SOME KOSZUL PROPERTIES OF STANDARD AND IRREDUCIBLE MODULES

BRIAN J. PARSHALL AND LEONARD L. SCOTT

Abstract. Let $G$ be a simple, simply connected algebraic group over an algebraically closed field of positive characteristic $p$. In recent work, [17], [18] and [20] the authors have studied a graded analogue of the category of rational $G$-modules. These gradings are not natural but are “forced” on related algebras though filtrations, often obtained from appropriate quantum structures. This paper presents new results on Koszul modules for the graded algebras obtained through this forced grading process. Most of these results require that the Lusztig character formula holds for all restricted $p$-regular weights, but the paper begins to investigate how these and previous results might be established when the Lusztig character formula is only assumed to hold on a proper poset ideal in the Jantzen region. This opens up the possibility of inductive arguments.

1. Introduction

Let $G$ be a simple, simply connected algebraic group over an algebraically closed field $k$ of characteristic $p > 0$. In a series of recent papers [17], [18], and [20], the authors have obtained results on the modular representation theory of $G$ using new, “forced grading,” methods. This work generally assumed that $p \geq 2h - 2$, though [19], which also employed forced grading techniques, suggests a method for removing that condition. In addition, it was required that the Lusztig character formula (LCF) hold for all restricted weights. Among the new results obtained by these methods was a verification of a conjecture of Jantzen [11] on $p$-Weyl filtrations of Weyl modules, assuming that the LCF holds (see [18]), and, in addition, many new results on the $\nabla$-filtrations (i.e., good filtrations) of the Ext-groups (after untwisting) for the first Frobenius kernel $G_1$ of $G$ (see [20]).

The representation theory of $G$ can be studied by means of quasi-hereditary algebras $A_\Gamma$ attached to a finite set $\Gamma$ of dominant weights which is a poset ideal in the dominance ordering or the Bruhat-Chevalley ordering. (See §2.1 below.) In turn, the algebra $A_\Gamma$ has a (forced) graded version, which we denote $B := \text{gr} A_\Gamma$, which is, remarkably, also quasi-hereditary. In addition, when the LCF holds for all restricted weights, the algebra $B$ has been shown to be a Q-Koszul algebra in the sense of [20, Defn. 3.6]. Koszul algebras are themselves examples of Q-Koszul algebras, and, when $\Gamma$ is contained inside the Jantzen region, the representation theory of $G$ can be studied by means of quasi-hereditary algebras $A_\Gamma$ attached to a finite set $\Gamma$ of dominant weights which is a poset ideal in the dominance ordering or the Bruhat-Chevalley ordering. (See §2.1 below.) In turn, the algebra $A_\Gamma$ has a (forced) graded version, which we denote $B := \text{gr} A_\Gamma$, which is, remarkably, also quasi-hereditary. In addition, when the LCF holds for all restricted weights, the algebra $B$ has been shown to be a Q-Koszul algebra in the sense of [20, Defn. 3.6]. Koszul algebras are themselves examples of Q-Koszul algebras, and, when $\Gamma$ is contained inside the Jantzen region, Let $G$ be a simple, simply connected algebraic group over an algebraically closed field of positive characteristic $p$. In recent work, the authors have studied

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a graded analogue of the category of rational $G$-modules. These gradings are not natural but are “forced” on related algebras through filtrations, often obtained from appropriate quantum structures. This paper presents new results on Koszul modules for the graded algebras obtained through this forced grading process. Most of these results require that the Lusztig character formula holds for all restricted $p$-regular weights, but the paper begins to investigate how these and previous results might be established when the Lusztig character formula is only assumed to hold on a proper poset ideal in the Jantzen region

$$\Gamma_{\text{Jan}} := \{ \lambda \in X(T)_+ \mid (\lambda + \rho, \alpha_0^\vee) \leq p(p - h + 2) \}.$$ 

This opens up the possibility of inductive arguments., $B$ turns out to be Koszul itself. But when $\Gamma$ moves outside the Jantzen region, Koszulity generally fails. However, $B$ remains Q-Koszul, and the notion of Q-Koszulity nicely captures some of the homological algebra of $G$, even for modules parameterized by dominant weights far from 0.

In the representation theory of Koszul algebras, there is an important notion of a linear (or “Koszul”) graded module, one which has a particularly nice minimal projective resolution. This concept readily extends to the case of Q-Koszul algebras, and the results of §4 provide new examples of these “Q-linear” modules, including, Q-analogues of maximal submodules of standard modules, or of any Q-linear module. In the Koszul case, we prove maximal submodules of linear modules often have stronger resolution properties, depending on the strong linearity of the original module.

One general aim of the project, of which this paper is a part, has been to keep conclusions, for a given algebra $A_\Gamma$ or $\text{gr} A_\Gamma$ or its representation theory, expressed solely in terms of $\Gamma$. This is desirable not only for aesthetic reasons, but for applications of the theory in inductive arguments. The papers listed above all fit this framework with the exception of [20], where hypotheses “external to $\Gamma$” were required. For instance, it was necessary throughout most of paper [20] to deal with primes $p$ sufficiently large so that LCF held for all irreducible modules $L(\gamma)$ with $\gamma$ a $p$-regular restricted weight.

Accordingly, we consider in this paper the possibility of eliminating such external hypotheses, and we make some progress in this direction. For example, for the poset ideals $\Gamma$ considered[1] in the key Theorem 5.2, when the LCF is assumed, it is only for characters of irreducible modules $L(\gamma)$, when $\gamma \in \Gamma$ is a $p$-restricted dominant weight. Nevertheless, this theorem shows that the irreducible modules $L(\gamma)$, $\gamma \in \Gamma$, behave homologically as if the (regular part of the) restricted enveloping algebra were Koszul (a known consequence of the LCF [2]). We apply Theorem 5.2 in §6. Here we achieve similar “relative to $\Gamma$” versions of the other results of §§3,4, though at the cost of assuming that $p$ is fairly large (but not huge). See the discussion at the end of §6.

In addition, though we continue here to use $p$-regular weights, the approach of this paper, together with the others above, could be applied in the singular case, once some basic questions have been answered. One of these simply asks for a small $p$ quantum version

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[1] We often consider only “stable” posets $\Gamma$, those such that whenever $\gamma = \gamma_0 + p\gamma_1 \in \Gamma$, with $\gamma_0$ restricted and $\gamma_1$ is dominant, it also true that $\gamma_0 \in \Gamma$. This is not a strong requirement.
(in characteristic 0, at a primitive \( p \)th root of 1) of Riche’s Koszulity theorem \([22]\), which applies for all weights. This is explicitly stated in §5.

Finally, in the course of proving Theorem \([5,2]\) it was necessary to sharpen the deformation result in \([18]\ Thm. 8.1\) which provided an integral form for the Koszul grading on the \( p \)-regular part of the small quantum enveloping algebra. See Theorem \([2,2]\) which shows that any grade in the form is a direct sum of its intersections with weight spaces. This result is interesting in its own right. The proof uses a corresponding result for the small quantum group itself in \([2]\ §18.21\), and we elaborate on part of its proof in §8 (Appendix).

2. Preliminaries

Unless otherwise noted, algebras over a field are assumed to be finite dimensional. Likewise, modules are generally taken to be finite dimensional. Let \( k \) be an algebraically closed field, generally of positive characteristic \( p \). In dealing with quantum enveloping algebras at a primitive \( p \)th root \( \zeta \) of unity, we will need a fixed \( p \)-modular system \(( K, \mathcal{O}, k)\). Thus, \( \mathcal{O} \) will be a DVR with maximal ideal \( m = (\pi) \), fraction field \( K \) and residue field \( k = \mathcal{O}/m \). It will be assumed that \( \pi = \zeta - 1 \). For more details, see \([13]\ §2\).

2.1 Some standard notation. Let \( G \) be a fixed simple, simply connected algebraic group defined over an algebraically closed field \( k \) of positive characteristic. We generally (but not always) follow the notation listed in \([12]\ pp. 569–572\).

Let \( T \) be a fixed maximal split torus and let \( R \) be the root system of \( G \). Fix a set \( R^+ \) of positive roots corresponding to a Borel subgroup \( B^+ \supset T \), and let \( B = B^- \supset T \) be the opposite Borel subgroup. Regard the weight lattice \( \Xi \) of \( T \) as a finite poset ideal in \( \Gamma \). A stronger partial order on \( \Xi \) is also a poset. A stronger partial order on \( \Xi \) is sometimes useful (as in the proof of Theorem \([5,2]\)). Namely, given a \( p \)-regular dominant weight, write \( \gamma = w \cdot \gamma_0 \) for a unique \( w \in W_p \) (the affine Weyl group) and \( \gamma_0 \in C^+ \) (the bottom dominant \( p \)-alcove). In this way, the intersection of any orbit with \( X_{\text{reg}}(T)_+ \) identifies with a subset of \( W_p \), and we partially order \( X_{\text{reg}}(T)_+ \) by using the Bruhat-Chevalley partial order on \( W_p \).

Let \( \mathcal{C} = G\text{-mod} \) be the category of finite dimensional rational \( G \)-modules. If \( \gamma \in X(T)_+ \), then \( L(\gamma) \) (resp., \( \Delta(\gamma) \), \( \nabla(\gamma) \)) denotes the irreducible (resp., Weyl module, dual Weyl module) of highest weight \( \gamma \). If \( \Gamma \) is a set of dominant weights, let \( \mathcal{C}[\Gamma] \) be the full subcategory of \( \mathcal{C} \) generated by the irreducible modules \( L(\gamma) \) of highest weight \( \gamma \in \Gamma \). If \( \Gamma \) is a finite poset ideal in \( X(T)_+ \) or in \( X_{\text{reg}}(T)_+ \), then \( \mathcal{C}[\Gamma] \) is a highest weight category with weight poset \( \Gamma \). Here \( \Gamma \) can be taken to be a poset using the dominant partial order \( \leq \), or any stronger order (e.g., the Bruhat-Chevalley order, or the \( \preceq \) order below.)

Let \( \Gamma \) a finite (\( \neq \emptyset \)) poset ideal in the set \( X_{\text{reg}}(T)_+ \) of \( p \)-regular dominant weights, satisfying the additional property that, if \( \gamma = \gamma_0 + p\gamma_1 \in \Gamma \) with \( \gamma_0 \in X_1(T) \) (the \( p \)-restricted dominant weights) and \( \gamma_1 \in X(T)_+ \), then \( \gamma_0 \in \Gamma \). In this case, \( \Gamma \) is called stable. For example, write \( \lambda \preceq \mu \) if and only if \( \mu - \lambda \in \mathbb{Q}^+ R^+ \). Then if \( \Gamma \) is a \( \preceq \)-ideal, it is stable.
Let $\text{Dist}(G)$ be the distribution algebra of $G$. For an poset ideal $\Gamma$ in the poset of $p$-regular weights, let $I_\Gamma$ be the annihilator in $\text{Dist}(G)$ of the category $\mathcal{C}[\Gamma]$. Then $A_\Gamma := \text{Dist}(G)/I_\Gamma$ is a quasi-hereditary algebra with weight poset $\Gamma$ such that $A_\Gamma\text{-mod} \cong \mathcal{C}[\Gamma]$. If $u'$ denotes the sum of the regular block subalgebras of the restricted enveloping algebra $u$ of $G$, there is a natural homomorphism $u' \to A_\Gamma$ arising because $u$ is a subalgebra of $\text{Dist}(G)$. If $M$ is a (finite dimensional) $A_\Gamma$-module, it can thus be regarded as a module for $u$, and hence as a $u'$-module, usually denoted $M|_{u'}$ or just $M$.

2.2 The length function. We will need a “length function” $\ell : X_{\text{reg}}(T) \to \mathbb{Z}$ defined, as described below, on the set of $p$-regular weights. For this, we follow [7] (3.12.3a), using Lusztig’s alcove distance function. That is, write $\tau = z \cdot \lambda$, for $\lambda \in C^+$, and set $\ell(\tau) := d(C^+, z \cdot C^+)$, which counts, with signs, the number of alcove geometry hyperplanes separating $C^+$ and $z \cdot C^-$. (A “+1” contribution occurs for a hyperplane separating alcoves $A, B$, in computing $d(A, B)$, when $A$ is on the negative side of the hyperplane, and a “−1” contribution is used in the opposite case.) In general, $\ell(\tau) \neq \ell(z)$ (the Bruhat-Chevalley length for the Coxeter group $W_p$) but the two lengths do agree if $z$ is dominant, i.e., $z \cdot C^+$ is a dominant alcove. If $z$ is any element of the affine Weyl group $\tilde{W}_p$, or extended affine Weyl group $\tilde{W}_p$ associated to $G$, written as the composition $t_{p\theta}w$ of a translation by $p\theta$ with $\theta \in X(T)$ and $w \in W$ (the Weyl group), then [7, Lemma 3.12.5]

$$d(C^+, z \cdot C^+) = -\ell(w) + 2\text{ht}(\theta),$$

where

$$2\text{ht}(\theta) = \sum_{\alpha > 0}(\theta, \alpha^\vee).$$

When $\theta$ is in the root lattice $\mathbb{Z}R$, the (integer) expression $2\text{ht}(\theta)$ is an even integer. (This is an easy calculation with the dual root system.) Consequently, for $\theta \in \mathbb{Z}R$ and $\tau \in X_{\text{reg}}(T)$,

$$\ell(\tau) \equiv \ell(\tau + p\theta) \mod 2 \quad (2.2.1)$$

2.3 The category of $G_1T$-modules. Following a notation used [2], let $\mathcal{C}_k$ be the category of finite dimensional rational $G_1T$-modules. It is a highest weight category with weight poset $X(T)$. For $\gamma \in X(T)$, let $L_k(\gamma)$ be the irreducible $G_1T$-module of highest weight $\gamma$. In case $\gamma \in X(T)_+$, write $\gamma = \gamma_0 + p\gamma_1$ with $\gamma_0 \in X_1(T)$ (the restricted weights) and $\gamma_1 \in X(T)_+$. Then $L_k(\gamma) \cong L(\gamma_0)|_{G_1T} \otimes p\gamma_1$. When $\gamma = \gamma_0$, we usually denote $L_k(\gamma)$ simply by $L(\gamma)$.

Suppose that $\Omega$ is a union of $W_p$-orbits in $X(T)$. Let $\mathcal{C}_k(\Omega)$ be the full subcategory of $\mathcal{C}_k$ generated by the irreducible modules $L_k(\gamma)$, $\gamma \in \Omega$. Then $\mathcal{C}_k(\Omega)$ is a highest weight category with standard (resp., costandard) modules denoted $Z_k(\gamma)$ (resp., $Z_k'(\gamma)$) defined by

$$(2.3.1) \quad \begin{cases} (1) \ Z_k'(\gamma) := \text{ind}_{G_1T}^{G_1T} \gamma; \\ (2) \ Z_k(\gamma) := \text{ind}_{G_1T}^{G_1T} (\gamma - 2(p - 1)\rho). \end{cases}$$
The category $C_k$ has a natural duality $D$ and $Z_k(\gamma) \cong DZ'_k(\gamma)$.

Now assume that $p > h$. Fix a $p$-regular weight $\lambda \in C^+$. A weight $\lambda \in X_{\text{reg}}(T)_+$ will be said to satisfy the Kazhdan-Lusztig property (with respect to the length function $\ell$, defined in §2.2), provided that

\[(2.3.2)\]
\[
\forall \mu \in X(T), n \in \mathbb{N}, \begin{cases}
\text{Ext}_{G_1T}^0(L(\lambda), Z_k'(\mu)) \neq 0 \implies n \equiv \ell(\lambda) - \ell(\mu) \mod 2; \\
\text{Ext}_{G_1T}^0(Z_k(\mu), L(\lambda)) \neq 0 \implies n \equiv \ell(\lambda) - \ell(\mu) \mod 2.
\end{cases}
\]

In fact, it is enough to check this for $\mu \in W_p \cdot \lambda$.

### 2.4 The modules $\Delta^{\text{red}}(\lambda)$, $\nabla^{\text{red}}(\lambda)$; character formulas.

Given $\gamma = \gamma_0 + p\gamma_1 \in X(T)_+$, with $\gamma_0 \in X_1(T)$ and $\gamma_1 \in X(T)_+$, define

\[(2.4.1)\]
\[
\begin{align*}
\Delta^p(\gamma) & := L(\gamma_0) \otimes \Delta(\gamma_1)^{[1]}; \\
\nabla_p(\gamma) & := L(\gamma_0) \otimes \nabla(\gamma_1)^{[1]}.
\end{align*}
\]

(Given $V \in C$, $V^{[1]} \in C$ denotes the twist of $V$ through the Frobenius morphism $F : G \rightarrow G$.) The module $\Delta^p(\gamma)$ (resp., $\nabla_p(\gamma)$) is a homomorphic image (resp., submodule) of $\Delta(\gamma)$ (resp., $\nabla(\gamma)$) and so is indecomposable with head (resp., socle) $L(\gamma)$.

There is another family of rational $G$-modules, denoted $\Delta^{\text{red}}(\gamma)$ and $\nabla^{\text{red}}(\gamma)$, $\gamma \in \Gamma$, which are closely related to the modules above. These modules are obtained, by a standard “reduction mod $p$,” from the irreducible type 1 modules $L_{\zeta}(\gamma)$, $\gamma \in X(T)_+$, for the quantum enveloping algebra $U_\zeta$ associated to the root system $R$ at a primitive $p$th root of unity $\zeta$. This is described in detail in [13], §2, which contains other references to the literature.

We do not repeat this, except to let $U_\zeta$ be the Lusztig $\mathcal{O}$-form associated to $R$ in which each $K_i = 1$. Then $\widetilde{U}_\zeta \otimes_{k} K \cong U_\zeta$. If $\overline{U}_\zeta := \widetilde{U}_\zeta / \pi \overline{U}_\zeta$ and if $I$ is the ideal of $R$ generated by the $K_i - 1$, $i = 1, \cdots, n$, where $n$ is the rank of $R$, then $U_\zeta / I \cong \text{Dist}(G)$. In this way, $U_\zeta$-modules which are integrable and of type 1, can be “reduced mod $p$” to obtain rational $G$-modules. Thus, given $\gamma \in X(T)_+$, $\Delta^{\text{red}}(\gamma)$ (resp., $\nabla^{\text{red}}(\gamma)$) is the rational $G$-module obtained by reduction mod $p$ of the irreducible module $L_{\zeta}(\gamma)$ for $U_\zeta$ using a minimal (resp., maximal) admissible lattice. (Sometimes, it will be convenient to write $L_K(\lambda)$ for $L_{\zeta}(\lambda)$, $\Delta_K(\lambda)$ for $\Delta_{\zeta}(\lambda)$, etc.)

In this paper, we will say that the Lusztig character formula (LCF) holds for a finite poset ideal $\Gamma$ of $p$-regular dominant weights provided that

\[(2.4.2)\]
\[
\Delta^{\text{red}}(\gamma) = \Delta^p(\gamma), \quad \forall \gamma \in \Gamma.
\]

(Equivalently, this means that $\nabla^{\text{red}}(\gamma) = \nabla_p(\gamma)$, for all $\gamma \in \Gamma$.) By [13, Cor. 2.5], if $\Gamma$ is the ideal generated by the $p$-regular restricted weights, then this condition implies that the characters of the irreducible modules $L(\gamma)$, for $\gamma \in X_1(T)$ are all given by Lusztig’s character formula [13]. If the poset $\Gamma$ is stable in the sense of §2.1, this just means that (2.4.2) holds for $\gamma \in \Gamma$ a restricted dominant weight or, equivalently, $L_{\zeta}(\lambda)$ and $L(\lambda)$ have
the same dimension for \( \lambda \in \Gamma \) restricted. Observe the existence of \( p \)-regular weighs means that \( p \geq h \). Often, we will assume that \( p \) is larger, e. g., \( p \geq 2h - 2 \).

Finally, we say the “LCF holds” (not mentioning any poset) to mean that (2.4.2) holds for all \( p \)-regular dominant weights, or, equivalently, for all restricted dominant weights.\(^2\)

2.5 Grading the restricted enveloping algebras. The category of rational \( G_1 \)-modules is equivalent to the category of modules for the restricted enveloping algebra \( u \) associated to \( G \). (For this reason, we freely identify \( G_1 \)-modules with \( u \)-modules in the discussion.) The maximal torus \( T \) acts rationally (via the adjoint action) on \( u \) as automorphisms \( x \mapsto t^x \) \( (t \in T, x \in u) \). This action induces a familiar (weight space) decomposition \( u = \bigoplus u_\nu \) in terms of all \( \nu \in \mathbb{Z}R \cup \{0\} \subseteq X \) where \( X := X(T) \). More generally, every rational \( T \)-action on a finite dimensional vector space \( V \) over \( k \) has a (direct sum) decomposition \( V = \bigoplus_{\tau \in X} V_\tau \).

There is, of course, a reverse, but that is not what we wish to emphasize. Following the work of Andersen-Jantzen-Soergel in [2, Appendix E], any decomposition \( V = \bigoplus_{\tau \in X} V_\tau \) is called an \( X \)-grading on a vector space \( V \). This makes sense for any abelian group \( X \), though we focus on the special case \( X = X(T) \). From this point of view, a \( G_1T \)-module is a \( u \)-module equipped with an \( X \)-grading that satisfies certain compatibility conditions. These are the multiplication conditions \( u_\nu V_\tau \subseteq V_{\nu + \tau} \) \( (\nu, \tau \in X) \) and another requirement\(^3\) which takes into account the fact that part of \( T \) is inside \( G_1 \). See [2, 2.4], which also gives a discussion for the analogous quantum situation (at a root of unity) using the \( X \)-grading terminology\(^4\). This gives, among other things, a useful uniformity of terminology. We are particularly concerned with (positive) \( \mathbb{Z} \)-gradings on either \( u_\zeta \) and \( u \) which might be compatible with their respective \( X \)-gradings. (To say that a space or algebra \( V \) over \( k \) with an \( X \)-grading has a compatible \( \mathbb{Z} \)-grading \( V = \bigoplus_{n \in \mathbb{Z}} V_n \) just means that each \( V_n \) is the sum of its intersections with the various spaces \( V_\tau \). This is equivalent to the compatibility notion in [2, F.8].) Also, [2, 18.21] observes that every block algebra component \( B \) of either algebra carries a natural \( X \)-grading. The same subsection shows in [2, Prop. 18.21 & Rem. 18.21(2)], under the validity of the LCF, that these block algebras, when \( p \)-regular, carry a compatible Koszul grading\(^5\). (The regularity requirement just means that the irreducible modules in the block are parameterized by \( p \)-regular weights.) We will need to quote this result and its proof in the proofs of Theorem 2.1 and Theorem 2.2 below.\(^6\) The following

\(^2\)This formulation makes sense for \( p \geq h \), though generally we assume \( p > h \), where [23, p. 273] guarantees that each \( L_\zeta(\lambda) \) satisfies Lusztig’s character formula for all dominant weights \( \lambda \).

\(^3\)Namely, it is required that \( h_\alpha \cdot v = (\tau, \alpha^\vee)v \) for \( \tau \in X, v \in V_\tau \).

\(^4\)In the quantum case, it is required that \( K_\alpha \cdot v = \zeta^{(\tau, \alpha^\vee)}v, v \in (u_\zeta)_\tau \). The small quantum group \( u_\zeta \) is also \( X \)-graded.

\(^5\)In fact, in the quantum cases, \( p \) is not required to be a prime, but does need to satisfying some other conditions—see [2, p. 231]—all of which hold if \( p \) is a prime > \( h \).

\(^6\)The brief proof given in [2 18.21] ignores the nontrivial relationship between the \( X \)-weights on \( B \) and those arising when \( B \) is considered as an endomorphism algebra. We supply the needed discussion in the Appendix below and explain how it completes the proof.
result for the quantum case is essentially a special case (for \( p \) a prime) of this result of [2, §18.21].

**Theorem 2.1.** Suppose \( p > h \) is a prime and \( \zeta \) is a primitive \( p \)-th root of unity. Then any regular block \( \mathcal{B} \) of \( u_\zeta \) has a Koszul grading compatible with its \( X \)-grading.

**Proof.** Given that the LCF always holds at the \( p \)-th roots of unity quantum case as long as \( p > h \) (see [23]), the theorem is an immediate consequence of [2, Prop.18.21 & Rem. 18.21(2)]. \( \square \)

As an easy consequence, the same theorem holds if \( \mathcal{B} \) is replaced by the sum \( u'_\zeta \) of all regular block components of the algebra \( u_\zeta \).

Let \((K, \mathcal{O}, k)\) be the \( p \)-modular system of §2.4. The usual \( \mathcal{O} \)-form \( \tilde{u}'_\zeta \) of \( u'_\zeta \) has a positive grading that base changes (i.e., by applying \( - \otimes_{\mathcal{O}} K \)) to a Koszul grading on \( u'_\zeta \) [19, Thm. 8.1]. The latter result states that the Koszul grading on \( u'_\zeta \) is that obtained in [2, §§17–18 & p. 231]. The reference to p. 231 implicitly refers to Conjecture 2 on that page which mentions the \( X \)-grading compatibility, established for all regular blocks of \( u'_\zeta \). The result in [2] which exhibits such an \( X \)-grading is [2, Prop. 8.21 & Rem. 18.21(2)]. The reader may confirm by comparing the proof of this latter result with that of [19, Thm. 8.1] that they use the same Koszul grading on \( u'_\zeta \). The discussion of \( X \)-gradings versus \( Y \)-gradings (\( Y := \mathcal{O} \mathcal{R} \)) above [2, Prop. 18.21] can be replaced by the Appendix to this paper.

Since the grading on \( \tilde{u}'_\zeta \) given in [19, §8] base changes to that on \( u'_\zeta \), it can itself be obtained by taking intersections with the grading on \( u'_\zeta \). This is also true for its natural \( X \)-grading. This proves the analog of Theorem 2.1 stated below as Theorem 2.2 for \( \tilde{u}'_\zeta \).

**Theorem 2.2.** Suppose \( p > h \) is a prime and \( \zeta \) is a primitive \( p \)-th root of unity. Then the algebra \( \tilde{u}'_\zeta \) has a positive integer grading, compatible with its \( X \)-grading, which bases changes to a Koszul grading on \( u'_\zeta \) also compatible with its \( X \)-grading. Applying \( - \otimes_{\mathcal{O}} k \), this grading on \( \tilde{u}'_\zeta \) also base changes to a positive grading on \( u' \) compatible with its \( X \)-grading (as induced by the adjoint action of \( T \)).

In the statement of the theorem, there is no claim about the Koszulity of the positive grading on \( u' \), though this will be true (as follows from [2]) when the LCF holds for \( p \)-restricted weights.

### 3. A review of some earlier results

In this section, we briefly review some results obtained in [18] and [20]. For convenience, let \( \tilde{a} = \tilde{u}'_\zeta \), and for any integer \( n \geq 0 \), define \( \text{rad}^n \tilde{a} := \tilde{a} \cap \text{rad}^n \tilde{a}_K \). Then we set, for any \( U_\zeta \)-module \( M \),

\[
\tilde{\text{gr}} M := \bigoplus_{n \geq 0} (\text{rad}^n \tilde{a}) M / (\text{rad}^{n+1} \tilde{a}) M.
\]
In particular, we can take \( M := \tilde{\alpha} \), to obtain an algebra \( \tilde{\alpha} \) over \( \mathcal{O} \), and, for any \( M, \tilde{\alpha} M \) is naturally a \( \tilde{\alpha} \)-module. More generally, let \( \Gamma \) be a finite poset ideal of \( p \)-regular dominant weights. If \( p \geq 2h - 2 \), by [17] Thm. 6.1, the graded algebra \( \tilde{\alpha} A_\Gamma \) is quasi-hereditary with weight poset \( \Gamma \). Additionally, \( \tilde{\alpha} A_\Gamma \) has standard (or Weyl) modules of the form \( \tilde{\alpha} \Delta(\gamma) \), \( \gamma \in \Gamma \). These were studied in [20] under stronger hypotheses which we will assume here:

A standing hypothesis in the remainder of this section is that the LCF (as recast in [2, 2]) holds and that \( p \geq 2h - 2 \) is odd. Throughout, \( \Gamma \) will be a fixed ideal of \( p \)-regular dominant weights.

**Theorem 3.1.** [18] §5] Given any dominant weight \( \lambda \), the Weyl module \( \Delta(\lambda) \) has a filtration by \( G \)-submodules with corresponding sections of the form \( \Delta^p(\gamma), \gamma \in X(T)_+. \) In case \( \lambda \) is \( p \)-regular, this filtration can be taken to be compatible with the \( G_1 \)-radical series of \( \Delta(\lambda) \), in the sense that each section of the radical series has a \( \Delta^p \)-filtration.

**Theorem 3.2.** [20] Thm. 5.3(a)] Let \( \lambda, \mu \in X_{\text{reg}}(T)_+ \). For any integer \( n \geq 0 \), the rational \( G \)-module \( \text{Ext}^n_{G_1}(\Delta^p(\lambda), \nabla(\mu))^{[-1]} \) has a \( \nabla \)-filtration.

Before stating the next result, we recall some standard terminology. In case \( B \) is a graded algebra and \( M, N \) are graded \( B \)-modules, \( \text{ext}^r_B(M, N) \) denotes the Ext-groups computed in the category of graded \( B \)-modules. If \( r \in \mathbb{Z} \), then \( N<\rangle r \) is the graded \( B \)-module obtained from \( N \) by shifting the grades \( r \)-steps to the right, i. e., \( N<\rangle r_i := N_{i-r} \), for all \( i \in \mathbb{Z} \). Therefore,

\[
\text{Ext}^n_B(M, N) = \bigoplus_{r \in \mathbb{Z}} \text{ext}^n_B(M, N<\rangle r).
\]

We will use these conventions in the next result.

**Theorem 3.3.** [20] Thm. 5.6] Let \( \lambda, \mu \in \Gamma \).

\[
\forall n \in \mathbb{N}, r \in \mathbb{Z}, \quad \text{ext}^n_{\tilde{\alpha} A}(\Delta^p(\lambda), \nabla(\mu)<\rangle r) \neq 0 \implies r = n.
\]

Recall from [20] Defn. 3.3 that a positively graded algebra \( B \) is called a \( Q \)-Koszul algebra provided that:

1. its grade 0 component \( B_0 \) is quasi-hereditary with poset \( \Gamma \) (and with standard and costandard modules denoted \( \Delta^0(\gamma) \) and \( \nabla_0(\gamma) \) (respectively); and
2. if \( \Delta^0(\gamma) \) and \( \nabla_0(\gamma) \) are given pure grade 0, as graded \( B \)-modules, then

\[
\forall n \in \mathbb{N}, r \in \mathbb{Z}, \lambda, \mu \in \Gamma, \quad \text{ext}^n_B(\Delta^0(\lambda), \nabla_0(\mu)<\rangle r) \neq 0 \implies n = r.
\]

\( ^7 \)The notation in [17] is slightly different than that used here in that we write \( \tilde{\alpha} A_\Gamma \) more simply as \( \text{gr} A_\Gamma \) (which has the danger of being confused with the radical series of \( A_\Gamma \), from which it may differ. Also, [17] proves a much stronger result which states that the quasi-hereditary algebra \( \tilde{\alpha} A_\Gamma \) arises through base change from a quasi-hereditary algebra \( A_\Gamma \) over \( \mathcal{O} \). We will not need that here.
Suppose that $B$ is Q-Koszul and a graded quasi-hereditary algebra with weight poset $\Gamma$, having graded standard (resp., costandard) modules $\Delta^B(\gamma)$ (resp., $\nabla_B(\gamma)$), $\gamma \in \Gamma$, with head (resp., socle) of grade 0. If

\[
(3.0.4) \quad \begin{cases}
\text{ext}^n_B(\Delta^B(\lambda), \nabla_0(\mu)\langle r \rangle) \neq 0 \implies n = r;
\text{ext}^n_B(\Delta^0(\mu), \Delta_B(\lambda)\langle r \rangle) \neq 0 \implies n = r,
\end{cases}
\]

then we say that $B$ is a standard Q-Koszul algebra.

The notions of Q-Koszul and standard Q-Koszul algebras are generalizations of the notions of Koszul and standard Koszul algebra. In the terminology above, a Koszul algebra $B$ is a Q-Koszul algebra in which the modules $\Delta^0(\gamma)$ and $\nabla_0(\gamma)$ are all irreducible. Equivalently, $B_0$ is semisimple. An algebra $B$ is a standard Koszul algebra if it is a Koszul algebra and a graded quasi-hereditary algebra, and if the conditions (3.0.4) hold. (See comments above [20, Defn. 3.6] for some history of the “standard” Koszul terminology. Our usage comes from Mazorchuk [14]. For more history of Koszul gradings, see [16].)

**Theorem 3.4.** [20] Thm. 6.2 For $\mu, \nu \in \Gamma$, the rational $G$-module $\text{Ext}^m_G(\Delta(\nu), \nabla_\text{red}(\mu))[-1]$ has a $\nabla$-filtration and the restriction natural map

\[
(3.0.5) \quad \text{Ext}^m_G(\Delta^\text{red}(\nu), \nabla_\text{red}(\mu)) \rightarrow \text{Ext}^m_G(\Delta(\nu), \nabla_\text{red}(\mu))
\]

is surjective.

Dually, the rational $G$-module $\text{Ext}^m_G(\Delta^\text{red}(\mu), \nabla(\lambda))[-1]$ has a $\nabla$-filtration and the natural map

\[
(3.0.6) \quad \text{Ext}^m_G(\Delta^\text{red}(\nu), \nabla_\text{red}(\mu)) \rightarrow \text{Ext}^m_G(\Delta^\text{red}(\nu), \nabla(\nu))
\]

is surjective.

The following theorem implies that $\text{gr} \, A_\Gamma$ is a standard Q-Koszul algebra.

**Theorem 3.5.** [20] Thm. 3.7 Let $\lambda, \mu \in \Gamma$. For any nonnegative integer $n$ and any integer $r$,

\[
\text{ext}^n_{\text{gr} \, A}(\text{gr} \, \Delta(\lambda), \nabla_\text{red}(\mu)\langle r \rangle) \neq 0 \implies r = n
\]

and

\[
\text{ext}^n_{\text{gr} \, A}(\Delta^\text{red}(\lambda), \nabla_{\text{gr} \, A}(\mu)\langle r \rangle) \neq 0 \implies r = n,
\]

where $\nabla_{\text{gr} \, A}(\mu)$ is the costandard object in the highest weight category corresponding to $\mu$.

If $\Gamma$ is a poset of $p$-regular weights contained in the Jantzen region $\Gamma_{Jan}$ and if the LCF holds, then $\text{gr} \, A_\Gamma \cong \text{gr} \, A_\Gamma$, the graded algebra obtained from the radical filtration of $A_\Gamma$. Similarly, $\text{gr} \, \Delta(\gamma) \cong \text{gr} \, \Delta(\gamma)$, the gr-$A_\Gamma$-module obtained from the radical series of $\Delta(\gamma)$.

**Corollary 3.6.** [20] Cor. 3.8 Now assume that $\Gamma$ is contained in the Jantzen region $\Gamma_{Jan}$. Then $\text{gr} \, \Delta(\lambda)$ is a linear module over $\text{gr} \, A$. Also, the graded quasi-hereditary algebra $\text{gr} \, A$-mod has a graded Kazhdan-Lusztig theory. In particular, $\text{gr} \, A$ is Koszul.

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8 A graded quasi-hereditary algebra is just a quasi-hereditary algebra with a positive grading. All irreducible, standard and costandard modules (and more) will have graded versions as above; see [5]. Here the positive grading is taken from the Koszul algebra.
4. SOME NEW PROPERTIES OF KOSZUL AND Q-KOSZUL ALGEBRAS

Some of the results of this section are quite general, and the characteristic of the underlying algebraically closed field \( k \) may be arbitrary, unless a prime \( p \) is mentioned (in which case, \( p \) is the characteristic of \( k \)). Our first result formalizes in the Q-Koszul algebra case a property observed for Koszul algebras in [1, Proof of Cor. 3.2]. Suppose that \( M \) is a non-negatively graded module for a Q-Koszul algebra \( B \) (as defined after Theorem 3.3 above). Then we will say that \( M \) is Q-linear provided that

\[
\forall n \in \mathbb{N}, r \in \mathbb{Z}, \gamma \in \Gamma, \quad \text{ext}^n_B(M, \nabla_0(\gamma)\langle r \rangle) \neq 0 \implies n = r.
\]

A non-positively graded \( B \)-module \( M \) is called Q-colinear provided that

\[
\forall n \in \mathbb{N}, r \in \mathbb{Z}, \gamma \in \Gamma, \quad \text{ext}^n_B(\Delta^0(\gamma)\langle -r \rangle, M) \neq 0 \implies n = r.
\]

**Proposition 4.1.** Let \( B \) be a Q-Koszul algebra with weight poset \( \Gamma \). Let \( M \) be a non-negatively graded Q-linear \( B \)-module. Assume for each \( i \geq 0 \), that \( M_i \) (regarded as a \( B_0 \cong B/B_{\geq 1} \)-module) has a \( \Delta^0 \)-filtration. Then each

\[
M_{\geq i}(-i) := \bigoplus_{j \geq i} M_j(-i)
\]

is a Q-linear \( B \)-module.

**Proof.** We proceed by induction on \( i \). Since \( M_{\geq 0}(0) = M \), the statement is true for \( i = 0 \) because \( M \) is assumed to be Q-linear. Now fix \( i \geq 0 \) and assume that \( M_{\geq i}(-i) \) is Q-linear. We will show that \( M_{\geq i+1}(-i - 1) \) is Q-linear. The short exact sequence \( 0 \to M_{\geq i+1} \to M_{\geq i} \to M_i \to 0 \) of \( B \)-modules gives, for any \( \mu \in \Gamma \), a long exact sequence

\[
\cdots \to \text{Ext}^n_B(M_i, \nabla_0(\mu)) \xrightarrow{\alpha} \text{Ext}^n_B(M_{\geq i}, \nabla_0(\mu)) \xrightarrow{\delta} \text{Ext}^n_B(M_{\geq i+1}, \nabla_0(\mu)) \to \cdots
\]

We claim the mapping \( \alpha \) is surjective, or equivalently \( \delta = 0 \). Assume not, so that

\[
\text{ext}^n_B(M_{\geq i+1}, \nabla_0(\mu)(n)) \neq 0,
\]

for some \( \mu \in \Gamma \). Because \( M_{\geq i+1} \) has a \( \Delta^0 \)-filtration, it follows, for some \( s \geq 1 \), that

\[
\text{ext}_B(\Delta_0(\gamma)\langle s \rangle, \nabla_0(\mu)(n)) \neq 0.
\]

Thus, \( \text{ext}_B(\Delta^0(\gamma), \nabla_0(\mu)(n-s)) \neq 0 \), contradicting the assumption that \( B \) is Q-Koszul.

We conclude that \( \text{Ext}^n_B(M_{\geq i+1}, \nabla_0(\mu)) \subseteq \text{Ext}^{n+1}_B(M_{\geq i}, \nabla_0(\mu)) \), for all \( \mu \in \Gamma \). Hence, for any integer \( m \geq 0 \), if

\[
\text{ext}_B^m(M_{\geq i+1}(-i-1), \nabla_0(\mu)(m)) \neq 0,
\]

then \( \text{ext}^{n+1}_B(M_{\geq i}(-i), \nabla_0(\mu)(m+1)) \neq 0 \). Thus, \( n = m \), as required. \( \square \)
A similar result holds for a non-positively graded $Q$-colinear module. The same is true for Corollaries 4.2 and 4.3 below, and their generalizations at the end of §6. We leave further details to the reader.

Now we return to the situation of the representation theory of our group $G$. Let $Γ$ be a finite poset ideal of $p$-regular weights, and consider the graded quasi-hereditary algebra $B := \text{gr} A_Γ$. It has standard modules $Δ^B(γ) := \text{gr} Δ(γ)$, $γ ∈ Γ$. Its costandard modules are denoted $∇_B(γ)$. In addition, the grade 0 component of $B_0$ of $B$ is quasi-hereditary with weight poset $Γ$ and with standard (resp., costandard) modules $Δ^0(γ) := Δ^\text{red}(γ)$ (resp., $∇_0(γ) := ∇_\text{red}(γ)$), $γ ∈ Γ$. If we assume the LCF holds, then the algebra $B$ is standard $Q$-Koszul. Put $Δ^B_i(γ) := Δ^B(γ)_{≥ i}$, for each integer $i ≥ 0$.

**Corollary 4.2.** Assume that $p ≥ 2h − 2$ is odd and that the LCF holds. If $γ ∈ Γ$ and $i ≥ 0$, $Δ^B(γ)(−i)$ is $Q$-linear. (Here $B := \text{gr} A_Γ$.)

**Proof.** By Theorem 3.3, $Δ^B(γ)$ is a $Q$-Koszul module. By Theorem 3.1, each $Δ^B_i(γ)/Δ^B_{i+1}(γ)$ has a $Δ^0$-filtration. Thus, the hypotheses of Proposition 4.1 hold, and the proof is complete. □

**Corollary 4.3.** Assume that $p ≥ 2h − 2$ is odd and that the LCF holds. Let $Γ$ be a poset ideal of $p$-regular dominant weights which is contained in the Jantzen region $Γ_{\text{Jan}}$.

Then $B := \text{gr} A_Γ$ is a Koszul algebra and, given any $γ ∈ Γ$ and $i ≥ 0$, the module $Δ^B_i(γ)$ is linear for $B$. In particular, both $\text{gr} Δ(γ)$ and its maximal submodule (shifted by $⟨−1⟩$) are linear modules for $\text{gr} A_Γ$.

Maximal submodules of standard modules are especially interesting for the study of the associated irreducible modules. The above corollary shows that their ext groups with coefficients in irreducible modules are especially well-behaved. We can prove a similar property for ext groups of these modules (and any term of their radical series) with coefficients in costandard modules. It is useful to discuss this before stating the next theorem.

Let us say that a graded module $M$ for a standard Koszul algebra $B$ (with weight poset $Γ$) is **strongly linear** if the following property holds:

\[(4.0.9) \quad \forall γ ∈ Γ, n ∈ \mathbb{N}, r ∈ \mathbb{Z}, \quad \text{ext}^0_B(M, ∇_B(γ)(r)) \neq 0 \implies n = r.\]

There is an evident dual notion of a strongly colinear module. By definition, the (purely graded) irreducible modules for $B$ are always strongly linear and strongly colinear.

If $Ω$ is a coideal in $Γ$, the strong linearity property of any module is preserved upon passage to (graded versions of) the natural highest weight category associated to $Ω$, and a similar statement holds for the strong colinearity property. In more detail, the passage is obtained by an exact additive functor $j^* : B\text{-mod} \rightarrow eBe\text{-mod}$, $M \mapsto j^*M = eM$, obtained by multiplication by a grade 0 idempotent $e ∈ B$. The functor $j^*$ maps standard (resp., costandard) modules $Δ^B(γ)$ (resp., $∇_B(γ)$) for $γ ∈ Ω$ to the corresponding standard

---

\[Γ_{\text{Jan}}\] contains all restricted weights if and only if $p ≥ 2h − 3$. 
and costandard modules in \(\text{e}B\text{-mod}\). In addition, the functor \(j^*\) admits a left exact right adjoint \(j_* := \text{Hom}_B(\text{e}B, -)\) which carries any costandard module \(\nabla_{\text{e}B}(\gamma), \gamma \in \Omega\), in \(\text{e}B\text{-mod}\) to the corresponding costandard module \(\nabla_B(\gamma)\) in \(B\text{-mod}\). Thus, for any \(B\text{-module} E\) and \(\gamma \in \Omega\), \(j^*\) induces an isomorphism

\[
\text{Ext}^n_B(E, \nabla_B(\gamma)) \sim \to \text{Ext}^n_{\text{e}B}(\text{e}E, \nabla_{\text{e}B}(\gamma))
\]

Dually, the strong colinearity property is similarly preserved by \(j_*\) (which admits a right exact adjoint \(j!\) taking costandard modules to costandard modules). As one consequence (using both strong linearity properties of irreducible modules), we can deduce that standard and costandard modules \(\text{e}B\text{-mod}\) are linear and colinear, respectively, which implies that the algebra \(\text{e}B\) is standard Koszul, and, in particular, Koszul.\(^{10}\)

As another consequence of the displayed isomorphism, we can deduce that, for any strongly linear \(B\text{-module} M\), the modules \(M_{\geq i}(-i)\) are also strongly linear. This is seen by choosing, for a given \(\gamma \in \Gamma\), a coideal \(\Omega\) with \(\gamma\) minimal in \(\Omega\). The minimality implies \(\nabla_{\text{e}B}(\gamma)\) is irreducible. Now the argument of Proposition 4.1 can be applied to \(\text{e}B\) for this fixed \(\gamma\), using the modules \(e(M_{\geq i}) = (eM)_{\geq i}\), to inductively deduce the strong linearity. The dual property, for strongly colinear \(B\text{-modules}, may be deduced by a dual argument.

In particular, Corollary 4.3 holds if “linear” is replaced by “strongly linear.” Explicitly,

**Proposition 4.4.** Assume the hypotheses of Corollary 4.3. For \(\gamma \in \Gamma\) and \(i \geq 0\), each \(\Delta^B_i(\gamma)\) is strongly linear. A dual statement holds for costandard modules, using strong colinearity.

This strong linearity property, for maximal submodules of standard modules (and their radical series, each appropriately shifted in grade) is new. Here is another new result for standard Koszul algebras, applicable to maximal submodules of standard modules and their radicals series on the “strong linearity” side, and to dual notions for costandard modules on the “strong colinearity” side.

**Theorem 4.5.** Suppose \(B\) is a standard Koszul algebra. Let \(M\) (resp., \(N\)) be a strongly linear (resp., strongly colinear) module for \(B\). Then, for all integers \(n\) and \(r\), \(\text{ext}^n_B(M, N(\langle r \rangle)) \neq 0 \implies n = r\).

**Proof.** We give the proof only in a special case, which is likely to be more familiar. First, assume that \(B\text{-mod}\) has a Kazhdan-Lusztig theory in the sense of [6]. If \(\Gamma\) is the poset for \(B\), this supposes there is a length function \(\ell : \Gamma \to \mathbb{Z}\), used mod 2 to assign parities to modules indexed by elements \(\gamma \in \Gamma\). More explicitly, it is required that

\[
\forall n \in \mathbb{N}, \gamma, \mu \in \Gamma, \begin{cases} \text{Ext}^n_B(\Delta^B(\gamma), L_B(\mu)) \neq 0 \implies n \equiv \ell(\gamma) - \ell(\mu) \mod 2; \\
\text{Ext}^n_B(L_B(\mu), \nabla_B(\mu)) \neq 0 \implies n \equiv \ell(\mu) - \ell(\gamma) \mod 2. 
\end{cases}
\]

\(^{10}\)This Koszulity implication goes back to Irving [10], as discussed in [15, p. 345]. It may also deduced from graded Grothendieck group arguments, as in [6, §3, appendix]. An ungraded analogue is given in [1, Thm. 1].
(In the presence of the Koszulity property for $B$, the existence of such a Kazhdan-Lusztig theory implies $B$ is a standard Koszul algebra, and many of the known examples arise this way. See [6], especially the appendix to §3 and the argument for Theorem 2.4.) Second, in addition to the Kazhdan-Lusztig theory, we will assume an additional property of $\ell$, namely, that all the irreducible constituents of the head of $M$ (which may be identified with $M_0$) all share a common parity (with regard to $\ell$), and that a similar parity sharing occurs for irreducible constituents of the socle $N_0$ of $N$.

These additional conditions can all be avoided by using the somewhat more sophisticated notion of a $\mathbb{Z}/2$-based Kazhdan-Lusztig theory in [15].

Returning to our chosen context, for $\gamma \in \Gamma$, observe that each map

$$\Ext_B^n(M_0, \nabla_B(\gamma)) \to \Ext_B^n(M, \nabla_B(\gamma))$$

is surjective. (Equivalently, the map $\Ext_B^n(M, \nabla_B(\gamma)) \to \Ext_B^n(M_{\geq 1}, \nabla_B(\gamma))$ is zero. But this can be deduced by passing to a suitable algebra $eBe$ with $e\nabla_B(\gamma)$ irreducible, and arguing with natural isomorphism induced by adjoint functors. See the argument in the paragraph preceding the theorem, and the proof of Proposition 4.1.) This gives the ungraded groups $\Ext_B^n(M, \nabla_B(\gamma))$ an even-odd vanishing property, the same as that possessed by $M_0$ or any of its irreducible constituents. A similar even-odd vanishing property is obtained dually for $N$. In particular, this yields the important conclusion (from the derived category arguments of [6]) that $M$ and $N$, respectively, belong to certain filtered derived subcategories, each associated with a particular parity of length function. $M_0$ and $M$ belong to the same subcategory $\mathcal{E}^L$ or $\mathcal{E}^L[1]$ and $N_0$ belongs to the subcategory, $\mathcal{E}^R$ or $\mathcal{E}^R[1]$, as $N$. However, $M_1$ and $M_{\geq 1}$ belong to the subcategory $\mathcal{E}^L[1]$ or $\mathcal{E}^L$ associated with the opposite parity to that of $M_0$ and $M$. (We know from above $M_{\geq 1}(-1)$ is strongly linear. Also, $\Ext^1$ nonvanishing between irreducible modules forces them to have opposite parity. Dual considerations apply for $N/N_0$ to give it a parity opposite to that of $N_0$.) We do not discuss in detail the meaning of these parity differences other than to note they imply $\Ext_B^n(M, N)$ and $\Ext_B^n(M_{\geq 1}, N)$ cannot be simultaneously nonzero. Also, $\Ext_B^n(M_0, N)$ and $\Ext_B^n(M_0, N/N_0)$ cannot be simultaneously nonzero.

Next, we prove the theorem for the case $M = M_0$. The theorem is certainly true in this case, if $N = N_0$. Suppose $\text{ext}_B^n(M_0, N(\ell)) \neq 0$ then $\text{ext}_B^n(M_0, N) \neq 0$, so $\Ext_B^n(M_0, N/N_0) = 0$. Consequently, the natural map $\Ext_B^n(M_0, N_0) \to \Ext_B^n(M_0, N)$ is surjective, inducing a surjection $\text{ext}_B^n(M_0, N_0(\ell)) \to \text{ext}_B^n(M_0, N(\ell))$. Hence, it follows that $\text{ext}_B^n(M_0, N_0(\ell)) \neq 0$, and so $n = \ell$, in this case.

Similarly, the theorem for general $M$ follows from the $M = M_0$ case, using the fact that $\Ext_B^n(M, N)$ and $\Ext_B^n(M_{\geq 1}, N)$ cannot be simultaneously nonzero.

This completes the proof for the case we have chosen. A general proof along roughly similar line, though working with parity considerations on ext groups, and appropriate categories $\mathcal{E}^L'$ and $\mathcal{E}^R'$ may be obtained using [15], but we omit further details. \qed

Several remarks are in order. First, it is interesting to note the above theorem implies that "strongly linear" modules are also "linear," with a similar property for "strongly colinear"
modules. (That is, these modules are also colinear.) Second, all of the above results for standard Koszul algebras appear to generalize to the standard Q-Koszul case, though we have not checked all details. Third, it is certainly not necessary to assume positive characteristic in the results above that are stated using a condition on \( p \), and these results hold, mutatis mutandis, for the BGG categories \( \mathcal{O} \). In fact, in that case, the algebra \( B = \text{gr} A \) is quasi-hereditary, because it is isomorphic to \( A \).

5. A graded Ext result for \( G_1T \)

In this section, \( \Gamma \) is a stable poset ideal of \( p \)-regular dominant weights. Suppose the Kazhdan-Lusztig property (2.3.2) holds for all \( \gamma \in \Gamma \). Assume that \( p > h \). Then, we can adopt an argument given in [7] to show that if \( \lambda, \mu \in \Gamma + pX \), then

\[
\text{Ext}^n_{G_1T}(L(\lambda), L(\mu)) \neq 0 \implies \ell(\lambda) - \ell(\mu) \equiv n \mod 2
\]

In more detail, the argument for [7, Thm. 5.6] is an inductive argument on lengths of restricted weights, starting with weights in the lowest dominant alcove \( C^+ \). Each \( W_p \)-orbit (under the “dot” action) of \( p \)-regular weights must contain such a weight, as will any nonempty intersection of such an orbit of \( \Gamma \). Then the inductive argument works entirely with restricted weights, increasing their lengths by 1 at each step of the argument. Every restricted weight is accessible in such a process. The stability assumption on \( \Gamma \) guarantees that, whenever any one of its restricted weights is accessed by such a sequence, each element \( \nu \) of the accessing sequence also belongs to \( \Gamma \). This just gets us to the combinatorial set-up, but we can also show inductively that each \( L(\nu) \) satisfies the necessary even-odd vanishing condition to define an element of the “enriched” Grothendieck group used in the proof: This is true for \( \nu \in C^+ \) by [7, Thm. 3.12.1]. If true for one \( L(\nu) \) in an ascending sequence, it will be true for the next, call it \( L(\nu') \), if the latter is a direct summand of the “middle” of a module obtained from a standard wall-crossing procedure. The latter is completely compatible with its analog for the larger group \( G \), see [7, Thm. 5.2(b)], but it is easier for it to be completely reducible for \( G_1T \) than for \( G \). If we assumed \( p \geq 2h - 3 \), we could argue that \( L(\nu) \) was the direct summand of the “middle” (which would even be completely reducible) from validity of the LCF for weights of \( \Gamma \) in the Jantzen region. We could then complete the induction and claim [7, Thm. 5.7] held for \( \Gamma \), and consequently equation (5.0.10) above (arguing further as in [7, Thm. 5.8]).

However, we will assume only that \( p > h \), and argue differently to obtain the same complete reducibility at the \( G_1T \) level. We broaden the induction, making use of a consequence of (5.0.10) in this section, namely (5.0.11) below. Let \( \Gamma_0 \) be the set of all weights in \( \Gamma \) whose lengths are at most that of \( \nu \) in the previous paragraph. We can assume that [7, Thm. 5.7] holds for all restricted irreducible modules for highest weights in \( \Gamma_0 \). It follows that [7, Thm. 5.8] and (5.0.10) hold for all \( p \)-translates of restricted weights in \( \Gamma_0 \), and further consequences noted in this section, such as (5.0.11). In particular, we can equate \( \text{Ext}^1 \)-calculations for \( \mathcal{O}_k \) and \( \mathcal{O}_K \) between irreducible modules with such highest weights. The character of the “middle” is the same for \( \mathcal{O}_k \) as that for its \( \mathcal{O}_K \) analog. Also, since the
LCF is assumed for $\Gamma$, the characters of irreducible $C_K$ modules appearing in the “middle” all reduce “mod $p$” to irreducible $C_k$ modules (even that of $L(\nu')$). Now for any $C_K$ irreducible module $L(\omega)$ appearing in the “middle”, its multiplicity can be determined as the dimension of $\text{Ext}^{1}_{C_K}(L(\nu), L(\omega))$, or of the same $\text{Ext}^1$ group with its two arguments reversed. For $\omega \neq \nu'$, this $\text{Ext}^1$ group has the same dimension, by application of (5.0.11) for $\Gamma_0$, as that for $C_k$. Consequently, all irreducible $C_k$ composition factors $L(\omega), \omega \neq \nu'$, of the “middle” appear with their full multiplicity in both its head and socle. For $\omega = \nu'$, the multiplicity of $L(\nu')$ is 1. It follows that the “middle” is completely reducible, and the induction is complete.

**Lemma 5.1.** (Even-odd vanishing) Assume that $p > h$ and let $\Gamma$ be a stable poset ideal of $p$-regular dominant weights. Assume that the Kazhdan-Lusztig property holds for all $\gamma \in \Gamma$; see (2.3.2). Then for restricted weights $\lambda, \mu \in \Gamma$ and $n \in \mathbb{Z}$,

$$\text{Ext}^n_{G_1}(L(\lambda), L(\mu)) \neq 0 \implies \text{Ext}^{n+1}_{G_1}(L(\lambda), L(\mu)) = 0.$$  

**Proof.** If $\text{Ext}^n_{G_1}(L(\lambda), L(\mu)) \neq 0$, then for some $\theta \in X(T)$, $\text{Ext}^n_{G_1}(L(\lambda), L(\mu + p\theta)) \neq 0$. Thus, $\ell(\lambda) - \ell(\mu + p\theta) \equiv n \mod 2$, using (5.0.10). Also, $\lambda$ and $\mu + p\theta$ are $W_\gamma$-conjugate, so that $\lambda - \mu - p\theta$ lies in the root lattice $\mathbb{Z}R$. For the same reason, if $\text{Ext}^{n+1}_{G_1}(L(\lambda), L(\mu)) \neq 0$, then for some $\theta' \in X(T)$, $\text{Ext}^{n+1}_{G_1}(L(\lambda), L(\mu + p\theta')) \neq 0$. This implies that $\ell(\lambda) - \ell(\mu + p\theta') \equiv n + 1 \mod 2$. Again, $\lambda - \mu - p\theta' \in \mathbb{Z}R$. Therefore, $p(\theta - \theta') \in \mathbb{Z}R$. Since $p > h$, $p$ is relatively prime to the index of connection of $R$, so that $\theta - \theta' \in \mathbb{Z}R$, and, therefore, by (2.2.1) above, $\ell(\mu + p\theta) \equiv \ell(\mu + p\theta') \mod 2$. Putting things together, we get that $n + 1 \equiv n \mod 2$, which is absurd. Thus, $\text{Ext}^{n+1}_{G_1}(L(\lambda), L(\mu)) = 0$. The same argument shows that $\text{Ext}^{n-1}_{G_1}(L(\lambda), L(\mu)) = 0$. \hfill $\square$

Let $A$ be a positively graded algebra. For a graded $A$-module $N$ and an integer $r$, let $N \langle r \rangle$ be the shifted graded $A$-module, obtained by putting $N \langle r \rangle_s := N_{s-r}$. If $M, N$ are graded $A$-modules, let $\text{ext}^n_A(M, N)$ be the $n$th Ext-group computed in the category of graded $A$-modules.

**Theorem 5.2.** Assume that $p > h$. Let $\Gamma$ be a stable poset ideal in $X_{\text{reg}}(T)_\uparrow$. Assume that if $\gamma \in \Gamma$ is $p$-restricted, then the LCF holds for $L(\gamma)$. If $\lambda, \mu \in \Gamma$ and $r \in \mathbb{Z}$, then

$$\text{ext}^n_{\tilde{L}}(L(\lambda), L(\mu)) \neq 0 \implies n = r.$$  

**Proof.** The hypothesis implies that $\Delta_{\text{red}}(\gamma) \cong \nabla_{\text{red}}(\gamma) = L(\gamma)$, for all $p$-restricted dominant weights $\gamma \in \Gamma$ (or, more generally, for all $\gamma \in \Gamma$). Thus, if $\gamma \in \Gamma$ is restricted, $L(\gamma) \cong k \otimes_{\mathfrak{g}} \tilde{L}_C(\gamma)$ as discussed right before the statement of the theorem.

Let $\lambda, \mu \in \Gamma$ be $p$-restricted. Form the short exact sequence

$$0 \to \tilde{L}_C(\mu) \to \tilde{L}_C(\mu) \to L(\mu) \to 0$$  

of $\tilde{u}'$-modules. Write $L = L(\lambda), L' = L(\mu)$, etc. and form the long exact sequence of Ext-groups

$$\cdots \to \text{Ext}^{n-1}_{\tilde{L}}(L, L') \to \text{Ext}^n_{\tilde{L}}(L, L') \to \text{Ext}_p^0(L, L') \to \text{Ext}^0_{\tilde{L}}(L, L') \to \text{Ext}^1_{\tilde{L}}(L, L') \to \cdots$$
Observe that $\text{Ext}_u^\bullet(\tilde{L}, L') \cong \text{Ext}_u^\bullet(L, L')$.

Now assume that $\text{Ext}_u^k(L, L') \neq 0$. Then, by Lemma 5.1, $\text{Ext}_u^{n+1}(L, L') = 0$ and Nakayama’s lemma, and the long exact sequence above force $\text{Ext}_u^n(\tilde{L}, L') = 0$. Thus,

$$\text{(5.0.11)} \quad \text{Ext}_u^n(\tilde{L}, L')/\pi \text{Ext}_u^n(\tilde{L}, L') \cong \text{Ext}_u^n(L, L'), \quad \forall n \geq 0.$$ 

In addition, $\text{Ext}_u^{n-1}(\tilde{L}, L') \cong \text{Ext}_u^{n-1}(L, L') = 0$, so that $\text{Ext}_u^n(\tilde{L}, L')$ is free of rank equal to the dimension of $\text{Ext}_u^n(L, L')$ or of $\text{Ext}_u^n(L_K, L'_K)$. In particular, $\text{Ext}_u^n(L_K, L'_K) \cong \text{Ext}_u^n(\tilde{L}, L)$. 

On the other hand, $u'$ inherits the structure of a positively graded algebra from the grading on $\tilde{u}$. We have

$$\begin{cases} 
\text{Ext}_u^n(\tilde{L}, L') \cong \bigoplus_r \text{Ext}_u^n(\tilde{L}, L'\langle r \rangle); \\
\text{Ext}_u^n(L, L') \cong \bigoplus_r \text{Ext}_u^n(L, L'\langle r \rangle). 
\end{cases}$$

It follows that the isomorphism (5.0.11) induces an isomorphism

$$\text{ext}_u^n(\tilde{L}, L'\langle r \rangle)/\pi \text{ext}_u^n(\tilde{L}, L'\langle r \rangle) \cong \text{ext}_u^n(L, L'\langle r \rangle).$$

Hence, $\text{ext}_u^n(\tilde{L}, L'\langle r \rangle) \neq 0$, as is $\text{ext}_u^n(L_K, L'_K\langle r \rangle) \neq 0$. Therefore, because $\tilde{u}'_K$ is a Koszul algebra, $n = r$, completing the proof. 

**Remark 5.3.** Consider the modules $Z_K(\lambda)$ and $Z'_K(\lambda)$ in the quantum category $C_K$ defined in [2] §2.11 $\lambda \in X$. By [2] §§8.8–8.12, these modules (for $p$-regular weights $\lambda$) are $\mathbb{Z}$-graded modules, denoted $\tilde{Z}_K(\lambda)$ and $\tilde{Z}'_K(\lambda)$, for the Koszul algebra $u'_K$. In fact, by [2] Prop. 18.19(b)), these graded modules are linear. It can be shown that the modules $\tilde{Z}_K(\lambda)$ and $\tilde{Z}'_K(\lambda)$ admit graded $\mathcal{O}$-forms $\tilde{Z}_\mathcal{O}(\lambda)$ and $\tilde{Z}'_\mathcal{O}(\lambda)$. Hence, base changing to the field $k$, we see that the classical modules $Z_k(\lambda)$ and $Z'_k(\lambda)$ have induced gradings. Then, it can be shown that, if $\lambda \in X(T)_+$ and $\mu \in \Gamma$, then $\text{ext}_u^n(Z_k(\lambda), L(\mu)\langle r \rangle) \neq 0$ implies $n = r$. A similar result holds for the $Z'_k(\lambda)$.

### 6. Some relative results

We begin with the following general result. It does not require any assumption of the LCF on $\Gamma$. We will work with the quasi-hereditary algebra $\tilde{\text{gr}} A_{\Gamma}$, which has weight poset $\Gamma$, standard modules $\Delta_{\tilde{\text{gr}} A_{\Gamma}}(\gamma)$, and costandard modules $\nabla_{\tilde{\text{gr}} A_{\Gamma}}(\gamma), \gamma \in \Gamma$. Also, $\Delta_{\tilde{\text{gr}} A_{\Gamma}}(\gamma) = \tilde{\text{gr}} \Delta(\gamma)$ and there is a dual construction (in the same spirit) for $\nabla_{\tilde{\text{gr}} A_{\Gamma}}(\gamma)$; see [18] (4.0.2), where $\nabla_{\tilde{\text{gr}} A_{\Gamma}}(\gamma) = \tilde{\text{gr}} \circ \nabla(\gamma)$.

**Theorem 6.1.** [20 Thm. 6.5] Assume that $p \geq 2h - 2$ is an odd prime, and let $\Gamma$ be a finite poset ideal of $p$-regular weights.

(a) For $\lambda, \mu \in \Gamma$ and any integer $n \geq 0$, there are natural vector space isomorphisms

$$\text{Ext}_u^n(\Delta_{\tilde{\text{gr}} A_{\Gamma}}(\lambda), \nabla_{\text{red}}(\mu)) \cong \text{Ext}_u^n(\Delta(\lambda), \nabla_{\text{red}}(\mu)) \cong \text{Ext}_G^n(\Delta(\lambda), \nabla_{\text{red}}(\mu))$$

(6.0.12)
and

\begin{equation}
\text{Ext}_{\gr \Lambda}^n(\Delta^{\text{red}}(\lambda), \nabla_{\gr \Lambda}(\mu)) \cong \text{Ext}_{\Lambda}^n(\Delta^{\text{red}}(\lambda), \nabla(\mu))
\end{equation}

(6.0.13)

\begin{equation}
\cong \text{Ext}_G^n(\Delta^{\text{red}}(\lambda), \nabla(\mu)).
\end{equation}

(b) For any integer \( n \geq 0 \), there are natural vector space isomorphisms

\begin{equation}
\text{Ext}_{\gr \Lambda}^n(\Delta^{\text{red}}(\lambda), \nabla_{\gr \Lambda}(\mu)) \cong \text{Ext}_{\Lambda}^n(\Delta^{\text{red}}(\lambda), \nabla_{\gr \Lambda}(\mu))
\end{equation}

(6.0.14)

\begin{equation}
\cong \text{Ext}_G^n(\Delta^{\text{red}}(\lambda), \nabla_{\gr \Lambda}(\mu)).
\end{equation}

for \( \lambda, \mu \in \Gamma \).

Now let \( \Gamma \) be a finite stable poset ideal of \( p \)-regular weights. Let \( a(\Gamma) \) be the number of \( p \)-alcoves \( C \) which intersect \( \Gamma \) non-trivially. By the argument for [16 Prop. 10.3], \( \Lambda_\Gamma \) has global dimension \( \leq 2a(\Gamma) \).

**Theorem 6.2.** Assume that the LCF holds on \( \Gamma \). Assume that \( p > 6a(\Gamma) + 3h - 4 \). For \( \gamma, \nu \in \Gamma, \ n \in \mathbb{N}, \ m \in \mathbb{Z} \), we have

\[
\begin{cases}
(1) \text{Ext}_{\gr \Lambda}^n(\Delta^{\text{red}}(\gamma), \nabla_{\gr \Lambda}(\nu)(m)) \neq 0 \implies n = m; \\
(2) \text{Ext}_{\gr \Lambda}^n(\gr \Delta(\gamma), \nabla_{\gr \Lambda}(\nu)(m)) \neq 0 \implies n = m; \\
(3) \text{Ext}_{\gr \Lambda}^n(\Delta^{\text{red}}(\gamma), \nabla(\nu)(m)) \neq 0 \implies n = m.
\end{cases}
\]

**Proof.** We first prove (1). Assume that \( 6a(\Gamma) + 3h - 4 \). Write \( A = A_\Gamma \). By Theorem 6.1

\begin{equation}
\text{Ext}_{\gr \Lambda}^n(\Delta^{\text{red}}(\gamma), \nabla_{\gr \Lambda}(\nu)) \cong \text{Ext}_{\Lambda}^n(\Delta^{\text{red}}(\gamma), \nabla_{\gr \Lambda}(\nu)).
\end{equation}

(6.0.15)

As noted above, \( A \) has global dimension at most \( 2a(\Gamma) \). Thus, the terms in (6.0.15) vanish if \( n > 2a(\Gamma) \).

On the other hand, there is a Hochschild-Serre spectral sequence

\begin{equation}
E_{s,t}^1 = H^s(G, \text{Ext}_{\Lambda}^t(\Delta^{\text{red}}(\gamma), \nabla_{\gr \Lambda}(\nu)^{-1})) \implies \text{Ext}_{\Lambda}^{s+t}(\Delta^{\text{red}}(\gamma), \nabla_{\gr \Lambda}(\nu)).
\end{equation}

(6.0.16)

We call a weight \( \lambda \in X(T) \), \( b \)-small provided that \( |(\lambda, \alpha^\vee)| \leq b \) for all positive roots \( \alpha \). If \( M \) is a finite dimensional rational \( G \)-module, then it is called \( b \)-small provided that its weights are all \( b \)-small. Write \( \gamma = \gamma_0 + p\gamma_1 \) and \( \nu = \nu_0 + p\nu_1 \), with \( \gamma_0, \nu_0 \in X_1(T) \) and \( \gamma_1, \nu_1 \in X(T)_+ \). Then

\[
\text{Ext}_{\Lambda}^t(\Delta^{\text{red}}(\gamma), \nabla_{\gr \Lambda}(\nu)^{-1}) \cong \text{Ext}_{\Lambda}^t(L(\gamma_0), L(\nu_0)^{-1}) \otimes \nabla(\gamma_1^\vee) \otimes \nabla(\nu_1).
\]

Here \( \gamma_1^\vee = -w_0\gamma_1 \) is the image of \( \gamma_1 \) under the opposition involution. Now [21 Cor. 3.6] implies that, if \( t \) is any non-negative integer, the modules \( \text{Ext}_{\Lambda}^t(\Delta^{\text{red}}(\lambda), L(\nu_0)^{-1}) \) are \((3t + 2h - 3)\)-small. Therefore, if \( t \leq 2a(\Gamma) \) and \( p > 6a(\Gamma) + 3h - 4 \), the dominant weights \( \xi \) in \( \text{Ext}_{\Lambda}^t(L(\gamma_0), L(\nu_0)^{-1}) \) lie in the bottom \( p \)-alcove \( C^+ \). Hence, in this case, \( \text{Ext}_{\Lambda}^t(L(\gamma_0), L(\nu_0)^{-1}) \) has a \( \nabla \)-filtration. Thus, \( \text{Ext}_{\Lambda}^t(\Delta^{\text{red}}(\lambda), \nabla_{\gr \Lambda}(\nu)^{-1}) \) has a \( \nabla \)-filtration. Therefore, as long as \( n := s + t \leq 2a(\Gamma) \), the hypotheses guarantee that the spectral sequence (6.0.16) collapses giving that

\begin{equation}
\text{Ext}_{\Lambda}^n(\Delta^{\text{red}}(\gamma), \nabla_{\gr \Lambda}(\nu)) \cong \text{Ext}_{\Lambda}^n(\Delta^{\text{red}}(\gamma), \nabla_{\gr \Lambda}(\nu))^G.
\end{equation}

(6.0.17)
This is because, if $M$ is a rational $G$-module with a $\nabla$-filtration, then $H^m(G, M) = 0$ for all $m > 0$. On the other hand, if $n \geq 2a(\Gamma)$, then, as pointed out in the previous paragraph, we have $\text{Ext}^n_{\mathfrak{g} \mathfrak{r},A}(\Delta_{\text{red}}(\gamma), \nabla_{\text{red}}(\nu)) = 0$.

Thus, if $\text{Ext}^n_{\mathfrak{g} \mathfrak{r},A}(\Delta_{\text{red}}(\gamma), \nabla_{\text{red}}(\nu)) \neq 0$, we obtain that

$$\text{Ext}^n_{\mathfrak{g} \mathfrak{r},A}(\Delta_{\text{red}}(\gamma), \nabla_{\text{red}}(\nu)) \cong \text{Ext}^n_{A}(\Delta_{\text{red}}(\gamma), \nabla_{\text{red}}(\nu))$$

injects into $\text{Ext}^n_H(\Delta_{\text{red}}(\gamma), \nabla_{\text{red}}(\nu))$. Returning to the level of graded modules, it follows easily that

$$\text{ext}^n_{\mathfrak{g} \mathfrak{r},A}(\Delta_{\text{red}}(\gamma), \nabla_{\text{red}}(\nu)(r)) \subseteq \text{ext}^n_H(\Delta_{\text{red}}(\gamma), \nabla(\nu)(r)), \quad \forall n.$$

Then (1) follows from Theorem 5.2.

Finally, we sketch the proof of (2), leaving the dual proof of case (3) to the reader. Because of the condition imposed on $p$, Theorem 6.1 implies that we can assume that $n \leq 2a(\Gamma)$ and then $\text{Ext}^n_{\mathfrak{g} \mathfrak{r},A}(\Delta_{\text{red}}(\nu), \nabla_{\text{red}}(\mu))[-1]$ has a $\nabla$-filtration. In addition, the rational $G$-module $\text{Ext}^n_{\mathfrak{g} \mathfrak{r},A}(\Delta(\nu), \nabla_{\text{red}}(\mu))[-1]$ has a $\nabla$-filtration. With this condition, the reader may check that the proof of Theorem 3.4 (namely, [20, Thm. 4.2]) remains valid. Hence, the natural map (3.0.5) is surjective. Therefore, passing to $	ext{ext}^n_H$, we obtain that, if $\text{ext}^n_{\mathfrak{g} \mathfrak{r},A}(\Delta(\gamma), \nabla_{red}(\nu)(m)) \neq 0$, then $\text{ext}^n_H(\Delta_{\text{red}}(\gamma), \nabla_{\text{red}}(\nu)(m)) \neq 0$. Hence, as before, $n = m$. \hfill \square

Scholia. In this section and the previous one we have proved results which enable the “relativization” of hypotheses on the underlying regular dominant weight posets $\Gamma$ of most results in §§3.4. By “relativization” we mean replacing any hypothesis that “The Lusztig character [4, 7], [1a] holds” with $\Gamma$ is stable and the Lusztig character formula holds for $\Gamma$ (see [2.4.2] for terminology). We explain how this works for each of the results in §§3.4:

Theorem 5.1 is already relativized in [20, Thm. 7.1]. In fact, the formulation does not even require that that $\Gamma$ be stable; and the LCF is effectively required only on the poset of non-maximal elements of $\Gamma$. Here it is required that $p \geq 2h - 2$ is odd. Theorem 3.3 is not yet fully relativized. Nevertheless, its conclusions hold without any LCF hypothesis, if $p$ is sufficiently large relative the Ext degree. This Ext group (in the statement of the theorem) is generally interesting only when the cohomological degree $n \leq \text{gl. dim. } A_\Gamma$. See the proof of Theorem 6.2. If we assume the LCF holds for $\Gamma$, this global dimension is at most $2a(\Gamma)$, where $a(\Gamma)$ is the number of alcoves intersecting $\Gamma$ nontrivially. For cohomological degree $n$ at most this value, the conclusion of the Theorem 5.2 holds if $p > 6a(\Gamma) + 3h - 4$. We regard such primes $p$ as “fairly large”, but not huge. Theorem 3.3 is relativized for $p$ fairly large. Explicitly, $p > 6a(\Gamma) + 3h - 4$ as above. This is established in Theorem 6.2. Theorem 3.4 is not yet fully relativized. Nevertheless, its $\nabla$-filtration conclusions hold for Ext with $n \leq 2a(\Gamma)$, provided $p$ is fairly large, as above. Again, if the LCF holds for $\Gamma$, this includes all cohomological degrees $n \leq \text{gl. dim. } A_\Gamma$. The surjectivity assertions of the theorem can be proved, under these assumptions on $\Gamma$ and $p$, as in the argument for Theorem 6.2 in [20]. Theorem 3.5 is relativized, as proved in Theorem 6.2 in this paper. Corollary 3.6 as a special case of Theorem 3.5 is also relativized. Corollaries 4.2
and 4.3 are relativized, using Theorem 3.5 and Corollary 3.6 discussed above. Similarly Proposition 4.4 is relativized. That is, in these results in §4, the assumption that the LCF holds may be replaced with the assumption that $\Gamma$ is stable and the LCF holds on $\Gamma$, as discussed above. All other results in §4 are stated in an abstract context, and so have no need of relativization. This concludes our discussion.

7. Open Questions

Many of the results in our “forced grading program” go back to the paper [16], entitled “A new approach to the Koszul property using graded subalgebras.” One of the subalgebras in the title is the $p$-regular part $u'_{\xi}$ of the small quantum group (associated to $G$). If $p > h$, it may be deduced from [2] §§§17–18 that $u'_{\xi}$ is a Koszul algebra. This fact needed for this deduction is the validity of the LCF for $u'_{\xi}$ when $p > h$. This is now known for $p > h$ [23], and the LCF holds also for $u_{\xi}$. Nevertheless, the corresponding Koszulity of $u_{\xi}$ is not known.

**Question 7.1.** Assume $p > h$. Is it true that the small quantum enveloping algebra $u_{\xi}$ at a $p$th root of unity is a Koszul algebra?

Curiously, such an extension of the work of [2] already exists in positive characteristic [22] (for “really” large $p$). But there is no extension so far in the (presumably easier) quantum case.

One can also hope that a similar result holds at the integral level.

**Question 7.2.** Assuming the answer to Question 7.1 is positive, does the graded algebra $u_{\xi}$ admit a compatible $O$-form as in Theorem 2.2 (or in [18, §8])?

It seems likely that, at least in classical types, these questions might also have positive answers for some small prime cases, i.e., $p \leq h$. Positive answers to these questions would likely lead to obtaining graded, integral quasi-hereditary algebras as in [17], and it could also lead to the results in [20], as well as to showing the result in §4 above are valid in the singular case.

Finally, we raise

**Question 7.3.** Do the modules $Z_K(\lambda)$ discussed in Remark 5.3 satisfy the linearity (or Koszul) condition when $\lambda$ is allowed to be singular? A dual property for the $Z'_{\Lambda}(\lambda)$ should also hold. (This question also appears to be open for the algebraic group schemes $G_1T$ in positive characteristic, even for very large $p$.)

8. Appendix: Comparison of Gradings

The aim of this appendix is to flesh out part of the argument for [2] Prop. 18.21] dealing with compatibilities of some $\mathbb{Z}$-gradings discussed there with natural weight gradings. We give two arguments. First, our own, is given after a general discussion of the context and issues, and deals fairly directly with the weight gradings involved. The second argument, which we give briefly, is our interpretation of the argument sketched in [2] 18.21] itself.
Let $Y$ be an abelian group. The appendix [2] Appendix E] defines a $Y$-category to be an additive category $C$ equipped with shift functors $\langle \xi \rangle : C \to C$, one for each $\xi \in Y$. Certain natural conditions must be satisfied. In a classical sense, there is also the notion of a $Y$-graded algebra $A$ and $Y$-graded modules for it, which [2] §E.3] notes gives rise to a $Y$-category (with obvious shift functors). Moreover, [2] E.3] defines the notion of a $Y$-generator for an abelian $Y$-category $C$ and shows, for any projective $Y$-generator $P$ that $\text{End}^Y_C(P)$ is a $Y$-graded algebra (and $E := \text{End}^Y_C(P)^{op}$ is $Y$-graded in the same way). The original category $C$ is equivalent to the module category $E$-$\text{grmod}$ of $Y$-graded $E$-modules. In case $C$ is the category of $Y$-graded modules for a $Y$-graded algebra $A$, then $\text{Hom}^Y_C(M, N) \cong \text{Hom}_A(M, N)$ (ungraded $\text{Hom}_A$).

Two important examples, introduced early in [2], of categories graded by an abelian group are the categories we call $C_k$ (namely, the category of finite dimensional rational $G_1T$-modules) and its quantum analogue, which we call $C_k$; see [2] §2.4]. The abelian group generally used is $Y := \mathbb{Z}R$, especially in studying blocks, though the larger group $pX$ can sometimes be used, where $X = X(T)$ as in §1 of this paper. If $M$ is any object in $C_k$, then $M \otimes p\xi$ is also in $C_k$. Setting $M(p\xi) := M \otimes p\xi$ gives $C_k$ the structure of a $pX$-category. These functors do not generally preserve blocks unless $p\xi \in Y$. However, any block is a $Y$-category.

Of course, $M$ has a classical weight space decomposition—namely, $M = \bigoplus_{\omega \in \lambda} M_\omega$. However, this decomposition of $M$ does not correspond to the weight space decomposition, using a graded endomorphism algebra as in the first paragraph above: Let $P \in C_k$ be such that $P|_u \cong u$ (as left $u$-modules) in Case 1 and $P|_{u\xi} \cong u\xi$ in Case 2. For instance, take $P := \bigoplus_{\lambda \in X_1(T)} \Phi_k(\lambda)$, where the construction $\Phi_k(\lambda)$ is described in [2] 2.6 (3)]. In Case 1, $\Phi_k(\lambda) := (\text{ind}_{T}^{G_1T} - \lambda)^* - \lambda$ is the module obtained by coiniding from $T$ to $G_1T$ the one-dimensional $T$-module defined by $\lambda$, and there is an analogous construction in Case 2. (Here $X_1(T)$ could be replaced by any collection $X'_1$ of coset representatives of $pX$ in $X$.) We have $\text{Hom}^{C_k}_k(\Phi_k(\lambda), M) \cong M_\lambda$, $\lambda \in X_1(T)$. The module $P$ is a projective $pX$-generator for $C_k$ in the sense of paragraph 1 above, giving an equivalence of $C_k$ with the category of $pX$-graded modules for a $pX$-graded algebra $E = \text{End}^Y_{C_k}(P)^{op}$. The equivalence is given by $M \mapsto \text{Hom}^{C_k}_k(P, M)$. The latter module is isomorphic to $M$ as a vector space and has decomposition $M = \bigoplus_{\theta \in X} M(p\theta)$ into $pX$-grades. We find that, in terms of the original weight space decomposition of $M$, that $M(p\theta) = \bigoplus_{\lambda \in X_1(T)} M_{\lambda+p\theta}$. Notice that the $\lambda$-weight spaces do not correspond exactly to the $pX$-weight spaces (which are made up of a sum of many of the former weight spaces). This suggests we cannot just use the results on graded categories in [2, Appendix E] to obtain $X$-weight space compatibility with the $Z$-gradings in [2, §§17,18].

11 Here $\text{Hom}^{C_k}_k(M, N) := \bigoplus_{\xi \in Y} \text{Hom}_C(M, N(\xi))$, for any pair of objects $M, N \in C$.
12 Both these categories are denoted $C_k$ in [2], distinguished only by a “Case 1” ($G_1T$ case) or “Case 2” (quantum analogue case) context.
13 One comes closer by thinking, in Case 1, of fully embedding the category of $G_1T$-modules into the category of rational modules for the semidirect product $G_1T := G_1 \rtimes T$ (using the group scheme surjection
However, the compatibility exists just the same, at least for modules \( M \) whose composition factors have \( p \)-regular highest weights. Before discussing this, we give more details on how the above formalism works for blocks. Let \( C_k(\Omega) \) be the block of \( C_k \) associated to a \( W_\rho \)-orbit \( \Omega \) of \( p \)-regular weights. (We will not use the \( p \)-regularity in this paragraph.) All the weights of any \( M \) in \( C_k(\Omega) \), \( \Omega = W_\rho \cdot \omega \), belong to a single coset of \( \omega + ZR \) of the root lattice \( ZR \). Working with any prime \( p \) relatively prime to \([X : ZR]\) (such as any prime \( p > h \)), choose a set of coset representatives \( X'_1 \) for \( pX \) in \( X \) such that \( X'_1 \subseteq \omega + ZR \). Construct \( P' = \bigoplus_{\lambda \in X'_1} \Phi_k(\lambda) \), analogous to the module \( P \) above. Let \( P'_\Omega \) be the projection of \( P' \) onto the block \( C_k(\Omega) \). Then \( P'_\Omega \) is a \( Y \)-generator for the \( Y \)-category \( C_k(\Omega) \). Also, \( P'_\Omega = \bigoplus_{\lambda \in X'_1} \Phi(\lambda) \), where \( \Phi(\lambda) \) is the projection of \( \Phi_k(\lambda) \) onto \( C_k(\Omega) \). Let \( Q_k(\nu) \), be the projective cover of the irreducible module \( L_k(\nu) \), \( \nu \in \Omega \); see [2] §4.15. The module \( \Phi(\lambda) \) is just the direct sum of copies of the \( \Phi_k(\lambda) \) onto \( C_k(\Omega) \). Let \( \Phi_k(\lambda) \) be any object in \( C_k(\Omega) \) and any \( \lambda \in X'_1 \), we have

\[
\text{Hom}_{C_k(\Omega)}(\Phi_k(\lambda), M) \cong \text{Hom}_{C_k}(\Phi_k(\lambda), M) \cong M_\lambda.
\]

We can recover any weight space \( M_\nu \) of \( M \) by writing \( \nu = \lambda + p\theta \), for some \( \lambda \in X'_1 \) and some \( \theta \in ZR \). (The fact that this can be done depends heavily on the construction of \( X'_1 \) described above: choose \( \lambda \in X'_1 \) so that \( \nu = \lambda + pX \). Since \( X'_1 \) and \( \nu \) belong to the same coset of \( ZR \), we have \( \lambda - \nu \in [X \cap ZR = pZR \text{ here.}] \) Then

\[
\text{Hom}_{C_k(\Omega)}(\Phi_k(\lambda), M(-p\theta)) \cong M_{\lambda+p\theta} = M_\nu.
\]

That is, we have completely recovered the \( X \)-grading of \( M \) using the \( Y \)-category \( C_k(\Omega) \).

Next, consider the issue of passing from \( Y \)-compatibility of a \( Z \)-graded version of \( C_k(\Omega) \) to \( X \)-compatibility. [2] §18] introduced various objects in \( C_k(\Omega) \) which have “graded forms,” putting them in an associated \( Y \times Z \)-category \( \tilde{C}_k(\Omega) \). Among these are the projective covers \( Q_k(\lambda) \) of the irreducible module \( L_k(\lambda) \), \( \lambda \in \Omega \); see [2] §18.16]. This gives a \( Y \times Z \)-graded version \( \tilde{Q}_k(\lambda) \) in \( \tilde{C}_k(\Omega) \) of \( Q_k(\lambda) \). For \( \lambda \in X'_1 \), define \( \tilde{\Phi}(\lambda) = \bigoplus_{\nu \in \Omega} \tilde{Q}_k(\nu)^{\Theta(\lambda:Q_k(\nu))} \). Thus, \( \Phi_k(\lambda) \) is obtained from \( \tilde{\Phi}(\lambda) \) in \( \tilde{C}_k(\Omega) \) by forgetting the \( Z \)-grading. We still have the isomorphism \( \Phi_k(\lambda) \cong \text{Hom}_{\tilde{C}_k(\Omega)}(\tilde{\Phi}(\lambda), \tilde{M}((-p\theta, -n))) \).

This establishes the needed \( X \)-compatibility from the \( Y \)-compatibility.

\[ G_1.T \to G_1.T \). Modules for \( G_1 \) are just \( X \)-graded modules for the \( X \)-graded algebra \( u \). In this category, it is possible to make sense of a tensor product \( M \otimes \xi \) for any \( \xi \in X \), thus fully embedding the \( Y \)-category \( C_k \) into an \( X \)-category in a natural way. See [2] 18.20] which takes a similar approach for blocks.
This passage from $Y$-compatibility to $X$-compatibility may be used in the discussion immediately above [2, Prop. 18.21] to complete the proof of that proposition and its quantum analogue. Essentially, we agree that the algebra $\text{End}^\mathbb{Z}_{\mathcal{C}}(P)^{\text{op}}$ discussed there is both $Y \times \mathbb{Z}$-graded and $X$-graded. However, it seems to us that additional detail is required to make it $X \times \mathbb{Z}$-graded, and a version of this has been supplied above. A second way to do this, perhaps implicit in [2], is to use the direct sum decomposition [2, 18.20] to define an $X \times \mathbb{Z}$-category, using shifts provided between summands. This process gives an $X \times \mathbb{Z}$-grading on $\text{End}^\mathbb{Z}_{\mathcal{C}}(P)^{\text{op}}$, and also leads to the desired $X$-compatibility.

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**Department of Mathematics, University of Virginia, Charlottesville, VA 22903**

*E-mail address*: bjp8w@virginia.edu (Parshall)

**Department of Mathematics, University of Virginia, Charlottesville, VA 22903**

*E-mail address*: lls2l@virginia.edu (Scott)