On the Spectrum of the Orr-Sommerfeld Equation on the semi-axis

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Abstract. The Orr-Sommerfeld equation is a spectral problem which is known to play an important role in hydrodynamic stability. For an appropriate operator theoretical realization of the equation, we will determine the essential spectrum, and calculate an enclosure of the set of all eigenvalues by elementary analytical means.

1. Introduction

The Orr-Sommerfeld equation is an non-selfadjoint general eigenvalue problem of the form

\[ (-D^2 + a^2)^2 u + iaR[V \cdot (-D^2 + a^2)u + V'' \cdot u] = \lambda (-D^2 + a^2)u \quad \text{on } I, \]

subject to Dirichlet boundary conditions for $u$ on the real interval $I$. Here, $D := d/dx$, $i$ is the imaginary unit, and $R > 0$ is the Reynolds number of an underlying fluid which moves in a stationary flow (perpendicular to $I$) with given real-valued flow profile $V \in C^2(I)$. This flow is perturbed by a single-mode perturbation with wave number $a > 0$, and the physical question of stability or instability of the underlying flow in response to this perturbation arises. This question is closely related to the spectrum of (an appropriate operator-theoretical realization of) the Orr-Sommerfeld equation (1.1). Essentially the flow is unstable (with respect to the wave number $a$) if (1.1) has an eigenvalue with negative real part.

In this article, we will exclusively be concerned with the case $I = [0, \infty)$ corresponding to the half-plane flow along a wall (an overview of results for this case can be found in [3]). So the boundary conditions to be added to (1.1) read

\[ u(0) = u'(0) = u(\infty) = u'(\infty) = 0. \]

A flow profile $V$ which is of particular interest in this case (but not under exclusive concern here) is the Blasius profile defined by $V := f'$, where $f$ is the solution of the
nonlinear boundary value problem (Blasius equation, a special case of the Falkner-Skan equation)

\begin{equation}
2f''' + ff'' = 0 \text{ on } [0, \infty), \quad f(0) = f'(0) = 0, \lim_{x \to \infty} f'(x) = 1,
\end{equation}

which can be shown to exist and to be unique, and moreover, to provide (for \( V = f' \))

\begin{equation}
V \to 1, \quad V' \to 0, \quad V'' \to 0 \text{ (exponentially) as } x \to \infty,
\end{equation}

\begin{equation}
V > 0, \quad V' > 0, \quad V'' < 0 \text{ on } (0, \infty).
\end{equation}

![Blasius profile together with its first two derivatives](image)

We choose the following operator theoretical setting for problem (1.1), (1.2), which we believe is the most natural and simple one:

Let \( X \) and \( Y \) denote the complex Banach spaces \( H^2(0, \infty) \cap H^1_0(0, \infty) \) and \( L^2(0, \infty) \), respectively, and let \( D(A) := H^4(0, \infty) \cap H^2_0(0, \infty) \). The operators \( A : D(A) \subset X \to Y \) and \( B : X \to Y \) are defined by \( Au \) and \( Bu \) denoting the left-hand and the right-hand side (without \( \lambda \)) of equation (1.1), respectively. It can be shown that \( A \) is closed (see [1, Theorem IX.9.5], plus the remark that the norm in \( X \) is stronger than the one in \( L^2(0, \infty) \)), and that \( B \) is one-to-one and onto, and bounded with bounded inverse. Our formulation of (1.1), (1.2) now reads

\begin{equation}
Au = \lambda Bu.
\end{equation}

So we choose a direct operator theoretical realization of the Orr-Sommerfeld problem, rather than a formulation via an auxiliary operator constructed by Riesz’s representation lemma, as done e.g. in [3].

Our goal in this article is twofold: In Section 2, we will determine the essential spectrum of problem (1.5) exactly; the main tool is the well-known theorem on preservation of the essential spectrum under relatively compact perturbations. In Section 3, an enclosure for the set of all eigenvalues of problem (1.5) is calculated by elementary
analytical means. Here, exploiting sign restrictions on the profile $V$ and its derivatives (such as (1.4)), we obtain results which extend enclosures known in the literature (see [2], [5]), even if those have been obtained by more involved methods. In [4] corresponding results have been obtained for the case of a compact interval $I$. For further results on the Orr–Sommerfeld equation on a compact interval see [6] and the literature cited there.

2. The essential spectrum

Since several definitions of essential spectra are around in the literature (see [1, Chapter IX] for an overview), we start with the definition which we are using.

With $X, Y$ denoting two complex Banach spaces, and $A : D(A) \subset X \to Y$ a closed linear and $B : X \to Y$ a bounded linear operator, we call the set

$$\sigma_{\text{ess}}(A, B) := \{ \lambda \in \mathbb{C} : A - \lambda B \text{ is not a Fredholm operator of index 0} \}$$

the essential spectrum of the problem $Au = \lambda Bu$ or of the pencil $(A, B)$. Here, a closed linear operator $T : D(T) \subset X \to Y$ is called a Fredholm operator if its range is closed and has finite codimension $d(T)$ in $Y$, and if its nullspace has finite dimension $n(T)$; it has index 0 if $d(T) = n(T)$. Furthermore,

$$\varrho(A, B) := \{ \lambda \in \mathbb{C} : A - \lambda B \text{ is one-to-one and onto} \}$$

denotes the resolvent set of the problem $Au = \lambda Bu$ (or of the pencil $(A, B)$), and $\sigma(A, B) := \mathbb{C} \setminus \varrho(A, B)$ its (total) spectrum. It is quite obvious that $\sigma_{\text{ess}}(A, B) \subset \sigma(A, B)$ and that $\sigma(A, B) \setminus \sigma_{\text{ess}}(A, B)$ consists only of the eigenvalues of finite geometric multiplicity.

We return to the specific choice for $X, Y, A, B$ introduced in Section 1, i.e., describing the Orr-Sommerfeld problem. Our main result of the present section is

**Theorem 2.1.** Suppose that the flow profile $V$ satisfies $V \to 1, V'' \to 0$ as $x \to \infty$ (compare (1.4)). With $R$ and $a$ denoting the parameters in (1.4), the essential spectrum of $(A, \lambda A)$ is given by

$$\sigma_{\text{ess}}(A, B) = \{ \mu + a^2 + i a R : \mu \in [0, \infty) \}.$$  

**Proof.** We introduce the auxiliary operators

$$A_0 : D(A) \to Y, \quad A_0 u := (-D^2 + a^2)^2 u + i a R (-D^2 + a^2) u,$$

$$K : D(A) \to Y, \quad K u := i a R [(V - 1) \cdot (-D^2 + a^2) u + V'' \cdot u]$$

so that $A = A_0 + K$. Since $V - 1$ and $V''$ tend to zero and $K$ is of lower order than $A_0$, it follows from [1, Theorem IX.8.2] that $K$ is relatively compact to $A_0$, i.e., for each $\| \cdot \|_X$-bounded sequence $(u_n)$ in $D(A)$ such that $\|A_0 u_n\|$ is bounded in $Y$, $(K u_n)$ contains a convergent subsequence. Strictly speaking, the cited Theorem in [1]
It remains to be shown that we call $\sigma_B$ a Fredholm operator of index 0 if and only if $B^{-1}$ is relatively compact to $B^{-1}A_0 : D(A) \to X$.

Now the invariance of the essential spectrum under relatively compact perturbations (see e.g. [1, Theorem IX.2.1]) provides

$$\sigma_{\text{ess}}(B^{-1}A + B^{-1}K, i\lambda) = \sigma_{\text{ess}}(B^{-1}A_0, i\lambda).$$

The cited theorem in [1] is formulated for densely defined operators, but the proof clearly does not use this assumption. (This density is a general assumption in [1, Chapter IX] to be able to work with adjoint operators, which however do not occur in Theorem IX.2.1 or in its proof.) Since $B$ and $B^{-1}$ are bounded, so that $A - \lambda B$ is a Fredholm operator of index 0 if and only if $B^{-1}A - \lambda id_X$ is, we obtain furthermore $\sigma_{\text{ess}}(A, B) = \sigma_{\text{ess}}(B^{-1}A, i\lambda)$, so that (2.2) and the identity $A_0 + K = A$ provide

$$\sigma_{\text{ess}}(A, B) = \sigma_{\text{ess}}(B^{-1}A_0, i\lambda).$$

It remains to be shown that $\sigma_{\text{ess}}(B^{-1}A_0, i\lambda)$ equals the right-hand side of (2.1) which we call $M$ from now on. This is done in two steps: We prove

a) $\mathcal{C}\setminus M \subset \varrho(B^{-1}A_0, i\lambda)$ (implying $M \supset \sigma(B^{-1}A_0, i\lambda) \supset \sigma_{\text{ess}}(B^{-1}A_0, i\lambda)$).

b) $M \subset \sigma_{\text{ess}}(B^{-1}A_0, i\lambda)$.

First we observe that, with $E(x) := e^{-ax}$,

$$B^{-1}A_0u = -u'' + (a^2 + iaR)u + u''(0) \cdot E \quad \text{for } u \in D(A),$$

since the expression on the right-hand side is in $X$ (for $u \in D(A)$) and $B$ applied to it equals $A_0u$.

ad a) Let $\lambda \in \mathcal{C}\setminus M$, so that $\lambda = \mu + a^2 + iaR$ with $\mu \in \mathcal{C}\setminus [0, \infty)$. Thus, $\mu \in \varrho(C, \varrho(id_X))$ where $C : D(C) \subset Y \to Y$ is given by $D(C) := H^2(0, \infty) \cap H^1(0, \infty)$, $C \mu := u''$. To prove that $B^{-1}A_0 - \lambda id_X : D(A) \to X$ is onto, let $r \in X$ and define $v := (C - \mu)^{-1}r \in D(C) = X, w := (C - \mu)^{-1}E \in D(C) = X$. It is easy to calculate $w$ in closed form and to show that $w'(0) \neq 0$. Therefore, $u := v - \frac{v'(0)}{w'(0)}w \in X$ solves the equation

$$-u'' - \mu u + \frac{v'(0)}{w'(0)}E = r$$

and moreover, $u(0) = u'(0) = 0$. Since $u, E, r \in H^2(0, \infty)$, (2.3) implies $u'' \in H^2(0, \infty)$. Therefore $u \in H^4(0, \infty)$. Altogether, $u \in D(A)$. Since $u(0) = r(0) = 0$ and $E(0) = 1$, (2.3) shows that $u''(0) = v'(0)/w'(0)$, and (2.5) then yield $(B^{-1}A_0 - \lambda)u = r$.

To prove that $B^{-1}A_0 - \lambda id_X$ is one-to-one, let $u \in D(A)$ satisfy $(B^{-1}A_0 - \lambda)u = 0$, i.e.,

$$-u'' - \mu u + u''(0)E = 0.$$
Thus, \( u \in D(C) \) and \((C-\mu)u = -u''(0)E \), i.e., \( u = -u''(0)w \). Since \( u'(0) = 0, u''(0) \neq 0 \), this implies \( u''(0) = 0 \) and therefore \( u = 0 \).

**ad b)** Let \( \lambda \in M \), so that \( \lambda = \mu + a^2 + iaR \) with \( \mu \in [0, \infty) \). Let \( \gamma \in \Phi \) and \( r := \gamma(1-E)E \in X \), and suppose that, for some \( u \in D(A) \), \((B^{-1}A_0 - \lambda)u = r \), so that \((2.4)\) yields
\[
-u'' - \mu u + u''(0)E = \gamma(1-E)E.
\]

We will prove that \((2.6)\), together with the fact \( u \in D(A) \), implies \( \gamma = 0 \) and \( u = 0 \), which shows that \( B^{-1}A_0 - \lambda id_X \) is one-to-one but not onto. Consequently, it is no Fredholm operator of index zero, which implies \( \lambda \in \sigma_{\text{ess}}(B^{-1}A_0, id_X) \).

The general solution of \((2.6)\) satisfies
\[
u = c_1 \varphi_1 + c_2 \varphi_2 + \frac{u''(0) - \gamma E}{a^2 + \mu} + \frac{\gamma}{4a^2 + \mu} E^2
\]
with \( c_1, c_2 \in \Phi \), where \( \varphi_1(x) = \cos(\sqrt{\mu}x) \) and \( \varphi_2(x) = \sin(\sqrt{\mu}x) \) if \( \mu > 0 \), \( \varphi_1(x) = 1 \) and \( \varphi_2(x) = x \) if \( \mu = 0 \). The condition \( u \in L^2(0, \infty) \) requires \( c_1 = c_2 = 0 \), and \( u(0) = u'(0) = 0 \) then implies
\[
\frac{\gamma - u''(0)}{a^2 + \mu} = \frac{\gamma}{4a^2 + \mu}, \quad a \frac{\gamma - u''(0)}{a^2 + \mu} = 2a \frac{\gamma}{4a^2 + \mu}
\]
which indeed yields \( \gamma = 0, u''(0) = 0 \), so that \((2.7)\) provides \( u \equiv 0 \). \( \square \)

It is easy to generalize Theorem 2.1 for flow profiles which satisfy \( V \to c \in \mathbb{R} \) instead of \( V \to 1 \).

**Corollary 2.2.** Suppose that the flow profile \( V \) satisfies \( V \to c \in \mathbb{R}, V'' \to 0 \) as \( x \to \infty \). Then the essential spectrum of \((L,A)\) is given by
\[
\sigma_{\text{ess}}(A,B) = \{ \mu + a^2 + iaR \, c : \mu \in [0, \infty) \}.
\]

### 3. An enclosure for the set of all eigenvalues

In this final section, we will calculate a set enclosing all eigenvalues of problem \((L,A)\).

Since this set will also contain the straight line which was identified as \( \sigma_{\text{ess}}(A,B) \) in Theorem 2.1, it therefore encloses the total spectrum \( \sigma(A,B) \).

We wish to put emphasize on the simplicity of the methods we use, which neverthless provide more accurate enclosures than those known in the literature [3, 4], if sign restrictions such as \((L,A)\) are exploited.

Let real constants \( V_{\min}, V_{\max}, |V'|_{\max}, V''_{\min}, V''_{\max} \) be given such that
\[
(3.1) V_{\min} \leq V(x) \leq V_{\max}, |V'(x)| \leq |V'|_{\max}, V''_{\min} \leq V''(x) \leq V''_{\max} \quad \text{for} \quad x \in [0, \infty)
\]
for the flow profile \( V \). Let \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) denote the usual inner product and norm in \( L^2(0, \infty) \).
For any eigenpair \((u, \lambda) \in D(A) \times \Phi\) of problem (1.5) we obtain (note that \(B\) is positive definite)

\[
\lambda = \frac{\langle Au, u \rangle}{\langle Bu, u \rangle}
\]

which is the basis of our enclosures. Partial integration and the condition \(u \in D(A) = H^4(0, \infty) \cap H^2_0(0, \infty)\) provide

\[
(Au, u) = \langle (-D^2 + a^2)u, u \rangle + i\lambda \langle V \cdot (-D^2 + a^2)u + V'' \cdot u, u \rangle
\]

so that (3.2) yields

\[
\lambda = \beta_1 + i\lambda \beta_2 - \beta_3
\]

where

\[
\beta_1 := \frac{\|Bu\|^2}{\langle Bu, u \rangle} \geq \frac{\|Bu\|}{\|u\|} \geq a^2,
\]

\[
\beta_2 := \frac{\langle V u', u' \rangle + a^2 \langle Vu, u \rangle}{\langle Bu, u \rangle} = V(\xi) \frac{\langle u', u' \rangle + a^2 \langle u, u \rangle}{\langle Bu, u \rangle}
\]

\[
\beta_3 := \frac{\langle V' u, u' \rangle}{\langle Bu, u \rangle}, \quad |\beta_3| \leq |V'|_{\max} \frac{\|u\| \|u'\|}{\|u'\|^2 + a^2 \|u\|^2} \leq \frac{|V'|_{\max}}{2a}.
\]

From (3.3) to (3.6) we obtain the following eigenvalue enclosure result, where sums and products of sets are to be understood in the canonical sense.

**Theorem 3.1.** All eigenvalues of problem (1.5) are contained in the set

\[
[a^2, \infty) + i\lambda R[V_{\min}, V_{\max}] + \frac{R}{2} |V'|_{\max} \cdot \Delta,
\]

with \(\Delta\) denoting the closed unit disc in \(\Phi\).

Of course, separate bounds for the real and imaginary parts of the eigenvalues can easily be extracted (with loss of information!) from Theorem 3.1:

**Corollary 3.2.** For each eigenvalue \(\lambda\) of problem (1.5),

\[
a^2 - \frac{R}{2} |V'|_{\max} \leq \Re \lambda \leq aRV_{\min} - \frac{R}{2} |V'|_{\max},
\]

\[
aRV_{\max} - \frac{R}{2} |V'|_{\max} \leq \Im \lambda \leq aRV_{\max} + \frac{R}{2} |V'|_{\max}.
\]

Figures 2 and 3 illustrate the results of Theorem 3.1, Corollary 3.2, and Theorem 3.3. Concretely, we have chosen \(V\) to be the Blasius profile and \(a = 0.179, R = 580\) here.
The bounds given in Corollary 3.2 have been obtained in [2] already, by more involved analytical means. These results can be substantially improved by the following simple calculation, if sign restrictions such as (1.4) are used:

\[
\text{Re} \, \langle V' u, u' \rangle = \frac{1}{2} (\langle V' u, u' \rangle + \langle V'' u, u' \rangle) = \frac{1}{2} \int_0^\infty V'(|u|^2)' \, dx = -\frac{1}{2} \int_0^\infty V'' |u|^2 \, dx = -\frac{1}{2} V''(\xi) \langle u, u \rangle \quad \text{(for some } \xi \in (0, \infty))
\]

so that \( \beta_3 \) defined in (3.6) satisfies

\[
\text{Re} \beta_3 = -\frac{1}{2} V''(\xi) \frac{\langle u, u \rangle}{(Bu, u)} \in -\frac{1}{2} [V''|u|^2] \cdot |0, \frac{1}{a^2}|
\]

which possibly restricts the enclosure set for \( \beta_3 \) used in Theorem 3.1 (which was obtained from (1.4)). Instead of formulating a corresponding improved theorem for the most general situation, we concentrate on the case where \( V'' \leq 0 \), which is true e.g. for the Blasius profile (see (1.4)). Then (3.7) yields \( \text{Re} \beta_3 \geq 0 \), which together with (3.3) to (3.6) provides the following improvement of Theorem 3.1:

**Theorem 3.3.** If \( V'' \leq 0 \), all eigenvalues of problem (1.4) are contained in the set

\[
[a^2, \infty) + iaR[V_{\min}, V_{\max}] + \frac{R}{2} |V'|_{\max} \cdot \Delta^-,
\]

where \( \Delta^- := \{ z \in \Delta : \text{Im } z \leq 0 \} \).

As in Corollary 3.2, we can extract separate bounds for real and imaginary parts of the eigenvalues, which provides here the improved bound

\[
aRV_{\min} - \frac{R}{2} |V'|_{\max} \leq \text{Im } \lambda \leq aRV_{\max}.
\]
In particular all eigenvalues lie "below" the essential spectrum, i.e. have imaginary parts less or equal $aR$.

![Illustration of Theorem 3.3](image)

**Figure 3: Illustration of Theorem 3.3**

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