Total Colorings-A Survey

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Abstract: The smallest integer $k$ needed for the assignment of colors to the elements so that the coloring is proper (vertices and edges) is called the total chromatic number of a graph. Vizing [86] and Behzad [4, 5] conjectured that the total coloring can be done using at most $\Delta(G) + 2$ colors, where $\Delta(G)$ is the maximum degree of $G$. It is not settled even for planar graphs. In this paper we give a survey on total coloring of graphs.

1 Introduction

Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. An element of $G$ is a vertex or an edge of $G$. The total coloring of a graph $G$ is an assignment of colors to the vertices and edges such that no two incident or adjacent elements receive the same color. The total chromatic number of $G$, denoted by $\chi''(G)$, is the least number of colors required for a total coloring. Clearly, $\chi''(G) \geq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of $G$.

Behzad [4, 5] and Vizing [86] independently posed a conjecture called Total Coloring Conjecture (TCC) which states that for any simple graph $G$, the total chromatic number is either $\Delta(G) + 1$ or $\Delta(G) + 2$. In [67], Molly and Reed gave a probabilistic approach to prove that for sufficiently large $\Delta(G)$, the total chromatic number is at most $\Delta(G) + 10^{26}$.

If $\chi''(G) = \Delta(G) + 1$ then $G$ is known as a type-I graph and if $\chi''(G) = \Delta(G) + 2$ then $G$ is a type-II graph. In this paper, we present a comprehensive survey on total coloring. Yap [107] gave a nice survey on total colorings that covers the results till 1995. Therefore, our survey cover results from 1996 onwards. There are four sections in this paper. In the second section, we focus on results on planar graphs. In [8], Borodin gave a survey of results on total colorings of planar graphs up to 2009. There are several improved results on planar graphs since then. Thus, we only present the results from 2010 onwards. For the earlier results we refer the readers to the earlier two excellent surveys mentioned above. Third section of this paper consists of results on non-planar graphs. In this paper, we also prove TCC holds for unitary cayley graphs, mock threshold graphs and odd graphs. In the last section, we survey the results on complexity aspects of total coloring.
2 Planar graphs

In this section, we consider the results related to planar graphs (graphs that have a plane embedding). Many of the results in total coloring of planar graphs are based on the maximum degree and the girth constraints.

One of the most yielding techniques on planar graphs is the Discharging Method. While one can say that the discharging method have been used in graph theory for more than 100 years, it came to prominence when it was used to prove the four color theorem by Appel and Hacken. Since then, the method has been applied to many types of problems (including graph embeddings and decompositions, spread of infections in networks, geometric problems, etc.). It is especially useful for dealing with planar graphs. An excellent guide to the method of discharging is given by Cranston and West [28].

A rough sketch of using the discharging method is as follows [70]:

**Charging phase:**
1. Assign initial charges to certain elements of a graph (vertices, edges, faces, etc.,).
2. Compute the total charge assigned to the whole graph (for planar graphs typically using Euler’s formula).

**Discharging phase:**
3. Redistribute charge in the graph according to a set of discharging rules.
4. Compute the total charge again (using the specific properties of the graph), and derive a conclusion.

A configuration in a graph $G$ can be any structure in $G$ (often a specified sort of subgraph). A configuration is reducible for a graph property $Q$ if it cannot occur in a minimal graph not having property $Q$. The method of discharging is used to show that a set of reducible configurations is unavoidable in the class of graphs being discussed which establishes that the property $Q$ cannot have a counterexample in the class. Let $d_G(v)$ or simply $d(v)$ denote the degree (number of neighbors) of vertex $v$ in $G$, and let $d(G)$ denote the average of the vertex degrees in $G$. Degree charging is the assignment to each vertex $v$ of an initial charge equal to $d(v)$. In the following, we present the total coloring results on planar graphs.

The total coloring conjecture was verified for planar graphs with $\Delta(G) \leq 5$ [55]. Sanders and Zhao [76] showed that every planar graph with $\Delta(G) \leq 7$ is 9-total colorable. Yap [107] verified TCC for planar graphs with $\Delta(G) = 8$. In [56], Kowalik et al. proved that any planar graph with $\Delta(G) \geq 9$ is type-I. Shen et al. [81] posed a conjecture on the total coloring of planar graph which states that “Planar graphs with $4 \leq \Delta(G) \leq 8$ are $\Delta(G)+1$-total colorable”. Wang et al. [90] proved that for a planar graph $G$ with maximum degree $\Delta(G)$ and girth $g$ such that $G$ has no cycles of length from $g + 1$ to $t$, $t > g$, the total chromatic number is $\Delta(G) + 1$ provided $(\Delta(G), g, t) \in \{(5, 4, 6), (4, 4, 17)\}$ or $\Delta(G) = 3$ and $(g, t) \in \{(5, 13), (6, 11), (7, 11), (8, 10), (9, 10)\}$, where each vertex is incident with at most one $g$-cycle. For more details we refer the excellent survey by Borodin [8].
First we start with results involving planar graphs with maximum degree at least six. As far as we know the first work on the total coloring with $\Delta(G) = 6$ was given by Wang et al. They verified TCC for planar graphs without 4-cycles. Shen et al. improved the result by showing that the planar graph without 4-cycles is type-I with the reducible configuration shown in Fig.1. Sun et al. proved that every planar graph $G$ with maximum degree 6 is totally 8-colorable if no two triangles in $G$ share a common edge (which implies that every vertex $v$ in $G$ is incident with at most $\left\lfloor \frac{d(v)}{2} \right\rfloor$ triangles. In other words, every vertex is missing either a 3-cycle or a 4-cycle). They also proved that a planar graph without adjacent triangles and without cycles of length $k$, $k \geq 5$ is type-I. Nicolas Roussel strengthened the result of Sun et al. by showing that a planar graph $G$ is total 8-colorable if every vertex of $G$ is missing some $k_v$-cycle for $k_v \in \{3, 4, 5, 6, 7, 8\}$. In 2017, Zhu and Xu improved the result of Sun et al. to show that TCC holds for planar graphs $G$ with $\Delta(G) = 6$, provided $G$ does not contain any subgraph isomorphic to a 4-fan.

Further improvements on the total coloring of planar graph with $\Delta \geq 6$ as follows. Zhang and Wang showed that every planar graph with $\Delta \geq 6$ and without adjacent short cycles (a cycle of length at most 4) is $\Delta + 1$-total-colorable. Hou et al. proved that a planar graph $G$ is type-I if $\Delta(G) \geq 5$ and $G$ contains neither 4-cycles nor 6-cycles or $\Delta(G) \geq 6$ and $G$ contains neither 5-cycles nor 6-cycles. A chordal $k$-cycle is a $k$-cycle with at least one chord. In Wu et al. improved the result of by showing that a planar graph with maximum degree $\Delta(G)$ is $M$-total-colorable if it contains neither chordal 5-cycle nor chordal 6-cycle, where $M = \max\{7, \Delta(G) + 1\}$. Dong et al. proved that a planar graph $G$ where no 6-cycles has a chord satisfies TCC provided $\Delta(G) \geq 6$. Li verified TCC for planar graphs with maximum degree six, if for each vertex $v$, there is an integer $k_v \in \{3, 4, 5, 6\}$ such that $G$ has no $k_v$-cycle containing $v$.

A notion closely related to total coloring of graphs is list total coloring of graphs. Suppose that a set $L(x)$ of colors, called a list of $x$, is assigned to each element $x \in V(G) \cup E(G)$. A total coloring $\phi$ is called a list total coloring of $G$ or $L$-coloring, if $\phi(G) \in L(x)$ for each element $x \in V(G) \cup E(G)$. If $|L(x)| = k$ for every $x \in V(G) \cup E(G)$, then a total $L$-coloring is called a list total $k$ coloring and we say that $G$ is totally $k$-choosable and the minimum integer $k$ for which $G$ is total $k$-choosable is the total choosability of $G$. Liu et al. proved that a planar graph $G$ is total $\Delta(G) + 2$-choosable ($\Delta(G) + 2$-total colorable) whenever (1) $\Delta(G) \geq 7$ and $G$ has no adjacent triangles or (2) $\Delta(G) \geq 6$ and $G$ has no intersecting triangles or (3) $\Delta(G) \geq 5$ and $G$ has no adjacent triangles and $G$ has no $k$-cycles for some integer $k \in \{5, 6\}$.
We now list the results on the total coloring of planar graph with maximum degree at least 7. The first work on this direction is due to Sanders and Zhao [76].

Fig. 2. Reducible configurations from [82]

Shen and Wang [82] showed that the planar graphs with maximum degree 7 and without 5-cycles are 8-totally colorable. Fig.2. shows the reducible configuration used in [82]. Chang et al. [18] proved that a planar graph $G$ with maximum degree 7, with the additional property that for every vertex $v$, there is an integer $k_v \in \{3, 4, 5, 6\}$ so that $v$ is not incident with any $k_v$-cycle, is type-I. Wang and Wu [87] proved that a planar graph of maximum degree $\Delta(G) \geq 7$ is $\Delta(G) + 1$-totally colorable if no 3-cycle has a common vertex with a 4-cycle or no 3-cycle is adjacent to a cycle of length less than 6. In [88], Wang et al. proved that for any planar graph with maximum degree $\Delta(G) \geq 7$ and without intersecting 3-cycles (two cycles of length 3 are not incident with a common vertex), and without intersecting 5-cycles, the total chromatic number is $\Delta(G) + 1$. [89]. Wu et al. [97] proved that for a planar graph with maximum degree at least 7 and without adjacent 4-cycles the total chromatic number is $\Delta(G) + 1$.

The total chromatic number of a planar graph with $\Delta(G) \geq 7$ and without chordal 6-cycles [91] and without chordal 7-cycles [10] is $\Delta(G) + 1$.

We now turn to the results on maximum degree at least 8. There are many works on planar graphs with maximum degree at least 8. Yap [107] verified the TCC for planar graphs with $\Delta(G) \geq 8$.

Roussel and Zhu [73] proved that for a planar graph $G$ with maximum degree 8, $\chi''(G) = 9$, if there is no $k_x$-cycle which contains $x$, where $x$ is a vertex in $G$ and $k_x \in \{3, 4, 5, 6, 7, 8\}$. This is an improvement over [49]. The reducible configurations used by Roussel and Zhu are given in Fig.3. Further, Wang et al. [95], strengthened the result and proved that for a planar graph with $\Delta(G) \geq 8$, if for every vertex $v \in V$, there exists two integers, $i_v, j_v \in \{3, 4, 5, 6, 7, 8\}$ such that $v$ is not incident with intersecting $i_v$-cycles and $j_v$-cycles, then the total chromatic number of $G$ is $\Delta(G) + 1$. Wang et al. [92] showed that the total chromatic number of planar graphs with $\Delta(G) \geq 8$ is $\Delta(G) + 1$, if for every vertex $v \in V(G)$,
there exists two integers $i_v, j_v \in \{3, 4, 5, 6, 7\}$ such that $v$ is not incident with adjacent $i_v-$cycles and $j_v-$cycles.

Xu and Wu [106] proved that if $G$ is a planar graph with maximum degree at least 8 and every 7-cycle of $G$ contains at most two chords, then $G$ has a $(\Delta(G) + 1)$-total-coloring. Wang et al. [96], considered planar graphs $G$ with maximum degree $\Delta(G) \geq 8$, and showed that if $G$ contains no adjacent $i, j$-cycles with two chords for some $i, j \in \{5, 6, 7\}$, then $G$ is total $(\Delta(G) + 1)$-colorable. Jian Chang et al. [19] proved that planar graphs with maximum degree at least 8 and without 5-cycles with two chords are $\Delta(G) + 1$ total colorable. In [105], Xu et al. proved that if $G$ is a planar graph with maximum degree at least 8 and every 6-cycle of $G$ contains at most one chord or any chordal 6-cycles are not adjacent, then $G$ has a $(\Delta(G) + 1)$-total coloring. In 2014, Wang et al. [94] showed that a planar graph with $\Delta(G) \geq 8$ and if $v \in V(G)$ is not incident with chordal 6-cycle, or chordal 7-cycle or 2-chordal 5-cycle, then it is type-I. This is a generalisation of [94].

There are some other classes of graphs which are similar to planar graphs. We discuss the total coloring of some of them.

A graph is called 1-planar, if it has at most one crossing edge. The following are some results on total coloring of 1-planar graphs. Zhang et al. [112] proved that each 1-planar graph with $\Delta(G) \geq 16$ is $(\Delta(G) + 2)$-total colorable and $(\Delta(G) + 1)$-total colorable if $\Delta(G) \geq 21$. Július Czap [29] studied the 1-planar graphs and gave some upper bound for the total chromatic number. He showed that a 1-planar graph with $\Delta(G) \geq 8$ satisfies TCC if $\chi(G) \leq 4$. He also proved that if $G$ is a 1-planar graph without adjacent triangles and with $\Delta(G) \geq 8$, then $\chi''(G) \leq \Delta(G) + 3$ and if $\chi(G) \leq 4$, then $\chi''(G) \leq \Delta(G) + 2$. Xin Zhang et al. [110] showed that for a 1-planar graph $G$, if $\Delta(G) \leq r$ and $r \geq 13$, where $r$ is an integer, then $\chi''(G) \leq r + 2$.

An outerplanar graph is a planar graph that has a plane embedding such that all vertices lie on the boundary of the outer face. In [100], Y. Wang and W. Wang characterized the adjacent vertex distinguishing total chromatic number of outerplanar graphs. They proved that, if $G$ is an outerplane graph with $\Delta(G) \geq 4$, then $\chi''(G) \leq \Delta(G) + 2$ and so the TCC is satisfied. They also proved that, if $G$ is an outerplane graph with $\Delta(G) \geq 4$ and without adjacent vertices of maximum degree, then $\chi''(G) = \Delta(G) + 1$ and hence $G$ is type-I graph.

In [20], Chang et al. proved that a planar graph with maximum degree 8 is 9-total colorable if for every vertex $v$, $v$ is incident with at most $d(v) - 2\lceil \frac{d(v)}{5} \rceil$ triangles. This is a generalisation of [11] and [19].

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A graph is pseudo-outerplanar if each of its blocks has an embedding in the plane so that the vertices lie on a fixed circle and the edges lie inside the disk of this circle with each of them crossing at most one another. In [111], Xin Zhang and Guizhen Liu verified TCC for pseudo-outerplanar graphs and proved that the total chromatic number of every outerplanar graph with $\Delta(G) \geq 5$ is $\Delta(G) + 1$. Xin Zhang [109] proved that every pseudo-outerplanar graph with $\Delta(G) \geq 5$ is totally $\Delta(G) + 1$-choosable and hence the total chromatic number also has this as upper bound.

Similar to planar graphs there is another type of graphs called toroidal graphs which are embedding of graphs on torus such that there is no crossing edges. Tao Wang [98] proved that if $G$ is a 1-toroidal graph with $\Delta(G) \geq 5$ is pseudo-outerplanar graphs and proved that the total chromatic number of every outerplanar graph with $\Delta(G) \geq 5$ is $\Delta(G) + 1$-choosable and hence the total chromatic number has this as upper bound.

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3 Non-planar graphs

3.1 Circulant Graph

For a sequence of positive integers $1 \leq d_1 < d_2 < \ldots < d_l \leq \lfloor \frac{n}{2} \rfloor$, the circulant graph $G = C_n(d_1, d_2, \ldots, d_l)$ has vertex set $V = Z_n = \{0, 1, 2, \ldots, n-1\}$, two vertices $x$ and $y$ being adjacent iff $x = (y \pm d_i) \mod n$ for some $i, 1 \leq i \leq l$.

Riadh Khennoufa and Olivier Togni [53] studied total colorings of circulant graphs and proved that every 4-regular circulant graphs $G = C_{5p}(1, k)$ and $C_{6p}(1, k)$ are type-I graphs for any positive integer $p$ and $k < \frac{5p}{2}$ with $k \equiv 2 \mod 5$ or $k \equiv 3 \mod 5$ and $p \geq 3$ and $k < 3p$ with $k \equiv 1 \mod 3$ or $k \equiv 2 \mod 3$ respectively.

A graph is a power of cycle, denoted $C_n^k$, $n$ and $k$ are integers, $1 \leq k < \lfloor \frac{n}{2} \rfloor$, if $V(C_n^k) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(C_n^k) = E^1 \cup E^2 \cup \ldots \cup E^k$, where $E^i = \{e^i_0, e^i_1, \ldots, e^i_{n-1}\}$ and $e^i_j = (v_j, v_{(j+i) \mod n})$ and $0 \leq j \leq n-1$, and $1 \leq i \leq k$.

Campos and de Mello [14] proved that $C_n^2, n \neq 7$, is type-I and $C_n^3$ is type-II. They [16] verified the TCC for powers of cycles $C_n^k, n$ even and $2 < k < \frac{n}{2}$ and also showed that one can obtain a $\Delta(G) + 2$-total coloring for these graphs in polynomial time. They also proved that $C_n^k$ with $n \equiv 0 \mod (\Delta(C_n^k) + 1)$ are type-I and they proposed the following conjecture

Conjecture 1. Let $G = C_n^k$, with $2 \leq k < \lfloor \frac{n}{2} \rfloor$. Then,

$$\chi''(G) = \begin{cases} 
\Delta(G) + 2, & \text{if } k > \frac{n}{3} - 1 \text{ and } n \text{ is odd} \\
\Delta(G) + 1, & \text{otherwise.}
\end{cases}$$

Geetha et al. [37] proved this conjecture for certain values of $n$ and $k$. They also verified TCC for the complement of powers of cycles $C_n^k$. In particular, they proved that $C_n^2$ is type-II for $n \leq 8$.

Cayley graphs are those whose vertices are the elements of groups and adjacency relations are defined by subsets of the groups. Cayley graphs contain long paths and have many other nice combinatorial properties. They have been used to construct other combinatorial structures. Also, for the constructions of various communication networks, and difference sets in design theory. Cayley graphs have been used to analyze algorithms for computing with groups.
Let $\Gamma$ be a multiplicative group with identity 1. For $S \subseteq \Gamma, 1 \notin S$ and $S^{-1} = \{s^{-1} : s \in S\} = S$ the Cayley Graph $X = \text{Cay}(\Gamma, S)$ is the undirected graph having vertex set $V(X) = \Gamma$ and edge set $E(X) = \{(a, b) : ab^{-1} \in S\}$.

For a positive integer $n > 1$ the unitary Cayley graph $X_n = \text{Cay}(\mathbb{Z}_n, U_n)$ is defined by the additive group of the ring $\mathbb{Z}_n$ of integer modulo $n$ and the multiplicative group $U_n$ of its units. If we represent the elements of $\mathbb{Z}_n$ by the integers $0, 1, ..., n-1$, then it is well known that

$$U_n = \{a \in \mathbb{Z}_n : \gcd(a, n) = 1\}.$$  

So $X_n$ has vertex set $V(X_n) = \mathbb{Z}_n = \{0, 1, 2, ..., n-1\}$ and edge set

$$E(X_n) = \{(a, b) : a, b \in \mathbb{Z}_n, \gcd(a - b, n) = 1\}.$$  

Boggess et al. [6] studied the structure of unitary cayley graphs. They have also discussed chromatic number, vertex and edge connectivity, planarity and crossing number. Klotz and Sander [54] have determined the clique number, the independence number and the diameter. They have given a necessary and sufficient condition for the perfection of $X_n$.

The graph $X_n$ is regular of degree $U_n = \varphi(n)$ denotes the Euler function. Let the prime factorization of $n$ be $p_1^{\alpha_1} p_2^{\alpha_2} ... p_t^{\alpha_t}$ where $p_1 < p_2 < ... < p_t$. If $n = p$ is a prime number, then $X_n = K_p$ is the complete graph on $p$ vertices. If $n = p^\alpha$ is a prime power then $X_n$ is a complete $p$-partite graph. In the following theorem we prove TCC holds for unitary Cayley graphs.

**Theorem 3.1.** A unitary Cayley graph $X_n$ is $(\Delta(X_n) + 2)$-total colorable.

**Proof.** We know that a unitary Cayley graph can be obtained from a balanced $r$ partite graph by deleting some edges. Suppose $n = p$ is a prime number, then $X_n$ is the complete graph on $p$ vertices. Also, if $n = p^\alpha$, a prime power, then $X_n$ is a complete $p$-partite graph and TCC holds for these two graphs [107].

When $n = 2k, k \in \mathbb{N}$, then the unitary Cayley graph is a bipartite graph and any bipartite graph is total colorable.

Suppose $n \neq 0 \mod 2$. As $p_1$ is the smallest prime, $kp_1, kp_1 + 1, ..., (k+1)p_1 - 1$, where $k = 0, 1, 2, ..., \frac{n}{p_1} - 1$ induces $\frac{n}{p_1}$ vertex disjoint cliques each of order $p_1$. Since $p_1$ is odd, we can color all the elements of these $\frac{n}{p_1}$ cliques using $p_1$ colors [107]. Now remove the edges of these cliques. The remaining graph is a $\varphi(n) - p_1 + 1$-regular graph where the vertices are already coloured. We color the edges of this resultant graph with $\varphi(n) - p_1 + 2$ colors. Thus we have used $\varphi(n) + 2$ colors for the total coloring of $X_n$. \qed

In the following section, we look at graph products and papers on total colouring of product graphs.

### 3.2 Product Graphs

Graph products were first defined by Sabidussi [74] and Vizing [85]. A lot of work has been done on various topics related to graph products, but on the other hand there are still many questions open. There are four standard graph products, namely, cartesian product ($G \Box H$),
direct product \((G \times H)\), strong product \((G \boxtimes H)\) and lexicographic product \((G \circ H)\). In [74], these products have been widely discussed with significant applications. The vertex sets of these products are same: \(V(G) \times V(H)\). The edge sets are \(E(G \square H) = \{(g, h), (g', h')\} \mid g = g', hh' \in E(H)\), or \(gg' \in E(G), \ h = h'\}, \ E(G \times H) = \{(g, h), (g', h')\} \mid gg' \in E(G)\) and \(hh' \in E(H)\}, \ E(G \boxtimes H) = E(G \square H) \cup E(G \times H)\) and \(E(G \circ H) = \{(g, h), (g', h')\} \mid g = g', hh' \in E(H)\), or \(gg' \in E(G)\}.

The first three products are commutative and the lexicographic product is associative but not commutative.

The total coloring conjecture was verified for the cartesian product of two graphs. Seoud et al. [79, 80] determined the total chromatic number of the join of two paths, the cartesian product of two paths, the cartesian product of a path and a cycle, certain classes of the corona of two graphs and the theta graphs. Kemnitz and Marangio [52] classified the corona product of two graphs and the theta graphs. Kemnitz and Marangio [52] classified the cartesian product of two cycles, the cartesian product of a path and a cycle, certain classes of the corona of two graphs and the theta graphs. Kemnitz and Marangio [52] classified the corona product of two graphs and the theta graphs. Kemnitz and Marangio [52] classified the cartesian product of two cycles, the cartesian product of a path and a cycle, certain classes of the corona of two graphs and the theta graphs.

Geetha and Somasundaram [39] proved that direct product \((g \times h)\), denoted by \((G \circ H)\) is also type-I. Zmasek and Žerovnik [114] proved that if TCC holds for graphs \(G\) and \(H\), then it holds for the cartesian product \(G \square H\). They also proved that if the factor with largest vertex degree is of type-I, then the product is also of type-I.

There are only a few results proved on total colorings of the other three product graphs. Katja Prnava and Blaž Zmasek [71] verified the conjecture for direct product of a path and any graph \(G\) with \(\chi'(G) = \Delta(G)\). Geetha and Somasundaram [39] proved that direct product of two even complete graphs are type-I and the direct product of two cycles \(C_m\) and \(C_n\) are type-I for certain values of \(m\) and \(n\). They also proved that if \(K_2 \boxtimes H\) (\(K_2 \circ H\)) satisfies TCC, then \(G \boxtimes H\) (\(G \circ H\)) satisfy TCC, where \(G\) is any bipartite graph. Mohan et al. [66] proved that the corona product of two graphs \(G\) and \(H\) is always type-I, provided \(G\) is total colorable and \(H\) is either a cycle, a complete graph or a bipartite graph. The deleted lexicographic product of two graphs \(G\) and \(H\), denoted by \(D_{lex}(G, H)\), is a graph with the vertex set \(V(G) \times V(H)\) and the edge set \(\{(g, h), (g', h')\} \mid g = g', hh' \in E(H)\), or \(hh' \in E(H)\) and \(gg' \in E(G)\}.

Similar to lexicographic product, \(D_{lex}(G, H)\) and \(D_{lex}(H, G)\) are not necessarily isomorphic. Recently, Vignesh et al. [54] proved that if \(G\) is a bipartite graph and \(H\) is any total colorable graph then \(G \circ H\) is also total colorable. They further show that for any class-I graph \(G\) and any graph \(H\) with at least 3 vertices, \(D_{lex}(G, H)\) is total colorable. In particular, if \(H\) is class-I then \(D_{lex}(G, H)\) is also type-I.
We present now the results on Sierpiński graphs.

### 3.3 Sierpiński Graphs

The Sierpiński graphs $S(n, K_k)$, $k, n \geq 1$, $k, n \in \mathbb{N}$ is defined on the vertex set $\{1, 2, ..., k\}^n$, where $K_k$ is complete graphs on $k$ vertices. Two different vertices $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ are adjacent if and only if there exists a $h \in \{1, 2, ..., n\}$ such that

- $u_t = v_t$ for $t = 1, 2, ..., h - 1$;
- $u_h \neq v_h$; and
- $u_t = v_h$ and $v_t = u_h$ for $t = h + 1, ..., n$.

Sierpiński gasket graphs $S_n$ were introduced by Scorer, Grundy and Smith [78]. The graph $S_n$ is obtained from the Sierpiński graphs $S(n, 3)$ by contracting every edge of $S(n, 3)$ that lies in no triangle. Marko Jakovac and Sandi Klavžar [51] generalized the graphs $S(n, 3)$ to Sierpiński graphs $S(n, k)$ for $k \geq 3$ and determined the total colorings of the Sierpiński gasket graphs $S_n$. In particular they proved that for any $n \geq 2$ and any odd $k \geq 3$, $S(n, k)$ and $S(n, 4)$ are type-I graphs. For the even values of $k \geq 6$, they believed that $S(n, k)$ is always type-II and hence they proposed a conjecture that $S(n, k)$ is type-II. After three years Andreas M. Hinz, Daniele Parisse [47] disproved the conjecture based on the canonical total colorings. Also they prove that the Hanoi graphs $H^n_\phi$ are type-I graphs. Geetha and Somasundaram [38] considered the generalized Sierpiński graphs $S(n, G)$ and proved that $S(n, G)$ is type-I for certain classes of $G$.

We now turn our attention to Chordal graphs.

### 3.4 Chordal Graphs

Chordal graphs are graphs in which every induced cycle is a 3-cycle. They form a very important class of graphs due to the fact that they have good algorithmic properties. The TCC is verified for several subfamilies of chordal graphs like interval graphs, split graphs and strongly chordal graphs. A graph $G$ is called split graph if its vertex set can be partitioned into two subsets $U$ and $V$ such that $U$ induces a clique and $V$ is independent set in $G$. A color diagram $C = \{R_1, R_2, ..., R_k\}$ of frame $d = (d_1, d_2, ..., d_k)$ is an ordered set of color arrays, where color array $R_i = \{c_{i,1}, c_{i,2}, ..., c_{i,d_i}\}$, of length $d_i$, consists of distinct colors for all $1 \leq i \leq k$. In [23], Chen et al. proved that the split graphs satisfies TCC. They also proved that if $G$ is a split graph with $\Delta(G)$ even, then $G$ is type-I. They extensively used the concept of color diagram to prove these results. Campos et al. [13] gave conditions for the split-indifference graph $G$ to be type-II and constructed a $\Delta(G) + 1$-total colorings for the remaining. Hilton [43] proved the following (it is known as Hilton’s condition): Let $G$ be a simple graph with an even number of vertices. If $G$ has a universal vertex, then $G$ is type-II if and only if $|E(\overline{G})| + \alpha(\overline{G}) < \frac{|V(G)|}{2}$, where $\alpha(G)$ is the cardinality of a maximum independent set of edges of $G$.

Three-clique graphs are generalization of the split-indifference graphs. Campos et al. [13] proposed a conjecture based on the Hilton’s condition.

**Conjecture 2.** A 3-clique graph is type-II if and only if it satisfies Hilton’s condition.
A graph is dually chordal if it is the clique graph of a chordal graph. The class of dually chordal graphs generalize known subclasses of chordal graphs such as doubly chordal graphs, strongly chordal graphs, interval graphs, and indifference graphs. Figueiredo et al. [34] proved that TCC holds for dually chordal graphs. A pullback from \( G \) to \( G' \) is a function \( f : V(G) \to V(G') \), such that: (i) \( f \) is a homomorphism and (ii) \( f \) is injective when restricted to the neighborhood of \( x, \forall x \in V(G) \). Based on this pullback method, they proved that if \( \Delta(G) \) is even, then \( G \) is type-I. A family of sets satisfies the Helly property if any subfamily of pairwise intersecting sets has nonempty intersection. A graph is neighborhood-Helly when the set \( \{N(v) : v \in V(G)\} \) satisfies the Helly property. A characterization of dually chordal graphs says that \( G \) is dually chordal if and only if \( G \) is neighborhood-Helly and \( G^2 \) is chordal. It is proved that [34] TCC holds for neighborhood-Helly graphs

They also proposed the following problem which is still open:

Determine the largest graph class for which all its odd maximum degree graphs are class-I and for which all its even maximum degree graphs are type-I.

A graph is weakly chordal if neither the graph nor the complement of the graph has an induced cycle on five or more vertices. A simple graph \( G \) is threshold if there is a vertex ordering \( v_1, ..., v_n \) such that for every \( i (1 \leq i \leq n) \) the degree of \( v_i \) in \( G : v_1, ..., v_i \) is 0, 1, \( i - 2 \), or \( i - 1 \). Threshold graphs are a simple generalization of threshold graphs that, like threshold graphs, are perfect graphs. Mock threshold graphs are perfect and indeed weakly chordal but not necessarily chordal [3]. Similarly, the complement of a mock threshold graph is also mock threshold.

In the following, we prove the TCC for Mock Threshold graphs.

**Note:** A total coloring of \( K_n \) can be constructed as follows: (This total coloring is due to Hinz and Parisse [47])

When \( n \) is even, we first construct an edge coloring of \( K_n \) and extend it. We denote \( [n]_0 = \{0, 1, 2, ..., n - 1\} \). For \( k \in [n]_0 \), let \( \tau_k \) be the transposition of \( k \) and \( n - 1 \) on \( [n]_0 \). For even \( n \), \( c_n(i, j) = (\tau_i(j) + \tau_j(i) + 2) \mod (n + 1) \), for \( i, j \in [n]_0, i \neq j \), defines a \( (n + 1) \)-edge coloring. In this coloring assignment line \( k \in [n]_0 \) will have the missing colors \( k \) and \( (k + 1) \mod n \). We color \( c_n(i) = i \) for all \( i \in [n]_0 \).

When \( n \) is odd, we use the same coloring of \( K_{n-1} \). In the coloring assignment of \( K_{n-1} \), still the color \( (k + 1) \mod n \) is missing in line \( k \in [n]_0 \). We use these colors to the edges incident with \( n^{th} \) vertex and color \( n \) to the \( n^{th} \) vertex.

**Theorem 3.2.** Total coloring conjecture holds for any Mock threshold graph \( G \).

**Proof.** Consider the Mock threshold graph \( G \) with vertex ordering \( v_1, v_2, ..., v_i, ..., v_k \).

We prove this theorem using mathematical induction on the induced subgraph \( G[v_1, v_2, ..., v_k] \).

For \( k \leq 4 \), the maximum degree of all the induced subgraphs is less than or equal to 3. We know that a graph with maximum degree less than or equal to 3 satisfies TCC [55].

Let us assume that \( G[v_1, v_2, ..., v_k], k \geq 5 \) satisfies TCC.
Claim: The graph $G[v_1, v_2, ..., v_k, v_{k+1}]$ satisfies TCC.

The degree of the vertex $v_{k+1}$ in $G[v_1, v_2, ..., v_{k+1}]$ can be $0, 1, k - 1$ or $k$.

Case-1: Suppose $d(v_{k+1}) = 0$.

In this case the vertex is $v_{k+1}$ is an isolated vertex. By the induction assumption, $G[v_1, v_2, ..., v_k, v_{k+1}]$ satisfies TCC.

Case-2: Suppose $d(v_{k+1}) = 1$.

In this case, the vertex $v_{k+1}$ is adjacent to a vertex, say $v_i$, in $G[v_1, v_2, ..., v_k]$. Since $G[v_1, v_2, ..., v_k]$ is total colorable graph with at most $\Delta(G[v_1, v_2, ..., v_k]) + 2$ colors, at each vertex there will be at least one missing color. We assign this missing color to the edge $(v_i, v_{k+1})$, and for the vertex $v_{k+1}$, we assign a color of a vertex which is not adjacent to $v_{k+1}$ and not the color of $v_i$. Therefore, $G[v_1, v_2, ..., v_{k+1}]$ satisfies TCC.

Case-3: Suppose $d(v_{k+1}) = k - 1$.

Let us assume that the vertex $v_{k+1}$ is not adjacent with $v_i$ and also assume that $\Delta(G[v_1, v_2, ..., v_{k+1}]) = k - 1$. We consider following two cases:

Subcase-1: $k$ is even.

Since $k$ is even, $k + 1$ is odd. Construct a complete graph induced by the vertices $v_1, v_2, ..., v_{i-1}, v_{i+1}, ..., v_{k+1}$. Now, color this complete graph using colors in the set $\{0, 1, ..., n + 1\}$ as given in the note. In this coloring assignment there will be one missing color at each of the vertices and they are distinct. Now, color the edges $(v_i, v_j), i \neq j, j = 1, 2, ..., v_{k+1}$, with the missing colors. Assign the color $n - 1$ to the vertex $v_i$. To get a total coloring of $G[v_1, v_2, ..., v_k, v_{k+1}]$, we remove these edges and there is no change in the maximum degree.

Subcase-2: $k$ is odd.

In this case $k + 1$ is even, say $2p$. It is known that a graph of order $2p$ with maximum degree $2p - 2$ satisfies TCC (see [41, 22]).

Case-4: Suppose $d(v_{k+1}) = k$.

The maximum degree of $G[v_1, v_2, ..., v_k, v_{k+1}]$ is $k$. Construct a complete graph on the vertex set $\{v_1, v_2, ..., v_k, v_{k+1}\}$. We know that the complete graph satisfies TCC. After removing the added edges we get a total coloring of $G[v_1, v_2, ..., v_k, v_{k+1}]$.

Hence, in all the cases, the mock threshold graph satisfies TCC.

\[\blacksquare\]

### 3.5 Multipartite Graphs

Graph amalgamation [1] is one of the powerful techniques for various graph problems. A graph $H$ is an amalgamation of a graph $G$ if there exists a function $\phi$ called an amalgamation function from $V(G)$ onto $V(H)$ and a bijection $\phi': E(G) \to E(H)$ such that $e$ joining $u$ and $v$ is in $E(G)$ if and only if $\phi'(e)$ joining $\phi(u)$ and $\phi(v)$ is in $E(H)$. Total coloring conjecture was verified for some classes of multipartite graphs using the amalgamation technique.

Dong and Yap [36] proved that the complete $p$-prartite graph $K = K(r_1, r_2, ..., r_p)$ is of type-I if $r_2 \leq r_3 - 2$ and $|V(K)| = 2n$, $r_1 \leq r_2 \leq ... \leq r_p$. Deficiency of a graph $G$ to be $\text{def}(G) = \sum_{e \in V(G)} (\Delta(G) - d(e))$. Dalal and Rodger [31] proved that $K = K(r_1, ..., r_5)$ is type-II if and only if $|V(K)| = 0(\text{mod} 2)$ and $\text{def}(K)$ is less than the number of parts in $K$ of odd size. Dalal et al. [30] proved that the complete $p$-partite graph $K = K(r_1, r_2, ..., r_p)$ is type-I if and only if $K \neq K_{r,r}$ and if $K$ has an even number of vertices then $\text{def}(K)$ is at
least the number of parts of odd size. Using graph amalgamations technique they showed that all complete multipartite graphs of the form $K(r, r, ..., r + 1)$ are type-I.

Chen et al. [21] proved that an $(n - 2)$-regular equi-bipartite graph $K_{n,n} - E(J)$ is type-I if and only if $J$ contains a 4-cycle. Campos and de Mello [15] determined the total chromatic number of some bipartite graphs like grids, near-ladders and $k$-dimensional cubes.

### 3.6 Cubic Graphs

In [33] Dantas et al. proved that for each integer $k \geq 2$, there exists an integer $N(k)$ such that, for any $n \geq N(k)$, the generalized Petersen graph $G(n, k)$ has total chromatic number 4.

Snarks are cyclically 4-edge-connected cubic graphs that do not allow a 3-edge-coloring. Cavicchioli et al. [17] proved that all the snarks of order less than 30 are of type-I (this was proved with the aid of a computer). Also they proposed a open problem “find (if any) the smallest snark (with respect to the order) which is of type-II”. Motivated by that question, Campos et al. [12] proved that all graphs in three infinite families of snarks, the Flower Snarks, the Goldberg Snarks, and the Twisted Goldberg Snarks are type-I. They gave recursive procedures to construct total-colourings that uses 4 colours in each case. Also they proposed an open problem that all snarks are type-I. Sasaki el al. [77] prove that the total chromatic number of some classes of Snarks like, Loupekline, Goldberg snarks both and Blannas families of graphs is 4. They observed that the total chromatic number seems to have no relation with the chromatic index for a cubic graph of cyclic-edge-connectivity less than 4. Also they proposed the following questions:

(i). What is the smallest type-II cubic graph without a square?

(ii). What is the smallest type-II snark?

Gunnar Brinkmann et al. [9] considered the problems posed by Campos et al. [12] and Sasaki el al. [77] and showed that there exists type-II snarks for each even order $n \geq 40$. They gave a computer search for which all the cubic graphs with girth 5 and up to 32 vertices are type-I. Also they proposed the following questions:

(i). Does there exist a type-II snark of order less than 40? (The only possible orders for which the existence is not yet known are 36 and 38.)

(ii). What is the smallest type-II cubic graph with girth at least 5?

(iii). Is there a girth $g$ so that all cubic graphs with girth at least $g$ are type-I?

### 3.7 Graphs with Degree Constraints

Being a very difficult conjecture, it makes sense to prove the conjecture either for known classes of graphs or graphs with some degree constrains. Hilton and Hind [45] showed that TCC holds for the graphs $G$ having $\Delta(G) \geq \frac{4}{7}|V(G)|$. Chetwynd et al. [25] gave a necessary and sufficient condition for $\chi''(G) = \Delta(G) + 1$, if $G$ is odd order and regular of degree $d \geq \frac{4}{7}\sqrt{7}|V(G)|$. 
Deficiency of a graph \( G \) to be \( \text{def}(G) = \sum_{v \in V(G)}(\Delta(G) - d(v)) \). A graph \( G \) is said to be conformable if \( G \) has a vertex colouring that uses \( \Delta(G) + 1 \) colours with \( \text{def}(G) > n \), where \( n \) is the number of colour classes with parity different from \( |V(G)| \). Chew \[26\] improved the previous result (Chetwynd et al. \[25\]) for \( d \geq \frac{\sqrt{37} - 1}{6}|V(G)| \). He proved that for any regular graph \( G \) of odd order and with \( d \geq \frac{\sqrt{37} - 1}{6}|V(G)| \), \( G \) is type-I if and only if \( G \) is conformable; otherwise type-II.

Dezheng Xie and Zhongshi He \[103\] showed that if \( G \) is a regular graph of even order and \( \delta(G) \geq \frac{2}{3}|V(G)| + \frac{23}{6} \), then \( \chi''(G) \leq \Delta(G) + 2 \). Later, Xie DeZheng and Yang WanNian \[104\] proved the same result for regular graph of odd order. Combining these two results, we conclude that if \( G \) regular graph with \( \delta(G) \geq \frac{2}{3}|V(G)| + \frac{23}{6} \) then \( G \) satisfies TCC. In \[62\] Machado and de Figueiredo proved that every non-complete \{square, unichord\}-free graph of maximum degree at least 4 is type-I. Also they proved that any \{square, unichord\}-free graph is total colorable. Using graph decompositions, the same authors \[61\] proved that the non-complete \{square, unichord\}-free graphs of maximum degree 3 are type-I.

A graph is said to be \( s \)-degenerate for an integer \( s \geq 1 \) if it can be reduced to a trivial graph by successive removal of vertices with degree \( \leq s \). For example, every planar graph is 5-degenerate. Shuji Isobe et al. \[50\] proved that an \( s \)-degenerate graph \( G \) has admits a total coloring with \( \Delta(G) + 1 \) colors if the maximum degree \( \Delta(G) \geq 4s + 3 \). The proof is based on Vizing’s and and Konig’s theorems on edge colorings. Further, they gave a linear time algorithm to find a total coloring of a graph \( G \) with minimum number of colors if \( G \) is a partial \( k \)-tree.

### 3.8 Other Classes of Graphs

Mycielski \[69\], introduced the graph Mycielskian graph \( \mu(G) \), to build a graph with a high chromatic number and a small clique number. Let \( G \) be a graph with vertex set \( V^0 = \{v^0_1, v^0_2, ..., v^0_n\} \) and edge set \( E^0 \). Given an integer \( m \geq 1 \), the \( m \)-Mycielskian of \( G \), denoted by \( \mu_m(G) \), is the graph with vertex set \( V^0 \cup V^1 \cup ... \cup V^m \cup \{u\} \), where \( V^i = \{v^i_j : v^0_j \in V^0\} \) is the \( i \)th distinct copy of \( V^0 \) for \( i = 1, 2, ..., m \), and the edge set \( E^0 \cup (\bigcup_{i=0}^{m-1} \{v^i_jv^{i+1}_j, v^0_jv^0_j \in E^0\}) \cup \{v^m_jv^m_j : v^m_j \in V^m\} \). Chen et al. \[24\] showed that the generalized Mycielski graphs satisfy TCC. Also they proved the total chromatic number of generalized Mycielski graphs \( \mu_m(G) \) is \( \Delta(\mu_m(G)) + 1 \) if \( \Delta(G) \leq \frac{|V(G)| - 1}{2} \).

Zhi-wen Wang et al. \[101\] proved that the vertex distinguishing total chromatic number and the total chromatic number are same for the graphs \( P_n \vee P_n \) and \( C_n \vee C_n \). Li and Zhang \[37\] proved that the join of a complete inequibipartite graph \( K_{n_1,n_2} \) and a path \( P_n \) is type-I. Hilton et al. \[46\], determined the total chromatic numbers of graphs of the form \( G_1 + G_2 \), where \( G_1 \) and \( G_2 \) are graphs of maximum degree at most two.

The line graph of \( G \), denoted by \( L(G) \), has the set \( E(G) \) as its vertex set and two distinct vertices \( e_1, e_2 \in V(L(G)) \) are adjacent if and only if they share a common vertex in \( G \). Vignesh et al. \[84\] showed in a direct manner that for \( n \leq 4 \), \( L(K_n) \) is type-I. They believe that \( L(K_n) \) is always type-I. Hence they proposed the following conjecture:

**Conjecture 3.** For any complete graph \( K_n \), \( \chi''(L(K_n)) = 2n - 3 \).
The double graph $D(G)$ of a given graph $G$ is constructed by making two copies of $G$ (including the initial edge set of each) and adding edges $((u, 1), (v, 2))$ and $((v, 1), (u, 2))$ for every edge $uv$ of $G$. Vignesh et al. [84] also prove that for any total colorable graph $G$, 

$$\chi''(D(G)) \begin{cases} = \Delta(D(G)) + 1 & \text{if } G \text{ is type I} \\ \leq \Delta(D(G)) + 2 & \text{if } G \text{ is type II.} \end{cases}$$

We know that middle graphs are subclasses of total graphs and super classes of line graphs. Muthuramakrishnan and Jayaraman [68] obtained the total chromatic number for line, middle and total graphs of star and bistars.

The Kneser graph $K(n, k)$ is the graph whose vertices correspond to the $k$-element subsets of a set of $n$ elements, and where two vertices are adjacent if and only if the two corresponding sets are disjoint. A vertex in the odd graph $O_n$ is a $(n-1)$-element subset of a $(2n-1)$-element set. Two vertices are connected by an edge if and only if the corresponding subsets are disjoint. Note that the odd graphs are particular case of Kneser graphs.

Theorem 3.3. Odd graph $O_n$ satisfies TCC.

Proof. Consider a $2n-1$ element set $X$. Let $O_n$ be an odd graph defined from the subsets of $X$. Let $x$ be any element of $X$. Then, among the vertices of $O_n$, exactly $\binom{2n-2}{n-2}$ vertices correspond to sets that contain $x$. Because all these sets contain $x$, they are not disjoint, and form an independent set of $O_n$. That is, $O_n$ has $2n-1$ different independent sets of size $\binom{2n-2}{n-2}$. Further, every maximum independent set must have this form, so, $O_n$ has exactly $2n-1$ maximum independent sets.

If $I$ is a maximum independent set, formed by the sets that contain $x$, then the complement of $I$ is the set of vertices that do not contain $x$. This complementary set induces a matching in $G$. Each vertex of the independent set is adjacent to $n$ vertices of the matching, and each vertex of the matching is adjacent to $n-1$ vertices of the independent set [40].

Based on the decomposition, we give a total coloring of $O_n$ in the following way:

Assign $n$ colors to the edges between the vertices in the maximum independent set $I$ and the vertices in the matching. Color the edges in matching and the vertices in $I$ with a new color. Color one set of vertices in the matching with another new color and the second set of vertices with the missing colors at these vertices. This will give a total coloring of $O_n$ using at most $n + 2 = \Delta(O_n) + 2$ colors.

In the next section we look at the algorithmic aspects of TCC that has been discussed in the literature.

4 Algorithms

It is known [75] that the problem of finding a minimal total coloring of a graph is in general case NP-hard. In the same paper, Sanchez-Arroyo also proved that the problem is NP-complete even for cubic bipartite graphs. For general classes of graphs, the total coloring would be harder than edge colouring. Due to its complexity, several authors aim to find
classes of graphs where there is a polynomial time algorithm for optimal total coloring. Boj- jarshinov [7] showed that the Behzad and Vizing conjecture holds for interval graphs. Also, he proved that every interval graph with even maximum degree can be totally colored in $\Delta(G) + 1$ colors in time $O(|V(G)| + |E(G)| + (\Delta(G))^2)$. This is the first known polynomial time algorithm for total colorings. Recently, Golumbic [41] showed that a rooted path graph $G$ is type-I if $\Delta(G)$ is even, otherwise it satisfies TCC. Also, he gave a greedy algorithm (very greedy neighborhood coloring algorithm) which takes $O(|V(G)| + |E(G)|)$ time.

Chordal graphs are a subclass of the perfect graphs. We know that linear time algorithms exist for vertex colorings of chordal graphs. Yet, the complexity of total coloring is open for the class of chordal graphs. The complexity is known for interval graphs [7], split graphs [23] and dually chordal graphs [34]. In [60], Machado and Celina de Figueiredo proved that the total coloring of bipartite unichord-free graphs is NP-complete using the concept of separating class. Machado et al. [64] used a decomposition result to establish that every chordless graph of maximum degree $\Delta(G) \geq 3$ has total chromatic number $\Delta(G) + 1$ and proved that this coloring can be obtained in time $O(|V(G)|^3|E(G)|)$. Machado et al. [65] discussed the time complexity of $\{\text{square, unichord}\}$-free graphs and showed that the total coloring can be obtained in polynomial time. In this case it is interesting to see that the edge coloring of this type of graphs is NP-complete.

| Class of graphs                  | Edge coloring | Total coloring |
|----------------------------------|---------------|----------------|
| Unichord free                    | NP-complete   | NP-complete    |
| Chordless                        | Polynomial    | Polynomial     |
| $\{\text{Square, unichord}\}$-free, $\Delta \geq 4$ | Polynomial    | Polynomial     |
| $\{\text{Square, unichord}\}$-free, $\Delta = 3$ | NP-complete   | Polynomial     |
| Bipartite unichord free          | NP-complete   | NP-complete    |
| Interval graphs                  | Polynomial    | Polynomial     |
| Some classes of circulant graphs | Polynomial    | Polynomial     |

Table 1: Computational complexity of edge and total colorings

Shuji Isobe et al. [50] proved that the total coloring problem for $s$-degenerate graph can be solved in time $O(n \log n)$ for a fairly large class of graphs including all planar graphs with sufficiently large maximum degree. Further, they showed that the total coloring problem can be solved in linear time for partial $k$-trees with bounded $k$.

Dantas et al. [32] proved that the problem of deciding whether the equitable total chromatic number is 4 is NP-complete for bipartite cubic graphs. They also found one family of type-I cubic graphs of girth 5 having equitable total chromatic number 4. There are several classes of graphs in which the complexity of total coloring are unknown. We conclude this survey with a listing of the computational complexity of edge and total colorings of certain classes of graphs in Table [1].
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