Massless finite and infinite spin representations of Poincaré group in six dimensions

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Abstract

We give a complete description of the massless irreducible representations of the Poincaré group in the six-dimensional Minkowski space. The Casimir operators are constructed and their eigenvalues are found. It is shown that the finite spin (helicity) representation is defined by two integer or half-integer numbers while the infinite spin representation is defined by the real parameter $\mu^2$ and one integer or half-integer number.

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1 Introduction

Study of the various aspects of field theory in higher dimensions attracts much attention due to the remarkable and sometimes even unexpected properties at classical and quantum levels. Many of such properties are closely related to superstring theory which may be treated as a theory of infinite number of higher spin fields in higher dimensional space-time (see e.g. [1]). In particular, the low-energy limits of superstring theory are supersymmetric gauge theories and supergravity in ten dimensions that after reduction yield to field models in dimensions from ten to four. Since the details of field theories are essentially defined by the space-time symmetry, it seems useful to focus an attention on studying the diverse specific properties of symmetry groups in the higher dimensions.

The fundamental space-time background in relativistic theory is Minkowski space where the basic symmetry is described by Poincaré group. Theory of unitary irreducible representations of Poincaré group in four dimensions was constructed in the pioneer papers [2, 4]. The aspects of unitary irreducible representations in higher dimensions and their applications are considered in the papers [5, 6, 7, 8, 9] and in lectures [10] (see also the recent paper [11]). Although the generic scheme of constructing the representations of the Poincaré group in any dimension seems can be realized on the base of known method of induced representations (see e.g. [12, 13]), many specific aspects important for classical and quantum field theory deserve a separate attention and require independent study. Some of such aspects are appropriate only for each concrete dimension and can not be formulated at once for all dimensions. For example, the spinor representations of the Lie algebra of multidimensional Lorenz group are defined independently for each space-time dimension. Therefore one can expect that a structure of relativistic symmetry representations in higher dimensions is much more reacher and more complicated then in the four-dimensional Minkowski space.

In this letter we construct the massless finite and infinite spin irreducible representations of the Lie algebra of the Poincaré group in six-dimensional Minkowski space. Some aspects of such representations are considered in papers [14, 15] however many issues, especially the infinite spin representations, were not addressed and complete analysis was not done. Recently there was the paper [16] where the unitary irreducible massless representations of the Poincaré group in five-dimensional Minkowski space were constructed and some issues related to representations in arbitrary dimensions were briefly studied and the representations of super Poincaré group were considered. The infinite spin representations were not addressed.

The letter is organized as follows. Section 2 is devoted to Casimir operators and their properties in the six-dimensional standard massless momentum reference frame. In section 3 we describe the massless finite spin irreducible representations and show that they are described by two integer or half-integer numbers. Section 4 is devoted to infinite spin representations which are described by arbitrary real parameter and a single integer or half-integer number. Section 5 is a summary of the results.

2 Poincaré algebra and light-cone reference frame

The generators $P_m$ and $M_{mn} = -M_{nm}$ of the Lie algebra $iso(1, D − 1)$ of the Poincaré group in $D$-dimensional space-time have the commutators

$$[P_n, P_k] = 0 , \quad [M_{mn}, P_k] = i (\eta_{mk}P_n - \eta_{nk}P_m) , \quad (2.1)$$

$$[M_{mn}, M_{kl}] = i (\eta_{mk}M_{nl} + \eta_{ml}M_{nk} - \eta_{nl}M_{mk} - \eta_{nk}M_{ml}) , \quad (2.2)$$
where the $D$-vector indices run the values $m, n = 0, 1, \ldots, D - 1$ and we use the space-time metric $\eta^{mn} = \text{diag}(+1, -1, \ldots, -1)$. We call the Lie algebra $\mathfrak{iso}(1, D - 1)$ of the Poincaré group as $D$-dimensional Poincaré algebra.

2.1 Casimir operators of 6-dimensional Poincaré algebra

We introduce the third rank tensor $W_{mnk}$ and the vector $\Upsilon_m$ as the elements of the enveloping algebra of $\mathfrak{iso}(1, 5)$.

\[ W_{mnk} = \varepsilon_{mnklpr} P^l M^{pr}, \quad (2.3) \]
\[ \Upsilon_m = \varepsilon_{mnklpr} P^m M^{kl} M^{pr}. \quad (2.4) \]

Here we use the totally antisymmetric tensor $\varepsilon_{mnklpr}$ and normalize it as $\varepsilon_{012345} = 1$. The operators (2.3) and (2.4) satisfy the equations

\[ P^m W_{mnk} = 0, \quad [P_l, W_{mnk}] = 0, \quad (2.5) \]
\[ P^m \Upsilon_m = 0, \quad [P_l, \Upsilon_m] = 0. \quad (2.6) \]

By using of these equations one can check that the operators

\[ C_2 := P^m P_m, \quad (2.7) \]
\[ C_4 := \frac{1}{24} W^{mnk} W_{mnk}, \quad (2.8) \]
\[ C_6 := \frac{1}{64} \Upsilon^m \Upsilon_m \quad (2.9) \]

are the Casimir operators of the Poincaré algebra $\mathfrak{iso}(1, 5)$. It is clear that $C_2$, $C_4$ and $C_6$ are second, fourth and sixth order operators in the Poincaré algebra generators, respectively.

Note that the quantity $\varepsilon_{mnklpr} W_{mnk} W_{lpr}$ could be an additional Casimir operator for $\mathfrak{iso}(1, 5)$ algebra. But it is identically equal to zero. This fact is a special case of the property of any rank $r$ antisymmetric tensor $W_{m_1 \ldots m_r}$ in $2r$-dimensional space, when $r$ is odd number. Indeed, in this case we have $\langle W, V \rangle_\varepsilon = (-1)^r \langle V, W \rangle_\varepsilon$, where $\langle W, V \rangle_\varepsilon := \varepsilon^{m_1 \ldots m_r n_1 \ldots n_r} W_{m_1 \ldots m_r} V_{n_1 \ldots n_r}$ and $\varepsilon^{m_1 \ldots m_r n_1 \ldots n_r} [W_{m_1 \ldots m_r}, V_{n_1 \ldots n_r}] = 0$. Thus, for antisymmetric tensor with components

\[ W_{m_1 \ldots m_r} = \varepsilon_{m_1 \ldots m_r n_1 \ldots n_r} P^{m_1} M^{n_2 n_3} \ldots M^{n_{r-1} n_r}, \]

which is defined only for odd $r$, we always have $\langle W, W \rangle_\varepsilon = 0$. In this case a Casimir operator for $\mathfrak{iso}(1, 2r - 1)$ algebra, of the second order in $W$, has the unique form

\[ W^2 = \frac{1}{(r + 1)!} W_{m_1 \ldots m_r} W_{m_1 \ldots m_r}. \]

Whereas for even $r$ we have antisymmetric tensor with components

\[ L_{m_1 \ldots m_r} = \varepsilon_{m_1 \ldots m_r n_1 \ldots n_r} M^{n_1 n_2} \ldots M^{n_{r-1} n_r}, \]

which yields for $\mathfrak{so}(\ell, 2r - \ell)$ algebra additional to $L^2 = L_{m_1 \ldots m_r} L_{m_1 \ldots m_r}$ Casimir operator $(L, L)_\varepsilon \neq 0$ (see below operator (3.4) written for the case of $\mathfrak{so}(4)$ algebra).
Taking into account the expressions (2.3), (2.4) we obtain explicit form of the Casimir operators (2.7), (2.8), (2.9):

\[ C_2 = P^m P_m, \]
\[ C_4 = \Pi^m \Pi_m - \frac{1}{2} M^{mn} M_{mn} C_2, \]
\[ C_6 = -\Pi^k M_{km} \Pi_l M^{lm} + \frac{1}{2} \left( M^{mn} M_{mn} - 8 \right) C_4 + \frac{1}{8} \left[ M^{kl} M_{kl} \left( M^{mn} M_{mn} - 8 \right) + 2 M^{mn} M_{nk} M^{kl} M_{lm} \right] C_2, \]

where we introduce new vector \( \Pi \) with components

\[ \Pi_m := P^k M_{km} = M_{km} P^k - 5i P_m, \]

which satisfy commutation relations (cf. (2.1))

\[ [\Pi_n, \Pi_k] = -i M_{nk} C_2, \quad [M_{mn}, \Pi_k] = i (\eta_{nk} \Pi_n - \eta_{nk} \Pi_m). \]

Further in this paper we consider the massless unitary representations of the algebra \( \text{iso}(1,5) \) when the quadratic Casimir operator (2.10) is fixed as following:

\[ C_2 \equiv P^2 = P^m P_m = 0. \]

### 2.2 Standard massless momentum reference frame

Let the algebra (2.1), (2.2) acts in the representation space \( \mathcal{H} \) with basis vectors \( |k, \sigma\rangle \), where \( \sigma \) is a set of eigenvalues of all operators commuting with \( P_m \) and \( P_m |k, \sigma\rangle = k_m |k, \sigma\rangle \). We take the light-cone reference frame for massless particle momentum \( k^m = (k^0, k^a, k^5) = (k, 0, 0, 0, 0, k) \) in which momentum operator (2.15) has the standard form

\[ P^0 = P^5 = k, \quad P^a = 0, \quad a = 1, 2, 3, 4. \]

We stress that all operator formulas presented in this Section (and written in the light-cone frame) should be understood as a result of their action on the subspace \( \mathcal{H}_k \subset \mathcal{H} \) spanned by vectors \( |k, \sigma\rangle \) with fixed light-cone momentum \( k_m \).

The transition to this light-cone reference frame is conveniently performed in the light-cone basis where any 6D vector \( X^m = (X^0, X^a, X^5) \) has the light-cone coordinates \( X^m = (X^+, X^-, X^a) \), where

\[ X^\pm = \frac{1}{\sqrt{2}} (X^0 \pm X^5), \quad X_\pm = \frac{1}{\sqrt{2}} (X_0 \pm X_5) \quad \Rightarrow \quad X^\pm = X_\mp. \]

Then, in the light-cone basis the contraction of two 6D vectors \( X^m \) and \( Y^m \) is

\[ X^m Y^a = X^+ Y_+ + X^- Y_+ + X^a Y_a = \eta^{--} X_0 Y_0 + \eta^{+-} X_+ Y_+ + \eta^{ab} X_b Y_a = X_+ Y_+ + X_+ Y_- - X_a Y_a, \]

\[ X^a Y^m = \eta^{ab} X_b Y_a. \]
where we use the light-cone metric $\eta^\pm \mp = \eta^\pm = 1$, $\eta^{\pm \pm} = 0$, $\eta^{ab} = -\delta_{ab}$. In the light-cone basis the total antisymmetric tensor $\varepsilon_{mklpr}$ has components

$$\varepsilon_{+abcd} = -\varepsilon_{-abcd} = \varepsilon^{+abcd} = -\varepsilon^{-+abcd} = \varepsilon_{abcd},$$

and we normalize the antisymmetric tensors $\varepsilon_{mklpr}$ and $\varepsilon_{abcd}$ as $\epsilon_{012345} = 1$ and $\epsilon_{1234} = 1$.

In the light-cone basis the total antisymmetric tensor $\varepsilon_{mnklpr}$ has components

$$\varepsilon^{\pm+abcd} = -\varepsilon^{-+abcd} = \varepsilon^{+-abcd} = -\varepsilon^{++abcd} = \varepsilon_{abcd},$$

and we normalize the antisymmetric tensors $\varepsilon_{mnklpr}$ and $\varepsilon_{abcd}$ as $\epsilon_{012345} = 1$ and $\epsilon_{1234} = 1$.

In the light-cone basis the standard momentum (2.16) has the components

$$P^+ = P^- = \sqrt{2}k, \quad P^0 = 0, \quad P^a = 0, \quad a = 1, 2, 3, 4.$$

Thus, in the light-cone reference frame (2.19) the Casimir operators (2.11), (2.12) take the form

$$\hat{C}_4 = -\hat{\Pi}_a\hat{\Pi}_a,$$

$$\hat{C}_6 = \hat{\Pi}_bM_{ba}\hat{\Pi}_cM_{ca} - \frac{1}{2}M_{bc}M_{bc}\hat{\Pi}_a\hat{\Pi}_a,$$

where we introduce Hermitian operators

$$\hat{\Pi}_a := \sqrt{2}kM_{+a}.$$

Formula (2.20) directly follows from (2.11), while derivation of (2.21) from (2.12) takes some efforts.

In view of (2.2) the operators $\hat{\Pi}_a$ (2.22) and $M_{ab}$, which generate (2.20) and (2.21), form the Lie algebra of ISO(4) group

$$\left[\hat{\Pi}_a, \hat{\Pi}_b\right] = 0, \quad \left[\hat{\Pi}_a, M_{bc}\right] = i \left(\delta_{ab}\hat{\Pi}_c - \delta_{ac}\hat{\Pi}_b\right),$$

$$\left[M_{ab}, M_{cd}\right] = i \left(\delta_{bc}M_{ad} - \delta_{bd}M_{ac} + \delta_{ac}M_{db} - \delta_{ad}M_{cb}\right),$$

and therefore generate the isometries of the four-dimensional Euclidean space. As a result, the operators $\hat{C}_4$ and $\hat{C}_6$ defined in (2.20) and (2.21) are the Casimir operators of the iso(4) algebra.

Six generators of rotations $M_{ab}$ in four-dimensional Euclidean space are decomposed into the sum

$$M_{ab} = M_{ab}^{(+) + M_{ab}^{(-)},}$$

where

$$M_{ab}^{(\pm)} := \frac{1}{2} \left( M_{ab} \pm \frac{1}{2} \epsilon_{abcd}M_{cd} \right),$$

are (anti)selfdual parts. They are satisfied the identities

$$M_{ab}^{(\pm)} = \pm \frac{1}{2} \epsilon_{abcd}M_{cd}^{(\pm)}.$$

The generators (2.23) form the algebra

$$\left[M_{ab}^{(\pm)}, M_{cd}^{(\pm)}\right] = i \left(\delta_{bc}M_{ad}^{(\pm)} - \delta_{bd}M_{ac}^{(\pm)} + \delta_{ac}M_{db}^{(\pm)} - \delta_{ad}M_{cb}^{(\pm)}\right), \quad \left[M_{ab}^{(+)}, M_{cd}^{(-)}\right] = 0,$$

1When we deduce (2.20) and (2.21) it is necessary, since we project all operator relations to the subspace $\mathcal{H}_k$, first move all operators $P_m$ in the expressions (2.11) and (2.12) to the right and only then perform the substitution (2.19).
which is direct sum of two algebras with three generators $M_{ab}^{(+)}$ and with three generators $M_{ab}^{(-)}$ respectively. Each of these algebras, containing three generators $M_{ab}^{(+)}$ or $M_{ab}^{(-)}$, is the $\mathfrak{su}(2)$ algebra.

This becomes clear (see e.g. [13]) after using the ‘t Hooft symbols [17]. The ‘t Hooft symbols $\eta_{ab}^i = -\eta_{ba}^i$, $i = 1, 2, 3$ and $\bar{\eta}_{ab}^{i'} = -\bar{\eta}_{ba}^{i'}$, $i = 1, 2, 3$ are (anti-)selfdual tensors with respect to the $SO(4)$ indices $a, b$:

\[
\eta_{ab}^i = \frac{1}{2} \epsilon_{abcd} \eta_{cd}^i, \quad \bar{\eta}_{ab}^{i'} = -\frac{1}{2} \epsilon_{abcd} \bar{\eta}_{cd}^{i'}. \tag{2.29}
\]

Below we use the following standard representations for the ‘t Hooft symbols

\[
\eta_{ab}^i = \begin{cases} 
\epsilon_{ab}^i & a, b = 1, 2, 3, \\
\delta_{ia} & b = 4,
\end{cases} \quad \bar{\eta}_{ab}^{i'} = \begin{cases} 
\epsilon_{i'ab} & a, b = 1, 2, 3, \\
-\delta_{i'a} & b = 4.
\end{cases} \tag{2.30}
\]

Due to the properties (2.29) the ‘t Hooft symbols connect (anti-)selfdual $SO(4)$ tensors $M_{ab}^{(\pm)}$ (2.26) with the $SO(3)$ vectors $M_{i}^{(+)}$, $M_{i'}^{(-)}$ by means of the following relations

\[
M_{ab}^{(+)} = -\eta_{ab}^i M_i^{(+)}, \quad M_{ab}^{(-)} = -\bar{\eta}_{ab}^{i'} M_{i'}^{(-)}. \tag{2.31}
\]

Such defined operators $M_{i}^{(+)}$ and $M_{i'}^{(-)}$ form two $\mathfrak{su}(2)$ algebras with standard form of the commutators

\[
[M_{i}^{(+)} , M_{j}^{(+)} ] = i\epsilon_{ijk} M_{k}^{(+)} , \quad [M_{i}^{(-)} , M_{j}^{(-)} ] = i\epsilon_{i'j'k'} M_{k'}^{(-)}, \quad [M_{i}^{(+)} , M_{j}^{(-)} ] = 0. \tag{2.32}
\]

In term of the operators (2.31) the Casimir (2.21) takes the form (we use the equalities $\eta_{ab}^i \eta_{ab}^{i'} = 4\delta^{ij}$, $\eta_{ab}^i \eta_{ab}^{i'} = 4\delta^{ij'}$ and $\eta_{ab}^i \eta_{ab}^{i'} = 0$)

\[
\hat{C}_6 = 2 M_{i}^{(+)} M_{j}^{(+)} \eta_{ab}^{i} \eta_{ab}^{i'} \hat{\Pi}_b \hat{\Pi}_a - \left( M_{i}^{(+)} M_{i}^{(+)} + M_{i'}^{(-)} M_{i'}^{(-)} \right) \hat{\Pi}_a \hat{\Pi}_a. \tag{2.33}
\]

Thus, in massless case (2.15) unitary irreducible representations are defined by the eigenvalues of the $\text{iso}(4)$ Casimir operators (2.20) and (2.21). In case of this noncompact symmetry there are two different cases defined the value of Casimir operator (2.20), i.e. square of “four-translation” generator $\hat{\Pi}_a$. So, in next sections we consider following unitary massless representations:

- **Finite spin (helicity) representations.**
  In these cases the $SO(4)$ four-vector $\hat{\Pi}_a$ has zero norm:

  \[
\hat{\Pi}_a \hat{\Pi}_a = 0. \tag{2.34}
\]

- **Infinite (continuous) spin representations.**
  In case of these representations the Euclidean four-vector $\hat{\Pi}_a$ has nonzero norm:

  \[
\hat{\Pi}_a \hat{\Pi}_a = \mu^2 \neq 0. \tag{2.35}
\]

In next sections we consider these massless representations in details.
3 Massless finite spin representations

This case is characterized by the fulfillment of condition (2.34), which implies that all components $\hat{\Pi}_a$ (since they are Hermitian operators) of the Euclidean vector are zero:

$$\hat{\Pi}_a = 0 \quad \text{at all} \quad a = 1, 2, 3, 4.$$  \hspace{1cm} (3.1)

As result, the Casimir operators (2.20) and (2.21) are vanish in this case: $\hat{C}_4 = 0$ and $\hat{C}_6 = 0$. In passing from this light-cone reference frame to an arbitrary frame, we get that all Casimir operators (2.11), (2.12) on the massless finite spin states take zero values (see also [14]):

$$C_4 = 0, \quad C_6 = 0,$$  \hspace{1cm} (3.2)

and in view of (2.15) we have $\Pi^k \Pi_k = 0$ and $\Pi^k M_{km} \Pi^\ell M^{\ell m} = 0$.

Due to (3.1) the Euclidean four-translations are realized trivially in case of these representations. As a result such representations of $ISO(1,5)$ are finite dimensional. Each such massless representation defines some 6D standard massless representation with finite number of massless particle states. As we saw above, such representations are induced from irreducible $SO(4)$ representations. Let us show below that the Casimir operators of the stability subgroup $SO(4)$ define the 6D helicity operators.

3.1 6D helicity operators

First, let us consider the vector $\Upsilon_m$ defined in (2.4). In the case $C_6 = 0$, according to (2.9), we have $\Upsilon_m \Upsilon^m = 0$ and, in the light-cone reference frame (2.16), (2.19), the components of 6D vector $\Upsilon$ are

$$\Upsilon^+ = \Lambda_1 P^+, \quad \Upsilon^- = \Upsilon_a = 0,$$  \hspace{1cm} (3.3)

where we have

$$\Lambda_1 := \epsilon_{abcd} M_{ab} M_{cd}.$$  \hspace{1cm} (3.4)

This operator is the Casimir operator of the $so(4)$ algebra.

The conditions (3.3) demonstrate that vectors $\Upsilon$ and $P$ are collinear in the light-cone reference frame and this property is conserved in any reference frame. Namely, the relations (2.6) show that the light like vector $\Upsilon$ is transverse to the vector $P$ and its components $\Upsilon_m$ commute with $P_k$. Therefore, the vector $\Upsilon_m$ is proportional to the vector $P_m$:

$$\Upsilon_m = \Lambda_1 P_m.$$  \hspace{1cm} (3.5)

This relation was also pointed out in [14,16]. Note that the operator (3.4) can be represented in the form

$$\Lambda_1 := \frac{\Upsilon_0}{P_0}.$$  \hspace{1cm} (3.6)

This expression appears for the 4D helicity operator when $\Upsilon_m$ is replaced by $W_m$. Due to the relations

$$[M_{0i}, \Lambda_1] = \frac{i}{P_0} (\Upsilon_i - \Lambda_1 P_i) = 0, \quad [M_{ik}, \Lambda_1] = 0 = [P_k, \Lambda_1], \quad (i, k = 1, \ldots, 5),$$  \hspace{1cm} (3.7)

we conclude that the operator (3.6) is invariant with respect to the 6D Poincare symmetry. Therefore, the operator $\Lambda_1$, defined in (3.6), is a 6D analog of the helicity operator and it coincides with one of $so(4)$ Casimir operators in the light-cone reference frame.
We note that irreducible so(4) representations are characterized by two quadratic Casimir operators. Another Casimir operator is appeared as helicity operator if we use the construction proposed in [16]. Indeed, by using the prescription of [16], one can construct another (third order in generators of \(\text{iso}(1,5)\)) vector with components:

\[
S_m := 3M^{nk}P_{m nk} = M^{nk}M_{nk}P_m - 2M^{kn}M_{mn}P_k .
\]

The square of this 6D vector is:

\[
S_m S_m = M^{nm}M_{nm}M^{tk}M_{lk}P^2 + 4\left[\Pi^k M_{km}\Pi^l M_{lm} - M^{nm}M_{nm}(\Pi^l P_l + P^2) + \Pi^l \Pi^l\right] .
\]

while its contraction with 6D vector momentum \(P_m\) gives:

\[
P_m S_m = M^{nm}M_{mn}P^2 - 2\Pi^m \Pi_m \equiv -2C_4 ,
\]

and the commutators of \(S_m\) and \(P_n\) are:

\[
[S_m, P_n] = 2iM_{mn}P^2 + 4i\Pi_{[m}P_{n]} .
\]

For the massless finite spin representations, defined by the conditions (2.15), (3.1) and (3.2), equations (3.9), (3.10) and (3.11) are reduced to:

\[
S_m S_m = 0, \quad P_m S_m = 0, \quad [S_m, P_n] = 0 ,
\]

which are the same as conditions (2.6) for light-like vectors \(\Upsilon\) and \(P\). So, in the case of massless finite spin representations, the vectors \(P_m\) and \(S_m\) are also proportional to each other. One can check this in the light-cone reference frame, when subject to the conditions (2.19) and (3.1) the components of the 6D vector (3.8) are equal to:

\[
S^+ = \Lambda_2 P^+ , \quad S^- = S_a = 0 ,
\]

where the operator

\[
\Lambda_2 := M_{ab}M_{ab}
\]

is second so(4) Casimir operator.

Due to the relations (3.12) in general frame the relations (3.13) take the form:

\[
S_m = \Lambda_2 P_m ,
\]

where the operator \(\Lambda_2\) (3.14) defines second helicity operator and has equivalent “covariant” form:

\[
\Lambda_2 := \frac{S_0}{P_0} .
\]

So these massless representations of finite spin are characterized by the pair \((\lambda_1, \lambda_2)\), where real numbers \(\lambda_{1,2}\) define the eigenvalue of the Casimir operators \(\Lambda_{1,2}\) presented in (3.6) and (3.16), respectively.

\[\text{In the definition of antisymmetrization of } n \text{ indices, we use the factor } n!, \text{ i.e.}\]

\[A_{[m B_n C_k]} = \frac{1}{3!}\left(A_mB_n C_k - A_mB_k C_n + \text{cyclic permutations}\right) .\]
Using (2.25) and (2.31) we represent helicity operators (3.4) and (3.14) in the form

\[
\Lambda_1 = 2 \left( M_{ab}^{(+)} M_{ab}^{(+)} - M_{ab}^{(-)} M_{ab}^{(-)} \right) = 8 \left( M_1^{(+) M_{1}^{(+)}} - M_1^{(-) M_{1}^{(-)}} \right),
\]

\[
\Lambda_2 = M_{ab}^{(+)} M_{ab}^{(+)} + M_{ab}^{(-)} M_{ab}^{(-)} = 4 \left( M_{i}^{(+) M_{i}^{(+)}} + M_{i}^{(-) M_{i}^{(-)}} \right).
\]

In case of unitary representations, the operators \( M_{i}^{(+) M_{i}^{(+)}} \) and \( M_{i}^{(-) M_{i}^{(-)}} \) equal \( j_{+}(j_{+} + 1) \) and \( j_{-}(j_{-} + 1) \) respectively. Therefore, the eigenvalues of the helicity operators (3.6) and (3.16) take the values

\[
\lambda_1 = 8j_{+}(j_{+} + 1) - 8j_{-}(j_{-} + 1),
\]

\[
\lambda_2 = 4j_{+}(j_{+} + 1) + 4j_{-}(j_{-} + 1),
\]

where \( j_{\pm} \) are integer or half-integer numbers in case of the unitary representations.

We note that the standard 4D helicity operator is invariant under proper \( SO(1, 3) \) rotations but changes its sign under improper \( O(1, 3) \) rotations (reflections). We have the same property for \( \Lambda_1 \) but it is not the case for \( \Lambda_2 \).

### 3.2 Examples

Here we will demonstrate the use of the obtained formulas for determining the helicities on the examples of some massless finite spin fields. To clarity and avoid technical complications, we will consider only bosonic integer-spin fields.

Since the irreducible massless representations of the 6D Poincaré group are induced by the irreducible \( SO(4) \) representations in the light-cone reference frame, we will use the following procedure.

Below, in all examples of this section, we first consider a fixed irreducible \( SO(4) \) representation and determine the values of the helicities. Here we will use the defining representation for the \( \mathfrak{so}(4) \) generators

\[
(M_{ab})_{eg} = i(\delta_{ae}\delta_{bg} - \delta_{ag}\delta_{be}).
\]

Then we reconstruct the corresponding 6D field, for which the equations of motion and gauge fixing show that the independent components are exactly those \( SO(4) \) fields which were considered earlier in the Euclidean four-dimensional picture.

#### 3.2.1 Vector field

Let us consider the \( SO(4) \) vector field \( A_a \). In this case the \( \mathfrak{so}(4) \) generators coincide with (3.21):

\[
(M_{ab})_{eg} = (M_{ab})_{eg}.
\]

Then, the \( SO(4) \) Casimir operators take the form

\[
(\Lambda_1)_{eg} = \epsilon_{abcd}(M_{ab}M_{cd})_{eg} = 0,
\]

\[
(\Lambda_2)_{eg} = (M_{ab}M_{ab})_{eg} = 6\delta_{eg}.
\]

When acting on the \( SO(4) \) vector field \( A_a \), the operators (3.23) give the following values of helicities:

\[
\lambda_1 = 0, \quad \lambda_2 = 6; \quad j_{+} = j_{-} = \frac{1}{2}.
\]
This Euclidean vector field $A_a$ describes physical components of the $6D$ vector gauge field $A_m$. In the momentum representation the $U(1)$ massless gauge field $A_m$ is described by the equations of motion

$$P^m F_{mn} = 0 ,$$

where $F_{mn} = i(P_m A_n - P_n A_m)$ is the field strength, and determined up to gauge transformations

$$\delta A_m = iP_m \varphi .$$

One of the possible gauge fixing for transformations (3.26) is the light-cone gauge (see e.g. [18])

$$A^+ = 0 .$$

Then in the light-cone frame (2.19), the equations of motion (3.25) give $A^- = 0$ and independent field is given by the transverse part $A_a$ of the $6D$ gauge field $A_m$.

### 3.2.2 Second rank symmetric tensor field

Now we consider the $SO(4)$ second rank tensors. In this case the $so(4)$ generators take the matrix form

$$(M_{ab})_{e_1 e_2, g_{12}} = ((M_{ab})_1 + (M_{ab})_2)_{e_1 e_2, g_{12}} = (M_{ab})_{e_1 g_1} \delta_{e_2 g_2} + \delta_{e_1 g_1} (M_{ab})_{e_2 g_2}$$

and the $SO(4)$ Casimir operators are

$$(\Lambda_1)_{e_1 e_2, g_{12}} = \epsilon_{abcd} (M_{ab} M_{cd} (M_{ab})_{e_1 e_2, g_{12}} = 2 \epsilon_{abcd} ((M_{ab})_1 (M_{cd})_2)_{e_1 e_2, g_{12}} = 8 \epsilon_{e_1 e_2 g_{12}} ,$$

$$(\Lambda_2)_{e_1 e_2, g_{12}} = (M_{ab} M_{ab})_{e_1 e_2, g_{12}} = ((M_{ab})_1 + (M_{ab})_2 + 2(M_{ab})_1 (M_{ab})_2)_{e_1 e_2, g_{12}} = 12 \delta_{e_1 g_1} \delta_{e_2 g_2} + 4(\delta_{e_1 g_2} \delta_{e_2 g_1} - \delta_{e_1 e_2} \delta_{g_1 g_2}) .$$

First, we consider the $SO(4)$ second rank tensor $\hat{h}_{ab}$, which is symmetric $\hat{h}_{ab} = \hat{h}_{ba}$ and traceless $\hat{h}_{aa}$. On this field the helicity operators (3.29) take the values

$$\lambda_1 = 0 ; \quad \lambda_2 = 16 ; \quad j_+ = j_- = 1 .$$

Let us show that this field $\hat{h}_{ab}$ describes the physical components of the $6D$ linearized gravitational field.

The $6D$ linearized gravitational field $h^{mn} = h^{nm}$ is determined by the well known equations of motion

$$P^2 h^{mn} - P^m P_k h^{nk} - P^n P_k h^{mk} + P^m P^n h_{k}^{k} = 0 ,$$

and has gauge invariance

$$\delta h^{mn} = iP^{(m} \varphi^{n)} .$$

For the transformations (3.32) we can put again the light-cone gauge (see also [18])

$$h^{+m} = 0 .$$

The equations of motion (3.31) produce $h^{-m} = 0, h_a^a = 0$ in the light-cone frame (2.19). As a result, nonvanishing physical components of the $6D$ gravity field $h_{mn}$ are given by the traceless part $\hat{h}_{ab}$ of its transverse components $h_{ab}$. 

9
3.2.3 Third rank (anti-)selfdual antisymmetric tensor fields

Now we consider the $SO(4)$ antisymmetric tensors of the second rank $B_{ab}^{(\pm)} = -B_{ba}^{(\pm)}$, which are (anti-)selfdual

$$B_{ab}^{(\pm)} = \pm \frac{1}{2} \epsilon_{abcd} B_{cd}^{(\pm)} .$$

(3.34)

These tensors form the spaces of two $SO(4)$ irreducible representations which make up the $SO(4)$ reducible representation in the space of all antisymmetric rank 2 tensors associated to Young diagram $[1^2] \equiv \begin{array}{c} 1 \\ 1 \end{array}$. In this case the $so(4)$ generators $M_{ab}$ and helicity operators $\Lambda_1, \Lambda_2$ have the same expressions (3.28) and (3.29). Then the eigenvalues of the operators $\Lambda_1, \Lambda_2$ and $(M_1^{(\pm)}M_1^{(\pm)})$ are given by numbers

$$\lambda_1 = 16 , \quad \lambda_2 = 8; \quad j_+ = 1, \quad j_- = 0$$

(3.35)

on the space of the selfdual fields $B_{ab}^{(+)\ },$ and by

$$\lambda_1 = -16 , \quad \lambda_2 = 8; \quad j_+ = 0, \quad j_- = 1$$

(3.36)

on the space of the anti-selfdual fields $B_{ab}^{(-)\ }$. It is clear that these $SO(4)$ (anti-)selfdual fields $B_{[ab]}^{(\pm)}$ are independent components of the $6D$ massless (anti-)selfdual 3-rank fields $B_{mnk}^{(\pm)}$ which satisfy the identities

$$B_{mnk}^{(\pm)} = \pm \frac{1}{3!} \epsilon_{mnklpr} B_{klpr}^{(\pm)\ } .$$

(3.37)

So, the equations of motion of the $6D$ massless fields $B_{mnk}^{(\pm)}$ are

a) $P^m B_{mnk}^{(\pm)\ } = 0 , \quad$ b) $P_{[m} B_{nkl]}^{(\pm)\ } = 0 , \quad$ c) $P^2 B_{mnk}^{(\pm)\ } = 0 .$$

(3.38)

Then in the light-cone frame (2.19) the equations (3.38a) give $B^{(\pm)m\ } = 0$ whereas the equations (3.38b) produce $B^{(\pm)a\ }bc = 0$. As a result, independent fields of the $6D$ tensors $B_{mnk}^{(\pm)}$ are the $SO(4)$ (anti-)selfdual fields $B_{mnk}^{(\pm)\ } = B^{(\pm)m\ }ab$ which are subjected the $SO(4)$ (anti-)selfdual conditions (3.34) due to the $6D$ (anti-)selfdual conditions (3.37).

Remark. One can generalize this example to the case of special $3n$-rank selfdual and anti-selfdual 6-dimensional tensor fields. These fields correspond to $SO(4)$ irreducible representations in spaces of $2n$-rank traceless selfdual and anti-selfdual tensors with components $B_{a_1\ldots a_{2n}}^{(\pm)}$ be symmetrized in accordance to the Young diagram $[n^2] \equiv \begin{array}{c} n \\ n \end{array}$. It is clear that for highest weights of such selfdual and anti-selfdual representations of $SO(4)$ we have respectively $j_+ = n, j_- = 0$ and $j_+ = 0, j_- = n$ and in view of (3.19) and (3.20) we obtain the eigenvalues of helicity operators $\lambda_1 = 8n(n+1), \lambda_2 = 4n(n+1)$ and $\lambda_1 = -8n(n+1), \lambda_2 = 4n(n+1)$ which is a generalization of (3.35) and (3.36).

4 Massless infinite (continuous) spin representations

In this case, when the condition (2.35) is satisfied and the Euclidean four-vector $\hat{\Pi}_a$ is nonzero. Then here the representations of the $ISO(4)$ group, which induce the $6D$ relativistic massless representations, are infinite dimensional.
In case of these representations the Casimir operator (2.20) has nonvanishing eigenvalue
\[ C_4 = \hat{C}_4 = -\mu^2, \quad \mu \neq 0. \] (4.1)
Moreover, for the orbits (2.35) we can take the basis with with nonzero only the fourth component:
\[ \hat{\Pi}_1 = \hat{\Pi}_2 = \hat{\Pi}_3 = 0, \quad \hat{\Pi}_4 = \mu. \] (4.2)
Then taking into account \( \eta_{\alpha 4} = \delta_{\alpha a} \) and \( \bar{\eta}_{\alpha 4} = -\delta_{\alpha a} \) (see (2.30)) we obtain from (2.33) the value of the Casimir operator (2.21):
\[ \hat{C}_6 = -\mu^2 J_i J_i, \] (4.3)
where
\[ J_i := M_i^{(+)} + M_i^{(-)} \] (4.4)
are the generators of the diagonal \( su(2) \) subalgebra of the \( so(4) = su(2) \oplus su(2) \) stability algebra. Using (2.26) and (2.31) and explicit expressions of the ‘t Hooft symbols (see e.g. Sect. 3.3.3 in [13]) we find
\[ J_i = -\frac{1}{2} \varepsilon_{ijk} M_{jk}, \quad i = 1, 2, 3. \] (4.5)
So the operators (4.4) are in fact the generators of the \( SO(3) \) subgroup of the \( SO(4) \) stability group. Therefore, in case of the unitary representations it is necessary to satisfy the equality
\[ J^2 = s(s + 1), \] (4.6)
where \( s \) is fixed integer or half-integer number.
So, in case of the irreducible representations of infinite (continuous) spin, the Casimir operator (2.12) takes the value
\[ C_6 = \hat{C}_6 = -\mu^2 s(s + 1), \] (4.7)
Such irreducible representations describe a tower of infinite number of massless states.
As a result, the massless infinite spin representations are characterized by the pair \((\mu, s)\), where the real parameter \( \mu \) defines the eigenvalue of the Casimir operator (4.1) and the (half-)integer number \( s \) defines the eigenvalue of the Casimir operator (4.7).
Let us examine in our consideration the \( D = 6 \) infinite integer spin system [19] which is higher dimension generalization of the \( D = 4 \) model [2], [3], [4]. This model [19] is described by the pair of the space-time phase operators
\[ x^m, p_m, \quad [x^m, p_k] = i\delta^m_k \] (4.8)
and two pairs of the additional bosonic phase vectors
\[ w^m, \xi_m, \quad [w^m, \xi_k] = i\delta^m_k; \quad u^m, \zeta_m, \quad [u^m, \zeta_k] = i\delta^m_k . \] (4.9)
These two pairs of vectors (4.9) are responsible for spinning degrees of freedom.
Infinite integer spin field \( \Psi \) in [19] is described by the \( D = 6 \) generalization of the Wigner-Bargmann equations
\[ p^2 \Psi = 0, \] (4.10)
\[ \xi \cdot p \Psi = 0, \] (4.11)
\[ (w \cdot p - \mu) \Psi = 0 , \] (4.12)
\[ (\xi \cdot \xi + 1) \Psi = 0, \] (4.13)
and additional equations with vectorial operators from the second pair (4.9)

\[ u \cdot p \Psi = 0 , \]  
\[ \zeta \cdot p \Psi = 0 , \]  
\[ \zeta \cdot \xi \Psi = 0 , \]  
\[ \zeta \cdot \zeta \Psi = 0 , \]  
\[ (u \cdot \zeta - s) \Psi = 0 , \]

where \( \xi \cdot p := \xi^m p_m \), etc.

Note that, in contrast to the four-dimensional case \[2, 3, 4\] with one pair of auxiliary variables \( w^m, \xi_m \) in the six-dimensional case it is necessary to use the second pair of auxiliary vector variables \( u^m, \zeta_m \) to describe arbitrary infinite spin representations.

In the light-cone frame (2.19), i.e. \( p^- = p_a = 0, p^+ = \text{const} \neq 0 \), and in the representation \( \xi_m = -i \partial / \partial w^m, \zeta_m = -i \partial / \partial u^m \) the equations (4.11)-(4.13) give the conditions

\[ \frac{\partial}{\partial w^+} \Psi = 0 , \]  
\[ (p^+ w^- - \mu) \Psi = 0 , \]  
\[ \left( \frac{\partial}{\partial w_a} \frac{\partial}{\partial w_a} + 1 \right) \Psi = 0 , \]

whereas (4.14)-(4.18) yield

\[ p^+ u^- \Psi = 0 , \]  
\[ \frac{\partial}{\partial u^+} \Psi = 0 , \]  
\[ \frac{\partial}{\partial u_a} \frac{\partial}{\partial w_a} \Psi = 0 , \]  
\[ \frac{\partial}{\partial u_a} \frac{\partial}{\partial u_a} \Psi = 0 , \]  
\[ \left( u_a \frac{\partial}{\partial u_a} - s \right) \Psi = 0 , \]

The solution of the equations (4.19)-(4.26) is the field

\[ \Psi = \delta(p^+ w^- - \mu) \delta(p^+ u^-) \Phi(w_a, u_a) , \]

where \( \Phi(w_a, u_a) \) is subjected (4.21), (4.24)-(4.26) and has series expansions presented in [19].

Now we can determine the values of the Casimir operators (2.20), (2.21) on the field (4.27).
For the field (4.27) the generators of the iso(4) algebra (2.23), (2.24) have the form

\[ M_{ab} = i \left( w_a \frac{\partial}{\partial w_b} - w_b \frac{\partial}{\partial w_a} + u_a \frac{\partial}{\partial u_b} - u_b \frac{\partial}{\partial u_a} \right), \quad \hat{\Pi}_a = -i\mu \frac{\partial}{\partial w_a}. \] (4.28)

As result, due to the equation (4.21), we obtain the fulfillment of the condition (4.1) for the Casimir operator \( C_4 \):

\[ C_4 = \hat{C}_4 = -\mu^2. \] Moreover, the representations (4.28) lead to the expression

\[ \hat{C}_6 = \mu^2 u_a \frac{\partial}{\partial u_a} \left( u_b \frac{\partial}{\partial u_b} + 1 \right) \frac{\partial}{\partial w_c} \frac{\partial}{\partial w_c} + \mu^2 \left( u_a \frac{\partial}{\partial w_a} u_b \frac{\partial}{\partial w_b} - u_a u_b \frac{\partial}{\partial w_a} \frac{\partial}{\partial w_b} \right) \frac{\partial}{\partial u_c} \frac{\partial}{\partial u_c} + \mu^2 \left( u_a u_a \frac{\partial}{\partial u_b} \frac{\partial}{\partial w_b} - 2 u_a \frac{\partial}{\partial u_a} u_b \frac{\partial}{\partial w_b} \right) \frac{\partial}{\partial u_c} \frac{\partial}{\partial w_c} \] (4.29)

for the sixth order Casimir operator. So, due to the equations (4.21), (4.24)-(4.26) the operator (2.21) takes the value \( C_6 = \hat{C}_6 = -\mu^2 s(s+1) \) on the field (4.27).

Thus, the infinite spin field with only one additional vector variables and obeying the Wigner-Bargmann equations (4.10)-(4.13) and additional equations (4.19)-(4.26) describes the irreducible \((\mu, s)\) infinite spin representation. The system with only one pair of auxiliary variables \( w^m, \xi_m \) in (4.9) (without using the second pair of auxiliary vector variables \( u^m, \zeta_m \)) and with only the equations of motion (4.11)-(4.13) describe the infinite spin representations at \( s = 0 \) [19].

5 Summary and outlook

We have studied the massless irreducible representations of the Poincaré group in six-dimensional Minkowski space and give full classification of all massless representations including infinite integer spin case. The representations are described by three Casimir operators written in the form (2.7), (2.8), (2.9) or in the equivalent form (2.10), (2.11), (2.12). The properties of these operators are explored in the standard massless momentum reference frame, where it is seen that the unitary representations of ISO(1,5) group are induced from representations of SO(4) and ISO(4) groups and correspondingly are divided into finite spin (helicity) and infinite spin representations. Both these representations are studied in details. It is proved that the finite spin representation is described by two integer or half-integer numbers while the infinite spin representation is described by one real parameter and one integer or half-integer number. In case of half-integer spin we should introduce an additional spinor or twistor variables like in [19].

As a continuation of this research it would be interesting to describe the massless representations with half-integer spin and massive irreducible representations of six-dimensional Poincaré group with both integer and half-integer spin. Another open problem is constructing the representations of the corresponding six-dimensional super Poincaré group. Also it would be useful to work out the field realizations of the massless representations considered in this paper (see, e.g., Remark at the end of Sect.3.2.3) and explore the new aspects of Lagrange
formulation for these fields in six-dimensional Minkowski space including infinite spin cases. We plan to study all these problems in the forthcoming papers.

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Currently, there is a fairly large literature on various aspects of infinite spin Lagrange formulation (see e.g., the recent paper [21] and the references therein).
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