Q-Exact Actions for BF Theories

ROGER BROOKS AND CLAUDIO LUCCHESI

Center for Theoretical Physics,
Laboratory for Nuclear Science
and Department of Physics,
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139 U.S.A.

Abstract

The actions for all classical (and consequently quantum) BF theories on n-manifolds is proven to be given by anti-commutators of hermitian, nilpotent, scalar fermionic charges with Grassmann-odd functionals. In order to show this, the space of fields in the theory must be enlarged to include “mass terms” for new, non-dynamical, Grassmann-odd fields. The implications of this result on observables are examined.

CTP # 2271 December 1993
hep-th/9401005

*This work is supported in part by funds provided by the U. S. Department of Energy (D.O.E.) under contract #DE-AC02-76ER03069.
†Supported by the Swiss National Science Foundation.
1 Introduction

Topological field theories (TFT’s) are usually classified as belonging to either one of two distinct classes. One such class called “topological quantum field theories” (TQFT’s)\(^1\), is characterized by the fact that the actions are expressible as the anti-commutator of a cohomology charge, \(Q\), with some field functional \(\mathbb{F}\) (we will refer to such actions as being \(Q\)-exact). Furthermore, the (formal) argument establishing the topological invariance of correlation functions of observables (for a review see ref. \(^2\)) is based on the existence of the cohomology operator. The other class of TFT’s is comprised of the \(BF\) theories. Until our work it was not understood how to express the entire quantum \(BF\) action as being \(Q\)-exact. However, we recently showed that two and three dimensional, abelian \(BF\) theories and TQFT’s are related via a map that inverts the Grassmann parity of the fields \(\mathbb{F}\). This yields a unified picture for the two classes of TFT’s outlined above. Hence, the cohomology structure present in TQFT’s should also exist in \(BF\) theories and it is an appealing task to derive it. In the present article, as part of establishing the \(Q\)-exactness of all \(BF\) theories, we give the explicit calculus for such a cohomology charge. We also show how this result is intertwined with the topological invariance\(^2\) of the theory.

Other (partial) results are known which indicate that such a cohomology should exist. In the two dimensional \(BF\) theory an equivariant cohomology operator was found \(^3\) by enlarging the space of fields to include a Grassmann-odd 1-form. However, in this extension and its higher dimensional analog \(^4\), position independence could not be proven for the correlation functions of operators which depend only on the \(A\) field. Furthermore, the classical action cannot be written as an anti-commutator with a nilpotent charge. Contrarily, the cohomology that we present in this paper for the \(BF\) theory has, as its elements, observables constructed most generally from the \(k\)-cycle integrals over the \(A\) (and \(B\)) field(s), their exterior derivatives and wedge products.

\(^1\)TQFT’s are sometimes referred to as Donaldson-Witten or cohomological field theories.

\(^2\)Let us recall that the \(BF\) action, as well as the correlation functions of the (Wilson loop) observables for the fields \(B\) and \(A\) [\(F\) being the (covariant) exterior derivative of \(A\)], are topological \(^3\). In fact, correlation functions of observables compute the intersection numbers of cycles on the space-time manifold.
As we are using TQFT’s as a paradigm, the cohomology operator we seek, call it $Q$, must be a scalar, hermitian, nilpotent symmetry generator. In addition, the (covariant) exterior derivative of $A$ must be $Q$-exact.

With the above as a background, we apply our results about $BF$ theories also to actions which are given by the square of $F$ and can be written as $BF$ actions by integrating out the $B$ field. For example, the action for the two-dimensional Yang-Mills theory ($YM_2$) can be written in this sense as $S = \int_M (B \wedge F - \frac{1}{2} B^2)$.

This paper is structured as follows. In Section 2, we shall first develop our ansatz based upon TQFT’s of constant maps from Riemann surfaces to $\mathbb{R}^d$. Next, in Section 3, these observations are used to motivate our derivation of a $Q$-exact action for the two dimensional non-abelian $BF$ theory. Then, in Section 4, we generalize the construction to higher dimensional theories and analyze consequences on the observables. In Section 5, we investigate the twisted and untwisted supersymmetries in the two-dimensional case. We conclude in Section 6. Throughout this work our notation is as follows:

| OBJECT | DEFINITION |
|--------|------------|
| $(\delta)\ d$ | (adjoint) exterior derivative |
| $d_A$ | covariant exterior derivative |
| $\ast$ | Hodge dual operator; $\ast\ast \equiv (-)^{k(n-k)}$ |
| $\alpha_0$ | parametrization from $BF$ to $F^2$ theories |
| $\Phi_{(k)}$ | a generic form of degree $k$. |

Table 1: Notation

2 A Hint From TQFT’s of Maps

Before starting the investigation of the gauge theory case, let us begin by recalling the construction of topological sigma models (see for instance [2]). We perform this exercise to motivate the transformations we shall be using in the sections ahead. The peculiarity here is that we gauge fix to a constant

\footnote{For example, in the two dimensional case, where $A = A$ is a gauge field, $F$ must be the (anti)commutator of $Q$ with some field.}
the map $X^I$ ($I = 1, \ldots, d$) from a Riemann surface $\Sigma$ to $\mathbb{R}^d$. That is, we gauge fix the local symmetry $X^I \rightarrow X^I + \epsilon^I_X$ via introducing the action

$$S^0_{\text{map}} = \{Q^0_{\text{map}}, \int_{\Sigma} \rho^I \wedge *(dX^I - \frac{1}{2} b^I)\}.$$  

(2.1)

It turns out that this action is further invariant under the local symmetry $\rho^I \rightarrow \rho^I + *d\epsilon^I_{\rho}$, which we gauge-fix by imposing $d\rho^I = 0$. The resulting action is

$$S_{\text{map}} = \{Q^{\text{map}}, \int_{\Sigma} (\rho^I \wedge *(dX^I - \frac{1}{2} b^I) + \varphi^I (d\rho^I + \frac{1}{2} \alpha^I_{0 \beta^I}))\}$$

$$= \int_{\Sigma} \left( \frac{1}{2} dX^I \wedge *dX^I + \frac{1}{2} d\varphi^I \wedge *d\varphi^I + \beta^I (d\rho^I + \frac{1}{2} \alpha^I_0 \beta^I) - \rho^I \wedge *d\lambda^I + \varphi^I d*d\tilde{\varphi}^I \right),$$

(2.2)

where the BRST charge $Q^{\text{map}}$ acts as

$$[Q^{\text{map}}, X^I] = \lambda^I, \quad [Q^{\text{map}}, \varphi^I] = \beta^I, \quad \{Q^{\text{map}}, \rho^I\} = b^I + *d\tilde{\varphi}^I.$$  

(2.3)

In the final expression for $S_{\text{map}}$ we have integrated out the auxiliary field $b^I$ yielding $b^I = dX^I$ and discarded a surface term. Notice that the $\beta^2$-term vanishes due to the Grassmann-odd nature of $\beta$; we have left it in for comparison with later expressions. The field content of the action $S_{\text{map}}$ is summarized in the following Table:

| FIELD | FORM DEGREE | GRASSMANN PARITY |
|-------|-------------|------------------|
| $\rho$ | 1           | odd             |
| $\beta$ | 0           | odd             |
| $\lambda$ | 0         | odd             |
| $\varphi$ | 0          | even            |
| $\tilde{\varphi}$ | 0        | even            |
| $X$ | 0           | even            |
| $b$ | 1           | even            |

Table 2: Form degrees and Grassmann parities of the fields in $S_{\text{map}}$.

Let us now consider this action irrespective of its origin and reverse the Grassmann parity of the fields. The first two terms in $S_{\text{map}}$ vanish. Next we
rename the fields as \((\rho, \beta, \lambda, \varphi, \tilde{\varphi}, X, b) \rightarrow (A, B, \Lambda, c, \tilde{c}, \chi, \psi)\). Having done this, we recognize (for the case \(d = 1\)) the Maxwell (or abelian BF) action in the Landau gauge. Hence, for the non-abelian case, this suggests that in order to obtain the gauge-fixed BF action as a \(Q^\text{map}\)-exact expression, we should enlarge the space of fields to include the Grassmann-odd counterpart of \(X\).

### 3 Q-Exact Two-dimensional BF Theory

In this section, we shall be concerned with the non-abelian theory in two dimensions, for which we now formalize the observations presented above. Consider a space of fields spanned by those appearing in the gauge fixed BF/YM\(_2\) theory and enlarge it to include two new fields, \(\chi\) and \(\eta\). These are Grassmann-odd and take value in the adjoint representation of the gauge group; they will turn out not to be dynamical. We summarize our space of fields in the following table:

| FIELD | FORM DEGREE | GRASSMANN PARITY |
|-------|-------------|------------------|
| \(A\) | 1           | even             |
| \(B\) | 0           | even             |
| \(\Lambda\) | 0        | even             |
| \(c\) | 0           | odd              |
| \(\tilde{c}\) | 0       | odd              |
| \(\chi\) | 0        | odd              |
| \(\eta\) | 2        | odd              |

Table 3: Enlarged space of fields for BF/YM\(_2\).

On this space of fields we define a scalar fermionic charge \(Q\) by:

\[
\{Q, \chi\} = \Lambda, \quad \{Q, c\} = B, \quad [Q, A] = \ast \tilde{d}c, \quad (3.1)
\]

and \([Q, (\eta, \Lambda, B, \tilde{c})] = 0\), as motivated by the transformations (2.3). Nilpotency is immediate: \(Q^2 = 0\). Furthermore, we take all the fields to be hermitian so that \(Q^\dagger = Q\). The action of \(Q\) on the fields suggests that it
is related to the abelian anti-BRST symmetry of the gauge-fixed BF theory \([4, 7]\). Using this cohomological structure, we define, in analogy with \(S^{\text{map}}\) of Section 3, the \(Q\)-exact action:

\[
S(\alpha_0) = \{Q, \int_\Sigma Tr (c(F + \frac{1}{2}\alpha_0 \ast B) + \chi \ast \delta A)\},
\]

(3.2)

where \(\alpha_0\) is an arbitrary parameter. Employing the transformation rules (3.1) we find, up to surface terms involving the fields \(c\) and \(\tilde{c}\):

\[
S(\alpha_0) = \int_\Sigma Tr (BF + \frac{1}{2}\alpha_0 B^2 + \Lambda \ast \delta A - \tilde{c} \ast \delta d_A c),
\]

(3.3)

where \(d_A\) is the covariant exterior derivative. Notice that \(\chi\) does not appear in the final form of the action, even in a purely auxiliary capacity. We recognize the first term as the usual pure BF action, the second is analogous to the \(\beta^2\) term in \(S^{\text{map}}\), the third implements the Landau gauge condition and the last term is the corresponding Faddeev-Popov action. The arbitrary parameter \(\alpha_0\) interpolates between YM\(_2\) and BF theories. Indeed, with \(\alpha_0 = 2\) and after integrating out the \(B\)-field, (3.3) is just the gauge-fixed YM\(_2\) action, whereas \(\alpha_0 = 0\) yields the gauge-fixed BF theory in two dimensions.

From (3.2), it would appear that the partition function of YM\(_2\) is metric independent following the usual arguments from TQFT’s. We know that this is not the case, however. Hence, it must be that the symmetry generated by \(Q\) is broken by an anomaly or a non-trivial measure. This is the case, indeed, and as we will now see.

From (3.1), it is natural to grade the space of fields by introducing a \(U(1)\) charge akin to ghost number, and assigned as follows: \((A, B, \Lambda, c, \tilde{c}, \chi, \eta) \leftrightarrow (0, 0, 0, 1, -1, 1, -1)\); \(Q\) itself carries charge \(-1\). Since \(\chi\) has charge 1, a partner \(\eta\) must be introduced in order to form a \(U(1)\) invariant functional measure. Although \(\chi\) appears in (3.2), neither field appears in the final form of the action, (3.3). However, as they were needed in establishing the \(Q\)-exactness of the action, we must integrate over them in the partition function. And since these fields are fermionic, in order for the partition function not to be zero, we must introduce a non-trivial measure. To wit, we construct the measure as

\[
\int_{\chi, \eta} \equiv \int [d\chi][d\eta] \exp \left(\int_\Sigma Tr(\eta \chi)\right).
\]

(3.4)

\[\text{Indeed, \(\chi\) only appears at an intermediate stage of the calculation in the vanishing term \(\chi \delta \ast \delta \tilde{c} = \chi \delta^2 \ast \tilde{c} = 0\).}\]
This measure explicitly breaks the $Q$ symmetry. We now use it to define the partition function on the enlarged space of fields (see Table 3) by simply adding the exponent in the measure (3.4) to the action. This yields:

$$Z = \int [dA][dB][d\Lambda][dc][\tilde{c}]d\chi]d\eta] \exp \left( - (S_0(\alpha_0) + S_{GF}) \right),$$  

where

$$S_0(\alpha_0) = \int_\Sigma Tr (\eta \chi + BF + \frac{1}{2} \alpha_0 B^2),$$

$$S_{GF} = \int_\Sigma Tr (\Lambda^* \delta A - \tilde{c}^* \delta d_A c).$$

(3.5)

$S_0(\alpha_0)$ is interpreted as the gauge invariant classical action and $S_{GF}$ arises from gauge-fixing the Yang-Mills symmetry.

The classical action $S_0(\alpha_0)$ is invariant under new sets of Grassmann-odd symmetries generated by $Q$ and $\tilde{Q}$:

$$[Q, B] = -\chi, \quad \{Q, \eta\} = F + \alpha_0^* B,$$

$$[\tilde{Q}, B] = *\eta, \quad \{\tilde{Q}, \chi\} = *F + \alpha_0 B.$$  

(3.6)

(Notice that there is no factor $\frac{1}{2}$ multiplying $\alpha_0$). Heuristically, $\tilde{Q}$ is dual to $Q$. Expressing $S_0(\alpha_0)$ in terms of these charges yields the following interesting results:

$$S_0(\alpha_0) = \{Q, \int_\Sigma Tr (B\eta)\} - \frac{1}{2} \alpha_0 \int_\Sigma Tr (B^2)$$

$$= \{\tilde{Q}, \int_\Sigma Tr (B\chi)\} - \frac{1}{2} \alpha_0 \int_\Sigma Tr (B^2),$$  

(3.7)

(3.8)

from which we immediately learn two things. First, the classical $BF$ action on the enlarged space of fields $S_0(\alpha_0 = 0)$ is $Q$- and/or $\tilde{Q}$-exact. Secondly, as expected, the $YM_2$ action ($\alpha_0 \neq 0$) is neither $Q$- nor $\tilde{Q}$-exact. This is consistent with the observation that the transformations $Q$ and $\tilde{Q}$ in (3.7) are nilpotent only when $\alpha_0 = 0$, that is only in the case of the $BF$ theory. The physical states of the $BF$ theory are hence elements of the $Q$-cohomology (and/or of the $\tilde{Q}$-cohomology), unlike the physical states of $YM_2$.  

6
4 Higher Dimensions and Observables

Generalization of the results presented above to higher dimensions is immediate. Given a $k$-form $A$ which transforms homogeneously in the adjoint representation of a gauge group and such that $F \equiv dA$ is gauge covariant ($A$ is a background gauge field), the analog of $S_0(\alpha_0)$ on an $n$-manifold $M$ is

$$S_0^{(n)}(\alpha_0) = \{Q, \int_M Tr (B_{(n-k-1)} \wedge \eta_{(k+1)})\} - \frac{1}{2} \alpha_0 \int_M Tr (B_{(n-k-1)} \wedge \star B_{(n-k-1)})$$,

and yields:

$$S_0^{(n)}(\alpha_0) = \int_M Tr \left( \eta_{(k+1)} \wedge \chi_{(n-k-1)} + B_{(n-k-1)} \wedge F_{(k+1)} + \frac{1}{2} \alpha_0 B_{(n-k-1)} \wedge \star B_{(n-k-1)} \right)$$,

where the generator $Q$ acts as

$$[Q, B_{(n-k-1)}] = -(n-k-1)(k-1) \chi_{(n-k-1)}$$,

$$\{Q, \eta_{(k+1)}\} = F_{(k+1)} + \alpha_0 \star B_{(n-k-1)}$$,

and we omit writing the corresponding transformations generated by $\tilde{Q}$. Based on the $Q$-exactness of the $BF$ action, we conclude that its partition function is independent of the background gauge field, $A$.

Using the results above, we now address the question of the observables in a $BF$ theory on an $n$-manifold. Let us start with $\mathcal{O}_A$, by which we denote gauge invariant operators constructed out of invariant polynomial functions $\mathcal{P}(F)$ of $F$ and generalized Wilson loop operators, $W_\Gamma(A)$ (where $\Gamma$ is a representative homology $k$-cycle). An example of such an operator $\mathcal{O}_A$ is $\mathcal{P}(F) \otimes W_\Gamma(A)$. Those $\mathcal{O}_A$ for which $\mathcal{P}$ is at least of degree one, will have vanishing correlation functions amongst themselves. This is due to the fact that $F$ is $Q$-exact and all functions of $A$ are $Q$-closed. This is not true, however, for observables composed of the gauge invariant polynomials of those $A$'s for which $F = 0$; for example, flat connections in the case $k \neq 1$. Derivatives of the latter with respect to the collective coordinates, on

\footnote{Of course, we can also take $A$ and $B$ not to take values in a gauge group, in which case, $F = dA$.}
which they may depend, can be shown to form gauge invariant polynomial observables (see ref. [8]). Our analysis does not preclude these from being non-trivial.

Restricting ourselves to the Wilson loop content of $\mathcal{O}_A$, we find that correlation functions of the observables $W_\Gamma(A)$ can only depend on the homotopy class of the $\Gamma$’s. The standard proof of this statement [9] makes explicit use of the restriction of the path integral to $F = 0$ solutions (arising from the integration over $B$)[1]. Given that under a homotopically trivial perturbation in $\Gamma$, the change in $W_\Gamma(A)$ depends on $F$, we conclude that such a change will be $Q$-exact. Consequently, its effect in a correlation function will vanish. Thus we have proven the statement about the dependence of Wilson loops on the homotopy class without appealing to the “on-shell” $F = 0$ condition.

Proceeding further, we can readily see that the $W_\Gamma(A)$ can only depend on the $A$ zero-modes (solutions of $F = 0$). This is due to the fact that since these do not contribute to the action, they do not appear in the transformations (4.3). In turn, this means that they are automatically in the $Q$-cohomology.

Next, we restrict our attention to the subset of observables $\mathcal{O}_B$, by which we denote operators which depend, in a gauge invariant manner, on $B$. We identify the zero-modes of $B$ as the solutions of $d_A B = 0$. The argument is identical to the above: since these do not contribute to the action, they are inert under the action of $Q$. This means that only gauge invariant functions of the $B$ zero-modes will be in the $Q$-cohomology.

Having studied separately the cases of $\mathcal{O}_A$ and $\mathcal{O}_B$, we now turn to observables which depend on both $A$ and $B$. Since all observables are in the $Q$-cohomology, our previous discussions regarding the $\mathcal{O}_A$ case applies also in the mixed case (remember that we did not have to appeal to the “on-shell” $F = 0$ or $d_A B = 0$ conditions). As a consequence, the observables of a $BF$ theory on an $n$-manifold can only depend on the zero-modes of $A$ and $B$ and on the homotopy classes of cycles over which they are integrated.

---

6Of course, this assumes that apart from gauge fixing terms, there is no additional $B$-dependence in the path integral under investigation.

7This can be found in equation (2.8) of ref. [9].
5 Twisting in Two Dimensions

Let us now return to the discussion of two dimensional space-times. The symmetries (4.3) hold on an arbitrary Riemann surface $\Sigma$. If the manifold is flat, however, we find that the action $S_0(\alpha_0)$ (3.8) is also invariant under

$$\{Q_1, \eta\} = dAB, \quad [Q_1, A] = \star \chi,$$

where $Q_1$ carries form-degree one and is not nilpotent. Of greater value is the fact that with $\alpha_0 = 0$, we have the algebra:

$$\{\tilde{Q}, Q_1\} = d_A, \quad \{Q, Q_1\} = 0. \quad (5.2)$$

Due to the vanishing of the anticommutator $\{Q, Q_1\}$, this is not a twisted $N = 2$ supersymmetry algebra. That this is indeed not the case may also be seen from the field content of the action $S_0$, and it is precisely by taking new fields into account that we shall obtain the twisted supersymmetric action. In ref. [5], an anti-commuting 1-form, $\psi$, was introduced in order to define the symplectic form on the space of gauge fields. This field transforms in the adjoint representation of the gauge group and forms a basis for 1-forms on that space. Let us add its action to $S_0$, and define our new action to be (on arbitrary Riemann surfaces)

$$S'_0(\alpha_0) = \int_\Sigma Tr \left( \eta \chi + BF + \frac{1}{2} \alpha_0 B^2 + \frac{1}{2} \psi \wedge \psi \right), \quad (5.3)$$

with four, real anti-commuting degrees of freedom. In addition to being invariant under the transformations (3.7), the action (5.3) possesses another set of symmetries [5] which do not affect the $\eta$ and $\chi$ fields:

$$[Q_\psi, A] = \psi, \quad \{Q_\psi, \psi\} = d_AB, \quad (5.4)$$

$$[\tilde{Q}_\psi, A] = \star \psi, \quad \{\tilde{Q}_\psi, \psi\} = \star d_AB.$$

The action $S'_0(\alpha_0 = 0)$ is actually the twisted form of the supersymmetric action

$$S_{\text{susy}} = \frac{i}{2} \int d^2z Tr \left( BF_{z\bar{z}} + \zeta_+ \bar{\zeta}_- + \zeta_- \bar{\zeta}_+ \right), \quad (5.5)$$

where $\zeta_{\pm}$ are the Weyl components of a complex spin-$1\over2$ field. Let us now prove this statement. We start from the fact that $S_{\text{susy}}$ is invariant under the
$N = 2$ supersymmetry transformations:

$$
\begin{align*}
[Q_+, A_\bar{z}] &= \bar{\zeta}_-, & [\bar{Q}_+, A_z] &= \zeta_-, \\
[Q_+, B] &= \zeta_+, & [\bar{Q}_+, B] &= \bar{\zeta}_+, \\
\{Q_+, \bar{\zeta}_-\} &= F_{z\bar{z}}, & \{\bar{Q}_+, \zeta_-\} &= F_{z\bar{z}}, \\
\{Q_+, \bar{\zeta}_+\} &= D_z B, & \{\bar{Q}_+, \zeta_+\} &= D_{\bar{z}} B, \\
[Q_-, A_\bar{z}] &= \zeta_+, & [\bar{Q}_-, A_z] &= \bar{\zeta}_+, \\
[Q_-, B] &= \zeta_-, & [\bar{Q}_-, B] &= \bar{\zeta}_-, \\
\{Q_-, \bar{\zeta}_-\} &= -D_{\bar{z}} B, & \{\bar{Q}_-, \zeta_-\} &= -D_z B, \\
\{Q_-, \bar{\zeta}_+\} &= F_{z\bar{z}}, & \{\bar{Q}_-, \zeta_+\} &= -F_{z\bar{z}},
\end{align*}
$$

(5.6)

where $D_z$ is the gauge covariant derivative. In addition to being Lorentz invariant ($[M, \zeta_{\pm}] = \frac{1}{2} \zeta_{\pm}$, where $M$ is the Lorentz generator), the action (5.5) is also symmetric under a fermion number $J$ for which the assignments $(\zeta_+, \bar{\zeta}_+, \zeta_-, \bar{\zeta}_-) \leftrightarrow (1, -1, 1, -1)$ are made. Twisting these fields by re-defining the Lorentz generator to be $M' \equiv M - \frac{1}{2} J$, we find that

$$
[M', (\zeta_+, \bar{\zeta}_+, \zeta_-, \bar{\zeta}_-)] = (0, \bar{\zeta}_+, -\zeta_-, 0). 
$$

(5.7)

That is, $\zeta_+$ transforms as the $z$-component of a vector whereas $\zeta_-$ behaves as its $\bar{z}$-component. Thus we rename

$$
(\zeta_+, \bar{\zeta}_+, \zeta_-, \bar{\zeta}_-) \longrightarrow (\chi, -\frac{1}{\sqrt{2}} \psi_{\bar{z}}, -\frac{1}{\sqrt{2}} \psi_z, -\eta_{z\bar{z}}),
$$

(5.8)

and the action becomes

$$
S^\text{twisted}_{\text{susy}} = \frac{i}{2} \int d^2z \text{Tr} \left( \eta_{z\bar{z}} \chi + BF_{z\bar{z}} + \frac{1}{2} \psi_{\bar{z}} \psi_z \right) = S'_0(\alpha_0 = 0). 
$$

(5.9)

This turns out to be the flat space-time version of $S'_0(\alpha_0 = 0)$ from (5.3), thus proving the above statement.

We now comment on the nature of the supersymmetries in the last two, untwisted and twisted, actions. First, $S'_0(\alpha_0 = 0)$ cannot be written as a $(Q + Q_\psi)$-exact expression [even though, as we have shown, $S_0(\alpha_0 = 0)$ can be written as a $Q$-exact expression]. This seems to be in contradiction with the fact that $S'_0(\alpha_0 = 0)$ is the twisted version of $S_{\text{susy}}$. Actually, it is possible
to show that $S_{\text{susy}}$ is itself given by the action of a spin-$\frac{1}{2}$ generator on some functional. This is most simply done in $N = 1$ superspace with two matter supermultiplets, the first one being the Yang-Mills supermultiplet and the second one being a spinor supermultiplet. The YM supermultiplet is given by the spinor superfield strength, $W_\alpha$ (greek letters are used as spinor indices) in which the lowest component is the gaugino $\lambda_\alpha$ and the middle component is the bosonic field strength, $F$. The spinor superfield, $\mathcal{B}_\alpha$, has components $(\kappa_\alpha, B, \beta_\alpha)$. The field $\zeta_\alpha$ which appears in $S_{\text{susy}}$ is the complex combination of $\beta_\alpha$ (the top component of $\mathcal{B}_\alpha$) and $\lambda_\alpha$. $\kappa_\alpha$ appears in the component form of the superspace lagrangian in a “kinetic” term for the gaugino $\lambda_\alpha$, as $i\kappa_\alpha \mathcal{P}_{\alpha\beta} \lambda_\beta$. This term is missing in $S_{\text{susy}}$. Likewise, its twisted version is missing in $S'_0(\alpha_0 = 0)$. Had we added such a term to the latter action, we would have obtained the twisted version of the $N = 2$ supersymmetric $BF$ action, which is a full fledged Donaldson-Witten theory. The lack of this term is the reason that $S'_0(\alpha_0 = 0)$ is not $(Q + Q_\psi)$-exact. Nevertheless, we reiterate that $S_0(\alpha_0 = 0)$ is $Q$-exact.

## 6 Conclusions

We have proven that all $BF$ theories on $n$-dimensional manifolds can be written as the anti-commutator of a Grassmann-odd, hermitian, nilpotent charge with some functional. This result was obtained by enlarging the space of fields to include two additional Grassmann-odd fields. The cohomology structure allows us to recover the known restrictions on the observables, namely that they may depend only on zero-modes of the fields and the homotopy classes of the homology cycles used in constructing them. Since our parametrization of the actions also includes the $F^2$-theories, for example Yang-Mills, it immediately follows that the latter theories are not $Q$-exact.

The $BF$ theory in the Landau gauge is perturbatively finite in dimensions less than six [7, 10, 11, 12]. The two-dimensional case, treated in [10], deserves special attention due to its IR singularities, but the proof of finiteness can nevertheless be maintained. Since the Grassmann-odd fields we added to the $BF$ action in order to prove its $Q$-exactness are non-propagating and uncoupled to the original fields, they are excluded from loop processes. Therefore, our result establishes that the $BF$ action is also perturbatively $Q$-exact.
References

[1] E. Witten, Commun. Math. Phys. 117 (1988) 353.

[2] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, Phys. Rep. 209 (1991) 129.

[3] G.T. Horowitz and M. Srednicki, Commun. Math. Phys. 130 (1990) 83; M. Blau and G. Thompson, Ann. Phys. 205 (1991) 130.

[4] R. Brooks, J.-G. Demers and C. Lucchesi, “Twisting to Abelian BF/Chern-Simons Theories”, MIT preprint # CTP 2237, bulletin board # hep-th/9308127, August 1993 (Nucl. Phys. B, in press).

[5] E. Witten, J. Geom. Phys. 9 (1992) 303.

[6] R. Brooks, “The Cosmological Constant and Volume-Preserving Diffeomorphism Invariants”, MIT preprint # CTP 2247, September 1993.

[7] C. Lucchesi, O. Piguet and S.P. Sorella, Nucl. Phys. B395 (1993) 325.

[8] R. Brooks and G. Lifschytz, in preparation.

[9] M. Blau and G. Thompson, “Lectures on Two-Dimensional Gauge Theories”, ICTP preprint # IC/93/356, bulletin board # hep-th/9310144, October 1993.

[10] A. Blasi and N. Maggiore, Class. Quan. Grav. 10 (1993) 37.

[11] N. Maggiore and S.P. Sorella, Nucl. Phys. B377 (1992) 236.

[12] E. Guadagnini, N. Maggiore and S.P. Sorella, Phys. Lett. 255B (1991) 65; N. Maggiore and S.P. Sorella, Intl. J. Mod. Phys. A8 (1993) 929.