A New Technique for Analyzing 6R Linkages: 
Quad Polynomials

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February 26, 2014

Abstract

A closed 6R linkage is generically rigid. In this paper we give, for the first time, equational conditions on the Denavit/Hartenberg parameters that are necessary for mobility. The conditions suggest to distinguish the movable 6R linkages into various types characterized by their bond diagrams.

1 Introduction

A closed \( n \)R linkage is uniquely determined by its set of Denavit/Hartenberg parameters \( \mathbb{R} \), which consists of \( 3n \) real numbers: the twist angles, the normal distances and the offsets. If \( n = 6 \), then a generic choice of these parameters leads to a rigid linkage. In the literature, there are many families of special choices of parameters such that the linkage is mobile, or in other words, several sufficient conditions for mobility are known in \( \mathbb{R} \). This paper has the purpose of deriving necessary conditions for mobility; up to our knowledge, not a single necessary equation condition has been known up to now.

Our method is based on bond theory, which has been introduced in \( \mathbb{R} \) in order to study overconstrained linkages. Intuitively, bonds are points in the algebraic closure of the configuration curve with complex coefficients where something degenerate happens. The bonds carry a combinatorial structure, which is visualized by the bond diagram: every bond connects two joints of the linkage. It is known that adjacent joints are never connected by a bond. If we number the joints cyclically, then a bond connects either joints \( i \) and \( i + 2 \) – then we speak of a near connection, – or it connects joints \( i \) and \( i + 3 \), and then we speak of a far connection.

In \( \mathbb{R} \), it was shown that the existence of a near connection implies the validity of a well-known condition which also arises frequently in many families, namely Bennett’s condition: \( s_i = 0 \), and \( \frac{d_i}{\sin(\phi_i)} = \frac{d_{i+1}}{\sin(\phi_{i+1})} \), where the \( s_i, d_i, \phi_i \) are Denavit/Hartenberg parameters (see section \( \mathbb{R} \) for the precise definitions). However, there are many mobile 6R linkages without near connections, for instance Bricard’s orthogonal linkage or Bricard’s line symmetric linkage. So, the Bennett conditions are not necessary for mobility.
In this paper, we introduce the quad polynomials: these are univariate polynomials of degree 2 with coefficients depending on the Denavit Hartenberg parameters by an explicit formula. Their most important feature is that the existence of a far connection implies a common root of two such quad polynomials. This gives rise to necessary equational conditions, because every mobile linkage has either near or far connections (or both).

It is convenient to classify linkages according to their bond diagram. For any bond diagram, we will derive a non-trivial system of algebraic equations, consisting of Bennett conditions for the near connections and quad polynomial conditions for the far connections. We will show that in some cases, but not in all, these equations do characterize the linkages with this particular bond diagram. As an application, we derive a new family of 6R linkages by solving the corresponding system of equations. This family is remarkable because two of its R-joints can be replaced by H-joints, and the linkage remains movable.

Structure of the paper  The remaining part of the paper is set up as follows. In section 2, we introduce all preliminary definitions we need. In section 3, we give the definition of the quad polynomial and its main property. Section 4 contains examples (old and new).

At several places, we used the computer algebra system Maple for more elaborate computations: examples, discussions of systems of equations, and the derivation of the quad polynomials. Because of the length of these computations, it is not reasonable to reproduce them in this paper, but they can be found in http://people.ricam.oeaw.ac.at/z.li/softwares/quadpolynomials.html. They can be read with any text editor and verified using Maple 16.

2 Preliminary Definitions

2.1 Computation of the Configuration Space

In this section we recall a method of computing the configuration space of a closed 6R linkage using dual quaternions and Denavit/Hartenberg parameters. First, we start by introducing the set of Denavit/Hartenberg parameters of a closed 6R linkage. For $i = 1, \ldots, 6$, let $l_i$ be the rotation axis of the $i$-th joint. The angle $\phi_i$ is defined as the angle of the direction vectors of $l_i$ and $l_{i+1}$ (with some choice of orientation). We also set $c_i := \cos(\phi_i)$ and $w_i = \cot(\frac{\phi_i}{2}) = \frac{\cos(\phi_i) + 1}{\sin(\phi_i)}$. The number $d_i$ is defined as the orthogonal distance of the lines $l_i$ and $l_{i+1}$. Note that $d_i$ may be negative; this depends on some choice of orientation of the common normal, which we denote by $n_i$.

From now on, we will always assume that there are no parallel adjacent lines, which means that the angles $\phi_1, \ldots, \phi_6$ are not equal to 0 or $\pi$. Then we may set $b_i := \frac{d_i}{\sin(\phi_i)}$ (Bennett ratios) as an abbreviation. Finally, we define the offset $s_i$ as the signed distance of the intersections of the common normals $n_{i-1}$ and $n_i$ with $l_i$. 

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The Denavit/Hartenberg parameters $\phi_i$, $d_i$, $s_i$ are invariant when the linkage is moving. Moreover, it is well-known that they form a complete system of invariants for all closed 6R linkages without adjacent parallel lines: if two such linkages share all parameters, then there is a configuration that moves the first into the second. (A description of invariant parameters for 6-bar linkages with adjacent parallel lines may be found in [2]).

The closure condition is an equation in the group $\text{SE}_3$ of Euclidean displacements. We give a formulation in the language of dual quaternions, based on the fact that $\text{SE}_3$ is isomorphic to the multiplicative group of dual quaternions with nonzero real norm modulo multiplication by nonzero real scalars. The set $\text{DH}$ is the 8-dimensional real vector-space generated by $1, i, j, k, \epsilon, \epsilon_i, \epsilon_j, \epsilon_k$ with multiplication inherited from quaternion multiplication and the rule $\epsilon^2 = 0$ (the dual number $\epsilon$ is considered as a scalar). The set of dual quaternions with real norm is also called the Study quadric and denoted by $S$; it is a quadric hypersurface in $\mathbb{R}^8$.

In the isomorphism described in [8, Section 2.4], the rotation with axis determined by $i$ and angle $\phi$ corresponds to the dual quaternion $(\cos(\phi/2) - \sin(\phi/2)i)$, which is projectively equivalent to $(\cot(\phi/2) - i)$. The translation parallel to $i$ by a distance $d$ corresponds to the dual quaternion $(1 - d^2/\epsilon i)$. So the closure equation is

$$(t_1 - i)g_1(t_2 - i)g_2 \cdots (t_6 - i)g_6 \in \mathbb{R}^*,$$  

where

$$g_i = \left(1 - \frac{s_i}{2} \epsilon i\right) (w_i - k) \left(1 - \frac{d_i}{2} \epsilon k\right),$$

for $i = 1, \ldots, 6$. This is just the reformulation of the well-known closure equations [3] in terms of dual quaternions.

**Remark 1.** In [6, 7, 9], we used a different formulation of the closure equation, namely

$$(t_1 - h_1)(t_2 - h_2) \cdots (t_6 - h_6) \in \mathbb{R}^*,$$

where $h_1, \ldots, h_6$ are dual quaternions specifying the rotation axes in some initial position of the linkage.

The set $K$ of all 6-tuples $(t_1, \ldots, t_6)$ fulfilling (1) is called the configuration set of the linkage $L$. The dimension of the configuration set is called the mobility of the linkage. We are mostly interested in linkages of mobility one.

For a fixed set of Denavit/Hartenberg parameters, the configuration set can be computed with the help of the computer algebra Maple. For example, let $L$ be the Bricard line symmetric linkage with parameters

$$(d_1, \ldots, d_6) = \left(\frac{3}{5}, \frac{24}{13}, \frac{72}{25}, \frac{3}{5}, \frac{24}{13}, \frac{72}{25}\right),$$

$$(w_1, \ldots, w_6) = \left(\frac{1}{3}, \frac{2}{3}, \frac{3}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}\right),$$

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Then all and a one-dimensional component: solution set has some zero-dimensional components, which are not interesting, namely the 5th (the coefficient of $t_1$). The reason is that the left hand side is always in the Study quadric $S$. These equations is redundant, namely the 5th (the coefficient of $\varepsilon$) is in $S$ then it follows that $b = 0$. In order to exclude unwanted solutions, we add the inequality $(t_1^2 + 1)(t_2^2 + 1)(t_3^2 + 1)(t_4^2 + 1)(t_5^2 + 1)(t_6^2 + 1) \neq 0$. In order to solve this system with the computer program Maple, we introduce an extra variable $u$ and add the equation $(t_1^2 + 1)(t_2^2 + 1)(t_3^2 + 1)(t_4^2 + 1)(t_5^2 + 1)(t_6^2 + 1)u - 1 = 0$, and compute a Gröbner basis that eliminates $u$ again. The solution set has some zero-dimensional components, which are not interesting, and a one-dimensional component:

$$171t_1^2t_2^2 - 134t_1^2t_2 + 40t_1t_2^2 + 49t_2^2 - 160t_1t_2 - 5t_2^2 - 24t_1 + 90t_2 - 255,$$

$$171t_1t_2^2 + 19t_2^3 - 134t_1t_2 + 40t_2^2 - 222t_2t_3 + 49t_1 - 288t_2 + 105t_3,$$

$$171t_1t_2 - 13st_1t_3 + 19t_2t_3 - 134t_1 + 40t_2 - 222t_3 - 323, t_1 - t_2 - t_5, t_3 - t_6.$$

## 2.2 The Bond Diagram

In this section we recall the fundamentals of bond theory introduced by [7].

Let $L$ be a closed 6R linkage with mobility 1. Let $K_C \subset (\mathbb{P}_C)^6$ be the Zariski closure of $K$ the configuration set, that is, the zero set of all polynomial equations that vanish on $K$, including complex points and points at infinity. The set of bonds is defined as

$$B := \{(t_1, \ldots, t_6) \in K_C \mid (t_1 - h_1)g_1(t_2 - h_2)g_2 \cdots (t_6 - h_6)g_6 = 0\}.$$ (4)

Let $\beta$ be a bond with coordinates $(t_1, \ldots, t_6)$. By Theorem 2 in [7], there exist indices $1 \leq i < j \leq 6$, such that $t_i^2 + 1 = t_j^2 + 1 = 0$. If there are exactly two coordinates of $\beta$ with values $\pm i$ (where $i$ denotes the imaginary unit in the field of complex numbers $\mathbb{C}$), then we say that $\beta$ connects joints $i$ and $j$. By [7, Corollary 12], we have

$$(t_i - h_i)g_i(t_{i+1} - h_{i+1})g_{i+1} \cdots (t_j - h_j) = 0.$$ (5)
In general, the situation is more complicated: a bond may connect several pairs of joints, or it may connect a single pair of joints with higher connection multiplicity; we refer to [7] for the technical details in these cases.

We visualize bonds and their connection numbers by bond diagrams. We start with the link diagram, where vertices correspond to links and edges correspond to joints. Then we draw connections between edges for the bonds connecting the corresponding joints. Note that bonds always come in pairs of complex conjugates, and conjugate bonds connect the same joints; so we draw only one line for each conjugate pair of bonds. In Figure 1, we show some known examples with bond diagrams.

Figure 1: Bond diagrams for the Cube linkage (a), the Bricard line symmetric linkage (b), the new linkage (c), the Waldron partially symmetric linkage (d), the Bricard plane symmetric linkage (e), the Hooke linkage (f), the Dietmaier linkage (g), and the Orthogonal Bricard linkage (h). The joints are labeled by $J_1, \ldots, J_6$. Each bond connects two joints.
We number the joints cyclically by $J_1,\ldots,J_6$. By [7, Theorem 3(c)], a bond cannot connect $J_i$ and $J_{i+1}$ (with addition modulo 6 because of cyclic indexing). We speak of a near connection if a bond connects joints $J_i$ and $J_{i+2}$, and of a far connection if a bond connects $J_i$ and $J_{i+3}$. For instance, the bond diagram in Figure 1(d) has 3 near connections and 6 far connections.

**Theorem 1.** A near connection implies a Bennett condition: if $J_i$ and $J_{i+2}$ are connected, then $b_i = \pm b_{i+1}$ and $s_{i+1} = 0$.

**Proof.** This is an immediate consequence of Theorem 1 and Corollary 2 of [7].

### 3 Quad Polynomials

In this section, we introduce a technique to derive algebraic equations on the Denavit/Hartenberg parameters which are necessary for the existence of a far connection. Because any mobile linkage has either a near or a far connection, this allows to deduce necessary conditions for mobility.

Assume that $\beta = (i,\alpha,\beta,i,\alpha',\beta')$ is a bond connecting $J_1$ and $J_4$ (note that the first and fourth coordinate must be $\pm i$ by the definition of a connection). We define $q := (i - i)g_1(\alpha - i)g_2(\beta - i)g_3$. Then we have

$$(i + i)q = 0 = q(i - i),$$

where the first equation follows from $i^2 - i^2 = 0$ and the second equation is a consequence of bond theory. All solutions of the two equations above are scalar multiples of $j + ik$, the scalar being an arbitrary dual number. Hence we may write $q = (a + be)(j + ik)$ with unique numbers $a, b \in \mathbb{C}$. If $a \neq 0$ (this is the case when both $\alpha$ and $\beta$ are not equal to $\pm i$), then $q$ is projectively equivalent (that is, up to multiplication by a nonzero complex number) to $(1 + x_0\epsilon)(j + ik)$ for some $x_0 \in \mathbb{C}$. Our goal is to define a polynomial $Q^+_1 \in \mathbb{C}[x]$, with coefficients depending on the Denavit/Hartenberg parameters, such that $Q^+_1(x_0) = 0$.

The triple of complex numbers $(\alpha,\beta,x_0)$ satisfies the following three equations:

1. the first coordinate (the coefficient of 1) of $q$ is zero;
2. the fifth coordinate (the coefficient of $\epsilon$) of $q$ is zero;
3. the product of the third coordinate (the coefficient of $j$) of $q$ and $x$ is equal to the seventh coordinate (the coefficient of $\epsilon j$) of $q$.

Conversely, if a triple $(\alpha,\beta,x_0)$ satisfies these three equations, it is straightforward to show that $q$ as defined above is projectively equivalent to $(1 + x_0\epsilon)(j + ik)$. We define now the quad polynomial $Q^+_1(x)$ as the resultant of the three polynomial equations above with respect to the variables $\alpha,\beta$. In order to achieve uniqueness, we assume that $Q^+_1$ is normed, that is, its leading coefficient is 1.
Using the computer algebra system Maple, the computation of the resultant can be done with symbolic expressions for the Denavit/Hartenberg parameters. The result is the quadric polynomial

\[ Q_1^+(x) = \left( x + \frac{b_3c_3 - b_1c_1}{2} - \frac{s_1}{2}i \right)^2 + \]

\[ \frac{i}{2} \left( b_1s_2 + b_3s_3 + s_2b_3c_2 + s_3b_1c_2 - \frac{b_1b_3c_2 - s_2s_3c_2}{2} + \frac{s_2^2 + s_3^2 - b_1^2 + b_2^2 - b_3^2 - b_2'c_2^2}{4} \right). \]

For \( i = 2, \ldots, 6 \), we define the quad polynomial \( Q_i^+(x) \) by a cyclic shift of indices that shifts 1 to \( i \). Finally, we define \( Q_i^-(x) \) by replacing the parameters \( c_1, \ldots, c_6, b_1, \ldots, b_6 \) and \( s_2, s_4, s_6 \) by their negatives, and leaving \( s_1, s_3, s_5 \) as they are. For instance,

\[ Q_2^-(x) = \left( x + \frac{b_3c_3 - b_1c_1}{2} - \frac{s_1}{2}i \right)^2 + \]

\[ \frac{i}{2} \left( b_1s_2 - b_3s_3 - s_2b_3c_2 + s_3b_1c_2 - \frac{-b_1b_3c_2 - s_2s_3c_2}{2} + \frac{s_2^2 + s_3^2 - b_1^2 + b_2^2 - b_3^2 - b_2'c_2^2}{4} \right). \]

**Theorem 2.** Let \( k \) be the number of bond connections of \( J_1 \) and \( J_4 \). Then

\[ k \leq \deg(\gcd(Q_1^+, Q_4^+)) + \deg(\gcd(Q_1^-, Q_4^-)). \]

**Proof.** Assume that \( \beta = (\pm i, \alpha, \pm \beta, \pm i, \alpha', \beta') \) is a bond connecting \( J_1 \) and \( J_4 \). We may assume that its first coordinate is \( i \); otherwise, we replace \( \beta \) by its complex conjugate. As above, we define \( q := (i - i)g_1(\alpha - i)g_2(\beta - i)g_3. \) By the construction of the quad polynomial \( Q_1^+ \), there exists \( x_0 \in \mathbb{C} \) such that \( q \) is projectively equivalent to \( (1 + x_0\epsilon)(j + ik) \) and \( Q_1^+(x_0) = 0. \) Now we apply a cyclic shift and obtain, in the same way, an \( x_1 \in \mathbb{C} \) such that \( q' := (i - i)g_1(\alpha' - i)g_3(\beta' - i)g_6 \) is projectively equivalent to \( (1 + x_1\epsilon)(j + ik) \) and \( Q_1^+(x_1) = 0. \) Now \( \beta \) satisfies all algebraic equations that are valid in the configuration set, in particular the equation expressing that \( (t_1 - i)g_1(t_2 - i)g_2(t_3 - i)g_4 \) is projectively equivalent to the quaternion conjugate of \( (t_4 - i)g_4(t_5 - i)g_5(t_6 - i)g_6. \) Hence \( q \) and \( q' \) are conjugate as dual quaternions, up to complex scalar multiplication. But the scalar parts of both \( q \) and \( q' \) vanish, hence \( q \) and \( q' \) are projectively equivalent. Hence \( x_0 = x_1, \) and we have derived the existence of a common zero of \( Q_1^+ \) and \( Q_4^+ \), under the assumption of the existence of a bond with \( t_1 = t_4 = i. \) Hence \( \deg(\gcd(Q_1^+, Q_4^+)) \) is an upper bound for the number of bond connections of \( J_1 \) and \( J_4 \) by bonds with \( t_1 = t_4 = i. \) Similarly, one shows that \( \deg(\gcd(Q_1^-, Q_4^-)) \) is an upper bound for the number of bond connections of \( J_1 \) and \( J_4 \) by bonds with \( t_1 = -t_4 = i. \) \( \square \)
It is well-known that two univariate polynomials have a greatest common divisor of positive degree if and only their resultant is zero. The resultant of two quad polynomials is a polynomial expression in the Denavit/Hartenberg parameters. Its vanishing gives rise to two equations, because the resultant has a real and an imaginary part. If the product of all these resultants times, say, the product of all offsets is not zero, then no bond can exist and the linkage is rigid.

However, the polynomial conditions obtained in this way are big and difficult to solve. It is therefore more promising to go through some case distinctions on the bond diagram. In order to obtain the strongest possible algebraic conditions, one should make the following assumptions on the bond structure.

- For each of the six pairs \((J_i, J_{i+2})\) of near joints, we make an assumption whether they are connected or not.
- For each pair of the three pairs \((J_i, J_{i+3})\) of far joints, we make an assumption on the number of connections by bonds with \(t_i = t_{i+3} = i\) and on the number of connections by bonds with \(t_i = -t_{i+3} = i\). In both cases, this number is in the set \(\{0, 1, 2\}\).
- The assumptions must be consistent with the condition that every joint is attached to at least one bond.

In the case when the number of connections of \((J_i, J_{i+3})\) by bonds with \(t_i = t_{i+3} = i\) is equal to 2, then the two polynomials \(Q_i^+(x)\) and \(Q_{i+3}^+(x)\) must be equal, because they are both quadratic and normed and have a quadratic greatest common divisor. This is equivalent to the vanishing of four polynomials in the Denavit/Hartenberg parameters, namely the real and the complex part of the linear and the constant coefficient of the difference polynomial.

4 Examples

In the first subsection, we apply the method of quad polynomials to several well-known families of mobile 6R linkages. The main purpose of this subsection is to show that our method is another way to “explain” already known equations with a unifying method. In the second subsection, we use the method to obtain a new family of mobile 6R linkages. This family is remarkable because two of its R-joints may be replaced by H-joints, and the linkage is still movable.

As in the section 2, we use the Bennett ratios \(b_1, \ldots, b_6\), the angle cosines \(c_1, \ldots, c_6\) and the offsets \(s_1, \ldots, s_6\). In addition, we also use the values \(f_k = c_kb_k\), \(k = 1, \ldots, 6\), as abbreviations; this leads to shorter formulas.

4.1 Some Known Examples

4.1.1 Bricard’s Line Symmetric Linkage

If \(b_i = b_{i+3}\), \(w_i = w_{i+3}\), and \(s_i = s_{i+3}\) for \(i = 1, 2, 3\), then there is a one-dimensional set of line symmetric positions which allow the link to move. Ap-
parently, we also have

\[ Q_1^+ = Q_4^+, \quad Q_2^+ = Q_5^+, \quad Q_3^+ = Q_6^+, \]

which means that the necessary conditions for a double connection between each pair of far joints are satisfied. As we saw in Figure 1(b), the bond diagram does indeed have these three double connections.

Conversely, it is not true that the 12 equations (obtained by \( Q_i^+ = Q_{i+3}^+ \) for \( i = 1, 2, 3 \)) imply that the linkage is line symmetric. A counterexample is Bricard’s orthogonal linkage, see section 4.1.4 below.

### 4.1.2 Hooke’s Double Spherical Linkage

By combining two spherical linkages with one joint in common, and then removing the common joint, we obtain a movable 6R linkage that has two triples of three joint axes meeting in a point (say the axes of \( J_6, J_1, J_2 \) and the axes of \( J_3, J_4, J_5 \)). In this case, it is easy to see that \( b_1 = b_3 = b_4 = b_6 = s_1 = s_4 = 0 \). Another equation, namely

\[ s_2 + s_3 + b_2 - f_2^2 + 2s_2s_3c_2 = s_2 + s_3 + b_5^2 + 2s_5s_6c_5 \]

can be derived by geometric considerations (see [4]). Alternatively, we consider the bond diagram of this linkage, which is shown in Figure 1(f). As we have a fourfold connection of \( J_1 \) and \( J_4 \), we get \( Q_1^+ = Q_4^+ \) and \( Q_1^- = Q_4^- \), and under the assumption that \( b_1 = b_3 = b_4 = b_6 = s_1 = s_4 = 0 \), this is equivalent to the above condition.

### 4.1.3 Dietmaier’s Linkage

In [4], Dietmaier describes a family of mobile 6R linkages, which he found by a computer-supported numerical search. It can be characterised by the equations

\[ b_6 = b_1, b_3 = b_4, b_2 = b_5, c_2 = c_3, f_6 + f_1 = f_3 + f_4, s_6 = s_2, s_3 = s_5, s_1 = s_4 = 0. \]

Its bond diagram is shown in Figure 1(g).

Starting from the assumption on the bond structure, we first obtain the conditions \( b_6 = b_1, b_3 = b_4, s_1 = s_4 = 0 \) as consequences of Bennett conditions implied by the existence of short connections. Since we have again a fourfold connection of \( J_1 \) and \( J_4 \), we again get \( Q_1^+ = Q_4^+ \) and \( Q_1^- = Q_4^- \). We added the inequality condition \( b_1b_4 \neq 0 \) and did a computer-supported analysis of the solution set using Maple. It turns out that there are two components. The first one is Dietmaier’s family. The second is given by the equations

\[ b_6 = b_1, b_3 = b_4, c_2 = c_5, f_6 + f_1 = f_3 + f_4, s_6 = s_2, s_3 = s_5, s_1 = s_4 = 0. \]

We computed the configuration set of a random instance of the second component. It appeared to be finite, hence the second component is not a family of mobile 6R linkages. However, the subset of solutions that also fulfill the condition \( f_1 + f_6 = 0 \) is a well-known family, namely the Bricard plane symmetric linkage. Its bond diagram is shown in Figure 1(e).
4.1.4 Bricard’s Orthogonal Linkage

The well-known family (see [1]) of orthogonal linkages can be described by the conditions

\[ s_1 = \cdots = s_6 = 0, \quad c_1 = \cdots = c_6 = 0, \]

\[ b_1^2 + b_2^2 + b_3^2 = b_4^2 + b_5^2 + b_6^2. \]

(The name of this family already tells the twist angles are right angles.) It is easy to prove that these equations imply

\[ Q_+^1 = Q_+^4, \quad Q_+^2 = Q_+^5, \quad Q_+^3 = Q_+^6, \quad Q^-_1 = Q^-_4, \quad Q^-_2 = Q^-_5, \quad Q^-_3 = Q^-_6, \quad (6) \]

which means that the necessary conditions for the existence of the maximal number of far connections is fulfilled. Indeed, the bond diagram has all these far connections; this can be seen in Figure 1(h).

The system of equations (6) has more solutions, leading to other linkages with the same bond diagram. This is studied in [5].

4.2 A New Example

We assume that there are no near connections, four connections of \( J_2 \) and \( J_5 \), four connections of \( J_3 \) and \( J_6 \), and two connections of \( J_1 \) and \( J_4 \) by bonds with \( t_1 = t_4 \). The bond diagram can be seen in Figure 1(c). Then we get the following equalities of polynomials in \( \mathbb{C}[x] \):

\[ Q_+^1 = Q_+^4, \quad Q_+^2 = Q_+^5, \quad Q_+^3 = Q_+^6, \quad Q^-_1 = Q^-_4, \quad Q^-_2 = Q^-_5. \quad (7) \]

Using the computer algebra system Maple, we obtained the following equivalent system of solutions:

\[ b_2 c_1 - b_3 c_5 - b_1 = b_2 c_3 - b_6 = b_5 c_6 - b_4 = 0, \]

\[ s_2 = s_3 = s_5 = s_6 = 0, s_1 = s_4. \]

The solution set is irreducible. Here is a random numerical example:

\[ b_1 = \frac{1}{3}, b_2 = -\frac{61}{33}, b_3 = \frac{305}{429}, b_4 = \frac{2000}{1001}, b_5 = \frac{2900}{1001}, b_6 = \frac{1740}{1001^1}; \]

\[ w_1 = \frac{2}{3}, w_2 = -4, w_3 = \frac{6}{5}, w_4 = \frac{1}{2}, w_5 = \frac{\sqrt{540838349}}{6619}, w_6 = \frac{3}{7}; \]

\[ s_2 = s_3 = s_5 = s_6 = 0, s_1 = s_4 = \frac{2}{3}. \]

We compute the configuration space of this example and it turns out that it is indeed a curve, hence the linkage is mobile.

One can observe, in addition, that all points in the configuration space satisfy the equation \( t_1 = t_4 \). When we vary \( s_1 = s_4 \), we get a configuration set of a CRRCCR linkage (2 cylindrical joints). This set is an infinite union of curves,
so its dimension is 2. Let \( a \in \mathbb{R} \). Then we may pose a new constraint like 
\[ t_1 = \cot(as_1), \]
and still have dimension 1; the second constraint 
\[ t_4 = \cot(as_4) \]
is implied because \( t_1 = t_4 \) and \( s_1 = s_4 \). This defines an HRRHRR linkage with mobility 1.

**Acknowledgements**

The research was supported by the Austrian Science Fund (FWF): W1214-N15, project DK9.

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