ON THE CONJECTURE OF GENERALIZED TRIGONOMETRIC
AND HYPERBOLIC FUNCTIONS

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Abstract. In this paper we prove the conjecture posed by Klén et al. in [13], and
give optimal inequalities for generalized trigonometric and hyperbolic functions.

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1. INTRODUCTION

In 1995, P. Lindqvist [15] studied the generalized trigonometric and hyperbolic
functions with parameter \( p > 1 \). Thereafter several authors became interested to
work on the equalities and inequalities of these generalized functions, e.g., see [4, 5, 6]
[8, 7, 10, 11, 18] and the references therein. Recently, Klén et al. [13] were motivated
by many results on these generalized trigonometric and hyperbolic functions, and
they generalized some classical inequalities in terms of generalized trigonometric and
hyperbolic functions, such as Mitrinović-Adamović inequality, Huygens’ inequality,
and Wilker’s inequality. In this paper we prove the conjecture posed by Klén et al.
in [13], and in Theorem 1.4 we generalize the inequality

\[
\frac{1}{\cosh(x)^a} < \frac{\sin(x)}{x} < \frac{1}{\cosh(x)^b},
\]

where \( a = \log(\pi/2)/\log(\cosh(\pi/2)) \approx 0.4909 \) and \( b = 1/3 \), due to Neuman and
Sándor [17, Theorem 2.1].

For the formulation of our main results we give the definitions of the generalized
trigonometric and hyperbolic functions as below.

The increasing homeomorphism function \( F_p : [0, 1] \rightarrow [0, \pi_p/2] \) is defined by

\[
F_p(x) = \arcsin_p(x) = \int_0^x (1 - t^p)^{-1/p} dt,
\]

and its inverse \( \sin_{p,q} \) is called generalized sine function, which is defined on the interval
\([0, \pi_p/2]\), where

\[ \arcsin_p(1) = \pi_p/2. \]
The function $\sin_p$ is strictly increasing and concave on $[0, \pi_p/2]$, and it is also called the eigenfunction of the Dirichlet eigenvalue problem for the one-dimensional $p-$Laplacian [9]. In the same way, we can define the generalized cosine function, the generalized tangent, and also the corresponding hyperbolic functions.

The generalized cosine function is defined by

$$\frac{d}{dx} \sin_p(x) = \cos_p(x), \quad x \in [0, \pi_p/2].$$

It follows from the definition that

$$\cos_p(x) = (1 - (\sin_p(x))^p)^{1/p},$$

and

$$|\cos_p(x)|^p + |\sin_p(x)|^p = 1, \quad x \in \mathbb{R}. \quad (1.1)$$

Clearly we get

$$\frac{d}{dx} \cos_p(x) = -\cos_p(x)^{2-p} \sin_p(x)^{p-1}.$$

The generalized tangent function $\tan_p$ is defined by

$$\tan_p(x) = \frac{\sin_p(x)}{\cos_p(x)}.$$

For $x \in (0, \infty)$, the inverse of generalized hyperbolic sine function $\sinh_p(x)$ is defined by

$$\text{arsinh}_p(x) = \int_0^x (1 + t^p)^{-1/p} dt,$$

and generalized hyperbolic cosine and tangent functions are defined by

$$\cosh_p(x) = \frac{d}{dx} \sinh_p(x), \quad \tanh_p(x) = \frac{\sinh_p(x)}{\cosh_p(x)},$$

respectively. It follows from the definitions that

$$(1.2) \quad |\cosh_p(x)|^p - |\sinh_p(x)|^p = 1.$$

From above definition and (1.2) we get the following derivative formulas,

$$\frac{d}{dx} \cosh_p(x) = \cos_p(x)^{2-p} \sin_p(x)^{p-1}, \quad \frac{d}{dx} \tanh_p(x) = 1 - |\tanh_p(x)|^p.$$

Note that these generalized trigonometric and hyperbolic functions coincide with usual functions for $p = 2$.

Our main result reads as follows:

1.3. Theorem. [13 Conjecture 3.12] For $p \in [2, \infty)$, the function

$$f(x) = \frac{\log(x/\sin_p(x))}{\log(\sinh_p(x)/x)}$$
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is strictly increasing from $\left(0, \pi_p/2\right)$ onto $(1, p)$. In particular,

$$\left(\frac{x}{\sinh_p(x)}\right)^p < \frac{\sin_p(x)}{x} < \frac{x}{\sinh_p(x)}.$$ 

1.4. Theorem. For $p \in [2, \infty)$, the function

$$g(x) = \frac{\log(x/\sin_p(x))}{\log(cosh_p(x))}$$

is strictly increasing in $x \in (0, \pi_p/2)$. In particular, we have

$$\frac{1}{\cosh_p(x)^\beta} < \frac{\sin_p(x)}{x} < \frac{1}{\cosh_p(x)^\alpha},$$

where $\alpha = 1/(1 + p)$ and $\beta = \log(\pi_p/2)/\log(cosh_p(\pi_p/2))$ are the best possible constants.

2. Preliminaries and proofs

The following lemmas will be used in the proof of main result.

2.1. Lemma. [2, Theorem 2] For $-\infty < a < b < \infty$, let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, and be differentiable on $(a, b)$. Let $g'(x) \neq 0$ on $(a, b)$. If $f'(x)/g'(x)$ is increasing (decreasing) on $(a, b)$, then so are

$$f(x) - f(a) \quad \text{and} \quad g(x) - g(a).$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

2.2. Lemma. For $p \in [2, \infty)$, the function

$$f(x) = \frac{p \sin_p(x) \log(x/\sin_p(x))}{\sin_p(x) - x \cos_p(x)}$$

is strictly decreasing from $(0, \pi_p/2)$ onto $(1, p \log(\pi_p/2))$. In particular,

$$\exp\left(\frac{1}{p} \left(\frac{x}{\tan_p(x)} - 1\right)\right) < \frac{\sin_p(x)}{x} < \exp\left(\left(\log \left(\frac{\pi_p}{2}\right)\right) \left(\frac{x}{\tan_p(x)} - 1\right)\right).$$

Proof. Write

$$f_1(x) = p \sin_p(x) \log(x/\sin_p(x)), \quad f_2(x) = \sin_p(x) - x \cos_p(x),$$

and clearly $f_1(0) = f_2(0) = 0$. Differentiation with respect $x$ gives

$$\frac{f_1'(x)}{f_2'(x)} = \frac{(\sin_p(x))/x + \cos_p(x)(\log(x/\sin_p(x)) - 1)}{x \cos_p(x) x^{2-p} \sin_p(x)^{p-1}}$$

$$= \frac{1}{x \tan_p(x)^{p-1}} \left(\frac{1}{x \cos_p(x)} + \log \left(\frac{x}{\sin_p(x)}\right) - 1\right),$$
which is the product of two decreasing functions, this implies that $f'_1/f'_2$ is decreasing. Hence the function $f$ is decreasing by Lemma 2.1. The limiting values follows from the l’Hôpital rule. □

### 2.3. Lemma

For $p \in [2, \infty)$ the function

$$g(x) = \frac{p \sinh_p(x) \log(\sinh_p(x)/x)}{x \cosh_p(x) - \sinh_p(x)}$$

is strictly increasing from $(0, \infty)$ onto $(1, p)$. In particular, we have

$$\exp\left(\frac{1}{p}\left(\frac{x}{\tanh_p(x)} - 1\right)\right) < \frac{\sinh_p(x)}{x} < \exp\left(\frac{x}{\tanh_p(x)} - 1\right).$$

**Proof.** Write

$$g_1(x) = \sinh_p(x) \log\left(\frac{\sinh_p(x)}{x}\right), \quad g_2(x) = x \cosh_p(x) - \sinh_p(x),$$

clearly $g_1(0) = g_2(0) = 0$. Differentiation with respect $x$ gives

$$g_1'(x) = \frac{\cosh_p(x)(1 + \log(\sinh_p(x)/x) - \sinh_p(x)/x)}{x \cosh_p(x)^{2-p} \sinh_p(x)^{p-1}},$$

$$g_2'(x) = \frac{\sinh_p(x) \cosh_p(x)(1 + \log(1 + \sinh_p(x)/x) - \sinh_p(x)/x)}{x \cosh_p(x) \tanh_p(x)^p},$$

which is increasing, this implies that $g$ is increasing. The limiting values follows from the l’Hôpital rule. □

### 2.4. Lemma

For all $x > 0$ and $p > 1$, we have

$$\log(\cosh_p(x)) > \frac{x}{p} \tanh_p(x)^{p-1}x.$$ 

**Proof.** Let

$$f(x) = \log(\cosh_p(x)) - \frac{x}{p} \tanh_p(x)^{p-1}.$$ 

A simple computation yields

$$f'(x) = \frac{\tanh_p(x)^{p-1}}{p} - \left(\frac{\tanh_p(x)^{p-2}}{p} + \frac{(p - 1) x \tanh_p(x)^{p-2}}{p \cosh_p(x)^p}\right),$$

$$= \frac{p - 1}{p} \tanh_p(x)^{p-2}\left(\tanh_p(x) - \frac{x}{\cosh_p(x)^p}\right).$$

which is positive because $\sinh_p(x) > x$ and $\cosh_p(x) > 1$ for all $x > 0$. Thus $f(x)$ is strictly increasing and $f(x) > f(0) = 0$, this implies the proof. □

**Proof of Theorem 1.3.** Write $f(x) = f_1(x)/f_2(x)$ for $x \in (0, \pi_p/2)$, where

$$f_1(x) = \log\left(\frac{x}{\sin_p(x)}\right), \quad f_2(x) = \log\left(\frac{\sinh_p(x)}{x}\right).$$
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For the proof of the monotonicity of the function $f$, it is enough to prove that

$$f'(x) = \frac{f'_1(x)f_2(x) - f_1(x)f'_2(x)}{f_2(x)^2}$$

is positive. After simple computation, this is equivalent to write

$$x (f_2(x))^2 f'(x) = \frac{\sin_p(x) - x \cos_p(x)}{\sin_p(x)} f_2(x) - \frac{x \cosh_p(x) - \sinh_p(x)}{\sinh_p(x)} f_1(x),$$

which is positive by Lemmas 2.2 and 2.3. Hence, $f$ is strictly increasing, and limiting values follow by applying the l'Hôpital rule. This completes the proof. □

Proof of Theorem 1.4. Write $g(x) = g_1(x)/g_2(x)$ for $x \in (0, \pi_p/2)$, where $g_1(x) = \log(x/\sin_p(x))$, $g_2(x) = \log(\cosh_p(x))$. Here we give the same argument as in the proof of the Theorem 1.3 and compute similarly

$$\left(\log(\cosh_p(x))\right)^2 g'(x) = \frac{\sin_p(x) - x \cos_p(x)}{x \sin_p(x)} \log \cosh_p(x) - \tanh_p(x)^{p-1} \log \left(\frac{x}{\sin_p(x)}\right)$$

$$> \frac{\sin_p(x) - x \cos_p(x)}{x \sin_p(x) \tanh_p(x)^{1-p}} p \frac{\sin_p(x) - x \cos_p(x)}{p \sin_p(x) \tanh_p(x)^{1-p}}$$

$$= 0,$$

by Lemmas 2.2 and 2.3. The limiting values follow from the l'Hôpital rule easily, hence the proof is obvious. □

The following corollary follows from [13, Lemma 3.3] and Theorem 1.4.

2.5. Corollary. For $p \in [2, \infty)$ and $x \in (0, \pi_p/2)$, we have

$$\cos_p(x)^\beta < \frac{1}{\cosh_p(x)^\beta} < \frac{\sin_p(x)}{x} < \frac{1}{\cosh_p(x)\alpha} < 1,$$

where $\alpha$ and $\beta$ are as in Theorem 1.4.

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