Supersymmetry method for interacting chaotic and disordered systems: the SYK model

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The nonlinear supermatrix σ-model is widely used to understand the physics of Anderson localization and the level statistics in noninteracting disordered electron systems. Here we develop a supersymmetry approach to the disorder averaging in the interacting models. In particular, we apply supersymmetry to study the Sachdev-Ye-Kitaev (SYK) model, where the disorder averaging has so far been performed only within the replica approach. We use a slightly modified, time-reversal invariant, version of the SYK model and perform calculations in real-time. As a demonstration of how the supersymmetry method works, we derive saddle point equations. In the semiclassical limit, we show that the results are in agreement with those found using the replica technique. We also develop the formally exact superbosonized representation of the SYK model. In the latter, the supersymmetric theory of original fermions and their superpartner bosons is reformulated as a model of unconstrained collective excitations. We argue that the superbosonized description of the model paves the way for the precise calculation of the window of universality in which random matrix theory is applicable to the chaotic SYK system, and for the derivation of the corresponding Wigner-Dyson eigenvalue statistics.

I. INTRODUCTION

Quantum phenomena in disordered or chaotic systems can efficiently be investigated analytically using methods of quantum field theory. Three most popular approaches are based on the replica trick, the Keldysh technique, and the supersymmetric σ-model approach originally developed by one of the authors. The necessity of applying these techniques stems from the fact that physical correlation functions of interest are expressed in terms of functional integrals containing weight denominators while averaging over quenched disorder has to be done at the end of calculations. This makes a direct application of methods of quantum field theory difficult. All the methods of Refs. allow one to eliminate the weight denominator – the partition function of the system – and average over disorder just at the beginning of all calculations. As a result of this manipulation, one obtains an effective field theory for “interacting” particles and application of well-developed methods and approximations become feasible.

Although the replica, Keldysh, and supersymmetry techniques look similar to each other, their efficiency when applying to different problems is very different. The replica approach allows one to avoid explicitly calculating Z by introducing an integer number of copies of the system and making use of the replica trick. It can be used for various systems of interacting particles, spins, etc., but the method requires an analytical continuation to non-integer numbers of replicas and assumes the existence of the replica limit when the number of copies approaches zero. A general procedure of this continuation does not exist, and one obtains very often unphysical results in certain situations, although one can also obtain important results using this method. Within the Keldysh technique, one doubles the degrees of freedom to obtain a normalized theory with partition function, Z = 1. The Keldysh sigma model representation of disordered systems is formally exact, but it can be quite complicated for some specific cases. Both approaches have been successfully applied to interacting theories with the disorder, but their efficiency in making essentially non-perturbative calculations is rather limited.

The supersymmetry approach makes use of the fact that the partition function of non-interacting fermions is always the inverse of that of the analogous bosonic theory. Therefore, if one introduces additional bosonic degrees of freedom that replicate the fermionic action, the overall partition function of the supersymmetric theory will be reduced to one. The approach is proven to be a handy tool for studies in various fields of physics and in particular, in models of quantum chaos involving random matrix theory and various models of disorder.

One of the prominent methods employing supersymmetry is the nonlinear supersymmetric sigma model description of disordered metallic conductors. According to this standard formalism, effective field theory is described by action with coordinate dependent supermatrix field, , obeying the constraint, . This method has a broad range of applications, including the study of Anderson localization, mesoscopic fluctuations, levels statistics in a limited volume, quantum chaos. The limitation of the supersymmetric approach was that it was deemed to be inapplicable to systems interacting particles.

However, it turns out that there are important non-trivial models of interacting particles with the disorder that can be written in a supersymmetric form, and one can average over the disorder at the beginning of calculations. The main goal of this paper is to identify such models and develop the supersymmetry approach to the disorder averaging. To be more specific, we will ap-
paly this approach to study the Sachdev-Ye-Kitaev (SYK) model\cite{10,11}, where the disorder averaging was so far performed only within the replica trick approach. Our mapping of the SYK model onto a supersymmetric model containing both fermion and boson degrees of freedom subsequent averaging over the disorder is exact. Moreover, we demonstrate that the new supersymmetric model with an effective particle-particle interaction can be reformulated in terms of some generalized supermatrix $\sigma$-model (superbosonization). This procedure is also exact. Leaving investigation of new non-trivial regimes of the SYK model for the future, we concentrate here on analyzing the semiclassical limit of the model. The results obtained in the semiclassical limit within this new approach are in agreement with those found earlier using the replica technique.

The SYK model exhibits inherently non-Fermi liquid behavior and quantum many-body chaotic eigenspectrum\cite{12,14,26,29}. This suggests that the two-point correlation function of the original fields of the model does not fully capture the many-body level statistics. The reason is that these are the many-body states that entirely determine the close energy levels. Thus, the many-body level statistics of the SYK model that follows the universal behavior of Wigner-Dyson random matrix ensembles is inaccessible to original single-particle fields. To account for many-body effects of the model, we perform the superbosonization transformation and rewrite the model in terms of the collective many-body excitations. To show the workability of the representation, we reproduce earlier established results. We argue that the developed superbosonized description of the SYK model is capable of producing novel non-perturbative many-body effects.

The paper is organized as follows. In Section II we introduce the SYK model. In Section III we develop a new, supersymmetric sigma-model representation for interacting disordered fermion systems and apply it to SYK model. To derive it, we decouple the interaction Hamiltonian using the conventional Hubbard-Stratonovich approach. Then we notice that the Hubbard-Stratonovich field can, in some situations, be gauged out from the denominator. This enables one to supersymmetrize the interacting theory. In Section IV, the new formalism is tested by calculating the fermion Green’s function in the SYK model at large times and is argued to be efficient for other interacting models with the disorder.

In Section V we rewrite the supersymmetric SYK model as a model describing unconstrained supermatrices representing collective many-body excitations. Such a representation, where the partition function is represented in a supermatrix action formulation without any constraints is dubbed superbosonization. Since the transformation is exact, it is fully capable describing the many-body modes instead of the original fermions of the SYK model. As such, it represents the first step towards derivation of the Wigner-Dyson eigenvalue statistics and the calculation of the Thouless time at which the universal random-matrix behavior sets in. Our conclusions and the possible directions for the future research are discussed in Section VI.

II. MODEL

The study of out-of-time correlation functions\cite{17} in the SYK model shows\cite{18,20,24,25}, that it exhibits chaotic behavior at all time scales. At short times it has exponentially decaying correlators, while at ultra-long times, otherwise nearly zero-temperatures, when the energy scale is less than the many-body level spacing, one has maximal chaoticity in the large system size limit. This happens because Lyapunov exponent saturates to the conjectured upper bound\cite{32}. One of the important problems here is the test of Eigenstate Thermalization Hypothesis (ETH)\cite{30,33}, which is a conjecture about the nature of matrix elements of physical observables that, if holds, reconciles the predictions of statistical physics of equilibrating states with those of quantum mechanics in the longtime limit.

The study of low energy (long time) scale\cite{34,35} shows ETH behavior because the eigenstates exhibit volume law entanglement\cite{34,35}, suggesting that system becomes ergodic. However, one of the major questions here is associated with finding the intermediate time/energy scale, at which the system transfers to a thermalized state. The characteristic time scale that leads to ergodicity in the SYK system is analogous to Thouless time in dirty metals, while the states are analogous to diffusive modes there. In the intermediate stage, one does not have an ergodic state. To study the latter, in Refs.\cite{33,36,38} the local two fermions hopping term (SYK$_2$) with random coupling was added to the four-fermion long-range randomly interacting SYK$_4$ Hamiltonian. Here the thermalization properties, including the Lyapunov exponent (or scrambling rate) and the so-called butterfly velocity, were analyzed. The butterfly speed is the speed at which the impacts of a local perturbation proliferate, while the scrambling rate is a proportion of the rate at which the local perturbation is mixed into non-local degrees of freedom. It has been demonstrated that in a general quantum framework, the Lyapunov exponent is limited by the temperature.

Another development in this direction was reported in Refs.\cite{11,17}, where d-dimensional generalization of SYK model was proposed by taking a number of SYK droplets in real space and including fermion hopping terms between them. This line of investigations is however out of the scope of the present project.

The level statistics in the generalized SYK$_4$ + SYK$_2$ model was studied recently using exact diagonalization\cite{36}. The results suggest that upon fixing the range of two-fermion hopping and keeping the four-fermion interaction sufficiently long-ranged, the spectral correlations will not change substantially compared to the random matrix prediction, which is typical for chaotic quantum sys-
tems. However, by reducing the range of the two-fermion terms, one will see a transition into an insulating state, characterized by Poisson statistics. It appeared, that in the vicinity of the many-body metal-insulator transition point, the spectral correlations share all the features that had been previously found in systems at the Anderson transition and in the proximity of the many-body localization transition. This indicates the potential relevance of generalized SYK models in the context of many-body localization and, also exhibits itself as a starting point for the exploration of a gravity-dual of this phenomenon.

An important demonstration of the SYK model being maximally chaotic is the fact of having a finite entanglement entropy at zero temperature22–25, indicating that at large time scales there is maximal mixing in the ground state. Basic features of the SYK model, that in turn support the existence of a gravity dual, include maximal chaos in the strong coupling limit, finite zero temperature entropy, linear specific heat in the low-temperature limit, the exponential growth of low-energy excitations, and the short-range spectral correlations given by random matrix theory.

In its simplified version12 the complex SYK model is a system of randomly interacting $N$ (originally Majorana) spinless fermions represented by their annihilation (creation) operators $\hat{c}_i$ ($\hat{c}_i^\dagger$), $i = 1, \ldots , N$, with random all-to-all interactions given by the Hamiltonian

$$\hat{H} = \sum_{ij,kl} J_{ij,kl} \hat{c}_i^\dagger \hat{c}_k \hat{c}_l \hat{c}_j - \mu \sum_i \hat{c}_i^\dagger \hat{c}_i. \quad (2.1)$$

The coupling constant $J_{ij,kl}$ were assumed to be random complex number

$$J_{ij,kl}^* = J_{lk,ji} \quad (2.2)$$

with a Gaussian distribution characterized by the following average and variance

$$\langle J_{ij,kl} \rangle = 0, \quad \langle J_{ij,kl} J_{i'j',kl'} \rangle = \frac{J^2}{N^3} \delta_{ii'} \delta_{jj'} \delta_{kk'} \delta_{ll'}. \quad (2.3)$$

Averages of the type $\langle J_{ij,kl} J_{i'j',kl'} \rangle$ are equal to zero unless they can be reduced to Eq. (2.3) using the symmetry relation (2.2).

At large time scales (low temperatures) the SYK model is conformal because the term that contains a time-derivative in the Lagrangian can be ignored. The action of the model can be written using the so-called $G, \Sigma$ representation, and the Schwarzian theory14–16 can describe its soft mode fluctuations. It has been shown that this theory is equivalent14–16 to two-dimensional dilaton gravity, the Jackiw-Teitelboim model17–20. This fact points out the link between AdS2 black hole physics and the SYK model.

The spectral form factor in the SYK model was studied numerically in Ref. 40 (analogous two-point correlation functions were studied using the random matrix approach in Refs. 58–59). Roughly, the spectral form factor is the Fourier transform of the connected two-point density-density correlation function $\langle \rho(E) \rho(E') \rangle$ in random matrix theory. The question of the precise window of universality in which random matrix theory is applicable is still unknown.

The supersymmetric reformulation of the SYK model we propose below, may give a possibility to derive it theoretically. The method may also allow one to study corrections beyond this universality regime. It is worth emphasizing that developing the supersymmetric representation we start with a fermionic SYK model analogous to the one given by Eq. (2.1). Bosons appear after certain transformations and are somehow "fictitious bosons" like those that appear in the supersymmetry technique for electron systems21–24.

All this clearly contrasts works on supersymmetric generalizations of the SYK model. For example, Ref. 60 reported a supersymmetric generalization of the SYK. In that model, however, the four fermion coupling constants $J_{ij,kl}$ are not entirely random (they are correlated and defined by free coupling constants in supercharge $Q$). The bosonic field appears here as a non-dynamical field to linearize the supersymmetry transformation and realize the supersymmetry algebra off-shell. Similar supersymmetric lattice models were reported in Refs. 61–67. Specific correlations of the random couplings of these models lead to $N = 1$ and $N = 2$ supersymmetry. Supersymmetric models with random couplings that include both bosons and fermions were considered in Refs. 68–71, while Ref. 72 explored the possibility of extending the 1+1 dimensional bosonization technique to $(0 + 1)$-dimensional SYK-type systems. Ref. 69 suggested that the SYK model with Majorana fermions and without fine-tuned couplings has the capacity of possessing some hidden supersymmetry, which may also be present in complex SYK model when the chiral symmetry is present70.

### III. SUPERSYMMETRY REFORMULATION OF SYK MODEL: AVERAGING OVER QUENCHED DISORDER

Now we apply the supersymmetry approach to the interacting SYK model. We believe that such an approach opens the door to analyzing the many-body effects and exponentially small bulk level spacing of the model. The formalism could also be adapted to study the effects in generalized SYK models such as SYK$_4$ + SYK$_2$ and establish a fruitful connection between complex and Majorana models.

Although the original model, Eq. (2.1), has been written in the Hamiltonian representation, it is more convenient to use the functional integral representation with fermionic fields $\chi_i (t), \chi_i^\dagger (t)$. They obey the anticommutation relations

$$\{\chi_i, \chi_j\} = \{\chi_i^\dagger, \chi_j^\dagger\} = \{\chi_i, \chi_j^\dagger\} = 0, \quad (3.1)$$

and we use the convention $(\chi^\dagger)^* = -\chi_i$.
In order to develop the supersymmetry approach for the model with the fermion-fermion interaction, we slightly modify the original model specified by Eqs. (3.1). Using the anticommuting Grassmann fields, \( \chi \), we write correlation functions in terms of a functional integral over these fields as

\[
G_{ij}(t, t') = \frac{i \int \chi_i(t) \chi_j^*(t') \exp(iS[\chi, \chi^*]) D\chi D\chi^*}{\int \exp(iS[\chi, \chi^*]) D\chi D\chi^*} \tag{3.2}
\]

In Eq. (3.2), the product of the fields \( \chi_i(t) \) and \( \chi_j^*(t) \) for arbitrary \( i, j \) and times \( t \) defines the Green’s function \( G_{ij} \).

Here we start with the action, \( S[\chi, \chi^*] \), which is slightly different from the field representation of the model given by Eq. (2.1). Namely, we consider

\[
S[\chi, \chi^*] = \int_{-\infty}^{\infty} \left[ \sum_{i=1}^{N} \chi_i^* (i\partial_t + \mu) \chi_i(t) \right.
- \sum_{i,j,k,l=1}^{N} J_{ijkl} \left( \chi_i^*(t) \chi_j(t) - \chi_j^*(t) \chi_i(t) \right)
\times (\chi_k^*(t) \chi_l(t) - \chi_l^*(t) \chi_k(t)) \big] dt.
\]

The random coupling constants \( J_{ijkl} \) in Eq. (3.3) are assumed to be real and obey the symmetry relations

\[
J_{ijkl} = -J_{ji,kl} = -J_{ij,ik} = J_{kl,ij}. \tag{3.4}
\]

Their distribution is Gaussian with zero average

\[
\langle J_{ijkl} \rangle = 0, \tag{3.5}
\]

and the variance

\[
\langle J_{ijkl} J_{ijkl'} \rangle = \frac{J^2}{8N^3}
\times \left( (\delta_{ii'} \delta_{jj'} - \delta_{ij'} \delta_{j'i'}) (\delta_{kk'} + \delta_{kk'}) \right)
\times \left( (\delta_{ii'} \delta_{jj'} - \delta_{ij'} \delta_{j'i'}) (\delta_{kk'} + \delta_{kk'}) \right). \tag{3.6}
\]

One can interpret the model described by Eq. (3.3) as a time-reversal invariant version of the SYK model.

First, under the functional integral, we introduce a time-dependent Hubbard-Stratonovich real antisymmetric matrix field, \( M^F_{ij}(t) \), and decouple the four-fermion interaction of the SYK Hamiltonian (3.18) by inserting identity operator,

\[
1 \equiv \int \frac{D M^F \exp \left\{ i \int dt \left[ \sum_{i,j,k,l=1}^{N} \sum_{i',j',k',l'=1}^{N} (M^F_{ij} - i (\chi_i^* \chi_k - \chi_k^* \chi_i) J_{ijkl}) 
\times (M^F_{ijkl'} - J_{ijkl'} \chi_i^* \chi_j(t) - \chi_j^* \chi_i(t)) \right] \right\}}{\int \exp \left\{ i S[\chi, \chi^*] \right\} D\chi D\chi^* D M^F} \tag{3.7}
\]

into the functional integrals over \( \chi, \chi^* \) in Eq. (3.2). Here \( (J^{-1})_{ijkl} \) is the inverse of \( J_{ijkl} \), namely

\[
\sum_{kl} (J^{-1})_{ijkl} J_{kl,mn} = \delta_{im} \delta_{jn}. \tag{3.8}
\]

Using the last property in Eqs. (2.2) of the coupling \( J_{ijkl} \) and the hermiticity of the matrix, \( M^F_{ij}(t) \), we see that the exponent in Eq. (3.7) is purely imaginary and the integral over matrix \( M^F_{ij}(t) \) converges. Then the action for the time-reversal symmetric modification of the SYK model is now equivalent to that of a system of electrons moving in a fluctuating real antisymmetric field \( M_{ij}(t) \) with random \( J_{ijkl} \):

\[
S[\chi, \chi^*, M^F] = \int_{-\infty}^{\infty} dt \sum_{ij=1}^{N} \left\{ \chi_i^*(t) \left[ (i\partial_t + \mu) \delta_{ij} - 2iM^F_{ij}(t) \right] \chi_j(t) + \sum_{ijkl} M^F_{ij}(t) (J^{-1})_{ijkl} M^F_{kl}(t) \right\}. \tag{3.9}
\]

This action therefore defines the Green’s function, \( G_{ij} \), of fermionic fields, \( \chi_i(t), \chi_j^*(t) \), as

\[
G_{ij}(t, t') = \frac{i \int \chi_i(t) \chi_j^*(t') \exp\left\{ iS[\chi, \chi^*, M^F] \right\} D\chi D\chi^* D M^F}{\int \exp\left\{ iS[\chi, \chi^*, M] \right\} D\chi D\chi^* D M}. \tag{3.10}
\]

The random coupling \( J_{ijkl} \) enters both the numerator and denominator in Eq. (3.10), and one cannot average over this coupling directly. This situation is typical for problems with quenched disorder. The standard supersymmetry approach of Refs. [1,7] relies on the fact that the system is initially non-interacting. In that case, one replaces the denominator by an integral over bosonic fields in the numerator. Here we deal with an inherently interacting system, we generate a field \( M_{ij} \) which enters both numerator and denominator in Eq. (3.10) and seemingly invalidates the possibility of supersymmetrizing the action.

Although this obstacle cannot be generally overcome, the SYK model considered here is in this respect exceptional. Now we make a crucial observation. We show now that the integral over the fermionic fields \( \chi, \chi^* \) in denominator of Eq. (3.10) does not, in fact, depend on the Hubbard-Stratonovich field \( M(t) \). The reason is that the real antisymmetric matrices \( M \) can be represented as a pure gauge field as \( 2M = -U^T \partial_t U \), where \( U \equiv U(t) \) is a real orthogonal matrix, \( U^TU = 1 \). This property helps us to drastically simplify the integral in the denominator:

\[
\int \exp\left\{ iS[\chi, \chi^*, M] \right\} D\chi D\chi^* = \det[U^T (i\partial_t + \mu) U] = \det[(i\partial_t + \mu)]. \tag{3.11}
\]

So, what we end up having in the denominator is just a determinant, \( \det[(i\partial_t + \mu)] \), which is independent of the fluctuating field \( M \) and random coupling constants \( J_{ijkl} \). This point is crucial and it allows one to express the integral over the fermionic fields in the denominator in Eq. (3.10) via additional bosonic superpartner fields. This
is a standard procedure of the supersymmetric approach
developed in Refs. [6,7].

Following this approach, we introduce complex bosonic
fields \( s_i(t) \), \( i = 1, 2, \ldots N \) and a new bosonic model with the
action

\[
S^B [s, s^*] = \int_{-\infty}^{\infty} \left[ \sum_{i=1}^{N} s_i^* (t) (i\partial_t + \mu) s_i (t) \right] \quad \text{(3.12)}
\]

where the action \( S^B [s, s^*] \) looks identical to action \( S [\chi, \chi^*] \),
Eq. (3.3), and it is real. Moreover, the coupling constant
\( J_{ij,kl} \) obeys the same symmetry relations (3.4). Now we write the bosonic partition function

\[
Z_B = \int \exp \left( i S^B [s, s^*, MB] \right) DsD s^* DM^B. \quad \text{(3.13)}
\]

The matrix \( S^B [s, s^*, MB] \) has the same symmetry as the matrix \( M^F (t) \) in Eqs. (3.7)
through (3.11), and we can calculate the Gaussian integrals over the bosonic field \( s (t) \) in the same manner as previously:

\[
\int \exp \left[ i \int_{-\infty}^{\infty} \left( s_i^* (t) (i\partial_t + \mu) \delta_{ij} - 2iM^B_{ij} (t) \right) s_j (t) dt \right] DsD s^* = (Det((i\partial_t + \mu) \delta_{ij} - 2iM^B_{ij}))^{-1}. \quad \text{(3.16)}
\]

The action \( S^B [s, s^*, MB] \) equals to

\[
S [s, s^*, MB] = \int_{-\infty}^{\infty} dt \sum_{i,j=1}^{N} \left( s_i^* (t) (i\partial_t + \mu) \delta_{ij} - 2iM^B_{ij} (t) \right) s_j (t) dt + \sum_{i,j,kl} M^B_{ij} (t) (J^{-1})_{ij,kl} M^B_{kl} (t) \right]. \quad \text{(3.15)}
\]

Combining the fermionic and bosonic degrees of freedom, one can form a supervector \( \Phi \equiv \{ \{ \chi_i \}; s \{ s_i \} \} \in U(N,1|N,1) \) and its Hermitian conjugate supervector \( \Phi^+ \in U(N,1|N,1) \). This allows us to write a supersymmetric action for the time-reversal invariant SYK model as

\[
\tilde{S} \left[ \Phi, \Phi^+, \hat{M} \right] = \int dt \left[ \sum_{i,a} \Phi_i^* (t) \left[ (i\partial_t + \mu) \delta_{ij} - 2M_{ij} (t) \right] \Phi_j (t) \right. \\
+ \left. \sum_{ij,kl} \text{Tr} \left( \hat{M}_{ij} (t) (J^{-1})_{ij,kl} \hat{M}_{kl} (t) \right) \right], \quad \text{(3.17)}
\]

where the two-component supervectors have the following structure

\[
\Phi_i (t) = \left( \begin{array}{c} \chi_i (t) \\ s_i (t) \end{array} \right), \quad \Phi_i^+ (t) = \left( \begin{array}{c} \chi_i^* (t) \\ s_i^* (t) \end{array} \right). \quad \text{(3.18)}
\]

Here the action \( S^B [s, s^*, MB] \) equals to

\[
S [s, s^*, MB] = \int_{-\infty}^{\infty} dt \left( \sum_{i,j=1}^{N} \left( s_i^* (t) (i\partial_t + \mu) \delta_{ij} - 2iM^B_{ij} (t) \right) s_j (t) dt \right.
+ \left. \sum_{i,j,kl} M^B_{ij} (t) (J^{-1})_{ij,kl} M^B_{kl} (t) \right]. \quad \text{(3.15)}
\]

The matrix \( M^B (t) \) in Eqs. (3.7) and (3.8) has the same symmetry as the matrix \( M^F (t) \) in Eqs. (3.7) through (3.11), and we can calculate the Gaussian integrals over the bosonic field \( s (t) \) in the same manner as previously:

\[
\int \exp \left[ i \int_{-\infty}^{\infty} \left( s_i^* (t) (i\partial_t + \mu) \delta_{ij} - 2iM^B_{ij} (t) \right) s_j (t) dt \right] DsD s^* = (Det((i\partial_t + \mu) \delta_{ij} - 2iM^B_{ij}))^{-1}. \quad \text{(3.16)}
\]

We see that the matrix \( M^B (t) \) is gauged out, and the result of the integration over \( s (t) \), \( s^* (t) \) is performed exactly in the same way as in the fermionic determinant. This matrix is also real and antisymmetric. However, in contrast to Eq. (3.11), one obtains the inverse of the determinant. It is this property of bosonic determinants that allows one to get rid of the denominator in Eq. (3.2).
Before performing disorder averaging, it is convenient to use more compact notations via introducing 4-component supervectors $\Psi(t)$ as

$$
\Psi_i(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_i^c(t) \\ \chi_i^s(t) \\ s_i^c(t) \\ s_i^s(t) \end{pmatrix}, \quad (3.23)
$$

The supervector $\Psi$ is related to $\Psi$ by a charge conjugation:

$$
\Psi = (C\Psi)^T, \quad (3.24)
$$

where “$T$” stands for transposition, and the matrix $C$ is given by

$$
C = \begin{pmatrix} c_2 & 0 \\ 0 & c_1 \end{pmatrix}, \quad c_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

One can notice that $\Psi$ has a simple connection to the hermitian conjugated supervector $\Psi^+$:

$$
\Psi = \Psi^+ \tau_3, \quad (3.25)
$$

where

$$
\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.26)
$$

is Pauli matrix in the “particle-hole” space of matrices $c_2$ and $c_1$.

Furthermore, the square of the modulus of the supervector $\Psi$ is equal to

$$
|\Psi|^2 = \Psi^+ \Psi = \Psi \tau_3 \Psi. \quad (3.27)
$$

It is also seen that

$$
\Psi_i \Psi_j = -\Psi_j \Psi_i. \quad (3.28)
$$

Substituting Eqs. (3.23) through (3.28) into Eqs. (3.21) and (3.22), we rewrite the fermion Green’s function in a more compact form

$$
G_{ij}(t,t') = \int \Psi_{2i}(t) \Psi_{2j}^+(t') \exp(iS[\Psi,\Psi^+]) D\Psi D\Psi^+. \quad (3.29)
$$

In Eq. (3.29), superscripts numerate blocks in the superspace, while first subscripts numerate elements in the particle-hole space. The action $S[\Psi,\Psi^+]$ entering Eq. (3.29) is given by

$$
S[\Psi,\Psi^+] = \int_{-\infty}^{\infty} dt \sum_i \Psi_i(t)(i\partial_t + \tau_3 \mu) \Psi_i(t) - 4 \sum_{a=1}^{2} \sum_{ij,kl} J_{ij,kl}(\Psi_i^a(t) \Psi_j^a(t)) (\Psi_k^b(t) \Psi_l^b(t)) \quad (3.30)
$$

where $\Psi^a(t)$, $a = 1, 2$ stand for the fermion and boson components of the supervectors $\Psi$. Substituting Eq. (3.30) into Eq. (3.29), one can easily average over the random $J_{ij,kl}$ using Eq. (3.24). The expression for disorder averaged Green’s function thus will read as

$$
\langle G_{ij}(t,t') \rangle = \int \Psi_{2i}(t) \Psi_{2j}^+(t') \exp(iS[\Psi,\Psi^+]) D\Psi D\Psi^+, \quad (3.31)
$$

where the non-local action action $S[\Psi,\Psi^+]$ equals

$$
S[\Psi,\Psi^+] = \int_{-\infty}^{\infty} dt \sum_{i=1}^{N} \Psi_i(t)(i\partial_t + \tau_3 \mu) \Psi_i(t) - 2 \frac{iN^2}{N^3} \sum_{a,b=1}^{4} \sum_{ij,kl}^{N} \int_{-\infty}^{\infty} \left( \Psi_i^a(t) \Psi_j^a(t) \right) \left( \Psi_k^b(t) \Psi_l^b(t) \right) \left( \Psi_i^b(t') \Psi_j^b(t') \right) \left( \Psi_k^a(t') \Psi_l^a(t') \right) dtdt'. \quad (3.32)
$$

We see that the action $S[\Psi,\Psi^+]$ in Eq. (3.32) does not contain disorder anymore, and the integral over the supervectors $\Psi_i^a(t)$ and $\Psi_j^a(t)$ in Eq. (3.31) is clearly convergent.

Of course, in Eq. (3.32) the addition of extra bosonic degrees of freedom comes at the price of introducing additional integrals. However, the resultant theory, Eqs. (3.31) and (3.32), does not contain disorder and is fully supersymmetric. As such, it has many simplifications. One simplification is the cancellation of a variety of Feynman diagrams in the perturbation theory in interactions due to the supersymmetry. Another simplification follows from the superbosonization of this supersymmetric action discussed in Section V. In the superbosonized representation, instead of the functional integral over supervectors, one deals with an integral over supermatrices.

In that approach, the number of integration variables can significantly be reduced upon the diagonalization of the supermatrices.

However, let us first make a saddle point approximation that has to become exact in the limit $N \to \infty$. This is done in the next section. Comparison of the hereby obtained results with those obtained within the replica approach in Refs. 10 and 12 can be done but one cannot expect a full coincidence because we consider a somewhat different model. In contrast to the calculations presented there we use the real-time representation.

IV. SADDLE-POINT APPROXIMATION

The saddle point approximation is expected to become exact in the limit $N \to \infty$. In order to see this property explicitly and proceed with the calculations, let us introduce $2 \times 2$ supermatrices, $W^{ab}(t,t')$ as

$$
W^{ab}(t,t') = \frac{2}{N} \sum_{i=1}^{N} \Psi_i^a(t) \Psi_i^b(t'), \quad (4.1)
$$

where
where supervectors $\Psi$ and $\bar{\Psi}$ are specified in Eq. (6.18). The supermatrix $W (t, t')$ has the evident symmetry

$$W^+ (t, t') = W (t', t).$$

Using Eqs. (2.22) and (2.23), and the disorder averaging procedure resulting in Eq. (6.33), we explicitly reduce Eq. (6.32) to a considerably more compact form

$$S [\Psi, \Psi^+] = \int_0^\infty dt \sum_{i=1}^N \bar{\Psi}_i (t) (i \partial_t + \tau_3 \mu) \Psi_i (t)$$

$$+ \frac{i N J^2}{2} \int_0^\infty \int_0^\infty dt dt' \left[ 2 \left( \text{Tr} (W^{21} (t, t') W^{12} (t', t)) \right)^2$$

$$+ \left( \text{Tr} (W^{11} (t, t') W^{11} (t', t)) \right)^2$$

$$\right] + \left( \text{Tr} (W^{22} (t, t') W^{22} (t', t)) \right)^2.$$  

where $2 \times 2$ matrices $W^{ab}$ have matrix-valued entries. Elements of matrices $W^{21} (t, t')$ and $W^{12} (t, t')$ are anticommuting fields, while those of the matrices $W^{11} (t, t')$ and $W^{22} (t, t')$ contain products of two anticommuting fields or are conventional complex functions.

Here we would like to invite the reader’s attention to the resemblance of the action (6.32) with the replicated imaginary time action of the SYK model outlined in Ref. 12 (see Eq. (16) there). However, now we have the formally exact supersymmetric representation of the model, where no replica limit, $n \to 0$ (see e.g., Refs. 79,82,83), has to be taken. It is also worth emphasizing that here we have 4 $\times$ 4 supermatrices $W (t, t')$ instead of $n \times n$ matrices in the replica approach. We emphasize that Eqs. (3.31) and (4.3) are still exact for any $N$.

Now one can explicitly see that the interaction term in Eq. (4.3) is proportional to $N$, and the accuracy of the saddle-point approximation should follow from the assumption that this number is large. Although details are different, we use the general chain of transformations suggested in Refs. 10 and 12 and analyze the behavior of the fermion Green’s function.

First, we decouple the interaction terms in Eq. (4.3) by introducing auxiliary functions $P^{ab} (t, t')$ and integrating over them. We write

$$S \left[ \Psi, \Psi^+ \right] = \int_0^\infty dt \sum_{i=1}^N \bar{\Psi}_i (t) (i \partial_t + \tau_3 \mu) \Psi_i (t)$$

$$+ \frac{i N J^2}{2} \int_0^\infty \int_0^\infty dt dt' \left[ 2 \left( \text{Tr} (W^{21} (t, t') W^{12} (t', t)) \right)^2$$

$$+ \left( \text{Tr} (W^{11} (t, t') W^{11} (t', t)) \right)^2$$

$$\right] + \left( \text{Tr} (W^{22} (t, t') W^{22} (t', t)) \right)^2.$$  

where

$$Z_0 = \int DP \exp \left[ -N \sum_{a,b=1}^N \int_{-\infty}^{+\infty} \frac{P^{ab} (t, t') P^{ba} (t', t)}{2 J^2} dt dt' \right].$$

In Eqs. (4.4) and (4.5), $P^{11} (t, t')$ and $P^{22} (t, t')$ are real symmetric functions, while $P^{12} (t, t') = (P^{21} (t', t))^\ast$. In the main approximation in $N$, contributions coming from $W^{aa} (t, t')$ are most important and we concentrate on them.

To simplify the action and analyze its equations of motion, we have to decouple the terms $\left( \text{Tr} (W (t, t') W (t', t)) \right)^2$ by one more gaussian decoupling. To do this we introduce a new diagonal matrix-field $Q^{aa} (t, t')$, $a = 1, 2$, and use the following identities

$$\exp \left[ N i \int_{-\infty}^{+\infty} P^{aa} (t, t') W^{aa} (t, t') W^{aa} (t', t) dt dt' \right]$$

$$= \int DQ \exp \left[ - N i \sum_{ij} \int_{-\infty}^{+\infty} P^{aa} (t, t') \left[ \text{Tr} (Q^{aa} (t, t') Q^{aa} (t', t))ight]$$

$$+ 2 Q^{aa} (t, t') \Psi^a (t) \Psi^a (t) \right] dt dt' \right] Z_a [P]$$

$$= Z_a [P] \int DQ \exp \left[ - N i \sum_{ij} \int_{-\infty}^{+\infty} \right.$$

$$\times \left[ \text{Tr} \left[ P^{aa} (t, t') Q^{aa} (t, t') Q^{aa} (t', t) \right]$$

$$- 2 (-1)^a \Psi^a (t) P^{aa} (t, t') Q^{aa} (t', t) \Psi^a (t') \right]$$

$$\times dt dt' \right], \quad (4.6)$$

where $a = 1, 2$, and we introduced the following notation:

$$Z_a [P] =$$

$$\int DQ \exp \left[ i N \sum_{ij} \int_{-\infty}^{+\infty} P^{aa} (t, t') Q^{aa} (t, t') Q^{aa} (t', t) \right].$$

In Eq. (4.6), the new matrices

$$Q (t, t') = \left( \begin{array}{cc} Q^{11} (t, t) & 0 \\ 0 & Q^{22} (t, t') \end{array} \right) \quad (4.8)$$

have the following symmetry

$$\bar{Q} (t, t') = C Q^T (t', t) C^T = Q^+ (t', t). \quad (4.9)$$

All these decouplings and notations allow us to write the full partition function, $Z$, of the model in the form

$$Z = \int \exp \left[ i S \left[ \Psi, \Psi^+, P, Q \right] \right] Z [P] D \Psi D P D \bar{Q}, \quad (4.10)$$

where the integrant contains a factor $Z [P]$ given by

$$Z [P] = Z_a [P] Z_0 [P] Z_0. \quad (4.11)$$
In Eq. (4.110), the functional $S[\Psi, \Psi^+, P, Q]$ is given by

$$S[\Psi, \Psi^+, P, Q] = \int_{-\infty}^{\infty} dt dt'$$

$$\times \left[ \sum_{i=1}^{N} \tilde{\Psi}_i(t) \left[ \delta_{i,t'} (i \partial_{t'} + \tau_3 \mu) + 2P(t, t') Q(t,t') \right] \tilde{\Psi}_i(t') \right]$$

$$- N \sum_{i,j=1}^{N} \text{Tr} \left[ P(t, t') Q(t, t') Q(t', t) \right] + \frac{iN}{2J^2} \text{Tr} \left[ P^2(t, t') \right],$$

where

$$P(t, t') = \begin{pmatrix} P^{11}(t, t') & 0 \\ 0 & P^{22}(t, t') \end{pmatrix}.$$

Integrating out the supervectors $\Psi, \Psi^+$, one obtains, using Eq. (4.110), the following formula for the partition function $Z$:

$$Z = \int Z[P, Q] DPDQ,$$

with the integrant $Z[P, Q]$ being equal to

$$Z[P, Q] = \exp \left[ N \int_{-\infty}^{\infty} dt dt' \left[ - \frac{\text{Tr} P^2(t, t')}{2J^2} \right. \right.$$

$$\left. + \text{Tr} \left[ \ln \left[ \delta(t-t') (i \partial_{t'} + \tau_3 \mu) + 2P(t, t') Q(t, t') \right] \right] \right.$$

$$\left. \left. - i \text{Tr} \left[ P(t, t') Q(t, t') Q(t', t) \right] \right] \right].$$

Here we introduced a $2 \times 2$ matrix, $k$, that differentiates between bosonic and fermionic superpartners,

$$k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Presence of the large $N$ in the exponential in Eq. (4.115) allows one to calculate the integral over $P(t, t')$ and $Q(t, t')$ using the saddle-point method. Minimizing action $-\ln Z[P, Q]$ with respect to the matrices $Q(t, t')$ and $P(t, t')$, we obtain the following saddle-point equations

$$Q(t, t') = -ik \left[ \delta(t-t') (i \partial_{t'} + \tau_3 \mu) + 2P(t, t') Q(t, t') \right]^{-1},$$

$$P(t, t') = -iJ^2 Q(t, t') Q(t', t)$$

$$+ 2J^2 Q(t, t') \left[ \right.$$

$$\left. \left. (i \partial_{t'} + \tau_3 \mu) \delta(t-t') \right] \right.$$

$$\left. \left. + 2P(t, t') Q(t, t') \right]^{-1}.$$

Using Eq. (4.116) we rewrite Eq. (4.17) in a simpler form

$$P(t, t') = iJ^2 Q(t, t') Q(t', t).$$

As the next step, substituting Eq. (4.18) into Eq. (4.16), one will obtain a closed equation for $Q(t, t')$

$$Q(t, t') = -ik \left[ \right.$$

$$\left. (i \partial_{t'} + \tau_3 \mu) \delta(t-t') \right]$$

$$\left. + 2iJ^2 Q(t, t') Q(t', t) \right]^{-1}.$$

Note that, Eq. (4.19) can also be written in a form of a differential equation,

$$\left[ \right.$$

$$\left. \left. (i \partial_{t'} + \tau_3 \mu) Q(t, t') \right]$$

$$\left. + 2iJ^2 \int Q(t, t'') Q(t', t) (t, t'') \right]$$

$$\times Q(t'', t') dt'' = -ik \delta(t-t').$$

As the function $Q(t, t')$ is diagonal, one can solve Eq. (4.20) separately for the fermion and boson parts. At small energies (large-time limit), one can neglect the first line in Eq. (4.20). Assuming that the solutions depend on the time difference, one comes to the following set of equations

$$2J^2 \int \left( Q^F(t-t'') \right)^2 \left. Q^B(t''-t) Q^F(t''-t') \right] dt''$$

$$= -\delta(t-t'),$$

$$2J^2 \int \left( Q^B(t-t'') \right)^2 \left. Q^B(t''-t) Q^B(t''-t') \right] dt''$$

$$= \delta(t-t'),$$

where $Q^F(t, t')$ and $Q^B(t, t')$ are fermion and boson parts the matrix $Q(t, t')$. The structure of Eqs. (4.21) is similar that of equations obtained in Ref. [12], although there are small differences due a fact we considered here a slightly different model, Eq. (4.33), written in real time.

From Eqs. (4.18, 4.20), we can find the Green’s function of fermions and bosons in energy space

$$G(\omega) = \left[ \omega + \mu - \Sigma(\omega) \right]^{-1},$$

where $\Sigma(\omega)$ is a Fourier image of the electron/boson self-energy,

$$\Sigma(t-t') = -2J^2 kG^2(t-t') G(t'-t).$$

In Eq. (4.22) $\Sigma$ and $G$ are $4 \times 4$ diagonal matrices. For fermionic part this relations fully coincide with ones obtained in Ref. [12], while bosonic self energy has the opposite sign, as it should be. One can see easily that this sign difference gives the unity partition function $Z[P, Q]$ given by Eqs. (4.16, 4.15). Indeed, writing the derivative of the logarithm of the partition function $Z$ and using the saddle point equations (4.16, 4.18), we obtain

$$-\frac{\partial}{\partial J} \ln Z[P, Q] = -N \frac{\text{Tr} P^2}{J^3}(t, t'),$$

where $Q_J$ and $P_J$ are solutions of the saddle point equation (4.16, 4.18). Using Eqs. (4.18, 4.21, 4.23) and
reconstructing the partition function $Z[P;j, Q;j]$ from its 
derivative we conclude that it is equal one. This con-
irms that the saddle point solution does not contradict 
the supersymmetry. Although study of the solution of 
Eqs. (4.21) at arbitrary time is also interesting, we do 
not perform it here.

In the scaling low energy limit $\omega \ll J$, and zero 
chemical potential, the expression for the Green’s func-
tion $G_a(\omega)$ (where $a = 1, 2$ corresponds to fermions and $a = 
3, 4$ to bosons) has one dimensional time reparametriza-
tion, $t = f(\sigma')$, and emergent $U(1)$ gauge invariance, 
defined by Sachdev in Ref. [12] for imaginary time:

$$G_a(t, t') = \left[f'(\sigma)f'(\sigma')\right]^{-1/4} \frac{g(\sigma)}{g(\sigma')}G_a(\sigma, \sigma')$$

$$\Sigma_a(t, t') = \left[f'(\sigma)f'(\sigma')\right]^{-3/4} \frac{g(\sigma)}{g(\sigma')}\Sigma_a(\sigma, \sigma').$$

(4.24)

Here $f(\sigma)$ and $g(\sigma)$ are arbitrary functions. These 
symmetries impose strong restrictions on $G$ and $\Sigma$ and 
lead to the following asymptotic expression for the Green’s 
function at zero temperature:

$$G_1(t) = \left\{ \begin{array}{ll}
C e^{\alpha/4} \sin[\pi/4 + \theta] & t >> 1/J \\
C e^{-3\alpha/4} \sin[\pi/4 - \theta] & -t >> 1/J,
\end{array} \right.$$ 

with constant $C$. These expressions were first obtained 
in Ref. [12]. Therefore at least in the asymptotic regime of 
large times (low energies), we do not expect a difference 
between our supersymmetric formulation of SYK model 
and the replica approach to it. However, at intermediate 
times, when we can not ignore the kinetic term for super-
symmetric (fermionic) fields in action (4.12), a difference 
may be essential.

V. SUPERBOSONIZATION OF SYK MODEL

In Section IV, we explicitly developed a supersymme-
try method for interacting SYK model, which produced 
non-perturbative results. Remarkably, in the above-
developed approach, the supersymmetry is explicit at the 
level of the saddle point equations. These equations are 
very interesting, and may potentially provide some more 
new information about the system behavior at various 
energy scales. At the same time, as we can see from sad-
dle point equations (4.21), the fermion and boson sectors 
of the diagonal matrix field $Q(t, t')$ are decoupled. Thus 
bosons and fermions do not interfere with each other in this 
formulation.

Interestingly enough, there is an alternative, conceptu-
ally similar but technically different, way of formulating 
the SYK model as a supersymmetric $\sigma$-model. It is the 
superbosonization procedure, which will be developed in 
this section. We will show that at the level of the sad-
dle point equations in the superbosonized description, 
bosonic degrees of freedom interfere with fermions. This 
interference effect can be accounted for analytically. It 
may potentially become crucial for revealing novel modes 
in correlation functions - the advantage of the supersym-
metric approaches as compared to replica and imaginary 
time methods is that they allow for controlled analysis of 
the intermediate time regime.

Consider a function $F(\Phi \otimes \Phi^+)$ of the tensor prod-
uct of a super-vector $\Phi$ and its conjugate $\Phi^+$ given by 
Eq. (4.15) or (3.15). Generally, after ensemble averaging of 
disordered single-particle systems, one deals with integrals 
of type $\int D\Phi D\Phi^+ F(\Phi \otimes \Phi^+)$. The super-bosoniza-
tion formula essentially allows evaluating such a supervector 
integral to an integral over a supermatrix $Q$, where $Q$ 
has no constraints (unlike direct product $\Phi \otimes \Phi^+$).

Being formally exact, the superbosonization approach [77, 78, 80] 
proved to be very efficient in producing nonperturbative results for example in the 
theory of almost diagonal random matrices [73, 74, 80], 
where the standard supersymmetry method was also instrumen-
tal [75, 76]. To derive the superbosonized repre-
sentation of the SYK model, here we will follow a 
slightly different path from the one outlined in Section III. 
In contrast with Eq. (3.17), wherein the joined 
fermion-boson action contained two different Hubbard-
Stratonovich fields, $M_F(t)$ and $M_B(t)$ defined in 
(3.19) for fermions and bosons respectively, here we introduce 
a unique field, $M(t)$ [11]. This procedure is allowed 
because of the property that the determinant in the 
numerator Eq. (3.11) is independent of the fluctuating 
Hubbard-Stratonovich field. Then this procedure will 
lead to the action

$$S = \int dt \left[ \sum_{i,a} \Phi^+_{i,a} \left[ -i\partial_t - \mu \right] \Phi_{i,a} - \sum_{ijkl, a,b} M_{ijl} [J^{-1}] M_{kl} \right].$$

(5.1)

Further, we integrate over the Gaussian fluctuating field, 
$M$. This procedure gives the following expression for the 
action

$$S = \int dt \sum_{i,a} \Phi^+_{i,a} \left[ -i\partial_t - \mu \right] \Phi_{i,a} + \sum_{ijkl, a,b} J_{ijkl} \Phi^+_{i,a}(t) \Phi_{j,a}(t) \Phi^+_{k,b}(t) \Phi_{l,b}(t).$$

(5.2)

As the next step, we perform disorder averaging. The 
integration measure of random couplings, $J_{ijkl}$, is Gauss-
ian: $\sim e^{-N^2 \sum_{ijkl} J_{ijkl}^2 / 8 J^2}$ with $J_{ijkl} = J_{jilk}$. How-
ever, since the couplings have a property of $J_{ijkl} = 
-J_{ikjl} = -J_{klij} = J_{klji}$, only half of them are indepen-
dent. We can select the independent part of couplings 
$J_{ijkl}$ by using the ordering of the indexes and choos-
ing $i > k, j > l$ term. Other terms with $i > k, j < l$, 
$i < k, j > l$, $i < k, j < l$ are equal to selected one with 
appropriate sign. The measure over independent cou-
plings thus becomes

$$W(J) = e^{-N^2 \sum_{i,j,l} J_{ijkl}^2}. \quad (5.3)$$
According to the ordering of indices described above, the interaction term in the right-hand-side of (5.2) is a sum of 4 independent terms \( J_{ijkl}, i > k, j > l \):

\[
\sum_{a,b} \sum_{ijkl} J_{ijkl} \Phi_{i,a}^+(t) \Phi_{j,a}(t) \Phi_{k,b}^+(t) \Phi_{l,b}(t) = \sum_{a,b} \sum_{ijkl} 2J_{ijkl} \left[ \Phi_{i,a}^+(t) \Phi_{j,a}(t) \Phi_{k,b}^+(t) \Phi_{l,b}(t) \right. \\
- \left. \Phi_{i,a}^+(t) \Phi_{l,a}(t) \Phi_{k,b}^+(t) \Phi_{j,b}(t) \right].
\] (5.4)

The disorder averaging (i.e., the integration over independent \( J_{ijkl} \)) thus produces an interacting theory with action that is similar to the one in Eq. (5.5):

\[
S = \int dt \left\{ \sum_{i,a} \Phi_{i,a}^+(t) \left( i \partial_t + \mu \right) \Phi_{i,a}(t) + \frac{2iJ^2}{N^3} \int dt dt' \sum_{ab,a'b'} \sum_{i,k,j,l} \left[ \Phi_{i,a}(t) \Phi_{j,a}(t) \Phi_{k,b}(t) \Phi_{l,b}(t) \right. \\
\left. \times \Phi_{i,a'}(t') \Phi_{j,a'}(t') \Phi_{k,b'}(t') \Phi_{l,b'}(t') \right. \\
\left. \times \Phi_{i,a'}^+(t') \Phi_{j,a'}^+(t') \Phi_{k,b'}^+(t') \Phi_{l,b'}^+(t') \right] \right\}.
\] (5.6)

Eq. (5.6) is invariant under the supersymmetry transformation \( \delta \chi_i = \epsilon \partial_t \chi_i, \delta \partial_t = -\epsilon \chi_i, i = 1 \cdots N \), where \( \epsilon \) is an infinitesimal Grassmann parameter. The reason for this is that the building block of the action, namely \( \Phi_{i,a}^+ \Phi_{i,a} \), is invariant.

There are two distinct approaches for superbosonization of the SYK model. A general approach is based on the introduction of identity into the partition function:

\[
1 = \int_{\mathbb{H}_n} dQ_{ia,jb}(t, t') \delta \left( Q_{ia,jb}(t, t') - \Phi_{ia}(t) \Phi_{jb}^+(t') \right)
\] (5.7)

Here \( Q_{ia,jb}(t, t') \), is a non-local supermatrix. The second, simpler way would be through introducing

\[
1 = \int_{\mathbb{H}_n} dQ'_{ia',jb'}(t, t') \delta \left( Q'_{ia',jb'}(t, t') - \Phi_{ia}(t) \Phi_{jb'}^+(t') \right).
\] (5.8)

imposed by the non-local matrix \( Q'_{ia',jb'}(t, t') \). Here \( \mathbb{H}_n \) is the linear space of Hermitian \( 2n \times 2n \) supermatrices. We recall, that formal sums of formal products \( \Phi \Phi^+ \), where \( \Phi \in U(n,1|n,1) \) and \( \Phi^+ \in U(n,1|n,1) \) are supervectors, constitute a vector space. This vector space is defined, up to isomorphism, by the condition that every antisymmetric, bilinear map \( f : U(n,1|n,1) \times U(n,1|n,1) \rightarrow \mathbb{G} \) determines a unique linear map \( g : U(n,1|n,1) \rightarrow \mathbb{G} \) with \( f(\Phi, \Phi^+) = g(\Phi \otimes \Phi^+) \). This implies that if we consider a map, \( \mathcal{F} : \mathbb{H}_n \rightarrow \mathbb{G} \), then the integral \( \int D\Phi D\Phi^+ F(\Phi \otimes \Phi^+) \) is now well defined. From now on we will restrict ourselves to the case of maps, \( \mathcal{F} \), such that the above integral is convergent.

The delta-function in Eqs. (5.7) and (5.8) is a functional defined as in Ref. 80. Namely, for all \( \mathcal{A} \in \mathbb{H}_n \) the convergent integral \( \delta(\mathcal{A}) = \lim_{\eta \rightarrow 0} \int_{\mathbb{H}_n} DB \exp \{ i \text{Str}[AB] - \eta \text{Str}[B^2] \} \), taken over \( \mathbb{H}_n \), with flat Berezin measure satisfies the condition \( \int_{\mathbb{H}_n} DA' \delta(\mathcal{A}' - \mathcal{A}) \equiv 1 \). Moreover, for any map, \( \mathcal{F} : \mathbb{H}_n \rightarrow \mathbb{G} \), that converges exponentially (or faster), the identity \( \mathcal{F}(Q) \equiv \int_{\mathbb{H}_n} DA \mathcal{F}(A) \delta(\mathcal{A} - Q) \) always holds.

Using the above expression for the delta-functional in Eq. (5.8), and inserting the identity to the partition function defined by Eq. (5.9), we obtain an effective action

\[
S = \int dtdt' \left\{ \sum_{i,a} \left[ - (i \partial_t + \mu) \delta_{ii'} \text{Str}[Q_i(t, t')] \right] + \Phi_{i,a}(t) B_{i,a}'(t, t') \Phi_{i,a'}(t') + \text{Str}[B^2(t, t') Q_i(t, t')] \right. \\
\left. - \eta \sum_{i} \text{Str}[B^2(t, t') B^i(t, t')] + \frac{2iJ^2}{N^3} \times \sum_{i > k, j > l} \left[ \text{Str}[Q_i(t, t')] Q_j(t, t') \text{Str}[Q_k(t, t') Q_l(t, t')] \right] \right\},
\] (5.9)

where symbol "Str" stands for supertrace. We see that the superfield \( \Phi_{i,a}(t) \) enters into this action only as a quadratic form with the matrix \( B^i(t, t') \). Therefore, the integral over superfields \( \Phi_{i,a} \) in the partition function can be exactly evaluated, producing the superdeterminant of \( B^i(t, t') \) in the denominator of the integrand. It is worth to mention that the supermatrix \( B \) should be considered as a matrix by its arguments \( B^i(t, t') = B_{i,a'}^{i,a} \). The partition function of the model thus becomes

\[
Z = \prod_i \text{Det} B(t, t) \text{Det} Q(t, t') \text{Det} B^{i, a'}(t, t') \exp \{ iS \},
\] (5.10)

where "Det" is superdeterminant.

Now, omitting for a while the first two terms in Eq. (5.9), we introduce a notation \( \tilde{S} \) for the remaining terms in the expression, and write is in the form

\[
\tilde{S} = \lim_{\eta \rightarrow 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dtdt' \left\{ \text{Str}[B^i(t, t') Q_i(t, t')] + i \eta \sum_i \text{Str}[B^i(t, t')] \right. \\
\left. + \frac{2iJ^2}{N^3} \times \sum_{i > k, j > l} \left[ \text{Str}[Q_i(t, t')] Q_j(t, t') \text{Str}[Q_k(t, t') Q_l(t, t')] \right] \right\},
\] (5.11)

Then following the method introduced in Ref. 80, we join \( Q \) with \( B \) by introducing a new supermatrix \( \tilde{B} = BQ \).
Here we note that the formal sums of Hermitian superbivectors (product of two supermatrices, each of them being from the linear space of complex Hermitian supermatrices $\mathbb{H}_n$, constitute a vector space $\Lambda^2(\mathbb{H}_n)$ called the second exterior power of $\mathbb{H}_n$. Then, integration over $B \in \Lambda^2(\mathbb{H}_n)$ decouples from the partition function and produces a constant

$$C_n = \int_{A^2(\mathbb{H}_n)} DB \text{SDet}[B] \exp\left\{ \int dt dt' \text{Str}[B(t, t')] \right\}. \quad (5.12)$$

The Berezinian of the transformation $\bar{B} = BQ$ is one. One can see this by explicitly writing the transformations using the matrix form of $\bar{B}$ (with first indices corresponding to fermion or boson fields). Namely, the Jacobian of the transformation $B_{Bc} = B B_{ac}$ is $\text{SDet}[Q]$, while for $B_{Fc} = B_{Fa}Q_{ac}$ the Jacobian is $1/\text{SDet}[Q]$. As a result of the transformation, these two terms cancel each other in the product. This happens because the fields $B_{Fc}$ and $B_{Bc}$ always have opposite fermionic parity.

Finally, the partition function $Z$ acquires the form

$$Z = \int_{\mathbb{H}_n} \prod_i DQ^i(t, t') \frac{1}{\text{SDet}[Q^i(t, t')]} \exp\{i\bar{S}(Q)\}, \quad (5.13)$$

with the action

$$\bar{S}(Q) = \int dt dt' \left\{ \sum_i \text{Str}[ - (i\partial_t + \mu)\delta_{tt'} Q^i(t, t') ] + \frac{2iJ^2}{N^3} \sum_{i,k,j,l} \left[ \text{Str}[Q^i(t,t') Q^j(t',t)] \text{Str}[Q^k(t,t') Q^l(t',t)] \right. \right.$$  

$$\left. - \text{Str}[Q^i(t, t') Q^j(t', t) Q^k(t, t') Q^l(t', t)] \right\}, \quad (5.14)$$

It is important now to observe that the $2 \times 2$ matrix Green’s function $G_{ab}^i(t, t')$ of two-component superfields $\Phi_{ia}(t)$ and $\Phi_{ib}^+ (t')$ (that contains the fermion propagator in its fermion-fermion block) is equal to the vacuum average of the superbosonization matrix field, $\langle Q_{ab}^i \rangle$. Indeed, using the identity Eq. (5.8), one can introduce $Q_{ab}^i (t, t')$ under the integral and obtain

$$G_{ab}^i(t, t') = -i\langle \Phi_{ia}(t)\Phi_{ib}^+(t') \rangle = -i \int D\Phi \Phi_{ia}(t)\Phi_{ib}^+(t') \exp\{iS\}$$  

$$= -i \int DQQ_{ab}^i(t, t') \exp\{i\bar{S}(Q)\} = -i\langle Q_{ab}^i(t, t') \rangle. \quad (5.13)$$

The functional $\text{SDet}[Q_i(t, t')]$ in the denominator of the expression [5.13] for $Z$ should be understood as the superdeterminant of the supermatrix $Q_{ia}^i(t, t')$, which acts linearly in the continuous space of time $t$. Namely, arguments $t, t'$ should be considered as matrix indexes. One can incorporate the pre-exponent $1/\text{SDet}[Q^i(t, t')]$ into the effective action, $S_{eff}$, that can be written as

$$S_{eff} = \int dt dt' \left\{ \sum_i \text{Str}[ - (i\partial_t + \mu)\delta_{tt'} Q^i(t, t') ] + \frac{2iJ^2}{N^3} \sum_{i,k,j,l} \left[ \text{Str}[Q^i(t,t') Q^j(t',t)] \text{Str}[Q^k(t,t') Q^l(t',t)] \right. \right.$$  

$$\left. - \text{Str}[Q^i(t, t') Q^j(t', t) Q^k(t, t') Q^l(t', t)] \right\} + i \sum_i \text{Str}[\text{log} \ Q^i(t, t') \delta(t - t') \right\}. \quad (5.15)$$

Here $\text{log} \ Q^i$ should be understood as the formal series $\text{log} \ Q^i = (Q^i - 1) + \frac{1}{2} (Q^i - 1) * (Q^i - 1) + \frac{1}{3} (Q^i - 1) * (Q^i - 1) * (Q^i - 1) + \cdots$, where the symbol $*$ stands for the convolution product $[A * B](t, t') = \int dt'' A(t, t') B(t'', t')$.

Now let us analyze the saddle point behavior of the field $Q_i^i(\tau, \tau')$ and compare it with the analysis performed in Section III. The crucial point is that we have additional log$Q^i(t, t')$ term, which can contribute in the asymptotic analysis. From Eq. (5.13) we see that $\langle Q^i(t, t') \rangle = \langle \Phi_{ia}(t)\Phi_{ia}^+(t') \rangle$ gives the Green function $G^i(t, t')$ and its asymptotic behavior at large time scale $t \to \infty$ is defined
by the equation of motion for matrix field $Q^i(t, t')$:
\[
\frac{\delta S_\text{eff}}{\delta Q^i(t, t')} = 0 = -(i\partial_t + \mu)\delta_{uv} + i\left[Q^i(t, t')\right]^{-1}
\]
\[
+ \frac{2i J^2}{N^3} \sum_{i>j,k,l>t} \text{Str} \left[Q^i(t, t') Q^j(t', t)\right] Q^k(t, t')
\]
\[
- \frac{2i J^2}{N^3} \sum_{i>j,k,l>t} Q^i(t, t') Q^j(t', t) Q^k(t, t').
\]
This equation shows that the solutions are independent of index, $i$, and therefore we can drop it. Putting now $-i\langle Q(t, t')\rangle \equiv \mathcal{G}(t, t')$ into the Eq. (5.16) and using $\mathcal{G}(t, t') = [(i\partial_t + \mu)\delta_{uv} - iK(t, t')]^{-1}$ with $K(t, t')$ being the self-energy, at the large time scale we obtain
\[
K(t, t') = 2J^2\text{Str}[\mathcal{G}(t, t')\mathcal{G}(t', t)]\mathcal{G}(t, t')
\]
(5.17)
\[
- 2J^2\mathcal{G}(t, t')\mathcal{G}(t', t)\mathcal{G}(t, t').
\]
In saddle point approximation, and due to supersymmetry, we expect that the fermion-fermion (F) and boson-boson (B) entries of the Green’s function are equal: $\mathcal{G}_F(t, t') = \mathcal{G}_B(t, t')$. The implication of this fact is that $\text{Str}[\mathcal{G}(t, t')\mathcal{G}(t', t)] = 0$, which leads to the relation between the self energy and Green’s functions for fermions and bosons
\[
K(t, t') = -2J^2\mathcal{G}(t, t')\mathcal{G}(t', t)\mathcal{G}(t, t').
\]
We note similarity with Eq. (4.22) and the similar relation for fermions obtained using the replica approach. Hence, at large time scales our supersymmetric model reproduces the same asymptotics for the Green’s function as the replica method provides. However, at intermediate times supersymmetric action is essentially different from the replica field theory and we expect that this method will provide new results at intermediate time scales. In order to see this we rewrite the supertrace over the supermatrices $Q_k$ in the interaction terms of the action $S_{\text{int}}$, defined by (5.14), using fermion-boson (FB) and boson-fermion (BF) components of the supermatrices:
\[
\text{Str}[Q^i(t, t') Q^j(t', t)] = Q_{BF}^i(t, t') Q_{FB}^j(t', t)
\]
\[
- Q_{FB}^i(t, t') Q_{BF}^j(t', t) - Q_{FF}^i(t, t') Q_{FF}^j(t', t)
\]
\[
+ Q_{BB}^i(t, t') Q_{BB}^j(t', t).
\]
Similarly for $\text{Str}[Q^k(t, t') Q^l(t', t)]$ part and $\text{Str}[Q^i(t, t') Q^j(t', t) Q^k(t, t') Q^l(t', t)]$. BF and FB components of supermatrices are Grassmann variables and the integration over them is easily performed. It will produce separate actions for fermions and bosons of the form of Eq. (5.15) and additional pre-exponential mixed polynomials from the BB and FF components. Appearance of these mixed polynomials is a result of the formally exact supersymmetric approach, and these terms are not captured within the replica approach. At large time scales, they have subleading contribution to the correlation function but will have essential contribution in the intermediate, finite time region. This fact is a major advantage of the supersymmetric method. More detailed and complete analysis of this effects is a subject of future investigations.

Another advantage of the superbosonized $\sigma$-model representation described above is that it is efficient for computation of correlation functions. The procedure, described in Ref. [30] consists of

1) Diagonalization of $m \times m$ supermatrix field, $Q$, as $Q = UQ_{\text{diag}}V$ with diagonalization matrices $U \in U(m|m)$ and $V \in U(m|m)/U^{2m}(1)$ restricted to the unitary supergroup and its subspace with removed phases.  

2) After the diagonalization of the supermatrix $Q$, one can integrate over $Q$ by integrating over its boson-boson eigenvalues in the interval $iR \equiv \{-\infty, \infty\}$, while the integration over the fermion-fermion eigenvalues should be performed in the interval $iR \equiv \{-i\infty, i\infty\}$.

We see that this procedure significantly reduces the number of integrations one has to perform to calculate correlation functions within superbosonized representation.

VI. CONCLUSIONS AND OUTLOOK

Despite being a standard tool for nonperturbative calculations in disordered and chaotic systems, the supersymmetric sigma model has rather poorly been understood for interacting systems. Historically, it was believed that the Hubbard-Stratonovich decoupling of the interaction Hamiltonian would not help to develop a supersymmetric description of the partition function of the model. The reason is that one has to introduce two different Hubbard-Stratonovich bosonic fields, $M_1$ and $M_2$, to decouple interaction terms both in the numerator and the denominator of the expression for any correlation function. It was believed, for about 40 years, that supersymmetric $\sigma$ model representation of interacting systems is impossible because fluctuating $M_1$ and $M_2$ fields are independent. And therefore, supersymmetry cannot become manifest in a theory that is disordered, interacting, and dynamical.

In this work, we have challenged this belief and have developed a mathematically rigorous supersymmetric $\sigma$-model framework for interacting disordered systems. The idea that helps to overcome the abovementioned problem of independence of fluctuating fields $M_1$ and $M_2$ is the following. The partition function of the system is calculated by the functional integration of an exponentiated action functional over the space of dynamical field configurations. We showed that for $(0+1)$ dimensional systems, such as quantum dots, the Hubbard-Stratonovich field in the denominator could be gauged out. It can also be reintroduced back to guarantee supersymmetry. In order to derive basic formulas of the supersymmetry method, we have introduced a new version of the SYK model. In contrast to the previous versions, the model is time-reversal
invariant. One of the main achievements of this paper is that we have given a supersymmetric sigma-model description of the SYK model. We have also developed its superbosonized description, where the functional integral is taken over unconstrained dynamical supermatrix fields representing collective many-body excitations.

It is now a conventional wisdom\textsuperscript{11} that the SYK model exhibits many-body chaotic properties at all time scales. At short times, chaos shows up in exponentially decaying correlations as manifested in out-of-time correlation functions\textsuperscript{12,13,14,15}. At large time scales, chaos manifests itself in a random matrix ensemble due to quantum energy level repulsion\textsuperscript{16}. However, the nature of the transition region from non-ergodic to ergodic regimes remains unclear. Moreover, the physics of non-ergodic states is not yet fully understood, and ”dirty” metals represent an excellent physically motivated playground for such studies. Here, an important development was made in Ref.\textsuperscript{17} where the theoretical description of non-ergodic extended states in a modified SYK model was put forward. The problem of finding the ergodic (Thouless) time in SYK model was considered in Ref.\textsuperscript{32} where the questions regarding the nature of the relaxation modes, their classification by certain effective quantum numbers, as well as the density of states, were addressed. An important correlation function, capable of detecting chaotic properties of the SYK model, is the spectral number variance $\Sigma_2(\epsilon)$. It represents the statistical variation in the number of many-body levels contained in an energy window of width $E$. The variance $\Sigma_2(\epsilon)$ was studied in Ref.\textsuperscript{33} where a deviation from the random matrix ensemble prediction was reported. This deviation demonstrates the possible breakdown of ergodicity, and this is one of the interesting points that can be investigated further using superbosonization.

The spectral form factor considered in Ref.\textsuperscript{40} representing the Fourier transform of the energy-dependent spectral two-point correlation function, $R_2(\epsilon)$, is yet another quantity of interest. While the longtime profile of it showed a ramp structure characteristic for random matrix theory ensembles, universal deviations from random matrix theory were observed for shorter times (see Refs.\textsuperscript{32,102} for related studies). Density-density correlators were studied in Ref.\textsuperscript{32} within the replica approach describing the quantum chaotic dynamics of the SYK model at large times. It was observed that there are non-ergodic collective modes, which relax in some time interval and become ergodic states by entering into the longtime regime. The latter modes can be described using the random matrix theory. These interesting modes share similar properties with the diffusion modes of dirty metals and have quantum numbers which have been identified as the generators of the Clifford algebra\textsuperscript{25}. There, each of the $2N$ different products formed from $N$ Majorana operators represents a mode.

Here we propose that the superbosonization approach to the SYK model will open new possibilities to study intermediate time regions and reveal new aspects of chaotic properties. In particular, it would be fascinating to (i) calculate one-point correlation function $\langle \rho(E) \rangle$ (the density of states) in superbosonized representation of SYK model and compare it with the universal random matrix prediction; (ii) calculate the two-point correlation function, $\langle \rho(E) \rho(E') \rangle$, in SYK model using its superbosonized representation and compare it with numerical calculations in Refs.\textsuperscript{32,40}; (iii) reveal the role of bosonic excitations presented in superbosonized representation and to detect their behavior at short times. Systematic deviations from the random matrix predictions, for sufficiently well-separated eigenvalues, imply that the model is not ergodic at short times. The point of departure from the results of random matrix theory increases with $N$, which is an indication of having a Thouless energy scale\textsuperscript{21,22} in the system. Detection of Thouless time within a superbosonized approach is yet another exciting project. It would be also interesting to calculate moments of the spectral density within the supersymmetric sigma model approach.

On another front, it is well-known that Anderson localization can be avoided under certain conditions for disorder potential supporting long-range\textsuperscript{23,24} or short-range\textsuperscript{25,26} correlations in low dimensions. It is thus very interesting to investigate the effect of introducing correlations to the disordered interaction constant, $J_{ijkl}$. We expect that such an analysis can also be performed using the technique outlined in the present work.

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