FIRST RETURN TIMES: MULTIFRACTAL SPECTRA AND DIVERGENCE POINTS

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Abstract. We provide a detailed study of the quantitative behavior of first return times of points to small neighborhoods of themselves. Let $K$ be a self-conformal set (satisfying a certain separation condition) and let $S : K \rightarrow K$ be the natural self-map induced by the shift. We study the quantitative behavior of the first return time,

$$\tau_{B(x,r)}(x) = \inf \{ 1 \leq k \leq n \mid S^k x \in B(x,r) \},$$

of a point $x$ to the ball $B(x,r)$ as $r$ tends to 0. For a function $\varphi : (0, \infty) \rightarrow \mathbb{R}$, let $A(\varphi(r))$ denote the set of accumulation points of $\varphi(r)$ as $r \searrow 0$. We show that the first return time exponent, $\frac{\log \tau_{B(x,r)}(x)}{-\log r}$, has an extremely complicated and surprisingly intricate structure: for any compact subinterval $I$ of $(0, \infty)$, the set of points $x$ such that for each $t \in I$ there exists arbitrarily small $r > 0$ for which the first return time $\tau_{B(x,r)}(x)$ of $x$ to the neighborhood $B(x,r)$ behaves like $1/r^t$, has full Hausdorff dimension on any open set, i.e.

$$\dim \left( \left\{ x \in K \mid A\left(\frac{\log \tau_{B(x,r)}(x)}{-\log r}\right) = I \right\} \right) = \dim K$$

for any open set $G$ with $G \cap K \neq \emptyset$. As a consequence we deduce that the so-called multifractal formalism fails comprehensively for the first return time multifractal spectrum. Another application of our results concerns the construction of a certain class of Darboux functions.

1. Introduction and Statement of Results. Let $X$ be a set and let $S : X \rightarrow X$ be a map from $X$ into itself. For $U \subseteq X$ and $x \in X$, let $\tau_U(x)$ denote the first return time of $x$ to $U$, i.e.

$$\tau_U(x) = \inf \{ 1 \leq k \leq n \mid S^k x \in U \}.$$

Poincaré’s classical recurrence theorem tells us that if $S : X \rightarrow X$ a measure preserving map on a probability measure space $(X, \mu)$, then $\mu$-a.a. points are infinitely recurrent to any set $U \subseteq X$ with positive measure. In particular, this implies that if $X$ is a metric space and the ball $B(x,r)$ with center $x$ and radius $r > 0$ has positive measure, then

$$\tau_{B(x,r)}(x) < \infty$$

for $\mu$-a.a. $x \in X$. In view of this, it is natural to ask for quantitative results regarding the dependence on $r$ of the first return time $\tau_{B(x,r)}(x)$ of $x$ to the ball $B(x,r)$, i.e. we ask the following question:

With which rate does a point $x$ return to a small neighborhood $B(x,r)$ of itself? (1.1)

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This question has recently attracted an increasing interest and certain results have been obtained within the past 10 years. In the early 1990’s Ornstein & Weiss [13] proved that if \( X \) is the full shift equipped with the natural metric and \( S : X \to X \) is the shift map, then for any ergodic shift invariant probability measure \( \mu \) on \( X \) we have

\[
\lim_{r \downarrow 0} \frac{\log \tau_{B(x,r)}(x)}{-\log r} = \dim_{loc}(\mu, x) = h(\mu) \tag{1.2}
\]

for \( \mu \)-a.a. \( x \in X \), where \( \dim_{loc}(\mu, x) \) denotes the local dimension of \( \mu \) at \( x \), i.e.

\[
\dim_{loc}(\mu, x) = \lim_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r}, \tag{1.3}
\]

and where \( h(\mu) \) denotes the entropy of \( \mu \); this result has later been generalized to more general dynamical systems [16]. The reader is referred to [1,4,8] for a different approach to the study of question (1.1).

The result in (1.2) immediately gives a lower bound for the Hausdorff dimension of the set of points \( x \) for which the first return time \( \tau_{B(x,r)}(x) \) of \( x \) to the neighborhood \( B(x,r) \) behaves like \( 1/r^h(\mu) \) for small values of \( r \), i.e. the set

\[
\left\{ x \in X \left| \lim_{r \downarrow 0} \frac{\log \tau_{B(x,r)}(x)}{-\log r} = h(\mu) \right. \right\}.
\]

Indeed, let \( \dim \) denote the Hausdorff dimension, and for a probability measure \( \mu \) on \( X \), let \( \dim \mu \) denote the Hausdorff dimension of \( \mu \), i.e. \( \dim \mu = \inf_{E=\mu} \dim E \). It follows from (1.2) that

\[
\dim \left\{ x \in X \left| \lim_{r \downarrow 0} \frac{\log \tau_{B(x,r)}(x)}{-\log r} = h(\mu) \right. \right\} \geq \dim \mu = h(\mu). \tag{1.4}
\]

The first inequality in (1.4) follows immediately from (1.2), and the second equality in (1.4) follows from the so-called mass distribution principle (cf. [3, p. 55, Theorem 4.2]) and the fact that \( \dim_{loc}(\mu, x) = h(\mu) \) for \( \mu \)-a.a. \( x \) by the Shannon-MacMillan-Breiman theorem. Unfortunately, inequality (1.4) only provides a partial and incomplete answer to problem (1.1). The purpose of this paper is to give a comprehensive answer to question (1.1).

We now present a few selected examples of the results in this paper. These examples illustrate the surprisingly rich and intricate structure first return times exhibit. Let \( N \) be a positive integer and let \( S : [0,1] \to [0,1] \) be defined by \( S(x) = Nx \mod 1 \). We prove that for any \( t > 0 \), the set of points \( x \in [0,1] \) for which the first return time \( \tau_{B(x,r)}(x) \) of \( x \) to the neighborhood \( B(x,r) \) behaves like \( 1/r^t \) for small values of \( r \), i.e. those \( x \in [0,1] \) such that \( \frac{\log \tau_{B(x,r)}(x)}{-\log r} \) is close to \( t \) for small values of \( r \), has full Hausdorff dimension on any open set.

**Theorem 1.** Let \( N \) be a positive integer and let \( S : [0,1] \to [0,1] \) be defined by \( S(x) = Nx \mod 1 \). For all \( t > 0 \) and for all open sets \( G \) with \( G \cap [0,1] \neq \emptyset \), we have

\[
\dim \left( G \cap \left\{ x \in [0,1] \left| \lim_{r \downarrow 0} \frac{\log \tau_{B(x,r)}(x)}{-\log r} = t \right. \right\} \right) = 1. \]
In fact, we prove a significantly more general and surprising result. For a function \( \varphi : (0, \infty) \to \mathbb{R} \), let \( A(\varphi(r)) \) denote the set of accumulation points of \( \varphi(r) \) as \( r \downarrow 0 \), i.e.

\[
A(\varphi(r)) = \{ x \in \mathbb{R} \mid \text{there exists a sequence } (r_n) \text{ with } r_n \downarrow 0 \text{ such that } \varphi(r_n) \to x \}.
\]

Even more surprisingly, we show that for any compact subinterval \( I \) of \((0, \infty)\), the set of points \( x \in [0, 1] \) for which the set of accumulation points of \( \frac{\log \tau_B(x,r)}{-\log r} = I \), has full Hausdorff dimension on any open set.

**Theorem 2.** Let \( N \) be a positive integer and let \( S : [0,1] \to [0,1] \) be defined by \( S(x) = Nx \mod 1 \). For all compact subintervals \( I \) of \((0, \infty)\) and for all open sets \( G \) with \( G \cap [0,1] \neq \emptyset \), we have

\[
\dim \left( G \cap \left\{ x \in [0,1] \mid A\left( \frac{\log \tau_B(x,r)}{-\log r} \right) = I \right\} \right) = 1.
\]

The result in Theorem 1 clearly has a multifractal flavor. In the study of geometric properties of dynamical systems or “fractal” measures one is often interested in the asymptotic behavior of various local quantities associated with the underlying dynamical or geometric structure. For example, one is often interested in the local dimension \( \lim_{r \downarrow 0} \frac{\log \mu(B(x,r))}{-\log r} \) of a measure \( \mu \) (cf. (1.3)) or the ergodic average of a continuous function. These quantities provide a description of various aspects of measures or dynamical systems, e.g. chaoticity, sensitive dependence, et c. All these quantities provide important information about the underlying geometric or dynamical structure. This idea leads to the notion of multifractal spectra. For a metric space \( X \) and a set \( Y \), we define the multifractal spectrum of a map \( \varphi : X \to Y \) by

\[
f(t) = \dim \{ x \in X \mid \varphi(x) = t \}, \quad t \in Y,
\]

where \( \dim \) denotes the Hausdorff dimension, cf. [9,10,11]. Motivated by this we define the first return time spectrum of a self map \( S : X \to X \) on a metric space \( X \) by

\[
f_{\text{return}}(t) = \dim \left\{ x \in X \left| \lim_{r \downarrow 0} \frac{\log \tau_B(x,r)}{-\log r} = t \right. \right\}, \quad t \in \mathbb{R}.
\]

Typically the multifractal spectrum \( f \) of a map \( \varphi : X \to Y \) satisfies the so-called multifractal formalism, i.e. \( f \) is strictly concave and there exists a “natural” function \( \tau : Y \to \mathbb{R} \) such that \( f \) equals the Legendre transform of \( \tau \), cf. [9,10,11,17] and the references therein. Theorem 1 shows that the first return time spectrum, in general, fails the multifractal formalism comprehensively: the first return time spectrum is not strictly convex, and since the first return time spectrum is constant, it is easily seen that there is no function \( \tau : \mathbb{R} \to \mathbb{R} \) such that the first return time spectrum equals the Legendre transform of \( \tau \).

Another application of Theorem 1 concerns the construction of a very irregular Darboux function of Baire class 3. Recall the a function \( \varphi : [0,1] \to \mathbb{R} \) is called
Durbin if it has the intermediate value property. Also recall that a function \( \varphi : [0, 1] \to \mathbb{R} \) is said to be of Baire class 0 if it is continuous, and that \( \varphi \) is said to be of Baire class \( n \) for some positive integer \( n \) if it is the pointwise limit of a sequence of functions of Baire class \( n - 1 \). Intuitively, one should think of a function of Baire class \( n \) as being \( n \) steps away from being continuous. Now, let \( N \) be a positive integer and define \( S : [0, 1] \to [0, 1] \) by \( S(x) = Nx \mod 1 \). For each positive integer \( n \), define \( f_n : [0, 1] \to \mathbb{R} \) by \( f_n(x) = \frac{\log \tau_{B(x,N^{-n})}(x)}{-\log N^{-n}} \) if the \( \tau_{B(x,N^{-n})}(x) < \infty \), and put \( f_n(x) \) equal to 0 otherwise. Since the restriction of \( S^k \) to \([0, 1] \setminus \mathbb{Q}\) is continuous for all \( k \), it is not difficult to see that \( f_n \) is of Baire class 1. Next, define \( f : [0, 1] \to \mathbb{R} \) by \( f(x) = \lim_n f_n(x) \) if the limit \( \lim_n f_n(x) \) exists, and and put \( f(x) \) equal to 0 otherwise. It is not difficult to check that \( f \) is of Baire class 3, and that the limit \( \lim_{r \to 0} \frac{\log \tau_{B(x,r)}(x)}{-\log r} \) exists and equals some real number \( a \), say, if and only if the limit \( \lim_n \frac{\log \tau_{B(x,N^{-n})}(x)}{-\log N^{-n}} \) exists and equals \( a \). Hence the following result follows immediately from Theorem 1.

Corollary 3. There exists a function \( f : [0, 1] \to \mathbb{R} \) of Baire class 3 such that if \( I \) is any non-empty open interval of \([0, 1]\) and if \( y \) is any real number, then

\[
\dim \{ x \in I \mid f(x) = y \} = 1
\]

Unfortunately, we have not been able to show that the function constructed in the proof of Corollary 3 is of Baire class 2, and we do not know if there exists a function of Baire class 2 satisfying the condition in Corollary 3. (Of course, no function of Baire class 0 (i.e. a continuous function) can satisfy the condition in Corollary 3. Also, since any function of Baire class 1 has a dense set of continuity points, no function of Baire class 1 can satisfy the condition in Corollary 3.) We therefore ask the following question.

Question 4. Does there exist a function \( f : [0, 1] \to \mathbb{R} \) of Baire class 2 such that if \( I \) is any non-empty open interval of \([0, 1]\) and if \( y \) is any real number, then

\[
\dim \{ x \in I \mid f(x) = y \} = 1
\]?

We emphasize that the above results are particular cases of the theory developed in this paper. In fact, our main results will be formulated in the setting of self-conformal sets. In Section 1.1 we define self-conformal sets and in Section 1.2 we state our main results. The proofs are given in Sections 2–6.

1.1. The setting: self-conformal sets. In this section we define self-conformal sets. A conformal iterated function system is a list \( (V, X, (S_i)_{i=1,...,N}) \) where

1. \( V \) is an open, connected subset of \( \mathbb{R}^d \).
2. \( X \subseteq V \) is a compact set with \( \text{int} X = X \) (where \( \text{int} X \) denotes the interior of \( X \)).
3. \( S_i : V \to V \) is a contractive \( C^{1+\gamma} \) diffeomorphism with \( 0 < \gamma < 1 \) such that \( S_i(X) \subseteq X \) for all \( i \).
4. The Conformality Condition: \( (DS_i)(x) \) is a contractive similarity map for all \( i \) and all \( x \in V \). (Here \( (DS_i)(x) \) denotes the derivative of \( S_i \) at \( x \).)

It follows from [5] that there exists a unique non-empty compact set \( K \subseteq X \) such that

\[
K = \bigcup_i S_i(K).
\]
The set $K$ is called the self-conformal set associated with the list $(V, X, (S_i)_{i=1,...,N})$.

There is a symbolic dynamic representation of $K$. Let $\Sigma = \{1,\ldots,N\}$. For a positive integer $n$, write $\Sigma^n = \{1,\ldots,N\}^n$ and $\Sigma^N = \{1,\ldots,N\}^\infty$, i.e. $\Sigma^n$ denotes the family of all finite strings $\omega = \omega_1\ldots\omega_n$ of length $n$ with entries $\omega_i \in \{1,\ldots,N\}$ and $\Sigma^N$ denotes the family of all infinite strings $\omega = \omega_1\omega_2\ldots$ with entries $\omega_i \in \{1,\ldots,N\}$. Let $S : \Sigma^N \to \Sigma^N$ denote the shift map. For $\omega = \omega_1\omega_2\ldots \in \Sigma^N$ and a positive integer $n$, let $\omega|n = \omega_n\ldots\omega_1$ denote the truncation of $\omega$ to the $n$th place.

If $\omega \in \Sigma^n$, we define the cylinder $[\omega]$ generated by $\omega$ by

$$[\omega] = \{ \sigma \in \Sigma^N | \sigma|n = \omega \}.$$  

Finally, if $\omega = \omega_1\ldots\omega_n \in \Sigma^n$, we will write $\omega_n \sigma = \omega_{n+1}\sigma \ldots \omega_n\sigma$ and $K_\omega = S_\omega K$.

Define $\pi : \Sigma^N \to \mathbb{R}^d$ by

$$\{\pi(\omega)\} = \bigcap_n K_{\omega|n} \quad (1.8)$$

for $\omega = \omega_1\omega_2\ldots \in \Sigma^N$. Then $K = \pi(\Sigma^N)$.

1.2. First return time spectra. In this section we state our main results. We will frequently assume that the list $(V, X, (S_i)_{i=1,...,N})$ satisfies certain “disjointness” conditions, viz. the Open Set Condition (OSC) or the Strong Separation Condition (SSC) defined below.

- **The Open Set Condition (OSC):** There exists an open non-empty and bounded subset $U$ of $V$ with $\cup_i S_i U \subseteq U$ and $S_i U \cap S_j U = \emptyset$ for all $i, j$ with $i \neq j$.
- **The Strong Separation Condition (SSC):** There exists an open non-empty and bounded subset $U$ of $V$ with $\cup_i S_i U \subseteq U$ and $S_i U \cap S_j U = \emptyset$ for all $i, j$ with $i \neq j$.

Clearly the SSC implies the OSC. If the SSC is satisfied then it is easily seen that there exists a continuous map $S : K \to K$ such that the diagram below commutes

$$
\begin{array}{ccc}
\Sigma^N & \xrightarrow{S} & \Sigma^N \\
\downarrow{\pi} & & \downarrow{\pi} \\
K & \xrightarrow{S} & K
\end{array}
\quad (1.9)
$$

In fact, $S(x) = S_i^{-1}(x)$ for $x \in S_i(K)$. In this case we consider the first return time of $x \in K$,

$$\tau_{B(x,r)}(x) = \inf \left\{ 1 \leq k \leq n \mid S^k x \in B(x,r) \right\},$$

defined with respect to the map $S$. However, if the SSC is not satisfied there is not necessarily a continuous map $S : K \to K$ that makes the diagram in (1.9) commutative. In this case we consider the natural analogue of $\tau_{B(x,r)}(x)$ in the shift space $\Sigma^N$. Namely, we consider the first return time

$$\tau_{\omega|n}(\omega) = \inf \left\{ 1 \leq k \leq n \mid S^k \omega \in [\omega|n] \right\}$$

of $\omega \in \Sigma^N$ with respect to the shift map. The return times $\tau_{\omega|n}(\omega)$ are significantly easier to analyze than their “geometrically” defined counterparts $\tau_{B(x,r)}(x)$. Also,
it is easily seen that if the SSC is satisfied, then the return times $\tau_B(x,r) \equiv \tau(\omega)$ are comparable for all $x = \pi(\omega)$ (cf. Section 6 for details), and results for the “symbolically” defined return times $\tau(\omega)$ can therefore be used to obtain precise information about the “geometrically” defined counterparts $\tau_B(x,r)$. In fact, in the main results below we will employ this fact. Indeed, we first state results for the “symbolically” defined return times $\tau(\omega)$ assuming the OSC and then, using these results, we obtain information about their “geometrically” defined counterparts $\tau_B(x,r)$ assuming the SSC.

**Theorem 5.**

1. Assume that the OSC is satisfied. For all $t > 0$ we have

$$\dim \left\{ \Omega \in \Sigma^N \bigg| \lim_{n} \frac{\log \tau(\omega)}{\log \text{diam } K_{\omega}[n]} = t \right\} = \dim K.$$  

2. Assume that the SSC is satisfied. For all $t > 0$ we have

$$\dim \left\{ x \in K \bigg| \lim_{r \downarrow 0} \frac{\log \tau_B(x,r)}{- \log r} = t \right\} = \dim K.$$  

In fact, we obtain significantly more general results providing precise information about the rate at which the first return exponent $\frac{\log \tau_B(x,r)}{- \log r}$ diverges as $r \downarrow 0$. Recall the notation from (1.5), i.e. if $\varphi : (0, \infty) \to \mathbb{R}$ is a real valued function, then $A(\varphi(r))$ denotes the set of accumulation points of $\varphi(r)$ as $r \downarrow 0$. Below we will use a similar notation for the set of accumulation points of a sequence, i.e. if $(x_n)_n$ is a sequence of real numbers then $A((x_n)_n)$ denotes the set of accumulation points of $(x_n)_n$. Using this notation it is seen that

$$\left\{ x \in K \bigg| \lim_{r \downarrow 0} \frac{\log \tau_B(x,r)}{- \log r} = t \right\} = \left\{ x \in K \bigg| A \left( \frac{\log \tau_B(x)}{- \log r} \right) = \{ t \} \right\},$$

and it is therefore natural to study the sets

$$\left\{ x \in K \bigg| A \left( \frac{\log \tau_B(x)}{- \log r} \right) = F \right\}$$

for closed subsets $F$ of $\mathbb{R}$.

**Theorem 6.**

1. Assume that the OSC is satisfied. For all compact subintervals $I$ of $(0, \infty)$ and for all open sets $G$ with $G \cap K \neq \emptyset$, we have

$$\dim \left( G \cap \pi \left\{ \Omega \in \Sigma^N \bigg| A \left( \frac{\log \tau(\omega)}{- \log \text{diam } K_{\omega}[n]} = I \right) \right\} \right) = \dim K.$$  

2. Assume that the SSC is satisfied. For all compact subintervals $I$ of $(0, \infty)$ and for all open sets $G$ with $G \cap K \neq \emptyset$, we have

$$\dim \left( G \cap \left\{ x \in K \bigg| A \left( \frac{\log \tau_B(x)}{- \log r} \right) = I \right\} \right) = \dim K.$$
We do not know if the results in Theorem 6 remain true if the compact subinterval $I$ of $(0, \infty)$ is replaced by an arbitrary closed subset $F$ of $(0, \infty)$, and we therefore ask the following question.

**Question 7.** Do the results in Theorem 6 remain true if the compact subinterval $I$ of $(0, \infty)$ is replaced by an arbitrary closed subset $F$ of $(0, \infty)$?

Observe that Theorem 2 follows immediately from Theorem 6 by considering the case where $V = \mathbb{R}$ and $X = [0, 1]$, and where the maps $S_1, \ldots, S_N : [0, 1] \to [0, 1]$ are given by $S_i(x) = \frac{x + i - 1}{N}$.

### 2. Proofs. Preliminary Results.

We begin by introducing some notation. Recall that $\Sigma = \{1, \ldots, N\}$, and that if $n$ is a positive integer then $\Sigma^n = \{1, \ldots, N\}^n$ denotes the family of all finite strings $\omega = \omega_1 \ldots \omega_n$ of length $n$ with entries $\omega_i \in \{1, \ldots, N\}$. Write

$$\Sigma^* = \bigcup_n \{1, \ldots, N\}^n,$$

i.e. $\Sigma^*$ is the family of all finite strings. Also, recall that $\Sigma^N = \{1, \ldots, N\}^N$ denotes the family of all infinite strings $\omega = \omega_1 \omega_2 \ldots$ with entries $\omega_i \in \{1, \ldots, N\}$, and that $S : \Sigma^N \to \Sigma^N$ denotes the shift. For $\omega \in \Sigma^n$, we write $|\omega| = n$. For $\omega = \omega_1 \ldots \omega_n \in \Sigma^n$ and a positive integer $m$ with $m \leq n$, or for $\omega = \omega_1 \omega_2 \ldots \in \Sigma^N$ and a positive integer $m$, let $\omega|_m = \omega_1 \ldots \omega_m$ denote the truncation of $\omega$ to the $m$th place. For $\omega = \omega_1 \ldots \omega_n \in \Sigma^n$ and $\sigma = \sigma_1 \ldots \sigma_m \in \Sigma^m$, we let $\omega \sigma = \omega_1 \ldots \omega_n \sigma_1 \ldots \sigma_m \in \Sigma^{n+m}$ denote the concatenation of $\omega$ and $\sigma$. Similarly, for $\omega = \omega_1 \ldots \omega_n \in \Sigma^n$ and $\sigma = \sigma_1 \sigma_2 \ldots \in \Sigma^N$, we let $\omega \sigma = \omega_1 \ldots \omega_n \sigma_1 \sigma_2 \ldots \in \Sigma^N$ denote the concatenation of $\omega$ and $\sigma$. Finally, recall that if $\omega = \omega_1 \ldots \omega_n \in \Sigma^n$, then we write $S_\omega = S_{\omega_1} \circ \cdots \circ S_{\omega_n}$ and $K_\omega = S_\omega K$.

Proposition 2.1 and Proposition 2.2 state that a conformal iterated function system distorts the geometry of sets in $\mathbb{R}^d$ in a uniformly bounded way. Both results are standard and their proofs can be found in many papers, cf. for example [2,7,14]. For a conformal iterated function system $(V, X, (S_i)_{i=1, \ldots, N})$, we define the map $\Phi : \Sigma^N \to \mathbb{R}$ as follows

$$\Phi(\omega) = \log |DS_{\omega_1}(\pi S_{\omega})|$$

for $\omega = \omega_1 \omega_2 \ldots \in \Sigma^N$ with $\omega_i \in \Sigma$.

**Proposition 2.1. The Principle of Bounded Distortion.** Let $(V, X, (S_i)_{i=1, \ldots, N})$ be a conformal iterated function system. Then there exists a constant $c \geq 1$ such that for all $n \in \mathbb{N}$ and all $\omega, \sigma \in \Sigma^N$ with $\omega|n = \sigma|n$ we have

$$c^{-1} \leq \frac{\exp \sum_{k=0}^{n-1} \Phi S_{\omega^k}}{\exp \sum_{k=0}^{n-1} \Phi S_{\omega^k}\sigma} \leq c.$$
Proposition 2.2. Let \((V, X, (S_i)_{i=1, \ldots, N})\) be a conformal iterated function system. Then there exists a constant \(c \geq 1\) such that the following statements hold.

(1) For all \(n \in \mathbb{N}\), \(\omega \in \Sigma^\mathbb{N}\) and \(x, y \in V\) we have
\[
 c^{-1} \exp \left( \sum_{k=0}^{n-1} \Phi_{S_i} \omega \right) |x - y| \leq |S_{\omega|n}x - S_{\omega|n}y| \leq c \exp \left( \sum_{k=0}^{n-1} \Phi_{S_i} \omega \right) |x - y|
\]

(2) For all \(n \in \mathbb{N}\), \(\omega \in \Sigma^\mathbb{N}\) and \(E \subseteq V\) we have
\[
 c^{-1} \exp \left( \sum_{k=0}^{n-1} \Phi_{S_i} \omega \right) \text{diam}(E) \leq \text{diam}(S_{\omega|n}E) \leq c \exp \left( \sum_{k=0}^{n-1} \Phi_{S_i} \omega \right) \text{diam}(E).
\]

Since the map \(x \to |(DS_i)(x)|, x \in V\), is continuous with \(0 < |(DS_i)(x)| < 1\) for all \(i\), and \(K\) is compact, we conclude that
\[
 r_{\text{min}} := \inf_{x \in K} |(DS_i)(x)| > 0, \quad r_{\text{max}} := \sup_{x \in K} |(DS_i)(x)| < 1. \tag{2.2}
\]

Also, observe that
\[
 \inf_{\sigma \in [\omega]} \exp \left( \sum_{k=0}^{\lfloor |\omega|/n \rfloor} \Phi_{S_i} \sigma \right) \geq r_{\text{min}} |\omega|, \tag{2.3}
\]
\[
 \sup_{\sigma \in [\omega]} \exp \left( \sum_{k=0}^{\lfloor |\omega|/n \rfloor} \Phi_{S_i} \sigma \right) \leq r_{\text{max}} |\omega|. \tag{2.4}
\]

for all \(\omega \in \Sigma^\ast\). Inequalities (2.3) and (2.4) will frequently be used tactically.

3. Proof of Theorem 6.(1): construction of the set \(Z\). Write
\[
 E = \left\{ \omega \in \Sigma \left| A \left( \frac{\log \tau_{\omega|n}(\omega)}{n} \right) = I \right. \right\}.
\]

We must now prove that \(\dim K \leq \dim(G \cap \pi(E))\). Fix \(\delta > 0\). The idea behind the proof is to construct a set \(Z \subseteq \Sigma\) such that
\[
 Z \subseteq E, \tag{3.1}
\]
\[
 \pi(Z) \subseteq G, \tag{3.2}
\]
and
\[
 \dim K - 2\delta \leq \dim \pi(Z). \tag{3.3}
\]

This proves Theorem 6.(1) since \(\delta > 0\) is arbitrary. In this section we construct the set \(Z\). In Section 4 we prove that \(Z \subseteq E\) and \(\pi(Z) \subseteq G\) (Proposition 4.3), and finally in Section 5 we prove that \(\dim K - 2\delta \leq \dim \pi(Z)\) (Proposition 5.2).
Let \( u, v \in \{1, \ldots, N\} \) with \( u \neq v \). For a positive integer \( M \) write
\[
\Gamma_M = \left\{ u \omega u \mid \omega \in \Sigma^{M-2} \right\}.
\]
Also write
\[
\Gamma^u_M = \{ \omega_1 \ldots \omega_n | \omega_i \in \Gamma_M \},
\]
\[
\Gamma^*_M = \bigcup_n \{ \omega_1 \ldots \omega_n | \omega_i \in \Gamma_M \},
\]
\[
\Gamma^n_M = \{ \omega_1 \omega_2 \ldots | \omega_i \in \Gamma_M \}.
\]
Let \( S_M : \Gamma^n_M \rightarrow \Gamma^n_M \) denote the shift map, and define \( \pi_M : \Gamma^n_M \rightarrow \mathbb{R}^d \) by
\[
\{ \pi_M(\omega) \} = \bigcap_n K_{\omega_1 \ldots \omega_n}
\]
for \( \omega = \omega_1 \omega_2 \ldots \in \Gamma^n_M \) with \( \omega_i \in \Gamma_M \). Next, we define \( \Phi_M : \Gamma^n_M \rightarrow \mathbb{R} \) by
\[
\Phi_M(\omega) = \log |D S_{\omega_1} (\pi_M S_M \omega)|
\]
for \( \omega = \omega_1 \omega_2 \ldots \in \Gamma^n_M \) with \( \omega_i \in \Gamma_M \). For a Hölder continuous function \( f : \Sigma^N \rightarrow \mathbb{R} \), we denote the topological pressure of \( f \) by \( P(f) \), i.e.
\[
P(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\omega \in \Sigma^n} \exp \left( \sup_{\sigma_1 \ldots \sigma_n \in \Sigma} \sum_{k=0}^{n-1} f(S^k \sigma) \right),
\]
f. \( 6 \). The topological pressure of a Hölder continuous function \( f : \Sigma^n \rightarrow \mathbb{R} \) is defined similarly. It is well-known that the function \( t \rightarrow P(t \Phi_M) \) is continuous with \( \lim_{t \rightarrow -\infty} P(t \Phi_M) = \infty \) and \( \lim_{t \rightarrow \infty} P(t \Phi_M) = -\infty \) (cf. \( 15 \)), and we therefore conclude that there exists a unique \( s_M \in [0, d] \) such that
\[
P(s_M \Phi_M) = 0.
\]

**Proposition 3.1.** Let \( s = \dim K \). Then \( s_M \rightarrow s \).

**Proof.** Recall that \( s = \dim K \) is determined as follows. The map \( \Phi : \Sigma^N \rightarrow \mathbb{R} \) is (cf. (2.1)) given by \( \Phi(\omega) = \log |D S_{\omega_1} (\pi_M S_M \omega)| \) for \( \omega = \omega_1 \omega_2 \ldots \in \Sigma^N \) with \( \omega_i \in \Sigma \). Then
\[
P(s \Phi) = 0,
\]
f. \( 6 \).

Let \( c \) denote the constant in Proposition 2.1. Observe that it follows from the principle of bounded variation (Proposition 2.1) that if \( n \) is a positive integer, then
\[
\frac{1}{n} \log \sum_{\omega \in \Gamma^n_M} \exp \left( \sup_{\sigma_1 \ldots \sigma_n \in \Sigma} \sum_{k=0}^{n-1} s_M \Phi_M(S^k_M \sigma) \right) = \frac{1}{n} \log \sum_{\omega \in \Gamma^n_M} \exp \left( \sup_{\sigma_1 \ldots \sigma_n \in \Sigma} \sum_{k=0}^{n-1} s_M \log |D S_{\sigma_{k+1}} (\pi_M S_M \sigma_k) | \right)
\]
\[
\frac{1}{n} \log \sum_{\omega \in \Gamma_M^n} \exp \left( \sum_{k=0}^{n-1} s_M \sup_{\sigma_1, \ldots, \sigma_M \in \Gamma_M} \log \prod_{i=1}^{M} |DS_{\sigma_{k+1,i}}(S_{\sigma_{k+1,i+1} \circ \cdots \circ S_{\sigma_{k+1,M}}(\pi_M S_M^{k+1} \sigma))| \right)
\]

\[
\leq \frac{1}{n} \log \sum_{\omega \in \Sigma^{n(M-2)}} c \exp \left( \sum_{k=0}^{n(M-2)-1} s_M \log |DS_{\tau_k+1}(\pi^{k+1} \tau)| \right) + 2ns_M \sup_{x \in K} \log |DS_u(x)|
\]

\[
= \frac{1}{n} \log \sum_{\omega \in \Sigma^{n(M-2)}} \exp \left( \sum_{k=0}^{n(M-2)-1} s_M \log |DS_{\tau_k+1}(\pi^{k+1} \tau)| \right) + 2s_M \inf_{x \in K} \log |DS_u(x)| - \frac{\log c}{n}.
\]

Similarly we obtain

\[
\frac{1}{n} \log \sum_{\omega \in \Gamma_M^n} \exp \left( \sum_{k=0}^{n-1} s_M \Phi_M(S^k_M \sigma) \right)
\]

\[
\geq \frac{1}{n} \log \sum_{\omega \in \Sigma^{n(M-2)}} \exp \left( \sum_{k=0}^{n(M-2)-1} s_M \log |DS_{\tau_k+1}(\pi^{k+1} \tau)| \right) + 2s_M \inf_{x \in K} \log |DS_u(x)| - \frac{\log c}{n}.
\]

Letting \(n \to \infty\) yields

\[
(M-2)P(s_M \Phi) + 2s_M \inf_{x \in K} \log |DS_u(x)| \leq P(s_M \Phi_M) \leq (M-2)P(s_M \Phi) + 2s_M \sup_{x \in K} \log |DS_u(x)|.
\]

(3.4)

However, since \(P(s_M \Phi_M) = 0\), we conclude from (3.4) that

\[
-\frac{2s_M \sup_{x \in K} \log |DS_u(x)|}{M-2} \leq P(s_M \Phi) \leq -\frac{2s_M \inf_{x \in K} \log |DS_u(x)|}{M-2}.
\]
This implies that $P(s_M \Phi) \rightarrow 0$. Since the function $t \rightarrow P(t \Phi)$ is strictly decreasing and continuous with $P(s \Phi) = 0$, we therefore conclude that $s_M \rightarrow s$. \hfill \Box

By Proposition 3.1 we can choose a positive integer $M$ such that

\[ s - \delta \leq s_M, \]

whence

\[ s - 2\delta \leq s_M - \delta. \]

Next, let $\tilde{\nu}$ denote the Gibbs state of the Hölder continuous function $s_M \Phi_M$, i.e. $\tilde{\nu}$ is the unique Borel probability measure on $\Gamma^\infty_M$ for which there exists a constant $C > 0$ such that

\[ C^{-1} \leq \frac{\tilde{\nu}([\omega[n]])}{e^{\sum_{k=0}^{n-1} s_M \Phi_M(S_{n}^k \omega)}} \leq C \quad (3.5) \]

for all positive integers $n$ and all $\omega \in \Gamma^\infty_M$.

To complete the construction of the set $Z$ we need the following elementary lemma.

**Lemma 3.2.** Let $0 < a \leq b < \infty$ and let $(t_n)_n$ be a sequence of real numbers such that $a \leq t_n \leq b$ and $|t_{n+1} - t_n| \leq \frac{a}{n+1}$ for all $n$. Then there exists a sequence $(k_n)_n$ of integers such that:

1. $\frac{k_n}{n} \rightarrow \infty$ for all $p > 0$;
2. $\log k_n \rightarrow 0$ for all $p > 0$;
3. $k_{n+1} - k_n \geq 4(n + M)$ for all $n$;
4. $\log k_n - t_n \rightarrow 0$.

**Proof.** Write $\delta_n = \frac{1}{\sqrt{n}}$ for a positive integer $n$. Next, let $\kappa_n = e^{n(t_n + \delta_n)}$. We clearly have

\[ \frac{\log \kappa_n}{n} \rightarrow \infty \quad \text{and} \quad \frac{\log \kappa_n}{n+1} = \frac{n(t_n + \delta_n)}{n+1} \leq \frac{b}{n+1} + \frac{\delta_n}{n+1} \rightarrow 0 \quad \text{for all } p > 0. \]

Also observe that $\frac{\log \kappa_n}{n} - t_n = \delta_n \rightarrow 0$. Finally, notice that

\[ \kappa_{n+1} - \kappa_n = e^{(n+1)(t_n+\delta_n+1)} - e^{n(t_n+\delta_n)} = e^{n(t_n+\delta_n)} (e^{(n+1)t_n+\delta_n} - e^{n(t_n+\delta_n)}) \]

\[ \geq e^{na} (e^{a(n+1)} - e^{nt_n+\sqrt{n+1}-\sqrt{n}} - 1) = e^{na} (e^{a - a + \sqrt{n+1} - \sqrt{n}} - 1) \]

\[ \geq e^{na} (\sqrt{n+1} - \sqrt{n}) = e^{na} \frac{1}{\sqrt{n+1} + \sqrt{n}}. \]

It follows from this that $\frac{\kappa_{n+1} - \kappa_n}{n} \rightarrow \infty$, and we can thus find a positive integer $n_0$ such that $\frac{\kappa_{n+1} - \kappa_n}{n} \geq 4(n + M)$ for all $n \geq n_0$. Now define the sequence $(k_n)_n$ by $k_n = \lceil \kappa_n \rceil$ (here $\lceil x \rceil$ denotes the integer part of $x \geq 0$). It is easily seen that the sequence $(k_n)_n$ has the desired properties. \hfill \Box

Write

\[ \chi = -\frac{1}{M} \int \Phi_M \, d\tilde{\nu}, \]

and observe that $\chi > 0$. Since $I$ is a compact subinterval of $(0, \infty)$, we may choose a countable dense sequence $(t_n)_n$ of $I$ such that $|\chi t_{n+1} - \chi t_n| \leq \inf \frac{\int I}{n+1}$ (recall, that
\(\chi > 0\). It therefore follows from Lemma 3.2 applied to the sequence \((\chi t_n)_n\) that

\[
\frac{k_n}{n^p} \to \infty \text{ as } n \to \infty \text{ for all } p > 0, \\
\frac{\log k_n}{n^{p+1}} \to 0 \text{ as } n \to \infty \text{ for all } p > 0, \\
k_{n+1} - k_n \geq 4(n + M) \text{ for all } n, \\
\frac{\log k_n}{n} - \chi t_n \to 0 \text{ as } n \to \infty.
\]

(3.6) (3.7) (3.8) (3.9)

Since \(G\) is open with \(G \cap K \neq \emptyset\), we can find \(\gamma \in \Sigma^*\) such that

\[K_\gamma \subseteq G.\]

(3.10)

We may clearly choose positive integers \(j_n\) satisfying

\[k_n \leq |\gamma| + M + M \sum_{i=1}^n j_i + \sum_{i=1}^n (i + 1) < k_n + M.
\]

Write

\[q_n = |\gamma| + M + M \sum_{i=1}^n j_i + \sum_{i=1}^{n-1} (i + 1), \]
\[Q_n = |\gamma| + M + M \sum_{i=1}^n j_i + \sum_{i=1}^n (i + 1).
\]

For \(i \in \{1, \ldots, N\}\), let \(i^+ = i + 1 \mod N\), and observe that \(i^+ \in \{1, \ldots, N\}\) with \(i \neq i^+\). Next, observe that \(q_n \geq k_n - (n + 1) \geq 4(n - 1 + M) - (n + 1) \geq n\), and we can therefore define \(\Xi_n: \Sigma^{q_n} \to \Sigma^{n+1}\), by

\[\Xi_n(\omega_1 \ldots \omega_{q_n}) = \omega_1 \ldots \omega_n \omega_{n+1}^+ .
\]

Next, we define \(\Lambda: \Gamma_M^N \to \Sigma^N\) by

\[\Lambda(\sigma_1 \sigma_2 \ldots) = \gamma v \sigma_1 \omega_1 \sigma_2 \omega_2 \ldots
\]

(3.11)

where \(\sigma_i \in \Gamma_M^N\), and \(v = \underbrace{v \ldots v}_{M \text{ times}}\) and

\[\omega_i = \Xi_i(\gamma v \sigma_1 \omega_1 \ldots \sigma_{i-1} \omega_{i-1} \sigma_i)
\]

(3.12)

for \(i \geq 1\). Observe that

\[q_n = |\gamma v \sigma_1 \omega_1 \ldots \sigma_{n-1} \omega_{n-1} \sigma_n|,
\]
\[Q_n = |\gamma v \sigma_1 \omega_1 \ldots \sigma_{n-1} \omega_{n-1} \sigma_n \omega_n| .
\]

Finally, write

\[X = \left\{ \sigma \in \Gamma_M^N \left| \lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} \Phi_M(S_M^k \sigma) = \int \Phi_M d\tilde{\nu} \right. \right\}.
\]
We can now define the set $Z$. We put

$$Z = \Lambda(X) = \left\{ \gamma \nu \sigma_1 \omega_1 \sigma_2 \omega_2 \ldots \mid \sigma_i \in \Gamma^i, \omega_i = \Xi_i(\gamma \nu \sigma_1 \omega_1 \ldots \sigma_{i-1} \omega_{i-1} \sigma_i), \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi_M(S_k M \sigma) = \int \Phi_M \, d\tilde{\nu} \text{ where } \sigma = \sigma_1 \sigma_2 \ldots \right\}$$

(3.13)

4. Proof of Theorem 6.1: $Z \subseteq E$ and $\pi(Z) \subseteq G$. Recall that

$$\chi = -\frac{1}{M} \int \Phi_M \, d\tilde{\nu}.$$ 

Proposition 4.1. For $\omega \in Z$, we have

$$\frac{1}{n} \log \operatorname{diam} K_{\omega|n} \to -\chi.$$ 

Proof.

Write $\omega = \gamma \nu \sigma_1 \omega_1 \sigma_2 \omega_2 \ldots \in \Sigma^\infty$ with $\sigma_i \in \Gamma^i_M$ and $\omega_i = \Xi_i(\gamma \nu \sigma_1 \omega_1 \ldots \sigma_{i-1} \omega_{i-1} \sigma_i)$ for $i \geq 1$. It follows from Proposition 2.2 that it suffices to show that

$$\frac{1}{n} \sum_{l=0}^{n-1} \Phi_{S_l \omega} \to -\chi. \quad (4.1)$$

We will now prove (4.1). Fix a positive integer $n$, and choose (unique) positive integers $m = m(n)$ and $k = k(n)$ with $0 \leq k \leq j_{m+1}$ such that

$$Q_m + kM \leq n \leq Q_m + (k + 1)M \quad \text{for } k < j_{m+1},$$

and

$$Q_m + kM \leq n \leq Q_m + kM + |\omega_{m+1}| \quad \text{for } k = j_{m+1}.$$ 

Write $\sigma = \sigma_1 \sigma_2 \ldots \in \Gamma^\infty_M$. Let $c$ be the constant in Proposition 2.1. The choice of $m$ and $k$, and the principle of bounded variation (Proposition 2.1) implies that

$$c^{-1} \exp \left( \sum_{l=0}^{j_1 + \ldots + j_m + \min(k+1,j_{m+1})-1} \Phi_M S_l M \sigma \right. \left. + \left| \gamma \right| + \left| \nu \right| + \left| \omega_1 \right| + \cdots + \left| \omega_{m+1} \right| \inf_{i} \inf_{x \in K} \log |DS_i(x)| \right)$$

$$\leq \exp \left( \sum_{l=0}^{n-1} \Phi_{S_l \omega} \right)$$

$$\leq c \exp \left( \sum_{l=0}^{j_1 + \ldots + j_m + k-1} \Phi_M S_l M \sigma \right),$$
whence
\[- \frac{\log c}{n} + \frac{J_m + \min(k + 1, j_{m+1})}{n} \frac{1}{J_m + \min(k + 1, j_{m+1})} \sum_{l=0}^{J_m + \min(k + 1, j_{m+1}) - 1} \Phi_M S_M^l \sigma \]
\[+ \frac{J_m + \min(k + 1, j_{m+1})}{n} \frac{1}{J_m + \min(k + 1, j_{m+1})} \sum_{l=0}^{J_m + \min(k + 1, j_{m+1}) - 1} \Phi_M S_M^l \sigma \]
\[\leq \frac{1}{n} \sum_{l=0}^{n-1} \Phi S_l^l \omega \]
\[\leq \frac{\log c}{n} + \frac{J_m + k}{n} \frac{1}{J_m + k} \sum_{l=0}^{J_m + k - 1} \Phi_M S_M^l \sigma , \quad (4.2)\]
where \( J_m = j_1 + \cdots + j_m \). However, since \( \omega \in \mathbb{Z} = \Lambda_X \), we conclude that \( \sigma \in X \).

This implies that
\[\frac{1}{J_m + \min(k + 1, j_{m+1})} \sum_{l=0}^{J_m + \min(k + 1, j_{m+1}) - 1} \Phi_M S_M^l \sigma \rightarrow \int \Phi_M d
\nu = -\chi M , \]
\[\frac{1}{J_m + \min(k + 1, j_{m+1})} \sum_{l=0}^{J_m + \min(k + 1, j_{m+1}) - 1} \Phi_M S_M^l \sigma \rightarrow \int \Phi_M d
\nu = -\chi M . \quad (4.3)\]

We also have (using (3.6))
\[|\gamma| + |\nu| + |\omega_1| + \cdots + |\omega_{m+1}| \leq |\gamma| + M + \sum_{i=1}^{m+1} (i + 1) \]
\[\leq |\gamma| + M + \frac{(m + 2)^2}{k_m} \rightarrow 0 , \]
\[\frac{1}{M} - \frac{J_m + k}{n} \leq \frac{|n - (J_m + k)M|}{nM} \leq \frac{|\gamma| + |\nu| + |\omega_1| + \cdots + |\omega_{m+1}|}{nM} \rightarrow 0 , \quad (4.4)\]
\[\frac{1}{M} - \frac{J_m + \min(k + 1, j_{m+1})}{n} \leq \frac{|n - (J_m + \min(k + 1, j_{m+1}))M|}{nM} \leq \frac{|\gamma| + |\nu| + |\omega_1| + \cdots + |\omega_{m+1}|}{nM} \rightarrow 0 . \]

Combining (4.2), (4.3) and (4.4) shows that \( \frac{1}{n} \sum_{l=0}^{n-1} \Phi S_l^l \omega \rightarrow -\chi \). \( \square \)

**Proposition 4.2.** For \( \omega \in \mathbb{Z} \), we have
\[\frac{1}{n} \log \tau_{\gamma|\nu} \omega \rightarrow -\chi t_n \rightarrow 0 . \]

**Proof.**

Since
\[\frac{1}{n} \log k_n - \chi t_n \rightarrow 0 \]
and
\[ k_n \leq q_n + n + 1 < k_n + M, \]

it suffices to prove that
\[ \tau_{|\omega|n}(\omega) = q_n \]
for \( n \geq |\gamma| + M \). Therefore, fix \( n \geq |\gamma| + M \). Write \( \omega = \gamma v \sigma_1 \omega_1 \sigma_2 \omega_2 \ldots \in \Sigma^N \) with \( \sigma_i \in \Gamma^M \) and \( \omega_i = \Xi_i(\gamma v \sigma_1 \omega_1 \ldots \sigma_{i-1} \omega_{i-1} \sigma_i) \) for \( i \geq 1 \).

We first prove that \( \tau_{|\omega|n}(\omega) \leq q_n \). Indeed, it follows from the definition of \( \omega_n \) that \( \omega_n \mid n = \omega \mid n \), whence \( S^q \omega = \omega_n \sigma_n \omega_{n+1} \sigma_{n+1} \ldots \in \sigma_n \subseteq [\omega_n] \subseteq [\omega_n] = [\omega] \). This implies that \( \tau_{|\omega|n}(\omega) \leq q_n \).

We must now prove that \( \tau_{|\omega|n}(\omega) \geq q_n \), i.e. we must prove that \( \omega \mid n \neq (S^k \omega) \mid n \) for all \( k < q_n \). Assume in order to obtain a contradiction that \( \omega \mid n = (S^k \omega) \mid n \) for some \( k < q_n \). We now introduce some terminology. Write
\[ S^k \omega = \tau_0 w_1 \tau_1 w_2 \tau_2 \ldots , \]
where \( w_i = v \ldots v \) for some \( n_i \geq M \), and where \( \tau_i \) does not contain \( v \) as a substring and the first and the last letter in \( \tau_i \) are both different from \( v \). In particular, we have
\[ S^{k+|\tau_0|} \omega = \tau_1 w_2 \tau_2 \ldots , \]
i.e. \( S^{k+|\tau_0|} \omega \) starts with the string \( v \). For a positive integer \( m \), we will say that \( S^{k+|\tau_0|} \omega \) starts inside \( \sigma_m \) if
\[ q_{m-1} + m \leq k + |\tau_0| < q_m , \]
and we will say that \( S^{k+|\tau_0|} \omega \) starts inside \( \omega_m \) if
\[ q_m \leq k + |\tau_0| < q_m + m + 1 . \]

Since \( n \geq |\gamma| + M = |\gamma v| \), we conclude that \( \omega \mid n \) begins with the string \( v \). Hence \( (S^k \omega) \mid n = \omega \mid n \) also begins with the string \( v \). Moreover, since each \( \sigma_i \) begins and ends with a letter different from \( v \) (namely \( u \)) and since the string \( v \) does not appear as a substring in any of the strings \( \sigma_1, \sigma_2, \ldots, \sigma_n \), we conclude that \( S^k \omega \) starts inside \( \omega_m \) for some \( m < n \). Write
\[ \omega_m = \gamma_0 v_1 \gamma_1 v_2 \gamma_2 \ldots v_l \gamma_l \]
where \( v_i = v \ldots v \) for some \( m_i \geq M \), and \( \gamma_j \) does not contain \( v \) as a substring and the first and the last letter in \( \gamma_i \) are both different from \( v \).
Since $S^k \omega$ starts inside $\omega_m$ and $(S^k \omega)|m = \omega|m$ begins with the substring $\tau_0 v$.
we conclude that
\[ S^k \omega = \tau_0 v_j \gamma_j v_{j+1} \gamma_{j+1} \ldots v_{m+1} \gamma_{m+2} \omega_{m+2} \ldots \] for some \( j > 1 \), (4.5)

cf. Figure 1. Indeed, if \( j = 1 \), then \( S^k \omega = \tau_0 v_1 \gamma_1 v_2 \gamma_2 \ldots v_{m+1} \gamma_{m+1} \omega_{m+2} \ldots \)
Since also \( (S^k \omega)m = \omega|m = \omega_m|m = (\gamma_0 v_1 \gamma_1 v_2 \gamma_2 \ldots v_{m+1} \gamma_{m+1})m \), we infer that \( \tau_0 = \gamma_0 \). Hence \( S^k \omega = \omega_m \sigma_{m+1} \omega_{m+2} \ldots \) and the fact that \( m + 1 \leq n \) therefore shows that \( \omega((m+1)) = (S^k \omega)((m+1)) = \omega_m \). But this contradicts the fact that the last letter in \( \omega_m \) is different from the \((m+1)\)th letter in \( \omega \). This proves (4.5).

Also, \( \omega|m = \omega_m|m \), whence
\[ (S^k \omega)m = \omega|m = \omega_m|m = \begin{cases} \gamma_0 v_1 \gamma_1 v_2 \gamma_2 \ldots v_{m+1} \gamma_{m+1} & \text{for } \gamma_l \neq \emptyset; \\ \gamma_0 v_1 \gamma_1 v_2 \gamma_2 \ldots v_{m+1} v_1 & \text{for } \gamma_l = \emptyset, \end{cases} \] (4.6)

where we have written \( \zeta^- = \zeta(i-1) \) for \( \zeta \in \Sigma^i \).

Since \( j > 1 \) and \( \sigma_{m+1} \) does not contain the substrings \( v^- \) and \( v \), and \( \omega_{m+1} \) contains the substring \( v \), we deduce from (4.5) and (4.6) (see also Figure 1), that

\[ \gamma_1 = \gamma_j, \]
\[ v_2 = v_{j+1}, \]
\[ \vdots \]
\[ v_{l-j+1} = v_l, \]
\[ \gamma_{l-j+1} = \gamma \sigma_{m+1}, \]
\[ v_{l-j+2} \text{ is a substring of } \omega_{m+1} \text{ for } \gamma_l \neq \emptyset, \]
\[ v_{l-j+2} \text{ is a substring of } \omega_{m+1} \text{ for } \gamma_l = \emptyset. \]

This shows that \( \sigma_{m+1} \) is a substring of \( \gamma_{l-j+1} \) and therefore, in particular, a substring of \( \omega_m \). Hence
\[ n \geq m + 1 = |\omega_m| \]
\[ \geq |\sigma_{m+1}| = M_{jm+1} \]
\[ = Q_{m+1} - Q_{m} - (n + 2) \]
\[ \geq k_{m+1} - (k_m + M) - (n + 2) \]
\[ = k_{m+1} - k_m - (n + M + 2). \]

Finally, since \( k_{m+1} - k_m \geq 4(n + M) > 2n + M + 2 \), this gives the desired contradiction. \( \square \)

**Proposition 4.3.** \( Z \subseteq E \) and \( \pi(Z) \subseteq G. \)

**Proof.**

We first prove that \( Z \subseteq E \). For \( \omega \in Z \), Proposition 4.1 and Proposition 4.2 imply that \( \frac{1}{n} \log \text{diam } K_{\omega|n} \to -\chi \) and \( \frac{1}{n} \log \tau_{|\omega|n}(\omega) - \chi n \to 0 \). It follows easily from this and the fact that \( 0 < \inf_n t_n \leq \sup_n t_n < \infty \), that
\[ \frac{\log \tau_{\omega|n}(\omega)}{-\log \text{diam } K_{\omega|n}} - t_n = \left( \frac{1}{n} \log \tau_{\omega|n}(\omega) + t_n \right) \to 0. \]
Finally, since \( \{t_n \mid n \in \mathbb{N} \} \) is a dense subset of \( I \), this implies that

\[
A \left( \frac{\log \tau_{[\omega]_n}(\omega)}{-\log \text{diam } K_{[\omega]_n}} \right) = I.
\]

This shows that \( Z \subseteq E \).

Next, we prove that \( \pi(Z) \subseteq G \). Indeed, since \( Z \subseteq \gamma \), it follows from the choice of \( \gamma \) (cf. (3.10)) that \( \pi(Z) \subseteq \pi(\gamma) \subseteq K\gamma \subseteq G \). \( \square \)

5. Proof of Theorem 6.(1): \( \dim K - 2\delta \leq \dim \pi(Z) \). We will now prove that

\[
\dim K - 2\delta \leq \dim \pi(Z). \tag{5.1}
\]

We will prove (5.1) by constructing a probability measure \( \mu \) on \( \pi(Z) \) for which there exists a constant \( C > 0 \) such that

\[
\mu(B(x, r)) \leq Cr^{\dim K - 2\delta} \tag{5.2}
\]

for all \( x \in \pi(Z) \) and all \( r > 0 \).

Recall that \( \bar{\nu} \) is the Gibbs state of \( s_M \Phi_M \) (cf. (3.5)), and recall that the map \( \Lambda : \Gamma_{M}^{\mathbb{N}} \to \Sigma^{\mathbb{N}} \) is defined in (3.11). Now define the probability measure \( \bar{\mu} \) on \( \Sigma^{\mathbb{N}} \) by

\[
\bar{\mu} = \bar{\nu} \circ \Lambda^{-1},
\]

and finally define the probability measure on \( K \) by

\[
\mu = \bar{\mu} \circ \pi^{-1}.
\]

The next lemma is a standard result due to Hutchinson [5].

**Lemma 5.1.** Let \( r, c_1, c_2 > 0 \), and let \( (V_i)_i \) be a family of open disjoint subsets of \( \mathbb{R}^d \) such that \( V_i \) contains a ball of radius \( c_1r \) and is contained in a ball of radius \( c_2r \). Then

\[
\left| \{ i \mid B(x, r) \cap V_i \neq \emptyset \} \right| \leq \left( \frac{1 + 2c_2}{c_1} \right)^d
\]

for all \( x \in \mathbb{R}^d \).

**Proposition 5.2.**

1. \( \mu(\pi(Z)) = 1 \).
2. There exists a constant \( C > 0 \) such that

\[
\mu(B(x, r)) \leq Cr^{\dim K - 2\delta} \tag{5.3}
\]

for all \( x \in \pi(Z) \) and all \( r > 0 \).
Proof.
(1) It follows from the ergodic theorem that \( \tilde{\nu}(X) = 1 \), whence \( 1 \geq \mu(\pi(Z)) = (\tilde{\nu} \circ \Lambda^{-1} \circ \pi^{-1})(\pi \Lambda(X)) \geq \tilde{\nu}(X) = 1 \).

(2) We first introduce some notation. For \( \gamma \nu \sigma_1 \omega_1 \ldots \sigma_n \omega_n \sigma_{n+1} \in \Sigma^* \) with \( \sigma_i \in \Gamma^i_M \) for \( i \leq n \) and \( \sigma_{n+1} = \gamma_1 \ldots \gamma_k \in \Gamma^k_M \) for some \( 1 \leq k \leq j_{n+1} \), and \( \omega_i = \Xi_i(\gamma \nu \sigma_1 \omega_1 \ldots \sigma_{i-1} \omega_{i-1} \sigma_i) \), we will write

\[
(\gamma \nu \sigma_1 \omega_1 \ldots \sigma_n \omega_n \sigma_{n+1})^\ast = \gamma \nu \sigma_1 \omega_1 \ldots \sigma_n \gamma_1 \ldots \gamma_k
\]

where \( \gamma_0 = \emptyset \). If \( n \) and \( k \) are positive integers and \( r > 0 \), then we will write

\[
X_{n,k,r} = \left\{ \omega = \gamma \nu \sigma_1 \omega_1 \ldots \sigma_n \omega_n \sigma_{n+1} \middle| \sigma_i \in \Gamma^i_M \text{ for } i \leq n, \right. \\
\sigma_{n+1} \in \Gamma^k_M, \\
\omega_i = \Xi_i(\gamma \nu \sigma_1 \omega_1 \ldots \sigma_{i-1} \omega_{i-1} \sigma_i), \\
\sup_{\sigma \in [\omega]} \exp \left( \sum_{k=0}^{[\omega]-1} \Phi^k \sigma \right) \leq r
\]

\[
X_r = \bigcup_n \bigcup_k X_{n,k,r}.
\]

Let \( c \) denote the maximum of the constants in Proposition 2.1, Proposition 2.2 and (3.5). Let \( \omega = \gamma \nu \sigma_1 \omega_1 \ldots \sigma_n \omega_n \sigma_{n+1} \in X_{n,k,r} \) where \( \sigma_i \in \Gamma^i_M \) for \( i \leq n \) and \( \sigma_{n+1} \in \Gamma^k_M \) for some \( 1 \leq k \leq j_{n+1} \), and \( \omega_i = \Xi_i(\gamma \nu \sigma_1 \omega_1 \ldots \sigma_{i-1} \omega_{i-1} \sigma_i) \). Write \( \sigma = \sigma_1 \ldots \sigma_n \sigma_{n+1} \) and \( \tau = \gamma \nu \omega_1 \ldots \omega_n \). Next observe that it follows from the principle of bounded distortion that

\[
\tilde{\mu}[\omega] = \tilde{\mu}[\gamma \nu \sigma_1 \omega_1 \ldots \sigma_n \omega_n \sigma_{n+1}]
\]

\[
= \tilde{\nu}\sigma_1 \ldots \sigma_n \sigma_{n+1}
\]

\[
\leq c \sup_{\zeta \in [\sigma]} \exp \left( \sum_{l=0}^{j_1+j_2+\ldots+j_n+k-1} s_M \Phi^l M(S^l_M \zeta) \right)
\]

\[
\leq c^3 \sup_{\zeta \in [\sigma]} \exp \left( \sum_{l=0}^{[\omega]-1} s_M \Phi^l M(S^l_M \zeta) \right)
\]

\[
\leq c^3 \frac{1}{r_{\min}^{\left( [\gamma]+[\nu]+[\omega_1]+[\omega_2]+\cdots+[\omega_n] \right) s_M}}
\]

\[
\leq c^3 \frac{1}{a n^2 r_{\min}^{s_M}}
\]
\[
\begin{align*}
&\leq c^3 \frac{1}{\delta \min} \left( \sup_{\xi \in \omega^*} \exp \left( \sum_{l=0}^{\omega^*-1} \Phi(S^l) \right) \right)^{1/2} \\
&\leq c^3 \frac{1}{\delta \min} \left( \sup_{\xi \in \omega^*} \exp \left( \sum_{l=0}^{\omega^*-1} \Phi(S^l) \right) \right)^{1/2} \\
&\leq c^3 \frac{1}{\delta \min} \left( \sup_{\xi \in \omega^*} \exp \left( \sum_{l=0}^{\omega^*-1} \Phi(S^l) \right) \right)^{1/2} \\
&\leq c^3 \frac{1}{\delta \min} \left( \sup_{\xi \in \omega^*} \exp \left( \sum_{l=0}^{\omega^*-1} \Phi(S^l) \right) \right)^{1/2} \\
&\leq c_1 \frac{b^{2k_n}}{a^{n^2}} r_{\dim K-2\delta}
\end{align*}
\]

where \( c_1 = c^3 r_{\min} (|\gamma|+M) \), \( a = \omega^4 r_M \) and \( b = \delta/2 \).

Since the OSC is satisfied there exists an open, non-empty and bounded set \( U \subseteq \mathbb{R}^d \) such that \( U_i, S_i \cup U \subseteq U \) for \( i \neq j \). For \( \omega \in \Sigma^* \) write \( U_\omega = S_\omega \cup U \).

Since \( U \) is open, non-empty and bounded, \( U \) contains a ball of radius \( r_1 > 0 \) and \( U \) is contained in a ball of radius \( r_2 \). Momentarily fix \( \omega \in X_{n,k,r} \).

Since \( U_\omega \) contains a ball of radius \( c^{-1} \sup_{\sigma \in [\omega]} \exp \left( \sum_{k=0}^{\omega^*} \Phi(S^k) \right) \rho_1 \) and \( \sup_{\sigma \in [\omega]} \exp \left( \sum_{k=0}^{\omega^*} \Phi(S^k) \right) \geq c^{-1} \sup_{\sigma \in [\omega]} \exp \left( \sum_{k=0}^{\omega^*} \Phi(S^k) \right) r^{M}_{\min} \geq r^{-1} r^{M}_{\min} \), we conclude that \( U_\omega \) contains a ball of radius \( r c^{-1} r^{M}_{\min} \rho_1 \).

Moreover, since the set \( U_\omega \) is contained in a ball of radius \( c \sup_{\sigma \in [\omega]} \exp \left( \sum_{k=0}^{\omega^*} \Phi(S^k) \right) \rho_2 \) and \( \sup_{\sigma \in [\omega]} \exp \left( \sum_{k=0}^{\omega^*} \Phi(S^k) \right) \leq r \), we conclude that \( U_\omega \) is contained in a ball of radius \( r c \rho_2 \).

It therefore follows from Lemma 5.1 that, if \( x \in \mathbb{R}^d \), then

\[
\left| \left\{ \omega \in X_{n,k,r} \ \mid \ U_\omega \cap B(x,r) \neq \emptyset \right\} \right| \leq c_2
\]

where \( c_2 = (\frac{1+2\rho_2}{r^{\min}})^d \).

Next, fix \( x \in \pi(Z) \) and \( r > 0 \). We now have (using (5.4) and (5.5), and the fact that \( K_\omega \subseteq U_\omega \)),

\[
\mu(B(x,r)) = \mu(\pi(E) \cap B(x,r))
\]

\[
\leq \mu \left( \bigcup_{\omega \in X_r} \pi[\omega] \right)
\]

\[
= \mu \left( \bigcup_{n} \bigcup_{k \leq n+1} \left( \bigcup_{\omega \in X_{n,k,r}} \pi[\omega] \right) \right)
\]

\[
\leq \sum_{n} \sum_{k \leq n} \sum_{\omega \in X_{n,k,r}} \mu[\omega]
\]

\[
\leq \sum_{n} \sum_{k \leq n} \sum_{\omega \in X_{n,k,r}} c_1 \frac{b^{2k_n}}{a^{n^2}} r_{\dim K-2\delta}
\]

\[
\leq \sum_{n} \sum_{k \leq n} \sum_{\omega \in X_{n,k,r}} c_1 \frac{b^{2k_n}}{a^{n^2}} r_{\dim K-2\delta}
\]
\[ \leq c_1 \sum_n k_{n+1} \left| \{ \omega \in X_{n,k,r} \mid K_\omega \cap B(x,r) \neq \emptyset \} \right| \frac{b^{2k_n}}{a^{n^2}} r^{\dim K - 2\delta} \]
\[ \leq c_1 c_2 \sum_n k_{n+1} \frac{b^{2k_n}}{a^{n^2}} r^{\dim K - 2\delta}. \]

Since \( 0 < b < 1 \), we deduce that \( \log b < 0 \), and (3.6) and (3.7) therefore imply that we can find a positive integer \( n_0 \) such that

\[ \frac{\log k_{n+1}}{n^2} + \frac{k_n}{n^2} \log b \leq \log a \]

for all \( n \geq n_0 \). Rearranging this inequality shows that \( \frac{k_{n+1}b^{k_n}}{a^{n^2}} \leq 1 \) for \( n \geq n_0 \), and we can thus find \( c_3 > 0 \) such that

\[ \frac{k_{n+1}b^{k_n}}{a^{n^2}} \leq c_3 \]

for all \( n \). Hence

\[ \mu(B(x,r)) \leq c_1 c_2 c_3 \left( \sum_{n_0 \leq n} b^{k_n} \right) r^{\dim K - 2\delta} \leq c_1 c_2 c_3 \left( \sum_n b^n \right) r^{\dim K - 2\delta}. \]

Since \( 0 < b < 1 \), this proves (5.3). \( \square \)

**Proposition 5.3.** \( \dim K - 2\delta \leq \dim \pi(Z) \).

**Proof.**

It follows from Proposition 5.2 and the mass distribution principle (cf. [3, p. 55, Theorem 4.2]) that \( \dim \pi(E) \geq \dim K - 2\delta \). \( \square \)

**Proof of Theorem 6.(1).**

It follows from Proposition 4.3 that \( \pi(Z) \subseteq \pi(E) \cap G \), and Proposition 5.2 therefore implies that \( \dim K - 2\delta \leq \dim(\pi(Z)) \leq \dim(\pi(E) \cap G) \). This concludes the proof since \( \delta > 0 \) was arbitrary. \( \square \)

**6. Proof of Theorem 6.(2).** In this section we will prove Theorem 6.(2). In fact, Theorem 6.(2) follows easily from Theorem 6.(1) and Proposition 6.1 below.

**Proposition 6.1.** Assume that the SSC is satisfied. For all \( \omega \in \Sigma^N \) we have

\[ A \left( \frac{\log \tau_{[\omega|n]}(\omega)}{n} \right) = A \left( \frac{\log \tau_{B(x,r)}(x)}{-\log r} \right) \]

where \( x = \pi(\omega) \).

**Proof.**

This result follows from standard arguments and the proof is therefore omitted. \( \square \)

**Proof of Theorem 6.(2).**

Theorem 6.(2) follows immediately from Theorem 6.(1) and Proposition 6.1. \( \square \)

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