In quantum metrology schemes, one generally needs to prepare \( m \) copies of \( N \) entangled particles, such as entangled photon state, and then they are detected in a destructive process to estimate an unknown parameter. Here, we present an experimental proposal for estimating this parameter by using repeated indirect quantum nondemolition measurements in the setup called “photon box”. This interaction-based scheme is able to achieve the Heisenberg limit scaling as \( 1/N \) of sensitivity with a Fock state of \( N \) independent photons. Moreover, we only need to prepare one initial \( N \)-photon state and it can be used repetitively for \( m \) trials of measurements.

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I. INTRODUCTION

Quantum parameter estimation, the emerging field of quantum technology, aims to use entanglement and other quantum resources to yield higher statistical precision of a parameter \( \theta \) than purely classical approaches \[1, 2\]. The precision of the estimation of \( \theta \) will depend on the available resources used in the measurement. It has been shown that standard quantum limit (SQL) or called shot noise limit scaling as \( \delta \theta \approx 1/\sqrt{N} \) with \( N \) the number of particles can be surpassed by using coherent light with squeezed vacuum [3]. It is also commonly considered that using NOON state \[4\] and quantum entanglement allows one to achieve the Heisenberg limit (HL) scaling as \( \delta \theta \approx 1/N \), which is the ultimate limit set by quantum mechanics. Recently, some works \[5–7\] have shown that, without prior information, sub-Heisenberg estimation strategies that are ineffective. There are also some papers showing that the Heisenberg limit can be saturated without the use of any exotic quantum entangled states \[8, 9\]. Practical quantum metrology considering the impact of noise have been considered and studied in Refs. \[10, 11\]. The technique of quantum parameter estimation figures in several metrology platforms, including optical interferometry \[12, 13\], atomic systems \[14, 15\], and Bose-Einstein condensates \[16, 17\]. In addition, it is at the heart of many modern technologies and researches, such as quantum clock synchronization \[18, 19\], quantum imaging \[20\], and gravitational wave observation \[21\].

General parameter estimation procedure can be divided into three distinct sections: probe preparations, interaction between the probe and the system, and the probe readouts \[22\]. These sections will be repeated many times before the final construction of the estimation of \( \theta \). Most of the quantum parameter estimation strategies require the preparations of \( m \) copies of entangled states. However, these states are extremely difficult to generate and fragile to the impact of decoherence. Therefore, the method of quantum nondemolition (QND) measurements \[23\] with entanglement-free states may be a suitable and practical way to overcome these challenges. The QND measurements dating back to as early as the 1920s realize ideal projective measurements that leave the system in an eigenstate of the measured observable \[24\]. With these ideal projective measurements performed on an initial coherent state, Fock states and “Schrödinger cat” states can be prepared and reconstructed \[25\]. Moreover, with appropriate feedback loops, it is possible to prepare on demand photon states and subsequently reverses the effects of decoherence \[26\]. With these merits, we can foresee the widespread applications of this techniques in quantum information and quantum metrology.

In this Letter, we present an practical proposal for realizing quantum parameter estimation in “photon box” \[27–30\] via repeated quantum nondemolition (QND) measurements. We show that, with single-mode Fock state of photons, this proposal can estimate the parameter \( \theta \) within a Heisenberg scaling accuracy. Unlike other quantum metrology strategies, our proposal has this advantage-the state of photons can be used circularly. We also review the stochastic method of preparation of this state in the “photon box” and demonstrate that it can be improved by using squeezed state instead of coherent state as an initial state. The experimental feasibility of our proposals can also be justified with current laboratory parameters.

II. HEISENBERG-ACCURACY PARAMETER ESTIMATION VIA QND MEASUREMENTS IN “PHOTON BOX”

In our proposal, the experimental setup is similar to the one discussed in Refs. \[27–30\] and is shown in Fig. 1(a). The core of this setup is a “photon box”, which is an open cavity \( C \) made up of two superconducting mirrors facing each other (the Fabry-Perot configuration). The QND probe atoms, generated form the atomic resource \( S \), are prepared in circularly and travel along the transverse direction of the cavity axis. The atoms cross the cavity \( C \) sandwiched between two auxiliary cavities \( R_1 \) and \( R_2 \) before being detected in the detector \( D \). The \( R_1\)-\( C\)-\( R_2 \) structure can be regarded as a Ramsey interferometry.

Suppose that the state of photons in the cavity \( C \) is in a superposition of Fock states with different photon number \( |\psi\rangle_S = \sum_n c_n |n\rangle \). One Rydberg atom is prepared in states \( |\varphi\rangle_P = |\uparrow_z\rangle \); and afterwards, for simplicity, we replace \( |\uparrow_z\rangle \) as the 1920s realize ideal projective measurements that leave the system in an eigenstate of the measured observable \[24\]. With these ideal projective measurements performed on an initial coherent state, Fock states and “Schrödinger cat” states can be prepared and reconstructed \[25\]. Moreover, with appropriate feedback loops, it is possible to prepare on demand photon states and subsequently reverses the effects of decoherence \[26\]. With these merits, we can foresee the widespread applications of this techniques in quantum information and quantum metrology.
is the general necessary and sufficient condition \([23]\) that the QND probe must satisfy.

Due to different outputs:

- the photon state in C is affected by this measurement
- the states of atoms are detected in the detector D.

(a) Diagram for our sequential strategy. \(m\) probe atoms are used. The Fock state of photons in C stays unchanged after each QND measurement.

(b) Diagram for performing Hadamard gate operation. The unknown parameter is \[\hat{\Theta} = \frac{\pi}{2}\] (to the lowest order for small \(\hat{\Theta}\)).

III. STOCHASTIC APPROACH FOR PREPARING FOCK STATES

In our proposal, although generating a single mode Fock state of \(N\) photons is a challenging task, it seems nowadays experimentally available \([30]\). For completeness, we review the stochastic method for generating a Fock state of \(N\) photons in the cavity by nondestructive atom-light interaction as before \([30]\) and then improve it with squeezed state. Note that an advanced method based on real-time quantum feedback has been reported in Refs. \([24, 33]\) which is an ideal technical means for generating and preserving Fock states for our metrology proposal. Preparing the single-mode number squeezing state is also helpful and urgent for ultrasensitive two-mode interferometry \([34]\).

We start with a superposition of Fock states with different photon number in the cavity \(C\): \[|\psi\rangle_C = \sum_n c_n |n\rangle\]. We set the interaction strength \(\theta_i\) at an appropriate and definite value such that \(q(i|n) \neq q(i|m)\) for all possible photon number \(n\) and \(m\), where \(q(i|n) = c_n^2|n\theta_i/2 - (i - 1)\pi/4\). In Fig. 2(c) and 2(d) we present the situation that saturates the condition \(q(i|n) = q(i|m)\) and obtain superposed Fock states as convergent states. As before, \(M\) atoms cross the R1-C-R2 interferometric layout and are measured with operator \(\sigma_z\). The sequence of measurement results for \(M\) probe atoms, called a readout, is expressed as \(\omega_M = \{i_1, \ldots, i_M\}\), where \(i_\mu = 1, -1\) and \(\mu = 1, 2, \ldots, M\). The photon number distribution of the final state in C can be calculated as

\[
P(n|\omega_M) = \frac{|c_n \cos^2(n\theta_s/2) \sin^2(n\theta_s/2)|^2}{Z(\omega_M)},
\]
FIG. 2: (color online). Simulation of the indirected measurement procedures (totally 100 atoms) via Monte Carlo method. The state of light in the cavity is initially chosen as a coherent state procedures (totally 100 atoms) via Monte Carlo method. The state represents the photon number and $(\text{c})$ and $(\text{d})$, the photon states converge to two different superposed Fock states.

where $Z(\omega_M) = \sum_{n} c_n \cos^{\eta}(n\theta_s/2) \sin^{\xi}(n\theta_s/2)^2$; $\eta$ and $\xi$ are the number of 1 and 1 in the event $\omega_M$, respectively. It should be noted that two events, $\omega_M = (i_1, \cdots, i_M)$ and $\omega'_M = (i_{\sigma_1}, i_{\sigma_2}, \cdots, i_{\sigma_M})$, are equivalent and exchangeable if $[\sigma_1, \sigma_2, \cdots, \sigma_M]$ is the permutation of $[1, 2, \cdots, M]$ [55]. It has been proved in Ref. [51] that (i) this photon number distribution converges as $M$ is infinity: $\lim_{M \to \infty} P(\eta|\omega_M) = \delta_{\eta,N}$, (ii) the probability for the state in cavity converges to a Fock state $|N\rangle$ is $P(N = |\eta\rangle = |c_N|^2$, and (iii) the convergence for $\delta_{\eta,N}$ is exponentially fast: $P(\eta \neq N|\omega_M) \approx \exp(-MS(N|n))$, where $S(N|n) = -\sum q(i|N) \ln[q(i|n)/q(i|N)]$ is the relative entropy of $q(N)$ relative to $q(n)$. Therefore, we can obtain a single Fock state by this approach with probability $|c_N|^2$ when $M$ is large enough. The final photon number $N$ can be determined via analyzing the spin measurement results of the probe atoms $\omega_M$. Any Fock states with more than one photons are available for the sub-shot-noise metrology.

Most commonly the initial state of the light in cavity C is a coherent state $|\psi\rangle = |\alpha\rangle_S$, where $\alpha = \sqrt{3}$ is a complex amplitude and $c_n = e^{-|\alpha|^2/2} \sqrt{\alpha^n}/\sqrt{n!}$ is a poissonian distribution of photon number. The probability distributions and $M$ QND measurements are numerical simulated via Monte Carlo method and plotted in Fig. 2. We observe the converging events of different photon numbers in Fig 2(a) and 2(b). We also numerically simulate the probability for the coherent state converging to a Fock state, which verifies the result (ii), see Fig. 3. These numerical results for simulating the preparation of a single-mode Fock state in a “photon box” conform with the experimental results in Ref. [30]. After generating a Fock state of a nonzero and known photon number, we can perform the parameter estimation without adjusting the experimental apparatus.

An efficient method for improving this strategy is to use the squeezed state as the initial state in the cavity C. A squeezed state may be generated by first acting with the squeeze operator $S(\zeta) = \exp(\frac{1}{2} \zeta \hat{a}^2 - \frac{1}{2} \zeta^\ast \hat{a}^\ast)^2$ with $\zeta = re^{i\theta}$ on the vacuum followed by the displacement operator $D(\alpha) = \exp(\alpha \hat{a}^\ast - \alpha^\ast \hat{a})$: $|\alpha, \zeta\rangle = D(\alpha)S(\zeta)|0\rangle$ [55]. The photon number distributions for coherent state $|\sqrt{3}\rangle$ and squeezed state $|\sqrt{3}, 0.5\rangle$ are compared in Fig. 3. We also test the result (ii) for squeezed state in Fig. 3. The average photon number in the squeezed state is $\langle \hat{n} \rangle = |\alpha|^2 + \sinh^2(\theta) = 3.27$ that approximates to the average photon number in the coherent state. It is shown that the photon number statistics of this squeezed state are super-poisonian and narrower. Thus, by using the squeezed state, we can obtain a higher success rate for generating a Fock state that is useful in sub-shot-noise metrology due to an increase in amplitude fluctuations. For example, as shown in Fig. 3 we have a higher chance of obtaining Fock states $|3\rangle$ and $|4\rangle$ and the probabilities for generating states $|0\rangle$ and $|1\rangle$ that are useless for sub-shot-noise metrology are reduced. The speed of convergence for squeezed state can be higher.
IV. CONCLUSION AND DISCUSSION

In conclusion, we have presented an experimental proposal for estimating an unknown parameter in "photon box" by using the method of QND measurements. We have shown that initially with Fock state of $N$ photons, the Heisenberg scaling accuracy of the estimation can be achieved. Moreover, we do not need to prepare $m$ copies of initial state as other metrology schemes. We have also studied the preparation of Fock state from squeezed state and classical input (coherent state). The feasibility of this sub-shot-noise estimation scheme can be met by the current laboratory achievements. Furthermore, this proposal with the help of QND measurements will also be an inspiration to other experimental platforms [36] for quantum metrology and quantum information techniques. In addition, our results should be of broad interest as many applications, such as clock synchronization, phase estimation and gravitational wave detection. The Bayesian approach of parameter estimation starting with a superposition of Fock states, such as coherent states and squeezed states, may be an open problem to be settled.

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