A remark on compact $H$-surfaces into $\mathbb{R}^3$

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1 Introduction

Let $\Omega$ be a smooth and bounded domain in $\mathbb{R}^2$. We consider the following system

$$\triangle u = u_x \wedge u_y, \quad \text{in } \Omega, \quad \text{(1.1)}$$

where $u \in C^2(\Omega; \mathbb{R}^3)$ and subscripts denote partial differentiation with respect to coordinates. This equation characterizes surfaces of constant mean curvature $H = \frac{1}{2}$ in $\mathbb{R}^3$ in conformal representation. More precisely, any non constant smooth map $u$ which is a solution of (1.1) and of the conformality condition

$$\omega := (|u_x|^2 - |u_y|^2 - 2i\langle u_x, u_y \rangle)dz \otimes dz = 0, \quad \text{in } \Omega, \quad \text{(1.2)}$$

(here $dz = dx + idy$) parametrizes a branched immersed constant mean curvature surface in $\mathbb{R}^3$. For that reason (1.1) is called the $H$-system. The complex tensor $\omega$ which appears in (1.2) is called the Hopf differential (see [8]). The first existence result for solutions to (1.1) and (1.2) was proved by H. Wente in [11].

In [7], the second author proposed a new variational approach for finding a solution to (1.1). For any pair of functions $a, b \in H^1(\Omega)$, we denote by $\varphi := \tilde{ab}$ the unique solution in $H^1(\Omega)$ of the Dirichlet problem

$$\begin{cases}
-\triangle \varphi = \{a, b\}, & \text{in } \Omega \\
\varphi = 0, & \text{on } \partial \Omega,
\end{cases} \quad \text{(1.3)}$$

where $\{a, b\} = a_x b_y - a_y b_x$. By a result of Brezis and Coron [3] based on an idea due to H. Wente [11] [12], we know that $\varphi$ is continuous on $\overline{\Omega}$ and

$$\|\varphi\|_{L^\infty(\Omega)} + \|
abla \varphi\|_{L^2(\Omega)} \leq C_0(\Omega) \|\nabla a\|_{L^2(\Omega)} \|\nabla b\|_{L^2(\Omega)}. \quad \text{(1.4)}$$

Thus the following energy functional makes sense

$$E(a, b, \Omega) = \frac{\|
abla a\|_{L^2(\Omega)}^2 + \|
abla b\|_{L^2(\Omega)}^2}{2\|
abla \varphi\|_{L^2(\Omega)}^2}, \quad \text{defined for } a, b \in H^1(\Omega) \setminus \{0\}.$$  

The Euler-Lagrange equation satisfied by the critical points of this functional was derived in [7]. Through the substitution $u := (\lambda a, \lambda b, \lambda^2 \varphi)$ for $\lambda = -\sqrt{\frac{\|
abla a\|_{L^2(\Omega)}^2 + \|
abla b\|_{L^2(\Omega)}^2}{2\|
abla \varphi\|_{L^2(\Omega)}^2}}$, this equation coincides with (1.1). The boundary conditions are

$$\varphi = \frac{\partial a}{\partial n} = \frac{\partial b}{\partial n} = 0 \quad \text{on } \partial \Omega, \quad \text{(1.5)}$$
where \( n = (n_1, n_2) \) is the normal vector on \( \partial \Omega \). Moreover, in [6] the first author showed that the Hopf differential \( \omega \) is holomorphic and satisfies the boundary condition \( Im(\omega \nu^2) = 0 \), where \( \nu = n_1 + in_2 \). This implies in particular that \( \omega \) vanishes - i.e. (1.2) is true - if \( \Omega \) is simply connected.

An important property of our problem is that the functional \( E \) and its critical points are preserved by conformal transformations of the domain \( \Omega \) (see e.g. [7]). So this variational problem depends only on the complex structure of \( \Omega \) and hence it also makes sense to consider the problem on a Riemann surface. The boundary conditions (1.5) allow us to construct a solution of (1.1) from a compact oriented Riemannian surface into \( \mathbb{R}^3 \) by gluing together two copies of \( \Omega \). More precisely, we construct \( N := \Omega \cup \partial \Omega \tilde{\Omega} \), where \( \tilde{\Omega} \) is a copy of \( \Omega \), provided with opposite orientation and define a \( C^\infty \) map \( \tilde{u} : N \to \mathbb{R}^3 \) by \( \tilde{u} = u \) on \( \Omega \) and \( \tilde{u} = (\lambda a, \lambda b, -\lambda^2 \varphi) \) on \( \tilde{\Omega} \). This map is a solution of the \( H \)-system (1.1) and its Hopf differential is holomorphic. Were this differential to vanish, i.e. if \( \omega = 0 \), then we would obtain a constant mean curvature branched immersion. Recall that Wente [13] (see also [1]) constructed an immersed constant mean curvature torus, which enjoys an invariance under an orthogonal symmetry with respect to a plane. Thus it has the form \( \Omega \cup \partial \Omega \tilde{\Omega} \) as above, where \( \Omega \) is some annulus. This motivates the search for critical points of \( E \). In [6], an existence result was derived for a perforated domain, provided the holes are small enough (in the same spirit as in [4]). Here we address the problem of a one-connected domain \( \Omega \), i.e. a domain of the form \( U \setminus \bar{V} \) where \( U \) and \( V \) are smooth bounded simply connected open sets and \( \bar{V} \subset U \), without smallness assumption on the hole \( V \). Because any such domain is conformally equivalent to a radially symmetric annulus [2] and thanks to the invariance of the variational problem under conformal transformations, we shall restrict ourself to annuli without loss of generality. Our method relies on a minimization procedure on a subset of \( H^1 \times H^1 \), which is equivariant with respect to some finite group.

Main Theorem. Consider the annulus \( \Omega := \{ (x, y) \in \mathbb{R}^2, \quad r_0 < r = \sqrt{x^2 + y^2} < 1 \} \) with \( 0 < r_0 < 1 \). Then, there exists a critical point \( (a, b) \in H^1(\Omega, \mathbb{R}^2) \setminus \{(0, 0)\} \) of the energy functional \( E \). Moreover \( (a, b) \) and \( \varphi := ab \) are smooth and satisfy the boundary conditions (1.5). Thus there exists a real number \( \lambda \neq 0 \) such that the map \( u := (\lambda a, \lambda b, \lambda^2 \varphi) \) is a solution of (1.1). Lastly the Hopf differential \( \omega \) of \( u \) has the form \( \omega = \frac{\tau}{\varphi}dz \otimes dz \) for some real number \( \tau \).

Unfortunately we are not able to prove that the map \( u \) is conformal. We expect that it should be so for some values of \( r_0 \). Indeed the parameter \( \tau \) characterizing \( \omega \) should vary with \( r_0 \) (since the set of holomorphic quadratic differentials is, roughly speaking, the dual space of Teichmüller space). But we are still far from understanding how \( \tau \) could be related to \( r_0 \).

2 The Euler-Lagrange Equation

First, we note that \( \Omega \) is invariant under rotations. We define \( F_m = \{ \Theta = (a, b) \in H^1(\Omega) \times H^1(\Omega), \quad \Theta \circ A = A \circ \Theta \} \) and we will prove existence of a minimum of \( E \) for \( m \) large enough, where \( A \) is the rotation of angle \( \frac{2\pi}{m} \) in \( \mathbb{R}^2 \).
Lemma 1 Assume that \((a, b) \in F_m\). Then, the unique solution \(\varphi\) of (1.3) is invariant under \(A\), that is,
\[
\varphi \circ A = \varphi.
\]

Proof. Clearly, we have
\[
d\Theta = A^{-1} \cdot (d\Theta) \circ A \cdot A.
\]
Thus,
\[
\{a, b\} = \det(d\Theta) = \det(A^{-1}) \cdot [\det(d\Theta)] \circ A \cdot \det(A) = [\det(d\Theta)] \circ A = \{a, b\} \circ A,
\]
since \(\det(A) = \det(A^{-1}) = 1\). On the other hand, the unique solution \(\varphi\) of (1.3) is also the unique minimum of the following energy functional \(E_1\):
\[
E_1(\psi) = \frac{1}{2} \int_{\Omega} |\nabla \psi|^2 - \int_{\Omega} \{a, b\} \psi, \quad \text{defined for all } \psi \in H^1_0(\Omega).
\]
Obviously, \(E_1(\varphi \circ A) = E_1(\varphi)\). By the uniqueness of the minimizer, we deduce that
\[
\varphi \circ A = \varphi.
\]

In order to get the Euler-Lagrange equation of \(E\), we first recall a technical lemma inspired by the work in [11] and proved in the Appendix in [3].

Lemma 2 If \(\varphi \in H^1(\Omega) \cap L^\infty(\Omega), a \in H^1(\Omega) \cap L^\infty(\Omega), b \in H^1(\Omega)\) and \(\varphi a = 0\) on \(\partial \Omega\), then we have
\[
\int_{\Omega} \varphi \{a, b\} = \int_{\Omega} a \{b, \varphi\}.
\]

The following result shows that critical points of \(E\) on \(F_m\) are also critical points of \(E\) on \(H^1 \times H^1\).

Lemma 3 Assume that \(H = (a, b) \in F_m\) is a minimizer of \(E\) on \(F_m\). Then
1) there exists \(\lambda \in \mathbb{R}^*\) such that \(\Psi = (a_1, b_1, \varphi_1) = (\lambda a, \lambda b, \lambda^2 \varphi)\) satisfies equation (1.1).
2) \(\Psi\) verifies the boundary conditions (1.5).
3) \(\int_{\Omega} \nabla a \cdot \nabla b = 0\).
4) \(\|\nabla a\|_{L^2} = \|\nabla b\|_{L^2}\).
5) there exists \(c \in \mathbb{R}\) such that
\[
\langle \partial_z \Psi, \partial_z \Psi \rangle = \frac{c}{z^2},
\]
where \(\partial_z = \frac{1}{2}(\partial_x - i\partial_y)\).
6) \(\Psi\) is regular on \(\Omega\).
**Proof.** The proof is very similar to the proof of Theorem 3.2 in [6]. We just need to adapt it to our equivariant setting.

Let \( \Lambda = (\alpha, \beta) \in F_m \). Denote by \( \psi \) the unique solution of the following equation
\[
\begin{aligned}
-\Delta \psi &= \{\alpha, b\} + \{a, \beta\}, & \text{in } \Omega, \\
\psi &= 0, & \text{on } \partial \Omega.
\end{aligned}
\] (2.6)

We claim that \( \psi \circ A = \psi \). Indeed, note that,
\[
\{\alpha, b\} + \{a, \beta\} = \det(dH + d\Lambda) - \det(dH) - \det(d\Lambda)
\]
and thus as in Lemma 1, we deduce that
\[
(\{\alpha, b\} + \{a, \beta\}) \circ A = \{\alpha, b\} + \{a, \beta\}.
\] (2.7)

Hence, by the same argument as before, we establish the claim. Now set \( \Theta_t = \Theta + t\Lambda \). Clearly, \( \Theta_t \in F_m \). A direct calculation leads to
\[
E(a_t, b_t, \Omega) = \frac{\|\nabla a\|_2^2 + \|\nabla b\|_2^2 + 2t \int_{\Omega} (\nabla a \cdot \nabla \alpha + \nabla b \cdot \nabla \beta) + O(t^2)}{2 \sqrt{\|\nabla \varphi\|_2^2 + 2t \int_{\Omega} \nabla \varphi \cdot \nabla \psi + O(t^2)}}
\]
\[
= \frac{\|\nabla a\|_2^2 + \|\nabla b\|_2^2 + 2t \int_{\Omega} (\nabla a \cdot \nabla \alpha + \nabla b \cdot \nabla \beta) + O(t^2)}{2 \left( \frac{t}{\|\nabla \varphi\|_2} \int_{\Omega} \varphi(\{\alpha, b\} + \{a, \beta\}) + O(t^2) \right)}
\]
\[
= E(a, b, \Omega) \left( 1 - \frac{t}{\|\nabla \varphi\|_2^2} \int_{\Omega} \varphi(\{\alpha, b\} + \{a, \beta\}) + O(t^2) \right)
\] (2.8)
\[
+ \frac{2t}{\|\nabla a\|_2^2 + \|\nabla b\|_2^2} \int_{\Omega} (\nabla a \cdot \nabla \alpha + \nabla b \cdot \nabla \beta) + O(t^2).
\]

For fixed \( \theta_0 \in [0, 2\pi] \), we consider the domain \( \Omega_{\theta_0} = \{(x, y), \quad r_0 < r < 1 \text{ and } \theta_0 < \theta < \theta_0 + \frac{2\pi}{m}\} \). It follows from (2.7) and Lemma 1 that
\[
[\varphi(\{\alpha, b\} + \{a, \beta\})] \circ A = \varphi(\{\alpha, b\} + \{a, \beta\}),
\]
which implies
\[
\int_{\Omega} \varphi(\{\alpha, b\} + \{a, \beta\}) = m \int_{\Omega_{\theta_0}} \varphi(\{\alpha, b\} + \{a, \beta\}).
\] (2.9)

On the other hand, we have
\[
(\nabla a \cdot \nabla \alpha + \nabla b \cdot \nabla \beta) \circ A = tr([d\Theta] \circ A \cdot [d\Lambda]^t \circ A)
\]
\[
= tr(A \cdot d\Theta \cdot A^t \cdot A \cdot d\Lambda^t \cdot A^t)
\]
\[
= tr(A \cdot d\Theta \cdot d\Lambda^t \cdot A^t)
\]
\[
= tr(d\Theta \cdot d\Lambda^t \cdot A^t \cdot A)
\]
\[
= \nabla a \cdot \nabla \alpha + \nabla b \cdot \nabla \beta.
\] (2.10)
that is,

$$\int_{\Omega} (\nabla a \cdot \nabla \alpha + \nabla b \cdot \nabla \beta) = m \int_{\Omega_0} (\nabla a \cdot \nabla \alpha + \nabla b \cdot \nabla \beta). \tag{2.11}$$

Combining (2.8) to (2.11), we obtain

$$\int_{\Omega_0} (\nabla a \cdot \nabla \alpha + \nabla b \cdot \nabla \beta) = \frac{\|\nabla a\|_2^2 + \|\nabla b\|_2^2}{2\|\nabla \varphi\|_2^2} \int_{\Omega_0} \varphi(\{a, b\} + \{a, \beta\}).$$

In particular, if we set $\alpha, \beta \in C^\infty(\Omega_{\theta_0})$, we deduce from Lemma 2 that

$$\begin{cases}
-\Delta a = \frac{\|\nabla a\|_2^2 + \|\nabla b\|_2^2}{2\|\nabla \varphi\|_2^2} \{b, \varphi\}, \text{ in } \Omega_{\theta_0} \\
-\Delta b = \frac{\|\nabla a\|_2^2 + \|\nabla b\|_2^2}{2\|\nabla \varphi\|_2^2} \{\varphi, a\}, \text{ in } \Omega_{\theta_0}.
\end{cases} \tag{2.12}$$

Setting $\lambda = -\sqrt{\frac{\|\nabla a\|_2^2 + \|\nabla b\|_2^2}{2\|\nabla \varphi\|_2^2}}$ and by arbitrariness of $\theta_0$, property 1) is demonstrated. Now, choosing $\alpha, \beta \in C^\infty(\Omega_{\theta_0})$ with $\alpha = \beta = 0$ on $\Gamma_1 = \{(r, \theta), \theta = \theta_0 \text{ or } \theta = \theta_0 + \frac{2\pi m}{m}\}$, it follows from Lemma 2

$$\int_{\partial \Omega_{\theta_0}} \frac{\partial a}{\partial n} \cdot \alpha + \frac{\partial b}{\partial n} \cdot \beta = 0,$$

that is, $\frac{\partial a}{\partial n} = \frac{\partial b}{\partial n} = 0$ on $\partial \Omega_{\theta_0} \setminus \Gamma_1$. By arbitrariness of $\theta_0$, we establish the property 2). The properties 3) and 4) are just results of 1) , 2) and Lemma 2.

Now, we choose any vector field $X \in C^\infty(\Omega, \mathbb{R}^2)$ such that $X \circ A = A \circ X$ and $X \cdot n = 0$ on $\partial \Omega$. Let $\sigma_t$ be the flow associated to $X$. Clearly,

$$\sigma_t \circ A = A \circ \sigma_t.$$

Therefore, $\Theta \circ \sigma_t \in F_m$. The proofs of 5) and 6) are the same as the proofs of Theorem 3.2 (vii) and (iv) in [6], respectively.

3 Study of a minimizing sequence

Through this section we analyze the behaviour of a minimizing sequence in the spirit of the theory developed in [6] (see also [5], [9] or [10]). First, we prove a useful fact.

**Lemma 4** Assume that $\Theta = (a, b) \in F_m$. Then we have

$$\int_{\Omega} a = \int_{\Omega} b = 0.$$

\[\square\]
Proof. By definition of \( F_m \), we have
\[
\int_{\Omega} \Theta = \int_{\Omega} A^{-1} \circ \Theta \circ A = A^{-1} \int_{\Omega} \Theta \circ A = A^{-1} \int_{\Omega} \Theta.
\]
We note that \( A^{-1} \) is a rotation. This implies
\[
\int_{\Omega} \Theta = (0,0).
\]

Now we consider the minimum of energy functional \( E \). Set \( G(\Omega) = \inf_{a,b \in H^1 \times H^1} E(a, b, \Omega) \) and \( G_m(\Omega) = \inf_{a,b \in F_m} E(a, b, \Omega) \). Let \((a_n, b_n, \varphi_n)\) be a minimizing sequence of \( E \) on \( F_m \), that is, \((a_n, b_n) \in F_m, (a_n, b_n, \varphi_n)\) satisfying equation (1.3) and
\[
E(a_n, b_n, \Omega) = G_m(\Omega) + o(1).
\] (3.13)
Without loss of generality, we can assume that \( \|\nabla \varphi_n\|_2 = 1 \). After extracting a subsequence, we may assume that
\[
a_n \rightarrow \alpha \text{ weakly in } H^1 \text{ and strongly in } L^2,
b_n \rightarrow \beta \text{ weakly in } H^1 \text{ and strongly in } L^2,
\varphi_n \rightarrow \psi \text{ weakly in } H^1 \text{ and strongly in } L^2.
\]
Obviously, \((\alpha, \beta) \in F_m\). First, we recall a technical lemma.

Lemma 5 \( \text{(see [11], [6] and also [3])} \) We assume that \( \varphi_n \) is a bounded sequence in \( H^1_0 \cap L^\infty \). Let \( a_n \rightarrow 0 \) weakly in \( H^1 \) and strongly in \( L^2 \). Then for every \( b \in H^1 \), we have
\[
\lim_{n \rightarrow \infty} \int \varphi_n \{a_n, b\} = 0.
\] (3.14)

We state the following result, analogous to Theorem 7.1 in [6].

Lemma 6 \( \text{Under the above assumptions, we have that:} \)
\( \text{(1)} \) if \( \psi = 0 \), then \( \alpha = \beta = 0 \);
or
\( \text{(2)} \) if \( \psi \neq 0 \), then \( (\alpha, \beta, \psi) \) is a minimum of the energy \( E \) on \( F_m \). Moreover, the following holds:
\[
a_n \rightarrow \alpha \text{ strongly in } H^1,
b_n \rightarrow \beta \text{ strongly in } H^1,
\varphi_n \rightarrow \psi \text{ strongly in } H^1.
\]
Proof. The proof is the same as the proof of Theorem 7.1 in [6], but here we work with equivariant maps.

In the following, we will suppose that \( \psi = \alpha = \beta = 0 \). Denote by \( M(\mathbb{R}^2) \) the space of non-negative measures on \( \mathbb{R}^2 \) with finite mass. Set \( \mu_n = \frac{1}{2}(|\nabla a_n|^2 + |\nabla b_n|^2) \) and \( \nu_n = |\nabla \phi_n|^2 dx \). We consider the extensions of \( \mu_n \) and \( \nu_n \) to all of \( \mathbb{R}^2 \) by valuing 0 in \( \mathbb{R}^2 \setminus \Omega \). Then \( \{\mu_n\} \) and \( \{\nu_n\} \) are bounded in \( M(\mathbb{R}^2) \). Modulo a subsequence, we may assume that \( \mu_n \rightharpoonup \mu, \nu_n \rightharpoonup \nu \) weakly in the sense of measures where \( \mu \) and \( \nu \) are bounded non-negative measure on \( \mathbb{R}^2 \).

Lemma 7 Under assumptions of Lemma 6, if \( \psi = \alpha = \beta = 0 \), then we have

\[
G_m(\Omega) \geq \sqrt{mG(\Omega)}.
\]

Proof. Clearly, \( \mu(\mathbb{R}^2 \setminus \bar{\Omega}) = \nu(\mathbb{R}^2 \setminus \bar{\Omega}) = 0 \). Choose \( \xi \in C^\infty(\mathbb{R}^2) \). Denote by \( \psi_n \) the unique solution of equation (1.3) for \( a = \xi a_n \) and \( b = \xi b_n \), that is

\[
\begin{aligned}
-\Delta \psi_n &= \{\xi a_n, \xi b_n\}, \quad \text{in } \Omega \\
\psi &= 0, \quad \text{on } \partial \Omega.
\end{aligned}
\]

A computation using the same arguments as in the proof of Lemma 7.5 in [6] gives

\[
\lim_{n \to \infty} \|
abla (\psi_n - \xi^2 \phi_n)
\|_2 = 0.
\]

Hence, we obtain

\[
G(\Omega)\|
abla (\xi^2 \phi_n)
\|_2 + o(1) = G(\Omega)\|
abla \psi_n
\|_2 \leq \frac{1}{2}(\|
abla (\xi a_n)
\|_2^2 + \|
abla (\xi b_n)
\|_2^2).
\]

Passing to the limit as \( n \to \infty \), there holds

\[
G(\Omega) \sqrt{\int \xi^4 d\nu} \leq \int \xi^2 d\mu, \quad \forall \xi \in C^\infty_0(\mathbb{R}^2).
\] (3.15)

By approximation, therefore,

\[
G(\Omega) \sqrt{\nu(E)} \leq \mu(E) \quad (E \subset \mathbb{R}^2, E \text{ Borel}).
\] (3.16)

Now since \( \mu \) is a finite measure, the set

\[
D \equiv \{x \in \bar{\Omega}, \mu(\{x\}) > 0\}
\]

is at most countable. We can therefore write \( D = \{x_j\}_{j \in J}, \mu_{x_j} = \mu(\{x_j\}) (j \in J) \) so that

\[
\mu \geq \sum_{j \in J} \mu_{x_j} \delta_{x_j}.
\]
Since (3.16) implies \( \nu \) is absolutely continuous relative to \( \mu \), we can write

\[
\nu(E) = \int_E h \, d\mu \quad (E \text{ Borel}),
\]

where

\[
h(x) \equiv \lim_{r \to 0} \frac{\nu(B(x, r))}{\mu(B(x, r))},
\]

this limit existing for \( \mu \)-a.e. \( x \in \mathbb{R}^2 \). On the other hand, from (3.16), we have

\[
(G(\Omega))^2 \frac{\nu(B(x, r))}{\mu(B(x, r))} \leq \mu(B(x, r)),
\]

provided \( \mu(B(x, r)) \neq 0 \). Thus we infer

\[
h = 0 \quad \mu\text{-a.e. } x \in \mathbb{R}^2 \setminus D.
\]

Finally, define \( \nu_{x_j} \equiv h(x_j)\mu_{x_j} \). Then we get

\[
\nu = \sum_{j \in J} \nu_{x_j} \delta_{x_j},
\]

and

\[
G(\Omega) \sqrt{\nu_{x_j}} \leq \mu_{x_j}.
\]

However, by symmetry of functions in \( F_m \), we have \( A^i x_j \subseteq D \) for \( i = 1, 2, ..., m-1 \) provided \( x_j \in D \). Consequently, by suitably relabelling the \( x_j \), we may assume that \( x_j \in \Omega_{\theta_0} \) (where \( \Omega_{\theta_0} \) is defined as in the proof of Lemma 3) for \( j \in \{1, ..., k\} = J' \) and \( k = \text{card}(J)m^{-1} \) and

\[
\nu = \sum_{i=0}^{m-1} \sum_{j \in J'} \nu_{x_j} \delta_{A^i x_j} \quad \text{and} \quad \mu \geq \sum_{i=0}^{m-1} \sum_{j \in J'} \mu_{x_j} \delta_{A^i x_j}.
\]

On the other hand, we have \( \nu(\bar{\Omega}) = 1 \) and \( \mu(\bar{\Omega}) = G_m(\Omega) \). This implies

\[
G_m(\Omega) = \mu(\bar{\Omega}) \geq m \sum_{j \in J'} \mu_{x_j} \geq m \sum_{j \in J'} G(\Omega) \sqrt{\nu_{x_j}} \geq mG(\Omega) \sqrt{\sum_{j \in J'} \nu_{x_j}} = \sqrt{mG(\Omega)}.
\]

4 Proof of the main Theorem

In view of Lemma 6, the result follows if there is no concentration (i.e. case (1) in Lemma 6 for minimizing sequences does not occur). By Lemma 7, a sufficient condition for that is to assume \( G_m(\Omega) < \sqrt{mG(\Omega)} \). For this purpose, we set \( a(x, y) = x \) and \( b(x, y) = y \). It
is obvious to see that \((a, b) \in F_m\) and \(E(a, b, \Omega) > 0\). For any fixed \(r_0 > 0\), we can choose some \(m \in \mathbb{N}\) such that

\[
\sqrt{m}G(\Omega) > E(a, b, \Omega) \geq G_m(\Omega).
\]

Thus, the main Theorem is proved.

**Acknowledgements.** The authors thank the referee for his valuable remarks on a first version of this paper.

**References**

[1] U. Abresh, *Constant mean curvature tori in term of elliptic functions*, J. Reine Angew. Math. **374**, 169-192 (1987).

[2] L. V. Ahlfors, *Complex analysis*, Mcgraw-Hill, New York (1966).

[3] H. Brezis and J. M. Coron, *Multiple solutions of H-systemes and Rellich’s conjecture*, Comm. Pure. Appl. Math, **37** (1984) 149-187.

[4] J. M. Coron, *Topologie et cas limite des injections de Sobolev*, C. R. Acad. Sc. Paris, **299**, Ser. I (1984) 209-212.

[5] L. C. Evans, *Weak convergence methods for nonlinear partial differential equations*, Regional conference series in mathematics, **74** (1990).

[6] Y. Ge, *Estimations of the best constant involving the \(L^2\) norm in Wente’s inequality and compact \(H\)-surfaces in Euclidean space*, Control, Optimisation and Calculus of Variations, Vol. **3**, (1998) 263-300.

[7] F. Hélein, *Applications harmoniques, lois de conservation et repère mobile*, Diderot éditeur, Paris-New York, 1996 (see Chapter III).

[8] J. Jost, *Two-dimensional geometric variational problems*, Wiley (1991).

[9] P. L. Lions, *The concentration-compactness principle in the calculus of variations: The limit case. Part I and Part II*, Rev. Mat. Ibero. **1**(1) (1985) 145-201 and **1**(2) (1985) 45-121.
[10] M. Struwe, *Variational Methods*, Springer, Berlin-Heidelberg-New York-Tokyo (1990).

[11] H. Wente, *An existence Theorem for surfaces of constant mean curvature*, J. Math. Anal. Appl. 26 (1969) 318-344.

[12] H. Wente, *Large solutions to the volume constrained Plateau problem*, Arch. rat. Mech. Anal. 75, 59-77 (1980).

[13] H. Wente, *Counter-example to a conjecture of H. Hopf*, Pacific J. Math. 121, 193-243 (1986).

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