Tail estimations for normed sums of centered exchangeable random variables.

M.R. Formica, E. Ostrovsky, L. Sirota.

Università degli Studi di Napoli Parthenope, via Generale Parisi 13, Palazzo Pacanowsky, 80132, Napoli, Italy.
e-mail: mara.formica@uniparthenope.it

Department of Mathematics and Statistics, Bar-Ilan University, 59200, Ramat Gan, Israel.
e-mail: eugostrovsky@list.ru

Department of Mathematics and Statistics, Bar-Ilan University, 59200, Ramat Gan, Israel.
e-mail: sirota3@bezeqint.net

Abstract.

We derive in this short report the exponential as well as power decreasing tail estimations for the sums of centered exchangeable random variables, alike ones for the sums of the centered independent ones.

Key words and phrases. Random variables (r.v.), exchangeability, distribution, tails, Lebesgue - Riesz and Grand Lebesgue norm and spaces, moments, estimations, de Finetti theorem and representation, permutation, conditional probability and moments, expectation and variance; estimations, auxiliary integral, saddle-point method, slowly varying function.

1 Statement of problem. Notations.

Let \( (\Omega = \{\omega\}, \mathcal{B}, \mathcal{P}) \) be certain non-trivial probability space with expectation \( E \) and variation \( \text{Var} \). Let also \( (Y, \mathcal{N}) \) be another measurable space.

Definition 1.1, see [1], [14], [15], [16], [23], [24] at all.

The sequence of r.v. \( \{\xi_i\}, i = 1, 2, \ldots, n; n \leq \infty \) with values in \( Y \) is said to be exchangeable, if for all the permutations \( \pi = (\pi(1), \pi(2), \ldots, \pi(n)) \) of the set of numbers \( N = (1, 2, \ldots, n) \), where \( n < \infty \), the distribution of the random
vector \((\xi_{\pi(1)}, \xi_{\pi(2)}, \ldots, \xi_{\pi(n)})\) coincides with the distribution of the source random vector \((\xi_1, \xi_2, \ldots, \xi_n)\):

\[
P\left(\xi_{\pi(1)} \in A_1, \xi_{\pi(2)} \in A_2, \ldots, \xi_{\pi(n)} \in A_n\right) = P\left(\xi_1 \in A_1, \xi_2 \in A_2, \ldots, \xi_n \in A_n\right) \quad (1)
\]

for all the tuples of the measurable sets \(A_j \in \mathcal{N}\). The infinity sequence of the r.v. \(\{\xi_i\}, \ i \in 1, 2, 3, \ldots\) is named exchangeable, if the relation \((1)\) is satisfied for all the finite values \(n\).

Many examples of these type of the random variables, aside from the classical case of independent identical distributed r.v., are described in the mentioned works; some applications in physic and statistic may be found in [20]. It was obtained in particular the classical limit theorems: LLN, CLT, LIL etc. for the exchangeable r.v., still with the remainder term evaluation.

Our aim in this short report is the non - asymptotic estimation of the tail distribution, power as well as exponential decreasing, for the classical normed sums of such a random variables

\[
R_n(t) = R[S(n)](t) \overset{df}{=} P \left( n^{-1/2} | \sum_{i=1}^{n} \xi_i | \geq t \right), \quad t \geq 1; \quad (2)
\]

and consequently

\[
\overline{R}(t) = \sup_n R[S(n)](t) \overset{df}{=} \sup_n P \left( n^{-1/2} | \sum_{i=1}^{n} \xi_i | \geq t \right), \quad t \geq 1; \quad (3)
\]

of course, when \( Y = R^1, \ E\xi_i = 0; \ Var(\xi_i) \in (0, \infty)\); in the spirit of the classical estimations belonging to Dharmadhikari - Jogdeo, Rosenthal, Nagaev, Pinelis, Schechtman at all. See also [5], [6], [12], [13], [25], chapters 1,2.

2 Auxiliary fact: de Finetti representation.

Let now in addition \( Y \) be metrisable separable complete space and \( \mathcal{N} \) be Borelian sigma field. We will use the following important fact: theorem (representation) belonging to de Finetti for the exchangeable r.v., see [14], [15]; and the more general proposition [1], [16].

**Proposition 2.1.** There exists a probability measure \( \mu \) defined on the Borelian sigma field \( \mathcal{N} \) such that
\[ P(\xi_1 \in A_1, \xi_2 \in A_2, \ldots, \xi_n \in A_n) = \int_{N} Q(A_1)Q(A_2)\ldots Q(A_n) \mu(dQ). \quad (4) \]

Here in turn, \( Q(\cdot) \) is certain probability measure.

On the other words, if we introduce the r.v. \( M \) having the distribution \( \mu \), then

\[ P(\xi_1 \in A_1, \xi_2 \in A_2, \ldots, \xi_n \in A_n / M = Q) = Q(A_1)Q(A_2)\ldots Q(A_n), \quad (5) \]

or in turn equally the r.v. \( \{\xi_i\} \) are independent and identical distributed under arbitrary fixed condition \( M = Q \); and in this condition they have the distribution \( Q(\cdot) \).

Herewith the measure \( M \) may be expressed as almost everywhere existing limit

\[ M(A) = \lim_{n \to \infty} n^{-1} \sum_{i=1}^{\infty} \chi_A(\xi_i), \quad (6) \]

where \( \chi_A(\cdot) \) is an indicator function for the Borelian set \( A \).

The inversely assertion to the proposition 2.1. is obviously also true. Indeed, if for some random vector \( \{\xi_i\}, i = 2, 3, \ldots, n \) there holds true the representation (4), this vector is exchangeable.

We suppose in addition that the r.v. \( \{\xi_j\} \) are conditional centered:

\[ \forall Q \in N \Rightarrow \mathbb{E}\xi_i/Q = 0, \quad (7) \]

and have a finite a.e. non-zero conditional variance (a second conditional moment)

\[ \forall Q \in N \Rightarrow \text{Var}(\xi_i)/Q \in (0, \infty). \quad (8) \]

Denote yet

\[ T[Q, n](t) \overset{def}{=} P \left( n^{-1/2} \sum_{j=1}^{n} \xi_j > t / Q \right), \quad Q \in N, \quad (9) \]

conditional tail of distribution if the normed sums, and consequently its uniform tail estimate

\[ \mathcal{T}[Q](t) \overset{def}{=} \sup_{n} T[Q, n](t). \quad (10) \]

Of course, under formulated above restrictions \( \mathcal{T}[Q](t) \to 0, \ t \to \infty. \)
3 Main result.

Proposition 3.1. It follows immediately from the well known complete probability formulae that

\[ R_n(t) = \int_N T[Q, n](t) \mu(dQ), \]  

(11)

and consequently

\[ \overline{R}(t) \leq \int_N \overline{T}[Q](t) \mu(dQ). \]  

(12)

4 Examples.

There are a huge number works devoted to the estimations of the probability \( T[Q, n](t), \) see e.g. \[2\], \[3\], \[4\], \[5\], \[6\], \[17\], \[18\], \[22\], \[25\], \[26\] etc. For instance, let \( \{\eta_i\}, i = 1, 2, \ldots, n \) be a sequence of independent identical distributed (i.i.d.) non trivial centered \( \mathbb{E}\eta_i = 0 \) random variables (r.v.). Set as before

\[ \Theta_n := n^{-1/2} \sum_{i=1}^{n} \eta_i, \]

and suppose in addition

\[ \exists m = \text{const} > 0 \Rightarrow \mathbb{P}( |\eta_i| \geq u) \leq \exp(-u^m), \ u \geq 0. \]

Then

\[ \exists c = c(m) \in (0, \infty) \Rightarrow \sup_n \mathbb{P}( |\Theta_n| > u) \leq \exp \left( -c(m) u^{\min(m, 2)} \right), \ u \geq 0, \]

and this estimate is essentially non-improvable as \( u \to \infty. \)

Example 4.1. So, it is reasonable to assume that the set \( \{Q\} \) may be identified with positive semi-axes \( \{Q\} = (0, \infty) \) and let

\[ T[Q, n](t) \leq \exp \left( -c_1 t^\alpha Q^\beta \right), \ \ c_1, \alpha, \beta \in (0, \infty); \]  

(13)

\[ \mu(A) = c_2 \int_A Q^\gamma \exp \left( -c_3 Q^\kappa \right) dQ, \ \gamma > -1, \ \kappa > 0, \ \ c_3 > 0; \]  

(14)

where \( A \) is Borelian subset of the positive half line \( R_1 : A \subset R_1 \) and of course (norming condition)

\[ c_2 = c_2(\kappa, c_3, \gamma) = \frac{\kappa c_3^{(\gamma+1)/\kappa}}{\Gamma((\gamma + 1)/\kappa)}, \]
$\Gamma(\cdot)$ is ordinary Gamma function.

We get to the following expression for the estimation of the value $\mathcal{R}(t)$ from (12) under formulated assumptions

$$\mathcal{R}(t) \leq c_2 \int_0^\infty Q^\gamma \exp \left( -c_1 t^\alpha Q^\beta - c_3 Q^\kappa \right) dQ. \quad (15)$$

**WE CAME TO THE NEED TO ESTIMATE THE FOLLOWING INTEREST AUXILIARY INTEGRAL**

$$I[\theta, g](t) \overset{\text{def}}{=} \int_0^\infty x^{\theta-1} \exp(-tx - g(x)) \, dx, \quad (16)$$

as $t \to \infty$, where $\theta = \text{const} > 0$, $g = g(x)$, $x \geq 0$ is non-negative measurable function such that $g(0) = g(0+) = 0$.

**Lemma 4.1.** We conclude under formulated conditions as $t \to \infty$

$$\lim_{t \to \infty} \left\{ t^\theta I[\theta, g](t) \right\} = \Gamma(\theta). \quad (17)$$

**Proof. Upper estimation.** We will use the classical saddle-point method, see e.g. [7], [30].

We have by virtue of the non-negativity of the function $g = g(x)$ for all the sufficient greatest values $t$; say, for the values $t \geq 1$

$$I[\theta, g](t) \leq \int_0^\infty x^{\theta-1} \exp(-tx) \, dx = \Gamma(\theta) \, t^{-\theta}. \quad (18)$$

**Lower estimate.** Let again $t \geq 1$. Let also $\Delta = \Delta(t)$ be certain numerical valued positive function such that $\lim_{t \to \infty} \Delta(t) = 0$; where $\Delta(t) := \sup_{s \geq t} \Delta(s)$.

We have as $t \to \infty$

$$I[\theta, g](t) \geq \int_0^\Delta \exp(-tx - g(x)) \, dx \geq \exp(-\Delta) \cdot \int_0^\Delta e^{-tx} x^{\theta-1} \, dx =$$

$$t^{-\theta} \exp(-\Delta) \int_0^t \Delta y^{\theta-1} e^{-y} \, dy.$$

It remains to choose for example $\Delta = \Delta(t) := t^{-1/2}$, to make sure of the fairness of our lemma.

**Remark 4.1.** One can generalize the proposition of this lemma: consider the following integral
\[ J[\theta, g, L](t) = \int_0^\infty x^{\theta - 1} e^{-tx-g(x)} L(x) \, dx, \]

where in addition \( L = L(x) \) is bounded non-negative continuous \textit{slowly varying} at origin function; then as \( t \to \infty \)

\[ \lim_{t \to \infty} \left\{ t^\theta J[\theta, g, L](t) \right\} = \int_0^\infty y^{\theta - 1} \exp(-y) L(y) \, dy. \]

Let us return to the source tail estimation (15). As long as, by virtue of the proposition of Lemma 4.1, we have as \( t \to \infty \)

\[ c_2 \int_0^\infty Q^\gamma \exp(-c_1 t^\alpha Q^\beta - c_3 Q^\kappa) \, dQ \sim \]

\[ c_1^{-(\gamma+1)/\beta} c_2 \beta^{-1} \Gamma((\gamma+1)/\beta) t^{-\alpha(\gamma+1)/\beta}, \]  

following

\[ \overline{R}(t) \leq C_4(\alpha, \beta, \kappa) t^{-\alpha(\gamma+1)/\beta}, \quad t \geq 1, \quad - \]  

the power tail decay.

**Example 4.2.** We assume here again that \( \{Q\} = (0, \infty) \) and let now

\[ T[Q, n](t) \leq \exp\left(-c_1 t^\alpha Q^{-\beta}\right), \quad c_1, \alpha, \beta \in (0, \infty); \]

and as above

\[ \mu(A) = c_2 \int_A Q^\gamma \exp(-c_3 Q^\kappa) \, dQ, \quad \gamma > -1, \quad \kappa > 0, \quad c_3 > 0; \]

\[ c_2 = c_2(\kappa, c_3, \gamma) = \frac{\kappa c_3^{(\gamma+1)/\kappa}}{\Gamma((\gamma+1)/\kappa)}. \]

We get to the following expression for the estimation of the new value \( \overline{R}(t) \) from (12) under formulated in this example assumptions \( \overline{R}(t) \leq R_0(t) = R_0[\alpha, \beta, \gamma, \kappa, c_1, c_2, c_3](t), \) where

\[ R_0(t) \overset{\text{def}}{=} c_2 \int_0^\infty Q^\gamma \exp\left(-c_1 t^\alpha Q^{-\beta} - c_3 Q^\kappa\right) \, dQ. \]

One can apply for the investigation of the last integral (24) as \( t \to \infty \) the classical saddle-point method, to obtain the logarithmical exact estimation

\[ \overline{R}(t) \leq c_6(\alpha, \beta, \kappa, \gamma) \exp\left\{ -c_7(\alpha, \beta, \kappa, \gamma) t^{\alpha/(\beta+\kappa)} \right\}, \quad t \geq 1, \]

for some finite non-zero "constants" \( c_6, c_7, \) the exponential tail decay.

More detail computations by means of the saddle-point (Laplace) method show us the following evaluation as \( t \to \infty \)
\[ R_0(t) \sim \sqrt{2\pi} \, c_2 \, c_{10} \, t^{c_{11}} \, (A + B)^{-1/2} \times \]
\[ \exp \left\{ -c_{10} \left( A^{-1} + B^{-1} \right) \, t^{\alpha \, \kappa / (\beta + \kappa)} \right\}, \]

where

\[ A = \frac{\beta}{\gamma + 1}, \quad B = \frac{\kappa}{\gamma + 1}, \]

\[ c_{10} = (\gamma + 1)^{-1} \cdot \left[ c_1^\kappa \, c_3^\beta \, \beta^\kappa \, \kappa^\beta \right]^{1/(\beta + \kappa)}, \quad c_{11} = \alpha \cdot \left[ \frac{2\gamma + 2 - \kappa}{2(\beta + \kappa)} \right] - 1. \]

5 Concluding remarks.

It is interest in our opinion to obtain the multivariate generalization of our results, as well as to consider another examples.

Acknowledgement. The first author has been partially supported by the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and by Università degli Studi di Napoli Parthenope through the project “sostegno alla Ricerca individuale”.

References

[1] Aldous D. (1985) Exchangeability and related topics. In: Hennequin P., editor; Ecole d’Ete de Probabilities de Saint - Flour, XIII, (1983), 1 - 198.

[2] V.V. Buldygin V.V., D.I.Mushtary, E.I.Ostrovsky, M.I.Pushalsky. New Trends in Probability Theory and Statistics. Mokslas, (1992), V.1, p. 78 - 92; Amsterdam, Utrecht, New York, Tokyo.

[3] Capone C, Formica M.R, Giova R. Grand Lebesgue spaces with respect to measurable functions. Nonlinear Analysis 2013; 85: 125 - 131.

[4] Capone C, and Fiorenza A. On small Lebesgue spaces. Journal of function spaces and applications. 2005; 3; 73 - 89.

[5] S. V. Ermakov, E. I. Ostrovsky. Central limit theorem for weakly dependent Banach - space valued random variables. Theory Probab. Appl., 30, 2, (1986), 391 - 394.

[6] S.V.Ermakov, and E. I. Ostrovsky. Continuity Conditions, Exponential Estimates, and the Central Limit Theorem for Random Fields. Moscow, VINITY, (1986), (in Russian).

[7] M.V.Fedoryuk. The saddle-point method. Moscow, Science, (1977) (In Russian).

[8] Fiorenza A., and Karadzhyov G.E. Grand and small Lebesgue spaces and their analogs. Consiglio Nazionale Delle Ricerche, Instituto per le Applicazioni del Calcoto Mauro Picone, Sezione di Napoli, Rapporto tecnicon. 272/03, (2005).
[9] A. Fiorenza, M. R. Formica and A. Gogatishvili. On grand and small Lebesgue and Sobolev spaces and some applications to PDE’s. Differ. Equ. Appl. 10, (2018), no. 1, 21–46.

[10] A. Fiorenza, M. R. Formica, A. Gogatishvili, T. Kopaliani and J. M. Rakotoson. Characterization of interpolation between grand, small or classical Lebesgue spaces. Preprint arXiv:1709.05892.

[11] A. Fiorenza, M. R. Formica and J. M. Rakotoson. Pointwise estimates for GT-functions and applications. Differential Integral Equations 30 (2017), no. 11-12, 809–824.

[12] M. R. Formica and R. Giova. Boyd indices in generalized grand Lebesgue spaces and applications. Mediterr. J. Math. 12 (2015), no. 3, 987–995.

[13] M.R.Formica, E.Ostrovsky, L.Sirota. Central Limit Theorem in Lebesgue-Riesz spaces for weakly dependent random sequences. arXiv:1912.00338v2 [math.PR] 3 Dec 2019

[14] Bruno de Finetti. Funzione caratteristica di un fenomeno aleatorio. Atti della R. Accademia Nazionale dei Lincei, Ser. 6, Memorie, Classe di Scienze Fisiche, Matematiche e Naturali vol. 4, (1931) , pp. 251 - 299.

[15] Bruno de Finetti. La prevision: ses lois logiques, ses sources subjectives. Annales de l'Institut Henri Poincare, vol. 7, (1937), pp. 1 - 68.

[16] Hewitt E., Savage L.J. Symmetric measures on Cartesian products. Trans. of AMS, 80, (1955), 470 - 501.

[17] Yu.V. Kozachenko and E.I. Ostrovsky. Banach spaces of random variables of subgaussian type. Theory Probab. Math. Stat., Kiev, (1985), v. 43, 42 - 56, (in Russian).

[18] Kozachenko Yu.V., Ostrovsky E., Sirota L. Relations between exponential tails, moments and moment generating functions for random variables and vectors. arXiv:1701.01901v1 [math.FA] 8 Jan 2017

[19] Kozachenko Yu.V., Ostrovsky E., Sirota L. Equivalence between tails, Grand Lebesgue Spaces and Orlicz norms for random variables without Kramer’s condition. Bulletin of KSU, Kiev, 2018, 4, pp. 20 - 29.

[20] Kingman J. Uses of exchangeability. Ann. of Probab. 6, (1978), 183 - 198.

[21] Werner Kirsch. An elementary proof of de Finetti’s Theorem. Preprint September 3, 2018, Preprint, Fern Universität in Hagen. https://www.researchgate.net/publication/327417122

[22] E.Liflyand, E.Ostrovsky, L.Sirota. Structural Properties of Bilateral Grand Lebesgue Spaces. Turk. J. Math.; 34, (2010), 207 - 219.

[23] Steffen Lauritzen. Exchangeability and de Finetti’s Theorems. University Oxford, April 26, 2007.

[24] Steffen Lauritzen. Extremal Families and Systems of sufficient Statistics. Lecture Notes in Statistics, 49, (1988).

[25] E.I. Ostrovsky. Exponential Estimations for Random Fields. Moscow - Obninsk, OINPE, (1999), in Russian.

[26] Pekshir G., Shiryaev A.N. Khintchine’s inequalities and martingale extension of scope of their actions. Uspekhi Mathem. Nauk, 1995, V. 50, Issue 5 (305), 3 - 63, (in Russian).

[27] E.M.Stein. Interpolation of linear operators. Trans. Amer. Math. Soc., 83; 482 - 492, 1956.

[28] V. I. Yudovich. Nonstationary flow of an ideal incompressible liquid. Zh. Vych. Mat., 3, (1963), 1032 - 1066.
[29] **V. I. Yudovich.** *Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid.* Mathematical Research Letters, **2.** (1995), 27 - 38.

[30] **R. Wong.** *Asymptotic approximations of integrals.*, Acad. Press. New York, London, Paris; (1989)