SPECIAL ELLIPTIC FIBRATIONS

by

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Abstract. — We construct examples of elliptic fibrations of orbifold general type (in the sense of Campana) which have no étale covers dominating a variety of general type.

Contents

1. Introduction .......................................................... 1
2. Generalities ........................................................... 2
3. Logarithmic transforms ............................................. 3
4. Construction .......................................................... 6
5. Holomorphic differentials .......................................... 8
References ................................................................. 13

To the memory of our friend and colleague Andrey Tyurin.

1. Introduction

Consider the following two classes of varieties:
– admitting an étale cover which dominates a (positive dimensional) variety of general type;
– admitting a nonconstant map with target an orbifold of general type (defined by taking into account possible multiple fibers of the map, see Section 2 for details).
In this note we construct examples of complex three-dimensional varieties in the second class which are not in the first class, answering a question of Campana (see [2]).

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2. Generalities

Throughout, let $X$ be a smooth projective algebraic variety over $\mathbb{C}$ with function field $\mathbb{C}(X)$, Pic$(X)$ its Picard group and $K_X$ its canonical class. For $D \in \text{Pic}(X)_\mathbb{Q}$ we let $\kappa(D)$ be the Kodaira dimension of $D$, $\kappa(X) = \kappa(K_X)$ the Kodaira dimension of $X$ and $\kappa(X,D) := \kappa(K_X + D)$ the corresponding log-Kodaira dimension. We denote by $\Omega^n_X$ the sheaf of differential $n$-forms, by $T_X$ the tangent bundle and by $\pi_1(X)$ the fundamental group.

We now recall some notions concerning fibrations following [2]. Let $\varphi : X \to B$ be a morphism between smooth algebraic varieties, such that the locus $D := \bigcup_j D_j \subset B$ over which the scheme-theoretic fibers of $\varphi$ are not smooth is a (strict) normal crossing divisor (with irreducible components $D_j$). For each $j$, let $n_j$ be the minimal (scheme-theoretic) multiplicity of a fiber-component over $D_j$ and

$$D(\varphi) := \sum_j (1 - 1/n_j) D_j \in \text{Pic}(B)_\mathbb{Q}$$

the multiplicity divisor of $\varphi$. The pair $(B, D(\varphi))$ will be called an orbifold associated to $\varphi$. It is called of an orbifold of general type if

$$\kappa(B, D(\varphi)) = \dim(B) > 0.$$

Example 2.1. — Let $\varphi : X \to B = \mathbb{P}^1$ be an elliptic fibration such that $D(\varphi) \neq \emptyset$. The degenerate fibers with $n_j \geq 2$ are multiple fibers. The associated orbifold $(B, D(\varphi))$ is of general type provided

$$\sum (1 - 1/n_j) > 2.$$
This condition implies that there exists a finite cover $\tilde{B} \to B$ ramified with multiplicity $n_j$ at points $D_j \subset D(\varphi)$ and of genus $\geq 2$. Let $\tilde{X}$ be the pullback of the elliptic fibration to $\tilde{B}$. Then $\tilde{X} \to X$ is étale and has a surjective map $\tilde{X} \to \tilde{B}$.

**Theorem 2.2.** — There exist smooth projective algebraic threefolds $X$ admitting an elliptic fibration $\varphi : X \to B$ such that

- $\pi_1(X) = 0$;
- $B$ is a smooth elliptic surface with $\kappa(B) = 1$;
- $D(\varphi) \subset B$ is a smooth irreducible divisor;
- the orbifold $(B, D(\varphi))$ is of general type.

**3. Logarithmic transforms**

We recall the construction of logarithmic transforms of elliptic fibrations due to Kodaira [4] (for more details see [3], Section 1.6).

Let $C$ be a smooth curve and $\eta : \mathcal{E} \to C$ a nonisotrivial elliptic fibration. Let $\Delta \subset C$ be a unit disc with center $p_0$ and smooth central fiber $E_0$ over $p_0$. For every $m \in \mathbb{N}$ consider the diagram

\[
\begin{array}{ccc}
\tilde{J} & \xrightarrow{\tilde{\eta}} & J \\
\downarrow{\tilde{\eta}} & & \downarrow{\eta} \\
\tilde{\Delta} & \xrightarrow{\imath_m} & \Delta
\end{array}
\]

where $J$ is the restriction of $\mathcal{E}$ to $\Delta$, $\imath_m$ is a cyclic cover of degree $m$ given by

$\tilde{z} \mapsto \tilde{z}^m = z$

(with $z$ a local analytic coordinate at $p_0$) and $\tilde{J}$ the pullback of $J$ to $\tilde{\Delta}$. After appropriate choices one has

$\tilde{J} = (\mathbb{C} \times \Delta)/\Lambda(z)$, \hspace{1em} $\tilde{\mathcal{F}} = (\mathbb{C} \times \Delta)/\Lambda(\tilde{z}^m)$

(where $\Lambda(z) \subset \mathbb{C}$ is a family of lattices) and

$J = \tilde{J}/C_m$,

where $C_m$ is a finite cyclic group generated by

$(s, \tilde{z}) \mapsto (s, \zeta_m \tilde{z}) \mod \Lambda(\tilde{z}^m)$
(and $\zeta_m$ is an $m$-th root of 1). Let $\omega_m(z)/m$ be a local $m$-torsion section of $J$ and define

$$J' := \tilde{J}/\mathcal{C}_m,$$

where $\mathcal{C}_m$ is a cyclic group generated by

$$(s, \tilde{z}) \mapsto (s + \frac{\omega_m(\tilde{z}^m)}{m}, \zeta_m \tilde{z}) \mod \Lambda(\tilde{z}^m).$$

We have an isomorphism

$$J' \setminus (E_0/\mathcal{C}_m) \simeq J \setminus E_0,$$

and we can extend $J'$ to an elliptic fibration $\eta' : \mathcal{E}' \to C$, called the \textit{logarithmic transform} (twist) of $\mathcal{E}$. In $\mathcal{E}'$ a cycle (circle) $S$ which was bounding a holomorphic section over a disc in $\mathcal{E}$ is homologous to a nontrivial cycle $S' \in \mathcal{E}_0$.

**Proposition 3.1.** — Assume that $\mathcal{E}$ is locally Jacobian and not locally isotrivial and that $\mathcal{E}'$ is obtained from $\mathcal{E}$ by a logarithmic transform at exactly one point $p_0 \in C$. Then

- $H^0(\mathcal{E}, K_\mathcal{E}) \simeq H^0(\mathcal{E}', K_{\mathcal{E}'})$;
- $\pi_1(\mathcal{E}) = 0 \Rightarrow \pi_1(\mathcal{E}') = 0$;
- $\mathcal{E}'$ is Kähler.

**Proof.** — Every form $w \in H^{2,0}(\mathcal{E})$ has a local representation as

$$w = dh \wedge d\log(s).$$

It is visibly invariant under translation by $s$ on $\mathcal{E} \setminus E_0$, is preserved under gluing and can be extended from $\mathcal{E} \setminus E_0$ to $\mathcal{E}'$. Moreover, on $\mathcal{E}'$ it has a zero of multiplicity $m - 1$ along $E_0/\mathcal{C}_m$. After twisting exactly one fiber, we have

$$K_{\mathcal{E}'} = K_\mathcal{E} + (1 - 1/m)E,$$

where $E$ is a (generic) fiber of $\eta$. Since $\mathcal{E}$ is locally Jacobian we have $K_\mathcal{E} = \eta^*L$, where $L \in \text{Pic}(C)$, and $H^0(\mathcal{E}, K_\mathcal{E}) = H^0(C, L)$. Similarly, we have an imbedding $K_{\mathcal{E}'} \hookrightarrow \eta^*(L + p_0)$ and

$$H^0(\mathcal{E}', K_{\mathcal{E}'}) \subset H^0(\mathcal{E}', \eta^*(L + p_0)) = H^0(C, (L + p_0)).$$

We have $h^0(C, (L + p_0)) \leq h^0(C, L) + 1$. An $f \in H^0(C, (L + p_0))$ which is not in the image of $H^0(C, L)$ is nonzero at $p_0$. The corresponding element in $H^0(\mathcal{E}', \eta^*(L + p_0))$ is also nonzero on the fiber over $p_0$. However, every
global section of \( K' \) vanishes on the central fiber. Thus \( f \notin H^0(E', K') \) so that every section of \( K' \) is an extension of a section of \( K \) (restricted to \( E \setminus E_0 \)):

\[
H^0(E, K) = H^0(E', K').
\]

Since \( \pi_1(E) = 0 \) we have \( C = \mathbb{P}^1 \). We claim that \( \pi_1(E \setminus E_0) = 0 \). Indeed, the fundamental group of the elliptic fibration \( (E \setminus E_0) \to (\mathbb{C} \setminus p_0) \) lies in the image of \( \pi_1(E) \), which is a finite abelian group since there are nontrivial vanishing cycles which are homotopic to zero. Since the global monodromy has finite index in \( \text{SL}_2(\mathbb{Z}) \) the group \( \pi_1(E \setminus E_0) \) is also finite abelian and the corresponding covering is fiberwise. Thus it extends as a finite étale covering of \( E \), contradicting the assumption that \( \pi_1(E) = 0 \).

Consider the (topological) quotient spaces \( E/E_0 \) and \( E'/E_0' \). They are naturally isomorphic and we have two exact homology sequences

\[
\ldots \to H_3(E, \mathbb{Q}) \to H_3(E/E_0, \mathbb{Q}) \xrightarrow{d} H_2(E_0, \mathbb{Q}) \to H_2(E, \mathbb{Q})
\]

and

\[
\ldots \to H_3(E', \mathbb{Q}) \to H_3(E'/E_0', \mathbb{Q}) \xrightarrow{d'} H_2(E_0', \mathbb{Q}) \to H_2(E', \mathbb{Q}).
\]

Since \( E \) is Kähler

\[
\mathbb{Q} = H_2(E_0, \mathbb{Q}) \hookrightarrow H_2(E, \mathbb{Q})
\]

and

\[
H_3(E, \mathbb{Q}) = H_3(E/E_0, \mathbb{Q}) = H_1(C, \mathbb{Q})^*.
\]

Here we used that \( H_1(E, \mathbb{Q}) = H_1(C, \mathbb{Q}) \) which follows from the local nonisotriviality of \( E \). Geometrically it means that every 3-cycle on \( E \) and \( E/E_0 \) can be realized as a product of a 1-cycle on \( C \) and an elliptic fiber.

Since \( E/E_0 = E'/E_0' \) the differential \( d' \) is also zero and

\[
H_2(E_0', \mathbb{Q}) \hookrightarrow H_2(E', \mathbb{Q}).
\]

Thus the class of the generic fiber \( E \) is nontrivial. This implies the existence of a Kähler metric (see [5]). Therefore, if \( E \) is Kähler then so is \( E' \).

**Corollary 3.2.** — *If \( E \) is algebraic and rational then \( E' \) is algebraic.*

**Proof.** — A smooth surface \( S \) is projective iff there is a class \( x \in H_2(S, \mathbb{Q}) \) with \( x^2 > 0 \) which is orthogonal to \( H^{2,0}(S) \subset H^2(S, \mathbb{C}) \). Since \( E' \) is Kähler and \( H^{2,0}(E) = H^{2,0}(E') = 0 \) there is such a class in \( H_2(E, \mathbb{Q}) \). 

Example 3.3. — Let \( \xi : \mathbb{P}^1 \to \mathbb{P}^1 \) be a polynomial map of degree \( n \geq 2 \) which is cyclically \( n \)-ramified over \( \infty \). Let \( \phi : \mathcal{E} \to \mathbb{P}^1 \) be a rational elliptic surface and \( \mathcal{E}' \) its logarithmic \( nm \)-twist over \( \infty \). Consider the diagram

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\eta} & \mathbb{P}^1 \\
\downarrow \xi & & \downarrow \xi' \downarrow \xi \\
\bar{\mathcal{E}} & \xrightarrow{\eta} & \mathbb{P}^1 \\
\end{array}
\]

The surface \( \mathcal{E}' \) (induced by \( \xi \)) is a logarithmic \( m \)-twist at \( \infty \) of \( \mathcal{E} \) (induced from \( \bar{\mathcal{E}} \)). We have \( h^0(\mathcal{E}, K_{\mathcal{E}}) = n - 1 \). Since \( \mathcal{E}' \) is algebraic (by Corollary 3.2), \( \mathcal{E}' \) is also algebraic.

For more details concerning algebraicity of elliptic fibrations obtained by logarithmic transformations we refer to [3], Section 1.6.2.

4. Construction

Consider the following diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & \mathcal{E}' \\
\downarrow \varphi & & \downarrow \eta' \\
B & \xrightarrow{\beta} & S & \xrightarrow{\psi} & C \\
\downarrow \gamma & & \downarrow \varphi_1 & & \downarrow C_1 \\
C_1 & & & & \\
\end{array}
\]

where
- \( C_1, C \) are \( \mathbb{P}^1 \);
- \( S \) is a nonisotrivial locally Jacobian elliptic surface with irreducible fibers, \( \pi_1(S) = 0 \) and \( \kappa(S) = 1 \);
- \( \psi : S \to C \) is a rational map with connected fibers defined by a generic line \( \mathbb{P}^1_{\psi} \subset \mathbb{P}(H^0(S, L)) \), where \( L \) is a polarization on \( S \);
- \( \beta : B \to S \) is a minimal blowup so that \( \gamma := \psi \circ \beta : B \to C \) is a fibration with irreducible fibers (it exists since \( L \) is very ample and \( \psi \) is generic, i.e., all singularities of \( \psi \) are simple and lie in different smooth fibers of \( \varphi_1 \)).
– $\eta' : E' \to C$ is the fibration from Example 3.3.
– $\varphi : X \to B$ is the pullback of $\eta'$ via $\gamma$.

**Lemma 4.1.** — Let $B$ be the surface above, $p \in C$ a generic point and $D = \gamma^{-1}(p)$. Then

– $\varphi_1 \circ \beta : B \to C_1$ is an elliptic fibration and $\kappa(B) = 1$;
– $\pi_1(B \setminus D) = 0$.

**Proof.** — The genericity of $L$ and $\psi$ implies that all fibers are irreducible and that $B$ is a blowup of $S$ in a finite number of distinct points in which the divisors from $P_1 \subset P(H_0(S, L))$ intersect transversally. Since $\pi_1(S) = 0$ we have
$$\pi_1(S \setminus D) = C_m$$
(by Lefschetz theorem), where $m$ is the largest integer dividing $L$ in $\text{Pic}(S)$. The corresponding cyclic covering of $S$ is $m$-ramified along $D$. We have
$$(B \setminus D) = (S \setminus D) \cup \bigcup_{i \in I} \ell_i$$
where $I$ is a finite set and $\ell_i$ are affine lines. A cycle generating $\pi_1(B \setminus D)$ is contracted inside one of these lines, which implies that the image of $\pi_1(S \setminus D)$ in $\pi_1(B \setminus D)$ is trivial. 

**Proof of Theorem 2.2** — The elliptic fibration $\varphi : X \to B$ satisfies the claimed properties.

First observe that $D$ intersects all components of the fibers $E$ of
$$\varphi_1 \circ \beta : B \to C_1.$$ 
Indeed, by genericity every such $E$ is either irreducible or a union of a smooth elliptic curve and a rational $(-1)$ curve $P$. If $E$ is irreducible the claim follows from the ampleness of $L$. For the same reason we have $\text{deg}(D_E) \geq 2$. Since $D \cdot P = 1$ there is a nontrivial intersection with another component.

Put $F := K_B + (1 - 1/m)D$. Since $\kappa(B) = 1$ and $K_B \cdot D > 0$ a subspace of sections in $H^0(B, aF)$ (for some $a \in \mathbb{N}$) gives a surjection $B \to C_1 \times C$, so that $\kappa(F) = 2$. Moreover, $F$ intersects positively every divisor in $B$ (except finitely many rational curves $P_i$ obtained by blowing
up $S$). It follows that $F = H + \sum m_i P_i$, where $H$ is a polarization on $B$ and $m_i \geq 0$. Thus $(B, D(\varphi))$ is an orbifold of general type. In particular,

$$\varphi^*K_B \subset \varphi^*F \subset \Omega^2_X,$$

where $\varphi^*F$ is saturated, and $\kappa(\varphi^*F) = 2$. Notice that $\kappa(X) = 2$ since

$$\varphi^*F \times \gamma^*K_{E'/C} \subset K_X \quad \text{and} \quad \kappa(K_{E'/C}) = 1.$$

The pullback $H'$ (to $X$) of a polarization on $E'$ is positive on the fibers of $\varphi$. Since $B$ is projective it has a polarization $H$ such that for some $a \in \mathbb{N}$ the divisor $a \varphi^*H + H'$ is positive on every curve in $X$ and is represented by a positive definite Kähler form. This implies that $X$ is projective.

We claim that $\pi_1(X) = 0$. We know that $\pi_1(B) = \pi_1(B \setminus D) = 0$. Hence $\pi_1(X)$ is in the image of $\pi_1(E)$ of a smooth (elliptic) fiber of $\varphi$. Since the monodromy of $\varphi$ is large it kills the fundamental group of the fiber. Indeed, the restriction of $\varphi$ to a $\mathbb{P}^1$ is isomorphic to $E'$. Since the complement of a multiple fiber in $E'$ has trivial fundamental group the same holds for $X$.

Thus $X$ admits a map onto an orbifold of general type but does not dominate a variety of general type nor has (any) étale covers.

**Remark 4.2.** — In fact, we have proved that $\pi_1(X \setminus \varphi^{-1}(D)) = 0$ so that no modification can yield an étale cover.

## 5. Holomorphic differentials

One of the features of the construction in Section 4 was the use of a 1-dimensional subsheaf of holomorphic forms with many sections. We have seen that such sheaves impose strong restrictions on the global geometry of the variety. Generalizing several results in [1], we now give an alternative proof of Campana’s theorem on the correspondence between such sheaves and maps onto orbifolds of general type (see [2]).

Let $X$ be a smooth Kähler manifold and $\omega \in \Omega^1_X$ a form. The *kernel* of $\omega$ is the subsheaf of $\mathcal{T}_X$ generated (locally) by sections $t$ such that for all $x \in \Lambda^{i-1}\mathcal{T}_X$

$$\omega(t \wedge x) = 0.$$
The kernel doesn’t change under multiplication of \( \omega \) by a nonzero (local) holomorphic section of the structure sheaf. This defines, for every subsheaf \( \mathcal{F} \subset \Omega^i_X \), its kernel \( \text{Ker}(\mathcal{F}) \) (a special case of the notion of support of a differential ideal).

**Definition 5.1.** — We say that \( \mathcal{F} \subset \Omega^i_X \) is \( k \)-monomial if at the generic point of \( X \) a nonzero local section \( f \) of \( \mathcal{F} \) is a product of local holomorphic 1-forms:

\[
f = q_1 \wedge \ldots \wedge q_k \wedge \omega,
\]

where \( 1 \leq k \leq i \) and \( \omega \) is a local \((i-k)\)-form. We call \( \mathcal{F} \) monomial if \( k = i \).

**Proposition 5.2.** — Let \( X \) be a smooth compact Kähler manifold and \( \mathcal{F} \subset \Omega^i_X \) a one-dimensional subsheaf such that

\[
h^0(X, \mathcal{F}^n) \geq an^k + b,
\]

where \( a > 0 \) and \( k \geq 1 \). Then

- \( k \leq i \);
- \( \mathcal{F} \subset \Omega^i_X \) is a \( k \)-monomial subsheaf;
- there exist an algebraic variety \( Y \) of dimension \( k \) and a meromorphic map

\[
\varphi = \varphi_\mathcal{F} : X \to Y
\]

with irreducible generic fibers such that the tangent space of the fiber of \( \varphi \) at a generic point coincides with \( \text{Ker}(\mathcal{F}) \).

**Proof.** — The ratios of sections \( s_l \in H^0(X, \mathcal{F}^n) \) generate a field of transcendence degree \( k \) (for some \( n \geq 1 \)). In particular, there is an \( x \in X \), with \( s_0(x) \neq 0 \), where the local coordinates

\[
f_l = s_l(x)/s_0(x), \quad l = 1, \ldots, k
\]

are independent. We know that \( s_0 \) is locally equal to \( w_0^n \), where \( w_0 \) is a local closed form nonvanishing at \( x \) (see [1]). Further, \( f_l w_0 \) is also a local closed form nonvanishing at \( x \). Since

\[
ds_0 = d(f_l w_0) = df_l \wedge w_0 = 0
\]

we obtain

\[
w_0 = df_l \wedge w'.
\]

Since the forms \( df_l \) are linearly independent we see that
Thus we have a meromorphic map
\[ \varphi : X \to Y, \quad \dim(Y) = k, \]
such that \( s_i \) are locally (at a generic point of \( X \)) products of a power of a volume form induced from \( Y \) under \( \varphi \) and a power of \( \omega \) which is nontrivial on the fiber of \( \varphi \). The map \( \varphi \) is holomorphic outside of the zero locus of the ring \( \oplus_n H^0(X, \mathcal{F}^n) \).

**Corollary 5.3.** — If \( k = i \) or \( k = i - 1 \) then \( \mathcal{F} \subset \Omega^i_X \) is monomial.

**Proof.** — It suffices to consider \( f \in \mathcal{F} \) at generic points. There are two cases:

- \( k = i \): then
  \[ f = df_1 \wedge \ldots \wedge df_k, \]
  (modulo multiplication by a function).
- \( k = i - 1 \): then
  \[ f = df_1 \wedge df_2 \ldots \wedge df_k \wedge q, \]
  where \( q \) is a closed 1-form.

**Remark 5.4.** — The map from (5.1) admits a bimeromorphic modification
\[ \varphi : X \to Y \]
such that

- \( \varphi \) is holomorphic with generically smooth and irreducible fibers;
- \( X \) and \( Y \) are smooth.

**Notations 5.5.** — For \( \varphi \) as in Remark 5.4 we define its degeneracy locus \( D = D_\varphi \) as the subset of all \( y \in Y \) such that \( d\varphi(x) = 0 \) for all \( x \in \varphi^{-1}(y) \).

**Remark 5.6.** — After another modification of \( \varphi \) we can achieve that

- \( \text{codim}(D) \geq 2 \) or
– $D = \cup_j D_j$ and each $\tilde{D}_j := \varphi^{-1}(D_j) = \cup_i \tilde{D}_{ij}$ is a normal crossing divisor.

Assumption 5.7. — The map $\varphi$ is as in Remarks 5.4 and 5.6.

Lemma 5.8. — If $k = i$ then either $\text{codim}(D) \geq 2$ and 
\[ F = \varphi^*K_Y \]

or there exist integers $n_j \geq 1$ such that

– $D(\varphi) := K_Y + \sum_j (1 - 1/n_j)D_j$ is big on $Y$;
– $\varphi$ has multiplicity $\geq n_j$ along every $\tilde{D}_{ij}$;
– $\varphi^*D(\varphi) \subset F$.

Proof. — Every $x$ with $d\varphi(x) \neq 0$ has a neighborhood $U$ such that the restriction of every section $s \in H^0(X, F^n)$ to $U$ is induced from a (unique) section $s_U \in H^0(\varphi(U), nK_Y)$. There is a unique holomorphic tensor $s_Y \in H^0(Y \setminus D, nK_Y)$ (where $D = D_\varphi$ is the degeneracy locus of $\varphi$) such that the restriction of $\varphi^*(s_Y)$ to $X \setminus \varphi^{-1}(D)$ coincides with $s$.

If $\text{codim}(D) \geq 2$ then $s_Y$ has a unique extension to a holomorphic tensor on $Y$ (since $Y$ is smooth). In this case, $Y$ is of general type. In case $\text{codim}(D) = 1$ we see (using Remark 5.6) that $s_Y$ is a well-defined tensor on $Y$ with poles along $D_j$, i.e.,
\[ s_Y \in H^0(Y, nK_Y + \sum_j d_jD_j), \]

for some $d_j \in \mathbb{N}$. Let $n_j$ be the minimal multiplicity of $\varphi$ on the components $\tilde{D}_{ij}$ which surject onto $D_j$ (for all $j$). Since $\varphi^*s_Y$ is holomorphic on $X$, a local computation shows that
\[ d_j \leq n(1 - 1/n_j) \]

(see, for example, [6] and [11]) and that
\[ K_Y + (1 - 1/n_j)D_j \]

is big. \qed

Remark 5.9. — This gives an alternative proof of Campana’s theorem characterizing fibrations over orbifolds of general type.
In the case \( i = k \) a section of \( \mathcal{F}^n \) (at a generic point of \( X \)) descends to the \( n \)th-power of a (local) volume form on \( Y \) but the corresponding global form on \( Y \) may have singularities. These singularities disappear after a finite local covering which is sufficiently ramified along the singular locus. This property can be defined for arbitrary tensors.

**Definition 5.10.** — A meromorphic tensor \( t \) on \( Y \) is locally integrable if for every point \( y \in Y \) there exist a neighborhood \( U = U_y \) and a (local) manifold \( V \) together with a proper finite map
\[
\lambda : \varphi^{-1}(U) \to U
\]
such that \( \lambda^* t \) is holomorphic on \( V \).

For \( k = i - 1 \) we have an analog of Lemma 5.8.

**Lemma 5.11.** — Let \( X \) be a smooth compact Kähler manifold, \( \mathcal{F} \subset \Omega^i_X \) a one-dimensional subsheaf such that
\[
a'n^{i-1} + b' > h^0(X, \mathcal{F}^n) \geq an^{i-1} + b,
\]
with \( a > 0 \), and \( \varphi = \varphi_{\mathcal{F}} \) (as in Proposition 5.2). Then there exist a nontrivial fibration
\[
\rho : A_Y \to Y
\]
(with fibers complex tori) and a map
\[
\alpha = \alpha_{\mathcal{F}} : X \to A_Y
\]
with connected fibers such that
- \( \varphi = \rho \circ \alpha \);
- the tangent space of the fiber of \( \alpha \) at a generic point is contained in \( \text{Ker}(\mathcal{F}) \);
- there is a divisor \( D \subset Y \) such that every section \( s \in \mathcal{F} \) is a lifting of a monomial locally integrable tensor on \( A_Y \).

**Proof.** — By Proposition 5.2 there is map \( \varphi : X \to Y \), where \( \dim(Y) = i - 1 \). It has a natural factorization
\[
\varphi : X \overset{\alpha}{\longrightarrow} A \overset{\rho}{\longrightarrow} Y,
\]
where the fiber $A_y$ of $\rho$ over a generic $y \in Y$ is the Albanese variety $\text{Alb}(X_y)$. Since $F \subset \Omega^i_X$ any section $s \in F^n$ (at a generic point of $X$) can be represented as
$$s = (df_1 \wedge \ldots \wedge q)^n,$$
where $q$ is a closed 1-form. The form $q$ defines a holomorphic form on a generic fiber $X_y$ of $\varphi$ so that $\rho$ is nontrivial and $\dim(\alpha(X_y)) \geq 1$. In particular, the restriction of $q$ to $X_y$ is induced from $A_y$.

It follows that there exists a sheaf $G \subset \Omega^i_A$ such that $F^n$ is a saturation of $\alpha^*G$. Moreover, all sections of $F^n$ are obtained as lifts of integrable meromorphic sections of $G$.

**Remark 5.12.** — Notice that if $\dim(A_y) = 1$ (for generic $y \in Y$) then $G = K_A$ (and $A \to Y$ is an elliptic fibration).

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