Località di meccanica quantistica e interpretazione del criterio di realtà EPR.

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Abstract
We prove that by adopting a strict interpretation of the Einstein-Podolsky-Rosen criterion of reality, the proofs of the known non-locality theorems fail in showing that quantum mechanics violates the principle of località and reality.

1 Introduction
The different non-località theorems appeared in the literature [1] - [6], starting from the pioneer theorem of Bell, were, and are, often interpreted as proofs of inconsistency between quantum mechanics and the principle of località and reality introduced by Einstein, Podolsky and Rosen [7], which consists of the following two statements.

(R) Criterion of reality. If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.

(L) Principle of locality. Let $R_1$ and $R_2$ be two space-like separated regions. The reality in $R_2$ is unaffected by operations performed in $R_1$.

In sections 3, 4, 5, we give equivalent reformulations of the main non-località theorems, which show that each of them proves is an inconsistency of quantum mechanics with the following assumption involving quantum correlations.

(EQC) Extension of quantum correlations. Let A and B be two observables whose measurements require operations in regions $R_1$ and $R_2$, respectively, space-like separated from each other. If quantum mechanics predicts correlations, in the state $\psi$, between the outcomes of actually performed measurements of A and B, then every specimen $x$ of the physical system in the state $\psi$ possesses objective values of A and B which satisfy these correlations.
The proved inconsistency unavoidably leads to conclude that quantum mechanics violates the principles of locality and reality if (EQC) could be inferred from the principle of locality and reality, and, eventually, from quantum theory. Such an inference can be established only if a wide interpretation of the criterion of reality (R) is adopted, expressed as law (wR) in section 2.

The strict interpretation of (R), expressed as (sR) in section 2, is not able to justify (EQC). Indeed, this last interpretation leads to an extension (sEQC) of quantum correlations which is strictly smaller than (EQC).

In this work we explore the possibility of interpreting the inconsistency proved by non-locality theorems as a failure of just (EQC), without provoking conflicts between the principle (R, L) and quantum mechanics. This program entails that interpretation (wR) must be abandoned, because it implies (EQC); but (sR) must be retained, because it is implied by (R, L). Furthermore, contrary to (EQC), statement (sEQC) should not conflict with quantum mechanics, because it is a consequence of (sR). In fact, we show that by replacing (EQC) by (sEQC) in the main non-locality theorems, no inconsistency between quantum theory and the principle of locality and reality can be proved. Thus, the meaning of the present work is that quantum mechanics can coexist with locality if the criterion of reality is interpreted according to its strict sense.

In section 2 we show how by strictly interpreting the criterion of reality (R), only a weaker form (sEQC) of the extension law (EQC) can be derived. Sections 3, 4, and 5 are devoted to show that the non-locality theorems fail when the strict interpretation is adopted, i.e. when (sEQC) replaces (EQC).

We have proved such a failure for the theorem [6] proved by Greenberger, Horne, Shimony, Zeilinger (GHSZ) in section 3, for the theorem of Hardy [4] in section 4 and for the classic Bell’s theorem [1] in section 5. In fact, these three theorems follow three different logical schemes, every non-locality theorem can be traced back to.

## 2 Weak extension of quantum correlations.

In this section we derive an extension of quantum correlations from (R, L), as it can be inferred by a strict interpretation of the criterion of reality (R). Our argument requires the formal introduction of terms to suitably represent the concepts to be handled. Given a quantum state vector \( \psi \) of the Hilbert space \( \mathcal{H} \) which describes the physical system, let \( S(\psi) \) be a support of \( \psi \), i.e. a concrete set of specimens of the physical systems whose quantum state is \( \psi \). Let \( A \) be any 1-0 observable, i.e. an observable having only the possible values 1 and 0, and hence represented by a projection operator \( \hat{A} \). In correspondence with \( A \) we introduce the following peculiar subsets of \( S(\psi) \). By \( A \) we denote the set of the specimens of \( S(\psi) \) which objectively possess a value of the observable \( A \). By \( A_1 \) (resp., \( A_0 \)) we denote the set of specimens of \( A \) which possess the objective value 1 (resp., 0) of \( A \); hence we can assume that \( A_1 \cup A_0 = A \) holds. By \( A \) we denote the concrete set of specimens of \( S(\psi) \) which actually undergo a measurement of \( A \). By \( A_1 \) (resp., \( A_0 \)) we denote the set of specimens of \( A \) for which the outcome 1 (resp., 0) of \( A \) has been obtained; hence we can assume that \( A_1 \cup A_0 = A \) holds. Moreover, we define the two mappings \( a : A \rightarrow \{1, -1\} \)
and \( a : \mathbb{A} \rightarrow \{1, -1\} \) as follows.

\[
a(x) = \begin{cases} 
1, & x \in A_1, \\
-1, & x \in A_0;
\end{cases} \quad a(x) = \begin{cases} 
1, & x \in A_1, \\
-1, & x \in A_0.
\end{cases}
\] (1)

According to standard quantum theory the following statement holds

\[
A_1 \subseteq A_1 \text{ and } A_0 \subseteq A_0,
\] (2.i)

because the reality of the outcome of an actually performed measurement cannot be denied. Moreover, two observables \( A \) and \( B \) can be measured together if and only if the corresponding operators commute with each other; therefore the following statements hold.

\[
[\hat{A}, \hat{B}] \neq 0 \quad \text{implies} \quad A \cap B = \emptyset \quad \text{for all } S(\psi). \quad (2.ii)
\]

\[
[\hat{A}, \hat{B}] = 0 \quad \text{implies} \quad \forall \psi \exists S(\psi) \quad \text{such that} \quad A \cap B \neq \emptyset. \quad (2.iii)
\]

Let \( A \) and \( B \) be two separated observables, written \( A \bowtie B \), i.e. observables whose measurements require operations to be performed in space-like separated regions \( R_1 \) and \( R_2 \). Since (L) applies, the following statement holds.

\[
A \bowtie B \quad \text{implies} \quad [\hat{A}, \hat{B}] = 0, \quad \text{hence } S(\psi) \text{ exists such that } A \cap B \neq \emptyset. \quad (3)
\]

The principle of locality and reality (R,L) leads to further implications in the case that the separated observables \( A \) and \( B \) are correlated. Let us consider the case that the correlation \( A \rightarrow B \) holds in the quantum state \( \psi \), which means that whenever both \( A \) and \( B \) are measured, i.e. if \( x \in A \cap B \), then \( a(x) = 1 \) implies \( b(x) = 1 \). Hence, the correlation \( A \rightarrow B \) holds if and only if \( A_1 \cap B \subseteq B_1 \) or if and only if \((a(x) + 1)(b(x) - 1) = 0\) for all \( x \in A \cap B \); equivalently, \( A \rightarrow B \) if and only if \( B_0 \cap A \subseteq A_0 \). Now, if \( A \bowtie B \) and \( A \) is measured on \( x \in A \) obtaining \( a(x) = 1 \), then the principle of locality (L) and the criterion of reality (R) imply \( x \in B \) and \( b(x) = 1 \). Therefore, \( A_1 \subseteq B_1 \) and the correlation \((a(x) = 1) \Rightarrow (b(x) = 1)\) holds for all \( x \in A_1 \). The reasoning repeated by exchanging \( A \) with \( B \) leads to conclude that \( B_0 \subseteq A_0 \) and that the correlation \((a(x) = 1) \Rightarrow (b(x) = 1)\) holds for every \( x \in B_0 \). Thus the correlation extends to \( A_1 \cup B_0 \). Hence from (R,L) and quantum mechanics we infer the following statement.

(sEQC) Weak extension of quantum correlations. Let \( A \) and \( B \) be space-like separated 1-0 observables. If \( A \rightarrow B \) then

\[
(a(x) + 1)(b(x) - 1) = 0, \quad \forall x \in (A_1 \cup B_0) \cup (A \cap B). \quad (4.i)
\]

The quantum correlation \( A \leftrightarrow B \), i.e. \( A \rightarrow B \) and \( B \rightarrow A \), means that the correlation \((a(x) = 1) \leftrightarrow (b(x) = 1)\) holds for all \( x \in A \cap B \). In this case, by (sEQC) we derive that \((a(x) = 1) \leftrightarrow (b(x) = 1)\) holds for all \( x \in (A_1 \cup B_0) \cup (B_1 \cup A_0) \cup (A \cap B) = A \cup B \). Hence, (sEQC) incorporates the following extension of quantum correlations.

\[
A \bowtie B \quad \text{and} \quad A \leftrightarrow B \quad \text{imply} \quad a(x) = b(x), \quad \forall x \in A \cup B. \quad (4.ii)
\]
We remark that in deriving (sEQC) we have applied the *strict* interpretation (sR) of the criterion of reality, according to which if \( A \triangleright B \) and \( A \rightarrow B \) we can predict with certainty the value of an eventual measurement of \( B \) (resp., \( A \)) only once a measurement of \( A \) with concrete outcome \( a(x) = 1 \) (resp., \( B \) with concrete outcome \( b(x) = 0 \)) is performed. If \( x \notin A_1 \) (resp., \( x \notin B_0 \)) no prediction of the value of \( B \) (resp., \( A \)) can be made by a strict application of (R).

The larger extension stated by (EQC) can be derived from (R,L) only if a wider interpretation (wR) is adopted, according to which for ascribing reality to \( B \) it is sufficient the “possibility” of performing a measurement whose outcome would allow for the prediction, with certainty, of the outcome of an eventual measurement of \( B \). Note 10 in [6] highlights the importance of this twofold possibility in interpreting the criterion of reality.

## 3  GHSZ theorem does not extend to (sEQC)

In this section we show that the argument of GHSZ cannot be used for proving inconsistence between quantum mechanics and statement (sEQC). In so doing, we first reformulate GHSZ proof to make clear the role of law (EQC).

The theorem of GHSZ makes use of seven 1-0 observables of a particular quantum system, separated into four classes

\[
\omega_A = \{A^\alpha, A^\beta\}, \quad \omega_B = \{B\}, \quad \omega_C = \{C^\alpha, C^\beta\}, \quad \omega_D = \{D^\alpha, D^\beta\}.
\]

These observables have been singled out by GHSZ in such a way that

\[\text{(5.i) two observables in two different classes are separated from each other.}\]

\[\text{(5.ii) } [\hat{A}^\alpha, \hat{A}^\beta] \neq 0, [\hat{C}^\alpha, \hat{C}^\beta] \neq 0, [\hat{D}^\alpha, \hat{D}^\beta] \neq 0.\]

The state vectors \( \psi \) is chosen so that the following correlations between actually measured outcomes hold, according to quantum mechanics.

\[
\begin{align*}
\text{i) } & a^\alpha(x)b(x) = -c^\alpha(x)d^\alpha(x) & \forall x \in (A^\alpha \cap B) \cap (C^\alpha \cap D^\alpha) \equiv X, \\
\text{ii) } & a^\beta(y)b(y) = -c^\beta(y)d^\beta(y) & \forall y \in (A^\beta \cap B) \cap (C^\beta \cap D^\alpha) \equiv Y, \\
\text{iii) } & a^\beta(z)b(z) = -c^\alpha(z)d^\beta(z) & \forall z \in (A^\beta \cap B) \cap (C^\alpha \cap D^\beta) \equiv Z, \\
\text{iv) } & a^\alpha(t)b(t) = c^\beta(t)d^\beta(t) & \forall t \in (A^\alpha \cap B) \cap (C^\beta \cap D^\beta) \equiv T.
\end{align*}
\]

In terms of 1-0 observables, equations (6.i), (6.ii), (6.iii), (6.iv) express the quantum correlations \( A^\alpha * B \leftrightarrow 1 - C^\alpha * D^\alpha, A^\beta * B \leftrightarrow 1 - C^\beta * D^\alpha, A^\alpha * B \leftrightarrow 1 - C^\alpha * D^\beta, A^\beta * B \leftrightarrow C^\beta * D^\beta \), respectively, where we have put \( A * B = 1 - (A - B)^2 \).

If (EQC) holds, then correlations (6) must be extended to the following correlations between objective values.

\[
\begin{align*}
\text{i) } & a^\alpha(x)b(x) = -c^\alpha(x)d^\alpha(x), \\
\text{ii) } & a^\beta(x)b(x) = -c^\beta(x)d^\alpha(x), \\
\text{iii) } & a^\beta(x)b(x) = -c^\alpha(x)d^\beta(x), \\
\text{iv) } & a^\alpha(x)b(x) = c^\beta(x)d^\beta(x),
\end{align*}
\]

\[\forall x \in S(\psi). \tag{7}\]
GHSZ prove that the correlations (7) are inconsistent because (i)-(iv) in (7) hold for a same \( x \in \mathcal{S}(\psi) \neq \emptyset \). Indeed, from (7.i) and (7.iv) we get

\[
c^\alpha(x)d^\alpha(x) = -c^\beta(x)d^\beta(x).
\]

On the other hand, from (7.ii) and (7.iii) the equality \( c^\alpha(x)d^\beta(x) = c^\beta(x)d^\alpha(x) \) follows, which is equivalent, since \( d^\beta(x)d^\beta(x) = d^\alpha(x)d^\alpha(x) = 1 \), to

\[
c^\alpha(x)d^\beta(x) = c^\beta(x)d^\alpha(x)
\]

which contradicts (8).

Now we prove that this proof of inconsistency does not work if we replace (EQC) by (sEQC). To this end, it is worth to remark that the contradiction between (8) and (9) cannot be derived from (6) alone, without the extension to (7) implied by (EQC), because (6.i)-(6.iv) cannot hold simultaneously for a same \( x = y = z = t \); indeed, \([\hat{A}^\alpha, \hat{A}^\beta] \neq 0\) by (5.ii); hence, according to quantum theory, \( A^\alpha \cap A^\beta = \emptyset\) by (2.ii), and therefore \( X \cap Y \cap Z \cap T = \emptyset \).

The extension of correlations (6) validated by (sEQC) in this case are obtained by applying (4.ii), i.e.

\[
\begin{align*}
\text{i)} & \quad a^\alpha(x)b(x) = -c^\alpha(x)d^\alpha(x) & \forall x \in (A^\alpha \cap B) \cup (C^\alpha \cap D^\alpha) \equiv \tilde{X}, \\
\text{ii)} & \quad a^\beta(y)b(y) = -c^\beta(y)d^\beta(y) & \forall y \in (A^\beta \cap B) \cup (C^\beta \cap D^\alpha) \equiv \tilde{Y}, \\
\text{iii)} & \quad a^\beta(z)b(z) = -c^\alpha(z)d^\beta(z) & \forall z \in (A^\beta \cap B) \cup (C^\alpha \cap D^\beta) = \tilde{Z}, \\
\text{iv)} & \quad a^\alpha(t)b(t) = c^\beta(t)d^\beta(t) & \forall t \in (A^\alpha \cap B) \cup (C^\beta \cap D^\beta) = \tilde{T}.
\end{align*}
\]

In order that the GHSZ argument – which leads to the contradiction from (7) to (9) through (8) – can be successfully repeated from (10), the first step requires that (10.i) and (10.iv) should hold for the same specimen \( x_0 \); therefore the condition \( \tilde{X} \cap \tilde{T} \neq \emptyset \) should hold; the second step requires that also (10.ii) and (10.iii) should hold for such a specimen. Thus, the condition

\[
\tilde{X} \cap \tilde{Y} \cap \tilde{Z} \cap \tilde{T} \neq \emptyset
\]

should be satisfied. Now, from (5.ii) and (2.ii) we derive

\[
\emptyset = (A^\alpha \cap B) \cap (A^\beta \cap B) = (C^\alpha \cap D^\alpha) \cap (C^\beta \cap D^\beta) =
\]

\[
= (C^\alpha \cap D^\alpha) \cap (C^\beta \cap D^\beta) = (C^\alpha \cap D^\alpha) \cap (C^\beta \cap D^\beta) = (C^\beta \cap D^\beta) = (C^\beta \cap D^\beta) = (C^\beta \cap D^\beta).
\]

To obtain the set \( \tilde{X} \cap \tilde{Y} \cap \tilde{Z} \cap \tilde{T} \) the distributive law for \( \cap \) and \( \cup \) of elementary set theory can be applied; in so doing, (12) imply

\[
\tilde{X} \cap \tilde{Y} \cap \tilde{Z} \cap \tilde{T} = \emptyset,
\]

which denies condition (11) necessary to prove the inconsistency. Thus, GHSZ proof cannot be extended to prove inconsistency between quantum mechanics and the strict interpretation of the principle of locality and reality.
4 Hardy’s theorem

The scheme of the theorem of Hardy involves four 1-0 observables $A^\alpha, B^\alpha, A^\beta, B^\beta$, chosen in such a way that

$$(13.i) \ A^\alpha \cong B^\alpha, A^\alpha \succeq B^\beta, A^\beta \cong B^\alpha, A^\beta \succeq B^\beta;$$

$$(13.ii) \ [A^\alpha, A^\beta] \neq 0 \text{ and } [B^\alpha, B^\beta] \neq 0.$$  

The state vector $\psi$ is chosen so that according to quantum theory the correlations $A^\alpha \rightarrow B^\alpha, B^\alpha \rightarrow A^\beta, A^\beta \rightarrow B^\beta$ hold, which can be equivalently expressed as follows.

i) \quad (a^\alpha(x) + 1)(b^\alpha(x) - 1) = 0, \ \forall x \in A^\alpha \cap B^\alpha

ii) \quad (b^\alpha(y) + 1)(a^\beta(y) - 1) = 0, \ \forall y \in A^\beta \cap B^\alpha

iii) \quad (a^\beta(z) + 1)(b^\beta(z) - 1) = 0, \ \forall z \in A^\beta \cap B^\beta.

A further constraint satisfied by the choice of $\psi$ in Hardy’s theorem is the following statistical prediction of quantum mechanics,

$$\langle \psi | \hat{A}^\alpha(1 - \hat{B}^\beta) \psi \rangle \neq 0.$$  \quad (15.i)

which means that there is a non-vanishing probability of obtaining $(1, 0)$ as pair of outcomes of a measurement of $A^\alpha$ and $B^\beta$. Hence, a support $S(\psi)$ exists such that $A^\alpha_1 \cap B^\beta_0 \neq \emptyset$, i.e. $S(\psi)$ and $x_0 \in S(\psi)$ exist such that $a^\alpha(x_0) = 1$ and $b^\beta(x_0) = -1$.  \quad (15.ii)

If (EQC) is assumed to hold, then from correlations (14) we infer that

i) \quad (a^\alpha(x) + 1)(b^\alpha(x) - 1) = 0,

ii) \quad (b^\alpha(x) + 1)(a^\beta(x) - 1) = 0,

iii) \quad (a^\beta(x) + 1)(b^\beta(x) - 1) = 0

are satisfied for any $x \in S(\psi)$, for every support $S(\psi)$. Now, if a specimen $x$ satisfies (16.i,ii,iii), then by using elementary algebra we imply that

$$(a^\alpha(x) + 1)(b^\beta(x) - 1) = 0$$  \quad (17)

holds for such $x$. Therefore (17) holds for every $x \in S(\psi)$, for every support $S(\psi)$. Thus, (15.ii) turns out to be contradicted, because $(a^\alpha(x_0) + 1)(b^\beta(x_0) - 1) = -4$.

Now we show that no contradiction arises if we replace (EQC) by (sEQC). The extension of correlations (14) obtained by applying (sEQC) is expressed by

i) \quad (a^\alpha(x) + 1)(b^\alpha(x) - 1) = 0, \ \forall x \in \mathbf{A}^\alpha \cup \mathbf{B}^\beta \cup (\mathbf{A}^\alpha \cap \mathbf{B}^\alpha) = \mathbf{X}

ii) \quad (b^\alpha(y) + 1)(a^\beta(y) - 1) = 0, \ \forall y \in \mathbf{A}^\beta \cup \mathbf{B}^\alpha \cup (\mathbf{A}^\beta \cap \mathbf{B}^\alpha) = \mathbf{Y}

iii) \quad (a^\beta(z) + 1)(b^\beta(z) - 1) = 0, \ \forall z \in \mathbf{A}^\beta \cup \mathbf{B}^\alpha \cup (\mathbf{A}^\beta \cap \mathbf{B}^\beta) = \mathbf{Z}$.

These extensions no longer imply (17). Indeed, equation $(a^\alpha(x) - 1)(b^\beta(x) - 1) = 0$ can be derived from (18) if all three equations therein hold for the same specimen $x$, i.e. if $x \in \mathbf{X} \cap \mathbf{Y} \cap \mathbf{Z}$. But this last set is empty; indeed $(\mathbf{A}^\alpha \cup \mathbf{B}^\alpha) \cap (\mathbf{A}^\beta \cup \mathbf{B}^\alpha) \cap (\mathbf{A}^\beta \cup \mathbf{B}^\beta) = \emptyset$ follows from (13.ii) and (2.ii); on the other hand, from the definition of $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ in (18) we have $\mathbf{X} \cap \mathbf{Y} \cap \mathbf{Z} \subseteq (\mathbf{A}^\alpha \cup \mathbf{B}^\alpha) \cap (\mathbf{A}^\beta \cup \mathbf{B}^\alpha) \cap (\mathbf{A}^\beta \cup \mathbf{B}^\beta)$, and thus $\mathbf{X} \cap \mathbf{Y} \cap \mathbf{Z} = \emptyset$.  

5 Bell’s theorem

Six 1-0 observables $A^α, A^β, A^γ, B^α, B^β, B^γ$ are involved in Bell’s theorem, which satisfy the following conditions

(19.i) Each $A$-observable is space-like separated from every $B$-observable;

(19.ii) $[\hat{A}^λ, \hat{A}^μ] \neq 0$ and $[\hat{B}^λ, \hat{B}^μ] \neq 0$ if $λ \neq μ$, where $λ, μ \in \{α, β, γ\}$.

The state vector $ψ$ is singled out so that quantum correlations $A^β \leftrightarrow 1 - B^β$ and $A^γ \leftrightarrow 1 - B^γ$ hold, i.e.

$$a^β(x) = -b^β(x) \quad \text{and} \quad a^γ(x) = -b^γ(x), \quad (20)$$

where (20.i) holds for all $x \in A^β \cap B^β$ and (20.ii) holds for all $x \in A^γ \cap B^γ$.

Following Bell’s proof, if $Y = \{x_1, x_2, \ldots, x_N\}$ is any finite set of specimens of the physical system such that both (20.i) and (20.ii) hold for every $x_k \in Y$, then the following (Bell’s) inequality

$$|a^αb^β - a^αb^γ| \leq 1 - a^βb^γ \quad (21)$$

can be derived for the mean values $\overline{a^αb^β}$, $\overline{a^αb^γ}$, $\overline{a^βb^γ}$ (for instance, $\overline{a^αb^β} = \frac{1}{N} \sum_{x_k \in Y} a^α(x_k)b^β(x_k)$, and so on), all computed on the same sample $Y$.

Quantum mechanics, by itself, does not conflict with Bell’s inequality (21), because according to quantum theory the correlations (20.i) and (20.ii) hold together only if $x \in A^β \cap B^β \cap A^γ \cap B^γ = ∅$.

But if (EQC) is assumed to hold, then (20) extends to

$$a^β(x) = -b^β(x), \forall x \in \mathcal{S}(ψ); \quad a^γ(x) = -b^γ(x), \forall x \in \mathcal{S}(ψ). \quad (22)$$

Therefore, (EQC) makes valid Bell’s inequality (21) whenever the involved mean values are evaluated for the “objective” values on any finite sample $Y \subseteq \mathcal{S}(ψ)$. Now, quantum theory cannot predict the three mean values in (21) evaluated on a same sample $Y$, because they refer to the three non-commuting (by 19.ii) observables $A^α*B^β$, $A^α*B^γ$, $A^β*B^γ$. Quantum mechanics can predict the mean values $\overline{a^αb^β}$, $\overline{a^αb^γ}$, $\overline{a^βb^γ}$ evaluated on different samples $Y_1 \subseteq A^α \cap B^β$, $Y_2 \subseteq A^α \cap B^γ$, $Y_3 \subseteq A^β \cap B^γ$, each of them contained in the domain of validity of (21); these mean values agree, according to quantum theory, with the quantum expectation values $⟨ψ | A^α*B^β | ψ⟩$, $⟨ψ | A^α*B^γ | ψ⟩$, $⟨ψ | A^β*B^γ | ψ⟩$. If the mean values are replaced by these expectation values, a violation of Bell’s inequality (21) is found.

In order that the mean values in (21) can be replaced by the quantum expectation values, the following further hypothesis has been assumed in Bell’s type theorems.

Fair sampling assumption. *The sample of physical systems which actually undergo a measurement fairly represent the entire population $\mathcal{S}(ψ)$.*

The validity of the fair sampling assumption has been submitted to deep investigations, as for instance in [8, 9] and references therein, which show that it can be seriously questioned without violating physical principles or statistical regularity. This assumption is not necessary in the theorems of GHSZ and Hardy which, for this reason, are more effective in showing inconsistency between quantum mechanics and (EQC).
Now we show that if (EQC) is replaced by (sEQC) then the domain of validity of (21) becomes smaller than \( S(\psi) \), so that the predictions of quantum theory about the mean values in (21) cannot longer apply, because they refer to samples \( Z \) which are outside of the domain of validity of (21).

By using (4.ii), we imply that (20.i) holds for all \( x \in A^\beta \cup B^\beta \), whereas (20.ii) is valid for all \( x \in A^\gamma \cup B^\gamma \); therefore both (20.i) and (20.ii) hold for all \( x \in X = (A^\beta \cup B^\beta) \cap (A^\gamma \cup B^\gamma) \). Since \( [\hat{A}^\beta, \hat{A}^\gamma] \neq 0 \) and \( [\hat{B}^\beta, \hat{B}^\gamma] \neq 0 \) we have \( X = (A^\beta \cup B^\beta) \cap (A^\gamma \cup B^\gamma) = (A^\beta \cap A^\gamma) \cup (A^\beta \cap B^\gamma) \cup (B^\beta \cap A^\gamma) \cup (B^\beta \cap B^\gamma) = (A^\beta \cap B^\gamma) \cup (B^\beta \cap A^\gamma) \) by (2.ii). On the other hand by (19.i), (3) and (2.iii) we infer that a support \( S(\psi) \) exists such that \( (A^\beta \cap B^\gamma) \neq \emptyset \) (or \( (A^\beta \cap B^\gamma) \neq \emptyset \)), and hence \( X \neq \emptyset \). Therefore, a result of the substitution of (EQC) with (sEQC) is that Bell inequality (21) holds only for samples \( Y \subseteq X = (A^\beta \cap B^\gamma) \cup (B^\beta \cap A^\gamma) \).

Such a limited validity does not violates quantum mechanics. Indeed, the quantum mechanical prediction \( \langle \psi | \hat{A}^\alpha \ast \hat{B}^\beta | \psi \rangle \) for \( a^\alpha b^\beta \) in (21) holds for samples \( Z_1 \) such that at least \( Z_1 \subseteq A^\alpha \) is satisfied. But since \( [\hat{A}^\alpha, \hat{A}^\beta] \neq 0 \) and \( [\hat{A}^\alpha, \hat{A}^\gamma] \neq 0 \), by (2.ii) we have
\[
A^\alpha \cap X = (A^\alpha \cap A^\beta \cap B^\gamma) \cup (A^\alpha \cap A^\gamma \cap B^\beta) = \emptyset.
\] (23)
Hence the quantum mechanical prediction for \( a^\alpha b^\beta \) refer to samples for which the limited Bell’s inequality does not hold.

To conclude, (sEQC) bounds the validity of Bell inequality in such a way that

(i) it holds for samples which make Bell inequality neither verifiable by experiment, nor comparable with the statistical predictions of quantum theory;

(ii) it is consistent with the principle of reality and locality and with the predictions of quantum theory.

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