Short Communication

One kind of construction on sunflower with two petals*

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A sunflower (or Δ-system) with k petals and a core Y is a collection of sets S₁, ..., Sₖ such that Sᵢ∩Sⱼ=Y for all i≠j; the sets S₁\Y, ..., Sₖ\Y are petals. In this paper, we first give a sufficient condition for the existence of a sunflower with 2 petals. Let F={A,B,C} be a family of subsets of a set {a₁, ..., aₙ, b₁, ..., bₙ, c₁, ..., cₙ} with \( \sum_{i=1}^{n} a_i = \sum_{r=1}^{m} b_r + \sum_{q=1}^{n} c_q \) and A={a₁, ..., aₘ}, B={b₁, ..., bₙ} and C={c₁, ..., cₙ} are non-increasing lists of nonnegative integers. Suppose that for each r with 1 ≤ r ≤ m, \( \sum_{q=1}^{n} \min[c_q, r] \leq \sum_{p=1}^{m} \min[b_p, r] \) and \( \sum_{q=1}^{n} \min[c_q, r] \), then the family F* contains a sunflower with two petals, where F'={G₁,G₂}, G₁=G[Y∪X] and G₂=[Z∪X] are the subgraphs induced respectively by Y∪X and Z∪X with d₁(v_j)=b_r for all v_j ∈ Y∪X and d₂(v_j)=c_q for all v_j ∈ Z∪X. Moreover, we generalize the consequence to the case of a much more general result.

Key words: Sunflower; family; tripartite graph.

INTRODUCTION

A non-increasing sequence \( π=(d₁, ..., d_n) \) of nonnegative integers is said to be graphic if it is the degree sequence of a graph G on n vertices and G is called a realization of \( π \). Many characterizations of graphic lists are known, of which one of the best explicit characterizations is that by Erdos and Gallai (1960). There have been several proofs of it, including a short constructive proof in Garg et al. (2011).

A k-partite graph is one whose vertex set can be partitioned into k subsets so that no edge has both ends in any one subset. In particular, 2-partite graph and 3-partite graph are also called bipartite graph and tripartite graph respectively.

Let A = (a₁, ..., aₘ) and B = (b₁, ..., bₙ) be two nonincreasing sequences of nonnegative integers. The pair S=(A,B) is said to be bigraphic if there exists a simple bipartite sets X={x₁, ..., xₘ} and Y={y₁, ..., yₙ} such that \( d_π(x_i)=a_i \) for 1≤i≤m and \( d_π(y_i)=b_i \) for 1≤i≤n. In this case, G is referred to as a realization of S. A well-known theorem due to Gale (1957) and Ryser (1957) independently gives a characterization of S that is bigraphic. In Gale (1957), Tripathi et al. (2010) generalized that theorem and provided a good characterization that is bigraphic on lists of intervals.

A sunflower (or Δ-system) with k petals and a core Y is a collection of sets S₁, ..., Sₖ such that \( Sᵢ∩Sⱼ=Y \) for all i≠j; the sets S₁\Y, ..., Sₖ\Y are k petals and we require that none of them is empty. About sunflower, Erdos and Rado

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discovered the so-called Sunflower Lemma (Erdős and Rado, 1960). In this paper, we present a special tripartite graph, which reduces to a sunflower with two petals and a core. Prior to our work, only the degree sequences of bipartite graphs were characterized.

A set system or a family of F is a collection of sets. Because of their intimate conceptual relation to graphs, a set system is often called a hypergraph. A family is k-uniform if all its members are k-element sets. Thus graphs are k-uniform families with k=2.

**THEOREM 1**

Let $F=\{A, B, C\}$ be a family of subsets of a set $\{a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_n\}$ with $\sum_{i=1}^{m} a_i = \sum_{p=1}^{n} b_p + \sum_{q=1}^{r} c_q$ and $A=\{a_1, \ldots, a_m\}$, $B=\{b_1, \ldots, b_n\}$ and $C=\{c_1, \ldots, c_n\}$ are non-increasing lists of nonnegative integers. Suppose that for each $r$ with $1 \leq r \leq m$,

$$\sum_{i=1}^{r} a_i \leq \sum_{p=1}^{n} \min\{b_p, r\} + \sum_{q=1}^{r} \min\{c_q, r\},$$

(1)

then the family $F'$ contains a sunflower with two petals, where $F'=\{G_1, G_2\}$, $G_1=GYUX$ and $G_2=[ZUX]$ are the subgraphs induced respectively by YUX and ZUX with $d_{G_1}(v_j) = b_j$ for all $v_j \in YUX$ and $d_{G_2}(v_j) = b_j$ for all $v_j \in ZUX$.

The consequence of Theorem 1 can be generalized to the case of a much more general result.

**THEOREM 2**

Theorem 2 can be proved by induction on $n(n\geq 3)$ and the proof technique closely follows that of Theorem 1. So a detailed proof will not be given here.

Let $F=\{A, B_1, B_2, \ldots, B_n\}$ be a family of subsets with $\sum_{i=1}^{m} a_i = \sum_{j=1}^{k_1} b_1^j + \cdots + \sum_{j=1}^{k_n} b_n^j$ and $A=\{a_1, \ldots, a_m\}$, $B_1=\{b_1^1, \ldots, b_{k_1}^1\}$, $B_2=\{b_2^1, \ldots, b_{k_2}^2\}$, $\ldots$, $B_n=\{b_n^{k_1}, \ldots, b_n^{k_n}\}$ are non-increasing lists of nonnegative integers. Suppose that for each $r$ with $1 \leq r \leq m$,

$$\sum_{i=1}^{r} a_i \leq \sum_{j=1}^{k_1} \min\{b_1^j, r\} + \cdots + \sum_{j=1}^{k_n} \min\{b_n^j, r\},$$

(2)

then the family $F'$ contains a sunflower with two petals, where $F'=\{G_1, G_2, \ldots, G_n\}$, $G_i=GYUX$ and $G_2=[ZUX]$ are the subgraphs induced respectively by YUX and ZUX with $d_{G_i}(v_j) = b_j$ for all $v_j \in YUX$, $i=1, \ldots, n$.

**Proof of Theorem 1**

For convenience, let $X=\{x_1, \ldots, x_n\}$, $Y=\{y_1, \ldots, y_n\}$ and $Z=\{z_1, \ldots, z_n\}$ be three sets of vertices. We shall construct a special tripartite graph $G$, which will yield the desired sunflower. In fact, $G$ is a realization of degree sequence $\pi=AUW$ with vertex-set $XUYZ$, i.e., $d(x_i)=a_i$ for $1 \leq i \leq m$, $d(y_j)=b_j$ for $1 \leq j \leq n$ and $d(z_j)=c_j$ for $1+n \leq j \leq 2n$. For convenience, we write $YUZ=W$ and maintain that $X$ and $W$ are independent sets. We first construct a graph $G'$ with partite sets $X$, $Y$ and $Z$ satisfying $d(x_i)=a_i$ for $1 \leq i \leq r$ and $d(x_i)<a_i$. We will iteratively remove the deficiency $a_i-d(x_i)$ at vertex $x_i$ while maintaining $d(x_i)=a_i$ for $1 \leq i \leq r$. $d(y_j)$ for $1 \leq j \leq n$ and $d(z_j)$ for $1+n \leq j \leq 2n$. Let $S=\{x_1, \ldots, x_n\}$. Note that there exists a vertex $v(e [X \cup \{x_i\}] \cap \{Y\})=c_i$ for some $X \cup \{x_i\}$, since $d(x_i)=a_i \geq a_i$. To prove the theorem we have to consider two cases depending on the degree of $v_i \in YUZ$ and its neighbourhood's intersection with $X$.

Case 1: Suppose, for some $j$, $v_i \rightarrow x_k$ for some $k>r$ and $v_i \in N(x_k)$ for some $s$. If $l=r$, replace $v_i x_k$ with $v_i x_r$. If $r<s$, replace $v_i x_k$ with $v_i x_r$. If $s=l$, then $v_i x_r$.

Case 2: Suppose, for some $j$, $d(y_j)=b_j$ or $d(y_j)=c_j$ and $v_i \in N(x_k)$ for some $s$. If $l=r$, add the edge $v_i x_k$. If $r<s$, replace $v_i x_k$ with $v_i x_l$. If none of the cases above arise, an application of (1) gives:

$$\sum_{i=1}^{r-1} a_i + d(x_i) = \sum_{i=1}^{r} d(x_i)$$

$$= \sum_{j=1}^{n} \min\{d(y_j), r\} + \sum_{j=n+1}^{2n} \min\{d(y_j), r\}$$

$$= \sum_{j=1}^{n} \min\{b_p, r\} + \sum_{j=n+1}^{2n} \min\{c_q, r\} \geq \sum_{i=1}^{r} a_i.$$ 

Hence $d(x_i) \geq a_i$. Furthermore, $d(x_i) \geq a_i$, and thus $d(x_i) = a_i$. Increasing $r$ by 1 and applying the similar steps leads to the required graph $G'$ with partite sets $X$ and $W$ (that is, $Y$ and $Z$) satisfying $d(x_i)=a_i$ for $1 \leq i \leq m$, $d(y_j)=b_j$ for $1 \leq j \leq n$ and $d(z_j)=c_j$ for $1+n \leq j \leq 2n$. On the other hand, since $W$ is an independent set and $\sum_{p=1}^{n} d(y_p) + \sum_{q=1}^{n} d(z_q) = \sum_{p=1}^{n} b_p + \sum_{q=1}^{n} c_q$, we have $\sum_{p=1}^{n} d(y_p) = \sum_{p=1}^{n} b_p + \sum_{q=1}^{n} d(z_q) = \sum_{q=1}^{n} c_q$. That is, $d(y_j)=b_j$ for $1 \leq j \leq n$ and $d(z_j)=c_j$ for $1+n \leq j \leq 2n$. Hence we construct a tripartite graph $G$ satisfying $d(x_i)=a_i$ for $1 \leq i \leq m$, $d(y_j)=b_j$ for $1 \leq j \leq n$ and $d(z_j)=c_j$ for $1+n \leq j \leq 2n$. Now let $F'=G$, $G_i=GYUX$ and $G_2=[ZUX]$ be the subgraphs...
induced by $V_1 = Y \cup X$ and $V_2 = Z \cup X$ in $G$, respectively, then $G_1$ and $G_2$ form a sunflower with two petals $G_1 = Y$ and $G_2 = Z$ and a core $X$.

**CONFLICT OF INTERESTS**

The authors have not declared any conflict of interests.

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