Supporting Information for

Topology, Vorticity and Limit Cycle in a Stabilized Kuramoto-Sivashinsky Equation

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The operations \( \text{fft} \) where \( \kappa \) Note that equation numbers without the prefix “S” in the SI refer to the main paper (1).

The two-dimensional fast Fourier transform MatLab \( \text{fft} \) routines in MatLab\( ^{\text{\textregistered}} \) with indices

\[ n = 0, 1, \ldots, N - 1 \]

\[ k = 0, 1, \ldots, N/2 - 1; -N/2, \ldots, -1 \]

and periodic boundary conditions \( x_N \equiv x_0 \) are imposed. Differential operators in \( x \) space are algebraic in \( k \) space so that the first derivative with respect to \( x \) maps to

\[ \partial_x \rightarrow d_k = (e^{i\kappa h} - e^{-i\kappa h})/2h = d_{-k} = -d_k \]

where \( \kappa = 2\pi k/N \) is the wavenumber of the \( k \)th Fourier component. The linear operator \( \hat{L}(x) \) of Eq. (2) maps to

\[ \hat{L}(x) \rightarrow L_k = \alpha + (e^{i\kappa h} - 2 + e^{-i\kappa h} \lambda^2 + (e^{2i\kappa h} - 4e^{i\kappa h} + 6 - 4e^{-i\kappa h} + e^{-2i\kappa h})/h^4 \]

Note that equation numbers without the prefix “S” in the SI refer to the main paper (1).

We define the Fourier transform of a square matrix \( M(x_n, x_{n'}) \) by discrete forward and inverse transforms in \( k \) space as

\[ M_{k, -k'} = h^2 \sum_{n, n'} M(x_n, x_{n'}) e^{-i2\pi(kn - k'n')/N} \]

\[ M(x_n, x_{n'}) = \frac{1}{L^2} \sum_{k, k'} M_{k, k'} e^{i2\pi(kn - k'n')/N} \]

Note that, in the transform, the column index \( k' \) has the opposite sign to that of the row index \( k \) in the exponents. In terms of the two-dimensional fast Fourier transform MatLab\( ^{\text{\textregistered}} \) routines \( \text{fft2}[\ldots] \) and its inverse \( \text{ifft2}[\ldots] \), Eqs. (S5) and (S6) are

\[ M_{k, -k'} = \text{fft2}[M(x_n, x_{n'}) h^2] \]

\[ M(x_n, x_{n'}) = \text{ifft2}[M_{k, -k'}/h^2] \]
In discrete $x$ space, the product of two matrices is weighted by $h$ and the unit matrix $I$ has elements $I_{n,n'} = \delta_{n,n'}/h$. In discrete $k$ space the product of two matrices is weighted by $1/L$ so that

$$K = MN \Rightarrow K_{kk'} = \frac{1}{2} \sum_{n,n'} M_{k,k'} N_{n,n'}.$$  \[S9\]

and the unit matrix $I$ becomes $I_{kk'} = L \delta_{k,k'}$. Note that matrix elements in Fourier space are complex numbers so that the Fourier transform of a real symmetric matrix in $x$ space is a Hermitian matrix in $k$ space. The transpose operation is followed by complex conjugation as usual. Sometimes, it is advantageous to work in $x$ space where the eigenvectors of real symmetric and anti-symmetric matrices are real.

### B. Stationary States of the SKS equation

To obtain stationary periodic solutions of the SKS equation in the noiseless limit, we use the semi-implicit algorithm of (3). The Fourier transform $N_k(t)$ of the nonlinear term $|u_x(x, t)|^2$ is found by combining the inverse and forward Fourier transforms as

$$N_k(t) = \text{fft}([\text{ifft}[d_k u_k(t)/h]^2 h](k),$$

so that, at time $t + \Delta t$

$$u_k(t + \Delta t) = \frac{u_k(t) + \Delta t N_k(t)}{1 + \Delta t L_k}.$$  \[S10\]

In the noiseless limit, a stationary state is achieved numerically in $N_k \sim 10^6$ iterations with a fairly small time step, $\Delta t \sim 3 \times 10^{-4}$ and this final state depends on the choice of the initial state. If this is dominated by a single wavenumber, $k = \pm k_0$, or $\kappa_0 = 2\pi k_0/N$, the final state consists of components $k_n = \pm n k_0, n = 0, 1, 2, \ldots$, where the harmonics decay rapidly for $n > 1$ and has the same periodicity as the initial state. When $\alpha < 0.25$, there is a band of stationary states clustered around the critical or linearly fastest growing wavenumber $\kappa_c = 1/\sqrt{2}$ which corresponds to $\min(L_k)$.

### C. Constructing the Global Potential

To avoid confusion, we drop subscripts indicating the order in powers of $\hat{u}$ when in Fourier space. The Fourier transform of $A_0$ of Eq. (17) is

$$A_{kk'} = 2d_{k-k'} a_{k-k'}; \quad a_k = \text{fft}[a(x)h](k).$$  \[S11\]

When $(k - k') \notin [-N/2, N/2 - 1], (k - k') \rightarrow (k - k' \pm N)$ to ensure that $(k - k')$ is inside the defined range. Note that there is an error in Eq. (34) of our earlier work (2) where there is an extra factor $h$. The consequences of this error are addressed and corrected in this SI. $Q_0$ is found numerically by solving Eq. (19) using a standard MatLab\textsuperscript{®} routine, $X = \text{lyap}(A, Q)$ which solves the continuous Lyapunov equation (4), \[AX + XA^T + Q = 0.\] When formatting the codes, special care must be taken in Fourier space with the $L^2$ norm associated with the product of two matrices in Eq. (S9). For example, the inverse of a matrix $C$ is calculated as $L^2 \text{inv}(C)$ by the matrix inversion routine of MatLab\textsuperscript{®} so that $[I + Q_0]^{-1} \Rightarrow L \text{inv}(I + Q_0/L)$. Note that the weight in $x$ space is $h$ so that $L \rightarrow 1/h$ when we transform matrices back to $x$ space. Eigenvalues and eigenvectors of $R_0$ in Eq. (18) are computed by another standard MatLab\textsuperscript{®} routine, $[V, D] = \text{eig}(A)$. The pair-wise potential differences between the stationary states are found by using these results and Eq. (22).

#### C.1. Eigenmodes, Eigenvalues, and Topology

With the corrected expression for the Fourier transform $A_0$ of Eq. (S11), we can correct some observations in our earlier work (2). It was stated that the number of unstable eigenmodes of $R_0$ is not changed by the nonlinearity for every stationary state. Only some of these become unstable when the wavenumber $\kappa \neq \kappa_c$ so that the size of the unstable region agrees with that predicted by the Eckhaus instability which is a secondary instability of the periodic stationary states (6, 7). The remaining observations in (2) remain correct. Modes with eigenvalues of small magnitude are found to lead to another stationary state inside the allowed range of $k$. This correspondence is identified by the dominant Fourier components as summarized in TABLE S1 (which should replace Table 1 in (2)). Note that a factor $1/L$ must be applied to the list as in Eq. (S9) when comparing these values to those of $L_q$ of Eq. (S4). Hence the low-lying eigenmodes with small negative eigenvalues are seen to connect stationary states so that the topology of the web of inter-connected fixed points remains robust. This phenomenon leads to useful information about the global potential landscape as discussed below.

#### C.2. Least Squares Polynomial Fitting

With this novel topology in mind, we first evaluate the pairwise potential difference $\Phi(\kappa)$ over the whole range of $\kappa$, assuming that this potential exists. However, this leads to a monotonically increasing potential as the wavenumber $\kappa$ increases which implies that we must consider the topology of the set of all states. Note that Fig. 3 in (2) is not accurate and is no longer meaningful. There are many paths in state space connecting any chosen pair of fixed points and each path involves different intermediate states. Different paths between the same two states yield different potential differences when the pair-wise potential formula is used. Therefore this procedure does not yield exact potential differences but rather is an attempt to obtain some estimate of the potential differences. The true potential difference must be path independent according to Eqs. (12) and (13) and one way...
This scheme is used to verify the theoretical predictions from the global potential landscape point of view. Technically, for every \( \xi \) where 4 of 9 Yong-Cong Chen, Chunxiao Shi, J. M. Kosterlitz, Xiaomei Zhu and Ping Ao potential differences. The resulting potential is insensitive to the order of the polynomial for potential that is parameterized by a low-order polynomial in the wavenumber to overcome this problem of path dependence is to weight the paths by a fitting procedure. We assume the existence of a

\[ \langle \Phi(t) \rangle = \frac{\langle F_k(t) \rangle}{1 + \Delta t L_k} \]

where \( \Phi(t) \) is the Fourier transform of a Gaussian distributed random variable with zero mean, \( \langle \xi(t) \rangle = 0 \), and uncorrelated in time and space so that

\[ \langle \xi_k(t) \xi_{k'}(t') \rangle = \delta_{k,k'} \delta_{t,t'} \].

Technically, for every \( k \neq 0 \), we use two sets of real normal noises \( \xi_k^{(1)}, \xi_k^{(2)} \) so that \( \xi_k = [\xi_k^{(1)} + i \text{sgn}(k) \xi_k^{(2)}]/\sqrt{2} \). Note that our Fourier transform convention of Eq. (S1) has an overall factor \( \sqrt{\lambda} \), so that the same factor appears in the numerator of Eq. (S12), rather than in the denominator of the noise term (cf. (3)).

D.2. Probability Distribution of Occupancies. In a simulation with noise, the probability of the system being in a given stationary state is a difficult question because, by definition, a stationary state is a solution of the noiseless SKS equation and, in a simulation without noise, the final stationary state is determined by the initial state of the system. To address this issue, we take a set of snapshots of the system evolving with external noise at periodic time intervals. To evolve these states to their associated stationary states, we use one of the snapshots as an initial state of the system which then evolves deterministically for time \( \Delta T_s \) to some stationary state which depends on the choice of \( \Delta T_s \). By performing this for every snapshot, we can define the probability distribution \( P_s(\kappa) \propto \exp[-\Phi_s(\kappa)/\epsilon] \) (but cf. below) of finding the system in the stationary state \( \kappa = 2\pi |k|/L \). This scheme is used to verify the theoretical predictions from the global potential landscape point of view.

The most stable state is expected to correspond to a minimum of the potential. The global potential \( \Phi(\kappa) \) obtained from the stochastic decomposition is compared to \( \Phi_s(\kappa) \) from simulations. We speculate that \( P_s(\kappa) \) is a standard Boltzmann distribution.
so that \( P_e(\kappa) = P_0(\kappa) \exp[-\Phi_\kappa(\kappa)/\epsilon] \) where \( \epsilon \) is the noise strength of Eq. (3) and \( P_0(\kappa) \) varies slowly with \( \kappa \). We now compare \( \Phi_\kappa(\kappa) \) with \( \Phi(\kappa) \).

It turns out that \( \Phi_\kappa(\kappa) \), shown in Fig. S2, does agree with the position of the minimum and also with the shape of \( \Phi(\kappa) \) when the noise strength \( \epsilon \) is replaced by a rescaled effective noise strength \( \epsilon_\kappa = \epsilon/\Delta T_\kappa \). This \( \Phi_\kappa(\kappa) \) yields correctly the most stable state and the shape of the potential up to some overall scale factor \( \Phi(\kappa)/\Phi_\kappa(\kappa) \sim 10 \). This holds independently of the individual values of \( \epsilon \) and the relaxation time \( \Delta T_\kappa \) which supports the theoretical analysis although neither theoretical nor numerical analysis is able to predict the absolute scale of the global potential landscape. A major reason for the large discrepancy between the magnitudes of the simulated and theoretical potentials is due to the numerical method of obtaining stationary states. An initial state obtained from a snapshot of the system evolving in the presence of external noise is not a single “pure” stationary state but is a group of stationary states. This also applies to the final state after deterministic relaxation for time \( \Delta T_\kappa \). In contrast, a theoretical stationary state is a single state so it is not surprising that the two estimates differ considerably. However, the selected wave numbers agree very well. The details of this discrepancy remain to be investigated.

Fig. S2. Simulated potentials \( \Phi_\kappa(\kappa) \) (left), obtained from the probability distribution \( P_e(\kappa) \) with \( \epsilon = 0.0 \) (right), for \( L = 512, h = 0.32, \alpha = 0.20 \), and different noise strengths \( \epsilon \).

E. Vortex Oscillation and Limit Cycle

Vortex-like oscillations and limit cycle motion of the eigenstates in \( \kappa \) space are predicted from general considerations based on the stochastic dynamics as discussed in the main work (1) and are observed near stationary points. The matrix \( Q \) is found from Eq. (19) in Fourier space (we drop the subscript 0 for simplicity) and, to obtain real matrices, we work in coordinate space by using the transform of Eq. (S8). The eigenmodes and eigenvalues of \( Q \) are computed by a standard routine in MatLab, \( \langle V, E \rangle = \text{eig}(A) \), which gives pairs of pure imaginary eigenvalues of the matrix \( E \). To obtain real eigenvalues in the form \( q_i(\sigma_y) \), which are real \( 2 \times 2 \) antisymmetric matrices used in the main work (1), we perform a rotation of \( V \) by \( Y = V \times P \) where \( P \) consists of \( N/2 \) diagonal \( 2 \times 2 \) matrix blocks of \( (1 - i \sigma_y)/\sqrt{2} \) and \( \sigma_y \) where \( \sigma_y \) are Pauli matrices. The eigenvectors obtained from this form is the basis for Eq. (25), labelled by the eigenvalues \( \{q_i\} \) in descending order as discussed in Eqs. (23)- (25). Each of these pairs constitutes a 2D subspace in which the system can have circular oscillations about a stationary state.

The first few pairs in each steady state are the most likely to exhibit this vorticity. We evolve the system from an initial state with a finite amplitude of such a pair, which is in turn a small perturbation on the underlying steady state and quasi-periodic oscillations or vortex motions around some fixed points are indeed observed. This behavior can occur at both stable and unstable fixed points. In the case of a fixed point that is unstable in both dimensions of the subspace, the oscillation moves away from the fixed point, leading to the possibility of a limit cycle when the nonlinearity becomes sufficiently large. An example of this is discussed in the main work (1) and more examples are discussed below.

E.1. Typical Oscillations and Regions with Vorticity. A typical vortex like motion is shown in Fig. S3 (a),(b) near a stationary periodic state inside the Eckhaus stable region, with \( L = 512, h = 0.32, \alpha = 0.20 \), and \( \kappa = 0.6995 \). Motion in the 15th subspace is described by Eq. (25) and we impose a small initial deviation from the stationary state in the direction of the eigenvector by \( \delta_0 = \pm \epsilon_\kappa \langle \sigma \rangle \) in Fourier space. Each state then evolves according to Eq. (S10) and the state vector is projected back onto the 1th subspace by \( P_e(t) = e^{\epsilon_\kappa t} \cdot \delta(t) \) (rename to \( x, y \) for convenience). When the exponential damping factors on the trajectories are removed, quasi-circular motions appear as shown in Fig. S3 (b). However, at high wave numbers, shown in
Fig. S3 (c) for the 1st subspace, the vorticity disappears so that trajectories in this subspace converge to the stable state but without vortex-like circulation.

Fig. S3. Trajectories $P_x(t) = e^{i\sigma} \tilde{u}(t)$ in 2D dimensional subspace for $L = 512, h = 0.32, \alpha = 0.20$ with different wavenumber. (a) Small deviations from a stable state decay back to the origin with vortex-like circulation for $\kappa = 0.6995$. (b) Vortex-like motion seen by removing attenuation from the trajectories with the same parameters as (a). (c) Small deviations from a stable state decay to zero without vortex like circulation for $\kappa = 0.7363$.

Fig. S4. Phase diagram for some parameters. In a vacillating-breathing (VB) mode, there will be quasi-periodic oscillation in the region far away from the fixed point, namely, limit cycle motion. In the range of stable region, the region below the green dotted-square (contains the points on this line) shows stable vortex-like circulation, which is absent at large wavenumber.

We next examine parameter regions that exhibit vorticity. We also investigate numerically the degree of isolation of pairs of eigenmodes from Eq. (28) and Fig. S4 gives an overall picture of this. In general, vorticity exists to the left of the dotted-square line. It is of interest to note that this dividing line coincides with the minima of mode isolation. The degree of isolation or the pair approximation improves as $\alpha$ decreases, even for unstable states. The same happens on the right-hand side of the line, although vorticity is absent for large wavenumbers. At present, there is no analytical understanding of this.

E.2. Vorticity Driven VB Modes and Pattern Drifting. A vacillating-breathing (VB) mode (3, 9–11) in which each cell oscillates out of phase by $\pi$ with its neighbors occurs around $\alpha \cong 0.1$ and $\kappa < \kappa_c$. This is the ideal region to show that the VB mode corresponds to the vortex motion discussed earlier and also to study the mechanism causing vorticity to drive the phase drifting of a periodic stationary state. A typical scenario of VB oscillation is shown in FIG. S5.

In a VB mode, when an unstable vortex-like circulation increases in size, eventually this growth ceases because of the nonlinearity and decays at some later time. We find that, simultaneously, the periodic cellular structure itself undergoes phase drifting, indicating that this is driven by the vorticity of the former. In fact, closer examination shows that, as the cells drift, the overlap of the growing eigenstate with a shrinking eigenstate of the drifted stationary state increases. When the drift phase reaches $\pi$, the overlap is a maximum. Clearly, the shrinking mode causes the vortex circulation to return to zero. Note we only look at one pair of modes. There are other modes acting so that the drifting continues and forms a limit cycle when the phase reaches $2\pi$. The complete dynamics is shown in Fig. S6. The system is initially in the 3rd subspace...
Vortex motions in VB oscillations decomposed into the eigenstates of $Q_0$ with $L = 512$, $h = 0.32$, $\alpha = 0.12$ and cellular wavenumber $\kappa = 0.6136$. (a) Spatiotemporal portrait of the oscillations, showing no apparent vorticity in the dynamics. (b) Decaying circulation in the stable 3rd subspace. (c) Growing circulation in the unstable 5th subspace. Here $P_{\sigma}(t) = e_{\sigma}^{\dagger} \cdot \hat{u}(t)$.

and evolves by Eq. (25) with $L = 512$, $h = 0.32$, $\alpha = 0.12$, and $\kappa = 0.6381$. The projection on to the subspace changes with time, $P_{\sigma}(t) = e_{\sigma}^{\dagger} \cdot \hat{u}(t)$ ($\sigma = x, y$ for convenience). From the perspective of the cross-section, this is a vortex-like motion in time with the time-step set to $dt = 0.003$, and data is recorded every 400 iterations. The saturation is clearly present when it reaches a certain magnitude and the motion becomes a quasi-limit cycle. As mentioned above, this happens while the cellular structure itself undergoes phase drifting in coordinate space. The behavior can be understood from the overlap, $P$, between the 4 dimensional degenerate subspace $[e_{3\sigma}, e_{4\sigma}]$ with the stable and unstable eigenmodes of the complete $(D + Q)R$ matrix calculated at respective stationary points with phase drift $\Delta \phi$. It is found that the overlaps between stable and unstable oscillate periodically with $\Delta \phi$.

F. Data Availability

The necessary formulae and the steps to perform the computations are detailed in the main work and the Supplementary Information. There is no need for a specific platform nor a specially purposed software package. The data used to justify the results and conclusions of this work are entirely presented within the body and supplementary information of the manuscript.

The code used in this paper is available on request from chenyongcong@shu.edu.cn.
Fig. S6. Vorticity driven VB oscillation explored in detail. (a) Periodic cell drifting in coordinate space for $u(x)$. (b) Vortex-like dynamics in a two-dimensional subspace. (c) The time-evolution of state vector in the initially unstable subspace. (d) The overlapping of the 4D degenerate subspace $[e_3, e_4, \sigma_3\sigma_4]$, as in Eq. (25), with corresponding unstable (solid blue line) and stable (dotted red line) eigenmodes of the full $N \times N$ matrix $(D + Q)R$. 
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