A generalization of the Ross symbols in higher $K$-groups and hypergeometric functions II

M. Asakura *

Abstract

This is a sequel of the paper [As2] where we introduced higher Ross symbols in higher $K$-groups of the hypergeometric schemes, and discussed the Beilinson regulators. In this paper we give its $p$-adic counterpart and an application to the $p$-adic Beilinson conjecture for K3 surfaces of Picard number 20.

1 Introduction

In [As2], we introduced a higher Ross symbol

$$\xi_{\text{Ross}} := \left\{ \frac{1-x_0}{1-\nu_0x_0}, \ldots, \frac{1-x_d}{1-\nu_dx_d} \right\}$$

in the Milnor $K$-group of the affine ring of a hypergeometric scheme

$$U_t : (1-x_0^{n_0}) \cdots (1-x_d^{n_d}) = t \quad (1.1)$$

where $\nu_k$ is a $n_k$-th root of unity (cf. §3.1 and §5.1). This is a generalization of the Ross symbol $\{1-z, 1-w\}$ in $K_2$ of the Fermat curve $z^n + w^n = 1$ introduced in [R1], [R2] (see [As2] 4.1 for the connection with our higher Ross symbols). In [As2] we discuss the Beilinson regulator map (cf. [S-Be])

$$\text{reg}_B : K_{d+1}(U_t)^{(d+1)} \to H^{d+1}_{\text{dR}}(U_t, \mathbb{Q}(d+1))$$

from Quillen’s higher $K$-group to the Deligne-Beilinson cohomology where $K_i(-)^{(j)} \subset K_i(-) \otimes \mathbb{Q}$ denotes the Adams weight piece. The main result [As2 Theorem 5.5] tells that $\text{reg}(\xi_{\text{Ross}})$ is a linear combination of complex analytic functions

$$F_a(t) := \sum_{k=0}^{d} (\psi(a_k) + \gamma) + \log(t) + a_0 \cdots a_d t F_\Delta(t) \quad (1.2)$$

*Department of Mathematics, Faculty of Sciences, Hokkaido University, Sapporo 060-0810, JAPAN. asakura@math.sci.hokudai.ac.jp
where \( a = (a_0, \ldots, a_d) \) and
\[
F_a(t) := d + Frac \left( \frac{a_0 + 1, \ldots, a_d + 1, 1, t}{2, \ldots, 2} \right),
\]
is the hypergeometric function (see [As2, 5.1] for \( \mathcal{F}_a(t) \), and [Sl] or [NIST, 15,16] for the general theory of hypergeometric functions).

The purpose of this paper is to provide the \( p \)-adic counterpart of \([As2]\), namely we prove similar theorems in \( p \)-adic situation by replacing the subjects as follows,

\[
\text{Beilinson regulator } \sim \text{ syntomic regulator } \mathcal{F}_a(t) \sim \mathcal{F}_a^{(\sigma)}(t)
\]

where \( \mathcal{F}_a^{(\sigma)}(t) \) is a certain \( p \)-adic convergent function introduced in \([As1]\). Let us explain it more precisely. Let \( W \) be the Witt ring of a perfect field of characteristic \( p > 0 \), and \( K = \text{Frac } W \) the fractional field. For a smooth projective variety \( X \) over \( W \), we denote by \( H^*_{\text{syn}}(X, \mathbb{Z}_p(j)) \) the syntomic cohomology of Fontaine-Messing (cf. \([Ka1\), Chapter I]). More generally, let \( U \) be a smooth \( W \)-scheme such that there is an embedding \( U \hookrightarrow X \) into a smooth projective \( W \)-scheme \( X \) with \( Z = X \setminus U \) a simple relative normal crossing divisor over \( W \). Then the log syntomic cohomology of \((X, Z)\) is defined (cf. \([As1\), §2.2]), which we denote by \( H^*_{\text{syn}}(U, \mathbb{Z}_p(j)) \). In their recent paper \([N-N]\), Nekovář and Niziol established the syntomic regulator maps
\[
\text{reg}_{\text{syn}}^{ij} : K_i(U) \otimes \mathbb{Q} \rightarrow H^2_{\text{syn}}(U, \mathbb{Q}_p(j)) := H^2_{\text{syn}}(U, \mathbb{Z}_p(j)) \otimes \mathbb{Q}
\]
from Quillen’s algebraic \( K \)-groups. These are the \( p \)-adic counterpart of the Beilinson regulator maps. Let us take \( U = U_\alpha \) the hypergeometric scheme \((1.1)\) for \( \alpha \in W \) and take the degrees \((i, j) = (d + 1, d + 1)\),
\[
\text{reg}_{\text{syn}} = \text{reg}_{\text{syn}}^{d+1, d+1} : K_{d+1}(U_\alpha)^{(d+1)} \rightarrow H_{\text{syn}}^{d+1}(U_\alpha, \mathbb{Q}_p(d + 1)) \cong H^{d}_{\text{dR}}(U_{\alpha, K}/K), \quad (1.3)
\]
where \( U_{\alpha, K} := U_\alpha \times W K \). We then discuss the element \( \text{reg}_{\text{syn}}(\xi_{\text{Ross}}) \) in the de Rham cohomology. Although the author does not know whether the higher Ross symbol \( \xi_{\text{Ross}} \in K_{d+1}(U_\alpha) \) lies in the image of \( K_{d+1}(X_\alpha) \) with \( X_\alpha \supset U_\alpha \) a smooth compactification (see \([As2\), 4.2] for more details), one can show that \( \text{reg}_{\text{syn}}(\xi_{\text{Ross}}) \) lies in \( W_\alpha H^{\text{dR}}_{\text{dR}}(U_{\alpha, K}/K) \) (Lemma \(5.2\)), where \( W_\alpha H^{\text{dR}}_{\text{dR}}(U_{\alpha, K}/K) \) denotes the weight filtration by Deligne. The main theorem of this paper (=Theorem 5.5) describes \( \text{reg}_{\text{syn}}(\xi_{\text{Ross}}) \) by a linear combination of
\[
\mathcal{F}_a^{(\sigma)}(t)
\]
introduced in \([As1\), §2\], which is the \( p \)-adic counterpart of \( \mathcal{F}_a(t) \) (see §2.3 below for the review on \( \mathcal{F}_a^{(\sigma)}(t) \)). The proof is based on the theory of \( F \)-isocrystals, especially the main result in \([AM\), and also uses the congruence relation for \( \mathcal{F}_a^{(\sigma)}(t) \) proven in \([As1\), 3.2\].

Our main theorem has an application to the study of the \( p \)-adic Beilinson conjecture by Perrin-Riou. Conceptually saying, the conjecture asserts that the special values of the \( p \)-adic
\textit{L}-functions are described by the syntomic regulators up to $\mathbb{Q}^\times$. There are several previous works by many people ([BK], [BD], [KLZ], [Ni], etc.). We also refer the recent paper [AC] where a number of numerical verifications for $K_2$ of elliptic curves over $\mathbb{Q}$ are given apart from the method of the Beilinson-Kato elements. In this paper we shall discuss the $p$-adic Beilinson conjecture for singular K3 surfaces over $\mathbb{Q}$ ("singular" means the Picard number 20). Let $h^2_{tr}(X) = h^2(X)/\text{NS}(X)$ denote the transcendental motive (cf. [KMP, 7.2.2]). If $X$ is a singular K3 surface over $\mathbb{Q}$, it is 2-dimensional, and there is a Hecke eigenform $f$ of weight 3 with complex multiplication such that $L(h^2_{tr}(X), s) = L(f, s)$ by a theorem of Livné [Li]. The $p$-adic Beilinson conjecture for $K_3(X)^{(3)}$ is formulated as the relation between the syntomic regulator and the special value of the $p$-adic $L$-function $L_p(f, \chi, s)$. See Conjecture 6.3 for the precise statement. Our hypergeometric schemes provide several explicit examples of singular K3 surfaces. The hypergeometric scheme

\[(1 - x_0^2)(1 - x_1^2)(1 - x_2^2) = \alpha, \quad \alpha \in \mathbb{Q} \setminus \{0, 1\}\]

is a K3 surface with the Picard number $\geq 19$. This is isogenous to the K3 surface

\[w^2 = u_1u_2(1 + u_1)(1 + u_2)(u_1 - \alpha u_2),\]

studied by Ahlgren, Ono and Penniston [AOP]. A complete list of $\alpha$’s such that the Picard number 20 (i.e. singular K3) is known. For such an example, one can employ the higher Ross symbol together with our main theorem (=Theorem 5.5), so that we have a formulation of the $p$-adic Beilinson conjecture in terms of our $p$-adic hypergeometric functions $\mathcal{F}_\omega^{(\sigma)}(t)$’s (Conjectures 6.8, 6.9). The advantage of our formulation is that it allows the numerical verifications, while we have not succeeded a theoretical proof. In §6.4 we construct other examples of singular K3 surfaces arising from the hypergeometric schemes, and give a description of the $p$-adic regulators in terms of $\mathcal{F}_\omega^{(\sigma)}(t)$ (Theorem 6.15).

\section{$p$-adic Hypergeometric Functions}

\subsection{Hypergeometric series}

Let $K$ be a field of characteristic zero. Let $(\alpha)_n = \alpha(\alpha+1) \cdots (\alpha+n-1)$ be the Pochhammer symbol. For $\underline{a} = (a_0, \ldots, a_d) \in K^{d+1}$, the power series

\[F_{\underline{a}}(t) = F_{d+1} \left( a_0, \ldots, a_d; t \right) = \sum_{n=0}^{\infty} \frac{(a_0)_n}{n!} \cdots \frac{(a_d)_n}{n!} t^n\]  

is called the hypergeometric series. In case $d = 1$, this is also referred to as the Gaussian hypergeometric series. If $a_i \notin \mathbb{Z}$ for all $i$, this is characterized as the unique solution (up to scalar) of the hypergeometric differential operator

\[P_{\text{HG}, \underline{a}} := D^{d+1} - t(D + a_0) \cdots (D + a_d), \quad D := t \frac{d}{dt} = \]  

in the ring $K[[t, t^{-1}]]$ of Laurent power series.
2.2 Dwork’s Hypergeometric Functions

Let \( p \) be a prime. For \( \underline{a} = (a_0, \ldots, a_d) \in \mathbb{Z}_p^{d+1} \), the hypergeometric series

\[
F_{\underline{a}}(t) = \sum_{n=0}^{\infty} \frac{(a_0)_n \cdots (a_d)_n}{n! \cdots n!} t^n
\]

has coefficients in \( \mathbb{Z}_p \). In his paper [Dw], Dwork discovered that a certain ratio of hypergeometric series is a uniform limit of a sequence of rational functions.

For \( \alpha \in \mathbb{Z}_p \), let \( \alpha' \) denote the Dwork prime, which is defined to be \((\alpha + k)/p\) where \( k \in \{0, 1, \ldots, p-1\} \) such that \( \alpha + k \equiv 0 \mod p \). Define the \( i\)-th Dwork prime by \( \alpha^{(i)} = (\alpha^{(i-1)})' \) and \( \alpha^{(0)} := \alpha \). Write \( \underline{a} = (a_1', \ldots, a_s') \) and \( \underline{a}^{(i)} = (a_1^{(i)}, \ldots, a_s^{(i)}) \). Dwork’s \( p\)-adic hypergeometric function is defined to be a power series

\[
\mathcal{F}_{\underline{a}}(t) := F_{\underline{a}}(t)/F_{\underline{a}}(t^p) \in \mathbb{Z}_p[[t]].
\]

Let \( W = W(\overline{\mathbb{F}_p}) \) be the Witt ring of \( \overline{\mathbb{F}_p} \). Let \( c \in 1 + pW \) and \( \sigma \) a \( p\)-th Frobenius on \( W[[t]] \) given by \( \sigma(t) = ct^p, c \in 1 + pW \). A slight modification of \( \mathcal{F}_{\underline{a}}(t) \) is

\[
\mathcal{F}_{\underline{a}, \sigma}(t) := F_{\underline{a}}(t)/F_{\underline{a}}(t^\sigma) \in W[[t]].
\]

Dwork discovered certain congruence relations which we refer to as the Dwork congruence. The precise statement is as follows. For a power series \( f(t) = \sum_{i \geq 0} a_i t^i \), we denote \( [f(t)]_{<k} = \sum_{0 \leq i < k} a_i t^i \) the truncated polynomial.

**Theorem 2.1 (Dw, p.37, Thm. 2, p.45)** For any \( n \geq 1 \), we have

\[
\mathcal{F}_{\underline{a}, \sigma}(t) \equiv \frac{[F_{\underline{a}}(t)]_{<p^n}}{[F_{\underline{a}}(t^\sigma)]_{<p^n}} \mod p^n W[[t]].
\]

The Dwork congruence implies that \( \mathcal{F}_{\underline{a}, \sigma}(t) \) is a \( p\)-adic analytic function in the following way. Put

\[
h_{\underline{a}}(t) := \prod_{i=0}^{N} [F_{\underline{a}^{(i)}}(t)]_{<p}
\]

with \( N \) sufficiently large such that \( \{[F_{\underline{a}^{(i)}}(t)]_{<p} \}_{i \geq 0} = \{[F_{\underline{a}^{(0)}}(t)]_{<p} \}_{0 \leq i \leq N} \) as subsets of \( \mathbb{F}_p[[t]] \) where \( \overline{f(t)} := f(t) \mod p. \) Let

\[
\mathbb{Z}_p[[t, h_{\underline{a}}(t)^{-1}]^\wedge := \lim_{n \to \infty} \left( \mathbb{Z}_p/p^n \mathbb{Z}_p[[t, h_{\underline{a}}(t)^{-1}]^\wedge \right)
\]

be the \( p\)-adic completion. The ring does not depend on the choice of \( N \). An immediate consequence of the Dwork congruence is

\[
\mathcal{F}_{\underline{a}, \sigma}(t) \in W \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[t, h_{\underline{a}}(t)^{-1}]^\wedge].
\]

As another application of the Dwork congruence, one can show

\[
\frac{\frac{d}{dt} F_{\underline{a}}(t)}{F_{\underline{a}}(t)} \equiv \frac{\frac{d}{dt} [F_{\underline{a}}(t)]_{<p^{n}}}{[F_{\underline{a}}(t)]_{<p^n}} \mod p^n \mathbb{Z}_p[[t]]
\]

for all \( n \geq 1 \) and \( j \geq 0 \) in the same way as the proof of [Dw] (3.14).
2.3 $p$-adic Hypergeometric Functions of logarithmic type [As1]

In [As1], we introduce a certain new $p$-adic hypergeometric function, which is different from Dwork’s one. We recall it here. Define a continuous function
\[
\tilde{\psi}_p(z) := \lim_{n \in \mathbb{Z}^+ \to z} \sum_{1 \leq k < n, p \nmid k} \frac{1}{k}
\]
on $\mathbb{Z}_p$ where “$n \to z$” means the limit with respect to the $p$-adic metric ([As1, 2.2]). Let $c \in 1 + pW$ and $\sigma(t) = ct^p$ be as before. Let
\[
G^{(\sigma)}(t) := \sum_{i=0}^{d} \tilde{\psi}_p(a_i) - p^{-1} \log(c) + \int_0^t (F_a(t) - F_a(t^p)) \frac{dt}{t}
\]
where $\log(z)$ is the Iwasawa logarithmic function. Then we define
\[
\mathcal{F}^{(\sigma)}(t) := G^{(\sigma)}(t)/F_a(t),
\]
and call the $p$-adic hypergeometric functions of logarithmic type. This is a power series with coefficients in $W$.

There are congruence relations that are similar to the Dwork congruence (Theorem 2.1).

**Theorem 2.2 ([As1 Theorem 3.3])** If $c \in 1 + pW$, then there are congruence relations
\[
\mathcal{F}^{(\sigma)}(t) \equiv \left[ \frac{G^{(\sigma)}(t)}{F_a(t^p)} \right]_{<p^n} \mod p^nW[[t]], \quad n \geq 1.
\]

If $p = 2$ and $c \in 1 + 2W$, then the congruence holds modulo $p^{n-1}W[[t]]$. Hence $\mathcal{F}^{(\sigma)}(t)$ belongs to the $p$-adic completion $W[t, h_a(t)^{-1}]$ of the ring $W[t, h_a(t)^{-1}]$.

The above congruence plays a key role in the proof of the main theorem (Theorem 5.3).

3 Hypergeometric Schemes

3.1 Review from [As2]

Let $A$ be a commutative ring. Let $d \geq 1$ and $n_i \geq 2$ ($i = 0, 1, \ldots, d$) be integers such that $n_0 \cdots n_d$ is invertible in $A$. We call an affine scheme
\[
U = \text{Spec} \ A[x_0, \ldots, x_d]/((1 - x_0^n_0) \cdots (1 - x_d^n_d) - t), \quad t \in A
\]
the hypergeometric scheme over $A$ ([As2 §2.1]). It is easy to see that $U$ is smooth over $A$ if $t(1 - t) \in A^\times$.

**Proposition 3.1** Assume that $t(1 - t) \in A^\times$ and $A$ is an integral domain. Then there is an open immersion $U \hookrightarrow X$ into a smooth projective $A$-scheme $X$ such that $Z = X \setminus U$ is a relative simple NCD over $A$.

**Proof.** [As2 Proposition 2.1].

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Hereafter we assume that $A$ is an integral domain and $t(1-t) \in A^\times$. Let $\mu_n = \mu_n(A)$ denote the group of $n$-th roots of unity in $A$. A finite abelian group $G = \mu_{n_0} \times \cdots \times \mu_{n_d}$ acts on $U$ in a way that $(x_0, \ldots, x_d) \mapsto (\nu_0 x_0, \ldots, \nu_d x_d)$ for $\nu = (\nu_0, \ldots, \nu_d) \in G$. In this way the de Rham cohomology groups $H^\bullet_{\text{dR}}(U/A)$ are endowed with the structure of $A[G]$-modules. For a $A[G]$-module $H$ and $(i_0, \ldots, i_d) \in \mathbb{Z}^{d+1}$ we denote by

$$H(i_0, \ldots, i_d) = \{ x \in H \mid \nu x = \nu_0^{i_0} \cdots \nu_d^{i_d} x, \forall \nu \in G \}. \quad (3.2)$$

the simultaneous eigenspace.

Let $A = K[t, (t - t^2)^{-1}]$ with $K$ a field of characteristic zero. Suppose that $K$ contains primitive $n_i$-th roots of unity for all $i$. We denote by $W \cdot H^\bullet_{\text{dR}}(U/A)$ the weight filtration. In particular $W_i H^i_{\text{dR}}(U/A) = \text{Im} [H^i_{\text{dR}}(X/A) \rightarrow H^i_{\text{dR}}(U/A)]$. Put $I := \{(i_0, \ldots, i_d) \in \mathbb{Z}^{d+1} \mid 0 \leq i_k < n_k \}$ and $I_+ := \{(i_0, \ldots, i_d) \in \mathbb{Z}^{d+1} \mid 0 < i_k < n_k \}$. There is the decomposition

$$H^i_{\text{dR}}(U/A) = \bigoplus_{(i_0, \ldots, i_d) \in I} H^i_{\text{dR}}(U/A)(i_0, \ldots, i_d)$$

of the de Rham cohomology group.

**Theorem 3.2 ([As2, §3.2 Summary (1), (2)])**

1. The $A$-submodule

$$\bigoplus_{(i_0, \ldots, i_d) \in I \setminus I_+} H^i_{\text{dR}}(U/A)(i_0, \ldots, i_d)$$

is generated by exterior products of $\{dx_k/(x_k - \nu_k) \mid \nu_k \in \mu_{n_k} \}$.

2. If $i < d$, then

$$\bigoplus_{(i_0, \ldots, i_d) \in I_+} H^i_{\text{dR}}(U/A)(i_0, \ldots, i_d) = 0.$$

In particular $W_i H^i_{\text{dR}}(U/A) = 0$ for $0 < i < d$ by this and (1).

3. \[
\bigoplus_{(i_0, \ldots, i_d) \in I_+} H^d_{\text{dR}}(U/A)(i_0, \ldots, i_d) = W_d H^d_{\text{dR}}(U/A).
\]

For each $(i_0, \ldots, i_d) \in I_+$, we put, cf. [As2, §2.4]

**Def.**

$$\omega_{i_0 \ldots i_d} := n_0^{-1} x_0^{i_0} n_0^{-1} x_1^{i_1-1} \cdots x_d^{i_d-1} \frac{dx_1 \cdots dx_d}{(1 - x_1^{n_1}) \cdots (1 - x_d^{n_d})} \quad (3.3)$$

a $d$-form in $\Gamma(X, \Omega^d_{X/A})(i_0, \ldots, i_d)$.

**Theorem 3.3 ([As2, Corollary 3.6])** Let $\mathcal{D} := K[t, (t - t^2)^{-1}, \frac{d}{dt}]$ be the Weyl algebra of Spec $A$, which acts on $H^d_{\text{dR}}(U/A)$. Let $(i_0, \ldots, i_d) \in I_+$ and put $a_k := 1 - i_k/n_k$. Let $P_{\text{HG},\omega}$ be the hypergeometric differential operator (2.2). Then $P_{\text{HG},\omega}(\omega_{i_0 \ldots i_d}) = 0$ and the homomorphism

$$\mathcal{D} \rightarrow P_{\text{HG},\omega} \cong H^d_{\text{dR}}(U/A)(i_0, \ldots, i_d), \quad P \mapsto P(\omega_{i_0 \ldots i_d})$$

of $\mathcal{D}$-modules is bijective. In particular, this is an irreducible $\mathcal{D}$-module.
In view of Theorem 3.3, we think the piece $H^d_{dR}(U/A)(i_0, \ldots, i_d)$ of being an hypergeometric motive associated to the hypergeometric function (3.1).

### Convention

For an integer $n > 0$, let $\mathbb{Z}_{(n)} = S^{-1}\mathbb{Z}$ denote the ring of fractions with respect to the multiplicative set $S = \{s \in \mathbb{Z} \mid \gcd(s, n) = 1\}$. For $j \in \mathbb{Z}_{(n)}$, let $[j]_n$ denote the unique integer such that $0 \leq [j]_n < n$ and $[j]_n \equiv j \mod n\mathbb{Z}_{(n)}$. We then extend the notation as follows,

$$H(i_0, \ldots, i_d) := H([i_0]_{n_0}, \ldots, [i_d]_{n_d}), \quad (i_0, \ldots, i_d) \in \prod_{k=0}^d\mathbb{Z}_{(n_k)};$$

$$\omega_{i_0(\ldots i_d) := \omega_{[i_0]_{n_0}(\ldots [i_d]_{n_d})}, \quad (i_0, \ldots, i_d) \in \prod_{k=0}^d(\mathbb{Z}_{(n_k)} \setminus n_k\mathbb{Z}_{(n_k)})}.$$

### 3.2 Pairing $Q$ on $W_dH^d_{dR}(U/A)$

Let $K$ be a field of characteristic zero, and $A$ an integral smooth $K$-algebra. Let $U$ be the hypergeometric scheme (3.1) over $A$ of relative dimension $d$ (we do not assume that $K$ contains primitive $n_1$-th roots of unity). We construct a natural pairing

$$Q : W_dH^d_{dR}(U/A) \otimes_A W_dH^d_{dR}(U/A) \longrightarrow A$$

in the following way. Let $X \supset U$ be a compactification such that $X \to \text{Spec} A$ is smooth projective with connected fibers (e.g. Proposition 3.4). We fix an ample class $[\omega] \in H^2_{dR}(X/A)$, and let $L$ be the Lefschetz operator on $H^\bullet_{dR}(X/A)$ i.e. the cup-product with $[\omega]$. Let

$$H^d_{dR}(X/A) = \bigoplus_{i \geq 0} L^iH^{d-2i}(X/A)_{prim}$$

be the Lefschetz decomposition by the primitive parts of cohomology. Let $j^* : H^\bullet_{dR}(X/A) \to H^\bullet_{dR}(U/A)$ be the pull-back by the immersion $j : U \to X$. Then, for $i > 0$, the image of the component $L^iH^{d-2i}(X/A)_{prim}$ vanishes by Theorem 3.2 (2), so that the map

$$j^* : H^d_{dR}(X/A)_{prim} \longrightarrow W_dH^d_{dR}(U/A)$$

is surjective. Let $T \subset H^d_{dR}(X/A)_{prim}$ be the kernel, and $T^\perp \subset H^d_{dR}(X/A)_{prim}$ the orthogonal complement with respect to the cup-product pairing $Q_X$ on $H^d_{dR}(X/A)_{prim}$. Since $Q_X$ induces the polarization on the primitive part $H^d_{dR}(X/A)_{prim}$, the restricted pairings on $T$ and $T^\perp$ are non-degenerate (e.g. [PS, Corollary 2.12]), and hence $T^\perp \cong W_dH^d_{dR}(U/A)$. Then we define the pairing (3.4) to be the induced one from $Q_X$ restricted on $T^\perp$.

**Proposition 3.4 (Q1)** The pairing $Q$ is $(-1)^d$-symmetric and non-degenerate,
(Q2) Suppose that $K$ contains primitive $n_i$-th roots of unity for all $i$. Then $Q(\sigma x, \sigma y) = Q(x, y)$ for $\sigma \in G$.

(Q3) $Q(\theta x, y) + Q(x, \theta y) = \theta(Q(x, y))$ for any derivative $\theta$ on $A$.

(Q4) $Q(F^p, F^q) = 0$ if $p + q > d$ and $Q(F^p, F^q) = A$ if $p + q = d$ and $p, q \geq 0$, where $F^*$ is the Hodge filtration.

When $K$ contains primitive $n_i$-th roots of unity for all $i$, the property (Q2) implies that $Q$ induces a perfect pairing

$$Q : W_d H^d_{\text{dR}}(U/A)(i_0, \ldots, i_d) \otimes_A W_d H^d_{\text{dR}}(U/A)(-i_0, \ldots, -i_d) \rightarrow A. \quad (3.5)$$

Proof. Everything but (Q2) is obvious from the construction. We should be careful of (Q2) as we do not assume that the action of $G$ extends on $X$. In proving (Q2), we give an alternative construction of $Q$. We may replace $K$ with $K$. Let $H^*_{\text{dR},c}(U/A)$ denote the de Rham cohomology with compact support, on which $G$ acts. There is the cup-product

$$H^d_{\text{dR},c}(U/A) \otimes_A H^d_{\text{dR}}(U/A) \rightarrow H^{2d}_{\text{dR},c}(U/A) \cong H^{2d}_{\text{dR}}(X/A) \cong A,$$

which induces a pairing

$$Q_c : \text{Gr}^W_d H^d_{\text{dR},c}(U/A) \otimes_A \text{Gr}^W_d H^d_{\text{dR},c}(U/A) \rightarrow A.$$

This is compatible with $Q_X$ under the natural map $\text{Gr}^W_d H^d_{\text{dR},c}(U/A) \rightarrow H^d_{\text{dR}}(X/A)$, and satisfies $Q_c(\sigma x, \sigma y) = Q_c(x, y)$ for $\sigma \in G$. We note that $W_d H^d_{\text{dR},c}(U/A) = H^d_{\text{dR},c}(U/A)$. If one can show that

$$\text{Im}(H^d_{\text{dR},c}(U/A)) \subset H^d_{\text{dR}}(X/A)_{\text{prim}}$$

and that the composition

$$u : H^d_{\text{dR},c}(U/A) \rightarrow H^d_{\text{dR}}(X/A) \rightarrow W_d H^d_{\text{dR}}(U/A)$$

is surjective, then the pairing induced from $Q_c$ agrees with $Q$, and hence (Q2) follows. The former is equivalent to that the composition $H^d_{\text{dR},c}(U/A) \rightarrow H^d_{\text{dR}}(X/A)$ is zero. This agrees with the composition $H^d_{\text{dR},c}(U/A) \rightarrow H^d_{\text{dR}}(X/A)$ where $L_U$ is the cup-product with $[\omega]_U \in W_2 H^2_{\text{dR}}(U/A)$. Therefore it is enough to show $[\omega]_U = 0$. If $d \neq 2$, this follows from the vanishing $W_2 H^2_{\text{dR}}(X/A) = 0$ (Theorem 3.2(2)). If $d = 2$, it follows from Theorem 3.2(3) and Theorem 3.3 that there is no non-zero element $z \in W_2 H^2_{\text{dR}}(U/A)$ such that $Dz = 0$, and in particular one has $[\omega]_U = 0$. We show that $u$ is surjective. This is equivalent to the surjectivity of the natural map

$$u_{i_0 \ldots i_d} : H^d_{\text{dR},c}(U/A)(i_0, \ldots, i_d) \rightarrow W_d H^d_{\text{dR}}(U/A)(i_0, \ldots, i_d)$$

for each $(i_0, \ldots, i_d) \in I_+$ by Theorem 3.2(3). Since this is a homomorphism of $\mathcal{D}$-modules, and the right hand side is irreducible by Theorem 3.3, it is enough to see the non-vanishing $u_{i_0 \ldots i_d} \neq 0$. To see this, it is enough to show that

$$F^d H^d_{\text{dR},c}(U/A) \rightarrow F^d H^d_{\text{dR}}(X/A) \cong \rightarrow F^d W_d H^d_{\text{dR}}(U/A)$$
is surjective. However this follows from the exact sequence

\[ H^d_{\text{dR}}(U/A) \longrightarrow H^d_{\text{dR}}(X/A) \longrightarrow H^d_{\text{dR}}(Z/A) \]

where \( Z = X \setminus U \), and the vanishing \( F^d H^d_{\text{dR}}(Z/A) = 0 \) as \( \dim(Z/A) = d - 1 \). This completes the proof. \( \square \)

**Lemma 3.5** The pairing \( Q \) does not depend on the choice of \( X \) and \([\omega]\).

**Proof.** The alternative construction of \( Q \) in the proof of Proposition 3.4 does not depend on \( X \) and \([\omega]\). \( \square \)

## 4 Unit root formula for Hypergeometric Schemes

Throughout this section, let \( W = W(\mathbb{F}) \) be the Witt ring of an algebraically closed field \( \mathbb{F} \) of characteristic \( p > 0 \), \( K := \text{Frac}(W) \) the fractional field and put \( A := W[t, (t - t^2)^{-1}] \).

### 4.1 Unit Root Vectors

Let \( n_0, \ldots, n_d > 1 \) be integers and \( p \) a prime such that \( p \nmid n_0 \cdots n_d \). Let

\[ U = \text{Spec} A[x_0, \ldots, x_d] / \langle (1 - x_0^{n_0}) \cdots (1 - x_d^{n_d}) - t \rangle \]

be the hypergeometric scheme over \( A \). We write \( A_K := K[t, (t - t^2)^{-1}] \), and \( U_K := U \times_A A_K \).

Let \( \mathcal{D} := K(t, (t - t^2)^{-1}, \frac{dt}{t}) \) be the Weyl algebra. Put \( D := \frac{dt}{t} \). Let \( (i_0, \ldots, i_d) \in \prod_{k=0}^d \mathbb{Z}_{(n_k)} \) satisfy \( [i_k]_{n_k} \neq 0 \) for all \( k \) (see **Convention** in §3.1 for the notation). Recall from Theorems 3.2 and 3.3 the eigenspace

\[ H^d_{\text{dR}}(U_K/A_K)(i_0, \ldots, i_d) = W_d H^d_{\text{dR}}(U_K/A_K)(i_0, \ldots, i_d) = \sum_{k=0}^d A_K D^k \omega_{i_0 \ldots i_d}, \]

which is stable under the action of \( \mathcal{D} \). We write \( H_{i_0 \ldots i_d}(U_K/A_K) = H^d_{\text{dR}}(U_K/A_K)(i_0, \ldots, i_d) \) for simplicity of notation.

**Proposition 4.1** Let \( H_{i_0 \ldots i_d}(U_K/A_K)_{K((t))} := K((t)) \otimes_{A_K} H_{i_0 \ldots i_d}(U_K/A_K) \) on which the action of \( \mathcal{D} \) extends in a natural way. Put \( a_k := 1 - [i_k]_{n_k}/n_k \). Let \( s_k \in \mathbb{Q} \) be defined by \( (t + a_0) \cdots (t + a_d) = t^{d+1} + s_1 t^d + \cdots + s_{d+1} \), and put \( q_{d-m} := -s_{m+1} t / (1 - t) \) for \( m = 0, 1, \ldots, d \). Put \( \hat{a} := (1 - a_0, \ldots, 1 - a_d) \) and

\[ y_d := (1 - t) F_{\hat{a}}(t) = (1 - t)_{d+1} F_d \left( \frac{1 - a_0, \ldots, 1 - a_d; t}{1, \ldots, 1} \right). \]
For $0 \leq i < d$ define $y_i$ by $y_i + D y_{i+1} = q_{i+1} y_d$. Put
\[
\hat{\eta}_{i_0 \ldots i_d} := y_0 \omega_{i_0 \ldots i_d} + y_1 D \omega_{i_0 \ldots i_d} + \cdots + y_d D^d \omega_{i_0 \ldots i_d} \in H_{i_0 \ldots i_d}(U_K/A_K) K((t)). \tag{4.1}
\]
Then
\[
\text{Ker}[D : H_{i_0 \ldots i_d}(U_K/A_K) K((t)) \to H_{i_0 \ldots i_d}(U_K/A_K) K((t))] = K \hat{\eta}_{i_0 \ldots i_d}. \tag{4.2}
\]

**Proof.** Recall Theorem [3.3] $H_{i_0 \ldots i_d}(U_K/A_K) K((t))$ is a free $K((t))$-module with basis \(\{D^k \omega_{i_0 \ldots i_d} | k = 0, 1, \ldots, d\}\) and the differential operator $P_{HG,\bar{a}} = D^{d+1} - t(D + a_0) \cdots (D + a_d) = (1 - t)(D^{d+1} + q_d D^d + \cdots + q_0)$ annihilates $\omega_{i_0 \ldots i_d}$. Therefore
\[
D \left( \sum_{k=0}^{d} z_k D^k \omega_{i_0 \ldots i_d} \right) = \sum_{k=0}^{d} D(z_k) D^k \omega_{i_0 \ldots i_d} + z_k D^{k+1} \omega_{i_0 \ldots i_d}
\]
\[
= \sum_{k=1}^{d} (z_{k-1} + D(z_k)) D^k \omega_{i_0 \ldots i_d} + D(z_0) \omega_{i_0 \ldots i_d} + z_d D^{d+1} \omega_{i_0 \ldots i_d}
\]
\[
= \sum_{k=1}^{d} (z_{k-1} + D(z_k) - q_k z_d) D^k \omega_{i_0 \ldots i_d} + (D(z_0) - q_d z_d) \omega_{i_0 \ldots i_d}
\]
vaneses if and only if
\[
z_i + D(z_{i+1}) = q_{i+1} z_d \ (0 \leq i \leq d - 1), \quad D(z_0) = q_d z_d. \tag{4.3}
\]
Put a differential operator
\[
P := D^{d+1} - D^d \ast q_d + \cdots + (-1)^d D \ast q_1 + (-1)^{d+1} q_0
\]
where $\ast$ denotes the composition of operators to make distinctions between $D \ast f \in \mathcal{D}$ and $D(f) \in K((t))$. Then (4.3) is equivalent to
\[
z_i + D(z_{i+1}) = q_{i+1} z_d \ (0 \leq i \leq d - 1), \quad P(z_d) = 0,
\]
so that the assertion is reduced to show that $y_d = (1 - t)F_{\bar{a}}(t)$ is the unique solution (up to scalar) in $K((t))$ of the differential equation $P(y) = 0$. One has
\[
P \ast (1 - t) = D^{d+1} \ast (1 - t) - \sum_{m=0}^{d} (-1)^m \left( \sum_{i_0 < i_1 < \cdots < i_m} a_{i_0} \cdots a_{i_m} \right) D^{d-m} \ast t
\]
\[
= D^{d+1} - t(D + 1)^{d+1} - \sum_{m=0}^{d} (-1)^m \left( \sum_{i_0 < i_1 < \cdots < i_m} a_{i_0} \cdots a_{i_m} \right) t(D + 1)^{d-m}
\]
\[
= D^{d+1} - t(D + 1 - a_0) \cdots (D + 1 - a_d)
\]
\[
P_{HG,\bar{a}}.
\]
Therefore $y_d = (1 - t)F_{\bar{a}}(t)$ is the unique solution for $P$. \qed

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We define a unit root vector

\[ \eta_{i_0\ldots i_d} := F_\partial(t)^{-1}\bar{i}_{i_0\ldots i_d} = \frac{y_0}{F_\partial(t)}\omega_{i_0\ldots i_d} + \frac{y_1}{F_\partial(t)}D\omega_{i_0\ldots i_d} + \cdots + (1-t)D^d\omega_{i_0\ldots i_d}. \]

(4.4)

Lemma 4.2 Let \( h_\partial(t) \) be the polynomial as in (2.3). Then we have

\[ \frac{y_i}{F_\partial(t)} \in \left( \mathbb{Z}_p[t, h_\partial(t)^{-1}] \right)[(t-t)^{-1}], \quad i = 0, 1, \ldots, d \]

where \((-)^\wedge\) denotes the \( p \)-adic completion, and hence

\[ \eta_{i_0\ldots i_d} \in A[h_\partial(t)^{-1}]^\wedge \otimes_A H_{i_0\ldots i_d}(U_K/A_K). \]

Proof. By the construction of \( y_i \), they are linear combinations of \( D^j(y_d) \) over a ring \( \mathbb{Z}_p[t, (1-t)^{-1}] \) and hence one can write

\[ y_i = \sum_j g_j \frac{d^j F_\partial(t)}{d^j} \]

by some \( g_j \in \mathbb{Z}_p[t, (t-t^2)^{-1}] \). Now the assertion follows from (2.5). \( \square \)

Let \( h(t) := \prod a_i h_\partial(t) \) where \( a = (i_0/n_0, \ldots, i_d/n_d) \) runs over all \((d+1)\)-tuple of integers \((i_0, \ldots, i_d)\) such that \( 0 < i_k < n_k \). Put \( \hat{B} := A[h(t)^{-1}], B_K := K \otimes_W \hat{B}, \) and

\[ \hat{B} := A[h(t)^{-1}]^\wedge = \lim_n \left( W/p^n W[t, (t-t^2)^{-1}, h(t)^{-1}] \right) \]

(4.5)

the \( p \)-adic completion, and \( \hat{B}_K := K \otimes_W \hat{B} \). Thanks to Lemma 4.2, the unit root vector \( \eta_{i_0\ldots i_d} \) belongs to \( H_{i_0\ldots i_d}(U_K/A_K)_{\hat{B}_K} := \hat{B}_K \otimes_{A_K} H_{i_0\ldots i_d}(U_K/A_K) \). We call the direct summand

\[ H_{i_0\ldots i_d}^{\text{unit}}(U_K/A_K)_{\hat{B}_K} := B_K \eta_{i_0\ldots i_d} \]

of \( H_{i_0\ldots i_d}(U_K/A_K)_{\hat{B}_K} \) the unit root subspace. Recall from §3.2 the perfect pairing (3.5)

\[ Q : H_{i_0\ldots i_d}(U_K/A_K) \otimes_{A_K} H_{-i_0,\ldots, -i_d}(U_K/A_K) \rightarrow A_K. \]

(4.6)

Tensoring with \( \hat{B}_K \), we have the pairing on the \( \hat{B}_K \)-modules, which we also write by \( Q \). Define a \( \hat{B}_K \)-submodule

\[ VH_{i_0\ldots i_d}(U_K/A_K)_{\hat{B}_K} \subset H_{i_0\ldots i_d}(U_K/A_K)_{\hat{B}_K} \]

(4.7)

to be the exact annihilator of the unit root part \( H_{-i_0,\ldots, -i_d}^{\text{unit}}(U_K/A_K)_{\hat{B}_K} \). By definition, the pairing

\[ H_{i_0\ldots i_d}(U_K/A_K)_{\hat{B}_K}/VH_{i_0\ldots i_d}(U_K/A_K)_{\hat{B}_K} \otimes H_{-i_0,\ldots, -i_d}^{\text{unit}}(U_K/A_K)_{\hat{B}_K} \rightarrow \hat{B}_K \]

(4.8)

is perfect. We shall later see \( \eta_{i_0\ldots i_d} \in VH_{i_0\ldots i_d}(U_K/A_K)_{\hat{B}_K} \) (Corollary 4.8). Let \( W((t))^\wedge \) be the \( p \)-adic completion and \( K((t))^\wedge := K \otimes_W W((t))^\wedge \). We write \( VH_{i_0\ldots i_d}(U_K/A_K)_{K((t))^\wedge} := K((t))^\wedge \otimes_{A_K} VH_{i_0\ldots i_d}(U_K/A_K) \). Note that \( \hat{B}_K \subset K((t))^\wedge \).
Lemma 4.3 \hspace{1cm} (1) $V H_{i_0...i_d}(U_K/A_K)\hat{\mathbb{B}}_K$ is stable under the action of $\mathcal{D}$.

(2) $H_{i_0...i_d}(U_K/A_K)\hat{\mathbb{B}}_K/V H_{i_0...i_d}(U_K/A_K)\hat{\mathbb{B}}_K \cong \hat{\mathbb{B}}_K$ is generated by $\omega_{i_0...i_d}$.

(3) $Q(\omega_{i_0...i_d}, \eta_{-i_0,...,-i_d}) = C \in \mathbb{Q}^\times$.

Proof. (1) is immediate from the fact that $D\hat{\eta}_{i_0...i_d} = 0$ (Proposition 4.1). To see (2), it is enough to show $Q(\omega_{i_0...i_d}, \eta_{-i_0,...,-i_d}) \neq 0$. Write $\omega = \omega_{i_0...i_d}$ and $\bar{\omega} = \omega_{-i_0,...,-i_d}$. Recall from [As2] Corollary 3.10 the fact that $\text{Gr}^i F H_{i_0...i_d}(U_K/A_K)$ is a free $A_K$-module with basis $D^{d-i}\omega$. Therefore, thanks to the property (Q4) in Proposition 3.4, we have

$$Q(D^i\omega, D^j\bar{\omega}) = 0$$

(4.9) for any $(i, j)$ such that $i + j < d$ and $i, j \geq 0$. In particular

$$Q(\omega_{i_0...i_d}, \eta_{-i_0,...,-i_d}) = (1 - t)Q(\omega, D^d\bar{\omega}).$$

(4.10)

If $Q(\omega_{i_0...i_d}, \eta_{-i_0,...,-i_d}) = 0$, then $Q(\omega, D^i\bar{\omega})$ vanishes for all $i \in \mathbb{Z}_{\geq 0}$, which contradicts with the fact that $Q$ is a perfect pairing. Hence $Q(\omega_{i_0...i_d}, \eta_{-i_0,...,-i_d}) \neq 0$. We show (3). Since

$$0 = DQ(D^i\omega, D^j\bar{\omega}) = Q(D^{i+1}\omega, D^j\bar{\omega}) + Q(D^i\omega, D^{j+1}\bar{\omega})$$

for $i + j = d - 1$ and $i, j \geq 0$ by (4.9), one has

$$(-1)^i Q(\omega, D^d\bar{\omega}) = Q(D^i\omega, D^{d-i}\bar{\omega}), \quad i = 0, 1, \ldots, d.\quad (4.11)$$

Applying $D$ on the both sides, one has

$$(-1)^i DQ(\omega, D^d\bar{\omega}) = Q(D^{i+1}\omega, D^{d-i}\bar{\omega}) + Q(D^i\omega, D^{d-i+1}\bar{\omega}).$$

Taking the alternating sum of the both sides, one has

$$(d + 1)DQ(\omega, D^d\bar{\omega}) = (-1)^d Q(D^{d+1}\omega, \bar{\omega}) + Q(\omega, D^{d+1}\bar{\omega}).\quad (4.12)$$

Using $P_{HG,\omega}\omega = 0$ and $P_{HG,\bar{\omega}}\bar{\omega} = 0$ together with (4.9), one has

$$Q(D^{d+1}\omega, \bar{\omega}) = (a_0 + \cdots + a_d)\frac{t}{1 - t}Q(D^d\omega, \bar{\omega})$$

$$= (-1)^d(a_0 + \cdots + a_d)\frac{t}{1 - t}Q(\omega, D^d\bar{\omega})$$

(by (4.11)),

$$Q(\omega, D^{d+1}\bar{\omega}) = (d + 1 - (a_0 + \cdots + a_d))\frac{t}{1 - t}Q(\omega, D^d\bar{\omega}).$$

Apply the above to (4.12), then

$$(d + 1)DQ(\omega, D^d\bar{\omega}) = (d + 1)\frac{t}{1 - t}Q(\omega, D^d\bar{\omega}) \iff \frac{d}{dt}Q(\omega, D^d\bar{\omega}) = \frac{1}{1 - t}Q(\omega, D^d\bar{\omega}).$$

This implies $Q(\omega, D^d\bar{\omega}) = C(1 - t)^{-1}$ with some $C \in K$. Hence $Q(\omega_{i_0...i_d}, \eta_{-i_0,...,-i_d}) = C$ by (4.10), and this is not zero by (2). Since the pairing $Q$ and the elements $\omega, \bar{\omega}$ are defined over a ring $\mathbb{Q}(t)$, it turns out that $C \in K^\times \cap \mathbb{Q}(t) = \mathbb{Q}^\times$. This completes the proof of (3). $\square$
4.2 Rigid cohomology of Hypergeometric schemes

We write \( X_K = X \times_W K \) and \( X_\mathbb{F} = X \times_W \mathbb{F} \) for a \( W \)-scheme \( X \). Let \( T \) be a smooth affine scheme over \( W \). Let \( \mathcal{O}(T) \) denote the weak completion of \( \mathcal{O}(T) \), and write \( \mathcal{O}(T)_K := \mathcal{O}(T) \otimes_W K \). We fix a \( p \)-th Frobenius \( \sigma \) on \( \mathcal{O}(T) \), namely it satisfies \( \sigma(x) \equiv x^p \mod p \), and \( \sigma \) on \( W \) agrees with the \( p \)-th Frobenius on the Witt ring. For a smooth morphism \( g : V \to T \), the \( i \)-th rigid cohomology group

\[
H^i_{\text{rig}}(V_\mathbb{F}/T_\mathbb{F}) := \Gamma(T \text{an}_K, R^i g_{\text{rig}} \mathcal{O}_{V_\mathbb{F}}),
\]

is defined (cf. [AM, Definition 2.12]). The \( p \)-th Frobenius \( \Phi \) that is \( \sigma \)-linear is defined on the rigid cohomology group in the canonical way. In this paper, we often employ the following fundamental fact.

**Theorem 4.4** Suppose that either of the following conditions (1) and (2) holds.

1. The morphism \( g : V \to T \) is smooth projective.

2. There is a completion \( T \hookrightarrow \overline{T} \) into a smooth projective \( W \)-scheme \( \overline{T} \) such that \( E := \overline{T} \setminus T \) is a smooth divisor over \( W \), and there are \( Y, Y \) and \( V \) which fit into commutative squares

\[
\begin{array}{ccc}
V & \xrightarrow{j} & Y \\
\downarrow{g} & & \downarrow{f} \\
T & \xrightarrow{\bar{j}} & \overline{T}
\end{array}
\]

satisfying the following conditions, where \( \square \) denotes a cartesian diagram.

2-1 \( Y \) is smooth projective over \( W \). The arrows \( j \) and \( j' \) are open embeddings, \( \bar{j} \) is projective, and \( f \) is smooth projective.

2-2 The scheme-theoretic inverse image \( D := \bar{j}^{-1}(E) \) is a relative simple normal crossing divisor over \( W \), and each multiplicity is prime to \( p \).

2-3 \( Z := Y \setminus V \) is a relative simple normal crossing divisor over \( T \).

2-4 Let \( \bar{Z} \) be the closure of \( Z \) in \( \overline{Y} \). Then, \( \bar{Z} + D \) is a relative simple normal crossing divisor over \( W \).

Then the \( i \)-th relative rigid cohomology sheaf \( R^i g_{\text{rig}} \mathcal{O}_{V_\mathbb{F}} \) is a coherent \( j^!_T \mathcal{O}_{\overline{T}_\mathbb{F}} \)-module for each \( i \). Consequently, \( H^i_{\text{rig}}(V_\mathbb{F}/T_\mathbb{F}) \) is a locally free \( \mathcal{O}(T)_K \)-module of finite rank, and the comparison map

\[
c : \mathcal{O}(T)^{\dagger}_K \otimes_{\mathcal{O}(T)_K} H^i_{\text{dR}}(V_K/T_K) \to H^i_{\text{rig}}(V_\mathbb{F}/T_\mathbb{F})
\]

is bijective for each \( i \) (see [AM, 2.5] for the construction of the comparison map).

**Proof.** See [AM] Propositions 2.15, 2.17] for the proof. We note that the essential point is Kiehl’s theorem for proper case and [Sh, Theorem 2.2] for non-proper case. \( \square \)
We turn to the setting in §4.1. We write \( A_F := \mathbb{F}[t, (t-t^2)^{-1}] \) and \( U_F := U \times_A A_F \) etc. For \( c \in 1 + pW \), let \( \sigma \) be a \( F \)-linear \( p \)-th Frobenius on \( A_K^r \) given by \( \sigma(t) = ct^p \). Let \( X \supset U \) be the compactification as in Proposition 3.1 and \( Z := X \setminus U \) the complements which is a relative simple normal crossing divisor over \( A \). We shall later see that \( U \to \text{Spec} \, A \) satisfies the condition Theorem 4.4(2) if \( p > \text{lcm}(n_0, \ldots, n_d) \) (Lemma 5.1). However in this section, we work under a weaker assumption that \( p \nmid n_0 \cdots n_d \).

Let \( Z = \cup_k Z_k \) be the irreducible decomposition. Then one has an exact sequence
\[
\bigoplus_{k} H_{\text{dir}}^{i-2}(Z_{k,K}/A_K) \rightarrow H_{\text{dir}}^{i}(X_{K}/A_K) \rightarrow W_i H_{\text{dr}}^{i}(U_{K}/A_K) \rightarrow 0,
\]
so that one has the Frobenius structure on
\[
A_K^r \otimes_A K W_i H_{\text{dr}}^{i}(U_{K}/A_K)
\]
compatible with the \( \mathcal{D} \)-module structure thanks to Theorem 4.4. Write
\[
H_{i_0 \cdots i_d}(U_{K}/A_K)_{A_K^r} := A_K^r \otimes_A K W_d H_{\text{dir}}^{i_d}(U_{K}/A_K)(i_0, \ldots, i_d) = A_K^r \otimes_A K H_{\text{dir}}^{i_d}(U_{K}/A_K)(i_0, \ldots, i_d).
\]
for \((i_0, \ldots, i_d) \in \prod_{k=0}^d \mathbb{Z}(n_k)\) satisfying \( i_k \not\equiv 0 \mod n_k \). Tensoring (4.6) with \( A_K^r \), we have a pairing
\[
H_{i_0 \cdots i_d}(U_{K}/A_K)_{A_K^r} \otimes A_K^r H_{-i_0 \cdots -i_d}(U_{K}/A_K)_{A_K^r} \rightarrow A_K^r \otimes A_K H_{e,\text{dir}}^{2d}(U_{K}/A_K) \cong A_K^r
\]
which we also write by \( Q \). Since the cup-product is compatible with the Frobenius, we have
\[
(\text{Q5}) \quad Q(\Phi(x), \Phi(y)) = p^d \sigma(Q(x, y)).
\]
The Frobenius \( \sigma \) extends on \( \hat{B}_K \) and \( K((t))^\wedge \) in a natural way. According to this, we extend \( \Phi \) to that on \( H_{i_0 \cdots i_d}(U_{K}/A_K)_{\hat{B}_K} \) and \( H_{i_0 \cdots i_d}(U_{K}/A_K)_{K((t))^\wedge} = K((t))^\wedge \otimes_A H_{i_0 \cdots i_d}(U_{K}/A_K) \), and denote by the same notation.

**Lemma 4.5** Let \( H_{i_0 \cdots i_d} \) denote \( H_{i_0 \cdots i_d}(U_{K}/A_K)_R \) where \( R \) is either of \( A_K^r, \hat{B}_K \) or \( K((t))^\wedge \), and \( V H_{i_0 \cdots i_d} \) denotes \( V H_{i_0 \cdots i_d}(U_{K}/A_K)_R \) (see (4.7) for the notation). Then
\[
(i) \quad \Phi(H_{p^{-1}i_0 \cdots p^{-1}i_d}) \subset H_{i_0 \cdots i_d}, \\
(ii) \quad \Phi(H_{p^{-1}i_0 \cdots p^{-1}i_d}^\text{unit}) \subset H_{i_0 \cdots i_d}^\text{unit}, \\
(iii) \quad \Phi(V H_{p^{-1}i_0 \cdots p^{-1}i_d}) \subset V H_{i_0 \cdots i_d}.
\]

**Proof.** Since each \( H_{i_0 \cdots i_d} \) is an irreducible \( \mathcal{D} \)-module, there is unique \((j_0, \ldots, j_d) \in \prod_{k=0}^d \mathbb{Z}(n_k)\) with \( j_k \not\equiv 0 \mod n_k \) such that
\[
\Phi(H_{i_0 \cdots i_d}) \subset H_{j_0 \cdots j_d}.
\]
The assertion (i) is equivalent to that \( H_{j_0 \cdots j_d} = H_{pi_0 \cdots pi_d} \). To show this, we restrict the cohomology at a fiber \( t = a \) where \( a \in W \) such that \( a \not\equiv 0, 1 \mod p \). We take \( ca^p = F(a) \) so
that one has $\sigma(t)|_{t=a} = F(a)$. Let $U_a := U \times_A \text{Spec } A/(t-a)$ and $X_a := X \times_A \text{Spec } A/(t-a)$. Then $\Phi$ induces the Frobenius

$$\Phi_a(H_{\text{dR}}^d(U_a, K/K)(i_0, \ldots, i_d)) \subset H_{\text{dR}}^d(U_a, K/K)(j_0, \ldots, j_d).$$

which agrees with the Frobenius on $H_{\text{rig}}^d(U_a, \mathbb{F}/\mathbb{F})$ under the isomorphism $H_{\text{dR}}^d(U_a, K/K) \cong H_{\text{rig}}^d(U_a, \mathbb{F}/\mathbb{F})$ (Theorem 4.4). Now the assertion follows from the fact that $g\Phi_a = \Phi_ag$ on $H_{\text{rig}}^d(U_a, \mathbb{F}/\mathbb{F})$ for $g \in G$ (see §3.1 for the group $G$). This completes the proof of (i).

We show (ii) and (iii). Since $\text{Ker } D$ in (4.2) is stable under the action of $\Phi$, we have

$$\Phi(\eta_{p^{-1}i_0 \ldots p^{-1}i_d}) = C\eta_{i_0 \ldots i_d} \tag{4.13}$$

for some $C \in K^\times$ ($C$ is not zero by (Q5) on noticing that $Q$ is a perfect pairing). This implies

$$\Phi(\eta_{p^{-1}i_0 \ldots p^{-1}i_d}) = C\frac{F_\tilde{\alpha}(t)}{F_\alpha(t)}\eta_0 \ldots i_d = C\mathcal{F}_{\tilde{\alpha}}^\text{Dr,}\eta_{i_0 \ldots i_d} \tag{4.14}$$

where $\tilde{\alpha} = (1 - a_0, \ldots, 1 - a_d)$ and $\tilde{\alpha}' = (1 - a_0', \ldots, 1 - a_d')$ denotes the Dwork prime. Hence (ii) follows, and (iii) is immediate from (ii) and the definition of $V H_{i_0 \ldots i_d}$. \hfill $\square$

**Theorem 4.6** Let $(i_0, \ldots, i_d)$ satisfy $i_k \not\equiv 0$ mod $n_k$ for all $k$. Then

$$\Phi(\omega_{p^{-1}i_0 \ldots p^{-1}i_d}) \equiv p^d\mathcal{F}_{\tilde{\alpha}}^\text{Dr,}\omega_{i_0 \ldots i_d} \mod VH_{i_0 \ldots i_d}(U_K/A_K)_{B/K}$$

**Proof.** Put $\tilde{\omega}_{i_0 \ldots i_d} := F_\tilde{\alpha}(t)^{-1}\omega_{i_0 \ldots i_d}$. By Lemma 4.3 (3), $Q(\tilde{\omega}_{i_0 \ldots i_d}, \tilde{\eta}_{-i_0 \ldots -i_d})$ is a non-zero constant. By (4.13) together with (Q5), we have

$$\Phi(\tilde{\omega}_{p^{-1}i_0 \ldots p^{-1}i_d}) \equiv p^dC\tilde{\omega}_{i_0 \ldots i_d} \iff \Phi(\omega_{p^{-1}i_0 \ldots p^{-1}i_d}) \equiv p^dC'\mathcal{F}_{\tilde{\alpha}}^\text{Dr,}\omega_{i_0 \ldots i_d} \mod VH_{i_0 \ldots i_d}$$

with some $C' \in K^\times$. We show $C' = 1$. To do this, we employ the log crystalline cohomology. We recall from [Aș2 Lemma 2.2] a cartesian diagram

$$\begin{array}{ccc}
  X & \xrightarrow{f} & Y \\
  \downarrow & & \downarrow \\
  \text{Spec } A & \xrightarrow{c} & \text{Spec } W[t]
\end{array}$$

where $Y$ is smooth over $W$, and the scheme theoretic fiber $D = f^{-1}(O)$ with $O := \text{Spec } W[t]/(t)$ is reduced and relative simple NCD over $W$. Let $\mathcal{Y} := X \times_A W[[t]] \rightarrow \text{Spec } W[[t]]$. We endow the log-structures on $\mathcal{Y}$ and $\text{Spec } W[[t]]$ by the divisors $D$ and $O$ respectively, and let

$$H_i^{\text{log-crys}}(\mathcal{Y}/\text{Spec } W[[t]], O)$$

be the $i$-th log crystalline cohomology group endowed with the $p$-th Frobenius $\Phi_{(\mathcal{Y}, D)}$ ([Ka2]). The cohomology is described by the relative log de Rham complex. Let

$$\omega_{\mathcal{Y}} := \Omega_{\mathcal{Y}/W}(\log D), \quad \omega_{W[[t]]} := \Omega_{W[[t]]/W}(\log O)$$



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denote the log de Rham complex, and
\[ \omega^\bullet_{/[W[t]]} := \text{Coker} \left[ \frac{dt}{t} \otimes \omega^{\bullet-1}_{/[W]} \rightarrow \omega^\bullet_{/[W]} \right]. \]

Then there is the canonical isomorphism \( H^1_{\log,\text{crys}}((\mathcal{Y}, D)/(W[[t]], O)) \cong H^1_{\text{zar}}(\mathcal{Y}, \omega^\bullet_{/[W[[t]]]} \}

\text{(\cite{Ka2} Theorem 6.4)). We have the canonical homomorphism
\[ H^1_\text{rig}(X_{\mathbb{F}}/A_{\mathbb{F}}) \cong A_k^1 \otimes_A K H^1_{\text{zar}}(\mathcal{Y}, \omega^\bullet_{/[W[t]]}) \]

where \((-)^\wedge\) denote the \(p\)-adic completion, and \(\Phi_\mathcal{Y}(D)\) are compatible. For \(\nu = (\nu_0, \ldots, \nu_d) \in G\), let \(P_{\nu}\) denote the subscheme of \(D\) defined by \(\{x_0 - \nu_0 = \cdots = x_d - \nu_d = 0\}\). Let \(R\) be the composition
\[ \omega^\bullet_{/[W[t]]} \xrightarrow{\mathcal{H}^\bullet + 1} \Omega^\bullet_{/[W]}(\log D) \xrightarrow{\text{Res}} \bigoplus_{\nu \in G} \mathcal{O}_{P_{\nu}}[-d] \]

of the complexes where \(\text{Res}\) is the Poincare residue map. This induces
\[ H^d_{\log,\text{crys}}((\mathcal{Y}, D)/(W[[t]], O)) \xrightarrow{R} \bigoplus_{\nu \in G} W \cdot P_{\nu}, \]

and it satisfies \(R \circ \Phi_\mathcal{Y}(D) = p^d F \circ R\). We claim that the submodules \(tH^d_{\text{zar}}(\mathcal{Y}, \omega^\bullet_{/[W[[t]]]} \}

\text{and } DH^d_{\text{zar}}(\mathcal{Y}, \omega^\bullet_{/[W[[t]]]} \}

\text{is annihilated by } R \text{ where } D := t \frac{dt}{dt}. \text{ The former is clear from the definition of the Poincare residue. To see the latter, we recall that } D \text{ is defined to be the connecting homomorphism arising from}
\[ 0 \rightarrow \frac{dt}{t} \otimes \omega^{\bullet-1}_{/[W[t]]} \rightarrow \omega^\bullet_{/[W]} \rightarrow \omega^\bullet_{/[W[[t]]]} \rightarrow 0. \]

Then a commutative diagram
\[ H^d_{\text{zar}}(\mathcal{Y}, \omega^\bullet_{/[W[[t]]]} \) \xrightarrow{D} H^d_{\text{zar}}(\mathcal{Y}, \omega^\bullet_{/[W[t]]}) \xrightarrow{\text{Res}} \bigoplus_{\nu \in G} W \cdot P_{\nu} \]

implies \(R \circ D = 0\).

We turn to the proof of \(C'\). Since \(D \widehat{\eta}_{i_0, \ldots, i_d} = 0\), we have \(Q(D^i \widehat{\omega}_{i_0, \ldots, i_d}, \widehat{\eta}_{i_0, \ldots, i_d}) = D^i Q(\omega_{i_0, \ldots, i_d}, \eta_{i_0, \ldots, i_d}) = 0\) by Lemma 4.3 (3). Therefore the submodule
\[ VH_{i_0, \ldots, i_d}(U_K/A_K)K((t))^\wedge \subset \bigoplus_{i=0}^d K((t))^\wedge \cdot D^i \widehat{\omega}_{i_0, \ldots, i_d} \]
is generated by \( \{ D^j \tilde{\omega}_{i_0 \ldots i_d} \}_{j > 0} \), and hence is annihilated by \( R \). To show \( C' = 1 \), it is enough to show that

\[
R \circ \Phi_{(\not\!y, D)}(\omega_{p^{-1}i_0 \ldots p^{-1}i_d}) = p^d R(\omega_{i_0 \ldots i_d}).
\]

Let \( j_k \) be the unique integer such that \( j_k \equiv p^{-1} i_k \mod n_k \) and \( 0 < j_k < n_k \). We have

\[
R(\omega_{p^{-1}i_0 \ldots p^{-1}i_d}) = \text{Res} \left( n_0^{-1} x_0^{n_0} x_1^{j_1} \ldots x_d^{j_d - 1} \frac{dx_1 \cdots dx_d}{(1 - x_1^{n_1}) \cdots (1 - x_d^{n_d})} \right)
\]

\[
= \frac{(-1)^d}{n_0 \cdots n_d} \left( \sum_{\nu \in G} \nu_0^{j_0} \cdots \nu_d^{j_d} P_{\nu} \right).
\]

Since \( R \circ \Phi_{(\not\!y, D)} = p^d F \circ R \), we have

\[
R(\omega_{p^{-1}i_0 \ldots p^{-1}i_d}) = p^d F(\omega_{p^{-1}i_0 \ldots p^{-1}i_d}) = p^d \frac{(-1)^d}{n_0 \cdots n_d} \left( \sum_{\nu \in G} \nu_0^{j_0} \cdots \nu_d^{j_d} P_{\nu} \right)
\]

\[
= p^d \left[ \frac{(-1)^d}{n_0 \cdots n_d} \left( \sum_{\nu \in G} \nu_0^{j_0} \cdots \nu_d^{j_d} P_{\nu} \right) \right]
\]

\[
= p^d R(\omega_{i_0 \ldots i_d})
\]

as required. \( \square \)

**Corollary 4.7 (Unit Root Formula)**

\[
\Phi(\eta_{p^{-1}i_0 \ldots p^{-1}i_d}) = \mathcal{F}^{Dw, \sigma}(t) \eta_{i_0 \ldots i_d}.
\]

**Proof.** Recall (4.14),

\[
\Phi(\eta_{p^{-1}i_0 \ldots p^{-1}i_d}) = C \mathcal{F}^{Dw, \sigma}(t) \eta_{i_0 \ldots i_d}.
\]

We want to show \( C = 1 \). By (Q5), we have

\[
Q(\Phi(\omega_{p^{-1}i_0 \ldots p^{-1}i_d}), \Phi(\eta_{p^{-1}i_0 \ldots p^{-1}i_d})) = p^d \sigma Q(\omega_{i_0 \ldots i_d}, \eta_{i_0 \ldots i_d}) = p^d Q(\omega_{i_0 \ldots i_d}, \eta_{i_0 \ldots i_d}),
\]

where the second equality follows from the fact \( Q(\omega_{i_0 \ldots i_d}, \eta_{i_0 \ldots i_d}) \in \mathbb{Q}^\times \) (Lemma 4.3 (3)). Applying Theorem 4.6 and (4.14) to this, we conclude \( C = 1 \). \( \square \)

**Corollary 4.8**  \( Q(\eta_{i_0 \ldots i_d}, \eta_{i_0 \ldots i_d}) = 0 \). In other words, \( \eta_{i_0 \ldots i_d} \in VH_{i_0 \ldots i_d}(U_K / A_K) \hat{B}_K^* \).

**Proof.** Applying Corollary 4.7 to the equality

\[
Q(\Phi(\eta_{p^{-1}i_0 \ldots p^{-1}i_d}), \Phi(\eta_{p^{-1}i_0 \ldots p^{-1}i_d})) = p^d \sigma Q(\eta_{i_0 \ldots i_d}, \eta_{i_0 \ldots i_d})
\]

in (Q5), one has

\[
\mathcal{F}^{Dw, \sigma}(t) \mathcal{F}^{Dw, \sigma}(t) Q(\eta_{i_0 \ldots i_d}, \eta_{i_0 \ldots i_d}) = p^d \sigma Q(\eta_{i_0 \ldots i_d}, \eta_{i_0 \ldots i_d}).
\]

Comparing the sup norm on both sides, it turns out that \( Q(\eta_{i_0 \ldots i_d}, \eta_{i_0 \ldots i_d}) = 0 \). \( \square \)

**Remark 4.9** There is an alternative proof of Corollary 4.8 with use of the Hodge theory (monodromy weight filtration).
5 Higher Ross symbols and Syntomic regulators

In this section we shall discuss the syntomic regulator of the higher Ross symbols. The main results are Theorems 5.3 and 5.5 which are the $p$-adic counterparts of [As2] Theorem 5.5, and also a generalization of the results in [As1] §4.4 in higher dimension.

5.1 Higher Ross symbols [As2]

Let $n_0, \ldots, n_d > 1$ be integers. Let $p$ be a prime number such that $p \nmid n_0 \cdots n_d$. Let $W = W(\overline{\mathbb{F}_p})$ be the Witt ring, and $F$ the $p$-th Frobenius on $W$. Let $A = W[t, (t - t^2)^{-1}]$, $S = \text{Spec } A$ and

$$U = \text{Spec } A[x_0, \ldots, x_d] / ((1 - x_0^{n_0}) \cdots (1 - x_d^{n_d}) - t)$$

(5.1)

the hypergeometric scheme over $A$. Then, for $\nu_k \in \mu_{n_k} (A)$ ($k = 0, 1, \ldots, d$), the higher Ross symbol is defined to be a Milnor symbol

$$\xi_{\text{Ross}} := \left\{ \begin{array}{c}
1 - x_0 \\
1 - \nu_0 x_0
\end{array}, \cdots, \begin{array}{c}
1 - x_d \\
1 - \nu_d x_d
\end{array} \right\} \in K_{d+1}^M(\mathcal{O}(U)),$$

(5.2)

in the Milnor $K$-group of $\mathcal{O}(U)$. We also think it of being an element of Quillen’s higher $K$-group $K_{d+1}(U)$ by the natural map $K_i^M(\mathcal{O}(U)) \rightarrow K_i(U)$, and the element is denoted by the same notation. Let $X \supset U$ be a smooth compactification $X \supset U$ such that $Z = X \setminus U$ is a relative simple NCD over $A$ ([As2 Proposition 2.1]). Then we expect

$$\xi_{\text{Ross}} \in \text{Im}[K_{d+1}(X)^{(d+1)} \rightarrow K_{d+1}(U)^{(d+1)}].$$

(5.3)

See [As2 4.2] for more details. This is true if $d \leq 2$ ([As2 Corollary 4.4]).

5.2 Category of filtered $F$-isocrystals [AM]

In §5.3 we shall employ the category of filtered $F$-isocrystals introduced in [AM] §2.1 as a fundamental material. We here recall the notation and some results which we shall need in below. For a moment, we work over an arbitrary smooth affine variety $S = \text{Spec}(B)$ over $W = W(\overline{\mathbb{F}_p})$. We denote by $B^\dagger$ the weak completion of $B$. Namely if $B = W[T_1, \ldots, T_n]/I$, then $B^\dagger = W[T_1, \ldots, T_n]^{\dagger}/I$ where $W[T_1, \ldots, T_n]^{\dagger}$ is the ring of power series $\sum a_{\alpha} T^\alpha$ such that for some $r > 1$, $|a_{\alpha}| r^{|\alpha|} \rightarrow 0$ as $|\alpha| \rightarrow \infty$. Let $K := \text{Frac}(W)$ be the fractional field, and write $B_K^\dagger = K \otimes_W B^\dagger$

Let $\sigma : B^\dagger \rightarrow B^\dagger$ be a $p$-th Frobenius compatible with the Frobenius $F$ on $W$. We define the category $\text{Fil-F-MIC}(S, \sigma)$ (which we call the category of filtered $F$-isocrystals on $S$) as follows. The induced endomorphism $\sigma \otimes_{\mathbb{Z}} \mathbb{Q} : B_K^\dagger \rightarrow B_K^\dagger$ is also denoted by $\sigma$. An object of $\text{Fil-F-MIC}(S, \sigma)$ is a datum $H = (H_{\text{dR}}, H_{\text{rig}}, c, \Phi, \nabla, \text{Fil}^*)$, where

- $H_{\text{dR}}$ is a coherent $B_K^\dagger$-module,
- $H_{\text{rig}}$ is a coherent $B_K^\dagger$-module,
• \( c : H_{\text{dR}} \otimes_{B_K} B_K^+ \overset{\cong}{\to} H_{\text{rig}} \) is a \( B_K^+ \)-linear isomorphism,

• \( \Phi : \sigma^* H_{\text{rig}} \overset{\cong}{\to} H_{\text{rig}} \) is an isomorphism of \( B_K^+ \)-algebra,

• \( \nabla : H_{\text{dR}} \to \Omega^1_{B_K} \otimes H_{\text{dR}} \) is an integrable connection and

• \( \text{Fil}^i \) is a finite descending filtration on \( H_{\text{dR}} \) of locally free \( B_K \)-module (i.e. each graded piece is locally free),

that satisfies \( \nabla(\text{Fil}^i) \subset \Omega^1_{B_K} \otimes \text{Fil}^{i-1} \) and the compatibility of \( \Phi \) and \( \nabla \), namely \( \Phi \nabla_{\text{rig}} = \nabla_{\text{rig}} \Phi \) where \( \nabla_{\text{rig}} : H_{\text{rig}} \to \Omega^1_{B_K} \otimes H_{\text{rig}} \) is the connection induced from \( \nabla \) under the comparison map \( c \). In what follows we write \( \nabla_{\text{rig}} = \nabla \) for simplicity of notation.

The category \( \text{Fil}^\bullet \text{-MIC}(S, \sigma) \) is an exact category (not an abelian category) in which the tensor product \( \otimes \) is defined in the customary way. There are the Tate objects \( \mathcal{O}_S(n) = (B_K, B_K^+, c, p^{-n} \sigma_B, d, \text{Fil}^\bullet) \), which is the counterpart of the \( l \)-adic sheaf \( \mathbb{Q}_l(n) \). We abbreviate \( \mathcal{O}_S = \mathcal{O}_S(0) \). We write \( H(n) := H \otimes \mathcal{O}_S(n) \) for an object \( H \in \text{Fil}^\bullet \text{-MIC}(S, \sigma) \).

Let \( u : U \to S = \text{Spec}(B) \) be a smooth morphism of smooth \( W \)-schemes of pure relative dimension. Assume that there is a projective smooth morphism \( u_X : X \to S \) that extends \( u \), and \( D := X \setminus U \) is a normal crossing divisor with \( S \)-smooth components (abbreviated relative simple NCD over \( S \)). Then the rigid cohomology \( H^i_{\text{rig}}(U_{\bar{\mathbb{F}}_p}/S_{\bar{\mathbb{F}}_p}) \) is defined. Let

\[
    c : B_K^+ \otimes H^i_{\text{dR}}(U_K/S_K) \longrightarrow H^i_{\text{rig}}(U_{\bar{\mathbb{F}}_p}/S_{\bar{\mathbb{F}}_p}). \tag{5.4}
\]

be the comparison map (e.g. [AM 2.5]). If this is bijective, then one can define an object

\[
    H^i(U/S) = (H^i_{\text{dR}}(U_K/S_K), H^i_{\text{rig}}(U_{\bar{\mathbb{F}}_p}/S_{\bar{\mathbb{F}}_p}), c, \nabla, \Phi, \text{Fil}^\bullet)
\]

of \( \text{Fil}^\bullet \text{-MIC}(S, \sigma) \), where \( \nabla \) is the Gauss-Manin connection, \( \Phi \) is the \( \sigma \)-linear \( p \)-th Frobenius on the rigid cohomology and \( \text{Fil}^\bullet H^i_{\text{dR}}(U_K/S_K) \) is the Hodge filtration.

Suppose that \( U \) is affine and the \( i \)-th relative rigid cohomology sheaf \( R^if_{\text{rig}}j_U^! \mathcal{O}_{U_K} \) is a coherent \( j^*_S \mathcal{O}_{S_K} \)-module for each \( i \). Then the comparison map (5.4) is bijective for all \( i \). Let \( n \geq 0 \) be an integer. If \( \text{Fil}^{n+1}H^{n+1}_{\text{dR}}(U_K/S_K) = 0 \), then it follows from [AM Theorem 2.23] that we have the symbol map

\[
    [-]_{U/S} : \mathbb{K}^M_{n+1}(\mathcal{O}(U)) \longrightarrow \text{Ext}^1_{\text{Fil}^\bullet \text{-MIC}(S, \sigma)}(\mathcal{O}_S, H^n(U/S)(n + 1)) \tag{5.5}
\]

to the group of 1-extensions in \( \text{Fil}^\bullet \text{-MIC}(S, \sigma) \).

### 5.3 Syntomic regulators of Higher Ross symbols

We turn to the setting in §5.1.

**Lemma 5.1** Suppose \( p > \text{lc}(n_0, \ldots, n_d) \). Then, for the hypergeometric scheme \( U \) in (5.1), \( R^if_{\text{rig}}j_U^! \mathcal{O}_{U_K} \) is a coherent \( j^*_S \mathcal{O}_{S_K} \)-module for each \( i \), and therefore the symbol map (5.5) is defined.
Proof. Let \( n := \text{lcm}(n_0, \ldots, n_d) \), and let \( U' \) be the hypergeometric scheme defined by 
\[
(1 - x_0^n_0) \cdots (1 - x_d^n_d) = t.
\]
Then there is a finite etale covering \( U' \to U \), so that the proof is reduced to the case of \( U' \) (e.g. \([CT\), 7.4.1\]). Therefore, we may assume \( n_0 = \cdots = n_d \).

We construct a diagram in Theorem 4.4 (2) for \( U/A \). It follows from \([As2\), Prop. 2.1, Lemmas 2.2, 2.3\] that there is a cartesian diagram

\[
\begin{array}{ccc}
U & \xrightarrow{c} & X \\
\downarrow & & \downarrow f \\
\text{Spec } A & \xrightarrow{c} & \mathbb{P}_W
\end{array}
\]

such that the following conditions hold. Let \( D_0, D_1, D_\infty \) denote the scheme-theoretic fiber of \( f \) at \( t = 0, t = 1, t = \infty \) respectively.

1. \( Y \) is smooth projective over \( W \).
2. \( D_0 \) is a reduced and relative simple NCD over \( W \).
3. \( D_\infty \) is a relative simple NCD over \( W \) with multiplicity \( \leq n \).
4. \( D_1 \) is a reduced and irreducible divisor which is smooth over \( W \) outside the point \( O := \{x_0 = \cdots = x_d = 0\} \).
5. Let \( \bar{Z} \) be the closure of \( Z := X \setminus U \) in \( Y \). Then \( D_0 + (D_1 \setminus O) + D_\infty + \bar{Z} \) is a relative simple NCD over \( W \).

Let \( \bar{X} \to Y \) be the blowing-up at \( O \). Then the inverse image of \( D_1 \) has two irreducible components, and they are a relative simple NCD over \( W \) (here we use the assumption \( n_0 = \cdots = n_d \)). Therefore the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{c} & X \\
\downarrow & & \downarrow \bar{f} \\
\text{Spec } A & \xrightarrow{c} & \mathbb{P}_W
\end{array}
\]

satisfies all conditions in Theorem 4.4 (2), and hence the bijectivity of (5.4) follows. This completes the proof. \( \square \)

By Lemma 5.1, one has the object \( H^i(U/S)(r) \) in Fil-F-MIC(S, \( \sigma \)), and the higher Ross symbol \( \xi_{\text{Ross}} \) defines a 1-extension

\[
0 \to H^d(U/S)(d + 1) \to M_{\xi_{\text{Ross}}}(U/S) \to \mathcal{O}_S \to 0 \tag{5.6}
\]

in Fil-F-MIC(S, \( \sigma \)) by the symbol map (5.5). In what follows we take \( \sigma \) to be the \( p \)-th Frobenius on \( W[[t]] \) given by \( \sigma(t) = ct^p \) with some \( c \in 1 + pW \). Let \( \Phi \) be the Frobenius on
$M_{ξ_{Ross}}(U_p/T_p)_{\rig} \cong M_{ξ_{Ross}}(U_K/S_K)_{\dR} \otimes_{A_K} A_K^{\dagger}$ that is $σ$-linear. Let $Φ_{U/S}$ be the Frobenius on $H^d(U_p/T_p)_{\rig}$ (without Tate twist). Notice that

$$Φ_{|H^d(U_p/T_p)_{\rig}} = p^{-d-1}Φ_{U/S}$$

by the definition of the Tate twist. Let $e_{ξ_{Ross}} \in \Fil^0 M_{ξ}(U/S)_{\dR}$ be the unique lifting of $1 \in O(S)$. Then one has a class $e_{ξ_{Ross}} - Φ(e_{ξ_{Ross}}) \in H^d_{\dR}(U/S) \otimes_A A_K^{\dagger}$.

**Lemma 5.2** $e_{ξ_{Ross}} - Φ(e_{ξ_{Ross}}) \in W_d H^d_{\dR}(U/S) \otimes A K^{\dagger}$.

**Proof.** Let $σ_i(ε_i) = (1, \ldots, ε_i, \ldots, 1) \in G = μ_{n_0} \times \cdots \times μ_{n_d}$ for a fixed primitive $n_i$-th root of unity $ε_i \in μ_{n_i}$. Put

$$h := \prod_{i=0}^d (1 + σ_i(ε_i) + \cdots + σ_i(ε_i)^{n_i-1}) \in \mathbb{Q}[G].$$

Then since

$$W_d H^d_{\dR}(U/S) = \text{Ker}(h : H^d_{\dR}(U/S) \to H^d_{\dR}(U/S))$$

by Theorem 3.2(3), it is enough to show $h(e_{ξ_{Ross}} - Φ(e_{ξ_{Ross}})) = 0$. However since the symbol map (5.5) is compatible with respect to the action of $\mathbb{Q}[G]$ and $Φ$ commutes with $G$, this agrees with $h(ξ_{Ross}) - Φ(h(ξ_{Ross}))$. One can directly show $h(ξ_{Ross}) = 0$ in $K^M_{d+1}(O(U)) \otimes \mathbb{Q}$ by definition, and hence the vanishing follows. \qed

**Theorem 5.3** Suppose $p > \text{lcm}(n_0, \ldots, n_d)$. Then

$$e_{ξ_{Ross}} - Φ(e_{ξ_{Ross}}) \equiv \sum_{i_0=1}^{n_0-1} \cdots \sum_{i_d=1}^{n_d-1} (1 - ν_0^{i_0}) \cdots (1 - ν_d^{i_d}) \mathcal{F}_a^{(σ)}(t) \omega_{i_0, \ldots, i_d} \mod \bigoplus_{i_0, \ldots, i_d} VH_{i_0, \ldots, i_d}(U/S)_{\rig}$$

where we put $a_k := 1 - i_k/n_k$ and $a := (a_0, \ldots, a_d)$.

**Proof.** One directly has (cf. [AS2] Lemma 4.1)

$$d\log(ξ_{Ross}) = (-1)^d \sum_{i_0=1}^{n_0-1} \cdots \sum_{i_d=1}^{n_d-1} (1 - ν_0^{i_0}) \cdots (1 - ν_d^{i_d}) \omega_{i_0, \ldots, i_d} \frac{dt}{t}.$$

Let $D := t^d \overset{d}{\partial}/\partial t$ be the differential operator acting on $M_{ξ}(U/S)_{\dR}$. It follows from [AM] (2.30) that we have

$$D(e_{ξ_{Ross}}) = \sum_{i_0=1}^{n_0-1} \cdots \sum_{i_d=1}^{n_d-1} (1 - ν_0^{i_0}) \cdots (1 - ν_d^{i_d}) \omega_{i_0, \ldots, i_d}.$$

Let us write

$$e_{ξ_{Ross}} - Φ(e_{ξ_{Ross}}) \equiv \sum_{i_0=1}^{n_0-1} \cdots \sum_{i_d=1}^{n_d-1} E_{i_0, \ldots, i_d}(t) \tilde{ω}_{i_0, \ldots, i_d} \mod VH_{i_0, \ldots, i_d}(U/S)_{\rig}$$

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where \( \tilde{\omega}_{i_0 \ldots i_d} := F_{\bar{a}}(t)^{-1}\omega_{i_0 \ldots i_d} \). Then

\[
D(e_{\xi_{\text{Ross}}} - \Phi(e_{\xi_{\text{Ross}}})) = D(e_{\xi_{\text{Ross}}}) - p\Phi D(e_{\xi_{\text{Ross}}})
\]

\[
= \sum_{0 < i_k < n_k} (1 - \nu_0^{i_k}) \cdots (1 - \nu_d^{i_k}) \left( \omega_{i_0 \ldots i_d} - p^{-d}\Phi_{U/S}(\omega_{p^{-1}i_0 \ldots p^{-1}i_d}) \right)
\]

\[
\equiv \sum_{0 < i_k < n_k} (1 - \nu_0^{i_k}) \cdots (1 - \nu_d^{i_k}) (1 - \mathcal{F}_{W,\sigma}(t)^{-1}) \omega_{i_0 \ldots i_d}
\]

\[
\mod \bigoplus VH_{i_0 \ldots i_d}(U/S)\tilde{B}_K \quad \text{(by Theorem 4.6)}
\]

\[
= \sum_{0 < i_k < n_k} (1 - \nu_0^{i_k}) \cdots (1 - \nu_d^{i_k}) (F_{\bar{a}}(t) - F_{\bar{a}}(t')) \tilde{\omega}_{i_0 \ldots i_d}
\]

where \( \bar{a}' = (a'_0, \ldots, a'_d) \) is the Dwork prime. Hence we have

\[
\sum_{0 < i_k < n_k} \left( \frac{t^d dt}{d} E_{i_0 \ldots i_d}(t) \right) \tilde{\omega}_{i_0 \ldots i_d} = \sum_{0 < i_k < n_k} (1 - \nu_0^{i_k}) \cdots (1 - \nu_d^{i_k}) (F_{\bar{a}}(t) - F_{\bar{a}}(t')) \tilde{\omega}_{i_0 \ldots i_d}
\]

as \( D\tilde{\omega}_{i_0 \ldots i_d} \equiv 0 \mod VH_{i_0 \ldots i_d}(U/S)\tilde{B}_K \). This implies

\[
E_{i_0 \ldots i_d}(t) = C + \int_0^t F_{\bar{a}}(t) - F_{\bar{a}}(t') \frac{dt}{t}
\]

with some constant \( C \). The rest is to show \( C = \psi_p(a_0) + \cdots + \psi_p(a_d) + (d+1)\gamma_p - p^{-1}\log(c) \).

Notice that \( E_{i_0 \ldots i_d}(t) \) satisfies that \( E_{i_0 \ldots i_d}(t)/F_{\bar{a}}(t) \) belongs to the ring \( \tilde{B}_K \) (Lemma 4.2). If \( C' = \psi_p(a_0) + \cdots + \psi_p(a_d) + (d+1)\gamma_p - p^{-1}\log(c) \), then this is satisfied (Theorem 2.2).

Suppose that there is another \( C' \) such that \( E_{i_0 \ldots i_d}(t)/F_{\bar{a}}(t) \) belongs to the ring \( \tilde{B}_K \). Then it follows that so does \( (C - C')/F_{\bar{a}}(t) \) and hence \( F_{\bar{a}}(t) \in \tilde{B}_K \). We show that this is impossible, which completes the proof of Theorem 5.3. Let \( \sigma_1 \) be the Frobenius given by \( t^p = t' \), and \( \Phi_{U/S,\sigma_1} \) the \( \sigma_1 \)-linear Frobenius on \( H_{i_0}^*(U_1^S/S^r) \). Let \( m > 0 \) be an integer such that \( p^m \equiv 1 \mod n_k \) for all \( k \). Note \( \bar{a}'(m) = a \). It follows from the unit root formula (Corollary 4.7) that we have

\[
(\Phi_{U/S,\sigma_1})^m(\eta_{-i_0 \ldots -i_d}) = (\Phi_{U/S,\sigma})^m(\eta_{-p^{-m}i_0 \ldots -p^{-m}i_d})
\]

\[
= \prod_{i=0}^{m-1} \mathcal{F}_{\bar{a}_{Dw,\sigma_1}}(t^{p^{m-i-1}}) \eta_{-i_0 \ldots -i_d}
\]

\[
= \prod_{i=0}^{m-1} \frac{F_{\bar{a}}(t^{p^{m-i-1}})}{F_{\bar{a}}(t^{p^{m-1}})} \eta_{-i_0 \ldots -i_d}
\]

Recall \( \tilde{B}_K = K \otimes W(W[t, (t - t^2)^{-1}, h(t)^{-1}]) \) (see 4.5 for the notation). Choose a \( \ell \)-th root \( \zeta \in W^\times \) of unity with \( p \nmid \ell \) such that \( (1 - \zeta)h(\zeta) \equiv 0 \mod p \). Let \( U_{S,\zeta} :=
\]
\( U_F \times \mathcal{S}_F \text{ Spec } \mathbb{F}[t]/(t - \zeta) \) be the fiber at \( t = \zeta \). Replacing \( m \) with \( m\varphi(\ell) \), we may assume \( \ell | p^m - 1 \) and hence \( \zeta p^m = \zeta \). Then the above formula implies that the evaluation

\[
\left. \frac{F_{\mathcal{A}}(t)}{F_{\mathcal{A}}(p^m)} \right|_{t=\zeta}
\]

is the eigenvalue of the \( p^m \)-th Frobenius on the rigid cohomology \( W_d H^d_{\text{rig}}(U_F, \zeta/\mathbb{F}) \) with respect to the unit root vector \( \eta_{-i_0, \ldots, -i_d} \). Now suppose \( F_{\mathcal{A}}(t) \in \hat{B}_K \). Then

\[
\left. \frac{F_{\mathcal{A}}(t)}{F_{\mathcal{A}}(p^m)} \right|_{t=\zeta} = \left. \frac{F_{\mathcal{A}}(t)}{F_{\mathcal{A}}(p^m)} \right|_{t=\zeta} = 1
\]

which contradicts with the Riemann-Weil hypothesis. This shows \( F_{\mathcal{A}}(t) \not\in \hat{B}_K \) as required.

\( \square \)

**Remark 5.4** The main result of [As2] is an explicit formula of the pairing

\[
\langle \text{reg}_B(\xi_{\text{Ross}}) \mid \Delta_\ell \rangle
\]

of the Beilinson regulator \( \text{reg}_B(\xi_{\text{Ross}}) \) with a certain homology cycle \( \Delta_\ell \). See loc. cit. Theorem 5.5 for the details. One can think

\[
Q(\text{reg}_{\text{syn}}(\xi_{\text{Ross}}), \hat{\eta}_{i_0, \ldots, i_d})
\]

of being a \( p \)-adic counterpart of \( \langle \text{reg}_B(\xi_{\text{Ross}}) \mid \Delta_\ell \rangle \) in the following way. Let \( F_{\mathcal{A}}^\text{an}(t) \) denote the complex analytic function defined by the hypergeometric series \( F_{\mathcal{A}}(t) \). Let \( \hat{\eta}_{i_0, \ldots, i_d} \) be the analytic functions defined in the same way as in Proposition 4.1, and put

\[
\hat{\eta}_{i_0, \ldots, i_d}^\text{an} := y_{i_0} D \omega_{i_0, \ldots, i_d} + \cdots + y_{i_d} D^d \omega_{i_0, \ldots, i_d},
\]

and \( \hat{\eta}_{i_0, \ldots, i_d}^\text{an} := F_{\mathcal{A}}^\text{an}(t)^{-1} \hat{\eta}_{i_0, \ldots, i_d}^\text{an} \). Proposition 4.1 asserts \( D \hat{\eta}_{i_0, \ldots, i_d}^\text{an} = 0 \). It follows from Lemma 4.3 (3) that \( Q(\omega_{i_0, \ldots, i_d}, \hat{\eta}_{i_0, \ldots, i_d}^\text{an}) \) is a constant and hence

\[
Q(D^j \omega_{i_0, \ldots, i_d}, \hat{\eta}_{i_0, \ldots, i_d}^\text{an}) = D^j Q(\omega_{i_0, \ldots, i_d}, \hat{\eta}_{i_0, \ldots, i_d}^\text{an}) = D^j F_{\mathcal{A}}^\text{an}(t) \times (\text{const.})
\]

for all \( j \geq 0 \). On the other hand, the homology cycle \( \Delta_\alpha \) satisfies ([As2 Thm.3.1])

\[
\langle \omega_{i_0, \ldots, i_d} \mid \Delta_\ell \rangle = F_{\mathcal{A}}^\text{an}(t) \times (\text{const.})
\]

which implies

\[
\langle D^j \omega_{i_0, \ldots, i_d} \mid \Delta_\ell \rangle = D^j F_{\mathcal{A}}^\text{an}(t) \times (\text{const.})
\]

for all \( j \geq 0 \). We thus have

\[
Q(-, \hat{\eta}_{i_0, \ldots, i_d}^\text{an}) = \langle - \mid \Delta_\ell \rangle \times (\text{const.})
\]

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**Theorem 5.5** Suppose that \( d = 1 \) and \( p \nmid n_0 \cdots n_d \) or that \( p > \text{lcm}(n_0, \ldots, n_d) \). Suppose further that \( p > d + 1 \). Let \( \alpha \in W \) satisfy \( \alpha \not\equiv 0, 1 \) and \( h(\alpha) \not\equiv 0 \) modulo \( p \), so that \( (t - \alpha) \) is a maximal ideal of \( \widehat{\mathcal{B}_K} \). Let \( \sigma \) be the \( p \)-th Frobenius on \( W[[t]] \) given by \( \sigma(t) = \alpha F_\alpha^{-ptp} \) where \( F \) is the \( p \)-th Frobenius on \( W \). Let

\[
\text{reg}_{\text{syn}} : K_{d+1}(U_\alpha) \rightarrow H_{\text{syn}}^{d+1}(U_\alpha, \mathbb{Q}_p(d+1)) \cong H_{dR}^d(U_{\alpha, K}/K)
\]

be the syntomic regulator map, cf. (1.3). Then \( \text{reg}_{\text{syn}}(\xi_{\text{Ross}}|_{U_\alpha}) \in W_d H_{dR}^d(U_{\alpha, K}/K) \) and

\[
\frac{Q(\text{reg}_{\text{syn}}(\xi_{\text{Ross}}|_{U_\alpha}), \eta_{-i_0, \ldots, -i_d})}{Q(\omega_{i_0, \ldots, i_d}, \eta_{-i_0, \ldots, -i_d})} = (1 - \nu_{i_0}^0) \cdots (1 - \nu_{i_d}^d) \mathcal{F}_\alpha^{(\sigma)}(t)|_{t=\alpha}.
\]

**Proof.** By [AM, Theorem 3.7], this is immediate from Theorem 5.3 together with Lemma 5.2. \( \square \)

### 6 \( p \)-adic Beilinson conjecture for \( K_3 \) of K3 surfaces

We discuss the syntomic regulators for singular K3 surfaces over \( \mathbb{Q} \). Here we mean by singular K3 a (smooth) K3 surface \( X \) over a field \( k \) such that the Picard number of \( X \) is maximal. For a singular K3 surface \( X \) over \( \mathbb{Q} \), there is a Hecke eigenform \( f \) which provides the \( L \)-function of \( X \). One can formulate the \( p \)-adic Beilinson conjecture using the \( p \)-adic \( L \)-function of \( f \).

For a smooth projective variety \( S \) over a field \( k \), we denote by \( \text{NS}(S) \) the Neron-Severi group. Note \( \text{NS}(S) \otimes \mathbb{Q} = (\text{NS}(S \times_k \overline{K} ) \otimes \mathbb{Q})^{\text{Gal}(\overline{K}/k)} \) the fixed part by the Galois group. Let \( h^2_{\text{tr}}(S, \mathbb{Q}(m)) = h^2(S, \mathbb{Q}(m))/\text{NS}(S) \otimes \mathbb{Q}(m-1) \) denote the transcendental part of the motive \( h^2(S, \mathbb{Q}(m)) \) (cf. [KMP, 7.2.2]). When \( k = \mathbb{Q} \), one can define the \( L \)-function of \( h^2_{\text{tr}}(S, \mathbb{Q}) \), which we denote by \( L(h^2_{\text{tr}}(S), s) \).

#### 6.1 Singular K3 surfaces over \( \mathbb{Q} \)

A K3 surface \( X \) over a field \( k \) is called singular if the rank of \( \text{NS}(X \times_k \overline{k}) \) is the largest. We work over the base field \( k = \mathbb{Q} \), then \( X \) is singular if and only if the rank is 20, and hence the transcendental motive \( h^2_{\text{tr}}(X) = h^2_{\text{tr}}(X, \mathbb{Q}) \) is 2-dimensional. Every such K3 admits the Shioda-Inose structure by elliptic curves \( E, E' \) with complex multiplications ([I-S], [Mo]),

\[
\begin{array}{ccc}
X & \xrightarrow{p} & E \times E' \\
\downarrow{p_\Sigma} & & \downarrow{p_{E \times E'}} \\
Z & & \\
\end{array}
\]

in which the arrows are rational dominant maps of degree 2, and \( Z \) is a K3 surface and the base field is \( \mathbb{Q} \). Moreover \( E \) and \( E' \) are isogenous. Thus the \( L \)-function \( L(h^2_{\text{tr}}(X), s) \) is essentially the \( L \)-function of the symmetric square of the elliptic curves. Although the Shioda-Inose structure usually requires the base field extension, the \( L \)-function has the descent to \( \mathbb{Q} \).
Theorem 6.1 (Livné, [Li]) Let $X$ be a singular K3 surface over $\mathbb{Q}$. Then there is a Hecke eigenform $f$ of weight 3 with complex multiplication such that $L(h_{\text{dR}}^2(X), s) = L(f, s)$.

The $p$-adic Beilinson conjecture asserts that the special values of $p$-adic $L$-functions are described by syntomic regulators. See [Pl 4.2.2] and also [Co, Conj.2.7]. Concerning a singular K3 surface, one can take the $p$-adic $L$-function to be the $p$-adic $L$-function

$$L_p(f, \chi, s)$$

of the associated Hecke eigenform $f$ and some Dirichlet character $\chi : \mathbb{Z}_p^\times \to \overline{\mathbb{Q}}_p^\times$ ([MTT]). See [AC, Conj.3.1] for the general statement of the $p$-adic Beilinson conjecture for a modular form. In case of singular K3 surfaces, one can write down (a part of) the conjecture as follows.

Let $X$ be a K3 surface over $\mathbb{Q}$. Let $p$ be a prime at which $X$ has a good reduction. Let $X_{\mathbb{Q}_p}$ be the smooth model over $\mathbb{Z}_p$ and put $X_{\mathbb{Q}_p} := X \times_{\mathbb{Q}} \mathbb{Q}_p$ and $X_{\mathbb{F}_p} := X_{\mathbb{Z}_p} \times_{\mathbb{Z}_p} \mathbb{F}_p$. Let

$$\text{reg}_{\text{syn}} : K_3(X_{\mathbb{Z}_p}) \to H^3_{\text{syn}}(X_{\mathbb{Z}_p}, \mathbb{Q}_p(3)) \cong H^2_{\text{dR}}(X_{\mathbb{Q}_p}/\mathbb{Q}_p)$$

be the syntomic regulator map ([N-N]). Let $X_{\text{dR}} := X_{\mathbb{Q}_p} \times_{\mathbb{Q}_p} \mathbb{Q}_p$ and put

$$\text{NS}_{\text{dR}}(X_{\mathbb{Q}_p}) := H^2_{\text{dR}}(X_{\mathbb{Q}_p}/\mathbb{Q}_p) \cap \text{NS}(X_{\mathbb{Q}_p}) \otimes \mathbb{Q}_p.$$ 

Let $\text{NS}_{\text{dR}}(X_{\mathbb{Q}_p})^\perp$ denote the orthogonal complement of $\text{NS}_{\text{dR}}(X_{\mathbb{Q}_p}) \subset H^2_{\text{dR}}(X_{\mathbb{Q}_p}/\mathbb{Q}_p)$ with respect to the cup-product. It follows

$$\text{NS}_{\text{dR}}(X_{\mathbb{Q}_p})^\perp \oplus \text{NS}_{\text{dR}}(X_{\mathbb{Q}_p}) \xrightarrow{\sim} H^2_{\text{dR}}(X_{\mathbb{Q}_p}/\mathbb{Q}_p).$$

The cup-product induces a non-degenerate pairing on

$$\left( H^2_{\text{dR}}(X_{\mathbb{Q}_p}/\mathbb{Q}_p)/\text{NS}_{\text{dR}}(X_{\mathbb{Q}_p}) \right) \otimes \text{NS}_{\text{dR}}(X_{\mathbb{Q}_p})^\perp \cong \text{NS}_{\text{dR}}(X_{\mathbb{Q}_p})^\perp \otimes \text{NS}_{\text{dR}}(X_{\mathbb{Q}_p})^\perp,$$

which we write by $\langle -, - \rangle$.

Lemma 6.2 Suppose that $X$ has a good ordinary reduction. Let $\alpha_p$ be the eigenvalue of the $p$-th Frobenius on $H^2_{\text{dR}}(X_{\mathbb{Q}_p}/\mathbb{Q}_p)$ which is a $p$-adic unit (such $\alpha_p$ is unique), and $\eta \in H^2_{\text{dR}}(X_{\mathbb{Q}_p}/\mathbb{Q}_p)$ the eigenvector. Then $\eta \in \text{NS}_{\text{dR}}(X_{\mathbb{Q}_p})^\perp$. Let $\omega \in \Gamma(X_{\mathbb{Q}_p}, \Omega^2_{X_{\mathbb{Q}_p}/\mathbb{Q}_p})$ be a non-zero regular 2-form. If $X$ is singular, then $\dim \text{NS}_{\text{dR}}(X_{\mathbb{Q}_p})^\perp = 2$ and hence

$$\text{NS}_{\text{dR}}(X_{\mathbb{Q}_p})^\perp = \mathbb{Q}_p \omega + \mathbb{Q}_p \eta.$$

Proof. We want to show that the cup-product $\eta \cup z$ vanishes for any $z \in \text{NS}_{\text{dR}}(X_{\mathbb{Q}_p})$. The eigenvalues of $\Phi$ on $\text{NS}_{\text{dR}}(X_{\mathbb{Q}_p})$ are $p \times$ (roots of unity). Take $m > 0$ such that $\Phi^m = p^m$ on $\text{NS}(X_{\mathbb{Q}_p})$. Since $\Phi(x) \cup \Phi(y) = \Phi(x \cup y) = p^2 x \cup y$, we have

$$\Phi^m(\eta) \cup \Phi^m(z) = p^{2m}(\eta \cup z) \iff \alpha_p^m p^m(\eta \cup z) = p^{2m}(\eta \cup z).$$

Since $\alpha_p \in \mathbb{Z}_p^\times$, it follows that $\eta \cup z = 0$. \qed
Conjecture 6.3 (weak \( p \)-adic Beilinson conjecture for ordinary singular K3 surfaces) Let \( X \) be a singular K3 surface over \( \mathbb{Q} \) and \( f \) the corresponding Hecke eigenform of weight 3. Fix \( \omega \in \Gamma(X, \Omega^2_{X/\mathbb{Q}}) \) a regular 2-form. Suppose that \( f \) has a good ordinary reduction at \( p \), which is equivalent to that \( p \nmid a_p \) where \( f = \sum_{n=1}^{\infty} a_n q^n \). Let \( \alpha_p \) be the unit root of \( T^2 - a_p T + p^2 \), and \( \eta \in \text{NS}(X_{\mathbb{Q}_p}) \) its eigenvector (note that \( \omega \) is unique up to \( \mathbb{Q}^\times \) and \( \eta \) is unique up to \( \mathbb{Q}_p^\times \)). Then there is an integral element \( \xi \in K_3(X)^{(3)} \) (cf. [S-Int]) and a constant \( C \in \mathbb{Q}^\times \) not depending on \( p \) such that
\[
L_p(f, \omega^{-1}, 0) = C(1 - p^2 \alpha_p^{-1}) \frac{\langle \text{reg}_{\text{syn}}(\xi), \eta \rangle}{\langle \omega, \eta \rangle}
\]
where \( \omega_{\text{Tei}} \) is the Teichmüller character.

Remark 6.4 For a singular K3 surface \( X \), one can expect \( K_i(X)^{(j)} = K_i(X)^{(j)}_{\mathbb{Z}} \) with \( i \neq 1 \) thanks to the Shioda-Inose structure (6.1). See [As2, Remark 6.8] for details.

6.2 Review of [AOP]

In [AOP], Ahlgren, Ono and Penniston study the \( L \)-function of a K3 surface \( Y_a \) over \( \mathbb{Q} \) defined by an affine equation
\[
w^2 = u_1 u_2 (1 + u_1)(1 + u_2)(u_1 - au_2), \quad a \in \mathbb{Q} \setminus \{0, 1\}.
\]

Their first main result is the following.

Theorem 6.5 ([AOP, Theorem 1.1]) Let
\[
E_a : y^2 = x \left( x^2 + 2x - \frac{a}{1 - a} \right)
\]
\[
E'_a : (1 - a)y^2 = x \left( x^2 + 2x - \frac{a}{1 - a} \right)
\]
be elliptic curves over \( \mathbb{Q} \). Then \( L(h^2_{\text{tr}}(Y_a), s) = L(h^2_{\text{tr}}(E_a \times E'_a), s) \).

The explicit correspondence between \( Y_a \) and \( E_a \times E'_a \) which gives an isomorphism \( h^2_{\text{tr}}(Y_a) \cong h^2_{\text{tr}}(E_a \times E'_a) \) of motives over \( \mathbb{Q} \) is constructed by van Geemen and Top [vGT, Theorem 1.2].

Following [AOP], we call \( Y_a \) modular if \( L(h^2_{\text{tr}}(Y_a), s) \) is the \( L \)-function of a Hecke eigenform of weight 3 with complex multiplication, or equivalently \( E_a \) has a complex multiplication ([vGT, Theorem 1.2]). The second main result of [AOP] gives the complete list of \( a \)'s for \( Y_a \) to be modular.

Theorem 6.6 ([AOP, Theorem 1.2]) The K3 surface \( Y_a \) is modular if and only if \( a = -1, 4^{\pm 1}, -8^{\pm 1}, 64^{\pm 1} \). In each case, \( E_a \) has complex multiplication.
The corresponding Hecke eigenforms are as follows ([AOP, p.366–367]). Let $\eta(z)$ be the Dedekind eta function. Let

$$
A = \eta^4(4z), \quad B = \eta^2(z)\eta(2z)\eta(4z)\eta^2(8z), \quad C = \eta^3(2z)\eta^3(6z), \quad D = \eta^3(z)\eta^3(7z)
$$

be weight 3 newforms of level 16, 8, 12, 7 respectively. Let $\chi_D$ denote the quadratic character associated to the quadratic field $\mathbb{Q}(\sqrt{D})$. Then the corresponding Hecke eigenforms are given as follows.

$$
\begin{array}{cccccccc}
& & & & & & & \\
Hecke eigenform & B \otimes \chi_4 & C & C \otimes \chi_4 & A & A \otimes \chi_8 & D & D \otimes \chi_4 \\
\hline
a & -1 & 4 & 1/4 & -8 & -1/8 & 64 & 1/64 \\
\end{array}
$$

(6.3)

where $f \otimes \chi$ denotes the $\chi$-twist of the modular form.

### 6.3 $p$-adic Beilinson conjecture for K3 surfaces in [AOP]

Let

$$U_a : (1 - x_0^2)(1 - x_1^2)(1 - x_2^2) = a$$

the hypergeometric scheme over $\mathbb{Q}$, and let $X_a$ be a smooth compactification of $U_a$ which is a K3 surface. Put $Z_a := X_a \setminus U_a$. We discuss the $p$-adic Beilinson conjecture for $X_a$. Let $p > 3$ be a prime at which $X_a$ has a good ordinary reduction. We take integral models $X_a, Z_a(p) \supset U_a, Z_a(p)$ which are smooth over $\mathbb{Z}_p$ (Proposition 3.1). For a $\mathbb{Z}_p$-ring $R$, we write $X_{a,R} := X_a, Z_a(p) \times Z_a(p) \times R$ etc. Recall the 2-forms

$$\omega_{1,1,1}, \eta_{1,1,1} \in W_2 H^2_{dR}(U_a/\mathbb{Q}_p) \cong H^2_{dR}(X_a/\mathbb{Q}_p)/H^2_{dR,Z_a}(X_a/\mathbb{Q}_p)$$

from (3.3) and (4.4). Let

$$\xi_{\text{Ross}} = \left\{ \frac{1 - x_0}{1 + x_0}, \frac{1 - x_1}{1 + x_1}, \frac{1 - x_2}{1 + x_2} \right\} \in K^M_3\left(\mathcal{O}(U_a, Z_a(p))\right)$$

be the higher Ross symbol. We think $\xi_{\text{Ross}}$ to be an element of $K_3(U_a, Z_a(p))^{(3)}$ under the natural map $K^M_3(\mathcal{O}(U_a, Z_a(p))) \to K_3(U_a, Z_a(p))^{(3)}$. There is the exact sequence

$$K^M_3(Z_a, Z_a(p))^{(3)} \to K_3(U_a, Z_a(p))^{(3)} \to K_3(U_a, Z_a(p))^{(3)}.$$
where \( \langle -, \eta_{1,1,1} \rangle \) is the cup-product pairing which is well-defined by Lemma 6.2. The diagram implies that \( \langle \text{reg}_{\text{syn}}(\xi_{\text{Ross}}), \eta_{1,1,1} \rangle \) does not depend on the choice of the lifting. Applying Theorem 5.5 we have the description of the right hand side of (6.2) in Conjecture 6.3 in terms of our \( p \)-adic function \( \mathcal{F}_2^\sigma(t) \).

**Theorem 6.7** Let \( \sigma(t) = a^{(1-p)p} \). Then
\[
\frac{\langle \text{reg}_{\text{syn}}(\xi_{\text{Ross}}), \eta_{1,1,1} \rangle}{\langle \omega_{1,1,1}, \eta_{1,1,1} \rangle} = 8 \mathcal{F}_2^\sigma(t)|_{t=a}.
\]

Next, we see the left hand side of (6.2) in Conjecture 6.3, namely the \( p \)-adic \( L \)-function. Recall the K3 surface \( Y_a \) from §6.2. There is a dominant rational map
\[
\rho : X_a \longrightarrow Y_a
\]
over \( \mathbb{Q} \) ([As2, (6.7)]), which induces an isomorphism
\[
h^2_{\text{tr}}(Y_a, \mathbb{Q}) \cong h^2_{\text{tr}}(X_a, \mathbb{Q})
\]
of motives over \( \mathbb{Q} \). Therefore the \( p \)-adic Beilinson conjecture for \( Y_a \) is equivalent to that for \( X_a \). The \( L \)-functions of \( X_a \) and \( Y_a \) agree, and if \( a = -1, 4^{\pm 1}, -8^{\pm 1}, 64^{\pm 1} \), then they are the \( L \)-functions of the Hecke eigenforms as in the table (6.3). Together with Theorem 6.7, Conjecture 6.3 for \( K_3(X_a) \) or \( K_3(Y_a) \) can be formulated as follows.

**Conjecture 6.8** Let \( a = -1, 4^{\pm 1}, -8^{\pm 1}, 64^{\pm 1} \). Let \( f_a \) be the Hecke eigenform corresponding to \( Y_a \), cf. table (6.3). Let \( p > 3 \) be a prime such that \( p \nmid a \) and let \( \alpha_p \) be the unit root of \( T^2 - a_p T + p^2 \). Let \( \sigma \) be the \( p \)-th Frobenius given by \( \sigma(t) = a^{1-pp} \). Then there is a constant \( C_a \in \mathbb{Q}^\times \) not depending on \( p \) such that
\[
L_p(f_a, \omega_{\text{Tel}}, 0) = C_a (1 - p^2 \alpha_p^{-1}) \mathcal{F}_2^\sigma(t)|_{t=a}.
\]

In view of [As2, Theorem 6.9], it is also plausible to expect that Conjecture 6.8 remains true for \( a = 1 \).

**Conjecture 6.9** Let \( p \equiv 1 \mod 4 \) be a prime. Let \( \alpha_p \in \mathbb{Z}_p \) be the root of \( T^2 - a_p T + p^2 \) such that \( \alpha_p \equiv a_p \mod p \) where \( A = \eta^6(4z) = \sum a_n q^n \). Then there is a constant \( C_1 \in \mathbb{Q}^\times \) not depending on \( p \) such that
\[
L_p(A, \omega_{\text{Tel}}, 0) = C_1 (1 - p^2 \alpha_p^{-1}) \mathcal{F}_2^\sigma(t)|_{t=1}
\]
where \( \sigma(t) = t^p \).
6.4 Some other elliptic K3 surfaces and $p$-adic regulators

Let $A = \mathbb{Q}[s, (s - s^2)^{-1}]$ and

$$V_n = \text{Spec } A[x_0, x_1]/((1 - x_0^n)(1 - x_1^n) - s)$$

a hypergeometric scheme for $n \geq 2$ an integer. Let $C_n \supset V_n$ be a smooth compactification over $A$ (Proposition 3.1). The relative dimension of $C_n/A$ is 1. Let $J(C_n) \to \text{Spec } A$ be the jacobian scheme. Put $H := H^1_{dR}(C_n/A)$. For $0 < i_0, i_1 < n$, we denote by $H_{\mathbb{Q}}(i_0, i_1) \subset H_{\mathbb{Q}} := H \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ the eigenspace defined in (3.2). Then the subspace

$$\sum_{\text{gcd}(r, n) = 1} H_{\mathbb{Q}}(ri_0, ri_1) \subset H_{\mathbb{Q}}$$

is endowed with $A$-module structure, which we denote by $H_A(i_0, i_1)$,

$$H_A(i_0, i_1) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = \sum_{\text{gcd}(r, n) = 1} H_{\mathbb{Q}}(ri_0, ri_1).$$

Note $H_A(i_0, i_1) = H_A(i_0', i_1')$ if and only if $i_0 \equiv ri_0'$ and $i_1 \equiv ri_1'$ mod $n$ for some $r$ prime to $n$. Let

$$H = \bigoplus_{i_0, i_1} H_A(i_0, i_1)$$

be the decomposition where $(i_0, i_1)$ runs over representatives of the set $\{(\tilde{i}_0, \tilde{i}_1) \in \mathbb{Z}/n\mathbb{Z}^2 \mid \tilde{i}_0 \neq 0, \tilde{i}_1 \neq 0\}/\sim$ with $(\tilde{i}_0, \tilde{i}_1) \sim (\tilde{i}_0', \tilde{i}_1') \iff (\tilde{i}_0, \tilde{i}_1) = (\tilde{i}_0', \tilde{i}_1')$ for some $r \in (\mathbb{Z}/n\mathbb{Z})^\times$. Thanks to the Poincare reducibility theorem ([Mu] §19 Thm. 1), the above decomposition induces an isogeny

$$J(C_n) \longrightarrow \prod_{i_0, i_1} J_{i_0, i_1}$$

of abelian $A$-schemes. We note that

$$\sum_{0 < r < n} H_{\mathbb{Q}}(ri_0, ri_1) = \sum_{0 < r < n} H_A(ri_0, ri_1) \otimes \overline{\mathbb{Q}}$$

is associated to the hypergeometric curve $g^n = x^{n-i_0}(1 - x)^{n-i_1}(1 - (1 - t)x)^i_1$ of Gauss type (cf. [As1] 4.6). Hereafter we consider the cases

$$n = 3, 4, 6.$$ 

Let $J_n$ be the associated abelian scheme to $H_A(1, n - 1)$. Since

$$H_A(1, n - 1) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = H_{\mathbb{Q}}(1, n - 1) \oplus H_{\mathbb{Q}}(n - 1, 1) \tag{6.7}$$

is of rank 4 (as $n = 3, 4, 6$), the relative dimension of $J_n/A$ is 2. Let $\tau$ be the involution on $V_n$ given by $(x_0, x_1) \mapsto (x_1, x_0)$. This acts on (6.7) such that $\tau(H_{\mathbb{Q}}(1, n - 1)) = H_{\mathbb{Q}}(n - 1, 1)$. Let $H_A(1, n - 1)^\pm$ denote the eigenspace of $\tau$ with eigenvalue $\pm 1$, and $E_n^\pm$ the
corresponding decomposition of $E_n$. Both of $E_n^\pm$ are elliptic curves over $A$. We thus have surjective morphisms

$$F_n^\pm : C_n \longrightarrow E_n^\pm$$

(6.8)

over $A$, which satisfies

$$H^1_{dR}(E_n^\pm/A) \xrightarrow{(F_n^\pm)^*} H_A(1, n - 1)^\pm \subset H^1_{dR}(C_n/A),$$

(6.9)

$$\begin{align*}
(i_0, i_1) \neq (1, n - 1), (n - 1, 1) &\implies (F_n^\pm)_*(H_{dR}(i_0, i_1)) = 0.
\end{align*}$$

(6.10)

Let $B = \mathbb{Q}[t, (t - t^2)^{-1}]$, and

$$U_n = \text{Spec } B[x_0, x_1, x_2]/((1 - x_0^n)(1 - x_1^n)(1 - x_2^2) - t).$$

Let $X_n \supset U_n$ be a smooth compactification over $B$. Let $S := \text{Spec } B[x_2, (1 - x_2^2)^{-1}, (1 - t - x_2^2)^{-1}]$. Let $\phi : S \to \text{Spec } A$ be the morphism given by $\phi^*(s) = t(1 - x_2^{-1})^{-1}$, and

$$V_{n,S} \subset C_{n,S} \xrightarrow{F_{n,S}^\pm} E_{n,S} \xleftarrow{h_{n,S}^\pm} S$$

the base change of (6.8) by $\phi$, where

$$V_{n,S} := V_n \times_A S$$

$$= \text{Spec } B[x_0, x_1, x_2, (1 - x_2^2)^{-1}, (1 - t - x_2^2)^{-1}]/((1 - x_0^n)(1 - x_1^n) - t(1 - x_2^2)^{-1})$$

$$= \text{Spec } B[x_0, x_1, x_2, (1 - t - x_2^2)^{-1}]/((1 - x_0^n)(1 - x_1^n)(1 - x_2^2) - t)$$

$$\hookrightarrow U_n.$$  

(6.11)

The morphisms $h_{n,S}^\pm$ give rise to relatively minimal elliptic fibrations $\mathcal{E}_n^\pm \to \mathbb{P}_{B}^1(x_2)$ over $B$, so that we have a commutative diagram

$$\begin{array}{ccc}
V_{n,S} & \xrightarrow{f_n^\pm} & \mathcal{E}_n^\pm \\
\downarrow g_n & & \downarrow h_n^\pm \\
\mathbb{P}_{B}^1(x_2) & & \\
\end{array}$$

(6.12)

of smooth $B$-schemes.

**Proposition 6.10** $\mathcal{E}_n^\pm (n = 3, 4, 6)$ are elliptic K3 surfaces over $B$.

**Proof.** Let $x \in \text{Spec } B$ be a closed point, and $k(x)$ the residue field. We denote by $\mathcal{E}_{n,x}^\pm$ the fiber at $x$. We first show that the proposition is reduced to show that $\mathcal{E}_{n,x}^\pm$ are K3 surfaces over $k(x)$ for any $x$. Indeed, suppose that they are true, namely $H^1(\mathcal{E}_{n,x}^\pm, \mathcal{O}) = 0$ and $\Omega^2_{\mathcal{E}_{n,x}^\pm/k(x)} \cong \mathcal{O}$. Then it follows from [Ha, III, 12.9] that one has the vanishing $H^1(\mathcal{E}_n^\pm, \mathcal{O}) = 0$. Moreover
$H^0(\Omega^2_{\mathcal{C}_n^\pm/B})$ is locally free $B$-module of rank one (and hence $H^0(\Omega^2_{\mathcal{C}_n^\pm/B}) \cong B$), and the natural map

$$H^0(\Omega^2_{\mathcal{C}_n^\pm/B}) \otimes_B k(x) \longrightarrow H^0(\Omega^2_{\mathcal{C}_n^\pm/k(x)})$$

is bijective. We have a commutative diagram

$$
\begin{array}{ccc}
(H^0(\Omega^2_{\mathcal{C}_n^\pm/B}) \otimes_B \mathcal{O}_{\mathcal{C}_n^\pm}) \otimes_B k(x) & \longrightarrow & \Omega^2_{\mathcal{C}_n^\pm/B} \otimes_B k(x) \\
\cong & & \cong \\
H^0(\Omega^2_{\mathcal{C}_n^\pm/k(x)}) \otimes k(x) & \longrightarrow & \Omega^2_{\mathcal{C}_n^\pm/k(x)}
\end{array}
$$

and the bottom arrow is bijective as $\Omega^2_{\mathcal{C}_n^\pm/k(x)} \cong \mathcal{O}$. Hence this implies an isomorphism

$$\mathcal{O}_{\mathcal{C}_n^\pm} \cong H^0(\Omega^2_{\mathcal{C}_n^\pm/B}) \otimes_B \mathcal{O}_{\mathcal{C}_n^\pm} \cong H^0(\Omega^2_{\mathcal{C}_n^\pm/B})$$

as required.

We may now replace the base ring $B$ with a field $k$ of characteristic zero. Then we want to show the vanishing $H^1(\mathcal{E}_n^\pm, \mathcal{O}) = 0$ and an isomorphism $\Omega^2_{\mathcal{C}_n^\pm/k} \cong \mathcal{O}$. To do this, we may further replace $k$ with $\mathbb{C}$. There is the injective map $(f_n)^* : H^1_{dR}(\mathcal{E}_n^\pm) \hookrightarrow H^1_{dR}(X_n) = W_1 H^1_{dR}(U_n)$. However since $W_1 H^1_{dR}(U_n) = 0$ (Theorem 3.2 (2)), it turns out that $H^1_{dR}(\mathcal{E}_n^\pm) = 0$ and hence the vanishing $H^1(\mathcal{E}_n^\pm, \mathcal{O}) = 0$ follows. The rest is to show an isomorphism $\Omega^2_{\mathcal{C}_n^\pm/\mathbb{C}} \cong \mathcal{O}$. To do this, we employ the canonical bundle formula. There is a $A$-rational point of $C_n/A$, and hence there is a section of the elliptic fibration $h_n^\pm : \mathcal{E}_n^\pm \rightarrow \mathbb{P}^1$. Hence it follows from the canonical bundle formula [BPV V, (12.3)] that one has

$$K_{\mathcal{C}_n^\pm} := \Omega^2_{\mathcal{C}_n^\pm/k} \cong (h_n^\pm)^* \mathcal{O}_{\mathbb{P}^1}(e) \quad (6.13)$$

with $e = \chi(\mathcal{O}_{\mathcal{C}_n^\pm}) - 2\chi(\mathcal{O}_{\mathbb{P}^1}) = \chi(\mathcal{O}_{\mathcal{C}_n^\pm}) - 2$. We show $e = 0$. We denote the topological Euler number of $X$ by $e(X)$.

$$\chi(\mathcal{O}_{\mathcal{C}_n^\pm}) = \frac{1}{12}(K_{\mathcal{C}_n^\pm}^2 + e(\mathcal{E}_n^\pm))$$

(Noether’s formula, [BPV I, (5.5)])

$$= \frac{1}{12}e(\mathcal{E}_n^\pm)$$

(by (6.13))

$$= \frac{1}{12}\sum_s e(Z_{n,s}^\pm) \quad (\text{[BPV III, (11.4)]})$$

where $Z_{n,s}^\pm$ runs over all singular fibers of $h_n^\pm : \mathcal{E}_n^\pm \rightarrow \mathbb{P}^1(x_2)$. To compute the last term, we write up the singular fibers for each $n = 3, 4, 6$. The singular fibers of $h_n^\pm$ appear at $x_2 = \pm 1, \pm \sqrt{1-t}$ and $x_2 = \infty$ (we think $t \in \mathbb{C} \setminus \{0, 1\}$ of being a constant). The Kodaira type of each singular fiber is determined by the local monodromy on $R^1(h_n^\pm)_* \mathbb{Q}$, and it can be computed from the local monodromy for the fibration $\mathcal{E}_n^\pm \rightarrow \text{Spec } A = \mathbb{A}^1(s) \setminus \{0, 1\}$. Here
we think $s$ of being a parameter). By virtue of (6.9), this is isomorphic to the monodromy of the Gaussian hypergeometric function $\,_{2}F_{1}\left(\frac{1}{4},1-\frac{1}{4};s\right)$, which is well-understood. In this way, we obtain the complete list of singular fibers of $h^\pm_n$,

| $x_2 = \pm 1$ | $x_2 = \pm \sqrt{1 - t}$ | $x_2 = \infty$ |
|----------------|----------------------------|-----------------|
| $h^+_n$        | IV*                        | I_1             |
| $h^-_n$        | III*                       | I_2             |
| $h^*_n$        | II*                        | I_1             |

It is now immediate to have $\sum_s e(Z^n_{n,s}) = 24$ in all cases. Hence we have $e = 0$ as required.

\[\square\]

**Lemma 6.11** Let $V_{n,S} = V_n \times_A S \hookrightarrow U_n$ be as in (6.11). One has

$$W_2 H^2_{\text{dR}}(U_n/B) = W_2 H^2_{\text{dR}}(V_{n,S}/B).$$

Hence the pull-back $(f^\pm_n)^* : H^2_{\text{dR}}(\mathcal{E}^\pm_n / B) \longrightarrow W_2 H^2_{\text{dR}}(U_n/B)$ is defined.

**Proof.** (This is implicitly proven in the proof of [As2, Theorem 3.4].) It is enough to show the lemma at each fiber, so that we may replace $B$ with $\mathbb{C}$ and may assume that $U_n$ and $V_{n,S}$ are smooth complex affine varities. Let $F_1, F_2 \subset U_n$ be the complement of $V_{n,S}$ which are irreducible affine curves with unique singular points $P_1 \in F_1$ and $P_2 \in F_2$. Put $U_n^0 := U_n \setminus \{P_1, P_2\}$ and $F_i^0 := F_i \setminus \{P_i\}$. Then there is an exact sequence

$$H^2(U_n) \rightarrow H^1(V_{n,S}) \xrightarrow{\text{Res}_1} H^0(F_i^0) \otimes \mathbb{Q}(-1) \rightarrow H^2(U_n^0) \rightarrow H^2(V_{n,S}) \xrightarrow{\text{Res}_2} H^1(F_i^0) \otimes \mathbb{Q}(-1)$$

of mixed Hodge structures, where $H^i(-) = H^i(-, \mathbb{Q})$ denotes the Betti cohomology. Since $\text{Res}_1$ is surjective and the weight of $H^1(F_i^0) \otimes \mathbb{Q}(-1)$ is $\geq 3$, we have $W_2 H^2(U_n) = W_2 H^2(V_{n,S})$ as required.

Put $H(U_n) := W_2 H^3_{\text{dR}}(U_n/B)$. The sum

$$\sum_r H(U_n)_{\mathbb{Q}}(r_{i_0}, r_{i_1}, 1) \subset H(U_n)_{\overline{\mathbb{Q}}} := H(U_n) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$$

of the subspaces is endowed with $B$-module structure, which we denote by $H(U_n)_B(i_0, i_1, 1)$,

$$H(U_n)_B(i_0, i_1, 1) \otimes \overline{\mathbb{Q}} = \sum_r H(U_n)_{\overline{\mathbb{Q}}}(r_{i_0}, r_{i_1}, 1),$$

$$H(U_n) = \bigoplus_{i_0, i_1} H(U_n)_B(i_0, i_1, 1).$$

Moreover we denote by $H(U_n)_B(i_0, i_1, 1)^+$ (resp. $H(U_n)_B(i_0, i_1, 1)^-$) the fixed part (resp. anti-fixed part) by the involution $\tau(x_0, x_1, x_2) = (x_1, x_0, x_2)$.
Proposition 6.12  The image of

\[(f_n^\pm)^* : H^2_{dR}(E_n^\pm/B) \longrightarrow H(U_n) = W_2 H^2_{dR}(U_n/B)\]

agrees with the component \(H(U_n)_B(1, n-1, 1)^\pm\).

Proof. There is a commutative diagram

\[
\begin{array}{ccc}
H^2_{dR}(E_n^\pm/B) & \xrightarrow{(f_n^\pm)^*} & W_2 H^2_{dR}(V_n,S/B) \\
\downarrow & & \downarrow_{\cong} \\
H^2_{dR}(E_{n,S}^\pm/B) & \xrightarrow{(F_{n,S}^\pm)^*} & H^2_{dR}(V_{n,S}/B) \\
\downarrow & & \downarrow_{\cong} \\
H^1_{dR}(S, R^1(h_n^\pm)_*\Omega^\bullet_{E_n^\pm,S/S}) & \xrightarrow{\cong} & H^1_{dR}(S, R^1(g_{n,S}^\pm)_*\Omega^\bullet_{V_{n,S}/S})
\end{array}
\]  \hfill (6.14)

where the bottom arrow is induced from \(F_{n,S}^\pm\) in (6.8) and \((\ast)\) follows from Lemma 6.11. The map \(i\) is injective and the cokernel of \(i\) is generated by the cycle class \(e\) of a section of \(E_{n,S}^\pm \to S\). One easily sees \((F_{n,S}^\pm)^*(e) = 0\). Therefore the image of \(F_{n,S}^\pm\) agrees with the image of \(H^1_{dR}(S, R^1(h_n^\pm)_*\Omega^\bullet_{E_n,S/S})\), and hence we have

\[(f_n^\pm)^* H^2_{dR}(E_n^\pm/B) \subset H(U_n)_B(1, n-1, 1)^\pm.\]

Since \(H(U_n)_B(1, n-1, 1)^\pm\) is an irreducible connection (Theorem 3.3), the equality holds in the above, as required. \(\square\)

Corollary 6.13  For a geometric point \(\bar{x} \to \text{Spec } B\), let \(E_{n,\bar{x}}^\pm\) be the fiber at \(\bar{x}\). Then the rank of the Neron-Sevart group \(\text{NS}(E_{n,\bar{x}}^\pm)\) is \(\geq 19\).

Proof. This follows from Proposition 6.12 and the fact \(\text{rank } H(U_n)_B(1, n-1, 1)^\pm = 3\). \(\square\)

Taking the Hodge filtration \(F^2\) of the diagram (6.14), we have a commutative diagram

\[
\begin{array}{ccc}
H^0(E_n^\pm, \Omega^2_{E_n^\pm/B}) & \xrightarrow{(f_n^\pm)^*} & H^0(X_n, \Omega^2_{X_n/B}) \\
\downarrow & & \downarrow_{\cong} \\
H^1_{dR}(S, R^1(g_{n,S}^\pm)_*\Omega^\bullet_{V_{n,S}/S}) & \xrightarrow{\cong} & F^2 H^1_{dR}(S, R^1(g_{n,S}^\pm)_*\Omega^\bullet_{V_{n,S}/S}) \\
F^2 H^1_{dR}(S, R^1(h_n^\pm)_*\Omega^\bullet_{E_{n,S}/S}) & \xrightarrow{\cong} & F^2 H^1_{dR}(S, R^1(g_{n,S}^\pm)_*\Omega^\bullet_{C_{n,S}/S})
\end{array}
\]  \hfill (6.15)

together with the push-forward maps.
Lemma 6.14 Let $\omega_{i_0,i_1} \in H^0(X_n, \Omega^2_{X_n/B})$ be the regular 2-forms in (3.3). Then

$$\left(f_n^+\right)^* H^0(\mathcal{E}_n^+ : \Omega^2_{\mathcal{E}_n^+ / B}) = B(\omega_{1,n-1,1} + \omega_n)$$

(6.16)

$$(i_0, i_1) \neq (1, n - 1), (n - 1, 1) \quad \implies \quad \left(f_n^+\right)_*(\omega_{i_0,i_1}) = 0,$$

(6.17)

$$\left(f_n^+\right)_*(\omega_{1,n-1,1}) = \mp \left(f_n^+\right)_*(\omega_n) \neq 0.$$  

(6.18)

Proof. Since the involution $\tau(x_0, x_1, x_2) = (x_1, x_0, x_2)$ on $U_n$ satisfies $\tau \omega_{1,n-1,1} = -\omega_n$, one has

$$F^2 H(U_n)B(1, n - 1, 1) = B(\omega_n).$$

Now (6.16) is immediate from Proposition 6.12. Moreover (6.17) follows from (6.10) by virtue of the diagram (6.15). By the construction of $F_n^+$, they satisfy $(F_n^+)_*(F_n^+)^* = (F_n^+)_*(F_n^+)^* = 0$, which implies $(f_n^+)_*(f_n^+)^* = (f_n^-)_*(f_n^-)^* = 0$. Therefore,

$$\left(f_n^+\right)_*(\omega_{1,n-1,1} + \omega_n) = 0, \quad \left(f_n^+\right)_*(\omega_{1,n-1,1} - \omega_n) = 0.$$

There remains to show the non-vanishing $(f_n^+)_*(\omega_n) \neq 0$. Suppose $(f_n^+)_*(\omega_n) = 0$ and hence $(f_n^+)_*(\omega_n) = 0$ as well. Then

$$\left(f_n^+\right)_* H^0(\Omega^2_{X_n/B}) = \left(f_n^+\right)_* F^2 W_2^2 H^2_{\text{dR}}(U_n / B) = 0$$

by (6.17). Since $(f_n^+)_*$ is surjective onto $H^0(\Omega^2_{\mathcal{E}_n^+ / B})$, this contradicts with that $\mathcal{E}_n^+$ are K3 surfaces (Proposition 6.10). This completes the proof. □

We discuss the $p$-adic Beilinson conjecture for $K_3(\mathcal{E}_n^-)$ with $n = 3, 4, 6$. Let $f_n^- : U_n \to \mathcal{E}_n^-$ be the morphism (6.12) (recall that we only consider the cases $n = 3, 4, 6$). For $a \in \mathbb{Q} \setminus \{0, 1\}$, we denote by $f_{n,a} : U_{n,a} \to \mathcal{E}_{n,a}$ and $X_{n,a}$ the fibers at the closed point $t = a$ of Spec $B$. Let $\zeta_n$ be a primitive $n$-th root of unity. Let $p > 3$ be a prime at which $X_{n,a}$ has a good ordinary reduction. Let $U_{a,n,\mathbb{Q}(\zeta_n)} = U_{a,n} \times_{\mathbb{Q}} \text{Spec} \mathbb{Q}(\zeta_n)$ and $U_{a,n,\mathbb{Z}(p)[\zeta_n]}$ a smooth model over $\mathbb{Z}(p)[\zeta_n]$. Let

$$\xi_{\text{Ross}}(\zeta_n) = \left\{ \frac{1 - x_0}{1 - \zeta_n x_0}, \frac{1 - x_1}{1 - \zeta_n x_1}, \frac{1 - x_2}{1 + x_2} \right\} \in K_3^M(\mathcal{O}(U_{a,n,\mathbb{Z}(p)[\zeta_n]}))$$

(6.19)

be the higher Ross symbol, which we think of being an element of Quillen’s $K_3$. Put

$$\xi_n := N_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\xi_{\text{Ross}}(\zeta_n)) \in K_3(U_{a,n,\mathbb{Z}(p)})$$

where $N_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}$ is the norm map in Quillen’s $K$-theory. Since $\xi_{\text{Ross}}(\zeta_n)$ lies in the image of $K_3(X_{n,a,\mathbb{Z}(p)[\zeta_n]})$ ([AS2, Corollary 4.4]), so does $\xi_n$, and hence there is a lifting $\tilde{\xi}_n \in K_3(X_{n,a,\mathbb{Z}(p)})$. Put

$$\omega_n := (f_n^-)_*(\omega_{1,n-1,1}), \quad \eta_n := (f_{n,a}^-)_*(\eta_{1,n-1,1}) \in H^2_{\text{dR}}(\mathcal{E}_{n,a}).$$

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Let \[ \text{reg}_{\text{syn}} : K_3(E_{n,a,Z(\mathbb{p})}) \to \mathbb{H}^3_{\text{syn}}(E_{n,a,Z(\mathbb{p})}, \mathbb{Q}_p(3)) \cong H^2_{dR}(E_{n,a}/\mathbb{Q}) \]

be the syntomic regulator map where \( E_{n,a,Z(\mathbb{p})} \) is a smooth integral model of \( E_{n,a} \) over \( \mathbb{Z}(\mathbb{p}) \).

Since \( \eta_n \) is a unit root vector, one can show that \( \langle \text{reg}_{\text{syn}}(f_{n,a} \xi_n), \eta_n \rangle \) does not depend on the choice of lifting in the same way as in [6.2](see the diagram [6.4]). We have

\[ \langle \text{reg}_{\text{syn}}((f_{n,a})_*, \xi_n), \eta_n \rangle = \langle \text{reg}_{\text{syn}}(\xi_n), (f_{n,a})_* \eta_n \rangle = \langle \text{reg}_{\text{syn}}(\zeta_{\text{Ross}}(\xi_n)), (f_{n,a})_* \eta_n \rangle + \langle \text{reg}_{\text{syn}}(\zeta_{\text{Ross}}(\xi_n)^{-1}), (f_{n,a})_* \eta_n \rangle. \]

Since

\[ \text{reg}_{\text{syn}}(\zeta_{\text{Ross}}(\xi_n^{\pm 1})) = (1 - \zeta_n)(1 - \zeta_n^{-1}) \langle \mathcal{F}_n^{(\sigma)} \rangle_{n-1,1,2} (t) \mid_{t=a} (\omega_{n-1,1,1} + \omega_{n-1,1,1}) + \text{(other terms)} \]

by Theorem 5.3, we have

\[ \langle \text{reg}_{\text{syn}}((f_{n,a})_*, \xi_n), \eta_n \rangle = 2(1 - \zeta_n)(1 - \zeta_n^{-1}) \langle \mathcal{F}_n^{(\sigma)} \rangle_{n-1,1,2} (t) \mid_{t=a} (\omega_{n-1,1,1} + \omega_{n-1,1,1}, (f_{n,a})_* \eta_n) \]

\[ = 4(1 - \zeta_n)(1 - \zeta_n^{-1}) \langle \mathcal{F}_n^{(\sigma)} \rangle_{n-1,1,2} (t) \mid_{t=a} (\omega_n, \eta_n) \]

Finally we note

\[ \langle \omega_n, \eta_n \rangle = \langle (f_{n,a})_*(f_{n,a})_* \omega_{n-1,1,1}, \eta_{1,n-1,1} \rangle \]

\[ = c \langle \omega_{n-1,1,1}, \omega_{n-1,1,1} \rangle \]

with \( c \neq 0 \) a constant, and this does not vanish by Lemma 4.3 (3). Summing up the above, we have the description of the \( p \)-adic regulator for \( K_3(E_{n,a}) \).

**Theorem 6.15** Let \( \sigma(t) = a^{1-p}t^p \) with \( p > 3 \). Suppose that \( E_{n,a} \) has a good ordinary reduction at \( p \). Then

\[ \frac{\langle \text{reg}_{\text{syn}}((f_{n,a})_*, \xi_n), \eta_n \rangle}{\langle \omega_n, \eta_n \rangle} = 4(1 - \zeta_n)(1 - \zeta_n^{-1}) \langle \mathcal{F}_n^{(\sigma)} \rangle_{n-1,1,2} (t) \mid_{t=a} \]

with \( n = 3, 4, 6 \).

**Conjecture 6.16** Suppose that \( E_{n,a} \) is a singular K3 surface over \( \mathbb{Q} \). Let \( A_{n,a} = \sum a_n q^n \) be the corresponding Hecke eigenform of weight 3, and \( \alpha_p \) the unit root of \( T^2 - a_p T + p^2 \). Then there is a constant \( C_{n,a} \in \mathbb{Q}^* \) not depending on \( p \) such that

\[ L_p(A_{n,a}, \omega_{1,0}^{-1}, 0) = C_{n,a}(1 - p^2 \alpha_p^{-1}) \langle \mathcal{F}_n^{(\sigma)} \rangle_{n-1,1,2} (t) \mid_{t=a}. \]

Unlike the K3 surfaces in [AOP], the author has not worked out on the \((p\text{-adic})\) \( L \)-functions of \( E_{n,a} \).
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