Weight parameterization of simple modules
for $p$-solvable groups

Lluis Puig

CNRS, Institut de Mathématiques de Jussieu
6 Av Bizet, 94340 Joinville-le-Pont, France
puig@math.jussieu.fr

1. Introduction

1.1. The weights for a finite group $G$ with respect to a prime number $p$
where introduced by Jon Alperin in [1] in order to formulate his celebra
ted conjecture. Explicitly, a weight of $G$ is a pair $(R, Y)$ formed by a $p$-subgroup
$R$ of $G$ and by an isomorphism class $Y$ of simple $kN_G(R)$-modules with
vertex $R$; then, Alperin’s Conjecture affirms that the number of $G$-conjugacy
classes of weights of $G$ coincides with the number of isomorphism classes of
simple $kG$-modules, where $k$ is an algebraically closed field of characteristic $p$.
More precisely, Alperin’s Conjecture involves the blocks of $G$ as we explain
below.

1.2. In the case that $G$ is $p$-solvable, thirty years ago Tetsuro Okuyama
[8] already proved that, for any $p$-subgroup $R$ of $G$, the number of isomor-
phism classes $Y$ of simple $kN_G(R)$-modules with vertex $R$ coincides with the
number of isomorphism classes of simple $kG$-modules of vertex $R$, which
clearly shows Alperin’s Conjecture restricted to $p$-solvable groups. Once
again, Okuyama’s result actually involves the blocks of $G$.

1.3. On the other hand, in [11, 6.4] we introduce a multiplicity module
for any indecomposable $kG$-module $M$, and in [11, Lemma 9.9] we prove that $M$
is determined by the triple formed by a vertex $R$, an $R$-source $E$ and a mul-
tiplicity module $V$ of $M$ — an indecomposable projective $k_\ast \hat{N}_G(R)_E$-module
where $\hat{N}_G(R)_E$ is the stabilizer of the isomorphism class of $E$ in $N_G(R)_E$,
$\hat{N}_G(R)_E$ is the central $k^*$-extension of $N_G(R)_E$ determined by the action on
End$_k(E)$, and $k_\ast \hat{N}_G(R)_E$ is the corresponding twisted group algebra (cf. 2.5
below) — and that this correspondence actually defines a bijection between
the set of isomorphism classes of indecomposable $kG$-modules and the set of
$G$-conjugacy classes of triples $(R, E, V)$ formed by a $p$-subgroup $R$ of $G$, an
indecomposable $kR$-module $E$ of vertex $R$ and an indecomposable projective
$k_\ast \hat{N}_G(R)_E$-module $V$.

1.4. Moreover, if $M$ is a simple $kG$-module then it follows from [9, Propo-
sition 1.6] that $V$ is actually a simple projective $k_\ast \hat{N}_G(R)_E$-module. But,
in the case that \( G \) is \( p \)-solvable and \( M \) is primitive — namely, not induced from any proper subgroup — it is well-known \([17, \text{Lemma 30.4}]\) that there is a \( G \)-stable finite \( p' \)-subgroup \( K \) of \( \text{End}_k(M) \) generating the \( k \)-algebra \( \text{End}_k(M) \). Consequently, in this case \( \text{End}_k(M) \) is actually a Dade \( R \)-algebra \([13, 1.3]\); in particular, \( N_G(R) \)-stabilizes the isomorphism class of \( E \) \([13, 1.8]\) and it follows from \([15, \text{Theorem 9.21}]\) that the central \( k^* \)-extension \( k_*N_G(R) \) above is split — we are more explicit from 2.13 to 2.17 below.

1.5. That is to say, if \( G \) is \( p \)-solvable and \( M \) a primitive simple \( kG \)-module, then the pair formed by a vertex \( R \) and by the isomorphism class of the restriction to \( N_G(R) \) of a multiplicity \( k^* \)-module \( V \) — after a choice of a splitting for the corresponding central \( k^* \)-extension — is actually a weight of \( G \). More generally, since any simple \( kG \)-module is certainly induced from a primitive simple \( kH \)-module for some subgroup \( H \) of \( G \), if \( G \) is \( p \)-solvable then \( \text{End}_k(E) \) is always a Dade \( R \)-algebra for any vertex \( R \) and any \( R \)-source \( E \) of \( M \); hence, in this case, the central \( k^* \)-extension \( k_*N_G(R)_E \) is always split and the corresponding multiplicity module \( V \) becomes a simple projective \( k_{\bar{N}_G(R)_E} \)-module.

1.6. In this paper, for a systematic choice of those splittings via a polarization \([15, 9.5]\), on the one hand we exhibit a natural bijection — namely compatible with the action of the group of outer automorphisms of \( G \) — between the sets of isomorphism classes of simple \( kG \)-modules \( M \) and of \( G \)-conjugacy classes of weights \((R,Y)\) of \( G \). On the other hand, we determine the relationship between a multiplicity \( k_*N_G(R)_E \)-module \( V \) and a simple \( kN_G(R) \)-module \( U \) with vertex \( R \) in the class \( Y \) of the corresponding weight of \( G \); explicitly, there is a subgroup \( N \) of \( N_G(R)_E \) containing \( R \), a simple \( kN \)-module \( W \) of vertex \( R \) and, setting \( \bar{N} = N/R \), a group homomorphism \( \theta : \bar{N} \to k^* \) in such a way that, denoting by \( \bar{W} \) the corresponding \( k\bar{N} \)-module and setting \( \bar{W}_\theta = k_\theta \otimes_k \bar{W} \), we have

\[
U \cong \text{Ind}_{N_G(R)}^N(W) \quad \text{and} \quad V \cong \text{Ind}_{\bar{N}}^{N_G(R)_E}(\bar{W}_\theta) \quad 1.6.1.
\]

The tools to carry out our purpose are mainly the Fong reduction theorems developed in \([16]\); as in that paper, it is handy — but not more general! — to work systematically with \( k^* \)-groups with finite \( k^* \)-quotient \( G \) \([11, \text{§5}]\) — namely, with central \( k^* \)-extensions of \( G \).

1.7. In 1994, when talking about this work at Beijing University, Zhang Jiping pointed out to us that Gabriel Navarro \([7]\) already had given a bijection between the above sets of isomorphism classes of simple \( kG \)-modules and of \( G \)-conjugacy classes of weights for finite groups of odd order, and therefore solvable. In our Appendix we show that Navarro’s bijection corresponds indeed to the bijection obtained for a particular choice of the splittings above, a choice which is only possible for groups of odd order.
2. Notations and quoted results

2.1. We fix a prime number $p$ and an algebraically closed field $k$ of characteristic $p$. We call $k^*$-group a group $X$ endowed with an injective group homomorphism $\theta : k^* \to Z(X)$ [11, §5], and call $k^*$-quotient of $(X, \theta)$ the group $X/\theta(k^*)$; we denote by $X^\circ$ the $k^*$-group formed by $X$ and by the composition of $\theta$ with the automorphism $k^* \cong k^*$ mapping $\lambda \in k^*$ on $\lambda^{-1}$; we say that a $k^*$-group is finite whenever its $k^*$-quotient is finite. Usually, we denote by $\hat{G}$ a $k^*$-group and by $G$ its $k^*$-quotient, and we write $\lambda \cdot x$ for the product of $x \in \hat{G}$ and the image of $\lambda \in k^*$ in $\hat{G}$.

2.2. If $\hat{G}'$ is a second $k^*$-group, we denote by $\hat{G} \times \hat{G}'$ the quotient of the direct product $\hat{G} \times \hat{G}'$ by the image in $\hat{G} \times \hat{G}'$ of the inverse diagonal of $k^* \times k^*$, which has an obvious structure of $k^*$-group with $k^*$-quotient $G \times G'$; moreover, if $G = G'$ then we denote by $\hat{G} \ast \hat{G}'$ the $k^*$-group obtained from the inverse image of $\Delta(G) \subset G \times G$ in $\hat{G} \times \hat{G}'$, which is nothing but the so-called sum of both central $k^*$-extensions of $G$; in particular, we have a canonical $k^*$-group isomorphism

$$\hat{G} \ast \hat{G}' \cong k^* \times G$$

2.2.1.

A $k^*$-group homomorphism $\varphi : \hat{G} \to \hat{G}'$ is a group homomorphism which preserves the $k^*$-multiplication; moreover, if $\hat{G}$ and $\hat{G}'$ are isomorphic then the group $\text{Hom}(G, k^*)$ acts regularly over the set of isomorphisms $\psi : \hat{G} \cong \hat{G}'$ and we denote by $\psi^\theta$ the $k^*$-group isomorphism determined by $\theta \in \text{Hom}(G, k^*)$ and $\psi$. We denote by $k^*-\mathfrak{Gr}$ the category of $k^*$-groups.

2.3. Note that for any $k$-algebra $A$ of finite dimension — just called $k$-algebra in the sequel — the group $A^*$ of invertible elements has a canonical $k^*$-group structure; we call point of $A$ any $A^*$-conjugacy class $\alpha$ of primitive idempotents of $A$ and denote by $A(\alpha)$ the simple quotient of $A$ determined by $\alpha$, and by $\mathcal{P}(A)$ the set of points of $A$. If $S$ is a simple algebra then $\text{Aut}_k(S)$ coincides with the $k^*$-quotient of $S^*$; in particular, any finite group $G$ acting on $S$ determines — by pull-back — a $k^*$-group $\hat{G}$ of $k^*$-quotient $G$, together with a $k^*$-group homomorphism [11, 5.7]

$$\rho : \hat{G} \to S^*$$

2.3.1.

2.4. If $\hat{G}$ is a finite $k^*$-group, we call $\hat{G}$-interior algebra any $k$-algebra $A$ endowed with a $k^*$-group homomorphism

$$\rho : \hat{G} \to A^*$$

2.4.1.

and, as usual, we write $\hat{x}a$ and $a \cdot \hat{x}$ instead of $\rho(\hat{x})a$ and $a\rho(\hat{x})$ for any $\hat{x} \in \hat{G}$ and any $a \in A$; we say that $A$ is primitive whenever the unity element is primitive in $A^\hat{G}$. A $\hat{G}$-interior algebra homomorphism from $A$ to another $\hat{G}$-interior algebra $A'$ is a not necessarily unitary algebra homomorphism
We denote by \( F \) then, it is quite clear that the (\( \ding{10} \))More generally, we say that an injective \( k \)-algebra embedding, a \( k \)-group isomorphism \( \alpha \in H \) on \( A \), so that \( A \) becomes an ordinary \( G \)-algebra; thus, all the pointed group language developed in [9] applies to \( \hat{G} \)-interior algebras.

2.5. Namely, for any \( k \)-subgroup \( \hat{H} \) of \( \hat{G} \), a point \( \alpha \) of \( \hat{H} \) on \( A \) is just a point of the \( k \)-algebra \( A^H \), and the pair \( \hat{H}_{\alpha} \) is a pointed \( k^* \)-group on \( A \); we denote by \( A(\hat{H}_{\alpha}) \) the simple quotient \( A^H(\alpha) \) and, setting

\[
\hat{N}_G(\hat{H}_{\alpha}) = N_G(\hat{H}_{\alpha})/H \quad \text{and} \quad A(\hat{H}_{\alpha}) = \text{End}_k(V_{\alpha})
\]

by \( \hat{N}_G(\hat{H}_{\alpha}) \) the \( k^* \)-group determined by the action of \( \hat{N}_G(\hat{H}_{\alpha}) \) on \( A(\hat{H}_{\alpha}) \), so that \( V_{\alpha} \) becomes a \( \hat{N}_G(\hat{H}_{\alpha}) \)-module called the multiplicity \( \hat{N}_G(\hat{H}_{\alpha}) \)-module of \( \hat{H}_{\alpha} \) [11, 6.4]. For any \( i \in \alpha \), \( iA_{\alpha} \) has an evident structure of \( H \)-interior algebra mapping \( \hat{x} \in H \) on \( \hat{x} \cdot i = i \cdot \hat{x} \) and we denote by \( A_{\alpha} \) one of these mutually \( (A^H)^* \)-conjugate \( H \)-interior algebras. If \( A' \) is another \( \hat{G} \)-interior algebra and \( f : A \to A' \) a \( \hat{G} \)-interior algebra embedding, \( f(\alpha) \) is contained in a unique point \( \alpha' \) of \( \hat{H} \) on \( A' \), usually identified with \( \alpha \), and \( f \) induces a \( k \)-algebra embedding, a \( k^* \)-group isomorphism and an \( H \)-interior algebra isomorphism

\[
A(\hat{H}_{\alpha}) \longrightarrow A'(\hat{H}_{\alpha'}) \quad , \quad \hat{N}_G(\hat{H}_{\alpha}) \cong \hat{N}_G(\hat{H}_{\alpha'}) \quad \text{and} \quad A_{\alpha} \stackrel{f_{\alpha'}}{\cong} A'_{\alpha'}
\]

2.6. A second pointed \( k^* \)-group \( \hat{K}_\beta \) on \( A \) is contained in \( \hat{H}_{\alpha} \) if \( \hat{K} \) is a \( k^* \)-subgroup of \( \hat{H} \) and, for any \( i \in \alpha \), there is \( j \in \beta \) such that \( ij = j = ji \); then, it is quite clear that the \( (A^K)^* \)-conjugation induces a \( \hat{K} \)-interior algebra embedding

\[
f^K_{\beta} : A_{\beta} \longrightarrow \text{Res}_{\hat{K}}^{\hat{H}}(A_{\alpha})
\]

More generally, we say that an injective \( k^* \)-group homomorphism \( \varphi : \hat{K} \to \hat{H} \) is an \( A \)-fusion from \( \hat{K}_{\beta} \) to \( \hat{H}_{\alpha} \) whenever there is a \( \hat{K} \)-interior algebra embedding

\[
f_{\varphi} : A_{\beta} \longrightarrow \text{Res}_{\varphi}(A_{\alpha})
\]

such that the inclusion \( A_{\beta} \subset A \) and the composition of \( f_{\varphi} \) with the inclusion \( A_{\alpha} \subset A \) are \( A^* \)-conjugate; then, the exterior embedding \( \tilde{f}_{\varphi} \) is uniquely determined [10, 2.8]. We denote by \( F_A(\hat{K}_{\beta}, \hat{H}_{\alpha}) \) the set of \( H \)-conjugacy classes of
$A$-fusions from $\hat{K}_x$ to $\hat{H}_\alpha$ [12, Definition 2.5] and we simply set

$$F_A(\hat{H}_\alpha) = F_A(\hat{H}_\alpha, \hat{H}_\alpha)$$  \quad 2.6.3;

note that the conjugation in $\hat{G}$ induces a canonical group homomorphism

$$\hat{N}_G(\hat{H}_\alpha) \rightarrow F_A(\hat{H}_\alpha)$$  \quad 2.6.4.

If $A'$ is another $\hat{G}$-interior algebra and $f : A \rightarrow A'$ a $\hat{G}$-interior algebra embedding, it follows from [10, Proposition 2.14] that we have

$$F_A(\hat{K}_x, \hat{H}_\alpha) = F_A'(\hat{K}_x, \hat{H}_\alpha)$$  \quad 2.6.5.

2.7. Note that any $p$-subgroup $P$ of $\hat{G}$ can be identified with its image in $G$ and determines the $k^*$-subgroup $k^*P \cong k^* \times P$ of $\hat{G}$; as usual, we consider the Brauer quotient and the Brauer algebra homomorphism

$$\text{Br}_P : A^P \rightarrow A(P) = A^P/\sum Q A_Q^P$$  \quad 2.7.1,

where $Q$ runs over the set of proper subgroups of $P$, and call local any point $\gamma$ of $P$ on $A$ not contained in $\text{Ker}(\text{Br}_P)$; recall that all the maximal local pointed groups $P_\gamma$ on $A$ contained in $\hat{H}_\alpha$ — called defect pointed groups of $\hat{H}_\alpha$ — are mutually $H$-conjugate [9, Theorem 1.2], and that the $k$-algebras $A_\alpha$ and $A_\gamma$ are Morita equivalent [9, Corollary 3.5]. If $A_\gamma = iAi$ for $i \in \gamma$, it follows from [10, Corollary 2.13] that we have a group homomorphism

$$F_A(\gamma) : N_{A_{\gamma}}((P \cdot i)/P \cdot (A_P^\gamma)^*)^*$$  \quad 2.7.2

and we consider the $k^*$-group $\hat{F}_A(\gamma)$ defined by the pull-back

$$\begin{array}{ccc}
F_A(\gamma) & \rightarrow & N_{A_{\gamma}}((P \cdot i)/P \cdot (A_P^\gamma)^*)^* \\
\uparrow & & \uparrow \\
\hat{F}_A(\gamma) & \rightarrow & N_{A_{\gamma}}((P \cdot i)/P \cdot (i + J(A_P^\gamma)))
\end{array}$$  \quad 2.7.3.

2.8. Then, from [11, Proposition 6.12] suitably extended to $k^*$-groups, it follows that the group homomorphism 2.6.4 can be lifted to a canonical $k^*$-group homomorphism

$$\hat{N}_G(\gamma) \ast \hat{N}_G(\gamma) \rightarrow \hat{F}_A(\gamma)$$  \quad 2.8.1

which, for any $\hat{x} \in \hat{N}_G(\gamma) = N_G(\gamma)/P$ and any $a \in (A_P^\gamma)^*$ having the same action on $A(\gamma)$, maps the element $(x, \hat{a}) \ast \hat{x}^{-1}$ of $\hat{N}_G(\gamma) \ast \hat{N}_G(\gamma)$ on the pair [11, Proposition 6.10]

$$(\hat{x}^{-1}, \hat{i}(\hat{x}^{-1} \cdot a)i) \in \hat{F}_A(\gamma)$$  \quad 2.8.2,

where $x$ denotes the image of $\hat{x}$ in $\hat{N}_G(\gamma)$, $\hat{a}$ the image of $a$ in $A(\gamma)$, $\hat{x}$ the image of $x$ in $F_A(\gamma)$ via homomorphism 2.6.4 and $\hat{i}(\hat{x}^{-1} \cdot a)i$ the image of $i(\hat{x}^{-1} \cdot a)i$ in the right-hand bottom of diagram 2.7.3.
2.9. If $A'$ is another $\hat{G}$-interior algebra and $f: A \to A'$ a $\hat{G}$-interior algebra embedding, it follows from [11, Proposition 6.8] that, denoting by $\gamma'$ the point of $P$ on $A'$ containing $f(\gamma)$, we have a canonical $k^*$-group isomorphism
\[
\hat{F} f(P_\gamma) : \hat{F}_A(P_\gamma) \cong \hat{F}_{A'}(P_{\gamma'})
\] (2.9.1)
which, according to [11, Proposition 6.21], is compatible with the corresponding $k^*$-group homomorphisms 2.8.1 and 2.5.2. More precisely, let $Q_\delta$ be another local pointed group on $A$ and denote by $\delta'$ the point of $Q$ on $A'$ containing $f(\delta)$; if there is a group isomorphism $\varphi : Q \cong P$ which is an $A$-fusion from $Q_\delta$ to $P_\gamma$, then, according to equality 2.6.5 above, $\varphi$ is also an $A'$-fusion from $Q_{\delta'}$ to $P_{\gamma'}$, so that we have two $Q$-interior algebra isomorphisms
\[
f_\varphi : A_\delta \cong \text{Res}_\varphi(A_\gamma) \quad \text{and} \quad f'_\varphi : A'_{\delta'} \cong \text{Res}_\varphi'(A'_{\gamma'})
\] (2.9.2)
and the uniqueness of the exterior isomorphisms $\hat{f}_\varphi$ and $\hat{f}'_\varphi$ forces the equality
\[
\hat{f}'_\varphi \circ \hat{f}_\delta = \text{Res}_{\varphi'}(\hat{f}'_{\gamma'}) \circ \hat{f}_\varphi
\] (2.9.3)
In particular, since by the very definition we have
\[
\hat{F}_{\text{Res}_\varphi(A_\gamma)}(Q_\delta) = \hat{F}_A(P_\gamma) \quad \text{and} \quad \hat{F}_{\text{Res}_\varphi'(A'_{\gamma'})}(Q_{\delta'}) = \hat{F}_{A'}(P_{\gamma'})
\] (2.9.4)
we get the following commutative diagram of $k^*$-group isomorphisms
\[
\begin{array}{ccc}
\hat{F}_A(Q_\delta) & \cong & \hat{F}_A(P_\gamma) \\
\hat{F}_f(Q_\delta) & \| & \hat{F}_f(P_\gamma) \\
\hat{F}_{A'}(Q_{\delta'}) & \cong & \hat{F}_{A'}(P_{\gamma'})
\end{array}
\] (2.9.5)

2.10. It is clear that the inclusion $k^* \subset k$ determines a $k$-algebra homomorphism to $k$ from the group algebra $kk^*$ of the group $k^*$, so that $k$ becomes a $kk^*$-algebra; for any finite $k^*$-group $\hat{G}$, it is clear that the group algebra $k\hat{G}$ of the group $\hat{G}$ is also a $kk^*$-algebra and then, we call $k^*$-group algebra of $\hat{G}$ the algebra
\[
k_\ast \hat{G} = k \otimes_{kk^*} k\hat{G}
\] (2.10.1)
note that the dimension of $k_\ast \hat{G}$ is equal to $|\hat{G}|$. Coherently, a block of $\hat{G}$ is a primitive idempotent $b$ of the center $Z(k_\ast \hat{G})$, so that $\alpha = \{b\}$ is a point of $\hat{G}$ on $k_\ast \hat{G}$; as usual, we denote by $\text{Irr}_b(\hat{G}, k)$ the set of Brauer characters of all the simple $k_\ast \hat{G}b$-modules, which corresponds bijectively with the set of points $\mathcal{P}(k_\ast \hat{G}b)$.

2.11. Recall that for any $p$-subgroup $P$ of $\hat{G}$ we have [11, 2.10.2 and Proposition 5.15]
\[
(k_\ast \hat{G})(P) \cong k_\ast C_\hat{G}(P)
\] (2.11.1)
in particular, if $P$ is normal in $G$, since the kernel of the obvious $k$-algebra homomorphism $k_\ast \hat{G} \to k_\ast (\hat{G}/P)$ is contained in the radical $J(k_\ast \hat{G})$ and contains $\text{Ker}(\text{Br}_P)$, this isomorphism implies that any point of $P$ on $k_\ast \hat{G}$ is local. Moreover, it follows from [10, Theorem 3.1] that we have

2.11.2 For any pair of local pointed groups $P_\gamma$ and $Q_\delta$ on $k_\ast \hat{G}$, a $k_\ast \hat{G}$-fusion from $Q_\delta$ to $P_\gamma$ coincides with the conjugation by an element $x \in G$ such that $Q_\delta \subset (P_\gamma)^x$.

2.12. If $\hat{G}$ is a finite $k^\ast$-group, $A$ a $\hat{G}$-interior algebra and $\hat{H}$ a $k^\ast$-subgroup of $\hat{G}$, as usual we denote by $\text{Res}\hat{G}_H(A)$ the corresponding $\hat{H}$-interior algebra. Conversely, for any $\hat{H}$-interior algebra $B$, we consider the induced $G$-interior algebra

$$\text{Ind}\hat{G}_H(B) = k_\ast \hat{G} \otimes_{k_\ast \hat{H}} B \otimes_{k_\ast \hat{H}} k_\ast \hat{G}$$

2.12.1 where the distributive product is defined by the formula

$$(\hat{x} \otimes \hat{b} \otimes \hat{y})(\hat{x}' \otimes \hat{b}' \otimes \hat{y}') = \begin{cases} \hat{x} \otimes \hat{b} \hat{y} \hat{x}' \otimes \hat{b}' \otimes \hat{y}' & \text{if } \hat{y} \hat{x}' \in \hat{H} \\ 0 & \text{otherwise} \end{cases}$$

2.12.2 for any $\hat{x}, \hat{y}, \hat{x}', \hat{y}' \in \hat{G}$ and any $\hat{b}, \hat{b}' \in B$, and where we map $\hat{x} \in \hat{G}$ on the element

$$\sum_{\hat{y}} \hat{x} \hat{y} \otimes 1_B \otimes \hat{y}^{-1} = \sum_{\hat{y}} \hat{y} \otimes 1_B \otimes \hat{y}^{-1} \hat{x}$$

2.12.3 $\hat{y} \in \hat{G}$ running over a set of representatives for $\hat{G}/\hat{H}$.

2.13. For a finite $p$-group $P$, we call Dade $P$-algebra [13, 1.3] a simple algebra $S$ endowed with an action of $P$ which stabilizes a basis of $S$ containing the unity element; actually, the action of $P$ on $S$ can be lifted to a unique group homomorphism $P \to S^\ast$ and usually we consider $S$ as a $P$-interior algebra; moreover, the Brauer quotient $S(P)$ is also a simple $k$-algebra [13, 1.8] which implies that $P$ has a unique local point $\rho$ on $S$ that very often we omit, respectively writing $F_S(P)$ and $F_S'(P)$ instead of $F_S(P_\rho)$ and $F_S'(P_\rho)$.

Recall that two Dade $P$-algebras $S$ and $S'$ are similar if $S$ can be embedded (cf. 2.4) in the tensor product $\text{End}(\mathcal{N}) \otimes_k S'$ for a suitable $kP$-module $\mathcal{N}$ with a $P$-stable basis [13, 1.5 and 2.5.1]; we denote by $\mathcal{D}_k(P)$ the set of similarity classes and the tensor product induces a group structure on $\mathcal{D}_k(P)$ — called the Dade group of $P$ — where the opposite $P$-algebra $S^\circ$ determines the inverse of the similarity class of $S$.

2.14. As in [15, 9.3], it is handy to consider the category $\mathcal{D}_k$ where the objects are the pairs $(P, S)$ formed by a finite $p$-group $P$ and by a Dade $P$-algebra $S$, and where a morphism from $(P, S)$ to a second $\mathcal{D}_k$-object
are the pairs \((\pi, f)\) formed by a surjective group homomorphism \(\pi : P \to P'\) such that \(\text{Ker}(\pi)\) is \(F_S(P)\)-stable, and by a \(P\)-interior algebra embedding

\[
f : \text{Res}_\pi(S') \to S
\]

Then, we have functors \(\hat{f}\) and \(\hat{f}'\) mapping \((P, S)\) on \(F_S(P)\) and \(\hat{F}_S(P)\) [15, 9.5], together a natural map \(\hat{f} \to f\) mapping \((P, S)\) on the structural homomorphism

\[
\hat{F}_S(P) \to F_S(P)
\]

2.14.1.

2.15. As in [15, 9.5], we call polarization any natural map \(\omega\) from the functor \(\hat{F} : \mathcal{D}_k \to k^*\text{-Gr}\) above to the trivial one — namely, to the functor mapping \((P, S)\) on \(k^*\) and \((\pi, f)\) on \(\text{id}_{k^*}\) — such that if \(T\) is a \(P\)-algebra with trivial \(P\)-action then \(\omega\) maps \((P, T)\) on the first projection in the isomorphism

\[
\hat{F}_T(P) \cong k^* \times F_T(P)
\]

2.15.1

obtained from the corresponding pull-back 2.7.3. The point is that, according to [15, Theorem 9.21], there exists such a natural map, and we will construct a bijection as announced above from any choice of a polarization \(\omega\), namely from any choice, in a coherent way, of a \(k^*\)-group homomorphism

\[
\omega(P, S) : \hat{F}_S(P) \to k^*
\]

2.15.2

for any \(\mathcal{D}_k\)-object \((P, S)\). A first application of this existence concerns the multiplicity modules of the indecomposable \(k^*\hat{G}\)-modules \(M\) having a vertex \(P\) and a \(P\)-source \(N\) such that \(\text{End}_k(N)\) is a Dade \(P\)-algebra.

Lemma 2.16. Let \(\hat{G}\) be a finite \(k^*\)-group, \(M\) an indecomposable \(k^*\hat{G}\)-module, \(P\) a vertex and \(N\) a \(P\)-source of \(M\); let us denote by \(P_N\) the local pointed group on the \(\hat{G}\)-interior algebra \(\text{End}_k(M)\) determined by the pair \((P, N)\).

If \(\text{End}_k(N)\) is a Dade \(P\)-algebra then the action of \(\tilde{N}_G(P_N)\) on the simple quotient \((\text{End}_k(M))(P_N)\) can be lifted to a \(k^*\)-group homomorphism

\[
\tilde{N}_G(P_N) \to (\text{End}_k(M))(P_N)^*
\]

2.16.1

Proof: In any case, this action determines a \(k^*\)-group \(\tilde{N}_G(P_N)\) and we have a canonical \(k^*\)-group homomorphism (cf. 2.8.1)

\[
\tilde{N}_G(P_N) * \tilde{N}_G(P_N)^* \to \hat{F}_{\text{End}_k(N)}(P_N)^*
\]

2.16.2

but, if \(\text{End}_k(N)\) is a Dade \(P\)-algebra, the existence of a polarization implies that, in particular, we have

\[
\hat{F}_{\text{End}_k(N)}(P) \cong k^* \times F_{\text{End}_k(N)}(P)
\]

2.16.3

consequently, we get \(\tilde{N}_G(P_N) \cong \tilde{N}_G(P_N)\). We are done.
2.17. More generally, if $S$ is a Dade $P$-algebra and $A$ a $P$-interior algebra, it follows from [12, Theorem 5.3] that, for any subgroup $Q$ of $P$, we have a canonical bijection between the sets of local points of $Q$ on $A$ and on $S \otimes_k A$; moreover, if $A$ admits a $P \times P$-stable basis by the multiplication on both sides, where $P \times \{1\}$ and $\{1\} \times P$ act freely, it follows from [6, Lemma 1.17] that, for any pair of local pointed groups $Q_\delta$ and $R_\varepsilon$ on $A$, we have

$$F_{S \otimes_k A}(R \times \varepsilon, Q \times \delta) = F_S(R, Q) \cap F_A(R_\varepsilon, Q_\delta)$$

2.17.1

where $S \times \varepsilon$ and $S \times \delta$ denote the corresponding local points of $R$ and $Q$ on $S \otimes_k A$; in this case, since the choice of a polarization $\omega$ determines a $k^*$-group homomorphism

$$\omega_{(Q, \Res^G_S(S))} : \hat{F}_S(Q) \to k^*$$

2.17.2

it follows from [12, Proposition 5.11] that the inclusion of $F_{S \otimes_k A}(Q \times \delta)$ in $F_A(Q_\delta)$ can be lifted to a $k^*$-group homomorphism determined by $\omega$

$$\Phi^\omega_S(Q_\delta) : \hat{F}_{S \otimes_k A}(Q \times \delta) \to \hat{F}_A(Q_\delta)$$

2.17.3

More precisely, as in 2.9 above, if $A'$ is a $P$-interior algebra and $f : A \to A'$ a $P$-interior algebra embedding, denoting by $\delta'$ the point of $Q$ on $A'$ containing $f(\delta)$, from [12, Proposition 5.11] we still get the following commutative diagram of $k^*$-group homomorphism

$$
\begin{array}{ccc}
\hat{F}_{S \otimes_k A}(Q \times \delta) & \xrightarrow{\Phi^\omega_S(Q_\delta)} & \hat{F}_A(Q_\delta) \\
\hat{F}_{\Res^G_S(S)(Q \times \delta)} & \parallel & \hat{F}_A(Q_\delta)
\end{array}
$$

2.17.4

3. The weights revisited

3.1. Let $\hat{G}$ be a finite $k^*$-group; we say that a local pointed group $Q_\delta$ on $k_*\hat{G}$ is selfcentralizing if $C_P(Q) = Z(Q)$ for any local pointed group $P_\gamma$ on $k_*\hat{G}$ containing $Q_\delta$, and that it is a radical whenever it is selfcentralizing and we have

$$\mathcal{O}_p(F_{k_*\hat{G}}(Q_\delta)) = \{1\}$$

3.1.1

Recall that, according to [15, 4.8 and Corollary 7.3], $Q_\delta$ is selfcentralizing if, denoting by $f$ the block of $C_{\hat{G}}(Q)$ determined by $\delta$, the image $\tilde{f}$ of $f$ in the $k^*$-group algebra of $C_{\hat{G}}(Q) = C_{\hat{G}}(Q)/Z(Q)$ is a block of defect zero; note that, in this case, $\delta$ is the unique local point of $Q$ on $k_*\hat{G}$ determining the block $f$.

3.2. As mentioned in 1.2 above, a weight $(R, Y)$ of $\hat{G}$ is formed by a $p$-subgroup $R$ of $\hat{G}$ and by the isomorphism class $Y$ of the restriction to $N_{\hat{G}}(R)$ of a simple projective $k_*\hat{N}_{\hat{G}}(R)$-module $V$, where we set $\hat{N}_{\hat{G}}(R) = N_{\hat{G}}(R)/R$.
let us denote by $\text{Wgt}_k(\hat{G})$ the set of $G$-conjugacy classes of weights of $\hat{G}$. Then, the restriction of $V$ to $\hat{C}_G(R) \triangleleft \hat{N}_G(R)$ is a semisimple projective $k_*\hat{C}_G(R)$-module and thus any simple direct summand $W$ of $\text{Res}_{\hat{C}_G(R)}(V)$ is also projective, so that it determines the unique local point $\varnothing$ of $R$ on $k_*\hat{G}$ (cf. 2.11.1) in a block $\bar{g}$ of defect zero of $\hat{C}_G(R)$; that is to say, $W$ determines a self-centralizing pointed group $R_\varnothing$ on $k_*\hat{G}$ and the stabilizer of the isomorphism class of $W$ in $N_G(R)$ coincides with $N_G(R_\varnothing)$.

3.3. Moreover, it follows from isomorphism 2.11.1 that we have

$$(k_*\hat{G})(R_{\varnothing}) \cong k_*\hat{C}_G(R) \bar{g} \cong \text{End}_k(W) \quad \text{3.3.1}$$

and from 2.5 we know that $W$ becomes an $\hat{N}_G(R_{\varnothing})$-module; then, since the $\hat{N}_G(R)$-interior algebra $\text{End}_k(V)$ is isomorphic to a suitable block algebra of $\hat{N}_G(R)$, and since we have (cf. 2.10)

$$N_G(R_{\varnothing})/C_G(R) \cong F_{k_*\hat{G}}(R_{\varnothing}) \quad \text{3.3.2},$$

if follows from [16, Theorem 3.7] and from 2.8 above that, for a suitable simple projective $k_*\hat{F}_{k_*\hat{G}}(R_{\varnothing})$-module $U$ restricted to $\hat{N}_G(R_{\varnothing})^\circ \ast \hat{N}_G(R_{\varnothing})$ via homomorphism 2.8.1, we obtain

$$V \cong \text{Ind}_{\hat{N}_G(R_{\varnothing})}^{N_G(R_{\varnothing})}(W \otimes_k U) \quad \text{3.3.3};$$

in particular, we get

$$\langle p(F_{k_*\hat{G}}(R_{\varnothing})) \rangle = \{1\} \quad \text{3.3.4},$$

so that $R_{\varnothing}$ is a radical pointed group.

3.4. Conversely, if $R_{\varnothing}$ is a radical pointed group on $k_*\hat{G}$ and $U$ a simple projective $k_*\hat{F}_{k_*\hat{G}}(R_{\varnothing})$-module, it is easily checked that the restriction of $U$ to $\hat{N}_G(R_{\varnothing})^\circ \ast \hat{N}_G(R_{\varnothing})$ throughout homomorphism 2.8.1, together with a multiplicity $\hat{N}_G(R_{\varnothing})$-module $W$ of $R_{\varnothing}$ define a simple projective $k_*\hat{N}_G(R_{\varnothing})$-module via the tensor product and the induction as in 3.3.3. In conclusion, we have proved that

3.4.1. **The above correspondence between the sets of $G$-conjugacy classes of weights $(R,Y)$ of $\hat{G}$ and of pairs $(R_{\varnothing},X)$ formed by a radical pointed group $R_{\varnothing}$ on $k_*\hat{G}$ and by an isomorphism class $X$ of simple projective $k_*\hat{F}_{k_*\hat{G}}(R_{\varnothing})$-modules is bijective.**

Let us call $b$-weight of $\hat{G}$ any pair $(R_{\varnothing},X)$ formed by a radical pointed group $R_{\varnothing}$ on $k_*\hat{G}b$ and by an isomorphism class $X$ of simple projective
$k, \hat{F}_{k,G}(R_{\varepsilon})$-modules, and let us denote by $Wgt_k(\hat{G}, b)$ the set of $G$-conjugacy classes of $b$-weights of $\hat{G}$; thus, statement 3.4.1 affirms that we have a canonical bijection

$$Wgt_k(\hat{G}) \cong \bigsqcup_b Wgt_k(\hat{G}, b)$$

where $b$ runs over the set of blocks of $\hat{G}$; in particular, any weight of $\hat{G}$ determines a block.

4. Fitting pointed groups

4.1. Let us say that a finite $k^*$-group $\hat{G}$ is $p$-solvable if the $k^*$-quotient $G$ of $\hat{G}$ is so; it is in this case that the following definition is actually useful. We call Fitting pointed group of $\hat{G}$ any radical pointed group $Q_\delta$ on $k, \hat{G}$ fulfilling the following condition

4.1.1. For any local pointed groups $P_\gamma$ and $R_\varepsilon$ on $k, \hat{G}$ such that $P_\gamma$ contains $Q_\delta$ and $R_\varepsilon$, any $k, \hat{G}$-fusion from $R_\varepsilon$ to $P_\gamma$ coincides with the conjugation by an element $x \in N_G(Q_\delta)$ fulfilling $R_\varepsilon \subset (P_\gamma)^x$.

Note that this condition implies that a Fitting pointed group $Q_\delta$ of $\hat{G}$ is normal in any local pointed group $P_\gamma$ containing $Q_\delta$.

**Proposition 4.2.** Let $\hat{G}$ be a finite $k^*$-group and $Q_\delta$ a Fitting pointed group of $\hat{G}$. If a local pointed group $P_\gamma$ on $k, \hat{G}$ contains both $Q_\delta$ and a radical pointed group $R_\varepsilon$ on $k, \hat{G}$, then $R_\varepsilon$ contains $Q_\delta$. In particular, $Q_\delta$ is the unique Fitting pointed group of $\hat{G}$ contained in $P_\gamma$.

**Proof:** We already know that $Q_\delta$ is normal in $P_\gamma$ and therefore the product $Q \cdot R$ is a subgroup of $P$; but, any element $y \in N_G(R_{\varepsilon})$ induces by conjugation a $k, \hat{G}$-fusion from $R_{\varepsilon}$ to $P_\gamma$ and therefore, according to condition 4.1.1, this $k, \hat{G}$-fusion is also induced by an element $x \in N_G(Q_\delta)$; in particular, the image of $N_G(R_{\varepsilon})$ in $F_{k,G}(R_{\varepsilon})$ is a normal $p$-subgroup and therefore it is trivial.

On the other hand, it follows from [2, Theorem 1.8] and from 3.1 above that $\varepsilon$ is the unique local point of $R$ on $k, \hat{G}$ such that $R_{\varepsilon} \subset P_\gamma$, and thus we have $N_Q(R_{\varepsilon}) = N_Q(R)$; moreover, since $R_{\varepsilon}$ is selfcentralizing, we still have $C_P(R) = Z(R)$ and therefore $N_{Q,R}(R)$ maps injectively into the group of outer automorphisms of $R$.

Consequently, we get $N_{Q,R}(R) = \{1\}$ which implies that $Q \cdot R = R$, so that $Q \subset R$; finally, once again it follows from [2, Theorem 1.8] and from 3.1 above that $Q_\delta \subset R_{\varepsilon}$. Since any Fitting pointed group is a radical, the last statement is now clear. We are done.
Corollary 4.3. Let \( \hat{G} \) be a finite \( k^* \)-group and \( P_\gamma \) a maximal local pointed group on \( k_s \hat{G} \). A radical pointed group \( Q_\delta \) on \( k_s \hat{G} \) contained in \( P_\gamma \) is a Fitting pointed group of \( \hat{G} \) if and only if it is contained in each radical pointed group on \( k_s \hat{G} \) contained in \( P_\gamma \).

Proof: It follows from Proposition 4.2 that this condition is necessary. Conversely, if \( Q_\delta \) is contained in any radical pointed group on \( k_s \hat{G} \) contained in \( P_\gamma \), it follows from [14, Theorem A.9] that, in particular, \( Q_\delta \) is contained in each essential pointed group \( R_\varepsilon \) contained in \( P_\gamma \); moreover, for any \( x \in G \) normalizing either \( R_\varepsilon \) or \( P_\gamma \), \( (Q_\delta)^x \) is a Fitting pointed group on \( k_s \hat{G} \) contained in \( P_\gamma \) and therefore it coincides with \( Q_\delta \); hence, \( N_{\hat{G}}(Q_\delta) \) contains \( N_{\hat{G}}(P_\gamma) \) and \( N_{\hat{G}}(R_\varepsilon) \) for each essential pointed group \( R_\varepsilon \) contained in \( P_\gamma \).

At this point, condition 4.1.1 follows from [14, Corollary A.12].

4.4. From now on, we assume that \( \hat{G} \) is a \( p \)-solvable finite \( k^* \)-group and let \( b \) be a block of \( \hat{G} \) and \( P_\gamma \) a maximal local pointed group on \( k_s \hat{G} b \); it follows from [16, Theorem 4.6] that there exists a \( P \)-source pair \( (S, \hat{L}) \), unique up to isomorphisms, formed by a primitive Dade \( P \)-algebra \( S \) and by a \( p \)-solvable finite \( k^* \)-group \( \hat{L} \) containing \( P \), which fulfills the following two conditions

4.4.1. \( C_L(\mathcal{O}_p(L)) = Z(\mathcal{O}_p(L)) \) where \( L \) denotes the \( k^* \)-quotient of \( \hat{L} \).

4.4.2. There is a \( P \)-interior algebra embedding \( e_\gamma : (k_s \hat{G})_{\gamma} \to S \otimes_k k_s \hat{L} \).

Note that, according to isomorphism 2.11.1, any \( p \)-subgroup of \( L \) containing \( \mathcal{O}_p(L) \) has a unique local point on \( k_s \hat{L} \) — actually, it coincides with \( \{1\} \) (cf. 2.10). In particular, \( P \) has a unique local point \( \dot{\gamma} = \{1\} \) on \( k_s \hat{L} \) and therefore it follows from [12, Theorem 5.3] that it has also a unique local point \( S \times \dot{\gamma} \) on \( S \otimes_k k_s \hat{L} \); then, the embedding above is equivalent to the existence of a \( P \)-interior algebra isomorphism

\[
(k_s \hat{G})_{\gamma} \cong (S \otimes_k k_s \hat{L})_{S \times \dot{\gamma}}
\]

4.4.3.

4.5. In particular, from isomorphism 4.4.3 and from [12, Theorem 5.3], any local pointed group \( Q_\delta \) on \( k_s \hat{G} \) contained in \( P_\gamma \) determines a local pointed group \( Q_\delta \) on \( k_s \hat{L} \) and this correspondence is bijective. Moreover, since we have \( P \)-interior algebra embeddings

\[
k_s \hat{L} \to S^o \otimes_k S \otimes_k k_s \hat{L} \leftarrow S^o \otimes_k (k_s \hat{G})_{\gamma}
\]

4.5.1 and \( (S^o \otimes_k S) \times \dot{\gamma} \) is the unique local point of \( P \) on \( S^o \otimes_k S \otimes_k k_s \hat{L} \), we still have a \( P \)-interior algebra embedding

\[
e_\gamma^* : k_s \hat{L} \to S^o \otimes_k (k_s \hat{G})_{\gamma}
\]

4.5.2 inducing the same bijection between the sets of local pointed groups on \( (k_s \hat{G})_{\gamma} \) and on \( k_s \hat{L} \); then, since \( (k_s \hat{G})_{\gamma} \) and \( k_s \hat{L} \) admit \( P \times P \)-stable bases
by the multiplication on both sides, where \( P \times \{1\} \) and \( \{1\} \times P \) act freely, it follows from 2.17 above applied twice that we have

\[
F_{k,\hat{G}}(R_{\varepsilon},Q_\delta) = F_{k,\hat{L}}(R_{\varepsilon},Q_\delta) \subset F_{S}(R,Q)
\]

for any pair of local pointed groups \( Q_\delta \) and \( R_{\varepsilon} \) on \( k,\hat{G} \) contained in \( P_\gamma \), and that the choice of a polarization \( \omega \) and of the embedding \( e_\gamma \) determine \( k^* \)-isomorphisms (cf. 2.8.3 and 2.17.3)

\[
\hat{F}_{k,\hat{G}}(Q_\delta) \cong \hat{F}_{S\otimes_k k,\hat{L}}(Q_{S\times\delta}) \cong \hat{F}_{k,\hat{L}}(Q_\delta)
\]

4.6. Set \( O = \mathcal{O}_p(L) \) and denote by \( \eta \) and by \( \eta \) the respective unique local points of \( O \) on \( k,\hat{L} \) and on \( (k,\hat{G})_\gamma \); since \( O_{\hat{G}} \) is clearly a Fitting pointed group of \( \hat{L} \), it follows from 4.5 above that \( O_{\hat{G}} \) is a Fitting pointed group of \( \hat{G} \). Moreover, from the \( k^* \)-group homomorphism 2.8.1 and from the last statement in 2.10, we get the \( k^* \)-group isomorphism

\[
\hat{L}/O \cong \hat{F}_{k,\hat{L}}(O_{\hat{G}})
\]

and therefore the choice of a polarization \( \omega \) determines a \( k^* \)-isomorphism

\[
\hat{L}/O \cong \hat{F}_{k,\hat{G}}(O_{\hat{G}})
\]

4.6.1

Remark 4.7. It follows from [16, 4.7] that the Dade \( P \)-algebra \( S \) above always come from a suitable nilpotent block admitting \( P \) as a defect group and therefore, according to [15, Theorem 7.8], the similarity class of \( S \) in the Dade group \( \mathcal{D}_k(P) \) is a torsion element (cf. 2.13). In particular, we can restrict our polarizations to the full subalgebra \( \mathcal{D}_{k}^{tor} \) of \( \mathcal{D}_k \) over the objects \((P,S)\) fulfilling this condition.

5. The key parameterizations

5.1. Let \( \hat{G} \) be a \( p \)-solvable finite \( k^* \)-group, \( b \) a block of \( \hat{G} \) and \( P_\gamma \) a maximal local pointed group on \( k,\hat{G} \), and denote by \((S,\hat{L})\) a \( P \)-source pair of this block and by \( O_\eta \) the Fitting pointed group of \( \hat{G} \) contained in \( P_\gamma \); in this section, our purpose is to show that the choice of a polarization \( \omega \) determines two bijections

\[
\Gamma_{(\hat{G},b)}^\omega : \text{Irr}_{k}(\hat{G},b) \cong \text{Irr}_{k}(\hat{F}_{k,\hat{G}}(O_\eta))
\]

\[
\Delta_{(\hat{G},b)}^\omega : \text{Wgt}_{k}(\hat{G},b) \cong \text{Wgt}_{k}(\hat{F}_{k,\hat{G}}(O_\eta))
\]

which are \emph{natural} with respect to the isomorphisms between blocks. We first need to know the group of \textit{exterior automorphisms} \( \text{Out}_P((k,\hat{G})_\gamma) \) (cf. 2.4) of the \( P \)-interior algebra \((k,\hat{G})_\gamma \); recall that, according to [11, Proposition 14.9],...
we have an injective group homomorphism
\[ \text{Out}_P((k_*\hat{G})_\gamma) \longrightarrow \text{Hom}(F_{k_*\hat{G}}(P_\gamma), k^*) \]
and therefore \( \text{Out}_P((k_*\hat{G})_\gamma) \) is Abelian.

**Proposition 5.2.** With the notation above, there are group isomorphisms
\[ \text{Out}_P((k_*\hat{G})_\gamma) \cong \text{Out}_P(k_*\hat{L}) \cong \text{Hom}(L, k^*) \]
mapping \( \bar{\sigma} \in \text{Out}_P((k_*\hat{G})_\gamma) \) on an element \( \hat{\sigma} \in \text{Out}_P(k_*\hat{L}) \) such that, for any \( P \)-interior algebra embedding \( e_\gamma : (k_*\hat{G})_\gamma \rightarrow S \otimes_k k_*\hat{L} \) we have
\[ \tilde{e}_\gamma \circ \bar{\sigma} = (\tilde{\text{id}}_S \otimes \hat{\sigma}) \circ \tilde{e}_\gamma \]
and mapping \( \zeta \in \text{Hom}(L, k^*) \) on the exterior class of the \( P \)-interior algebra automorphism of \( k_*\hat{L} \) sending \( \hat{y} \in \hat{L} \) to \( \zeta(y) \cdot \hat{y} \) where \( y \) is the image of \( \hat{y} \) in \( L \). Moreover, \( \text{Out}_P((k_*\hat{G})_\gamma) \) acts regularly over the set of exterior embeddings from \( (k_*\hat{G})_\gamma \) to \( S \otimes_k k_*\hat{L} \).

**Proof:** Since \( S^\circ \times \gamma \) is the unique local point of \( P \) on \( S^\circ \otimes_k (k_*\hat{G})_\gamma \), embedding 4.5.2 induces a \( P \)-interior algebra isomorphism
\[ k_*\hat{L} \cong (S^\circ \otimes_k (k_*\hat{G})_\gamma)_{S^\circ \times \gamma} \]
and therefore, for a representative \( \sigma \) of \( \bar{\sigma} \), the automorphism \( \text{id}_S \otimes \sigma \) of \( S^\circ \otimes_k (k_*\hat{G})_\gamma \), composed with a suitable inner automorphism, induces an automorphism \( \hat{\sigma} \) of \( k_*\hat{L} \) and it is quite clear that the exterior class \( \hat{\sigma} \) of \( \hat{\sigma} \) does not depend on our choices, and fulfills
\[ \tilde{e}_\gamma^o \circ \hat{\sigma} = (\tilde{\text{id}}_S \otimes \hat{\sigma}) \circ \tilde{e}_\gamma^o \]

Tensoring embedding 4.5.2 by \( S \) and arguing as in 4.5 above, it is not difficult to prove that equality 5.2.2 also holds. Similarly, since this correspondence comes from “conjugation” via the exterior class of isomorphisms 4.4.3 and 5.2.3, it is clear that it is a group isomorphism; actually, this argument also proves the last statement.

On the other hand, for any \( \zeta \in \text{Hom}(L, k^*) \), it is clear that the map sending \( \hat{y} \in \hat{L} \) to \( \zeta(y) \cdot \hat{y} \) defines an automorphism of the \( k^* \)-group \( \hat{L} \) inducing the identity on \( P \) and thus, it determines a \( P \)-interior algebra automorphism of \( k_*\hat{L} \); moreover, since \( y \) is also the image of \( \zeta(y) \cdot \hat{y} \) in \( L \), we clearly get a group homomorphism
\[ \text{Hom}(L, k^*) \longrightarrow \text{Aut}_P(k_*\hat{L}) \]
Conversely, any $P$-interior algebra automorphism $\hat{\sigma}$ of $k_s\hat{L}$ stabilizes the *Fitting pointed group* $O_{\hat{\eta}}$, acting trivially on $O$; hence, it acts on the $k^*$-group $\hat{F}_{k_s\hat{L}}(O_{\hat{\eta}})$ acting trivially on its $k^*$-quotient $F_{k_s\hat{L}}(O_{\hat{\eta}}) \subset \text{Out}(O)$ and therefore, according to isomorphism 4.6.1 above, it determines an element of
\[
\text{Hom}(L/\hat{\eta}, k^*) = \text{Hom}(L, k^*)
\]
clearly, any inner $P$-interior algebra automorphism of $k_s\hat{L}$ determines the trivial element of $\text{Hom}(L, k^*)$ and thus, we easily get the second isomorphism in 5.2.1.

5.3. We are ready to define the first bijection in 5.1.1. Since the restriction determines a *Morita equivalence* between the $k$-algebras $k_s\hat{G}b$ and $(k_s\hat{G})_{\gamma}$ (cf. 2.7), we certainly have a *natural* bijection (cf. 2.10)
\[
\text{Irr}_k(\hat{G}, b) \cong \mathcal{P}((k_s\hat{G})_{\gamma})
\]
and any embedding $e_{\gamma} : (k_s\hat{G})_{\gamma} \to S \otimes_k k_s\hat{L}$ induces an injective map and a $k^*$-group isomorphism (cf. 2.5 and 2.8.3)
\[
\begin{align*}
\mathcal{P}(\hat{e}_{\gamma}) : & \mathcal{P}((k_s\hat{G})_{\gamma}) \longrightarrow \mathcal{P}(S \otimes_k k_s\hat{L}) \\
\hat{F}_{\hat{e}_{\gamma}}(O_{\hat{\eta}}) : & \hat{F}_{k_s\hat{G}}(O_{\hat{\eta}}) \cong \hat{F}_{S \otimes_k k_s\hat{L}}(O_s \otimes_{\hat{\eta}})
\end{align*}
\]
then, the existence of embedding 4.5.2 proves that the map $\mathcal{P}(e_{\gamma})$ is actually bijective. On the other hand, the choice of a *polarization* $\omega$ determines a $k^*$-group isomorphism
\[
\Phi_{\omega}(O_{\hat{\eta}}) : \hat{F}_{S \otimes_k k_s\hat{L}}(O_s \otimes_{\hat{\eta}}) \cong \hat{F}_{k_s\hat{L}}(O_{\hat{\eta}})
\]
Finally, isomorphism 4.6.1 determines a canonical bijection
\[
\Gamma_{\hat{L}} : \text{Irr}_k(\hat{L}) \cong \text{Irr}_k(\hat{F}_{k_s\hat{L}}(O_{\hat{\eta}}))
\]

**Corollary 5.4.** With the notation and the choice above, there is a bijection
\[
\Gamma_{(\hat{G}, b)}^\omega : \text{Irr}_k(\hat{G}, b) \cong \text{Irr}_k(\hat{F}_{k_s\hat{G}}(O_{\hat{\eta}}))
\]
such that, for any embedding $e_{\gamma} : (k_s\hat{G})_{\gamma} \to S \otimes_k k_s\hat{L}$, we have the commutative diagram
\[
\begin{align*}
\text{Irr}_k(\hat{G}, b) & \cong \mathcal{P}(S \otimes_k k_s\hat{L}) \cong \text{Irr}_k(\hat{L}) \\
\text{Irr}_k(\hat{F}_{k_s\hat{G}}(O_{\hat{\eta}})) & \cong \text{Irr}_k(\hat{F}_{S \otimes_k k_s\hat{L}}(O_s \otimes_{\hat{\eta}})) \cong \text{Irr}_k(\hat{F}_{k_s\hat{L}}(O_{\hat{\eta}}))
\end{align*}
\]

**Proof:** It is clear that, for a choice of an embedding
\[
e_{\gamma} : (k_s\hat{G})_{\gamma} \longrightarrow S \otimes_k k_s\hat{L}
\]
and (5.2.1).
the bijections 5.3.1 and \( \mathcal{P}(\tilde{e}_\gamma) \), and the \( k^* \)-group isomorphism \( \hat{F}_{k_\gamma}((O_{\hat{\eta}}) \) above determine the horizontal left-hand bijections in diagram 5.4.2; the top horizontal right-hand bijection follow from 2.10 and 2.17, and the bottom horizontal right-hand bijection from isomorphism 5.3.3 up to the choice of \( \omega \); then, the bijection \( \Gamma_\delta \) and the commutativity of the diagram define the bijection \( \Gamma_{(\tilde{G},b)}^{\omega} \).

We claim that this bijection does not depend on the choice of \( e_\gamma \); indeed, for another choice \( e_\gamma' \) of this embedding, it follows from Proposition 5.2 that there is \( \tilde{\sigma} \in \text{Out}_P((k_s \hat{G})_{\gamma}) \) fulfilling
\[
\tilde{e}_\gamma' = \tilde{e}_\gamma \circ \tilde{\sigma} = (\tilde{id}_S \otimes \tilde{\hat{\sigma}}) \circ \tilde{e}_\gamma
\]
and therefore, with obvious notation, we get the following commutative diagrams

\[
\begin{align*}
\text{Irr}_k(\hat{G}, b) & \cong \mathcal{P}((k_s \hat{G})_{\gamma}) \cong \mathcal{P}(S \otimes_k k_s \hat{L}) \\
\text{Irr}_k(\hat{G}, b) & \cong \mathcal{P}((k_s \hat{G})_{\gamma}) \cong \mathcal{P}(S \otimes_k k_s \hat{L}) \\
\text{Irr}_k(\hat{F}_{k_s \hat{G}}(O_\eta)) & \cong \text{Irr}_k(\hat{F}_{S \otimes_k k_s \hat{L}}(O_{S \times \eta})) \cong \text{Irr}_k(\hat{F}_{S \otimes_k k_s \hat{L}}(O_{S \times \eta}))
\end{align*}
\]

Moreover, we have the evident commutative diagram
\[
\begin{align*}
\mathcal{P}(S \otimes_k k_s \hat{L}) & \cong \text{Irr}_k(\hat{L}) \\
\mathcal{P}(\tilde{id}_S \otimes \tilde{\sigma}) & \cong \text{Irr}_k(\tilde{\hat{L}})
\end{align*}
\]
on the other hand, since the groups of \( k^* \)-group automorphisms of \( \hat{F}_{k_s \hat{L}}(O_{\hat{\eta}}) \) and \( \hat{F}_{S \otimes_k k_s \hat{L}}(O_{S \times \eta}) \) which induce the identity over (cf. 2.17 applied twice)
\[
F_{k_s \hat{L}}(O_{\eta}) = F_{S \otimes_k k_s \hat{L}}(O_{S \times \eta})
\]
are both canonically isomorphic to the Abelian group \( \text{Hom}(F_{k_s \hat{L}}(O_{\hat{\eta}}), k^*) \), we still have the commutative diagram
\[
\begin{align*}
\text{Irr}_k(\hat{F}_{S \otimes_k k_s \hat{L}}(O_{S \times \eta})) & \cong \text{Irr}_k(\hat{F}_{k_s \hat{L}}(O_{\eta})) \\
\text{Irr}_k(\hat{F}_{S \otimes_k k_s \hat{L}}(O_{S \times \eta})) & \cong \text{Irr}_k(\hat{F}_{k_s \hat{L}}(O_{\eta}))
\end{align*}
\]
Finally, from isomorphism 4.6.2 we obviously get the following commutative diagram

\[
\begin{array}{ccc}
\text{Irr}_k(\hat{L}) & \xrightarrow{\text{Irr}(\hat{\sigma})} & \text{Irr}_k(\hat{L}) \\
\sigma_L & \cong & \sigma_L \\
\text{Irr}_k(\hat{F}_{k,\hat{L}}(O_\eta)) & \xrightarrow{\text{Irr}(\hat{F}_k(O_\eta))} & \text{Irr}_k(\hat{F}_{k,\hat{L}}(O_\eta))
\end{array}
\]

5.4.10;

now, our claim follows from putting together all these commutative diagrams.

5.5. In order to define the second bijection in 5.1.1, let \((R_\varepsilon, X)\) be a \(b\)-weight of \(\hat{G}\); for our purposes, we may assume that \(P_\gamma\) contains \(R_\varepsilon\); then, \(R_\varepsilon\) and \(R_\varepsilon\) respectively contain \(O_\eta\) and \(O_\eta\); recall that, with the notation and the choice above, we have \(k\)-group isomorphisms

\[
\hat{F}_{k,\hat{L}}(R_\varepsilon) \cong \hat{F}_{k,\hat{L}}(R_\varepsilon) \cong \hat{F}_{k,\hat{L}}(R_\varepsilon)
\]

5.5.1

and, in this case, \(X\) determines an isomorphism class \(X\) of simple projective \(k\)-\(\hat{F}_{k,\hat{L}}(R_\varepsilon)\)-modules; moreover, we clearly have \(N_\ell(R_\varepsilon) = N_\ell(R)\) and from 2.11.2 it is easily checked that the \(k\)-group homomorphism 2.8.3 induces a \(k\)-group isomorphism

\[
\hat{N}_\ell(R_\varepsilon) \cong \hat{N}_\ell(R_\varepsilon)
\]

5.5.2;

consequently, the pair \((R, \hat{X})\) is a weight of \(\hat{L}\).

**Proposition 5.6.** With the notation and the choice above, let \((R_\varepsilon, X)\) and \((R_\varepsilon', X')\) be \(b\)-weights of \(\hat{G}\) such that \(P_\gamma\) contains \(R_\varepsilon\) and \(R_\varepsilon\). If \((R_\varepsilon, X)\) and \((R_\varepsilon', X')\) are \(G\)-conjugate then the corresponding weights \((R, \hat{X})\) and \((R', \hat{X}')\) of \(L\) are \(L\)-conjugate. In particular, this correspondence induces a bijection

\[
\text{Wgt}_k^w(e_\gamma) : \text{Wgt}_k(\hat{G}, b) \cong \text{Wgt}_k(\hat{L})
\]

5.6.1.

**Proof:** Assume that \((R_\varepsilon', X')^x = (R_\varepsilon, X)\) for some \(x \in G\); then, the conjugation by \(x\) determines a \((k, \hat{G}_x)\)-fusion \(\varphi\) from \(R_\varepsilon\) to \(R_\varepsilon'\) (cf. 2.6), and the corresponding \(R\)-interior algebra isomorphism

\[
f_\varphi : (k, \hat{G})_x \cong \text{Res}_x((k, \hat{G})_x)
\]

5.6.2

induces a \(k\)-group isomorphism (cf 2.8.3)

\[
\hat{F}_{f_\varphi}(R_\varepsilon) : \hat{F}_{(k, \hat{G})_x}(R_\varepsilon) \cong \hat{F}_{(k, \hat{G})_x}(R_\varepsilon')
\]

5.6.3;

actually, we have \(X = \text{Res}_{\hat{F}_{f_\varphi}}(R_\varepsilon)(X')\).
But, according to equality 4.5.3, the group homomorphism \( \varphi \) is also a \( k_s \cdot \hat{L} \)-fusion from \( R_\varepsilon \) to \( R'_{\varepsilon'} \), so that \( \varphi : R \cong R' \) is also induced by some element \( \hat{\varepsilon} \in \hat{L} \) (cf. statement 2.11.2); moreover, we have the corresponding \( R \)-interior algebra isomorphism

\[
\hat{f}_{\varphi} : (k_s \hat{L})_\varepsilon \cong \text{Res}_{\varphi}((k_s \hat{L})_{\varepsilon'})
\]

inducing a \( k^* \)-group isomorphism (cf 2.8.3)

\[
\hat{F}_{\hat{f}_{\varphi}}(R_\varepsilon) : \hat{F}_{(k_s \hat{L})_\varepsilon}(R_\varepsilon) \cong \hat{F}_{(k_s \hat{L})_{\varepsilon'}}(R'_{\varepsilon'})
\]

Then, the commutativity of diagrams 2.9.5 and 2.17.4 applied here yields the following commutative diagrams of \( k^* \)-group isomorphisms

\[
\begin{align*}
\hat{F}_{(k_s \hat{G})_{\gamma}}(R_\varepsilon) & \cong \hat{F}_{S \otimes k_s \hat{L}}(R_{S \times \hat{G}}) \cong \hat{F}_{k_s \hat{G}}(R_\varepsilon) \\
\hat{F}_{\hat{f}_{\varphi}}(R_\varepsilon) & \cong \hat{F}_{S \otimes k_s \hat{L}}(R_{S \times \hat{G}}) \cong \hat{F}_{k_s \hat{G}}(R_\varepsilon)
\end{align*}
\]

Consequently, we also have \( \hat{X} = \text{Res}_{\hat{F}_{\hat{f}_{\varphi}}(R_\varepsilon)}(\hat{X}') \) and therefore we get

\[
(R'_{\varepsilon'}, \hat{X}')^x = (R_\varepsilon, \hat{X})
\]

that is to say, the correspondence above induces a map

\[
\text{Wgt}^w_k(e_\gamma) : \text{Wgt}_k(\hat{G}, b) \rightarrow \text{Wgt}_k(\hat{L})
\]

which is quite clear that it is a bijection. We are done.

**Proposition 5.7.** With the the notation above, the canonical \( k^* \)-group isomorphism \( \hat{L}/O \cong \hat{F}_{k_s \hat{L}}(O_\eta) \) induces a bijection

\[
\Delta_L : \text{Wgt}_k(\hat{L}) \cong \text{Wgt}_k(\hat{F}_{k_s \hat{L}}(O_\eta))
\]

**Proof:** Let \( (R, Y) \) be a weight of \( \hat{L} \); since the unity element in \( k_s \hat{L} \) is a block of \( \hat{L} \) and condition 4.4.1 holds, \( R \) has a unique local point \( \hat{\varepsilon} \) on \( k_s \hat{L} \) and \( R_\varepsilon \) is a radical pointed group which contains \( O_\eta \) (cf. Corollary 4.3); moreover, since we have the \( k^* \)-group isomorphism \( \hat{L}/O \cong \hat{F}_{k_s \hat{L}}(O_\eta) \) (cf. 4.6.2), setting \( \hat{R} = R/O \) and identifying \( \hat{R} \) with its image in \( \hat{F}_{k_s \hat{L}}(O_\eta) \), the normalizer \( \hat{N}_L(R_\varepsilon) = N_L(R) \) is just the converse image in \( \hat{L} \) of \( \hat{N}_{\hat{F}_{k_s \hat{L}}(O_\eta)}(\hat{R}) \) and therefore we have the canonical \( k^* \)-group isomorphism

\[
\hat{N}_L(R) \cong \hat{N}_{\hat{F}_{k_s \hat{L}}(O_\eta)}(\hat{R})
\]
in particular, $Y$ determines an isomorphism class $\bar{Y}$ of simple $N_{\tilde{F}_{k,\ast L}(\Omega_\eta)}(\bar{R})$-modules of vertex $\bar{R}$, so that the pair $(\bar{R}, \bar{Y})$ is a weight of $\tilde{F}_{k,\ast L}(\Omega_\eta)$.

Conversely, if we start with a weight $(R, Y)$ of $\tilde{F}_{k,\ast L}(\Omega_\eta)$, it is clear that, for the converse image $R$ of $\bar{R}$ in $\hat{L}$, isomorphism 5.6.2 still holds and therefore $\bar{Y}$ determines an isomorphism class $Y$ of simple $k, \hat{L}$-modules of vertex $R$, so that the pair $(R, Y)$ is a weight of $\hat{L}$. Since this correspondence is compatible with the $L$-conjugation, we get the announced bijection 5.6.1.

**Corollary 5.8.** With the notation and the choice above, there is a bijection

$$\Delta^\omega_{(G,b)} : \text{Wgt}_k(\hat{G}, b) \cong \text{Wgt}_k(\tilde{F}_{k,\ast G}(O_\eta))$$

such that, for any embedding $e_\gamma : (k, \hat{G}) \rightarrow S \otimes_k k, \hat{L}$, we have the commutative diagram

$$\begin{array}{ccc}
\text{Wgt}_k(\hat{G}, b) & \xrightarrow{\text{Wgt}_k^\gamma(e_\gamma)} & \text{Wgt}_k(\hat{L}) \\
\Delta^\gamma_{(G,b)} & \| & \Delta_L \\
\text{Wgt}_k(\tilde{F}_{k,\ast G}(O_\eta)) & \xrightarrow{\text{Wgt}_k^\gamma(\bar{e}_\gamma)} & \text{Wgt}_k(\tilde{F}_{k,\ast L}(O_\eta))
\end{array}$$

where the bottom bijection is induced by the $k^*$-group isomorphisms

$$\tilde{F}_{k,\ast G}(O_\eta) \cong \tilde{F}_{S \otimes \hat{k}, \ast L}(O_{S \otimes \hat{k}}) \Phi^\omega(O_\eta) \cong \tilde{F}_{k,\ast L}(O_\eta)$$

**Proof:** It is clear that, for a choice of $e_\gamma$, Propositions 5.6 and 5.7, and the commutativity of the diagram define the bijection $\Delta^\omega_{(G,b)}$. We claim that this bijection does not depend on this choice; indeed, for another choice $e'_\gamma$ of this embedding, it follows from Proposition 5.2 that there is $\bar{e} \in \text{Out}_P((k, \hat{G})_\gamma)$ fulfilling

$$e'_\gamma = e_\gamma \circ \bar{e} = (\bar{id}_S \otimes \bar{e}) \circ e_\gamma$$

in particular, if $(R_\varepsilon, X)$ is a $b$-weight of $\hat{G}$ and $(R_\varepsilon, \hat{X})$ the corresponding weight of $\hat{L}$ in 5.5 above — $\hat{X}$ is the isomorphism class of a simple projective $k, \hat{L}$-module $V$ restricted to $N_L(R)$ — then $\text{Wgt}_k^\gamma(e'_\gamma)$ sends the $G$-conjugacy class of $(R_\varepsilon, X)$ to the $L$-conjugacy class of $(R_\varepsilon, \hat{X}')$ where $\hat{X}'$ is the isomorphism class of corresponding the simple projective $k, \tilde{F}_{k,\ast L}(R_\varepsilon)$-module $\text{Res}_{\tilde{F}_{k,\ast L-1}(R_\varepsilon)}(V)$, since $\text{Hom}(L, k^*)$ clearly acts trivially on the set of local pointed groups on $k, \hat{L}$ and we have the commutative diagram (cf. 2.17.4)

$$\begin{array}{ccc}
\tilde{F}_{S \otimes \hat{k}, \ast L}(R_{S \times \varepsilon}) & \xrightarrow{\Phi^\omega_{(R_\varepsilon)}} & \tilde{F}_{k,\ast L}(R_\varepsilon) \\
\tilde{F}_{S \otimes \hat{k}, \ast L}(R_{S \times \varepsilon}) & \xrightarrow{\Phi^\omega_{(R_\varepsilon)}} & \tilde{F}_{k,\ast L}(R_\varepsilon)
\end{array}$$
Now, setting $\hat{R} = R/O$, since we have (cf. isomorphisms 4.6.1 and 5.5.2)
\[
\hat{F}_{k, L}(\hat{R}_\epsilon) \cong \hat{N}_{L}(\hat{R}_\epsilon) = \hat{N}_{L}(\hat{R}) \cong \hat{N}_{\hat{F}_{k, L}(O_\eta)}(\hat{R})
\]
5.8.6, $V$ determines a simple projective $\hat{N}_{\hat{F}_{k, L}(O_\eta)}(\hat{R})$-module $\hat{V}$; moreover, since Hom($L, k^*$) clearly stabilizes $\hat{L}$ and it acts trivially on $P$, the corresponding representative $\hat{\sigma}$ of $\hat{\sigma}$ induces a $k^*$-group automorphism $\hat{\sigma}$ of $\hat{F}_{k, L}(O_\eta)$ (cf. isomorphism 4.6.1) which stabilizes $\hat{N}_{\hat{F}_{k, L}(O_\eta)}(\hat{R})$, and it is quite clear that, with obvious notation, we get the following commutative diagram
\[
\begin{array}{c}
\hat{F}_{k, L}(\hat{R}_\epsilon) \\ \hat{F}_{\hat{\sigma}}(\hat{R}_\epsilon) | | & \hat{\eta} | | & \hat{N}_{\hat{\sigma}}(\hat{R}) \\
\hat{F}_{k, L}(\hat{R}_\epsilon) & \cong & \hat{N}_{\hat{F}_{k, L}(O_\eta)}(\hat{R})
\end{array}
\]
5.8.7; hence, via isomorphisms 5.6.7, Res$^{\hat{F}_{\hat{\sigma}}}_{\hat{\sigma}}(\hat{R}_\epsilon)(V)$ determines the simple projective $\hat{N}_{\hat{F}_{k, L}(O_\eta)}(\hat{R})$-module Res$^{\hat{\eta}}_{\hat{\sigma}}(\hat{R})(\hat{V})$.

At this point, denoting by $\hat{X}$ and $\hat{X}'$ the respective isomorphism classes of the $\hat{N}_{\hat{F}_{k, L}(O_\eta)}(\hat{R})$-modules $\hat{V}$ and Res$^{\hat{\eta}}_{\hat{\sigma}}(\hat{R})(\hat{V})$, it follows from Proposition 5.7 that $\Delta_{\hat{L}}$ maps $(\hat{R}_\epsilon, \hat{X})$ on $(\hat{R}, \hat{X})$, and $(\hat{R}_\epsilon, \hat{X}')$ on $(\hat{R}, \hat{X}')$. But, we also have the following commutative diagram (cf. 2.17.4)
\[
\begin{array}{c}
\hat{F}_{S \otimes k, L}(O_{S \times \eta}) \phi_{\phi}(O_\eta) \cong \hat{F}_{k, L}(O_\eta) \\
\hat{F}_{\hat{\sigma} \otimes S \otimes \hat{\eta}}(O_{S \times \eta}) & \cong & \hat{F}_{k, L}(O_\eta)
\end{array}
\]
and we consider its restriction to all the normalizers of $\hat{R}$. Consequently, since we have (cf. 5.8.4)
\[
\hat{F}_{\hat{\sigma}}(O_\eta) = \hat{F}_{\hat{\sigma} \otimes \hat{\eta}}(O_{S \times \eta}) \circ \hat{F}_{\hat{\sigma}}(O_\eta)
\]
5.8.9, the corresponding bottom bijections in diagram 5.8.2 maps the weights $(\hat{R}, \hat{X})$ and $(\hat{R}, \hat{X}')$ of $\hat{F}_{k, L}(O_\eta)$ on the same weight of $\hat{F}_{k, L}(O_\eta)$. We are done.

6. The Fitting block sequences

6.1. In order to exhibit bijections between the sets of isomorphism classes of simple $k_\hat{G}$-modules and of $G$-conjugacy classes of weights of $\hat{G}$, we need a third set, namely the set of $G$-conjugacy classes of Fitting block sequences of $\hat{G}$. We call Fitting block sequence of $\hat{G}$ any sequence $B = \{ (\hat{G}_n, b_n) \}_{n \in \mathbb{N}}$ of pairs formed by a $k^*$-group $\hat{G}_n$ and by a block $b_n$ of $\hat{G}_n$, such that $\hat{G}_0 = \hat{G}$ and that, for any $n \in \mathbb{N}$, we have $\hat{G}_{n+1} = \hat{F}_{k, \hat{G}_n}(O_{b_n}^\eta)$ for some Fitting pointed
group $O^n_{\eta_n}$ of $\hat{G}_n$. Note that, since clearly $|G_{n+1}| = |G_n|$, such a sequence stabilizes, and actually we have $|G_{n+1}| = |G_n|$ if and only if $b_n$ is a block of defect zero of $\hat{G}_n$ (cf. statement 4.4.1). Moreover, for any $h \in \mathbb{N}$, the sequence $B_h = \{(\hat{G}_{h+n}, b_{h+n})\}_{n \in \mathbb{N}}$ is clearly a Fitting block sequence of $\hat{G}_h$.

6.2. If $\hat{G}'$ is a $k^*$-group isomorphic to $\hat{G}$ and $\theta : \hat{G} \cong \hat{G}'$ a $k^*$-group isomorphism of $\hat{G}$, it is quite clear that, from any Fitting block sequence $B = \{((\hat{G}_n, b_n))\}_{n \in \mathbb{N}}$ of $\hat{G}$, we are able to construct a Fitting block sequence $B' = \{((\hat{G}'_n, b'_n))\}_{n \in \mathbb{N}}$ of $\hat{G}'$ inductively defining a sequence of $k^*$-group isomorphisms $\theta_n : \hat{G}_n \cong \hat{G}'_n$ by $\theta_0 = \theta$ and, for any $n \in \mathbb{N}$, by (cf. 2.9)

$$\theta_{n+1} = \hat{F}_{\theta_n}(O^n_{\eta_n}) : \hat{F}_{k, \hat{G}_n}(O^n_{\eta_n}) \cong \hat{F}_{k, \hat{G}'_n}(\theta_n(O^n) \theta_0(\eta_n))$$

6.2.1, where we sill denote by $\theta_n : k_n \hat{G}_n \cong k_n \hat{G}'_n$ the corresponding $k$-algebra isomorphism, and setting

$$b'_n = \theta_n(b_n) \quad \text{and} \quad \hat{G}_{n+1} = \hat{F}_{k, \hat{G}'_n}(\theta_n(O^n) \theta_0(\eta_n))$$

6.2.2 for any $n \in \mathbb{N}$. In particular, the group of inner automorphisms of $G$ acts on the set of Fitting block sequences of $\hat{G}$ and then we denote by $\text{Fbs}_k(\hat{G})$ the set of “$G$-conjugacy classes” of the Fitting block sequences of $\hat{G}$, and by $N_G(\mathcal{B})$ the stabilizer of $\mathcal{B}$ in $G$.

6.3. In this section, our purpose is to show that the choice of a polarization \( \omega \) determines two bijections

$$\text{Fbs}_k(\hat{G}) \cong \text{Irr}_k(\hat{G}) \quad \text{and} \quad \text{Fbs}_k(\hat{G}) \cong \text{Wgt}_k(\hat{G})$$

6.3.1 which are natural with respect to the $k^*$-group isomorphisms, the composition of the inverse of the first one with the second one being our announced parameterization.

6.4. Let $\mathcal{B} = \{(\hat{G}_n, b_n)\}_{n \in \mathbb{N}}$ be a Fitting block sequence of $\hat{G}$, so that we have $\hat{G}_{n+1} = \hat{F}_{k, \hat{G}_n}(O^n_{\eta_n})$ for some Fitting pointed group $O^n_{\eta_n}$ of $\hat{G}_n$ and, choosing a polarization $\omega$, we denote by

$$\Gamma_{(\hat{G}_n, b_n)}^\omega : \text{Irr}_k(\hat{G}_n, b_n) \cong \text{Irr}_k(\hat{G}_{n+1})$$

$$\Delta_{(\hat{G}_n, b_n)}^\omega : \text{Wgt}_k(\hat{G}_n, b_n) \cong \text{Wgt}_k(\hat{G}_{n+1})$$

6.4.1 the bijections coming from Corollaries 5.4 and 5.8. Let us call character sequence \( \omega \)-associated to $\mathcal{B}$ any sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ where $\varphi_n$ belongs to $\text{Irr}_k(\hat{G}_n, b_n)$ in such a way that we have

$$\Gamma_{(\hat{G}_n, b_n)}^\omega(\varphi_n) = \varphi_{n+1}$$

6.4.2
for any \( n \in \mathbb{N} \). Similarly, let us call weight sequence \( \omega \)-associated to \( B \) any sequence \( \{(R^n, Y^n)\}_{n \in \mathbb{N}} \) where \((R^n, Y^n)\) is the \( G_n \)-conjugacy class of a weight \((R^n, Y^n)\) of \( G_n \), determining a \( G_n \)-conjugacy class \((R^n, X^n)\) of \( b_n \)-weights of \( G_n \) (cf. statement 3.4.1) in such a way that we have

\[
\Delta_{\omega(G_n, b_n)} ((R^n_x, X^n)) = (R^{n+1}, Y^{n+1}) \tag{6.4.3}
\]

for any \( n \in \mathbb{N} \).

**Theorem 6.5.** With the notation and the choice above, any Fitting block sequence \( B = \{ (G_n, b_n) \}_{n \in \mathbb{N}} \) of \( \hat{G} \) admits a unique character sequence \( \{ \varphi_n \}_{n \in \mathbb{N}} \) and a unique weight sequence \( \{(R^n, Y^n)\}_{n \in \mathbb{N}} \) \( \omega \)-associated to \( B \). Moreover, the correspondences mapping \( B \) to \( \varphi_0 \) and to \((R^n, Y^n)\) induce two natural bijections

\[
\text{Fbs}_k(\hat{G}) \cong \text{Irr}_k(\hat{G}) \quad \text{and} \quad \text{Fbs}_k(\hat{G}) \cong \text{Wgt}_k(\hat{G}) \tag{6.5.1}
\]

**Proof:** Since the sequence \( B \) stabilizes, we can argue by induction on the “length to stabilization”. If this length is zero then the block \( b_0 \) is already of defect zero and therefore \( \text{Irr}_k(\hat{G}_0, b_0) \) has a unique element \( \varphi_0 \) and, setting \( \varphi_n = \varphi_0 \) for any \( n \in \mathbb{N} \), we get a character sequence \( \omega \)-associated to \( B \); similarly, \( \text{Wgt}_k(\hat{G}_0, b_0) \) has a unique element and the corresponding constant sequence defines a weight sequence \( \omega \)-associated to \( B \).

If the “length to stabilization” is not zero then the Fitting block sequence \( B_1 = \{ (\hat{G}_{n+1}, b_{n+1}) \}_{n \in \mathbb{N}} \) of \( \hat{G}_1 \) already admits a character sequence \( \{ \varphi_{n+1} \}_{n \in \mathbb{N}} \) and a weight sequence \( \{(R^{n+1}, Y^{n+1})\}_{n \in \mathbb{N}} \) \( \omega \)-associated to \( B_1 \); then, in order to get a character sequence and a weight sequence \( \omega \)-associated to \( B \), it suffices to define (cf. 6.4.1)

\[
\varphi_0 = (\Gamma_{(G_0, b_0)}^{-1}(\varphi_1)) \tag{6.5.2}
\]

\[
(R^n_{G_0}, X^0) = (\Delta_{(G_0, b_0)}^{-1}((R^1, Y^1))
\]

and to consider the \( G \)-conjugacy class \((R^n, Y^n)\) of weights of \( \hat{G} \) determined by \((R^n_{G_0}, X^0)\) (cf. statement 3.4.1).

On the other hand, since the maps \( \Gamma_{(G_n, b_n)} \) and \( \Delta_{(G_n, b_n)} \) are bijective, equalities 6.4.2 and 6.4.3 show that a character sequence \( \{ \varphi_n \}_{n \in \mathbb{N}} \) and a weight sequence \( \{ (R^n, Y^n) \}_{n \in \mathbb{N}} \) \( \omega \)-associated to \( B \) are uniquely determined by one of their terms; but, for \( n \) big enough, we know that \( b_n \) is a block of defect zero of \( \hat{G}_n \) and then \( \varphi_n \) and \((R^n, Y^n)\) are uniquely determined; consequently, \( \{ \varphi_n \}_{n \in \mathbb{N}} \) and \( \{ (R^n, Y^n) \}_{n \in \mathbb{N}} \) are uniquely determined and it is quite clear that they only depend on the \( G \)-conjugacy class of \( B \); thus, since \( \Gamma_{(G_n, b_n)} \) and \( \Delta_{(G_n, b_n)} \) are natural, we have obtained two natural maps

\[
\text{Fbs}_k(\hat{G}) \longrightarrow \text{Irr}_k(\hat{G}) \quad \text{and} \quad \text{Fbs}_k(\hat{G}) \longrightarrow \text{Wgt}_k(\hat{G}) \tag{6.5.3}
\]
We claim that they are both bijective; actually, we will define the inverse maps. For any \( \varphi \in \text{Irr}_k(\hat{G}) \), we inductively define two sequences \( \{ \varphi_n \}_{n \in \mathbb{N}} \) and \( \{(\hat{G}_n, b_n)\}_{n \in \mathbb{N}} \) by setting \( \varphi_0 = \varphi, \hat{G}_0 = \hat{G} \) and by denoting by \( b_0 \) the block of \( \varphi \), and further, for any \( n \in \mathbb{N} \), by setting
\[
\varphi_{n+1} = \Gamma_{(\hat{G}_n, b_n)}(\varphi_n), \quad \hat{G}_{n+1} = \hat{F}_{k, \hat{G}_n}(O^n_{b_n})
\]
for some Fitting pointed group \( O^n_{b_n} \) on \( k, \hat{G}_n b_n \), and by denoting by \( b_{n+1} \) the block of \( \varphi_{n+1} \); then, it is clear that \( B = \{(\hat{G}_n, b_n)\}_{n \in \mathbb{N}} \) is a Fitting block sequence of \( \hat{G} \) and that \( \{\varphi_n\}_{n \in \mathbb{N}} \) becomes the character sequence \( \omega \)-associated to \( B \); note that, our construction only depends on the choice of the Fitting pointed group \( O^n_{b_n} \) on \( k, \hat{G}_n b_n \) for a finite set of values of \( n \). Moreover, since all the Fitting pointed group on \( k, \hat{G}_n b_n \) are mutually \( G \)-conjugate (cf. Proposition 4.2), \( \varphi \) determines a unique \( G \)-conjugacy class of Fitting block sequence of \( \hat{G} \). That is to say, we have obtained a map
\[
\text{Irr}_k(\hat{G}) \longrightarrow \text{Fbs}_k(\hat{G})
\]
and it is easily checked that it is the inverse of the left-hand map in 6.5.3.

Analogously, for any \((\hat{R}, \hat{Y}) \in \text{Wgt}_k(\hat{G})\), we inductively define two sequences \( \{(\hat{R}^n, Y^n)\}_{n \in \mathbb{N}} \) and \( \{(\hat{G}_n, b_n)\}_{n \in \mathbb{N}} \) by setting \((\hat{R}^0, Y^0) = (\hat{R}, \hat{Y})\), \( \hat{G}_0 = \hat{G} \) and by denoting by \( b_0 \) the block of \( \hat{G}_0 \) determined by \((\hat{R}, \hat{Y})\) (cf. bijection 3.4.2), and further, for any \( n \in \mathbb{N} \), by setting
\[
(\hat{R}^{n+1}, Y^{n+1}) = \Delta_{(\hat{G}_n, b_n)}((\hat{R}^n, X^n)), \quad \hat{G}_{n+1} = \hat{F}_{k, \hat{G}_n}(O^n_{b_n})
\]
where \((\hat{R}^n, X^n)\) is the \( b_n \)-weight of \( \hat{G}_n \) determined by \((\hat{R}^n, Y^n)\) and \( O^n_{b_n} \) a Fitting pointed group on \( k, \hat{G}_n b_n \), and by denoting by \( b_{n+1} \) the block of determined by the weight \((\hat{R}^{n+1}, Y^{n+1})\) (cf. bijection 3.4.2); then, it is clear that \( B = \{(\hat{G}_n, b_n)\}_{n \in \mathbb{N}} \) is a Fitting block sequence of \( \hat{G} \) and that \( \{(\hat{R}^n, Y^n)\}_{n \in \mathbb{N}} \) becomes the weight sequence \( \omega \)-associated to \( B \). As above, our construction only depends on the choice of the Fitting pointed group \( O^n_{b_n} \) on \( k, \hat{G}_n b_n \) for a finite set of values of \( n \) and therefore we have obtained a map
\[
\text{Wgt}_k(\hat{G}) \longrightarrow \text{Fbs}_k(\hat{G})
\]
which is the inverse of the right-hand map in 6.5.3.

7. Vertex, sources and multiplicity modules

7.1. Let \( \hat{G} \) be again a \( p \)-solvable finite \( k^* \)-group and choose a polarization \( \omega \); then, it follows from Theorem 6.5 above that any simple \( k, \hat{G} \)-module \( M \) determines a \( G \)-conjugacy class \((\hat{R}, \hat{Y})\) of weights of \( \hat{G} \) and in this section we discuss the relationship between this \( G \)-conjugacy class \((\hat{R}, \hat{Y})\) and the \( G \)-conjugacy class of the triples formed by a vertex \( Q \), a \( Q \)-source \( E \) and a multiplicity \( \hat{N}_G(Q)_E \)-module \( V \) of \( M \) (cf. 2.5).
7.2. Actually, $M$ also determines a $G$-conjugacy class of Fitting block sequences $\mathcal{B} = \{(\hat{G}_n, b_n)\}_{n \in \mathbb{N}}$; let us denote by $\{O^*_{\eta_n}\}_{n \in \mathbb{N}}$ the corresponding sequence of Fitting pointed groups $O^*_{\eta_n}$ on $k_\star \hat{G}_n b_n$, so that for any $n \in \mathbb{N}$ we have

$$\hat{G}_{n+1} = \hat{F}_{k_\star \hat{G}}(O^*_{\eta_n})$$  \hspace{0.5cm} \text{(7.2.1)}$$

let $P^n_\gamma$ be a maximal local pointed group on $k_\star \hat{G}_n$ containing $O^*_{\eta_n}$ and $(S_n, \hat{L}_n)$ a $P^n$-source pair for $k_\star \hat{G}_n b_n$ (cf. 4.4); note that, according to statement 2.11.2 above and to [12, Lemma 3.10], up to a suitable identification, $P^n = P^n/O^n$ is a Sylow $p$-subgroup of $G_{n+1}$ and therefore there is $\bar{x} \in G_{n+1}$ such that $(\bar{P}^n)^\bar{x}$ contains $P_{n+1}$; thus, since the sequence $\mathcal{B}$ “stabilizes”, up to finite number of choices we may assume that $\bar{P}_n$ contains $P_{n+1}$ for any $n \in \mathbb{N}$.

7.3. Moreover, from Theorem 6.5 we still obtain a weight sequence $\{(\hat{R}^n, Y^n)\}_{n \in \mathbb{N}}$ $\omega$-associated to $\mathcal{B}$ starting on $(R, Y) = (\hat{R}^0, Y^0)$, and from Corollary 5.4 we get a simple sequence $\{M_n\}_{n \in \mathbb{N}}$ $\omega$-associated to $M$ of simple $k_\star \hat{G}_n$-modules $M_n$ inductively defined by $M_0 = M$ and, denoting by $\varphi_n$ the Brauer character of $M_n$, by $\varphi_{n+1} = \Gamma^\omega_{(G_n, b_n)}(\varphi_n)$ for any $n \in \mathbb{N}$; explicitly, the Morita equivalence between $k_\star \hat{G}_n b_n$ and $(k_\star \hat{G}_n)_{\gamma_n}$ determines a simple $(k_\star \hat{G}_n)_{\gamma_n}$-module $(M_n)_{\gamma_n}$ and let us set

$$\text{End}_k(M_n)_{\gamma_n} = \text{End}_k((M_n)_{\gamma_n})$$  \hspace{0.5cm} \text{(7.3.1)}$$

then, choosing an embedding (cf. statement 4.4.2)

$$e_{\gamma_n} : (k_\star \hat{G}_n)_{\gamma_n} \rightarrow S_n \otimes_k k_\star \hat{L}_n$$  \hspace{0.5cm} \text{(7.3.2)}$$

the restriction via the embedding 4.5.2 determines a simple $k_\star \hat{L}_n$-module $\hat{M}_n$ which becomes a simple $k_\star \hat{F}_{k_\star \hat{L}}(O^*_{\eta_n})$-module (cf. isomorphism 4.6.1); finally, we may assume that we have (cf. isomorphism 4.5.4)

$$M_{n+1} = \text{Res}_{\varphi_{\eta_n}}^{\Gamma^\omega_{(G_n, b_n)}}(O^*_{\eta_n})(M_n)$$  \hspace{0.5cm} \text{(7.3.3)}$$

7.4. For any $n \in \mathbb{N}$, let $Q^n$ be a vertex and $E_n$ a $Q^n$-source of $M_n$; denoting by $Q^n_{\delta_n}$ the corresponding local pointed group on $\text{End}_k(M_n)$, it is clear that there is a local point $\delta_n$ of $Q^n$ on $k_\star \hat{G}_n b_n$ which has a nonzero image in $(\text{End}_k(M_n))(Q^n_{\delta_n})$; thus, we may assume that $P^n_{\delta_n}$ contains $Q^n_{\delta_n}$ and it follows easily from [9 Proposition 1.6] applied to $\text{End}_k(M)$ that $Q^n_{\delta_n}$ is a radical pointed group on $k_\star \hat{G}_n b_n$, so that $Q^n_{\delta_n}$ contains $O^*_{\eta_n}$ (cf. Proposition 4.2).

**Lemma 7.5.** With the notation above and up to a suitable identification, the quotient $Q^n = Q^n/O^n$ is a vertex of $M_{n+1}$. In particular, there is $\bar{x} \in G_{n+1}$ such that $(Q^n)^\bar{x} = Q^{n+1}$.
Proof: Since $Q^n$ is a vertex of $M_n$ and $P^n_{\gamma_n}$ contains $Q^n_{\delta_n}$, we have
\[
\text{End}_k(M_n)_{\gamma_n}(Q^n) \neq \{0\}
\]
and therefore we still have
\[
(\text{End}_k(M_{n+1}))(\bar{Q}^n) \cong (\text{End}_k(M_n))(\bar{Q}^n) \neq \{0\}
\]
so that there is $\bar{x} \in G_{n+1}$ such that $(\bar{Q}^n)^{\bar{x}} \subset Q^{n+1}$.

Conversely, denoting by $\bar{Q}^{n+1}$ the converse image of $Q^{n+1}$ in $P^n$ and by $x$ a lifting of $\bar{x}$ to $N_{G_n}(O_{\gamma_n})$, we have $(Q^n)^x \subset \bar{Q}^{n+1}$; moreover, since $Q^{n+1} \subset P^n$, it is clear that $S_n(\bar{Q}^{n+1}) \neq \{0\}$ and therefore we get
\[
(S_n \otimes_k \text{End}_k(M_n))(\bar{Q}^{n+1}) \cong S_n(\bar{Q}^{n+1}) \otimes_k (\text{End}_k(M_{n+1}))(Q^{n+1}) \neq \{0\}
\]
which implies that $\text{End}_k(M_n)_{\gamma_n}(\bar{Q}^{n+1}) \neq \{0\}$ and a fortiori that
\[
(\text{End}_k(M_n))(\bar{Q}^{n+1}) \neq \{0\}
\]
thus, $\bar{Q}^{n+1}$ is contained in a vertex of $M_n$ and thus we have $(\bar{Q}^n)^{\bar{x}} = Q^{n+1}$.

7.6. Once again, since the sequence $\mathcal{B}$ “stabilizes”, up to finite number of choices we may assume that $\bar{Q}^n = Q^{n+1}$ for any $n \in \mathbb{N}$; at this point, setting $Q = Q^0$, these equalities determine group homomorphisms
\[
\rho_n : Q \rightarrow Q^n \subset P^n
\]
and therefore we have a Dade $Q$-algebra $\text{Res}_{\rho_n}(S_n)$ for any $n \in \mathbb{N}$; then, since all but a finite number of these Dade $Q$-algebras are isomorphic to $k$, it makes sense to define the Dade $Q$-algebra
\[
T = \bigotimes_{n \in \mathbb{N}} \text{Res}_{\rho_n}(S_n)
\]
more generally, we denote by $T_h$ the Dade $Q^h$-algebra obtained from the tensor product $\bigotimes_{n \in \mathbb{N}} \text{Res}_{\rho_{h+n}}(S_{h+n})$ for any $h \in \mathbb{N}$. We are ready to describe a vertex $Q$ and a $Q$-source $E = E_0$ of $M$; as it could be expected, our parameterizations agree with the correspondence exhibited by Okuyama in [8].

**Proposition 7.7.** With the notation and the choice above, $R$ is a vertex of $M$ and, assuming that $Q = R$, an $R$-source $E$ of $M$ is determined by an $R$-interior algebra embedding $\text{End}_k(E) \rightarrow T$.

Proof: We argue by induction on the “length to stabilization” of $\mathcal{B}$; if this length is zero then we have $Q = \{1\} = R$ and $T \cong k$, so that everything is clear. Otherwise, considering the $k^*$-group $\hat{G}_1$, the simple $k, \hat{G}_1$-module $M_1$, the Fitting block sequence $\mathcal{B}_1 = \{(\hat{G}_{1+n}, b_{1+n})\}_{n \in \mathbb{N}}$ and the $G_1$-conjugacy
class \((R^1, Y^1)\) of weights of \(\hat{G}_1\) determined by \(M_1\), it follows from the induction hypothesis that we may assume that \(Q^1 = R^1\) and that an \(R^1\)-source \(E_1\) of \(M_1\) is determined by an \(R^1\)-interior algebra embedding \(\text{End}_k(E_1) \to T_1\).

But, it follows from Lemma 7.5 that we may assume that \(Q\) is the converse image of \(Q^1\) in \(P\), and from Corollary 5.7 that \(R\) is \(G\)-conjugate to the converse image of \(R^1\); consequently, \(R\) is also a vertex of \(M\) and we may assume that \(R = Q\). Moreover, we clearly have a \(P^0\)-interior algebra embedding (cf. 7.3)

\[
\text{End}_k(M_0)_{\gamma_0} \to S_0 \otimes_k \text{End}_k(M_0)
\]

and therefore, since we have \(Q_0 \subset P_0^0\) (cf. 7.4), we can choose an \(R\)-source \(E\) of \(M_0 = M\) such that embedding 7.7.2 determines an \(R\)-interior algebra embedding

\[
\text{End}_k(E) \to \text{Res}_{\rho_0}(S_0) \otimes_k \text{End}_k(E) \to \text{Res}_{\rho_0}(S_0) \otimes_k T_1 = T
\]

7.7.3. We are done.

7.8. From now on, we assume that \(Q^n = R^n\) for any \(n \in \mathbb{N}\) and, as in Lemma 2.16 above, we denote by \(R^n_{E_n}\) the corresponding local pointed group on \(\text{End}_k(M_n)\); let us consider a multiplicity \(k_*\hat{N}_{G_n}(R^n_{E_n})\)-module \(V_n\) of \(M_n\); since by Proposition 7.7 we already know that \(\text{End}_k(E_n)\) is a Dade \(R^n\)-algebra, it follows from Lemma 2.16 that there exists a \(k^*\)-group isomorphism

\[
\hat{N}_{G_n}(R^n_{E_n}) \cong \hat{N}_{\hat{G}_n}(R^n_{E_n})
\]

7.8.1 which, according to the \(k^*\)-group homomorphism 2.8.1, depends on the choice of a splitting for the \(k^*\)-group (cf. 2.9 and Proposition 7.7)

\[
\hat{F}_{\text{End}_k(M_n)}(R^n_{E_n}) \equiv \hat{F}_{\text{End}_k(E_n)}(R^n_{E_n}) \equiv \hat{F}_{T_n}(R^n)
\]

7.8.2 and indeed, from our choice of the polarization \(\omega\), we have the splitting

\[
\omega(R^n, T_n) : \hat{F}_{T_n}(R^n) \to k^*
\]

7.8.3.

7.9. On the other hand, for any \(n \in \mathbb{N}\), let \(W_n\) be the restriction to the stabilizer \(\hat{N}_{G_n}(R^n_{\delta_n})_{E_n}\) in \(\hat{N}_{G_n}(R^n_{\delta_n})\) of the isomorphism class of \(E_n\), of a multiplicity \(k_*\hat{N}_{G_n}(R^n_{\delta_n})\)-module of \(R^n_{\delta_n}\) (cf. 2.5); more explicitly, denoting by \(\hat{b}(\delta_n)\) the block of \(\hat{C}_{\hat{G}_n}(R^n)\) determined by \(\delta_n\), since \(R^n_{\delta_n}\) is a radical pointed group on \(k_*\hat{G}_n\), we have (cf. 2.11.1 and 3.1)

\[
k_*\hat{C}_{\hat{G}_n}(R^n)\hat{b}(\delta_n) \cong \text{End}_k(W_n)
\]

7.9.1.
Consequently, since \((\text{End}_k(M_n))(R^n_{E_n})\cong \text{End}_k(V_n)\) has a \(\hat{C}_G(R^n)\)-interior algebra structure, it makes sense to consider \(\tilde{b}(\delta_n)\cdot(\text{End}_k(M_n))(R^n_{E_n})\cdot \tilde{b}(\delta_n)\) as a \(\hat{N}_{G_n}(R^n_{E_n})\delta_n\)-interior algebra and, from the structural homomorphism above, we get an injective \(k\)-algebra homomorphism

\[
(k,\hat{G})(R^n_{E_n}) \rightarrow \tilde{b}(\delta_n)\cdot(\text{End}_k(M_n))(R^n_{E_n})\tilde{b}(\delta_n)
\]

then, denoting by \(\hat{N}_{G_n}(R^n_{E_n})\delta_n\) the stabilizer of \(\delta_n\) in \(\hat{N}_{G_n}(R^n_{E_n})\) and setting

\[
\hat{N}^\delta_n = \hat{N}_{G_n}(R^n_{E_n})\delta_n \ast (\hat{N}_{G_n}(R^n_{E_n})E_n)^\circ
\]

it follows from [9, Proposition 2.1] that, for a suitable primitive \(\hat{N}^\delta_n\)-interior algebra \(B_n\), we have a \(\hat{N}_{G_n}(R^n_{E_n})\)-interior algebra isomorphism

\[
\text{End}_k(V_n) \cong \text{Ind}_{\hat{N}_{G_n}(R^n_{E_n})\delta_n}^\hat{N}_{G_n}(R^n_{E_n}) (k,\hat{C}_G(R^n)\tilde{b}(\delta_n) \otimes_k B_n)
\]

actually, it is easily checked that the subgroup

\[
\hat{C}_G(R^n) \cong \hat{C}_G(R^n) \ast \hat{C}_G(R^n)^\circ \subset \hat{N}^\delta_n
\]

has a trivial image in \(B_n\) so that, up to an obvious identification, \(B_n\) becomes an \(\hat{N}^\delta_n/\hat{C}_G(R^n)\)-interior algebra.

7.10. Similarly, denoting by \(U_n\) a simple projective \(k_*\hat{N}_{G_n}(R^n)\)-module which restricted to \(\hat{N}_{G_n}(R^n)\) belongs to the isomorphism class \(Y^n\), it follows from [16, Proposition 3.2] applied to the primitive \(\hat{N}_{G_n}(R^n)\)-interior algebra \(\text{End}_k(U_n)\) that, setting

\[
\hat{N}_n = \hat{N}_{G_n}(R^n) \ast \hat{N}_{G_n}(R^n)^\circ
\]

for a suitable primitive \(\hat{N}_n\)-interior algebra \(D_n\) we have

\[
\text{End}_k(U_n) \cong \text{Ind}_{\hat{N}_{G_n}(R^n)\delta_n}^\hat{N}_{G_n}(R^n) (k_*\hat{C}_G(R^n)\tilde{b}(\delta_n) \otimes_k D_n)
\]

actually, it is clear from its very definition that \(D_n\) becomes a \(\hat{N}_n/\hat{C}_G(R^n)\)-interior algebra and note that, according to homomorphism 2.8.1, we have a canonical \(k^*\)-group isomorphism

\[
\hat{N}_n/\hat{C}_G(R^n) \cong \hat{F}_{k_*G_n}(R^n_{\delta_n})
\]

In order to relate \(D_n\) with \(U_{n+1}\), we have to consider the following \(k^*\)-group isomorphism.
Proposition 7.11. With the notation and the choice above, there is a $k^*$-group isomorphism

$$\omega \phi_n : \hat{F}_{k, G_n}(R^n_{\delta_n}) \cong \tilde{N}_{G_{n+1}}(R^{n+1})$$

such that, for any $P^n$-interior algebra embedding

$$e_{\gamma_n} : (k_* \hat{G}_n)_{\gamma_n} \to S_n \otimes_k k_* \hat{L}_n$$

we have the commutative diagram

$$\begin{align*}
\hat{F}_{k, G_n}(R^n_{\delta_n}) &\cong \hat{F}_{S_n \otimes_k k_* \hat{L}_n}(R^n_{S_n \times \delta_n}) &\cong \hat{F}_{k_* \hat{L}_n}(R^n_{\delta_n}) \\
\omega \phi_n &\cong \tilde{N}_{L_n}(R^n) \\
N_{G_{n+1}}(R^{n+1}) &\cong \tilde{N}_{F_{S_n \otimes k_* \hat{L}_n}(O_{S_n \times \eta_n})}(\tilde{R}^n) &\cong \tilde{N}_{F_{k_* \hat{L}_n}(O_{\eta_n})}(\tilde{R}^n)
\end{align*}$$

Proof: Choosing a $P^n$-interior algebra embedding $e_\gamma : (k_* \hat{G})_{\gamma} \to S \otimes_k k_* \hat{L}$, we have $k^*$-group isomorphisms (cf. 5.5.1)

$$\hat{F}_{k, G_n}(R^n_{\delta_n}) \cong \hat{F}_{S_n \otimes_k k_* \hat{L}_n}(R^n_{S_n \times \delta_n}) \cong \hat{F}_{k_* \hat{L}_n}(R^n_{\delta_n})$$

similarly, the $k^*$-group isomorphisms 4.5.4 applied to $O_{\eta_n}^n$ yield

$$\hat{F}_{k, G_n}(O^n_{\eta_n}) \cong \hat{F}_{S_n \otimes_k k_* \hat{L}_n}(O^n_{S_n \times \eta_n}) \cong \hat{F}_{k_* \hat{L}_n}(O^n_{\eta_n})$$

furthermore, setting $\tilde{R}^n = R^n / O^n$, we have canonical $k^*$-group isomorphisms (cf. isomorphisms 4.6.2 and 5.5.2)

$$\hat{F}_{k, L_n}(R^n_{\delta_n}) \cong \tilde{N}_{L_n}(R^n) \cong \tilde{N}_{F_{k_* \hat{L}_n}(O_{\eta_n})}(\tilde{R}^n)$$

Now, it is clear that the commutativity of the corresponding diagram above defines a $k^*$-group isomorphism $\omega \phi_n$.

We claim that this $k^*$-group isomorphism does not depend on the choice of $e_{\gamma_n}$; indeed, for another choice $e'_{\gamma_n}$ of this embedding, it follows from Proposition 5.2 that there is $\tilde{\sigma}_n \in \text{Out}_{P^n}((k_* \hat{G}_n)_{\gamma_n})$ fulfilling

$$e'_{\gamma_n} = \tilde{e}_{\gamma_n} \circ \tilde{\sigma}_n = (\tilde{id}_{S_n} \otimes \tilde{\sigma}_n) \circ \tilde{e}_{\gamma_n}$$
and therefore, with obvious notation, we get the following commutative diagrams

\[
\begin{align*}
\hat{\mathbf{F}}_{k, \mathcal{G}}(R^n_{k\mathbb{A}}) & \cong \hat{\mathbf{F}}_{S_n \otimes k, L_n}(R^n_{S_n \times \mathbb{A}}) \cong \hat{\mathbf{F}}_{k, L_n}(R^n_{\mathbb{A}}) \\
\| & \| \quad \| & \| \quad \| & \| \\
\hat{\mathbf{F}}_{k, \mathcal{G}}(O^n_{\mathbb{A}}) & \cong \hat{\mathbf{F}}_{S_n \otimes k, L_n}(O^n_{S_n \times \mathbb{A}}) \cong \hat{\mathbf{F}}_{k, L_n}(O^n_{\mathbb{A}})
\end{align*}
\]

7.11.8

Now, the commutativity of the corresponding diagram above follows from these commutative diagrams and from the naturality of the right-hand vertical isomorphisms in the diagram. We are done.

**Corollary 7.12.** With the notation and the choice above, we have an 
\[D_n \cong \text{Res}_{\phi_n}(\text{End}_k(U_{n+1}))\] 7.12.1.

**Proof:** Since \(\text{End}_k(U_n)\) is actually isomorphic to a block of defect zero of the \(k^n\)-group \(\bar{N}_{\mathcal{G}}(R^n)\) (cf. 7.10), it follows from [16, Theorem 3.7] and from isomorphism 7.10.2 above that \(D_n\) is isomorphic to a block of defect zero of the \(k^n\)-group \(\hat{\mathcal{F}}_{k, \mathcal{G}}(R^n_{\mathcal{A}})\) and then isomorphism 7.12.1 easily follows from the commutativity of diagram 7.11.3 above.

7.13. On the other hand, according to Proposition 7.7, we have \(R^n\) and \(R^{n+1}\)-interior algebra embeddings

\[\text{End}_k(E_n) \hookrightarrow T_n \quad \text{and} \quad \text{End}_k(E_{n+1}) \hookrightarrow T_{n+1}\] 7.13.2;

but, denoting by \(t_n\), \(t_{n+1}\) and \(s_n\) the respective similarity classes in the Dade group \(D_{\alpha}(R^n)\) of \(T_n\), of the restriction to \(R^n\) of \(T_{n+1}\), and of \(S_n\) (cf. 2.13), we clearly have \(t_n = s_n + t_{n+1}\) (cf. 7.6) and therefore any automorphism of \(R^n\) stabilizing \(s_n\) stabilizes \(t_n\) if and only if it stabilizes \(t_{n+1}\); moreover, it follows from the inclusion in 4.5.3 that \(F_{k, \mathcal{G}}(R^n_{\mathcal{A}})\) stabilizes \(s_n\). Consequently, since \(t_n\) and \(t_{n+1}\) respectively determine the isomorphism classes of \(E_n\) and of the restriction to \(R^n\) of \(E_{n+1}\), the stabilizers in \(F_{k, \mathcal{G}}(R^n_{\mathcal{A}})\) of these isomorphism classes coincide with each other, and therefore we have a canonical surjective homomorphism

\[\nu_n : \bar{N}_{\mathcal{G}}(R^n_{\mathcal{A}})E_n = \bar{N}_{\mathcal{G}}(R^n_{E_n})_{\delta_n} \rightarrow \bar{N}_{G_{n+1}}(R^{n+1}_{E_{n+1}})\] 7.13.3.
Proposition 7.14. With the notation and the choice above, there are a $k^*$-group homomorphism and a $\hat{\mathcal{N}}_{E_n}^*$-interior algebra isomorphism

$$\hat{\nu}_n : \hat{\mathcal{N}}_{E_n}^* \rightarrow \hat{\mathcal{N}}_{E_{n+1}}^*(R_{E_{n+1}}^{n+1})$$

$$f_n : B_n \cong \text{Res}_{\nu_n}\left(\left(\text{End}_k(M_{n+1})\right)(R_{E_{n+1}}^{n+1})\right)$$

such that, for any $P^n$-interior algebra embedding

$$e_{\gamma_n} : (k_*\hat{G}_n)_{\gamma_n} \rightarrow S_n \otimes_k k_*\hat{L}_n$$

we have the commutative diagram

$$\begin{array}{ccc}
B_n & \cong & (\text{End}_k(M_n))(R_{E_n}^n) \\
\text{Res}_{\nu_n} \downarrow \quad \downarrow & & \downarrow \\
(\text{End}_k(M_{n+1}))(R_{E_{n+1}}^{n+1}) & \cong & (\text{End}_k(M_n))(\hat{R}_{E_n}^n) \\
\end{array}$$

**Proof:** We have a structural injective $k$-algebra homomorphism (cf. 7.3.1)

$$(k_*\hat{G})_{\gamma_n}(R_{\delta_n}^n) \rightarrow (\text{End}_k(M_n))(R_{E_n}^n)_{\gamma_n}$$

and, as in 7.9 above, denoting by $C_n$ the centralizer in $\text{End}_k(M_n)_{\gamma_n}(R_{E_n}^n)$ of the image of $(k_*\hat{G}_n)_{\gamma_n}(R_{\delta_n}^n)$, it follows from [9, Proposition 2.1] that we have a $k$-algebra isomorphism

$$\text{End}_k(M_n)_{\gamma_n}(R_{E_n}^n) \cong (k_*\hat{G})_{\gamma_n}(R_{E_n}^n) \otimes_k C_n$$

Moreover, always according to [9, Proposition 2.1], the obvious commutative diagram

$$\begin{array}{ccc}
(k_*\hat{G})_{\gamma_n}(R_{\delta_n}^n) & \rightarrow & (k_*\hat{G})(R_{\delta_n}^n) \\
\downarrow & & \downarrow \\
\text{End}_k(M_n)_{\gamma_n}(R_{E_n}^n) & \rightarrow & \hat{b}(\delta_n) \cdot (\text{End}_k(M_n))(R_{E_n}^n) \cdot \hat{b}(\delta_n) \\
\end{array}$$

induce a canonical $k$-algebra isomorphism $C_n \cong B_n$ which allows us to identify both centralizers.

Choosing a $P^n$-interior algebra embedding (cf. statement 4.4.2)

$$e_{\gamma_n} : (k_*\hat{G}_n)_{\gamma_n} \rightarrow S_n \otimes_k k_*\hat{L}_n$$

note that $(k_*\hat{L}_n)(R_{\delta_n}^n) \cong k$ since $R_{\delta_n}^n$ and $R_{\delta_n}^n$ are radical and therefore they are selfcentralizing; then, the corresponding commutative diagram

$$\begin{array}{ccc}
(k_*\hat{G}_n)_{\gamma_n}(R_{\delta_n}^n) & \rightarrow & S_n(R_{\delta_n}^n) \\
\downarrow & & \downarrow \\
(\text{End}_k(M_n))_{\gamma_n}(R_{E_n}^n) & \rightarrow & S_n(R_{E_n}^n) \otimes_k (\text{End}_k(M_n))(R_{E_n}^n) \\
\end{array}$$
and the argument above yield the top \( k \)-algebra isomorphism
\[
B_n \cong \left( \text{End}_k(\hat{M}_n) \right) \left( R_{E_n}^n \right) 
\]
7.14.9
in the diagram 7.14.3 above, and the \( k^* \)-group isomorphism
\[
\hat{N}_n^{E_n}/C_G(R^n) \cong \hat{N}_{L_n}(R_{E_n}^n) 
\]
7.14.10.

Moreover, according to 7.3, we get
\[
\text{End}_k(\hat{M}_n) \cong \text{End}_k(\hat{M}_n) \cong \text{End}_k(M_{n+1}) 
\]
7.14.11
with the interior structures coming from the \( k^* \)-group isomorphisms (cf. 4.5.4 and 4.6.1)
\[
\hat{L}_n/O^n \cong \hat{F}_{k\ast \hat{L}_n}(O^n_{\eta_n}) \cong \hat{G}_{n+1} 
\]
7.14.12;
consequently, we still get
\[
(\text{End}_k(\hat{M}_n))(R_{E_n}^n) \cong (\text{End}_k(\hat{M}_n))(R_{E_n}^n) 
\]
\[
\cong (\text{End}_k(\hat{M}_{n+1}))(R_{E_{n+1}}^{n+1}) 
\]
7.14.13
with the interior structures coming from the \( k \)-group isomorphisms
\[
\hat{N}_{L_n}(R_{E_n}^n) \cong \hat{N}_{F_{k\ast \hat{L}_n}}(O^n_{\eta_n})(\hat{R}_{E_n}^n) \cong \hat{N}_{G_{n+1}^{n+1}}(R_{E_{n+1}}^{n+1}) 
\]
7.14.14.

Finally, for a particular choice of \( e_{\gamma_n} \), the commutativity of diagram 7.14.3 induce the isomorphism \( f_n \), and the \( k^* \)-group isomorphisms 7.14.10 and 7.14.14 determine the \( k^* \)-group homomorphism \( \nu_n \). Once again \( f_n \) and \( \nu_n \) do not depend on the choice of \( e_{\gamma_n} \); indeed, for another choice \( e'_{\gamma_n} \) of this embedding, it follows from Proposition 5.2 that there is \( \hat{s}_n \in \text{Out}_{\nu_n}(\{k\ast \hat{G}_n\}_{\gamma_n}) \) fulfilling
\[
e'_{\gamma_n} = e_{\gamma_n} \circ \hat{s}_n = (\text{id}_{S_n} \otimes \hat{s}_n) \circ e_{\gamma_n} 
\]
7.14.15
and we get commutative diagrams as above.

7.15. We are ready to describe the multiplicity \( \hat{N}_G(R_E) \)-module of \( M \); first of all, from the very definition of \( N_G(B) \) (cf. 6.2), we get a sequence of \( k^* \)-groups \( \hat{N}_G(B) \), with the same \( k^* \)-quotient \( N_G(B) \), and of \( k^* \)-group homomorphisms \( \hat{\mu}_n : \hat{N}_G^n(B) \to \hat{G}_n \) inductively defined as follows; the \( k^* \)-group \( \hat{N}_G^n(B) \) is just the converse image of \( N_G(B) \) in \( \hat{G} \) and \( \hat{\mu}_n \) the inclusion map; then, for any \( n \geq 1 \), arguing by induction on \( n \) it is easily checked that the image of \( \hat{\mu}_{n-1} \) normalizes the pointed group \( O_{\eta_n-1}^{n-1} \) on \( k\ast \hat{G}_{n-1} \) and therefore \( \hat{\mu}_{n-1} \) induces a group homomorphism \( \mu_{n-1} \) from \( N_G(B) \) to \( N_G(B) \) of \( O_{\eta_n-1}^{n-1} \):
since we have (cf. statement 2.11.2)

\[ N_{G_{n-1}}(O_{\eta_{n-1}}^{n-1})/O_{\eta_{n-1}}^{n-1} \cdot C_{G_{n-1}}(O_{\eta_{n-1}}^{n-1}) \cong F_{k_* G_{n-1}}(O_{\eta_{n-1}}^{n-1}) \]

we define \( \hat{N}_G^0(B) \) and \( \hat{\mu}_n \) by the following pull-back

\[
\begin{align*}
N_G(B) & \longrightarrow F_{k_* G_{n-1}}(O_{\eta_{n-1}}^{n-1}) \\
\hat{N}_G^0(B) & \xrightarrow{\hat{\mu}_n} \hat{G}_n
\end{align*}
\]

7.15.2.

7.16. Now, consider the pointed vertex sequence \( \mathcal{R} = \{ R^n_{\delta_n} \}_{n \in \mathbb{N}} \) of \( M \) associated to \( \omega \) (cf. 7.4 and 7.8), and denote by \( N_G(\mathcal{R}) \) the stabilizer of \( \mathcal{R} \) in \( G \); clearly, \( N_G(B) \) contains \( N_G(\mathcal{R}) \) and we denote by \( \hat{N}_G^n(\mathcal{R}) \) the corresponding \( k^* \)-subgroup of \( \hat{N}_G^0(B) \). Moreover, arguing by induction on \( n \), it is easily checked that the subgroup \( \mu_n(N_G(\mathcal{R})) \subset G_n \) normalizes the pointed group \( R^n_{\delta_n} \) on \( k_* \hat{G}_n \); thus, setting \( \hat{N}_G(\mathcal{R}) = N_G(\mathcal{R})/R \), we get a \( k^* \)-group \( \hat{N}_G^n(\mathcal{R}) \) from the following pull-back

\[
\begin{align*}
\hat{N}_G^0(\mathcal{R}) & \xrightarrow{\hat{\mu}_n} \hat{N}_G^n(R^n_{\delta_n}) \\
\hat{N}_G^n(\mathcal{R}) & \xrightarrow{\hat{\mu}_n} \hat{N}_G^n(R^n_{\delta_n})
\end{align*}
\]

7.16.1

and then we have the \( k_* \hat{N}_G^n(\mathcal{R}) \)-module \( \text{Res}_{\hat{\mu}_n}^n(W_n) \) for any \( n \in \mathbb{N} \) (cf. 7.9.1).

7.17. On the other hand, from the \( k^* \)-group homomorphism 2.8.1 and from Proposition 7.11, for any \( n \in \mathbb{N} \) we get the \( k^* \)-group homomorphisms

\[
\begin{align*}
\hat{N}_G^n(R^n_{\delta_n}) \ast \hat{N}_G^n(R^n_{\delta_n}) & \longrightarrow \hat{F}_{k_* G_n}(R^n_{\delta_n}) \cong \hat{N}_G^{n+1}(R^{n+1}) \\
\hat{N}_G^n(\mathcal{R}) \ast \hat{N}_G^n(\mathcal{R}) & \longrightarrow \hat{N}_G^{n+1}(\mathcal{R})
\end{align*}
\]

7.17.1

and therefore, since all the bottom \( k^* \)-groups admit the same \( k^* \)-quotient, we still get a \( k^* \)-group isomorphism

\[
\omega \Psi_n : \hat{N}_G^n(\mathcal{R}) \cong \hat{N}_G^n(\mathcal{R}) \ast \hat{N}_G^{n+1}(\mathcal{R})
\]

7.17.2.

But, for \( n \) big enough we have

\[
W_n \cong k \quad \text{and} \quad \hat{N}_G^0(B) \cong k^* \times N_G(B)
\]

7.17.3;

moreover, note that \( \hat{N}_G^0(B) \) coincides with the converse image \( N_G(B) \) of \( N_G(B) \) in \( \hat{G} \) and similarly we set \( N_G(\mathcal{R}) = \hat{N}_G^0(\mathcal{R}) \) and \( \hat{N}_G(\mathcal{R}) = N_G(\mathcal{R})/R \);
in particular, the following tensor product

\[ \omega W = \bigotimes_{n \in \mathbb{N}} \text{Res}_{\hat{\mu}^n_{\delta_n} \circ (\Psi_n)^{-1}}(W_n) \]  

makes sense and it is clearly a \( k_* \hat{N}_G(R) \)-module.

7.18. Finally, it follows from 4.5.3 that \( F_{k_* \hat{G}_n}(R^n_{\delta_n}) \) stabilizes the isomorphism class of \( \text{Res}_{R^n_{\delta_n}}(S_n) \) and therefore \( \hat{N}_G(R) \) stabilizes the similarity class of \( T_n \) and, in particular, the isomorphism class of \( E_n \) for any \( n \in \mathbb{N} \) (cf. 7.6 and Proposition 7.7); hence, we get again a \( k^* \)-group \( \hat{N}_{E_n}(R) \) from the following pull-back

\[
\begin{array}{c}
\hat{N}_G(R) \xrightarrow{\hat{\mu}_n} \hat{N}_{G_n}(R^n_{\delta_n})_{E_n} \\
\uparrow \\
\hat{N}_{E_n}(R) \xrightarrow{\hat{\mu}^n_{E_n}} \hat{N}_{G_n}(R^n_{E_n})_{\delta_n}
\end{array}
\]  

Similarly, from Proposition 7.14, for any \( n \in \mathbb{N} \) we get the \( k^* \)-group homomorphisms

\[
\begin{array}{c}
\hat{N}_{G_n}(R^n_{E_n})_{\delta_n} \ast \hat{N}_{G_n}(R^n_{\delta_n})_{E_n} \xrightarrow{\hat{\mu}^n_{E_n} \ast \hat{\mu}^n_{\delta_n}} \hat{N}_{G_{n+1}}(R^{n+1}_{E_{n+1}}) \\
\uparrow \\
\hat{N}_{E_n}(R) \ast \hat{N}_{G_n}(R) \xrightarrow{\hat{\mu}^n_{E_n} \ast \hat{\mu}^n_{\delta_n}} \hat{N}_{G_{n+1}}(R)
\end{array}
\]  

and therefore, since all the bottom \( k^* \)-groups admit the same \( k^* \)-quotient, we still get a \( k^* \)-group isomorphism

\[ \Psi_n : \hat{N}_G(R) \cong \hat{N}_{E_n}(R) \ast \hat{N}_{E_{n+1}}(R) \]  

thus, the following tensor product

\[ W = \bigotimes_{n \in \mathbb{N}} \text{Res}_{\hat{\mu}^n_{\delta_n} \circ (\Psi_n)^{-1}}(W_n) \]  

makes sense and it is clearly a \( k_* \hat{N}_G(R) \)-module. As above, we set \( R = R^0 \), \( E = E_0 \), \( V = V_0 \) and \( U = U_0 \).

**Theorem 7.19.** With the notation and the choice above, we have natural \( k_* \hat{N}_G(R) \)- and \( k_* \hat{N}_G(R_E) \)-module isomorphisms

\[ U \cong \text{Ind}_{\hat{N}_G(R)}^{\hat{N}_G(R)}(\omega W) \]  

and

\[ V \cong \text{Ind}_{\hat{N}_G(R_E)}^{\hat{N}_G(R)}(W) \]
Proof: Once again, we can argue by induction on the “length to stabilization” of $B$. If this length is zero then the block $b_0$ is already of defect zero and therefore everything is trivial so that isomorphisms 7.19.1 above are trivially true.

If the “length to stabilization” is not zero then we consider the Fitting block sequence $B_1 = \{(G_1, b_1 + n)\}_{n \in \mathbb{N}}$ of $\hat{G}_1$ and the corresponding weight sequence $\{([1 + n], [1 + n])\}_{n \in \mathbb{N}}$ and simple sequence $\{M_{1 + n}\}_{n \in \mathbb{N}}$ $\omega$-associated to $B_1$; mutatis mutandis, we consider the corresponding pointed vertex sequence $R_1 = \{R_{1+n}\}_{n \in \mathbb{N}}$ of $M_1$ associated to $\omega$, and denote by $N_{G_1}(R_1)$ and $N_{G_1}(R_1)$ the respective stabilizers of $R_1$ in $G_1$ and in $\hat{G}_1$.

Moreover, from the corresponding pull-back 7.16.1, for any $n \geq 1$ we get a $k^*$-group $\hat{N}_{G_1}(R_1)$ of $k^*$-quotient $\hat{N}_{G_1}(R_1)$ and a $k^*$-group homomorphism

$$\hat{N}_{G_1}(R_1) \xrightarrow{\mu_{1,n}} \hat{N}_{G_1}(R_{1+n})$$

7.19.2,

so that we still get the $k_*,\hat{N}_{G_1}(R_1)$-module $\text{Res}_{\mu_{1,n}}(W_n)$. Analogously, for any $n \geq 1$ we still get the corresponding $k^*$-group isomorphisms 7.17.2 and 7.18.3

$$\omega \psi_{1,n} : \hat{N}_{G_1}(R_1) \cong \hat{N}_{G_1}(R_1) \xrightarrow{\mu_{1,n}} \hat{N}_{G_1}(R_{1+n}) \circ \hat{N}_{G_1}(R_1)$$

7.19.3,

and, once again, the following tensor products

$$\omega W^1 = \bigotimes_{n \geq 1} \text{Res}_{\mu_{1,n}}(\omega \psi_{1,n})^{-1}(W_n)$$

7.19.4

$$W^1 = \bigotimes_{n \geq 1} \text{Res}_{\mu_{1,n}}(\psi_{1,n})^{-1}(W_n)$$

make sense and respectively become $k_*,\hat{N}_{G_1}(R_1)$- and $k_*,\hat{N}_{G_1}(R_1)$-modules.

At this point, it follows from the induction hypothesis that we have natural $k_*,\hat{N}_{G_1}(R_1)$- and $k_*,\hat{N}_{G_1}(R_{1,E_1})$-module isomorphisms

$$U_1 \cong \text{Ind}_{\hat{N}_{G_1}(R_1)}(\omega W^1) \quad \text{and} \quad V_1 \cong \text{Ind}_{\hat{N}_{G_1}(R_{1,E_1})}(W^1)$$

7.19.5.

But, it follows from isomorphisms 7.9.4 and 7.10.2, and from Corollary 7.12 and Proposition 7.14 that, considering the surjective $k^*$-group homomorphism (cf. 2.8.1 and Propositions 7.11 and 7.14)

$$\hat{N}_0 = \hat{N}_G(R_0) \rightarrow \hat{N}_G(R_0) \circ \hat{F}_{k,G}(R_0) \cong \hat{N}_G(R_1)$$

7.19.6
and denoting by $\hat{U}_1$ and $\hat{V}_1$ the corresponding restrictions of $U_1$ to $\hat{N}_0$ and of $V_1$ to $\hat{N}_0^E$, we have

$$U \cong \text{Ind}_{N_G(R)}^{N_G(R_0)}(W_0 \otimes_k \hat{U}_1) \quad \text{and} \quad V \cong \text{Ind}_{\hat{N}_G(R_0)}^{\hat{N}_G(R_0^E)}(W_0 \otimes_k \hat{V}_1) \quad 7.19.7.$$ 

Furthermore, it is easily checked that the image of $\hat{N}_G(R) \subset \hat{N}_G(R_0)$ in $\hat{N}_G(R^1)$ throughout the $k^*$-quotient of homomorphism $7.19.2$ is contained in $\hat{N}_G(R_1)$ and that this $k^*$-group homomorphism induces $k^*$-group homomorphisms (cf. 7.17.1 and 7.18.2)

$$\hat{N}_G(R) \star \hat{N}_G(R) \cong \hat{N}_G(R) \longrightarrow \hat{N}_G(R_1) \quad 7.19.8;$$

$$\hat{N}_G(R^1) \star \hat{N}_G(R^1) \cong \hat{N}_G(R) \longrightarrow \hat{N}_G(R_1)$$

thus, denoting by $\omega \hat{W}^1$ and by $\hat{W}^1$ the corresponding restrictions of $\omega W^1$ to $\hat{N}_G(R)$ and of $W^1$ to $\hat{N}_G(R_1)$ we have (cf. 7.19.5)

$$\hat{U}_1 = \text{Ind}_{\hat{N}_G(R)}^{\hat{N}_G(R_1)}(\omega \hat{W}^1) \quad \text{and} \quad \hat{V}_1 = \text{Ind}_{\hat{N}_G(R)}^{\hat{N}_G(R_1)}(\hat{W}^1) \quad 7.19.9.$$ 

More explicitly, for any $n \geq 1$ the following diagrams of $k^*$-group homomorphisms

$$\hat{N}_G(R_1) \star \hat{N}_G(R_1) \cong \hat{N}_G(R_1) \longrightarrow \hat{N}_G(R_0^1)$$

are commutative since all the vertical arrows are defined by pull-back via the group homomorphism $N_G(R) \to N_G(R_1)$ determined by the $k^*$-quotient of homomorphism $7.19.2$; hence, we actually get a $k^*$-quotient of homomorphism $7.19.2$- and a $k^*$-quotient of homomorphism $7.19.2$-module isomorphisms

$$\omega \hat{W}^1 \cong \bigotimes_{n \geq 1} \text{Res}_{\hat{N}_G(R^1)}^{\hat{N}_G(R_0^1)}(W_n) \quad 7.19.12$$

$$\hat{W}^1 \cong \bigotimes_{n \geq 1} \text{Res}_{\hat{N}_G(R^1)}^{\hat{N}_G(R_0^1)}(W_n)$$
Consequently, from the Frobenius property, we get a $k_∗\tilde{N}_G(R_δ)$-module isomorphism

$$W_0 \otimes_k \bar{U}_1 \cong \text{Ind}_{\tilde{N}_G(R_δ)}^{\tilde{N}_G(R_δ)}(\text{Res}_{\hat{\rho}_0(\Psi_0)^{-1}}(W_0) \otimes_k \omega\bar{W})$$  \hspace{1cm} 7.19.13

and therefore from the left-hand isomorphism in 7.19.7 we obtain the left-hand isomorphism in 7.19.1. Similarly, we get a $k_∗\tilde{N}_G(R_δ_{E})$-module isomorphism

$$W_0 \otimes_k \bar{V}_1 \cong \text{Ind}_{\tilde{N}_G(R_δ_{E})}^{\tilde{N}_G(R_δ_{E})}(\text{Res}_{\hat{\rho}_0(\Psi_0)^{-1}}(W_0) \otimes_k \omega\bar{W})$$  \hspace{1cm} 7.19.14

and therefore from the right-hand isomorphism in 7.19.7 we obtain the right-hand isomorphism in 7.19.1. We are done.

7.20. In order to compare both isomorphisms in 7.19.1, note that from homomorphism 2.8.2 and from our choice of a polarization $\omega$ we have a $k^*$-group homomorphism

$$\tilde{N}_G(R_{E})^\circ \star \tilde{N}_G(R_δ) \longrightarrow \bar{F}_T(R) \xrightarrow{\omega(R,T)} k^*$$  \hspace{1cm} 7.20.1

which determines a $k^*$-group isomorphism $\tilde{N}_G(R_{E}) \cong \hat{\tilde{N}}_G(R_{E})$; let us denote by $^*V$ the restriction of $V$ throughout this isomorphism. Similarly, for any $n \in \mathbb{N}$, the $k_∗(\hat{\tilde{N}}_G^n(\mathcal{R}) \star \hat{\tilde{N}}_G^{n+1}(\mathcal{R})^\circ)$-module $\text{Res}_{\hat{\rho}_{n}(\Psi_0)^{-1}}(W_n)$ restricted throughout the composed $k^*$-group isomorphism

$$\hat{\tilde{N}}_G^n(\mathcal{R}) \star \hat{\tilde{N}}_G^{n+1}(\mathcal{R})^\circ \xrightarrow{(-\Psi_0)^{-1}} \hat{\tilde{N}}_G^n(\mathcal{R}) \cong \hat{\tilde{N}}_G(n+1)(\mathcal{R})^\circ \hat{\tilde{N}}_G(n)(\mathcal{R}) \cong \hat{\tilde{N}}_G(\mathcal{R}) \star \hat{\tilde{N}}_G^{n+1}(\mathcal{R})^\circ$$  \hspace{1cm} 7.20.2

coincides with $^*W_n$.

7.21. But, according to the right-hand $k^*$-group isomorphism in 4.5.4, the corresponding splitting

$$\left(\hat{\tilde{N}}_G^n(\mathcal{R}) \star \hat{\tilde{N}}_G^{n+1}(\mathcal{R})^\circ\right) \star \left(\hat{\tilde{N}}_G^n(\mathcal{R}) \star \hat{\tilde{N}}_G^{n+1}(\mathcal{R})^\circ\right) \longrightarrow k^*$$  \hspace{1cm} 7.21.1

comes from $\omega(R^*,\text{Res}_{\hat{\rho}_{n}(S_{n})}) : \hat{F}_S(R^*) \rightarrow k^*$ and needs not coincide with the splitting

$$\left(\hat{\tilde{N}}_G^n(\mathcal{R}) \star \hat{\tilde{N}}_G^{n+1}(\mathcal{R})^\circ\right) \star \left(\hat{\tilde{N}}_G^n(\mathcal{R}) \star \hat{\tilde{N}}_G^{n+1}(\mathcal{R})^\circ\right) \longrightarrow k^*$$  \hspace{1cm} 7.21.2
coming from (cf. 2.8.1)
\[ \omega_{(R^n, T^n)} : \hat{F}_{T^n}(R^n) \to k^* \quad \text{and} \quad \omega_{(R^{n+1}, T^{n+1})} : \hat{F}_{T^{n+1}}(R^{n+1}) \to k^* \]
that is to say, this splitting determines a new \( k^* \)-group isomorphism between 
\[ \hat{N}_G(R) \star \hat{N}_G(R) \] and \[ \hat{N}_G(R) \star \hat{N}_G(R) \] ; thus, this isomorphism and \( \psi_n \circ (\omega \psi_n)^{-1} \) determine an automorphism \( \omega \theta \) of \( \hat{N}_G(R) \star \hat{N}_G(R) \). Then, it is clear that the product of all these automorphisms defines an automorphism \( \omega \theta \) of \( \hat{N}_G(R) \) and that the right-hand isomorphism in 7.19.1 implies the following result.

**Corollary 7.22.** With the notation and the choice above, we have a natural \( k^* \)-module isomorphism
\[ \omega V \cong \text{Ind}_{\hat{N}_G(R)}^{\hat{N}_G(R)} (\text{Res}_{\omega \theta} (\omega W)) \]

**Appendix: The odd order case**

A.1. Assume that \( p \neq 2 \) and let \( \hat{G} \) be a \( k^* \)-group with finite \( k^* \)-quotient \( G \) of odd order. In this case, by the fundamental Feit-Thompson Theorem [3], \( G \) is solvable and therefore, for any choice of a polarization \( \omega \), Theorem 6.5 above supplies a natural bijection
\[ \text{Irr}_k(\hat{G}) \cong \text{Wgt}_k(\hat{G}) \]

actually, it suffices to consider \( \omega \) over the torsion subcategory \( \mathcal{D}_k^\text{tor} \) (cf. Remark 4.7); further, the oddness of our groups only demands the choice of a splitting for the \( k^* \)-subgroup \( \mathcal{O}^2(\hat{F}_S(P)) \) of \( \hat{F}_S(P) \) for any \( \mathcal{D}_k^\text{tor} \)-object \((P, S)\).

A.2. That is to say, in the present situation we can replace \( \mathcal{D}_k, \hat{f} \) and \( \omega \) (cf. 2.5) by the full subcategory \( \mathcal{D}_k^\text{tor} \) of \( \mathcal{D}_k \), by the subfunctor of \( \hat{f} \)
\[ 2\hat{f} : \mathcal{D}_k^\text{tor} \to k^* \text{-Gr} \]

mapping any \( \mathcal{D}_k^\text{tor} \)-object \((P, S)\) on the \( k^* \)-group \( \mathcal{O}^2(\hat{F}_S(P)) \), and finally by a natural map \( 2\omega : 2\hat{f} \to k^* \) fulfilling the condition in 2.15.1 — called an odd-polarization. Although any odd-polarization can be easily extended to a polarization, the point is that there is a unique odd-polarization compatible with the tensor product of Dade \( P \)-algebras. We borrow the notation from [15, Chap. 9] and denote by \( \mathcal{D}_k^\text{tor}(P) \) the subgroup of torsion elements of \( \mathcal{D}_k(P) \); actually, it is known that all the nontrivial torsion elements of \( \mathcal{D}_k(P) \) have order 2 [15, 8.16 and Corollary 8.22] or, equivalently, that \( S \cong S^o \) for any \( \mathcal{D}_k^\text{tor} \)-object \((P, S)\).
Theorem A.3. There is a unique odd-polarization \( ^2\omega \) such that the following diagram is commutative

\[
\begin{array}{c}
\mathcal{O}^2(\hat{F}_S(P)) \cap \mathcal{O}^2(\hat{F}_{S'}(P)) \xrightarrow{\beta_{P,S,S'}} \mathcal{O}^2(\hat{F}_{S \otimes_k S'}(P)) \\
^2\omega(P,S) \times ^2\omega(P,S') \quad \forall \quad \vee \quad ^2\omega(P,S \otimes_k S')
\end{array}
\]

\[ k^* \]

for any pair of \( \mathcal{O}_k^{\text{tor}} \)-objects \((P, S)\) and \((P, S')\). Moreover, for any normal subgroup \( Q \) of \( P \), setting \( T = \text{Res}_Q^P(S) \) and \( \bar{P} = P/Q \), and denoting by \( \hat{F}_S(P)_Q \) the stabilizer of \( Q \) in \( \hat{F}_S(P) \), the following diagram is also commutative

\[
\begin{array}{c}
\mathcal{O}^2(\hat{F}_S(P)_Q) \xrightarrow{\Delta_{P,S,Q}} \mathcal{O}^2(\hat{F}_T(Q)) \times \mathcal{O}^2(\hat{F}_{S}(\bar{P})) \\
^2\omega(P,S) \quad \vee \quad ^2\omega(Q,T) \times ^2\omega(P,S(Q))
\end{array}
\]

\[ k^* \]

Proof: Let \( \omega \) be a polarization [15, Theorem 9.21]; for any \( \mathcal{O}_k^{\text{tor}} \)-object \((P, S)\), it is clear that there is a group homomorphism \( \beta_{P,S} : \mathcal{O}^2(F_S(P)) \rightarrow k^* \) fulfilling

\[ (\omega(P,S) \times \omega(P,S))(\hat{\varphi} \cdot \hat{\varphi}) = \beta_{P,S}(\varphi)\omega(P,S \otimes_k S)(\hat{\varphi} \cdot \hat{\varphi}) \]

\[ A.3.3 \]

for any \( \hat{\varphi} \in \mathcal{O}^2(\hat{F}_S(P)) \), where \( \hat{\varphi} \cdot \hat{\varphi} \) denotes the image of \((\hat{\varphi}, \hat{\varphi})\) in the \( k^* \)-group (cf. 2.2)

\[ \hat{F}_S(P) \cap \hat{F}_S(P) = \hat{F}_S(P) \ast \hat{F}_S(P) \]

\[ A.3.4 \]

and \( \varphi \) is the image of \( \hat{\varphi} \) in \( \mathcal{O}^2(F_S(P)) \); then, there is a unique group homomorphism \( \alpha_{P,S} : \mathcal{O}^2(F_S(P)) \rightarrow k^* \) fulfilling \((\alpha_{P,S})^2 = \beta_{P,S}\) and we claim that it suffices to define

\[ ^2\omega(P,S)(\hat{\varphi}) = \alpha_{P,S}(\varphi)^{-1}\omega(P,S)(\hat{\varphi}) \]

\[ A.3.5 \]

for any \( \hat{\varphi} \in \mathcal{O}^2(\hat{F}_S(P)) \).

In any case, note that the uniqueness of \(^2\omega(P,S)\) follows from the uniqueness of \( \alpha_{P,S} \). The commutativity of diagram A.3.1 for \( S' = S \) follows from our very definition; otherwise, the diagrams corresponding to the pairs of Dade \( P \)-algebras \((S \otimes_k S', S \otimes_k S')\), \((S, S)\) and \((S', S')\) are certainly commutative and then the commutativity of diagram A.3.1 follows.

Moreover, once again it is clear that there is a group homomorphism

\[ \gamma_{P,S,Q} : \mathcal{O}^2(F_S(P)_Q) \rightarrow k^* \]

\[ A.3.6 \]

such that, for any \( \hat{\varphi} \in \mathcal{O}^2(\hat{F}_S(P)_Q) \), we have

\[ ^2\omega(P,S)(\hat{\varphi}) = \gamma_{P,S,Q}(\varphi)(^2\omega(Q,T) \times ^2\omega(\bar{P},S(Q)))(\Delta_{P,S,Q}(\hat{\varphi})) \]

\[ A.3.7 \]
But, it follows from [15, Proposition 9.16] that the diagram
\[ \hat{F}_S(P_Q) \star \hat{F}_S(P_Q) \to (\hat{F}_T(Q) \star \hat{F}_T(Q)) \hat{\times} (\hat{F}_S(Q) \star \hat{F}_S(Q)) \]
\[ \hat{\nu}_{P,S,S} \downarrow \hat{\nu}_{Q,T,T} \hat{\times} \hat{\nu}_{P,S(Q),S(Q)} \]
\[ \hat{F}_{S \otimes_k S}(P_Q) \to \hat{F}_{T \otimes_k T}(Q) \hat{\times} \hat{F}_{(S \otimes_k S)(Q)}(\hat{P}) \]
is commutative; moreover, since the Dade -algebra
\[ S \otimes_k S \cong S \otimes_k S^\circ \cong \text{End}_k(S) \]
is similar to \( k \), the corresponding diagram A.3.2 is clearly commutative. Consequently, for any \( \hat{\varphi} \in O^2(\hat{F}_S(P_Q)) \), the element \( (2^\omega(P,S) \hat{\times} 2^\omega(P,S))(\hat{\varphi} \cdot \hat{\varphi}) \) coincides with the image of \( \Delta_{P,S,Q}(\hat{\varphi}) \cdot \Delta_{P,S,Q}(\hat{\varphi}) \) throughout the map
\[ (2^\omega(Q,T) \hat{\times} 2^\omega(Q,T)) \hat{\times} (2^\omega(P,S(Q)) \hat{\times} 2^\omega(P,S(Q))) \]
and therefore we get \( \gamma_{(P,S,Q)}(\hat{\varphi})^2 = 1 \) which forces \( \gamma_{(P,S,Q)}(\hat{\varphi}) = 1 \). We are done.

A.4. Since the unique odd-polarization \( 2^\omega \) in Theorem 4.3 can be easily extended to a polarization, if follows from Theorem 6.5 above that it supplies a natural bijection
\[ \text{Irr}_k(\hat{G}) \cong \text{Wgt}_k(\hat{G}) \]
and we claim that this bijection coincides with the bijection defined by Gabriel Navarro in [7, Theorem 4.3] for \( \pi = \{ p \} \). First of all, borrowing all the notation in §7, suitably translated to our present situation, and choosing this odd-polarization \( 2^\omega \), we claim that the corresponding automorphism \( 2^\omega \theta \) of \( \hat{N}_G(R) \) in 7.21 above is the identity map and therefore, according to Theorem 7.19 and Corollary 7.22, in this case we have
\[ U \cong \text{Ind}_{\hat{N}_G(R)}^{N_G(R_E)}(2^\omega V) \]
That is to say, in the bijection A.4.1 determined by \( 2^\omega \) the image of any simple \( k_\pi \hat{G} \)-module \( M \) can be directly computed from the triple formed by a vertex \( R \), an R-source \( E \) and a multiplicity \( \hat{N}_G(R_E) \)-module \( V \) of \( M \).

A.5. More precisely, for any \( n \in \mathbb{N} \), we claim that the automorphism \( 2^\omega \theta_n \) of \( \hat{N}_G(R) \star \hat{N}_G(R)^\circ \) is the identity map; indeed, since (cf. 7.13)
\[ \text{Res}_{\rho_n}(T_n) \cong \text{Res}_{\rho_n}(S_n) \otimes_k \text{Res}_{\rho_{n+1}}(T_{n+1}) \]
up to a suitable identification, from Theorem 7.3 above we get the following commutative diagram
\[ \mathbb{G}^2(\hat{F}_S(R^n)) \cap \mathbb{G}^2(\hat{F}_{T_{n+1}}(R^n)) \to \mathbb{G}^2(\hat{F}_T(R^n)) \]
\[ 2^\omega(R^n,S_n) \hat{\times} 2^\omega(R^n,T_{n+1}) \]
which, according to the very definition of \( 2^\omega \theta_n \), proves our claim.
A.6. In particular, if $\hat{G}'$ is a $k^*$-subgroup of $\hat{G}$ and $M'$ a $k_\ast\hat{G}'$-module such that $M \cong \text{Ind}_{\hat{G}'}^{\hat{G}}(M')$, then $M'$ is clearly a simple $k_\ast\hat{G}'$-module and it is easily checked that a vertex $R'$ and an $R'$-source $E'$ of $M'$ are also a vertex and an $R'$-source of $M$; moreover, we claim that if $V'$ is a multiplicity $\hat{N}_{G'}(R_{E'})$-module of $M'$ then the $k_\ast\hat{N}_G(R_{E'})$-module

$$V = \text{Ind}_{\hat{N}_{G'}(R_{E'})}^{\hat{N}_G(\hat{R}_{E'})}(V')$$  \hspace{1cm} A.6.1

is a multiplicity $\hat{N}_G(R_{E'})$-module of $M$. Indeed, recall that we have (cf. 2.12)

$$\text{Ind}_{\hat{G}'}^{\hat{G}}(\text{End}_k(M')) \cong \text{End}_k(M)$$  \hspace{1cm} A.6.2

and that $\text{id}_M$ is the image of $\text{Tr}_{\hat{G}'}^{\hat{G}}(1 \otimes \text{id}_{M'} \otimes 1)$, and denote by $G'_M$, the pointed group on $\text{End}_k(M)$ determined by the group $G'$ and the idempotent $1 \otimes \text{id}_{M'} \otimes 1$. Since $R'_{E'}$ is a local pointed group on $\text{End}_k(M)$, the unity element in $(\text{End}_k(M))(R'_{E'})$ coincides with the sum $\sum_x x \otimes \text{id}_{M'} \otimes x^{-1}$ where $x$ runs over the elements fulfilling $(R'_{E'})^x \subset G'_M$, in a set of representatives for $\hat{G}'/\hat{G}'$ in $\hat{G}$ and, for such an element $x$, $x \otimes \text{id}_{M'} \otimes x^{-1}$ denotes the image of $x \otimes \text{id}_{M'} \otimes x^{-1}$in $(\text{End}_k(M))(R'_{E'})$ [9, Proposition 1.3]. But, it is clear that $(R'_{E'})^x$ is also a maximal local pointed group on $\text{End}_k(M')$ and therefore there is $x' \in \hat{G}'$ such that $(R'_{E'})^x = (R'_{E'})^{x'}$. Consequently, it follows from [6, statement 2.13.2] that we get an $\hat{N}_G(R_{E'})$-interior algebra isomorphism

$$\text{Ind}_{\hat{N}_{G'}(R_{E'})}^{\hat{N}_G(\hat{R}_{E'})}(\text{End}_k(M'))(R'_{E'}) \cong (\text{End}_k(M))(R'_{E'})$$  \hspace{1cm} A.6.3

which proves our claim.

A.7. Moreover, by the very definitions of $\hat{\omega}V$ and $\hat{\omega}V'$ in 7.20 above, then we still have

$$\hat{\omega}V = \text{Ind}_{\hat{N}_{G'}(R_{E'})}^{\hat{N}_G(\hat{R}_{E'})}(\hat{\omega}V')$$  \hspace{1cm} A.7.1

and therefore it follows from isomorphism A.4.2 that we have a $k_\ast\hat{N}_G(R')$-module isomorphism

$$U \cong \text{Ind}_{\hat{N}_G(\hat{R}')}(U')$$  \hspace{1cm} A.7.2

where $U'$ is a simple projective $\hat{N}_{G'}(R')$-module which, together with $R'$, determines the $G'$-conjugacy class of weights of $\hat{G}'$ determined by $M'$ via the corresponding bijection A.4.1.

A.8. In conclusion, in order to prove that Navarro’s correspondence in [7, Theorem 4.3] also maps $M$ on the $G$-conjugacy class of the weight of $\hat{G}$ determined by $R$ and $U$, we may assume that $M$ is primitive — namely,
that it is *not* induced from any proper \( k^* \)-subgroup of \( \hat{G} \). In this case, as we mention in 1.4 above, it follows from [17, Lemma 30.4] that there is a \( G \)-stable finite \( p \)-subgroup \( K \) of \( \text{End}_k(M)^* \) which generates the \( k \)-algebra \( \text{End}_k(M) \); in particular, \( \text{End}_k(M) \) is actually a Dade \( R \)-algebra [13, 1.3], \( R \) is a Sylow \( p \)-subgroup of \( G \) and \( \tilde{N}_G(R) \)-stabilizes the isomorphism class of \( E \) [13, 1.8], so that \( U \cong \omega \cdot V \) (cf. A.4.2). At this point, a careful inspection of the origin of Navarro’s correspondence in [7, Theorem 3.1] shows that it maps \( M \) on the \( G \)-conjugacy class of the weight of \( \hat{G} \) determined by \( R \) and \( U \) if, for a suitable Brauer character \( \psi \) over \( \tilde{N}_G(R) \), we have

\[
\text{Res}_{\tilde{N}_G(R)}^\hat{G}(\varphi_M) = \varphi_U + 2\psi
\]

where \( \varphi_M \) and \( \varphi_U \) respectively denote the Brauer characters of \( M \) and of the \( k\cdot\tilde{N}_G(R) \)-module \( U \cong \omega \cdot V \). Then, the fact that in our situation such an equality holds is more or less a consequence of [5, Theorem 5.3] but here we give a direct proof.

**Proposition A.9.** Let \( M \) be a simple primitive \( k\cdot\hat{G} \)-module, \( R \) a vertex, \( E \) an \( R \)-source and \( V \) a multiplicity \( k\cdot\tilde{N}_G(R_E) \)-module of \( M \). Consider the unique odd-polarization \( \omega \) such that diagram 4.3.1 is commutative and denote by \( \omega \cdot V \) the restriction of \( V \) throughout the isomorphism \( \tilde{N}_G(R) \cong \hat{N}_G(R_E) \) determined by \( \omega \), and by \( \varphi_M \) and \( \varphi_{\omega \cdot V} \) the respective Brauer characters of \( M \) and of \( \omega \cdot V \) considered as a \( k\cdot\tilde{N}_G(R) \)-module. Then, for a suitable Brauer character \( \psi \) over \( \tilde{N}_G(R) \), we have

\[
\text{Res}_{\tilde{N}_G(R)}^\hat{G}(\varphi_M) = \varphi_{\omega \cdot V} + 2\psi
\]

**Proof:** Arguing by induction on \( |G| \), we may assume that \( M \) is a faithful \( k\cdot\hat{G} \)-module, then identifying \( \hat{G} \) with its image in \( \text{End}_k(M) \); moreover, the case where \( \dim_k(M) = 1 \) being clear, we assume that \( \dim_k(M) \neq 1 \). Then, a minimal normal nontrivial subgroup \( K \) of \( G \) is an Abelian \( \ell \)-elementary group for a prime number \( \ell \neq p \) and the primitivity of \( M \) forces the converse image \( \hat{K} \) of \( K \) in \( \hat{G} \) to be the *central* product of \( k^* \) by an *extra-special* normal \( \ell \)-subgroup of \( \hat{G} \) [4, Ch. 5, §5].

Let \( S \) be the \( k \)-subalgebra of \( \text{End}_k(M) \) generated by \( \hat{K} \); once again, the *primitivity* of \( M \) forces \( S \) to be a simple \( k \)-algebra and then the \( k^* \)-quotient \( G \) of \( \hat{G} \) acts on \( S \) determining a \( k^* \)-group \( \hat{G} \) together with a \( k^* \)-group homomorphism \( \hat{G} \to S^* \) (cf. 2.3), and we set

\[
G = \hat{G} \ast (\hat{G})^\circ
\]
then, it follows from [16, Proposition 3.2] that there exists a $k, G$-module $\hat{M}$ such that we have a $\hat{G}$-interior algebra isomorphism

$$\text{End}_k(M) \cong S \otimes_k \text{End}_k(\hat{M})$$

A.9.3;

actually, $\hat{K}$ is canonically isomorphic to the converse image of $K$ in $G$ and therefore $K$ lifts to a normal subgroup of $G$ acting trivially on $\hat{M}$; thus, up to suitable identifications, setting $\hat{G} = G/K$ and $S = \text{End}_k(N)$, $\hat{M}$ becomes a $k, \hat{G}$-module, we have a $k, \hat{G}$-module isomorphism

$$M \cong N \otimes_k \hat{M}$$

A.9.4

and, denoting by $\varphi_N$ the Brauer character of $N$ and by $\hat{\pi}: G \to \hat{G}$ the canonical $k^*$-group homomorphism, we have

$$\varphi_M = \varphi_N \cdot \text{Res}_\hat{G}(\varphi_{\hat{M}})$$

A.9.5.

Now, it is clear that $\hat{M}$ is a simple primitive $k, \hat{G}$-module, that the image $\hat{R}$ of $R$ in $G = G/K$ is a vertex of $\hat{M}$ (actually, it is a Sylow $p$-subgroup of $\hat{G}$), that we have a canonical $R$-interior algebra embedding (cf. 2.4)

$$\text{End}_k(E) \longrightarrow S \otimes_k \text{End}_k(\hat{E})$$

A.9.6

where $\hat{E}$ denotes an $\hat{R}$-source of $\hat{M}$, and that we still have a $\hat{N}_G(R_E)$-interior algebra isomorphism [12, Proposition 5.6]

$$(\text{End}_k(M))(R_E) \cong S(R) \otimes_k (\text{End}_k(\hat{M}))(\hat{R}_E)$$

A.9.7,

together with a $k^*$-group isomorphism [12, Proposition 5.11]

$$\hat{N}_G(R_E) \cong \hat{N}_G^S(R) \ast \text{Res}_\pi(\hat{N}_G(\hat{R}_E))$$

A.9.8

where $\hat{N}_G^S(R)$ and $\text{Res}_\pi(\hat{N}_G(\hat{R}_E))$ respectively denote the $k^*$-groups coming from the action of $\hat{N}_G(R)$ on the simple $k$-algebra $S(R)$ [13, 1.8], and obtained by pull-back from the canonical group homomorphism $\pi: \hat{N}_G(R) \to \hat{N}_G(\hat{R})$.

On the one hand, denoting by $\hat{V}$ a multiplicity $\hat{N}_G(\hat{R}_E)$-module of $\hat{M}$, so that we have

$$\text{End}_k(\hat{V}) \cong (\text{End}_k(\hat{M}))(\hat{R}_E)$$

A.9.9,

it follows from the induction hypothesis that, for a suitable Brauer character $\psi$ over $N_G(\hat{R})$, we have

$$\text{Res}_{\hat{N}_G(\hat{R})}(\varphi_{\hat{M}}) = \varphi_{\hat{V}} + 2 \cdot \hat{\psi}$$

A.9.10.
On the other hand, denoting by $W$ a multiplicity $\tilde{\mathbb{N}}_G(R)$-module of $N$, it follows from isomorphisms $A.9.7$ and $A.9.8$ that we have a $k_N\tilde{\mathbb{N}}_G(R_E)$-module isomorphism

$$ V \cong W \otimes_k \text{Res}_\pi(\hat{V}) \quad A.9.11; $$

moreover, it follows from the commutativity of diagram $A.3.1$ that we still have a $k_N\tilde{\mathbb{N}}_G(R)$-module isomorphism

$$ ^2\omega V \cong ^2\omega W \otimes_k \text{Res}_{\hat{\pi}}(\hat{^2\omega V}) \quad A.9.12 $$

where $\hat{\pi}$ denotes the restriction to $\tilde{\mathbb{N}}_G(R_E)$ of $\hat{\pi}$; consequently, with evident notation, we get

$$ \varphi_2^\omega V = \varphi_2^\omega W \cdot \text{Res}_{\hat{\pi}}(\varphi_2^\omega V) \quad A.9.13. $$

But, according to Theorem $A.10$ below, we also have

$$ \text{Res}_{\tilde{\mathbb{N}}_G(R)}^\tilde{\mathbb{G}}(\varphi_N) = \varphi_2^\omega W + 2\eta \quad A.9.14 $$

for a suitable Brauer character $\eta$ over $\tilde{\mathbb{N}}_G(R)$. In conclusion, from equalities $A.9.5$, $A.9.10$ and $A.9.14$ we get

$$ \text{Res}_{\tilde{\mathbb{N}}_G(R)}^\tilde{\mathbb{G}}(\varphi_M) = \text{Res}_{\tilde{\mathbb{N}}_G(R)}^\tilde{\mathbb{G}}(\varphi_N) \cdot \text{Res}_{\hat{\pi}}(\text{Res}_{\tilde{\mathbb{N}}_G(R)}^\tilde{\mathbb{G}}(\varphi_M))$$

$$ = (\varphi_2^\omega W + 2\eta) \cdot \text{Res}_{\hat{\pi}}(\varphi_2^\omega V + 2\psi) \quad A.9.15 $$

$$ = \varphi_2^\omega V + 2\psi $$

where $\psi = \eta \cdot \text{Res}_{\hat{\pi}}(\varphi_2^\omega V + 2\hat{\psi}) + \varphi_2^\omega W \cdot \text{Res}_{\hat{\pi}}(\hat{\psi})$. We are done.

**Theorem A.10.** Let $M$ be a $k_G$-module such that $\text{End}_k(M)$ is generated by a $G$-stable $k^*$-subgroup $\hat{K}$ of $\text{End}_k(M)^*$ which is the central product of $k^*$ by an extra-special $\ell$-subgroup for an odd prime number $\ell \neq p$. For any local pointed group $R_E$ on $\text{End}_k(M)$, denoting by $V$ a multiplicity $\tilde{\mathbb{N}}_G(R_E)$-module of $R_E$ and by $^2\omega V$ the restriction of $V$ via the isomorphism $\tilde{\mathbb{N}}_G(R) \cong \tilde{\mathbb{N}}_G(R_E)$ determined by the unique odd-polarization $^2\omega$ such that diagram $4.3.1$ is commutative, we have

$$ \text{Res}_{\tilde{\mathbb{N}}_G(R)}^\tilde{\mathbb{G}}(\varphi_M) = \varphi_2^\omega V + 2\psi \quad A.10.1 $$

where $\varphi_M$ and $\varphi_2^\omega V$ denote the respective Brauer characters of $M$ and of $^2\omega V$ considered as a $k_N\tilde{\mathbb{N}}_G(R)$-module, and $\psi$ is a Brauer character over $\tilde{\mathbb{N}}_G(R)$.

**Proof:** We actually may assume that $M$ is faithful and that $\hat{G} = \tilde{\mathbb{N}}_G(R)$; then, $G$ stabilizes the decomposition $[4, \text{Ch. 5, Theorem 2.3}]

$$ K = C_K(R) \times [K, R] \quad A.10.2 $$
of the $k^*$-quotient of $\hat{K}$; thus, setting $S = \text{End}_k(M)$ and denoting by $S'$ and $S''$, the $k$-subalgebras of $S$ generated by the respective converse images $\hat{K}'$ of $C_K(R)$ and $\hat{K}''$ of $[K,R]$, we have $S = S' \otimes_k S''$, $\hat{K}'$ and $\hat{K}''$ are also central products of $k^*$ by extra-special $\ell$-subgroups (here we also consider $\mathbb{Z}/\ell\mathbb{Z}$ as an extra-special $\ell$-group) and $G$ still stabilizes

$$S' = \text{End}_k(M') \quad \text{and} \quad S'' = \text{End}_k(M'') \quad \text{A.10.3.}$$

Consequently, as in the proof above, it follows from the commutativity of diagram A.3.1 that it suffices to prove the theorem for $M'$ and for $M''$. That is to say, we may assume that either $K = C_K(R)$ or $K = [K,R]$; in the first case, $R$ centralizes $\hat{K}$ [4, Ch. 5, Theorem 1.4], so that it centralizes $S$ which forces $R = \{1\}$; then, we have $M = V$, $E = k$ and $\hat{F}_S(R) = k^*$, and by the very definition of $\hat{N}_G(R_E)$ (cf. 2.5) we get an isomorphism $\hat{N}_G(R_E) \cong \hat{N}_G(R) = \hat{G}$ compatible with the canonical $k^*$-group homomorphism 2.8.1, so that equality A.10.1 is trivially true with $\psi = 0$.

Following the notation in A.11 and according to isomorphism A.12.2 below, let us denote by $H$ the image of $Sp(K,\kappa)$ in $N_{S^*}(\hat{K})$; in particular, the nontrivial element in $Z(H)$ is an involution $s \in S$ which stabilizes $\hat{K}$ and induces $-\text{id}_k$ over $K$; note that, if $s' \in S$ is such an involution then $s'$ stabilizes $\hat{K}$ acting trivially on $K$ and therefore, according again to isomorphism A.12.2 below, $s'$ belongs to $\hat{K} \cdot (s)$, so that we have $s' \in \{s^z, -s^z\}$ for a suitable $z \in \hat{K}$.

In the second case above, we have $C_K(R) = \{1\}$ and therefore $R$ fixes a unique pair of such involutions $\{s, -s\}$ which by oddness forces

$$\hat{G} = N_{\hat{G}}(R) \subset C_{S^*}(s) \quad \text{A.10.4;}$$

consequently, $\hat{G}$ is contained in the intersection

$$N_{S^*}(\hat{K}) \cap C_{S^*}(s) = k^* \times H \quad \text{A.10.5.}$$

Moreover, since $K$ indexes an $R$-stable basis of $S = k_\cdot \hat{K}$ (cf. A.11 below), we have $S(R) \cong k$ which forces $V \cong k$; hence, by the very definition of the $k^*$-group $\hat{N}_G(R_E)$, in this case we get a $k^*$-group isomorphism (cf. 2.5)

$$\hat{N}_G(R_E) \cong k^* \times \hat{N}_G(R) \quad \text{A.10.6.}$$

At this point, it follows from Lemma A.14 below that the decomposition

$$\hat{G} = N_{\hat{G}}(R) \cong \hat{N}_G(R_E) \cong k^* \times G \quad \text{A.10.7}$$

determined by the $k^*$-group homomorphism $^2\omega_{(R,S)}: \hat{F}_S(R) \to k^*$ coincides
with the decomposition
\[ \hat{G} = k^* \times (\hat{G} \cap H) \]
A.10.8

obtained from the inclusion \( \hat{G} \subset N_{S'}( \hat{K}) \cap C_{S'}(s) \). In particular, the restriction of \( \varphi_{2\omega} \) to \( \hat{G} \cap H \) is just the trivial character.

On the other hand, for any \( y \in \hat{G} \cap H \), acting over \( K \) the product \( sy \) only fix the trivial element 1; indeed, if \( syx(sy)^{-1} = x \) for some \( x \in K \) then \( yx^{-1}y^{-1} = x \) and therefore \( \{x, x^{-1}\} \) is an orbit of \( (y) \) which forces \( x = x^{-1} \), so that \( x = 1 \); in particular, we get
\[ \text{tr}_M(sy) \cdot \text{tr}_M^*(sy) = \text{tr}_S(sy) = 1 \]
A.10.9.

But, we clearly have
\[ k(s) = k \cdot \text{id}_M + k \cdot s = k \cdot i + k \cdot i' \]
A.10.10

for suitable mutually orthogonal idempotents \( i \) and \( i' \) of \( S \), and we choose the notation in such a way that \( \dim(i(M)) \geq \dim(i'(M)) \).

Then, denoting by \( \varphi_M, \varphi_{i(M)} \) and \( \varphi_{i(M') \prime} \) the respective Brauer characters of \( M, i(M) \) and \( i'(M) \), let us consider the ordinary characters \( \chi_M, \chi_{i(M)} \) and \( \chi_{i(M') \prime} \) over \( \hat{G} \cap H \) which respectively lift the restrictions to \( \hat{G} \cap H \) of \( \varphi_M, \varphi_{i(M)} \) and \( \varphi_{i(M') \prime} \) to the set of characters \( \chi \) fulfilling \( \chi(y) = \chi(y_{p'}) \) for any \( y \in \hat{G} \cap H \); consequently, we clearly have \( \chi_M = \chi_{i(M)} + \chi_{i(M') \prime} \) and moreover
\[ 1 = (\chi_{i(M)}(y) - \chi_{i(M') \prime}(y))(\tilde{\chi}_{i(M)}(y) - \tilde{\chi}_{i(M') \prime}(y)) \]
A.10.11

for any \( y \in \hat{G} \cap H \) (cf. A.10.9); in particular, the norm of \( \chi_{i(M)} - \chi_{i(M') \prime} \) is equal to 1 and, according to our choice of notation, we still have
\[ 1 = \chi_{i(M)}(1) - \chi_{i(M') \prime}(1) \]
A.10.12

hence, for a suitable linear character \( \lambda \) of \( \hat{G} \cap H \), we get \( \chi_{i(M)} = \lambda + \chi_{i(M') \prime} \) or, equivalently,
\[ \chi_M = \lambda + 2 \cdot \chi_{i(M') \prime} \]
A.10.13.

Now, it suffices to prove that \( \lambda \) is the trivial character. Note that, denoting by \( k' \) the subfield of \( k \) generated by the \( \ell \)-th roots of unity, we still can define a \( k'^* \)-group \( \hat{K}' = k'^* \times K \) as in A.11.1 below and, setting \( S' = k' \cdot \hat{K}' \), we have \( S = k \otimes_k S' \) and \( H \) is contained in \( 1 \otimes S' \), so that \( i \) and \( i' \) also belong to \( 1 \otimes S' \); hence, the values of the ordinary characters \( \chi_M, \chi_{i(M)} \) and \( \chi_{i(M') \prime} \) are contained in the extension of \( \mathbb{Q} \) by the \( \ell \)-th roots of unity. Consequently, it suffices to prove that the restriction of \( \lambda \) to a Sylow \( \ell \)-subgroup \( L \) of \( \hat{G} \cap H \) is trivial.
But, it is well-known that for a maximal Abelian $k^*$-subgroup $\hat{A}$ of $\hat{K}$ and a $k^*$-group homomorphism $\zeta:\hat{A} \to k^*$, denoting by $k_\zeta$ the corresponding $k_\zeta\hat{A}$-module, we have

$$M \cong \text{Ind}^{\hat{K}}_{\hat{A}}(k_\zeta) \quad \text{A.}10.14;$$

moreover, since $H$ acts over $\hat{K} \cong k^* \times K$ stabilizing $1 \times K$, it is easily checked that $L$ stabilizes a suitable choice of $\text{Ker}(\zeta) \subset 1 \times K$ and therefore, choosing a complement $X$ of $\text{Ker}(\zeta)$ in $1 \times K$, it stabilizes the basis $\{(1, x) \otimes 1\}_{x \in X}$ of $M$; then, $L$ fixes $(1, 1) \times 1$ and, for any $L$-orbit $O$ in $X - \{1\}$, $\{(1, x) \otimes 1\}_{x \in O}$ and $\{(1, x^{-1}) \otimes 1\}_{x \in O}$ are different orbits of $L$ in this basis, since $|O|$ is odd; that is to say, the number of orbits of $L$ in this basis is odd.

In conclusion, since $L$ is an $\ell$-group and $\ell \neq p$, the multiplicity of $k$ in $M$ considered as a $kL$-module is an odd number; then, the restriction of equality A.10.13 to $L$ proves that the restriction of $\lambda$ to $L$ is trivial. We are done.

A.11. Let $\ell$ be an odd prime number different from $p$ and $\hat{K}$ a $k^*$-group which is the central product of $k^*$ by an extra-special $\ell$-group and, for our purposes, we also consider $\mathbb{Z}/\ell\mathbb{Z}$ as an extra-special $\ell$-group. Denote by $\kappa$ the non-singular skew symmetric scalar product over the $k^*$-quotient $K$ induced by the commutator in $\hat{K}$; thus, we have $|K| = \ell^{2n}$ and note that the case $n = 0$ is not excluded. Then, it is easily checked that $\hat{K}$ is isomorphic to $k^* \times K$ endowed with the product defined by

$$(\lambda, x) \cdot (\lambda', x') = (\lambda \lambda' \kappa(x, x')^{1/2}, xx') \quad \text{A.11.1},$$

for any $\lambda, \lambda' \in k^*$ and any $x, x' \in K$, and with the group homomorphism $k^* \to k^* \times K$ mapping $\lambda \in k^*$ on $(\lambda, 1)$. It is quite clear that the corresponding symplectic group $Sp(K, \kappa)$ acts over this $k^*$-group and we actually have

$$\text{Aut}_{k^*}(\hat{K}) \cong K \rtimes Sp(K, \kappa) \quad \text{A.11.2}.$$

A.12. Moreover, it is well-known that $S = k, \hat{K}$ is a simple $k$-algebra and $Sp(K, \kappa)$ clearly acts over this $k$-algebra stabilizing $\hat{K}$; thus, since any central $k^*$-extension of $Sp(K, \kappa)$ is trivial, this action can be lifted to a group homomorphism

$$Sp(K, \kappa) \to N_{S^*}(\hat{K}) \quad \text{A.12.1};$$

then, since $C_{S^*}(\hat{K}) = k^* \text{id}_S$, from isomorphism A.11.2 we easily get

$$N_{S^*}(\hat{K}) \cong \hat{K} \rtimes Sp(K, \kappa) \quad \text{A.12.2};$$

let us identify $Sp(K, \kappa)$ with its image in $N_{S^*}(\hat{K})$ (for the choice of a $k^*$-group isomorphism $\hat{K} \cong k^* \times K$).
A.13. Identifying \( \tilde{K} \) with \( k^* \times K \), it is clear that a \( p \)-subgroup \( R \) of \( Sp(K, \kappa) \) stabilizes the basis \( \{(1, x)\}_{x \in K} \) of \( S \) and therefore \( S \) becomes a \textit{Dade} \( R \)-\textit{algebra}; moreover, it is easily checked that the restriction \( \kappa_R \) of \( \kappa \) to \( C_K(R) \) remains a \textit{non-singular skew symmetric scalar product} and therefore \( C_K(R) \cong k^* \times C_K(R) \) is also the central product of \( k^* \) by an extra-special \( \ell \)-group. Then, it is clear that the Brauer homomorphism induces a \textit{k}-\textit{algebra isomorphism} [11, statement 2.8.4]

\[ k_*C_K(R) \cong S(R) \quad A.13.1 \]

and it is easily checked that the action of \( N_{Sp(K, \kappa)}(R) \) over \( C_K(R) \) is contained in the corresponding symplectic group \( Sp(C_K(R), \kappa_R) \); that is to say, the Brauer homomorphism can be extended to a group homomorphism

\[ Br^*_R : N_{Sp(K, \kappa)}(R) \rightarrow Sp(C_K(R), \kappa_R) \subset S(R)^* \quad A.13.2 \]

such that the action of \( x \in N_{Sp(K, \kappa)}(R) \) coincides with the conjugation by \( Br^*_R(x) \) on \( S(R) \); thus, choosing an element \( a \in S^R \) lifting \( Br^*_R(x) \) and an idempotent \( i \) in the unique local point of \( R \) on \( S \), and denoting by \( \bar{x}^R \) the image of \( x \) in \( F_S(R) \) and by \( ixa^{-1}i^S \) the image of the product \( ixa^{-1}i \) in the quotient

\[ N_{(iSi)}^*(Ri) \left/ \left( i + J((iSi)^R) \right) \right. \quad A.13.3, \]

the pair \( (\bar{x}^R, ixa^{-1}i^S) \) is an element of \( F_S(R) \) [11, Proposition 6.10].

\textbf{Lemma A.14.} With the notation above, denote by \( 2\omega \) the unique odd-polarization such that diagram 4.3.1 is commutative and let \( R \) be a \( p \)-subgroup of \( Sp(K, \kappa) \). For any \( x \in N_{Sp(K, \kappa)}(R) \) of odd order, choosing an element \( a \in S^R \) lifting \( Br^*_R(x) \) and an idempotent \( i \) in the unique local point of \( R \) on \( S \), and denoting by \( x^R \) the image of \( x \) in \( F_S(R) \) and by \( ixa^{-1}i^S \) the image of the product \( ixa^{-1}i \) in the quotient

\[ N_{(iSi)}^*(Ri) \left/ \left( i + J((iSi)^R) \right) \right. \quad A.14.1, \]

we have

\[ 2\omega_{(R,S)}(\bar{x}^R, ixa^{-1}i^S) = 1 \quad A.14.2. \]

\textbf{Proof:} Arguing by induction on \( |R| \), set \( Z = \Omega_1(Z(R)) \), \( \bar{R} = R/Z \) and \( T = \text{Res}_{Z}^{R}(S) \); it follows from Theorem A.3 above that we have the commutative diagram

\[ \begin{array}{ccc}
\mathcal{O}^2(\hat{F}_S(R)Z) & \overset{\Delta_{R,S,Z}}{\longrightarrow} & \mathcal{O}^2(\hat{F}_T(Z)) \times \mathcal{O}^2(\hat{F}_S(\bar{R})) \\
2\omega_{(R,S)} \times & & \sqrt[2]{2\omega_{(Z,T)} \times 2\omega_{(\bar{R},S(Z))}} \\
k^* & & \\
\end{array} \quad A.14.3. \]
But, choosing an element \( c \in S^Z \) lifting \( x = \text{Br}_Z^*(x) \) and an idempotent \( j \) in the unique local point of \( Z \) on \( S \) fulfilling \( ji = j = ij \), and setting \( \bar{a} = \text{Br}_Z(a) \) and \( \bar{i} = \text{Br}_Z(i) \), it is easily checked from [15, Proposition 9.11] that we have

\[
\Delta_{R,S,Z}(\bar{x}^R, jxa^{-1}j^T) = (\bar{x}^Z, jxc^{-1}jc^T) \cdot (\bar{x}^R, jxa^{-1}j^T)^{S(Z)} \tag{A.14.4}
\]

hence, since we clearly have

\[
\text{Br}_R^*(\text{Br}_Z^*(x)) = \text{Br}_R^*(x) \tag{A.14.5}
\]

if \( Z \neq R \) then from the induction hypothesis we get

\[
2^\omega_{(Z,T)}(\bar{x}^Z, jxc^{-1}jc^T) = 1 = 2^\omega_{(R,S(Z))}(\bar{x}^R, jxa^{-1}j^T)^{S(Z)} \tag{A.14.6}
\]

Now, equality A.14.2 follows from the commutativity of diagram A.14.3.

From now on, we assume that \( R \) is \( p \)-elementary Abelian. Arguing by induction on \( |K| \), if \( K \) decomposes on a direct orthogonal sum of two \( R\langle x \rangle \)-stable nontrivial subspaces

\[
K = K' \perp K'' \tag{A.14.7}
\]

then \( \bar{K} \) is the central product of the converse images \( \bar{K}' \) of \( K' \) and \( \bar{K}'' \) of \( K'' \), and, setting

\[
S' = k_* \bar{K}' \quad \text{and} \quad S'' = k_* \bar{K}'' \tag{A.14.8}
\]

\( S' \) and \( S'' \) are also Dade \( R \)-algebras and we have \( S \cong S' \otimes_k S'' \); in particular, it follows from Theorem A.3 above that we have the commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}^2(\hat{F}_{S'}(R)) \cap \mathcal{O}^2(\hat{F}_{S''}(R)) & \xrightarrow{\iota_{R,S',S''}} & \mathcal{O}^2(\hat{F}_{S}(R)) \\
2^\omega_{(R,S')} \times 2^\omega_{(R,S'')} \downarrow & \text{and} & \downarrow 2^\omega_{(R,S)} \\
k^* & & \\
\end{array} \tag{A.14.9}
\]

But, denoting by \( \kappa' \) and \( \kappa'' \) the respective restrictions of \( \kappa \) to \( K' \) and \( K'' \), it is clear that \( x \) is the image of \( x' \otimes x'' \) for suitable elements \( x' \in \text{Sp}(K', \kappa') \) and \( x'' \in \text{Sp}(K'', \kappa'') \) normalizing the respective images \( R' \subset \text{Sp}(K', \kappa') \) and \( R'' \subset \text{Sp}(K'', \kappa'') \) of \( R \). Moreover, choosing elements \( a' \in S'^{R'} \) and \( a'' \in S''^{R''} \) respectively lifting \( \text{Br}_{R'}^*(x') \) and \( \text{Br}_{R''}^*(x'') \), and idempotents \( i' \) and \( i'' \) in the respective unique local points of \( R \) on \( S' \) and on \( S'' \), we clearly may choose the element \( a \) equal to the image of \( a' \otimes a'' \) and the idempotent \( i \) in an orthogonal decomposition of the image of \( i' \otimes i'' \), so that \( \text{Br}_R(x) \) is equal to the corresponding image of \( \text{Br}_{R'}(x') \otimes \text{Br}_{R''}(x'') \) via the isomorphism

\[
S'(R') \otimes_k S''(R'') \cong S(R) \tag{A.14.10}
\]
Then, it easily follows from [15, 9.15] that we have
\[ \vartheta_{R, S', S''}(\{ x^{R'}, \xi x'^{a'}^{-1}S' \} \cdot \{ x^{R''}, \xi'' x''^{a''}^{-1}S'' \}) = (x^{R}, \xi x a^{-1}S) \]  
A.14.11.

Now, since the induction hypothesis implies that
\[ 2^{\omega_{\langle R', S' \rangle}}(x^{R'}, \xi x'^{a'}^{-1}S') = 1 = 2^{\omega_{\langle R'', S'' \rangle}}(x^{R''}, \xi'' x''^{a''}^{-1}S'') \]  
A.14.12,
equality A.14.2 follows from the commutativity of diagram A.14.9.

Thus, we may assume that \( K \) does not admit a decomposition on a direct orthogonal sum of two \( R \langle x \rangle \)-stable nontrivial subspaces. Denoting by \( \mathbb{F} \) the field of cardinal \( \ell \), if \( L \) is a simple \( FR \)-submodule of \( K \) then the restriction of \( \kappa \) to \( L \) is either non-singular or, denoting by \( L^\perp \) the orthogonal space of \( L \), \( L^\perp \) contains \( L \) and we have a canonical isomorphism \( K/L^\perp \cong L^* \), so that, if \( L' \) is an \( FR \)-complement of \( L^\perp \) in \( K \), the restriction of \( \kappa \) to \( L \oplus L' \) is non-singular. Consequently, the dimensions of all the simple \( FR \)-submodules of \( K \) have the same parity.

Firstly assume that the dimensions of all the simple \( FR \)-submodules \( L \) of \( K \) are odd; in this case, \( L^* \) is also a \( FR \)-submodule of \( K \) not isomorphic to \( L \). Thus, since \( \langle x \rangle \) is odd, the group \( \langle x \rangle \) has exactly two orbits in the set of isotypic components of the \( FR \)-module \( K \) and then, denoting by \( A \) and \( B \) the sums of isotypic components in each \( \langle x \rangle \)-orbit, \( A \) and \( B \) are maximal totally singular subspaces fulfilling \( K = A \oplus B \). Hence, the converse images \( \hat{A} \) of \( A \) and \( \hat{B} \) of \( B \) in \( K \) are maximal Abelian subgroups and it is well-known that, for a \( k^* \)-group homomorphism \( \zeta : \hat{A} \to k^* \) that we may choose \( R \langle x \rangle \)-stable (cf. A.11.1), we have
\[ M \cong \text{Ind}_{\hat{A}}^K(k_{\zeta}) \]  
A.14.13
where \( k_{\zeta} \) denotes the corresponding \( k_{\zeta} \hat{A} \)-module.

In this situation, the group \( R \langle x \rangle \) stabilizes the basis \( \{ (1, y) \otimes 1 \}_{y \in B} \) of \( M \), so that the Dade \( R \)-algebra \( S \) is similar to \( k \); in particular, identifying \( S \) with the induced \( \hat{K} \)-interior algebra \( \text{Ind}_{\hat{A}}^K(k_{\zeta}) \) (cf. 2.12) where \( k_{\zeta} \) still denotes the corresponding \( \hat{A} \)-interior algebra, the primitive idempotent
\[ i = (1, 1) \otimes 1 \otimes (1, 1) \]  
A.14.14
actually belongs to the unique local point of \( R \) on \( S \); now, \( x \) and \( Br_R^*(x) \) respectively centralize \( i \) and \( Br_R(i) \), and, with the notation above, it is easily checked that \( \overline{\xi x a^{-1}S} = \overline{i} \) which proves equality A.14.2 in this case.

Finally assume that the dimensions of all the simple \( FR \)-submodules \( L \) of \( K \) are even; in this case, the image of \( FR \) in \( \text{End}_{\mathbb{F}}(L) \) is an extension \( F_L \) of \( \mathbb{F} \) of even degree and therefore it contains a primitive fourth root \( \tau_L \) of unity;
moreover, since $|\langle x \rangle|$ is odd, the stabilizer in $\langle x \rangle$ of the isotypic component containing $L$ acts on $F_L$ fixing $\tau_L$. Consequently, considering all the orbits of $\langle x \rangle$, we get a self-adjoint endomorphism $\tau$ of $K$ which centralizes $R\langle x \rangle$ and fulfills $\tau^2 = -\text{id}_K$.

At this point, we consider the central product $\hat{K} \times \hat{K}$, and in the $k^*$-quotient $\hat{K} \times \hat{K}$ we set

$$A = \{(y, \tau(y))\}_{y \in \hat{K}} \quad \text{and} \quad B = \{(-y, \tau(y))\}_{y \in \hat{K}}$$

A.14.15;

as above, we have $K = A \oplus B$, $A$ is totally singular since

$$(\kappa \times \kappa)((y, \tau(y)), (y', \tau(y'))) = \kappa(y, y')\kappa(\tau(y), \tau(y'))$$

$$= \kappa(y, y')\kappa(\tau^2(y), y')$$

$$= \kappa(y, y')\kappa(y, y')^{-1} = 1$$

A.14.16

for any $y \in \hat{K}$ and, similarly, $B$ is totally singular too. Once again, the converse images $\hat{A}$ of $A$ and $\hat{B}$ of $B$ in $\hat{K} \times \hat{K}$ are maximal Abelian subgroups; hence, the argument above applied to the $p$-subgroup $\Delta(R) = \{u \otimes u\}_{u \in R}$ and to the element $x \otimes x$ of $Sp(K \times K, \kappa \times \kappa)$ proves that

$$2^{\omega(\Delta(R), S \otimes kS)}(x \otimes x, j(x \otimes x)(a \otimes a)^{-1}S \otimes kS) = 1$$

A.14.17

for the choice of an idempotent $j$ in the unique local point of $\Delta(R)$ on $S \otimes kS$.

Consequently, since it follows again from Theorem A.3 that we have the commutative diagram

$$
\begin{array}{ccc}
\mathbb{Q}^2(\hat{F}_S(R)) \ast \mathbb{Q}^2(\hat{F}_S(R)) & \xrightarrow{\nu_{R,S,S}} & \mathbb{Q}^2(\hat{F}_{S \otimes kS}(R)) \\
2^{\omega(R,S)} \times 2^{\omega(R,S)} & \xrightarrow{\times} & 2^{\omega(R,S \otimes kS)} \\
\end{array}
$$

$k^*$

and since we clearly have

$$
\nu_{R,S,S}(\langle x^R, ixa^{-1}\hat{S} \rangle, \langle x^R, ixa^{-1}\hat{S} \rangle) = (x \otimes x, j(x \otimes x)(a \otimes a)^{-1}S \otimes kS)
$$

A.14.19

from equality A.14.17 we actually get

$$
(2^{\omega(R,S)}(x^R, ixa^{-1}\hat{S}))^2 = 1
$$

A.14.20

which forces $2^{\omega(R,S)}(x^R, ixa^{-1}\hat{S}) = 1$ since $x$ has odd order. We are done.
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