ON THE INTERPRETATION OF $\pi N \rightarrow \pi \pi N$ DATA NEAR THRESHOLD

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Abstract

Near threshold pion production experiments have been recently carried out and used to extract S–wave $\pi\pi$ scattering lengths. We emphasize here that at present these processes are related only at the tree level (and its first correction) in chiral perturbation theory. Higher order corrections (including loops) must be evaluated before rigorous claims concerning S–wave $\pi\pi$ scattering lengths can be made.

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1 Introduction

In the last few years an impressive series of experiments have measured the total cross section for the processes $\pi N \rightarrow \pi\pi N$ quite close to threshold \cite{1, 2, 3, 4, 5}. The modulus of the threshold amplitude is then found by the extrapolation

$$|A(\pi\pi N)|^2 = \lim_{T_\pi \rightarrow T_{\pi}^{th}} \frac{\sigma(\pi N \rightarrow \pi\pi N)}{C S (T_\pi - T_{\pi}^{th})^2}$$  \hspace{1cm} (1)$$

where $T_\pi$ is the incident laboratory pion kinetic energy, $S$ is a Bose symmetry factor ($S = 1/2$ if the final two pions are identical, otherwise it is unity), and

$$C = M_\pi^2 \left( \frac{1}{128\pi^2} \right) \sqrt{3} (2 + \mu)^{1/2} (2 + 3\mu)^{1/2} (1 + 2\mu)^{-11/2}$$  \hspace{1cm} (2)$$

where $\mu = M_\pi/m$ the ratio of the pion to nucleon mass. The threshold modulus has been obtained in this way for the five charge states initiated by $\pi^\pm p$ \cite{6}. Explicit isospin violation due to the electromagnetic mass differences has been removed through the kinematics of the threshold $T_{\pi}^{th}$ value and the threshold amplitude modulus is assumed to be isospin invariant.

By Watson’s theorem \cite{7} the threshold amplitude has the phase of the initial elastic $J^P = \frac{1}{2}^+$ amplitude (up to an overall sign). The threshold production amplitude complex phase is then $\delta_{31} \simeq -4^\circ$ for initial isospin $3/2$ and $\delta_{11} \simeq 2^\circ$ for total isospin $1/2$. The threshold production amplitude is thus nearly real. At threshold the final $\pi\pi$ state must have isospin 0 or 2 by extended Bose symmetry and hence there are only two independent threshold amplitudes; called $A_{2I,I_{\pi\pi}}$ (with $I$ the total isospin of the incident $\pi N$ system and $I_{\pi\pi}$ the isospin of the two–pion system in the final state).

| process amplitude | $A_{32}(\pi\pi N)$ | $A_{10}(\pi\pi N)$ |
|-------------------|-------------------|-------------------|
| $\pi^+ p \rightarrow \pi^+\pi^+ n$ | $\frac{2}{\sqrt{5}}$ | 0 |
| $\rightarrow \pi^+\pi^0 p$ | $-\frac{1}{\sqrt{10}}$ | 0 |
| $\pi^- p \rightarrow \pi^+\pi^- n$ | $\frac{1}{3\sqrt{5}}$ | $-\frac{\sqrt{2}}{3}$ |
| $\rightarrow \pi^0\pi^0 n$ | $\frac{2}{3\sqrt{5}}$ | $\frac{\sqrt{2}}{3}$ |
| $\rightarrow \pi^-\pi^0 p$ | $-\frac{1}{\sqrt{10}}$ | 0 |

Table 1: Clebsch-Gordan coefficients

In Table 1 we list the five measured process amplitudes in terms of the two independent isospin amplitudes $A_{32}(\pi\pi N)$ and $A_{10}(\pi\pi N)$. From the measured process amplitude moduli a unique value of $A_{10}$ and $A_{32}$ can be found up to an overall sign.
2 Relation to $\pi\pi$ scattering

The purest process to test the chiral dynamics of QCD is the reaction $\pi\pi \rightarrow \pi\pi$ in the threshold region. The pertinent partial waves admit an energy expansion of the type

$$t_I^l(s) = q^{2l} \{ a_I^l + q^2 b_I^l + \ldots \}$$

(3)

with $q$ the modulus of the pion three–momentum, $s = 4(M_\pi^2 + q^2)$ the cms energy squared and $l (I)$ denote the angular momentum (isospin) of the $\pi\pi$ system. As first pointed out by Weinberg [8], the $\pi\pi$ scattering amplitude can be written in terms of one invariant function $A(s, t, u)$ which takes the form

$$A(s, t, u) = \frac{s - M_\pi^2}{F_\pi^2} + \mathcal{O}(E^4)$$

(4)

where $F_\pi \simeq 93$ MeV is the pion decay constant and $\mathcal{O}(E^4) = \mathcal{O}(s^2, sM_\pi^2, M_\pi^4, \ldots)$ are corrections which can not be calculated from current algebra. Consequently, the S–wave scattering lengths

$$a_0^0(\pi\pi) = \frac{7M_\pi^2}{32\pi F_\pi^2}, \quad a_2^0(\pi\pi) = -\frac{2M_\pi^2}{32\pi F_\pi^2},$$

(5)

vanish in the chiral limit, $M_\pi \rightarrow 0$, and are therefore particularly sensitive to the explicit chiral symmetry breaking in QCD. Furthermore, the one–loop corrections to the Weinberg result have been worked out [9] and rather accurate predictions could be given, i.e. $a_0^0(\pi\pi) = 0.20 \pm 0.01$ [10]. It is, however, not straightforward to determine these fundamental quantities experimentally. Therefore, any option to do this is highly welcome (for a review, see e.g. [11]). In what follows, we will be concerned with one possible candidate, namely the reaction $\pi N \rightarrow \pi\pi N$ at threshold.

From the effective Lagrangian formulation [8] of PCAC and the algebra of currents Olsson and Turner (OT) [12] showed in effect that

$$\mathcal{A}_{32}(\pi\pi) = -2\sqrt{10\pi} \frac{g_{\pi N}}{m} \left[ \frac{a_0^0(\pi\pi)}{M_\pi^2} + d_2 \right]$$

$$\mathcal{A}_{10}(\pi\pi) = 4\pi \frac{g_{\pi N}}{m} \left[ \frac{a_0^0(\pi\pi)}{M_\pi^2} + d_0 \right]$$

(6)

with $g_{\pi N} = 13.4$ the strong pion–nucleon coupling constant. The above result is a consequence of the dominance of the pion exchange and contact diagrams shown in Fig. 1(a),1(b). To lowest order, the two “shift” constants $d_I$ arise from the sub-leading diagrams of Fig. 1(c). The $d_I$ are of order $\mathcal{O}(M_\pi)$. A remark on the diagram 1(a)

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[3] Here, we are not considering $\pi N \rightarrow \pi\pi N$ data at higher energy which might be analyzed with the help of Chew–Low type techniques to give the $\pi\pi$ phases.
is in order here. One frequently finds in the literature the erroneous statement that the threshold $\pi N \rightarrow \pi \pi N$ amplitudes can not be directly related to the $\pi \pi$ phase shifts since the exchanged pion in the pion pole graph is off mass–shell. However, a general argument invoking only unitarity tells us that the residue of the pion–pole term must factor into the product of the on-shell $\pi \pi$ scattering amplitude times the pion–nucleon vertex function (in the $t$–channel) \[13\].

Fig. 1: Pion pole (a) and contact (b) diagrams which lead to the OT relation \[6\]. The consecutive pion emission (c) contributes to the shift constants $d_{0,2}$.

The OT relation \[6\] predates QCD and its expression through chiral perturbation theory (ChPT). Nevertheless, the OT production amplitude is equivalent to the tree level ChPT result if the $a_0^i(\pi \pi)$ are the tree level $\pi \pi$ S–wave scattering lengths. In the original formulation, the OT relation contains a parameter called $\xi$. Its meaning and relevance for present day data analysis is discussed in the next section.

3 The $\xi$ parameter

Let us elaborate on the OT $\xi$ parameter \[12\]. $\xi$ described the pattern of chiral symmetry breaking in the pre QCD era of the effective Lagrangian. Only $\xi = 0$ is consistent with QCD. To see this in more detail, consider the so–called $\sigma$–commutator, i.e. the commutator between an axial charge $Q_5^a$ and the divergence of the axial current, $D^b = \partial^\mu A_5^b$.

\begin{align*}
    i \left[ Q_5^a, D^b \right] &= \sigma^{ab} = -F_\pi M_\pi^2 \left\{ \delta^{ab} \left( F_\pi - \frac{\pi^2}{2F_\pi} \right) + \frac{\xi}{4F_\pi} (\delta^{ab} + 2\pi^a \pi^b) \right\} \\
    \text{where the first term is an isoscalar and the second an isotensor. The corresponding } \\
    \pi \pi \text{ scattering amplitude to lowest order is then given by} \\
    A_\xi(s, t, u) &= \frac{1}{F_\pi^2} \left[ s - M_\pi^2 \left( 1 + \frac{\xi}{2} \right) \right], \\
    \text{and the scattering lengths } a_0^0(\pi \pi) \text{ and } a_0^2(\pi \pi) \text{ depend on } \xi, \\
    a_0^0(\pi \pi) &= \frac{M_\pi^2}{32\pi F_\pi^2} \left( 7 - \frac{5}{2} \xi \right), \quad a_0^2(\pi \pi) = -\frac{M_\pi^2}{32\pi F_\pi^2} (2 + \xi). \quad (9)
\end{align*}
These forms are frequently used in the literature\[2\]\[6\] to determine the S–wave $\pi\pi$ scattering lengths from the measured and extrapolated $\pi N \to \pi\pi N$ data. However, in QCD, the $\sigma$–commutator stems from the explicit chiral symmetry breaking quark mass term, i.e. (in the isospin limit $m_u = m_d = \hat{m}$)

$$\sigma^{ab} = \delta^{ab} \hat{m} (\bar{u}u + \bar{d}d)$$  \hspace{1cm} (10)

which is purely isoscalar and thus $\xi_{QCD} = 0$. It is worth to stress that $\xi = 0$ also holds in the so–called ‘generalized ChPT’\[14, 15\]. In that scheme, the symmetry–breaking terms are subject to another counting which for example modifies even the lowest order (Weinberg) expression for the elastic $\pi\pi$ scattering amplitude,

$$A_{G\text{ChPT}}(s, t, u) = \frac{1}{F^2_\pi} s - M^2_\pi(1 - \frac{\chi}{3}) + \mathcal{O}(E^3),$$ \hspace{1cm} (11)

Notice that the corrections start at order $E^3$ in contrast to the standard scenario, cf. eq.(4). The new parameter $\chi$ measures the deviation from the conventionally adopted (and presumably correct) quark mass ratio $m_s/\hat{m} = 2M^2_K/M^2_\pi - 1 \simeq 25$, i.e.

$$\chi = 6 \frac{(2M^2_K/M^2_\pi - 1) - (m_s/\hat{m})}{(m_s/\hat{m})^2 - 1},$$ \hspace{1cm} (12)

neglecting the small OZI violation in the $0^{++}$ channel. Nevertheless, the explicit symmetry breaking is still purely isoscalar, i.e. $\xi = 0$.

4 An improved low–energy representation

We observe from (6) that the effect of the constants $d_I$ is to shift the $\pi\pi$ scattering length values relative to the measured production amplitudes. It is crucial therefore to reliably estimate their values. We point out here that the loop corrections, counter terms, and other contributions of higher order ChPT will alter both the scattering lengths and the $\pi\pi N$ threshold amplitudes.

The leading order ChPT corrections\[17\] are shown in Fig. 2. Other corrections which must be considered in the shift parameters involve intermediate $\Delta_{33}$ and possibly other resonances in diagrams similar to Fig. 1(c). Such tree contributions are implicitly contained e.g. in the model of Oset and Vicente–Vacas\[16\] which is intended to describe the data over a wide range of energies.

In ref.[17], the corrections to the OT relation of order $M_\pi$ were worked out. This

\#6 While in standard ChPT, one has $m_s/B \ll 1$, in GChPT one assumes $m_s \simeq B$, with $B$ the order parameter of the chiral symmetry breaking, $B = - \langle 0|\bar{q}q|0 \rangle / F^2_\pi$. 

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leads to an improved low–energy representation of the form\[7\]

\[
\begin{align*}
\mathcal{A}_{32}(\pi\pi N) &= -2\sqrt{10}\pi (1 + \frac{7}{2}\mu) \left[ \frac{a_0^2(\pi\pi)}{M^2_\pi} + \tilde{d}_2 M^2_\pi \right] \\
\mathcal{A}_{10}(\pi\pi N) &= 4\pi (1 + \frac{37}{14}\mu) \left[ \frac{a_0^0(\pi\pi)}{M^2_\pi} + \tilde{d}_0 M^2_\pi \right]
\end{align*}
\]

(13)

where the new shift constants \(\tilde{d}_{0,2}\) have the form

\[
\tilde{d}_I = c^0_I + c^1_I M_\pi + c^2_I M^2_\pi + \ldots, \quad I = 0, 2
\]

modulo logs. One notices that the correction of order \(M_\pi\) is comparable in size to the leading term (approximately 40% and 50% for \(\mathcal{A}_{10}\) and \(\mathcal{A}_{32}\), respectively). Therefore, it is mandatory to calculate (at least) the coefficients \(c^0_I\). Also, at that order the one–loop corrections to the S–wave \(\pi\pi\) scattering lengths appear \cite{9, 10}. One can, however, estimate the \(\mathcal{O}(M^2_\pi)\) corrections by calculating the unambiguous absorptive parts of the one–loop diagrams \cite{17}. The corresponding corrections are small for \(\mathcal{A}_{32}(\pi\pi N)\) and of the order of 30% for \(\mathcal{A}_{10}(\pi\pi N)\). Such a pattern is expected, the \(\pi\pi\) interactions are small for \(I = 2\) but sizeable for \(I = 0\). Such estimates should only be considered indicative and can not substitute for the complete calculation of the shift constants \(\tilde{d}_I\).

\[\text{Fig. 2: Diagrams which give the contributions to } \mathcal{A}_{10}(\pi\pi N) \text{ and } \mathcal{A}_{32}(\pi\pi N) \text{ up-to-and-including } \mathcal{O}(M_\pi). \text{ The circle–cross denotes an insertion from the next-to-leading order chiral effective Lagrangian } \mathcal{L}^{(2)}_{\pi N}.\]

Before discussing the influence of the new corrections in eq.\((13)\) on the extraction of the S-wave \(\pi\pi\) scattering lengths, let us comment on the extraction of the threshold amplitudes in ref.\cite{6}. As pointed out in refs.\cite{17, 18}, only in the channels \(\pi^+ p \to \pi^+ \pi^+ n\) and \(\pi^- p \to \pi^0 \pi^0 n\) are the data close enough to threshold to allow for an extraction of the threshold amplitudes \(\mathcal{A}_{10}\) and \(\mathcal{A}_{32}\). A global fit to all five channels as in \cite{3} gives insufficient weight to the threshold region. Correspondingly, one finds \cite{18}

\[
\mathcal{A}_{10} = (8.01 \pm 0.64) M^{-3}_\pi, \quad \mathcal{A}_{32} = (2.53 \pm 0.14) M^{-3}_\pi,
\]

(15)

\[\#7\text{Notice that the overall sign of the } \mathcal{A}_{10,32} \text{ at threshold is fixed by the chiral expansion.}\]
which differ somewhat from the values given in [6]. Ignoring for the moment $O(M^2)$ contributions (i.e. setting $\tilde{d}_0 = \tilde{d}_2 = 0$) and inserting on the left hand side of eq.(13) the result of the fit in eq.(15) (for more details, see ref.[18]), one extracts $a_0^0(\pi\pi) = 0.23 \pm 0.02$ and $a_0^2(\pi\pi) = -0.042 \pm 0.002$, which are quite close to the CHPT prediction at next-to-leading order. We stress, however, that these numbers should only be considered indicative since the corrections to eq.(13) are not yet fully under control.

Finally, we would like to make a few comments on the work of Sossi et al. [19]. There, the next–to–leading order $\pi\pi$ amplitude was combined with the Oset and Vicente–Vacas model [16] and a comparison with the existing data (for $T_\pi \leq 400$ MeV) was made. This procedure is, as should be clear from the previous discussions, not consistent since the $\pi\pi$ and $\pi\pi N$ amplitudes should be treated at the same order in the chiral expansion. Besides, in ref.[19] the mesonic low–energy constants based on the work of ref.[20] are used. These are, however, determined from a fit to $\pi\pi$ data over an energy range which clearly exceeds the range of validity of the one–loop calculation (see also the discussion in section 4.1 of ref.[11]).

The threshold pion production amplitude plays an important role in low energy hadron physics because of its relationship to $\pi\pi$ scattering. The precise nature of this relationship must be explored by an examination of the effect of higher order ChPT contributions. In principle, these corrections are manageable. However, the appearance of novel counter terms with a priori unknown coefficients introduces uncertainties. It remains to be seen whether the relation between the $\pi\pi$ S–wave scattering lengths and the threshold $\pi\pi N$ amplitudes can be formulated with a sufficient numerical accuracy to pin down $a_0^0(\pi\pi)$ and $a_0^2(\pi\pi)$ with an uncertainty comparable to the one of theoretical predictions [10].

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References

[1] G. Kernel et al., Z. Phys. C48, 201 (1990); M. Sevior et al., Phys. Rev. Lett. 66, 2569 (1993); G. Smith et al., (CHAOS at TRIUMF, 1994). [$\pi^+ p \rightarrow \pi^+ \pi^+ n$]

[2] D. Počanić et al., Phys. Rev. Lett. 72, 1156 (1993); G. Smitt et al., (CHAOS at TRIUMF, 1994). [$\pi^+ p \rightarrow \pi^+ \pi^0 n$]
[3] G. Kernel et al., Phys. Lett. B216, 244 (1989); G. Smitt et al., (CHAOS at TRIUMF, 1994); G. Rebka et al., (LAMPF, 1994). \[ \pi^-p \rightarrow \pi^-\pi^+n \]

[4] G. Kernel et al., Phys. Lett. B225, 198 (1989); G. Smitt (CHAOS at TRIUMF), 1994). \[ \pi^-p \rightarrow \pi^-\pi^0p \]

[5] J. Lowe et al., Phys. Rev. C44, 956 (1991).

[6] H. Burkhard and J. Lowe, Phys. Rev. Lett. 67, 2622 (1991).

[7] K.H. Watson, Phys. Rev. 95, 228 (1954).

[8] S. Weinberg, Phys. Rev. Lett. 18, 188 (1967); Phys. Rev. Lett. 18, 507 (1967); Phys. Rev. 166, 1568 (1968).

[9] J. Gasser and H.Leutwyler, Phys. Lett. 125B, 325 (1983).

[10] J. Gasser and H.Leutwyler, Ann. Phys. (NY) 158, 142 (1984).

[11] Ulf–G. Meißner, Rep. Prog. Phys. 56, 903 (1993).

[12] M.G. Olsson and Leaf Turner, Phys. Rev. Lett 20, 1127 (1968); Phys. Rev. 181, 2141 (1969); Phys. Rev. Lett. 38, 296 (1977).

[13] R. Dashen and M. Weinstein, Phys. Rev. 183, 1261 (1969).

[14] M. Knecht and J. Stern, Orsay preprint IPNO-TH-94-53, to be published in the second edition of the DAPHNE physics handbook, eds. L. Maiani, G. Pancheri and N.Paver.

[15] J. Stern, H. Sazdijan, and N. Fuchs, Phys. Rev. D38, 2195 (1988).

[16] E. Oset and M.J. Vicente–Vacas, Nucl. Phys. A446, 584 (1985)

[17] V. Bernard, N. Kaiser and Ulf-G. Meißner, Phys. Lett. B332, 415 (1994); (E) B338, 520 (1994).

[18] V. Bernard, N. Kaiser and Ulf-G. Meißner, “Chiral Dynamics in Nucleons and Nuclei”, preprint CRN 95/3 and TK 95 1, to appear in Int. J.Mod. Phys. E (1995).

[19] V. Sossi, N. Fazel, R.R. Johnson and M.J. Vicente–Vacas, Phys. Lett. B298, 287 (1993).

[20] J.F. Donoghue, C. Ramirez and G. Valencia, Phys. Rev. D38, 2195 (1988).