Characteristics of Black Hole Entropy

Thesis submitted in partial fulfillment of the requirements for the degree of "DOCTOR OF PHILOSOPHY"

by

Judith Kupferman

Submitted to the Senate of Ben-Gurion University of the Negev

November 18, 2013
Beer-Sheva
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Approved by the advisor
Approved by the Dean of the Kreitman School of Advanced Graduate Studies

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Abstract

Black hole entropy has been studied for the past forty years, but there is still a lack of clarity as to its meaning, and the various interpretations this term is given in the fields of thermodynamics, quantum information and geometry. In the hope of attaining understanding of the concept of entropy in black holes I study specific characteristics of this entropy, and compare these characteristics with respect to the various fields. In this research I focused on the following definitions of entropy: thermodynamic and statistical entropy, entropy of entanglement, Wald entropy as a Noether charge, and the entropy developed by Carlip from conformal symmetry.

First I studied the question of divergence of entropy on the black hole horizon. It is known that the statistical entropy of a matter field in the black hole metric diverges on the horizon, whereas entanglement entropy does not diverge. Thus it appears that the two entropies differ. I show that it is possible that the divergence is the result of inappropriate treatment of the question, and so it may simply stem from quantum uncertainty between position and momentum, and is not an inherent characteristic of statistical entropy. Thus it is possible that statistical entropy and entanglement entropy are related to the same quantity.

I then turned to the question of observer dependence of entropy. Wald’s entropy is defined in terms of intrinsic curvature. I clarify the definition of statistical entropy in curved relativistic space, and I find a transformation which leaves the entropy invariant while the curvature changes. There is a subgroup in which the transformation of the number of states preserves invariant curvature, but the counter example presented here proves that it is not possible to assume that a given number of states and the statistical entropy derived from it relate to a space time with a given curvature.

Finally, I examined a geometric variation of the statistical entropy of a matter field near a black hole, and compared this to the variation of Wald's entropy. I find that the statistical entropy contains terms that coincide with the variation of Wald's entropy but can have additional terms as well related to the energy momentum tensor.
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Part I

Aspects of entropy
Chapter 1

Introduction

1.1 Overview

The field of black hole thermodynamics was born forty years ago, when black holes were shown to have entropy [1, 2]. However it is not clear just what this entropy is. Entropy appears in various contexts and meanings. In thermodynamics the change in entropy for a canonical ensemble is related to the change in energy, if volume and other factors are constant. Statistical mechanics in treating the microcanonical ensemble defines entropy as the number of accessible states. In classical information theory entropy quantifies the amount of information in a message, while in quantum information theory, entanglement entropy is a measure of quantum correlations between parts of a system. Black hole entropy was originally discussed in the thermodynamic context. However since then it has been explored in other contexts, and to this day when referring to black hole entropy there is still a lack of clarity as to what exactly the term refers to. If, as in statistical mechanics, we are counting microstates, just what are we counting in the black hole? If quantum correlations, between what? The vacuum within and without the black hole? Hawking particles? Matter within and without?

In this thesis I focus on specific aspects of black hole entropy with the aim of clarifying the similarities and differences in the various interpretations of the term. After an introductory review of selected treatments of black hole entropy, I examine the issue of divergence on the black hole horizon, in an attempt to clarify the distinction between statistical and entanglement entropy. I then focus on the treatment of the horizon as boundary, as a partition in space, and on the
distinction between treatment that imposes boundary conditions and treatment that does not do so. Another aspect is observer dependence: can entropy be a function of an invariant such as scalar curvature or is it observer dependent? The third part of the thesis examines a variation of entropy, in an effort to clarify the distinction between Wald entropy and statistical mechanical entropy. If the variations were shown to be identical that could indicate that the two kinds of entropy originate in a common source.

In focusing on these specific issues I find that entanglement entropy and the entropy of statistical mechanics may refer to the same phenomenon, but that these differ from geometrical concepts such as Wald’s Noether charge entropy and Carlip’s derivation of entropy from conformal field theory. A general conclusion is that in determining the real meaning of black hole entropy, a careful and unambiguous description of these aspects is necessary in order to dispel confusion and to distinguish the particular theory under discussion.

Remark on notation: Throughout this work natural units are used, so that $\hbar, c,$ and the Newton and Boltzmann constants are taken as unity, except where explicitly quoting previous work. The personal pronoun varies between “I” and “we”, not the royal plural but rather due to discomfort at seeing one hundred pages full of myself alone.

1.2 Background

Black hole entropy has had a rich and varied history over the past forty years. Below I review only a few examples which are relevant to the work in this thesis. These include examples of thermodynamic and statistical mechanical entropy, entanglement entropy, the geometric formulations of Wald and Carlip, and the mapping of the gravitational equations of motion to the first law of thermodynamics. In cases where extensive details of the calculations are relevant to what follows, these can be found in Part II of the thesis.

1.2.1 Thermodynamics

Black hole entropy was first discussed in the context of thermodynamics. In 1973 Bardeen, Carter and Hawking \[\text{[1]}\] formulated the four laws of black hole mechanics, analogous to the laws of thermodynamics, where the surface gravity
played the role of temperature, and the horizon area that of entropy. They viewed this strictly in terms of an analogy, taking pains to emphasize that \( \frac{\kappa}{8\pi} \) and \( A \) are distinct from the temperature and entropy of the black hole...In this sense a black hole can be said to transcend the second law of thermodynamics.\(^1\)

Hawking had shown that the area of a black hole cannot decrease \(^3\). Bekenstein \(^2\) reasoned that just as area increases so does thermodynamic entropy, and so the area of a black hole can be interpreted as its entropy. The generalized second law of thermodynamics holds that the sum of the entropy of matter outside a black hole together with the area of the black hole never decreases.

In \(^1\) the authors had taken care to point out that the parallel between the laws of black hole mechanics and thermodynamics is purely an analogy, but that the two though similar are distinct. This approach underwent a startling change in 1974, when Hawking showed that black holes emit radiation \(^4\). Using quantum field theory he showed that this radiation is thermal, and obtained the temperature, which could then be used in the thermodynamic formulation.

Gibbons and Hawking \(^5\) used the path integral formalism to obtain the partition function for a black hole. The partition function is taken as

\[
Z = \int DgD\phi e^{iI[g,\phi]} \tag{1.2.1}
\]

where \( Dg \) is a measure on the space of metrics, and \( D\phi \) on that of matter fields, and \( I[g,\phi] \) the action. The result gives the probability of the occurrence in the vacuum state of a black hole with the relevant parameters that appear in the action. The gravitational action has a surface term chosen so that variation of the action gives the Einstein equations.

In order to avoid singularities the authors define a new coordinate, \( \tau = it \) so that the metric becomes Euclidean. This procedure can then be extended to non Euclidean systems because the quantities in the integral, which include the Ricci scalar, the electromagnetic field tensor and the extrinsic curvature, are holomorphic functions on the complexified spacetime except at the singularities. Thus the action integral is in fact a contour integral, and it will have the same value on any section of the complexified spacetime corresponding to the Euclidean section even though the induced metric on this section may be complex.

In the path integral approach to quantization of a field, going to a Euclidean...
system gives the thermodynamic partition function. Thus

$$Tr \left(e^{-\beta H}\right) = \int D\phi e^{i\mathcal{H}[\phi]}$$

(1.2.2)

where the integral is taken over all fields which are periodic with period $\beta$ in imaginary time. Gibbons and Hawking use this to obtain thermodynamic quantities from the gravitational and matter action. Since the dominant contribution to the path integral comes from metrics $g$ and matter fields $\phi$ which are near background fields $g_0$ and $\phi_0$ they expand the action in a Taylor series around the background fields, neglect higher order terms, and write the resulting partition function $Z$. From this they obtain the free energy using the thermodynamic relation $ln Z = -\beta F$. In a specific example for a rotating black hole they identify the temperature with the surface gravity, and write the free energy in terms of the black hole mass, charge and angular rotation, obtaining

$$\frac{1}{2} M = TS + \frac{1}{2} \Phi Q + \Omega J$$

(1.2.3)

and plugging in the generalized Smarr formula [6]

$$\frac{1}{2} M = \frac{\kappa}{8\pi} A + \frac{1}{2} \Phi Q + \Omega J$$

(1.2.4)

they find

$$S = \frac{A}{4}.$$  

(1.2.5)

Thus the path integral is used in conjunction with a Wick rotation to give the thermodynamic partition function, and the entropy is derived from this by plugging in thermodynamic relations.

1.2.2 Statistical mechanics

Statistical and thermodynamic entropy are of course interrelated, but here we use the term statistical entropy when the computation is the number of accessible states, while thermodynamics is used in the context of the canonical ensemble and based on macroscopic properties of black holes.

A seminal work obtaining black hole entropy from the number of states was that of 't Hooft [7]. He calculates the number of states of a quantum matter field
in the region of a black hole, and from the number of states he obtains the free
energy and the entropy. To find the number of states for a particle in the black
hole metric, t’Hooft uses a one dimensional WKB approximation. He takes the
contribution to energy of the transverse momenta as an effective radial potential
\( V_{\text{eff}} = l(l + 1)/r^2 \), since it behaves as a centrifugal potential.

The one-dimensional WKB approximation gives the number of radial modes
\( n \) thus:

\[
\pi n = \int_{2M+h}^{R} dr \sqrt{g_{rr}} k(r)
\]  

(1.2.6)

The integral ought to be taken from the horizon, but to avoid divergence ‘t Hooft
goes a short distance \( h \) from the horizon, known as the “brick wall.”

From the wave equation one obtains the square of the radial eigenfunction

\[
(k(r))^2 = g_{rr} \left( g^{00} E^2 - g^{\theta \theta} (k(\theta))^2 - g^{\phi \phi} (k(\phi))^2 - m^2 \right)
\]  

(1.2.7)

where \( k(\theta), k(\phi) \) denote the eigenfunctions of the angular components of the
Laplacian. In the Schwarzschild metric this becomes

\[
(k(r))^2 = \frac{1}{1 - \frac{2M}{r}} \left( \frac{1}{1 - \frac{2M}{r}} E^2 - \frac{1}{r^2} (l(l + 1)) - m^2 \right)
\]  

(1.2.8)

The number of radial modes is then summed over the angular degrees of freedom,

\[
N \pi = \int_{2M+h}^{R} dr \int_{0}^{E^2/2} d(l+1) \sqrt{E^2 - \left( \frac{1}{1 - \frac{2M}{r}} \right) \left( \frac{l(l+1)}{r^2} + m^2 \right)}
\]  

(1.2.9)

where the upper limit of the second integral is in order to ensure a positive root.
Integration gives two terms. One is the contribution from the vacuum surrounding
the system at large distances and ‘t Hooft discards it. The second term is the
horizon contribution, which diverges as \( h \to 0 \). From this he obtains the free
energy and the entropy, which also diverge on the horizon. Further details of the
calculation appear in Sec. 6.1.

The motivation for this work is to reconcile black hole physics with quantum
mechanics. At the time it was written black holes were understood to be in a
quantum mechanically mixed state, and 't Hooft attempted to describe them as pure states resembling ordinary particles. Thus black holes inhabit an extension of Hilbert space with an according Hamiltonian. This system is sensitive to observer dependence: the free falling observer perceives matter, and 't Hooft writes that it is this matter which he considers in this paper. The distinction between vacuum and matter is assumed to be observer dependent when considering coordinate transformations with a horizon. The brick wall model is intended to show that the horizon itself rather than the black hole as a whole determines its quantum properties. This seems to have a relation to entanglement entropy, considered in the following section, and that is not surprising since it is 't Hooft’s explicit intention to reconcile gravity and quantum mechanics.

The number of black hole microstates was computed much later using string theory, beginning in 1996 when Strominger and Vafa computed the entropy of an extremal supersymmetric black hole in string theory and obtained the Bekenstein Hawking entropy [8]; this approach was extended to a wide variety of black holes. Another approach in loop quantum gravity considers black hole states as spin networks on the horizon. These treatments are interesting and productive, but they are beyond the scope of this thesis.

1.2.3 Entanglement entropy

Several hints lead us to consider a relationship between quantum entanglement entropy and the geometry of space-time. First, the fact that entanglement entropy arises as a consequence of a partitioning of subsystems. Such a partition is located in space in the case of a black hole horizon, for example: if there were no such barrier the entanglement entropy would be zero, and it is possible that barrier width or extent determine the amount of entanglement entropy. Second, the fact that entropy can be related to energy. Einstein’s equations point at a relationship between geometry and energy, and it is clearly of interest whether and how this relationship can be extended to entropy. A relation has been shown when the entropy is thermodynamic (see Sec. 1.2.6) but not for entropy as a measure of the amount of entanglement. Third, it has been shown that entanglement affects the speed of evolution [9] but [10], indicating that there is a relationship between entanglement and time. In a relativistic context this too points to a relationship between entanglement and geometry.
Entanglement entropy differs by definition and by characteristics from the entropy defined in statistical physics. The latter is the logarithm of the number of states, and is extensive. Entanglement entropy is a measure of quantum correlations between states which are part of a composite system, and has been found to be proportional to area \[^{11, 12}\]. Since Bekenstein showed that black hole entropy is proportional to area, many people view this as a hint that BH entropy is entanglement entropy \[^{13, 14, 15}\]. Entanglement entropy is formally defined as 
\[ -\text{Tr} (\rho \ln \rho) \]
where \( \rho \) is the partial trace of the system, that is, the reduced density matrix of part of the composite system. Tractable calculation of the entropy for bipartite states is with eigenvalues \(- \sum \lambda \ln \lambda \) in the Schmidt basis, in which the reduced density matrices are diagonalized, and their eigenvalues are found to be identical.

The first major papers on entanglement entropy were by Bombelli et al. in 1986 \[^{11}\] and Srednicki in 1993 \[^{12}\]. Both showed entanglement entropy is proportional to area by treating discrete coupled harmonic oscillators and then taking the continuum limit. Srednicki’s result is more powerful for two reasons: first, unlike Bombelli he explicitly calculates it for a sphere and so it can be related to a black hole. Second, and this is probably a reason for its greater impact, he presents a convincing justification for the area law: his logic is that surface area is the only thing the inside and outside of the black hole have in common, so their entanglement entropy, which is the same for inside and outside, must depend on that. He assumes entropy depends on area and since entropy is dimensionless, looks for a dimensionful parameter to cancel out area. He uses the square of the lattice spacing. The area law is arrived at by numerical means.

Srednicki first calculates the entropy for a pair of discrete harmonic oscillators in the ground state. This is

\[
S = -\log(1 - \xi) - \frac{\xi}{1 - \xi} \log \xi
\]

where \( \xi = f(k_1/k_0) \) (the interaction potential of each oscillator is \( -k_i x_i \)). When summed over \( n \), a number of harmonic oscillators, the entropy becomes a function of \( n \) and thus of lattice length. He defines the radius of the region he has specified as \( R = (n + \frac{1}{2})a \), where \( a \) denotes the lattice spacing. He obtains the entropy as

\[^{2}\text{These widely cited papers generated much further work, but the idea and calculations were first given three years earlier in a talk by Sorkin}\[^{16}\].]
$S = 0.30 M^2 R^2$. This looks neatly proportional to area. However he has defined $M = \frac{1}{a}$. When writing the entropy explicitly with $M, R$ in terms of $a$, the spatial component cancels out and we are left with a function only of the number of oscillators, $S = 0.30(n + \frac{1}{2})^2$. That is really where the “area law” comes from. It is obtained numerically.

Srednicki’s entropy diverges because it sums over $n$, the number of lattice sites, which is then taken to continuum. But there is no particular divergence at the boundary of the sphere which he examines. This and other general treatments of entanglement entropy [17] all calculate it for discrete lattices and then take it to continuum. In all these cases the entropy does not diverge at the boundary unless - since it is proportional to area - the boundary area is infinite. Divergence of entanglement entropy in the general treatment is a result of having an infinite number of modes, so that a UV cutoff is necessary.

A different treatment of black hole entropy as entanglement entropy was that of Ryu and Taskayanagi [18], who calculate entropy in a $d+1$ dimensional conformal field theory from the area of a $d$ dimensional minimal surface in $AdS_{d+2}$, and show that the result reproduces the Bekenstein Hawking formula. This will not be discussed in the thesis.

1.2.4 Entropy as a Noether charge

Wald [19, 20] obtains black hole entropy for generalized theories of gravity by requiring diffeomorphism invariance in conjunction with the first law of thermodynamics. In accordance with Noether’s theorem that to every continuous symmetry a conserved current and charge can be associated, diffeomorphism symmetry has such a current and charge. Wald uses the abstract formalism of Hamiltonian mechanics [21], defining the phase space of a linear dynamical system as a symplectic vector space. The solutions of the equations of motion are mapped to points on a symplectic manifold. This is done in the language of differential forms. The advantage of this approach is that the equations of motion are independent of choice of a coordinate system.

Wald sets out by writing the most general possible Lagrangian for diffeomorphism invariant theories. This is a function of the Riemann tensor and its derivatives, as well as the metric and any other fields. A variation of the Lagrangian by varying the fields gives the equations of motion as well as the exterior derivative
of a symplectic potential form $\Theta$. For $\xi^a$ any smooth vector field on the spacetime manifold, that is $\xi^a$ the infinitesimal generator of a diffeomorphism, and for any field configuration $\phi$ (not necessarily a solution to the equations of motion) the Noether current is defined by

$$J = \Theta (\phi, L_\xi \phi) - \xi \cdot L$$

where the dot denotes contraction of the vector field $\xi$ into the first index of the differential form of the Lagrangian $L$. One finds that $dJ = -E L_\xi \phi$ where $E$ is the equations of motion. On shell one sees that $J$ is a closed form and thus there is a Noether charge $Q$ such that $J = dQ$.

Wald applies this to a stationary black hole solution with bifurcate Killing horizon, where $\xi^a$ is the Killing field vanishing on the bifurcation surface. A variation of the fields away from the background solution provides an equation relating the surface term at infinity to a surface term on the horizon. The infinity terms give the mass and angular momentum. These are equated to the horizon term. This equation has the form of the first law of thermodynamics, where

$$\delta \int_\Sigma Q (\xi) = \frac{\kappa}{2\pi} \delta S$$

where $\Sigma$ is the bifurcation surface. He then shows that

$$S = -2\pi \int_\Sigma E^a_{\text{abcd}} \epsilon_{ab} \epsilon_{cd}$$

where $E^a_{\text{abcd}}$ is a tensor field obtained by taking the functional derivative of $L$ with respect to $R_{abcd}$ (viewed as independent of $g_{ab}$) and $\epsilon_{ab}$ is the binormal to the bifurcation surface, $\epsilon_{cd}$ is the area form. Thus the black hole entropy is proportional to the Noether charge coming from the boundary at the black hole horizon.

Wald’s entropy is sometimes referred to as geometric. It originates in diffeomorphism symmetry, but also from the first law of thermodynamics. The purely geometric derivation gives the energy at the horizon, in an equation corresponding to the first law. The entropy is obtained by assuming the relation between mass, angular momentum, energy and temperature which is given by the first law of
thermodynamics. (Further details are in Sec.6.2)

1.2.5 Entropy from conformal field theory

Carlip [22, 23] attempted to obtain black hole entropy from considerations of symmetry. General relativity is diffeomorphism invariant, and this symmetry is expressed in an algebra, which, when space has a boundary, can be shown to have a central extension. In a different context Cardy [24] obtained a formula for statistical entropy as a function of the central charge of a conformal theory, and Carlip makes use of the Cardy formula to obtain black hole entropy. He first did this using the ADM formalism. In this Hamiltonian form, general relativity is constrained with constraints $H_\mu$ and these constraints generate the symmetries of the theory. The generators obey an algebra, and if space has a boundary the algebra has been shown to be isomorphic to a pair of Virasoro algebras, with a central extension. The boundaries in the Schwarzschild metric, for example, are taken, as with Wald, to be infinity at one end, and the black hole horizon at the other. In further work, Carlip uses Wald’s covariant phase space formalism and concept of entropy as a Noether charge as a point of departure to write the algebra for general relativity and compute the central charge. He plugs in Cardy’s formula to obtain the entropy as a function of the central charge, and shows that in the case of a black hole, this gives the known result. This is explained in detail in Sections 2.6 and 6.2.

1.2.6 The first law of thermodynamics

In contrast to the derivation by Bardeen et al. of the four laws of black hole mechanics from the equations of gravity, Jacobson [25] derived the Einstein equations of state from the first law of thermodynamics. This was done by taking the energy flow across a causal horizon - not necessarily a black hole - and arguing that the entropy of the system beyond the horizon is proportional to horizon area. The first law is an equilibrium relation, whereas the horizon may not be in equilibrium but rather contracting, expanding or shearing. Therefore Jacobson takes a small neighborhood of a point on the horizon. This neighborhood is locally flat by the equivalence principle, and there is a Killing field $\chi^a$ generating boosts orthogonal
to the point $P$ in question. The energy flow across the horizon is

$$\delta Q = \int_{\mathcal{H}} T_{ab} \chi^a d\Sigma^b$$

(1.2.14)

where the integral is over a “pencil” of generators of the inside past horizon of $P$. Temperature is taken as the Unruh temperature, $\kappa/2\pi$ where $\kappa$ is the acceleration of the Killing orbit. Taking the entropy variation $\delta S$ as a variation of area $\delta A$, he writes the area variation as an integral of the expansion $\theta$ of the horizon generators. The Raychauduri equation relates this to the Ricci tensor, thus giving $\delta S$ as an integral over a quantity including the Ricci tensor. Plugging in the first law of thermodynamics, $\delta Q = T\delta S$, the Einstein equations are obtained.

This remarkable work was later generalized to higher order theories of gravity ([26, 27, 28] and see Sec.9.1).

Here I have detailed only a selection of a wide variety of treatments of black hole entropy. The question remains: when discussing the entropy of a black hole, what are we actually talking about? In the following chapters I examine specific aspects of black hole entropy, in the hope that clarifying the similarities and differences between the various treatments outlined above will shed some light on this question. Part I of the thesis treats behavior at the boundary, observer dependence and variations of entropy. Part II contains supplementary details, proofs and calculations.
Chapter 2

The black hole boundary

If the universe is perceived as a pure state composed of the part within the black hole and the part outside, then either of the parts will have entanglement entropy since by themselves each is not pure but mixed. Therefore a black hole has entanglement entropy by definition. A more physical understanding of entanglement entropy in the black hole context may be entanglement of Hawking pairs. This hints that entanglement entropy of a black hole may be the same as thermodynamic entropy, since Hawking radiation is primarily thermal. This idea is strengthened by Bekenstein’s observation that black hole entropy is proportional to area, just as is entanglement entropy. The question is whether the statistical entropy of a black hole coincides with entanglement entropy.

In order to explore this we focus on behavior of entropy at the boundary. When t’Hooft investigated black holes in a statistical mechanics framework he found divergence on the horizon [7]. Later work [29, 30] assumed that the divergence is a direct result of the black hole redshift. However entanglement entropy in the general case does not show divergence at the barrier, but rather diverges as a function of high momentum cutoff. This might indicate a difference between black hole entropy and entanglement entropy. The question is whether this divergence is related to the horizon itself as a causal barrier, or whether it is due to simple quantum uncertainty, in which case the divergence does not point to an essential difference between the two entropies.

General calculations of entanglement entropy do not show divergence at the barrier. In Bombelli’s calculation divergence comes from high frequency modes and so he inserts a UV cutoff, but there is no other cause of divergence. Srednicki
defines $R = (n + \frac{1}{2})a$, where $a$ is lattice spacing and $n$ the number of discrete oscillators. The expression for entropy which he obtains numerically is $S = 0.30M^2R^2$ which does not diverge for a particular radius. In fact he has defined $M$ as the inverse lattice spacing $a^{-1}$ so that the actual expression is $S = 0.30(n + \frac{1}{2})^2$ which diverges for an infinite number of oscillators, but again, not at a particular location. Srednicki also performs a perturbative approximation for $l \gg N$, where $l$ is the usual quantum number for angular momentum and $N$ the total number of lattice sites, it diverges for infinite lattice sites but not at a particular location. Plenio in his review of entanglement entropy in lattice systems $^{17}$ has no divergence because an upper bound for entanglement entropy is known to be logarithmic negativity, $E_n > S, E_n = \ln \|\rho_{PT}\|$, that is, the logarithm of the trace normed partial transpose matrix. This would diverge if the matrix were infinite, but not with respect to any particular location.

In $^{31}$ divergence at the barrier subdividing a system is explored in a non-relativistic context, rather than in the specific context of a black hole. Taking a system and looking at a subvolume, energy fluctuations in the subvolume are proportional to the surface area of the subvolume. If the fluctuations were thermal they should have been extensive (proportional to volume) but they are not. The claim is that they are not thermal but rather caused by the division into a subvolume, and thus by entanglement.

A high momentum cutoff is necessary to prevent divergence of $\Delta E^2$. Such a cutoff is equivalent to smearing the boundary which divides off the subvolume. In $^{31}$ this was done explicitly for particles in a box. A more general treatment appears in $^{13}$ where half of Minkowski space is treated as a thermal ensemble in Rindler space. If a pure state is subdivided into two mixed states, $\rho_A, \rho_B$, then the equation $Tr(\rho_A \hat{O}_A) = \langle \psi_{AB} | \hat{O}_A | \psi_{AB} \rangle$ allows to obtain expectation values of operators in an entangled pure state by calculating expectation values of one of the mixed states. In this way entanglement entropy of the pure state is found to be the thermal entropy of the mixed state. It diverges at the horizon, scales as area, and needs a UV cutoff.
2.1 Divergence on the horizon

Calculations of black hole entropy on the horizon give rise to divergence. We will show that this divergence is not unique to a black hole, nor is it a UV divergence found in field theory which requires appropriate renormalization. Rather it can be seen as a result of quantum x/p uncertainty because the horizon is defined as a perfectly sharp boundary dividing spacetime into an observable and an unobservable region. A similar divergence arises for any quantum mechanical system when a sharp boundary divides the whole system into an observed and an unobserved regions. This is also the case with a coordinate system which truncates part of flat space, as with Rindler coordinates. The divergence is tamed by smoothing over the boundary, rather than by renormalizing the theory. The same is true for black hole entropy.

In quantum mechanics we know that there are questions which can, but should not be asked. If we insist on asking them, the theory itself lets us know in a clear way by giving us a senseless answer. For example, if we ask “what is the typical momentum of a perfectly localized particle?” the formal answer will be infinite because of the position/momentum uncertainty relation. Of course, this just means that the momentum fluctuations will become larger as the particle is localized in a sharper way. Here the observer needs to change the question to “what is the typical momentum of a particle whose wave function has a small finite width in space?” and treat the concept of a sharply localized particle as a limit.

In quantum field theory we are familiar with questions involving infinity. Some of these indicate real problems with divergence, but others are meaningless just as with momentum of a localized particle. An example of a real problem is to ask “what is the charge of the electron?” where the answer comes out infinite. In this case the infinite answer does not mean that we should not have asked the question. Rather, it means that we have misidentified a microscopic parameter in the theory and that this parameter should be “renormalized”. After a redefinition of the “bare” (correct) microscopic theory we can ask the question and get a finite answer. However, in other cases the divergence can not be corrected by modifying the theory because the question itself does not make sense.

An example of this second type of divergence would be to look at a non-relativistic particle in a finite box and ask: What is the energy in the left hand
side of the box. If we approach this problem using second quantization and field operators, we will see that the problem may then be extended to relativistic fields, and that there too the difficulty arises from an ill posed question. We ask “what is a typical energy or momentum in the left half of the box.” Note that this does not involve putting a real partition into the box; that would simply give two smaller boxes, with finite energy, of course. However if the partition is imposed by limiting the possibility of observations to only half the box without imposing new boundary conditions, then if the partition is sharp the answer will be infinite because the fluctuations of momentum and energy are infinite.

How should we interpret the infinite answer when we know that in fact the energy is finite? In [31] it was shown that the reason for the senseless answer is that the question is inappropriate. The insistence on an infinitely sharp division between the (observable) left region and the (unobservable) right region is the cause of the divergence. In this case the sensible question should involved a smoothed division of the box, allowing the boundary between the observable and unobservable domains to be smoothed. If the resolution with which the box is divided into the observable and unobservable halves is limited, then the answer is finite and inversely proportional to the smoothing width, exactly as in the case of the localized particle.

The distinction between the two classes of divergences is the distinction between an ultraviolet (UV) divergence and an ill posed question. We would like to know to which of these two classes black hole entropy belongs. Are its divergences inherent to the system and requiring some knowledge of the UV properties of the theory, a theory of quantum gravity, or both? Or are they, rather, similar to those one obtains when dividing space into two regions, one observable and the other unobservable, and tracing over the unobservable region?

In this chapter we will consider some generic wave function and apply a “window-function" to it, leaving the boundary conditions exactly the same as they were initially. The window function will allow us to impose a smooth division between the observable and unobservable regions. When the width of the window function is taken to zero, a sharp division between the regions is obtained. This set up is different than setting up the quantum system with boundary conditions that would have made the wave function vanish outside a certain region.

We will show that the problem of divergence at the dividing boundary can be resolved for a quantum mechanical system by asking the right question, namely
smoothing the division between the two regions. We will then argue that the origin of the divergences encountered for black holes is similar. This will allow us to argue that such divergences are not a unique black hole characteristic but rather a result of quantum uncertainty, and the correct expression must involve smearing out the boundary. In fact Bekenstein noted in 1994 that if the boundary of the region being traced out were absolutely sharp, the energy would be very large due to the uncertainty principle, and so the boundary must be thought of as "slightly fuzzy" [32], and we will show in detail that this is the case.

This chapter is organized as follows. First we briefly review the behavior of black hole entropy at the horizon. Then we show the relationship between entropy and energy near the horizon of a black hole. Next, we clarify the concept of partitioning and define an operator which may smooth a partition. This is then used to examine behavior of energy at a boundary between two subsystems, first for the non-relativistic and then for the relativistic case, and to show that in both cases energy diverges as the boundary becomes sharp. We extend this to Rindler space, as a partitioning of Minkowski space. Finally, we examine ’t Hooft’s calculation of black hole entropy, and find that his relocation of the boundary to avoid divergence is equivalent to smearing out the boundary. Therefore here too the divergence is related to sharpness of the boundary and is not unique to a black hole.

2.2 Thermal and entanglement entropy

’t Hooft calculated thermodynamic characteristics of a black hole, among them entropy, and in doing so found a divergence of the density of states and hence of the entropy density at the horizon. He overcame the problem by adjusting the limits of integration to a “brick wall” a finite infinitesimal distance from the horizon. Entanglement entropy also diverges, but the divergence appears to be an ultraviolet divergence that does not seem to diverge at any particular location.

For a BH in equilibrium, the space just outside the hole near the horizon can be treated as a thermal state in Rindler space [13,33]. In this case entanglement entropy coincides with thermal entropy, as follows. To find entanglement entropy we take the trace of part of the system. If that part of the system is a thermal
state, the partial trace is a thermal density matrix,

\[ \rho_{\text{part}} = \frac{1}{Z} \sum_i e^{-\beta E_i} |E_i\rangle\langle E_i| . \]  

(2.2.1)

Entanglement entropy is given by

\[ S = -Tr(\rho_{\text{part}} \ln \rho_{\text{part}}) \]  

(2.2.2)

and the energy is given by

\[ \langle E \rangle = \frac{1}{Z} \sum_i E_i e^{-\beta E_i} \]  

(2.2.3)

It follows that

\[ S = -\frac{1}{Z} \sum_i e^{-\beta E_i} \times \left( -\beta \sum_i E_i - \ln Z \right) \]

\[ = \beta \langle E \rangle + \ln Z . \]  

(2.2.4)

For a scalar field at a finite temperature \( \ln Z \) is a constant, so the entropy is linear to the expectation value of the energy. Therefore in the case of a black hole the entanglement entropy behaves as does the energy. Thus instead of examining entropy at a barrier dividing the two subsystems, which is a complicated non-local quantity, we can calculate the reduced density matrix of a subsystem and look at the behavior of its energy which is a simpler local quantity.

### 2.3 Momentum fluctuations, energy and entropy for smooth partitions

#### 2.3.1 Partitioning a subvolume

We will examine various examples of partitioning: first we take a single non-relativistic particle in a box, then a relativistic field, then an entire region of Minkowski space and finally a black hole. The first case is clearest but our claim is that the others are essentially the same.

It is crucial to clarify that the partitioning corresponds to limiting the ob-
servability to a subvolume. If we were to take a box and place an actual physical partition in the middle, this would impose new boundary conditions and we would simply have two smaller boxes with observables appropriate to the new boundary conditions. Instead we leave the particle in the original box, but consider only a subvolume of the box. An example of this would be to work out the probability of finding the particle. Had we actually partitioned the box and looked for the particle in the left half, we would find a probability of one or zero to find it there. But we do not actually do this; rather than making the actual observation, we just calculate the probability to find the particle on the left, and then we will obtain a probability of one half. Similarly in what follows we will calculate expectation values for part of a system without actually imposing a partition with new boundary values.

This kind of partitioning is equivalent to tracing out part of the system. The mathematical operation of tracing defines in a clear way the kind of partitioning of the quantum system that we have in mind. We do not impose new boundary conditions, but rather we restrict the domain of observability to a limited region of the total volume. This will be implemented by a window operator, as described below. If the partitioning is done at a sharply localized point we will see divergence of momentum and energy, even though in fact obviously the particle itself has the same finite energy it had initially. If the partition is not sharply localized we will no longer see a divergence.

We are interested in the expectation value for the reduced energy in the case where we look at a subvolume of the entire system. This can be expressed in two ways. We can rewrite the state so that it is multiplied by a window function: $|\psi\rangle \text{window} = f(\vec{r}, w) |\psi\rangle$. Thus the expectation value for the reduced energy in this restricted system will be $\langle \psi | fHf | \psi \rangle$. Rather than regarding the window function as part of the state, we can treat as part of the operator, so that we define the restricted Hamiltonian as $H^V = fHf$. A striking equation relates quantum expectation values of operators that act on part of a system to the statistical averages for a reduced density matrix of the subsystem. Writing the density matrix for the subsystem as $\rho^V$, 

$$\langle \psi | H^V | \psi \rangle = Tr \left( \rho^V H^V \right). \quad (2.3.1)$$

Therefore we can calculate the reduced energy in the subsystem by taking the
expectation value of the restricted Hamiltonian in the entire system.

The restricted Hamiltonian may be smoothed so that the partition into sub-systems is not completely sharp. This is equivalent to giving the window function varying width. For details see Sec.7.1. It is possible to define a smoothing function that is strictly zero on the left and continuous at any fixed desired order at the boundary. When we discuss the horizon in Rindler and Schwarzschild metrics, we will see that this is the form which can be given to a smoothing function operating on the redshift.

The function \( f(\vec{r}, w) \) behaves as a window enclosing part of space, and thus it mimics the horizon by “truncating” part of space for the field. We provide it with a varying width, and examine energy as a function of its width. Our aim is to see how sharp localization affects the reduced energy divergence.

### 2.3.2 Energy and momentum fluctuations in a restricted non relativistic system

We now write the reduced density matrix for nonrelativistic bosons restricted (in the sense defined above) to one part of space. We emphasize again: this restriction is related to limiting the region in which observations can be made, without imposing new boundary conditions. In practical terms it could mean adding a Heaviside step function as our window operator, thus integrating only up to a defined point. We calculate the energy we as observers will measure. We take free spinless bosons and consider states that are created by the field operator \( \Psi \) acting on the vacuum:

\[
|\Psi\rangle = \Psi^{\dagger}(\vec{r}) |0\rangle = \sum_{p} e^{-ip\vec{r}} g(p) a_p^\dagger |0\rangle.
\]  

(2.3.2)

The function \( g(p) \) will not be particularly relevant for us and in most cases we will ignore it by setting \( g(p) = 1 \). All our results can be easily generalized for the case \( g(p) \neq 1 \).

The Hamiltonian is given by

\[
H = \sum_{p} \frac{p^2}{2m} a_p^\dagger a_p.
\]  

(2.3.3)
The energy of a state $|\psi\rangle$ is given by

$$E = \langle \psi | H | \psi \rangle = \langle 0 | \Psi H \Psi^\dagger | 0 \rangle. \quad (2.3.4)$$

In configuration space the energy is given by

$$E = \int_{-\infty}^{\infty} d^3r \frac{1}{2m} \langle 0 | \nabla_r \Psi (\vec{r}) \nabla_r \Psi^\dagger (\vec{r}) | 0 \rangle. \quad (2.3.5)$$

We calculate the energy corresponding to the restricted Hamiltonian $E^V_{\psi} = \langle \psi | H^V | \psi \rangle = Tr (\rho^V H^V)$. We replace the restricted Hamiltonian $H^V$ by its smoothed counterpart with the help of a window function $f(\vec{r}, w)$, as discussed above. Alternatively, we can use a restricted smoothed field operator (here we set $g = 1$)

$$\Psi^V_{\text{smoothed}} = \int d^3r f(\vec{r}) \Psi^\dagger (\vec{r}) = \int d^3r f(\vec{r}, w) \sum_{\vec{p}} \frac{e^{-i\vec{p}\cdot\vec{r}}}{\sqrt{V}} a^\dagger_{\vec{p}} = \sum_{\vec{p}} f(\vec{p}, w) a^\dagger_{\vec{p}}, \quad (2.3.6)$$

with $f(\vec{p}, w)$ being the Fourier transform of $f(\vec{r}, w)$. Because $f(\vec{r}, w)$ is a smooth function its Fourier transform suppresses large momenta and acts effectively as a high momentum cutoff. The result of Eq. (2.3.6) is substituted into Eq. (2.3.4). The creation operators on the vacuum give delta functions, resulting in

$$E^V_{\text{smoothed}} = \frac{1}{2m} \int_{-\infty}^{\infty} d^3r \nabla \cdot \nabla f(\vec{r}, w) \quad (2.3.7)$$

In Sec. 7.2 we evaluate explicitly a related case, the restricted smoothed momentum squared $\langle \psi | (P^2_{\text{smooth}})^V | \psi \rangle$.

For specific window functions the smoothed restricted energy can be evaluated explicitly. Consider, for example, a one dimensional case with

$$f(x, w) = \frac{1}{2} + \frac{1}{\pi} \arctan \left( \frac{x}{w} \right). \quad (2.3.8)$$

The function is depicted by the dashed line in Fig. 2.3.1. In momentum space (ignoring the singularity at $p = 0$)

$$f(p, w) = \frac{1}{\sqrt{2\pi} p} e^{-|p|w}. \quad (2.3.9)$$
So $1/w$ acts as a high momentum cutoff suppressing any momentum components of the smoothed wavefunction with $|p| > 1/w$.

The value of the restricted energy, the restriction being the positive half of the x-axis, can be calculated analytically in this case

$$E^V_{\text{smoothed}} = \frac{1}{2m} \frac{1}{2\pi w}.$$  \hspace{1cm} (2.3.10)

This is also shown in Fig.2.3.1 taking $m = 1/2$. As $w \to 0$, so that the partition becomes sharper, the energy increases, and it diverges for an infinitely sharp partition.\(^2\)

Figure 2.3.1: Shown is a one dimensional example of a smooth window function (left) and the corresponding restricted energy as a function of barrier width (right).

Other smoothing functions yield very similar results. The restricted energy is inversely proportional to the smoothing width $w$ and diverges in the limit $w \to 0$.

For the nonrelativistic case, $E^V_{\text{smoothed}} = \frac{1}{2m} \langle \psi | (P^2_{\text{smooth}})^V | \psi \rangle$. Since $\langle \psi | (P^2_{\text{smooth}})^V | \psi \rangle = 0$ it follows that $\langle \psi | (P^2_{\text{smooth}})^V | \psi \rangle = (\Delta P^V_{\text{smooth}})^2$ so the divergence of the energy is equal to the divergence of the momentum fluctuations. The divergence should not be confused with a UV divergence; the two are unrelated. The boundary behaves as if it is a localized particle. Given a function describing barrier slope, energy increases as the barrier grows sharper. That is, the more sharply the position of the dividing barrier is specified, the larger the energy. In the limit that the width tends to zero $w \to 0$ the energy diverges. This is the same phenomenon found in quantum mechanical uncertainty, where the more sharply we specify the position

\(^2\)We note that this term represents the contribution of the partitioning to the energy. A full calculation would include the wave function for the particle, $g(\vec{p})$ as explained in the previous note.
of a particle, the greater the uncertainty of its momentum. The energy in this case is a simple function of momentum and linearly related to momentum uncertainty, so that as the momentum fluctuations diverge so will the energy. Thus the energy divergence here is an indication of position/momentum uncertainty.

2.3.3 Relativistic smoothed restricted energy

We extend the previous computation from the case of non-relativistic fields to the case of relativistic fields. It is not immediately clear what position uncertainty means in the case of a relativistic field because the position operator is not defined in a clear way for this case.

The momentum operator, on the other hand, can be defined in a straightforward way from the energy-momentum tensor $P_j = T_{0j} = \int \frac{d^3k}{(2\pi)^3} k_j a_k^\dagger a_k$ taking $c, \hbar = 1$. Using the momentum operator we can have a practical definition of the uncertainty relations based on evaluation of the momentum fluctuations in a localized state corresponding to excitation of the field in a limited region of space. This is what we will use in the following, leaving the formal definitions and the deeper meaning of this definition for more philosophical discussions. Let us consider a single particle state $\int d^3 x g(\vec{x} - \vec{x}_0, w) \Psi^\dagger(\vec{x}) |0\rangle$. The wavefunction of this state $g(\vec{x} - \vec{x}_0, w)$ is localized at $x = x_0$ with $w$ being the scale on which the state is spread. For example, we can take $g(x - x_0, w) \propto e^{-\frac{(x-x_0)^2}{2w^2}}$. We then evaluate the momentum fluctuations in this state. They will grow in an inverse proportionality to the localization scale $w$ of the state. Similarly, if we have an $n$-particle state $\int \prod_1^n d^3 x_i g(x_i - \vec{x}_0, w) \prod_1^n \Psi_j^\dagger(\vec{x}_j) |0\rangle$ and we evaluate the fluctuations of the total momentum of the state, they will grow in an inverse proportionality to the localization scale $w$. Obviously, the state can have several localization scales. In that case the smallest one will be the most significant. The generalization to an arbitrary state should be clear by now.

In this context, formally, the only difference between a relativistic field and the non-relativistic field treated with second quantization is that both creation and destruction operators appear in the field operator. The formal analogy between the relativistic case and the non-relativistic one is clear and must point to a real correspondence between the two cases when considering the position/momentum uncertainty relation despite the inability to define a covariant position operator for the relativistic case.
In a relativistic system the energy operator is taken from the energy momentum tensor: \( H = T_{00} = \int \frac{d^3k}{(2\pi)^3} k_0 a_k^\dagger a_k \).

In order to look for the various expectation values we recall the relativistic scalar product:

\[
\langle \varphi | \phi \rangle = -i \int d^3x [\varphi \partial_t \phi^* - (\partial_t \varphi) \phi^*] \tag{2.3.11}
\]

and the expression for the expectation value of the Hamiltonian in a state \( |\varphi\rangle \)

\[
\langle \varphi | H | \varphi \rangle = -i \int d^3x [\varphi \partial_t (H \varphi)^* - (\partial_t \varphi) (H \varphi)^*]. \tag{2.3.12}
\]

A smoothed state with window function, as before, can be defined as before \( \int d^3r f(\vec{r}) \Psi^\dagger(\vec{r}) |0\rangle \) where the field operator here is the relativistic one. The resulting smoothed restricted energy is

\[
E_{\text{smooth}}^V = \langle \psi | (H_{\text{smooth}})^V | \psi \rangle = \int d^3p f(\vec{p},w) p f(-\vec{p},w) = \int d^3r f(\vec{r},w) \sqrt{\nabla^2} f(\vec{r},w) \tag{2.3.13}
\]

The details of the derivation are given in Sec.7.3. The result clearly has the same behavior as in the non relativistic case. Alternately, since \( E^2 \sim P^2 \) we may calculate \( \langle P^2 \rangle \) and obtain

\[
\frac{1}{2} \int_{-\infty}^{\infty} d^3r \nabla f(\vec{r},w) \cdot \nabla f(\vec{r},w) \tag{2.3.14}
\]

This is identical to the non relativistic result, and equals \( (\Delta P_{\text{smooth}}^V)^2 \).

We saw that in the nonrelativistic treatment energy tends to diverge the more sharply the boundary between the different parts of space is specified. The relativistic case shows the same phenomenon. Here too, the smoothing function \( f(\vec{r},w) \) acts as a momentum cutoff. In both cases the energy increases as the barrier width becomes narrower, and diverges for a completely sharp barrier with zero width. In the relativistic case \( E^2 \sim P^2 \) rather than \( E \sim P^2 \) but we still obtain \( E \sim \Delta p \). As before, the energy is proportional to the momentum uncertainty, and just as in the previous section, it diverges when the barrier is made sharp. This can be seen as an example of position/momentum uncertainty.


2.4 Restricted energy and statistical entropy of the black hole

So far we have discussed restricted operators in flat spacetime. The restriction was implemented in an ad-hoc way by a choice of a (smoothed) theta function. In the case of the BH, spacetime is restricted in a different way. For example, in the Schwarzschild geometry, the metric \( ds^2 = -(1 - \frac{r_s}{r})dt^2 + \frac{1}{1 - \frac{r_s}{r}} dr^2 + r^2 d\Omega^2 \) is used to treat the region of space outside the horizon \( r > r_s \). So all the operators used in Schwarzschild geometry are restricted operators. One can view the redshift factor \( \frac{1}{1 - \frac{r_s}{r}} \) as implementing the restriction by becoming infinite at the horizon \( r = r_s \).

We will try to explain how the redshift, acting as a restriction, creates an infinitely sharp boundary that results in divergence of the reduced energy and reduced entropy. We begin with the simpler case of Rindler spacetime, that is the spacetime of an accelerated observer in Minkowski space. Rindler space has the advantage that it is equivalent to a restriction to half of Minkowski space so this example allows us to explicitly compare the two restriction mechanisms. We will explain how we can implement the ideas of smoothing the boundary by restricting the maximal value of the redshift, and show that when smoothing is implemented all quantities are rendered finite with magnitude inversely proportional to the smoothing parameter, exactly as in the cases that we have encountered before. This will allow us to show that a similar phenomenon occurs for BH’s.

2.4.1 The uncertainty principle in Rindler spacetime

We use the Minkowski space metric:

\[
    ds^2 = -dt^2 + dz^2 + d\vec{x}_\perp^2,
\]

(2.4.1)

where \( z \) is the coordinate that will be used to separate space into the left and right halves \( z < 0 \) and \( z > 0 \) and \( \vec{x}_\perp \) stands for the transverse coordinates. An accelerated observer whose acceleration is \( a/2\pi \) lives in Rindler space whose metric is

\[
    ds^2 = -e^{2a\xi} d\eta^2 + e^{2a\xi} d\xi^2 + d\vec{x}_\perp^2.
\]

(2.4.2)
The Minkowski coordinates and Rindler coordinates are related by:

\[ t(\xi, \eta) = \frac{1}{a} e^{a\xi} \sinh a\eta \]  
(2.4.3)

\[ z(\xi, \eta) = \frac{1}{a} e^{a\xi} \cosh a\eta \]  
(2.4.4)

\[ \vec{x}_\perp = \vec{x}_\perp. \]  
(2.4.5)

Choosing a fixed Rindler time, for example, \( \eta = 0 \), we see that the \( \xi \) coordinate only covers the \( z > 0 \) half of space. The restriction is implemented by the redshift factor \( e^{-a\xi} \) which diverges for \( \xi \to -\infty \), corresponding to \( z = 0 \).

As it stands, the restriction implemented by the redshift is infinitely sharp. The Rindler observer does not see the region \( z < 0 \). We wish to understand how to implement a smoothed restriction rather than an infinitely sharp one. So we analyze just how the redshift leads to divergence of \( (\Delta p)^2 \) and vanishing of \( (\Delta z)^2 \), in order to consider how the divergence may be tamed. We consider a non-relativistic particle whose wave function has some spread \( \Delta z \) in Minkowski space. For example,

\[ \psi(z) = \frac{1}{\sqrt{2\pi(\Delta z)^2}} e^{-\frac{1}{2} \frac{z^2}{(\Delta z)^2}}. \]  
(2.4.6)

In momentum space the spread of the wave function is inversely proportional to \( \Delta z \), \( (\Delta p)^2 \sim 1/(\Delta z)^2 \). Viewed by an accelerated observer, the wave function at the origin \( z = 0 \) corresponding to \( \xi \to -\infty \) would be squeezed in the \( \xi \) direction: \( \Delta \xi = e^{a\xi} \Delta z \). As required by the uncertainty principle the spread in momentum would increase, \( \Delta p_\xi = e^{-a\xi} \Delta p_z \). Thus finite \( \Delta z \) and \( \Delta p \) in Minkowski space are adjusted by the Rindler metric, so that to the Rindler observer the position fluctuations at the origin will vanish and momentum fluctuations will diverge.

By our choice the particle is localized at the origin (any other choice would simply require a shift in the Rindler time \( \eta \)), so in the limit \( \xi \to -\infty \) the momentum fluctuations diverge because the wave function has been squeezed in space. This divergence obviously does not signal a breakdown of physics. It just means that considering the classical Rindler geometry when viewing a quantum particle requires closer thought. Rindler geometry imposes a restriction on Minkowski space. When the restriction is sharp, equivalent to localizing a particle at the origin, the momentum fluctuations diverge. Limiting the Rindler redshift factor tames the divergence and increases position fluctuations, thus softening the
localization, and smoothing the restriction.

### 2.4.2 Momentum fluctuations and redshift in Rindler spacetime

In view of the previous discussion, and in preparation for the reinterpretation of the 't Hooft calculation, let us consider a (massless) scalar field $\phi$ that satisfies the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}} \left( \partial_{\mu} \sqrt{-g} g^{\mu\nu} \partial_{\nu} \right) \phi = 0. \quad (2.4.7)$$

In Minkowski spacetime there is an exact solution to the Klein-Gordon equation. The $z$ dependent part of the solution is given by

$$\phi(z) = e^{\pm ipz}. \quad (2.4.8)$$

However, for the purpose of making the calculation more similar to the 't Hooft calculation we can rewrite the solution in a WKB form, where the WKB solution is

$$\phi_{WKB}(z) = e^{\pm \frac{i}{\hbar} p(z) dz}. \quad (2.4.9)$$

Obviously, in Minkowski space $p(z)$ is a constant and the WKB solution reduces to the exact solution. The WKB momentum can be expressed as

$$p^2(z) = E^2 - p_\perp^2. \quad (2.4.10)$$

In Rindler spacetime the WKB wave function is

$$\phi_{WKB}(\xi) = e^{\pm \frac{i}{\hbar} \int d\xi \sqrt{g_{\xi\xi}} p(\xi)} \quad (2.4.11)$$

with

$$p^2(\xi) = g^{\eta\eta} E^2 - p_\perp^2 \quad (2.4.12)$$

which is space varying. So the WKB wave function is

$$\phi_{WKB}(\xi) = e^{\pm \frac{i}{\hbar} \int d\xi \sqrt{g_{\xi\xi}} \sqrt{g^{\eta\eta} E^2 - p_\perp^2}}. \quad (2.4.13)$$
Near the horizon $p(\xi)$ diverges as $\sqrt{g^{\mu\nu}E^2} = e^{-a\xi}E$ and the proper length $\tilde{d}\xi = d\xi/\sqrt{g_{\xi\xi}} = d\xi e^{a\xi}$ vanishes. This is a manifestation of the position/momentum uncertainty relation caused by the redshift.

Rindler space implements a sharp division of Minkowski space. That is, the Rindler observer sees a sharp cutoff at the horizon $\xi \to -\infty$. Smoothing this cutoff in momentum space means restricting the momentum $p(\xi)$ near the horizon. We saw in the previous section that restricting the redshift widens $\Delta x$ and shrinks $\Delta p$. Therefore restricting the redshift $g^{\mu\nu}$, $g^{\xi\xi}$ will smooth the cutoff.

In 't Hooft’s black hole calculation the energy and entropy diverge due to a diverging density of states. In Rindler space too the density of states diverges, and we will see that this divergence is due to the uncertainty principle. We define the density of states near energy $E$ in Rindler space and evaluate it by counting the number of WKB solutions

$$\pi n = \int d\xi e^{a\xi} \int \frac{d^2 p_\perp}{(2a)^2} p(\xi, E, p_\perp)$$
$$= 2\pi \int d\xi e^{a\xi} \int \frac{dp_\perp}{(2a)^2} p_\perp \sqrt{e^{-2a\xi}E^2 - p_\perp^2}$$
$$= -\frac{2}{3} \frac{\pi}{(2a)^2} E^3 \int d\xi e^{-2a\xi} \quad (2.4.14)$$

where we have performed first the angular integral of $p_\perp$ and then the radial part. This integral diverges because of the diverging redshift factor at the horizon. So the density of states, the entropy and energy are divergent for the same reason and if the redshift factor is restricted, they all become finite.

We can smooth the partition by limiting the redshift, or alternately, by implementing a smoothing function on states of the system. This equivalent procedure will also tame the divergence. The smoothed functions that we need to count are obtained by multiplying the original unsmoothed function by the smoothing function, $\psi(\xi) \to \psi(\xi)f(\xi, w)$, or in Fourier space $\phi(p) \to \phi(p)f(p, w)$. Recall that in momentum space the function $f(p, w)$ acted as a high momentum cutoff for $p > 1/w$. Then for wavefunctions with energy $E$ we need to effectively restrict the Rindler momentum $p(\xi) = e^{-a\xi}\sqrt{E}$ to be $p(\xi) < 1/w$. In this context it simply means that the redshift factor is limited to some maximal value which can always be expressed as $e^{-a\xi_{\text{max}}}$. The “brick wall” model of 't Hooft in this
context amounts to a sharp cutoff on the momentum \( p(\xi) \). However, clearly, any other cutoff schemes will do the same job. The density of states of smoothed wavefunctions is of course finite,

\[
\pi n = \int_{\xi_{\min}} d\xi e^{a\xi} \int \frac{d^2p_\perp}{(2a)^2} p(\xi, E, p_\perp)
= 2\pi \int_{\xi_{\min}} d\xi e^{a\xi} \int \frac{dp_\perp}{(2a)^2} p_\perp \sqrt{e^{-2a\xi E^2} - p_\perp^2}
= \frac{2}{3} \frac{\pi}{(2a)^3} E^3 e^{-2a\xi_{\min}}. \tag{2.4.15}
\]

This makes the energy and entropy finite and inversely proportional to the maximal redshift which determines the smoothing width of the division in Rindler space.

### 2.4.3 Momentum fluctuations and entanglement entropy in Schwarzschild spacetime

‘t Hooft solves the wave equation in the Schwarzschild metric, identifies \( p \), the wave number, and using a WKB approximation he obtains the density of states. However the redshift leads this to diverge at the horizon. The region near the black hole horizon is a thermal state in Rindler space, and indeed just as in Rindler space, limiting the redshift will prevent the divergence.

We recall the calculation in Schwarzschild coordinates. For simplicity we have chosen the scalar field to be massless. The Klein-Gordon equation in these coordinates is

\[
\left(1 - \frac{2M}{r}\right)^{-1} E^2 \phi + \frac{1}{r^2} \partial_r (r^2 (r - 2M) \partial_r) \phi - \left(\frac{l(l+1)}{r^2}\right) \phi = 0. \tag{2.4.16}
\]

The wave number can be defined as

\[
p^2 = g^{tt} E^2 - \left(\frac{l(l+1)}{r^2}\right) \tag{2.4.17}
\]

Using a WKB approximation the density of states for a massless scalar field is

\footnote{This differs by a a factor \( g_{rr} \) from ‘t Hooft’s original defintion.}
given by

\[ \pi n = \sum_{l,m} \int_{2M} dr \sqrt{g_{rr}} p(r, l, m) \]  

(2.4.18)

\[ = \int_{2M} dr \sqrt{g_{rr}} \int (2l + 1) dl \sqrt{g_{tt} E^2 - \frac{l(l + 1)}{r^2}} \]

where \( l, m \) are the angular parameters. Evaluating the integral over \( l \) we find

\[ \pi n = -\frac{2}{3} \int_{2M} dr \sqrt{g_{rr}} r^2 \left( g_{tt} E^2 \right)^{3/2} \]

\[ = -\frac{2}{3} E^3 \int_{2M} dr \frac{r^2}{(1 - \frac{2M}{r})^2} \]  

(2.4.19)

This integral diverges at the horizon. If we were to limit the redshift, as we did with Rindler space, there would be no divergence. Apparently 't Hooft does otherwise: he takes the lower limit a slight distance away from the horizon, his well known “brick wall,” so that the lower limit becomes \( 2M + h \). From this expression he obtains the energy and entropy, which diverge as \( h \to 0 \).

In fact 't Hooft’s adjustment of the lower limit of the integral from \( 2M \) to \( 2M + h \) is equivalent to a change of variable which leaves the lower limit at \( 2M \) but changes the redshift:

\[ \int_{2M+h} dr \left( 1 - \frac{2M}{r} \right)^{-2} \]

\[ = \int_{2M} d\tilde{r} \left( 1 - \frac{2M}{\tilde{r} + h} \right)^{-2} \]  

(2.4.20)

This clearly does not diverge at the horizon. The new expression is always finite and is limited by \( \left( 1 - \frac{2M}{2M+h} \right)^{-2} \lesssim (2M/h)^2 \) for \( h \ll M \).

The altered redshift is equivalent to multiplication of the original redshift in the \( \tilde{r} \) system by a smoothing function:

\[ \left( 1 - \frac{2M}{\tilde{r} + h} \right)^{-1} = \left( 1 - \frac{2M}{\tilde{r}} \right)^{-1} f(\tilde{r}, h) \]  

(2.4.21)
with

\[ f(\tilde{r}, h) = \frac{(\tilde{r} + h)(\tilde{r} - 2M)}{\tilde{r}(\tilde{r} - 2M + h)} \quad (2.4.22) \]

Thus the change of variable implemented by the brick wall has the effect of multiplying the redshift by a smoothing function.

The original divergent integral in eq.2.4.20 can be expressed in terms of a sharp step function \( \int_{2M}^{\infty} dr \left( 1 - \frac{2M}{r} \right)^{-2} = \int_{0}^{\infty} dr \ \Theta(r - 2M) \left( 1 - \frac{2M}{r} \right)^{-2} \). The altered integral can be expressed in terms of a smoothed step function

\[ \int_{0}^{\infty} dr \left( 1 - \frac{2M}{r} \right)^{-2} f^{2}(r, h) \ \Theta(r - 2M) = \int_{0}^{\infty} dr \left( 1 - \frac{2M}{r} \right)^{-2} \tilde{\Theta}(r - 2M, h) \]

Thus we see that 't Hooft’s changed lower limit is exactly equivalent to smoothing the step function to a new one \( \tilde{\Theta}(r - 2M, h) = f^{2}(r, h) \ \Theta(r - 2M) \) with width \( h \). Formally the brick wall can be seen as either changing the redshift or smoothing the step function and thus modifying the sharp partitioning of the region. Obviously, any other limiting procedure of the maximal redshift will render the integral finite and make the energy and entropy finite.

Figure 2.4.1: Smoothed step function as function of \( r/2M \). Curves have \( h = 0 \) (sharp step), \( h = 0.1 \) and \( h = 0.9 \) (lowest).

### 2.5 Summary of results

Energy has been shown to diverge as the boundary between two quantum subsystems, an observable subsystem and an unobservable subsystem, becomes sharp. The divergence is due to the fact that the energy is a simple function of the mo-
momentum fluctuations. These diverge in the presence of a sharp boundary because of the uncertainty principle, much in the same way that they diverge for a sharply localized particle. For the nonrelativistic case $\langle E \rangle = \frac{1}{2m} (\Delta P)^2$. In the relativistic case $\langle E \rangle = \Delta P$ so in both cases energy divergence at an infinitely sharp boundary is clearly a consequence of position/momentum uncertainty.

In a coordinate system which implements a sharply localized boundary, the density of states and thus energy and entropy diverge at the boundary. Limiting the redshift tames this divergence. We have shown that limiting the redshift smoothes the boundary by widening $\Delta x$ and limiting $\Delta p$. Therefore the smoothing cutoff prevents the energy from diverging. This implies that the divergence of the energy and entropy was a result of the sharp localization of the boundary, and was due to the uncertainty principle.

The region near the boundary of a black hole is a thermal state, where the entropy is linear to energy. Therefore black hole entropy will diverge at the boundary as well. We have shown that regardless of any other cause, there would be divergence at the infinitely sharp boundary as a result of the uncertainty principle. We have also shown that 't Hooft’s divergence at the black hole is an example of momentum/position uncertainty, as seen by the fact that the “brick wall” which corrects it in fact smoothes the sharp boundary.

This result raises the question whether the entanglement and statistical mechanics definitions of black hole entropy might refer to the same quantity. Both are proportional to area. The UV divergence may be tamed with a UV cutoff, and the boundary divergence by smearing out the boundary (both procedures might turn out to be equivalent). So the two expressions could be expressing the same quantity. If this is the case, then the microscopic counting of the number of states becomes tantamount to counting the correlations between the observed and unobserved regions of spacetime. Black hole entropy has also been shown, from thermodynamic considerations as well as explicit calculations in string theory, to equal one fourth of the horizon area. An open problem is to obtain the factor of 1/4 in either of these definitions of black hole entropy.
2.6 Entropy from conformal field theory

In the previous section we showed that the divergence of entropy at the horizon may be due to quantum uncertainty. We explicitly refrained from imposing boundary conditions at the horizon. The boundary served to trace out part of the system by limiting the possibility of observations, but did not imposing new boundary conditions on the system.

Carlip [22] derived black hole entropy from arguments of symmetry. This was motivated by earlier work of Brown and Henneaux [34] showing that 2+1 dimensional gravity with a negative cosmological constant has an asymptotic symmetry consisting of a pair of Virasoro algebras, so that a microscopic quantum black hole theory should be a conformal field theory. In conformal theory entropy can be obtained with the Cardy formula. Strominger [35] made use of this to obtain entropy of a 2+1 dimensional black hole.

Carlip proposed a scheme applicable in any dimension. When the horizon is treated as a boundary, the algebra of constraints in general relativity acquires a central extension. With appropriate boundary conditions this extended algebra contains a Virasoro subalgebra and the Cardy formula can be used to obtain entropy. In [23] this was done using the ADM formula, and then in [22] using covariant phase space methods. We focus on the second approach here. We note that Carlip’s entropy does NOT diverge at the boundary, and discuss the reason.

Below we outline Carlip’s scheme and focus on his treatment of the boundary. He works with a stretched horizon, such that \( \chi^2 = \epsilon \) where \( \chi \) is a Killing vector at the boundary, and at the end of the calculation he takes \( \epsilon \to 0 \). He does so in order to vary the Noether charge at the boundary. To do this he must go a short distance away from the exact boundary. There he decomposes \( \chi \) (no longer a Killing vector at that point), into two orthogonal vectors, as the “r,t plane.” It’s as if in order to wiggle something one needs to stretch it a bit. If one sits exactly on the boundary there is no wiggle room. The question we examine is - what happens to Carlip’s entropy when taking \( \chi^2 \) to 0, that is, exactly at the boundary? Does it diverge as in the previous section? We find that the scheme breaks down at the boundary itself, and discuss the implications.
2.6.1 Background

In 1986 Brown and Henneaux [34] showed that in the canonical (ADM) formalism general relativity has a central extension, so that \( \{ H [\xi_1], H_2 [\xi_2] \} = H \{ \xi_1, \xi_2 \} + K \{ \xi_1, \xi_2 \} \) where \( \xi \) is a surface deformation vector, \( H [\xi] \) is its symmetry generator and \( K \) is the central charge. In the ADM formalism the vector fields \( \xi \) which preserve the spacetime metric under Lie transport become deformations of a spacelike surface described by the canonical variables \( g_{ij}, \pi^{ij} \). Brown and Henneaux show this symmetry group is isomorphic to the two dimensional conformal group, its central charge is non trivial and its algebra a direct sum of two Virasoro algebras. This was done for \( 2 + 1 \) dimensional gravity with a negative cosmological constant, but they write that it can easily be generalized to higher dimensions. In 1991 Barnich, Henneaux and Schomblond showed that the Hamiltonian Poisson bracket structure is equivalent to the covariant phase space formalism which deals with the Lagrangian [38]. Therefore the latter too can be used in the calculation of a central charge for the algebra of general relativity.

Cardy [24] studied statistical systems at a critical point, and the consequences of their conformal invariance. He related the free energy of such a system to the central charge of the Virasoro algebra. He thus obtained the partition function as a function of the central charge. The number of states can be extracted from the partition function [39], and the entropy is the logarithm of the number of states:

\[
S(\Delta) = 2\pi \sqrt{\frac{c_{\text{eff}} \Delta}{6}}
\]  

(2.6.1)

where \( \Delta \) is the eigenvalue of Virasoro generator \( L_0 \), and the effective central charge takes different forms depending on the particular conformal field theory under discussion. In cases where the lowest eigenvalue of \( L_0 \) does not vanish, \( \Delta_0 \neq 0 \), \( c_{\text{eff}} = (c - 24\Delta_0) \). For explicit examples see [39].

In [22] Carlip uses the covariant phase space formalism, and obtains the central charge for general relativity. The scheme is detailed in Sec 6.2, but here we give a brief outline. Since he is interested in the entropy associated with the black hole horizon, he attempts to specify boundary conditions that will reflect the presence of the horizon. He works with a stretched horizon. In that case the spacetime has no Killing vector, so he requires boundary conditions which preserve the asymptotic horizon structure, that is, as \( \chi^2 \to 0 \) they ensure that the boundary will be
a null surface. He focuses on vector fields $\xi^a$ which generate a diffeomorphism, and decomposes them into a linear combination of orthogonal vectors in the “r-t plane.” $\xi^a = R\rho^a + T\chi^a$. At the horizon $\rho^a$ and $\chi^a$ coincide. One finds that $R \sim \chi^a \nabla_a T$, so that $\xi^a = \text{const} \cdot \chi^b \nabla_b T \rho^a + T\chi^a$, and that closure of the algebra requires $\rho^a \nabla_a T = 0$ at the horizon.

Writing $\chi^a \nabla_a T \equiv DT$, the central charge is shown to be

$$K[\xi_1, \xi_2] = \frac{1}{16\pi G} \int_H \hat{\epsilon}_{a_1...a_{n-2}} \frac{1}{\kappa} (DT_1 D^2 T_2 - DT_2 D^2 T_1). \quad (2.6.2)$$

The $T$ are written as periodic functions

$$T_n (\nu, \theta^i) = \frac{1}{\kappa} e^{ink\nu} f_n (\theta^i) \quad (2.6.3)$$

where $\nu$ is a parameter along the orbits of the Killing vector $\chi$, and $\theta^i$ are angular coordinates. Implementing previously derived orthogonality constraints, the central term is

$$K[T_m, T_n] = -\frac{iA}{8\pi G} m^3 \delta_{m+n,0} \quad (2.6.4)$$

and plugging this into the Cardy formula, with $\Delta$ as a given eigenvalue of $J[T_0]$, gives the black hole entropy with the known value of $\frac{A}{4G}$.

### 2.6.2 The horizon

Carlip’s derivation was done using the stretched horizon. To find the entropy at the horizon one then takes $\chi^2 \to 0$. However this is problematic.

First, at the horizon the two orthogonal vectors to which the diffeomorphism generator was decomposed coincide. Since $\rho = \chi$ at the horizon this means that on the horizon the central charge vanishes. A key requirement for closure of the algebra was $\rho^a \nabla_a T = 0$. When $\rho = \chi$ then also $\chi^a \nabla_a T = 0$ and clearly eq.$(2.6.2)$ equals zero and the central charge vanishes, as does the entropy. Therefore either the entropy at the horizon vanishes or the entire scheme is not valid exactly at the horizon. This could be meshed with the previous section of this chapter, as showing that it is not possible to calculate entropy at a defined point in space. Cardy’s entropy is in fact that of statistical mechanics and is proportional to energy. The problem with this idea is that Carlip’s entropy does not diverge on
the boundary if the central charge vanishes, but rather the entropy vanishes as well.

To further investigate this we change to horizon crossing coordinates. Since away from the horizon, $\chi, \rho$ are the “r,t plane,” then $\chi$ is in fact the timelike vector $\partial_t$ and $\rho$ is the radial vector $\partial_r$. To enforce closure of the algebra, the constraint is $\rho^a \nabla_a T = 0$. So $T$ is only a function of $\chi$ (that is, of $t$ not of $r$). Writing them as Kruskal coordinates we will see that for $T$ a function of $t$,

$$T = f(t) \rightarrow T = f \left( \frac{u}{v} \right) = f \left( e^{t/2M} \right).$$

(2.6.5)

Then $T = f \left( \frac{u}{v} \right)$ at the horizon will be either diverge or be trivial, depending whether $u$ or $v$ goes to 0 faster. In detail: We write the vector $\xi = ADT \hat{\rho} + T \hat{\chi}$ where $A \equiv \frac{1}{\kappa} \chi^2$. We use tortoise coordinates:

$$r^* = r + 2M \log \left| \frac{r - 2M}{2M} \right| \approx 2M \log \left| \frac{r - 2M}{2M} \right|,$$

(2.6.6)

at the horizon.

$$U = -e^{\exp \left\{ \frac{r^* - t}{4M} \right\}} = -e^{\exp \left\{ \frac{2M \log \left| \frac{ADT - 2M}{2M} \right| - T}{4M} \right\}} = -\sqrt{\frac{ADT - 2M}{2M}} e^{-T/4M},$$

$$V = e^{\exp \left\{ \frac{r^* - t}{4M} \right\}} = \sqrt{\frac{ADT - 2M}{2M}} e^{T/4M}.$$

(2.6.7)

So

$$\xi = -\sqrt{\frac{ADT - 2M}{2M}} e^{-T/4M} \hat{U} + \sqrt{\frac{ADT - 2M}{2M}} e^{T/4M} \hat{V},$$

(2.6.8)

or in a basis of $(U,V,x_\perp)$

$$\xi = \left( -\sqrt{\frac{ADT - 2M}{2M}} e^{-T/4M}, \sqrt{\frac{ADT - 2M}{2M}} e^{T/4M}, 0 \right).$$

(2.6.9)

To work out the central charge we write the modes
\[ T_n = \frac{1}{k} e^{iknt} f_n(\theta), \quad \frac{dT}{dt} = -iknT, \quad \frac{d^2T}{dt^2} = -k^2n^2T \]
\[ -\frac{U}{V} = e^{-t/(2M)} \rightarrow t = \log \left( -\frac{U}{V} \right)^{-2M} \]
\[ T = \frac{1}{k} \left[ -\frac{U}{V} \right]^{-ikn2M} \]

Then the central charge works out to be

\[ K = \frac{1}{16\pi G} \int \hat{f}_n \hat{f}_m (-inm^2 + imn^2) \left[ \frac{U}{V} \right]^{-i4kM(n+m)} \]
\[ = \frac{A}{16\pi G} \delta_{m+n,0} (-inm^2 + imn^2) \left[ \frac{U}{V} \right]^{-i4kM(n+m)} \]
\[ = \frac{A}{8\pi G} \delta_{m+n,0} (in^3) \left[ \frac{U}{V} \right]^{-i4kM(n+m)} \] (2.6.10)

If the integration is done before going to the horizon, this term is finite thanks to the delta function. However the calculation cannot be performed on the horizon itself, where if \( V = 0, U \neq 0 \), the expression will vanish or diverge at the horizon. The limiting process must be done at the end of the calculation.

The divergence seems to recall the divergence we found in the previous section, in particular because it arises as the stretched horizon is taken back to the precise location of the horizon. However it results from the constraint for closure of the algebra rather than from quantum uncertainty.

We see that the very constraint which ensures the existence of the algebra of diffeomorphism invariance prevents calculation of entropy at the horizon. This makes intuitive sense, since the horizon limits diffeomorphism invariance by defining a border beyond which no continuous translation is possible. In Chapter 4 we vary the entropy using a Lie derivative of the partition function, and we see that in a space without a horizon it would vanish, as a result of diffeomorphism invariance, but once the space is bounded it does not vanish. In Carlip’s scheme the existence of a boundary of space gives rise to the central charge from which the entropy is calculated; the weak point in this scheme is that the algebra itself does not hold at the boundary, and thus the central charge itself cannot be calculated.
this differs from statistical and entanglement entropy in two aspects. First, in statistical and entanglement entropy, the boundary is perceived as a “virtual wall” allowing us to look at part of the system, rather than an actual wall imposing boundary conditions. Second, problematic behavior at the boundary in statistical entropy was shown to be a possible artifact of quantum uncertainty, whereas in Carlip’s scheme it results from constraints on the classical algebra.

2.7 Discussion

Entanglement entropy does not necessarily diverge at the horizon, whereas thermodynamic entropy has been shown to do so. However this divergence can be seen as a result of quantum uncertainty rather than related to some new unknown black hole physics.

In this chapter I have looked at boundary behavior of different treatments of black hole entropy. In the first part, I claimed that boundary conditions must not be changed when tracing out part of the system: imposing new boundary conditions would mean changing the subsystem itself. In contrast, Carlip’s treatment imposes boundary conditions and his derivation of the entropy is a direct result of the existence of the boundary, and of his imposition of boundary conditions.

Carlip’s entropy does not result from the state being thermal but rather from conformal symmetry. His imposition of boundary conditions is related to this symmetry. One effect of this is that there is no horizon divergence. But the lack of divergence is essentially related to the entropy’s source in symmetry, as opposed to thermodynamic entropy which is proportional to energy and thus diverges from \( x/p \) uncertainty. On the face of it horizon divergence appears to be related to treatment of boundary conditions, but in fact it is related to the question whether we deal with a thermal state or not, because in that case entropy is proportional to energy, which was shown to diverge at a sharply localized boundary.

There is a possibility that a stretched horizon bears resemblance to ’t Hooft’s brick wall, which we have here shown may be a method of delocalizing the horizon. In neither case can the entropy be calculated exactly at the horizon. However the source of the problem in these two cases is different: in statistical mechanics it results from quantum uncertainty and in Carlip’s scheme it results from the
classical algebraic constraint on diffeomorphism invariance.

Entanglement entropy and statistical entropy of a black hole may refer to the same phenomenon. While entanglement entropy does not diverge at the horizon, the divergence of statistical entropy may be due to $x/p$ uncertainty. If they are the same, the degrees of freedom that are entangled would be the Hawking pairs, since it is they that lead to a thermal state at the horizon. In both cases boundary conditions are not imposed at the border. This is essentially different from Carlip’s entropy from conformal symmetry. It is clear from all this that when considering black hole entropy one must specify the treatment of boundary conditions in the scheme under discussion.
Chapter 3

Curvature independence

In this chapter I focus on statistical entropy. I examine the number of states, as given by the volume of phase space. Statistical mechanical entropy is then calculated from the number of states, following ’t Hooft. Inspired by ’t Hooft’s treatment of a single particle in a black hole background, I consider whether statistical entropy of a particle in a black hole background has any unique dependence on curvature, for the following reason:

Wald entropy is defined in terms of the curvature tensor. Since Wald used the first law of thermodynamics to obtain the entropy, it should be related to the entropy defined in statistical mechanics. However if one calculates statistical entropy of a test particle in a region of Minkowski space and then Rindler space one obtains two different results (Sec.2.4.2 and see also [36]). Thus though the curvature in both Minkowski and Rindler spaces is identical (and vanishes), the statistical entropy will be different. In a space with non vanishing curvature, could statistical entropy be uniquely dependent on curvature?

For a stationary black hole solution in Einstein gravity, Wald entropy coincides with the original Bekenstein/Hawking entropy and is proportional to area[19, 20]. The statistical entropy of a particle in a black hole background was also shown to be dependent on area [36, 37]. This too leads one to wonder about a connection between the two formulations of entropy.

We will show that the number of states is an explicit function of the metric. Since the curvature derives from the metric, it would seem that the number of states is related to curvature. However we find that for certain transformations of the metric, the number of states is preserved. These transformations do not
preserve curvature. This is shown only for a diagonal metric, but it serves as a counter example showing that in the most general case the number of states is not uniquely dependent on the spacetime curvature scalar, although it may depend on the curvature together with other factors that at times cancel the effect of the curvature.

This chapter is organized as follows. First we establish the definition of the number of states. We then give an example that indicates observer dependence. This is followed by a more thorough exploration of the number of states, and of different methods of calculating the volume of phase space. We then ask under what conditions transformation of the metric will leave this volume invariant. We obtain a general transformation of any metric which displays a clear constraint on the preservation of the number of states. We examine characteristics of this transformation and look for a possible relationship to curvature. We find that in general it need not preserve curvature. That is, the number of states and thus the entropy will remain the same for systems with different curvature.

The constraint in our proof holds for a conformal transformation, but in general other transformations of the metric will not preserve the number of states. This proves that the number of states, and the statistical entropy derived from it, are observer dependent quantities.

In [40] numerical computation showed that for specific examples of curved space entanglement entropy does not depend on curvature. This section of the thesis we prove analytically that statistical entropy is not uniquely dependent on curvature. This gives further weight to the idea that statistical and entanglement entropy represent the same entity.

### 3.1 Definition of number of states

The system we treat here is that of a classical, massless free particle, confined to a spatial volume V. We want to calculate the number of states up to a given energy E. In statistical mechanics the number of states of a single particle in a nonrelativistic system is defined as follows: Take an integral over the volume of phase space \((d^3x d^3p)\), restrict it to values of momenta which fit the energy
eigenvalues of the system and to obtain

\[ N = g \int d^3x \frac{d^3p}{(2\pi\hbar)^3} \]

\[ = gV \int \frac{d^3p}{(2\pi\hbar)^3} \]

(3.1.1)

where \( g \) is a numerical factor related to the degeneracy (eg., for spins with Dirichlet boundary conditions \( g = 2(1/8) \) for the positive octant and spin degeneracy). This will be taken as 1 is the system under consideration, and in what follows \( g \) will refer to the determinant of the metric. Integration over the volume in phase space gives \( V = (\pi/L)^3 \) for a potential well, \((2\pi/L)^3\) for periodic boundary conditions. Dividing by a unit of volume in momentum space, \( 2\pi\hbar \), gives the number of states in phase space with the given energy, per unit volume of phase space. Since we do not limit ourselves to nonrelativistic thermodynamics, nor to three space dimensions, a more general definition is necessary.

The number of states, that is, the phase space volume of a single particle, is then defined as

\[ N(E,V,1) = \int d^dx \frac{d^dp}{(2\pi\hbar)^d} dE \delta(E - E(p)). \]

(3.1.2)

where \( d \) denotes the number of space dimensions\(^1\). Without loss of generality we are taking a constant time hypersurface (Sec.8.1.1). For \( n \) particles

\[ N(E,V,n) = \prod_i^n N_i \]

where \( N_i = N(E,V,1) \), and the particles are bosons (this would be more complicated for fermions). For \( n \) identical particles this is \( (N_i)^n / n! \), where the \( \frac{1}{n!} \) comes from Bose statistics, and the entropy for \( n \) particles is

\[ S(E,V,n) = k\log N(E,V,n) \]

where \( k \) is Boltzmann’s constant, and from here on we take \( k = 1 \). The number of states is Lorentz invariant. For a proof see Sec.8.1.

\(^1\)This integral is actually \( \int d^dx\delta(t - t_0) \frac{d^dp}{(2\pi\hbar)^d} dE \left[ 2\pi\hbar \delta(E - E(p)) \right] \).
In order to apply this definition to curved space, we need to clarify what momentum and energy refer to for a matter field or gas of particles in curved space. The discussion of the number of states assumes equilibrium or near equilibrium thermodynamics. Introducing time dependence involves complications on two levels: the spacetime may not be stationary, and the number of states may be time dependent. Here we dispense with these complications: we assume a stationary spacetime and take a constant time slice. This is because our argument will depend on showing counter examples to the claim that the number of states depends on the scalar curvature. If a simpler case provides the counter example, it should suffice.

There are (at least) two possible ways to approach the issue. One is that of 
\cite{7}, who took $\psi(x)$ a scalar wave function for a light spinless particle of mass $m$ in the Schwarzschild metric, $m \ll 1 \ll M$ where $M$ is the BH mass, used a WKB approximation, wrote the wave equation and defined the spatial momentum $k(r)$ in terms of the eigenvalues of the Laplacian operator while taking energy as the eigenvalue of the time component of the Laplacian. He obtained the number of states by calculating $\int k(r) dr$ and then summing over angular degrees of freedom.

Another possibility is that of \cite{36,37} who treated a relativistic gas of particles, and rather than the wave equation, used the scalar invariance of the squared momentum four-vector of the particle, while the covariant energy of a particle is the projection of the timelike Killing vector on the four momentum, $\xi^a p_a$. Both approaches give the same relationship between energy and momentum, which for a static diagonal metric is

$$g^{00} E^2 = \sum_i g^{ii} (p_i)^2$$  

(3.1.3)

where we are taking a massless particle for simplicity.

Remark on notation: for simplicity of notation in this section, $g_{00}$ refers to the positive value of the time-time component of the metric, except where explicitly stated otherwise. The minus sign appears in the form of the equation.
3.2 Dependence of N on time-time component of metric

As a warm up, we show heuristically that for constant volume and energy (for an observer at infinity) the number of states $N$ depends on the time-time component of the metric. For 3+1 dimensions we show that

$$N \sim V_\perp E^3 \int dr \sqrt{g_{rr}} (g^{00})^{3/2} \quad (3.2.1)$$

where $V_\perp$ refers to volume transverse to the radial coordinate. Here $E$ refers to energy perceived by an observer at infinity, while $\sqrt{g^{00}E}$ is just the energy of a local observer. Thus $N$ is found to be observer dependent.

Proof of this is as follows: In general $N \sim \int d^3x d^3p \equiv V_x V_p$, the product of the volume of configuration and momentum space for a given energy. We may write this as $|X|^3 |P|^3$ where we take space as a box with sides of equal length $|X|$, and analogously for momentum with length $|P|$. (This is just handwaving; a precise treatment will be given later.)

Now

$$V_x = \int \sqrt{g_3} d^3x = |X|^3 \quad (3.2.2)$$

where $g_3$ is the determinant of the spatial metric, and similarly

$$V_p = \int \sqrt{1/g_3} d^3p = |P|^3 \quad (3.2.3)$$

since the metric in a cotangent space is given by the inverse matrix of that for configuration space.

To find $|P|^3$ we use the wave equation: $\Box \phi = 0$. Then

$$g^{00} E^2 - g^{xx} p_x^2 - g^{yy} p_y^2 - g^{zz} p_z^2 = 0 \quad (3.2.4)$$

and transferring the momenta to the RHS and taking the root of both sides

$$\sqrt{g_{00} E^2} = \sqrt{g^{xx} p_x^2 + g^{yy} p_y^2 + g^{zz} p_z^2} = |P| \quad (3.2.5)$$
so that

\[ (\sqrt{g_{00}}E^2)^3 = |P|^3 = V_p. \] (3.2.6)

Since we are interested in black holes, we assume that \( g_{00} \) depends on one space coordinate \( x \). This has no impact on the argument up till now. In that case

\[ N \sim V_x V_p = \int \sqrt{g_{ii}} d^3x \int \sqrt{g^{ii}} d^3p = \int \sqrt{g_{ii}} d^3x \left( \sqrt{g_{00}}E^2 \right)^3 \]

\[ = V_\perp E^3 \int \sqrt{g_{xx}} dx \left( \sqrt{g_{00}} \right)^3. \] (3.2.7)

This shows that the volume of phase space in this simple case is determined by the energy as perceived by the observer in the specific metric background under discussion.

### 3.3 Calculation of \( N \)

#### 3.3.1 Calculation by the definition:

The number of states given by the volume of phase space is the product of the volume of position and momentum space. The momentum component of the number of states belongs to a constrained region in the cotangent space of the region of configuration space in question, obtained from the wave equation. For example, in Cartesian coordinates in flat space a Fourier transform of the wave equation gives

\[ E^2 - p_x^2 - p_y^2 - p_z^2 = 0 \] (3.3.1)

and this defines a sphere of radius \( E \):

\[ 1 = \frac{p_x^2}{E^2} + \frac{p_y^2}{E^2} + \frac{p_z^2}{E^2}. \] (3.3.2)

In statistical physics we take all energies up to a given energy, and so we look for the volume enclosed by this sphere, \( \frac{4}{3} \pi E^3 \). If the metric is not flat, the volume will be an ellipsoid.
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The wave equation for a general diagonal metric is

\[ g^{00} E^2 = \sum_i g^{ii} (p_i)^2 \]

\[ 1 = \sum_i \frac{g^{ii} (p_i)^2}{g^{00} E^2} = \sum_i \frac{p_i^2}{g^{ii} g^{00} E^2} \]

(3.3.3)

where \( p_i \) the spatial momenta are summed in all space directions. This is the formula for volume of an ellipsoid with axes \( \sqrt{g_{ii} g^{00} E} \), which encloses a region whose volume in three space dimensions would be \( \frac{4}{3} \pi \sqrt{g_{xx} g_{yy} g_{zz}} \left( \sqrt{g^{00} E} \right)^3 \). In \( d + 1 \) spacetime dimensions this becomes

\[ C_d \sqrt{g_d} \left( \sqrt{g^{00} E} \right)^d \]

(3.3.4)

where \( g_d \) denotes the determinant of the spatial part of the metric and \( C_d \) is the volume enclosed by the \( d \)-dimensional unit ball. One then integrates over all momentum space. Since the measure in the momentum integral includes the root of inverse metric \( g^d \), that is,

\[ \int \frac{d^d p}{\sqrt{g_d}} \]

(3.3.5)

then the space determinant in eq.(3.3.4) cancels out, and the integral over momentum space gives the volume of a sphere with radius \( \sqrt{g^{00} E} \).

Therefore the number of states of a test particle in \( d + 1 \) dimensions (\( d \) space dimensions) for a diagonal metric in a given volume \( V \) is

\[ N(E, V, d, 1) = C E^d \int_V d^d x \sqrt{g_d} \left( g^{00} \right)^{\frac{d}{2}} \]

\[ C = \frac{\pi^{\frac{d}{2}}}{\Gamma \left( \frac{d}{2} + 1 \right)} \].

(3.3.6)

and given \( n \) identical particles \( N(E, V, d, n) = (N(E, V, d, 1))^n \). An explicit proof for \( 3 + 1 \) and \( 4 + 1 \) dimensions appears in Sec.8.3.1. For explicit calculations of this quantity for a gas of particles in various spacetimes please see [36].
3.3.2 Calculation with the WKB approximation:

Another method of calculating the number of states is with the WKB approximation, which for a free particle in one dimension is exact. WKB quantization gives the number of modes in one dimension \( n \) as\(^{2}\)

\[
    n = \frac{1}{\pi \hbar} \int dx p(x). \tag{3.3.7}
\]

In flat space the WKB term can be extended to \( d \) dimensions thus:

\[
    n_i = \frac{1}{\pi \hbar} \int dx_i p(x)
    \quad \text{and}
    \quad n \equiv \left[ \sum_{i=1}^{d} n_i^2 \right]^{1/2} \tag{3.3.8}
\]

and one then continues the calculation by relating \( n \) to the energy (see Sec.8.2). In curved space problems arise.

In a spherical coordinate system relating \( n \) to the energy is problematic. For a particle within sphere of radius \( R \), \( k = \beta_{n,l} / R \) where \( \beta_{n,l} \) are the zeros of the Bessel function: \( j_l(kR) = 0 \). But unlike the Cartesian case one cannot extract \( E(n) \) in neat analytical form, because the \( \beta_{n,l} \) must be calculated numerically, and each \( l \) has its own \( \beta_{n,l} \).\(^{3}\) Instead one follows ’t Hooft and obtains the radial modes from the wave equation. Essentially this means incorporating energy of transverse degrees of freedom into a radial potential. One then sums the radial modes over angular degrees of freedom.

3.4 Invariance of number of states under transformation of metrics

We wish to examine a general transformation which changes the metric while leaving the number of states invariant. We find that such a transformation exists, but does not preserve curvature. We give details of the transformation, followed by examples of the relation to curvature.

\(^{2}\)Note that for periodic BCs \( k = 2n\pi/L \) whereas WKB used for a potential well has \( k = n\pi/L \).

\(^{3}\)For example see \([11]\).
We begin with conformal rescaling. If a $3$-dimensional metric changes by $	ilde{g} = a(x)Ig$, then the number of states is
\[ N_0 = \int \sqrt{g_0} d^3x d^3p = \int \sqrt{g_3} \frac{4\pi E^3}{3} (g^{00})^{3/2} d^3x. \]
\[ \tilde{N} = \int \sqrt{\tilde{g}} d^3x d^3p \]
\[ = \int a^{3/2} \sqrt{g} \frac{4\pi E^3}{3} \left( \frac{1}{a} g^{00} \right)^{3/2} d^3x \]
\[ = \int \sqrt{g} \frac{4\pi E^3}{3} (g^{00})^{3/2} d^3x = N \] (3.4.1)
since $\tilde{g}_{00} = a(x)g_{00}$ and so $\tilde{g}^{00} = \frac{1}{a(x)} g^{00}$. This only works if the metric is uniformly rescaled, so that $a_0 = a_i$. Thus conformal rescaling preserves the number of states.

We conclude that preservation of the number of states requires a constraint on the relationship between the time and space components of the metric.

In search of a more general transformation we take a general diagonal metric in $1 + 3$ dimensions. Generalization to more space dimensions will be simple.

\[
\begin{pmatrix}
  g_{00} & & \\
  & g_{xx} & \\
  & & g_{yy} \\
  & & & g_{zz}
\end{pmatrix}
\] (3.4.3)

The volume of space in this metric:
\[ \int_V \sqrt{g_{xx}g_{yy}g_{zz}} d^3x \] (3.4.4)

where the integral is over a given volume $V$. The volume of momentum space is
\[ \int_{V_p} \frac{d^3p}{\sqrt{g_{xx}g_{yy}g_{zz}}} \] (3.4.5)

where $V_p$ is the volume in momentum space. As explained above, from the wave equation
\[ 1 = \frac{1}{g_{xx}g^{00}E^2} p_x^2 + \frac{1}{g_{yy}g^{00}E^2} p_y^2 + \frac{1}{g_{zz}g^{00}E^2} p_z^2 \] (3.4.6)

which is the equation for volume of ellipsoid with axes $\sqrt{g_{xx}g^{00}E}$, $\sqrt{g_{yy}g^{00}E}$, $\sqrt{g_{zz}g^{00}E}$. 
The momentum volume is obtained by integration, or more simply by just plugging in the formula for volume of ellipsoid in 3 dimensions: 
\[ \frac{4}{3} \pi abc = \frac{4}{3} \pi \sqrt{g_{xx}g_{yy}g_{zz}} (g^{00}E^2)^{3/2} . \]

Phase space is given as
\[ N = \int_{v} d^3x \int_{v_p} d^3p. \]  
(3.4.7)

We now transform the metric in arbitrary way but keeping it diagonal:
\[
\begin{pmatrix}
  a(\vec{x})g_{00} \\
  b(\vec{x})g_{xx} \\
  c(\vec{x})g_{yy} \\
  d(\vec{x})g_{zz}
\end{pmatrix}
\]
(3.4.8)

We plug this into the term for phase space. First we calculate the volume of momentum space for the transformed metric. The wave equation is now
\[
\frac{1}{a(\vec{x})}g^{00}E^2 = \frac{1}{b(\vec{x})}g^{xx}p_x^2 + \frac{1}{c(\vec{x})}g^{yy}p_y^2 + \frac{1}{d(\vec{x})}g^{zz}p_z^2
\]  
(3.4.9)

and using eq. (3.4.6)
\[
1 = \frac{a(\vec{x})}{b(\vec{x})}g_{xx}g^{00}E^2p_x^2 + \frac{a(\vec{x})}{c(\vec{x})}g_{yy}g^{00}E^2p_y^2 + \frac{a(\vec{x})}{d(\vec{x})}g_{zz}g^{00}E^2p_z^2
\]  
(3.4.10)

so that the volume becomes
\[
V_p = \frac{4}{3} \pi \sqrt{b(\vec{x})c(\vec{x})d(\vec{x})g_{xx}g_{yy}g_{zz}} \left( \frac{g^{00}}{a(\vec{x})} E^2 \right)^{3/2} .
\]  
(3.4.11)

This will equal the volume before the transformation if
\[
b(\vec{x})c(\vec{x})d(\vec{x}) = a(\vec{x})^3.
\]  
(3.4.12)

Thus we have identified the constraint for an arbitrary transformation to preserve the volume of phase space.

\footnote{This can have a prefactor of \((2\pi)^{-3}\) when calculating the density of modes per unit of phase space.}
3.4.1 Characteristics of the transformation

We looked for some kind of general algebraic characterization for this kind of matrix, but found none. It belongs to \( GL(n, R) \) but does not represent a particular symmetry. Conformal transformations are a subgroup of our transformation, as shown in Sec[3.4.3]. Certain non conformal transformations also preserve the number of states. This holds if the determinants cancel out: That is, for \( d \) space dimensions, the time part \( a(x) \) when raised to the \( d^{th} \) power, has to equal the determinant of the space part. Take

\[
A = \begin{pmatrix}
    a(x) & 0 & 0 & 0 \\
    0 & a(x)^2 & 0 & 0 \\
    0 & 0 & a(x) & 0 \\
    0 & 0 & 0 & 1
\end{pmatrix}
\]  

(3.4.13)

As with the conformal transformation, we still have \( \sqrt{\tilde{g}_3} = a^{3/2} \sqrt{g_3} \) and and so \( \tilde{N} = N_0 \).

So in general our constraint is:

\[
(g_{00})^d = \det g_{space}
\]  

(3.4.14)

where \( d \) is the number of space dimensions and \( g_{space} \) is the determinant of the spatial part of the metric.

We can regard the transformation matrix, labeled \( A \), as two blocks, separating the time and space components:

\[
A = \begin{pmatrix}
    T \\
    S
\end{pmatrix}
\]  

(3.4.15)

where \( T \) is a 1x1 matrix, and \( S \) is a diagonal matric of rank \( d \) where \( d \) is the dimension of space (rank of \( A \) is the dimension of space-time, \( d + 1 \)). Then the constraint requires

\[
\det(S) = \det(T)^d \\
\det(A) = \det(T)^{2d}.
\]  

(3.4.16)
3.4.2 Relation to curvature

In $d + 1$ dimensions

$$N \sim \int d^3 x \sqrt{\frac{g_d}{(g_{00})^d}}$$  \hspace{1cm} (3.4.17)

where $g_d$ denotes the determinant of the space part of metric. To preserve the number of states we have to preserve the ratio $g_d / g_{00}^d$, which entails the constraint on the determinant as detailed above. The question becomes: given a change of metric for which this constraint holds, will such a constraint ensure preservation of scalar curvature? If so preservation of the number of states would entail preservation of curvature, which is an observer independent characteristic.

We take two matrices representing two possible transformations of a given 3-dimensional metric:

$$A = \frac{1}{L} \begin{pmatrix} x \\ x \\ x \end{pmatrix}$$

$$B = \frac{\sqrt{2} x}{L} \begin{pmatrix} 2 \\ \frac{x^2}{L^2} \end{pmatrix}$$  \hspace{1cm} (3.4.18)

where $L$ is a constant with dimension of length. Both transformation matrices preserve the constraint given above, while their intrinsic curvature differs: the first has $R = \frac{3L^2}{2x^3}$, the second has $R = \frac{L}{\sqrt{2}x^3}$. This is because the second one has fewer nonzero components of the Christoffel connection, since the derivative must be $\partial_x$, and $\partial_x g_{yy} = 0$. Therefore clearly imposing the constraint on a metric transformation will not necessarily preserve the curvature of the original metric.

Curvature in these examples is affected by the number of terms with an $x$ derivative, while the determinant is not. Thus the constraint on the determinant does NOT preserve curvature. This is intuitively understandable: the determinant indicates volume but gives no information as to the spatial distribution of the volume.

\[5\] This reason is purely formal, in terms of the coordinates chosen here; $R$ is not dependent on coordinate system of course. Just for that reason, a different choice of coordinates would also involve different scalar curvature for the two cases.
3.4.3 Examples in various dimensions

In $1 + 1$ dimensions a transformation that preserves $N$ must be conformal: $g_{00} = g_{xx}$ since $\det g_{\text{space}} = g_{xx}$. In $2+1$ dimensions we give two examples of transformations that preserve $N$. One is conformal, the other non conformal but symmetric:

We give the matrices for the transformations, and the scalar curvature when they are applied to a flat Lorentzian metric:

Conformal:

$$A = \frac{1}{L} \begin{pmatrix} -x & x \\ x & x \end{pmatrix}, \quad R = -\frac{3L}{2x^3} \quad (3.4.19)$$

Symmetric:

$$B = \frac{1}{L} \begin{pmatrix} -\sqrt{xy} & x \\ x & y \end{pmatrix}, \quad R = L \left( \frac{-6y^2 + 5x\sqrt{xy}}{8[xy]^{5/2}} \right) \quad (3.4.20)$$

An asymmetric example is like the one given in the previous section for a Euclidean metric.

Note that plugging in the value $x = y$ after deriving $R$ for matrix $B$ does not give the curvature of matrix $A$. This is because the derivation of $R$ takes into account the direction of each component as well as its numerical value. If one plugs in $y = x$ before deriving $R$ all the derivatives $\partial_y$ vanish, giving the different result.

We next look at $3+1$ dimensions. The constraint requires $\left| (g_{00})^3 \right| = \det g_3$. Comparing several matrices that obey this constraint and inspecting their curva-
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The transformation:

\[
A = \begin{pmatrix}
-x/L^3 & \frac{x^3}{L^2} & 1 \\
\frac{x^3}{L^2} & 1 & 1 \\
1 & 1 & 1
\end{pmatrix},
\quad R = \frac{2L^3}{x^5}
\]

\[
B = \frac{1}{L} \begin{pmatrix}
-(xyz)^{\frac{1}{3}} & x & y \\
x & y & z \\
z & z & z
\end{pmatrix},
\quad R = \frac{4L}{9} \left( \frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{z^3} \right)
\]

\[
C = \frac{1}{L} \begin{pmatrix}
-x & x & x \\
x & x & x \\
x & x & x
\end{pmatrix},
\quad R = \frac{3L}{2x^3}
\]  

(3.4.21)

A few comments: 1) The curvature for the third transformation is the same as for the conformal matrix in 1+2 dimensions. 2) As before, setting \( x = y = z \) after calculating the curvature for matrix \( B \) does not give the same result as the curvature for matrix \( C \). Again, this is because the direction of the variable contributes in calculating \( R \), and not just its numeric value. This sheds light on the fact that the number of states, which is proportional to the volume of phase space, is different from curvature, which incorporates information on the distribution of that volume. A constraint on the determinant, representing Euclidean volume, is not the same as that on Ricci curvature, which in fact represents the amount by which the volume of a geodesic ball in a curved Riemannian manifold deviates from that of the standard ball in Euclidean space.

3.4.3.1 Rindler vs Schwarzschild:

The transformation from Minkowski metric to Rindler metric is not diagonal. Rindler coordinates mix time and space coordinates of the Minkowski metric, and that is why \( N \) in the Rindler calculation is different from flat space. We cannot conclude from this that curvature is irrelevant to statistical entropy. That conclusion can only be drawn from the general proof given above.

The Schwarzschild calculation gives divergence at the boundary and we found that the number of states (and thus the entropy) is different from that of Minkowski space as shown in the previous chapter. This is not the same as the difference between the number of states in Rindler and Minkowski spaces. The basic difference
is that Schwarzschild and Minkowski spaces have different geometry, whereas the transformation to Rindler space is a coordinate change from Minkowski space. In terms of the calculation given above, in the Schwarzschild case the argument here given does apply, since the transformation metric from Minkowski to Schwarzschild metric is diagonal. The Schwarzschild number of states differs from that of Minkowski because of the redshift on energy: $g_{00}(r)$.

### 3.4.4 Discussion

Our transformation leaves $N$ invariant because it preserves the relationship between the volume of momentum space and of position space. $(g^{00})^{3/2}$ is the variable part of momentum space, and $\sqrt{g_d}$ is the variable part of position space. $N$ is invariant so long as the relation between the two is preserved, so that if position space shrinks, momentum space grows and vice versa: $a(x)^d$ multiplying $\sqrt{g_{\text{space}}}$ equals $1/a(x)^d$ multiplying momentum space.

It would seem that a proof of curvature independence must show that there are no cases at all where curvature is preserved under a transformation that preserved the number of states. In fact it is quite possible that in some case curvature might be preserved. We claim that this must be seen as a coincidence because the constraint on preservation of the number of states relates to the determinant. By definition, there is a difference between the determinant, which represents Euclidean volume and does not depend on directions in space, and curvature which does depend on directions in space. The number of states does not depend on directions in space and so it can be preserved even if the directional characteristics and thus the curvature are changed. There will be a subgroup where transformation of the number of states will indeed preserve curvature. But one cannot assume that any given number of states, and the entropy derived from it, relate in a unique manner to a spacetime with a given curvature. Curvature may affect the number of states but there is no one to one correspondence between the two.

We examined the question whether in curved space the number of states, and the statistical entropy derived from this, is observer dependent or is related to a physical quantity such as curvature. We found that - as with any measurement of length - the result just depends on your ruler. Finding $N$ is actually just a generalization of finding a length: it involves finding a volume in space and momentum space. Obviously people with different rulers will give you a different
answer for the length. The statistical number of states is essentially like a scaled ruler and thus is observer dependent and not related to the intrinsic geometry. This reflects the scheme in the previous chapter, in that boundary conditions are imposed on the observer rather than on the state. This differs from the work of Wald and Carlip.

In addition, in [40] it was shown for explicit examples that entanglement entropy does not depend on curvature. For a discretized region in curved space it was found that even when the space is large enough for the effects of curvature to be noticeable, entropy remains proportional to area and is not affected by the curvature of the background. This qualitative similarity to our result reinforces the idea that entanglement and statistical entropy are one and the same thing, and that they possibly differ from Noether charge entropy.

The results in this chapter apply to a diagonal metric only. A general metric could only be diagonalized locally, and that would not be relevant to a discussion of curvature. However since a diagonal metric is seen to be curvature independent, and there is certainly at least one diagonal metric, this serves as a counter example to the claim that the number of states is uniquely dependent on curvature.

In conclusion, we have shown that the number of states is a function of the metric and is preserved under specific transformations of the metric, which do not preserve curvature. Therefore the number of states calculated with the accepted definition of phase space does not depend uniquely on curvature. In this it appears to differ from Wald entropy. However, as discussed in Sec. 2.5 for general theories of gravity a wider definition of phase space may be necessary.
Chapter 4

Entropy variation

4.1 Introduction

In this chapter we write the variation of the statistical entropy of matter fields outside a black hole along a Killing vector, and we compare this to the variation of Wald’s entropy. The calculation applies to any generalized theory of gravity. If the variations are equivalent, this would imply a natural connection between the two entropies, where one is derived from the presence of matter in the region of a black hole and the other derived exclusively from spacetime geometry. Such a connection would relate statistical mechanics to the gravity field equations, with profound implications for the idea of gravity as an emergent phenomenon. We find that under certain conditions for a stationary black hole the two entropy variations do in fact coincide.

Wald [19, 20] has studied black holes in generalized theories of gravity and proposed that the correct dynamical entropy of stationary black hole solutions with bifurcate Killing horizons is a Noether charge entropy. The microphysical understanding of Wald’s entropy is unknown.

Another approach to the calculation of black hole entropy is that of statistical mechanics, as discussed above. Since statistical entropy is calculated using the partition function of the fields outside the horizon, its microphysical origin is understood. However, as statistical entropy is known to depend on the number of fields in the theory, whereas Wald’s entropy does not, it seems that the two entropies may be essentially different.

We note that Wald’s calculation was for general (higher derivative) theories
of gravity. Therefore we attempt to relate these two entropies for generalized
theories of gravity. A connection between the two entropies could indicate a
relationship between the entropy of microstates within the black hole and that
of matter outside. This would be similar to the concept of black hole entropy as
entanglement entropy between the states inside and outside the black hole.

The variation of Wald’s entropy was calculated in [26] (see also [27]). In [26]
the authors differentiate Wald’s entropy along a Killing vector to obtain $\delta S$. They
use the generalized gravitational field equations to derive a term for the energy
flow across the black hole horizon, and find that the first law $\delta Q = T\delta S$ is fulfilled.
Thus the entropy variation along a Killing vector is:

$$\delta S_{Wald} = 2\pi \int T_{ab}\chi^a\epsilon^b$$

(4.1.1)

where $\chi^a$ is the Killing field and $\epsilon^b$ is a $(D - 1)$ volume form. (Details of the
derivation are in Section 9.1). We here obtain a similar relationship for the varia-
tion of statistical entropy and $T_{ab}$, where our variation will be along an arbitrary
vector, which on the horizon we will take to be a Killing vector. Thus we will compare the variation of the two different entropies along a Killing vector on the
black hole horizon.

Note: This chapter, unlike the rest of this thesis, is written using differential
forms because it follows the variation of Wald entropy [26] which was calculated
in that language.

4.2 Preliminaries

For the following treatment we will need the partition function. In the path
integral approach the amplitude to go from a field configuration $\phi_1$ at a time $t_1$
to a field configuration $\phi_2$ at time $t_2$ is given

$$\langle \phi_2, t_2 | \phi_1, t_1 \rangle = \int D\phi e^{i[I[\phi]]}$$

(4.2.1)

where the path integral is over all field configurations which take the values $\phi_1$ at
time $t_1$ and $\phi_2$ at time $t_2$. However if $H$ is the Hamiltonian,

$$\langle \phi_2, t_2 | \phi_1, t_1 \rangle = \langle \phi_2 | e^{-iH(t_2-t_1)} | \phi_1 \rangle .$$

(4.2.2)
One defines the time period \( t_2 - t_1 = -i\tau \), and in going to Euclidean time one takes the time as periodic with period \( \beta \), and one identifies the fields \( \phi_1 = \phi_2 \), sums over all \( \phi_1 \) and thus obtains the partition function for the canonical ensemble of field \( \phi \) at inverse temperature \( \beta \),

\[
Z = Tr(e^{-\beta H}) = \int D\phi e^{-I_E[\phi]} \tag{4.2.3}
\]

and \( I_E[\phi] \) is the matter action as defined above, but now with Euclidean time. (see for instance \[45\], Chap.9).

### 4.3 Variation of statistical entropy

The entropy is given by

\[
S_{st} = \beta \langle E \rangle + \ln Z. \tag{4.3.1}
\]

since, as discussed in Chapter 2, the region near a black hole is a thermal state.

We now perform a variation of the entropy. Since we are interested in comparing to Wald entropy, we look for the effect on the statistical entropy of a change in the geometry. A natural way to vary the geometry is to push the metric forward with the flow of a vector field \( \xi \). The corresponding variation of the metric is given by its Lie derivative along \( \xi \), therefore it is this derivative that will appear in the entropy variation below.

Since it has been shown \[26\] \[27\] that the variation of Wald’s entropy is related to the energy momentum tensor, we would like to relate the variation of the statistical entropy to \( T_{ab} \) as well. We label by \( \delta_g S_{st} \) the variation of the entropy we have just described. From eq.(4.3.1) we have

\[
\delta_g S_{st} = \delta_g \langle \beta \langle E \rangle \rangle + \delta_g \ln Z. \tag{4.3.2}
\]

We want to examine the effect of a change only in the metric on the expression for the entropy. To do this, we take the variation as described above, where the metric is varied by taking its Lie derivative along a vector field \( \xi \), as follows.
We begin with the second term on the RHS.

\begin{equation}
\delta g \ln Z = \int \frac{\delta \ln Z}{\delta g^{ab}} \mathcal{L}_{\xi} g^{ab} = \frac{1}{Z} \int \frac{\delta Z}{\delta g^{ab}} \mathcal{L}_{\xi} g^{ab} = \frac{1}{Z} \int \frac{\epsilon}{2} \langle T_{ab} \rangle \mathcal{L}_{\xi} g^{ab} \tag{4.3.5}
\end{equation}

where \( \epsilon \) is a D-dimensional volume form, and the last line was obtained as in \([43]\), Sec.6.1. Note that \( \langle T_{ab} \rangle \) is not normalized.

We express the Lie derivative of the metric in terms of \( \xi^a \):

\[ \mathcal{L}_{\xi} g^{ab} = \nabla^a \xi^b + \nabla^b \xi^a \]

and the variation of the action becomes:

\begin{equation}
\delta g \ln Z = \frac{1}{Z} \int_{V} \frac{\epsilon}{2} \left( \nabla^a \xi^b + \nabla^b \xi^a \right) \left( \langle T_{ab} \rangle \right)
= \frac{1}{Z} \int_{V} \epsilon \langle T_{ab} \rangle \nabla^a \xi^b
= \frac{1}{Z} \left[ \int_{V} \epsilon \nabla^a \left( \langle T_{ab} \rangle \xi^b \right) - \int_{V} \epsilon \nabla^a \langle T_{ab} \rangle \xi^b \right] \tag{4.3.6}
\end{equation}

where we have used symmetry of \( T_{ab} \) and integration by parts. Since \( \nabla^a \langle T_{ab} \rangle = 0 \) we obtain

\begin{equation}
\delta g \ln Z = \frac{1}{Z} \int_{V} \epsilon \nabla^a \left( \langle T_{ab} \rangle \xi^b \right) \tag{4.3.7}
\end{equation}

From the generalized Stokes law we have \( \int_{V} \left( \nabla^c \omega_c \right) \epsilon = \int_{H} \omega^c \epsilon_c \) and using \( \omega_c = \langle T_{ab} \rangle \xi^b \) we have

\begin{equation}
\delta g \ln Z = \frac{1}{Z} \int_{H} \langle T_{ab} \rangle \xi^a \epsilon^b \tag{4.3.8}
\end{equation}

where \( \epsilon^b \) is a (D-1) volume form and \( H \) denotes the boundary of \( V \). The volume is bounded by the black hole horizon and at infinity. We assume space is asymptotically flat and so \( T_{ab} \) vanishes at infinity, and the contribution in eq.(4.3.8) comes only from the black hole horizon.[2]
Eq.(4.3.8) gives part of the variation of the matter action of the fields found in the volume outside the black hole when the metric changes along some arbitrary vector field. We now specialize to the horizon. At the horizon we require $\xi \to \chi$ to be a Killing vector which fulfills $\chi^a \nabla_a \chi^b = \chi^b$. Since $\chi$ vanishes on the horizon, we take a region just outside the horizon and take the limit to the horizon, as Carlip does (Chap.2, and see [25, 26, 27]). The second term in the variation of the matter action along the Killing vector is thus

$$\delta g \ln Z = \frac{1}{Z} \int_H \langle T_{ab} \rangle \chi^a \epsilon^b. \quad (4.3.9)$$

If we had taken $\xi$ to be a Killing vector throughout the volume in eq.(4.3.3), eq.(4.3.9) would vanish as $L_{\xi} g_{ab}$ would vanish. However eq.(4.3.3) is an integral over volume, and $\xi$ is a Killing vector only on the boundary of this volume, not in the bulk. So eq.(4.3.3) does not vanish, and from Stokes’ law eq.(4.3.9) does not vanish.\(^3\)

We now turn to the first term in eq.(4.3.2) $\langle E \rangle = -\partial \ln Z / \partial \beta$. In this case $\beta$ is constant, as the period of Euclidean time (and as is the temperature for a stationary black hole) and will not itself be varied. Thus using eq.(4.3.9) we find

$$\delta_g (\beta \langle E \rangle) = -\beta \delta_g \frac{\partial}{\partial \beta} \ln Z = -\beta \frac{\partial}{\partial \beta} \delta g \ln Z$$

$$= -\beta \frac{\partial}{\partial \beta} \left( \frac{1}{Z} \int_H \langle T_{ab} \rangle \chi^a \epsilon^b \right). \quad (4.3.10)$$

Eqs.(4.3.9),\(^4\)(4.3.10) give

$$\delta_g S_{st} = -\beta \frac{\partial}{\partial \beta} \left( \frac{1}{Z} \int_H \langle T_{ab} \rangle \chi^a \epsilon^b \right) + \frac{1}{Z} \int_H \langle T_{ab} \rangle \chi^a \epsilon^b. \quad (4.3.11)$$

First we take $\langle T_{ab} \rangle$ as well as $Z$ as independent of Euclidean time, in this case the first term on the RHS vanishes. We define $\epsilon^b = k^b d\tau dA$ where $k^b$ is the tangent to the vectors generating the horizon for parameter $\tau$, and $dA$ is the area element

\(^3\)For example, take the following integral over a volume with radius $R$: $\int (r - 1) dV$. The integrand vanishes at the surface, but clearly the integral over the volume does not vanish, and so the surface integral obtained by Stokes law will not vanish either.
of a cross section of the horizon. Then

\[ \delta_g S_{st} = \frac{1}{Z} \int_H \langle T_{ab} \rangle \chi^a \epsilon^b \]

\[ = \frac{1}{Z} \int_{\tau_2}^{\tau_1} d\tau \int_A \langle T_{ab} \rangle \chi^a k^b dA. \tag{4.3.12} \]

Integration over Euclidean time gives

\[ \delta_g S_{st} = \frac{\beta}{Z} \int_A \langle T_{ab} \rangle \chi^a k^b dA \tag{4.3.13} \]

We normalize \( \langle T_{ab} \rangle \) with the partition function, obtaining

\[ \delta_g S_{st} = \beta \int_A \langle \tilde{T}_{ab} \rangle \chi^a k^b dA \tag{4.3.14} \]

where the tilde represents normalization, and taking \( \beta = 2\pi \) this is equal to the variation of Wald entropy. If we do not integrate over the entire time period but rather take a finite interval we obtain a term proportional to Wald entropy. In either case it appears that for time independent \( \langle \tilde{T}_{ab} \rangle \) the variation of Wald’s entropy is equivalent up to a constant to a purely geometric variation of statistical entropy. The factor of \( 2\pi \) in the variation of Wald’s entropy comes from the original term for Wald entropy, eq.1.2.13 and not from the variation, and represents the temperature in the first law of thermodynamics, rather than being inherent to the derivation of the Noether charge for diffeomorphism invariance.

If \( \langle \tilde{T}_{ab} \rangle \) is not independent of time the first term of eq.(4.3.11) does not vanish and becomes

\[ \delta_g (\beta \langle E \rangle) = -\beta \frac{\partial}{\partial \beta} \left( \frac{1}{Z} \int_H \langle T_{ab} \rangle \chi^a \epsilon^b \right). \tag{4.3.15} \]

where we have abandoned normalization in order to explore the role of the partition function. The next question is whether to take \( Z \) as a function of \( \beta \). If not, we obtain

\[ \delta_g (\beta \langle E \rangle) = -\frac{\beta}{Z} \int_H \frac{\partial}{\partial \beta} \langle T_{ab} \rangle \chi^a \epsilon^b \tag{4.3.16} \]
for which one needs to know the precise dependence of the stress energy tensor on the Euclidean time period. If $Z$ is a function of $\beta$ we obtain two terms,

$$
\delta_g (\beta \langle E \rangle) = -\frac{\beta}{Z} \int_H \frac{\partial}{\partial \beta} \langle T_{ab} \rangle \chi^a \epsilon^b + \frac{\beta}{Z^2} \int H D\phi(-L)e^{-\mathcal{E}} \int_H \langle T_{ab} \rangle \chi^a \epsilon^b
$$

(4.3.17)

This may also be written

$$
\delta_g (\beta \langle E \rangle) = -\beta \int_H \frac{\partial}{\partial \beta} \langle T_{ab} \rangle \chi^a \epsilon^b - \beta \frac{\langle L \rangle}{Z} \int_H \langle T_{ab} \rangle \chi^a \epsilon^b
$$

(4.3.18)

The expectation value of the Lagrangian is not normalized just as $\langle T_{ab} \rangle$ is not (see [43] Chap. 6). Normalization gives

$$
\delta_g (\beta \langle E \rangle) = -\beta \int_H \frac{\partial}{\partial \beta} \langle \tilde{T}_{ab} \rangle \chi^a \epsilon^b - \beta \langle \tilde{L} \rangle \int_H \langle \tilde{T}_{ab} \rangle \chi^a \epsilon^b.
$$

(4.3.19)

### 4.4 Summary

We have three possibilities for this variation: one is just the same as the variation of Wald entropy, resulting from a Lie derivative of the metric, and the other two contain additional terms which derive from time dependence of $T_{ab}$ and $Z$ for the matter fields in question. Thus the existence of matter fields are responsible for the difference between even a purely geometric variation as above, and the variation of Wald entropy.

The first possibility is this: If the expectation value of the stress energy tensor of the matter fields is time independent then we obtain a term proportional to the variation of Wald’s entropy. Integration over all Euclidean time gives the exact same term as the variation of Wald entropy. We are treating a stationary black hole, so that it seems reasonable to assume time independence. (We recall that $\beta$ appears as inverse temperature in the partition function, but is also the period of Euclidean time.)

If the expectation value of the stress energy tensor is time dependent, the question then arises whether $Z$ is as well. If not, then the entropy variation is

$$
\delta_g S = \beta \int_{\partial V} \langle \tilde{T}_{ab} \rangle \chi^a \epsilon^b dA - \beta \int_H \frac{\partial}{\partial \beta} \langle \tilde{T}_{ab} \rangle \chi^a \epsilon^b
$$
which includes one term which is equal to the variation of Wald entropy, and a second which is not. If $Z$ is taken as dependent on the time period,

$$\delta g S = \beta \int_A \langle \tilde{T}_{ab} \rangle \chi^a k^b dA - \beta \int_H \frac{\partial}{\partial \beta} \langle \tilde{T}_{ab} \rangle \chi^a \epsilon^b - \beta \frac{Z^2}{2} \langle \tilde{L} \rangle \int_H \langle \tilde{T}_{ab} \rangle \chi^a \epsilon^b.$$

In conclusion, if one assumes a stationary black hole with time independent $\langle T_{ab} \rangle$ and $Z$, one obtains a variation according to a change in the geometry which is just that of the variation of Wald’s entropy. It is not surprising that the variation of statistical entropy due to a slight change in the geometry should be similar to the variation of the Noether charge for diffeomorphism invariance. If we were to perform a complete variation of the statistical entropy including variation by the fields, and not just by the geometry, the two terms would differ.

If we do not make the above assumptions, the variation due to a change in the geometry does not coincide with the variation of Wald entropy, but is affected by the nature of the stress energy tensor and the partition function, that is, by the nature of the matter fields in the black hole metric.

Thus we have shown that along a Killing vector at the horizon, the variation of the statistical entropy of the matter fields caused by variation of the matter Lagrangian density due to a change of the metric differs from the variation of Wald’s entropy, which is derived from the diffeomorphism invariance of spacetime. This is true for any generalized theory of gravity. In previous work, it is not clear what Wald’s entropy counts: it could be quanta of Planck length, or Planck area, some geometrical or other quantity. The statistical entropy term in our work is explicitly taken only from the matter Lagrangian. In all the above possibilities one finds that the two variations coincide in the purely geometrical aspect. The variation itself must be only according to a change in the geometry and not in the matter fields. In addition, for the variations to coincide one must ignore dependence of the matter fields and partition function on Euclidean time. This makes it seem that the Wald’s entropy isolates the geometric characteristics of space time at the black hole horizon, which also affect the matter fields in its vicinity, but do not consitute their statistical entropy.
Chapter 5

Conclusions and discussion

In this thesis we discussed several issues which differentiate between different conceptions of black hole entropy: divergence at the horizon, imposition of boundary conditions, observer dependence and behavior under variation.

We saw that divergence at the boundary as in ’t Hooft’s model may result from simple x/p uncertainty, since thermodynamic entropy is linear to energy and energy diverges as \( \Delta x \) vanishes. Therefore statistical entropy may be identified in the case of the black hole with entanglement entropy even though entanglement entropy does not diverge at the boundary, because this divergence may simply be a result of quantum uncertainty. Entanglement entropy expresses correlations between two different parts of a system, while statistical entropy counts degrees of freedom of one of those two parts. Counting correlations is thus the same as counting the number of degrees of freedom in one of the subsystems, since each degree of freedom in one subsystem is correlated with one in the other subsystem.

Another criterion for distinguishing between entropies is the imposition of boundary conditions. Carlip’s treatment of entropy as resulting from symmetry necessitates the imposition of boundary conditions and must be distinguished from statistical/entanglement entropy for this reason. This is also true of Wald entropy, which is derived from surface terms at the boundaries at infinity and at the horizon. In contrast, entanglement entropy traces out one part of a system, that is, looks at only part of a system without imposing boundary conditions. In the example given of a potential well, the probability to find a particle in the left side of the well would be one half, whereas if we imposed boundary conditions, it would be either 1 or 0 because we would have two smaller wells. However we have
not divided the system into two smaller wells, but rather we look only at one half of the entire well.

It is possible that a stretched horizon, like 't Hooft's brick wall, is just a method of delocalizing the horizon. In that case Carlip's central charge would not vanish despite the problem raised by imposing boundary conditions. Further work is needed to test this possibility.

A third characteristic that must be taken into account is curvature independence of the entropy. It has been shown that statistical entropy is not uniquely related to the curvature scalar, although the curvature may be one of a number of factors affecting it. It is possible that the particular method used to explore statistical entropy is not sufficiently general. In particular it holds only for a diagonal metric. However since this method provides a counter example, it is difficult to see how in a more general framework statistical entropy would not be observer dependent. Since statistical entropy is derived from the number of states, whereas Wald's entropy is defined using the curvature tensor, this seems to indicate a difference between these two concepts of entropy. However the issue may not be so simple. Phase space is defined as the product of spatial volume and its canonical conjugate. This definition arose in a context where the canonical conjugate of the variable in the Lagrangian was its time derivative. However the gravitational Lagrangian includes the Ricci scalar, and Ricci tensors as well in the generalized theories of gravity with which Wald dealt. The Lagrangian of a particle in a gravitational background will include at least two terms, the matter Lagrangian and the gravitational term. Each will have a generalized momentum conjugate to the dynamical variable in the Lagrangian. Therefore it may be necessary to redefine statistical entropy to take into account a more general formulation of phase space.

In conclusion, we have shown that the number of states is a function of the metric and is preserved under specific transformations of the metric, which do not preserve curvature. Therefore the number of states calculated with the accepted definition of phase space does not depend on curvature. For general theories of gravity it may be necessary to redefine statistical entropy taking into account a more general concept of phase space.

Variation of statistical entropy resulting from a change in the metric was shown to differ from the variation of Wald's entropy. The two variations coincide only in the purely geometrical aspect. To coincide, the variation of statistical entropy must be only according to a change in the geometry and not a change in the
matter fields. In addition, one must ignore dependence of the matter fields and partition function on Euclidean time. This makes it seem that the Wald’s entropy isolates the geometric characteristics space time at the black hole horizon, which also affect the matter fields in its vicinity, but do not constitute their statistical entropy.

It appears from all the above that statistical entropy and entanglement entropy may refer to the same thing, whereas Wald’s geometric entropy, as well as Carlip’s attempt to obtain entropy from conformal symmetry, refer to a different entity. This does not explain what the entropy actually means, or what degrees of freedom statistical entropy represents. But it does draw a significant distinction between these two basic types of entropy, and that is a necessary first step in understanding what black hole entropy refers to.

Further investigation is necessary to clarify these points. In particular it is necessary to explore higher theories of gravity. The main objective of this thesis was to implement the idea that in trying to understand what the various concepts of black hole entropy refer to, and which of them coincide, one should focus on specific aspects of this entropy. Here I have pointed out several crucial aspects: behavior at the boundary, observer dependence, imposition of boundary conditions, and behavior of a variation of this entropy. This investigation appears to lead to the conclusion that statistical entropy and entanglement entropy may coincide, and differ from the entropy of Wald and Carlip which derives from spacetime symmetry.
Part II

Supplementary Details and Calculations
Chapter 6

Supplement to background

6.1 ’t Hooft’s calculation of statistical entropy

To find the number of modes for a particle in the black hole metric, t’Hooft uses a one dimensional WKB approximation. He takes the contribution to energy of the transverse momenta as an effective radial potential $V_{\text{eff}} = \frac{l(l+1)}{r^2}$ (since it behaves as a centrifugal potential).

The wave equation for a free massive particle in a diagonal metric is

$$(g^{00}E^2 - g^{rr}k^2_r - g^{\theta\theta}k^2_{\theta} - g^{\phi\phi}k^2_{\phi} - m^2) \psi = 0. \quad (6.1.1)$$

The one-dimensional WKB approximation has

$$n\pi = \int_0^R dr \sqrt{g_{rr}} k(r). \quad (6.1.2)$$

From the wave equation we obtain the radial eigenfunction

$$k^2_l(r) = g_{rr} \left( g^{00}E^2 - g^{\theta\theta}k(\theta)^2 - g^{\phi\phi}k(\phi)^2 - m^2 \right) \quad (6.1.3)$$

for each eigenfunction $k_l$. In the Schwarzschild metric this becomes

$$k^2_l(r) = -\frac{1}{1 - \frac{2M}{r}} \left( \frac{1}{1 - \frac{2M}{r}} E^2 - \frac{1}{r^2} (l(l+1)) - m^2 \right). \quad (6.1.4)$$
The number of radial modes is then summed over the angular degrees of freedom,

\[
N\pi = \int_0^R dr \frac{1}{1 - \frac{2M}{r}} \int_0^{E^2 r^2} dl (2l + 1) \sqrt{E^2 - \left(1 - \frac{2M}{r}\right) \left(\frac{l(l + 1)}{r^2} + m^2\right)}
\]  

(6.1.5)

where the upper limit of the second integral is in order to ensure a positive root.

The free energy is obtained from \(N\) as follows:

\[
F = -T \ln Z = \langle E \rangle - TS
\]

\[
S = \beta \langle E \rangle + \ln Z = -\frac{\partial F}{\partial T}
\]

\[
e^{-\beta F} = \sum e^{-\beta E} = \Pi_{n,l,l} \frac{1}{1 - e^{-\beta E}}
\]

(6.1.6)

\[
\beta F = \sum \log(1 - e^{-\beta E}).
\]

(6.1.7)

Now, taking the sum to an integral

\[
\pi \beta F = \pi \int dN \log(1 - e^{-\beta E})
\]

(6.1.8)

and \(\pi N = g(E)\) so integrating by parts

\[
\pi \beta F = -\int_0^{\infty} dE \frac{\beta g(E)}{e^{\beta E} - 1}
\]

(6.1.9)

because \(dN = dg(E), \frac{d}{dE} \log(1 - e^{-\beta E}) = \frac{\beta}{1 - e^{-\beta E}}\), and plugging in \(\pi N = g(E)\) gives

\[
\pi \beta F = -\beta \int_0^{\infty} dE \frac{1}{e^{\beta E} - 1} \int_0^{L} dr \frac{1}{1 - \frac{2M}{r}} \int (2l + 1) dl \sqrt{E^2 - \left(1 - \frac{2M}{r}\right)(m^2 + \frac{l(l + 1)}{r^2})}
\]

(6.1.10)

so

\[
F = -\frac{1}{\pi} \int_0^{\infty} \frac{dE}{1 - e^{-\beta E}} N.
\]

(6.1.11)

Integration gives two terms, one of which is the contribution from the vacuum surrounding the system at large distances and 't Hooft discards it. The second
term is the horizon contribution:

\[ F_{\text{horizon}} = -\frac{2\pi^3}{45h} \left( \frac{2M}{\beta} \right)^4 \]  

which diverges as \( h \to 0 \). Then taking \( U = \frac{\partial}{\partial \beta} (\beta F) \)

\[ S = \beta (U - F) = \frac{8\pi^3}{45h} 2M \frac{3}{\beta} Z \]  

where \( Z \) is the total number of particle types. 't Hooft then adjusts parameters of the model, such that \( S = 4M^2/\lambda \), in accordance with area dependence derived previously. The main point of this derivation is that the entropy diverges on approaching the horizon.

### 6.2 Wald and Carlip

Carlip has worked on obtaining black hole entropy from conformal theory over a period of years, in the ADM formalism as well as Wald’s canonical phase space formalism. This section is taken from [22].

#### 6.2.1 Canonical phase space formalism

Wald [54] based his work on the abstract formalism of Hamiltonian mechanics where states of a system with \( n \) degrees of freedom are represented by points in a \( 2n \) dimensional manifold referred to as phase space, on which a symplectic form is defined. Wald mapped the space of the solutions of the field equations of motion to phase space. Then variation of the Lagrangian is

\[ \delta L = E \cdot \delta \phi + d\Theta \]  

where \( L \) is an \( n \) form Lagrangian and \( \phi \) are fields (taken as scalar in this treatment, but could be otherwise), \( E = 0 \) are the Euler-Lagrange equations, and \( \Theta \) is an \( n-1 \) form given by surface terms. The symplectic current obtained from symplectic
variation of $\Theta$ is

$$\omega [\phi, \delta_1 \phi, \delta_2 \phi] = \delta_1 \Theta [\phi, \delta_2 \phi] - \delta_2 \Theta [\phi, \delta_1 \phi]$$

$$\Omega = \int_C \omega$$  \hspace{1cm} (6.2.2)$$

where $C$ is a Cauchy surface\(^\dagger\).

Given a diffeomorphism generated by vector field $\xi$ we can define a Noether current (n-1 form)

$$J [\xi] = \Theta [\phi, L_\xi \phi] - \xi \cdot L$$

$$dJ = d\Theta - d[\xi \cdot L].$$  \hspace{1cm} (6.2.3)

We have $d\Theta = \delta L - E \cdot \delta \phi$ but $E = 0$ on shell.

Note that $\delta L = L_\xi L$ and that the Lie derivative of a form $\Lambda$ is given by $L_\xi \Lambda = \xi \cdot d\Lambda + d(\xi \cdot \Lambda)$. Plugging this in:

$$dJ = \delta L - d(\xi \cdot L)$$

$$= \xi \cdot dL + d(\xi \cdot L) - d(\xi \cdot L).$$  \hspace{1cm} (6.2.4)

$\xi \cdot dL = 0$ since $L$ is an n form, and so

$$dJ = 0, \rightarrow J = dQ.$$  \hspace{1cm} (6.2.5)

Given $\xi^a$ and a generator of diffeomorphism $H [\xi]$, Hamilton’s equations are:

$$\delta H [\xi] = \Omega = \int_C \omega [\phi, \delta \phi, L_\xi \phi].$$  \hspace{1cm} (6.2.6)

By definition, $\omega [\phi, \delta \phi, L_\xi \phi] = \delta_\phi \Theta - L_\xi \Theta$. Plug that in and we find

$$\omega = \delta J [\xi] - d(\xi \cdot [\phi, \delta \phi])$$

$$\delta H = \int_C \delta J - d(\xi \cdot \Theta).$$  \hspace{1cm} (6.2.7)

\(^\dagger\) $\Theta$ is like $\frac{\partial \mathcal{L}}{\partial \phi^\mu} \delta \phi^\mu$, which is a vector quantity. So one then needs to integrate over all the vectors.
and since \( J = dQ \)

\[
H[\xi] = \int_{\partial C} (Q[\xi] - \text{“undelta” of } d(\xi \cdot \Theta))
= \int_{\partial C} (Q[\xi] - \xi \cdot B) \tag{6.2.8}
\]

where \( B \) is defined so that \( \delta \int_{\partial C} \xi \cdot B = \int_{\partial C} \xi \cdot \Theta \), thus explaining the unprofessional use of the term “undelta.”

### 6.2.2 Carlip’s implementation of this

Using this formalism as his basis, Carlip writes out \( L, \Theta, Q \) specifically for general relativity.

The brackets of generators \( H[\xi] \) form an algebra. If the spacetime has a boundary we get a central term because a generator is unique up to a constant, and if space has a boundary the constants of the \( H_i \) might not match (and won’t be absorbed by infinity if there is a defined boundary). So

\[
\{ H[\xi_1], H[\xi_2] \} = H[\{\xi_1, \xi_2\}] + K[\{\xi_1, \xi_2\}] . \tag{6.2.9}
\]

Take 2 vector fields, \( \xi_1, \xi_2 \) and a field \( \phi \) that fulfills the equations of motion. The diffeomorphism along \( \xi_2 \) of \( J[\xi_1] \) (of the Noether current generated by \( \xi_1 \)) is given by the Lie derivative

\[
\delta_{\xi_2} J[\xi_1] = \mathcal{L}_{\xi_2} J[\xi_1] = \xi_2 \cdot dJ[\xi_1] + d(\xi_2 \cdot J[\xi_1]) . \tag{6.2.10}
\]

The first term on the RHS vanishes so

\[
\mathcal{L}_{\xi_2} J[\xi_1] = d(\xi_2 \cdot J[\xi_1])
= d(\xi_2 \cdot (\Theta - \xi_1 \cdot L)) . \tag{6.2.11}
\]
Plug into $\delta H$:

$$\delta_{\xi_2} H [\xi_1] = \int_C \delta_{\xi_2} J [\xi_1] - d (\xi_1 \cdot \Theta (\phi, \mathcal{L}_{\xi_2} \phi))$$

$$= \int_e d (\xi_2 \cdot (\Theta (\phi, \mathcal{L}_{\xi_1} \phi) - \xi_1 \cdot \mathcal{L})) - (\xi_1 \cdot \Theta (\phi, \mathcal{L}_{\xi_2} \phi))$$

$$= \int_{\partial C} \xi_2 \cdot (\Theta (\phi, \mathcal{L}_{\xi_1} \phi) - \xi_2 \xi_1 \mathcal{L} - \xi_1 \cdot \Theta (\phi, \mathcal{L}_{\xi_2} \phi))$$

(6.2.12)

(and $\mathcal{L} = 0$ on shell). In the bulk $\delta_{\xi_2} H = 0$ because constraints hold on shell. Then $\delta_{\xi_2} H$ is the variation $\delta_{\xi_2} J [\xi_1]$ where $J$ is the boundary term (as in the $\delta L$ variation it’s the boundary term, and also Carlip shows this explicitly in Appendix B of the paper).

By definition

$$\delta_{\xi_2} J [\xi_1] = \{ J [\xi_1], J [\xi_2] \}, \quad (6.2.13)$$

the change in $J [\xi_1]$ under surface deformation generated by $J [\xi_2]$.

Using the algebra we get

$$\{ J [\xi_1], J [\xi_2] \} = J \{ \xi_1, \xi_2 \} + K \{ \xi_1, \xi_2 \}$$

(6.2.14)

where $K$ the central term can be found. Carlip then plugs the general relativity expressions into this.

### 6.2.3 Boundary conditions

For a Killing vector $\chi^a$, $\chi^2 = 0$ on a Killing horizon. Carlip works with a stretched horizon such that $\chi^2 = \epsilon$, and at the end of the calculation takes $\epsilon \to 0$. Near the boundary he defines an orthogonal vector $\rho$:

$$\nabla_a \chi^2 = -2\kappa \rho_a, \quad \chi^a \rho_a = 0.$$  

(6.2.15)

At the horizon $\chi$ is a Killing vector:

$$\chi^a \nabla_a \chi_b = \kappa \chi_b$$

(6.2.16)
and then
\[ \kappa \rho_a = -\frac{1}{2} \nabla_a \chi^2 = -\frac{1}{2} 2 \chi^b \nabla_a \chi_b = \chi^b \nabla_b \chi_a = \kappa \chi_b \] (6.2.17)
so at the horizon \( \rho_a \to \chi_a \) (and thus the normal and tangent vectors are the same at the horizon). Away from horizon they are orthogonal.

If one varies the metric there is no Killing vector. But the scheme requires boundary conditions such that \( \chi^2 = 0 \). He therefore imposes boundary conditions:

\[ \delta \chi^2 = 0, \delta \rho_a = 0, \chi^a \delta g_{ab} = 0 \text{ for } \chi^2 = 0 \]
\[ (\frac{\chi^a \chi^b}{\chi^2} \delta g_{ab} = 0). \] (6.2.18)
This guarantees a boundary where \( \chi^2 = 0 \) remains null, \( \chi^a \) is a null normal vector.

### 6.2.4 Decomposition of \( \xi \):

A crucial step in this scheme is the decomposition of \( \xi^a \) (which becomes \( \chi^a \) at the boundary) into components

\[ \xi^a = R \rho^a + T \chi^a \] (6.2.19)
as Carlip says “deformations in r-t plane.” One finds that \( R \sim \chi^a \nabla_a T \equiv DT \). In order for diffeomorphisms along such a \( \xi^a \) to form a closed algebra one requires \( \rho^a \nabla_a T = 0 \). (To see this work out: \( \{ \xi_1, \xi_2 \} = \xi_3 \), substituting the decomposition into orthogonal components, and require that \( \xi_3 \) too has the form \( DT \rho + T \chi \) just like \( \xi_1, \xi_2 \).

There is an additional subtlety: these boundary conditions don’t guarantee the existence of \( H[\xi] \). You need another. Define

\[ \tilde{\kappa}^2 = -\frac{a^2}{\chi^2}, \ a^a = \chi^b \nabla_b \chi^a. \] (6.2.20)

At horizon \( \tilde{\kappa} \to \kappa \), as \( \chi^2 \to 0 \). But away from the horizon \( \tilde{\kappa} = \frac{\kappa a}{|\chi|} \).

If you vary the metric \( \tilde{\kappa} \) is not constant. We require an average value over the
cross section of the horizon:

$$\delta \int \frac{\partial}{\partial C} \varepsilon \left( \kappa - \frac{\kappa \rho}{|\chi|} \right) = 0.$$  \hspace{1cm} (6.2.21)

This guarantees existence of $H$, AND in order to obtain it we require:

$$\int \varepsilon D^3 T = 0$$  \hspace{1cm} (6.2.22)

(this results from calculation of variation of the metric),

$$\delta = \frac{\partial}{\partial g_{ab}} \delta g_{ab} = \nabla_{(a} \xi_{b)} = \nabla_{(a} (DT \rho + T \chi)_{b)}$$  \hspace{1cm} (6.2.23)

If $DT = \lambda T$ then $T_a \sim e^{i\lambda_a T}$ and $\int e^{i(a+b)k}dk = \delta_{a+b}$.

**Result:**

For such a $\xi$

$$\{\xi_1, \xi_2\}^a = (T_1 DT_2 - T_2 DT_1) \chi^a + \frac{1}{\kappa} \rho^2 D (T_1 DT_2 - T_2 DT_1) \rho^a$$  \hspace{1cm} (6.2.24)

or labelling the term in parenthesis $A$, this is $A \chi^a + \text{const} \cdot D (A) \rho^a$, that is, some other vector $\xi_3$. This is isomorphic to the algebra of diffeomorphisms of $S^1$ or $R$.

Next we want to compute $K$.

$$K \{\xi_1, \xi_2\} = \{J [\xi_1], J [\xi_2]\} - J \{[\xi_1, \xi_2]\}.$$  \hspace{1cm} (6.2.25)

For general relativity Carlip had calculated:

$$\{J [\xi_1], J [\xi_2]\}^* = \frac{1}{16\pi G} \int \varepsilon_{bca_1...a_{n-2}} \left[ \delta_{b}^{d} \nabla_{d} \left( \nabla_{c} \xi_{a_1} - \nabla_{a_1} \xi_{c} \right) - [1 \leftrightarrow 2] \right].$$  \hspace{1cm} (6.2.26)

**6.2.5 Central charge as function of $T, DT$:**

First he deals with the measure in eq.(6.2.26), $\varepsilon_{bca_1...a_{n-2}}$. The integral itself is over the horizon ($n - 2$ dimensions) and the $b, c$ contract with the integrand. The
aim is to obtain an expression written in terms of $\chi, \rho$ (that is, $T, DT$.) This is done by writing the $bc$ part as $\chi_{[b}N_{c]}$ where $N$ is the other null normal to the horizon besides $\chi$, $N^a\chi_a = -1$. The measure becomes $\hat{\epsilon}(\chi_{[b}N_{c]})$ where $\hat{\epsilon}$ is the measure over the horizon ($\epsilon_{a_1...a_{n-2}}$). So the integral is $\int \hat{\epsilon}\chi_{[b}N_{c]}A^bB^c$ where $A, B$ are the other indexed components in the integral.

After similar manipulations he obtains the term $\xi^b\epsilon_{bc}$ decomposed into $\chi$ and $\rho$ components. That is, the integrand includes $\xi^b\epsilon_{bca_1...a_{n-2}}$ expressed as the induced metric $\hat{e}^{a_1...a_{n-2}}$ multiplied by a $T\rho_c$ and an $R\chi_c$ component. He plugs this into the term for $\{J[\xi_1], J[\xi_2]\}$ and gets

$$\{J[\xi_1], J[\xi_2]\} = \frac{1}{16\pi G} \int_{\partial C} \hat{\epsilon}_{a_1...a_{n-2}} \left[ -\frac{1}{\kappa} (T_1 D^3 T_2) + 2\kappa (T_1 DT_2) - (1 \leftrightarrow 2) \right]$$  
(6.2.27)

where he omitted terms of $O(\chi^2)$. Then

$$Q_{a_1...a_{n-2}} = \frac{1}{16\pi G} \hat{\epsilon}_{a_1...a_{n-2}} \left( 2\kappa T - \frac{1}{\kappa} D^2 T \right)$$  
(6.2.28)

(also up to $O(\chi^2)$).

He needs $J[[\xi_1, \xi_2]]$, the ”surface term of H” where $H = \int_{\partial C} Q[\xi] - \xi \cdot B$, but the $B$ term doesn’t contribute (as shown in his appropriately named Appendix B). Recall that $J = \int_{\partial C} Q$. Thus the central term is

$$K\{\xi_1, \xi_2\} = \frac{1}{16\pi G} \int_{\partial C} \hat{\epsilon}_{a_1...a_{n-2}} (DT_1 D^2 T_2 - (1 \leftrightarrow 2)) .$$  
(6.2.29)

Measures

A Virasoro algebra has $K = \frac{c}{24} \int \frac{dz}{2\pi i} \left( \xi' \xi'' - \xi'' \xi' \right)$. Here we have $K \sim \int \hat{\epsilon} \left( T_1 T_2 - T_2 T_1 \right)$. It’s the same form of integrand as Virasoro algebra, but the measures are different. $\hat{\epsilon}$ that appears in Carlip’s $K$ has dimension $d-2$ (or $2d$ in $R^4$). Integration is over cross section of $H$ (that is you integrate over all dimensions except the radial and time directions.) Whereas the complex integral is only over one complex variable. Carlip needs to reduce his integral to one variable.

Taking $v$ as a parameter along $\chi^a$, $Dv = 1$, we have $T_i$ as functions of $v$ and
of horizon angular variable $\theta$, we require
\begin{equation}
\int_{\partial C} \hat{\epsilon} T_1 (v, \theta) T_2 (v, \theta) = \text{const} \int dv T_1 (v, \theta) T_2 (v, \theta).
\end{equation}

But there is a mismatch: LHS is over horizon. RHS is over parameter of Killing orbits, which are NOT one of the angles of horizon. Since he can’t fix this, he adds variables to $T$ so that he can integrate over horizon, and then use a Kronecker delta to adjust the result.

Since $\chi^a$ are over the horizon, he takes $v$, their parameter, as periodic with period $2\pi/\kappa$. (Writing it as an exponent with discrete coefficient $n$ means it’s periodic. The only notable thing here is that as a period, he takes $\kappa/2\pi$ as natural to a BH horizon.)

He uses the orthogonality condition obtained earlier. The $T_i$ are orthogonal from before (the eq. $DT = \lambda T$ gives $T_i$ as exponents) and are now given the specific form
\begin{equation}
T_n (v, \theta^i) = \frac{1}{k} e^{inkv} f_n (\theta^i),
\end{equation}
\begin{equation}
\int_{\partial C} \hat{\epsilon} f_m f_n \sim \delta_{m+n,0}.
\end{equation}

Thus the angles in the horizon will give a Kronecker delta times the horizon area.

\begin{equation}
T_m = \frac{1}{k} e^{inkv} f (\theta) \equiv \tilde{T}_m f_m (\theta)
\end{equation}

where $\int \hat{\epsilon} f_m f_n \sim \delta_{m+n,0}$. (the $f$s need a Kronecker delta because they are indeed angular.) $v$ is a parameter along $\chi^a$ and so $DT = \chi^a \nabla_a T = \partial_v T$. Integral (4.21)
of his paper is over the horizon

\[
K [\xi_m, \xi_n] = \frac{1}{16\pi G} \int_\mathcal{H} \hat{\epsilon}_{a_1 \ldots a_{n-2}} \frac{1}{k} (D T_m D^2 T_n - D T_n D^2 T_m)
\]

\[
= \frac{1}{16\pi G} \int_\mathcal{H} \hat{\epsilon}_{a_1 \ldots a_{n-2}} \frac{1}{k} \left( \partial_v \left( f_m e^{imkv} \right) \partial_v \left( f_n e^{inkv} \right) - [m \leftrightarrow n] \right)
\]

\[
= \frac{1}{16\pi G} \int_\mathcal{H} \hat{\epsilon}_{a_1 \ldots a_{n-2}} \frac{1}{k} f_m f_n \left( \left( \frac{imk}{k} e^{imkv} \right) \left( \frac{-n^2k^2}{k} e^{inkv} \right) - [m \leftrightarrow n] \right)
\]

\[
= -iA \frac{\delta_{m+n} (mn^2 - nm^2)}{16\pi G} e^{(m+n)kv}
\]

\[
= -iA \frac{\delta_{m+n} m^3}{16\pi G}. \quad (6.2.32)
\]

Note that the integral is only over area. The exponents from the \( v \) dependence of \( T \) vanish because the delta function makes them equal 1.

6.2.6 Counting states:

For a rotating BH, \( \chi^a = t^a + \sum \Omega_{(\alpha)} \psi^a_{(\alpha)} \), where \( t^a \) is a time Killing vector, \( \Omega \) angular velocity and \( \psi^a \) the rotational Killing vector. Then

\[
T_n = \frac{1}{k} e^{\sum \left( k \phi_{(\alpha)} - \Omega_{(\alpha)} v \right)} \quad (6.2.33)
\]

Plug this into the commutation relations for \( \{\xi_m, \xi_n\}^a \) and indeed it gives a new vector \( \xi_{m+n}^a \sim T_{m+n} \chi^a + DT_{m+n} \rho^a \).

Then:

\[
\{T_m, T_n\} = -i (m - n) T_{m+n}
\]

\[
K \{T_m, T_n\} = -iA \frac{\delta_{m+n,0}}{8\pi G}
\]

\[
i \{J [T_m], J [T_n]\} = (m - n) J [T_{m+n}] + A \frac{m^2 \delta_{m+n,0}}{8\pi G}
\]

\[
\frac{c}{12} = \frac{A}{8\pi G}. \quad (6.2.34)
\]

Boundary term; \( J [T_0] \equiv \Delta = \frac{A}{8\pi G} \).

Cardy formula:

\[
\rho = \exp \left\{ 2\pi \sqrt{\frac{c}{6}} \left( \Delta - \frac{c}{24} \right) \right\}. \quad (6.2.35)
\]
Taking the log and plugging in $c, \Delta$

$$S = 2\pi \sqrt{\frac{A}{4\pi G} \left( \frac{A}{8\pi G} - \frac{A}{16\pi G} \right)} = \frac{A}{4G}. \tag{6.2.36}$$

### 6.2.7 Extra term

Later work \[55,56,57\] showed that an extra term was necessary in Carlip’s scheme. The bracket of two Noether fields

$$[J_m, J_n] = \delta_{\xi_m} J_n \tag{6.2.37}$$

gives the Lie derivative of $J_n$ along vector field $\xi_m(x)$, where $J_m$ is the Noether current associated with the diffeomorphism generated by $\xi_m$. In Carlip’s work

$$[J_m, J_n] = \int_B \frac{\sqrt{g}}{16\pi G} \{ \xi^\mu \nabla_\rho (\nabla^\nu \xi^\rho - \nabla^\nu \xi^\rho_\nu) -$$

$$- \xi^\mu_\nu \nabla_\rho (\nabla^\nu \xi^\rho - \nabla^\nu \xi^\rho_\nu) + \xi^\mu \xi^\nu \mathcal{L} - (\mu \leftrightarrow \nu) \} dS_{\mu\nu} \tag{6.2.38}$$

where the integral is over the boundary. However Silva \[56\] gives this as

$$[J_m, J_n] = \int_B \frac{\sqrt{g}}{16\pi G} \{ \nabla^\mu \xi^\nu \nabla_\rho s_{m \nu} - \nabla^\mu \xi^\nu \nabla_\rho s_{m \nu} -$$

$$- 2\nabla_\rho s_{m \nu} \nabla^\rho \xi^\mu - (R^\mu_{\rho \nu} - 2\delta^\mu_{\rho} R^\nu_{\sigma}) \xi^\rho_\nu s_{m \sigma} - (\mu \leftrightarrow \nu) \} dS_{\mu\nu} \tag{6.2.39}$$

The difference in these two terms up to a total derivative is

$$\int_B \frac{\sqrt{g}}{16\pi G} \{ \nabla^\mu (\xi^\rho_\nu \nabla_\rho s_{n \nu} - \xi^\rho_\nu \nabla_\rho s_{m \nu}) - (\mu \leftrightarrow \nu) \} dS_{\mu\nu}. \tag{6.2.40}$$

This is actually $J[\{\xi_m, \xi_n\}]$ where

$$J[\xi] = \int_B \frac{\sqrt{g}}{16\pi G} (\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu) dS_{\mu\nu}. \tag{6.2.41}$$

Carlip varies the metric, not $\xi$: his variation acts on the metric but not on the parameters. Therefore it is necessary to subtract this from his result.
Chapter 7

Boundary divergence

7.1 Smooth restricted operators

We will now define a smoothing function which, when applied to an operator that is restricted to a sub-volume, will soften the sharp partition and serve as a momentum cutoff. Let us discuss a quantum system in a volume $\Omega$ which is initially prepared in a pure state $|\psi\rangle$ defined in $\Omega$. We divide the total volume into some sub-volume $V$, and its complement $\hat{V}$ so that $\Omega = V \oplus \hat{V}$. The Hilbert space inherits a natural product structure $\mathcal{H}_\Omega = \mathcal{H}_V \otimes \mathcal{H}_{\hat{V}}$. We are interested in states $|\psi\rangle$ that are entangled with respect to the Hilbert spaces of $V$ and $\hat{V}$ so that they cannot be brought into a product form $|\psi\rangle = |\psi\rangle_V \otimes |\psi\rangle_{\hat{V}}$ in terms of a pure state $|\psi\rangle_V$ that belongs to the Hilbert space of $V$, and another pure state $|\psi\rangle_{\hat{V}}$ that belongs to the Hilbert space of $\hat{V}$.

The total density matrix is defined in terms of the total state $|\psi\rangle$

$$\rho = |\psi\rangle\langle\psi|.$$  \hfill (7.1.1)

The partition of the total volume of the system into two parts

$$\Omega = V \oplus \hat{V}$$ \hfill (7.1.2)

induces a product structure on the Hilbert space and allows defining the reduced density matrix by performing a trace over part of the Hilbert space

$$\rho^V = Tr_{\hat{V}}\rho.$$ \hfill (7.1.3)
Operators that act on part of the Hilbert space are defined as integrals over densities in a part of space

\[ O^V = \int_V d^3r \mathcal{O}(\vec{r}) \]  

(7.1.4)

or alternatively in terms of a theta function

\[ \Theta^V(\vec{r}) = \begin{cases} 
1 & \vec{r} \in V \\
0 & \vec{r} \in \hat{V} 
\end{cases} \]  

(7.1.5)

\[ O^V = \int_\Omega d^3r \mathcal{O}(\vec{r}) \Theta^V(\vec{r}). \]  

(7.1.6)

The relation between quantum expectation values of operators that act on part of the Hilbert space to the statistical averages with a reduced density matrix is given by

\[ \langle \psi | O^V | \psi \rangle = \text{Tr} \left( \rho^V O^V \right). \]  

(7.1.7)

We can also define a smoothed operator

\[ O_{\text{smooth}}^V = \int_\Omega d^3r \mathcal{O}(\vec{r}) \Theta^{V}_{\text{smooth}}(\vec{r}, w) \]  

(7.1.8)

where \( \Theta^{V}_{\text{smooth}}(\vec{r}, w) \) represents a smoothed step function that rather than changing in a discontinuous way from zero to unity on the boundary of \( V \) changes in a smooth way over a region of width \( w \) near the boundary of \( V \). Expressing \( \Theta^{V}_{\text{smooth}} \) as the product of a step function and an auxiliary smoothing function \((f(\vec{r}, w))^2\) (the reason for the square will become clear in what follows):

\[ \Theta^{V}_{\text{smooth}}(\vec{r}, w) = (f(\vec{r}, w))^2 \Theta^V(\vec{r}) = \begin{cases} 
1 & \vec{r} \in V \\
0 \to 1 & \vec{r} \in \partial V \text{ with width } w \\
0 \to 0 & \vec{r} \in \hat{V} 
\end{cases} \]  

(7.1.9)

The smooth theta function defined in this way can be made continuous to any fixed desired order in derivatives. So if a class of operators has at most a given order of derivatives it is possible to define a smooth theta function that will be
effectively analytic for this class. For example, the one dimensional function

\[ \Theta_{\text{smooth}}^{V}(x, w) = \begin{cases} 
  x^n & x \geq 0 \\
  0 & x \leq 0 
\end{cases} \quad (7.1.10) \]

has \( n - 1 \) continuous derivatives at \( x = 0 \).

Rather than using the smoothed step function to modify the operators \( O^{V} \), we can view the smoothing function \( f(\vec{r}, w) \) as modifying the wave function (or state) in which the operator is being evaluated

\[ \langle \psi | O_{\text{smooth}}^{V} | \psi \rangle = \langle \psi | (f(\vec{r}, w))^2 O^{V} | \psi \rangle = \langle f(\vec{r}, w) \psi | O^{V} | f(\vec{r}, w) \psi \rangle \quad (7.1.11) \]

Defining

\[ | \psi_{\text{smooth}} \rangle = f(\vec{r}, w) | \psi \rangle \quad (7.1.12) \]

we may express the expectation value of the smoothed operator in the original state \( | \psi \rangle \) in terms of an expectation value of the original operator in a smoothed state

\[ Tr (\rho^{V}_{\text{smooth}} O^{V}) = Tr (\rho_{\text{smooth}}^{V} O^{V}) \quad (7.1.13) \]

where

\[ \rho_{\text{smooth}}^{V} = | \psi_{\text{smooth}} \rangle \langle \psi_{\text{smooth}} | \quad (7.1.14) \]

In momentum space

\[ | \psi_{\text{smooth}} \rangle = \int d^3p f(\vec{p}, w) \psi(\vec{p}) e^{-ip \cdot \vec{r}} \quad (7.1.15) \]

Here the smoothing function \( f(\vec{p}, w) \) looks as if it is a UV cutoff suppressing the high momentum components of the wave function.

### 7.2 Details of nonrelativistic smoothed momentum fluctuations

We now calculate the expectation value of the smoothed operators \((P^2)^V\) which can be used to evaluate \( H^V \) and other smooth operators. The partial volume \( V \) is defined by a window function as described in the text.
The operator $P^2$ is given by

$$P^2 = \sum_{\vec{p}} \vec{p} \cdot \vec{k} \ a^\dagger_{\vec{p}} (a^\dagger_{\vec{k}} a_{\vec{p}} + [a_{\vec{p}}, a^\dagger_{\vec{k}}]) a_{\vec{k}},$$

$$= \sum_{\vec{p}, \vec{k}} \vec{p} \cdot \vec{k} \ a^\dagger_{\vec{p}} (a^\dagger_{\vec{k}} a_{\vec{p}} + \delta_{\vec{p}\vec{k}} a^\dagger_{\vec{p}} a_{\vec{k}}).$$

$$= \sum_{\vec{p}, \vec{k}} \vec{p} \cdot \vec{k} \ (a^\dagger_{\vec{p}} a^\dagger_{\vec{k}} a_{\vec{p}} a_{\vec{k}}) + \sum_{\vec{p}} p^2 a^\dagger_{\vec{p}} a_{\vec{p}}.$$  \quad (7.2.1)

Evaluating the expectation value:

$$\langle \psi | (P^2_{\text{smooth}}) V | \psi \rangle = \int d^3 r_1 d^3 r_2 \langle 0 | \Psi (\vec{r}_1, w) \sum_{\vec{p}, \vec{k}} \vec{p} \cdot \vec{k} \ (a^\dagger_{\vec{p}} a^\dagger_{\vec{k}} a_{\vec{p}} a_{\vec{k}}) f (\vec{r}_2, w) \Psi^\dagger (\vec{r}_2) | 0 \rangle$$

$$= \int d^3 r_1 d^3 r_2 f (\vec{r}_1, w) f (\vec{r}_2, w) \sum_{\vec{q}, \vec{s}} e^{i \vec{q} \cdot \vec{r}_1} e^{-i \vec{s} \cdot \vec{r}_2} \langle 0 | \sum_{\vec{p}, \vec{k}} \vec{p} \cdot \vec{k} \ a^\dagger_{\vec{p}} a^\dagger_{\vec{k}} a_{\vec{p}} a_{\vec{k}} + \sum_{\vec{p}} p^2 a^\dagger_{\vec{p}} a_{\vec{p}} | 0 \rangle. \quad (7.2.2)$$

Since

$$\langle 0 | a^\dagger_{\vec{q}} a^\dagger_{\vec{p}} a_{\vec{p}} a_{\vec{q}} | 0 \rangle = \delta_{\vec{p}\vec{q}} \delta_{\vec{p}\vec{q}}$$ \quad (7.2.3)

and

$$\langle 0 | a^\dagger_{\vec{q}} a^\dagger_{\vec{p}} a_{\vec{p}} a_{\vec{q}} | 0 \rangle = 0, \quad (7.2.4)$$

the expectation value of the smooth operator is then

$$\langle \psi | (P^2_{\text{smooth}}) V | \psi \rangle = \int d^3 r d^3 r_2 f (\vec{r}, w) f (\vec{r}_2, w) \sum_{\vec{q}, \vec{s}} e^{i \vec{q} \cdot \vec{r}_1} e^{-i \vec{s} \cdot \vec{r}_2} \sum_{\vec{p}} p^2 \delta_{\vec{p}\vec{q}} \delta_{\vec{p}\vec{q}}$$

$$= \int d^3 r \vec{\nabla} f (\vec{r}, w) \cdot \vec{\nabla} f (\vec{r}, w)$$

$$= \sum_{\vec{p}} p^2 f (\vec{p}, w) f (-\vec{p}, w). \quad (7.2.5)$$
7.3 Details of relativistic smoothed energy

In a relativistic theory the Hamiltonian is given by \( \hat{H} = \int \frac{d^3k}{(2\pi)^3} k_0 a_k^\dagger a_k \) in momentum space. In configuration space, the expectation value of the smoothed restricted Hamiltonian is given by the relativistic scalar product,

\[
\langle \psi \mid (H_{\text{smooth}})^V \mid \psi \rangle = -i \int d^3r_1 d^3r_2 \left[ \left. \left( \Psi (\vec{r}_1, t_1) f (\vec{r}_2, w) \partial_{t_2} \left( H f (\vec{r}_2, w) \Psi^\dagger (\vec{r}_2, t_2) \right) - \partial_{t_1} \left( \Psi (\vec{r}_1, t_1) f (\vec{r}_1) \right) H f (\vec{r}_2) \Psi^\dagger (\vec{r}_2) \right) \right| \right| t_1 = t_2 \equiv A - B \quad (7.3.1)
\]

The first term \( A \) is given by

\[
A = \int d^3r_1 d^3r_2 202 f (\vec{r}_1, w) f (\vec{r}_2, w) \left( \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \left( \int \frac{d^3q}{(2\pi)^3} \right) \left| a_{\vec{r}_1} a_{\vec{q}} a_{\vec{k}} \right| \right) \times
\]

\[
\int \frac{d^3q}{(2\pi)^3} q_0 a^\dagger_{\vec{q}} a_{\vec{q}} \times -i \partial_{t_2} \int \frac{d^3k}{(2\pi)^3} \left( \int \frac{d^3q}{(2\pi)^3} \right) \left( a_{\vec{k}} e^{i\vec{k} \cdot \vec{r}_2 - ik_0 t_2} + a^\dagger_{\vec{k}} e^{-i\vec{k} \cdot \vec{r}_2 + ik_0 t_2} \right) \left| 0 \right\rangle \right| \right| t_1, t_2 = 0 \quad (7.3.2)
\]

where \( \vec{p}_0^2 = \vec{p}^2 \), \( k_0^2 = \vec{k}^2 \). The second term \( B \) can be expressed in a similar straightforward manner.

We first perform the momentum integrals and evaluate the expectation value. This integral includes the following sets of operators:

\[
a_{\vec{r}} a^\dagger_{\vec{q}} a_{\vec{k}}, \ a_{\vec{r}} a^\dagger_{\vec{q}} a_{\vec{k}}, \ a_{\vec{p}} a^\dagger_{\vec{q}} a_{\vec{k}}, \ a_{\vec{p}} a^\dagger_{\vec{q}} a_{\vec{k}}.
\]

but only the second term yields a non-vanishing contribution,

\[
\int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} q_0 k_0 \left( \left| a_{\vec{r}} e^{i\vec{p} \cdot \vec{r}_1 - ip_0 t_1} + a^\dagger_{\vec{p}} e^{-i\vec{p} \cdot \vec{r}_1} \right| \right) a^\dagger_{\vec{q}} a_{\vec{k}} \left( a_{\vec{k}} e^{i\vec{k} \cdot \vec{r}_2} + a^\dagger_{\vec{k}} e^{-i\vec{k} \cdot \vec{r}_2} \right) \left| 0 \right\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} q_0 k_0 \left( \left| a_{\vec{r}} a^\dagger_{\vec{q}} a_{\vec{k}} \right| 0 \right) = \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} q_0 k_0 \left( e^{i\vec{p} \cdot \vec{r}_1 - ip_0 t_1} \right) \left( \delta(\vec{p} - \vec{q}) \delta(\vec{k} - \vec{q}) \right) \delta(\vec{p} - \vec{q}). \quad (7.3.3)
\]
Substituting the result of eq. (7.3.3) into eq. (7.3.2) we find

\[
A = \int d^3r_1 d^3r_2 f(\vec{r}_1, w) f(\vec{r}_2, w) \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{q_0k_0}{\sqrt{2k_0\sqrt{2p_0}}} e^{i\vec{p}\cdot\vec{r}_1 - i\vec{k}\cdot\vec{r}_2} \delta(\vec{p} - \vec{q}) \delta(\vec{k} - \vec{q})
\]

\[
= \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{q_0k_0}{\sqrt{2k_0\sqrt{2p_0}}} f(\vec{p}, w) f(-\vec{k}, w) \delta(\vec{p} - \vec{q}) \delta(\vec{k} - \vec{q})
\]

\[
= \frac{1}{2} \int d^3p \ f(\vec{p}, w) \ f(-\vec{p}, w), \quad (7.3.4)
\]

where \( p^2 = \vec{p}^2 \), and \( f(\vec{p}, w) \) is the Fourier transform of \( f(\vec{r}, w) \). Repeating the same steps for \( B \) we find \( B = -A \) so that

\[
\langle \psi | (H_{\text{smooth}})^V | \psi \rangle = \int d^3p \ f(\vec{p}, w) \ f(-\vec{p}, w)
\]

\[
= \int d^3r \ f(\vec{r}, w) \sqrt{\nabla^2} f(\vec{r}, w). \quad (7.3.5)
\]
Chapter 8

Number of states

8.1 Lorentz invariance of phase space

For simplicity of notation we take $\hbar, c = 1$. In classical thermodynamics the density of states is defined for a non relativistic system as follows: Take an integral over the volume of phase space $(d^3x d^3p)$, restrict it to values of $p$ which fit the energy eigenvalues that solve the Schroedinger equation: $p^2 = 2mE$, and divide by a unit of volume in momentum space: This gives the number of states in phase space with the given energy, per unit volume of phase space, and one multiplies this by $g$, a numerical factor related to the degeneracy (eg., for spins with Dirichlet BCs $g = 2(1/8)$ for the positive octant and spin degeneracy), and integration over the volume in phase space would give $V = (\pi/L)^3$ for a potential well, $(2\pi/L)^3$ for periodic BCs.

We will show that the number of states is Lorentz invariant by explicit calculation. For simplicity we will calculate this in 1+1 dimensions, and assume the transformed system moves in the x direction relative to original system. Generalization to more dimensions is transparent. We take $g = 1$ and assume $E = p^2/2m$:

$$N = \int dx \frac{dp}{dE} dE.$$  (8.1.1)

8.1.1 Special relativity: phase space

The question of invariance arises because we are restricting 8 dimensional space to the product of two 3 dimensional hypersurfaces, in both volume and momentum, by choosing $t = 0$ and $E = p^2/2m$ which is the energy that solves the Schroedinger
equation, or \( E^2 = p^2 \) for the relativistic wave equation. However we will impose the restriction afterwards and first perform the transformation.

If we were not restricting space to a hypersurface, the transformation would be as follows (in \( 1+1 \) dimensions): Going to frame moving with velocity \( \beta \) relative to the original one, the transformed momenta is

\[
\tilde{\mathbf{p}} = \Lambda \mathbf{p}
\]

where

\[
\Lambda = \begin{pmatrix}
\gamma & \gamma \beta \\
\beta & \gamma
\end{pmatrix}
\]

and so the differential of the new time component is

\[
d\tilde{p}^0 = d\left( \Lambda_0^0 p^0 + \Lambda_i^0 \beta p^i \right).
\]

and plugging in the explicit values of the transformation matrix elements this is

\[
d\tilde{p}^0 = d\left( \gamma p^0 + \gamma \beta p^i \right).
\]

In what follows, for clarity, we write \( p^0 \) as \( E \), \( p^i \) as \( \vec{p} \), \( \sqrt{p^i p^i} \) as \( p \).

**Volume elements**

Recall that the volume form is invariant. Since \( dx dt \) actually refers to \( dx \wedge dt \).

\[
dx \wedge dt = (dxdt - dtdx).
\]

\[
d\tilde{x} \wedge d\tilde{t} = ((\gamma dx + \gamma \beta dt)(\gamma dt + \gamma \beta dx) - (\gamma dt + \gamma \beta dx)(\gamma dx + \gamma \beta dt))
\]
\[
= \gamma^2 (dxdt + \beta^2 dtdx + \beta dx^2 + dt^2) - (dtdx + \beta^2 dxdt + \beta dx^2 + dt^2)
\]
\[
= \gamma^2 (dxdt (1 - \beta^2) + dtdx (\beta^2 - 1)) = \gamma^2 (1 - \beta^2)(dxdt - dtdx)
\]
\[
= dx \wedge dt.
\]

(8.1.2)

**Energy momentum relation**

\[
E^2 = m^2 + \vec{p}^2, \quad p = \sqrt{E^2 - m^2}, \quad \frac{dp}{dE} = \frac{E}{\vec{p}}.
\]

(8.1.3)

In this case \( \vec{p} = \vec{p}, p = \sqrt{\vec{p}^2} \).
After imposing the restriction on energy we get

\[ d\tilde{E} = \frac{1}{p} \left( \gamma E + \beta \sqrt{E^2 - m^2} \right) dE = \gamma \left( 1 + \frac{\beta E}{p} \right) dE. \]

Instead we can write

\[ d\tilde{p} = \frac{1}{p} \left( \gamma px + \gamma \beta E \right) = \gamma \left( 1 + \frac{\beta E}{p} \right) dp = \gamma \left( 1 + \frac{\beta p}{E} \right) dp. \]

Density of states

In original 4-dimensional frame, taking hypersurface \( t = 0 \) and the energy momentum relation, we have

\[ N = \int dx \int \! dt \delta(t) \int dp \int \! dE \delta(p^2 - E^2) = \int dx \int \! dt \delta(t) \int \! \frac{dp}{2E}. \quad (8.1.4) \]

First we deal with the space-time part. After Lorentz transformation, and taking into account the invariance of \( dx \wedge dt \), this becomes

\[ \int d\tilde{x} d\tilde{t} \delta(\tilde{t}) = \int dx dt \delta(\gamma t + \gamma \beta x) = \int dx dt \frac{\delta(t)}{\gamma} = \frac{1}{\gamma} \int dx. \quad (8.1.5) \]

\[ \gamma \frac{L}{\gamma} = \gamma \frac{L}{\gamma} = L \quad (8.1.6) \]

and so

\[ \int_0^{\tilde{L}} d\tilde{x} \int_{-\infty}^{\infty} d\tilde{t} \delta(t) = \int_0^L dx \int_{-\infty}^{\infty} dt \delta(t) = L. \quad (8.1.7) \]

For the momentum integral the delta function on the energy is scalars, so as we know from field theory, the form of the integral is invariant:

\[ \int \frac{dp}{2E} = \int \frac{dp}{2E}. \quad (8.1.8) \]
and writing the integrand explicitly:

\[
\int \frac{dp}{2E} = \frac{1}{2} \int \gamma dp + \gamma \beta dE = \frac{1}{2} \int \frac{dp + \beta dE}{E + \beta p}.
\]

\[
dE = \frac{p}{E} dp.
\]

\[
\frac{1}{2} \int \frac{dp + \beta dE}{E + \beta p} = \frac{1}{2} \int \frac{dp + \frac{\beta}{E} dp}{E + \beta p} = \frac{1}{2} \int \frac{(E + \beta p) dp}{E(E + \beta p)}.
\]

\[
\int \frac{dp}{2E} = \int \frac{dp}{2E}.
\]

Therefore

\[
\int d\tilde{t} d\delta(t) d\tilde{p} d\delta(\tilde{E}^2 - \tilde{p}^2) = \int dx dt \delta(t) dp d\delta(E^2 - p^2)
\]

and the density of states is Lorentz invariant.

Extension to more space dimensions is simple but long. A general if somewhat tricky proof can also be found in [58], p.36.

8.2 Equivalence of WKB and formal calculations of the number of states

In this section we calculate the number of states for non-relativistic massless particles in a box in several different ways and show the different methods of calculation are equivalent. We do this for a cube and a sphere, but one can see that the result would be the same for other geometries.

8.2.1 Photons in a cubic box

8.2.1.1 Formal method

First we use the textbook method, obtained by solving wave equation:

Take a box with each side of length L, and Dirichlet boundary conditions. The wave equation gives

\[
\frac{\pi^2}{L^2} \left( n_x^2 + n_y^2 + n_z^2 \right) = \frac{\omega^2}{c^2}
\]

where \( n_i \) are integers that index the modes, and of course one can generalize to
more dimensions.

\[ n \equiv \left[ \sum_{i=1}^{d} n_i^2 \right]^{1/2}, \quad \omega = \frac{n \pi c}{L}. \quad (8.2.2) \]

Taking the sum to an integral in the space of mode indices

\[ \sum_n (\ldots) \to \frac{1}{8} \int_{0}^{\infty} 4\pi n^2 dn \quad (8.2.3) \]

using only the positive octant. Usually one also multiplies by 2 for spin or polarization but we don’t do so here for simplicity. Plugging in eq. (8.2.2) we get

\[ N = \frac{\pi}{2} \int_{0}^{\infty} \left( \frac{L}{\pi c} \right)^3 \omega^2 d\omega = \frac{V}{2\pi^2 c^3} \int_{0}^{\infty} \omega^2 d\omega \quad (8.2.4) \]

and since \( \omega = E/\hbar \),

\[ N = \frac{V}{2\pi^2 c^3 \hbar^3} \int_{0}^{\infty} E^2 dE \quad (8.2.5) \]

which is dimensionless as it should be. For photons in finite temperature this will be multiplied by the Bose Einstein distribution. The density of states is

\[ g(E) = \frac{VE}{2\pi^2 c^3 \hbar^3}. \quad (8.2.6) \]

and number of states for fixed energy is

\[ N = \frac{VE^3}{6\pi^2 c^3 \hbar^3} = \frac{1}{(2\pi)^3 c^3 \hbar^3} \frac{4}{3} \pi E^3 V^3. \quad (8.2.7) \]

### 8.2.1.2 WKB method for a cubic box

WKB is generally used in one dimension, so we begin with a one dimensional system. For such a system the textbook method gives \( n = Lk/\pi \) and using \( E = h\omega = hkc \) we obtain \( dn = (L/h\pi c) dE \) and \( g(E) = L/h\pi c. \)

Using the WKB approximation \( n\pi h = \int_{0}^{L} pdx = \int_{0}^{L} hkd\xi \). This gives \( n = kL/\pi \), and taking \( E = pc = hkc \),

\[ N = \frac{E L}{\hbar c \pi}, \quad \frac{dn}{dE} = \frac{L}{\pi ch} \quad (8.2.8) \]
which is the same as the “textbook” result.

We get \(1/\pi \hbar\) rather than \(2/\pi \hbar\) as in the general definition of \(N\) because WKB is defined for Dirichlet boundary conditions rather than periodic.

Using WKB for a 3D system, solving the wave equation by separation of variables, \(\psi(xyz) = X(x)Y(y)Z(z)\),

\[
\psi_{WKB} \sim e^{i \vec{k} \vec{x}/\hbar} = e^{ik_x x/\hbar}e^{ik_y y/\hbar}e^{ik_z z/\hbar}.
\]

(8.2.9)

For each exponent we take the one dimensional calculation as above, and so for each direction we get \(k_i = n_i \pi / L\) and

\[
n_i = \frac{k_i L}{\pi}.
\]

(8.2.10)

As before,

\[
n \equiv \left[ \sum_{i=1}^{d} n_i^2 \right]^{1/2}, \quad \omega = \frac{n \pi c}{L}
\]

and the rest of the calculation is identical to the exact one in the previous section, eqs. (8.2.5), (8.2.6).

8.2.1.3 Phase space calculation for cubic box

We have

\[
E^2 - p_x^2 - p_y^2 - p_z^2 = 0
\]

\[
p_x^2 = E^2 - p_y^2 - p_z^2
\]

\[
N = \frac{1}{(2\pi)^3} \int_0^L dx \int_0^L dy \int_0^L dz \int d^3p = \frac{L^3}{(2\pi)^3} \int d^3p
\]

(8.2.11)
Dealing with the momentum integral we have

\[
\int_{-E}^{E} dp_y \int_{-\sqrt{E^2-p_y^2}}^{\sqrt{E^2-p_y^2}} dp_z \int_{-\sqrt{E^2-p_y^2-p_z^2}}^{\sqrt{E^2-p_y^2-p_z^2}} dp_x = 2 \int_{0}^{E} dp_y \sqrt{E^2-p_y^2} \int_{-\sqrt{E^2-p_y^2}}^{\sqrt{E^2-p_y^2}} dp_z
\]

\[
E^2 - p_z^2 = A^2
\]

\[
2 \int_{-A}^{A} dp_y \sqrt{A^2 - p_y^2} = 2A \int_{-A}^{A} dp_y \sqrt{1 - \frac{p_y^2}{A^2}} = A^2 \int_{0}^{1} du \sqrt{1 - u^2} = 2A^2 \frac{\pi}{2}
\]

where we substituted \( u = \cos \theta \). We need \( E^2 - p_z^2 \geq 0 \) for the above equations to be real, so the momentum integral is now

\[
2 \int_{-E}^{E} dp_z A^2 \frac{\pi}{2} = \pi \int_{-E}^{E} dp_z (E^2 - p_z^2) = \pi \left[ E^2 p_z - \frac{p_z^3}{3} \right]_{-E}^{E} = \pi \frac{4E^3}{3}
\]

(8.2.12)

and so

\[
N = \frac{1}{(2\pi)^3} \frac{4\pi V E^3}{3}
\]

(8.2.13)

as before.

### 8.2.2 Spherical “box”

Definition of number of states:

\[
N = \frac{1}{(2\pi)^3} \int d^3x d^3p.
\]

(8.2.14)

This is the product of volume of two spheres, one with radius \( R \) and the other with radius \( E \), so it has to be

\[
N = \frac{1}{(2\pi)^3} \frac{4\pi R^3}{3} \frac{4\pi E^3}{3} = \frac{2R^3E^3}{9\pi}.
\]

(8.2.15)

Solving in detail: The wave equation in its most general form is

\[
- g^{tt} E^2 - g^{rr} p_r^2 - g^{\perp \perp} p_\perp^2 = 0.
\]

(8.2.16)
For spherical coordinates we have $g^{tt} = -1$, $g^{rr} = 1$, $g^{\perp\perp} = r^{-2}$. and so the wave equation is

$$E^2 - p_r^2 - \frac{p_\perp^2}{r^2} = 0.$$  

$$p_r = \sqrt{E^2 - \frac{p_\perp^2}{r^2}}$$

$$\int d^3 x d^3 p = 4\pi \int dr r^2 \int dp_r d^2 p_\perp \frac{1}{r^2}$$

where the space integral has a factor of $\sqrt{g_{\perp\perp}} = r^2$, and the momentum integral has a factor of $\sqrt{g_{\perp\perp}} = \frac{1}{r}$, these cancel and won’t appear in what follows.

$$N = \frac{1}{8\pi^3} \int_0^R dr \int_0^{4\pi} d\Omega \int_0^\infty dp_r \int_0^{\infty} d^2 p_\perp$$

$$= \frac{1}{2\pi^2} \int_0^R dr \int_0^\infty d^2 p_\perp 2\sqrt{E^2 - \frac{p_\perp^2}{r^2}}$$

$$= \frac{1}{2\pi^2} \int_0^R dr \int_0^{E_r} 2\pi p_\perp dp_\perp 2\sqrt{E^2 - \frac{p_\perp^2}{r^2}}$$

$$= \frac{2}{\pi} \int_0^R dr \int_0^{E_r} dp_\perp \sqrt{E^2 - \frac{p_\perp^2}{r^2}}.$$  \hspace{1cm} (8.2.17)

Substituting $E^2 - p_\perp^2 = u$, the momentum integral is

$$- \frac{1}{2} \int du \sqrt{u} = - \frac{1}{3} u^{3/2}$$  \hspace{1cm} (8.2.18)

Plugging in the limits of the $p_\perp$ integral

$$N = \frac{2E^3}{3\pi} \int_0^R dr r^2 = \frac{2E^3 R^3}{9\pi}.$$  \hspace{1cm} (8.2.19)
In terms of volume
\[ N = \frac{E^3 V}{6\pi^2} = \frac{1}{(2\pi)^3} V \left(\frac{4\pi E^3}{3}\right) = \frac{1}{(2\pi)^3} V_x V_p. \] (8.2.20)

### 8.2.2.1 WKB calculation for spherical box

To find the number of modes we sum over the number of radial modes \( \int_0^R dr k(r, l) \) where \( R \) is an IR cutoff. This is multiplied by the sum over angular momentum since each level is degenerate due to the angular momentum:

\[
k(r, l, E) = \sqrt{\frac{E^2}{c^2} - \frac{l(l+1)}{r^2}}
\]

\[
N\pi = \int_0^R dr \int_0^\infty d(l(l+1)) \sqrt{\frac{E^2}{c^2} - \frac{l(l+1)}{r^2}}
\] (8.2.21)

The root has to be positive. This determines the upper limit for \( l(l+1) \):

\[
\frac{E^2 r^2}{c^2} = l(l+1)_{\text{max}}
\]

\[
\int_0^{E^2 r^2/c^2} d(l(l+1)) \sqrt{\frac{E^2}{c^2} - \frac{l(l+1)}{r^2}} = -\frac{2}{3} r^2 \left( \left[ \frac{E^2}{c^2} - \frac{E^2 r^2/c^2}{r^2} \right]^{3/2} - \frac{E^3}{c^3} \right)
\]

\[
= \frac{2}{3} \frac{E^3}{c^3} r^2.
\]

\[
N\pi = \frac{2}{3} \frac{E^3}{c^3} R \int_0^R dr r^2.
\]

\[
N = \frac{2}{9} \frac{R^3 E^3}{\pi c^3}
\] (8.2.22)
as before.

### 8.2.3 Comparisons

Comparing the sphere to the calculation for a cube, setting \( \hbar = 1 \):

\[
g(E)_{\text{sphere}} = \frac{VE^2}{2\pi^2 c^3}, \quad g(E)_{\text{cube}} = \frac{VE^2}{\pi^2 c^3 \hbar^3}.
\] (8.2.23)
This is approximately the ratio of their volumes. Taking length of side of cube as $2R$,

$$V_{cube} = 8R^3, V_{sphere} = \frac{\pi}{3} 4R^3 \approx \frac{1}{2} V_{cube}.$$  \hspace{1cm} (8.2.24)

and as expected, $N \sim V_x V_p$.

### 8.3 Phase space in 3+1 and in 4+1 dimensions

The number of states in $d+1$ dimensions (d space dimensions) for a diagonal metric works out to be

$$N = C E^d \int d^d x \sqrt{g_d} \left( g^{00} \right)^{\frac{d}{2}}$$

$$C = \frac{\pi^{\frac{d}{2}}}{\Gamma \left( \frac{d}{2} + 1 \right)}$$  \hspace{1cm} (8.3.1)

and $g_d$ is the determinant of the spatial components of the metric.

#### 8.3.1 Proof in 3+1 dimensions:

The wave equation:

$$g^{00} E^2 - g^{xx} p_x^2 - g^{yy} p_y^2 - g^{zz} p_z^2 = 0$$

$$p_x = \sqrt{g_{xx}} \sqrt{g^{00} E^2 - g^{yy} p_y^2 - g^{zz} p_z^2}$$

$$\int d^3 p = \sqrt{g_{yy} g^{00} E} \int dp_y \sqrt{g^{00} E^2 - g^{yy} p_y^2 - g^{zz} p_z^2}$$

$$- \sqrt{g_{yy} g^{00} E} \int dp_y \sqrt{g^{00} E^2 - g^{yy} p_y^2 - g^{zz} p_z^2}$$

$$= 2 \sqrt{g_{xx}} \sqrt{g_{yy} g^{00} E} \int dp_y \sqrt{g^{00} E^2 - g^{yy} p_y^2 - g^{zz} p_z^2}$$

$$- \sqrt{g_{yy} g^{00} E} \int dp_y \sqrt{g^{00} E^2 - g^{yy} p_y^2 - g^{zz} p_z^2}$$
We label $g^{00}E^2 - g^{yy}p_y^2 \equiv A^2$. Then the integral over $p_z$ becomes

$$\int_{-\sqrt{gzzA}}^{\sqrt{gzzA}} dp_z \sqrt{A^2 - g^{zz}p_z^2} = A \int_{-\sqrt{gzzA}}^{\sqrt{gzzA}} dp_z \sqrt{1 - \frac{p_z^2}{gzzA^2}} = A^2 \sqrt{gzz \frac{\pi}{2}} \quad (8.3.2)$$

Plugging this in we get

$$\int d^3p = 2\sqrt{gxxgzz\pi} \int_{-\sqrt{gyyg00E}}^{\sqrt{gyyg00E}} dp_y (g^{00}E^2 - g^{yy}p_y^2)$$

$$= \sqrt{gxxgzz\pi} \left[ 2\sqrt{gyy} (g^{00}E^2)^{3/2} - \frac{2}{3}g^{yy} \left( \sqrt{gyyg00E} \right)^3 \right]$$

$$= \sqrt{gxxgzzgyy} \frac{4}{3} \pi \left( \sqrt{g00E} \right)^3. \quad (8.3.3)$$

### 8.3.2 One more dimension:

$$g^{00}E^2 - g^{xx}p_x^2 - g^{yy}p_y^2 - g^{zz}p_z^2 - g^{ww}p_w^2 = 0$$

$$p_w = \sqrt{gww} \sqrt{g^{00}E^2 - g^{xx}p_x^2 - g^{yy}p_y^2 - g^{zz}p_z^2}$$
\[
\int d^3p = \int dp_y \int dp_z \int dp_x \times \left( \sqrt{g_{yy}g^{00}E} \sqrt{g_{zz}g^{00}E - g^{yy}p_y^2} \right)
\]
\[
\times \left( \sqrt{g_{ww}g^{00}E - g^{zz}p_z^2} \times \sqrt{g_{xx}g^{00}E - g^{yy}p_y^2} \right)
\]
\[
\left( g^{00}E^2 - g^{yy}p_y^2 - g^{zz}p_z^2 \right) \quad (8.3.4)
\]

We label \( g^{00}E^2 - g^{yy}p_y^2 - g^{zz}p_z^2 \equiv A^2 \). Then the integral over \( p_x \) becomes
\[
\int dp_x \sqrt{A^2 - g^{xx}p_x^2} = A \int dp_x \sqrt{1 - \frac{p_x^2}{g_{xx}A^2}}
\]
\[
= A^2 g_{xx} \int_1^{-1} du \sqrt{1 - u^2} = A^2 \sqrt{g_{xx} \frac{\pi}{2}}. \quad (8.3.6)
\]

Plugging this in we get
\[
\int d^3p = 2 \sqrt{g_{yy}g_{ww} \frac{\pi}{2}} \int dp_y \int dp_z \left( g^{00}E^2 - g^{yy}p_y^2 - g^{zz}p_z^2 \right)
\]
\[
\left( g^{00}E^2 - g^{yy}p_y^2 \right) \times \sqrt{g_{xx}g^{00}E - g^{yy}p_y^2} \times \sqrt{g_{zz}g^{00}E - g^{yy}p_y^2}
\]
\[
\left( g^{00}E^2 - g^{yy}p_y^2 - g^{zz}p_z^2 \right) \quad (8.3.7)
\]

Let us label \( g^{00}E^2 - g^{yy}p_y^2 \equiv B^2 \). Then the \( p_z \) integral becomes
\[
\int dp_z \left( B^2 - g^{zz}p_z^2 \right) = \frac{4}{3} \sqrt{g_{zz}B^3} = \frac{4}{3} \sqrt{g_{zz} \left( g^{00}E^2 - g^{yy}p_y^2 \right)^{3/2}}. \quad (8.3.7)
\]
Integrate over $p_y$:

$$\sqrt{g_{yy}g^{00}} E \int_{-\sqrt{g_{yy}g^{00}} E} d p_y \left( g^{00} E^2 - g_{yy} p_y^2 \right)^{3/2} = \sqrt{g_{yy}} \frac{3}{8} \pi \left( \sqrt{g^{00}} E \right)^4. \quad (8.3.8)$$

(this was done with Mathematica, you get a result containing $\arctan[\infty] = \frac{\pi}{2}$).

Plugging this back in,

$$\int d^3 p = \frac{4}{3} \left( \frac{3}{8} \right) \pi^2 \sqrt{g_{xx} g_{ww} g_{yy} g_{zz}} \left( g^{00} \right)^2 E^4$$

$$= \frac{\pi^2}{2} \sqrt{g_4} \left( g^{00} \right)^2 E^4$$

and so

$$N = \frac{\pi^2}{2} E^4 \int_V d^4 x \sqrt{g_4} \left( g^{00} \right)^2$$

$$= CE^4 \int_V d^4 x \sqrt{g_4} \left( g^{00} \right)^{\frac{4}{2}} \left( C = \frac{\pi^{\frac{4}{2}}}{\Gamma \left( \frac{4}{2} + 1 \right)} \right) \quad (8.3.9)$$

just as we claimed. It would be good to be able to prove by induction that if it’s true for $N_d$ it’s true for $N_{d+1}$ but (so far) I have not been able to generalize the integration: the integrand becomes $\left( g^{00} E^2 - g^{(d+1,d+1)} p_{d+1} \right)^{d/2}$.

### 8.4 Number of states in Rindler space

We here explicitly calculate the number of states in Rindler space. The number of states can be obtained by writing out the wave equation for Rindler space:

$$-e^{-2a\xi} \partial_{tt} \psi + e^{-2a\xi} \partial_{\xi\xi} \psi + \partial_{\perp\perp} \psi = 0$$

where $\partial_{\perp\perp}$ is the second derivative over transverse degrees of freedom.

$$e^{-2a\xi} E^2 - e^{-2a\xi} p_\xi^2 - p_\perp^2 = 0$$

$$p_\xi = \sqrt{E^2 - e^{2a\xi} p_\perp^2}. \quad (8.4.1)$$
\[ N = \frac{1}{(2\pi)^3} \int d^2 x_\perp d\xi \sqrt{g_{\xi\xi}} \frac{\sqrt{E^2 - e^{2\alpha E}p_\perp^2}}{-\sqrt{E^2 - e^{2\alpha E}p_\perp^2}}. \] (8.4.2)

Integration of \( dp_\xi \) within these limits gives
\[ \int_{-\sqrt{E^2 - e^{2\alpha E}p_\perp^2}}^{\sqrt{E^2 - e^{2\alpha E}p_\perp^2}} dp_\xi = 2\sqrt{E^2 - e^{2\alpha E}p_\perp^2}. \] (8.4.3)

\[ N = \frac{1}{(2\pi)^3} \int d^2 x_\perp d\xi \sqrt{g_{\xi\xi}} \frac{e^{-\alpha E}}{0} \int dp_\perp 2\pi p_\perp \sqrt{E^2 - e^{2\alpha E}p_\perp^2} \] (8.4.4)

\[ = \frac{2V_\perp}{(2\pi)^2} \int d\xi \sqrt{g_{\xi\xi}} \int_{0}^{\infty} dp_\perp \sqrt{E^2 - e^{2\alpha E}p_\perp^2} \] (8.4.5)

where we integrated over \( x_\perp \) and limits of \( p_\perp \) are to ensure the root is positive. (it can’t be \(-e^{-\alpha E}E\) because \( p_\perp \) here is like a radial coordinate). To integrate over \( p_\perp \) we substitute \( u = E^2 - e^{2\alpha E}p_\perp^2 \) and obtain
\[ -\frac{1}{2} e^{-2\alpha E} \int du \sqrt{u} = -\frac{1}{3} e^{-2\alpha E} u^{3/2}. \] (8.4.6)

Putting back \( p_\perp \), we have
\[ N = \frac{2V_\perp}{(2\pi)^2} \int d\xi \sqrt{g_{\xi\xi}} \left( -\frac{1}{3} e^{-2\alpha E} \left[ 0 - (E^2)^{3/2} \right] \right) = \frac{2V_\perp E^3}{3 (2\pi)^2} \int_{-\infty}^{\infty} d\xi \sqrt{g_{\xi\xi}} e^{-2\alpha E} \] (8.4.7)

and since \( \sqrt{g_{\xi\xi}} = e^{\alpha E} \), this becomes
\[ N = \frac{2V_\perp E^3}{3 (2\pi)^2} \int_{-\infty}^{\infty} d\xi e^{-\alpha E} \]
which diverges at the lower boundary (as expected). Putting in a lower limit of \( \xi_{\text{min}} \) we have

\[
N = \frac{V_\perp E^3}{6\pi^2 a} e^{-a\xi_{\text{min}}}. \tag{8.4.8}
\]

Checking units, this is dimensionless as it should be.

\[
\frac{dN}{dE} = \frac{V_\perp E^2}{2\pi^2 a} e^{-a\xi_{\text{min}}}. \tag{8.4.9}
\]

This differs from the result in Sec.\[2.1\]. In that work we took \( V_\perp = (2a)^{-2} \) by comparing Unruh temperature with Hawking temperature and relating it to the Schwarzschild radius.

\[
T_H = \frac{1}{8\pi M}, \quad T_{\text{Rind}} = \frac{a}{2\pi} \rightarrow 2a = \frac{1}{2M}
\]

\[
R_{\text{horizon}} = 2M = \frac{1}{2a}
\]

\[
\frac{1}{(2a)^2} = \xi_\perp^2. \tag{8.4.10}
\]

This was then incorporated into a WKB approximation following 't Hooft, rather than the exact calculation presented here. However the dependence on energy is the same, as is the divergence at the horizon.

### 8.4.1 Comparisons

Photons in a cube in Minkowski space are proportional to the entire volume. Comparing the density of modes ot our result:

\[
\frac{dN}{dE} (\text{Mink cube}) = \frac{VE^2}{\pi^2}, \quad \frac{dN}{dE} (\text{Rind}) = \frac{V_\perp E^2 e^{-a\xi_{\text{min}}}}{\pi^2 2a}. \tag{8.4.11}
\]

The density of states in spherical and Schwarzschild metrics are

\[
\frac{dN}{dE} (\text{sphere}) = \frac{2R^3 E^2}{3\pi},
\]

\[
\frac{dN}{dE} (\text{Schwarz}) = \frac{2E^2}{\pi} \int_{r_0}^{R} dr \frac{r^{9/2}}{(r - 2M)^{5/2}}. \tag{8.4.12}
\]
8.4.2 Momentum only in $\xi$ direction:

If there is no momentum in transverse directions we will have

$$n = \frac{1}{\pi} \int_{-\infty}^{\infty} d\xi e^{a\xi} \sqrt{e^{-2a\xi} E^2} = \frac{1}{\pi} \int_{-\infty}^{\infty} d\xi E = \frac{E}{\pi} L$$

where $L$ is the length of all space on the $\xi$ axis. If we had a partition, the length factor would be shorter accordingly, but the momentum still isn’t a function of $\xi$. Thus we note that: **for divergence at the boundary you need transverse momenta as well.**
Chapter 9

Variations of entropy

9.1 Variation of Wald’s entropy

In [26] the authors calculated the variation of Wald’s entropy. They write the variation of energy as

$$\delta Q = \int_{\mathcal{H}} T_{ab} \chi^a \epsilon^d$$  \hspace{1cm} (9.1.1)

where $\chi^a$ is a Killing vector and $\epsilon^d$ a $(D - 1)$ volume form. (See also [25]). The generalized gravitational field equations give

$$T_{ab} = 2 \left[ -2 \nabla_p \nabla_q \frac{\partial \mathcal{L}}{\partial R_{pabq}} + \frac{\partial \mathcal{L}}{\partial R_{pqra}} R_{b}^{\quad pqra} \right] - g_{ab} \mathcal{L}.$$ \hspace{1cm} (9.1.2)

This term for $T_{ab}$ is then substituted into $\delta Q$, thus writing it as a function of the Riemann tensor and its derivatives.

Wald entropy is

$$S_W = -\frac{1}{T} \oint_{\mathcal{H}} \frac{\partial \mathcal{L}}{\partial R_{abcd}} \hat{\epsilon}_{ab} \epsilon_{cd}$$ \hspace{1cm} (9.1.3)

where $\epsilon_{cd}$ is a $(D - 2)$ volume form and $\hat{\epsilon}_{ab}$ is the binormal vector to the area element. Since any $W^{cd}$ satisfies $d \left( W^{cd} \epsilon_{cd} \right) = 2 \nabla_c W^{cd} \epsilon_d$ this can be written

$$S_W = -\frac{2}{T} \int_{\mathcal{H}} \nabla_c \left( \frac{\partial \mathcal{L}}{\partial R_{abcd}} \hat{\epsilon}_{ab} \right) \epsilon_d$$ \hspace{1cm} (9.1.4)
and then differentiated along a Killing vector to obtain $\delta S$

$$
\delta S_W = -\frac{2}{T} \int_\mathcal{H} \chi_m \nabla^m \nabla_c \left( \frac{\partial \mathcal{L}}{\partial R_{abcd}} \hat{\epsilon}_{ab} \right) \epsilon_d. 
$$

(9.1.5)

Using $T = \frac{\kappa}{2\pi}$ they find that this fulfills the first law of thermodynamics, $\delta Q = T \delta S$ and thus they show that

$$
\delta S_W = 2\pi \int T_{ab} \chi^a \epsilon^b.
$$

(9.1.6)
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