AN EXPLICIT FORMULA FOR THE GENUS 3 AGM

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Abstract. Given a smooth non-hyperelliptic curve $C$ of genus 3 and a maximal isotropic subgroup (w.r.t. the Weil pairing) $L \subset \text{Jac}(C)[2]$, there exists a smooth curve $C'$ s.t. $\text{Jac}(C') = \text{Jac}(C)/L$. This construction is symmetric, i.e. if we start with $C'$ and the dual flag on it, we get $C$. A previous less explicit approach was taken by Donagi and Livné (see [DL]). The advantage of our construction is that it is explicit enough to describe the isomorphism $H^0(C, \Omega_C) \cong H^0(C', \Omega_{C'})$.

1. Introduction

1.1. Gauss’s arithmetic geometric mean (agM) can be viewed as an iterative process of dividing a given elliptic curve $E$, by an element $\alpha \in \text{Pic}(E)[2]$, yielding a new elliptic curve $E'$ (a full treatment is given in [Cox]). This process can be generalized to the following question:

1.2. Question. For what genera $g$ does there exist a process that, given a curve $C$ of genus $g$, and a maximal isotropic subgroup (with respect to the Weil pairing) $L \subset \text{Pic}(C)[2]$, gives a curve $C'$ s.t. $\text{Jac}(C') \cong \text{Jac}(C)/L$ ?

1.3. This question was answered affirmatively on the case $g = 2$ by Richelot [Ric] and Humbert [Hum] (for a classical treatment, see [BM], and for a more modern one, see [DL]). Donagi and Livné proved in [DL], that no such process can be given for $g > 3$, and presented such a process for $g = 3$. The Donagi-Livné construction has four “problems”:

- It is not symmetric.
- There is a 1-parameter choice for $f$, which is unsatisfactory.
- It is not easy to give coordinates to the spaces & functions involved.
- It is not obvious how one can Track the canonical class.

Our new agM construction avoids these problems. Thus, tracing our construction one can obtain integration identities (in the spirit of the original agM).

1.4. Notation. In what follows, we fix a ground field $F$ such that $\text{char} F \neq 2, 3$. Fix a smooth non-hyperelliptic curve $C$ of genus 3. Lastly fix a maximal...
isotropic flag (under the Weil pairing) in \( \text{Jac}(C)[2] \) denoted by
\[
\mathcal{L} = \{ L_1 \subset L_2 \subset L_3 \}.
\]
Denote by \( \alpha, \alpha' \) the unique non-zero elements in \( L_1, L_2/L_3 \) respectively.

1.5. We summarize the new agM construction in the diagram:

where:
- The zero divisor of the map \( f \) is \( (f)_0 = K_C + \alpha \)
- The zero divisor of the map \( f' \) is \( (f')_0 = K_{C'} + \alpha' \)
- The faces \( T, T' \) are trigonal. (see [Do] p. 74)
- The construction is symmetric.
- The face B is bigonal (see [Do] p. 68-69) with normalization.
- The map \( q_E : E \rightarrow \mathbb{P}^1 \) identifies two of the branch points of \( Y/E \) thus describing \( L_2 \) (see Theorem 2.11).

The paper is organized as follows: In section 2 we give the necessary technical background, recalling how the double covers \( Y \rightarrow E \) and \( Y' \rightarrow E' \) are explicitly determined by the curve \( C \) and the element \( \alpha \), and vice-versa. In section 3 we present the construction, prove the isomorphism:

\[
\text{Jac}(C)/L_3 \cong \text{Jac}(C'),
\]

and show that the isomorphism
\[
H^0(C, \Omega_C) \cong H^0(C', \Omega_{C'})
\]
is explicitly determined by its projectivization
\[
|K_C| \cong |K_{C'}|.
\]
In section 4 we construct an explicit isomorphism
\[
|K_C| \cong |K_Y|_{\text{odd}}
\]
(for the definition of \( |K_Y|_{\text{odd}} \) see Notation 4.22). In section 5 we show how to construct the double cover \( Y'/E' \) from the double cover \( Y/E \) and the non
zero element in $\alpha^\perp/L_2^\perp$, and construct an explicit isomorphism

$$|K_Y|_{\text{odd}} \cong |K_{Y'}|_{\text{odd}}.$$  

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2. Technical Background

We review below the background and necessary results from [Le]. A proof for all of the assertions in the rest of this section is given there.

2.1. Notation. Let $f : C \to \mathbb{P}^1$ be a function s.t. $(f)_0 = K + \alpha$. Denote by $Z, X$ the curves appearing in the reverse trigonal construction on $f$ (See the [Do] p. 74 and the diagram at 1.5).

2.2. Theorem. The following properties hold:

1. There is a natural isomorphism $Z \cong \Theta_C \cap (\Theta_C + \alpha)$.
2. There are 3 natural commuting involutions on $Z$:
   
   $i : d \mapsto d - \alpha$
   $j : d \mapsto K_C - d$
   $\sigma := i \circ j : d \mapsto K_C + \alpha - d$.

3. There is a natural isomorphism: $X \cong Z/\sigma$.
4. The fixed points of the involution $j$ are exactly the theta characteristics of $C$ in $Z$. Moreover:
   
   (a) They are all odd.
   (b) They are coupled by $i$ into pairs of the form $z, z + \alpha$.
   (c) There are 12 such points.

2.3. Notation. Denote

$$E := Z/(i,j), \quad F := Z/j, \quad Y := Z/i.$$  

2.4. Proposition. The genera of $F, Y, E$ are 1, 4, 1 respectively. The curves $E, F$ smooth.

2.5. Theorem. The trigonal construction gives a coarse moduli space isomorphism between the moduli of the following sets of data:

- Pairs $C, \alpha$ such that the linear system $|K_C + \alpha|$ has no base points, up to isomorphisms.
- Ramified double covers $Y/E$ of irreducible curves of genera 4, 1 respectively, such that the curve $E$ is smooth, the curve $Y$ has at most simple nodes, with a choice of an element $\beta \in \text{Pic}(E)/[2] - \{0\}$, up to isomorphisms.

This isomorphism is given by the following maps (which are inverses of one another):
The reverse trigonal construction on the map $f : C \to \mathbb{P}^1$ (the map $f$ is defined as in Notation 2.1) is the double cover $Z \to X$, which induces the double covers $Y \to E$ and $F \to E$.

Given the double covers $Y \to E, F \to E$ define $Z := Y \times_E F$, and define $X$ to be the third quotient of $Z$, and perform the trigonal construction on one of the rulings of $X$ (they are conjugate under the involution on $X$).

Theorem 2.5 can be viewed through the canonical embedding of the curve $C$ (if the curve $C$ is not hyperelliptic then the divisor $|K_C + \alpha|$ has no base points):

2.6. Theorem. The curve $Z$ can be naturally identified with the set of pairs of points $\{p_1, p_2\}$ on the curve $C$ such that $p_1 + p_2 \leq K_C + \alpha$. Under this identification, the following assertions hold:

1. The involution $\sigma$ takes each such pair to the residual pair of points in $K_C + \alpha$.
2. The involution $j$ takes each such pair to the residual pair in $C \cap \overline{p_1p_2}$. Hence:
3. The curve $F$ embeds naturally to $\mathbb{P}^2$ as the set of lines through each of these pairs.
4. The involution $i_F$ acts on the points of $F$ in the following way: Let $\{p_1, p_2\}$ be a point in $Z$, and let $\overline{p_1p_2}$ be the corresponding line in $F$. There is a unique pair of points $\{p_3, p_4\}$ s.t. $K_C + \alpha \sim p_1 + p_2 + p_3 + p_4$. We then have
$$i(\overline{p_1p_2}) = \overline{p_3p_4}$$
5. $E \hookrightarrow \mathbb{P}^2$ naturally as the Hessian construction (see [Sa] chapter V section V) of $i_F$ and the involution induced on the curve $F$ by the involution $i$, (as the intersection point of each of the pairs of lines $\{l, i(l)\}$).
6. There are 6 pairs of bitangents $\{l_{i,1}, l_{i,2}\}_{1 \leq i \leq 6}$ such that $\frac{1}{3}(l_{i,1} - l_{i,2}) = \alpha$. These bitangents match (under the canonical embedding) the 12 effective theta characteristics in $Z$. (see Lemma 2.2). They are the ramification points of $Z \to F$.
7. The 6 intersection points of the pairs $\{l_{i,1}, l_{i,2}\}$ (recall that $l_{i,2} = i(l_{i,1})$) sit on a smooth conic $Q \subset \mathbb{P}^2$ (note that these points are naturally identified with the $q_i$s).

2.7. Notation. Choose homogeneous coordinates $x_0, x_1, x_2$ on $\mathbb{P}^2$. Denote by $f_E, f_Q \in \mathbb{F}[x_0; x_1; x_2]$ the homogeneous functions in $x_0, x_1, x_2$ of degrees 3, 2 respectively such that
$$\text{Nulls}(f_E) = E, \quad \text{Nulls}(f_Q) = Q.$$
Denote by $Q_2, Q_3$ respectively the nulls of $f_Q + x_3^2, f_E$ in $\mathbb{P}^3$ (constructed as a cone over $\mathbb{P}^2$). By abuse of notation we denote the nulls of $Q_2, Q_3$ by $Q_2, Q_3$ (if there is no danger of confusion).
2.8. **Theorem.** The curve $Y$ is canonically embedded in $\mathbb{P}^3$ as the complete intersection $Q_2 \cap Q_3$.

2.9. **Theorem.** The trigonal construction and the norm map $\text{Jac} \mathbb{Z} \rightarrow \text{Jac}(\mathbb{Z}/i)$ induce an isogeny $\text{Jac}(C) \rightarrow \text{Prym}((\mathbb{Z}/i)/E)$. The kernel of this isogeny is $\alpha^\perp$.

2.10. **Notation.** Denote by $q_1, \ldots, q_6$ the points of $\Sigma/i$ (The $q_i$’s are naturally identified with the 6 ramification points of the double cover $Y \rightarrow E$).

2.11. **Theorem.** Considering isotropic (under the Weil pairing) subgroups of $\text{Jac}(C)[2]$, there are bijections between the following objects:

- Isotropic groups $L_2 \supset \alpha$ of order 4, and pairs of $q_i$’s.
- Lagrangians $L_3 \supset \alpha$ and partitions of the $q_i$’s to 3 pairs.
- Full flags $L_3 \supset L_2 \supset \langle \alpha \rangle$, and partitions of the $q_i$s into pairs, with one (of the three) distinguished pair.

2.12. **Remark.** The techniques we use here are algebraic. The only requirements we make are those that Donagi makes in [Do]: char $\mathbb{F} \neq 2, 3$. Although the statement of the results in [Do] is for $\mathbb{C}$, we are using only the parts that rely on char $\mathbb{F} \neq 2, 3$.

3. **The Construction**

3.1. **Notation.** Let $q_1, q_2$ be the two $q_i$s (see Notation 2.10) that correspond to the isotropic subgroup $L_2 \subset \text{Jac}(C)[2]$ (see Theorem 2.11). Let $\tau : E \rightarrow \mathbb{P}^1$ be the unique double cover such that $\tau(q_1) = \tau(q_2)$. Denote by $\tilde{H} \rightarrow H$ the bigonal construction on the tower $Y \rightarrow E \rightarrow \mathbb{P}^1$.

Denote by $Y', E'$ the normalizations of the curves $\tilde{H}, H$ respectively.

3.2. **Theorem.** Assume that the ramification pattern of the tower $Y \rightarrow E \rightarrow \mathbb{P}^1$ is generic, i.e. the ramification pattern (see the dictionary at [Do] page 74) is the following:

| $n$(type) | $Y/E$ |
|-----------|-------|
| 1         | $\mathbb{C} \subset \subset \subset$ |
| 4         | $\mathbb{C} \subset \subset$ |
| 4         | $\mathbb{C} = / \subset$ |

where $n$(type) is the number of points in $\mathbb{P}^1$ with a given non trivial ramification type. In this case, there exists a curve $C'$ of genus 3, and an element $\alpha' \in \text{Jac}(C')[2]$ such that:

1. $\text{Jac}(C') = \text{Jac}(C)/L_3$.
2. $\langle \alpha' \rangle = L_2^2/L_3$.
3. The double cover $Y' \rightarrow E'$ is the double cover related to $C', \alpha'$ in terms of Theorem 2.3.
Proof. The ramification pattern (see [Do] p. 68-69) of the bigonal construction on \(Y \to E \to \mathbb{P}^1\) is:

| \(n/\text{type}\) | \(Y/E\) | \(\tilde{H}/H\) |
|----------------|--------|----------------|
| 1 | \(\subset\subset / = \subset\subset / \times\) | |
| 4 | \(\subset\subset / \subset\subset / = / = \) | |
| 4 | \(\subset\subset / \subset\subset / \subset\subset / \subset\subset /\) | |

By [Pa] Proposition 3.1 (page 307) The Abelian variety \(\text{Prym}(\tilde{H}/H)\) is isomorphic to the dual of the Abelian variety \(\text{Prym}(Y/E)\). Since the curve \(E\) has only one singular point, the normalizations of the curves \(\tilde{H}, H\) induce the isomorphism (See [DL] Lemma 1):

\[\text{Prym}(Y'/E') \cong \text{Prym}(\tilde{H}/H).\]

By the Riemann-Hurwitz formula, the genera of the curves \(Y', E'\) are 4,1 respectively. The choice of a partition to two pairs on the points of type \(\subset\subset / = / \) in the tower \(Y \to E \to \mathbb{P}^1\), is equivalent to a partition to two pairs of the ramification points of \(E'/\mathbb{P}^1\), which is equivalent to a choice of an unramified cover \(F'/E'\). Define \(Z' := Y' \times_{E'} F'\).

By this definition \(\text{Gal}(Z'/E') = (\mathbb{Z}/2\mathbb{Z})^2\). Define the curve \(X'\) to be third quotient of \(Z'\) (the one that is not \(F'\) or \(Y'\)). By the Riemann-Hurwitz formula, the genera of the curves \(Z', X'\) are 7,4 respectively. Perform the trigonal construction on the double cover \(Z' \to X'\) to get \(f': C' \to \mathbb{P}^1\). By Theorem 2.5 there exists an \(\alpha'' \in \text{Jac}(C')\) such that the zero divisor of the map \(f'\) satisfies

\[(f')_0 = K_{C'} + \alpha''.\]

By Theorem 2.11 applied to \(C', \alpha''\) the equality

\[\langle \alpha'' \rangle = \frac{L_2}{L_3}\]

holds, thus \(\alpha'' = \alpha'\).\qed

3.3. Remark. One can describe analogous results for the other (hence, non-generic) ramification patterns of the tower

\(Y \to E \to \mathbb{P}^1\).

3.4. Notation. Denote by \(i\) the isomorphism

\[i : H^0(C, \Omega_C) \to H^0(C', \Omega_{C'}).\]

Denote the projectivization of this isomorphism by \(\mathbb{P}i\). Denote by \((\mathbb{P}i)^*\) the induced isomorphism

\[H^*(|K_C|, -) \to H^*(|K'_{C'}|, \mathbb{P}i(-)).\]

Denote the canonical maps of \(C, C'\) by \(k_C, k_{C'}\) respectively. Fix a line \(V_\infty \in |K_C|\). Denote by \(z_1, z_2\) affine coordinates on \(\mathbb{P}^2 - V_\infty\).
3.5. **Notation.** Denote by $P_C, P_{C'}$ the maps

$$P_C : H^0(\mathbb{P}^2, V_\infty) \to H^0(C, \Omega_C)$$

$$l \mapsto \frac{ldz_1}{\partial k_C / \partial z_2}$$

$$P_{C'} : H^0(\mathbb{P}^2, \mathbb{P}i(V_\infty)) \to H^0(C', \Omega_{C'})$$

$$l \mapsto \frac{l\mathbb{P}i(dz_1)}{\partial k_{C'} / \partial (\mathbb{P}i(z_2))}$$

3.6. **Proposition.** The maps $P_C, P_{C'}$ are isomorphisms.

**Proof.** The map

$$H^0(\mathbb{P}^2, C - 3V_\infty) \to H^0(\mathbb{P}^2, V_\infty)$$

$$\omega \mapsto (4V_\infty / k_C)\omega$$

is an isomorphism since the degree map $\text{deg} : \text{Pic}(\mathbb{P}^2) \to \mathbb{Z}$ is an isomorphism. Up to this map, the map $P_C$ is the Poincaré residue map

$$H^0(\mathbb{P}^2, C - K_{\mathbb{P}^2}) \to H^0(C, K_C),$$

which is an isomorphism in this case (see [GH] p. 221).

3.7. **Corollary.** The equality

$$i = P_{C'} \circ (\mathbb{P}i)^* \circ P_{-1}^{-1}$$

holds

**Proof.** This follows from Proposition 3.6.

4. **Analysis of $|K_C|$ and $|K_Y|$**

4.1. **Lemma.** Let $\omega \in H^0(Y, \Omega_Y)$ be a differential such that the zero divisor $\omega_0$ is symmetric under the involution $i$. Then the differential $\omega$ is either symmetric or antisymmetric under the involution $i$.

**Proof.** The zero divisor of the differential $\omega + i\omega$ satisfies

$$(\omega + i\omega)_0 \supset (\omega)_0.$$ 

Hence, there exists some $k \in \mathbb{C}$ such that

$$k(i\omega) = k\omega = \omega + i\omega.$$ 

So either $k = 0$ or $k = 2$.

4.2. **Notation.** Denote the the odd (respectively even) part of the cohomology group $H^0(Y, \Omega_Y)$ under the double cover $Y \to E$ by $H^0(\Omega_Y, Y)_{\text{odd}}$ (respectively $H^0(\Omega_Y, Y)_{\text{even}}$). Denote by $|(K_Y)_{\text{odd}}$ the projectivization of the vector space $H^0(\Omega_Y, Y)_{\text{odd}}$. Given a cover $A \to B$ denote the ramification divisor of the cover by $R_{A/B}$. 
4.3. Remark. Recall (see Theorem 2.8) that under the canonical map, the curve \( Y \) is mapped to the complete intersection \( Q_2 \cap Q_3 \), where \( Q_3 \) is a cone over \( E \subset H \) and \( H \subset \mathbb{P}^3 \) is a plane.

4.4. Lemma. The following properties hold:

1. \( H^0(Y, \Omega_Y)_{\text{even}} \) is one dimensional, and its divisor is \( R \).
2. \( \mathbb{P}(H^0(Y, \Omega_Y)_{\text{even}}) \subset |K_Y| \) is the point dual to \( H \).
3. \( H^0(Y, \Omega_Y)_{\text{odd}} \) is 3 dimensional.
4. \( \mathbb{P}(H^0(Y, \Omega_Y)_{\text{odd}}) \subset |K_Y| \) is the dual space of the hyperplanes in \( \mathbb{P}^3 \) through the vertex of \( Q_3 \).

Proof. The first two claims follow from Theorem 2.8. The third claim comes from the decomposition

\[
H^0(Y, \Omega_Y) = H^0(Y, \Omega_Y)_{\text{odd}} \oplus H^0(Y, \Omega_Y)_{\text{even}}.
\]

We are left with the last claim. Let \( H' \subset \mathbb{P}^3 \) be a plane through the vertex of the cubic surface \( Q_3 \). Denote \( D := Y \cap H' \). Since \( \deg(Y) = 6 \), the number of points in the set \( D \) (counting multiplicities) is 6. Let \( \omega \in H^0(Y, \Omega_Y) \) be a differential on the curve \( Y \) such that the zero divisor \( (\omega)_0 \) satisfies

\[
(\omega)_0 = D.
\]

Since the set \( D \) is invariant under the involution \( \sigma \), Lemma 4.1 implies that the differential \( \omega \) is either symmetric or anti-symmetric. As the even differentials are analyzed above, the differential \( \omega \) is odd. Since the space of such planes \( H' \subset \mathbb{P}^3 \) is 2 dimensional, this space is all of \( \mathbb{P}(H^0(\Omega_Y, Y)_{\text{odd}})^* \). \( \square \)

4.5. Definition. Denote by \( \phi_Y \) the map

\[
\phi_Y : H^* \longrightarrow \mathbb{P}(H^0(Y, \Omega_Y)_{\text{odd}})
\]

\[
L^* \mapsto q_Y^* L \cap E.
\]

4.6. Corollary. The map \( \phi_Y \) is a natural isomorphism.

Proof. This follows from composing the isomorphism of claim 4 in Lemma 4.4 with the natural isomorphism

\[
H^* \longrightarrow \text{planes through the vertex of } Q_3
\]

\[
L^* \mapsto \text{the unique plane through the vertex and } L.
\]

\( \square \)

4.7. Proposition. The norms in the trigonal construction on \( f \) induce an isomorphism

\[
H^0(C, \Omega_C) \xrightarrow{\cong} H^0(Z, \Omega_Z)/H^0(X, \Omega_X).
\]

Proof. By [Do] Theorem 2.11 (p. 76) the norms in the trigonal construction induce an isomorphism:

\[
\text{Jac}(C) \cong \text{Prym}(Z/X).
\]

Whence, they induce an isomorphism on the universal covers. \( \square \)
4.8. **Notation.** Denote by
\[ t : H^0(C, \Omega_C) \longrightarrow H^0(Y, \Omega_Y)_{\text{odd}} \]
The isomorphism given by the composition of the norms at diagram in 1.3 (it is an isomorphism since the map \( \text{Jac}(C) \to \text{Prym}(Y/E) \) is an isogeny, see Theorem 2.9). Denote by \( \psi \) the map
\[ \psi : H^* \longrightarrow \mathbb{P}(H^0(C, \Omega_C)) \]
\[ L^* \mapsto L \cap C. \]
Denote by \( m_{C,\alpha} \) the automorphism of the plane \( H \) dual to the automorphism
\[ \phi_Y^{-1} \circ \mathbb{P}(t) \circ \psi \]
of the dual plane \( H^* \).

4.9. **Proposition.** The following properties hold:
- The inequality \( R_W/C > R_W/Z \) holds (in \( \text{Div}(W) \)).
- The divisor \( R_W/Z \) is invariant under the involution \( \tau \).

**Proof.** The first claim follows from the trigonal construction dictionary (see [Do] p. 74). The second claim is true since the curve \( Z \) is defined as the quotient of \( W \) by the involution \( \tau \). \( \square \)

4.10. **Theorem.** The automorphism \( m_{C,\alpha} \) is the identity.

**Proof.** We will prove that the automorphism \( m_{C,\alpha} \) fixes each of the 6 points of the set \( Q \cap E \). Since the set \( Q \cap E \) is not contained in a line, this will prove our claim. Let \( L \) be one of the 12 bitangents \( l_{ij} \). Let \( \omega \in H^0(C, \Omega_C) \) be a representative of \( \psi(L) \in \mathbb{P}(H^0(C, \Omega_C)) \). Denote the two tangency points of the line \( L \) and the curve \( C \) by \( p, q \). Denote by \( r, t \) the two residual points in the divisor class \( |K_C + \alpha| \). We will calculate the zero divisor of the differential:
\[ \xi := s_*(\pi^*\omega + s\pi^*\omega) \in H^0(\Omega_Z, Z). \]
By [Ha] proposition IV.2.1,
\[ (\pi^*\omega)_0 = \pi^*((\omega)_0) + R_{W/C}, \]
and \( (\pi^*\omega + s\pi^*\omega)_0 = s^*((\xi)_0) + R_{W/Z} \).

Whence,
\[ \pi^*((\omega)_0) = \{ x, \tilde{x} \mid x \in L \cap C \text{ and } x + \tilde{x} \geq K_C + \alpha \} \]
\[ = 2((p, q) + (p, r) + (p, t) + (q, p) + (q, r) + (q, t)) \]
\[ \Rightarrow \]
\[ (\pi^*\omega)_0 = \pi^*((\omega)_0) + R_{W/C} \geq R_{W/Z} + 2((p, q) + (q, p)) \]
\[ \Rightarrow \]
\[ (\pi^*\omega + s\pi^*\omega)_0 \geq R_{W/Z} + 2((p, q) + (q, p)) \]
\[ \Rightarrow \]
\[ (\xi)_0 \geq 2\{p, q\} \]

By Proposition [L7], the set \( (\xi)_0 \) is invariant under the involution \( \sigma \). Since
\[ K + \alpha - (p + q) = r + t, \]
we have:

\[(\xi)_0 \supset 2\{p, q\} + 2\{r, t\}. \Rightarrow \]

\[(q_Y^*(\xi)_0 \supset 2\{\{p, q\}, \{r, t\}\}). \]

The point \(\{\{p, q\}, \{r, t\}\}\) is one of the ramification points of the double cover \(Y \rightarrow E\). Under the canonical embedding of the curve \(Y\) (see Theorem 2.8) this point is mapped to the point \(\overline{pq} \cap \overline{rt} \in H\). i.e. the line \(L = \overline{pq}\) is mapped under \(m_{C, \alpha}\) to a line through the point \(\overline{pq} \cap \overline{rt}\). By symmetry considerations, \(\overline{pq} \cap \overline{rt}\) is mapped to itself. Since this applies to any one of the 6 points in the set \(Q \cap E\), we have proved the claim. \(\square\)

5. Analysis of \(|K_Y|\) and \(|K_{Y'}|\)

5.1. Notation. Denote \(M := Y \times_{\mathbb{P}^1} E'\). Denote by \(\mu, \mu'\) the quotients

\[\mu : M \rightarrow Y, \quad \mu' : M \rightarrow Y'.\]

Denote by \(\tau'\) the involution on \(E'\) s.t. \(E/\tau' = \mathbb{P}^1\). Denote by \(q_E, q_{E'}\) the quotients by the involutions \(\tau, \tau'\) respectively. Denote by \(q_Y, q_{Y'}\) the quotient by the double covers \(Y \rightarrow E, Y' \rightarrow E'\). Recall Notation 4.2. We use the analog notations \(H^0(\Omega_{Y'}, Y')_{\text{odd}}, H^0(\Omega_{Y'}, Y')_{\text{even}}, |(K_{Y'})_{\text{odd}}|\) for the curve \(Y'\).

5.2. Lemma. The norms in the bigonal face of the diagram in \([L]:\) (face “B”) induce an isomorphism

\[(1) \quad \mu'_* \circ \mu^* : H^0(Y, \Omega_Y)_{\text{odd}} \cong H^0(Y', \Omega_{Y'})_{\text{odd}}.\]

Proof. By \([Pr]\) Proposition 3.1 (p. 307) the norms in the bigonal construction induce an isomorphism:

\[\text{Prym}(Y/E) \cong \text{Prym}(Y'/E')^\wedge.\]

Whence, they induce an isomorphism on the universal covers. \(\square\)

5.3. Notation. Denote by \(P\) the isomorphism

\[P : |(K_Y)_{\text{odd}}| \rightarrow |(K_{Y'})_{\text{odd}}|\]

induced from the isomorphism in Equation \(1\).

5.4. We will describe the isomorphism \(P\) with two goals in mind:

- Combined with the isomorphisms \n
  \[|K_C| \cong |(K_Y)_{\text{odd}}|, \text{ and } |K'_C| \cong |(K_{Y'})_{\text{odd}}|\]

  (described in Notation 4.3) we will get an explicit description of the isomorphism

  \[\mathbb{P}H^0(C, \Omega_C) = |K_C| \cong |K'_C| = \mathbb{P}H^0(C', \Omega_{C'}).\]

- Using this isomorphism can calculate the plane configuration of \(Y'\), from the plane configuration of \(Y\).

5.5. Notation. We will use the notation \(\phi_Y\), presented in Definition 4.5. Define \(\phi_{Y'}\) analogy.
5.6. **Definition.** Considering the curves $E, E'$ embedded in $\mathbb{P}^2$, the maps $\tau, \tau'$ are given as projections from points on 

$$t \in E \subset \mathbb{P}^2, \quad t' \in E' \subset \mathbb{P}^2$$

respectively. For any point $p \in \mathbb{P}^2$, denote by $L_p$ (respectively $L'_p$) the line between the two points of $q_{E}^{-1}(p)$ (respectively $q_{E'}^{-1}(p)$).

5.7. **Proposition.** The following equalities hold:

$$L_p \cap E = q_{E}^{-1}(p) + t, \quad L'_p \cap E' = q_{E'}^{-1}(p) + t.$$  

**Proof.** This follows from Definition 5.6. \qed

5.8. **Theorem.** Let $p$ be a point in $\mathbb{P}^1$, then 

$$P(\phi_\mathcal{E}(L_p)) = \phi_\mathcal{E}'(L'_p).$$

**Proof.** Let $\omega \in H^0(Y, \Omega_Y)_{\text{odd}}$ be a representative of $\phi_\mathcal{E}(L_p) \in P(H^0(Y, \Omega_Y)_{\text{odd}})$.

By [Ha] proposition IV.2.1:

$$(\mu^*\omega)_0 \supset \mu^*((\omega)_0) = \mu^*(q_{E}(L_p \cap E)) \supset (q_{E} \circ q_{Y} \circ \mu)^{-1}(p).$$

All the points in the effective divisor $(q_{E} \circ q_{Y} \circ \mu)^{-1}(p)$ are moving with $p$. Whence, generically,

$$R_{M/Y'} \cap (q_{E} \circ q_{Y} \circ \mu)^{-1}(p) = \emptyset.$$  

By [Ha] proposition IV.2.1:

$$2(\mu^*\omega)_0 = \mu'_*(\mu^*\omega)_0 + R_{M/Y'} \Rightarrow 2(\mu'_*(\mu^*\omega)_0 = \mu'_*((\mu^*\omega)_0) - \mu'_*(R_{M/Y'}) \supset \mu'_*((q_{E} \circ q_{Y} \circ \mu)^{-1}(p))$$

$$= \mu'_*((q_{E'} \circ q_{Y'} \circ \mu')^{-1}(p)) = 2(q_{E'} \circ q_{Y'})^{-1}(p).$$

By Lemma 5.2, the differential $\mu'_*((\mu^*\omega)$ is odd, and by proposition 5.7, the differential

$$\mu'_*(\mu^*\omega) \in H^0(Y', \Omega_{Y'})_{\text{odd}}$$

is a representative of $\phi_{Y'}(L'_p)$. \qed

5.9. **Corollary.** The equality

$$P(t) = t'$$

hold.

**Proof.** This follows from the definition of $t$ and $t'$ (Definition 5.6). \qed

5.10. **Lemma.** There are no multiple points in the ramification divisor $R_{M/\mathbb{P}^1}$.

**Proof.** By the bigonal construction dictionary (see [De] p. 68-69) for any point $p \in \mathbb{P}^1$, if the cover $Y' \rightarrow \mathbb{P}^1$ is ramified, then the double cover $E \rightarrow \mathbb{P}^1$ is etale. The result follows since $M = Y' \times_{\mathbb{P}^1} E$, and there are no multiple points in $R_{Y'/\mathbb{P}^1}$. \qed
5.11. **Theorem.** The following equalities (of sets of points in $\mathbb{P}^2$) hold:
\[ P(Q \cap E \setminus \{q_1, q_2\}) = B_{E'/\mathbb{P}^1}, \quad P^{-1}(Q' \cap E' \setminus \{q'_1, q'_2\}) = B_{E/\mathbb{P}^1}. \]

**Proof.** Let $p \in \mathbb{P}^1$ be a branch point of the double cover $q_E : E \to \mathbb{P}^1$. Denote by $q$ the ramification point of the map $q_E$ above $p$. Let $\omega$ be an odd differential on the curve $Y$ such that $(\omega)_0 \supset q_Y^{-1}(q)$. i.e. the line $\phi_Y^{-1}(\mathbb{P}(\omega)) \subset \mathbb{P}^2$ (see Definition 4.5) passes through the point $q \in \mathbb{P}^2$.

By the bigonal construction dictionary (see [Do] p. 68-69) there is a unique point $q' \in q_{E'}^{-1}(p)$ such that the double $q_{Y'} : Y' \to E'$ is branched at $q'$. By [Ha] proposition IV.2.1:
\[
(\mu^*\omega)_0 \supset \mu^*((\omega)_0) \supset \frac{1}{2}(q_E \circ q_Y \circ \mu)^{-1}(p) = \frac{1}{2}(q_{E'} \circ q_{Y'} \circ \mu')^{-1}(p)
\]
\[
\supset \frac{1}{2}(q_{Y'} \circ \mu')^{-1}(q').
\]

However, by Lemma 5.10
\[
(q_{Y'} \circ \mu')^{-1}(q') \cap R_{M/Y'} = \emptyset.
\]

By Lemma 5.2
\[
2\mu^*\omega = \mu'^*\mu_\ast\mu^*\omega.
\]

Whence, by [Ha] proposition IV.2.1:
\[
(\mu^*\omega)_0 = \mu'^* (\mu_\ast (\mu^*\omega)_0) + R_{M/Y'} \quad \Rightarrow
\]
\[
2(\mu_\ast (\mu^*\omega)_0) = \mu'^* (\mu^*\omega)_0 - \mu'^*(R_{M/Y'}) \supset \frac{1}{2}\mu'^*((q_{Y'} \circ \mu)^{-1}(q'))
\]
\[
= q_Y^{-1}(q').
\]

By Lemma 5.2 the differential $\mu'^* (\mu^*\omega) \in H^0(Y', \Omega_{Y'})$ is odd. Whence, the point $q' \in \mathbb{P}^2$ sits on the line
\[
\phi_Y^{-1}(\mu'^* (\mu^*\omega))
\]
(see Notation 5.3 and Definition 4.3 for the definition of $\phi_Y$). Thus:
\[
P(q) = q'.
\]

The second assertion is symmetric. \qed

5.12. **Corollary.** 5.8 and Theorem 5.11 enable us to calculate directly the plane configuration of the double cover $Y' \to E'$ from the plane configuration of the double cover $Y' \to E'$. The curve $E' \subset \mathbb{P}^2$ is the unique cubic that passes through the following 9 points

- The point $t$ (see Definition 5.6).
- The four ramification points of the double cover $q_E : E \to \mathbb{P}^1$.
- The four points of $Q \cap E \setminus \{q_1, q_2\}$.

such that the lines through the point $t$ are tangent to $E'$ at the four points of $Q \cap E \setminus \{q_1, q_2\}$. The conic $Q' \subset \mathbb{P}^2$ is the unique conic that passes through the following 6 (note that there are 6, and not only 5 known points):

- The four ramification points of the double cover $q_E : E \to \mathbb{P}^1$. 
• The two points of $\mathcal{C}_{12} \cap E'$.

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