Lax pair and first integrals for two of nonlinear coupled oscillators

Nikolay A. Kudryashov

National Research Nuclear University MEPhI (Moscow Engineering Physics Institute), 31 Kashirskoe Shosse, 115409 Moscow, Russian Federation

Abstract

The system of two nonlinear coupled oscillators is studied. As partial case this system of equation is reduced to the Duffing oscillator which has many applications for describing physical processes. It is well known that the inverse scattering transform is one of the most powerful methods for solving the Cauchy problems of partial differential equations. To solve the Cauchy problem for nonlinear differential equations we can use the Lax pair corresponding to this equation. The Lax pair for ordinary differential or systems or for system ordinary differential equations allows us to find the first integrals, which also allow us to solve the question of integrability for differential equations. In this report we present the Lax pair for the system of coupled oscillators. Using the Lax pair we get two first integrals for the system of equations. The considered system of equations can be also reduced to the fourth-order ordinary differential equation and the Lax pair can be used for the ordinary differential equation of fourth order. Some special cases of the system of equations are considered.

Key words: System of equations, Oscillator, Lax pair, First integral.

1 Introduction

It is known that Gardner, Green, Kruskal and Miura first opened the inverse scattering transform \[1\-3\] for solving the Cauchy problem of the Korteweg-de Vries equation \[4\]. Using a linear system of equations of the above-mentioned authors Peter Lax in 1968 introduced a new concept \[5\-7\] now called the Lax pair, which allows to solve the Cauchy problem by means of the inverse scattering transform for a certain class of nonlinear evolution equations.

Five years later, in 1973 four young graduates from the Potsdam University Mark Ablowitz, David Kaup, Alain Newell and Harvi Segur suggested to look for nonlinear evolution equations for which the Cauchy problems can be solved by the inverse scattering transform taking into account the operator equation.

Using the power dependencies of the matrix elements on the spectral parameter and on the function and their derivatives, from the operator equation for the AKNS scheme e dependencies of the matrix elements and the evolutionary
equations are sequentially looked for which the Cauchy problem is solved by the inverse scattering transform. Certainly we assume that if a nonlinear differential equation passes the Painlevé test, then the necessary condition for the integrability of an ordinary nonlinear differential equation is satisfied [8–13].

The disadvantage of using the Painlevé test for nonlinear differential equations is that despite the useful information contained in the Fuchs indices and in the expansions of the General solutions to the Laurent series found in the Painlevé test, as a result we obtain neither a general solution of the differential equation nor its first integrals [14–16].

The aim of this paper is to look for nonlinear integrable ordinary differential equations and the first integrals using the modified AKNS scheme for nonlinear ordinary differential equations.

The rest of this work is organized as follows. In Section 2, we discuss the Lax pair associated with the system of nonlinear ordinary differential equations. In Section 3, using the Lax pair we find the system of two nonlinear ordinary differential equations of the second order. We also find the first integrals for this system of equations and discuss the partial cases.

2 The pair for the system of nonlinear coupled equations

Let us consider the following system of nonlinear differential equations

\[ a_1 q_{tt} + b_1 q_t + c_1 p q^2 + d_1 q = 0 \]  \hspace{1cm} (1)

\[ a_2 p_{tt} + b_2 p_t + c_2 q p^2 + d_2 p = 0, \]  \hspace{1cm} (2)

where \( p(t) \) and \( q(t) \) are unknown functions and \( t \) is independent variable, \( a_1, a_2, b_1, b_2, c_1, c_2, d_1 \), and \( d_2 \) are parameters of mathematical models.

Let us look for the Lax pairs for the system of equations in the form

\[ \mathbf{A} \psi = \lambda \psi, \]

\[ \psi_t = \mathbf{B} \psi, \]  \hspace{1cm} (3)

where \( \psi, \mathbf{A} \) and \( \mathbf{B} \) are matrices in the form [8, 9, 13]

\[ \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -i \lambda & q(t) \\ p(t) & i \lambda \end{pmatrix} \]  \hspace{1cm} (4)

We look for the Lax pair for traveling wave reduction of the KdV hierarchy taking into account the equation

\[ \frac{d\mathbf{A}}{dt} = \mathbf{B} \mathbf{A} - \mathbf{A} \mathbf{B}. \]  \hspace{1cm} (5)

Note that equation (5) similar to the Lax pair for the KdV hierarchy if we write this one using the traveling wave solutions.
From equation (5) we have four ordinary differential equation for matrix elements $a_{11}, a_{12}, a_{21}$ and $a_{22}$ in the form

$$\frac{da_{11}}{dt} = wa_{21} - wa_{12},$$

$$\frac{da_{12}}{dt} = -2i\lambda a_{12} + wa_{22} - wa_{11},$$

$$\frac{da_{21}}{dt} = wa_{11} - wa_{22} + 2i\lambda a_{21},$$

$$\frac{da_{22}}{dt} = wa_{12} - wa_{21}.$$

Adding equations (6) and (9) we have

$$\frac{d}{dt}(a_{11} + a_{22}) = 0.$$

From the last equality follows that we get

$$a_{11} = -a_{22}.$$

Taking into account equations (7) and (8) we obtain

$$\frac{d}{dt}(a_{12} + a_{21}) = 2i\lambda(a_{21} - a_{12}).$$

Let us look for the dependence of elements $a_{11}, a_{12}, a_{21}$ and $a_{22}$ in the form

$$a_{11} = \sum_{k=0}^{n} a_k(w, w_2, \ldots)\lambda^{n-k}, \quad a_{12} = \sum_{k=0}^{n-1} b_k(w, w_2, \ldots)\lambda^{n-1-k},$$

$$a_{21} = \sum_{k=0}^{n-1} c_k(w, w_2, \ldots)\lambda^{n-1-k}, \quad a_{22} = -a_{11}.$$

The matrix elements $a_{11}, a_{12}, a_{21}$ and $a_{22}$ of the matrix $A$ can be used for finding the first integrals of the system of equations. It is known that if the matrix $A$ satisfies equation (5) then the first integrals corresponding to the original equation can be obtained by means of calculating of traces $\text{tr}A^k$.

We can use the consequence of this proposition. If the matrix elements $a_{11} = -a_{22}$ then $\text{tr}A^2 = -2\det A$. Let us note that in case $a_{11} = -a_{22}$ we have the following equality

$$\text{tr}A^2 = a_{11}^2 + 2a_{12}a_{22} + a_{22}^2 = (a_{11} + a_{22})^2 +$$

$$+ 2a_{12}a_{21} - 2a_{11}a_{22} = 0 + 2(a_{12}a_{21} - a_{11}a_{22}) = -2\det A.$$

So, to look for the first integrals of the system of equations (11) and (2) we have to calculate the determinant of matrix $A$. 

3
3 Two nonlinear coupled oscillators and their first integrals

Let us assume in (13) $n = 2$. In this case we have

$$a_{11}(t) = a_2(t) + a_1(t)\lambda + a_0(t)\lambda^2, \quad a_{22} = -a_{11},$$

$$a_{12}(t) = b_1(t) + b_0(t)\lambda, \quad a_{21}(t) = c_1(t) + c_0(t)\lambda.$$  \hspace{1cm} (15)

Substituting (15) into equations (6), (7), (8) and (9) we have after calculations the following values of the matrix elements

$$a_{11} = -\alpha p(t) q(t) - C_0 - i\beta \lambda - 2\alpha \lambda^2, \quad a_{22} = -a_{11},$$

$$a_{12} = \alpha q_t + \beta q - 2i\alpha q\lambda, \quad a_{21} = -\alpha p_t + \beta p - 2i\alpha p\lambda.$$ \hspace{1cm} (16)

We also have the system of equations in the form

$$\alpha q_{tt} + \beta q_t - 2\alpha p q^2 - 2C_0 q = 0$$ \hspace{1cm} (18)

and

$$\alpha p_{tt} - \beta p_t - 2\alpha p^2 q - 2C_0 p = 0,$$ \hspace{1cm} (19)

where $p(t)$ and $q(t)$ are unknown functions and $\alpha$, $\beta$ and $C_0$ are parameters of the system of equations.

To look for the first integrals for the system of equations from the Lax pair we have to calculate the determinant of matrix $A$. Determinant of matrix $A$ takes the form

$$\det A = \alpha^2 p_t q_t - \alpha^2 p^2 q^2 - 2C_0 \alpha p q - C_0^2 - \beta^2 p q +$$

$$+ \alpha \beta (q p_t - p q_t) + 2i (\alpha \beta p q - \beta C_0 + \alpha^2 p q_t - \alpha^2 q p_t)\lambda +$$

$$(\beta^2 - 4\alpha C_0)\lambda^2 - 4i\alpha \beta \lambda^3 - 4\alpha^2 \lambda^4.$$ \hspace{1cm} (20)

From expression (20) we obtain two first integral for the system of equations (19) and (18) in the form

$$I_1 = \alpha \beta p q - \beta C_0 + \alpha^2 p q_t - \alpha^2 q p_t$$ \hspace{1cm} (21)

and

$$I_2 = \alpha^2 p_t q_t - \alpha^2 p^2 q^2 - 2C_0 \alpha p q - \beta^2 p q + \alpha \beta (q p_t - p q_t) - C_0^2.$$ \hspace{1cm} (22)

Now let us consider the partial cases of the system of equations (19) and (18) with obtained lax pair.
Assuming $\alpha = 1$, $\beta = 0$ and $p = q$ we have the well-known second-order nonlinear differential equation

$$q_{tt} - 2q^3 - 2C_0 q = 0.$$  

(23)

As this takes place integral (21) is generated and integral (22) is transformed to the well-known integral for equation (23) in the form

$$I_2^{(1)} = q_t^2 - q^4 - C_0 q^2 = C_2.$$  

(24)

The general solution of equation (24) is expressed by means of the Jacobi elliptic function.

Equation (24) can be written in the form

$$q_t^2 = (q - \alpha)(q - \beta)(q - \gamma)(q - \delta).$$  

(25)

where $\alpha$, $\beta$, $\gamma$ and $\delta$ ($\alpha \geq \beta \geq \gamma \geq \delta$) are real roots of the algebraic equation

$$q^4 + \frac{1}{2}C_0 q^2 + C_2 + 0.$$  

(26)

Equation (25) can be transformed to the following form

$$v_t^2 = (1 - v^2)(1 - k^2 v^2),$$  

$$v^2 = \frac{(\beta - \delta)(q - \alpha)}{(\alpha - \delta)(y - \beta)}, \quad k^2 = \frac{(\beta - \gamma)(\alpha - \delta)}{(\alpha - \gamma)(\beta - \delta)}.$$  

(27)

The general solution of (26) is expressed via the elliptic function in the form

$$v(t) = \text{sn} (\chi t, k), \quad \chi^2 = \frac{1}{4}(\beta - \delta)(\alpha - \gamma)$$  

(28)

where $\text{sn} (\chi t, k)$ is the elliptic sine.

The general solution of equation (28) takes the form

$$y(t) = \frac{\beta(\alpha - \delta)\text{sn}^2 (\chi t, k) - \alpha(\beta - \delta)}{(\alpha - \delta)\text{sn}^2 (\chi t, k) - \beta + \delta}.$$  

(29)

From equation (28) we get

$$p \equiv \frac{\frac{q_{ttt}}{2q^2} + \frac{\beta q_t}{2\alpha q^2} - \frac{C_0}{\alpha q}}{2q^2}.$$  

(30)

Substituting $p$ from (30) into equation (23) we have the fourth-order differential equation in the form

$$\alpha^2 \left(q^2 q_{tttt} - 4q q_{tt} q_{ttt} - 3q q_{ttt}^2 + 6q^2 q_{ttt} + \right)$$

$$+ \alpha \left(6\beta q_t^3 - 6\beta q_t q_{tt} + 4C_0 q^2 q_{ttt} - 4C_0 q q_{ttt} \right)$$

$$+ \beta^2 q_t^2 - \beta^2 q_t q_{ttt} = 0.$$  

(31)
From (22) and (21) we have the first integrals for (31) in the form

\[ I_1 = \alpha^2 (q q_{ttt} - 3 q_t q_{tt}) + \alpha \left( 4C_0 q_{tt} - 3 \beta q_t^2 \right) + 4 \beta C_0 q^2 - \beta^2 q_t \]  

(32)

and

\[ I_2 = \alpha^3 \left( 2 q q_{ttt} - 4 q_t^2 q_{tt} - q_q^2 \right) + 4 \beta^2 C_0 q^3 - 2 \beta^3 q^2 q_t + \alpha \beta \left( 8C_0 q_{tt}^2 - 7 \beta q_q^2 \right). \]  

(33)

Assuming \( \beta = 0 \) and \( \alpha = 1 \) we obtain from (18) and (19) the system of equations for description in the form

\[ p_{tt} - 2 p^2 q - 2 C_0 p = 0 \]  

(34)

and

\[ q_{tt} - 2 q^2 p - 2 C_0 q = 0 \]  

(35)

with Hamiltonian that follows from the first integral (22) in the form

\[ H = p_t q_t - q^2 p^2 - 2 C_0 q p = 0. \]  

(36)

The first integrals \( I_1 \) and \( I_2 \) at \( \alpha = 1 \) and \( \beta \) take the form

\[ I_1 = -p q_t - q p_t \]  

(37)

and

\[ I_2 = p_t q_t - p^2 q^2 - 2 C_0 p q - C_0^2. \]  

(38)

Assuming

\[ q_1 = q, \quad p_1 = p, \quad q_2 = q, \quad p_2 = q_t \]  

(39)

we obtain that system equations (34) and (35) are the Hamilton system of equations

\[ \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad (i = 1, 2). \]  

(40)

We also obtain that the first integrals (21) and (22) satisfy to the involution, as this take we have

\[ \{ I_1, I_2 \} = 0 \]  

(41)

At \( \beta \neq 0 \) the system equations (18) and (19) is the Hamilton system too. Hamiltonian for this system of equation can be found form the first integrals (21) and (22) using the same variables \( q_i \) and \( p_i \), where \( (i = 1, 2) \).

We have at \( \alpha = 1 \) and \( \beta = 0 \) the following integrable differential equation of fourth order

\[ q^2 q_{tttt} - 4 q q_{tt} q_{ttt} - 3 q q_{tt}^2 + 6 q_t^2 q_{tt} + 4 C_0 q^2 q_{tt} - 4 C_0 q q_t^2 = 0 \]  

(42)

with two first integrals in the form

\[ I_1 = q q_{ttt} - 3 q_t q_{tt} + 4 C_0 q q_t \]  

(43)
and
\[ I_2 = 2 \frac{q q_t q_{ttt}}{q_t^2} - 4 q_t q_t^2 - q_{ttt} q_t^2 + 4 C_0 q_t^2. \]  
(44)

Equation (43) can be iterated with respect to \( z \). It takes the form
\[ q q_t - 2 q_t^2 + 2 C_0 q^2 = I_1 t + I_3 \]  
(45)

Taking into account the new variable \( q = \frac{V}{2} \) we have from (45) the second order differential equation in the form
\[ V_{tt} - 2 C_0 V + (I_3 + I_1 t) V^3 = 0, \]  
(46)

where \( I_1 \) and \( I_3 \) are arbitrary constants.

At \( I_1 = 0 \) we obtain after integration the equation for the elliptic function Jacobi in the form
\[ V_{tt}^2 - 2 C_0 V^2 + \frac{1}{2} I_3 V^4 = C_4, \]  
(47)

where \( C_4 \) is arbitrary constant.

Let us use the new variable \( q(t) = -\frac{1}{V(t)} \) again in equation (44). We have
\[ 2 V_t^2 V_{tt} - V V_t V_{ttt} + \frac{1}{2} V V_{tt}^2 - 4 C_0 V V_x^2 - I_2 V^6 = 0 \]  
(48)

Substituting \( V_{tt} \) from (46) we get the first-order nonlinear equation in the form
\[ (I_1 t + I_3) V_t^2 + I_1 V V_t - 2 C_0^2 - (2 I_1 C_0 t + I_3 C_0) V^2 - \]  
\[ -\frac{1}{2} I_2 V^3 + \left( \frac{1}{2} I_3^2 + I_1 I_3 t + \frac{1}{2} I_1^2 t^2 \right) V^4 = 0. \]  
(49)

From (49) at \( I_1 = 0 \) we obtain the equation
\[ I_3 V_t^2 + 2 C_0^2 - 2 I_3 C_0 V^2 - \frac{1}{2} I_2 V^3 + \frac{1}{2} I_3^2 V^4 = 0. \]  
(50)

The general solution of equation (50) is expressed via the Jacobi elliptic function.

## 4 Conclusion

In this report we have considered the system of two nonlinear differential equations. We have found the Lax pair for this system. Using this one we have obtained the first integrals for the system of equations.

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