From constructive field theory to fractional stochastic calculus. (II) Constructive proof of convergence for the Lévy area of fractional Brownian motion with Hurst index $\alpha \in \left(\frac{1}{8}, \frac{1}{4}\right)$

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Let $B = (B_1(t), \ldots, B_d(t))$ be a $d$-dimensional fractional Brownian motion with Hurst index $\alpha < 1/4$, or more generally a Gaussian process whose paths have the same local regularity. Defining properly iterated integrals of $B$ is a difficult task because of the low Hölder regularity index of its paths. Yet rough path theory shows it is the key to the construction of a stochastic calculus with respect to $B$, or to solving differential equations driven by $B$.

We intend to show in a series of papers how to desingularize iterated integrals by a weak, singular non-Gaussian perturbation of the Gaussian measure defined by a limit in law procedure. Convergence is proved by using ”standard” tools of constructive field theory, in particular cluster expansions and renormalization. These powerful tools allow optimal estimates, and call for an extension of Gaussian tools such as for instance the Malliavin calculus.

After a first introductory paper [39], this one concentrates on the details of the constructive proof of convergence for second-order iterated integrals, also known as Lévy area. A summary in French may be found in [53].

Keywords: fractional Brownian motion, stochastic integrals, rough paths, constructive field theory, Feynman diagrams, renormalization, cluster expansion.

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Let $B = (B_1,\ldots,B_d)$ be a fractional Brownian motion of Hurst index
$
\alpha \in (0,1)\n$ with $d$ independent, identically distributed components. The
paths of this Gaussian process are continuous but very "rough", actually
$\alpha$-Hölder, or more precisely $\alpha^-$-Hölder for every $\alpha^- < \alpha$. This makes the
very definition of stochastic integration along $B$ or of solutions of stochastic
differential equations driven by $B$ a difficult problem, the solution of which
is gradually emerging, with deep connections to sub-Riemannian geometry
\cite{[17]}, combinatorial Hopf algebras of trees \cite{[51],[50],[14]}, and quantum field
theory, more specifically renormalization \cite{[49]}. Contrary to the case of usual
Brownian motion (given by $\alpha = 1/2$), stochastic integrals may not be de-
fin for small $\alpha$ by straightforward, e.g. piecewise linear approximations.
Rough path theory \cite{34,35} shows that the key problem lies in a proper definition of iterated integrals of $B$ of order $2, 3, \ldots, N$, with $N = \lfloor 1/\alpha \rfloor$, \footnote{\[ \lfloor . \rfloor = \text{integer part}, \]} making up together what is called a rough path over $B$. Definitions may be found in the previous article \cite{39}. Let us simply say here that a rough path over $B$ is a limit in appropriate Hölder norms of iterated integrals of order $2, 3, \ldots, N$ of a sequence of approximations of $B$ converging to $B$ in $\alpha$-Hölder norm. Other more geometric or algebraic definitions exist, which are shown to be equivalent by using piecewise sub-Riemannian geodesic approximations, the natural (but far less explicit, especially for large $N$, due to the notorious difficulty of construction of geodesics in this setting) generalization of piecewise linear approximations. Despite an abstract (non constructive) proof of existence \cite{36}, and several recent investigations \cite{51, 50, 14} yielding a sort of general classification of rough paths in the algebraic sense, this series of papers gives the first construction of a rough path over $B$ for $\alpha \leq 1/4$ by means of an explicit sequence of approximations. The barrier at $\alpha = 1/4$ has been recognized by several authors using different approaches \cite{11, 42, 47, 48}, and shown to extend to other models as well \cite{26}.

Our solution relies on the previously mentioned algebraic investigations, which have brought to the light the crucial importance of the use of skeleton integrals instead of iterated integrals and of the concept of Fourier normal ordering, and most essentially on the reformulation of this problem in the language of quantum field theory. We shall concentrate here on the construction of second-order iterated integrals of $fBm$ with $\alpha \in (1/8, 1/4)$ and $d = 2$. Skeleton integrals and Fourier normal ordering have been discussed at length in the previous paper \cite{39}. Let us simply state that the singular part of the Lévy area of $fBm$,

$$\text{Area}(s, t) := \int_s^t dB_1(t_1) \int_s^{t_1} dB_2(t_2) - \int_s^t dB_2(t_2) \int_s^{t_2} dB_1(t_1) \quad (0.1)$$

– a second-order iterated integral of $B$ measuring the signed area generated by the path –, is the sum of two terms, $A^\pm(t) - A^\pm(s)$, which are simply increments of two functions $A^\pm$. These diverge in the ultra-violet limit. In other words, $A^\pm$ diverges because of the contribution of highest frequency components of $B$. Precise statements may be given if one decomposes the "signal" $B$ into its different "scales" by using a dyadic Fourier partition of unity. This is well-known to those acquainted either to Besov spaces, wavelets or quantum field theory. Replacing $B$ with the cut-off field $B^{\to \rho} = \sum_{j=-\infty}^{\rho} B^j$, with $\mathcal{F}B^j$ supported on $[M^{j-1}, M^{j+1}] \cup [-M^{j+1}, -M^{j-1}]$ for some fixed base $M > 1$, one obtains cut-off functions $A^{\to \rho}$, $A = A^\pm$, whose variance
diverges like $M^\rho (1-4\alpha)$ when $\rho \to \infty$. This quantity may be expressed as an ultra-violet diverging Feynman diagram, see Fig. 1. Pursuing this reinterpretation, it is tempting to consider the entire bubble series instead of the single bubble diagram. By inserting thin lines with the correct scaling dimension between the bubbles, and considering vertices with an imaginary coupling constant $i\lambda$, one obtains a geometric series (see Fig. 2) which formally sums up to a finite quantity, with the correct degree of homogeneity.

Formally again, $1-\lambda^2 \left( \frac{Me}{\xi} \right)^{1-4\alpha} + \lambda^4 \left( \frac{Me}{\xi} \right)^{2(1-4\alpha)} + \cdots = \frac{1}{1+\lambda^2 (Me/\xi)^{1-4\alpha}}$, a very small quantity (for $\rho \to \infty$), measuring an almost insensitive interaction but sufficient to make the Lévy area converge. As explained in [39], section 3, this may be implemented (in theory at least) by multiplying the statistical weight of the Gaussian paths by the exponential

$$e^{-\frac{1}{2}c'_\alpha \lambda^2 \int dt_1 dt_2 |t_1-t_2|^{-4\alpha} \left( \partial A^+(t_1) \partial A^+(t_2) + \partial A^-(t_1) \partial A^-(t_2) \right)}.$$ Mathematically this sounds like a joke, since we are well beyond the radius of convergence of the series, even for small $\lambda$. But such summations may be performed rigorously scale after scale in a finite time horizon $V = [-T,T]$, going down from scale $\rho$ to scale $-\infty$, uniformly in $\rho$ and $V$. A quantum field theoretic model underlying this may be defined, yielding a sequence of Gibbs measures $P_{\lambda,V,\rho}$ which converges weakly to a unique probability measure $P_\lambda$ when $|V|, \rho \to +\infty$. The law of the process $B$ under this measure is the same

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\[ \text{Figure 1: Bubble diagram. Boldface lines scale as } 1/|\xi|^{1+2\alpha}. \]

\[ \text{Figure 2: Bubble series. Thin lines scale as } 1/|\xi|^{1-4\alpha}. \]

\[ 1-\lambda^2 \left( \frac{Me}{\xi} \right)^{1-4\alpha} + \lambda^4 \left( \frac{Me}{\xi} \right)^{2(1-4\alpha)} + \cdots = \frac{1}{1+\lambda^2 (Me/\xi)^{1-4\alpha}}, \]

\[ \text{The unessential constant } c'_\alpha \text{ is fixed e. g. by demanding that the Fourier transform of the kernel } c'_\alpha |t_1-t_2|^{-4\alpha} \text{ be the function } |\xi|^{4\alpha-1}. \]
as its initial, Gaussian measure, but the cut-off singular quantities $A^\to\rho$ in the interacting measures $P_{\lambda,V,\rho}$ converge to give ultimately a finite rough path over the limit process $B$.

The precise statement is as follows. As a rule, we denote in this article by $\mathbb{E} [...]$ the Gaussian expectation and by $\langle [...] \rangle_{\lambda,V,\rho}$ the expectation with respect to the $\lambda$-weighted interaction measure with scale $\rho$ ultraviolet cut-off restricted to a compact interval $V$, so that in particular $\mathbb{E} [...] = \langle [...] \rangle_{0,R,\infty}$. In the following theorem, $\phi = (\phi_1,\phi_2)$ is the stationary process naturally associated to $B$, see §1.2, whose increments $\phi(t) - \phi(s)$ coincide with those of $B$.

**Theorem 0.1** Assume $\alpha \in (\frac{1}{8}, \frac{1}{4})$. Consider for $\lambda > 0$ small enough the family of probability measures (also called: $(\phi,\partial\phi,\sigma)$-model)

\[
P_{\lambda,V,\rho}(\phi_1,\phi_2) = \frac{1}{Z_{\lambda,V,\rho}} \exp \left\{ -\frac{1}{2} c'_\alpha \lambda^2 \int \int dt_1 dt_2 |t_1 - t_2|^{-4\alpha} \right.  \\
\left. (\partial A^+(t_1) \partial A^+(t_2) + \partial A^-(t_1) \partial A^-(t_2)) + \int \mathcal{L}_{\text{bdry}}^{\to\rho} d\mu^{\to\rho}(\phi_1) d\mu^{\to\rho}(\phi_2), \right. \\
(0.2)
\]

where $d\mu^{\to\rho}(\phi_i) = d\mu(\phi_i^\to\rho)$ is a Gaussian measure obtained by an ultraviolet cut-off at Fourier momentum $|\xi| \approx M^\rho (M > 1)$, see Definition 1.2, and $Z_{\lambda,V,\rho}$ is a normalization constant. Then $(P_{\lambda,V,\rho})_{V,\rho}$ converges in law when $|V|,\rho \to \infty$ to some measure $P_\lambda$, and the associated Lévy area processes $\text{Area}^{\to\rho}(s,t) := \int_s^t d\phi_1^\to\rho(t_1) \int_s^{t_1} d\phi_2^\to\rho(t_2)$ converge in law to some process $\text{Area}(s,t)$.

The purpose of this article is to show this theorem rigorously by using constructive arguments. Perturbative ”arguments” for this – which we have just tried to summarize – have been presented in great details in the previous article [39]. Some brief indications on how to transform these informal arguments into a rigorous proof have been given at the very end of that article. The probability measures $P_{\lambda,V,\rho}$ (including the somewhat mysterious boundary term $\mathcal{L}_{\text{bdry}}^{\to\rho}$) are actually written explicitly in terms of four fields, $\phi_1, \phi_2, \sigma_+$ and $\sigma_-$, where $\sigma_\pm$ are Gaussian fields ”conjugate” to $A_\pm$; integrating out the intermediate fields $\sigma_\pm$ yields eq. (0.2). The exponential weight quadratic in $A$ is equivalent to the imaginary exponential weight $e^{-\int \mathcal{L}_4(\phi,\sigma)(t) dt}$, with $\mathcal{L}_4(\phi,\sigma)(t) = i\lambda (\partial A^+(t) \sigma_+(t) - \partial A^-(t) \sigma_-(t))$.  

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Constructive field theory has a long history, see e.g. the monographies [1, 40, 45]. It is a program originally advocated in the sixties by A. S. Wightman [56], the aim of which was to give explicit examples of field theories with a non-trivial interaction; see [19] and references therein for an extensive bibliography. A short guide to the history of the subject may be found in the survey article [53], a "user’s guide on constructive field theory", so to speak, which also gives some hints on how constructive arguments may be implemented in the particular case of this rough path model. The traditional perturbative approach consists in expanding into series the exponential weight of an interacting theory, $e^{-\lambda \int L_{\text{int}}}$, say. The outcome, represented graphically as a sum over Feynman diagrams, is a formal series in $\lambda$, which diverges due to the accumulation of vertices in small regions of space. This problem is cured by introducing cluster expansions, based on a simplified wavelet decomposition $(\psi^j_\Delta)$ of the fields $\psi = \phi_1, \phi_2, \sigma_{\pm}$ where $j$ is a vertical (Fourier) scale index, and $\Delta$ a horizontal (i.e. in direct space) interval of size $M^{-j}$ around the center of the wavelet component. Each $\psi^j_\Delta$ is to be seen as a degree of freedom of the theory, relatively independent from the others, so that the interaction may be expressed as a divergent doubly-infinite vertical and horizontal sum. Then, instead of a blind series expansion, one Taylor-expands $e^{-\lambda \int L_{\text{int}}}$ to a finite order in each interval, leaving out a Taylor remainder. The outcome may be represented graphically as a sum over polymers, which are connected sums of intervals extending over several scales. Renormalization is performed by computing inductively, scale after scale, scale-dependent counterterms in the interaction, which are not associated to Feynman diagrams but to polymers of a given lowest scale.

It is also the hope of the authors that this article may serve as a reference for anyone willing to learn about constructive field theory in general, since – despite the variety of models – the same core arguments are used again and again, in particular cluster expansions.

Here is the outline of the article. Section 1 contains the definition of multiscale Gaussian fields, including fBm, and many useful notations and general results concerning the scale (wavelet) decompositions. Section 2 is on cluster expansions, section 3 on renormalization. Sections 1 to 3 are extremely general, valid also for fields living on $\mathbb{R}^D$, in the hope that they may serve as a basis for future work, possibly also for $D > 1$-models. Section 4, on the contrary, concentrates on the definition of our specific $(\phi, \partial \phi, \sigma)$-model. The proof of finiteness of $n$-point functions of the Lévy area and freeness of the $\phi$-field is given in section 5. It depends on Gaussian bounds which hold in great generality, and on domination arguments which are very
specific of our model. Finally, the reader may find a list of notations and a glossary in the appendix.

1 Multiscale Gaussian fields

1.1 Scale decompositions

We fix some constant $M > 1$. The next definition is borrowed from [46].

Definition 1.1 (Fourier partition of unity) Let $\chi : \mathbb{R} \to [0, 1]$ be an even, $C^\infty$ function such that $\chi|_{[-1,1]} \equiv 1$ and $\text{supp} \chi \subset [-M,M]$, and

$$\chi^j(\xi) := \chi(M^{-j}\xi) - \chi(M^{1-j}\xi), \quad j \in \mathbb{Z}$$

so that $\text{supp} \chi^j \subset [M^j-1, M^j+1] \cup [-M^{j+1}, -M^j+1]$.

The $(\chi_j)_{j \in \mathbb{Z}}$ define a $C^\infty$ partition of unity, namely, $\sum_{j \in \mathbb{Z}} \chi_j \equiv 1$.

Note that $\text{supp} \chi_j \cap \text{supp} \chi_{j'}$ has empty interior if $|j - j'| = 2$, and is empty if $|j - j'| \geq 3$.

Definition 1.2 (ultra-violet cut-off) 1. Let $\rho \in \mathbb{Z}$. Then the ultra-violet cut-off at scale $\rho$ of a function $f : \mathbb{R}^D \to \mathbb{R}^d$ is

$$f^{-\rho} := \mathcal{F}^{-1} \left( \xi \mapsto \left[ \sum_{\rho j = 0}^{\rho} \chi_j(\xi) \right] \mathcal{F}f(\xi) \right),$$

where $\mathcal{F}$ is the Fourier transformation. Roughly speaking, the ultra-violet cut-off cuts away Fourier components of momentum $\xi$ such that $|\xi| > M^\rho$.

2. Let $C_\psi(x,y) := C_\psi(x-y)$ be the covariance of a stationary Gaussian field $\psi : \mathbb{R}^D \to \mathbb{R}$. Then $\psi$ has same law as the series of independent Gaussian fields $\sum_{j \in \mathbb{Z}} \psi^j$, where $\psi^j$ has covariance kernel $C_\psi^j := \mathcal{F}^{-1} \left( \xi \mapsto \chi^j(\xi) \mathcal{F}C_\psi(\xi) \right)$. The ultra-violet cut-off at scale $\rho$ of the Gaussian field $\psi$ is then $\psi^{-\rho} := \sum_{j = -\infty}^{-\rho} \psi^j$, with covariance $C_\psi^{-\rho} := \sum_{j = -\infty}^{-\rho} C_\psi^j$.

3. The low-momentum field of scale $k$ associated to $\psi$ is $\psi^{-k} := \sum_{j \leq k} \psi^j$. The high-momentum field of scale $j$ associated to $\psi$ is $\psi^j := \sum_{k = j}^{+\rho} \psi^k$ (depending on the cut-off).
Definition 1.3 (phase space) Let $\mathbb{D}^j := \{(kM^{-j}, (k + 1)M^{-j}), k \in \mathbb{Z}\}$, $j \geq 0$ be the set of $M$-adic intervals of scale $j$, and $\mathbb{D} := \cup_{j \geq 0} \mathbb{D}^j$ the disjoint union of these sets over all scales, also called phase space. The set $\mathbb{D}$ is a tree with links (called: inclusion links) connecting each interval $\Delta \in \mathbb{D}^j$ to the unique interval $\Delta' \in \mathbb{D}^{j-1}$ such that $\Delta \subset \Delta'$ (see Definition 2.7 below, and left part of Fig. 2.3 in subsection 3.3). An element of $\mathbb{D}^j$ is usually denoted by $\Delta^j$, or simply $\Delta$ if no confusion may arise. The volume $|\Delta^j|$ is simply $M^{-j}$. If $\Delta \in \mathbb{D}^j$, then one denotes by $j(\Delta) = j$ the scale of $\Delta$.

If $x \in \mathbb{R}$, then $x$ belongs to a single $M$-adic interval of scale $j$, denoted by $\Delta^j$.

If $\Delta^j \in \mathbb{D}^j$, then the set of intervals $\Delta \in \psi_{h<j} \mathbb{D}^h$ such that $\Delta$ lies below $\Delta^j$, i.e. $\Delta \supset \Delta^j$, is denoted by $(\Delta^j)^\uparrow$.

We denote by $d^j(\Delta, \Delta')$, $\Delta, \Delta' \in \mathbb{D}^j$, the distance in terms of number of $M$-adic intervals of scale $j$ between $\Delta$ and $\Delta'$, namely,

$$d^j([kM^{-j}, (k + 1)M^{-j}), [k'M^{-j}, (k' + 1)M^{-j})]) = |k' - k|. \quad (1.2)$$

By extension, one may also define the $d^j$-distance of two points or two $M$-adic intervals of scale $j' > j$, namely, $d^j(x, y) = M^j|x - y|$ and

$$d^j(\Delta, \Delta') := M^{j-j'}d^j'(\Delta, \Delta'), \quad \Delta, \Delta' \in \mathbb{D}^{j'}. \quad (1.3)$$

Remark. It is preferable not to define $d^j(\Delta, \Delta')$ for $\Delta, \Delta' \in \mathbb{D}^{j'}$ with $j' < j$, since $d^j(x, x')$, $x \in \Delta, x' \in \Delta'$ depends strongly on the choice of the points $x, x'$ then.

The definition extends in a natural way to a $D$-dimensional setting by decomposing $\mathbb{R}^D$ into a disjoint union of hypercubes of size side $M^{-j}$.

Now comes a general remark. Let $\hat{f} = \mathcal{F} f(\xi)$ be some function with support in $|\xi| \leq M^j$ such that $|\mathcal{F}(f')(\xi)| = |\xi \hat{f}(\xi)|$ is bounded. Then

$$|\mathcal{F}^{-1} \hat{f}(x) - \mathcal{F}^{-1} \hat{f}(y)| \leq |x - y| \int_{0}^{M^j} |\xi||\hat{f}(\xi)| d\xi \leq K \cdot M^j |x - y|, \quad (1.4)$$

so $\psi^j(x)$ or $\psi^{-j}(x)$ varies slowly inside intervals of scale $k$ if $k > j$. Hence it makes sense in first approximation to consider $\psi^j(x)$ or $\psi^{-j}(x)$ to be approximately equal to the averaged, locally constant function

$$\psi_{av}^j(x) := \sum_{\Delta^j \in \mathbb{D}^j} 1_{x \in \Delta^j} \frac{1}{|\Delta^j|} \int_{\Delta^j} \psi^j(y)dy = \frac{1}{|\Delta^j|} \int_{\Delta^j} \psi^j(y)dy, \quad (1.5)$$

or similarly for the low-momentum field $\psi^{-j}$

$$\psi_{av}^{-j}(x) := \sum_{\Delta^j \in \mathbb{D}^j} 1_{x \in \Delta^j} \frac{1}{|\Delta^j|} \int_{\Delta^j} \psi^{-j}(y)dy = \frac{1}{|\Delta^j|} \int_{\Delta^j} \psi^{-j}(y)dy. \quad (1.6)$$
Summing the $\psi^j_{av}$ over $j$ would give a new function $\sum_{j\geq 0} \psi^j_{av}(x)$ which is a sort of “naive” wavelet expansion of the original function $\psi$; with a little extra care, one could arrange that the two functions be equal, but we shall not need to do so.

Conversely, if $\psi$ is a ‘reasonable’ random Gaussian field, then the covariance $\langle \psi^j(x)\psi^j(y) \rangle$ – or, more generally, $\langle \psi^{j\rightarrow}(x)\psi^{j\rightarrow}(y) \rangle$ – is usually small if the corresponding $M$-adic intervals are far apart, i.e. if $d^j(\Delta^j_x, \Delta^j_y) \gg 1$.

These two remarks may be made precise in the case when $\psi$ is a multi-scale Gaussian field, see Definition 1.4 below. Then, for any scale $j$,

- the field $\psi^{j\rightarrow}$ may be decomposed into the sum of the locally averaged field at scale $j$, namely, $\psi_{av}^j(x)$, and a secondary field, denoted by $\delta \psi^{j\rightarrow}$, whose low momentum components of scale $h < j$ decrease like $M^{-\gamma(j-h)}$ for some $\gamma > 0$, see Lemma 1.6 and Corollary 1.7. For reasons explained below, it is customary to use the nickname of spring factor for a decrease factor of the type $M^{-\gamma(j-h)}$, $\gamma > 0$.

- since the covariance decreases with the distance in terms of number of $M$-adic intervals of scale $j$, it makes sense to try and find some expansion of the functional $L(\psi)$ in which the field values over far enough $M$-adic intervals have been made independent. This is called a cluster expansion (see section 2).

**Definition 1.4 (multiscale Gaussian field)** A multiscale Gaussian field with scaling dimension $\beta < 1$ is a field $\psi = \psi(x)$ such that, for every $r \geq 0$ and $\tau, \tau' = 0, 1, 2, \ldots$,

$$|\langle \partial^{\tau} \psi^j(x) \partial^{\tau'} \psi^j(y) \rangle| \leq K_{\tau, \tau', r} M^{(r + \tau' - 2\beta)j} (1 + M^j |x - y|)^r$$  \hspace{1cm} (1.7)

with some constant $K_{\tau, \tau', r}$ depending only on $\tau, \tau'$ and $r$.

**Remark.** A multiscale Gaussian field $\psi$ with scaling dimension $\beta \in (0, 1)$ has almost surely $\beta$-Hölder paths. Namely, $E|\psi^j(x) - \psi^j(y)|^2 = \int_x^y \int_x^y dz dz' \langle \partial \psi^j(z) \partial \psi^j(z') \rangle$ is bounded by $K |x - y|^2 M^{(2-2\beta)j}$ if $|x - y| < M^{-j}$, and by

$$2 \int_{|x - y|}^{x - y} dz' \langle \psi^j(0) \partial \psi^j(z') \rangle \leq K \int_0^{x - y} dz' \frac{M^{(1-2\beta)j}}{(1 + M^j z')^2} \leq K'M^{-2\beta j}$$ \hspace{1cm} (1.8)
otherwise. Summing over $j$ yields

$$\mathbb{E}|\psi(x) - \psi(y)|^2 \leq K \left( (x-y)^2 \sum_{j \leq -\log|x-y|} M^{2-2\beta}j + \sum_{j \geq -\log|x-y|} M^{-2\beta}j \right) \leq K'|x-y|^{2\beta}, \quad (1.9)$$

and one concludes by using the well-known Kolmogorov-Centsov lemma [44].

As we shall see in the next paragraph, fractional Brownian motion with Hurst index $\alpha$ is the paramount example of multiscale Gaussian field with scaling dimension $\beta = \alpha \in (0, 1)$.

**Definition 1.5 (averaged and secondary fields)** Choose some scale $k \geq 0$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function, typically, $f(x) = \psi^{-k}(x)$ or $\psi^j(x)$ for some $j < k$. Then:

(i) the averaged field at scale $k$ is the locally constant field

$$f_{av}(x) := \sum_{\Delta^k \in \mathbb{D}^k} \mathbf{1}_{\Delta^k} \frac{1}{|\Delta^k|} \int_{\Delta^k} f(y)dy. \quad (1.10)$$

It is more convenient to consider $f_{av}$ as some function on $\mathbb{D}^k$, still denoted by $f$,

$$f(\Delta^k) := \frac{1}{|\Delta^k|} \int_{\Delta^k} f(y)dy, \quad \Delta^k \in \mathbb{D}^k, \quad (1.11)$$

so that $f_{av}(x) = f(\Delta^k_x)$. This notation fixes unambiguously the scale of the average.

(ii) the secondary field at scale $k$ is the difference between the original field and the averaged field at scale $k$, namely,

$$\delta^k f(x) := f(x) - f(\Delta^k_x). \quad (1.12)$$

These definitions imply the following easy Lemma:

**Lemma 1.6**

$$\delta^k f(x) = \frac{1}{|\Delta^k|} \int_{\Delta^k_x} \left( \int_u^x f'(v)dv \right) du = \int_{\Delta^k_x} dv f'(v) \delta^k(x; v) \quad (1.13)$$
where
\[
\delta^k(x; v) := \frac{1}{|\Delta^k_x|} \left\{ (v - \inf \Delta^k_x)1_{v < x} + (v - \sup \Delta^k_x)1_{v > x} \right\} \in [-1, 1] \quad (1.14)
\]
is a signed distance from \(v\) to the boundary of the interval \(\Delta^k_x = [\inf \Delta^k_x, \sup \Delta^k_x]\) measured in terms of the rescaled \(d^k\)-distance.

**Proof.** Straightforward.

**Corollary 1.7** Let \(k, k' > j\). Assume \(\psi = \psi(x)\) is a multiscale Gaussian field with scaling dimension \(\beta < 1\). Then

\[
|\langle \delta^k\psi_j(x)\delta^{k'}\psi_j(y) \rangle| \leq K_r M^{-\beta(k+k')} \frac{M^{-(1-\beta)(k-j)}M^{-(1-\beta)(k'-j)}}{(1 + M^j|x-y|)^r} \quad (1.15)
\]

**Proof.** Straightforward.

Eq. (1.15) emphasizes the “spring-factor” \(M^{-(k-j)}\), resp. \(M^{-(1-\beta)(k-j)}\) gained for each secondary field with respect to the covariance of a multiscale Gaussian field with scaling dimension \(\beta\) at scale \(k\). Had one considered directly \(|\langle \psi_j(x)\psi_j(y) \rangle|\), the spring factor – called rescaling spring factor – in (1.15) would have been simply \(M^{\beta(k-j)}\), see introduction to §5.1.2.

**Remarks.**

1. The same spring factors appear in \(D\) dimensions. It is sometimes useful to take a cleverer definition of the secondary field by replacing the simple average \(f(\Delta^k_x)\) with a wavelet component of \(f\), where the wavelets have vanishing first moments up to order \(\tau \geq 1\). This allows one to enhance the spring-factor from \(M^{-(k-j)}\) to \(M^{-(\tau+1)(k-j)}\), resp. from \(M^{-(1-\beta)(k-j)}\) to \(M^{-(\tau+1-\beta)(k-j)}\).

2. The separation of low-momentum fields into a sum (field average)+(secondary field) is required only for fields with scaling dimension \(\beta > -D/2\), and is not performed otherwise (see explanation after Definition 2.9 and in subsection 5.1).

Consider conversely high-momentum fields \(\psi^j(x), \psi^j(x'), j > h\) with \(x, x' \in D^h\) but \(d^j(x, x') \gg 1\). Then \(\psi^j(x)\) and \(\psi^j(x')\) are almost decorrelated if \(x', x'' \in D^h\) but \(d^j(x, x') \gg 1\). Hence it makes sense to restrict \(\psi^j\) over each sub-interval \(\Delta^j \subset D^h\) of scale \(j\):
Definition 1.8 (restriction of high-momentum fields) Let $\Delta^h \in \mathbb{D}^h$ and $j > h$. Then the high-momentum field $\psi^j(x), x \in \Delta^h$, splits into

$$\psi^j(x) = \sum_{\Delta^j \in \mathbb{D}^j, \Delta^j \subset \Delta^h} \text{Res}_{h\Delta^j} \psi^j(x), \quad x \in \mathbb{D}^h$$

(1.16)

where $\text{Res}_{h\Delta^j} \psi^j(x) := 1_{x \in \Delta^j} \psi^j(x)$.

1.2 Multiscale Gaussian fields in one dimension

We introduce here more specifically the infra-red divergent stationary field $\phi$ associated to fBm $B$, the singular fields $A_{\pm}$ associated to its Lévy area, and the fields $\sigma = \sigma_{\pm}$ conjugate to $A_{\pm}$ (see Introduction or [39]). In this paragraph $D = 1$. All fields come implicitly with an ultra-violet cut-off at scale $\rho$, so that $\psi = \phi, \sigma$ should be understood as $\psi^{\rightarrow \rho}$, $\psi^{\rightarrow \rho}$ as $\psi^{\leftarrow \rho}$, and so on.

Definition 1.9 (Harmonizable representation of fBm) Let $W(\xi), \xi \in \mathbb{R}$ be a complex Brownian motion such that $W(-\xi) = \overline{W(\xi)}$, and

$$B_t := (2\pi c_{\alpha})^{-\frac{1}{2}} \int_{-\infty}^{+\infty} e^{i\xi t} \frac{1}{i\xi} |\xi|^{-\frac{1}{2} - \alpha} dW(\xi), \quad t \in \mathbb{R}$$

(1.17)

The field $B_t, t \in \mathbb{R}$ is called fractional Brownian motion. Its paths are almost surely $\alpha^{-}$ Hölder, i.e. $(\alpha-\varepsilon)$-Hölder for every $\varepsilon > 0$. It has dependent but identically distributed (or in other words, stationary) increments $B_t - B_s$.

In order to gain translation invariance, we shall rather use the closely related stationary process

$$\phi(t) := \int_{-\infty}^{+\infty} e^{i\xi t} |\xi|^{-\frac{1}{2} - \alpha} dW(\xi), \quad t \in \mathbb{R}$$

(1.18)

– with covariance

$$\langle \phi(x)\phi(y) \rangle = \int e^{i\xi(x-y)} \frac{1}{|\xi|^{1+2\alpha}} d\xi$$

(1.19)

– which is infrared divergent. However, the increments $\phi(t) - \phi(s)$ are well-defined for any $(s, t) \in \mathbb{R}^2$.

\footnote{The constant $c_{\alpha}$ is conventionally chosen so that $\mathbb{E}(B_t - B_s)^2 = |t - s|^{2\alpha}$.}
Definition 1.10 1. Let
\[ C_j^\phi(x,y) := \int e^{i(\xi - y)} \frac{\chi^j(\xi)}{|\xi|^{1+2\alpha}} d\xi, \quad j \in \mathbb{Z}. \] (1.20)
Then \( C_{\phi} := \sum_{j \in \mathbb{Z}} C_j^\phi \) is the covariance of the field \( \phi \). We denote by \( \phi := \sum_{j \in \mathbb{Z}} \phi^j \) the corresponding multiscale decomposition of the field \( \phi \) into independent components \( \phi^j, j \in \mathbb{Z} \).

2. Let \( \phi^{\rightarrow j} = \sum_{h=-\infty}^{j} \phi^h \) and \( \phi^{\leftarrow j} = \sum_{k=j}^{\rho} \phi^k \). The covariance of \( \phi \), resp. \( \phi^{\rightarrow j} \), resp. \( \phi^{\leftarrow j} \) is \( D(\chi^j)C_{\phi} \), resp. \( D(\chi^{\rightarrow j})C_{\phi} \), resp. \( D(\chi^{\leftarrow j})C_{\phi} \), where \( \chi^{\rightarrow j} := \sum_{h=-\infty}^{j} \chi^h \), \( \chi^{\leftarrow j} = \sum_{k=j}^{\rho} \chi^k \).

Lemma 1.11 The stationary field \( \phi \) associated to fBm is a multiscale Gaussian field with scaling dimension \( \alpha \).

Proof. A simple scaling of the variable of integration yields
\[ C_j^\phi(x,y) = M^{-2\alpha(j-1)} \int e^{iMj^{-1}(x-y)\xi'} \frac{1}{|\xi'|^{1+2\alpha}} \chi^1(\xi') d\xi' \] (1.21)
with \( \text{supp} \chi^1 \subset [1, M^2] \cup [-M^2, -1] \) bounded away from 0, hence \( |C_j^\phi(x,y)| \lesssim M^{-2\alpha j} \), and more precisely (by integrating by parts \( n \) times) \( |C_j^\phi(x,y)| = O(M^{-2\alpha j} |Mj|x - y|^{-n}) \) when \( |Mj|x - y| \to \infty \). The bound for \( \langle \partial^{\tau} \phi^j(x) \partial^{\tau'} \phi^j(y) \rangle \) is obtained in the same way (simply multiply by \( |\xi'|^{\tau + \tau'} \) in Fourier coordinates). \( \square \)

Definition 1.12 (singular part of the Lévy area) Let
\[ A^+(t) := \frac{1}{2} \sum_{j \in \mathbb{Z}} \partial \phi^j_1 (t) \phi^j_2 (t) + \sum_{j<k} \partial \phi^j_1 (t) \phi^k_2 (t), \] (1.22)
\[ A^-(t) := \frac{1}{2} \sum_{j \in \mathbb{Z}} \partial \phi^j_1 (t) \phi^j_2 (t) + \sum_{j<k} \partial \phi^j_2 (t) \phi^k_1 (t). \] (1.23)

In both cases the derivative acts on the field with lower scale. The reasons underlying this definition may be found in [51, 39]. We shall accept it as it is. Let us simply state that the Lévy area of \( B \) is equal up to a constant coefficient to \( (A^+(t) - A^+(s)) - (A^-(t) - A^-(s)) \), plus a sum of terms (called boundary terms) which are immediately seen to be \( 2\alpha^{-1\text{-Hölder}} \).
for any $\alpha \in (0,1)$. Hence we may forget altogether about these boundary terms.

We may now define the intermediate field $\sigma = (\sigma_+, \sigma_-)$. By definition, the Fourier transform of its bare covariance $\langle \sigma_i \sigma_{i'} \rangle$, $i, i' = \pm$ is

$$\frac{\delta_{i,i'}}{|\xi|^{1-4\alpha}}.$$  

On the other hand, the renormalized covariance (up to the overlap between the supports of the Fourier multipliers $\chi^j$) is essentially

$$\frac{1}{|\xi|^{1-4\alpha}} \text{Id} + \sum_{j=-\infty}^{\rho} \frac{b_j}{|\chi^j(\xi)|},$$

where the scale $j$ mass counterterm $b^j$ for diagrams of scale $j$ with two low-momentum external legs $\sigma \rightarrow (j-1)$ – a two-by-two, positive matrix – is defined inductively (see subsection 2.5, section 3 and subsection 5.3) and shown to be of order $\lambda^2 M^{j(1-4\alpha)}$. For technical reasons one chooses to retain in the covariance of $\sigma$ only a simplified version of the counterterm (essentially the term of highest scale $j$), which is of the same order as the sum of all mass counterterms.

**Definition 1.13**

(i) Let $\sigma$ be the stationary two-component massive Gaussian field with covariance kernel

$$C\quad C_{\sigma}^{\tau}(x,y) := \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{|\xi|^{1-4\alpha} \text{Id} + b^\rho} \chi^\rho(\xi) \, d\xi. \quad (1.24)$$

(ii) Decompose $C_{\sigma}^{\tau}$ into $\sum_{j \in \mathbb{Z}} C_{\sigma}^j$ and $\sigma$ into a sum of independent fields $\sum_{j \in \mathbb{Z}} \sigma^j$ as in Definition 1.10 by setting

$$C_{\sigma}^j(x,y) = \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{|\xi|^{1-4\alpha} \text{Id} + b^\rho} \chi^j(\xi) \, d\xi. \quad (1.25)$$

**Lemma 1.14** $\sigma$ is a multiscale Gaussian field with scaling dimension $-2\alpha$. More precisely,

$$\left| \langle \partial^\tau \sigma^i(x) \partial^{\tau'} \sigma^j(y) \rangle \right| \leq K_{\tau,\tau',r} \frac{M^{(\tau+\tau'-4\alpha)j}}{(1 + M^j|x-y|)^r} \cdot \inf(1, \frac{M^j(1-4\alpha)}{b^\rho}). \quad (1.26)$$

For $\rho$ large enough, $\inf(1, \frac{M^j(1-4\alpha)}{b^\rho}) \approx \lambda^{-2} M^{-(\rho-j)(1-4\alpha)}$. Thus $\sigma$ vanishes in the limit $\rho \to \infty$ because of the infinite mass counterterm.

**Proof.** Same as for Lemma 1.11. Note that the rescaled denominator $\frac{\chi^j(\xi)}{|\xi|^{1-4\alpha} + b^\rho M^{j(1-4\alpha)}}$ is bounded by $\inf(1, \frac{M^j(1-4\alpha)}{b^\rho})$.  \hfill $\square$
2 Cluster expansions: an outline

The horizontal (H) and vertical (V) cluster expansions allow to rewrite the partition function \( Z_V^{\rightarrow \rho} \) over a finite volume, with ultraviolet truncation at scale \( \rho \), as a sum,

\[
Z_V^{\rightarrow \rho} = \sum_{n} \frac{1}{n!} \sum_{\mathbb{P}_1, \ldots, \mathbb{P}_n \text{ non-overlapping}} F_{HV}(\mathbb{P}_1) \ldots F_{HV}(\mathbb{P}_n),
\]  

(2.1)

where:

- \( \mathbb{P}_1, \ldots, \mathbb{P}_n \) are disjoint *polymers*, i.e. sets of intervals \( \Delta \) connected by vertical and horizontal links; during the course of the expansion, the Gaussian measure has been modified so that the field components belonging to different polymers have become independent;

- \( F_{HV}(\mathbb{P}) \), \( \mathbb{P} = \mathbb{P}_1, \ldots, \mathbb{P}_n \) is the \( \lambda \)-weighted expectation value, \( F_{HV}(\mathbb{P}) = \langle f_{HV}(\mathbb{P}) \rangle_\lambda \), of some function \( f_{HV} \) depending only on the field components located in the support of \( \mathbb{P} \).

The fundamental idea is that (i) the polymer evaluation function \( F(\mathbb{P}) \) is all the smaller as the polymer \( \mathbb{P} \) is large, due to the polynomial decrease of correlation at large distances (for the horizontal direction), and to power-counting arguments developed in section 4 for the vertical direction, leading to the image of horizontal islands maintained together by vertical *springs*; (ii) the horizontal and vertical links in \( \mathbb{P} \) (once one interval belonging to \( \mathbb{P} \) has been fixed) suppress the invariance by translation, which normally leads to a divergence when \( |V| \rightarrow \infty \). A classical combinatorial trick, called *Mayer expansion*, allows one to rewrite eq. (2.1) as a similar sum over *trees of polymers*, also called *Mayer-extended polymers* and denoted by the same letter \( \mathbb{P} \), but *without non-overlap conditions*, \( Z_V^{\rightarrow \rho} = \sum_{n} \frac{1}{n!} \sum_{\mathbb{P}_1, \ldots, \mathbb{P}_n} F(\mathbb{P}_1) \ldots F(\mathbb{P}_n) \), where \( F = F_{HV,M} \) is the Mayer-extended polymer evaluation function, so that \( \ln Z_V^{\rightarrow \rho} = \sum_{\mathbb{P}} F(\mathbb{P}) \). In the process, local parts of diverging graphs have been resummed into an exponential, leading to a *counterterm in the interaction*; this is the essence of renormalization. On the whole, one finds that in the limit \( |V|, \rho \rightarrow \infty \), the free energy \( \ln Z_V^{\rightarrow \rho} \) is a sum over each scale of scale-dependent extensive quantities, i.e. \( \ln Z_V^{\rightarrow \rho} = |V| \sum_{j=-\infty}^{\rho} M_j f_{V}^{\rightarrow \rho} \), where \( f_{V}^{\rightarrow \rho} \) converges when \( |V| \rightarrow \infty \) to a finite quantity of order \( O(\lambda) \). One retrieves the idea that each interval of scale \( j \) contains one degree of freedom. Finally, \( n \)-point functions are computed as derivatives of an external-field dependent version of the free energy. The (non restricted) horizontal cluster expansion has been given by D. Brydges and T. Kennedy \[9\], and later on A. Abdesselam and V. Rivasseau \[1, 2, 3\], a beautiful combinatorial structure.
in terms of forests of intervals (see §2.1 and 2.2). The vertical or momentum-decoupling expansion, in terms of $t$-parameters, is somewhat looser, relying on a Taylor expansion to some order $\tau_\Delta$ in each interval $\Delta$. Putting together these two expansions, one obtains so-called polymers (see §2.3). Polymers with too few external legs must still be renormalized by resumming the local part of diverging graphs into an exponential making up a scale-dependent interaction counterterm (see section 3 for detailed explanations); a Mayer expansion makes it possible to get rid of their non-overlap constraints with the other polymers, and finally to divide out the polymers with no external legs (called: vacuum polymers) when computing $n$-point functions (see §2.4). All these provide series depending on the small parameter $\lambda$ which will be shown to converge in section 5.

Summarizing, the general expansion scheme which one must have in mind is made up of a sequence of transformations for each scale, starting from highest scale $\rho$, namely

- Horizontal cluster expansion at scale $\rho \rightarrow$ Vertical cluster expansion at scale $\rho \rightarrow$ separation of local part of diverging graphs of lowest scale $\rho \rightarrow$ Mayer expansion at scale $\rho \rightarrow$ resummation of local parts of diverging graphs of lowest scale $\rho \rightarrow$ ... \rightarrow Horizontal cluster expansion at scale $j \rightarrow$ Vertical cluster expansion at scale $j \rightarrow$ separation of local part of diverging graphs of lowest scale $j \rightarrow$ Mayer expansion at scale $j \rightarrow$ resummation of local parts of diverging graphs of lowest scale $j \rightarrow$ ... down to lowest scale $j = -\infty$.

2.1 The general Brydges-Kennedy-Abdesselam-Rivasseau formula

Let us start with the following general definition.

**Definition 2.1** Let:

(i) $\mathcal{O}$ be an arbitrary set, whose elements are called objects;

(ii) $L(\mathcal{O})$ be the set of links of the total graph associated to $\mathcal{O}$, or in other words, the set of pairs of objects, so that $\ell \in L(\mathcal{O})$ is represented as a pair, $\ell \sim \{o_\ell, o'_\ell\} \subset \mathcal{O}$, $o_\ell \neq o'_\ell$;

(iii) $[0,1]^{L(\mathcal{O})} := \{z = (z_\ell)_{\ell \in L(\mathcal{O})}, 0 \leq z_\ell \leq 1\}$ be the convex set of link weakenings of $\mathcal{O}$;

(iv) $\mathcal{F}(\mathcal{O})$ be the set of forests connecting (some, not necessarily all) vertices of $\mathcal{O}$. A typical element of $\mathcal{F}(\mathcal{O})$ is denoted by $\mathcal{F}$, and its set of links by $L(\mathcal{F}) \subset L(\mathcal{O})$. 

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Assume finally that link weakenings have been made to act in some smooth way on $Z$ (a functional depending on some extra parameters), thus defining a $C^\infty$ link-weakened functional $Z : [0,1]^{L(O)} \to \mathbb{R}, z = (z_\ell)_{\ell \in L(O)} \mapsto Z(z)$ on the set of pairs of objects, still denoted by $Z$ by a slight abuse of notation, such that $Z$ (the original functional) is equal to $Z(1,\ldots,1) = Z(1_{L(O)})$.

**Definition 2.2 (one step of BKAR expansion)** The Brydges-Kennedy-Abdesselam-Rivasseau decoupling expansion consists in the following steps:

(i) Taylor-expand $Z(1,\ldots,1)$ with respect to the parameters $(z_\ell)$ simultaneously, namely,

$$Z(1,\ldots,1) = Z((z_\ell)_{\ell \in L(O)} = 0) + \sum_{\ell_1 \in L(O)} \int_0^1 dw_1 \partial z_{\ell_1} Z((z_\ell)_{\ell \in L(O)} = w_1) . \quad (2.2)$$

(ii) Choose some link $\ell_1$ in the above sum. Draw a link of strength $w_1$ between $o_{\ell_1}$ and $o'_{\ell_1}$ and consider the new set $O_1$ made up of the simple objects $o \in O \setminus \{o_{\ell_1}, o'_{\ell_1}\}$ and of the composite object $\{(o_{\ell_1}, o'_{\ell_1})\}$, with set of links $L(O_1) = L(O) \setminus \{\ell_1\}$. Then $\partial z_{\ell_1} Z(z_{\ell_1} = w_1, (z_\ell)_{\ell \in L(O_1)})$ must now be considered as a functional of $(z_\ell)_{\ell \in L(O_1)}$.

When iterating this procedure, composite objects grow up to be trees. This leads to the following result:

**Proposition 2.3 (Brydges-Kennedy-Abdesselam-Rivasseau or BKAR1 cluster formula)** $Z(1,\ldots,1)$ may be computed as an integral over weakening parameters $w_\ell \in [0,1]$, where $\ell$ does not range over the links of the total graph, but, more restrictively, over the links of a forest on $O$:

$$Z(1,\ldots,1) = \sum_{F \in \mathcal{F}(O)} \left( \prod_{\ell \in L(F)} \int_0^1 dw_\ell \right) \left( \prod_{\ell \in L(F)} \frac{\partial}{\partial z_\ell} Z \right) (z(w)) \quad (2.3)$$

where $z_\ell(w), \ell \in L(O)$ is the infimum of the $w_\ell'$ for $\ell'$ running over the unique path from $o_\ell$ to $o'_{\ell}$ if $o_\ell$ and $o'_{\ell}$ are connected by $F$, and $z_\ell(w) = 0$ otherwise.
If it is not desirable to make a cluster expansion with respect to the links between certain objects (of type 2 in Proposition 2.4 below), then it is sufficient to consider all these objects as belonging to the same composite object. This yields:

**Proposition 2.4 (restricted 2-type cluster or BKAR2 formula)** Assume $\mathcal{O} = \mathcal{O}_1 \sqcup \mathcal{O}_2$. Choose as initial object an object $o_1 \in \mathcal{O}_1$ of type 1, and stop the Brydges-Kennedy-Abdesselam-Rivasseau expansion as soon as a link to an object of type 2 has appeared. Then choose a new object of type 1, and so on. This leads to a restricted expansion, for which only the link variables $z_\ell$, with $\ell \notin \mathcal{O}_2 \times \mathcal{O}_2$, have been weakened. The following closed formula holds. Let $\mathcal{F}_{res}(\mathcal{O})$ be the set of forests $\mathcal{F}$ on $\mathcal{O}$, each component of which is (i) either a tree of objects of type 1, called unrooted tree; (ii) or a rooted tree such that only the root is of type 2. Then

$$Z(1,\ldots,1) = \sum_{\mathcal{F} \in \mathcal{F}_{res}(\mathcal{O})} \left( \prod_{\ell \in L(\mathcal{F})} \int_0^1 dw_\ell \right) \left( \prod_{\ell \in L(\mathcal{F})} \frac{\partial}{\partial S_\ell} \right) Z(S_\ell(w)),$$

(2.4)

where $S_\ell(w)$ is either 0 or the minimum of the $w$-variables running along the unique path in $\overline{\mathcal{F}}$ from $o_\ell$ to $o'_\ell$, and $\overline{\mathcal{F}}$ is the forest obtained from $\mathcal{F}$ by merging all roots of $\mathcal{F}$ into a single vertex.

This restricted cluster expansion will be useful for the Mayer expansion (see section 3.4).

**2.2 Single scale cluster expansion**

**Definition 2.5** (i) A horizontal cluster forest of level $\rho' \leq \rho$, associated to a $d$-dimensional vector Gaussian field $\psi = (\psi_1(x),\ldots,\psi_d(x))$, is a finite number of $M$-adic intervals in $D^{\rho'}$, seen as vertices, connected by links, without loops. Any link $\ell$ connects $\Delta_\ell$ to $\Delta'_\ell$ ($\Delta_\ell, \Delta'_\ell \in D^{\rho'}$), bears a double index, $(i_\ell,i'_\ell) \in \{1,\ldots,d\} \times \{1,\ldots,d\}$, and may be represented as two half-propagators or simply half-segments put end to end, one starting from $\Delta_\ell$ with index $i_\ell$, and the other starting from $\Delta'_\ell$ with index $i'_\ell$.

The set of all horizontal cluster forests of level $\rho'$ is denoted by $\mathcal{F}^{\rho'}$. If $\mathcal{F}' \in \mathcal{F}^{\rho'}$, then $L(\mathcal{F}')$ is its set of links.

(ii) A horizontal cluster tree is a connected horizontal cluster forest. Any horizontal cluster forest decomposes into a product of cluster trees which are its connected components.
(iii) If there exists a link between $\Delta$ and $\Delta'$ ($\Delta, \Delta' \in D^{\rho'}$) then we shall write $\Delta \sim_{\rho'} \Delta'$, or simply (if no ambiguity may arise) $\Delta \sim \Delta'$.

The following result is an easy consequence of Proposition 2.3, see [2].

**Proposition 2.6 (single-scale cluster expansion)** Let

$$Z_V(\lambda) := \int e^{-f_\nu L(\psi)(x)dx}d\mu(\psi),$$

(2.5)

where $d\mu$ is the Gaussian measure associated to a Gaussian field with $d \geq 1$ components, $\psi(x) = (\psi_1(x), \ldots, \psi_d(x))$, with covariance matrix $C = C(i, x; j, y)$, and $L$ is a local functional.

Choose $\rho' \leq \rho$.

Let $d\mu_s(\psi), s = (s_{\Delta, \Delta'}(\Delta, \Delta') \in \mathbb{D}^{\rho'} \times \mathbb{D}^{\rho'} \in [0, 1]$ such that $s_{\Delta, \Delta'} = s_{\Delta', \Delta}$ and $s_{\Delta, \Delta} = 1$, be the Gaussian measure with covariance kernel (if positive definite)

$$C_s(i, x; i', x') = s_{\Delta, \Delta'}C(i, x; i', x')$$

(2.6)

if $\Delta \ni x$, resp. $\Delta' \ni x'$ ($\Delta, \Delta' \in \mathbb{D}^{\rho'}$) are the intervals of size $M^{-\rho'}$ containing $x$, resp. $y$.

Then

$$Z_V(\lambda) := \sum_{\mathbb{F}' \in \mathbb{F}^{\rho'}} \left[ \prod_{\ell \in L(\mathbb{F}^{\rho'})} \int_0^1 dw_\ell \int_{\Delta_\ell} dx_\ell \int_{\Delta'_\ell} dx'_\ell C_s(w_{\ell})(i_\ell, x_\ell; i'_\ell, x'_\ell) \right]$$

$$\int d\mu_{s(w)}(\psi) \text{Hor}^{\rho'}(e^{-f_\nu L(\psi)(x)dx})$$

(2.7)

where:

$$\text{Hor}^{\rho'}(R^{\rho'}; (x_\ell)_{\ell \in L(\mathbb{F}^{\rho'})), (x'_\ell)_{\ell \in L(\mathbb{F}^{\rho'})}) := \prod_{\ell \in L(\mathbb{F}^{\rho'})} \left( \frac{\delta}{\delta \psi_{i_\ell}(x_\ell)} \frac{\delta}{\delta \psi'_{i'_\ell}(x'_\ell)} \right);$$

$$s(w) = (s_{\Delta, \Delta'}(w))_{\Delta, \Delta' \in \mathbb{D}^{\rho'}}, s_{\Delta', \Delta}(w), \Delta \neq \Delta'$$ being the infimum of the $w_\ell$ for $\ell$ running over the unique path from $\Delta$ to $\Delta'$ in $R^{\rho'}$ if $\Delta \sim_{\rho'} \Delta'$, and $s_{\Delta, \Delta'}(w) = 0$ else.

Note that $C_{s(w)}$ is positive-definite with this definition [2], as a convex sum of evidently positive-definite kernels.
2.3 Multi-scale cluster expansion

As explained in the Introduction, cluster expansion has two main objectives. The first one is to express the partition function as a sum over quantities depending essentially on a finite number of degrees of freedom. For a given scale $j$, the horizontal cluster expansion perfectly meets this aim. The second one is to get rid of the invariance by translation, which necessarily produces divergent quantities.

Multi-scale cluster expansion fulfills this program by building inductively, starting from the highest scale $\rho$ and going down to scale $-\infty$, connected multi-scale clusters called polymers. These are finite, connected subsets of $D$, extended over several scales, and containing at least one fixed interval $\Delta^j$ at the bottom which breaks invariance by translation. Like the horizontal cluster expansion, they are obtained by a Taylor expansion with respect to some $t$-parameters $(t_\Delta)_\Delta$ depending on the interval.

Let us give a precise definition of such an object (see left part of Fig. 3).

**Definition 2.7**

(i) A polymer of scale $j$ (typically denoted by $P^j$) is a non-empty tree of $M$-adic intervals of scale $j$, i.e. a connected element of $F^j$. The set of all polymers of scale $j$ is denoted by $P^j$.

(ii) A polymer down to scale $j$ (typically denoted by $P^{j\rightarrow}$) is a connected
graph connecting \(M\)-adic intervals of scale \(k = j, j + 1, \ldots, \rho\), whose links are of two types:

- horizontal links connecting \(M\)-adic intervals of the same scale; the restriction of \(P^j\) to any given scale, \(P^j \cap D^k, k \geq j\), is required to be a disjoint union of polymers of scale \(k\), or simply in other words, an element of \(F^k\); furthermore, \(P^j \cap D^j\) is assumed to be non-empty;

- vertical links or more explicitly inclusion links connecting an \(M\)-adic interval \(\Delta \in D^k, k > j\) to the unique interval \(\Delta' \in D^{k-1}\) below \(\Delta\), i.e. such that \(\Delta \subset \Delta'\). These may be multiple links, with degree of multiplicity \(\tau_\Delta\).

A polymer \(P^j\) is also allowed to have an external structure, characterized by a subset \(\Delta^j \subset P^j \cap D^j\) of intervals of scale \(j\), such that each \(\Delta^j \in \Delta^j\) is connected below by an external inclusion link to \(D^{j-1}\).

(iii) The horizontal skeleton of a polymer \(P^j\) is the disjoint union of the single-scale cluster forests \(P^j \cap D^k, k \geq j\) (all vertical links removed).

One denotes by \(P^j\) the set of polymers down to scale \(j\), and \(P^j_{\text{ext}} \subset P^j\) the subset of polymers with \(N_{\text{ext}} := \sum_{\Delta^j \in P^j \cap D^j} \tau_\Delta\) external links.

In particular, polymers in \(P^j_{\text{ext}}\), without external links, are called vacuum polymers.

The easiest example is that of so-called full-inclusion polymers \(P^j \subset P^j_{\text{ext}}\) containing all inclusion links. Such polymers may be obtained from arbitrary multiscale horizontal cluster forests \(F^j = (F^j, F^{j+1}, \ldots, F^\rho)\) by linking all pairs \((\Delta, \Delta'), \Delta, \Delta' \in F^j\) such that \(\Delta'\) is the unique interval lying below \(\Delta\).

In general, if \(\Delta^k \subset \Delta^{k-1}\), \(\Delta^k \in P^j\), then there is an inclusion link from \(\Delta^k\) to \(\Delta^{k-1}\) inside \(P^j\) (by definition) if and only if \(\tau_{\Delta^k} \neq 0\).

The integer \(\tau_\Delta\) corresponds in the momentum-decoupling expansion to the number of derivatives \(\partial / \partial t_\Delta\).

Whereas horizontal links are built in by independent horizontal cluster expansions at each scale, the construction of vertical links depends on a procedure called momentum decoupling (or sometimes vertical cluster) expansion, which we now set about to describe.

Let us first give some definitions.

Assume positive numbers \(t'_x, j \leq \rho\), depending on \(x\) but locally constant in each \(M\)-adic interval \(\Delta^j \in D^j\), have been defined. Let us introduce
operators $T_x^{-k}$ ($k \leq \rho, x \in \mathbb{R}$) acting on the low-momentum components of the fields at the point $x$, i.e. on $\phi^j(x)$ and $\sigma^j(x)$, $j < k$, by

$$(T^{-k}\psi)^j(x) = T^{-k}\psi^j(x) = t_x^{k} \cdots t_x^{j+1} \psi^j(x), \quad k > j, \quad (2.9)$$

$$(T^{-j}\psi)^j(x) = T^{-j}\psi^j(x) = \psi^j(x), \quad (2.10)$$

where $\psi = \phi, \partial\phi$ or $\sigma$.

The shorthand $(T^{-k}\psi)^j$ emphasizes the idea that $T^{-k}$ is not simply a multiplication by some product of $t$-variables, but an operator acting diagonally on the whole field $\psi$, or rather on $\psi^{-k}$ (seen as a vector).

In other words, writing $t_x^j = t_{\Delta^j} x \in \Delta^j$, so that $t$ is a real-valued function on $\mathbb{D} \rightarrow \rho = \cup_{0 \leq j \leq \rho} \mathbb{D}^j$, one has for $k \geq j$: $(T^{-k}\psi)^j(x) = \prod_{k'=j+1}^{k} t_{\Delta^j} \cdot \psi^j(x)$.

We shall also use the following notation:

$$(T\psi)^{-k} := \sum_{j \leq k} (T^{-k}\psi)^j. \quad (2.11)$$

This weakened field, depending on the reference scale $k$, will be called the dressed low-momentum field at scale $k$. Separating the $\psi^k$-component and the $t^k$-variables from the others yields equivalently:

$$(T\psi)^{-k}(x) = \psi^k(x) + t_x^k \sum_{j < k} (T^{-k}(t^{j-1})\psi)^j(x). \quad (2.12)$$

Dressed interactions may be built quite generally out of dressed low-momentum fields. Let us give a general definition.

**Definition 2.8 (dressed interaction)** (i) Let $\mathcal{L}_q^{-\rho}(\psi)(x) = (\lambda^\rho)^{\kappa_q} \prod_{i \in I_q} \psi_i(x)$ be some arbitrary local functional built out of a product of fields, with bare coupling constant $\lambda^\rho$ to the power $\kappa_q$, called: bare interaction. Then the momentum decoupling of $\mathcal{L}_q$, or simply dressed interaction, is the following quantity,

$$\mathcal{L}_q^{-\rho}(\cdot ; t)(x) := (\lambda^\rho)^{\kappa_q} \prod_{i \in I_q} (T\psi_i)^{-\rho}(x) + \sum_{\rho' \leq \rho} (\lambda^{\rho'-1})^{\kappa_q} (1- (t_x^{\rho'})^{I_q}) \prod_{i \in I_q} (T\psi_i)^{-\rho'(1-1)}(x). \quad (2.13)$$

where $\lambda^{\rho'}, \rho' \leq \rho$ are the renormalized (or running) coupling constants.

(ii) (generalization to scale-dependent case)
Let $L_{q}^{\rho}(\psi)(x) := (\lambda^{\rho})^{\kappa_{q}} \sum_{(j_{i}) \in I_{q}} K_{q}^{(j_{i})} \prod_{i \in I_{q}} \psi_{i}^{j_{i}}(x)$ be a local functional with bare coupling constant $\lambda^{\rho}$ and coefficients $K_{q}^{(j_{i})}$ depending on the scales. Then one defines

$$L_{q}^{\rho}(\cdot; t)(x) := (\lambda^{\rho})^{\kappa_{q}} \sum_{-\infty < (j_{i}) \in I_{q} \leq \rho} K_{q}^{(j_{i})} \prod_{i \in I_{q}} (T^{\to \rho} \psi_{i})^{j_{i}}(x) + \sum_{\rho' \leq \rho} (\lambda^{\rho'-1})^{\kappa_{q}} (1 - (\rho')^{1/|I_{q}|}) \sum_{-\infty < (j_{i}) \in I_{q} \leq \rho'-1} K_{q}^{(j_{i})} \prod_{i \in I_{q}} (T^{\to (\rho'-1)} \psi_{i})^{j_{i}}(x).$$

(2.14)

The general mechanism which causes the initial (so-called bare) coupling constant $\lambda^{\rho}$ to vary with the scale is called renormalization. For some generalities on the subject, one may turn to [39] or to [53]. See section 2.4 for the inductive determination of the running coupling constants $\lambda^{j}$.

Summing up in general the contribution of the different interaction terms, $L_{\text{int}} := \sum_{q=1}^{p} K_{q} L_{q}$, this defines a dressed interaction $L_{\text{int}}^{\to \rho}(\cdot; t)$ and a dressed partition function

$$Z_{V}^{\to \rho}(\lambda; t) = \int \cdots \int Z_{\text{int}}^{\to \rho}(\cdot; t)(x) dx d\mu(\psi).$$

(2.15)

For the moment, vertical links have not been constructed, and nothing prevents a priori the different horizontal clusters to move freely in space one with respect to the other. Translation invariance at scale $j$ produces a factor $O(|V|/M^{j})$ per cluster, due to the choice of one fixed interval for each. As we shall now see, the momentum-decoupling expansion provides the mechanism responsible for the translation-invariance breaking. The net result of the expansion is a separation of phase space into a disjoint union of polymers.

Since this is an inductive procedure, let us first consider the result of horizontal cluster expansion at highest scale $\rho$. It may be expressed as a sum of monomials split over each connected component $T_{1}^{\rho}, \ldots, T_{c_{\rho}}^{\rho}$ of $F^{\rho}$,
with generic term (called $G$-monomial or simply monomial)

$$G^\rho_c := \prod_{t \in L(\mathcal{T}_c^\rho)} \left( \psi_{i_t}(x_t) \psi_{i_t'}(x'_t) \right) \cdot \prod_{t \in L(\mathcal{T}_c^\rho)} \left( t^\rho_{x_t}(T\psi_{j_t}) \rightarrow (\rho^{-1})(x_t) \right) \cdot \left( t^\rho_{x'_t}(T\psi_{j'_t}) \rightarrow (\rho^{-1})(x'_t) \right)$$

(2.16)

for some (possibly empty) index subsets $I_t, I'_t, J_t, J'_t \subseteq \{1, \ldots, d\}$, multiplied by a product of propagators as in eq. (2.7); the $t^\rho$-variables dressing low-momentum components have been written explicitly as in eq. (2.12).

Let us draw an oriented, downward dashed line from $\Delta^\rho \in \mathcal{T}_c^\rho$ to some $\Delta'^\rho \in (\Delta^\rho)^{\downarrow\downarrow}$, $\rho' < \rho$ below $\Delta^\rho$ if $G^\rho_c$ contains some low-momentum field $\psi_{i_t}(x_t)$ with $x_t \in \Delta^\rho$. Either $\Delta'^\rho \not\in \mathbb{F}'$, or $\Delta'^\rho$ belongs to some cluster component $\mathcal{T}^\rho_{\rho'}$ of $\mathbb{F}'$. In the latter case, one has attached $\mathcal{T}^\rho_{\rho'}$ to some cluster tree below, $\mathcal{T}^\rho_{\rho'}$, by the inclusion constraint $\Delta^\rho \subset \Delta'^\rho$, which prevents $\mathcal{T}^\rho_c$ from moving freely in space with respect to $\mathcal{T}^\rho_{\rho'}$. It may also happen that $G^\rho_c$ contains no low momentum field component $\psi_{i_t'}$, $\rho' < \rho$, so that $\mathcal{T}^\rho_c$ looks isolated; unfortunately, nothing prevents some horizontal cluster expansion at a lower scale $\rho'$ from generating some high-momentum field component $\psi_{i_t}(x), x \in \Delta'^\rho \in (\Delta^\rho)^{\downarrow\downarrow}$, which may be represented as a reversed upward dashed line from $\Delta'^\rho$ to $\Delta^\rho$.

In order to have an effective mechanism of separation of scales, we shall make a Taylor expansion to order 1 with respect to the $(t^\rho_{\Delta})_{\Delta \in \mathcal{T}_c^\rho}$-variables of the product (G-monomial) $\times (t^\rho_{\Delta})_{\Delta \in \mathcal{T}_c^\rho}$-dependent part of the dressed interaction), $\tilde{G}^\rho_c := G^\rho_c \ e^{-\int_{\mathcal{T}_c^\rho} L_{\text{int}}(x) dx}$, namely (splitting $\tilde{G}^\rho_c$ into a product $\prod_{\Delta} \tilde{G}^\rho_{\Delta} = \prod_{\Delta} G^\rho_{\Delta} e^{-\int_{\Delta} L_{\text{int}}(x) dx}$ over the fields located in each interval $\Delta$ of scale $\rho$ in $\mathcal{T}_c^\rho$)

$$\tilde{G}^\rho_{\Delta}(t^\rho_{\Delta} = 1) = \tilde{G}^\rho_{\Delta}((t^\rho_{\Delta})_{\Delta \in \mathcal{T}_c^\rho} = 0) + \int_0^1 dt^\rho_{\Delta} \partial_{t^\rho_{\Delta}} \tilde{G}^\rho_{\Delta}(t^\rho_{\Delta}),$$

(2.17)

thus producing a new set of monomials multiplied by the interaction.

Setting all $(t^\rho_{\Delta})_{\Delta \in \mathcal{T}_c^\rho}$ to zero has the effect, see eq. (2.13) and (2.16), of killing in the interaction – and hence in $G^\rho_c$ which is a derivative of the interaction – all mixed terms containing both $\psi_{i_t}(x)$ and $\psi_{i_t'}(x')$, with $x \in$
However, there isn’t necessarily a dashed line from $\Delta \rho ^4$). The number of external legs is does not need to be renormalized. On the other hand, a polymer whose total term for some $\Delta$ leads to a polymer with $\rho$ number of external legs per interval. Choosing the Taylor integral remainder $N$ up to order (see section 4) and to the domination problem (see §6.2), one may need to Taylor expand to higher order, up to order $N_{\text{ext,max}} + O(n(\Delta))$. One obtains thus a polymer with a certain number of external legs per interval. Choosing the Taylor integral remainder term for some $\Delta$ leads to a polymer with $\geq N_{\text{ext,max}}$ external legs, which does not need to be renormalized. On the other hand, a polymer whose total number of external legs is $< N_{\text{ext,max}}$ requires renormalization (see section 4).

Due to the necessity of renormalization (see section 4) and to the domination problem (see §6.2), one may need to Taylor expand to higher order, up to order $N_{\text{ext,max}} + O(n(\Delta))$. One obtains thus a polymer with a certain number of external legs per interval. Choosing the Taylor integral remainder term for some $\Delta$ leads to a polymer with $\geq N_{\text{ext,max}}$ external legs, which does not need to be renormalized. On the other hand, a polymer whose total number of external legs is $< N_{\text{ext,max}}$ requires renormalization (see section 4).

Let us emphasize at this point that high-momentum fields have two scales attached to them: $j$ and $k$ for a field $\psi^h_i(x)$, $x \in \Delta^j$ ($k > j$), produced by the horizontal/vertical cluster expansion at scale $j$; but low-momentum fields $\psi^h_i$ have three scales. There are in fact two cases:

(i) If $\beta_1 < -D/2$ then $\psi_i$ is not separated into a sum (low-momentum field average)+(secondary field). The genesis of $\psi^h_i$ (contrary to the high-momentum case) is actually a process which may not be understood apart from the multi-scale cluster expansion. At their production scale $k$, low-momentum fields are of the form $(T\psi_i)\rightarrow^{(k-1)}(x) = \sum_{j=-\infty}^{k-1} \prod_{x'=y_j+1}^{x'_{k-1}} t_{\Delta'} \psi^j(x)$. Successive $t$-derivations of scale $k-1$, $k-2, \ldots$ ”push” $(T\psi_i)\rightarrow^{(k-1)}(x)$ downward like a down-going elevator, in the sense that $\partial_t^{\Delta_{k-1}}(T\psi_i)\rightarrow^{(k-1)}(x)$ is by construction of scale $\leq k-2$, $\partial_t^{\Delta_{k-1}}(T\psi_i)\rightarrow^{(k-1)}(x)$ of scale $\leq k-3$ and so on. But of course, $t$-derivations may act on other fields instead. The last $t$-derivation acting on $(T\psi_i)\rightarrow^{(k-1)}(x)$ drops $(T\psi_i)\rightarrow^{(k-1)}(x)$ at a certain scale $j < k$. Then $(T\psi_i)\rightarrow^{(j-1)}(x)$ leaves the elevator and is torn apart into its scale components $((T\psi_i)\rightarrow^{(j-1)}h < j)$ through free falling. Thus $\psi^h_i$ has a production scale $k$ and a dropping scale $j$, while $h$ itself may be called
its free falling scale or simply its scale.

(ii) If \( \beta_i \geq -D/2 \), then, at the dropping scale, \((T\psi_i)^{(j-1)}(x)\) is separated from its average \((T\psi_i)^{(j-1)}(\Delta_j^i)\) which must be dominated apart, while \((T\psi_i)^{(j-1)}(x) - (T\psi_i)^{(j-1)}(\Delta_j^i)\) splits into its scale components \(((T^{(j-1)}\delta^j\psi_i)^h)_{h<j}\) through free falling as in case (i).

The extension of the above procedure to lower scales is straightforward and leads to the following result.

**Definition 2.9 (multi-scale cluster expansion)**  
1. Fix a multi-scale horizontal cluster expansion \( \mathbb{F}^{j\rightarrow} = (\mathbb{F}_j^j, \ldots, \mathbb{F}_0^0) \) and consider a polymer down to scale \( j, \mathbb{P}^{j\rightarrow} \), with horizontal skeleton \( \mathbb{F}^{j\rightarrow} \) (see Definition 2.7). To such a polymer is associated a sum of products \( (G_{\text{monomial}}) \times (\text{dressed interaction} \mathcal{L}_{\text{int}}(\cdot ; t)) \), where all \( t_{\Delta} \)-variables such that \( \Delta \in \mathbb{D}^{j\rightarrow} \) and \( \tau_{\Delta} < N_{\text{ext,max}} + O(n(\Delta)) \) have been set to 0, and \( G \) is one of the monomials obtained by expanding \( \prod_{k \geq j} \text{Vert}^k \cdot \text{Hor}^k \).

\[
\text{Vert}^k = \prod_{\Delta \in \mathbb{P}^k} \left( \sum_{\tau_{\Delta} = 0}^{N_{\text{ext,max}}^k - 1} \partial_{t_{\Delta}}^{\tau_{\Delta}} |_{t_{\Delta} = 0}^1 \int_0^1 dt_{\Delta} \frac{(1 - t_{\Delta})^{N_{\text{ext,max}}^k - 1}}{(N_{\text{ext,max}}^k - 1)!} \partial_{t_{\Delta}}^{N_{\text{ext,max}}^k} \right).
\]

2. (definition of \( n(\Delta) \)) Fix \( \mathbb{F}^{j\rightarrow} \) and let \( \Delta \in \mathbb{D}^{j\rightarrow} \). Then \( n(\Delta) \) is the number of intervals of scale \( j(\Delta) \) connected to \( \Delta \) by the forest \( \mathbb{F}_j(\Delta) \).

3. (definition of \( N(\Delta) \)) Fix \( \mathbb{F}^{j\rightarrow} \) and some G-monomial, and let \( \Delta \in \mathbb{D}^{j\rightarrow} \). Then \( N(\Delta) \) is the number of fields \( \psi^{j(\Delta)}(x), x \in \Delta \) of scale \( j(\Delta) \) lying in the interval \( \Delta \).

One must actually consider somewhat separately polymers made up of one isolated interval \( \Delta_j \) where no vertex has been produced; this means that \( t_{\Delta_j+1} = 0 \) for all intervals \( \Delta_{j+1} \subset \Delta_j \); \( t_{\Delta_j} = 0 \); and \( s_{\Delta_j, \Delta'} = 0 \) if \( \Delta' \in \mathbb{D}^j \setminus \{\Delta_j\} \). Write as usual \( \mathcal{L}_{\text{int}} := \sum_{q=1}^p K_q \lambda^q \mathcal{L}_q \). Their contribution
to the partition function reads simply
\[
\int d\mu(\psi) e^{-\int_{\Delta^j} L_{\text{int}}(\cdot; t=0)(x) dx} = 1 - \sum_{q=1}^{p} \kappa_q K_q \lambda^{\kappa_q} \int d\mu(\psi) \\
\int_0^1 dv \int L_{-\Delta^j}^\rho(\cdot; t=0)(x) dx \cdot e^{-v \int_{\Delta^j} L_{\text{int}}(\cdot; t=0)(x) dx} =: 1 - F^j(\emptyset) \quad (2.19)
\]
by using a Taylor expansion to order 1. Note that, in the above expression, all fields have same scale \( j \) since all \( t \)-coefficients connecting \( \Delta^j \) have been set to zero. Now, one eliminates the factor 1 by releasing the constraint that the disjoint union of all polymers must span the whole volume \( V \), and ends up with a polymer whose evaluation is of order \( O(\lambda^\kappa) \). The integral remainder \( F^j(\emptyset) = O(\lambda^{\kappa_q}) \) may now be seen as an isolated polymer with one vertex, instead of an empty polymer, and may be bounded exactly like all other vacuum polymers.

Remark. Note that \( N(\Delta) \) is at most of order \( O(n(\Delta)) \) for a single-scale cluster expansion. This is not true for a multi-scale cluster expansion. Namely, fix \( \Delta^h \in D^h \). Assume a low-momentum field \( \psi^h_i(x) (\beta_i < -D/2) \) or \( \delta^j \psi^h_i(x) (\beta_i \geq -D/2) \) has been produced at scale \( k \) and dropped inside \( \Delta^j \) at scale \( j \), with \( k > j > h \). Although the number of fields produced at scale \( k \) in an interval \( \Delta^k \subset \Delta^j \) increases exponentially with \( k - j \), there are \( \leq N_{\text{ext,max}} + O(n(\Delta^j)) \) low-momentum fields dropped inside \( \Delta^j \) for a given \( G \)-monomial, since \( \partial_{\Delta^j} \) occurs at a power \( \leq N_{\text{ext,max}} + O(n(\Delta^j)) \). On the other hand, the number of low-momentum fields \( \psi^h_i(x), x \in \Delta^j \) with \( \Delta^j \subset \Delta^h \) originating from vertices of scale \( j \) may be of order \( \#\{\Delta^j \in D^j; \Delta^j \subset \Delta^h\} = M^{D(2j-h)} \). This is a well-known phenomenon, called accumulation of low-momentum fields. This "negative" spring-factor must be combined with the rescaling spring factor \( M^{2\beta(j-h)} \), see Corollary 1.7, resulting in \( M^{(D+2\beta)(j-h)} \), a positive spring-factor if \( \beta < -D/2 \). This accounts for the (already mentioned) fact that secondary fields need not be produced for fields with scaling dimension \( < -D/2 \), see subsection 5.1 for details.

At a given scale \( j \), composing the horizontal cluster and momentum-decoupling expansions at all scales \( \geq j \) yields the following result, easy to show by induction:

**Lemma 2.10 (result of the expansion above scale \( j \))**

\[
Z_V^\rho(\lambda; t \rightarrow (j-1)) = \int d\mu(\psi \rightarrow (j-1)) Z_V^{j \rightarrow \rho}(\lambda; (\psi^h)_h \leq j-1; t \rightarrow (j-1)), \quad (2.20)
\]
with

\[
Z_{\lambda}^{[j\rightarrow\rho]}(\ldots) = \left( \sum_{\mathcal{P} \in \mathcal{P}} \prod_{\ell \in L(\mathcal{P})} \int_{0}^{1} dw_{\ell} \int_{\Delta_{t_{\ell}}} dw_{t_{\ell}} \sum_{C_{\ell}(w_{\ell}^{t_{\ell}}, \ldots, \Delta_{t_{\ell}}^{\rho})} \right)
\]

\[
\prod_{\ell \geq j} \int_{\text{Vert}^{k} \text{ Hor}^{k}} e^{-iV_{\ell}^{[j\rightarrow\rho]}(x) dx}
\]

(\ref{eq:2.21})

where \(d\mu_{j\rightarrow\rho}(\psi^{j\rightarrow\rho})\) is a short-hand for \(\prod_{k=1}^{\rho} d\mu_{s(k}(\psi^{k})\).

2. The right-hand side \((\ref{eq:2.21})\) depends on the low-momentum fields \((\psi_{i}^{\lambda})_{h\leq j}\) only through the dressed fields \((T\psi_{i})^{\rightarrow(j-1)}\), since the \(t\)-variables of scale \(\leq j-1\) have not been touched. Hence it makes sense to consider the quantity \(Z_{\lambda}^{[j\rightarrow\rho]}(\lambda) := Z_{\lambda}^{[j\rightarrow\rho]}(\psi^{\lambda}_{h \leq j-1} = 0; t^{\rightarrow(j-1)} = 1)\).

3. The partition function \(Z_{\lambda}^{[j\rightarrow\rho]}(\lambda)\) writes

\[
Z_{\lambda}^{[j\rightarrow\rho]}(\lambda) = \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{\text{non–overlapping } \mathcal{P}_{1}, \ldots, \mathcal{P}_{N} \in \mathcal{P}^{[j\rightarrow\rho]}_{0}} \prod_{n=1}^{N} F_{HV}(\mathcal{P}_{n})
\]

\[
= \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{\mathcal{P}_{1}, \ldots, \mathcal{P}_{N} \in \mathcal{P}^{[j\rightarrow\rho]}_{0}} \prod_{n=1}^{N} F_{HV}(\mathcal{P}_{n}) \cdot \prod_{\ell=\{\mathcal{P}, \mathcal{P}'\}} 1_{\mathcal{P}, \mathcal{P}'} \text{ non–overlapping}
\]

(\ref{eq:2.22})

where \(F_{HV}\), called polymer functional associated to horizontal (H) and vertical (V) cluster expansion, is the contribution of each polymer \(\mathcal{P}^{j\rightarrow\rho}\) to the right-hand side of \((\ref{eq:2.21})\).

Later on, the polymer functional \(F_{HV}\) will be replaced with the renormalized (R) polymer functional \(F_{HV,R}\), or simply \(F\), hence eq. \((\ref{eq:2.22})\) shall not be used in this form. The general discussion in the next subsection, which does not depend on the precise form of \(F\), shows how to get rid of the non-overlapping conditions.

### 2.4 Mayer expansion and renormalization

Recall the polynomial evaluation function \(F_{HV}\) introduced in the expansion of \(Z_{\lambda}^{[j\rightarrow\rho]}\), see eq. \((\ref{eq:2.22})\). It is unsatisfactory in this form because of the non-overlapping conditions which make it impossible to compute directly a
finite quantity out of it, see Introduction. Were this the only source of trouble, it would suffice to make a global Mayer expansion for $Z^{0\rightarrow\rho}(\lambda)$ after the multi-scale cluster expansion has been completed. However, it is also unsatisfactory when renormalization is required; local parts of diverging graphs (for reasons accounted for in section 3) must be discarded and resummed into an exponential, thus leading to a counterterm in the interaction and to the running coupling constants of Definition 2.8. This forces upon us a sequence of three moves at each scale $j$, starting from highest scale $\rho$: (1) a separation of the local part of diverging graphs; (2) the Mayer expansion proper, at scale $j$; (3) the construction of the interaction counterterm (also called renormalization phase).

Let us formalize this into the following:

**Induction hypothesis at scale $j$.** After completing all expansions of scale $\geq j + 1$ and the horizontal/vertical cluster expansion at scale $j$, $Z^{\rho\rightarrow\rho}(\lambda; t)$ has been rewritten as

$$Z^{\rho\rightarrow\rho}(\lambda; t^{\rightarrow(j-1)}) = \prod_{k=j+1}^{\rho} e^{\int_{V|M^k} f^{k\rightarrow\rho}(\lambda)} \cdot \int d\mu(\psi^{\rightarrow(j-1)}) Z^{j\rightarrow\rho}(\lambda; t^{\rightarrow(j-1)}; \psi^{\rightarrow(j-1)}),$$

(2.23)

where $f^{k\rightarrow\rho}(\lambda)$ may be reinterpreted as a scale $k$ free energy density per degree of freedom, with

$$Z^{j\rightarrow\rho}(\lambda; t^{\rightarrow(j-1)}; \psi^{\rightarrow(j-1)}) =$$

$$\sum_{N=1}^{\infty} \frac{1}{N!} \sum_{\text{non}-j-\text{overlapping } \mathcal{P}_1, \ldots, \mathcal{P}_N \in \mathcal{P}^{j\rightarrow}} \int d\mu_s(\psi^{\rightarrow}) \prod_{n=1}^{N} F^{j\rightarrow}_{HV}(\mathcal{P}_n; \psi),$$

(2.24)

see Lemma 2.10 for notations, where "non-$j$-overlapping" means that $\mathcal{P}_1 \cap \mathcal{D}^j, \ldots, \mathcal{P}_N \cap \mathcal{D}^j$ are non-overlapping single-scale polymers, and $F^{j\rightarrow}_{HV}(\mathcal{P}_n; \psi)$ depends only on the values of $\psi$ on the support of $\mathcal{P}_n$.

For $j = \rho$, this formula is equivalent to the single-scale expansion of $Z^{\rho\rightarrow\rho}(\lambda)$ of eq. (2.22).

According to the general expansion scheme (see introduction to section 3), we must now perform three tasks.

1. **Separation of local part of diverging polymers**

Consider the polymer evaluation function $F^{j\rightarrow}_{HV}(\mathcal{P}_n; \psi)$. If $\mathcal{P}_n$ has $N_{\text{ext}} = \sum_{\Delta j \in \mathcal{P}_n \cap \mathcal{D}^j} \tau_{\Delta j} < N_{\text{ext},\text{max}}$ external legs, situated in (possibly coinciding) intervals $\Delta j$, it may be superficially divergent, in which case one
separates its local part according to the rule explained in section 3, by displacing its external legs at the same point: letting \((\psi_i \rightarrow (j-1)(x_i))_{i \in I_q}\), with \(x_i \in \Delta^j_i\), be its external structure, \(F_{HV}^j(P_n; \psi)\) is rewritten as

\[
\prod_{i \in I_q} \int_{\Delta^j_i} dx_i F_{HV, \text{amputated}}^j(P_n; \psi; (x_i)) \prod_{i \in I_q} \psi_i \rightarrow (j-1)(x_i)
= F_{HV, \text{local}}^j(P_n; \psi) + \delta F_{HV}^j(\delta N_{\text{ext}, \text{max}} - N_{\text{ext}} P_n; \psi),
\]

(2.25)

where

\[
F_{HV, \text{local}}^j(P_n; \psi) = \prod_{i \in I_q} \int_{\Delta^j_i} dx_i F_{HV, \text{amputated}}^j(P_n; \psi; (x_i)) \cdot \left( \frac{1}{|I_q|} \sum_{i \in I_q} \psi_i \rightarrow (j-1)(x_i) \right)
\]

(2.26)

and \(F_{HV}^j(P_n; \psi)\) minus its local part is now thought (because it is a Taylor remainder whose degree of divergence has been made negative) as if it had \(N_{\text{ext}, \text{max}} - N_{\text{ext}}\) supplementary external legs shared in an arbitrary way among the intervals \(\Delta^j_i\), which produces a polymer denoted by \(\delta N_{\text{ext}, \text{max}} - N_{\text{ext}} P_n\) belonging to \(P_{N_{\text{ext}, \text{max}}}^j\).

Equivalently (see subsection 3.1), considering only local parts, according to our new convention,

\[
F_{HV}^j(P_n; \psi) = \prod_{i \in I_q} \int_{\Delta^j_i} dx_i F_{HV, \text{amputated}}^j(P_n; \psi; (x_i)) \cdot \prod_{i \in I_q} \psi_i \rightarrow (j-1)(x_i),
\]

(2.27)

where now

\[
F_{HV, \text{amputated}}^j(P_n; \psi; (x_i)) = \frac{1}{|I_q|} \sum_{i \in I_q} \int \prod_{i' \neq i} dx_i' F_{HV}^j(P_n; \psi; (x_i)).
\]

(2.28)

2. Mayer expansion at scale \(j\)

We shall now apply the restricted cluster expansion, see Proposition 2.4, to the functional \(Z_{V}^{j-p}(\lambda; t; \psi^{-(j-1)})\). The ”objects” are multi-scale polymers \(P\) in \(O = \{P_1, \ldots, P_N\}\) (see induction hypothesis above); a link \(\ell \in L(O)\) is a pair of polymers \(\{P_n, P_{n'}\}\), \(n \neq n'\). Objects of type 2 are polymers with \(\geq N_{\text{ext}, \text{max}}\) external legs, whose non-overlap conditions we shall not remove at this stage, because these
polymers are already convergent, hence do not need to be renormalized. All other objects are of type 1, they belong to $\mathcal{P}^j \rightarrow N_{\text{ext,max}} := \bigcup_{N=0}^{N_{\text{ext,max}}} \mathcal{P}^j_N$; they are either vacuum polymers, i.e. polymers with no external legs, or superficially divergent polymers whose contribution to the interaction counterterm one would like to compute.

Link weakenings $S = (S_{\{P, P'\}})_{\{P, P'\} \in L(O)} \in [0, 1]^L(O)$ act on $Z^{j \rightarrow \rho}(\lambda; t \rightarrow (j-1); \psi \rightarrow (j-1))$ by replacing the non-overlapping condition

$$\text{NonOverlap}(\mathbb{P}_1, \ldots, \mathbb{P}_N) := \prod_{(\mathbb{P}_n, \mathbb{P}_n')} 1_{\mathbb{P}_n, \mathbb{P}_n' \text{ non}-j-\text{overlapping}}$$

$$= \prod_{(\mathbb{P}_n, \mathbb{P}_n')} \prod_{\Delta \in \mathbb{P}_n \cap D_j, \Delta' \in \mathbb{P}_n' \cap D_j} (1 + (1_{\Delta \neq \Delta'} - 1))$$

(2.29)

with a weakened non-overlapping condition

$$\prod_{(\mathbb{P}_n, \mathbb{P}_n')} \prod_{\Delta \in \Delta_{\text{ext}}(\mathbb{P}_n), \Delta' \in \Delta_{\text{ext}}(\mathbb{P}_n')} 1_{\Delta \neq \Delta'} \cdot$$

$$\left(1 + S_{\{\mathbb{P}_n, \mathbb{P}_n'\}} \left( \prod_{\Delta \in \mathbb{P}_n \cap D_j, \Delta' \in \mathbb{P}_n' \cap D_j, (\Delta, \Delta') \notin \Delta_{\text{ext}}(\mathbb{P}_n) \times \Delta_{\text{ext}}(\mathbb{P}_n')} 1_{\Delta \neq \Delta'} - 1 \right) \right),$$

(2.30)

where $\Delta_{\text{ext}}(\mathbb{P}) \subset \mathbb{P} \cap D_j$ is the subset of intervals $\Delta_j$ with external legs, i.e. such that $\tau_{\Delta_j} \neq 0$. Choosing the factor 1 in the right-hand side of (2.30) means suppressing the non-overlap conditions. Choosing the factor $\prod 1_{\Delta \neq \Delta'} - 1$ instead makes some overlap between $\mathbb{P}_n$ and $\mathbb{P}_n'$ compulsory.

Note that each factor in (2.30) ranges in $[0, 1]$. We ask the reader to accept this definition as it is and wait till the remark after Proposition 2.11 for explanations.

Let us now give some necessary precisions. The Mayer expansion is really applied to the non-overlap function NonOverlap and not to $Z^{j \rightarrow \rho}(\lambda; t \rightarrow (j-1); \psi \rightarrow (j-1))$. Hence one must still extend the function

$$\int d\mathbf{w}^j \int d\mathbf{t}^j \int d\mathbf{u}^j \int d\mathbf{v}^j \prod_{i=1}^N P_{H^1, j}(\mathbb{P}_n; \psi)$$

to the case when the $\mathbb{P}_n$, $n = 1, \ldots, N$ have some overlap. The natural way to do this is to assume
that the random variables \((\psi^j_{|\mathcal{P}_n})_{n=1,\ldots,N}\) remain independent even when they overlap. This may be understood in the following way. Choose a different color for each polymer \(\mathcal{P}_n = \mathcal{P}_1, \ldots, \mathcal{P}_N\), and paint with that color all intervals \(\Delta^j \in \mathcal{P}_n \cap D^j\). If \(\Delta^j \in \Delta_{\text{ext}}(\mathcal{P}_n)\), then its external links to the interval \(\Delta^{j-1}\) below it are left in black. The rules (2.30) imply that intervals with different colors may superpose; on the other hand, external inclusion links may not, so that: (i) low-momentum fields \(\psi^{(j-1)}(x), x \in \Delta^j\) with \(\Delta^j \in \Delta_{\text{ext}}(\mathcal{P}_n)\), do not superpose and may be left in black (till the next expansion stage at scale \(j-1\) at least); (ii) the color of high-momentum fields of scale \(j\) created at a later stage may be determined without ambiguity. Hence one must see \(\psi^j\) as living on a two-dimensional set, \(D^j \times \{\text{colors}\}\), so that copies of \(\psi^j\) with different colors are independent of each other. This defines a new, extended Gaussian measure \(d\tilde{\mu}^j_{\lambda}(\tilde{\psi}^j)\) associated to an extended field \(\tilde{\psi}^j : \mathbb{R}^D \times \{\text{colors}\} \to \mathbb{R}\), and Mayer-extended polymers. By abuse of notation, we shall skip the tilde in the sequel, and always implicitly extend the fields and the measures of each scale. Mayer-extended polymers shall be considered as (colored) polymers in section 5.

This gives the following expansion for \(Z^{j\rightarrow \rho}_{\psi^j}(\lambda; t^{\rightarrow (j-1)}; \psi^{(j-1)})\).

**Proposition 2.11 (Mayer expansion)** Let \(\mathcal{F}(\mathcal{P}^{j\rightarrow})\) be the set of all forests of polymers whose each component \(T\) is (i) either a tree of polymers of type 1 (called: unrooted tree); (ii) or a rooted tree of polymers such that only the root is of type 2. Then

\[
Z^{j\rightarrow \rho}_{\psi^j}(\lambda; t^{\rightarrow (j-1)}; \psi^{(j-1)}) = \sum_{\mathcal{F} \in \mathcal{F}(\mathcal{P}^{j\rightarrow})} \text{Mayer}(Z^{j\rightarrow \rho}_{\psi^j}(\lambda; t^{\rightarrow (j-1)}; \psi^{(j-1)}); \mathcal{F}),
\]

with

\[
\text{Mayer}(Z^{j\rightarrow \rho}_{\psi^j}(\lambda; t^{\rightarrow (j-1)}; \psi^{(j-1)}); \mathcal{F}) = \left( \prod_{\ell \in L(\mathcal{F})} \int_0^1 dW_{\ell} \right) \left( \prod_{\ell \in L(\mathcal{F})} \frac{\partial}{\partial S_{\ell}} \right) Z^{j\rightarrow \rho}_{\psi^j}(\lambda; t^{\rightarrow (j-1)}; \psi^{(j-1)}) \left(S(W)\right)
\]

(2.32)
where \( S_\ell(W) \) is either 0 or the minimum of the \( W \)-variables running along the unique path in \( \overline{\mathcal{F}} \) from \( o_\ell \) to \( o'_\ell \), and \( \overline{\mathcal{F}} \) is the forest obtained from \( \mathcal{F} \) by merging all roots in \( \mathcal{P}_{\geq N_{\text{ext,max}}} \) into a single vertex.

As a result, (i) polymers \( P_\ell, P'_\ell \) linked by a Mayer link are \( j \)-overlapping (otherwise the derivative \( \partial S_\ell \) would produce a zero factor); (ii) pairs of vacuum polymers \( P, P' \) belonging to different Mayer trees come with the factor 1: they have lost their non-overlap conditions and may superpose each other freely (in other words, they have become transparent to each other). Hence \( Z_{\gamma}^{j\rightarrow\rho}(\lambda; t^{(j-1)}; \psi) \) factorizes as a product

\[
Z_{\gamma}^{j\rightarrow\rho}(\lambda; t^{(j-1)}; \psi) = e^{F_{HVM}^j} \int d\mu(\psi^{(j-1)}) \cdot
\sum_{N=1}^{\infty} \frac{1}{N!} \sum_{\text{non-}j\text{-overlapping}} \prod_{n=1}^{N} F_{HVM}^j(P'_n; \psi)
\]

(2.33)

where \( F_{HVM}^j \) is the contribution of all (unrooted) trees of vacuum polymers. Denote by \( f_j^{\rightarrow\rho}(\lambda) \) the quantity obtained by fixing one interval \( \Delta^j \) of scale \( j \) belonging to one of the polymers of the tree. Summing over all \( \Delta^j \), one obtains an overall factor \( e^{V|M_{\rho}f_j^{\rightarrow\rho}(\lambda)} \).

3. Renormalization phase

For simplicity we shall assume that only 2-point functions \( \langle \psi_i \psi_{i'} \rangle \), \( 1 \leq i, i' \leq d \) need to be renormalized (which is the case for the \( (\phi, \partial \phi, \sigma) \)-model). Denote by \( -\frac{1}{2} \int_{\Delta^j} dx b_{i,i'}^{j}(x) \psi_i^{(j-1)}(x) \psi_{i'}^{(j-1)}(x) \) the contribution to \( F_{HVM}^j \) of all unrooted trees of polymers containing exactly one polymer with 2 external legs in \( \Delta^j \), plus a cloud of vacuum polymers, attached to it, directly or indirectly, by Mayer links. The intervals without external legs of the different unrooted trees of polymers have become transparent to each other, and to the rooted trees too, hence \( b_{i,i'}^{j} \) may be computed by considering one unrooted tree of polymers with 2 external legs, irrespectively of the position of the other trees of polymers. By translation invariance, \( b_{i,i'}^{j} \) is a constant, which is fixed by induction on \( j \) by demanding that

\[
0 \equiv \int dy \frac{\partial^2}{\partial \psi_i^{(j-1)}(x) \partial \psi_{i'}^{(j-1)}(y)} Z_{\gamma}^{j\rightarrow\rho}(\lambda; \psi^{(j-1)}) \bigg|_{\psi^{(j-1)}=0},
\]

(2.34)
or better (so as to eliminate higher scale polymers which depend only on $b^{(j+1)\to}_{i,i'}$)

$$0 \equiv \int dy \frac{\partial^2}{\partial \psi_{i}^{(j-1)}(x) \partial \psi_{i'}^{(j-1)}(y)} \left(Z^{\to \rho} - Z^{(j+1)\to \rho}\right) (\lambda; \psi^{(j-1)})|_{\psi^{(j-1)}=0}.$$  

(2.35)

Hence eq. (2.35) yields $b^{j}_{i,i'}$ as a sum over polymers (containing at least one interval of scale $j$) of an expression depending itself on $b^{j}_{i,i'}$, – an implicit equation.

Local parts of 2-point functions are generated by the exponential over the whole volume, $e^{-\frac{1}{2} \int_{V} dx \int_{i,i'} b^{j}_{i,i'} \psi_{i}^{(j-1)}(x) \psi_{i'}^{(j-1)}(x)}$, which may be seen as a counterterm in the interaction of scale $j$. This counterterm disappears by renormalizing the Gaussian covariance kernel $C_{\psi} = K^{-1}_{0}$ – previously renormalized down to scale $j+1$ – to $K^{-1}$, where

$$K(\psi,\psi) = K_{0}(\psi,\psi) + \delta K^{j}(\psi,\psi) = K_{0}(\psi,\psi) + \int_{V} dx b^{j}_{i,i'} \psi_{i}^{(j-1)}(x) \psi_{i'}^{(j-1)}(x).$$  

(2.36)

Note that the renormalization of $C_{\sigma}$ in the case of the $(\phi, \partial \phi, \sigma)$ shall only be performed at scale $\rho$, while the other counterterms shall be left out as a counterterm in the interaction (see section 4).

After applying a horizontal/vertical expansion at scale $(j-1)$ to $Z^{(j-1)\to \rho}$, we are finally back to the induction hypothesis, at scale $(j-1)$ this time.

Let us now show the following single scale Mayer bounds.

**Proposition 2.12 (Mayer bounds)**  
1. (vacuum polymers) Let $P^{j\to}_{0}(\Delta^{j})$ be the set of vacuum polymers down to scale $j$ containing some fixed interval $\Delta^{j} \in \mathbb{D}^{j}$. Assume

$$\sum_{P \in P^{j\to}_{0}(\Delta^{j})} e^{\left|P\right|-1} \left|F^{j}_{HV}(P)\right| \leq K' \lambda, \quad (2.37)$$

where $F^{j}_{HV}(P) = \int d\mu(\psi^{j\to}) F^{j}_{HV}(P; \psi^{j\to})$. Then

$$\left|f^{j\to \rho}(\lambda)\right| \leq K' \lambda(1 + O(\lambda)) \quad (2.38)$$

Note that $K$ – contrary to the original bare kernel – is only almost-diagonal, namely, $\int \psi_{i}^{j}(x) \psi_{i'}^{j}(x) dx = 0$ by momentum conservation if $|\text{supp}(\chi^{j}) \cap \text{supp}(\chi^{j'})| = 0$, i.e. if $|j-j'| \geq 2$.

*Since $P$ is a vacuum polymer, no high-momentum fields of scale $j$ may be produced at a later stage, hence one may integrate out the field components $\psi^{j\to}$.*
with the same constant $K'$.

2. (counterterm) Let $P_{i,i'}^J(\Delta^j)$ be the set of polymers down to scale $j$ with exactly two external legs, $\psi_j(i)\rightarrow(i')$ and $\psi_j'(i)\rightarrow(i')$, displaced into the same point in a fixed interval $\Delta^j$. Rewrite $F_{HV,amputated}^J(\mathbb{P}) := \int d\mu(\psi_j\rightarrow) F_{HV,amputated}^J(\mathbb{P}; \psi_j\rightarrow)$, $\mathbb{P} \in P_{i,i'}^J(\Delta^j)$ as

$$M^j(1+\beta_i+\beta_i') F_{HV,amputated}^{rescaled}(\lambda, \mathbb{P}; \psi_j\rightarrow(j-1)), \quad (2.39)$$

with:

- $\sum_{\mathbb{P} \in P_{i,i'}^J(\Delta^j)} \#\{\text{vertices of } \mathbb{P}\} = 2 F_{HV,amputated}^{rescaled}(\lambda, \mathbb{P}) = -K\lambda^2(1+O(\lambda));$

- $\sum_{\mathbb{P} \in P_{i,i'}^J(\Delta^j), \#\{\text{vertices of } \mathbb{P}\} \geq 3} e^{\mathbb{P}} F_{HV,amputated}^{rescaled}(\lambda, \mathbb{P})$

$$= -K'\lambda^3(1 + O(\lambda) + O(M^{-j(1+\beta_i+\beta_i')}b^j_{i,i'})). \quad (2.40)$$

Then

$$b^j_{i,i'} = K\lambda^2 M^j(1+\beta_i+\beta_i')(1 + O(\lambda)). \quad (2.41)$$

We shall see in section 5.1 how to obtain estimates which are valid for a succession of Mayer expansions at each scale.

**Proof.**

1. Fix some interval $\Delta^j \in \mathbb{D}^j$ and compute $f^{j\rightarrow \rho}(\lambda)$ using Proposition 2.11 as

$$\sum_{N \geq 1} \sum_{P_i \in \mathbb{P}_0^J(\Delta^j), P_2, \ldots, P_N \in \mathbb{P}_0^J} \text{Mayer}(Z^{j\rightarrow \rho}_V(\lambda); \mathcal{T}), \quad (2.42)$$

where

$$|\text{Mayer}(Z^{j\rightarrow \rho}_V(\lambda); \mathcal{T})| \leq \frac{1}{N!} \prod_{n=1}^{N} |F_{HV}^j(\mathbb{P}_n)|. \quad (2.43)$$

The $\frac{1}{N!}$ factor is matched by Cayley’s theorem, which states that the number of trees over $\mathbb{P}_1, \ldots, \mathbb{P}_N$ with fixed coordination number $(n(\mathbb{P}_i))_{i=1}^{N}$ equals $\prod_{i=1}^{N} (n(\mathbb{P}_i)-1)!$. Recall that $\mathbb{P}_\ell$ and $\mathbb{P}_\ell'$ are necessarily overlapping if $\ell \in L(\mathcal{T})$. Start from the leaves and go down
the branches of the tree inductively. Let \( P_1, \ldots, P_{n(P')} \) be the leaves attached onto one and the same vertex \( P' \) of \( T \). Choose \( n(P') - 1 \) (possibly non-distinct) vertices of \( P' \cap D^j \) (there are \( |P' \cap D^j|^{n(P')-1} \) possibilities), fix their spatial location, \( \Delta_1^{j}, \ldots, \Delta_{n(P')-1}^{j} \), and assume that \( \Delta_i^{j} \in P_i \cap D^j \). For each choice of polymer \( P' \), this gives a supplementary factor \( \leq (K' \lambda |P' \cap \Delta^j|)^{n(P')-1} \), to be multiplied by \( 1/(n(P')-1)! \) coming from Cayley’s theorem. Summing over \( n(P') = 2, 3, \ldots \), yields \( e^{K' \lambda |P' \cap \Delta^j| - 1} \), which is \( \leq 2^{\sum_{h \geq 1}} (K' \lambda)^h = K' \lambda (1 + O(\lambda)) \), where \( h \) is the height of the tree.

2. The definition of \( b_{i,i'}^j \) and arguments analogous to those used in (1) yield (letting \( \tilde{b}_{i,i'}^j := \lambda^{-2} M^{-j(1+\beta_i+\beta_{i'})} b_{i,i'}^j \))

\[
\tilde{b}_{i,i'}^j = -K (1 + O(\lambda)) + O(\lambda^2 \tilde{b}_{i,i'}^j), \tag{2.44}
\]

hence the result by the implicit function theorem.

3  Power-counting and renormalization

This section is divided into two parts. The first one gives a quick overview of how divergences in quantum field theory are discarded by extracting the local part of diverging diagrams; these ideas come from classical power-counting arguments which are recalled here. The whole idea of renormalization is then to transfer the sum of these local parts to the interaction as a counterterm (see §2.4). The second one is an informal discussion of the domination problem of low-momentum field averages, in particular in the case of the \((\phi, \partial \phi, \sigma)\)-model. The full treatment of this problem is postponed to §5.2.

3.1  Power-counting and diverging graphs

Definition 3.1 (Feynman diagrams) Let \( \psi = (\psi_1(x), \ldots, \psi_d(x)) \) be a multiscale Gaussian field with covariance kernel \( C_\psi \), and \( \mathcal{L}_{\text{int}} = \sum_{q=1}^{d} K_q \lambda^{q} \psi I_q \) be an interaction. Then:

1. a Feynman diagram for this theory is a connected graph \( \Gamma \) (whose lines, resp. vertices are generically denoted by \( \ell \), resp. \( z \)) with
(i) external vertices $y = (y_1, \ldots, y_n)$ of type $1, \ldots, p$;
(ii) internal vertices $x$ of type $1, \ldots, p$;
(iii) internal lines $\ell$ connecting $z_\ell$ to $z'_\ell$ with double index $(i_\ell, i'_\ell)$.
Since the lines do not have a preferred orientation, and in order to avoid confusion, one writes $\ell \simeq (i_\ell, z_\ell; i'_\ell, z'_\ell)$ or indifferently $(i'_\ell, z'_\ell; i_\ell, z_\ell)$. For every vertex $z$ of type $q$, one may order the lines leaving or ending in $z$ as $\ell_z, 1 \simeq (i_1, z; i'_1, z'_1), \ldots, \ell_{z,n(z)} \simeq (i_{n(z)}, z; i'_{n(z)}, z'_n(z))$ so that $n(z) = q$ and $(i_1, \ldots, i_q) = I_q$ if $z$ is an internal vertex, and $n(z) < q$, $(i_1, \ldots, i_q) \subsetneq I_q$ if $z$ is external.
(iv) external lines $\ell \simeq (i_\ell, y_\ell; i'_\ell, y'_\ell)$, where $y'_\ell \in \mathbb{R}^D$ is an external point not belonging to $\Gamma$; $|I_q| - n(y)$ of them per external vertex $y$ of type $q$. The total number of external lines is $N_{\text{ext}}(\Gamma) := \sum_q \sum_y n(y) |I_q| - n(y)$.

The evaluation $A(\Gamma)$ of an amputated Feynman diagram is given by

$$A(\Gamma)(y) = \int \prod_x dx \left( \prod_{\ell \in L_{\text{int}}(\Gamma)} C_\psi(i_\ell, z_\ell; i'_\ell, z'_\ell) \right), \quad (3.1)$$

where $x$ ranges over all internal vertices, and $L_{\text{int}}(\Gamma)$ is the set of all internal lines.

The evaluation $A(\Gamma)$ of a full Feynman diagram (i.e. including its external legs) is given by

$$A(\Gamma)(y) = \int \prod_y dy \left( \prod_{\ell \in L_{\text{ext}}(\Gamma)} C_\psi(i_\ell, y_\ell; i'_\ell, y'_\ell) \right) A(\Gamma)(y), \quad (3.2)$$

where $L_{\text{ext}}(\Gamma)$ is the set of all external lines.

2. A quasi-local multi-scale Feynman diagram $(\Gamma; (j(\ell)))$ is obtained from $\Gamma$ by choosing a scale $j(\ell)$ for each (internal or external) line and splitting each vertex into these different scales, with the following constraint,

$$\text{height}(\Gamma) := \min\{j(\ell); \ell \in L_{\text{int}}(\Gamma)\} - \max\{j(\ell); \ell \in L_{\text{int}}(\Gamma)\} \geq 0. \quad (3.3)$$

Note that $\text{height}(\Gamma)$ measures the height of internal lines of $\Gamma$ with respect to the external lines it is attached to.
The evaluation $A(\Gamma)$ of a multi-scale Feynman diagram is given by

$$A(\Gamma; (j(\ell))_{\ell \in L_{\text{int}}(\Gamma)})(y) = \int \prod_x dx \left( \prod_{\ell \in L_{\text{int}}(\Gamma)} C_{\psi}^{j(\ell)}(i_\ell, z_\ell; i'_\ell, z'_\ell) \right),$$

(3.4)

while

$$\tilde{A}(\Gamma; (j(\ell))_{\ell \in L(\Gamma)})(y') = \int \prod_y dy \left( \prod_{\ell \in L_{\text{ext}}(\Gamma)} C_{\psi}^{j(\ell)}(i_\ell, y_\ell; i'_\ell, y'_\ell) \right) A(\Gamma; (j(\ell)))(y).$$

(3.5)

Cluster expansions produce single-scale (horizontal) Feynman trees, multiplied by some $G$-polynomial and by the exponential of the interaction, up to the modification of the measure by the weakening coefficients $s$. In the final bounds, see section 5, one bounds the exponential by a constant and applies the Cauchy-Schwarz inequality to the rest, yielding a quasi-local multi-scale Feynman diagram. The scale $j(z)$ of a vertex $z$ is then the scale at which the vertex has been produced.

Let us from now on assume that the interaction is just renormalizable. This means that, for every $q = 1, \ldots, p$, $\sum_{i \in I_q} \beta_i = -D$, or in other words, $\int \psi_{I_q}(x)dx$ is homogeneous of degree 0. In that case, the following very simple power-counting rules hold.

**Proposition 3.2 (power-counting)**

1. The power-counting of an amputated Feynman diagram $\Gamma$ is the product $\Lambda^{\omega(\Gamma)} := \Lambda^D \prod_z \Lambda^{-D} \prod_{\ell \in L_{\text{int}}(\Gamma)} \Lambda^{-(\beta_{i\ell} + \beta_{j\ell})}$, see Definition 3.1 for notations, where $\Lambda$ is some large, indefinite constant representing an ultra-violet cut-off.

Then the degree of divergence $\omega(\Gamma)$ of $\Gamma$ is equal to $D + \sum_{\ell \in L_{\text{ext}}(\Gamma)} \beta_{i\ell}$.

2. The power-counting of a full quasi-local multi-scale Feynman diagram $\Gamma$ is the product

$$M^{\omega_{\text{full}}(\Gamma)} := M^{D \cdot \text{height}(\Gamma)} \prod_z M^{-Dj(z)} \prod_{\ell \in L(\Gamma)} M^{-j(\ell)(\beta_{i\ell} + \beta_{j\ell})}. \quad (3.6)$$
The multi-scale degree of divergence $\omega_{m.s.}(\Gamma)$ is equal to

$$\omega_{m.s.}(\Gamma) = D \cdot \text{height}(\Gamma) + \sum_{z} \sum_{\ell \in L_z} \beta_i (j(z) - j(\ell)), \quad (3.7)$$

where $L_z$ is the set of internal or external lines leaving or ending in $z$.

Let us give brief explanations. The power-counting in 1. may be obtained from Definition 1.4 by assuming all internal lines of $\Gamma$ to be of scale $\frac{\ln \Lambda}{\ln M}$ and summing over all vertices except one, due to overall (approximate or exact) translation invariance, see [55]. The multi-scale power counting is a refined version of the previous one, which takes into account the scale of the external legs; the definition of the height of $\Gamma$ is somewhat ad-hoc and measures the horizontal freedom of movement of $\Gamma$ with respect to its external legs if these are supposed to be fixed. More precise computations in the spirit of cluster expansion appear in the final bounds in section 5.

The principle of the power-counting rule explained in section §6.1.2 is to rescale all fields produced at scale $j$ of a multiscale Feynman diagram as if they were of scale $j$. The production scale of a given vertex $z$ being $j(z)$, this leads to a rescaled degree of divergence,

$$\omega_{m.s.}^{\text{rescaled}}(\Gamma) := \omega_{m.s.}(\Gamma) - \sum_{z} \sum_{\ell \in L_z, \text{int}} \beta_i (j(z) - j(\ell)), \quad (3.8)$$

where the sum on $\ell$ ranges over all internal lines leaving or ending in $x$. Hence

$$\omega_{m.s.}^{\text{rescaled}}(\Gamma) = D \cdot \text{height}(\Gamma) + \sum_{y} \sum_{\ell \in L_y, \text{ext}} \beta_i (j(y) - j(\ell)), \quad (3.9)$$

where the sum on $\ell$ ranges over all external lines leaving or ending in $y$.

If $\text{height}(\Gamma)$ and all $j(y) - j(\ell)$ in (3.9) are equal to the same positive constant, then $\omega_{m.s.}^{\text{rescaled}}(\Gamma)$ is proportional to the naive degree of divergence defined in Proposition 3.2 (1).

**Definition 3.3 (renormalization)** If the degree of divergence $\omega(\Gamma)$ of a Feynman diagram, resp. multiscale Feynman diagram, is $\geq 0$, then it needs to be renormalized.

The local part of an amputated Feynman diagram or quasi-local multiscale Feynman diagram is obtained by integrating over all external vertices.
except one (in order to take into account the global invariance by translation),

\[ A(\Gamma; (j(\ell)))(\mathbf{y} = (y_1, \ldots, y_n)) \sim \int dy_2 \ldots dy_n A(\Gamma; (j(\ell)))(\mathbf{y}) =: \text{Local}(A(\Gamma; (j(\ell)))). \]  

(3.10)

By invariance by translation again, it does not depend on \( y_1 \).

Integrating over external points, one obtains the local part of a full Feynman diagram,

\[ \bar{A}(\Gamma; (j(\ell)))(\mathbf{y}') \sim \text{Local}(\bar{A}(\Gamma; (j(\ell)))(\mathbf{y}')) := \text{Local}(A(\Gamma; (j(\ell)))) \int \prod_{\mathbf{y}} dy \prod_{\ell \in L_{\text{ext}}(\Gamma)} C_{\psi, j}^{(\ell)}(i_\ell, y_\ell; i'_\ell, y'_\ell) \bigg|_{A(\Gamma; (j(\ell)))(\mathbf{y})}, \]

(3.11)

which may be rewritten as

\[ \text{Local}(\bar{A}(\Gamma; (j(\ell)))(\mathbf{y}')) = \int \prod_{\mathbf{y}} dy \prod_{\ell \in L_{\text{ext}}(\Gamma)} C_{\psi, j}^{(\ell)}(i_\ell, y_1; i'_\ell, y'_\ell) A(\Gamma; (j(\ell)))(\mathbf{y}). \]  

(3.12)

In other terms, all external legs of \( \Gamma \) have been displaced at the same arbitrary external vertex, here \( y_1 \).

The renormalized amplitude \( R A \) or \( \bar{R} \bar{A} \) of a Feynman diagram or quasi-local multi-scale Feynman diagram is the difference between its amplitude and its local part.

Remark. Using a Fourier transform, the local part is equivalent to the classical evaluation at zero external momenta.

Let us from now on consider only quasi-local multiscale Feynman diagrams. The idea is that a quasi-local diagram should look almost like a point viewed from the scale of its external legs, hence almost equal to its local part, which means that the difference should be of lower order. Coming to the facts, a simple Taylor expansion of order 1 yields

\[ R A(\Gamma; (j(\ell)))(\mathbf{y}') = \int \prod_{\mathbf{y}} dy \sum_{\ell \in L_{\text{ext}}(\Gamma)} \left( \prod_{\ell' \in L_{\text{ext}}(\Gamma), \ell' \neq \ell} C_{\psi, j}^{(\ell')}(i_\ell, y_1; i'_{\ell'}, y'_{\ell'}) \right) \int_0^1 dt (y_\ell - y_1) \mathbb{E} \left[ \partial_{\psi_{i_\ell}}^{(\ell)}(y_1 + t(y_\ell - y_1)) \psi_{i'_{\ell'}}^{(\ell)}(y'_{\ell'}) \right] \cdot A(\Gamma; (j(\ell)))(\mathbf{y}). \]  

(3.13)
In principle, this formula shows that the renormalized diagram amplitude \( R(\Gamma; (j(\ell))) \) comes with a supplementary spring factor \( M^{-\text{height}(\Gamma)} \), leading to an equivalent degree of divergence \( \omega^*(\Gamma) := \omega(\Gamma) - 1 \). Namely, \( y_\ell - y_1 \) should be at most of order \( M^{-\min\{j(\ell); \ell \in L_{\text{int}}(\Gamma)\}} \), while (by Definition 1.4)

\[
\mathbb{E} \left[ \partial \psi^{j(\ell)}_x(y_1 + t(y_\ell - y_1)) \psi^{j(\ell)}_y(y'_\ell) \right] \text{ is of order } M^{j(\ell)} M^{-\beta_{i\ell} + \beta_{i'\ell}} j(\ell).
\]

If \( \omega(\Gamma) < 1 - D \), then \( \omega^*(\Gamma) < -D \) and the diagram has become convergent.

Let us give a formal proof of this general statement. This requires a little care because the covariance is not exponentially decreasing at large distances in our setting, so using systematically the Taylor expansion (3.13) does not work. Consider a given quasi-local multi-scale diagram \((\Gamma; (j(\ell)))\).

We consider only divergent two-point subdiagrams for the sake of notations. Start from the divergent diagram of lowest scale, \( j_{\text{min}} \), say, with external legs \( y, y' \), and use the Taylor expansion (3.13). Choose a tree of propagators of scale \( \geq j_{\text{min}} \) connecting \( y \) to \( y' \) through intermediary points \( x_2, \ldots, x_n \), and set \( x_1 = y, x_{n+1} = y' \), so that \( |y - y'| \leq \sum_{m=1}^n |x_m - x_{m+1}| \). The divergent diagram with external legs \( y, y' \) has been renormalized as explained above. Letting \( j_m \) be the scale of the link \((x_m, x_{m+1})\), one has lost a rescaled factor \( \sum_{m=1}^n d_{\text{min}}(x_m, x_{m+1}) = \sum_{m=1}^n M^{-(j_m - j_{\text{min}})} d_{\text{min}}(x_m, x_{m+1}) \). Each term in this sum has obtained a spring factor \( M^{-\beta_{i\ell} + \beta_{i'\ell}} j(\ell) \), which shows that the corresponding diagram with external legs \( x_m, x_{m+1} \) is already de facto superficially renormalized, although subdiagrams of higher scale may still need renormalization. In the process, it is clear that every propagator of the diagram may appear only in one tree of propagators at most. In the bounds of section 5, this yields an overall factor per polymer \( P \) which is bounded by \( \prod_{\ell \in L(\Gamma)} (1 + d^{(\ell)}(x_{\ell}, x'_{\ell})) \), which is easily controlled by the polynomial decrease of the covariance at large distances.

Let us summarize our brief discussion.

**Definition 3.4 (diverging graphs)** A Feynman graph or multi-scale Feynman graph \( \Gamma \) is divergent if and only if

\[
\omega(\Gamma) := D + \sum_{\ell \in L_{\text{ext}}(\Gamma)} \beta_{i\ell} \geq 0.
\]  

We shall call \( N_{\text{ext}, \max} \) the minimum value of \( N_{\text{ext}} \) such that every diagram with \( \geq N_{\text{ext}, \max} \) external legs is convergent.

\(^{5}\)Otherwise one should renormalize to a higher order by removing the beginning of the Taylor expansion of the diagram in the external momenta. We shall not describe this straightforward extension of the procedure since we shall not use it in our context.
Renormalizing the evaluation of a quasi-local multiscale Feynman graph \( \Gamma \) yields a quantity \( \mathcal{R}(\Gamma) \) for which the displaced external legs have obtained a supplementary spring factor, which is equivalent to replacing one of the scaling dimensions \( \beta_{i_\ell} \), \( \ell \in L_{\text{ext}}(\Gamma) \) with \( \beta^*_{i_\ell} := \beta_{i_\ell} - 1 \), or globally \( \omega(\Gamma) \) by \( \omega^*(\Gamma) := \omega(\Gamma) - 1 \).

Let \( \omega_{\text{max}}^* < 0 \) the maximal value of the set \( \{ \omega^*(\Gamma) \} \), where \( \Gamma \) ranges over the set of all Feynman graphs.

**Example** \(((\phi, \partial \phi, \sigma))-\text{model})

In our case \( \beta_\phi = \alpha \), \( \beta_{\partial \phi} = \alpha - 1 \), \( \beta_\sigma = -2\alpha \). Note however that the interaction \( L_4 \) (see section 4) decomposes into a sum over scales \( \sum_{j \leq k, j' \neq k} \partial \phi^j \phi^k \sigma^{k'} \), with \( k' = k \) or \( k \pm 1 \) (by momentum conservation, the two highest lines have essentially the same scale), hence (assuming \( k' = k \) which does not change anything) \( \phi \) may not be an external leg of a multi-scale Feynman diagram (in particular, the vertex \( (\partial \phi) \phi \sigma \) is not renormalized). Let \( N_{\partial \phi} \), resp. \( N_\sigma \) be the number of external \( \partial \phi \)- or \( \sigma \)-legs of such a diagram \( \Gamma \). The power-counting rule yields \( \sum_{\ell \in L_{\text{ext}}(\Gamma)} \beta_{i_\ell} = (\alpha - 1) N_{\partial \phi} - 2\alpha N_\sigma < -1 \) as soon as \( N_{\partial \phi} \geq 2 \) or \( N_\sigma \geq 4 \), so \( N_{\text{ext,max}} = 4 \). However, by symmetry arguments, local parts of diagrams with \( N_{\partial \phi} \) or \( N_\sigma \) odd vanish, so there remains only the \( \sigma \)-propagator \( \langle \sigma_\pm(x) \sigma_\pm(y) \rangle \) to renormalize.

Let us write down precisely these power-counting rules in the framework of multi-scale cluster expansion, where Feynman diagrams are replaced by polymers. Recall from §2.3 that a low momentum field \( (T^{\rightarrow(j-1)} \psi_i)^h(x) \) – or \( (T^{\rightarrow(j-1)} \delta^j \psi_i)^h(x) \) – has three scales attached to it. It is the difference between the two highest ones – the production scale \( k \) and the dropping scale \( j \), with \( k > j \) – that counts here. Namely, considering \( h < j' < j \), there are two cases:

(i) either \( \tau_{\Delta_x^{j'}} \) (the number of \( t \)-derivatives produced inside \( \Delta_x^{j'} \)) is \( \geq N_{\text{ext,max}} \). Then rescaling the fields to which the \( t \)-derivatives are applied is enough to ensure the convergence of the polymer \( P_{j' \rightarrow} \) containing \( \Delta_x^{j'} \);

(ii) or \( \tau_{\Delta_x^{j'}} < N_{\text{ext,max}} \). Then (by definition) \( t_{\Delta_x^{j'}} \) is set to 0, which kills \( (T^{\rightarrow(j-1)} \psi_i)^h(x) \) or \( (T^{\rightarrow(j-1)} \delta^j \psi_i)^h(x) \). Hence one must count only on the derived fields for the power-counting. As explained above, this may give a non-negative degree of divergence, in which case one must renormalize by removing the local part of the polymer.
This means that the rescaling spring factor $M^{\beta_i(k-h)}$ of these low-momentum fields must be split into $M^{\beta_i(k-j)}$ – ensuring the horizontal fixing of the polymer, and counting in the right-hand side of eq. (3.9) – and $M^{\beta_i(j-h)}$ – which helps control the accumulation of low-momentum fields as explained briefly in §2.3.

3.2 Domination problem and boundary term in the interaction

Assume $\psi_i$ is a multi-scale Gaussian field with scaling dimension $\beta_i \in (-D/2, 0)$. Then $2\beta_i > -D$ so, by Definition 3.4, the associated two-point functions must be renormalized. If $\psi_i$ occurs in some bare interaction term $(\lambda \rho)^{\nu_i} \psi_{I_q}$, i.e. $i \in I_q$, then renormalization produces a scale-dependent counterterm of the form

$$\delta L(\psi_i; x) = \sum_j (\lambda^j)^{2\nu_j} C_j M^{(D+2\beta_j)j} \left( (T\psi_i)^{-j} \right)^2 (x), \quad (3.15)$$

where $\lambda^j$ is the renormalized coupling constant, and $C_j$ is a scale-dependent constant. In this paper we generally assume that $\lambda^j = \lambda$ is not renormalized. (In the case of our $(\phi, \partial \phi, \sigma)$-model, $\delta L$ is of the form $\sum_{j \geq 0} b_j ((T\sigma)^{-j})^2$, with $b_j \approx \lambda^2 M^{1-4\alpha} j$ and $C_j \approx 1$. See the counterterm $\delta L_4$ in section 4 below for details.) Separating the low-momentum field $\psi_i^{(j-1)}(\Delta^j) + \delta^j \psi_i^{(j-1)}(x)$ as in Definition 1.5 produces a secondary field $\delta^j \psi_i^{(j-1)}$ whose contributions to the partition function are easily bounded thanks to the "spring factor" (see subsection 5.1). Unfortunately, the averaged fields $\psi_i^{(j-1)}(\Delta^j)$ do not come with such a spring factor and (unless $\beta_i < -D/2$, see subsection 5.1 again) must be dominated apart, using the positivity of the interaction. We assume that $C_j > 0$. Then Lemma 5.10 (1) – simply based on the trivial inequality $ae^{-a} \leq 1$ – shows that

$$| \lambda^{\nu_i} (T\psi_i)^{(j-1)}(\Delta^j) | n(\Delta) e^{-\int \delta L(\psi_i; x) dx} \text{ is bounded by } O \left( Kn(\Delta)^{1/2} C_j^{-1/2} M^{-j\beta_i} \right)^{n(\Delta)}/2,$$

which agrees – up to unessential local factorials $n(\Delta)^{n(\Delta)/2}$, see §5.1 – with the correct power-counting, but the "petit facteur" $\lambda^{\nu_i}$ has been entirely used, in contradiction with the general guideline of cluster expansions. So – unless $C_j^{-1/2}$ is small – one must use a different strategy.

Several strategies have been used, depending on the model. Here things are particularly simple because the mass counterterm of highest scale, $b^\rho \approx \lambda^2 M^{\rho(1-4\alpha)}$, couples with all scales of the field $\sigma$ and is much better than the
term of order $\lambda^2 M^j(1-4\alpha)$ appearing in the right-hand side of (3.15). This is
the reason why this counterterm has been set apart from the counterterm
and put into the covariance of $\sigma$ right from the beginning. The supplen-
tary spring factor $M^{-\frac{1}{2}(1-4\alpha)(\rho-j)}$ per field plays the rôle of a “petit
facteur”, except in a small scale interval $\rho - q, \ldots, \rho$, where $q \approx \ln(1/\lambda)$
is $\rho$-independent. Add now e.g. to the interaction a term of the form

$$
M^{-(4n\alpha-1)\rho} \lambda^\kappa \sum_{\rho' \leq \rho} \| (T \sigma)^{\rho'}(x) \|^{2n}
$$

with $2n \geq 4$, which is homogeneous
of degree 0, and totally negligible away from the highest scales because
of the evanescent coupling coefficient $M^{-4(\alpha-1)\rho} \leq M^{-(8\alpha-1)\rho} < 1$. If
$\kappa < 4n$, then each $\sigma$-field in the interaction is coupled to $\lambda^{\kappa/2n} \gg \lambda$, which
makes it possible to dominate the low-momentum fields for the highest scales
$\rho - q + 1, \ldots, \rho$. In this sense this term is a (Fourier) boundary term. The
term $\delta L_{12}$ in section 4 below plays this rôle. The choice of $4n = 12$ is
arbitrary.

4 Definition of the model

As a general rule, we shall denote by $\mu_K$ the Gaussian measure with covari-
ance $K^{-1}$ if $K$ is a positive-definite kernel.

**Definition 4.1 (Gaussian covariance kernels)**  
(i) Let $d\mu^{\rho\phi}(\phi) = d\mu_{[\xi]^{1+2\alpha}}(\phi^{\rho})$
be the Gaussian measure associated to the field $\phi^{\rho}$ defined in section
2.2.

(ii) Let $d\mu^{\rho}(\sigma_{\pm}) = d\mu_{[\xi]^{1-4\alpha}Id+b\rho}(\sigma_{\pm}^{\rho})$
be the Gaussian measure associ-
ated to the fields $\sigma_{\pm}^{\rho}$ defined in section 2.2.

The two-by-two matrix coefficient $b^{\rho}$ is called the
renormalized mass co-
efficient of the $\sigma$-field at scale $\rho$. It is equal to the local part at scale $\rho$ of the
two-point function of the $\sigma$-field (see precise definition below). Note that
d$\mu_{[\xi]^{1-4\alpha}Id+b\rho}(\sigma^{\rho\rho}) = \frac{1}{Z'} e^{-\frac{1}{2} b^{\rho} \int (\sigma^{\rho\rho})^2(x) dx} d\mu_{[\xi]^{1-4\alpha}}$, where $Z'$ is a normaliza-
tion constant.

In the sequel, expressions such as $b\sigma^2$ are to be understood as a scalar
product $(b\sigma, \sigma) = b_{++}(\sigma^+)^2 + 2b_{+-}\sigma^+\sigma^- + b_{-+}(\sigma^-)^2$, whereas $||\sigma|| = 
\sqrt{\sigma^2_+ + \sigma^2_-}$ simply.

**Definition 4.2 (bare interaction)** Let $\int L^{\rho\phi}(\phi, \sigma)(x) dx = -\frac{b^{\rho}}{2} \int (\sigma^{\rho\rho}(x))^2 dx +$
\[ \int \mathcal{L}^{\rightarrow \rho}_{4}(\phi, \sigma)(x)dx + \int \mathcal{L}_{12}(\sigma)(x)dx, \]
where
\[ \int \mathcal{L}^{\rightarrow \rho}_{4}(\phi, \sigma)(x)dx = i \lambda \int (\partial \mathcal{A}^{+})^{-\rho}(x) \sigma^{-\rho}(x) - (\partial \mathcal{A}^{-})^{-\rho}(x) \sigma^{-\rho}(x) dx; \]  
(4.1)
\[ \int \mathcal{L}^{\rightarrow \rho}_{12}(\sigma)(x)dx = M^{-((12 \alpha - 1)\rho \lambda^{3})} \int ||\sigma^{-\rho}(x)||d^{6}dx \]  
(4.2)

and \( \mathbb{P}_{\lambda, V, \rho}(\phi, \sigma) = \frac{1}{Z_{\lambda, V, \rho}} e^{-\int_{V} \mathcal{L}^{\rightarrow \rho}_{int}(\phi, \sigma)(x)dx} d\mu(\phi) d\mu(\sigma) \) be the associated probability measure.

The lonely term \(-\frac{\rho^{6}}{2}(\sigma^{-\rho}(x))^{2}\) compensates the scale \(\rho\) renormalization of the Gaussian covariance kernel of the \(\sigma\)-field, so one may also write (in agreement with the introduction)
\[ \mathbb{P}_{\lambda, V, \rho}(\phi, \sigma) = \frac{1}{Z_{\lambda, V, \rho}} e^{-\int_{V} \mathcal{L}^{\rightarrow \rho}_{int}(\phi, \sigma)(x)dx} d\mu(\phi) d\mu_{bare}(\sigma), \]  
(4.3)

where \(d\mu_{bare}(\sigma)\) is the measure associated to the bare covariance kernel \(\langle |\mathcal{F}_{\sigma}(\xi)|^{2} \rangle = \frac{1}{|\xi|^{8}}\).

Integrating out the field \(\sigma\) yields a measure
\[ \mathbb{P}_{\lambda, V, \rho}(\phi) = \frac{1}{Z_{\lambda, V, \rho}} e^{-F(\phi)} d\mu(\phi), \]  
(4.4)
where
\[ F(\phi) = -\log \int e^{-\int_{V} \mathcal{L}^{\rightarrow \rho}_{int}(\phi, \sigma)(x)dx} d\mu(\sigma) \int e^{-\int_{V} \mathcal{L}^{\rightarrow \rho}_{12}(\sigma)(x)dx} d\mu(\sigma) \]  
(4.5)
is the generating function of connected correlation functions (see [30], eq. (5.3.2)), a real expression by parity.

Note that we have not yet defined the local functional \(\mathcal{L}^{\rightarrow \rho}_{int}(\phi, \sigma)\), only its integral on the volume. We may freely choose to throw out terms with vanishing integral. Quite generally, momentum conservation imply constraints on the scales of fields \(\psi_{i}, i \in I_{q}\) when integrating \(\mathcal{L}_{q} = \psi_{I_{q}}\) over \(\mathbb{R}\). However, after putting in the momentum-decoupling \(t\)-coefficients, these constraints disappear because the interaction is no more translation-invariant. If needed, however, one may directly discard the scales which are incompatible with momentum conservation before the momentum-decoupling expansion. In particular, for a vertex of order three, the two highest scales are equal up to \(\pm 1\). We shall do so for the vertex \(\mathcal{L}_{4}\), in order to avoid artificial vertices \((\partial \phi)(\phi)\sigma\) with two low-momentum fields \((\partial \phi, \phi)\sigma\) – in other words, a renormalization of the coupling constant \(\lambda\).
\[ \sum_{j_1,j_2,k} \text{adm} \] means that the sum is restricted to this admissible subset; explicitly, \[ \sum_{(j_1,j_2,k) \in I_{\text{adm}}} = \sum_{(j_1,j_2,k) \in I \setminus I_{\text{adm}}} \] with \[ I_{\text{adm}} = \{(j_1,j_2,k); j_1 \simeq j_2 \geq k \text{ or } j_2 \simeq k \geq j_1 \text{ or } k \simeq j_1 \geq j_2\} \], where \( j \simeq k \) means \( j = k \) or \( k \pm 1 \).

**Definition 4.3 (dressed interaction)** Let

\[ L_{\text{int}}^{-\rho}(\phi, \sigma; t)(x) := L_4^{-\rho}(\cdot ; t)(x) - \frac{b^{-\rho}}{2} (t_x^\rho)^2 \| (T_\sigma)^{-\rho}(x) \|^2 + \delta L_4^{-\rho}(\cdot ; t)(x) + L_{12}^{-\rho}(\cdot ; t)(x), \] (4.6)

where:

\[ L_4^{-\rho}(\cdot ; t)(x) = L_{4+}^{-\rho}(\cdot ; t)(x) - L_{4-}^{-\rho}(\cdot ; t)(x), \] (4.7)

\[ L_{4+}^{-\rho} := i\lambda D^+ \left( \sum_{j_1,j_2,k \leq \rho} \partial^{(T^{-x} \phi_1)^j(x)(T^{-x} \phi_2)^k(x)}(T^{-x} \rho_+)^k(x) ight) 
+ \sum_{j_1,j_2,k \leq \rho} \sum_{j_1,j_2,k \leq \rho-1} \partial^{(T^{-x} \phi_1)^j(x)(T^{-x} \phi_2)^k(x)}(T^{-x} \rho_+)^k(x) \right) \] (4.8)

and similarly for \( L_{4-}^{-\rho} \) (the Fourier projections \( D_{\pm} \) have been defined in section 1);

\[ \delta L_4^{-\rho}(\cdot ; t)(x) := \frac{1}{2} \sum_{\rho' \leq \rho} b^{-\rho' - 1} \sum_{\rho'' \leq \rho'} \left( 1 - (t_x^\rho')^2 \right) \left( (T_\sigma)^{-\rho''}(x) \right)^2 \] (4.9)

\[ L_{12}^{-\rho}(\cdot ; t)(x) := M^{-(12\alpha - 1)\rho} \lambda^3 \left\{ \| (T_\sigma)^{-\rho}(x) \|^6 + \sum_{\rho' \leq \rho} (1 - (t_x^\rho')^6) \int_{\Delta} \| (T_\sigma)^{-\rho'}(x) \|^6 dx \right\} \] (4.10)

By construction \( L_{\text{int}}^{-\rho}(\phi, \sigma) = L_{\text{int}}^{-\rho}(\phi, \sigma; t = 1) \).

Let

\[ Z_{\nu}^{-\rho}(\lambda) := \int e^{-\int_{\nu} L_{\text{int}}^{-\rho}(\phi, \sigma; t = 1)(x) dx} d\mu^{-\rho}(\phi) d\mu^{-\rho}(\sigma), \] (4.11)
and, more generally,

\[
Z_{V}^{\rho'\rightarrow\rho}(\lambda; (\phi^{h})_{h<\rho'}, (\sigma^{h})_{h<\rho'}) = \int e^{-\int_{V} L_{ini}^{\rho'}(\phi,\sigma; t=1)(x)d\mu^{\rho'\rightarrow\rho}(\phi)d\mu^{\rho'\rightarrow\rho}(\sigma)}
\]

which is a function of the low-momentum components of the fields, considered as external fields.

**Definition 4.4 (renormalized mass coefficient \( b^{\rho'} \))** Fix \( b^{\rho'} \) by requiring that

\[
0 = \int dy \partial_{\sigma^{\rightarrow(\rho'-1)}}^{2}(x)\partial_{\sigma^{\rightarrow(\rho'-1)}}(y) Z_{V}^{\rho'\rightarrow\rho}(\lambda; (\phi^{h})_{h<\rho'}, (\sigma^{h})_{h<\rho'})\big|_{\sigma^{\rightarrow(\rho'-1)}=\phi^{\rightarrow(\rho'-1)}=0}.
\]

(4.13)

Note that \( \delta L_{4}^{\rightarrow\rho}(\cdot : t)(x) \) may be seen as a dressed interaction as in Definition 2.8 if one sets \( b^{\rho'} := 0 \) (the counterterm of scale \( \rho \) has been treated separately). The interaction before dressing therefore vanishes, which is coherent with the fact that it has not been put into the model from the beginning, but built inductively to compensate local parts of diverging graphs.

5 **Bounds**

5.1 **Gaussian bounds**

This paragraph is the backbone of the section, since it provides a means (i) to bound the sum all possible Wick pairings of all possible \( G \)-monomial associated to a given polymer; (ii) to bound the sum over all possible polymers containing some fixed interval \( \Delta \) at its lowest scale. The general idea (as explained in the introduction) is that a polymer is connected either by horizontal cluster links which are polynomially decreasing at large distances, or by vertical inclusion links which create spring factors. The computations below for a polymer \( \mathbb{P} \) (see §5.1.2 and 5.1.3) give in the end a bound which is of order \( \prod_{v} \lambda^{\kappa} \prod_{\Delta} M^{-\varepsilon} \) for some positive exponents \( \kappa, \varepsilon \), where the product ranges over all vertices \( v \) and intervals \( \Delta \) of \( \mathbb{P} \). This is the general principle of the bounds for cluster expansions. Both terms \( \lambda^{\kappa} \) and \( M^{-\varepsilon} \) are called a “petit facteur par carré” (small factor per interval, in French). It does not include combinatorial factors (see §5.1.4), dominated averaged low-momentum fields (see §5.2), and possibly some other terms (see first step in the Proof of Theorem 5.2 in §5.3), but all these are proved to give for each choice of cluster expansion and \( G \)-polynomial a supplementary factor of the
type $\prod_\Delta (1 + O(\lambda^{\kappa'}))$ for some exponent $\kappa' > 0$, which is eaten up by the factor $\prod_\Delta M^{-\varepsilon}$.

Since all computations are Gaussian in this paragraph, we shall take the liberty to write $\langle (\cdots) \rangle$ instead of $E[ (\cdots) ]$, without any risk of confusion.

5.1.1 Wick’s formula and applications

We first recall the classical Wick formula.

**Proposition 5.1 (Wick’s formula)** Let $(X_1, \ldots, X_{2N})$ be a (centered) Gaussian vector. Pair the indices $1, \ldots, 2N$; the result may be represented as a graph $F$ with $n$ connected components, linking the vertices $1, \ldots, 2N$ two by two. As in Definition 2.5, we use the pair notation $\ell = \{i_\ell, i'_\ell\}$ for links. Then

$$\langle X_1 \cdots X_{2N} \rangle = \sum_{F \text{ pairing of } \{1, \ldots, 2N\}} \prod_{\ell \in F} \langle X_{i_\ell} X_{i'_\ell} \rangle.$$  \hspace{1cm} (5.1)

**Proof.** see e.g. [30], §5.1.2. \hfill \Box

**Corollary 5.2 (simple Wick bound)** Let $(X_1, \ldots, X_{2N})$ be a Gaussian vector. Then, for every $K > 0$,

$$|\langle X_1 \cdots X_{2N} \rangle| \leq K^{-N} \prod_{i=1}^{2N-1} \left[ 1 + K \sum_{j > i} |\langle X_i X_j \rangle| \right].$$  \hspace{1cm} (5.2)

In particular,

$$|\langle X_1 \cdots X_{2N} \rangle| \leq \prod_{i=1}^{2N} \left[ 1 + \sum_{j \neq i} |\langle X_i X_j \rangle| \right].$$  \hspace{1cm} (5.3)

**Proof.** Expand the right-hand side and use Wick’s formula to get eq. (5.2) with $K = 1$. The bound with $K \neq 1$ may be obtained from the previous one by a simple rescaling $X_i \to \sqrt{K} X_i$, $i = 1, \ldots, 2N$. \hfill \Box

The above bound eq. (5.2) depends on the ordering of the variables $X_1, \ldots, X_{2N}$, although $\langle X_1 \cdots X_{2N} \rangle$ doesn’t, of course. The idea conveyed by this bound is that it may be important to choose the right order. Similarly, eq. (5.2) is clearly optimal when the factors $K \sum_{j > i} |\langle X_i X_j \rangle|$, $1 \leq i \leq 2N - 1$, are of order 1.
However this bound is too simple to apply in most cases, and we shall need refined versions of it using the spatial structure of the Gaussian variables. The following lemmas, for the reasons we have just explained, are to be used after a suitable rescaling.

**Corollary 5.3 (Wick bound with spatial structure)**  

1. *(single-scale bound)* Let \((X(\Delta,n))_{\Delta \in \mathbb{D}^j, 1 \leq n \leq N_{\text{max}}(\Delta)}\) be a Gaussian vector indexed by \(M\)-adic intervals of scale \(j\). Denote by \(I = \{(\Delta, n); \Delta \in \mathbb{D}^j, 1 \leq n \leq N(\Delta)\}\) the total set of indices. Call connecting pairing a partial pairing \(F\) of the indices \((\Delta, n)\) such that its spatial projection \(\overline{F}\) with vertices \(\{\Delta \in \mathbb{D}^j; \exists n \leq N_{\text{max}}(\Delta) \mid (\Delta, n) \in F\}\) and links \(\{(\Delta, \Delta ') \in \mathbb{D}^j \times \mathbb{D}^j; \exists n \leq N_{\text{max}}(\Delta), n' \leq N_{\text{max}}(\Delta') \mid (\Delta, n) \sim_F (\Delta', n')\}\) is connected. Fix some \(M\)-adic interval \(\Delta_1 \in \mathbb{D}^j\). Then

\[
\sum_{F \text{ connecting pairing of } I, |F| = 2N, \Delta_1 \in \overline{F}} \prod_{\ell \in F} |\langle X(\Delta_\ell, n_\ell) X(\Delta'_\ell, n'_\ell) \rangle| \leq \left(1 + \sup_{\Delta \in \mathbb{D}^j} \left(\sum_{\Delta' \in \mathbb{D}^j} \sum_{i=1}^{d} |\langle X(\Delta, i) X(\Delta', i') \rangle|\right)\right)^{3N}.
\]

(5.4)

2. *(single-scale bound, improved version)*

More generally \(^7\) assume \(N_{\text{max}}(\Delta) = \infty\) and \(K : (\mathbb{D}^j \times \{1, \ldots, d\}) \times (\mathbb{D}^j \times \{1, \ldots, d\}) \rightarrow \mathbb{R}_+\) is some kernel, copied an infinite number of times, so that \(K((\Delta, pd + i), (\Delta', p'd + i')) = K((\Delta, i), (\Delta', i'))\), with \(p, p' = 1, 2, \ldots\) (see Remark below). Let \(I = \mathbb{D}^j \times \{1, 2, \ldots\}\). Connecting pairings of \(I\) must be understood modulo the pair identifications \(((\Delta, pd + i), (\Delta', p'd + i')) \sim ((\Delta, i), (\Delta', i'))\).

Then, for any \(\gamma \geq 1\), letting \(N(\Delta) := \#\{(\Delta, n); (\Delta, n) \in F\}\) to be interpreted as the number of fields lying in a given interval \(\Delta\),

\[
\sum_{F \text{ connecting pairing of } I, |F| = 2N, \Delta_1 \in \overline{F}} \prod_{\ell \in F} (N(\Delta_\ell) N(\Delta'_\ell))^{-\gamma} K((\Delta_i, i), (\Delta'_i, i')) \leq \left(1 + \sup_{\Delta \in \mathbb{D}^j} \sum_{\Delta' \in \mathbb{D}^j} \sum_{i, i'=1}^{d} K((\Delta, i), (\Delta', i'))\right)^{3N}.
\]

(5.5)

\(^7\)This version allows an unlimited number of fields per interval; cluster expansions with arbitrarily high connectivity numbers \(n(\Delta)\) do produce such a situation (see Remark after the corollary).
3. (multi-scale bound)

Let \( K : (\mathbb{D}^j \times \{1, \ldots, d\}) \times (\mathbb{D}^j \times \{1, \ldots, d\}) \rightarrow \mathbb{R}_+ \) be some kernel indexed by \( M \)-adic intervals of scale \( \geq j \), copied an infinite number of times as in 2. We denote once again by \( I \) the total set of indices. Let \( I^k := \{ (\Delta, n) \in I; \Delta \in \mathbb{D}^k \} \), \( k \geq j \) and \( I^{k'} := \{ j = j, k' \geq j \}. \) Fix a certain number of (non necessarily distinct) \( M \)-adic intervals for each scale \( k = j, j+1, \ldots, \rho \), say, \( \Delta^k_1, \ldots, \Delta^k_c \) \((k \geq j)\), and \( c_k \geq 0 \) \((k > j)\) and \( c_j = 1 \); write for short \( \Delta^j \rightarrow = \{ (\Delta^k)_{k \geq j, 1 \leq \ell \leq c_k} \} \subset \mathbb{D}^j \rightarrow \). Let \( \mathcal{F}^j \rightarrow (\Delta^j \rightarrow) \) be the set of multi-scale cluster forests \( \mathcal{F}^j \rightarrow \) (called: \( \Delta^j \rightarrow \)-connected multiscale cluster forests) such that, for each \( j' \geq j \), each vertex of \( \mathcal{F}^j \rightarrow \) is connected by horizontal cluster links or inclusion links to some (possibly many) of the selected intervals \((\Delta^k)_{k \geq j', 1 \leq \ell \leq c_{k'}}\), and the intervals \( \Delta^j_1', \ldots, \Delta^j_{c_j'} \) are not connected within \( \mathcal{F}^j \rightarrow \) by horizontal cluster nor inclusion links. (In other words, each selected interval \( \Delta^j_1', \ldots, \Delta^j_{c_j'} \) lies within a different horizontal cluster and inclusion connected component of \( \mathcal{F}^j \rightarrow \), and these \( c_j' \) connected components exhaust the set of horizontal cluster and inclusion connected components of \( \mathcal{F}^j \rightarrow \) which contain at least one \( M \)-adic interval of scale \( j' \)). Call \( \Delta^j \rightarrow \)-connecting pairing a partial pairing \( \mathcal{P} \) of the indices \((\Delta, n)\) such that its spatial projection \( \mathcal{P} \) has the same set of vertices and links as some \( \Delta^j \rightarrow \)-connected forest, plus possibly some supplementary links, possibly reducing the number of connected components. Then:

\[
\sum_{\mathcal{P} \Delta^j \rightarrow \text{-connecting pairing of } I, |\mathcal{P}|=2N} \prod_{\ell \in \mathcal{P}} (N(\Delta_\ell)N(\Delta'_\ell))^{-\gamma} K((\Delta_\ell, i_\ell), (\Delta'_\ell, i'_\ell))
\leq \left(1 + \max_{k \geq j} \sup_{\Delta^k \in \mathbb{D}^k} \left[ \sum_{\Delta^j \in \mathbb{D}^j \rightarrow k} \sum_{i, i'=1}^d K((\Delta^k, i), (\Delta'_j, i')) \right]^{k-1} \right)^N \sum_{k'=j} \sum_{\Delta''^k \in \mathbb{D}^j \rightarrow k'} \sum_{i, i'=1}^{d} K((\Delta^k, i'), (\Delta''^k, i''))^{3N},
\]

(5.6)

where \( \Delta^k \) is the unique interval of scale \( k' \) such that \( \Delta^k \subset \Delta^k \).

**Remark.** When \( N_{\text{max}}(\Delta) = \infty \), which is due to the fact that the total number of fields in a given \( M \)-adic interval, \( N(\Delta) \), may be of order \( n(\Delta) \), hence unbounded, the bound in 1. is infinite. In practice, a cluster expansion generates – thanks to the polynomial decrease in the distance of
the covariance of multi-scale Gaussian fields – extremely small factors per interval when \( n(\Delta) \) is large. The idea is then to bound \( |\langle \psi_i^j(x)\psi_i^j(x') \rangle| \), \( x \in \Delta, x' \in \Delta' \), by \( \frac{1}{[1+\phi((\Delta,\Delta'))^p]} K((\Delta,i),(\Delta',i')) \), where some of the polynomial decrease in the distance has been retained in the kernel \( K \) (see §5.1.2).

**Proof.**

1. Consider first the left-hand side of (5.4). Consider a connecting pairing \( \mathcal{F} \) such that \( |\mathcal{F}| = 2n \) and containing the \( M \)-adic interval \( \Delta_1 \), and a spanning tree \( \mathcal{T} \) of \( \mathcal{F} \) containing \( \Delta_1 \). Associate to \( \mathcal{F} \) the following sequence of links and of factors 1:

   – consider all the pairings of the indices \((\Delta_1,n)\), \(1 \leq n \leq N_{\text{max}}(\Delta_1)\) among themselves and with indices \((\Delta',n')\), \(\Delta' \neq \Delta_1\); say (in some arbitrary order), \((\Delta_1,n_1)\) pairs with \((\Delta'_1,n'_1)\), \ldots, \((\Delta_1,n_{N_1-1})\) pairs with \((\Delta'_{N_1-1},n'_{N_1-1})\). Insert after these \( N_1-1 \) links a factor 1, signifying that all paired Gaussian variables lying in the interval \( \Delta_1 \) have been exhausted;

   – continue to explore new vertices of \( \mathcal{F} \) by going along the branches of \( \mathcal{T} \). Always insert a factor 1 after all the pairings of the Gaussian variables lying in a given interval have been exhausted.

Since \( \mathcal{T} \) is connected, all \( M \)-adic intervals in \( \mathcal{F} \) and all indices in \( \mathcal{F} \) will eventually have been explored. The number of factors is \( \ell(\mathcal{F}) + |\mathcal{F}| \leq N + |\mathcal{F}| = 3N \), to the completed by the required number of factors 1 so that there are exactly 3N factors.

Consider now the right-hand side. Let

\[ K_\Delta := \sum_{1 \leq n \leq N_{\text{max}}(\Delta)} \left\{ \sum_{(\Delta',n') \neq (\Delta,n)} |\langle X_{(\Delta,n)}X_{(\Delta',n')} \rangle| \right\} \quad (5.7) \]

and \( K_\emptyset = 1 \), and expand \( K^{3N} := (K_\emptyset + \sup_{\Delta \in \mathcal{D}} K_\Delta)^{3N} \). One gets

\[
K^{3N} = \sum_p \sum_{N_1 \ldots N_p} (\sup_{\Delta} K_\Delta)^{N_1-1} \cdot 1 \cdot (\sup_{\Delta} K_\Delta)^{N_2-N_1-1} \cdot 1 \ldots 1 \cdot (\sup_{\Delta} K_\Delta)^{N_p-N_{p-1}-1} \cdot 1 \ldots 1.
\]

(5.8)

Replace the first sequence of \( N_1-1 \) factors by \( K_{\Delta_1}^{N_1-1} \leq (\sup_{\Delta} K_\Delta)^{N_1-1} \) and expand them, which encodes in particular all possible pairings of the Gaussian variables lying in \( \Delta_1 \). Consider now an interval \( \Delta_2 \neq \Delta_1 \).
linked by $\bar{T}$ to $\Delta_1$, and replace the second sequence of factors by $K_{\Delta_2}^{N_2-N_1-1}$, and so on. Thus one has encoded all possible connecting pairings containing the interval $\Delta_1$.

2. Considering as in the previous case all the pairings of the indices $(\Delta_1, n)$, $n = 1, 2, \ldots$ for a given pairing $F$, one gets the factor

$$K_{\Delta_1} := \sum \left( (N(\Delta_1)N(\Delta'))^{-\gamma}K((\Delta_1, i_1), (\Delta'_1, i'_1)) \ldots \right.$$  

$$\left( (N(\Delta_1)N(\Delta'_{N_1-1}))^{-\gamma}K((\Delta_1, i_{N_1-1}), (\Delta'_1, i'_{N_1-1})) \right),$$  

where the sum ranges over all possible $(N_1-1)$-uple couplings $((\Delta_1, i), (\Delta', i'))$ with multiplicities (note that $\frac{N(\Delta_1)}{2} \leq N_1 - 1 \leq N(\Delta_1)$, depending on the number of couplings of fields inside $\Delta_1$). Then

$$K_{\Delta_1} \leq \left( \sum_{\Delta' \in D^j} \sum_{1 \leq i, i' \leq d} K((\Delta_1, i), (\Delta', i')) \sum_{n=1}^{N(\Delta_1)} \sum_{n'=1}^{N(\Delta')} \left( (N(\Delta_1)N(\Delta'))^{-\gamma} \right)^{N_1-1} \right)^{N_1-1}$$  

$$\leq \left( \sum_{\Delta' \in D^j} \sum_{i, i'=1}^d K((\Delta_1, i), (\Delta', i')) \right)^{N_1-1}. \quad (5.10)$$

Apart from this slight difference, the exploration procedure is the same.

3. Choose a spanning tree of $\bar{\mathbb{F}} \cap D^\rho$, complete it into a spanning tree of $\bar{\mathbb{F}} \cap D^{(\rho-1)\rightarrow}$, and so on. As in 1., explore the horizontal cluster connected components of scale $\rho$ starting from the selected intervals $\Delta^\rho_c$, $1 \leq c \leq c_\rho$, then the connected components of scale $\rho - 1$, and so on, down to scale $j$. The only difference is that two different horizontal cluster connected components of $\bar{\mathbb{F}}$ of the same scale $j'$ may be connected from above by inclusion links and horizontal cluster links of higher scale; in this case, this procedure may not explore all vertices of $\bar{\mathbb{F}}$. Fortunately, the bound in eq. (5.6) gives the possibility, starting from some interval $\Delta^k \in D^k$, to go on to explore all the Gaussian variables located below $\Delta^k$, i.e. in some interval $\Delta^{k'} \supset \Delta^k$ with $k' < k$. 

\qed
5.1.2 Gaussian bounds for cluster expansions

We assume here that \( \mathcal{L}_{\text{int}} \) is just renormalizable, so that (assuming just for simplicity of notations that its coefficients are scale-independent) \( \mathcal{L}_{\text{int}} = K_1 \lambda^{\kappa_1} \psi_{I_1} + \ldots + K_p \lambda^{\kappa_p} \psi_{I_p} \), where \( \kappa_1, \ldots, \kappa_p > 0 \) and \( \sum_{i \in I_1} \beta_i = \ldots = \sum_{i \in I_p} \beta_i = -D \). Each term \( \psi_{I_1}, \ldots, \psi_{I_p} \) is called a vertex by reference to the Feynman diagram representation (see section 4). Let us recall briefly that the \( G \)-monomials are produced:

- either by horizontal cluster expansions; if \( i_\ell \in I_q, \ell \) being a link at scale \( j \), then \( \frac{\partial}{\partial \psi_{i_\ell}^{(x)}} e^{-\lambda x q} \int \psi_{I_q}(x) dx \) produces \( \lambda^{\kappa_q} \psi_{I_q\setminus\{i_\ell\}}(x_\ell) \). On the other hand, \( \frac{\partial}{\partial \psi_{i_\ell}^{(x)}} \) may derivate the low-momentum components of monomials produced at scales \( \geq j + 1 \), which lowers the degree of \( G \);
- or by \( t \)-derivations acting on \( e^{-\int \mathcal{L}_{\text{int}}(x) dx} \), yielding (up to \( t \)-coefficients) some (scale components) of the \( \lambda^{\kappa_q} \psi_{I_q}(x_\Delta) \), \( 1 \leq q \leq p \), integrated over some \( M \)-adic interval \( \Delta \). Again, \( t \)-derivations may derivate the monomials produced at scales \( \geq j \), which does not change the degree of \( G \).

The above products of fields must now be split according to their scale decomposition. Thus one obtains a certain number \( \nu \) of vertices split into different scales.

We use the following notations in order to avoid the proliferation of indices. Make a list \( (\psi_1, \ldots, \psi_d) \), \( d = |I_1| + \ldots + |I_p| \), of all the fields involved in the interaction, possibly with repetitions. Thus the cluster expansion at scale \( j \) generates at the same scale \( \lambda^{\kappa_q} \psi_{I_q}^{\psi_j}(x_\ell) \), where \( I_\ell \subset I_q \setminus \{i_\ell\} \subset \{1, \ldots, d\} \setminus \{i_\ell\} \) for some \( q \leq p \); on the other hand, each \( t \)-derivation in an interval \( \Delta \in \mathbb{D}^j \) generates (up to \( t \)-coefficients) some \( \lambda^{\kappa_q} \psi_{I_q}^{\psi_j}(x_\Delta) \), \( I_\Delta \subset I_q \), or (with an extra index \( \tau_\Delta \) for the order of derivation) \( (\lambda^{\kappa_q} \psi_{I_q}^{\psi_j}(x_\Delta))_{\tau_1, \ldots, \tau_\Delta} \).

But other field components of scale \( j \), lying in some fixed interval \( \Delta^j \in \mathbb{D}^j \), are produced, either at an earlier stage \( k > j \), in the form of a low-momentum field, \( \psi_j(x) \) or \( \delta^k \psi_j(x) \) (secondary field) with \( x \in \Delta^k, \Delta^k \in \mathbb{D}^k, \Delta^k \subset \Delta^j \), or at a later stage \( h < j \), in the form of a high-momentum field, \( \text{Res}_{\Delta^h}^h, \psi_j(x), x \in \Delta^h \), where \( \Delta^h \in \mathbb{D}^h, \Delta^h \supset \Delta^j \).

The general principle of bounds for cluster expansions in quantum field theory (as explained at the beginning of §5.1) is to (1) use the polynomial decrease in the distance of the covariance of the field components; (2) find out a “petit facteur par carré” (small factor per cube, or rather per interval in one dimension). This means essentially the following: chose some possibly derivated interaction term \( \lambda^{\kappa_q} \psi_{I_q\setminus\{i_\ell\}}(x_\ell), x_\ell \in \Delta^j \) or \( \lambda^{\kappa_q} \psi_{I_q}(x_\Delta) \) coming
from a vertex at scale \( j \); the fields \( \psi^j_i, i \in I_q \) scale like \( M^{-\beta j} \), and the integration over the interval \( \Delta^j \in D^j \) produces a factor \( M^{-j} \) (or \( M^{-D_j} \) in general). As for the cluster expansion at scale \( j \), it has produced a factor \( C_{\psi}^j(i,t,x;i',t',y') \) which scales like \( M^{-\beta_{i,j}} \), times the same quantity with a prime. Supposing one chooses a splitting of the vertex such that all fields are of scale \( j \), then the product of these factors is \( \lambda \kappa \sum_{i \in I_q} \beta_i = \lambda \kappa q \ll 1 \), which is the “petit facteur”. Unfortunately the splittings of the vertex produce much more complicated situations; however, the guideline is to compare the scalings of the high-momentum fields (rewritten as a sum of restricted fields) and of the low-momentum fields (possibly rewritten as secondary fields, modulo averaged fields) with the scaling they would produce if they were of scale \( j \). In other words, the rescaled fields

\[
\psi^{k, \text{rescaled}}_i(x) = M^{\beta_{i,j}} \psi^k_i(x) \quad (k > j), \quad \text{resp.} \quad \psi^{h, \text{rescaled}}_i(x) = M^{\beta_{i,j}} \psi^h_i(x) \quad (h < j),
\]

give – when computing their Gaussian pairings using Wick’s formula – a factor of order \( M^{-\beta_{i,j} (k-j)} \), resp. \( M^{\beta_{i,j} (j-h)} \). The factor \( \lambda \kappa q \) is split into the different scales of the vertex, so that each field \( \psi_i(x), i = i_1, \ldots, i_q \) is accompanied by a small factor \( \leq \lambda \kappa_0 / |I_q| \) for some \( \kappa_0 > 0 \).

Summarizing, one has the following picture:

**Proposition 5.4 (spring factors)**

(i) each high-momentum field of scale \( k \) produced at scale \( j \) comes with a spring factor \( M^{-\beta_{i,j} (k-j)} \).

Most of this spring factor, \( M^{-\gamma_{i,j} (k-j)} \), shall be used in Lemma 5.9 for the Gaussian bounds of scale \( k \), see eq. (5.30). A small part of it, \( M^{-\epsilon/q (k-j)} \), shall be used in Theorem 5.1 to assemble the polymer.

(ii) each low-momentum field (or secondary field if required) of scale \( h \), produced at scale \( k \) and dropped at scale \( j \) (see subsection 2.3) comes with a spring factor \( M^{\beta_{i,j} (j-h)} \) (see Definition 5.6), used in Lemma 5.9 for the Gaussian bounds of scale \( h \), see eq. (5.29). The remaining part of the spring factor, \( M^{\beta_{i,j} (k-j)} \), shall be used in Theorem 5.1 to assemble the polymer.

Averaged low-momentum fields must be treated apart and account for the so-called domination problem; bounding them may require part of the
small factor $\lambda^\kappa$, so that, generally speaking, the “petit facteur” is of order $\lambda^\kappa$, for some $\kappa > 0$ but small.

Let us first consider the following single scale situation, throwing away all low- or high-momentum fields for the time being.

**Lemma 5.5** Let $\psi = (\psi_1(x), \ldots, \psi_d(x))$ be a Gaussian field with $d$ components such that

$$|C^j_\psi(i, x; i', x')| = |\langle \psi^j_i(x) \psi^j_{i'}(x') \rangle| \leq K_r \frac{M^{-j(\beta_i + \beta_{i'})}}{(1 + M_j|x - x'|)^r}$$

(5.13)

for every $r \geq 0$, with some constant $K_r$ depending only on $r$; these bounds hold in particular if $\psi_1, \ldots, \psi_d \subset \{(\partial^n \tilde{\psi}_1)_{n \geq 0}, \ldots, (\partial^n \tilde{\psi}_d)_{n \geq 0}\}$ are derivatives of some independent multiscale Gaussian fields $\tilde{\psi}_1, \ldots, \tilde{\psi}_d$.

Consider a horizontal cluster forest $\mathbb{F}^j \in \mathcal{F}^j$ of scale $j$, and associated cluster points $x_\ell, x'_\ell$, $\ell \in \mathbb{L}(\mathbb{F}^j)$, $x_\ell \in \Delta_\ell$, $x'_\ell \in \Delta'_\ell$. Choose:

- for each link $\ell \in \mathbb{L}(\mathbb{F}^j)$, a subset $I_\ell$ of $\{1, \ldots, d\}$ \{\{i_\ell\}$ and a subset $I'_{\ell}$ of $\{1, \ldots, d\}$ \{\{i'_{\ell}\}$;

- for each $M$-adic interval $\Delta \in \mathbb{F}^j$, $\tau_\Delta \leq N_{\text{ext}, \max} + O(n(\Delta))$ subsets $(I_{\Delta, \tau})_{\tau = 1, \ldots, \tau_\Delta}$ of $\{1, \ldots, d\}$, and additional integration points $(x_{\Delta, \tau})_{\tau = 1, \ldots, \tau_\Delta}$ in $\Delta$.

Such a choice defines uniquely a monomial $G^{i,j} = G^{i,j}(\mathbb{F}^j; (I_\ell), (I'_\ell); (I_{\Delta, \tau}), (x_{\Delta, \tau}))$ in the fields $\psi^j_i$, $i = 1, \ldots, d$ taken at the cluster points $(x_\ell)$, $(x'_\ell)$ and the $t$-derivation points $(x_{\Delta, \tau})$, namely,

$$G^{i,j} := \lambda^v(\mathbb{F}^j; G^{i,j}) \left[ \prod_{\ell \in \mathbb{L}(\mathbb{F}^j)} \psi^j_{I_\ell}(x_\ell) \psi^j_{I'_\ell}(x'_\ell) \right] \cdot \left[ \prod_{\Delta \in \mathbb{F}^j} \prod_{\tau = 1}^{\tau_\Delta} \psi^j_{I_{\Delta, \tau}}(x_{\Delta, \tau}) \right],$$

(5.14)

where $v(\mathbb{F}^j; G^{i,j}) := 2L(\mathbb{F}^j) + \sum_{\Delta \in \mathbb{F}^j} \tau_\Delta$ is the total number of vertices obtained from the horizontal cluster and the $t$-derivations is (two per horizontal cluster link, one per $t$-derivation acting on the exponential of the interaction).

Denote by $N_i^j(G^{i,j}; \Delta), i = 1, \ldots, d$ the number of fields $\psi^j_i(x)$, $x \in \Delta$ occurring in $G^{i,j}$ if $\Delta \in \mathbb{F}^j$, summing up to $N^j(G^{i,j}; \Delta) := \sum_{i=1}^d N_i^j(G^{i,j}; \Delta)$, and by $N^j(G^{i,j}) := \sum_{\Delta \in \mathbb{F}^j} N_i^j(G^{i,j}; \Delta)$ the total number of fields $\psi^j_i$ occurring in $G^{i,j}$. Similarly, we denote by $N_i^j(\mathbb{F}^j)$ the number of half-propagators of type $i$ in $\mathbb{F}^j$. Note that $N^j(G^{i,j}; \Delta) = O(n(\Delta))$, see (5.14).

1. Let

$$I^j_{\text{Gaussian}}(\mathbb{F}^j; G^{i,j}) := \int \mu(\psi^j) \prod_{\ell \in \mathbb{L}(\mathbb{F}^j)} C^j_{\psi}(i_\ell, x_\ell; i'_\ell, x'_\ell) \cdot G^{i,j}. \quad (5.15)$$

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Then

\[
|I_{\text{Gaussian}}^j(\mathbb{F}^j; G^{j,j})| \leq K |\mathbb{F}| \lambda^{\alpha_0 v(\mathbb{F}^j; G^{j,j})} \prod_{i=1}^d M^{-j \beta_i (N_i^j(G^{j,j}) + N_i^j(\mathbb{F}^j))}.
\]

(5.16)

2. Fix the total number of vertices, \( v = v(\mathbb{F}^j; G^{j,j}) \), and fix one \( M \)-adic interval \( \Delta^j_1 \in \mathbb{D}^j \). Let \( \mathcal{F}^j_{\Delta^j_1} \subset \mathcal{F}^j \) be the subset of connected horizontal cluster forests of scale \( j \) containing \( \Delta^j_1 \). Consider the rescaled quantity

\[
I_{\text{Gaussian}}^{\text{rescaled}}(v; \Delta^j_1) := \prod_{i=1}^d M^{-j \beta_i (N_i^j(G^{j,j}) + N_i^j(\mathbb{F}^j))} I_{\text{Gaussian}}^j(\mathbb{F}^j; G^{j,j})
\]

(5.17)

and:

- sum over all \( \ell \in \mathcal{F}^j_{\Delta^j_1} \), \( (I_\ell) \subset \{1, \ldots, d\} \setminus \{i_\ell\} \), \( (I'_\ell) \subset \{1, \ldots, d\} \setminus \{i'_\ell\} \), \( (I_{\Delta,\tau}) \subset \{1, \ldots, d\} \);

- maximize for \( (x_\ell), (x'_\ell), (x_{\Delta,\tau}) \), each one ranging over its associated interval in \( \mathbb{D}^j \).

Call \( I_{\text{Gaussian}}^{\text{rescaled}}(v; \Delta^j_1) \) the result. Then

\[
I_{\text{Gaussian}}^{\text{rescaled}}(v; \Delta^j_1) \leq (K \lambda^{\alpha_0})^v.
\]

(5.18)

Proof.

1. The integral \( I_{\text{Gaussian}}^j(\mathbb{F}^j; G^{j,j}) \) may be evaluated by using Wick’s lemma. Each choice of contractions leads, using the numerator in the right-hand side of eq. (5.13), to some term with the correct homogeneity factor, \( \lambda^{\alpha_0 v(\mathbb{F}^j; G^{j,j})} \prod_{i=1}^d M^{-j \beta_i (N_i^j(G^{j,j}) + N_i^j(\mathbb{F}^j))} \). Consider the rescaled fields \( \psi_i^{\text{rescaled}} := M^{j \beta_i} \psi_i^j \). For reasons to be discussed presently, we shall apply Corollary 5.2 to the rescaled fields \( n(\Delta)^{-\gamma} \psi_i^{\text{rescaled}}(x) \), for some power \( \gamma \geq 1 \).

The possibility to introduce this supplementary scaling factor \( n(\Delta)^{-\gamma} \) comes from the following argument. Split \( 1 \) into \( \frac{1}{1+M |x-x'|^r} \) and \( \frac{1}{1+M |x-x'|^{r''}} \), with \( r = r' + r'' \) and \( r', r'' > 0 \). The product of propagators \( \prod_{\ell \in L(\mathbb{F}^j)} C^j_{\psi}(i_\ell, x_\ell; i'_\ell, x'_\ell) \) contributes, see denominator in the right-hand side of eq. (5.13), a convergence factor \( K_{L(\mathbb{F}^j)}^{|L(\mathbb{F}^j)|} \).
\[ \prod_{\Delta \in F_j} \prod_{\Delta' \sim \Delta} \left( d^j(\Delta, \Delta') \right)^{-r''/2}, \] for some constant \( K_{r''} \) depending only on \( r'' \). Since the number of intervals \( \Delta' \in \mathbb{D}^j \) such that \( d^j(\Delta, \Delta') \leq n(\Delta)/4 \leq n(\Delta)/2 \), this means that at least half of the intervals \( \Delta' \sim \Delta \) are at a \( d^j \)-distance > \( n(\Delta)/4 \) from \( \Delta \), so that

\[ \prod_{\Delta' \sim \Delta} \left( d^j(\Delta, \Delta') \right)^{-r''/2} \leq \left[ (n(\Delta)/4)^{-r''/2} \right]^{n(\Delta)/2} = K^{n(\Delta)} n(\Delta)^{-K' r'' n(\Delta)}. \]

(5.19)

On the other hand, taking into account the \( n(\Delta'_\ell) \gamma \), resp. \( n(\Delta'_\ell) \gamma \) factors separated from the rescaled fields in cluster intervals contributes

\[ \prod_{\Delta \in F_j} n(\Delta)^{\gamma N} \leq \prod_{\Delta \in F_j} n(\Delta)^{\gamma O(n(\Delta))}, \]

a product of so-called local factorials, which is compensated by the above convergence factor as soon as \( r'' \) is chosen large enough.

We may now apply Corollary 5.2, eq. (5.3) to the rescaled fields, which yields, using once again \( N_j(\gamma, G^j, G^j, \Delta) = O(n(\Delta)) \),

\[ I_{\text{rescaled}}^{j,G^j} \leq K \sum_{\Delta \in F_j} n(\Delta) \lambda^{N_{\text{tot}}(F^j, G^j, \Delta)} \left[ 1 + (n(\Delta))^{-\gamma} \sum_{\Delta' \in F_j} (n(\Delta'))^{-\gamma} \frac{O(n(\Delta'))}{(1 + d^j(\Delta, \Delta'))^{r''}} \right]^{O(n(\Delta))}. \]

(5.20)

Now the sum \( \sum_{\Delta' \in \mathbb{D}^j} \frac{1}{(1 + d^j(\Delta, \Delta'))^{r''}} \) converges as soon as \( r'' > D \). Hence each term between square brackets is bounded by a constant. Since \( \sum_{\Delta \in F_j} n(\Delta) = 2|F^j| - 2 = O(|F^j|) \), one gets:

\[ I_{\text{rescaled}}^{j,G^j} \leq K |F^j| \lambda^{N_{\text{tot}}(F^j, G^j, \Delta)}. \]

(5.21)

2. Associate to a connected forest \( F^j \in F^j \) and a monomial \( G^{j, \bar{j}} \) as in \( (5.14) \) its Wick expansion, represented as a sum over a set of connecting pairings of \( \mathbb{D}^j \) as in Corollary 5.3 (1), except that \( N(\Delta) \leq d(n(\Delta) + \tau_{\Delta}) + n(\Delta) - \) the number of fields and half-propagators in the \( M \)-adic interval \( \Delta \) – depends on \( F^j \) and \( G^{j, \bar{j}} \), and is unbounded since \( n(\Delta) \) may be arbitrarily large. Hence (to get a finite bound for our sum of Gaussian integrals) we shall use the \( F^j \)-dependent rescaling by \( n(\Delta)^{-\gamma} = O(N(\Delta)^{-\gamma}) \) of the fields defined in 1., at the price of

\[ \text{In } D \text{ dimensions, } n(\Delta)^{1/D} \text{ becomes } Kn(\Delta)^{1/D} \text{ and eq. (5.19) holds, with different constants.} \]
the extension of the exploration procedure described in the proof of
Corollary 5.3 (2). Note however that the mapping \((F^i, G^i, j) \mapsto \) connecting pairing is not one-to-one, since a link of the resulting pairing \(F\) may come either from the links of \(F^i\) or from the pairings of \(G^i, j\); this contributes at most a factor 2 per pairing, hence at most \(2^{d_i/2}\).

Now, the factor \(\sum_{\Delta \in D^i} \sum_{i,j=1}^{d_i} K((\Delta, i), (\Delta', i'))\) of Corollary 5.3 (2) associated to the rescaled fields defined in 1. is bounded up to a constant by \(\sum_{\Delta \in D^i} 1/(1 + d_i(\Delta, \Delta'))^r < \infty\), hence the result.
\[\square\]

The above arguments extend easily to single scale Mayer trees of polymers of scale \(j\). The new rules are:

(i) there may be some undetermined number of copies of each interval \(\Delta^i\), each with a different color;

(ii) fields in intervals with different colors are uncorrelated;

(iii) each cluster forest of a given color is connected; one of them (the red one, say) contains a fixed interval, \(\Delta^i\);

(iv) the different cluster forests are connected by Mayer links. These define a tree structure on the set of colors, and imply for each link between 2 colors, say, red and blue, an overlap between one red interval and one blue interval (chosen at random if they have several overlaps).

The proof of Lemma 5.5 (2) is the same as before, except that the exploration procedure must now take into account Mayer links. Let \(n_{\text{Mayer}}(P')\) be the coordination number of a (red, say) polymer \(P'\) in the Mayer tree. The overlap constraint between \(P'\) and its neighbours \(P_1, \ldots, P_{n_{\text{Mayer}}(P')-1}\) in the tree splits into multiple overlaps of order \(n_i = n_1, \ldots, n_c \geq 1\) between an interval \(\Delta_i\) in \(P'\) and \(n_i\) intervals in \(n_i\) neighbouring trees, \(P_i, 1, \ldots, P_i, n_i\), with \(n_1 + \ldots + n_c = n_{\text{Mayer}}(P') - 1\). The exploration procedure at the red interval \(\Delta_i\) adds \(n_i\) to \(K_{\Delta_i}\), see eq. (5.7), corresponding to the number of possible choices of neighbouring trees, but Cayley’s theorem, see proof of Proposition 2.12, yields a factor \(1/n_{\text{Mayer}}(P')! \leq \frac{1}{n_1! \ldots n_c!}\). Summing over all possible values of \(n_i\) leads to replacing \(K_{\Delta_i}\) by \(\sum_{n_i \geq 0} \frac{K_{\Delta_i} + n_i}{n_i!} = O(1)\).

The whole procedure must be slightly amended to take into account rooted Mayer trees (see §2.4) connecting possibly an interval \(\Delta \in \Delta_{\text{ext}}(\mathbb{P})\), where \(\mathbb{P} \in P^{j \to}\) is a polymer with \(\geq N_{\text{ext, max}}\) external legs, to intervals

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without external legs of polymers of type 1. Then one should associate some small power of $\lambda$, say $\lambda^\kappa$ with $\kappa \ll \kappa_0$ (at the price of reducing slightly $\kappa_0$ in eq. (5.18)) to each interval with a field lying in it, while the intervals $\Delta$ of the above type and containing moreover no field define intervals of a new type (of type 2, say), with no small factor attached to them. The whole discussion is very similar to the Remark after Proposition 2.12. For such intervals, $K_\Delta \equiv 1$ must be replaced with $1 + \sum_{n \geq 1} \lambda^n \frac{n!}{n^n} = 1 + O(\lambda^\kappa)$. As explained at the end of this paragraph, such a factor is not a problem.

We may now give a more general, multiscale bound which takes into account secondary fields and high-momentum fields. We rescale the low-momentum fields by reference to their dropping scale $j$, and not to their production scale $k$, see §2.3, which leaves outside a supplementary spring factor that will be used to fix the horizontal motion of the polymers as in §4.1. As mentioned in the Remarks following Corollary 1.7 and Definition 2.9, we shall not split low-momentum fields $\psi_i \rightarrow \tilde{\delta}_j \tilde{\psi}_h i$ into a sum (field average)+(secondary field) if $\beta_i < -D/2$. The following definition is valid in all cases:

**Definition 5.6 (spring factors)** Let $\tilde{\delta}_j \tilde{\psi}_h^i := \delta_j \psi_h^i$ if $\beta_i \geq -D/2$, with $\delta_j$ defined by means of a wavelet admitting $|\beta_i + \frac{D}{2}|$ vanishing moments, and $\tilde{\delta}_j \psi_h^i := \psi_h^i$ if $\beta_i < -D/2$, so that, by Corollary 1.7,

$$|\langle \tilde{\delta}_j, \text{rescaled} \tilde{\psi}_h^i (x) \tilde{\delta}_j', \text{rescaled} \tilde{\psi}_h^i (x') \rangle| \leq K_r M^{\tilde{\beta}_i-j-h} M^{\tilde{\beta}_i' - (j'-h)} (1 + d(h(\Delta_j, \Delta_j'))) r, \quad (5.22)$$

where $\tilde{\delta}_j, \text{rescaled} \tilde{\psi}_h^i := M^{\tilde{\beta}_i} \tilde{\delta}_j \tilde{\psi}_h^i$ is the rescaled field, and

$$\tilde{\beta}_i = \beta_i \quad (\beta_i < -D/2), \quad \tilde{\beta}_i = \beta_i - |\beta_i + \frac{D}{2} + 1| \in [-1-D/2, -D/2) \quad (\beta_i \geq -D/2). \quad (5.23)$$

**Example.** The $\sigma$-field in the $(\phi, \partial \phi, \sigma)$-model has $\beta_{\sigma} = -2 \alpha > -1/2$. Thus low-momentum fields are severed from their averages, and $\tilde{\beta}_\sigma = \beta_{\sigma} - 1 = -1 - 2\alpha$.

**Hypothesis 5.7 (high-momentum fields)** Assume

(i) either that all $\beta_i$, $i = 1, \ldots, d$ are $< 0$; \footnote{which is the case of $\phi^4$-theory for $D > 2$ for instance}
(ii) or, more generally, that there is a scale constraint on \( L_{\text{int}} \) of the following form: rewriting \( L_{\text{int}} \) as
\[
L_{\text{int}}(x) = \sum_{q \geq 2} \sum_{1 \leq i_1, \ldots, i_q \leq q} \sum_{j_1 \leq \ldots \leq j_q} K_{i_1, \ldots, i_q}^{j_1, \ldots, j_q} \psi_{i_1}^{j_1}(x) \cdots \psi_{i_q}^{j_q}(x),
\]
then
\[
(K_{i_1, \ldots, i_q}^{j_1, \ldots, j_q} \neq 0) \implies (\beta_{i_1} < 0, \beta_{i_1} + \beta_{i_2} < 0, \ldots, \beta_{i_1} + \ldots + \beta_{i_q} < 0).
\]

This condition on the scales of the low-momentum fields is of course equivalent to a condition on the scales of the high-momentum fields due to the homogeneity of the vertices.

Note that Hypothesis (5.25) holds true for our \((\phi, \partial \phi, \sigma)\)-model, since splitting a vertex leads to one low-momentum field, either \( \partial \phi \) or \( \sigma \), with respective scaling exponents \( \alpha - 1 \), \(-2 \alpha < 0 \). In general, it has the following obvious consequence.

**Lemma 5.8** Assume Hypothesis (5.25) holds, and fix \( I_q = (i_1, \ldots, i_q) \). Choose \( \varepsilon > 0 \) such that
\[
\varepsilon < \min (|\beta_{i_1}|, |\beta_{i_1} + \beta_{i_2}|, \ldots, |\beta_{i_1} + \ldots + \beta_{i_q}|)
\]
whenever there exists \( j_1 \leq \ldots \leq j_q \) such that \( K_{i_1, \ldots, i_q}^{j_1, \ldots, j_q} \neq 0 \), and let
\[
\gamma_i := \frac{\varepsilon}{q} - \beta_i, \quad i = i_1, \ldots, i_q; \quad \gamma_I := \sum_{i \in I} \gamma_i (I \subset I_q).
\]
Then \( \beta_i + \gamma_i > 0 \) for all \( i \), and \( \gamma_{i_1} + \ldots + \gamma_{i_q} < D \) for every \( q' = 1, \ldots, q \).

**Proof.** Since \( \beta_{i_1} + \ldots + \beta_{i_q} = -D \),
\[
\gamma_{i_1} + \ldots + \gamma_{i_q} < \varepsilon + (D + \beta_{i_1} + \ldots + \beta_{i_{q-1}}) < D.
\]

Under the above Hypothesis, one has a multiscale generalization of Lemma 5.5 by considering the contribution of all fields of some fixed scale \( j \). We adopt the following convention. If a vertex \( \psi_{I_q}(x) \) is split into \( i \) fields of scale \( j \) and \( |I_q| - i \) fields of scale \( \neq j \), then it contributes a (fractional number) of vertices, \( \frac{i}{|I_q|} \), of scale \( j \). This implies a small factor \( \lambda^{\alpha \nu} \) for the Gaussian bounds at scale \( j \), where \( \nu \) (the total number of vertices at scale \( j \)) is a fraction with bounded denominator.
Lemma 5.9 (multiscale generalization) Assume 
\( \psi_1, \ldots, \psi_d \subset \{(\partial^n \psi_1)_{n \geq 0}, \ldots, (\partial^n \psi_d)_{n \geq 0}\} \) are derivatives of some independent multiscale Gaussian fields \( \tilde{\psi}_1, \ldots, \tilde{\psi}_d \). Fix some constant \( \kappa_0 \in (0, 1) \) as in the previous lemma, as well as some reference scale \( j_{\min} \leq j \).

1. For each \( k \geq j \), consider a horizontal cluster forest \( F^k \in F \) and associated cluster points \( x_{\ell k}, x'_{\ell k} \), and choose subsets \( (I_{\ell k}), (I'_{\ell k}), (I_{\Delta k, \tau}), (x_{\Delta k, \tau}) \) as in the previous lemma. Do the same for each \( h < j \), and choose for each point \( x = x_{\ell k}, x'_{\ell k} \) or \( x_{\Delta k, \tau} \) a restriction interval \( \Delta^j = \Delta^j_{\ell k}, \Delta^j_{\ell k}, \Delta^j_{\Delta k, \tau} \) such that \( \Delta^j \subset \Delta^j_{\ell k}, \Delta^j_{\ell k} \) or \( \Delta^j_{\Delta k, \tau} \) respectively.

Such as choice defines uniquely a monomial \( G^{j,k} \) as before, and one more monomial per different scale,

\[
G^{j,k} := \lambda_{\kappa_0 \tau}(F^{j,k}) \left[ \prod_{\ell k \in L(F^k)} \delta^j_{\ell k} (x_{\ell k}) \delta^j_{\ell k} (x'_{\ell k}) \right] \prod_{\Delta^k \in F^k} \bigg[ \prod_{\tau = 1}^{\tau_{\Delta k}} \sum_{\Delta^j_{\Delta^k, \tau}} M^{-\gamma \tau_{\Delta^k, \tau}} \Res^j_{\Delta^j_{\Delta^k, \tau}} (x_{\Delta^j_{\Delta^k, \tau}}) \bigg] \]  

(5.29)

for \( k > j \), and, see eq. (5.27) for notations,

\[
G^{j,h} := \lambda_{\kappa_0 \tau}(F^{j,h}) \left[ \prod_{\ell k \in L(F^h)} \sum_{\Delta^j \subset \Delta^j_{\ell k}} \sum_{\Delta^j \subset \Delta^j_{\ell k}} \sum_{\Delta^j \subset \Delta^j_{\ell k}} M^{-\gamma \tau_{\Delta^j_{\ell k}}} \Res^j_{\Delta^j_{\Delta^k, \tau}} (x_{\ell k}) M^{-\gamma \tau_{\Delta^j_{\Delta^k, \tau}}} \Res^j_{\Delta^j_{\Delta^k, \tau}} (x_{\Delta^j_{\Delta^k, \tau}}) \right] \]  

(5.30)

for \( h < j \), where the \( \Delta^j, \Delta^j_{\ell k} \) are restriction intervals.

Let \( v(F, G^i) := v(F^i; G^{i,j}) + \sum_{k > j} v(F^k; G^{i,k}) + \sum_{h < j} v(F^j; G^{i,h}) \) be the total number of vertices. Let finally \( G^j := G^{i,j} \prod_{k > j} G^{j,k} \prod_{h < j} G^{i,h} \) and

\[
I^j_{\text{Gaussian}}(F; G^j) := \int d\mu(\psi^j) \prod_{\ell \in L(F^j)} C^j_{\psi}(i_{\ell j}, x_{\ell j}, i'_{\ell j}, x'_{\ell j}) G^j.  
\]

(5.31)

Then

\[
I^j_{\text{Gaussian}}(F; G^j) \leq K^{\text{dim}} M^{-(D + \omega_{\text{max}})} n_{\Delta^j_{\min}} \prod_{a \geq j_{\min}} \prod_{i = 1}^d M^{-a \beta_i (N_i^a (G^{j,a}) + N_i^a (F^a))},  
\]

(5.32)
where \( a \) stands either for a low-momentum scale \( h \leq j \) or a high-momentum scale \( k > j \), and \( \omega_{\text{max}} < 0 \) is an in Definition 3.4.

2. Fix the total number of vertices, \( v := v(\mathcal{F}, G_j) \). Define \( \mathcal{F}^{j_{\min} \rightarrow}(\Delta_{j_{\min} \rightarrow}) \) as in Corollary 5.3 (3). Consider, similarly to the previous lemma, the rescaled quantity

\[
I_{\text{Gaussian}}^j (\mathcal{F}; G_j) := \prod_{i=1}^d M^j \beta_i (N_i^j (G_j^i); N_i^j (\mathcal{F}^i)) \prod_{a \geq j_{\min}, a \neq j} M^a \beta_i (N_i^j (G_j^a)) I_{\text{Gaussian}}^j (\mathcal{F}; G_j).
\]

(5.33)

and:

- sum over all \( \mathcal{F} \in \mathcal{F}^{j_{\min} \rightarrow}(\Delta_{j_{\min} \rightarrow}), (I_{\mathcal{F}^k}) \subset \{1, \ldots, d\} \setminus \{i_{\mathcal{F}^k}\}, (I_{\mathcal{F}^k}) \subset \{1, \ldots, d\} \setminus \{i_{\mathcal{F}^k}\}, (I_{\Delta_{k, \tau}}) \subset \{1, \ldots, d\} (k \geq j) \), and similarly for \( h < j \);

- sum over all possibly choices of the restriction intervals \( \Delta_j \);

- maximize for \( (x_{\mathcal{F}^k}), (x_{\mathcal{F}^k}'), (x_{\Delta_{k, \tau}}) \), each one over its associated interval in \( \mathbb{D}_k \), \( k \geq j \), and for \( (x_{\mathcal{F}^k}), (x_{\mathcal{F}^k}'), (x_{\Delta_{h, \tau}}) \), \( h < j \), each one ranging over its associated restriction interval in \( \mathbb{D}_j \) (and not \( \mathbb{D}_h \)).

Call \( I_{\text{Gaussian}}^j (v; \Delta_{j_{\min} \rightarrow}) \) the result. Then

\[
I_{\text{Gaussian}}^j (v; \Delta_{j_{\min} \rightarrow}) \leq M^{-(D+|\omega_{\text{max}}|)\#(\Delta_{j_{\min} \rightarrow})} (K\lambda_0^\nu)^v. \tag{5.34}
\]

**Proof.**

Consider first the factor \( M^{-(D+|\omega_{\text{max}}|)\#(\Delta_{j_{\min} \rightarrow})} \). It comes from the part of the rescaling spring factors used for fixing horizontally the polymers (see §4.1). Let \( \mathbb{P}_{\mathcal{F}}^{j_{\min} \rightarrow} \) be one of the connected components of the multi-scale forest at scale \( j \), and \( \Delta_j^j \subset \Delta_{j_{\min} \rightarrow} \cap \mathbb{D}_j \) be its intervals of scale \( j \) with external legs. Then the rescaling spring factors of the corresponding low-momentum fields \( (T_{\psi_{\text{next}}}^{i_{\text{next}}} \rightarrow(j-1)(x_{\text{next}}))\text{, } n_{\text{ext}} = 1, 2, \ldots, N_{\text{ext}}(\mathbb{P}_{\mathcal{F}}^{j_{\min} \rightarrow}) \), yield a factor \( \prod_{n_{\text{ext}}=1}^{N_{\text{ext}}(\mathbb{P}_{\mathcal{F}}^{j_{\min} \rightarrow})} M^{\beta_{n_{\text{ext}}}^*} \) when going down from scale \( j \) to scale \( j - 1 \), where \( \beta_{n_{\text{ext}}}^* = \beta_{n_{\text{ext}}} \) or \( \beta_{n_{\text{ext}}} - 1 \), see §4.1, are such that \( \sum_{n_{\text{ext}}=1}^{N_{\text{ext}}(\mathbb{P}_{\mathcal{F}}^{j_{\min} \rightarrow})} \beta_{n_{\text{ext}}}^* \leq -D - |\omega_{\text{max}}^*| \).

Let us now prove 2. directly, since 1. is a weaker form of 2. Note that by the Cauchy-Schwarz type inequality \( \int |fgh| \leq (\int |f|^3 \int |g|^3 \int |h|^3)^{1/3} \), one
may separate low-momentum fields from high-momentum fields and from the fields produced at scale $j$, to which the previous lemma applies.\(^\text{10}\)

Let us first consider low-momentum fields. We use the same rescaling as in the proof of the previous lemma, namely, we consider the rescaled fields $n(\Delta^k_x)^{-\gamma}\tilde{\psi}_j^i$, with $k > j$. The $\Delta_{\text{min}}^{j-\to}$-connecting pairing associated to $(\mathcal{F}, \mathcal{G}^j)$ has links of two types:

(i) links due to pairings $\langle \psi^i_j \psi^{i'}_j \rangle$, or, more or less equivalently, cluster links $\mathcal{C}_\psi^j(i_\ell^a, x_{\ell^a}; i'_{\ell^a}, x'_{\ell^a})$ of scale $j$;

(ii) cluster links to the propagators $\mathcal{C}_\psi^a(i_\ell^a, x_{\ell^a}; i'_{\ell^a}, x'_{\ell^a})$ of scale $a \neq j$, $a = k > j$ in the specific case of low-momentum fields.

Cluster links of scale $k > j$ (or $h < j$) contribute a factor $\frac{1}{(1 + d^k(\Delta^k, (\Delta^k)^r))}$, which is required both for the bound on $I^j$ and for that on $I^k$. Since $r$ is arbitrary, one chooses it large enough and splits the above factor among the different scales of the vertices. On the other hand, the scaling of the propagators $\mathcal{C}_\psi^k$ or $\mathcal{C}_\psi^h$ is left for the computation of $I^k$ or $I^h$. Note the possible existence of chains of propagators of scale $k$ connecting two vertices with low-momentum fields of scale $j$; summing over all possible chains yields the same factor of order $\frac{1}{(1 + d^k(\Delta^k, (\Delta^k)^r))}$ as soon as $r > D$.

By Definition 5.6, the term between square brackets in eq. (5.6) is bounded up to a constant (see proof of Corollary 5.3 (3)) by $A_{\text{cluster}}(\Delta^k) + A_{\text{low}}(\Delta^k)$, where

\[
A_{\text{cluster}}(\Delta^k) := \sum_{i,i'=1}^d \sum_{k'=j}^k M\tilde{\beta}(k'-j) M\tilde{\beta}^r(k'-j) \sum_{\Delta' \in k'} \frac{1}{(1 + d^k(\Delta^k, (\Delta^k)^r))} \tag{5.35}
\]

where $\Delta^k'$ is the unique interval of scale $k'$ such that $\Delta^k' \supset \Delta^k$, and

\[
A_{\text{low}}(\Delta^k) := \sum_{i,i'=1}^d \sum_{k'=j}^k M\tilde{\beta}(k'-j) \sum_{k''=j}^{k'} M\tilde{\beta}^r(k''-j) \sum_{\Delta'' \in k''} \frac{1}{(1 + d^k(\Delta^k, (\Delta^k)^r))} \tag{5.36}
\]

The above sum, $\sum_{\Delta'' \in k''} \frac{1}{(1 + d^k(\Delta^k, (\Delta^k)^r))}$, is of order $M^D(k''-j)$, which

\(^{10}\)Of course, fields produced at scale $j$ may also be treated on an equal foot with high-momentum fields for instance.
yields

\[ A^{low}(\Delta^k) \leq K \sum_{i,i'=1}^d A^{low}_{\tilde{\beta}_i, \tilde{\beta}_{i'}} \cdot A^{low}_{\tilde{\beta}_i, \tilde{\beta}_{i'}} := \sum_{k'=j}^\rho M^{\tilde{\beta}_{(k'-j)} \sum_{k''=j}^{k'} M^{(\tilde{\beta}_{i'} + D)(k''-j)}, \]

(5.37)

while clearly \( A_{\text{cluster}}(\Delta^k) \) is finite since \( \tilde{\beta}_i, \tilde{\beta}_{i'} < 0 \).

Let us finish the proof with the assumption that \( D = 1 \). There are 3 different cases:
- either \( \beta_{i'} < -1 \); then \( \tilde{\beta}_{i'} = \beta_{i'} \) and \( \sum_{k''=j}^{k'} M^{(\beta_{i'}+1)(k''-j)} = O(1), \)
- or \( -1 \leq \beta_{i'} < -\frac{1}{2} \), resp. \( 0 \leq \beta_{i'} < \frac{1}{2} \); then \( \tilde{\beta}_{i'} = \beta_{i'} \), resp. \( \beta_{i'} - 1 \), and \( \sum_{k''=j}^{k'} M^{(\beta_{i'}+1)(k''-j)} = O(1 + \sum_{k''=j}^{k'} (k''-1)) \) otherwise;
- or \( -\frac{1}{2} \leq \beta_{i'} < 0 \), resp. \( \frac{1}{2} \leq \beta_{i'} < 1 \); then \( \tilde{\beta}_{i'} = \beta_{i'} - 1 \), resp. \( \beta_{i'} - 2 \), and \( \sum_{k''=j}^{k'} M^{(\beta_{i'}+1)(k''-j)} = O(1), \) while the sum over \( k' \) converges as in the first case.

The simpler case \( D \geq 2 \) is left to the reader (there are only 2 subcases: \( \beta < -D/2 \), or \( \beta \geq -D/2 \), the latter subcase to be split according to the value of \( |\beta + D/2| \)).

Consider now high-momentum fields, produced at a scale \( h < j \) (or \( h \leq j \)). By the same method, one ends up with the following quantity to bound instead of eq. (5.36):

\[ A^{high}(\Delta^j) := \sum_{i,i'=1}^d \sum_{h'=j_{\min}}^h M^{-(\beta_{i'}+\gamma_{i'})_{(j-h')}} \sum_{h''=j_{\min}}^{h'} M^{-(\beta_{i'}+\gamma_{i'})_{(j-h'')}} \sum_{\Delta'j \in \Delta^j} \frac{1}{(1 + d'((\Delta_{i}, \Delta_{i'})))^{r'}}, \]

(5.38)

where \( \Delta_{i}, \Delta_{i'} \) range over restriction intervals, plus the finite term \( A_{\text{cluster}}(\Delta^h) \) due to cluster links of scale \( h \) as before. Now \( \sum_{\Delta'j \in \Delta^j} \frac{1}{(1 + d'((\Delta_{i}, \Delta_{i'})))^{r'} < K} \) and \( \beta_{i} + \gamma_{i}, \beta_{i'} + \gamma_{i'} > 0 \) by Lemma 5.8 hence \( A^{high}(\Delta^j) \) is bounded by a constant.

\[ \square \]
Note that a small part of the rescaling spring factor $M \tilde{\beta}_i (k-j) (\tilde{\beta}_i < 0)$ for low-momentum fields may be used to obtain a factor $M^{-\varepsilon} < 1$ for $\varepsilon > 0$ small enough per interval $\Delta$ belonging to a fixed polymer $P$ – in particular in empty intervals where there is no field. This simple remark is essential in the sequel since various estimates yield a factor $1 + O(\lambda^\kappa)$ per interval, which may be compensated by this (not so) small factor $M^{-\varepsilon}$. On the other hand, to each vertex or equivalently to each non-empty interval – or to each field – is associated a factor of order $\lambda^\kappa$, which may be arbitrarily small. This is a general principle for the bounds to come now.

5.1.3 Gaussian bounds for polymers

It remains to be seen how these Gaussian estimates, valid for each scale, combine to give estimates for the (Mayer-extended) polymer evaluation functions defined in subsection 2.4. Note that (rescaled) low-momentum field averages have been left out; the domination estimates in subsection 5.2 prove that it is possible in the case of the $(\phi, \partial \phi, \sigma)$-model to bound them while leaving a small factor per field, $\lambda^\kappa_0 + \kappa^\prime_0 > 0$, negligible with respect to $\lambda^\kappa_0$.

We take this into account for our next Theorem by choosing a small enough exponent $\kappa_0$.

**Theorem 5.1** Fix some reference scale $j_{\text{min}} \geq 0$ and some exponent $\kappa > 0$.

Let $\mathcal{P}_0^{j_{\text{min}}}(\Delta_{j_{\text{min}}})$ be the set of vacuum (Mayer-extended) polymers down to scale $j_{\text{min}}$ containing some fixed interval $\Delta_{j_{\text{min}}}$ of scale $j_{\text{min}}$. Let $\text{Eval}_{\kappa_0} (P)$, $P \in \mathcal{P}_0^{j_{\text{min}}}(\Delta_{j_{\text{min}}})$ be the sum over all multi-scale splitting of vertices into cluster forests $F^j$ extending over $P$ and monomials $G^j$ of the product integrals $\prod_{j=j_{\text{min}}}^0 \int d\mu(\psi^j) \prod_{\ell \in L(F^j)} C^j(\psi^j; i^j_x, x^j_x; i^{j^\prime}_x, x^{j^\prime}_x) G^j$, all fields in $G^j$ being accompanied as before with the factor $\lambda^\kappa_0$. Then

$$\sum_{P \in \mathcal{P}_0^{j_{\text{min}}}(\Delta_{j_{\text{min}}}); \# \{\text{vertices of } P=\emptyset\}} \left| \text{Eval}_{\kappa_0 + \kappa_0^\prime} (P) \right| \leq (K \lambda^\kappa_0)^v. \quad (5.39)$$

In particular, for $\lambda$ small enough,

$$\sum_{P \in \mathcal{P}_0^{j_{\text{min}}}(\Delta_{j_{\text{min}}})} \left| \text{Eval}_{\kappa_0 + \kappa_0^\prime} (P) \right| < \infty. \quad (5.40)$$

**Proof.**

Let $\mathcal{P}_0^{j_{\text{min}}}(\Delta_{j_{\text{min}}})$ be the set of vacuum (Mayer-extended) polymers down to scale $j_{\text{min}}$ with fixed set of intervals $\Delta_{j_{\text{min}}}$ as in the previous Lemma. Note that the horizontal fixing scaling factor $M^{-(D+|\omega^*_{\text{max}}|)} \# \Delta_{j_{\text{min}}}$

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makes it possible to sum over all inclusion links of the polymers. Namely, each inclusion link $\Delta^j \subset \Delta^{j-1}$—implying necessarily some $\ell$-derivative in $\Delta^j$—produces a factor $M^D$ due to the choice of $\Delta^j$ among the $M^D$ intervals $\Delta \in \mathbb{D}^j$ such that $\Delta \subset \Delta^{j-1}$, which is compensated by the factor $M^{-(D+|\omega_{\text{max}}^0|)}$ attached to $\Delta^j$.

Hence it suffices to prove eq. (5.39) for $\mathcal{P}$ ranging in the set $\mathcal{P}_0^{\text{jmin} \rightarrow (\Delta_1^{\text{jmin} \rightarrow})}$.

Let $A_0^{\text{jmin} \rightarrow (\Delta_1^{\text{jmin} \rightarrow})} := \sum_{\mathcal{P} \in \mathcal{P}_0^{\text{jmin} \rightarrow (\Delta_1^{\text{jmin} \rightarrow})}} \#\{\text{vertices of } \mathcal{P}\} \cdot \left| \text{Eval}_{\kappa_0 + \kappa_0^0}(\mathcal{P}) \right|$. Split the small factor per vertex into $\lambda^{\kappa_0} \lambda^{\kappa_0^0}$, and set apart $\lambda^{\kappa_0}$ to get a global homogeneity factor $\lambda^{\kappa_0^0}$. Then

$$A_0^{\text{jmin} \rightarrow (\Delta_1^{\text{jmin} \rightarrow})} \leq \lambda^{\kappa_0^0} \sum_{\mathcal{P} \in \mathcal{P}_0^{\text{jmin} \rightarrow (\Delta_1^{\text{jmin} \rightarrow})}} \left| \text{Eval}_{\kappa_0}(\mathcal{P}) \right|,$$

(5.41)

where the total number of vertices is now unrestricted.

Let $\mathcal{P} \in \mathcal{P}_0^{\text{jmin} \rightarrow (\Delta_1^{\text{jmin} \rightarrow})}$. Consider some product of fields of type $q$ produced in some interval $\Delta^j$ of scale $j$, $\psi_{I_q}(x_{\Delta^j}, \tau)$ or $\psi_{I_q \setminus (\{i_t\})}(x_{\ell})$, interpreted as some pairing of $\psi_{I_q}(x_{\ell})$ with $\psi_{I_q^l}(x_{\ell})$, and:

- choose some non-empty subset $I_{\text{high}}^j = (i_{q'}, \ldots, i_q) \subset I_q$;
- choose some high-momentum scales $(k_i)_{i \in I_{\text{high}}^j}$, with $j \leq k_i \leq \cdots \leq k_{i_q}$ as in Hypothesis 5.7, and restriction intervals $(\Delta^j_i)_{i \in I_{\text{high}}^j}$, $\Delta^j_i \subset \Delta^j$;
- letting $I_{\text{low}}^j := I_q \setminus I_{\text{high}}^j$, choose some low-momentum scales $(h_i)_{i \in I_{\text{low}}^j}$, $h_i < j$.

Then

$$\psi_{I_q} := \sum_{I_{\text{high}}^j \subset I} \sum_{(k_i)} \sum_{\Delta^j_i} \left[ \prod_{i \in I_{\text{high}}^j} \text{Res}^{\Delta^j_i}_{k_i} \psi_{I_q^l}^{k_i} \right] \cdot \left[ \prod_{i \in I_{\text{low}}^j} \psi_{I_q^l}^{h_i} \right]$$

(5.42)

is the decomposition of $\psi_{I_q}$ into all possible splittings. Any given splitting of $\psi_{I_q}$ is supported on an $M$-adic interval $\cap_{i \in I_{\text{high}}^j} \Delta^j_i$ of size bounded by

$$M^{-k_{i_q}^D} = M^{-D} \cdot M^{-(k_{i_{q'}} - j)^D} \cdots M^{-(k_{i_q} - k_{i_{q-1}} - 1)^D} \leq M^{-D}M^{-(\gamma_{i_{q'}} + \cdots + \gamma_{i_q})(k_{i_{q'}} - j)} \cdots M^{-(\gamma_{i_q}(k_{i_q} - k_{i_{q-1}}))} = M^{-D}M^{-\sum_{i \in I_{\text{high}}^j} \gamma_i(k_i - j)}$$

(5.43)

by Lemma 5.8.
Fixing $k_i$, letting $j$ range over all scales $\leq k_i$ and changing notations $(k_i, j) \to (j, h)$ yields the spring factors $M^{-\gamma^{th}_{j}(j-h)}$, $M^{-\gamma^{th}_{j}(j-h)}$, $M^{-\gamma^{th}_{j}(j-h)}$ of eq. (5.30). The remaining factor $M^{-jD} = |\Delta^j|$ may be rewritten as $M^{j} \sum_{r<|\Delta^j|} \delta^{r}$ which is distributed between the different fields, $\psi^{j_i} \to \psi^{j_i}_{\text{rescaled}} = M^{j_i} \psi^{j_i}_{\text{rescaled}}$ as in §5.1.2. Recall from the end of the preceding paragraph that each interval of $\mathbb{P}$ comes with a factor $M^{-\varepsilon} < 1$. Hence, letting $j_{\text{max}}$ be the maximal scale of a given polymer $\mathbb{P}$ one has by Lemma 5.9

$$\sum_{\mathbb{P} \in \mathbb{P}_{0}^{j_{\text{min}} \to j_{\text{max}}}} \left| \text{Eval}_{\Omega}^{\varepsilon} (\mathbb{P}) \right| \leq \sum_{j_{\text{max}} = j_{\text{min}}}^{j_{\text{max}} \to j_{\text{min}}} \prod_{j=j_{\text{min}}}^{j_{\text{max}}} \left( M^{-\varepsilon} + \sum_{v \geq 1} H^{\text{rescaled}}_{j} (v; \Delta_{j_{\text{min}}} \to j_{\text{max}}) \right) \leq \sum_{j_{\text{max}} = j_{\text{min}}}^{j_{\text{max}} \to j_{\text{min}}} (M^{-\varepsilon} + K' \lambda \kappa_{0}^{\varepsilon} j_{\text{max}} - j_{\text{min}} + 1 < \infty \quad (5.44)$$

if $\lambda$ if chosen small enough so that in particular (with the constant $K$ of eq. (5.34)) $\sum_{v \geq 1} (K \lambda \kappa_{0}^{\varepsilon})^v \leq K' \lambda \kappa_{0}^{\varepsilon} < 1 - M^{-\varepsilon}$.

5.1.4 Combinatorial factors

The last point – for the Gaussian part of the final bounds – is to control the combinatorial factors due to the horizontal and vertical cluster expansions. Let us show briefly how to do this.

As a general rule, differentiating a product of $n$ fields yields $n$ terms (by the Leibniz formula). This implies supplementary combinatorial factors when estimating the polymer evaluation functions $F(\mathbb{P})$, compared to the estimates of Theorem [5.1]. Consider the $O(n(\Delta^j))$ derivations in a given interval $\Delta^j$ due to horizontal/vertical cluster expansion at scale $j$. A field produced by one such derivation may be acted upon by another one in the same interval, yielding a local factorial $(O(n(\Delta^j)))^{O(n(\Delta^j))}$. Otherwise, derivations in $\Delta^j$ act on fields produced at an earlier stage in some interval $\Delta^k \subset \Delta^j$ belonging to the same polymer $\mathbb{P}$ as $\Delta^j$. Integrating over these, and borrowing some small power of $\lambda$ from one of the differentiated fields, yields at most an averaged factor in $\Delta^j$ of order $K := \frac{1}{|\Delta^j|} \lambda^\kappa \sum_{k>j} \sum_{\Delta^h \subset \Delta^j \cap \Delta^k} \int_{\Delta^h} dx$. Once again, there are $O(n(\Delta^j))$ such derivations. One may always gain a local factorial $\frac{1}{n(\Delta^j)^{\kappa}}$ to some arbitrary power (see proof of Lemma [5.5]); using $\frac{1}{n(\Delta^j)^{\kappa}} \leq e^K$, and
multiplying over all scales $j$ and all intervals $\Delta^j \in \mathbb{P} \cap \mathbb{D}^j$, one obtains

$$\exp \lambda^\kappa \sum_k \sum_{\Delta^k \in \mathbb{P} \cap \mathbb{D}^k} \sum_{j<k} M^{-(k-j)},$$

hence a factor of order $1 + O(\lambda^\kappa)$ per interval $\Delta \in \mathbb{P}$, compensated by some factor $M^{-\varepsilon}$ as explained at the end of §5.1.2.

5.2 Domination bounds

Unlike Gaussian bounds, which are rather sophisticated, these are essentially based on the simple fact that

$$|x|e^{-A|x|} = A^{-1}(A|x|)e^{-A|x|} \leq A^{-1}$$ if $A > 0$.

Lemma 5.10 (domination) Let $\psi$ be a multiscale Gaussian field with scaling dimension $\beta$. Then

$$|(T\psi)^{(k-1)}(\Delta^k)|^n \exp -\lambda^\kappa M^{m\beta k} \cdot \frac{1}{|\Delta^k|} \int_{\Delta^k} ((T\psi)^{(k-1)})^m (x) \, dx \leq K^n n^{n/m} \lambda^{-\kappa n/m} M^{-n\beta k}.$$  \hfill (5.46)

Proof. Let $u := (T\psi)^{(k-1)}(\Delta^k)$ and $v := \frac{1}{|\Delta^k|} \int_{\Delta^k} ((T\psi)^{(k-1)})^m (x) \, dx$; by H"older’s inequality, $|u| \leq v^{1/m}$, so that

$$|u|^n e^{-\lambda^\kappa M^{m\beta k} v} \leq \left( \frac{\lambda^\kappa}{n} M^{m\beta k} \right)^{-n/m} \cdot \left( \frac{\lambda^\kappa}{n} M^{m\beta k} v \right)^{1/m} \exp -\lambda^\kappa M^{m\beta k} v \leq K^n n^{n/m} \lambda^{-\kappa n/m} M^{-n\beta k}. $$ \hfill (5.47)

Example \((\phi, \partial \phi, \sigma)\)-model. Lemma 5.10 implies in particular the following four kinds of low-momentum field domination. The notation $\text{Av}_{\mathcal{L}<\mathcal{L}'}$ means that averaged low-momentum fields coming from a vertex of $\mathcal{L}$ are dominated by exp $\mathcal{L}'$.

(i) $\text{Av}_{\mathcal{L}_4<\delta \mathcal{L}_4}$ terms

Consider low-momentum fields $\sigma$ produced at a scale $k$ by letting some derivation $\frac{\delta}{\delta \sigma}$ or $\partial_t$ (due resp. to horizontal and vertical cluster expansions of scale $k$) act on $\mathcal{L}_4$. When $\rho - k$ is large enough, they will be dominated by the part of the counterterm $\delta \mathcal{L}_4$ which is coupled to $b^{\rho-1}$. Eq. (5.46) yields, for $n \geq N$ (using $1 - t^2 \geq (1 - t)^2$ for $t \in [0, 1]$)
The lonely term one dominates by some fraction (say, one tenth) of the boundary term, \( \rho \), when replacing in the exponential \( \lambda^2 M^{(1-\alpha)k} \) in the Taylor-Lagrange expansion as in §3.3, with \( n = N_{\text{ext,max}} + O(n(\Delta)) \). The number of fields \( N = N_\sigma(\Delta) \) is \( O(n(\Delta)) \) too, so (with some care) \( n \geq N \). This is precisely the reason why we chose to Taylor expand to order \( N_{\text{ext,max}} + O(n(\Delta)) \) and not simply to order \( N_{\text{ext,max}} \). The other terms in the Taylor-Lagrange expansion have \( t^k_\Delta = 0 \).

Replacing in the exponential \( \rho \lambda^2 M^{(1-4\alpha)k} \) by the term \( b^\rho - 1 \approx \lambda^2 M^{(1-4\alpha)(\rho-1)} \), one gains a supplementary “petit facteur” \( (M^{-1/2}(1-4\alpha)(\rho-1-k)) \).

In the following lines, we shall simply set \( n = N \).

(ii) \( \Lambda \mathcal{L}_4 \ll \mathcal{L}_{12} \) and \( \Lambda \nu b^\rho(T\sigma) \ll \mathcal{L}_{12} \) terms

When \( \rho - k \) is too small, the “petit facteur” is not small any more. One dominates by some fraction (say, one tenth) of the boundary term, \( \frac{1}{10} \delta \mathcal{L}_{12} \) instead. Again, eq. (5.46) yields (using \( 1 - (t^k_\Delta)^6 \geq (1 - t^k_\Delta)^6 \))

\[
\frac{(1 - t^k_\Delta)^n}{n!} \left| \lambda(T\sigma)^\rightarrow(k-1)(\Delta^k) \right|^N e^{-\lambda^2(1-(t^k_\Delta)^2)M^{(1-4\alpha)k} \int_{\Delta^k} (T\sigma)^\rightarrow(k-1)|^2(x)dx} \leq \left( KM^{2\alpha k} n^{-\frac{1}{2}} \right)^N.
\]

(5.48)

Note the factor \( \frac{(1 - t^k_\Delta)^n}{n!} \) coming from the rest term in the Taylor-Lagrange expansion as in §3.3, with \( n = N_{\text{ext,max}} + O(n(\Delta)) \). The number of fields \( N = N_\sigma(\Delta) \) is \( O(n(\Delta)) \) too, so (with some care) \( n \geq N \). This is precisely the reason why we chose to Taylor expand to order \( N_{\text{ext,max}} + O(n(\Delta)) \) and not simply to order \( N_{\text{ext,max}} \). The other terms in the Taylor-Lagrange expansion have \( t^k_\Delta = 0 \).

Replacing in the exponential \( \rho \lambda^2 M^{(1-4\alpha)k} \) by the term \( b^\rho - 1 \approx \lambda^2 M^{(1-4\alpha)(\rho-1)} \), one gains a supplementary “petit facteur” \( (M^{-1/2}(1-4\alpha)(\rho-1-k)) \).

In the following lines, we shall simply set \( n = N \).

The lonely term \( b^\rho(t^k_\Delta)^2((T\sigma)^\rightarrow\rho)^2(x) \) in the interaction may be similarly dominated by using the exponential of the term of scale \( \rho \) in \( \mathcal{L}_{12} \), \( M^{-(12\alpha-1)\rho} \lambda^3 ||(T\sigma)^\rightarrow\rho(x)||^6 \).

Replacing in the exponential \( M^{(1-12\alpha)k} \) by the term \( M^{(1-12\alpha)\rho} \) present in \( \delta \mathcal{L}_{12} \) (note that \( 1 - 12\alpha < 0 \)), one loses this time a supplementary large factor \( M^{\frac{12\alpha-1}{\rho}(\rho-k)} \). However, it is accompanied by a small factor \( \lambda^\frac{1}{2} \) per field.

Assume \( \rho - k = 0, 1, \ldots, q \). We fix \( q \) such that \( \lambda^\frac{1}{2} M^{\frac{12\alpha-1}{6}} = \lambda^{1/4} \), i.e. \( q = 3\frac{\ln(1/\lambda)}{2(12\alpha-1)} \ln M \). Thus the ”petit facteur” in the preceding \( \Lambda \mathcal{L}_4 \ll \delta \mathcal{L}_4 \) case – where \( \rho - k > q \) – is at most \( \lambda^k \), where \( \kappa = \frac{3(1-4\alpha)}{4(12\alpha-1)} \).
(iii) $\text{Av}_{\mathcal{L}_{12}<\mathcal{L}_{12}}$-terms

Consider low-momentum fields $\sigma$ produced from some vertex $\Delta \rho'$ by letting some derivation act on $\delta \mathcal{L}_{12}$ this time. It produces $i \leq 5$ low-momentum $\sigma$-fields, accompanied by $\lambda^3$. Again, eq. (5.46) yields, by dominating by one tenth of the boundary term, $\frac{1}{10} \delta \mathcal{L}_{12}$, and leaving out once and for all the $t$-coefficients,

$$\frac{1}{n!} \left| \lambda^3 ((T\sigma)^{-(\rho'-1)}(\Delta \rho'))^i \right| e^{-\frac{1}{n!} \lambda^3 M^{(1-12\alpha)} \rho \int_{\Delta \rho'}^\rho |(T\sigma)^{-(\rho'-1)}(\rho')| dx} \leq \left( K' \lambda^{3/4} (M^{\frac{12\alpha-1}{6}} \rho') \right)^{\frac{1}{6n}}.$$

(5.50)

The corresponding vertex produces after integration a factor $|\Delta \rho'| M^{-(12\alpha-1)} \rho = M^{-12\alpha}.M^{1-\rho'}$, multiplied by $M^{2\alpha} \rho M^{\frac{1}{6}} (\rho') M^{\frac{12\alpha-1}{6}} (\rho')$ per high-momentum field, and by

$$M^{2\alpha} \rho M^{\frac{12\alpha-1}{6}} (\rho') \lambda^{3/4-1/2} = M^{2\alpha} \rho M^{\frac{1}{6}} (\rho') \lambda^{3/4-1/2}$$

per low-momentum field. Thus all together one has obtained a small factor $\leq \lambda^{1/2}$ per vertex, to be shared between the six fields, and $M^{-\frac{12\alpha-1}{6}} (\rho') \leq 1$ per high-momentum field.

(iv) $\text{Av}_{\delta \mathcal{L}_{4} < \delta \mathcal{L}_{4}}$

Consider low-momentum fields $\sigma$ produced in an interval $\Delta^k$ of scale $k$ by letting some derivation act on $\delta \mathcal{L}_{4}$ or on the counterterm of scale $\rho$, $-\frac{b^\rho}{2} (T(\rho'))^2 ((T\sigma)^{-(\rho')}(\rho'))^2$. It produces at most one low-momentum $\sigma$-field, accompanied by $b^\rho = (b^\rho / \lambda) \cdot \lambda$ instead of $\lambda$.

Again, eq. (5.46) yields, by dominating as in (i) by the part of the counterterm $\delta \mathcal{L}_{4}$ which is coupled to $b^\rho$, a factor $O(M^{2\alpha} \cdot M^{-\frac{1}{2}} (1-4\alpha) (\rho-k))$ per field, alas multiplied by $b^\rho / \lambda \approx \lambda M^{\rho}(1-4\alpha)$. The rest of the argument goes as in subsection 5.1.4 – a general argument called ”aplatissement du fortement connexe” in (colloquial) French. The factor $M^{-\frac{1}{2}} (1-4\alpha) (\rho-k)$ may be simply bounded by 1.

Such terms may be produced only in intervals $\Delta \rho' \subset \Delta^k$ such that $\Delta \rho' \in \mathbb{P}$. Integrating over all such intervals – and taking into account the $M^{-2\alpha}$-scaling of the high-momentum field $\sigma^k$ left behind – yields at most $\lambda K := \lambda \sum_{\rho' \geq k} M^{-4\alpha} (\rho'-k) \# \{ \mathbb{P} \cap D \rho' \}$ per low-momentum
field produced, all together \((\lambda^{1/2})^n \cdot (\lambda^{1/2})^n\). One may always gain a local factorial \(\frac{1}{n!}\); using \(\frac{K^n}{n!} \leq e^K\) and multiplying over all scales \(k\) and all intervals \(\Delta \in \mathbb{P} \cap \mathbb{D}^k\), one gets

\[
\lambda^{v/2} \cdot \exp \lambda^{1/2} \sum_{\rho'} \sum_{\Delta' \in \mathbb{P}} \sum_{k \leq \rho'} M^{-4\alpha(k')^{-k}}
\]

where \(v\) is the number of vertices where such low-momentum fields have been produced, hence a factor \(\lambda^{1/2}\) per vertex and a factor of order \(1 + O(\lambda^{1/2})\) per interval \(\Delta \in \mathbb{P}\).

### 5.3 Final bounds

The main Theorem is the following.

**Theorem 5.2**

1. There exists a constant, positive two-by-two matrix \(K\) such that
\[
b^j = K\lambda^{2}M^j(1-4\alpha)(1 + O(\lambda)).
\]

2. There exists a constant \(K'\) such that the Mayer bound of Proposition 2.12 for the scale \(j\) free energy,
\[
|f^{j \rightarrow \rho}(\lambda)| \leq K'(1 + O(\lambda))
\]

holds uniformly in \(j\).

**Proof.**

Let us first prove eq. (5.52) for \(b^j\).

Consider a product \((G\text{-monomial}) \times \text{product of propagators}\) as in subsection 3.3, written generically as \(GC\). Multiply it by a product of averaged low-momentum fields of one of the four types \(Av\), \(Av_{\mathcal{L}_1<\mathcal{L}_4}\), \(Av_{\mathcal{L}_1<\mathcal{L}_4}\), or \(Av_{\mathcal{L}_4<\mathcal{L}_4}\), see subsection 5.2, generically written as \(Av_{\text{low}}\).

We make the following induction hypothesis:

**Induction hypothesis.** \(\tilde{b}^k = K(1 + O(\lambda))\) for all \(k > j\) for some scale-independent constant \(K\), where \(\tilde{b}^k := \lambda^{-2}M^{-1-4\alpha)k}b^k\) is the rescaled mass counterterm of scale \(k\).
We shall soon see how to compute the constant $K$. For the time being, we must bound
\[
\int d\mu_s(\phi) d\mu_s(\sigma) \sum_{P \in \mathcal{P}_{\sigma,\sigma}^{j,\Delta}} \sum (G,C) \frac{\sum_{P \in \mathcal{P}_{\sigma,\sigma}^{j,\Delta}} (\int d\mu_s(\phi) d\mu_s(\sigma) |GC|^2)^{1/2}}{\lambda^2 \sum_{k \geq j} \int V C_{\phi_2}^k (x,y) (-\partial^2) C_{\phi_1}^j (x,y) dy}.
\]

where $|\mathcal{P}|$ is the support of the polymer $\mathcal{P}$ with two external $\sigma$-legs, in the notation of Proposition 2.11.

**First step (domination of the mass counterterm).**

Note first that the term between square brackets in eq. (5.54) is negative when $X := \lambda M^{-2\kappa} \| (T\sigma) \rightarrow^\rho \|$ is small. Up to unessential coefficients, it is equal to $M^\rho (\lambda^{-3} X^6 - (t_x^0)^2 X^2)$, which is minimal, of order $M^\rho \lambda^{9/2}$ for $t_x^0 \approx 1$, which is the worst case – when $X$ is of order $\lambda^{3/4}$. The factor $t_x^0$ in front of $(T\sigma) \rightarrow^\rho$ selects the intervals in $\mathbb{D}^\rho$ belonging to the polymer. Hence the exponential in eq. (5.54) is bounded by $e^{K \int |P \cap \mathcal{D}_\rho| M^\rho \lambda^{9/2} / 2 dx}$, all together a factor of order $1 + O(\lambda^{9/2})$ per interval $\Delta \in \mathcal{P} \cap \mathbb{D}^\rho$ of the polymer with $t_\Delta^0 \neq 0$.

**Second step (domination of low-momentum fields).**

Split $Av_{\text{low}} e^{-\int |P| (L_4 + \delta L_4 + \frac{1}{2} L_{12})( ; t)(x) dx}$ from the expression in eq. (5.54). As shown in subsection 5.2, these produce a small factor of order $\lambda^\kappa$, $\kappa > 0$ per field. More precisely, any $\kappa < \inf \left( \frac{1}{4}, \frac{3(1-4\alpha)}{4(12\alpha-1)} \right)$ is suitable. There remains (by using the Cauchy-Schwarz inequality) to bound $\sum_{P \in \mathcal{P}_{\sigma,\sigma}^{j,\Delta}} (G,C) \left( \int d\mu_s(\phi) d\mu_s(\sigma) |GC|^2 \right)^{1/2}$.

**Third step (computation of $b^j$).**

Let us now estimate $b^j$ by means of our induction hypothesis and of the Gaussian bounds of subsection 5.1. Consider for instance the diagonal term $b^j_{+,+}$. The terms with the fewest number of vertices are:

- the term with 0 vertex obtained by applying twice $\partial \sigma$ to the counterterm $\delta L_4$, namely, $b^j_{+,+}$ (by definition);
- the polymers with 2 vertices, see Fig. 4 which sum up to

\[
\lambda \sum_{k \geq j} \int V C_{\phi_2}^k (x,y) (-\partial^2) C_{\phi_1}^j (x,y) dy =: \lambda^2 M^j (1-4\alpha) K_V,
\]

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Figure 4: Main part of the mass counterterm of scale $j$.

where

$$K_V := M^{-j(1-4\alpha)} \int_V C_{\phi^j}^{j \to \rho}(x,y)(-\partial^2)C_{\phi^j}(x,y)dy$$

$$\to |V| \to \infty K := M^{-j(1-4\alpha)} \int |\xi|^{-4\alpha} \chi^j(\xi)\chi^{j \to \rho}(\xi)d\xi = \int |\xi|^{-4\alpha} \chi^1(\xi)\chi^{1 \to 2}(\xi)d\xi,$$

(5.55)

a scale-independent quantity. As in section 1, the non-diagonal counterterm $b^j_{+,+}$ or $b^j_{-,+}$ may be computed in the same way, yielding in the end a scale-independent positive two-by-two matrix.

More complicated polymers are of order $M^j(1-4\alpha)\lambda^2 g(\lambda^2 b^j_{k>j},\lambda)$, where $\tilde{b}^j \equiv \lambda^2 M^{-j(1-4\alpha)} b^j$ is the rescaled mass counterterm as in the induction hypothesis. The function $g$ is $C^\infty$ in a neighbourhood of 0 and vanishes at 0. Hence, by the implicit function theorem, $b^j = KM^j(1-4\alpha)(1+O(\lambda))$ as in Proposition 2.11.

The bound for the scale $j$ free energy $f^{j \to \rho}$ is now straightforward. □

Bounds for $n$-point functions are easy generalizations of the preceding Theorem. Consider for instance the 2-point function $\langle |F_{\phi^j_1}(\xi)|^2 \rangle_\lambda$, with $M^j \leq |\xi| \leq M^{j+1}$. By momentum conservation, and by definition of the Fourier partition of unity, see subsection 211, this is equal to the sum over $j_1, j_2 = j, j \pm 1$ of $\langle F_{\phi^{j_1}_{1}}(\xi)F_{\phi^{j_2}_{1}}(-\xi) \rangle_\lambda$. The term of order 0 in $\lambda$ is given by the Gaussian evaluation $\mathbb{E} \left[ F_{\phi^{j_1}_{1}}(\xi)F_{\phi^{j_2}_{1}}(-\xi) \right]$. Further terms involve at least one $\sigma$-propagator with a small factor (see Lemma 1.14) of order $\inf(1, \frac{M^{j(1-4\alpha)}}{b^j}) \leq K \inf(1, \lambda^{-2}M^{-(\rho-j)(1-4\alpha)})$ which goes to 0 when $\rho \to \infty$. 73
6 List of notations and glossary

Cluster expansions imply the use of many indices and letters. Let us summarize here some of our most important conventions, in the hope that this will help the reader not to get lost.

1. Quite generally, $\psi = (\psi_1(x), \ldots, \psi_d(x))$ is a $d$-dimensional field living on $\mathbb{R}^D$, $D \geq 1$, with scaling dimensions $\beta_1, \ldots, \beta_d$. Roughly speaking, for probabilists, $\beta$ is the Hölder continuity index, at least when $\beta > 0$; physicists usually call scaling dimension $-\beta$. Most of the model-independent results here are valid for arbitrary $D$. The Fourier transform is denoted by $F$.

2. If $\psi = \psi(x)$ is a Gaussian field, then its scale decomposition (see section 1) reads $\psi = \sum_{j \geq 0} \psi^j$. The low-momentum, resp. high-momentum field with respect to scale $j$ is denoted by $\psi^j = \sum_{h \leq j} \psi^h$, resp. $\psi^j = \sum_{h \geq j} \psi^h$. All through the text, we observe the following convention: if $h, j, k$ are scale indices, then $h \leq j \leq k$; any primed scale index (for instance $j', \rho', \ldots$) is less than the original scale index $j, \rho, \ldots$ Secondary fields (i.e. low-momentum fields $\psi$ minus their average) are denoted by $\delta \psi$. We also introduce restricted high-momentum fields, see section 1, denoted by $\text{Res} \psi$.

3. (products of fields) If $\psi = (\psi_1(x), \ldots, \psi_d(x))$ is a $d$-dimensional field, and $I = (i_1, \ldots, i_n) \in \{1, \ldots, d\}^n$ ($n \geq 1$) is a multi-index, we denote by $\psi_I$ the product of fields $\psi_I(x) := \psi_{i_1}(x) \ldots \psi_{i_n}(x)$. The interaction Lagrangian $L_{\text{int}}$ is written in general as $\sum_{q=1}^{\gamma} K_q \lambda^{\kappa_q} \psi_{I_q}$ for some constants $K_q$ and exponents $\kappa_q$. Cluster expansions produce propagators and products of fields which are linear combinations of monomials (also called: $G$-monomials), generically denoted by $G$.

4. (constants) $K$ is a constant depending only (possibly) on the details of the model, such as the degree of the interaction, the scaling dimension of the fields... It may vary from line to line. $M > 1$ is the base of the scale decomposition; it is an absolute constant, whose value is unimportant. $\gamma$ is some constant $> 1$, or sometimes simply $\geq 1$. $N_{\text{ext,max}}$ is such that every Feynman diagram with $\geq N_{\text{ext,max}}$ external legs is superficially convergent; this perturbative notion also plays a central rôle in constructive field theory. Our model $(\phi, \partial \phi, \sigma)$ has $N_{\text{ext,max}} = 4$.

5. (variables) The maximum scale index is $\rho$. Other scale indices are denoted by $h, j, k$, or any of those with primes. Indices $i$ are summation
indices, with finite range, used in various contexts, for instance for the field components. The parameters of the horizontal, resp. Mayer, resp. vertical (also called momentum-decoupling) cluster expansion, are denoted by $s$, resp. $S$, resp. $t$. The scale of an interval $\Delta$ is denoted by $j(\Delta)$. If $\Delta$ is an interval, then $n(\Delta)$ is the coordination number of $\Delta$ inside the tree defined by the horizontal cluster expansion, while $N(\Delta)$, resp. $N_i(\Delta)$ is the total number of fields, resp. the number of fields $\psi_i$ located in the interval $\Delta$. $\tau$ stands for a number of derivations, usually with respect to the $t$-parameters, in which case it is assumed to be $\leq N_{\text{ext,max}} + O(n(\Delta))$. 
Glossary

accumulation of low-momentum fields, §2.3
averaged low-momentum field, §1.1
boundary term (interaction), §4
colored polymer,
    see: Mayer-extended polymer
counterterm, §2.4
degree of divergence, §3.1
dropping scale, §2.3
G-monomial, §2.2
high-momentum field, §1.1
inclusion link, §2.3
low-momentum field, §1.1
Mayer-extended polymer, §2.4
multiscale cluster expansion, §2.3
overlap, §2.4
polymer, §2.3
production scale, §2.3
renormalized coupling constant, §2.4
rescaling spring factor, §1.1, §3.1, §5.1.2
secondary field, §1.1
split vertex, §5.1.2
vertical link, see: inclusion link
Wick’s formula, §5.1.1

aplatissement du fortement connexe,
see: combinatorial factors
Brydges-Kennedy-Abdesselam-
Rivasseau formula, §2.1
combinatorial factors, §5.1.4
domination, §3.2
free falling scale, §2.3
harmonizable representation of fBm, §1.2
horizontal cluster expansion, §2.1
local part, §2.4
Mayer expansion, §2.4
momentum-decoupling expansion, §2.3
multiscale Gaussian field, §1.1
petit facteur par carré, §5.1
power-counting, §3.1
quasi-local multiscale Feynman diagram, §3.1
renormalized diagram, §3.1
restricted high-momentum field, §1.1
singular part of the Lévy area $A^\pm$, §1.2
vertical cluster expansion,
    see: momentum-decoupling expansion

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