Duality in multi-channel Luttinger Liquid with local scatterer

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Abstract - We have devised a general scheme that reveals multiple duality relations valid for all multi-channel Luttinger Liquids. The relations are universal and should be used for establishing phase diagrams and searching for new non-trivial phases in low-dimensional strongly correlated systems. The technique developed provides universal correspondence between scaling dimensions of local perturbations in different phases. These multiple relations between scaling dimensions lead to a connection between different inter-phase boundaries on the phase diagram. The dualities, in particular, constrain phase diagram and allow predictions of emergence and observation of new phases without explicit model-dependent calculations. As an example, we demonstrate the impossibility of non-trivial phase existence for fermions coupled to phonons in one dimension.

Solid-state systems like Fractional Quantum Hall systems [1], Quantum Spin Quantum Hall or topological insulators [2,3], carbon nanotubes [4] and quantum wires are now routinely described in terms of few (chiral or not) channels with intra- and inter-channel interactions. A similar description is also applied to cold atoms mixtures [5], ballistic quasi–one-dimensional waveguides [6], hollow-core fibers [7] and one-dimensional electron-phonon systems [8,9]. The intra-channel interactions being taken into account lead to the formation of Luttinger Liquid [10] for each individual channel. The perturbations that scatter or tunnel particles within the same channel are either irrelevant (the scaling dimension is higher than physical dimension which is unity for a local perturbation) or relevant (the scaling dimension is lower than one) in terms of the renormalization group analysis [11]. This means that a single channel is to be found in one of the following two states (depending on material parameters): perfectly conducting or insulating.

The inter-channel interactions make scaling dimensions of all perturbation inter-dependent. A new and rich phase diagram emerges as the result. Since the interaction in a single channel makes it either perfectly conducting or completely insulating, the best starting point to examine the effect of a local scatterer is to assume that the state of a \(N\)-channel liquid has \(n\) insulating and \((N-n)\) conducting channels. Each particular realization of such configuration is called a phase. This is a natural generalization of phases observed in two-channel problems (like spin and charge channels). The phase is stable if all allowed perturbations are irrelevant. The condition that all scaling dimensions of local perturbations are higher than one (irrelevant perturbations) defines a region of physical parameters where this particular phase can be observed. Intersections of different regions correspond to unstable fixed points meaning that there is no unique phase for those system parameters and which phase is realized depends on bare values of perturbations (multiple attraction basins). On the other hand, if the union of all those regions does not cover the whole space it means that there is a range of system parameters where none of bare phases is stable, there must be a new stable fixed point corresponding to a new phase of matter. Such situation is known to occur for one-dimensional electrons with spin but without \(SU(2)\) symmetry [11] or in topological insulators at strong interactions [2]. It occurs that when two-particle local scattering is taken into account all bare phases become unstable in some region of material parameters.

The scaling dimensions of two operators defining the instability of two phases of a single-channel Luttinger Liquid are known to be inversely proportional to each other [11]. This relation between scaling dimensions is the consequence of the duality between weak- and strong-scatterer limits. Recently, it was also observed in [8] that
the coupling of interacting electrons (the Luttinger Liquid) to acoustic phonons did not change the duality. The fact that the scaling dimensions of the operators determining the instability of the fermionic channel were changed essentially but stayed inversely proportional to each other came as a surprise. The scaling dimensions were derived explicitly as the result of lengthy calculations and the reason behind duality was unclear. It will be shown in this letter that the duality observed in [8] is just one particular example of a universal property which is a generalized duality relation valid for arbitrary multi-channel Luttinger Liquid.

We develop a generic scheme of calculation of scaling dimensions of all symmetry-allowed local perturbations not just because it gives a machinery of dealing with multi-component phases. We do it to reveal hidden symmetries reflected in relations between scaling dimensions of perturbations in different phases. We will show below that the matrix $\Delta$ which has scaling dimensions of different perturbations as entries is given by

$$\Delta = \Delta_I \oplus \Delta_C$$

for each phase characterized by a set of insulating states and the complementary set of conducting states. The introduction of two sectors for the multi-channel Luttinger Liquid is a natural generalization of the now accepted in two-channel problems nomenclature: $I$ or $CC$ stand for both channels being insulating or conducting, $CI$ and $IC$ for one conducting and one insulating channel. The matrices $\Delta_I$ and $\Delta_C$ are defined on two orthogonal subspaces of insulating ($I$) and conducting ($C$) channels. Their matrix elements are not independent though. To construct them one has to find the matrix $\Delta_\theta$ of scaling dimensions in an ideally conducting phase. The projection of this matrix onto the $I$-subspace and inverting the projection gives $\Delta_I$. The projection of the inverse matrix $\Delta^{-1}_\theta$ and the following inversion of the projection gives $\Delta_C$. The common source for both matrices $\Delta_I$ and $\Delta_C$ implies relations between scaling dimensions in conducting and insulating sectors. These relations impose restrictions on inter-phase boundaries in material parameter space and can be used to predict (without system-dependent calculations) whether a new phase is expected or not on a quite generic basis.

The multi-channel Luttinger Liquid is a set of $N$ interacting individual liquids. Each channel is labeled with an index $i = 1, \ldots, N$ and described in terms of density, $n_i = \partial_x \theta_i / \pi$, and current, $j_i = \partial_x \phi_i / \pi$, fields. Each channel is characterized by its own velocity $v_i$ and the Luttinger parameter $K_i$ reflecting the strength and statistics of underlying particles. The Lagrangian written in terms of the bosonic fields $\theta_i(x,t)$ and $\phi_i(x,t)$,

$$\mathcal{L} = \frac{1}{\pi} \delta_{ij} \partial_x \phi_i - H[\theta, \phi]$$

contains a Hamiltonian part which is (neglecting backscattering) a sum of two quadratic forms for density-density and current-current interactions:

$$H[\theta, \phi] = \frac{1}{2\pi} \partial_x \theta_i \left( \frac{v_i}{K_i} \delta_{ij} + \delta_{ij} \right) \partial_x \theta_j + \frac{1}{2\pi} \partial_x \phi_i \left( v_i K_i \delta_{ij} + \phi' \right) \partial_x \phi_j. \quad (3)$$

The diagonal terms describe individual channels while inter-channel interactions are included into the interaction matrices $\tilde{\gamma}' \phi$ and $\tilde{\gamma}'$. There are two complementary representations either in terms of density, $\theta^T = (\theta_1, \ldots, \theta_N)$, and current, $\phi^T = (\phi_1, \ldots, \phi_N)$, vector fields or in terms of chiral right-moving, $\theta_R = \phi + \theta$, and left-moving, $\theta_L = \phi - \theta$, fields. We will be switching between these two representations because they both have advantages when performing different tasks. The first step will be the reduction of the Lagrangian to a diagonal form and it is much easier in $(\theta, \phi)$ representation because we have the Hamiltonian part which is the sum of two quadratic forms in this representation. The transformation matrices

$$\theta = M_\theta \tilde{\theta}, \quad \phi = M_\phi \tilde{\phi} \quad (4)$$

must diagonalize the Hamiltonian part of the action and also preserve the structure of the first term in eq. (2) (which is the same as the preservation of the commutation relations in the operator formulation). The later requirement is imposing connection between those transformations:

$$M^{\theta T}_\theta M_\phi = 1. \quad (5)$$

These transformations always exist and can be constructed as a four-steps procedure with each step preserving the scalar product $\theta^T \phi = \tilde{\theta}^T \tilde{\phi}$. First, one can apply unitary transformation to diagonalize the quadratic form in $\phi$ and simultaneously apply the same unitary transformation to the $\theta$-vector. Second, one rescales each component of the new $\phi$-field to absorb the eigenvalues and turn the quadratic form into a scalar product while the $\theta$-vector is subject to inverse rescaling. Now, we may again apply identical unitary transformations to both vectors choosing them to diagonalize the quadratic form in the new $\theta$-fields (the kernel stays real and symmetric during all transformations). Finally, we can rescale each component of $\theta$- and inversely rescale $\phi$-vectors in such a way that the coefficients in front of either $i$-th component are the same, $u_i$ (they will be new velocities). The resulting Lagrangian in terms of new fields is given by the expression

$$\mathcal{L} = \frac{1}{\pi} \partial_t \partial_x \tilde{\phi}_i - \frac{u_i}{2\pi} \left( (\partial_x \tilde{\phi}_i)^2 + (\partial_x \tilde{\phi}_i)^2 \right). \quad (6)$$

In a translational invariant system the transformations (4) relate the Green functions of interacting Luttinger Liquids and the Green functions of uncoupled liquids. Since the latter is well known one can easily find, for example, the local Green functions $iG^{\theta \phi}_{ij}(t; t') = \langle \theta_i(x, t) \theta_j(x, t') \rangle$ and $iG^{\phi \phi}_{ij}(t; t') = \langle \phi_i(x, t) \phi_j(x, t') \rangle$. In matrix form the
 retarded components can be written as
\[
\hat{G}_R^R(\omega) = -\frac{i\pi}{2}\frac{\Delta_\theta}{\omega + i0}, \quad \hat{G}_R^R(\omega) = -\frac{i\pi}{2}\frac{\Delta_\phi}{\omega + i0},
\]
(7)
with matrices
\[
\Delta_\theta = M_\theta M_\theta^T, \quad \Delta_\phi = M_\phi M_\phi^T
\]
(8)
being inversely proportional to each other
\[
\Delta_\theta \Delta_\phi = 1.
\]
(9)

In the presence of a scatterer at the origin \( x = 0 \) the transformation (4) should be performed on the left and on the right of the scatterer separately because the fields are no longer continuous across the origin. To take into account boundary conditions that relate those fields it is now convenient to switch to right- and left-movers combining them into incoming (\( \text{in} \)) and outgoing (\( \text{out} \)) fields:
\[
\Theta_{\text{out}} = \begin{pmatrix} \theta_{\text{out}}^R(t) \\ \theta_{\text{out}}^L(t) \end{pmatrix}, \quad \Theta_{\text{in}} = \begin{pmatrix} \theta_{\text{in}}^R(t) \\ \theta_{\text{in}}^L(t) \end{pmatrix}.
\]
(10)

Here we have used the notations \( \theta_{\text{out}}^R = \theta_R(x = +0), \theta_{\text{out}}^L = \theta_L(x = -0) \) and \( \theta_{\text{in}}^R = \theta_R(x = -0), \theta_{\text{in}}^L = \theta_L(x = +0) \). The same definitions are used to construct the new (transformed) fields \( \Theta_{\text{in}} \) and \( \Theta_{\text{out}} \). The boundary conditions for the original and new fields can be written using the \( S \)-matrix
\[
\Theta_{\text{out}} = \hat{S} \Theta_{\text{in}}, \quad \Theta_{\text{out}} = \tilde{S} \Theta_{\text{in}}.
\]
(11)

While the scattering matrix for the new fields has yet to be found the scattering matrix for the original fields is known and it depends on the bare phase of the Luttinger Liquid whose stability against scattering we would like to try. The phase under investigation can be presented by a Liquid whose stability against scattering we would like to know and it depends on the bare phase of the LuttingerLiquid whose stability against scattering we would like to try.

The rotation with the matrix \( \hat{L} \) is equivalent to the redefinition of right- and left-moving incoming fields in terms of new uncorrelated fields \( \theta^R_{\text{in}} = \theta_R \pm \theta_L \).

To find scaling dimensions of perturbations it is sufficient to know the correlation functions of incoming fields (the rest can be restored using the scattering matrix, if necessary). The Green function of the new incoming fields is essentially trivial because the scatterer is located “downstream” for incoming fields and there is no interaction between incoming and outgoing fields. The Green function \( \hat{G}_{\text{in}} \) is simply given by the expression valid in the translatational invariant problem
\[
\hat{G}_{\text{in}}^R(\omega) = -\frac{2\pi i}{\omega + i0} \hat{1}.
\]
(18)
The Green functions of incoming original, \( \hat{G}_{\text{in}} \), and new, \( \hat{G}_{\text{in}} \), fields are related to each other (eq. (13)):
\[
\hat{L}^{-1}\hat{G}_{\text{in}}^R(\omega)\hat{L} = -\frac{2\pi i}{\omega + i0} \begin{pmatrix} \hat{\Delta} & 0 \\ 0 & \hat{\Delta}_\phi \end{pmatrix}.
\]
(19)

Here the matrix \( \hat{\Delta} \) is a direct sum of two matrices, each being projected onto \( I \) or \( C \) subspaces:
\[
\hat{\Delta} = \hat{R} \hat{\delta}^{-1} \hat{R} + \hat{T} \hat{\delta}^{-1} \hat{T}.
\]
(20)
The matrix \( \hat{\delta} \) has a similar structure:
\[
\hat{\delta} = \hat{\Delta}_\phi \hat{R} + \hat{T} \hat{\Delta}_\phi \hat{T}.
\]
(21)
The inverse matrix is also a direct sum with elements defined in two orthogonal to each other subspaces \( I \) of insulating channels (with the projector \( \hat{R} \)) and \( C \) of conducting channels (with the projector \( \hat{T} \)):
\[
\hat{\Delta} = \hat{\Delta}_I \oplus \hat{\Delta}_C.
\]
(22)
These matrices are defined as follows. To construct $\hat{\Delta}_I$ one has to project $\Delta_\theta$ onto the subspace $I$ of insulating channels or, in other words, consider all non-zero matrix elements of the matrix $\hat{R} \Delta_\theta \hat{R}$ because $\hat{R}$ is the projector onto that subspace. The inversion of the projected matrix also belongs to the $I$-subspace of insulating channels and is called $\hat{\Delta}_I$. Analogously, to construct $\Delta_C$ one has to project $\Delta_\theta$ onto the $C$-subspace of conducting channels or, in other words, consider all non-zero matrix elements of the matrix $\hat{T} \Delta_\theta \hat{T}$ because $\hat{T}$ is the projector onto that subspace. The inversion of the projected matrix also belongs to the subspace $C$ of conducting channels and is called $\Delta_C$.

There is an obvious duality: interchanging $\hat{R} \leftrightarrow \hat{T}$ and $\theta \leftrightarrow \phi$ is equivalent to interchange

$$\Delta_C \leftrightarrow \Delta_I. \quad (23)$$

This is the generalization of the well-known duality in the single-channel problem between weak link (insulating phase) and weak scatterer (conducting phase) [12,13]. In a generic situation of a multi-channel Luttinger Liquid the generalized duality relates two phases: one of them has $n$ insulating and $N - n$ conducting channels, another ("partner") phase has $N - n$ insulating and $n$ conducting channels. The simplest example is a perfectly conducting phase (all channels are conductors) with $\Delta_{\text{cond}} = \Delta_\theta$ against a perfectly insulating phase (all channels are insulators) with $\Delta_{\text{ins}} = \Delta = \hat{\Delta}_{\text{cond}}^{-1}$.

This duality connects some "partner" phases but does not relate scaling dimensions of similar perturbations in seemingly unrelated phases. Nevertheless, as we will show below there are intimate relations between scaling dimensions beyond the one discussed above. We will leave a general analysis for future investigations. In this letter we will focus on a two-channel liquid to reveal the universality of the relation between scaling dimensions which was observed in [8].

To analyze the stability of a phase, first of all, we have to parameterize perturbations in terms of the incoming fields. It is easy since we provide the description of the stability of the phases where each channel is either perfectly conducting or insulating and all inter- and intra-channel particle transfers can be written in terms of incoming fields only. An arbitrary process of simultaneous transfer of few particles with $n_{ij}$ between channels $i$ to channel $j$ is described by the perturbation

$$T = v \cos \left(\sum_{ij} \left(n_{ij} \theta^+_{ij} \pm n_{ij} \theta^-_{ij}\right)\right), \quad (24)$$

where $\theta^+_{ij} = \theta^+_{in,i} - \theta^+_{in,j}$ and $\theta^-_{ij} = \theta^-_{in,i} - (R_i - T_j) \theta^-_{in,j}$ are built from the fields defined after eq. (17). The scaling dimension of this perturbation is given by the sum of two dimensions since $\theta^+_{ij}$ and $\theta^-_{ij}$ are independent of each other. According to eq. (19) the scaling dimension related to $\theta^+_{in}$ is the same for all phases while the other one is governed by the matrix $\hat{\Delta}$ and depends on the phase which is uniquely defined by the reflection matrix $\hat{R}$.

For the particular problem of a two-channel Luttinger Liquid with no particle transfers between channels due to the distinct nature of the particles (fermions and phonons in the paper [8]) the only perturbations allowed are intra-channel ones. The most generic perturbation intra-channel transfer $T = \cos \left(n_1 \theta^+_{in} + n_2 \theta^-_{in}\right)$ has the scaling dimension

$$D_{n_1, n_2}(R_1, R_2) = n_1^2 \Delta_{11} + n_2^2 \Delta_{22} + 2n_1n_2 \Delta_{12} \quad (25)$$

that depends on the phase which is tried for stability. The phase is uniquely described by the diagonal matrix elements of the matrix $\hat{R} = \text{diag}(R_1, R_2)$. The scaling dimension of the phase where both channels are conducting (CC-phase corresponding to $R_1 = R_2 = 0$) is given by

$$D_{n_1, n_2}(0, 0) = n_1^2 \Delta_{11} + n_2^2 \Delta_{22} + 2n_1n_2 \Delta_{12} \quad (26)$$

The I1-phase ($R_1 = R_2 = 1$) is restored by the duality

$$D_{n_1, n_2}(1, 1) = n_1^2 \Delta_{11} + n_2^2 \Delta_{22} + 2n_1n_2 \Delta_{12} \quad (27)$$

For IC-phases ($R_1 = 1, R_2 = 0$) and by duality for CI-phases ($R_1 = 0, R_2 = 1$) we get

$$D_{n_1, n_2}(1, 0) = n_1^2 \frac{\Delta_{11}}{\Delta_{12}} + n_2^2 \frac{\Delta_{22}}{\Delta_{12}}, \quad D_{n_1, n_2}(0, 1) = n_1^2 \frac{\Delta_{11}}{\Delta_{12}} + n_2^2 \frac{\Delta_{22}}{\Delta_{12}}. \quad (28)$$

In the paper [8] only one-particle perturbations in the fermion channel ($n_1 = 1, n_2 = 0$) were considered. The second channel did not contain renormalizable perturbation since phonon scattering is given by a local quadratic perturbation which at low energy leads to either $R_2 = 0$ (local mass distortion) or $R_2 = 1$ (local pinning). The weak-scatterer and weak-link scaling dimensions were defined as

$$\Delta_{ws}(R_2) = D_{10}(0, R_2), \quad \Delta_{wl}(R_2) = D_{10}(1, R_2). \quad (29)$$

One can see from eqs. (26)–(28) that

$$\Delta_{ws}(R_2) \Delta_{wl}(R_2) = 1 \quad (30)$$

and the duality observed in [8] is just one particular case of general relations, eqs. (26)–(28).

In conclusion, we have devised a new approach to calculate the scaling dimensions of all local perturbations in multi-channel Luttinger Liquids. Since the scaling dimension of perturbation defines the inter-phase boundary on the phase diagram, the duality relations play the role of constraints for the phase diagram. The phase diagram for a low-dimensional strongly correlated system must be universal in the space of the “generalized Luttinger parameters” $\Delta_{ij}$ and $\theta_{ij}$. Using the duality in two-channel liquids enabled us to prove the absence of a new phase for fermions coupled to phonons [8]. Although the analysis of
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numerous practically important realizations was beyond the scope of this letter we believe that the scheme will prove very useful and will be used to determine the existence of new stable phases of multi-channel Luttinger Liquids.

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