Designing Two-Dimensional Complete Complementary Codes for Omnidirectional Transmission in Massive MIMO Systems

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Abstract—This paper presents an efficient construction of two-dimensional (2D) complete complementary codes (CCCs) for their modern application as omnidirectional precoding matrices in massive MIMO systems to attain enhanced cell coverage. Unlike the traditional 1D CCCs, little progress has been made on efficient and systematic constructions of the 2D counterpart. In contrast to the existing recursive constructions with the aid of various sequence operations, certain 1D seed sequences or 2D arrays, we propose to use 2D generalized Boolean functions for direct synthesis of 2D CCCs. Simulation results show that the proposed 2D CCCs appear to be good candidates for precoding matrices to achieve omnidirectional transmission in massive MIMO systems.

Index Terms—Generalized Boolean Function, Complete Complementary Code (CCC), Omnidirectional MIMO Transmission, Precoding Matrices, Uniform Rectangular Arrays (URAs).

I. INTRODUCTION

The study of complementary codes (CCs) dates back to the early 1950s in Golay’s celebrated publications on a class of sequence pairs (with impulse-like aperiodic autocorrelation sums) for the design of infrared multislit spectrometry [11]. Over the years, this topic has been gaining increasing research attention as diverse applications of CCs in engineering especially in wireless communications [3]–[8] are found. Due to the limitation on the set sizes of CCs, the CCCs were extended to the Z-complementary sets (ZCSs) to have larger set sizes [9]–[11]. In the literature, extensive research works are concerned with CCs/ZCSs whose correlations are 1D functions of the time-shifts only [12]–[17]. By contrast, this paper focuses efficient design of 2D complete complementary codes (CCCs) which enjoy the maximum set size as well as ideal aperiodic auto-correlations and cross-correlation properties in 2D domain.

In recent years, a major driving force of 2D CCCs is for omnidirectional transmission in massive multi-input and multi-output (MIMO) systems [18], [19]. Such an application is desired when a base-station (BS) intends to broadcast certain common messages to all the randomly distributed user equipments (UEs) within the current cell, including those at the cell edges, through a public signaling channel. A promising approach is to send through a stream of space-time block coded (STBC) symbols from one or more uniform rectangular arrays (URAs) of the BS such that all the UEs can receive identical signal power for any angle of incidence. It has been shown that such an optimal omnidirectional transmission can be attained when 2D CCCs are adopted as precoding matrices in the URAs [20]–[23]. Other applications of 2D CCCs include measuring and identification in higher dimensional systems [24], image change and motion detection [25], 2D multi-carrier code-division multiple access [26], [27].

In most of the current state-of-the-art, 2D CCCs are obtained by recursively applying various sequence operations to certain seed 1D sequences or 2D arrays [19]–[24], [28]. In [29], [30], we presented constructions of 2D mutually orthogonal Golay complementary array pairs (GCAP) which are essentially 2D CCCs with the set size of two only. Recently, [31] constructed 2D Z-complementary array sets (ZCASs) through recursive product of certain matrices of generating polynomials, which may include the 2D CCCs as a special case when the set size equals the number of constituent arrays and the zero-correlation zone covers the entire 2D shift domain. Very recently, [32] presented a method to construct higher-dimensional CCCs based on para-unitary (PU) matrices and explicit Boolean functions can be extracted from the seed PU matrix. Then, 2D CCCs can be obtained by applying the projection mapping method in [33]. Besides, the design of 2D Z-complementary array pair (ZCAP) has also attracted increasing research attention [34]–[36].

Direct constructions of 2D CCCs can lead to reduced hardware storage and faster code generation. Motivated by the above background, we propose to use 2D generalized Boolean functions (GBFs) for efficient synthesis of 2D CCCs whose array parameters (e.g., array size, set size, number of constituent arrays) take any power-of-two integer values. This settles our open problem raised in [29] on direct constructions of 2D CCCs. Moreover, the proposed 2D CCCs include those from [29] Th. 6 and Th. 7] as a special case. Furthermore, simulation results validate that our proposed 2D CCCs are
II. PRELIMINARIES AND NOTATIONS

The following notations will be used throughout this paper:

- $(\cdot)^*$ denotes the complex conjugation.
- $\mathbb{Z}_q = \{0, 1, \ldots, q-1\}$ is the ring of integers modulo $q$.
- Let $\xi = e^{2\pi \sqrt{-1}/q}$.
- $q$ is an even integer.

A. Definition of 2D Complete Complementary Code (2D CCC)

Let $C = (C_{g,i})$ be a $q$-ary array of size $L_1 \times L_2$ where $0 \leq g < L_1, 0 \leq i < L_2$.

**Definition 1:** The 2-D aperiodic cross-correlation function of arrays $C$ and $D$ at shift $(u_1, u_2)$ is defined as

$$\rho(C, D; u_1, u_2) = \begin{cases} L_1-1-u_1 & L_2-1-u_2 \sum_{i=0}^{L_1-1} \sum_{i=0}^{L_2-1} \xi^{D_{g,i}+u_1,-i+u_2-C_{g,i}}, 0 \leq u_1 < L_1, \\ L_2-1-u_2 \sum_{i=0}^{L_2-1} \sum_{i=0}^{L_1-1} \xi^{D_{g,i},-i+u_2-C_{g,i}}, 0 \leq u_2 < L_2; \\ L_1-1-u_1 & L_1-1-u_2 \sum_{i=0}^{L_1-1} \sum_{i=0}^{L_1-1} \xi^{D_{g,i}-u_1,-i-u_2}, -L_1 < u_1 < 0, \\ -L_2 < u_2 < 0; \\ L_2-1-u_2 \sum_{i=0}^{L_2-1} \sum_{i=0}^{L_2-1} \xi^{D_{g,i},-i-u_2-C_{g,i}}, -L_1 < u_1 < 0, \\ -L_2 < u_2 < 0; \\ 0 \leq u_2 \leq L_2. \\ 0 \leq u_1 \leq L_1. \end{cases}$$

(1)

If $C = D$, $\rho(C, C; u_1, u_2)$ is called the 2-D aperiodic autocorrelation function of $C$ and referred to as $\rho(C; u_1, u_2)$. Note that $\rho(C; u_1, -u_2) = \rho(C; -u_1, u_2)$.

**Definition 2:** For a set of $M$ sets of arrays $G = \{G^p| p = 0, 1, \ldots, M-1\}$, each set $G^p = \{C^p_{0}, C^p_{1}, \ldots, C^p_{N-1}\}$ is composed of $N$ arrays and the array size is $L_1 \times L_2$. Let

$$\rho(G^p, G^p'; u_1, u_2) = \sum_{t=0}^{N-1} \rho(C^p_t, C^p_{t'}; u_1, u_2)$$

(2)

$$= \begin{cases} NL_1L_2, & (u_1, u_2) = (0, 0), \ p = p'; \\ 0, & (u_1, u_2) \neq (0, 0), \ p = p'; \\ 0, & p \neq p'. \end{cases}$$

where $C^p_{g,i} = (C^p_{g,i})_g$ for $0 \leq g < L_1$, $0 \leq i < L_2$. When $M = N$, the set $G$ is called a 2D CCC, denoted by 2D $(M, N, L_1, L_2)$-CCC. Actually, each set $G^p$ forms a 2D Golay Complementary Array Set (GCAS), referred to as $(N, L_1, L_2)$-GCAS. Note that a 2D CCC can be reduced to a conventional 1-D $(M, N, L_2)$-CCC by taking $L_1 = 1$. Likewise, a 1D $(M, N, L_2)$-CCC comprises $M$ distinct 1D Golay complementary sets (GCSSs), where each GCS is denoted by $(N, L_2)$-GCS.

B. 2D Generalized Boolean Functions (2D GBFs)

In this section, we will introduce a concept of a 2D GBFs [29]. A 2D GBF is composed of $n + m$ variables $y_1, y_2, \ldots, y_n, x_1, x_2, \ldots, x_m$, mapping from $\mathbb{Z}_q^{n+m}$ to $\mathbb{Z}_q$, where $y_s, x_i \in \{0, 1\}$ for $s = 1, 2, \ldots, n$ and $l = 1, 2, \ldots, m$. We denote a product of $r$ variables by the monomial of degree $r$. For instance, $x_1x_2y_1y_2$ is a monomial of degree 4. The array $f$ linked with the GBF $f$ of $n + m$ variables can be written as

$$f = \begin{pmatrix} f_{0,0} & f_{0,1} & \cdots & f_{0,2^{n-1}-1} \\ f_{1,0} & f_{1,1} & \cdots & f_{1,2^{n-1}-1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{2^n-1,0} & f_{2^n-1,1} & \cdots & f_{2^n-1,2^{m-1}-1} \end{pmatrix}$$

(3)

by letting $f_{g,i} = f((y_1, g_2, \ldots, g_n), (i_1, i_2, \ldots, i_m))$, where $(y_1, g_2, \ldots, g_n)$ and $(i_1, i_2, \ldots, i_m)$ are binary representation of integers $g = \sum_{h=1}^{n} g_h 2^{h-1}$ and $i = \sum_{j=1}^{m} i_j 2^{j-1}$, respectively. For example, if $q = 2, n = 2$, and $m = 3$, the associated array of the GBF $f = x_1x_3 + x_2y_1 + y_2$ is

$$f = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

(4)

C. Omnidirectional Precoding based on 2D Arrays

We briefly introduce the application of the 2D CCC to omnidirectional transmission in massive MIMO systems with a URA. For the system model, we consider a downlink transmission from a BS equipped with a URA to single-antenna UEs within a cell [22, 23]. Assume that the URA comprises $L_1L_2$ antennas with $L_1$ rows and $L_2$ columns. Let $A(\varphi, \theta)$ be the steering matrix of size $L_1 \times L_2$ at the direction $(\varphi, \theta)$ and $A(\varphi, \theta) = (A(\varphi, \theta))_{g,i}$ for $g = 0, 1, \ldots, L_1-1, i = 0, 1, \ldots, L_2-1$. From [23], we have

$$A(\varphi, \theta)_{g,i} = e^{-j\varphi g_d \sin \theta} e^{-j\varphi d_y \sin \varphi \cos \theta}$$

(5)

$$\varphi \in [0, \pi/2], \theta \in [0, 2\pi],$$

where $\lambda$ is the wavelength of the carrier, $d_x$ and $d_y$ represent the inter-elements along the vertical and horizontal directions of the URA, respectively. In this paper, we consider an orthogonal STBC $S$ of size $N \times N$. Denote the $(n, t)$-th entry of $S$ by $s_n(t)$ which is beamformed by the precoding matrix $W_n$ of size $L_1 \times L_2$. Therefore, the transmitted signal mapped onto the $L_1L_2$ antennas of the URA is given by

$$\sum_{n=0}^{N-1} W_n \cdot s_n(t), \ t = 0, 1, \ldots, N-1.$$  

(6)
In the line-of-sight (LOS) channel without multipaths, the received signal for a user at direction \((\varphi, \theta)\) is

\[
\sum_{n=0}^{N-1} \left[ \text{vec}(\mathbf{A}(\varphi, \theta))^T \text{vec}(\mathbf{W}_n) \right] \cdot s_n(t) + w(t), \quad t = 0, \ldots, N - 1
\]

(7)

where \(\text{vec}(\cdot)\) denotes the vectorization of the matrix by vertically stacking the columns into a column vector and \(w(t)\) is the complex additive white Gaussian noise (AWGN).

**Lemma 1:** \([23]\) If the precoding matrices \(\mathbf{W}_0, \mathbf{W}_1, \ldots, \mathbf{W}_{N-1}\) of size \(L_1 \times L_2\) are constituent arrays of a 2D \((N, L_1, L_2)\)-GCAS, then the received power \(\sum_{n=0}^{N-1} \left[ \| \text{vec}(\mathbf{A}(\varphi, \theta))^T \text{vec}(\mathbf{W}_n) \| \right]^2\) is independent of \((\varphi, \theta)\).

**Lemma 7** states that the 2D GCASs can be utilized as the precoding matrices to attain omnidirectional transmission. Note that the 2D CCC can be regarded as a collection of 2D GCASs as mentioned in **Definition** 2.

### III. Proposed Direct Construction of 2D CCCs

Motivated by the construction of 1D CCCs in [14], we propose a direct construction of 2D CCCs based on 2D GBFs.

**Theorem 1:** For positive integers \(m, n, k\) and \(n \geq k\), we let nonempty sets \(I_1, I_2, \ldots, I_k\) be a partition of \(\{1, 2, \ldots, m\}\) and also let another nonempty sets \(I_1', I_2', \ldots, I_k'\) be a partition of \(\{1, 2, \ldots, n\}\). We take \(m_\alpha\) and \(n_\alpha\) as the orders of \(I_\alpha\) and \(I_\alpha'\), respectively. Then we define \(\pi_\alpha\) as a bijection from \(\{1, 2, \ldots, m_\alpha\}\) to \(I_\alpha\) and let \(\sigma_\alpha\) be a bijection from \(\{1, 2, \ldots, n_\alpha\}\) to \(I_\alpha'\). The generalized Boolean function is

\[
f = \frac{q}{2} \left( \sum_{i=1}^{k} m_i^{-1} x_{\pi_i(1)} x_{\pi_i(l+1)} + \sum_{i=1}^{k} n_i^{-1} y_{\sigma_i(1)} y_{\sigma_i(l+1)} \right) + \frac{q}{2} \sum_{i=1}^{k} x_{\pi_i(m_\alpha)} y_{\sigma_i(n_\alpha)} + \sum_{i=1}^{m} d_i x_i + \sum_{i=1}^{n} w_i y_i + w_0
\]

(8)

where \(d_i, w_i \in \mathbb{Z}_q\). If we let

\[
G^p = \left\{ f + \frac{q}{2} \sum_{\alpha=1}^{k} t_\alpha x_{\pi_\alpha(1)} + \frac{q}{2} \sum_{\alpha=1}^{k} p_\alpha y_{\sigma_\alpha(n_\alpha)} : t_\alpha \in \{0, 1\} \right\}
\]

(9)

where \(p_\alpha \in \{0, 1\}\) and \(p = \sum_{i=1}^{k} p_i 2^{k-1}\). Then the set \(\{C_0, G_1, \ldots, G_{2^k-1}\}\) forms a \((2^k, 2^k, q, 2^m)-CCC\). The power radiation pattern, i.e., \(\sum_{i=0}^{N-1} \left[ \| \text{vec}(\mathbf{A}(\varphi, \theta))^T \text{vec}(\mathbf{W}_n) \| \right]^2\), is depicted in Fig. [I] which can achieve omnidirectional transmission by using our proposed 2D CCC as precoding matrices. Furthermore,


V. CONCLUSION

In this work, driven by their modern application for omnidirectional transmission in massive MIMO systems, we have introduced a novel direct design of 2D CCCs based on 2D GBFs in Theorem 1. The proposed 2D \((2^k, q^k, 2^m, 2^m)\)-CCC s have flexible set sizes and array sizes, which includes our earlier result in [29, Th. 7] as a special case. Furthermore, we have illustrated that the proposed 2D CCCs are good omnidirectional precoding matrices in massive MIMO systems.

![Fig. 2. BER performance comparison for different precoding schemes with an 8 × 16 URA.](image)

TABLE I

| Method | Parameters | Based on | Note |
|--------|------------|----------|------|
| [19] Th. 3 | \((N, L_1, L_2)\) | 1D \((N, L_1)\)-GCSs and 1D \((2, L_2)\)-GCSs | \(N, L_1, L_2 \geq 2\) |
| [20] Lemma III.3 | \((N_1, N_2, L_1, L_2)\) | 1D \((N_1, L_1)\)-GCSs and 1D \((N_2, L_2)\)-GCSs | \(N_1, N_2, L_1, L_2 \geq 2\) |
| [23] Th. III.3 | \((2, 2 L_1, L_2)\) | 1D \((2, L_1)\)-GCSs and 1D \((2, L_2)\)-GCSs | \(L_1, L_2 \geq 2\) |
| [24] Th. 7 | \((M, K, M^2, K^2)\) | 1D \((M, M^2)\)-CCC s and 1D \((K, K^2)\)-CCC s | \(M, K \geq 2\) |
| [25] (Proposed) | \((2^n, 2^n, 2^m)\) | 2D Generalized Boolean functions | \(n, m > 0\) |
| [26] Th. 1 | \((M, M, L, M)\) | BH matrices of array size \(M \times M\) | \(M \geq 2\) |

![Table II](image)

### Table II: Comparisons of 2D \((M, N, L_1, L_2)\)-CCC s

| Method | Parameters | Based on | Note |
|--------|------------|----------|------|
| [19] Th. 3 | \((N, L_1, L_2)\) | 1D \((N, L_1)\)-GCSs and 1D \((2, L_2)\)-GCSs | \(N, L_1, L_2 \geq 2\) |
| [20] Lemma III.3 | \((N_1, N_2, L_1, L_2)\) | 1D \((N_1, L_1)\)-GCSs and 1D \((N_2, L_2)\)-GCSs | \(N_1, N_2, L_1, L_2 \geq 2\) |
| [23] Th. III.3 | \((2, 2 L_1, L_2)\) | 1D \((2, L_1)\)-GCSs and 1D \((2, L_2)\)-GCSs | \(L_1, L_2 \geq 2\) |
| [24] Th. 7 | \((M, K, M^2, K^2)\) | 1D \((M, M^2)\)-CCC s and 1D \((K, K^2)\)-CCC s | \(M, K \geq 2\) |
| [25] (Proposed) | \((2^n, 2^n, 2^m)\) | 2D Generalized Boolean functions | \(n, m > 0\) |
| [26] Th. 1 | \((M, M, L, M)\) | BH matrices of array size \(M \times M\) | \(M \geq 2\) |

![Fig. 2. BER performance comparison for different precoding schemes with an 8 × 16 URA.](image)

given in [23] (95), which needs four precoding matrices of size \(8 \times 16\) obtained from two ZC sequences of lengths 8 and 16, respectively. The random-matrix-based scheme utilizes four \(8 \times 16\) random matrices with values drawn from \(\{+1, -1\}\). It can be observed that the performance of the 2D CCC-based scheme enjoys about 4 dB gain for BER at \(10^{-6}\) compared to the ZC-based scheme. Hence, our proposed 2D CCCs are good candidates for omnidirectional precoding matrices in massive MIMO systems.
APPENDIX

PROOF OF THEOREM [1]

Proof: For the array \( C_t^p \) constructed from Theorem [1] it can be expressed as

\[ C_t^p = f + \frac{q}{2} \sum_{\alpha=1}^{k} t_{n} \alpha_{\pi_{\alpha}(1)} + \frac{q}{2} \sum_{\alpha=1}^{k} p_{\alpha} u_{\sigma_{\alpha}(n_{\alpha})}, \quad (11) \]

where

\[ C_t^p = \begin{pmatrix} (C_t^p)_{0,0} & (C_t^p)_{0,1} & \cdots & (C_t^p)_{0,2^m-1} \\ (C_t^p)_{1,0} & (C_t^p)_{1,1} & \cdots & (C_t^p)_{1,2^m-1} \\ & & \ddots & \vdots \\ (C_t^p)_{2^{m-1},0} & (C_t^p)_{2^{m-1},1} & \cdots & (C_t^p)_{2^{m-1},2^m-1} \end{pmatrix} \]

for \( 0 \leq t, p < 2^k \).

In the first part, we would like to prove that every \( G_P = \{ C_t^p, C_t^p, \ldots, C_t^p \} \) is a 2D GCAS for \( p = 0, 1, \ldots, 2^k - 1 \). Namely,

\[
\sum_{t=0}^{2^k-1} \sum_{g=0}^{2^m-1} \sum_{i=0}^{2^m-1} \left( C_t^g \right)^{\pi_{u_{1},i}+u_{2}-(C_t^g)_{g,i}} = 0
\]

(13)

for \( 0 \leq u_1 < 2^m \), \( 0 \leq u_2 < 2^m \), and \((u_1, u_2) \neq (0, 0)\). Then we let \( h = g + u_1, j = i + u_2 \) and also let \( (g_1, g_2, \ldots, g_m), (h_1, h_2, \ldots, h_m), (i_1, i_2, \ldots, i_m) \), and \((j_1, j_2, \ldots, j_m)\) be the binary representations of \( g, h, i \), and \( j \), respectively. Next we consider four cases to demonstrate that [13] holds.

Case 1: If \( u_1 \geq 0 \) and \( u_2 \geq 0 \), we suppose \( i_{\pi_{\alpha}(1)} \neq j_{\pi_{\alpha}(1)} \) for some \( \alpha \in \{1, 2, \ldots, k\} \). Then for any array \( C_t^p \in G'_P \), we can find an array \( C_t^p = C_t^p + (q/2) i_{\pi_{\alpha}(1)} \in G'_P \) satisfying \((C_t^p)_{h,j} - (C_t^p)_{g,i} = (C_t^p)_{h,j} + (C_t^p)_{g,i} \equiv \frac{q}{2} \) (mod \( q \)). Thus, we have

\[
\sum_{t=0}^{2^k-1} \left( (C_t^g)_{h,j} - (C_t^g)_{g,i} \right) = 0
\]

(14)

Since \( i_{\pi_{\alpha}(\beta-1)} = j_{\pi_{\alpha}(\beta-1)} \) and \( i_{\pi_{\alpha}(\beta-2)} = j_{\pi_{\alpha}(\beta-2)} \), we have

\[
(C_t^g)_{h,j} - (C_t^g)_{g,i} = \frac{q}{2} \left( i_{\pi_{\alpha}(\beta-2)} - j_{\pi_{\alpha}(\beta-2)} \right) + \frac{q}{2} \left( i_{\pi_{\alpha}(\beta)} - j_{\pi_{\alpha}(\beta)} \right) + d_{\pi_{\alpha}(\beta)} (j_{\pi_{\alpha}(\beta-1)} - 2i_{\pi_{\alpha}(\beta-1)}) \quad (15)
\]

implying \( \sum_{t=0}^{2^k-1} (C_t^g)_{h,j} - (C_t^g)_{g,i} = 0 \).

Case 2: If \( u_1 > 0 \) and \( u_2 = 0 \), we consider \( g_{\pi_{\alpha}(1)} = h_{\pi_{\alpha}(1)} \) for some \( \alpha \in \{1, 2, \ldots, k\} \). Using the similar logic, we set \( (C_t^g)_{h,j} \equiv \frac{q}{2} \) (mod \( q \)). By following the similar arguments as given in Case 2, we can obtain \( \sum_{t=0}^{2^k-1} (C_t^g)_{h,j} - (C_t^g)_{g,i} = 0 \).

Case 3: If \( u_1 < 0 \) and \( u_2 = 0 \), we consider \( g_{\pi_{\alpha}(1)} = h_{\pi_{\alpha}(1)} \) for some \( \alpha \in \{1, 2, \ldots, k\} \). Then using the similar argument as mentioned in Case 2, we can obtain \( \sum_{t=0}^{2^k-1} (C_t^g)_{h,j} - (C_t^g)_{g,i} = 0 \).

Similarly, we let \( h = g + u_1, j = i + u_2 \), and consider five cases below.

Case 1: Consider \( u_1 \geq 0, u_2 > 0 \) and \( i_{\pi_{\alpha}(1)} \neq j_{\pi_{\alpha}(1)} \) for some \( \alpha \in \{1, 2, \ldots, k\} \). By following a similar derivation in Case 1 of the first part, we can obtain \( \sum_{t=0}^{2^k-1} (C_t^g)_{h,j} - (C_t^g)_{g,i} = 0 \).

Case 2: Taking \( u_1 > 0 \) and \( u_2 > 0 \), we suppose \( i_{\pi_{\alpha}(1)} = j_{\pi_{\alpha}(1)} \) for all \( \alpha \in \{1, 2, \ldots, k\} \). The similar result can be obtained as mentioned in Case 2 of the first part. Thus, we have \( \sum_{t=0}^{2^k-1} (C_t^g)_{h,j} - (C_t^g)_{g,i} = 0 \).

Case 3: For \( u_1 > 0 \) and \( u_2 = 0 \), we assume \( g_{\pi_{\alpha}(1)} \neq h_{\pi_{\alpha}(1)} \) for some \( \alpha \in \{1, 2, \ldots, k\} \). We can obtain the similar result \( \sum_{t=0}^{2^k-1} (C_t^g)_{h,j} - (C_t^g)_{g,i} = 0 \) as argued in Case 3 of the first part.

Case 4: Suppose \( u_1 > 0, u_2 = 0 \), and \( g_{\pi_{\alpha}(1)} = h_{\pi_{\alpha}(1)} \) for all \( \alpha \in \{1, 2, \ldots, k\} \). Following the similar argument provided in Case 4 of the first part, we have \( \sum_{t=0}^{2^k-1} (C_t^g)_{h,j} - (C_t^g)_{g,i} = 0 \).

Case 5: Lastly, it remains to show that [16] holds for \((u_1, u_2) = (0, 0)\). From [11], we can obtain

\[
(C_t^g)_{g,i} - (C_t^g)_{g,i} = \frac{q}{2} \sum_{\alpha=1}^{k} (s_{\alpha} - v_{\alpha}) g_{\pi_{\alpha}(n_{\alpha})} \quad (17)
\]
where $s_n$, $v_n$, and $g_{\sigma,(n_n)}$ are the $\alpha$-th bit and the $\sigma$-th bit of binary representations of $s$, $v$, and $g$, respectively. We can observe that (17) is the linear combination of the term $g_{\sigma,(n_n)}$. Note that $g = \sum_{i=1}^{n} g_i 2^{i-1}$. For $g = 0, 1, \ldots, 2^n - 1$, there are $2^n - 1$ pairs fulfilling $\sum_{i=1}^{l} (C_{g,i}^{(t)}) = \sum_{j=1}^{n} (C_{g,j}^{(s)}) = \xi^t = -1$ and $2^n - 1$ pairs satisfying $\sum_{i=1}^{l} (C_{g,i}^{(t)}) = \sum_{j=1}^{n} (C_{g,j}^{(s)}) = \xi^s = 1$. Therefore, we have $\sum_{i=1}^{l} \sum_{j=1}^{n} (\xi^{(C_{g,i}^{(t)})(C_{g,j}^{(s)})}) = 0$.

Combining these five cases, we complete the proof.

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