Topology-optimization based design of multi-degree-of-freedom compliant mechanisms (mechanisms with multiple pseudo-mobility)

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Abstract
Unlike conventional mechanisms, compliant mechanisms produce the desired deformations by exploiting elastic strain. The mobility of a conventional mechanism is the number of independent coordinates needed to define a configuration of the mechanism. The corresponding concept applicable to compliant mechanisms is the so-called pseudo-mobility defined as the number of scalar parameters needed to identify one single desired deformation of a compliant mechanism. In the case of compliant mechanisms with multiple pseudo-mobility, only synthesis approaches for relatively simple mechanisms exist so far, while systems for more complex tasks like shape adaptation are not covered. In addition, only a limited choice of transverse loads are considered in those approaches. In this paper, a novel optimization algorithm is presented that addresses these two shortcomings. This algorithm is based on a two-step iterative procedure in which the load-case dependency of the deformation is minimized and desired deformations are imposed. The algorithm is tested on mechanisms of different complexity. It could be demonstrated that the new procedure is well suited for the synthesis of different types of compliant mechanisms.

Keywords
Compliant mechanisms, topology optimization, selective compliance, multi-degree-of-freedom, compliant parallel mechanism, design of compliant mechanisms

1. Introduction
This paper focuses on the design of so-called multiple-degree of freedom compliant mechanism. We included this terminology in the paper’s title for the sake of uniformity with many published studies of other authors, but we consider the use of the term multiple-degree of freedom, in this context, an unfortunate choice. For this reason we decided to begin the paper with a discussion on it, which, by the way, will provide a good introduction to the paper’s content.

There are two main reasons why the mentioned use of the term “degree of freedom” in conjunction with compliant mechanisms is problematic. As a term, it is used incorrectly or at least ambiguous; as a concept, it suffers from the fact of being arbitrary: no number of degrees of freedom can be assigned to a given compliant mechanism in an objective and rigorous way.

Conventional mechanisms can be idealized as systems of rigid bodies (links) connected by coupling elements (joints). For a given mechanism, the number of degrees of freedom (DoF) or mobility $n$ can be uniquely determined as a function of the number of rigid bodies as well as the number and kind of joints (Gogu, 2005). In the sense of conventional mechanisms, the term DoF is unambiguous. After defining as deformation of the mechanism any motion of the links that does not corresponds to a global rigid-body motion, it can be stated that the mobility expresses the number of kinematic quantities that need to be specified in order to determine a particular deformation of the mechanism. Since any single deformation is associated with the choice of $n$ scalar quantities, there are $\infty^n$ possible deformations.

Compliant mechanisms need elastic strain to produce their functional deformations. This imply that
they, in the general case, have to be idealized as continua, that is, as systems with a theoretically infinite number of DoFs in the structural sense. Even when a discretized model is used (e.g. a FE-Model) they usually possess a very large number of DoFs. This also apply for instance to the mechanism of Figure 1(a), which is, however, commonly seen as a single-DoF compliant mechanism, since it resembles a single-DoF conventional mechanism. Analogously, the mechanism of Figure 1(b), which is provided with an infinite number of DoFs in the structural sense as well, is denoted as a two-DoF mechanism. This shows the formal part of the problem: the concept of DoFs is used improperly in the “new” (kinematic) sense. The term DoF is thus ambiguous for compliant mechanisms. Therefore, DoFs in the structural sense will be referred to as DoFs, while DoFs in the kinematic sense will be referred to as “kinematic DoFs” or pseudo-mobility. Beyond this question of terms, there is a substantial one: The number of “kinematic” DoFs cannot be used as a characteristic of a compliant mechanism, since no way exists to determine such number in an objective and rigorous way for a given mechanism, and so this quantity is, in an analysis framework, arbitrary.

Two of the authors (Hasse and Campanile, 2009) addressed this question for linearized mechanisms on the basis of an eigenvalue analysis. Compliant mechanisms like the ones in Figure 1, for which a number of kinematic DoFs or pseudo-mobility is assumed, show a pronounced step between the $m$-th and the $(m+1)$-th eigenvalue of the stiffness matrix. Even on this basis, a quantitative and unambiguous criterion to define the pseudo-mobility of a conventional mechanism requires a convention on how large the increase between two subsequent eigenvalues must be to consider it as “pronounced step.” And such convention would be, in turn, arbitrary.

In a synthesis framework, however, the concept of kinematic DoFs or pseudo-mobility can be used in an objective and exact sense. If a compliant mechanism is designed to mimic a two-DoF conventional mechanism, specific requirements are included in the design procedure to allow $\approx^2$ deformations, each of them assigned to two scalar parameters. In this case, it can be objectively asserted that the mechanism to be designed possesses a pseudo-mobility of two. From now on, we will avoid the term DoFs because of the above described formal ambiguity.

But the pseudo-mobility remains an attribute of the design and not of the mechanism: “the mechanism is designed for a pseudo-mobility of two” is an objective statement, while “the mechanism has a pseudo-mobility of two” is not. In the following, when speaking about a compliant mechanism with a certain value of pseudo-mobility, we will mean that the considered mechanism was designed for that value of pseudo-mobility and/or fulfills corresponding, properly specified design restrictions.

In general, we can say that if a compliant mechanism is designed for $\approx^m$ desired deformations, associated to a $m$-dimensional vector space, it is designed for a pseudo-mobility of $m$.

In this paper, the modal procedure for the topology-optimization based synthesis of compliant mechanisms (Hasse, 2016; Hasse and Campanile, 2009; Kirmse et al., 2021) is extended to the case of $m>1$ (multiple pseudo-mobility). Like the original procedure, the presented procedure is valid for the geometrically linear case. The desired deformations can then be expressed as a linear combination of $m$ desired deformation modes.

2. Overview of the state of the art

Compliant mechanisms with lumped compliance (Kota and Ananthasuresh, 1995) of the kind shown in Figure 1 can be efficiently designed on the basis of the pseudo-rigid-body model. With the pseudo-rigid-body model, the solid-state joints of the compliant mechanism are replaced by classical joints (Howell and Midha, 1994, 1996). The elastic restoring effect of the deformed material is simulated by a spring that acts between the two links connected through the solid-state joint. With the help of this model, the compliant mechanism can first be designed as a conventional mechanism and then the conventional joints can be converted into suitable solid-state joints (Lu et al., 2004; Tang and Chen, 2006; Wang and Zhang, 2017). This approach, called rigid-body replacement method, is one of the two main schools in synthesis.
of compliant mechanisms. It has the advantage of translating the main part design problem into the design of a conventional mechanism and therefore accessing an established methodology. However, it is strongly limited in the kind of designs it can handle. The position of the center of relative rotation between two members changes, with respect to the members themselves, as a function of the load (Cammarata et al., 2016). In compliant mechanisms with distributed compliance (Kota and Ananthasuresh, 1995), where this effect is more pronounced, this problem has been solved for certain simple mechanisms with selected load cases (e.g. constant bending moment) (Valentini and Pennestrì, 2017; Valentini et al., 2019). In more complex mechanisms with selective compliance the replacement with conventional hinges can reveal impracticable.

If the whole potential of compliant mechanisms is to be exploited, with no a-priori restriction in the layout, the second design school, the structural-optimization based approach, is the proper choice. In the structural optimization based approach, synthesis is performed by topology optimization. This distributes material in a previously defined design space according to a given objective function and proper restrictions. In this way, any distribution of flexible and rigid regions compatible with the chosen parametrization is a possible solution, which makes this method potentially very powerful. The main disadvantage of the structural-optimization approach is that it does not allow a kinematic design. While the design of conventional mechanisms (and, as a consequence, the rigid-body replacement approach) operates on kinematic constraints which restrict the possible deformations to a clearly defined subset without the need of taking into account the loads acting on the mechanism, the structural-optimization approach operates on the functional relationship between forces and displacements. Every deformation of the mechanism is related to a specific load, which means that the structural-optimization methods are, as a matter of principle, load-case specific. A subset of the system’s DoF is used to control the mechanism by given forces or displacements (input DoFs). To address all desired deformations, the number of input DoFs must be greater than or equal to \( m \). The output DoFs are defined as the ones of relevance for the mechanism’s function and can be subjected to external forces (usually called transverse loads).

A successful optimization formulation (objective function and restrictions) must include a proper number of load cases and can become quite complex. Beyond the pseudo-mobility and the number of input DoFs, the number of output DoFs plays a particular role in this sense, since it determines the number of possible load configurations. Independently on the load-case issue, a large number of output DoF also implies that the desired deformations are defined over a large number of DoF, which must properly be processed by the optimization formulation. Hence, a large portion of published studies deals with the simplest option, provided by the so-called Single-Input/Single-Output (SISO) mechanisms. Under linear assumptions, just two independent load cases are present (with or without loads at the output port) and the optimization logic can be straightforwardly formulated by maximizing the output displacement under the action of the sole input force and minimizing it (in a multiobjective sense) when the input DoF is blocked and a force acts on the output DoF. In Frecker et al. (1997), these requirements are translated into an energy criterion that involves the so-called mutual potential energy, related to the first load case, and the strain energy, related to the second one. Owing to the presence of just one input DoF, a SISO design has a pseudo-mobility of one.

Another concept which is often used for the synthesis of SISO mechanisms is based on the maximization of the mechanical advantage. This describes the ratio between input and output forces of a compliant mechanism (see e.g. Sigmund, 1997). However, this quantity can only be considered when the mechanism is loaded at the output. The loading results from coupling with a spring of given spring constant. The two load cases described above are implicitly considered in this formulation; the requirement for the lowest possible input force corresponds to maximizing flexibility under no-load conditions, and the requirement for the highest possible output force corresponds to maximizing stiffness under transverse loads. By choosing the spring constant at the output, a weighting of the two goals is obtained. The possibility of combining a transverse load with the no-load case by coupling with a spring is often used in optimization-based synthesis methods. We will use the term “load springs” in this context in the following.

Another function used in SISO structural optimization design approaches is the geometric advantage, which is described as the ratio between the input and the output displacement of the mechanism. It is maximized, for example, in Lau et al. (2001). One more concept used is to maximize the displacement of the output DoF of the mechanism coupled with a load spring while the input DoF is loaded (Bendsøe and Sigmund, 2003a).

When leaving the SISO-terrain, the availability of structural-optimization based procedures becomes quite sparse. For mechanisms with single pseudo-mobility in the geometrically linear case, in Frecker et al. (1999) the ratio of mutual potential energy to strain energy is maximized for one input DoF, where the output displacements are included into a combined mutual potential energy by so-called virtual loads, or a weighted sum of the individual mutual potential energies is used. In Li et al. (2014), a multi-objective optimization minimizes the “mean compliance” which, like the strain energy, is a measure of the structural stiffness of the mechanism,
while maximizing a weighted sum of the output displacements. Here, transverse loads are included with the help of one load spring applied per output DoF. Another approach is to maximize the displacements of several output DoFs with load springs, as described in Bendsoe and Sigmund (2003b). Optimization of shape-adaptive mechanisms often involves minimizing the deviation of the desired shape change from the existing shape change for a given actuator force. A popular example here is the morphing wing, discussed for instance in Santer and Pellegrino (2009). In the optimization, this is additionally subjected to a constant pressure distribution as a transverse load. In Kumar et al. (2021), no transverse loads occur in the optimization, but contact forces act on the mechanism. The minimization of the error between the desired and actual shape described by Fourier descriptors is performed. For the synthesis of an antenna reflector in Lu and Kota (2003) two possible objective functions are used. One was based on a modified Fourier transformation, which includes a coordinate transformation and a sampling rate conversion, the other is the sum of the error squares related to the deviation between the desired and actual shape. Here, the deformation is controlled with a single input displacement. A load case with transverse loads is considered by applying a force of 1 N to each output node. Several comprehensive review papers have been written in the area of synthesis of compliant mechanisms with single pseudo-mobility, which include additional synthesis approaches beyond those cited here. For an even wider overview of these, the reader is referred to Zhu et al. (2020), Arumugam and Kumar (2016), and Albanesi et al. (2010).

Concerning the case of multiple pseudo-mobility, several synthesis approaches exist specifically for the synthesis of compliant parallel mechanisms consisting of a rigid working platform placed centrally in a defined design space and several, similar compliant structures arranged around it. The multiple pseudo-mobility of the considered compliant parallel mechanisms results from the fact that the working platform retains more than one rigid-body DoF. In some approaches, the compliant structures are designed separately and the result is then combined into a complete compliant parallel mechanism (Lum et al., 2013, 2015; Pham et al., 2017). In Lum et al. (2013, 2015), the compliant structures are loaded with unit loads to compute a compliance matrix. This compliance matrix refers to the six rigid-body DoFs in space. All entries of the compliance matrix are then integrated into the objective function in such a way that its minimum search maximizes the entries belonging to the desired directions of motion and minimizes the remaining entries in the multi-objective sense. Suitable assembly of the compliant structures computed in this way subsequently results in a compliant parallel mechanism with pseudo-mobility of three. A parallel mechanism with pseudo-mobility of three is also designed in Pham et al. (2017). Here, the main-diagonal elements of the stiffness matrix are integrated into the objective function. The ratio between the product of the elements belonging to desired directions of motion and the product of the elements belonging to the undesired directions of motion is minimized.

In Zhang and Zhu (2018) and Jin and Zhang (2016), the geometrical advantages of all the combinations between an input and an output DoF are collected in a Jacobi matrix. However, this only applies to the no-load case. In the objective function, the entries of the Jacobi matrix related to the desired motion are maximized or set to a given value and, in addition, the characteristic input and output stiffnesses are minimized. The characteristic input and output stiffnesses are the elements of the stiffness matrix belonging to the main diagonal and associated with the input and output DoFs. This formulation is used for parallel mechanisms with pseudo-mobility of two and three.

Further synthesis approaches were developed for compliant mechanisms of general kind. In Wang and Tai (2010), a multi-objective optimization is performed for the synthesis of a “grip-and-move manipulator” with a pseudo-mobility of two and two output DoFs, which describes on the one hand the size of the working space and on the other hand the maximum motion of the output DoFs in one desired direction by defined applied forces. No transverse loads are assumed. In Zhu et al. (2018), the weighted sum of the displacements caused by given applied forces is maximized. The approach is tested on compliant mechanisms with pseudo-mobility of two and three. The approach is suitable for synthesizing mechanisms with a number of output DoF equal to the pseudo-mobility. The transverse loads in the optimization formulation are generated by one load spring per output DoF. In Zhan and Zhang (2010), the ratio of a total mutual potential energy to the strain energy is maximized. The mutual potential energy and strain energy are calculated for each output DoF, and then combined in the objective function. Load springs act on the output DoF while computing the strain energy. This approach is tested on a mechanism with pseudo-mobility of two. In Alonso et al. (2014), mutual potential energy is weighted and summed for each input-DoF/output-DoF combination in the objective function. Then, the objective function is maximized. Virtual loads are applied to the output DoFs. This optimization approach is used, for example, to synthesize a mechanism that combines a crunching mechanism and an inverter mechanism.

In some of the presented synthesis approaches, the synthesis additionally requires that “decoupled” mechanisms result (Du et al., 2016; Hao and Li, 2015; Tang and Chen, 2006; Wang and Zhang, 2008; Zhan and Zhang, 2010; Zhu et al., 2018). According to Du et al. (2016), decoupling can be divided into output decoupling and input decoupling. The base for the
A first general limitation of state-of-the-art, optimization-based approaches is the way to cope with transverse loads. When they are considered, they enter the optimization formulation in combination with other load cases (no loads or loads at other outputs) and so they can be taken into account only in form of a weighted average, where the weights are arbitrary. For instance, when the entries of the stiffness matrix are used in the objective function, like in Pham et al. (2017), the single transverse loads are weighted with the same coefficient. There is, anyway, mostly no request that the system is insensitive to transverse loads independently on the particular load case. However, the request for this insensitivity is important for the compliant mechanism to reliably perform its function. This enables an approximation to the load independence of conventional mechanisms explained above. Insensitivity to transverse loads is advantageous in all applications, in which the transverse loads acting later on the compliant mechanism are unknown at the time of synthesis and/or variable. A popular example of this kind of application is the morphing wing. In the conventional wing with stiff control surfaces, the geometry change does not essentially depend on the external load. In a compliant solution for a morphing wing, the same should be achieved. Thus, it is not enough to design it for a limited number of load cases. A second limitation, which applies specifically to the multiple pseudo-mobility case, is that only simple kinematic specifications are treated. The focus is on decoupled mechanisms as defined above, which implies an equal number of input and output DoFs and excludes deformation modes involving more than one DoF (see Zhu et al., 2020, subsection 5.2). Methods for multiple pseudo-mobility which allows for a large number of output DoFs (typically much larger than the pseudo-mobility) would be, however, very advantageous, particularly in the area of shape-control (e.g. morphing wings). The purpose of this publication is to present a suitable approach that overcomes these two limitations in the state of the art. The goal is to enable the synthesis of mechanisms with multiple pseudo-mobility that are insensitive to transverse loads and can fulfill complex kinematic specifications in the sense specified above. Owing to this, the decoupling requirement is not further considered.

The so-called modal approach (Hasse and Campanile, 2009) distinguishes itself from other published structural-optimization based methods through its essential kinematic nature. It operates on the system’s eigenproperties, which—under certain conditions—describes preferred deformations independently on the acting loads. In this sense, and owing to its generality and applicability to systems with a large number of input and output DoF (collectively called “active DoF”—see below) (Hasse, 2016; Hasse and Campanile, 2009; Kirmse et al., 2021), the modal approach represents a paradigm change in the area of compliant mechanism synthesis. It has the potential of covering a large part of the gaps left by the present state of the art. Up to now, the modal approach was valid for design problems with single pseudo-mobility. According to the definition given in Section 1, in such a problem it is required that the deformation of the mechanisms is limited to a one-dimensional family of deformations (with each member of the family associated to a particular value of a scalar parameter). In the linear case, this reduces to the requirement that the deformation vectors are all obtainable by scaling a given vector (desired deformation mode) (Hasse, 2016; Hasse and Campanile, 2009; Kirmse et al., 2021). In this paper, the extension to multiple pseudo-mobility is described and documented, which handle kinematics described by multidimensional families of deformations or, in the linear case, deformations expressed by linear combinations of a freely selectable number of desired deformation modes.

3. Design problem

Owing to the essentially load-case independent nature of the modal approach, the distinction between input- and output-DoF falls away. Any DoF that can be subjected to loads and/or is involved in the definition of desired deformations is considered as active. The remaining DoFs are passive. As explained above, compliant mechanisms have a theoretically infinite number of DoFs. For the following considerations, they are discretized to \( p \) DoFs—for example by the finite element method. These are then divided into \( q \) active (subscript \( a \)) and \( p - q \) passive DoFs (subscript \( c \)). Due to the fact that no external forces are allowed to act on the passive DoFs, the stiffness matrix \( K \), the displacement vector \( u \) and the force vector \( f \) can be partitioned as follows:

\[
Ku = f \Leftrightarrow \begin{bmatrix} K_{aa} & K_{ac} \\ K_{ca} & K_{cc} \end{bmatrix} \begin{bmatrix} u_a \\ u_c \end{bmatrix} = \begin{bmatrix} f_a \\ 0 \end{bmatrix}
\]  

(1)

Then, \( K \) can be condensed to its active DoF’s according to Gasch and Knothe (1989):

\[
(K_{aa} - K_{ac}K_{cc}^{-1}K_{ca})u_a = \hat{K}u_a = f_a
\]  

(2)

In equations (1) and (2), a proper support of the system is already considered, so that global rigid-body displacements are suppressed. All vectors \( u_a \in \mathbb{R}^q \) are then...
possible deformations of the mechanism. The vector space $\mathbb{R}^p$ is now subdivided into two subsets: the $m$-dimensional subspace of desired deformations and its complement. This is done by specifying a vector base for the subspace of desired deformations:

$$\Phi_i = [\varphi_1 \varphi_2 \ldots \varphi_m]; \quad (3)$$

The vectors $\varphi_i, i = 1 \ldots m$ are the above introduced desired deformation modes. The eigenmodes $X_j, j = 1 \ldots q$ and the corresponding eigenvalues $\lambda_j, j = 1 \ldots q$ are determined for a compliant mechanism via the solution of the eigenvalue problem:

$$\lambda \chi = K \chi \quad (4)$$

As usual, the eigenvalues are sorted in ascending order and the eigenmodes correspondingly. The first $m$ eigenmodes are collected into the subset of kinematic eigenmodes $X_d$ and the remaining $q - m$ eigenmodes in the subset of the parasitic eigenmodes $X_{ad}$:

$$X = [X_d \mid X_{ad}] \quad (5)$$

$$X_d = [X_1 X_2 \ldots X_m]; X_{ad} = [X_{m+1} X_{m+2} \ldots X_q]; \quad (6)$$

For the following, it is required that:

$$X^T_i X_i = 1, \ i = 1 \ldots q \quad (7)$$

The eigenmodes $X_j, j = 1 \ldots q$ are, in addition to orthogonality with respect to the unit matrix $I$, orthogonal with respect to the stiffness matrix, such that holds:

$$K_{ij} = \lambda_j X_i^T X_j = 0 \quad (8)$$

With the conditions (7), the quantities

$$K_{ij}(K) = \lambda_j = X_j^T K X_i, \ i = 1 \ldots m \quad (9)$$

can be called primary stiffnesses of the compliant mechanism (Kirmse et al., 2021). The secondary stiffness is calculated as:

$$K_s(K) = \lambda_m + 1 = X_m^T K X_{m+1} \quad (10)$$

The primary stiffnesses correspond to the eigenvalues $\lambda_1 \ldots \lambda_m$ associated to the kinematic eigenmodes. The secondary stiffness corresponds to the smallest eigenvalue $\lambda_{m+1}$ associated to the parasitic eigenmodes. The eigenvalues are a measure of the strain energy required for an eigenmode under the normalization condition (7). Without such condition, the strain energy levels of two eigenmodes would not be comparable. While in ideal conventional mechanisms the deformation is a linear combination of $p$ desired modes, in compliant mechanisms it usually consists of a combination of kinematic and parasitic eigenmodes. The influence of the parasitic eigenmodes on the deformation response is increasingly reduced with an increasing value of the selectivity:

$$S = \frac{\lambda_m + 1}{\lambda_m} \quad (11)$$

The higher the selectivity, the more accurately a deformation $\bar{u}_d$ of the mechanism can be expressed as a linear combination of the kinematic eigenmodes:

$$\bar{u}_d = \alpha_1 \cdot X_1 + \alpha_2 \cdot X_2 + \ldots + \alpha_m \cdot X_m \quad (12)$$

The scaling factors $\alpha$ have the dimension of a length.

The main logistics of the modal approach for the multiple pseudo-mobility case consists in requiring:

- that the subspace of desired deformations is spanned by the kinematic eigenmodes;
- that the selectivity reaches a high value.

Up to now, three different optimization approaches have been published that allow for the synthesis of compliant mechanisms with single pseudo-mobility using the modal procedure. In its first version, the basic principle of the modal synthesis procedure is to maximize the selectivity while fixing the desired deformation as the first eigenmode (Hasse and Campanile, 2009). In a modified version of the procedure (Hasse, 2016), the volume of the stiffness ellipsoid is minimized. As a constraint, the ellipsoid is required to have a certain minimum extension in the direction described by the desired deformation. However, these procedures were found to have convergence problems or very long computation times for mechanisms with a large number of active DoFs, so a two-step iterative procedure was developed where the secondary stiffness is maximized while the primary stiffness is constrained using a restriction (Kirmse et al., 2021). This optimization procedure serves as a starting point for the extension to the case of multiple pseudo-mobility.

4. Optimization formulation

4.1. Overview

As mentioned earlier, the optimization procedure described here is based on the procedure presented in Kirmse et al. (2021) for the single pseudo-mobility case.

As mentioned, the first goal of the optimization is to determine the design variables $x_i, i = 1 \ldots r$ such that the vector space $\mathbb{K}^{kle}$ spanned by the first $m$ eigenmodes of the condensed stiffness matrix $K$ resembles as closely as possible the space $\mathbb{K}^p$ spanned by the desired deformation modes:
Planar structures are assumed for the following considerations. Extension to non-planar mechanisms can be achieved by a proper adjustment of the parametrization, that is of the functional relationship between stiffness matrix and design variables. In the present case, the ground structure method (Bendsøe and Sigmund, 2003c) was used for this, which suits the nature of compliant systems as an arrangement of flexural elements. Beams with combined axial and bending behavior are considered as an arrangement of flexural elements. Extension to non-planar mechanisms can be indirectly achieved by maximizing the secondary stiffness matrix and design variables. In the present case, for simplicity, all limits to the primary stiffnesses are set to the same value. Since the goal of the optimization algorithm involves that (13) holds, the secondary stiffnesses are fixed and the undesired modes are variable. The orthonormal base is then expanded to all structural DoFs. This step is followed by a step of the second subprocedure, in which \( \mathbf{K}(\mathbf{x}) \) is varied and improved by linear optimization. Both steps are performed alternately within a global iterative procedure, described in detail in the next subsections.

**4.2. Sub-problem 1: Calculation of the orthonormal base and expansion**

The variable part of the orthonormal base \( \mathbf{\Psi} \) for a given \( \mathbf{K} \) is calculated using the following optimization formulation:

\[
\min f(\mathbf{\Psi}_j) = \mathbf{\phi}_j^T \mathbf{K} \mathbf{\phi}_j
\]

s.t.:

\[
g_1(\mathbf{\Psi}_j) = \mathbf{\phi}_j^T \mathbf{K} \mathbf{\phi}_j = 0 \\
\vdots\\
g_m(\mathbf{\Psi}_j) = \mathbf{\phi}_m^T \mathbf{K} \mathbf{\phi}_j = 0 \\
g_{m+1}(\mathbf{\Psi}_j) = \mathbf{\phi}_1^T \mathbf{K} \mathbf{\phi}_j = 0, \quad j = 2 \ldots m
\]

The lower bound in (19) is used to prevent numerical problems (a limit greater than zero avoids singularities of the stiffness matrix) and the upper bound is introduced for physical reasons.

The presented optimization problem (16)–(19) is difficult to solve due to its nonlinearity. Therefore, it is divided into two subproblems, which renders a stepwise linear solution possible. This leads to a fast and efficient synthesis procedure, which also allows parametrizations with very large numbers of beam elements.
This optimization formulation is to be solved recursively for \( \Phi_j \) for increasing \( j \) from 1 to \( q - m \). This corresponds to the restricted search for the extreme values of the quadratic form on the right-hand side of (21) with the constraints (22) and (24). While in Kirmse et al. (2021) only one constraint was necessary for (22), there is now a set of constraints, which includes \( m \) equations. A computationally efficient solution to this problem is described in Golub (1973). In contrast to Kirmse et al. (2021), an extended formulation for this problem is described in Golub (1973). In contrast to Kirmse et al. (2021), an extended formulation for this solution has been used here which is suitable in order to take into account the constraints (22). With this approach, all \( q - m \) vectors of the orthonormal base are computed using a substitute eigenvalue problem. For the correct positioning in \( \Psi \), the vectors of the orthonormal base are then to be ordered according to the function values computed from (21) in ascending order. Then, all vectors of \( \Psi \) are expanded to all structural DoF’s as follows:

\[
\varphi_i = \begin{bmatrix} \varphi_i \\ -K_{aa}^{-1}K_{ai}\varphi_i \end{bmatrix}, \quad i = 1 \ldots m; \\
\psi_j = \begin{bmatrix} \psi_j \\ -K_{aa}^{-1}K_{aj}\psi_j \end{bmatrix}, \quad j = 1 \ldots q - m
\]

(25)

\[ g_m + j(\psi_j) = \psi_j^T K \psi_j = 0 \]

\[ g_m + j - 1(\psi_j) = \psi_{j-1}^T K \psi_j = 0 \]

\[ h(\psi_j) = \psi_j^T \psi_j = 1 \]

(24)

4.3. Sub-problem 2: Updating the design variables

The entries in the orthonormal base \( \Psi \) are used to solve the second sub-problem, where the maximization of the objective function (16) with the constraints (17), (18) and (19) is performed. The second sub-problem is as follows:

\[
\begin{align*}
\text{max } f(x) &= \psi_j^T K(x) \psi_1 \\
\text{s.t.:} & \\
g_1(x) &= \varphi_i^T K \varphi_i - 2\mu \leq 0 \\
& \vdots \\
g_m(x) &= \varphi_m^T K \varphi_m - 2\mu \leq 0 \\
h_1(x) & \ldots h_{m(2)}(x) = \varphi_i^T K \varphi_j = 0, \quad i \neq j, i = 1 \ldots m, j = 1 \ldots m \\
k_1(x) &= \psi_i^T K(x) \psi_1 - \psi_j^T K(x) \psi_2 \leq 0 \\
& \vdots \\
k_n(x) &= \psi_i^T K(x) \psi_1 - \psi_j^T K(x) \psi_n \leq 0 
\end{align*}
\]

(26)

(27)

(28)

(29)

(30)

(31)

The minimization (26) corresponds to (16), as mentioned. The constraint (27) corresponds to the constraint (17), rewritten for the uncondensed system. The constraints (30) and (31) coincide with (18) and (19).

The conditions (29) ensure that the vector which should approximate the \( m + 1 \) eigenmode of the stiffness matrix (ordered by increasing eigenvalue) does not swap its rank with another undesired deformation mode, as this could destabilize the procedure. All \( q - m \) undesirable deformation modes can be included in the optimization. In systems with many active DoFs, this can lead to an inconvenient increase of the computational cost. In this case, a smaller number of undesired modes can be used as a compromise between stability and low computational cost.

Due to its linearity, this subproblem can be solved very efficiently. This is achieved by the dual simplex algorithm (Nelder and Mead, 1965).

As a further fundamental difference to the single pseudo-mobility case the fact should be discussed that the synthesis of selective compliant mechanisms with multiple pseudo-mobility can lead to solutions in which the first \( m \) eigenmodes do not approximate the desired deformation modes. This is not of relevance, as long as the first \( m \) eigenmodes span the same space as \( \mathbb{R}^{\text{dim}} \). If the same limit \( \mu \) is set for the primary stiffness of all desired deformation modes, the solutions tend to an eigenvalue with geometric multiplicity \( m \). The space \( \mathbb{R}^{\text{dim}} \) can then be described by different sets of \( m \) linearly independent vectors, which are all eigenmodes of the mechanism’s stiffness matrix.

4.4. Global iteration procedure

It has already been mentioned that subproblems 1 and 2 must be solved step by step alternately in an iterative manner. The output of subproblem 1, that is, the extended orthonormal base, serves as the input for subproblem 2. The computed set of design variables from subproblem 2, in turn, represents the input for subproblem 1. At convergence, the final solution is found for the vector of design variables \( x \). In the following, two
consecutive steps of subproblems 1 and 2 are referred to as iteration step.

In order to stabilize the global iteration procedure, the variation of the design variables between two iteration steps must be restricted. Therefore, a restriction is added to the optimization problem, which acts in addition to restriction (31). The additionally inserted lower and upper bounds restrict the possible range of the design variables and thus the possible change of $K$. The bounds for the next iteration step are set up based on the currently calculated values for $x$ as follows:

$$x_l^i(s + 1) = x_t^i(s) + v$$
$$x_u^i(s + 1) = x_t^i(s) - v$$  \(32\)

The variable $s$ defines the current iteration step and $v$ is a process parameter. How large the elements of $v$ can be chosen depends on the one hand on the number of design variables and on the other hand on the number of stabilizing deformation modes. If the limits per step are too large and no restriction (29) is used, the optimization algorithm may not converge to a useful result because the change of the stiffness matrix between both subproblems may become too large. More information about the stabilization of the optimization procedure can be found in Kirmse et al. (2021).

For the optimization procedure, a starting vector $x_0$ must be chosen. Depending on this, the optimization can converge to different solutions since the global problem is non-convex. The procedure is illustrated in Figure 2.

### 5. Design examples

The presented optimization algorithm was implemented in MATLAB and tested for different design examples. Two mechanisms with simple kinematics as well as a shape-adaptive structure serve as design examples. The first design example is a mechanism with a combination of a rotation and straight translation as desired kinematics, corresponding to a pseudo-mobility of two. The number of active DoFs is very small here. The second example is a compliant parallel mechanism with a pseudo-mobility of two. Here, a larger number of active DoFs was chosen for the synthesis, and the desired kinematics describes translational rigid-body displacements of the platform. The last example is a shape-adaptive structure with a pseudo-mobility of three, which allows a combination of a sinusoidal deformation, a deformation in the form of a parabola and a translational displacement of its contour. Here the number of active DoFs is very large. For all examples, beam elements with the following material and geometry parameters were chosen for parametrization:

$$A = 20 \text{ mm}^2, E = 210 \text{ GPa}, I = 6.66 \text{ mm}^4$$  \(33\)

where $A$ is the cross-sectional area, $E$ the elastic modulus, and $I$ the area moment of inertia.

In the implementation of the procedure discussed in this work, all design variables were bound by the same limits according to (31):

$$x_l^i = x_l^i, i = 1...r$$
$$x_u^i = x_u^i, i = 1...r$$  \(34\)

$$v_i = v, i = 1...r$$  \(36\)

In all examples $x_l = 1 \cdot 10^{-8}$ and $x_u = 1$ were chosen. Similarly, only vectors $v$ with the same value for all entries are considered.

---

**Figure 2.** Global iteration procedure.
In all examples $\nu$ was chosen equal to 0.001.

Since, as mentioned, the solution depends on the starting value, it is necessary to perform several calculations starting from different vectors $x_0$ to get close enough to the best possible solution. Uniformly distributed random numbers between $1 \cdot 10^{-8}$ and 1 were chosen as entries in $x_0$. In addition, $\mu$ should be varied, as the achieved selectivity may vary depending on it. A detailed consideration of the influence of $\mu$ on the optimization result can be found in Kirmse et al. (2021). Here, however, $\mu$ influences not only one kinematic eigenmode, but $m$ kinematic eigenmodes. If equal values for $\mu$ are chosen, as in this case, the stiffness will be similar for all kinematic eigenmodes. The quality of the solutions is assessed according to two criteria: the level of selectivity achieved and the similarity of $K^{xd}$ to $K^d$. In each of the following sections, solutions with high selectivity are shown. In contrast to Kirmse et al. (2021), here the similarity between the desired and the obtained kinematics cannot be just assessed on the basis of the comparison of two vectors (the desired deformation mode and the kinematic eigenmode), since there are now several desired deformation modes as well as kinematic eigenmodes. The design examples show that the desired deformation modes do not always match the kinematic eigenmodes. This is not relevant, as long as the vector spaces $K^{xd}$ and $K^d$, spanned by the respective vectors, are similar. For the comparison among the vector spaces it was necessary to develop and apply new methods.

In order to evaluate the similarity of the two vector spaces to each other, the best approximation—in the least squares sense—of a desired deformation mode $\phi_i$ by a linear combination of the kinematic eigenmodes can be determined:

$$\phi_i' = \alpha_{i1} \cdot \bar{x}_1 + \alpha_{i2} \cdot \bar{x}_2 + \ldots + \alpha_{im} \cdot \bar{x}_m \ i = 1 \ldots m$$

and compared with $\phi_i$. A quantitative comparison criterion is the extended cosine similarity presented in Campanile et al. (2021), which provides one single scalar value as a similarity measure for the whole subspaces. This criterion was developed by the authors specifically for this application. It was published separately because it can also be applied to more general cases and in order not to burden this paper with mathematical details. For this, the following eigenvalue problem has to be solved first:

$$\Phi^T X_d X_d^T \Phi b = \beta b$$

Then, the extended cosine similarity is calculated as follows using the smallest eigenvalue $\beta_1$:

$$\delta_c = \sqrt{\beta_1}$$

The extended cosine similarity produces values between zero and one, with higher values for increasing similarity. It reaches the value of one if the compared subspaces coincide. If two one-dimensional spaces are compared, it supplies the cosine similarity (Singhal, 2001) of the two basis vectors.

In the following figures of the resulting designs there are some “unconnected” beam elements or beam elements which generally do not contribute to the behavior of the mechanism. In reality, however, these elements are connected to the rest of the structure. It is only for clarity that beam elements with a design variable size $< 0.001$ are not shown in the figures but are included in the final analysis of the design examples. However, if the designs shown are implemented in real structures, these beam elements can be neglected.

Since the deformation of the resulting designs is tendentially load-case independent, random load cases could be used for the figures of the deformed structures. Load cases were chosen that can well show the deformation according to the desired modes individually and linear combinations of them.

### 5.1. Mechanism with a combination of a rotation and a translation with double pseudo-mobility

To analyze the dependency of the solutions on the chosen parameterization, optimizations for two different parameterizations (labeled as parametrization 1 and 2) were performed for this design example. The design spaces chosen for this example and their dimensions can be seen in Figure 3. Here, for parametrization 1, 796 and for parametrization 2, 1770 beam elements were used. In total, the structure with parametrization 1 has 663 and with parametrization 2, 1425 structural DoFs. It was clamped at its lower side. Four active DoFs were defined, provided by the $x$- and $y$-displacements of the two green points marked in Figure 3. The first desired deformation mode $\bar{\phi}_1$ (orange solid line) describes a translation. The two selected points move horizontally by the same amount in the same direction. The second of the two chosen deformation modes $\bar{\phi}_2$ (orange dashed line) describes a rotation. Thereby the two selected points shall move vertically by the same amount in opposite directions.

$$\bar{\phi}_1 = \begin{bmatrix} \phi_{11x} \\ \phi_{11y} \\ \phi_{12x} \\ \phi_{12y} \end{bmatrix} = \begin{bmatrix} 0.707 \\ 0 \\ 0.707 \\ 0 \end{bmatrix}$$

$$\bar{\phi}_2 = \begin{bmatrix} \phi_{21x} \\ \phi_{21y} \\ \phi_{22x} \\ \phi_{22y} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.707 \\ 0 \\ 0.707 \end{bmatrix}$$
The difference between the number of active DoF (four) and the pseudo-mobility (two) provides the number of undesired modes (two). \( V \) is equal to 636.8 for parametrization 1 and to 1416 for parametrization 2, which corresponds to 80% of the maximum allowable volume. Seven uniformly distributed values of \( \mu \) between 1000 and 4000 were chosen and 100 calculations each were performed. The computation time for one example of parametrization 1 amounted about 1 min and about 5 min for parametrization 2. It must be remarked, at this point, that no speed-optimized hardware and software was used. The provided calculation times are therefore not significant in the absolute sense and are provided only to comparatively evaluate the computational costs among different design examples. The condition (29) was applied for both undesired modes.

One of the resulting designs for each parametrization and the evolution of the objective function value during the iterative process can be seen in Figure 4. It can be seen that the solutions are similar despite the different parametrization. The eigenmodes of the structure and the corresponding eigenvalues can be seen in Table 1.

The results show a large difference between the second and third eigenvalue. As described in section 1, this confirms the value of two for the pseudo-mobility. The selectivity is 27.0 for parametrization 1, 44.2 for parametrization 2 and the first two eigenvalues are very similar to another within each parametrization. According to (37), the best least square approximation for the first desired mode is:

\[
\begin{bmatrix}
0.7079 \\
-0.0019 \\
0.7055 \\
-0.0019
\end{bmatrix}
\]

\[ \Phi_1 = -0.9995 \cdot \bar{x}_1 + 0.0001 \cdot \bar{x}_2 \]

(43)

and for the second desired mode:

\[
\begin{bmatrix}
-0.0002 \\
-0.7072 \\
0.0002 \\
0.7070
\end{bmatrix}
\]

\[ \Phi_2 = -0.60 \cdot \bar{x}_1 - 0.80 \cdot \bar{x}_2 \]

(44)

Comparison with (40) and (41) show that the similarity of the vector spaces \( \mathbb{R}^{\mathcal{X}} \) and \( \mathbb{R}^{\mathcal{F}} \) is very high. The extended cosine similarity according to Equation 39 with 0.9999997 for parametrization 1 and 0.9999949 for parametrization 2 confirms this. The volume is 531.27 for parametrization 1 and 1118.9 for parametrization 2. In Figure 5 the deformed structure for parametrization 1 can be seen under different load cases. Both of the desired deformation modes can be addressed individually (Figure 5(a) and (b)), as well as a linear combination (Figure 5(c)).
5.2. Parallel mechanism with double pseudo-mobility

The synthesis of compliant parallel mechanisms is a popular topic in the literature. Therefore, this case was also chosen here to show the suitability of the presented optimization algorithm for such mechanisms. The selected design space can be seen in Figure 6. The mechanism was fixed at all four boundaries. By a suitable choice of nodes with active DoFs, a rigid platform was defined in the center of the design space, which is required to be able to perform any translation, but no rotation. The nodes with active DoFs were again highlighted with green points. Two desired deformation modes ($\phi_1$, $\phi_2$) were specified, which include all $x$- and $y$- displacements of the points bounding the platform. This results in a total of 64 active DoFs. The first desired deformation mode (orange solid line) describes the horizontal motion of the platform. This means that the entries of the $x$-displacements of the nodes all have the same value, while the $y$-displacements vanish. The second desired deformation mode (orange dashed line) describes a vertical motion of the platform. Here, all $y$-displacements have the same value, while the $x$-displacements are zero. The design space was parametrized with 6480 beam elements and 5043 structural DoFs.

Figure 4. Double pseudo-mobility mechanism with a combination of a rotation and a translation: Solutions for (a) parametrization 1 with $\mu = 1500$ and (b) parametrization 2 with $\mu = 1000$ and objective function value during the iterative process.

Table 1. Mechanism with a combination of a rotation and a translation: eigenvalue analysis.

| $i$ | 1         | 2         | 3         | 4         |
|-----|-----------|-----------|-----------|-----------|
| $\lambda_i$ | 2999.8 | 2999.9 | 80,973.6 | 368,667.9 |
| $\bar{\lambda}_i$ | 0.565 | -0.426 | -0.132 | 0.694 |
| $\bar{x}_i$ | 0.564 | -0.425 | 0.132 | -0.695 |
| $\bar{\lambda}_i$ | -0.425 | -0.565 | -0.695 | -0.132 |

Parametrization 2

| $i$ | 1         | 2         | 3         | 4         |
|-----|-----------|-----------|-----------|-----------|
| $\lambda_i$ | 1556.0 | 1999.1 | 88,267.9 | 262,347.4 |
| $\bar{\lambda}_i$ | -0.708 | -0.001 | 0.218 | -0.671 |
| $\bar{x}_i$ | 0.002 | -0.707 | 0.672 | 0.217 |
| $\bar{\lambda}_i$ | -0.706 | 0.001 | -0.215 | 0.675 |
| $\bar{\lambda}_i$ | 0.002 | 0.707 | 0.674 | 0.216 |
Here, 20 undesired modes were used to stabilize the procedure according to equation (29). The allowed volume $V$ was set to the value 2592, which corresponds to 40% of the maximum allowable volume. Five calculations each were performed with four different $\mu$ equally spaced between 500 and 2000. The computation time for one example amounted about 120 min. The selected result can be seen in Figure 7. The eigenvalues belonging to the first five eigenmodes can be seen in Table 2. Again, it can be seen that the first two eigenvalues are very similar. All other eigenvalues are much larger, and the selectivity amounts to 108.4. The first two eigenmodes are shown graphically in Figure 8(a). Again, it can be seen that they do not correspond to the desired deformation modes. However, also in this case proper similarity can be reached by linear recombination:

$$\bar{\psi}_1' = 0.72 \cdot \bar{x}_1 - 0.70 \cdot \bar{x}_2$$  \hspace{1cm} (46)$$

$$\bar{\psi}_2' = -0.70 \cdot \bar{x}_1 - 0.72 \cdot \bar{x}_2$$  \hspace{1cm} (47)$$

The similarity between $\bar{\psi}_i'$ and $\bar{\psi}_i$ can be seen in Figure 8(b) on the right. The vectors agree well, as also confirmed by the extended cosine similarity of 0.999993. The volume is 2592.

Figure 9 shows the mechanism under different loads. The platform defined by the active DoFs is represented by a white square.

5.3. Shape-adaptive structure with triple pseudo-mobility

The third example is a mechanism with triple pseudo-mobility and complex kinematics. The active DoFs, the dimensions of the structure, the parametrization, as well as the desired deformation modes can be seen in Figure 10. The parametrization consists of 4870 beam elements and the number of structural DoFs amounts to 3813, 82 of them are defined as active DoFs. These are all $x$ and $y$ displacements of the nodes on the free contour. The structure was fixed on one side. The
The optimization algorithm was stabilized with 20 modes. The allowed volume $V$ was 3409, which corresponds to 70% of the total possible volume. Five calculations each were performed with nine different values for $m$, equally spaced between 100 and 900. The computation time for one example amounted about 210 min. One of the resulting designs can be seen in Figure 11. The eigenvalues of the first 5 eigenmodes are shown in Table 3. As can be seen, here the first 3 eigenvalues are very similar, as desired. A selectivity of 12.1 was achieved.

The first three eigenmodes of the structure are shown in Figure 12(a). The second and third eigenmodes show little similarity to the desired deformation modes. However, also in this case the eigenmodes can be well approximated using a linear combination, as shown in Figure 12(b). The approximations can be achieved by the following linear combinations:

$$
\bar{\phi}_1 = -0.26 \cdot \bar{x}_1 - 0.35 \cdot \bar{x}_2 - 0.90 \cdot \bar{x}_3
$$

$$
\bar{\phi}_2 = 0.96 \cdot \bar{x}_1 - 0.13 \cdot \bar{x}_2 - 0.23 \cdot \bar{x}_3
$$

$$
\bar{\phi}_3 = -0.04 \cdot \bar{x}_1 - 0.92 \cdot \bar{x}_2 - 0.38 \cdot \bar{x}_3
$$

The approximation here is not as good as in the previous examples. Also, the extended cosine similarity of 0.9984 is slightly lower than in the previous examples. The selectivity is also lower. The volume is 2737.73. In Figure 13, selected load cases can be seen. In Figure 13(a) it can be seen that the second desired deformation mode (sinusoidal) can be addressed well, but a second force is needed to suppress the components of the first desired deformation mode (parabolic). In the deformation shown in Figure 13(b) it can be seen that the third desired deformation mode (horizontal displacement) can be well addressed with one force. Figure 13(c) shows the deformation according to the second desired deformation mode. Figure 13(d) shows a combination of all three desired deformation modes. Here it is shown that the loads can in principle be chosen arbitrarily in order to achieve a linear combination of the desired deformation modes. However, the loads

| $\lambda_i$ | 991.2 | 993.0 | 107,688.2 | 223,390.0 | 108.4 |
|------------|-------|-------|---------|----------|------|

Table 2. Compliant parallel mechanism: Eigenvalue analysis.
Figure 8. Compliant parallel mechanism: (a) Kinematic eigenmodes $\chi_i$, (b and c) comparison of the best linear combination of the kinematic eigenmodes with the desired deformation modes $\phi_i$.

Figure 9. Compliant parallel mechanism: Different load cases, (a) favorable load case for deformation of the mechanism according to $\phi_1$, (b) favorable load case for deformation of the mechanism according to $\phi_2$. 
should be chosen specifically in order to achieve a deformation corresponding to only one desired deformation mode or a linear combination of two desired deformation modes. Like the optimization algorithm presented in Kirmse et al. (2021), the optimization algorithm for mechanisms with multiple pseudo-mobility is thus also suitable for the synthesis of compliant mechanisms with a large number of active DoFs.

6. Conclusion and future work

The modal approach for the synthesis of compliant mechanisms provides a load-case independent design strategy with high potential of filling numerous gaps in the state of the art, since it inherently considers transverse load and is able to operate with a large number of output DoFs. In the paper, the extension of the modal approach to design tasks with multiple pseudo-mobility (usually referred to as design multi-DoF compliant mechanisms) is described and applied to several examples. The results show that the method is able to successfully synthesize mechanisms for a pseudo-mobility of two and three and implement a kinematics defined over a large number of DoF. The synthesis procedure presented in this paper was developed assuming small deformations. However, many compliant mechanisms perform large deformations, which is why we intend to apply this synthesis method to the case of large deformations (geometric nonlinearity) in future. This will require an appropriate extension of the procedure. It would be very interesting to combine this synthesis approach with motion control procedures, as already mentioned in Sachse and Bischoff (2021).

Table 3. Shape-adaptive structure: Eigenvalue analysis.

| i  | 1   | 2   | 3   | 4   | 5   | 5   |
|----|-----|-----|-----|-----|-----|-----|
| $\lambda_i$ | 147.9 | 154.0 | 164.8 | 1992.2 | 3917.5 | 12.1 |

Figure 10. Shape-adaptive structure: Parametrized design space and desired deformation modes $\varphi_i$.

Figure 11. Shape-adaptive structure: Solution for $\mu = 100$. 
Figure 12. Shape-adaptive structure: (a) Kinematic eigenmodes $\chi_i$, (b–d) comparison of the linear combination of kinematic eigenmodes with the desired deformation modes $\phi_i$.

Figure 13. Shape-adaptive structure: Different load cases, (a) favorable load case for deformation of the mechanism according to $\phi_2$, (b) favorable load case for deformation of the mechanism according to $\phi_3$, (c) favorable load case for deformation of the mechanism according to $\phi_1$, (d) favorable load case for deformation of the mechanism according to a linear combination of $\phi_1$, $\phi_2$, and $\phi_3$. 
A MATLAB-code of the algorithm can be found at https://www.tu-chemnitz.de/mb/mp/jimss.

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**Appendix**

**Notation**

| Symbol | Description |
|--------|-------------|
| $A$ | cross-sectional area |
| $a$ | subscript of active DoFs |
| $b$ | eigenmodes |
| $c$ | subscript of passive DoFs |
| $E$ | elastic modulus |
| $f$ | force vector |
| $f(x)$ | objective function |
| $g_i(x), i = 1 \ldots m$ | restriction functions |
| $I$ | area moment of inertia |
| $I$ | unit matrix |
| $i$ | numbering index |
| $j$ | numbering index |
| $K$ | stiffness matrix |
| $K_{pi}, i = 1 \ldots m$ | primary stiffnesses |
| $K_s$ | secondary stiffness |
| $K^\Phi$ | vector space spanned by the desired deformation modes |
| $K^x_d$ | vector space spanned by first $m$ eigenmodes of $K$ |
| $m$ | pseudo-mobility |
| $m(x)$ | restriction function |
| $n$ | mobility |
| $p$ | number of structural degrees of freedom |
| $q$ | number of active degrees of freedom |
| $r$ | number of design variables |
| $S$ | selectivity |
| $s$ | iteration step index |
| $u$ | displacement vector |
| $u_i$ | deformation of the mechanism |
| $V$ | volume limit |
| $x$ | vector of design variables |
| $x_0$ | starting vector of the design variables |
| $x_1, i = 1 \ldots r$ | elements of $x$ |
| $x_1', x_2', i = 1 \ldots r$ | limits of the design variables |
| $X_1', X_2'$ | vectors of temporary additional |
| $\alpha$ | limits of the design variables |
| $\beta$ | scaling factor |
| $\beta_1$ | eigenvalues of the extended cosine similarity problem |
| $\delta_c$ | smallest value for beta |
| $\lambda_j, j = 1 \ldots q$ | extended cosine similarity |
| $\mu$ | eigenvalues of the condensed stiffness matrix |
| $\nu$ | parameter of the primary stiffness restriction |
| $\eta$ | stabilization parameter |
\(\Phi\)
\(\varphi_i, i = 1 \ldots m\)  
subspace of desired deformations
expanded desired deformation modes

\(\bar{\varphi}_i, i = 1 \ldots m\)  
desired deformation modes
Elements of \(\bar{\varphi}\) in the design examples of 5.1 and 5.2

\(\varphi'_i\)
best approximation of a desired deformation mode

\(\bar{\Phi}\)
\(\varphi'_i, i = 1 \ldots m\)  
expanded desired deformation modes

\(\bar{\varphi}_i, i = 1 \ldots m\)  
desired deformation modes

\(\bar{\varphi}_{1x}, \bar{\varphi}_{1y}, \bar{\varphi}_{2x}, \bar{\varphi}_{2y}\)

\(i = 1 \ldots m\)

\(\bar{\varphi}'_i\)
best approximation of a desired deformation mode

\(\bar{\Phi}_{1x}, \bar{\Phi}_{1y}, \bar{\Phi}_{2x}, \bar{\Phi}_{2y}\)

\(i = 1 \ldots m\)

\(\bar{\varphi}'_i\)
best approximation of a desired deformation mode

\(\bar{\Phi}_{1x}, \bar{\Phi}_{1y}, \bar{\Phi}_{2x}, \bar{\Phi}_{2y}\)

\(i = 1 \ldots m\)

\(\bar{\varphi}'_i\)
best approximation of a desired deformation mode

\(\tilde{X}_d, \chi_j, j = 1 \ldots q, \tilde{X}_{ud}\)  
kinematic eigenmodes
eigenmodes of the condensed stiffness matrix
parasitic eigenmodes
orthonormal base
expanded undesired modes
undesired modes

\(\Psi, \psi_j, j = 1 \ldots q - m\)  
expanded undesired modes