TANGENTIAL VARIETIES OF SEGRE VARIETIES

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ABSTRACT. We determine the generators of the ideal of the tangential variety of a Segre variety, confirming a conjecture of Landsberg and Weyman.

1. Introduction

For a projective algebraic variety $X \subset \mathbb{P}^N$, the tangential variety $\tau(X)$ (also known as the tangent developable or the first osculating variety of $X$) is the union of all points on all embedded tangent lines to $X$. The points in $\tau(X)$ together with those lying on the secant lines to $X$ form the (first) secant variety of $X$, denoted $\sigma_2(X)$. Tangential and secant varieties were studied classically, among others by Terracini, and were brought into a modern light by F. L. Zak [Zak93]. The relationship between the tangential and secant varieties can be described as follows, as a consequence of Zak’s “Theorem on Tangencies”: the two varieties are either equal (the degenerate situation), or they both have the expected dimension and the tangential variety is a hypersurface in the secant variety (the typical situation). In this sense, a Segre variety with at least three factors is typical, while one with two factors is degenerate [JML12].

Besides understanding the dimensions of $\tau(X)$ and $\sigma_2(X)$, a basic problem is to understand their defining ideals. In [Rai10], the second author computes the ideal of $\sigma_2(X)$ in the case when $X$ is a Segre–Veronese variety. The purpose of this paper is to solve the similar problem for the tangential variety of a Segre variety. This problem is the content of the Landsberg–Weyman Conjecture stated below, after introducing some notation. We think of the Segre variety as the image $X$ of the embedding

$$\text{Seg} : \mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^* \longrightarrow \mathbb{P}(V_1^* \otimes \cdots \otimes V_n^*),$$

$$([e_1], \ldots, [e_n]) \longmapsto [e_1 \otimes \cdots \otimes e_n],$$

for $e_i \in V_i^*$. The equations of degree $k$ of $\tau(X)$ are $GL$-submodules of $\text{Sym}^k(V_1 \otimes \cdots \otimes V_n)$, and as such they decompose into irreducible representations $S_{\lambda^1}V_1 \otimes \cdots \otimes S_{\lambda^n}V_n$, where $S_{\lambda^i}$ is the Schur functor associated to the partition $\lambda^i$. We write $\Lambda^k = S_{(1^k)}$ for the exterior power functor.

Conjecture 1.1 ([LW07, Conjecture 7.6]). When $X$ is a Segre variety, $I(\tau(X))$ is generated by the submodules of quadrics which have at least four $\Lambda^2$ factors, the cubics with four $S_{(2,1)}$ factors.

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\footnote{For convenience we dualize vector spaces here so that our modules of polynomials may be written without the dual.}
factors and all other factors $S_{(3)}$, the cubics with at least one $\bigwedge^3$ factor, and the quartics with three $S_{(2,2)}$’s and all other factors $S_{(4)}$.

We call the equations appearing in the above conjecture the Landsberg–Weyman equations. The role of the cubics with at least one $\bigwedge^3$ factor, the equations of the subspace variety, is to reduce the problem to the case when $\dim(V_i) = 2$ (see [LW07b, Theorem 3.1]). We won’t then be concerned with these equations in the rest of the paper, and the reader should feel free to assume that the Segre variety is in fact a product of $\mathbb{P}^1$’s. This is in fact the context in which the conjecture is formulated in [LW07]. While the language of modules provides an efficient way to describe sets of equations that are invariant under a group action, in order to practically use these equations, one needs an explicit realization. This is described in [Oed11, Section 3.2].

The equations of degree 2 and 3 in Conjecture 1.1 are built from minors of matrices of flattenings, which were the main players in the case of the secant variety of the Segre. The quartics however are the new interesting equations for the tangential variety, constructed out of Cayley’s hyperdeterminant of a $2 \times 2 \times 2$ tensor [GKZ94]. The presence of these equations lead to an unexpected connection to the variety of principal minors of symmetric matrices (studied in [Oed11b], [HS07]), allowing the first author to prove that a subset of the Landsberg–Weyman equations define $\tau(X)$ set-theoretically [Oed11], a weaker version of Conjecture 1.1. Our main result confirms the Landsberg–Weyman conjecture in its strong, ideal theoretic, form:

**Theorem 4.1.** Let $X = \text{Seg}(\mathbb{P}V_1^* \times \mathbb{P}V_2^* \times \cdots \times \mathbb{P}V_n^*)$ be a Segre variety, where each $V_i$ is a vector space of dimension at least 2 over a field $K$ of characteristic zero. The ideal of $\tau(X)$ is generated by the Landsberg–Weyman equations, and moreover, for every nonnegative integer $r$ we have the decomposition of the degree $r$ part of its homogeneous coordinate ring

$$K[\tau(X)]_r = \bigoplus_{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathfrak{S}^\lambda V_1 \otimes \cdots \otimes \mathfrak{S}^\lambda V_n} m_\lambda,$$

where $m_\lambda$ is either 0 or 1, obtained as follows. Set

$$f_\lambda = \max_{i=1,\ldots,n} \lambda_i, \quad e_\lambda = \lambda_1 + \cdots + \lambda_n.$$

If some partition $\lambda_i$ has more than two parts, or if $e_\lambda < 2f_\lambda$, or if $e_\lambda > r$, then $m_\lambda = 0$. Otherwise $m_\lambda = 1$.

We note that the description of the coordinate ring of the tangential variety in the above theorem is a special case of [LW07, Theorem 5.2].

Our methods are to apply the techniques of the second author, which were used to determine the ideal of the secant variety to the Segre variety [Rai10]. This represents a departure from how this ideal was studied in the past in that we study the ideal via combinatorial properties of representations of the symmetric group. Since these techniques were successful in studying the ideal of the secant varieties to Segre variety, it is natural to expect that they would also apply to the tangential variety, and this is what we show in the remainder of the paper. The main new ingredient in our work is Proposition 3.19, which sheds some light on the multiplicative structure of the generic polynomials: we do know how to add tableaux, but we don’t yet have a really good grasp on how to multiply them.
This tool wasn’t necessary in [Rai10] because of the simplicity of the equations coming from minors of flattenings, but we expect it to be relevant in other situations where the structure of the generating set of the ideal is more involved.

For any variety, the ideal of defining equations allows one to test membership on that variety. This problem for various types of varieties is addressed in depth in [JML12], and the tangential variety is discussed in Chapter 8 in particular. In [Oed11], the first author pointed out applications of the tangential variety where the equations allow one to answer the question of membership for the following sets: the set of tensors with border rank 2 and rank $k \leq n$ (the secant variety is stratified by such tensors [BB]), a special Context–Specific Independence model, and a certain type of inverse eigenvalue problem. Another recent instance of the tangential variety is in [SZ11, §4], where the authors showed that after a non–linear change of coordinates to cumulant coordinates, the tangential variety becomes a toric variety, and they computed its ideal in the case $n = 5$ in cumulant coordinates.

We summarize the structure of the paper as follows. In Section 2 we give an explicit description of the tangential variety $\tau(X)$ of a Segre variety $X$, and characterize its equations in terms of linear algebra. In Section 3 we set up the generic case: we translate the descriptions of the Landsberg–Weyman equations and of the equations of $\tau(X)$ into a more combinatorial framework in Sections 3.1 and 3.2, following the methods of [Rai10]. This allows us to reduce Conjecture 1.1 to proving that certain representations of a product of symmetric groups coincide (Conjecture 3.12). In Section 3.4 we recall the language of graphs and tableaux from [Rai10], and we give a graphical description of the generic Landsberg–Weyman equations in Section 3.5. In Section 3.6 we collect a series of results on the generic equations of $\sigma_2(X)$ that will be relevant to the study of $\tau(X)$. Finally in Section 4 we put all the ingredients together to prove the Landsberg–Weyman Conjecture.

Notation. We denote the set $\{1, 2, \cdots, r\}$ by $[r]$. If $\mu = (\mu_1 \geq \mu_2 \geq \cdots)$ is a partition of $r$ (written $\mu \vdash r$) and $W$ a vector space, then $S_\mu W$ (resp. $[\mu]$) denotes the irreducible representation of the general linear group $GL(W)$ (resp. of the symmetric group $S_r$) corresponding to $\mu$. If $\mu = (r)$, then $S_\mu W$ is $Sym^r(W)$ and $[\mu]$ is the trivial $S_r$–representation. The $GL(W)$– (resp. $S_r$–) representations $U$ that we consider decompose as $U = \bigoplus U_\mu$ where $U_\mu \simeq (S_\mu W)^{m_\mu}$ (resp. $U_\mu \simeq [\mu]^{m_\mu}$) is the $\mu$–isotypic component of $U$. We make the analogous definitions when we work over products of general linear (resp. symmetric) groups, replacing partitions by $n$–tuples of partitions (called $n$–partitions and denoted by $\vdash^n$). For an introduction to the representation theory of general linear and symmetric groups, see [FH91]. For a short account of the relevant facts needed in this paper, see [Rai10, Section 2.3]. We will sometimes write $a \vdash r$ for a partition of $r$, to mean an $n$–tuple $a = (a_1, \cdots, a_n)$ with $a_1 + \cdots + a_n = r$. In contrast, $r$ will always mean the $n$–tuple $(r, r, \cdots, r)$.

2. Equations of the tangential variety of a Segre variety

For the rest of the paper, let $X$ denote the Segre variety, the image of the embedding

$$\text{Seg} : \mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^* \longrightarrow \mathbb{P}(V_1^* \otimes \cdots \otimes V_n^*)$$
defined above, where $V_i$ are vector spaces over a field of characteristic zero. The cone $\bar{\tau}(X)$ over the tangential variety $\tau(X)$ is the set of tensors obtained as

$$\lim_{t \to 0} \frac{(e_1 + tf_1) \otimes \cdots \otimes (e_n + tf_n) - e_1 \otimes \cdots \otimes e_n}{t}$$

$$= f_1 \otimes e_2 \otimes \cdots \otimes e_n + e_1 \otimes f_2 \otimes \cdots \otimes e_n + e_1 \otimes \cdots \otimes e_{n-1} \otimes f_n,$$

where $e_i, f_i \in V_i$, i.e. $\bar{\tau}(X)$ is the image of the map

$$s : (V_1^* \times \cdots \times V_n^*) \times (V_1^* \times \cdots \times V_n^*) \to V_1^* \otimes \cdots \otimes V_n^*,$$

$$(e_1, \ldots, e_n, f_1, \ldots, f_n) \mapsto \sum_{i=1}^n e_1 \otimes \cdots \otimes f_i \otimes \cdots \otimes e_n.$$

\$s$ corresponds to a ring map

$$s^\#: \text{Sym}(V_1 \otimes \cdots \otimes V_n) \to (\text{Sym}(V_1) \otimes \cdots \otimes \text{Sym}(V_n))^{\otimes 2},$$

which on generators acts as

$$v_1 \otimes \cdots \otimes v_n \mapsto \sum_{i=1}^n (v_1 \otimes \cdots \otimes 1 \otimes \cdots \otimes v_n) \otimes (1 \otimes \cdots \otimes v_i \otimes \cdots \otimes 1).$$

In degree $r$, $s^\#$ restricts to a map $s^\#_r$ between

$$S_{(r)}(V_1 \otimes \cdots \otimes V_n) \to \bigoplus_{a_1 + \cdots + a_n = r} (S_{(r-a_1)}V_1 \otimes \cdots \otimes S_{(r-a_n)}V_n) \otimes (S_{(a_1)}V_1 \otimes \cdots \otimes S_{(a_n)}V_n).$$

If we write $\underline{a} \vdash r$ to indicate the partition $a_1 + \cdots + a_n$ of $r$, it follows that $s^\#_r$ decomposes as

$$s^\#_r = \bigoplus_{\underline{a} \vdash r} \pi_{\underline{a}}(V),$$

where

$$\pi_{\underline{a}} = \pi_{\underline{a}}(V) : S_{(r)}(V_1 \otimes \cdots \otimes V_n) \to (S_{(r-a_1)}V_1 \otimes \cdots \otimes S_{(r-a_n)}V_n) \otimes (S_{(a_1)}V_1 \otimes \cdots \otimes S_{(a_n)}V_n)$$

is described in more detail below. We write $\pi_{\underline{a}}(V)$ to distinguish these maps from their generic versions which are introduced in Section 3.2. We write $\pi_{\underline{a}}$ for $\pi_{\underline{a}}(V)$ when there is no danger of confusion.

Let $m_j = \dim(V_j)$ and let $B_j = \{x_{i,j} : i \in [m_j]\}$ be a basis for $V_j$. For $a_1, \ldots, a_n$ positive integers, the vector space

$$S_{(a_1)}V_1 \otimes \cdots \otimes S_{(a_n)}V_n$$

has a basis $B = B_{a_1, \ldots, a_n}$ consisting of tensor products of monomials in the elements of the bases $B_1, \ldots, B_n$. We write this basis, suggestively, as

$$B = \text{Sym}^{a_1} B_1 \otimes \cdots \otimes \text{Sym}^{a_n} B_n.$$ 

We can index the elements of $B$ by $n$-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ of multisets $\alpha_i$ of size $a_i$ with entries in $\{1, \ldots, m_i = \dim(V_i)\}$, as follows. The $\alpha$–th element of the basis $B$ is

$$z_\alpha = (\prod_{i_1 \in \alpha_1} x_{i_1,1}) \otimes \cdots \otimes (\prod_{i_n \in \alpha_n} x_{i,n,n}).$$
When \( a_1 = \cdots = a_n = 1 \), we think of \( z_\alpha \) as a linear form in \( S = \text{Sym}(V_1 \otimes \cdots \otimes V_n) \), so that \( S = K[z_\alpha] \) is a polynomial ring in the variables \( z_\alpha \). We identify each \( z_\alpha \) with an \( 1 \times n \) block with entries \( \alpha_1, \cdots, \alpha_n \):

\[
\begin{bmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_n
\end{bmatrix}.
\]

We represent a monomial \( m = z_{\alpha_1} \cdots z_{\alpha_r} \) of degree \( r \) as an \( r \times n \) block \( M \), whose rows correspond to the variables \( z_{\alpha_i} \) in the way described above.

\[
m \equiv M = \begin{bmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_n \\
\alpha_1' & \alpha_2' & \cdots & \alpha_n' \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1 & \alpha_2 & \cdots & \alpha_n
\end{bmatrix}.
\]

Note that the order of the rows is irrelevant, since the \( z_{\alpha_i} \)'s commute.

For \( \underline{a} \vdash r \), we represent a monomial \( z_\beta \otimes z_\gamma \) in the target of the map \( \pi_{\underline{a}} \) as a \( 2 \times n \) block

\[
\begin{bmatrix}
\beta_1 & \beta_2 & \cdots & \beta_n \\
\gamma_1 & \gamma_2 & \cdots & \gamma_n
\end{bmatrix}
\]

where \( \beta_i \) are multisets of size \( r - a_i \) and \( \gamma_i \) are multisets of size \( a_i \) (the order of the rows is now important!). The map \( \pi_{\underline{a}} \) can then be written in terms of blocks as

\[
M \rightarrow \sum_{A_1 \cup \cdots \cup A_n = [r]} \begin{bmatrix}
\cdots & \{\alpha_{i_1} : i \notin A_j\} & \cdots \\
\cdots & \{\alpha_{i_2} : i \in A_j\} & \cdots
\end{bmatrix} (2.1)
\]

**Example 2.1.** Assume that \( r = 3, n = 4 \), and \( V_i \) are vector spaces of dimensions at least three. Let \( \underline{a} \vdash r \) with \( a_1 = 2, a_2 = a_3 = 0, a_4 = 1 \), and consider the monomial

\[
m = z_{1,1,2,3} \cdot z_{1,2,1,2} \cdot z_{2,3,2,1} \in S_{(3)}(V_1 \otimes V_2 \otimes V_3 \otimes V_4).
\]

We can represent it as the \( 3 \times 4 \) block

\[
M = \begin{bmatrix}
1 & 1 & 2 & 3 \\
1 & 2 & 1 & 2 \\
2 & 3 & 2 & 1
\end{bmatrix}.
\]

The map \( \pi_{\underline{a}} \) sends

\[
M \rightarrow \begin{bmatrix}
2 & 1,2,3 & 1,2,2 & 2,3 \\
1,1 & 1,2,2 & 1 & 1
\end{bmatrix} + \begin{bmatrix}
1 & 1,2,3 & 1,2,2 & 1,3 \\
1,2 & 1,2,2 & 2 & 1
\end{bmatrix} + \begin{bmatrix}
1 & 1,2,3 & 1,2,2 & 1,2 \\
1,2 & 1,2,2 & 3 & 1
\end{bmatrix}.
\]

The above discussion implies the following

**Proposition 2.2.** The equations of degree \( r \) of the tangential variety \( \tau(X) \) of the Segre variety \( X \) are precisely those elements of \( S_{(\underline{a})}(V_1 \otimes \cdots \otimes V_n) \) contained in the intersection of the kernels of the maps \( \pi_{\underline{a}} \) as \( \underline{a} \) ranges over the set of partitions of \( r \) with \( n \) terms.
3. The generic case

The material in this section is based on [Rai10, Section 3.2]. For a nonnegative integer \( r \) we write \( r^n \) for the \( n \)-tuple \((r, \ldots, r)\). We write \( \mathcal{S}_r = \mathcal{S}_r^n \) for the product of \( n \) copies of the symmetric group \( \mathcal{S}_r \). If \( A_1, \ldots, A_n \) are sets of size \( r \), we write \( A \) for \((A_1, \ldots, A_n)\), and \( \mathcal{S}_A \) for the product \( \mathcal{S}_{A_1} \times \cdots \times \mathcal{S}_{A_n} \), where \( \mathcal{S}_{A_i} \) is the group of permutations of \( A_i \).

Definition 3.1. Let \( A = (A_1, \ldots, A_n) \), \(|A_j| = r\), as above. We denote by \( U_A \) the vector space with basis consisting of monomials

\[
m = z_{\alpha^1} \cdots z_{\alpha^r},
\]

where \( \alpha^i \) are \( n \)-tuples and for each \( j \) we have \( \{\alpha^1_j, \ldots, \alpha^r_j\} = A_j \). Alternatively, \( U_A \) has a basis consisting of \( r \times n \) blocks \( M \), where each column of \( M \) yields a permutation of the elements of the set \( A_j \). When all \( A_j \) coincide with the set \([r]\), we write \( U_r \) for \( U_A \). As before, we identify two blocks if they differ by permutations of their rows. \( U_r \) is the generic version of the representation \( U_r(V) = S_{(r)}(V_1 \otimes \cdots \otimes V_n) \): when \( \dim(V_i) = r \) for \( i = 1, \ldots, n \), \( U_r \) is precisely the zero–weight space of \( U_r(V) \) with respect to the \( \mathfrak{sl} \)-action.

Example 3.2. For \( n = 3 \), \( r = 4 \), \( A_1 = A_3 = \{1, 2, 3, 4\} \), \( A_2 = \{2, 5, 7, 8\} \), a typical element of \( U_A \) is

\[
\begin{bmatrix}
1 & 2 & 1 \\
2 & 8 & 4 \\
4 & 7 & 2 \\
3 & 5 & 3
\end{bmatrix}
= z_{2,1} \cdot z_{2,8,4} \cdot z_{4,7,2} \cdot z_{3,5,3} = z_{3,5,3} \cdot z_{1,2,1} \cdot z_{4,7,2} \cdot z_{2,8,4} = 
\begin{bmatrix}
3 & 5 & 3 \\
1 & 2 & 1 \\
4 & 7 & 2 \\
2 & 8 & 4
\end{bmatrix}
\]

Consider now positive integers \( r \) and \( r' \), and let \( A, B \) be \( n \)-tuples with \(|A_i| = r\), \(|B_i| = r'\) and \( A_i \cup B_i = [r + r'] \). We have a natural multiplication map

\[
p_{A,B} : U_A \otimes U_B \rightarrow U_{r+r'}.
\]

The images of the maps \( p_{A,B} \) generate together the vector space \( U_{r+r'} \). The collection of all the maps \( p_{A,B} \) should be thought of as the generic analogue of the multiplication map

\[
S_{(r)}(V_1 \otimes \cdots \otimes V_n) \otimes S_{(r')} (V_1 \otimes \cdots \otimes V_n) \rightarrow S_{(r+r')} (V_1 \otimes \cdots \otimes V_n).
\]

We also need the following noncommutative version of the representations \( U_A \), which will only be used in the proof of Proposition 3.19.

Definition 3.3. With the above notation, we denote by \( U_{\text{nc}}^A \) the vector space with basis consisting of monomials

\[
m = z_{\alpha^1} \otimes \cdots \otimes z_{\alpha^r},
\]

where for each \( j \) we have \( \{\alpha^1_j, \ldots, \alpha^r_j\} = A_j \). Alternatively, \( U_{\text{nc}}^A \) can be seen as a space of \( r \times n \) blocks, as in Definition 3.1 where we no longer identify two blocks that differ by permutations of the rows.

Example 3.4. With the notation in Example 3.2 a typical element of \( U_{\text{nc}}^A \) is

\[
\begin{bmatrix}
1 & 2 & 1 \\
2 & 8 & 4 \\
4 & 7 & 2 \\
3 & 5 & 3
\end{bmatrix}
= z_{2,1} \otimes z_{2,8,4} \otimes z_{4,7,2} \otimes z_{3,5,3} \neq z_{3,5,3} \otimes z_{1,2,1} \otimes z_{4,7,2} \otimes z_{2,8,4} = 
\begin{bmatrix}
3 & 5 & 3 \\
1 & 2 & 1 \\
4 & 7 & 2 \\
2 & 8 & 4
\end{bmatrix}
\]
There is a natural right action of \( S_r \) on \( U_{nc} \), which we denote by \( * \) (see also [Rai10, Section 3.3]), where the \( j \)-th copy of \( S_r \) acts by permuting the positions in the \( j \)-th column of a block, rather than permuting the entries. The right action of \( S_r \) commutes with the left action of \( S_{2r} \). If we regard \( S_r \) as a subgroup of \( S_{2r} \) diagonally, then there is a natural identification

\[
U_{nc} = U_{nc}^* (\sum_{\sigma \in S_r} \sigma).
\]

**Remark 3.5.** The advantage of \( U_{nc} \) over \( U_{nc}^* \) is that it admits both a left and right action of \( S_r \), which will be exploited in the proof of Proposition 3.19. The multiplication maps \( p_{AB} \) defined in (3.1) have a natural analogue in the noncommutative case, which we denote by \( p_{AB}^{nc} \). The maps \( p_{AB}^{nc} \) are easily seen to be injective, so we will regard \( U_{nc} \otimes U_{nc} \) as subsets of \( U_{nc}^{2r} \) via these maps.

### 3.1. The generic Landsberg–Weyman equations.

**Definition 3.6.** For positive integers \( k \leq r \) and \( n \)-partition \( \mu \vdash n \), we consider the subrepresentation \( I_r(\mu) \subset U_r \) defined as

\[
I_r(\mu) = \sum_{|A_i| = k} p_{AB}((U_A)_{\mu} \otimes U_B),
\]

where \( A \cup B = [r] \) is shorthand for \( A_i \cup B_i = [r] \) for every \( i = 1, \ldots, n \), and \( (U_A)_{\mu} \) is the \( \mu \)-isotypic component of the representation \( U_A \). \( I_r(\mu) \) is the generic version of the degree \( r \) part of the ideal in \( \text{Sym}(V_1 \otimes \cdots \otimes V_n) \) generated by the polynomials in the \( \mu \)-isotypic component of \( S_k(V_1 \otimes \cdots \otimes V_n) \).

We can define the noncommutative version of \( I_r(\mu) \) analogously, and denote it by \( I_r^{nc}(\mu) \).

**Definition 3.7** (Generic Landsberg–Weyman equations). For a positive integer \( r \), we define the \( X \)-part of \( U_r \), denoted \( X(U_r) \), by

\[
X(U_r) = \sum_{\mu} I_r(\mu) \subset U_r,
\]

where the sum ranges over \( n \)-partitions \( \mu \vdash n \) with at least four \( \mu^i \)'s equal to \( (1,1) \). \( X(U_r) \) is the generic version of the degree \( r \) part of the ideal generated by the equations of degree 2 in Conjecture 1.1.

Similarly, the \( Y \)-part of \( U_r \), denoted \( Y(U_r) \) is the subrepresentation of \( U_r \) defined by

\[
Y(U_r) = \sum_{\mu} I_r(\mu) \subset U_r,
\]

where the sum is over \( \mu \vdash n \) with exactly four of the \( \mu^i \)'s equal to \( (2,1) \), and the rest equal to \( (3) \). The \( Z \)-part of \( U_r \), denoted \( Z(U_r) \), is defined analogously, but now \( \mu \) runs over \( n \)-partitions \( \mu \vdash n \) with exactly three of the \( \mu^i \)'s equal to \( (2,2) \), and the rest equal to \( (4) \).

We define the \( XY \)-part of \( U_r \) by

\[
XY(U_r) = X(U_r) + Y(U_r),
\]
and in a similar vein we obtain the $XZ$-, $YZ$- and $XYZ$- parts of $U_r$. Thus the $XYZ$-part of $U_r$ (i.e. $X(U_r) + Y(U_r) + Z(U_r)$) is the generic version of the degree $r$ part of the ideal generated by the Landsberg–Weyman equations. We call an element of $XYZ(U_r)$ a generic Landsberg–Weyman equation of degree $r$.

The above notation is not meant to be intuitive, but rather concise and easy to remember: $X$, $Y$, resp. $Z$ correspond to the degrees 2, 3, resp. 4 of the Landsberg–Weyman equations.

### 3.2. The generic equations of the tangential variety.

**Definition 3.8.** Given a positive integer $r$ and a partition $a \vdash r$, we write $U_a$ for the vector space with basis consisting of monomials

$$m = z_\beta \otimes z_\gamma,$$

where $\beta = (\beta_1, \ldots, \beta_n)$, $\gamma = (\gamma_1, \ldots, \gamma_n)$, with $\beta_i \cup \gamma_i = [r]$, $|\beta_i| = r - a_i$, $|\gamma_i| = a_i$. We can represent $m$ as a $2 \times n$ block

$$M = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n \end{bmatrix}.$$

$U_a$ is an $\mathfrak{S}_r$–representation, where the $j$–th copy of $\mathfrak{S}_r$ acts on the $j$–th column of a block $M$. $U_a$ is the generic version of

$$(S_{(r-a_1)}V_1 \otimes \cdots \otimes S_{(r-a_n)}V_n) \otimes (S_{(a_1)}V_1 \otimes \cdots \otimes S_{(a_n)}V_n).$$

**Definition 3.9 (Generic $\pi_a$).** For $a \vdash r$ as above, we consider the map

$$\pi_a : U_r \longrightarrow U_a$$

defined on blocks according to the formula (2.1).

**Example 3.10.** Assume that $r = 3$, $n = 4$, and let $a \vdash r$ with $a_1 = 2$, $a_2 = a_3 = 0$, $a_4 = 1$. Consider the monomial

$$m = z_{1,1,3,3} \cdot z_{3,2,1,2} \cdot z_{2,3,2,1} \in U_3.$$

We can represent it as the $3 \times 4$ block

$$M = \begin{bmatrix} 1 & 1 & 3 & 3 \\ 3 & 2 & 1 & 2 \\ 2 & 3 & 2 & 1 \end{bmatrix}.$$

The map $\pi_a$ sends

$$M \mapsto \begin{bmatrix} 2 & 1,2,3 & 1,2,3 & 2,3 \\ 1,3 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 1,2,3 & 1,2,3 & 1,3 \\ 1,2 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1,2,3 & 1,2,3 & 1,2 \\ 3,2 & 1,2 & 3 \end{bmatrix}.$$

**Definition 3.11 (Generic equations of tangential variety).** For a positive integer $r$, the generic version of the degree $r$ part of the ideal of the tangential variety consists of the intersection of the kernels of the maps $\pi_{\underline{a}}$, as $\underline{a}$ ranges over partitions of $r$. 
3.3. **The generic version of the Landsberg–Weyman conjecture.** The polarization–
specialization technique described in [Rai10] allows us to reformulate Conjecture 1.1 as

**Conjecture 3.12** (Generic Landsberg–Weyman conjecture). For any positive integer $r$, the $XYZ$–part of the $\mathcal{S}_\mathcal{A}^r$–module $U_r$ (the generic Landsberg–Weyman equations) coincides with the module of generic equations of the tangential variety.

The equivalence between Conjectures 1.1 and 3.12 is deduced just as [Rai10, Prop. 3.27].

3.4. **$n$–tableaux and graphs.** Fix a positive integer $r$ and an $n$–partition $\lambda \vdash^n r$. For each $j = 1, \ldots, n$, choose an indexing of the boxes of the Young diagram of shape $\lambda^j$ with index set $A_j$, with $|A_j| = r$. Most of the time we will choose $A_j = [r]$ and the canonical indexing of the boxes: increasingly from left to right and top to bottom. The choice of $A_j$ and indexing yields a Young symmetrizer $c_\lambda$ in the group algebra $K[\mathcal{S}_\mathcal{A}]$ of $\mathcal{S}_\mathcal{A}$. For a $\mathcal{S}_\mathcal{A}$–representation, we write $\text{hwt}_\lambda(U)$ for the vector subspace $c_\lambda \cdot U \subset U$, the “highest–weight space” of the representation $U$. It has dimension equal to the multiplicity in $U$ of the irreducible representation $[\lambda]$ of $\mathcal{S}_\mathcal{A}$ corresponding to the $n$–partition $\lambda$.

Given a block $M \in U$ (where $U$ is one of $U_r, U^{\text{nc}}_r, U_{\mathcal{A}}, U^{\text{nc}}_{\mathcal{A}}, U_{\mathcal{A}}$), we associate to the element $c_\lambda \cdot M \in \text{hwt}_\lambda(U)$ the $n$–tableau

$$T = T^1 \otimes \cdots \otimes T^n$$

of shape $\lambda$, obtained as follows (see also [Rai10, Definition 3.14]). Suppose that the block $M$ has the element $\alpha^j_i$ (or the set $\alpha^j_i$, if $U = U_{\mathcal{A}}$) in its $i$–th row and $j$–th column. Then we set equal to $i$ the entry (entries) in the box (boxes) of $T^j$ indexed by $\alpha^j_i$ (the elements of $\alpha^j_i$). In the “commutative” case ($U = U_r$ or $U_{\mathcal{A}}$), we identify two $n$–tableaux that are obtained from blocks differing only by permutations of their rows.

**Example 3.13.** We use the notation in Example 3.10. We let $\lambda \vdash^4 3$ be the 4–partition with $\lambda^3 = (2,1)$, and $M \in U_3$. The indexing of the boxes is the canonical one. We get

$$c_\lambda \cdot M = c_\lambda \cdot \begin{bmatrix} 1 & 1 & 3 & 3 \\ 3 & 2 & 1 & 2 \\ 2 & 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix} \otimes \begin{bmatrix} 2 & 3 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 3 & 2 \\ 1 \end{bmatrix}.$$
the same if they differ by a permutation of $A$, we shall also identify two graphs if they differ by a relabeling of their nodes. Any such graph $G$ defines an element in $\text{hwt}_\lambda(U_r)$.

All the graphs we consider in this paper will be oriented graphs with colored edges, associated to some $n$–tableaux as explained above.

**Example 3.14.** For the 3-tableau

$$T = \begin{array}{ccc}
3 & 5 & 1 \\
2 & 4 & 7 \\
6 & 7 & 5
\end{array} \otimes \begin{array}{ccc}
3 & 1 & 2 \\
4 & 7 & 6 \\
5 & 3 & 7
\end{array} \otimes \begin{array}{ccc}
1 & 5 & 3 \\
2 & 6 & 4
\end{array}$$

the associated graph is

$$G = \begin{array}{cccc}
1 & & & 2 \\
& 3 & \cdots & 7 \\
& & 4 & \cdots 5 \\
& & & 6 \\
& & & 7 \\
& & & 8
\end{array}$$

where color 1 corresponds to $\longrightarrow$, color 2 to $\cdots \longrightarrow$, and color 3 to $\cdots \longrightarrow$.

**Example 3.15.** Here are examples of the Landsberg–Weyman equations expressed as $n$–tableaux (see [Oed11, § 3.2]) and graphs for $n \leq 4$. Colors 1, 2, 3 are as before, and color 4 corresponds to $\cdots \longrightarrow$.

$$T = \begin{array}{ccc}
1 & 2 \\
3 & 4
\end{array} \otimes \begin{array}{ccc}
1 & 2 \\
3 & 4
\end{array} \otimes \begin{array}{ccc}
1 & 2 \\
3 & 4
\end{array}$$

is associated to

$$G = \begin{array}{cccc}
1 & & & 2 \\
& 3 & \cdots & 7 \\
& & 4 & \cdots 5 \\
& & & 6 \\
& & & 7 \\
& & & 8
\end{array}$$

Finally, we point out a non–example.

$$T = \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6
\end{array}$$

is associated to

$$G = \begin{array}{cccc}
1 & & & 2 \\
& 3 & \cdots & 7 \\
& & 4 & \cdots 5 \\
& & & 6 \\
& & & 7 \\
& & & 8
\end{array}$$

This graph is easily seen to be the zero polynomial, using Lemma 3.16a) below.
Lemma 3.16 ([Rai10, Lemma 4.7]). The following relations between $n$–tableaux hold (we suppress from the notation the parts of the $n$–tableaux that don’t change, and only illustrate the relevant subtableaux)

\[ a) \begin{array}{ccc} x & y \\ y & x \end{array} = - \begin{array}{ccc} y & x \\ x & y \end{array}, \text{ in particular } \begin{array}{ccc} x & x \\ x & x \end{array} = 0. \]

\[ b) \begin{array}{ccc} x & z \\ y & x \end{array} = \begin{array}{ccc} x & y \\ z & x \end{array} + \begin{array}{ccc} z & x \\ x & x \end{array}. \]

Relations like these are sometimes called straightening laws, or shuffling relations. On graphs, part (a) says that reversing an arrow changes the sign, and part (b) can be depicted as follows:

Example 3.17. Here’s an example of a graph containing a triangle, written as a linear combination of Landsberg–Weyman equations (see Lemma 3.24): the 4–tableau

\[ T = \begin{array}{ccc} 1 & 2 \\ 3 & 3 \\ 2 & 2 \end{array} \otimes \begin{array}{ccc} 1 & 2 \\ 3 & 3 \\ 2 & 2 \end{array} \otimes \begin{array}{ccc} 3 & 1 \\ 3 & 3 \\ 2 & 2 \end{array} \]

can be written as a linear combination of two 4–tableaux

\[ \begin{array}{ccc} 1 & 2 \\ 3 & 3 \\ 2 & 2 \end{array} \otimes \begin{array}{ccc} 1 & 2 \\ 3 & 3 \\ 2 & 2 \end{array} \otimes \begin{array}{ccc} 1 & 3 \\ 2 & 2 \\ 3 & 3 \end{array} \]

using Lemma (3.16). The associated graphs are

3.5. A graphical description of the Landsberg–Weyman equations. In this section we give a series of sufficient conditions for a graph to represent a generic Landsberg–Weyman equation. The main technical result is Proposition 3.19.

Definition 3.18. We consider the (lexicographic) partial order on $n$–partitions of $k$ given by

\[ \delta \prec \mu \]

if for each $j$, the smallest $i = i_j$ for which $\delta^j_i \neq \mu^j_i$ has the property $\delta^j_i < \mu^j_i$.  

When $r \geq k$ and $\lambda \vdash^n r$, $\delta \vdash^n k$, we write

\[ \delta \subset \lambda \]

to indicate that $\delta^j_i \leq \lambda^j_i$ for all $i, j$.  

Proposition 3.19. Let \( k \leq r \) be positive integers, let \( \mu \vdash n \), \( \lambda \vdash r \), \( \mu \subseteq \lambda \), and let \( I_{r}^{nc} \) be as in Definition 3.6. If \( T^{nc} \in U_{r}^{nc} \) is an \( n \)-tableau of shape \( \lambda \), containing a subtableau \( S \) of shape \( \mu \) with all its entries in \( \{1, \ldots , k\} \), then
\[
T^{nc} \in \sum_{\delta \subseteq \mu} I_{r}^{nc}(\delta)
\]
If we write \( T \) for the “commutative” tableau in \( U_{r} \) corresponding to \( T^{nc} \), then
\[
T \in \sum_{\delta \subseteq \mu} I_{r}(\delta)
\]
Proof. The last part of the proposition follows from the first by symmetrization: multiply everything on the right by \( \sum_{\sigma \in S_{r}} \sigma \). It is then enough to treat the noncommutative case.

For \( \delta \vdash k \), we denote by \( W(\delta) \) the subspace of \( \text{hwt}_{\lambda}(U_{r}^{nc}) \) spanned by \( n \)-tableaux containing a subtableau of shape \( \delta \) with all its entries in \( \{1, 2, \ldots , k\} \). We let
\[
W_{<\mu} = \sum_{\delta <\mu} W(\delta),
\]
and prove by induction that
\[
W(\mu) \in I_{r}(\mu) + W_{<\mu},
\]
from which the conclusion follows easily. Fix \( T^{nc} \in W(\mu) \). We show that for an appropriately chosen Young symmetrizer \( c_{\mu} \) in the group algebra \( K[\mathfrak{S}_{k}] \) of \( \mathfrak{S}_{k} \) (where we regard \( \mathfrak{S}_{k} \) as a subgroup of \( \mathfrak{S}_{r} \) by letting each copy of \( \mathfrak{S}_{k} \) act on the subset \([k] \subseteq [r]\)), there exists a positive integer \( n_{\mu} \) such that
\[
T^{nc} * c_{\mu} = n_{\mu} * T^{nc} \mod W_{<\mu}.
\]
(3.2)
Since \( T^{nc} * c_{\mu} \in I_{r}(\mu) \), the conclusion follows after dividing by \( n_{\mu} \).

We take \( c_{\mu} \) to be the Young symmetrizer corresponding to the indexing of the boxes of the Young diagram of shape \( \mu \) induced by the filling of the \( n \)-tableau \( S \): the \( i \)-th box of \( \mu^{j} \) is precisely the box whose entry in \( S^{j} \) is equal to \( i \). The reason why the relation \( \text{(3.2)} \) holds is roughly speaking the fact that the shuffling relations inside the subdiagram \( \mu \) induce relations on \( n \)-tableaux of shape \( \lambda \), modulo the subspace \( W_{<\mu} \). The following simple example will make the proof of the general statement clear.

Example 3.20. Take \( n = 1 \), \( r = 3 \), \( \lambda = (2, 1) \), \( k = 2 \), \( \mu = (2) \), and
\[
T^{nc} = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix} = z_{1} \otimes z_{2} \otimes z_{3} - z_{3} \otimes z_{2} \otimes z_{1} + z_{2} \otimes z_{1} \otimes z_{3} - z_{3} \otimes z_{1} \otimes z_{2}.
\]
We have \( S = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix} \) and \( c_{\mu} = Id + (1, 2) \in K[\mathfrak{S}_{3}] \), where \( Id \) denotes the identity permutation, and \( (1, 2) \) is the transposition of 1 and 2. Using \( \text{(3.16b)} \), we get
\[
T^{nc} * c_{\mu} = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix} + 2 \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix} + \left( \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix} \right) = 2 \cdot \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix},
\]
where
\[
T^{nc} * c_{\mu} = (z_{1} \otimes z_{2} + z_{2} \otimes z_{1}) \otimes z_{3} - (z_{3} \otimes z_{2} + z_{2} \otimes z_{3}) \otimes z_{1} + (z_{2} \otimes z_{1} + z_{1} \otimes z_{2}) \otimes z_{3} - (z_{3} \otimes z_{1} + z_{1} \otimes z_{3}) \otimes z_{2}.
\]
Each of the terms on the RHS are easily seen to be contained in \((U_A)_\mu \otimes U_B\) for an appropriate choice of a partition \(A \cup B = [3]\), with \(|A| = 2, |B| = 1\).

The only shuffling relation for tableaux of shape \(\mu = (2)\) is given by

\[
1\ 2 = 2\ 1.
\]

and this extends to a relation between tableaux of shape \(\lambda = (2, 1)\) but only modulo the subspace \(W_{\prec \mu}\) of tableaux of shape \(\lambda\) that contain

\[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} = \begin{bmatrix}
2 \\
1 \\
3
\end{bmatrix} + \begin{bmatrix}
1 \\
3 \\
2
\end{bmatrix}.
\]

Going back to the general \(\mu\) and \(\lambda\), we note that every shuffling relation (see [Wey03, Section 2.1] or [Rai10, Lemma 3.16]) on a subtableau \(S\) of an \(n\)-tableau \(T^{nc} \in W(\mu)\) extends to a shuffling relation where all the new tableaux are in \(W_{\prec \mu}\). Since the shuffling relations generate all relations between \(n\)-tableaux [Wey03, Proposition 2.1.9], and

\[
S * c_\mu = n_\mu \cdot S,
\]

where \(n_\mu = |\mathfrak{S}_k|/\dim(\mu)\) (see [FH91, Lemma 4.26]), it follows that this relation can be lifted modulo \(W_{\prec \mu}\) to a relation involving \(T^{nc}\), that is (3.2) holds.

For the rest of this section we fix a positive integer \(r\), an \(n\)-partition \(\lambda \vdash n\) with each \(\lambda^i\) having at most two parts, and consider \(T\) to be an \(n\)-tableau of shape \(\lambda\). We write \(G\) for the associated graph.

**Lemma 3.21.** If \(T\) contains three columns of size two with entries \(\{1, 2\}\), then \(T \in X(U_r)\). Equivalently, \(G \in X(U_r)\) if it contains

\[
\begin{bmatrix}
1 \\
2
\end{bmatrix}
\]

**Proof.** It follows by Proposition 3.19 that

\[
T \in X(U_r) + \sum_{\delta} I_r(\delta),
\]

where \(\delta\) runs over a subset of \(n\)-partitions of \(2\) with precisely three \(\delta^i\)'s equal to \((1, 1)\). But for such \(\delta\), \(I_r(\delta) = 0\), because \((U_2)_\delta = 0\). This can be seen for example from the formula of the decomposition of the plethysm \(S(2)(V_1 \otimes \cdots \otimes V_n)\), or by a quick calculation using Lemma 3.16a). \(\square\)

**Lemma 3.22.** If \(T\) contains five columns of size two with entries in the set \(\{1, 2, 3\}\), then \(T \in X(U_r)\).

**Proof.** Using the relation

\[
\begin{bmatrix}
2 \\
1 \\
3
\end{bmatrix} = \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} + \begin{bmatrix}
2 \\
3 \\
1
\end{bmatrix}
\]

of Lemma 3.16 we may assume that the five columns of \(T\) have entries \(\{1, i\}\), with \(i = 2\) or \(3\). It follows that (up to interchanging the labels 2 and 3) we may assume that at least three of these columns contain the entries \(\{1, 2\}\). The conclusion then follows from Lemma 3.21. \(\square\)
Lemma 3.23. If $T$ contains four columns of size two with entries in the set $\{1, 2, 3\}$, then $T \in XY(U_r)$. In particular, $G \in XY(U_r)$ if it contains

$$\begin{array}{c}
\{1, 2, 3\} \\
\{1, 2, 3\}
\end{array}$$

Proof. It follows by Proposition 3.19 that

$$T \in Y(U_r) + \sum_\delta I_r(\delta),$$

where $\delta$ runs over a subset of $n$-partitions of 3 with at least five $\delta^j$s equal to (2, 1). Using Lemma 3.22, we obtain

$$\sum_\delta I_r(\delta) \subset X(U_r),$$

i.e. $T \in Y(U_r) + X(U_r) = XY(U_r)$. \qed

Lemma 3.24. If $T$ contains the columns

$$\begin{array}{c}
1 \\
2 \\
3
\end{array} \quad \begin{array}{c}
1 \\
2 \\
3
\end{array} \quad \begin{array}{c}
2 \\
3
\end{array}$$

then $T \in XY(U_r)$. Equivalently, if $G$ contains a triangle then $G \in XY(U_r)$.

Proof. If $T$ contains one other column with entries $\{i, j\} \subset \{1, 2, 3\}$, then the conclusion follows from Lemma 3.23. We let $W$ be the span of the tableaux containing $\begin{array}{c}
1 \\
2 \\
3
\end{array}$ and $\begin{array}{c}
2 \\
3
\end{array}$ Assuming $T$ only contains three columns of size two with all entries in $\{1, 2, 3\}$, we will show that $T \in W$. We may assume that $T^1 \otimes T^2 \otimes T^3$ contains the subtableau

$$S = \begin{array}{c}
1 \\
2 \\
3
\end{array} \otimes \begin{array}{c}
2 \\
1 \\
3
\end{array} \otimes \begin{array}{c}
3 \\
2 \\
1
\end{array}$$

The rest of the argument goes along the ideas in the proof of Proposition 3.19. $S = 0$ as an element of $U_3$, and this is a consequence of the shuffling relations; but these relations can be lifted to relations involving $T$, modulo the subspace $W$. We make this argument explicit for the reader’s convenience. Using the relations

$$\begin{array}{c}
x \\
y
\end{array} = \begin{array}{c}
y \\
x
\end{array} + \begin{array}{c}
x \\
y
\end{array}$$

in the tableaux $T^i, i > 3$, with $\{x, y\} \subset \{1, 2, 3\}$, we see that, modulo $W$, we can perform any permutation of boxes labeled $\{1, 2, 3\}$ in these tableaux. This implies that a permutation of the labels $\{1, 2, 3\}$ in $T^1, T^2, T^3$ doesn’t change $T$ (modulo $W$). We thus have

$$\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3
\end{array} \otimes \begin{array}{c}
1 \\
2 \\
3
\end{array} \otimes \begin{array}{c}
2 \\
3 \\
1
\end{array} = \begin{array}{c}
1 \\
2 \\
3
\end{array} \otimes \left( \begin{array}{c}
1 \\
2 \\
3
\end{array} + \begin{array}{c}
2 \\
3 \\
1
\end{array} \right) \otimes \left( \begin{array}{c}
2 \\
3 \\
1
\end{array} + \begin{array}{c}
3 \\
1 \\
2
\end{array} \right)
\end{array}$$

$$= \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3
\end{array} \otimes \begin{array}{c}
1 \\
2 \\
3
\end{array} \otimes \begin{array}{c}
2 \\
3 \\
1
\end{array} + \begin{array}{c}
1 \\
2 \\
3
\end{array} \otimes \begin{array}{c}
1 \\
2 \\
3
\end{array} \otimes \begin{array}{c}
2 \\
3 \\
1
\end{array} + \begin{array}{c}
1 \\
2 \\
3
\end{array} \otimes \begin{array}{c}
2 \\
3 \\
1
\end{array} \otimes \begin{array}{c}
3 \\
1 \\
2
\end{array} + \begin{array}{c}
1 \\
2 \\
3
\end{array} \otimes \begin{array}{c}
2 \\
3 \\
1
\end{array} \otimes \begin{array}{c}
3 \\
1 \\
2
\end{array}
\end{array}.$$
We write $A, B, C, D$ for the four $n$–tableaux above. We have $B \equiv -T \mod W$ (permute the labels $\{1, 2, 3\}$), and $C \equiv -C \mod W$ (permute $\{1, 2\}$). We have

$$D = \begin{array}{ccc}
3 & 1 & 2 \\
2 & 3 & 1 \\
\end{array} \otimes \begin{array}{ccc}
2 & 1 & 3 \\
3 & 2 & 1 \\
\end{array} + \begin{array}{ccc}
1 & 2 & 3 \\
2 & 1 & 3 \\
\end{array} \otimes \begin{array}{ccc}
3 & 1 & 2 \\
1 & 3 & 2 \\
\end{array} \equiv -A - T \mod W$$

It follows that

$$T = A + B + C + D \equiv A - T + 0 + (-A - T) \mod W,$$

i.e. $3T \in W$, thus $T \in W$. □

**Lemma 3.25.** If $T = (T_1, \ldots, T_n)$ is such that three of the $T_i$’s contain a subtableau $\begin{array}{llll}
a & b \\
c & d \\
\end{array}$ with $\{a, b, c, d\} = \{1, 2, 3, 4\}$, then $T \in XYZ(U_r)$. In fact, the same conclusion holds if two of the $T_i$’s contain a subtableau as before, and a third contains a column consisting of two elements of the set $\{1, 2, 3, 4\}$. In particular, $G \in XYZ(U_r)$ if it contains

\[
\begin{array}{ccc}
1 & \rightarrow & 2 \\
3 & \rightarrow & 4 \\
\end{array}
\]

**Proof.** Note first that the multiplicity of $S(2,2)_1 \otimes S(2,2)_2 \otimes S(3,1)_3 \otimes \bigotimes_{i>3} S(4)_i$ in $S(4)(V_1 \otimes \cdots \otimes V_n)$ (and of all its permutations) is equal to zero: to prove this, it suffices to assume that $n = 3$ [Oed11b, Lem. 5.4], in which case the SchurRings package [RS11] in Macaulay2 [GS] can be used to check the assertion. Alternatively, use the character table of $S_4$ [FH91, Ex. 2.22] to show that $\langle \chi^2_{(2,2)}, \chi_{(3,1)} \rangle = 0$, where $\chi_\mu$ is the character of the $S_4$–representation $[\mu]$.

Using this observation together with Proposition 3.19, it follows in both situations described in the statement of the lemma that

$$T \in Z(U_r) + \sum_\delta I_r(\delta),$$

where $\delta$ runs over a subset of $n$–partitions of 4 with at least two $\delta^i$’s (say $\delta^3$ and $\delta^2$) equal to $(2,2)$, and at least two other $\delta^j$’s ($\delta^3, \delta^4$) having precisely two parts (i.e. equal to $(3,1)$, because if one of $\delta^3, \delta^4$ equals $(2,2)$, then the corresponding tableau is by definition contained in $Z(U_r)$). Fix one such $\delta$, and consider an $n$–tableau $S$ of shape $\delta$. We show that $S \in XYZ(U_4)$, which yields the desired conclusion.

Using Lemma 3.16b) in the form

$$\begin{array}{llll}
a & 1 & 1 & a \\
b & 1 & a & 1 \\
\end{array} = \begin{array}{llll}
a & 1 & b & a \\
b & 1 & b & 1 \\
\end{array} = \begin{array}{llll}
a & 1 & b & a \\
b & 1 & b & 1 \\
\end{array} - \begin{array}{llll}
a & 1 & b & a \\
b & 1 & b & 1 \\
\end{array}$$

we may assume that $S^3$ and $S^4$ have the first column equal to $\begin{array}{llll}
1 & * \\
2 & 2 \\
\end{array}$. After relabeling, the first columns of $S^3$ and $S^4$ are $\begin{array}{llll}
1 & 1 & 1 & 2 \\
2 & 2 & 1 & 3 \\
\end{array}$ or $\begin{array}{llll}
1 & 1 & 1 & 2 \\
2 & 2 & 1 & 3 \\
\end{array}$. Now if $S^1$ or $S^2$ contains the column $\begin{array}{llll}
1 & 1 & 1 & 2 \\
2 & 2 & 1 & 3 \\
\end{array}$, then its other column has to be $\begin{array}{llll}
2 & 3 & 4 & 1 \\
3 & 4 & 1 & 2 \\
\end{array}$. Using Lemma 3.16b) again as above, with $a = 2, b = 3$, we see that in fact we may assume that 1 only occurs together with 2 or 3 in the columns.
of size two of $S^1, \cdots, S^4$. We can then apply Lemma 3.23 to conclude that $S \in XY(U_4)$, which concludes the proof of the lemma.

3.6. MCB graphs and graphs containing triangles.

Definition 3.26 (MCB graphs, [Rai10, Section 4.2.3]). A maximally connected bipartite (MCB) graph $G$ is one (oriented graph with colored edges as in the previous section) that is either bipartite and connected, or is the union of a tree and a collection of isolated nodes. The type $(a \geq b; \lambda)$ (or just $(a, b)$ when $\lambda$ is understood) of $G$ is a pair of integers representing the sizes of the sets $A, B$ in the bipartition of its maximal connected component, together with an $n$–partition $\lambda$ specifying the number of edges of $G$ of every given color. $G$ is canonically oriented if all edges have endpoints in the smaller set of the bipartition (i.e. in $B$). If $a = b$, there are two canonical orientations.

Example 3.27. The graph in example 3.14 is a canonically oriented MCB–graph of type $(3, 3)$. The graph $G'$ obtained by reversing the directions of all arrows of $G$ is also a canonically oriented MCB–graph of type $(3, 3)$.

The following proposition collects a series of facts that will be used freely throughout the proof of Theorem 4.1.

Proposition 3.28 ([Rai10, Section 4.2]). Fix a positive integer $r$ and an $n$–partition $\lambda \vdash n$ with each $\lambda^i$ having at most two parts. With the usual identification of $n$–tableaux, graphs, and elements of $\text{hwt}_\lambda(U_r)$, and letting $f_\lambda = \max\{\lambda^i_2\}$ (as in Theorem 4.1), we have:

1. There exists an MCB–graph of type $(a, b)$ iff $b \geq f_\lambda$ and $e_\lambda \geq 2f_\lambda - 1$.
2. The space $\text{hwt}_\lambda(U_r)$ is spanned by MCB–graphs and graphs containing a triangle.
3. Any two canonically oriented MCB–graphs of the same type differ by a linear combination of graphs containing a triangle.
4. In particular, an MCB–graph of type $(a, a)$ with an odd number of edges (such as the graphs in part (1) having $e_\lambda = 2f_\lambda - 1$) is a linear combination of graphs containing a triangle.

4. The proof of the Landsberg–Weyman conjecture

The main result of our paper confirms the Landsberg–Weyman conjecture:

Theorem 4.1. Let $X = \text{Seg}(\mathbb{P}V_1^* \times \mathbb{P}V_2^* \times \cdots \times \mathbb{P}V_n^*)$ be a Segre variety, where each $V_i$ is a vector space of dimension at least 2 over a field $K$ of characteristic zero. The ideal of $\tau(X)$ is generated by the Landsberg–Weyman equations, and moreover, for every nonnegative integer $r$ we have the decomposition of the degree $r$ part of its homogeneous coordinate ring

$$K[\tau(X)]_r = \bigoplus_{\lambda=(\lambda^1, \cdots, \lambda^n)}^\lambda (S_{\lambda^1} V_1 \otimes \cdots \otimes S_{\lambda^n} V_n)^{m_\lambda},$$

where $m_\lambda$ is either 0 or 1, obtained as follows. Set

$$f_\lambda = \max_{i=1, \cdots, n} \{\lambda^i\}, \quad e_\lambda = \lambda^1_2 + \cdots + \lambda^n_2.$$

If some partition $\lambda^i$ has more than two parts, or if $e_\lambda < 2f_\lambda$, or if $e_\lambda > r$, then $m_\lambda = 0$. Otherwise $m_\lambda = 1$. 
Remark 4.2. In terms of graphs, \( r \) is the number of vertices, \( e_\lambda \) is the total number of edges, and \( f_\lambda \) is the maximum number of edges of a single color.

**Proof of Theorem 4.1.** We can reduce to the generic case (Section 3), using the polarization–specialization technique described in [Rai10], so it suffices to prove Conjecture 3.12.

By Lemma 3.24, the graphs containing a triangle are in the span of the generic Landsberg–Weyman equations, so we only need to focus on MCB–graphs. To prove Theorem 4.1 it suffices to show that

(i) When \( e_\lambda > r \), for any given type \((a, b)\) of which there exist nonzero MCB–graphs, one can produce one such graph which is in the \( XYZ \)–part of \( U_r \).

(ii) When \( e_\lambda = r \), the space of MCB–graphs is spanned, modulo the Landsberg–Weyman equations, by those of type \((a, b)\) with \( a - b \leq 1 \).

(iii) When \( e_\lambda < r \), the space of MCB–graphs is spanned, modulo the Landsberg–Weyman equations, by those of type \((a, b)\) with \( a - b \leq 2 \).

Proving (i) suffices to conclude that \( m_\lambda = 0 \) when \( e_\lambda > r \): this follows from Proposition 3.28(2) and the fact that graphs containing triangles are generic Landsberg–Weyman equations (Lemma 3.24). When \( e_\lambda \leq r \) and \( e_\lambda < 2f_\lambda \), there are no MCB–graphs, so \( m_\lambda = 0 \). We may then assume that \( e_\lambda \geq 2f_\lambda \), in which case [LW07, Theorem 5.2] states that \( m_\lambda = 1 \) (we will give an independent proof that \( m_\lambda \geq 1 \) in Lemma 4.5). To finish the proof of Theorem 4.1, it suffices to show that \( m_\lambda \leq 1 \), or equivalently that the spaces of MCB–graphs, modulo the Landsberg–Weyman relations, are at most one–dimensional.

To finish the proof of Theorem 4.1 it remains to support the assertions (i), (ii) and (iii).

### 4.1. Proof of (i).

Fix a type \((a, b)\) for which there exists a nonzero MCB–graph of type \((a, b)\). Since \( e_\lambda > r \), then we need to have edges of at least three different colors \((f_\lambda \leq r/2)\). Assume that \( \lambda_2^1 \geq \lambda_2^2 \geq \cdots \). We have that \( 2\lambda_2^1 = 2f_\lambda \leq r \leq e_\lambda - 1 \). The equality \( 2f_\lambda = e_\lambda - 1 \) can only occur if \( r = 2f_\lambda \) and \( e_\lambda = r + 1 \), in which case every MCB–graph is a Landsberg–Weyman equation (Proposition 3.28(4) and Lemma 3.24). We may thus assume that \( 2f_\lambda \leq e_\lambda - 2 \).

If \( r = 2 \), then \( e_\lambda \geq 3 \), so any MCB–graph is in \( X(U_2) \), by Lemma 3.21. We may thus assume that \( r \geq 3 \).

Assume first that \( \lambda_2^1 = 0 \), i.e. precisely three colors occur \((1, 2, 3, \text{corresponding to arrows as in Example 3.14})\). We construct an MCB–graph containing

\[
\begin{array}{cccc}
1 & \sim & 2 \\
\downarrow & & \downarrow \\
3 & \sim & 4
\end{array}
\]

which will have to be contained in \( XYZ(U_r) \) by Lemma 3.25. To construct such a graph, note first that \( e_\lambda \geq 4 \), so that there are at least two edges of the same color. Since any
such two edges are disjoint, we must have \( r \geq 4 \). The inequality \( 2f_\lambda \leq e_\lambda - 2 \) implies that \( \lambda_2^3 \geq 2 \). Since \( 2(f_\lambda - 2) \leq e_\lambda - 6 \) and \( 2(f_\lambda - 2) \leq r - 4 \), there exists an MCB–graph \( G' \) of type \((a - 2, b - 2)\) with \( r - 4 \) vertices and \( \lambda_2^2 - 2 \) edges of color \( i \), for \( i = 1, 2, 3 \). Joining \( G' \) with the graph in (4.1) by an edge of color 3, we get the desired MCB–graph. This is always possible unless all vertices of \( G' \) are incident to an edge of color 3, which can happen only if \( \lambda_2^1 = \lambda_2^2 = \lambda_2^3 = 1 \) and \( r = 2l \). In this case, we take \( G \) to be the graph with double edges \((1, 2), (3, 4), \ldots, (r - 1, r)\) of colors 1 and 2, and edges \((1, 4), (3, 6), (5, 8), \ldots, (r - 1, 2)\) of color 3. For example, when \( r = 4 \), we get the hyperdeterminantal Landsberg–Weyman equation given by the graph

![Graph Example](image1)

and for \( r = 6 \) we get

![Graph Example](image2)

Assume now that \( \lambda_2^4 \geq 1 \), i.e. at least four colors occur. We construct an MCB–graph of type \((a, b)\) containing

![Graph Example](image3)

where \( \longrightarrow \) corresponds to color 4. Such a graph is contained in \( X(U_r) \), by Lemma 3.21.

Let’s first consider the case when \( a = b = f_\lambda \). If \( f_\lambda = 1 \), then \( r = 2 \) and \( G \) is just the graph with two vertices and one edge of color \( i \) for each \( i \) with \( \lambda_2^i = 1 \). Suppose now that \( f_\lambda > 1 \), so that \( r = 2f_\lambda \geq 4 \), and begin building \( G \) from the graph \( G_0 \)

![Graph Example](image4)

We draw an edge of color 1 between each of the pairs of vertices \((5, 6), (7, 8), (9, 10) \cdots \), and one edge of any (available) color between each of the pairs \((3, 6), (5, 8), (7, 10) \cdots \). This is possible because the total number of edges is \( e_\lambda \geq 2f_\lambda + 1 \). In this way we get a connected graph, so to obtain the desired MCB–graph, we need to add the remaining (up to \( \lambda_2^i \)) edges of each color \( i \). We do this in steps: for each \( i \) for which we haven’t yet used \( \lambda_2^i \) edges of color \( i \) we find \(\mathcal{B}, \mathcal{B}'\) with \( x \) odd, \( y \) even, such that no edge of color \( i \) is incident to either of \(\mathcal{B}, \mathcal{B}'\) (this is possible because \( f_\lambda \geq \lambda_2^i \)). We draw an edge \((x, y)\) of color \( i \) and proceed to the next step. It is clear that this construction provides an MCB–graph as desired.

**Example 4.3.** Suppose \( r = 8 \), \( \lambda_2^1 = 4 \), \( \lambda_2^2 = 3 \), \( \lambda_2^3 = 2 \) and \( \lambda_2^4 = 1 \), and start with \( G_0 \) above. We first add the arrows of color 1 between the pairs of vertices \((5, 6)\) and \((7, 8)\), then draw
arrows of color 2 between the pairs (3, 6) and (5, 8). Then we add in the remaining edge of color 3.

For the general case of an MCB–graph of type \((a, b)\), with \(r = a + b\) vertices, \(a \geq b \geq \lambda\), we start with an MCB–graph \(G^0\) of type \((f_\lambda, f_\lambda)\) and \(2f_\lambda\) vertices, as constructed in the previous paragraph. We construct a sequence of MCB–graphs \(G^1, \ldots, G^{r-2f_\lambda}\), where \(G^j\) has type \((a', b')\) with \(f_\lambda \leq a' \leq a\), \(f_\lambda \leq b' \leq b\), and \(a' + b' = 2f_\lambda + j\) vertices. Let \(E\) be the set of three edges (of colors 1, 3 and 4) appearing in the subgraph \(1.2\) of \(\tilde{G}\). Suppose we’ve already constructed \(G^j\), of type \((a', b')\), and assume without loss of generality that \(a' < a\). There exist a nondisconnecting edge \((x, y)\) of \(G^j\) different from those in \(E\) (because the genus of the graph \(G^j\) is \(e_\lambda - a' - b' + 1 > e_\lambda - r + 1 \geq 2\), so we can replace \((x, y)\) with an edge \((2f_\lambda + j, y)\) of the same color to obtain an MCB–graph \(G^{j+1}\) with one more vertex than \(G^j\). The graph \(G = G^{r-2f_\lambda}\) is the desired MCB–graph of type \((a, b)\).

**Example 4.4.** Assume now that \(\lambda_1 = 4\), \(\lambda_2 = 3\), \(\lambda_3 = 2\) and \(\lambda_4 = 1\), as in Example 4.3 but now \(r = 9\) and \((a, b) = (5, 4)\). We write \(G^0\) for the graph constructed in Example 4.3 and note that the edge (3, 6) is a nondisconnecting edge. We construct the graph \(G = G^1\) by removing it from \(G^0\), and replacing it by an edge (9, 6) of the same color:

4.2. **Proof of (ii).** Consider a type \((a, b)\), with \(a - b \geq 2\), for which there exists a nonzero MCB–graph. Note that MCB–graphs with edges of only two colors can exists only when \(a - b \leq 1\). We show that, modulo the Landsberg–Weyman equations, an MCB–graph \(G\) of type \((a, b)\) coincides with one of type \((a-1, b+1)\). We assume as before that \(\lambda_2^1 \geq \lambda_2^2 \geq \cdots\).
Suppose first that $\lambda_4^2 = 0$. If $\lambda_3^3 = 1$, then removing the unique edge of color 3 from the MCB–graph $G$ yields a disjoint union of a chain and an even cycle (which might be empty). In particular, the type $(a, b)$ has to satisfy $a - b \leq 1$. We may thus assume that $\lambda_3^3 \geq 2$ and that $G$ contains

$$G_0 = \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array}
\begin{array}{c}
4 \\
5
\end{array}
\end{array}$$

with (1) and (2) leaves. To see this, note that there exists an MCB–graph (tree) of type $(a - 3, b - 2)$ with $\lambda_i^2 - 2$ edges of color $i = 1, 2, 3$. Since $a - 3 > b - 2$, one of the $a - 3$ vertices “on the left” has no incident edge of color 3. We can therefore join this vertex with the vertex (5) of (4.3) by an edge of color 3 to obtain an MCB–graph $G$ of type $(a, b)$.

The relation

$$G_0 = \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array}
\begin{array}{c}
4 \\
5
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array}
\begin{array}{c}
4 \\
5
\end{array}
\end{array}$$

realizes $G$ as a sum between an element of $XYZ(U_r)$ (Lemma 3.25) and an MCB–graph of type $(a - 1, b + 1)$, as desired.

Suppose now that $\lambda_4^2 \geq 1$. We may assume that $G$ contains

$$G_0 = \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array}
\begin{array}{c}
4 \\
5
\end{array}
\end{array}$$

where (1) is a leaf and (2) has no incident edge of color 1. To see this, note that if $b > f_\lambda$ then $a - 3 \geq b - 1 \geq f_\lambda$, thus there exists an MCB–graph $\tilde{G}$ of type $(a - 3, b - 1)$ with $\lambda_i^2 - 1$ edges of color $1, 2, 3, 4$ and $\lambda_i^2$ edges of color $i$, for $i > 4$. Since $\tilde{G}$ has the same number $(r - 4)$ of vertices and edges, it must contain a cycle. Take an edge $(x, y)$ of this cycle, of color different from 1, and replace it with an edge $(2, y)$ connecting the vertex (2) of $G_0$ with the vertex (2) of $\tilde{G}$. This produces the desired MCB–graph $G$. If $b = f_\lambda$, then we write $k$ for the maximal index $i > 4$ for which $f_\lambda = \lambda_i^2$ (if no such index exists, take $k = 4$). Consider the graph $G'$ obtained from $G_0$ by adjoining an edge $(i, 4)$ of color $i$, for each $i > 4$ with the property that $f_\lambda = \lambda_i^2$. $G'$ is then an MCB–graph (tree) of type $(k - 1, 1)$. Since $b - 1 \geq f_\lambda - 1$, there exists an MCB–graph $\tilde{G}$ of type $(a - k + 1, b - 1)$
with $\lambda_i^2 - 1$ edges of color $i$, for $1 \leq i \leq k$, and $\lambda_i^2$ edges of color $i$ for $i > k$. To see that $a - k + 1 \geq b - 1 = f_\lambda - 1$ (so that the type $(a - k + 1, b - 1)$ makes sense), note that $a + f_\lambda = a + b = e_\lambda \geq k \cdot f_\lambda$, hence in order to prove the inequality $a - k + 1 \geq f_\lambda - 1$, which is the same as $a = e_\lambda - k \geq f_\lambda$, it is enough to show that $k \cdot f_\lambda \geq f_\lambda + k - 2$, which is equivalent to $(b - 2) \cdot (f_\lambda - 1) \geq 0$, which is clear. Since $G$ has $a + b - k = r - k$ vertices and $e_\lambda - k = r - k$ edges, it must contain a cycle. We can now argue as in the case $b > f_\lambda$ to produce the desired MCB–graph $G$.

The relation

realizes $G$ as a sum between an element of $XY(U_r)$ (Lemma 3.23) and an MCB–graph of type $(a - 1, b + 1)$, as desired.

4.3. Proof of (iii). Consider a type $(a, b)$, with $a - b \geq 3$, for which there exists a nonzero MCB–graph. We show that, modulo the Landsberg–Weyman equations, an MCB–graph $G$ of type $(a, b)$ is a combination of MCB–graphs of type $(a - 1, b + 1)$ and $(a - 2, b + 2)$ (or $(b + 2, a - 2)$ if $a - b = 3$). This suffices to prove assertion (iii). We assume as before that there exist edges of at least three distinct colors, and that $\lambda_1^2 \geq \lambda_2^2 \geq \cdots$.

If $\lambda_2^4 = 0$, then we may assume as in Section 4.2 that $\lambda_2^3 \geq 2$. We leave it as an exercise for the reader to check that we may assume that $G$ contains

where the only vertex of $G$ in (4.5) that could have incident edges not shown in the picture is $\overrightarrow{5}$. $G$ is the sum between an MCB–graph of type $(a - 1, b + 1)$ (the first term) and a
graph containing

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 6 & 2 & 6 \\
3 & = & 3 + 3 \\
4 & 7 & 4 & 7 \\
5 & 5 & 5 & 5 \\
\end{array}
\]

which in turn is the sum between an element of \( XYZ(U_r) \) (Lemma 3.23) and a graph containing

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 6 & 2 & 6 \\
3 & = & 3 + 3 \\
4 & 7 & 4 & 7 \\
5 & 5 & 5 & 5 \\
\end{array}
\]

i.e. a graph that’s the sum of an \( MCB \)–graph of type \((a - 2, b + 2)\) (or \((b + 2, a - 2)\) if \(a - b = 3\)) and one of type \((a - 1, b + 1)\).

Suppose now that \( \lambda_2^1 \geq 1 \), and choose \( G \) to be an \( MCB \)–graph of type \((a, b)\) containing

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 5 & 2 & 5 \\
3 & = & 3 + 3 \\
4 & 4 & 4 & 4 \\
\end{array}
\]

such that \(1, 2, 3\) are leaves (it’s an easy exercise that this can be done). \( G \) is the sum between an \( MCB \)–graph of type \((a - 1, b + 1)\) (the first term) and a graph containing

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 5 & 2 & 5 \\
3 & = & 3 + 3 \\
4 & 4 & 4 & 4 \\
\end{array}
\]
which in turn is the sum between an element of $X(U_r)$ (Lemma 3.21) and a graph containing

\[
\begin{array}{ccc}
1 & 3 \\
2 & 5 \\
4 &
\end{array}
\quad \quad \quad \quad
\begin{array}{ccc}
1 & 3 \\
2 & 5 \\
4 &
\end{array}
\quad \quad \quad \quad
\begin{array}{ccc}
1 & \\
2 & 5 \\
4 &
\end{array}
\]

i.e. a graph that’s the sum of an MCB–graph of type $(a - 1, b + 1)$ and one of type $(a - 2, b + 2)$ (or $(b + 2, a - 2)$ if $a - b = 3$).

We end with the promised

**Lemma 4.5.** If $2f_\lambda \leq e_\lambda \leq r$, then there exists a partition $\lambda \vdash r$ and an MCB–graph $G$ such that $\pi_\lambda(G) \neq 0$.

**Proof.** If $e_\lambda < r$ is even then we take $G$ to be an MCB–graph without vertices of degree greater than two (a path–graph). If $e_\lambda < r$ is odd, then we take $G$ to be an MCB–graph with precisely one vertex $v$ of degree three, and all other vertices of degree at most two, such that (at least) two of the neighbors of $v$ are leaves. If $e_\lambda = r$ is even then we take $G$ to be an MCB–graph which is a cycle of length $r$. If $e_\lambda = r$ is odd then we take $G$ to be an MCB–graph consisting of a cycle $C$ of length $r - 1$, together with one other vertex connected to one of the vertices of $C$ by an edge. It is an easy exercise, left to the reader, that such MCB–graphs exist.

We assume that $f_\lambda = \lambda_1 \geq \lambda_2 \geq \cdots$. We define the partition $\lambda \vdash r$ by $a_i = \lambda_i^2$ for $i > 1$, and $a_1 = r - (a_2 + \cdots + a_n)$. It follows that $a_1 > \lambda_2$ unless $e_\lambda = r$, in which case $a_1 = \lambda_2^2$.

Write $T$ for the $n$–tableau associated to $G$.

We have

$$\pi_\lambda(G) = \pi_\lambda(T) = \sum_{A_1 \sqcup \cdots \sqcup A_n = [r]} \sum_{|A_j| = a_j} T(A_1, \ldots, A_n), \quad (4.6)$$

where $T(A_1, \ldots, A_n)$ is defined as the $n$–tableau with entries 1 and 2 (without the symmetry between the labels 1 and 2!) obtained from $T$ as follows: for each $j = 1, \ldots, n$, replace the entries of $T_j$ in the set $A_j$ by 2’s, and the entries of $T_j$ outside $A_j$ by 1’s (see Example 3.13). By Lemma 3.16, $T(A_1, \ldots, A_n) = 0$ if for some $j$ there exists an edge of color $j$ with both endpoints in $A_j$, or both endpoints outside $A_j$.

If we think of the partitions $A_1 \sqcup \cdots \sqcup A_n = [r]$ as colorings of the vertices of the graph $G$, it follows that $T(A_1, \ldots, A_n) = 0$ unless each edge of $G$ of color $c$ has one endpoint of color $c$ and the other endpoint of color different from $c$ (the fact that $T(A_1, \ldots, A_n) \neq 0$ in this case is essentially the last part of the argument in [Rai 10, Lemma 4.20], and is in fact easier, because of the absence of symmetry between the labels 1 and 2).

Consider first the case $e_\lambda = r$ even. We write $G = E_1 E_2 \cdots E_r$, where $E_i = (x_i, x_{i+1})$ ($x_{r+1} = x_1$) is the $i$–th edge in the cycle $G$. We write $e_i$ for the color of $E_i$. We claim that there are exactly two colorings of the vertices of $G$ for which $T(A_1, \ldots, A_n) \neq 0$. To see this, consider a coloring of the vertices of $G$ for which $T(A_1, \ldots, A_n) \neq 0$, and denote by $c(x_i)$ the color of the vertex $x_i$ ($x_i \in A_{c(x_i)}$). If $c(x_1) \neq e_1$, then the remark in the previous paragraph forces $c(x_2) = e_1$, which in turn forces $c(x_3) = e_2, \ldots, c(x_r) = e_{r-1}$ and therefore $c(x_1) = e_r$. 

\[\sum_{|A_j| = a_j} T(A_1, \ldots, A_n), \quad (4.6)\]
If \( c(x_1) = e_1 \), then \( c(x_r) = e_r \), which in turn implies \( c(x_{r-1}) = e_{r-1}, \ldots, c(x_2) = e_2 \). Note that the two colorings we obtain in this way yield two (nonzero) \( n \)-tableaux \( T_1, T_2 \) that differ by an even number of column transpositions, hence their sum is \( T_1 + T_2 = 2T_1 \neq 0 \).

Consider now the case \( e_\lambda = r \) odd. \( G \) consists of a cycle \( E_1 \cdots E_{r-1} \), where \( E_i = (x_i, x_{i+1}) \) \((x_{r-1} = x_1)\), together with a vertex \( x_0 \) connected to \( x_1 \) by an edge \( E_0 \). Forgetting the edge \( E_0 \), the argument in the previous paragraph implies that there are precisely two interesting colorings of \( x_1, \ldots, x_{r-1} \), characterized by \( c(x_1) = e_1 \) and \( c(x_1) = e_{r-1} \) respectively. In both cases, \( c(x_1) \neq e_0 \), hence \( c(x_0) = e_0 \). As before, the two colorings yield two nonzero \( n \)-tableaux representing the same element of \( U_r \), hence \( \pi(a)(G) \neq 0 \).

Consider now the case \( e_\lambda < r \). The graph \( G \) has type \((a+1,a-1)\) if \( e_\lambda \) is odd \((a = (1+e_\lambda)/2)\), and type \((a+1,a)\) if \( e_\lambda \) is even \((a = e_\lambda/2)\). Using the relation

\[
\begin{array}{c}
\begin{array}{c}
1 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
3 \\
2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
3 \\
2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
3 \\
2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\]
\]

we can write \( G = G_0 + G' \), where \( G_0 \) has type \((a,a)\) when \( e_\lambda \) odd (and therefore \( G_0 \) is a Landsberg–Weyman equation by Proposition 3.28(4) and Lemma 3.24), and it has the same type (but opposite orientation) as \( G \) when \( e_\lambda \) is even. In the latter case, Proposition 3.28(3) implies that \( G_0 \equiv -G \) modulo the Landsberg–Weyman equations. In both cases, the graph \( G' \), which is a path–graph with a double edge, is then a positive multiple of \( G \), so it suffices to show that \( \pi(a)(G') \neq 0 \). Since \( G' \) has two edges, \( E_1, E_2 \) joining the same pair of vertices \((x_1, x_2)\), it follows that any interesting coloring must have \( c(x_1) = e_1 \) and \( c(x_2) = e_2 \), or \( c(x_1) = e_2 \) and \( c(x_2) = e_1 \). The coloring of \( x_1, x_2 \) determines the colorings of the other vertices. The same arguments as before apply to conclude that \( \pi(a)(G') = T_1 + T_2 = 2T_1 \neq 0 \). □

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