Research Article

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Third-order differential subordination and superordination involving a fractional operator

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Abstract: The third-order differential subordination and the corresponding differential superordination problems for a new linear operator convoluted the fractional integral operator with the Carlson-Shaffer operator, are investigated in this study. The new operator satisfies the required first-order differential recurrence (identity) relation. This property employs the subordination and superordination methodology. Some classes of admissible functions are determined, and these significant classes are exploited to obtain fractional differential subordination and superordination results. The new third-order differential sandwich-type outcomes are investigated in subsequent research.

Keywords: Fractional calculus, Fractional integral operator, Subordination, Superordination, Unit disk, Analytic function

MSC: 30C45

1 Introduction

Integral operators defined on various classes of holomorphic functions are vital in studying numerous problems related to geometric function theory. Since the beginning of the previous century, several types of the well-known classical integral operators have been introduced by prominent authors, such as Alexander [1], Libera [2], Bernardi [3], Miller and Mocanu [4, 5], Reade [4], Singh [6], and Pascu and Pescar [7]. Nowadays, new frontiers of integral operators are designed to raise the stimulus for many young researchers who are willing to study this area. Numerous renowned mathematicians have designed these operators after utilizing different methods in their investigation, such as Breaz et al. [8–10] as well as Darus and Ibrahim [11, 12]. The present study investigates some types of well-known integral operators for normalized holomorphic function \( f \) in the unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). In 1915, Alexander [1] introduced the first integral operator in the following form:

\[
A(z) = \int_0^z \frac{f(t)}{t} dt.
\]

In 1965, Libera [2] presented an integral operator, which is defined by the following formula:

\[
L(z) = \frac{2}{z} \int_0^z f(t) dt.
\]
In 1969, Bernardi [3] investigated the more general integral operator in the following form:

\[ B(z) = \frac{1 + y}{z^\gamma} \int_0^z f(t) t^{\gamma - 1} dt. \]

In [5] several studies concerning other types of operators are pointed out. These types are known as the integrals of the first type

\[ F_\alpha(z) = \int_0^z \left( \frac{f(t)}{t} \right)^\alpha dt, \quad \alpha \in \mathbb{C}. \]

In 1978, Miller together with Mocanu and Reade [4] considered a more general operator in the following form:

\[ G(z) = \left[ \frac{\beta + y}{z^\gamma \Phi(z)} \int_0^z f^{\alpha}(t) \phi(t) t^{\delta - 1} dt \right]^{1/\beta}. \]

The present study deals with another type of the fractional integral operators, namely, integrals of the second type defined by (see [13, 14]):

\[ G_\alpha(z) = \int_0^z (f'(t))^\alpha dt, \quad \alpha \in \mathbb{C}. \]

(1)

Significant and interesting problems in the geometric function theory are studied using third-order differential subordination and superordination for functions, which are analytic in the unit disk. In 1935, Goluzin [15] considered the simple first-order differential subordination

\[ z p'h(z) < h(z) \]

and showed that if \( h \) is convex, then \( p(z) < q(z) = \int_0^z h(t)^{-1} dt \), and this \( q \) is the best dominant. In 1970, Suffridge [16, p. 777] improved the Goluzin’s result. In 1947, Robinson [17, p. 22] considered the differential subordination

\[ p(z) < q(z) = z^{-1} \int_0^z h(z) dt \]

are univalent, then \( q \) is the best dominant, at least for \( |z| < 1/5 \). In 1975, Hallenbeck and Rusheiewy [18] considered the differential subordination

\[ p(z) + \frac{z p'(z)}{\gamma} < h(z) \quad (\gamma \neq 0, \text{Re} \gamma \geq 0) \]

and proved that if \( h \) is convex, then \( p(z) < q(z) = \gamma z^{-\gamma} \int_0^z h(t)^{-1} dt \), and this is the best dominant.

The theory of differential subordination in \( \mathbb{C} \) is the complex analogue of differential inequality in \( \mathbb{R} \). This theory of differential subordination was initiated by the works of Miller and Mocanu in 1981 [19], which was developed in other studies in 1987 [20] and 1989 [21]. Many significant works on differential subordination were pioneered by Miller and Mocanu, and their monograph (2000) [22] compiled their considerable efforts in introducing and developing the same. In 2003, Miller and Mocanu [23] investigated the dual problem of differential superordination, whereas Bulboaca (2005) [24] investigated both subordination and superordination. The theory of first and second order differential subordination and superordination has been used by numerous authors to solve problems in this field (see [25–30]). By contrast, few articles mentioned third-order inequalities and subordination. The first authors investigated the third order, and Ponnusamy et al. [31] published in 1992. In 2011, Antonino and Miller [32] extended the theory of second order differential subordination in the open unit disk \( U \) introduced by Miller and Mocanu [22] to the third order case. They determined the properties of \( p \) functions that satisfy the following third-order differential subordination:

\[ \{\psi(p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z) : z \in \mathbb{U} \} \subset \Omega. \]

They also obtained the differential subordination and the corresponding differential superordination implications for meromorphically multivalent functions, which are defined by convolution operators involving the Liu-Srivastava
operator by determining certain classes of admissible functions. In 2014, Tang et al. [35] investigated some third-order differential subordination results for analytic functions involving the generalized Bessel functions. In 2014, Tang et al. [36] studied the differential superordination based on analytic functions involving the generalized Bessel functions. In 2014, Farzana et al. [37] discussed some third-order differential subordination results for analytic functions involving the fractional derivative operator.

The present study utilized the methods of the third-order differential subordination and superordination results of Antonino and Miller [26] and Tang et al. [34], respectively. Certain suitable classes of admissible functions are considered in this study, and some applications of the third-order differential subordination and superordination of analytic functions associated with the new operator are investigated. Several interesting examples are also discussed.

2 Preliminaries

Let \( \mathcal{H}(\mathcal{U}) \) be the class of all holomorphic functions \( f(z) \) which are defined in the unit disk \( \mathcal{U} \). For \( a \in \mathbb{C} \) and \( n \in \mathbb{N} \), let

\[
\mathcal{H}[a, n] = \{ f \in \mathcal{H}(\mathcal{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \}
\]

and also let \( \mathcal{H}_0 = \mathcal{H}[0, 1] \) and \( \mathcal{H}_1 = \mathcal{H}[1, 1] \). Let \( f \) and \( F \) be members in \( \mathcal{H}(\mathcal{U}) \), the function \( f \) is said to be subordinate to \( F \), or \( F \) is superordinate to \( f \), if there is an analytic function \( g(z) \) in \( \mathcal{U} \) with \( g(0) = 0 \) and \( |g(z)| < 1 \) for all \( z \in \mathcal{U} \), such that \( f(z) = F(g(z)) \). In this case, we write \( f \prec F \), or \( f(z) \prec F(z) \). Furthermore, if the function \( F \) is univalent in \( \mathcal{U} \), then \([22]:(\)

\[
f(z) \prec F(z) \text{ (} z \in \mathcal{U} \text{)} \iff f(0) = F(0), \ f(\mathcal{U}) \subset F(\mathcal{U})
\]

Let \( \mathcal{A} \) denote the well-known class of all normalized holomorphic functions \( f(z) \) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathcal{U}).
\]

The integrals of the second type \( G_\alpha(z) \) \((\alpha \in \mathbb{C})\) defined in (1), can be written as follows: for \( f \in \mathcal{A}, z \in \mathcal{U} \)

\[
G_\alpha f(z) = \int_0^z \left[ f'(t) \right]^{\alpha} dt = \int_0^z \left[ 1 + \sum_{n=2}^{\infty} n a_n t^{n-1} \right]^{\alpha} dt = \int_0^z \sum_{k=0}^{\infty} \left( \begin{array}{c} \alpha \\ k \end{array} \right) \left[ \sum_{n=2}^{\infty} n a_n t^{n-1} \right]^k dt
\]

\[
= \int_0^z \left( \begin{array}{c} \alpha \\ 0 \end{array} \right) + \left( \begin{array}{c} \alpha \\ 1 \end{array} \right) \sum_{n=2}^{\infty} n a_n t^{n-1} + \left( \begin{array}{c} \alpha \\ 2 \end{array} \right) \left[ \sum_{n=2}^{\infty} n a_n t^{n-1} \right]^2 + \ldots \right) dt
\]

\[
= \left( \begin{array}{c} \alpha \\ 0 \end{array} \right) z + \sum_{n=2}^{\infty} \left( \begin{array}{c} \alpha \\ 1 \end{array} \right) a_n z^n + \ldots,
\]

where \( \mathcal{A}_\alpha \) is the combination of \( \alpha \) and \( a_n \).

Corresponding to the Carlson-Shaffer operator \( L(\alpha, c)f(z) \) defined by [38]

\[
L(\alpha, c)f(z) = z + \sum_{n=1}^{\infty} \frac{(\alpha)_n}{(c)_n} a_n z^n + \ldots, \quad (z \in \mathcal{U})
\]

and the integrals of the second type \( G_\alpha f(z) \) defined in (1), we consider a linear operator \( T_\alpha \) \((\alpha \in \mathbb{C})\) by

\[
T_\alpha f(z) = L(\alpha, c)(G_\alpha f(z)) = z + \sum_{n=1}^{\infty} \frac{(\alpha)_n}{(c)_n} \mathcal{A}_\alpha z^n + \ldots \quad (z \in \mathcal{U}),
\]

\[
= z + \sum_{n=2}^{\infty} \frac{(\alpha)_n}{(c)_{n-1}} \mathcal{A}_\alpha z^n
\]
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satisfying the following first-order differential recurrence (identity) relation:

$$z(T_\alpha f(z))' = c_\alpha T_{\alpha+1} f(z) - (c_\alpha - 1) T_\alpha f(z)$$

(3)

for some $c_\alpha \in \mathbb{C}$.

The development by which we attain at fractional operators is somewhat similar to what was done for numbers. First, we had positive integers, and then tailed the zero, fractions, irrational, negative, and complex numbers. Nevertheless, the utility of fractional operators and derivatives is wide-ranging. Especially, we will constrain ourselves to the areas of geometric function theory and univalent function theory. For convenience, we will also assume that the numbers and functions treated here are generalizations to complex numbers. The applications of the fractional operators have appeared in various fields such as control theory [39], image processing [40–42] and diffusion concept [43].

In this section, we offer each of the essential definitions and fundamental theorems in theory of the third-order differential subordination and superordination which will deal with to derive our major results. We first recall the basic concepts in theory of the third-order differential subordination due to Antonino and Miller [32].

**Definition 2.1** ([32, Definition 1, p. 440]). Let $\psi : \mathbb{C}^4 \times \mathcal{U} \rightarrow \mathbb{C}$ and the function $h(z)$ be univalent in $\mathcal{U}$. If the function $p(z)$ is analytic in $\mathcal{U}$ and satisfies the following third-order differential subordination

$$\psi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z) < h(z),$$

(4)

then $p(z)$ is called a solution of the differential subordination.

A univalent function $q(z)$ is called a dominant of the solutions of the differential subordination, or, more simply, a dominant if $p(z) < q(z)$ for all $p(z)$ satisfying (5). A dominant $\bar{q}(z)$ that satisfies $\bar{q}(z) < q(z)$ for all dominants $q(z)$ of (5) is said to be the best dominant.

**Definition 2.2** ([32, Definition 2, p. 441]). Let $Q$ denote the set of functions $q$ that are analytic and univalent on the set $\mathcal{U} \setminus E(q)$, where $E(q) = \{\xi : \xi \in \partial \mathcal{U} : \lim_{z \to \xi} q(z) = \infty\}$, is such that $\min |q'(\xi)| = \rho > 0$ for $\xi \in \partial \mathcal{U} \setminus E(q)$. Further, let the subclass of $Q$ for which $q(0) = a$ be denoted by $Q(a)$, $Q(0) = Q_0$ and $Q(1) = Q_1$.

The subordination methodology is applied to an appropriate class of admissible functions. The following class of admissible functions was given by Antonino and Miller [32].

**Definition 2.3** ([32, Definition 3, p. 449]). Let $\Omega$ be a set in $\mathbb{C}$ and $q \in Q$ and $n \in \mathbb{N}\setminus\{1\}$. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times \mathcal{U} \rightarrow \mathbb{C}$ achieving the following admissibility condition:

$$\phi(r,s,t,u;z) \notin \Omega$$

whenever

$$r = q(\xi), \quad s = k\xi q'(\xi), \quad \Re\left(\frac{t}{s} + 1\right) \geq k \Re \left(\frac{\xi q''(\xi)}{q'(\xi)} + 1\right),$$

and

$$\Re\left(\frac{u}{s}\right) \geq k^2 \Re \left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right),$$

where $z \in \mathcal{U}$, $\xi \in \partial \mathcal{U} \setminus E(q)$, and $k \geq n$.

The next theorem is the foundation result in the theory of third-order differential subordination.

**Theorem 2.4** ([32, Theorem 1, p. 449]). Let $p \in \mathcal{H}[a,n]$ with $n \geq 2$, and $q \in Q(a)$ achieving the following conditions:

$$\Re \left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0, \text{ and } \left|\frac{zp'(z)}{q'(\xi)}\right| \leq k,$$
where \( z \in \mathcal{U}, \xi \in \partial \mathcal{U} \setminus E(q) \), and \( k \geq n \). If \( \Omega \) is a set in \( \mathbb{C} \), \( \psi \in \Psi_n[\Omega, q] \) and

\[
\psi \left( p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z \right) \in \Omega,
\]

then

\[
p(z) < q(z) \quad (z \in \mathcal{U}).
\]

**Definition 2.5** ([34]). Let \( \psi : \mathbb{C}^4 \times \mathcal{U} \to \mathbb{C} \) and the function \( h(z) \) be analytic in \( \mathcal{U} \). If the functions \( p(z) \) and

\[
\psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z)
\]

are univalent in \( \mathcal{U} \) and satisfy the following third-order differential superordination:

\[
h(z) < \psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z), \quad (5)
\]

then \( p(z) \) is called a solution of the differential superordination. An analytic function \( q(z) \) is called a subordinant of the solutions of the differential superordination, or more simply a subordinant, if \( q(z) < p(z) \) for all \( p(z) \) satisfying (6).

A univalent subordinant \( \tilde{q}(z) \) that satisfies the condition \( q(z) < \tilde{q}(z) \) for all subordinants \( q(z) \) of (6) is said to be the best subordinant, (see [34]). Tang et al. [34] considered the following class of admissible functions related to differential superordination.

**Definition 2.6** ([34]). Let \( \Omega \) be a set in \( \mathbb{C}, q \in \mathcal{H}[a, n] \) and \( q'(z) \neq 0 \). The class of admissible functions \( \Psi_n[\Omega, q] \) consists of those functions \( \psi : \mathbb{C}^4 \times \overline{\mathcal{U}} \to \mathbb{C} \) that satisfy the following admissibility condition:

\[
\phi(r, s, t, u; \xi) \in \Omega
\]

whenever

\[
r = q(z), \quad s = \frac{zq'(z)}{m}, \quad \Re \left( \frac{t}{s} + 1 \right) \leq \frac{1}{m} \Re \left( \frac{zq''(z)}{q'(z)} + 1 \right),
\]

and

\[
\Re \left( \frac{u}{s} \right) \leq \frac{1}{m^2} \Re \left( \frac{z^2 q'''(z)}{q'(z)} \right).
\]

where \( z \in \mathcal{U}, \xi \in \partial \mathcal{U}, \) and \( m \geq n \geq 2 \).

**Theorem 2.7** ([34]). Let \( q \in \mathcal{H}[a, n] \) and \( \psi \in \Psi_n[\Omega, q] \). If

\[
\psi \left( p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z \right)
\]

is univalent in \( \mathcal{U} \) and \( p \in \mathcal{Q}(a) \) satisfy the following conditions:

\[
\Re \left( \frac{zq''(z)}{q'(z)} \right) \geq 0, \quad \text{and} \quad \left| \frac{zp'(z)}{q'(z)} \right| \leq m, \quad (6)
\]

where \( z \in \mathcal{U}, \xi \in \partial \mathcal{U}, \) and \( m \geq n \geq 2 \), then

\[
\Omega \subset \{ \psi \left( p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z \right) : z \in \mathcal{U} \}
\]

implies that

\[
q(z) < p(z) \quad (z \in \mathcal{U}).
\]


3 Subordination of the integral operator $T_\alpha f(z)$

In this section, the following class of admissible functions is defined, which is required to prove the main third-order differential subordination theorem for the operator $T_\alpha f(z)$ defined by (3).

**Definition 3.1.** Let $\Omega$ be a set in $\mathbb{C}$, $\alpha$, $\alpha+1$, $\alpha+2 \in \mathbb{C}\setminus\{0\}$ and $q \in \mathbb{Q}_0 \cap \mathbb{H}_0$. The class of admissible functions $\Phi_T[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathcal{U} \rightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(u, w, x, y; z) \not\in \Omega$$

whenever

$$v = q(\zeta), \quad w = \frac{kq'(\zeta) + (c_\alpha - 1)q(\zeta)}{c_\alpha},$$

$$\Re\left\{\frac{(c_\alpha c_{\alpha+1} x - (c_\alpha - 1)(c_{\alpha+1} - 1)v}{c_\alpha w - (c_\alpha - 1)v} - (c_\alpha + c_{\alpha+1} - 2)\right\} \geq k\Re\left(\frac{q''(\zeta)}{q'(\zeta) + 1}\right),$$

and

$$\Re\left\{\frac{(c_\alpha c_{\alpha+1} x - (c_\alpha - 1)(c_{\alpha+1} - 1)(c_{\alpha+2} - 1)v}{c_\alpha w - (c_\alpha - 1)v} - (c_\alpha + c_{\alpha+1} - 1)\right\} \geq k^2\Re\left(\frac{q''(\zeta)}{q'(\zeta)}\right),$$

where $z \in \mathcal{U}$, $\zeta \in \partial \mathcal{U}\setminus E(q)$ and $k \geq 2$.

**Theorem 3.2.** Let $\phi \in \Phi_T[\Omega, q]$. If the function $f \in \mathcal{A}$ and $q \in \mathbb{Q}_0$ satisfy the following conditions:

$$\Re\left(\frac{q''(\zeta)}{q'(\zeta)}\right) \geq 0, \quad \left|\frac{T_{\alpha+1} f(z)}{q'(\zeta)}\right| \leq k,$$

and

$$\{|\phi(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z) : z \in \mathcal{U}\} \subset \Omega,$$

then

$$T_\alpha f(z) < q(z) \quad (z \in \mathcal{U}).$$

**Proof.** Define the analytic function $p(z)$ in $\mathcal{U}$ by

$$p(z) = T_\alpha f(z).$$

From equations (4) and (10), we have

$$T_{\alpha+1} f(z) = \frac{zp'(z) + (c_\alpha - 1)p(z)}{c_\alpha}.$$  \hspace{1cm} (10)

Further computations show that

$$T_{\alpha+2} f(z) = \frac{z^2 p''(z) + (c_\alpha + c_{\alpha+1} - 1)zp'(z) + (c_\alpha - 1)(c_{\alpha+1} - 1)p(z)}{c_\alpha c_{\alpha+1}}.$$  \hspace{1cm} (11)

and

$$T_{\alpha+3} f(z) = \frac{z^3 p''(z) + (c_\alpha + c_{\alpha+1} + c_{\alpha+2})z^2 p'(z) + [c_{\alpha+2}(c_\alpha + c_{\alpha+1} - 1) + (c_\alpha - 1)(c_{\alpha+1} - 1)]zp'(z) + (c_\alpha - 1)(c_{\alpha+1} - 1)(c_{\alpha+2} - 1)p(z)}{c_\alpha c_{\alpha+1} c_{\alpha+2}}.$$  \hspace{1cm} (12)
Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$
v(r, s, t, u) = r, \quad w(r, s, t, u) = \frac{s + (c\alpha - 1)r}{c\alpha},
$$

(13)

and

$$
y(r, s, t, u) = \frac{u + (c\alpha + c\alpha + 1 + c\alpha + 2)I}{c\alpha c\alpha + 1} + \frac{[c\alpha + 2(c\alpha + c\alpha + 1 - 1) + (c\alpha - 1)(c\alpha + 1 - 1)]s + (c\alpha - 1)(c\alpha + 1 - 1)(c\alpha + 2 - 1)r}{c\alpha c\alpha + 1}.
$$

(14)

Let

$$
\psi(r, s, t, u; z) = \phi(v, w, x, y; z) = \frac{s + (c\alpha - 1)r}{c\alpha}, \quad \frac{t + (c\alpha + c\alpha + 1 - 1)s + (c\alpha - 1)(c\alpha + 1 - 1)r}{c\alpha c\alpha + 1}, \quad \frac{u + (c\alpha + c\alpha + 1 + c\alpha + 2)I}{c\alpha c\alpha + 1} + \frac{[c\alpha + 2(c\alpha + c\alpha + 1 - 1) + (c\alpha - 1)(c\alpha + 1 - 1)]s + (c\alpha - 1)(c\alpha + 1 - 1)(c\alpha + 2 - 1)r}{c\alpha c\alpha + 1}.
$$

(15)

The proof will make use of Theorem 2.4. Using equations (10) to (13), and from (16), we have

$$
\psi \left( p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z \right) = \phi \left( T_\alpha f(z), T_\alpha + 1 f(z), T_\alpha + 2 f(z), T_\alpha + 3 f(z); z \right).
$$

(16)

Hence, (9) becomes

$$
\psi \left( p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z \right) \in \Omega.
$$

Note that

$$
\frac{t}{s} + 1 = \frac{c\alpha c\alpha + 1 x - (c\alpha - 1)(c\alpha + 1 - 1)v}{c\alpha w - (c\alpha - 1)v} - (c\alpha + c\alpha + 2),
$$

and

$$
\frac{u}{s} = \frac{(c\alpha + c\alpha + 1 c\alpha + 2)y - (c\alpha - 1)(c\alpha + 1 - 1)v}{c\alpha w - (c\alpha - 1)v} - (c\alpha + c\alpha + 1 + c\alpha + 2)
$$

Thus, the admissibility condition for $\phi \in \Phi_T[\Omega, q]$ in Definition 3.1 is equivalent to the admissibility condition for $\psi \in \Psi_2[\Omega, q]$ as given in Definition 2.3 with $n = 2$. Therefore, by using (8) and Theorem 2.4, we have

$$
p(z) = T_\alpha f(z) < q(z).
$$

The next result is an extension of Theorem 3.2 to the case where the behavior of $q(z)$ on $\partial U$ is not known.

**Corollary 3.3.** Let $\Omega \subset \mathbb{C}$ and let the function $q$ be univalent in $U$ with $q(0) = 0$. Let $\phi \in \Phi_T[\Omega, q_\rho]$ for some $\rho \in (0, 1)$, where $q_\rho(z) = q(\rho z)$. If the function $f \in A$ and $q_\rho$ satisfy the following conditions:

$$
\Re \left( \frac{\xi q_\rho'''(\xi)}{q_\rho'(\xi)} \right) \geq 0, \quad \left| \frac{T_\alpha + 1 f(z)}{q_\rho'(\xi)} \right| \leq k \quad (z \in U, \xi \in \partial U \setminus \E(q_\rho)),
$$

and

$$
\phi \left( T_\alpha f(z), T_\alpha + 1 f(z), T_\alpha + 2 f(z), T_\alpha + 3 f(z); z \right) \in \Omega.
$$

then

$$
T_\alpha f(z) < q(z) \quad (z \in U).
$$
Proof. From Theorem 3.2, yields \( T_\alpha f(z) < q_\rho(z) \) \((z \in U)\). The result asserted by corollary 3.3 is now deduced from the following subordination property: \( q_\rho(z) < q(z) \) \((z \in U)\).

If \( \Omega \neq \mathbb{C} \) is a simply connected domain, then \( \Omega = h(U) \), for some conformal mapping \( h(z) \) of \( U \) onto \( \Omega \). In this case, the class \( \Phi_T[h(U), q] \) is written as \( \Phi_T[h, q] \). The following result follows immediately as a consequence of Theorem 3.2.

Theorem 3.4. Let \( \phi \in \Phi_T[h, q] \). If the function \( f \in A \) and \( q \in Q_\alpha \) satisfy the following conditions (8) and

\[
\phi(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z) < h(z), \tag{17}
\]

then

\( T_\alpha f(z) < q(z) \) \((z \in U)\).

The next result is an immediate consequence of Corollary 3.3.

Corollary 3.5. Let \( \Omega \subset \mathbb{C} \) and let the function \( q \) be univalent in \( U \) with \( q(0) = 0 \). Let \( \phi \in \Phi_T[h, q_\rho] \) for some \( \rho \in (0, 1) \), where \( q_\rho(z) = q(\rho z) \). If the function \( f \in A \) and \( q_\rho \) satisfy the following conditions:

\[
\Re \left( \frac{\zeta q_\rho'((\zeta)}{q_\rho'((\zeta)} \right) \geq 0, \quad \left| \frac{T_{\alpha+1} f(z)}{q_\rho'(z)} \right| \leq k \quad (z \in U, \zeta \in \partial U \setminus E(q_\rho)),
\]

and

\[
\phi(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z) < h(z),
\]

then

\( T_\alpha f(z) < q(z) \) \((z \in U)\).

The following result yields the best dominant of the differential subordination (18).

Theorem 3.6. Let the function \( h \) be univalent in \( U \) and let \( \phi : \mathbb{C}^4 \times U \rightarrow \mathbb{C} \) and \( \psi \) be given by (16). Suppose that the differential equation

\[
\psi \left( q(z), zq'(z), z^2 q''(z), z^3 q'''(z); z \right) = h(z), \tag{18}
\]

has a solution \( q(z) \) with \( q(0) = 0 \), which satisfies condition (8). If the function \( f \in A \) satisfies condition (18) and

\[
\phi(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z)
\]

is analytic in \( U \), then

\( T_\alpha f(z) < q(z) \)

and \( q(z) \) is the best dominant.

Proof. From Theorem 3.2, we have \( q \) is a dominant of (18). Since \( q \) satisfies (19), it is also a solution of (18) and therefore \( q \) will be dominated by all dominants. Hence \( q \) is the best dominant. \( \square \)

In view of Definition 3.1, and in the special case \( q(z) = Mz \), \( M > 0 \), the class of admissible functions \( \Phi_T[\Omega, q] \), denoted by \( \Phi_T[\Omega, M] \), is expressed as follows.

Definition 3.7. Let \( \Omega \) be a set in \( \mathbb{C} \), \( c_\alpha, c_{\alpha+1}, c_{\alpha+2} \in \mathbb{C} \setminus \{0\} \) and \( M > 0 \). The class of admissible functions \( \Phi_T[\Omega, M] \) consists of those functions \( \phi : \mathbb{C}^4 \times U \rightarrow \mathbb{C} \) such that

\[
\phi \left( M e^{i\theta}, \frac{(k + c_\alpha - 1)Me^{i\theta}}{c_\alpha}, \frac{L + [(c_\alpha + c_{\alpha+1} - 1)k + (c_\alpha - 1)(c_{\alpha+1} - 1)]Me^{i\theta}}{c_\alpha c_{\alpha+1}}, \right.
\]

\[
\left. \left( N + (c_\alpha + c_{\alpha+1} + c_{\alpha+2})L + [(c_{\alpha+2}(c_\alpha + c_{\alpha+1} - 1) + (c_\alpha - 1)(c_{\alpha+1} - 1)]k \right.
\]

\[
+ (c_{\alpha+1} - 1)(c_{\alpha+2} - 1)(c_{\alpha+2} - 1))Me^{i\theta} \right) \times \left( c_\alpha c_{\alpha+1} c_{\alpha+2} \right)^{-1}(z) \notin \Omega. \tag{19}
\]
whenever $z \in \mathcal{U}$. \(\Re(Le^{-i\theta}) \geq (k - 1)kM\), and \(\Re(Ne^{-i\theta}) \geq 0\) for all \(\theta \in \mathbb{R}\) and \(k \geq 2\).

**Corollary 3.8.** Let \(\phi \in \Phi_{\mathcal{F}}[\Omega, M]\). If the function \(f \in \mathcal{A}\) satisfies
\[
|T_{\alpha+1}f(z)| \leq kM \quad (k \geq 2; M > 0),
\]
and
\[
\phi(T_{\alpha}f(z), T_{\alpha+1}f(z), T_{\alpha+2}f(z), T_{\alpha+3}f(z); z) \in \Omega,
\]
then
\[
|T_{\alpha}f(z)| < M.
\]

In the special case \(\Omega = q(\mathcal{U}) = \{w : |w| < M\}\), the class \(\Phi_{\mathcal{F}}[\Omega, M]\) is simply denoted by \(\Phi_{\mathcal{F}}[M]\). Corollary 3.6 can now be written in the following form:

**Corollary 3.9.** Let \(\phi \in \Phi_{\mathcal{F}}[M]\). If the function \(f \in \mathcal{A}\) satisfies
\[
|T_{\alpha+1}f(z)| \leq kM \quad (k \geq 2; M > 0),
\]
and
\[
|\phi(T_{\alpha}f(z), T_{\alpha+1}f(z), T_{\alpha+2}f(z), T_{\alpha+3}f(z); z)| < M,
\]
then
\[
|T_{\alpha}f(z)| \leq M.
\]

**Example 3.10.** Let \(\Re(\alpha) \geq \frac{1-k}{2} \), \(0 \neq c_{\alpha} \in \mathbb{C}, k \geq 2\) and \(M > 0\). If the function \(f \in \mathcal{A}\) satisfies
\[
|T_{\alpha+1}f(z)| < M,
\]
then
\[
|T_{\alpha}f(z)| < M.
\]

**Proof.** By taking \(\phi(v, w, x, y; z) = w\) in Corollary 3.9, we have to find the condition so that \(\phi \in \Phi_{\mathcal{F}}[M]\), that is, the admissibility condition (20) is satisfied. This follows from
\[
|\phi(v, w, x, y; z)| \geq M,
\]
which implies
\[
\left|\frac{(k + c_{\alpha} - 1)Me^{i\theta}}{e^{i\theta}}\right| \geq M
\]
or
\[
|k + c_{\alpha} - 1| \geq |c_{\alpha}| \quad (20)
\]
Preceding inequality (21), shows that
\[
\Re(\alpha) \geq \frac{1-k}{2}.
\]
Then it is sufficient to write
\[
\Re(\alpha) \geq \frac{1-k}{2}.
\]
for (21) holds true. Hence, from Corollary 3.9, if \(\Re(\alpha) \geq \frac{1-k}{2} \), \(k \geq 2\) and \(|T_{\alpha+1}f(z)| < M\), then \(|T_{\alpha}f(z)| < M\).

**Example 3.11.** Let \(k \geq 2\), \(0 \neq c_{\alpha} \in \mathbb{C}\) and \(M > 0\). If the function \(f \in \mathcal{A}\) satisfies
\[
|T_{\alpha+1}f(z)| \leq kM,
\]
and

\[ |T_{\alpha+1} f(z) - T_\alpha f(z)| < \frac{M}{|c_\alpha|}, \]

then

\[ |T_\alpha f(z)| < M. \]

Proof. Let

\[ \phi(v, w, x, y; z) = w - v, \quad \Omega = h(U), \]

where

\[ h(z) = \frac{M z}{|c_\alpha|} \quad (M > 0). \]

In order to use Corollary 3.8, we need to show that \( \phi \in \Phi_T[\Omega, M] \), that is, the admissibility condition (20) is satisfied. This follows since

\[ \left| \phi(v, w, x, y; z) \right| = \left| \frac{(k - 1)M e^{i\theta}}{c_\alpha} \right| \geq \frac{M}{|c_\alpha|}, \]

whenever \( z \in U, \theta \in \mathbb{R} \) and \( k \geq 2 \). The required result now follows from Corollary 3.8.

\[ \square \]

Definition 3.12. Let \( \Omega \) be a set in \( \mathbb{C} \), \( q \in \mathbb{Q}_1 \cap \mathbb{H}_1 \) and \( c_\alpha, c_\alpha+1, c_\alpha+2 \in \mathbb{C} \setminus \{0\} \). The class of admissible functions \( \Phi_{T,1}[\Omega, q] \) consists of those functions \( \phi : \mathbb{C}^4 \times U \rightarrow \mathbb{C} \) that satisfy the following admissibility condition:

\[ \phi(v, w, x, y; z) \notin \Omega \]

whenever

\[ v = q(\zeta), \quad w = \frac{k \zeta q(\zeta)}{c_\alpha} + \frac{c_\alpha q(\zeta)}{c_\alpha}, \]

\[ \Re \left( \frac{c_\alpha + 1}{w - v} \right) \geq k \Re \left( \frac{c_\alpha - 1}{q(\zeta)} + 1 \right), \]

\[ \Re \left( \frac{c_\alpha + 1}{w - v} \right) \geq k \Re \left( \frac{c_\alpha - 1}{q(\zeta)} + 1 \right), \]

\[ \Re \left( \frac{(c_\alpha + 1)(c_\alpha + 2)(y - v)}{w - v} \right) - (c_\alpha + c_\alpha + 1 + c_\alpha + 2 + 3) \left[ \frac{c_\alpha + 1}{w - v} \right] \geq k \Re \left( \frac{q''(\zeta)}{q(\zeta)} \right), \]

where \( z \in U, \zeta \in \partial U \setminus E(q) \) and \( k \geq 2 \).

Theorem 3.13. Let \( \phi \in \Phi_{T,1}[\Omega, q] \). If the function \( f \in \mathcal{A} \) and \( q \in \mathbb{Q}_1 \) satisfy the following conditions:

\[ \Re \left( \frac{\zeta q''(\zeta)}{q'(\zeta)} \right) \geq 0, \quad \left| \frac{T_{\alpha+1} f(z)}{z q'(\zeta)} \right| \leq k, \]

and

\[ \left\{ \phi \left( \frac{T_\alpha f(z)}{z}, \frac{T_{\alpha+1} f(z)}{z}, \frac{T_{\alpha+2} f(z)}{z}, \frac{T_{\alpha+3} f(z)}{z} \right) : z \in U \right\} \subset \Omega, \]

then

\[ \frac{T_\alpha f(z)}{z} < q(z) \quad (z \in U). \]

Proof. Define the analytic function \( p(z) \) in \( U \) by

\[ p(z) = \frac{T_\alpha f(z)}{z}. \]

By using equation (4) and (24), we get

\[ \frac{T_{\alpha+1} f(z)}{z} = \frac{zp'(z) + c_\alpha p(z)}{c_\alpha}, \]

(24)
Further computations show that

\[
\frac{T_{\alpha+2}f(z)}{z} = \frac{z^2p''(z) + (c_\alpha + c_{\alpha+1} + 1)zp'(z) + c_\alpha c_{\alpha+1}p(z)}{c_\alpha c_{\alpha+1}},
\]

and

\[
\frac{T_{\alpha+3}f(z)}{z} = \left(3^3p'''(z) + (c_\alpha + c_{\alpha+1} + c_{\alpha+2} + 3)z^2p''(z) + [(c_\alpha + c_{\alpha+1} + 1)
\]

\[
(c_{\alpha+2} + 1) + c_\alpha c_{\alpha+1} \right)p'(z) + c_\alpha c_{\alpha+1}c_{\alpha+2}p(z) \right) \times \left(c_\alpha c_{\alpha+1}c_{\alpha+2} \right)^{-1}.
\]

Define the transformation from \( \mathbb{C}^4 \) to \( \mathbb{C} \) by

\[
v(r,s,t,u) = r, \quad w(r,s,t,u) = \frac{s + c_\alpha r}{c_\alpha},
\]

\[
x(r,s,t,u) = \frac{t + (c_\alpha + c_{\alpha+1} + 1)s + c_\alpha c_{\alpha+1}r}{c_\alpha c_{\alpha+1}},
\]

and

\[
y(r,s,t,u) = \left( u + (c_\alpha + c_{\alpha+1} + c_{\alpha+2} + 3)t + [(c_\alpha + c_{\alpha+1} + 1)(c_{\alpha+2} + 1) + c_\alpha c_{\alpha+1}] s
\]

\[
+ (c_\alpha c_{\alpha+1}c_{\alpha+2})r \right) \times \left(c_\alpha c_{\alpha+1}c_{\alpha+2} \right)^{-1}.
\]

Let

\[
\psi(r,s,t,u;z) = \phi(v,w,x,y;z) = \phi\left( \frac{r, s + c_\alpha r, t + (c_\alpha + c_{\alpha+1} + 1)s + c_\alpha c_{\alpha+1}r}{c_\alpha c_{\alpha+1}} \right),
\]

\[
\left( u + (c_\alpha + c_{\alpha+1} + c_{\alpha+2} + 3)t + [(c_\alpha + c_{\alpha+1} + 1)(c_{\alpha+2} + 1) + c_\alpha c_{\alpha+1}] s + (c_\alpha c_{\alpha+1}c_{\alpha+2})r \right)
\]

\[
\times \left(c_\alpha c_{\alpha+1}c_{\alpha+2} \right)^{-1}; z \right).
\]

The proof will make use of Theorem 2.4. Using equations (24) to (27), and from (30), we obtain

\[
\psi \left( p(z), zp'(z), z^2p''(z), z^3p'''(z); z \right) = \phi \left( \frac{T_\alpha f(z)}{z}, \frac{T_{\alpha+1}f(z)}{z}, \frac{T_{\alpha+2}f(z)}{z}, \frac{T_{\alpha+3}f(z)}{z}; z \right).
\]

Hence, (23) becomes

\[
\psi \left( p(z), zp'(z), z^2p''(z), z^3p'''(z); z \right) \in \Omega.
\]

Note that

\[
\frac{t}{s} + 1 = \frac{c_\alpha + 1(x - v)}{w - v} - (c_\alpha + c_{\alpha+1}),
\]

and

\[
\frac{u}{s} = \frac{c_{\alpha+1}c_{\alpha+2}(y - v)}{w - v} - (c_\alpha + c_{\alpha+1} + c_{\alpha+2} + 3) \left[ \frac{c_{\alpha+1}(x - v)}{w - v} - (c_\alpha + c_{\alpha+1} + 1) \right]
\]

\[
- \left( (c_\alpha + c_{\alpha+1} + 1)(c_{\alpha+2} + 1) + c_\alpha c_{\alpha+1} \right).
\]

Thus, the admissibility condition for \( \psi \in \Phi_{T,1}[\Omega, q] \) in Definition 3.12 is equivalent to the admissibility condition for \( \psi \in \Psi_{2}[\Omega, q] \) as given in Definition 2.3 with \( n = 2 \). Therefore, by using (22) and Theorem 2.4, we have

\[
p(z) = \frac{T_\alpha f(z)}{z} < q(z).
\]

If \( \Omega \neq \mathbb{C} \) is a simply connected domain, then \( \Omega = h(\mathcal{U}) \), for some conformal mapping \( h(z) \) of \( \mathcal{U} \) onto \( \Omega \). In this case, the class \( \Phi_{T,1}[h(\mathcal{U}), q] \) is written as \( \Phi_{T,1}[h, q] \). The following result follows immediately as a consequence of Theorem 3.13.
Theorem 3.14. Let $\phi \in \Phi_{T,1}[h,q]$. If the function $f \in A$ and $q \in Q_1$ satisfy the following conditions (22) and
\[
\phi \left( \frac{T_\alpha f(z)}{z}, \frac{T_{\alpha+1} f(z)}{z}, \frac{T_{\alpha+2} f(z)}{z}, \frac{T_{\alpha+3} f(z)}{z}; z \right) < h(z),
\]
then
\[
\frac{T_\alpha f(z)}{z} < q(z) \quad (z \in \mathcal{U}).
\]
In view of Definition 3.12, and in the special case $q(z) = Mz$, $M > 0$, the class of admissible functions $\Phi_{T,1}[\Omega, q]$, denoted by $\Phi_{T,1}[\Omega, M]$, is expressed as follows.

Definition 3.15. Let $\Omega$ be a set in $\mathbb{C}$, $c_\alpha$, $c_{\alpha+1}$, $c_{\alpha+2} \in \mathbb{C} \setminus \{0\}$ and $M > 0$. The class of admissible functions $\Phi_{T,1}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathcal{U} \rightarrow \mathbb{C}$ such that
\[
\phi \left( M^i \theta, \frac{(k + c_\alpha) Me^{i \theta}}{c_\alpha}, \frac{c_\alpha c_{\alpha+1}}{c_\alpha} \frac{k + c_\alpha c_{\alpha+1} + 1}{c_\alpha} \right) L + \left[ \frac{c_\alpha + c_{\alpha+1} + 1}{c_\alpha c_{\alpha+1}} c_\alpha c_{\alpha+2} + 3 \right] M + \left( c_\alpha + c_{\alpha+1} + c_{\alpha+2} \right) L
\]
whenever $z \in \mathcal{U}$, $\Re \left( Me^{-i \theta} \right) \geq (k - 1) k M$, and $\Re \left( Ne^{-i \theta} \right) \geq 0$ for all $\theta \in \mathbb{R}$ and $k \geq 2$.

Corollary 3.16. Let $\phi \in \Phi_{T,1}[\Omega, M]$. If the function $f \in A$ satisfies
\[
\left| \frac{T_{\alpha+1} f(z)}{z} \right| \leq kM \quad (k \geq 2; M > 0),
\]
and
\[
\phi \left( \frac{T_\alpha f(z)}{z}, \frac{T_{\alpha+1} f(z)}{z}, \frac{T_{\alpha+2} f(z)}{z}, \frac{T_{\alpha+3} f(z)}{z}; z \right) \in \Omega,
\]
then
\[
\left| \frac{T_\alpha f(z)}{z} \right| < M.
\]
In the special case $\Omega = q(\mathcal{U}) = \{w : |w| < M\}$, the class $\Phi_{T,1}[\Omega, M]$ is simply denoted by $\Phi_{T,1}[M]$, and Corollary 3.16 has the following form:

Corollary 3.17. Let $\phi \in \Phi_{T,1}[M]$. If the function $f \in A$ satisfies
\[
\left| \frac{T_{\alpha+1} f(z)}{z} \right| \leq kM \quad (k \geq 2; M > 0),
\]
and
\[
\phi \left( \frac{T_\alpha f(z)}{z}, \frac{T_{\alpha+1} f(z)}{z}, \frac{T_{\alpha+2} f(z)}{z}, \frac{T_{\alpha+3} f(z)}{z}; z \right) \in M,
\]
then
\[
\left| \frac{T_\alpha f(z)}{z} \right| < M.
\]

Example 3.18. Let $\alpha \in \mathbb{C}$ with $\Re(\alpha) \geq \frac{-k}{2}$, $k \geq 2$ and $M > 0$. If the function $f \in A$ satisfies
\[
\left| \frac{T_{\alpha+1} f(z)}{z} \right| < M
\]
then
\[
\left| \frac{T_\alpha f(z)}{z} \right| < M.
\]

Proof. By taking $\phi(v,w,x,y;z) = w = \frac{(k + c_\alpha) Me^{i \theta}}{c_\alpha}$ in Corollary 3.17, the result is obtained.
Example 3.19. Let $M > 0$, $c_{\alpha}$, $c_{\alpha+1} \in \mathbb{R}$ and the function $f \in A$ satisfies
\[
\left| \frac{T_{\alpha+1} f(z)}{z} \right| \leq kM,
\]
and
\[
\left| c_{\alpha} c_{\alpha+1} \frac{T_{\alpha+2} f(z)}{z} - c_{\alpha} \frac{T_{\alpha+1} f(z)}{z} \right| < M[2 + c_{\alpha+1}(2 + c_{\alpha})].
\]
then
\[
\left| \frac{T_{\alpha} f(z)}{z} \right| < M.
\]

Proof. Let
\[
\phi(v, w, x, y; z) = c_{\alpha} c_{\alpha+1} x + c_{\alpha} w
\]
and
\[
\Omega = h(\mathcal{U}),
\]
where
\[
h(z) = M[2 + c_{\alpha+1}(2 + c_{\alpha})]z \quad (M > 0).
\]
In order to use Corollary 3.16, we need to show that $\phi \in \Phi_{\mathcal{T},1}[\Omega, M]$, that is, the admissibility condition (33) is satisfied. This follows since
\[
\begin{align*}
\phi\left( M e^{i\theta}, \frac{k + c_{\alpha}) M e^{i\theta}}{c_{\alpha}}, L + \left[ (c_{\alpha} + c_{\alpha+1} + 1) k + c_{\alpha} c_{\alpha+1} \right] M e^{i\theta} \right) & \geq \Re \left( L e^{-i\theta} \right) + c_{\alpha+1} k \geq k(k-1)M + c_{\alpha+1} k \geq M[2 + c_{\alpha+1}(2 + c_{\alpha})],
\end{align*}
\]
whenever $z \in \mathcal{U}$, $\theta \in \mathbb{R}$ and $k \geq 2$. \hfill \Box

Definition 3.20. Let $\Omega$ be a set in $\mathbb{C}$, $c_{\alpha}$, $c_{\alpha+1}$, $c_{\alpha+2}$, $c_{\alpha+3} \in \mathbb{C} \setminus \{0\}$ and $q \in Q_{1} \cap \mathcal{H}_{1}$. The class of admissible functions $\Phi_{\mathcal{T},2}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathcal{U} \rightarrow \mathbb{C}$ that satisfy the following admissibility condition:
\[
\phi(v, w, x, y; z) \notin \Omega
\]
whenever
\[
v = q(\zeta), \quad w = \frac{1}{c_{\alpha+1} \left[ \frac{k q'(\zeta)}{q(\zeta)} + c_{\alpha} q(\zeta) + c_{\alpha+1} - c_{\alpha} \right].}
\]
\[
\Re \left( c_{\alpha+1} (w-1) - c_{\alpha} (2 w - 1) + \frac{c_{\alpha+1} w [c_{\alpha+2} (x-1) - c_{\alpha+1} (w-1)]}{c_{\alpha+1} (w-1) - c_{\alpha} (v-1)} \right) \geq \Re \left( \frac{\xi q''(\zeta)}{q'(\zeta)} + 1 \right),
\]
and
\[
\Re \left( \left[ c_{\alpha+3} (y-1) - c_{\alpha+2} - c_{\alpha+1} (w-1) - \frac{c_{\alpha+1} w [c_{\alpha+2} (x-1) - c_{\alpha+1} (w-1)]}{c_{\alpha+2} + c_{\alpha+1} (w-1)} \right] \right. \left[ \frac{c_{\alpha+2} + c_{\alpha+1} (w-1) + \frac{c_{\alpha+1} w [c_{\alpha+2} (x-1) - c_{\alpha+1} (w-1)]}{c_{\alpha+2} + c_{\alpha+1} (w-1)}}{c_{\alpha+1} (w-1) - c_{\alpha} (v-1)} \right] \right.
\]
\[
- c_{\alpha+1} w^2 \left[ c_{\alpha+2} (x-1) - c_{\alpha+1} (w-1) \right] + \left[ c_{\alpha+1} w \left[ c_{\alpha+2} (x-1) - c_{\alpha+1} (w-1) \right] \right] \right)^2 \right)
\]
\[
\times \left( \frac{(c_{\alpha+1} w)^{-1} v}{c_{\alpha+1} (w-1) - c_{\alpha} (v-1)} \right) - (c_{\alpha} v)^2 + 3 \left[ c_{\alpha+1} (w-1) - c_{\alpha} (2 w - 1) - 1 + \frac{c_{\alpha+1} w [c_{\alpha+2} (x-1) - c_{\alpha+1} (w-1)]}{c_{\alpha+1} (w-1) - c_{\alpha} (v-1)} \right] \right) + 3 c_{\alpha+1} w v \left[ c_{\alpha+2} (x-1) - c_{\alpha+1} (w-1) \right]
\]
\[
- v(1 + c_\alpha v) - 3c_\alpha v^2 [c_{\alpha+1}(w - 1) - c\alpha(v - 1)] + 5v [c_{\alpha+1}(w - 1) - c\alpha(v - 1)]^2
\geq k^2 \left( \frac{\zeta^2 q''''(\zeta)}{q'(\zeta)} \right),
\]

where \( z \in U, \xi \in \partial U \setminus E(q) \) and \( k \geq 2 \).

**Theorem 3.21.** Let \( \phi \in \Phi_{T,2}[\Omega, q] \). If the function \( f \in A \) and \( q \in Q_1 \) satisfy the following conditions:

\[
y(\frac{\zeta q''''(\zeta)}{q'(\zeta)}) \geq 0, \quad \left| \frac{T_{\alpha+2}f(z)}{T_{\alpha+1}f(z)q'(\zeta)} \right| \leq k,
\]

and

\[
\left\{ \phi \left( \frac{T_{\alpha+1}f(z)}{T_{\alpha}f(z)}, \frac{T_{\alpha+2}f(z)}{T_{\alpha+1}f(z)}, \frac{T_{\alpha+3}f(z)}{T_{\alpha+2}f(z)}, \frac{T_{\alpha+4}f(z)}{T_{\alpha+3}f(z)} \right) : z \in U \right\} \subset \Omega,
\]

then

\[
\frac{T_{\alpha+1}f(z)}{T_{\alpha}f(z)} < q(z) \quad (z \in U).
\]

**Proof.** Define the analytic function \( p(z) \) in \( U \) by

\[
p(z) = \frac{T_{\alpha+1}f(z)}{T_{\alpha}f(z)}.
\]

By using equation (4) and (36), we get

\[
\frac{T_{\alpha+2}f(z)}{T_{\alpha+1}f(z)} = \frac{1}{c_{\alpha+1}} \left[ \frac{z'p'(z)}{p(z)} + c_\alpha p(z) + c_{\alpha+1} - c_\alpha \right].
\]

Further computations show that

\[
\frac{T_{\alpha+3}f(z)}{T_{\alpha+2}f(z)} = \frac{1}{c_{\alpha+2}} \left[ c_{\alpha} p(z) + c_{\alpha+2} - c_\alpha + \frac{z'p'(z)}{p(z)} + \frac{z^2 p''(z)}{p(z)} + \frac{z p'(z)}{p(z)} + \frac{z p'(z)}{p(z)} - \left( \frac{z p'(z)}{p(z)} \right)^2 + c_\alpha z p'(z) \right].
\]

and

\[
\frac{T_{\alpha+4}f(z)}{T_{\alpha+3}f(z)} = \frac{1}{c_{\alpha+3}} \left[ c_{\alpha} p(z) + c_{\alpha+3} - c_\alpha + \frac{z'p'(z)}{p(z)} + \frac{z^2 p''(z)}{p(z)} + \frac{z p'(z)}{p(z)} - \left( \frac{z p'(z)}{p(z)} \right)^2 + c_\alpha z p'(z) \right] + \left( \frac{z^2 p''''(z)}{p(z)} + \frac{z p'(z)}{p(z)} - \left( \frac{z p'(z)}{p(z)} \right)^2 + c_\alpha z p'(z) \right) + \left( \frac{z^2 p''''(z)}{p(z)} + \frac{z p'(z)}{p(z)} - \left( \frac{z p'(z)}{p(z)} \right)^2 + c_\alpha z p'(z) \right) - \left( \frac{z p'(z)}{p(z)} + c_\alpha p(z) + c_{\alpha+1} - c_\alpha \right)^2.
\]

Define the transformation from \( \mathbb{C}^4 \) to \( \mathbb{C}^4 \) by

\[
v(r, s, t, u) = r, \quad w(r, s, t, u) = \frac{1}{c_{\alpha+1}} \left[ \frac{s}{r} + c_\alpha r + c_{\alpha+1} - c_\alpha \right]
\]

\[
x(r, s, t, u) = \frac{1}{c_{\alpha+2}} \left[ c_\alpha r + c_{\alpha+2} - c_\alpha + \frac{s}{r} + \frac{t + \frac{s}{r} - (\frac{s}{r})^2 + c_\alpha s}{r + c_\alpha r + c_{\alpha+1} - c_\alpha} \right].
\]

\[
(38)
\]

\[
(39)
\]
and

\[
y(r, s, t, u) = \frac{1}{c_{\alpha+3}} \left[ c_{\alpha} r + c_{\alpha+3} - c_{\alpha} + \frac{s}{r} + \frac{t + s - \left(\frac{s}{r}\right)^2 + c_{\alpha} s}{\frac{t}{r} + \frac{s}{r} - \left(\frac{s}{r}\right)^2 + c_{\alpha} s} \right] \\
\times \left[ \frac{s}{r} + c_{\alpha} r + c_{\alpha+1} - c_{\alpha} \right] \left[ \frac{u}{r^2} + \frac{3t}{r^2} + \frac{s}{r} - \frac{3ts}{r^2} - 3\left(\frac{s}{r}\right)^2 + c_{\alpha} s \right]^{-2} \\
\times \left[ \frac{s}{r} + c_{\alpha} r + c_{\alpha+1} - c_{\alpha} \right]^{-2} \left[ \frac{t}{r} + \frac{s}{r} - \left(\frac{s}{r}\right)^2 + c_{\alpha} s \right]^{-2} \left[ \frac{s}{r} + c_{\alpha} r + c_{\alpha+1} - c_{\alpha} \right]^{-1}
\]

(40)

Let

\[
\psi(r, s, t, u; z) = \phi(v, w, x, y; z)
\]

\[
= \phi \left( r, \frac{1}{c_{\alpha+1}} \left[ \frac{s}{r} + c_{\alpha} r + c_{\alpha+1} - c_{\alpha} \right], \frac{1}{c_{\alpha+2}} \left[ c_{\alpha} r + c_{\alpha+2} - c_{\alpha} + \frac{s}{r} + \frac{t + s - \left(\frac{s}{r}\right)^2 + c_{\alpha} s}{\frac{t}{r} + \frac{s}{r} - \left(\frac{s}{r}\right)^2 + c_{\alpha} s} \right] \right)
\]

\[
= \frac{1}{c_{\alpha+3}} \left[ c_{\alpha} r + c_{\alpha+3} - c_{\alpha} + \frac{s}{r} + \frac{t + s - \left(\frac{s}{r}\right)^2 + c_{\alpha} s}{\frac{t}{r} + \frac{s}{r} - \left(\frac{s}{r}\right)^2 + c_{\alpha} s} \right] \left[ \frac{s}{r} + c_{\alpha} r + c_{\alpha+1} - c_{\alpha} \right] \left[ \frac{u}{r^2} + \frac{3t}{r^2} + \frac{s}{r} - \frac{3ts}{r^2} - 3\left(\frac{s}{r}\right)^2 + c_{\alpha} s \right]^{-2} \left[ \frac{s}{r} + c_{\alpha} r + c_{\alpha+1} - c_{\alpha} \right]^{-2} \left[ \frac{s}{r} + c_{\alpha} r + c_{\alpha+1} - c_{\alpha} \right]^{-1}
\]

(41)

The proof will make use of Theorem 2.4. Using equations (36) to (39), and from (42), we obtain

\[
\psi \left( p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z \right) = \phi \left( \frac{T_{\alpha+1} f(z)}{T_{\alpha} f(z)}, \frac{T_{\alpha+2} f(z)}{T_{\alpha+1} f(z)}, \frac{T_{\alpha+3} f(z)}{T_{\alpha+2} f(z)}, \frac{T_{\alpha+4} f(z)}{T_{\alpha+3} f(z)} ; z \right).
\]

(42)

Hence, (35) becomes

\[
\psi \left( p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z \right) \in \Omega.
\]

Note that

\[
\frac{t}{s} + 1 = c_{\alpha+1} (w - 1) - c_{\alpha} (2w - 1) + \frac{c_{\alpha+1} w \left[ c_{\alpha+2} (x - 1) - c_{\alpha+1} (w - 1) \right]}{c_{\alpha+1} (w - 1) - c_{\alpha} (v - 1)}.
\]

and

\[
\frac{u}{s} = \left[ c_{\alpha+3} (y - 1) - c_{\alpha+2} - c_{\alpha+1} (w - 1) - \frac{c_{\alpha+1} w \left[ c_{\alpha+2} (x - 1) - c_{\alpha+1} (w - 1) \right]}{c_{\alpha+2} + c_{\alpha+1} (w - 1)} \right] \\
\times \left[ c_{\alpha+2} + c_{\alpha+1} (w - 1) + \frac{c_{\alpha+1} w \left[ c_{\alpha+2} (x - 1) - c_{\alpha+1} (w - 1) \right]}{c_{\alpha+2} + c_{\alpha+1} (w - 1)} \right] \\
- c_{\alpha+1} w^2 \left[ c_{\alpha+2} (x - 1) - c_{\alpha+1} (w - 1) \right] + \left[ c_{\alpha+1} w \left[ c_{\alpha+2} (x - 1) - c_{\alpha+1} (w - 1) \right] \right] \right)^2
\]

\[
\times \left( \frac{c_{\alpha+1} w^{-1} v}{c_{\alpha+1} (w - 1) - c_{\alpha} (v - 1)} \right) - \left( c_{\alpha} v^2 + 3 \left[ c_{\alpha+1} (w - 1) - c_{\alpha} (2w - 1) - 1 + \frac{c_{\alpha+1} w \left[ c_{\alpha+2} (x - 1) - c_{\alpha+1} (w - 1) \right]}{c_{\alpha+1} (w - 1) - c_{\alpha} (v - 1)} \right] + 3 c_{\alpha+1} w v \left[ c_{\alpha+2} (x - 1) - c_{\alpha+1} (w - 1) \right] \\
- v (1 + c_{\alpha} v) - 3 c_{\alpha} v^2 \left[ c_{\alpha+1} (w - 1) - c_{\alpha} (v - 1) \right] + 5 v \left[ c_{\alpha+1} (w - 1) - c_{\alpha} (v - 1) \right]^2. \]
Thus, the admissibility condition for $\phi \in \Phi_{\Omega, 2}[\Omega, q]$ in Definition 3.20 is equivalent to the admissibility condition for $\psi \in \Psi_2[\Omega, q]$ as given in Definition 2.3 with $n = 2$. Therefore, by using (34) and Theorem 2.4, we have

$$p(z) = \frac{T_{\alpha+1}f(z)}{T_\alpha f(z)} < q(z).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$, for some conformal mapping $h(z)$ of $U$ onto $\Omega$. In this case, the class $\Phi_{\Omega, 2}[h(U), q]$ is written as $\Phi_{\Omega, 1}[h, q]$. The following result follows immediately as a consequence of Theorem 3.21.

**Theorem 3.22.** Let $\phi \in \Phi_{\Omega, 2}[h, q]$. If the function $f \in A$ and $q \in \mathcal{Q}_1$ satisfy the following conditions (34) and

$$\phi \left( \frac{T_{\alpha+1}f(z)}{T_\alpha f(z)}, \frac{T_{\alpha+2}f(z)}{T_\alpha f(z)}, \frac{T_{\alpha+3}f(z)}{T_\alpha f(z)}, \frac{T_{\alpha+4}f(z)}{T_\alpha f(z)} \right) < h(z),$$

then

$$\frac{T_{\alpha+1}f(z)}{T_\alpha f(z)} < q(z) \quad (z \in U).$$

In view of Definition 3.20, and in the special case $q(z) = 1 + Mz, M > 0$, the class of admissible functions $\Phi_{\Omega, 2}[\Omega, q]$, denoted by $\Phi_{\Omega, 2}[\Omega, M]$, is expressed as follows.

**Definition 3.23.** Let $\Omega$ be a set in $\mathbb{C}, c_\alpha, c_{\alpha+1}, c_{\alpha+2}, c_{\alpha+3} \in \mathbb{C} \setminus \{0\}$ and $M > 0$. The class of admissible functions $\Phi_{\Omega, 2}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ such that

$$\phi \left( 1 + Me^{i\theta}, 1 + \frac{(1 + Me^{i\theta}) c_\alpha + k}{c_{\alpha+1} + (1 + Me^{i\theta})} Me^{i\theta}, 1 + \frac{(1 + Me^{i\theta}) c_\alpha + k}{c_{\alpha+2} + (1 + Me^{i\theta})} Me^{i\theta} + \frac{(M + e^{-i\theta}) (Le^{-i\theta} + kM(c_\alpha + 1) + c_\alpha k^2 M^2 e^{i\theta}) - k^2 M^2}{c_{\alpha+2} (M + e^{-i\theta}) (c_\alpha M^2 e^{i\theta} + c_{\alpha+1} e^{-i\theta} + M(c_\alpha + c_\alpha + 1 + k))}, \frac{(M + e^{-i\theta}) (Le^{-i\theta} + kM(c_\alpha + 1) + c_\alpha k M^2 e^{i\theta}) - k^2 M^2}{c_{\alpha+2} (M + e^{-i\theta}) (c_\alpha M^2 e^{i\theta} + c_{\alpha+1} e^{-i\theta} + M(c_\alpha + c_\alpha + 1 + k))} \right)$$

$$\times \left( \frac{(M + e^{-i\theta}) (Le^{-i\theta} + kM(c_\alpha + 1) + c_\alpha k M^2 e^{i\theta}) - k^2 M^2}{c_{\alpha+2} (M + e^{-i\theta}) (c_\alpha M^2 e^{i\theta} + c_{\alpha+1} e^{-i\theta} + M(c_\alpha + c_\alpha + 1 + k))} \right)^{-2} \right) \neq \Omega,$$

wherever $z \in U, \Re (Le^{-i\theta}) \geq (k - 1)kM$, and $\Re (Ne^{-i\theta}) \geq 0$ for all $\theta \in \mathbb{R}$ and $k \geq 2$.

**Corollary 3.24.** Let $\phi \in \Phi_{\Omega, 2}[\Omega, M]$. If the function $f \in A$ satisfies

$$\frac{T_{\alpha+2}f(z)}{T_{\alpha+1}f(z)} \leq kM \quad (k \geq 2; M > 0),$$

whenever $z \in U, \Re (Le^{-i\theta}) \geq (k - 1)kM$, and $\Re (Ne^{-i\theta}) \geq 0$ for all $\theta \in \mathbb{R}$ and $k \geq 2$. 

$$\phi \left( 1 + Me^{i\theta}, 1 + \frac{(1 + Me^{i\theta}) c_\alpha + k}{c_{\alpha+1} + (1 + Me^{i\theta})} Me^{i\theta}, 1 + \frac{(1 + Me^{i\theta}) c_\alpha + k}{c_{\alpha+2} + (1 + Me^{i\theta})} Me^{i\theta} + \frac{(M + e^{-i\theta}) (Le^{-i\theta} + kM(c_\alpha + 1) + c_\alpha k^2 M^2 e^{i\theta}) - k^2 M^2}{c_{\alpha+2} (M + e^{-i\theta}) (c_\alpha M^2 e^{i\theta} + c_{\alpha+1} e^{-i\theta} + M(c_\alpha + c_\alpha + 1 + k))}, \frac{(M + e^{-i\theta}) (Le^{-i\theta} + kM(c_\alpha + 1) + c_\alpha k M^2 e^{i\theta}) - k^2 M^2}{c_{\alpha+2} (M + e^{-i\theta}) (c_\alpha M^2 e^{i\theta} + c_{\alpha+1} e^{-i\theta} + M(c_\alpha + c_\alpha + 1 + k))} \right)$$

$$\times \left( \frac{(M + e^{-i\theta}) (Le^{-i\theta} + kM(c_\alpha + 1) + c_\alpha k M^2 e^{i\theta}) - k^2 M^2}{c_{\alpha+2} (M + e^{-i\theta}) (c_\alpha M^2 e^{i\theta} + c_{\alpha+1} e^{-i\theta} + M(c_\alpha + c_\alpha + 1 + k))} \right)^{-2} \right) \neq \Omega,$$
and
\[
\phi \left( \frac{T_{\alpha+1} f(z)}{T_{\alpha} f(z)}, \frac{T_{\alpha+2} f(z)}{T_{\alpha} f(z)}, \frac{T_{\alpha+3} f(z)}{T_{\alpha} f(z)}, \frac{T_{\alpha+4} f(z)}{T_{\alpha} f(z)} ; z \right) \in \Omega,
\]
then
\[
\left| \frac{T_{\alpha+1} f(z)}{T_{\alpha} f(z)} - 1 \right| < M.
\]

In the special case \( \Omega = q(\mathcal{U}) = \{ w : |w - 1| < M \} \), the class \( \Phi_{T,2}[\Omega, M] \) is simply denoted by \( \Phi_{T,2}[M] \), and Corollary 3.24 has the following form:

**Corollary 3.25.** Let \( \phi \in \Phi_{T,2}[M] \). If the function \( f \in \mathcal{A} \) satisfies
\[
\left| \frac{T_{\alpha+1} f(z)}{T_{\alpha+2} f(z)} - 1 \right| < M,
\]
then
\[
\left| \phi \left( \frac{T_{\alpha+1} f(z)}{T_{\alpha} f(z)}, \frac{T_{\alpha+2} f(z)}{T_{\alpha} f(z)}, \frac{T_{\alpha+3} f(z)}{T_{\alpha} f(z)}, \frac{T_{\alpha+4} f(z)}{T_{\alpha} f(z)} ; z \right) \right| < M.
\]

## 4 Superordination of the integral operator \( T_{\alpha} f(z) \)

In this section, the third-order differential subordination theorem for the operator \( T_{\alpha} f(z) \) defined by (3) is investigated. For this purpose, the class of admissible functions is given in the following definition.

**Definition 4.1.** Let \( \Omega \) be a set in \( \mathbb{C} \). \( c_{\alpha}, c_{\alpha+1}, c_{\alpha+2} \in \mathbb{C} \setminus \{0\} \) and \( q \in \mathcal{H}_0 \) with \( q'(z) \neq 0 \). The class of admissible functions \( \Phi_{T}[\Omega, q] \) consists of those functions \( \phi : \mathbb{C} \times \mathcal{U} \longrightarrow \mathbb{C} \) that satisfy the following admissibility condition:
\[
\phi(v, w, x, y; \xi) \in \Omega
\]
whenever
\[
v = q(z), \ w = \frac{zq'(z) + m(c_{\alpha} - 1)q(z)}{mc_{\alpha}},
\]
\[
\Re \left( \frac{c_{\alpha}c_{\alpha+1}x - (c_{\alpha} - 1)(c_{\alpha+1} - 1)u}{c_{\alpha}w - (c_{\alpha} - 1)u} - (c_{\alpha} + c_{\alpha+1} - 2) \right) \leq \frac{1}{m} \Re \left( \frac{zq''(z)}{q'(z)} + 1 \right),
\]
and
\[
\Re \left( \frac{c_{\alpha}(c_{\alpha+1}c_{\alpha+2})y - (c_{\alpha} - 1)(c_{\alpha+1} - 1)(c_{\alpha+2} - 1)u}{c_{\alpha}w - (c_{\alpha} - 1)u} - (c_{\alpha} + c_{\alpha+1} + c_{\alpha+2}) \left( \frac{c_{\alpha}c_{\alpha+1}x - (c_{\alpha} - 1)(c_{\alpha+1} - 1)u}{c_{\alpha}w - (c_{\alpha} - 1)u} - (c_{\alpha} + c_{\alpha+1} - 1) \right) \right. \\
\left. - (c_{\alpha} + c_{\alpha+1} + c_{\alpha+2}) \left( \frac{c_{\alpha}c_{\alpha+1}x - (c_{\alpha} - 1)(c_{\alpha+1} - 1)u}{c_{\alpha}w - (c_{\alpha} - 1)u} - (c_{\alpha} + c_{\alpha+1} - 1) \right) \right) \leq \frac{1}{m^2} \Re \left( \frac{z^2q'''(z)}{q'(z)} \right),
\]
where \( z \in \mathcal{U}, \ \zeta \in \partial \mathcal{U} \setminus E(q) \) and \( m \geq 2 \).

**Theorem 4.2.** Let \( \phi \in \Phi_{T}[\Omega, q] \). If the function \( f \in \mathcal{A}, T_{\alpha} f(z) \in \mathcal{Q}_0 \) and \( q \in \mathcal{H}_0 \) with \( q'(z) \neq 0 \) satisfy the following conditions:
\[
\Re \left( \frac{zq''(z)}{q'(z)} \right) \geq 0, \quad \left| \frac{T_{\alpha+1} f(z)}{q'(z)} \right| \leq m,
\]
and
\[
\phi \left( T_{\alpha} f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z \right)
\]
is univalent in \( U \), then
\[
\Omega \subset \{ \phi(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z) : z \in U \}.
\] (46)

implies that
\[
q(z) < T_\alpha f(z) \quad (z \in U).
\]

**Proof.** Let the function \( p(z) \) be defined by (10) and \( \psi \) by (16). Since \( \phi \in \Phi_2', \psi \in \Psi_2' \), (16) and (47) yield
\[
\Omega \subset \{ \psi(p(z)), z p'(z), z^2 p''(z), z^3 p'''(z); z \in U \}.
\]

From equations (14) and (15), we see that the admissible condition for \( \phi \in \Phi_2' \) in Definition 4.1 is equivalent to the admissible condition for \( \psi \) as given in Definition 2.6 with \( n = 2 \). Hence \( \psi \in \Psi_2' \), and by using (46) and Theorem 2.7, we have
\[
q(z) < p(z) = T_\alpha f(z).
\]

If \( \Omega \neq \mathbb{C} \) is a simply connected domain, then \( \Omega = h(U) \), for some conformal mapping \( h(z) \) of \( U \) onto \( \Omega \). In this case, the class \( \Phi_2'[h(U), q] \) is written as \( \Phi_2'[h, q] \). The following result follows immediately as a consequence of Theorem 4.2.

**Theorem 4.3.** Let \( \phi \in \Phi_2' \) and the function \( h \) be analytic in \( U \). If the function \( f \in A \), \( T_\alpha f(z) \in \mathcal{Q}_0 \) and \( q \in \mathcal{H}_0 \) with \( q'(z) \neq 0 \) satisfy the following conditions (46) and
\[
\phi(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z).
\]
is univalent in \( U \), then
\[
h(z) < \phi(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z)
\] (47)

implies that
\[
q(z) < T_\alpha f(z) \quad (z \in U).
\]

Theorem 4.2 and 4.3 can only be used to obtain subordinations of the third-order differential superordination of the forms (47) or (48). The following theorem proves the existence of the best subordinant of (48) for a suitable chosen \( \phi \).

**Theorem 4.4.** Let the function \( h \) be analytic in \( U \) and let \( \phi : \mathbb{C}^4 \times U \to \mathbb{C} \) and \( \psi \) be given by (16). Suppose that the differential equation
\[
(\psi(q(z)), z q'(z), z^2 q''(z), z^3 q'''(z); z) = h(z).
\] (48)

has a solution \( q(z) \in \mathcal{Q}_0 \). If the function \( f \in A \), \( T_\alpha f(z) \in \mathcal{Q}_0 \) and \( q \in \mathcal{H}_0 \) with \( q'(z) \neq 0 \) satisfy the following conditions (46) and
\[
\phi(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z).
\]
is univalent in \( U \), then
\[
h(z) < \phi(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z)
\]
implies that
\[
q(z) < T_\alpha f(z) \quad (z \in U).
\]

and \( q \) is the best subordinant.

**Proof.** In view of Theorem 4.1 and Theorem 4.3 we deduce that \( q \) is a subordinant of (48). Since \( q \) satisfies (49), it is also a solution of (48) and therefore \( q \) will be subordinated by all subordinants. Hence \( q \) is the best subordinant.

Combining Theorems 3.4 and 4.3, we obtain the following sandwich-type theorem.
Corollary 4.5. Let \( h_1 \) and \( q_1 \) be analytic functions in \( U \). \( h_2 \) be univalent function in \( U \), \( q_2 \in Q_0 \) with \( q_1(0) = q_2(0) = 0 \) and \( \phi \in \Phi_T[h_2, q_2] \cap \Phi_T[h_1, q_1] \). If the function \( f \in A \), \( T_a f(z) \in Q_0 \cap H_0 \), and

\[
\phi \left( T_a f(z), T_{a+1} f(z), T_{a+2} f(z), T_{a+3} f(z) ; z \right),
\]

is univalent in \( U \), and the condition (8) and (46) are satisfied, then

\[
h_1(z) < \phi \left( T_a f(z), T_{a+1} f(z), T_{a+2} f(z), T_{a+3} f(z) ; z \right) < h_2(z)
\]

implies that

\[
q_1(z) < T_a f(z) < q_2(z) \quad (z \in U).
\]

Definition 4.6. Let \( \Omega \) be a set in \( \mathbb{C} \), \( c', c_{a+1}, c_{a+2} \in \mathbb{C} \setminus \{0\} \) and \( q \in \mathcal{H}_1 \) with \( q'(z) \neq 0 \). The class of admissible functions \( \Phi'_{T,1}[\Omega, q] \) consists of those functions \( \phi : \mathbb{C}^4 \times U \rightarrow \mathbb{C} \) that satisfy the following admissibility condition:

\[
\phi(v, w, x, y; \zeta) \in \Omega
\]

whenever

\[
v = q(z), \quad w = \frac{zq'(z) + mc_a q(z)}{mc_a},
\]

\[
\Re \left( \frac{c_{a+1} (x - v)}{w - v} - (c_a + c_{a+1}) \right) \leq \frac{1}{m} \Re \left( \frac{zq''(z)}{q'(z)} + 1 \right),
\]

and

\[
\Re \left( \frac{c_a + c_{a+2}}{w - v} - (c_a + c_{a+1} + c_{a+2} + 3) \left[ \frac{c_{a+1} (x - v)}{w - v} - (c_a + c_{a+1} + 1) \right] 
\]

\[
- \left[ (c_a + c_{a+1} + 1)(c_{a+2} + 1) + c_a c_{a+1} \right] \right) \leq \frac{1}{m^2} \Re \left( \frac{z^2 q'''(z)}{q'(z)} \right),
\]

where \( z \in U \), \( \zeta \in \partial U \) and \( m \geq 2 \).

Theorem 4.7. Let \( \phi \in \Phi'_{T,1}[\Omega, q] \). If the function \( f \in A \), \( T_a f(z) \in Q_1 \) and \( q \in \mathcal{H}_1 \) with \( q'(z) \neq 0 \) satisfy the following conditions:

\[
\Re \left( \frac{zq''(z)}{q'(z)} \right) \geq 0, \quad \left| \frac{T_{a+1} f(z)}{zq'(z)} \right| \leq m,
\]

and

\[
\phi \left( \frac{T_a f(z)}{z}, \frac{T_{a+1} f(z)}{z}, \frac{T_{a+2} f(z)}{z}, \frac{T_{a+3} f(z)}{z} ; z \right)
\]

is univalent in \( U \), then

\[
\Omega \subset \left\{ \phi \left( \frac{T_a f(z)}{z}, \frac{T_{a+1} f(z)}{z}, \frac{T_{a+2} f(z)}{z}, \frac{T_{a+3} f(z)}{z} ; z \right) : z \in U \right\},
\]

implies that

\[
q(z) < \frac{T_a f(z)}{z} \quad (z \in U).
\]

Proof. Let the function \( p(z) \) be defined by (24) and \( \psi \) by (30). Since \( \phi \in \Phi'_{T,1}[\Omega, q] \), (31) and (51) yield

\[
\Omega \subset \{ \psi \left( p(z), zp'(z), z^2 p''(z), z^3 p'''(z) ; z \right) : z \in U \}.
\]

From equations (28) and (29), we see that the admissible condition for \( \phi \in \Phi'_{T,1}[\Omega, q] \) in Definition 4.6 is equivalent to the admissible condition for \( \psi \) as given in Definition 2.6 with \( n = 2 \). Hence \( \psi \in \Psi_2[\Omega, q] \), and by using (50) and Theorem 2.7, we have

\[
q(z) < p(z) = \frac{T_a f(z)}{z}.
\]
If \( \Omega \neq \mathbb{C} \) is a simply connected domain, then \( \Omega = h(\mathbb{U}) \), for some conformal mapping \( h(z) \) of \( \mathbb{U} \) onto \( \Omega \). In this case, the class \( \Phi_{T,1}[h(\mathbb{U}), q] \) is written as \( \Phi_T'[h, q] \). The following result follows immediately as a consequence of Theorem 4.7.

**Theorem 4.8.** Let \( \phi \in \Phi_T'[h, q] \) and the function \( h \) be analytic in \( \mathbb{U} \). If the function \( f \in \mathcal{A} \), \( \frac{T_\alpha f(z)}{z} \in \mathcal{Q}_1 \) and \( q \in \mathcal{H}_1 \) with \( q'(z) \neq 0 \) satisfy the following conditions (50) and

\[
\phi \left( \frac{T_\alpha f(z)}{z}, \frac{T_{\alpha+1} f(z)}{z}, \frac{T_{\alpha+2} f(z)}{z}, \frac{T_{\alpha+3} f(z)}{z}; z \right) ,
\]

is univalent in \( \mathbb{U} \), then

\[
h(z) < \phi \left( \frac{T_\alpha f(z)}{z}, \frac{T_{\alpha+1} f(z)}{z}, \frac{T_{\alpha+2} f(z)}{z}, \frac{T_{\alpha+3} f(z)}{z}; z \right)
\]

implies that

\[
q(z) < \frac{T_\alpha f(z)}{z} \quad (z \in \mathbb{U}) .
\]

Combining Theorems 3.14 and 4.8, we obtain the following sandwich-type theorem.

**Corollary 4.9.** Let \( h_1 \) and \( q_1 \) be analytic functions in \( \mathbb{U} \), \( h_2 \) be univalent function in \( \mathbb{U} \), \( q_2 \in \mathcal{Q}_1 \) with \( q_1(0) = q_2(0) = 1 \) and \( \phi \in \Phi_{T,1}[h_2, q_2] \cap \Phi_T'[h_1, q_1] \). If the function \( f \in \mathcal{A} \), \( \frac{T_\alpha f(z)}{z} \in \mathcal{Q}_1 \cap \mathcal{H}_1 \), and

\[
\phi \left( \frac{T_\alpha f(z)}{z}, \frac{T_{\alpha+1} f(z)}{z}, \frac{T_{\alpha+2} f(z)}{z}, \frac{T_{\alpha+3} f(z)}{z}; z \right) ,
\]

is univalent in \( \mathbb{U} \), and the condition (22) and (50) are satisfied, then

\[
h_1(z) < \phi \left( \frac{T_\alpha f(z)}{z}, \frac{T_{\alpha+1} f(z)}{z}, \frac{T_{\alpha+2} f(z)}{z}, \frac{T_{\alpha+3} f(z)}{z}; z \right) < h_2(z)
\]

implies that

\[
q_1(z) < \frac{T_\alpha f(z)}{z} < q_2(z) \quad (z \in \mathbb{U}) .
\]

**Definition 4.10.** Let \( \Omega \) be a set in \( \mathbb{C}, c_\alpha, c_\alpha+1, c_\alpha+2, c_\alpha+3 \in \mathbb{C} \setminus \{0\} \) and \( q \in \mathcal{H}_1 \) with \( q'(z) \neq 0 \). The class of admissible functions \( \Phi_T'[\Omega, q] \) consists of those functions \( \phi : \mathbb{C}^4 \times \overline{\mathbb{U}} \rightarrow \mathbb{C} \) that satisfy the following admissibility condition:

\[
\phi(v, w, x, y; \xi) \in \Omega
\]

whenever

\[
v = q(z), \quad w = \frac{1}{c_\alpha+1} \left[ \frac{z q'(z)}{mq(z)} + c_\alpha q(z) + c_\alpha+1 - c_\alpha \right],
\]

\[
\left[ c_\alpha+1(w-1) - c_\alpha(2w-1) + \frac{c_\alpha+1 w [c_\alpha+2(x - 1) - c_\alpha+1(w - 1)]}{c_\alpha(w - 1) - c_\alpha(v - 1)} \right] \leq \frac{1}{m} \left[ \frac{z q''(z)}{q'(z)} + 1 \right] ,
\]

and

\[
\left[ c_\alpha+3(y - 1) - c_\alpha+2 - c_\alpha+1(w - 1) - \frac{c_\alpha+1 w [c_\alpha+2(x - 1) - c_\alpha+1(w - 1)]}{c_\alpha+2 + c_\alpha+1(w - 1)} \right] \left[ c_\alpha+2 + c_\alpha+1(w - 1) + \frac{c_\alpha+1 w [c_\alpha+2(x - 1) - c_\alpha+1(w - 1)]}{c_\alpha+2 + c_\alpha+1(w - 1)} \right] \left[ c_\alpha+1 w^2 [c_\alpha+2(x - 1) - c_\alpha+1(w - 1)] + [c_\alpha+1 w [c_\alpha+2(x - 1) - c_\alpha+1(w - 1)]]^2 \right] \\
\times \left( \frac{(c_\alpha+1 w)^{-1} v}{c_\alpha+1(w - 1) - c_\alpha(v - 1)} \right) - (c_\alpha v^2 + 3) [c_\alpha+1(w - 1) - c_\alpha(2w - 1) - 1 +
\]
\[
\frac{c_{\alpha+1} w [c_{\alpha+2}(w-1) - c_{\alpha+1}(w-1)]}{c_{\alpha+1}(w-1) - c_{\alpha}(w-1)} + 3c_{\alpha+1} w v [c_{\alpha+2}(w-1) - c_{\alpha+1}(w-1)] \\
- v(1 + c_{\alpha} v) - 3c_{\alpha} v^2 [c_{\alpha+1}(w-1) - c_{\alpha}(w-1)] + 5v [c_{\alpha+1}(w-1) - c_{\alpha}(w-1)]^2
\leq \frac{1}{m^2} \mathfrak{H} \left( \frac{z^2 q'''(z)}{q'(z)} \right),
\]

where \( z \in \mathcal{U} \), \( \zeta \in \partial \mathcal{U} \setminus \mathcal{E}(q) \), \( \alpha \in \mathbb{C} \), \( n \geq 1 \) and \( m \geq 2 \).

**Theorem 4.11.** Let \( \phi \in \Phi'_{T,2}[\Omega, q] \). If the function \( f \in \mathcal{A} \), \( \frac{T_{\alpha+1} f(z)}{T_{\alpha} f(z)} \in \mathcal{Q}_1 \) and \( q \in \mathcal{H}_1 \) with \( q'(z) \neq 0 \) satisfy the following conditions:

\[
\mathfrak{H} \left( \frac{zq''(z)}{q'(z)} \right) \geq 0, \quad \left| \frac{T_{\alpha+2} f(z)}{T_{\alpha+1} f(z)} \right| \leq m, \tag{51}
\]

and

\[
\phi \left( \frac{T_{\alpha+1} f(z)}{T_{\alpha} f(z)} \cdot \frac{T_{\alpha+2} f(z)}{T_{\alpha+1} f(z)} \cdot \frac{T_{\alpha+3} f(z)}{T_{\alpha+2} f(z)} \cdot \frac{T_{\alpha+4} f(z)}{T_{\alpha+3} f(z)} ; z \right)
\]

is univalent in \( \mathcal{U} \), then

\[
\Omega \subset \left\{ \phi \left( \frac{T_{\alpha+1} f(z)}{T_{\alpha} f(z)} \cdot \frac{T_{\alpha+2} f(z)}{T_{\alpha+1} f(z)} \cdot \frac{T_{\alpha+3} f(z)}{T_{\alpha+2} f(z)} \cdot \frac{T_{\alpha+4} f(z)}{T_{\alpha+3} f(z)} ; z \right) : z \in \mathcal{U} \right\}, \tag{52}
\]

implies that

\[
q(z) < \frac{T_{\alpha+1} f(z)}{T_{\alpha} f(z)} \quad (z \in \mathcal{U}).
\]

**Proof.** Let the function \( p(z) \) be defined by (36) and \( \psi \) by (42). Since \( \phi \in \Phi'_{T,2}[\Omega, q] \), (43) and (53) yield

\[
\Omega \subset \{ \psi \left( p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z \right) : z \in \mathcal{U} \}.
\]

From equations (40) and (41), we see that the admissible condition for \( \phi \in \Phi'_{T,2}[\Omega, q] \) in Definition 4.10 is equivalent to the admissible condition for \( \psi \) as given in Definition 2.6 with \( n = 2 \). Hence \( \psi \in \Psi'_{2}[\Omega, q] \), and by using (52) and Theorem 2.7, we have

\[
q(z) < p(z) = T_{\alpha} f(z).
\]

If \( \Omega \neq \mathbb{C} \) is a simply connected domain, then \( \Omega = h(\mathcal{U}) \), for some conformal mapping \( h(z) \) of \( \mathcal{U} \) onto \( \Omega \). In this case, the class \( \Phi'_{T,2}[h(\mathcal{U}), q] \) is written as \( \Phi'_{T}[h, q] \). The following result follows immediately as a consequence of Theorem 4.11.

**Theorem 4.12.** Let \( \phi \in \Phi'_{T,2}[h, q] \) and the function \( h \) be analytic in \( \mathcal{U} \). If the function \( f \in \mathcal{A} \), \( \frac{T_{\alpha+1} f(z)}{T_{\alpha} f(z)} \in \mathcal{Q}_1 \) and \( q \in \mathcal{H}_1 \) with \( q'(z) \neq 0 \) satisfy the following conditions (52) and

\[
\phi \left( \frac{T_{\alpha+1} f(z)}{T_{\alpha} f(z)} \cdot \frac{T_{\alpha+2} f(z)}{T_{\alpha+1} f(z)} \cdot \frac{T_{\alpha+3} f(z)}{T_{\alpha+2} f(z)} \cdot \frac{T_{\alpha+4} f(z)}{T_{\alpha+3} f(z)} ; z \right)
\]

is univalent in \( \mathcal{U} \), then

\[
h(z) < \phi \left( \frac{T_{\alpha+1} f(z)}{T_{\alpha} f(z)} \cdot \frac{T_{\alpha+2} f(z)}{T_{\alpha+1} f(z)} \cdot \frac{T_{\alpha+3} f(z)}{T_{\alpha+2} f(z)} \cdot \frac{T_{\alpha+4} f(z)}{T_{\alpha+3} f(z)} ; z \right)
\]

implies that

\[
q(z) < \frac{T_{\alpha+1} f(z)}{T_{\alpha} f(z)} \quad (z \in \mathcal{U}).
\]

Combining Theorems 3.22 and 4.12, we obtain the following sandwich-type theorem.
Corollary 4.13. Let $h_1$ and $q_1$ be analytic functions in $\mathcal{U}$, $h_2$ be univalent function in $\mathcal{U}$, $q_2 \in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{T,2}[h_2,q_2] \cap \Phi_{T,2}[h_1,q_1]$. If the function $f \in A$, $\frac{T_{\alpha+1} f(z)}{T_{\alpha} f(z)} \in Q_1 \cap H_1$, and
\[
\phi \left( \frac{T_{\alpha+1} f(z)}{T_{\alpha} f(z)}, \frac{T_{\alpha+2} f(z)}{T_{\alpha+1} f(z)}, \frac{T_{\alpha+3} f(z)}{T_{\alpha+2} f(z)}, \frac{T_{\alpha+4} f(z)}{T_{\alpha+3} f(z)}, z \right),
\]
is univalent in $\mathcal{U}$, and the condition (34) and (52) are satisfied, then
\[
h_1(z) \prec \phi \left( \frac{T_{\alpha+1} f(z)}{T_{\alpha} f(z)}, \frac{T_{\alpha+2} f(z)}{T_{\alpha+1} f(z)}, \frac{T_{\alpha+3} f(z)}{T_{\alpha+2} f(z)}, \frac{T_{\alpha+4} f(z)}{T_{\alpha+3} f(z)}, z \right) \prec h_2(z)
\]
implies that
\[
q_1(z) \prec \frac{T_{\alpha+1} f(z)}{T_{\alpha} f(z)} \prec q_2(z) \quad (z \in \mathcal{U}).
\]

5 Conclusion

In term of the fractional calculus in a complex domain, we defined a new fractional integral. The above operator is a generalization of several integral operators such as the Carlson-Shaffer operator. This fractional operator may be used to obtain new classes of analytic functions in the open unit disk. Moreover, we introduced a new application of the third-order differential subordination and superordination to obtain a sandwich theorem involving the new fractional integral operator. Fractional inequalities are suggested in this work by utilizing the fractional integral operator of different order. These inequalities have been shown the upper and lower cases of this operator in the unit disk.

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