1. Introduction

1.1. Virasoro constraints. Let \( X \) be smooth projective variety of dimension \( r \). The descendent Gromov-Witten invariants \( \langle \tau_{k_1}(\gamma_{a_1}) \ldots \tau_{k_n}(\gamma_{a_n}) \rangle_{g,\beta}^X \) of \( X \) may be assembled into generating functions:

\[
\langle \langle \tau_{k_1}(\gamma_{a_1}) \ldots \tau_{k_n}(\gamma_{a_n}) \rangle \rangle_{g,\beta}^X = \sum_{\beta \in H_2(X,\mathbb{Z})} q^{\beta} \sum_{N \geq 0} \frac{1}{N!} \sum_{\ell_1, \ldots, \ell_N} t_{\ell_1}^{b_{1,N}} \ldots t_{\ell_N}^{b_{N}} \tau_{\ell_1}(\gamma_{b_1}) \ldots \tau_{\ell_N}(\gamma_{b_N}) \tau_{k_1}(\gamma_{a_1}) \ldots \tau_{k_n}(\gamma_{a_n})_{g,\beta}^X.
\]

In particular, we may form the exponential generating function for all of the descendent Gromov-Witten invariants

\[
Z^X = \exp \left( \sum_{g \geq 0} h^{g-1} \langle \langle \rangle \rangle_g^X \right).
\]

It has been conjectured by Eguchi, Hori, and Xiong \cite{eguchi_hori_xiong1, eguchi_hori_xiong2} that \( Z^X \) is annihilated by formal differential operators \( L_k, k \geq -1 \), on the affine space with coordinate \( \{ t^k \mid k \geq 0 \} \), whose definition (which we recall in Section 3) depends only on the inner product space \( H^\bullet(X,\mathbb{C}) \), its Hodge decomposition, and the endomorphism of multiplication by the anticanonical class \( c_1(X) \), and which satisfy the commutation relations

\[
[L_k, L_\ell] = (k - \ell) L_{k+\ell}.
\]

Note that this is a representation of the Lie subalgebra \( v_+ \subset v \) of the Virasoro algebra spanned by \( L_k, k \geq -1 \); it is isomorphic to the Lie algebra of polynomial coefficient vector fields on the line, by the map \( L_k \mapsto -x^k \partial/\partial x \).

If \( X \) is a point, the generating function \( Z = Z^X \) equals

\[
Z = \exp \left( \sum_{g \geq 0} h^{g-1} \frac{1}{n!} \sum_{k_1 \ldots k_n} t_{k_1} \ldots t_{k_n} \langle \tau_{k_1} \ldots \tau_{k_n} \rangle_g \right).
\]

where

\[
\langle \tau_{k_1} \ldots \tau_{k_n} \rangle_g = \int_{\mathcal{M}_{g,n}} \psi_1^{k_1} \ldots \psi_n^{k_n}.
\]
A conjecture of Witten, proved by Kontsevich [8], asserts the annihilation of $Z$ by the operators

$$
L_k = \begin{cases}
  \sum_{m=1}^{\infty} (t_m - \delta m_1) \partial_{m-1} + \frac{1}{2\hbar} t_0^2, & k = -1, \\
  \sum_{m=0}^{\infty} (m + \frac{1}{2}) (t_m - \delta m_1) \partial_m + \frac{1}{16}, & k = 0, \\
  \sum_{m=0}^{\infty} \frac{\Gamma(k+m+\frac{1}{2})}{\Gamma(m+\frac{1}{2})} (t_m - \delta m_1) \partial_{m+k} + \frac{\hbar}{2} \sum_{m=0}^{k-1} (-1)^{m+1} \frac{\Gamma(k-m+\frac{1}{2})}{\Gamma(-m-\frac{1}{2})} \partial_m \partial_{k-m-1}, & k > 0.
\end{cases}
$$

The Virasoro conjecture for arbitrary smooth projective varieties $X$ may be viewed as a generalization of this conjecture of Witten.

The Virasoro conjecture differs in some respects from Witten’s conjecture: whereas Witten’s conjecture determines all intersection numbers $\langle \tau_{k_1} \ldots \tau_{k_n} \rangle_0$ in terms of the basic one $\langle \tau_3 \rangle_0 = 1$, the Virasoro conjecture does not appear to suffice to determine the descendant Gromov-Witten invariants of positive-dimensional $X$. Furthermore, although Gromov-Witten invariants may be defined for any compact symplectic manifold $X$, the Virasoro conjecture depends on the Hodge decomposition of $H^\bullet(X, \mathbb{C})$, and thus does not appear to generalize beyond Kähler manifolds. Furthermore, the action of $v_+$ only extends to an action of $v$ when $X$ is even dimensional (in which case it has central charge the Euler characteristic $\chi(X)$ of $X$).

In [4], the authors outline a proof of the Virasoro conjecture in genus 0 using the genus 0 topological recursion relation. Throughout this paper, we will assume that the Virasoro conjecture holds in genus 0.

There is a natural conjecture lying between those of Witten and of Eguchi-Hori-Xiong. The Virasoro conjecture for $X$ implies that the generating function for degree 0 descendant Gromov-Witten invariants

$$
Z^X_0 = \exp \left( \sum_{g \geq 0} h^{g-1} F^X_{g,0} \right)
$$

is also annihilated by the representation $\rho^X$. It is the implications of this degree 0 Virasoro conjecture, in genus $g > 0$, that we investigate here.

1.2. Chern classes of the Hodge bundle. Let $g > 1$. The zero degree Gromov-Witten invariants which we study are obtained by integrating against a cycle of dimension $(r - 3)(1 - g) + n$ in the moduli stack $\overline{M}_{g,n}(X,0)$ of stable maps, called the virtual fundamental class. Axiom IV for Gromov-Witten invariants of Behrend [1] states that this virtual fundamental class is the flat pullback by the map $\overline{M}_{g,n}(X,0) \to \overline{M}_{g,0}(X,0)$ of the virtual fundamental class of $\overline{M}_{g,0}(X,0)$; in particular, it vanishes if $r > 3$. Thus, there are only three cases to be considered: $X$ is a curve, a surface or a threefold. (The case where $X$ is zero-dimensional is precisely Witten’s conjecture.)

Let $\overline{M}_{g,n+1}/\overline{M}_{g,n}$ be the universal curve $n$-pointed curve of genus $g$, let $\pi_{g,n}$ be the projection from $\overline{M}_{g,n+1}$ to $\overline{M}_{g,n}$, and let $\omega_{g,n} = \omega_{\overline{M}_{g,n+1}/\overline{M}_{g,n}}$ be the relative dualizing sheaf. The Hodge bundle on $\overline{M}_{g,n}$ is the vector bundle $E_{g,n} = \pi_{g,n}^* \omega_{g,n}$. Note that

$$
\pi_{g,n}^* E_{g,n} \cong E_{g,n+1}.
$$
Fix a genus $g$, let $\lambda_i$ be the $i$th Chern class $c_i(E)$ of $E = E_g$, and let $c_t(E)$ be the total Chern class of $E$:

$$c_t(E) = \sum_{i=0}^{g} t^i \lambda_i.$$  

The omission of the number of marked points $n$ from the notation for $\lambda_i$ is justified, since $\pi^*_{g,n} c_t(E_{g,n}) = c_t(E_{g,n+1})$ by (4).

Mumford [10] has proved the relation $c_t(E)c_{-t}(E) = 1$. Extracting the coefficients of $t^{2g}$ and $t^{2g-1}$, we see in particular that

$$\begin{cases} 
\lambda_g^2 = 0, \\
\lambda_{g-1}^2 = 2\lambda_g \lambda_{g-2}.
\end{cases}$$

Equation (5) generalizes the well-known multinomial formula for the $\psi$ integrals in genus 0. The constants $b_g$, $g > 0$, are calculated in [7] using algebro-geometric techniques:

$$\sum_{g=0}^{\infty} b_g t^{2g} = \frac{t/2}{\sin(t/2)},$$

or, in terms of Bernoulli numbers,

$$b_g = \frac{2^{2g-1} - 1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}.$$  

They do not appear to be constrained by the Virasoro conjecture, degree 0 or otherwise.

The degree 0 Virasoro conjecture for a curve is equivalent to (6) together with an explicit recursion relation (see Section 4) for the intersection numbers

$$\begin{cases} 
\lambda_g^2 = 0, \\
\lambda_{g-1}^2 = 2\lambda_g \lambda_{g-2}.
\end{cases}$$

Equation (6) generalizes the well-known multinomial formula for the $\psi$ integrals in genus 0. The degree 0 Virasoro conjecture for $\mathbb{P}^1$ implies that if $2g - 3 + n = k_1 + \cdots + k_n$, then

$$\int_{\overline{M}_{g,n}} \psi_{k_1}^{k_1} \cdots \psi_{k_n}^{k_n} \lambda_g = (2g + n - 3) b_g,$$

where

$$b_g = \begin{cases} 1, & g = 0, \\
\int_{\overline{M}_{g,1}} \psi_{2g-2}^{2g-2} \lambda_g, & g > 0. 
\end{cases}$$

The constants $b_g$, $g > 0$, are calculated in [7] using algebro-geometric techniques:

$$\sum_{g=0}^{\infty} b_g t^{2g} = \frac{t/2}{\sin(t/2)},$$

or, in terms of Bernoulli numbers,

$$b_g = \frac{2^{2g-1} - 1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}.$$  

They do not appear to be constrained by the Virasoro conjecture, degree 0 or otherwise.

The degree 0 Virasoro conjecture for a curve is equivalent to (6) together with an explicit recursion relation (see Section 4) for the intersection numbers

$$\int_{\overline{M}_{g,n}} \psi_{k_1}^{k_1} \cdots \psi_{k_n}^{k_n} \lambda_{g-1},$$

* C. Faber helped us to find this explicit expression.
for which we do not know a closed solution. In particular, in combination with (7), the conjecture implies that

\[ c_g = \int_{\overline{M}_{g,1}} \psi_1^{2g-1} \lambda_{g-1} = \left( \sum_{k=1}^{2g-2} \frac{1}{k} \right) b_g - \frac{1}{2} \sum_{g=g_1+g_2} \frac{(2g_1-1)!(2g_2-1)!}{(2g-1)!} b_{g_1} b_{g_2}. \]  

All integrals of \( \psi \) and \( \lambda \) classes over \( \overline{M}_{g,n} \) may in principle be calculated by an algorithm of Faber [5], which he has implemented in Maple. This algorithm uses Mumford’s Grothendieck-Riemann-Roch formulas to replace factors of \( \lambda_i \) in the integrand by combinations of boundary divisor and \( \psi \) classes. The resulting integrals may then be reduced to pure \( \psi \) integrals, which may be calculated by Witten’s conjectures. Unfortunately, it appears to be impractical to prove the degree 0 Virasoro conjecture using this algorithm.

We have verified (8) up to genus 5 using Faber’s program, obtaining the following results:

| \( g \) | \( b_g \) | \( c_g \) |
|-----|-----|-----|
| 1   | 1/24 | 1/24 |
| 2   | 7/5760 | 1/480 |
| 3   | 31/967680 | 41/580608 |
| 4   | 127/154828800 | 13/6220800 |
| 5   | 73/3503554560 | 21481/367873228800 |

1.4. The degree 0 Virasoro conjecture for surfaces. The degree 0 Virasoro conjecture for \( \mathbb{P}^2 \) implies that if \( g - 1 + n = k_1 + \cdots + k_n \) and \( k_i > 0 \),

\[ \int_{\overline{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_g \lambda_{g-1} = \frac{(2g+n-3)!(2g-1)!!}{(2g-1)!(2k_1-1)!! \cdots (2k_n-1)!!} \int_{\overline{M}_{g,1}} \psi_1^{g-1} \lambda_g \lambda_{g-1}. \]  

The constant

\[ \int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g \lambda_{g-1} = \frac{1}{2^{2g-1}(2g-1)!!} \frac{|B_{2g}|}{2g} \]  

has been calculated by Faber [6], who shows that it follows from Witten’s conjecture.

Remarkably, (8) is part of a deep conjecture of Faber [4] concerning the so-called tautological Chow ring \( R^*(M_g) \) generated over \( \mathbb{Q} \) by the classes \( \kappa_i \). Combining results of Looijenga [9] and Faber [5], we know that if \( i_1 + \cdots + i_m = g - 2 \), then

\[ \kappa_{i_1} \cdots \kappa_{i_m} = \int_{\overline{M}_g} \kappa_{i_1} \cdots \kappa_{i_m} \lambda_g \lambda_{g-1} \]

\[ \int_{\overline{M}_g} \kappa_{g-2} \lambda_g \lambda_{g-1} \in A^{g-2}(M_g)_\mathbb{Q}. \]

By the formula

\[ \int_{\overline{M}_{g,n}} \psi_1^{k_1+1} \cdots \psi_n^{k_n+1} \lambda_g \lambda_{g-1} = \sum_{\sigma \in S_n} \int_{\overline{M}_g} \kappa_\sigma \lambda_g \lambda_{g-1}, \]

where \( \kappa_\sigma \) is the product of \( \kappa_{|O|} \), one for each cycle \( O \) of \( \sigma \), and \( |O| = \sum_{i \in O} k_i \), the integrals in this expression may be calculated from (8). In [4], Faber proves (8) for genus \( g \leq 15 \), and conjectures that it holds in all genera.
Assuming (8), the degree 0 Virasoro conjecture for surfaces is equivalent to a complicated recursion relation for the intersection numbers
\[ \int_{\mathcal{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_g \lambda_{g-2}, \]
for which we do not know a closed solution.

1.5. The degree 0 Virasoro conjecture for threefolds. The degree 0 Virasoro conjecture for threefolds is already implied by the string and dilaton equations: the constants in this case are the intersection numbers
\[ \int_{\mathcal{M}_g} \lambda_3^{g-1} = \frac{1}{(2g-2)!} \frac{|B_{2g-2}|}{2g-2} \frac{|B_{2g}|}{2g}, \]
whose values were conjectured by Faber [4], and are calculated in [7].

2. The Euler characteristic of the obstruction bundle

The moduli space of degree 0 maps to \( X \) has a very simple form:

\[ \overline{\mathcal{M}}_{g,n}(X,0) = X \times \overline{\mathcal{M}}_{g,n}. \]

The virtual fundamental class \( [\overline{\mathcal{M}}_{g,n}(X,0)]_{\text{vir}} \) is equal to \( e(T_X \boxtimes E^\vee) \cap [X \times \overline{\mathcal{M}}_{g,n}] \) via the isomorphism (13), where \( e(T_X \boxtimes E^\vee) = c_{rg}(T_X \boxtimes E^\vee) \) denotes the Euler class, or top Chern class, of the vector bundle \( T_X \boxtimes E^\vee \). Let \( \{ \gamma_a \} \) be a basis of \( H^*(X, \mathbb{Q}) \). Let \( \psi_i \) denote the first Chern class of the \( i \)th cotangent line bundle on the moduli space of maps. The degree 0 gravitational descendents of \( X \) are the integrals:

\[ \langle \tau_{k_1}(\gamma_{a_1}) \cdots \tau_{k_n}(\gamma_{a_n}) \rangle^X_{g,0} = \int_{X \times \overline{\mathcal{M}}_{g,n}} \gamma_{a_1} \cdots \gamma_{a_n} \psi_1^{k_1} \cdots \psi_n^{k_n} \cup e(T_X \boxtimes E^\vee). \]

These descendents involve the cohomology ring of \( X \) and the integrals over \( \overline{\mathcal{M}}_{g,n} \) of the \( \psi \) and \( \lambda \) classes. The degree 0 Virasoro conjectures imply relations among the latter set of integrals on \( \overline{\mathcal{M}}_{g,n} \).

In this section, we calculate the Euler class \( e(\mathcal{E}) \) of the obstruction bundle \( \mathcal{E} = T_X \boxtimes E^\vee \) on the moduli space \( \overline{\mathcal{M}}_{g,n}(X,0) \cong X \times \overline{\mathcal{M}}_{g,n} \) of degree 0 stable maps in the three cases in which \( X \) is a curve, a surface and a threefold.

The case \( g = 1 \) is exceptional, since the Euler class \( e(\mathcal{E}) \) may be nonzero for \( X \) of any dimension: it is easily seen that
\[ e(\mathcal{E}) = c_r(X) - c_{r-1}(X) \lambda_1. \]

2.1. \( X \) a curve. Observe that if \( L \) is a line bundle,
\[ c(L \boxtimes E^\vee) = \sum_{i=0}^{g} c_1(L)^i c_{g-i}(E^\vee) = \sum_{i=0}^{g} (-1)^{g-i} c_1(L)^i \lambda_{g-i}. \]

If \( X \) is a curve, we may set \( L = T_X \), so that \( c_1(L) = c_1(X) \). Since \( c_1(L)^i \) vanishes if \( i > 1 \), we conclude that
\[ (-1)^g e(\mathcal{E}) = \lambda_g - c_1(X) \lambda_{g-1}. \]
2.2. \textbf{\textit{X a surface.}} In this case, by the splitting principle, we may suppose that $T_X \cong L_1 \oplus L_2$ is the sum of two line bundles. We see that
\[
e(\mathcal{E}) = e(L_1 \boxtimes \mathbb{E}^\vee)e(L_2 \boxtimes \mathbb{E}^\vee)
\]
\[
= (\lambda_g - c_1(L_1)\lambda_{g-1} + c_1(L_1)^2\lambda_{g-2}) (\lambda_g - c_1(L_2)\lambda_{g-1} + c_1(L_2)^2\lambda_{g-2})
\]
\[
= \lambda_g^2 - c_1(X)\lambda_g \lambda_{g-1} + (c_1(X)^2 - 2c_2(X))\lambda_g \lambda_{g-2} + c_2(X)\lambda_{g-1}^2.
\]
Applying Mumford’s relations $\lambda_g^2 = 0$ and $\lambda_{g-1}^2 = 2\lambda_g \lambda_{g-2}$, we see that
\[
e(\mathcal{E}) = -c_1(X)\lambda_g \lambda_{g-1} + c_1(X)^2 \lambda_g \lambda_{g-2}.
\]

2.3. \textbf{\textit{X a threefold.}} By the splitting principle, we may suppose that the tangent bundle $T_X \cong L_1 \oplus L_2 \oplus L_3$ is the sum of three line bundles. We see that
\[
e(\mathcal{E}) = e(L_1 \boxtimes \mathbb{E}^\vee)e(L_2 \boxtimes \mathbb{E}^\vee)e(L_3 \boxtimes \mathbb{E}^\vee)
\]
\[
= (-1)^g (\lambda_g - c_1(L_1)\lambda_{g-1} + c_1(L_1)^2\lambda_{g-2} - c_1(L_1)^3\lambda_{g-3})
\]
\[
(\lambda_g - c_1(L_2)\lambda_{g-1} + c_1(L_2)^2\lambda_{g-2} - c_1(L_2)^3\lambda_{g-3})
\]
\[
(\lambda_g - c_1(L_3)\lambda_{g-1} + c_1(L_3)^2\lambda_{g-2} - c_1(L_3)^3\lambda_{g-3}).
\]
Since $\lambda_g^2 = 0$ and $\lambda_g \lambda_{g-2} = 2\lambda_g^2 \lambda_{g-2} = 0$, many terms of the expansion of this product drop out, and we see that
\[
(-1)^g e(\mathcal{E}) = - \sum_{i \neq j} c_1(L_i)c_1(L_j)^2\lambda_g \lambda_{g-1} \lambda_{g-1} - c_3(X)\lambda_{g-1}^3
\]
\[
= (3c_3(X) - c_2(X)c_1(X))\lambda_g \lambda_{g-1} \lambda_{g-2} - c_3(X)\lambda_{g-1}^3
\]
\[
= \frac{1}{2} (c_3(X) - c_2(X)c_1(X))\lambda_{g-1}^3.
\]

3. \textbf{\textit{The Virasoro conjecture.}}

Let $X$ be a smooth projective variety of dimension $r$, and let $\gamma_a$ be a basis for $H^\bullet(X, \mathbb{C})$; we suppose that the cohomology classes are homogeneous with respect to the Hodge decomposition, so that there exist integers $p_a$ and $q_a$ such that $\gamma_a \in H^{p_a,q_a}(X)$. Let $b_a = p_a + (1 - r)/2$.

In the following formulas, we use the Einstein summation convention over indices $a$ and $b$, making use of the non-degenerate inner product
\[
\eta_{ab} = \int_X \gamma_a \cup \gamma_b
\]
and its inverse $\eta^{ab}$ to raise and lower indices as needed. Let $C^b_a$ be the matrix of the first Chern class of $X$:
\[
C^b_a \gamma_b = c_1(X) \cup \gamma_a.
\]

Introduce an affine space with coordinates $\{t^a_k \mid k \geq 0\}$, called the large phase space; the full Gromov-Witten potential \([\mathcal{P}]\) is a formal function on this space. Let $\partial_{a,k} = \partial/\partial t^a_k$, and let $\bar{t}^a_k = t^a_k - \delta_{a0}\delta_{k1}$. Let
\[
[x]^k_a = e_{k+1-i}(x, x+1, \ldots, x+k),
\]
where $e_k$ is the $k$th elementary symmetric function of its arguments; thus,
\[
\sum_{i=0}^{k+1} [x]^i t^i = (t + x)(t + x + 1) \ldots (t + x + k).
\]

Following Eguchi, Hori and Xiong [2, 3], we introduce differential operators $L_k$, $k \geq -1$, by the formulas
\[
L_k = \sum_{m=0}^\infty \sum_{i=0}^{k+1} \left( [b_a + m]^i (C^i)_{a,b,m+i} + \frac{h}{2} (-1)^{m+1} [b_a - m - 1]^i (C^i)_{a,b,m-k-m-i-1} \right) + \frac{1}{2\hbar} (C^{k+1})_{ab} \epsilon_m \epsilon_0 + \frac{\delta \epsilon_0}{48} \int_X ((3 - r) c_r(X) - 2c_1(X) c_{r-1}(X)),
\]
where it is understand that $\epsilon_m$ and $\partial_{a,m}$ vanish if $m < 0$. Note that the conjecture of Eguchi-Hori-Xiong is for projective varieties such that $p_a = q_a$ for all $a$; the extension of their conjecture to general smooth projective varieties is due to Katz (private communication, March 1997).

4. The degree 0 Virasoro conjecture for a curve

Let $X$ be a smooth projective curve of genus $\gamma$. Choose dual bases $(e^1, \ldots, e^9)$ and $(f^1, \ldots, f^9)$ of $H^{0,1}(X)$ and $H^{1,0}(X)$. Denote by $t_{k}$, $k \geq 0$, the coordinates on the large phase space dual to the descendents $\tau_k(1)$ of $1 \in H^0(X)$, by $\alpha_k = (\alpha_k^1, \ldots, \alpha_k^9)$ and $\beta_k = (\beta_k^1, \ldots, \beta_k^9)$ the coordinates dual to the descendents $\tau_k(e^i)$ and $\tau_k(f^i)$, and by $s_k$ the coordinates dual to the descendents $\tau_k(\omega)$ of the class $\omega \in H^2(X)$ Poincaré dual to a point.

If $\alpha$ is a cohomology class on $\overline{M}_g$ (and hence, by pullback, on the moduli spaces $\overline{M}_{g,n}$), introduce the generating functions
\[
\langle \langle \tau_1 \ldots \tau_n \mid \alpha \rangle \rangle_g = \sum_{N=0}^\infty \frac{1}{N!} \sum_{l_1 \ldots l_N} t_{l_1} \ldots t_{l_N} \int_{\overline{M}_{g,n+N}} \psi^{k_1} \ldots \psi^{k_n} \alpha.
\]

**Theorem 1.** We have
\[
\frac{L_k Z^X_0}{Z^X_0} = \sum_{g=0}^{\infty} h^{g-1} (-1)^g \left( (2\gamma - 2)x^g_k(t) + \sum_{\ell=0}^\infty \sum_{m=0}^\infty \alpha_{\ell} \cdot \beta_m \partial_{m+\ell} \right) y^g_{k,\ell}(t),
\]
where
\[
x^g_k(t) = -[1]^k_0 \langle \langle \tau_{k+1} \mid \lambda_{g-1} \rangle \rangle_g + \sum_{m=0}^\infty t_m [m]^k_0 \langle \langle \tau_{k+m} \mid \lambda_{g-1} \rangle \rangle_g
\]
\[
+ [1]^k_1 \langle \langle \tau_k \mid \lambda_g \rangle \rangle_g - \sum_{m=0}^\infty t_m [m]^k_1 \langle \langle \tau_{k+m-1} \mid \lambda_g \rangle \rangle_g
\]
\[
- \frac{1}{2} \sum_{m=0}^{k-2} \sum_{g_1 + g_2} (-1)^{m+1} [-m-1]^k_1 \langle \langle \tau_m \mid \lambda_{g_1} \rangle \rangle_g \langle \langle \tau_{k-m-2} \mid \lambda_{g_2} \rangle \rangle_g
\]
\[
y^g_{k,\ell}(t) = -[1]^k_0 \langle \langle \tau_{k+1} \tau_\ell \mid \lambda_g \rangle \rangle_g + \sum_{m=1}^\infty [m]^k] t_m \langle \langle \tau_{k+m} \tau_\ell \mid \lambda_g \rangle \rangle_g + [\ell+1]^k_0 \langle \langle \tau_{k+\ell} \mid \lambda_g \rangle \rangle_g
\]
Proof. For \( k > 0 \), \( L_k \) is given by the formula

\[
L_k = -[1]_0^k \partial_{t_{k+1}} + \sum_{m=0}^{\infty} \left( [m]_0^k (t_m \partial_{t_{k+m}} + \alpha_m \cdot \partial_{\alpha_{k+m}}) + [m+1]_0^k (s_m \partial_{s_{k+m}} + \beta_m \cdot \partial_{\beta_{k+m}}) \right) + (2 - 2\gamma) \left( -[1]_1^k \partial_{s_k} + \sum_{m=0}^{\infty} [m]_1^k t_m \partial_{s_{k+m-1}} + \frac{\hbar}{2} \sum_{m=0}^{k-2} (-1)^{m+1} \frac{m+1}{m} \partial_{s_m} \partial_{s_{k-m-2}} \right)
\]

Using the notation \((13)\), \( Z_0^X \) is given by the formula

\[
Z_0^X = \exp \left( (2\gamma - 2) \sum_{g=1}^{\infty} (-1)^g \hbar^{g-1} \langle \langle | \lambda_{g-1} \rangle \rangle_g \right) + \sum_{g=0}^{\infty} (-1)^g \hbar^g \left( \sum_{m=0}^{\infty} s_m \langle \langle \tau_m | \lambda_g \rangle \rangle_g + \sum_{\ell,m=0}^{\infty} \alpha_\ell \cdot \beta_m \langle \langle \tau_\ell \tau_m | \lambda_g \rangle \rangle_g \right)
\]

The theorem follows on combining these formulas. \( \square \)

Corollary 2. The degree 0 Virasoro conjecture for algebraic curves is equivalent to the vanishing of the expressions \( x_g^k(t) \) and \( y_g^k(t) \). In particular, if the degree 0 Virasoro conjecture holds for \( \mathbb{P}^1 \), then it holds for all curves.

The vanishing of \( y_g^k(t) \) for \( k \geq 1 \) and \( \ell \geq 0 \) is equivalent to the formula \((13)\) for the generating function \( \langle \langle | \lambda_g \rangle \rangle_g \). To see this, note that

\[
\frac{1}{(k+1)!} \partial_{t_{k_1}} \cdots \partial_{t_{k_n}} y_{g,k_0}^k(0) = -\langle \tau_{k+1} \tau_{k_0} \cdots \tau_{k_n} | \lambda_g \rangle_g + \left( \begin{array}{c} k_0 + k + 1 \\ k_0 \end{array} \right) \langle \tau_{k_0+k} \tau_{k_1} \cdots \tau_{k_n} | \lambda_g \rangle_g + \sum_{i=1}^{n} \left( \begin{array}{c} k_i + k \\ k_i - 1 \end{array} \right) \langle \tau_{k_0} \cdots \tau_{k_i+k} \cdots \tau_{k_n} | \lambda_g \rangle_g,
\]

where it is understood that \( \left( \begin{array}{c} a \\ 0 \end{array} \right) = 0 \) for \( a \) a natural number.

Theorem 3. The recursion given by the vanishing of \((13)\) has the unique solution

\[
\langle \tau_{k_1} \cdots \tau_{k_n} | \lambda_g \rangle_g = \begin{cases} \left( \begin{array}{c} n - 3 \\ k_1, \ldots, k_n \end{array} \right) \langle \tau_0^3 \rangle_0, & g = 0, \\ \left( \begin{array}{c} 2g + n - 3 \\ k_1, \ldots, k_n \end{array} \right) \langle \tau_{2g-2} | \lambda_g \rangle, & g > 0. \end{cases}
\]

Proof. We prove the theorem by induction on \( n \); in the cases \( n = 3 \) for \( g = 0 \) and \( n = 1 \) for \( g > 0 \), the formula is a tautology. Thus, we must prove that

\[
\left( \begin{array}{c} 2g + n - 1 \\ k_0, \ldots, k_n, k + 1 \end{array} \right) = \left( \begin{array}{c} k_0 + k + 1 \\ k_0 \end{array} \right) \left( \begin{array}{c} 2g + n - 2 \\ k_0 + k, k_1, \ldots, k_n \end{array} \right) + \sum_{i=1}^{n} \left( \begin{array}{c} k_i + k \\ k_i - 1 \end{array} \right) \left( \begin{array}{c} 2g + n - 2 \\ k_0, \ldots, k_i + k, \ldots, k_n \end{array} \right).
\]
This follows from the equation
\[ 2g + n - 1 = (k_0 + k + 1) + \sum_{i=1}^{n} k_i, \]
on multiplication of both sides by \( \frac{(2g + n - 2)!}{k_0! \ldots k_n!(k + 1)!}. \)

In particular, the well-known formulas for the intersection numbers \( \langle \tau_{k_1} \ldots \tau_{k_n} \rangle_0 \) are seen to be special cases of the conjectured formulas for the intersection numbers \( \langle \tau_{k_1} \ldots \tau_{k_n} | \lambda_g \rangle_g \).

The same technique applied to \( x_k^2(t) \) leads to a recursion for the intersection numbers
\[ \langle \tau_{k_1} \ldots \tau_{k_n} | \lambda_{g-1} \rangle_g \]
which expresses them in terms of the numbers \( b_h, h \leq g \). We will only discuss the simplest case \( n = 1 \). Taking the relation \( x_k^2(t) = 0 \), we obtain the formula
\[ (2g - 1)! c_g = s(2g, 2)b_g - \frac{1}{2} \sum_{g=g_1+g_2} (2g_1 - 1)!(2g_2 - 1)!b_{g_1}b_{g_2}, \]
where \( s(2g, 2) \) is the Stirling number of the first kind
\[ s(2g, 2) = [1]_1^{2g-2} = (2g - 1)! \sum_{k=1}^{2g-1} \frac{1}{k}. \]
We have not been able to find a solution of this recursion, or its generalizations to larger \( n \), in closed form.

5. The degree 0 Virasoro conjecture for a surface

The discussion of the degree 0 Virasoro conjecture for a surface runs along the same lines as for a curve, although the details are a little more complicated. To simplify notation, we restrict attention to simply-connected surfaces.

Let \( X \) be a smooth simply-connected projective surface. Choose dual bases \( (e_1, \ldots, e_p) \) and \( (f_1, \ldots, f_p) \) of \( H^{0,2}(X) \) and \( H^{2,0}(X) \), and a basis \( \omega_i, 1 \leq i \leq \ell \) of \( H^{1,1}(X) \). Denote by \( t_k, s_k = (s^1_k, \ldots, s^d_k), r_k, a_k = (a^1_k, \ldots, a^p_k) \) and \( b_k = (b^1_k, \ldots, b^p_k) \) the coordinates on the large phase space dual respectively to the descendents of \( 1 \in H^0(X, \mathbb{Z}) \), of \( \omega_i, 1 \leq i \leq d \), of the class in \( H^4(X, \mathbb{Z}) \) Poincaré dual to a point, and of \( e_i \) and \( f_i, 1 \leq i \leq p \), respectively. Let \( c = (c_1, \ldots, c_d) \) be the vector in the vector space dual to \( H^{1,1}(X) \) such that \( c_1(X) = c \cdot \omega \).

**Theorem 4.** We have
\[
\frac{L_k Z_0^X}{Z_0^X} = \sum_{g=1}^{\infty} h^{g-1} (-1)^g \left( |c|^2 x^k_0(t) - \sum_{\ell=0}^{\infty} c \cdot s_{\ell} y^k_{\ell}(t) \right) + \frac{1}{\hbar} w(r, a, s, b, t)
\]
where

\[
x_g^k(t) = -\left[\frac{1}{2}\right]^k_0 \langle \tau_{k+1} \mid \lambda g \lambda g-2 \rangle_g - \sum_{m=0}^{\infty} t_m \left[ m - \frac{1}{2}\right]^k_0 \langle \tau_{k+m} \mid \lambda g \lambda g-2 \rangle_g \\
+ \sum_{m=0}^{k-1} (-1)^{m+1} \left[ -m - \frac{3}{2}\right]^k_0 \langle \tau_m \rangle_0 \langle \tau_{k-m-1} \mid \lambda g \lambda g-2 \rangle_g \\
+ \left[ \frac{1}{2}\right]^k_1 \langle \tau_k \mid \lambda g \lambda g-1 \rangle_g + \sum_{m=0}^{\infty} t_m \left[ m - \frac{1}{2}\right]^k_1 \langle \tau_{k+m} \mid \lambda g \lambda g-1 \rangle_g \\
+ \sum_{m=0}^{k-2} (-1)^{m+1} \left[ -m - \frac{3}{2}\right]^k_1 \langle \tau_m \rangle_0 \langle \tau_{k-m-2} \mid \lambda g \lambda g-1 \rangle_g \right] \\
y_{g,\ell}^k = -\left[\frac{1}{2}\right]^k_0 \langle \tau_{k+1} \tau_\ell \mid \lambda g \lambda g-1 \rangle_g + \sum_{m=0}^{\infty} t_m \left[ m - \frac{1}{2}\right]^k_0 \langle \tau_{k+m} \tau_\ell \mid \lambda g \lambda g-1 \rangle_g \\
+ \left[ \ell + \frac{1}{2}\right]^k_0 \langle \tau_{k+\ell} \mid \lambda g \lambda g-1 \rangle_g \\
+ \sum_{m=0}^{k-1} (-1)^{m+1} \left[ -m - \frac{3}{2}\right]^k_0 \langle \tau_m \rangle_0 \langle \tau_{k-m-1} \tau_\ell \mid \lambda g \lambda g-1 \rangle_g \\
+ \left[ -m - \frac{1}{2}\right]^k_0 \langle \tau_m \tau_\ell \rangle_0 \langle \tau_{k-m-1} \mid \lambda g \lambda g-1 \rangle_g \right]
\]

We have omitted the explicit expression for \( w(r, a, s, b, t) \), because of its greater complexity, because it differs in nature from the higher genus coefficients, and because in any case we are assuming that it vanishes by the genus 0 Virasoro conjecture.
Proof. For \( k > 0 \), \( L_k \) is given by the formula

\[
L_k = -\left[ \frac{1}{2} \right] \partial_{t_{k+1}} + \sum_{m=0}^{\infty} \left( [m - \frac{1}{2}] t_m \partial_{b_{m+1}} + b_{m} \cdot \partial b_{k+m} \right) + [m + \frac{1}{2}] s_m \cdot \partial b_{k+m} \\
+ \sum_{m=0}^{\infty} \left( [m - \frac{3}{2}] \partial_{r_{m}} \partial_{b_{k+m-1}} + \frac{1}{2} [m - \frac{3}{2}] \partial_{s_{m}} \cdot \partial b_{k+m-1} \right) \\
+ \sum_{m=0}^{\infty} \left( [m - \frac{1}{2}] t_m \partial_{b_{k+m-1}} + [m + \frac{1}{2}] s_m \partial_{r_{k+m-1}} \right) \\
+ \frac{1}{h} \sum_{m=0}^{\infty} \left( -[m - \frac{3}{2}] \partial_{b_{m+1}} \partial_{b_{k+m-1}} \right) \\
+ \frac{1}{h} \sum_{m=0}^{\infty} \frac{(2k + 2k_0 + 1)!}{(2k + 1)!} (2k_0 - 1)! \sum_{g=1}^{\infty} h^{g-1} \sum_{m=0}^{\infty} \frac{c \cdot s_m \langle \tau_m | \lambda_g \lambda_{g-1} \rangle}{g}
\]

The generating function \( Z_0^X \) is given by the formula

\[
Z_0^X = \exp \left( \frac{1}{h} \sum_{m=0}^{\infty} r_m \langle \tau_m \rangle_0 + \frac{1}{h} \sum_{\ell, m} \left( \frac{1}{2} s_{\ell} \cdot s_{m} + a_{\ell} \cdot b_{m} \right) \langle \tau_\ell \tau_m \rangle_0 \right) \\
+ \frac{1}{h} \sum_{g=1}^{\infty} h^{g-1} \sum_{m=0}^{\infty} \frac{c \cdot s_m \langle \tau_m | \lambda_g \lambda_{g-1} \rangle}{g}
\]

The theorem follows on combining these formulas. \qed

**Corollary 5.** The degree 0 Virasoro conjecture for surfaces is equivalent to the vanishing of the expressions \( x_g^k(t) \) and \( y_g^k(t) \). In particular, if the degree 0 Virasoro conjecture holds for \( \mathbb{P}^2 \), then it holds for all simply connected surfaces.

The vanishing of \( y_g^k(t) \) for \( k \geq 0 \) and \( \ell > 0 \) implies formula (15) for the generating function \( \langle \langle | \lambda_g \lambda_{g-1} \rangle \rangle \). To see this, note that if \( k_1, \ldots, k_n > 0 \),

\[
\frac{1}{(2k + 2k_0 + 1)!} (2k_0 - 1)! \langle \tau_{k_0 + \ell} \tau_{k_1} \cdots \tau_{k_n} | \lambda_g \lambda_{g-1} \rangle_0 = -\langle \tau_{k+1} \tau_{k_0} \cdots \tau_{k_n} | \lambda_g \lambda_{g-1} \rangle_0
\]

The proof of the following theorem is close to that of Theorem 3.
Theorem 6. The recursion given by the vanishing of \ref{eq:15} has the unique solution
\[ \langle \tau_{k_1} \cdots \tau_{k_n} | \lambda_g \lambda_{g-1} \rangle_g = \frac{(2g+n-3)!(2g-1)!!}{(2g-1)!(2k_1-1)!! \cdots (2k_n-1)!!} \langle \tau_{g-1} | \lambda_g \lambda_{g-1} \rangle \]

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