A remark on the convergence of inverse $\sigma_k$-flow

JIAN XIAO

Abstract

We study the positivity of related cohomology classes concerning the convergence problem of inverse $\sigma_k$-flow in the conjecture proposed by Lejmi and Székelyhidi.

1 Introduction

We aim to study the positivity of related cohomology classes in the following conjecture proposed by Lejmi and Székelyhidi [LS15]. We generalize their conjecture by weakening the numerical condition on $X$ a little bit.

Conjecture 1.1. (see [LS15, Conjecture 18]) Let $X$ be a compact Kähler manifold of dimension $n$, and let $\omega, \alpha$ be two Kähler metrics over $X$ satisfying

$$\int_X \omega^n - \frac{n!}{k!(n-k)!} \omega^{n-k} \wedge \alpha^k \geq 0.$$ (1.1)

Then there exists a Kähler metric $\omega' \in \{\omega\}$ such that

$$\omega'^{n-1} - \frac{(n-1)!}{k!(n-k-1)!} \omega'^{n-k-1} \wedge \alpha^k > 0$$ (1.2)

as a smooth $(n-1,n-1)$-form if and only if

$$\int_V \omega^p - \frac{p!}{k!(p-k)!} \omega^{p-k} \wedge \alpha^k > 0$$ (1.3)

for every irreducible subvariety of dimension $p$ with $k \leq p \leq n-1$.

For the previous works closely related to this conjecture, we refer the reader to [Don99], [Che00, Che04], [SW08] and [FLM11]. And in this note we mainly concentrate on the case when $k = 1$ and $k = n-1$.

For $k = 1$, [CS14, Theorem 3] confirmed this conjecture for toric manifolds. Over a general compact Kähler manifold, it is not hard to see the implication (1.2) $\Rightarrow$ (1.3) holds. In the reverse direction, we prove $\{\omega - \alpha\}$ must be a Kähler class under the numerical conditions in Conjecture 1.1 for $k = 1$; indeed, this is a necessary condition of (1.2) and [LS15, Proposition 14] proved this over Kähler surfaces.
**Theorem 1.1.** Let $X$ be a compact Kähler manifold of dimension $n$, and let $\omega, \alpha$ be two Kähler metrics over $X$ satisfying the numerical conditions in Conjecture 1.1 for $k = 1$. Then \( \{\omega - \alpha\} \) is a Kähler class.

For $k = n - 1$, we have the following similar result.

**Theorem 1.2.** Let $X$ be compact Kähler manifold of dimension $n$, and let $\omega, \alpha$ be two Kähler metrics over $X$ satisfying the numerical conditions in Conjecture 1.1 for $k = n - 1$. Then the class \( \{\omega^{n-1} - \alpha^{n-1}\} \) lies in the closure of the Gauduchon cone, i.e. it has nonnegative intersection number with every pseudoeffective $(1,1)$-class.

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## 2 Proof of the main results

In this section, we give the proofs of Theorem 1.1 and Theorem 1.2.

### 2.1 Theorem 1.1

**Proof.** The first observation is that, when $k = 1$, the inequalities in the numerical conditions are just the right hand side in weak transcendental holomorphic Morse inequalities. Recall that Demailly’s conjecture on weak transcendental holomorphic Morse inequalities (see e.g. [BDPP13, Conjecture 10.1]) is stated as following:

Let $X$ be a compact complex manifold of dimension $n$, and let $\gamma, \beta$ be two nef classes over $X$. Then we have

\[
\text{vol}\ (\gamma - \beta) \geq \gamma^n - n\gamma^{n-1} \cdot \beta.
\]

In particular, $\gamma^n - n\gamma^{n-1} \cdot \beta > 0$ implies the class $\gamma - \beta$ is big, that is, $\gamma - \beta$ contains a Kähler current.

Note that the last statement has been proved for Kähler manifolds [Pop14] (see also [Xia13]), that is, if $X$ is a compact Kähler manifold then $\gamma^n - n\gamma^{n-1} \cdot \beta > 0$ implies there exists a Kähler current in the class $\gamma - \beta$.

We apply this bigness criterion to the classes $\{\omega\}$ and $\{\alpha\}$, then the numerical condition (1.3) implies $\{\omega - \alpha\}_{|_V}$ is a big class on every proper irreducible subvariety $V$. More precisely, if $V$ is singular then by some resolution of singularities we have a proper modification $\pi: \hat{V} \to V$ with $\hat{V}$ smooth, and by (1.3) we know

\[
\pi^*\{\omega\}_{|_V}^p - p\pi^*\{\omega\}_{|_V}^{p-1} \cdot \pi^*\{\alpha\}_{|_V} > 0,
\]

\[
\pi^*\{\omega\}_{|_V}^p - p\pi^*\{\omega\}_{|_V}^{p-1} \cdot \pi^*\{\alpha\}_{|_V} > 0.
\]
thus the class $\pi^*\{\omega - \alpha\}|_V$ contains a Kähler current over $\hat{V}$. So by the push-forward map $\pi_*$ we obtain that the class $\{\omega - \alpha\}|_V$ is big over $V$.

In particular, by (1.1) and (1.3) the restriction of the class $\{\omega - (1 - \epsilon)\alpha\}$ is big on every irreducible subvariety (including $X$ itself) for any sufficiently small $\epsilon > 0$.

We claim this yields $\{\omega - (1 - \epsilon)\alpha\}$ is a Kähler class over $X$ for any $\epsilon > 0$ small. Indeed, our proof implies the following fact.

- Assume $\beta$ is a big class over a compact complex manifold (or compact complex space) and its restriction to every irreducible subvariety is also big, then $\beta$ is a Kähler class over $X$.

To this end, we will argue by induction on the dimension of $X$. If $X$ is a compact complex curve, then this is obvious. For the general case, we need a result of Mihai Păun (see [Pău98b, Pău98a]):

Let $X$ be a compact complex manifold (or compact complex space), and let $\beta = \{T\}$ be the cohomology class of a Kähler current $T$ over $X$. Then $\beta$ is a Kähler class over $X$ if and only if the restriction $\beta|_Z$ is a Kähler class on every irreducible component $Z$ of the Lelong sublevel set $E_c(T)$.

As $\{\omega - (1 - \epsilon)\alpha\}$ is a big class on $X$, by Demailly’s regularization theorem [Dem92] we can choose a Kähler current $T \in \{\omega - (1 - \epsilon)\alpha\}$ such that $T$ has analytic singularities on $X$. Then the singularities of $T$ are just the Lelong sublevel set $E_c(T)$ for some positive constant $c$. For every irreducible component $Z$ of $E_c(T)$, by (1.3) the restriction $\{\omega - (1 - \epsilon)\alpha\}|_Z$ is a big class. After resolution of singularities of $Z$ if necessary, we obtain a Kähler current $T_Z \in \{\omega - (1 - \epsilon)\alpha\}|_Z$ over $Z$ with its analytic singularities contained in a proper subvariety of $Z$, and for every irreducible subvariety $V \subseteq Z$ the restriction $\{\omega - (1 - \epsilon)\alpha\}|_V$ is also a big class. By induction on the dimension, we get that $\{\omega - (1 - \epsilon)\alpha\}|_Z$ is a Kähler class over $Z$. So the above result of [Pău98b, Pău98a] implies $\{\omega - (1 - \epsilon)\alpha\}$ is a Kähler class over $X$, finishing the proof our claim.

By the arbitrariness of $\epsilon > 0$, we get $\{\omega - \alpha\}$ is a nef class on $X$. Next we prove $\{\omega - \alpha\}$ is a big class. By [DP04, Theorem 2.12], we only need to show

$$\text{vol}(\{\omega - \alpha\}) = \int_X (\omega - \alpha)^n > 0.$$ 

Since $\{\omega - \alpha\}$ is nef, we can compute the derivative of the function $\text{vol}(\omega - t\alpha)$ for any $t \in [0, 1)$. Thus we have

$$\text{vol}(\{\omega\} - \{\alpha\}) - \text{vol}(\{\omega\}) = \int_0^1 \frac{d}{dt} \text{vol}(\{\omega\} - t\{\alpha\}) dt$$

$$= - \int_0^1 n(\omega - t\alpha)^{n-1} \cdot \{\alpha\} dt,$$

which implies

$$\text{vol}(\{\omega\} - \{\alpha\}) = \text{vol}(\{\omega\}) - \int_0^1 n(\omega - t\alpha)^{n-1} \cdot \{\alpha\} dt$$

$$\geq \int_0^1 n(\omega)^{n-1} - (\omega - t\alpha)^{n-1} \cdot \{\alpha\} dt.$$
Here the last line follows from the equality (1.1). Since \( \omega, \alpha \) are Kähler metrics, this shows
\[ \text{vol}(\{\omega - \alpha\}) > 0. \]
Thus \( \{\omega - \alpha\} \) is a big and nef class on \( X \) with its restriction to every irreducible subvariety being big and nef. By the arguments before, we know \( \{\omega - \alpha\} \) must be a Kähler class.

Finally, we give an alternative proof of the fact that the class \( \{\omega - \alpha\} \) is nef using the main result of [CT13] instead of using [Pâuf98b, Pâuf98a]. (I would like to thank Tristan C. Collins who pointed out this to me.) Since \( \{\omega\} \) is a Kähler class, the class \( \{\omega - t\alpha\} \) is Kähler for \( t > 0 \) small. Let \( s \) be the largest number such that \( \{\omega - s\alpha\} \) is nef. We prove that \( s \geq 1 \). Otherwise, suppose \( s < 1 \). Then by the numerical conditions (1.1) and (1.3), the bigness criterion given by transcendental holomorphic Morse inequalities implies that the class \( \{\omega - s\alpha\} \) is big if \( s < 1 \), and furthermore, this holds for all irreducible subvarieties in \( X \). Thus \( \{\omega - s\alpha\} \) is big and nef on every irreducible subvariety \( V \) in \( X \). This means the null locus of the big and nef class \( \{\omega - s\alpha\} \) is empty, and then the main result of [CT13] implies that \( \{\omega - s\alpha\} \) is a Kähler class. This contradicts with the definition of \( s \), so we get \( s \geq 1 \), or equivalently, \( \{\omega - \alpha\} \) must be a nef class.

Remark 2.1. If \( X \) is a smooth projective variety of dimension \( n \) and \( \{\omega\} \) and \( \{\alpha\} \) are the first Chern classes of holomorphic line bundles, then the nefness of the class \( \{\omega - \alpha\} \) just follows from Kleiman’s ampleness criterion, since the numerical condition (1.3) for \( p = 1 \) implies the divisor class \( \{\omega - \alpha\} \) has non-negative intersection against every irreducible curve.

2.2 Theorem 1.2

Next we give the proof of Theorem 1.2.

Proof. The proof mainly depends on Boucksom’s divisorial Zariski decomposition for pseudo-effective \((1,1)\)-classes [Bou04] and the bigness criterion for the difference of two movable \((n-1, n-1)\)-classes [Xia14].

Through a sufficiently small perturbation of the Kähler metric \( \alpha \), e.g. replace \( \alpha \) by
\[ \alpha_\epsilon = (1 - \epsilon)\alpha \]
with \( \epsilon \in (0, 1) \), we can obtain that the inequality in (1.1) is strict for the classes \( \{\omega\} \) and \( \{\alpha_\epsilon\} \).

We claim that in this case the \((n-1, n-1)\)-class \( \{\omega^{n-1} - \alpha_\epsilon^{n-1}\} \) has nonnegative intersections with all pseudo-effective \((1,1)\)-classes. Then let \( \epsilon \) tends to zero, we conclude the desired result for the class \( \{\omega^{n-1} - \alpha^{n-1}\} \). Thus we can assume the inequality in (1.1) is strict for the classes \( \{\omega\} \) and \( \{\alpha\} \) at the beginning.

Let \( \beta \) be a pseudoeffective \((1,1)\)-class over \( X \). By [Bou04, Section 3], \( \beta \) admits a divisorial Zariski decomposition
\[ \beta = Z(\beta) + N(\beta). \]
Note that $N(\beta)$ is the class of some effective divisor (may be zero) and $Z(\beta)$ is a modified nef class. In particular, we have
\[
\{\omega^{n-1} - \alpha^{n-1}\} \cdot N(\beta) \geq 0. \tag{2.1}
\]
For any $\delta > 0$, we have
\[
Z(\beta) + \delta\{\omega\} = \pi_*\{\hat{\omega}\}
\]
for some modification $\pi : \hat{X} \to X$ and some Kähler metric $\hat{\omega}$ on $\hat{X}$ (see [Bou04, Proposition 2.3]).

By our assumption on (1.1), we have
\[
\int_{\hat{X}} \pi^*\omega^n - n\pi^*\omega \wedge \pi^*\alpha^{n-1} > 0. \tag{2.2}
\]
By [Xia14, Theorem 3.3] (or [Xia13, Remark 3.1]), the inequality (2.2) implies that the class $\{\pi^*\omega^{n-1} - \pi^*\alpha^{n-1}\}$ contains a strictly positive $(n-1, n-1)$-current. This implies
\[
\{\omega^{n-1} - \alpha^{n-1}\} \cdot (Z(\beta) + \delta\{\omega\}) \\
= \{\omega^{n-1} - \alpha^{n-1}\} \cdot \pi_*\{\hat{\omega}\} \\
= \pi^*\{\omega^{n-1} - \alpha^{n-1}\} \cdot \{\hat{\omega}\} \\
> 0.
\]
By the arbitrariness of $\delta$, we get $\{\omega^{n-1} - \alpha^{n-1}\} \cdot Z(\beta) \geq 0$. With (2.1), we show that
\[
\{\omega^{n-1} - \alpha^{n-1}\} \cdot \beta \geq 0.
\]
Since $\beta$ can be any pseudoeffective $(1,1)$-class, this implies $\{\omega^{n-1} - \alpha^{n-1}\}$ lies in the closure of the Gauduchon cone by [Xia15, Proposition 2.1] (see also [Lam99, Lemma 3.3]).

\begin{remark}
We expect $\{\omega^{n-1} - \alpha^{n-1}\}$ should have strictly positive intersection numbers with nonzero pseudoeffective $(1,1)$-classes. To show this, one only need to verify this for modified nef classes.
\end{remark}

\begin{remark}
Let $X$ be a smooth projective variety, and assume $\{\omega^{n-1} - \alpha^{n-1}\}$ is a curve class. Then the numerical condition (1.3) in Theorem 1.2 implies that $\{\omega^{n-1} - \alpha^{n-1}\}$ is a movable class by [BDPP13, Theorem 2.2].
\end{remark}

3 Further discussions

In analogue with Theorem 1.1 and Theorem 1.2, one would like to prove similar positivity of the class $\{\omega^k - \alpha^k\}$. To generalize our results in this direction, one can apply [Xia13, Remark 3.1]. By [Xia13, Remark 3.1], we know that the condition
\[
\int_V \omega^p - \frac{p!}{k!(p-k)!} \omega^{p-k} \wedge \alpha^k > 0
\]
implies that the class \( \{\omega^k - \alpha^k\}_{|V} \) contains a strictly positive \((k, k)\)-current over every irreducible subvariety \( V \) of dimension \( p \) with \( k < p \leq n - 1 \). However, the difficulties appear as we know little about the singularities of positive \((k, k)\)-currents for \( k > 1 \). We have no analogues of Demailly's regularization theorem for such currents.

Inspired by the prediction of Conjecture 1.1, we propose the following question on the positivity of \((k, k)\)-currents.

**Question 3.1.** Let \( X \) be a compact Kähler manifold (or general compact complex manifold) of dimension \( n \). Let \( \Omega \in H^{k, k}(X, \mathbb{R}) \) be a big \((k, k)\)-class, i.e. it can be represented by a strictly positive \((k, k)\)-current over \( X \). Assume the restriction class \( \Omega_{|V} \) is also big over every irreducible subvariety \( V \) with \( k \leq \dim V \leq n - 1 \), then does \( \Omega \) contain a smooth strictly positive \((k, k)\)-form in its Bott-Chern class? Or does \( \Omega \) contain a strictly positive \((k, k)\)-current with analytic singularities of codimension at least \( n - k + 1 \) in its Bott-Chern class?

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INSTITUTE OF MATHEMATICS, FUDAN UNIVERSITY, 200433 SHANGHAI, CHINA

CURRENT ADDRESS:
INSTITUT FOURIER, UNIVERSITÉ JOSEPH FOURIER, 38402 SAINT-MARTIN D’HÈRES, FRANCE
Email: jian.xiao@ujf-grenoble.fr