Least squares estimator of fractional Ornstein Uhlenbeck processes with periodic mean

Salwa Bajja, Khalifa Es-Sebaiy and Lauri Viitasaari
Cadi Ayyad University and Aalto University

Abstract

We first study the drift parameter estimation of the fractional Ornstein-Uhlenbeck process (fOU) with periodic mean for every $\frac{1}{2} < H < 1$. More precisely, we extend the consistency proved in [6] for $\frac{1}{2} < H < \frac{3}{4}$ to the strong consistency for any $\frac{1}{2} < H < 1$ on the one hand, and on the other, we also discuss the asymptotic normality given in [6]. In the second main part of the paper, we study the strong consistency and the asymptotic normality of the fOU of the second kind with periodic mean for any $\frac{1}{2} < H < 1$.

Keywords: Fractional Ornstein-Uhlenbeck processes; Least squares estimator; Malliavin calculus.

1 Introduction

Consider the fractional Ornstein-Uhlenbeck process (fOU) $X = \{X_t, t \geq 0\}$ given by the following linear stochastic differential equation

$$dX_t = -\alpha X_t dt + dB_t^H, \quad X_0 = 0,$$

where $\alpha$ is an unknown parameter, and $B_t^H = \{B_t^H, t \geq 0\}$ is a fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$.

The drift parameter estimation problem for the fOU $X$ observed in continuous time and discrete time has been studied by using several approaches (see [15, 12, 13, 4, 9, 10]). In a general case when the process $X$ is driven by a Gaussian process, [7] studied the non-ergodic case corresponding to $\alpha < 0$. They provided sufficient conditions, based on the properties of the driving Gaussian process, to ensure that least squares estimators-type of $\alpha$ are strongly consistent and asymptotically Cauchy. On the other hand, using Malliavin-calculus advances (see [17], [11]) provided new techniques to statistical inference for stochastic differential equations related to stationary Gaussian processes, and they used their result to study drift parameter estimation problems for some stochastic differential

---

1National School of Applied Sciences - Marrakesh, Cadi Ayyad University, Marrakesh, Morocco. Email: salwa.bajja@gmail.com
2National School of Applied Sciences - Marrakesh, Cadi Ayyad University, Marrakesh, Morocco. Email: k.essebaiy@uca.ma
3Department of Mathematics and System Analysis, Aalto University School of Science, P.O. Box 11100, FIN-00076 Aalto, Finland. E-mail:lauri.viitasaari@aalto.fi
equations driven by fractional Brownian motion with fixed-time-step observations (in particular for the fOU \(X\) given in (1) with \(\alpha > 0\)). Similarly, in [21] the authors studied an ergodicity estimator for the parameter \(\alpha\) in (1), where the fractional Brownian motion is replaced with a general Gaussian process having stationary increments.

Recently, [6] studied a drift parameter estimation problem for the above equation (1) with slight modifications on the drift. More precisely, they considered the following fractional Ornstein-Uhlenbeck process with periodic mean function

\[
dX_t = \left(\sum_{i=1}^{p} \mu_i \varphi_i(t) - \alpha X_t\right) dt + dB_t^H, \quad X_0 = 0
\]  

where \(B^H\) is a fBm with Hurst parameter \(\frac{1}{2} < H < \frac{3}{4}\), the functions \(\varphi_i, i = 1, \ldots, p\) are bounded by a constant \(C > 0\) and periodic with the same period \(\nu > 0\), and the real numbers \(\mu_i, i = 1, \ldots, p\) together with \(\alpha > 0\) are considered unknown parameters.

The motivation comes from the fact that such equation can be used to model time series which are a combination of a stationary process and periodicities. In [6] the authors proposed the least squares estimator (LSE) to estimate \(\theta := (\mu_1, \ldots, \mu_p, \alpha)\) based on the continuous-time observations \(\{X_t, 0 \leq t \leq n\nu\}\) as \(n \to \infty\). For the sake of simplicity, we assume that the functions \(\varphi_i, i = 1, \ldots, p\) are orthonormal in \(L^2([0, \nu], \nu^{-1} dt)\), i.e. \(\int_0^\nu \varphi_i(t)\varphi_j(t)\nu^{-1} dt = \delta_{ij}\). We also choose \(\nu = 1\).

Let us consider the LSE \(\hat{\theta}_n\) of \(\theta\) given in [6] by

\[
\hat{\theta}_n := Q_n^{-1} P_n
\]  

where

\[
P_n := \left(\int_0^n \varphi_1(t) dX_t, \ldots, \int_0^n \varphi_p(t) dX_t, -\int_0^n X_t dX_t\right)^\top, \quad Q_n = \begin{pmatrix} G_n & -a_n \\ -a_n^\top & b_n \end{pmatrix}
\]

with

\[
G_n := \left(\int_0^n \varphi_i(t)\varphi_j(t) dt\right)_{1 \leq i, j \leq p};
\]

\[
a_n^\top := \left(\int_0^n \varphi_1(t) X_t dt, \ldots, \int_0^n \varphi_p(t) X_t dt\right); \quad b_n := \int_0^n X_t^2 dt.
\]

Let us describe what is known about the asymptotic behavior of \(\hat{\theta}_n\): if \(\frac{1}{2} < H < \frac{3}{4}\), then

- as \(n \to \infty\),

\[
\hat{\theta}_n \longrightarrow \theta,
\]  

in probability, see [6] Theorem 1;

- as \(n \to \infty\),

\[
n^{1-H} (\hat{\theta}_n - \theta) \text{ converges in law to a normal distribution,}
\]  

see [6] Theorem 2.
In the first main part of our paper we extend the convergence in probability (4) proved when \( \frac{1}{2} < H < \frac{3}{4} \) to the almost sure convergence for every \( \frac{1}{2} < H < 1 \). More precisely, we establish the strong consistency for the LSE \( \hat{\theta}_n \) for every \( \frac{1}{2} < H < 1 \). On the other hand, in Theorem 3 we correct the covariance matrix of the normal limit distribution given in [6, Theorem 1] because the proof of [6, Proposition 5.1] relies on a possibly flawed technique in line -2 page 13.

Our second main interest in this paper is to estimate the drift parameters of the fractional Ornstein Uhlenbeck process of the second kind with periodic mean, that is the solution of the following equation

\[
dX^{(1)}_t = \left( \sum_{i=1}^{p} \mu_i \varphi_i(t) - \alpha X^{(1)}_t \right) dt + dY^{(1)}_t, \quad X_0 = 0
\]

where \( Y^{(1)}_t := \int_{0}^{t} e^{-s} dB_{a_t}^H \) with \( a_t = H e^{t \hat{\pi}} \) and \( B^H \) is a fBm with Hurst parameter \( \frac{1}{2} < H < 1 \). The parameter estimation for the fOU of the second kind without periodicities is well studied in several recent papers (see [11, 2, 3, 8]). Let \( \tilde{\theta}_n \) be the LSE of \( \theta \) defined by

\[
\tilde{\theta}_n = \tilde{Q}_n^{-1} \tilde{P}_n
\]

where

\[
\tilde{P}_n := \left( \int_{0}^{n} \varphi_1(t) dX^{(1)}_t, \ldots, \int_{0}^{n} \varphi_p(t) dX^{(1)}_t, - \int_{0}^{n} X^{(1)}_t dX^{(1)}_t \right) \top; \quad \tilde{Q}_n = \begin{pmatrix} G_n & -\tilde{a}_n \\ -\tilde{a}_n \top & \tilde{b}_n \end{pmatrix}
\]

with \( G_n \) is given as in above, and

\[
\tilde{a}_n \top := \left( \int_{0}^{n} \varphi_1(t) X^{(1)}_t dt, \ldots, \int_{0}^{n} \varphi_p(t) X^{(1)}_t dt \right); \quad \tilde{b}_n := \int_{0}^{n} (X^{(1)}_t)^2 dt.
\]

Let us now describe the results we establish for the asymptotic behavior of the LSE \( \tilde{\theta}_n \): if \( \frac{1}{2} < H < 1 \), then

- as \( n \to \infty \),
  \[
  \tilde{\theta}_n \to \theta,
  \]
  almost surely, see Theorem 5

- as \( n \to \infty \),
  \[
  \sqrt{n}(\tilde{\theta}_n - \theta) \xrightarrow{law} \mathcal{N}(0, M \top \Sigma M)
  \]
  see Theorem 8

Our article is structured as follows. In section 2, we establish the strong consistency for the LSE \( \hat{\theta}_n \) for every \( \frac{1}{2} < H < 1 \). Moreover, we discuss the asymptotic normality given in [6]. Section 3 is devoted to study the strong consistency and the asymptotic normality for the LSE \( \tilde{\theta}_n \) for any \( \frac{1}{2} < H < 1 \). Finally, some basic elements of Malliavin calculus with respect to fBm which are helpful for some of the arguments we use, and some of the technical results used in various proofs are in the Appendix.
2 LSE for fOU with periodic mean

From (3) and (2) we can write (see [6] for details)

\[ \hat{\theta}_n = \theta + Q_n^{-1} R_n \]  

(8)

with an explicit expression of the matrix \( Q_n^{-1} \)

\[ Q_n^{-1} = \frac{1}{n} \left( \begin{array}{cc} I_p + \gamma_n \Lambda_n \Lambda_n^\top & -\gamma_n \Lambda_n \\ -\gamma_n \Lambda_n^\top & \gamma_n \end{array} \right) \]

and

\[ R_n := \left( \int_0^n \varphi_1(t) dB_t^H, \ldots, \int_0^n \varphi_p(t) dB_t^H, - \int_0^n X_t \delta B_t^H \right)^\top, \]

where

\[ \Lambda_n = (\Lambda_{n,1}, \ldots, \Lambda_{n,p})^\top := \left( \frac{1}{n} \int_0^n \varphi_1(t) X_t dt, \ldots, \frac{1}{n} \int_0^n \varphi_p(t) X_t dt \right)^\top \]

and

\[ \gamma_n := \left( \frac{1}{n} \int_0^n X_t^2 dt - \sum_{i=1}^p \Lambda_{n,i}^2 \right)^{-1}. \]

On the other hand, it is readily checked that we have the following explicit expression for the solution \( X \) of (2)

\[ X_t = h(t) + Z_t, \quad t \geq 0, \]  

(9)

where

\[ h(t) := e^{-\alpha t} \sum_{i=1}^p \mu_i \int_0^t e^{\alpha s} \varphi_i(s) ds, \quad Z_t := e^{-\alpha t} \int_0^t e^{\alpha s} dB_s^H. \]  

(10)

Moreover the process \( Z \) is a fOU, that is solution of the following equation

\[ dZ_t = -\alpha Z_t dt + dB_t^H, \quad Z_0 = 0. \]  

(11)

The following result establishes the strong consistency of the LSE \( \hat{\theta}_n \).

**Theorem 1** Assume that \( \frac{1}{2} < H < 1 \). Then

\[ \hat{\theta}_n \longrightarrow \theta \]

almost surely as \( n \to \infty \).
Proof. Using the decomposition (8) we can write \( \hat{\theta}_n = \theta + (nQ^{-1}_n) \left( \frac{1}{n} R_n \right) \). By combining this with Propositions 2 and 10 below the result follows at once.

**Proposition 2** Assume that \( \frac{1}{2} < H < 1 \). Then, as \( n \to \infty \)

\[
\frac{1}{n} R_n \to 0
\]

almost surely.

**Proof.** Since

\[
\sup_{t \geq 0} |\varphi_i(t)| \leq C < \infty, \quad i = 1, \ldots, p
\]

we have

\[
E \left[ \left( \int_0^n \varphi_i(t) dB_t^H \right)^2 \right] = H(2H - 1) \int_0^n \int_0^n \varphi_i(u) \varphi_i(v) |u - v|^{2H-2} du dv
\]

\[
\leq H(2H - 1) C^2 \int_0^n \int_0^n |u - v|^{2H-2} du dv = C^2 n^{2H}.
\]

Then

\[
\left\| \frac{1}{n} \int_0^n \varphi_i(t) dB_t^H \right\|_{L^2(\Omega)} \leq C n^{H-1}.
\]

Combining this with the fact that \( \int_0^n \varphi_i(t) dB_t^H \) is Gaussian and Lemma 12 in the Appendix, we obtain for every \( i = 1, \ldots, p \)

\[
\frac{1}{n} \int_0^n \varphi_i(t) dB_t^H \to 0
\]

almost surely as \( n \to \infty \).

Let us now compute the limit for the last component of \( \frac{1}{n} R_n \). Using the link between the divergence integral and the path-wise integral we have

\[
\int_0^n X_t \delta B_t^H = \int_0^n X_t dB_t^H - H(2H - 1) \int_0^n \int_0^t D_s X_t(t-s)^{2H-2} ds dt.
\]

By (9) and (11) we can write

\[
\frac{1}{n} \int_0^n X_t dB_t^H = \frac{1}{n} \int_0^n (h(t) + Z_t)(dZ_t + \alpha Z_t dt)
\]

\[
= \frac{1}{n} \int_0^n h(t) dZ_t + \frac{\alpha}{n} \int_0^n h(t) Z_t dt + \frac{\alpha}{n} \int_0^n Z_t^2 dt + \frac{1}{n} \int_0^n Z_t dZ_t
\]

\[
= \frac{Z_n h(n)}{n} - \frac{1}{n} \int_0^n h'(t) Z_t dt + \frac{\alpha}{n} \int_0^n h(t) Z_t dt + \frac{\alpha}{n} \int_0^n Z_t^2 dt + \frac{Z_n^2}{2n}.
\]
Furthermore

\[ h(t) = \tilde{h}(t) - e^{-at}\tilde{h}(0) \]  

and

\[ Z(t) = \tilde{Z}(t) - e^{-at}\tilde{Z}(0) \]  

where

\[ \tilde{h}(t) := e^{-at} \sum_{i=1}^{p} \mu_i \int_{-\infty}^{t} e^{\alpha s} \varphi_i(s) ds \]  

which is periodic with period 1, and

\[ \tilde{Z}_t := e^{-at} \int_{-\infty}^{t} e^{\alpha s} dB_t^H \]  

which is a stationary and ergodic process (see [5]). Then the ergodic theorem implies that, almost surely

\[ \lim_{n \to \infty} \frac{1}{n} \int_{0}^{n} h'(t)Z_t dt = \lim_{n \to \infty} \frac{Z_n h(n)}{n} = \lim_{n \to \infty} \frac{\alpha}{n} \int_{0}^{n} h(t)Z_t dt = \lim_{n \to \infty} \frac{Z_n^2}{2n} = 0; \]

\[ \lim_{n \to \infty} \frac{\alpha}{n} \int_{0}^{n} Z_t^2 dt = \alpha^{1-2H} \Gamma(2H). \]

Thus, almost surely

\[ \lim_{n \to \infty} \frac{1}{n} \int_{0}^{n} X_t dB_t^H = \alpha^{1-2H} \Gamma(2H). \]

Combining this with (13) and

\[ \lim_{n \to \infty} \frac{H(2H-1)}{n} \int_{0}^{n} \int_{0}^{t} D_sX_t(t-s)^{2H-2} ds dt = \alpha^{1-2H} \Gamma(2H) \]

we deduce that, almost surely

\[ \lim_{n \to \infty} \frac{1}{n} \int_{0}^{n} X_t dB_t^H = 0 \]

which completes the proof. ■

Let us now discuss the asymptotic normality of the LSE $\hat{\theta}_n$ of $\theta$.  

6
Theorem 3 Assume that $1/2 < H < 3/4$. Then

$$n^{1-H}(\hat{\theta}_n - \theta) \xrightarrow{law} \mathcal{N}(0, M^\top \Sigma M) \quad (17)$$

where the matrix $M$ is defined in Proposition 10 and

$$\Sigma := \left( \begin{array}{cc} G & -a \\ -a^\top & b \end{array} \right)$$

with

$$a^\top := \left( H(2H - 1) \int_0^1 \int_0^1 \varphi_i(u) \tilde{h}(v) |v - u|^{2H-2} dudv \right)_{1 \leq i \leq p};$$

$$G := \left( \int_0^1 \int_0^1 \varphi_i(u) \varphi_j(v) dudv \right)_{1 \leq i, j \leq p};$$

$$b := H(2H - 1) \int_0^1 \int_0^1 \tilde{h}(u) \tilde{h}(v) |v - u|^{2H-2} dudv.$$

Proof. From (5) we have

$$n^{1-H} \left( \hat{\theta}_n - \theta \right) = \left( nQ_n^{-1} \right) \left( n^{-H} R_n \right).$$

From Proposition 10 we have $nQ_n^{-1} \to M$ almost surely. Then, to prove (17) it is sufficient to show that, as $n \to \infty$

$$n^{-H} R_n = \left( n^{-H} \int_0^n \varphi_1(t) dB_t^H, \ldots, n^{-H} \int_0^n \varphi_p(t) dB_t^H, -n^{-H} \int_0^n X_t \delta B_t^H \right) \xrightarrow{law} \mathcal{N}(0, \Sigma).$$

According to (9)

$$n^{-H} \int_0^n X_t \delta B_t^H = n^{-H} \int_0^n Z_t \delta B_t^H + n^{-H} \int_0^n h(t) dB_t^H.$$

Moreover, it follows from [12] that if $1/2 < H < 3/4$, $n^{-1} E \left[ \left( \int_0^n Z_t dB_t^H \right)^2 \right]$ converges to a positive constant as $n \to \infty$. This implies that, as $n \to \infty$

$$E \left[ \left( n^{-H} \int_0^n Z_t dB_t^H \right)^2 \right] \longrightarrow 0.$$

It is also clear that for every $1 \leq i \leq p$

$$E \left[ \left( \int_0^n \varphi_i(t) dB_t^H \right) \left( \int_0^n Z_t \delta B_t^H \right) \right] = 0.$$
Indeed, this follows from the fact that the first integral can be viewed as an element in the first Wiener chaos and the second integral as an element in the second Wiener chaos. Hence it remains to check
\[
\left( n^{-H} \int_0^n \varphi_1(t) dB_t^H, \ldots, n^{-H} \int_0^n \varphi_p(t) dB_t^H, -n^{-H} \int_0^n h(t) dB_t^H \right) \xrightarrow{\text{law}} \mathcal{N}(0, \Sigma).
\]
By using (14) and the fact that the functions $\tilde{h}, \varphi_i, i = 1, \ldots, p$ are periodic with period 1 it is enough to prove that if $f_k, k = 1, \ldots, q$ are periodic real valued functions with period 1, then for every $H > 1/2$ we have, as $n \to \infty$
\[
\left( n^{-H} \int_0^n f_k(t) dB_t^H \right)_{1 \leq k \leq q} \xrightarrow{\text{law}} \mathcal{N} \left( 0, \left( \int_0^1 \int_0^1 f_k(x) f_l(y) dxdy \right)_{1 \leq k,l \leq q} \right).
\]
Because the left-hand side is a Gaussian vector it is sufficient to check the convergence of its covariance matrix. Since the functions $f_k, k = 1, \ldots, q$ are periodic with period 1, we have for every $1 \leq k, l \leq q, i \geq 1$
\[
E \left[ \int_{i-1}^i f_k(t) dB_t^H \int_{i-1}^i f_l(t) dB_t^H \right] = H(2H-1) \int_0^1 \int_0^1 f_k(t) f_l(t) |t-s|^{2H-2} dtds.
\]
Hence, for every $1 \leq k, l \leq q$
\[
E \left( \int_0^n f_k(s) dB_s^H \int_0^n f_l(t) dB_t^H \right) = \sum_{i,j=1}^n E \left( \int_{i-1}^i f_k(s) dB_s^H \int_{j-1}^j f_l(t) dB_t^H \right)
= H(2H-1) \left[ n \int_0^1 \int_0^1 f_k(x) f_l(y) |y-x|^{2H-2} dxdy 
+ \sum_{i \neq j=1}^n \int_0^1 \int_0^1 f_k(x) f_l(y) |j - i + y - x|^{2H-2} dxdy \right].
\]
Furthermore, for every $x, y \in [0, 1]$
\[
\sum_{i<j=1}^n |j - i + y - x|^{2H-2} = \sum_{i<j=1}^n (j - i + y - x)^{2H-2}
= \sum_{i=1}^{n-1} \sum_{j=i+1}^n (j - i + y - x)^{2H-2}
= \sum_{m=1}^{n-1} (n - m)(m + y - x)^{2H-2}.
\]
We have $(m + y - x)^{2H-2} \sim m^{2H-2}$, and $m (m + y - x)^{2H-2} \sim m^{2H-1}$. Hence, since $H > \frac{1}{2}$, we get
\[
n \sum_{m=1}^{n-1} (m + y - x)^{2H-2} \sim \frac{n^{2H}}{2H-1}; \quad \sum_{m=1}^{n-1} m (m + y - x)^{2H-2} \sim \frac{n^{2H}}{2H}.
\]
This implies that, as $n \to \infty$
\[
n^{-2H} \sum_{i<j=1}^{n} |j - i + y - x|^{2H-2} \longrightarrow \frac{1}{2H - 1} - \frac{1}{2H} = \frac{1}{2H(2H - 1)}.
\]
Similarly,
\[
n^{-2H} \sum_{j>i=1}^{n} |j - i + y - x|^{2H-2} \longrightarrow \frac{1}{2H(2H - 1)}.
\]
As a consequence, as $n \to \infty$
\[
n^{-2H} \mathbb{E} \left( \int_{0}^{n} f_{k}(s)dB^{H}_{s} \int_{0}^{n} f_{l}(t)dB^{H}_{t} \right) \longrightarrow \int_{0}^{1} \int_{0}^{1} f_{k}(x)f_{l}(y)dxdy
\]
which implies the desired result. □  

**Remark 4** It seems challenging to obtain the limiting behaviour of our estimator in the case $H \geq \frac{3}{4}$, and the same phenomena is present even in the case of fOU-process without periodicities (see, e.g. [12, 21]). On the other hand, this is in analogue with the quadratic variations of the fractional Brownian motion in which case the limit distribution is not normal in the case $H > \frac{3}{4}$.

### 3 LSE for fOU of second kind with periodic mean

From (6) and (7) we can write
\[
\tilde{\theta}_{n} = \theta + \tilde{Q}_{n}^{-1} \tilde{R}_{n},
\]
where
\[
\tilde{R}_{n} := \left( \int_{0}^{n} \varphi_{1}(t) dY_{t}^{(1)}, \ldots, \int_{0}^{n} \varphi_{p}(t) dY_{t}^{(1)}, - \int_{0}^{n} X_{t}^{(1)} \delta Y_{t}^{(1)} \right)^{\top},
\]
and
\[
\tilde{Q}_{n}^{-1} = \frac{1}{n} \left( I_{p} + \eta_{n} \Gamma_{n} \Gamma_{n}^{\top} - \eta_{n} \Gamma_{n} \right)
\]
with
\[
\Gamma_{n} = (\Gamma_{n,1}, \ldots, \Gamma_{n,p})^{\top} := \left( \frac{1}{n} \int_{0}^{n} \varphi_{1}(t)X_{t}^{(1)} dt, \ldots, \frac{1}{n} \int_{0}^{n} \varphi_{p}(t)X_{t}^{(1)} dt \right)^{\top}
\]
and
\[
\eta_{n} := \left( \frac{1}{n} \int_{0}^{n} (X_{t}^{(1)})^{2} dt - \sum_{i=1}^{p} \Gamma_{n,i}^{2} \right)^{-1}.
\]
Theorem 5 Assume that $1/2 < H < 1$. Then

$$\tilde{\theta}_n \rightarrow \theta$$

(19)

almost surely as $n \rightarrow \infty$.

Proof. By (18) we have $\tilde{\theta}_n - \theta = \left( n\tilde{Q}_n^{-1} \right) \left( \frac{1}{n} \tilde{R}_n \right)$. Thus the convergence (19) is a direct consequence of Propositions 6 and 7 below. ■

Proposition 6 Assume that $\frac{1}{2} < H < 1$. Then the sequence $\frac{1}{n} \tilde{R}_n$ almost surely to 0 as $n \rightarrow \infty$.

Proof. Applying Proposition 13 and (12), we have for every $i = 1, \ldots, p$

$$E \left[ \left( \int_0^n \varphi_i(t) dY_t^{(1)} \right)^2 \right] \leq C^2 \int_0^n \int_0^n r_H(u,v) du dv$$

$$= C^2 E[(Y_n^{(1)})^2].$$

Thanks to [5, 8],

$$E \left[ (Y_n^{(1)})^2 \right] = O(n) \quad \text{as } n \rightarrow \infty. \quad \text{(20)}$$

Thus

$$\left\| \frac{1}{n} \int_0^n \varphi_i(t) dY_t^{(1)} \right\|_{L^2(\Omega)} = O(n^{-1/2}).$$

Hence we can apply Lemma 12 to obtain, as $n \rightarrow \infty$

$$\frac{1}{n} \int_0^n \varphi_i(t) dY_t^{(1)} \rightarrow 0$$

(21)

almost surely for every $i = 1, \ldots, p$.

In order to compute the variance of the last component of $\frac{1}{n} \tilde{R}_n$, observe that we may write the solution of (6) as follows

$$X_t^{(1)} = h(t) + Z_t^{(1)}$$

(22)

where the function $h$ is defined in (10), and

$$Z_t^{(1)} := e^{-\alpha t} \int_0^t e^{\alpha s} dY_s^{(1)}.$$ 

(23)

Hence

$$\int_0^n X_t^{(1)} \delta Y_t^{(1)} = \int_0^n h(t) dY_t^{(1)} + \int_0^n Z_t^{(1)} \delta Y_t^{(1)}.$$
As in (21),
\[ \frac{1}{n} \int_0^n h(t) dY_t^{(1)} \rightarrow 0 \]
almost surely as \( n \to \infty \).

From [2, Lemma 3.1] we have
\[ \int_0^n Z_t^{(1)} \delta Y_t^{(1)} \xrightarrow{law} \int_0^n \tilde{Z}_t^{(1)} \delta \tilde{G}_t \]  
(24)
where the processes \( \tilde{Z}^{(1)} \) and \( \tilde{G} \) are well defined in (34) and (35) respectively. Moreover, it follows from [2, Theorem 3.2] that there exists a positive constant \( \lambda(\theta, H) > 0 \) such that, as \( n \to \infty \)
\[ E \left[ \left( \frac{1}{\sqrt{n}} \int_0^n \tilde{Z}_t^{(1)} \delta \tilde{G}_t \right)^2 \right] \rightarrow \lambda(\theta, H). \]  
(25)
Combining (24), (25), (33) and Lemma 12 we conclude that, almost surely
\[ \lim_{n \to \infty} \frac{1}{n} \int_0^n Z_t^{(1)} \delta Y_t^{(1)} = 0 \]
which finishes the proof. \[ \blacksquare \]

**Proposition 7** Assume that \( \frac{1}{2} < H < 1 \). Then, as \( n \to \infty \)
\[ n \tilde{Q}_n^{-1} \rightarrow \bar{M} := \begin{pmatrix} I_p + \eta \Gamma \Gamma^t & -\eta \Gamma \\ -\eta \Gamma^t & \eta \end{pmatrix}, \]  
(26)
almost surely, where
\[ \Gamma = (\Gamma_1, \ldots, \Gamma_p)^t := \left( \int_0^1 \varphi_1(t) \tilde{h}(t) dt, \ldots, \int_0^1 \varphi_p(t) \tilde{h}(t) dt \right)^t; \]
\[ \eta := \left( \int_0^1 \tilde{h}^2(t) dt + \frac{(2H - 1)H^{2H}}{\alpha} (\beta ((\alpha - 1)H + 1, 2H - 1) - \sum_{i=1}^p \Lambda_i^2) \right)^{-1}, \]
with \( \tilde{h} \) is given in (10).

**Proof.** Define
\[ \tilde{X}_t^{(1)} = \tilde{h}(t) + \tilde{Z}_t^{(1)} \]
where
\[ \tilde{Z}_t^{(1)} = e^{\alpha t} \int_{-\infty}^t e^{(\alpha - 1)s} dB_s^{(1)} = Z_t^{(1)} + e^{\alpha t} \tilde{Z}_0^{(1)}. \]
Since the process $\bar{Z}^{(1)}$ is ergodic (see [14]), as $n \to \infty$

$$\frac{1}{n} \int_0^n (\bar{Z}^{(1)}_t)^2 dt \to E(\bar{Z}^{(1)}_0)^2$$

almost surely. Hence

$$\frac{1}{n} \int_0^n (Z^{(1)}_t)^2 dt \to E(\bar{Z}^{(1)}_0)^2.$$  \hspace{1cm} (27)

almost surely as $n \to \infty$. Moreover

$$E(\bar{Z}^{(1)}_0)^2 = H^{-2(\alpha-1)H} E\left(\int_0^{a_0} s^{(\alpha-1)H} dB_s\right)^2 = H^{-2(\alpha-1)H} H(2H-1) \int_0^{a_0} \int_0^{a_0} s^{(\alpha-1)H} t^{(\alpha-1)H} |s-t|^{2H-2} ds dt = \frac{(2H-1)H^{2H}}{\alpha} \beta((\alpha-1)H + 1, 2H - 1).$$

Thus,

$$\lim_{n \to \infty} \frac{1}{n} \int_0^n (X^{(1)}_t)^2 dt = \lim_{n \to \infty} \frac{1}{n} \int_0^n (\bar{X}^{(1)}_t)^2 dt = \int_0^1 \tilde{h}(t)^2 dt + \frac{(2H-1)H^{2H}}{\alpha} \beta((\alpha-1)H + 1, 2H - 1),$$

On the other hand, we also have

$$\lim_{n \to \infty} \Gamma_{n,i} = \lim_{n \to \infty} \frac{1}{n} \int_0^n X^{(1)}_t \varphi_i(t) dt = \lim_{n \to \infty} \frac{1}{n} \int_0^n \bar{X}^{(1)}_t \varphi_i(t) dt$$

$$= E \left[ \int_0^1 X^{(1)}_t \varphi_i(t) dt \right]$$

$$= \int_0^1 \tilde{h}(t) \varphi_i(t) dt + \int_0^1 E \left[ \int_{-\infty}^t \varphi_i(t)e^{-(t-s)} dY^{(1)}_s \right] dt,$$

$$= \int_0^1 \tilde{h}(t) \varphi_i(t) dt$$

Furthermore,

$$\lim_{n \to \infty} \eta_n = \lim_{n \to \infty} \left( \frac{1}{n} \int_0^n (X^{(1)}_t)^2 dt - \sum_{i=1}^p \Gamma_{n,i}^2 \right)$$

$$= \left( \int_0^1 \tilde{h}(t)^2 dt + \frac{(2H-1)H^{2H}}{\alpha} B((\alpha-1)H + 1, 2H - 1) - \sum_{i=1}^p \Gamma_i^2 \right)^{-1}$$

$$= \eta.$$
Using the fact that the functions \( \varphi_i; \ i = 1, \ldots, p \) are orthonormal in \( L^2[0, 1] \) and the Bessel inequality we get
\[
\sum_{i=1}^{p} \Gamma_i^2 = \sum_{i=1}^{p} \left( \int_0^1 \varphi_i(t) \tilde{h}(t) dt \right)^2 \leq \int_0^1 \tilde{h}^2(t) dt.
\]
This implies that the limit \( \eta \) is well defined and finite, which completes the proof.

Let us now study the asymptotic normality of the LSE \( \tilde{\theta}_n \) of \( \theta \).

**Theorem 8** Assume that \( H \in (1/2, 1) \). Then, as \( n \to \infty \)
\[
\sqrt{n}(\tilde{\theta}_n - \theta) \xrightarrow{\text{law}} \mathcal{N}(0, \overline{M}^T \Sigma \overline{M}) \tag{28}
\]
where the matrix \( \overline{M} \) is defined in Proposition 7 and
\[
\Sigma := \begin{pmatrix}
\overline{G} & -\overline{a} \\
-\overline{a}^T & \overline{b}
\end{pmatrix}
\]
with
\[
\overline{a}^T := \left( \int_0^1 \int_0^1 \varphi_i(x) \tilde{h}(y) \sum_{m \in \mathbb{Z}} r_H(x, y + m) dx dy \right)_{1 \leq i \leq p};
\]
\[
\overline{G} := \left( \int_0^1 \int_0^1 \varphi_i(x) \varphi_j(y) \sum_{m \in \mathbb{Z}} r_H(x, y + m) dx dy \right)_{1 \leq i, j \leq p};
\]
\[
\overline{b} := \int_0^1 \int_0^1 \tilde{h}(x) \tilde{h}(y) \sum_{m \in \mathbb{Z}} r_H(x, y + m) dx dy + \sigma^2.
\]

**Proof.** We can write
\[
\sqrt{n}(\tilde{\theta}_n - \theta) = \left( n \tilde{Q}_n^{-1} \right) \left( \frac{1}{\sqrt{n}} \tilde{R}_n \right)
\]
From (26) we have \( n \tilde{Q}_n^{-1} \to \overline{M} \) almost surely as \( n \to \infty \). Then, to prove (28) it is sufficient to show that, as \( n \to \infty \)
\[
\frac{1}{\sqrt{n}} \tilde{R}_n \xrightarrow{\text{law}} \mathcal{N}(0, \Sigma).
\]
Hence by using the main results of [20] and [19] together with the fact that \( \frac{1}{\sqrt{n}} \tilde{R}_n \) is a vector of multiple integrals it is sufficient to check the convergence of the covariance matrix of \( \frac{1}{\sqrt{n}} \tilde{R}_n \) as \( n \to \infty \).
Since \( X^{(1)} \) admits the decomposition [22], and
\[
\frac{1}{\sqrt{n}} \int_0^n Z_t^{(1)} \delta Y_t^{(1)} \xrightarrow{\text{law}} \mathcal{N}(0, \sigma^2) \quad (\text{see [2]}),
\]
and for every $1 \leq i \leq p$
\[
E\left(\int_0^n \varphi_i(t)dY_t^{(1)} \int_0^n Z_t^{(1)}\delta Y_t^{(1)}\right) = E\left(\int_0^n h(t)dY_t^{(1)} \int_0^n Z_t^{(1)}\delta Y_t^{(1)}\right) = 0,
\]
it remains to prove that, if $f$ and $g$ are two periodic functions with period 1 then, as $n \to \infty$,
\[
\frac{1}{n}E\left(\int_0^n f(s)dY_s^{(1)} \int_0^n g(t)dY_t^{(1)}\right) \quad \to \quad \int_0^1 \int_0^1 f(x)g(y) \sum_{m \in \mathbb{Z}} r_H(x, y + m) dx dy. \quad (29)
\]

Thanks to (36),
\[
E\left(\int_0^n f(s)dY_s^{(1)} \int_0^n g(t)dY_t^{(1)}\right) = \sum_{i,j=1}^n E\left(\int_{i-1}^i f(s)dY_s^{(1)} \int_{j-1}^j g(t)dY_t^{(1)}\right)
\]
\[
= n \int_0^1 \int_0^1 f(x)g(y)r_H(x, y) dx dy
\]
\[
+ \sum_{i \neq j=1}^n \int_0^1 \int_0^1 f(x)g(y)r_H(x + i, y + j) dx dy.
\]

We also have for every $x, y \in [0, 1]$
\[
\frac{1}{n} \sum_{i<j=1}^n r_H(x + i, y + j) = \frac{1}{n} \sum_{i<j=1}^n r_H(x, y + j - i)
\]
\[
= \frac{1}{n} \sum_{m=1}^{n-1} (n - m)r_H(x, y + m)
\]
\[
= \sum_{m=1}^{n-1} r_H(x, y + m) - \sum_{m=1}^{n-1} mr_H(x, y + m).
\]

Since $r_H(x, y + m) \sim \frac{H(2H - 1)H^{2(H-1)}}{e^{-\left(\frac{1}{H-1}\right)(m+y-x)}}$ as $m \to \infty$, we deduce that for every fixed $x, y \in [0, 1]$
\[
\sum_{m=1}^\infty r_H(x, y + m) < \infty,
\]
and as $n \to \infty$,
\[
\frac{1}{n} \sum_{m=1}^\infty mr_H(x, y + m) \to 0.
\]

Combining these convergences with the fact that $r_H$ is symmetric we conclude (29), which completes the proof.
4 Appendix

In this section, we briefly recall some basic elements of Malliavin calculus with respect to fBm which are helpful for some of the arguments we use. For more details we refer to [1, 17, 18]. We also give here some of the technical results used in various proofs of this paper.

Let \( B^H = \{ B^H_t, t \geq 0 \} \) be a fBm with Hurst parameter \( H \in (0, 1) \) that is a centered Gaussian process with the covariance function

\[
R_H(t, s) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right).
\]

It is well-known that the covariance function \( R_H \) can be represented as

\[
R_H(t, s) = \int_0^{t \wedge s} K_H(t, u)K_H(s, u)du
\]

where, in the case when \( H > \frac{1}{2} \), the kernel \( K_H \) has a explicit expression given by

\[
K_H(t, s) = c_H s^{\frac{1}{2} - H} \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{1}{2}} du, \quad s < t,
\]

where \( c_H = (H - \frac{1}{2}) \left( \frac{2H^2 - H}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)} \right)^{\frac{1}{2}} \).

We denote by \( \mathcal{E} \) the set of step \( \mathbb{R} \)-valued functions on \([0,T]\). Let \( \mathcal{H} \) be the Hilbert space defined as the closure of \( \mathcal{E} \) with respect to the scalar product

\[
\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).
\]

We denote by \( | \cdot |_{\mathcal{H}} \) the associated norm. The mapping \( \mathbf{1}_{[0,t]} \mapsto B_t \) can be extended to an isometry between \( \mathcal{H} \) and the Gaussian space associated with \( B \). We denote this isometry by

\[
\varphi \mapsto B(\varphi) = \int_0^T \varphi(s)dB_s. \tag{30}
\]

When \( H \in (\frac{1}{2}, 1) \), it is well known that the elements of \( \mathcal{H} \) may not be functions but distributions of negative order. It will be more convenient to work with a subspace of \( \mathcal{H} \) which contains only functions. Such a space is the set \( |\mathcal{H}| \) of all measurable functions \( \varphi \) on \([0,T]\) such that

\[
|\varphi|^2_{|\mathcal{H}|} := H(2H-1) \int_0^T \int_0^T |\varphi(u)||\varphi(v)||u-v|^{2H-2}du dv < \infty.
\]

If \( \varphi, \psi \in |\mathcal{H}| \) then

\[
E[B(\varphi)B(\psi)] = H(2H-1) \int_0^T \int_0^T \varphi(u)\psi(v)|u-v|^{2H-2}du dv. \tag{31}
\]
We know that \(|\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}}|\) is a Banach space, but that \(|\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}}|\) is not complete. We have the dense inclusions  

\[ L^2([0, T]) \subset L^\infty([0, T]) \subset |\mathcal{H}| \subset \mathcal{H}. \]

Let us introduce the linear operator \(K^*_H\) between \(\mathcal{E}\) and \(L^2[0, T]\) defined by

\[ (K^*_H \varphi)(s) = \int_s^T \varphi(t) \frac{\partial K_H}{\partial t}(t, s) dt. \]

The operator \(K^*_H\) is an isometry that can be extended to \(\mathcal{H}\). Moreover, the process \(W = \{W_t, t \in [0, T]\}\) given by

\[ W_t := B^H((K^*_H)^{-1}(1_{[0,t]})) \tag{32} \]

is a Brownian motion. In addition, the processes \(B^H\) and \(W\) are related through the integral representation

\[ B^H_t = \int_0^t K_H(t, s) dW_s. \]

Let \(\mathcal{S}\) be the set of all smooth cylindrical random variables, which can be expressed as

\[ F = f(B^H(\phi_1), \ldots, B^H(\phi_n)) \]

where \(n \geq 1\), \(f : \mathbb{R}^n \to \mathbb{R}\) is a \(C^\infty\)-function such that \(f\) and all its derivatives have at most polynomial growth, and \(\phi_i \in \mathcal{H}, i = 1, \ldots, n\). The Malliavin derivative of \(F\) with respect to \(B^H\) is the element of \(L^2(\Omega, \mathcal{H})\) defined by

\[ D_s F = \sum_{i=1}^n \frac{\partial f}{\partial \phi_i}(B^H(\phi_1), \ldots, B^H(\phi_n)) \phi_i(s), \quad s \in [0, T]. \]

In particular \(D_s B^H_t = 1_{[0,t]}(s)\). As usual, \(\mathbb{D}^{1,2}\) denotes the closure of the set of smooth random variables with respect to the norm

\[ \|F\|^2_{1,2} = E[F^2] + E[|DF|^2_{\mathcal{H}}]. \]

Moreover, for any \(F \in \mathbb{D}^{1,2}_W\),

\[ K^*_H DF = D^W F, \]

where \(D^W\) denotes the Malliavin derivative operator with respect to \(W\), and \(\mathbb{D}^{1,2}_W\) the corresponding Hilbert space.

The Skorohod integral with respect to \(B^H\) denoted by \(\delta\) is the adjoint of the derivative operator \(D\). If a random variable \(u \in L^2(\Omega, \mathcal{H})\) belongs to the domain of the Skorohod integral (denoted by \(\text{dom}\delta\)), that is, if it verifies

\[ |E\langle DF, u \rangle_{\mathcal{H}}| \leq c_u \sqrt{E[F^2]} \quad \text{for any } F \in \mathcal{S}, \]

then \(\delta(u)\) is defined by the duality relationship

\[ E[F\delta(u)] = E[\langle DF, u \rangle_{\mathcal{H}}], \]
for every $F \in \mathbb{D}^{1,2}$. In the sequel, when $t \in [0, T]$ and $u \in \text{dom} \delta$, we shall sometimes write $\int_0^t u_s \delta B^H_s$ instead of $\delta(u1_{[0,t]})$. If $g \in \mathcal{H}$, notice moreover that $\int_0^T g_s \delta B^H_s = \delta(g) = B^H(g)$. It is known that the multiple Wiener integrals satisfy a hypercontractivity property, which implies that for any $F$ having the form of a finite sum of multiple integrals, we have
\[
\left( E[|F|^p] \right)^{1/p} \leq c_{p,q} \left( E[|F|^2] \right)^{1/2} \text{ for any } p \geq 2.
\]
(33)

One can also develop a Malliavin calculus for any continuous Gaussian process $G$ of the form (see [1])
\[
G_t = \int_0^t K(t,s) dW_s
\]
where $W$ is a Brownian motion and the kernel $K$ satisfying $\sup_{t \in [0, T]} \int_0^t |K(t,s)|^2 ds < \infty$.

Consider the linear operator $K^*$ from $\mathcal{E}$ to $L^2([0, T])$ defined by
\[
(K^* \varphi)(s) = \varphi(s)K(T,s) + \int_s^T [\varphi(t) - \varphi(s)] K(dt,s).
\]

The Hilbert space $\mathcal{H}_G$ generated by covariance function of the Gaussian process $G$ can be represented as $\mathcal{H}_G = (K^*)^{-1}(L^2([0, T]))$ and $\mathbb{D}^{1,2}(\mathcal{H}_G) = (K^*)^{-1}(\mathbb{D}^{1,2}_W(L^2([0, T])))$. For any $n \geq 1$, let $\mathcal{S}_n$ be the $n$th Wiener chaos of $G$, i.e. the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(G(\varphi)), \varphi \in \mathcal{H}, \|\varphi\|_{\mathcal{H}} = 1\}$, and $H_n$ is the $n$th Hermite polynomial. It is well known that the mapping $I^G_n(\varphi^\otimes n) = n! H_n(G(\varphi))$ provides a linear isometry between the symmetric tensor product $\mathcal{H}^\otimes n$ and subspace $\mathcal{S}_n$. Specifically, for all $f \in \mathcal{H}_G^\otimes p$, $f \in \mathcal{H}_G^\otimes q$ and $p, q \geq 1$, one has
\[
E[I^G_p(f)I^G_q(g)] = \delta_{pq} q! (f, g)_{\mathcal{H}_G^\otimes q}.
\]

We say that the kernel $K$ is regular if for all $s \in [0, T)$, $K(.,s)$ has bounded variation on the interval $(s, T]$, and
\[
\int_0^T |K|(s, T],s)^2 ds < \infty.
\]

For regular kernel $K$, put $K(s^+,s) := K(T,s) - K((s, T], s)$. For any $\varphi \in \mathcal{E}$, define the seminorm
\[
\|\varphi\|_{Kr}^2 = \int_0^T \varphi(s)^2 K(s^+, s)^2 + \int_0^T \left( \int_s^T |\varphi(t)||K|(dt, s) \right)^2 ds.
\]

Denote by $\mathcal{H}_{Kr}$ the completion of $\mathcal{E}$ with respect to seminorm $\|\|_{Kr}$.

The following proposition establishes the relationship between path-wise integral and Skorokhod integral.
Proposition 9 Assume $K$ is a regular kernel with $K(s^+, s) = 0$ and $u$ is a process in $D^{1/2}_G(\mathcal{H}_K)$. Then the process $u$ is Stratonovich integrable with respect to $G$ and
\[
\int_0^T u_t dG_t = \int_0^T u_t \delta G_t + \int_0^T \left( \int_s^T D_s u_t K(dt, s) \right) ds.
\]

Proposition 10 Assume that $\frac{1}{2} < H < 1$. Then, as $n \to \infty$ we obtain that $n Q_n^{-1}$ converges almost surely to $M := \left( \begin{array}{cc} I_p + \gamma \Lambda \Lambda^t & -\gamma \Lambda \\ -\gamma \Lambda^t & \gamma \end{array} \right)$
with
\[
\Lambda = (\Lambda_1, \ldots, \Lambda_p)^t := \left( \int_0^1 \varphi_1(t) \tilde{h}(t) dt, \ldots, \int_0^1 \varphi_p(t) \tilde{h}(t) dt \right)^t
\]
and
\[
\gamma := \left( \int_0^1 \tilde{h}^2(t) dt + \alpha^{-2H} H \Gamma(2H) - \sum_{i=1}^p \Lambda_i^2 \right)^{-1},
\]
where $\tilde{h}(t) := e^{-\alpha t} \sum_{i=1}^p \mu_i \int_{-\infty}^t e^{\alpha s} \varphi_i(s) ds$.

Lemma 11 Let $\tilde{B}_t = B_{t+H} - B_H$ be the shifted fractional Brownian motion. Then there exists a regular Volterra-type kernel $\tilde{L}$, in above sense, so that for the solution of the following stochastic differential equation
\[
d\tilde{Z}_t^{(1)} = -\alpha \tilde{Z}_t^{(1)} dt + d\tilde{G}_t, \quad \tilde{Z}_0^{(1)} = 0,
\]
where the Gaussian process
\[
\tilde{G}_t = \int_0^t \left( K_H(t, s) + \tilde{L}(t, s) \right) d\tilde{W}_s
\]
with $\tilde{W}$ is a Brownian motion as in (32).
In addition, $\{Z_t^{(1)}, t \in [0, T]\} \overset{law}{=} \{\tilde{Z}_t^{(1)}, t \in [0, T]\}$ with $Z^{(1)}$ is given in (23).

We also need the following technical results.

Lemma 12 Let $\gamma > 0$ and $p_0 \in \mathbb{N}$. Moreover let $(Z_n)_{n \in \mathbb{N}}$ be a sequence of random variables. If for every $p \geq p_0$ there exists a constant $c_p > 0$ such that for all $n \in \mathbb{N}$,
\[
(E |Z_n|^p)^{1/p} \leq c_p n^{-\gamma},
\]
then for all $\varepsilon > 0$ there exists a random variable $\eta_\varepsilon$ such that
\[
|Z_n| \leq \eta_\varepsilon n^{-\gamma + \varepsilon} \text{ almost surely}
\]
for all $n \in \mathbb{N}$. Moreover, $E|\eta_\varepsilon|^p < \infty$ for all $p \geq 1$.
Proposition 13  Assume that $\frac{1}{2} < H < 1$. Let $f : [0, \infty) \to \mathbb{R}$ be a function of class $C^1$. Then
\[
\int_s^t f(r) dY_r^{(1)} = \int_{a_s}^{a_t} f(a_u^{-1}) e^{a_u^{-1}} dB_u
\]
where $a_u^{-1} = H \log(u/H)$. Moreover, for every $f, g$ of $C^1$
\[
E \left( \int_s^t f(r) dY_r^{(1)} \int_u^v g(r) dY_r^{(1)} \right) = H(2H - 1) \int_{a_s}^{a_t} \int_{a_u}^{a_v} f(a_x^{-1}) g(a_y^{-1}) e^{a_x^{-1} - a_y^{-1}} |x - y|^{2H-2} dxdy
\]
In particular, we obtain
\[
E \left( (Y_t^{(1)} - Y_s^{(1)})(Y_v^{(1)} - Y_u^{(1)}) \right) = \int_s^t \int_u^v r_H(w, z) dwdz
\]
where $r_H(x, y)$ is a symmetric kernel given by
\[
r_H(w, z) = H(2H - 1) H^{2(H-1)} \frac{e^{-(1-H)(w-z)/H}}{(1 - e^{-(w-z)/H})^{2(1-H)}}.
\]

Proof. Combining integration by parts and change of variable $u = a_r$ we obtain
\[
\int_s^t f(r) dY_r^{(1)} = \int_s^t f(r) e^{-r} dB_a
\]
\[
= f(t) e^{-t} B_{at} - f(s) e^{-s} B_{as} - \int_{s}^{t} \left( f(r) e^{-r} \right)' (r) B_{ar} dr
\]
\[
= f(t) e^{-t} B_{at} - f(s) e^{-s} B_{as} - \int_{a_s}^{a_t} \left( f(a_u^{-1}) e^{-a_u^{-1}} \right)' (u) B_{ar} du
\]
\[
= f(t) e^{-t} B_{at} - f(s) e^{-s} B_{as} - \int_{a_s}^{a_t} f(a_u^{-1}) e^{-a_u^{-1}} dB_u
\]
which proves (36). Using (36) we get
\[
E \left( \int_s^t f(r) dY_r^{(1)} \int_u^v g(r) dY_r^{(1)} \right) = H(2H - 1) \int_{a_s}^{a_t} \int_{a_u}^{a_v} f(a_x^{-1}) g(a_y^{-1}) e^{a_x^{-1} - a_y^{-1}} |x - y|^{2H-2} dxdy
\]
Making now change of variable $w = a_x^{-1}$ and $z = a_y^{-1}$ we obtain (36). □
References

[1] Alos, E., Mazet, O. and Nualart, D. (2001). *Stochastic Calculus with respect to Gaussian processes*. Ann. Probab. 766-801.

[2] Azmoodeh, E. and Morlanes, G. I. (2013). *Drift parameter estimation for fractional Ornstein-Uhlenbeck process of the second kind*. Statistics. DOI: 10.1080/02331888.2013.863888.

[3] Azmoodeh, E. and Viitasaari, L. (2015). *Parameter estimation based on discrete observations of fractional Ornstein-Uhlenbeck process of the second kind*. Statist. Infer. Stoch. Proc. 18, no. 3, 205-227.

[4] Brouste, A. and Iacus, S. M. (2012). *Parameter estimation for the discretely observed fractional Ornstein-Uhlenbeck process and the Yuima R package*. Comput. Stat. 28, no. 4, 1529-1547.

[5] P. Cheridito and H. Kawaguchi and M. Maejima. *Fractional Ornstein-Uhlenbeck processes*. Electronic Journal of Probability, 8, 1-14, 2003.

[6] H. Dehling, B. Franke and J.H.C. Woerner. *Estimating drift parameters in a fractional Ornstein Uhlenbeck process with periodic mean*. Statist. Infer. Stoch. Proc., 1-14, 2016.

[7] El Machkouri, M., Es-Sebaiy, K. and Ouknine, Y. (2015). *Least squares estimator for non-ergodic Ornstein-Uhlenbeck processes driven by Gaussian processes*. Journal of the Korean Statistical Society, DOI: 10.1016/j.jkss.2015.12.001 (In press).

[8] El Onsy, B., Es-Sebaiy, K. and Tudor, C. (2014). *Statistical analysis of the non-ergodic fractional Ornstein-Uhlenbeck process of the second kind*. Preprint.

[9] El Onsy, B., Es-Sebaiy, K. and Viens, F. (2014). *Parameter Estimation for Ornstein-Uhlenbeck driven by fractional Ornstein-Uhlenbeck processes*. Preprint, in revision for Stochastics.

[10] Es-Sebaiy, K., Ndiaye, D. (2014). *On drift estimation for discretely observed non-ergodic fractional Ornstein-Uhlenbeck processes with discrete observations*. Afr.Stat. 9, 615-625.

[11] Es-Sebaiy, K., Viens, F. (2016). *Optimal rates for parameter estimation of stationary Gaussian processes*. Preprint: [http://arxiv.org/pdf/1603.04542.pdf](http://arxiv.org/pdf/1603.04542.pdf)

[12] Hu, Y. and Nualart, D. (2010). *Parameter estimation for fractional Ornstein Uhlenbeck processes*. Statistics and Probability Letters, 80, 1030-1038.

[13] Hu, Y. and Song, J. (2013). *Parameter estimation for fractional Ornstein-Uhlenbeck processes with discrete observations*. F. Viens et al (eds), Malliavin Calculus and Stochastic Analysis: A Festschrift in Honor of David Nualart, 427-442, Springer.
[14] Kaarakka, T. and Salminen, P. (2011). On Fractional Ornstein-Uhlenbeck process. Communications on Stochastic Analysis, 5, 121-133.

[15] Kleptsyna, M. and Le Breton, A. (2002). Statistical analysis of the fractional Ornstein-Uhlenbeck type process. Statist. Infer. Stoch. Proc., 5, 229-241.

[16] Kloeden, P. and Neuenkirch, A. (2007). The pathwise convergence of approximation schemes for stochastic differential equations. LMS J. Comp. Math, 10, 235-253.

[17] Nourdin, I. and Peccati, G. (2012). Normal Approximations with Malliavin calculus: From Stein’s method to Universality. Cambridge Tracts in Mathematics. Cambridge University.

[18] Nualart, D. (2006). The Malliavin calculus and related topics. Springer-Verlag, Berlin, 2nd edition.

[19] Peccati, G. (2007). Gaussian Approximations of Multiple Integrals. Electronic communications in probability, 12, 350-364.

[20] Peccati, G. and Tudor, C.A. (2005). Gaussian limits for vector-valued multiple stochastic integrals. In: Séminaire de Probabilités XXXVIII, 247-262, Springer Verlag.

[21] Sottinen, T. and Viitasaari, L. (2016). Parameter Estimation for the Langevin Equation with Stationary-Increment Gaussian Noise. Preprint, in revision for Statist. Infer. Stoch. Proc.