Weighted Intriguing Sets in Finite Polar Spaces

John Bamberg, Jan De Beule, and Ferdinand Ihringer

ABSTRACT. We develop a theory for ovoids and tight sets in finite classical polar spaces, and we illustrate the usefulness of the theory by providing new proofs for the non-existence of ovoids of particular finite classical polar spaces, including $\mathbb{Q}^+(9, q)$, $q$ even, and $H(5, 4)$. We also improve the results of A. Klein on the non-existence of ovoids of $H(2n + 1, q^2)$ and $\mathbb{Q}^+(2n + 1, q^2)$.

1. Introduction

Finite polar spaces (of which finite classical polar spaces are a subclass), are the natural geometries for the finite classical simple groups. In the history of polar spaces, the work of F. Veldkamp [33] and J. Tits [32] plays a fundamental role. The main geometric property of polar spaces, as shown by Buekenhout and Shult [8], is the so-called one or all axiom. A polar space is a point-line geometry with the property that if $P$ is a point and $\ell$ a line not incident with $P$, then $P$ is collinear with either all points of $\ell$ or with exactly one point of $\ell$. Polar spaces are some of the most important examples in the theory of incidence geometries and provide a rich class of spherical buildings. We will be interested in finite classical polar spaces which arise from equipping a finite vector space with a sesquilinear or quadratic form. Moreover, these are the only finite polar spaces of rank at least 3 by a theorem of Tits and Veldkamp [32].

According to Dembowski [14, footnote page 48], an ovoid of a projective space is defined for the first time by Tits in [31], where Tits began his introduction with reference to the works of Barlotti and Segre ([5] and [26, 25]) on arcs and caps of finite projective spaces. The connection with polar spaces is due to Tits’ geometric construction of the Suzuki groups $2B_2(2^{2h+1})$ (see also [31]), where Tits’ ovoids can also be realised in a second sense, as ovoids of a particular finite classical polar space (i.e., rank 2 symplectic spaces). Furthermore, the Ree groups $2G_2(3^{2h+1})$ can also be naturally constructed from ovoids of a certain family of finite classical polar spaces (i.e., rank 3 parabolic quadrics). In 1974, J. A. Thas [32] synthesised these objects as ovoids of polar spaces; a set $O$ of points such that every generator of $P$ meets $O$ in exactly one point. (See the next section for necessary definitions and background).

To use the words of J. A. Thas [30], ovoids of polar spaces have “many connections with and applications to projective planes, circle geometries, generalised polygons, strongly regular graphs, partial geometries, semi-partial geometries, codes, designs.” Recently, there has been an interesting development in the theory of permutation groups that has links with ovoids of polar spaces. A permutation group $G$ acting on $\Omega$ is separating if there do not exist subsets $A$, $B$ of $\Omega$ such that $|A|, |B| > 1$ and $|A^g \cap B| = 1$ for all $g \in G$. A classical group, acting naturally
on the points of its polar space $P$, is non-separating if and only if $P$ possesses an ovoid (see the proof of [9, Theorem 3.5]).

In [29], the existence and non-existence of ovoids of polar spaces was first investigated. A striking consequence is that the existence of ovoids of finite polar spaces simulates the results of Tits for ovoids of projective spaces; that is, that ovoids exist only when the rank of the geometry\(^2\) is small. Shult [27, §2.2] all but conjectures that if a polar space of rank $r$ possesses an ovoid, then $r \leq 4$. Thas [29] showed that this property holds true for the symplectic and elliptic polar spaces, but also for Hermitian polar spaces in even dimension. The seminal work of Thas inspired vibrant investigations into the existence and non-existence of ovoids on many diverse fronts, with widely different perspectives and techniques. However, some notorious cases remain open and are some of the most prominent open problems in finite geometry. These cases are: existence of ovoids of the Hermitian polar space $H(2n + 1, q^2)$ for general $q$, existence of ovoids of the polar space $Q(6, q)$, $q$ not prime, $q$ not even and $q \neq 3^h$, and existence of ovoids of $Q^+(2n + 1, q)$, for general $q$ and $n > 3$. We will survey what is known in the next section.

In this paper, it is our main objective to describe a unified approach to show the non-existence of ovoids of finite classical polar spaces. Our approach is based on the study of weighted intriguing sets of finite classical polar spaces, of which ovoids are a particular example. In some cases, this yields a new and shorter proof. We give short non-existence proofs for ovoids of $Q^-(5, q)$, $W(5, q)$ and $H(4, q^2)$; we provide a geometric proof for the non-existence of ovoids of $Q^+(9, q)$, $q$ even; we provide a proof for the non-existence of ovoids of $H(5, 4)$, based on the use of particular intriguing sets, and we improve results of A. Klein on the non-existence of ovoids of $H(2n + 1, q^2)$.

2. Preliminary definitions and survey of (non)-existence results

Throughout, we will use the symbol $q$ for a prime power $q := p^h$, $p$ prime and $h \geq 1$, and we will denote the the finite field of order $q$ as $\text{GF}(q)$. The vector space of dimension $d$ over $\text{GF}(q)$ will be written as $V(d, q)$, and $\text{PG}(n, q)$ will denote the projective space with underlying vector space $V(n + 1, q)$. Let $f$ be a (reflexive) sesquilinear or quadratic form on $V(n + 1, q)$. The elements of the finite classical polar space $P$ associated with $f$ are the totally singular or totally isotropic subspaces of $\text{PG}(n, q)$ with relation to $f$, according to whether $f$ is a quadratic or sesquilinear form. The Witt index of the form $f$ determines the the dimension of the subspaces of maximal dimension contained in $P$; the rank $P$ equals the Witt index of its form, and the (projective) dimension of generators will be one less than the Witt index. Hence, a finite classical polar space of rank $r$ embedded in $\text{PG}(n, q)$ has an underlying form of Witt index $r$, and contains points, lines, ..., $(r - 1)$-dimensional subspaces. The elements of maximal dimension are called its generators.

We will use projective notation for polar spaces so that they differ from the standard notation for their collineation groups. For example, we will use the notation $W(d - 1, q)$ to denote the symplectic polar space coming from the vector space $V(d, q)$ equipped with a non-degenerate alternating form. Here is a summary of the notation we will use for polar spaces, together with their ovoid numbers (which we define below).
Table 1. Notation for the finite classical polar spaces, together with their ovoid numbers.

| Polar Space               | Notation | Collineation Group | Ovoid Number | Type $e$ |
|--------------------------|----------|--------------------|--------------|---------|
| Symplectic               | $W(d-1,q)$, $d$ even | $P^GSp(d,q)$ | $q^{d/2} + 1$ | 1       |
| Hermitian                | $H(d-1,q^2)$, $d$ odd | $P^GU(d,q)$ | $q^d + 1$ | 3/2     |
| Hermitian                | $H(d-1,q^2)$, $d$ even | $P^GU(d,q)$ | $q^{d-1} + 1$ | 1/2     |
| Orthogonal, elliptic     | $Q^-(d-1,q)$, $d$ even | $P^GO^-(d,q)$ | $q^{d/2} + 1$ | 2       |
| Orthogonal, parabolic    | $Q(d-1,q)$, $d$ odd | $P^GO(d,q)$ | $q^{(d-1)/2} + 1$ | 1       |
| Orthogonal, hyperbolic   | $Q^+(d-1,q)$, $d$ even | $P^GO^+(d,q)$ | $q^{d/2-1} + 1$ | 0       |

Throughout, two sets of subspaces $S_1$ and $S_2$ of a common polar space will be said to be equivalent if there exists a collineation of the polar space mapping $S_1$ onto $S_2$.

Let $\mathcal{P}$ be a polar space defined by a sesquilinear or quadratic form $f$. Let $X$ be a point of the ambient projective space. Then $X^\perp$ is the set of projective points whose coordinates are orthogonal to $X$ with respect to the form $f$. Note that when $f$ is a quadratic form, it determines a (possibly degenerate when $q$ is even), symplectic form $g$, and two projective points $X$ and $Y$ are orthogonal with relation to $f$ if, by definition, they are orthogonal with relation to $g$.

The set of points $X^\perp$ is a hyperplane, and when $X$ is a point of $\mathcal{P}$, the hyperplane $X^\perp$ is the tangent hyperplane at $X$ to $\mathcal{P}$. For any set $A$ of points, $A^\perp := \cap_{X \in A} X^\perp$. The following result is fundamental in the theory of finite classical polar spaces.

**Lemma 2.1.** Let $\mathcal{P}_r$ be a polar space of rank $r$, $r \geq 2$, and let $X$ be a point of $\mathcal{P}_r$. Then the set of points $X^\perp \cap \mathcal{P}_r$ is a cone with vertex $X$ and base a polar space $\mathcal{P}_{r-1}$ of rank $r-1$, of the same type as $\mathcal{P}_r$.

As a consequence, the elements of a polar space $\mathcal{P}_r$ incident with a point $X \in \mathcal{P}_r$ induce a polar space of rank $r-1$ of the same type, which is called the quotient space, and which is sometimes denoted as $X^\perp / X$. This is equivalent to saying that projecting the elements from $X$ onto any hyperplane $\pi$ not on $X$, will yield a polar space isomorphic with $\mathcal{P}_{r-1}$ embedded in the subspace $\pi \cap X^\perp$, [18, p. 3].

**Definition 2.2.** Suppose that $\mathcal{P}$ is a finite polar space. An ovoid is a set $O$ of points of $\mathcal{P}$ such that every generator of $\mathcal{P}$ meets $O$ in exactly one point.

The non-existence of ovoids in higher rank is implied by the non-existence of ovoids in low rank. Projection from a point $X$ not in an ovoid is well-known to produce an ovoid of the quotient polar space (see [20, §2]), which we reproduce below.

**Lemma 2.3.** Let $O$ be an ovoid of a polar space $\mathcal{P}_r$ of rank $r \geq 3$ and embedded in $PG(d,q)$, and let $X$ be a point of $\mathcal{P}$ not in $O$. Then $O$ induces an ovoid $O_X$ in a polar space embedded in $PG(d-2,q)$ of the same type but of rank $r-1$.

**Proof.** The quotient space $X^\perp / X$ is a polar space $\mathcal{P}_{r-1}$ of rank $r-1$. Each generator of $\mathcal{P}_r$ meets $O$ in exactly one point, so after projection from $X$, the induced generators of $\mathcal{P}_{r-1}$ meet the projection of $O$ in exactly one point. So $O$ induces an ovoid $O_X$ of $\mathcal{P}_{r-1}$.

Table 2 is the summary of Table 1.2 in [10]. For each polar space, we indicate if examples of ovoids are known, if the non-existence is shown, or if the existence of ovoids is open. More information on particular examples (and references for these examples) can be found in [10].

---

3When $f$ is a quadratic form, $g(v, w) := f(v + w) - f(v) - f(w)$ is an alternating form.
4If $(V, f)$ is a formed space, and $X$ is a totally isotropic subspace of $V$, then we can equip the quotient vector space $X^\perp / X$ with the form $f'$ defined by $f'(X + u, X + v) := f(u, v)$, which is the algebraic counterpart of the geometric statement.
| Polar Space | Non-existence shown / Known examples | References |
|-------------|-------------------------------------|------------|
| $Q(2n+1,q)$ | $n > 1$: non-existence shown        | [29]       |
| $Q(4,q)$    | examples known for all $q$         |            |
| $W(3,q)$    | examples known for $q = 2^h$       |            |
|             | non-existence shown for $q$ odd    | [26]+[28]  |
| $Q(6,q)$    | $q$ even; $q > 3$, $q$ prime: non-existence shown $q = 3^h$; examples known other values of $q$: (non)-existence is open | [29]; [23]+[1] |
| $Q(2n,q)$   | $n \geq 4$: non-existence shown    | [17]; [29] |
| $Q^+(2n+1,q)$ | examples known for all $q$ and $n = 1,2$ |            |
| $Q^+(7,q)$  | examples known for $q = 2^h,3^h$, $q = p^h$, $p \equiv 2 \mod 3$, $p$ prime, $h$ odd, $q \geq 5$ prime |            |
| $Q^+(2n+1,q)$ | $q = p^h$, $p$ prime, non-existence shown if $p^n > (2n+p)/(2n+1)$ | [6]         |
| $W(2n+1,q)$ | non-existence shown for $q$ odd and $n = 1$; and for all $q$, $n > 1$ | [29]        |
| $H(4,q^2)$  | non-existence shown                | [29]       |
| $H(3,q^2)$  | examples known                     |            |
| $H(2n+1,q^2)$ | $q = p^h$, $p$ prime, non-existence shown if $p^{2n+1} > (2n+p)^2 - (2n+p-1)^2$ | [22]        |
| $H(5,4)$    | no                                 | [11]       |

### 3. Intriguing Sets of Strongly Regular Graphs

This section repeats the theory of intriguing sets of polar spaces as considered by [3, 4, 15] in the more general context of strongly regular graphs. All the results in this section are due to Delsarte [12, 13].

Let $\Gamma = (X, \sim)$ be a graph, where $X$ is a set of vertices, and $\sim$ is a symmetric relation with $\sim \subseteq X \times X$. We say that two vertices are adjacent if they are in relation $\sim$. We say that $\Gamma$ is strongly regular with parameters $(n, k, \lambda, \mu)$ if all of the following holds:

(i) The number of vertices is $n$.
(ii) Each vertex is adjacent with $k$ vertices.
(iii) Each pair of adjacent vertices is commonly adjacent to $\lambda$ vertices.
(iv) Each pair of non-adjacent vertices is commonly adjacent to $\mu$ vertices.

The adjacency matrix $A$ of $\Gamma$ is matrix over $\mathbb{C}$ indexed by the vertices $X$ with $(A)_{xy} = 1$ if the vertex $x$ and the vertex $y$ are adjacent and $(A)_{xy} = 0$ if the vertex $x$ and the vertex $y$ are non-adjacent. If $0 < k < n - 1$, then the adjacency matrix of $A$ has exactly three eigenvalues $k$, $r$, and $s$ [7, Theorem 1.3.1 (i)]. The eigenvalue $k$ has multiplicity 1, and the all-ones vector $j$ is one of its eigenvectors. The other eigenvalues satisfy

\begin{align*}
    r &= \frac{\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2} \\
    s &= \frac{\lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2}
\end{align*}

Let $V_+$, respectively, $V_-$ the eigenspace corresponding to $r$, respectively, $s$. Since $A$ is a symmetric matrix over $\mathbb{C}$, the eigenspaces of $A$ yield an orthogonal direct sum decomposition of $\mathbb{C}^n$:

\[ \mathbb{C}^n = (j) \perp V_+ \perp V_- \]
The parameters of a strongly regular graph are constrained by a wealth of well-known equations. For our purposes, we need the following three equations [7, Theorem 1.3.1]:

\begin{align}
(3) \quad n\mu &= (k - r)(k - r), \\
(4) \quad sr &= \mu - k, \\
(5) \quad k(k - \lambda - 1) &= (n - k - 1)\mu. 
\end{align}

Following the theme of [3], we will introduce the notion of a weighted intriguing set of a polar space.

**Definition 3.1.** Let $\chi \in \mathbb{C}^n$ and $\epsilon \in \{-, +\}$. We say that $\chi$ is a weighted intriguing set if $\chi \in \langle j \rangle \perp V^\epsilon$. If $\chi$ is a 0-1-vector, then we say that $\chi$ is an intriguing set. We call a weighted intriguing set $\chi$ in $\langle j \rangle \perp V_-$ a weighted ovoid. We call a weighted intriguing set $\chi$ in $\langle j \rangle \perp V_+$ a weighted tight set. We say that a collection of intriguing sets have the same type if they are either all weighted oovoids, or if they are all weighted tight sets.

In the language of Delsarte, an intriguing set is a design for the association scheme arising from the strongly regular graph. The following result is due to Delsarte, and we repeat its short proof to make this paper more self-contained.

**Lemma 3.2.** Let $\Gamma = (X, \sim)$ be a strongly regular graph with parameters $(n, k, \lambda, \mu)$, $0 < k < n - 1$ and adjacency matrix $A$. Let $r$ and $s$ be the eigenvalues of $A$ different to $k$. Let $\chi \in \mathbb{C}^n$. Then

\[
(j\chi^\top)^2k + s(n\chi \chi^\top - (j\chi^\top)^2) \leq n\chi A\chi^\top \leq (j\chi^\top)^2k + r(n\chi \chi^\top - (j\chi^\top)^2).
\]

Equality holds on the left hand side if and only if $\chi$ is a weighted ovoid. Equality holds on the right hand side if and only if $\chi$ is a weighted tight set. In particular, if $\chi$ is a 0-1-vector, then

\[
(j\chi^\top)^2k + sj\chi^\top(n - j\chi^\top) \leq n\chi A\chi^\top \leq (j\chi^\top)^2k + rj\chi^\top(n - j\chi^\top).
\]

**Proof.** We can write $\chi$ as a sum of eigenvectors, so $\chi = \alpha j + \chi_+ + \chi_-$ with $\alpha \in \mathbb{C}$, $\chi_+ \in V_+$, $\chi_- \in V_-$. Recall that $j$, $\chi_+$, and $\chi_-$ are pairwise orthogonal. Then

\[
(6) \quad \chi A\chi^\top = (\alpha j + \chi_+ + \chi_-)A(\alpha j + \chi_+ + \chi_-)^\top = nk\alpha^2 + r\chi_+ \chi_+^\top + s\chi_- \chi_-^\top,
\]

and

\[
(7) \quad 0 = \chi_+ \chi_+^\top + \chi_- \chi_-^\top = (\chi_+ + \chi_-)(\chi_+ + \chi_-)^\top = \chi\chi^\top - \alpha^2 jj^\top = \chi\chi^\top - n\alpha^2.
\]

By (6), (7), $r > 0$, and $s < 0$, we have

\[
\begin{align*}
    nk\alpha^2 + s(\chi\chi^\top - n\alpha^2) &\leq \chi A\chi^\top \leq nk\alpha^2 + r(\chi\chi^\top - n\alpha^2).
\end{align*}
\]

We have $j\chi^\top = \alpha jj^\top$. Hence, $\alpha = j\chi^\top / n$. This yields the first part of the assertion. If $\chi$ is a 0-1-vector, then $\chi\chi^\top = j\chi^\top$ yields the remaining claim. \qed

If the vector $\chi$ is the characteristic vector of a set $Y$,\footnote{This means $\chi_x = 1$ if $x \in Y$, and $\chi_x = 0$ if $x \in X \setminus Y$.} then $\chi A\chi^\top / j\chi^\top = \chi A\chi^\top / |Y|$ is the average number of adjacent vertices in $Y$. A *coclique* of a graph is a set of pairwise non-adjacent matrices (i.e. $\chi A\chi^\top = 0$), while a *clique* is a set of pairwise adjacent vertices (i.e. $\chi A\chi^\top = x(x - 1)$). In the point graph of a polar space, cocliques are traditionally called *ovoids* if they satisfy the first inequality of Lemma 3.2, and cliques are traditionally called *tight sets* if they satisfy the second inequality of Lemma 3.2. Lemma 3.2 makes it clear that the characteristic vector $\chi$ of an ovoid is a weighted ovoid with $j\chi^\top = \frac{ns}{k - s}$, and that the...
characteristic vector of a tight set is a weighted tight set\(^6\) with \(j\chi^\top = 1 - \frac{k}{s}\). Consistent with this definition, we call a weighted ovoid \(\chi\) a \textit{weighted }\(m\)-ovoid if \(j\chi^\top = m\frac{s}{k} = \frac{1}{s}\), and we call a weighed tight set \(\psi\) a \textit{weighted }\(i\)-tight set if \(j\psi^\top = i(1 - \frac{k}{s})\). If a weighted \(m\)-ovoid \(\chi\) is a 0-1-vector, then we say that \(\chi\) is an \(m\)-ovoid. If a weighted \(i\)-tight set \(\psi\) is a 0-1-vector, then we say that \(\psi\) is an \(i\)-tight set. We identify 0-1-vectors with the corresponding sets of vertices. Then this is consistent with the usual definitions of \(m\)-ovoids and \(i\)-tight sets which can be found in the literature [4].

The following result has been known for a long time and is crucial for the investigation of intriguing sets.

**Lemma 3.3.** Let \(\chi\) be a weighted \(m\)-ovoid. Let \(\psi\) be a weighted \(i\)-tight set. Then

\[\chi\psi^\top = mi.\]

**Proof.** By definition, there exist \(\chi_- \in V_-\) and \(\psi_+ \in V_+\) such that \(\chi = m\frac{s}{k} - s\chi_- + \frac{s}{k}\psi_+\) and \(\psi = i(1 - \frac{k}{s})j/n + \psi_+\). The vectors \(\chi\) and \(\psi\) are orthogonal, hence

\[
\chi\psi^\top = (m\frac{s}{k} - s\chi_-) + (i(1 - \frac{k}{s})j/n + \psi_+) \]

\[
= mi\frac{s}{k} - s(1 - \frac{k}{s})jj^\top = mi.
\]

\[\Box\]

Let \(G\) be a group of automorphisms of the graph \(\Gamma\). We say that \(G\) acts \textit{generously transitively} on \(\Gamma\), if for all vertices \(x, y \in \Gamma\), there exists an automorphism \(g \in G\) such that \(x^g = y\) and \(y^g = x\).

We say that a weighted intriguing set is \textit{non-trivial} if it is not in the span of \(j\). The following characterisation of intriguing sets, which follows immediately from the definitions, is very helpful.

**Lemma 3.4.** Let \(G\) be a group which acts generously transitively on \(\Gamma\). Let \(\chi, \psi \in \mathbb{C}^n\) with \(\chi, \psi \notin \langle j \rangle\). Then the following statements are equivalent.

(a) One of the vectors \(\chi\) and \(\psi\) is a non-trivial weighted \(m\)-ovoid, and one of the vectors is a non-trivial weighted \(i\)-tight set.

(b) For all \(g \in G\), we have \(\psi^g\chi^\top = mi\).

Lemma 3.4 implies the following.

**Corollary 3.5.** Let \(G\) be a group which acts generously transitively on \(\Gamma\). Let \(\chi \in \mathbb{C}^n\).

(a) If \(\chi\) is a non-trivial weighted \(m\)-ovoid with \(m \neq 0\), then \(\langle j \rangle \perp V_- = \langle \chi^g : g \in G \rangle\).

(b) If \(\chi\) is a non-trivial weighted \(i\)-tight set with \(i \neq 0\), then \(\langle j \rangle \perp V_+ = \langle \chi^g : g \in G \rangle\).

(c) If \(\chi\) is a non-trivial weighted \(0\)-ovoid, then \(V_- = \langle \chi^g : g \in G \rangle\).

(d) If \(\chi\) is a non-trivial weighted \(0\)-tight set, then \(V_+ = \langle \chi^g : g \in G \rangle\).

**Proof.** Let \(\chi\) be a weighted \(0\)-ovoid. Obviously, \(\langle \chi^g : g \in G \rangle \subseteq V_-\). Suppose contrary to our claim that \(\langle \chi^g : g \in G \rangle \neq V_-\). Then there exists a weighted \(0\)-ovoid \(\chi' \in V_- \setminus \langle \chi^g : g \in G \rangle\) orthogonal to \(\langle \chi^g : g \in G \rangle\). Hence by Lemma 3.4, \(\chi'\) is a weighted tight set. Contradiction.

The other cases follow similarly. \[\Box\]

Lemma 3.4 is an excellent tool for showing that a set is intriguing: If one knows a weighted ovoid \(\chi\), then one can see if a vector \(\psi\) is a weighted tight set just by considering \(\chi^g\psi^\top\). If one knows a weighted tight set \(\psi\), then one can see if a vector \(\chi\) is a weighted ovoid set just

\[\text{This equality is not obvious, since Lemma 3.2 only shows } j\chi^\top = \frac{n(r+1)}{n-k+r}. \text{ One can use the well-known identities } n\mu = (k-r)(k-r) \text{ and } sr = \mu - k \text{ [7, Theorem 1.3.1 (iii)] to prove the claim.}\]
by considering $\chi(\psi^g)^\top$ (for each $g \in G$). One needs some general examples for non-trivial intriguing sets to do so, and we shall construct some simple ones in the future. For a point $x \in X$ we write $x^\sim$ for the set of all vertices adjacent with $x$. We write $\chi_M$ for the characteristic vector of a set of vertices $M \subseteq X$.

**Example 3.6.** Let $\Gamma = (X, \sim)$ be a strongly regular graph with parameters $(n, k, \lambda, \mu)$. Let $\alpha = \frac{k(n-k+\mu)}{ns+k-s}$. Let $x$ be a vertex of $\Gamma$. Then
\[
\chi = \alpha \chi_\{x\} + \chi_{x^\sim}
\]
is a non-trivial weighted $(\alpha+k)(1-k)\frac{\epsilon}{n}$-ovoid.

**Proof.** We have to show that $\chi$ satisfies the lower bound on $\chi A \chi^\top$ from Lemma 3.2 with equality. By the definition of $\chi$, we have
\[
j \chi^\top = \alpha + k, \quad \chi^\top = \alpha^2 + k, \quad \chi A \chi^\top = k(2\alpha + \lambda).
\]
Together with the equations (2) and (5) one can easily check that $\chi$ reaches the lower bound in Lemma 3.2. Hence, $\chi$ is a weighted ovoid. The weight of the weighted ovoid follows from the definition. 

**Example 3.7.** Let $\Gamma = (X, \sim)$ be a strongly regular graph with parameters $(n, k, \lambda, \mu)$. Let $\alpha = \frac{k(n-k+r)}{nr+k-r}$. Let $x$ be a vertex of $\Gamma$. Then
\[
\chi = \alpha \chi_\{x\} + \chi_{x^\sim}
\]
is a non-trivial weighted $\alpha+k\frac{1-k}{1-\frac{\epsilon}{2}}$-tight set.

The definition of weighted intriguing sets obviously implies that linear combinations of weighted intriguing sets of the same type are again weighted intriguing sets. This motivates the following lemma, whose proof is straightforward and we leave (as a simple exercise) for the reader.

**Lemma 3.8.** Let $G$ be a group which acts generously transitively on $\Gamma$. Let $U$ be a subgroup of $G$. Let $\chi$ be a non-trivial weighted intriguing set of $\Gamma$. Define $I$ as
\[
\left\{ \sum_{u \in U} (\chi^g)^u : g \in G \right\}.
\]
Then the following holds.
\[(a)\] The elements of $I$ are intriguing sets of the same type as $\chi$.
\[(b)\] Let $\psi$ be a weighted intriguing set of the same type as $\chi$, fixed by $U$ (i.e., $\psi^u = \psi$ for all $u \in U$). Then $\psi \in \langle I \rangle$.

4. Intriguing Sets of Polar Spaces

Recall from Table 1 that all polar spaces have a type, and the type $e$ can be easily discerned from the rank 1 example, having $q^e + 1$ singular points, where $q$ is the order of the defining field. So for example, the hyperbolic quadrics have $e = 0$ since $Q^\top(1, q)$ consists of two singular points (on the projective line), yet the Hermitian spaces of even dimension have type $e = 3/2$ as $H(2, q^2)$ has $q^{3/2} + 1$ singular points. In general, $e \in \{0, 1/2, 1, 3/2, 2\}$. This number will be useful for summarising the essential combinatorial information of all polar spaces in this section. The
collineation graph of the point set of a polar space of rank $d$ and type $e \in \{0, 1/2, 1, 3/2, 2\}$ over $\text{GF}(q)$ is a strongly regular graph with the parameters [4, Section 4]

\[
\begin{align*}
    n &= \frac{(q^{d+1+e} + 1)(q^d - 1)}{q - 1}, \\
    k &= q \frac{(q^{d-2+e} + 1)(q^{d-1} - 1)}{q - 1}, \\
    \lambda &= q^{d-1} - 1 + q \frac{(q^{d-2+e} + 1)(q^{d-2} - 1)}{q - 1}, \\
    \mu &= \frac{(q^{d-2+e} + 1)(q^{d-1} - 1)}{q - 1}, \\
    r &= q^{d-1} - 1, \text{ and } s = -1 - q^{d-2+e} \\
    m_r &= q^e \frac{(q^d - 1)(q^{d-2+e} + 1)}{(q - 1)(q^{e-1} + 1)}, \text{ and } m_s = q \frac{(q^{d-1} - 1)(q^{d-1+e} + 1)}{(q - 1)(q^{e-1} + 1)}.
\end{align*}
\]

Let $P$ be a point of the polar space. We write $P^\sim$ for all points collinear, but not equal to $P$. Then Example 3.6 equals

\[-(q^{d-1} - 1)\chi_P + \chi_P^\sim\]

and is a non-trivial weighted $\frac{q^{d-1} - 1}{q - 1}$-ovoid. Example 3.7 equals

\[(q^{d-2+e} + 1)\chi_P + \chi_P^\sim\]

and is a non-trivial weighted $(q^{d-2+e} + 1)$-tight set. Let $T$ be the points of a generator of the considered polar space. Then $\chi_T$ is a well-known 1-tight set [4, §3]. We will only use these three intriguing sets to show that other vectors are intriguing sets. The collineation group of a polar space acts generously transitively on the point set of the polar space, so we can use Lemma 3.4 and Lemma 3.8.

The following result is well known for intriguing sets [4, Lemma 7 ff.].

**Lemma 4.1.** Let $P_d$ be a finite classical polar space of rank $d$. Let $H$ be a non-degenerate hyperplane.

(a) If $\chi$ is a weighted $m$-ovoid of $P_d$ and $P_d \cap H$ is a polar space of rank $d$, then $\chi$ restricted to $H$ is a weighted $m$-ovoid of $P_d \cap H$.

(b) If $\chi$ is a weighted $i$-tight set of $P_d$, then $\chi$ restricted to $H$ is a weighted $i$-tight set of $P_d \cap H$.

(c) If $\chi$ is a weighted $m$-ovoid of $P_d \cap H$ and $P_d \cap H$ is a polar space of rank $d - 1$, then $\chi$ is a weighted $m$-ovoid of $P_d$.

(d) If $\chi$ is a weighted $i$-tight set of $P_d \cap H$ and $P_d \cap H$ is a polar space of rank $d$, then $\chi$ is a weighted $i$-tight set of $P_d$.

**Proof.** Recall that generators are 1-tight sets of polar spaces. By Corollary 3.5, every weighted $i$-tight set is a linear combination of generators where the coefficients of the linear combination sum up to $i$. This shows the assertions for weighted tight sets. Lemma 3.4 shows the assertions for weighted ovoids. 

Here are some more useful weighted intriguing sets which exist in all polar spaces.

**Lemma 4.2.** Let $0 \leq s < d - 1$. Let $S$ be a totally isotropic $s$-dimensional subspace of a polar space of rank $d$ and type $e$ over a finite field with $q$ elements. Then

\[(q^{d+e-2-s} + 1)\chi_S + \chi_S^\sim\]

is a weighted $(q^{d+e-2-s} + 1)$-tight set.
Proof. Recall that a generator is a 1-tight set and that a sum of tight sets is again a tight set. There are
\[ \prod_{i=0}^{d-2-s} (q^{i+e} + 1) \]
generators through \( S \). There are
\[ \prod_{i=0}^{d-3-s} (q^{i+e} + 1) \]
generators through \( \langle S, P \rangle \) if \( P \in S^\perp \). Hence,
\[ \left( \prod_{i=0}^{d-2-s} (q^{i+e} + 1) \right) \chi_S + \left( \prod_{i=0}^{d-3-s} (q^{i+e} + 1) \right) \chi_{S^\perp} \]
is a weighted \((\prod_{i=0}^{d-2-s} q^{i+e} + 1)\)-tight set. Dividing by \((\prod_{i=0}^{d-3-s} q^{i+e} + 1)\) shows the claim. \( \square \)

5. Non-existence results

5.1. A short non-existence proof of ovoids in some polar spaces. In this section we focus on the generalised quadrangles \( W(3, q) \), \( Q^-_5(q) \), \( H(4, q^2) \) and the polar space \( W(5, q) \). The following 2-tight set of \( W(3, q) \) was described already in [24, II.4].

Lemma 5.1. Let \( l \) be a line of \( PG(3, q) \setminus W(3, q) \). Then \( l \cup l^\perp \) is a 2-tight set of \( W(3, q) \).

Proof. Call \( \mathcal{M} \) the set of lines of \( W(3, q) \) meeting \( l \) in one point. Clearly \( |\mathcal{M}| = (q+1)^2 \). A generator is a 1-tight set, and the sum of the generators in the set \( \mathcal{M} \) is a weighted \((q+1)^2\)-tight set. We will compute the weight of each of the points to have an alternative description of this weighted tight set.

Consider a point \( P \in l^\perp \). The line \( \langle P, Q \rangle \), \( Q \in l \), is a line of \( W(3, q) \). Hence, a point of \( l^\perp \) lies on \((q+1)\) generators of \( \mathcal{M} \). Call \( \mathcal{R} \) the set of points not on \( l \) or not in \( l^\perp \). Consider a point \( P \in \mathcal{R} \). Then \( P^\perp \cap l = \{Q\} \), and the line \( \langle P, Q \rangle \) is a generator of \( W(3, q) \). Hence, a point not on \( l \) and not in \( l^\perp \) lies on exactly one line of \( \mathcal{M} \). Hence the weighted \((q+1)^2\)-tight set induced as the sum of generators in \( \mathcal{M} \) has characteristic vector
\[ \chi := (q+1)\chi_l + (q+1)\chi_{l^\perp} + \chi_\mathcal{R} \]
Note that \( j \) is a \((q^2+1)\)-tight set. Hence,
\[ (\chi - j)/q = \chi_l + \chi_{l^\perp} \]
is a 2-tight set as we have \((|\mathcal{M}| - (q^2+1))/q = 2 \). \( \square \)

Theorem 5.2. The generalised quadrangle \( W(3, q) \) has ovoids if and only if \( q = 2^h \).

Proof. Assume that \( \mathcal{O} \) is an ovoid of \( W(3, q) \). Let \( l \) be a line of \( W(3, q) \) and let \( P \in l \cap \mathcal{O} \), the unique point of \( \mathcal{O} \) on \( l \). Consider a point \( R \in l \setminus \{P\} \), and a second point \( S \in l \setminus \{P, R\} \). Let \( \mathcal{T} \) be the \( q\)-tight set which is the sum of the \( q \) lines of \( W(3, q) \) on \( R \) different from \( l \). Call \( \pi \) the plane containing \( \mathcal{T} \) (this is a plane since all lines of \( W(3, q) \) on a point are contained in a plane). Consider the \( q \) lines \( m_i \) of \( PG(3, q) \) on \( S \) contained in \( \pi \). None of these \( q \) lines is a line of \( W(3, q) \), each set \( \mathcal{T}_i := m_i \cup m_i^\perp \) is a 2-tight set of \( W(3, q) \). Since \( R \notin \mathcal{O} \), the \( q \) points of \( \mathcal{O} \cap \mathcal{T} \) are contained in \( \pi \setminus l \). Since \( S \notin \mathcal{O} \), the \( q \) lines \( m_i \) partition the set \( \mathcal{O} \cap \mathcal{T} \). But the two points of each \( \mathcal{T}_i \cap \mathcal{O} \) must be contained in either \( m_i \) or \( m_i^\perp \). So if \( x \) denoted the number of lines \( m_i \) such that \( m_i \cap \mathcal{O} \neq \emptyset \), then \( 2x = q \). Hence, if \( W(3, q) \) has ovoids, then \( q \) is even. Note that \( W(3, q) \) has ovoids if \( q \) is even (see Table 2). This completes the proof. \( \square \)
For the generalised quadrangles, $Q^-(5, q)$, $H(4, q^2)$, a very short non-existence proof for ovoids is inspired by [3]. In Section 4.7, it is shown that a linear combination of a hyperbolic line and its perp yield a weighted tight set. For two non-collinear points $x$ and $y$ of a generalised quadrangle, the hyperbolic line on $X$, $Y$ is the set of points $\{X, Y\}^\sim$. Note that for two non-collinear points $X$, $Y$ of $Q^-(5, q)$, $\{X, Y\}^\sim = \{X, Y\}$.

Theorem 4.4 of [3] translates, in our notation, to the following two lemmas.

**Lemma 5.3.** Let $X$ and $Y$ be two non-collinear points of $Q^-(5, q)$. Then $q\chi_{\{X,Y\}} + \chi_{\{X,Y\}^\sim}$ is a $(q + 1)$-tight set.

**Lemma 5.4.** Let $X$ and $Y$ be two non-collinear points of $H(4, q^2)$. Then $q\chi_{\{X,Y\}^\sim} + \chi_{\{X,Y\}^\sim}$ is a $(q + 1)$-tight set.

A one-line proof of the non-existence of ovoids can be produced now for these two generalised quadrangles.

**Theorem 5.5.** No ovoids of the generalised quadrangle $Q^-(5, q)$ exist.

**Proof.** Let $O$ be an ovoid of $S$. Choose two points $X, Y \in S$. Then $X \not\sim Y$, and $\chi_T = q\chi_{\{X,Y\}} + \chi_{\{X,Y\}^\sim}$ is a $(q + 1)$-tight set, so $\chi_T \cdot \chi_O = q + 1$. But clearly $\chi_{\{X,Y\}} \cdot \chi_O = 2$ and $\chi_{\{X,Y\}^\sim} \cdot \chi_O = 0$. Hence, $(q + 1) = 2q$, a contradiction.

**Theorem 5.6.** No ovoids of the generalised quadrangle $H(4, q^2)$ exist.

**Proof.** Let $O$ be an ovoid of $S$. Choose a point $X \in O$ and $Y \in S \setminus O$ such that $X \not\sim Y$. There are exactly $q^6$ hyperbolic lines through one point, since $O$ contains exactly $q^5 + 1$ points, $Y$ can be chosen in such a way that the hyperbolic line through $X$ and $Y$ meets $O$ only in the point $X$, and $\chi_T = q\chi_{\{X,Y\}^\sim} + \chi_{\{X,Y\}^\sim}$ is a $(q + 1)$-tight set, so $\chi_T \cdot \chi_O = q + 1$. But clearly $\chi_{\{X,Y\}^\sim} \cdot \chi_O = 1$ and $\chi_{\{X,Y\}^\sim} \cdot \chi_O = 0$. Hence, $(q + 1) = q$, a contradiction.

A suitable tight set to show the non-existence of ovoids of $W(5, q)$ in one line is based on the tight set used in the proof for $W(3, q)$. Note that the proof will work for all $q$ now.

**Lemma 5.7.** Let $l$ be a line of $PG(5, q)$ that is not a line of $W(5, q)$. Then $\chi_T := q\chi_l + \chi_{l^\perp}$ is a $(q + 1)$-tight set.

**Proof.** Call $M$ the set of generators of $W(5, q)$ meeting $l$ in one point. Clearly, $|M| = (q + 1)^2(q^2 + 1)$, since each point of $l$ lies on exactly $(q + 1)(q^2 + 1)$ generators of $W(5, q)$. A generator is a 1-tight set, and the sum of the generators in the set $M$ is a weighted $(q + 1)^2(q^2 + 1)$-tight set. We will compute the weight of each of the points to have an alternative description of this weighted tight set.

Consider a point $P \in l^\perp$. The line $\langle P, Q \rangle$, $Q \in l$, lies on $q + 1$ generators of $W(5, q)$. Hence, a point of $l^\perp$ lies on $(q + 1)^2$ generators of $M$. Call $R$ the set of points not on $l$ or not in $l^\perp$. Consider a point $P \in R$. Then $P^\perp \cap l = \{Q\}$, and the line $\langle P, Q \rangle$ lies on $(q + 1)$ generators of $W(5, q)$. Hence, a point not on $l$ and not in $l^\perp$ lies on exactly $q + 1$ generators of $M$. Hence the weighted $(q + 1)^2(q^2 + 1)$-tight set induced as the sum of generators in $M$ has characteristic vector $\chi := (q + 1)(q^2 + 1)\chi_l + (q + 1)^2\chi_{l^\perp} + (q + 1)\chi_R$.

Note that $j$ is a $(q^3 + 1)$-tight set. Hence, $(\chi - (q + 1)j)/(q(q + 1)) = q\chi_l + \chi_{l^\perp}$ is a $(q + 1)$-tight set as we have $(|M| - (q + 1)(q^3 + 1))/(q(q + 1)) = q + 1$.

**Theorem 5.8.** No ovoids of the polar space $W(5, q)$ exist.
Proof. Suppose that $O$ is an ovoid of $W(5,q)$, and let $P$ be a point of $O$. Since there exist $q^4$ different projective lines on $P$ not in $W(5,q)$, there is at least one line $l$, on $P$, that is not a line of $W(5,q)$. By Lemma 5.7, $\chi_l := q\chi + \chi_l$ is a $(q + 1)$-tight set of $W(5,q)$. Clearly $\chi_l \cdot \chi_l = q\chi \cdot \chi_l$, but $\chi_l \cdot \chi_l = q + 1$ by Lemma 3.3, a contradiction (as $q$ does not divide $q + 1$ for $q > 1$).

5.2. The non-existence proof of ovoids of $Q^+(9,q)$, $q$ even.

Lemma 5.9. Let $\pi_1, \pi_2, \pi_3$ be disjoint planes of $Q(6,q)$. Then there exist exactly $q + 1$ lines meeting $\pi_1, \pi_2, \pi_3$. These lines span a subspace of dimension 3 meeting $Q(6,q)$ in a hyperbolic quadric $Q^+(3,q)$.

Proof. The planes $\pi_1$ and $\pi_2$ span a 5-dimensional subspace $\alpha$ meeting $Q(6,q)$ in a hyperbolic quadric $Q^+(5,q)$. Clearly $\alpha \cap \pi_3$ is a line $l$. A line meeting $\pi_1$, $\pi_2$, and $\pi_3$ in a point, meets $l$ in exactly one point. Hence all these lines are contained in the 4-dimensional space $(l, \pi_2)$, which meets $\pi_1$ in a line $m$. It is not possible that two such lines $g, h$ meet $l$ in the same point $p$, since then the plane $(g, h)$ meets both $\pi_2$ and $\pi_3$ in a line, a contradiction since $\pi_1$ and $\pi_2$ are disjoint. Consequently, the number of such lines is exactly $q + 1$, and they span a 3-dimensional subspace, necessarily meeting $Q(6,q)$ in a hyperbolic quadric $Q^+(3,q)$. □

Let $O$ be an ovoid of $Q^+(9,q)$. Let $P_1, P_2 \in O$. Denote $Q^+(5,q) := P_1^+ \cap P_2^+ \cap Q^+(9,q)$. $Q^+(5,q)$ is a hyperbolic quadric $Q^+(7,q)$. Let $\sigma_1$ be a generator of $Q^+_7$ (hence $\sigma_1$ is a solid). Consider an elliptic quadric $Q^-(3,q)$ embedded in $\sigma_1$, denote the point set of this quadric as $Q^-_3$. Let $\Pi$ be the set of $q^2 + 1$ tangent hyperplanes at the points of $Q^-_3$ in $\sigma_1$. So $\Pi$ is a set of planes. Let $\sigma_2 \subseteq Q^+_7$ be a generator of $Q^+_7$ disjoint to $\sigma_1$. Note that for any plane $\pi \in \Pi$, $\pi^\perp$ is a 6-dimensional space, meeting $Q^+_7$ in a cone with vertex $\pi$ and base a hyperbolic quadric $Q^+(1,q)$. Furthermore, $\pi^\perp$ meets $\sigma_2$ in a point. Define

$$L = \langle \pi^\perp \cap \sigma_2, \pi \cap Q^-_3 : \pi \in \Pi \rangle.$$

So $L$ is a set of $q^2 + 1$ lines.

Lemma 5.10. Let $l_1, l_2 \in L$ with $l_1 \neq l_2$. Let $P \in l_1$ and $Q \in l_2$ be collinear points with $P, Q \notin \sigma_1$. Then $P \notin l_2^\perp$.

Proof. Suppose that $P \in l_2^\perp$. Hence, $l_2 \subseteq P^\perp$. Then $l_2 \cap \sigma_1 \subseteq (l_2 \cap \sigma_1, Q)^\perp$. By $P \notin \sigma_1$, $(l_1 \cap \sigma_1, P) = l_1$. Hence, $l_2 \cap \sigma_1 \subseteq l_1^\perp$. This contradicts the definition of $L$. □

Lemma 5.11. Let $l_1, l_2, l_3 \in L$ pairwise different. Let $P_1 \in l_1 \setminus (\sigma_1 \cup \sigma_2)$. Define $P_2 = P_1^\perp \cap l_2$ and $P_3 = P_1^\perp \cap l_3$. Then $P_2$ and $P_3$ are collinear.

Proof. Let $P_1 \in l_1 \setminus (\sigma_1 \cup \sigma_2)$, and $P_3 = P_1^\perp \cap l_3$. Suppose that $P_2$ and $P_3$ are non-collinear. Set $P_k^i = P_k \cap (P_3^{i-1})^\perp \cap l_1$ for $i > 0$, $P_k^0 = (P_3^0)^\perp \cap l_2$ for $i = 0$, and $P_k^i = (P_3^i)^\perp \cap l_3$ for $i > 0$. By Lemma 5.10, all $P_k^i$ are points. By $P_2$ and $P_3$ non-collinear, $P_3^0 \neq P_3^j$. Let $i$ be the smallest number with $P_k^i = P_k^j$ for some $k$ and $j < i$. Then we have $P_1^i \supseteq (P_3^{i-1}, P_3^{i-1}) = l_3$ if $k = 1$ (and similar for $k = 2, 3$) which contradicts Lemma 5.10. Hence, $P_k^i \neq P_k^j$ for all $i \neq j$. This is a contradiction as the number of points on $l_k$ is finite. □

Lemma 5.12. Given the set of lines $L$, there exists a third generator $\sigma_3$ of $Q^+_7$ disjoint from $\sigma_1$ and $\sigma_2$, which meets all lines of $L$.

Proof. Let $l \in L$. Let $P \in l \setminus (\sigma_1, \sigma_2)$. By Lemma 5.11, the points $P = \{P \cup \{P^\perp \cap l' : l' \in L \setminus \{l\}\}$ are pairwise collinear. Hence, there exists a generator $\sigma_3$ which contains $P$. □

Theorem 5.13. Let $q$ be even, then $Q^+(9,q)$ has no ovoids.
PROOF. Let $O$ be an ovoid of $Q^+(9,q)$. Consider two points $P_1, P_2 \in O$, and consider the set $L$. Define $P$ as the set of points $X \in Q^+(9,q) \setminus (P_1 \cup P_2)$ such that $(X,l) \subseteq Q^+(9,q)$ for at least one line $l \in L$. We show first that for each point $X \in P$, there are exactly two lines $l_i \in L$ such that $l_i \in X$.

Consider $P \in P$, then $P \cap P_1 \cap P_2 \cap Q^+(9,q) = Q_6$, i.e. a parabolic quadric $Q(6,q)$, and the hyperplane $P^\perp$ meets $Q_3^\perp$ in 1 or $q+1$ points.

Assume that $|P^\perp \cap Q_3^\perp| = 1$. Then there exists exactly one line $l \in L$ such that $l \in P^\perp$.

Recall that $l_1 \cap l_2 = \text{a tangent plane of } Q_3^\perp$. Then $|P^\perp \cap Q_3^\perp| = 1$ implies $P^\perp \cap l_1 = l^\perp$. Hence $P^\perp \cap Q_3^\perp$ contains the 3-space $l \cap P^\perp \cap l_1$, a contradiction since $P^\perp \cap Q_3^\perp$ is a parabolic quadric $Q(6,q)$.

Assume that $|P^\perp \cap Q_3^\perp| = q + 1$. Consider the three planes $P^\perp \cap l_1, P^\perp \cap l_2$, and $P^\perp \cap l_3$, which are three planes of $Q_6 = P^\perp \cap Q_3^\perp$. There are exactly $q + 1$ lines meeting these three planes in a point by Lemma 5.9, and these lines span a 3-space $\alpha$ meeting $Q_6$ in a hyperbolic quadric $Q^+(3,q)$. Since $\alpha \cap l_1 \cap P^\perp$ is a line, meeting $Q_3^\perp$ in 0, 1 or 2 points, if a line $l \in L$ is contained in $P^\perp$, then $\alpha \cap l \cap P^\perp$ meets $Q_3^\perp$. Hence, if $P^\perp$ contains exactly one line $l \in L$, $l' := P^\perp \cap l \cap l_1$ is a tangent line to the conic $Q_3^\perp \cap P^\perp$. Consider $r \in (l \cap l') \setminus P^\perp$. Then $l \cap r$ is a plane contained in $R^+/R$, a quotient geometry isomorphic with a parabolic quadric $Q(4,q)$, a contradiction. Hence, for each point $P \in P$, there are exactly two lines $l_i \in L$ such that $l_i \in P^\perp$.

Count now the pairs $\{(l, P) : l \in L, P \in P \cap O\}$. There are $k := |O \cap P|$ choices for $P$. By the above argument, there are 2 choices for $l$ given a point $P \in P$, so we find $2k$ pairs. On the other hand, there are $q^2 + 1$ choices for a line $l \in L$. For each line $l \in L$, $l_1 \cap l_2$ is a quotient geometry isomorphic to $Q^+(5,q)$, so $l_1$ contains exactly $q^2 + 1$ points of $O$, always including $P_1, P_2$. Hence there are $(q^2 + 1)(q^2 - 1) = q^4 - 1$ pairs, so $2k = q^4 - 1$. With $k$ a natural number, this is a contradiction when $q$ is even.

We conclude this part with two short remarks.

REMARK 5.14. We have to clear the connection of the given proof with tight sets. Consider the situation of the previous proof, and the weighted tight set that consists of all generators that contain a line of $L$, but not $P_1$ or $P_2$. There are $2(q + 1)(q^2 + 1)(q^2 - 1)$ such generators. Hence, we obtain a weighted $2(q + 1)(q^2 + 1)(q^2 - 1)$-tight set as every line of $L$ is contained in $2(q + 1)(q^2 - 1)$ such generators. By the arguments of the previous proof, we have weight $4(q + 1)$ on the points of $P$, but the other points of the tight set have either weight 0 or are in $P_1^\perp \cup P_2^\perp$. Hence,

$$4(q + 1)|P \cap O| = 2(q + 1)(q^2 + 1)(q^2 - 1).$$

The proof is based on a simplification of this tight set, which was found for small $q = 2, 4$ with variants of the approach described in Section 7.

REMARK 5.15. For $q$ even, there exists a natural embedding $Q^-(3,q) \subseteq W(3,q) \subseteq H(3,q^2)$. By field reduction one can map $H(3,q^2)$ onto $Q^+(7,q)$. Hereby, $W(3,q)$ gets mapped onto the Segre variety $S_{3,1} = W(3,q) \otimes W(1,q)$. The image of $Q^-(3,q)$ gives us the $q^2 + 1$ lines of $L \subseteq S_{3,1}$.

5.3. The non-existence proof of ovoids of $H(5,4)$. Recall that an ovoid of $H(5,q^2)$ has size $q^5 + 1$. A subspace $\pi$ of $PG(5,q)$ meets the Hermitian variety $H(5,q^2)$ in a set of points of a (possibly) degenerate hermitian variety, in other words, the underlying hermitian form induces a (possibly) degenerate Hermitian form on the subspace $\pi$ Such a degenerate Hermitian variety is a cone with vertex a subspace $\alpha$ of $\pi$ and base a non-degenerate Hermitian variety in the complement of $\alpha$ in $\pi$. When we say, for example, that a line $l$ is isomorphic to $H(1,q^2)$, it means that the line meets $H(5,q^2)$ in the point set of the Hermitian variety $H(1,q^2)$. Likewise,
when we say, a plane $\pi$ is isomorphic to $pH(1, q^2)$, we mean that $\pi$ meets $H(5, q^2)$ in the cone $pH(1, q^2)$, $p \in l$ and $H(1, q^2) \in p_\perp$.

**Lemma 5.16.** Let $\mathcal{O}$ be an ovoid of $H(5, q^2)$. Then there exists a line $\ell$ isomorphic to $H(1, q^2)$ with $2 \leq |\ell \cap \mathcal{O}| < q + 1$.

**Proof.** Assume that all lines isomorphic to $H(1, q^2)$ meet $\mathcal{O}$ in $0$, $1$, or $q + 1$ points. Call a line $\ell$ a Baer subline of $\mathcal{O}$ if $|\ell \cap \mathcal{O}| = q + 1$.

**Case 1:** All Baer sublines of $\mathcal{O}$ are contained in a plane (necessarily isomorphic to $H(2, q^2)$). This is not possible, since $|H(2, q^2)| = q^3 + 1 < |\mathcal{O}|$.

**Case 2:** Two Baer sublines $\ell, \ell'$ of $\mathcal{O}$ span a solid. Consider the following set of $(q + 1)^2$ lines isomorphic to $H(1, q^2)$.

$$\mathcal{L} = \{PQ : P \in \ell \cap \mathcal{O}, Q \in \ell' \cap \mathcal{O}\}.$$

Let $s \in \mathcal{L}$. By $s \cap \ell \in \mathcal{O}$ and $s \cap \ell' \in \mathcal{O}$, we have $|s \cap \mathcal{O}| \geq 2$. Hence,

$$|s \cap \mathcal{O} \setminus (\ell \cup \ell')| = q - 1.$$

Let $s' \in \mathcal{L}$ with $s' \neq s$, $s \cap \ell \neq s' \cap \ell$ and $s \cap \ell' \neq s' \cap \ell'$. Suppose for the sake of a contradiction that $s \cap s'$ is not empty. Then $\ell, \ell' \subseteq \langle s, s' \cap \ell, s \cap \ell', s' \cap \ell' \rangle$ is a plane. This contradicts our assumption that $\ell$ and $\ell'$ span a solid. Hence,

$$\left| \bigcup_{s \in \mathcal{L}} s \cap \mathcal{O} \right| = |\ell \cap \mathcal{O}| + |\ell' \cap \mathcal{O}| + \bigcup_{s \in \mathcal{L}} |s \cap \mathcal{O} \setminus (\ell \cup \ell')|$$

$$= 2(q + 1) + (q + 1)^2(q - 1)$$

$$= q^3 + q^2 + q + 1.$$

The subspace $S = \langle \ell, \ell' \rangle$ is isomorphic to $H(3, q^2)$. Hence, $S_\perp$ is isomorphic to $H(1, q^2)$ and contains a point $P \in H(5, q^2) \setminus \mathcal{O}$. Then $|P_\perp \cap \mathcal{O}| \geq q^3 + q^2 + q + 1$. This contradicts Lemma 4.2, that implies $|P_\perp \cap \mathcal{O}| = q^3 + 1$.

**Case 3:** All Baer sublines of $\mathcal{O}$ meet in a point $x$ of $PG(5, q^2)$ and are not contained in a plane. Let be $\ell, \ell', \ell''$ Baer sublines of $\mathcal{O}$ that span a 3-space. Choose $y \in \ell$, $z \in \ell'$ with $x \neq y, z$. Then $yz$ is isomorphic to $H(1, q^2)$ with $|yz| \geq 2$. Hence, $yz$ is a Baer subline of $\mathcal{O}$ with $x \notin yz$. This contradicts the assumption.

□

**Lemma 5.17** ([4, Theorem 8]). Let $r > 1$ and let $W_r$ be a subgeometry of $H(2r - 1, q^2)$ isomorphic to $W(2r - 1, q)$. Then the set of points of $W_r$ is a $(q + 1)$-tight set of $H(2r - 1, q^2)$.

**Lemma 5.18.** Let $\mathcal{O}$ be an ovoid of $H(5, 4)$. Then exist $P, Q, R \in \mathcal{O}$ such that $\langle P, Q, R \rangle$ is isomorphic to $pH(1, q^2)$.

**Proof.** By Lemma 5.16, there exists a line $\ell$ isomorphic to $H(1, 4)$ with $|\ell \cap \mathcal{O}| = 2$. Let $W_5$ be a subgeometry of $H(5, 4)$ isomorphic to $W(5, 2)$ with $\ell \subseteq W_5$. By Lemma 5.17, $|W_5 \cap \mathcal{O}| = 3$. Hence, $\pi = \langle W_5 \cap \mathcal{O} \rangle$ is a plane. The plane $\pi$ is isomorphic to $pH(1, 4)$ as $W(5, 2)$ does not contain non-degenerate planes.

Define the following objects.

(1) Let $P$ be a point of $H(5, 4)$
(2) Let $\ell \subseteq P_\perp$ be a line of $H(5, 4)$ isomorphic to $H(1, 4)$.
(3) Let $W_2$ be the set of $(q^2 - 1)/(q - 1) = 3$ subgeometries $W_2$ of $P\ell$ isomorphic to $pW(1, 2)$ with $\ell \cap H(5, 4) \subseteq W_2$.
(4) Let $W_5$ be a subgeometry of $H(5, 4)$ isomorphic to $W(5, 2)$ with $W_2^0 \subseteq W_5$ for one fixed $W_2^0 \in W_2$.

□
(5) Define $W_3 = W_5 \cap \ell^\perp$.

(6) Fix a subgeometry $Q_3^-$ isomorphic to $Q^- (3,2)$ in $W_5$ with $P \in Q_3^-$.

(7) Let $U$ be the intersection of the setwise stabiliser of $W_5$, the element-wise stabiliser of $W_2$, and the setwise stabiliser of $Q_3^-$. The group $U$ has size 144.

Let $W_2 \in W_2$ with $W_2 \notin W_5$. Let $Q_3^- = \{ Q_3^-, W_3 \setminus (P^\perp \cup Q_3^-) \}$; a partition of the points of $W_3 \setminus P^\perp$. Let $S \in Q_3^-$ and let $O_{W_2,S}$ be the set of points $Q$ of $H(5,4)$ with

1. $Q \notin W_5$,
2. $Q^\perp \cap \ell$ is a point $R$ of $H(5,4)$,
3. $P \notin Q^\perp$,
4. $|QR \cap S| = 1$,
5. $Q^\perp \cap P \ell \cap W_2 \cong W(1,2)$.

Let $O$ be an ovoid of $H(5,4)$. By Lemma 5.18, we suppose without loss of generality $|W_0 \cap O| = q + 1$ and $|\ell \cap O| = 0$. By Lemma 5.17, $|W_5 \cap O \setminus P \ell| = 0$. Under this assumption, the equations defined by Lemma 3.8 with $U$ as a group imply

$$|O_{W_2,S} \cap O| = \frac{3}{2}.$$  

This is clearly a contradiction.

**Lemma 5.19.** The Hermitian polar space $H(5,4)$ does not posses an ovoid.

We provide a coordinatised description of $U$ and the used tight set in the appendix. It is also possible to provide a geometrical description of the involved weighted tight sets without too much effort. As this argument heavily relies on Lemma 5.18, so $q = 2$, we see no point in doing so. We do hope that it will be possible to generalise the given construction in the future.

### 5.4. The non-existence proof of ovoids of $Q(6,q)$, $q$ prime.

This section presents the non-existence result by O’Keefe and Thas for ovoids in $Q(2n,q)$ [23].

Let $S$ and $T$ be sets of points of a polar space $P$ with polarity $\perp$. Define $S^\perp = \bigcap_{P \in S} P^\perp$.

Define $C(S,T)$ by

$$C(S,T) = \{ P \in P \setminus (S \cup S^\perp) : P \in RQ \text{ for some } R \in S, Q \in T \}.$$  

Define $C(S)$ by

$$C(S) = C(S,S^\perp).$$

**Lemma 5.20.** Let $d > 2$. Let $Q_s$ be a subgeometry of $Q(2d,q)$ isomorphic to $Q^- (s,q)$, $1 < s < 2d - 2$. Then

$$(1 - q^{2d-3})\chi_{Q_s} + (1 - q^{\frac{s-1}{2}})\chi_{Q_s^\perp} + \chi_{C(Q_s)},$$

is a weighted $(1 - q^{\frac{s-1}{2}})(1 - q^{2d-3}))$-tight set.

**Proof.** This follows immediately from Lemma 6.5 and Lemma 4.1.

**Lemma 5.21.** Let $d > 2$. Let $Q_s$ be a subgeometry of $Q(2d,q)$ isomorphic to $Q^+(s,q)$, $1 < s < 2d - 2$. Then

$$(q^{2d-3} + 1)\chi_{Q_s} + (q^{\frac{s-1}{2}} + 1)\chi_{Q_s^\perp} + \chi_{C(Q_s)},$$

is a weighted $(q^{\frac{s-1}{2}} + 1)(q^{2d-3} + 1)$-tight set.

**Proof.** This follows immediately from Lemma 6.7 and Lemma 4.1.

This following argument is due to O’Keefe and Thas. We reformulate it with tight sets as a divisibility argument.
Lemma 5.22 (O’Keefe, Thas [23]). Let $\mathcal{C}$ be a conic in $Q(6, q)$. Let $O$ be an ovoid of $Q(6, q)$. Suppose that we have
\[(q^2 + 1)\chi_{\{P\}} + \chi_{Q'} + \chi_{C((P), Q')})\chi^T_O \in \{0, 1, \ldots, q + 1, q^2 + 1\}\]
for all points $P$ of $Q(6, q)$ and all $Q'$ isomorphic to $Q^{-}(3, q)$ with $Q^- \subset P^\perp \setminus \{P\}$. Then $|C \cap O| < q + 1$ or $q = \{2, 3\}$.

Proof. Let $O$ be an ovoid of $Q(6, q)$. Let $\mathcal{C}$ be a conic in $Q(6, q)$. Let $\pi$ be a hyperplane isomorphic to $Q^+(5, q)$ with $\mathcal{C} \subseteq \pi$. Suppose $|C \cap O| = q + 1$. We want to show that this show $q \in \{2, 3\}$.

Let $P \in (\pi \cap O) \setminus \mathcal{C}$. Since neither $Q^+(3, q)$, nor $P Q(2, q)$ possess $q + 2$ pairwise non-collinear points, we find that $(\langle P, C \rangle)$ is isomorphic to $Q^-(3, q)$. By Lemma 5.20 and $|O \cap \langle P, C \rangle^\perp| = 0$,
\[(q - 1)^2 = ((1 - q)\chi_{\langle P, C \rangle} + (1 - q)\chi_{\langle P, C \rangle^\perp} + \chi_{C((P), C)})\chi^T_O \]
\[= (1 - q)\chi_{\langle P, C \rangle}\chi^T_O + \chi_{C((P), C))}\chi^T_O.\]

Let $Q \in \langle P, C \rangle^\perp$. By assumption $Q \notin O$ and $|\langle P, C \rangle \cap O| \geq q + 2$, so $(\chi_{\langle P, C \rangle} + \chi_{C((Q), \langle P, C \rangle)})\chi^T_O = q^2 + 1$. Hence with $x_P$ defined as $|\langle P, C \rangle \cap O|$
\[(q + 1)(q^2 + 1) = \sum_{Q \in \langle P, C \rangle^\perp} (\chi_{\langle P, C \rangle} + \chi_{C((Q), \langle P, C \rangle)})\chi^T_O \]
\[= (q + 1)\chi_{\langle P, C \rangle}\chi^T_O + \chi_{C((P), C))}\chi^T_O \]
\[= (q + 1)x_P + (q - 1)^2 + (q - 1)x_P \]
\[= 2qx_P + (q - 1)^2.\]

This shows $x_P = \frac{q^2 + 3}{2}$.

Let $k$ be the number of pairwise different subspaces of $Q(6, q)$ of the form $\langle P, C \rangle$. By Lemma 4.1, $\pi$ is a $(q^2 + 1)$-tight set, hence
\[q(q - 1) = |(\pi \cap O) \setminus \mathcal{C}| \]
\[= \sum_{\langle P, C \rangle \text{ with } P \in (\pi \cap O) \setminus \mathcal{C}} (x_P - q - 1) \]
\[= k(q - 1)^2/2.\]

This shows $(q - 1)k = 2q$. This implies $q \in \{2, 3\}$, since $k$ is an integer. \qed

It follows from results by O’Keefe, Thas [23] and by Ball, Govaerts, and Storme [1] that Lemma 5.22 implies the non-existence of ovoids of $Q(6, q)$, $q \neq 3$ prime.

6. An application to Andreas Klein non-existence proofs

Andreas Klein showed in [21] the non-existence of ovoids in $H(2d - 1, q^2)$ if $d > q^3 + 1$. This result shows the non-existence of ovoids in certain cases where the result of E. Moorhouse in [22] does not show it and vice versa. The approach in [22] is based on the computation of the $p$-rank of a generator matrix associated to a hypothetical ovoid, while the approach in [21] is purely combinatorial. With the systematic approach followed in this paper we can now improve Klein’s result.

Lemma 6.1. Let $d > 2$. Let $H_s$ be a subgeometry of $H(2d - 1, q^2)$ isomorphic to $H(s, q^2)$, $1 < s < 2d - 2$, $s$ even. Then
\[(1 - q^{2d-2-s})\chi_{H_s} + (1 - q^s)\chi_{H_s^\perp} + \chi_{C(H_s)},\]
is a weighted $(q^s - 1)(q^{2d-2-s} - 1)$-tight set.
Proof. By Lemma 3.3, Lemma 3.4, and Lemma 3.6, we only have to show
\[(1 - q^{2d-2})\chi_{\{P\}} + \chi_{P^\perp}((1 - q^{2d-2-s})\chi_{H_s} + (1 - q^s)\chi_{H^\perp} + \chi_{C(H_s)})^\top = (q^s - 1)(q^{2d-2-s} - 1)\frac{q^{2d-2} - 1}{q^2 - 1},\]
for all points $P \in H(2d - 1, q^2)$. We have only four possible choices for $P$. Either $P \in H_s$, $P \in H^\perp_s$, $P \in C(H_s)$, or $P \notin H_s \cup H^\perp_s \cup C(H_s)$.

If $P \in H_s$, then
\[
|P^\perp \cap H_s| = q^2|H(s - 2, q^2)| = q^2(q^{s-1} + 1)(q^{s-2} - 1)
\]
\[
|P^\perp \cap H^\perp_s| = (q^{2d-1-s} + 1)(q^{2d-2-s} - 1)
\]
\[
|P^\perp \cap C(H_s)| = (q^s - 1)(q^{2d-2-s} - 1)(1 + q^2(q^{s-1} + 1)(q^{s-2} - 1))
\]
Hence,
\[
((1 - q^{2d-2})\chi_{\{P\}} + \chi_{P^\perp}((1 - q^{2d-2-s})\chi_{H_s} + (1 - q^s)\chi_{H^\perp} + \chi_{C(H_s)})^\top
\]
\[
= (1 - q^{2d-2})(1 - q^{2d-2-s}) + (1 - q^{2d-2-s})|P^\perp \cap H_s| + (1 - q^s)|P^\perp \cap H^\perp_s| + |P^\perp \cap C(H_s)|
\]
\[
= (q^s - 1)(q^{2d-2-s} - 1)\frac{q^{2d-2} - 1}{q^2 - 1}.
\]
If $P \in H^\perp_s$, then the intersection numbers are the same as in the case $P \in H_s$ only with $(H_s, s)$ replaced by $(H^\perp_s, 2d - 2 - s)$. If $P \in C(H_s)$, then
\[
|P^\perp \cap H_s| = 1 + q^2|H(s - 2, q^2)| = 1 + q^2(q^{s-1} + 1)(q^{s-2} - 1)
\]
\[
|P^\perp \cap H^\perp_s| = 1 + q^2|H(2d - 4 - s, q^2)| = 1 + q^2(q^{2d-3-s} + 1)(q^{2d-4-s} - 1)
\]
\[
|P^\perp \cap C(H_s)| = (|P^\perp \cap H_s| - 1) + (|H(s, q^2)| - |P^\perp \cap H_s|)(|H(2d - s - 2, q^2)| - |P^\perp \cap H^\perp_s|).
\]
This shows (8) in this case. If $P \in C(H_s)$, then
\[
|P^\perp \cap H_s| = |H(s - 1, q^2)|,
\]
\[
|P^\perp \cap H^\perp_s| = |H(2d - s - 3, q^2)|,
\]
\[
|P^\perp \cap C(H_s)| = |H(s - 1, q^2)| \cdot |H(2d - s - 3, q^2)|.
\]
This shows (8) in this case. \qed
and, since $H^s_1$ is empty,

$$\chi_0 \chi^T = (1 - q^{2d-2-s})|H_s \cap \mathcal{O}| + |C(H_s) \cap H_s|$$

Hence $|H_s \cap \mathcal{O}| + |C(H_s) \cap \mathcal{O}| = (q^s - 1)(q^{2d-2-s} - 1) + q^{2d-2-s}|H_s \cap \mathcal{O}|$. Hence,

$$(q^s - 1)(q^{2d-2-s} - 1) + q^{2d-2-s}|H_s \cap \mathcal{O}| \leq |\mathcal{O}| = q^{2d-1}.$$ 

□

**Lemma 6.3.** Let $d > 1$. Let $H_s$ be a subgeometry of $H(2d - 1, q^2)$ isomorphic to $H(s, q^2)$, $0 < s < 2d - 1$, $s$ odd. Then

$$(q^{2d-2-s} + 1)\chi_{H_s} + (q^s + 1)\chi_{H^s} + \chi_{C(H_s)},$$

is a weighted $(q^s + 1)(q^{2d-2-s} + 1)$-tight set.

Now we come to the improvement of Andreas Klein’s result.

**Lemma 6.4.** The polar space $H(2d - 1, q^2)$, with $d > q^3 - q^2 + 2$, does not possess an ovoid.

**Proof.** Suppose that $H(2d - 1, q^2)$, $d > 3$, possesses an ovoid $\mathcal{O}$. We can easily find a subgeometry $H_1$ isomorphic to $H(1, q^2)$ with $|H_1 \cap \mathcal{O}| \geq 2$. We want to show that we can find a subgeometry $H_2$ isomorphic to $H(2, q^2)$. If $|H_1 \cap \mathcal{O}| \geq 3$, then we are done. Hence suppose to the contrary that $|H_1 \cap \mathcal{O}| = 2$ and that all subgeometries $H_2$ isomorphic to $H(2, q^2)$ with $H_1 \subseteq H_2$ satisfy $|H_2 \cap \mathcal{O}| = 2$. Then $|H_1 \cap \mathcal{O}| + |C(H_1) \cap \mathcal{O}| = |\mathcal{O}| = q^{2d-1} + 1$. Hence by Lemma 6.3,

$$(q + 1)(q^{2d-3} + 1) = (q^{2d-3} + 1)|H_1 \cap \mathcal{O}| + |C(H_1) \cap \mathcal{O}| = 2(q^{2d-2-s} + 1) + q^{2d-1} - 1.$$ 

This is a contradiction.

Let $H_2$ a subgeometry isomorphic to $H(2, q^2)$ with $|H_2 \cap \mathcal{O}| \geq 3$. Let $\chi_T$ be the weighted $(q^2 - 1)(q^{2d-4} - 1)$-tight set of Lemma 6.1. Then

$$(q^2 - 1)(q^{2d-4} - 1) = \chi_0 \chi^T = (1 - q^{2d-4})|H_2 \cap \mathcal{O}| + |C(H_2) \cap H_2| \geq 3(1 - q^{2d-4}) + |C(H_2) \cap H_2|.$$ 

By $d > 3$,

$$|C(H_2) \cap \mathcal{O}| \geq (q^2 + 2)(q^{2d-4} - 1) > 0.$$ 

Hence, we find a point $P$ in $H^s_2$ such that in the induced ovoid of the quotient polar space of $P$ meets the projection of $H_2$ in at least $|H_2 \cap \mathcal{O}| + 1$ points.

By Corollary 6.2, $|H_2 \cap \mathcal{O}| \leq q^3 - q^2 + 2$ for $H(5, q^2)$. We can project $(d - 3)$-times before reaching $H(5, q^2)$. Therefore, ovoids can only exist if $d \leq q^3 - q^2 + 2$. □

The same arguments yield a similar bound for $\mathcal{O}^+(2d - 1, q)$, which was not considered by Andreas Klein in [21].

**Lemma 6.5.** Let $d > 2$. Let $Q_s$ be a subgeometry of $\mathcal{O}^+(2d - 1, q)$ isomorphic to $Q^-(s, q)$, $1 < s < 2d - 3$. Then

$$(1 - q^{2d-s-4})\chi_{H_s} + (1 - q^{s-1})\chi_{H^s} + \chi_{C(H_s)},$$

is a weighted $(q^{s-1} - 1)(q^{2d-s-3} - 1)$-tight set.

**Corollary 6.6.** Let $d > 2$. Let $Q_s$ be a subgeometry of $\mathcal{O}^+(2d - 1, q)$ isomorphic to $Q^-(s, q)$, $1 < s < 2d - 3$. Let $\mathcal{O}$ be an ovoid of $\mathcal{O}^+(2d - 1, q)$. Then

$$|Q_s \cap \mathcal{O}| \leq q^{s-1} - q^{s-1} + q^{s+1-d} + 1.$$
Lemma 6.7. Let $d > 1$. Let $Q_s$ be a subgeometry of $Q^+(2d-1,q)$ isomorphic to $Q^+(s,q)$, $1 < s < 2d - 2$. Then
\[(q^{2d-\frac{s+1}{2}} + 1)\chi_{Q_s} + (q^{2d-\frac{s+1}{2}} + 1)\chi_{Q^s} + \chi_{C(Q_s)},\]
is a weighted $(q^{2d-\frac{s+1}{2}} + 1)(q^{2d-\frac{s+1}{2}} + 1)$-tight set.

Lemma 6.8. The polar space $Q^+(2d-1,q)$, with $d > q^2 - q + 3$, does not possess an ovoid.

The best known bounds on ovoids of $H(2d-1,q^2)$ or $Q^+(2d-1,q)$ are due to Blokhuis and Moorhouse [6]. In contrast to the results here their proof is purely algebraic and gives no information on the hypothetical structure of an ovoid. The same arguments together with the weighted tight set of Lemma 5.20 also give the following existence conditions on ovoids of parabolic quadrics, but in this case better geometric results are known [1, 17, 29].

Lemma 6.9. The polar space $Q(2d,q)$, with $d > \frac{q^2 + 3}{2}$, does not possess an ovoid.

7. Computer searches for non-existence proofs

The non-existence proofs for $Q^+(9,q)$, $q$ even, and $H(5,4)$ were found with the help of a computer search for suitable weighted tight sets for small $q$. The authors used several algorithms to do so. In the following we describe one basic method. It can be easily generalised to similar problem such as the existence of $x$-ovoids or $i$-tight sets for given $x$ or $i$.

For each subgroup $U$ of a group $G$ as in Lemma 3.8, Lemma 3.8 (b) defines a set of linear equations over the integers which have to be satisfied by an ovoid.

Let $Y$ be a set of vertices of $\Gamma$. Let $G$ be a group acting generously transitively on $\Gamma$. Let $U$ be a subgroup of $G$. Let $I = \{\psi_1, \ldots, \psi_m\}$ be the family of weighted $z_j$-tight sets defined by Lemma 4.2. Let $\{M_1, \ldots, M_{m'}\}$ be the set of point orbits of $U$. Suppose that $\chi$ is the characteristic vector of an ovoid $O$. Define the nonnegative integers $x_1, \ldots, x_{m'}$ and $y_{1,1}, \ldots, y_{m,m'}$ by
\[x_i = |M_i \cap O|,\]
\[\psi_j = \sum_{i=1}^{m'} y_{ij} \chi_{M_i}.\]

By Lemma 3.8 (b), the integer program defined by
\[(9)\]
\[z_j = \sum_{i=1}^{m'} x_i y_{ij} \quad \text{and} \quad x_i \geq |M_i \cap Y|\]
has a solution if an ovoid $O$ with $Y \subseteq O$ exists. The most elementary way to use this integer program is Algorithm 1.

Algorithm 1 Testing for the non-existence of ovoids.

for all subgroups $U$ of $G$ do
    Let $C$ be the integer program defined by (9) for $Y$ and $U$.
    if $C$ has no solution then
        print There exists no ovoid $O$ with $Y \subseteq O$.
        return $U$
    end if
end for
print There exists an ovoid $O$ with $Y \subseteq O$.  

Recall that if $U$ is the trivial subgroup of $G$ and (9) and has solution, then an ovoid exists by Lemma 3.4. If one finds a large subgroup $U$ of $G$ for which (9) has no solution, then this simplifies the problem significantly. Our implementations of the algorithm start with large subgroups of $U$ before considering the small ones. Besides, finding a solution to the integer program becomes futile for small $U$ for all interesting examples. Furthermore, it makes sense to assume $Y$ to be a configuration for which $Y \subseteq O$ can be easily guaranteed if $O$ exists.

The non-existence results in Section 5 for $H(4, q^2)$, $Q^-(5, q)$, $Q^+(9, q)$, $H(5, 4)$ can be found with a variant of Algorithm 1 for small $q$. While we were able to generalise the new nonexistence proof for $Q^+(9, q)$ to all $q$ even, we failed at doing so for our new non-existence proof for $H(5, 4)$.

The authors used various versions of GAP [16] and FinInG [2] to implement variants of algorithm 1 with the exception of the integer programs which where solved using Gurobi [19].

We conclude by posing a problem. From the algebraic graph theory point of view, an ovoid is essentially a $0-1$-vector $\chi$ of $(V_+)^+$ such that $j\chi^\top = \theta$ where $\theta$ is the ovoid number of the ambient polar space. For the spaces where ovoids do not exist, our method has been to find a subspace $M$ of $V_+$ such that $M^\perp$ does not contain such $0-1$-vectors $\chi$, yet $M$ is simpler to understand geometrically; that is, it has a nice spanning set and is invariant under a large subgroup of isometries. Therefore, we have the following problem:

**Generalised ovoid existence problem, first version:** Let $M \subseteq V^+$. Does $M^\perp$ contain a $0$-$1$-vector $\chi$ with $j\chi^\top = \theta$? If not, what is the smallest dimension of such a subspace $M$?

Additionally, we made a general assumption on how the ovoid intersects the weighted tight set. This translates to the fact that we assume that some entries of $\chi$ are 0, respectively, 1. Therefore, we also have the following generalisation of the first problem:

**Generalised ovoid existence problem, second version:** Let $M \subseteq V^+$. Let $Y, Z \subseteq X$. Does $M^\perp$ contain a $0$-$1$-vector $\chi$ with $j\chi^\top = \theta$, $\chi_i = 1$ for all $i \in Y$ and $\chi_i = 0$ for all $i \in Z$? If not, what is the smallest dimension of such a subspace $M$?

**References**

[1] S. Ball, P. Govaerts, and L. Storme. On ovoids of parabolic quadrics. Des. Codes Cryptogr., 38(1):131–145, 2006.

[2] J. Bamberg, A. Betten, P. Cara, J. De Beule, M. Lavrauw, and M. Neunhöffer. FinInG – Finite Incidence Geometry, Version 1.0, 2014. http://cage.ugent.be/fining/.

[3] J. Bamberg, A. Devillers, and J. Schillewaert. Weighted intriguing sets of finite generalised quadrangles. J. Algebraic Combin., 36(1):149–173, 2012.

[4] J. Bamberg, S. Kelly, M. Law, and T. Penttila. Tight sets and $m$-ovoids of finite polar spaces. J. Combin. Theory Ser. A, 114(7):1293–1314, 2007.

[5] A. Barlotti. Un’estensione del teorema di Segre-Kustaanheimo. Boll. Un. Mat. Ital. (3), 10:498–506, 1955.

[6] A. Blokhuis and G. E. Moorhouse. Some $p$-ranks related to orthogonal spaces. J. Algebraic Combin., 4(4):295–316, 1995.

[7] A. E. Brouwer, A. M. Cohen, and A. Neumaier. Distance-regular graphs, volume 18 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1989.

[8] F. Buekenhout and E. Shult. On the foundations of polar geometry. Geometriae Dedicata, 3:155–170, 1974.

[9] P. J. Cameron and P. A. Kazanidis. Cores of symmetric graphs. J. Aust. Math. Soc., 85(2):145–154, 2008.

[10] J. De Beule, A. Klein, and K. Metsch. Substructures of finite classical polar spaces. In Current research topics in Galois geometry, chapter 2, pages 33–59. Nova Sci. Publ., New York, 2012.

[11] J. De Beule and K. Metsch. The Hermitian variety $H(5, 4)$ has no ovoid. Bull. Belg. Math. Soc. Simon Stevin, 12(5):727–733, 2005.

[12] P. Delsarte. An algebraic approach to the association schemes of coding theory. Philips Res. Rep. Suppl., (10):vi+97, 1973.

[13] P. Delsarte. Pairs of vectors in the space of an association scheme. Philips Res. Rep., 32(5-6):373–411, 1977.
Appendix A. The weighted tight set used for $H(5, 4)$

We use the Hermitian form $x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3$. Let $\{0, 1, a, a^2\}$ be the elements of $\text{GF}(4)$.

Let $U_1$ be the setwise stabiliser of the points spanned by the following eight vectors:

\[(0, 1, 1, 0, a, a), \quad (1, a^2, 1, 0, 0, 0), \]
\[(1, 0, 0, 1, a^2, a), \quad (0, 0, 1, a, a, a^2), \]
\[(1, a^2, 1, 0, 0, 1), \quad (1, a^2, a^2, a, a, 1), \]
\[(1, 1, 0, 1, 0, 1), \quad (1, 1, a, a, 1, a^2). \]
The group $U$ described in Subsection 5.3 is isomorphic to $U_1 \cap \mathbf{PGU}(6,2)$. The group has 34 point orbits, here given by their representatives.

$$(1, 1, 0, 0, 0, 0) \quad (0, 0, 0, 0, 1, 1)$$
$$(0, 0, 0, 0, 1, a) \quad (0, 0, 0, 0, 1, a^2)$$
$$(0, 0, 0, 1, 0, 1) \quad (0, 0, 0, 1, 0, a^2)$$
$$(0, 0, 0, 1, 1, 0) \quad (0, 0, 0, 1, a, 0)$$
$$(0, 0, 0, 1, a^2, 0) \quad (0, 0, 1, 0, 0, 1)$$
$$(0, 0, 1, 0, 0, a^2) \quad (0, 0, 1, 0, a^2, 0)$$
$$(0, 0, 1, 1, 1, 1) \quad (0, 0, 1, 1, 1, a)$$
$$(0, 0, 1, 1, 1, a^2) \quad (0, 0, 1, a, a, 1)$$
$$(0, 0, 1, a, a, a) \quad (0, 0, 1, a, a, a^2)$$
$$(0, 0, 1, a, a^2, 1) \quad (0, 0, 1, a, a^2, a)$$
$$(0, 0, 1, a^2, 0, 0) \quad (0, 0, 1, a^2, a^2, 1)$$
$$(0, 0, 1, a^2, a^2, a) \quad (0, 1, 0, 1, a, 1)$$
$$(0, 1, 0, a, 0, 0) \quad (0, 1, 0, a, a, 1)$$
$$(0, 1, 0, a, a, a) \quad (0, 1, 0, a^2, 1, a^2)$$
$$(0, 1, 0, a, a^2) \quad (0, 1, a, a, 1, 0)$$
$$(0, 1, a, a^2, 0) \quad (1, 0, a, 0, 1, a^2)$$
$$(1, 0, a^2, a, 0, a^2) \quad (1, a^2, 1, 0, 0, a)$$

Put the following weights on these orbits (with the same order) to obtain a weighted 36-tight set.

$$(5, 0, -2, 0, -7, 3, 16, 0, 2, 0, 10, 0, 0, 0, 24, 12, 0, 36, 0, 0, -9, 6, -6, -6, 0, 0, 0, 2, -8, 6, 24, -12, 0, 0, 0)$$

In the notation of Subsection 5.3, $\ell$ is the 30th orbit, $P$ is the 34th orbit, and the 9 remaining totally isotropic points of $\langle P, \ell \rangle$ are in the 14th, 31st and 33rd orbit. The assumption says that there are 3 non-collinear points of the ovoid in the 14th orbit. All orbits except 2, 4, 8, 10, 12, 13, 14, 15, 16, 18, 19, 25, 26, 32 contain no point of the ovoid, since their points are in the perp of one point of the 14th orbit. The orbits 16, 18, 25, 32 are empty, since they are subsets of $W_5$ and all 3 points of the ovoid in $W_5$ are in the 14th orbit. Let $O_i$ denote the $i$-th point orbit. Hence,

$$24|O_{15} \cap O| = 36.$$ 

This is a contradiction, since $|O_{15} \cap O|$ is an integer.

Centre for the Mathematics of Symmetry and Computation, School of Mathematics and Statistics, The University of Western Australia, 35 Stirling Highway, Crawley, W.A. 6009, Australia.

E-mail address: John.Bamberg@uwa.edu.au

Department of Mathematics, Ghent University, Krijgslaan 281 - S22, 9000 Ghent, Belgium.

E-mail address: jdebeule@cage.ugent.be

Mathematisches Institut, Justus Liebig University Giessen, Arndtstrasse 2, 35392 Giessen, Germany.

E-mail address: Ferdinand.Ihringer@math.uni-giessen.de