Topological 1-soliton solutions to some conformable fractional partial differential equations

Gökhan Koyunlu *

Department of Computer Engineering
Nile University of Nigeria. Abuja, Nigeria.

September 11, 2017

Abstract

Topological 1-soliton solutions to various conformable fractional PDEs in both one and more dimensions are constructed by using simple hyperbolic function ansatz. Suitable traveling wave transformation reduces the fractional partial differential equations to ordinary ones. The next step of the procedure is to determine the power of the ansatz by substituting it into the ordinary differential equation. Once the power is determined, if possible, the power determined form of the ansatz is substituted into the ordinary differential equation. Rearranging the resultant equation with respect to the powers of the ansatz and assuming the coefficients are zero leads to an algebraic system of equations. The solution of this system gives the relation between the parameters used in the ansatz.

Keywords: Conformable derivative, fractional modified EW equation, fractional Klein-Gordon equation, fractional potential Kadomtsev-Petviashvili equation, topological 1-soliton solution.

PACS: 02.30.Jr, 02.70.Wz, 47.35.Fg.

AMS2010: 5C07;35R11;35Q53.

*g.koyunlu@nileuniversity.edu.ng
1 Introduction

In the literature of fractional partial differential equations, several definitions are used to construct the equations. One of the recent definitions is the conformable fractional derivative. The main description and some useful properties of this derivative is given in the next section.

This new definition of the fractional derivative has been used to derive new forms, probably a more general ones, of the nonlinear partial differential equations. Thus, exact solutions of these equations have become more significant in both theoretical studies and real world applications. There exist plenty of techniques in the literature to solve nonlinear partial differential equations. Lately, most of them have been adapted to solve fractional partial differential equations when possible. When the exact solutions to nonlinear partial differential equations are examined, one recognize that a method may be sufficient to generate an exact solutions to some equations but does not guarantee exact solutions to all nonlinear partial differential equations. The non linearity is a general concept and characteristics of each non linearity can be completely different from the other. This perspective of non linearity forces researchers to implement the known methods of solution to each non linear partial differential equations.

In the literature, there are many classical methods to solve non linear partial differential equations to set up exact solutions of various types. Simple equation methods [1-4], first integral method [5-7], variations of Kudryashov methods [8-10], \((G'/G)\)-expansion methods [11-13], and various hyperbolic function ansatz techniques [14-16] can be listed as some of well known methods to develop exact solutions to non linear partial differential equations. These techniques can also be implemented to fractional partial differential equations for some types of equations. This study aims to implement hyperbolic tangent ansatz methods to develop topological 1-soliton solutions to some conformable time fractional partial differential equations in one and two space dimensions, namely the fractional modified Equal-Width (fmEW), the fractional Klein-Gordon (fKG), and the fractional potential Kadomtsev-Petviashvili (fpKP) equations. All the fractional derivatives used in these equations are chosen as newly defined conformable derivative. In order not to make restrictions in space domain, the only time fractional forms of the equations are considered throughout the study.

Before setting up the topological 1-soliton solutions, some significant properties of conformable fractional derivative are briefly described in the next section. The considered equations are described briefly and topological 1-soliton solutions are developed in the following sections.
2 Conformable Derivative

The conformable fractional derivative definition is given by Khalil \[17\] as

\[
D^\alpha(u(t)) = \lim_{h \to 0} \frac{u(t + ht^{1-\alpha}) - u(t)}{h}
\]  

(1)

where \(\alpha \in (0, 1]\) and \(u : [0, \infty) \to \mathbb{R}\) in the half space \(t > 0\). This fractional derivative supports plenty of properties given below under the assumptions that the order is \(\alpha \in (0, 1]\) and that \(u = u(t)\) and \(v = v(t)\) are sufficiently \(\alpha\)-differentiable for all \(t > 0\). Then,

- \(D^\alpha(a_1u + a_2v) = a_1D^\alpha(u) + a_2D^\alpha(v)\)
- \(D^\alpha(t^k) = kt^{k-\alpha}, \forall k \in \mathbb{R}\)
- \(D^\alpha(\lambda) = 0, \text{ for all constant } \lambda\)
- \(D^\alpha(uv) = uD^\alpha(v) + vD^\alpha(u)\)
- \(D^\alpha\left(\frac{u}{v}\right) = \frac{vD^\alpha(u) - uD^\alpha(v)}{v^2}\)
- \(D^\alpha(u)(t) = t^{1-\alpha} \frac{du}{dt}\)

for \(\forall a_1, a_2 \in \mathbb{R} \[18,19\]\). The conformable derivative gives support to Laplace transformations, exponential function properties, chain rule, Taylor Series expansion, etc. \[20\]. Probably the most useful property indicates the relation between the conformable derivative and common derivative.

**Theorem 1.** Let \(u\) be an \(\alpha\)-conformable differentiable function, and \(v\) is also differentiable function defined in the range of \(u\). Then,

\[D^\alpha(u \circ v)(t) = t^{1-\alpha} v'(t)u'(v(t))\]  

(2)

3 The Main Frame of Method

Consider a fractional order partial differential equation in a general form

\[F_1(u, D^\alpha_t u, u_x, D^\alpha_t u_x, u_{xx}, \ldots) = 0\]  

(3)

The traveling wave transformation

\[u(x, y, t) = U(\xi), \xi = ax + by - \frac{\nu}{\alpha} t^\alpha\]  

(4)
reduces the fractional partial differential equation (3) to
\[ F_3(U, U', U'', \ldots) = 0 \] (5)
where \( \prime \) denotes the ordinary derivative. In the traveling wave transformation (4), \( a, b, \) and \( \nu \) are assumed as constants. The next step of the method is to suppose that (5) has a solution of the form
\[ U(\xi) = A \tanh B \xi \] (6)
where \( A \neq 0 \) and \( B \in \mathbb{Z}^+ \) are constants to be determined. Substituting this solution into (5) and rearranging the resultant algebraic form for the powers of \( \tanh \) function gives an equation. The power constant \( B \) is determined by equating the powers of the \( \tanh \) function, if possible. After determination of \( B \), the solution (6) is substituted into (5) with writing the value of \( B \). Rearranging the resultant equation for powers of \( \tanh \) and equating the coefficients of them to zero give an algebraic equation system to be solved for \( A, \nu, a, b \).

Note that when the dimension of space domain of \( u \) is 2, then, one of \( a, b \) is removed from the wave transformation (4). Similarly, the wave transformation (4) is modified in a compatible form while studying in one space dimension.

4 Topological 1-Soliton Solutions to Time Fractional Modified EW (fmEW) Equation

The fmEW equation is given as
\[ D_t^\alpha u(x, t) + pu(x, t)u_x(x, t) + qD_t^\alpha u_{xx}(x, t) = 0 \] (7)
where \( D_t^\alpha \) is the \( \alpha \)th order conformable fractional derivative and the subscripts denote the ordinary derivative. The traveling wave transformation (4) for \( b = 0 \) converts (7) to
\[ -\nu U' + paU^2U' - qva^2U''' = 0 \] (8)
Integrating this equation once leads
\[ -\nu U + \frac{1}{3}paU^3 - qva^2U'' = K \] (9)
where \( K \) is constant of integration. Substituting the hyperbolic tangent ansatz \( \tanh^B(\xi) \) into (9) gives
\[ \frac{1}{3}A^3ap\tanh^3B \xi + (2 \nu qa^2 AB - \nu A) \tanh^B \xi + (-\nu qa^2 AB^2 + ABa^2 \nu q) \tanh^{B-2} \xi + (-\nu qa^2 AB^2 - ABa^2 \nu q) \tanh^{B+2} \xi = K \] (10)
Equating the powers $3B = B + 2$ results $B = 1$. Thus, substituting tanh $\xi$ into (9) results

$$\left(\frac{1}{3} p a A^3 - 2 a^2 \nu Aq\right) \tanh^3 \xi + \left(2 a^2 \nu Aq - \nu A\right) \tanh \xi - K = 0 \quad (11)$$

Since tanh $\xi$ is a nonzero solution, then, the coefficients of tanh $\xi$ and tanh$^3 \xi$ should be zero in addition to the zero integration constant, $K = 0$. Thus, the solution of the system of algebraic equations

$$\begin{align*}
1/3 p a A^3 - 2 a^2 \nu Aq &= 0 \\
2 a^2 \nu Aq - \nu A &= 0
\end{align*} \quad (12)$$

gives

$$A = \pm \sqrt{3 \frac{\sqrt{2} \sqrt{q^{-1}} \nu q}{p}} \quad \text{(13)}$$

$$a = 1/2 \sqrt{2} \sqrt{q^{-1}}$$

and

$$A = \pm \sqrt{\pm 3 \frac{\sqrt{2} \sqrt{q^{-1}} \nu q}{p}} \quad \text{(14)}$$

$$a = -1/2 \sqrt{2} \sqrt{q^{-1}}$$

for arbitrarily chosen $\nu$. These values of $A$ and $a$ gives several solutions to (9) as

$$\begin{align*}
U_{1,2}(\xi) &= \pm \sqrt{3 \frac{\sqrt{2} \sqrt{q^{-1}} \nu q}{p}} \tanh \xi \\
U_{3,4}(\xi) &= \pm \sqrt{-3 \frac{\sqrt{2} \sqrt{q^{-1}} \nu q}{p}} \tanh \xi
\end{align*} \quad (15)$$

Returning the original variables makes the solutions

$$\begin{align*}
u_{1,2}(x,t) &= \pm \sqrt{3 \frac{\sqrt{2} \sqrt{q^{-1}} \nu q}{p}} \tanh \left(1/2 \sqrt{2} \sqrt{q^{-1}} x - \nu \frac{t}{\alpha}\right) \\
u_{3,4}(x,t) &= \pm \sqrt{-3 \frac{\sqrt{2} \sqrt{q^{-1}} \nu q}{p}} \tanh \left(-1/2 \sqrt{2} \sqrt{q^{-1}} x - \nu \frac{t}{\alpha}\right)
\end{align*} \quad (16)$$

for $p \neq 0$ and $q \neq 0$.

The graphical illustration of $u_1(x,t)$ for particular choices of $p$, $q$ and $\nu$ is given in Fig 1(a)-1(d) for various values of $\alpha$. 

5
Figure 1: Illustration of the solutions $u_1(x, t)$ for $p = q = 1$ and $\nu = 3$
5 Topological 1-Soliton Solutions to Time Fractional Klein-Gordon Equation

The conformable fractional Klein-Gordon equation

\[ D_t^{2\alpha} u - u_{xx} - pu - qu^3 = 0 \]  

(17)

can transformed into

\[ \nu^2 U'' - a^2 U'' - pU - qU^3 = 0 \]  

(18)

via the compatible wave transformation \( \xi = ax - \nu \frac{\nu}{\alpha} \). Substituting \( \tanh B \xi \) into (18) gives

\[-A^3 q \tanh^3 B \xi + (-A B^2 a^2 + A B^2 \nu^2 + A B a^2 - AB \nu^2) \tanh^{B-2} \xi \\
+ (2 AB^2 a^2 - 2 AB^2 \nu^2 - pA) \tanh^{B} \xi + (-A B^2 a^2 + A B^2 \nu^2 - A B a^2 + AB \nu^2) \tanh^{B+2} \xi = 0 \]

(19)

Equating \( 3B = B + 2 \) gives \( B = 1 \). When the solution \( \tanh \xi \) is substituted into (18), it takes the form

\[-A^3 q - 2 A a^2 + 2 A \nu^2 \tanh^3 \xi + (2 A a^2 - 2 A \nu^2 - pA) \tanh \xi = 0 \]  

(20)

When the coefficients are equated to zero, the algebraic system

\[-A^3 q - 2 A a^2 + 2 A \nu^2 = 0 \\
2 A a^2 - 2 A \nu^2 - pA = 0 \]  

(21)

Solving this system for \( A \) and \( a \)

\[ A = \pm \sqrt{-\frac{p}{q}} \]

\[ a = \pm \sqrt{\nu^2 + \frac{1}{2} p} \]  

(22)

Thus, the solutions to (18) is determined as

\[ U_{5,6}(\xi) = \pm \sqrt{-\frac{p}{q}} \tanh(\xi) \]  

(23)

Therefore, topological 1-soliton solutions to the fKG equation become

\[ u_{5,6,7,8}(x, t) = \pm \sqrt{-\frac{p}{q}} \tanh(\pm \sqrt{\nu^2 + \frac{1}{2} px - \nu \frac{\nu}{\alpha}}) \]  

(24)
where \( q \neq 0 \). The graphical illustration of \( u_5(x, t) \) for various choices of \( p, q \) and \( \nu \) is given in Fig 2(a)–2(d) for different values of \( \alpha \).

(a) \( \alpha = 0.25 \)

(b) \( \alpha = 0.50 \)

(c) \( \alpha = 0.75 \)

(d) \( \alpha = 1.00 \)

Figure 2: Illustration of the solutions \( u_5(x, t) \) for \( p = -1, q = 1 \) and \( \nu = 2 \)
6 Topological 1-Soliton Solutions to the fractional potential Kadomtsev-Petviashvili (fpKP) Equation

The time fractional potential Kadomtsev-Petviashvili (fpKP) equation of the form

\[ D_t^\alpha u_x + pu_{xx}u_x + qu_{xxxx} + ru_{yy} = 0 \]  \hfill (25)

is reduced to the ordinary differential equation

\[-\nu a U'' + p a^3 U'' U' + q a^4 U''' + r b^2 U'' = 0 \]  \hfill (26)

by using the wave transformation (4). Assuming (26) has a solution of the form \( \tanh \xi \). Therefore, the algebraic system of equations is obtained

\[
\begin{align*}
(6 AB^4 a^4 q + 10 AB^2 a^4 q - 2 AB^2 b^2 r + 2 AB^2 a v) \tanh^6 \xi \\
+ (AB^4 a^4 q - 6 AB^2 a^4 q + 11 AB^2 a^4 q - 6 AB a^4 q) \tanh^4 \xi \\
+ (-4 AB^4 a^4 q + 12 AB^2 a^4 q - 16 AB^2 a^4 q + 8 AB a^4 q + AB b^2 r - AB^2 a v - AB b^2 r + AB a v) \tanh^2 \xi \\
+ (-4 AB^4 a^4 q - 12 AB^2 a^4 q - 16 AB^2 a^4 q - 8 AB a^4 q + AB b^2 r - AB^2 a v + AB b^2 r - AB a v) \tanh^0 \xi \\
+ (AB^4 a^4 q + 6 AB^2 a^4 q + 11 AB^2 a^4 q + 6 AB a^4 q) \tanh^0 \xi \\
+ (A^2 B^3 a^3 p - A^2 B^2 a^3 p) \tanh^2 \xi \\
+ (-3 A^2 B^3 a^3 p + A^2 B^2 a^3 p) \tanh^2 \xi \\
+ (3 A^2 B^3 a^3 p + A^2 B^2 a^3 p) \tanh^2 \xi \\
+ (-A^2 B^3 a^3 p - A^2 B^2 a^3 p) \tanh^2 \xi = 0
\end{align*}
\]  \hfill (27)

Equating the power \( 2B - 3 \) to \( B - 2 \) gives \( B = 1 \). Thus, the predicted solution becomes \( \tanh^B \xi \). Substituting this solution into (26) gives

\[
(\begin{align*}
-2 A^2 a^3 p + 24 A a^4 q & = 0 \\
4 A^2 a^3 p - 40 A a^4 q - 2 A b^2 r - 2 A a v & = 0 \\
-2 A^2 a^3 p + 16 A a^4 q - 2 A b^2 r + 2 A a v & = 0
\end{align*}
\]  \hfill (28)

Since we seek a nonzero solution, the only coefficients of the powers of \( \tanh \xi \) should be zero. Therefore, the algebraic system of equations is obtained

\[
\begin{align*}
-2 A^2 a^3 p + 24 A a^4 q & = 0 \\
4 A^2 a^3 p - 40 A a^4 q - 2 A b^2 r - 2 A a v & = 0 \\
-2 A^2 a^3 p + 16 A a^4 q - 2 A b^2 r + 2 A a v & = 0
\end{align*}
\]  \hfill (29)

The solutions of this system for \( A, \nu, a \) and \( b \) gives

\[
A = \frac{12 a q}{p} \quad \nu = \frac{4 a^4 q + b^2 r}{a}
\]  \hfill (30)

9
for arbitrarily chosen $a \neq 0$ and $b$ where $p \neq 0$. Thus, the solution to (26) is constructed as

$$U(\xi) = \frac{12aq}{p} \tanh \xi$$

(31)

Returning to the original variables gives the solution to the fpKP as

$$u_9(x, y, t) = \frac{12aq}{p} \tanh \left( ax + by - \frac{4a^4q + b^2r t^\alpha}{a} \frac{1}{\alpha} \right)$$

(32)

where $p \neq 0$ and $a \neq 0$. The graphical illustration of $u_p(x, t)$ for various choices of $p, q, r, a$ and $b$ is given in Fig 3(a)-3(d) for different values of $\alpha$. 
Figure 3: Projection of the solution $u_9(x, t)$ for $p = q = a = 1$, $r = -1$ and $y = 0$. 
7 Conclusion

In the present study, a simple hyperbolic tangent ansatz is used to derive topological 1-soliton solutions to some one and two dimensional fractional nonlinear partial differential equations. The conformable derivative supports the chain rule. Compatible traveling wave transformation reduces the fractional partial differential equations to some ordinary differential equations. The hyperbolic tangent ansatz is a predicted solution and includes some parameters to be determined in an order. The first parameter to be determined is the positive integer power parameter. The determination of this parameter is followed by the other parameters in the solution by solving some algebraic equation systems. The time fractional modified EW, Klein-Gordon and potential Kadomtsev-Petviashvili equations are solved by using the hyperbolic tangent ansatz. Explicit solutions are derived and graphical illustrations are represented in 3D plots by assist of computer.

References

[1] Zayed, E. M., & Al-Nowehy, A. G. (2017). Solitons and other exact solutions for variant nonlinear Boussinesq equations. Optik-International Journal for Light and Electron Optics.

[2] Roshid, H. O., Roshid, M. M., Rahman, N., & Pervin, M. R. (2017). New solitary wave in shallow water, plasma and ion acoustic plasma via the GZK-BBM equation and the RLW equation. Propulsion and Power Research.

[3] Kaplan, M., & Bekir, A. (2016). The modified simple equation method for solving some fractional-order nonlinear equations. Pramana, 87(1), 1-5.

[4] Kaplan, M., Bekir, A., Akbulut, A., & Aksoy, E. (2015). The modified simple equation method for nonlinear fractional differential equations. Romanian J. Phys, 60(9-10), 1374-1383.

[5] Eslami, M., Mirzazadeh, M., Vajargah, B. F., & Biswas, A. (2014). Optical solitons for the resonant nonlinear Schrödinger’s equation with time-dependent coefficients by the first integral method. Optik-International Journal for Light and Electron Optics, 125(13), 3107-3116.

[6] Lu, B., Zhang, H., & Xie, F. (2010). Travelling wave solutions of nonlinear partial equations by using the first integral method. Applied Mathematics and Computation, 216(4), 1329-1336.

[7] Bekir, A., & Ünsal, Ö. (2012). Analytic treatment of nonlinear evolution equations using first integral method. Pramana, 79(1), 3-17.
[8] Tuluce Demiray, S., Pandir, Y., & Bulut, H. (2014, July). Generalized Kudryashov method for time-fractional differential equations. In Abstract and applied analysis (Vol. 2014). Hindawi Publishing Corporation.

[9] Aksoy, E., Çevikel, A. C., & Bekir, A. (2016). Soliton solutions of (2+ 1)-dimensional time-fractional Zoomeron equation. Optik-International Journal for Light and Electron Optics, 127(17), 6933-6942.

[10] Korkmaz, A. (2017). Exact Solutions to (3+1) Conformable Time Fractional Jimbo-Miwa,Zakharov-Kuznetsov and Modified Zakharov-Kuznetsov Equations, Communications in Theoretical Physics, 67(5), 479-482.

[11] Guner, O., Atik, H., & Kavyrzhanovich, A. A. (2017). New exact solution for space-time fractional differential equations via (G'/G)-expansion method. Optik-International Journal for Light and Electron Optics, 130, 696-701.

[12] Ganji, D. D., Hosseini, M., Talarposhti, R. A., Pourmousavik, S. I., & Sheikholeslami, M. (2014). The (G'/G)-Expansion Method for Magnetohydrodynamics Jeffery-Hamel Nanofluid Flow. Journal of Nanofluids, 3(1), 60-64.

[13] Naher, H., & Abdullah, F. A. (2014). New generalized and improved (G'/G)-expansion method for nonlinear evolution equations in mathematical physics. Journal of the Egyptian Mathematical Society, 22(3), 390-395.

[14] Guner, O., Bekir, A., & Korkmaz, A. (2017). Tanh-type and sech-type solitons for some space-time fractional PDE models. The European Physical Journal Plus, 132, 1-12.

[15] Korkmaz, A. (2017). Exact solutions of space-time fractional EW and modified EW equations. Chaos, Solitons & Fractals, 96, 132-138.

[16] Guner, O., Korkmaz, A., & Bekir, A. (2017). Dark Soliton Solutions of Space-Time Fractional Sharma-Tasso-Olver and Potential Kadomtsev-Petviashvili Equations. Communications in Theoretical Physics, 67(2), 182.

[17] Khalil, R., Al Horani, M., Yousef, A., & Sababheh, M. (2014). A new definition of fractional derivative. Journal of Computational and Applied Mathematics, 264, 65-70.

[18] Atangana, A., Baleanu, D., & Alsaedi, A. (2015). New properties of conformable derivative. Open Mathematics, 13(1), 1-10.

[19] Çenesiz, Y., Baleanu, D., Kurt, A., & Tasbozan, O. (2016). New exact solutions of Burgers’ type equations with conformable derivative. Waves in Random and Complex Media, 1-14.
[20] Abdeljawad, T. (2015). On conformable fractional calculus. Journal of computational and Applied Mathematics, 279, 57-66.