Stability of Twisted States in the Kuramoto Model on Cayley and Random Graphs

Georgi S. Medvedev · Xuezhi Tang

Abstract The Kuramoto model of coupled phase oscillators on complete, Paley, and Erdős–Rényi (ER) graphs is analyzed in this work. As quasirandom graphs, the complete, Paley, and ER graphs share many structural properties. For instance, they exhibit the same asymptotics of the edge distributions, homomorphism densities, graph spectra, and have constant graph limits. Nonetheless, we show that the asymptotic behavior of solutions in the Kuramoto model on these graphs can be qualitatively different. Specifically, we identify twisted states, steady-state solutions of the Kuramoto model on complete and Paley graphs, which are stable for one family of graphs but not for the other. On the other hand, we show that the solutions of the initial value problems for the Kuramoto model on complete and random graphs remain close on finite time intervals, provided they start from close initial conditions and the graphs are sufficiently large. Therefore, the results of this paper elucidate the relation between the network structure and dynamics in coupled nonlinear dynamical systems. Furthermore, we present new results on synchronization and stability of twisted states for the Kuramoto model on Cayley and random graphs.

Keywords Kuramoto model · Twisted state · Synchronization · Quasirandom graph · Cayley graph · Paley graph

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Georgi S. Medvedev
medvedev@drexel.edu

Xuezhi Tang
xt32@drexel.edu

1 Department of Mathematics, Drexel University, 3141 Chestnut Street, Philadelphia, PA 19104, USA
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1 Introduction

Collective dynamics of large systems of coupled oscillators feature prominently in the mathematical modeling of many physical, biological, social, and technological networks. Examples include regulatory and neuronal networks (Bressloff 2012; Medvedev and Zhuravytska 2012), Josephson junctions and coupled lasers (Watanabe and Strogatz 1994), power networks and consensus protocols (Dorfler and Bullo 2012; Medvedev 2012), to name a few. Mathematical models of individual oscillators comprising real-world networks can be quite complex, which can obstruct the view of global mechanisms controlling collective dynamics. For weakly coupled networks, the problem can be substantially simplified by recasting the coupled model as a system of equations for the phase variables only (Malkin 1949; Kuramoto 1984b; Hoppensteadt and Izhikevich 1997). This is the basis of the following model of coupled phase oscillators due to Kuramoto.

Let $\Gamma = (V, E)$ be an undirected graph. Denote the neighborhood of $x \in V$ by

$$N(x) = \{y \in V : xy \in E\}.$$ 

The Kuramoto model of coupled phase oscillators on $\Gamma$ has the following form

$$\frac{\partial}{\partial t} u(x, t) = \omega(x) + \frac{(-1)^\alpha}{|N(x)|} \sum_{y \in N(x)} \sin(2\pi(u(y, t) - u(x, t))),$$ (1.1)

where $u(x, t)$ denotes the phase of the oscillator at $x \in V$ at time $t \in \mathbb{R}$ and $\omega(x) \in \mathbb{R}$ is its intrinsic frequency. The coupling in (1.1) is called attractive if $\alpha = 0$ and repulsive if $\alpha = 1$. The derivation of (1.1) assumes that the frequencies $\omega(x), x \in V,$ are close to each other (cf. Kuramoto 1984b). In this paper, we study (1.1) for $\omega(x) = \text{const}$. In this case, after recasting (1.1) into a moving frame of coordinates, we have

$$\frac{\partial}{\partial t} u(x, t) = \frac{(-1)^\alpha}{|N(x)|} \sum_{y \in N(x)} \sin(2\pi(u(y, t) - u(x, t))).$$ (1.2)

The Kuramoto model (1.1) and its numerous variations have formed an influential framework for studying synchronization (Kuramoto 1984a,b; Hoppensteadt and Izhikevich 1997). More recently, new spatiotemporal patterns in the Kuramoto model with nonlocal coupling have been discovered and received a great deal of attention. These are first of all chimera states, spectacular patterns combining regions of coherent and incoherent oscillations (Kuramoto and Battogtokh 2002; Abrams and Strogatz 2006; Wolfrum et al. 2011; Omelchenko 2013), and so-called twisted or splay states (Wiley et al. 2006). The latter are defined as follows.
Twisted states were identified as an important family of steady-state solutions in the Kuramoto model on \( k \)-nearest-neighbor graphs by Wiley et al. (2006). They have been studied in the Kuramoto model with repulsive coupling in Girnyk et al. (2012), in that on small-world graphs in Medvedev (2014c), and in the Kuramoto model with randomly distributed frequencies (Omelchenko et al. 2014). The stability of twisted states, which is intimately related to the network connectivity (Wiley et al. 2006), provides valuable insights into the structure of the phase space of the Kuramoto model and is important for understanding more complex spatial patterns such as chimera states (Xie et al. 2014). In this paper, we study stability of twisted states in the Kuramoto model on certain important regular and random graphs. Our goal is to elucidate the link between the network structure and emergent spatiotemporal patterns in coupled nonlinear dynamical systems such as the Kuramoto model. To this end, the problem of stability of twisted states provides a convenient setting for highlighting a subtle relation between the network topology and dynamics.

Dynamics in coupled networks is shaped by the interplay of the properties of the local dynamical systems at the individual nodes of the graph and by the structure of connections between them. The connectivity patterns that are of interest in such applications as power grids, neuronal networks, or the Internet can be extremely complex. Therefore, one has to understand what structural features of large networks are important for various aspects of its dynamics. As a first step in this direction, we ask the following question. Do networks with similar structural properties exhibit similar dynamics? Large graphs that have similar combinatorial properties do not have to look alike. This is perhaps best illustrated by quasirandom graphs (Thomason 1987; Chung et al. 1988; Krivelevich and Sudakov 2006; Alon and Spencer 2008). Roughly speak-
Quasirandom graphs are the graphs that behave like truly random Erdős–Rényi (ER) graphs (Alon and Spencer 2008). Surprisingly, some very regular graphs like complete and Paley graphs turn out to be quasirandom (see Fig. 2). To quantitatively compare connectivity of disparate graphs such as complete, Paley, and ER graphs, one can use the edge distributions, or employ the densities of the homomorphisms from simpler graphs (e.g., triangles and \( n \)-cycles) into these (large) graphs, or compare the eigenvalues of the corresponding adjacency matrices. Surprisingly, all these tests turn out to be equivalent for quasirandom graphs, i.e., it is sufficient to use any of them to determine whether a given graph is quasirandom (Chung et al. 1988). In particular, the infinite sequences of the complete, Paley, and ER graphs exhibit the same (up to rescaling by degree) asymptotics of all of the above quantities. Thus, all three graph sequences have a great deal of similarity from a combinatorial viewpoint. But what does this say about the dynamics on these graphs? To tackle this question, in this paper we undertake the study of stability of twisted states in the Kuramoto model on these three families of graphs.

In Sect. 3, we show that twisted states are steady-state solutions of the Kuramoto model on any Cayley graph generated by a cyclic group. This, in particular, includes the models on Paley and complete graphs. Furthermore, one can show that the Kuramoto model on ER graphs supports twisted states almost surely, albeit in the limit as the number of oscillators tends to infinity (cf. Theorem 7.4, see also Medvedev 2014c). Thus, by comparing stability of twisted states on the complete, Paley, and ER graphs, we can distill the effects of the finer structural features of these graphs on the dynamics of the Kuramoto model. The analysis of this paper shows that while there is a significant overlap in the sets of stable twisted states in the Kuramoto models on the complete and Paley graphs, surprisingly there are twisted states with opposite stability properties. This indicates that the stability of steady-state solutions may differ even in networks with such close asymptotic properties as the families of Paley and complete graphs. On the other hand, we show that for the repulsively coupled Kuramoto model on finite (albeit large) ER graphs, all nontrivial twisted states are metastable in the following sense. The solutions of the IVP with the initial data near nontrivial twisted states remain near them on for a long time. The same conclusion obviously holds for models on Paley and complete graphs by the continuous dependence on initial values. Thus, the solutions of the IVPs for all three models that started sufficiently close to a given twisted state remain close on finite time intervals, but not necessarily asymptotically.

\[\text{For convenience, we consider complete graphs with odd number of vertices, so that they can be interpreted as Cayley graphs.}\]
The organization of the paper is as follows. Before turning to stability of twisted states, in the preliminary Sect. 2, we review necessary background information from graph theory. Our goal here is to explain the similarity between the infinite families of complete, Paley, and ER graphs. In addition, to the results about edge distributions and the eigenvalues of quasirandom graphs, in this section we include several basic facts about graph limits, which provide another way to highlight the similarities between different quasirandom graphs. In Sect. 3, we develop two methods for studying stability of twisted states on Cayley graphs. The first approach is based on linearization. To study the stability of the linearized system, we use the discrete Fourier transform (Terras 1999). This leads to a sufficient condition for stability of twisted states generalizing the condition for stability of twisted states in the $k$-nearest-neighbor coupled Kuramoto model in Wiley et al. (2006). In the same section, we present an alternative variational method for studying stability of twisted states. Specifically, we show that stable steady states in the Kuramoto model with attractive (repulsive) coupling are local minima (maxima) of the quadratic form for the graph Laplacian $L \in \mathbb{R}^{n}$ on the $n$-torus

$$Q(z) = z^* L z, \quad z = (z_1, z_2, \ldots, z_n)^T, \quad z_i \in \mathbb{C}, \quad |z_i| = 1, i \in [n].$$

We use this observation to show that synchrony is always stable in an attractively coupled model, whereas twisted states corresponding to the largest eigenvalue of $L$ are stable in the model with repulsive coupling. This provides an interesting relation between the spectral properties of the graph $\Gamma$ and stability of twisted states. Furthermore, the variational interpretation helps to determine stability of twisted states when the spectrum of the linearized problem has multiple zero eigenvalue\(^2\), and the linear stability analysis is inconclusive. In Sect. 4, we take a closer look at synchronization. We prove that the synchronization subspace is asymptotically stable in the attractively coupled model (cf. Theorem 4.2). Here, we use the gradient structure of the Kuramoto model to construct a Lyapunov function. Likewise, we show that twisted states corresponding to the largest eigenvalue of the graph Laplacian are stable in the model with repulsive coupling. In Sect. 5, we study twisted states in the Kuramoto model on the complete graph. For this system, the variational approach of Sect. 3 implies that all nontrivial twisted states are stable (unstable) if the coupling is repulsive (attractive). To get a more detailed picture of the flow near twisted states, we use linearization. The linearized problem for the model on the complete graph on $n$ nodes, $K_n$, is highly degenerate: The spectrum contains two negative eigenvalues and $n - 2$ zero eigenvalues. We show that each nontrivial $q$-twisted state ($q \neq 0$) is stable, because it lies in an $(n - 2)$-dimensional smooth manifold formed by the equilibria of the Kuramoto model. In Sect. 6, we turn to the Kuramoto model on Paley graphs. Here, we use the combination of the linear stability analysis and the variational analysis to determine stability for most of the twisted states in the Kuramoto model with attractive and repulsive coupling. Interestingly, we find many unstable twisted states in the repulsively coupled model. Recall that in the same model on $K_n$, all nontrivial twisted states are stable. In Sect. 7, we study the Kuramoto model on the ER random graphs. We first establish that the solutions of the IVPs for the Kuramoto model on ER and complete

\(^2\) One zero eigenvalue is always present due to the translational invariance of twisted states.
graphs on \( n \) nodes that started from the same initial condition remain \( O(n^{-1/2}) \) close in the appropriate metric on finite time intervals with probability tending to 1 as the graphs’ size tends to infinity. This result is an analog of homogenization in the discrete setting. Next, we show that the Kuramoto model on \( G(n, p) \) supports twisted states almost surely as \( n \to \infty \). Finally, we prove that in finite random networks solutions starting near twisted states remain near them for a long time provided the network is sufficiently large. In other words, in the Kuramoto model on ER graphs twisted states are metastable. Sect. 8 offers concluding remarks.

2 The Graphs

In this section, we present the background material from graph theory, which is meant to explain why we study the Kuramoto model on the complete, Paley, and ER graphs in the remainder of this paper. The facts collected below show that these three families of graphs are very similar from the combinatorial, probabilistic, and algebraic viewpoints. In particular, all three graph sequences yield the same (up to rescaling) formal mean field limit for the Kuramoto model (see Fact 3, Sect. 2.4). Nonetheless, the dynamics of the Kuramoto model on these graphs may be qualitatively different, as follows from the stability analysis of the twisted states in the second half of this paper.

2.1 Preliminaries

Let \( \Gamma = \langle V, E \rangle \) be an undirected graph. Here, \( V \) stands for the set of nodes and \( E \) denotes the set of edges, i.e., unordered pairs from \( V \). An edge joining \( x \in V \) and \( y \in V \) is denoted by \( xy \). Note that \( xy \) and \( yx \) mean the same edge. We assume that \( \Gamma \) does not have multiple edges and loops, i.e., it is a simple graph. For dynamical systems defined on graphs, edges represent connections between the local dynamical systems located at nodes of \( \Gamma \). Below we consider dynamical systems on several different types of graphs. We start with symmetric Cayley graphs defined on the additive cyclic group \( \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}, n \in \mathbb{N} \), with respect to a symmetric subset \( S \subset \mathbb{Z}_n \), i.e., \( s \in S \iff -s \in S \).

**Definition 2.1** \( \Gamma = \langle V, E \rangle \) is called a Cayley graph of \( \mathbb{Z}_n \) with respect to \( S \) and denoted \( \text{Cay}(\mathbb{Z}_n, S) \) if \( a, b \in \mathbb{Z}_n \), \( ab \in E \) if \( b - a \in S \).

We restrict to Cayley graphs of finite cyclic groups (i.e., circulant graphs), because they provide a natural setting for studying twisted states in the Kuramoto model. This is a special class of Cayley graphs \(^3\) (see Magnus et al. 1966 for a general definition). The following examples of graphs are used throughout this paper.

**Definition 2.2(A)** Let \( n \geq 3, r \leq \lfloor n/2 \rfloor \) and \( B(r) = \{ \pm 1, \pm 2, \ldots, \pm r \} \). \( \Gamma = \text{Cay}(\mathbb{Z}_n, B(r)) \) is called a Cayley graph based on the ball \( B(r) \) (Terras 1999).

\(^3\) Note, however, that any finite abelian group is isomorphic to a direct product of cyclic groups. This can be used to extend the linear stability analysis of this section to the Kuramoto model on Cayley graphs of abelian groups (cf. Terras 1999, Chapter 10).
(B) Let $n = 1 \pmod{4}$ be a prime and denote

$$Z_n^\times = \mathbb{Z}_n/\{0\} \quad \text{and} \quad Q_n = \{x^2 \pmod{n} : x \in \mathbb{Z}_n^\times\}.$$ 

$Q_n$ is viewed as a set (not multiset), i.e., each element has multiplicity 1. Then, $Q_n$ is a symmetric subset of $Z_n^\times$ and $|Q_n| = 2^{-1}(n-1)$ (cf. Krebs and Shaheen 2011, Lemma 7.22). $P_n = \text{Cay}(\mathbb{Z}_n, Q_n)$ is called a Paley graph. For the details of construction of Paley graphs and the discussion of their properties, we refer an interested reader to Krebs and Shaheen (2011, Chapter 7, §6).

(C) A simple graph $K_n = \langle V, E \rangle$ on $n$ with $V = [n]$ and $E = \{ij : i, j \in [n] \text{ and } i \neq j\}$ is called the complete graph.

**Remark 2.3** For analytical convenience, in this paper we consider complete graph $K_n$ on the odd number of nodes. In this case, $K_{2r+1}$ can be viewed as a Cayley graph $\text{Cay}(\mathbb{Z}_{2r+1}, B(r))$.

Along with highly regular Cayley graphs, we consider random graphs. The ER graphs defined below do not look like Cayley graphs, but it turns out that they have much in common with $K_n$ and $P_n$, $n \gg 1$.

**Definition 2.4** (Krivelevich and Sudakov 2006) Let $n \in \mathbb{N}$ and $p \in (0, 1]$. The ER graph $G(n, p)$ is the probability space of all labeled graphs on vertex set $V = [n]$ such that every pair $(i, j)$ $(1 \leq i < j \leq n)$ forms an edge with probability $p$ independently from any other pair.

Next, we review some tools for describing connectivity of graphs. The structure of $\Gamma = \langle [n], E \rangle$ is encoded in its adjacency matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ defined as follows

$$a_{ij} = \begin{cases} 1, & \text{if } ij \in E, \\ 0, & \text{otherwise}. \end{cases} \quad (i, j) \in [n]^2. \quad (2.1)$$ 

Throughout this paper, we will often refer to the following geometric realization of the adjacency matrix $A$. Consider a $\{0, 1\}$-valued function $W_\Gamma$ defined on the unit square $[0, 1]^2$ as follows

$$W_\Gamma(x, y) = \begin{cases} 1, & \text{if } ij \in E \text{ and } (x, y) \in \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right], \\ 0, & \text{otherwise}. \end{cases} \quad (2.2)$$ 

The plot of the support of $W_\Gamma$ is called the pixel picture of $\Gamma$ (Lovász 2012). It provides a convenient visualization of the structure of the graph. Figure 3 presents a schematic representation of the Cayley graph based on a ball (a) and its pixel picture (b). In this and other pixel pictures of graphs, we place the origin at the top left corner of the plot to emphasize the relation between $W_\Gamma$ and the adjacency matrix $A$.

The adjacency matrix and the pixel plot yield algebraic and geometric means for comparing connectivity of distinct graphs. A complementary analytic way is provided by the edge distribution of $\Gamma = \langle V, E \rangle$, $e_U : 2^V \to \mathbb{Z}$. This function for every $U \subset V$ yields the number of edges in $U$ as an induced subgraph of $\Gamma$. With these tools at
Fig. 3  a A Cayley graph based on a ball.  b The pixel picture of the graph in a

Fig. 4  Pixel pictures of the adjacency matrices of a Paley (a) and an ER (b) graphs

hand, we are now in a position to discuss the similarities the complete, Paley, and ER random graphs.

2.2 The Edge Distributions of $K_n$, $P_n$, and $G(n, p)$

At first glance, the pixel pictures of the large Paley and ER graphs shown in Fig. 4 look quite different. The former shows distinct band structure, while the latter appears scrambled. Nonetheless, what the two plots have in common is that the pixels are distributed approximately uniformly. This becomes more evident for pixel plots of graphs of larger size. The widths of the black and white bands in the pixel plots of $P_n$ decrease with increasing $n$. Thus, for large $n$ the pixel plots for $P_n$ and $G(n, 1/2)$ look increasingly more alike at least if viewed from a distance. The uniform pixel distribution obviously holds for the pixel plots of the complete graphs. These observations suggest that all three families of graphs may in fact have qualitatively similar edge distributions. This brings us to the first result that highlights the similarity of $K_n$, $P_n$ and $G(n, p)$.  

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Fact 1 Let $p \in (0, 1)$, $n \gg 1$, and $\Gamma \in \{K_n, P_n, G(n, p)\}$. Then
\[
e_{\Gamma}(U) = \frac{\gamma_{\Gamma}}{2} |U|^2 + o(n^2) \quad \forall U \subset V(\Gamma),
\]
where $\gamma_{\Gamma} = |E(\Gamma)| / |(V(\Gamma))^2|$ stands for the edge density of $\Gamma$. In the case $\Gamma = G(n, p)$, (2.3) holds with probability 1.

Remark 2.5 Graphs satisfying (2.3) are called quasirandom or pseudorandom graphs (Chung et al. 1988; Krivelevich and Sudakov 2006).

Asymptotic relation (2.3) obviously holds for complete graphs $K_n$. Clearly, $\gamma_{K_n} = 1$ and
\[
e_{K_n}(U) = \binom{|U|}{2},
\]
in this case. For the random graph, $e_{G(n,p)}(U)$ is a binomial random variable with parameters $p$ and $\binom{|U|}{2}$. Thus,
\[
\mathbb{E}e_{G(n,p)}(U) = p\binom{|U|}{2} \quad \text{and} \quad \mathbb{E}\gamma_{G(n,p)} = p.
\]

Using the estimates for the tail of the binomial distribution, one can further show that (2.3) with even smaller error term holds for $G(n, p)$ almost surely. As to Paley graphs, we do not know any simple way of verifying (2.3) directly. However, for $P_n$, (2.3) can be shown using an equivalent to (2.3) spectral characterization of quasirandom graphs (Chung et al. 1988; Alon and Spencer 2008), which we discuss in the next paragraph.

2.3 The Graph Eigenvalues

The graph Laplacian of $\Gamma$ is defined as follows
\[
L = D - A,
\]
where $A$ is the adjacency matrix and
\[
D = \text{diag}(d_1, d_2, \ldots, d_n)
\]
is the degree matrix. Here, $d_i = |N(i)|$ is the degree of node $i \in [n]$. The normalized graph Laplacian of $\Gamma$ is defined by
\[
\tilde{L} = I - D^{-1/2}AD^{-1/2}.
\]

If the degree of every node of $\Gamma$ is equal to $d$, i.e., $\Gamma$ is a $d$-regular graph, then the normalized graph Laplacian of $\Gamma$ is equal to $d^{-1}L$. The eigenvalues of $L$ and $\tilde{L}$ carry important information about $\Gamma$ (Chung 1997). In this subsection, we review certain
facts about the eigenvalues of Cayley and quasirandom graphs that will be needed below.

Let $\Gamma = \text{Cay}(\mathbb{Z}_n, S)$ for some symmetric subset $S \subset \mathbb{Z}_n$. Then, the eigenvalues of $L$ can be computed using characters of $\mathbb{Z}_n$

$$e_x(y) = \exp \left\{ \frac{2\pi i xy}{n} \right\}, \quad y \in \mathbb{Z}_n. \quad (2.4)$$

Let $e_x : \mathbb{Z}_n \to \mathbb{C}$, $x \in \mathbb{Z}_n$, be a complex-valued function on $\mathbb{Z}_n$:

$$e_x = (e_x(0), e_x(1), \ldots, e_x(n-1))^T, \quad x \in \mathbb{Z}_n. \quad (2.5)$$

The space of all complex-valued functions on $\mathbb{Z}_n$ is denoted by $L^2(\mathbb{Z}_n, \mathbb{C})$.

**Lemma 2.6** The eigenvalues of the graph Laplacian of $\Gamma = \text{Cay}(\mathbb{Z}_n, S)$ are given by

$$\lambda_x = |S| - \sum_{y \in S} e_x(y) = |S| - \sum_{y \in S} \cos \left( \frac{2\pi xy}{n} \right), \quad x \in \mathbb{Z}_n. \quad (2.6)$$

The corresponding eigenvectors are $e_x$, $x \in \mathbb{Z}_n$.

**Proof** For any $x, y \in \mathbb{Z}_n$, we have

$$(Ae_x)(y) = \sum_{s \in S} e_x(y + s) = \left( \sum_{s \in S} e_x(s) \right) e_x(y).$$

Thus,

$$Ae_x = \mu_x e_x, \quad \mu_x = \sum_{s \in S} e_x(s). \quad (2.7)$$

Since characters $e_x$, $x \in \mathbb{Z}_n$, are mutually orthogonal, (2.7) gives the full spectrum of $A$. The statement of the lemma follows from (2.7) and $L = |S|I - A$. □

**Corollary 2.7** The spectrum of $L$ contains a simple zero eigenvalue $\lambda_0 = 0$ corresponding to the constant eigenvector $1_n = (1, 1, \ldots, 1)^T$.

At the level of eigenvalues, the similarity between the sequences of complete, Paley, and ER graphs can be expressed as follows.

**Fact 2** The $n-1$ nonzero eigenvalues of the normalized Laplacian of $\Gamma \in \{K_n, P_n, G(n, p)\}$ are equal to $1 + o(1)$ for $n \gg 1$.

From Lemma 2.6, for complete graphs, we immediately have

$$\lambda_x(K_n) = n - 1, \quad x \in \mathbb{Z}_n^X. \quad (2.8)$$
By (2.6), the eigenvalues of $P_n^4$ are
\[
\lambda_x(P_n) = \frac{n-1}{2} - 2^{-1} \sum_{k=1}^{n-1} e_x(k^2) = 2^{-1}(n - G_n(x)), \quad x \in \mathbb{Z}_n^\times, \quad (2.9)
\]

where
\[
G_n(x) = \sum_{k=0}^{n-1} e_x(k^2) \quad (2.10)
\]

stands for the Gauss sum (Krebs and Shaheen 2011, §7.6). By the well-known properties of the Gauss sum (see, e.g., Krebs and Shaheen 2011),
\[
G_n(x) = \begin{cases} 
0 & x \equiv 0 \pmod{n}, \\
\sqrt{n} & y^2 = x \pmod{n} \text{ has a solution}, \\
\sqrt{n} & y^2 = x \pmod{n} \text{ does not have a solution}.
\end{cases} \quad (2.11)
\]

If for a given $x$ equation $y^2 = x \pmod{n}$ is solvable, we say that $x$ is a quadratic residue (QR) modulo $n$. Thus, for $x \in \mathbb{Z}_n^\times$, we have
\[
\lambda_x(P_n) = \begin{cases} 
2^{-1}(n - \sqrt{n}) & x \text{ is a QR } \pmod{n}, \\
2^{-1}(n + \sqrt{n}) & x \text{ is not a QR } \pmod{n}.
\end{cases} \quad (2.12)
\]

From (2.8) and (2.12), it follows that the nonzero eigenvalues of the normalized Laplacians of the complete and Paley graphs are all $1 + o(1)$. Furthermore, Theorem 4 in Chung and Radcliffe (2011) implies that $n - 1$ eigenvalues of the normalized Laplacian of $G(n, p)$ converge to 1 in probability. Thus, the spectra of all three families of graphs coincide to leading order for $n \gg 1$. The localization of the nonzero eigenvalues of the normalized graph Laplacian is one of several equivalent characterization of quasirandom graphs. In particular, the formulas for the nonzero eigenvalues (2.12) imply (2.3) for the sequence of Paley graphs (cf. Chung et al. 1988; see also Alon and Spencer 2008, §9.3).

2.4 Graph Limits

In the previous subsections, we explored the similarities between the sequences of complete, Paley, and ER graphs through the prisms of the edge distributions and graph eigenvalues. Here, we will use graph limits as another way for comparing these graph sequences. Below, we review some facts about graph limits that will be needed in the remainder of this paper. For more details, we refer the interested reader to Lovász and Szegedy (2006), Borgs et al. (2008), Lovász (2012).

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4 Whenever we refer to $P_n$, we assume implicitly that $n$ is a prime and $n \equiv 1 \pmod{4}$. 
Let $\Gamma_n = \langle V(\Gamma_n), E(\Gamma_n) \rangle$, $n \in \mathbb{N}$, be a sequence of dense (simple) graphs, i.e., $|E(\Gamma_n)| = O(|V(\Gamma_n)|^2)$. The convergence of the graph sequence $\{\Gamma_n\}$ is defined in terms of the homomorphism densities

$$t(F, \Gamma_n) = \frac{\text{hom}(F, \Gamma_n)}{|V(\Gamma_n)|^{|V(F)|}}.$$  

(2.13)

Here, $F = \langle V(F), E(F) \rangle$ is a simple graph and $\text{hom}(F, \Gamma_n)$ stands for the number of homomorphisms (i.e., adjacency preserving maps $V(F) \to V(\Gamma_n)$). In other words, (2.13) is the probability that a random map $h: V(F) \to V(\Gamma_n)$ is a homomorphism.

**Definition 2.8** (Lovász and Szegedy 2006; Borgs et al. 2008) The sequence of graphs $\{\Gamma_n\}$ is called convergent if $t(F, \Gamma_n)$ is convergent for every simple graph $F$.

**Remark 2.9** Sometimes it is more convenient instead of the homomorphism density (2.13) to use the injective or induced homomorphism densities defined, respectively, by

$$t_0(F, \Gamma_n) = \frac{\text{inj}(F, \Gamma_n)}{(n)_k} \quad \text{and} \quad t_1(F, \Gamma_n) = \frac{\text{ind}(F, \Gamma_n)}{(n)_k},$$

(2.14)

where $(n)_k = n(n - 1) \ldots (n - k + 1)$, $t_1(F, \Gamma_n)$ is the density of the imbeddings of $F$ into $\Gamma_n$ as an induced subgraph (i.e., it preserves adjacency as well as nonadjacency of nodes). Asymptotically for $n \gg 1$, all three densities are equivalent in the sense that $|t(F, \Gamma_n) - t_{0,1}(F, \Gamma_n)| = o(1)$ (Lovász and Szegedy 2006; Lovász 2012).

For a convergent graph sequence, the limiting object is represented by a measurable symmetric function $W: I^2 \to I$. Here and below, $I$ stands for $[0, 1]$. Such functions are called graphons. The set of all graphons is denoted by $\mathcal{W}_0$.

**Theorem 2.10** (Lovász and Szegedy 2006) For every convergent sequence of simple graphs, there is $W \in \mathcal{W}_0$ such that

$$t(F, G_n) \to t(F, W) := \int_{[V(F)]} \prod_{(i,j) \in E(F)} W(x_i, x_j) dx_1 dx_2 \ldots dx_{|V(F)|}$$

(2.15)

for every simple graph $F$. Moreover, for every $W \in \mathcal{W}_0$ there is a sequence of graphs $\{G_n\}$ satisfying (2.15).

Graphon $W$ in (2.15) is the limit of the convergent sequence $\{\Gamma_n\}$. It is unique up to measure preserving transformations, meaning that for any other limit $W' \in \mathcal{W}_0$ there are measure preserving maps $\phi, \psi: I \to I$ such that $W(\phi(x), \phi(y)) = W'(\psi(x), \psi(y))$. Geometrically, graphon $W$ representing the limit of a convergent sequence of simple graphs $\{\Gamma_n\}$ can be interpreted as the limit of pixel pictures of the adjacency matrices of $\Gamma_n$, $A(\Gamma_n)$ (cf. 2.2). After relabeling the nodes of $\{\Gamma_n\}$ if
necessary, \( \{W_{\Gamma_n}\} \) converges to \( W \) in the cut norm.\(^5\) Note that the pixel picture of \( A(\Gamma_n) \) coincides with the plot of the support of \( W_{\Gamma_n} \) (see Fig. 4 for the pixel pictures of \( A(P_n) \) and \( A(G(n, 1/2)) \)). Thus, the graph limit \( W \) can be thought as the limit of the pixel pictures of adjacency matrices \( A(\Gamma_n) \) in the appropriate metric. The similarity of the pixel pictures of the adjacency matrices of \( K_n \), \( P_n \), and \( G(n, p) \) reflects the following fact about their limits.

**Fact 3** Graph sequences \( \{K_n\} \), \( \{P_n\} \), and \( \{G(n, p)\} \) converge to the constant graphons \( \text{Const}(1) \), \( \text{Const}(1/2) \), and \( \text{Const}(p) \), respectively.

Clearly, \( \{K_n\} \) is a convergent sequence as \( t_0(F, K_n) = 1 \) for any simple graph \( F \) and sufficiently large \( n \). It is not hard to see that the expected value \( \mathbb{E} t_0(F, G(n, p)) = p|E(F)| \) for \( n \geq |V(F)| \). From this using concentration inequalities, one can further show that \( t(F, G(n, p)) \to p|E(F)| \) almost surely (Lovász and Szegedy 2006, Corollary 2.6). The sequence of Paley graphs \( P_n \), as quasirandom graphs, satisfies

\[
t_1(F, \Gamma_n) = (1 + o(1))n|V(F)|2^{-\frac{|V(F)|}{2}}, \quad n \gg 1 \tag{2.16}
\]

(Alon and Spencer 2008, Property \( P_1(s) \), §9.3). This property, follows the concentration of eigenvalues (cf. 2.12) (Alon and Spencer 2008, Property \( P_3 \), §9.3). Thus, all three sequences converge to constant graphons. Furthermore, any quasirandom sequence with edge density \( p \) converges to \( \text{Const}(p) \) (Borgs et al. 2008). In particular, \( \{P_n\} \) converges to \( \text{Const}(1/2) \).

### 3 Twisted States in the Kuramoto Model on Cayley Graphs

In this section, we formulate the Kuramoto model and show that on Cayley graphs it has a family of steady-state solutions called twisted states. We present two methods for studying stability of twisted states. The first method uses linearization and the Fourier transform on \( \mathbb{Z}_n \), while the second relies on a Lyapunov function. Both methods elucidate the link between the structure of a Cayley graph and stability of the twisted states.

Suppose \( \Gamma = \text{Cay}(\mathbb{Z}_n, S) \). Then, (1.2) can be rewritten as follows

\[
\frac{\partial}{\partial t} u(x, t) = \frac{(-1)^g}{|S|} \sum_{y \in S} \sin(2\pi(u(x + y, t) - u(x, t))) , \quad x \in V(\Gamma), \quad t \in \mathbb{R} \tag{3.1}
\]

\(^5\) The cut norm of graphon \( W \in \mathcal{W}_0 \) is defined by

\[
\|W\|_{\square} = \sup_{S, T \in \mathcal{L}_I} \left| \int_{S \times T} W(x, y) dx dy \right|
\]

where \( \mathcal{L}_I \) stands for the set of all Lebesgue measurable subsets of \( I \).
3.1 Stability of Twisted States via Linearization

**Lemma 3.1** Twisted states (1.3) are steady-state solutions of (3.1).

**Proof** Using the symmetry of $S$, we have

$$
\sum_{y \in S} \sin \left( 2\pi (u_n^{(q)}(x + y, t) - u_n^{(q)}(x, t)) \right)
= 2^{-1} \sum_{y \in S} \sin \left( 2\pi (u_n^{(q)}(x + y, t) - u_n^{(q)}(x, t)) \right)
+ 2^{-1} \sum_{y \in -S} \sin \left( 2\pi (u_n^{(q)}(x + y, t) - u_n^{(q)}(x, t)) \right)
= 2^{-1} \sum_{y \in S} \left\{ \sin (2\pi(u(x + y, t) - u(x, t))) \sin (2\pi(u(x - y, t) - u(x, t))) \right\}. \quad (3.2)
$$

By plugging (1.3) into (3.2), we obtain

$$
\sum_{y \in S} \sin \left( 2\pi (u_n^{(q)}(x + y, t) - u_n^{(q)}(x, t)) \right)
= 2^{-1} \sum_{y \in S} \left\{ \sin \left( \frac{2\pi q y}{n} \right) - \sin \left( \frac{2\pi q y}{n} \right) \right\}
= 0.
$$

□

Next, we turn to the stability of the twisted states. By plugging $u = u_n^{(q)} + \eta(x, t)$ into (3.1), we have

$$
\frac{\partial}{\partial t} \eta(x, t) = \frac{(-1)^n 2\pi}{|S|} \sum_{y \in S} \cos \left( \frac{2\pi q y}{n} \right) [\eta(y + x, t) - \eta(y, t)] + O(\|\eta(\cdot, t)\|^2).
$$

(3.3)

The linearized equation can be rewritten as

$$
\frac{\partial}{\partial t} \eta = \delta_S \ast \eta - m_S \eta, \quad (3.4)
$$

where $f \ast g$ stands for the convolution of two functions from $L_2(\mathbb{Z}_n, \mathbb{C})$

$$
f \ast g = \sum_{y \in \mathbb{Z}_n} f(y)g(x - y),
$$

$$
\delta_S(x) = \begin{cases} 
\frac{(-1)^n 2\pi}{|S|} \Re \exp \left( \frac{2\pi i q x}{n} \right), & x \in S, \\
0, & \text{otherwise,} 
\end{cases}
$$

and $m_S = \sum_{y \in S} \delta_S(y). \quad (3.5)$

**Lemma 3.2** Let $B: L_2(\mathbb{Z}_n, \mathbb{C}) \to L_2(\mathbb{Z}_n, \mathbb{C})$ be the linear operator defined by the right-hand side of (3.4). The eigenvalues of $B$ are
\begin{equation}
\lambda_x(B) = \frac{(-1)^\alpha 2\pi}{|S|} \sum_{y \in S} \left\{ \cos \left( \frac{2\pi (q + x) y}{n} \right) - 2 \cos \left( \frac{2\pi q y}{n} \right) + \cos \left( \frac{2\pi (q - x) y}{n} \right) \right\} , \quad x \in \mathbb{Z}_n.
\end{equation}

**Remark 3.3** It follows from (3.6) that the spectrum of \( B \) always has a zero eigenvalue \( \lambda_0 = 0 \). The corresponding eigenspace

\begin{equation}
\mathcal{D} = \text{span}\{1_n\}.
\end{equation}

The presence of the zero eigenvalue reflects the fact that the set of \( q \)-twisted states is invariant under spatial translations.

**Proof** We compute the eigenvalues of \( B \) using the discrete Fourier transform. For \( f \in L^2(\mathbb{Z}_n, \mathbb{C}) \), the latter is defined by

\begin{equation}
\mathcal{F} f(x) = \hat{f}(x) := \sum_{y \in \mathbb{Z}_n} f(y) e_x(-y),
\end{equation}

where \( e_x, x \in \mathbb{Z}_n \), stand for the characters of \( \mathbb{Z}_n \) (cf. Terras 1999, Equation (2.4)). By applying the Fourier transform to the right-hand side of (3.4), we have

\begin{equation}
\mathcal{F} B \eta(x) = \mathcal{F} (\delta_S \ast \eta - m_S \eta) = \left( \hat{\delta}_S(x) - m_S \right) \mathcal{F} \eta(x).
\end{equation}

Setting \( h = \mathcal{F} \eta \), we rewrite (3.8) as

\begin{equation}
\left[ \mathcal{F} B \mathcal{F}^{-1} (h) \right](x) = \left( \hat{\delta}_S(x) - m_S \right) h(x).
\end{equation}

Thus, the Fourier transform diagonalizes \( B \). The eigenvalues of \( B \) are

\begin{equation}
\lambda_x(B) = \hat{\delta}_S(x) - m_S, \quad x \in \mathbb{Z}_n,
\end{equation}

and \( e_x, x \in \mathbb{Z}_n \), are the corresponding eigenvectors. Finally, from

\begin{equation}
\hat{\delta}_S(x) = \frac{(-1)^\alpha 2\pi}{|S|} \sum_{y \in S} \cos \left( \frac{2\pi q y}{n} \right) \exp \left( -\frac{2\pi i x y}{n} \right)
\end{equation}

we obtain

\begin{equation}
\text{Re} \hat{\delta}_S(x) = \frac{(-1)^\alpha 2\pi}{|S|} \sum_{y \in S} \cos \left( \frac{2\pi q y}{n} \right) \cos \left( \frac{2\pi x y}{n} \right) = \frac{(-1)^\alpha \pi}{|S|} \sum_{y \in S} \left\{ \cos \left( \frac{2\pi (q + x) y}{n} \right) + \cos \left( \frac{2\pi (q - x) y}{n} \right) \right\},
\end{equation}

\begin{equation}
\text{Im} \hat{\delta}_S(x) = 0,
\end{equation}
where the symmetry of $S$ was used to obtain (3.13). The expressions for the eigenvalues in (3.6) follow from (3.10), (3.11), (3.12), and (3.13).

**Theorem 3.4** A $q$-twisted state $u^{(q)}_n, q \in \mathbb{Z}_n$, is a stable equilibrium of (3.1) provided

\[
(-1)^\alpha \sum_{y \in S} \left\{ \cos \left( \frac{2\pi(q + x)y}{n} \right) - 2 \cos \left( \frac{2\pi qy}{n} \right) + \cos \left( \frac{2\pi(q - x)y}{n} \right) \right\} < 0 \quad \forall x \in \mathbb{Z}_n^\times.
\]

(3.14)

**Proof** The spectrum of the linearized problem about $q$-twisted state $u^{(q)}_n$ contains the zero eigenvalue (see Remark 3.3). Below, we show that $u^{(q)}_n$ is a stable twisted state provided the remaining eigenvalues are negative (cf. 3.14).

Recall that $\mathcal{D} = \text{span}\{1_n\}$. Suppose $u(x, t)$ is a solution of (3.1). Then,

\[
\frac{d}{dt} \langle u(\cdot, t), 1_n \rangle = \frac{(-1)^\alpha}{|S|} \sum_{xy \in E} \{ \sin (2\pi(u(y, t) - u(x, t))) \\
+ \sin (2\pi(u(x, t) - u(y, t))) \} = 0,
\]

(3.15)

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{Z}_n, \mathbb{C})$. Thus, $\mathcal{D}^\perp$ is an invariant subspace of the nonlinear system (3.1).

Further, recall the linearization Ansatz $u = u^{(q)}_n + \eta(x, t)$ (cf. 3.3). Decompose

$$
\eta = c_1 1_n + \tilde{\eta}, \quad \tilde{\eta} \in \mathcal{D}^\perp.
$$

Then,

$$
u(x, t) = u^{(q)}_n(x) + \eta(x, t) = \tilde{u}^{(q)}_n(x) + \tilde{\eta}(x, t),
$$

where $\tilde{u}^{(q)}_n$ is a shifted twisted state

$$
\tilde{u}^{(q)}_n := u^{(q)}_n + c_1 1_n \quad \text{(mod 1)}
$$

and $\tilde{\eta}(\cdot, 0) \in \mathcal{D}^\perp$. Note that $\tilde{u}^{(q)}_n$ depends continuously on $\|\eta(\cdot, 0)\|$. Further, $\tilde{\eta}(\cdot, t) \in \mathcal{D}^\perp$ for all $t > 0$, because by (3.15),

$$
\langle u(\cdot, t), 1_n \rangle = \langle u^{(q)}_n, 1_n \rangle + \langle \tilde{\eta}(\cdot, t), 1_n \rangle = \text{const},
$$

and therefore,

$$
\langle \tilde{\eta}(\cdot, t), 1_n \rangle = \langle \tilde{\eta}(\cdot, 0), 1_n \rangle = 0.
$$

This shows that for studying stability of $q$-twisted state $u^{(q)}_n$, we can restrict to perturbations from $\mathcal{D}^\perp$. □
The linearization about $\tilde{u}_n^{(q)}$ yields
\[
\frac{\partial}{\partial t} \tilde{\eta}(\cdot, t) = B \tilde{\eta}(\cdot, t) + O(\|\tilde{\eta}(\cdot, t)\|^2).
\tag{3.16}
\]
since $\tilde{\eta}(\cdot, t) \in D^\perp$ for all $t > 0$, we restrict (3.16) to $D^\perp$. The spectrum of $B$ restricted to $D^\perp$ consists of the eigenvalues $\lambda_x(B)$, $x \in \mathbb{Z}^\times_n$, because the eigenvector corresponding to the zero eigenvalue $e_0 \in D$. By the assumption (3.14), the eigenvalues of the linear operator $B$ restricted to $D^\perp$ are negative. Therefore, $\tilde{u}_n^{(q)}$ is asymptotically stable to small perturbations from $D^\perp$, and $u_n^{(q)}$ is a stable steady state of (3.1). \[\square\]

### 3.2 The Variational Approach to Stability of Twisted States

In this section, we present a variational interpretation of stability and instability of certain twisted states in the Kuramoto model. In particular, we establish the correspondence between twisted states of the Kuramoto model and the eigenvalues of the graph Laplacian $L$. We show that twisted states corresponding to the smallest (largest) eigenvalue are stable (unstable) if the coupling is attractive. The converse relations hold for systems with repulsive coupling. In particular, synchrony is stable for the Kuramoto models with attractive coupling and is unstable for those with repulsive coupling.

First, we rewrite the Kuramoto model on $\Gamma = \langle E, V \rangle$ in the form convenient for the analysis of this section:
\[
\dot{\theta}_i = \frac{(-1)^\alpha}{4\pi} \sum_{j : ij \in E} \sin(2\pi(\theta_j - \theta_i)), \quad i \in [n].
\tag{3.17}
\]

**Remark 3.5** For convenience, in (3.17) we use a different scaling on the right-hand side [compare (3.17) with (3.1)]. Clearly, this does not affect existence and stability of twisted state. After rescaling time (3.17) covers Kuramoto models on Cayley graphs (3.1), as well as on other undirected graphs.

The coboundary matrix of $\Gamma = \langle V, E \rangle$ ($m = |E|$, $n = |V|$), $H \in \mathbb{R}^{m \times n}$, is defined as follows (Biggs 1993). For each edge $e := ij \in E$, we choose the starting node $s(e) \in \{i, j\}$ and the terminal node $t(e) = \{i, j\}/s(e)$. Then, the entries of $H$ are given by
\[
(H)_{ev} = \begin{cases} 
1, & \text{if } t(e) = v, \\
-1, & \text{if } s(e) = v, \\
0, & \text{otherwise},
\end{cases}
\tag{3.18}
\]

Consider
\[
\Phi(\theta) = \langle He^{2\pi i \theta}, He^{2\pi i \theta} \rangle,
\tag{3.19}
\]
where $\theta = (\theta_1, \theta_2, \ldots, \theta_n)^\top \in \mathbb{R}^n$ and $e^{2\pi i \theta} := (e^{2\pi i \theta_1}, e^{2\pi i \theta_2}, \ldots, e^{2\pi i \theta_n})^\top$. Since $\Phi(\theta)$ is a 1-periodic function, we consider $\theta$ as an element of $\mathbb{T}^n$. On the other hand, $e^{2\pi i \theta}$ is convenient to interpret as an element of $\mathbb{T}_C = \mathbb{T}_C \times \mathbb{T}_C \times \cdots \times \mathbb{T}_C$, where $\mathbb{T}_C = \{z \in \mathbb{C} : |z| = 1\}$. The exponential map

$$\phi \mapsto R(\phi) = e^{2\pi i \phi}$$

provides a bijection between $\mathbb{T}$ and $\mathbb{T}_C$.

**Lemma 3.6** $\Phi$ is a nonnegative bounded function

$$0 \leq \Phi(\theta) \leq 4m \quad \forall \theta \in \mathbb{T}^n.$$  \hspace{1cm} (3.21)

Moreover, (3.17) can be rewritten as a gradient system

$$\dot{\theta} = -\nabla U_\alpha(\theta), \quad U_\alpha(\theta) := (-1)^\alpha \Phi(\theta),$$

and thus, for any solution $\theta(t)$ of (3.17), we have

$$\Phi(\theta) := \frac{d}{dt} \Phi(\theta(t)) = (-1)^{\alpha+1} \|\nabla \Phi(\theta)\|^2.$$  \hspace{1cm} (3.23)

**Proof**

$$0 \leq \Phi(\theta) = \|He^{i2\pi \theta}\|^2
= \sum_{kj \in E} \left\{ \left( \cos(2\pi \theta_j) - \cos(2\pi \theta_k) \right)^2 + \left( \sin(2\pi \theta_j) - \sin(2\pi \theta_k) \right)^2 \right\}
= 2 \sum_{kj \in E} \left( 1 - \cos(2\pi(\theta_j - \theta_k)) \right) \leq 4m.$$  \hspace{1cm} (3.24)

This shows (3.21). By differentiating (3.24), we have

$$\frac{\partial}{\partial \theta_j} \Phi(\theta) = -4\pi \sum_{k: k \neq j \in E} \sin(2\pi(\theta_k - \theta_j)), \quad j \in [n].$$  \hspace{1cm} (3.25)

Thus, (3.22) follows. $\square$

Since the coupled system (3.17) is a gradient system with the real analytic potential function $U_\alpha$ (3.22), the set of stable equilibria of (3.17) coincides with the set of local minima of $U_\alpha$ (Absil and Kurdyka 2006, Theorem 3). The latter are the local minima of $\Phi$ if the coupling is attractive and are the local maxima of $\Phi$ if the coupling is repulsive. Thus, the problem of locating the attractors for (3.17) for both attractive and repulsive coupling is reduced to the variational problem

$$\Phi(\theta) \to \min / \max, \quad \theta \in \mathbb{T}^n.$$  \hspace{1cm} (3.26)
We consider the minimization problem first

$$\Phi(\theta) = \langle He^{2\pi i \theta}, He^{2\pi i \theta} \rangle \rightarrow \min, \quad \theta \in \mathbb{T}^n. \quad (3.27)$$

This problem can be rephrased as the minimization problem for the quadratic form

$$Q(z) = z^* L z \rightarrow \min, \quad z \in \mathbb{T}_C^n. \quad (3.28)$$

where $L$ is the Laplacian of $\Gamma$. By the variational characterization of the eigenvalues of Hermitian matrices (Horn and Johnson 2013), the minimum of $Q$ on $\mathbb{T}_C^n$ is achieved on the intersection of $\mathbb{T}_C^n$ and the one-dimensional eigenspace corresponding to the smallest eigenvalue of $L$, $\lambda_{\text{min}} = 0$:

$$z_{\text{min}} = e^{2\pi i c} 1_n, \quad c \in \mathbb{R}.$$ 

Therefore, the minimum of $\Phi$ on $\mathbb{T}^n$ is achieved on

$$u_{\text{min}} = \frac{1}{2\pi i} \ln z_{\text{min}} = c 1_n \pmod{1},$$

where $\ln z$ stands for the univalent branch of the complex logarithm with values in the strip $0 \leq \Im \ln z < 2\pi$. Thus, we have proved the following theorem.

**Theorem 3.7** A synchronous solution $c1_n$ is a stable (unstable) equilibrium of the Kuramoto model (3.17) with attractive (repulsive) coupling on any undirected graph $\Gamma$ and for any $c \in \mathbb{T}$.

Next, we turn to the repulsive coupling case, which leads us to consider the maximization problem

$$\Phi(\theta) = \langle He^{2\pi i \theta}, He^{2\pi i \theta} \rangle \rightarrow \max, \quad \theta \in \mathbb{T}^n, \quad (3.29)$$

which, in turn, is equivalent to the following problem

$$Q(z) = z^* L z \rightarrow \max, \quad z \in \mathbb{T}_C^n. \quad (3.30)$$

For the remainder of this section, we assume that $\Gamma$ is a Cayley graph on a cyclic group so that its eigenvalues and the corresponding eigenvectors are given in Lemma 2.6. Denote the largest EV of $L$

$$\bar{\lambda} = \max\{\lambda_x : x \in \mathbb{Z}_n\}. \quad (3.31)$$

Since the corresponding eigenvectors

$$e_m, \ m \in M = \{j \in \mathbb{Z}_n : \lambda_j = \bar{\lambda}\}, \quad (3.32)$$
belong to $\mathbb{T}^n_C$, $e^{2\pi i c}e_m$ maximize $Q$ on $\mathbb{T}^n_C$, for any $c \in \mathbb{R}$ and $m \in M$. By arguing as in the attractive coupling case, we conclude that the corresponding twisted states

$$u_m = (u_m(0), u_m(1), \ldots, u_m(n-1)), \quad u_m(x) = \frac{mx}{n} + c \pmod{1}, \quad c \in \mathbb{R}, m \in M$$

(3.33)

are stable equilibria of (3.1). Thus, we arrive at the following result.

**Theorem 3.8** Suppose $\Gamma$ is a Cayley graph of a cyclic group and $L$ is the graph Laplacian of $\Gamma$. Then, twisted states (3.33) corresponding to the largest eigenvalue of $L$ (3.31) are stable (unstable) equilibria of the Kuramoto model on $\Gamma$ is the coupling is repulsive (attractive).

### 4 Synchronization

In this section, we study the stability of the family of synchronous solutions of (3.17) in more detail. Theorem 3.7 shows that synchronous solutions are stable equilibria of (3.17) with attractive coupling. Below, we show that the invariant subspace formed by such solutions is, in fact, asymptotically stable. To this end, we use the properties of the system at hand rather than relying on the general stability result for gradient systems in Absil and Kurdyka (2006). Specifically, the analysis below uses on the Lyapunov function $U_\alpha$.

Let

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$$

denote the eigenvalues of the Laplacian $L = H^TH$ and let $v_1, v_2, \ldots, v_n$ be an orthonormal set of the corresponding eigenvectors. We choose $v_1 = n^{-1/2}1_n$, where $1_n = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$. Recall

$$D = \text{span}\{c1_n \pmod{1}, \quad c \in \mathbb{R}\},$$

the invariant subspace of (3.22) corresponding to the synchronous dynamics. Next, we construct a system of tubular neighborhoods of $D$.

Let $0 < \delta < 1/2$ and denote

$$E_\delta = \left\{ \theta = c_1v_1 + V\tilde{c} \pmod{1}: \ c_1 \in \mathbb{R}, \ \sum_{i=2}^n \lambda_i c_i^2 < \frac{\delta^2}{(2\pi)^2} \right\},
\text{ (4.1)}$$

and its boundary

$$\partial E_\delta = \left\{ \theta = c_1v_1 + V\tilde{c} \pmod{1}: \ c_1 \in \mathbb{R}, \ \sum_{i=2}^n \lambda_i c_i^2 = \frac{\delta^2}{(2\pi)^2} \right\}.
\text{ (4.2)}$$
Here, $\bar{c} = (c_2, c_3, \ldots, c_n)$ and $V = \text{col}(v_2, v_3, \ldots, v_n)$. The following lemma describes the level sets of $\Phi$.

**Lemma 4.1** There exists $\delta_0 > 0$ such that

$$
\Phi \Big|_{\partial E_\delta} \leq \delta^2, \quad (4.3)
$$

$$
\Phi \Big|_{\partial E_{2\delta}} \geq 2\delta^2, \quad (4.4)
$$

provided $0 < \delta \leq \delta_0$.

**Proof** Recall (3.24)

$$
\Phi(\theta) = 2 \sum_{k:j} \left(1 - \cos(2\pi(\theta_k - \theta_j))\right). \quad (4.5)
$$

After expanding the right-hand side of (4.5) into Taylor series and using the remainder estimates for the resultant alternating series, we have

$$
\sum_{k:j} (2\pi(\theta_k - \theta_j))^2 - \frac{1}{12} \sum_{k:j} (2\pi(\theta_k - \theta_j))^4 \leq \Phi(\theta) \leq \sum_{k:j} (2\pi(\theta_k - \theta_j))^2. \quad (4.6)
$$

Using the definitions of $H$ and $L$, we have

$$
\sum_{k:j} (\theta_k - \theta_j)^2 = \langle H\theta, H\theta \rangle = \theta^T L\theta. \quad (4.7)
$$

By plugging $\theta = c_1 1_n + V\bar{c}$ into (4.6) and using (4.7), we have

$$
(2\pi)^2 \sum_{i=2}^n \lambda_i c_i^2 - \psi(\bar{c}) \leq \Phi(\theta) \leq (2\pi)^2 \sum_{i=2}^n \lambda_i c_i^2. \quad (4.8)
$$

where

$$
0 \leq \psi(\bar{c}) = \frac{(2\pi)^4}{12} \sum_{k:j} ((V\bar{c})_k - (V\bar{c})_j)^4 \leq C_1 \|\bar{c}\|^4. \quad (4.9)
$$

for some $C_1 > 0$.

The right-hand side inequality in (4.8) and the definition of $\partial E_\delta$ (cf. 4.2) imply (4.3). Further, suppose

$$
0 < \delta < \delta_0 = \min \left\{ \frac{\lambda_2}{2\sqrt{2}C_1}, \frac{1}{2} \right\}. \quad (4.10)
$$

Then, from the left-hand side inequality in (4.8), we have

$$
\Phi \big|_{\partial E_{2\delta}} \geq 4\delta^2 - \psi \big|_{\partial E_{2\delta}}. \quad (4.11)
$$
For $\theta \in \partial E_{2\delta}$,
\[
\lambda_2 \|\bar{c}\|^2 \leq \sum_{i=2}^{n} \lambda_i c_i^2 = 4\delta^2,
\]
and thus,
\[
\|\bar{c}\|^4 \leq \frac{16\delta^4}{\lambda_2^2}.
\]
(4.12)

The combination of (4.11), (4.12), and (4.9) yields
\[
\Phi_1|_{\partial E_{2\delta}} \geq 4\delta^2 - \frac{16C_1\delta^2}{\lambda_2} \geq 2\delta^2,
\]
where we used $0 < \delta \leq \delta_0$ to derive the last inequality (cf. 4.10).

**Theorem 4.2** Invariant subspace $\mathcal{D}$ of (3.22) is asymptotically stable if $\alpha = 0$ and is unstable if $\alpha = 1$.

**Proof** We consider the case of $\alpha = 0$ first. Let $\theta_0 \in E_{\delta_1}$ for some $0 < \delta_1 \leq \delta_0$. We want to show that $\omega$-limit set of $\theta_0$, $\omega(\theta_0) \in \mathcal{D}$. This is clearly true for $\theta_0 \in \mathcal{D}$. We, therefore, assume $\theta_0 \in E_{\delta_1}/\mathcal{D}$. Then, $\theta_0 \in \partial E_{\delta}$ for some $0 < \delta < \delta_1$.

By (4.3) and (3.23),
\[
\Phi(\theta(t)) \leq \delta^2, \quad t \geq 0.
\]
Thus, $\theta(t) \in E_{2\delta}$ for all $t \geq 0$. By Hartman (2002, Lemma 11.1), $\omega(\theta_0) \in E_0 = \{\theta \in \mathbb{T}^n: \Phi(\theta) = 0\} \cap E_{\delta}$.

It remains to show that $E_0 = \mathcal{D}$. To this end, let $\theta = c_1 \mathbf{1}_n + V\bar{c}$ and note that
\[
\Phi(\theta) = (2\pi)^2 \sum_{i=2}^{n} \lambda_i c_i^2 + O(\|\bar{c}\|^4),
\]
by (4.8) and (4.9). It follows from (4.13) that there are no critical points of $\Phi$ outside $\mathcal{D}$ in $E_{\delta_1}$ for sufficiently small $\delta_1 > 0$. Thus,
\[
\nabla \Phi(\theta) \neq 0, \quad \theta \in E_{\delta}/\mathcal{D} \forall 0 < \delta < \delta_1.
\]
(4.14)

By combining (4.14) and (3.23), we have $E_0 = \mathcal{D}$. Thus, $\omega(\theta_0) \in \mathcal{D}$.

Next, we consider the repulsive coupling case. Set $\alpha = 1$ in (3.22) and fix $0 < \delta_2 < \delta_1$ such that (4.14) holds. Let $\theta_0 \in E_{\delta_2}/\mathcal{D}$. Then, $\theta_0 \in \partial E_{\delta}$ for some $0 < \delta < \delta_2$. Function $\|\nabla \Phi(\theta)\|^2$ is bounded away from zero in $E_{\delta_2}/E_{\delta}$. By (3.23), the trajectory starting from $\theta_0$ leaves $E_{\delta_2}$ in finite time. Thus, $\mathcal{D}$ is unstable. $\square$
We now turn to the repulsive coupling case. Here, we will need an additional assumption that $\Gamma$ is a Cayley graph on the cyclic group $\mathbb{Z}_n$. Thus, in the remainder of this section, instead of (3.17) we consider (3.1). Recall that $u_m, m \in M$, denote twisted states corresponding to the largest eigenvalue of $L$ (see 3.33). Below, we show that $u_m, m \in M$ are stable twisted steady states of the Kuramoto model with repulsive coupling under an additional nondegeneracy condition on the Hessian matrix

$$h_{ij} := (H(\Phi)(u_m))_{ij} = \frac{\partial^2 \Phi}{\partial \theta_i \partial \theta_j}(u_m), \quad m \in M. \quad (4.15)$$

By differentiating (3.25) with respect to $\theta_i$ and using (3.33), we have

$$h_{ij} = \begin{cases} -4\pi \sin \left( \frac{2\pi m(j-i)}{n} \right), & ij \in E, \\ 0, & \text{otherwise} \end{cases} \quad (4.16)$$

Since sin is an odd function and $\Gamma$ is a Cayley graph, we have $h_{ij} = -h_{ji}$. Thus, the Hessian matrix $H(\Phi)(u_m)$ has zero row sums. Therefore, $-H(\Phi)(u_m)$ has a zero eigenvalue $\tilde{\lambda}_1 = 0$ with the corresponding eigenvector $\tilde{v}_1 = n^{-1/2}1_n$. Denote the remaining eigenvalues of $-H(\Phi)(u_m)$ arranged in the increasing order by $\tilde{\lambda}_k, k = 2, 3, \ldots, n$. Choose an orthonormal basis in $\mathbb{R}^n$ from the eigenvectors of $H(\Phi)(u_m)$, $\{v_k, k \in [n]\}$. Because $u_m$ is a point of maximum of $\Phi$, all eigenvalues $\tilde{\lambda}_k, k \in [n]$, are nonnegative:

$$0 = \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots \leq \tilde{\lambda}_n. \quad (4.17)$$

**Theorem 4.3** Let $u_m$ be a twisted state corresponding to the largest eigenvalue of $L$, i.e., $m \in M$. Suppose that $\tilde{\lambda}_0 = 0$ is a simple eigenvalue of $-H(\Phi)(u_m)$. Then,

$$D_m = \{\theta = u_m + c1_n \pmod{1}: c \in \mathbb{R}\} \quad (4.18)$$

is asymptotically stable.

**Proof** Let

$$\tilde{\Phi}(\theta) = \Phi(u_m) - \Phi(\theta) \quad \text{and} \quad \tilde{E}_\delta$$

$$= \left\{ \theta = u_m + c_11_n + V\tilde{c} \pmod{1}: c_1 \in \mathbb{R}, \sum_{i=2}^n \tilde{\lambda}_i c_i^2 < \frac{\delta^2}{(2\pi)^2} \right\}. \quad (4.19)$$

Then, $\tilde{\Phi}$ is nonnegative and is monotonically decreasing along the trajectories of (3.22) ($\alpha = 1$). Following the lines of the proof of Lemma 4.2, one can show that for initial condition $\theta_0 \in \tilde{E}_\delta$, the trajectory $\theta(t)$ remains in $\tilde{E}_{2\delta}$, provided $\delta > 0$ is sufficiently small. Furthermore, for small $\delta > 0$, there are no zeros of $\tilde{\Phi}$ in $\tilde{E}_{2\delta}$ outside $D_m$. From this, we conclude that $\omega(\theta_0)$ is contained in maximal invariant subspace contained in the set of zeros of $\tilde{\Phi}$ in $\tilde{E}_\delta$, i.e., in $D_m$. □
5 The Kuramoto Model on $K_n$

In this and in the following sections, we apply the results of Sects. 3 and 4 to study stability of twisted states on complete and Paley graphs. We first focus on the Kuramoto model on the family of complete graphs $K_n$. To simplify the presentation, we consider $K_n$ for odd $n$, so that they can be viewed as Cayley graphs.

By (2.8), the graph Laplacian of $K_n$ has a simple zero eigenvalue and $n-1$ multiple eigenvalues equal to $n-1$. This immediately implies that all nontrivial twisted states ($q = 0$) are unstable if the coupling is attractive and are stable for repulsive coupling (cf. Theorem 3.8). Thus, the stability of all twisted states is determined completely by the spectrum of the graph Laplacian. In the remainder of this section, we develop a complementary approach to studying stability of twisted states using linearization. The main difficulty in implementing this approach is dealing with the highly degenerate matrix of the linearized system about the twisted states. This problem does not come up in the stability analysis of phase-locked solutions in the Kuramoto model with distributed intrinsic frequencies $\omega(x) \neq \text{const}$ (Mirollo and Strogatz 2005; Bronski et al. 2012). Our analysis shows that this degeneracy is due to the fact that nontrivial twisted states lie in an $(n-2)$-dimensional manifold formed by the equilibria of the Kuramoto model. Therefore, in addition to recovering the stability results, known from the variational properties of the Kuramoto model, the second approach yields a detailed picture of the phase flow near the nontrivial twisted states.

We start by computing the eigenvalues of the linearized problem (3.4).

**Lemma 5.1** The eigenvalues of the linearization of the Kuramoto model on $K_n$ about the $q$-twisted state $u_{n(q)}$ are

$$\begin{align*}
q = 0: \\
\lambda_0(0) &= 0 \quad \text{and} \\
\lambda_x(0) &= (-1)^{q+1}4\pi \left(1 + (n-1)^{-1}\right), \quad x \in \mathbb{Z}_n^\times, \\
q \in \mathbb{Z}_n^\times:
\lambda_x(q) &= \left\{ \begin{array}{ll}
(-1)^q 2\pi [1 + (n-1)^{-1}], & \quad x \in \{q, n-q\}, \\
0, & \quad \text{otherwise}.
\end{array} \right.
\end{align*}$$

where $n \in \mathbb{N}$ is odd.

**Proof** We compute the eigenvalues for the Kuramoto model on Cayley graphs generated by the ball $B(r)$, Cay($Z_n, B(r)$), for any $r \in [n-1/2]$ first. We then use this result to compute the eigenvalues of $K_{2r+1} = \text{Cay}(Z_{2r+1}, B(r))$.

By Lemma 3.2,

$$\begin{align*}
\lambda_x(q) &= \frac{(-1)^q \pi}{r} \sum_{y=-r}^{r} \left\{ \cos \left(\frac{2\pi(q + x)y}{n}\right) - 2 \cos \left(\frac{2\pi q y}{n}\right) \right. \\
&\quad \left. + \cos \left(\frac{2\pi(q - x)y}{n}\right) \right\}, \quad q \in \mathbb{Z}_n.
\end{align*}$$

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Using the formula for the partial sum of the geometric series, we obtain
\[
\sum_{k=-r}^{r} w^q k = \frac{w^q - w^{-q}}{1 - w^q} = \frac{w^q(r+1) - 1}{w^q/2 - w^{-q/2}},
\]
which for \( w = e^{\frac{2\pi i}{n}} \) becomes
\[
\sum_{k=-r}^{r} \cos \left( \frac{2\pi q k}{n} \right) = \begin{cases} 
\frac{\sin(\pi q (2r+1)/n)}{\sin(\pi q/n)} - 2r + 1, & q = 0, \\
\frac{\sin(\pi q (2r+1)/n)}{\sin(\pi q/n)}, & q \neq 0.
\end{cases}
\]
(5.4)

The combination of (5.3) and (5.4) yields
\[
\lambda_x(q) = \frac{(-1)^\alpha \pi}{r} \left[ S(q + x, 2r + 1) - 2S(q, 2r + 1) + S(q - x, 2r + 1) \right], \quad x, q \in \mathbb{Z}_n.
\]
(5.5)

where
\[
S(q, m) = \begin{cases} 
\frac{\sin(\pi q m / n)}{\sin(\pi q/n)}, & q \in \mathbb{Z}_n^\times, \\
\sin(\pi q/n), & q = 0.
\end{cases}
\]

The formulas for the eigenvalues of the linearized problem in (5.1) and (5.2) follow from (5.5) with \( n = 2r + 1 \).

By Theorem 4.2, the synchronous state \( q = 0 \) is stable when the coupling is attractive. By (5.2), there are no other stable twisted states in the attractive coupling case. Thus, in the remainder of this section we focus on the repulsive coupling case. Theorem 4.2 implies that the synchronous solutions are unstable if the coupling is repulsive. If \( q \neq 0 \) and \( \alpha = 1 \), (5.2) shows that the spectrum of the linearized problem contains two negative eigenvalues and \( n - 2 \) zero eigenvalues. The linearization alone, therefore, does not resolve stability of the twisted states. The argument below deals with the presence of the zero eigenvalues.

**Lemma 5.2** Consider \( F: \mathbb{T}^n \to \mathbb{C} \) defined by
\[
F(u) = \sum_{k=1}^{n} e^{2\pi i u_k}, \quad u = (u_1, u_2, \ldots, u_n).
\]
(5.6)

Then, every element in \( F^{-1}(0) = \{ u \in \mathbb{T}^n: F(u) = 0 \} \) is a steady-state solution of (1.2). Furthermore, \( q \)-twisted states \( u_n^{(q)} \in F^{-1}(0) \) for all \( q \in \mathbb{Z}_n \).
Proof From \( F(u) = 0 \), it follows

\[
\text{Im} \ F(u) e^{-2\pi i u_j} = 0 \quad \forall j \in [n], \tag{5.7}
\]

which, in turn, implies

\[
\sum_{k=1}^{n} \sin(2\pi (u_k - u_j)) = 0 \quad \forall j \in [n]. \tag{5.8}
\]

Thus, \( F^{-1}(0) \) is formed by steady states of \( (1.2) \).

Next, we show that \( u^{(q)}_n \) is a stable steady state of the Kuramoto model on \( K_n \) with repulsive coupling for any odd \( n \in \mathbb{N} \) and \( q \in [n - 1] \) or \( \{0\} \).

Theorem 5.3 The twisted state \( u^{(q)}_n \) is a stable steady state of the Kuramoto model on \( K_n \) with repulsive coupling for any odd \( n \in \mathbb{N} \) and \( q \in [n - 1] \).

Proof By (5.2), the spectrum of the linearized problem in the repulsive coupling case contains two negative eigenvalues and \( n - 2 \) zero eigenvalues. Therefore, the stability of the equilibrium is decided by the dynamics on the center manifold. Below, we show that \( u^{(q)}_n \) lies in the \((n - 2)\)-dimensional smooth manifold of equilibria. From this, we will derive stability.

Let \( \mathbb{R} F(u) \) stand for the realification of \( F(u) \):

\[
\mathbb{R} F(u) = \left( \sum_{k=0}^{n-1} \cos(2\pi u_k) \sum_{k=0}^{n-1} \sin(2\pi u_k) \right).
\]

From the definition of \( u^{(q)}_n \) and (5.4), we have

\[
\mathbb{R} F \left( u^{(q)}_n \right) = 0. \tag{5.9}
\]

Further,

\[
\frac{\partial}{\partial u} \mathbb{R} F \left( u^{(q)}_n \right) = 2\pi \begin{pmatrix} -\sin(2\pi q/n) & -\sin(4\pi q/n) & \cdots & -\sin(2(n-1)\pi q/n) \\ \cos(2\pi q/n) & \cos(4\pi q/n) & \cdots & \cos(2(n-1)\pi q/n) \end{pmatrix} \tag{5.10}
\]

and

\[
\begin{vmatrix} -\sin(2\pi q/n) & -\sin(4\pi q/n) \\ \cos(2\pi q/n) & \cos(4\pi q/n) \end{vmatrix} = -\sin(2\pi q/n) \neq 0. \tag{5.11}
\]
because \( n \) is odd. This implies rank \( \frac{\partial}{\partial u} F(u_n^{(q)}) = 2 \). Thus, 0 is a regular value of \( \mathbb{R} F \). By the Regular Value Theorem (Hirsch 1994, Theorem 3.2), near \( u_n^{(q)} \), \( \mathbb{R} F^{-1}(0) \) is an \((n - 2)\)-dimensional differentiable manifold, which we denote by \( \mathcal{M} \). Let \( \psi \) be a local parametrization of \( \mathcal{M} \) near \( u_n^{(q)} \).

Consider the change of variables \( v = \psi^{-1}(u) \). Then,

\[
\dot{v} = \frac{\partial}{\partial u} \psi^{-1}(u) f(\psi(u)) = 0,
\]

because \( \psi(U) \subset \mathcal{M} \) and \( f|_{\mathcal{M}} = 0 \). Equation (5.12) represents the flow on the \((n - 2)\)-dimensional center manifold of \( u_n^{(q)} \). Since the spectrum of the linearized problem has two negative eigenvalues, by the Reduction Theorem (cf. Arnold et al. 1999), in a small neighborhood of \( u_n^{(q)} \), the flow is topologically equivalent to the standard saddle suspension over its restriction to the center manifold

\[
\dot{x} = -x, \\
\dot{y} = 0, (x, y) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}.
\]

This shows that \( u_n^{(q)} \) is stable. \( \square \)

### 6 The Kuramoto Model on \( P_n \)

In this section, we focus on stability of twisted states in the Kuramoto model on Paley graphs \( P_n \). Thus, we consider (3.1) with \( \Gamma = P_n \). It is instructive to discuss the linearization about the twisted state \( u_n^{(q)} \), \( q \in \mathbb{Z}_n \), first. By Lemma 3.2, the eigenvalues of the linearized problem are

\[
\lambda_x(q) = \frac{(-1)^q 4\pi}{n - 1} \sum_{k=1}^{n-1} \cos(2\pi(q + x)k^2/n) - 2\cos(2\pi qk^2/n) \\
+ \cos(2\pi(q - x)k^2/n), \quad x \in \mathbb{Z}_n.
\]

We rewrite (6.1) in terms of the Gauss sums (2.10) as follows

\[
\lambda_x(q) = \frac{(-1)^q 4\pi}{n - 1} \{G_n(q + x) - 2G_n(q) + G_n(q - x)\}, \quad x \in \mathbb{Z}_n.
\]

The value of \( G_n(q) \) in (6.2) depends on whether or not \( q \) is a QR modulo \( n \). This leads to several cases for stability of twisted states on Paley graphs. We summarize the information about the spectrum of the linearized problem in the following lemma.

**Lemma 6.1** The eigenvalues of the linearization of the Kuramoto model (3.1) with \( \Gamma = P_n \) about \( q \)-twisted state \( u_n^{(q)} \), \( q \in \mathbb{Z}_n \), are as follows. For all \( q \in \mathbb{Z}_n \), \( \lambda_0(q) = 0 \). The remaining eigenvalues \( \lambda_x(q) \), \( x \in \mathbb{Z}_n^\times, q \in \mathbb{Z}_n \), are given in Table 1.
Table 1  The spectrum of the linearized equation

| (A) $q = 0$ | $\lambda_x(q) \leq -4\pi(1 + O(n^{-1/2}))$ | $\lambda_x(q) > 0$ |
| (B) $q$ is a QR (mod n) | $\lambda_x \leq 0$ | $\lambda_x(q) \geq 0$ |
| (C) $q$ is a not QR (mod n) | $\lambda_x \geq 0$ | $\lambda_x(q) \leq 0$ |

In each of the above cases, the spectrum contains a nonzero eigenvalue

Table 2  Stability of twisted states

| (A) $q = 0$ | Stable (V,L) | Unstable (VL) |
| (B) $q$ is a QR | Unknown | Unstable (L) |
| (C) $q$ is a not QR (mod n) | Unstable (V) | Stable (V) |

We used L and V to indicate that the conclusion about the stability of a twisted state was based on the linearization or the variational argument, respectively

Corollary 6.2  If $q$ is not a QR modulo n, then $q$-twisted states are unstable in the Kuramoto model (3.1) with $\Gamma = P_n$ with repulsive coupling.

Proof (Lemma 6.1) Consider (C1) first: $\alpha = 0$ and $q$ is not a QR. Then, $G_n(q) = -\sqrt{n}$ by (2.11). Further, pick $y \in \mathbb{Z}_n^\times$ such that $y$ is a QR. Using $x = |q - y|$ in (6.2), we have $\lambda_x(q) \geq \pi n^{-1/2} > 0$. This shows (C1) in Table 1. The results for all other case follow similarly from (6.2) and (2.11).

Aside from the stability of the synchronous solutions in (A) and instability result in (C1), the linear stability analysis of $q$-twisted states on Paley graphs is inconclusive, because we cannot exclude the presence of zero eigenvalues. Thus, we cannot decide on stability in (B1) and in (C2) based on linearization. To clarify stability of twisted states in these cases, we employ the variational principle of Sect. 3.2. To this end, we compute the eigenvalues of the graph Laplacian of $P_n$. Using (2.12), we compute

$$\lambda_x(P_n) = \begin{cases} 2^{-1}(n - \sqrt{n}), & x \text{ is a QR (mod n),} \\ 2^{-1}(n + \sqrt{n}), & x \text{ is not a QR (mod n),} \end{cases}$$

for $x \in \mathbb{Z}_n^\times$ (cf. 2.12). Theorem 3.8 implies that if $q$ is not a QR (mod n), the $q$-twisted states are stable (unstable) if the coupling is repulsive (attractive). Likewise, Theorem 3.7 yields stability (instability) of synchronous solutions ($q = 0$) for attractive (repulsive) coupling. Thus, we have resolved stability of the $q$-twisted states in (A), (C), and (B2). For part (B1), we have to resort to numerics, which suggests that twisted states are unstable in this case. We summarize information about stability of twisted states on Paley graphs in Table 2.
7 The Kuramoto Model on $G(n, p)$

In this section, we study the Kuramoto model on the ER graph $G(n, p)$. The random graph $G(n, p)$ is not a Cayley graph. Therefore, neither of the techniques of Sect. 3 applies to the analysis of the Kuramoto model on $G(n, p)$. Moreover, in contrast to the graphs that we considered above, $G(n, p)$ in general does not support twisted states as steady-state solutions of the Kuramoto model for finite $n$. Nonetheless, in this section, we show that the dynamics of the Kuramoto model on random and complete graphs are closely related. Specifically, for large $n$ the solutions of the IVP for the Kuramoto model on $G(n, p)$ can be approximated by those for the Kuramoto model on $K_n$. This result is a discrete version of the homogenization, because the Kuramoto model on the complete graph can be viewed as an averaged model on the ER graph. Having established the link between the Kuramoto models on ER and complete graphs, we use it to elucidate the dynamics of the former model. To this effect, we show that twisted states become steady-state solutions of the Kuramoto model on $G(n, p)$ almost surely in the limit as $n \to \infty$. Further, we show that all nontrivial twisted states in the Kuramoto model on $G(n, p)$ with repulsive coupling are metastable for finite albeit large $n$.

7.1 Discrete Homogenization

Our next goal is to show that the solutions of the IVPs for coupled dynamical systems on the large complete and ER graphs remain close on finite time intervals with high probability, provided they start from close initial data. To achieve this, we employ the method developed in Medvedev (2014b).

Let $A_n = (a_{ni,j})$ be the adjacency matrix of the ER random graph $G(n, p)$. $A_n$ is a symmetric matrix with zero diagonal. Entries $a_{ni,j}$, $1 \leq i < j \leq n$, are independent identically distributed random variables (RVs) from the binomial distribution with parameter $p \in (0, 1)$. The homogenization result, that we prove below, holds for the following class of models, which covers the Kuramoto model. Specifically, for a fixed $n \in \mathbb{N}$ assume that $u_{ni}, i \in [n]$ satisfy the following system of differential equations

$$\frac{d}{dt} u_{ni}(t) = (np)^{-1} \sum_{i=1}^{n} a_{ni,j} D(u_{nj} - u_{ni}), \quad i \in [n],$$

(7.1)

where $D$ is a Lipschitz continuous function with constant $L > 0$

$$|D(u) - D(v)| \leq L|u - v|, \quad u, v \in \mathbb{R}$$

(7.2)

Along with (7.1), we consider the averaged model on the complete graph $K_n$

$$\frac{d}{dt} v_{ni}(t) = n^{-1} \sum_{i=1}^{n} D(v_{nj} - v_{ni}), \quad i \in [n].$$

(7.3)

Remark 7.1 Note that we are using a different scaling compared with that in (1.2). The right-hand side of (7.1) is scaled by the expected value of the degree of a node of
The analysis of the model scaled by the actual degree can be done similarly. We choose the former scaling for analytical convenience. For the same reason, we scale the right-hand side of the Kuramoto model on $K_n$ (7.3) by $n$ rather than by $n - 1$, the degree of $K_n$.

The following weighted Euclidean inner product
\[ (u, v)_n = \frac{1}{n} \sum_{i=1}^{n} u_i v_i, \quad u = (u_1, u_2, \ldots, u_n)^T, \quad v = (v_1, v_2, \ldots, v_n)^T, \] (7.4)
and the corresponding norm $\|u\|_{n,2} = \sqrt{(u, u)_n}$ will be used to measure the distance between solutions of the dynamical equations on random and complete graphs. We are ready to state our first result.

**Theorem 7.2** Let
\[ u_n(t) = (u_{n1}(t), u_{n2}(t), \ldots, u_{nn}(t)) \text{ and } v_n(t) = (v_{n1}(t), v_{n2}(t), \ldots, v_{nn}(t)) \]
denote the solutions of (7.1) and (7.3) subject to the same initial condition $u_n(0) = v_n(0)$. Assume that for a given $T > 0$, there are constants $0 < C_1 \leq C_2$ such that
\[ C_1 \leq \lim \inf_{n \to \infty} \min_{t \in [0,T]} n^{-2} \sum_{i,j=1}^{n} D(v_{nj} - v_{ni})^2 \]
\[ \leq \lim \sup_{n \to \infty} \max_{t \in [0,T]} n^{-2} \sum_{i,j=1}^{n} D(v_{nj} - v_{ni})^2 \leq C_2. \] (7.5)

Then,
\[ \lim_{n \to \infty} \mathbb{P} \left\{ \max_{t \in [0,T]} \|u_n(t) - v_n(t)\|_{n,2} \geq Cn^{-1/2} \right\} = 0, \] (7.6)
where a positive constant $C = C(L, T)$ depends on $L$ and $T$ but not on $n$.

For the proof of Theorem 7.2, we will need the following application the Central Limit Theorem, which follows from Medvedev (2014b, Lemma 4.4 and Corollary 4.5).

**Lemma 7.3** Let $p \in (0, 1)$, $T \in [0, \infty)$ and $f_{nij} \in L^\infty([0, T])$. Suppose that RVs $\xi_{nij}, i, j \in [n], n \in \mathbb{N}$, are such that for fixed $n \in \mathbb{N}$ and $i \in [n], \{\xi_{nij}, j \in [n]\}$ are independent identically distributed binomial RVs with parameter $p \in (0, 1)$.

Denote
\[ \sigma_{ni}^2(t) = n^{-1} \sum_{i=1}^{n} f_{nij}^2(t) p(1 - p), \quad i \in [n], \quad \sigma_n^2(t) = n^{-1} \sum_{i=1}^{n} \sigma_{ni}^2(t), \]
\[ z_{ni}^2(t) = \frac{1}{n} \sum_{j=1}^{n} (\xi_{nij} - p) f_{nij}(t), \quad S_n(t) = \sum_{i=1}^{n} z_{ni}^2(t). \]
and assume that for some $0 < C_3 \leq C_4$

$$C_3 \leq \lim \inf_{n \to \infty} \min_{t \in [0,T]} \sigma_n^2(t) \leq \lim \sup_{n \to \infty} \max_{t \in [0,T]} \sigma_n^2(t) \leq C_4. \quad (7.7)$$

Then,

$$\frac{S_n(t) - \sigma_n^2(t)}{n^{-1/2} \sqrt{5\sigma_n^4(t) + O(n^{-1})}} \overset{d}{\to} N(0, 1) \text{ as } n \to \infty,$$  

where the convergence in (7.8) is in distribution.

Proof (Theorem 7.2) Denote $w_n = u_n - v_n$. By subtracting Equation $i$ in (7.3) from the corresponding equation in (7.1), we have

$$\frac{d}{dt} w_{ni} = \frac{1}{np} \sum_{j=1}^{n} a_{nij} \left[ D(u_{nj} - u_{ni}) - D(v_{nj} - v_{ni}) \right] + z_{ni}, \quad i \in [n], \quad (7.9)$$

where

$$z_{ni} = (np)^{-1} \sum_{j=1}^{n} (a_{nij} - p) D(v_{nj} - v_{ni}). \quad (7.10)$$

By multiplying both sides of (7.9) by $n^{-1} w_{ni}$ and summing over $i$, we have

$$\frac{1}{2} \frac{d}{dt} \| w_n \|_{2,n}^2 = \frac{1}{n^2} \sum_{i,j=1}^{n} a_{ij} \left[ D(u_{nj} - u_{ni}) - D(v_{nj} - v_{ni}) \right] w_{ni} + (z_n, w_n)_n. \quad (7.11)$$

We bound the first term on the right-hand side of (7.11) using the Lipschitz continuity of $D$, $|a_{ij}| \leq 1$, the Cauchy–Schwarz inequality, and the triangle inequality

$$\frac{1}{n^2} \sum_{i,j=1}^{n} a_{ij} \left[ D(u_{nj} - u_{ni}) - D(v_{nj} - v_{ni}) \right] w_{ni} \leq \frac{L}{n^2} \sum_{i,j=1}^{n} (|w_{nj}| + |w_{ni}|) |w_{ni}| \leq 2L \| w_n \|_{2,n}^2. \quad (7.12)$$

We bound the second term using the Cauchy–Schwarz inequality

$$|(z_n, w_n)_n| \leq \| z_n \|_{2,n} \| \eta_n \|_{2,n} \leq \frac{1}{2} \left( \| z_n \|_{2,n}^2 + \| w_n \|_{2,n}^2 \right), \quad (7.13)$$

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where \( z_n = (z_{n1}, z_{n2}, \ldots, z_{nn}) \). The combination of (7.11), (7.12), and (7.13) yields

\[
\frac{d}{dt} \| w_n \|_{2,n}^2 \leq (4L + 1) \| w_n \|_{2,n}^2 + \| z_n \|_{2,n}^2. 
\]  

(7.14)

By Gronwall’s inequality,

\[
\max_{t \in [0, T]} \| w \|_{2,n}^2 \leq \max_{t \in [0, T]} \| z_n(t) \|_{2,n}^2 \sqrt{(4L + 1)^T}. 
\]  

(7.15)

Thus,

\[
\max_{t \in [0, T]} \| w \|_{2,n} \leq \max_{t \in [0, T]} \| z_n(t) \|_{2,n} \sqrt{(2L + 1)^T}. 
\]  

(7.16)

It remains to estimate \( \| z_n(t) \|_{2,n} \) (see 7.10). To this end, we use Lemma 7.3 with \( f_{nij}(t) := D(v_{nj}(t) - v_{ni}(t)) \).

Using the assumption (7.5), we verify that (7.7) holds. Thus, we apply Lemma 7.3, to show that

\[
P\{n \| z_n(t) \|_{2,n}^2 - \sigma_n^2(t) \leq 1 \} \leq \begin{cases} 
\mathbb{P}\left( \left| n \| z_n(t) \|_{2,n}^2 - \sigma_n^2(t) \right| > \frac{n^{1/2}}{\sqrt{5\sigma_n^4(t) + O(n^{-1})}} \right) \\ \leq \mathbb{P}\left( \left| n \| z_n(t) \|_{2,n}^2 - \sigma_n^2(t) \right| > \frac{n^{1/2}}{\sqrt{5\sigma_n^4(t) + O(n^{-1})}} \right) \to 0 \text{ as } n \to \infty 
\end{cases} 
\]  

(7.17)

uniformly in \( t \in [0, T] \).

Using (7.5), from (7.17), we have

\[
P \left( \| z_n(t) \|_{2,n}^2 \leq (C_4 + 1) n^{-1} \right) \leq P\{n \| z_n(t) \|_{2,n}^2 - \sigma_n^2(t) \leq 1 \} \to 0 \text{ as } n \to \infty
\]  

(7.18)

uniformly in \( t \in [0, T] \). Thus,

\[
\lim_{n \to \infty} P \left\{ \max_{t \in [0, T]} \| z_n(t) \|_{2,n} \leq C_5 n^{-1/2} \right\} = 0, \quad C_5 = \sqrt{C_4 + 1}. 
\]  

(7.19)
7.2 Twisted States on the Random Network

In this subsection, we prove two results on the existence of the twisted states for the Kuramoto model on ER graphs. First, we show that in the limit of large $n$, twisted states become steady-state solutions of the model on random graph and study the stability of twisted states. Furthermore, for finite albeit large $n \in \mathbb{N}$, we show that the solution of the IVP for the Kuramoto model on $G(n, p)$ with the initial condition sufficiently close to a given twisted state remains near this twisted state for a long time. This means that while twisted generically are not the equilibria of the Kuramoto model on $G(n, p)$ for finite $n$, they can be interpreted as metastable states of the random network provided $n$ is sufficiently large. Both results elucidate the relation between the Kuramoto models on $K_n$ and $G(n, p)$.

The existence of twisted states for the Kuramoto model on $G(n, p)$ in the limit as $n \to \infty$ follows from the argument that was used in the analysis of the Kuramoto model on small-world networks (Medvedev 2014c).

**Theorem 7.4** Let $\{G(n, p)\}$ be a sequence of the ER random graphs with adjacency matrices $A_n = (a_{ni})$, and

$$R_{ni}(u_n) = \lim_{n \to \infty} \frac{1}{np} \sum_{j=1}^{n} a_{nj} \sin(2\pi(u_{nj} - u_{ni})), \quad u_n \in \mathbb{R}^n, \quad i \in [n]. \quad (7.20)$$

Then, for any $q \in \mathbb{N} \cup \{0\}$ and $i \in \mathbb{N}$

$$\lim_{n \to \infty} R_{ni}(u_n^{(q)}) = 0 \quad (7.21)$$

almost surely.

**Proof** Let $i \in \mathbb{N}$ be arbitrary but fixed and $n \geq i$. Recall that $a_{nj}$, $1 \leq i < j \leq n$, are independent identically distributed binomial (with parameter $p$) RVs and consider

$$\eta_{ni} = a_{nj} \sin(2\pi(u_{nj}^{(q)} - u_{ni}^{(q)})) = a_{nj} \sin\left(\frac{2\pi q(j - i)}{n}\right), \quad (7.22)$$

$$S_{ni} = \sum_{j=1}^{n} \eta_{ni}. \quad (7.23)$$

Using (7.20) and the definitions of $\{\eta_{ni}\}$, we have

$$R_{ni}(u_n^{(q)}) = (np)^{-1}S_{ni}. \quad (7.24)$$
Using the binomial distribution of $a_{nj}$ and (7.23), it is straightforward to compute
\[
E \eta_{nj}^2 = \sin \left( \frac{2\pi q(j - i)}{n} \right)^2 p \leq 1, \quad (7.25)
\]
\[
E \eta_{nj}^4 = \sin \left( \frac{2\pi q(j - i)}{n} \right)^4 p \leq 1, j \in [n]. \quad (7.26)
\]
Further, using (7.23), we have
\[
E S_{ni} = p \sum_{j=1}^{n} \sin \left( \frac{2\pi q(j - i)}{n} \right) = 0, \quad (7.27)
\]
and
\[
E S_{ni}^4 = \sum_{j_1, j_2, j_3 \in [n]} E \left[ \eta_{n j_1} \eta_{n j_2} \eta_{n j_3} \eta_{n j_4} \right] \\
= \sum_{j \in [n]} E \left[ \eta_{n j}^4 \right] + \binom{4}{2} \sum_{j_1 < j_2, j_1, j_2 \in [n]} E \left[ \eta_{n j_1}^2 \eta_{n j_2}^2 \right] \\
\leq n + 3n(n - 1) < 3n^2, \quad (7.28)
\]
where we used independence of $\{\eta_{n j}: j \in [n]\}$, (7.25), (7.26), and (7.27).

By Markov inequality, using (7.27) and (7.28), for any $\epsilon > 0$, we have
\[
\mathbb{P}\{|S_{ni}| \geq n\epsilon\} \leq 3\epsilon^{-4}n^{-2}, \quad (7.29)
\]
which in turn implies via the first Borel–Cantelli lemma (Billingsley 1995, Theorem 4.3)
\[
\mathbb{P}\{|n^{-1}S_{ni}| \geq \epsilon \text{ holds for infinitely many } n\} = 0,
\]
By Theorem 5.2(i) in Billingsley (1995), the last statement is equivalent to convergence of $n^{-1}S_{ni}$ to 0 almost surely.

Next, we show that in the Kuramoto model on $G(n, p)$ with finite $n$, twisted states are metastable.

**Theorem 7.5** For any $\epsilon_1, \epsilon_2 \in (0, 1)$, $T > 0$ and $q \in \mathbb{N} \cup \{0\}$ there exists $N = N(\epsilon_1, \epsilon_2)$ such that for every $n \geq N$ one can find $\delta = \delta(n, \epsilon_2)$ so that
\[
\|u_n(0) - u_n^{(q)}\|_{n, 2} < \delta \quad \implies \quad \mathbb{P}\{\max_{t \in [0, T]} \|u_n(t) - u_n^{(q)}\|_{n, 2} > \epsilon_1\} < \epsilon_2.
\]
Here,
\[ u_n^{(q)} = \left( u_n^{(q)}(0), u_n^{(q)}(1), \ldots, u_n^{(q)}(n-1) \right), \]
\[ u_n^{(q)}(x) = \frac{2\pi q x}{n} + c \pmod{n}, \quad x \in \mathbb{Z}_n. \] (7.30)

denotes a q-twisted state for some \( c \in \mathbb{R} \).

**Remark 7.6** For \( q = 0 \), the (in)stability follows from Theorem 3.7: a synchronous solution is stable if the coupling is attractive and is unstable otherwise.

**Proof** Let \( \epsilon_1, \epsilon_2 \in (0, 1) \), \( T > 0 \) and \( q \in \mathbb{N}/\{0\} \) be arbitrary but fixed.

Since \( u_n^{(q)}(n > q) \) is an equilibrium of (7.3), by continuous dependence on initial data, there exists \( \delta = \delta(n, \epsilon_2) \) such that
\[
\left\| v_n(0) - u_n^{(q)} \right\|_{2,n} < \delta \quad \Rightarrow \quad \max_{i \in [n]} \max_{t \in [0,T]} \left| v_{ni}(t) - u_n^{(q)}(i-1) \right| < \epsilon_1/8.
\] (7.31)

Using (7.31) and the definition of \( u_n^{(q)} \) (7.30), we have
\[
\frac{2\pi q(i - j)}{n} - \frac{\epsilon_1}{4} \leq v_{ni}(t) - v_{nj}(t) \leq \frac{2\pi q(i - j)}{n} + \frac{\epsilon_1}{4}, \quad t \in [0,T],
\]
and
\[
\sin \left( \frac{2\pi q(i - j)}{n} \right)^2 - \epsilon_1/2 \leq \sin \left( 2\pi (v_{ni} - v_{nj}) \right)^2 \leq \sin \left( \frac{2\pi q(i - j)}{n} \right)^2 + \epsilon_1/2.
\]

Thus,
\[
\frac{1}{n} \sum_{j=1}^{n} \sin \left( 2\pi (v_{ni} - v_{nj}) \right)^2 \geq \frac{1}{n} \sum_{j=1}^{n} \sin \left( \frac{2\pi q(i - j)}{n} \right)^2 - \epsilon_1/2
\]
\[
\geq \frac{1}{2n} \sum_{j=1}^{n} \left( 1 - \cos \left( \frac{4\pi q(i - j)}{n} \right) \right) - \epsilon_1/2 = (1 - \epsilon_1)/2.
\] (7.32)

Similarly, we show that
\[
\frac{1}{n} \sum_{j=1}^{n} \sin \left( 2\pi (v_{ni} - v_{nj}) \right)^2 \leq (1 + \epsilon_1)/2.
\] (7.33)

Let \( u_n(t) \) and \( v_n(t) \) denote the solutions of (7.1) and (7.3), respectively, that start from the same initial condition as in (7.31) \( u_n(0) = v_n(0) \). By Theorem 7.2, there...
exists \( N = N(\epsilon_1, \epsilon_2) \) such that

\[
P \left\{ \max_{t \in [0,T]} \| u_n(t) - u_n(t) \|_{n,2} > \epsilon_1/2 \right\} < \epsilon_2 \quad \text{for } n > N(\epsilon_1, \epsilon_2).
\]

(7.34)

Using (7.31), (7.34), and the triangle inequality, we have

\[
\| u_n(t) - u^{(q)}_n \|_{n,2} \leq \| u_n(t) - v_n(t) \|_{n,2} + \epsilon_1/8.
\]

Thus,

\[
P \left\{ \max_{t \in [0,T]} \| u_n(t) - u^{(q)}_n \|_{n,2} > \epsilon_1 \right\} \leq P \left\{ \max_{t \in [0,T]} \| v_n(t) - u^{(q)}_n \|_{n,2} > \epsilon_1/2 \right\} < \epsilon_2.
\]

\[\square\]

To illustrate Theorem 7.5 with numerical results, we integrated the repulsively coupled Kuramoto model on a large random graph \( G(2000, 0.5) \). The results in Fig. 5a–f show a very slow evolution starting from a 1-twisted state. The initial pattern gradually coarsens until the oscillators fill up the phase space randomly. Plots g–l show that the evolution starting from a 2-twisted state follows a similar pattern. Theorem 7.5 applies to both models with repulsive and attractive coupling. In the latter case, the trajectory spends a long time near a twisted state, its initial position, before converging to a synchronous solution.

8 Discussion

In this paper, we studied stability of twisted states on certain Cayley and random graphs. The motivation for this study was twofold. First, we wanted to extend the stability analysis in Wiley et al. (2006) to the Kuramoto model on Cayley graphs generated by cyclic groups and to find out what can be said about twisted states on random graphs. Our second goal was to compare dynamics of the Kuramoto model on the families of complete, Paley, and ER graphs, which are structurally very similar. In particular, these graph sequences exhibit asymptotically equivalent edge distributions, graph spectra, and have the same limits (cf. Sect. 2). Nonetheless, as our results show, the relation between network structure and dynamics of coupled nonlinear systems can be quite subtle. On the one hand, we found that the dynamics on these graph sequences are similar in many respects. All three models have twisted states as steady-state solutions, albeit for random graphs \( G(n, p) \) this statement holds almost surely in the limit as \( n \to \infty \). Further, the synchronization subspace is asymptotically stable for all three models with attractive coupling. Moreover, the rates of convergence to the synchronization subspace, which are determined by the first nonzero eigenvalues of the graph Laplacians (at least when coupling is sufficiently strong, Medvedev 2012), are approximately the same for all three graph sequences. On the other hand, the stability of the same twisted states on large complete and Paley graphs may be different despite
Fig. 5 Numerical solution of the IVP for the Kuramoto model with repulsive coupling on $G(2000, 0.5)$. Plots a–f show the solution of the problem initialized with a 1-twisted state at times $t = 20, 40, 80, 160, 200,$ and 300, respectively. The plots in g–l show the results for the problem initialized by a 2-twisted state at the same times.

the strong similarity between these graphs (see Fig. 6). In particular, we found that in the Kuramoto model with repulsive coupling on $K_n$, all nontrivial twisted states are stable. In the same model on $P_n$, $q$-twisted states are unstable if $q$ is not a QR modulo $n$. Thus, half of all twisted states are unstable in the Kuramoto model with repulsive
Fig. 6 Numerical solutions of the IVP for the Kuramoto model with repulsive coupling on Paley (a–c) and complete (d–f) graphs. Each model involves 17 coupled oscillators. The initial conditions for each problem are taken near a 2-twisted state (a, d). These numerics illustrate that the 2-twisted state is unstable for the Paley graph and stable for the complete graph.

coupling on \( P_n \).\(^6\) This example shows that the asymptotic behavior of solutions of coupled models on graphs with close structural properties may still be very different. This example also cautions about the validity of conclusions; one can draw from the analysis of formal continuum limits of large networks. Recall that the sequences of complete, Paley, and ER random graphs have the same graph limits (see Sect. 2.4). Thus, one might expect that in the limit as the size of the network goes to infinity, the dynamics of all three models are approximated by the same continuum model. In fact, for the Kuramoto models on \( K_n \) and \( G(n, p) \) such limit was established in Medvedev (2014a, b), respectively. It was shown that the solutions of the IVPs for these models for large \( n \) are approximated by the solutions of the IVP for the continuum equation

\[
\frac{\partial}{\partial x} u(x, t) = (-1)^\alpha \int_I \sin(u(y, t) - u(x, t)) dy.
\] (8.1)

The justification of the continuum limit for \( P_n \) is not covered by the analysis in these papers. However, even for the Kuramoto models on \( K_n \) and \( G(n, p) \), the results in Medvedev (2014a, b) establish the proximity of solutions of the IVPs for discrete and continuum models only on finite time intervals, which is not sufficient to guarantee that the solutions of the discrete and continuum models have the same asymptotic behavior.

\(^6\) Note, however, that the positive eigenvalues in the spectrum of the linearized problem in this case are all \( o(1) \). Thus, the instability is rather weak for large \( n \).
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