Dynamical properties of the weak stability boundary and associated sets

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Abstract. In this contribution the weak stability boundary algorithmic definition was numerically accomplished with the inclusion of lunar and earth collisional sets and a subclassification of the unstable set. Then, the associated sets to WSB definition were analyzed and characterized according to relevant dynamical properties in order to clarify their applicability in earth-moon transfer orbit design. The obtained stable, unstable, and collisional sets are defined as a function of the osculating ellipse eccentricity for prograde and retrogade initial conditions. The stable sets, candidates to ballistic capture transfers, are subclassified according to chosen specific criteria, namely, the Jacobi constant intervals defined by distinct classes of Hill regions, the location of the final state after a complete cycle with respect to the Hill sphere, the permanence in the lunar sphere of influence in a full cycle around the moon, and exit basins for retrograde evolution. By the first time, with this investigation, elucidative criteria based on three-body problem elements are employed to identify initial condition subsets with required properties to design ballistic capture transfers.

1. Introduction
The Weak Stability Boundary (WSB) concept was heuristically proposed by E. Belbruno (1987 and beyond) [1] in the context of ballistic lunar capture in low energy earth-moon transfer orbit design. Traditional techniques usually rely on two-body dynamics, building transfer trajectory approximations by patching several segments of conic solutions obtained by considering successive two-body interactions, with one body always being the spacecraft. As space missions became more demanding, alternative design methods have been proposed aiming to employ the nonlinear dynamical characteristics of three and four body systems in a more natural way. By doing so, total fuel consumption may be reduced, allowing a higher diversity of mission profiles.

Particularly, a class of low energy lunar transfers associated with the WSB concept was introduced, including the gravitational effects of the earth, the moon and the sun in modeling the motion of the spacecraft [2]. In the final portion of the transfer trajectory, the spacecraft approaches the moon in a state defined as lunar ballistically captured, thus reducing the propellant mass needed to stabilize its motion into the final prescribed selenocentric orbit. Some expressions found in literature to describe the WSB concept are “regions in the phase space where the perturbative effects of the earth-moon-sun acting on the spacecraft tend to balance” [3] and “a location near the moon where the spacecraft lies in the transition between ballistic capture and ejection” [4].
The WSB concept was successfully employed for the first time in the Japan’s Hiten spacecraft rescue to the moon in 1991 [3]. Originally, the Hiten mission, known before launch as Muses-A, had been designed to orbit the earth, serving as a communication relay for Hagoromo, a second spacecraft, that would be send to the moon. However, Hagoromo failed due to loss of communication, so it was proposed to send Hiten to the moon. The low amount of available on board propellant (10% of necessary fuel for a classical transfer design) required an alternative transfer orbit design technique and WSB strategy was able to carry Hiten from an earth orbit to a lunar orbit. This successful application motivated a number of papers regarding WSB earth-moon transfer projects [5–8].

The WSB algorithmic definition [1, 9] was recently refined by F. García and G. Gómez [6]. In this approach, the WSB concept is established in the framework of the Planar Circular Restricted Three-Body Problem (PCRTBP). Departing from an initial condition set, trajectories are generated and classified according to a prescribed stability criterion, in such a way that the WSB set is defined as the boundary set between stable and unstable sets.

This contribution reports a modified numerical implementation of the usual algorithmic definition and a series of dynamical analyses performed in order to investigate the applicability aspects of the generated sets in ballistic capture transfer projects. In this work, the collision of the third body with the surfaces of the primaries, namely, the earth and the moon, is taken into account with the inclusion of the mean radius of each primary since the beginning of the implementation. The algorithmic definition numerical implementation is performed for distinct osculating ellipse eccentricity values and for prograde and retrograde initial conditions. Moreover, the unstable set is explicitly generated and subclassified according to the associated unstable behavior of each initial condition, revealing the existence of subcases for the stable-unstable boundary.

The preliminary analysis classifies the full initial condition set according to the integral of motion of the PCRTBP, the Jacobi constant, in order to verify which possible transport channels are available for the set. Then, we focus our attention on the stable set analysis, since its elements are natural candidates to be employed in ballistic capture transfers. The stable set is subclassified according to convenient specific criteria, namely, the Jacobi constant intervals defined by distinct classes of Hill regions, the location of the final state after a complete cycle in relation to the Hill sphere, the permanence in the lunar sphere of influence in a full cycle around the moon, and exit basins for retrograde evolution. The purpose of this investigation is to elucidate the applicability characteristics of the trajectories associated to the defined set of initial conditions in ballistic capture transfers.

The present paper is organized as follows: in Section 2, we present the theoretical framework upon which the work is based on, namely, we describe the dynamical model, the PCRTBP, several capture definitions, and the revised version of the WSB algorithmic definition, as in [6]. Our particular implementation, analyses and results are delineated in Section 3. The last section is devoted to the final remarks and conclusions.

2. Theoretical Framework

We proceed with the description of the Planar Circular Restricted Three-Body Problem (PCRTBP), since it provides the dynamical framework in which the WSB has been defined. Then some definitions related to capture are stated in order to set ground to define a special kind of capture in the PCRTBP, called ballistic capture, related to the WSB [9]. Finally, definitions for the WSB, proposed in the literature, are stated.

2.1. Dynamical Model

The Restricted Three-Body Problem (RTBP) describes the motion of a particle \( P_3 \) of negligible mass moving under the gravitational influence of two bodies \( P_1 \) and \( P_2 \), called the primaries,
of masses $m_1$ and $m_2$, respectively. The motion of $P_1$ and $P_2$ is not perturbed by $P_3$, being a solution of a Kepler Problem, with two bodies moving under mutual gravitational influence. In the planar circular version of the RTBP, the PCRTBP, $P_1$ and $P_2$ describe circular coplanar orbits around the barycenter of this two-body system and the motion of the third body is restricted to the orbital plane of the primaries [10]. This model is suited to describe the earth-moon-spacecraft system (earth ⇔ $P_1$, moon ⇔ $P_2$, spacecraft ⇔ $P_3$), given that the moon’s orbit eccentricity around the earth is 0.05490 and its inclination with respect to the orbit of the planet is 5°09′. This dynamical model is expressed in non-dimensional variables, in such a way that the distance between $P_1$ and $P_2$, the sum of their masses and their angular velocity around the barycenter are normalized to one. The normalized masses of $P_1$ and $P_2$ are given, respectively, by $1 - \mu$ and $\mu$, with $\mu = m_2/(m_1 + m_2)$, $m_1 > m_2$, being the only parameter of the model. For the earth-moon system, $\mu = 0.0121506683$.

In the synodic coordinate system $x$-$y$, with origin in the barycenter of $P_1$-$P_2$ and that rotates with the primaries with respect to an inertial frame $X$-$Y$, $P_1$ and $P_2$ are located at $(\mu, 0)$ and $(\mu - 1, 0)$, respectively, as shown in Figure 1.

Let $(x, y, \dot{x}, \dot{y})$ represent the state of $P_3$ in the synodic reference system, the particle’s equations of motion are given by

$$\begin{align*}
\ddot{x} - 2\dot{y} &= \Omega_x, \\
\dot{y} + 2\dot{x} &= \Omega_y,
\end{align*}$$

(1)

where

$$\Omega(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu(1 - \mu)}{2},$$

(2)

is the effective potential, shown in Figure 2, for $\mu = 0.3$, and $r_1 = [(x - \mu)^2 + y^2]^{1/2}$ and $r_2 = [(x + 1 - \mu)^2 + y^2]^{1/2}$ are the distances from $P_1$ and $P_3$, respectively. Both in Figures 2 and 3, the value of $\mu = 0.3$ was adopted for the sake of visualizing clarity.

This system has an integral of motion given by

$$J(x, y, \dot{x}, \dot{y}) = 2\Omega(x, y) - (\dot{x}^2 + \dot{y}^2) = C,$$

(3)

where $C$ is the Jacobi constant. The conservation associated to $J$ defines a three-dimensional invariant manifold immersed in the four-dimensional phase space by

$$\mathcal{M}(\mu, C) = \{ (x, y, \dot{x}, \dot{y}) \in \mathbb{R}^4 | J(x, y, \dot{x}, \dot{y}) = \text{constant} \}.$$
The dynamical model has five equilibria, called libration points or Lagrangian points, defined by
\[
\frac{\partial J}{\partial x} = \Omega_x = 0, \quad \frac{\partial J}{\partial y} = \Omega_y = 0, \quad \frac{\partial J}{\partial \dot{x}} = 0 \Rightarrow \dot{x} = 0, \quad \frac{\partial J}{\partial \dot{y}} = 0 \Rightarrow \dot{y} = 0. \tag{5}
\]
Three of them, the collinear equilibria \(L_k, k = 1, 2, 3\), are located on the \(x\)-axis and are saddle-center points. The other two, the triangular points \(L_k, k = 4, 5\), are located at the vertices of equilateral triangles formed with the positions of primaries and are stable if \(m_1/m_2 > 24.96\), as it is the case for the earth-moon system. The Jacobi constant values evaluated in these static solutions are denoted by \(C_k, k = 1, 2, 3, 4, 5\).

The regions obtained by the projection of the \(M\) surface onto the position space \(x-y\) are called the Hill regions, \(M\), defined by
\[
M(\mu, C) = \{(x, y)|\Omega(x, y) \geq C/2\} \tag{6}
\]
and bounded by the zero-velocity curves, the locus of points in the \(x-y\) plane where the kinetic energy vanishes. The Hill regions constitute the accessible areas to the trajectories for each given \(C\), or analogously, energy value. For a given \(\mu\), there are five basic configurations for these regions, defined by the \(C_k\) values. The first four cases are shown in Figure 3. In the fifth case, motion over the entire \(x-y\) plane is possible.

![Figure 3](image-url)

**Figure 3.** Four possible Hill regions cases for \(\mu = 0.3\): the white areas correspond to the Hill regions. The inaccessible areas and the zero velocity curves are shown in grey and black, respectively. The blue points correspond to the primaries and the red ones to the equilibrium points \(L_k, k = 1, 2, 3, 4, 5\). (a) Case 1 (\(C > C_1\)): no transit orbits between the primaries are possible. (b) Case 2 (\(C_1 > C > C_2\)): the realms around \(P_1\) and \(P_2\) are connected by transit orbits through the neck region around \(L_1\). (c) Case 3 (\(C_2 > C > C_3\)): the same as in Case 2, besides the possibility of transit to the external realm of motion through the neck region around \(L_2\). (d) Case 4 (\(C_3 > C > C_4 = C_5\)): there is a neck region also around \(L_3\). In the fifth case (\(C < C_4 = C_5\)), motion over the entire \(x-y\) plane is possible.

There are periodic orbits, called Lyapunov orbits, around each collinear Lagrangian point [11]. The unstable and stable invariant manifolds associated to a Lyapunov orbit are locally
homeomorphic to two-dimensional cylinders and act as separatrices of the phase space, defining four categories of trajectories in the neck region near each equilibrium [12], namely transit, nontransit and asymptotic orbits, besides the periodic orbit itself. The $x$-$y$ projection $R$ of the region $\mathcal{R}$ around a generic equilibrium $L_k$, $k = 1, 2, 3$ is shown in Figure 4.

![Figure 4. The four categories of orbits are shown in the $x$-$y$ projection $R$ of the neck region $\mathcal{R}$ around an equilibrium $L_k$, $k = 1, 2, 3$. The periodic orbit is the black curve around $L_k$ and the transit, nontransit and asymptotic orbits are represented by the red, blue and green curves, respectively.](image)

2.2. Capture

Let $q = (x, y) \in \mathbb{R}^2$ be the position vector of $P_3$. In general, a capture state of $P_3$ is defined when $P_3$ is somehow geometrically bounded to $P_1$ and (or) $P_2$. Following [9], we state the definitions of some relevant types of capture.

**Definition 1 (Permanent capture)** $P_3$ is permanently captured into the $P_1$-$P_2$ system in forward time if $|q|$ is bounded as $t \to \infty$, and $|q| \to \infty$ as $t \to -\infty$. $P_3$ is permanently captured in backward time if $|q|$ is bounded as $t \to -\infty$, and $|q| \to \infty$ as $t \to \infty$.

For the General Problem of Three Bodies, it was proved by Chazy that the set of initial values leading to permanent capture comprises a set of measure zero. The existence of such orbits was later shown by Alekseev. For further discussion we refer the reader to [13].

**Definition 2 (Temporary capture)** $P_3$ has temporary capture at $t = t^*$, $|t^*| < \infty$, if $|q(t^*)| < \infty$ and

$$\lim_{t \to \pm \infty} |q(t)| = \infty.$$

The temporary capture state implies the ejection of the particle as $t \to \pm \infty$, that is, $P_3$ is neither permanently captured nor describes an unbounded oscillatory orbit.

Definition 2 can be modified to finite temporary capture considering that $P_3$ achieves a prescribed finite distance $|q(t)| = d > 0$ from $P_2$, for instance, at time $t^*$. In this case, the time interval $\Delta t = |t^* - T^*| > 0$ represents the duration of the capture relative to a reference time $T^*$. This definition measures the time that $P_3$ remains within a given neighborhood of $P_2$.

An alternative type of capture, called ballistic capture, said to be distinguished from the previously geometrically defined types of capture, is presented in [9].

Let $(\tilde{x}, \tilde{y}, \dot{\tilde{x}}, \dot{\tilde{y}})$ be the state of $P_3$ in an inertial reference frame with origin in $P_2$. The Keplerian (two-body) energy of $P_3$ w.r.t. $P_2$ is

$$h_K = \frac{1}{2} (\dot{\tilde{x}}^2 + \dot{\tilde{y}}^2) - \frac{Gm_2}{\tilde{r}_2}, \quad (7)$$

where $\tilde{r}_2 = \sqrt{\tilde{x}^2 + \tilde{y}^2}$ is the distance between $P_3$ and $P_2$, $m_2$ is the mass of the primary and $G$ stands for the universal gravitational constant [14].
Definition 3 (Ballistic capture) $P_3$ is ballistically captured by $P_2$ at time $t = t_c$ if

$$h_K(\varphi(t_c)) \leq 0$$

for a solution $\varphi(t) = (x(t), y(t), \dot{x}(t), \dot{y}(t))$ of the RTBP.

We note that this definition is not unique. In [15], for example, ballistic capture by the moon refers to an orbit which under natural dynamics of the RTBP gets within a 20,000 km radius region around the moon and performs at least one revolution around that primary.

2.3. The Weak Stability Boundary
Let $(x, y, \dot{x}, \dot{y})$ be the state of $P_3$ in the synodic reference frame and $J(x, y, \dot{x}, \dot{y})$ be the integral of Jacobi (Eq.(3)). Consider the sets

$$J^{-1}(C) = \{(x, y, \dot{x}, \dot{y})|J(x, y, \dot{x}, \dot{y}) = C\},$$

$$\Sigma = \{(x, y, \dot{x}, \dot{y})|h_K(x, y, \dot{x}, \dot{y}) \leq 0\},$$

$$\sigma = \{(x, y, \dot{x}, \dot{y})|\dot{r}_2 = 0\},$$

where $\dot{r}_2$ is the radial velocity of $P_3$ relative to $P_2$, so that $\sigma$ is the set of points corresponding to local periapsis or apoapsis of osculating ellipses, and $h_K$ is the Keplerian energy of $P_3$ w.r.t. $P_2$.

Definition 4 (WSB analytically defined) The set $W$, associated to a certain value of the Jacobi constant $C$, given by

$$W = J^{-1}(C) \cap \Sigma \cap \sigma,$$

is called the Weak Stability Boundary, a special set where ballistic capture occurs in the RTBP.

There is also an algorithmic definition of the WSB [3,9], discussed and revised by F. García and G. Gómez [6]. Following this reference, consider a radial segment $l(\theta)$ departing from $P_2$ and making an angle $\theta$ with the $x$-axis. Take trajectories for $P_3$, starting on $l(\theta)$ which satisfy:

(i) $P_3$ starts its motion at the periapsis of an osculating ellipse around $P_2$, at a certain point on $l(\theta)$. If $r_2$ denotes the distance between $P_3$ and $P_2$ at this point, then $r_2 = a(1 - e)$, where $a$ and $e$ are, respectively, the semi-major axis and the eccentricity of the ellipse.

(ii) The initial velocity vector of the trajectory is perpendicular to the radial segment $l(\theta)$.

(iii) The initial Kepler energy $h_K$ of $P_3$ w.r.t. $P_2$ is negative, which occurs for $e \in [0, 1)$, since the two-body energy computed at the periapsis is

$$h_K = \frac{\mu(e - 1)}{2r_2}.$$  \hfill (10)

(iv) The eccentricity of the initial Keplerian motion is held fixed along $l(\theta)$. According to Eq.(10), this condition is equivalent to fix the value of $h_K$ for each value of $r_2$ and $\forall \theta \in [0, 2\pi]$.

The following definition of stability and instability is given for the trajectories with the initial conditions specified above.

Definition 5 The motion of a particle is said to be stable about $P_2$ if after leaving $l(\theta)$ it makes a full cycle about $P_2$ without going around $P_1$ and returns to $l(\theta)$ at a point with negative Kepler energy w.r.t. $P_2$. Otherwise, the motion will be unstable. (See Figure 5.)

As the initial conditions vary along $l(\theta)$, there is a distance $r^*(\theta, e)$ such that:
• If \( r_2 < r^* \), the motion is stable.
• If \( r_2 > r^* \), the motion is unstable.

Furthermore, \( r^*(\theta, e) \) is a smooth function of \( \theta \) and \( e \) which defines the WSB as the locus of all points \( r^*(\theta, e) \) along \( l(\theta) \), \( \theta \in [0, 2\pi] \), for which there is a change of stability \([4, 9]\).

**Definition 6 (WSB algorithmic definition)** The Weak Stability Boundary is given by the set

\[
\mathcal{W} \equiv \{ r^*(\theta, e) | \theta \in [0, 2\pi], e \in [0, 1) \}.
\]

As pointed in \([6]\) this definition is not accurate in the following points:

- The requirements on the initial conditions fix the modulus and direction of the velocity but not its sense. So, for a fixed position on \( l(\theta) \) there are two different initial velocities, fulfilling the four restrictions, which can produce orbits with different stability behavior. The modulus of the initial velocity of \( P_3 \), is given by

\[
\nu^2 = \mu \left( \frac{2}{r_2^2} - \frac{1}{a} \right) = \mu \left( 1 + e \right) \frac{r_2^2}{a}.
\]

The initial conditions with positive velocity (prograde osculating motions about \( P_2 \)) are

\[
\begin{align*}
x &= -1 + \mu + r_2 \cos \theta, \\
\dot{x} &= r_2 \sin \theta - \nu \sin \theta, \\
y &= r_2 \sin \theta, \\
\dot{y} &= -r_2 \cos \theta + \nu \cos \theta,
\end{align*}
\]

and the initial conditions with negative velocity (retrograde osculating motions about \( P_2 \)) are

\[
\begin{align*}
x &= -1 + \mu + r_2 \cos \theta, \\
\dot{x} &= r_2 \sin \theta + \nu \sin \theta, \\
y &= r_2 \sin \theta, \\
\dot{y} &= -r_2 \cos \theta - \nu \cos \theta.
\end{align*}
\]

- Contrary to what is suggested by the definition of \( \mathcal{W} \), for fixed values of \( \theta \) and \( e \), there may be several transitions from stability to instability.
- Some maximum time interval must be fixed for the numerical integration. In \([6]\), \( T_{\text{max}} \) was taken as 80 non-dimensional time units. If after this time interval, a trajectory does not cross again the segment \( l(\theta) \), the orbit is considered unstable.

The algorithmic definition is extended in \([6]\), considering the stability criterion after \( n \) revolutions of \( P_3 \) about \( P_2 \) without going around \( P_1 \). Also some slight modification in the definition is introduced in \([8]\) to study the behavior in the first stable-unstable transition.
3. Results and dynamical analysis

3.1. Results of the considered WSB algorithmic definition implementation

Once that this work deals with the applicability of the WSB and associated sets in realistic mission design involving ballistic capture transfers, the punctual mass idealization of the mathematical model is substituted by finite bodies with the explicit inclusion of the mean radii of the earth and the moon. Being so, this WSB algorithmic definition implementation generates, besides stable and unstable sets, the moon and earth collisional sets, i.e., sets of initial conditions of trajectories which collide with each primary. Moreover, the unstable set was subclassified according to five proposed instability criteria: (i) unstable due to negative Keplerian energy, when trajectories return to \( l(\theta) \) after one turn around the moon, (ii) primary interchange through \( L_1 \) Lagrangian point with \( C > C_3 \), (iii) primary interchange through \( L_2 \) Lagrangian point with \( C > C_3 \), (iv) geometric escape with \( C < C_3 \), and (v) unstable due to exceeding the maximum integration time, without returning to \( l(\theta) \). With the unstable set subclassification, the WSB can be decomposed in distinct subsets of stable-unstable transitions.

![Figure 6](image-url)

**Figure 6.** Projection onto the \( x-y \) plane of the sets generated by the WSB algorithmic definition with positive initial velocity (osculating prograde motion about \( P_2 \)) for: (a) \( e = 0.0 \), (b) \( e = 0.3 \), (c) \( e = 0.6 \), and (d) \( e = 0.9 \). **Black:** \( S \), stable; **Red:** \( C^M \), collision with the moon; **Yellow:** \( C^E \), collision with the earth; **Green:** \( E \), unstable due to negative Kepler energy; **Dark grey:** \( G^1 \), primary interchange through \( L_1 \); **Grey:** \( G^2 \), primary interchange through \( L_2 \); **Light grey:** \( G^3 \), geometric escape with \( C > C_3 \).
The implementation was performed holding fixed the osculating ellipse eccentricity value, in such a way that the investigated sets are given as functions of \( r \) and \( \theta \).

Figure 6 presents the projection onto the \( x-y \) plane of the \( \mathcal{W}^e \) associated sets generated by the prescribed WSB algorithmic definition with initial positive velocity (osculating prograde motion about \( P_2 \)) and for the eccentricity values (a) \( e = 0.0 \), (b) \( e = 0.3 \), (c) \( e = 0.6 \), and (d) \( e = 0.9 \). The same is shown by Figure 7 for initial conditions with negative velocities (osculating retrograde motion about \( P_2 \)). The ranges of the spatial coordinates were chosen following [6]. The generated sets of initial conditions correspond to arrival states of the complete transfer orbits, and it is desirable that these states are in a near enough neighborhood of the moon. So, the presence of initial conditions outside the lunar Hill region seems not to be appropriate for transfer orbit design from the beginning.

Figure 7. Projection onto the \( x-y \) plane of the sets generated by the WSB algorithmic definition with negative initial velocity (osculating retrograde motion about \( P_2 \)) for: (a) \( e = 0.0 \), (b) \( e = 0.3 \), (c) \( e = 0.6 \), and (d) \( e = 0.9 \). **Black:** \( S \), stable; **Red:** \( C^M \), collision with the moon; **Yellow:** \( C^E \), collision with the earth; **Green:** \( \mathcal{E} \), unstable due to Kepler energy; **Dark grey:** \( G^1 \), primary interchange through \( L_1 \); **Grey:** \( G^2 \), primary interchange through \( L_2 \); **Light grey:** \( G^3 \), geometric escape with \( C > C_3 \).

The prograde and retrograde initial conditions generate completely different resulting sets. The stable set for low eccentricity usually is localized in a large region around the moon. From this central region, thin structures spread in the vertical direction. As the eccentricity increases, this core region diminishes and for negative velocities, its shape becomes irregular. The lunar collisional set, in general, spreads along the longitudinal direction of the stable set boundary.

The ballistic capture definition is quantitatively based on a two-body scalar, the Kepler
energy, whose validity of application in a many-body context is restricted to the inner region of the moon sphere of influence (SOI) [16], here approximated by the region delimited by the Hill radius (about 64,483 km around the moon). The details of the central regions of Figure 6(a) and (d) and Figure 7(a) and (d) are presented, respectively, in Figures 8 and 9, at which the lunar Hill region boundary is depicted by a white curve. The portion of the stable set contained in the lunar SOI diminishes drastically with increasing eccentricity values for both signs of initial velocity.

Figure 8. Detail of the central region of Figure 6(a) and (d), exhibiting the lunar Hill radius depicted by the white curve.

Figure 9. Detail of the central region of Figure 7(a) and (d), exhibiting the lunar Hill radius depicted by the white curve.

There are important remarks to be pointed out here: (i) It is claimed by [9] that the ideal transfer trajectories lie on the WSB set, i.e., at the boundary between sets with different stability behavior. These figures illustrate that just a reduced fraction of the full WSB set is interior to the lunar SOI for the considered full initial condition sets, as defined by [6]. (ii) Due to the restricted adequacy of the two-body quantifier, the exterior lunar SOI region of initial conditions can produce highly nonlinear trajectories for which many-problem effects are not negligible. (iii) Our new approach for the numerical implementation reveals that a large portion of the WSB set, mathematically defined, is contained in the moon collisional set, thus restricting the practical applicability of these trajectories to the circularization at the initial condition.
3.2. Dynamical investigation of the generated sets

This subsection presents a series of dynamical analyses of the obtained associated sets. The first analysis is applied for sets of both senses of initial velocity aiming their general characterization. However the other analyses were applied just to the prograde initial condition sets due to their great interest for practical applicability.

It is worthy of notice that Belbruno’s original WSB algorithmic definition $W^e$ holds fixed a two-body problem quantity, the osculating ellipse eccentricity, and maintains free a relevant three-body problem quantity: the invariant of the PCRTBP, the Jacobi constant. A very elucidative clarifying preliminary analysis is to evaluate the Jacobi constant for each initial condition of the algorithmic definition, once that transport properties in the PCRTBP is intimately related with the five possible Hill region profiles which are delimited by the critical Jacobi constants, $C_k$. Different possibilities of transport channels may occur between the realms of each primary, and the external region (lunar, earth, and exterior realms, respectively) as a function of $C$.

![Figure 10](image)

**Figure 10.** Classification of the prograde initial condition set according to the Jacobi constant for (a) $e = 0.0$, (b) $e = 0.3$, (c) $e = 0.6$, and (d) $e = 0.9$. **Black:** Case 1, $C > C_1$; **Red:** Case 2, $C_1 > C > C_2$; **Green:** Case 3, $C_2 > C > C_3$; **Blue:** Case 4, $C_3 > C > C_4 = C_5$; **Cyan:** Case 5, $C < C_4 = C_5$.

In the first case, $C > C_1$, no transit orbits between the primaries are possible. In the second case, $C_1 > C > C_2$, the realms around $P_1$ and $P_2$ are connected by transit orbits through the neck region around $L_1$, while for the third case, $C_2 > C > C_3$, the same as in the second case occurs plus the possibility of transit to the external realm of motion through the neck region around $L_2$. In the fourth case, $C_3 > C > C_4 = C_5$, there is a neck region also around $L_3$. In the
fifth case \( C < C_4 = C_5 \), motion over the entire \( x-y \) plane is possible. The availability of these transport channels is essential in the capture process.

**Figure 11.** Classification of retrograde initial condition set according to the Jacobi constant for (a) \( e = 0.0 \), (b) \( e = 0.3 \), (c) \( e = 0.6 \), and (d) \( e = 0.9 \). **Black:** Case 1, \( C > C_1 \); **Red:** Case 2, \( C_1 > C > C_2 \); **Green:** Case 3, \( C_2 > C > C_3 \); **Blue:** Case 4, \( C_3 > C > C_4 = C_5 \); **Cyan:** Case 5, \( C < C_4 = C_5 \).

Figures 10 and 11 exhibit this Hill region classification of the full initial condition set, respectively, for prograde and retrograde initial velocities, for four eccentricity values, revealing that the distinct five classes of Hill regions occur for these initial condition sets. As it can be seen, a large amount of initial conditions belongs to the first case at which direct transfer orbits do not exist. In this case, the three distinct accessible isolated Hill regions, namely, the moon (in the central region), earth (at the right side region), and exterior (at the left side region) realms, are present in the full set. This observation corroborates that there is no reason or convenience to consider such extended initial condition ranges.

The same analysis was performed for the stable set with prograde and retrograde initial conditions as shown in Figures 12 and 13. The stable sets were subclassified in subsets \( S_j^k \), \( k = 1,2,3,4,5 \) according to the Hill region cases, and \( j = 1,2 \), respectively, for prograde and retrograde initial velocities. The first subset, \( S_1^1 \), is predominant in the prograde stable low altitude initial conditions for low eccentricity values. As the eccentricity increases, \( S_1^1 \) shrinks, being almost inexistent for higher eccentricities. For retrograde stable set, most of the initial conditions belongs to the high energy cases. In particular, for \( e = 0.9 \) the first two subsets, \( S_2^1 \) and \( S_2^2 \), are completely absent in the stable set, and \( S_2^5 \) is prevalent. For usual altitudes of selenocentric final orbits considered in realistic projects, initial conditions located far away from
Figure 12. Classification of the prograde stable set of initial conditions according to the Jacobi constant for (a) $e = 0.0$, (b) $e = 0.3$, (c) $e = 0.6$, and (d) $e = 0.9$. **Black:** $S_1^1$ subset ($C > C_1$); **Red:** $S_2^1$ subset ($C_1 > C > C_2$); **Green:** $S_3^1$ subset ($C_2 > C > C_3$); **Blue:** $S_4^1$ subset ($C_3 > C > C_4 = C_5$); and **Cyan:** $S_5^1$ subset ($C < C_4 = C_5$).

the moon, such as those of $S_1^1$ and $S_2^1$, correspond to solutions of high energetic cost.

In the second and third analyses of the stable set, the permanence in the lunar sphere of influence in a full cycle around the moon and the location of the final state after a complete cycle around the moon inside or outside the lunar sphere of influence (or Hill sphere) are investigated. Second and third analyses for prograde initial condition sets are illustrated by Figures 14 and 15 for various eccentricities.

The purpose of the second analysis is to identify orbits which during a full cycle remain in a near enough vicinity of the moon. As expected, the remnant subset prevails on the stable core region for all the considered cases, but most of them are contained in the first Hill region case, that is useless for direct transfer feasibility. For $e = 0.0$ the remnant subset is about 80% of the stable set. This value decreases to about 74%, 57% and 5%, respectively, when the eccentricity assumes 0.3, 0.6 and 0.9 values.

The third investigation restricts the previous one to the analysis of the final state after a full cycle. Its aim is to verify the adequacy of the stability criterion based on the Kepler energy computation when the trajectory returns to $l(\theta)$. The quantity of final states inside the lunar sphere of influence is very high for all the eccentricities. The combination of these two analyses reveals that only the observation of the final state does not guarantee the fulfillment of qualitative aspects of the full cycle orbits which can be required in a ballistic capture transfer.

In the next analyses, the retrograde time evolution of each initial condition of the stable
Figure 13. Classification of the retrograde stable set of initial conditions according to the Jacobi constant for (a) $e = 0.0$, (b) $e = 0.3$, (c) $e = 0.6$, and (d) $e = 0.9$. **Black**: $S^1_2$ subset ($C > C_1$); **Red**: $S^2_2$ subset ($C_1 > C > C_2$); **Green**: $S^3_2$ subset ($C_2 > C > C_3$); **Blue**: $S^4_2$ subset ($C_3 > C > C_4 = C_5$); and **Cyan**: $S^5_2$ subset ($C < C_4 = C_5$).

$S^2_2$ and $S^3_2$ subsets is performed in order to classify it according to its exit basin [17] (entrance basin for direct time evolution). Aiming to build lunar capture solutions, this dynamical aspect investigation is an essential requirement to define which kind of transfer possibility is available for the trajectories of each set.

For the $S^2_2$ subset, three possible retrograde exit basins are defined, namely, the earth exit basin of trajectories which escape through the neck around $L_1$ stable solution; the lunar collisional set, and the bounded set of trajectories which do not escape from the lunar realm in the considered time. The retrograde time evolution of the population percentage of each of these basins for the $S^2_2$ subset is shown in Figure 16 from one month to one year, for eccentricity values (a) $e = 0.0$, (b) $e = 0.3$, (c) $e = 0.6$, and (d) $e = 0.9$. For low eccentricity values, the convergence of the populations occurs slowly with time, while it becomes fast for higher $e$ values. The earth exit basin population assumes higher asymptotic values for low eccentricities. This limit value diminishes with increasing $e$, due to the growth of the asymptotic value of the lunar collisional basin population (from 10% for $e = 0.0$ to 60% for $e = 0.9$). For the considered time scale, the bounded set converges to a population of about 10% for low $e$ and goes to zero for higher eccentricities, as it occurs for $e = 0.9$. Among the considered cases, the numerical experiment for $e = 0.9$ has an outstanding behavior since the asymptotic earth exit basin population value is practically reached for one month.

A new basin must be introduced for the $S^3_2$ subset, i.e., the exterior basin of trajectories which
The black points correspond to the remnant subset, and the orange one to the complementary subset.

pass through the neck around $L_2$ fixed point. Figure 17 presents the retrograde time evolution of the population percentage of each of the $S_3^1$ subset escape basins. The main dynamical process observed in this subset is the population interchange between the exterior basin and the bounded set for all the considered eccentricities, while the earth basin and the collisional set reach their limit values in a short span of time, one or two months. In general, for each eccentricity, the bounded basin set is already small at the first month and vanishes very quickly. Unlike the $S_2^1$ case, the asymptotic collisional set population does not grow with the eccentricity increasing.

Specifically, for direct interior transfers, it is important to investigate the spacecraft’s maximum approximation to the earth. With this goal, we computed the minimum distance $r_1^*$ of the spacecraft from the earth for all the obtained trajectories generated with the earth exit basin for both $S_2^1$ and $S_3^1$ subsets as a function of the time of the retrograde evolution. This analysis results is presented by Figure 18. The minimum approximation occurs for the $S_3^1$ subset and seems not to depend strongly on the eccentricity value. Typical high and medium altitudes of usual earth parking orbits, HEO and MEO, are achieved in four months in the $S_3^1$ subset.

This retrograde investigation was restricted to $S_2^1$ and $S_3^1$ stable subsets once that $S_1^1$ can not produce any transfer and $S_4^1$ and $S_5^1$ are typically associated to high energy demands for the stabilization of the final usual altitude selenocentric orbit in ballistic capture transfer context.
Figure 15. Classification of the final state after a full cycle around the moon with respect to the moon’s sphere of influence for the prograde stable set, for (a) $e = 0.0$, (b) $e = 0.3$, (c) $e = 0.6$, and (d) $e = 0.9$. The black points correspond to interior subset, and the orange one to the external final state subset.

Figure 16. Retrograde time evolution of the population percentage of each of the $S^2_1$ subset escape basins for (a) $e = 0.0$, (b) $e = 0.3$, (c) $e = 0.6$, and (d) $e = 0.9$. The black, red, and green curves stand for bounded, lunar collisional, and earth basins, respectively.
Figure 17. Retrograde time evolution of the population percentage of each of the $S^3_1$ subset escape basins for (a) $e = 0.0$, (b) $e = 0.3$, (c) $e = 0.6$, and (d) $e = 0.9$. The black, red, green, and blue curves stand for bounded, lunar collisional, earth, and exterior basins, respectively.

Figure 18. Retrograde time evolution of the maximum earth approximation of (a) $S^2_1$ and (b) $S^3_1$ subsets for four eccentricities values, namely, $e = 0.0$, $e = 0.3$, $e = 0.6$, and $e = 0.9$, calculated for all the trajectories belonging to the earth exit basin.

4. Conclusions and final remarks
In this contribution the numerical implementation of the WSB algorithmic definition was rebuilt with the inclusion of lunar and earth collisional sets and the classification of the unstable set according to the instability criteria based on the PCRTBP dynamical behavior. Several analyses were performed in order to clarify the applicability of the WSB associated sets in the context of low energy earth-moon ballistic capture transfer design.
The Jacobi constant interval criterion classifies the associated sets according to possible available transport channels. As expected, the initial conditions external to the lunar Hill region are not convenient. Besides of the fact they require a high energetic cost for stabilization on usual low lunar altitudes, a large portion of them belongs to the earth or external isolated Hill region for \( C > C_1 \).

The verification of the validity of the use of the stability two-body quantifier at the final state after a full cycle is performed for positive initial velocities, showing that a large amount of the initial conditions satisfy this criterion. The stable subset which remains in the lunar sphere of influence for a full cycle around the moon is detected in order to guarantee the permanence in a near enough neighborhood of the moon. The observation of orbits which evolve outside the nearby region of the central body, \( P_2 \), reveals that highly nonlinear behavior due to the third-body gravitational effect can be produced, what is improper for a selenocentric final orbit if no control is applied at the initial condition.

The dynamical transport investigation is complemented by examining the exit basins for retrograde temporal evolution of two suitable stable subsets which are the best candidates for low energy projects with prograde initial conditions, allowing the verification of the existence of initial conditions which can provide interior and external transfers.

In particular, the numerical implementation of the associated WSB sets of this contribution was able to reveal that a large portion of the stable-unstable boundary set is contained in the lunar collisional set, restricting its practical applicability under natural dynamics.

These original analyses exhibit the dynamical characteristics of trajectories associated to the WSB algorithmic definition, establishing criteria to determine possible adequate final portions of ballistic capture transfer trajectories, and setting ground to a future refined investigation of the boundary set between the several possible stable-unstable transitions. A detailed investigation of the retrograde initial condition sets still deserves attention in order to verify their adequacy in transfer orbit design. This approach is valuable even when four or more body models are considered to design a complete transfer mission.

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References
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[1] Belbruno E 1987 Proceedings of AIAA/DGLR/JSASS International Electric Propulsion Conference, Paper No. AIAA 87-1054
[2] Belbruno E 1994 Journal of The British Interplanetary Society 47 73–80
[3] Belbruno E and Miller J 1993 Journal of Guidance, Control and Dynamics 16 770–775
[4] Belbruno E, Topputto F and Gidea M 2008 Advances in Space Research 42 1330–1351
[5] Circi C and Teofilatto P 2001 Celestial Mechanics and Dynamical Astronomy 79 41–72
[6] García F and Gómez G 2007 Celestial Mechanics and Dynamical Astronomy 97 87–100
[7] Topputto F and Belbruno E 2009 Celestial Mechanics and Dynamical Astronomy 105 3–17
[8] Romagnoli D and Circi C 2009 Celestial Mechanics and Dynamical Astronomy 103 79–103
[9] Belbruno E 2004 Capture Dynamics and Chaotic Motions in Celestial Mechanics (Princeton University Press)
[10] Szefelehely V 1967 Theory of Orbits (Academic Press)
[11] Moser J 1958 Communications on Pure and Applied Mathematics 11 257–271
[12] Conley C 1969 Journal of Differential Equations 5 136–158
[13] Siegel C and Moser J 1971 Lectures on Celestial Mechanics vol 187 (Springer-Verlag)
[14] Bate R, Mueller D and White J 1971 Fundamentals of Astrodynamics (Dover Publications)
[15] Koon W, Lo M, Marsden J and Ross S 2001 Celestial Mechanics and Dynamical Astronomy 81 63–73
[16] Roy A 2005 Orbital Motion 4th ed (Institute of Physics Publishing)
[17] Aguirre J, Viana R and Sanjuan M 2009 Reviews of Modern Physics 81 333–386