QUOTIENTS OF AN AFFINE VARIETY
BY AN ACTION OF A TORUS

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Abstract. Let $X$ be an affine $T$-variety. We study two different quotients for the action of $T$ on $X$: the toric Chow quotient $X/C^T$ and the toric Hilbert scheme $H$. We introduce a notion of the main component $H_0$ of $H$ which parameterizes general $T$-orbit closures in $X$ and their flat limits. The main component $U_0$ of the universal family $U$ over $H$ is a preimage of $H_0$. We define an analogue of a universal family $W_X$ over the main component of the $X/C^T$. We show that the toric Chow morphism restricted on the main components lifts to a birational projective morphism from $U_0$ to $W_X$. The variety $W_X$ also provides a geometric realization of the Altmann-Hausen family. In particular, the notion of $W_X$ allows us to provide an explicit description of the fan of the Altmann-Hausen family in the toric case.

1. Introduction

An important problem in algebraic geometry is to introduce a good notion of a quotient for an action of a reductive algebraic group $G$ on a variety $X$. For many actions there exists an open subset $U \subset X$ where $G$ acts freely, such that a variety $U/G$ exists as a geometric quotient. Constructing a quotient $X/G$ is thus choosing a compactification of $U/G$.

In the case when $X$ is projective, one approach to this problem is provided by the geometric invariant theory (GIT) developed by Mumford [MFK94]. Given a $G$-equivariant embedding of $X$ in the projectivisation of a $G$-module, the GIT-quotient is the projective spectrum of the subring of $G$-invariants in the homogeneous coordinate ring on $X$. There are two another natural ways to speak about the compactification which are provided by appropriate Chow varieties of algebraic cycles and Hilbert schemes. The Chow quotient of a projective variety $X$ parameterizes the closures of $G$-orbits in $X$ having the same dimension and degree and their limits in the Chow variety of all algebraic cycles having these parameters. The Chow quotient of a toric projective variety by a subtorus action was studied in [KSZ91]. The invariant Hilbert scheme classifies closed $G$-invariant subschemes $Z \subset X$ such that the $G$-module $O(Z)$ has prescribed multiplicities (see [Br11]). The $G$-Hilbert scheme is a particular case of the invariant Hilbert scheme. It arises in the case of a finite group $G$ and is considered in [Be08]. The main component of the $G$-Hilbert scheme parameterizes regular $G$-orbits in $X$ and their flat limits. Another particular case of an invariant Hilbert scheme is the toric Hilbert scheme which will be considered below.

We are interested in the following case. Let $X$ be an affine variety and $G = T$ be an algebraic torus. Denote by $X^s$ the subset of stable points in $X$ under the torus action which is an open subset in $X$ where $T$ acts freely (see Section 3.2.2 for the precise definition).
The toric Hilbert scheme $H$ is defined as the invariant Hilbert scheme parameterizing $T$-invariant ideals in $k[X]$ having the same Hilbert function as the toric variety $TX$, where $x \in X^*$. Its main component $H_0$ is irreducible component of $H$ defined as the closure of $X^*/T$ via canonical embedding. In the case of a toric variety $X$, the toric Hilbert scheme was studied in [Ch08]. The author considered the toric Chow morphism from $H_0$ to the main component of the inverse limit of GIT-quotients $(X/cT)_0$. We include this morphism into a commutative square

\[
\begin{array}{ccc}
U_0 & \longrightarrow & W_X \\
\downarrow & & \downarrow p_C \\
H_0 & \longrightarrow & (X/cT)_0,
\end{array}
\]

and generalize it to the $T$-variety case (see Theorem 2). Here $U_0$ is the component of the universal family of the Hilbert scheme $H$ that lies over the main component $H_0$, the variety $W_X$ is an analogue of universal family over $(X/cT)_0$ which is defined as the closure of the graph of the canonical rational map $q : X \dashrightarrow (X/cT)_0$, and $p_C$ is a projection on $(X/cT)_0$.

Now let us additionally assume that the affine $T$-variety $X$ is normal. A combinatorial-geometrical description of such varieties is given in [AH06]. One can associate to $X$ a pair $(Y, \mathcal{D})$, such that $Y$ is a normal semiprojective variety (geometrical datum) and $\mathcal{D}$ is a proper polyhedral divisor on $Y$ (combinatorial datum). As an intermediate step of this correspondence there arises a normal irreducible variety $\tilde{X}$ with a faithful action of the torus $T$, together with the morphisms $\psi : \tilde{X} \rightarrow Y$, and $\varphi : \tilde{X} \rightarrow X$, such that $\psi$ is a good quotient under the action of $T$ and $\varphi$ is proper and birational (see Proposition 5):

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\varphi} & X \\
\downarrow \psi & & \downarrow \varphi \\
Y. & & 
\end{array}
\]

If one additionally requires, that divisor $\mathcal{D}$ is minimal (see [AH06, Definition 8.7]), than the variety $\tilde{X}$ is uniquely defined by the variety $X$. Under such assumption we refer to the family of $T$-varieties $\psi : \tilde{X} \rightarrow Y$ as Altmann-Hausen family. The canonical construction of the Altmann-Hausen family given in [AH06, Section 6] will be discussed in Section 2.1.3.

The variety $W_X$ provides a geometrical realization of $\tilde{X}$. The Altmann-Hausen family $\psi$ is a normalization of the morphism $p_C$ from commutative diagram (1) (see Proposition 6). This immediately gives us the canonical morphism $U_0^{\text{norm}} \rightarrow \tilde{X}$ and a description of the fan of $\tilde{X}$ in the toric case, see Corollary 1 and Theorem 3 respectively.

Authors are grateful to Ivan V. Arzhantsev for his attention to this work and fruitful discussions. We also thank Dmitri A. Timashev for many important comments. The first author is thankful to Michel Brion for stimulating ideas and useful discussions.

2. Preliminaries

We consider the category of schemes over an algebraically closed field $k$ of characteristic zero. Our main references on schemes are [EH00] and [Ha77].
A variety is a separated reduced scheme of finite type. We denote by $\mathcal{O}_Z$ the structure sheaf of a scheme $Z$, and $k[Z]$ denotes the algebra of sections of $\mathcal{O}_Z$ over $Z$.

2.1. Basic facts from toric geometry. An $n$-dimensional torus $T$ is an algebraic group isomorphic to the direct product of $n$ copies of the multiplicative group $k^\times$. For the lattices of characters and one-parameter subgroups of $T$, we use the notations $\mathfrak{X}(T) = \text{Hom}(T, k^\times)$ and $\Lambda(T) = \text{Hom}(k^\times, T)$ respectively. Denote by $\mathfrak{X}(T)_\mathbb{Q} := \mathfrak{X}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\Lambda(T)_\mathbb{Q} := \Lambda(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ the corresponding $\mathbb{Q}$-vector spaces.

For any affine scheme $X$ with an action of a torus $T$, its algebra of regular functions $k[X]$ is graded by the group $\mathfrak{X}(T)$ of characters of $T$:
\[
k[X] = \bigoplus_{\chi \in \mathfrak{X}(T)} k[X]_\chi,
\]
where $k[X]_\chi$ is the subspace of $T$-semi-invariant functions of weight $\chi$. Let
\[
\mathfrak{X}^T_X := \{ \chi \in \mathfrak{X}(T) : k[X]_\chi \neq 0 \}.
\]
If $X$ is an irreducible variety, then $\mathfrak{X}^T_X$ is a finitely generated monoid called the weight monoid. If $T$ acts on $X$ faithfully, then $\mathfrak{X}^T_X$ generates $\mathfrak{X}(T)$.

A $T$-variety is a normal variety endowed with a faithful regular action of $T$. Let $X$ be a $T$-variety. A morphism $\pi : X \to Y$ is called a good quotient for this action if $\pi$ is affine, $T$-invariant and the canonical map $\pi^*: \mathcal{O}_Y \to \pi_*(\mathcal{O}_X)^T$ is an isomorphism.

Given a scheme $S$, a family of affine $T$-schemes over $S$ is a scheme $X$ equipped with an action of $T$ and with a morphism $p : X \to S$ such that $p$ is affine, of finite type, and $T$-invariant. Then the sheaf of $\mathcal{O}_S$-algebras $p_*(\mathcal{O}_X)$ is equipped with a compatible grading by $\mathfrak{X}(T)$.

A toric variety under the torus $T$ is an irreducible $T$-variety $X$ that contains an open orbit isomorphic to $T$. We do not require $X$ to be normal. We will consider only those toric varieties $X$ that admit an open covering by affine $T$-invariant charts (all normal toric varieties satisfy this condition). Our main references on toric varieties are [CLS11], [Fu93] and [Od88].

Given a toric variety $X$, we denote by $\mathcal{C}_X \subset \Lambda(T)_\mathbb{Q}$ the associated fan. The $T$-orbits on $X$ are in order-reversing one-to-one correspondence with the cones of $\mathcal{C}_X$. If $\sigma(Z)$ is the cone in $\mathcal{C}_X$ corresponding to a $T$-orbit $Z$, then a one-parameter subgroup $\lambda \in \Lambda(T)$ lies in the interior of $\sigma(Z)$ if and only if $\lim_{s\to 0} \lambda(s)$ exists and lies in $Z$. A toric variety is determined by its fan up to normalization.

Given two toric varieties: $X_1$ under a torus $T_1$ and $X_2$ under a torus $T_2$. A morphism $\varphi : X_1 \to X_2$ is toric if $\varphi$ maps the torus $T_1$ into $T_2$ and $\varphi|_{T_1} : T_1 \to T_2$ is a group homomorphism. For the toric morphism $\varphi$ we have the following commutative diagram:
\[
\begin{array}{ccc}
T_1 \times X_1 & \xrightarrow{\varphi|_{T_1} \times \varphi} & X_2 \\
\downarrow & & \downarrow \\
T_2 \times X_2 & \xrightarrow{\varphi} & X_2.
\end{array}
\]

The morphism of algebraic groups $\varphi : T_1 \to T_2$ induces a $\mathbb{Z}$-linear map $\overline{\varphi} : \Lambda(T_1) \to \Lambda(T_2)$ and $\mathbb{Q}$-linear map $\varphi_\mathbb{Q} : \Lambda(T_1)_\mathbb{Q} \to \Lambda(T_2)_\mathbb{Q}$.

In the following two propositions $X_1$ and $X_2$ are normal toric varieties.
Proposition 1. [CLS11, Theorem 3.3.4] A morphism $\varphi : X_1 \rightarrow X_2$ is toric if and only if the corresponding $\mathbb{Z}$-linear map $\varphi^*$ is compatible with fans $\mathcal{C}_{X_1}$ and $\mathcal{C}_{X_2}$. It means that for every cone $\varsigma \in \mathcal{C}_{X_1}$ there exists a cone $\tilde{\varsigma} \in \mathcal{C}_{X_2}$ such that $\varphi^*(\varsigma) \subset \tilde{\varsigma}$.

Denote by $|\mathcal{C}|$ the support of the fan $\mathcal{C}$.

Proposition 2. [CLS11, Theorem 3.4.11] A toric morphism $\varphi : X_1 \rightarrow X_2$ is proper if and only if $\varphi^{-1}(|\mathcal{C}_{X_2}|) = |\mathcal{C}_{X_1}|$.

If $X$ is toric variety under a factortorus $T/T$, then we can consider the fan of this variety as a quasifan in $\Lambda(T)_{\mathbb{Q}}$ whose cones include the subspace $\Lambda(T)_{\mathbb{Q}} \subset \Lambda(T)$.

Proposition 3. Let $X = T x$, $Y = T y$ be toric varieties. Then the fan associated to the toric variety $T(x,y) \subset X \times Y$ is the coarsest common refinement of fans $\mathcal{C}_X$ and $\mathcal{C}_Y$ (in particular, its support is equal to $|\mathcal{C}_X| \cap |\mathcal{C}_Y|$).

Note that the action of the torus $T$ on the toric varieties $X$ and $Y$ is not required to be faithful here. So the fans $\mathcal{C}_X$ and $\mathcal{C}_Y$ should be considered as quasifans in $\Lambda(T)$.

**Proof.** Let $\lambda \in \Lambda(T)$. The limit $\lim_{t \rightarrow 0} \lambda(t)x, y)$ exists if and only if both limits $\lim_{t \rightarrow 0} \lambda(t)x$ and $\lim_{t \rightarrow 0} \lambda(t)y$ exist. The limits of the point $(x, y)$ with respect to one-parameter subgroups $\lambda_1$ and $\lambda_2$ coincide if and only if the limits with respect to $\lambda_1$ and $\lambda_2$ of both points $x$ and $y$ coincide, i.e. $\lambda_1$ and $\lambda_2$ lie in the same cone of the fans $\mathcal{C}_X$ and $\mathcal{C}_Y$. $\square$

2.2. Basic facts on the functor of points. Recall that any scheme $Z$ is characterized by its functor of points that is contravariant functor from the category of schemes to the category of sets:

$$
\mathcal{Z} : (\text{Sch})^\circ \rightarrow (\text{Set}), \quad \mathcal{Z}(X) := \text{Mor}(X, Z),
$$

where $\text{Mor}(X, Z)$ is the set of morphisms of schemes from $X$ to $Z$ over $k$ (we denote the functor of points of a scheme by the corresponding underlined letter). Each $f \in \text{Mor}(X, Y)$ defines a morphism of sets $\mathcal{Z}(f) : \mathcal{Z}(Y) \rightarrow \mathcal{Z}(X)$. For $g \in \text{Mor}(Y, Z)$ we have $\mathcal{Z}(f)(g) := g \circ f \in \text{Mor}(X, Z)$.

Let $F : (\text{Sch})^\circ \rightarrow (\text{Set})$ be an arbitrary functor. We say that scheme $Z$ represents the functor $F$ if there exists an isomorphism of functors $\mathcal{Z} \cong F$ (scheme $Z$ is also called the fine moduli space of the functor $F$). Denote by $(\text{Fun}((\text{Sch})^\circ, (\text{Set})))$ the category of contravariant functors from the category of schemes to the category of sets.

The covariant functor

$$
\underline{\text{z}} : (\text{Sch}) \rightarrow (\text{Fun}((\text{Sch})^\circ, (\text{Set})))
$$

is defined by $X \mapsto \underline{X}$. By Yoneda’s lemma, the functor $\underline{z}$ is an equivalence between the category of schemes and the full subcategory of the category of functors. In particular, it defines a natural bijection between the sets $\text{Mor}(X, Y)$ and $\text{Mor}(\underline{X}, \underline{Y})$. 
3. General constructions

3.1. The toric Chow quotient and the Almann-Hausen theory.

3.1.1. Combinatorial description of affine normal $T$-varieties. In this section we recall the description of normal affine $T$-varieties in terms of proper polyhedral divisors on a normal semiprojective variety given in [AH06].

Let $M$ be a lattice, $N$ be its dual lattice, $M_Q := M \otimes \mathbb{Z} \mathbb{Q}$, $N_Q := N \otimes \mathbb{Z} \mathbb{Q}$. Denote by $(\cdot, \cdot) : M_Q \times N_Q \to \mathbb{Q}$ the natural duality pairing, by $\sigma$ a pointed polyhedral cone in $N_Q$, by $\sigma^\vee \subset M_Q$ its dual cone and by $\text{Pol}_\sigma(N_Q)$ the set of all $\sigma$-polyhedra, i.e. polyhedra in $N_Q$ with the recession cone $\sigma$. Let $T := \text{Spec } k[M]$ be an algebraic torus with its lattice of characters equal to $M$.

To a $\sigma$-polyhedron $\Delta \in \text{Pol}_\sigma(N_Q)$ we associate its support function $h_\Delta : \sigma^\vee \to \mathbb{Q}$, defined by

$$h_\Delta(m) = \min(\langle m, \Delta \rangle) = \min(m, p).$$

**Definition 1.** A variety is called **semiprojective** if it is projective over an affine variety. Let $Y$ be a normal semiprojective variety. A $\sigma$-**polyhedral divisor** on $Y$ is a formal sum $D = \sum_z D_z \cdot Z$, where $Z$ runs over all prime divisors on $Y$, $D_z \in \text{Pol}_\sigma(N_Q)$, and $D_z = \sigma$ for all but finitely many $Z$. For $m \in \sigma^\vee$ we define the $\mathbb{Q}$-divisor $D(m) := \sum_z h_{D_z}(m) \cdot Z$ on $Y$. A $\sigma$-polyhedral divisor $D$ is called **proper** if the following two conditions hold:

1. $D(m)$ is semiample and $\mathbb{Q}$-Cartier for all $m \in \sigma^\vee$,
2. $D(m)$ is big for all $m \in \text{relint}(\sigma^\vee)$.

A $\mathbb{Q}$-Cartier divisor $D$ is called **semistable** if there exists $r > 0$, such that the linear system $|rD|$ is base point free, and **big** if there exists a divisor $D_0 \in |rD|$, for some $r > 0$, such that the complement $Y \setminus \text{Supp}(D_0)$ is affine.

For a $\mathbb{Q}$-divisor $D = \sum_z a_z \cdot Z$ let $[D] := \sum_z [a_z] \cdot Z$ be the round-down divisor of $D$, and $\mathcal{O}(D) := \mathcal{O}(\lfloor D \rfloor)$ be the corresponding sheaf of $\mathcal{O}_Y$-modules. Any $\sigma$-polyhedral divisor $D$ defines an $M$-graded quasicoherent sheaf of algebras on $Y$ as follows:

$$\mathcal{A}[Y, D] := \bigoplus_{m \in \sigma^\vee \cap M} \mathcal{O}(m).$$

Let $\mathcal{A}[Y, D] := \Gamma(Y, \mathcal{A}[Y, D])$ be the $M$-graded algebra corresponding to $D$.

**Theorem 1.** To any proper $\sigma$-polyhedral divisor $D$ on a normal semiprojective variety $Y$ one can associate a normal affine $T$-variety of dimension $\dim Y + \dim T$ defined by $X := \text{Spec } \mathcal{A}[Y, D]$.

Conversely, any normal affine $T$-variety is isomorphic to $X[Y, D]$ for some semiprojective variety $Y$ and some proper $\sigma$-polyhedral divisor $D$ on it.

We will discuss the converse correspondence in Section 3.1.3.

3.1.2. Ingredients from GIT. Let $X$ be an affine $T$-variety. Let us recall the construction of the inverse limit of GIT-quotients.

Denote $\Sigma := X^T_X$ the weight monoid. The cone $\omega$ generated by $\Sigma$ in the vector space $X(T)_Q := X(T) \otimes \mathbb{Q}$ is called the **weight cone**. For any $\chi \in \Sigma$, consider the set of its semistable points

$$X^{ss}_\chi := \bigcup_{r > 0} \bigcup_{f \in k[X]_\chi} X_f.$$
Two characters \( \chi_1 \in \Sigma \) and \( \chi_2 \in \Sigma \) are called equivalent if \( X_{\chi_1}^{ss} = X_{\chi_2}^{ss} \). Under this equivalence \( \Sigma \) decomposes into finitely many equivalence classes which are forming the GIT-fan \( Q \subset X(T)_Q \) with \( \text{supp}(Q) = \omega \) (see [BH06, Section 2]). Recall some definitions giving us construction of GIT-fan.

**Definition 2.** [BH06, Definition 2.1] For any \( x \in X \), the orbit cone \( w_x \) associated to \( x \) is the following convex cone in \( X(T) \):

\[
w_x := \text{cone}\{ \chi \in \Sigma : \exists f \in k[X]_\chi \text{ such that } f(x) \neq 0 \}.
\]

**Definition 3.** [BH06, Definition 2.8] For any character \( \chi \in \Sigma \), the associated GIT-cone \( \sigma_\chi \) is the intersection of all orbit cones containing \( \chi \):

\[
\sigma_\chi = \bigcap \{ x \in X : \chi \in w_x \}.
\]

By [BH06, Theorem 2.11], the collection of GIT-cones forms a fan \( Q \) having \( \omega \) as its support. Moreover, the following statement holds.

**Proposition 4.** [BH06, Proposition 2.9] Let \( \chi_1, \chi_2 \in \Sigma \). Then \( X_{\chi_1}^{ss} \subseteq X_{\chi_2}^{ss} \) if and only if \( \sigma_{\chi_1} \supseteq \sigma_{\chi_2} \).

For any cone \( \lambda \in Q \), denote \( X_\lambda^{ss} := X_\lambda^{ss} \), where \( \chi \) is an arbitrary character in \( \text{relint}(\lambda) \). Let \( X/\lambda T := X_\lambda^{ss}/T \) be the good quotient under the action of \( T \). Varieties \( X/\lambda T \) are called GIT-quotients. Notice also that

\[
X/\lambda T = \text{Proj} \ k[X]^{(\chi)}
\]

for arbitrary \( \chi \in \text{relint}(\lambda) \), where

\[
k[X]^{(\chi)} := \bigoplus_{r=0}^{\infty} k[X]_{r\chi}.
\]

In particular, \( X/\lambda_1 T = X//T = \text{Spec} \ k[X]^T \).

Denote by \( q_\lambda : X_\lambda^{ss} \rightarrow X/\lambda T \) the quotient map. We consider natural morphisms between GIT-quotients. Namely, if \( \lambda_1 \supseteq \lambda_2 \), where \( \lambda_1, \lambda_2 \in Q \), then we have the following commutative diagram:

\[
\begin{array}{ccc}
X_{\lambda_1}^{ss} & \xrightarrow{q_{\lambda_1}} & X_{\lambda_2}^{ss} \\
\downarrow & & \downarrow \quad q_{\lambda_2} \\
X/\lambda_1 T & \xrightarrow{p_{\lambda_1,\lambda_2}} & X/\lambda_2 T
\end{array}
\]

So the quotient maps \( q_\lambda : X_\lambda^{ss} \rightarrow X/\lambda T \) form a finite inverse system with \( q_0 : X \rightarrow X//T \) sitting at the end. We have the morphism of inverse limits

\[
q : X^{ss} := \bigcap_{\chi \in \Sigma} X_\chi^{ss} \rightarrow \lim_{\leftarrow} X/\lambda T =: X/cT.
\]

The variety \( X/cT \) is called GIT-limit.

**Definition 4.** The main component \((X/cT)_0\) of the GIT-limit \( X/cT \) is the closure of the image \( q(X^{ss}) \subset X/cT \).
The main component of the GIT-limit is also called the \emph{toric Chow quotient} (see e.g. \cite{CM07}). This terminology corresponds to results of work \cite{KSZ91}, where it was proved that in the case of a projective toric variety $X$ the main component of GIT-limit is indeed isomorphic to the Chow quotient by the action of a subtorus.

3.1.3. The Altmann-Hausen family. Now we give a construction of the Altmann-Hausen family of an affine normal $T$-variety $X$ following \cite{AH06} Section 6.

Let $Y := (X/cT)^{norm}$ be the normalization of the main component of the GIT-limit, $q^{norm} : X^{ss} \rightarrow Y$ be the normalization of morphism $q$, and $p_{\lambda} : Y \rightarrow X/\lambda T$ be the normalization of canonical morphisms from GIT-limit to the elements of the inverse system, restricted on the main component. For $\lambda_1, \lambda_2 \in \mathbb{Q}$, $\lambda_1 \geq \lambda_2$ we have the following commutative diagram:

$$
\begin{array}{cccc}
X^{ss} & \xrightarrow{q^{norm}} & X_{\lambda_1}^{ss} & \xrightarrow{q_{\lambda_1}} & Y \\
\downarrow{p_{\lambda_1}} & & \downarrow{q_{\lambda_1}} & & \downarrow{p_{\lambda_2}} \\
X/\lambda_1 T & \xrightarrow{p_{\lambda_1} \lambda_2} & X/\lambda_2 T & \xrightarrow{p_{\lambda_2}} & X \\
\end{array}
$$

As it was shown in \cite{AH06} Lemma 6.1], the morphisms $p_{\lambda_1}$ and $p_{\lambda_1 \lambda_2}$ are projective surjections with connected fibers. In particular, $Y$ is projective over the affine variety $X/\lambda T = \text{Spec } k[X]^T$. Moreover, morphisms $p_{\lambda_1}$ are birational when $\lambda \cap \text{relint}(\sigma) \neq \emptyset$ and consequently $\dim Y = \dim X - \dim T$ and $k(Y) = k(X)^T$.

A character $\chi \in \Sigma$ is called \emph{saturated}, if the algebra $k[X]^{(\chi)}$ is generated by elements of degree one. It is well known that for any $\chi \in \Sigma$ there exists an integral $n_\chi > 0$ such that $kn_\chi \chi$ is saturated for all integral $k > 0$. For multigraded version of this notion and an algebraic characterization of GIT-fan see \cite{AH07}.

For any $\lambda \in \mathbb{Q}$ and $\chi \in \text{relint}(\lambda)$ we have a sheaf of $\mathcal{O}_{X/\lambda T}$-modules: $A_{\lambda, \chi} := (q_{\lambda})_* (\mathcal{O}_{X_{\lambda}^{ss}})^{\chi}$. It is easy to see that the defined sheaf coincides with the twisting sheaf of Serre, if the variety $X/\lambda T$ is considered as $\text{Proj } k[X]^{(\chi)}$. In particular, the sheaf $A_{\lambda, \chi}$ is invertible if $\chi$ is saturated. For a saturated $\chi$ we denote by $A_{\chi} := (p_{\lambda(\chi)})^* (A_{\lambda(\chi), \chi})$ the invertible sheaves of $\mathcal{O}_Y$-modules.

Firstly, assume that $\chi$ is saturated. If $f \in k[X]_{\chi}$, $\lambda := \sigma_{\chi} \in Q$, then $X_f/\lambda T$ is an open affine subset in $X/\lambda T$ and $A_{\lambda, \chi}|_{X_f/\lambda T} = f \cdot \mathcal{O}_{X_f/\lambda T}$ (see \cite{AH06} Lemma 6.3(ii)). Denote $Y_f := p_{\lambda}^{-1}(X_f/\lambda T)$. The open sets $Y_f$ cover $Y$, and, by definition of the inverse image, we have (see \cite{AH06} Lemma 6.4(ii))

$$A_{\chi}|_{Y_f} = f \cdot \mathcal{O}_{Y_f} \subset f \cdot k(Y) = f \cdot k(X)^T = k(X)_{\chi}.$$ 

Consequently, we can consider sheaves $A_{\chi}$ as subsheaves of the constant sheaf $k(X)$ on $Y$.

For unsaturated $\chi$ we define

$$A_{\chi}(U) := \{ f \in k(X) : f^{n_\chi} \in A_{n_\chi, \chi}(U) \}.$$
The defined sheaves together constitute the $\mathcal{X}(T)$-graded sheaf of $\mathcal{O}_Y$-algebras
\[ A := \bigoplus_{\chi \in \Sigma} A_\chi. \]

From [AH06, Lemma 6.4] we can see that the multiplication is defined correctly and preserves the grading.

**Definition 5.** The Altmann-Hausen family is a family of $T$-schemes $\psi : \tilde{X} \to Y$, such that $\tilde{X} := \text{Spec}_Y A$ and the $T$-action on $\tilde{X}$ is defined by the $\mathcal{X}(T)$-grading of $A$.

By [AH06, Lemma 6.4(ii)], we have $X = \text{Spec} \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}})$.

Summarizing all that we done in this section, note that from a $T$-variety $X$ we constructed a semiprojective variety $Y$ equipped with a $\mathcal{X}(T)$-graded sheaf of $\mathcal{O}_Y$-algebras $A$, such that $X$ can be restored from this data.

Denote $\Lambda(T)_Q := \Lambda(T) \otimes}\mathbb{Z} Q$ and $\sigma := \omega^Y \subset \Lambda(T)_Q$. One can easily pass from the sheaf of algebras $A$ to the proper $\sigma$-polyhedral divisor $\mathcal{D}$ on $Y$ such that
\[ A = A[Y, \mathcal{D}], \]

To do this, let’s choose a (non-canonical) homomorphism $s : \mathcal{X}(T) \to k(X)$ such that for every $\chi \in \mathcal{X}(T)$ the function $s(\chi)$ is homogeneous of degree $\chi$. Such homeomorphisms always exist since $T$ acts on $X$ faithfully. Next, for a saturated $\chi \in \omega \cap \mathcal{X}(T)$ there exists an unique Cartier divisor $D(\chi)$ on $Y$ such that
\[ \mathcal{O}(D(\chi)) = \frac{1}{s(\chi)} \cdot A_\chi \subset k(Y). \]

For unsaturated $\chi$ define $D(\chi) := \frac{1}{n_\chi} D(n_\chi \chi)$. Now one can check that $\mathbb{Q}$-Cartier divisors $D(\chi)$ can be “glued” together to form a proper $\sigma$-polyhedral divisor $\mathcal{D}$ on $Y$ satisfying condition (2) (see [AH06, Section 6]). Divisor $\mathcal{D}$, constructed in such way, is called minimal.

We see that the variety $\tilde{X}$ appears as a middle step in the correspondence between normal affine $T$-varieties and combinatorial data of this action $(Y, \mathcal{D})$ from Theorem 1 with an additional minimality condition on $\mathcal{D}$:
\[ T : X \quad \quad T : \tilde{X} \quad \quad \quad (Y, \mathcal{D}), \]

where one of arrows is dashed, because here the correspondence is not canonical.

The main result illustrating the role of the Altmann-Hausen family in this assignment is the following one.

**Proposition 5.** [AH06, Theorem 3.1] The morphism $\psi$ is a good quotient for $T$-action. The canonical morphism $\varphi : \tilde{X} \to X$ is $T$-equivariant, birational and proper. We have the following commutative diagram:

\[
\begin{array}{ccc}
X^\text{ss} & \xrightarrow{\varphi} & X \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\tilde{\varphi}} & \tilde{X}
\end{array}
\]
3.1.4. A geometric description of the Altmann-Hausen family. Define
\[ W_X := \{(x, q(x)) : x \in X^{ss}\} \subset X \times (X/cT)_0, \]
and let \( p_C \) be projection on the second component.

Next result modulo normalization morphism was stated in [Vo10, Lemma 1] where actions of complexity one were considered, but the same proof is valid for a general case.

**Proposition 6.** The Altmann-Hausen family \( \psi : \tilde{X} \to Y = (X/cT)_0^{\text{norm}} \) is isomorphic to the normalization of the family of \( T \)-schemes \( p_C : W_X \to (X/cT)_0 \).

**Proof.** Define a morphism \( \theta : \tilde{X} \to (X/cT)_0 \) as a composition \( \iota \circ \psi \) of the morphism \( \psi : \tilde{X} \to (X/cT)_0^{\text{norm}} \) and the normalization morphism \( \iota : (X/cT)_0^{\text{norm}} \to (X/cT)_0 \). Due to commutativity of the diagram from Proposition 5 the mappings \( \varphi : \tilde{X} \to X \) and \( \theta : \tilde{X} \to (X/cT)_0 \) define a morphism \( \alpha := (\varphi, \theta) : \tilde{X} \to W_X \).

The morphism \( \iota \) is affine as a normalization morphism, \( \psi \) is affine as a good quotient, so \( \theta \) is affine as a composition of affine morphisms. The morphism \( \varphi \) is proper, so \( \alpha \) is proper. Further, \( \alpha \) is finite, because it is both proper and affine. But \( \alpha \) is also birational, hence it is the normalization morphism. \( \square \)

Note that variety \( W_X \) is equipped with the natural faithful action of the torus \( T \), which lifts to an action of \( T \) on \( \tilde{X} = W_X^{\text{norm}} \) and defines a \( \mathfrak{X}(T) \)-grading of the sheaf \( \psi_*(\mathcal{O}_{\tilde{X}})_X \). By formula \( \mathcal{A}_X = \psi_*(\mathcal{O}_{\tilde{X}})_X \), one can recover the \( \mathfrak{X}(T) \)-graded sheaf of \( \mathcal{O}_Y \)-algebras \( \mathcal{A} \), and, as well, the proper polyhedral divisor on \( Y \) (see Section 3.1.3).

Denote by \( p_C \) the projection \( W_X \to (X/cT)_0 \). Note that \( p_C \) is the good quotient and we have the following commutative diagram:

\[
\begin{array}{ccc}
W_X & \xrightarrow{p^W_X} & X \\
\downarrow{p_C} & & \downarrow\pi \\
(X/cT)_0 & \xrightarrow{p^{\text{norm}}_X} & X//T,
\end{array}
\]

where \( p^W_X \) denotes the projection on \( X \).

Conclude this section with the following example.
Example 1. Let $X = \mathbb{A}^n$, $T = k^\times$ and $T$ act on $X$ by scalar multiplication, i.e. $t \cdot (x_1, \ldots, x_n) = (tx_1, \ldots, tx_n)$. Then $\mathbb{Z}$-grading of the algebra $k[X] = k[x, y]$ is given by weights of variables $\deg(x_i) = 1$.

For this action GIT-fan consists of two cones $\lambda_0$ and $\lambda_1$:

|   | $\lambda_0$ | $\lambda_1$ |
|---|-------------|-------------|
|   | $-1$        | $0$         | $1$         |

We have

\begin{align*}
X_{ss}^{\lambda_0} &= 0, \\
X/\lambda_0 T &= X/\{x_1 = \ldots = x_n = 0\}, \\
X_{ss}^{\lambda_1} &= X\{x_1 = \ldots = x_n = 0\}, \\
X/\lambda_1 T &= \text{Proj} k[x_1, \ldots, x_n] = \mathbb{P}^{n-1}.
\end{align*}

The inverse limit $X/\lambda T$ of GIT-system is equal to the variety $X/\lambda_1 T$, so

\begin{align*}
Y &= (X/cT)_0 = X/cT = X/\lambda_1 T = \mathbb{P}^{n-1}, \\
Y^{ss} &= X^{ss}/\lambda_1 = X\{x_1 = \ldots = x_n\}.
\end{align*}

Choosing coordinate system, the map $q^\text{norm} : X^{ss} \to Y$ could be written in the following way: $q(x_1, \ldots, x_n) = [x_1 : \ldots : x_n]$. The variety $W_X \subseteq X \times (X/cT)_0 = \mathbb{A}^n \times \mathbb{P}^{n-1}$ is defined by the equations $x_i y_j = x_j y_i$, where $(x_1, \ldots, x_n)$ are coordinates on $\mathbb{A}^n$, and $[y_1 : \ldots : y_n]$ are homogeneous coordinates on $\mathbb{P}^{n-1}$, i.e. $W_X$ is the blowing-up of $\mathbb{A}^n$ at the point $(0, \ldots, 0)$. On the other hand

\begin{align*}
\tilde{X} &= \text{Spec}_Y \mathcal{A} = \text{Spec}_Y \bigoplus_{n \in \mathbb{Z}_{>0}} \mathcal{A}_n = \\
&= \text{Spec}_Y \mathcal{O}_Y \oplus \bigoplus_{n \in \mathbb{Z}_{>0}} \mathcal{A}_{\lambda_1, n} = \text{Spec}_Y \bigoplus_{n \in \mathbb{Z}_{>0}} \mathcal{O}(n) = \text{Tot}(\mathcal{O}(-1))
\end{align*}

is the total space of the tautological line bundle on $\mathbb{P}^{n-1}$. It is well-known that these varieties are isomorphic.

3.2. The toric Hilbert scheme and its main component. Let $X$ be an irreducible affine $T$-variety. Denote by $\Sigma$ the weight monoid $\mathfrak{X}^T_Y \subseteq \mathfrak{X}(T)$. In the following section we give an exposition of basic properties on multigraded Hilbert schemes and study some particular cases of this notion. See [Br11, Section 3] for some generalizations of these results on the case of $G$-variety $X$ with a reductive algebraic group $G$.

3.2.1. Definitions and basic facts on multigraded Hilbert schemes.

Definition 6. Given a function $h : \mathfrak{X}(T) \to \mathbb{N}$, a family $p : F \to S$ of affine $T$-schemes has Hilbert function $h$ if every sheaf of $\mathcal{O}_S$-modules $p_* (\mathcal{O}_F)$ is locally free of constant rank $h(\chi)$, $\chi \in \mathfrak{X}(T)$.

Note that the morphism $p$ is flat. If $h(0) = 1$, then $p$ is the good quotient of $X$ by the action of $T$.

The following definition was introduced in [HS04].

Definition 7. Given a function $h : \mathfrak{X}(T) \to \mathbb{N}$, the **Hilbert functor** is the contravariant functor $\mathcal{H}^h_{X,T}$ from the category of schemes to the category of sets assigning to any scheme $S$ the set of all closed $T$-stable subschemes $Z \subseteq S \times X$ such that the projection $p : Z \to S$ is a (flat) family of affine $T$-schemes with Hilbert function $h$. 


In [HS04, Theorem 1.1] it was proved that there exists a quasiprojective scheme $H^h_{V,T}$ which represents this functor in the case when $X$ is a finite-dimensional $T$-module $V$; the scheme $H^h_{V,T}$ is called the multigraded Hilbert scheme. In the case of an arbitrary $X$ there exists a $T$-equivariant closed immersion $X \hookrightarrow V$, where $V$ is a finite-dimensional $T$-module. Then the Hilbert functor $H^h_{X,T}$ is represented by a closed subscheme $H^h_{X,T}$ of $H^h_{V,T}$ (see [AB05, Lemma 1.6]). The scheme $H^h_{X,T}$ is called the invariant Hilbert scheme.

Recall that the universal family $U^h_{X,T}$ is an element of $H^h_{X,T}(H^h_{X,T})$ corresponding to the identity map $\{Id : H^h_{X,T} \to H^h_{X,T}\} \in H^h_{X,T}(H^h_{X,T}) := \text{Mor}(H^h_{X,T}, H^h_{X,T})$. So $U^h_{X,T}$ is the closed subscheme of $H^h_{X,T} \times X$ such that for any $Z \in H^h_{X,T}(S)$ we have $Z = U^h_{X,T} \times H^h_{X,T}S$.

In particular, for any $T$-equivariant closed immersion of $X$ in a finite-dimensional $T$-module $V$, we have the following cartesian diagram:

$$
\begin{array}{ccc}
U^h_{X,T} & \longrightarrow & U^h_{V,T} \\
\downarrow & & \downarrow \\
H^h_{X,T} & \longrightarrow & H^h_{V,T}.
\end{array}
$$

**Lemma 1.** Assume that $h(0) = 1$. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
U^h_{X,T} & \overset{p_X}{\longrightarrow} & X \\
\downarrow & & \downarrow \pi \\
H^h_{X,T} & \overset{p_X/T}{\longrightarrow} & X/T,
\end{array}
$$

where $p_X$ is the projection, and $p_X/T$ assigns to any family its quotient by $T$. The morphisms $p_X$ and $p_X/T$ are projective.

**Proof.** The commutativity of the diagram can be seen easily by considering the corresponding morphisms of functors of points. The morphism $p_X/T$ is projective by [Ch08, Lemma 3.3]. The morphism $U^h_{X,T} \to H^h_{X,T} \times X/T \times X$ is a closed embedding, so the morphism $p_X$ is projective. See also [Br11, Proposition 3.15].

3.2.2. **Definition of main component.** Let

$$X^s := \bigcap_{x \in \Sigma} \bigcup_{f \in k[x]} X_f.$$

Note that, if $x \in X^s$, then $\overline{Tx}$ has the following Hilbert function:

$$h^s_{\Sigma}(x) := \begin{cases} 1 & \text{if } x \in \Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Let $X_\Sigma$ denote the affine toric $T$-variety such that $X^*_\Sigma = \Sigma$; then $\overline{Tx} \simeq X_\Sigma$ for $x \in X^s$. We denote by $H$ the invariant Hilbert scheme $H^h_{X,T}$; it is called the toric Hilbert scheme [PS02]. So for any $x \in X^s$ we have the $k$-rational point $\overline{Tx} \in H$. Also denote $U := U^h_{X,T}$, $H = H^h_{X,T}$.

Let $X^s \times^T X_\Sigma$ be the quotient of the variety $X^s \times X_\Sigma$ by the natural action of $T$. 
Lemma 2. There exists the geometric quotient $\pi^* : X^* \to X^*/T$ and an open embedding of $X^*/T$ in the toric Hilbert scheme $H$. Moreover, we have the following Cartesian diagram:

$$
\begin{array}{ccl}
X^* \times^T X_\Sigma & \to & U \\
\downarrow & & \downarrow \text{p}_H \\
X^*/T & \to & H.
\end{array}
$$

Proof. Step 1. There exists the categorical quotient $\pi^* : X^* \to X^*/T$. Moreover, $\pi^*$ is a locally trivial bundle. Indeed, let $\chi_1, \ldots, \chi_r \in \Sigma$ be such that the algebra $k[X]$ is generated by $k[X]_{\chi_i}$, $i = 1, \ldots, r$. Then $X^*$ is covered by $T$-invariant open affine subschemes

$$
X_{f_1, \ldots, f_r} = \text{Spec} \left( k[X]_{f_1, \ldots, f_r} \right),
$$

where $f_i \in k[X]_{\chi_i}$, $f_i \neq 0$. Fix a basis $e_1, \ldots, e_d \in \mathfrak{X}(T)$. Since the characters $\chi_1, \ldots, \chi_r$ generate $\Sigma$ and, consequently, $\mathfrak{X}(T)$, we can choose non-zero elements $h_i \in k[X_{f_1, \ldots, f_r}]_{e_i}$. Then we have

$$
k[X_{f_1, \ldots, f_r}] \simeq k[X_{f_1, \ldots, f_r}]^T \otimes k[h_1^{\pm 1}, \ldots, h_d^{\pm 1}] \simeq k[X_{f_1, \ldots, f_r}]^T \otimes k[T].
$$

Note that these isomorphisms satisfy the compatibility conditions and the variety $X^*/T$ obtained by gluing of affine charts $X_f/T$ is separated. Indeed, we have to show that the diagonal morphism

$$
X_{fg}/T \to X_f/T \times X_g/T
$$

is closed for any $X_f/T = \text{Spec} \left( k[X_f]^T \right)$ and $X_g/T = \text{Spec} \left( k[X_g]^T \right)$, where $f = f_1 \cdot \ldots \cdot f_r$, $g = g_1 \cdot \ldots \cdot g_r$ and $f_i, g_i \in k[X]_{\chi_i}$. This is equivalent to show that the corresponding homomorphism of algebras

$$
k[X_f]^T \otimes k[X_g]^T \to k[X_{fg}]^T
$$

is surjective. But this is clear since for any element $\frac{h}{(fg)^r} \in k[X_{fg}]^T$ we have $\frac{h}{(fg)^r} = \frac{h}{f^r} \frac{g^r}{1}$.

Step 2. Let us prove that $X^*/T$ represents an open subfunctor of $H$. Consider the family

$$
p^* : X^* \times^T X_\Sigma \to X^*/T.
$$

We shall show that this is the universal family over $X^*/T$. Indeed, $X^* \times^T X_\Sigma$ is a locally trivial bundle over $X^*/T$ with fiber $X_\Sigma$. Further, there is a canonical closed embedding $X^* \times^T X_\Sigma \subset X \times (X^*/T)$, which is locally given by the surjective homomorphisms of algebras

$$
k[X] \otimes k[X_{f_1, \ldots, f_r}]^T \to \bigoplus_{\chi \in \Sigma} k[X_{f_1, \ldots, f_r}]_{\chi}
$$

$(X^* \times^T X_\Sigma$ is covered by the open affine subschemes $\text{Spec} \bigoplus_{\chi \in \Sigma} k[X_{f_1, \ldots, f_r}]_{\chi}$). This gives us an element in $H(X^*/T)$, i.e., we have the following cartesian diagram:

$$
\begin{array}{ccl}
X^* \times^T X_\Sigma & \to & U \\
\downarrow & & \downarrow \text{p}_H \\
X^*/T & \to & H.
\end{array}
$$

Step 3. It is clear that the image of $X^*/T$ lies in the locus $H^*$ of $H$ where the fibers of the universal family are irreducible and reduced. By [Gr67, Theorem 12.1.1], it follows
that \( p^{-1}_H(H^*) \) is an open subscheme of \( U \). Since \( p^{-1}_H(H^*) \) is \( T \)-invariant and \( p_H \) maps closed \( T \)-invariant subsets to closed subsets, it follows that \( H^* \) is an open subscheme of \( H \). We shall show that the morphism \( \Phi : X^*/T \rightarrow H^* \) is an isomorphism. Given a morphism \( \phi : S \rightarrow X^*/T \), we have the following commutative diagram:

\[
\begin{array}{ccc}
Z & \longrightarrow & X^* \times^T X_\Sigma \\
\downarrow & & \downarrow \\
S & \longrightarrow & X^*/T \longrightarrow H^*,
\end{array}
\]

where \( U^* := H^* \times_H U \) and \( Z := S \times_H U^* \) is the image of \( \phi \) in \( H^*(S) \). Note that all the squares of this diagram are cartesian. In particular, it follows that \( Z = Z^* \times^T X_\Sigma \), where \( Z^* := S \times_{X^*/T} X^* = Z \cap (S \times X^*) \).

Conversely, let us construct the inverse morphism \( \Phi' : H^* \rightarrow X^*/T \). Let \( Z \in H^*(S) \). Consider \( Z^* := Z \cap (S \times X^*) \); then \( Z^*/T = S \). So the projection \( Z^* \rightarrow X^* \) defines a morphism of quotients \( \{S \rightarrow X^*/T \} \subseteq X^*/T(S) \). Moreover, \( Z^* = S \times_{X^*/T} X^* \) and \( Z = Z^* \times^T X_\Sigma \). Thus we see that \( \Phi \circ \Phi' = \text{Id}_{H^*} \) and \( \Phi' \circ \Phi = \text{Id}_{X^*/T} \).

**Definition 8.** The main component \( H_0 \) of the toric Hilbert scheme \( H \) is the closure in \( H \) of the image of \( X^*/T \).

By Lemma 2 it follows that the main component \( H_0 \) is an irreducible component of \( H \). Consider also the main component of the universal family \( U_0 := p^{-1}_H(H_0) \). Denote by \( p_0 \) the restriction of \( p_H \) on \( U_0 \).

**Lemma 3.** The main component \( U_0 \) is an irreducible component of \( U \).

**Proof.** It is sufficient to show that \( U_0 \) is irreducible. By \([Ch08, Lemma 3.5]\), the dimension of any irreducible component of the fibre \( p^{-1}(x) \) equals \( \dim T \) for any \( x \in H \). Since \( p_0 \) is flat, this implies that the dimension of any irreducible component \( Z \) of \( U_0 \) is equal to \( \dim X \) \([Ha77, Corollary 9.6]\). It follows that \( Z \) dominates \( H_0 \). Consequently, the intersection \( Z \cap (X^* \times^T X_\Sigma) \) is non-empty and \( Z = Z \cap (X^* \times^T X_\Sigma) \). This implies that \( U_0 = X^* \times^T X_\Sigma \) is irreducible.

### 3.3. The toric Chow morphism

First of all, let us recall the construction of the toric Chow morphism \( H \rightarrow X/cT \). This construction was given in \([HS04, Section 5]\) for the case when \( X \) is a finite-dimensional \( T \)-module, but it is almost the same in the case of an affine \( T \)-variety.

For any character \( \chi \in \Sigma \) there exists \( n > 0 \) such that the \( \mathbb{Z} \)-graded algebra \( R_{n\chi} := k[X]^{(\alpha\chi)} \) is generated by its elements of degree one. Note that the scheme \( X/\sigma_cT := \text{Proj} R_{n\chi} \) represents the functor \( \mathcal{H}_{\text{Spec} R_{n\chi}, k^*}^h \), where

\[
h(m) := \begin{cases} 
1 & \text{if } r \geq 0, \\
0 & \text{otherwise}
\end{cases}
\]

(see \([Ch08, Corollary 3.5]\)). The natural morphism of functors \( \mathcal{H} \rightarrow \mathcal{H}_{\text{Spec} R_{n\chi}, k^*}^h \) induces the canonical morphism of the schemes representing this functors \( H \rightarrow X/\sigma_cT \) and it does not depend on the choice of \( n \). Moreover, by \([HS04, Lemma 7.5]\), these morphisms commute with the morphisms of the inverse system, so they define a canonical morphism \( \Psi : H \rightarrow X/cT \).
Lemma 4. There exists an open embedding $X^s/T \subset (X/cT)_0$. Moreover, we have the following cartesian diagram:

$$
\begin{array}{ccc}
X^s \times T & \xrightarrow{X \Sigma} & W_X \\
\downarrow & & \downarrow p_C \\
X^s/T & \xrightarrow{X/cT} & (X/cT)_0
\end{array}
$$

Proof. Note that in the inverse limit of GIT-quotients $X^\lambda/T$, $\lambda \in \Lambda$, it is sufficient to take the limit over $\lambda$ lying in $\Lambda^0$. Moreover, for any such $\lambda$ the quotient $X^s/T$ is an open subscheme in $X^\lambda_s/T$ and $X^s = q_\lambda^{-1}(X^s/T) \subset X^\lambda_s$. Indeed, $X^s$ is covered by its open affine $T$-stable subschemes $X_{f_1 \ldots f_r}$ (with the notation of the proof of Lemma 2). There exist $c_i > 0$ such that $\sum c_i x_i = \chi$ for some $n > 0$ and $\chi \in \text{relint} \lambda$. Then $f = f_1^{c_1} \ldots f_r^{c_r} \in k[X]_\chi$, and $X_{f_1 \ldots f_r} = X_f = q_\lambda^{-1}(X_f/T) \subset X^\lambda_s$.

This implies that we have a Cartesian diagram

$$
\begin{array}{ccc}
X^s & \xrightarrow{X^s} & X^{ss} \\
\downarrow & & \downarrow \\
X^s/T & \xrightarrow{X/cT} & X/cT.
\end{array}
$$

This means that $X^s = X^{ss} \cap (X \times X^s/T) \subset X \times X/cT$. Note also that $X^s \times T X_\Sigma = \overline{X^s} \subset X \times X^s/T$. Finally, this implies that $X^s \times T X_\Sigma = W_X \cap (X \times X^s/T) \subset X \times X/cT$.

In the following theorem we show that the toric Chow morphism $\Psi$ restricted on the main component lifts to a birational projective morphism $U^0 \to W_X$.

Theorem 2. We have the following commutative diagram:

$$
\begin{array}{ccc}
U_0 & \xrightarrow{\Phi} & W_X \\
\downarrow p^0_X & & \downarrow p^W_X \\
X & \xrightarrow{\pi} & (X/cT)_0 \\
\downarrow & & \downarrow \\
H_0 & \xrightarrow{\Psi^0} & X/T
\end{array}
$$

where $\Psi^0$ is the restriction of $\Psi$ on $H_0$ and $\Phi$ is the restriction of the morphism $\Psi^0 \times \text{Id}_X$ on $U_0 \subset H_0 \times X$. The morphisms $\Psi^0$, $\Phi$, $p^W_X$ and $p^W_X//T$ are projective. First three are also birational.

Proof. The restriction of $\Psi^0$ on $X^s/T \subset H_0$ is the identity map on $X^s/T \subset (X/cT)_0$, thus $\Psi^0$ is a birational morphism. Also, the restriction of $\Psi^0 \times \text{Id}_X$ on $X^s \subset U_0$ is the identity map on $X^s \subset W_X$, so $\Psi^0 \times \text{Id}_X$ maps $U_0$ birationally on $W_X$. By construction of
the morphisms $\Psi$ and $\Phi$, it follows that $p^W_X = p^W_X \circ \Phi$ and $p^W_X // T = p^W_X // T \circ \Psi^0$. Further, by Lemma 1, the morphisms $p^0_X$ and $p^W_X // T$ are projective. It follows that $\Phi$ and $\Psi^0$ are projective. Other assertions are obvious. □

Corollary 1. There exists a canonical birational projective morphism $U_0^\text{norm} \to \tilde{X}$.

4. Subtorus actions on affine toric varieties

In this section we start with an assumption that $X = X_\sigma$ is a normal affine toric variety under the torus $T$, $\sigma \subset \Lambda(T)_Q$ is a corresponding cone, and consider the action on $X$ of the subtorus $T \subset T$. The natural homomorphism $T \to T/T$ induces the surjective map of vector spaces that are generated by the corresponding lattices of one-parameter subgroups $\alpha : \Lambda(T)_Q \to \Lambda(T/T)_Q$. The embedding of torus $T \hookrightarrow T$ induces the surjective map of vector spaces $\beta : \mathfrak{X}(T)_Q \to \mathfrak{X}(T)_Q$, such that $\beta(\sigma^\perp) = \omega$, where $\omega \subset \mathfrak{X}(T)_Q$ is the weight cone.

4.1. The fan representation of the Altmann-Hausen family. Let $\psi : \tilde{X} \to Y$ be the Altmann-Hausen family of the action $X : T$, and let $\varphi : \tilde{X} \to X$ be the canonical morphism. We follow the notations of Section 3.1.3 related to the construction of $\psi$.

Let $\mathcal{T}x$ be the open orbit of the action $T : X$. It can be easily seen that $X^{ss} \supset \mathcal{T}x$. Denote by $x_x, x_C$ and $y$ the images of $x$ in $X/\Lambda T, (X/cT)_0$ and $Y$ respectively. Quotient maps $q_\lambda$ induce actions of the torus $T$ on GIT-quotients $X/\Lambda T$. So, $X/\Lambda T = \mathcal{T}x_\lambda$ is the toric variety under the torus $T/T_{x_\lambda}$, where $T_{x_\lambda}$ is the stabilizer. If $\lambda$ lies in the interior of $\omega$, general fibers of the morphism $q_\lambda$ contain a unique dense $T$-orbit, and consequently $\mathcal{T}x_\lambda = T$. Note that morphisms $p_{\lambda_1\lambda_2}$ in the GIT-system are $T$-equivariant, so $Y = \mathcal{T}y$ is a normal toric variety under the torus $T/T$ (see [CM07, Proposition 3.8]), and morphisms $p_\lambda$ are $T$-equivariant. Also $W_X = \mathcal{T}(x, x_C) \subset X \times (X/cT)_0$ is a toric (not necessarily normal) variety with the torus $T$.

Lemma 5. The fan $\mathcal{C}_{W_X}$ of the toric variety $W_X = \mathcal{T}(x, x_C) \subset X \times (X/cT)_0$ is the coarsest common refinement of the cone $\sigma$ and the quasifan $\alpha^{-1}(\mathcal{C}_{(X/cT)_0})$.

Proof. Note that we can consider a fan of the toric variety $W_X$ (which is not necessarily normal), because there exists a $T$-invariant affine covering $W_X = X \times U_i$, where $(X/cT)_0 = \bigcup U_i$ is an affine $T$-invariant covering of the variety $(X/cT)_0$. The existence of the last one immediately follow from the existence of a $T$-equivariant closed embedding of variety $(X/cT)_0$ in the direct product of GIT-factors.

Lemma follows now from Proposition 3 and the definition of the variety $W_X$. □

Let $\Lambda \subset \mathfrak{X}(T)_Q$ be the GIT-fan. Denote by $\Lambda^0$ the set of $\lambda \in \Lambda$ satisfying the condition $\lambda \cap \text{relint}(\omega) \neq \emptyset$. We have already seen that for $\lambda \in \Lambda^0$ the corresponding GIT-quotients $X/\lambda T$ are toric varieties under the torus $T/T$. The variety $Y$ is also toric under the torus $T/T$. The construction of the fans of these varieties is described in [CM07]. In the framework of that paper a more general case of a semiprojective toric variety $X$ corresponding to the normal fan of some polyhedra $P \subset \mathfrak{X}(T)_Q$. The case of an affine toric variety $X = X_\sigma$ considered in [Ch08, Section 5] (in this case $P = \sigma^\perp$). We shall continue our exposition following [Ch08].

For all $\chi \in \omega$ denote $P_\chi = \beta^{-1}(\chi) \cap \sigma^\perp$. Let $\lambda \in \Lambda^0$. Then for all $\chi \in \text{relint}(\lambda)$ polyhedra $P_\chi$ have the same normal fan $\mathcal{N}_\lambda \subset \Lambda(T/T)$ (see [Ch08, Remark 4.12]) which coincides with the fan of the toric variety $X/\lambda T$. The fan of the toric variety $Y$ is the
coarsest common refinement of all fans $N_\lambda$ where $\lambda \in \Lambda^0$, which is the normal fan of the Minkowski sum

$$\sum_{\lambda \in \Lambda^0} P_{\chi_\lambda},$$

where $\chi_\lambda \in \text{relint}(\lambda)$.

Having the description of the fan $C_Y = C_{(X/cT)_0}$, the following result allows us to describe the Altmann-Hausen family in the toric case.

**Theorem 3.** Let $X = X_\sigma$ be the toric variety with a big torus $T$, and let $T \subset T$ be a subtorus. Then the variety $\tilde{X}$ constructed for the action $T : X$ is a normal toric variety with the torus $T$. Its fan $C_{\tilde{X}} \subset \Lambda(T)$ is the coarsest common refinement of the quasifan $\alpha^{-1}(C_{(X/cT)_0})$ and the cone $\sigma$, where $\alpha : \Lambda(T) \to \Lambda(T/T)$ is the natural map.

**Proof.** By Proposition 6, the toric variety $\tilde{X}$ is isomorphic to the normalization of the variety $W_X$, so their fans $C_{\tilde{X}}$ and $C_{W_X}$ coincide. Now our statement follows directly from Lemma 5. □

The fact that fans $C_{\tilde{X}}$ and $C_{W_X}$ coincide (and so Theorem 3) could be proved by methods of toric geometry, without using Proposition 6. We shall give this proof below. Moreover, one can prove Proposition 6 in the toric case by using Theorem 3 and then pass on to general case.

Another proof of Theorem 3: The morphism $\varphi : \tilde{X} \to X$ is proper, so by Proposition 2 the support of the fan $C_{\tilde{X}}$ coincides with $\sigma$. The morphism $\psi : \tilde{X} \to Y$ is toric, hence by Proposition 1 the image $\alpha_Q(\tau)$ of any cone $\tau \in C_{\tilde{X}}$ is contained in some cone $\delta \in C_Y$. Also the morphism $\psi$ is a good quotient, so by [Sw99, Theorem 4.1] we have that $\delta = \alpha(\tau)$ and $\tau = \alpha^{-1}(\delta) \cap \sigma$, which exactly means that the fan $C_{\tilde{X}}$ is the coarsest common refinement of a quasifan $\alpha^{-1}(C_{(X/cT)_0})$ and the cone $\sigma$. By Lemma 5 it follows that fans $C_{W_X}$ and $C_{\tilde{X}}$ coincide. □

**Example 2.** Let us consider the action of the one-dimensional torus $T$ on the four-dimensional affine space $X = \mathbb{A}^4$, given by $t \cdot (x_1, x_2, x_3, x_4) = (tx_1, tx_2, t^{-1}x_3, t^{-1}x_4)$. Our main purpose is to construct the fan $C_{\tilde{X}}$ of the toric variety $\tilde{X}$. By the way we will construct the fans of the toric varieties $X/\lambda T$ and $Y$, where $\lambda \in \Lambda^0$.

In this case the GIT-fan consists of the three cones $\lambda_{-1}$, $\lambda_0$, and $\lambda_1$:

| cone    | $\lambda_{-1}$ | $\lambda_0$ | $\lambda_1$ |
|---------|----------------|-------------|-------------|
| $\lambda_0$ | $-1$       | $0$         | $1$         |

We will consider the torus $T$ as the subtorus of the four-dimensional torus $T$, that acts on $X$ by the rescaling of coordinates. Then $X = X_\sigma$, where $\sigma \subset \Lambda(T)_Q$ is the cone generated by the standard basis $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 0, 1)$ of $\Lambda(T)_Q$.

There is a natural immersion $\Lambda(T)_Q \hookrightarrow \Lambda(T)_Q$. We will identify the space $\Lambda(T/T)_Q$ with the orthogonal complement of $\Lambda(T)_Q$ in the space $\Lambda(T)_Q$. Then the natural map $\alpha : \Lambda(T)_Q \to \Lambda(T/T)_Q$ will be the orthogonal projection.

In the coordinates of the standard basis of $\Lambda(T)_Q$ denote $\rho_1 = (3, -1, 1, 1)$, $\rho_2 = (1, 1, -1, 3)$, $\rho_3 = (-1, 3, 1, 1)$, $\rho_4 = (1, 1, 3, -1)$ and $\rho = (1, 1, 1, 1)$. The fans of the toric varieties $X/\lambda_0 T$, $X/\lambda_{-1} T$ and $X/\lambda_1 T$ are the normal fans of the polyhedra $P_0$, $P_{-1}$ and $P_1$.
and $P_1$ respectively. After some calculations one can verify that the fan $\mathcal{C}_{X/\lambda_0 T} = \mathcal{N}_{\lambda_0}$ consists of one maximal cone
\[ \langle p_1, p_2, p_3, p_4 \rangle_{Q \geq 0}, \]
the fan $\mathcal{C}_{X/\lambda_{-1} T} = \mathcal{N}_{\lambda_{-1}}$ consists of two maximal cones
\[ \langle p_1, p_2, p_3 \rangle_{Q \geq 0} \text{ and } \langle p_3, p_4, p_1 \rangle_{Q \geq 0} \]
and the fan $\mathcal{C}_{X/\lambda_1 T} = \mathcal{N}_{\lambda_1}$ consists of two maximal cones
\[ \langle p_4, p_1, p_2 \rangle_{Q \geq 0} \text{ and } \langle p_2, p_3, p_4 \rangle_{Q \geq 0}. \]

The fan $\mathcal{C}_Y$, as the coarsest common refinement of $\mathcal{C}_{X/\lambda_0 T}$, $\mathcal{C}_{X/\lambda_{-1} T}$ and $\mathcal{C}_{X/\lambda_1 T}$, consists of four maximal cones
\[ \langle p_1, p_2, p \rangle_{Q \geq 0}, \langle p_2, p_3, p \rangle_{Q \geq 0}, \langle p_3, p_4, p \rangle_{Q \geq 0}, \text{ and } \langle p_4, p_1, p \rangle_{Q \geq 0}. \]

**GIT-system**

\[ \mathcal{C}_{X/\lambda_0 T} \quad \mathcal{C}_{X/\lambda_{-1} T} \quad \mathcal{C}_{X/\lambda_1 T} \quad \mathcal{C}_X \]

The fan $\tilde{\mathcal{C}}_X$ is the coarsest common refinement of the quasifan $\alpha^{-1}(\mathcal{C}_Y)$ and the cone $\sigma$. Denote $\mu_1 = (1, 1, 0, 0)$, $\mu_2 = (0, 0, 1, 1)$. Calculations show that $\tilde{\mathcal{C}}_X$ consists of four maximal cones
\[ \langle \mu_1, e_2, e_3 \rangle_{Q \geq 0}, \langle e_1, e_4, \mu_1, \mu_2 \rangle_{Q \geq 0}, \langle e_2, \mu_1, \mu_2, e_3 \rangle_{Q \geq 0}, \text{ and } \langle e_2, e_4, \mu_1, \mu_2 \rangle_{Q \geq 0}. \]

On the picture below you can see polyhedral slice of this fan by the hyperplane $x_1 + x_2 + x_3 + x_4 = 1$. 
4.2. **The fan of the main component of the toric Hilbert scheme.** Finally, we complete a description of all the fans of the toric varieties from commutative diagram (1) by recalling some results from [Ch08].

Consider the main component $H_0$ of the toric Hilbert scheme and the main component $U_0$ of the universal family in the toric situation with their structure of reduced schemes. As shown in [Ch08, Proposition 3.6], these varieties are toric under the tori $T/T$ and $T$ respectively. Our purpose is to recall the construction of the fans corresponding to toric varieties $H_0$ and $U_0$.

For all $\chi \in \omega \cap \mathfrak{X}(T)$ denote $P^I_\chi := \text{conv}(\beta^{-1}(\chi) \cap \mathfrak{X}(T))$.

**Remark 1.** The equality $P^I_\chi = P_\chi$ holds if and only if the polyhedra $P_\chi$ have integer vertices. Such characters $\chi$ are called integer.

We call two polyhedra equivalent if they have the same normal fan.

**Proposition 7.** [Ch08, Proposition 4.9(2)] There are only finitely many non-equivalent polyhedra $P^I_\chi$ for $\chi \in \omega$. The fan $C_{H_0}$ corresponding to the variety $H_0$ is the normal fan of the Minkowski sum of representatives of the equivalent classes.

Of course it is again equivalent to say that $C_{H_0}$ is the coarsest common refinement of the normal fans of all $P_\chi$.

The variety $U_0$ coincides with $\overline{\mathfrak{X}(y, x)} \subset H_0 \times X$, where $x \in X$ lies in the open $T$-orbit and $y := \overline{Tx} \in H_0$. From Proposition 5 it follows that the fan $C_{U_0}$ is the coarsest common refinement of the fan $C_{H_0}$ and $\sigma$.

Now we know the fans of all varieties placed in the vertices of the square from the commutative diagram (1). By Remark 1 it follows that the fan $C_{H_0}$ refines the fan $C_{(X/T)}$ and the fan $C_{U_0}$ refines the fan $C_{W_X}$, what illustrates that morphisms $\Psi^0$ and $\Phi$ are toric.

**Remark 2.** From construction of all these fans or straightforwardly from the properties of morphisms $\Psi^0$, $\Phi$ and $p^W_X$ it follows that $|C_{H_0}| = |C_{(X/T)}|$ and $|C_{U_0}| = |C_{W_X}| = \sigma$.

In Example 2 all characters $\chi \in \omega \cap \mathfrak{X}(T)$ are integer, since the fans of the toric varieties $C_{H_0}$ and $C_{(X/T)}$ coincide (and so, certainly, the fans $C_{U_0}$ and $C_{W_X}$ coincide). But this is not true in general. The example below was considered in [Ch08]. There were constructed the fans $C_{H_0}$ and $C_{(X/T)}$, and it was shown that they do not coincide in the considered situation. We shall complete this example constructing the fans $C_{U_0}$ and $C_{W_X} = C_X$.

**Example 3.** [Ch08, Example 5.3] Let $X_\sigma = \mathbb{A}^3$, $T = (k^\times)^3$ act by rescaling of coordinates, and $T = k^\times$ act by $t \cdot (x_1, x_2, x_3) = (tx_1, tx_2, t^2x_3)$. Denote by $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ the standard basis of $\Lambda(T) \subseteq \mathbb{Q}$, and let $\nu_1$, $\nu_2$, $\nu_3$ be its dual basis in $\mathfrak{X}(T) \subseteq \mathbb{Q}$.

\[
\begin{array}{ccc}
\mathfrak{X}(T) & \xrightarrow{\beta} & \mathfrak{X}(T) \\
\nu_1 & \beta & \nu_2 \\
\nu_3 & \beta & \nu_2 \\
\end{array}
\]

It is easy to see that the fan $C_{H_0}$ is the normal fan of the trapezoid $P_3$, whereas the fan $C_{(X/T)}$ is the normal fan of the triangle $P_2$. As in Example 2 we will identify the space $\Lambda(T) \subseteq \mathbb{Q}$ with the orthogonal complement of $\Lambda(T) \subseteq \mathbb{Q}$ in the space $\Lambda(T) \subseteq \mathbb{Q}$. 
Denote \( \rho_1 = (1, 1, -1), \rho_2 = (-5, 1, 2), \rho_3 = (1, -5, 2) \) in the coordinates of the standard basis of \( \Lambda(T) \). All this vectors are lying in \( \Lambda(T/T) \), and it is easy to verify that \( C_{(X/C)\otimes} \) consists of three maximal cones

\[
\langle \rho_1, \rho_2 \rangle_{\mathbb{Q} \geq 0}, \langle \rho_2, \rho_3 \rangle_{\mathbb{Q} \geq 0}, \text{ and } \langle \rho_1, \rho_3 \rangle_{\mathbb{Q} \geq 0},
\]

and \( C_{H_0} \) consists of four maximal cones

\[
\langle \rho_1, \rho_2 \rangle_{\mathbb{Q} \geq 0}, \langle \rho_1, \rho_3 \rangle_{\mathbb{Q} \geq 0}, \langle -\rho_1, \rho_2 \rangle_{\mathbb{Q} \geq 0}, \text{ and } \langle -\rho_1, \rho_3 \rangle_{\mathbb{Q} \geq 0}.
\]

We see that the fan \( C_{H_0} \) subdivides the fan \( C_{(X/C)\otimes} \) and in particular they do not coincide. If we intersect corresponding quasifans in \( \Lambda(T) \) with the cone \( \sigma = \langle e_1, e_2, e_3 \rangle_{\mathbb{Q} \geq 0} \) we get the fans \( C_{U_0} \) and \( C_{WX} \) respectively. Denote \( \kappa = (1, 1, 2), \mu_1 = (1, 1, 0), \mu_2 = (1, 0, 2), \) and \( \mu_3 = (0, 1, 2) \). Then \( C_{Wx} \) consists of three maximal cones

\[
\langle \kappa, \mu_1, e_1, \mu_2 \rangle_{\mathbb{Q} \geq 0}, \langle \kappa, \mu_1, e_2, \mu_3 \rangle_{\mathbb{Q} \geq 0}, \text{ and } \langle \kappa, \mu_2, e_3, \mu_3 \rangle_{\mathbb{Q} \geq 0},
\]

and \( C_{U_0} \) consists of four maximal cones

\[
\langle \kappa, \mu_1, e_1, \mu_2 \rangle_{\mathbb{Q} \geq 0}, \langle \kappa, \mu_1, e_2, \mu_3 \rangle_{\mathbb{Q} \geq 0}, \langle \kappa, e_3, \mu_2 \rangle_{\mathbb{Q} \geq 0}, \text{ and } \langle \kappa, e_3, \mu_3 \rangle_{\mathbb{Q} \geq 0}.
\]
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