Orbits in Non-Supersymmetric Magic Theories

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Abstract

We determine and classify the electric-magnetic duality orbits of fluxes supporting asymptotically flat, extremal black branes in $D = 4, 5, 6$ space-time dimensions in the so-called non-supersymmetric magic Maxwell-Einstein theories, which are consistent truncations of maximal supergravity and which can be related to Jordan algebras (and related Freudenthal triple systems) over the split complex numbers $\mathbb{C}_s$ and the split quaternions $\mathbb{H}_s$. By studying the stabilizing subalgebras of suitable representatives, realized as bound states of specific weight vectors of the corresponding representation of the electric-magnetic duality symmetry group, we obtain that, as for the case of maximal supergravity, in magic non-supersymmetric Maxwell-Einstein theories there is no splitting of orbits, namely there is only one orbit for each non-maximal rank element of the relevant Jordan algebra (in $D = 5$ and 6) or of the relevant Freudenthal triple system (in $D = 4$).

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1 Introduction

The so-called *magic* Maxwell-Einstein supergravity theories (MESGT’s) were discovered in [1–3]; they are endowed with eighth-maximal ($N = 2$) local supersymmetry, and they can be described in terms of Euclidean Jordan algebras $J^A_3$ of rank 3, generated by $3 \times 3$ Hermitian matrices over the four normed division algebras $A = \mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{H}$ (quaternions), and $\mathbb{O}$ (octonions). The “magic” of these theories can be traced back to the fact that their electric-magnetic (U-)duality Lie groups in $D = 3, 4, 5$ Lorentzian space-time dimensions can respectively be arranged into the fourth, the third, and the second row of the single split (non-symmetric) real form of the magic square of Freudenthal, Rozenfeld and Tits ([9]; see also [4]); it is worth noting that the fourth row is made up only of exceptional Lie groups: $F_4(4)$, $E_6(2)$, $E_7(-5)$ and $E_8(-24)$, respectively.

The remarkable relation of magic MESGT’s to simple Jordan algebras has allowed for a quite elegant algebraic classification of the non-transitive action of their global duality symmetries on the representation spaces of fluxes supporting asymptotically flat black brane solutions; this has been exploited firstly in [10], and then in a number of papers (see e.g. [11] for a comprehensive treatment, and for list of Refs.). In recent years, other approaches yielded to deep insights in the orbit stratifications; for instance, in [12] the duality orbits were investigated by studying the stabilizing subalgebras of suitable representatives, realized as bound states of specific weight vectors of the corresponding duality representations.

For what concerns the massless spectrum of magic MESGT’s, in [13] its bosonic sector (also including the $(D - 1)$-forms and the $D$-forms) was obtained from the very-extended Kac-Moody algebras $g^{+++}$ (where $g$ denotes the suitable non-compact, real form of the Lie algebra of the $D = 3$ U-duality group). More specifically, the deletion of a suitable node in the Tits-Satake diagram of $g^{+++}$ determines the global symmetry of the theory in all the dimensions in which the corresponding MESGT can be consistently defined; the fact that the nodes of the Tits-Satake diagrams corresponding to compact Cartan generators cannot be deleted explains why the magic MESGT’s can be defined only up to $D = 6$.

One can also consider the split versions of complex numbers, quaternions and octonions, respectively denoted by $\mathbb{C}_s$, $\mathbb{H}_s$, and $\mathbb{O}_s$; such normed composition algebras are no more division, because they contain non-trivial zero divisors. Correspondingly, a doubly split (symmetric) magic square can be constructed using $\mathbb{R}$, $\mathbb{C}_s$, $\mathbb{H}_s$, $\mathbb{O}_s$ ([14] also cfr. [4]), and it is given in Table 1. The theories based on the corresponding Euclidean simple cubic Jordan algebras $J^A_3$ are an interesting class of Maxwell-Einstein theories; in fact, they all share the feature that upon dimensional reduction to $D = 3$, the resulting $U$-duality group (which can be realized as quasiconformal [15] symmetry of the corresponding $J^A_3$) is real.

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1 The Lorentzian version of the exceptional, simple, Lorentzian cubic Jordan algebras can also be defined; for its definition and symmetries, see [4], and for its relation to MESGT’s, see [5].

2 Here duality is referred to as the analogue in a non-supersymmetric context of the “continuous” symmetries of [6,7], whose discrete versions in the supersymmetric case are the $U$-dualities of non-perturbative string theory introduced by Hull and Townsend [8].
Table 1: The doubly split magic square [14].

|   | $\mathbb{R}$ | $\mathbb{C}_s$ | $\mathbb{H}_s$ | $\mathbb{O}_s$ |
|---|--------------|----------------|--------------|--------------|
| $\mathbb{R}$ | $SO(3)$      | $SL(3, \mathbb{R})$ | $Sp(6, \mathbb{R})$ | $F_{4(4)}$ |
| $\mathbb{C}_s$ | $SL(3, \mathbb{R})$ | $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$ | $SL(6, \mathbb{R})$ | $E_{6(6)}$ |
| $\mathbb{H}_s$ | $Sp(6, \mathbb{R})$ | $SL(6, \mathbb{R})$ | $SO(6, 6)$ | $E_{7(7)}$ |
| $\mathbb{O}_s$ | $F_{4(4)}$ | $E_{6(6)}$ | $E_{7(7)}$ | $E_{8(8)}$ |

and maximally non-compact (i.e., split), and of exceptional type $E_{n(n)}$, namely $E_{6(6)}$ for $\mathbb{C}_s$, $E_{7(7)}$ for $\mathbb{H}_s$ and $E_{8(8)}$ for $\mathbb{O}_s$. The theory based on split octonions $\mathbb{O}_s$ (and thus on the corresponding split form $J^O_3$ of the Albert algebra) has $E_{8(8)}$ U-duality symmetry in $D = 3$, and it is maximal supergravity. Maximally supersymmetric MESGT’s in any dimension are related to the very-extended Kac-Moody algebra $e^{++8}$ (also dubbed $e_{11}$) [15]; since this is a split form, the Tits-Satake diagram coincides with the Dynkin diagram, with all non-compact white nodes (corresponding to non-compact Cartan generators), as given in fig. 1. The supergravity theory in $D$ Lorentzian space-time dimensions is related to the decomposition of $e^{++8}$ in which the “gravity line” is identified with the $\mathfrak{a}_{D-1} \approx \mathfrak{sl}(D, \mathbb{R})$ subalgebra containing node 1; on the other hand, the portion of the diagram not connected to the $\mathfrak{a}_{D-1}$ subalgebra gives rise to the internal ($U$-duality) symmetry. Thus, from fig. 1 one can realize that maximal supergravity can consistently be defined for any $3 \leq D \leq 11$, with two different theories existing in $D = 10$, namely the IIA non-chiral theory (with $\mathfrak{a}_9$ given by nodes from 1 to 9) and the IIB chiral theory (with $\mathfrak{a}_9$ given by nodes from 1 to 8, plus node 11) [17]. Moreover, the resulting decomposition of $e^{++8}$ yields all the $p$-forms of the maximally supersymmetric theory in the corresponding dimension $D$, sitting in the pertaining representation(s) of the $U$-duality group, as given in Table 2 [18, 19].

The Maxwell-Einstein theories based on split quaternions $\mathbb{H}_s$ and on split complex (also named hypercomplex) numbers $\mathbb{C}_s$ (and thus on $\mathcal{J}_{3}^{\mathbb{H}_s}$ resp. $\mathcal{J}_{3}^{\mathbb{C}_s}$) are consistent truncations [20] of maximal supergravity, but they are non-supersymmetric: namely, their field content cannot be regarded as the bosonic sector of a theory endowed with any amount of local supersymmetry; while maximal supergravity has been the object of intense study along the years (for an analysis of its $U$-orbit structure, see [21–25]), its $\mathbb{H}_s$- and $\mathbb{C}_s$- based truncations are much less known, and they have been analyzed in some detail only recently, in [26]. Remarkably, the electric-magnetic ($U$-)duality groups of such theories in $D = 3, 4, 5$
Lorentzian space-time dimensions correspond to the (complex, quaternionic and octonionic columns of the) fourth, the third and the second rows of the aforementioned doubly split (symmetric) magic square, given in Table 1. Thus, such theories will be named magic non-supersymmetric Maxwell-Einstein theories. Analogously to their aforementioned maximal Os-based counterpart, magic non-supersymmetric theories correspond to the very-extended Kac-Moody algebras e+++7(7) resp. e+++6(6)[20, 27]; such split forms were firstly investigated in [27], and a progressive deletion of nodes of the corresponding Tits-Satake diagrams (respectively given by figs. 2 and 3), it turns out that they can be consistently defined up to D = 10 resp. D = 8; in such maximal dimensions, the bosonic spectrum of these theories has been determined in [20].

An analysis of the Hs- and Cs-based magic non-supersymmetric theories has been carried out in [26], in which it was shown that they arise as “Ehlers” sl(2, R)- and sl(3, R)-truncations [28] of maximal supergravity.

The magic non-supersymmetric Maxwell-Einstein theory based on Hs (and thus on J3Hs), is related to e+++7(7), whose Dynkin diagram is drawn in fig. 2. As mentioned above, the D-dimensional theory corresponds to an sl(D, R) symmetry in the diagram, which involves the nodes with labels 1 to D − 1, whereas those nodes which are not connected to any of the sl(D, R) nodes determine the global, electric-magnetic duality Lie algebra.

Table 3 lists all p-forms (fitting into representations of the electric-magnetic duality)
of the split quaternionic magic non-supersymmetric theories in any dimension \[26\]. In order to get the full bosonic spectrum, one has to add (Einstein) gravity as well the scalar fields, coordinatizing a target space which is a symmetric manifold \(G/H\), where \(G\) is the electric-magnetic duality Lie group and \(H\) its maximal compact subgroup. Note that the aforementioned fact that the \(e^{+++}_{7(7)}\) theory is an \(\mathfrak{sl}(2,\mathbb{R})\)-truncation of the maximal \(e^{+++}_{8(8)}\) theory also explains the occurrence of two different theories (8A and 8B) in \(D = 8\) \[26\]; moreover, it should be noted that in all dimensions \(3 \leq D \leq 10\), the representations of Table 3 result from \(\mathfrak{sl}(2,\mathbb{R})\)-invariant truncations of the representations of Table 2 with the only exceptions being given by \(D\)-forms in \(D\) dimensions and by 2-forms in \(D = 3\), in which cases the \(\mathfrak{sl}(2,\mathbb{R})\)-invariant truncation overestimates the number of forms \[26\]. As explained in \[26\], the perturbative symmetry of the \(\mathbb{H}_s\)-based magic theory in \(D\) dimensions is \(\mathfrak{so}(8 - D, 8 - D) \oplus \mathfrak{sl}(2,\mathbb{R})\), where \(\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{tri}(\mathbb{H}_s) \oplus \mathfrak{so}(\mathbb{H}_s)\), with \(\mathfrak{tri}(\mathbb{H}_s)\) and \(\mathfrak{so}(\mathbb{H}_s)\) respectively denoting the \textit{triality} and the \textit{norm-preserving} symmetries of \(\mathbb{H}_s\) \[29\].

A similar analysis can be done for the magic non-supersymmetric Maxwell-Einstein theory based on \(\mathbb{C}_s\) (and thus on \(J^C_3\)), which is related to \(e^{+++}_{6(6)}\), whose Dynkin diagram is drawn in fig. 3. In any dimension, the \(p\)-form massless spectrum of the theory can thus be obtained from the very extended Kac-Moody algebra \(e^{+++}_{6(6)}\) \[20,27\], whose Dynkin diagram is given in fig. 3. Table 4 lists all \(p\)-forms (fitting into representations of the electric-

\[
\begin{align*}
\text{Figure 2: The } e^{+++}_{7(7)} \text{ Dynkin diagram.}
\end{align*}
\]

magnetic duality) of the split complex magic non-supersymmetric theories in any dimension \[26\]. As for the split quaternionic theories treated above, the full bosonic massless spectrum is then obtained by adding (Einstein) gravity as well the scalar fields, whose target space is the symmetric coset \(G/H\). Note that in all dimensions \(3 \leq D \leq 8\), the representations of Table 4 result from \(\mathfrak{sl}(3,\mathbb{R})\)-invariant truncations of the representations of Table 2 again with the only exceptions being given by \(D\)-forms in \(D\) dimensions and by 2-forms in \(D = 3\), in which cases the \(\mathfrak{sl}(3,\mathbb{R})\)-invariant truncation overestimates the number of forms \[26\]. Again, as explained in \[26\], the perturbative symmetry of the \(\mathbb{C}_s\)-based magic
The present paper, expanding on [26], is devoted to the detailed analysis of the $U$-duality orbits of asymptotically flat, extremal black branes in the aforementioned magic non-supersymmetric Maxwell-Einstein theories, based on $\mathbb{H}_s$ and on $\mathbb{C}_s$, in $D = 4, 5, 6$ Lorentzian space-time dimensions. Such theories, despite being present in the classification of symmetric non-linear sigma models coupled to Maxwell-Einstein gravity (cfr. Table 2 of [30]), did not receive much attention in literature, as mentioned above. Symmetries of Freudenthal triple systems and cubic Jordan algebras defined over split algebras were studied in [15] and [31] (see also table 1 of [32], and Refs. therein); theories over split algebras have been quite recently considered, in a different context, in [33], while $\mathbb{C}_s$- and $\mathbb{H}_s$- valued scalar fields have also been recently considered in cosmology [34]. However, to the best of our knowledge, magic non-supersymmetric theories were investigated only recently, in [26]. In such a paper, an analysis of the $U$-duality orbit stratification was

| Dim | Symmetry          | $p = 1$ | $p = 2$ | $p = 3$ | $p = 4$ | $p = 5$ | $p = 6$ | $p = 7$ | $p = 8$ |
|-----|-------------------|---------|---------|---------|---------|---------|---------|---------|---------|
| 10  | -                 |         |         |         | 1       |         |         |         |         |
| 9   | $\mathbb{R}^+$   | 1       | 1       | 1       |         |         | 1       | 1       |         |
| 8A  | $GL(2, \mathbb{R})$ | 2       | 1       | 2       | 2       | 3       | 1       | $2 \times 2$ | 3       |
| 8B  | $SL(3, \mathbb{R})$ | 3       | 3       | $\overline{3}$ | $\overline{3}$ | 8       | 8       | $\overline{6}$ | $\overline{6}$ |
| 7   | $GL(3, \mathbb{R})$ | 1       | 3       | $\overline{3}$ | $\overline{3}$ | 1       | 1       | 3       | $2 \times 3$ |
| 6   | $SL(4, \mathbb{R}) \times SL(2, \mathbb{R})$ | (4, 2) | (6, 1) | (4, 2) | (15, 1) | (20, 2) | (64, 1) | (16, 3) | (6, 3) |
| 5   | $SL(6, \mathbb{R})$ | 15      | $\overline{15}$ | 35      | $105$   | $384$   | $2079$  | $105$   | $15$    |
| 4   | $SO(6, 6)$       | 32      | 66      | 352     | $462$   | $66$    |         |         |         |
| 3   | $E_{7(7)}$       | 133     | 1539    | 40755   | 1539    | 1       | 1       | 1       |         |

Table 3: All the $p$-forms of the $\mathfrak{e}^{+++}_{7(7)}$ theory in any dimension.
performed only for the $\mathbb{C}_s$-based theory in $D = 4$ (for the electric-magnetic charges of dyonic extremal black holes). The present investigation details the study of the $\mathbb{C}_s$-based magic non-supersymmetric theory in $D = 5$ and 6, as well as of the $\mathbb{H}_s$-based magic non-supersymmetric theory in $D = 4, 5$ and 6. This will result in a detailed algebraic classification of the asymptotically flat, extremal black brane solutions of such theories.

The results of our analysis rigorously prove and illustrate in detail a conjecture made in [26]: as for the case of maximal supergravity (whose the considered theories are non-supersymmetric truncations related to suitable Jordan subalgebras of $J_3(\mathbb{O}_s)$ and $J_2(\mathbb{O}_s)$), in magic non-supersymmetric Maxwell-Einstein theories there is no splitting of orbits, namely there is only one orbit for each non-maximal rank element of the relevant algebraic structure (i.e., of $J_2$ - or of its non-singlet algebraic complement to $J_3$ - in $D = 6$, of $J_3$ in $D = 5$, and of the Freudenthal triple system over $J_3$ in $D = 4$). This to be contrasted to the magic quarter-maximal Maxwell-Einstein supergravity theories [12], in which the orbit splitting for non-maximal-rank elements generally takes place, depending on the real form and on the relevant representations of the duality group$^3$.

\[3\text{Note that the orbit of rank-1 elements is generally the highest weight orbit, which is unique, regardless}\]
The plane of the paper is as follows.

In Sec. 2, we start and analyze the $U$-orbit structure of the $\mathbb{C}_s$- (Sec. 2.1) and $\mathbb{H}_s$- (Sec. 2.2) based magic non-supersymmetric theories in $D = 6$, which, as pointed out above, is the highest dimension in which they can be consistently defined (as such they can be regarded as non-supersymmetric truncations of the non-chiral $(2,2)$ maximal supergravity in $D = 6$); the analysis is based on quadratic (i.e., rank-2) Euclidean Jordan algebras over $\mathbb{C}_s$ and $\mathbb{H}_s$ and on the non-transitive action of their reduced structure symmetry. Then, in Sec. 3, we determine the $U$-orbit structure of the $\mathbb{C}_s$- (Sec. 3.1) and $\mathbb{H}_s$- (Sec. 3.2) based magic non-supersymmetric theories in $D = 5$ (they can be regarded as non-supersymmetric truncations of maximal $\mathcal{N} = 8$ supergravity); in this case, the analysis is based on cubic (i.e., rank-3) Euclidean Jordan algebras over $\mathbb{C}_s$ and $\mathbb{H}_s$ and on the non-transitive action of their reduced structure symmetry. Finally, in Sec. 4, we determine the $U$-orbit structure of the $\mathbb{C}_s$- (Sec. 4.1) and $\mathbb{H}_s$- (Sec. 4.2) based magic non-supersymmetric theories in $D = 4$, which is the lowest dimension in which asymptotically flat black branes exist; also in this case, such theories can be regarded as truncations of the maximal $\mathcal{N} = 8$ supergravity. In $D = 4$, our analysis is based on (reduced) Freudenthal triple systems \cite{35} based on cubic Euclidean Jordan algebras over $\mathbb{C}_s$ and $\mathbb{H}_s$ and on the non-transitive action of their derivations (which act as conformal symmetry of the corresponding cubic Jordan algebra, as well). Three Appendices, containing technical details, conclude the paper.

## 2 $D = 6$

In $D = 6$ dimensions, the asymptotically flat branes are black holes (electric 0-branes), their duals, the magnetic black 2-branes (they are respectively associated to vectors and their duals), and the dyonic black strings (1-branes, associated to tensors).

### 2.1 $\mathbb{C}_s$

The $U$-duality\footnote{As noted, this is the unique case considered in \cite{26}.} group is

\[
SO(2,2) \times SO(1,1) \cong SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SO(1,1) \tag{1}
\]

\[
\cong Str_0 (J_{2^c}^s) \times \frac{Tri (\mathbb{C}_s)}{SO(\mathbb{C}_s)} \cong SL(2,\mathbb{C}_s) \times SO(1,1), \tag{2}
\]

where $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \simeq SL(2,\mathbb{C}_s)$ \footnote{Due to their non-supersymmetric nature, both $\mathbb{H}_s$- and $\mathbb{C}_s$- based non-supersymmetric magic theories are generally anomalous in $D = 6$.}, and $Tri (\mathbb{C}_s)$ and $SO(\mathbb{C}_s)$ respectively denote the triality and norm-preserving symmetries of $\mathbb{C}_s$ (see e.g. \cite{29}). Therefore, the scalar manifold reads

\[
\frac{Str_0 (J_{2^c}^s)}{mcs (Str_0 (J_{2^c}^s))} = \frac{SO(2,2)}{SO(2) \times SO(2)} \times SO(1,1). \tag{3}
\]

\[
\text{the theory under consideration.}
\]

\[
\text{As noted, this is the unique case considered in \cite{26}.}
\]
$Str_0 \left( J_2^{C^*} \right) \simeq SL(2, \mathbb{C}_s)$ is the reduced structure group of the quadratic Jordan algebra $J_2^{C^*}$.

The electric 0-brane and magnetic 2-brane irreps. are the $(2, 1)_1 + (1, 2)_{-1}$, and its conjugate $(2, 1)_{-1} + (1, 2)_1$, of $SO(2, 2) \times SO(1, 1)$ respectively, whereas the dyonic 1-brane rep. is the $(2, 2)_0$. While the $(2, 2)_0$ admits a unique independent quadratic invariant polynomial $I_2$, the $(2, 1)_1 + (1, 2)_{-1}$ and $(2, 1)_{-1} + (1, 2)_1$ do not admit any quadratic invariant polynomial. Consequently, while (dyonic) strings can be “large” (i.e., with non-vanishing entropy), electric black holes and magnetic 2-branes are necessarily “small” (i.e., with a vanishing entropy), at least at Einsteinian (two-derivative) level. In this framework, black strings are thus the unique asymptotically flat objects which may exhibit an attractor behaviour [50] in their near-horizon region (mutatis mutandis, the same considerations holds for the $\mathbb{H}_s$-based theory; see further below).

### 2.1.1 1-Branes

While the non-linear action of $SO(2, 2) \times SO(1, 1)$ on the scalar manifold (eq. (3)) is transitive, the linear action of $SO(2, 2) \times SO(1, 1)$ on the $(2, 2)_0$ trivially determines the stratification into the following orbits, classified in terms of invariant constraints on $I_2$, or equivalently in terms of the rank of the corresponding Jordan algebra $J_2^{C^*}$ (cfr. e.g. [47,49], and Refs. therein).

By exploiting the method developed in [12], we are now going to explicitly determine the U-duality orbits, by studying the stabilizers of bound states of the weights of the $(2, 2)_0$ of the U-duality Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(1, 1)$.

The action of the Cartan involution, $\theta$, on the simple roots $\alpha_1$ and $\alpha_2$ of the two $\mathfrak{sl}(2, \mathbb{R})$ is given by

$$\theta \alpha_i = -\alpha_i \quad i = 1, 2.$$  \hfill (4)

Our analysis of the orbits starts by reporting the Dynkin tree of the $(2, 2)_0$ of $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(1, 1)$ in fig. (4). We note that all the weights in this representation have the same length and are noncompact.

![Dynkin Tree](image)

Figure 4: Dynkin tree of the $(2, 2)_0$ of $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(1, 1)$. We denote with different color the Dynkin labels corresponding to the two $\mathfrak{sl}(2, \mathbb{R})$’s.

Subscripts denote weights with respect to $SO(1, 1)$ throughout.
To find the different orbits for single brane and bound states we should consider single weight or linear combinations of them and study their behaviors under the action of the algebra.

1-weight

To determine the 1-weight stabilizer, one simply can look at the Dynkin tree in fig. (4) and the results, listed in tab. (10) immediately follow. The corresponding 1-weight orbit reads

\[ I_2 = 0 : \quad \mathcal{O}_{I_2=0,q=2} := \frac{SO(2,2) \times SO(1,1)}{(SO(1,1)_* \times SO(1,1)) \times \mathbb{R}^2} = \frac{SO(2,2)}{SO(1,1) \times \mathbb{R}^2}, \tag{5} \]

where \( \mathbb{R}^2 \simeq 2 \) of \( SO(1,1)_* \subset SO(2,2) \). We also reported the corresponding \( (SO(2,2) \times SO(1,1)) \) - invariant constraint on the quadratic invariant \( I_2 \) of the \( (2,2)_0 \); note that the orbit (eq. (5)) is the orbit of rank-1 elements (lightlike vectors) of the quadratic Jordan algebra \( J_2^{CS} \).

2-weights

In order to define a bound state we have to identify two weights not connected by the action of a single generator in the algebra. Looking at fig. (4) we realize that we could take \( \Lambda_1^+ \) and \( \Lambda_2^- \). These weights could be combined into two different bound states, whose stabilizers are listed in tab. (6).

| Common  | Conjunction |
|---------|-------------|
| \( \Lambda_1^+ \), \( \Lambda_2^- \) | \( \Lambda_1^+ + \Lambda_2^- \) | \( \Lambda_1^+ - \Lambda_2^- \) |
| \( H_{\alpha_1} - H_{\alpha_2} \) | \( E_{\alpha_1} - E_{-\alpha_2} \) | \( E_{\alpha_1} + E_{-\alpha_2} \) |
| | \( E_{\alpha_2} - E_{-\alpha_1} \) | \( E_{\alpha_2} + E_{-\alpha_1} \) |

Table 6: \( \Lambda_1^+ \pm \Lambda_2^- \) stabilizers.
The corresponding 2-weights bound states orbit reads\(^7\) for both choices “±”:

\[
I_2 \neq 0 : \mathcal{O}_{I_2 \neq 0, q=2} := \frac{SO(2, 2) \times SO(1, 1)}{SO(2, 1) \times SO(1, 1)} \cong \frac{SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SO(1, 1)}{SL(2, \mathbb{R})_d \times SO(1, 1)} = \frac{SL(2, \mathbb{R}) \times SL(2, \mathbb{R})}{SL(2, \mathbb{R})_d}. \tag{6}
\]

Note that this is a symmetric manifold. We also reported the corresponding \((SO(2, 2) \times SO(1, 1))\)-invariant constraint on the quadratic invariant \(I_2\) of the \((2, 2)_0\); note that the orbit \((\text{eq. (6)})\) is the orbit of rank-2 elements (non-lightlike vectors) of the quadratic Jordan algebra \(J_2^{*\alpha}\).

### 2.1.2 0- and 2- Branes

On the other hand, the linear action of \(SO(2, 2) \times SO(1, 1)\) on the \((2, 1)_1 + (1, 2)_{-1}\) and \((2, 1)_{-1} + (1, 2)_1\) determines the stratification into orbits, classified in terms of invariant constraints on the \((2, 2)_0\)-covariant vector\(^8\)

\[
V^I := (\gamma^I)^{\alpha \beta} q_{\alpha} q_{\beta}, \tag{7}
\]

\((I = 1, ..., 4, \alpha, \beta = 1, ..., 4)\), where \(\gamma^I\) here denotes the gamma matrices of \(SO(2, 2)(\times SO(1, 1))\).

As a consequence of the result of \(\text{[51]}\) and of the algebraic embedding \(\mathbb{C}_s \subset \mathbb{O}_s\), it should be remarked that both the \((2, 1)_1 + (1, 2)_{-1}\) and \((2, 1)_{-1} + (1, 2)_1\) of \(\text{Str}_0(J_2^{\gamma s}) \times \frac{\text{Tr}(\mathbb{C}_s)}{\text{SO}(\mathbb{C}_s)}\) = \(SO(2, 2) \times SO(1, 1)\) can be represented as a pair of split complex numbers \(\mathbb{C}_s\).

By applying the same methods used in the analysis of the black 1-brane orbits above, the following stratification is determined\(^9\).

**1-weight orbit** \((V^I = 0, \forall I)\):

\[
\mathcal{O}_{V^I = 0, q=2} := \frac{SO(2, 2) \times SO(1, 1)}{SO(2, 1) \times SO(1, 1) \times \mathbb{R}}, \tag{8}
\]

where \(\mathbb{R} \cong \mathbb{1}\) is a singlet of \(SL(2, \mathbb{R})\). This is the orbit of pure, non-generic\(^{10}\) spinors.

**2-weights orbit** \((V^I \neq 0 \text{ for some } I)\):

\[
\mathcal{O}_{V^I \neq 0, q=2} := \frac{SO(2, 2) \times SO(1, 1)}{SO(1, 1) \times \mathbb{R}^2}. \tag{9}
\]

This is the generic orbif of pure, generic spinors.

---

\(^7\) The subscript “d” stands for diagonal throughout.

\(^8\) We consider throughout only the case of electric black holes \((q_0 \in (2, 1)_1 + (1, 2)_{-1}, \text{ in this case}); the treatment and results for magnetic black 2-branes \((p^0 \in (2, 1)_{-1} + (1, 2)_1, \text{ in this case})\) are identical.

\(^9\) These orbits recently appeared in \(\text{[53]}\) (apart from \(SO(1, 1) \simeq \frac{\text{Tr}(\mathbb{C}_s)}{\text{SO}(\mathbb{C}_s)}\) - see further below - in the numerator therein).

\(^{10}\) The orbits \((\text{eq. (8)})\) and \((\text{eq. (15)})\) are the same as the ones obtained in \(\text{[46]}\).
2.2 \( \mathbb{H}_s \)

The U-duality group is

\[
SO(3, 3) \times SO(2, 1) \simeq SL(4, \mathbb{R}) \times SL(2, \mathbb{R}) \\
= \text{Str}_0 \left( J^\mathbb{H}_s_2 \right) \times \frac{\text{Tri}(\mathbb{H}_s)}{SO(\mathbb{H}_s)} \simeq SL(2, \mathbb{H}_s) \times SL(2, \mathbb{R}),
\]

where \( SL(4, \mathbb{R}) \simeq SL(2, \mathbb{H}_s) \) \[10\], and \( \text{Tri}(\mathbb{H}_s) \) and \( SO(\mathbb{H}_s) \) respectively denote the triality and norm-preserving symmetries of \( \mathbb{H}_s \) (see e.g. \[29\], and Refs. therein). Therefore, the scalar manifold reads

\[
\text{Str}_0 \left( J^\mathbb{H}_s_2 \right)_{mcs} \left( \text{Str}_0 \left( J^\mathbb{H}_s_2 \right) \right) = \mathbb{H}_s \times \text{Tri}(\mathbb{H}_s) \simeq \text{SO}(3, 3) \times \text{SO}(2, 1),
\]

\[
\text{Str}_0 \left( J^\mathbb{H}_s_2 \right) \simeq \text{SL}(2, \mathbb{H}_s) \times \text{SL}(2, \mathbb{R}),
\]

\[
\text{Str}_0 \left( J^\mathbb{H}_s_2 \right) \text{ is the reduced structure group of the quadratic Jordan algebra } J^\mathbb{H}_s_2. \text{ The electric 0-brane and magnetic 2-brane irreps. are the chiral bi-spinor } (4, 2), \text{ and its conjugate } (4', 2) \text{ of } \text{SO}(3, 3) \times \text{SO}(2, 1) \text{ respectively, whereas the dyonic 1-brane sit in the irrep } (6, 1). \text{ While the } (6, 1) \text{ admits a unique independent quadratic invariant polynomial } I_2, \text{ the } (4, 2) \text{ and } (4', 2) \text{ do not admit any quadratic invariant polynomial.}
\]

Our conventions for the simple roots of \( \mathfrak{sl}(4, \mathbb{R}) \) appear in fig. (5).

\[\alpha_1 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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Figure 6: Dynkin tree of the \((6,1)\) of \(SL(4,\mathbb{R}) \times SL(2,\mathbb{R})\). \(\alpha_1, \alpha_2\) and \(\alpha_3\) are the three simple roots of \(\mathfrak{sl}(4,\mathbb{R})\).

Table 7: Stabilizers for the weights in the \((6,1)\) of \(SL(4,\mathbb{R}) \times SL(2,\mathbb{R})\)

| \(\Lambda_1\) | \(\Lambda_2\) |
|---|---|
| \(\alpha_1 + \alpha_2 + \alpha_3\) | \(\alpha_1\) |
| \(\alpha_1 + \alpha_2\) \(\alpha_2 + \alpha_3\) | \(\alpha_3\) |
| \(\alpha_1\) \(\alpha_2\) \(\alpha_3\) | \(H_{\alpha_1}\) \(H_{\alpha_3}\) |
| \(-\alpha_1\) \(-\alpha_3\) | \(-\alpha_1\) \(-\alpha_2\) \(-\alpha_3\) |
| \(-\alpha_1\) \(-\alpha_2\) \(-\alpha_3\) | \(-\alpha_1\) \(-\alpha_2\) \(-\alpha_3\) |

where \(\mathbb{R}^4 \simeq 4\) of \(SO(2,2)\), or equivalently \(\mathbb{R}^{(2,2,1)} \simeq (2,2)\) of \(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SL(2,\mathbb{R})\). We also reported the corresponding \((SL(4,\mathbb{R}) \times SL(2,\mathbb{R}))\)-invariant constraint on the quadratic invariant \(I_2\) of the \((6,1)\); note that the orbit (eq. (13)) is the orbit of rank-1 elements (lightlike vectors) of the quadratic Jordan algebra \(\mathcal{J}_2^H\).

**Rank-2 (2-weights orbit)** the rank-2 orbit could be defined by combining the weight \(\Lambda_1\) and \(\Lambda_2\), with the notation of fig. (6). Their common and conjunction stabilizers
Table 8: Stabilizers for the weights in the $(6,1)$ of $SL(4,\mathbb{R}) \times SL(2,\mathbb{R})$. We list only the elements in $SL(4,\mathbb{R})$.

| States | $\theta$ | Semisimple Stabilizer | Stabilizer | rank |
|--------|---------|----------------------|-----------|------|
| 1-w    | $\Lambda_1$ | $SO(3,2) \times SO(2,1)$ | $SO(3,2) \times SO(2,1)$ | 1 |
| 2-w    | $\Lambda_1 \pm \Lambda_2$ | $Sp(4,\mathbb{R}) \times Sp(2,\mathbb{R})$ | $[Sp(4,\mathbb{R}) \times Sp(2,\mathbb{R})] \ltimes \mathbb{R}^{(4,2)}$ | 2 |

Table 9: Summary of the orbits in the $(6,1)$ of $SL(4,\mathbb{R}) \times SL(2,\mathbb{R})$

2.2.2 0- and 2- Branes

On the other hand, the linear action of $SO(3,3) \times SO(2,1)$ on the $(4,2)$ and $(4',2)$ determines the stratification into orbits, classified in terms of invariant constraints on the $(6,1)$-covariant vector $V^I$ [eq. (7)] (where in this case $I = 1,...,6,\alpha,\beta = 1,...,8$, with the $\gamma_I$’s now denoting the gamma matrices of $SO(3,3) \times SO(2,1)$. As a consequence of the result of [51] and of the algebraic embedding $\mathbb{H}_{s} \subset \mathbb{O}_{s}$, it should be remarked that both
the \((4, 2)\) and \((4', 2)\) of \(Str_0 \left( J^H_2 \right) \times \frac{Tri(H_2)}{SO(H_2)} = SO(3, 3) \times SO(2, 1)\) can be represented as a pair of split quaternions \(\mathbb{H}_s\).

By exploiting the same methods used above for the \(C_s\)-based theory, the following stratification is determined:

**1-weight orbit** \((V^I = 0, \forall I)\):

\[
\mathcal{O}_{V^I=0,q=4} := \frac{SO(3, 3) \times SL(2, \mathbb{R})}{{SO(3, 3) \times SO(1, 1)} \times (\mathbb{R}^3 \times \mathbb{R})} \cong \frac{SO(3, 3) \times SL(2, \mathbb{R})}{{SO(1, 1) \times \mathbb{R}}},
\]

where \(\mathbb{R}^3 \cong 3'\) denotes the rank-2 antisymmetric irrep. of \(SL(3, \mathbb{R})\), and \(SO(1, 1) \times \mathbb{R}\) is the maximal triangular subgroup of \(SL(2, \mathbb{R})\) itself. This is the orbit of pure, non-generic spinors.

**2-weights orbit** \((V^I \neq 0 \text{ for some } I)\):

\[
\mathcal{O}_{V^I \neq 0,q=4} := \frac{SO(3, 3) \times SL(2, \mathbb{R})}{{SO(2, 1) \times SL(2, \mathbb{R})} \times \mathbb{R}^4},
\]

where \(\mathbb{R}^4 \cong 4\) denotes the spinor of \(SO(2, 1)\). This is the generic orbit of pure, generic spinors.

### 2.3 Remark on the Orbit Dimensions

From the above analysis (as well as from the result of \ref{21} \ref{25} concerning the \(U\)-orbit stratification in maximal supergravity) and \ref{51}, one can conclude that for \((q := \dim_{\mathbb{R}} A_s = 8, 4, 2 \text{ for } \mathbb{O}_s, \mathbb{H}_s \text{ and } \mathbb{C}_s, \text{ respectively}; \text{ also recall that } SO(5, 5) \cong SL(2, \mathbb{O}_s) \ref{49})\)

\[
SL(2, A_s) \times \frac{Tri(A_s)}{SO(A_s)} \cong Spin \left( \frac{q}{2} + 1, \frac{q}{2} + 1 \right) \times \frac{Tri(A_s)}{SO(A_s)},
\]

it holds that

\[
\begin{align}
\dim (\mathcal{O}_{I^I \neq 0,q}) &= q + 2 \text{ [non-lightlike vectors = non-lightlike elements of } J^A_s]; \\
\dim (\mathcal{O}_{I^I = 0,q}) &= q + 2 \text{ [lightlike vectors = lightlike elements of } J^A_s]; \\
\dim (\mathcal{O}_{V^I \neq 0,q}) &= 2q \text{ [generic spinors = generic elements of } A^2_s]; \\
\dim (\mathcal{O}_{V^I = 0,q}) &= \frac{3}{2} q - 1 \text{ [non-generic spinors = non-generic elements of } A^2_s].
\end{align}
\]

### 3 \(D = 5\)

In \(D = 5\) dimensions, the asymptotically flat branes are black holes (electric 0-branes), and their duals, the black strings (magnetic 1-branes).
3.1 $\mathbb{C}_s$

The $U$-duality group is $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$ (split form of $SU(3) \times SU(3)$), and the scalar manifold reads

$$\frac{\text{Str}_0 \left(J_3^{C_s}\right)}{\text{mcs} \left(\text{Str}_0 \left(J_3^{C_s}\right)\right)} = \frac{SL(3, \mathbb{R}) \times SL(3, \mathbb{R})}{SO(3) \times SO(3)},$$

where $\text{Str}_0 \left(J_3^{C_s}\right)$ is the reduced structure group of the cubic Jordan algebra $J_3^{C_s}$ (cfr. e.g. [47], and Refs. therein).

The electric 0-brane (black hole) and magnetic 1-brane (black string) irreps. are the $(3, 3')$, and its conjugate $(3', 3)$, respectively. They are both characterized by a unique independent cubic invariant polynomial, which we will denote by $I_{3, \text{el}}$ and $I_{3, \text{mag}}$, respectively [39, 41, 43, 43].

While the non-linear action of $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$ on the scalar manifold (eq. (22)) is transitive, the linear action of $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$ on the $(3, 3')$ and $(3', 3)$, classified in terms of invariant constraints on $I_{3, \text{el}}$ and $I_{3, \text{mag}}$, respectively, or equivalently in terms of the rank of the corresponding Jordan algebra $J_3^{C_s}$ [44, 45].

We are now going to determine such orbits, by studying the stabilizers of bound states of the weights of the $(3', 3)$ of the $U$-duality Lie algebra $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$, whose Tits-Satake diagram is sketched in fig. (7). Correspondingly, the action of the Cartan involution

$$\alpha_1 \alpha_2 \alpha_3 \alpha_4$$

Figure 7: Tits-Satake diagram of $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$.

is non-trivial, and it is given again by (eq. (4)).

We start by reporting the Dynkin tree of the $(3', 3)$ of $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$ in (8a). All the weights in this representation have the same length and are real.

The action of the generators on the weights appearing above is described in detail in subsection A.1

3.1.1 1-weight

The 1-weight stabilizers are listed in (tab. (10)).

Correspondingly, the orbit of a real weight reads

$$\partial I_{3, \text{mag}} = 0 : \frac{SL(3, \mathbb{R}) \times SL(3, \mathbb{R})}{[SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SO(1, 1)] \rtimes \mathbb{R}^{(2, 2)}},$$

where $\mathbb{R}^{(2, 2)} \simeq (2, 1)_1 + (1, 2)_{-1}$ of the split form $SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SO(1, 1)$. Note that $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ is the $U$-duality group of the corresponding theory uplifted in

$^{12}$In the following treatment, we will consider only magnetic black strings; the treatment and the results are identical for electric black holes.
\( D = 6 \), and \( (2,1)_1 + (1,2)_{-1} \) is the irrep. relevant to non-dyonic asymptotically flat branes (black holes and black 2-branes, respectively) in \( D = 6 \). We also reported the corresponding \( (SL(3, \mathbb{R}) \times SL(3, \mathbb{R})) \)-invariant constraint on the cubic (magnetic) invariant \( I_{3, \text{mag}} \) of the \( (3', 3) \); note that the orbit \( \text{(eq. (23))} \) is the orbit of rank-1 elements of the cubic Jordan algebra \( J_{3}^{\mathbb{C}} \).

3.1.2 2-weights

Without loss of generality, in this case a bound state can be obtained as combination of \( \Lambda_1 \pm \Lambda_2 \). The stabilizers are listed in tab. (11) the conjunctions can be easily visualized by looking at the overlaps of the orbits sketched in 8b.

The orbit for both \( \Lambda_1 \pm \Lambda_2 \) read

\[
I_{3, \text{mag}} = 0 : \quad \frac{SL(3, \mathbb{R}) \times SL(3, \mathbb{R})}{[SL(2, \mathbb{R})_d \times SO(1, 1)] \rtimes \mathbb{R}^{(2,2)}},
\]

where \( SL(2, \mathbb{R})_d \) is diagonally embedded into the \( SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \) (non-maximal) subgroup of the e.m. duality, and \( \mathbb{R}^{(2,2)} \simeq (2, 2) \) denotes the bi-fundamental of \( SL(2, \mathbb{R})_d \times SO(1, 1) \). We also reported the corresponding \( (SL(3, \mathbb{R}) \times SL(3, \mathbb{R})) \)-invariant constraint on the cubic (magnetic) invariant \( I_{3, \text{mag}} \) of the \( (3', 3) \); note that the orbit \( \text{(eq. (24))} \) is the orbit of rank-2 elements of the cubic Jordan algebra \( J_{3}^{\mathbb{C}} \).
### Stabilizer

|   | \(\Lambda_1\) | \(\Lambda_2\) | \(\Lambda_3\) |
|---|---|---|---|
| \(\alpha_1 + \alpha_2\) | \(\alpha_1 + \alpha_2\) | \(\alpha_2\) |
| \(\alpha_1\) | \(\alpha_1\) | \(\alpha_3\) |
| \(\alpha_2\) | \(\alpha_4\) | \(\alpha_3\) |
| \(H_{\alpha_1}\) | \(H_{\alpha_2} + H_{\alpha_3}\) | \(H_{\alpha_2} - H_{\alpha_3}\) |
| \(H_{\alpha_4}\) | \(-\alpha_2 - \alpha_3\) | \(-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4\) |
| \(-\alpha_1 - \alpha_4\) | \(-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4\) |

Table 10: Stabilizers for the weights in the \((3',3)\) of \(\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R})\).

| Common | Conjunction |
|---|---|
| \(\Lambda_1, \Lambda_2\) | \(\Lambda_1 + \Lambda_2\) | \(\Lambda_1 - \Lambda_2\) |
| \(E_{\alpha_1 + \alpha_2}\) | \(E_{\alpha_2 - \alpha_3}\) | \(E_{\alpha_2 + \alpha_3}\) |
| \(E_{\alpha_3 + \alpha_4}\) | \(E_{\alpha_3 - \alpha_2}\) | \(E_{\alpha_3 + \alpha_2}\) |
| \(E_{\alpha_1}\) | \(E_{\alpha_3}\) | \(E_{\alpha_3}\) |
| \(E_{\alpha_4}\) | \(E_{\alpha_3}\) | \(E_{\alpha_3}\) |
| \(H_{\alpha_1} - H_{\alpha_4}\) | \(H_{\alpha_3} - H_{\alpha_2}\) | \(H_{\alpha_3} - H_{\alpha_2}\) |
| \(H_{\alpha_2} - H_{\alpha_3}\) | \(H_{\alpha_3} - H_{\alpha_2}\) | \(H_{\alpha_3} - H_{\alpha_2}\) |

Table 11: \(\Lambda_1 \pm \Lambda_2\) stabilizers.

#### 3.1.3 3-weights

For the 3-weights bound states, we consider \(\Lambda_1 + \Lambda_2 \pm \Lambda_3\), with stabilizers reported in tab. (12).

Thus, for both \(\Lambda_1 + \Lambda_2 \pm \Lambda_3\), we obtain the following 3-weights bound state orbit:

\[
I_{3,\text{mag}} \neq 0 : \quad \frac{SL(3,\mathbb{R}) \times SL(3,\mathbb{R})}{SL(3,\mathbb{R})_d},
\]

where the stabilizer \(SL(3,\mathbb{R})_d\) is embedded in the e.m. duality group through the (maximal and symmetric; \textit{cfr.} \textit{e.g.} [48]) diagonal embedding (with simple roots \(\alpha_2 - \alpha_3\) and \(\alpha_1 - \alpha_4\)); thus, this orbit is a symmetric manifold. We also reported the corresponding \((SL(3,\mathbb{R}) \times SL(3,\mathbb{R}))\)-invariant constraint on the cubic (magnetic) invariant \(I_{3,\text{mag}}\) of the \((3',3)\); note that the orbit (eq. (25)) is the orbit of rank-3 elements of the cubic Jordan algebra \(J^C_3\).

We summarize the stratification of the \((3',3)\) of the split form \(\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R})\) of the Lie algebra \(a_2 \oplus a_2\) in tab. (13).
Table 12: $\Lambda_1 + \Lambda_2 \pm \Lambda_3$ stabilizers.

| State | $\theta$ | Semisimple Stabilizer | Stabilizer | Rank |
|-------|---------|------------------------|------------|------|
| 1-w   | $\Lambda_1$ | $SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SO(1,1)$ | $[SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SO(1,1)] \ltimes \mathbb{R}^{(2,2)}$ | 1 |
| 2-w   | $\Lambda_1 \pm \Lambda_2$ | $SL(2, \mathbb{R})_d \times SO(1,1)$ | $[SL(2, \mathbb{R}) \times SO(1,1)] \ltimes \mathbb{R}^{(2,2)}$ | 2 |
| 3-w   | $\Lambda_1 + \Lambda_2 \pm \Lambda_3$ | $SL(3, \mathbb{R})$ | $SL(3, \mathbb{R})_d$ | 3 |

Table 13: Orbit stabilizers in the $(3', 3)$ of $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$.

3.2 $H_s$

The $U$-duality group is $SL(6, \mathbb{R})$, and the scalar manifold reads (cfr. eq. (33))

$$\frac{Str_0 \left( J^R_3 \right)}{mcs \left( Str_0 \left( J^R_3 \right) \right)} = \frac{SL(6, \mathbb{R})}{SO(6)} \sim \frac{Conf \left( J^E_3 \right)}{mcs \left( Conf \left( J^E_3 \right) \right)}.$$  

(26)

The electric 0-brane and magnetic 1-brane irreps. are the rank-2 antisymmetric $15$, and its conjugate (rank-4 antisymmetric) $15'$, respectively. They are both characterized by a unique independent cubic invariant polynomial, which we will denote by $I_{3,el}$ and $I_{3,mag}$, respectively \cite{39, 41, 43, 43}.

While the non-linear action of $SL(6, \mathbb{R})$ on the scalar manifold (eq. (26)) is transitive, the linear action of $SL(6, \mathbb{R})$ on the $15$ and $15'$ determines the stratification into three orbits, classified in terms of invariant constraints on $I_3$, or equivalently in terms of the rank of the corresponding Jordan algebra $J^R_3 \cite{44, 45}.$

Figure 8: Tits-Satake diagram of $\mathfrak{sl}(6, \mathbb{R})$.

In fig. (9) and we show the Dynkin tree of the $15$ and the orbits of the weights $\Lambda_1, \Lambda_2$ and $\Lambda_3$ we have chosen as our representatives. The simple roots have been denoted as in fig. (8). All the weights in the representation have the same length.
(a) 15 of $SL(6, \mathbb{R})$. All the weights in the representation have the same length and we refer to the weights without label as $\Sigma_i$ with $i$ ranging from 1 to 12 and increasing from the top to the bottom from the left to the rights of the diagram.

(b) Orbits in the 15 of $SL(6, \mathbb{R})$.

Figure 9: Dynkin tree of the 15 of $\mathfrak{sl}(6, \mathbb{R})$ and the orbits of the weights $\Lambda_1$, $\Lambda_2$ and $\Lambda_3$.

**Rank-1 (1-weight orbit)** the rank-one orbit could be identified looking at tab. (14) and corresponds to

$$\partial I_{3, \text{mag}} = 0 : \quad \frac{SL(6, \mathbb{R})}{[SL(4, \mathbb{R}) \times SL(2, \mathbb{R})] \rtimes \mathbb{R}^{(4,2)}},$$

where $\mathbb{R}^{(4,2)} \simeq (4, 2)$ real bi-fundamental of the split form $SL(4, \mathbb{R}) \times SL(2, \mathbb{R})$. Note that $SL(4, \mathbb{R}) \times SL(2, \mathbb{R})$ is the $U$-duality group of the corresponding theory in $D = 6$, and $(4, 2)$ ($(4', 2)$) is the irrep. relevant to non-dyonic asymptotically flat branes (black holes and black 2-branes, respectively) in $D = 6$.

**Rank-2 (2-weights orbit)** to study the rank 2 orbit we select the combinations $\Lambda_1 \pm \Lambda_3$. Their common stabilizers, appearing in tab. (15) together with the conjunction listed
The orbit reads

\[ [SL(2, \mathbb{R}) \times SL(2, \mathbb{R})] \rtimes \mathbb{R}^{(4,2)} \]  \hspace{1cm} (28)

where the \( SO(2, 3) \) has simple roots

\[ \beta_1 = \alpha_1 \quad \beta_2 = \frac{\alpha_5 - \alpha_1}{2} \]  \hspace{1cm} (29)

in [tab. (16)] define the full set of generators annihilating these linear combinations. Table 14: Stabilizers for the weights in the 15 of \( \mathfrak{sl}(6, \mathbb{R}) \). Table 15: Stabilizers for the weights in the 15 of \( \mathfrak{sl}(6, \mathbb{R}) \).
### Conjunction Stabilizers

| Combination | Stabilizer | Result |
|-------------|------------|--------|
| $\Lambda_1 + a\Lambda_3$ | $E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5} - aE_{-\alpha_2-\alpha_3-\alpha_4}$ | - |
| | $E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} - aE_{-\alpha_2-\alpha_3-\alpha_4-\alpha_5}$ | - |
| | $E_{\alpha_2+\alpha_3+\alpha_4+\alpha_5} - aE_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$ | - |
| | $E_{\alpha_2+\alpha_3+\alpha_4} - aE_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5}$ | - |
| $\Lambda_2 + a\Lambda_3$ | $E_{\alpha_3+\alpha_4+\alpha_5} - aE_{-\alpha_4}$ | - |
| | $E_{\alpha_4+\alpha_5} - aE_{-\alpha_3-\alpha_4}$ | - |
| | $E_{\alpha_3+\alpha_4} - aE_{-\alpha_4-\alpha_5}$ | - |
| | $E_{\alpha_4} - aE_{-\alpha_3-\alpha_4-\alpha_5}$ | - |

Table 16: Stabilizers for the weights in the 15 of $\mathfrak{sl}(6, \mathbb{R})$. For any pair of weights appearing in the combination $\Lambda_1 + \Lambda_2 + a\Lambda_3$ we have identified the corresponding conjunction stabilizers in the first column. In the second column the action of each of them has been evaluated on the remaining weight of the bound state.

and $SL(2, \mathbb{R})$ is associated with the simple roots $\alpha_3$. The orbit is

$$I_{3,\text{mag}} = 0 : \quad \frac{SL(6, \mathbb{R})}{[Sp(4, \mathbb{R}) \times Sp(2, \mathbb{R})] \times \mathbb{R}^{(4,2)^4}}$$

(30)

where $\mathbb{R}^{(4,2)}$ denotes the real bi-fundamental of the split form $Sp(4, \mathbb{R}) \times Sp(2, \mathbb{R}) \simeq SO(3, 2) \times SL(2, \mathbb{R}) \simeq SO(3, 2) \times SO(2, 1)$.

**Rank-3 (3-weights orbit)** for the rank-three orbit the $[SL(2, \mathbb{R})]^3$ defined by the common stabilizers, tab. (15) is promoted by the twelve conjunction stabilizers appearing...
in tab. (16) to the algebra $\mathfrak{sp}(6, \mathbb{R})$ with simple roots

$$\beta_1 = \frac{\alpha_1 - \alpha_3}{2}, \quad \beta_2 = \frac{\alpha_3 - \alpha_5}{2}, \quad \beta_3 = \alpha_5. \quad (31)$$

The orbit is

$$I_{3, \text{mag}} \neq 0 : \quad \frac{SL(6, \mathbb{R})}{Sp(6, \mathbb{R})}. \quad (32)$$

Note that this is a symmetric manifold. We summarize our results for the five-dimensional $\mathbb{H}_3$ case in tab. (17).

| States | $\theta_+$ | $\theta_-$ | semisimple stabilizer | stabilizer       | rank |
|--------|--------|--------|---------------------|------------------|------|
| 1-w    | $\Lambda_1$ | 7   | 11                 | $SL(4, \mathbb{R}) \times SL(2, \mathbb{R})$ | [11] |
|        |          |      |                    | $[SL(4, \mathbb{R}) \times SL(2, \mathbb{R})] \ltimes \mathbb{R}^{(4,2)}$ |      |
| 2-w    | $\Lambda_1 \pm \Lambda_2$ | 5   | 8                  | $Sp(4, \mathbb{R}) \times Sp(2, \mathbb{R})$ | [8]  |
|        |          |      |                    | $[Sp(4, \mathbb{R}) \times Sp(2, \mathbb{R})] \ltimes \mathbb{R}^{(4,2)}$ |      |
| 3-w    | $\Lambda_1 + \Lambda_2 \pm \Lambda_3$ | 9   | 12                 | $Sp(6, \mathbb{R})$ | [12] |
|        |          |      |                    | $Sp(6, \mathbb{R})$ |      |

Table 17: Summary of the orbits in the 15 of $\mathfrak{sl}(6, \mathbb{R})$

4 \hspace{1cm} D = 4

In $D = 4$ dimensions, the unique asymptotically flat branes are black holes (0-branes), which can be dyonic.

4.1 $\mathbb{C}_s$

The $U$-duality group is $SL(6, \mathbb{R})$ (split form of $SU(6)$), and the scalar manifold reads

$$\frac{Conf(J_3^{C_s})}{mcs(Conf(J_3^{C_s}))} = \frac{SL(6, \mathbb{R})}{SO(6)}, \quad (33)$$

where $Conf(J_3^{C_s}) \simeq Aut(\mathfrak{F}(J_3^{C_s}))$ is the conformal group [15] of the cubic Jordan algebra $J_3^{C_s}$, or equivalently, the automorphism group of the Freudenthal triple system (FTS) $\mathfrak{F}$ over $J_3^{C_s}$ [35] ($mcs$ stands for maximal compact subgroup throughout). The 0-brane dyonic irrep. is the rank-3 antisymmetric self-dual (real) 20, such that the pair $(SL(6, \mathbb{R}), 20)$ defines a group “of type $E_7$”, characterized by a unique independent quartic invariant
polynomial $I_4$. \(39, 41, 43, 43\).

While the non-linear action of $SL(6, \mathbb{R})$ on the scalar manifold (eq. (33)) is transitive, the linear action of $SL(6, \mathbb{R})$ on the 20 representation space determines the stratification into orbits, classified in terms of invariant constraints on $I_4$, or equivalently in terms of the rank \(44, 45\) of the corresponding representative in the Freudenthal triple system $\mathfrak{F}(J_3^C)$.

In order to define the U-duality orbits now we are going to study the stabilizers of bound states of the weights of the 20 of the U-duality Lie algebra $\mathfrak{sl}(6, \mathbb{R})$, whose Tits-Satake diagram is the same as the one appearing in fig. (8), along the lines defined in the previous sections. First of all we note that the action of the Cartan involution is, as expected for a split real form,\(\theta(\alpha_i) = -\alpha_i\) (34) for any root of the algebra.

The Dynkin tree of the irrep 20 of $\mathfrak{sl}(6, \mathbb{R})$ is depicted in fig. (10), while in fig. (10a) we sketchily represent the orbits of the four weights $\Lambda_1, \Lambda_4, \Lambda_6, \Lambda_7$ and their overlaps.

4.1.1 1-weight

To determine the 1-weight stabilizer, one simply can look at the Dynkin tree in fig. (10a) and the results listed in tab. (18) immediately follow.

Consequently, the orbit of a real weight is

$$\partial^2 I_4 \big|_{35} = 0 : \quad \frac{SL(6, \mathbb{R})}{[SL(3, \mathbb{R}) \times SL(3, \mathbb{R})] \times \mathbb{R}^{(3,3')}},$$

(35)

where $\mathbb{R}^{(3,3')}$ isomorphic to $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$, which is the U-duality group in $D = 5$; furthermore, $(3, 3') ((3', 3))$ is the irrep. relevant to asymptotically flat branes (black holes and black strings, respectively) in $D = 5$. We also reported the corresponding $SL(6, \mathbb{R})$-invariant constraint on the quartic invariant $I_4$ of the 20 (with 35 denoting the adjoint of $SL(6, \mathbb{R})$ itself); note that the orbit (eq. (35)) is the orbit of rank-1 elements of the FTS $\mathfrak{F}$ over $J_3^C$.

4.1.2 2-weights

We consider the 2-weights bound states $\Lambda_1 \pm \Lambda_4$. The corresponding stabilizers are listed in tab. (19). The conjunctions can be visualized in fig. (10b) in which the orbits of the real weights $\Lambda_1, \Lambda_4, \Lambda_6$ and $\Lambda_7$ are drawn according to the notation of fig. (8).

The generators in tab. (19) give rise to an algebra $\mathfrak{so}(2, 3) \oplus \mathfrak{so}(1, 1)$ in both the “±” branches, with the $\mathfrak{so}(1, 1)$ generated by the Cartan $H_{\alpha_1} - H_{\alpha_5}$. The simple roots of such a
stabilizing algebra are $\alpha_4$ and $\frac{\alpha_2 - \alpha_4}{2}$. Correspondingly, the resulting 2-weights orbit reads

$$\partial I_4 = 0 : \quad \frac{SL(6, \mathbb{R})}{[Sp(4, \mathbb{R}) \times SO(1, 1)] \ltimes (\mathbb{R}^{(4, 2)} \times \mathbb{R})},$$ (36)

where $\mathbb{R}^{(4, 2)} \simeq (4, 2)$ denotes the real bi-fundamental\(^{13}\) of the split form $Sp(4, \mathbb{R}) \times SO(1, 1) \simeq SO(3, 2) \times SO(1, 1)$. We also reported the corresponding $SL(6, \mathbb{R})$-invariant constraint on the quartic invariant $I_4$ of the 20; note that the orbit (eq. (36)) is the orbit of rank-2 elements of the FTS $\mathfrak{F}$ over $J_5^{C\ast}$.

There is another non-isomorphic 2-weights orbit in the 20 of $SL(6, \mathbb{R})$, named dyonic orbit. Without loss of generality, it can be realized as the orbit of the bound state $\Lambda_1 + \Lambda_8$; in this case, there are only common stabilizers, and these are listed in table (20). The resulting stabilizing algebra is $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$, which - as mentioned before - is nothing but the $U$-duality Lie algebra of the corresponding theory uplifted to $D = 5$. The dyonic orbit reads

$$I_4 < 0 : \quad \frac{SL(6, \mathbb{R})}{SL(3, \mathbb{R}) \ltimes SL(3, \mathbb{R})},$$ (37)

where we also reported the corresponding $SL(6, \mathbb{R})$-invariant constraint on the quartic invariant $I_4$ of the 20; consequently, the orbit (eq. (37)) is the orbit of rank-4 elements with $I_4 < 0$ of the FTS $\mathfrak{F}$ over $J_5^{C\ast}$. We note that the dyonic orbit is realized as a bound states of two weights but it corresponds to rank-4 elements of the FTS $\mathfrak{F}$ over $J_5^{C\ast}$. The characterization of these weights $\Lambda_1$ and $\Lambda_8$ with respect to other possible choices is that their single orbits do not overlap. Furthermore fixed one weight in the representation then there is a unique choice of a second weight with this property.

No other choice of 2-weights bound states yields other, non-isomorphic orbits.

### 4.1.3 3-weights

With no loss of generality, we consider the 3-weights bound states $\Lambda_1 + \Lambda_4 \pm \Lambda_6$. For such bound states, we identify the the stabilizers reported in table (21) and the corresponding orbit reads

$$I_4 = 0 : \quad \frac{SL(6, \mathbb{R})}{SL(3, \mathbb{R}) \ltimes \mathbb{R}^8},$$ (38)

where $\mathbb{R}^8 \simeq 8$ denotes the adjoint of $SL(3, \mathbb{R})$. We also reported the corresponding $SL(6, \mathbb{R})$-invariant constraint on the quartic invariant $I_4$ of the 20; note that the orbit (eq. (38)) is the orbit of rank-3 elements of the FTS $\mathfrak{F}$ over $J_5^{C\ast}$. In subsection A.2, we describe explicitly how the stabilizers generates the semisimple part of the full stabilizer.

\(^{13}\)The real fundamental irrep. of $Sp(4, \mathbb{R})$ is the real spinor of $SO(3, 2)$.
4.1.4 4-weights

Finally, the analysis of the orbit stratification in the 20 is completed by considering the orbits of 4-weights bound states. Once chosen $\Lambda_1$, $\Lambda_4$ and $\Lambda_6$, these orbits can be built adding to them $\Lambda_7$, which is the only weight disconnected from such weights. The independent combinations one can construct are $\Lambda_1 + \Lambda_4 \pm \Lambda_6 \pm \Lambda_7$. Their stabilizers are listed in tab. (22) in which the various combinations are parametrized by the constants $a = \pm 1$ and $b = \pm 1$. There are two common stabilizers and fourteen conjunctions.

Changing the values of $a$ and $b$, one obtains two different real forms for the stabilizers, namely $\mathfrak{sl}(3, \mathbb{C})_R$ and $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$ where signatures, stabilizing algebras and their Tits-Satake diagrams are shown. For the explicit realization of the stabilizing algebra we refer to subsection A.2. Summarizing, the 4-weights bound states orbits are given by

\[ I_4 > 0 : \quad \frac{SL(6, \mathbb{R})}{SL(3, \mathbb{C})_R} \quad (39) \]

and by another orbit isomorphic to the dyonic orbit (eq. (37)). Consequently, the orbits (eq. (37)) and (eq. (39)) are the orbits of rank-4 elements of the FTS $\mathfrak{F}$ over $J_3^{\mathbb{C}^*}$.

We summarize the stratification of the 20 of the split form $\mathfrak{sl}(6, \mathbb{R})$ of the Lie algebra $\mathfrak{a}_5$ in tab. (23).

\[ \text{tab. (22)} \]

\[ \text{tab. (23)} \]

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\[^{14}\text{The subscript \textquotedblright R\textquotedblright denotes the Lie algebra to be considered as an algebra over the reals.}\]
(a) Dynkin tree of the 20 of sl(6, R).

(b) Orbits of the real weights $\Lambda_1$, $\Lambda_4$, $\Lambda_6$ and $\Lambda_7$ in the 20 of sl(6, R). The red circles denote the starting points of the corresponding orbit.

Figure 10: In the figure are sketched the structure of representation 20 of SL(6, R) and the orbits of four different weights.
|   | $\lambda_1$ | $\lambda_4$ | $\lambda_6$ | $\lambda_7$ |
|---|-------------|-------------|-------------|-------------|
|   | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ |
|   | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ |
|   | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ |
|   | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ |
|   | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ |
|   | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ |
|   | $H_{\alpha_4}$ | $H_{\alpha_4}$ | $H_{\alpha_4}$ | $H_{\alpha_4}$ |
|   | $H_{\alpha_5}$ | $H_{\alpha_5}$ | $H_{\alpha_5}$ | $H_{\alpha_5}$ |
|   | $H_{\alpha_1} + H_{\alpha_2} + H_{\alpha_3} + H_{\alpha_4}$ | $H_{\alpha_1} + H_{\alpha_2} + H_{\alpha_3} + H_{\alpha_4}$ | $H_{\alpha_1} + H_{\alpha_2} + H_{\alpha_3} + H_{\alpha_4}$ | $H_{\alpha_1} + H_{\alpha_2} + H_{\alpha_3} + H_{\alpha_4}$ |
|   | $-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$ | $-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$ | $-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$ | $-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$ |
|   | $-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$ | $-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$ | $-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$ | $-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$ |
|   | $-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$ | $-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$ | $-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$ | $-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$ |
|   | $-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$ | $-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$ | $-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$ | $-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$ |

Table 18: Stabilizers for the weights in the 20 of $\mathfrak{sl}(6, \mathbb{R})$. 
### Table 19: Stabilizers of $\Lambda_1 \pm \Lambda_4$.

| Common | $\Lambda_1 \pm \Lambda_4$ | $\Lambda_1 - \Lambda_4$ |
|--------|---------------------------|-------------------------|
| $E_{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}$ | $E_{\alpha_2 + \alpha_3 + \alpha_4} - E_{-\alpha_3}$ | $E_{\alpha_2 + \alpha_3 + \alpha_4} + E_{-\alpha_3}$ |
| $E_{\alpha_1 + \alpha_2 + \alpha_3}$ | $E_{\alpha_3} - E_{-\alpha_2 - \alpha_3 - \alpha_4}$ | $E_{\alpha_3} + E_{-\alpha_2 - \alpha_3 - \alpha_4}$ |
| $E_{\alpha_1 + \alpha_2}$ | $E_{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5} - E_{-\alpha_3}$ | $E_{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5} + E_{-\alpha_3}$ |
| $E_{\alpha_1}$ | $E_{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - \alpha_5}$ | $E_{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}$ |
| $E_{\alpha_2}$ | $E_{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5} - E_{-\alpha_3}$ | $E_{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5} + E_{-\alpha_3}$ |
| $H_{\alpha_2}$ | $E_{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5} - E_{-\alpha_3}$ | $E_{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}$ |
| $H_{\alpha_4}$ | $E_{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}$ | $E_{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}$ |
| $H_{\alpha_1 - \alpha_5}$ | $E_{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}$ | $E_{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}$ |
| $E_{-\alpha_2}$ | $E_{\alpha_2} - E_{\alpha_3}$ | $E_{\alpha_2}$ |
| $E_{-\alpha_4}$ | $E_{\alpha_3}$ | $E_{\alpha_3}$ |

### Table 20: Stabilizers of $\Lambda_1 + \Lambda_8$.

| Common | $\Lambda_1 + \Lambda_8$ |
|--------|-------------------------|
| $E_{\alpha_1 + \alpha_2}$ | $E_{\alpha_4 + \alpha_5}$ |
| $E_{\alpha_1}$ | $E_{\alpha_2}$ |
| $H_{\alpha_1}$ | $H_{\alpha_2}$ |
| $E_{-\alpha_1}$ | $E_{-\alpha_2}$ |
| $E_{-\alpha_1 - \alpha_2}$ | $E_{-\alpha_4 - \alpha_5}$ |

### Table 21: Stabilizers of $\Lambda_1 + \Lambda_4 \pm \Lambda_6$.

| Common | $\Lambda_1 + \Lambda_4 \pm \Lambda_6$ | $\Lambda_1 + \Lambda_4 - \Lambda_6$ |
|--------|--------------------------------------|----------------------------------|
| $E_{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}$ | $E_{\alpha_2 + \alpha_3 + \alpha_4} - E_{-\alpha_3}$ | $E_{\alpha_2 + \alpha_3 + \alpha_4} - E_{-\alpha_3}$ |
| $E_{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}$ | $E_{\alpha_3} - E_{-\alpha_2 - \alpha_3 - \alpha_4}$ | $E_{\alpha_3} - E_{-\alpha_2 - \alpha_3 - \alpha_4}$ |
| $E_{\alpha_1 + \alpha_2}$ | $E_{\alpha_3 + \alpha_4} - E_{-\alpha_2 - \alpha_3}$ | $E_{\alpha_3 + \alpha_4} - E_{-\alpha_2 - \alpha_3}$ |
| $E_{\alpha_2}$ | $E_{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5} - E_{-\alpha_3}$ | $E_{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5} + E_{-\alpha_3}$ |
| $H_{\alpha_2}$ | $E_{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - \alpha_5}$ | $E_{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}$ |
| $H_{\alpha_4}$ | $E_{\alpha_3 + \alpha_4 + \alpha_5} - E_{-\alpha_1 - \alpha_2 - \alpha_3}$ | $E_{\alpha_3 + \alpha_4 + \alpha_5}$ |
| $H_{\alpha_1 - \alpha_5}$ | $E_{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}$ | $E_{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}$ |
| $E_{-\alpha_2}$ | $E_{\alpha_2} - E_{\alpha_3}$ | $E_{\alpha_2}$ |
| $E_{-\alpha_4}$ | $E_{\alpha_3}$ | $E_{\alpha_3}$ |

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Common | Conjunction
---|---
\(\Lambda_1, \Lambda_4, \Lambda_6, \Lambda_7\) | \(\Lambda_1 + \Lambda_4 + a\Lambda_6 + b\Lambda_7\)

| \(H_{a_2} - H_{a_4}\) | \(H_{a_1} - H_{a_5}\) |
|---|---|
| \(E_{a_2+a_3} - E_{-a_3-a_4}\) | \(E_{a_2+a_3} - E_{-a_3-a_4}\) |
| \(E_{a_3+a_4} - E_{-a_2-a_3}\) | \(E_{a_3+a_4} - E_{-a_2-a_3}\) |
| \(E_{a_1+a_2+a_3} - aE_{-a_3-a_4-a_5}\) | \(E_{a_1+a_2+a_3} - aE_{-a_3-a_4-a_5}\) |
| \(E_{a_3+a_4+a_5} - aE_{-a_1-a_2-a_3}\) | \(E_{a_3+a_4+a_5} - aE_{-a_1-a_2-a_3}\) |
| \(E_{a_5} - aE_{-a_1}\) | \(E_{a_5} - aE_{-a_1}\) |
| \(E_{a_1} - aE_{-a_5}\) | \(E_{a_1} - aE_{-a_5}\) |
| \(E_{a_1+a_2+a_3+a_4} - bE_{-a_2-a_3-a_4-a_5}\) | \(E_{a_1+a_2+a_3+a_4} - bE_{-a_2-a_3-a_4-a_5}\) |
| \(E_{a_2+a_3+a_4+a_5} - bE_{-a_1-a_2-a_3-a_4}\) | \(E_{a_2+a_3+a_4+a_5} - bE_{-a_1-a_2-a_3-a_4}\) |
| \(E_{a_1+a_2} - bE_{-a_4-a_5}\) | \(E_{a_1+a_2} - bE_{-a_4-a_5}\) |
| \(E_{a_4+a_5} - bE_{-a_1-a_2}\) | \(E_{a_4+a_5} - bE_{-a_1-a_2}\) |
| \(E_{a_2} - abE_{-a_4}\) | \(E_{a_2} - abE_{-a_4}\) |
| \(E_{a_4} - abE_{-a_2}\) | \(E_{a_4} - abE_{-a_2}\) |
| \(F_{-ab} + abF_{a_2+a_3+a_4}\) | \(F_{-ab} + abF_{a_2+a_3+a_4}\) |
| \(F_{-ab} + bF_{a_1+a_2+a_3+a_4+a_5}\) | \(F_{-ab} + bF_{a_1+a_2+a_3+a_4+a_5}\) |

Table 22: Stabilizers of \(\Lambda_1 + \Lambda_4 + a\Lambda_6 + b\Lambda_7\).

\[ \begin{array}{cccc}
\beta_1 & \beta_2 & \beta_3 & \beta_4 \\
\end{array} \]

\[ a = \pm 1 \quad b = \pm 1 \]

\((8|8)\) \[\mathfrak{sl}(3, \mathbb{C})_{\mathbb{R}}\]

\[ \begin{array}{cccc}
\beta_1 & \beta_2 & \beta_3 & \beta_4 \\
\end{array} \]

\[ a = \pm 1 \quad b = \mp 1 \]

\((8|8)\) \[\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})\]

31
| States | $\theta$ | Semisimple Stabilizer | Stabilizer | Rank |
|--------|---------|-----------------------|------------|------|
| 1-w    | $\Lambda_1$ | 6 10 $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$ | $[SL(3, \mathbb{R}) \times SL(3, \mathbb{R})] \ltimes \mathbb{R}^{3,3}$ | 1 |
| 2-w    | $\Lambda_1 \pm \Lambda_4$ | 4 7 $SO(2, 3) \times SO(1, 1)$ | $[SO(2, 3) \times SO(1, 1)] \ltimes (\mathbb{R} \times \mathbb{R}^{4,2})$ | 2 |
|        | $\Lambda_1 + \Lambda_8$ | 6 10 $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$ | $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$ | dyonic |
| 3-w    | $\Lambda_1 + \Lambda_4 \pm \Lambda_6$ | 3 5 $SL(3, \mathbb{R})$ | $SL(3, \mathbb{R}) \times \mathbb{R}^8$ | 3 |
| 4-w    | $\Lambda_1 + \Lambda_4 \pm \Lambda_6 \pm \Lambda_7$ | 8 8 $SL(3, \mathbb{C})$ | $SL(3, \mathbb{C})$ | 4 |
|        | $\Lambda_1 + \Lambda_4 \pm \Lambda_6 \mp \Lambda_7$ | 6 10 $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$ | $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$ | 4 |

Table 23: Orbit stabilizers in the 20 of $\mathfrak{sl}(6, \mathbb{R})$. 

32
4.2 \( \mathbb{H}_s \)

The \( U \)-duality group is \( SO(6, 6) \) (split form of \( SO(12) \)), and the scalar manifold reads

\[
\frac{\text{Conf} \left( J_3^{\mathbb{H}_s} \right)}{\text{mcs} \left( \text{Conf} \left( J_3^{\mathbb{H}_s} \right) \right)} = \frac{SO(6, 6)}{SO(6) \times SO(6)}.
\]

(40)

Note that the coset eq. (40) also characterizes 6 self-dual and 6 anti-self-dual 3-form field strengths in \( D = 6 \) (see e.g. \[37\], and Refs. therein).

The 0-brane dyonic irrep. is the Majorana-Weyl spinor \( 32 \) (or its conjugate \( 32' \)), such that the pair \((SO(6, 6), 32)\) defines a group “of type \( E_7 \)”, \[38\], characterized by a unique independent quartic invariant polynomial \( I_4 \) \[39, 41–43\].

While the non-linear action of \( SO(6, 6) \) on the scalar manifold (eq. (40)) is transitive, the linear action of \( SO(6, 6) \) on the \( 32 \) determines the stratification into five orbits, classified in terms of invariant constraints on \( I_4 \), or equivalently in terms of the rank \[44, 45\] of the corresponding representative in the Freudenthal triple system \( \mathfrak{F} \left( J_3^{\mathbb{H}_s} \right) \). We consider the representation \( 32 \) of \( SO(6, 6) \). In fig. (11) we report the Tits-Satake diagram for \( \mathfrak{so}(6, 6) \), fixing the conventions for the simple roots. In this section we only sketch the the Dynkin tree and the orbits with their overlaps in fig. (12) and fig. (13) respectively, referring to subsection A.3 for the detailed analysis of the orbits.

Figure 11: Tits-Satake diagram for \( \mathfrak{so}(6, 6) \).

The weights in fig. (12) with no explicit labels are denoted with \( \Sigma_i \) with \( i \) that goes from 1 to 28 increasing from the top to the bottom from the left to the right. The weights \( \Lambda_i \) with \( i = 1, 2, 3, 4 \) are the ones we choose to study the orbits. These are not connected by the action of any generator.

By exploiting the same methods used in the analysis of the \( C_s \)-based theory in the previous subsection, in this case, the following stratification is determined:

**The rank-1 (1-weight orbit)** orbit could be easily deduced by the stabilizers of \( \Lambda_1 \):

\[
\partial^2 I_4 \bigg|_{66} = 0 : \quad \frac{SO(6, 6)}{SL(6, \mathbb{R}) \times \mathbb{R}^{15}},
\]

(41)

\[15\]It should be remarked that the orbit classification done on this Section can be regarded as the real split form, worked out with completely different methods, of the classification made by Igusa in \[40\], with whom ours agrees.
Figure 12: Dynkin tree of the representation $32$ of $so(6,6)$. All the weights have the same length. We denote with $\Lambda_i$, with $i=1,2,3,4$ four weights not connected by algebra transformations. We choose this weights to study the orbits and we refer to the other weight as $\Sigma_i$ with $i$ ranging from 1 to 28 and increasing, in the diagram from the top to the bottom from the left to the right.

where $\mathbb{R}^{15} \simeq 15$ and $66$ respectively denote the rank-2 antisymmetric irrep. of $SL(6,\mathbb{R})$ and the rank-2 antisymmetric (adjoint) irrep. of $SO(6,6)$. The orbit
Figure 13: Diagram of the orbits in the 32 of the four weights picked as starting points for our analysis. It is possible to appreciate the overlaps of different orbits. There is a triple overlap on four weights and any two orbits overlap on exactly six weights. These will define the conjunction stabilizers.

Eq. (41) is the pure spinor orbit of $SO(6,6)$ [46]. Note that $SL(6,\mathbb{R})$ is the $U$-duality group in $D = 5$ (as well as the $U$-duality group of the $D = 4$ theory on $\mathbb{C}_x$; see [eq. (33)]), and $15 (15')$ is the irrep. relevant to asymptotically flat branes (black
holes and black strings, respectively) in $D = 5$.

**Rank-2 (2-weights non-dyonic orbit)** this is the orbits of the combinations $\Lambda_1 \pm \Lambda_2$. These bound states have stabilizer made by common and conjunction stabilizers. The two bound state have the same rank-2 orbit

$$
\partial I_4 = 0 : \left. \frac{SO(6, 6)}{[SO(4, 3) \times SL(2, \mathbb{R})] \ltimes (\mathbb{R}^{8, 2} \times \mathbb{R})} \right. \quad (42)
$$

where $\mathbb{R}^{8, 2} \simeq (8, 2)$ denotes the real bi-spinor of the split form $SO(4, 3) \times SL(2, \mathbb{R})$. The $SO(3, 4)$ has simple roots

$$
\beta_1 = \alpha_4 \quad \beta_2 = \alpha_3 \quad \beta_3 = \frac{\alpha_5 - \alpha_3}{2},
$$

while $SL(2, \mathbb{R})$ has simple root $\alpha_1$.

**Rank-3 (3-weights orbit)** the rank three orbit, corresponding to the combinations $\Lambda_1 + \Lambda_2 \pm \Lambda_3$. This bound states has stabilizer of the common and conjunction types. In particular these lasts act on two of the three states stabilizing also the remaining one. The semisimple part of the stabilizer is an $Sp(6, \mathbb{R})$ with simple roots recognized as

$$
\beta_1 = \frac{\alpha_1 - \alpha_3}{2} \quad \beta_2 = \frac{\alpha_3 - \alpha_5}{2} \quad \beta_3 = \alpha_5
$$

The orbit reads

$$
I_4 = 0 : \left. \frac{SO(6, 6)}{Sp(6, \mathbb{R}) \ltimes \mathbb{R}^{14}} \right. \quad (45)
$$

where $\mathbb{R}^{14} \simeq 14$ denotes the rank-2 antisymmetric irrep. of $Sp(6, \mathbb{R})$.

**Rank-4 (4-weights and the 2-weights dyonic orbits)** ($I_4 \neq 0$, these comprise the 4-weights orbits, and the 2-weights dyonic orbit; this latter is isomorphic to the $I_4 < 0$ one). The Dyonic orbit could be realized as the orbit of the highest weight $\Lambda_1$ and the lowest weight $\Sigma_{28}$. These two weights have two not overlapping orbits thus there are no conjunction stabilizers. The common stabilizers correspond to all the generators of $SO(6, 6)$ except the one containing $\alpha_6$. The result is a stabilizer $SL(6, \mathbb{R})$. The rank-4 orbits could be seen as the orbits of the combinations $\Lambda_1 + \Lambda_2 + a \Lambda_3 + b \Lambda_4$ with $a, b$ that can take values $\pm 1$. The common stabilizers define an algebra $[SL(2, \mathbb{R})]^3$ that enlarged by the twenty-four conjunction stabilizers. In particular twenty-two of them have the same compactness properties despite of $a, b$ while there are two of them (for further details see [subsection A.3]),

$$
F_{ab}^{\alpha_6} = abF_{\alpha_1 + 2\alpha_4 + \alpha_5 + \alpha_6}^{\alpha_6}
$$

$$
F_{ab}^{\alpha_6} = bF_{\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6}^{\alpha_6},
$$

(46a)

(46b)
where

\[ F_\alpha^\pm = E_\alpha \pm E_{-\alpha}, \] (47)

that are compact if \( ab = -1 \), non-compact if \( ab = +1 \). This means there are two independent orbits

\[ \mathcal{O}(\Lambda_1 + \Lambda_2 \pm \Lambda_3 \mp \Lambda_4) = \frac{SO(6, 6)}{SL(6, \mathbb{R})} \] (48a)

\[ \mathcal{O}(\Lambda_1 + \Lambda_2 \pm \Lambda_3 \mp \Lambda_4) = \frac{SO(6, 6)}{SU(3, 3)} \] (48b)

corresponding to

\[ I_4 > 0 : \quad \frac{SO(6, 6)}{SU(3, 3)}; \] (49a)

\[ I_4 < 0 : \quad \frac{SO(6, 6)}{SL(6, \mathbb{R})}. \] (49b)

We summarize our results in Table 24.

| States | \( \theta \) | semisimple stabilizer | stabilizer | rank |
|--------|-------------|-----------------------|------------|------|
|        | + | - | SL(6, \mathbb{R}) | SL(6, \mathbb{R}) \times \mathbb{R}^{15} | 1 |
| 1-w    | \( \Lambda_1 \) | 15 | 20 | |
| 2-w    | \( \Lambda_1 \pm \Lambda_2 \) | 10 | 14 | \([SO(3, 4) \times SL(2, \mathbb{R})] \times (\mathbb{R} \times \mathbb{R}^{(8, 2)})\) | 2 |
| 3-w    | \( \Lambda_1 + \Lambda_2 \pm \Lambda_3 \) | 9 | 12 | Sp(6, \mathbb{R}) \times \mathbb{R}^{14} \times \mathbb{R} \times \mathbb{R}^{(8, 2)} | 3 |
| 4-w    | \( \Lambda_1 + \Lambda_2 + a\Lambda_3 - a\Lambda_4 \) | 17 | 18 | Sp(6, \mathbb{R}) \times \mathbb{R}^{14} \times \mathbb{R} \times \mathbb{R}^{(8, 2)} | 4 |
|        | \( \Lambda_1 + \Lambda_2 + a\Lambda_3 + a\Lambda_4 \) | 15 | 20 | \( SL(6, \mathbb{R}) \times \mathbb{R}^{14} \times \mathbb{R} \times \mathbb{R}^{(8, 2)} \) | 4 |

Table 24: Summary of the orbits in the 32 of \( SO(6, 6) \)

Acknowledgments

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A Orbits in Detail

In this appendix we collect some relevant details on the results obtained for the U-duality orbits in four and five dimensions

A.1 $(3,3')$ of $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$ in $D = 5$

We study the action of the generators of $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$ on the $(3,3')$ irrep. In particular we could consider the two components separately. We describe the action of $\mathfrak{sl}(3, \mathbb{R})$ on the $3'$ in tab. (25). Analogously, with another suitable choice of the structure constants, the action on the $3$ irrep. can be simplified to the form appearing in the table, namely each raising and lowering generators acts on any weights annihilating it or connecting it to another weights without changing sign. This implies that there is a (at least one) choice of the structure constants for $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$ such that its action on the $(3',3)$ is analogous to the one described above.

|               | $\Lambda_1$ | $\Sigma_2$ | $\Sigma_4$ |
|---------------|-------------|------------|------------|
| $E_{\alpha_1+\alpha_2}$ | $\Lambda_1$ |            |            |
| $E_{\alpha_2}$   | $\Lambda_1$ | $\Sigma_2$ |            |
| $E_{\alpha_1}$   | $\Lambda_1$ |            | $\Sigma_4$ |
| $H_{\alpha_1}$   | 1           | -1         |            |
| $H_{\alpha_2}$   | 1           | -1         |            |
| $E_{-\alpha_2}$  | $\Sigma_2$ |            |            |
| $E_{-\alpha_1}$  | $\Sigma_4$ |            |            |
| $E_{-\alpha_1-\alpha_2}$ | $\Sigma_4$ |            |            |

Table 25: Action of the generators of $\mathfrak{sl}(3, \mathbb{R})$ on the $3'$. The blue entries contains an ambiguity since the action is defined up to a sign that could be arbitrarily fixed. Once these ambiguities are fixed the action on the all the other weights is uniquely fixed.

A.2 20 of $SL(6, \mathbb{R})$ in $D = 4$

In order to compute the orbits of one or bound states of weights the first thing we have to do is to fix the ambiguities in the structure constants of the algebra. In particular we do this as follows. We define the signs of the extraspecial pairs\footnote{Extraspecial pairs are pairs of root vectors $(\alpha, \beta)$ for which, in the Chevalley basis, the signs of the structure constants appearing in commutators $[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}$ could be chosen arbitrarily.} (cfr. e.g. [12], and Refs. therein) in such a way to have $\theta E_\alpha = -E_{\theta \alpha}$ for all the raising and lowering operators and
this relation holds only if the structure constants satisfy
\[ N_{\alpha,\beta} = -N_{\theta\alpha,\theta\beta}. \quad (50) \]
In the split case (as it is the case for \( \mathfrak{sl}(6,\mathbb{R}) \)) the previous relation automatically holds, since from \( \theta\alpha = -\alpha \) we get
\[ N_{\alpha,\beta} = -N_{-\alpha,-\beta}, \quad (51) \]
that is the basic relation satisfied by the structure constants.

The structure constants for \( \mathfrak{sl}(6,\mathbb{R}) \) are listed in [tab. (26)]. In order to study the orbits, one must define the action of the generators on this representation. We note from \( ?? \) that all loops are made of simple roots such that their sum is never a root; this means that one can set all the arbitrariness associated to simple root vectors to one, and the others follow directly from the structure constants.

. Once the action of each generator on each weight has been defined one can just define the set of stabilizers taking the most general linear combination of generators and acting on a given state. Doing this in the 3-weights case presented in [subsection 4.1] we obtain the semisimple part of stabilizer and define the following identification

\[ H_{\alpha_2} - H_{\alpha_4} \rightarrow H_{\frac{1}{2}(\alpha_2 - \alpha_4)}, \quad (52a) \]
\[ H_{\alpha_1} - H_{\alpha_5} \rightarrow H_{\frac{1}{2}(\alpha_1 - \alpha_5)}, \quad (52b) \]
\[ E_{\alpha_1 + \alpha_2 + \alpha_3} \pm E_{-(\alpha_3 + \alpha_4 + \alpha_5)} \rightarrow E_{\frac{1}{2}(\alpha_1 + \alpha_2 - \alpha_4 - \alpha_5)}, \quad (52c) \]
\[ E_{\alpha_2 + \alpha_3 + \alpha_4} \pm E_{-(\alpha_1 + \alpha_2 + \alpha_4)} \rightarrow E_{\frac{1}{2}(\alpha_4 + \alpha_5 - \alpha_1 - \alpha_2)}, \quad (52d) \]
\[ E_{\alpha_5} \pm E_{-\alpha_1} \rightarrow E_{\frac{1}{2}(\alpha_5 - \alpha_1)}, \quad (52e) \]
\[ E_{\alpha_1} \pm E_{-\alpha_5} \rightarrow E_{\frac{1}{2}(\alpha_1 - \alpha_5)}, \quad (52f) \]
\[ E_{\alpha_2 + \alpha_3} - E_{-(\alpha_3 + \alpha_4)} \rightarrow E_{\frac{1}{2}(\alpha_2 - \alpha_4)}, \quad (52g) \]
\[ E_{\alpha_3 + \alpha_4} - E_{-(\alpha_2 + \alpha_3)} \rightarrow E_{\frac{1}{2}(\alpha_4 - \alpha_2)}, \quad (52h) \]

This identification makes manifest that the semisimple part of the stabilizer corresponds to an algebra \( \mathfrak{sl}(3,\mathbb{R}) \) with simple roots \( \frac{\alpha_2 - \alpha_4}{2} \) and \( \frac{\alpha_1 - \alpha_5}{2} \).

For the 4-weight orbit we could proceed by the same way above obtaining for the linear combinations discussed in [subsection 4.1] a set of stabilizers parameterized by the variables
$a,b$. The complexification of the stabilizing algebra gives an $\mathfrak{sl}(3, \mathbb{C})$ with generators

\[
H_{\beta_1} = \frac{1}{2} \left[ H_{\alpha_1} - H_{\alpha_5} + \sqrt{-ab} \left( F_{-a_2+a_3+a_4}^{ab} - aF_{a_1+a_2+a_3+a_4+a_5}^{ab} \right) \right]; \\
H_{\beta_2} = \frac{1}{2} \left[ H_{\alpha_2} - H_{\alpha_4} - \sqrt{-ab} \left( F_{-a_2+a_3+a_4}^{ab} + abF_{a_2+a_3+a_4}^{ab} \right) \right], \\
H_{\beta_3} = \frac{1}{2} \left[ H_{\alpha_4} - H_{\alpha_2} - \sqrt{-ab} \left( F_{-a_2+a_3+a_4}^{ab} + abF_{a_2+a_3+a_4}^{ab} \right) \right], \\
H_{\beta_4} = \frac{1}{2} \left[ H_{\alpha_5} - H_{\alpha_1} + \sqrt{-ab} \left( F_{-a_2+a_3+a_4}^{ab} - aF_{a_1+a_2+a_3+a_4+a_5}^{ab} \right) \right],
\]

and simple roots $\beta_1, \beta_2, \beta_3$ and $\beta_4$. The two resulting real forms $\mathfrak{sl}(3, \mathbb{C})_{\mathbb{R}}$ and $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$ of $\mathfrak{sl}(3, \mathbb{C})$ have the same signature, but they are discriminated by looking at the imaginary units appearing in the Chevalley basis (in particular, for $\mathfrak{sl}(3, \mathbb{C})_{\mathbb{R}}$ there are no imaginary units in the stabilizing algebra).
Table 26: Structure constants of α5. The grey-shaded boxes denote extraspecial pairs (cfr. e.g. [12], and Refs. therein).
A.3 32 of $SO(6,6)$ in $D = 4$

In this subsection we describe in detail the structure of the orbits discussed in subsection 4.2, in particular identifying explicitly the stabilizers. We refer to fig. (12) and fig. (13) for the weight labels. In tab. (27) and tab. (28) we have identified explicitly the stabilizers for the weights $\Lambda_1, \Lambda_2, \Lambda_3$ and $\Lambda_4$. From this one can immediately deduce the 1-weight orbit.

To construct the stabilizers for multi-weight states we need to identify generators of the algebras annihilating at the same time more than one of weights we have chosen as our single brane. The sets of the common stabilizers appear in tab. (29). It is already interesting to note that for the four weights the common stabilizers form an $\mathfrak{sl}(2,\mathbb{R})^3$ algebra.

Finally in order to define the full sets of stabilizers for the different bound states we should consider the conjunction stabilizers. These are studied for the general combination $\Lambda_1 + \Lambda_2 + a\Lambda_3 + b\Lambda_4$ (where $a, b = \pm 1$) are listed in tab. (30) and tab. (31). Collecting all the information we have defined the orbits appearing in subsection 4.2.
Table 27: Stabilizer for the weights $\Lambda_1$ and $\Lambda_2$ in the \textbf{32} of $\mathfrak{so}(6, 6)$. In the first column we report $\Lambda_1 (= [-\Sigma_{28}])$ meaning that $\Sigma_{28}$, the lowest weight, has the same stabilizers of the $\Lambda_1$ with different simple root signs. The dyonic orbit is built as combination of these two weights.
Table 28: Stabilizer for the weights $\Lambda_3$ and $\Lambda_4$ in the 32 of $\mathfrak{so}(6,6)$
Table 29: Common stabilizer in the $\mathbf{32}$ of $\mathfrak{so}(6,6)$
Table 30: Conjunction stabilizers for the weights in the $\mathbf{32}$ of $\mathfrak{so}(6, 6)$. In the first column we list the combination of generators annihilating the corresponding pair of state, in the last two columns their effect on the other weights we have chosen to study the orbit.

$$
\begin{array}{|c|c|c|}
\hline
\text{Conjunction Stabilizers} & a\Lambda_3 & b\Lambda_4 \\
\hline
\Lambda_1 + \Lambda_2 & a\Lambda_3 & b\Lambda_4 \\
\hline
E_{\alpha_3+2\alpha_4+\alpha_5+\alpha_6} - E_{-\alpha_6} & - & b(\Sigma_{20} - \Sigma_{28}) \\
E_{\alpha_3+\alpha_4+\alpha_5+\alpha_6} - E_{-\alpha_4-\alpha_6} & - & - \\
E_{\alpha_3+\alpha_4+\alpha_6} - E_{-\alpha_4-\alpha_5-\alpha_6} & - & - \\
E_{\alpha_4+\alpha_5+\alpha_6} - E_{-\alpha_4-\alpha_5-\alpha_6} & - & - \\
E_{\alpha_4+\alpha_6} - E_{-\alpha_3-\alpha_4-\alpha_5-\alpha_6} & - & - \\
E_{\alpha_6} - E_{-\alpha_3-2\alpha_4-\alpha_5-\alpha_6} & a(\Sigma_{20} - \Sigma_{28}) & - \\
\hline
\Lambda_1 + a\Lambda_3 & \Lambda_2 & b\Lambda_4 \\
\hline
E_{\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6} - aE_{-\alpha_6} & - & b\Sigma_7 - ab\Sigma_{28} \\
E_{\alpha_1+2\alpha_2+3\alpha_3+\alpha_4+\alpha_5+\alpha_6} - aE_{-\alpha_2-\alpha_3-\alpha_4-\alpha_6} & - & - \\
E_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6} - aE_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_6} & - & - \\
E_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6} - aE_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6} & - & - \\
E_{\alpha_6} - aE_{-\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4-\alpha_5-\alpha_6} & \Sigma_7 - a\Sigma_{28} & - \\
E_{\alpha_1+2\alpha_3+\alpha_4+\alpha_6} - aE_{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6} & - & - \\
\hline
\Lambda_1 + b\Lambda_4 & \Lambda_2 & a\Lambda_3 \\
\hline
E_{\alpha_3+2\alpha_4+\alpha_5+\alpha_6} - bE_{-\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4-\alpha_5-\alpha_6} & \Sigma_1 - b\Sigma_{28} & - \\
E_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6} - bE_{-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4-\alpha_5-\alpha_6} & - & - \\
E_{\alpha_1+\alpha_2+3\alpha_4+\alpha_5+\alpha_6} - bE_{-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4-\alpha_5-\alpha_6} & - & - \\
E_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6} - bE_{-\alpha_1-\alpha_2-\alpha_3-2\alpha_4-\alpha_5-\alpha_6} & - & - \\
E_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6} - bE_{-\alpha_2-\alpha_3-2\alpha_4-\alpha_5-\alpha_6} & - & - \\
E_{\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6} - bE_{-\alpha_3-2\alpha_4-\alpha_5-\alpha_6} & - & a\Sigma_1 - ab\Sigma_{28} \\
\hline
\end{array}
$$
| Conjunction Stabilizers |
|-------------------------|
| $\Lambda_2 + a\Lambda_3$ | $\Lambda_1$ | $b\Lambda_4$ |
| $E_{a_2} - aE_{-a_1-a_2-a_3}$ | - | - |
| $E_{a_2+a_3} - aE_{-a_1-a_2}$ | - | - |
| $E_{a_1+a_2} - aE_{-a_2-a_3}$ | - | - |
| $E_{a_1+a_2+a_3} - aE_{-a_2}$ | - | - |
| $E_{a_3+2a_4+a_5+a_6} - aE_{a_1+2a_2+2a_3+2a_4+a_5+a_6}$ | $b\Sigma_{20} - ab\Sigma_7$ | - |
| $E_{-a_1-2a_2-2a_3-2a_4-a_5-a_6} - aE_{-a_1-2a_4-a_5-a_6}$ | $\Sigma_{20} - a\Sigma_7$ | - |
| $\Lambda_2 + b\Lambda_4$ | $\Lambda_1$ | $a\Lambda_3$ |
| $E_{a_2+a_3+a_4} - bE_{-a_1-a_2-a_3-a_4}$ | - | - |
| $E_{a_1+a_2+a_3+a_4} - bE_{-a_2-a_3-a_4}$ | - | - |
| $E_{a_1+a_2+a_3+a_4+a_5} - E_{-a_2-a_3-a_4}$ | - | - |
| $E_{a_2+a_3+a_4+a_5} - bE_{-a_1-a_2-a_3-a_4}$ | - | - |
| $E_{-a_6} - bE_{-a_1-2a_2-2a_3-2a_4-a_5-a_6}$ | $\Sigma_1 - b\Sigma_{20}$ | - |
| $E_{a_6} - bE_{a_1+2a_2+2a_3+2a_4+a_5+a_6}$ | $a\Sigma_{20} - ba\Sigma_1$ | - |
| $a\Lambda_3 + b\Lambda_4$ | $\Lambda_1$ | $\Lambda_2$ |
| $aE_{-a_6} - bE_{-a_3-2a_4-a_5-a_6}$ | $a\Sigma_1 - b\Sigma_7$ | - |
| $aE_{a_4} - bE_{-a_3-a_4-a_5}$ | - | - |
| $aE_{a_3+a_4} - bE_{-a_4-a_5}$ | - | - |
| $aE_{a_4+a_5} - bE_{-a_3-a_4}$ | - | - |
| $bE_{a_3+a_4+a_5} - aE_{-a_4}$ | - | - |
| $aE_{a_6} - bE_{a_3+2a_4+a_5+a_6}$ | $a\Sigma_7 - b\Sigma_1$ | - |

Table 31: Conjunction stabilizers for the weights in the 32 of $\mathfrak{so}(6,6)$. In the first column we list the combination of generators annihilating the corresponding pair of state, in the last two columns their effect on the other weights we have chosen to study the orbit.
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