The renormalization group is applied to the $\phi^4$ model in the symmetry broken phase in order to identify different scaling regimes. The new scaling laws reflect nonuniversal behavior at the phase transition. The extension of the analysis to finite temperature is briefly outlined. It is mentioned that the coupling constants can be found in the mixed phase by taking into account the saddle points of the blocking procedure.

1 Introduction

In the search of the quark confinement mechanism the haaron model has been proposed\cite{1} because it comprises the lesson to be learned from lattice QCD. The characteristic feature of the model is that the linearly rising potential between static color charges arises from a simple sine-Gordon type effective model after a partial resummation of the Haar-measure vertices of the path integral. This was a rather puzzling result since it was difficult to accept that the leading long range force comes from vertices which are nonrenormalizable and irrelevant. But one can easily find the explication of this apparent paradox: On the one hand, the renormalizability stands for the relevant or marginal behavior of the operators in the ultraviolet scaling regime. On the other hand, the confining forces are observed beyond this scaling regime, where new scaling laws arise at the low energies. There is no reason to expect that the relevant operator set agree for the ultraviolet and the infrared scaling regime. Thus one is left with a more general question whether the set of relevant operators in a model may differ in different scaling regimes and what consequences such a phenomenon might have.

When the mass scale is explicitly given in the lagrangian as in the massive
$\phi^4$ model then the simple perturbation expansion is sufficient to study the different scaling regimes. There are cases when a partial resummation of the perturbation expansion is needed to generate the mass gap, like for photons at finite temperature. In these cases where perturbation expansion applies there is no transmutation of the degrees of freedom, i.e. one finds the same particles at every energy and each fixed point is Gaussian. When the low energy scaling law is accompanied by the appearance of new particles, such as in the two dimensional Gross-Neveu model or in QCD then new relevant operators are expected which are responsible for the formation of the new composite particles and the low energy fixed point is not Gaussian anymore. The generic mechanism for the nonperturbative modification of the scaling law and the generation of new relevant operators is the condensation. This phenomenon may occur either at the low or at the high energy scaling regimes. Since the mass generation is usually achieved in High Energy Physics by the help of the condensates one may find similar complication in the unified models, as well, despite their perturbative appearance. The present contribution is a brief summary of some results obtained in this direction.

We shall first argue in Section 2 that the appearance of several fixed points and scaling regimes is a rule rather than an exception in High Energy Physics. The powerful Wegner-Haughton form of the renormalization group equation is introduced in Section 3 as the method to tackle our problem in the symmetry broken phase of the $\phi^4$ model. A diverging and a focusing effect of the renormalization group flow is discussed in Section 4. Section 5 is a brief digression into the structure of the mixed phase. The generalization of our results to finite temperature is the subject of Section 6. Finally Section 7 is for the summary.

2 Multiple Fixed Points

The models with intrinsic mass scales possess at least two distinct scaling regions, one at the ultraviolet and another one at the infrared side of the mass scale. The infrared scaling is usually called trivial because it can be proven that there is no non-Gaussian relevant or marginal operator. In fact, for models with finite correlation length the fluctuations are exponentially suppressed at large distances and the evolution of the running coupling constants slows down in the infrared limit. The manifold of the possible attractive infrared fixed points is parametrized by the initial values of the relevant coupling constants of the ultraviolet scaling regime given at the scale of the ultraviolet cutoff. When the excitation spectrum has no gap above the vacuum or there is an instability then the long range interactions might be so strong as to change this simple situation. The result is that divergences might pile up and drive
Figure 1: The renormalized trajectory of the Theory of Everything (TOE) starts at the (supposed) ultraviolet fixed point at $k = \Lambda$, it passes by the fixed points of the Grand Unified Models (GUT), the unified Electro-Weak theory (EW), the strong interactions (QCD), the electromagnetic interactions (QED) certain fixed points of the Solid State and Condensed Matter Physics (CM) and finally approaches the ultimate IR fixed point, $k \to 0$. The circles denote the domains of linearizability, the asymptotic scaling regions.

The presence of several scaling regimes is easily recognizable at the Theory of Everything. Whatever theory will proven to be that, its renormalized trajectory should be traced down in a space which contains all coupling constants what is used in physics at finite energies. From the coupling constants of possible composite models at so far unexplored high energies through the parameters of the Standard Model down to the coupling constants in Solid State Physics one includes several axes in this space. On the renormalized trajectory depicted in fig. 1 one observes the scaling laws characteristic of different interactions in the energy regime where the trajectory is in the linearizability region of a fixed point. Note that the trajectory may be influenced by the environment and bifurcates into different thermodynamical phases in the IR regime in different laboratories.

It is mathematically certainly correct to say that the renormalized trajectory, the set of the physical "constants", is given by the initial condition at the TOE. In this respect the High Energy physicist seeks the few ultimate constants of Nature. But the same algebra of observables is classified at each scaling region according to the actual scaling laws and there is no guarantee that the set of relevant and marginal operators is found to be always the same.
The renormalizable (relevant or marginal) operators of a scaling regime are usually nonrenormalizable (irrelevant) at the higher energy fixed points such as the quark-gluon QCD vertex appears as a nonrenormalizable one in a composite model for the quarks and gluons. Consider now a coupling constant, denoted by $g_{n-r}(k)$ what is nonrenormalizable at high energy and becomes relevant at a lower energy scaling regime. Then this coupling constant undergoes a suppression at the high energy scaling regime. How can it became a relevant coupling constant at the low energy scaling regime? What is its role during the evolution? According to the usual scenario the initial value of the irrelevant coupling constant, $g_{n-r}(\Lambda)$, modifies the theory in an overall scale and the relation between observables of the same dimension is given by the initial value of the renormalizable coupling constants only. This is expressed by the condition for the beta function for any coupling constant $g$ as

$$\lim_{\Lambda/k \to \infty} \frac{\partial \beta_g(k)}{\partial g_{n-r}(\Lambda)} = \lim_{\Lambda/k \to \infty} k \frac{\partial^2 g(k)}{\partial k \partial g_{n-r}(\Lambda)} = 0,$$

where the coupling constants are made dimensionless by the help of the running cutoff, $k$.

One can imagine the following two, opposite possibilities as far as the coupling constant $g_{n-r}$ is concerned:

1. **Divergence:** The universality, eq. (1) is violated because the trajectories with slightly differing initial conditions for $g_{n-r}$ diverge from each other. The initial value of $g_{n-r}$ must then be specified at the TOE and it becomes an independent free parameter.

2. **Focusing:** The dimensionless quantities at low energy are (at least locally) independent of the initial values of the coupling constants of the TOE and are determined by one of the lower energy fixed points. This is a strong version of the universality because the coupling constants at low energy are independent of the initial values of the renormalizable coupling constants.

The existence of different scaling regimes may lead to serious problems in indentifying the important parameters of the theory. In both cases mentioned above the determination of the physical content of the theory at different energies in terms of the initial values of the relevant coupling constants of the TOE is, though being mathematically possible, unfeasible by means of measurements with small but finite errors.

It is instructive to consider a simpler model with two scaling regimes whose generic scaling patterns are listed in table 1. Consider the theory of photons,
Table 1: The four classes of the coupling constants in QED.

| U.V.  | I.R.       | Fig. 2 | example                |
|-------|------------|--------|------------------------|
| relevant | relevant | (a)    | $m_e \bar{\psi}_e \psi_e$ |
| relevant | irrelevant | (b)    | $m_\mu \bar{\psi}_\mu \psi_\mu$ |
| irrelevant | relevant | (c)    | $G_4 (\bar{\psi}_e \psi_e)^2$ |
| irrelevant | irrelevant | (d)    | $G_6 (\bar{\psi}_e \psi_e)^3$ |

Figure 2: The qualitative dependence of the running coupling constants in the function of the cutoff, $a = 2\pi/\Lambda$. The UV and the IR scaling regimes are shown. The coupling constant is supposed to be constant in between for simplicity. See Table 1 for the details.
electrons and nuclei in the presence of a chemical potential for the baryon number. For certain values of the chemical potential the ground state is a solid state lattice. We identify two scaling regimes, the ultraviolet one characteristic of QED in the trivial, homogeneous vacuum and an infrared one which is governed by the lattice effects. The electron mass, $m_e$, is a relevant parameter in each scaling regime since it is a renormalizable coupling constant of QED and appears in the equations of Solid State Physics. The muon mass, $m_\mu$, is as renormalizable parameter at high energy as $m_e$ but drops out from the physics of the solids because the processes (without neutrinos) at the energies of eV the muon contributions are always dominated by the contributions from the electron. The six-electron vertex is irrelevant everywhere. The four electron vertex with the coupling constant $G_4$ is the most interesting interaction. It is a nonrenormalizable, irrelevant vertex of QED. But it becomes relevant at low energies since it is the effective vertex which is generated by the attractive force between the Cooper-pairs and drives the transition into the superconducting state. In the simple perturbative treatment of QED $G_4$ is suppressed in the ultraviolet scaling regime so its high energy initial value in the QED lagrangian is set to zero. It was only after the experimental discovery of the superconducting phase and its explication by the BCS ground state that the importance of $G_4$ at lower energies was demonstrated by means of the partial resummation of the perturbation expansion. This is an example where the renormalization group was used to find a coupling constant what appears to be unnecessary at high energies but nevertheless is important at low energy. Such parameters will be called hidden coupling constants after their undetectable small values at intermediate energies.

The divergence, case 1 mentioned above, requires that the suppression of $g_{n-r}$ in the ultraviolet regime is weaker than the amplification during the low energy scaling. If the fixed point of the infrared scaling has a certain region of the ultraviolet coupling constants in its attractive zone then focusing, the case 2, is realized. We present here a study of the four dimensional $\phi^4$ model in search for the manifestations of these phenomena.

3 Wegner-Haughton Equations

In order to study the role of nonrenormalizable operators on the evolution one has to be able to handle the mixing of a large number of operators during the renormalization group transformation. This is achieved in an elegant manner by the Wegner-Haughton equation[1]. This is a functional differential equation describing the evolution of the renormalized, blocked bare action under the change of the cutoff. According to the usual strategy the running cou-
pling constants are identified with the bare ones and the observational scale with the cutoff so the evolution of the bare coupling constants qualitatively reproduces the trajectory for the running coupling constants. We shall be satisfied here to indicate the derivation of the leading order equation in the gradient expansion, the preliminary results indicate that the higher orders do not change our conclusions.

Let us write the action corresponding to the ultraviolet cutoff \( k \) as

\[
S_k[\phi] = \int d^d x \left[ \frac{1}{2} (\partial_\mu \phi(x))^2 + U_k(\phi(x)) \right].
\]

According to the usual Wilson-Kadanoff blocking procedure we write

\[
e^{-\frac{1}{\hbar} S_{k'}[\phi]} = \int D[\phi'] e^{-\frac{1}{\hbar} S_k[\phi + \phi']}
\]

where \( k' < k \) in the Euclidean space-time. The Fourier transform of the field variable \( \phi \) and \( \phi' \) are nonvanishing for \( p < k' \) and \( k' < p < k \), respectively. The right hand side is evaluated by means of the loop expansion,

\[
S_{k'}[\phi] = S_k[\phi + \phi'_0] + \frac{\hbar}{2} \text{tr} \log \delta^2 S + O(\hbar^2),
\]

where

\[
\delta^2 S(x, y) = \frac{\delta^2 S_k[\phi + \phi'_0]}{\delta \phi'(x) \delta \phi'(y)},
\]

and the saddle point, \( \phi'_0 \), is defined by the extremum condition

\[
\frac{\delta S_k[\phi + \phi'_0]}{\delta \phi'(p)} = 0,
\]

in which the infrared background field, \( \phi(x) \), is held fixed. Eq. (4) is the generalization of the Wegner-Haughton equation for condensates. One can prove that the saddle point is trivial, \( \phi'_0 = 0 \), so long as the matrix \( \delta^2 S(x, y) \) is invertible and the infrared background field is homogeneous, \( \phi(x) = \Phi \). What is remarkable in this equation is that each successive loop integral brings a suppression factor

\[
\frac{k^d - k'^d}{k'^d} = O \left( \frac{k - k'}{k'} \right)
\]

due to the integration volume in the momentum space. By choosing an infinitesimal fraction of the degrees of freedom to be eliminated in a step we
find a new small parameter, $\delta k/k' = k - k'/k'$, suppressing the higher loop contributions in the blocking relation.

In the leading order of the gradient expansion, the so-called local potential approximation, the only function characterizing the action is the local potential, $U_k(\phi)$, so we choose the infrared background field homogeneous, $\phi(x) = \Phi$, and obtain from (1)

$$
U_{k-\delta k}(\Phi) = U_k(\Phi) + \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \log \left[ p^2 + U_k''(\Phi) \right] + O(\delta k^2),
$$

(8)

where we introduced the notation

$$
U_k''(\Phi) = \frac{\partial^2 U_k(\Phi)}{\partial \Phi^2}.
$$

(9)

In the limit $\delta k \to 0$ one ends up with the differential equation

$$
k \frac{\partial}{\partial k} U_k(\Phi) = - \frac{\Omega_d k^d}{2(2\pi)^d} \log \left[ k^2 + U_k''(\Phi) \right]
$$

(10)

with

$$
\Omega_d = \frac{\Gamma \left( \frac{d}{2} \right)}{2^d \pi^{d/2}},
$$

(11)

the projection of the functional equation (1) onto the functional form (2). It is instructive to expand this equation in $U_k''(\Phi) - m_k^2$, where

$$
m_k^2 = U''(0),
$$

(12)

when we find

$$
k \frac{\partial}{\partial k} U_k(\Phi) = - \frac{\Omega_d k^d}{2(2\pi)^d} \left[ \log(k^2 + m_k^2) + \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{U_k''(\Phi) - m_k^2}{k^2 + m_k^2} \right)^n \right].
$$

(13)

One recovers here the usual one loop resummation of the effective potential except that the loop momentum is now restricted into the subspace of the modes to be eliminated. Note that the derivation of eq. (10) shows that the restoring force for the fluctuations into the equilibrium is proportional to the argument of the logarithm function. Thus when

$$
k^2 + U_k''(\Phi) = 0
$$

(14)

then nontrivial saddle point should be used. The original equation, (1), remains always valid because the action is bounded from below.
We define the coupling constant and the beta function to the \( n \)-th order vertex for the vacuum \( \phi(x) = \Phi \) as

\[
 g_n(k) = \frac{\partial^n}{\partial \Phi^n} U_k(\Phi),
\]

\[
 \beta_n = k \frac{\partial}{\partial k} g_n(k) = \frac{\partial^n}{\partial \Phi^n} k \frac{\partial}{\partial k} U_k(\Phi),
\]

where in the last equation we assumed the analyticity of the potential in \( k \) and \( \Phi \), what holds except at the singular points. By taking the successive derivatives of the renormalization group equation (10) one obtains

\[
 \beta_n = -\frac{\Omega d k^d}{2(2\pi)^d} \mathcal{P}_n(G_2, \cdots, G_{n+2}),
\]

where

\[
 G_n = -\frac{g_n}{k^2 + g_2}
\]

and

\[
 \mathcal{P}_n = \frac{\partial^n}{\partial \Phi^n} \log \left[ k^2 + U_k''(\Phi) \right]
\]

is a polynomial of order \( n/2 \) in the variables \( G_j, j = 2, \cdots, n+2 \),

\[
 \begin{align*}
 \beta_1 &= G_3, \\
 \beta_2 &= G_4 - G_3^2, \\
 \beta_3 &= G_5 - 3G_3G_4 + 2G_3^3, \\
 \beta_4 &= G_6 - 4G_5G_3 - 3G_4^2 + 12G_3^2G_4 - 6G_3^4, \\
 \beta_5 &= G_7 - 5G_6G_3 - 4G_5G_4 + 20G_3^2G_5 - 6G_4^2G_3 + 30G_3^2G_5 - 60G_3^3G_4 + 24G_3^5.
\end{align*}
\]

etc. It is interesting to verify that \( \mathcal{P}_n \) contains the integrand of all one loop graphs that contribute to \( \beta_n \).

We change now to dimensionless parameters,

\[
 k \to \frac{k}{\Lambda}, \quad U \to k^d U, \quad \phi \to k^{\frac{d}{2} - 1} \phi, \quad g_n \to k^{n(1 - \frac{d}{2}) - d} g_n,
\]

and

\[
 G_n \to \frac{g_n}{1 + g_2},
\]

what will be used below.
One can distinguish an ultraviolet and an infrared scaling regime, for 
$k^2 \gg |m_k^2|$ and for $k^2 \ll |m_k^2|$, respectively. In the former one recovers the
usual renormalization group coefficient functions used in studying the asymptotic scaling. The latter is trivial for $m_0^2 > 0$ when the factor $k^d$ suppresses the evolution at the infrared fixed point. The infrared scaling law is presumable trivial in the case $m^2(0) = 0$ because the vacuum of this model is supposed to be\footnote{It could naively be identified with $m^2(0) < 0$ but we should not forget the Maxwell construction what sets $m^2(0) = 0.$} at $\Phi \neq 0$ where the fluctuations become massive again.

The symmetry broken phase\footnote{another distinguishing feature of the blocking procedure employed in this work is that the momentum space cutoff is imposed in a sharp manner. The smooth cutoff is believed to be superior and its use is more widespread. We shall argue below that the problems with the sharp cutoff are not incurable with careful methods and that actually no smooth cutoff is applicable for models} is characterized by the condition that there is a scale, $k_{cr}(\Phi)$, where the spinodal instability occurs and the restoring force of the fluctuations is vanishing,

$$k_{cr}^2(\Phi) = -U''_{k_{cr}}(\Phi)$$

in a region around $\Phi = 0$. We shall call the line $k_{cr}^2(\Phi)$ on the plane $(\Phi, k^2)$ critical. The renormalized trajectory has discontinuous derivatives along the curve\footnote{It could naively be identified with $m^2(0) < 0$ but we should not forget the Maxwell construction what sets $m^2(0) = 0.$} and the system is in a mixed phase for $k^2 < k_{cr}^2(\Phi)$. The tree level instability induces nontrivial saddle points for the blocking\footnote{another distinguishing feature of the blocking procedure employed in this work is that the momentum space cutoff is imposed in a sharp manner. The smooth cutoff is believed to be superior and its use is more widespread. We shall argue below that the problems with the sharp cutoff are not incurable with careful methods and that actually no smooth cutoff is applicable for models} and eq. (10), being based on the vanishing of the saddle point, is no longer valid. As the critical curve is approached in decreasing $k$ during the blocking the denominator $1 + g_2(k)$ becomes small in the beta functions and new scaling laws develop as a precursor of the mixed phase. There is no reason to expect that these new scaling laws should be trivial. We shall study the renormalization group flow in the vicinity and below the critical line.

One can nowadays experience a renaissance of the infinitesimal renormalization group methods. The so called exact renormalization group\footnote{another distinguishing feature of the blocking procedure employed in this work is that the momentum space cutoff is imposed in a sharp manner. The smooth cutoff is believed to be superior and its use is more widespread. We shall argue below that the problems with the sharp cutoff are not incurable with careful methods and that actually no smooth cutoff is applicable for models} is similar in spirit to the Wegner-Haughton equation but it follows the evolution of the generator functional for the connected or the 1PI vertices. The advantage of this method is that it produces directly the particle physics motivated running coupling constants which are based on the scattering amplitudes. The bare renormalization group, (3), yields simpler expressions for the evolution of the bare coupling constants of the action. We found this latter method more attractive since, as mentioned at the beginning of this section, the evolution of the bare and the running, renormalized coupling constants is qualitatively similar.

Another distinguishing feature of the blocking procedure employed in this work is that the momentum space cutoff is imposed in a sharp manner. The smooth cutoff is believed to be superior and its use is more widespread. We shall argue below that the problems with the sharp cutoff are not incurable with careful methods and that actually no smooth cutoff is applicable for models
with instabilities. The sharp cutoff induces diffraction integrals during the blocking what represent oscillating forces at large distances. The oscillations cast doubt on the physical significance of the running parameters of the blocked action and were blamed for the occurrence of the infrared singularities in the renormalized trajectory. Attempts to eliminate the oscillating part from the blocked action lead to the introduction of a smooth cutoff. Let us deal with the cases where the singularity occurs at $k_{cr}^2 = 0$ and $k_{cr}^2 > 0$ separately and consider a physical quantity obtained in the framework of the loop expansion,

$$P(\epsilon, k) = \sum_{n=1}^{\infty} \hbar^n I_n(\epsilon, k), \quad (23)$$

where $I_n(\epsilon, k)$ stands for the $n$-th order loop integral with the range of integration $\epsilon \leq |p_j| \leq k, j = 1, \ldots, n$. In the case of the infrared unstable theories it is necessary to introduce the infrared cutoff, $\epsilon$, what is removed after the computation is completed. The blocking transformation with sharp cutoff can be used to obtain the right hand side of the differential equation

$$\frac{\partial P}{\partial \epsilon} = B(\epsilon). \quad (24)$$

The integration of this equation yields the quantity sought when $\epsilon \to 0$. So long as the nontrivial vacuum, e.g. the condensate, what shields the infrared divergences is properly incorporated in the computation the thermodynamical limit is well defined and the $\epsilon$-dependence is continuous at $\epsilon = 0$. Moreover any infrared cutoff should yield the same thermodynamical limit. When the singularity occurs at finite scale then the quantity $P(0, k)$ can be thought as it had been obtained in the effective theory with a sharp ultraviolet cutoff at $\Lambda = k$. The bare parameters of this effective theory have singular cutoff dependence at $k = k_{cr}$. One might argue that this singularity is an artifact of the cutoff employed because the observables in the effective theory, being renormalization group invariant, show no singular $k$-dependence. But we see no conceptual problem with singular, i.e. nondifferentiable renormalized trajectories if the singularity corresponds to a real physical effect, i.e. some instability what shows up at a well defined, sharp value of the momentum. One should simply make sure that the physical effects behind the singularity have properly been accounted for during the solution of the effective theory or the continuation of the blocking procedure.

The argument prohibiting the application of any smooth cutoff in theories with condensate or other instabilities is the following. In order to show the exactness of the infinitesimal renormalization schemes one has to assume first
the validity of the loop expansion. In fact, the counting of the power of δk is
done in the loop expansion where one has to integrate around the saddle point.
Without the proper choice of the saddle point the formal steps in arguing about
the exactness are not valid. The point is that the subsequent elimination of
the degrees of freedom modifies the integrand for the unstable mode. Due to
the factor $\bar{h}^{-1}$ coming from the nontrivial saddle point the loop corrections
of the stable modes what are computed after the elimination of the unstable
mode yields the contribution $O(h^0)$. Thus all stable modes should completely
be eliminated before one arrives at the unstable sector of the theory. The
models with condensate require a blocking method where the stable modes
are eliminated completely before arriving at the instability and the tree level
structure of the saddle point expansion must be retained. Finally we note that
this type of cutoff poses no problem in going to higher orders in the gradient
expansion.

4 Zooming into a fixed point

Our goal is to follow the evolution of the blocked action from the initial condi-
tion set at $k = \Lambda = 1$ towards the infrared regime. We shall consider the one
component four dimensional $\phi^4$ model in the symmetry broken phase. Thus
the integration of the differential equation (10) in $k$ runs into a singularity
for $\Phi = 0$ at $k = k_{cr}(0) > 0$. When the evolution in the outer, stable re-
region is approximated by the tree level expression, $U_k(\Phi) = U_\Lambda(\Phi)$, then
the singular line, $k_{cr}^2(\Phi)$, is an upside down parabola, $k_{cr}^2 = -m^2_\Lambda - g_4(\Lambda)\Phi^2/2$,
on the plane $(\Phi, k^2)$. But this is an inconsistent approximation for the tree
level saddle point structure is nontrivial in the unstable region for $k < k_{cr}(\Phi)$.
Since the input for the elimination of a mode at $k$ is the potential $U_{k+\delta k}(\Phi)$
for $-\infty < \Phi < \infty$ the result of the naive blocking what does not take into
account the nontrivial saddle point structure is built on the wrong potential
and is incomplete by a term $O(h)$ even in the outer, stable region. We shall
avoid the problem of the singularity by assuming that the potential $U_k(\Phi)$ is
analytical inside and outside of the unstable region and its only nonanalytical
behavior is confined on the curve $k_{cr}^2(\Phi)$. This assumption allows us to study
the evolution equation locally in $\Phi$, i.e. by expanding the potential as

$$U_k(\Phi) = \sum_{n=0}^{N} \frac{1}{n!} g_n(\Phi_0)(\Phi - \Phi_0)^n$$

and following the evolution of the coupling constants $g_n(\Phi_0)$ what obeys the
loop expansion with the trivial saddle point in the outer, stable region.
We note that the Taylor expansion of the potential motivated here by the decoupling of the stable and the unstable region proved to be necessary in the numerical integration of eq. (10). The numerical integration of this equation by simply discretizing the variables \( \Phi \) and \( k \) becomes highly unstable at \( k^2 \approx k^2_{cr}(0) \) because the logarithm function amplifies the numerical errors in computing \( U_k''(\Phi) \). The smoothing or interpolating techniques we tried were unable to stabilize the solution. Thus we finally integrated numerically the coupled differential equations imposed at \( \Phi_0 = 0 \),

\[
k \frac{\partial g_n}{\partial k} = -(n - 4)g_n - \frac{k^4}{16\pi^2} P_n(G_2, \cdots, G_{n+2}),
\]

for \( n = 1, \cdots, N \).

We can present here only some preliminary numerical results, the detailed account will be given elsewhere. They suggest the existence of two distinct scaling regimes, an ultraviolet, \( k^2 \gg k^2_{cr}(0) \), and another one in the vicinity of the singular line. Though the singular line is not a fixed point since \( k \neq 0 \) or \( \infty \) nevertheless the singularities suggest to parametrize the renormalized trajectory by

\[
k^2 = \begin{cases} 
 k^2 - k^2_{cr}(0) & \text{if } k^2 > k^2_{cr}(0), \\
 0 & \text{if } k^2 < k^2_{cr}(0). 
\end{cases}
\]

This parametrization possess scaling region at at \( \tilde{k} = 0 \) and \( \infty \). Furthermore we shall argue below that the potential is renormalization group invariant in the interior, unstable region. In this manner the whole unstable region represents a single fixed point. Note that though the critical point belongs to finite scale, \( k = k_{cr}(0) \neq 0 \), the singularities presented below require a dense enough spectrum for the momentum operator, the execution of the thermodynamical limit.

The result of the numerical integration indicates an attractive fixed point at \( k = k_{cr}(0) \) for \( N = 10 \), i.e. by truncating the potential at \( O(\Phi^{20}) \). The value of \( \delta k \) was adjusted during the integration in the range \( 10^{-18} < \delta k < 10^{-2} \). We compared the third and the fourth order Runge-Kutta approximation for the coupling constants and \( \delta k \) was chosen to keep the relative local error on them less than \( 10^{-15} \). After the system leaves the ultraviolet scaling regime the higher order coupling constants undergo oscillations with large amplitude and a new scaling law is found in the vicinity of the instability. For example \( g_{20} \) reaches the range of \( 10^{-12} \) for \( k \approx k_{cr}(0) \) after having gone through the peaks at \( 10^{23} \). The fixed point at \( \tilde{k} = 0 \) corresponds to the potential

\[
U_{k_{cr}(0)}(\Phi) = -\frac{1}{2} k^2_{cr}(0) \Phi^2.
\]
The infrared fixed point is trivial and its attractive zone seemed to extend over the whole symmetry broken phase of the renormalizable $\phi^4$ model. The approach to the fixed point is such that $G_n \to 0$, $n > 2$ as $k^2 \to k_{cr}^2(0)$. This is consistent with the observation that the fourth equation of (19) excludes any finite, nonzero value for $g_4$ at the critical point what belongs to the trivial root of $P_n$, $n > 2$. Thus the scaling at this critical point shows strong universality, its result does not depend even on the renormalizable coupling constants. The symmetry broken $\phi^4$ model when its potential is truncated at $O(\phi^{20})$ realizes an example of the case 2 mentioned in Section 2.

When the potential is truncated at $O(\Phi^{22})$ or at higher order the qualitative behavior of the solution is different. At a certain point the trajectory with $N \geq 11$ suddenly turns away from the solution with $N \leq 10$ and the coupling constants start to diverge violently towards $-\infty$. The qualitative behavior seen on fig. 3 remains the same except the singularity becomes stronger for $N > 11$. The sudden departure of the trajectories with $N = 10$ and 11 can be traced back to the contribution of $g_{22}$ to $\beta_{20}$ which happens to be exceedingly large at $k \approx k_{cr}(0)$ and starts to push the lower order coupling constants one after the other towards $-\infty$. The integration of such a singular curve requires extreme numerical accuracy. The quadruple precision was used in the codes and the relative precision of the beta functions was kept below $10^{-11}$. The values of $\delta k$ were between $10^{-17}$ and $10^{-13}$ at $k \approx k_{cr}(0)$.

It is interesting to observe the derivative of few beta functions with respect to the initial value of a nonrenormalizable coupling constant, $g_6(\Lambda)$,

$$\frac{\partial \beta_n(k)}{\partial g_6(\Lambda)}$$

what is presented in fig. 3. The derivatives are small in the ultraviolet scaling regime and diverge at the instability. This divergence shows the presence of a strong amplification mechanism due to the instability what lets the small details of the microscopic interaction felt in the vicinity of $k = k_{cr}(0)$. By assuming the singular behavior $\tilde{k}^{-\nu_n}$ for (22) as far as the $k$ dependence is concerned at $k \approx k_{cr}(0)$ and the perturbative $O(\Lambda^2)$ suppression mechanism in the ultraviolet scaling regime for $g_6$, one arrives at

$$\frac{\partial \beta_n(k)}{\partial g_6(\Lambda)} \approx \tilde{k}^{-\nu_n} \Lambda^{-2},$$

what reveals a sensitivity of the dynamics at the scale $k$ to the values of the nonrenormalizable parameters at

$$\Lambda \approx \tilde{k}^{-\nu_n/2}.$$
Figure 3: The evolution of $\log |g_n|$ for some values of $n$ at $N = 11$, with the initial conditions $g_2(\Lambda) = -0.1 \Lambda^2$, $g_4(\Lambda) = 0.01$, $g_6(\Lambda) = 10^{-4} \Lambda^{-2}$ and $g_j(\Lambda) = 0.0$ for $j = 4, \ldots, 11$. Where $g_n(k)$ changes sign during the oscillations for $n > 4$ there log $|g_n(k)|$ shows a cusp whose altitude is finite due to the finite resolution of the $k$ values on the plot. The points shown in the more detailed curves are separated by $10^3$ iterations.

We note that there is no difficulty to extend the analysis by including the complete $O(\partial^2)$ order in the gradient expansion. The preliminary numerical results show no qualitative modification of the behavior presented above. The divergence of the coupling constants defined by expanding $U_k(\Phi)$ and $Z_k(\Phi)$ increases with the order of $\Phi$ what suggests that the relevant operator of this scaling regime might be nonlocal.

We close this Section by few remarks about the interior, unstable region. The problem in this region is related to the appearance of the saddle points at the blocking transformation what correspond to plane waves and make the renormalization group step very involved. Apart from the technical complications due to the inhomogeneous background field a conceptual problem arises, namely the renormalization at the tree level. This happens when there is a saddle point with nonvanishing length scale what gives an $O(h^0)$ contribution to the blocking relations. The novel feature of the tree level renormalization is that it does not fit into the usual classification scheme of the coupling constants what is based on the inspection of the loop integrals. The classical differential equations may produce richer and more singular dependence in the coupling constants than the polynomials of the perturbation expansion. This actually happens in the unstable region and the resulting blocking relation offers no guarantee that change in the blocked action will be small, infinitesimal if the the cutoff is decreased in a small, infinitesimal amount. By assuming that the renormalized trajectory, i.e. the path integral possesses finite derivative with respect the cutoff within the unstable region we have an additional con-
Figure 4: The evolution of $\log |\frac{\partial \beta_n(b)}{\partial \theta_0(\Lambda)}|$ with $N = 11$. 

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sistency relations. It simplifies the problem and the important saddle point contributions to the blocking can be resummed with the result

\[ U_k(\Phi) = -\frac{1}{2} k^2 \Phi^2 + c(k), \quad (32) \]

where \( c(k) \) is chosen to have continuous potential at the singular curve \( k^2 = k_{cr}^2(\Phi) \). Observe that this potential is indeed a "fixed point", i.e. stays invariant under blocking. For each plane wave saddle point there is a zero mode what is related to the translation in the direction of the wave vector. The integration over the zero modes finally restores the translation invariance.

Another, independent support of this form comes from a simple exact relation given in terms of the variables in their original, dimensional form

\[
\frac{\partial V_k(\Phi)}{\partial k} \frac{\partial W_k(\Phi)}{\partial \Phi} = \frac{\partial W_k(\Phi)}{\partial k} \frac{\partial V_k(\Phi)}{\partial \Phi}, \quad (33)
\]

where

\[
V_k(\Phi) = k^{1-d} \frac{\partial U_k(\Phi)}{\partial k},
\]

\[
W_k(\Phi) = k^2 + U_k''(\Phi). \quad (34)
\]

It was derived by using the gradient expansion only, without making reference to the loop expansion. This is a higher order differential equation which has spurious solutions but it is at least satisfied by (32). We believe that the simplicity of our result, \( (32) \), originates from the same balance between the energy and entropy as in the case of the Maxwell construction except that the role of the domains is played by the zero modes of each saddle point plane wave with \( k < k_{cr}(0) \).

5 Finite Temperature

The vacuum of the symmetry broken theory is in the stable region so the singularities of the coupling constants studied in this work are not obviously important for the fluctuations around the vacuum. But the singularity reappear when the symmetry is broken by controlling external environment variables, such as the temperature. Imagine the cooling of a ferromagnet slightly above the Curie point or the hot Universe before the spontaneous breakdown of a global symmetry group. The order parameter is at its symmetrical value \( \Phi = 0 \) in the high temperature phase and the fluctuations are characterized by the coupling constants at \( \Phi = 0 \) what diverge and reflect the nonuniversal behavior investigated above. When the system arrives at the symmetry broken
phase then the order parameter remains close to the minimum of the effective potential what is outside of the spinodial unstable region so long as the time evolution is adiabatic. When \( T << T_{cr} \) the vacuum is far from the unstable region what is detected by the large amplitude fluctuations only. Thus the saddle points of the inner, unstable region influence the evolution around \( T \approx T_{cr} \), new scaling law can be observed in the vicinity of the phase transition.

To outline the procedure at finite temperature we note that the blocking should be made in three space only in order to preserve the value of the physical temperature on the renormalized trajectory. Such an anisotrop blocking in the Euclidean space-time where the cutoff \( k \) refers to the three space and the fluctuations in time are eliminated at the one loop level yields the equation:

\[
k \frac{\partial}{\partial k} U_k(\Phi) = -\frac{T \Omega_0 k^3}{2(2\pi)^3} \sum_{n=-\infty}^{\infty} \log \left[ \omega_n^2 + k^2 + U''_k(\Phi) \right]
\]

\[
= -\frac{T \Omega_0 k^3}{2(2\pi)^3} \left\{ \frac{1}{T} \sqrt{k^2 + U''_k(\Phi)} + 2 \log \left[ 1 - e^{-\frac{1}{T} \sqrt{k^2 + U''_k(\Phi)}} \right] \right\},
\]

where \( \omega_n = 2\pi n T \). Another simpler strategy is to eliminate the \( |n| > n_0 \) Matsubara modes perturbatively and to apply the blocking for \( |n| \leq n_0 \) only. The beta functions derived by either method diverge in the vacuum at \( \Phi = k = 0 \) as \( T \to T_{cr} \) from above and one expects a singular structure to appear what is similar to what was presented above in four dimensions at \( k = k_{cr}(0) \).

## 6 Summary

We presented evidences that radically different scaling laws are present in the \( \phi^4 \) model with spontaneous symmetry breaking at high and low energies. The system appears nonuniversal at the phase transition where the influence of the nonrenormalizable coupling constants what is suppressed at the ultraviolet scaling regime can be compensated for by the singular tree level structure of the condensate formation. It was mentioned that the renormalization group method can successfully be applied to the spinodial instability, the mixed phase. The saddle points of the blocking procedure which are plane waves can be taken into account and the integration over the zero modes restores the homogeneity of the mixed phase, what is the reminiscent of the Maxwell construction.

It is worthwhile noting that as one increases the number of terms retained in the potential then the qualitative behavior changes completely when the potential was retained up to \( O(\Phi^{22}) \). We believe that the divergences occurring in the gradient expansion suggests the presence of nonlocal relevant operators at the low energy scaling regime.
There are several questions left open by these results. What we find the most pressing is the classification of the possible nonrenormalizable parameters what influences the seemingly nonuniversal dynamics of the phase transition. A related issue is the more useful application of the amplification mechanism of the divergences generated by the instability as a "renormalization group microscope" in the coupling constant space to discover the microscopic parameters from the long distance observables.

1. K. Johnsson, L. Lellouch, J. Polonyi, *Nucl. Phys.* B **367**, 675 (1991).
2. S.B. Liao, J. Polonyi, *Phys. Rev.* D **51**, 4474 (1995).
3. V. Branchina, H. Mohrbach, J. Polonyi, *The Antiferromagnetic $\phi^4$ model, I., II.*, submitted to *Nucl. Phys.*
4. J. Polonyi, "Non-Perturbative External Field Effects in QED", in *Vacuum Structure in Intense Fields*, H. Fried, B. Muller eds. Plenum Press, 1990.
5. F. J. Wegner, A. Haughton, *Phys. Rev.* A **8**, 40 (1973).
6. S. B. Liao, J. Polonyi, *Ann. Phys.* **222**, 122 (1993).
7. K. Wilson and J. Kogut, *Phys. Rep.* C **12**, 75 (1974); K. Wilson, *Rev. Mod. Phys.* **47**, 773 (1975).
8. A. Hasenfratz, P. Hasenfratz, *Nucl. Phys.* B **270**, 685 (1986).
9. A. Hasenfratz, P. Hasenfratz, *Nucl. Phys.* B **295**, 1 (1988).
10. S. Coleman, E. Weinberg, *Phys. Rev.* D **7**, 1888 (1973).
11. J. Polchinski, *Nucl. Phys.* B **231**, 269 (1984); C. Wetterich, *Nucl. Phys.* B **352**, 529 (1991);
12. V. Branchina, J. Alexandre, J. Polonyi, in preparation.
13. S.B. Liao, J. Polonyi, D. Xu, *Phys. Rev.* D **51**, 748 (1995).
14. A. Patkos, P. Petreczky, J. Polonyi, "Renormalization Group Aided Finite Temperature Reduction in Quantum Field Theories", *Ann. Phys.* in print.