itô, On the Probability Distribution on a Compact Group. I, Proceedings of the Physico-Mathematical Society of Japan. 3rd Series 22 (1940), no. 3, 977–998.
[30] J.L. Kelley, Averaging Operators on $C^∞(X)$, Illinois J. Math. 2 (1958).
[31] J. Kraus and Z.-J. Ruan, Approximation Properties for Kac Algebras, Indiana University Mathematics Journal 48 (1999), no. 2.
[32] J. Kustermans, Locally Compact Quantum Groups in the Universal Setting, International Journal of Mathematics 12 (2001), no. 3.
[33] J. Kustermans and S. Vaes, Locally Compact Quantum Groups, Ann. Sci. Éc. Norm. Supér 33 (2000), no. 6, 837–934.
[34] , Locally Compact Quantum Groups in the von Neumann Algebraic Setting, Mathematica Scandinavica 92 (2003), no. 1.
[35] A. T. Lau, Analysis on a Class of Banach Algebras with Applications to Harmonic Analysis on Locally Compact Quantum Groups and Semigroups, Fund. Math. 118 (1983), 161–175.
[36] V. Losert, Properties of the Fourier Algebra that are Equivalent to Amenability, Proc. Amer. Math. Soc. 92 (1984), no. 3.
[37] M. Mbekhta and N. Neufang, On the Structure of Ideals and Multipliers: A Unified Approach, Proceedings of the American Mathematical Society 147 (2019), no. 11.
[38] A. McKee, Weak Amenability for Dynamical Systems, Studia Mathematica (2016).
[39] C. Nebbia, Multipliers and Asymptotic Behaviour of the Fourier Algebra of Nonamenable Groups, Proc. Amer. Math. Soc. 84 (1982).
[40] M. Neufang, P. Salmi, A. Skalski, and N. Spronk, Contractive Idempotents on Locally Compact Quantum Groups, Indiana University Mathematics Journal 62 (2013), no. 6.
[41] M. Neufang, N. Spronk, A. Skalski, and P. Salmi, Fixed Points and Limit of Convolution Powers of Contractive Idempotent Measures, arXiv:1907.07337 (2019).
[42] A. Pal, A Counterexample on Idempotent States on a Compact Quantum Group, Letters in Mathematical Physics 37 (1996), no. 1, 75–77.
[43] P Podleś, Quantum Spheres, Letters in Mathematical Physics 14 (1987).
[44] Z.-J. Ruan, Amenability of Hopf von Neumann Algebras and Kac Algebras, Journal of Functional Analysis 139 (1996), no. 2, 466–499.
[45] V. Runde, Characterizations of Compact and Discrete Quantum Groups Through Second Duals, Journal of Operator Theory 60 (2008), no. 2, 415–428.
[46] , Uniform Continuity Over Locally Compact Quantum Groups, Journal of the London Mathematical Society 80 (2009), no. 1.
[47] M. Takesaki, *Theory of Operator Algebras II*, Encyclopaedia of Mathematical Sciences, Springer-Verlag Berlin Heidelberg 125 (2003).

[48] T. Timmermann, *An Invitation to Quantum Groups and Duality*, European Mathematical Society (2008).

[49] R. Tomatsu, *Amenable Discrete Quantum Groups*, Journal of the Mathematical Society of Japan 58 (2006), no. 4, 949–964.

[50] S. Wang, *Tensor Products and Crossed Products of Compact Quantum Groups*, Proceedings of the London Mathematical Society 71 (1995), no. 3, 695–720.

[51] , *Simple Compact Quantum Groups I*, Journal of Functional Analysis 256 (2009), 3313–3341.

[52] , *Equivalent Notions of Normal Quantum Subgroups, Compact Quantum Groups with Properties F and FD, and Other Applications*, Journal of Algebra 397 (2013).

[53] , *Some Problems in Harmonic Analysis on Quantum Groups*, Ph.D Thesis, Université de Franche-Comté - École Doctorale Carnot-Pasteur and Institute of Mathematics, Polish Academy of Sciences, Besançon, France and Warszawn, Poland (2016).

[54] J. White, *Left Ideals of Banach Algebras and Dual Banach Algebras*, Proceedings of the 23rd International Conference on Banach Algebras and Applications, De Gruyter Proceedings in Mathematics (De Gruyter 2020) (2020).

[55] S. L. Woronowicz, *Compact Quantum Groups*, Symétries quantiques (Les Houches, 1995), Amsterdam (1998), 845–884.