Scaling Properties of Long-Range Correlated Noisy Signals

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Abstract

The Hurst coefficient $H$ of a stochastic fractal signal is estimated using the function $\sigma_{MA}^2 = \frac{1}{N_{\text{max}} - n} \sum_{i=n}^{N_{\text{max}}} (y(i) - \bar{y}_n(i))^2$, where $\bar{y}_n(i)$ is defined as $1/n \sum_{k=0}^{n-1} y(i - k)$, $n$ is the dimension of moving average box and $N_{\text{max}}$ is the dimension of the stochastic series. The ability to capture scaling properties by $\sigma_{MA}^2$ can be understood by observing that the function $C_n(i) = y(i) - \bar{y}_n(i)$ generates a sequence of random clusters having power-law probability distribution of the amplitude and of the lifetime, with exponents equal to the fractal dimension $D$ of the stochastic series.
Power-law distributions are important manifestations of scale invariance as observed in fractal, percolating structures and self-organized dynamical systems. The scaling exponents of power law statistics are indeed related to the universality class of the system and thus are helpful to understand the fundamental processes ruling the dynamics of complex systems. The scaling parameters have been practically deployed in fields as different as biology, geophysics, solar physics, social science. When random time records should be analyzed the Hurst exponent $H$, which is related to the fractal dimension of the time series by $D = 2 - H$, is usually evaluated. For example, healthy and sick heart beat rate can be distinguished on the basis of the value of $H$. Financial series with degree of persistence higher than that of the price series have been encountered. A number of frequency, time and, even, integrated domain approaches have been thus developed to gain as accurate as possible estimate of these exponents. Such procedures generally consist in calculating appropriate statistical functions from the whole signal: in the time domain, Detrended Fluctuation Analysis (DFA) and Rescaled Range Analysis (R/S) are the most popular scaling methods to extract power-law correlation exponents from the random signals $y(i)$ [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

A generalized variance $\sigma^2_{MA}$ of the noisy signal $y(i)$ with respect to $\tilde{y}_n(i)$, defined by:

$$\sigma^2_{MA} = \frac{1}{N_{max} - n} \sum_{i=n}^{N_{max}} C_n(i)^2,$$

(1)

with:

$$C_n(i) = y(i) - \tilde{y}_n(i),$$

(2)

has been introduced in [9]. In Eq.1, $N_{max}$ is the size of the series, $\tilde{y}_n(i)$ is the moving average defined by $\tilde{y}_n(i) = \frac{1}{n} \sum_{k=0}^{n-1} y(i - k)$ and $n$ is the moving average window.

The function $\sigma_{MA}$ corresponding to each $\tilde{y}_n(i)$ has been calculated for random series $y(i)$ with different Hurst exponent $H$. The values of $\sigma_{MA}$ are plotted as a function of $n$ on log-log axes shown in Fig.[1], for a series with $N_{max} = 2^{19}$ and $10 \leq n \leq 10^4$. The most remarkable property of the curves plotted in Fig.[1] is their power-law dependence on $n$, i.e.:

$$\sigma_{MA} \approx n^H.$$  

(3)

The structure of the algorithm based on the $\sigma_{MA}$ function is analogous to the DFA technique, but a higher accuracy has been observed. This may be due to the better smogthing
and detrending action of the moving average with respect to the linear or to the polynomial fit. Furthermore, the moving average filter dynamically detrends the series. Every time the discrete index \( i \) increases by a unity, the box window \( n \) switches its position accordingly, allowing also for online applications of the present technique. In order to gain a deeper insight of the function \( \sigma_{MA} \), we will show that \( C_n(i) \) generates, for each \( \bar{y}_n(i) \), a sequence of clusters with amplitude and lifetime distributed as power-laws. Furthermore the exponent of these power-laws is equal to the fractal dimension \( D = 2 - H \) of the time series. These properties allow to understand the ability of \( \sigma_{MA} \) to estimate the Hurst exponent \( H \) of the stochastic series.

Consider thus the sub-set of \( y(i) \) corresponding to the region delimited by two consecutive intersections between \( y(i) \) and \( \bar{y}_n(i) \). Let us refer to such regions as cluster. In Fig.\[2\], one of these clusters is shown. For each cluster \( j \), the size \( l_j \):

\[
l_j = \sum_{i=i_c(j)}^{i_c(j+1)} y(i) \quad (4)
\]

and the duration \( \tau_j \):

\[
\tau_j = i_c(j+1) - i_c(j). \quad (5)
\]

can be defined. In the previous relationships, \( i_c(j) \) and \( i_c(j+1) \) are the values taken by the index \( i \) in correspondence of two subsequent intersections between \( \bar{y}_n(i) \) and \( y(i) \). The size \( l_j \) and the duration \( \tau_j \) of the clusters are shown in Fig.\[2\] for a series with \( N_{\text{max}} = 2^{19}, H = 5.3 \) and \( n = 30 \).

The probability distribution \( P(z) \) of the cluster sizes has been determined by counting the number of clusters with length \( l \). Fig.\[3\] shows the log-log plot of the distribution \( P(l) \) for series having different \( H \) (respectively \( H = 0.2, H = 0.5 \) and \( H = 0.8 \)). The clusters are obtained by the intersection of the series with a moving average with a window amplitude \( n = 60 \). The curves are consistent with a straight line, indicating a power-law behavior:

\[
P(l) \approx l^{-\alpha} \quad (6)
\]

The linear behavior over two decades indicates a scaling distribution of clusters. The exponent \( \alpha \) has been plotted against \( H \) can the inset of Fig.\[3\].
The probability distribution \( P(\tau) \) of the cluster lifetime has been determined by counting the number of clusters with lifetime \( \tau \). Fig.\[4\] shows the log-log plot of the distribution \( P(\tau) \) for series having different \( H \) (respectively \( H = 0.2, H = 0.4 \) and \( H = 0.8 \)). The clusters are obtained by the intersection of the series with a moving average with a window amplitude \( n = 60 \). This leads to another line indicating a distribution of lifetimes of the form:

\[
P(\tau) \approx \tau^{-\beta}.
\] (7)

The values of \( \alpha \) and \( \beta \), for \( H \) varying from 0.10 to 0.90, are reported in Table (1). The distributions \( P(l) \) and \( P(\tau) \) have been calculated for a wide range of values of \( N_{\text{max}} \) and \( n \) (\( 2^{18} < N_{\text{max}} < 2^{21} \) and \( 60 < n < 1000 \)), the exponents \( \alpha \) and \( \beta \) have been found to be independent of \( N_{\text{max}} \) and \( n \). It can be concluded that \( \alpha \) and \( \beta \) vary as \( 2 - H \), i.e. as the fractal dimension \( D \) of the noisy signal.

These results can be understood keeping in mind the box-counting method to estimate the fractal dimension of a random signal is recalled (Ch.10 of \[1\]). The stochastic signal is covered with boxes of width \( b\tau \) (in time) and of length \( ba \) (in amplitude), with \( \tau \cdot a \) the minimum box. The box dimension \( D_B \) is related to the number \( N(b; a, \tau) \) of the boxes needed to cover the record by the relationship: \( N(b; a, \tau) \approx b^{-D_B} \), where \( D_B = 2 - H \) for self-affine records.

We have demonstrated that the Hurst exponent \( H \) of long-range correlated stochastic series \( y(i) \) can be calculated using the function \( \sigma_{MA} \) defined by the Eq.(1). We have found indeed the remarkable result that the function \( \sigma_{MA} \) varies as a power-law of the amplitude \( n \) of the moving average box. This work shed more light on the results of the papers \[5, 9\] and reinforce the general idea that a deeper physical insight may add new perspectives to the practical methods of financial analysis.

These results, appeared in very preliminary form in \[9\], have been here validated in terms of the statistical properties of the random clusters tracked by \( \tilde{y}_n(i) \) over the random time series. Furthermore, the power-law behavior of the distributions of cluster size and lifetime, just as expected from general arguments about dynamical system, clarifies the intrinsic capability of the function \( \sigma_{MA} \) to capture scaling exponents of the series. Furthermore, the exponents \( \alpha \) and \( \beta \) of Eqs.(6,7) are insensitive to the parameters characterizing \( \tilde{y}_n(i) \) and \( y(i) \). This robustness is essential to the assessment of the proposed scaling technique. Last but not least, the \( \sigma_{MA} \) algorithm yields in higher accuracy, speed of execution and possibility
of online application.

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FIG. 1: Log-log plot of the function $\sigma_{MA}$ vs. the moving average box $n$. The curves have been obtained using the computational algorithm described in the text. The curves refers to artificially generated series having $N_{\text{max}} = 2^{19}$ and Hurst coefficient $H$ varying between 0.10 and 0.90. A power-law dependence ($\sigma_{MA} \propto n^H$) is found. The results are independent of the series dimension $N_{\text{max}}$. 
FIG. 2: Stochastic series $y_n(i)$ obtained by the Random Midpoint Displacement algorithm with $H = 0.3$. The size of the series is $N_{max} = 2^{19}$. The moving average $\tilde{y}_n(i)$, with box dimension $n = 30$, is also shown. The total length of the segment and the time interval between two subsequent crossing points represent respectively the size and the duration (lifetime) of the cluster according to the definition given in the text (Eqs. (4,5)).
FIG. 3: Log-log plot of distribution $P(l)$ of the cluster size $l$ for series having different $H$ (respectively $H = 0.2$, $H = 0.5$ and $H = 0.8$). The clusters are obtained by the intersection of the series with a moving average with a window amplitude $n = 60$. A power-law dependence $P(l) \approx l^{-\alpha}$ has been observed. It is worth-mentioning that the exponent $\alpha$ depends on the Hurst exponent of the series, but is independent of the series dimension $N_{\text{max}}$ and of the moving average box $n$. 
FIG. 4: Log-log plot of the distribution $P(\tau)$ of the cluster lifetime $\tau$ for series having different $H$ (respectively $H = 0.2$, $H = 0.5$ and $H = 0.8$). The clusters are obtained by the intersection of the series with a moving average with a window amplitude $n = 60$. The lifetime of the cluster corresponds to the time interval between two consecutive intersection points, as defined by the Eq. (5). A power-law dependence $P(\tau) = \tau^{-\beta}$ has been observed. It is worth-mentioning that the exponent $\beta$ depends on the Hurst exponent of the series, but is independent of the dimension $N_{\text{max}}$ and of the moving average box $n$. 