ASYMPTOTIC BEHAVIOUR OF SINGULAR SOLUTION OF THE FAST DIFFUSION EQUATION IN THE PUNCTURED EUCLIDEAN SPACE

KIN MING HUI AND JINWAN PARK

ABSTRACT. We study the existence, uniqueness, and asymptotic behaviour of the singular solution of the fast diffusion equation \( u_t = \frac{n-1}{m} \Delta u^m, \ u > 0, \) in \((\mathbb{R}^n \setminus \{0\}) \times (0, \infty), \) \( u(x,0) = u_0(x) \) in \( \mathbb{R}^n \setminus \{0\}, \) which blows up at the origin for all time \( t > 0. \) For \( n \geq 3, 0 < m < \frac{n-2}{2}, \) \( \beta < 0 \) and \( \alpha = \frac{2d}{n}, \) we prove the existence and asymptotic behaviour of singular eternal self-similar solution of the fast diffusion equation in \((\mathbb{R}^n \setminus \{0\}) \times \mathbb{R} \) of the form \( U_f(x, t) = e^{-\lambda t} f_\beta (e^{\beta t} x), \ x \in \mathbb{R}^n \setminus \{0\}, t \in \mathbb{R}, \) where \( f_\beta \) is a radially symmetric function satisfying

\[
\frac{n-1}{m} \Delta f^m + \alpha f + \beta x \cdot \nabla f = 0 \text{ in } \mathbb{R}^n \setminus \{0\},
\]

with \( \lim_{r \to 0} \frac{f(\rho r)}{\rho^{n-2}} = \frac{2n-4+2m}{n-2} \) and \( \lim_{r \to \infty} f(r) = \lambda \frac{1}{2} e^{-\frac{n-2}{2}}, \) for some constant \( \lambda > 0. \)

As a consequence, we prove the existence and uniqueness of solution of Cauchy problem for the fast diffusion equation in \( \mathbb{R}^n \setminus \{0\} \times (0, \infty) \) with initial value \( u_0 \) satisfying \( f_{2t}(x) \leq u_0(x) \leq f_{2t}(x) \) \( \forall x \in \mathbb{R}^n \setminus \{0\}, \) which satisfies \( U_{f_\beta}(x, t) \leq u(x, t) \leq U_{f_\beta}(x, t), \ \forall x \in \mathbb{R}^n \setminus \{0\}, t \geq 0, \) for some constants \( \lambda_1 > \lambda_2 > 0. \)

For \( n = 3,4 \) and \( \frac{n-2}{2} \leq m < \frac{n-4}{2}, \) under appropriate condition on the initial value \( u_0, \) we prove the asymptotic large time behaviour, the rescaled function \( \tilde{u}(x, t) := e^{\omega t} u(e^{\beta t} x, t) \) converges uniformly on every compact subset of \( \mathbb{R}^n \setminus \{0\} \) to \( f_{2t} \) as \( t \to \infty, \) for some \( \omega_1 > 0. \)

Furthermore, for the radially symmetric initial value \( u_0, 3 \leq n < 8, 1 - \sqrt{\frac{n}{2}} \leq m \leq \min \{ \frac{2n-2}{3n}, \frac{n-2}{n+2} \}, \) we also have the asymptotic large time behaviour.

CONTENTS

1. Introduction
2. Existence and Uniqueness of Radially Symmetric Eternal Self-similar Solutions
   2.1. Solution of the Associated Inversion Problem
   2.2. Existence and Uniqueness of Radially Symmetric Self-similar Solution
3. Higher Order Asymptotic Behaviour of Self-similar Profile
4. Existence, Uniqueness and Asymptotic Large Time Behaviour of Singular Solutions
   4.1. Existence and Uniqueness of Singular Solutions
   4.2. Asymptotic Large Time Behaviour of Singular Solutions
   4.3. Asymptotic Large Time Behaviour of Radially Symmetric Singular Solutions
5. References

2010 Mathematics Subject Classification. Primary: 35B40, 35B44, 35K55, 35K65.

Key words and phrases: asymptotic behavior of solutions, blow-up, fast diffusion equation, singular solution, \( L^1 \)-contraction, radially symmetric self-similar solution.

Date: July 15, 2020.
1. Introduction

Recently there is a lot of study on the equation
\[ u_t = \frac{n-1}{m} \Delta u^m, \quad u > 0, \quad (1.1) \]
in \( \mathbb{R}^n \times (0, T) \) for some constant \( T \in (0, \infty) \) by D.G. Aronson [A], P. Daskalopoulos, J. King, M. del Pino, N. Sesum, M. Sáez, [DKS], [DPS], S.Y. Hsu [Hs1], [Hs2], [Hs3], K.M. Hui [Hui1], [Hui2], [Hui3], M. Filá, M. Winkler, E. Yanagida, J.L. Vázquez [FWY], [FW1], [FW2], [FW3], [VW], [V1], etc. We refer the readers to the survey paper [A] and the books [DK], [V2] on the recent results of (1.1).

For \( m > 1 \), (1.1) arises in the flow of gases through porous media or oil passing through sand, etc., and it is called the porous medium equation. For \( m = 1 \), (1.1) is the heat equation. For \( 0 < m < 1 \), (1.1) is called the fast diffusion equation. If \( g = u^{\frac{n}{2}} \text{d}x^2 \) is a metric on \( \mathbb{R}^n \), \( n \geq 3 \), then \( g \) satisfies the Yamabe flow,
\[ \partial g / \partial t = -Rg \quad \text{in} \quad \mathbb{R}^n \times (0, T), \]
if and only if \( u \) satisfies (1.1) in \( \mathbb{R}^n \times (0, T) \) with \( m = n - 2 \).

As observed by J.L. Vázquez [V1] and others there is a considerable difference in the behaviour of the solutions of (1.1) for the cases \( 0 < m < \frac{n-2}{n} \) and \( \frac{n-2}{n} < m < 1 \), and \( m > 1 \). Now numerous research of (1.1) for the case \( 0 < m < \frac{n-2}{n} \), \( n \geq 3 \) is conducted by P. Daskalopoulos, J. King, M. del Pino, N. Sesum [DKS], [DPS], S.Y. Hsu [Hs1], [Hs2], [Hs3], K.M. Hui [Hui1], [Hui2], [Hui3], J.L. Vázquez [V1], etc. On the other hand, various singular solutions of (1.1) for the case \( 0 < m < \frac{n-2}{n} \), \( n \geq 3 \), were studied by K.M. Hui, Soojung Kim and Sunghoon Kim [HK], [HKs].

In this paper, we study the existence, uniqueness, and asymptotic behaviour of the singular solution of the Cauchy problem,
\[
\begin{cases}
  u_t = \frac{n-1}{m} \Delta u^m & \text{in} \quad (\mathbb{R}^n \setminus \{0\}) \times (0, \infty) \\
  u(\cdot, 0) = u_0 & \text{in} \quad \mathbb{R}^n \setminus \{0\},
\end{cases}
\]
(1.2)
which blows up at the origin for all time \( t > 0 \). The main difficulty of the theory is to find appropriate \( L^1 \)-contraction result which can be used in proof of the uniqueness and asymptotic behaviour of the solution of (1.2).

First, for \( 0 < m < \frac{n-2}{n} \), \( n \geq 3 \), we prove the existence and uniqueness of radially symmetric solution \( f \) of the equation
\[
\frac{n-1}{m} \Delta f^m + \alpha f + \beta x \cdot \nabla f = 0, \quad f > 0, \quad \text{in} \quad \mathbb{R}^n \setminus \{0\},
\]
or equivalently,
\[
\frac{n-1}{m} \left( f^m \right)_r + \frac{n-1}{r} \left( f^m \right)_r + \alpha f + \beta r f_r = 0, \quad f > 0, \quad \forall r > 0
\]
(1.3)
which blows up at the origin, where
\[
\beta < 0 \quad \text{and} \quad \alpha = \frac{2\beta}{1-m},
\]
(1.4)
Note that (1.3) arises in the study of eternal self-similar solution of (1.1) in \( \mathbb{R}^n \setminus \{0\} \times (-\infty, \infty) \) which blows up at the origin for all time \( t \in \mathbb{R} \). More precisely, we prove the following results in this paper.
Theorem 1.1 (Existence of self-similar profile). Let $n \geq 3$ and $0 < m < \frac{2}{n-2}$. Suppose $\alpha$ and $\beta$ satisfy (1.4). Then, for any constant $A > 0$, there exists a unique solution $f = f_{\beta, A}$ of (1.3) which satisfies
\begin{equation}
\lim_{r \to 0} \frac{r^2 f(r)^{1-m}}{\log r^{-1}} = \frac{2(n-1)(n-2-nm)}{\beta(1-m)}
\end{equation}
and
\begin{equation}
\lim_{r \to 0} r^{\frac{2}{1-m}} f(r) = A.
\end{equation}
Moreover,
\begin{equation}
\alpha f(r) + \beta r f_\gamma(r) > 0 \quad \forall r > 0
\end{equation}
holds.

Remark 1.2. Let $n \geq 3$, $0 < m < \frac{2}{n-2} \text{ and } \alpha, \beta \text{ be given by (1.4)}$. Let $f_1$ be the unique solution of (1.3) satisfying (1.5) and (1.6) with $A = 1$ and let
\begin{equation}
f_1(x) = \lambda^{\frac{1}{\gamma_1}} f_1(\lambda x) \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \lambda > 0.
\end{equation}
Then, $f_1$ satisfies (1.3) with
\begin{equation}
\lim_{r \to 0} \frac{r^2 f_1(r)^{1-m}}{-\log r} = \lim_{r \to 0} \frac{(\lambda r)^2 f_1(\lambda r)^{1-m}}{-\log \lambda - \log(\lambda r)} = \frac{2(n-1)(n-2-nm)}{-\beta(1-m)}
\end{equation}
and
\begin{equation}
\lim_{r \to \infty} r^{\frac{2}{1-m}} f_1(r) = \lim_{r \to \infty} r^{\frac{2}{1-m}} \lambda^{\frac{2}{\gamma_1}} f_1(\lambda r) = \lim_{r \to \infty} \lambda^{\frac{2}{\gamma_1}} r^{\frac{2}{1-m}} f_1(\lambda r) = \lambda^{\frac{2}{\gamma_1}} r^{\frac{2}{1-m}}.
\end{equation}
Hence by Theorem 1.1 $f_1 = f_{\beta, A}$ is the unique solution of (1.3) which satisfies (1.5) and (1.6) with $A = \lambda^{-\gamma_1}$, where
\begin{equation}
\gamma_1 = \frac{n-2}{m} - \frac{2}{1-m} > 0.
\end{equation}
Since $f_1$ satisfies (1.3) with $A = 1$, by (1.8) there exists a constant $r_1 > 0$ such that
\begin{equation}
\frac{1}{2} \leq r^{\frac{2}{1-m}} f_1(r) \leq \frac{3}{2} \quad \forall r \geq r_1
\end{equation}
\begin{equation}
\Rightarrow \quad \frac{1}{2} \lambda^{\frac{2}{\gamma_1}} \leq r^{\frac{2}{1-m}} f_1(r) \leq \frac{3}{2} \lambda^{\frac{2}{\gamma_1}} \quad \forall r \geq r_1/\lambda.
\end{equation}
Note that for any $\lambda > 0$, the function
\begin{equation}
U_\lambda(x, t) := e^{-\lambda t} f_1(e^{-\lambda t} x) \quad \forall x \in \mathbb{R}^n \setminus \{0\}, t \in \mathbb{R}
\end{equation}
is an eternal self-similar solution of (1.1) in $\mathbb{R}^n \times \mathbb{R}$.

Next, we will study the existence and uniqueness of the singular solution of (1.2) with initial value $u_0$ satisfying
\begin{equation}
f_{\beta_1} \leq u_0 \leq f_{\beta_2} \quad \text{in } \mathbb{R}^n \setminus \{0\},
\end{equation}
for some constants $\lambda_1 > \lambda_2 > 0$ such that
\begin{equation}
U_{\beta_1}(x, t) \leq u(x, t) \leq U_{\beta_2}(x, t) \quad \text{in } (\mathbb{R}^n \setminus \{0\}) \times (0, \infty).
\end{equation}
The existence of such solution follows by an argument similar to the proof of Theorem 1.3 of [HK]. On the other hand, for the uniqueness, the blow-up rate (1.5) at the origin is relatively high to have an appropriate $L^1$-contraction. Hence, we need higher order asymptotic behaviour of self-similar profile near the origin:
Theorem 1.3 (Higher order asymptotic behaviour of self-similar profiles). Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $\lambda > 0$ and $\alpha, \beta$ satisfy (1.4). Let

$$a_1 := \frac{(n-2-(n+2)m)^2}{4(n-2-nm)^2} - \frac{(1-m)^2a_2(1,1)}{4(n-1)(n-2-nm)^2} \quad (1.15)$$

and

$$K_0 = \frac{(1-m)K(1,1)}{2(n-1)(n-2-nm)} \quad (1.16)$$

where

$$a_2(\eta, \tilde{\beta}) = \frac{2(1-2m)(n-2-nm)}{(1-m)^2} + \frac{(n-1)(n-2-(n+2)m)^2}{(1-m)^2} - \frac{(n-2-(n+2)m)}{(1-m)K(\eta, \tilde{\beta})} \quad (1.17)$$

and $K(\eta, \tilde{\beta})$ is given by (1.17) for $m \neq \frac{n-2}{n+2}$ and by (1.10) for $m = \frac{n-2}{n+2}$ respectively. Let $f_\lambda(r)$ be given by (1.4). Then, the following holds.

$$f_{1-m}(r) = \frac{2(n-1)(n-2-nm)}{(1-m)(-\beta)r^2} \left\{ -\log r + \frac{(n-2-(n+2)m)}{2(n-2-nm)} \log(\log r^{-1}) + \frac{m}{n-2-nm} \log(-\beta) + \frac{a_1}{\log r^{-1}} \right. \quad (1.18)$$

$$+ K_0 - \log \lambda + \frac{(n-2-(n+2)m)^2}{4(n-2-nm)^2} \frac{\log(\log r^{-1})}{\log r^{-1}} + o\left(\frac{1}{\log r^{-1}}\right) \right\} \text{ as } r \to 0.$$

Thus, Theorem 1.3 implies an estimate on the blow-up rate of the difference $f_{\lambda_0} - f_{\lambda_1}$ at the origin, for any $\lambda_1 > \lambda_0 > 0$ (Lemma 4.1) and a weighted $L^1$-contraction with a weight function $r^{-\mu}$:

Proposition 1.4 (Weighted $L^1$-contraction with a weight function $|x|^{-\mu}$). Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $\mu_1 = n - \frac{2}{n+2}$, and $\beta < 0$, $\alpha$ be given by (1.4). Let $\lambda_1 > \lambda_2 > 0$ and $f_{\lambda_1}$ and $U_{\lambda_1}$, $i = 1, 2$ be given by (1.8) and (1.12), respectively. Let $u_0$ and $v_0$ satisfy

$$f_{\lambda_1}(x) \leq u_0(x), v_0(x) \leq f_{\lambda_2}(x) \quad \text{in } \mathbb{R}^n \setminus \{0\} \quad (1.19)$$

and let $u$ and $v$ be the solutions of (1.2) with initial values $u_0$ and $v_0$, respectively, which satisfy

$$U_{\lambda_1}(x,t) \leq u(x,t), v(x,t) \leq U_{\lambda_2}(x,t) \quad \text{in } (\mathbb{R}^n \setminus \{0\}) \times (0, \infty). \quad (1.20)$$

Suppose that

$$|u_0 - v_0| \in L^1(|x|^{-\mu}; \mathbb{R}^n) \quad (1.21)$$

holds for some constant $\mu \in (0, \mu_1)$. Then for

$$\mu < \mu_1 \quad \text{or} \quad \left\{ \begin{array}{l}
0 < m < \min\left(\frac{n-2}{n}, \frac{1}{2}\right),
\end{array} \right.$$

$$\int_{\mathbb{R}^n} |u - v|(x,t)|x|^{-\mu} \, dx \leq \int_{\mathbb{R}^n} |u_0 - v_0|(x)|x|^{-\mu} \, dx \quad \forall t > 0 \quad (1.22)$$

and

$$\int_{\mathbb{R}^n} (u - v)_+(x,t)|x|^{-\mu} \, dx \leq \int_{\mathbb{R}^n} (u_0 - v_0)_+(x)|x|^{-\mu} \, dx \quad \forall t > 0 \quad (1.23)$$

hold.
Therefore, by an argument similar to the proof of Theorem 1.3 of [HK] and the weighted $L^1$-contraction with the weight function $|x|^{-\alpha}$ (Proposition 1.4) gives the uniqueness of the singular solution of (1.2).

**Theorem 1.5** (Existence and Uniqueness). Let $n \geq 3, 0 < m < \frac{n+2}{n} , \beta < 0$ and $\alpha$ be given by (1.4). Let $\lambda_1 > \lambda_2 > 0$ and $f_{\lambda_i}, i = 1, 2$ be given by (1.8) with $\lambda = \lambda_1, \lambda_2$. Suppose $u_0$ satisfies $(1.13)$ and $v_0$ satisfies $(1.14)$, then there exists a unique solution $u$ of (1.2) which satisfies (1.14) and

$$u_i \leq \frac{u}{(1-m)t} \quad \text{in} \ (\mathbb{R}^n \setminus \{0\}) \times (0, \infty). \quad (1.25)$$

The last topic of this paper is the asymptotic large time behaviour of the solution to the fast diffusion equation. Precisely, for any solution $u$ of (1.2), let

$$\tilde{u}(x, t) = e^{\alpha t}u(e^{\beta t}x, t) \quad \forall x \in \mathbb{R}^n \setminus \{0\}, t \geq 0. \quad (1.26)$$

Then, $\tilde{u}$ satisfies

$$\begin{cases}
\tilde{u}_t = \frac{n-1}{m} \Delta \tilde{u} + \alpha \tilde{u} + \beta y \cdot \nabla \tilde{u} & \text{in} \ (\mathbb{R}^n \setminus \{0\}) \times (0, \infty) \\
\tilde{u}(x, 0) = u_0 & \text{in} \ \mathbb{R}^n \setminus \{0\}. \quad (1.27)
\end{cases}$$

We prove that under appropriate condition on the initial value $u_0$, the rescaled function $\tilde{u}$ will converge uniformly on every compact subset of $\mathbb{R}^n \setminus \{0\}$ to $f_{\lambda_0}$ as $t \to \infty$ for some $\lambda_0 > 0$.

Note that if $u, v$ are as given by Proposition 1.4 then by direct computation,

$$\int_{\mathbb{R}^n} |u - \tilde{v}|(x, t)|x|^{-\mu} \, dx = \int_{\mathbb{R}^n} e^{\alpha t}|u - v|(e^{\beta t}x, t)|x|^{-\mu} \, dx$$

$$= e^{\alpha t} \int_{\mathbb{R}^n} |u - v(y, t)|y|^{-\mu} e^{-\beta|y|} \, dy = e^{(\alpha - \beta n + \beta \mu)t} \int_{\mathbb{R}^n} |u - v(y, t)|y|^{-\mu} \, dy \quad \forall t > 0, 0 < \mu \leq \mu_1.$$ 

Hence Proposition 1.4 implies

$$\int_{\mathbb{R}^n} |\tilde{u} - \tilde{v}|(x, t)|x|^{-\mu} \, dx \leq e^{(\alpha - \beta n + \beta \mu)t} \int_{\mathbb{R}^n} |u_0 - v_0(x)|x|^{-\mu} \, dx \quad \forall t > 0, 0 < \mu \leq \mu_1. \quad (1.28)$$

Since $\mu \in (0, \mu_1), \alpha - \beta n + \beta \mu \geq 0$ and it is not clear that the right hand side of (1.28) converges to 0 as $t$ goes to $\infty$. Thus, it is not appropriate to apply Proposition 1.4 to have the asymptotic behaviour. Therefore, a new $L^1$-contraction result for the solutions of (1.2) is needed which is the following:

**Proposition 1.6** (Weighted $L^1$-contraction with a weight function $f_{\lambda_0}^{my}$). Let $n = 3, 4, \frac{n+2}{n+2} \leq m < \frac{n+2}{n}, \beta < 0, \alpha$ be given by (1.4) and

$$\gamma := \frac{1 - m}{2m} \left( n - \frac{2}{1 - m} \right). \quad (1.29)$$

Let $\lambda_1 > \lambda_2 > 0, \lambda_3 > 0$, and $f_{\lambda_i}, i = 1, 2, 3$ be given by (1.8) with $\lambda = \lambda_1, \lambda_2, \lambda_3$. Let $u_0$ and $v_0$ satisfies (1.19) and

$$u_0 - v_0 \in L^1 \left( f_{\lambda_0}^{my}; \mathbb{R}^n \setminus \{0\} \right). \quad (1.30)$$

Let $u$ and $v$ be the solutions of (1.2) with initial values $u_0, v_0$, respectively which satisfy, (1.20) where $U_{\lambda_i}, i = 1, 2$, are given by (1.2) with $\lambda = \lambda_1, \lambda_2$. Then

$$\int_{\mathbb{R}^n \setminus \{0\}} |u - v|(x, t)f_{\lambda_i}(x)^{my} \, dx \leq \int_{\mathbb{R}^n \setminus \{0\}} |u_0 - v_0(x)f_{\lambda_i}(x)^{my} \, dx \quad \forall t > 0. \quad (1.31)$$

Moreover if

$$0 \neq u_0 - v_0 \in L^1 \left( f_{\lambda_0}^{my}; \mathbb{R}^n \setminus \{0\} \right), \quad (1.32)$$
then
\[ \int_{\mathbb{R}^n \setminus \{0\}} |u - v(x, t)f_{\lambda}(x)^{my} - u_0 - v_0(x)f_{\lambda}(x)^{my}| \, dx < \int_{\mathbb{R}^n \setminus \{0\}} |u_0 - v_0(x)f_{\lambda}(x)^{my}| \, dx \quad \forall t > 0. \]  
(1.33)

We note that (1.31) implies
\[ \int_{\mathbb{R}^n \setminus \{0\}} |\tilde{u} - \tilde{v}(y, t)f_{\lambda}(y)^{my} - u_0(y) - v_0(y)\tilde{f}_{\lambda}(y)^{my}| \, dy \leq \int_{\mathbb{R}^n \setminus \{0\}} |u_0(y) - v_0(y)\tilde{f}_{\lambda}(y)^{my}| \, dy \quad \forall t > 0 \]  
(1.34)

and the right hand side converges to 0 as \( t \) goes to \( \infty \). It is the main difference between (1.34) and the same inequality with the weight function \( |x|^{-\mu} \), (1.28). The property for (1.34) is crucially used in the proof of the asymptotic large time behaviour:

**Theorem 1.7** (Asymptotic behaviour). Let \( n = 3, 4, \frac{2}{n+2} \leq m < \frac{2}{n}, \beta < 0, \) and \( \alpha \) be given by (1.4), and \( \gamma \) be given by (1.29). Let \( \lambda_1 \geq \lambda_0 \geq \lambda_2 > 0, \lambda_3 > 0, \) and \( f_{\lambda_i}, i = 0, 1, 2, 3 \) be given by (1.5) with \( \lambda = \lambda_0, \lambda_1, \lambda_2, \lambda_3 \). Let \( u_0 \) satisfy (1.13) and \( u_0 - f_{\lambda_0} \in L^1(\mathbb{R}^n) \).

Let \( \tilde{u} \) be a solution of (1.2) which satisfies
\[ f_{\lambda_1}(x) \leq \tilde{u}(x, t) \leq f_{\lambda_2}(x) \quad \forall x \in \mathbb{R}^n \setminus \{0\}, t > 0. \]  
(1.35)

Then, \( \tilde{u}(\cdot, t) \) converges uniformly in \( C^{2,1}(K) \) for any compact subset \( K \) of \( \mathbb{R}^n \setminus \{0\} \) to \( f_{\lambda_0} \) and in \( L^1(\mathbb{R}^n) \) as \( t \to \infty \).

The last theorem of this paper is the asymptotic behaviour of the solution \( u \) of (1.2) with radially symmetric initial value \( u_0(x) = u_0(r) \), where \( r := |x| \). First, we consider a weighted \( L^1 \)-contraction, Proposition 4.9, of the inversion problem
\[
\begin{cases}
\bar{u}_i = \frac{n-1}{m} r^m \Delta \bar{u}^m \\
\bar{u}_0 = r^{-\frac{n-2}{2}} u_0(r^{-1}) =: \bar{u}_0(r) 
\end{cases}
\]  
(1.37)

with initial values \( \bar{u}_0, \bar{u}_0 \) respectively which satisfy
\[ \bar{U}_{\lambda_1} \leq \bar{u} \leq \bar{U}_{\lambda_2} \quad \text{in} \ (\mathbb{R}^n \setminus \{0\}) \times (0, \infty), \]  
(1.38)

where \( \bar{U}_{\lambda_i}, i = 1, 2 \), are given by
\[
\bar{U}_i(r, t) := e^{-\lambda_1 t} g_1(e^{-\lambda_1 t} r) 
\]  
(1.39)

and
\[
g_1(r) := r^{\frac{n}{\lambda_2}} f_{\lambda_1}(r^{-1}) \quad \forall r > 0 
\]  
(1.40)

with \( \lambda = \lambda_1, \lambda_2 \). Note that for radially symmetric initial value \( u_0 \), \( u \) is radially symmetric solution of (1.2) that satisfies (1.20) if and only if \( \bar{u}(r, t) = r^{-\frac{n}{\lambda_2}} u(r^{-1}, t) \) is a solution of (1.37) that satisfies (1.40). Then, the change of variable gives a weighted \( L^1 \)-contraction with a weight function \( |x|^{-\beta} r^{n-2m+2(n-2)\mu} f_{\lambda}(x)^{my} \) for radially symmetric solutions of (1.2). Proposition 4.9 Then, the asymptotic behaviour for radially symmetric solutions of (1.2) follows as the same method for Theorem 1.7. This asymptotic large time behaviour of \( u \) is equivalent to asymptotic behaviour large time behaviour of \( \bar{u} \).

**Theorem 1.8** (Asymptotic behaviour of radially symmetric solutions). Let \( 3 \leq n < 8, 1 - \sqrt{\frac{2}{n}} \leq m \leq \min \left\{ \frac{2(n-2)}{3n}, \frac{n-2}{n+2} \right\}, \beta < 0, \) and \( \alpha \) be given by (1.4), and \( \gamma' := \frac{1}{m} \left( \frac{\beta}{\alpha} n - 1 \right) \).

\[ (1.41) \]
Let $\lambda_1 \geq \lambda_0 \geq \lambda_2 > 0$, $\lambda_3 > 0$, and $f_{\lambda_3}$, $i = 0, 1, 2, 3$ be given by (1.8) with $\lambda = \lambda_0, \lambda_1, \lambda_2, \lambda_3$. Let $u_0$ satisfy (1.13) and

$$u_0 - f_{\lambda_3} \in L^1 \left( \left| x \right|^{\frac{m}{2} - 2n+(n-2)n'} \right)^{\text{even}}; \mathbb{R}^n \right),$$

(1.42)

Let $u$ be a solution of (1.2) which satisfies

$$f_{\lambda_3}(x) \leq u(x, t) \leq f_{\lambda_3}(x) \quad \forall x \in \mathbb{R}^n \setminus \{0\}, t > 0.$$  

(1.43)

Then, $u(t)$ converges uniformly in $C^{1,1}(K)$ for any compact subset $K$ of $\mathbb{R}^n \setminus \{0\}$ to $f_{\lambda_3}$ and in $L^1(\left| x \right|^{\frac{m}{2} - 2n+(n-2)n'} \right)^{\text{even}}; \mathbb{R}^n)$ as $t \to \infty$.

We leave for a future work for the asymptotic behaviour of the general cases which are not containing the cases of Theorems (1.7) and (1.8).

In Section 2 we will use the technique of K.M. Hui and Soojung Kim [HK] to prove the existence of eternal self-similar solution of (1.1) which blows up at the origin for all time $t \in \mathbb{R}$. We will also prove the blow-up rate of the eternal self-similar solutions at the origin and near infinity in Section 2. In Section 3 we will use the technique of [1.8] to prove the higher order asymptotic of the solution $f$ of (1.3) that blow-up at the origin. In Section 4 we will prove the existence, uniqueness and the asymptotic large time behaviour of the singular solution of (1.2) that satisfies (1.14) for some constants $\lambda_1 > \lambda_2 > 0$. The $L^1$-contractions, which play important roles in the theory, are discussed in detail in the subsection.

1.1. Definitions. We start with some definitions. For any $0 \leq u_0 \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, we say that $u$ is a solution of (1.2) if $u > 0$ in $(\mathbb{R}^n \setminus \{0\}) \times (0, \infty)$ satisfies (1.1) in $(\mathbb{R}^n \setminus \{0\}) \times (0, \infty)$ in the classical sense and

$$\|u(\cdot, t) - u_0\|_{L^1(K)} \to 0 \quad \text{as } t \to 0$$

for any compact set $K \subset \mathbb{R}^n \setminus \{0\}$. For any $x_0 \in \mathbb{R}^n$ and $R > 0$, we let $B_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R\}$ and $B_R = B_R(0)$. For any $R > \varepsilon > 0$, let $A_{R, \varepsilon} = \{x \in \mathbb{R}^n : \varepsilon < |x| < R\}$. For any function $k : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, let

$$L^1(k; \mathbb{R}^n \setminus \{0\}) := \left\{ h : \int_{\mathbb{R}^n \setminus \{0\}} |h(x)|k(x) \, dx < \infty \right\}$$

with the norm

$$\|h\|_{L^1(k; \mathbb{R}^n \setminus \{0\})} := \int_{\mathbb{R}^n \setminus \{0\}} |h(x)|k(x) \, dx.$$

2. Existence and Uniqueness of Radially Symmetric Eternal Self-similar Solutions

In this section, we prove the existence and uniqueness of solution $f$ of (1.3), which satisfies (1.5) and (1.6). First, we introduce an inversion problem:

$$\frac{n-1}{m} \left( g''_{rr} + \frac{n-1}{r} \left( g'' \right) \right) + r^{\frac{2\alpha}{n-2}} \left( \tilde{a} g + \tilde{\beta} g r^\epsilon \right) = 0, \quad g > 0, \quad \text{in } (0, \infty),$$

(2.1)

where

$$\tilde{a} := \alpha - \frac{n-2}{m} \beta \quad \text{and} \quad \tilde{\beta} := -\beta.$$  

(2.2)

By (2.3) and (2.2),

$$\tilde{a} = \frac{2}{1-m} \frac{n-2}{m} \varepsilon \left( 0, \frac{n-2}{m} \right), \quad \tilde{\beta} > 0, \quad \text{and} \quad \tilde{a} > 0.$$  

(2.3)

The existence and uniqueness of the solution $g$ of (2.1) follow Theorem 2.4 of [HK].
In Subsection 2.1 by using argument in [HS1], we have the decay rate of \( g \) at \( \infty \) by using \( w(r) = r^{-\frac{2}{n}(1-m)}g(r)^{1-m} \), Theorem 2.4.

In Subsection 2.2 by using the existence and uniqueness of a solution \( g \) to (2.1) (Theorem 2.4 of [HK]) and methods in [HK], we have the existence and uniqueness of solution \( f \) to (1.3), Theorem 1.1. Precisely, by using the relationships
\[
\begin{align*}
    g(r) &:= r^{-\frac{2}{n}} f(r^{-1}) \quad \forall r > 0, \\
    f(r) &:= r^{-\frac{2}{n}} g(r^{-1}) \quad \forall r > 0,
\end{align*}
\]
and
\[
\text{the existence of the solution } f \text{ with the given growth rates (1.5) and (1.6) is equivalent to the existence of the solution } g \text{ of the inversion problem (2.1) with appropriate growth rate at the origin and at infinity. For the uniqueness of } f, \text{ (1.7) will be checked.}
\]

2.1. Solution of the Associated Inversion Problem. In this subsection, we discuss the decay rate of \( g \) at \( \infty \), Theorem 2.1. First, we recall a result of [HK].

**Theorem 2.1** (Theorem 2.4 of [HK]). Let \( n \geq 3, \ 0 < m < \frac{n^2}{n+2}, \ \bar{\alpha} > 0, \ \bar{\beta} > 0, \ \frac{2}{\bar{\beta}} \leq \frac{n^2}{2m} \), and \( \eta > 0 \).

(a) If \( 0 < m < \frac{n^2}{n+2} \), then there exists a unique solution \( g \in C^1([0, \infty); \mathbb{R}) \cap C^2((0, \infty); \mathbb{R}) \) of (2.1) which satisfies
\[
g(0) = \eta \quad \text{and} \quad g_r(0) = 0.
\]

(b) If \( \frac{n^2}{n+2} \leq m < \frac{n^2}{2} \), then there exists a unique solution \( g \in C^0,0([0, \infty); \mathbb{R}) \cap C^2((0, \infty); \mathbb{R}) \) of (2.1) which satisfies
\[
g(0) = \eta \quad \text{and} \quad \lim_{r \to 0} r^{\delta_1} g_r(r) = -\frac{\bar{\alpha} \eta^{2-m}}{n-2-2m},
\]
where
\[
\delta_1 = 1 - \frac{n-2-nm}{m} \in [0, 1) \quad \text{and} \quad \delta_0 = \frac{1-\delta_1}{2} = \frac{n-2-nm}{2m} \in (0, 1/2].
\]

Unless stated otherwise we will now let \( g = g_{\bar{\beta},\eta}(r) \) be the unique solution of
\[
\begin{cases}
    \frac{n-1}{m} \left( (g_m)_r + \frac{n-1}{r} (g^m)_r \right) + r^{-\frac{n-2}{m}} (\bar{\alpha} g + \bar{\beta} g_r) = 0, & g > 0, \text{ in } (0, \infty) \\
    g(0) = \eta
\end{cases}
\]
for some constant \( \eta > 0 \) given by Theorem 2.1. Furthermore, in order to have the decay rate of \( g \) at \( \infty \), we define
\[
w(r) := r^{\frac{2}{\bar{\beta}}(1-m)} g(r)^{1-m},
\]
where \( \bar{\alpha} \) and \( \bar{\beta} \) satisfy (2.3), for the rest of the paper.

**Lemma 2.2.** Let \( n \geq 3, \ 0 < m < \frac{n^2}{n+2}, \ \eta > 0 \) and \( \bar{\alpha}, \bar{\beta} \) be given by (2.3). Then,
\[
w_r(r) > 0 \quad \forall r > 0.
\]

**Proof.** We first observe that by Lemma 2.1 of [HK],
\[
g(r) + \frac{\bar{\beta}}{\bar{\alpha}} g_r(r) > 0 \quad \forall r > 0.
\]

Hence, by direct computation,
\[
w_r(r) = (1-m)\frac{\bar{\alpha}}{\bar{\beta}} r^{\frac{2}{\bar{\beta}}(1-m)-1} g(r)^{-m} \left( g(r) + \frac{\bar{\beta}}{\bar{\alpha}} g_r(r) \right) > 0 \quad \forall r > 0
\]
and the lemma follows. \( \square \)
Let \( q(r) = r^{\tilde{a}} \tilde{g}(r) \), \( s = \log r \), and \( \tilde{g}(s) = q(r) \). Then
\[
 r^2(q''(r)) + \left( n - 1 - \frac{2\tilde{m}}{\tilde{\beta}} \right) r(q''(r)) - \frac{\tilde{m}}{\tilde{\beta}} \left( n - 2 - \frac{\tilde{m}}{\tilde{\beta}} \right) q'' + \frac{m_0}{n-1}r_{q'} = 0 \quad \forall r > 0
\]
and
\[
 (\tilde{g}'')_n + \left( n - 2 - \frac{2\tilde{m}}{\tilde{\beta}} \right) (\tilde{g}'')_n - \frac{\tilde{m}}{\tilde{\beta}} \left( n - 2 - \frac{\tilde{m}}{\tilde{\beta}} \right) \tilde{g}'' + \frac{m_0}{n-1}\tilde{q}_s = 0 \quad \text{in } (-\infty, \infty). \quad (2.13)
\]
Let \( \tilde{w}(s) = \tilde{q}^{1-m}(s) \). Then \( w(r) = \tilde{w}(s) \) and
\[
 \tilde{w}_{ss} = -2m w_{ss} = \frac{\tilde{m}}{\tilde{\beta}} \left( n - 2 - \frac{\tilde{m}}{\tilde{\beta}} \right) \tilde{w}_s + \frac{2(n - 2 - nm) w}{1-m} \tilde{w} - \frac{\tilde{\beta}}{n-1} \tilde{w}_s \quad (2.14)
\]
in \((0, \infty)\). Hence
\[
 w_{rr} + \left( n - 1 - \frac{2\tilde{m}}{\tilde{\beta}} \right) w_{r} + \frac{1 - 2m w_{r}}{1-m} - \frac{\tilde{\beta}}{1-m} \tilde{w}_r - \frac{2(n - 2 - nm) w}{1-m} \tilde{w} = 0 \quad (2.15)
\]
in \((0, \infty)\). Let
\[
b_0 = \frac{(n + 2)m - (n - 2)}{1-m} \quad \text{and} \quad b_1 = \frac{2(n - 2 - nm)}{1-m}. \quad (2.16)
\]
Since (2.14) and (2.15) are of the same form as (3.7) and (3.8) of \([Hs1]\), by the same argument as the proof of Lemma 3.2 of \([Hs1]\) but with (2.14) and (2.15) replacing (3.7), (3.8) in the proof there, we have the following result:

**Lemma 2.3.** Let \( n \geq 3 \), \( 0 < m < \frac{n-2}{n} \) and \( \tilde{a}, \tilde{\beta} \) satisfy (2.3). Then, there exist positive constants \( C_1, C_2 \) and \( C_3 \) such that
\[
 \frac{rw_r(r)}{w(r)} \leq C_1, \quad \forall r \geq 0 \quad (2.17)
\]
and
\[
 C_2 \leq rw_r(r) \leq C_3, \quad \forall r \geq 1. \quad (2.18)
\]
Moreover,
\[
 w(r) \to \infty \quad \text{as } r \to \infty. \quad (2.19)
\]

Now, we prove the decay rate of \( w \) at \( \infty \).

**Theorem 2.4.** Let \( n \geq 3 \), \( 0 < m < \frac{n-2}{n} \) and \( \tilde{a}, \tilde{\beta} \) satisfy (2.3). Then
\[
 \lim_{r \to \infty} \frac{w(s)}{s} = \lim_{r \to \infty} \frac{w(r)}{\log r} = \lim_{r \to \infty} rw_r = \lim_{s \to \infty} \tilde{w}_s(s) = \frac{2(n - 1)(n - 2 - nm)}{-\beta(1-m)}. \quad (2.20)
\]

**Proof.** We will use a modification of the proof of Theorem 1.3 of \([Hs1]\) to prove this theorem. Let \( \tilde{b}_0, \tilde{b}_1 \) be given by (2.16) and let \( v(r) = rw_r(r) \)
\[
a_0 = \frac{n-1}{\tilde{\beta}} \tilde{b}_1, \quad \text{and} \quad v_1(r) = v(r) - a_0. \quad (2.21)
\]
Let \( \{r_i\}_{i=1}^{\infty} \) be a sequence of positive numbers such that \( r_i \to \infty \) as \( i \to \infty \). By Lemma 2.3 there exist positive constants \( C_1, C_2 \) and \( C_3 \) such that (2.17) and (2.18) hold. Then by (2.18), the sequence \( \{v(r_i)\}_{i=1}^{\infty} \) has a subsequence which we may assume without loss of generality to be the sequence itself that converges to some constant \( v_\infty \) as \( i \to \infty \) and
\[
 C_2 \leq v_\infty \leq C_3. \quad (2.22)
\]

Let
\[
f_1(r) := \exp \left( \frac{\tilde{\beta}}{n-1} \int_1^r \rho^{-1}w(r) \, \mathrm{d}\rho \right).
\]
By (2.19) there exists a constant \( r_0 > 1 \) such that
\[
w(r) > \frac{n-1}{\beta} (|b_0| + 2) \quad \forall r \geq r_0.
\]
Hence
\[
f_1(r) \geq \left( \frac{r}{r_0} \right)^{|b_0|+2} \quad \forall r \geq r_0
\]
\[
= r_0^{-|b_0|+1} f_1(r) \geq \frac{\rho r_0^{b_0+1}}{r_0^{|b_0|+2}} \geq r_0^{-2} \quad \forall r \geq r_0
\]
\[
\Rightarrow r_0^{-2} f_1(r) \to \infty \quad \text{and} \quad r \to \infty.
\]
(2.23)

By (2.18), (2.19), (2.23), and the l’Hospital rule,
\[
\lim_{r \to \infty} r^{b_0-1} f_1(r) = \lim_{r \to \infty} \frac{r^{b_0-1} f_1(r)}{w(r)} = \infty
\]
\[
\Rightarrow \int_{1}^{\infty} \frac{r^{b_0-1} f_1(r)}{w(r)} \, d\rho \to \infty \quad \text{as} \quad r \to \infty.
\]
(2.24)

On the other hand by (2.19), (2.23) and the l’Hospital rule,
\[
\lim_{r \to \infty} \int_{1}^{r} \frac{r^{b_0-1} f_1(r)}{w(r)} \, d\rho = \lim_{r \to \infty} \frac{r^{b_0-1} f_1(r)}{b_0 r^{b_0-1} f_1(r) + \frac{\beta}{n} r^{b_0-1} w(r) f_1(r)}
\]
\[
= \lim_{r \to \infty} \frac{1}{b_0 + \frac{\beta}{n} r w(r)} = 0.
\]
(2.25)

Observe that by the same argument as the proof of (3.24) of [Hs1] but with (2.15) replacing (3.8) of [Hs1] in the proof there, we have
\[
\left( r^{b_0} v(r) w(r) \right)' = \frac{\beta}{n-1} \cdot \frac{w(r)}{v^{1-b_0}} (v(r) - a_0) \quad \forall r > 0
\]
\[
\Rightarrow v_1(r) = \frac{b_0}{r} v_1 + \frac{\beta}{n-1} \cdot \frac{w(r)}{r} v_1 = \frac{1 - 2m}{1 - m} \cdot \frac{v(r)^2}{rw(r)} - \frac{b_0 a_0}{r} \quad \forall r > 0
\]
\[
\Rightarrow r^{b_0} f_1(r) v_1(r) = f_1(1) v_1(1) - a_0 b_0 \int_{1}^{r} \rho^{b_0-1} f_1(\rho) \, d\rho
\]
\[
+ \frac{1 - 2m}{1 - m} \int_{1}^{r} \rho^{b_0-1} v(r)^2 f_1(\rho) w(\rho)^{-1} \, d\rho
\]
(2.26)

holds for any \( r \geq 1 \). By (2.19), (2.22), (2.23), (2.24), (2.25), (2.26), and the l’Hospital rule,
\[
\lim_{r \to \infty} v_1(r) = \lim_{r \to \infty} f_1(1) v_1(1) - a_0 b_0 \int_{1}^{r} \rho^{b_0-1} f_1(\rho) \, d\rho + \frac{1 - 2m}{1 - m} \int_{1}^{r} \rho^{b_0-1} v(r)^2 f_1(\rho) w(\rho)^{-1} \, d\rho
\]
\[
= 1 - 2m \lim_{r \to \infty} \frac{\rho^{b_0-1} v(r)^2 f_1(\rho) w(\rho)^{-1}}{b_0 \rho^{b_0-1} f_1(\rho) + \frac{\beta}{n-1} \rho^{b_0-1} w(\rho) f_1(\rho)}
\]
\[
= 1 - 2m \lim_{r \to \infty} \frac{v(r)^2 w(\rho)^{-1}}{b_0 + \frac{\beta}{n-1} w(\rho)} = 0
\]
(2.27)

Hence \( \lim_{r \to \infty} v(r) = a_0 \). Since the sequence \( \{r_i\}_{i=1}^{\infty} \) is arbitrary, \( \lim_{r \to \infty} v(r) = a_0 \) and the theorem follows.
2.2. Existence and Uniqueness of Radially Symmetric Self-similar Solution. In this subsection, we will prove the existence and uniqueness of radially symmetric self-similar profile \( f \) of (1.3) with growth rates (1.5) and (1.6) at the origin and infinity, respectively. As discussed in the introduction of this section, the existence of the solution \( g \) of the inversion problem (2.1) with appropriate growth rate at the origin and at infinity implies the existence of the solution \( f \) with the given growth rates (1.5) and (1.6).

In order to use the uniqueness of \( g \) to have the uniqueness of \( f \), we need an additional property (1.7) for \( f \), due to \( g_r(0) = 0 \) for \((a)\) in Theorem 2.1 and the second condition of \((b)\) in Theorem 2.1. Hence, in Theorem 2.5 and Lemma 2.6 we obtain (1.7) for \( f \) and prove the uniqueness of the solution \( f \), by a method in [HK].

**Theorem 2.5.** Let \( n \geq 3, 0 < m < \frac{n-2}{n} \), and \( \alpha, \beta \) be given by (1.4). Then for any \( A > 0 \), there exists a solution \( f \) of (1.3) satisfying (1.5), (1.6), and (1.7).

**Proof.** Let \( \tilde{\alpha}, \tilde{\beta} \) be given by (2.2) and \( g \) be the unique solution of (2.9) given by Theorem 2.1 with \( \eta = A \). Then, \( f(r) = \frac{C}{r^{\tilde{\alpha}} g(r^{-1})} \) satisfies (1.3). By (2.3), (2.9), (2.5) and Theorem 2.4, (1.5) and (1.6) hold. By (2.11) and the same argument as the proof of Lemma 3.1 of [HK], (1.7) holds and the lemma follows.

**Lemma 2.6** (Lemma 3.2 and Lemma 3.3 of [HK]). Let \( n \geq 3, 0 < m < \frac{n-2}{n} \), and \( \alpha, \beta \) be given by (1.4). Let \( f \) be a solution of (1.3) satisfying (1.5) and (1.6) for some positive constant \( A \). Let \( g, \tilde{\alpha}, \tilde{\beta} \) be given by (2.4) and (2.2), respectively. Then, \( f \) satisfies (1.7) and \( g \) is equal to the solution of (2.9) given by Theorem 2.1 with \( \eta = A \).

Now we are ready for the proof of Theorem 1.1

**Proof of Theorem 1.1**. By Theorem 2.5 for any \( A > 0 \), there exists a solution \( f = f_{\beta, A} \) of (1.3) which satisfies (1.5), (1.6), and (1.7). It remains to prove the uniqueness of solution of (1.3) that satisfies (1.5) and (1.6). Suppose (1.5) has two solutions \( f_1 \) and \( f_2 \), which satisfies (1.5) and (1.6). Then by Lemma 2.6 both \( f_1 \) and \( f_2 \) satisfies (1.7). Let \( g_1, g_2 \) be given by (2.4) with \( f = f_1, f_2 \). By Lemma 2.6 and Theorem 2.1 \( g_1 \equiv g_2 \) on \([0, \infty)\) and it is the unique solution of (2.9) with \( \eta = A \). Hence \( f_1 \equiv f_2 \) on \([0, \infty)\) and the theorem follows.

By (1.7), (1.9), (1.10), and an argument similar to the proof of Remark 2 of [HK], we have the following result.

**Remark 2.7** (cf. Remark 2 of [HK]).

\[
\frac{d}{d\lambda} f_\lambda(r) < 0 \quad \forall r > 0, \lambda > 0.
\]

Moreover, for any \( \lambda_1 > \lambda_2 > 0 \), there exists a constant \( c_0 > 0 \) such that

\[
c_0 f_{\lambda_1}(r) \leq f_{\lambda_2}(r) < c_0 f_{\lambda_2}(r) \quad \forall r > 0.
\]

3. Higher Order Asymptotic Behaviour of Self-similar Profile

In this subsection, we obtain the higher order asymptotic behaviour of the self-similar profile \( f_\lambda \) near the origin, Theorem 1.3, by using the methods in [CD] and [Hs5]. Since the proofs of lemmas and corollaries in this subsection are similar to that of [CD] and [Hs5], we will only sketch the proofs here.
By (2.14), we have
\[
\hat{w}_s = \frac{1-2m}{1-m} \cdot \hat{w}_s + \frac{n-2-(n+2)m}{1-m} \hat{w}_s + \beta \frac{2(n-1)(n-2-nm)}{(1-m)\beta - \hat{w}_s} \hat{w} \quad \forall s \in \mathbb{R}.
\] (3.1)

Note that this is the same as (2.2) of [Hs5] after the \( \beta \) there being replaced by \( \tilde{\beta} \) and a change in the sign of the second term on the right hand side of (3.1). Let
\[
h(s) = \hat{w}(s) - \frac{2(n-1)(n-2-nm)}{(1-m)\beta} s.
\] (3.2)

Then, by (3.1),
\[
h_s + \left( \frac{2(n-2-nm)}{1-m} \right) s + \beta \frac{2(n-1)(n-2-nm)}{(1-m)\beta} h_s = \frac{1-2m}{1-m} \cdot \hat{w}_s + b_2 \quad \forall s \in \mathbb{R},
\] (3.3)

where
\[
b_2 = \frac{2(n-1)(n-2-nm)(n-2-(n+2)m)}{(1-m)^2 \beta}.
\]

Then, by an argument similar to the proof of Lemma 2.3 of [Hs5] but with (3.3) and Theorem 2.4 replacing (2.4) and Theorem 2.1 in the proof there, we get the following result.

**Lemma 3.1.** Let \( n \geq 3, 0 < m < \frac{\alpha^2}{n}, m \neq \frac{\alpha^2}{n+2} \) and \( \bar{\alpha}, \tilde{\alpha} \) satisfy (2.3). Then, \( h \) satisfies
\[
\lim_{s \to \infty} \frac{h(s)}{\log s} = \lim_{s \to \infty} \frac{1}{\log s} \cdot h_s(s) = \frac{1}{(n-1)(n-2-(n+2)m)} \cdot \frac{(1-m)b_2}{2(n-2-nm)} = (n-1)(n-2-(n+2)m) (1-m)\beta.
\]

By an argument similar to the proof Proposition 3.1 of [Hs5] or Proposition 2.3 of [CD], we have the following lemma for the case \( m = \frac{\alpha^2}{n+2} \).

**Lemma 3.2.** [CD] [Hs5] Let \( n \geq 3, 0 < m < \frac{\alpha^2}{n}, m \neq \frac{\alpha^2}{n+2} \) and \( \bar{\alpha}, \tilde{\alpha} \) satisfy (2.3). Then
\[
\lim_{s \to \infty} s^2 h_s(s) = \frac{(n-1)(1-2m)}{(1-m)\beta}
\] (3.4)

holds.

For \( m \neq \frac{\alpha^2}{n+2} \), let
\[
h_1(s) = h(s) - \frac{(n-1)(n-2-(n+2)m)}{(1-m)\beta} \log s.
\] (3.5)

Then, \( h_1 \) satisfies
\[
h_{1,s} + \left( \frac{2(n-2-nm)}{1-m} \right) s + \beta \frac{2(n-1)(n-2-nm)}{(1-m)\beta} h_{1,s} = \frac{1-2m}{1-m} \cdot \hat{w}_s + a_3 \left[ \frac{1}{s^2} - \left( \frac{\hat{w}}{n-1} - \frac{2(n-1)(n-2-(n+2)m)}{(1-m)\beta} \right) \frac{1}{s} \right] \quad \text{in} \mathbb{R},
\] (3.6)

where
\[
a_3 = \frac{(n-1)(n-2-(n+2)m)}{(1-m)\beta}.
\]

By Lemmas 3.1, 3.3 and an argument similar to the proof of Lemma 2.5 of [Hs5], we have the following lemma.

**Lemma 3.3.** Let \( n \geq 3, 0 < m < \frac{\alpha^2}{n}, m \neq \frac{\alpha^2}{n+2} \) and \( \bar{\alpha}, \tilde{\alpha} \) satisfy (2.3). Then,
\[
\lim_{s \to \infty} \frac{s^2 h_1(s)}{\log s} = \frac{(n-1)(n-2-(n+2)m)^2}{2(n-2-nm)(1-m)\beta}.
\]
Corollary 3.4. Let \( n \geq 3, 0 < m < \frac{n-2}{n}, m \neq \frac{n-2}{n^2}, \eta > 0, \) and \( \alpha, \beta \) satisfy (2.3). Then

\[
K(\eta, \beta) := \lim_{s \to \infty} h_1(s) \in \mathbb{R} \quad \text{exists}
\]  

(3.7)

and

\[
h_1(s) = K(\eta, \beta) + \frac{(n-1)(2-2(n+2)m)^2}{2(n-2-nm)(1-m)\beta} \left( \frac{1 + \log s}{s} \right) + o\left( \frac{1 + \log s}{s} \right)
\]  

(3.8)

as \( s \to \infty \).

Proof. By Lemma 3.3 there exists a constant \( C_1 > 0 \) such that

\[
\left| \frac{s^2 h_{1,s}(s)}{\log s} \right| \leq C_1 \quad \forall s \geq 2
\]

\[
\Rightarrow |h_1(s_1) - h_1(s_2)| \leq \int_{s_1}^{s_2} |h_{1,s}(z)| dz \leq \int_{s_1}^{s_2} \frac{1}{z^{3/2}} dz \leq C_1 \int_{s_1}^{s_2} \frac{1}{z^{3/2}} dz \leq \frac{C'}{\sqrt{s_1}}
\]  

(3.9)

for any \( s_2 > s_1 \geq 2 \). Hence (3.7) holds. Then by Lemma 3.3, (3.7) and an argument similar to the proof of Corollary 2.6 of [Hs5], we get that (3.8) holds and the corollary follows.

Corollary 3.5. Let \( n \geq 3, 0 < m < \frac{n-2}{n}, m = \frac{n-2}{n^2}, \eta > 0, \) and \( \alpha, \beta \) satisfy (2.3). Then,

\[
K(\eta, \beta) := \lim_{s \to \infty} h(s) \in \mathbb{R} \quad \text{exists}
\]  

(3.10)

and

\[
h(s) = K(\eta, \beta) - \frac{(n-1)(1-2m)}{(1-m)\beta} \cdot \frac{1}{s} + o(s^{-1}) \quad \text{as } s \to \infty.
\]  

(3.11)

Proof. By (3.2) and an argument similar to the proof of Corollary 3.4 (3.10) holds. We now let (cf. proof Corollary 2.4 of [CD]),

\[
h_3(s) = h(s) - K(\eta, \beta) + \frac{(n-1)(1-2m)}{(1-m)\beta} \cdot \frac{1}{s}
\]  

(3.12)

and

\[
\lim_{s \to \infty} s^2 h_{3,s}(s) = 0.
\]  

(3.13)

Hence \( h_{3,s}(s) = o(s^{-1}) \) and the corollary follows.

For \( m \neq \frac{n-2}{n^2} \), let

\[
h_2(s) = h_1(s) - K(\eta, \beta) - \frac{(n-1)(2-2(n+2)m)^2}{2(n-2-nm)(1-m)\beta} \left( \frac{1 + \log s}{s} \right).
\]  

(3.14)

Then,

\[
h_{2,s} + \left( \frac{2(n-2-nm)}{(1-m)s} + \frac{\beta}{n-1} \right) h_{2,s} = -\frac{1-2m}{1-m} \frac{\eta}{\tilde{w}} + \frac{(n-1)(n-2-(n+2)m)^2}{(1-m)^2\beta} \cdot \frac{1}{s} - \frac{(n-2-(n+2)m)h_1}{1-m} \frac{a_3}{s^3} + \frac{(n-1)(n-2-(n+2)m)^2}{2(n-2-nm)(1-m)\beta} \left( \frac{1 - 2 \log s}{s^3} \right) + \left( \frac{\beta}{n-1} - \frac{n-2-(n+2)m}{1-m} \right) \frac{\log s}{s^2}.
\]  

(3.15)

By (3.15), Theorem 2.4 and an argument similar to the proof of Lemma 2.7 of [Hs5], we get the following result.
Lemma 3.6. Let \( n \geq 3 \), \( 0 < m < \frac{2n^2}{n+1} \), \( m \neq \frac{2n^2}{n+1} \) and \( \tilde{\alpha}, \tilde{\beta} \) satisfy (2.3). Then,

\[
\lim_{s \to \infty} s^2 h_{2,s}(s) = \frac{(1-m)a_2(\tilde{\alpha}, \tilde{\beta})}{2(n-2-nm)\beta^2}
\]

where \( a_2(\tilde{\alpha}, \tilde{\beta}) \) is given by (1.17) and \( K(\tilde{\alpha}, \tilde{\beta}) \) given by (1.7).

By (3.7), (3.14), Lemma 3.6 and an argument similar to the proof of Corollary 3.5 we have the following result.

Corollary 3.7. Let \( n \geq 3 \), \( 0 < m < \frac{2n^2}{n+1} \), \( m \neq \frac{2n^2}{n+1} \) and \( \tilde{\alpha}, \tilde{\beta} \) satisfy (2.3). Then,

\[
h_2(s) = -\frac{(1-m)a_2(\tilde{\alpha}, \tilde{\beta})}{2(n-2-nm)\beta^2} - \frac{1}{s} + o\left(\frac{1}{s}\right) \quad \text{as } s \to \infty,
\]

where \( a_2(\tilde{\alpha}, \tilde{\beta}) \) is given by (1.17).

Let

\[
h_3(s) = h_2(s) + \frac{(1-m)a_2(\tilde{\alpha}, \tilde{\beta})}{2(n-2-nm)\beta^2}.
\]

Lemma 3.8. Let \( n \geq 3 \), \( 0 < m < \frac{2n^2}{n+1} \) and \( m \neq \frac{2n^2}{n+1} \). \( \eta > 0 \) and \( \tilde{\beta} > 0 \). Then, for \( \eta, \tilde{\beta} \)

\[
\lim_{s \to \infty} s^2 h_{3,s}(s) = 0
\]

and

\[
h_3(s) = o\left(\frac{1}{s}\right) \quad \text{as } s \to \infty.
\]

Proof. By Lemma 3.6 and the l'Hospital rule,

\[
\lim_{s \to \infty} \frac{h_3(s)}{1/s} = \lim_{s \to \infty} \frac{h_{3,s}(s)}{1/(1/s^2)} = \lim_{s \to \infty} s^2 h_{3,s}(s) = 0
\]

and the lemma follows. \( \Box \)

Since \( \tilde{\nu}(s) = w(r) \), where \( s = \log r \), by (2.3), (2.10), (3.2), (3.5), (3.14), (3.17), Corollary 3.5 we have the following proposition.

Proposition 3.9. Let \( n \geq 3 \) and \( 0 < m < \frac{2n^2}{n+1} \). Let \( a_1 \) by given by (1.15) and \( K_0 \) given by (1.16). Then the following holds.

\[
g_{1,1}(r)^{1-m} = \frac{2(n-1)(n-2-nm)}{(1-m)r^{\frac{n+1}{n}}} \left\{ \log r + \frac{(n-2-(n+2)m)}{2(n-2-nm)} \log(\log r) + K_0 \right. \\
+ \frac{a_1}{\log r} + \frac{(n-2-(n+2)m)}{4(n-2-nm)^2} \cdot \frac{\log(\log r)}{\log r} + o\left(\frac{1}{\log r}\right) \Bigg\} \quad \text{as } r \to \infty,
\]

where \( a_1 \) is given by (1.15).

Lemma 3.10. Let \( n \geq 3 \), \( 0 < m < \frac{2n^2}{n+1} \), \( \alpha, \beta \) satisfy (1.4) and \( \gamma_1 \) be given by (1.11). Let \( f_{\beta, A}(r) \) be the unique solution of (1.3) satisfying (1.5) and (1.6) with a positive constant \( A > 0 \) given by Theorem 1.4. Then

\[
f_{\beta_1 A_1}(r) = (A_2/A_1)^{1/m} f_{\beta_1 A_1} \left( (A_2/A_1)^{\frac{1}{m}} r \right),
\]

where

\[
f_{\beta_1 \gamma_1 \beta_2 \gamma_1}(r) = (\beta_2/\beta_1)^{\frac{1}{m}} f_{\beta_1 A_1}(r),
\]

for any \( \beta_1 < 0, \beta_2 < 0 \) and \( A_1 > 0, A_2 > 0 \).
Proof. Let

\[ F_\mu = \mu^{\frac{1}{\beta_1}} f_{\beta_1, A_1}(\mu r) \quad \forall \mu > 0, r \geq 0. \]

Then \( F_\mu \) satisfies (1.13). We now let \( \mu = (A_2/A_1)^{\frac{1}{\beta_1}} \). Then

\[ r^{\frac{1}{\beta_1}} F_\mu(r) = (\mu r)^{\frac{1}{\beta_1}} f_{\beta_1, A_1}(\mu r) \mu^{\frac{1}{\beta_1}} \sim A_2 \mu^{\frac{1}{\beta_1}} \sim A_1 \quad \text{as } r \to \infty \]

and

\[ \lim_{r \to 0} r^2 F_\mu(r)^{1-m} \sim \lim_{r \to 0} \frac{(\mu r)^2 f_{\beta_1, A_1}(\mu r)^{1-m}}{\log \mu - \log(\mu r)} = \frac{2(n-1)(n-2-nm)}{\beta(1-m)}. \]

Hence by Theorem (1.1),

\[ F_\mu(r) = f_{\beta_1, A_1}(r) \quad \forall r > 0 \]

and (3.22) follows.

For the second equality, we let

\[ \tilde{F}_{\mu_1}(r) = \mu_1 f_{\beta_1, A_1}(r) \quad \text{with} \quad \mu_1 = (\beta_2/\beta_1)^{\frac{1}{\beta_1}}. \]

Then \( \tilde{F}_{\mu_1} \) satisfies (1.13), (1.5), with

\[ r^{\frac{1}{\beta_1}} \tilde{F}_{\mu_1}(r) = (\beta_2/\beta_1)^{\frac{1}{\beta_1}} r^{\frac{1}{\beta_1}} f_{\beta_1, A_1}(r) \to (\beta_2/\beta_1)^{\frac{1}{\beta_1}} A_1 \quad \text{as } r \to \infty. \]

Hence by Theorem (1.1),

\[ \tilde{F}_{\mu_1}(r) = f_{\beta_1, \beta_2/\beta_1, A_1}(r) \quad \forall r > 0 \]

and (3.23) follows.

Proof of Theorem (1.3) Let \( \gamma_1 \) be given by (1.11) and \( A = \lambda^{\gamma_1} \). By Proposition 3.9

\[ f_{\gamma_1, 1, 1}(r) = r^{\frac{1}{\gamma_1}} f_{\gamma_1, 1, 1}(r) \]

\[ = \frac{2(n-1)(n-2-nm)}{(1-m)r^2} \left( - \log r + \frac{(n-2-(n+2)m)}{2(n-2-nm)} \log(-\log r) + K_0 ight) + a_1 \frac{-\log r}{4(n-2-nm)^2} \log(-\log r) + a \left( \frac{1}{\log(-\log r)} \right) \quad \text{as } r \to 0. \]

Hence by (3.24), Remark (1.2) and Lemma (3.10)

\[ f_{\gamma_1, 1, 1}(r)^{1-m} = f_{\beta A}(r)^{1-m} = (A^{-\gamma_1})^{\frac{1}{\gamma_1}} f_{\beta \gamma_1, A}(r) \sim \left( A^{-\gamma_1} \right)^{\frac{1}{\gamma_1}} r^{\frac{1}{\gamma_1}} \]

\[ = \left( A^{-\gamma_1} \right)^{\frac{1}{\gamma_1}} (-\beta)^{-1} f_{\beta, 1, 1} \left( (A^{-\gamma_1})^{\frac{1}{\gamma_1}} \right)^{1-m} \]

\[ = \frac{2(n-1)(n-2-nm)}{(1-m)(-\beta)r^2} \left( - \log r + \frac{(n-2-(n+2)m)}{2(n-2-nm)} \log(-\log r) ight) + K_0 + \frac{1}{\gamma_1} \log A + \frac{1}{\gamma_1} \log(-\log r) + a_1 \frac{-\log r}{4(n-2-nm)^2} \log(-\log r) + a \left( \frac{1}{\log(-\log r)} \right) \quad \text{as } r \to 0. \]

and (1.18) follows.
Lemma 3.11. Let \( n \geq 3 \) and \( 0 < m < \frac{n-2}{n} \). Then
\[
g_{\tilde{\beta}_1, \eta_1}(r) = (\eta_1/\eta_2)g_{\tilde{\beta}_1, \eta_2} \left( (\eta_1/\eta_2)^{\frac{m-1}{n-m}} r \right) \quad \forall r \geq 0
\] (3.25)
and
\[
g_{\tilde{\beta}_1, \tilde{\eta}_1}(r) = (\tilde{\beta}_2/\tilde{\beta}_1) \frac{r^{\frac{1}{m-1}}}{r^{\frac{1}{n-1}}} g_{\tilde{\beta}_2, \eta_1}(r) \quad \forall r > 0,
\] (3.26)
hold for any \( \tilde{\beta}_1 < 0, \tilde{\beta}_2 < 0 \) and \( \eta_1 > 0, \eta_2 > 0 \).

Proof. By direct computation the function
\[
G_1(r) := (\eta_1/\eta_2)g_{\tilde{\beta}_1, \eta_2} \left( (\eta_1/\eta_2)^{\frac{m-1}{n-m}} r \right) \quad \forall r \geq 0
\]
satisfies \( (2.9) \) with \( \eta = \eta_1 \) and \( \tilde{\alpha} = \tilde{\alpha}_1, \tilde{\beta} = \tilde{\beta}_1 \), given by \((2.3)\). Hence by Theorem \( 2.1 \)
\[G_1(r) = g_{\tilde{\beta}_1, \eta_1}(r) \quad \forall r > 0 \]
and \((3.25)\) follows. Similarly, the function
\[
G_2(r) := (\tilde{\beta}_2/\tilde{\beta}_1) \frac{r^{\frac{1}{m-1}}}{r^{\frac{1}{n-1}}} g_{\tilde{\beta}_2, \eta_1}(r) \quad \forall r \geq 0
\]
satisfies \((2.9)\) with \( \eta = (\tilde{\beta}_2/\tilde{\beta}_1) \frac{r^{\frac{1}{m-1}}}{r^{\frac{1}{n-1}}} \eta_1 \) and \( \tilde{\alpha} = \tilde{\alpha}_1, \tilde{\beta} = \tilde{\beta}_1 \), given by \((2.3)\). Hence by Theorem \( 2.1 \)
\[G_2(r) = g_{\tilde{\beta}_1, \tilde{\eta}_1}(r) \quad \forall r > 0 \]
and \((3.26)\) follows. \( \Box \)

Proposition 3.12. Let \( n \geq 3, 0 < m < \frac{n-2}{n}, \eta > 0 \) and \( \tilde{\alpha}, \tilde{\beta} \) satisfy \((2.3)\). Let \( a_1 \) by given by \((1.15)\) and \( K_0 \) given by \((1.16)\). Then the following holds.

\[
g_{\tilde{\beta}, \eta} \left( r^m \right) = \frac{2(n-1)(n-2-nm)}{(1-m)\tilde{\beta}r^{\frac{n-1}{n-m}}} \left\{ \log r + \frac{(n-2-(n+2)m)}{2(n-2-nm)} \log(\log r) \right. \\
+ K_0 + \frac{m(1-m)}{n-2-nm} \log \eta + \frac{m}{n-2-nm} \log \tilde{\beta} \\
+ \frac{a_1}{\log r} + \frac{(n-2-(n+2)m)^2}{4(n-2-nm)^2} \cdot \frac{\log(\log r)}{\log r} + o\left( \frac{1}{\log r} \right) \left\} \right. \quad \text{as } r \to \infty.
\] (3.27)

Proof. By Lemma 3.11
\[
g_{\tilde{\beta}, \eta} \left( r^m \right) = (\eta \tilde{\beta}^\frac{1}{n-m})g_{\tilde{\beta}, \eta} \left( (\eta \tilde{\beta}^\frac{1}{n-m}) \frac{r^{\frac{n-1}{n-m}}}{r^{\frac{n-1}{n-m}}} \right) = \eta g_{\tilde{\beta}, \eta} \left( (\eta \tilde{\beta}^\frac{1}{n-m}) \frac{r^{\frac{n-1}{n-m}}}{r^{\frac{n-1}{n-m}}} \right).
\] (3.28)

By \((3.21)\) and \((3.28)\), we get \((3.27)\) and the proposition follows. \( \Box \)

4. Existence, Uniqueness and Asymptotic Large Time Behaviour of Singular Solutions

In this section, we prove existence, uniqueness and asymptotic large time behaviour of singular solutions of \((1.2)\), Theorems \((1.5)\) and \((1.7)\). As discussed in the introduction section the \( L^1 \)-contractions, Propositions \((1.4)\) and \((1.6)\) play an important role in the theory.
4.1. **Existence and Uniqueness of Singular Solutions.** In this subsection, we prove the existence and uniqueness of solution of (1.2) which satisfies (1.14). Note that the existence of a solution of (1.2) which satisfies (1.14) and (1.25) in Theorem 1.5 follows by an argument similar to the proof of Theorem 1.3 of [HK]. Hence, we will focus on the proof of the uniqueness of the solution of (1.2) in this subsection.

The main proposition for the uniqueness of the solution is the $L^1$-contraction with a weight function $r^{-\mu}$ in (1.22), Proposition 1.4. In the proof of the $L^1$-contraction, an estimate on the difference of the self-similar profiles, (4.2) is essential. Thus, for a first step, we formulate the function $f_1$ with $h_3$ which is defined in Corollary 3.5 and Lemma 3.8 instead of the little-O notation.

Indeed, Proposition 3.9 with $h_3(\log r)$ instead of $o\left(\frac{1}{\log r}\right)$ gives

\[
f_{1,1}(r) = \frac{c_0}{r^2} \left\{ -\log r + c_1 \log(-\log r) + K_0 + \frac{a_1}{\log r} + c_2 \frac{\log(-\log r)}{-\log r} + h_3(-\log r) \right\},
\]

for sufficient large $r$, where $a_1$ and $K_0$ are given by (1.15) and (1.16) respectively and $c_1 := \frac{(n-2-(n+2)\beta)}{2(n-2-\mu)}$ and $c_2 := \frac{(n-2-(n+2)\mu)}{4(n-2-\mu)}$. Then, since

\[
A = \lambda^{-\gamma_1} \quad \text{and} \quad f_{1,1}^{-m}(r) = \left( A(-\beta) \frac{\hat{m}}{n} \right)^{\frac{1}{\gamma_1}} \left( -\beta \right)^{\frac{1}{\gamma_1}} f_{1,1}^{-m}(A(-\beta) \frac{\hat{m}}{n} \left( -\beta \right)^{-1} f_{1,1}^{-m}(A(-\beta) \frac{\hat{m}}{n}), \right.
\]

we have

\[
f_{1}^{-m}(r) = \left( A(-\beta) \frac{\hat{m}}{n} \right)^{\frac{1}{\gamma_1}} \left( -\beta \right)^{\frac{1}{\gamma_1}} f_{1,1}^{-m}(A(-\beta) \frac{\hat{m}}{n} \left( -\beta \right)^{-1} f_{1,1}^{-m}(A(-\beta) \frac{\hat{m}}{n}) \right.
\]

\[
= \frac{\tilde{c}_0}{r^2} \left\{ -\log r + c_1 \log \left( \log \left( \frac{c_1}{r} \right) \right) + K_0 + \log \frac{1}{\lambda} + \frac{1}{\gamma_1(1-m)} \log(-\beta) \right\} \\
+ \frac{a_0}{\log \left( \frac{c_1}{r} \right)} + c_1 \frac{\log \left( \frac{c_1}{r} \right)}{\log \left( \frac{c_1}{r} \right)} + h_3 \left( \log \left( \frac{c_1}{r} \right) \right),
\]

where $\tilde{c}_0 := \frac{c_0}{\beta}$ and $c_1 := \frac{(-\beta)^{\gamma_1+\frac{1-m}{\lambda}}}{\lambda}$. In the following lemma, we will use (4.1) instead of (1.18) in Theorem 1.3.

**Lemma 4.1.** Let $n \geq 3$, $0 < m < \frac{n-2}{n}$ and $\lambda_1 > \lambda_0 > 0$. Then there is $\varepsilon_1 = \varepsilon_1(\lambda_0, \lambda_1)$ such that

\[
f_{h_0}(r) - f_{h_1}(r) \leq \frac{C}{r^{\frac{1}{\gamma_1}}} \left( \log r^{-1} \right)^{\frac{n}{\gamma_1}}, \quad \text{for} \ r \leq \varepsilon_1.
\]
Proof. By the mean value theorem and (5.11),

\[
    f_{\delta_0}(r) - f_{\delta_1}(r) = \frac{\varepsilon_0^m}{r^{1-m}} \left\{ \left( \frac{c^2}{\varepsilon_0} f_{\delta_0}^{1-m}(r) \right)^{\frac{1}{1-m}} - \left( \frac{r^2}{\varepsilon_0} f_{\delta_1}^{1-m}(r) \right)^{\frac{1}{1-m}} \right\} \\
    \leq \frac{\varepsilon_0^m}{r^{1-m}} \frac{1}{1-m} \max \left\{ \left( \frac{c^2}{\varepsilon_0} f_{\delta_0}^{1-m}(r) \right)^{\frac{1}{1-m}}, \left( \frac{r^2}{\varepsilon_0} f_{\delta_1}^{1-m}(r) \right)^{\frac{1}{1-m}} \right\} \left( \frac{c^2}{\varepsilon_0} f_{\delta_0}^{1-m}(r) - \frac{r^2}{\varepsilon_0} f_{\delta_1}^{1-m}(r) \right) \\
    \leq \frac{\varepsilon_0^m}{r^{1-m}} \frac{1}{1-m} \max \left\{ \left( \frac{c^2}{\varepsilon_0} f_{\delta_0}^{1-m}(r) \right)^{\frac{1}{1-m}}, \left( \frac{r^2}{\varepsilon_0} f_{\delta_1}^{1-m}(r) \right)^{\frac{1}{1-m}} \right\} \\
    \times \left\{ \log \frac{\delta_1}{\delta_0} + c_1 \left( \log \left( \frac{c_0}{\varepsilon_0} \right) - \log \left( \frac{c_1}{r} \right) \right) \right\} \\
    + \frac{a_0}{\log \left( \frac{c_0}{\varepsilon_0} \right)} - \frac{a_0}{\log \left( \frac{c_0}{r} \right)} + c_2 \left( \log \left( \frac{c_1}{\varepsilon_0} \right) - \frac{a_0}{\log \left( \frac{c_0}{\varepsilon_0} \right)} \right) \\
    + h_3 \left( \log \left( \frac{c_0}{\varepsilon_0} \right) - h_3 \left( \log \left( \frac{c_0}{r} \right) \right) \right). 
\]

By the mean value theorem and (3.19), there is \( \varepsilon_1 \) such that for \( r \leq \varepsilon_1 \),

\[
    \left| h_3 \left( \log \left( \frac{c_0}{\varepsilon_0} \right) \right) - h_3 \left( \log \left( \frac{c_0}{r} \right) \right) \right| \leq \frac{1}{2} \frac{a_0}{\log \left( \frac{c_0}{\varepsilon_0} \right)} \log \frac{\delta_1}{\delta_0}. 
\]

By (1.25) and the mean value theorem on \( \log \left( \frac{c_0}{\varepsilon_0} \right) - \log \left( \frac{c_0}{r} \right) \), \( \frac{a_0}{\log \left( \frac{c_0}{\varepsilon_0} \right)} - \frac{a_0}{\log \left( \frac{c_0}{r} \right)} \), and \( \log \left( \frac{c_0}{\varepsilon_0} \right) - \log \left( \frac{c_0}{r} \right) \), we have

\[
    f_{\delta_0}(r) - f_{\delta_1}(r) \leq \frac{\varepsilon_0^m}{r^{1-m}} \frac{1}{1-m} \max \left\{ \left( \frac{c^2}{\varepsilon_0} f_{\delta_0}^{1-m}(r) \right)^{\frac{1}{1-m}}, \left( \frac{r^2}{\varepsilon_0} f_{\delta_1}^{1-m}(r) \right)^{\frac{1}{1-m}} \right\} \\
    \times \log \frac{\delta_1}{\delta_0} \left\{ 1 + \frac{c_1}{\log \left( \frac{c_0}{\varepsilon_0} \right)} + \frac{2a_0}{\log \left( \frac{c_0}{\varepsilon_0} \right)} + c_2 \frac{\log \left( \frac{c_0}{\varepsilon_0} \right) - 1}{\log \left( \frac{c_0}{\varepsilon_0} \right)^2} \right\} \\
    \leq \frac{C}{r^{1-m}} \left( \log r^{-1} \right)^{\frac{m+1}{m}},
\]

for \( r \leq \varepsilon_1 \). \( \square \)

**Proof of Proposition 5.2** We will use a modification of the proof of Lemma 4.1 of [DS1] and Theorem 1.2 of [HK] to prove the theorem. We choose \( \eta \in C_0^\infty(\mathbb{R}^n) \) such that \( 0 \leq \eta \leq 1 \), \( \eta = 1 \) for \( |x| \leq 1 \), and \( \eta = 0 \) for \( |x| \geq 2 \). For any \( R > 2 \), and \( 0 < \varepsilon < 1 \), let \( \eta_\varepsilon(x) := \eta(x/R) \), \( \eta_\varepsilon(x) := \eta(x/\varepsilon) \), and \( \eta_{\varepsilon,R}(x) = \eta_\varepsilon(x) - \eta_\varepsilon(x) \). Then \( |\nabla \eta_{\varepsilon,R}|^2 + |\Delta \eta_{\varepsilon,R}| \leq C \varepsilon^{-2} \) for \( \varepsilon \leq |x| \leq 2 \varepsilon \), and \( |\nabla \eta_{\varepsilon,R}|^2 + |\Delta \eta_{\varepsilon,R}| \leq C R^{-2} \) for \( R \leq |x| \leq 2R \). By the Kato inequality ([DK], [K]),

\[
    \frac{\partial}{\partial t} [u - v] \leq \frac{n-1}{m} \Delta (|u^m - v^m|) \quad \text{in} \quad \mathcal{D} ((\mathbb{R}^n \setminus \{0\}) \times (0, \infty)).
\]
Then
\[
\frac{d}{dt} \int_{\mathbb{R}^n \setminus \{0\}} |u - v|(x, t) \eta_{E,R}(x)|x|^{-\mu} \, dx
\]
\[
\leq \int_{\mathbb{R}^n \setminus \{0\}} |u^n - v^n|(x, t) \Delta (\eta_{E,R}(x)|x|^{-\mu}) \, dx
\]
\[
= \int_{\mathbb{R}^n \setminus \{0\}} |u^n - v^n|(x, t) \{ |x|^{-\mu} \Delta \eta_{E,R}(x) + 2\nabla \eta_{E,R} \cdot \nabla |x|^{-\mu} + \eta_{E,R} \Delta |x|^{-\mu} \} \, dx. \quad (4.3)
\]
Let \( r_1 \) be as given in Remark 1.2, \( 0 < \varepsilon_0 < 1 \) and \( R_0 = \max(2, r_1 / \lambda_2) \). Since \( 0 < \mu \leq \mu_1 < n - 2 \),
\[
\Delta |x|^{-\mu} = \mu \{ \mu - (n - 2) \} |x|^{-\mu - 2} \leq 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}.
\]
Let \( T_0 > 0 \). By integrating (4.3) over \((0, t)\), \( 0 < t < T_0 \), by (1.12), (1.20) and Remark 1.2 for any \( 0 < t < T_0 \), \( 0 < \varepsilon \leq \varepsilon_0 \), and \( R \geq R_0 \),
\[
\int_{\mathbb{R}^n \setminus \{0\}} |u - v|(x, t) \eta_{E,R}(x)|x|^{-\mu} \, dx - \int_{\mathbb{R}^n \setminus \{0\}} |u_0(x) - v_0(x)| \eta_{E,R}(x)|x|^{-\mu} \, dx
\]
\[
\leq CR^{-2\mu} \int_0^t \int_{B_{2R} \setminus B_R} U_{I_3}^m(x, s) \, dx \, ds + Ce^{-2\mu} \int_0^t \int_{B_{2R} \setminus B_R} \left( U_{I_3}^m(x, s) - U_{I_3}^m(x, s) \right) \, dx \, ds
\]
\[
\leq CR^{-2\mu} \int_0^t \int_{B_{2R} \setminus B_R} f_{I_3}(e^{-\beta s}x)^m \, dx \, ds
\]
\[
+ Ce^{-2\mu} \int_0^t \int_{B_{2R} \setminus B_R} \left( f_{I_3}(e^{-\beta s}x)^m - f_{I_3}(e^{-\beta s}x)^m \right) \, dx \, ds
\]
\[
\leq CR^{-2\mu} \int_0^t \int_{B_{2R} \setminus B_R} |e^{-\beta s}x|^m \, dx \, ds
\]
\[
+ Ce^{-2\mu} \int_0^t \int_{B_{2R} \setminus B_R} \left( f_{I_3}(e^{-\beta s}x)^m - f_{I_3}(e^{-\beta s}x)^m \right) \, dx \, ds
\]
\[
\leq CR^{-2\mu} + Ce^{-2\mu} \int_0^t \int_{B_{2R} \setminus B_R} \left( f_{I_3}(e^{-\beta s}x)^m - f_{I_3}(e^{-\beta s}x)^m \right) \, dx \, ds
\]
\[
\leq CR^{-m} + I_1. \quad (4.4)
\]
Since by Theorem 1.3 Lemma 4.1 and the mean value theorem,
\[
f_{I_3}^m(r) - f_{I_3}^m(r) \sim r^{-\frac{m}{\alpha}} \left( \log r^{-1} \right)^{\frac{m}{\alpha} - 1} \quad \forall 0 < r \leq \varepsilon \quad (4.5)
\]
for sufficiently small \( \varepsilon \in (0, \varepsilon_0) \), hence for sufficiently small \( \varepsilon \in (0, \varepsilon_0) \),
\[
I_1 \leq Ce^{-2\mu} \int_0^t \int_{B_{2R} \setminus B_R} |x|^{-\mu} \left( \log((e^{-\beta s}r)^{-1}) \right)^{\frac{m}{\alpha} - 1} \, dx \, ds
\]
\[
\leq Ce^{-\frac{m}{\alpha} - \mu} \left( \log \varepsilon \right)^{\frac{m}{\alpha} - 1}. \quad (4.6)
\]
Since \( T_0 \) is arbitrary, letting \( \varepsilon \to 0 \) and \( R \to \infty \) in (4.4), by (1.22) and (4.6), (1.25) follows. By an argument similar to the proof of (1.22) we get (1.24).

By Proposition 1.4 and an argument similar to the proof of Theorem 1.3 of [HK], we have the existence and uniqueness of the solution to (1.2) which satisfies (1.14), Theorem 1.5.
4.2. Asymptotic Large Time Behaviour of Singular Solutions. In this subsection, we discuss the asymptotic large time behaviour of singular solutions, Theorem 1.7 for $n = 3, 4$ and $\frac{n-2}{n} \leq m < \frac{n-2}{n-1}$. Since the $L^1$-contraction with the weigh function $r^{-\mu}$, Proposition 1.7 is not appropriate to the asymptotic behaviour, as discussed in the introduction of this paper, we will use the $L^1$-contraction with the weigh function of the form $f^{\alpha r}$, for some $\lambda > 0$ and $\gamma > 0$, Proposition 1.2, to have the asymptotic behaviour.

**Lemma 4.2.** Let $n \geq 3$, $0 < m < \frac{n-2}{n}$ and $g, w$ be given (2.4) and (2.10), where $f = f_1$ is the solution of (1.3) which satisfies (1.6) with $A = 1$. Then

$$\lim_{r \to \alpha} \frac{rg_r}{g} = -\frac{\alpha}{\beta}$$

(4.7)

and

$$\lim_{r \to 0} \frac{rf_r}{f} = -\frac{2}{1-m}$$

(4.8)

Hence, there exists constants $r_2 > 0$ and $C_1 > C_2 > 0$ such that

$$-C_1rf_r(r) \leq f(r) \leq -C_2rf_r(r) \ \forall 0 < r \leq r_2.$$  

(4.9)

**Proof.** By Lemma 2.3, Theorem 2.4, and a direct computation,

$$rw_r = (1-m)w\left(\frac{\alpha}{\beta} + \frac{rg(r)}{g}\right) \ \forall r > 0$$

$$\Rightarrow \ (1-m) \lim_{r \to 0} \left(\frac{\alpha}{\beta} + \frac{rg(r)}{g}\right) = \lim_{r \to 0} \frac{rw_r}{w} = \lim_{r \to 0} \left(\frac{rg_r}{g}\right) = 0$$

and (4.7) follows. By (2.4), (2.5), (4.7), and a direct computation,

$$\frac{rf_r}{f} = \frac{-n-2}{m} - \frac{r^{-1}g(r^{-1})}{g(r^{-1})} \ \forall r > 0$$

$$\Rightarrow \ \lim_{r \to 0} \frac{rf_r}{f} = \frac{-n-2}{m} - \lim_{r \to 0} \frac{r^{-1}g(r^{-1})}{g(r^{-1})} = \frac{-n-2}{m} + \frac{\alpha}{\beta} = \frac{-2}{1-m}$$

and (4.8) follows. By (4.8) there exists constants $r_2 > 0$ and $C_2 > C_1 > 0$ such that (4.9) holds. \hfill \Box

By (1.8) and Lemma 4.2, we have the following corollary.

**Corollary 4.3.** Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $\lambda > 0$, $f = f_1$ be given by (1.8) and $r_2 > 0$, $C_1 > C_2 > 0$ be as in Lemma 4.2. Then

$$\lim_{r \to 0} \frac{rf_{1,r}}{f_1} = -\frac{2}{1-m}$$

(4.10)

and

$$-C_1rf_{1,r} \leq f_1(r) \leq -C_2rf_{1,r} \ \forall 0 < r \leq r_2/\lambda.$$  

(4.11)

**Lemma 4.4.** Let $n \geq 3$, $0 < m < \frac{n-2}{n}$ and let $f = f_1$ be the solution of (1.3) which satisfies (1.6) with $A = 1$. Then

$$\lim_{r \to \infty} \frac{rf_r}{f} = \frac{-n-2}{m}.$$  

(4.12)

Hence there exists constants $r_3 > 0$, $C_3 > C_4 > 0$, such that

$$-C_3rf_r \leq f(r) \leq -C_4rf_r \ \forall r \geq r_3.$$  

(4.13)
Proof. By (1.6), (4.10) and Theorem 2.1
\[ \lim_{r \to 0} r g_r(r) = 0 \quad \text{and} \quad g(0) = 1 \]
\[ \Rightarrow \lim_{r \to 0} \frac{r g_r(r)}{g(r)} = 0 \]
\[ \Rightarrow \lim_{r \to \infty} \frac{r f_r(r)}{f(r)} = \frac{n - 2}{m} - \lim_{r \to \infty} \frac{r^{-1} g_r(r^{-1})}{g(r^{-1})} = -\frac{n - 2}{m} \]
and (4.11) follows. By (4.11), there exists constants \( r_1 > 0, C_4 > C_3 > 0 \), such that (4.12) holds.

By (1.8) and Lemma 4.4 we have the following corollary.

Corollary 4.5. Let \( n \geq 3, 0 < m < \frac{n^2}{n}, \lambda > 0, f = f_\lambda \) be given by (1.3) and \( r_3 > 0, C_3 > C_4 > 0 \) be as in Lemma 4.4. Then,
\[ \lim_{r \to \infty} \frac{r f_{3r}(r)}{f_r(r)} = -\frac{n - 2}{m} \]
and
\[ -C_3 r f_{3r}(r) \leq f_3(r) \leq -C_4 r f_{3r}(r) \quad \forall r \geq r_3/\lambda. \]

Proof of Proposition 4.6. We will use a modification of the proof of Lemma 4.1 of [DS1] and Theorem 1.2 of [HK] to prove the proposition. Let \( q(x, t) = |u(x, t) - \psi(x, t)| \) and \( \eta_{\epsilon, R} \) be as in the proof of Proposition 1.4. By the Kato inequality ([DK], K),
\[ q_\epsilon \leq \frac{n - 1}{m} \Delta (a(x, t)q) \quad \text{in} \ (\mathbb{R}^n \setminus \{0\}) \times (0, \infty), \quad \text{(4.13)} \]
where
\[ m U_{\lambda_1}^{m-1}(x, t) \leq a(x, t) := \int_0^1 \frac{m \, ds}{su + (1 - s)v} \leq m U_{\lambda_1}^{m-1}(x, t) \quad \forall x \in \mathbb{R}^n \setminus \{0\}, t > 0. \quad \text{(4.14)} \]
By (4.13),
\[ \frac{d}{dt} \int_{\mathbb{R}^n \setminus \{0\}} q(x, t) \eta_{\epsilon, R}(x) f_{\lambda_1}(x)^m \, dx \]
\[ \leq \frac{n - 1}{m} \int_{\mathbb{R}^n \setminus \{0\}} a(x, t) q(x, t) \left[ f_{\lambda_1}^m \Delta \eta_{\epsilon, R} + 2 \nabla f_{\lambda_1}^m \cdot \nabla \eta_{\epsilon, R} + \eta_{\epsilon, R} \Delta f_{\lambda_1}^m \right] \, dx. \quad \text{(4.15)} \]
Since \( \frac{n^2}{n+2} \leq m \leq \frac{n^2}{n} \) implies \( 0 < \gamma \leq 1 \), by (1.3) and (1.7), for \( h(x) := f_{\lambda_1}(x)^m \) we have
\[ \Delta h^r = (h^r)_r + \frac{n - 1}{r - (h^r)_r} \]
\[ = \gamma (\gamma - 1) h^{r-2} (h_r)^2 + \gamma h^{r-1} h_r + \frac{n - 1}{r} \gamma h^{r-1} h_r, \]
\[ \leq \gamma h^{r-1} \left[ \left( f_{\lambda_1}^m \right)_r + \frac{n - 1}{r} \left( f_{\lambda_1}^m \right)_r \right] = -\frac{m y}{n - 1} \gamma h^{r-1} \left( \alpha f_{\lambda_1} + \beta r f_{\lambda_1} \right) \]
\[ < 0 \quad \text{in} \ \mathbb{R}^n \setminus \{0\}. \quad \text{(4.16)} \]
Let \( r_1, r_2, r_3 \) be as given in Remark 4.2, Corollary 4.3 and Corollary 4.5. Let
\[ \epsilon_0 = \frac{r_2}{\max(\lambda_1, \lambda_3)} \quad \text{and} \quad R_0 = \frac{\max(r_1, r_3)}{\min(\lambda_2, \lambda_3)}. \]
Then by (1.12), (1.20), (4.14), (4.15), (4.16), Corollary 4.3, and Corollary 4.5 for any \(0 \leq \varepsilon \leq \varepsilon_0/2, R \geq R_0\) and \(t > 0\),

\[
\frac{d}{dt} \int_{E^+(0)} q(x, t) \eta_{\varepsilon, R}(x) f_{A_1}(x)^m y^r \, dx \\
\leq \frac{n-1}{m} \int_{E^+(0)} a(x, t) q(x, t) \left[ f_{A_1}^{m y} \Delta \eta_{\varepsilon, R} + 2 \nabla f_{A_1}^{m y} \cdot \nabla \eta_{\varepsilon, R} \right] \, dx \\
\leq CR^{-2} \int_{B_{2R} \setminus B_0} U_{A_1}(x, t)^{m-1} U_{A_1}(x, t) f_{A_1}(x)^{m y} \, dx \\
+ CE^{-2} \int_{B_{2R} \setminus B_0} U_{A_1}(x, t)^{m-1} (U_{A_1}(x, t) - U_{A_1}(x, t)) f_{A_1}(x)^{m y} \, dx
\]

(4.17)

Integrating (4.17) over \((0, t)\),

\[
\int_{E^+(0)} q(x, t) \eta_{\varepsilon, R}(x) f_{A_1}(x)^{m y} \, dx - \int_{E^+(0)} q(x, 0) \eta_{\varepsilon, R}(x) f_{A_1}(x)^{m y} \, dx \\
\leq CR^{-2} \int_{0}^{t} e^{-\max} \int_{B_{2R} \setminus B_0} f_{A_1}(e^{-\beta s} x)^{m-1} f_{A_1}(e^{-\beta s} x) f_{A_1}(x)^{m y} \, dx \, ds \\
+ CE^{-2} \int_{0}^{t} e^{-\max} \int_{B_{2R} \setminus B_0} f_{A_1}(e^{-\beta s} x)^{m-1} (f_{A_1}(e^{-\beta s} x) - f_{A_1}(e^{-\beta s} x)) f_{A_1}(x)^{m y} \, dx \, ds
\]

\[=: I_1 + I_2 \quad \forall 0 \leq \varepsilon \leq \varepsilon_0, R \geq R_0, t > 0. \tag{4.18}\]

Since \(m < \frac{n}{n+2}\) and \(\alpha, \beta\) satisfy (1.4), we have \((n-2)\beta - ma = \frac{(n-2-2m)\gamma}{1-m} < 0\). Hence, by Remark 1.2 for any \(t > 0, R \geq R_0\),

\[
I_1 \leq CR^{-2} \int_{0}^{t} e^{-\max} (e^{-\beta s} R)^{2-n} R^{-(n-2)\gamma} R^n \, ds \\
\leq CR^{-(n-2)\gamma} \int_{0}^{t} e^{(n-2)\beta e^{-ms}} \, ds \leq CR^{-(n-2)\gamma} t. \tag{4.19}\]

We note that by (1.9) and Lemma 4.1

\[
f_{A_1}(r)^{m y} \sim r^2 (\log r^{-1})^{-1} \quad \text{and} \quad f_{A_2}(r) - f_{A_1}(r) \sim r^{-\frac{m}{1-m}} (\log r^{-1})^{\frac{m}{1-m}},
\]

for all \(0 < r < \varepsilon\) as \(\varepsilon \to 0\). Hence, for fixed \(T_0 > 0\), for any \(0 < t \leq T_0\) and sufficiently small \(\varepsilon > 0\),

\[
I_2 \leq CE^{-2} \int_{0}^{t} e^{-\max} e^{-2\beta s} e^2 \left( \log(e^{-\beta s} \varepsilon)^{-1} \right)^{-1} e^{\frac{2m}{1-m}} e^{-\frac{m}{1-m}} (\log(e^{-\beta s} \varepsilon)^{-1})^{\frac{m}{1-m}} \cdot e^{-\frac{2m}{1-m} (\log e^{-1})^{\frac{m}{1-m}}} e^{\frac{m}{1-m}} \, ds \\
\leq C \int_{0}^{t} (\log e^{-1})^{1+\frac{m}{1-m}+\frac{m}{1-(n-2)\gamma}} \, ds = C (\log e^{-1})^{1+\frac{m}{1-m}+\frac{m}{1-(n-2)\gamma}} t. \tag{4.20}\]
Since \( n \leq 4, -1 + \frac{m}{1-m} + \frac{1}{2}(n - \frac{2}{n+2}) = \frac{1}{2}n - 2 \leq 0 \). Hence, by (4.18), (4.19) and (4.20), for any \( 0 < t \leq T_0, R \geq R_0, \) and \( \varepsilon \) sufficiently small,
\[
\int_{\mathbb{R}^n \setminus \{0\}} q(x, t) \eta_{\varepsilon,R}(x)f_{\lambda_1}(x)^{m*} dx \leq \int_{\mathbb{R}^n \setminus \{0\}} q(x, 0)\eta_{\varepsilon,R}(x)f_{\lambda_1}(x)^{m*} dx
\]
\[
\leq C \left( R^{-(n-2)m} + (\log R^{-1})^{\frac{1}{n-2}} \right) t \leq Ct
\]
and
\[
\int_{\mathbb{R}^n \setminus \{0\}} q(x, t)f_{\lambda_1}(x)^{m*} dx \leq Ct + \int_{\mathbb{R}^n \setminus \{0\}} q(x, 0)f_{\lambda_1}(x)^{m*} dx.
\]
Since \( u_0 - v_0 \in L^1(f_{\lambda_1}^{m*} ; \mathbb{R}^n \setminus \{0\}) \), we have \( u - v \in L^1(f_{\lambda_1}^{m*} ; \mathbb{R}^n \setminus \{0\}) \) and
\[
\int_0^t \int_{\mathbb{R}^n \setminus \{0\}} q(x, s)f_{\lambda_1}(x)^{m*} dx ds \leq Ct^2 + t \int_{\mathbb{R}^n \setminus \{0\}} q(x, 0)f_{\lambda_1}(x)^{m*} dx, \quad \forall 0 < t \leq T_0.
\]
By the same argument as in (4.17) and the growth rate of \( f_{\lambda_1} \), (4.21).

Since \( T_0 \) is arbitrary, we have (1.31).

Suppose that (1.32) holds. Integrating (4.15) over \((0, t)\) and letting \( R \to \infty \) and \( \varepsilon \to 0 \), by an argument similar to the proof in the previous paragraphs, we have
\[
\int_{\mathbb{R}^n \setminus \{0\}} q(x, t)f_{\lambda_1}(x)^{m*} dx \leq \limsup_{R \to \infty, \varepsilon \to 0} \int_0^t \int_{\mathbb{R}^n \setminus \{0\}} a(x, t)q(x, t)\eta_{\varepsilon,R}\Delta f_{\lambda_1}(x)^{m*} dx ds
\]
\[
< 0 \quad \forall 0 < t < T_0.
\]
Since \( T_0 \) is arbitrary, (1.33) follows. \( \square \)

**Corollary 4.6.** Let \( n = 3, 4, \frac{n-2}{n+2} \leq m < \frac{n-2}{n}, \beta < 0, \) and \( \alpha, \gamma \) be given by (1.4) and (1.29) respectively. Let \( \lambda_1 > \lambda_2 > 0, \lambda_3 > 0, \) and \( f_{\lambda_1} \) be given by (1.8) with \( \lambda = \lambda_1, \lambda_2, \lambda_3. \) Let \( u_0, v_0 \) satisfy (1.19), (1.30), and \( \bar{u}, \bar{v} \) be solutions of
\[
\begin{align*}
\frac{\partial \bar{u}}{\partial t} &= \frac{n-1}{m} \Delta \bar{u} + \alpha \bar{u} + \beta \bar{v} \cdot \nabla \bar{u} \quad \text{in } (\mathbb{R}^n \setminus \{0\}) \times (0, \infty) \\
\bar{u}(\cdot, 0) &= u_0 \quad \text{in } \mathbb{R}^n \setminus \{0\},
\end{align*}
\]
with initial values \( u_0, v_0, \) respectively, which satisfies
\[
f_{\lambda_1}(x) \leq \bar{u}(x, t), \bar{v}(x, t) \leq f_{\lambda_3}(x) \quad \text{in } (\mathbb{R}^n \setminus \{0\}) \times (0, \infty).
\] (4.22)
Then
\[ \int_{\mathbb{R}^n \setminus [0]} |\bar{u} - \bar{v}||x,t)f_{\lambda_3}(x)^{my} \, dx \leq \int_{\mathbb{R}^n \setminus [0]} |u_0 - v_0||x,t)f_{\lambda_3}(x)^{my} \, dx \quad \forall t > 0. \tag{4.23} \]

If \((1.32)\) holds, then
\[ \int_{\mathbb{R}^n \setminus [0]} |\bar{u} - \bar{v}||x,t)f_{\lambda_3}(x)^{my} \, dx < \int_{\mathbb{R}^n \setminus [0]} |u_0 - v_0||x,t)f_{\lambda_3}(x)^{my} \, dx \quad \forall t > 0. \tag{4.24} \]

**Proof.** Let
\[ u(x,t) := e^{-\alpha t}\bar{u}(e^{-\beta}x,t) \quad \text{and} \quad v(x,t) := e^{-\alpha t}\bar{v}(e^{-\beta}x,t). \]

Then \(u\) and \(v\) are solutions of \((1.2)\) with initial values \(u_0\) and \(v_0\), respectively, which satisfy \((1.20)\). By using the fact that \(\alpha - n\beta = -\frac{2\gamma\nu}{1-m}\) and \((1.8)\), we have
\[ \int_{\mathbb{R}^n \setminus [0]} |\bar{u} - \bar{v}||x,t)f_{\lambda_3}(x)^{my} \, dx = e^{\alpha t} \int_{\mathbb{R}^n \setminus [0]} |u - v||x,t)f_{\lambda_3}(x)^{my} \, dx = \int_{\mathbb{R}^n \setminus [0]} |u - v||y,t)(e^{-\beta y})^{my} \, dy = \int_{\mathbb{R}^n \setminus [0]} |u - v||y,t)f_{e^{-\beta \gamma}}(y)^{my} \, dy. \tag{4.25} \]

On the other hand, by Remark \((2.7)\) and \((1.30)\),
\[ u_0 - v_0 \in L^1(f_{e^{-\beta \gamma}}; \mathbb{R}^n) \quad \forall t > 0. \]

This together with Proposition \((1.6)\) implies that
\[ \int_{\mathbb{R}^n \setminus [0]} |u - v||y,t)f_{e^{-\beta \gamma}}(y)^{my} \, dy \leq \int_{\mathbb{R}^n \setminus [0]} |u_0(y) - v_0(y)|f_{e^{-\beta \gamma}}(y)^{my} \, dy. \tag{4.26} \]

By \((4.23)\), \((4.26)\), and Remark \((2.7)\) the desired inequality \((4.23)\) follows. If \((1.32)\) holds, then the same argument with \((1.32)\) implies \((4.24)\). \(\square\)

By the same argument as the proof of Lemma 4.3 of \([\text{HK}]\) but with Corollary \((4.6)\) replacing Lemma 4.2 in the proof there, we have the following result.

**Lemma 4.7** (cf. Lemma 1 of \([\text{OR}]\) and Lemma 4.3 of \([\text{HK}]\)). Let \(n = 3, 4, \frac{2m}{n-m} \leq m < \frac{n-2}{n} \) and \(\beta < 0, \alpha\) be given by \((1.4)\) and \(\gamma\) be given by \((1.29)\). Let \(\lambda_1 \geq \lambda_0 \geq \lambda_2 > 0, \lambda_3 > 0\), and \(f_{\lambda_i}, i = 0, 1, 2, 3\) be given by \((1.3)\). Let \(u_0\) satisfy \((1.13)\) and \((1.42)\). Let \(\bar{u}\) be a solution of \((1.2)\) which satisfies \((1.27)\) which satisfies \((4.22)\). Suppose that \(t_i \rightarrow \infty\), \(\bar{u}(t_i) \rightarrow v_0\) in \(L^1(f_{\lambda_3}; \mathbb{R}^n)\) as \(i \rightarrow \infty\). Suppose \(v\) is the solution of \((1.2)\) with initial value \(v_0\) which satisfies \((1.14)\) and \((1.25)\) and \(\bar{v}(x,t) := e^{\beta t}\bar{v}(e^{\beta}x,t)\). Then
\[ \|\bar{v}(\cdot,t) - f_{\lambda_0}\|_{L^1(f_{\lambda_3}; \mathbb{R}^n)} = \|v_0 - f_{\lambda_0}\|_{L^1(f_{\lambda_3}; \mathbb{R}^n)} \quad \forall t > 0. \tag{4.27} \]

**Proof of Theorem 4.7** Since \(\bar{u}\) satisfies \((4.22)\), the equation \((1.2)\) is uniformly parabolic on \(A_{2R,1}^\infty \times (\frac{1}{2}, \infty)\) for any \(0 < R < \infty\). By the Schauder’s estimates \(\bar{u}\) is equi-Hölder continuous on \(A_{2R,1}^\infty \times [1, \infty)\) for any \(0 < R < \infty\). By the Ascoli Theorem and a diagonalization argument, any sequence \((\bar{u}(\cdot, t_i))\), \(t_i \rightarrow \infty\) as \(i \rightarrow \infty\), of \((\bar{u}(\cdot, t_i))\), has a subsequence which we may assume without loss of generality to be the sequence itself that converges uniformly on every compact subset of \(\mathbb{R}^n \setminus \{0\}\) as \(i \rightarrow \infty\).
Let \( v_0(x) = \lim_{t \to \infty} \bar{u}(x, t) \). By (4.25) and Proposition 1.6
\[
\int_{\mathbb{R}^n \setminus \{0\}} |\bar{u}(x, t) - f_{\lambda_0}(x)|^{\|m\|_{L^1(R^n)}} \, dx = \int_{\mathbb{R}^n \setminus \{0\}} |u(x, t) - f_{\lambda_0}(x)|^{\|m\|_{L^1(R^n)}} \, dx \\
\leq \int_{\mathbb{R}^n \setminus \{0\}} |u_0(x) - f_{\lambda_0}(x)|^{\|m\|_{L^1(R^n)}} \, dx \quad \forall t > 0, i \in \mathbb{Z}^+. \quad (4.28)
\]
Putting \( t = t_i \) and letting \( i \to \infty \) in (4.28), by Remark 2.7 and the Lebesgue monotone convergence theorem,
\[
\int_{\mathbb{R}^n \setminus \{0\}} |v_0(x) - f_{\lambda_0}(x)|^{\|m\|_{L^1(R^n)}} \, dx = 0 \quad \Rightarrow \quad v_0(x) \equiv f_{\lambda_0}(x) \quad \forall x \in \mathbb{R}^n \setminus \{0\}
\]
Since \( t_i \) is arbitrary, \( \bar{u}(\cdot, t) \) converges to \( f_{\lambda_0} \) uniformly on each compact subset of \( \mathbb{R}^n \setminus \{0\} \) as \( t \to \infty \). Letting \( t \to \infty \) in (4.28), by Remark 2.7 and the Lebesgue monotone convergence theorem,
\[
\lim_{t \to \infty} \int_{\mathbb{R}^n \setminus \{0\}} |\bar{u}(x, t) - f_{\lambda_0}|^{\|m\|_{L^1(R^n)}} \, dx = 0.
\]
Hence \( \bar{u}(\cdot, t) \to f_{\lambda_0} \) in \( L^1(f_{\lambda_0}^{\|m\|_{L^1(R^n)}}; \mathbb{R}^n) \) as \( t \to \infty \) and the theorem follows. \( \square \)

4.3. Asymptotic Large Time Behaviour of Radially Symmetric Singular Solutions. In this subsection, we study the asymptotic behaviour of the solution \( u \) of (1.2) with radially symmetric initial value \( u_0(x) = u_0(r) \). First, we consider a weighted \( L^1 \)-contraction of the problem (1.3).

**Proposition 4.8.** Let \( 3 \leq n < 8 \) and \( 1 - \frac{1}{\frac{n-2}{n}} \leq m \leq \min \left\{ \frac{2(n-2)}{3n}, \frac{n-2}{n+2} \right\} \). \( \bar{\alpha}, \bar{\beta}, \gamma \) be given by (2.2) and (1.41), respectively. Let \( \lambda_1 > \lambda_2 > \lambda_3 > 0 \), and \( g_{\lambda, i} = 1, 2, 3 \) be given by (1.40) with \( \lambda = \lambda_1, \lambda_2, \lambda_3 \). Let \( \overline{u}_0 \) and \( \overline{v}_0 \) and satisfy
\[
g_{\lambda, 1}(x) \leq \overline{u}_0(x), \overline{v}_0(x) \leq g_{\lambda, 2}(x) \quad \text{in} \quad \mathbb{R}^n \setminus \{0\} \quad (4.29)
\]
and
\[
\overline{u}_0 - \overline{v}_0 \in L^1 \left( g_{\lambda, 1}^{\|m\|_{L^1(R^n)}}; \mathbb{R}^n \right) \setminus \{0\}.
\]
Let \( \overline{u} \) and \( \overline{v} \) be the solutions of (1.3) with initial values \( \overline{u}_0, \overline{v}_0 \), respectively which satisfy
\[
\overline{U}_{\lambda, 1} \leq \overline{u}, \overline{v} \leq \overline{U}_{\lambda, 2} \quad \text{in} \quad (\mathbb{R}^n \setminus \{0\}) \times (0, \infty), \quad (4.30)
\]
where \( \overline{U}_{\lambda, i}, i = 1, 2, \) are given by (1.39) with \( \lambda = \lambda_1, \lambda_2 \). Then,
\[
\int_{\mathbb{R}^n} (\overline{u} - \overline{v})(x, t) g_{\lambda, 3}^{\|m\|_{L^1(R^n)}} \, dx \leq \int_{\mathbb{R}^n} (\overline{u}_0 - \overline{v}_0)(x) g_{\lambda, 3}^{\|m\|_{L^1(R^n)}} \, dx \quad \forall t > 0. \quad (4.31)
\]
Suppose that
\[
0 \neq \overline{u}_0 - \overline{v}_0 \in L^1(g_{\lambda, 3}^{\|m\|_{L^1(R^n)}}, \mathbb{R}^n).
\]
Then
\[
\int_{\mathbb{R}^n} (\overline{u} - \overline{v})(x, t) g_{\lambda, 3}^{\|m\|_{L^1(R^n)}} \, dx < \int_{\mathbb{R}^n} (\overline{u}_0 - \overline{v}_0)(x) g_{\lambda, 3}^{\|m\|_{L^1(R^n)}} \, dx \quad \forall t > 0. \quad (4.32)
\]

**Proof.** We will use a modification of the proof of Theorem 1.2 of [HK] to prove this theorem. We choose \( \eta \in C_0^\infty(\mathbb{R}^n) \) such that \( 0 \leq \eta \leq 1, \eta = 1 \) for \( |x| \leq 1 \), and \( \eta = 0 \) for \( |x| \geq 2 \). For any \( R > 2 \), and \( 0 < \varepsilon < 1 \), let \( \eta_{\varepsilon, R}(x) := \eta(x/R) \), \( \eta_{\varepsilon, \varepsilon}(x) := \eta(x/\varepsilon) \), and \( \eta_{\varepsilon, \varepsilon}(x) := \eta(x - \varepsilon \eta_{\varepsilon, \varepsilon}(x)) \). Then \( |\nabla \eta_{\varepsilon, R}|^2 + |\Delta \eta_{\varepsilon, R}| \leq C\varepsilon^{-2} \) for \( \varepsilon \leq |x| \leq 2\varepsilon \), and \( |\nabla \eta_{\varepsilon, R}|^2 + |\Delta \eta_{\varepsilon, R}| \leq CR^{-2} \) for \( R \leq |x| \leq 2R \).
By Kato’s inequality,

\[ \frac{d}{dt} \int_{\mathbb{R}^n} |u - v| \phi \, dx \leq \int_{\mathbb{R}^n} |u'' - v''| |\Delta (f^{n+2-\frac{2}{m}} \phi) \, dx, \]  

(4.33)

for any nonnegative function \( \phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\}).

Let \( h = g_A^m \) and claim that

\[
\Delta \left( f^{n+2-\frac{2}{m}} g_A^{m'y} \right) = \Delta f^{(n+2-\frac{2}{m})} h^{y'} + 2 \nabla f^{(n+2-\frac{2}{m})} \cdot \nabla h^{y'} + f^{n+2-\frac{2}{m}} \Delta h^{y'}
\]

\[= \left( n + 2 - \frac{n - 2}{m} \right) \left( 2n - \frac{n - 2}{m} \right) f^{n+2-\frac{2}{m}} h^{y'} + 2 \left( n + 2 - \frac{n - 2}{m} \right) \gamma r^{(n-\frac{2}{m})+1} h^{y-1} h_r
\]

\[+ \left( \gamma' (y' - 1) h^{y-2} (h_r)^2 + \gamma' h^{y-1} h_r + \frac{n - 1}{r} \gamma' h^{y-1} h_r \right) f^{n+2-\frac{2}{m}} \]

\[< 0, \forall x \in \mathbb{R}^n \setminus \{0\}, \]

where \( r := |x| \). Since \( m \leq \frac{n^2}{n^2 + 2} \) and \( g_A + \frac{\beta}{r} (g_A)_r > 0 \) for all \( r > 0 \) (Lemma 4.32), we have

\[
\left( n + 2 - \frac{n - 2}{m} \right) \left( 2n - \frac{n - 2}{m} \right) f^{n+2-\frac{2}{m}} h^{y'} + 2 \left( n + 2 - \frac{n - 2}{m} \right) \gamma' r^{(n-\frac{2}{m})+1} h^{y-1} h_r
\]

\[= \left( n + 2 - \frac{n - 2}{m} \right) f^{n+2-\frac{2}{m}} h^{y'} \left( 2n - \frac{n - 2}{m} \right) h + 2 \gamma' r h_r
\]

\[= \left( n + 2 - \frac{n - 2}{m} \right) f^{n+2-\frac{2}{m}} h^{y-1} g_A - \left( 2n - \frac{n - 2}{m} \right) g_A - 2 \gamma' m r (g_A)_r
\]

\[< \left( n + 2 - \frac{n - 2}{m} \right) f^{n+2-\frac{2}{m}} h^{y-1} g_A \left( 2n - \frac{n - 2}{m} \right) + 2 \gamma' m r (g_A)_r
\]

\[= \left( n + 2 - \frac{n - 2}{m} \right) f^{n+2-\frac{2}{m}} h^{y-1} g_A \left( \frac{4}{1 - m} - \frac{n - 2}{m} \right)
\]

\[< 0.
\]

Since \( \frac{n+1}{m} \Delta g_A^m + f^{(n+2-\frac{2}{m})} (\tilde{a} g_A + \tilde{b} r (g_A)_r) = 0 \) and \( \tilde{a} g_A + \tilde{b} r (g_A)_r > 0. \)

\[
\gamma' h^{y-1} h_r + \frac{n - 1}{r} \gamma' h^{y-1} h_r = \gamma' h^{y-1} \left( h_r + \frac{n - 1}{r} h_r \right) = \frac{n - 1}{m} \gamma y h^{y-1} \Delta g_A^m < 0, \quad x \in \mathbb{R}^n \setminus \{0\}.
\]

Furthermore, \( 1 - \sqrt{\frac{2}{n}} \leq m \leq \frac{2n^2}{n^2 + 2} \) implies \( 0 \leq \gamma' \leq 1 \). Therefore, \( \Delta \left( f^{n+2-\frac{2}{m}} g_A^{m'y} \right) < 0 \), for all \( x \in \mathbb{R}^n \setminus \{0\}. \)
Take a test function $\phi$ as $\eta_{\epsilon,R}^{\mu+2-n/m_1} g_{\lambda_1}^{\gamma}$ in (4.33), then by (4.34), (1.40), Corollaries 4.3 and (4.30),
\[
\frac{d}{dt} \int_{\mathbb{R}^n \setminus |0|} \left[ \overline{\nabla}^m - \nabla^m \right] \eta_{\epsilon,R}(x) g_{\lambda_1}^{\gamma} \, dx \leq \int_{\mathbb{R}^n \setminus |0|} \left[ \overline{\nabla}^m - \nabla^m \right] \left( \eta_{\epsilon,R}(x) \rho^{\mu+2-n/m_1} g_{\lambda_1}^{\gamma} \right) \, dx
\]

\[
\leq \int_{\mathbb{R}^n \setminus |0|} \left[ \overline{\nabla}^m - \nabla^m \right] \left( \rho^{\mu+2-n/m_1} g_{\lambda_1}^{\gamma} \Delta \eta_{\epsilon,R}(x) + 2 \nabla \eta_{\epsilon,R} \cdot \nabla (\rho^{\mu+2-n/m_1} g_{\lambda_1}^{\gamma}) \right) \, dx.
\]

\[
\leq CR^{-2} \int_{B_{2R} \setminus B_{R}} \left( \overline{U}_{\lambda_1}^m(x,t) - \overline{U}_{\lambda_1}^m(x,t) \right) \rho^{\mu+2-n/m_1} g_{\lambda_1}^{\gamma} \, dx
\]

\[
+ CE^{-2} \int_{B_{2R} \setminus B_{R}} \left( \overline{U}_{\lambda_1}^m(x,t) - \overline{U}_{\lambda_1}^m(x,t) \right) \rho^{\mu+2-n/m_1} g_{\lambda_1}^{\gamma} \, dx.
\]

\[
= I + II
\]

By (4.40), Remark 1.12 and Lemma 4.4, $g_{\lambda_1}(r) \sim r^{-\frac{\gamma}{2}} (\log r)^{\frac{m_1}{m}}$, $g_{\lambda_1}(r) - g_{\lambda_2}(r) \leq CR^{-\frac{\gamma}{2}} (\log r)^{\frac{m_1}{m}}$, for sufficiently large $r$. Hence, for fixed $T_0 > 0$, for any $0 < t < T_0$, $0 < \epsilon \leq \epsilon_0$, and $R \geq R_0$,

\[
I \leq CR^{-2} \int_{B_{2R} \setminus B_{R}} \rho^m (x,t) \left( g_{\lambda_1}(r) - g_{\lambda_2}(r) \right) \rho^{\mu+2-n/m_1} g_{\lambda_1}^{\gamma} \, dx
\]

\[
\leq CR^{-2} \int_{0}^{1} \int_{B_{2R} \setminus B_{R}} \rho^{-\frac{\gamma}{2}(m-1)} (\log r)^{-1} \rho^{\mu+2-n/m_1} \rho^{\frac{m_1}{m}} (\log r)^{\frac{m_1}{m}} \rho^{\mu} \, dx \, ds
\]

\[
\leq CR^{2n-\frac{m_1}{m} - \frac{m_1}{2}(m-1)} (\log R)^{\frac{m_1}{m} - 1} \frac{1}{\gamma} = C (\log R)^{\frac{m_1}{m} - 1} \frac{1}{\gamma}.
\]

Furthermore,

\[
II \leq CE^{-2} \rho^{\mu+2-n/m_1} \int_{B_{2R} \setminus B_{R}} \left( g_{\lambda_1}^m (e^{-\beta x} - \rho_0^m (e^{-\beta x}) \right) \rho^{\mu} \, dx \leq CE^{-2} \rho^{\mu+2-n/m_1} \rho^{\mu}.
\]

Then,

\[
\int_{\mathbb{R}^n} \left[ \overline{\nabla}^m - \nabla^m \right] \eta_{\epsilon,R}(x) \rho^{\mu} \, dx - \int_{\mathbb{R}^n} \left[ \overline{\nabla}^m - \nabla^m \right] \eta_{\epsilon,R}(x) \rho^{\mu} \, dx \leq C \left( (\log R)^{\frac{m_1}{m} - 1} \frac{1}{\gamma} + e^{2n-\frac{m_1}{m} \frac{1}{2}} \right).
\]

Since $1 - \frac{\sqrt{n}}{m} < m < \frac{2(n-2)}{3m}$, we have $\frac{m_1}{m} - 1 + \frac{m_1}{2} m < 0$ and $2n - \frac{m_1}{m} \geq 0$. Letting $\epsilon \to 0$ and $R \to \infty$ in (4.36) implies the desired inequality. \qed

**Proposition 4.9** (Weighted $L^1$-contraction with a weight function $|x|^{\frac{n_1}{m} - 2n+(n-2)\gamma} f_i(x)\rho^{\gamma'}$).

Let $3 \leq n < 8$, $1 - \frac{\sqrt{n}}{m} \leq m \leq \min \left( \frac{2(n-2)}{3n}, \frac{n-2}{n+2} \right)$ and $\beta < 0$, $\alpha$, $\gamma'$ be given by (1.4), (1.41), respectively. Let $\lambda_1 > \lambda_2 > 0$, $\lambda_3 > 0$, and $f_i$, $i = 1, 2, 3$ be given by (1.8) with $\lambda = \lambda_1, \lambda_2, \lambda_3$. Let $\nu_0$ and $\nu_0$ be radially symmetric and satisfy (1.19) and

\[
u_0 - \nu_0 \in L^1 \left( \left( |x|^{\frac{n_1}{m} - 2n+(n-2)\gamma} f_i(x)\rho^{\gamma'} ; \mathbb{R}^n \setminus \{0\} \right) \right).
\]

Let $u$ and $v$ be the solutions of (1.2) with initial values $u_0$, $v_0$, respectively which satisfy (1.20), where $U_{\lambda, i} i = 1, 2$, are given by (1.12) with $\lambda = \lambda_1, \lambda_2$. Then

\[
\int_{\mathbb{R}^n \setminus |0|} \left( |x|^{\frac{n_1}{m} - 2n+(n-2)\gamma} f_i(x)\rho^{\gamma'} \right) \, dx \leq \int_{\mathbb{R}^n \setminus |0|} \left( u_0 - v_0 \right) \left( |x|^{\frac{n_1}{m} - 2n+(n-2)\gamma} f_i(x)\rho^{\gamma'} \right) \, dx \forall t > 0.
\]

Moreover if

\[
u_0 - \nu_0 \in L^1 \left( \left( |x|^{\frac{n_1}{m} - 2n+(n-2)\gamma} f_i(x)\rho^{\gamma'} ; \mathbb{R}^n \setminus \{0\} \right) \right).
\]
then
\[ \int_{\mathbb{R}^n} |u-v| f_A(x)|x|^{-2n+2(1-\gamma)|y|} \, dx < \int_{\mathbb{R}^n} |u_0-v_0| f_A(x)|x|^{-2n+2(1-\gamma)|y|} \, dx \quad \forall t > 0. \tag{4.40} \]

**Proof.** Since \( u_0 \) and \( v_0 \) are radially symmetric, \( u \) and \( v \) are radially symmetric for any \( t \), i.e., \( u(x,t) = u(r,t) \) for all \( t \), where \( r := |x| \). Let \( \rho := r^{-1} \) and \( \tilde{u}(\rho,t) := \rho^{-\frac{n-1}{2}} u(\rho^{-1},t) \) and \( \tilde{v}(\rho,t) := \rho^{-\frac{n-1}{2}} v(\rho^{-1},t) \). Then \( \tilde{u} \) and \( \tilde{v} \) are solutions of (1.37) with (4.29). Since \( u \) and \( v \) satisfy (1.20), \( \tilde{u} \) and \( \tilde{v} \) satisfy (4.30).

\[ \int_{\mathbb{R}^n} |\tilde{u}_0-\tilde{v}_0| g_A^{my'} \, dx = \int_{\mathbb{R}^n} |u_0-v_0|(r^{-1})\rho^{-\frac{n-1}{2}} g_A^{my'} \, dx = \int_{\mathbb{R}^n} |u_0-v_0|(\rho)\rho^{-\frac{n-1}{2}} g_A^{my'} (\rho^{-1})\rho^{-2n} \, dy \]
\[ = \int_{\mathbb{R}^n} |u_0-v_0|(\rho)\rho^{-\frac{n-1}{2}} (\rho^{-2n+2(1-\gamma)|y|}) f_A(x)^{my'} (\rho) \, dy. \tag{4.41} \]

Then, the \( L^1 \)-contraction for weight \( |x|^{-2n+2(1-\gamma)|y|} f_A(x)^{my'} \), (4.40), holds. \( \square \)

**Corollary 4.10.** Let \( 3 \leq n < 8 \), \( 1 - \sqrt{\frac{2}{n}} \leq m \leq \min \{ \frac{2n-2}{3n}, \frac{2-\gamma}{\gamma-1} \} \), \( \beta < 0 \), and \( \alpha, \gamma' \) be given by (1.4) and (1.41) respectively. Let \( \lambda_1 > \lambda_2 > 0 \), \( \lambda_3 > 0 \), and \( f_A \) be given by (1.8) with \( \lambda = \lambda_1, \lambda_2, \lambda_3 \). Let \( u_0, v_0 \) be radially symmetric and satisfy (1.19), (4.37), and \( \tilde{u}, \tilde{v} \) be solutions of (1.20) with initial values \( u_0, v_0 \), respectively, which satisfies (4.22). Then
\[ \int_{\mathbb{R}^n} |\tilde{u}-\tilde{v}| f_A(x)^{my'} \, dx \leq \int_{\mathbb{R}^n} |u_0-v_0| f_A(x)^{my'} \, dx \quad \forall t > 0. \tag{4.42} \]

If (4.39) holds, then
\[ \int_{\mathbb{R}^n} |\tilde{u}-\tilde{v}| f_A(x)^{my'} \, dx < \int_{\mathbb{R}^n} |u_0-v_0| f_A(x)^{my'} \, dx \quad \forall t > 0. \tag{4.43} \]

**Proof.** Let
\[ u(x,t) := e^{-\alpha t} \tilde{u}(e^{-\beta t} x, t) \quad \text{and} \quad v(x,t) := e^{-\alpha t} \tilde{v}(e^{-\beta t} x, t). \]

Then \( u \) and \( v \) are solutions of (1.2), with initial values \( u_0 \) and \( v_0 \), respectively, which satisfy (1.20). Let \( -\delta := \frac{n-2}{m} - 2n + (n-2)\gamma' \). By using the fact that \( \alpha - n\beta + \beta \delta + \frac{2m\gamma'}{1-m} = 0 \) and (1.8), we have
\[ \int_{\mathbb{R}^n} |\tilde{u}-\tilde{v}| f_A(x)^{my'} \, dx \]
\[ = e^{\alpha t} \int_{\mathbb{R}^n} |u-v| (e^{\alpha t} x, t) |x|^{-\delta} f_A(x)^{my'} \, dx = e^{(\alpha-n\beta) t} \int_{\mathbb{R}^n} |u-v| (y,t) (e^{\beta t} y)^{-\delta} f_A(x)^{my'} \, dy \]
\[ = e^{(\alpha-n\beta) t} \int_{\mathbb{R}^n} |u-v| (y,t) |y|^{-\delta} e^{\frac{2m}{1-m} (e^{\frac{2m}{1-m}} f_A(x)^{my'})} \, dy \]
\[ = \int_{\mathbb{R}^n} |u-v| (y,t) |y|^{-\delta} f_{e^{\beta t} A}(y)^{my'} \, dy. \tag{4.44} \]

where \( y := e^{\beta t} x. \)

On the other hand, by Remark 2.7 and (4.37),
\[ u_0 - v_0 \in L^1 \left( |x|^{\frac{2n-2+2(1-\gamma)|y|}{m}} f_{e^{\alpha t} A}; \mathbb{R}^n \right) \quad \forall t > 0. \]
This together with Proposition 4.9 implies that
\[
\int_{\mathbb{R}^n \setminus \{0\}} |\mu - v(y, t)|^{2n/(n-2)} \, dx \leq \int_{\mathbb{R}^n \setminus \{0\}} |u_0(y) - v_0(y)|^{2n/(n-2)} \, dy.
\]
By (4.44), (4.45), and Remark 2.7, the desired inequality (4.32) follows. If (4.39) holds, then the same argument with (4.39) implies (4.43).

The proof of Theorem 1.8 follows by Corollary 4.10 and an argument similar to the proof of Theorem 1.7 and the result follows.

References

[A] D.G. Aronson, The porous medium equation, CIME Lectures, in Some problems in Nonlinear Diffusion, Lecture Notes in Mathematics 1224, Springer-Verlag, New York, 1986.

[CD] B. Choi and P. Daskalopoulos, Yamabe flow: steady solutions and type II singularities, Nonlinear Analysis 173 (2018), 1–18.

[DK] P. Daskalopoulos and C.E. Kenig, Degenerate diffusion-initial value problems and local regularity theory, Tracts in Mathematics 1, European Mathematical Society, 2007.

[DKS] P. Daskalopoulos, J. King and N. Sesum, Extinction profile of complete non-compact solutions to the Yamabe flow; arXiv:1306.0859

[DPS] P. Daskalopoulos, M.del Pino and N. Sesum, Type II ancient compact solutions to the Yamabe flow, J. Reine Angew. Math. 622 (2008), 95–119.

[DS1] P. Daskalopoulos and N. Sesum, On the extinction profile of solutions to fast diffusion, J. Reine Angew. Math. 204 (2013), 346–369.

[DS2] P. Daskalopoulos and N. Sesum, The classification of locally conformally flat Yamabe solitons, Adv. Math. 240 (2013), 346–369.

[FVWY] M. Fila, J.L. Vazquez, M. Winkler and E. Yanagida, Rate of convergence to Barenblatt profiles for the fast diffusion equation, Arch. Rational Mech. Anal. 184 (2012), no. 2, 599–625.

[FW1] M. Fila and M. Winkler, Optimal rates of convergence to the singular Barenblatt profile for the fast diffusion equation, Proc. Roy. Soc. Edinburgh Sect. A 146 (2016), no. 2, 309–324.

[FW2] M. Fila and M. Winkler, Rate of convergence to separable solutions of the fast diffusion equation, Israel J. Math. 213 (2016), no. 1, 1–32.

[FW3] M. Fila and M. Winkler, Slow growth of solutions of super-fast diffusion equations with unbounded initial data. arXiv:1605.04150v1.

[HP] M.A. Herrero and M. Pierre, The Cauchy problem for \( u_t = \Delta u^m \) for \( 0 < m < 1 \), Trans. Amer. Math. Soc. 291 (1985), no. 1, 145–158.

[Hs1] S.Y. Hsa, Singular limit and exact decay rate of a nonlinear elliptic equation, Nonlinear Analysis TMA 75 (2012), no. 7, 3443–3455.

[Hs2] S.Y. Hsa, Existence and asymptotic behaviour of solutions of the very fast diffusion, Manuscripta Math. 140 (2013), no. 3–4, 441–460.

[Hs3] S.Y. Hsa, Some properties of the Yamabe soliton and the related nonlinear elliptic equation, Calc. Var. Partial Differential Equations 49 (2014), no. 1-2, 307–321.

[Hs4] S.Y. Hsa, Exact decay rate of a nonlinear elliptic equation related to the Yamabe flow, Proc. Amer. Math. Soc. 142 (2014), no. 12, 4239–4249.

[Hs5] S.Y. Hsa, Global behaviour of solutions of the fast diffusion equation, Manuscripta Math. 158 (2019), no. 1–2, 103–117.

[Hui1] K.M. Hui, On some Dirichlet and Cauchy problems for a singular diffusion equation, Differential Integral Equations 15 (2002), no. 7, 769–804.

[Hui2] K.M. Hui, Singular limit of solutions of the very fast diffusion equation, Nonlinear Anal. TMA 68 (2008), 1120–1147.

[Hui3] K.M. Hui, Asymptotic behaviour of solutions of the fast diffusion equation near its extinction time, J. Math. Anal. Appl. 454 (2017), no. 2, 695–715.

[HK] K.M Hui and Soojung Kim, Asymptotic large time behavior of singular solutions of the fast diffusion equation, Discrete Contin. Dyn. Syst. Series A 37 (2017), no. 11, 5943–5977.

[HK1] K.M Hui and Sunghoon Kim, Existence and large time behaviour of finite points blow-up solutions of the fast diffusion equation, Calculus of Variations and PDE 57, Article 112 (2018).

[K] T. Kato, Schrödinger operators with singular potentials, Israel J. Math. 13 (1973), 135–148.
[LSU] O.A. Ladyzenskaya, V.A. Solonnikov and N.N. Uraltceva, Linear and quasilinear equations of parabolic type, Transl. Math. Mono. vol. 23, Amer. Math. Soc., Providence, R.I., U.S.A., 1968.

[OR] S.J. Osher and J.V. Ralston, $L^1$ stability of traveling waves with applications to convective porous media flow, Comm. Pure Appl. Math. 35 (1982), 7371–7749.

[PS] M.del Pino and M. Sáez, On the extinction profile for solutions of $u_t = \Delta u^{\frac{N-2}{N+2}}$, Indiana Univ. Math. J. 50 (2001), no. 1, 611–628.

[V1] J.L. Vazquez, Nonexistence of solutions for nonlinear heat equations of fast-diffusion type, J. Math. Pures Appl. 71 (1992), 503–526.

[V2] J.L. Vazquez, Smoothing and decay estimates for nonlinear diffusion equations, Oxford Lecture Series in Mathematics and its Applications 33, Oxford University Press, Oxford, 2006.

[VW] J.L. Vázquez and M. Winkler, The evolution of singularities in fast diffusion equations: infinite time blow-down, SIAM J. Math. Anal. 43 (2011), no. 4, 1499–1535.

Institute of Mathematics, Institute of Mathematics, Academia Sinica, Taipei, Taiwan, R. O. C.

E-mail address: kmhui@gate.sinica.edu.tw

Institute of Mathematics, Institute of Mathematics, Academia Sinica, Taipei, Taiwan, R. O. C.

E-mail address: jinwan@gate.sinica.edu.tw