Hole probability for noninteracting fermions in a d-dimensional trap

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Abstract – The hole probability, i.e., the probability that a region is void of particles, is a benchmark of correlations in many-body systems. We compute analytically this probability \(P(R)\) for a sphere of radius \(R\) in the case of \(N\) noninteracting fermions in their ground state in a \(d\)-dimensional trapping potential. Using a connection to the Laguerre-Wishart ensembles of random matrices, we show that, for large \(N\) and in the bulk of the Fermi gas, \(P(R)\) is described by a universal scaling function of \(k_F R\), for which we obtain an exact formula (\(k_F\) being the local Fermi wave vector). It exhibits a super-exponential tail \(P(R) \propto e^{-\kappa_d (k_F R)^{d+1}}\) where \(\kappa_d\) is a universal amplitude, in good agreement with existing numerical simulations. When \(R\) is of the order of the radius of the Fermi gas, the hole probability is described by a large deviation form which is not universal and which we compute exactly for the harmonic potential. Similar results also hold in momentum space.

Since the seminal works of Wigner and Dyson [1,2], the study of the correlations of the eigenvalues of random matrices has played a major role in characterizing the statistics of random collections of points, called “point processes”, beyond the well-known (uncorrelated) Poisson statistics [3]. Besides the original applications to the energy level statistics of heavy nuclei [2], more recent applications include low-dimensional chaotic systems [4,5], mesoscopic disordered conductors [6] or localization/delocalization transitions in disordered quantum systems [7–10]. The eigenvalues of random matrices in the Wigner-Dyson class exhibit level repulsion and spectral rigidity. A benchmark to quantify this effect is the distribution of the spacing \(s\) between two consecutive eigenvalues, approximately given by the famous Wigner surmise, \(i.e., p(s) \sim s^2e^{-cs^2}\) for Hermitian random matrices from the Gaussian unitary ensemble (GUE) [2,11]. This spacing distribution can be computed from the hole probability, \(i.e., the probability that a given interval contains no eigenvalue and it has been calculated for various random matrix ensembles [11–18].

Another interesting example of a random point process is the set of positions of noninteracting fermions in their ground state in a confining potential. These can be measured in quantum microscopes in cold atoms experiments with traps of tuneable shapes [19–24]. In this case, the randomness originates from quantum fluctuations and the Pauli principle, which leads to a Slater determinant form for the many-body wave function. It turns out that in one dimension, \(d = 1\), this point process can be mapped, in some cases, onto the statistics of the eigenvalues of random matrices. Indeed, both are examples of determinantal point processes (DPPs) on the line [25,26]. DPPs are defined by the property that their many-body correlations are given by determinants built from a central object called the kernel [27–29]. In \(d = 1\) the hole probability for fermions can thus often be obtained from random matrix theory (RMT).

Recently the connections to DPP have been much exploited to describe Fermi gases in trapping potentials in any dimension [30–32]. In higher dimension there are some results for the hole probability in two specific examples in \(d = 2\): i) in mathematics, for the zeroes of random series in the complex plane [33–35], ii) in the Ginibre ensemble of random matrices, where there are exact formulae for the probability that there is no eigenvalue inside a disk [36–40]. The latter result can be transposed in terms of the hole probability for noninteracting fermions in a
harmonic trap rotating at a critical frequency such that the problem can be mapped onto the lowest Landau levels of a quantum Hall system [41,42].

Besides these two examples, obtaining the hole probability for $d > 1$ for a general model of trapped noninteracting fermions remains an outstanding question. In the context of hyperuniform systems [43,44], this observable was studied numerically and using dimensional arguments [45,46] in the case of free fermions, i.e., in the absence of a trapping potential. However there is presently no analytical calculation of this quantity.

In this letter, we consider noninteracting fermions in $d$ dimensions described by the single particle Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + V(r),$$

where $V(r)$ is a central external potential, $r = |\mathbf{x}|$, $\mathbf{p}$ is the momentum in $d$ dimensions, and here we work in units such that $m = \hbar = 1$. We first obtain, in the case of free fermions (i.e., $V(r) = 0$), an exact formula for the probability $P(R) = P_d(k_F R)$ that a spherical domain of radius $R$ contains no fermion (in their ground state). Here $k_F = \sqrt{2\mu}$ is the Fermi wave vector and $\mu$ is the Fermi energy. The scaling function $P_d(z)$ can be expressed as a product of Fredholm determinants associated to the so-called hard edge Bessel kernel, well known in RMT, see eqs. (11), (13), (14) below. Its asymptotic behavior at small distance is given as $z \to 0$ by

$$P_d(z) = 1 - B_d z^d + \frac{d}{(d+2)^2} B_d^2 z^{2d+2} + O(z^{3d+2}, z^{2d+4}),$$

with $B_d = \frac{1}{2\Gamma(1+d/2)^2}$ (see the Supplementary Material Supplementary material.pdf (SM)¹ for higher orders), which generalizes the result for $d = 1$ [12–14,31] in which case the level spacing distribution is given by $p(s) \propto P_1'(s)$ (see, e.g., chapter 8 of [11]).

At large distance one finds that the hole probability decays super-exponentially,

$$P_d(z) \sim \exp(-\kappa_d z^{d+1}), \quad \kappa_d = \frac{2}{(d+1)^2\Gamma(d+1)},$$

This result agrees for $d \to 1$ with $\kappa_1 = \frac{1}{2}$ obtained in [12].

The plot of $\frac{d}{2}P_d(z)$ is shown in fig. 1 and represents the scaled probability density of the position of the fermion closest to the origin. Note that for $d = 2$ the power of the exponential is cubic $z^3$ here, which is at variance with both the fermion models related to the Ginibre ensemble, and the random series, for both of which it is $z^4$ [33,35,37,39, 40]. We can also compare these results with the numerical data analysis of [45,46] for free fermions. In that work the power law $z^{d+1}$ in the exponential was conjectured to hold in all dimensions, and the coefficient $\kappa_d$ was measured numerically. The comparison with our analytic prediction is presented in table 1. Although the agreement is quite good the exact values lie somewhat outside of the error bars, which suggests that obtaining numerically the true asymptotics requires larger values of $z$.

Next we consider the hole probability for $N$ noninteracting fermions in their ground state and in the presence of a smooth confining central potential $V(r)$. We consider here the limit of large $N$, which corresponds to large Fermi energy $\mu$. In the typical case, for instance for a harmonic potential $V(r) = \frac{1}{2}r^2$, the fermion density, $\rho(r)$, has a bounded support $r < R_e$ and vanishes beyond the edge at $r = R_e$. For large $N$ it is given by the local density approximation (LDA) (or semi-classical) expression $\rho(r) = c_d k_F r^d$, with $k_F = \sqrt{2\mu}$ the Fermi wave vector and $c_d$ a constant given below. We consider a sphere of radius $R$ around the origin.

We find that there are two regimes for the hole probability $P(R)$ depending on whether $R$ is of microscopic size $R = O(1/k_F(0))$ (typical interparticle distance), or $R$ is macroscopic, i.e., $O(R_e)$.

In the first regime (microscopic scales) we find that the hole probability takes the scaling form $P(R) \sim P_d(k_F(0)R)$, where $P_d(z)$ is the same universal scaling function as obtained above for free fermions, with a nonuniversal scale $1/k_F(0)$. This universality extends to any microscopic sphere located anywhere inside the bulk in the presence of a general smooth potential².

In the case of a hole of macroscopic size, $R = O(R_e)$ the probability $P(R)$ is very small and is characterized by a large deviation form

$$P(R) \sim \exp\left( - (k_F(0)R_e)^{d+1} \Psi(\tilde{R} = R/R_e) \right),$$

¹The SM also cites refs. [47–53].

²For a sphere centered around $x$, $k_F(0)$ is replaced by $k_F(r = |x|)$.  

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where the rate function $\Psi(\tilde{R})$ is not universal and depends on some details of the potential $V(r)$. Here we calculate it explicitly in the case of the potential $V(r) = \frac{1}{r^2}$, which case $R_c = \sqrt{2}\mu = k_F(0)$. The function $\Psi(\tilde{R})$ is given in (20) and plotted in the SM. It is related to the large deviations in the Wishart-Laguerre ensemble of random matrices [54]. Its behavior at small argument is found to be $\Psi(\tilde{R}) \sim \kappa_d \tilde{R}^{d-1}$ which matches smoothly with the large distance behavior from microscopic scales, see eq. (3). For large $\tilde{R}$ it behaves as $\Psi(\tilde{R}) \sim \tilde{R}^2$ and in $d = 1$ it is exactly $\Psi(\tilde{R}) = \frac{1}{2} \tilde{R}^2$, as found for GUE matrices (see the SM and [16]).

Let us consider $N$ spinless noninteracting fermions in a central potential $V(r)$ in space dimension $d$. The ground state is obtained as a Slater determinant where all the eigenstates of the single particle Hamiltonian $\hat{H}$ in (1) are occupied up to the Fermi energy $\mu$. We use the spherical coordinates $x = (r, \theta)$ where $\theta$ is a $(d-1)$-dimensional angular vector. The Hamiltonian $\hat{H}$ can be written as $\hat{H} = -\frac{1}{r^{d-1}} \partial_r (r^{d-1} \partial_r) + \frac{\mu}{r^2} + V(r)$, and commutes with the angular momentum $L$. The eigenfunctions of $H$ thus take the form $\psi_n L(r, \theta) = r^{\frac{d-1}{2}} \chi_n, L(r) Y_L(\theta)$ where the $d$-dimensional spherical harmonics $Y_L(\theta)$, labeled by the set of angular quantum numbers $L$, are eigenfunctions of $L^2$ with eigenvalues $\ell(\ell + d - 2)$, $\ell = 0, 1, \ldots$, which defines the angular sector. The radial parts $\chi_n, L(r)$ are the eigenfunctions of a collection of 1D radial Hamiltonians $\hat{H}_\ell = -\frac{1}{2} \partial_r^2 + V(r)$, $r \geq 0$, with potentials (see, e.g., [55])

$$V_\ell(r) = V(r) + \frac{a^2 - \frac{1}{2}}{2r^2}, \quad a = \ell + \frac{d - 1}{2},$$

with eigenenergies $\epsilon_{n, \ell}$, $n = 0, 1, \ldots$, each with degeneracy $g_\ell(\ell) = \frac{2n + \ell + d - 2}{\ell + 1}$, $\ell \geq 1$ and $g_\ell(0) = 1$. In the ground state of the $N$ fermions, for each angular sector the lowest $m_\ell$ energy levels are occupied, i.e., $n = 0, \ldots, m_\ell - 1$, such that $\epsilon_{n, \ell} \leq \mu$ and $N = \sum_{\ell} g_\ell(\ell) m_\ell$.

Let us now focus on the example of the harmonic oscillator, $V(r) = \frac{1}{2} r^2$. In that case the eigenfunctions of (5) can be computed exactly, and are given by Laguerre polynomials $\chi_{n, \ell}(r) \propto r^{\frac{\ell}{2}} L_n^{\frac{d-2}{2}}(r^2) e^{-r^2/2}$ with eigenenergies $\epsilon_{n, \ell} = 2n + a + 1$, where $a = \ell + \frac{d - 1}{2} - 1$. The number of occupied states in the ground state within the $\ell$ sector is thus $m_\ell = \text{Int}(\frac{d + a}{2}) + 1$, where $\text{Int}(z)$ denotes the integer part of $z$ (i.e., the floor function). Note that $m_\ell = 0$ for $\ell > \ell_{\text{max}}(\mu) = \frac{d + \frac{1}{2}}{2} - \frac{d - 1}{2}$ (where $\mu$ is integer for even $d$ and half-integer for odd $d$). The ground state wave function is given by $\Psi_0(x_1, \ldots, x_N) = \frac{1}{\sqrt{\text{det}(1_{i,j} \leq N) [\epsilon_k(x_i)]}}$, where $\epsilon_k = (n_k, L_k)$ labels the single particle eigenfunction of the occupied eigenstates. We assume here that the ground state is nondegenerate (i.e., the last level is fully occupied, see discussion in [56]).

We now compute the hole probability $P(R)$ as the probability that there is no fermion in the sphere of radius $R$ centered on the origin. It is given by

$$P(R) = \frac{N}{2\pi} \int_{|x_i| > R} d^d x_i \left| \Psi_0(x_1, \ldots, x_N) \right|^2.$$  

Using the Andreev-Cauchy-Binet formula (see, e.g., [56, 57]) it can be written as a determinant

$$P(R) = \det_{1 \leq i,j \leq N} \left[ \delta_{ij} - \mathbb{A}_{ij} \right]$$

in terms of the overlap matrix $\mathbb{A}_{ij} = \int_{r = |x| \leq R} d^d x \psi^*_n L_i(r) \psi_n L_j(r)$. Using the orthogonality of the spherical harmonics, the angular integration gives $\mathbb{A}_{ij} = \delta_{i,j} \lambda_n L_i \lambda_n L_j$, with $\lambda_n L_i = \int_0^R d r \chi_n, L_i(r) \chi_n, L_j(r)$. Hence the matrix $\mathbb{A}$ is diagonal in the variables $L_i$, and the determinant factorizes over the angular sectors [56]

$$P(R) = \prod_{\ell = 0}^{\ell_{\text{max}}(\mu)} P_\ell(R)^{g_\ell(\ell)},$$

where $P_\ell(R)$ is the probability that the interval $[0, R]$ is empty in the ground state of $m_\ell$ noninteracting fermions described by the single particle Hamiltonian $H_\ell$. Note that formulae (7) and (8) are valid for any central potential $V(r)$.

We start by studying the 1D radial problem to obtain $P_\ell(R)$ within each $\ell$ sector, and then we evaluate the product (8). For a general potential $V(r)$ this is a difficult problem, however in the case of the harmonic oscillator we can make further progress by using a connection to the complex Wishart-Laguerre (WL) ensemble of random matrix theory [11,58,59]. It is defined by the following joint probability distribution function (PDF) for a set of $m$ eigenvalues $\lambda_i$:

$$p_\text{WL}(m)(\lambda) \propto e^{-\sum_{i=1}^{m} \lambda_i} \prod_{i=1}^{m} \lambda_i^{\nu} \prod_{1 \leq j,k \leq m} \left( \lambda_j - \lambda_k \right)^2,$$  

which depends on the continuous parameter $\nu > -1$. In the case where $\nu$ is a positive integer, it describes the eigenvalues of complex Wishart random matrices. They are of

Table 1: Comparison between our exact result (3) for $\kappa_d$ (last column) and the numerical estimates of ref. [46].

| Dimension $d$ | Numerics [46] | Exact result |
|---------------|---------------|--------------|
| $d = 1$       | 0.5           | $\frac{1}{2}$ |
| $d = 2$       | 0.1175 ± 0.0007 | $\frac{1}{9} = 0.1111$ |
| $d = 3$       | 0.02287 ± 0.0003 | $\frac{1}{48} = 0.02083$ |
| $d = 4$       | 0.00392 ± 0.00015 | $\frac{1}{300} = 0.00333$ |
Indeed, the centrifugal energy \( a \) can approximate \( m \ell \) sectors with \( \nu \) eigenvalues of the WL ensemble of parameter \( W \) the form of process order \( \nu = a = \ell + d/2 - 1 \), described by the kernel

\[
K^B_{\nu}(b, b') = \frac{1}{4} \int_{0}^{1} dJ_{\nu}(\sqrt{z}b)J_{\nu}(\sqrt{z}b'),
\]

where \( J_{\nu}(z) \) is the Bessel function of index \( \nu \). Using standard results for DPP the hole probability in the \( \ell \) sector is given as a Fredholm determinant [28,29],

\[
F_{\nu}(b) := \text{Prob}(b_{\text{min}} > b) = \text{Det}(I - P_{[0,b]}K^B_{\nu}),
\]

where \( b_{\text{min}} = \min b, I \) and \( P_{[0,b]} \) is the projector on the interval \([0,b] \). This Fredholm determinant (FD) can be expressed as log \( F_{\nu}(b) = -\int_{0}^{b} d\alpha \frac{1}{\alpha} \) from the solution \( \sigma(s) \) of the Painlevé III equation [62],

\[
(s\sigma'')^2 + \sigma'(s - s\sigma')(4s' - 1) - \nu^2(s')^2 = 0,
\]

where \( \sigma(s) \approx \frac{\mu^{\nu+1} + \nu^{\nu+1}}{\mu^{\nu+1} + \nu^{\nu+1}} \) at small \( s \). For even space dimension \( d \), \( \nu \), integer \( \nu \), as discussed above, there are other representations for the hole probability, which in this microscopic regime lead to the remarkably simple formula [61] (see also refs. [67,68]),

\[
F_{\nu}(b) = e^{-b^4/4} \text{det}_{1 \leq i, k \leq \nu} I_{j-k}(\sqrt{b}),
\]

where \( I_{n}(x) \) is the modified Bessel function. Using formula (8), we obtain that in this scaling regime, the hole probability \( P(R) \) for the fermions in any dimension \( d \) takes the scaling form \( P(R) \approx P_d(kR(0)R) \), where the scaling function is given as an infinite product,

\[
P_d(z) = \prod_{\ell=0}^{+\infty} F_{\ell+\frac{d}{2}-1}(z^2)^{\nu_{\ell+\frac{d}{2}}}. 
\]

This result, which we derived for the harmonic oscillator (with \( kR(0) = \sqrt{2} \nu \)) holds asymptotically for large \( N \) for any smooth trapping potential. In addition, it is exact for free fermions in \( d \) dimensions, with \( kR = \sqrt{2} \nu \). Note that for free fermions, since their positions form a \( d \)-dimensional determinantal point process with a known kernel \( K_d(r, r') = \frac{\delta_{d/2}(|r-r'|)}{(2\pi)^{d/2}|r-r'|^{d/2}} \) (i.e., the \( d \)-dimensional extension of the sine-kernel [31,45]), there exists an alternative formula for the hole probability in terms of another Fredholm determinant,

\[
P_d(z) = \text{Det}(I - P_zK_d),
\]

where \( P_z \) is the projector on \( |r| < z \). The formulae (14) and (15) are in fact equivalent (which is not a trivial property). Both formulæ can be expanded at small \( z \), leading to (2) and pushed to higher orders in the SM. While the expansion of (15) is straightforward, expanding (14) requires to solve the Painlevé III equation at small arguments. The formula (14) however allows to study the asymptotic behavior of \( P_d(z) \) at large \( z \), as we now show.
Hole probability for noninteracting fermions in a $d$-dimensional trap

In the large $z$ limit one needs the asymptotics of $F_v(b)$ at large $b$. One can check (see the SM) that the asymptotics of the infinite product in (14) is dominated by large values of $\ell$, for which the decay of $F_v(b)$ occurs on scale $b \sim \ell^2$. This double limit for (13) was studied, using Coulomb gas techniques, in the context of lattice QCD in [69,70] and later in the study of the longest increasing subsequence of random permutations [71] (see also [54]) and it was shown to take the scaling form (with $v \sim \ell$)

$$F_v(b) \sim \exp \left[ -\ell^2 \phi_+ \left( \gamma = \sqrt{\frac{b}{\ell}} \right) \right],$$

$$\phi_+(\gamma) = \theta(\gamma - 1) \left( \gamma^2 - \gamma + \frac{1}{2} \log \gamma + \frac{3}{4} \right),$$

where $\theta(x)$ is the Heaviside function. Inserting this expression into (14), approximating the sum over $\ell$ by an integral, using $g_d(\ell) \simeq \ell^{d-2} \exp[-\ell^2]$ at large $\ell$, one obtains

$$P_d(z) \sim \exp \left[ -\frac{2}{\Gamma(d-1)} \int_0^{+\infty} d\ell \ell^d \phi_+ \left( \frac{z}{\ell} \right) \right],$$

leading to our main result (3), with $\kappa_d = \frac{2}{\Gamma(d-1)} \frac{d}{d+1} \frac{d}{d+1} \phi_+ (\gamma =) \frac{2^2}{(d+1)^2}$. Note that the calculations of [69,71] use the formula (13) valid only for even $d$, however one can also obtain (16) in any $d$ using the Painlevé equation (see the SM).

**Macroscopic regime.** We recall that for the harmonic potential $V(r) = \frac{1}{2} \mu r^2$, in the large $N$ limit, the density has an edge at $r = R_e = \sqrt{2} \mu$. The macroscopic regime corresponds to $R = O(R_e) = O(\sqrt{N})$. Since $\mu$ is large, in each angular sector the number of fermions is again $n_\ell \simeq (\mu - \ell)/2$, however in this regime the product in (8) is controlled by the values of $\ell = O(\mu)$. In the language of the WL ensemble (9) this corresponds to the limit of large matrix size $m = m_\ell \to +\infty$, and large index $n = a = \ell + \frac{d}{2} - 1 \to +\infty$, with $\alpha = \frac{d}{2}$ fixed of order unity. Using the correspondence with fermions discussed above, $\lambda_\ell = r_\ell^2$, one has $\alpha = \frac{d}{2} \simeq \frac{m_0}{m} = O(1)$. It is known that in this regime the spectrum of the WL matrices has support on the interval $[\lambda_-, \lambda_+]$ with $\lambda_\ell = m_\ell \zeta_\ell$ and $\zeta_\ell = (1 \pm \sqrt{1 + \alpha^2})^2$. The correspondence with fermions shows that within each $\ell$ sector, the mean fermion density $\rho(r)$ has support $[r_-, r_+](\ell)$ with

$$r_\pm(\ell)^2 = \lambda_\pm = m_\ell \zeta_\pm \simeq \mu \pm \sqrt{\mu^2 - \ell^2},$$

which coincides with the result obtained using the LDA, i.e., $\rho(r) = \frac{1}{\sqrt{2} (\mu - V(r)) + }$, where $V(r)$ is given in (5).

For each sector $\ell$, there are three priori different scaling regimes for $P_d(R) = \text{Prob}(\lambda_{\text{min}} > R^2)$ when $R = O(\sqrt{N})$ and $\ell = O(\mu)$. Indeed, it is known that the smallest eigenvalue $\lambda_{\text{min}}$ of a WL random matrix (9) exhibits three regimes [72]:

i) A typical fluctuation regime around the lower edge for $\lambda_{\text{min}} - \lambda_- = O(m^{1/3})$, described by the “soft-edge” Tracy-Widom distribution $F_2$. This regime will not play a role here.

ii) A “pulled” large deviation regime to the left of $\lambda_-$, i.e., $\lambda_{\text{min}} - \lambda_- = O(m) < 0$, which in terms of the fermions reads [72] $P_d(R) \sim 1 - e^{-2m u_+ \left( \frac{m^{1/2} - R^2}{m} \right)}$.

iii) A “pushed” large deviation regime to the right of $\lambda_-$, i.e., $\lambda_{\text{min}} - \lambda_- = O(m^2) > 0$, which in terms of the fermions reads

$$P_d(R) \sim e^{-2m u_+ \left( \frac{m^{1/2} + R^2}{m} \right)},$$

The rate functions $\Phi_\pm(z, \alpha)$ were computed in ref. [54] using Coulomb gas methods and are recalled in the SM.

Consider now the expression for the logarithm of the hole probability $\log P(R)$ expressed as a sum over $\ell$ from (8). As $\ell$ increases in $[0, \ell_{\text{max}}(\mu) \simeq \mu]$, the edge $r_-(\ell)$ in (18) increases from 0 to $\sqrt{\mu}$. There are two cases. For $R^2 > \mu$ one has $R > r_-(\ell)$ for all $\ell < \mu$, and only the regime iii) applies. For $R^2 < \mu$, a priori the three regimes apply, i.e., regime ii) for the sectors with $\ell > \ell_{\text{max}}(R) = 2R\sqrt{2\mu - R^2}$ and regime iii) for the sectors $\ell < \ell_{\text{max}}(R)$. However, one sees that regime ii) contributes only to exponentially small corrections. Hence the leading contributions come from regime iii) in (19) in all cases. In computing the logarithm of (8) we can approximate the sum over $\ell$ by an integral and $g_d(\ell) \simeq \frac{2^d}{(d-1)!}$. Performing the change of variable $v = \ell/\mu$ we obtain the large deviation formula for the hole probability in the form (4) with $k_F(0) R_e = 2\mu$ and $\bar{R} = R/\sqrt{2\mu}$ with the rate function

$$\Psi(\bar{R}) = \int_0^{v_{\text{max}}(\bar{R})} \frac{dv}{2^{d+1} \Gamma(d-1)} \left( v^d - 3(v^d - 1) \right) \times \Phi_+ \left( \frac{4\bar{R}^2}{1 - v} - 1 - \sqrt{1 + \alpha v^2} \right),$$

where $\alpha = \frac{2\sqrt{2}}{\sqrt{m}}$ and $v_{\text{max}}(\bar{R}) = 2\bar{R}\sqrt{1 - \bar{R}^2}$ for $\bar{R}^2 < 1/2$ and $v_{\text{max}}(\bar{R}) = 1$ for $\bar{R}^2 > 1/2$. The function $\Phi_+(z, \alpha)$ being quite complicated, the integral in (20) has been evaluated numerically in the SM. It exhibits a transition of high order at $\bar{R} = 1/\sqrt{2}$ (see footnote 3) and the asymptotics of $\Psi(\bar{R})$ can be extracted (see the SM).

In conclusion, we have computed analytically the hole probability for noninteracting fermions in a $d$-dimensional central trapping potential. We have obtained an exact formula for the universal scaling function $P_d(z)$ which describes holes of size of the order of the interparticle distance, in an arbitrary smooth potential. It characterizes the rigidity of the Fermi gas, a generalization of level repulsion in random matrix theory. In addition we have obtained, for the harmonic oscillator the full large deviation function for macroscopic holes. Interestingly, our results also apply to the hole probability in momentum space (see

3A detailed calculation shows that the transition is of sixth order, i.e., the sixth derivative is discontinuous.
the SM) which can be measured from time of flight experiments [23]. The method introduced here could allow to predict a larger variety of probes of fermion correlations in traps [24]. It is also possible to incorporate finite temperature effects [31], and we hope that our results can be compared with cold atom experiments [24]. It would be interesting to also study the hole probability for interacting systems for which very few results exist [73–76].

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