An ansatz for the exclusion statistics parameters in macroscopic physical systems described by fractional exclusion statistics

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Abstract – I introduce an ansatz for the exclusion statistics parameters of fractional-exclusion-statistics (FES) systems and I apply it to calculate the statistical distribution of particles from both, bosonic and fermionic perspectives. To check the applicability of the ansatz, I calculate the FES parameters in three well-known models: a Fermi-liquid-type system, a one-dimensional quantum system described in the thermodynamic Bethe ansatz and quasi-particle excitations in a fractional quantum Hall (FQH) system. The FES parameters of the first two models satisfy the ansatz, whereas those of the third model, although close to the form given by the ansatz, represent an exception. In this case I also show that the general properties of the FES parameters, deduced elsewhere (EPL, 87 (2009) 60009), are satisfied also by the parameters of the FQH liquid.

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Introduction. – Haldane’s concept of fractional exclusion statistics (FES) [1] has recently been amended in a series of publications [2–5]. In these publications I showed that in the original formulation of FES some basic properties have been overlooked. The situation was corrected first by introducing a conjecture [2] and then by deducing the general, basic properties of the FES parameters [5]; it turned out that the conjecture of ref. [2] is just a special case of the general conditions deduced in ref. [5], which allows one to write down an explicitly consistent system of equations for the statistical distribution of particles in a FES system.

Now that the general properties of the FES parameters are deduced [5], the conjecture of ref. [2] looses its status and becomes simply an ansatz. This ansatz seems to be quite general and applies to most of the macroscopic (i.e. quasi-continuous) systems. Nevertheless, by the end of this letter I will show an exception.

In this letter, using the general properties of the direct exclusion statistics parameters (see eq. (2c) below) I propose an even more restrictive form of this ansatz (which is therefore more convenient for applications), by specifying also the form of the direct FES parameters. This new ansatz allows me to write the system of equations for the statistical distribution of particles in a clearer form and to single out the direct exclusion statistics parameters which are most often used in FES calculations.

In general, the FES equations for the particle distribution are used in the bosonic formulation. Here I shall write and use these equations in both the bosonic and fermionic formulations. This allows for direct application of the formalism to systems of either bosons or fermions.

In the end I will give three examples of systems in which FES is manifest. After calculating their parameters, I show that two of them satisfy the ansatz, whereas the third, although quite similar, constitutes an exception.

The general properties of the exclusion statistics parameters. – Let us assume that we have a system of particles which we divide into species indexed by i = 0, 1, …, each of the species containing N_i particles and having G_i “available single-particle states”. Then the number of microscopic configurations for species i is

\[ W_i = \frac{(G_i + N_i)!}{N_i!G_i!}. \]  

(1)

To introduce FES into the system, we define the exclusion statistics parameters, \( \tilde{\alpha}_{ij} \), such that for a variation \( \delta N_j \) of \( N_j \), the number \( G_i \) changes by \( \delta G_i = -\tilde{\alpha}_{ij} \delta N_j \). The
diagonal elements, \( \tilde{\alpha}_{ii} \), are called direct exclusion statistics parameters, whereas the nondiagonal ones, \( \tilde{\alpha}_{ij}, i \neq j \), are called mutual exclusion statistics parameters.

In ref. [5] I showed that if one of the particle species, say species \( j \), is divided into the subspecies \( j_0, j_1, \ldots \), then the new statistics coefficients should satisfy the relations

\[
\tilde{\alpha}_{ij} = \tilde{\alpha}_{ij_0} = \tilde{\alpha}_{ij_1} = \ldots \quad \text{for any } i, \quad i \neq j, \quad (2a)
\]

\[
\tilde{\alpha}_{ji} = \tilde{\alpha}_{j_0 i} + \tilde{\alpha}_{j_1 i} + \ldots \quad \text{for any } i, \quad i \neq j, \quad (2b)
\]

\[
\tilde{\alpha}_{jj} = \tilde{\alpha}_{j_0 j_0} + \tilde{\alpha}_{j_1 j_0} + \ldots = \tilde{\alpha}_{j_0 j_1} + \tilde{\alpha}_{j_1 j_1} + \ldots = \ldots \quad (2c)
\]

Here I use the notations \( \tilde{\alpha}_{ij} \), like in refs. [2–5], to make the difference between the “extensive” and “intensive” FES parameters that are going to be defined below.

The conjecture introduced in ref. [2] stated that in a macroscopic physical system described by FES there is a division of the system into species, say \( \{ (G_i, N_i) \}_{i=0,1, \ldots} \), so that no matter how we further divide these species into subspecies, the mutual exclusion statistics parameters are always proportional to the dimension of the space on which they act. Concretely, this means that for any \( i, j \), with \( i \neq j \), we can write \( \tilde{\alpha}_{ij} = \tilde{\alpha}_{ij_0} G_i \), where \( \alpha_{ij} \) are constants that depend on the species \( i \) and \( j \), and at any further division, say species \( i \) is divided into the subspecies \( i_0, i_1, \ldots \), the new mutual exclusion statistics parameters satisfy \( \tilde{\alpha}_{i_0 i_j} = \tilde{\alpha}_{i_0 i_j_0} G_i \alpha_{ij} \) for any \( i \neq j \) [2]. The parameters \( \alpha_{ij} \) and \( \alpha_{ij} \) are called extensive and intensive parameters, respectively. It is easy to check that the extensive parameters satisfy the general conditions (2a) and (2b).

In this paper I will extend the ansatz to the direct FES parameters, which should satisfy (2c).

**Ansatz for the FES parameters.** – We can make the form of the exclusion statistics parameters that satisfy eqs. (2) more specific and easier to apply to FES calculations if we decompose \( \tilde{\alpha}_{ij} \) into a sum of two different types of parameters, \( \alpha_{ij}^{(s)} \) and \( \alpha_{ij}^{(e)} \), by the relation

\[
\tilde{\alpha}_{ij} = \alpha_{ij}^{(s)} + \alpha_{ij}^{(e)} \delta_{ij}. \quad (3)
\]

The parameters \( \alpha_{ij}^{(s)} \) are the “extensive” ones discussed above, but now, by separating \( \tilde{\alpha}_{ij} \), we can extend the condition

\[
\alpha_{ij}^{(s)} = \alpha_{ij} G_i, \quad (4)
\]

also to the case \( i = j \).

The additional parameters, \( \alpha_{ij}^{(s)} \), always refer to only one species of particles and are not extensive. If we split the species \( i \) into the sub-species \( i_0, i_1, \ldots \), then by eqs. (2c), (3) and the extensivity property of \( \alpha_{ij}^{(s)} \), we obtain

\[
\tilde{\alpha}_{i_0 i_i} = G_i \alpha_{i_0 i_i} + \alpha_{i_0 i_i}^{(s)} \delta_{i_0 i_i}. \quad (5)
\]

Typically, in the literature we find exclusion statistics parameters of the \( (s) \) type (see, e.g., [6–13]). Therefore in general \( \tilde{\alpha}_{ij} = 0 \) for any \( i \neq j \), so there is no mutual statistics in the system. In such a case the thermodynamic calculations simplify considerably. Note also that the ideal Fermi gas corresponds to \( \tilde{\alpha}_{ij}^{(s)} = 1 \) for any \( i \).

**Particle population in the bosonic formulation.** – Let us now deduce the equations for the particle population. To avoid unphysical (negative or divergent weights) I write the number of microscopic configurations as [2]

\[
W = \prod_i \frac{(G_i + N_i - 1 + (1 - \tilde{\alpha}_{ii}^{(s)}) \delta N_i - \sum_j \tilde{\alpha}_{ij} \delta N_j)!}{(N_i + \delta N_i)! (G_i - 1 - \tilde{\alpha}_{ii}^{(s)} \delta N_i - \sum_j \tilde{\alpha}_{ij} \delta N_j)!}
\]

\[
\approx \prod_i \frac{(G_i + N_i - 1 + (1 - \tilde{\alpha}_{ii}^{(s)}) \delta N_i - \sum_j \alpha_{ij} \delta N_j)!}{(N_i + \delta N_i)! (G_i - \sum_j \alpha_{ij} \delta N_j)!}
\]

which then I plug into the expression of the grand canonical partition function,

\[
Z = \sum_{\{N_i\}} W(\{N_i\}) \exp \left[ \sum_i \beta N_i (\mu_i - \epsilon_i) \right], \quad (7)
\]

where \( \beta = 1/k_B T \), \( T \) is the temperature of the system, whereas \( \mu_i \) and \( \epsilon_i \) are the chemical potential and the single-particle energy for the particles of species \( i \). Maximizing \( Z \) with respect to the populations \( n_i = N_i/G_i \), I obtain the system of equations

\[
\beta (\mu_i - \epsilon_i) + \ln \frac{1 + n_i}{n_i} - \frac{1}{n_i} \sum_j G_j a_{ij} \ln [1 + n_j] = 0. \quad (8)
\]

Notice that eq. (8) is similar to eq. (18) of ref. [2], except that by singling out the coefficients \( \alpha_{ij}^{(s)} \), we could extend the summation on the r.h.s. to include also the terms \( i = j \). This makes the application of eq. (8) more straightforward.

If \( \tilde{\alpha}_{ij} = 0 \) for any \( i \) and \( j \), we recover the typical formulas for the calculation of particle population without mutual exclusion statistics [6,14,15],

\[
n = w(\zeta) + \tilde{\alpha}_{ii}^{(s)} \zeta = w(\zeta) \left[ 1 + \tilde{\alpha}_{ii}^{(s)} \right]^{-1}, \quad (9)
\]

with \( w \) and \( \zeta \) defined by

\[
w^{\alpha_{ij}^{(s)}}(\zeta) [1 + w(\zeta)]^{1 - \tilde{\alpha}_{ij}^{(s)}} = \zeta \equiv \exp[\beta (\epsilon_i - \mu_i)]. \quad (10)
\]

In the quasi-continuous limit, in a phase-space spanned by the single-particle states of quantum numbers \( k \) (\( k \) is not necessarily the wave number), of density of states \( \sigma(k) \), eq. (8) transforms into

\[
\beta (\mu_k - \epsilon_k) + \ln \frac{1 + n_k}{n_k} - \frac{1}{n_k} \int \sigma(k') \ln [1 + n_{k'}] a_{kk'} dk'. \quad (11)
\]

Equation (11) is similar to eq. (19) of ref. [2], if we identify, say \( \tilde{\alpha}_{kk}^{(s)} = \tilde{\alpha}_{kk}^{(s)} \) of [2]. The identification is natural, since the interval \( G_i \) becomes smaller, \( \tilde{\alpha}_{ii} \) converges to \( \tilde{\alpha}_{ii}^{(s)} \), which stays constant while \( \tilde{\alpha}_{ii}^{(s)} \) decreases to zero.
Particle population in the fermionic formulation.

Formula (1) represents the number of configurations of particles of species \( i \) in the bosonic formulation of FES [1]. This description is not the most convenient for example when one describes systems of (interacting) fermions in the FES formalism, since in such a case \( G_i \) represents the difference between the number of single-particle states and the number of fermions. Therefore in such cases it is easier to work directly with the total number of states, \( T_i \equiv G_i + N_i - 1 \), in the fermionic description. Although the two descriptions are equivalent, let me write down the system of equations for the \( n_i \)'s in the fermionic description. For this, I first write the number of configurations for the species \( i \),

\[
W_{i}^{(f)} = \frac{T_i!}{N_i!(T_i-N_i)!}.
\]

I add again small perturbations to the particle numbers and I write

\[
W^{(f)} = \prod_i \left\{ \frac{(T_i - \tilde{\alpha}_i^{(s)} \delta N_i - G_i \sum_j a_{ij} \delta N_j)!}{[T_i - N_i - (1 + \tilde{\alpha}_i^{(s)} \delta N_i - T_i \sum_j a_{ij} \delta N_j)!} \times \frac{1}{(N_i + \delta N_i)!} \right\}.
\]

I plug (13) into the expression (7) for \( Z \) and, by maximization, I get the equations for the particle population, \( f_i \equiv N_i / T_i \),

\[
\beta(\mu_i - \epsilon_i) + \ln \frac{[1 - f_i]^{1+\tilde{\alpha}_i^{(s)}}}{f_i} = -\sum_j G_j a_{ji} \ln[1 - f_j].
\]

Introducing the density of states \( \sigma^{(f)}(k) \) I write eq. (14) in the quasi-continuous limit,

\[
\beta(\mu_k - \epsilon_k) + \ln \frac{[1 - f_k]^{1+\tilde{\alpha}_k^{(s)}}}{f_k} = -\int \sigma^{(f)}(k') \ln[1 - f_{k'}] a_{kk'} \, dk'.
\]

Applications.

Let me analyse now three models of interacting-particle systems in which FES is manifest and compare the results with the ansatz proposed here.

FES in a system with Fermi-liquid-type interaction. I take again the model of ref. [3], which is a generalization of the Murthy and Shankar model [7], widely used in FES [7–10,12,16,17]. In this model the total energy of the system,

\[
E = \sum_i \epsilon_i n_i + \frac{1}{2} \sum_{ij} V_{ij} n_i n_j,
\]

is split into “quasi-particle energies”, \( E \equiv \sum_i \tilde{\epsilon}_i n_i \), with

\[
\tilde{\epsilon}_i = \epsilon_i + \sum_{j=0}^{i-1} V_{ij} n_j + \frac{1}{2} V_{ii} n_i.
\]

In eqs. (16) and (17) \( \epsilon_i \) and \( n_i \) \((i = 0, 1, \ldots)\) are the free-particle energy and population of level \( i \), respectively, whereas \( V_{ij} \) represent the interaction energy between a particle in the state \( i \) and a particle in the state \( j \).

Going to the quasi-continuous limit, assuming that the single-particle energy spectrum has density of states \( \sigma(\epsilon) \) and that the interaction energy depends only on the energies, we replace the indices \( i \) and \( j \) by the energies \( \epsilon \) and \( \epsilon' \) to write

\[
\tilde{\epsilon} = \epsilon + \int_0^\epsilon V(\epsilon, \epsilon') \sigma(\epsilon') n(\epsilon') \, d\epsilon'.
\]

In what follows I shall assume that the function \( \tilde{\epsilon}(\epsilon) \) is bijective and therefore I shall use interchangeably, whenever necessary, both \( \tilde{\epsilon}(\epsilon) \) and \( \epsilon(\tilde{\epsilon}) \).

The FES is manifest in this case in the quasi-particle energies, \( \tilde{\epsilon} \). In order to describe FES and to calculate its parameters, I split the \( \tilde{\epsilon} \) axis into small intervals. In general I shall denote such interval by \( \delta \tilde{\epsilon} \) and by this notation I shall assume that it contains the quasi-particle energy level \( \tilde{\epsilon} \).

From ref. [3] I can immediately identify the direct exclusion statistics parameters of \( (s) \) type,

\[
\tilde{\alpha}_s^{(s)} = \frac{V(\epsilon, \epsilon') \sigma(\epsilon)}{1 + \int_0^\epsilon \frac{\partial V(\epsilon, \epsilon')}{\partial \epsilon'} \sigma(\epsilon') n(\epsilon') \, d\epsilon'},
\]

where \( \epsilon \equiv \epsilon(\tilde{\epsilon}) \).

The mutual exclusion statistics parameters are [3]

\[
\delta \tilde{\epsilon} \delta \tilde{\epsilon}_i = \frac{\delta \tilde{\epsilon} \delta \tilde{\epsilon}_i}{1 + \int_0^\epsilon \frac{\partial V(\epsilon, \epsilon')}{\partial \epsilon'} \sigma(\epsilon') n(\epsilon') \, d\epsilon'} \int \frac{\partial \sigma(\epsilon')}{\partial \epsilon'} \sigma(\epsilon') n(\epsilon') \, d\epsilon',
\]

where \( \epsilon \) and \( \epsilon_i \) are actually \( \epsilon \equiv \epsilon(\tilde{\epsilon}) \) and \( \epsilon_i \equiv \epsilon(\tilde{\epsilon}_i) \), respectively, and moreover, \( \tilde{\epsilon} \in \delta \tilde{\epsilon} \), and \( \tilde{\epsilon}_i \in \delta \tilde{\epsilon}_i \). Equation (20) refers to the process in which quasi-particles are inserted into the interval \( \delta \tilde{\epsilon}_i \) and the variation of the number of states is observed in the interval \( \delta \tilde{\epsilon} \). The function \( f(\tilde{\epsilon}, \tilde{\epsilon}_i) \) is

\[
f(\tilde{\epsilon}, \tilde{\epsilon}_i) = \int_\epsilon^{\epsilon_i} \frac{\partial V(\epsilon, \epsilon')}{\partial \epsilon'} \sigma(\epsilon') n(\epsilon') \left[ \frac{\delta \epsilon'}{\delta \rho(\epsilon')} \right] \rho(\tilde{\epsilon}) \, d\epsilon',
\]

where \( \rho(\tilde{\epsilon}) = \sigma(\epsilon)n(\tilde{\epsilon}) \) I denoted the particle density along the \( \tilde{\epsilon} \) axis and the notation \( [\delta \epsilon'/\delta \rho(\epsilon')]_{\rho(\tilde{\epsilon})} \) represents the functional derivative of \( \epsilon' \) with respect to the particle density at energy \( \tilde{\epsilon}_i \), when we keep \( \tilde{\epsilon}(\epsilon') \) fixed.

Since the number of states in the interval \( \delta \tilde{\epsilon} \) is \( \tilde{\sigma}(\delta \tilde{\epsilon}) \tilde{\epsilon} \), the coefficients \( a_{\tilde{\epsilon} \tilde{\epsilon}_i} \) are

\[
a_{\tilde{\epsilon} \tilde{\epsilon}_i} = \frac{V(\epsilon, \epsilon_i) + f(\tilde{\epsilon}, \tilde{\epsilon}_i)}{1 + \int_0^\epsilon \frac{\partial V(\epsilon, \epsilon')}{\partial \epsilon'} \sigma(\epsilon') n(\epsilon') \, d\epsilon'} \int \frac{\partial \log \sigma(\epsilon')}{\partial \epsilon'} \sigma(\epsilon') n(\epsilon') \, d\epsilon'.
\]

We are now left with the calculation of \( \tilde{\alpha}_s^{(s)} \). For this we first note that eqs. (20) and (22) are valid for any two
disjoint intervals, so let us divide the interval $\delta \varepsilon$ into the sub-intervals $\delta \varepsilon_i$, $i = 0, 1, \ldots, M$, of dimensions $\delta \varepsilon \varepsilon_i$, where we always maintain the notation convention that $\varepsilon_i$ belongs to the interval $\delta \varepsilon$. We pick the energy level $\varepsilon_k$ from the interval $\delta \varepsilon_k$ and apply eq. (2c):
\[
\tilde{\alpha}_{\delta \varepsilon k} = \bar{\sigma} (\varepsilon_0) \delta \varepsilon_0 a_{\varepsilon_0} + \ldots + \bar{\sigma} (\varepsilon_M) \delta \varepsilon_M a_{\varepsilon_M} \varepsilon_k.
\]
(23)

Making the interval $\delta \varepsilon_k$ small enough as compared to $\delta \varepsilon$ and assuming that $\delta \varepsilon$ is also small, so that we can use the approximations $a_{\varepsilon_i} \varepsilon_i = a_{\varepsilon i}^s$ and $\bar{\sigma} (\varepsilon_i) \equiv \bar{\sigma} (\varepsilon)$ for any $k, l \in \{1, \ldots, M\}$, we can simplify eq. (23) to write the general expression
\[
\tilde{\alpha}_{\delta \varepsilon k} = a_{\varepsilon i} \bar{\sigma} (\varepsilon) \delta \varepsilon + \tilde{\alpha}_{\varepsilon i} = a_{\varepsilon i} \bar{\sigma} (\varepsilon) \delta \varepsilon + \delta (\varepsilon - \varepsilon_i) \alpha_{\varepsilon i}^s,
\]
(24)

which has the form of the ansatz (3). Note that in $a_{\varepsilon i}$ of eq. (24), $\bar{\sigma} (\varepsilon)$, $\varepsilon$ = $\theta$. In the simplified models of refs. [7–10,12,16,17], only the direct exclusion statistics parameters have been used and $a_{\varepsilon i}$, was identically zero for any $\varepsilon$ and $\varepsilon_i$. We observe now, from eq. (20), that this happens whenever $\delta \sigma / \delta \varepsilon = 0$.

Having calculated all the exclusion statistics parameters, one can in principle apply eq. (11) or (15) — depending on the type of particles we have in the system — to calculate the particle distribution.

**FES in 1D quantum gas in the thermodynamic Bethe ansatz.** The 1D gas of quantum particles in the thermodynamic Bethe ansatz (TBA) has been analysed before (see, e.g., [9,15,18,19]) and is recognized in general as being a system which can be described by FES. The only reason why I discuss it again here is to show that its FES parameters are indeed of the type (3) and also because in general a confusion is made in the literature between the intensive $a_{\varepsilon i}$ and extensive $\tilde{\alpha}_{\varepsilon i}$ parameters and this has to be clarified.

Therefore, let us consider the typical gas of $N$ spinless particles, bosons or fermions, on a ring of circumference $L$. We assume that the system is non-diffusive [20] and the asymptotic particle number, $k$, is determined by the equation [21]
\[
Lk - \sum_{k'} \theta (k - k') = L k^{(0)},
\]
(25)

where $k^{(0)} = 2 \pi I (k)/L$ is the free-particle wave number, $I (k)$ an integer that depends on $k$, and $\theta (k - k')$ is the phase-shift due to the interaction.

To simplify the notations and to be also in accordance with ref. [20,21], we set the units so that $\hbar = m = 1$, where $m$ is the mass of the particle. In these units the total number of particles, momentum and energy of the system are
\[
N = \sum_k 1, \quad P = \sum_k k, \quad \text{and} \quad E = \frac{1}{2} \sum_k k^2.
\]
(26)

For large systems we transform the summations into integrals and define the densities of states, $\sigma (k)$ and $\sigma_0 (k^{(0)})$, by the relations
\[
D (\delta k) \equiv \sigma (k) \delta k \quad \text{and} \quad D (\delta k^{(0)}) \equiv \sigma_0 (k^{(0)}) \delta k^{(0)},
\]
(27)

where $D (\delta k)$ and $D (\delta k^{(0)})$ are the numbers of states in the small intervals $\delta k$ and $\delta k^{(0)}$, respectively. If $\delta k$ and $\delta k^{(0)}$ are related by eq. (25), then
\[
D (\delta k) = D (\delta k^{(0)}),
\]
(28)

Obviously, $\sigma_0 (k^{(0)}) = L / (2 \pi)$ (if we impose periodic boundary conditions on $k^{(0)}$) [20,21].

The populations of the single-particle levels, $n_0 (k^{(0)})$ and $n (k)$, are defined as
\[
N (\delta k) \equiv n (k) \sigma (k) \delta k = N (\delta k^{(0)}) \equiv n_0 (k^{(0)}) \sigma_0 (k^{(0)}) \delta k^{(0)},
\]
(28)

where $N (\delta k) = N (\delta k^{(0)})$ is the number of particles which is the same in the two intervals $\delta k$ and $\delta k^{(0)}$, that are related by eq. (25). Moreover, since both the number of particles and the number energy levels, are the same in the intervals $\delta k$ and $\delta k^{(0)}$, we have the identity $n (k) = n_0 (k^{(0)}) (k)$. In accordance with the notations in the literature [18,21] I introduce here also the particle density, $\rho (k) \equiv \sigma (k) n (k) / L = N (\delta k) / L \delta k$.

In the new notations, eq. (25) becomes a self-consistent equation for $k$,
\[
k = k^{(0)} (k) + \int \theta (k - k') \rho (k') \, dk',
\]
(29)

from which we can calculate [21]
\[
\frac{dk^{(0)}}{dk} \equiv 1 - \int \theta '(k - k') \rho (k') \, dk',
\]
(30)

where $\theta '(k) \equiv d \theta (k) / dk$. Plugging eq. (30) into (27), we get the density of states
\[
\sigma (k) = \frac{L}{2 \pi} \left\{ 1 - \int \theta ' (k - k') \rho (k') \, dk' \right\}.
\]
(31)

The FES is manifest in the system because of the dependence of $\sigma$ on $\rho$; the variation of $\rho (k_i)$ produces, in principle, a change of the density of states $\sigma (k)$, at any $k$. To determine the coefficients of the exclusion statistics we calculate the variation of $\sigma (k)$ at the variation of $\rho (k_i)$, i.e. we calculate the functional derivative
\[
\frac{d \sigma (k)}{L \delta \rho (k_i)} \delta k = \frac{1}{2 \pi} \theta ' (k - k_i) \delta k \delta N (\delta k_i).
\]
(32)

Therefore if we split now the $k$ axis into small intervals, with $\delta k$ the interval around $k$ and $\delta k_i$ the interval around $k_i$, then the variation $\delta N (\delta k_i)$ of $N (\delta k_i)$ produces a variation of $D (\delta k)$ equal to
\[
\delta D (\delta k) \equiv \delta \sigma (k) / L \delta \rho (k_i) = \frac{1}{2 \pi} \theta ' (k - k_i) \delta k \delta N (\delta k_i).
\]
(33)
From the FES formula, \( \delta D(\delta k) = -\bar{\alpha}_{\delta k,\delta k} \delta N(\delta k_i) \), we obtain immediately

\[
\bar{\alpha}_{\delta k,\delta k} = \frac{1}{2\pi} \theta'(k - k_i) \delta k
\]

and therefore the exclusion statistics parameter \( \bar{\alpha}_{\delta k,\delta k} \) is proportional to the dimension of the space on which it acts, \( D(\delta k) \).

Comparing eq. (34a) with eq. (4), we get

\[
a_{kk} = \frac{\theta'(k - k_i)}{2\pi \sigma(k)}. \tag{34b}
\]

From eqs. (34) all the TBA thermodynamics follows in general, by direct application of the formalism presented before, in either bosonic or fermionic formulations, as we shall see immediately.

If the particles are bosons, we plug \( a_{kk} \) into (11), setting \( \bar{\alpha}^{(s)}_k = 0 \), and we obtain after some simplifications

\[
\epsilon(k) = \epsilon_k^{(0)} - \frac{k_BT}{2\pi} \int \log[1 - e^{-\beta \epsilon(k')}]\theta'(k' - k) \, dk'. \tag{35}
\]

In eq. (35) I used the notation \( \epsilon_k^{(0)} = k^2/2 \) and I defined the quasi-particle energy, \( \epsilon(k) \), by \( n_k \equiv \{\exp[\beta \epsilon(k)]\}^{-1} \).

If the particles are fermions, we define the quasi-particle energy by \( \exp[-\beta \epsilon(k)] = n_k/(1 - n_k) \) and we plug eq. (34b) into eq. (15). In this way we recover immediately the TBA equation,

\[
\epsilon(k) = \epsilon_k^{(0)} - \mu + \frac{k_BT}{2\pi} \int \log[1 + e^{-\beta \epsilon(k')}\theta'(k' - k)] \, dk', \tag{36}
\]

similarly to eq. (35).

For the delta-function interaction potential between the particles, \( V(x) = 2\delta(x) \) (\( x \) being the distance between the particles), the phase shift is \( \theta(k) = -2\arctan(k/c) \), which gives

\[
a_{kk} = -\frac{1}{\pi \sigma(k)} \frac{c}{c^2 + (k - k_i)^2} \tag{37}
\]

and therefore

\[
\bar{\alpha}^{(e)}_{\delta k,\delta k} = \frac{\delta k}{\pi} \frac{c}{c^2 + (k - k_i)^2}, \tag{38}
\]

while

\[
\bar{\alpha}^{(s)}_{\delta k} = 0 \tag{39}
\]

for any \( k \).

If the particle-particle interaction is \( V(x) = \lambda(x-1)/x^2 \), then the phase shift is \( \theta(k) = \pi(x-1)\text{sgn}(k) \), where \( \text{sgn} \) is the sign function. From this we obtain

\[
a_{kk} = \frac{(x-1)}{\sigma(k)} \delta(k - k_i). \tag{40}
\]

Note that in this case we do not have any “extensive” mutual exclusion statistics parameters and therefore we may write \( \bar{\alpha}^{(e)}_{\delta k,\delta k} \equiv \bar{\alpha}^{(s)}_{\delta k,\delta k} = (x-1)\delta(k - k_i). \)

In both cases we recover the ansatz (3).

**Fractional quantum Hall effect.** Another system which is traditionally related to the FES is the fractional quantum Hall effect (FQHE) [1,6,14,22,23].

In a Laughlin 1/m-liquid, with \( m \) an odd integer, at any finite temperature there are quasi-particle vortex-like excitations, \( N_+ \) and \( N_- \), corresponding to quasi-electrons and quasi-holes. The numbers of quasi-excitations are related to the number of flux quanta in the system, \( N_\phi = eBA/hc \) (where \( B \) is the magnetic flux and \( A \) is the area of the sample), and the electron number, \( N_e \), by the relation [22,23],

\[
N_\phi = mN_e + N_- - N_+. \tag{41}
\]

For single-particle occupancy (only one quasi-excitation in the system), the number of available states for each of these types of excitations is \( G_+ = G_+ = N_e \), while for general \( N_+ \) and \( N_- \) we have

\[
G_+ = \frac{1}{m} N_\phi - \alpha_+ + N_+ - \alpha_- N_-, \tag{42a}
\]

\[
G_- = \frac{1}{m} N_\phi - \alpha_- + N_+ - \alpha_- N_-, \tag{42b}
\]

where \( \alpha_\pm(i,j = +,-) \) are the FES parameters. Although eventually there is still no consensus regarding the values of \( \alpha_\pm \), which differ for the different liquid models used to describe the FQHE (see, e.g., [23]), for concreteness I shall adopt here the bosonic vortex scheme [22-24], with

\[
\alpha_+ = 2 - 1/m, \quad \alpha_- = 1/m, \tag{43a}
\]

\[
\alpha_- = -1/m, \quad \alpha_+ = 1/m, \tag{43b}
\]

although this is not important for our discussion.

The point is that although the fractional quantum Hall liquid (FQHL) is a macroscopic system, apparently the FES parameters of this system do not obey the general relations (2): there are only two species of quasi-particles —the quasi-electrons and the quasi-holes, with degenerate energy levels—and the \( \alpha \)'s are fixed by (43), so we cannot (apparently) split the species into further sub-species. Still, relations (2) are deduced on very general grounds, so they should be valid.

We have a puzzle.

The solution of this puzzle is straightforward and may be obtained by macroscopic considerations. For this we observe that FQHL being a macroscopic system, one can always divide its area, \( A \), into smaller (but still macroscopic) areas, \( A_i \), \( i = 0,1,\ldots \), and in each of the smaller subsystems the FQHL has the same properties, but with scaled number of electrons, \( N_{ei} \), flux quanta, \( N_{\phi i} \), quasi-electrons, \( N_{\phi i} \), quasi-holes, \( N_{\phi i} \), and \( G_{\phi i} = G_{\phi i} \cdot A_i/A \). In equilibrium, \( N_{ei} = N_{ei} \cdot A_i/A, N_{\phi i} = N_{\phi i} \cdot A_i/A, N_{\phi i} = N_{\phi i} \cdot A_i/A, N_{\phi i} = N_{\phi i} \cdot A_i/A \). In each of these subsystems, the same relations, (42) and (43), are valid for the quantities \( N_{ei}, N_{\phi i}, N_{ei}, N_{\phi i}, \) and \( G_{\phi i} = G_{\phi i} \cdot A_i/A \),
We may eventually end up with a coarse-grained surface of elementary areas, \( \delta A(x, y) \), where the \( x \) and \( y \) are the two-dimensional coordinates on the surface of the FQHL, and with the FES parameters, \( \alpha_{ij}[\delta A(x, y)] \) (\( i, j = +, - \)) that act always on the same elementary area, \( (\delta A(x, y)) \).

Therefore the puzzle was only apparent. Since the system is macroscopic, therefore extensive, the quantities \( \alpha_{ij} \) are local and should be rigorously written as \( \alpha_{ij}(r, r') \equiv \alpha_{ij}\delta(r - r') \), for any \( i, j = +, - \), where by \( r \) and \( r' \) I denoted two position vectors, \((x, y)\) and \((x', y')\), in the plane. This form of FES parameters obeys the general relations (2), as it should, but constitutes an exception to the ansatz (3) proposed in the beginning, due to the fact that the off-diagonal elements \( \alpha_{+-} \) and \( \alpha_{-+} \) are not extensive, but are also proportional to \( \delta(r - r') \).

In the reverse process, if we “glue” all the elementary areas, \( \delta A(r) \), together, we re-obtain the original system, of area \( A \), two species of particles, \( N_+ \) and \( N_- \), and the overall FES parameters (43).

Conclusions. – In this paper I introduced an ansatz for the FES parameters and with it I calculated the statistical distribution of particles from both the bosonic and fermionic perspectives. This ansatz allowed me to write the system of equations for the statistical distribution of particles in a clearer form and to single out the direct exclusion statistics parameters which are most often used in FES calculations.

Then I took three examples: a Fermi-liquid–type system, a one-dimensional integrable quantum system and a fractional quantum Hall (FQH) system. I calculated the FES parameters for these systems and I showed that in the first two cases the parameters obey the ansatz proposed here, whereas for the third system they do not. In this case I also showed that if one takes properly into account the extensivity of a FQH system, then its FES parameters also obey the general conditions deduced in ref. [5].

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