Dwell-time control sets and applications to the stability analysis of linear switched systems

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Outline

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2. Invariant control sets of general nonlinear systems
3. Invariant control sets for projected linear switched systems and periodization
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Linear switched systems and their Lyapunov exponents
Consider a switched system on $\mathbb{R}^d$ of the type

$$\dot{x} = A(t)x, \quad A(\cdot) \in S$$

($\Sigma$)

$S$ class of signals from $\mathbb{R}$ to some set $S \subset M_d(\mathbb{R})$.

Examples:

- $S_0 = \{ A : \mathbb{R} \to S \mid A$ piecewise constant $\}$ $\rightarrow$ arbitrary switching

- $S_\tau = \{ A : \mathbb{R} \to S \mid A$ piecewise constant, discontinuities at distance $\geq \tau$ $\}$ $\rightarrow$ switching with (guaranteed) dwell-time $\tau$

- other classes can be introduced in terms of average dwell-time constraints, persistence of excitation, Lipschitz constraints, ...$

Crucial property of switched systems: uniform asymptotic stability with respect to $A \in S$
Measures of stability of linear switched systems

$A \in S \rightarrow$ fundamental matrix $\Phi_A(\cdot)$ solution to

$$\frac{d}{dt} \Phi_A(t) = A(t) \Phi_A(t), \quad \Phi_A(0) = \text{Id}_d$$

- $S$-attractive: $\Phi_A(t) \rightarrow 0$ for every $A \in S$
- $S$-uniform exponential stability: $\exists C, \lambda > 0$ s.t. $\forall A \in S$
  \[
  \|\Phi_A(t)\| \leq Ce^{-\lambda t}, \quad \forall t \geq 0 \tag{\star}
  \]
Measures of stability of linear switched systems

\( A \in S \quad \overset{\text{fundamental matrix}}{\rightarrow} \quad \Phi_A(\cdot) \) solution to

\[
\frac{d}{dt} \Phi_A(t) = A(t)\Phi_A(t), \quad \Phi_A(0) = \text{Id}_d
\]

- **S-attractive:** \( \Phi_A(t) \to 0 \) for every \( A \in S \)
- **S-uniform exponential stability:** \( \exists C, \lambda > 0 \) s.t. \( \forall A \in S \)

\[
\| \Phi_A(t) \| \leq Ce^{-\lambda t}, \quad \forall t \geq 0
\]

(\( \star \))

- **uniform exponential rate:**

\[
\lambda(S) = \limsup_{t \to +\infty} \sup_{A \in S} \frac{\log(\| \Phi_A(t) \|)}{t} = \inf\{ \lambda \mid \exists C \text{ s.t. } (\star) \forall A \in S \}
\]

- **S-uniform exponential stability** \( \Leftrightarrow \lambda(S) < 0 \)
Measures of stability of linear switched systems

\[ A \in \mathcal{S} \longrightarrow \text{fundamental matrix } \Phi_A(\cdot) \text{ solution to} \]

\[ \frac{d}{dt} \Phi_A(t) = A(t)\Phi_A(t), \quad \Phi_A(0) = \text{Id}_d \]

- **\(\mathcal{S}\)-attractive:** \(\Phi_A(t) \to 0\) for every \(A \in \mathcal{S}\)
- **\(\mathcal{S}\)-uniform exponential stability:** \(\exists C, \lambda > 0\) s.t. \(\forall A \in \mathcal{S}\)

\[ \|\Phi_A(t)\| \leq Ce^{-\lambda t}, \quad \forall t \geq 0 \]  \((\star)\)

- uniform exponential rate:

\[ \lambda(\mathcal{S}) = \limsup_{t \to +\infty} \sup_{A \in \mathcal{S}} \frac{\log(\|\Phi_A(t)\|)}{t} = \inf\{\lambda \mid \exists C \text{ s.t. } (\star) \forall A \in \mathcal{S}\} \]

- **\(\mathcal{S}\)-uniform exponential stability** \(\iff\) \(\lambda(\mathcal{S}) < 0\)
- maximal Lyapunov exponent:

\[ \hat{\lambda}(\mathcal{S}) = \sup_{A \in \mathcal{S}} \limsup_{t \to +\infty} \frac{\log(\|\Phi_A(t)\|)}{t} = \inf\{\lambda \mid \forall A \in \mathcal{S}, \exists C \text{ s.t. } (\star) \} \]

- \(\hat{\lambda}(\mathcal{S}) \leq \lambda(\mathcal{S})\)
Equality between $\lambda(S_\tau)$ and $\hat{\lambda}(S_\tau)$

**Lemma (Fenichel)**

Let $S = S_\tau$ for $\tau \geq 0$. Then $(\Sigma)$ is $S$-attractive if and only if it is $S$-uniformly exponentially stable.

**Corollary**

For every $\tau \geq 0$, $\lambda(S_\tau) = \hat{\lambda}(S_\tau)$.

From now on

$$\lambda_\tau(S) := \lambda(S_\tau)$$

**Our aim:** give a useful characterization of $\lambda_\tau(S)$.
Let $S = \{1, \ldots, N\}$ and $Q = (q_{ij})_{i,j=1}^{N}$ be Markov transition matrix $(q_{ij} \geq 0, \sum_{j=1}^{N} q_{ij} = 1)$

A trajectory is a random variable, as well as its switching law $(i_k, t_k)_{k \in \mathbb{N}}$:

- the initial index $i_1$ in $S$ is a random variable
- transition $A_{i_k} \rightarrow A_{i_{k+1}}$ at time $t_k$ with probability $q_{i_k i_{k+1}}$
- we can introduce a dwell-time:

$$P(\{t_{k+1} - t_k \leq \theta\}) = \begin{cases} 0 & \text{if } \theta < \tau \\ \nu \int_{\tau}^{\theta} e^{-\nu(t-\tau)} dt & \text{if } \theta \geq \tau \end{cases}$$

- duration of each interval between switching times:

$$P(\{t_{k+1} - t_k \leq \theta\}) = \nu \int_{0}^{\theta} e^{-\nu t} dt$$

Furstenberg–Kesten theorem: if $Q$ is strongly connected, then, with probability one \( \exists \lim_{t \to \infty} \frac{1}{t} \| \Phi_A(t) \| = \chi(\tau) \)
Invariant control sets of general nonlinear systems
Invariant control sets

$M$ manifold, $F$ family of smooth complete vector fields on $M$
$f(\cdot) \in \mathcal{F}_0$ piecewise constant with values in $F$, $q$ initial condition
$\rightarrow$ solution $t \mapsto \phi(t, q, f)$
Attainable set from $q \in M$: $A(q) = \{ \phi(t, q, f) \mid t \geq 0, \; f \in \mathcal{F}_0 \}$

Definition

$\emptyset \neq D \subset M$ invariant control set (ICS) if $D = \overline{A(q)}$ for every $q \in D$

Example: $A_1, \ldots, A_m \in M_d(\mathbb{R}), \, x_1, \ldots, x_m \in \mathbb{R}^d$.
If

- $\dot{x} = A_{i(t)}x$ is asymptotically stable
  (with arbitrary switching)

then

$\overline{A(x_1)} = \bigcap_{\Omega \neq \emptyset} \Omega$ compact invariant $\Omega$

is a ICS for $\dot{x} = A_{i(t)}(x - x_{i(t)})$
Existence of invariant control sets

**Theorem (see, e.g., Colonius–Kliemann, 2000)**

Let $M$ be compact. For each $q \in M$ there exists a nonempty ICS $D_q$ contained in $\overline{A(q)}$. Assume, moreover, that $F$ has the Lie algebra rank condition (LARC). Then

- $D_q$ has nonempty interior
- there exists $\mathcal{C}_q \subset D_q$ open and dense in $D_q$ such that $A(q') = \mathcal{C}_q$ for every $q' \in \mathcal{C}_q$
- there exist finitely many distinct ICS

- existence by Zorn lemma: $\overline{A(q')} \subset \overline{A(q)}$ if $q' \in \overline{A(q)}$
- nonempty interior by Krener theorem ($D_q = \overline{A(q')} \cap \overline{A(q)}$ for $q' \in D_q$)
Invariant control sets for projected linear switched systems and periodization
Some link between linear switched systems and invariant control sets

Interesting properties on the behavior of a linear switched system can be deduced from its angular component:

\[ x(t) \rightarrow (\|x(t)\|, [x(t)]) =: (r(t), s(t)) \in (0, +\infty) \times \mathbb{RP}^{d-1} \]

Using local identification \([x] = \frac{\dot{x}}{\|x\|}, \dot{x} = Ax\) can be rewritten as

\[ \frac{\dot{r}}{r} = \langle s, As \rangle, \quad \dot{s} = (A - \langle s, As \rangle \text{Id}_d)s =: (\pi_*A)s \]

\((\pi \Sigma)\) projected linear system on \(\mathbb{RP}^{d-1}\) associated with \(F := \pi_*S\)
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\((\pi \Sigma)\) projected linear system on \(\mathbb{RP}^{d-1}\) associated with \(F := \pi_\ast S\)

- **[Arnold, Kliemann, Oeljeklaus, 1986]** → if \(F\) LARC on \(\mathbb{RP}^{d-1}\), then \((\pi \Sigma)\) has a unique ICS \(D\) and \(\text{int}(D) \neq \emptyset\)
- **[Colonius, Kliemann, 1993]** → if \(F\) LARC on \(\mathbb{RP}^{d-1}\), then \(\lambda_0(S)\) is equal to

\[ \lambda_0^{\text{per}}(S) := \sup \left\{ \limsup_{t \to +\infty} \frac{\log(\|\Phi_A(t)x_0\|)}{t} \mid A \in S_0, (A(\cdot), \pi\Phi_A(\cdot)x_0) \text{ periodic} \right\} \]

- For PDMP ICSs characterize support of invariant measures
  **[Benaïm, Colonius, Lettau, 2017]**
Interest of periodization

The identity $\lambda_0(S) = \lambda_0^{\text{per}}(S)$:

- provides a monotone finite horizon approximation scheme
- proves the Gelfand-like formula

$$\lambda_0(S) = \limsup_{t \to +\infty} \sup_{A \in S, \ x_0 \neq 0} \frac{\log(\rho(\Phi_A(t)))}{t}$$

with $\rho$ spectral radius

- can be used to show continuity of $S \mapsto \lambda_0(S)$
- first introduced to bound large deviations for Piecewise Deterministic Markov Processes [Arnold, Kliemann, 1987]
Let $x(t) = \Phi_A(t)x_0$ be (quasi-)maximizing for $\lambda_0(S)$

In order to prove that $\lambda_0^{\text{per}}(S) \geq \lambda_0(S) - \varepsilon$ we should be able to close the loop and, for $t$ large, use $(\pi \Sigma)$ to go from $[x(t)]$ to $[x_0]$
Periodization (proof by Colonius and Kliemann)

\[ \lambda_0(S) = \sup \left\{ \limsup_{t \to +\infty} \frac{\log(\|\Phi_A(t)x_0\|)}{t} \mid A \in S_0, x_0 \neq 0 \right\} \]

\[ \lambda_{0}^{\text{per}}(S) = \sup \left\{ \limsup_{t \to +\infty} \frac{\log(\|\Phi_A(t)x_0\|)}{t} \mid A \in S_0, (A(\cdot), \pi \Phi_A(\cdot)x_0) \text{ periodic} \right\} \]

Let \( x(t) = \Phi_A(t)x_0 \) be (quasi-)maximizing for \( \lambda_0(S) \)

In order to prove that \( \lambda_{0}^{\text{per}}(S) \geq \lambda_0(S) - \varepsilon \) we should be able to close the loop and, for \( t \) large, use \((\pi \Sigma)\) to go from \([x(t)]\) to \([x_0]\)

- **Step 1:** Choose \( x_0 \) appropriately. Take \( D \) the unique ICS for \((\pi \Sigma)\), fix \( v_1, \ldots, v_d \) linearly independent in \( \text{int} D \). Since 
  \[ \| M \| = \max_{i=1}^{d} \| Mv_i \| \] is a norm on \( M_d(\mathbb{R}) \), we can take as \( x_0 \) one of the \( v_i \)

- **Step 2:** guarantee that there exist a uniform controllability time \( T \) for driving \((\pi \Sigma)\) from any point in \( D \) to any of the \( v_i \) within time \( T \)
Dwell-time invariant control sets for general nonlinear systems
Goal: extend control sets analysis to the dwell-time case

- The definition of invariant control sets does not suit the dwell-time case (invariance fails to see dwell-time)
- Mathematically, the difficulty come from non-concatenability of the class of admissible signals
- Equivalently, the family of admissible flows is not a semigroup
- Idea: recover main geometric properties by looking not at attainable sets (built with entire trajectories issuing from a point) but only at points which are attainable in a *concatenable* manner

Dwell-time attainable set: \( A_\tau(q) = \{ \phi(T, q, f) | f|_{[0,T]} \in \mathcal{F}_\tau \} \)

with

\[ \mathcal{F}_\tau = \{ f_1 \ast \cdots \ast f_m | m \in \mathbb{N}, f_i \text{ constant on a interval of length } \geq \tau \} \]

Note: \( \mathcal{F}_\tau \) not shift invariant!
Semigroups of concatenable flows

\[ \mathcal{S}_\tau = \{ \phi(T, \cdot, f) \mid f|_{[0,T]} \in \mathcal{F}_\tau \} \]

Then \( A_\tau(q) = \mathcal{S}_\tau(q) \).

**Definition**

*\( D \) is a dwell-time invariant control set (\( \tau \)-ICS) if \( D = \overline{A_\tau(q)} \) for every \( q \in D \).*

**Remark**

[San Martin, 1993] already studied control sets for orbits of not necessarily connected semigroups, in a setting which does not directly applies here (semigroup with nonempty interior in a Lie group \( G \) and action on some \( X/G \)).
Basic properties of dwell-time attainable and control sets

**Theorem**

Let $M$ be compact, $\tau \geq 0$. For each $q \in M$ there exists a $\tau$-ICS $D_q$ contained in $\overline{A_\tau(q)}$. If, moreover, $F$ has the LARC, then $\text{int} D_q \neq \emptyset$

Remark: if there exists $\bar{q}$ such that $\bar{q} \in D_q$ for every $q \in M$, then there exists a unique $\tau$-ICS ($= \overline{A_\tau(\bar{q})}$)

**Lemma**

Let $F$ satisfy LARC and assume that $D \subset M$ is a $\tau$-ICS. Then

(i) $\overline{\text{int}(D)} = D$

(ii) $\Phi(\text{int}(D)) \subset \text{int}(D)$ for every $\Phi \in \mathcal{S}_\tau$

(iii) There exists an open and dense subset $\mathcal{C}$ of $D$ such that $\mathcal{C} = \mathcal{S}_\tau(q)$ for all $q \in \mathcal{C}$
Dwell-time invariant control sets and linear switched systems with dwell-time
Example of dwell-time control set for projected linear switched system

\[ f_1, f_2 \text{ vector fields on } \mathbb{RP}^1, \text{ conjugate to } \dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x \]

\[ A' := e^{\tau f_1}(B), \quad B' := e^{\tau f_2}(A) \]

- \[ D = \hat{AA'} \cup \hat{B'B} \text{ unique } \tau\text{-ICS} \]
- if \( \tau_* \) is such that \( e^{\tau_* f_1}(B) = e^{\tau_* f_2}(A) \) then \( \tau \leq \tau_* \rightarrow D = \hat{AB}; \quad \tau > \tau_* \rightarrow D \text{ disconnected} \)
- \( C = \text{int}(\hat{AA'}) \cup \text{int}(\hat{B'B}) \) (\( \neq \text{int}D \) for \( \tau = \tau_* \))
Theorem (F. Boarotto, M.S., JDE, to appear)

Let $S \subset M_d(\mathbb{R})$ and $\tau \geq 0$. Then $\lambda_\tau(S) = \lambda_\tau^{\text{per}}(S)$

Idea: restrict the projected system to some orbit for the family $\pi_* S$

$O([x_0]) = \{ [x(t)] \mid \dot{x} = A(t)x, \, x(0) = x_0, \, A(t) \in S \cup -S \}$

$(M, F) = (O([x_0]), \pi_* S)$

Advantages

- $O([x_0])$ has the structure of smooth manifold (Orbit theorem)
- LARC of the system restricted to $M$ is for free
Periodization without LARC condition

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**Advantages**

- $O([x_0])$ has the structure of smooth manifold (Orbit theorem)
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**Difficulties**

- we should guarantee that the orbit carries all informations about asymptotic behavior $\rightarrow$ reduction to irreducible case
- existence of $\tau$-ICS requires compactness of orbits
Existence of a closed orbit

Theorem

Let $B$ be the group generated by $\{e^{tA_j} \mid t \in \mathbb{R}, j = 1, \ldots, m\}$ (any connected Lie subgroup of $GL(\mathbb{R}, d)$). Then the action

$$\varphi : B \times \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}, \quad \varphi(b, x) = \frac{bx}{\|bx\|},$$

induced by $B$ on the $(d - 1)$-dimensional unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ admits at least one closed orbit in $\mathbb{S}^{d-1}$ (and the same is true for $\mathbb{RP}^{d-1}$)

Existence (and even uniqueness) of $\tau$-ICS is obtained and Colonius–Kliemann’s periodization argument can be performed, proving $\lambda_\tau(S) = \lambda_\tau^{\text{per}}(S)$
We proved that the maximal Lyapunov exponent of linear switched systems with dwell-time can be characterized using only trajectories with periodic angular component (new also in the case $\tau = 0$ when the LARC does not hold).
Conclusions and perspectives

- We proved that the maximal Lyapunov exponent of linear switched systems with dwell-time can be characterized using only trajectories with periodic angular component (new also in the case $\tau = 0$ when the LARC does not hold).

- This gives an alternative prove of the Gelfand formula and of the continuity of maximal Lyapunov exponent with respect to the family of matrices and the dwell-time [Wirth, 2005].
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- Existence of a compact orbit for a projected linear system could be useful for other control problems.
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- this gives an alternative prove of the Gelfand formula and of the continuity of maximal Lyapunov exponent with respect to the family of matrices and the dwell-time \cite{Wirth, 2005}
- existence of a compact orbit for a projected linear system could be useful for other control problems
- $\tau$-ICS to characterizes support of the invariant measure for piecewise deterministic random process with dwell time
Conclusions and perspectives

- We proved that the maximal Lyapunov exponent of linear switched systems with dwell-time can be characterized using only trajectories with periodic angular component (new also in the case \( \tau = 0 \) when the LARC does not hold)
- This gives an alternative prove of the Gelfand formula and of the continuity of maximal Lyapunov exponent with respect to the family of matrices and the dwell-time \([\text{Wirth, 2005}]\)
- Existence of a compact orbit for a projected linear system could be useful for other control problems
- \( \tau \)-ICS to characterizes support of the invariant measure for piecewise deterministic random process with dwell time
- **Ongoing work:** adapt our technique to a more abstract setting applying to other non-concatenable classes of switching signals