Delayed choice of paths selected by grin and snarl of quantum Cheshire Cat

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The quantum Cheshire Cat is a mysterious phenomenon that temporarily strips a system - a “cat” - of its property - its “grin”. A corollary of this effect is the decoupling of two properties of the same system, which we call the grin and the snarl. This can in principle be realized by detecting two components of polarization of a photon in two different arms of an interferometer. We introduce a mechanism by which we can interchange the positions of these two properties. By doing this we uncover a phenomenon that decouples the grin and the snarl, and puts them in their place even before they encounter the mechanism.

I. INTRODUCTION

The measurement process is one of the least understood aspects in quantum mechanics. In its simplest description, when a measurement of an observable $A$ is performed on a quantum system in a pure state $|\Psi\rangle$, the outcome is one of the eigenvalues $a_i$ with a probability $|\langle a_i|\Psi\rangle|^2$. Further, the state collapses to the corresponding eigenstate $|a_i\rangle$. As a result of this, one can at best obtain an average of the eigenvalues, known as the expectation value of the observable $A$, denoted as $\langle A \rangle$.

In order to extract information about a quantum system, without significantly altering it, a technique known as weak value measurement was developed in Ref. [1]. The quantum system is initially prepared in a pure state $|\Psi_{in}\rangle$, also known as the preselected state. The weak measurement is executed by weakly coupling the system and a meter. After performing the weak measurement of the observable $A$, a projective or strong measurement of a second observable $B$, which in general does not commute with $A$, is performed and one of the outcomes, $|\Psi_f\rangle$ is selected. This process is known as postselection. The average of the shift in the meter readings, for the weak measurement, corresponding to the postselected state, is known as the weak value of the observable $A$. The weak value $A_w$ is interpreted as the value of an observable $A$, between two strong measurements, one giving rise to the preselected state $|\Psi_{in}\rangle$ and the other producing the postselected state $|\Psi_f\rangle$. It is defined as

$$A_w = \frac{\langle \Psi_f | A | \Psi_{in} \rangle}{\langle \Psi_f | \Psi_{in} \rangle}. \quad (1)$$

Although interpreted as a value of an observable, the weak value can lie outside the eigenvalue spectrum [1–3] and can even be complex [4] with the imaginary part being related to the shift in the momentum of the pointer. Weak values have been experimentally observed in Ref. [5][12].

While there has been a great deal of debates and discussions on the meaning and interpretation of weak value and on the implications it can have on the foundations of quantum mechanics [3][13][14], the concept has found myriad applications, including signal amplification [15], spin Hall effect [6], quantum state tomography [16][17], geometric description of quantum states [18], state visualization [19], directly measuring the wave function of a photon [7][20], measuring the expectation value of non-Hermitian operators [21][22], and quantum thermometry [23]. Weak values have also led to unearthing the possibilities of a number of counter-intuitive results such as the Hardy’s paradox [24] and the three-box paradox [3]. The quantum Cheshire Cat is one such phenomenon [25] that betrays commonsense perception.

Experience in the everyday world leads us to believe that an entity and its property are inseparable. However, the idea of quantum Cheshire Cat is exactly to demonstrate that in the realm of quantum mechanics, a particle can be decoupled from its property, under certain conditions. Using a setup, based on the Mach-Zehnder interferometer and a single photon, it can be shown that for a certain combination of preselected and postselected states of the photon, it can be made to traverse one arm of the interferometer, while its polarization traverses through the other arm [25]. This means that the polarization, a property of the photon, can exist independent of the photon itself. Clearly, this is not what we experience in the classical domain. The phenomenon resonates with an episode in the famous novel, Alice in Wonderland, where Alice remarks, “‘Well! I’ve often seen a cat without a grin, but a grin without a cat! It’s the most curious thing I ever saw in my life!’” [26], after her encounter with a character named Cheshire Cat. It is of no surprise that the this phenomenon was therefore christened “Quantum Cheshire Cat”. It is to be noted that to detect the photon or its polarization in an arm of the interferometer, it is necessary to perform weak measurements and not strong measurements, as the latter would destroy the preselected state completely and modify the probability distribution leading to the postselected state.

The quantum Cheshire Cat has been observed experimentally using neutron interferometry [8] as well as photon interferometry [9][11][12]. A recent work deals with the phenomenon in presence of decoherence [27]. A refined version of the original proposal has been suggested in Ref. [25] that decouples all the components of the po-
laboration from the photon. Another proposal that deals with the separation of two degrees of freedom belonging to the same photon can be found in Ref. 29. Some of the recent works in this area include the teleportation of the decoupled circular polarization without the photon 30 and exchanging the decoupled circular polarizations of two quantum Cheshire photons 31. An interesting case of the quantum Cheshire Cat arising from the three-box paradox in Section IV.

In this paper, we unveil yet another counter-intuitive aspect of the phenomenon. Instead of a photon and a polarization component, we deal with two different components of the polarization of the photon, which are known to traverse two different arms of the Mach-Zehnder interferometer. A mechanism to tune the linear polarization of the photon is introduced in the two arms of the interferometer. We show that for two configurations of this tuner, i.e., for two different linear polarizations, the presence of the two components of the linear polarization is interchanged in the two arms. The element of surprise in this is the fact that while the two components have to separate in the two arms at a beam-splitter, the phase-shifting tuners are deciding their fates, regarding which path each should take, at a different point in space, and later in time. Yet each component somehow knows which way it should go.

The paper is organized as follows. In Section II, we lay out the basic principles and methods used in the original quantum Cheshire Cat and discuss how the same idea can be applied to separating two different components of the linear polarization. In Section III, we present our thought experiment that exposes the paradox of the splitting of the components of polarization with the which path decision being made at a different location and time. We conclude with some discussion on the implications of this paradox in Section IV.

II. QUANTUM CHESHIRE CAT

The phenomenon known as the quantum Cheshire Cat can be best realized using a modified version of a Mach-Zehnder interferometer, as outlined in Ref. 25. A photon with a linear horizontal polarization $|H\rangle$ is incident on a beam-splitter $BS_1$. Following this, it can either traverse the left or the right arms of the interferometer. Let us denote the corresponding path degrees of freedom as two orthogonal states, $|L\rangle$ and $|R\rangle$, respectively. Thus the photon can be prepared in a state given by

$$|\Psi_{in}\rangle = \frac{1}{\sqrt{2}} (i |L\rangle + |R\rangle) |H\rangle,$$

which is treated as the preselected state. The required postselected state, on the other hand, is given by

$$|\Psi_f\rangle = \frac{1}{\sqrt{2}} (|L\rangle |H\rangle - i |R\rangle |V\rangle),$$

where $|V\rangle$ denotes the vertically polarized state. The postselection can be achieved using an arrangement of a half-waveplate $HWP$, a phase-shifter $PS$, a beam-splitter $BS_2$, a polarization beam-splitter $PBS$ and three detectors $D_1$, $D_2$ and $D_3$. See Fig 1. The half waveplate $HWP$, that flips horizontal polarization into vertical polarization, and vice-versa, is placed on the right arm, followed by the phase-shifter $PS$ that adds a phase-factor to the photon. The beam-splitter $BS_2$, where the left and the right arms of the interferometer meet again, and the phase-shifter $PS$ are chosen so that the state $\frac{1}{\sqrt{2}}(|L\rangle + i |R\rangle)$ always goes towards the polarization beam-splitter $PBS$ and never towards the detector $D_2$. The $PBS$ transmits the horizontal polarization $|H\rangle$, towards the detector $D_1$, and reflects the vertical polarization $|V\rangle$, towards the detector $D_3$. This ensures that if the state entering the postselection arrangement, starting with the $HWP$, is $|\Psi_f\rangle$, then only $D_1$ will click. Conversely, by selecting the clicks of the detector $D_1$ only, one can postselect the state $|\Psi_f\rangle$.

The whole exercise is to detect the path of the photon and the path of a polarization component, between the preselection and the postselection. To detect the photon, without appreciably disturbing the state, weak measurements of the observables $\Pi_L = |L\rangle \langle L|$ and $\Pi_R = |R\rangle \langle R|$ are performed in the left and the right arm, respectively. Similarly, weak measurements of certain observables must be performed to detect the polarization component in the two arms of the interferometer. These observables can be defined as

$$\sigma_{x}^{L} = \Pi_{L} \otimes \sigma_{x},$$

$$\sigma_{x}^{R} = \Pi_{R} \otimes \sigma_{x},$$

where $\sigma_{x} = |H\rangle \langle V| + |V\rangle \langle H|$. 

FIG. 1. The Quantum Cheshire Cat. The schematic diagram shows the requirements for a potential experimental realization of the quantum Cheshire Cat. See text for more details. From 31 with permission.
The corresponding weak values for the preselected state $|\Psi_{in}\rangle$ and the postselected state $|\Psi_f\rangle$ are thus measured to be

$$ (\Pi_L)_w = 1 \quad \text{and} \quad (\Pi_R)_w = 0, \quad (5) $$

and

$$ (\sigma_z^L)_w = 0 \quad \text{and} \quad (\sigma_z^R)_w = 1, \quad (6) $$

which indicate that the photon passed through the left arm while the $x$-component of its polarization passed through the right arm. Thus, the polarization component can exist without the presence of the photon in the right arm, a situation best described by the analogy with ‘a grin without a cat!...’ from Alice in Wonderland [26].

A. The grin and the snarl of the Cat

So far we have concentrated on decoupling a property (polarization-component/ grin) of a system from the system (photon/ cat) itself. Let us now look at two different properties of the same system, the $x$-component of polarization and the $z$-component of the polarization, labelled as the grin and the snarl of the cat, respectively. Notice that the presence of these two properties can be detected in the left and right arms of the interferometer by weakly measuring the following observables. For the grin, we must weakly measure

$$ \sigma_x^L = \Pi_L \otimes \sigma_x \quad \text{and} \quad \sigma_x^R = \Pi_R \otimes \sigma_x. \quad (7) $$

On the other hand, for the snarl, we must perform weak measurements of

$$ \sigma_z^L = \Pi_L \otimes \sigma_z \quad \text{and} \quad \sigma_z^R = \Pi_R \otimes \sigma_z, \quad (8) $$

where $\sigma_z = |H\rangle \langle H| - |V\rangle \langle V|$. Under the same preselection $|\Psi_{in}\rangle$ and the postselection $|\Psi_f\rangle$, the weak values for these operators are measured to be

$$ (\sigma_z^L)_w = 0 \quad \text{and} \quad (\sigma_z^R)_w = 1, \quad (9) $$

$$ (\sigma_x^L)_w = 1 \quad \text{and} \quad (\sigma_x^R)_w = 0. \quad (10) $$

Therefore, while the $x$-component (the grin) of polarization is detected in the left arm, the $z$-component (the snarl) is detected in the right arm. Thus two non-commutative properties ($\sigma_x$ and $\sigma_z$) of the same system stand separated. In the succeeding section, we add a fresh twist to the tale by arguing that this separation can be controlled non-locally, by preparing the preselected state after the point of separation.

III. DELAYED CHOICE OF POLARIZATION AND ITS EFFECT ON THE GRIN AND THE SNARL OF THE QUANTUM CHESHIRE CAT

We have seen that the preselected state preparation and the decoupling of the grin and the cat or the separation of the grin and the snarl, occur at a particular point in space i.e., at the beam-splitter $BS_1$. In this section, we relocate the preselection beyond the point in the interferometer where the separation of the two degrees of freedom occurs. To make this happen, we introduce tunable polarization phase-shifters in the two arms of the interferometer, which are synchronized and can be adjusted simultaneously. See Fig. 2 If we consider that the photon entering $BS_1$ has a polarization state $|H\rangle$, then the phase-shifters $P_1$ cause a transformation $|H\rangle \rightarrow \cos \frac{\theta}{2} |H\rangle + \sin \frac{\theta}{2} |V\rangle$. The state of the photon, post these phase-shifters, is given by

$$ |\Psi_{in}\rangle = \frac{1}{\sqrt{2}}(|L\rangle + e^{i\phi} |R\rangle)(\cos \frac{\theta}{2} |H\rangle + \sin \frac{\theta}{2} |V\rangle), \quad (10) $$

where we have installed a second adjustable phase-shifter $P_2$ in the right arm that causes a total phase-difference of $\phi$ between the left and the right path degrees of freedom. Note that an extra phase of $\frac{\pi}{2}$ is added to the state asso-
cated with the left path, due to reflection from $BS_1$. The phase $\phi$ is the sum of this phase and the phase-difference we would like to introduce using $P_2$.

The postselection is carried out in the state

$$|\Psi_f\rangle = \frac{1}{\sqrt{2}} (|L\rangle |H\rangle + |R\rangle |V\rangle).$$  \hfill (11)

Any weak value measurement after this point and before the postselection, chooses the state given in Eq. (10) as the preselected state. Accordingly, the modified weak values are given by

$$\langle \sigma^L_w \rangle = \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2} + \sin \frac{\theta}{2} e^{i \phi}},$$

$$\langle \sigma^R_w \rangle = \frac{\cos \frac{\theta}{2} e^{i \phi}}{\cos \frac{\theta}{2} + \sin \frac{\theta}{2} e^{i \phi}},$$  \hfill (12)

and

$$\langle \sigma^z_w \rangle = \frac{\cos \frac{\theta}{2}}{\cos \frac{\theta}{2} + \sin \frac{\theta}{2} e^{i \phi}},$$

$$\langle \sigma^x_w \rangle = -\frac{e^{i \phi} \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} + \sin \frac{\theta}{2} e^{i \phi}}.$$  \hfill (13)

Suppose that the tuner $P_1$ has been set to $\theta = \pi$. We can immediately see from Eq. (13), that for an arbitrary value of $\phi$,

$$\langle \sigma^L_w \rangle = 0 \text{ and } \langle \sigma^R_w \rangle = -1.$$  \hfill (14)

For the same value of $\theta$, and $\phi$ set to 0, we can see from Eq. (12),

$$\langle \sigma^L_w \rangle = 1 \text{ and } \langle \sigma^R_w \rangle = 0.$$  \hfill (15)

It is clear that for the above choice of the phases, the $z$-component of the polarization can be detected only in the right arm, while simultaneously, the $x$-component can be detected only in the left arm. Now consider the choice of the phases as $\theta = 0$ and $\phi = 0$. The corresponding weak values are

$$\langle \sigma^L_w \rangle = 1 \text{ and } \langle \sigma^R_w \rangle = 0,$$  \hfill (16)

$$\langle \sigma^Z_w \rangle = 0 \text{ and } \langle \sigma^X_w \rangle = 1.$$  \hfill (17)

Clearly, the $z$-component of polarization can now be detected only in the left arm and the $x$-component of polarization can be detected only in the right arm.

The two configurations of the tunable phase-shifters can be used to flip the $x$ and $z$-components of the polarization of the photon between the two arms of the interferometer. What is perplexing is that the components know which arm to enter even before they encounter the phase-shifters $P_1$ and $P_2$ which actually dictate this very thing! Thus the grin and the snarl of the quantum Cheshire Cat, not only can travel independent of each other, for the given combination of pre and postselected states, they also seem to be affected non-locally and from the future, by the configurations of the phase-tuners.

**IV. CONCLUSION**

The quantum Cheshire Cat is an intriguing phenomenon, in which a property of a physical system can be temporarily decoupled, i.e. separated, from the system. The phenomenon produces ripples in foundational aspects of quantum mechanics as well as provides interesting applications. A corollary of the effect is that two properties, corresponding to non-commuting observables, of the same system can be decoupled as well. Within a related but different setting, we have proposed a gendanken experiment in which we show that the decoupling and the eventual temporary locations of the properties in separate regions can be affected by an operation at a different location and time.

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