A Gauge Field Model On
$SU(2)_L \times SU(2)_R \times U(1)_Y \times \pi_4(G_{YM})$

Bin Chen  Han-Ying Guo  Hong-Bo Teng

Institute of Theoretical Physics, Academia Sinica, P. O. Box 2735, Beijing 100080, China.

Abstract

We construct a gauge field model based on $SU(2)_L \times SU(2)_R \times U(1)_Y \times \pi_4(SU(2)_L \times SU(2)_R \times U(1)_Y)$ from the principle that both the original gauge group $G_{YM}$ and the discrete group $\pi_4(G_{YM})$ should be taken as gauge groups in the sense of non-commutative geometry. We show that the Yukawa coupling and the Higgs mechanism appear as natural results.
1 Introduction

Very recently, an $SU(2)$ generalized gauge field model has been constructed\[1]. In this model, the Yang-Mills gauge group $SU(2)$ and its fourth homotopy group $\pi_4(SU(2)) = \mathbb{Z}_2$ are dealt with on the equal footing in the sense of non-commutative differential geometry\[3\]. It is remarkable that not only the Higgs mechanism is automatically included in the model but also it survives quantum correlations since the spontaneous symmetry breaking breaks down both $SU(2)$ and $\pi_4(SU(2))$. The later is different from Connes’ NCDG approach to the particle model building\[3\]. In \[2\], this model is generalized to the Weinberg-Salam model and the standard model. The reason why the fourth homotopy group plays this important role lies in the fact that in these cases the base manifold is the 4−dimensional spacetime which may be compactified to $S^4$ and there is a kind of non-trivial $SU(2)_L$ gauge transformations which are topologically inequivalent to the identity\[16\]. That means there does exist an internal gauge symmetry which we used to neglect by considering the infinitesimal transformation of a Lie gauge group. Taking into account this fact, both $G_{YM}$ and $\pi_4(G_{YM})$ should be taken as gauge groups on the equal footing, where $G_{YM}$ is the original gauge group and $\pi_4(G_{YM})$ is the fourth homotopy group of $G_{YM}$. Then using the mathematic structure in\[3\], \[4\], the gauge fields will have two parts: the gauge fields on $G_{YM}$ which are the same as before and the gauge fields on discrete group $\pi_4(G_{YM})$ which appear as the Higgs fields. And the Yukawa coupling and Higgs mechanism will appear as a natural result.

It should be mentioned that since the discrete symmetry is spontaneously broken down at the same time with the continuous gauge symmetry, there is no need to concern about this discrete symmetry when we quantize the theory. On the other hand, other approaches\[5, 6, 7, 8, 9, 10, 11, 12\] does not survive the quantum correlation.

If this principle is true, it should be applicable to all $4−D$ gauge theory models with non-trivial fourth homotopy groups of the gauge groups. One of them is of left-right symmetric model. As is well known, a missing link in the standard model is that the $V−A$ structure of currents is put by hand. In the middle 70’s, Pati, Salam and Mohapatra\[13,14\] proposed a gauge theory model based on $SU(2)_L \times SU(2)_R \times U(1)_Y$ which is totally left-right symmetric.
before symmetry breaking. They showed the right-handed charged gauge meson $W_R^+$ can be made much heavier than the left-handed $W_L^+$ and the $V - A$ structure of weak interaction can be regarded as a low energy phenomenon which should disappear at $10^3 GeV$ or higher. And in the limit of infinitely heavy $W_R^+$, the predictions are the same as the standard model, as far as the charged and neutral currents interactions are concerned. In other words, such theories are indistinguishable from the standard model at low energies.

In this paper, we study such a model of left-right symmetric based on $SU(2)_L \times SU(2)_R \times U(1)_Y \times \pi_4(SU(2)_L \times SU(2)_R \times U(1)_Y)$. We choose the minimum Higgs assignment which is necessary to break the gauge group down to $U(1)_{em}$. Then we show that the three Higgs fields included can be regarded as gauge fields along the three directions of the tangent space of $\pi_4(SU(2)_L \times SU(2)_R \times U(1)_Y) = Z_2 \oplus Z_2$. We first show the differential calculus on $Z_2 \oplus Z_2$ in section two. Then in section three we regain the Lagrangian of this model from the generalized gauge principle mentioned above. Finally we end with conclusions and remarks.

2 Differential Calculus on $Z_2 \oplus Z_2$

Since $\pi_4(SU(2)_L \times SU(2)_R \times U(1)_Y) = Z_2 \oplus Z_2$, we need to clarify the differential calculus on $Z_2 \oplus Z_2$. Let's write the four elements of $Z_2 \oplus Z_2$ as

$$(e_1, e_2), (r_1, e_2), (e_1, r_2), (r_1, r_2).$$

And the group multiplication is

$$(g_1, g_2)(h_1, h_2) = (g_1h_1, g_2h_2).$$ \hspace{1cm} (1)

Let $\mathcal{A}$ be the algebra of complex valued functions on $Z_2 \oplus Z_2$. The derivative on $\mathcal{A}$ is defined as

$$\partial_g f = f - R_g f \quad g \in Z_2 \oplus Z_2, \ f \in \mathcal{A}$$ \hspace{1cm} (2)

with $R_g f(h) = f(hg)$. We will write $\partial_i$ and $R_i$ for convenience where $i = 1, 2, 3$ refers to $(r_1, e_2), (e_1, r_2), (r_1, r_2)$ respectively.

The basis of space of one forms are $\chi^1, \chi^2, \chi^3$ which are defined with

$$\chi^i(\partial_j) = \delta^i_j \quad i, j = 1, 2, 3.$$ \hspace{1cm} (3)
One can easily find that the following relations hold

\[ \partial_1 \partial_2 = \partial_1 + \partial_2 - \partial_3 \]
\[ \partial_1 \partial_1 = 2 \partial_1 \]
\[ \partial_1 \partial_2 = \partial_2 \partial_1 \]

(4)

\[ d\chi^1 = -\chi^1 \otimes \chi^2 - \chi^1 \otimes \chi^3 + \chi^2 \otimes \chi^3 - 2\chi^1 \otimes \chi^1 - \chi^2 \otimes \chi^1 - \chi^3 \otimes \chi^1 + \chi^3 \otimes \chi^2 \]

\[ \vdots \quad \text{and similar eqs under permutations (1,2,3) and (2,1,3)}. \]

In order to get a left-right symmetric Lagrangian before symmetry breaking, we let the metric to be symmetric under 1 ↔ 2

\[ <\chi^1,\chi^1> = <\chi^2,\chi^2> = \eta ; \quad <\chi^3,\chi^3> = \eta' \]
\[ <\chi^i,\chi^j> = 0, \quad i \neq j. \]

(5)

And let

\[ <\chi^i \otimes \chi^j, \chi^k \otimes \chi^l> = a <\chi^j,\chi^k> <\chi^i,\chi^l> + b <\chi^i,\chi^k> <\chi^j,\chi^l> \]

(6)

with \(a, b\) two constants.

### 3 Gauge Theory

In this model we take in three Higgs fields

\[ \Phi = \begin{pmatrix} \phi_0^0 & \phi_1^+ \\ \phi_2^- & \phi_2^0 \end{pmatrix}; \Delta_L = \begin{pmatrix} \chi_L^+ \\ \chi_L^0 \end{pmatrix}; \Delta_R = \begin{pmatrix} \chi_R^+ \\ \chi_R^0 \end{pmatrix} \]

(7)

which belong to \((\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, 1)\) and \((0, \frac{1}{2}, 1)\) respectively. This is the minimum choice to give necessary symmetry breaking. And we choose the Lagrangian to be invariant under \(L \leftrightarrow R\) transformation so we let \(g_L = g_R = g\) in this case.

All fields are regarded as elements of function space on \(SU(2)_L \times SU(2)_R \times U(1)_Y \times \pi_4(SU(2)_L \times SU(2)_R \times U(1)_Y)\). We reasonably postulate that a \(SU(2)\) singlet is invariant under \(\pi_4(SU(2))\), that means \(L \overset{R}{\rightarrow} L^*\), \(L \overset{R}{\rightarrow} L\), etc. Here \(L\) is left-hand fermion doublet. And we will not need any detail information about how the fields transform.
We write, for fermion

\[
\begin{align*}
L(x, e_1, e_2) &= L; & L(x, r_1, e_2) &= R_1L \equiv L^{r_1} \\
L(x, e_1, r_2) &= R_2L = L; & L(x, r_1, r_2) &= R_3L = L^{r_1} \\
R(x, e_1, e_2) &= R; & R(x, r_1, e_2) &= R_1R = R \\
R(x, e_1, r_2) &= R_2R \equiv R^{r_2}; & R(x, r_1, r_2) &= R_3R = R^{r_2}
\end{align*}
\]

(8)

where \( L = \begin{pmatrix} \nu_l \\ l \end{pmatrix} \) or \( \begin{pmatrix} u^i \\ d^i \end{pmatrix} \).

for gauge fields

\[
\begin{align*}
L_\mu(x, e_1, e_2) &= L_\mu; & L_\mu(x, r_1, e_2) &= R_1L_\mu \equiv L^{r_1}_\mu \\
L_\mu(x, e_1, r_2) &= R_2L_\mu = L_\mu; & L_\mu(x, r_1, r_2) &= R_3L_\mu = L^{r_1}_\mu \\
R_\mu(x, e_1, e_2) &= R_\mu; & R_\mu(x, r_1, e_2) &= R_1R_\mu = R_\mu \\
R_\mu(x, e_1, r_2) &= R_2R_\mu \equiv R^{r_2}_\mu; & R_\mu(x, r_1, r_2) &= R_3R_\mu = R^{r_2}_\mu
\end{align*}
\]

(9)

where \( L_\mu = -ig_\mu^L W^L_\mu - ig_\mu^Y B_\mu \), \( R_\mu = -ig_\mu^L W^R_\mu - ig_\mu^Y B_\mu \).

for Higgs fields

\[
\begin{align*}
\Phi(x, e_1, e_2) &= \Phi; & \Phi(x, r_1, e_2) &= R_1\Phi \equiv \Phi^{r_1}; \\
\Phi(x, e_1, r_2) &= R_2\Phi \equiv \Phi^{r_2}; & \Phi(x, r_1, r_2) &= R_3\Phi \equiv \Phi^{r_3}
\end{align*}
\]

(10)

\[
\begin{align*}
\Delta_L(x, e_1, e_2) &= \Delta_L; & \Delta_L(x, r_1, e_2) &= R_1\Delta_L \equiv \Delta^{r_1}_L; \\
\Delta_L(x, e_1, r_2) &= R_2\Delta_L = \Delta_L; & \Delta_L(x, r_1, r_2) &= R_3\Delta_L = \Delta^{r_1}_L \\
\Delta_R(x, e_1, e_2) &= \Delta_R; & \Delta_R(x, r_1, e_2) &= R_1\Delta_R \equiv \Delta^{r_2}_R; \\
\Delta_R(x, e_1, r_2) &= R_2\Delta_R \equiv \Delta^{r_2}_R; & \Delta_R(x, r_1, r_2) &= R_3\Delta_R = \Delta^{r_2}_R
\end{align*}
\]

(10)

Now we assign these fields into three sectors. On point \((e_1, e_2)\), they can be written as

\[
\Psi(x, e_1, e_2) = \begin{pmatrix} L \\ R \\ 0 \end{pmatrix} = \Psi(x); \quad A_\mu(x, e_1, e_2) = \begin{pmatrix} L_\mu \\ R_\mu \\ 0 \end{pmatrix} = A_\mu(x)
\]

\[
\phi_1 = \begin{pmatrix} \alpha_1 & -\Delta_L \\ -\bar{\Delta}_L^{r_1} & \alpha_1 \end{pmatrix}; \quad \phi_2 = \begin{pmatrix} \alpha_2 & -\Delta_R \\ -\bar{\Delta}_R^{r_2} & \alpha_2 \end{pmatrix}; \quad \phi_3 = \begin{pmatrix} \alpha_3 & -\Phi^{r_2} \\ -\bar{\Phi}^{r_1} & \alpha_3 \end{pmatrix}
\]

(11)

Fields on other points of \( Z_2 \oplus Z_2 \) can be easily written out according to \( \text{(8), (9), (10)} \). It should be mentioned that assignment \( \text{(11)} \) not only assigns the fields to the points of \( Z_2 \oplus Z_2 \) but also
gives certain matrices arrangement. Such an arrangement is only a working hypothesis which has nothing to do with non-commutative geometry and sometimes one should avoid certain extra constraints coming from this arrangement.

It is easy to see

\[ \phi_i^\dagger = \phi_i^{r_i}, \quad i = 1, 2, 3. \]  \(12\)

Here \(\phi_1, \phi_2, \phi_3\) are Higgs fields assigned to three directions of \(\Omega^1\), the space of one-forms. That is, in framework of [4], the connection one-form on \(SU(2)_L \times SU(2)_R \times U(1)_Y \times \pi_4(SU(2)_L \times SU(2)_R \times U(1)_Y)\) reads

\[ A(x, h) = A_\mu(x, h)dx^\mu + \sum_{i=1}^3 \frac{1}{\alpha_i} \phi_i(x, h)\chi^i \]  \(13\)

We will later let \(\alpha_1 = \alpha_2\) in order to get an left-right symmetric Lagrangian.

The generalized curvature two-form reads

\[ F(h) = \frac{1}{2} F_{\mu\nu}dx_\mu \wedge dx_\nu + \sum_{i=1}^3 \frac{1}{\alpha_i} F_{r_i, \mu} \chi^i dx^\mu + \sum_{i,j=1}^3 \frac{1}{\alpha_i \alpha_j} F_{r_i, r_j} \chi^i \chi^j. \]  \(14\)

After some calculation, we get

\[ F_{\mu\nu} = \begin{pmatrix} L_{\mu\nu} & 0 \\ R_{\mu\nu} & 0 \end{pmatrix} \]

\[ F_{r_1, \mu} = D_\mu \Phi_1 = \begin{pmatrix} D_\mu \Delta_L & 0 \\ (D_\mu \Delta_L)^{r_1} & 0 \end{pmatrix} \]  \(15\)

\[ F_{r_2, \mu} = D_\mu \Phi_2 = \begin{pmatrix} 0 & D_\mu \Delta_R \\ (D_\mu \Delta_R)^{r_2} & 0 \end{pmatrix} \]

\[ F_{r_3, \mu} = D_\mu \Phi_3 = \begin{pmatrix} (D_\mu \Phi)^{r_1} & D_\mu \Phi^{r_2} \\ (D_\mu \Phi)^{r_1} & 0 \end{pmatrix} \]

And

\[ F_{r_1, r_1} = \Phi_1 \Phi_1^\dagger - \alpha_1^2 \]

\[ F_{r_1, r_2} = \Phi_1 \Phi_2^\dagger - \frac{\alpha_1 \alpha_2}{\alpha_3} \Phi_3 \]  \(16\)

\[ \vdots \quad \text{and similar eqs under permutations } (1,2), (1,3), (2,3), (1,2,3), (1,3,2). \]
where $\Phi_i = \alpha_i - \phi_i$, $L_{\mu
u} = -ig_2 W^L_{\mu\nu} - ig_2^{R} B_{\mu\nu}$, $R_{\mu\nu} = -ig_2^{R} W^R_{\mu\nu} - ig_2^{L} B_{\mu\nu}$, and

\[
\begin{align*}
D_\mu \Delta_L &= \partial_\mu \Delta_L + L_\mu \Delta_L \\
D_\mu \Delta_R &= \partial_\mu \Delta_R + R_\mu \Delta_R \\
D_\mu \Phi &= \partial_\mu \Phi + L_\mu \Phi - \Phi R_\mu.
\end{align*}
\] (17)

Using (18), (19) and

\[\mathcal{L} = \langle F(h), \bar{F}(h) \rangle\] (18)

we have the bosonic sector of the Lagrangian

\[
\mathcal{L}_{Y_{M-H}}(x) = -\frac{1}{4N_L} Tr L_{\mu\nu} L_{\mu\nu} - \frac{1}{4N_R} Tr R_{\mu\nu} R_{\mu\nu}
\]

\[
-\frac{2}{N_L} \frac{\eta}{\alpha^2} [Tr(D_\mu \Delta_L)(D_\mu \Delta_L)^\dagger + Tr(D_\mu \Delta_R)(D_\mu \Delta_R)^\dagger] - \frac{2}{N_R} \frac{\eta'}{\alpha'^2} Tr(D_\mu \Phi)(D_\mu \Phi)^\dagger
\]

\[
-\frac{2}{N_L} \frac{\eta}{\alpha^2} [Tr(\Delta_L \Delta_L^\dagger - \alpha^2)^2 + Tr(\Delta_R \Delta_R^\dagger - \alpha^2)^2] - \frac{2}{N_R} \frac{\eta'}{\alpha'^2} Tr(\Phi \Phi^\dagger - \alpha'^2)^2
\]

\[
-\frac{1}{N_L} \frac{\eta}{\alpha^2} \left\{ 2a \Delta_L \Delta_L^\dagger \Delta_R \Delta_R - 2(a + b) \frac{\eta}{\alpha^2} (\Delta_L \Phi \Delta_R + \Delta_L \Phi^\dagger \Delta_L) + 4(a + b) \frac{\eta^2}{\alpha^2} Tr(\Phi \Phi^\dagger) \right\}
\]

\[
-\frac{1}{N_L} \frac{\eta}{\alpha^2} \left\{ 2a \Delta_L^\dagger \Phi \Phi^\dagger \Delta_L - 2(a + b) \alpha' (\Delta_L \Phi \Delta_R + \Delta_L \Phi^\dagger \Delta_L) + 4(a + b) \alpha'^2 \Delta_L \Delta_L \right\}
\]

\[
-\frac{1}{N_L} \frac{\eta}{\alpha^2} \left\{ 2a \Delta_R \Phi \Phi^\dagger \Delta_R - 2(a + b) \alpha' (\Delta_R \Phi \Delta_R + \Delta_R \Phi^\dagger \Delta_L) + 4(a + b) \alpha'^2 \Delta_R \Delta_L \right\}
\]

+const. (19)

Here we have set $\alpha_1 = \alpha_2 = \alpha$, $\alpha_3 = \alpha'$ in order to get a left-right symmetric Lagrangian and inserted some normalization constants $N_L, N_R, N, N', N_1, N_1'$ to avoid extra constraints coming from our arrangement of the fields. Since $\mathcal{L}$ is gauge invariant and so that is independent of the elements of $Z_2 \oplus Z_2$, there is no need to take the Haar integral.

As to the fermionic sector of the Lagrangian, we have

\[
\mathcal{L}_F(x) = -\bar{L}\gamma_\mu (\partial_\mu + L_\mu)L - R\gamma_\mu (\partial_\mu + R_\mu)R - \lambda (\bar{L} \Phi R + R \Phi^\dagger L)
\] (20)

where $\lambda$ is the Yukawa coupling constant. We see that only $\Phi$ has contribution to fermions masses.
After choosing proper normalization constants, we rewrite the above Lagrangian as

\[ \mathcal{L}_{Y-M-H}(x) = -\frac{1}{4} W_{\mu\nu}^{L} W_{\mu\nu}^{L} - \frac{1}{4} W_{\mu\nu}^{R} W_{\mu\nu}^{R} - \frac{1}{4} B_{\mu\nu} B_{\mu\nu} \]

\[ -\left[ \text{Tr}(D_{\mu}\Delta_{L})(D_{\mu}\Delta_{L})^{\dagger} + \text{Tr}(D_{\mu}\Delta_{R})(D_{\mu}\Delta_{R})^{\dagger} \right] - \text{Tr}(D_{\mu}\Phi)(D_{\mu}\Phi)^{\dagger} \]

\[ -V(\Phi, \Delta_{L}, \Delta_{R}) \]

and

\[ V(\Phi, \Delta_{L}, \Delta_{R}) = \rho_{1}(\Delta_{L}^{\dagger}\Delta_{L}\Delta_{L}^{\dagger}\Delta_{L} + \Delta_{R}^{\dagger}\Delta_{R}\Delta_{R}^{\dagger}\Delta_{R}) - \mu_{2}^{2}(\Delta_{L}^{\dagger}\Delta_{L} + \Delta_{R}^{\dagger}\Delta_{R}) \]

\[ + \rho_{3} Tr(\Phi\Phi^{\dagger}\Phi\Phi^{\dagger}) - \mu_{2}^{2} Tr(\Phi\Phi^{\dagger}) \]

\[ + \rho_{4} Tr(\Delta_{L}^{\dagger}\Phi\Phi^{\dagger}\Delta_{L}^{\dagger} + \Delta_{R}^{\dagger}\Phi\Phi^{\dagger}\Delta_{R}) \]

\[ - \mu_{3}^{2}(\Delta_{L}^{\dagger}\Phi\Delta_{R}^{\dagger} + \Delta_{R}^{\dagger}\Phi\Delta_{L}) \]

(21)

The breaking pattern of such a Higgs potential is well-known. For a detail discussion, we refer the readers to [15] by G. Senjanovic.

4 Conclusions and Remarks

Let us summarize what we have done. Based on a generalized gauge principle, we have constructed an SU(2)$_L \times$ SU(2)$_R \times$ U(1)$_Y$ model with π$_4$(SU(2)$_L \times$ SU(2)$_R \times$ U(1)$_Y$) taken as discrete gauge symmetry. The Higgs mechanism is automatically included in this generalized gauge theory model.

There are several advantages in this approach compared to others. First, the homotopy group of the original gauge group is a most natural and meaningful internal symmetry. Besides, if we take the homotopy group as generalized gauge group, then the discrete symmetry is broken synchronously with the continuous symmetry. So we get the same version as an ordinary Yang-Mills model. That means we do not need to concern about this discrete symmetry when we quantize the model.

References

[1] H.Y. Guo and J.M. Li, preprint AS-ITP-94-25.
[2] H.Y. Guo, J.M. Li and K. Wu, preprint AS-ITP-94-26.

[3] A. Connes, *Non-commutative Geometry* English translation of Geometrie Non-commutative, IHES Paris, Interedition.

[4] A. Sitarz, Non-commutative Geometry and Gauge Theory on Discrete Groups, preprint TPJU-7/1992.

[5] H.G. Ding, H.Y. Guo, J.M. Li and K. Wu, Comm. Theor. Phys. *bf* 21 (1994) 85-94.

[6] H.G. Ding, H.Y. Guo, J.M. Li and K. Wu, J.Phys. *A* 27 (1994) L75-L79; ibid. L231-L236.

[7] B. Chen and K. Wu, preprint AS-ITP-93-64.

[8] A. Connes, in: The Interface of Mathematics and Particle Physics, eds. D. Quillen, G. Segal and S. Tsou (Oxford U. P, Oxford 1990).

[9] A. Connes and J. Lott, Nucl. Phys. (Proc. Suppl.) *B18*, 44(1990).

[10] D. Kastler, Marseille, CPT preprint CPT-91/P.2610, CPT-91/P.2611.

[11] A. H Chamseddine, G Felder and J. Frohlich, Phys. Lett. *296B* (1993) 109, Zurich preprint ZU-TH-30/92 and Zurich preprint ETH-TH/92/44.

[12] R. Coquereaux, G. Esposito-Farese and G. Vailant, Nucl. Phys. *B353* 689 (1991).

[13] J.C. Pati and Salam, Phys. Rev. D10 (1974) 275.

[14] R.N. Mohapatra and J.C. Pati, Phys. Rev. D11(1975) 566, 2558.

[15] G. Senjanovic, Nucl. Phys. *B153* 334 (1979).

[16] S.T. Hu, Homotopy Theory, Academic Press, New York, 1959.