A FREE BOUNDARY PROBLEM OF SOME MODIFIED
LESLIE-GOWER PREDATOR-PREY MODEL WITH NONLOCAL
DIFFUSION TERM

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Abstract. This paper is mainly considered a Leslie-Gower predator-prey model with nonlocal diffusion term and a free boundary condition. The model describes the evolution of the two species when they initially occupy the bounded region \([0, h_0]\). We first show that the problem has a unique solution defined for all \(t > 0\). Then, we establish the long-time dynamical behavior, including Spreading-vanishing dichotomy and Spreading-vanishing criteria.

1. Introduction. The dynamical relationship between predator and prey is an important subject in mathematical ecology. Recently, many authors have studies the Leslie-Gower predator-prey model, which is the following form

\[
\begin{align*}
\frac{du}{dt} &= r_1 u \left(1 - \frac{u}{K}\right) - \delta uv, \\
\frac{dv}{dt} &= r_2 v \left(1 - \frac{hv}{u}\right), \quad u(0) > 0, v(0) > 0,
\end{align*}
\]

where \(u\) and \(v\) denote the populations of the prey and predator, respectively. All parameters are assumed to be positive, \(r_1, r_2\) represent intrinsic growth rates for prey and predator respectively, \(K\) stands for environmental carrying capacity, \(\delta\) denotes the per capita capturing rate of the prey by a predator per unit time \(\frac{hv}{u}\) represents Leslie-Gower terms, which means the carrying capacity of the predator is proportional to the population size of the prey. The model (1) has been extensively studied in [3, 45, 49].

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In order to establish the long time behavior of the Leslie-Gower predator-prey model with a free boundary, Liu et al. in [30] have considered the following model

\[
\begin{align*}
    \frac{\partial u}{\partial t} &= u_{xx} + u(1-u) - \delta uv, & \quad t > 0, 0 < x < h(t), \\
    \frac{\partial v}{\partial t} &= Dv_{xx} + \kappa v \left( 1 - \frac{v}{u + \alpha} \right), & \quad t > 0, 0 < x < h(t), \\
    h'(t) &= -\mu(u_x(t, h(t)) + \rho v_x(t, h(t))), & \quad t > 0, \\
    h(0) &= h_0, \\
    u_x(t, 0) &= v_x(t, 0) = u(t, h(t)) = v(t, h(t)) = 0, & \quad t > 0, \\
    u(0, x) &= u_0(x), v(0, x) = v_0(x), & \quad x \in [0, h_0].
\end{align*}
\] (2)

They have proved a spreading-vanishing dichotomy for the long-time dynamical behavior. The unique solution \((u, v, h)\) of (2) is satisfied one of the following conditions.

(i) Spreading: if \(\lim_{t \to \infty} h(t) = +\infty\), then \(\lim_{t \to \infty} u(t, x) = p_0\) and \(\lim_{t \to \infty} v(t, x) = q_0\),

where \((p_0, q_0)\) is the unique interior equilibrium of (2);

(ii) Vanishing: if \(\lim_{t \to \infty} h(t) < +\infty\), then \(\lim_{t \to \infty} ||u(t, \cdot)||_{C[0, h(t)]} = 0\) and \(\lim_{t \to \infty} ||v(t, \cdot)||_{C[0, h(t)]} = 0\).

The problem (2) is a variation of the Leslie-Gower predator-prey model, which is considered over a bounded spatial domain with suitable boundary conditions or considered over the entire space \(\mathbb{R}^n\) (see [19, 28]). For the initial value problem of the Leslie-Gower predator-prey model, A. Ducrot in [23] has studied some spreading properties of modified Leslie-Gower predator-prey reaction-diffusion system. In [9], we have considered the spreading speed properties for the Leslie-Gower predator-prey model with the fractional diffusion term \(\Delta^\alpha\) \((\alpha \in (0, 1))\). In [12, 13], we have showed the existence and stability of the Leslie-Gower predator-prey model with nonlocal diffusion. In [14, 34], we have also obtained the asymptotic behavior of two species evolving in a domain with a free boundary in higher dimensional homogeneous and heterogeneous environment.

The free boundary problem (2) describes the dynamical process of an introduced predator species with population density \(v(t, x)\) invading into the habitat of a native prey species with population density \(u(t, x)\). The initial function \(v_0(x)\), which occupies in \(\{x < h_0\}\), stands for the population of the predator in the beginning stage of its introduction. The predator population, which initially exists in \(\{0 < x < h_0\}\), disperses through random diffusion over \(\{0 < x < h(t)\}\), whose boundary \(\{x = h(t)\}\) is the spreading front and satisfies the free boundary condition \(h'(t) = -\mu v_x(t, h(t))\). This is well-known as the Stefan condition. The ecological background and derivation of the free boundary problems can also refer to [17, 31, 32, 33] and their references.

If the prey are only as food for the predator, that is, \(\delta = 0\), the system reduces to the logistic diffusive equation with free boundary condition, which has been studied in [20] for the one dimensional case, in [16] for the radially symmetric case, in [18] for the time periodic environment, in [50] for the heterogeneous environment and in [15] for the heterogeneous time-periodic environment. The behavior of the solution for these cases is characterized by a spreading-vanishing dichotomy. Moreover, when spreading occurs, it is showed that \(\frac{h(t)}{t} \to k_0 \in (0, 2)\) as \(t \to \infty\) and \(k_0\) is called the asymptotic spreading speed of \(v\). We can find the further discussion of the spreading speed and a deduction of the free boundary condition based on ecological assumption in [6]. They have presented a new approach to describe the front propagation for population species, which is different from the classical method for traveling waves.
of the diffusive logistic equation on the entire space $\mathbb{R}^n$ with $n \geq 1$ (such as in [1, 2]). The free boundary problems have been studied by many authors, refer to [22, 27, 36, 37, 38, 39] and references cited therein.

Recently, the free boundary problems for the predator-prey model have been studied in several earlier papers. For example, many authors have considered the competition model in the one dimensional and homogeneous environment over a bounded spatial interval in [25, 29], over the half spatial line in [24], and over the half spatial line with zero Dirichlet boundary or zero Neuman boundary condition in [41, 42] and with double free boundaries in [43]. For the higher space dimension case, Du et al. [21] have studied the diffusive competition model in the homogeneous environment and Zhao et al. deduced the spreading and vanishing properties of the predator-prey model for the heterogeneous environment in [47]. Zhang and Wang in [46] have considered the free boundary problem of the ratio-dependent prey-predator model. They have all established the spreading-vanishing dichotomy, the long time behavior of the solution and sharp criteria for spreading and vanishing.

For the model (2), the dispersal of the species is described by the local diffusion operator. Although the standard Laplacian operator can described the movement of individuals under a Brownian process, the movement of individuals which cannot be limited in a small area is often free and random. Recently, various integral operators have been widely used to model the nonlocal diffusion phenomena. For example, an operator of the form

$$K[u](x) = \int_{\mathbb{R}} k(x, y)u(y)dy - u(x)$$

appears in the theory of phase transition, ecology, genetics and neurology, see [4, 26, 40]. Many authors have obtained the properties of the solution of the equation with nonlocal term, for example in [10, 11, 44]. As we all know, the free boundary problem of Fisher-KPP nonlocal diffusion model has been considered extensively in [7, 8]. In [48], Zhao et al. have considered the dynamics of a degenerate epidemic model with nonlocal diffusion and free boundaries.

In this paper, we will consider the model of (2) with nonlocal diffusion term

$$\begin{align*}
    u_t &= \int_0^{h(t)} J(x - y)u(t, y)dy - u(t, x) + u(1 - u) - \delta uv, \quad t > 0, 0 < x < h(t), \\
v_t &= D \int_0^{h(t)} J(x - y)v(t, y)dy - v(t, x) + \kappa v \left(1 - \frac{v}{u + \alpha}\right), \quad t > 0, 0 < x < h(t), \\
h'(t) &= \mu \left(\int_0^{h(t)} \int_{h(t)}^{+\infty} J(x - y)u(t, x)dydx + \right. \\
    &\quad \left. + \rho \int_0^{h(t)} \int_{h(t)}^{+\infty} J(x - y)v(t, x)dydx\right), \quad t > 0, \\
    u_x(t, 0) = v_x(t, 0) = u(t, h(t)) = v(t, h(t)) = 0, \quad t > 0, \\
    u(0, x) = u_0(x), v(0, x) = v_0(x), h(0) = 0, \quad x \in [0, h_0] .
\end{align*}$$

The parameters $\mu, \rho, \kappa, D, h_0$ are positive constants and $\delta, \alpha$ are satisfied $0 < \delta \alpha + \alpha < 1$. The initial functions $(u_0(x), v_0(x))$ are satisfied
We assume that the kernel function $J : \mathbb{R} \to \mathbb{R}$ is continuous, nonnegative and satisfied

\begin{enumerate}[(J)]  \item $J(0) > 0$, $\int_{\mathbb{R}} J(x) dx = 1$, $J$ is symmetric and $\text{supp}_2 J < \infty$. \end{enumerate}

In (3), the free boundary condition

\[ h'(t) = \mu \left( \int_0^{h(t)} \int_{h(t)}^{+\infty} J(x-y)u(t,x)dydx + \rho \int_0^{h(t)} \int_{h(t)}^{+\infty} J(x-y)v(t,x)dydx \right) \]

denotes that the expanding rate of the range $[0, h(t)]$ is proportional to the outward flux of the population. In [7], some reasons of these conditions were given in the context of a single species population model. For most species, we would like to emphasize that its living environment involves many factors, such as food or nutrient supplies. Thus it is reasonable to assume that this survival rate is roughly a constant for given species.

The main results of this paper are the following theorems.

**Theorem 1.1.** (Global existence and uniqueness) Suppose that (J) holds. Then for any given $h_0 > 0$ and $(u_0(x), v_0(x))$ satisfying the condition (4), the problem (3) admits a unique solution $(u, v, h)$ defined for all $t > 0$.

**Theorem 1.2.** (Spreading-vanishing dichotomy) Assume the conditions of Theorem 1.1 hold and $(u, v, h)$ be the unique solution of the problem (3). Then one of the following alternatives must happen.

(i) Spreading: $\lim_{t \to +\infty} h(t) = +\infty$ and $\lim_{t \to +\infty} (u(t, x), v(t, x)) = (p_0, q_0)$ locally uniformly in $[0, +\infty)$, where $(p_0, q_0) = \left( \frac{1 - \delta \alpha}{1 + \delta}, \frac{1 + \alpha}{1 + \delta} \right)$ is the unique interior equilibrium of (3);

(ii) Vanishing: $\lim_{t \to +\infty} h(t) = h_\infty < \infty$ and $\lim_{t \to +\infty} u(t, x) = 0$, $\lim_{t \to +\infty} v(t, x) = 0$ uniformly in $[0, h_\infty]$.

**Theorem 1.3.** (Spreading-vanishing criteria) Under the condition of Theorem 1.2, if $\kappa \geq D$, then spreading always happens. If $0 < \kappa < D$, there exists a unique $h_* > 0$ such that when $h_0 \geq h_*$, spreading always happens; when $0 < h_0 < h_*$, then there exist some positive constants $\mu$ and $\bar{\mu}$ such that spreading happens for $\mu > \bar{\mu}$ and vanishing happens for $0 < \mu \leq \bar{\mu}$.

The rest of the paper is organized as follows. In Section 2, we will prove that (3) has a unique solution for $t > 0$. In Section 3, we will study the long-time dynamical behavior of the solution for the model (3), where Theorem 1.2 and 1.3 will be established.

**2. Global existence and uniqueness.** In this section, we assume that $h_0 > 0$ and $(u_0, v_0)$ satisfies (4). For any given $T > 0$, we introduce the following notation

\[ A := \max \left\{ \frac{1}{2}, \|u_0\|_{\infty}, 1 - \|u_0\|_{\infty} \right\}, \quad B := \max \left\{ \|v_0\|_{\infty}, \frac{1 - \|u_0\|_{\infty}}{\delta} \right\}, \]

\[ H_{h_0, T} := \left\{ h \in C([0, T]) : h(0) = h_0, \inf_{0 \leq t_1 < t_2 \leq T} \frac{h(t_2) - h(t_1)}{t_2 - t_1} > 0 \right\}, \]

\[ D_{h, T} := \left\{ (t, x) \in \mathbb{R}^2 : 0 < t \leq T, 0 < x < h(t) \right\}, \]
Lemma 2.1. Assume that (J) holds and \( h \in H_{h_0,T} \) for some \( T > 0 \). Suppose that \( c_{ii} \in L^{\infty}(D_{h,T}) \), \( i = 1, 2 \), \((u(t,x),v(t,x))\) and \((u(t,x),v_t(t,x))\) are continuous in \( D_{h,T} \) and satisfy

\[
\begin{cases}
u_t \geq \int_{0}^{h(t)} J(x-y) u(t,y)dy - u(t,x) + c_{11} u, & 0 < t \leq T, x \in (0, h(t)), \\
u_t \geq D \int_{0}^{h(t)} J(x-y) v(t,y)dy - v(t,x) + c_{22} v, & 0 < t \leq T, x \in (0, h(t)), \\
u(t,h(t)) \geq 0, v(t,h(t)) \geq 0, & 0 < t \leq T, \\
u(0,x) \geq 0, v(0,x) \geq 0, & x \in [0, h_0].
\end{cases}
\]

Then \( (u(t,x),v(t,x)) \geq (0,0) \) for all \( 0 \leq t \leq T \) and \( 0 \leq x \leq h(t) \). Moreover, if we assume additionally \( u(0,x) \neq 0 \) in \([0,h_0]\), then \( u(t,x) > 0 \) in \( D_{h,T} \).

Proof. Let \( w(t,x) = e^{kt} u(t,x) \) and \( z = e^{kt} v(t,x) \), where \( k \) is large enough such that

\[
k > 1 + \|c_{11}\|_{\infty} + \|c_{22}\|_{\infty},
\]

then

\[
p(t,x) := k + c_{22}(t,x) \geq k - \|c_{22}\|_{\infty} > 0 \text{ for all } (t,x) \in D_{h,T}.
\]

By directed calculation, we have that \( w(t,x) \) and \( z(t,x) \) satisfy

\[
\begin{cases}
w_t \geq \int_{0}^{h(t)} J(x-y) w(t,y)dy + (k-1 + c_{11})w, & 0 < t \leq T, \\
z_t \geq D \int_{0}^{h(t)} J(x-y) z(t,y)dy + (k - D + c_{22})z.
\end{cases}
\]

Denote

\[
p_0 = \sup_{(t,x) \in D_{h,T}} p(t,x) \text{ and } T^* = \min \left\{ T, \frac{1}{4 \left( k - \frac{1}{2} + \|c_{11}\|_{\infty} \right)}, \frac{1}{4 \left( p_0 - \frac{D}{2} \right)} \right\}.
\]

Now we prove \( w \geq 0 \) and \( z \geq 0 \) in \( D_{h,T^*} \). Suppose that

\[
m := \min \left\{ \inf_{(t,x) \in D_{h,T^*}} w(t,x), \inf_{(t,x) \in D_{h,T^*}} z(t,x) \right\} < 0.
\]

By (2.1), \( w \geq 0 \) and \( z \geq 0 \) on the boundary of \( D_{h,T^*} \). Hence, there exists \( (t^*,x^*) \in D_{h,T^*} \) such that \( \frac{m}{2} = w(t^*,x^*) < 0 \) or \( \frac{m}{2} = z(t^*,x^*) < 0 \). Define

\[
t_0 = t_0(x^*) := \begin{cases} 0, & \text{if } x^* \in [0, h_0], \\ t^h_{x^*}, & \text{if } x^* \in (h_0, h(t^*)) \text{ and } x^* = h(t^h_{x^*}).
\end{cases}
\]

Clearly, \( u(t_0,x^*) \geq 0 \) and \( v(t_0,x^*) \geq 0 \).

If \( \frac{m}{2} = w(t^*,x^*) < 0 \), then it follows from the choice of \( k \) and the first equation of (2.2) that

\[
w(t^*,x^*) - w(t_0,x^*)
\]
which is a contradiction with
For any given
Lemma 2.2.

\[ \begin{align*}
    \geq & t^* \int_0^{h(t)} J(x^* - y)w(t, y)dydt + \int_{t_0}^{t^*} (k_1 - 1 + c_{11})w(t, x^*)dt \\
    \geq & t^* \int_0^{h(t)} J(x^* - y) \cdot mdydt + \int_{t_0}^{t^*} (k_1 - 1 + c_{11}) \cdot mdt \\
    \geq & m(k_1 - \frac{1}{2} + \|c_{11}\|_\infty)(t^*-t_0).
\end{align*} \]

Since \( w(t_0, x^*) = e^{kh(t)} u(t_0, x^*) \geq 0 \), we deduce
\[ \frac{m}{2} \geq m \left( k_1 - \frac{1}{2} + \|c_{11}\|_\infty \right) (t^*-t_0) \geq m \left( k_1 - \frac{1}{2} + \|c_{11}\|_\infty \right) T^* \geq \frac{m}{4}, \]
which is a contradiction with \( m < 0 \).

If \( \frac{m}{2} = z(t^*, x^*) < 0 \), then it follows from the choice of \( k \) and the second equation of (2.2) that
\[ z(t^*, x^*) - z(t_0, x^*) \]
\[ \geq D \int_{t_0}^{t^*} \int_0^{h(t)} J(x^* - y)z(t, y)dydt + \int_{t_0}^{t^*} (k_1 - D + c_{22})z(t, x^*)dt \]
\[ \geq D \int_{t_0}^{t^*} \int_0^{h(t)} J(x^* - y) \cdot mdydt + \int_{t_0}^{t^*} (k_1 - D + c_{22}) \cdot mdt \]
\[ \geq m \left( p_0 - \frac{D}{2} \right) (t^*-t_0). \]

Since \( z(t_0, x^*) = e^{kh(t)} v(t_0, x^*) \geq 0 \), we deduce
\[ \frac{m}{2} \geq m \left( p_0 - \frac{D}{2} \right) (t^*-t_0) \geq m \left( p_0 - \frac{D}{2} \right) T^* \geq \frac{m}{4}, \]
which is a contradiction with \( m < 0 \).

If \( T^* = T \), then \( (u(t, x), v(t, x)) \geq (0, 0) \) for all \( 0 \leq t \leq T \) and \( 0 \leq x \leq h(t) \). If \( T^* < T \), we may repeat this process with \( (u_0(x), v_0(x)) \) replaced by \( (u(T^*, x), v(T^*, x)) \), and \([0, T]\) replaced by \([T^*, T]\). After repeating this process finitely times, we will obtain \( (u(t, x), v(t, x)) \geq (0, 0) \) for all \( 0 \leq t \leq T \) and \( 0 \leq x \leq h(t) \).

Due to \( v(t, x) \geq 0 \) for all \( 0 \leq t \leq T \) and \( 0 \leq x \leq h(t) \), we have that \( u \) satisfies
\[ \begin{align*}
    u_0 & \geq \int_0^{h(t)} J(x - y)u(t, y)dy - u(t, x) + c_{11}u, \quad 0 < t \leq T, x \in (0, h(t)), \\
    u(t, h(t)) & \geq 0, u(t, 0) \geq 0, \quad 0 < t \leq T, \\
    u(0, x) & \geq 0, \quad x \in [0, h_0].
\end{align*} \]

If \( u(0, x) \neq 0 \) in \([0, h_0]\), then it follows directly from Lemma 2.2 of [7] that \( u(t, x) > 0 \) in \( D_{h, T} \).

The following result will play a crucial role in the proof of Theorem 1.1.

**Lemma 2.2.** For any given \( T > 0, h \in H_{h_0, T}, \) and \( v \in \mathcal{K}_{v_0, T} \), the problem
\[ \begin{align*}
    u_t & = \int_0^{h(t)} J(x - y)u(t, y)dy - u(t, x) + u(1 - u) - \delta uv, \quad 0 \leq t \leq T, x \in (0, h(t)), \\
    u(t, h(t)) & = 0, \quad 0 < t \leq T, \\
    u(0, x) & = u_0(x), \quad x \in [0, h_0],
\end{align*} \]
admits a unique solution $u^* \in C(\overline{D_{h,T}})$. Moreover,
\begin{equation}
0 < u^* \leq A \text{ for any } (t,x) \in D_{h,T}.
\end{equation}

Proof. We break the proof into three steps.

**Step 1.** A parameterized problem. For any given $x \in [0,h(T)]$, define
\[
\hat{u}_0(x) := \begin{cases}
  u_0(x), & x \in [0,h_0], \\
  0, & x \notin [0,h_0],
\end{cases}
\]
and
\[
t_x := \begin{cases}
  0, & x \in [0,h_0], \\
  t^h_x, & x \in (h_0,h(T)] \text{ and } x = h(t^h_x).
\end{cases}
\]
Clearly, $t_x = T$ for $x = h(T)$ and $0 \leq t_x < T$ for $x \in (0,h(T))$.

For any given $s \in (0,T]$ and $\phi \in X_{u_0,s}$, we fix $x \in (0,h(s))$ and consider the following initial value problem
\begin{equation}
\begin{cases}
  u_t(t,x) = F(t,x,u), & t_x < t \leq s, \\
  u(t_x,x) = \hat{u}_0(x),
\end{cases}
\end{equation}
with
\[
F(t,x,u) := \int_0^{h(t)} J(x-y)\phi(t,y)dy - u + u(1-u) - \delta uv.
\]
Set $L_1 := 1 + \max\{\|\phi\|_{C(\overline{D_{h,T}})}, A\}$. For any $u_1,u_2 \in [0,L_1],
\begin{align*}
|F(t,x,u_1) - F(t,x,u_2)| &= \left| \int_0^{h(t)} J(x-y)\phi(t,y)dy - u_1 + u_1(1-u_1) - \delta u_1 v \\
&\quad - \int_0^{h(t)} J(x-y)\phi(t,y)dy + u_2 - u_2(1-u_2) + \delta u_2 v \right| \\
&\leq (2A + \delta B)|u_1 - u_2|.
\end{align*}

Hence, $F(t,x,u)$ is Lipschitz continuous in $u$ for $u \in [0,L_1]$ with Lipschitz constant $(2A + \delta B)$, uniformly for $t \in [0,s]$ and $x \in (0,h(s))$. Additionally, $F(t,x,u)$ is continuous in all its variables in this range. By the fundamental theorem, the problem (9) admits a unique solution $U_\phi(t,x)$ defined in some interval $[t_x,s_x)$ of $t$ and $U_\phi(t,x)$ is continuous.

To see that $t \to U_\phi(t,x)$ can be uniquely extended to $[t_x,s]$, it is sufficient to show that if $U_\phi$ is uniquely defined for $t \in [t_x,\hat{t}]$ with $\hat{t} \in (t_x,s)$, then
\begin{equation}
0 \leq U_\phi(t,x) \leq L_1 \text{ for } t \in [t_x,\hat{t}].
\end{equation}

It is easy to check that
\[
F(t,x,L_1) = \int_0^{h(t)} J(x-y)\phi(t,y)dy - L_1 + L_1(1-L_1) - \delta L_1 v
\]
\[
\leq \|\phi\|_{\infty} - L_1 + L_1(1-L_1) - \delta L_1 v
\]
\[
\leq 0,
\]
and
\[
L_1 > \|\phi\|_{C(\overline{D_{h,T}})} \geq \|u_0(x)\|_{C([0,h_0])} = \|\hat{u}_0\|_{\infty}.
\]
Now a simple comparison argument gives \( U(t, x) \leq L_1 \) for \( t \in [t_x, \hat{t}] \). This proves the second inequality in (10). The first inequality there can be obtained similarly by using \( F(t, x, 0) \geq 0 \).

**Step 2.** A fixed point problem. For \( s \in (0, T) \), we denote

\[
D_s := D_{h, s}, X_s := \mathbb{X}_{u_{0, s}}.
\]

By Step 1, for any \( \phi \in X_s \), we can find a unique \( U_\phi(t, x) \) satisfying (9) for \( t \in [0, s] \) and \( U_\phi(t, x) \) is continuous in \( \bar{D}_s \). Hence \( U_\phi \in X_s \). Note that \( X_s \) is a completed space equipped with the norm

\[
d(\phi_1, \phi_2) = \| \phi_1 - \phi_2 \|_{C(\bar{D}_s)}.
\]

Then we define the mapping \( \Gamma : X_s \rightarrow X_s \) by

\[
\Gamma(\phi) = U_\phi.
\]

Clearly, if \( \Gamma(\phi) = \phi \), then \( \phi \) solves (7) for \( t \in [0, s] \).

Fix \( M = \max\{\|u_0\|_\infty, A\} \) and define

\[
\mathbb{X}_s^M := \{ \phi | \phi \in X_s, \phi(t, x) \leq M \}.
\]

In the following, we will show that \( \Gamma \) has a unique fixed point in \( \mathbb{X}_s^M \) for small \( s \) by the contraction mapping theorem.

We first claim that there exists sufficiently small \( s^* \) such that \( \Gamma \) maps \( \mathbb{X}_s^M \) into itself for any \( s \in (0, s^*]. \) Let \( \phi \in \mathbb{X}_s^M \), we show that \( U_\phi(t, x) \leq M \) for \( (t, x) \in D_s \).

By the first equation of (9) for \( t_x < t \leq s \), we have

\[
(U_\phi)_1(t, x) \leq \int_0^{h(t)} J(x-y) \phi(t, y) dy + u(1-u)
\]

\[
\leq \| \phi \|_{C(\bar{D}_s)} + A(1-A)
\]

\[
\leq \| \phi \|_{C(\bar{D}_s)} + A
\]

and then

\[
U_\phi(t, x) \leq U_\phi(t_x, x) + (\| \phi \|_{C(\bar{D}_s)} + A)(t - t_x)
\]

\[
\leq \| u_0 \|_\infty + (\| \phi \|_{C(\bar{D}_s)} + A)s
\]

\[
\leq \frac{M}{2} + 2Ms.
\]

If we choose \( s^* \) small enough such that \( 0 < s^* \leq \frac{1}{3} \), then \( U_\phi(t, x) \leq M \) for \( s \in (0, s^*] \). This implies that \( U_\phi \in \mathbb{X}_s^M \) for \( s \in (0, s^*] \). The claim is now proved.

Next, we show that \( \Gamma \) is a contraction map for \( s \in (0, s^*] \). Namely, there exists some \( \beta < 1 \) such that

\[
\| U_{\phi_1} - U_{\phi_2} \|_{C(\bar{D}_s)} \leq \beta \| \phi_1 - \phi_2 \|_{C(\bar{D}_s)}
\]

for any given \( \phi_i \in \mathbb{X}_s^M, i = 1, 2 \). Denote \( W = U_{\phi_1} - U_{\phi_2} \), we have

\[
\begin{cases}
W_t + c_1(t, x)W = \int_0^{h(t)} J(x-y)(\phi_1 - \phi_2)(t, y) dy, & t_x < t \leq s, \\
W(t_x, x) = 0, & x \in (0, h(s)),
\end{cases}
\]

where \( c_1(t, x) = U_{\phi_1} + U_{\phi_2} + v \) and \( \| c_1 \|_\infty \leq 2M + \delta B := K_1 \). It follows that

\[
W(t, x) = e^{-\int_{t_x}^{t} c_1(t, x) dt} \int_{t_x}^{t} e^{\int_{\tau}^{t} c_1(t, x) dt} \int_0^{h(\xi)} J(x-y)(\phi_1 - \phi_2)(\xi, y) dy d\xi
\]

\[
\leq \int_0^{h(t)} J(x-y)(\phi_1 - \phi_2)(t, y) dy
\]

\[
\leq \| \phi_1 - \phi_2 \|_{C(\bar{D}_s)} + A(1-A)
\]

\[
\leq \| \phi_1 - \phi_2 \|_{C(\bar{D}_s)} + A
\]

\[
\leq \frac{M}{2} + 2Ms.
\]
for \( t_x < t \leq s \) and \( x \in (0, h(t)) \). Thus we deduce that

\[
|W(t, x)| \leq e^{K_1(t-t_x)}\|\phi_1 - \phi_2\|_{C(\bar{D}_x)} \int_{t_x}^t e^{K_1(\xi-t_x)} d\xi \\
\leq e^{K_1s}\|\phi_1 - \phi_2\|_{C(\bar{D}_x)}(t-t_x)e^{K_1s} \\
\leq se^{2K_1s}\|\phi_1 - \phi_2\|_{C(\bar{D}_x)}
\]

for \( t_x < t \leq s \) and \( x \in (0, h(t)) \). Therefore we can choose small enough \( s^* \) satisfied \( s^*e^{2K_1s^*} \leq \frac{1}{2} \) such that

\[
\|\Gamma(\phi_1) - \Gamma(\phi_2)\|_{C(\bar{D}_x)} \leq \frac{1}{2}\|\phi_1 - \phi_2\|_{C(\bar{D}_x)} \text{ for } s \in (0, s^*].
\]

Hence, \( \Gamma \) is a contraction map. For any \( s \in (0, s^*] \), applying the contraction mapping theorem, we can obtain a unique fixed point \( u^* \) of \( \Gamma \) in \( X^*_s \). Clearly, \( u^* \) is a solution of (7) for \( t \in [0, s] \).

In order to prove that \( u^* \) is the unique solution of (7) for \( t \in [0, s] \) with \( s \in (0, s^*] \), it remains to show that any solution \( u \) of (7) for \( t \in [0, s] \) belongs to \( X^*_s \). Next we establish the following estimate

\[
0 \leq u(t, x) \leq A \text{ for } t \in [0, s] \text{ and } x \in [0, h(t)].
\]

(11)

It is easy to check that

\[
\int_0^{h(t)} J(x-y)u(t, y)dy - u(1-u) - \delta uv \\
\leq \int_0^{h(t)} J(x-y) \cdot Ady - A + A(1-A) \\
\leq A \left( \frac{1}{2} - A \right) < 0.
\]

Therefore, in view of \( A \geq \|u_0\|_{\infty} \), a simple comparison argument yields \( u(t, x) \leq A \) for \( t \in [0, s] \) and \( x \in [0, h(t)] \). Using \( u(0, x) = u_0(x) \geq 0 \) for \( x \in [0, h_0] \), we can similarly see from Lemma 2.2 of [48] that \( u(t, x) \geq 0 \) for \( t \in [0, s] \) and \( x \in [0, h(t)] \). Hence, (11) holds. Now we have proved the fact that for any \( s \in (0, s^*] \), (7) has a unique solution for \( t \in [0, s] \).

**Step 3.** Extension of the solution. From Step 2, we know that (7) has a unique solution for \( t \in [0, s] \) with \( s \in (0, s^*] \). Applying Step 2 to (7) with the initial time \( t = 0 \) replaced by \( t = s \), we see that the unique solution can be extended to \( t \in [0, s] \) for any \( s \in (0, \min\{2s^*, T\}] \). Moreover, the extended solution \( u \) still satisfies (11). By repeating this process finitely times, the solution of the problem (7) is uniquely extended to \([t_x, T]\), and (8) is a consequence of (11) obtained in each step of the extension.

**Proof of Theorem 1.1.** For any given \( T > 0 \), \( h^* \in H_{h_0,T} \) and \( v^* \in X_{v_0,T} \), it follows from Lemma 2.2 that the problem (7) with \( (v, h) = (v^*, h^*) \) has a unique solution \( u^* \). For such \((u^*, h^*) \), we can define \( \check{v}_0(x) \) as the zero extension of \( v_0(x) \) to \( x \in \mathbb{R}^+ \setminus [0, h_0] \) and then define \( t_x^* \) as \( t_x \) in Step 1 of the proof of Lemma 2.2, with \( h \) replaced by \( h^* \).
For each \( x \in (0, h^*(T)) \), we will consider the initial value problem
\[
\begin{aligned}
v_t &= D \left[ \int_0^{h(t)} J(x-y)v(t,y)dy - v(t,x) \right] + \kappa v \left( 1 - \frac{v}{u+\alpha} \right), \quad t^*_x < t \leq T, \\
v(t^*_x, x) &= v_0(x).
\end{aligned}
\] (12)

By the Fundamental Theorem of ordinary differential equation and the comparison principle, it can be easily shown that (12) has a unique continuous solution \( \tilde{v}^*(t, x) \) which is satisfied
\[ 0 \leq \tilde{v}^*(t, x) \leq B \] for \( t \in (0, T], x \in [0, h^*(t)] \).

Therefore \( \tilde{v}^* \in \mathcal{X}_{v_0,T} \).

Next we define \( h^* \) by
\[
\begin{aligned}
\hat{h}^*(t) &:= h_0 + \mu \left[ \int_0^{h^*(t)} \int_0^{h^*(\tau)} J(x-y)u^*(\tau, x)dydx \right. \\
&\quad \left. + \rho \int_0^{h^*(t)} \int_0^{h^*(\tau)} J(x-y)v^*(\tau, x)dydx \right] for t \in [0, T].
\end{aligned}
\] (13)

Since \( u^* \) is the solution of (7), we obtain that
\[
\begin{aligned}
u^*_x(t, x) &\geq -u - \delta Bu, \quad 0 < t \leq T, x \in (0, h(t)), \\
u^*_x(h(t)) &= 0, \quad 0 < t \leq T, \\
u^*(0, x) &= u^*_0(x), \quad x \in [0, h_0].
\end{aligned}
\]

It follows that
\[
u^*(t, x) \geq e^{-(1+\delta B)t} u^*_0(x) \geq e^{-(1+\delta B)T} u^*_0(x) \text{ for } x \in [0, h_0], t \in (0, T].
\]

Similarly, we can obtain
\[
u^*(t, x) \geq e^{-(D + \frac{\mu}{\alpha})t} v^*_0(x) \geq e^{-(D + \frac{\mu}{\alpha})T} v^*_0(x) \text{ for } x \in [0, h_0], t \in (0, T].
\]

Due to the assumption (J), there exist constants \( \epsilon_0 \in (0, \frac{h_0}{4}) \) and \( \delta_0 \) such that
\[ J(x) \geq \delta_0 \text{ if } |x| \leq \epsilon_0.
\]

We assume that \( h^*(T) \leq h_0 + \frac{\epsilon_0}{4} \). Using (13), we can easily see
\[
\hat{h}^*(t) \leq h_0 + \mu T (A + \rho B) \left( h_0 + \frac{\epsilon_0}{4} \right) \leq h_0 + \frac{\epsilon_0}{4} \text{ for } t \in [0, T],
\]

where \( T > 0 \) is small enough depending on \( (\mu, A, B, h_0, \epsilon_0) \). For \( t \in (0, T] \), we can obtain
\[
\begin{aligned}
&\int_0^{h^*(t)} \int_0^{h^*(T)} J(x-y)u^*(t, x)dydx + \rho \int_0^{h^*(t)} \int_0^{h^*(T)} J(x-y)v^*(t, x)dydx \\
&\geq \int_{h^*(t)-\frac{\epsilon_0}{4}}^{h^*(t)} \int_{h^*(t)-\frac{\epsilon_0}{4}}^{h^*(t)+\frac{\epsilon_0}{4}} J(x-y)u^*(t, x)dydx \\
&\quad + \rho \int_{h^*(t)-\frac{\epsilon_0}{4}}^{h^*(t)} \int_{h^*(t)-\frac{\epsilon_0}{4}}^{h^*(t)+\frac{\epsilon_0}{4}} J(x-y)v^*(t, x)dydx \\
&\geq e^{-(1+\delta B)t} \int_{h_0-\frac{\epsilon_0}{4}}^{h_0} u^*_0(x) \left( \int_{h_0-\frac{\epsilon_0}{4}}^{h_0+\frac{\epsilon_0}{4}} J(x-y)dy \right) dx
\end{aligned}
\]
and define the mapping fixed point in $\Sigma$ with $c$

Step 1.

We will complete this task in several steps.

Thus, for sufficiently small $T_0 = T_0(\mu, A, B, h_0, \epsilon_0)$

> 0 and any $T \in (0, T_0)$,

Let

$$\Sigma_T := \left\{ (v, h) \in X_{\rho v}, T \times \mathbb{H}_{h_0,T} : \inf_{0 \leq t_1 < t_2 \leq T} \frac{h^*(t_2) - h^*(t_1)}{t_2 - t_1} \geq \mu c_0, \quad h^*(t) \leq h_0 + \frac{\epsilon_0}{4} \text{ for } t \in [0, T] \right\},$$

and define the mapping

$$F(v^*, h^*) = (\tilde{v}^*, \tilde{h}^*).$$

Then the above analysis indicates that

$$F(\Sigma_T) \subset \Sigma_T \text{ for } T \in (0, T_0].$$

In the following, we show that for sufficiently small $T \in (0, T_0]$, $F$ has a unique fixed point in $\Sigma_T$, which clearly is a solution of (3) for $t \in [0, T]$. Then we will show that this is the unique solution of (3) and it can be extended uniquely to all $t > 0$. We will complete this task in several steps.

**Step 1.** We will show that for sufficiently small $T \in (0, T_0]$, $F$ has a unique fixed point in $\Sigma_T$ by the contraction mapping theorem.

For $T \in (0, T_0]$ and any given $(v_i^*, h_i^*) \in \Sigma_T$ ($i = 1, 2$), denote

$$(\tilde{v}_i^*, \tilde{h}_i^*) = F(v_i^*, h_i^*), \quad i = 1, 2.$$ Define

$$h_m(t) := \min\{h_1^*(t), h_2^*(t)\}, \quad h_M(t) := \max\{h_1^*(t), h_2^*(t)\}.$$ In the following, we will show that there exists $\beta \in (0, 1)$ such that, for all small $T \in (0, T_0]$ and $(v_i^*, h_i^*) \in \Sigma_T$ ($i = 1, 2$),

$$\|\tilde{v}_1^* - \tilde{v}_2^*\|_{C([0,T] \times \mathbb{R}^+)} + \|\tilde{h}_1^* - \tilde{h}_2^*\|_{C([0,T])} \leq \beta \|v_1^* - v_2^*\|_{C([0,T] \times \mathbb{R}^+)} + \|h_1^* - h_2^*\|_{C([0,T])}. \quad (14)$$

Clearly this implies that $F$ is a contraction mapping on $\Sigma_T$.

In order to prove (14), we first estimate $\|\tilde{v}_1^* - \tilde{v}_2^*\|_{C([0,T] \times \mathbb{R}^+)}$. Let

$$\tilde{V}(t, x) := \tilde{v}_1^*(t, x) - \tilde{v}_2^*(t, x), \quad U(t, x) := u_1^*(t, x) - u_2^*(t, x), \quad V(t, x) := v_1^*(t, x) - v_2^*(t, x).$$

**Claim 1.** There exist positive constants $\bar{T}$ and $C$ such that for any $T \in (0, \bar{T}]$ and $(t^*, x^*) \in D_{T,h_M},$

$$\|V(t^*, x^*)\| \leq CT\|V\|_{C([0,T] \times \mathbb{R}^+)} + \|h_1^* - h_2^*\|_{C([0,T])}. \quad (15)$$

Now, $X$ is defined by

$$X = \{u(x) \mid u(x) : \mathbb{R} \rightarrow \mathbb{R} \text{ is uniformly continuous and bounded} \}.$$ $T$ is an analytic semigroup on $X$. Therefore, by [35], we can get existence result of mild solution for (12). To prove Claim 1, we proceed according to three cases.
Case 1. $x^* \in [0, h_0]$. Using Theorem 3.1 of [35] and $\tilde{V}(0, x^*) = 0$ for (12), we obtain
\[
|\tilde{V}(t^*, x^*)| = \left| \int_0^{t^*} T(t^* - \tau) \left[ \kappa \tilde{v}_1^* \left( 1 - \frac{\tilde{v}_1^*}{u_1^* + \alpha} \right) - \kappa \tilde{v}_2^* \left( 1 - \frac{\tilde{v}_2^*}{u_2^* + \alpha} \right) \right] d\tau \right|
\leq T \left| \kappa \tilde{v}_1^* u_1^* + \alpha - \tilde{v}_1^* \right| - \kappa \tilde{v}_2^* u_2^* + \alpha - \tilde{v}_2^* \left| \frac{u_1^* + \alpha}{u_2^* + \alpha} \right|.
\]
From the first equation of problem (7) with $(v, h) = (v_1^*, h_1^*)$ and $U(0, x^*) = 0$, we obtain
\[
U(t^*, x^*) = \int_0^{t^*} T(t^* - \tau) [u_1^* (1 - u_1^*) - \delta u_1^* v_1^* - u_2^* (1 - u_2^*) + \delta u_2^* v_2^*] d\tau,
\]
and
\[
|U(t^*, x^*)| \leq \int_0^{t^*} T(t^* - \tau) \left[ (u_1^* - u_2^*) - (u_1^* + u_2^*)(u_1^* - u_2^*) - \delta u_1^* (v_1^* - v_2^*) - \delta v_2^* (u_1^* - u_2^*) \right] d\tau
\leq T[(2A + 1 + \delta B)||U||_{C([0,T] \times \mathbb{R}^+)} + \delta A||V||_{C([0,T] \times \mathbb{R}^+)}].
\]
It follows that
\[
|U(t^*, x^*)| \leq C_1 T[||U||_{C([0,T] \times \mathbb{R}^+)} + ||V||_{C([0,T] \times \mathbb{R}^+)}],
\]
where $C_1$ depends only on $(A, B, \delta, T)$.

Case 2. $x^* \in (h_0, h_m(t^*))$. In this case, there exist $t_1^*, t_2^* \in (0, t^*)$ such that $h_1^*(t_1^*) = h_2^*(t_2^*) = x^*$. Without loss of generality, we may assume $0 \leq t_1^* \leq t_2^*$. We first prove that
\[
t_2^* - t_1^* \leq \frac{1}{\mu_0} ||h_1^* - h_2^*||_{C([0,T])}.
\]
If $t_1^* = t_2^*$, then (18) clearly hold. If $t_1^* < t_2^*$, using $\frac{h_1^*(t_2^*) - h_1^*(t_1^*)}{t_2^* - t_1^*} \geq \mu_0$, we can obtain
\[
0 < t_2^* - t_1^* \leq (\mu_0)^{-1} [h_1^*(t_2^*) - h_1^*(t_1^*)] = (\mu_0)^{-1} [h_2^*(t_2^*) - h_2^*(t_1^*)] \leq (\mu_0)^{-1} ||h_1^* - h_2^*||_{C([0,T])}.
\]
We next estimate $\tilde{V}(t^*, x^*) = (\tilde{v}_1^* - \tilde{v}_2^*)(t^*, x^*)$. Since $\tilde{v}_2^*(t_2^*, x^*) = 0$,
\[
|\tilde{V}(t^*, x^*)| = \left| T(t^*) \tilde{v}_1^*(t_2^*, x^*) + \int_0^{t^*} T(t^* - \tau) \left[ \kappa \tilde{v}_1^* \left( 1 - \frac{\tilde{v}_1^*}{u_1^* + \alpha} \right) - \kappa \tilde{v}_2^* \left( 1 - \frac{\tilde{v}_2^*}{u_2^* + \alpha} \right) \right] d\tau \right|
\leq ||\tilde{v}_1^*(t_2^*, x^*)|| + \frac{\kappa T}{\alpha} \left[ B||U||_{C([0,T] \times \mathbb{R}^+)} + (\alpha + A + 2B)||V||_{C([0,T] \times \mathbb{R}^+)} \right].
\]
Using (18) and \( \tilde{v}_1^*(t_1^*, x^*) = 0 \), we have

\[
0 \leq \tilde{v}_1(t_2^*, x^*) = \int_{t_1^*}^{t_2^*} T(t_2^* - \tau) \kappa \tilde{v}_1^* \left( 1 - \frac{\tilde{v}_1^*}{u_1^* + \alpha} \right) d\tau \\
\leq \kappa B(t_2^* - t_1^*) \\
\leq \frac{\kappa B}{\mu c_0} \| h_1^* - h_2^* \|_{C([0,T])}
\]

for \( t \in [t_1^*, t_2^*] \). Therefore

\[
|\tilde{V}(t^*, x^*)| \leq \frac{\alpha \kappa B}{\mu c_0 [\alpha - \kappa (\alpha + A + 2B)]} \| h_1^* - h_2^* \|_{C([0,T])} \\
+ \frac{\kappa TB}{\alpha - \kappa (\alpha + A + 2B)} \| U \|_{C([0,T] \times \mathbb{R}^+)}. \tag{19}
\]

Moreover, we have

\[
U(t^*, x^*) = TU(t_2^*, x^*) + \int_{t_2^*}^{t^*} T(t^* - \tau)[u_1^*(t^*, x^*) - \delta u_1^* v_1^* - u_2^*(1 - u_2^*) + \delta u_2^* v_2^*] d\tau
\]

for \( t \in [t_1^*, T] \). By the first equation of (7) and \( u_1^*(t_1^*, x^*) = 0 \), we have

\[
u_1^*(t, x^*) = \int_{t_1^*}^{t} \int_{0}^{h_1^*(\tau)} J(x^* - y)u_1^*(\tau, y) dy - u_1^*(1 - u_1^*) - \delta u_1^* v_1^* \]

\[
\leq C_2(t - t_1^*) \text{ for } t \in [t_1^*, t_2^*],
\]

with \( C_2 \) depending only on \( (A, B, \delta) \). Due to \( u_2^*(t_2^*, x^*) = 0 \), we have \( U(t_2^*, x^*) = u_1^*(t_2^*, x^*) \). Thus making use of (18), we obtain

\[
|U(t_2^*, x^*)| \leq C_2(t_2^* - t_1^*) \leq \tilde{C}_2 \| h_1^* - h_2^* \|_{C([0,T])},
\]

with \( \tilde{C}_2 \) depending on \( C_2 \) and \( (\mu, c_0) \). Substituting this into (20), we obtain

\[
|U(t^*, x^*)| \leq \tilde{C}_2 \| h_1^* - h_2^* \|_{C([0,T])} + C_1 T[\| U \|_{C([0,T] \times \mathbb{R}^+)} + \| V \|_{C([0,T] \times \mathbb{R}^+)}. \tag{21}
\]

**Case 3.** \( x^* \in [h_m(t^*), h_M(t^*)] \). Without loss of generality, we assume \( h_2^*(t^*) < h_1^*(t^*) \). In this case, there exists \( t_1^* \in (0, t^*) \) such that \( h_1^*(t_1^*) = x^* \). Then

\[
\tilde{v}_1^*(t^*, x^*) = u_1^*(t^*, x^*) = 0, u_1^*(t_1^*, x^*) = 0,
\]

and

\[
h_m(t^*) = h_2^*(t^*), h_M(t^*) = h_1^*(t^*), h_1^*(t_1^*) = x^* \leq h_m(t^*) = h_2^*(t^*). \]

Hence,

\[
\tilde{V}(t^*, x^*) = \tilde{v}_1^*(t^*, x^*), U(t^*, x^*) = u_1^*(t^*, x^*)
\]

We first prove

\[
t^* - t_1^* \leq \frac{1}{\mu c_0} \| h_1^* - h_2^* \|_{C([0,T])}. \tag{22}
\]

In fact,

\[
0 < t^* - t_1^* \leq \frac{1}{\mu c_0} [h_1^*(t^*) - h_1^*(t_1^*)] \leq \frac{1}{\mu c_0} [h_1^*(t^*) - h_2^*(t^*)] \leq \frac{1}{\mu c_0} \| h_1^* - h_2^* \|_{C([0,T])}.
\]
By the first equation of (7) and \( u_1^*(t_1^*, x^*) = 0 \), we have
\[
\begin{aligned}
u_1^*(t, x^*) &= \int_{t_1^*}^t \left[ \int_0^{h_1^*(\tau)} J(x^* - y)u_1^*(\tau, y)dy - u_1^* + u_1^*(1 - u_1^*) - \delta u_1^*v_1^* \right] d\tau \\
&\leq C_3(t - t_1^*),
\end{aligned}
\]
where \( C_3 \) depending only on \((A, B, \delta)\) and \( t \in [t_1^*, t^*]\). By (22) and (23), we have
\[
u_1^*(t^*, x^*) \leq C_3(t^* - t_1^*) \leq (\mu_0)^{-1}C_3\|h_1^* - h_2^*\|_{C([0, T])}.
\]
Thus, we obtain
\[
|U(t^*, x^*)| \leq (\mu_0)^{-1}C_3\|h_1^* - h_2^*\|_{C([0, T])},
\]
(24)

We next estimate
\[
\begin{aligned}
\hat{v}_1^*(t^*, x^*) &= \int_{t_1^*}^{t^*} T(t^* - \tau)\kappa \hat{v}_1^* \left( 1 - \frac{\hat{v}_1^*}{\hat{u}_1^* + \alpha} \right) d\tau \\
&\leq \kappa B(t^* - t_1^*) \leq \frac{\kappa B}{\mu_0} \|h_1^* - h_2^*\|_{C([0, T])}.
\end{aligned}
\]
Therefore,
\[
|\hat{V}(t^*, x^*)| \leq \frac{\kappa B}{\mu_0} \|h_1^* - h_2^*\|_{C([0, T])} \leq \tilde{C}_3\|h_1^* - h_2^*\|_{C([0, T])},
\]
(25)

with \( \tilde{C}_3 \) depending on \((A, B, \kappa, \delta, \mu, \mu_0)\).

Without loss of generality, we assume \( T \leq 1 \). Then by the inequalities (17), (21) and (24), we have
\[
|U(t^*, x^*)| \leq C_4[T|U|_{C([0, T] \times \mathbb{R}^+)} + |V|_{C([0, T] \times \mathbb{R}^+)} + \|h_1^* - h_2^*\|_{C([0, T])}]\] for \( x^* \in [0, h_M(t^*)] \),
where \( C_4 \) does not depend on \( T \) and \((t^*, x^*)\). Since \( |U(t^*, x^*)| = 0 \) for \( 0 \leq t^* \leq T \) and \( x^* \in \mathbb{R}^+ \setminus [0, h_M(t^*)] \), it follows that
\[
|U|_{C([0, T] \times \mathbb{R}^+)} \leq C_4[T|U|_{C([0, T] \times \mathbb{R}^+)} + |V|_{C([0, T] \times \mathbb{R}^+)} + \|h_1^* - h_2^*\|_{C([0, T])}].
\]
(26)

Hence if \( C_4T < \frac{1}{2} \), then we have
\[
|U|_{C([0, T] \times \mathbb{R}^+)} \leq 2C_4[|V|_{C([0, T] \times \mathbb{R}^+)} + \|h_1^* - h_2^*\|_{C([0, T])}].
\]
(27)

It follows from the inequalities (16), (19), (25) and (27) that
\[
|\hat{V}(t^*, x^*)| \leq C_5 T[|V|_{C([0, T] \times \mathbb{R}^+)} + \|h_1^* - h_2^*\|_{C([0, T])}]\]
for \( 0 \leq t^* \leq T \) and \( x^* \in [0, h_M(t^*)] \). This proves Claim 1.

From (15), we can obtain
\[
|\hat{V}|_{C([0, T] \times \mathbb{R}^+)} \leq C_5 T[|V|_{C([0, T] \times \mathbb{R}^+)} + \|h_1^* - h_2^*\|_{C([0, T])}]
\]
(28)

for \( 0 < T \leq \min\{\frac{1}{2C_4}, T_0\} \). To derive (14), we still need to estimate \( \|\hat{h}_1^* - \hat{h}_2^*\|_{C([0, T])} \).

Claim 2. For \( 0 < T \leq \min\{\frac{1}{2C_4}, T_0\} \), we have
\[
\|\hat{h}_1^* - \hat{h}_2^*\|_{C([0, T])} \leq C_6 T[|V|_{C([0, T] \times \mathbb{R}^+)} + \|h_1^* - h_2^*\|_{C([0, T])}],
\]
(29)

where \( C_6 \) depends on \((\mu, \rho, h_0, A, B, C_4)\).

For \( 0 \leq t \leq T \),
\[
\hat{h}_1^*(t) = h_0 + \mu \int_0^t \int_0^{h_1^*(\tau)} J(x - y)u_1^*(\tau, x)dydxdt
\]
Moreover, it follows from the definition of $\Sigma_T$ that

$$h_i(t) \leq h_0 + \frac{\epsilon_0}{4} \leq 2h_0.$$ 

By the calculation, we obtain

$$|\tilde{h}_1(t) - \tilde{h}_2(t)|$$

$$\leq \mu \int_0^t \left| \int_0^{h_1^i(\tau)} \int_{h_1^i(\tau)}^{+\infty} J(x-y)v_1^i(\tau, x) dy dx \right| d\tau$$

$$- \int_0^{h_2^i(\tau)} \int_{h_2^i(\tau)}^{+\infty} J(x-y)v_2^i(\tau, x) dy dx \right| d\tau$$

$$+ \mu \rho \int_0^t \left| \int_0^{h_1^i(\tau)} \int_{h_1^i(\tau)}^{+\infty} J(x-y)v_1^i(\tau, x) dy dx - \int_0^{h_2^i(\tau)} \int_{h_2^i(\tau)}^{+\infty} J(x-y)v_2^i(\tau, x) dy dx \right| d\tau$$

$$\leq 2h_0\mu T\|U\|_{C([0,T] \times \mathbb{R}^+)} + 2h_0\mu T\|V\|_{C([0,T] \times \mathbb{R}^+)}$$

$$+ \mu \int_0^t \left( \int_0^{h_2^i(\tau)} \int_{h_2^i(\tau)}^{+\infty} J(x-y)v_2^i(\tau, x) dy dx \right| d\tau$$

$$\leq 2h_0\mu T\|U\|_{C([0,T] \times \mathbb{R}^+)} + 2h_0\mu T\|V\|_{C([0,T] \times \mathbb{R}^+)}$$

$$+ 2\mu T(\mu + B\rho)\|h_1^i - h_2^i\|_{C([0,T])}.$$

Combining these estimates with (27), we can obtain (29). This proves Claim 2.

From (28) and (29), we deduce

$$\|\tilde{V}\|_{C([0,T] \times \mathbb{R}^+)} + \|\tilde{h}_1^i - \tilde{h}_2^i\|_{C([0,T])} \leq (C_5 + C_6)T\|\tilde{V}\|_{C([0,T] \times \mathbb{R}^+)} + \|h_1^i - h_2^i\|_{C([0,T])}.$$

Therefore, if we choose $\tilde{T}$ such that

$$0 < \tilde{T} \leq \min \left\{ T_0, 1, \frac{1}{2C_4}, \frac{1}{2(C_5 + C_6)} \right\}.$$
It follows that $\beta = \frac{1}{2}$ for any $T \in (0, \bar{T}]$. Thus $\mathcal{F}$ is a contraction mapping on $\Sigma_T$. Hence $\mathcal{F}$ has a unique fixed point $(v, h)$ in $\Sigma_T$, which gives a nonnegative solution $(u, v, h)$ of (3) for $t \in (0, T]$.

**Step 2.** We show that the solution $(u, v, h)$ of (3) for $t \in (0, T]$ is the unique nonnegative solution.

Let $(\bar{u}, \bar{v}, \bar{h})$ be an arbitrary solution of the problem (3) for $t \in (0, T]$. Since $(v, h)$ is the unique fixed point of $\mathcal{F}$ in $\Sigma_T$, the uniqueness conclusion will follow if we can show $(\bar{v}, \bar{h}) \in \Sigma_T$. We first give

$$\bar{u}(t, x) \leq A \text{ for } (t, x) \in \tilde{D}_{h, T}.$$  \hfill (30)

It is only need to show that the above inequality holds with $A$ replaced by $A + \epsilon$ for any given $\epsilon > 0$. Suppose it is not true. Due to $\bar{u}(0, x) < A + \epsilon =: A_\epsilon$, there exist $t^* \in (0, T]$ and $x^* \in (0, \bar{h}(t^*))$ such that

$$\bar{u}(t^*, x^*) = A_\epsilon, \bar{u}_t(t^*, x^*) \geq 0,$$

and

$$\bar{u}(t, x) < A_\epsilon \text{ for } t \in [0, t^*), x \in [0, \bar{h}(t)].$$

Define

$${\bar{\ell}}_\epsilon := \left\{ \begin{array}{l} 0, \quad x^* \in [0, h_0], \\
\bar{t}_x^* \quad x \in (h_0, \bar{h}(t^*)) \text{ and } x^* = \bar{h}(\bar{t}_x^*). \end{array} \right.$$ \hfill (31)

Then for $\bar{\ell}_\epsilon < t \leq t^*$, $\bar{v}(t, x^*)$ satisfies the problem

$$\left\{ \begin{array}{l} v_t \leq D \left[ \int_0^{\bar{h}(t)} J(x - y)v(t, y)dy - v(t, x) \right] + \kappa v \left( 1 - \frac{v}{A_\epsilon + \alpha} \right), \\
v(\bar{\ell}_\epsilon) = \bar{v}_0(x^*). \end{array} \right.$$ \hfill (31)

By a simple comparison argument, we obtain

$$\bar{v}(t, x^*) \leq B_\epsilon := \max \left\{ \frac{1 - A_\epsilon}{\delta}, \|v_0\|_\infty \right\} \text{ for } t \in [\bar{\ell}_\epsilon, t^*].$$

It follows that $\bar{u}(1 - \bar{u}) - \delta \bar{u} \bar{u} \leq A_\epsilon(1 - A_\epsilon) - \delta A_\epsilon \bar{v} \leq 0$. Hence

$$0 \leq \bar{u}_t(t^*, x^*) \leq \int_0^{\bar{h}(t^*)} J(x^* - y)\bar{u}(t^*, y)dy - \bar{u}(t^*, x^*).$$

Since $\bar{u}(t^*, \bar{h}(t^*)) = 0$, $\bar{u}(t^*, y) < A_\epsilon$ for $y \in (0, \bar{h}(t^*))$. It follows that

$$A_\epsilon = \bar{u}(t^*, x^*) \leq \int_0^{\bar{h}(t^*)} J(x^* - y)\bar{u}(t^*, y)dy < A \int_0^{\bar{h}(t^*)} J(x^* - y)dy \leq A_\epsilon.$$ 

This is a contradiction. So (30) hold.

Now we use (31) with $(t^*, x^*)$ replaced by $(t, x) \in D_{h,T}$ and $A_\epsilon$ replaced by $A$, to deduce

$$\bar{v}(t, x) \leq B \text{ for } (t, x) \in D_{h,T}.$$ 

In addition, for $t \in (0, T]$

$$\bar{h}'(t) = \mu \left[ \int_0^{\bar{h}(t)} \int_0^{+\infty} J(x - y)\bar{u}(t, x)dydx + \rho \int_0^{\bar{h}(t)} \int_0^{+\infty} J(x - y)\bar{v}(t, x)dydx \right] \leq \mu \bar{h}(t)(A + \rho B).$$

Thus we obtain

$$\bar{h}(t) \leq \bar{h}_0 e^{\mu(A + \rho B)t}. \hfill (32)$$
Therefore if we choose small enough $T$ so that
\[ \bar{h}_0e^{\mu(A+\rho)T} \leq \bar{h}_0 + \frac{\epsilon_0}{4}, \]
then
\[ h(t) \leq \bar{h}_0 + \frac{\epsilon_0}{4} \text{ for } t \in [0, T]. \]
Moreover, $h'(t) \geq \mu \epsilon_0$. Thus $(\bar{u}, \bar{v}, \bar{h}) \in \Sigma_T$. This proves the local existence and uniqueness of the solution to (3).

**Step 3.** Extension of the solution of (3) to $t \in (0, \infty)$.

By Step 2, we see the model (3) has a unique solution $(u, v, h)$ for some initial time interval $(0, T]$. For any $s \in (0, \tilde{T})$, $(u(s, \cdot), v(s, \cdot))$ satisfies (4) with $(0, h_0)$ replaced by $(0, h(s))$. This implies that we can treat $u(s, x)$ and $v(s, x)$ as the initial functions and use the above argument to extend the solution from $t = s$ to some $T' \geq \tilde{T}$. Suppose that $(0, \tilde{T})$ is the maximal interval that the solution $(u, v, h)$ of (3) can be defined through this extension process.

We will show that $\tilde{T} = \infty$. Otherwise $\tilde{T} \in (0, \infty)$ and we will derive a contradiction. Firstly, as above, we can similarly show that
\[ 0 \leq u \leq A, 0 \leq v \leq B \text{ for } (t, x) \in D_{h, \bar{T}}. \]

For $t \in (0, \tilde{T})$, since
\[
h'(t) = \mu \left[ \int_0^{h(t)} \int_0^{+\infty} J(x-y)u(t,x)dydx + \rho \int_0^{h(t)} \int_0^{+\infty} J(x-y)v(t,x)dydx \right],
\]
then
\[ h(t) \leq h_0e^{\mu(A+\rho)T} \text{ for } t \in (0, \tilde{T}). \]

Since $h(t)$ is monotone function in $(0, \tilde{T})$, we may define
\[ h(\tilde{T}) = \lim_{t \to \tilde{T}} h(t) \text{ with } h(\tilde{T}) \leq h_0e^{\mu(A+\rho)\tilde{T}}. \]

Denote
\[ \Omega_{\tilde{T}} = \{(t, x) : t \in (0, \tilde{T}), x \in (0, h(t))\}. \]
By (33) and $0 \leq u(t, x) \leq A, 0 \leq v(t, x) \leq B$ in $\Omega_{\tilde{T}}$, we know $h' \in L^\infty([0, \tilde{T})$) and $h \in C([0, \tilde{T}])$. It is easy to see that the right-hand sides of the first and the second equations in (3) belong to $L^\infty(\Omega_{\tilde{T}})$. It follows that $u_t, v_t \in L^\infty(\Omega_{\tilde{T}})$. Thus for each $x \in (0, h(\tilde{T}))$,
\[ u(\tilde{T}, x) = \lim_{t \to \tilde{T}} u(t, x) \text{ and } v(\tilde{T}, x) = \lim_{t \to \tilde{T}} v(t, x) \text{ exist.} \]

Next we show that $u(\tilde{T}, \cdot), v(\tilde{T}, \cdot)$ are continuous for $x \in (0, h(\tilde{T}))$. For any given constant $\varepsilon_0 > 0$ small enough, let
\[ \theta_1 = \sup \{\theta : [0, h(\tilde{T}) - \varepsilon_0] \subset (0, h(\tilde{T}) - \theta)\}. \]
For any $\varepsilon > 0$, we take $\theta_2 = \min \left\{ \frac{\varepsilon_1}{4C}, \theta_1 \right\}$. For any $x \in [0, h(\tilde{T}) - \varepsilon_0]$, we have
\[ |u(\tilde{T}, x) - u(\tilde{T} - \theta_2, x)| = \left| \int_{\tilde{T} - \theta_2}^{\tilde{T}} u_\tau(x) d\tau \right| \leq C\theta_2 \leq \frac{\varepsilon_1}{4}. \]
Lemma 3.1. Let \( 1.2 \) and \( 1.3 \). We start with two comparison results.

3. The long-time dynamical behavior. In this section, we will prove Theorems 1.2 and 1.3. We start with two comparison results.

Lemma 3.1. Let \( h_0, T > 0 \) and \( \Omega_0 := [0, T] \times [0, h_0] \). Suppose \((u(t, x), v(t, x))\) and \((u_1(t, x), v_1(t, x))\) are continuous in \( \Omega_0 \) and satisfy

\[
\begin{cases}
    u_t(t, x) \geq \int_{0}^{h_0} J(x - y)u(t, y)dy - u(t, x) + u(t, x), & 0 < t \leq T, x \in [0, h_0], \\
    v_t(t, x) \geq D \left[ \int_{0}^{h_0} J(x - y)v(t, y)dy - v(t, x) \right] + \kappa v(t, x), & 0 < t \leq T, x \in [0, h_0], \\
    u(0, x) \geq 0, v(0, x) \geq 0, & x \in [0, h_0].
\end{cases}
\]

Then \( u(t, x), v(t, x) \geq 0 \) for all \( 0 \leq t \leq T \) and \( 0 \leq x \leq h_0 \).

Proof. This results can be proved by the arguments in the proof of Lemma 2.1, the situation here is actually much simpler. We omit the details. \( \square \)
Lemma 3.2. Suppose that \( h, \bar{h} \in C([0,T]) \), \( \bar{u}, \bar{v} \in C(\bar{D}_{\bar{h}}, \bar{v}) \), \( u, v \in C(D_{h,T}) \), \( 0 \leq \bar{u}, \bar{v} \leq A \) and \( 0 \leq u, v \leq B \). If \( (\bar{u}, \bar{v}, \bar{h}) \) satisfies

\[
\begin{aligned}
\bar{u}_t & \geq \int_0^{\bar{v}(t)} J(x-y) \bar{u}(t,y) dy - \bar{u}(t,x) + \bar{u}(1-\bar{\alpha}), \quad 0 < t \leq T, x \in (0, \bar{h}(t)), \\
\bar{v}_t & \geq D \left[ \int_0^{\bar{h}(t)} J(x-y) \bar{v}(t,y) dy - \bar{v}(t,x) \right] \\
& \quad + \kappa \bar{v} \left( 1 - \frac{\bar{v}}{A + \alpha} \right), \quad 0 < t \leq T, x \in (0, \bar{h}(t)), \\
\bar{h}'(t) & \geq \mu \left[ \int_0^{\bar{h}(t)} \int_{\bar{h}(t)}^{+\infty} J(x-y) \bar{u}(t,y) dy dx \right] \\
& \quad + \rho \int_0^{\bar{h}(t)} \int_{\bar{h}(t)}^{+\infty} J(x-y) \bar{v}(t,y) dy dx \right], \quad 0 < t \leq T,
\end{aligned}
\]

and the couple \((u, h)\) satisfies

\[
\begin{aligned}
u & \leq \int_0^{h(t)} J(x-y) u(t,y) dy - u(t,x) + u(1-\delta B - u), \quad t > 0, x \in (0, h(t)), \\
h'(t) & \leq \mu \int_0^{h(t)} \int_{h(t)}^{+\infty} J(x-y) u(t,y) dy dx, \quad t > 0, \\
u_x(t, 0) & \geq 0, u(t, \bar{h}(t)) = 0, \quad t > 0, \\
0 & < u(0, x) \leq u_0(x), \bar{h}(0) \leq h_0, \quad x \in [0, h_0],
\end{aligned}
\]

and the couple \((v, h)\) satisfies

\[
\begin{aligned}
v & \leq D \left[ \int_0^{h(t)} J(x-y) v(t,y) dy - v(t,x) \right] + \kappa v \left( 1 - \frac{v}{\alpha} \right), \quad t > 0, x \in (0, h(t)), \\
v'(t) & \leq \mu \rho \int_0^{h(t)} \int_{h(t)}^{+\infty} J(x-y) v(t,y) dy dx, \quad t > 0, \\
v_x(t, 0) & \geq 0, v(t, \bar{h}(t)) = 0, \quad t > 0, \\
0 & < v(0, x) \leq v_0(x), \bar{h}(0) \leq h_0, \quad x \in [0, \bar{h}(0)].
\end{aligned}
\]

Then the unique solution \((u, v, h)\) of (3) satisfies

\[
0 \leq u \leq \bar{u}, \quad 0 \leq v \leq \bar{v}, \quad \bar{h}(t) \leq h(t) \leq \bar{h}(t) \text{ for } 0 < t \leq T, 0 \leq x \leq \bar{h}(t).
\]

Proof. Due to (4) and Lemma 2.1, we have \( \bar{u} > 0 \) for \( 0 < t \leq T \) and \( 0 < x < \bar{h}(t) \). For small \( \epsilon > 0 \), let \((u_\epsilon, v_\epsilon, h_\epsilon)\) denote the unique solution of (3) with \( h_0 \) replaced by \( h_0' := h_0(1 - \epsilon) \), \( \mu \) replaced by \( \mu_\epsilon := \mu(1 - \epsilon) \) and \((u_0', v_0')\) replaced by \((u_0', v_0')\) which satisfies

\[
0 \leq u_0'(x) < u_0(x), 0 \leq v_0'(x) < v_0(x) \text{ in } [0, h_0'],
\]

and

\[
(u_\epsilon \left( \frac{h_0'}{h_0} x \right), v_\epsilon \left( \frac{h_0'}{h_0} x \right)) \to (u_0(x), v_0(x)) \text{ as } \epsilon \to 0 \text{ in the } C([0, h_0]).
\]

We claim that \( h_\epsilon(t) < \bar{h}(t) \) for all \( t \in (0, T] \). Suppose that there exists \( t_1 \leq T \) such that

\[
h_\epsilon(t) < \bar{h}(t) \text{ for } t \in (0, t_1) \text{ and } h_\epsilon(t_1) = \bar{h}(t_1).
\]
Now we compare \((u_\epsilon, v_\epsilon)\) and \((\bar{u}, \bar{v})\) on \(\Omega_{\epsilon, t_1} := \{(t, x) | 0 < t \leq t_1, 0 < x < h_\epsilon(t)\}\). Let \(w = e^{\epsilon t}(\bar{u} - u_\epsilon)\) and \(z = e^{\epsilon t}(\bar{v} - v_\epsilon)\), then \((w, z)\) satisfies

\[
\begin{aligned}
  w_t &\geq \int_0^{h_\epsilon(t)} J(x-y)w(t, y)dy - w(t, x) \\
  z_t &\geq D \left[ \int_0^{h_\epsilon(t)} J(x-y)z(t, y)dy - z(t, x) \right] \\
  w(t, h_\epsilon(t)) &\geq 0, z(t, h_\epsilon(t)) \geq 0, \\
  w(0, x) &> 0, z(0, x) > 0,
\end{aligned}
\]

where \(C_i(t, x) \in L^\infty (i = 1, 2)\) and \(k_1, k_2\) large enough such that

\[
p_1 := k_1 - 1 + C_1(t, x) > 0, p_2 := k_2 - D + C_2(t, x) > 0
\]

for all \((t, x) \in \Omega_{\epsilon, t_1}\). By Lemma 2.1, it follows that \(\bar{u} > u_\epsilon\) and \(\bar{v} > v_\epsilon\) in \(\Omega_{\epsilon, t_1}\).

Furthermore, according to the definition of \(t_1\), we have \(h'_\epsilon(t_1) \geq \bar{h}'(t_1)\). Thus

\[
0 \geq \bar{h}'(t_1) - h'_\epsilon(t_1)
\]

\[
\geq \mu \left[ \int_0^{h_\epsilon(t_1)} \int_{h_\epsilon(t_1)}^{+\infty} J(x-y)\bar{u}(t_1, x)dydx + \rho \int_0^{h_\epsilon(t_1)} \int_{h_\epsilon(t_1)}^{+\infty} J(x-y)\bar{v}(t_1, x)dydx \right]
\]

\[
- \mu \left[ \int_0^{h_\epsilon(t_1)} \int_{h_\epsilon(t_1)}^{+\infty} J(x-y)u_\epsilon(t_1, x)dydx \right]
\]

\[
+ \rho \left[ \int_0^{h_\epsilon(t_1)} \int_{h_\epsilon(t_1)}^{+\infty} J(x-y)v_\epsilon(t_1, x)dydx \right]
\]

\[
\geq \mu \left[ \int_0^{h_\epsilon(t_1)} \int_{h_\epsilon(t_1)}^{+\infty} J(x-y)(\bar{u}(t_1, x) - u_\epsilon(t_1, x))dydx \right]
\]

\[
+ \rho \left[ \int_0^{h_\epsilon(t_1)} \int_{h_\epsilon(t_1)}^{+\infty} J(x-y)(\bar{v}(t_1, x) - v_\epsilon(t_1, x))dydx \right]
\]

\[
> 0,
\]

which is a contradiction. Then we have \(h_\epsilon(t) < \bar{h}(t)\) for all \(t \in (0, T]\). Moreover, we also have \(\bar{u} > u_\epsilon, \bar{v} > v_\epsilon\) in \(\Omega_{\epsilon, T}\).

Since the unique solution of (3) depends continuously on the parameters, when \(\epsilon \to 0\), we can obtain \(u \leq \bar{u}, v \leq \bar{v}\) and \(h(t) \leq \bar{h}(t)\) for \(0 < t \leq T, 0 \leq x \leq h(t)\).

Moreover, the proof of (35) and (36) is similar to the proof of Theorem 3.1 of [7]. Hence we omit the details. \(\Box\)

By the comparison principle, we can get the following result. To stress the dependence on the parameter \(\mu\), we use \((u_\mu, v_\mu, h_\mu)\) to denote the solution of (3).

**Corollary 1.** Assume that (J) hold. If \(\mu_1 \leq \mu_2\), we have \(u_{\mu_1}(t, x) \leq u_{\mu_2}(t, x), v_{\mu_1}(t, x) \leq v_{\mu_2}(t, x), h_{\mu_1}(t) \leq h_{\mu_2}(t)\) for \(0 < t \leq T, 0 \leq x \leq h_{\mu_1}(t)\).

Define the operator \(\Xi_I + a : C(I) \to C(I)\) by

\[
(\Xi_I + a)[\phi](x) =: d \left[ \int_I J(x-y)\phi(y)dy - \phi(x) \right] + a(x)\phi(x),
\]
where $I$ is an open bounded interval in $[0, +\infty)$, $a \in C(\bar{I})$. The generalized principal eigenvalue of $\mathcal{L}_I + a$ is given by

$$\lambda_p(\mathcal{L}_I + a) := \inf\{\lambda \in \mathbb{R} : (\mathcal{L}_I + a)[\phi] \leq \lambda \phi \text{ in } I \text{ for some } \phi \in C(\bar{I}), \phi > 0\}.$$ 

Here we considered the properties of $\lambda_p(\mathcal{L}_I + a_0)$ for $a_0$ a positive constant. By the proposition 3.4 of [7], we have the following lemma.

**Lemma 3.3.** Assume that the kernel function $J$ satisfied (J), $a_0$ is a positive constant, then we have that

(i) if $0 < a_0 < d$, there exists $l^*(d, a_0) > 0$ such that

$$\lambda_p(\mathcal{L}_I + a_0) = 0 \text{ if } |I| = l^*(d, a_0),$$

$$\lambda_p(\mathcal{L}_I + a_0) < 0 \text{ if } |I| < l^*(d, a_0),$$

$$\lambda_p(\mathcal{L}_I + a_0) > 0 \text{ if } |I| > l^*(d, a_0),$$

where $|I|$ denotes its length.

(ii) If $a_0 \geq d$, $\lambda_p(\mathcal{L}_I + a_0) > 0$ for any finite open interval $I \subseteq [0, +\infty)$.

Next we give some well known conclusions for nonlocal diffusion equations on some fixed spatial domains. Considering the following model

$$\begin{align*}
    u_t &= d \left[ \int_0^l J(x-y)u(t,y)dy - u(t,x) \right] + u(a - bu), \quad t > 0, x \in (0, l), \\
    u(0,x) &= u_0(x), \quad x \in (0, l),
\end{align*}$$

(37)

we have the following important proposition.

**Proposition 1.** (Proposition 3.5 of [7]) Denote $I = (0, l)$ and suppose (J) hold. Then (37) admits a unique positive steady state $u_I$ if and only if $\lambda_p(\mathcal{L}_I + a) > 0$.

Moreover, for $u_0(x) \in C(\bar{I})$, $u_0 \geq \gamma, \not\equiv 0$, (37) has a unique solution $u(t,x)$ defined for all $t > 0$, and it converges to $u_I$ as $t \to +\infty$ when $\lambda_p(\mathcal{L}_I + a) > 0$; when $\lambda_p(\mathcal{L}_I + a) \leq 0$, $u(t,x)$ converges to $0$ as $t \to +\infty$.

In order to discuss the spreading of the species, we will use Proposition 3.6 of [7]. We restate these results here for the reader’s convenience.

**Proposition 2.** (Proposition 3.6 of [7]) Assume (J) hold. Then there exists $L > 0$ such that $\lambda_p(\mathcal{L}_{(0,l)} + a) > 0$ for $l > L$ and hence (37) admits a unique positive steady state $u_{(0,l)}$. Moreover,

$$\lim_{t \to +\infty} u_{(0,l)} = \frac{a}{b} \text{ locally uniformly in } [0, \infty).$$

By the proposition, we will establish the following lemmas.

**Lemma 3.4.** Let $M \geq 0$ and (J) hold. For any given $\epsilon > 0$ and $l_\epsilon > 0$, there exist $l > l_\epsilon$ such that, if the continuous and non-negative function $U(t,x)$ satisfies

$$\begin{align*}
    U_t - d \left[ \int_0^l J(x-y)U(t,y)dy - U(t,x) \right] &\geq U(a - bU), \quad t > 0, 0 < x < l, \\
    U_x(t,0) = 0, U(t,l) &\geq M, \quad t > 0,
\end{align*}$$

(38)

and $U(0,x) > 0$ for $x \in [0, l)$, then

$$\liminf_{t \to +\infty} U(t,x) > \frac{a}{b} - \epsilon \text{ uniformly on } [0, l_\epsilon].$$
Lemma 3.5. Assume (J) hold. For any given $\epsilon > 0$ and $l_\epsilon > 0$, there exist $l > l_\epsilon$ such that, if the continuous and non-negative function $V(t, x)$ satisfies

$$\begin{align*}
V_t - d \int_0^t J(x - y) V(t, y) dy - V(t, x) &\leq V(a - b V), \quad t > 0, 0 < x < l, \\
V_x(t, 0) = 0, V(t, l) = 0, &\quad t > 0,
\end{align*}$$

and $V(0, x) > 0$ for $x \in [0, l)$, then

$$\limsup_{t \to +\infty} V(t, x) < \frac{a}{b} + \epsilon \text{ uniformly on } [0, l].$$

It is clearly seen that $h'(t) > 0$ for all $t > 0$. Therefore, we note $\lim_{t \to +\infty} h(t) = h_\infty \in (h_0, +\infty)$. Furthermore, we will discuss the spreading-vanishing dichotomy.

Lemma 3.6. Suppose that $(u, v, h(t))$ is the solution of (3). If $h_\infty = +\infty$, then we have

$$\lim_{t \to +\infty} u(t, x) = p_0 \text{ and } \lim_{t \to +\infty} v(t, x) = q_0 \text{ locally uniformly in } [0, +\infty).$$

Proof. Since $h_\infty = +\infty$, then for any $l_\epsilon$, there exists $T_1 > 0$ and $l_1 > 0$ such that $l_1 > l_\epsilon$ for $t > T_1$ and then $u$ satisfies

$$\begin{align*}
\begin{cases}
u_t - \int_0^{l_1} J(x - y) u(t, y) dy - u(t, x) &\leq u(1 - u), \quad t > T_1, 0 < x < l_1, \\
u_x(t, 0) = 0, u(t, l_1) &\leq M, \quad t > T_1, \\
u(T_1, x) &> 0, \quad x \in [0, l_1),
\end{cases}
\end{align*}$$

where $M = \max\{A, B\}$. Applying Lemma 3.5, we can obtain that

$$\limsup_{t \to +\infty} u(t, x) < 1 + \epsilon \text{ uniformly in } [0, l_\epsilon].$$

Since $\epsilon$ and $l_\epsilon$ are arbitrary small, then $\limsup_{t \to +\infty} u(t, x) \leq 1 =: \bar{u}_1$ uniformly in $[0, +\infty)$.

Set $l_2 > l_\epsilon$. By the last conclusion, there exists $T_2 > T_1$ such that $u(t, x) < \bar{u}_1 + \epsilon$ for $t > T_2$ and $0 < x < l_2$. Then we can deduce that $v$ satisfies

$$\begin{align*}
\begin{cases}
u_t - D \int_0^{l_2} J(x - y) v(t, y) dy - v(t, x) &\leq \kappa v \left(1 - \frac{v}{\bar{u}_1 + \epsilon + \alpha}\right), \quad t > T_2, 0 < x < l_2, \\
u_x(t, 0) = 0, v(t, l_2) &\leq M, \quad t > T_2, \\
v(T_2, x) &> 0, \quad x \in [0, l_2),
\end{cases}
\end{align*}$$

We can use Lemma 3.5 again to obtain $\limsup_{t \to +\infty} v(t, x) < \bar{u}_1 + \alpha + 2\epsilon$ uniformly in $[0, l_\epsilon]$.

By the arbitrariness of $\epsilon$ and $l_\epsilon$, we can deduce that $\limsup_{t \to +\infty} v(t, x) \leq \bar{u}_1 + \alpha =: \bar{v}_1$ uniformly in $[0, +\infty)$.

Let $l_3 > l_\epsilon$. From the above discussion, there exists $T_3 > T_2$ such that $v(t, x) < \bar{v}_1 + \epsilon$ and $u(t, x) > 0$ for $t > T_3$ and $0 < x < l_3$. Then $u$ satisfies

$$\begin{align*}
\begin{cases}
u_t - \int_0^{l_3} J(x - y) u(t, y) dy - u(t, x) &\geq u(1 - u) - \delta u(\bar{v}_1 + \epsilon), \quad t > T_3, 0 < x < l_3, \\
u_x(t, 0) = 0, u(t, l_3) &> 0, \quad t > T_3, \\
u(T_3, x) &> 0, \quad x \in [0, l_3).
\end{cases}
\end{align*}$$
By Lemma 3.4, we can get $\liminf_{t \to +\infty} u(t, x) > 1 - \delta \bar{v}_1 - \delta \epsilon - \epsilon$ uniformly in $[0, l_\epsilon]$. According the arbitrariness of $\epsilon$ and $l_\epsilon$, it follows that $\liminf_{t \to +\infty} u(t, x) \geq 1 - \delta \bar{v}_1 =: \bar{u}_1 > 0$ because of the hypothesis $\delta \alpha + \delta < 1$.

Choose $l_4 > l_\epsilon$. In view of the above result, there exists $T_4 > T_3$ such that $u(t, x) > \bar{u}_1 - \epsilon$ for $t > T_4$ and $0 < x < l_4$. Then $v$ satisfies

$$
\begin{aligned}
&v_t - D \left[ \int_0^{l_4} J(x-y) v(t, y) dy - v(t, x) \right] \geq \kappa v \left( 1 - \frac{v}{\bar{u}_1} - \epsilon + \alpha \right), & t > T_4, 0 < x < l_4, \\
v_x(t, 0) = 0, v(t, l_4) = 0, & t > T_4, \\
v(T_4, x) > 0, & x \in [0, l_4).
\end{aligned}
$$

(43)

Using Lemma 3.4 again, we have $\liminf_{t \to +\infty} v(t, x) > \bar{u}_1 + \alpha$ uniformly in $[0, l_\epsilon]$. Since $\epsilon$ and $l_\epsilon$ are arbitrary, we obtain $\liminf_{t \to +\infty} v(t, x) \geq \bar{u}_1 + \alpha =: \bar{v}_1$.

Denote $l_5 > l_\epsilon$. By the above conclusion, there exists $T_5 > T_4$ such that $v(t, x) > \bar{u}_1 - \epsilon$ for $T_5$ and $0 < x < l_5$. Then we can establish that $u$ satisfies

$$
\begin{aligned}
&u_t - \left[ \int_0^{l_5} J(x-y) u(t, y) dy - u(t, x) \right] \leq u(1 - u) - \delta u(\bar{u}_1 - \epsilon), & t > T_5, 0 < x < l_5, \\
u_x(t, 0) = 0, u(t, l_5) \leq M, & t > T_5, \\
u(T_5, x) > 0, & x \in [0, l_5).
\end{aligned}
$$

(44)

According to Lemma 3.5, we have $\limsup_{t \to +\infty} u(t, x) < 1 - \delta \bar{u}_1 + \delta \epsilon + \epsilon$ uniformly in $[0, l_\epsilon]$. Using the arbitrariness of $\epsilon$ and $l_\epsilon$, it follows that $\limsup_{t \to +\infty} u(t, x) \leq 1 - \delta \bar{u}_1 =: \bar{u}_2 > 0$ uniformly in $[0, +\infty)$.

Set $l_6 > l_\epsilon$, there exists $T_6 > T_5$ such that $u(t, x) < \bar{u}_2 + \epsilon$ for $T_6$ and $0 < x < l_6$ and then $v$ satisfies

$$
\begin{aligned}
&v_t - D \left[ \int_0^{l_6} J(x-y) v(t, y) dy - v(t, x) \right] \leq \kappa v \left( 1 - \frac{v}{\bar{u}_2 + \epsilon + \alpha} \right), & t > T_6, 0 < x < l_6, \\
v_x(t, 0) = 0, v(t, l_6) \leq M, & t > T_6, \\
v(T_6, x) > 0, & x \in [0, l_6).
\end{aligned}
$$

(45)

Applying Lemma 3.5 again, we have $\limsup_{t \to +\infty} v(t, x) < \bar{u}_2 + \alpha + 2 \epsilon$ uniformly in $[0, l_\epsilon]$. Considering the arbitrariness of $\epsilon$ and $l_\epsilon$, we have $\limsup_{t \to +\infty} v(t, x) \leq \bar{u}_2 + \alpha =: \bar{v}_2$ uniformly in $[0, +\infty)$.

Furthermore, choose $l_7 > l_\epsilon$, we know that there exists $T_7 > T_6$ such that $v(t, x) < \bar{v}_2 + \epsilon$ and $u(t, x) > 0$ for $T_7, 0 < x < l_7$. Then we can obtain that $u$ satisfies

$$
\begin{aligned}
&u_t - \left[ \int_0^{l_7} J(x-y) u(t, y) dy - u(t, x) \right] \geq u(1 - u) - \delta u(\bar{v}_2 + \epsilon), & t > T_7, 0 < x < l_7, \\
u_x(t, 0) = 0, u(t, l_7) = 0, & t > T_7, \\
u(T_7, x) > 0, & x \in [0, l_7).
\end{aligned}
$$

(46)

By Lemma 3.4, we have $\liminf_{t \to +\infty} u(t, x) > 1 - \delta \bar{v}_2 - \epsilon$ uniformly in $[0, l_\epsilon]$. Again using the arbitrariness of $\epsilon$ and $l_\epsilon$, it follows that $\liminf_{t \to +\infty} u(t, x) \geq 1 - \delta \bar{v}_2 =: \underline{u}_2$ uniformly in $[0, +\infty)$. 
We continue to use the above approach and give $l_8 > l_ε$. In view of above result, there exists $T_8 > T_7$ such that $u(t, x) > \frac{u_2}{2} - ε$ for $t > T_8, 0 < x < l_8$ and $v$ satisfies

$$\begin{cases} v_t - D \int_0^{l_8} J(x - y) v(t, y) dy - v(t, x) \geq κv \left(1 - \frac{v}{\frac{u_2}{2} - ε + α}\right), & t > T_8, 0 < x < l_8, \\ v_x(t, 0) = 0, v(t, l_8) = 0, \\ v(T_8, x) > 0, \end{cases}$$

Thus, $\bar{v}$ from below and the sequences $\{v_i\}$ converge to the unique steady state $W$. According to Lemma 3.4 again, we have $\liminf_{t \to +∞} v(t, x) > \frac{u_2}{2} + α$ uniformly in $[0, l_ε]$. Due to the arbitrariness of $ε$ and $l_ε$, it follows that $\liminf_{t \to +∞} v(t, x) ≥ \frac{u_2}{2} + α =: \bar{v}_2$ uniformly in $[0, +∞)$.

Furthermore, we continue the above way to obtain the following sequences

$$u_1 ≤ · · · ≤ u_i ≤ · · · ≤ \liminf_{t \to +∞} u(t, x) ≤ \limsup_{t \to +∞} u(t, x) ≤ · · · ≤ \bar{u}_i ≤ · · · ≤ \bar{u}_1,$$

and

$$\bar{v}_1 ≤ · · · ≤ \bar{v}_i ≤ · · · ≤ \liminf_{t \to +∞} v(t, x) ≤ \limsup_{t \to +∞} v(t, x) ≤ · · · ≤ \bar{v}_i ≤ · · · ≤ \bar{v}_1,$$

where $u_1 = 1 - δ\bar{v}_i, \bar{u}_i = 1 - δ\bar{u}_{i-1}, \bar{v}_i = \bar{u}_i + α$ and $\bar{v}_i = \bar{u}_i + α$ for $i = 1, 2, · · ·$.

Since the sequences $\{\bar{u}_i\}$ and $\{\bar{v}_i\}$ are monotone non-increasing and bounded from below and the sequences $\{u_i\}$ and $\{v_i\}$ are monotone non-decreasing and bounded from above, the limits of these sequences exist as $i \to +∞$ and denote $\bar{u}, \bar{v}, \bar{u}$ and $\bar{v}$, respectively. Then we have

$$\bar{u} = 1 - δ\bar{v}, \bar{u} = 1 - δ\bar{u}, \bar{v} = \bar{u} + α \text{ and } \bar{v} = \bar{u} + α.$$

Thus,

$$\begin{cases} \bar{u} = 1 - δ(\bar{u} + α), \\ \bar{u} = 1 - δ(\bar{u} + α). \end{cases}$$

Since $δα + δ < 1$, we can easily get that $\bar{u} = \bar{u} = p_0, \bar{v} = \bar{v} = q_0$, and it implies that

$$\liminf_{t \to +∞} u(t, x) = \limsup_{t \to +∞} u(t, x) = p_0 \text{ and } \liminf_{t \to +∞} v(t, x) = \limsup_{t \to +∞} v(t, x) = q_0.$$

This completes the proof.

Lemma 3.7. If $h_∞ < +∞$, then we have $\lim_{t \to +∞} ||u(t, ·)||_{C((0, h(t)))} = 0$, $\lim_{t \to +∞} ||v(t, ·)||_{C((0, h(t)))} = 0$ and $\lambda_p(\mathcal{L}(0, h_∞) + 1) < 0$, $\lambda_p(\mathcal{L}(0, h_∞) + 1) < 0$.

Proof. We first prove that

$$\lambda_p(\mathcal{L}(0, h_∞) + 1) < 0 \text{ and } \lambda_p(\mathcal{L}(0, h_∞) + 1) < 0.$$

Suppose that $\lambda_p(\mathcal{L}(0, h_∞) + 1) > 0$ and $\lambda_p(\mathcal{L}(0, h_∞) + 1) > 0$. Then $\lambda_p(\mathcal{L}(0, h_∞) + 1) > 0$ and $\lambda_p(\mathcal{L}(0, h_∞) + 1) > 0$ for small $ε ∈ (0, 1)$. Moreover, there exists $T_ε > 0$ such that $h(t) > h_∞ - ε$ for $t > T_ε$. Considering the following model

$$\begin{cases} w_t = \int_{h_∞ - ε}^{h_∞} J(x - y) w(t, y) dy - w(t, x) + w(1 - w), & t > T_ε, x ∈ [ε, h_∞ - ε], \\ w(T_ε, x) = u(T_ε, x), & x ∈ [ε, h_∞ - ε]. \end{cases}$$

Since $\lambda_p(\mathcal{L}(0, h_∞) + 1) > 0$, Proposition 1 indicates that the solution $w_ε(t, x)$ converge to the unique steady state $W_ε(x)$ uniformly in $[ε, h_∞ - ε]$ as $t \to +∞$. By
Thus there exists $T_{1,\epsilon} > T_1$ such that
\[ u(t, x) \geq \frac{1}{2} W_\epsilon(x) > 0 \text{ for } t > T_{1,\epsilon}, x \in [\epsilon, h_\infty - \epsilon]. \]

By the similar argument, due to $\lambda_p(\mathcal{L}(\epsilon, h_\infty - \epsilon) + \kappa) > 0$, we can deduce that
\[ v(t, x) \geq \frac{1}{2} Z_\epsilon(x) > 0 \text{ for } t > T_{2,\epsilon}, x \in [\epsilon, h_\infty - \epsilon]. \]

Since $\mathcal{J}(0) > 0$, there exist $\epsilon_0 > 0$ and $\delta_0 > 0$ such that $\mathcal{J}(x) > \delta_0$ if $|x| < \epsilon_0$.

Thus for $0 < \epsilon < \min \left\{ \epsilon_1, \frac{1}{2}\epsilon_0 \right\}$ and $t > \max\{T_{1,\epsilon}, T_{2,\epsilon}\}$, we have
\[
h'(t) = \mu \left[ \int_0^{h(t)} \int_0^{+\infty} J(x - y)u(t, x)dydx + \rho \int_0^{h(t)} \int_0^{+\infty} J(x - y)v(t, x)dydx \right] \\
\geq \mu \left[ \int_\epsilon^{h_\infty - \epsilon} \int_\epsilon^{+\infty} J(x - y)u(t, x)dydx + \rho \int_\epsilon^{h_\infty - \epsilon} \int_\epsilon^{+\infty} J(x - y)v(t, x)dydx \right] \\
\geq \mu \left[ \int_{h_\infty - \frac{\epsilon}{2}}^{h_\infty} \int_{h_\infty - \frac{\epsilon}{2}}^{h_\infty} \delta_0 \cdot \frac{1}{2} W_\epsilon(x)dydx + \rho \int_{h_\infty - \frac{\epsilon}{2}}^{h_\infty} \int_{h_\infty - \frac{\epsilon}{2}}^{h_\infty} \delta_0 \cdot \frac{1}{2} Z_\epsilon(x)dydx \right] > 0.
\]

This implies $h_\infty = +\infty$, it is a contradiction with the assumption $h_\infty < \infty$. So we have
\[ \lambda_p(\mathcal{L}(0, h_\infty)) + 1 \leq 0 \text{ and } \lambda_p(\mathcal{L}(0, h_\infty) + \kappa) \leq 0. \]

Now we establish $\lim_{t \to +\infty} ||u(t, \cdot)||_{C((0, h(t))]} = 0$ and $\lim_{t \to +\infty} ||v(t, \cdot)||_{C((0, h(t))]} = 0$.

Let $\bar{u}(t, x), \bar{v}(t, x)$ denote the unique solution of
\[
\begin{align*}
\bar{u}_t &= \int_0^{h_\infty} J(x - y)\bar{u}(t, y)dy - \bar{u}(t, x) + \bar{u}(1 - \bar{u}), & t > 0, x \in [0, h_\infty], \\
\bar{v}_t &= D \int_0^{h_\infty} J(x - y)\bar{v}(t, y)dy - \bar{v}(t, x) + \kappa \bar{v} \left( 1 - \frac{\bar{v}}{A + \alpha} \right), & t > 0, x \in [0, h_\infty], \\
\bar{u}(0, x) &= \bar{u}_0(x), \bar{v}(0, x) = \bar{v}_0(x), & x \in [0, h_\infty],
\end{align*}
\]

where
\[ \bar{u}_0(x) = u_0(x), \bar{v}_0(x) = v_0(x) \text{ if } 0 \leq x \leq h_0, \]
\[ \bar{u}_0(x) = 0, \bar{v}_0(x) = 0 \text{ if } x > h_0. \]

By Lemma 3.2, it follows that $u(t, x) \leq \bar{u}(t, x)$ and $v(t, x) \leq \bar{v}(t, x)$ for $t > 0, x \in [0, h(t)]$.

Since
\[ \lambda_p(\mathcal{L}(0, h_\infty)) + 1 \leq 0, \lambda_p(\mathcal{L}(0, h_\infty) + \kappa) \leq 0. \]

Proposition 1 implies that $\bar{u}(t, x) \to 0$ and $\bar{v}(t, x) \to 0$ as $t \to +\infty$ uniformly in $x \in [0, h_\infty]$. Thus $\lim_{t \to +\infty} ||u(t, \cdot)||_{C((0, h(t))]} = 0$ and $\lim_{t \to +\infty} ||v(t, \cdot)||_{C((0, h(t))]} = 0$.

This completes the proof.

Combining Lemma 3.6 and Lemma 3.7, we can establish Theorem 1.2. Next we will show some criteria about the spreading and vanishing for the free boundary problem (3).

From Lemma 3.3, we can see that if $\kappa \geq D$, then
\[ \lambda_p(\mathcal{L}(0, h_\infty)) + 1 > 0, \lambda_p(\mathcal{L}(0, h_\infty) + \kappa) > 0. \]
According to Lemma 3.7, we can deduce the following conclusion.

**Theorem 3.8.** When $\kappa \geq D$, spreading always happens for (3).

**Lemma 3.9.** Suppose $0 < \kappa < D$. If $h_\infty < \infty$, then $h_\infty \leq \min\{l^*(1,1-\epsilon), l^*(D, \kappa)\} := h_*$ for some small enough $\epsilon > 0$. For the case, two species will vanish eventually. Furthermore, $h_0 \geq h_*$ implies that $h_\infty = +\infty$, that is to say, spreading will always happens.

**Proof.** By Lemma 3.7, if $h_\infty < \infty$, then

$$\lim_{t \to +\infty} ||u(t, \cdot)||_{C((0, h(t)))} = 0 \quad \text{and} \quad \lim_{t \to +\infty} ||v(t, \cdot)||_{C((0, h(t)))} = 0. \quad (48)$$

In the following, we suppose $h_\infty > \min\{l^*(1,1-\epsilon), l^*(D, \kappa)\}$ to get some contradictions.

If $h_\infty > l^*(1,1-\epsilon)$, there exists $T > 0$ such that $h(T) > l^*(1,1-\epsilon)$ and $v(t, x) \leq \frac{\epsilon}{\delta}$ for $t > T$ and $x \in [0, h(t)]$. Let $u(t, x)$ be the solution of the following problem

\[
\begin{aligned}
&u_t - \int_0^{h(T)} J(x-y)u(t,y)dy - u(t, x) = u(1-\epsilon, u), & t > T, x \in (0, h(T)), \\
&u(x, 0) = u(t, h(T)) = 0, & t > T, \\
&u(T, x) = u(T, x), & 0 < x < h(T).
\end{aligned}
\]

By the comparison principle, we have $u(t, x) \leq u(t, x)$ for all $t > T$ and $0 < x < h(T)$. Since $0 < 1 - \epsilon < 1$ and $h(T) > l^*(1,1-\epsilon)$, the Proposition 3.6 of [7] yields $\lim_{t \to +\infty} u(t, x) \geq \lim_{t \to +\infty} u(t, x) > 0$, which is a contradiction with the first equality of (48).

When $h_\infty > l^*(D, \kappa)$, there exists $T > 0$ such that $h(T) > l^*(D, \kappa)$ and $u(t, x) \geq 0$ for all $t > T$ and $0 < x < h(T)$. Let $v(t, x)$ be the solution of the following equation

\[
\begin{aligned}
&v_t - D \left[ \int_0^{h(T)} J(x-y)v(t,y)dy - v(t, x) \right] = \kappa v \left( 1 - \frac{v}{\alpha} \right), & t > T, x \in (0, h(T)), \\
v(x, 0) = v(t, h(T)) = 0, & t > T, \\
v(T, x) = v(T, x), & 0 < x < h(T).
\end{aligned}
\]

By the comparison principle, we have $v(t, x) \leq v(t, x)$ for all $t > T$ and $0 < x < h(T)$. Since $0 < \kappa < D$ and $h(T) > l^*(D, \kappa)$, by the Proposition 3.6 of [7], we have $\lim_{t \to +\infty} v(t, x) \geq \lim_{t \to +\infty} v(t, x) > 0$, which is a contradiction with the second equality of (48).

Finally, by the above arguments, we can see that if $h_0 \geq \min\{l^*(1,1-\epsilon), l^*(D, \kappa)\}$ and $h'(t) > 0$ for all $t > 0$, then $h_\infty = +\infty$. \(\Box\)

**Lemma 3.10.** If $0 < \kappa < D$ and $h_0 < h_*$, then there exists $\mu > 0$ such that $h_\infty < \infty$ for $0 < \mu < \mu$.

**Proof.** Fix $h_1 \in (h_0, h_*)$, we consider the following problem

\[
\begin{aligned}
&u_t - \int_0^{h_1(T)} J(x-y)u(t,y)dy - u(t, x) = u(1-\epsilon, u), & t > T, x \in (0, h_1(T)), \\
&u(x, 0) = u(t, h_1(T)) = 0, & t > T, \\
&u(T, x) = u(T, x), & 0 < x < h_1(T).
\end{aligned}
\]
and denote its unique solution by \((\hat{u}(t, x), \hat{v}(t, x))\). We choose \(h_1\) to guarantee
\[
\lambda_1 := \lambda_0(\Omega_{(0,h_1)} + 1) < 0 \text{ and } \lambda_2 := \lambda_0(\Omega_{(0,h_1)} + \kappa) < 0.
\]
Set \(\phi_1, \phi_2 > 0\) be the corresponding normalized eigenfunction of \(\lambda_1, \lambda_2\) and
\[
(\Omega_{(0,h_1)} + 1)[\phi_1](x) = \lambda_1 \phi_1(x) \text{ and } (\Omega_{(0,h_1)} + \kappa)[\phi_2](x) = \lambda_2 \phi_2(x) \text{ for } x \in [0,h_1].
\]
Furthermore,
\[
\hat{u}_t = \int_0^{h_1} J(x-y)\hat{u}(t, y)dy - \hat{u} + \hat{u}(1 - \hat{u})
\]
\[
\leq \int_0^{h_1} J(x-y)\hat{u}(t, y)dy - \hat{u}(t, x) + \hat{u}.
\]
Similarly,
\[
\hat{v}_t = D \left[ \int_0^{h_1} J(x-y)\hat{v}(t, y)dy - \hat{v} \right] + \kappa \hat{v} \left( 1 - \frac{\hat{v}}{A + \alpha} \right)
\]
\[
\leq D \left[ \int_0^{h_1} J(x-y)\hat{v}(t, y)dy - \hat{v}(t, x) \right] + \kappa \hat{v}.
\]
Next we define \(u_1 = C_1e^{\lambda_1 t/4} \phi_1\) and \(v_1 = C_2e^{\lambda_2 t/4} \phi_2\) for \(C_1, C_2 > 0\). It is easy to check that
\[
\int_0^{h_1} J(x-y)u_1(t, y)dy - u_1 + u_1 - u_{1t}
\]
\[
=C_1e^{\lambda_1 t/4} \left[ \int_0^{h_1} J(x-y)\phi_1(y)dy - \phi_1(x) + \phi_1 - \frac{\lambda_1}{4} \phi_1 \right]
\]
\[
< \frac{3\lambda_1}{4} C_1 e^{\lambda_1 t/4} \phi_1 < 0.
\]
Similarly,
\[
D \left[ \int_0^{h_1} J(x-y)v_1(t, y)dy - v_1 \right] + \kappa v_1 - v_{1t} < \frac{3\lambda_2}{4} C_2 e^{\lambda_2 t/4} \phi_2 < 0.
\]
Choosing \(C_1, C_2 > 0\) large such that \(C_1 \phi_1 > u_0, C_2 \phi_2 > v_0\) in \([0,h_1]\). Then by Lemma 3.1, we can deduce that
\[
\hat{u}(t, x) \leq u_1(t, x) \leq C_1e^{\lambda_1 t/4}, \hat{v}(t, x) \leq v_1(t, x) \leq C_2e^{\lambda_2 t/4} \text{ for } t > 0, x \in [0,h_1]
\]
Now define
\[
\hat{h}(t) = h_0 + \mu \left[ h_1 C_1 \int_0^t e^{\lambda_1 s/4} ds + \rho h_1 C_2 \int_0^t e^{\lambda_2 s/4} ds \right] \text{ for } t \geq 0.
\]
For \(0 < \mu \leq \mu := \frac{-\lambda_1 \lambda_2 (h_1 - h_0)}{4h_1 (C_1 \lambda_2 + \rho C_2 \lambda_1)}\), we can obtain that for \(t > 0\),
\[
\hat{h}(t) = h_0 - \mu h_1 C_1 \frac{4}{\lambda_1} \left(1 - e^{\lambda_1 t/4}\right) - \mu \rho h_1 C_2 \frac{4}{\lambda_2} \left(1 - e^{\lambda_2 t/4}\right) \leq h_0 - \mu h_1 C_1 \frac{4}{\lambda_1} - \mu \rho h_1 C_2 \frac{4}{\lambda_2} \leq h_1.
\]
Then by (49), \(\hat{u}(t, x)\) and \(\hat{v}(t, x)\) satisfies
\[
\begin{cases}
\hat{u}_t \geq \int_0^{\hat{h}(t)} J(x-y) \hat{u}(t, y)dy - \hat{u} + \hat{u}(1-\hat{u}), & t > 0, x \in [0, \hat{h}(t)], \\
\hat{v}_t \geq D \left[ \int_0^{\hat{h}(t)} J(x-y) \hat{v}(t, y)dy - \hat{v}\right] + \kappa \hat{v} \left(1 - \frac{\hat{v}}{A + \alpha}\right), & t > 0, x \in [0, \hat{h}(t)].
\end{cases}
\]
Due to (50), it is easy to check that
\[
\begin{align*}
\int_0^{\hat{h}(t)} \int_0^{+\infty} J(x-y) \hat{u}(t, x)dydx + \rho \int_0^{\hat{h}(t)} \int_0^{+\infty} J(x-y) \hat{v}(t, x)dydx < h_1 C_1 e^{\lambda_1 t/4} + \rho h_1 C_2 e^{\lambda_2 t/4}.
\end{align*}
\]
Thus,
\[
\hat{h}'(t) = \mu h_1 C_1 e^{\lambda_1 t/4} + \mu \rho h_1 C_2 e^{\lambda_2 t/4} = \mu \left(h_1 C_1 e^{\lambda_1 t/4} + \rho h_1 C_2 e^{\lambda_2 t/4}\right) \geq \mu \int_0^{\hat{h}(t)} \int_0^{+\infty} J(x-y) \hat{u}(t, x)dydx + \rho \int_0^{\hat{h}(t)} \int_0^{+\infty} J(x-y) \hat{v}(t, x)dydx.
\]
Combining (51), (52), \(\hat{u}(0, x) = u_0(x), \hat{v}(0, x) = v_0(x), \hat{h}(0) = h_0\) and using Lemma 3.2, we have
\[
u(t, x) \leq \hat{u}(t, x), v(t, x) \leq \hat{v}(t, x) \text{ and } h(t) \leq \hat{h}(t) \text{ for } t > 0, x \in [0, \hat{h}(t)].
\]
Thus, \(h_\infty \leq \lim_{t \to +\infty} \hat{h}(t) \leq h_1 < +\infty\). This completes the proof. \(\square\)

**Lemma 3.11.** Assume \(0 < \kappa < D\) and \(h_0 < h_\ast\), there exists \(\tilde{\mu} > 0\) such that \(h_\infty = +\infty\) for \(\mu > \tilde{\mu}\).

**Proof.** Considering the following problem
\[
\begin{align*}
\begin{cases}
\psi_t = D \left[ \int_0^{\hat{h}(t)} J(x-y) \psi(t, y)dy - \psi\right] + \kappa \psi \left(1 - \frac{\psi}{A}\right), & t > 0, x \in (0, \hat{h}(t)), \\
h'(t) = \mu \int_0^{\hat{h}(t)} \int_0^{+\infty} J(x-y) \psi(t, y)dydx, & t > 0, \\
\psi(0, t) = 0, \psi(t, \hat{h}(t)) = 0, & t > 0, \\
\hat{h}(0) = h_0, \psi(0, x) = v_0(x), & x \in [0, \hat{h}(0)].
\end{cases}
\end{align*}
\]
By Lemma 3.2, we have that \(\hat{h}(t) \leq h(t)\) and \(\psi(t, x) \leq v(t, x)\) for \(t > 0, 0 < x < \hat{h}(t)\).
Using Theorem 3.13 of [7], if \(\hat{h}(0) = h_0 < h_\ast \leq l'(D, \kappa)\), then there exists a constant \(\mu_1 > 0\) such that \(\hat{h}(t) = +\infty\) for \(\mu > \mu_1\). Then it follows \(h_\infty = +\infty\).
Assume \( \|v_0\| \leq \frac{\epsilon}{\delta} \). From the following model

\[
\begin{align*}
\frac{u}{t} &= \int_{0}^{h(t)} J(x-y)w(t,y)dy - u + u(1 - \epsilon - u), \quad t > 0, \ x \in (0, h(t)), \\
\frac{h'}{t} &= \mu \int_{0}^{h(t)} \int_{h(t)}^{+\infty} J(x-y)w(t,y)dydx, \quad t > 0, \\
\frac{w_x}{t}(0,0) &= 0, \frac{w}{t}(0, h(t)) = 0, \quad t > 0, \\
\frac{w}{t}(0, x) &= w_0(x), \ h(0) = h_0, \quad x \in [0, h_0],
\end{align*}
\]

and Lemma 3.2, we can deduce that \( h(t) \leq \bar{h}(t) \) and \( y(t, x) \leq v(t, x) \) for \( t > 0, 0 < x < \bar{h}(t) \). Since \( \bar{h}(0) = h_0 < h_* \leq \hat{l}^*(1, 1 - \epsilon) \) and Theorem 3.13 of [7], we know that \( h(t) = +\infty \) for some \( \mu > \mu_2 \). Therefore when \( \bar{\mu} > \min\{\mu_1, \mu_2\} \), we have \( \bar{h}_\infty = +\infty \).

By Theorem 3.8 and Lemma 3.9-3.11, we can establish Theorem 1.3, that is the spreading-vanishing dichotomy. At the same time, we can obtain the critical length of the habitat which can be dominated by the spreading-vanishing dichotomy. At the same time, we can obtain the critical length of the habitat which can be dominated by \( h_* \). For \( h_0 \geq h_* \), or \( h_0 < h_* \) but \( \mu > \bar{\mu} \), the two species will spread successfully. When \( h_0 < h_* \) and \( \mu \leq \bar{\mu} \), the two species will disappear eventually.

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