Real Submanifolds in Complex Spaces

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Abstract Let \((z_1, \ldots, z_N, \ldots, z_{m1}, \ldots, z_{mN}, w_{11}, \ldots, w_{mm})\) be the coordinates in \(\mathbb{C}^{mN+m^2}\). In this note we prove the analogue of the Theorem of Moser in the case of the real-analytic submanifold \(M\) defined as follows
\[
W = Z\overline{Z} + O(3),
\]
where \(W = \{w_{ij}\}_{1 \leq i, j \leq m}\) and \(Z = \{z_{ij}\}_{1 \leq i \leq m, 1 \leq j \leq N}\). We prove that \(M\) is biholomorphically equivalent to the model \(W = Z\overline{Z}\) if and only if is formally equivalent to it.

Keywords Real submanifold, equivalence problem, fischer decomposition

MR(2010) Subject Classification 32V40, 32H02, 32V99

1 Introduction and Main Result

One of the most beautiful problems in complex analysis is the equivalence problem between two real analytic submanifolds in complex spaces. Chern–Moser [5] proved that any formal holomorphic equivalence defined between two pseudoconvex real-analytic hypersurfaces is convergent. Versions of this result have been proven by Mir [22, 23] in the CR finite type case using the Artin Approximation Theorem [1]. However, in the infinite type case it has been shown very recently by Kossovskiy–Shafikov [20] that there exist real-analytic hypersurfaces which are formally, but not holomorphically equivalent. Kossovskiy–Lamel [21] proved a similar result for two formally CR-equivalent real-analytic holomorphically nondegenerate CR-submanifolds. The analogous problem in the CR singular case [2] has been studied by Moser–Webster [25] and Gong [14]. They constructed real analytic submanifolds in the complex space which are formally equivalent, but not holomorphically equivalent.

Let \((z, w)\) be the coordinates in \(\mathbb{C}^2\). We consider the following analytic surface in \(\mathbb{C}^2\) defined near \(p = 0\) as follows
\[
w = z\overline{z} + O(3). \tag{1.1}
\]
Moser [26] proved that (1.1) is holomorphically equivalent to the quadric model \(w = z\overline{z}\) if and only if it is formally equivalent to it. This result is known as the Theorem of Moser [26]. An higher dimensional analogue version of the Theorem of Moser [26] has been obtained by Huang–Yin [15] for the real-analytic submanifold of codimension 2 in \(\mathbb{C}^{N+1}\) defined as follows
\[
w = z_1 \overline{z}_1 + \cdots + z_N \overline{z}_N + O(3), \tag{1.2}
\]
Received March 25, 2015, revised April 2, 2016, accepted May 11, 2016
Supported partially by CAPES
where \((w, z_1, \ldots, z_N)\) are the coordinates in \(\mathbb{C}^{N+1}\). Huang–Yin [15] proved that the real-analytic submanifold defined in (1.2) is biholomorphically equivalent to the model \(w = z_1\bar{z}_1 + \cdots + z_N\bar{z}_N\) if and only if it is formally equivalent to it.

The purpose of this note is to prove the analogue of the Theorem of Moser [26] in the case of the real-analytic submanifolds in the complex space defined near \(p = 0\) as follows:

\[
W = \mathcal{Z}^t + O(3),
\]

where

\[
W = \{w_{ij}\}_{1 \leq i, j \leq m}, \quad Z = \{z_{ij}\}_{1 \leq i \leq m, 1 \leq j \leq N}
\]

and \((z_{11}, \ldots, z_{1N}, \ldots, z_{m1}, \ldots, z_{mN}, w_{11}, \ldots, w_{mm})\) are the coordinates in \(\mathbb{C}^{mN+m^2}\). The main result of this note is the following

**Theorem 1.1** Let \(\mathbb{C}^{mN+m^2}\) be the real-analytic submanifold defined near \(p = 0\) by (1.3). Then \(M\) is biholomorphically equivalent to the model

\[
W = \mathcal{Z}^t
\]

if and only if it is formally equivalent to it.

This result can be seen as a generalization of the Theorem of Moser [26] in the case when the codimension is different from 2. If in the case (1.2) of Huang–Yin [15] the model “arises” from the sphere using the classical Cayley transformation, in our case (1.4) the model “arises” by transforming the Shilov boundary of the bounded symmetric domain of first kind [18] using a generalized Cayley type transformation [11]. This can be seen as analogue of the Theorem of Moser [26] when the real submanifold is “modelled” by the Shilov boundary [11] of an bounded symmetric domain of first kind [18]. We have to mention that Kaup–Zaitsev [17] observed other cases of real submanifolds in complex spaces derived from the Shilov boundary of a bounded and symmetric domain [19] of first kind [18].

We prove Theorem 1.1 using the lines of the proof of the Theorem of Moser [26] and of the proof of the Generalization of Huang–Yin [15] of the Theorem of Moser [26]. We firstly develop a partial normal form using techniques based on the Fischer decomposition [27] applied by Zaitsev [28–30] and the author in [3, 4]. Once the partial normal form is constructed, we bring our case (1.3) in a similar situation to the case (1.2) of Huang–Yin [15] in order to make suitable estimations and then we apply methods based on rapid convergence arguments used by Moser [26]. We mention that similar methods have been used by Coffman [6, 7] and by Gong [12, 13]. In particular, our proof gives a different proof using rapid convergence arguments of the Generalization of Huang–Yin [15] of the Theorem of Moser [26]. The difference of our case is given by the application of the Fischer decomposition [27] and of the orthogonality properties of the Fischer inner product [27] in order to make suitable estimates of the \(G\)-part of the formal transformation map in the local defining equations. These estimates are used in order to adapt proof of Moser [26] and the proof of Huang–Yin [15] in our case and the our proof follows exactly as in their cases. As in [3], the main ingredient is the Fischer decomposition [27] and its properties. We would like to mention that the Fischer decomposition [27] and its properties has been used also by Ebenfelt–Render [9, 10] in order to study various partial differential equations problems.
2 The Partial Normal Form

In order to prove Theorem 1.1, we follow the lines of the proof of the Theorem of Moser [26] and we construct a partial normal form for the real submanifolds defined by (1.3) using the strategy used by the author in [3]. In order to prove Theorem 1.1, it is enough to consider the particular case when \( m = 2 \) in (1.3). The general case can be studied using similar computations and methods.

Throughout this note, we consider the following notations:

\[
W := \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \equiv (w_{11}, w_{12}, w_{21}, w_{22}),
\]

\[
Z := \begin{pmatrix} z_{11} & \cdots & z_{1N} \\ \vdots & \ddots & \vdots \\ z_{21} & \cdots & z_{2N} \end{pmatrix} = \begin{pmatrix} z_1^1 \\ \vdots \\ z_2^1 \end{pmatrix} \equiv (z_{11}, \ldots, z_{1N}, z_{21}, \ldots, z_{2N}), \tag{2.1}
\]

where \((z_{11}, \ldots, z_{1N}, z_{21}, \ldots, z_{2N}, w_{11}, w_{12}, w_{21}, w_{22})\) are the coordinates in \( \mathbb{C}^{2N+4} \) and \( z^1, z^2 \) are the lines of the matrix \( Z \).

Let \( \mathbb{C}^{2N+4} \) be a real formal submanifold defined near \( p = 0 \) as follows:

\[
M : W = ZZ' + \varphi_3 (Z, Z), \tag{2.2}
\]

where we have used the following notation:

\[
\varphi_3 (Z, Z) = \sum_{m+n \geq 3} \varphi_{m,n} (Z, Z),
\]

such that \( \varphi_{m,n} (Z, Z) := \begin{pmatrix} \varphi_{m,n}^{1,1} (Z, Z) & \varphi_{m,n}^{1,2} (Z, Z) \\ \varphi_{m,n}^{2,1} (Z, Z) & \varphi_{m,n}^{2,2} (Z, Z) \end{pmatrix}, \tag{2.3}
\]

where \( \varphi_{m,n} (Z, Z) \) is a matrix of bihomogeneous polynomials of bidegree \((m, n)\) in \((Z, Z)\) for all \(m, n \in \mathbb{N} \) with \(m + n \geq 3\).

Let now \( M' \subset \mathbb{C}^{2N+4} \) be another real formal submanifold defined near \( p = 0 \) as follows:

\[
M' : W' = Z'Z'' + \sum_{m+n \geq 3} \varphi'_{m,n} (Z', Z''), \tag{2.4}
\]

where \( \varphi'_{m,n} (Z', Z'') \) is a matrix of bihomogeneous polynomial of bidegree \((m, n)\) in \((Z', Z'')\) defined similarly as in (2.3) for all \(m, n \in \mathbb{N} \) with \(m + n \geq 3\). Substituting a formal transformation \((Z', W') = (F(Z, W), G(Z, W))\), fixing the point \( 0 \in \mathbb{C}^{2N+4} \) that transforms \( M \) defined by (2.2) into \( M' \) defined by (2.4), we obtain the following:

\[
G(Z, W) = (F(Z, W)) (F(Z, W))^\dagger + \sum_{m+n \geq 3} \varphi'_{m,n} (F(Z, W), F(Z, W)), \tag{2.5}
\]

where \( W \) satisfies (2.2). We now write the following formal expansions:

\[
F(Z, W) = \sum_{m,n \geq 0} F_{m,n} (Z, W) = \begin{pmatrix} \sum_{m,n \geq 0} F_{m,n}^{1,1} (Z, W) \\ \sum_{m,n \geq 0} F_{m,n}^{1,2} (Z, W) \\ \sum_{m,n \geq 0} F_{m,n}^{2,1} (Z, W) \\ \sum_{m,n \geq 0} F_{m,n}^{2,2} (Z, W) \end{pmatrix}, \tag{2.6}
\]
\[ G(Z, W) = \sum_{m,n \geq 0} G_{m,n}(Z, W) = \left( \sum_{m,n \geq 0} G^1_{m,n}(Z, W) \right) \left( \sum_{m,n \geq 0} G^2_{m,n}(Z, W) \right) \]

where \( G_{m,n}(Z, W) \) and \( F_{m,n}(Z, W) \) are homogeneous matrix polynomials of degree \((m, n)\) in \((Z, W)\), where \(m, n \in \mathbb{N}\). For \(W\) satisfying (2.2), it follows by (2.3), (2.5) and (2.6) that

\[
\sum_{m,n \geq 0} G_{m,n}(Z, ZZ' + \varphi_{\geq 3}(Z, \bar{Z})) = \left( \sum_{m,n \geq 0} F_{m,n}(Z, ZZ' + \varphi_{\geq 3}(Z, \bar{Z})) \right) \left( \sum_{m,n \geq 0} F_{m,n}(Z, ZZ' + \varphi_{\geq 3}(Z, \bar{Z})) \right) + \varphi'_{\geq 3} \left( \sum_{m,n \geq 0} F_{m,n}(Z, ZZ' + \varphi_{\geq 3}(Z, \bar{Z})), F_{m,n}(Z, ZZ' + \varphi_{\geq 3}(Z, \bar{Z})) \right). \tag{2.7}
\]

Since our map fixes the point \(0 \in \mathbb{C}^{2N+4}\), it follows that \(G_{0,0}(Z) = 0, F_{0,0}(Z) = 0\). Changing linearly the coordinates in \((w_{11}, w_{12}, w_{21}, w_{22})\), we can assume that \(G_{0,1}(W) = (w_{11}, w_{12}, w_{21}, w_{22})\). Continuing as in [3], we collect the terms of bidegree \((1, 1)\) in \((Z, \bar{Z})\) in (2.7) and we obtain the following:

\[ ZZ' = (F_{1,0}(Z))(\overline{F_{1,0}(Z)}). \tag{2.8} \]

After a composition with a linear automorphism of the model manifold \(W = ZZ'\), we can assume that \(F_{1,0}(Z) = Z\).

In order to construct the partial normal form we use the following matrix quadratic model

\[
\begin{pmatrix}
  w_{11} & w_{12} \\
  w_{21} & w_{22}
\end{pmatrix}
= \begin{pmatrix}
  z_{11} \overline{z}_{11} + z_{12} \overline{z}_{12} + \cdots + z_{1n} \overline{z}_{1N} & z_{11} \overline{z}_{21} + z_{12} \overline{z}_{22} + \cdots + z_{1N} \overline{z}_{2N} \\
  z_{21} \overline{z}_{12} + z_{22} \overline{z}_{22} + \cdots + z_{2N} \overline{z}_{2N} & z_{21} \overline{z}_{21} + z_{22} \overline{z}_{22} + \cdots + z_{2N} \overline{z}_{2N}
\end{pmatrix}
= \begin{pmatrix}
  \langle l_1, l_1 \rangle & \langle l_1, l_2 \rangle \\
  \langle l_2, l_1 \rangle & \langle l_2, l_2 \rangle
\end{pmatrix}, \tag{2.9}
\]

where \(l_1 := (z_{11}, z_{12}, \ldots, z_{1N}), \ l_2 := (z_{21}, z_{22}, \ldots, z_{2N})\). The hermitian product \(\langle \cdot, \cdot \rangle\) is defined canonically as follows

\[ \langle a, b \rangle := a_1 \overline{b_1} + \cdots + a_N \overline{b_N}, \]

where \(a = (a_1, \ldots, a_N)\) and \(b = (b_1, \ldots, b_N) \in \mathbb{C}^N\).

Before going further, we recall the following notation of Fischer [27]:

\[
P^* (Z, \bar{Z}) = \sum_{|I|+|J| = k_0} p_{I,J} \frac{\partial^{k_0}}{\partial z^I \partial \bar{z}^J}, \quad \text{if} \ P(Z, \bar{Z}) = \sum_{|I|+|J| = k_0} p_{I,J} z^I \bar{z}^J, \quad k_0 \in \mathbb{N}. \tag{2.10}
\]

If \(\mathbb{H}_k\) is the space of all homogeneous polynomials of degree \(k\) in \(Z\), we recall also the Fischer inner product [27] defined as follows:

\[
\langle Z^\alpha; Z^\beta \rangle_F = \begin{cases}
0, & \alpha \neq \beta, \\
\alpha!, & \alpha = \beta,
\end{cases} \tag{2.11}
\]

where \(Z\) is defined by (2.1). We make the following observation:
Lemma 2.1  Let $P (Z, \overline{Z})$ be a bihomogeneous polynomial of bidegree $(m, n)$ in $(Z, \overline{Z})$ with $m > n$, and we denote with $\mathcal{I}_n$ the set of the all multi-indexes $I := (i_1, i_2, i_3, i_4) \in \mathbb{N}^4$ such that $|I| := i_1 + i_2 + i_3 + i_4 = n$. Then there exist $\{Q_I (\overline{Z})\}_{I \in \mathcal{I}_n}$ and $R (Z, \overline{Z})$ uniquely determined polynomials such that the following holds:

$$P (Z, \overline{Z}) = \sum_{I \in \mathcal{I}_n} Q_I (Z) \langle l_1, l_1 \rangle^{i_1} \langle l_1, l_2 \rangle^{i_2} \langle l_2, l_1 \rangle^{i_3} \langle l_2, l_2 \rangle^{i_4} + R (Z, \overline{Z}),$$

$$R (Z, \overline{Z}) \in \bigcap_{j=1}^{N} \bigcap_{I \in \mathcal{I}_n} \ker \left( \langle l_1, l_1 \rangle^{i_1} \langle l_1, l_2 \rangle^{i_2} \langle l_2, l_1 \rangle^{i_3} \langle l_2, l_2 \rangle^{i_4} \right)^*.$$  \hfill (2.12)

If $P' (Z, \overline{Z})$ is a bihomogeneous polynomial of bidegree $(m, n)$ in $(Z, \overline{Z})$ with $m < n$, then there exist $\{Q_I (Z)\}_{I \in \mathcal{I}_{m-1}}$ and $R' (Z, \overline{Z})$ uniquely determined polynomials such that the following holds:

$$P' (Z, \overline{Z}) = \sum_{j=1}^{N} (z_{1j} + z_{2j}) \sum_{I \in \mathcal{I}_{m-1}} Q^j_I (Z) \langle l_1, l_1 \rangle^{i_1} \langle l_1, l_2 \rangle^{i_2} \langle l_2, l_1 \rangle^{i_3} \langle l_2, l_2 \rangle^{i_4} + R' (Z, \overline{Z}),$$

$$R' (Z, \overline{Z}) \in \bigcap_{j=1}^{N} \bigcap_{I \in \mathcal{I}_{m-1}} \ker \left( (z_{1j} + z_{2j}) \langle l_1, l_1 \rangle^{i_1} \langle l_1, l_2 \rangle^{i_2} \langle l_2, l_1 \rangle^{i_3} \langle l_2, l_2 \rangle^{i_4} \right)^*.$$  \hfill (2.13)

Proof  The existence of the both Fischer decompositions are obtained from the classical generalized Fischer decomposition [27]. The uniqueness follows immediately by the fact that if $\mathbb{N}^4 \ni I := (i_1, i_2, i_3, i_4) \neq I' := (i'_1, i'_2, i'_3, i'_4) \in \mathbb{N}^4$, the following holds:

$$\langle \langle l_1, l_1 \rangle^{i_1} \langle l_1, l_2 \rangle^{i_2} \langle l_2, l_1 \rangle^{i_3} \langle l_2, l_2 \rangle^{i_4}, \langle l_1, l_1 \rangle^{i'_1} \langle l_1, l_2 \rangle^{i'_2} \langle l_2, l_1 \rangle^{i'_3} \langle l_2, l_2 \rangle^{i'_4} \rangle = 0,$$

with respect to the inner Fischer product defined previously in (2.11).

We recall also from [27] the following orthogonal decomposition

$$\mathbb{H}_{k+p} = \mathbb{T} \oplus \ker \left( P^* (D) \right)$$  \hfill (2.14)

If $f (Z) = \sum_{k \geq 0} f_k (Z)$ is the formal power series expansion of a smooth function $f (Z)$, the Fischer norm [27] is defined as follows:

$$\| f_k (Z) \|_F := \sum_{|I| = k} |I! |c_I|^2, \text{ if } f_k (Z) := \sum_{|I| = k} c_I Z^I.$$  \hfill (2.15)

As a corollary of the previous orthogonal decomposition (2.14), we obtain easily the following lemma.

Lemma 2.2  Let $f (Z), g (Z) \in \mathbb{H}_k$ defining the orthogonal decomposition $f (Z) = g (Z) + h (Z)$. Then $\| f (Z) \|_F = \| g (Z) \|_F + \| h (Z) \|_F$.

We are ready now to prove the main result of this section.

Proposition 2.3  Let $M \subset \mathbb{C}^{2N+4}$ be the real-formal submanifold defined near $0 \in M$ by (2.2). Then there exists a unique formal transformation of the following type:

$$(Z', W') = \left( Z + \sum_{m+n \geq 2} F_{m,n} (Z, W), W + \sum_{m+n \geq 2} G_{m,n} (Z, W) \right),$$  \hfill (2.16)
where \( F_{m,n}(Z, W) \), \( G_{m,n}(Z, W) \) are homogeneous polynomials in \( Z \) of degree \( m \) and degree \( m \) in \( W \) normalized as follows:

\[
F_{0,n+1}(Z, W) = 0, \quad F_{1,n}(Z, W) = 0, \quad \text{for all } n \geq 1,
\]

which transforms \( M \) into the following partial normal form:

\[
W' = Z'\overline{Z}' + \sum_{m+n \geq 3, m, n \neq 0} \varphi_{m,n}'(Z', \overline{Z}') + 2\Re \left\{ \sum_{k \geq 3} \varphi_{k,0}'(Z') \right\},
\]

where \( \varphi_{m,n}'(Z, \overline{Z}) \) are matrix bihomogeneous polynomials of bidegree \((m, n)\) in \((Z, \overline{Z})\) for all \( m, n \geq 0 \), which satisfy for \( n \geq m - 1 \) the following Fischer normalization conditions:

\[
(\varphi_{m,n}^{1,1} + \varphi_{m,n}^{1,1})'(Z, \overline{Z}), (\varphi_{m,n}^{2,2} + \varphi_{m,n}^{2,2})'(Z, \overline{Z}) \in \bigcap_{j=1}^{N} \bigcap_{j \in I_{n-1}} \ker \left( (z_{1j} + z_{2j}) \langle l_{1,1}, l_{1,2} \rangle^{i_1} \langle l_{2,1}, l_{2,2} \rangle^{i_2} \langle l_{1,1}, l_{1,2} \rangle^{i_3} \langle l_{2,1}, l_{2,2} \rangle^{i_4} \right) \ast,
\]

and respectively for \( m \geq n \) the following Fischer normalization conditions:

\[
(\varphi_{m,n}^{1,2} + \varphi_{m,n}^{1,2})'(Z, \overline{Z}), (\varphi_{m,n}^{2,1} + \varphi_{m,n}^{2,1})'(Z, \overline{Z}) \in \bigcap_{l \in I_1} \ker \left( (l_{1,1}, l_{1,2})^{i_1} (l_{2,1}, l_{2,2})^{i_2} (l_{1,1}, l_{1,2})^{i_3} (l_{2,1}, l_{2,2})^{i_4} \right) \ast,
\]

where \( I_n \) is the set of the all multi-indexes \( I := (i_1, i_2, i_3, i_4) \in \mathbb{N}^4 \) such that \( |I| := i_1 + i_2 + i_3 + i_4 = n \).

**Proof** In order to prove the statement, we follow the proof of Extended Moser Lemma [3] and we collect the terms of bidegree \((m, n)\) in \((Z, \overline{Z})\) with \( T = m + n \) in (2.7). We obtain, by (2.1) and (2.9), the following:

\[
\left( (\varphi_{m,n}^{1,1})' - \varphi_{m,n}^{1,1} (\varphi_{m,n}^{1,2})' - \varphi_{m,n}^{1,2} (\varphi_{m,n}^{2,1})' - \varphi_{m,n}^{2,1} (\varphi_{m,n}^{2,2})' - \varphi_{m,n}^{2,2} \right)(Z, \overline{Z}) = \sum_{i_1 + i_2 + i_3 + i_4 = n} \left( \langle l_{1,1}, l_{1,2} \rangle^{i_1} \langle l_{2,1}, l_{2,2} \rangle^{i_2} \langle l_{1,1}, l_{1,2} \rangle^{i_3} \langle l_{2,1}, l_{2,2} \rangle^{i_4} \right) \langle F_{m-n+1,1}(Z), z_1^{i_1} \rangle <br>\langle F_{m-n+1,1}(Z), z_2^{i_2} \rangle <br>\langle F_{m-n+1,1}(Z), z_1^{i_3} \rangle <br>\langle F_{m-n+1,1}(Z), z_2^{i_4} \rangle <br>\langle F_{m-n+1,1}(Z), z_1^{i_1} \rangle <br>\langle F_{m-n+1,1}(Z), z_2^{i_2} \rangle <br>\langle F_{m-n+1,1}(Z), z_1^{i_3} \rangle <br>\langle F_{m-n+1,1}(Z), z_2^{i_4} \rangle + \cdots,
\]

where “…” represents terms which depend on the polynomials \( G_{k,l}(Z) \) with \( k + 2l < T \), \( F_{k,l}(Z) \) with \( k + 2l < T - 1 \) and on \( \varphi_{k,l}(Z, \overline{Z}), \varphi_{k,l}'(Z, \overline{Z}) \) with \( k + l < T = m + n \). We compute then the
polynomials $F_{m',n'}(Z)$ with $m' + 2n' = T - 1$, and respectively $G_{m',n'}(Z)$ with $m' + 2n' = T$ using induction depending on $T = m' + 2n'$. We assume that we have computed the polynomials $F_{k,l}(Z)$ with $k + 2l < T - 1$, $G_{k,l}(Z)$ with $k + 2l < T$.

The computation of $F_{k,l}(Z,W)$ for $k + 2l = T$. Collecting the terms of bidegree $(m,n)$ in $(Z,Z)$ in (2.21) with $m < n - 1$ and $m, n \geq 1$, and then by making the sum between the $(1,1)$-position terms and the $(1,2)$-position terms in (2.21), we obtain the following

\[
((\varphi_{m,n}^{'})^{1,1} + (\varphi_{m,n}^{'})^{1,2})(Z,Z) = - \sum_{j_1+j_2+j_3+j_4=n-1} (z^1 + z^2, F_{m-n+1,(j_1,j_2,j_3,j_4)}^{1})(Z) \left( (l_1,l_1)^{j_1} (l_1,l_2)^{j_2} (l_2,l_1)^{j_3} (l_2,l_2)^{j_4} + ((\varphi_{m,n})^{1,1} + (\varphi_{m,n})^{1,2})(Z,Z) + \cdots \right). \quad (2.22)
\]

By the second part of Lemma 2.1, we obtain the following generalized Fischer-decomposition:

\[
((\varphi_{m,n})^{1,1} + (\varphi_{m,n})^{1,2})(Z,Z) + \cdots = - \sum_{j=1}^{N} (z_{1j} + z_{2j}) \left( \sum_{j_1+j_2+j_3+j_4=n-1} Q_{j_1}^{i}(Z) (l_1,l_1)^{j_1} (l_1,l_2)^{j_2} (l_2,l_1)^{j_3} (l_2,l_2)^{j_4} + R_{1}^{i}(Z,Z) \right), \quad (2.23)
\]

where the following generalized Fischer-normalization condition is satisfied:

\[
R_{1}^{i}(Z,Z) \in \bigcap_{j=1}^{N} \bigcap_{j_1+j_2+j_3+j_4=n-1} \ker ((z_{1j} + z_{2j}) (l_1,l_1)^{j_1} (l_1,l_2)^{j_2} (l_2,l_1)^{j_3} (l_2,l_2)^{j_4})^{*}.
\]

By imposing the corresponding generalized Fischer-normalization condition on $((\varphi_{m,n}^{'})^{1,1} + (\varphi_{m,n}^{'})^{1,2})(Z,Z)$ defined by (2.19), and then by the uniqueness of the Fischer decomposition, we obtain by (2.23) and (2.22) the following:

\[
F_{m-n+1,J}(Z,W) = \sum_{j_1+j_2+j_3+j_4=n-1} (Q_{j_1}^{1}, \ldots, Q_{j_4}^{N})(Z) w_{11}^{j_1} w_{12}^{j_2} w_{21}^{j_3} w_{22}^{j_4}. \quad (2.24)
\]

We compute analogously $F_{m-n+1,J}(Z,W)$ for all $J \in \mathbb{N}^4$ with $|J| = n - 1$.

We assume that $T$ is odd. Collecting the terms of bidegree $(n, n+1)$ in $(Z,Z)$ in (2.21) with $n \geq 2$, and then by making the sum between the $(1,1)$-position terms and the $(1,2)$-position terms in (2.21), we obtain the following:

\[
((\varphi_{n,n+1}^{'})^{1,1} + (\varphi_{n,n+1}^{'})^{1,2})(Z,Z) = - \sum_{j_1+j_2+j_3+j_4=n-1} (z^1 + z^2, F_{1,(j_1,j_2,j_3,j_4)}^{1})(Z) (l_1,l_1)^{j_1} (l_1,l_2)^{j_2} (l_2,l_1)^{j_3} (l_2,l_2)^{j_4} - \sum_{j_1+j_2+j_3+j_4=n-1} (F_{0,(j_1,j_2,j_3,j_4)}^{1}(Z), z^1 (l_1,l_1)^{j_1} (l_1,l_2)^{j_2} (l_2,l_1)^{j_3} (l_2,l_2)^{j_4} + ((\varphi_{n,n+1})^{1,1} + (\varphi_{n,n+1})^{1,2})(Z,Z) + \cdots \right). (2.25)
\]
By Lemma 2.1, we obtain the following generalized Fischer-decomposition:

\[ ((\varphi_{n,n+1})^{1,1} + (\varphi_{n,n+1})^{1,2})(Z, \overline{Z}) + \cdots \]

\[ = - \sum_{j=1}^{N} (z_{1j} + z_{2j}) \left( \sum_{j_1+j_2+j_3+j_4=n-1} C_j^{j_1,j_2,j_3,j_4}(Z) \right) \]

\[ \cdot \langle l_1, l_1 \rangle^{j_{1}} \langle l_1, l_2 \rangle^{j_{2}} \langle l_2, l_1 \rangle^{j_{3}} \langle l_2, l_2 \rangle^{j_{4}} + R_2'(Z, \overline{Z}), \quad (2.26) \]

where the following generalized Fischer-normalization condition is satisfied:

\[ R_2'(Z, \overline{Z}) \in \bigcap_{j=1}^{N} \bigcap_{j_1+j_2+j_3+j_4=n-1} \ker((z_{1j} + z_{2j}) \langle l_1, l_1 \rangle^{j_{1}} \langle l_1, l_2 \rangle^{j_{2}} \langle l_2, l_1 \rangle^{j_{3}} \langle l_2, l_2 \rangle^{j_{4}})^{\ast}. \]

Imposing the corresponding generalized Fischer-normalization condition defined by (2.19) on

by (2.26), (2.25), (2.17) and using the uniqueness of the Fischer decomposition, we obtain the following

\[ F_{2,n-1}(Z, W) = \sum_{j_1+j_2+j_3+j_4=n-1} (C_{1}^{j_1}, \ldots, C_{N}^{j_{4}})(Z)w_{11}^{j_{1}}w_{12}^{j_{2}}w_{21}^{j_{3}}w_{22}^{j_{4}}. \quad (2.27) \]

We compute \( F_{2,n}(Z, W) \) analogously. We also can study similarly the case when \( T \) is even using the normalization conditions (2.17) on the formal transformation (2.16) and the corresponding Fischer normalization conditions (2.19). This situation is similar to the one from [3].

The computation of \( G_{k,l}(Z, W) \) for \( k + 2l = T \). Collecting the terms of bidegree \( (m, n) \) in

\( (Z, \overline{Z}) \) in (2.21) with \( m \geq n \) and \( m, n \geq 1 \), we obtain the following

\[ \varphi_{m,n}(Z, \overline{Z}) = \sum_{j_1+j_2+j_3+j_4=n} G_{m-n,(j_1,j_2,j_3,j_4)}(Z)\langle l_1, l_1 \rangle^{j_{1}} \langle l_1, l_2 \rangle^{j_{2}} \langle l_2, l_1 \rangle^{j_{3}} \langle l_2, l_2 \rangle^{j_{4}} \]

\[ - \sum_{i_{1}+i_{2}+i_{3}+i_{4}=n} \left( \langle F_{m-n+1,1}^{1}(Z), z_{1}^{i} \rangle \langle F_{m-n+1,1}^{1}(Z), z_{2}^{i} \rangle \right) \]

\[ \cdot \langle l_1, l_1 \rangle^{i_{1}} \langle l_1, l_2 \rangle^{i_{2}} \langle l_2, l_1 \rangle^{i_{3}} \langle l_2, l_2 \rangle^{i_{4}} + \varphi_{m,n}(Z, \overline{Z}) + \cdots \quad (2.28) \]

By the first part of Lemma 2.1 we obtain the following generalized Fischer-decomposition

\[ \varphi_{m,n}(Z, \overline{Z}) + \sum_{i_{1}+i_{2}+i_{3}+i_{4}=n} \left( \langle F_{m-n+1,1}^{1}(Z), z_{1}^{i} \rangle \langle F_{m-n+1,1}^{1}(Z), z_{2}^{i} \rangle \right) \]

\[ \cdot \langle l_1, l_1 \rangle^{i_{1}} \langle l_1, l_2 \rangle^{i_{2}} \langle l_2, l_1 \rangle^{i_{3}} \langle l_2, l_2 \rangle^{i_{4}} \]

\[ = \sum_{j_1+j_2+j_3+j_4=n} E_{m,(j_1,j_2,j_3,j_4)}(Z)\langle l_1, l_1 \rangle^{j_{1}} \langle l_1, l_2 \rangle^{j_{2}} \langle l_2, l_1 \rangle^{j_{3}} \langle l_2, l_2 \rangle^{j_{4}} \]

\[ + \begin{pmatrix} R_{11}(Z, \overline{Z}) & R_{12}(Z, \overline{Z}) \\ R_{21}'(Z, \overline{Z}) & R_{22}'(Z, \overline{Z}) \end{pmatrix}, \quad (2.29) \]
where the following generalized Fischer-normalization condition is satisfied

\[ R_{11}^1(Z, \overline{Z}), R_{12}^1(Z, \overline{Z}), R_{21}^2(Z, \overline{Z}), R_{22}^2(Z, \overline{Z}) \in \bigcap_{j_1+j_2+j_3+j_4=n} \ker \left( (l_1, l_1)^{j_1} (l_1, l_2)^{j_2} (l_2, l_1)^{j_3} (l_2, l_2)^{j_4} \right)^*. \]

Imposing the corresponding generalized Fischer-normalization condition defined by (2.20) on \( \phi_{m,n}' \), it follows by (2.17), (2.28), (2.29) and by the uniqueness of the Fischer decomposition, the following

\[ G_{m-n,n}(Z, W) = \sum_{j_1+j_2+j_3+j_4=n} E_{m,(j_1, j_2, j_3, j_4)}(Z) w_1^{j_1} w_2^{j_2} w_3^{j_3} w_4^{j_4}. \quad (2.30) \]

The computation of \( G_{T,0}(Z, W) \). Collecting the terms of bidegree \((T, 0)\) and \((0, T)\) in \((Z, \overline{Z})\) in (2.21), we obtain the following:

\[ G_{T,0}(Z) + \phi_{T,0}'(Z) = \phi_{T,0}(Z) + A(Z), \quad \phi_{0,T}'(Z) = \phi_{0,T}(Z) + B(Z), \quad \text{where} \]

\[ A(Z), B(Z) \] are the sums of terms that are determined by the induction hypothesis. By imposing the reality normalization condition \( \phi_{0,T}(Z) = \overline{\phi_{T,0}'(Z)} \), we obtain the following:

\[ G_{T,0}(Z) = \phi_{T,0}(Z) - \overline{\phi_{0,T}(Z)} + A(Z) - B(Z). \quad (2.32) \]

Proposition 2.3 provides us a partial normal form for the real submanifolds defined by (1.3) using the Fischer normalization conditions [27]. The chosen Fischer normalization conditions (2.19) and (2.20) are motivated by how the formal transformation (2.16) appears in the local defining equation and this motivation is given partially by the partial normal form from [3]. The chosen normalization conditions (2.19) seem to be more appropriate in our situation. Other partial normal forms may be possibly considered in our case using other normalization conditions and the Fischer normalization conditions (2.19) are just a choice in our case.

Proposition 2.3 leaves undetermined an infinite number of parameters (2.17) making the formal transformation possibly divergent similarly to the case of Moser [26] and similarly to the case of Huang–Yin [15]. In order to prove our result, we cancel its undetermined part by composing the formal transformation with an automorphism of the model (1.4). This is done using the general formula of an automorphism of the model (1.4) in \( \mathbb{C}^{N+1} \) computed by Huang–Yin [15] which helps us to fabricate automorphisms for the model (1.4) and then to follow the ideas of Moser [26] in order to find the desired automorphism. In order to prove Theorem 1.1, we need use the following lemma.

**Lemma 2.4** There exist \( T \in \text{Aut}_0(W = \mathbb{Z}^d) \) such that \( T \circ F \) is normalized as in (2.17).

**Proof** In order to produce automorphisms of the model \( W = \mathbb{Z}^d \) for the normalization of the formal transformation (2.16), we follow Proposition 3.1 of Huang–Yin [15] and we consider different types of formal transformations leaving the model \( W = \mathbb{Z}^d \). For instance, we consider the formal transformations class

\[ (Z', W') = B(W) (ZU(W), \overline{B'}(W) W) \quad (2.33) \]

Here \( B(W) \) is holomorphic in \( W \) near \( 0 \in \mathbb{C}^4 \) and \( U(W) \) is holomorphic in \( W \) near \( 0 \in \mathbb{C}^4 \) such that \( U(W + \overline{W}) \) is a transformation leaving invariant the model \( W = \mathbb{Z}^d \). This transformation
(2.33) helps us to impose partially the desired normalizations. Another useful transformation class leaving invariant the model manifold $W = Z \bar{Z}$ is the following

$$(Z', W') = \left( \frac{WA(W) - D(W)\bar{Z}(W)A(W) + C(W)(Z - D(W)\bar{Z}(W)A(W))U(W), W}{I_2 - \bar{Z}(W)} \right).$$

(2.34)

Here the holomorphic matrix-function $V(W)$ defined near $0 \in \mathbb{C}^4$ satisfies

$$V(Z \bar{Z} \bar{Z} \bar{Z}^t) = I_2 - Z \bar{Z} A(Z \bar{Z} A(Z \bar{Z}^t)\bar{A}(Z \bar{Z}^t),$$

and we have

$$A(W) = \begin{pmatrix} a_{11}(W) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_1(W) = \begin{pmatrix} a_{11}(W) \\ 0 \end{pmatrix}, \quad D(W) = \frac{1}{a_{11}(W)} a_{11}(W) I_2.$$

The fact that the transformation (2.34) defines a class of formal self-transformations of the model $W = Z \bar{Z}$ can be easily checked using matrices computations. The position $(1,1)$ of $a_{11}(W)$ in the matrix $A(Z)$ can be changed with any other position obtaining new classes of transformations leaving invariant the model manifold $W = Z \bar{Z}$.

In order to continue the proof, we firstly assume that $B(W) = I_2$, $U(W) = \text{Id}$ and we introduce the following notation

$$T_1(Z, W) := (I_2 - Z \bar{Z}(W))^{-1} (z_1 - W a_1(W), V(W) z_2, \ldots, V(W) z_n, w_1, w_2), \quad (2.35)$$

where $z_1$ is the first column of the matrix $Z$ given by (2.1), $\ldots$, $z_N$ is the last column of the matrix $Z$ given by (2.1), $w_1$ is the first column of the matrix $W$ and $w_2$ is the first column of the matrix $W$ given by (2.1). Following Moser [26], in order to normalize as in (2.17), we have in our view the following equations system:

$$\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \begin{pmatrix} a_{11}(W) \\ 0 \end{pmatrix} = \begin{pmatrix} F^{11}(0, W) \\ F^{12}(0, W) \end{pmatrix},$$

where

$$\begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} + \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} (0, w_{11}, w_{12}, w_{21}, w_{22}). \quad (2.36)$$

By the Implicit Function Theorem, we obtain the following:

$$\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} (W_{11}, W_{12}, W_{21}, W_{22}).$$

We write the following unique decompositions:

$$F^{11}(0, W) = w_{11} G_{11}(w_{12}, w_{21}, w_{22}, w_{11}) + H_{11}(w_{12}, w_{21}, w_{22}),$$

$$F^{12}(0, W) = w_{21} G_{12}(w_{12}, w_{21}, w_{22}, w_{11}) + H_{12}(w_{11}, w_{21}, w_{22}). \quad (2.37)$$

If we would have that $G_{11}(w_{12}, w_{21}, w_{22}, w_{11}) = G_{12}(w_{12}, w_{21}, w_{22}, w_{11})$, then we would be able to find $a_{11}(W)$ immediately by (2.36). Contrary, we firstly find $a_{11}(W)$ giving the previous property and then we find easily another automorphism of the type of (2.35) giving us partially the normalization condition (2.17). We continue by composing those two automorphisms. Then, we repeat this procedure until the normalization conditions (2.17) are fulfilled taking instead
of $F^{11}(0, W)$ and $F^{12}(0, W)$, the reminders $H_{11}(w_{12}, w_{21}, w_{22})$ and $H_{12}(w_{11}, w_{21}, w_{22})$. In this situation we replace $A(Z)$ with the following matrix
\[
A_1(Z) = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
\alpha_{22}(W) & 0 & \cdots & 0
\end{pmatrix},
\]
and we repeat the previous procedure. In order to finish this part of the proof we chose $U(W)$ such that it changes the position $(1, 1)$ with the position $(1, 2)$ and we apply similar arguments. Composing the last four considered automorphisms we obtain an automorphism $T_1(Z, W)$ which gives the first normalization condition in (2.17) for $F^{11}(Z, W)$ and $F^{21}(Z, W)$. We define analogously the automorphisms $T_2(Z, W), \ldots, T_n(Z, W)$ and then we consider the following composition $T(Z, W) := T_1(Z, W) \circ \cdots \circ T_n(Z, W)$ which gives the first normalization condition in (2.17) for $F(Z, W)$ by a composition on the left side.

We consider now the transformation (2.33) and we find $B(W)$ such that the second normalization condition in (2.17) holds. Following Moser [26] we obtain the following system of equations:
\[
\begin{pmatrix}
z_{11} + \tilde{F}^{11}(Z, W) & \cdots & z_{1N} + \tilde{F}^{1N}(Z, W)
z_{21} + \tilde{F}^{21}(Z, W) & \cdots & z_{2N} + \tilde{F}^{2N}(Z, W)
\end{pmatrix} = B(W) \begin{pmatrix}
z_{11} + F^{11}(Z, W) & \cdots & z_{1N} + F^{1N}(Z, W)
z_{21} + F^{21}(Z, W) & \cdots & z_{2N} + F^{2N}(Z, W)
\end{pmatrix},
\]
where we have
\[
W = \begin{pmatrix}
w_{11} & w_{12} \\
w_{21} & w_{22}
\end{pmatrix},
\]
\[
B(W) = \begin{pmatrix}
b_{11}(W) & 0 \\
0 & b_{22}(W)
\end{pmatrix}.
\]
In order to simply the computations we assume $N = 2$. We take $U(W) = \text{Id}$, and then following Moser [26], we compute $b_{11}(W)$ by collecting the terms which depend on $z_{11}$. It follows that $b_{11}(W)(z_{11} + F^{11}(Z, W)) = z_{11} + \tilde{F}^{11}(Z, W)$ and by taking the derivative with $z_{11}$ and setting $Z = 0$ we compute $b_{11}(W)$. We compute analogously $b_{22}(W)$ in order to eliminate the coefficient of $z_{21}$ depending on $W$ in the Taylor expansion of $F^{21}(Z, W)$. We consider another automorphism of the model in order to eliminate the coefficient depending smoothly on $W$ of $z_{12}$ in the Taylor expansion of $F^{12}(Z, W)$, and respectively in order to eliminate the coefficient depending on $W$ of $z_{22}$ in the Taylor expansion of $F^{22}(Z, W)$. Because the normalization conditions (2.17) are not affected if we multiply $F(Z, W)$ with scalar matrices on the left side, we continue the proof taking the composition of the previous two automorphisms. In order to eliminate the coefficients depending on $W$ of $z_{12}$ in the Taylor expansion of $F^{11}(Z, W)$, and respectively the coefficient of $z_{11}$ in the Taylor expansion of $F^{21}(Z, W)$, we chose a new automorphism of the model defined by the matrix $U(W)$ sending $(z_{11}, z_{12})$ into $(\alpha_{11}z_{11} + \alpha_{12}z_{12}, \alpha_{21}z_{11} + \alpha_{22}z_{12})$, where $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} \neq 0$. By compositions with scalar matrices we can repeat the procedure from above obtaining the desired automorphism. In order to finish imposing the normalization
conditions (2.17), we continue the proof by modifying again \( U(W) \). We continue this procedure and we find the desired automorphism \( T_1 \). We finish the proof by taking \( T = T \circ T_1 \).

We finally have to observe that the formal automorphism \( T \) defined by the previous lemma may not be unique as in the classical cases of Huang–Yin [15] and Moser [26] due to the restriction to the scalar matrices in the previous proof.

3 Notations

The proof of Theorem 1.1 has as model the proof of the Generalization [15] of Huang-Yin [15] of the Theorem of Moser [26]. We consider \( R := (r, \ldots, r) \) and we define the following domains

\[
\Delta_r = \{(Z, W) \in \mathbb{C}^{2N+4}; |z_{ij}| < r, |w_{1,1}|^2, |w_{1,2}|^2, |w_{2,2}|^2 < N r^2, \text{ for all } (i, j) \in \{1, \ldots, N\} \times \{1, 2\}\},
\]

\[
D_r = \{(Z, \xi) \in \mathbb{C}^{2N} \times \mathbb{C}^{2N}; |z_{ij}| < r, |\xi_{ij}| < r, \text{ for all } (i, j) \in \{1, \ldots, N\} \times \{1, 2\}\},
\]

where \( Z \) and \( W \) are defined by (2.1) and \( \xi \) is defined similarly as \( Z \). Throughout the rest of this paper, we use the following notations

\[
\|E\|_r := \sup_{(Z, W) \in \Delta_r} |E(Z, \xi)|, \quad |h|_r = \sup_{(Z, \xi) \in \Delta_r} |h(Z, \xi)|,
\]

where \( E(Z, \xi) \) is a holomorphic function defined over \( D_r \), and where respectively \( h(Z, W) \) is a holomorphic function defined over \( \Delta_r \). In the case of a matrix \( E(Z, Z) \) defined as follows

\[
E(Z, Z) = \begin{pmatrix}
E^{1,1}(Z, Z) & E^{1,2}(Z, Z) \\
E^{2,1}(Z, Z) & E^{2,2}(Z, Z)
\end{pmatrix},
\]

we use the following notation

\[
\|E\|_r = \max_{i,j \in \{1,2\}} \|E^{i,j}\|_r.
\]

These domains and notations are used later in order to apply use the methods based on Moser’s rapid convergence arguments. Following Moser [26] we define also the following real numbers:

\[
\frac{1}{2} < r' < \sigma < \rho < r \\leq 1, \quad \rho = \frac{2r' + r}{3}, \quad \sigma = \frac{2r' + \rho}{3}.
\]

We also recall here Lemma 4.5 of Huang–Yin [15] that will be applied later.

Lemma 3.1 Suppose that there exists \( C > 0 \) and a number \( a > 1 \) such that \( d_n \geq Ca^n \). Then we have that

\[
\sum_{n=1}^{\infty} n^{m_3} d_n^{m_3} \left(1 - \frac{1}{n^{m_2}} \right) d_n = 0,
\]

for any integers \( m_1, m_2, m_3 > 0 \).

4 Proof of Theorem 1.1

We consider the real formal submanifold \( M \subset \mathbb{C}^{2N+4} \) defined near \( p = 0 \) as follows

\[
W = \Phi(Z, Z) = ZZ^d + E(Z, Z),
\]

(4.1)
where \( E(Z, \xi) = O(3) \) is holomorphic matrix near \( Z = \xi = 0 \). We consider now the following formal transformation:

\[
H(Z, Wt) = (Z + F(Z, W), W + G(Z, W)),
\]

which satisfies the normalization conditions (2.17) sending \( M \) defined near \( p = 0 \) by (1.3) into the model manifold defined by (1.4) up to the degree \( d \geq 3 \). By the Fischer-normalization conditions (2.19) and (2.20), we find the following pair of polynomials

\[
(F_{d-1}^\text{nor}(Z, W), G_{d}^\text{nor}(Z, W)),
\]

where we have used the notations from [3]. Following the strategy of Huang–Yin [15], we define the following transformation:

\[
\Theta(Z, W) : = (Z + \widehat{F}(Z, W), W + \widehat{G}(Z, W))
\]

\[
= (Z + F_{d-1}^\text{nor}(Z, W) + O_{wt}(d), W + G_{d}^\text{nor}(Z, W) + O_{wt}(d + 1)).
\]

which sends \( M \) up to the degree \( d \) into the model manifold \( M_{\infty} \) defined by (1.4):

\[
M' = \Theta(M) : W' = Z\mathbb{Z}^t.
\]

In order to apply later the rapid iteration procedure of Moser [26], we need to understand how the degree of the remaining terms is changing by the assumption of Theorem 1.1 and using the transformation (4.3). We need to prove the following lemma.

**Lemma 4.1** Let \( M \subset \mathbb{C}^{2N+4} \) be a real-analytic submanifold defined near \( p = 0 \) by (4.1) such that \( \text{Ord} (E(Z, \xi)) \geq d \). If \( \Theta \) is defined in (4.3) and \( M' \) is defined in (4.4), then \( \text{Ord} (E'(Z, \xi)) \geq 2d - 2 \), where

\[
M' : W' = Z\mathbb{Z}^t + E'(Z, \overline{Z}).
\]

**Proof** By (4.3) and (4.4), it follows that

\[
E'(Z', \overline{Z}') = G(Z, \Phi(Z, \overline{Z})) - G(Z, W_0) - 2\mathcal{R}\{Z(F(Z, \Phi(Z, \overline{Z})) - F(Z, W_0))\} \\
+ (F(Z, \Phi(Z, \overline{Z}))(F(Z, \Phi(Z, \overline{Z})))^t \\
- (\varphi'(\widehat{F}(Z, \Phi(Z, \overline{Z})), F(Z, \Phi(Z, \overline{Z})t)t) - \varphi'(Z, \overline{Z})) \\
- (J^{2d-3}(E(Z, \overline{Z}))) - E(Z, \overline{Z})),
\]

where \( J^{2d-3} (E(Z, \overline{Z})) \) represents as in [15] the polynomial defined by the Taylor expansion of \( E(Z, \overline{Z}) \) up to the degree \( 2d - 3 \). Previously \( \Phi(Z, \overline{Z}) \) is given by (4.1) and we have used the following notation

\[
W_0 = Z\overline{Z}^t.
\]

Writing in the following way

\[
(\Phi(Z, \overline{Z}))^J - \langle l_1, l_1 \rangle_{j_1} \langle l_1, l_2 \rangle_{j_12} \langle l_2, l_1 \rangle_{j_21} \langle l_2, l_2 \rangle_{j_22} \\
= (\Phi(Z, \overline{Z}))^J - (\Phi(Z, \overline{Z})_{11})^{j_11} (\Phi(Z, \overline{Z})_{12})^{j_12} (\Phi(Z, \overline{Z})_{21})^{j_21} (\Phi(Z, \overline{Z})_{22})^{j_22} \\
+ \cdots - \langle l_1, l_1 \rangle_{j_11} \langle l_1, l_2 \rangle_{j_12} \langle l_2, l_1 \rangle_{j_21} \langle l_2, l_2 \rangle_{j_22},
\]

(4.8)
where $\Phi(Z, \overline{Z}) = (\Phi_{1,1}(Z, \overline{Z}), \Phi_{1,2}(Z, \overline{Z}), \Phi_{2,1}(Z, \overline{Z}), \Phi_{2,2}(Z, \overline{Z}))$ and $J = (j_{11}, j_{12}, j_{21}, j_{22}) \in \mathbb{N}^4$. Then if $I = (i_{11}, i_{12}, i_{21}, i_{22}) \in \mathbb{N}^4$ is an multi-index such that $|I| + 2|J| = d$, it follows that the following degree estimate holds

$$\text{Ord}\{z^I((\Phi(Z, \overline{Z}))^J - w_0^I)\} \geq 2d - 2. \quad (4.9)$$

By (4.9), we obtain easily the following degree estimates

$$\text{Ord}_{\mu}(G(Z, \Phi(Z, \overline{Z}))) \geq 2d - 2,$$
$$\text{Ord}_{\mu}(\hat{F}(Z, \Phi(Z, \overline{Z}))) \geq 2d - 3,$$  
$$\text{Ord}\{\hat{F}(Z, \Phi(Z, \overline{Z}))\hat{F}(Z, \Phi(Z, \overline{Z})) t\},$$
$$\text{Ord}\{\varphi'((\hat{F}(Z, \Phi(Z, \overline{Z}))), \hat{F}(Z, \Phi(Z, \overline{Z})))\} \geq 2d - 3,$$  

which together with (4.6) gives us the desired degree estimate in (4.5).

In order to apply Moser’s iteration arguments [26] in our case (1.3), we need to make firstly suitable estimations for the solution (4.2). The $F$-part of the transformation is computed by the general transformation equation (2.21). The only difficulty that occurs here is that we can not make directly suitable estimations on the $G$-part of the solution (4.2) because of the non-triviality of the Fischer inner product [27]. Following Huang–Yin [15] and Moser [26], we need to prove the following lemma.

**Lemma 4.2** Assume that the real-analytic submanifold $M$ defined in (4.1) is formally equivalent to $M_\infty$ defined in (2.9) with $E(z, \xi)$ holomorphic over $\overline{D}_r$ and $\text{Ord}\{E(Z, \xi)\} \geq d$, such that the following estimates hold:

$$\|E(Z, \xi) - J^{2d-3}(E(Z, \xi))\|_\rho \leq \frac{(2d)^{4N}\|E\|_r}{(r - \rho)^{2N}} \left(\frac{\rho}{r}\right)^{2d-2},$$

$$|F_{k,l}(Z, W)|_\rho \leq \frac{4}{N}(2d)^{4N}\|E\|_r \left(\frac{\rho}{r}\right)^{2d-3},$$

$$|\nabla \hat{F}_{k,l}(Z, W)|_\rho \leq \left(\frac{36}{r - \rho} + 2N\right)(2d)^{4N}\|E\|_r \left(\frac{\rho}{r}\right)^{2d-3},$$

$$|\hat{G}_{\alpha,\beta}(Z, W)|_\rho \leq ((2d)^{4N} + (2d)^{6N})\|E\|_r \left(\frac{\rho}{r}\right)^{2d-2},$$

$$|\nabla \hat{G}_{\alpha,\beta}(Z, W)|_\rho \leq \left(\frac{36(1 + (2d)^{2N})}{r - \rho} + 6N(1 + (2d)^{2N})\right)(2d)^{4N}\|E\|_r \left(\frac{\rho}{r}\right)^{d-1},$$

for all $k \in \{1, \ldots, N\}$, $\alpha, \beta, l \in \{1, 2\}$, where $J^{2d-3}(E(z, \xi))$ is the polynomial defined by the Taylor expansion of $E(Z, \xi)$ up to the degree $2d - 3$ and $\nabla$ represents the gradient.

**Proof** Following Huang–Yin [15] and applying the Cauchy estimates for (3.1), we obtain by (3.2)–(3.4) the following

$$\|E(Z, \xi) - J^{2d-3}(E(Z, \xi))\|_\rho \leq \left\| \sum_{|I|+|J| \geq 2d+2, I,J \in \mathbb{N}^{2N}} a_{I,J}Z^IZ^J \right\|_\rho \leq \sum_{|I|+|J| \geq 2d+2, I,J \in \mathbb{N}^{2N}} \|E\|_r \left(\frac{R'}{R}\right)^{I+J}$$
By (2.22) together with (2.23), (2.24) and (2.25), (2.27), we obtain the following

applying the Cauchy estimates and using the second inequality of our statement, we obtain the

Because of the following fact

Then the following holds:

Proof

By the Cauchy inequality, we obtain the first inequality. By applying the Cauchy
formulas using the domain

and as well the following Cauchy estimates using the domain

where we have used the following notations

\[ R' := (\rho, \ldots, \rho), \quad R := (r, \ldots, r). \]

By (2.22) together with (2.23), (2.24) and (2.25), (2.27), we obtain the following

Because of the following fact

applying the Cauchy estimates and using the second inequality of our statement, we obtain the following

for all \( k = 1, \ldots, N \) and \( l \in \{1, 2\} \).

The main ingredient for computing the \( G \)-part of our transformation is the following remark

**Remark 4.3** Let \( S(Z, \overline{Z}) \) be a homogeneous polynomial of degree \( k \) in \((Z, \overline{Z})\) written as follows

Then the following holds:

\[
|S(Z, \overline{Z})|^2 \leq \frac{\|S(Z, \overline{Z})\|^2}{k!}(|z_1|^2 + \cdots + |z_{1N}|^2 + |z_{21}|^2 + \cdots + |z_{2N}|^2 t)^{2k},
\]

and as well the following Cauchy estimates using the domain \( D_r \) defined in (3.1)

\[
\|S(Z, \overline{Z})\|^2_r \leq \frac{k!(k+1)^{2N}}{r^{2k}}\|S\|^2_r.
\]

**Proof** By the Cauchy inequality, we obtain the first inequality. By applying the Cauchy formulas using the domain \( D_r \) defined in (3.1), we obtain following Shapiro [27] the following:

\[
\|S(Z, \overline{Z})\|^2 = \sum_{|I|+|J|=k} I!J! |c_{I,J}|^2 \leq \frac{\|S\|^2_r}{r^{2k}} \left( \sum_{|I|+|J|=k} I!J! \right) \leq \frac{k!(k+1)^{2N}}{r^{2k}}\|S\|^2_r.
\]

By (2.28) using the previous remark together with Lemma 2.2, we obtain by (2.31) the following

\[
|\hat{F}_{k,l}(Z, W)|_{\rho} \leq \frac{(2d)^{4N} \|E\|_r}{(r-\rho)^{2N}} \left( \frac{\rho}{r} \right)^{2d-2}, \quad \text{for all } k = 1, \ldots, N \text{ and } l \in \{1, 2\}.
\]
and immediately we obtain the following
\[
\left| \frac{\partial \hat{G}_{\alpha,\beta}}{\partial z_{i,j}} (Z, W) \right|_{\rho} \leq \frac{3(2d)^4 N (1 + 2d)^2 N}{\rho (r - \rho)} \left( \frac{\rho}{r} \right)^{d-1},
\]
\[
\left| \frac{\partial \hat{G}_{\alpha,\beta}}{\partial w_{\alpha,\beta}} (Z, W) \right|_{\rho} \leq \frac{9(2d)^4 N (1 + 2d)^2 N}{\rho (r - \rho)^2} \left( \frac{\rho}{r} \right)^{d-1}
\]
(4.18)
for all \( i = 1, \ldots, N \) and \( \alpha, \beta, j \in \{1, 2\} \). Now the third inequality in (4.12) follows easily by (4.18) and (4.14).

In order to use the iteration procedure of Moser [26], we follow Huang–Yin [15] and we prove the following.

Proposition 4.4 There exist a constant \( \delta_0(d) > 0 \) depending on \( n \) and independent on \( E(Z, \xi) \) and \( r, \sigma, \rho, r' \) defined by (3.5) such that if the following inequality holds
\[
\left( \frac{36(1 + 2d)^2 N}{r - \rho} \right) (2d)^4 N \left( \frac{\rho}{r} \right)^{d-1} < \delta_0(d),
\]
(4.19)
we have that the mapping \( \Psi(Z', W') := H^{-1}(Z', W') \) is well defined in \( \Delta_\sigma \). Furthermore, it follows that \( \Psi(\Delta_r) \subset \Delta_\sigma, \Psi(\Delta_\rho) \subset \Delta_\rho, E'(Z, \xi) \) is holomorphic in \( \Delta_\sigma \) and also the following inequality holds:
\[
\|E'\|_{r'} \leq \|E\|_{r} \frac{2N (2d)^4 N}{(r - r')^{2N}} \left( \frac{r'}{r} \right)^{d-1}
\]
\[
+ \|E\|_{r} \left( \frac{2d}{N(r - r')} \left( \frac{A(n)}{r - r'} + B(n) \right) \left( \frac{r'}{r} \right)^{d-1} + \left( \frac{108}{r - r'} + D(n) \right) \left( \frac{r'}{r} \right) \left( \frac{r'}{r} \right)^{2d-3} \right) + E(n) \left( \frac{r'}{r} \right)^{2d-3},
\]
(4.20)
where we have used the following notations
\[
A(n) = 324(1 + 2d)^2 N, \quad B(n) = 18N(1 + 2d)^2 N, \quad D(n) = 6N, \quad E(n) = \frac{48 N}{(2d)^4 N}.
\]

Proof Following Huang–Yin [15] and Moser [26], we need to prove that for each \((Z', W') \in \Delta_\sigma\) we can uniquely solve the system
\[
(Z', W') = (Z + F(Z, W), W + G(Z, W)),
\]
where \((Z, W) \in \Delta_\rho\). Following Moser [26] and by (4.19), we can chose \( \delta_0(n) > 0 \) depending on \( n \) and independent on \( r, r', E(z, \xi) \), such that
\[
|\nabla \hat{F}|_\rho + |\nabla \hat{G}|_\rho < \frac{1}{2},
\]
where \( |\nabla \hat{F}(Z, W)|_\rho = \sum_{k=1}^{N} \sum_{l=1}^{2} |\nabla \hat{F}_{k,l}(Z, Wt)|_\rho \). Taking \((Z^{[1]}, W^{[1]}) := (Z', W')\) we define the point \((Z^{[j]}, W^{[j]})\) inductively as follows:
\[
(Z^{[j+1]}, W^{[j+1]}) = (Z + F(Z^{[j]}, W^{[j]}), W + G(Z^{[j]}, W^{[j]})).
\]
Following Huang–Yin [15] and Moser [26], we find using the classical Picard iteration procedure a point \((Z, W) \in \Delta_\rho\) such that
\[
\hat{H}(Z, W) = (Z', W').
\]
Similarly as in the case studied by Huang–Yin [15] and as in the case of Moser [26], we can assume that $\psi(\Delta, r) \subset \Delta_\sigma$ implying that $E'(Z', \xi')$ is holomorphic in $\Delta_\sigma$. We obtain then by (3.2)–(3.4) and following Huang–Yin [15] that $\|E'(Z', \xi')\|_{\sigma} \leq \|Q(Z, \xi)\|_{\sigma}$, where we have used the following notations:

$$Q(Z, \xi) = \langle \tilde{G}(Z, \Phi(Z, \xi)) - \tilde{G}(Z, \tilde{U}) \rangle - 2\Re\{\xi(F(Z, \Phi(Z, \xi)) - F(Z, \tilde{U})) \} \quad - F(Z, W)F(Z, W)' + (E - J^{2d-3}(E))(Z, \xi),$$

$$\tilde{U} = Z\xi', \quad \xi := \left(\begin{array}{cccc}
\xi_{11} & \xi_{12} & \cdots & \xi_{1N} \\
\xi_{21} & \xi_{22} & \cdots & \xi_{2N}
\end{array}\right). \quad (4.22)$$

Following Huang–Yin [15] and Moser [26], we obtain by Lemma 4.2 and by (3.2)–(3.4) the following inequalities:

$$\|E(Z, \xi) - J^{2d-3}(E(Z, \xi))\|_{\sigma} \leq \frac{3^{2N} (2d)^{4N} \|E\|_{\sigma}}{(r - r')^{2N}} \left(\frac{r'}{r}\right)^{d-1},$$

$$\left| \sum_{l=1}^{N} F_{k, l}(\xi, \Phi(Z, \xi))F_{k', l}(\xi, \Phi(\tilde{Z}, \xi)) \right|_{\sigma} \leq \frac{48}{N} (2d)^{8N} \|E_t\|_{\sigma}^2 \left(\frac{r'}{r}\right)^{2d-3},$$

$$|G_{\alpha, \beta}(\xi, \Phi(Z, \xi)) - G_{\alpha, \beta}(\xi, \tilde{U})|_{\sigma} \leq 3 \left(\frac{36(1 + (2d)^{2N})}{r - r'} + 6N(1 + (2d)^{2N}) \right) \left(\frac{2d}{N} \|E\|_{\sigma}^2 \left(\frac{r'}{r}\right)^{d-\frac{1}{2}} \right),$$

$$|F_{k, l}(Z, \Phi(Z, \xi)) - F_{k, l}(Z, \tilde{U})|_{\sigma}, \quad |F_{k, l}(\xi, \Phi(Z, \xi)) - F_{k, l}(\xi, \tilde{U})|_{\sigma} \leq 3 \left(\frac{36}{r - r'} + 2N \right) \left(\frac{2d}{N} \|E\|_{\sigma}^2 \left(\frac{r'}{r}\right)^{2d-3} \right), \quad (4.23)$$

for all $k \in \{1, \ldots, N\}$, $\alpha, \beta, l \in \{1, 2\}$, where $J^{2d-3}(E(Z, \xi))$ is the polynomial defined by the Taylor expansion of $E(Z, \xi)$ up to the degree $2d - 3$. The estimate (4.20) follows by (4.23) and (4.22).

The main ingredients of the proof of Proposition 4.4 are borrowed from Moser [26] and Huang–Yin [15]. Also, the proof of Theorem 1.1 is principally motivated by Moser [26] and by Huang–Yin [15] and it uses rapid convergence arguments applied also by Coffman [6, 7] and Gong [12, 13] in other situations. In order to prove the convergence of a formal transformation between real-analytic manifolds various methods have been applied beside the rapid convergence arguments and the Approximation Theorem of Artin [1]. We would like to mention that Huang–Yin [16] used recently new convergence arguments in the literature based on notions of hyperbolic geometry.

4.1 Proof of Theorem 1.1

Following Huang–Yin [15] and Moser [26], we define the following sequence of real analytic submanifolds:

$$M_n : \quad W = Z\bar{Z} + E_n(Z, \bar{Z}),$$

as follows $M_0 := M$, $M_{n+1} := \Psi_n^{-1}(M_n)$, for all $n \in \mathbb{N}$. Here $\Psi_n$ is the holomorphic mapping between $\Delta_{\sigma_n}$ and $\Delta_{\rho_n}$. It is clear that $d_n := \text{Ord}(E_n(Z, \bar{Z})) \geq 2^n + 2$ for all $n \in \mathbb{N}$. Following
Moser [26], we define the following sequences of numbers
\[ r_n := \frac{1}{2} \left( 1 + \frac{1}{n+1} \right), \quad \rho_n = \frac{r_{n+1} + 2r_n}{3}, \quad \sigma_n = \frac{\rho_n + 2r_n}{3}, \]
and we apply the estimations with \( r = r_n, \rho = \rho_n, r' = r_{n+1}, \psi = \psi_n \), for all \( n \in \mathbb{N} \). Following Moser [26] we have that
\[ \frac{r_{n+1}}{r_n} = 1 - \frac{1}{(n+1)^2}, \quad \frac{1}{r_n - r_{n+1}} = (n+1)(n+2). \tag{4.24} \]

We define the following sequence of real numbers
\[ \epsilon_n := \frac{\|E\|}{n(r_n - r_{n+1})^2}, \]
and by (4.20) we obtain the following:
\[ \epsilon_{n+1} \leq \epsilon_n \left( \frac{r_n - r_{n+1}}{r_{n+1} - r_n} \right)^2 \left( \frac{2N(2d_n)^4N}{n} \right) \left( \frac{r_{n+1}}{r_n} \right)^{d_n^{-1}} + \epsilon_n \cdot \left( \frac{r_n - r_{n+1}}{r_{n+1} - r_n} \right)^4 \left( \frac{2d_n}{N} \right)^{d_n^{-3}}, \]
\[ \epsilon_{n+1} \leq \epsilon_n \left( \frac{r_n - r_{n+1}}{r_{n+1} - r_n} \right)^2 \left( \frac{2N(2d_n)^4N}{n} \right) \left( \frac{r_{n+1}}{r_n} \right)^{d_n^{-1}} + \epsilon_n \cdot \left( \frac{r_n - r_{n+1}}{r_{n+1} - r_n} \right)^4 \left( \frac{2d_n}{N} \right)^{d_n^{-3}}, \tag{4.25} \]

where \( A(n), B(n), D(n), E(n) \) are defined by (4.21). By (4.24) and by Lemma 3.1 it follows easily that
\[ \lim_{n \to \infty} \left( A(n)(n+2)(n+1) + B(n) \right) \left( \frac{2d_n}{N} \right)^{d_n^{-1}} = 0, \]
\[ \lim_{n \to \infty} \left( 108(n+2)(n+1) + D(n) \right) \left( \frac{2d_n}{N} \right)^{d_n^{-3}} = 0, \]
\[ \lim_{n \to \infty} \left( 3^{2N} \left( \frac{2d_n}{N} \right)^{d_n^{-1}} \right) = 0, \tag{4.26} \]
where \( A(n), B(n), D(n), E(n) \) are defined by (4.21). By (4.26) and (4.25) using the standard arguments of Moser [26] and Huang–Yin [15] we obtain the convergence of \( \Psi_n = \psi \circ \cdots \circ \psi_n \) in \( \Delta_{\frac{1}{2}} \). The proof of Theorem 1.1 is completed now.

5 Some Open Problems

One question that appears naturally is if we can prove an nonequidimensional analogue of the Theorem of Moser [26] working with formal transformations \( F(z,w) = (g(z,w), f_1(z,w), \ldots, f_{N'}(z,w)) : \mathbb{C}^{N+1} \to \mathbb{C}^{N'+1} \) satisfying the following properties
\[ \frac{\partial g}{\partial w} \neq 0, \quad F(M) \subset M'_\infty \subset \mathbb{C}^{N'+1}, \quad M'_\infty : w' = z_1'z_1 + \cdots + z_{N'}z_{N'}, \quad N < N', \]
where $M$ is defined by (1.2). By [8] a real submanifold in the complex space defined near a CR singularity [2] can have in its local defining equation the quadratic model containing also pure terms, and therefore this question can be reformulated in a more general setting and as well in cases when the right-side quadratic model is of higher codimension. We have to mention that convergence problems in the non-equidimensional case have been studied by Mir [24] in the CR situation.

Acknowledgements  This project is based on projects initiated by me when I was working as Ph.D. student of Prof. Dmitri Zaitsev in School of Mathematics, Trinity College Dublin, Ireland. I would like to thank him for suggesting to me the Fischer decomposition [27] in [3]. I would like to thank also Prof. Xiaojun Huang for useful discussions regarding the Generalization [15] of the Theorem of Moser [26]. I thank also for hospitality to the Department of Mathematics of the Federal University of Santa Catarina in Brazil while I was working as postdoctoral researcher. I would like also to thank Jiří Lebl for interesting conversations.

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