INPUT-TO-STATE STABILITY OF CONTINUOUS-TIME SYSTEMS VIA FINITE-TIME LYAPUNOV FUNCTIONS

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Abstract. In this paper, input-to-state stability (ISS) of continuous-time systems is analyzed via finite-time Lyapunov functions. ISS of a continuous-time system is first proved via finite-time robust Lyapunov functions for an introduced auxiliary system of the considered system. It is then obtained that the existence of a finite-time ISS Lyapunov function implies that the continuous-time system is ISS. The converse finite-time ISS Lyapunov theorem is proposed. Furthermore, we explore the properties of finite-time ISS Lyapunov functions for the continuous-time system on a bounded and compact set without a small neighborhood of the origin. The effectiveness of our results is illustrated by four examples.

1. Introduction. Sontag [20] in the late 1980s first introduced the concept of input-to-state stability (ISS) for nonlinear continuous-time systems. The concept defines a stability property of state trajectories with respect to initial states and inputs and implies that bounded inputs lead to bounded outputs. Many results about ISS for continuous-time systems have been obtained, see [20, 21, 18, 22]. In [22], it describes different equivalent formulations of ISS. It is especially shown that the ISS property is equivalent to the existence of an ISS Lyapunov function. The ISS notion plays an important role in stability analysis of large scale systems. It is possible to analyze stability of large interconnected networks of systems via ISS small gain theorems if all the subsystems are ISS [3, 4, 5, 6, 11]. Therefore, many researchers are very interested in ISS and ISS Lyapunov functions for nonlinear continuous-time systems.

In [20, 21, 18, 22], the time derivative of an ISS Lyapunov function along the trajectories of the considered continuous-time system without inputs is assumed to be negative. In order to relax the constraint, in this paper, we introduce the

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concept of finite-time ISS Lyapunov function, where the function decreases along the trajectories of system (without inputs) after a certain time, and not at every time. This approach was first used in [1] where the author proposed a new sufficient condition for asymptotic stability of an equilibrium of a time-varying system without inputs via a Lyapunov function whom the time derivative of along the trajectories of the state may have negative and positive values. This result was extended to continuous-time systems in [12] via Lyapunov functions with non sign-definite derivative. In [14], using Lyapunov functions with relaxed constraints the authors stated sufficient and necessary conditions for asymptotic stability of an equilibrium of a discrete-time homogeneous dynamical system without inputs. In [8, 9], the authors discussed similar results for discrete-time systems without inputs. An alternative converse Lyapunov theorem for discrete-time systems without inputs was stated based on Lyapunov functions with relaxed conditions called finite-step Lyapunov functions by the authors. The new converse Lyapunov theorem was extended to continuous-time systems without inputs in [7]. The authors proposed two methods to compute Lyapunov functions with relaxed conditions called non-monotonic Lyapunov functions for continuous-time systems without inputs in [15]. Moreover, in [10, 8], the authors discussed a relaxation of ISS Lyapunov functions for discrete-time systems. Inspired by results of the papers [10, 8], in this paper, we are interested in ISS Lyapunov functions with relaxed constraints, later named finite-time ISS Lyapunov functions for continuous-time systems. Some interesting results are obtained. The existence of a finite-time ISS Lyapunov function for a continuous-time system implies that the system is ISS (Theorem 3.3). It is pointed out that under certain conditions if system is ISS, then any scaled norm is a finite-time ISS Lyapunov function (Theorem 3.5). Moreover, it is discussed that the properties of finite-time ISS Lyapunov functions for a continuous-time system on a bounded and compact set excluding a small neighborhood of the origin (Lemma 3.7, Lemma 3.11, Proposition 3.10). Finite-time ISS Lyapunov functions for the system on a bounded and compact set without a small neighborhood of the origin satisfy linear inequalities.

The outline of this paper is as follows. In Section 2, the notations and preliminaries are given. The concept of a finite-time ISS Lyapunov function is introduced. The problem we investigate is stated. In Section 3, we mainly discuss how to prove that the system is ISS by finite-time Lyapunov functions and propose a converse finite-time ISS Lyapunov theorem. Via introducing an auxiliary system of the considered system, it is proved that the system is ISS via finite-time robust Lyapunov functions for the auxiliary system. Furthermore, it is proved that the system is ISS via finite-time ISS Lyapunov functions without introducing an auxiliary system. These two results are obtained via different techniques. If system is ISS and Assumption 1 holds, then any scaled norm is a finite-time ISS Lyapunov function. Especially for an expISS system, any scaled norm is a finite-time ISS Lyapunov function. Without Assumption 1, we further prove that if a system is ISS in a bounded, compact and positively $T$-invariant set $\Omega$, then any scaled norm is a finite-time ISS Lyapunov function for the system on $\Omega$ without a small neighborhood of the origin. We study the properties of finite-time ISS Lyapunov functions for the continuous-time system on a bounded and compact set without a small neighborhood of the origin (See Lemma 3.7, Lemma 3.11 and Remark 3.12). Four examples are presented to demonstrate the effectiveness of our results in Section 4. We close this paper in Section 5 with some concluding remarks.
2. Notations and preliminaries. Let \( \mathbb{R}, \mathbb{R}^+, \mathbb{Z} \) and \( \mathbb{N} \) represent the real numbers, the nonnegative real numbers, the integers and the nonnegative integers, respectively. Given a vector \( x \in \mathbb{R}^n \), let \( x^T \) denote its transpose. The standard inner product of \( x, y \in \mathbb{R}^n \) is noted by \( \langle x, y \rangle \). For a set \( \Omega \subset \mathbb{R}^n \), the boundary, the closure and the complement of \( \Omega \) are denoted by \( \partial \Omega, \overline{\Omega} \) and \( \Omega^C \) respectively. For a vector \( x \in \mathbb{R}^n \), let \( |x| \) denote its Euclidean norm. Given a positive constant \( r \in \mathbb{R}^+ \) and a vector \( x_0 \in \mathbb{R}^n \), \( B(x_0, r) = \{ x \in \mathbb{R}^n \ | \ |x - x_0| < r \} \) denotes the open ball of radius \( r \) around \( x_0 \) in the norm of \( | \cdot | \). The induced matrix norm is defined by \( |A| := \max_{|x|=1} |Ax| \). By \( |u|_\infty := \sup_{t \in \mathbb{R}^+} |u(t)| \) we denote the supremum norm of a function \( u : \mathbb{R}^+ \to \mathbb{R}^m \).

We recall comparison functions which are very useful in stability analysis. We say a continuous function \( \alpha : \mathbb{R}^+ \to \mathbb{R}^+ \) is positive definite if it satisfies \( \alpha(0) = 0 \) and \( \alpha(s) > 0 \) for all \( s > 0 \). A positive definite function is of class \( \mathcal{K} \) if it is strictly increasing and of class \( \mathcal{K}_\infty \) if it is of class \( \mathcal{K} \) and unbounded. A continuous function \( \gamma : \mathbb{R}^+ \to \mathbb{R}^+ \) is of class \( \mathcal{L} \) if \( \gamma(r) \) is strictly decreasing to 0 as \( r \to \infty \) and we call a continuous function \( \beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) of class \( \mathcal{KL} \) if it is of class \( \mathcal{K}_\infty \) in the first argument and of class \( \mathcal{L} \) in the second argument. More detail of comparison functions are discussed in [13].

In this paper we consider a continuous-time system described by

\[
\dot{x}(t) = f(x(t), u(t)) \tag{1}
\]

with vector field \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \), \( f(0, 0) = 0 \), state \( x(t) \in \mathbb{R}^n \), and perturbation input \( u(t) \in \mathbb{R}^m \), \( t \geq 0 \). The admissible input values are given by \( U_R := \overline{B(0, R)} \subset \mathbb{R}^m \) for a constant \( R > 0 \) and the admissible input functions by \( u \in \mathcal{U}_R := \{ u : \mathbb{R}^+ \to \mathbb{R}^m \ \text{measurable} \ | \ |u|_\infty \leq R \} \). We assume the map \( f \) is locally Lipschitz continuous, and for a compact set \( \Omega \subset \mathbb{R}^n \) with \( 0 \in \text{int} \Omega \) let real numbers \( L_1, L_2 > 0 \) denote the constants such that

\[
|f(x_1, u_1) - f(x_2, u_2)| \leq L_1|x_1 - x_2| + L_2|u_1 - u_2|, \tag{2}
\]

for \( x_1, x_2 \in \Omega, u_1, u_2 \in \mathcal{U}_R \). The solution of (1) corresponding to an initial condition \( x(0) = x_0 \) and an input \( u \in \mathcal{U}_R \) is denoted by \( x(t, x_0, u(\cdot)) \).

**Definition 2.1.** System (1) is input-to-state stable (ISS) in a compact set \( \Omega \subset \mathbb{R}^n \) with \( 0 \in \text{int} \Omega \) if there exist functions \( \beta \in \mathcal{KL}, \zeta \in \mathcal{K}_\infty \) such that for all \( x_0 \in \Omega, u \in \mathcal{U}_R \)

\[
|x(t, x_0, u(\cdot))| \leq \beta(|x_0|, t) + \zeta(|u|_\infty), \text{ for all } t \geq 0. \tag{3}
\]

**Remark 2.2.**

(i) If \( u \equiv 0 \) for system (1), then ISS implies the origin is an equilibrium which is asymptotically stable.

(ii) If \( f(0, u) \equiv 0 \) for all \( u \in \mathcal{U}_R \), and the function \( \zeta(\cdot) \equiv 0 \) in Definition 2.1, then the origin of system (1) is robustly asymptotically stable in \( \Omega \).

(iii) If the function \( \beta \in \mathcal{KL} \) from (3) can be chosen as

\[
\beta(s, t) = C\kappa^t s \tag{4}
\]

with \( C \geq 1 \) and \( \kappa \in [0, 1) \), then system (1) is exponentially input-to-state stable (expISS) in \( \Omega \).

It is proved in [23] that the ISS property of system (1) is equivalent to the existence of a smooth, i.e. \( C^\infty \), ISS Lyapunov function for system (1).
**Proof.** robustly asymptotically stable in if there exists a function \( \gamma \in K_\infty \) such that
\[
\alpha_1(|x|) \leq W(x) \leq \alpha_2(|x|),
\]
\[
< \nabla W(x), f(x,u) > \leq -\alpha(|x|) + \gamma(|u|_\infty)
\] hold for all \( x \in \Omega, u \in U_R \).

**Remark 2.4.** If \( f(0,u) \equiv 0 \) for all \( u \in U_R \), and in Definition 2.3 the function \( \gamma(\cdot) \equiv 0 \), then \( W \) is called a robust Lyapunov function for system (1) on \( \Omega \).

**Lemma 2.5.** System (1) is ISS in a compact set \( \Omega \subset \mathbb{R}^n \) with \( 0 \in \text{int } \Omega \) if and only if there exists a function \( \eta \in K_\infty \) such that the origin of system described by (7) is 
\[\dot{x}(t) = f(x,\eta(|x|)u), \ |u|_\infty \leq 1, \ u \in \mathbb{R}^m.\] 
**Proof.** \( \Rightarrow \) We assume a differentiable function \( W : \Omega \to \mathbb{R}_+ \) is an ISS Lyapunov function for system (1) on \( \Omega \), i.e., there exist functions \( \alpha_1, \alpha_2, \alpha, \gamma \in K_\infty \) such that the inequalities (5), (6) hold. For the inequality (6), we have that
\[
< \nabla W(x), f(x,u) > \leq -\alpha(|x|) + \gamma(|u|_\infty)
\]
\[
= -1/2\alpha(|x|) - 1/2\alpha(|x|) + \gamma(|u|_\infty).
\] For a function \( \gamma \in K_\infty \), Proposition 3 of [9] shows that \( \gamma^{-1} \in K_\infty \). If \( |u|_\infty \leq \gamma^{-1}(1/2\alpha(|x|)) \), then
\[
< \nabla W(x), f(x,u) > \leq -1/2\alpha(|x|).
\] Using (10), we attain that \( W \) is a robust Lyapunov function for system (11) on \( \Omega \)
\[\dot{x}(t) = f(x,\eta(|x|)u), \ u \in \mathbb{R}^m \text{ with } |u|_\infty \leq 1,\] 
where \( \eta(s) = \gamma^{-1}(1/2\alpha(s)) \) for \( s \geq 0 \). Based on [16, Theorem 2.9], we have that the origin of system (7) is robustly asymptotically stable in \( \Omega \).

\( \Leftarrow \) According to the assumption, the solution \( x_a(t,x_0,u(\cdot)) \) of system (7) corresponding to an initial condition \( x_0 \in \Omega \) and an input \( u \in U_1 \) satisfies the inequality (12) in \( \Omega \)
\[
|x_a(t,x_0,u(\cdot))| \leq \beta(x_0,t), \ t \in \mathbb{R}_+, \ \beta \in KL.
\] Then we obtain that for system (1)
\[
|x(t,x_0,u(\cdot))| \leq \beta(x_0,t) + \eta^{-1}(|u|_\infty), \ t \in \mathbb{R}_+, \ u \in U_R.
\] Therefore, system (1) is ISS in \( \Omega \).

**Remark 2.6.** The trick of concluding ISS of system (1) from robust stability of the origin of system (7) is classical. It was utilized in [23].

In order to obtain our main results, we introduce the concept of positively \( T \)-invariant set for system (1).

**Definition 2.7.** Let \( T > 0 \) be a constant. A compact set \( \Omega \subset \mathbb{R}^n \) is said to be a positively \( T \)-invariant set for system (1) if for any \( x(t,x_0,u(\cdot)) \in \Omega \) it holds that \( x(t+T,x_0,u(\cdot+T)) \in \Omega \) for all \( t \in \mathbb{R}_+, \ u \in U_R \).

**Remark 2.8.** In Remark 2.7, let \( T = 0 \), then \( \Omega \) is called a positively invariant set for system (1).
In this paper, we are interested in a relaxation of ISS Lyapunov function described by the following definition 2.9.

**Definition 2.9.** Let $\Omega \subset \mathbb{R}^n$ with $0 \in \text{int} \Omega$ be a compact set. A continuous function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is called a finite-time ISS Lyapunov function for system (1) on $\Omega$ if there exist functions $\alpha_1, \alpha_2, \gamma \in \mathcal{K}_\infty$ and a positive definite function $\rho$ with $(\text{id} - \rho) \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad \text{for } x \in \Omega,$$

$$V(x(T, x_0, u(\cdot))) \leq \rho(V(x_0)) + \gamma(|u|_\infty), \quad \text{for } x_0 \in \Omega, u \in U_R$$

(14) (15)

**Remark 2.10.**
(i) If $f(0, u) \equiv 0$ for all $u \in U_R$, and in Definition 2.9 the function $\gamma(\cdot) \equiv 0$, then $V$ is called a finite-time robust Lyapunov function for system (1) on $\Omega$.
(ii) According to the condition (15) in Definition 2.9, it is clear that the finite-time ISS Lyapunov function decreases along the trajectories of system (1) (without input) after a certain time $T$, and not at every time. Thus the constraints on finite-time ISS Lyapunov functions are more relaxed than that on ISS Lyapunov functions.
(iii) If the function $\rho$ from (15) satisfies $\rho(a + b) \leq \rho(2a) + \rho(2b)$ for $a, b \in \mathbb{R}_+$, then (15) holds with all $MT(0 < M < +\infty, M \in \mathbb{N})$.

3. **Main results.** In this section, our main aim is to prove system (1) is ISS in a compact and positively $T$-invariant set $\Omega \subset \mathbb{R}^n$ with $0 \in \text{int} \Omega$ via finite-time Lyapunov functions, and to derive a constructive converse finite-time ISS Lyapunov theorem. By introducing an auxiliary system of system (1), we prove that if there exists a finite-time robust Lyapunov function for the auxiliary system, then system (1) is ISS. Furthermore, if there exists a finite-time ISS Lyapunov function for system (1) on $\Omega$, then system (1) is ISS in $\Omega$. Conversely, under certain conditions, we prove that if system (1) is ISS in $\Omega$ then any scaled norm is a finite-time ISS Lyapunov function. Moreover, it is shown that if system (1) is ISS on $\Omega$, then any scaled norm is a finite-time ISS Lyapunov function for system (1) on $\Omega$ without a small neighborhood of the origin. Under certain constraints, we conclude system (1) is ISS in $\Omega$ via finite-time ISS Lyapunov functions for system (1) on $\Omega$ excluding a small neighborhood of the origin.

**Theorem 3.1.** Let $\eta \in \mathcal{K}_\infty$ be Lipschitz continuous with a Lipschitz constant $L > 0$, $T$ and $R$ be positive constants, and a compact set $\Omega \subset \mathbb{R}^n$ with $0 \in \text{int} \Omega$ be a positively $T$-invariant set for system (1). Assume the inequality $\eta(\max_{x \in \Omega} |x|) \leq R$ holds. If there exists a finite-time robust Lyapunov function $V$ with $T$ for system (7) on $\Omega$, then the origin of system (7) is robustly asymptotically stable in $\Omega$. Moreover, it is obtained that system (1) is ISS in $\Omega$, and $V$ is a finite-time ISS Lyapunov function for system (1) on $\Omega$.

**Proof.** Based on the assumption, there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a positive definite function $\rho$ with $(\text{id} - \rho) \in \mathcal{K}_\infty$ such that the inequalities (14) and (16) hold.

$$V(x(T, x_0, u(\cdot))) \leq \rho(V(x_0)), \quad \text{for } x_0 \in \Omega, u \in U_1.$$
For $t \in \mathbb{R}_+$, there exist an integer $N \geq 0$, $j \in \mathbb{R}_+$, $j < T$ such that $t = NT + j$. According to the constraints, for $x_0 \in \Omega$, $u \in \mathcal{U}_1$ we have
\[
V(x(t, x_0, u(\cdot))) = V(x(NT + j, x_0, u(\cdot)))
= V(x(T + (N - 1)T + j, x_0, u(\cdot)))
= V(x(T, x((N - 1)T + j, x_0, u(\cdot)), u(\cdot)))
\leq \rho(V(x((N - 1)T + j, x_0, u(\cdot))))
\]
\[
\vdots
\leq \rho^N(V(x(j, x_0, u(\cdot))))
\leq \rho^N(\alpha_2(|x(j, x_0, u(\cdot))|)),
\]
where $\alpha_2$ is from (14), and $\rho^N$ denotes the $N$-times composition of $\rho$.

In the following, we estimate $|x(j, x_0, u(\cdot))|$ for any $j \geq 0$. The solution of (7) at time $t = j$ corresponding to an initial condition $x_0 \in \Omega$ and an input $u \in \mathcal{U}_1$ is given by
\[
x(j, x_0, u(\cdot)) = x_0 + \int_0^j f(x(s), \eta(|x(s)|)u(s))ds,
\]
for any $j \geq 0$. We abbreviate the state $x(j) := x(j, x_0, u(\cdot))$. Then we have
\[
|x(j) - x(0)| \leq \int_0^j |f(x(s), \eta(|x(s)|)u(s))|ds
\leq \int_0^j |f(x(s), \eta(|x(s)|)u(s)) - f(x_0, \eta(|x_0|)u(s))|ds
+ \int_0^j |f(x_0, \eta(|x_0|)u(s)) - f(0, 0)|ds.
\]
Using Lipschitz conditions of the functions $f, \eta$, we obtain that
\[
|x(j) - x(0)| \leq \int_0^j (L_1 + LL_2)|x(s) - x(0)|ds + \int_0^j (L_1 + LL_2)|x_0|ds.
\]
Utilizing Bellman-Gronwall inequality ([19, Lemma 3.1]), it is attained that
\[
|x(j) - x(0)| \leq e^{(L_1 + LL_2)j} \int_0^j (L_1 + LL_2)|x_0|ds.
\]
Thus it holds that for all $j \in [0, T)$
\[
|x(j)| \leq e^{(L_1 + LL_2)j} \int_0^j (L_1 + LL_2)|x_0|ds + |x_0|
\leq e^{(L_1 + LL_2)j} \int_0^T (L_1 + LL_2)|x_0|ds + |x_0|
\leq ((L_1 + LL_2)T e^{(L_1 + LL_2)T} + 1)|x_0| := P(|x_0|)
\]
It is clear that the function $P : \mathbb{R}_+ \to \mathbb{R}_+$ defined by (20) is a $\mathcal{K}_\infty$ function.
\[
P(r) = ((L_1 + LL_2)T e^{(L_1 + LL_2)T} + 1)r, \quad \text{for } r \geq 0.
\]
Thus
\[
V(x(t)) \leq \rho^N(\alpha_2(P(|x_0|)))
= \rho^{\frac{NT}{j}}(\alpha_2(P(|x_0|))).
\]
The following argument of deducing the existence of the function \( \hat{\rho} \) is similar as that of [7, Theorem 2.1] and [15, Lemma 12]. Since \( \rho \) is positive definite, without loss of generality it is possible to suppose that \( \rho \) is invertible, that is, \( \rho \) is a one-to-one and onto function. Since \( \rho \) is continuous, then by [2, Theorem 3.16] we have that \( \rho^{-1} \) is continuous and \( \rho^{-1}(0) = \rho^{-1}(\rho(0)) = 0 \). Therefore, there exists a function \( \hat{\rho} \in \mathcal{K}_\infty \) such that \( \rho^{-1}(s) \leq \hat{\rho}(s) \) for all \( s \geq 0 \). Therefore,

\[
V(x(t)) \leq \rho^{T} \circ \hat{\rho} \circ \alpha_2 \circ P(|x_0|) := \hat{\beta}(|x(0)|, t). \tag{21}
\]

It is evident that \( \rho^{T} \in \mathcal{L}, \hat{\rho} \circ \alpha_2 \circ P \in \mathcal{K}_\infty \). Thus \( \hat{\beta} \in \mathcal{K}\mathcal{L} \).

Finally, we get that

\[
|x(t)| \leq \alpha_1^{-1}(\hat{\beta}(|x(0)|, t)) := \beta(|x_0|, t). \tag{22}
\]

holds for all \( x_0 \in \Omega, u \in U_\Omega \). It is clear that \( \beta \in \mathcal{K}\mathcal{L} \). Then it is concluded that the origin of system (7) is robustly asymptotically stable in \( \Omega \). Based on Lemma 2.5, we obtain that system (1) is ISS in \( \Omega \). Furthermore, for system (1) we have that

\[
V(x(T, x_0, u(\cdot))) \leq \rho(V(x_0)) + \alpha_2 \circ \eta^{-1}(|u|_\infty), \quad \text{for } x \in \Omega, u \in U_R. \tag{23}
\]

Hence, the function \( V \) is a finite-time ISS Lyapunov function for system (1) on \( \Omega \).

\[\square\]

**Remark 3.2.** (1) From Theorem 3.1, it is known that if a function \( V : \mathbb{R}^n \to \mathbb{R}_+ \) is a finite-time robust Lyapunov function for the auxiliary system (7) of system (1), then \( V \) is a finite-time ISS Lyapunov function for system (1).

(2) In order to ensure \( x(T, x_0, u(\cdot)) \in \Omega \) for \( x \in \Omega, u \in U_R \), it is necessary to require that \( \Omega \) is a positively \( T \)-invariant set for system (1).

(3) It is proved that system (1) is ISS using finite-time robust Lyapunov functions for the auxiliary system (7). In the following, we prove that without introducing an auxiliary system system (1) is ISS via finite-time ISS Lyapunov functions.

**Theorem 3.3.** Let \( T, R \) be positive constants, and a compact set \( \Omega \subset \mathbb{R}^n \) with \( 0 \in \text{int} \, \Omega \) a positively \( T \)-invariant set for system (1). If there exists a finite-time ISS Lyapunov function \( V \) with \( T > 0 \) for system (1) on \( \Omega \), then system (1) is ISS in \( \Omega \).

The idea of the proof of Theorem (3.3) is inspired by the proof of [10, Theorem 4.1] which states that the existence of finite-step ISS Lyapunov function implies ISS of discrete-time systems.

**Proof.** Since \( V \) is a finite-time ISS Lyapunov function for system (1) on \( \Omega \), then there exist functions \( \alpha_1, \alpha_2, \gamma \in \mathcal{K}_\infty \) and a positive definite function \( \rho \) with \( (\text{id} - \rho) \in \mathcal{K}_\infty \) such that the inequalities (14) and (15) hold. For an initial condition \( x_0 \in \Omega \) and an input \( u \in U_R \), we abbreviate the state \( x(t) := x(t, x_0, u(\cdot)) \). Let \( \mu \in \mathcal{K}_\infty \) be such that \( \rho + \mu < \text{id} \). According to the assumption, it is evident that such a function \( \mu \in \mathcal{K}_\infty \) exists. We consider the set

\[
S := \{ x_0 \in \Omega \subset \mathbb{R}^n : V(x_0) \leq \delta := \mu^{-1} \circ \gamma(|u|_\infty) \}. \]
We now prove that for any \( t \in \mathbb{R}_+ \) with \( x(t) \in S \) we have \( x(t + NT) \in S \) for all \( N \in \mathbb{N} \). By (15), we get that

\[
V(x(t + T)) \leq \rho(V(x(t))) + \gamma(|u|_{\infty}) \leq \rho(\delta) + \gamma(|u|_{\infty})
\]

\[
= \rho(\delta) + \mu(\delta) - \mu(\delta) + \gamma(|u|_{\infty})
\]

\[
= (\rho + \mu)(\delta) \leq \delta.
\]

Thus \( x(t + T) \in S \) and by induction we have \( x(t + NT) \in S \) for all \( N \in \mathbb{N} \).

Define \( \tau_0 := \inf \{ t \in \mathbb{R}_+ : x(\tau) \in S, \tau \in [t, t+T) \} \). It is clear that \( \tau_0 \in \mathbb{R}_+ \cup \{ +\infty \} \). According to the definition of \( \tau_0 \) and the above discussion, we attain that \( x(t) \in S \) for all \( t \geq \tau_0 \), i.e.,

\[
V(x(t)) \leq \mu^{-1} \circ \gamma(|u|_{\infty}) =: \tilde{\gamma}(|u|_{\infty}), \forall t \geq \tau_0.
\]

(24)

For \( t < \tau_0 \), we need to consider two cases.

First, if \( x(t) \in S \) then by the definition of \( S \) we have \( V(x(t)) \leq \tilde{\gamma}(|u|_{\infty}) \). Secondly, if \( x(t) \notin S \), then there exist an integer \( N \geq 0, j \in \mathbb{R}_+, j < T \) such that \( t = NT + j \).

Because \( x(j) \in S \) implies that \( x(j + NT) \in S \), we conclude \( x(j) \notin S \). By definition of \( S \), it holds that \( V(x(j)) > \mu^{-1} \circ \gamma(|u|_{\infty}) \), i.e., \( \gamma(|u|_{\infty}) \leq \mu \circ V(x(j)) \). Hence we have

\[
V(x(j + T)) \leq \rho(V(x(j))) + \gamma(|u|_{\infty})
\]

\[
< \rho(V(x(j))) + \mu \circ V(x(j))
\]

\[
= (\rho + \mu) \circ V(x(j)).
\]

It is pointed out that the positive definite function \( \varphi := (\rho + \mu) \) satisfies \( \varphi < \text{id} \).

Define \( \bar{N} := \sup \{ N \in \mathbb{N} : V(j + NT) \notin S \} \). Then it is obtained that for all \( N \in [0, \bar{N}] \cap \mathbb{N} \)

\[
V(x(j + (N + 1)T)) \leq \varphi(V(x(j + NT)))).
\]

For \( N \in \mathbb{N} \cap [0, \bar{N}] \), using (14) we have

\[
V(x(j + NT)) \leq \varphi^N(\alpha_2(|x(j)|)).
\]

(25)

Utilizing Lipschitz continuity of the function \( f \) and Bellman-Gronwall inequality ([19, Lemma 3.1]), we get that

\[
|x(j) - x(0)| \leq \int_0^j L_1|x(s) - x(0)|ds + \int_0^j (L_1|x_0| + L_2|x|_{\infty})ds.
\]

(26)

Then

\[
|x(j)| \leq (L_1|x_0| + L_2|x|_{\infty})T e^{L_1T} + |x_0|.
\]

(27)

Thus using the inequalities (25) and (27), we have that for \( t = j + NT \), \( x(j) \notin S \),

\[
V(x(t)) \leq \varphi^N \circ \alpha_2((L_1|x_0| + L_2|x|_{\infty})e^{L_1T} + |x_0|)
\]

\[
\leq \varphi^N \circ \alpha_2(2(L_1e^{L_1T} + 1)|x_0|) + \varphi^N \circ \alpha_2(2L_2|x|_{\infty} e^{L_1T})
\]

\[
\leq \varphi^{\frac{t}{T}} \circ \alpha_2(2(L_1e^{L_1T} + 1)|x_0|) + \alpha_2(2L_2|x|_{\infty} e^{L_1T}).
\]

Since \( \varphi \) is positive definite, without loss of generality it is possible to suppose that \( \varphi \) is invertible, that is, \( \varphi \) is a one-to-one and onto function. Since \( \varphi \) is continuous, then by [2, Theorem 3.16] we have that \( \varphi^{-1} \) is continuous and \( \varphi^{-1}(0) = \varphi^{-1}(\varphi(0)) = 0 \). Hence we obtain that for \( t = j + NT \), \( x(j) \notin S \),

\[
V(x(t)) \leq \varphi^{\frac{t}{T}} \circ \varphi^{-1} \circ \alpha_2(2(L_1e^{L_1T} + 1)|x_0|) + \alpha_2(2L_2e^{L_1T}|u|_{\infty}).
\]

(28)
For $N > \bar{N}$, we get that $x(j + NT) \in S$ and
\[
V(x(j + NT)) \leq \hat{\gamma}(|u|_{\infty}).
\] (29)

From the above discussion, we attain that for $x(t) \in S$ or $x(t) \notin S$
\[
V(x(t)) \leq \varphi^{1/T} \circ \varphi^{-1} \circ \alpha_2(2(L_1 e^{L_1 T} + 1)|x_0|) + \alpha_2(2L_2 e^{L_2 T}|u|_{\infty})
\]
\[
+ \hat{\gamma}(|u|_{\infty}).
\]
It is evident that there exists a function $\varphi_1 \in \mathcal{K}_\infty$ such that for $s \geq 0$, $\varphi^{-1}(s) \leq \varphi_1(s)$. Thus it holds that
\[
V(x(t)) \leq \varphi^{1/T} \circ \varphi_1 \circ \alpha_2(2(L_1 e^{L_1 T} + 1)|x_0|) + \alpha_2(2L_2 e^{L_2 T}|u|_{\infty})
\]
\[
+ \hat{\gamma}(|u|_{\infty}).
\]
Let $\hat{\beta}(|x_0|, t) := \varphi^{1/T} \circ \varphi_1 \circ \alpha_2(2(L_1 e^{L_1 T} + 1)|x_0|)$, $\hat{\gamma}(|u|_{\infty}) := \alpha_2(2L_2 e^{L_2 T}|u|_{\infty}) + \hat{\gamma}(|u|_{\infty})$. It is clear that $\hat{\beta} \in \mathcal{K}$ and $\hat{\gamma} \in \mathcal{K}_\infty$ are satisfied. From the above discussion, and using the inequality (14) we obtain that for $x_0 \in \Omega$, $u \in U_R$
\[
|x(t)| \leq \alpha_1^{-1}(\hat{\beta}(|x_0|, t) + \hat{\gamma}(|u|_{\infty})).
\]
hold.

Since $\alpha_1 \in \mathcal{K}_\infty$ is satisfied, $\alpha_1^{-1} \in \mathcal{K}_\infty$ holds. Then it is attained that
\[
|x(t)| \leq \alpha_1^{-1}(2\hat{\beta}(|x_0|, t)) + \alpha_1^{-1}(2\hat{\gamma}(|u|_{\infty})).
\] (30)

Therefore, system (1) is ISS in $\Omega$. \qed

**Theorem 3.4.** Assume $T, R$ are positive constants, and a compact set $\Omega \subset \mathbb{R}^n$ with $0 \in \text{int} \; \Omega$ is a positively $T$-invariant set for system (1). If there exists a finite-time ISS Lyapunov function $V$ for system (1) satisfying for any $x_0 \in \Omega \subset \mathbb{R}^n$ and $u \in U_R$
\[
a|x|^\lambda \leq V(x) \leq b|x|^\lambda,
\] (31)
\[
V(x(T, x_0, u(\cdot))) \leq cV(x_0) + d|u|_{\infty}
\] (32)
with $0 < a \leq b$, $c \in [0, 1)$ and $d, \lambda > 0$, then system (1) is expISS.

**Proof.** The proof follows the lines of the proof of Theorem 3.3. Thus we omit the detail of the proof. \qed

Next we investigate converse finite-time ISS Lyapunov theorem. In order to obtain the result, the following assumption 1 is necessary.

**Assumption 1.** There exists a function $\beta \in \mathcal{KL}$ satisfying (3) for system (1) such that
\[
\beta(s, T) < 1/2s
\] (33)
for some positive $T \in \mathbb{R}_+$ and $s > 0$.

**Theorem 3.5.** If system (1) is ISS and Assumption 1 is satisfied, then for any function $\eta \in \mathcal{K}_\infty$, the function $V : \mathbb{R}^n \to \mathbb{R}$ with
\[
V(x) := \eta(|x|), \; \forall x \in \mathbb{R}^n
\] (34)
satisfies (14) and (15) with $T$ from Assumption 1.
Proof. Based on Assumption 1, there exists a constant $T > 0$ such that (33) holds. According to the conditions, we have that
\[
\eta(|x(t + T, x(t), u(\cdot))|) \leq \eta(\beta(|x(t), T|) + \zeta(|u|_\infty)) \\
\leq \eta(2\beta(|x(t), T|)) + \eta(2\zeta(|u|_\infty)) \\
= \eta(2\beta(\eta^{-1}(V(x(t)), T)) + \eta(2\zeta(|u|_\infty)).
\]
Let $\rho := \eta(2\beta(\eta^{-1}(\cdot), T))$, $\gamma := \eta(2\zeta(\cdot))$. It is clear that $\gamma \in \mathcal{K}_\infty$ is satisfied. Utilizing Assumption 1, we obtain that $\rho < \eta(\eta^{-1}(\cdot)) = \text{id}$ holds and the function $\rho$ is positive definite. Then we get that
\[
V(x(t + T, x(t), u(\cdot))) \leq \rho(V(x(t))) + \gamma(|u|_\infty).
\] (35)
Therefore, the function $V$ defined by (34) is a finite-time ISS Lyapunov function for system (1).

Remark 3.6. Based on Theorem 3.5, it is evident that if system (1) is expISS, then any scaled norm is a finite-time ISS Lyapunov function for system (1).

Lemma 3.7. Let a function $\beta \in \mathcal{KL}$ and the constants $a, b$ satisfy $0 < a < b < +\infty$. Then there exists a constant $T > 0$ such that (33) holds for any $s \in [a, b]$.

Proof. According to Sontag’s lemma on $\mathcal{KL}$-estimate [17, Proposition 7], for $\beta$ there exist functions $\varphi_1, \varphi_2 \in \mathcal{K}_\infty$ such that, for all $s, t \in \mathbb{R}_+$
\[
\varphi_1(\beta(s, t)) \leq \varphi_2(s) \exp^{-t}.
\] (36)
In order to get the desired $T$, we have to require that
\[
\varphi_2(s) \exp^{-t} \leq \varphi_1(s/2), \quad \text{for } t = T, s \geq 0.
\] (37)
Now, we prove such a $T$ exists in two cases.

Case 1. $\varphi_2(b) \leq \varphi_1(a/2)$. In this case, for $T > 0$, the inequality (33) holds for any $s \in [a, b]$.

Case 2. $\varphi_2(b) > \varphi_1(a/2)$. For this case, for $T > \ln \frac{\varphi_2(b)}{\varphi_1(a/2)}$, the inequality (33) holds for any $s \in [a, b]$.

Therefore, it is proved the existence of $T$ ensuring (33) holds.

Remark 3.8. (1) From the proof of Lemma 3.7, it is clear that we can choose any $a, b$ satisfying $0 < a < b < +\infty$. However, the chosen $T$ depends on $a, b$ (see Case 1, Case 2).

(2) Given a compact, bounded and positively $T$-invariant set $\Omega \subset \mathbb{R}^n$ with $0 \in \text{int } \Omega$ and a small real number $\epsilon > 0$. Based on Lemma 3.7, via similar technique used in the proof of Theorem 3.5 we can get that if system (1) is ISS, then any scaled norm is a finite-time ISS Lyapunov function for system (1) on $\Omega \setminus B(0, \epsilon)$.

The following Propositions 3.9-3.10 describe some properties of finite-time ISS Lyapunov functions for system (1) on a compact, bounded and positively $T$-invariant set $\Omega \subset \mathbb{R}^n$ with $0 \in \text{int } \Omega$ excluding a small neighborhood of the origin. It is emphasized that finite-time ISS Lyapunov functions satisfy linear inequalities (38), (39) for the state in $\Omega$ without a small neighborhood of the origin.
Proposition 3.9. Let a set \( \Omega \subset \mathbb{R}^n \) with \( 0 \in \text{int } \Omega \) be bounded, compact and positively \( T \)-invariant, and a small real number \( \epsilon \) be positive. If a function \( V : \mathbb{R}^n \to \mathbb{R} \) is a finite-time ISS Lyapunov function for system (1) on \( \Omega \), then for any constant \( \sigma > 0 \) there exist positive constants \( C, r \) such that \( V_1(x) = CV(x) \) satisfies
\[
V_1(x) \geq |x|, \text{ for } x \in \Omega \setminus B(0, \epsilon)
\]
and
\[
V_1(x(T, x_0, u(\cdot)))) - V_1(x_0) \leq -\sigma|x| + r|u|_{\infty}
\]
for \( x_0 \in \Omega \setminus B(0, \epsilon) \) and \( u \in \mathcal{U}_R \).

Proof. Since \( V \) is a finite-time ISS Lyapunov function for system (1) on \( \Omega \), then there exist functions \( \alpha_1, \alpha_2, \gamma \in \mathcal{K}_\infty \) and a positive definite function \( \rho \) with \( (id - \rho) \in \mathcal{K}_\infty \) such that the inequalities (14) and (15) hold. Define
\[
C = \min\{C \in \mathbb{R}_+ \mid C\alpha_1(|x|) \geq |x|, \eta \}
\]
and
\[
\begin{align*}
\alpha_2(\omega(x)) &= \min\{C \in \mathbb{R}_+ \mid C\alpha_2(|x|) \geq \sigma|x|, \epsilon \}, \\
V_1(x(T, x_0, u(\cdot)))) - V_1(x_0) &\leq -\sigma|x_0| + \epsilon + r|u|_{\infty} \\
&\leq -\sigma|x_0| + r|u|_{\infty}.
\end{align*}
\]
Therefore, \( V_1(x) = CV(x) \) satisfies (38) and (39).

Proposition 3.10. Suppose a set \( \Omega \subset \mathbb{R}^n \) with \( 0 \in \text{int } \Omega \) is compact, bounded and positively \( T \)-invariant. Let a small real number \( \epsilon \) be positive. If a function \( V : \mathbb{R}^n \to \mathbb{R}_+ \) satisfies (14) and there exist positive constants \( T, \sigma, r \) such that
\[
V(x(T, x_0, u(\cdot)))) - V(x_0) \leq -\sigma|x_0| + r|u|_{\infty}
\]
for \( x_0 \in \Omega \setminus B(0, \epsilon) \) and \( u \in \mathcal{U}_R \), then \( V \) is a finite-time ISS Lyapunov function for system (1) on \( \Omega \setminus B(0, \epsilon) \).

Proof. Since \( \Omega \) is bounded, there exists a constant \( \delta > \sigma \) such that the function \( \tilde{\alpha}_2 : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( \tilde{\alpha}_2(s) = \delta s + s \) satisfying
\[
\tilde{\alpha}_2(|x|) > \sigma|x|, \text{ for } x \in \Omega \setminus B(0, \epsilon),
\]
\[
\tilde{\alpha}_2(|x|) > \alpha_2(|x|), \text{ for } x \in \Omega, \alpha_2 \text{ from (14)}.
\]
Let
\[
\alpha_3(s) = \begin{cases} \frac{\delta s + \epsilon}{\alpha_2(\epsilon)} \alpha_2(s), & 0 \leq s < \epsilon, \\ \delta s + s, & \epsilon \leq s. \end{cases}
\]
Then we have $\alpha_3 \in \mathcal{K}_\infty$ such that

\begin{equation}
\alpha_3(|x|) > \sigma|x|, \text{ for } x \in \Omega \setminus B(0, \epsilon),
\end{equation}

\begin{equation}
\alpha_3(|x|) \geq \alpha_2(|x|) \geq V(x), \text{ for } x \in \Omega.
\end{equation}

Define a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ with $\rho(s) = \frac{1 + \delta - \sigma}{1 + \delta - \sigma} s$ for all $s \geq 0$. Using the inequalities (42), (46) and (47), we obtain for $x \in \Omega \setminus B(0, \epsilon)$ and $u \in \mathcal{U}_R$

\begin{equation}
V(x(T, x_0, u(\cdot))) - V(x_0) \leq -\sigma\alpha_3^{-1}(V(x_0)) + r|u|_{\infty},
\end{equation}

\begin{equation}
V(x(T, x_0, u(\cdot))) \leq (\id - \sigma\alpha_3^{-1})(V(x_0)) + r|u|_{\infty}.
\end{equation}

It is clear that $\rho(\cdot) = (\id - \sigma\alpha_3^{-1})(\cdot)$ and $\rho < \id$. Thus $V$ is a finite-time ISS Lyapunov function for system (1) on $\Omega \setminus B(0, \epsilon)$.

**Lemma 3.11.** Given a constant $0 < T < +\infty$ and a small real number $\epsilon > 0$. Let a bounded and compact set $\Omega \subset \mathbb{R}^n$ with $B(0, \epsilon) \subset \Omega$ be a positively $T$-invariant set for system (1). Assume there exists a finite-time ISS Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ defined by $V(x) = \eta(|x|)$ with $\eta \in \mathcal{K}_\infty$ for system (1) on $\Omega \setminus B(0, \epsilon)$, and $V$ satisfies (15) with $T$. Let $\Omega_1 \subset \mathbb{R}^n$ be a bounded and compact set such that $\overline{B}(0, \epsilon) \subset \int \Omega_1, \Omega_1 \subset \int \Omega$. We further suppose that there exists a finite-time ISS Lyapunov function $W : \mathbb{R}^n \to \mathbb{R}_+$ defined by $W(x) = \eta_1(|x|)$ with $\eta_1$ positive definite for system (1) on $\Omega_1$, and $W$ satisfies (15) with $T$. Then system (1) is ISS in $\Omega$.

**Proof.** Under the assumption, it is obvious that there exist functions $\overline{\alpha}_1, \overline{\alpha}_2 \in \mathcal{K}_\infty$ such that

\begin{equation}
\overline{\alpha}_1(|x|) \leq V(x) \leq \overline{\alpha}_2(|x|),
\end{equation}

\begin{equation}
\overline{\alpha}_1(|x|) \leq W(x) \leq \overline{\alpha}_2(|x|).
\end{equation}

Since $V, W$ are finite-time ISS Lyapunov functions for system (1) on $\Omega \setminus B(0, \epsilon)$, $\Omega_1$, respectively, there exist positive functions $\rho_1, \rho_2$ and $\mathcal{K}_\infty$ functions $\gamma_1, \gamma_2$ such that for $V, W$ with $T$

\begin{equation}
V(x(T, x_0, u(\cdot))) \leq \rho_1(V(x_0)) + \gamma_1(|u|_{\infty})
\end{equation}

for $x_0 \in \Omega \setminus B(0, \epsilon), u \in \mathcal{U}_R$, and

\begin{equation}
W(x(T, x_0, u(\cdot))) \leq \rho_2(W(x_0)) + \gamma_2(|u|_{\infty})
\end{equation}

for $x_0 \in \Omega_1, u \in \mathcal{U}_R$.

It follows that

\begin{equation}
V(x(T, x_0, u(\cdot))) - V(x_0) \leq -(\id - \rho_1)(V(x_0)) + \gamma_1(|u|_{\infty})
\end{equation}

for $x_0 \in \Omega \setminus B(0, \epsilon), u \in \mathcal{U}_R$.

Let $Z = \max_{x \in \Omega} |x|$. It is clear that $Z$ is positive and bounded. Since $(\id - \rho_1) \in \mathcal{K}_\infty$ holds, it is easy to see that there exists a constant $0 < l < 1$ such that $(\id - \rho_1)(s) \geq ls$ for $Z \geq s \geq \epsilon$. Then we get that

\begin{equation}
V(x(T, x_0, u(\cdot))) - V(x_0) \leq -lV(x_0) + \gamma_1(|u|_{\infty})
\end{equation}

for $x_0 \in \Omega \setminus B(0, \epsilon), u \in \mathcal{U}_R$.

Define $C := \frac{V(x)}{W(x)}$ for $|x| = \epsilon$. According to the condition, it is easy to see that such a $C$ exists and $0 < C$ is satisfied. Define a function $V_1 : \mathbb{R}^n \to \mathbb{R}$ by $V_1(x) = CV(x)$, for $x \in \Omega$. For the function $V_1$, it holds that

\begin{equation}
V_1(x(T, x_0, u(\cdot))) - V_1(x_0) \leq -lV_1(x_0) + C\gamma_1(|u|_{\infty})
\end{equation}
for \( x_0 \in \Omega \setminus B(0, \varepsilon) \), \( u \in \mathcal{U}_R \). It is obvious that \( V_1(x) \) is a finite-time ISS Lyapunov function for system (1) on \( \Omega \setminus B(0, \varepsilon) \).

Define
\[
V_2(x) = \begin{cases} 
W(x), & \text{for } |x| \leq \varepsilon, \\
V_1(x), & \text{for } |x| > \varepsilon. 
\end{cases}
\]

(57)

From the above discussion, it satisfies that \( V_2 \) is positive definite and continuous. For the function \( W \), we have that
\[
W(x(T, x_0, u(\cdot))) - W(x_0) \leq -(\text{id} - \rho_2)(W(x_0)) + \gamma_2(|u|_{\infty})
\]
holds for \( x_0 \in \Omega_1 \), \( u \in \mathcal{U}_R \) and \( \text{id} - \rho_2 \in \mathcal{K}_\infty \). In the following, we prove that \( V_2 \) satisfies (15) for \( x \in \Omega \), i.e., there exist functions \( \gamma \in \mathcal{K}_\infty \) and \( \rho \) positive definite with \( \text{id} - \rho \in \mathcal{K}_\infty \) such that
\[
V_2(x(T, x_0, u(\cdot))) - V_2(x_0) \leq -(\text{id} - \rho)(V_2(x_0)) + \gamma(|u|_{\infty})
\]
for \( x_0 \in \Omega \), \( u \in \mathcal{U}_R \).

**case 1**. \((\text{id} - \rho_2)(s) \leq \varepsilon s \) for \( s = \varepsilon \). Define \( d = \min\{s \mid (\text{id} - \rho_2)(s) = \varepsilon s, Z \geq s \geq \varepsilon \} \). If \( d \) exists, then we let
\[
(\text{id} - \rho)(s) = \begin{cases} 
(\text{id} - \rho_2)(s), & \text{for } 0 \leq s \leq d, \\
\varepsilon s, & \text{for } d \leq s.
\end{cases}
\]

(60)

If \( d \) does not exist, then we define \( \text{id} - \rho \) by \((\text{id} - \rho)(s) = (\text{id} - \rho_2)(s) \) for \( s \geq 0 \).

**case 2**. \((\text{id} - \rho_2)(s) > \varepsilon s \) for \( s = \varepsilon \). Let \( y = \max\{s \mid (\text{id} - \rho_2)(s) = \varepsilon s, 0 \leq s < \varepsilon \} \). It is clear that such a \( y \) exists. Then we let
\[
(\text{id} - \rho)(s) = \begin{cases} 
(\text{id} - \rho_2)(s), & \text{for } 0 \leq s \leq y, \\
\varepsilon s, & \text{for } y \leq s.
\end{cases}
\]

(61)

We choose a function \( \gamma \in \mathcal{K}_\infty \) such that \( \gamma(s) \geq C\gamma_1(s), \gamma(s) \geq \gamma_2(s), \) for \( s \geq 0 \). It is easy to know that such a function \( \gamma \in \mathcal{K}_\infty \) exists. Based on the above analysis, it holds that
\[
V_2(x(T, x_0, u(\cdot))) \leq \rho(V_2(x_0)) + \gamma(|u|_{\infty})
\]
for \( x_0 \in \Omega \), \( u \in \mathcal{U}_R \). Therefore, the function \( V_2(x) \) is a finite-time ISS Lyapunov function for system (1) on \( \Omega \). According to Theorem 3.3, we conclude that system (1) is ISS on \( \Omega \). \( \square \)

**Remark 3.12.**

(1) Consider the linear system described by
\[
\dot{x} = Ax + Bu
\]
where \( x \in \mathbb{R}^n \), \( u \in \mathcal{U}_R \), \( A = \frac{\partial f(x, u)}{\partial x}(0, 0) \), \( B = \frac{\partial f(x, u)}{\partial u}(0, 0) \). If the eigenvalues of \( A \) are negative, then system (63) is expISS. According to Theorem 3.5, we have that the function \( W \) satisfying the conditions of Lemma 3.11 exists. For system (63), it is easy to verify that \( W(x) = |x| \) is a finite-time ISS Lyapunov function satisfying (15) with a linear function \( \rho \). Based on Theorem 3.5 andLemma 3.7, it is known that if system (1) is ISS in \( \Omega \setminus B(0, \varepsilon) \), then \( V \) satisfying the conditions of Lemma 3.11 exists. Based on Remark 2.10 (iii), if the functions \( V, W = |x| \) satisfy (15) with \( T_1, T_2 \), respectively, then \( T \) may be chosen as \( \min\{T \in \mathbb{R} \mid \frac{T}{T_1} > 0 \text{ and } \frac{T}{T_2} \in N, \frac{T}{T_2} > 0 \text{ and } \frac{T}{T_2} \in N\} \).
Propositions 3.9-3.10 show that we may compute finite-time ISS Lyapunov functions satisfying (38), (39) for system (1) on $\Omega \setminus B(0, \epsilon)$. If the constraints of Lemma 3.11 hold, then it is concluded that system (1) is ISS in $\Omega$. The computation of finite-time ISS Lyapunov functions for system (1) on $\Omega \setminus B(0, \epsilon)$ is our future research topic.

4. Examples. In this section, we present four examples to illustrate how to analyze stability of continuous-time systems with finite-time Lyapunov functions. According to the definition of finite-time ISS Lyapunov function (see Definition 2.9), in order to check if a continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ is a finite-time ISS Lyapunov function for system (1), we have to calculate $V(x(T, x_0, u(\cdot)))$ in (15) from Definition 2.9.

For a constant $0 < T < +\infty$, the value of $x(T, x_0, u(\cdot))$ of system (1) with respect to an initial condition $x_0 \in \Omega$, and an input $u \in \mathcal{U}$ is computed by Euler method. Let $h_t$ denote the time step used in the computation of the value $x(T, x_0, u(\cdot))$. In order to make the computation easy, for the following examples we use 1-norm $|\cdot|_1$ for the state and let $u_0$ denote $u(0)$ for the input.

4.1. Example 1. Consider system described by

$$\dot{x} = -x + u,$$

where $x \in \mathbb{R}, u \in \mathbb{R}$.

Let $V_1(x) = |x|$ be a finite-time ISS Lyapunov function candidate and we choose $h_t = 1/2$. Then we have that $x(1/2, x_0, u(\cdot)) = 1/2x_0 + 1/2u_0$. Thus it holds that

$$V_1(x(1/2, x_0, u(\cdot))) = |x(1/2, x_0, u)| \leq 1/2V_1(x_0) + 1/2|u|_\infty.$$

(65)

It is clearly that $V_1$ satisfies constraints of Definition 2.9. Then $V_1$ is a finite-time ISS Lyapunov function with $T = 1/2$. Based on Theorem 3.3 we conclude that system (64) is ISS. Furthermore, it is easy to verify that $V_1$ is a finite-time ISS Lyapunov functions with any $0 < T < +\infty$.

4.2. Example 2. We study the following system described by

$$\dot{x} = -x + 2xu,$$

where $x \in \mathbb{R}, u \in \mathbb{R}$.

Let

$$V_2(x) = \begin{cases} |x|, & \text{for } |x| \leq 1, \\ |x|^2, & \text{for } |x| > 1. \end{cases}$$

(67)

We assume that $V_2$ is a finite-time ISS Lyapunov function candidate and that $h_t = 1$. Then we have that $x(1, x_0, u(\cdot)) = 2x_0u_0$. Thus it holds that

$$V_2(x(1, x_0, u(\cdot))) = |x(1, x_0, u)| \leq V_2(x_0) + |u|_\infty^2.$$

(68)

Via the above inequality (68) we can not conclude that $V_2$ is a finite-time ISS Lyapunov function. However, we can verify that $V_2$ is a finite-time ISS Lyapunov function by an auxiliary system (66) described by

$$\dot{x} = -x + 2x(1/4|x|u),$$

(69)

where $|u|_\infty \leq 1$.

We assume that $V_2$ is a finite-time robust Lyapunov function candidate for system (69). Let $h_t = 1$. Then we have that for system (69) with respect to an initial
condition $x_0 \in \Omega$, and an input $u \in \mathcal{U}_1$ the value $x(1, x_0, u(\cdot)) = 1/2x_0|x_0|u_0$. Thus it holds that

$$|x(1, x_0, u(\cdot))| \leq 1/2|x_0|^2 \leq 1/2V_2(x_0).$$

Therefore $V_2$ is a finite-time robust Lyapunov function for system (69). According to Theorem 3.1, we conclude that the origin of system (69) is robustly asymptotically stable. Based on Theorem 3.1, it is concluded that system (66) is ISS and $V$ is a finite-time ISS Lyapunov function for system (66).

4.3. Example 3. Consider a nonlinear system described by

$$\dot{x} = -2x + x^3 + u,$$

(71)

where $x \in \Omega \subset \mathbb{R}$ and $\Omega = \{x \in \mathbb{R} \mid |x| < 1\}$, $u \in \mathbb{R}$ and $|u| \leq 1$.

Let $V_3(x) = |x|$ be a finite-time ISS Lyapunov function candidate. We choose $h_t = 1$. It is calculated that $x(1, x_0, u(\cdot)) = -x_0 + x_0^3 + u_0$. Then it satisfies that

$$V_3(x(1, x_0, u(\cdot))) \leq (1 - x^2)V_3(x_0) + |u|_\infty.$$  

(72)

Based on the above analysis, if a small neighborhood of the origin $B(0, \epsilon)$ with $0 < \epsilon < \max |x|$ is excluded, then $V_3$ is a finite-time ISS Lyapunov function with $T = 1$ for system (71) on the set $\{x \in \mathbb{R} \mid \epsilon \leq |x| < 1\}$.

In order to analyze ISS of system (71) in $\Omega$, we consider the linear system described by

$$\dot{x} = -2x + u.$$  

(73)

It is obvious that system (74) is expISS via the finite-time ISS Lyapunov function $V_3 = |x|$ with $T_1 = 1/4$. From the above analysis, $V_3$ is a finite-time ISS Lyapunov function for system (71) with $T = 1$ via Lemma 3.11 and Remark 3.12. Then we conclude that system is ISS on $\Omega$ via Theorem 3.3.

4.4. Example 4. Consider a two dimensions system described by

$$\begin{cases} 
\dot{x}_1 = -0.2x_2 + u, \\
\dot{x}_2 = x_1 - x_2, 
\end{cases}$$

(74)

where $x = (x_1, x_2)^T \in \mathbb{R}^2$, $u \in \mathbb{R}$.

Let $V_4(x) = |x|_1$ be a finite-time ISS Lyapunov function candidate and $h_t = 1$. In the following we calculate the values of $x(h_t, x_0, u(\cdot))$, $x(2h_t, x_0, u(\cdot))$, $\ldots$, $x(5h_t, x_0, u(\cdot))$. We use the following notations: $x_0 = (x_1(0), x_2(0))^T$, $x_{10} = x_1(0)$, $x_{20} = x_2(0)$, $x_1 = (x_{11}, x_{21})^T = x(h_t, x_0, u)^T$, $x_2 = (x_{12}, x_{22})^T = x(2h_t, x_0, u)^T$, $\ldots$, $x_5 = (x_{15}, x_{25})^T = x(5h_t, x_0, u)$, and $u_0 = u(0)$, $u_1 = u(h_t)$, $\ldots$, $u_4 = u(4h_t)$.

We calculate that

$$\begin{cases} 
x_{11} = x_{10} - 0.2x_{20} + u_0, \\
x_{21} = x_{10}, \\
\ldots, \\
x_{15} = 0.32x_{10} - 0.088x_{20} + 1.64u_0 + 1.4u_1 + 1.2u_2 + u_3 + u_4, \\
x_{25} = 0.44x_{10} - 0.12x_{20} + 1.4u_0 + 1.2u_1 + u_2 + u_3.
\end{cases}$$

(75)

Then we have that

$$|x_5|_1 \leq 0.76|x_0|_1 + 10.84|u|_\infty.$$  

(76)
Hence it holds that
\[ V_4(x(5,x_0,u)) \leq 0.76V_4(x_0) + 10.84|u|_{\infty}. \] (77)
Therefore, \( V_4(x) \) is a finite-time ISS Lyapunov function with \( T = 5 \) for system (74).
Based on Theorem 3.3, we conclude that system (74) is ISS.

**Remark 4.1.** According to Definition 2.3, it is obvious that \( V_1, V_2, V_3 \) and \( V_4 \) are not ISS Lyapunov functions for systems (64), (66), (71) and (74), respectively. However, \( V_1, V_2, V_3 \) and \( V_4 \) are finite-time ISS Lyapunov functions for systems (64), (66), (71) and (74), respectively. We conclude from results of the examples that it is comparatively easier to construct finite-time ISS Lyapunov functions than ISS Lyapunov functions. The results of Example 2 show that we may analyze ISS property of system (1) via introducing an auxiliary system and constructing a finite-time robust Lyapunov functions for the auxiliary system. Example 3 demonstrates that we may use Lemma 3.11 and Remark 3.12 to explore ISS of systems.

5. **Conclusion.** In this paper, we discussed finite-time ISS Lyapunov functions for system (1) and converse finite-time ISS Lyapunov theorem. It is proved that if there exists a finite-time ISS Lyapunov function for system (1) on a compact and positively \( T \)-invariant set \( \Omega \) with \( 0 \in \text{int} \Omega \), then system (1) is ISS on \( \Omega \) (Theorem 3.3). Conversely, if system (1) is ISS on \( \Omega \) and Assumption 1 holds, then any scaled norm is a finite-time ISS Lyapunov function for system (1) (Theorem 3.5). Theorem 3.5 and Lemma 3.7 show that if system (1) is ISS, any scaled norm is a finite-time ISS Lyapunov function for system (1) on \( \Omega \) excluding a small neighborhood of the origin. We further proved that if system (63) is expISS, and the conditions of Lemma 3.11 are satisfied, then system (1) is ISS. It becomes comparatively easier to construct ISS Lyapunov functions for continuous-time systems which is demonstrated by four examples.

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