\textbf{Z}_2\text{-reductions of spinor models in two dimensions}

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We propose new types of integrable spinor models, generalizing the well known ones of: i) Nambu–Jona-Lasinio–Vaks–Larkin models, related to $SU(N)$; ii) the Gross–Neveu models – $SP(2N)$; and the iii) Zakharov–Mikhailov models – $SO(N)$. We propose a method for constructing their Lax representation and outline the spectral properties of the Lax operators.

PACS numbers: 02.30.Ik, 02.30.Jr, 02.30.Zz

\section{I. INTRODUCTION}

Spinor models play important role in contemporary theoretical physics. The famous Nambu–Jona-Lasinio–Vaks–Larkin models\textsuperscript{7,9} and Gross–Neveu models\textsuperscript{5,8} have been proposed initially as models for describing the strong interactions. Later, with the development of the inverse scattering method (ISM)\textsuperscript{2,10} it was proven that the two-dimensional versions of these models are integrable\textsuperscript{12}. In the same paper Zakharov and Mikhailov propose a third class of spinor models related to the orthogonal groups.

The aim of the present paper is to derive new types of integrable spinor models by applying additional $\text{Z}_2$-reductions to their Lax representations. In doing this we will be using the reduction group\textsuperscript{6}. Next we describe the spectral properties of the Lax operators.

We start in Section II by some preliminaries concerning the spinor models and the reduction group of Mikhailov\textsuperscript{6}. In Section III we outline the spectral theory of the unreduced Lax operators. In Section IV we derive the $\text{Z}_2$-reduced spinor models. Section V is devoted to the spectral properties of the $\text{Z}_2$-reduced Lax operators. More specifically we treat 4 different cases in each of which we specify the continuous spectrum and the symmetries of the discrete eigenvalues. We end with brief conclusions.
II. PRELIMINARIES

The integrability of the 2-dimensional versions of the Nambu–Jona-Lasinio–Vaks–Larkin (NJLVL) and the Gross–Neveu model (GN) was discovered by Zakharov and Mikhailov in \[11\]. They showed that NJLVL models are related to \( su(N) \) algebras, while the Gross–Neveu models are related to the \( sp(N) \). In the same paper an additional type of spinor models related to the algebras \( so(N) \) was discovered; we will call them Zakharov–Mikhailov (ZM) models.

Let us first outline the Lax representations of these models \[12\].

\[ \Psi_\xi = U(\xi, \eta, \lambda) \Psi(\xi, \eta, \lambda), \quad \Psi_\eta = U(\xi, \eta, \lambda) \Psi(\xi, \eta, \lambda), \]

\[ U(\xi, \eta, \lambda) = \frac{U_1(\xi, \eta)}{\lambda - a}, \quad V(\xi, \eta, \lambda) = \frac{V_1(\xi, \eta)}{\lambda - a}, \]  

(1)

where \( \eta = t + x, \xi = t - x \) and \( a \) is a real number.

We also impose the \( \mathbb{Z}_2 \)-reduction:

\[ U^\dagger(x, t, \lambda) = -U(x, t, \lambda^*), \quad V^\dagger(x, t, \lambda) = -V(x, t, \lambda^*). \]  

(2)

The compatibility condition of the above linear problems reads:

\[ U_\eta - V_\xi + [U, V] = 0, \]  

(3)

which is equivalent to

\[ U_{1,\eta} + \frac{1}{2a} [U_1, V_1(\xi, \eta)] = 0, \quad V_{1,\xi} - \frac{1}{2a} [V_1, U_1(\xi, \eta)] = 0. \]  

(4)

From these equations, fixing up properly the gauge, (see \[11\]) there follows that

\[ U_1(\xi, \eta) = -i \phi J_1^0 \phi^{-1}, \quad V_1(\xi, \eta) = i \psi I_1^0 \psi^{-1}, \]  

(5)

where \( J_1^0 \) and \( I_1^0 \) are properly chosen constant elements (choice of the gauge) of the corresponding simple Lie algebra \( g \). In what follows we fix up \( J_1^0 = -I_1^0 = J \) and choose \( J \) for each of the above mentioned models accordingly. The matrix valued functions \( \phi(\xi, \eta) \) and \( \psi(\xi, \eta) \) take values in the corresponding simple Lie group and are fundamental solutions of the following ODE’s:

\[ \psi_\xi \equiv -\frac{U_1(\xi, \eta)}{2a} \psi(\xi, \eta) = \frac{i}{2a} \phi J \hat{\phi} \psi(\xi, \eta), \]

\[ \phi_\eta \equiv \frac{V_1(\xi, \eta)}{2a} \phi(\xi, \eta) = \frac{i}{2a} \psi J \hat{\psi} \phi(\xi, \eta). \]  

(6)
Here and below by ‘hat’ we will denote the inverse matrix, i.e. $\hat{\psi} \equiv (\psi)^{-1}$.

In this way we get three classes of spinor models. Below, following [12] we briefly outline their derivation.

i) Nambu-Jona-Lasinio-Vaks-Larkin models. Here we choose $\mathfrak{g} \simeq su(N)$. Then $\psi(\xi, \eta)$ and $\phi(\xi, \eta)$ are elements of the group $SU(N)$ and by definition $\hat{\psi}(\xi, \eta) = \psi^\dagger(\xi, \eta)$, $\hat{\phi}(\xi, \eta) = \phi^\dagger(\xi, \eta)$. Next we choose $J = \text{diag}(1, 0, \ldots, 0)$ and as a result only the first columns $\phi^{(1)}$, $\psi^{(1)}$ and the first rows $\hat{\phi}^{(1)}$, $\hat{\psi}^{(1)}$ enter into the systems (6). If we introduce the notations:

$$
\phi_\alpha(\xi, \eta) = \phi^{(1)}_{\alpha, 1}, \quad \psi_\alpha(\xi, \eta) = \psi^{(1)}_{\alpha, 1},
$$

then the explicit form of the system is:

$$
\frac{\partial \phi_\alpha}{\partial \eta} = \frac{i}{2a} \psi_\alpha \sum_{\beta=1}^{N} \psi^*_\beta \phi_\beta,
$$

$$
\frac{\partial \psi_\alpha}{\partial \xi} = \frac{i}{2a} \phi_\alpha \sum_{\beta=1}^{N} \phi^*_\beta \psi_\beta.
$$

The functional of the action is:

$$
A_{NJLVL} = \int_{-\infty}^{\infty} dx \, dt \, \left( \sum_{\alpha=1}^{N} \left( \chi_\alpha \frac{\partial \phi_\alpha}{\partial \eta} + \psi^*_\alpha \frac{\partial \psi_\alpha}{\partial \xi} \right) - \frac{1}{2a} \left| \sum_{\alpha=1}^{N} \left( \psi^*_\alpha \phi_\alpha \right) \right|^2 \right).
$$

ii) Gross-Neveu models. Here we choose $\mathfrak{g} \simeq sp(2N, \mathbb{R})$; then $\psi(\xi, \eta)$ and $\phi(\xi, \eta)$ are elements of the group $\mathfrak{g} \simeq SP(2N, \mathbb{R})$. Following [12] we use the standard definition of symplectic group elements:

$$
\hat{\psi}(\xi, \eta) = \mathfrak{J} \psi^T(\xi, \eta) \mathfrak{J}, \quad \hat{\phi}(\xi, \eta) = \mathfrak{J} \phi^T(\xi, \eta) \mathfrak{J}, \quad \mathfrak{J} = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}.
$$

Then the corresponding Lie algebraic elements acquire the following block-matrix structure:

$$
U_1(\xi, \eta) = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix},
$$

where $A$, $B$, and $C$ are matrices of appropriate dimensions.
where $A, B, C$ are arbitrary real $N \times N$ matrices. Next we choose

$$J = \begin{pmatrix} 0 & B_0 \\ 0 & 0 \end{pmatrix}, \quad B_0 = \text{diag} \,(1, 0, \ldots, 0, 0).$$

(12)

As a consequence again only the first columns $\phi^{(1)}, \psi^{(1)}$ and the first rows $\hat{\phi}^{(1)}, \hat{\psi}^{(1)}$ enter into the systems (6). If we introduce the $N$-component complex vectors:

$$\phi_{\alpha}(\xi, \eta) = \frac{1}{2}(\phi_{\alpha,1}^{(1)} + i\phi_{N+\alpha,1}^{(1)}), \quad \psi_{\alpha}(\xi, \eta) = \frac{1}{2}(\psi_{\alpha,1}^{(1)} + i\psi_{N+\alpha,1}^{(1)})$$

(13)

then the explicit form of the system is:

$$\begin{align*}
\frac{\partial \phi_{\alpha}}{\partial \eta} &= \frac{i}{a} \psi_{\alpha} \sum_{\beta=1}^{N} (\psi_{\beta} \phi^{*}_{\beta} - \psi^{*}_{\beta} \phi_{\beta}), \\
\frac{\partial \psi_{\alpha}}{\partial \xi} &= -\frac{i}{a} \phi_{\alpha} \sum_{\beta=1}^{N} (\phi_{\beta} \psi^{*}_{\beta} - \phi^{*}_{\beta} \psi_{\beta}).
\end{align*}$$

(14)

The functional of the action is:

$$A_{GN} = \int_{-\infty}^{\infty} dx \, dt \left( i \sum_{\alpha=1}^{N} \left( \phi^{*}_{\alpha} \frac{\partial \phi_{\alpha}}{\partial \eta} + \psi^{*}_{\alpha} \frac{\partial \psi_{\alpha}}{\partial \xi} \right) - \frac{1}{2a} \left( \sum_{\alpha=1}^{N} (\psi^{*}_{\alpha} \phi_{\alpha} - \phi^{*}_{\alpha} \psi_{\alpha}) \right)^{2} \right).$$

(15)

iii) Zakharov–Mikhailov models. Now we choose $g \simeq so(N, \mathbb{R})$; then $\psi(\xi, \eta)$ and $\phi(\xi, \eta)$ are elements of the group $\mathfrak{g} \simeq SO(N, \mathbb{R})$. Following [12] we use the standard definition of orthogonal group elements:

$$\hat{\psi}(\xi, \eta) = \psi^{T}(\xi, \eta), \quad \hat{\phi}(\xi, \eta) = \phi^{T}(\xi, \eta).$$

(16)

Now we choose

$$J = E_{1,N} - E_{N,1},$$

(17)

where the $N \times N$ matrices $E_{kp}$ are defined by $(E_{kp})_{nm} = \delta_{kn} \delta_{pm}$.

As a consequence now the first and the last columns $\phi^{(1)}, \phi^{(N)}, \psi^{(1)}, \psi^{(N)}$ and the first and the last rows $\hat{\phi}^{(1)}, \hat{\phi}^{(N)}, \hat{\psi}^{(1)}, \hat{\psi}^{(N)}$ enter into the systems (10). If we introduce the $N$-component complex vectors:

$$\phi_{\alpha}(\xi, \eta) = \frac{1}{2}(\phi_{\alpha,1}^{(1)} + i\phi_{\alpha,N}^{(N)}), \quad \psi_{\alpha}(\xi, \eta) = \frac{1}{2}(\psi_{\alpha,1}^{(1)} + i\psi_{\alpha,N}^{(N)})$$

(18)
then the explicit form of the system becomes:

\[
\begin{align*}
\frac{i}{a} \partial \psi_\alpha & = \frac{i}{a} \sum_{\beta=1}^{N} (\phi_\alpha^* \phi_\beta \psi_\beta - \phi_\alpha \phi_\beta^*) \psi_\beta, \\
\frac{i}{a} \partial \phi_\alpha & = \frac{i}{a} \sum_{\beta=1}^{N} (\psi_\alpha^* \psi_\beta \phi_\beta - \psi_\alpha \psi_\beta^*) \phi_\beta,
\end{align*}
\]

(19)

The functional of the action is:

\[
A_{\text{ZM}} = \int_{-\infty}^{\infty} dx \, dt \left( \frac{i}{a} \sum_{\alpha=1}^{N} \left( \phi_\alpha^* \partial \phi_\alpha + \psi_\alpha \partial \psi_\alpha \right) - \frac{1}{2a} \left( \sum_{\alpha,\beta=1}^{N} (\phi_\alpha^* \phi_\beta - \phi_\beta^* \phi_\alpha)(\psi_\alpha^* \psi_\beta - \psi_\beta^* \psi_\alpha) \right) \right).
\]

(20)

For more details of deriving the models see [12].

III. SPECTRAL PROPERTIES OF THE LAX OPERATOR

Here we briefly outline the construction of the fundamental analytic solutions of the Lax operator \( L \).

First we fix up the class of potentials \( U_1(\xi, \eta) \) and \( V_1(\xi, \eta) \) by assuming that \( U_1(\xi, \eta) + iJ \) and \( V_1(\xi, \eta) - iJ \) are Schwartz-type functions of \( \xi \) and \( \eta \). We also assume that \( J \in \mathfrak{h} \) is a real element of the Cartan subalgebra of \( \mathfrak{g} \).

Remark 1 These conditions are compatible with two of the classes of spinor models listed above. These are, first of all the NJLVL models for which \( J \), up to a trivial term \( 1/N \mathbb{I} \), belongs to the Cartan subalgebra of \( \mathfrak{su}(n) \). For the ZM models \( J \) belongs to the Cartan subalgebra, which in that case consists of off-diagonal matrices. However, there is a simple similarity transformation which takes \( J \) of eq. (17) into \( J = i \text{diag}(1, 0, \ldots, 0, -1) \).

For the GN models the choice of \( J \) is nilpotent. The spectral problem for such Lax operators is singular and will not be discussed here.

In what follows we will consider the spectral problem for Lax operators of the type:

\[
L \Psi(\xi, \eta, \lambda) \equiv \frac{\partial \Psi}{\partial \xi} + i \frac{\phi^J \hat{\phi}}{\lambda - a} \Psi(\xi, \eta, \lambda) = 0,
\]

(21)
where $J \in \mathfrak{h}$ and $\phi(\xi, \eta) \in \mathfrak{g}$ and $\lim_{\xi \to \pm \infty} \phi(\xi, \eta) = 1$. The Jost solutions of $L$ are defined by:

$$
\lim_{\xi \to \infty} \Psi_{\pm}(\xi, \eta, \lambda) \hat{\mathcal{E}}(\xi, \lambda) = 1, \quad \lim_{\xi \to -\infty} \Psi_{-}(\xi, \eta, \lambda) \hat{\mathcal{E}}(\xi, \lambda) = 1,
$$

(22)

where

$$
\mathcal{E}(\xi, \lambda) = \exp \left( -i \frac{J \xi}{\lambda - a} \right).
$$

(23)

The scattering matrix is introduced by:

$$
T(\lambda, \eta) = \hat{\Psi}_{+}(\xi, \eta, \lambda) \Psi_{-}(\xi, \eta, \lambda)
$$

(24)

The continuous spectrum of $L$ is located on a line of the complex $\lambda$-plane on which $\mathcal{E}(\xi, \lambda)$ oscillates. In our case the continuous spectrum of $L$ fills up the real axis on the complex $\lambda$-plane. The discrete eigenvalues $\lambda_{k}^{\pm} \in \mathbb{C}_{\pm}$ come in pairs, which due to the reduction (2) are mutually conjugate $\lambda_{k}^{\pm} = (\lambda_{k}^{-})^*$, see fig. 1.

The next step is to construct the fundamental analytic solutions (FAS) of $L$. Their construction is done analogously to the case of the generalized Zakharov-Shabat system, see [3]. To this end we need the generalized Gauss decomposition of $T(\lambda, \eta)$ compatible with $J$:

$$
T(\lambda, \eta) = T_{j} D_{j}^{+} \hat{S}_{j}^{+}, \quad T(\lambda, \eta) = T_{j}^{+} D_{j}^{-} \hat{S}_{j}^{-},
$$

(25)
If $J = \text{diag}(1, 0, \ldots, 0)$ then

$$S^+_j(\eta, \lambda) = \begin{pmatrix} 1 & s^{+T} \\ 0 & I \end{pmatrix}, \quad S^-_j(\eta, \lambda) = \begin{pmatrix} 1 & 0 \\ \bar{s}^- & I \end{pmatrix},$$

$$T^+_j(\eta, \lambda) = \begin{pmatrix} 1 & \tau^{+T} \\ 0 & I \end{pmatrix}, \quad S^-_j(\eta, \lambda) = \begin{pmatrix} 1 & 0 \\ \bar{\tau}^- & I \end{pmatrix},$$

$$D^+_j(\lambda) = \begin{pmatrix} d^+_1 \\ 0 & d^+_2 \end{pmatrix}, \quad D^-_j(\lambda) = \begin{pmatrix} d^-_1 \\ 0 & d^-_2 \end{pmatrix}. \quad (26)$$

For $J = \text{diag}(1, 0, \ldots, 0, -1)$ and $g \simeq so(N)$ we have:

$$S^+_j(\eta, \lambda) = \begin{pmatrix} 1 & \bar{s}^{+T} \\ 0 & I \end{pmatrix}, \quad S^-_j(\eta, \lambda) = \begin{pmatrix} 1 & 0 \\ \bar{s}^- & I \end{pmatrix},$$

$$T^+_j(\eta, \lambda) = \begin{pmatrix} 1 & \tau^{+T} \\ 0 & I \end{pmatrix}, \quad S^-_j(\eta, \lambda) = \begin{pmatrix} 1 & 0 \\ \bar{\tau}^- & I \end{pmatrix},$$

$$D^+_j(\lambda) = \begin{pmatrix} d^+_1 \\ 0 & d^+_2 \end{pmatrix}, \quad D^-_j(\lambda) = \begin{pmatrix} d^-_1 \\ 0 & d^-_2 \end{pmatrix}, \quad (27)$$

where $s_0 = \sum_{j=1}^{N} E_{j,N+1-j}$.

Then the FAS analytic for $\lambda \in \mathbb{C}_\pm$ are related to Jost solutions by:

$$\chi^\pm(\xi, \eta, \lambda) = \Psi_-(\xi, \eta, \lambda)S^+_j(\eta, \lambda) = \Psi_+(\xi, \eta, \lambda)T^+_j(\eta, \lambda)D^+_j(\lambda). \quad (28)$$

The FAS (28) satisfy a Riemann-Hilbert problem (RHP) with canonical normalization at $\lambda \to a$:

$$\chi^+(\xi, \eta, \lambda) = \chi^-(\xi, \eta, \lambda)G_{J}(\lambda, \eta), \quad G_{J}(\lambda, \eta) = \hat{S}_J(\lambda, \eta)S^+_j(\lambda, \eta),$$

$$\lim_{\lambda \to a} \chi^+(\xi, \eta, \lambda) = I. \quad (29)$$

The canonical normalization of the RHP means that the FAS allow asymptotic expansions over the powers of $\lambda - a$:

$$\chi^\pm(\xi, \eta, \lambda) = I + \sum_{s=1}^{\infty} \chi^\pm_s(\xi, \eta)(\lambda - a)^{-s}. \quad (30)$$
Therefore if we are given a solution of the RHP $\chi^+(\xi, \eta, \lambda)$ then the corresponding potential of $L$ can be recovered from

$$U_1(\xi, \eta) \equiv igJg^{-1}(\xi, \eta) = i\frac{\partial X^\pm_1}{\partial \xi}.$$  

(31)

We finish this Section with the obvious remark, that the RHP formulation allows one to derive the $N$-soliton solutions of the corresponding model via the Zakharov-Shabat dressing procedure $[12]$. 

IV. $\mathbb{Z}_2$-REDUCTIONS OF THE SPINOR MODELS 

Here we combine the construction of spinor models in two dimensions $[11]$ with the idea of the reduction group $[6]$. Thus we intend to construct new types of spinor models generalizing the ones in Section II.

Start with the Lax representation:

$$\Psi_\xi = U_R(\xi, \eta, \lambda)\Psi(\xi, \eta, \lambda), \quad U_R(\xi, \eta, \lambda) = \frac{U_1(\xi, \eta)}{\lambda-a} + \frac{CU_1(\xi, \eta)C^{-1}}{\epsilon\lambda^{-1}-a},$$

$$\Psi_\eta = V_R(\xi, \eta, \lambda)\Psi(\xi, \eta, \lambda), \quad V_R(\xi, \eta, \lambda) = \frac{V_1(\xi, \eta)}{\lambda+a} + \frac{CV_1(\xi, \eta)C^{-1}}{\epsilon\lambda^{-1}+a},$$

(32)

where $\epsilon = \pm 1$, $a \neq 1$ is a real number and $C$ is an involutive automorphism of $\mathfrak{g}$. Obviously this Lax representation along with the typical reduction $[2]$ satisfy also:

$$U_R(\xi, \eta, \lambda) = CU_R(\xi, \eta, \epsilon\lambda^{-1})C^{-1}, \quad V_R(\xi, \eta, \lambda) = CV_R(\xi, \eta, \epsilon\lambda^{-1})C^{-1},$$

(33)

which is automatically compatible with the Lax representation $[6]$.

The new Lax representation is:

$$\frac{\partial U_R}{\partial \eta} - \frac{\partial V_R}{\partial \xi} + [U_R, V_R] = 0,$$

(34)

which is equivalent to

$$U_1,\xi + [U_1, V_R(\xi, \eta, a)] = 0, \quad V_1,\xi + [V_1, U_R(\xi, \eta, -a)] = 0.$$  

(35)

Next we apply the same way of deriving the models as in Section II; obviously, due to the additional terms in $U_R$ and $V_R$ we get additional terms in the models. In what follows we also list some typical choices for the automorphism $C$. Skipping the details we get:
i) $\mathbb{Z}_2$-NJLVL models. Here $\mathfrak{G} \simeq SU(N)$ and the system takes the form:

$$
\frac{i}{\partial \eta} \frac{\partial \tilde{\phi}}{\partial \eta} + \frac{1}{2a} \tilde{\psi} (\tilde{\psi}^\dagger \tilde{\phi}) + \frac{1}{\epsilon a^{-1} + a} C \tilde{\psi} (\tilde{\psi}^\dagger \hat{C} \tilde{\phi}) (\xi, \eta) = 0, \\
\frac{i}{\partial \xi} \frac{\partial \tilde{\phi}}{\partial \xi} + \frac{1}{2a} \tilde{\phi} (\tilde{\phi}^\dagger \tilde{\psi}) + \frac{1}{\epsilon a^{-1} + a} C \tilde{\phi} (\tilde{\phi}^\dagger \hat{C} \tilde{\psi}) (\xi, \eta) = 0.
$$

(36)

where $\tilde{\psi} = (\psi_{a,1}, \ldots, \psi_{a,N})^T$ and $\tilde{\phi} = (\phi_{a,1}, \ldots, \phi_{a,N})^T$.

For the automorphism $C$ of the $SU(N)$ group we may have

a) $C_N = \text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_N), \quad \epsilon_j = \pm 1$, \quad b) $C'_N = \begin{pmatrix} 1 & 0 \\ 0 & C_{N-1} \end{pmatrix}.$

(37)

where $C_{N-1}$ belongs to the Weyl group of $SU(N - 1)$ and is such that $C_{N-1}^2 = \mathbb{1}$. These two special choices of $C$ are such that $\lim_{\xi \to \pm \infty} U_R (\xi, \eta) = \lim_{\xi \to \pm \infty} C U_R (\xi, \eta) \hat{C}$.

ii) $\mathbb{Z}_2$-GN models. Here $\mathfrak{G} \simeq SP(2N, \mathbb{R})$ and the form of the reduced system depends on the choice of the automorphism $C$. Two typical choices of $C$ are given by:

a) $C = \begin{pmatrix} C_1 & 0 \\ 0 & C_1 \end{pmatrix}$, \quad b) $C' = \begin{pmatrix} 0 & C_2 \\ C_2 & 0 \end{pmatrix}$.

(38)

where $C_1^2 = C_2^2 = \mathbb{1}$. In this way we obtain two different systems of GN-type. Using the $N$-component vectors $\tilde{\psi}$ and $\tilde{\phi}$ we can write them down in compact form:

$$
\frac{\partial \tilde{\phi}}{\partial \eta} = -\frac{i}{a} \tilde{\psi} \left( (\tilde{\psi}^\dagger \tilde{\phi}) - (\tilde{\phi}^\dagger \tilde{\psi}) \right) - \frac{2i}{a + \epsilon a^{-1}} C_1 \tilde{\psi} \left( (\tilde{\psi}^\dagger C_1 \tilde{\phi}) - (\tilde{\phi}^\dagger C_1 \tilde{\psi}) \right), \\
\frac{\partial \tilde{\psi}}{\partial \xi} = \frac{i}{a} \tilde{\phi} \left( (\tilde{\psi}^\dagger \tilde{\phi}) - (\tilde{\phi}^\dagger \tilde{\psi}) \right) + \frac{2i}{a + \epsilon a^{-1}} C_1 \tilde{\phi} \left( (\tilde{\psi}^\dagger C_1 \tilde{\phi}) - (\tilde{\phi}^\dagger C_1 \tilde{\psi}) \right).
$$

(39)

The corresponding action can be written as follows:

$$
A_{\mathbb{Z}_2,\text{GNa}} = \int_{-\infty}^{\infty} dx \ dt \ \left( i \left( \frac{\partial \tilde{\phi}}{\partial \eta} + \tilde{\psi}^\dagger \frac{\partial \tilde{\psi}}{\partial \xi} \right) - \frac{1}{2a} \left( (\tilde{\psi}^\dagger \tilde{\phi}) - (\tilde{\phi}^\dagger \tilde{\psi}) \right)^2 \right.
\left. - \frac{1}{\epsilon a^{-1} + a} \left( (\tilde{\psi}^\dagger C_1 \tilde{\phi}) - (\tilde{\phi}^\dagger C_1 \tilde{\psi}) \right)^2 \right).
$$

(40)

The second $\mathbb{Z}_2$-reduced GN-system is:

$$
\frac{\partial \tilde{\phi}}{\partial \eta} = -\frac{i}{a} \tilde{\psi} \left( (\tilde{\psi}^\dagger \tilde{\phi}) - (\tilde{\phi}^\dagger \tilde{\psi}) \right) + \frac{2i}{a + \epsilon a^{-1}} C_2 \tilde{\psi}^* \left( (\tilde{\psi}^T C_2 \tilde{\phi}) + (\tilde{\phi}^T C_2 \tilde{\psi}) \right),
\frac{\partial \tilde{\psi}}{\partial \xi} = \frac{i}{a} \tilde{\phi} \left( (\tilde{\psi}^\dagger \tilde{\phi}) - (\tilde{\phi}^\dagger \tilde{\psi}) \right) + \frac{2i}{a + \epsilon a^{-1}} C_2 \tilde{\phi}^* \left( (\tilde{\phi}^T C_2 \tilde{\psi}) + (\tilde{\psi}^T C_2 \tilde{\phi}) \right).
$$

(41)
These equations can be obtained from the action:

$$A_{Z_2, \text{GNb}} = \int_{-\infty}^{\infty} dx \, dt \left( i \left( \vec{\phi}^\dagger \frac{\partial \vec{\phi}}{\partial \eta} + \vec{\psi}^\dagger \frac{\partial \vec{\psi}}{\partial \xi} \right) - \frac{1}{2a} \left( (\vec{\psi}^\dagger, \vec{\phi}) - (\vec{\phi}^\dagger, \vec{\psi}) \right)^2 
- \frac{1}{\epsilon a^{-1} + a} \left( \left( \vec{\phi}^\dagger C_2 \vec{\psi}^* \right) + \left( \vec{\phi}^T C_2 \vec{\psi} \right) \right)^2 \right). \quad (42)$$

iii) $Z_2$-ZM models. Here $\mathfrak{g} \simeq SO(N, \mathbb{R})$. Again we used $N$-component vectors to cast the $Z_2$-reduced ZM systems in the form:

$$\frac{\partial \vec{\psi}}{\partial \xi} = \frac{i}{a} \left( \vec{\phi}^* (\vec{\phi}^T, \vec{\psi}) - \vec{\phi} (\vec{\phi}^\dagger, \vec{\psi}) \right) + \frac{2i}{a + \epsilon a^{-1}} C \left( \vec{\phi}^* (\vec{\phi}^T C \vec{\psi}) - \vec{\phi} (\vec{\phi}^\dagger C \vec{\psi}) \right), \quad (43)$$

where the involutive automorphism $C$ can be chosen as one of the type:

$$C = \text{diag} (\epsilon_1, \epsilon_2, \ldots, \epsilon_2, \epsilon_1), \quad \epsilon_j = \pm 1, \quad b) \quad C' = \begin{pmatrix} 1 & 0 \\ 0 & C_3 \end{pmatrix}, \quad (44)$$

with $C_3^2 = 1$. For these choices of $C$ we have $\lim_{\xi \to \pm \infty} U_R(\xi, \eta) = \lim_{\xi \to \pm \infty} C U_R(\xi, \eta) \hat{C}$.

The action for the reduced ZM models is provided by:

$$A_{Z_2, \text{ZM}} = \int_{-\infty}^{\infty} dx \, dt \left( i \left( \vec{\phi}^\dagger \frac{\partial \vec{\phi}}{\partial \eta} + \vec{\psi}^\dagger \frac{\partial \vec{\psi}}{\partial \xi} \right) + \frac{1}{a} \left( (\vec{\psi}^\dagger, \vec{\phi}^*) (\vec{\phi}^T, \vec{\psi}) - (\vec{\phi}^\dagger, \vec{\psi}) (\vec{\psi}^\dagger, \vec{\phi}) \right) 
+ \frac{2}{\epsilon a^{-1} + a} \left( \left( \vec{\psi}^\dagger C \vec{\phi}^* \right) (\vec{\phi}^T C \vec{\psi}) - \left( \vec{\phi}^\dagger C \vec{\psi} \right) (\vec{\psi}^\dagger C \vec{\phi}) \right) \right). \quad (45)$$

V. SPECTRAL PROPERTIES OF THE REDUCED LAX OPERATORS

Here we briefly outline the construction of the fundamental analytic solutions of the Lax operator $L_R$. First we introduce the Jost solutions:

$$\lim_{\xi \to \infty} \Psi_{R,+}(x, t, \lambda) E_R^{-1}(x, t, \lambda) = 1, \quad \lim_{\xi \to -\infty} \Psi_{R,-}(x, t, \lambda) E_R^{-1}(x, t, \lambda) = 1,$$

$$E_R(x, t, \lambda) = \exp \left( -i \frac{J \xi}{\lambda - a} - i \frac{CJC^{-1} \xi}{\epsilon \lambda^{-1} - a} \right), \quad (46)$$

The scattering matrix is defined by:

$$T_R(\lambda, \eta) = \Psi_{R,+}(x, t, \lambda) \Psi_{R,-}(x, t, \lambda) \quad (47)$$
Again we will need the generalized Gauss decomposition compatible with $J$:

$$T_R(\lambda, t) = T_{jR}^{-}D_{jR}^{+}S_{jR}^{+}, \quad T_R(\lambda, t) = T_{jR}^{+}D_{jR}^{-}S_{jR}^{-},$$

(48)

Their block-matrix form is the same like in eq. (26) or (27); the only difference is that they should satisfy the additional symmetry condition with respect to the second involution of $L_R$.

The continuous spectrum of $L_R$ fills up the curves on the complex $\lambda$-plane on which

$$\text{Re} \left( -i \frac{J \xi}{\lambda - a} - i \frac{CJC^{-1} \xi}{\epsilon \lambda^{-1} - a} \right) = \text{Im} \left( \frac{J \xi}{\lambda - a} + \frac{CJC^{-1} \xi}{\epsilon \lambda^{-1} - a} \right) = 0. \quad (49)$$

Below we consider four different cases depending on the choice of $\epsilon$ and $C$. For convenience we denote $\lambda = \lambda_0 + i\lambda_1$ where $\lambda_0$ and $\lambda_1$ are real.

**Case a):** $CJ\hat{C} = J$ and $\epsilon = 1$. Condition (49) becomes:

$$\lambda_1 (\lambda_0^2 + \lambda_1^2 - 1) = 0, \quad (50)$$

Thus the continuous spectrum of $L_{ii}$ consists of $\mathbb{R} \cup S^1$, where $S^1$ is the unit circle with center at the origin.

The discrete spectrum of $L_R$ contains quadruplets of discrete eigenvalues. The generic quadruplet of eigenvalues consists of $\lambda_k, \lambda_k^*, 1/\lambda_k$ and $1/\lambda_k^*$, see fig. 2a).

**Case b):** $CJ\hat{C} = J$ and $\epsilon = -1$. The analog of eq. (49) is:

$$\text{Im} \left( \frac{J \xi}{\lambda - a} + \frac{CJC^{-1} \xi}{-\lambda^{-1} - a} \right) = 0. \quad (51)$$

Its solution is

$$\lambda_1 (\lambda_0^2 + \lambda_1^2 + 1) = 0, \quad (52)$$

The second factor $\lambda_0^2 + \lambda_1^2 + 1$ is always positive, therefore in this case the continuous spectrum of $L_{ii}$ consists of the real axis $\mathbb{R}$ only.

The discrete spectrum of $L_R$ consists of quadruplets and doublet discrete eigenvalues. The generic quadruplet of eigenvalues consists of $\lambda_k, \lambda_k^*, -1/\lambda_k$ and $-1/\lambda_k^*$. These quadruplets do not degenerate even on the unit circle. Doublet eigenvalues takes place only at $i$ and $-i$, see fig. 2b).
Case c): $CJ\hat{C} = -J$ and $\epsilon = 1$. From eq. (49) we get:

$$\lambda_1 \left( \left( \lambda_0 - \frac{2a}{1 + a^2} \right)^2 + \lambda_1^2 + c_0 \right) = 0, \quad c_0 = \frac{(1 - a^2)^2}{(1 + a^2)^2}. \quad (53)$$

Again the second factor $\lambda_0^2 + \lambda_1^2 + c_0$ is always positive, and therefore the continuous spectrum of $L_R$ consists of the real axis $\mathbb{R}$ only.

The discrete spectrum of $L_R$ consists of quadruplets and doublet discrete eigenvalues. The generic quadruplet eigenvalues consists of $\lambda_k, \lambda_k^*, 1/\lambda_k$ and $1/\lambda_k^*$. The doublet eigenvalues take place if $|\lambda_k| = 1$, i.e. they lie on the unit circle, see fig. 2c).

Case d): $CJ\hat{C} = -J$ and $\epsilon = -1$. From eq. (51) we find:

$$\lambda_1 \left( \left( \lambda_0 - \frac{2a}{1 - a^2} \right)^2 + \lambda_1^2 - c_1^2 \right) = 0, \quad c_1 = \frac{a^2 + 1}{|a^2 - 1|}. \quad (54)$$

Thus the continuous spectrum of $L_{iv}$ consists of $\mathbb{R} \cup S^1$, where $S^1$ is a circle with center on the real axis at $2a/(1 - a^2)$ and radius $c_1$.

The discrete spectrum of $L_R$ consists of quadruplets. The generic quadruplet eigenvalues consists of $\lambda_k, \lambda_k^*, -1/\lambda_k$ and $-1/\lambda_k^*$. The only possible doublet eigenvalues at $\pm i$ are ruled out because they lie on the continuous spectrum of $L_R$, see fig. 2d).

**Remark 2** Note that for the NJLVL models with $g \simeq su(N)$ and $J = \text{diag}(1, 0, \ldots, 0)$ only cases a) and b) are relevant. Indeed, there are no automorphisms of $su(N)$ that transform $J$ into $-J$.

In all the cases described above one should avoid discrete eigenvalues lying on the continuous spectrum of $L_R$.

Now we construct the FAS using the Gauss factors in (18):

$$\chi^\pm(x, t, \lambda) = \Psi_-(x, t, \lambda)S^\pm_J(t, \lambda) = \Psi_+(x, t, \lambda)T^\pm_J(t, \lambda)D^\pm_J(\lambda). \quad (55)$$

For the cases b) and c) $\chi^+(x, t, \lambda)$ and $\chi^-(x, t, \lambda)$ are analytic for $\lambda \in \mathbb{C}_+$ and $\lambda \in \mathbb{C}_-$ respectively. For the cases a) and d) $\chi^+(x, t, \lambda)$ is analytic for $\lambda \in \Omega_1 \cup \Omega_3$ and $\chi^-(x, t, \lambda)$ – for $\lambda \in \Omega_2 \cup \Omega_4$. 
The FAS (55) satisfy a RHP on a contour in \( \mathbb{C} \) which coincides with the continuous spectrum of \( L_R \):

\[
\chi^+(x, t, \lambda) = \chi^-(x, t, \lambda)G_J(\lambda, t), \quad G_J(\lambda, t) = \hat{S}_J^-(\lambda, t)S_J^+(\lambda, t), \quad \lambda \in \mathcal{S}, \quad (56)
\]

where \( \mathcal{S} \) is the continuous spectrum of \( L_R \), see fig. 2. This fact allows one to apply the Zakharov-Shabat dressing method for constructing the soliton solutions of the \( \mathbb{Z}_2 \)-reduced spinor models, very much along the ideas of [12]. Unfortunately now there is no natural point in \( \mathbb{C} \) at which the RHP can be normalized, which presents an additional difficulty in applying the dressing method.
VI. CONCLUSION

We have proposed a new class of $\mathbb{Z}_2$-reduced spinor models. The spectral properties and the construction of the FAS for their reduced Lax operators $L_R$ are outlined.

Other important developments are related to the interpretation of the ISM as a generalized Fourier transform [1, 4]. This can be done using the Wronskian relations to analyze the mapping between the potential $U_R$ and the scattering data. The soliton solutions of these models can be calculated using the method of [12] and will be published elsewhere.

New classes of generalized GN-type spinor models can be constructed choosing appropriate rank-2 matrices for $J$ instead of eq. (12). Such models will have $4N$ independent components and the inverse scattering problem for their Lax operators will be regular.

One can also consider reductions with automorphisms $C$ such that $CJ\hat{C} \neq \pm J$.

Another important problem will be to explore the supersymmetric generalizations of the above models.

Acknowledgements

I am grateful to Professor A. V. Mikhailov and Professor A. S. Sorin for useful suggestions and discussions. I also acknowledge a grant with the JINR, which allowed me to work on the topic 01-3-1073-2009/2013 of Dubna scientific plan and to participate in the XV SYMPHYS conference in Dubna.

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