OPTIMAL CONTROL FOR THE INFINITY OBSTACLE PROBLEM

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Abstract. In this note, we show that a natural optimal control problem for the \( \infty \)-obstacle problem admits an optimal control which is also an optimal state. Moreover, we show the convergence of the minimal value of an optimal control problem for the \( p \)-obstacle problem to the minimal value of our optimal control problem for the \( \infty \)-obstacle problem, as \( p \to \infty \).

1. Introduction

The obstacle problem corresponding to an obstacle \( f \) in
\[
W_g^{1,2}(\Omega) = \{ u \in W^{1,2}(\Omega) : u = g \text{ on } \partial \Omega \}
\]
consists of minimizing the Dirichlet energy
\[
\int_\Omega |Du(x)|^2 \, dx
\]
over the set
\[
K_{f,g}^2 = \{ u \in W^{1,2}_g(\Omega) : u(x) \geq f(x) \text{ in } \Omega \}
\]
where \( \Omega \subset \mathbb{R}^n \) is a bounded and smooth domain, \( Du \) is the gradient of \( u \), and \( g \in tr(W^{1,2}(\Omega)) \) with \( tr \) the trace operator. In [1], the equality \( u = g \) on \( \partial \Omega \) is in the sense of trace. This problem is used to model the equilibrium position of an elastic membrane whose boundary is held fixed at \( g \) and is forced to remain above a given obstacle \( f \). It is known that the obstacle problem admits a unique solution \( v \in K_{f,g}^2 \). That is, there is a unique \( v \in K_{f,g}^2 \) such that
\[
\int_\Omega |Dv(x)|^2 \, dx \leq \int_\Omega |Du(x)|^2 \, dx, \quad \forall u \in K_{f,g}^2.
\]

In [3] Adams, Lenhart and Yong introduced an optimal control problem for the obstacle problem by studying the minimizer of the functional
\[
J_2(\psi) = \frac{1}{2} \int_\Omega (|T_2(\psi)|^2 + |D\psi|^2) \, dx.
\]
In the above variational problem, following the terminology in control theory [10], \( \psi \) is called the control variable and \( T_2(\psi) \) is the corresponding state. The control \( \psi \) lies in the space \( W_0^{1,2}(\Omega) \), the state \( T_2(\psi) \) is the unique solution for the obstacle problem corresponding to the obstacle \( \psi \) and the profile \( z \) is in \( L^2(\Omega) \). The authors proved that there exists a unique minimizer \( \bar{\psi} \in W_0^{1,2}(\Omega) \) of the functional \( J_2 \). Furthermore, they showed that \( T_2(\bar{\psi}) = \psi \).

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Following suit, for $1 < p < \infty$, and $z \in L^p(\Omega)$, Lou in [17] considered the variational problem of minimizing the functional

\[
(P_p) \quad \bar{J}_p(\psi) = \frac{1}{p} \int_{\Omega} |T_p(\psi) - z|^p + |D\psi|^p \, dx
\]

for $\psi \in W^{1,p}_0(\Omega) := \{ u \in W^{1,p}(\Omega) : u = 0 \text{ on } \partial\Omega \}$ and established that the problem admits a minimizer $\bar{\psi}$. Here $T_p(\psi)$ is the unique solution for the $p-$obstacle problem with obstacle $\psi \in W^{1,p}_0(\Omega)$, see [6] and references therein for discussions about the $p$-obstacle problem. We remind the reader that the $p$–obstacle problem with obstacle $f \in W^{1,p}_0(\Omega)$ refers to the problem of minimizing the $p$–Dirichlet energy

\[
\int_{\Omega} |Du(x)|^p \, dx
\]

among all functions in the class

\[
\mathbb{K}^p_{f,g} = \{ u \in W^{1,p}(\Omega) : u \geq f \text{ in } \Omega \text{ and } u = g \text{ on } \partial\Omega \},
\]

with $g \in tr(W^{1,p}(\Omega))$. It is further shown in [17] that, as in the case of $p = 2$, $T_p(\bar{\psi}) = \bar{\psi}$.

For the boundary data $g \in Lip(\partial\Omega)$, letting $p \to \infty$, one obtains a limiting variational problem of $L^\infty$-type which is referred in the literature as the infinity obstacle problem or $\infty$-obstacle problem (see [20]). That is, given an obstacle $f \in W^{1,\infty}_g(\Omega)$ one considers the minimization problem:

(1.3) \quad Finding $u_{\infty} \in \mathbb{K}^\infty_{f,g}$: $||Du_{\infty}||_{L^\infty} = \inf_{u \in \mathbb{K}^\infty_{f,g}} ||Du||_{L^\infty},$

where

\[
\mathbb{K}^\infty_{f,g} = \{ u \in W^{1,\infty}(\Omega) : v \geq f \text{ in } \Omega \text{ and } u = g \text{ on } \partial\Omega \},
\]

and $|| \cdot ||_{L^\infty} := ess sup | \cdot |$.

It is established in [20] that the minimization problem (1.3) has a solution

(1.4) \quad u_{\infty} := u_{\infty}(f) \in \mathbb{K}^\infty_{f,g}

which verifies

(1.5) \quad - \Delta u_{\infty} \geq 0 \text{ in } \Omega \text{ in a weak sense}.

More importantly, the authors in [20] characterize $u_{\infty}$ as the smallest infinity superharmonic function on $\Omega$ that is larger than the obstacle $f$ and equals $g$ on the boundary. Thus for a fixed $F \in Lip(\partial\Omega)$, this generates an obstacle to solution operator

\[
T_{\infty} : W^{1,\infty}_F(\Omega) \longrightarrow W^{1,\infty}_F(\Omega)
\]

defined by

(1.6) \quad T_{\infty}(f) := u_{\infty}(f) \in W^{1,\infty}_F(\Omega), \quad f \in W^{1,\infty}_F(\Omega),

where

\[
W^{1,\infty}_F(\Omega) := \{ u \in W^{1,\infty}(\Omega) : u = F \text{ on } \partial\Omega \}.
\]

In this note, we consider a natural optimal control problem for the infinity obstacle problem. More precisely, for $F \in Lip(\partial\Omega)$ and for $z \in L^\infty(\Omega)$ fixed, we introduce the functional

\[
J_{\infty}(\psi) = \max \{ ||T_{\infty}(\psi) - z||_{L^\infty}, ||D\psi||_{L^\infty} \}, \quad \psi \in W^{1,\infty}_F(\Omega)
\]
and study the problem of existence of $\psi_\infty \in W^{1,\infty}_F(\Omega)$ such that:

$$(P_\infty) \quad J_\infty(\psi_\infty) \leq J_\infty(\psi), \quad \forall \ \psi \in W^{1,\infty}_F(\Omega).$$

In deference to optimal control theory, a function $\psi_\infty$ satisfying $(P_\infty)$ is called an optimal control and the state $T_\infty(\psi_\infty)$ is called an optimal state.

Several variants of control problems where the control variable is the obstacle have been studied by different authors since the first of such works appeared in [3]. The literature is vast, but to mention a few, in [2] the authors studied a generalization of [3] by adding a source term. In [1] a similar problem is studied when the state is a solution to a parabolic variational inequality. In [18] the author studied regularity of the optimal state obtained in [3]. When the state is governed by a bilateral variational inequality, results are obtained in [9], [10], [11] and [12]. Optimal control for higher order obstacle problems appears in [5] and [14]. Related works where the control variable is the obstacle are also studied in [13, 21] and the references therein.

In this note, we prove that the optimal control problem $(P_\infty)$ associated to $J_\infty$ is solvable. Precisely we show the following result:

**Theorem 1.1.** Assuming that $\Omega \subset \mathbb{R}^n$ is a bounded and smooth domain, $F \in \text{Lip}(\partial \Omega)$, and $z \in L^\infty(\Omega)$, $J_\infty$ admits an optimal control $u_\infty \in W^{1,\infty}_F(\Omega)$ which is also an optimal state, i.e

$$u_\infty = T_\infty(u_\infty).$$

Using also arguments similar to the ones used in the proof of Theorem 1.1, we show the convergence of the minimal value of an optimal control problem associated to $\bar{J}_p$ to the minimal value of the optimal control problem corresponding to $J_\infty$ as $p$ tends to infinity. Indeed we prove the following result:

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain, $F \in \text{Lip}(\partial \Omega)$, and $z \in L^\infty(\Omega)$. Then setting

$$J_p = (p \bar{J}_p)^{\frac{1}{p}}, \quad C_p = \min_{\psi \in W^{1,p}_F(\Omega)} J_p(\psi) \text{ for } 1 < p < \infty, \quad \text{and} \quad C_\infty = \min_{\psi \in W^{1,\infty}_F(\Omega)} J_\infty(\psi),$$

where $\bar{J}_p$ is as in $(P_p)$, we have

$$\lim_{p \to \infty} C_p = C_\infty.$$

In the proofs of the above results, we use the $p$-approximation technique as in the study of the $\infty$-obstacle problem combined with the classical methods of weak convergence in Calculus of Variations. As in the study of the $\infty$-obstacle problem, here also the key analytical ingredients are the $L^q$-characterization of $L^\infty$ and Hölder’s inequality. The difficulty arises from the fact that the unicity question for the $\infty$-obstacle problem is still an open problem to the best of our knowledge. To overcome the latter issue, we make use of the characterization of the solution of the $\infty$-obstacle problem by Rossi-Teixeira-Urbano [20].
2. Preliminaries

One of the most popular ways of approaching problems related to minimizing a functional of $L^\infty$-type is to follow the idea first introduced by Aronsson in [7] and which involves interpreting an $L^\infty$-type minimization problem as a limit when $p \to \infty$ of an $L^p$-type minimization problem. In this note, this $p$-approximation technique will be used to show existence of an optimal control for $J_\infty$. In order to prepare for our use of the $p$-approximation technique, we are going to start this section by discussing some related $L^p$-type variational problems.

Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain and $g \in \text{Lip}(\partial \Omega)$. Moreover let $\psi \in W^{1,\infty}_g(\Omega)$ be fixed and $1 < p < \infty$. Then as described earlier the $p$-obstacle problem with obstacle $\psi$ corresponds to finding a minimizer of the functional

\[
I_p(v) = \int_{\Omega} |Dv(x)|^p \, dx
\]

over the space $K^p_{\psi,g} = \{v \in W^{1,p}(\Omega) : v \geq \psi, \text{ and } v = g \text{ on } \partial \Omega\}$. The energy integral (2.1) admits a unique minimizer $u_p \in K^p_{\psi,g}$. The minimizer $u_p$ is not only $p$-superharmonic, i.e $\Delta_p u_p \leq 0$, but is also a weak solution to the following system

\[
\begin{cases}
-\Delta_p u \geq 0 & \text{in } \Omega \\
-\Delta_p u (u - \psi) = 0 & \text{in } \Omega \\
u \geq \psi & \text{in } \Omega
\end{cases}
\]

(2.2)

where $\Delta_p$ is the $p$-Laplace operator given by

\[
\Delta_p u := \text{div}(|Du|^{p-2}Du).
\]

Moreover, it is known that the $p$-obstacle problem is equivalent to the system (2.2) (see [16] or [19]) and hence we will refer to (2.2) as the $p$-obstacle problem as well. On the other hand, by the equivalence of weak and viscosity solutions established in [19] (and [15]) $u_p$ is also a viscosity solution of (2.2) according to the following definition.

**Definition 2.1.** A function $u \in C(\Omega)$ is said to be a viscosity subsolution (supersolution) to

\[
F(x,u,Du,D^2u) = 0 \quad \text{in } \Omega
\]

\[
u = 0 \quad \text{in } \partial \Omega
\]

(2.3)

if for every $\phi \in C^2(\Omega)$ and $x_0 \in \Omega$ whenever $\phi - u$ has a minimum (resp. maximum) in a neighborhood of $x_0$ in $\Omega$ we have:

\[
F(x,u,D\phi,D^2\phi) \leq 0 \quad (\text{resp. } \geq 0).
\]

The function $u$ is called a viscosity solution of (2.3) in $\Omega$ if $u$ is both viscosity subsolution and viscosity supersolution of (2.3) in $\Omega$.

The asymptotic behavior of the sequence of minimizers $(u_p)_{p>1}$ as $p$ tends to infinity has been investigated in [20]. In fact, in [20], it is established that for a fixed $\psi \in W^{1,\infty}_g(\Omega)$, there exists $u_\infty = u_\infty(\psi) \in K^\infty_{\psi,g} = \{v \in W^{1,\infty}_g(\Omega) : v \geq \psi\}$ such that $u_p \to u_\infty$ locally uniformly...
in $\Omega$, and that for every $q \geq 1$, $u_p$ converges to $u_\infty$ weakly in $W^{1,q}(\Omega)$. Furthermore, $u_\infty$ is a solution to the $\infty$-obstacle problem
\begin{equation}
\min_{v \in K_{\infty,\psi,g}} \|Dv\|_\infty \tag{2.4}
\end{equation}

For $\Omega$ convex (see [8]), the variational problem (2.4) is equivalent to the minimization problem
\[ \min_{v \in K_{\infty,\psi,g}} \mathcal{L}(v), \]
where
\[ \mathcal{L}(v) = \inf_{(x,y) \in \Omega^2, x \neq y} \frac{|v(x) - v(y)|}{|x - y|}. \]
Moreover, in [20], it is show that $u_\infty$ is a viscosity solution to the following system.
\[ \begin{cases}
-\Delta_\infty u \geq 0 & \text{in } \Omega \\
-\Delta_\infty u (u - \psi) = 0 & \text{in } \Omega \\
u \geq \psi & \text{in } \Omega
\end{cases} \]
where $\Delta_\infty$ is the $\infty$-Laplacian and is defined by
\[ \Delta_\infty u = \langle D^2 u D u, Du \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j}. \]

Recalling that $u$ is said to be infinity superharmonic or $\infty$-superharmonic, if $-\Delta_\infty u \geq 0$ in the viscosity sense, we have the following characterization of $u_\infty$ in terms of infinity superharmonic functions and it is proven in [20]. We would like to emphasize that this will play an important role in our arguments.

**Lemma 2.2.** Setting
\[ \mathcal{F}^+ = \{ v \in C(\Omega), -\Delta_\infty v \geq 0 \text{ in } \Omega \text{ in the viscosity sense} \} \]
and
\[ \mathcal{F}^+_\psi = \{ v \in \mathcal{F}^+, v \geq \psi \text{ in } \Omega, and v = \psi \text{ on } \partial \Omega \}, \]
we have
\begin{equation}
T_\infty(\psi) = u_\infty = \inf_{v \in \mathcal{F}_\psi^+} v, \tag{2.5}
\end{equation}
with $T_\infty$ as defined earlier in (1.6).

Lemma 2.2 implies the following characterization of infinity superharmonic functions as fixed points of $T_\infty$. This characterization plays a key role in our $p$-approximation scheme for existence.

**Lemma 2.3.** Assuming that $u \in W^{1,\infty}_g(\Omega)$, $u$ being infinity superharmonic is equivalent to $u$ being a fixed point of $T_\infty$, i.e.
\[ T_\infty(u) = u. \]
Proof. Let \( u \in W^{1,\infty}(\Omega) \) be an infinity superharmonic function and \( v \) be defined by \( v = T_\infty(u) \). Then clearly the definition of \( v \) and lemma 2.2 imply \( v \geq u \). On the other hand, since \( u \in W^{1,\infty}(\Omega) \) and is an infinity superharmonic function, we deduce from lemma 2.2 that \( u \geq T_\infty(u) = v \). Thus, we get \( T_\infty(u) = u \). Now if \( u = T_\infty(u) \), then using again lemma 2.2 or (1.4)-(1.6), we obtain \( u \) is an infinity superharmonic function. Hence the proof of the lemma is complete.

To run our \( p \)-approximation scheme for existence, another crucial ingredient that we will need is an appropriate characterization of the limit of sequence of solution \( w_p \) of the \( p \)-obstacle problem (2.2) with obstacle \( \psi_p \) under uniform convergence of both \( w_p \) and \( \psi_p \). Precisely, we will need the following lemma.

Lemma 2.4. If \( w_p \) is a solution to the \( p \)-obstacle problem (2.2) with obstacle \( \psi_p \) that is, \( w_p \) satisfies

\[
\begin{align*}
-\Delta_p w_p & \geq 0 & \text{in } & \Omega \\
-\Delta_p w_p (w_p - \psi_p) & = 0 & \text{in } & \Omega \\
w_p & \geq \psi_p & \text{in } & \Omega 
\end{align*}
\]

in the viscosity sense and if also that \( w_p \rightarrow u_\infty \) and \( \psi_p \rightarrow \psi_\infty \) locally uniformly in \( \overline{\Omega} \), then \( u_\infty \) is a solution in the viscosity sense of the following system

\[
\begin{align*}
-\Delta_\infty u_\infty & \geq 0 & \text{in } & \Omega \\
-\Delta_\infty u_\infty (u_\infty - \psi_\infty) & = 0 & \text{in } & \Omega \\
u_\infty & \geq \psi_\infty & \text{in } & \Omega.
\end{align*}
\]

Proof. First of all, note that since \( w_p \geq \psi_p \), \( -\Delta_p w_p \geq 0 \) in the viscosity sense in \( \Omega \) for every \( p \), \( w_p \rightarrow u_\infty \), and \( \psi_p \rightarrow \psi_\infty \) both locally uniformly in \( \overline{\Omega} \), and \( \overline{\Omega} \) is compact, we have \( w_\infty \geq \psi_\infty \) and \( -\Delta_\infty w_\infty \geq 0 \) in the viscosity sense in \( \Omega \). It thus remains to prove that \( -\Delta_\infty u_\infty (u_\infty - \psi_\infty) = 0 \) in \( \Omega \) which (because of \( w_\infty \geq \psi_\infty \) in \( \Omega \)) is equivalent to \( -\Delta_\infty u_\infty = 0 \) in \( \{ w_\infty > \psi_\infty \} \). Thus to conclude the proof, we are going to show \( -\Delta_\infty w_\infty = 0 \) in \( \{ w_\infty > \psi_\infty \} \). To that end, fix \( y \in \{ w_\infty > \psi_\infty \} \). Then, by continuity there exists an open neighborhood \( V \) of \( y \) in \( \Omega \) such that \( V \) is a compact subset of \( \Omega \), and a small real number \( \delta > 0 \) such that \( w_\infty > \delta > \phi_\infty \) in \( V \). Thus, from \( w_p \rightarrow w_\infty \), \( \psi_p \rightarrow \psi_\infty \) locally uniformly in \( \overline{\Omega} \), and \( V \) compact subset of \( \Omega \), we infer that for sufficiently large \( p \)

\[
w_p > \delta > \psi_p \quad \text{in } \quad V.
\]

On the other hand, since \( w_p \) is a solution to the \( p \) obstacle problem (2.2) with obstacle \( \psi_p \), then clearly \( -\Delta_p w_p = 0 \) in \( \{ w_p > \psi_p \} := \{ x \in \Omega : \ w_p(x) > \psi_p(x) \} \). Thus, (2.8) imply \( -\Delta_p w_p = 0 \) in the sense of viscosity in \( V \). Hence, recalling that \( w_p \rightarrow w_\infty \) locally uniformly in \( \overline{\Omega} \) and letting \( p \rightarrow \infty \), we obtain

\[
-\Delta_\infty w_\infty = 0 \quad \text{in the sense of viscosity in } \quad V.
\]

Thus, since \( y \in V \) is arbitrary in \( \{ w_\infty > \psi_\infty \} \), then we arrive to

\[
-\Delta_\infty w_\infty = 0 \quad \text{in the sense of viscosity in } \quad \{ w_\infty > \psi_\infty \}.
\]
thereby ending the proof of the lemma.

On the other hand, to show the convergence of the minimal values of $J_p$ to that of $J_\infty$, we will make use of the following elementary results.

**Lemma 2.5.** Suppose $\{a_p\}$ and $\{b_p\}$ are nonnegative sequences with

$$\liminf_{p \to \infty} a_p = a \quad \text{and} \quad \liminf_{p \to \infty} b_p = b.$$  

Then

$$\liminf_{p \to \infty} \max\{a_p, b_p\} = \max\{a, b\}.$$  

**Proof.** Let $\{b_{p_k}\}$ be a subsequence converging to $b = \liminf_{p \to \infty} b_p$. Then

$$\lim_{k \to \infty} \max\{a_{p_k}, b_{p_k}\} = \max\{a, b\}.$$  

Since the lim inf is the smallest limit point we have

$$\liminf_{p \to \infty} \max\{a_p, b_p\} \leq \max\{a, b\}. \quad (2.9)$$  

On the other hand

$$a_p, b_p \leq \max\{a_p, b_p\}, \quad \text{for all} \quad p.$$  

Thus

$$b = \liminf_{p \to \infty} b_p \leq \liminf_{p \to \infty} \max\{a_p, b_p\},$$  

and likewise

$$a \leq \liminf_{p \to \infty} \max\{a_p, b_p\}.$$  

Consequently

$$\liminf_{p \to \infty} \max\{a_p, b_p\} \geq \max\{a, b\}. \quad (2.10)$$  

Finally (2.9) and (2.10) conclude the proof of the lemma.

**Lemma 2.6.** Suppose $\{a_p\}$ and $\{b_p\}$ are nonnegative sequences with

$$\liminf_{p \to \infty} a_p = a \quad \text{and} \quad \liminf_{p \to \infty} b_p = b.$$  

Then

$$\liminf_{p \to \infty} (a_p^p + b_p^p)^{1/p} = \max\{a, b\}.$$  

**Proof.** It follows directly from the trivial inequality

$$2^{1/p} \max\{a_p, b_p\} \geq (a_p^p + b_p^p)^{1/p} \geq \max\{a_p, b_p\}, \quad \forall p \geq 1,$$

lemma 2.5 and the fact that $\liminf (a_n b_n) = (\lim_n a_n)(\liminf_n b_n)$ if $\lim_n a_n > 0$.

3. **Existence of Optimal Control for $J_\infty$ and Limit of $C_p$**

In this section, we show the existence of an optimal control for $J_\infty$ and show that $C_p$ converges to $C_\infty$ as $p \to \infty$. We divide it in two subsections. In the first one we show existence of an optimal control for $J_\infty$ via the $p$-approximation technique, and in the second one we show that $C_p$ converges to $C_\infty$ as $p$ tends to infinity.
3.1. Existence of optimal control. In this subsection, we show the existence of a minimizer of $J_\infty$ via the $p$-approximation technique using solutions of the optimal control for $J_p$. For this end, we start by recalling some optimality facts about $J_p$ inherited from $\bar{J}_p$ (see (P) for its definition) and mentioned in the introduction. For $\Omega \subset \mathbb{R}^n$ a bounded and smooth domain, $z \in L^\infty(\Omega)$, $F \in Lip(\partial\Omega)$, and $1 < p < \infty$, we recall that the functional $J_p$ is defined by the formula

$$J_p(\psi) = \left[ \int_\Omega |T_p(\psi) - z|^p + |D\psi|^p dx \right]^{1/p}, \quad \psi \in W^{1,p}_F(\Omega)$$

and that the optimal control problem for $J_p$ is the variational problem of minimizing $J_p$, namely

$$\inf_{\psi \in W^{1,p}_F(\Omega)} J_p(\psi)$$

over $W^{1,p}_F(\Omega)$, where

$$W^{1,p}_F(\Omega) = \{ \psi \in W^{1,p}(\Omega) : \psi = F \text{ on } \partial\Omega \},$$

and $T_p(\psi)$ is the solution to the $p$-obstacle problem with obstacle $\psi$. Moreover, as for the functional $\bar{J}_p$, $J_p$ also admits a minimizer $\psi_p \in W^{1,p}_F(\Omega)$ verifying

$$T_p(\psi_p) = \psi_p.$$

As mentioned in the introduction, for more details about the latter results, see [3] for $p = 2$ and see [17] for $p > 2$.

To continue, let us pick $\eta \in W^{1,\infty}_F(\Omega)$. Since $\eta$ competes in the minimization problem (3.2), we have

$$\int_\Omega |D\eta|^p dx \leq J_p(\eta) = \int_\Omega |T_p(\eta) - z|^p + |D\eta|^p dx.$$

Since $\overline{\Omega}$ is compact and $T_p(\eta) \to T_\infty(\eta)$ as $p \to \infty$ locally uniformly on $\overline{\Omega}$ (which follows from the definition of $T_\infty(\eta)$), we deduce that for $p$ very large

$$\int_\Omega |D\eta|^p dx \leq M\Omega$$

for some $M$ which depends only on $||\eta||_{W^{1,\infty}}$, $||T_\infty(\eta)||_{C^0}$ and $||z||_\infty$. Furthermore, let us fix $1 < q < p$. Then by using Holder’s inequality, we can write

$$\int_\Omega |D\eta|^q dx \leq \left\{ \int_\Omega (|D\eta|^q)^{p/q} dx \right\}^{q/p} \Omega^{\frac{p-q}{p}}$$

and we obtain by using (3.4) that for $p$ very large

$$\int_\Omega |D\eta|^q dx \leq M^q \Omega^{\frac{q}{p}} \Omega^{\frac{p-q}{p}}$$

and raising both sides to $1/q$, we derive that for $p$ very large, there holds

$$||D\eta||_{L^q} \leq M^{1/q} \Omega^{1/q},$$

with $|| \cdot ||_{L^q}$ denoting the classical $L^q(\Omega)$-norm. This shows, that the sequence $\{\psi_p\}$ is bounded in $W^{1,q}_F(\Omega)$ in the gradient norm for every $q$ with a bound independent of $q$, ...
and by Poincaré’s inequality, that for every \( 1 < q < \infty \), the sequence \( \{ \psi_p \} \) is bounded in \( W_{1,q}^1(\Omega) \) in the standard \( W_{1,q}^1(\Omega) \)-norm. Therefore, by classical weak compactness arguments, we have that, up to a subsequence,

\[
\psi_p \rightarrow \psi_\infty, \quad \text{as } p \rightarrow \infty \quad \text{locally uniformly in } \overline{\Omega} \quad \text{and weakly in } W_{1,q}^1(\Omega) \quad \forall \ 1 < q < \infty.
\]

Notice that consequently \( ||D\psi_\infty||_{L^q} \leq M|\Omega|^{1/q} \) for all \( 1 < q < \infty \). Thus, we deduce once again by Poincaré’s inequality that

\[
(3.6)
\psi_\infty \in W_{1,\infty}^1(\Omega).
\]

We want now to show that \( \psi_\infty \) is a minimizer of \( J_\infty \). To that end, we make the following observation which is a consequence of lemma 2.4.

**Lemma 3.1.** The function \( \psi_\infty \) is a fixed point of \( T_\infty \), namely

\[
T_\infty(\psi_\infty) = \psi_\infty,
\]

and the solutions \( T_p(\psi_p) \) of the \( p \)-obstacle problem with obstacle \( \psi_p \) verify: as \( p \rightarrow \infty \),

\[
T_p(\psi_p) \rightarrow T_\infty(\psi_\infty) \quad \text{locally uniformly in } \overline{\Omega} \quad \text{and weakly in } W_{1,q}^1(\Omega) \quad \forall \ 1 < q < \infty.
\]

**Proof.** We know that \( T_p(\psi_p) = \psi_p \) (see (3.3)) Thus using (3.6) and Lemma 2.4 with \( \phi_p = \psi_p \) and \( w_p = T_p(\psi_p) = \psi_p \), we have \( T_p(\psi_p) \rightarrow \psi_\infty \) locally uniformly in \( \overline{\Omega} \), weakly in \( W_{1,q}^1(\Omega) \) for every \( 1 < q < \infty \), and \( \psi_\infty \) is an infinity superharmonic. Thus, recalling (3.7), we have lemma 2.3 implies \( T_\infty(\psi_\infty) = \psi_\infty \). Hence the proof of the lemma is complete. \( \square \)

Now, with all the ingredients at hand, we are ready to show that \( \psi_\infty \) is a minimizer of \( J_\infty \). Indeed, we are going to show the following proposition:

**Proposition 3.2.** Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded and smooth domain, \( F \in \text{Lip}(\partial \Omega) \) and \( z \in L^\infty(\Omega) \). Then \( \psi_\infty \) is a minimizer of \( J_\infty \) on \( W_{1,\infty}^1(\Omega) \). That is:

\[
J_\infty(\psi_\infty) = \min_{\eta \in W_{1,\infty}^1(\Omega)} J_\infty(\eta)
\]

**Proof.** We first introduce for \( n < p < \infty \) and \( \psi \in W_{1,p}^1(\Omega) \)

\[
H_p(\psi) = \max\{||T_p(\psi) - z||_{\infty}, ||D\psi||_{\infty}\}
\]

which is well defined by Sobolev Embedding Theorem. Then for any \( \eta \in W_{1,\infty}^1(\Omega) \)

\[
\int_{\Omega} |D\psi_p|^p dx \leq J_p^p(\eta) = \int_{\Omega} ((|T_p(\eta) - z|^p + |D\eta|^p) dx.
\]

Therefore, using the trivial identity \( (|a|^p + |b|^p)^{\frac{1}{p}} \leq 2^{\frac{1}{p}} \max\{|a|, |b|\} \), we get

\[
\left( \int_{\Omega} |D\psi_p|^p dx \right)^{1/p} \leq 2^{1/p}|\Omega|^{1/p} H_p(\eta).
\]

If we now set

\[
(3.8) \quad I_p = \inf_{\eta \in W_{1,\infty}^1(\Omega)} H_p(\eta),
\]

...
we deduce that
\[
\left( \int_{\Omega} |D\psi_p|^p \, dx \right)^{1/p} \leq 2^{1/p} |\Omega|^{1/p} I_p.
\]
Let us fix \( q \) such that \( n < q < \infty \). Then for \( q < p < \infty \), by proceeding as in \(3.5\), we obtain
\[
||D\psi_p||_{L^q} \leq 2^{1/p} I_p |\Omega|^{1/q}.
\]
Similarly,
\[
||T_p(\psi_p) - z||_{L^q} \leq 2^{1/p} I_p |\Omega|^{1/q}.
\]
Thus
\[
(3.9) \quad \max\{||T_p(\psi_p) - z||_{L^q}, ||D\psi_p||_{L^q}\} \leq 2^{1/p} I_p |\Omega|^{1/q}.
\]
For any \( \eta \in W^{1,\infty}_F(\Omega) \) we also have \( I_p \leq H_p(\eta) \) and \( \lim_{p \to \infty} I_p \leq \liminf_{p \to \infty} H_p(\eta) \). Thus, since \( \psi_p \) converges weakly in \( W^{1,q}(\Omega) \) to \( \psi_\infty \) as \( p \to \infty \) and \( (3.9) \) holds, then by weak lower semicontinuity, we conclude that
\[
||D\psi_\infty||_{L^q} \leq \liminf_{p \to \infty} ||D\psi_p||_{L^q} \leq |\Omega|^{1/q} \liminf_{p \to \infty} H_p(\eta).
\]
Moreover, since \( T_p(\eta) \) converges locally uniformly on \( \overline{\Omega} \) to \( T_\infty(\eta) \) as \( p \to \infty \) and \( \overline{\Omega} \) is compact, then clearly
\[
\lim_{p \to \infty} H_p(\eta) = J_\infty(\eta),
\]
and hence
\[
||D\psi_\infty||_{L^q} \leq \inf_{\eta \in W^{1,\infty}_F(\Omega)} J_\infty(\eta) \leq J_\infty(\psi_\infty).
\]
(3.10)
Since this holds for any element \( \eta \) of \( W^{1,\infty}_F(\Omega) \), we conclude that by taking the infimum over \( W^{1,\infty}_F(\Omega) \) and letting \( q \to \infty \)
\[
||D\psi_\infty||_{L^q} \leq J_\infty(\psi_\infty).
\]
Using lemma \(3.1\) and equation \(3.9\) combined with Rellich compactness Theorem or the continuous embedding of \( L^\infty \) into \( L^q \), we conclude that
\[
||T_\infty(\psi_\infty) - z||_{L^q} = \lim_{p \to \infty} ||T_p(\psi_p) - z||_{L^q} \leq |\Omega|^{1/q} \liminf_{p \to \infty} H_p(\eta).
\]
Thus, as above letting \( q \) goes to infinity and taking infimum in \( \eta \) over \( W^{1,\infty}_F(\Omega) \), we also have
\[
(3.11) \quad ||T_\infty(\psi_\infty) - z||_{L^q} \leq \inf_{\eta \in W^{1,\infty}_F(\Omega)} J_\infty(\eta) \leq J_\infty(\psi_\infty).
\]
Finally, from \(3.7\), \(3.10\) and \(3.11\) we deduce
\[
J_\infty(\psi_\infty) = \min_{\eta \in W^{1,\infty}_F(\Omega)} J_\infty(\eta),
\]
as desired.
3.2. Convergence of Minimum Values. In this subsection, we show the convergence of the minimal value of the optimal control problem of $J_p$ to the one of $J_\infty$ as $p \to \infty$, namely Theorem 1.2 via the following proposition:

**Proposition 3.3.** Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain, $F \in \text{Lip}(\partial \Omega)$ and $1 < p < \infty$. Then recalling that $C_p = \min_{\psi \in W^{1,p}_F(\Omega)} J_p(\psi)$ and $C_\infty = \min_{\psi \in W^{1,\infty}_F(\Omega)} J_\infty(\psi)$, we have

$$\lim_{p \to \infty} C_p = C_\infty.$$

**Proof.** Let $\psi_p \in W^{1,p}_F(\Omega)$ and $\psi_\infty \in W^{1,\infty}_F(\Omega)$ be as in subsection 3.1. Then they satisfy $J_p(\psi_p) = C_p$ and $J_\infty(\psi_\infty) = C_\infty$. Moreover, up to a subsequence, we have $\psi_p$ and $\psi_\infty$ verify (3.6) and the conclusions of lemma 3.1. On the other hand, by minimality and Hölder’s inequality, we have

$$J_p(\psi_p) \leq J_\infty(\psi_\infty) \leq 2^{1/p} |\Omega|^{1/p} \max\{||T_p(\psi_\infty) - z||_{\infty}, ||D\psi_\infty||_{\infty}\}.$$

Thus

$$\limsup_{p \to \infty} J_p(\psi_p) \leq J_\infty(\psi_\infty). \tag{3.12}$$

Now we are going to show the following

$$J_\infty(\psi_\infty) \leq \liminf_{p \to \infty} J_p(\psi_p). \tag{3.13}$$

To that end observe that by definition of $J_\infty$, we have

$$J_\infty(\psi_\infty) = \max\{||T_\infty(\psi_\infty) - z||_{\infty}, ||D\psi_\infty||_{\infty}\}. \tag{3.14}$$

Thus, using the $L^q$-characterization of $L^\infty$, we have that (3.14) imply

$$J_\infty(\psi_\infty) = \max\{\lim_{q \to \infty} ||T_\infty(\psi_\infty) - z||_{L^q}, \lim_{q \to \infty} ||D\psi_\infty||_{L^q}\}, \tag{3.15}$$

and by using lemma 2.5, we get

$$J_\infty(\psi_\infty) = \lim_{q \to \infty} \max\{||T_\infty(\psi_\infty) - z||_{L^q}, ||D\psi_\infty||_{L^q}\}. \tag{3.16}$$

On the other hand, by weak lower semicontinuity, and corollary 3.1, we have

$$||D\psi_\infty||_{L^q} \leq \liminf_{p \to \infty} ||D\psi_p||_{L^q}. \tag{3.17}$$

Now, combining (3.16) and (3.17), we obtain

$$J_\infty(\psi_\infty) \leq \liminf_{q \to \infty} \max\{||T_\infty(\psi_\infty) - z||_{L^q}, \liminf_{p \to \infty} ||D\psi_p||_{L^q}\}. \tag{3.18}$$

Next, using lemma 2.6, corollary 3.1 and (3.18), we get

$$J_\infty(\psi_\infty) \leq \liminf_{q \to \infty} \liminf_{p \to \infty} \left\{||T_p(\psi_p) - z||_{L^q} + ||D\psi_p||_{L^q}\right\}^{1/p}. \tag{3.19}$$
To continue, we are going to estimate the right hand side of (3.19). Indeed, using Hölder’s inequality, we have

\[
(\|T_p(\psi_p) - z\|_{L^q})^p = \left\{ \int_{\Omega} |T_p(\psi_p) - z|^q \, dx \right\}^{p/q} \\
\leq \left\{ \int_{\Omega} |T_p(\psi_p) - z|^p \, dx \right\} |\Omega|^{(1-q/p)p/q} \\
= \left\{ \int_{\Omega} |T_p(\psi_p) - z|^p \, dx \right\} |\Omega|^{(1-q/p)p/q}.
\]

Similarly, we obtain

\[
(\|D\psi_p\|_{L^q})^p \leq \left\{ \int_{\Omega} |D\psi_p|^p \, dx \right\} |\Omega|^{(1-q/p)p/q}.
\]

By using the latter two estimates in (3.19), we get

\[
J_\infty(\psi_\infty) \leq \liminf_{q \to \infty} \liminf_{p \to \infty} \left[ \int_{\Omega} \left( |T_p(\psi_p) - z|^p + |D\psi_p|^p \right) \, dx \right]^{1/p} |\Omega|^{(1-q/p)q(1/p)} \\
= \liminf_{q \to \infty} \liminf_{p \to \infty} \left[ \int_{\Omega} \left( |T_p(\psi_p) - z|^p + |D\psi_p|^p \right) \, dx \right]^{1/p} |\Omega|^{1 - \frac{1}{p}} \\
= \lim_{q \to \infty} \left[ |\Omega|^{\frac{q}{1}} \liminf_{p \to \infty} J_p(\psi_p) \right] = \liminf_{p \to \infty} J_p(\psi_p) \\
\leq J_\infty(u_\infty),
\]

proving claim (3.13). Combining (3.12) with (3.20) we obtain

\[
\lim_{p \to \infty} J_p(\psi_p) = J_\infty(u_\infty),
\]

and recalling that we were working with a possible subsequence, then we have that up to a subsequence

\[
\lim_{p \to \infty} C_p = C_\infty.
\]

Hence, since the limit is independent of the subsequence, we have

\[
\lim_{p \to \infty} C_p = C_\infty
\]

as required. \(\square\)

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