GLOBAL CLASSICAL SOLUTIONS TO THE RELATIVISTIC BOLTZMANN EQUATION WITHOUT ANGULAR CUT-OFF

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Abstract. We prove the unique existence and exponential decay of global in time classical solutions to the special relativistic Boltzmann equation without any angular cut-off assumptions with initial perturbations in some weighted Sobolev spaces. We consider perturbations of the relativistic Maxwellian equilibrium states. We work in the case of a spatially periodic box. We consider the general conditions on the collision kernel from Dudyński and Ekiel-Jeżewska (Commun Math Phys 115(4):607–629, 1985). Additionally, we prove sharp constructive upper and coercive lower bounds for the linearized relativistic Boltzmann collision operator in terms of a geometric fractional Sobolev norm; this shows that a spectral gap exists and that this behavior is similar to that of the non-relativistic case as shown by Gressman and Strain (Journal of AMS 24(3), 771–847, 2011). Lastly, we derive the relativistic analogue of Carleman dual representation of Boltzmann collision operator. This is the first global existence and stability result for relativistic Boltzmann equation without angular cutoff and this resolves the open question of perturbative global existence for the relativistic kinetic theory without the Grad’s angular cut-off assumption.

1. Introduction

In 1872, Boltzmann [12] derived an equation which mathematically models the dynamics of a gas represented as a collection of molecules. This was a model for the collisions between non-relativistic particles. For the collisions between relativistic particles whose speed is comparable to the speed of light, Lichnerowicz and Marrot [39] have derived the relativistic Boltzmann equations in 1940. This is a fundamental model for fast moving particles. Understanding the nature of relativistic particles is crucial in describing many astrophysical and cosmological processes [38]. Although the classical non-relativistic Boltzmann kinetic theory has been widely and heavily studied, the relativistic kinetic theory has received relatively less attention because of its complicated structure and computational difficulty on dealing with relativistic post-collisional momentums. The relativistic Boltzmann equation is written as

\begin{equation}
 p^\mu \partial_\mu f = p^0 \partial_t f + cp \cdot \nabla_x f = C(f, f),
 \end{equation}

where the collision operator \( C(f, f) \) can be written as

\begin{equation}
 C(f, h) = \int_{\mathbb{R}^3} \frac{dq}{q^0} \int_{\mathbb{R}^3} \frac{dq'}{q'^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} W(p, q|p', q')[f(p')h(q') - f(p)h(q)],
 \end{equation}

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Here, the transition rate $W(p, q|p', q')$ is
\begin{equation}
W(p, q|p', q') = \frac{c}{2} \sigma(g, \theta) s^{(4)} (p^\mu + q^\mu - p'^\mu - q'^\mu),
\end{equation}
where $\sigma(g, \theta)$ is the scattering kernel measuring the interactions between particles and the Dirac $\delta$ function expresses the conservation of energy and momentum.

1.1. Notation. The relativistic momentum of a particle is denoted by a 4-vector representation $p^\mu$ where $\mu = 0, 1, 2, 3$. Without loss of generality we normalize the mass of each particle $m = 1$. We raise and lower the indices with the Minkowski metric $g_{\mu\nu} = g_{\mu\nu} p^\nu$, where the metric is defined as $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. The signature of the metric through this paper is $(-+++)$.

We also define the quantities $s$ and $g$ which respectively stand for the square of the energy and the relative momentum in the center-of-momentum system, $p + q = 0$, as
\begin{equation}
s = s(p^\mu, q^\mu) = -(p^\mu + q^\mu)(p_\mu + q_\mu) = 2(-p^\mu q_\mu + 1) \geq 0,
\end{equation}
and
\begin{equation}
g = g(p^\mu, q^\mu) = \sqrt{(p^\mu - q^\mu)(p_\mu - q_\mu)} = \sqrt{2(-p^\mu q_\mu - 1)}.
\end{equation}
Note that $s = g^2 + 4c^2$.

Conservation of energy and momentum for elastic collisions is described as
\begin{equation}
p^\mu + q^\mu = p'^\mu + q'^\mu.
\end{equation}
The scattering angle $\theta$ is defined by
\begin{equation}
\cos \theta = \frac{(p^\mu - q^\mu)(p'_\mu - q'_\mu)}{g^2}.
\end{equation}
Together with the conservation of energy and momentum as above, it can be shown that the angle and $\cos \theta$ are well-defined [24].

Here we would like to introduce the relativistic Maxwellian which models the steady state solutions or equilibrium solutions also known as Jüttner solutions. These are characterized
as a particle distribution which maximizes the entropy subject to constant mass, momentum, and energy. They are given by

\begin{equation}
J(p) = \frac{e^{-\frac{p^0}{4\pi}}}{4\pi c k_B T K_2(\frac{c^2 k_B T}{\pi m p^0})},
\end{equation}

where \( k_B \) is Boltzmann constant, \( T \) is the temperature, and \( K_2 \) stands for the Bessel function \( K_2(z) = \frac{z^2}{2} \int_1^\infty dt e^{-zt}(t^2 - 1)^{\frac{3}{2}}. \) Throughout this paper, we normalize all physical constants to 1, including the speed of light \( c = 1. \) Then we obtain that the relativistic Maxwellian is given by

\begin{equation}
J(p) = \frac{e^{-p^0}}{4\pi}.
\end{equation}

We now consider the center-of-momentum expression for the relativistic collision operator as below. Note that this expression has appeared in the physics literature; see [14]. For other representations of the operator such as Glassey-Strauss coordinate expression, see [1], [27], and [25]. Also, see [50] for the relationship between those two representations of the collision operator. As in [47] and [14], one can reduce the collision operator (2) using Lorentz transformations and get

\begin{equation}
Q(f, h) = \int_{\mathbb{R}^3} dq \int_{S^2} d\omega \ v_\phi \sigma(g, \theta) [f(p')h(q') - f(p)h(q)],
\end{equation}

where \( v_\phi = v_\phi(p, q) \) is the Møller velocity given by

\begin{equation}
v_\phi(p, q) = \sqrt{\left| \frac{p - q}{p^0 - q^0} \right|^2 - \left| \frac{p}{p^0} \times \frac{q}{q^0} \right|^2} = \frac{g\sqrt{s}}{p^0 q^0}.
\end{equation}

Comparing with the reduced version of collision operator in [1], [27], and [25], we can notice that one of the advantages of this center-of-momentum expression of the collision operator is that the reduced integral (11) is written in relatively simple terms which only contains the Møller velocity, scattering kernel, and the cancellation between gain and loss terms.

The post-collisional momentums in the center-of-momentum expression are written as

\begin{equation}
p' = \frac{p + q}{2} + \frac{g}{2} \left( \omega + (\gamma - 1)(p + q) \frac{(p + q) \cdot \omega}{|p + q|^2} \right),
\end{equation}

and

\begin{equation}
q' = \frac{p + q}{2} - \frac{g}{2} \left( \omega + (\gamma - 1)(p + q) \frac{(p + q) \cdot \omega}{|p + q|^2} \right).
\end{equation}

The energy of the post-collisional momentums are then written as

\begin{equation}
p'^0 = \frac{p^0 + q^0}{2} + \frac{g}{2\sqrt{s}} (p + q) \cdot \omega,
\end{equation}

and

\begin{equation}
q'^0 = \frac{p^0 + q^0}{2} - \frac{g}{2\sqrt{s}} (p + q) \cdot \omega.
\end{equation}

These can be derived by using the conservation of energy and momentum (7); see [49]. As in (267) in the Appendix, we can show that the angle can be written as \( \cos \theta = k \cdot \omega \) with \( k = k(p, q) \) and \( |k| = 1. \)
For \( f, g \) smooth and small at infinity, it turns out [24] that the collision operator satisfies
\[
\int Q(f, g) dp = \int pQ(f, g) dp = \int \rho^0 Q(f, g) dp = 0
\]
and
\[
\int Q(f, f)(1 + \log f) dp \leq 0.
\]

Using (18), we can prove the famous Boltzmann H-theorem that the entropy of the system \(- \int f \log f dp dx\) is a non-decreasing function of \( t \). The expression \(- f \log f\) is called the entropy density.

1.2. A brief history of previous results in relativistic kinetic theory. The full relativistic Boltzmann equation appeared first in the paper by Lichnerowicz and Marrot [39] in 1940. In 1967, Bichteler [11] showed the local existence of the solutions to the relativistic Boltzmann equation. In 1989, Dudynski and Ekiel-Jezewiska [20] showed that there exist unique \( L^2 \) solutions to the linearized equation. Afterwards, Dudynski [17] studied the long time and small-mean-free-path limits of these solutions. Regarding large data global in time weak solutions, Dudynski and Ekiel-Jezewiska [19] in 1992 extended DiPerna-Lions renormalized solutions [16] to the relativistic Boltzmann equation using their causality results from 1985 [18]. Here we would like to mention the work by Alexandre and Villani [4] on renormalized weak solutions with non-negative defect measure to non-cutoff non-relativistic Boltzmann equation. In 1996, Andreasson [1] studied the regularity of the gain term and the strong \( L^1 \) convergence of the solutions to the Jüttner equilibrium which were generalizations of Lions’ results [40, 41] in the non-relativistic case. He showed that the gain term is regularizing. In 1997, Wennberg [53] showed the regularity of the gain term in both non-relativistic and relativistic cases.

Regarding the Newtonian limit for the Boltzmann equation, we have a local result by Cercignani [13] and a global result by Strain [50]. Also, Andreasson, Calogero and Illner [2] proved that there is a blow-up if only with gain-term in 2004. Then, in 2009, Ha, Lee, Yang, and Yun [34] provided uniform \( L^2 \)-stability estimates for the relativistic Boltzmann equation. In 2011, Speck and Strain [45] connected the relativistic Boltzmann equation to the relativistic Euler equation via the Hilbert expansions.

Regarding problems with the initial data nearby the relativistic Maxwellian, Glassey and Strauss [25] first proved there exist unique global smooth solutions to the equation on the torus \( T^3 \) for the hard potentials in 1993. Also, in the same paper they have shown that the convergence rate to the relativistic Maxwellian is exponential. Note that their assumptions on the differential cross-section covered the case of hard potentials. In 1995 [26], they extended their results to the whole space and have shown that the convergence rate to the equilibrium solution is polynomial. Under reduced restrictions on the cross-sections, Hsiao and Yu [35] gave results on the asymptotic stability of Boltzmann equation using energy methods in 2006. Recently, in 2010, Strain [48] showed that unique global-in-time solutions to the relativistic Boltzmann equation exist for the soft potentials which contains more singular kernel and decay with any polynomial rate towards their steady state relativistic Maxwellian under the conditions that the initial data starts out sufficiently close in \( L^\infty \).

In addition, we would like to mention that Glassey and Strauss [27] in 1991 computed the Jacobian determinant of the relativistic collision map. Also, we notice that there are results by Guo and Strain [51, 52] on global existence of unique smooth solutions which are initially close to the relativistic Maxwellian for the relativistic Landau-Maxwell system in 2004 and
for the relativistic Landau equation in 2006. In 2009, Yu [55] proved the smoothing effects for relativistic Landau-Maxwell system. In 2010, Yang and Yu [54] proved time decay rates in the whole space for the relativistic Boltzmann equation with hard potentials and for the relativistic Landau equation.

2. Statement of the Main Results and Remarks

2.1. Linearization and reformulation of the Boltzmann equation. We will consider the linearization of the collision operator and perturbation around the relativistic Jüttner equilibrium state

\[ F(t, x, p) = J(p) + \sqrt{J(p)} f(t, x, p). \] (19)

Without loss of generality, we suppose that the mass, momentum, energy conservation laws for the perturbation \( f(t, x, p) \) holds for all \( t \geq 0 \) as

\[ \int_{\mathbb{R}^3} dp \int_{\mathbb{T}^3} dx \left( \begin{array}{c} 1 \\ p \\ p^0 \end{array} \right) \sqrt{J(p)} f(t, x, p) = 0. \] (20)

We linearize the relativistic Boltzmann equation around the relativistic Maxwellian equilibrium state (19). By expanding the equation, we obtain that

\[ \partial_t f + \hat{p} \cdot \nabla x f + L(f) = \Gamma(f, f), \quad f(0, x, v) = f_0(x, v), \] (21)

where the linearized relativistic Boltzmann operator \( L \) is given by

\[ L(f) \overset{\text{def}}{=} -J^{-1/2}Q(J, \sqrt{J}f) - J^{-1/2}Q(\sqrt{J}f, J) \]

(22)

\[ = \int_{\mathbb{R}^3} dq \int_{S^2} d\omega v_\sigma(g, \omega) \left( f(q) \sqrt{J(p)} \right. \]

\[ + f(p) \sqrt{J(q)} - f(q') \sqrt{J(p')} - f(p') \sqrt{J(q')} \left. \right) \sqrt{J(q)}, \]

and the bilinear operator \( \Gamma \) is given by

\[ \Gamma(f, h) \overset{\text{def}}{=} J^{-1/2}Q(\sqrt{J}f, \sqrt{J}h) \]

(23)

\[ = \int_{\mathbb{R}^3} dq \int_{S^2} d\omega v_\sigma(g, \theta) \sqrt{J(q)}(f(q')h(p') - f(q)h(p)). \]

Then notice that we have

\[ L(f) = -\Gamma(f, \sqrt{J}) - \Gamma(\sqrt{J}, f). \] (24)

We further decompose \( L = N + K \). We would call \( N \) as norm part and \( K \) as compact part. First, we define the weight function \( \tilde{\zeta} = \zeta + \zeta_K \) such that

\[ \Gamma(\sqrt{J}, f) = \left( \int_{\mathbb{R}^3} dq \int_{S^2} d\omega v_\sigma(g, \theta)(f(p') - f(p))\sqrt{J(q')}\sqrt{J(q)} \right) - \tilde{\zeta}(p)f(p), \] (25)
This norm will appear in the process of linearization of the collision operator. For the second
then, this piece satisfies that

This completes our main splitting of the linearized relativistic Boltzmann collision operator.

We can also think of the spatial derivative of \( \Gamma \) which will be useful later. Recall that the
linearization of the collision operator is given by (23) and that the post-collisional variables
$p'$ and $q'$ satisfies (13) and (14). Then, we can define the spatial derivatives of the bilinear collision operator $\Gamma$ as

$$\partial^\alpha \Gamma(f, h) = \sum_{\alpha_1 \leq \alpha} C_{\alpha, \alpha_1} \Gamma(\partial^{\alpha - \alpha_1} f, \partial^{\alpha_1} h),$$

where $C_{\alpha, \alpha_1}$ is a non-negative constant.

2.2. Main Hypothesis on the collision kernel $\sigma$. The Boltzmann collision kernel $\sigma(g, \theta)$ is a non-negative function which only depends on the relative velocity $g$ and the scattering angle $\theta$. Without loss of generality, we may assume that the collision kernel $\sigma$ is supported only when $\cos \theta \geq 0$ throughout this paper; i.e., $0 \leq \theta \leq \frac{\pi}{2}$. Otherwise, the following symmetrization [24] will reduce the case:

$$\bar{\sigma}(g, \theta) = [\sigma(g, \theta) + \sigma(g, -\theta)]1_{\cos \theta \geq 0},$$

where $1_A$ is the indicator function of the set $A$.

Throughout this paper we assume the collision kernel satisfies the following growth/decay estimates:

$$\sigma(g, \theta) \lesssim (g^a + g^{-b})\sigma_0(\theta)$$

$$\sigma(g, \theta) \gtrsim (\frac{g}{\sqrt{s}})g^a\sigma_0(\theta)$$

Additionally, the angular function $\theta \mapsto \sigma_0(\theta)$ is not locally integrable; for $c > 0$, it satisfies

$$\frac{c}{g^{1+\gamma}} \leq \sin \theta \cdot \sigma_0(\theta) \leq \frac{1}{cg^{1+\gamma}}, \quad \gamma \in (0, 2), \quad \forall \theta \in (0, \frac{\pi}{2}].$$

Here we have that $a + \gamma \geq 0$ and $\gamma < b < \frac{3}{2} + \gamma$.

Note that we do not assume any cut-off condition on the angular function.

The assumptions on our collision kernel have been motivated from many important physical interactions; the Boltzmann cross-sections which satisfy the assumptions above can describe many interactions such as short range interactions [22, 44] which describe the relativistic analogue of hard-sphere collisions, Möller scattering [14] which describes electron-electron scattering, Compton scattering [14] which is an approximation of photon-electron scattering, neutrino gas interactions [15], and the interactions of Israel particles [36] which are the relativistic analogue of the interactions of Maxwell molecules. Some of the collision cross-sections of those important physical interactions have high angular singularities, so the non-cutoff assumptions on the angular kernel are needed.

2.3. Spaces. As will be seen, our solutions depend heavily on the following weighted geometric fractional Sobolev space:

$$I^{a, \gamma} \overset{\text{def}}{=} \{ f \in L^2(\mathbb{R}_p^3) : |f|_{I^{a, \gamma}} < \infty \},$$

where the norm is described as

$$|f|^2_{I^{a, \gamma}} \overset{\text{def}}{=} |f|_{L^{2+\gamma}}^2 + \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dp' \frac{(f(p') - f(p))^2}{\bar{g}^{3+\gamma}}(p'^\mu p_{\mu} - p^\mu p_{\mu})^{\frac{a+\gamma}{2}}1_{\bar{g} \leq 1},$$

where $\bar{g}$ is the relative momentum between $p'^\mu$ and $p^\mu$ in the center-of-momentum system and is defined as

$$\bar{g} = g(p'^\mu, p^\mu) = \sqrt{(p'^\mu - p^\mu)(p'_\mu - p_\mu)} = \sqrt{2(-p'^\mu p_\mu - 1)}.$$
Here, we also define another relative momentum between \( p^\mu \) and \( q^\mu \) as
\[
\bar{g} = g(p^\mu, q^\nu) = \sqrt{(p^\mu - q^\mu)(p_\mu - q_\mu)} = \sqrt{2(-p^\mu q_\mu - 1)}.
\]
Note that this space \( \mathcal{I}^{a,\gamma} \) is included in the following weighted \( L^2 \) space given by
\[
||f||^2_{L^2} \overset{\text{def}}{=} \int_{\mathbb{R}^3} dp \ (p^0)^{\frac{a_1}{2}} |f(p)|^2.
\]
The analogous norm acting on \( \mathbb{T}_3^3 \times \mathbb{R}^3_p \) is denoted by \( || \cdot || \). So, we have
\[
||f||^2_{\mathcal{I}^{a,\gamma}} \overset{\text{def}}{=} ||f||^2_{\mathcal{I}^{a,\gamma}} (\mathbb{T}^3) ,
\]
The multi-indices \( \alpha = (\alpha^1, \alpha^2, \alpha^3) \) will be used to record spatial derivatives. For example, we write
\[
\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}.
\]
If each component of \( \alpha \) is not greater than that of \( \alpha_1 \), we write \( \alpha \leq \alpha_1 \). Also, \( \alpha < \alpha_1 \) means \( \alpha \leq \alpha_1 \) and \( |\alpha| < |\alpha_1| \) where \( |\alpha| = \alpha^1 + \alpha^2 + \alpha^3 \).
We define the space \( H^N = H^N (\mathbb{T}^3 \times \mathbb{R}^3) \) with integer \( N \geq 0 \) spatial derivatives as
\[
||f||^2_{H^N} = ||f||^2_{H^N (\mathbb{T}^3 \times \mathbb{R}^3)} = \sum_{|\alpha| \leq N} ||\partial^\alpha f||^2_{L^2 (\mathbb{T}^3 \times \mathbb{R}^3)} .
\]
We sometimes denote the norm \( ||f||^2_{H^N} \) as \( ||f||^2_H \) for simplicity.
We also define the derivative space \( \mathcal{I}^{a,\gamma}_N (\mathbb{T}^3 \times \mathbb{R}^3) \) whose norm is given by
\[
||f||^2_{\mathcal{I}^{a,\gamma}_N (\mathbb{T}^3 \times \mathbb{R}^3)} = \sum_{|\alpha| < N} ||\partial^\alpha f||^2_{\mathcal{I}^{a,\gamma}_N (\mathbb{T}^3 \times \mathbb{R}^3)} .
\]
Lastly, we would like to mention that we denote \( B_R \subset \mathbb{R}^3 \) to be the Euclidean ball of radius \( R \) centered at the origin. The space \( L^2 (B_R) \) is the space \( L^2 \) on this ball and similarly for other spaces.
Now, we state our main result as follows:

**Theorem 1.** (Main Theorem) Fix \( N \geq 2 \), the total number of spatial derivatives. Choose \( f_0 = f_0 (x, p) \in H^N (\mathbb{T}^3 \times \mathbb{R}^3) \) in (19) which satisfies (20). There is an \( \eta_0 > 0 \) such that if \( ||f_0||_{H^N (\mathbb{T}^3 \times \mathbb{R}^3)} \leq \eta_0 \), then there exists a unique global strong solution to the relativistic Boltzmann equation (1), in the form (19), which satisfies
\[
f(t, x, p) \in L^\infty ((0, \infty); H^N (\mathbb{T}^3 \times \mathbb{R}^3)) \cap L^2 ((0, \infty); \mathcal{I}^{a,\gamma}_N (\mathbb{T}^3 \times \mathbb{R}^3)).
\]
Furthermore, we have exponential decay to equilibrium. For some fixed \( \lambda > 0 \),
\[
||f(t)||_{H^N (\mathbb{T}^3 \times \mathbb{R}^3)} \lesssim e^{-\lambda t} ||f_0||_{H^N (\mathbb{T}^3 \times \mathbb{R}^3)} .
\]
We also have positivity: \( F = J + \sqrt{J} f \geq 0 \) if \( F_0 = J + \sqrt{J} f_0 \geq 0 \).

2.4. Remarks and possibilities for the future. Our main theorem assumes that the initial function has at least \( N \) spatial derivatives. The minimum number of spatial derivatives \( N \geq 2 \) is needed to use the Sobolev embedding theorems that \( L^\infty (\mathbb{T}_3^3) \supset H^2 (\mathbb{T}_3^3) \). Note that if the number of spatial derivatives is \( N \geq 4 \), the strong solutions in the existence theorem are indeed classical solutions by the Sobolev lemma [23] that if \( N > 1 + \frac{a}{2} \) then \( H^N (\mathbb{T}^3 \times \mathbb{R}^3) \subset C^1 (\mathbb{T}^3 \times \mathbb{R}^3) \). For the lowest number of spatial derivatives, \( N \geq 2 \), we obtain that the equation is satisfied in the weak sense; however, the weak solution is also a strong solution to the equation because we show that the solution is unique.
Cancellation estimates. Here we want to record one of the main computational and technical difficulties which arise in dealing with relativistic particles. While one of the usual techniques to deal with the cancellation estimates which contains $|f(p) - f(p')|$ is to use the fundamental theorem of calculus and the change of variables in the non-relativistic settings, this method does not give a favorable output in the relativistic theory because the momentum derivative on the post-collisional variables (13) and (14) creates additional high singularities which are tough to control in the relativistic settings. Even with the other different representation of post-collisional variables as in [25], it is known in much earlier work [27] that the growth of momentum derivatives is large enough and this high growth prevents us from using known the non-relativistic method from [32]. It is also worth it to mention that the Jacobian which arises in taking the change of variables from $p$ to $u = \theta p + (1 - \theta)p'$ for some $\theta \in (0,1)$ has a bad singularity at some $\theta = \theta(p, p')$. Even if we take a non-linear path from $p$ to $p'$, the author has computed that the Jacobian always blows up at a point on the path and has concluded that there exists a 2-dimensional hypersurface between the momentums $p$ and $p'$ on which the Jacobian blows up. This difficulty led the author to deal with the cancellation estimate by avoiding the change of variables technique; see Section 4.

Non-cutoff results. Regarding non-relativistic results with non-cutoff assumptions, we would like to mention the work by Alexandre and Villani [4] from 2002 on renormalized weak solutions with non-negative defect measure. Also, we would like to record the work by Gressman and Strain [30, 31] in 2010-2011. We also want to mention that Alexandre, Morimoto, Ukai, Xu, and Yang [5, 7, 8, 9, 10] obtained a proof, using different methods, of the global existence of solutions with non-cutoff assumptions in 2010-2012. Lastly, we would like to mention the recent work by the same group of Alexandre, Morimoto, Ukai, Xu, and Yang [6] from 2013 on the local existence with mild regularity for the non-cutoff Boltzmann equation where they work with an improved initial condition and do not assume that the initial data is close to a global equilibrium.

We also want to remark that Theorem 1 is the first global existence and stability proof in the relativistic kinetic theory without angular cutoff conditions. This solves an open problem regarding global existence and stability for the relativistic Boltzmann equations without cutoff assumption.

Future possibilities: We believe that our method can be useful for making further progress on the non-cutoff relativistic kinetic theory. Note that our kernel assumes the hard potential interaction. We can use the similar methods to prove another open problem on the global stability of the relativistic Boltzmann equations for the soft potentials without angular cutoff. We will soon address in a future work the generalization to the soft potential interaction which assumes $-b + \gamma < 0$ and $-\frac{3}{2} < -b + \gamma$ in a subsequent paper [37]. For more singular soft potentials $-b + \gamma \leq -\frac{3}{2}$, we need to take the velocity-derivatives on the bilinear collision operator $\partial_2 \Gamma$ which is written in the language of the derivatives of the post-collision maps of (13) and (14) and the estimates on those terms need some clever choices of splittings of kernels so that we reduce the complexity of the derivatives. This difficulty on the derivatives is known and expected in the relativistic kinetic theory, for the representations of the post-collisional momentums in the center-of-momentum expression in (13) and (14) contain many non-linear terms.

Furthermore, we expect to generalize our result to the whole space case $\mathbb{R}_x^3$ by combining our estimates with the existing cut-off technology in the whole space.
It is also possible that our methods could help to prove the global existences and stabilities for other relativistic PDEs such as relativistic Vlasov-Maxwell-Boltzmann system for hard potentials without angular cut-off.

2.5. Outline of the article. In the following subsection, we first introduce the main lemmas and theorems that are needed to prove the local existence in Section 8.

In Section 3, some simple size estimates on single decomposed pieces will be introduced. We first start by introducing our dyadic decomposition method of the angular singularity and start making an upper bound estimate on each decomposed piece. Some proofs will be based on the relativistic Carleman-type dual representation which is introduced in the Appendix. Note that some proofs on the dual representation require the use of some new Lorentz frames.

In Section 4, we estimate the upper bounds of the difference of the decomposed gain and loss pieces for the $k \geq 0$ case.

In Section 5, we develop the Littlewood-Paley decomposition and prove estimates connecting the Littlewood-Paley square functions with our weighted geometric fractional Sobolev norm $\| \cdot \|_I$.

In Section 6, we first split the main inner product of the non-linear collision operator $\Gamma$ which is written as a trilinear form. Then, we use the upper bound estimate on each decomposed piece, upper bound estimates on the difference terms, and the estimates on the Littlewood-Paley decomposed piece that were proven in the previous sections to prove the main upper bound estimates.

In Section 7, we use the Carleman dual representation on the trilinear form and find the coercive lower bound. We also show that the norm part $\langle Nf, f \rangle$ is comparable to the weighted geometric fractional Sobolev norm $\| \cdot \|_I$.

In Section 8, we finally use the standard iteration method and the uniform energy estimate for the iterated sequence of approximate solutions to prove the local existence. After this, we derive our own systems of macroscopic equations and the local conservation laws and use these to prove that the local solutions should be global by the standard continuity argument and the energy estimates.

In the Appendix, we mainly derive the relativistic Carleman-type dual representation of the gain and loss terms and obtain the dual formulation of the trilinear form which is used in many places from the previous sections.

2.6. Main estimates. Here we would like to record our main upper and lower bound estimates of the inner products that involve the operators $\Gamma$, $L$, and $N$. The proofs for the estimates are introduced in Section 3 through 7.

**Theorem 2.** We have the basic estimate

\begin{equation}
|\langle \Gamma(f, h), \eta \rangle| \lesssim |f|_{L^2} |h|_{I^{s, \gamma}} |\eta|_{I^{s, \gamma}}.
\end{equation}

**Lemma 3.** Suppose that $|\alpha| \leq N$ with $N \geq 2$. Then we have the estimate

\begin{equation}
|\langle \partial^\alpha \Gamma(f, h), \partial^\alpha \eta \rangle| \lesssim |f|_{H^N} |h|_{I^{s, \gamma}} |\partial^\alpha \eta|_{I^{s, \gamma}}.
\end{equation}

**Lemma 4.** We have the uniform inequality for $K$ that

\begin{equation}
|\langle Kf, f \rangle| \leq \epsilon |f|_{L^2}^2 + C_\epsilon |f|_{L^2(B_{C_\epsilon})}^2
\end{equation}

where $\epsilon$ is any positive small number and $C_\epsilon > 0$. 


Lemma 5. We have the uniform inequality for \( N \) that
\[
\langle |Nf,f| \rangle \lesssim |f|_{L^a}^2.
\]

Lemma 6. We have the uniform coercive lower bound estimate:
\[
\langle Nf,f \rangle \gtrsim |f|_{L^a}^2.
\]

Lemma 5 and Lemma 6 together implies that the norm piece is comparable to the fractional Sobolev norm \( I^a \gamma \) as
\[
\langle Nf,f \rangle \approx |f|_{I^a}^2.
\]

Finally, we have the coercive inequality for the linearized Boltzmann operator \( L \):

Lemma 7. For some \( C > 0 \), we have
\[
\langle Lf,f \rangle \gtrsim |f|_{I^a}^2 - C|f|_{L^2(\mathcal{B}_C)}^2.
\]

Note that this lemma is a direct consequence of Lemma 4 and Lemma 6 because \( L = K + N \).

3. Estimates on the Single Decomposed Piece

In this chapter, we mainly discuss about the estimates on the decomposed pieces of the trilinear product \( \langle \Gamma(f,h),\eta \rangle \). Each decomposed piece can be written in two different representations: one with the usual 8-fold reduced integral in \( \int dp \int dq \int d\omega \) and the other in Carleman-type dual representation as introduced in the Appendix.

For the usual 8-fold representation, we recall (23) and obtain that
\[
\langle \Gamma(f,h),\eta \rangle = \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{S^2} d\omega \; v_{\phi} \sigma(g,\theta)\eta(p)\sqrt{J(q)}(f(q')h(p') - f(q)h(p))
\]
\[
= T_+ - T_-
\]
where the gain term \( T_+ \) and the loss term \( T_- \) are defined as
\[
T_+(f,h,\eta) \overset{\text{def}}{=} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{S^2} d\omega \; v_{\phi} \sigma(g,\theta)\eta(p)\sqrt{J(q)}f(q')h(p')
\]
\[
T_-(f,h,\eta) \overset{\text{def}}{=} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{S^2} d\omega \; v_{\phi} \sigma(g,\theta)\eta(p)\sqrt{J(q)}f(q)h(p)
\]

In this chapter, we would like to decompose \( T_+ \) and \( T_- \) dyadically around the angular singularity as the following. We let \( \{\chi_k\}_{k=-\infty}^{\infty} \) be a partition of unity on \((0, \infty)\) such that \( |\chi_k| \leq 1 \) and \( \text{supp}(\chi_k) \subset [2^{-k-1}, 2^{-k}] \). Then, we define \( \sigma_k(g,\theta) \overset{\text{def}}{=} \sigma(g,\theta)\chi_k(\tilde{g}) \) where \( \tilde{g} \overset{\text{def}}{=} g(p',p''\nu) \). The reason that we dyadically decompose around \( \tilde{g} \) is that we have \( \theta \approx \frac{2}{\tilde{g}} \) for small \( \theta \). Then we write the decomposed pieces \( T^+_k \) and \( T^-_k \) as
\[
T^+_k(f,h,\eta) \overset{\text{def}}{=} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{S^2} d\omega \; v_{\phi} \sigma_k(g,\theta)\eta(p)\sqrt{J(q)}f(q')h(p')
\]
\[
T^-_k(f,h,\eta) \overset{\text{def}}{=} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{S^2} d\omega \; v_{\phi} \sigma_k(g,\theta)\eta(p)\sqrt{J(q)}f(q)h(p)
\]

For some propositions, we utilize the Carleman-type dual representation and write the operator \( T_+ \) as
\[
T_+(f,h,\eta) \overset{\text{def}}{=} \frac{c}{2} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \eta(p') \int_{\mathbb{R}^3} dq \frac{q^0}{q^0} f(q) \int_{E_{q-p'}^+} \frac{d\pi_p}{p^0} s_{\phi}(g,\theta)\sqrt{J(q')}h(p).
\]
We also take the dyadic decomposition on those integral above. Again, we let \( \{\chi_k\}_{k=-\infty}^{\infty} \) be a partition of unity on \( (0, \infty) \) such that \( |\chi_k| \leq 1 \) and \( \text{supp}(\chi_k) \subset [2^{-k-1}, 2^{-k}] \). Then, we define the following integral

\[
T^k_+(f, h, \eta) \overset{\text{def}}{=} \frac{c}{2} \int_{\mathbb{R}^3} \frac{dp'}{p'^{\alpha}} \eta(p') \int_{\mathbb{R}^3} \frac{dq}{q^{\beta}} f(q) \int_{E^2_{q'p'}} \frac{d\pi_p}{p'} \hat{\sigma}_k \sqrt{J(q')} h(p),
\]

where

\[
\hat{\sigma}_k \overset{\text{def}}{=} \frac{s\sigma(g, \theta)}{g} \chi_k(\tilde{g}), \quad \tilde{g} \overset{\text{def}}{=} g(p^\mu, p'^\mu), \quad \hat{g} \overset{\text{def}}{=} g(p'^\mu, q^\mu).
\]

Thus, for \( f, h, \eta \in S(\mathbb{R}^3) \),

\[
(\Gamma(f, h, \eta)) = \sum_{k=-\infty}^{\infty} \{T^k_+(f, h, \eta) - T^k_-(f, h, \eta)\}
\]

Now, we start making some size estimates for the decomposed pieces \( T^k_+ \) and \( T^k_- \).

**Proposition 1.** For any integer \( k, l \), and \( m \geq 0 \), we have the uniform estimate:

\[
|T^k_+(f, h, \eta)| \lesssim 2^{k} |f|_{L^2_m} |h|_{L^2_{\frac{m}{2}}} |\eta|_{L^2_{\frac{m}{2}}}
\]

**Proof.** The term \( T^k_+ \) is given as:

\[
T^k_+(f, h, \eta) = \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{S^2} d\omega \sigma_k(g, \omega) v_\phi f(q) h(p) \sqrt{J(q')} \eta(p),
\]

where \( \sigma_k(g, \omega) = \sigma(g, \omega) \chi_k(\tilde{g}) \). Since \( \cos \theta = 1 - 2\tilde{g}^2 \), we have that \( \tilde{g} = g \sin \frac{\theta}{2} \). Therefore, the condition \( \tilde{g} \approx 2^{-k} \) is equivalent to say that the angle \( \theta \) is comparable to \( 2^{-k}g^{-1} \). Given the size estimates for \( \sigma(g, \omega) \) and the support of \( \chi_k \), we obtain

\[
\int_{S^2} d\omega \sigma_k(g, \omega) \lesssim (g^a + g^{-b}) \int_{S^2} d\omega \sigma_0(\cos \theta) \chi_k(\tilde{g})
\]

\[
\lesssim (g^a + g^{-b}) \int_{2^{-k-1}}^{2^{-k}} d\theta \sin \theta
\]

\[
\lesssim (g^a + g^{-b}) \int_{2^{-k-1}}^{2^{-k}} d\theta \frac{1}{\theta^{1+\gamma}}
\]

\[
\lesssim (g^a + g^{-b}) 2^{k \gamma} \tilde{g}^\gamma.
\]

Thus,

\[
|T^k_-(f, h, \eta)| \lesssim 2^{k \gamma} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq (g^{a+\gamma} + g^{-b+\gamma}) v_\phi |f(q)||h(p)| \sqrt{J(q')} |\eta(p)|
\]

\[
= I_1 + I_2.
\]

Note that \( a + \gamma \geq 0 \) and \( -b + \gamma < 0 \). We first estimate \( I_1 \). Since \( g \lesssim \sqrt{p^\mu q^\mu} \) and \( v_\phi \lesssim 1 \), we obtain

\[
I_1 \lesssim 2^{k \gamma} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq (p^\mu q^\mu)^{\frac{a+\gamma}{2}} |f(q)||h(p)| \sqrt{J(q')} |\eta(p)|.
\]
By Cauchy-Schwarz inequality,

\[
I_1 \lesssim 2^{k\gamma} \left( \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \, |f(q)|^2 |h(p)|^2 \sqrt{J(q)p^0} \psi(q) \right)^{1/2}
\]

(61)

\[
\times \left( \int_{\mathbb{R}^3} dp \, |\eta(p)|^2 p^0 \psi^{1/2}(p^0 a + 2\gamma) \int_{\mathbb{R}^3} dq \sqrt{J(q)q^{0a+\gamma}} \right)^{1/2}.
\]

Since \( \int_{\mathbb{R}^3} dq \sqrt{J(q)q^{0a+\gamma}} \approx 1 \), we have

\[
I_1 \lesssim 2^{k\gamma} \left( \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \, |f(q)|^2 |h(p)|^2 \sqrt{J(q)p^0} \psi^{1/2}(p^0 a + 2\gamma) \int_{\mathbb{R}^3} dp \, |\eta(p)|^2 p^0 \psi^{1/2}(p^0 a + 2\gamma) \right)^{1/2}
\]

(62)

\[
\lesssim 2^{k\gamma} |f|_{L^2_{-m_1}} |h|_{L^2_{\frac{1}{2}(-b+\gamma)}} |\eta|_{L^2_{\frac{1}{2}(-b+\gamma)}} \quad \text{for } m_1 \geq 0.
\]

For \( I_2 \), we have

\[
I_2 = 2^{k\gamma} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \, g^{-b+\gamma} |f(q)||h(p)| \sqrt{J(q)}|\eta(p)|.
\]

(63)

Since \( g \geq \frac{|p-q|}{|p^0 q^0|} \) and \( -b + \gamma < 0 \), this is

\[
I_2 \lesssim 2^{k\gamma} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \, |p-q|^{-b+\gamma} (p^0 q^0)^{1/2} |f(q)||h(p)| \sqrt{J(q)}|\eta(p)|.
\]

(64)

With Cauchy-Schwarz,

\[
I_2 \lesssim 2^{k\gamma} \left( \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \, |f(q)|^2 |h(p)|^2 \sqrt{J(q)p^0} \psi^{1/2}(p^0 a + 2\gamma) q^{0b-\gamma} \right)^{1/2}
\]

(65)

\[
\times \left( \int_{\mathbb{R}^3} dp \, |\eta(p)|^2 p^0 \psi^{1/2}(p^0 a + 2\gamma) \int_{\mathbb{R}^3} dq \sqrt{J(q)}|p-q|^2(p^0 a + 2\gamma) \right)^{1/2}.
\]

Since \( \int_{\mathbb{R}^3} dq \sqrt{J(q)}|p-q|^m \approx p^0m \) if \( m > -3 \) and since \( 2(-b + \gamma) > -3 \), we have

\[
I_2 \lesssim 2^{k\gamma} \left( \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \, |f(q)|^2 |h(p)|^2 \sqrt{J(q)p^0} \psi^{1/2}(p^0 a + 2\gamma) q^{0b-\gamma} \right)^{1/2}
\]

(66)

\[
\times \left( \int_{\mathbb{R}^3} dp \, |\eta(p)|^2 p^0 \psi^{1/2}(p^0 a + 2\gamma) \right)^{1/2}
\]

\[
\lesssim 2^{k\gamma} |f|_{L^2_{-m_2}} |h|_{L^2_{\frac{1}{2}(-b+\gamma)}} |\eta|_{L^2_{\frac{1}{2}(-b+\gamma)}} \quad \text{for some } m_2 \geq 0.
\]

This completes the proof. \( \square \)

Before we do the size estimates for \( T^k \) terms, we first prove a useful inequality as in the following proposition.

**Proposition 2.** On the set \( E_{q-p'}^p \), we have that

\[
\int_{E_{q-p'}^p} \frac{d\pi_p}{p^0} \tilde{g}(\tilde{g})^{-2-\gamma} \chi_k(\tilde{g}) \lesssim 2^{k\gamma} \sqrt{q^0},
\]

(67)

where \( d\pi_p \) is the Lebesgue measure on the set \( E_{q-p'}^p \) and is defined as

\[
d\pi_p = dp \, u(p^0 + q^0 - p'^0) \delta \left( \frac{\tilde{g}^2 + 2p^0(q_n - p'_n)}{2\tilde{g}} \right).
\]

(68)
\textbf{Proof.} We first introduce our 4-vectors $\bar{p}^\mu$ and $\tilde{p}^\mu$ defined as
\begin{equation}
\bar{p}^\mu = p^\mu - p'^\mu \quad \text{and} \quad \tilde{p}^\mu = p'^\mu - q^\mu. 
\end{equation}

Then, notice that the Lorentzian inner product of the two 4-vectors are given by
\begin{equation}
\bar{p}^\mu \bar{p}_\mu = \tilde{g}^2 \quad \text{and} \quad \tilde{p}^\mu \tilde{p}_\mu = \tilde{g}^2.
\end{equation}

Similarly, we define some other 4-vectors which will be useful:
\begin{equation}
\underline{p}^\mu = p^\mu + p'^\mu \quad \text{and} \quad \bar{p}^\mu = p'^\mu + q^\mu.
\end{equation}

The product is then given by
\begin{equation}
- \underline{p}^\mu \underline{p}_\mu = \bar{s} \quad \text{and} \quad - \bar{p}^\mu \bar{p}_\mu = \bar{s}.
\end{equation}

Note that the four-dimensional delta-function occurring in the measure is derived from the following orthogonality equation
\begin{equation}
(p^\mu - q'^\mu)(p_\mu + q_\mu') = 0
\end{equation}
which tells that the total momentum is a time-like 4-vector orthogonal to the space-like relative momentum 4-vector. This orthogonality can be obtained from the following conservation laws
\begin{equation}
p^\mu + q^\mu = p'^\mu + q'^\mu.
\end{equation}

We start with expanding the measure as
\begin{equation}
I \overset{\text{def}}{=} \int_{\mathbb{R}^3} \frac{d\pi_p}{p^0} g(\bar{g})^{-2-\gamma} \chi_k(\bar{g})
\end{equation}
\begin{equation}
= \int_{\mathbb{R}^3} \frac{dp}{p^0} u(p^0 + q^0 - p'^0) \delta \left( \frac{\bar{g}^2 + 2p^\mu(q_\mu - p'_\mu)}{2\bar{g}^2} \right) \bar{g}^{-2-\gamma} \chi_k(\bar{g})
\end{equation}
where $u(x) = 1$ if $x \geq 1$ and 0 otherwise.

Here, the numerator in the delta function can be rewritten as
\begin{equation}
\bar{g}^2 + 2p^\mu(q_\mu - p'_\mu)
\end{equation}
\begin{equation}
= (q^\mu - p'^\mu + 2p^\mu)(q_\mu - p'_\mu)
\end{equation}
\begin{equation}
= q'^\mu q_\mu + p'^\mu p'_\mu - 2p'^\mu q_\mu + 2p^\mu q_\mu - 2p^\mu p'_\mu
\end{equation}
\begin{equation}
= 2(p'^\mu p'_\mu - p'^\mu q_\mu + p^\mu q_\mu - p^\mu p'_\mu)
\end{equation}
\begin{equation}
= 2(p^\mu - p'^\mu)(p'_\mu - q_\mu).
\end{equation}

Now, define $\bar{p} = p - p' \in \mathbb{R}^3$ and $\bar{p}^0 = p^0 - p'^0 \in \mathbb{R}$. We denote the 4-vector $\bar{p}^\mu = (\bar{p}^0, \bar{p}) = p^\mu - p'^\mu$. We now apply the change of variables $p \in \mathbb{R}^3 \rightarrow \bar{p} \in \mathbb{R}^3$. Note that our kernel $I$ will be estimated inside the integral of $\int \frac{d\bar{p}}{\bar{p}^0} \int \frac{dp}{\bar{p}^0}$ in the next propositions and this change of variables is indeed $(p', p) \rightarrow (\bar{p}', \bar{p}) = (p' - p, \bar{p})$.

With this change of variables the integral becomes
\begin{equation}
I = \int_{\mathbb{R}^3} \frac{d\bar{p}}{\bar{p}^0 + p'^0} u(p^0 + q^0) \delta \left( \frac{p^\mu(p_\mu - q_\mu)}{\bar{g}^2} \right) \bar{g}^{-2-\gamma} \chi_k(\bar{g}).
\end{equation}

The remaining part of this estimate will be performed in the \textit{center-of-momentum} system where $p + p' = 0$; i.e., we take a Lorentz transformation such that $p'^\mu = (\sqrt{3}, 0, 0, 0)$ and $
\bar{p}^\mu = (0, \bar{p}) = (0, \bar{p}_x, \bar{p}_y, \bar{p}_z)$. Note that this gives us that $|\bar{p}| = \tilde{g}$. Also, we choose the z-axis
parallel to \( \vec{p} \in \mathbb{R}^3 \). Then, we have \( \vec{p}_x = \vec{p}_y = 0 \) and \( \vec{p}_z = \vec{g} \). Additionally, we introduce a polar-coordinates for \( \vec{p} \), taking the polar-axis along the z-direction:

\[
\vec{p} = |\vec{p}|(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).
\]

Note that \( \vec{g} \) and the measure \( \frac{d\vec{p}}{p^0 + p'0} \) are Lorentz invariant because

\[
\frac{d\vec{p}}{p^0 + p'^0} = 2d\vec{p}'u(\vec{p}^0 + p'^0)\delta(\vec{p}' \cdot \vec{p}_\mu + 2\vec{p}'_\mu) = 2d\vec{p}'u(\vec{p}^0 + p'^0)\delta((\vec{p}'^0 + \vec{p}'^\mu)((\vec{p}_\mu + \vec{p}'_\mu) + 1)
\]

and these are Lorentz invariant. Then the measure of the integral is now

\[
d\vec{p} = |\vec{p}|^2d|\vec{p}|d(\cos \theta)d\phi
\]

(79)

We now write the terms in the delta function in these variables and perform the integration with respect to \( \cos \theta \). The delta function is now written as

\[
\delta\left(\frac{2\vec{p}'(\vec{p}'_0 - q_0)}{\vec{g}'^2}\right) = \frac{\vec{g}^2}{2|\vec{p}||\vec{p}'|}\delta(\cos \theta) = \frac{\vec{g}^2}{2\vec{g}' - q}|\delta(\cos \theta).
\]

(80)

After we evaluate the integral by reducing this delta function, we obtain that our integral is now

\[
I = \int_0^\infty d\vec{g}(\vec{g})^{-\gamma}\chi_k(\vec{g}) \frac{\vec{g}^2}{2\vec{g}' - q} - \frac{\vec{g}^2}{2\vec{g}' - q} \int_0^\infty d\vec{g}(\vec{g})^{-1-\gamma}\chi_k(\vec{g}).
\]

(81)

We recall the inequality that \( \vec{g} \leq |\vec{p}' - q| \) and that \( \vec{g} \lesssim \sqrt{p'^0q^0} \). Using this inequality and the support condition of \( \chi \), we obtain that the integral is bounded above by

\[
I \lesssim \sqrt{q^0} \int_0^\infty d\vec{g}(\vec{g})^{-1-\gamma}\chi_k(\vec{g}) \lesssim 2^{k\gamma}\sqrt{q^0}.
\]

(82)

This completes the proof for the proposition. \( \square \)

**Proposition 3.** Fix an integer \( k \). Then, we have the uniform estimate:

\[
|T^k_+(f, h, \eta)| \lesssim 2^{k\gamma} |f|_{L^2} |h|_{L^2_{\overline{g}}} |\eta|_{L^2_{\overline{g}}}.
\]

(83)

**Proof.** The term \( T^k_+(f, h, \eta) \) is defined as:

\[
T^k_+(f, h, \eta) = \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \sigma_k(g, \omega)v_\phi f(q)h(p)\sqrt{J(q')}\eta(p'),
\]

where \( \sigma_k(g, \omega) = \sigma(g, \omega)\chi_k(\vec{g}) \). Thus,

\[
|T^k_+(f, h, \eta)| \lesssim \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega (g^\alpha + g^{-\beta})v_\phi \sigma_0 \chi_k(\vec{g})|f(q)||h(p)|\sqrt{J(q')}|\eta(p')|
\]

(85)

def \( \overset{\text{def}}{=} I_1 + I_2 \).

We estimate \( I_2 \) first. By Cauchy-Schwarz,

\[
I_2 \lesssim \left( \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega v_\phi g^{-\beta} \sigma_0 \chi_k(\vec{g})|f(q)|^2|h(p)|^2 \sqrt{J(q')}|p'^\alpha| \right)^{\frac{1}{2}}
\]

(86)

\[
\times \left( \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega v_\phi g^{-\beta} \sigma_0 \chi_k(\vec{g})g^{-\beta\gamma} |\eta(p')|^2 \sqrt{J(q')}|p'^\alpha| \right)^{\frac{1}{2}}
\]

\[
= I_{21} \cdot I_{22}.
\]
For $I_{21}$, we split the region of $p'$ into two: $p'^0 \leq \frac{1}{2}(p^0 + q^0)$ and $p'^0 \geq \frac{1}{2}(p^0 + q^0)$.

If $p'^0 \leq \frac{1}{2}(p^0 + q^0)$, $p'^0 + q^0 - q'^0 \leq \frac{1}{2}(p^0 + q^0)$ by conservation laws. Thus, $-q'^0 \leq -\frac{1}{2}(p^0 + q^0)$ and $J(q'^0) \leq \sqrt{J(p)} \sqrt{J(q)}$. Since $(p^0)\frac{1}{2}(-b+\gamma) \lesssim 1$ and the exponential decay is faster than any polynomial decay, we have

$$ (p^0)\frac{1}{2}(-b+\gamma) \sqrt{J(q')} \lesssim (p^0)-m(q^0)-m $$

for any fixed $m > 0$.

On the other region, we have $p'^0 \geq \frac{1}{2}(p^0 + q^0)$ and hence $p'^0 \approx (p^0 + q^0)$ because $p'^0 \leq (p^0 + q^0)$.

Also, we have $(p^0)\frac{1}{2}(-b+\gamma) \lesssim (p^0)\frac{1}{2}(-b+\gamma)$ because $-b + \gamma < 0$. Thus, we obtain

$$ (p^0)\frac{1}{2}(-b+\gamma) \sqrt{J(q')} \lesssim (p^0)\frac{1}{2}(-b+\gamma). $$

In both cases, we have

$$ I_{21} \lesssim \left( \int_{R^3} dp' \int_{R^3} dq' \int_{S^2} dw \ \varphi g^{-b} \sigma_0 \chi_k(\hat{g}) g^{-b+\gamma} |\eta(p')|^2 \sqrt{J(q')} (p'^0)^{-\frac{b+\gamma}{2}} \right) $$

(89)

$$ \lesssim \left( \int_{R^3} dp' \int_{R^3} dq' \frac{g \sqrt{s}}{p'^0 q^0} 2^{b+\gamma} g^{-b+\gamma} |\eta(p')|^2 \sqrt{J(q')} (p'^0)^{-\frac{b+\gamma}{2}} \right)^{\frac{1}{2}} $$

by (58) and Cauchy-Schwarz inequality.

Now we estimate $I_{22}$. Note that $v_\phi = \frac{g \sqrt{s}}{p'^0 q^0}$. Then, by (58),

$$ I_{22} = \left( \int_{R^3} dp' \int_{R^3} dq' \frac{g \sqrt{s}}{p'^0 q^0} 2^{b+\gamma} g^{-b+\gamma} |\eta(p')|^2 \sqrt{J(q')} (p'^0)^{-\frac{b+\gamma}{2}} \right) $$

(90)

$$ \lesssim \left( \int_{R^3} dp' \int_{R^3} dq' \frac{g \sqrt{s}}{p'^0 q^0} 2^{b+\gamma} g^{-b+\gamma} |\eta(p')|^2 \sqrt{J(q')} (p'^0)^{-\frac{b+\gamma}{2}} \right)^{\frac{1}{2}} $$

By pre-post collisional change of variables, we have

$$ I_{22} \lesssim \left( \int_{R^3} dp' \int_{R^3} dq' \frac{g \sqrt{s}}{p'^0 q^0} 2^{b+\gamma} g^{-b+\gamma} |\eta(p')|^2 \sqrt{J(q')} (p'^0)^{-\frac{b+\gamma}{2}} \right)^{\frac{1}{2}} $$

(91)

Note that, by conservation laws,

$$ v_\phi = \frac{g \sqrt{s}}{p'^0 q^0} = \frac{g(p'_\mu, q'_\mu) \sqrt{s(p_\mu, q_\mu)}}{p'^0 q^0} \lesssim 1. $$

Since $g \geq \frac{|p' - q'|}{p'^0 q^0}$ and $-b + \gamma < 0$,

$$ I_{22} \lesssim \left( \int_{R^3} dp' \int_{R^3} dq' 2^{b+\gamma} |\eta(p')|^2 \sqrt{J(q')} (p'^0)^{-\frac{b+\gamma}{2}} \right)^{\frac{1}{2}}. $$

(92)

Note that $(q'^0)^{b-\gamma} \sqrt{J(q')} \lesssim \sqrt{J^\alpha(q')}$ for some $\alpha > 0$. Thus,

$$ I_{22} \lesssim \int_{R^3} dp' \int_{R^3} dq' 2^{b+\gamma} |\eta(p')|^2 (p'^0)^{\frac{1}{2}} \left( \int_{R^3} dq' \frac{\sqrt{J^\alpha(q')}}{|p' - q'|^{2(b-\gamma)}} \right)^{\frac{1}{2}} $$

(93)

$$ \lesssim \int_{R^3} dp' \int_{R^3} dq' 2^{b+\gamma} |\eta(p')|^2 (p'^0)^{\frac{1}{2}} (p'^0)^{2(-b+\gamma)} \frac{2^{b+\gamma}}{S^{b+\gamma}}. $$

(94)
Together, we obtain that
\begin{equation}
I_2 \lesssim 2^{k\gamma} |f|_{L^2} |h|_{L^2_{\frac{1}{1-b-\gamma}}} |\eta|_{L^2_{\frac{1}{b-\gamma}}}.
\end{equation}

Now, we estimate $I_1$. By Cauchy-Schwarz,
\begin{equation}
I_1 \lesssim \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} d\rho \int_{\mathbb{S}^2} d\omega \; v_0 \frac{g^a \sigma_0 \chi \left( \tilde{g} \right) \tilde{g}^{a+\gamma}}{\tilde{g}^{a+\gamma}} |f(q)|^2 |\eta(p')|^2 \sqrt{J(q')(p')^{a+\gamma}} \right)^{\frac{1}{2}}
\end{equation}
\begin{equation}
\times \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} d\rho \int_{\mathbb{S}^2} d\omega \; v_0 g^a \sigma_0 \chi \left( \tilde{g} \right) \tilde{g}^{a+\gamma} |h(p)|^2 \sqrt{J(q')(p')^{-a-\gamma}} \right)^{\frac{1}{2}}
= I_{11} \cdot I_{12}.
\end{equation}

As before, we split the region of $p'$ into two: $p'^0 \leq \frac{1}{2}(p^0 + q^0)$ and $p'^0 \geq \frac{1}{2}(p^0 + q^0)$.

If $p'^0 \leq \frac{1}{2}(p^0 + q^0)$, we have
\begin{equation}
(p'^0)^{-a-\gamma} \sqrt{J(q')} \lesssim (p^0)^{-m} (q^0)^{-m}
\end{equation}
for any fixed $m > 0$.

On the other region, we have $p'^0 \geq \frac{1}{2}(p^0 + q^0)$ and hence $p'^0 \approx (p^0 + q^0)$ because $p'^0 \leq (p^0 + q^0)$.

Also, we have $(p'^0)^{-a-\gamma} \lesssim (p^0)^{-a-\gamma}$ because $-a - \gamma \leq 0$. Thus, we obtain
\begin{equation}
(p'^0)^{-a-\gamma} \sqrt{J(q')} \lesssim (p^0)^{-a-\gamma} (q^0)^{-1}.
\end{equation}

Thus, in both cases, we have
\begin{equation}
I_{11} \lesssim \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} d\rho \int_{\mathbb{S}^2} d\omega \; v_0 \frac{g^a \sigma_0 \chi \left( \tilde{g} \right) \tilde{g}^{a+\gamma}}{\tilde{g}^{a+\gamma}} |f(q)|^2 |\eta(p')|^2 (p'^0)^{a+\gamma} (q^0)^{-1} \right)^{\frac{1}{2}}.
\end{equation}

By the Carleman dual representation, this is
\begin{equation}
I_{11} \approx \left( \int_{\mathbb{R}^3} \frac{dp'}{p'^0} |\eta(p')|^2 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} \frac{dq}{q^0} |f(q)|^2 \right)^{\frac{1}{2}}
\cdot (p'^0)^{a+\gamma} (q^0)^{-1} \int_{E^p_{q-\rho'}} \frac{d\pi_p}{p'^0} \frac{s}{g} \frac{g^a \sigma_0 \chi \left( \tilde{g} \right) \tilde{g}^{a+\gamma}}{\tilde{g}^{a+\gamma}}
\end{equation}
where $d\pi_p = dp \cdot u(p^0 + q^0 - p'^0) \cdot \delta \left( \frac{s^2 + 2p^a (q_a - p_a')}{2g} \right)$.

Note that $\sigma_0(\theta) \approx \theta^{2-2-\gamma} \approx \left( \frac{2}{9} \right)^{2-2-\gamma}$ and $g \approx \tilde{g}$ on the set $E^p_{q-\rho'}$.

By the inequality (67) and $s \approx \tilde{s} \lesssim p'^0 q^0$, we have
\begin{equation}
\int_{E^p_{q-\rho'}} \frac{d\pi_p}{p'^0} \frac{s}{g} \frac{g^a \sigma_0 \chi \left( \tilde{g} \right) \tilde{g}^{a+\gamma}}{\tilde{g}^{a+\gamma}} \approx \int_{E^p_{q-\rho'}} \frac{d\pi_p}{p'^0} (\tilde{g})^{-2-2-\gamma} \chi \left( \tilde{g} \right) \tilde{s} \tilde{g} \lesssim 2^{k\gamma} p'^0 q^0 \tilde{g}^{\frac{1}{2}}.
\end{equation}

Thus,
\begin{equation}
I_{11} \lesssim \left( \int_{\mathbb{R}^3} \frac{dp'}{p'^0} |\eta(p')|^2 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} \frac{dq}{q^0} |f(q)|^2 (p'^0)^{a+\gamma} (q^0)^{-1} 2^{k\gamma} p'^0 q^0 \tilde{g}^{\frac{1}{2}} \right)^{\frac{1}{2}}
\end{equation}
\begin{equation}
\lesssim \left( \int_{\mathbb{R}^3} \frac{dp'}{|\eta(p')|^2} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |f(q)|^2 (p'^0)^{a+\gamma} 2^{k\gamma} \right)^{\frac{1}{2}}
\lesssim 2^{\frac{a}{2}} w^{t+2} |\eta|_{L^2} |h|_{L^2_{\frac{1}{2}}}.
\end{equation}

On the other hand, by taking pre-post collisional change of variables and by the relativistic
Carleman dual representation of $I_{12}$, we have

$$I_{12} = \left( \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^2} d\omega \, v_0 g^a \sigma_0 \chi_k(\bar{g}) \tilde{g}^{a+\gamma} |h(p)|^2 \sqrt{J(q')}(p^0)^{-\alpha-\gamma} \right)^{1/2}$$

$$= \left( \int_{\mathbb{R}^3} dp' \int_{\mathbb{R}^3} dq' \int_{\mathbb{R}^2} d\omega \, \frac{g \sqrt{S}}{p^0 q^{a+\gamma}} g^a \sigma_0 \chi_k(\bar{g}) |h(p')|^2 \sqrt{J(q')}(p^0)^{-\alpha-\gamma} \right)^{1/2}$$

$$\simeq \left( \int_{\mathbb{R}^3} dp' \int_{\mathbb{R}^3} dq' \int_{\mathbb{R}^2} d\omega \, \frac{g \sqrt{S}}{p^0 q^{a+\gamma}} g^a \sigma_0 \chi_k(\bar{g}) \int_{E_{q',-p}^p} \frac{d\pi'}{p^0}(g_{p^0}g^a s) \sigma_0 \chi_k(\bar{g}) \right)^{1/2}$$

where $d\pi' = dp \cdot u(p^0 + q^0 - p^0) \cdot \delta \left( \frac{(p',-p^0) - (q',-p^0)}{2g} \right)$. Following the same proof of Proposition 2 with the roles of $q$ and $q'$ reversed, we obtain the corollary with respect to the measure $d\pi'$ which tells

$$\int_{E_{q',-p}^p} \frac{d\pi'}{p^0} g(\bar{g})^{-2-\gamma} \chi_k(\bar{g}) \lesssim 2^k \sqrt{q^0}.$$  

Together with this corollary, $s \lesssim p^0 q^0$, and that $g \lesssim \sqrt{p^0 q^0}$, we finally obtain the following inequality:

$$I_{12} \lesssim \left( \int_{\mathbb{R}^3} dp' \left( p^0 \right)^{a+\gamma} |h(p')|^2 \right) \left( \int_{\mathbb{R}^3} dq' \sqrt{J(q')}2^{k\gamma}(q^0)^{-\frac{1}{2}} \right)^{1/2}$$

$$\lesssim 2^{k\gamma} |h|_{L^2} \frac{\sqrt{2}}{\sqrt{2}}.$$  

Thus,

$$I_1 \lesssim 2^{k\gamma} |f|_{L^2} |h|_{L^2} \frac{\sqrt{2}}{\sqrt{2}} |\eta|_{L^2} \frac{\sqrt{2}}{\sqrt{2}}.$$  

This completes the proof.}

4. Cancellation with hard potential kernels

Our goal in this section is to establish an upper bound estimate for the difference $T^k_+ - T^k_-$ for the case that $k \geq 0$. We would like it to have a dependency on the negative power of $2^k$ so we have a good estimate after summation in $k$. Note that $k \geq 0$ also implies that $\bar{g} \leq 1$. Firstly, we define paths from $p'$ to $p$ and from $q'$ to $q$. Fix any two $p, p' \in \mathbb{R}^3$ and consider $\kappa : [0, 1] \to \mathbb{R}^3$ given by

$$\kappa(\theta) \equiv \theta p + (1 - \theta)p'.$$

Similarly, we define the following for the path from $q'$ to $q$;

$$\kappa_q(\theta) \equiv \theta q + (1 - \theta)q'.$$

Then we can easily notice that $\kappa(\theta) + \kappa_q(\theta) = p' + q' = p + q$.

We define the length of the gradient as:

$$|\nabla^i H(p) | \equiv \max_{0 \leq j \leq 3} \sup_{|\chi| \leq 1} \left( \chi \cdot \nabla \right)^j H(p) \bigg|, \quad i = 0, 1, 2,$$

where $\chi \in \mathbb{R}^3$ and $|\chi|$ is the usual Euclidean length. Note that we have $|\nabla^0 H | = |H|$. 


Now we start estimating the term \(|T^k_+ - T^k_-|\) under the condition \(\tilde{\gamma} \leq 1\). We recall from (57) and (84) that \(|(T^k_+ - T^k_-)(f, h, \eta)|\) is defined as

\[
|(T^k_+ - T^k_-)(f, h, \eta)| = \left| \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{S^2} d\omega \sigma_k(g, \omega) v_\alpha f(q) h(p) \left( \sqrt{J(q')} \eta(p') - \sqrt{J(q)} \eta(p) \right) \right|,
\]

(108)

The key part is to estimate \(|\sqrt{J(q')} \eta(p') - \sqrt{J(q)} \eta(p)|\).

We have the following proposition for the cancellation estimate:

**Proposition 4.** Suppose \(\eta\) is a Schwartz function on \(\mathbb{R}^3\). Then, for any \(k \geq 0\) and for \(0 < \gamma < 2\) and \(m \geq 0\), we have the uniform estimate:

\[
|(T^k_+ - T^k_-)(f, h, \eta)| \lesssim 2^{(\gamma - 2)k} |f|_{L^2_m} |h|_{L^{2, \gamma + 1}} |\eta|_{L^{2, \gamma}} + 2^{(\gamma - 2)k} |f|_{L^2_m} |h|_{L^{2, \gamma + 1}} |\eta|_{L^{2, \gamma}}.
\]

(109)

We observe that the weighted fractional Sobolev norm \(|\eta|_{I^\gamma, \gamma}\) is greater than or equal to \(|\eta|_{L^{2, \gamma} + 1}\). Therefore, the direct consequence of this proposition is that

\[
|(T^k_+ - T^k_-)(f, h, \eta)| \lesssim \max \left\{ 2^{(\gamma - 2)k}, 2^{(\gamma - 3)k} \right\} |f|_{L^2_m} |h|_{L^{2, \gamma + 1}} |\eta|_{I^\gamma, \gamma}.
\]

(110)

**Proof.** Note that \(0 < \gamma < 2\). We want our kernel has a good dependency on \(2^{-k}\) so we end up with the negative power on \(2\) as \(2^{(\gamma - 2)k}\). Note that under \(\tilde{\gamma} \leq 1\), we have \(p^0 \approx p^0\) and \(q^0 \approx q^0\). Thus, it suffices to estimate \(|\sqrt{J(q')} \eta(p') - \sqrt{J(q)} \eta(p)|\) only. We now split the term into three parts as

\[
|\sqrt{J(q')} \eta(p') - \sqrt{J(q)} \eta(p)|
\leq |\sqrt{J(q')} \eta(p') - \eta(p)| + |\eta(p)| \left| \sqrt{J(q')} - \sqrt{J(q)} - (\nabla \sqrt{J})(q) \cdot (q' - q) \right|
+ |\eta(p)| \left| (\nabla \sqrt{J})(q) \cdot (q' - q) \right|
= I + II + III.
\]

(111)

We estimate the part II first. By the mean-value theorem for multi-variables on \(\sqrt{J}\), we have

\[
\sqrt{J(q')} - \sqrt{J(q)} = (q' - q) \cdot (\nabla \sqrt{J})(\kappa_q(\theta_1))
\]

for some \(\theta_1 \in (0, 1)\). Now with the fundamental theorem of calculus, we obtain

\[
(\nabla \sqrt{J})(\kappa_q(\theta_1)) - (\nabla \sqrt{J})(q) = \left( \int_0^{\theta_1} D(\nabla \sqrt{J})(\kappa_q(\theta')) d\theta' \right) \cdot (\kappa_q(\theta_1) - q),
\]

(112)

for some \(\theta_1 \in (0, 1)\). Now with the fundamental theorem of calculus, we obtain
where \( D(\nabla \sqrt{J}) \) is the 3\( \times \)3 Jacobian matrix of \( \nabla \sqrt{J} \). With the definition on \( |\nabla| \) from (107), we can bound the part \( II \) by

\[
II \leq |\eta(p)||q' - q| \left( \int_0^{\theta_1} \frac{d\theta'}{\theta_1} \right) \cdot (\kappa_4(\theta') - q) \cdot \left( \int_0^{\theta_1} \frac{d\theta'}{\kappa_4(\theta')} \right) \cdot \left( \frac{1}{\theta_1} \right)
\]

(114)

\[
\leq |\eta(p)||q' - q| \left( \int_0^{\theta_1} |\nabla| \sqrt{J}(\kappa_4(\theta')) d\theta' \right)
\]

\[
\leq |\eta(p)||q' - q|^2 \left( \int_0^{\theta_1} |\nabla|^2 \sqrt{J}(\kappa_4(\theta')) d\theta' \right).
\]

Note that

(115)

\[ |\nabla|^2 \sqrt{J} \lesssim \sqrt{J}. \]

Also, we have that

\[ \sqrt{J}(\kappa_4(\theta')) \lesssim (J(q)J(q'))^\epsilon \]

for sufficiently small \( \epsilon \). Thus, the estimate for the integral with this kernel \( II \) follows exactly the same as in the proposition for \( |T_{+k}^k| \) as in (59) and we get the first term in the right-hand side of the proposition.

For the part \( III \), as (299) in the Appendix, we reduce this integral to the integral on the set \( E_{\mu+\eta}^\nu \) as the following:

(116)

\[
|T_{+,III}^k - T_{-,III}^k| \leq \left| \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^4} dE \frac{d\pi'}{\sqrt{S_{p,\eta}^0}} \sigma(g, \omega) f(q)(p - p') \cdot (\nabla \sqrt{J})(q) \eta(p) \right|
\]

where

\[ d\pi' = dp' \cdot u(p^0 + q^0 - p'^0) \delta \left( -\frac{x}{\sqrt{\nu}} - \frac{p''(p_{\mu} + q_{\mu})}{\sqrt{\nu}} \right). \]

As it is expressed in the measure \( d\pi' \), we can see that \((p'' - p')(p_{\mu} + q_{\mu}) = 0 \) on the set \( E_{\mu+\eta}^\nu \). In this integral as \( p' \) varies on hyperboloids of constant \( \bar{g} \), the integral is constant except for \((p - p') \cdot (\nabla \sqrt{J})(q) \). Write the term as:

\[
(p - p') \cdot (\nabla \sqrt{J})(q) = (p'' - p')(\nabla \sqrt{J})(\mu)(q),
\]

where we define \((\nabla \sqrt{J})(\mu)(q) \) \( \overset{\text{def}}{=} (0, (\nabla \sqrt{J})(q)) \). Then this term is linear in \((p'' - p')(\mu)\) and hence this whole integral vanishes by the symmetry of the set \( E_{\mu+\eta}^\nu \).

For the part \( I \), we define \( \tilde{\eta}(p, p') = \eta(p') - \eta(p) \). Since \( \bar{g} \leq 1 \), we have \( q^0 \approx q' \) and this gives that there is some uniform constant \( c > 0 \) such that \( \sqrt{J}(q') \lesssim (J(q))^c \). Thus, we have

\[
|T_{+,I}^k - T_{-,I}^k| \leq \left| \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^4} dE \frac{d\pi'}{\sqrt{S_{p,\eta}^0}} \sigma(g, \omega) f(q)(p - p') \cdot (\nabla \sqrt{J})(q) \eta(p) \right|
\]

(117)

\[
\leq \left| \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^4} dE \frac{d\pi'}{\sqrt{S_{p,\eta}^0}} \sigma(g, \omega) f(q)(p - p') \tilde{\eta}(p, p')(J(q))^c \right|.
\]

Now, we use Cauchy-Schwarz inequality and obtain

\[
|T_{+,I}^k - T_{-,I}^k| \leq \left( \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^4} dE \frac{d\pi'}{\sqrt{S_{p,\eta}^0}} \sigma(g, \omega) [f(q)|^2 |h(p)|^2 (J(q))^c] \right)^{\frac{1}{2}}
\]

\[
\times \left( \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^4} dE \frac{d\pi'}{\sqrt{S_{p,\eta}^0}} \sigma(g, \omega) [\tilde{\eta}(p, p')|^2 (J(q))^c] \right)^{\frac{1}{2}}.
\]

(118)
The first part on the right-hand side is bounded by $2\frac{k^2}{\gamma} |f|_{L^2_{\gamma_\bot}} |h|_{L^2_{\frac{\gamma}{2\gamma_\bot}}}^2$ for some $m \geq 0$ as in (61) and (62). For the second part, we rewrite this 8-fold integral as the following 12-fold integral:

$$
\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \, v_p \sigma_k(g, \omega) |\hat{\eta}(p, p')|^2 (J(q))^c
$$

$$
= \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dp' \int_{\mathbb{R}^3} dq' \, \sigma(g, \omega) \chi_k(\hat{g}) |\hat{\eta}(p, p')|^2 (J(q))^c \delta^{(4)}(p^{\mu} + q^{\mu} - p^{\mu} - q^{\mu}).
$$

As in (303), we reduce this integral to the integral on the set $E^q_{\rho' - p}$ as the following:

$$
\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dp' \int_{\mathbb{R}^3} dq' \, \sigma(g, \omega) \chi_k(\hat{g}) |\hat{\eta}(p, p')|^2 (J(q))^c.
$$

Therefore, we finally obtain that

$$
|T_{\rho+1} - T_{\rho-1}| \leq 2^{2(2-\gamma)k} |f|_{L^2_{\gamma_\bot}} |h|_{L^2_{\frac{\gamma}{2\gamma_\bot}}} |\eta|_{H^{r/\gamma}}.
$$

Together with the previous estimates on part II and III, we obtain the proposition.

5. Littlewood-Paley decompositions

In this section, we would like to decompose our function further so that each decomposed piece has a support of disjoint annulus in the frequency space. We will see that the sum of decomposed pieces depends on the negative power of 2 and the sum is bounded well to obtain the main upper bound estimate by the Littlewood-Paley theory in this and the
next chapters. This standard Littlewood-Paley decompositions will allow us to make sharp estimates on the linearized relativistic Boltzmann operator.

5.1. **Main Estimates on the Littlewood-Paley Decomposition.** Our purpose in this section is to decompose $f$ into infinitely many pieces $f_j$ for $j \geq 0$ such that

$$f = \sum_{j=0}^{\infty} f_j \quad (124)$$

and that each $f_j$ corresponds to the usual projection onto frequencies comparable to $2^j$ which corresponds to the scale $2^{-j}$ in physical space. From here, $\hat{f}(\xi)$ will denote the Fourier transform of $f(p)$.

We first choose and fix any $C^\infty$-function $\phi$ supported on the unit ball of $\mathbb{R}^3$. Then, we define the difference kernel $\psi$ as

$$\psi(w) \overset{\text{def}}{=} \phi(w) - 2^{-3}\phi(w/2)$$

so that its Fourier transform satisfies $\hat{\psi}(\xi) = \hat{\phi}(\xi) - \hat{\phi}(2\xi)$. We also abbreviate $\hat{\phi}_j(w) \overset{\text{def}}{=} 2^{3j}\phi(2^jw)$ and likewise for $\psi_j$ so that their Fourier transform satisfies $\hat{\phi}_j(\xi) = \hat{\phi}(2^{-j}\xi)$ and likewise for $\hat{\psi}_j$. Then we have

$$\hat{\phi}(\xi) + \sum_{j=1}^{l} \hat{\psi}_j(\xi) = \hat{\phi}(\xi) + \sum_{j=1}^{l} (\hat{\phi}(2^{-j}\xi) - \hat{\phi}(2^{-j+1}\xi)) = \hat{\phi}(2^{-l}\xi) \to 1 \text{ as } l \to \infty. \quad (125)$$

Now define the partial sum operator

$$S_j(f) \overset{\text{def}}{=} f \ast \phi_j = \int_{\mathbb{R}^3} 2^{3j}\phi(2^j(p-q))f(q)dq \quad (126)$$

and the difference operator

$$\Delta_j(f) \overset{\text{def}}{=} f \ast \psi_j = \int_{\mathbb{R}^3} 2^{3j}\psi(2^j(p-q))f(q)dq \quad (127)$$

where we define $\Delta_0 = S_0$. Then we notice that $\Delta_j$ satisfies

$$\Delta_j(1)(p) = (1 \ast \psi_j)(p) = \int_{\mathbb{R}^3} \psi_j(q)dq = \hat{\psi}_j(0) = 0. \quad (128)$$

Throughout this section, the variables $p$ and $p'$ are considered to be independent vectors in $\mathbb{R}^3$ and we will not assume the variables $p$ and $p'$ are related by the collision geometry. We will, however, see that the estimates on these Littlewood-Paley projections will be used in later chapters for the estimates that involve the relativistic collisional geometry.

Note that we have

$$\hat{S}_j f(\xi) = \hat{f}(\xi) \hat{\phi}_j(\xi) \quad (129)$$

and

$$\hat{\Delta}_j f(\xi) = \hat{f}(\xi) \hat{\psi}_j(\xi). \quad (130)$$

Remark that, if $\int_{\mathbb{R}^3} \phi \, dx = 1$, we have that

$$S_j f(p) \to f(p) \quad (131)$$

as $j \to \infty$ for all sufficiently smooth $f$ and that

$$\left( \int_{\mathbb{R}^3} dp |S_j f(p)| \bar{p}(p^0)^{\rho} \right)^{\frac{1}{\rho}} \lesssim \left( \int_{\mathbb{R}^3} dp |f(p)| \bar{p}(p^0)^{\rho} \right)^{\frac{1}{\rho}}. \quad (132)$$
uniformly in $j \geq 0$ for any fixed $\rho \in \mathbb{R}$ and any $\bar{p} \in [1, \infty]$. This $L^p$-boundedness property also holds for the operators $\Delta_j$.

We are interested in making an upper bound estimate for

$$
\sum_{j=0}^{\infty} 2^{\gamma j} \int_{\mathbb{R}^3} dp \, |\Delta_j f|^2 (p^0)^\rho
$$

when $0 < \gamma < 2$ and $\rho \geq 0$.

Here we state our first main proposition of this chapter:

**Proposition 5.** For any $\gamma \in (0, 2)$ and any $\rho \in \mathbb{R}$, the following inequality holds:

$$
\sum_{j=0}^{\infty} 2^{\gamma j} \int_{\mathbb{R}^3} dp \, |\Delta_j f|^2 (p^0)^\rho
$$

$$(133) \quad \lesssim |f|_{L^2}^2 + \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dp' (p^0 p'^0)^\gamma \frac{(f(p) - f(p'))^2}{p^{3+\gamma}} 1_{\bar{p} \leq 1}.
$$

This holds for any smooth $f$.

We denote the right-hand side of the inequality as $|f|_{J^\gamma}^2$.

**Proof.** For any $j \geq 1$, we have

$$
\frac{1}{2} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dp' \int_{\mathbb{R}^3} dz (f(p) - f(p'))^2 \psi_j(z - p) \psi_j(z - p')(z^0)^\rho
$$

$$(134) \quad = - \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dp' \int_{\mathbb{R}^3} dz f(p) f(p') \psi_j(z - p) \psi_j(z - p')(z^0)^\rho + \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dz f(p) \psi_j(z - p) \Delta_j(1)(z)(z^0)^\rho
$$

because $\Delta_j(1)(z) = \int_{\mathbb{R}^3} dp' \psi_j(z - p')$ and the left-hand side of the equality is equal to

$$(\text{LHS}) = \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dz (f(p)^2 \psi_j(z - p)(z^0)^\rho \int_{\mathbb{R}^3} dp' \psi_j(z - p') - \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dp' \int_{\mathbb{R}^3} dz f(p) f(p') \psi_j(z - p) \psi_j(z - p')(z^0)^\rho
$$

$$(135) \quad = \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dz (f(p)^2 \psi_j(z - p)(z^0)^\rho \Delta_j(1)(z) - \int_{\mathbb{R}^3} dz (z^0)^\rho \left( \int_{\mathbb{R}^3} dp f(p) \psi_j(z - p) \right)^2 = (\text{RHS}).
$$

Since $\Delta_j(1)(p) = 0$ from (128), we have

$$
\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dz (f(p)^2 \psi_j(z - p)(z^0)^\rho \Delta_j(1)(z) = 0.
$$

On the other hand, from the support condition for $\psi_j(z - p) \psi_j(z - p')$ on $z$, we have $p^0 \approx p'^0 \approx z^0$. Also notice that $|\psi_j(z - p')| \lesssim 2^{3j}$ because $|\psi_j(z - p)| = 2^{3j} |\psi(2^j(z - p))| \lesssim 2^{3j}$. Thus,

$$
\int_{\mathbb{R}^3} dz |\psi_j(z - p')||\psi_j(z - p)|(z^0)^\rho
$$

$$(136) \quad \lesssim 2^{3j} \int_{\mathbb{R}^3} dz |\psi_j(z - p)|(z^0)^\rho \lesssim 2^{3j}(p^0 p'^0)^\frac{\gamma}{2}.
$$
Note that the integral is supported only when $|p - p'| \leq 2^{-j+1}$ because
$$|p - p'| \leq |z - p| + |z - p'| \leq 2^{-j+1}.$$ Therefore, we have

$$2^{3j} \int_{\mathbb{R}^3} dz |\psi_j(z - p')||\psi_j(z - p)|(z^0)^p \lesssim 2^{(3+\gamma)j} (p^0 p^0) \frac{\tilde{\gamma}}{1}|_{|p - p'| \leq 2^{-j+1}}. \quad (138)$$

Since there exists $j_0 > 0$ such that $2^{-j_0} < |p - p'| \leq 2^{-j_0+1}$, we have

$$\sum_{j=1}^{\infty} 2^{(3+\gamma)j} 1_{|p - p'| \leq 2^{-j+1}} \lesssim 2^{(3+\gamma)j_0} 1_{|p - p'| \leq 2^{-j+1}} \lesssim \frac{1_{\tilde{g} \leq 1}}{|p - p'|^{\gamma+3}} \quad (139)$$

because $\tilde{g} \leq |p - p'|$. We recall that $\tilde{g} = g(p^\mu, p^\nu) = (p^\mu - p^\nu)(p^\mu - p^\nu)$. If $j = 0$, the term $\int_{\mathbb{R}^3} dp |\Delta_0 f^2(p^0)^p|$ is bounded above by $|f|^2_{L^2}$.

Finally, we obtain that

$$\sum_{j=0}^{\infty} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dp' \int_{\mathbb{R}^3} dz (f(p) - f(p'))^2 \psi_j(z - p) \psi_j(z - p')(z^0)^p \quad (140)$$

because $\tilde{g} \leq |p - p'|$. Therefore, by the first equality in this proof, we obtain the proposition. \hfill \Box

5.2. Estimates on the Derivatives. We would also need to establish a similar inequality when $\Delta_j$’s are replaced by $2^{-kj}\nabla \Delta_j$, where $\nabla$ is the spatial gradient. We consider the estimates of the spatial derivatives of our operators. Although we have high angular singularity on the collision kernel, we do not use any momentum derivative for our proof throughout this paper.

Recall our notation that $\nabla^\alpha = (\partial_{x_1}^{\alpha_1}, \partial_{x_2}^{\alpha_2}, \partial_{x_3}^{\alpha_3})$. Note that, for any partial derivative $\frac{\partial}{\partial x_j} \Delta_j f$, there holds $\frac{\partial}{\partial x_j} \Delta_j f = 2^j \tilde{\Delta}_j f$ where $\tilde{\Delta}_j$ is the $j^{th}$-Littlewood-Paley cut-off operator associated to a new cut-off function $\tilde{\psi}$ which also satisfies the cancellation property (128) that $\tilde{\Delta}_j(1)(p) = 0$. Thus, we can write

$$2^{-|\alpha|/2} \nabla^\alpha \Delta_j f(p) = \Delta_j^\alpha(f)(p) \quad (141)$$

where $\Delta_j^\alpha$ is the cut-off operator associated to some $\psi^\alpha$ and $\nabla$ is the usual 3-dimensional spatial gradient.

Then, we can repeat the similar proof as that for Proposition 5 by considering the following integral instead to make an upper-bound estimate on the weighted $L^2$-norm of the derivatives of each Littlewood-Paley decomposed piece:

$$\frac{1}{2} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dp' \int_{\mathbb{R}^3} dz (f(p) - f(p'))^2 \psi_j^0(z - p) \psi_j^0(z - p')(z^0)^p. \quad (142)$$
Then we show that the integral above is equal to

\[- \int_{\mathbb{R}^3} dp |\Delta_j^\alpha(f)(p)|^2 (p^0)^\rho + \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dz (f(p))^2 \psi^\alpha_j(z - p) \Delta_j^\alpha(1)(z)(z^0)^\rho.\]

Together with the same condition as in (128) that \(\Delta_j^\alpha(1)(p) = 0\), these estimates can be multiplied by \(2^\gamma j\) and summed over \(j\) to get

\[
\sum_{j=0}^{\infty} 2^{\gamma j} \int_{\mathbb{R}^3} dp |\Delta_j^\alpha(f)(p)|^2 (p^0)^\rho \lesssim |f|_{L_2}^2 + \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dp' (p^0 p'^0)^{\frac{\alpha}{2}} \frac{(f(p) - f(p'))^2}{q^{3+\gamma}} 1_{\bar{g} \leq 1}.
\]

Therefore, it follows that

\[
\sum_{j=0}^{\infty} 2^{(\gamma - |\alpha|)j} \int_{\mathbb{R}^3} dp |\nabla^\alpha \Delta_j f(p)|^2 (p^0)^{\frac{\alpha}{2}} \lesssim |f|_{L_2}^2.
\]

These two inequalities hold for any multi-index \(\alpha\) and any fixed \(l \in \mathbb{R}\).

6. Main upper bound estimates

In this section, we finally establish the main upper bound estimates with the hard potential collision kernel.

We write

\[
h = \Delta_0 h + \sum_{j=1}^{\infty} \Delta_j h = \sum_{j=0}^{\infty} h_j
\]

where we denote \(h_j = \Delta_j h\) for \(j \geq 0\). Then, the trilinear product can be written as

\[
(\Gamma(f, h), \eta) = \sum_{j=0}^{\infty} (\Gamma(f, h_j), \eta).
\]

We consider the dyadic decomposition of gain and loss terms as the following,

\[
\sum_{j=0}^{\infty} (\Gamma(f, h_j), \eta) = \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \{ T_k^+(f, h_j, \eta) - T_k^-(f, h_j, \eta) \}
\]

\[
= \sum_{j=0}^{\infty} \sum_{k=-\infty}^{0} \{ T_k^+(f, h_j, \eta) - T_k^-(f, h_j, \eta) \}
\]

\[
+ \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \{ T_k^+(f, h_j, \eta) - T_k^-(f, h_j, \eta) \}
\]

\[
+ \sum_{j=0}^{\infty} \sum_{k=\lfloor \frac{\gamma}{4} \rfloor + 1}^{\infty} \{ T_k^+(f, h_j, \eta) - T_k^-(f, h_j, \eta) \}
\]

\[
def = S_1 + S_2 + S_3.
\]
We first compute the upper bound for the sum $S_3$. In this sum, we note that $k \geq 0$ and $0 < \gamma < 2$. Then, by (110), we obtain
\begin{equation}
|S_3| \lesssim \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \max\{2^{(\gamma-2)k}, 2^{\frac{\gamma-3}{2}k}\} |f|_{L^2_m} |h_j|_{L^2_{\frac{3+\gamma}{2}}} |\eta|_{I^m,\gamma} \tag{148}
\end{equation}
\begin{equation}
\lesssim \sum_{j=0}^{\infty} \max\{2^{(\gamma-2)\frac{3}{4}}, 2^{\frac{\gamma-3}{2}k}\} |f|_{L^2_m} |h_j|_{L^2_{\frac{3+\gamma}{2}}} |\eta|_{I^m,\gamma} \tag{149}
\end{equation}

Then, we impose (133) to obtain that
\begin{equation}
|S_3| \lesssim |f|_{L^2_m} |h|_{I^{m,\gamma}} |\eta|_{I^{m,\gamma}}. \tag{149}
\end{equation}

For the sum $S_2$, we use (83) and (59). Then, we have
\begin{equation}
|S_2| \lesssim \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} 2^{\gamma j} |f|_{L^2_m} |h_j|_{L^2_{\frac{3+\gamma}{2}}} |\eta|_{L^2_{\frac{3+\gamma}{2}}} \tag{150}
\end{equation}
\begin{equation}
\lesssim \sum_{j=0}^{\infty} 2^{\gamma j} |f|_{L^2_m} \left( \sum_{j=0}^{\infty} 2^{\gamma j} |h_j|_{L^2_{\frac{3+\gamma}{2}}} \right)^{\frac{1}{2}} \tag{150}
\end{equation}
\begin{equation}
\lesssim |f|_{L^2_m} \left| h \right|_{I^{m,\gamma}} |\eta|_{L^2_{\frac{3+\gamma}{2}}} \tag{150}
\end{equation}

where the last inequality is by (133) and the third inequality is by
\begin{equation}
\sum_{j=0}^{\infty} 2^{\gamma j} |h_j|_{L^2_{\frac{3+\gamma}{2}}} \lesssim \sum_{j=0}^{\infty} \left( \sum_{j=0}^{\infty} 2^{\gamma j} |h_j|_{L^2_{\frac{3+\gamma}{2}}} \right)^{\frac{1}{2}} \lesssim \sum_{j=0}^{\infty} 2^{\gamma j} \tag{150}
\end{equation}

For the sum $S_1$, we note that $\sum_{k=-\infty}^{0} 2^{k\gamma} \leq 1$. Then, by (83) and (59), we obtain that
\begin{equation}
|S_1| \lesssim \sum_{j=0}^{\infty} \sum_{k=-\infty}^{0} 2^{\gamma j} |f|_{L^2_m} |h_j|_{L^2_{\frac{3+\gamma}{2}}} |\eta|_{L^2_{\frac{3+\gamma}{2}}} \tag{151}
\end{equation}
\begin{equation}
\lesssim \sum_{j=0}^{\infty} |f|_{L^2_m} |h_j|_{L^2_{\frac{3+\gamma}{2}}} |\eta|_{L^2_{\frac{3+\gamma}{2}}} \tag{151}
\end{equation}
\begin{equation}
\lesssim |f|_{L^2_m} \left( \sum_{j=0}^{\infty} 2^{\gamma j} |h_j|_{L^2_{\frac{3+\gamma}{2}}} \right)^{\frac{1}{2}} \tag{151}
\end{equation}
\begin{equation}
\lesssim |f|_{L^2_m} |h|_{I^{m,\gamma}} |\eta|_{I^{m,\gamma}} \tag{151}
\end{equation}

Thus, we can collect the estimates on $S_1, S_2, S_3$ and conclude that
\begin{equation}
|\langle \Gamma(f, h), \eta \rangle | \lesssim |f|_{L^2_m} |h|_{I^{m,\gamma}} |\eta|_{I^{m,\gamma}}. \tag{152}
\end{equation}

This proves Theorem 2. Note that this immediately implies Lemma 3 by taking the spatial derivatives on the functions.
Here we also would like to mention a proposition that is used to prove other further compact estimates.

**Proposition 6.** Let $\phi(p)$ be an arbitrary smooth function which satisfies for some positive constant $C_\phi$ and $c$ that

$$|\phi(p)| \leq C_\phi e^{-cp^0}.$$  

Then we have that

$$|\langle \Gamma(\phi, f), h \rangle| \lesssim |f|_{L^2_{\alpha + \gamma}} |h|_{L^2_{\alpha + \gamma}}.$$  

If $\phi$ further satisfies a more smoothing condition that for some positive constant $C_\phi$ and $c$

$$|\nabla|^2 \phi \leq C_\phi e^{-cp^0},$$

then we have

$$|\langle \Gamma(f, \phi), h \rangle| \lesssim |f|_{L^2_{\alpha - \delta}} |h|_{L^2_{\alpha - \delta}}.$$  

Additionally, for any $m \geq 0$, we have

$$|\langle \Gamma(f, h), \phi \rangle| \lesssim |f|_{L^2_{\alpha - m}} |h|_{L^2_{\alpha - m}}.$$  

**Proof.** For (154), we expand the trilinear form as in (146) and use Sobolev embeddings on the $L^2$-norm of $\phi$ to bound it by $L^\infty$-norm with some derivatives which are also bounded uniformly. For (156), we use that

$$|\langle \Gamma(f, \phi), h \rangle| \lesssim |f|_{L^2_{\alpha - \delta}} |h|_{L^2_{\alpha - \delta}} \sum_{j=0}^\infty \sum_{k=-\infty}^\infty \min\{2^{(\gamma-2)j}, 2^{\gamma k}\} 2^{-2j}.$$  

Similar proof works for (157). \qed

Note that (154) implies Lemma 5. Also, this proposition further implies the following lemma:

**Lemma 8.** For any $l \in \mathbb{R}$, we have the uniform estimate

$$|\langle K f, h \rangle| \lesssim |f|_{L^2_{\alpha + \gamma - \delta}} |h|_{L^2_{\alpha + \gamma - \delta}}$$

where $\delta = \min\{\gamma, 2\}$.

An immediate consequence of this lemma is Lemma 4 by letting $h = f$. More precisely, we use that the upper bound of the inequality in the lemma is bounded above by

$$|f|_{L^2_{\alpha + \gamma - \delta}}^2 \leq \frac{\epsilon}{2} |f|_{L^2_{\alpha + \gamma - \delta}}^2 + C_\epsilon |f|_{L^2_{\alpha + \gamma - \delta}}^2.$$  

For the term $C_\epsilon |f|_{L^2_{\alpha + \gamma - \delta}}^2$, we split the region into $|p| \leq R$ and $|p| \geq R$. We choose $R > 0$ large enough so that $C_\epsilon R^{-\delta} \leq \frac{\epsilon}{2}$. Then we obtain Lemma 4.
7. Main coercive estimates

In this section, for any Schwartz function $f$, we consider the quadratic difference arising in the inner product of the norm part $Nf$ with $f$. The main part is to estimate the norm $|J_{B}|_{\dot{B}}^{2}$ which arises in the inner product and will be defined as follows.

\[
|J_{B}|_{\dot{B}}^{2} \overset{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}^{3}} dp \int_{\mathbb{R}^{3}} dq \int_{\mathbb{S}^{2}} d\omega \ v_{p} \sigma(g, \theta)(f(p') - f(p))^2 \sqrt{J(q)J(q')} \nonumber \\
\geq \frac{1}{2} \int_{\mathbb{R}^{3}} dp \int_{\mathbb{R}^{3}} dq \int_{\mathbb{S}^{2}} d\omega \ v_{p} \sigma(g, \theta)(f(p') - f(p))^2 \sqrt{J(q)J(q')} 1_{\bar{g} \leq 1} \quad (160) 
\]

Note that if $\bar{g} \leq 1$, we have $\bar{q}^{0} \approx q^{0}$ as well as $p^{0} \approx p^{0}$. Thus, we can bound $\sqrt{J(q)J(q')}$ below as $\sqrt{J(q)J(q')} \geq e^{-Cq^{0}}$ for some uniform constant $C > 0$.

By the alternative Carleman-type dual representation of the integral operator as in (309), it is possible to write the lower bound of the norm as an integral of some kernel $K(p, p')$ as follows:

\[
|J_{B}|_{\dot{B}}^{2} \gtrsim \int_{\mathbb{R}^{3}} dp \int_{\mathbb{R}^{3}} dq \int_{\mathbb{S}^{2}} d\omega \ v_{p} \sigma(g, \theta)(f(p') - f(p))^2 e^{-Cq^{0}} 1_{\bar{g} \leq 1} \nonumber \\
\approx \int_{\mathbb{R}^{3}} \frac{dp}{p^{0}} \int_{\mathbb{R}^{3}} \frac{dp'}{p'^{0}} (f(p') - f(p))^2 1_{\bar{g} \leq 1} \int_{\mathbb{R}^{3}} \frac{dq_{s}}{\sqrt{|q_{s}|^{2} + 8}} \delta(q_{s}((p'_{\mu} - p_{\mu}))s\sigma(g, \theta)e^{-Cq^{0}} \nonumber \\
\overset{\text{def}}{=} \int_{\mathbb{R}^{3}} \frac{dp}{p^{0}} \int_{\mathbb{R}^{3}} \frac{dp'}{p'^{0}} (f(p') - f(p))^2 1_{\bar{g} \leq 1} K(p, p'), \quad (161) 
\]

where the kernel $K(p, p')$ is defined as

\[
K(p, p') \overset{\text{def}}{=} \int_{\mathbb{R}^{3}} \frac{dq_{s}}{\sqrt{|q_{s}|^{2} + 8}} \delta(q_{s}((p'_{\mu} - p_{\mu}))s\sigma(g, \theta)e^{-Cq^{0}}. 
\]

Our goal in this section is to make a coercive lower bound of this kernel and hence the norm $|J_{B}|$. First of all, the delta function in (162) implies that $(p'^{\mu} - p^{\mu})(p'_{\mu} - p_{\mu} + 2q'_{\mu}) = 0$. Then this implies that

\[
2(p'^{\mu} - p^{\mu})(p'_{\mu} - p_{\mu}) = 2p'^{\mu}q'_{\mu} - 2p^{\mu}p_{\mu} - 2p'^{\mu}q'_{\mu} + 2p^{\mu}p_{\mu} \quad (163) 
\]

Then, we obtain that

\[
\bar{g}^{2} + \bar{q}^{2} = (p'^{\mu} - p^{\mu})(p'_{\mu} - p_{\mu}) - 2(p'^{\mu} - p^{\mu})(p'_{\mu} - p_{\mu}) + (q'^{\mu} - p^{\mu})(q'_{\mu} - p_{\mu}) \nonumber \\
= (p'^{\mu} - p^{\mu})(p'_{\mu} - p_{\mu}) \overset{\text{def}}{=} g'^{2}, \quad (164) 
\]

and we have $\bar{g}^{2} + \bar{q}^{2} = g^{2}$ on this hyperplane as expected where $g'^{\text{def}} = g(p'^{\mu}, q'^{\mu})$. Note that, from the assumptions on the collision kernel, we have $\sigma(g', \theta) = \Phi(g')\sigma_{0}(\theta)$ and

\[
\sigma_{0}(\theta) \approx \frac{1}{\sin \theta \cdot \theta^{1+\gamma}} \approx \frac{1}{\bar{g}^{2+\gamma}} \approx \left( \frac{g'}{g} \right)^{2+\gamma} \quad (165) 
\]

Thus,

\[
\sigma(g', \theta) \approx \Phi(g')\left( \frac{g'}{g} \right)^{2+\gamma} \quad (166) 
\]

Together with this, we have
\[
K(p,p') \approx \int_{\mathbb{R}^3} \frac{dq_s}{\sqrt{|q_s|^2 + s}} \delta(q_\alpha^\mu (p_\mu' - p_\mu)) s \Phi(g') \left( \frac{g'}{g} \right)^{2+\gamma} e^{C q^\alpha} \\
\geq \int_{\mathbb{R}^3} \frac{dq_s}{\sqrt{|q_s|^2 + s}} \delta(q_\alpha^\mu (p_\mu' - p_\mu)) s \left( \frac{g'}{g} \right)^{2+\gamma} e^{C q^\alpha} \frac{g'^{a+\gamma}}{g^{2+\gamma}} \\
\geq \int_{\mathbb{R}^3} \frac{dq_s}{q^0} \delta(q_\alpha^\mu (p_\mu' - p_\mu)) e^{-C q^\alpha} \frac{g'^{4+a+\gamma}}{g^{2+\gamma}},
\]
(167)
where the first inequality is from the assumption on the collision kernel (34) that \( \Phi(g') \geq \frac{q^\alpha}{g^2} g^a \) and that \( s = g^2 + 4 > g^2 \), and the last inequality is by that \( \sqrt{|q_s|^2 + s} \lesssim q^0 \) if \( \bar{g} \leq 1 \) by the geometry.

Here, we have the following lower bound for the kernel \( K(p,p') \).

**Proposition 7.** If \( \bar{g} \leq 1 \), the kernel \( K(p,p') \) is bounded uniformly from below as
\[
K(p,p') \gtrsim \frac{(p^0)^{2+\frac{4+a+\gamma}{2+\gamma}}}{\bar{g}^{3+\gamma}}.
\]
(168)

With this proposition, we can obtain the uniform lower bound for the norm \( |f|_B \) as below.
\[
|f|_B^2 \gtrsim \int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} (f(p') - f(p))^2 (p^0)^{2+\frac{4+a+\gamma}{2}} 1_{\bar{g} \leq 1} \\
\geq \int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} (f(p') - f(p))^2 (p'^0)^{2+\frac{4+a+\gamma}{2}} 1_{\bar{g} \leq 1} \\
\geq \int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} (f(p') - f(p))^2 (p'^0 p^0)^{2+\gamma} 1_{\bar{g} \leq 1}
\]
(169)
Thus, the proof for our main coercive inequality is complete because we have that
\[
|f|_{L^2_{\frac{1}{2+\gamma}}}^2 + |f|_B^2 \gtrsim |f|_{L^{1,\gamma}}^2.
\]
(170)

**Proof.** Here we prove Proposition 7. We begin with
\[
K(p,p') \gtrsim \int_{\mathbb{R}^3} \frac{dq_s}{q^0} \delta(q_\alpha^\mu (p_\mu' - p_\mu)) e^{-C q^\alpha} \frac{g'^{4+a+\gamma}}{g^{2+\gamma}}.
\]
(171)
First, we take a change of variables from \( q_s = p' - p + 2q' \) to \( q' \). Then we obtain that
\[
K(p,p') \gtrsim \int_{\mathbb{R}^3} \frac{dq'}{q'^0} \delta((p'^\mu - p^\mu + 2q'^\mu)(p_\mu' - p_\mu)) e^{-C q'^\alpha} \frac{g'^{4+a+\gamma}}{g^{2+\gamma}}
\]
(172)
Now we take a change of variables on \( q' \) into polar coordinates as \( q' \in \mathbb{R}^3 \rightarrow (r, \theta, \phi) \) and choose the z-axis parallel to \( p' - p \) such that the angle between \( q' \) and \( p' - p \) is equal to \( \phi \).

Then we obtain that
\[
K(p,p') \gtrsim \int_{\frac{1}{2+\gamma}}^{\infty} dq'^0 \int_0^{2\pi} d\theta \int_0^{\pi} d\phi \ r^2 \sin \phi \\
\times \frac{g'^{4+a+\gamma}}{g^{2+\gamma}} \delta(g^2 + 2q'^\mu (p_\mu' - p_\mu)) \delta(r^2 + 1 - (q'^0)^2) e^{-C q'^\alpha}.
\]
(173)
The terms in the first delta function in (173) can be written as
\[ g^2 - 2q^0(p'_\mu - p_\mu) = g^2 - 2q^0(p^0 - p^0) + 2q' \cdot (p' - p) = g^2 - 2q^0(p^0 - p^0) + 2r|p' - p| \cos \phi. \]

Also, note that the second delta function is
\[ \delta(r^2 + 1 - (q^0)^2) = \delta((r - \sqrt{(q^0)^2} - 1)(r + \sqrt{(q^0)^2} - 1)) = \frac{\delta(r - \sqrt{(q^0)^2} - 1)}{2 \sqrt{(q^0)^2} - 1}, \]
because \( r > 0 \). Now we reduce the integration against \( r \) using this delta function and get
\[
K(p, p') \gtrsim \int_1^\infty dq^0 \int_0^{2\pi} d\theta \int_0^\pi d\phi \frac{(q^0)^2 - 1}{2 \sqrt{(q^0)^2} - 1} \sin \phi
\times \delta(g^2 - 2q^0(p^0 - p^0) + 2\sqrt{(q^0)^2} - 1|p' - p| \cos \phi) g^{4+a+\gamma} e^{-Cq^0}.
\]

Now, let \( v = \cos \phi \). Then, \( dv = -\sin \phi d\phi \) and the integration is now rewritten as
\[
K(p, p') \gtrsim \int_1^\infty dq^0 \int_0^{2\pi} d\theta \int_{-1}^1 dv \frac{(q^0)^2 - 1}{2 \sqrt{(q^0)^2} - 1}
\times \delta(g^2 - 2q^0(p^0 - p^0) + 2\sqrt{(q^0)^2} - 1|p' - p|v) g^{4+a+\gamma} e^{-Cq^0}.
\]

Note that
\[
\delta(g^2 - 2q^0(p^0 - p^0) + 2\sqrt{(q^0)^2} - 1|p' - p|v) = \frac{\delta(v + \frac{g^2 - 2q^0(p^0 - p^0)}{2\sqrt{(q^0)^2} - 1}|p' - p|)}{2 \sqrt{(q^0)^2} - 1|p' - p|}.
\]

We remark that \( \frac{|\frac{g^2 - 2q^0(p^0 - p^0)}{2\sqrt{(q^0)^2} - 1}|p' - p|}{2 \sqrt{(q^0)^2} - 1|p' - p|} \leq 1 \). Then we further reduce the integration on \( v \) by removing this delta function and get
\[
K(p, p') \gtrsim \int_1^\infty dq^0 \int_0^{2\pi} d\theta \frac{1}{|p' - p|} g^{4+a+\gamma} e^{-Cq^0}
\gtrsim \int_1^\infty dq^0 e^{-Cq^0} g^{4+a+\gamma}
\gtrsim \int_1^\infty dq^0 e^{-Cq^0} \frac{|p^0| - q^0|4+a+\gamma}{g^{3+\gamma}(|q^0|^2 + 4 + a + \gamma q^0)}
\gtrsim \frac{1}{g^{3+\gamma}(p^0)^2 + \frac{4a+\gamma}{p^0}} \int_1^\infty dq^0 e^{-Cq^0} \frac{|p^0| - q^0|4+a+\gamma}{(q^0)^2 + \frac{4a+\gamma}{q^0}}
\gtrsim \frac{(p^0)^{4+a+\gamma}}{g^{3+\gamma} (p^0)^2 + \frac{4a+\gamma}{p^0}}
\]

where \( q = p' + q' - p \) and the second inequality is by \( \frac{|p' - p|}{q^0} \approx \frac{|q - q'|}{\sqrt{q^0 q'}} \leq \tilde{g}(q', q'^\prime) = g' \), the third inequality is by \( \frac{|p^0 - q^0|}{\sqrt{p^0 q^0}} \leq g' \), and the last equivalence is by \( \int_1^\infty d(q^0)e^{-Cq^0} \frac{|p^0 - q^0|4+a+\gamma}{(q^0)^2} \approx (p^0)^4 + \gamma \) for any \( k \in \mathbb{R} \). This proves the proposition.

Note that Lemma 6 has been proven in this proof above.
8. Global existence

8.1. Local existence. In this section, we use the estimates that we made in the previous sections to show the local existence results for small data. We use the standard iteration method and the uniform energy estimate for the iterated sequence of approximate solutions. The iteration starts at \( f^0(t, x, p) = 0 \). We solve for \( f^{m+1}(t, x, p) \) such that

\[
(\partial_t + \hat{p} \cdot \nabla_x + N)f^{m+1} + Kf^m = \Gamma(f^m, f^{m+1}), \quad f^{m+1}(0, x, p) = f_0(x, p).
\]

Using our estimates, it follows that the linear equation (180) admits smooth solutions with the same regularity in \( H^N \) as a given smooth small initial data and that the solution also has a gain of \( L^2((0, T); I_N^0) \). We will set up some estimates which is necessary to find a local classical solution as \( m \to \infty \).

We first define some notations. We will use the norm \( \| \cdot \|_H \) for \( \| \cdot \|_{H^N} \) for convenience and also use the norm \( \| \cdot \|_{\tilde{I}^{\alpha, \gamma}} \) for \( \| \cdot \|_{I_N^0} \). Define the total norm as

\[
M(f(t)) = \| f(t) \|_H^2 + \int_0^t d\tau \| f(\tau) \|_H^2.
\]

We will also use \( |f|_{\alpha, \gamma} \) for \( \langle Nf, f \rangle \).

Here we state a crucial energy estimate:

**Lemma 9.** The sequence of iterated approximate solutions \( \{f^m\} \) is well defined. There exists a short time \( T^* = T^*(\|f_0\|_H) > 0 \) such that for \( \|f_0\|_H \) sufficiently small, there is a uniform constant \( C_0 > 0 \) such that

\[
\sup_{m \geq 0} \sup_{0 \leq \tau \leq T^*} M(f^m(\tau)) \leq 2C_0\|f_0\|_H^2.
\]

**Proof.** We prove this lemma by induction over \( m \). If \( m = 0 \), the lemma is trivially true. Suppose that the lemma holds for \( m = k \). Let \( f^{k+1} \) be the solution to the linear equation (180) with given \( f^k \). We take the spatial derivative \( \partial^\alpha \) on the linear equation (180) and obtain

\[
(\partial_t + \hat{p} \cdot \nabla_x)\partial^\alpha f^{m+1} + K(\partial^\alpha f^m) = \partial^\alpha \Gamma(f^m, f^{m+1}).
\]

Then, we take a inner product with \( \partial^\alpha f^{m+1} \). The trilinear estimate of Lemma (3) implies that

\[
\frac{1}{2} \frac{d}{dt} \| \partial^\alpha f^{m+1}(t) \|^2_{L^2_x L^2_p} + \| \partial^\alpha f^{m+1} \|^2_{\tilde{I}^0} + (K(\partial^\alpha f^m), \partial^\alpha f^{m+1})
\]

\[
= (\partial^\alpha \Gamma(f^m, f^{m+1}), \partial^\alpha f^{m+1}) \lesssim \| f^m \|_H \| f^{m+1} \|^2_{L^2}.
\]

We integrate over \( t \) we obtain

\[
\frac{1}{2} \| \partial^\alpha f^{m+1}(t) \|^2_{L^2_x L^2_p} + \int_0^t d\tau \| \partial^\alpha f^{m+1}(\tau) \|^2_{\tilde{I}^0} + \int_0^t d\tau (K(\partial^\alpha f^m), \partial^\alpha f^{m+1})
\]

\[
\leq \frac{1}{2} \| \partial^\alpha f_0 \|^2_{L^2_x L^2_p} + C \int_0^t d\tau \| f^m \|_H \| f^{m+1} \|^2_{L^2}.
\]

From the compact estimate (154), for any small \( \varepsilon > 0 \) we have

\[
\left| \int_0^t d\tau (K(\partial^\alpha f^m), \partial^\alpha f^{m+1}) \right| \leq \int_0^t d\tau \left( \frac{1}{2} \| \partial^\alpha f^{m+1}(\tau) \|^2_{L^2_x L^2_p} + C \| \partial^\alpha f^{m+1}(\tau) \|^2_{L^2} \right)
\]

\[
+ \varepsilon \int_0^t d\tau \| \partial^\alpha f^{m+1}(\tau) \|^2_{L^2_x L^2_p} + C \varepsilon \int_0^t d\tau \| \partial^\alpha f^{m}(\tau) \|^2_{L^2}.
\]
We use this estimate for (185) and take a sum over all the derivatives such that $|\alpha| \leq N$ to obtain

\[
M(f^{m+1}(t)) \leq C_0\|f_0\|_H^2 + \int_0^t \,d\tau (C\|f^{m+1}\|_H(\tau) + C\|f^m(\tau)\|_H^2) \\
+ C\epsilon \int_0^t \,d\tau \|f^m\|_H^2(\tau) + C\epsilon \sup_{0 \leq \tau \leq t} M(f^{m+1}(\tau)) \sup_{0 \leq \tau \leq t} M^{1/2}(f^m(\tau)) \\
\leq C_0\|f_0\|_H^2 + C\epsilon \tau + C\epsilon \sup_{0 \leq \tau \leq t} M(f^{m+1}(\tau)) \sup_{0 \leq \tau \leq t} M^{1/2}(f^m(\tau)).
\]

(187)

Then by the induction hypothesis, we obtain that

\[
M(f^{m+1}(t)) \leq C_0\|f_0\|_H^2 + C\epsilon \tau + C\epsilon \sup_{0 \leq \tau \leq t} M(f^{m+1}(\tau)) \sup_{0 \leq \tau \leq t} M^{1/2}(f^m(\tau)) \\
\leq C_0\|f_0\|_H^2 + C\epsilon \tau + C\epsilon \sup_{0 \leq \tau \leq t} M(f^{m+1}(\tau)) + \sup_{0 \leq \tau \leq t} M(f^m(\tau)) \\
+ C\epsilon \sup_{0 \leq \tau \leq t} M(f^m(\tau)) + C\epsilon \sup_{0 \leq \tau \leq t} M(f^{m+1}(\tau)) \sup_{0 \leq \tau \leq t} M^{1/2}(f^m(\tau)).
\]

Then by the induction hypothesis, we obtain that

\[
M(f^{m+1}(t)) \leq C_0\|f_0\|_H^2 + C\epsilon \tau + C\epsilon \sup_{0 \leq \tau \leq t} M(f^{m+1}(\tau)) + 2C_0\|f_0\|_H^2 \\
+ 2C\epsilon C_0\|f_0\|_H^2 + C\epsilon \sup_{0 \leq \tau \leq t} M(f^{m+1}(\tau)) \|f_0\|_H \\
\leq C_0\|f_0\|_H^2 + C\epsilon \tau + C\epsilon \sup_{0 \leq \tau \leq t} M(f^{m+1}(\tau)) + 2C_0\|f_0\|_H^2 \\
+ 2C\epsilon C_0\|f_0\|_H^2 + C\epsilon \sup_{0 \leq \tau \leq t} M(f^{m+1}(\tau)) \|f_0\|_H,
\]

where $C' = \sqrt{2C_0}$. Then we obtain that

\[
(1 - C'\|f_0\|_H - C\epsilon T^*) \sup_{0 \leq \tau \leq t} M(f^{m+1}(t)) \leq (C_0 + 2C\epsilon C_0 T^* + 2C\epsilon C_0)\|f_0\|_H^2.
\]

Then, for sufficiently small $\epsilon$, $T^*$ and $\|f_0\|_H$, we obtain that

\[
\sup_{0 \leq \tau \leq t} M(f^{m+1}(t)) \leq 2C_0\|f_0\|_H^2.
\]

This proves the Lemma by the induction argument. \qed

Now, we prove the local existence theorem with the uniform control on each iteration.

**Theorem 10.** For any sufficiently small $M_0 > 0$, there exists a time $T^* = T^*(M_0) > 0$ and $M_1 > 0$ such that if $\|f_0\|_H^2 \leq M_1$, then there exists a unique solution $f(t, x, p)$ to the linearized relativistic Boltzmann equation (21) on $[0, T^*) \times \mathbb{T}^3 \times \mathbb{R}^3$ such that

\[
\sup_{0 \leq t \leq T^*} M(f(t)) \leq M_0.
\]

(191)

Also, $M(f(t))$ is continuous on $[0, T^*)$. Furthermore, we have the positivity of the solutions; i.e., if $F_0(x, p) = J + \sqrt{T} f_0 \geq 0$, then $F(t, x, p) = J + \sqrt{T} f(t, x, p) \geq 0$.

**Proof.** Existence and Uniqueness. By letting $m \to \infty$ in the previous lemma, we obtain sufficient compactness for the local existence of a strong solution $f(t, x, p)$ to (21). For the uniqueness, suppose there exists another solution $h$ to the (21) with the same initial data satisfying $\sup_{0 \leq t \leq T^*} M(h(t)) \leq \epsilon$. Then, by the equation, we have

\[
\{\partial_t + \bar{p} \cdot \nabla f\}(f - h) + L(f - h) = \Gamma(f - h, f) + \Gamma(h, f - h).
\]

(192)

Then, by Sobolev embedding $H^2(\mathbb{T}^3) \subset L^\infty(\mathbb{T}^3)$ and Theorem 2, we have

\[
||\{\Gamma(f - h, f) + \Gamma(h, f - h)\}, f - h|| \leq ||h||_{L^2_x H^2_t} ||f - h||_{L^{2, \gamma}} \\
+ ||f - h||_{L^2_x H^2_t} ||f||_{H^2_t L^{1, \gamma}} ||f - h||_{L^{1, \gamma}}
\]

\[
= T_1 + T_2.
\]

(193)
For $T_1$, we have
\begin{equation}
\int_0^t d\tau \, T_1(\tau) \leq \sqrt{\varepsilon} \int_0^t d\tau \|f(\tau) - h(\tau)\|_{I_{p,x}}^2
\end{equation}
because we have $\sup_{0 \leq \tau \leq T} M(h(t)) \leq \varepsilon$. For $T_2$, we use Cauchy-Schwarz inequality and obtain
\begin{equation}
\int_0^t d\tau \, T_2(\tau) \leq \sqrt{\varepsilon} \left( \sup_{0 \leq \tau \leq t} \|f(\tau) - h(\tau)\|_{L_{p,x}}^2 \int_0^t d\tau \|f(\tau) - h(\tau)\|_{I_{p,x}}^2 \right)^{1/2}
\end{equation}
\begin{equation}
\lesssim \sqrt{\varepsilon} \left( \sup_{0 \leq \tau \leq t} \|f(\tau) - h(\tau)\|_{L_{p,x}}^2 + \int_0^t d\tau \|f(\tau) - h(\tau)\|_{I_{p,x}}^2 \right)
\end{equation}
because $f$ also satisfies $\sup_{0 \leq \tau \leq T} M(f(t)) \leq \varepsilon$. For the linearized Boltzmann operator $L$ on the left-hand side of (192), we use Lemma 7 to obtain
\begin{equation}
(L(f - h), f - h) \leq c\|f - h\|_{L^2(I_a, \gamma)} - C\|f - h\|_{L^2(I_a, \gamma)}
\end{equation}
for some small $c > 0$. We finally take the inner product of (192) and $(f - h)$ and integrate over $[0, t] \times I^3 \times \mathbb{R}^3$ and use the estimates above to obtain
\begin{equation}
\frac{1}{2} \|f(t) - h(t)\|_{L_{p,x}}^2 + c \int_0^t d\tau \|f(\tau) - h(\tau)\|_{I_{p,x}}^2
\end{equation}
\begin{equation}
\lesssim \sqrt{\varepsilon} \left( \sup_{0 \leq \tau \leq t} \|f(\tau) - h(\tau)\|_{L_{p,x}}^2 + \int_0^t d\tau \|f(\tau) - h(\tau)\|_{I_{p,x}}^2 \right)
\end{equation}
\begin{equation}
+ \int_0^t d\tau \|f(\tau) - h(\tau)\|_{I_{p,x}}^2 (I_a, \gamma) .
\end{equation}
By the Gronwall’s inequality, we obtain that $f = h$ because $f$ and $h$ satisfies the same initial conditions. This proves the uniqueness of the solution.

**Continuity.** Let $[a, b]$ be a time interval. We follow the similar argument as in (185) and (187) with the time interval $[a, b]$ instead of $[0, t]$ and let $f^m = f^{m+1} = f$ and obtain that
\begin{equation}
|M(f(b)) - M(f(a))| = \left| \frac{1}{2} \|f(b)\|_H^2 - \frac{1}{2} \|f(a)\|_H^2 + \int_a^b d\tau \|f(\tau)\|_H^2 \right|
\end{equation}
\begin{equation}
\lesssim \left( \int_a^b d\tau \|f(\tau)\|_H^2 \right) \left( 1 + \sup_{a \leq \tau \leq b} M^{1/2}(f(\tau)) \right).
\end{equation}
As $a \to b$, we obtain that $|M(f(b)) - M(f(a))| \to 0$ because $\|f\|_{L_2}$ is integrable in time. This proves the continuity of $M$.

**Positivity** For the proof of positivity of the solution, we recall the old paper [3] where we see the positivity of strong solutions to the non-relativistic Boltzmann equations without angular cut-off with the initial data $f_0 \in H^M$ for $M \geq 5$ and with moderate singularity $0 \leq \gamma \leq 1$. Similar to this proof, we consider the cut-off approximation $F^\varepsilon$ to the relativistic Boltzmann equation except that the kernel $\sigma$ has been replaced by $\sigma_\varepsilon$ where the angular singularity has been removed and $\sigma_\varepsilon \to \sigma$ as $\varepsilon \to 0$. We obtain that $F^\varepsilon$ is positive. If our initial data is nice enough to be in $H^M$ for $M > 5$, we conclude that $F = J + \sqrt{T}f \geq 0$ using the compactness argument from the uniqueness of the solution. If our initial solution is not regular enough, then we use the density argument that $H^M$ is dense in $H(I^3 \times \mathbb{R}^3)$ and the approximation arguments and the uniqueness to show the positivity. If the angular cutoff is more singular as $1 \leq \gamma < 2$, then the positivity can be obtained by using higher
derivative estimates and following the same compactness argument as in the case with lower singularity.

We notice that if the number of spatial derivatives is large enough, then we have the existence of a classical solution. For the lowest number of spatial derivatives, \( N \geq 2 \), we obtain the existence of a strong solution to the equation.

8.2. Global existence. In this section, we would like to derive the systems of macroscopic equations and balance laws with respect to the coefficients appearing in the expression for the hydrodynamic part \( Pf \) and prove an coercive inequality of the microscopic part \( \{I - P\} f \). With this coercivity estimates for the non-linear local solutions to the relativistic Boltzmann system, we will show that these solutions must be global with the standard continuity argument and by proving energy inequalities. We will also show rapid time decay of the solutions.

For the relativistic Maxwellian solution \( J \), we have normalized so that \( \int_{\mathbb{R}^3} J(p) dp = 1 \). Here we introduce the following notations for the integrals:

\[
\lambda_0 = \int_{\mathbb{R}^3} p^0 J dp, \quad \lambda_{00} = \int_{\mathbb{R}^3} (p^0)^2 J dp, \quad \lambda_1 = \int_{\mathbb{R}^3} (p_1)^2 J dp,
\]

\[
\lambda_{10} = \int_{\mathbb{R}^3} \frac{p_1^2}{p^0} J dp, \quad \lambda_{12} = \int_{\mathbb{R}^3} \frac{p_1^2 p_2^2}{p^0} J dp, \quad \lambda_{11} = \int_{\mathbb{R}^3} \frac{p_1^4}{p^0} J dp,
\]

\[
\lambda_{100} = \int_{\mathbb{R}^3} \frac{p_1^2 p_0^2}{p^0} J dp.
\]

We also mention that the null space of the linearized Boltzmann operator \( L \) is given by

\[
N(L) = \text{span}\{\sqrt{J}, p_1 \sqrt{J}, p_2 \sqrt{J}, p_3 \sqrt{J}, p^0 \sqrt{J}\}.
\]

Then we define the orthogonal projection from \( L^2(\mathbb{R}^3) \) onto \( N(L) \) by \( P \). Then we can write \( Pf \) as a linear combination of the basis as

\[
Pf = \left( A^f(t,x) + \sum_{i=1}^3 B_i^f(t,x)p_i + C^f(t,x)p^0 \right) \sqrt{J}
\]

where the coefficients are given by

\[
A^f = \int_{\mathbb{R}^3} f \sqrt{J} dp - \lambda_0 c^f, \quad B_i^f = \frac{\int_{\mathbb{R}^3} f p_i \sqrt{J} dp}{\lambda_i}, \quad C^f = \frac{\int_{\mathbb{R}^3} f (p^0 \sqrt{J} - \lambda_0 \sqrt{J})}{\lambda_{00} - \lambda_0^2}.
\]

Then we can decompose \( f(t,x,p) \) as

\[
f = Pf + \{I - P\} f.
\]

We start from plugging the expression (202) into (21). Then we obtain

\[
\{ \partial_t + \hat{p} \cdot \nabla_x \} Pf = -\partial_t \{I - P\} f - (\hat{p} \cdot \nabla_x + L) \{I - P\} f + \Gamma(f,f).
\]

Note that we have expressed the hydrodynamic part \( Pf \) in terms of the microscopic part \( \{I - P\} f \) and the higher-order term \( \Gamma \). We define an operator \( l = - (\hat{p} \cdot \nabla_x + L) \) here. Using the expression (201) of \( Pf \) with respect to the basis elements, we obtain that the left-hand
side of the (203) can be written as

\[ \partial_t A \sqrt{J} + \sum_{i=1}^{3} \partial_i (A + C p^0) \frac{p_i}{p^0} \sqrt{J} + \partial_t C p^0 \sqrt{J} + \sum_{i=1}^{3} \partial_i B_i p_i \sqrt{J} + \sum_{i=1}^{3} \partial_i B_i \frac{p_i p_j}{p^0} \sqrt{J} \]

Then, we can rewrite the left-hand side as

\[ \sqrt{J}, \left( \frac{p_i}{p^0} \sqrt{J} \right)_{1 \leq i \leq 3}, p^0 \sqrt{J}, \left( p_i \sqrt{J} \right)_{1 \leq i \leq 3}, \left( \frac{p_i p_j}{p^0} \sqrt{J} \right)_{1 \leq i, j \leq 3}. \]

Then, we can rewrite the left-hand side as

\[ \partial_t A \sqrt{J} + \sum_{i=1}^{3} \partial_i A \frac{p_i}{p^0} \sqrt{J} + \partial_t C p^0 \sqrt{J} + \sum_{i=1}^{3} (\partial_i C + \partial_i (B)) p_i \sqrt{J} + \sum_{i=1}^{3} \sum_{j=1}^{3} \left( (1 - \delta_{ij}) \partial_i B_j + \partial_j B_i \right) \frac{p_i p_j}{p^0} \sqrt{J}. \]

Then we obtain a system of macroscopic equations

\[ \partial_t A = -\partial_t m_a + l_a + G_a, \]
\[ \partial_t A = -\partial_t m_{ia} + l_{ia} + G_{ia}, \]
\[ \partial_t C = -\partial_t m_c + l_c + G_c, \]
\[ \partial_t C + \partial_t B_i = -\partial_t m_{ic} + l_{ic} + G_{ic}, \]
\[ (1 - \delta_{ij}) \partial_i B_j + \partial_j B_i = -\partial_t m_{ij} + l_j + G_{ij}, \]

where the indices are from the index set defined as \( D = \{ a, i a, c, i c, i j | 1 \leq i \leq j \leq 3 \} \) and \( m_{i \mu}, l_{i \mu}, \) and \( G_{\mu} \) for \( \mu \in D \) are the coefficients of \( \{ I - P \} f, l(I - P) f, \) and \( \Gamma(f, f) \) with respect to the basis \( \{ e_k \}_{k=1}^{13} \) respectively.

We also derive a set of equations from the conservation laws. For the perturbation solution \( f \), we multiply the linearized Boltzmann equation by \( \sqrt{J}, p_i \sqrt{J}, p^0 \sqrt{J} \) and integrate over \( \mathbb{R}^3 \) to obtain that

\[ \partial_t \int_{\mathbb{R}^3} f \sqrt{J} dp + \int_{\mathbb{R}^3} \hat{p} \cdot \nabla_x f \sqrt{J} dp = 0 \]

\[ \partial_t \int_{\mathbb{R}^3} f \sqrt{J} p_i dp + \int_{\mathbb{R}^3} \hat{p} \cdot \nabla_x f \sqrt{J} p_i dp = 0 \]

\[ \partial_t \int_{\mathbb{R}^3} f \sqrt{J} p_i dp + \int_{\mathbb{R}^3} \hat{p} \cdot \nabla_x f \sqrt{J} p_i dp = 0. \]

These hold because \( 1, p_i, p^0 \) are collisional invariants and hence

\[ \int_{\mathbb{R}^3} q(f, f) dp = \int_{\mathbb{R}^3} q(f, f)p_i dp = \int_{\mathbb{R}^3} q(f, f)p^0 dp = 0. \]
We will plug the decomposition $f = Pf + \{I - P\}f$ into (208). We first consider the microscopic part. Note that

$$
\int_{\mathbb{R}^3} \hat{p} \cdot \nabla x \{I - P\}f \sqrt{\mathcal{J}} \left( \begin{array}{c} 1 \\
1 \\
p_i \\
p_0 \end{array} \right) dp = \sum_{j=1}^{3} \int_{\mathbb{R}^3} \frac{p_j}{p^0} \partial_j \{I - P\}f \sqrt{\mathcal{J}} \left( \begin{array}{c} 1 \\
1 \\
p_i \\
p_0 \end{array} \right) dp
$$

(209)

Also, we have that

$$
\partial_t \int_{\mathbb{R}^3} \{I - P\}f \sqrt{\mathcal{J}} \left( \begin{array}{c} 1 \\
1 \\
p_i \\
p_0 \end{array} \right) = \partial_t \{\{I - P\}f, \sqrt{\mathcal{J}} \left( \begin{array}{c} 1 \\
1 \\
p_i \\
p_0 \end{array} \right) \} = 0.
$$

On the other hand, the hydrodynamic part $Pf = (A + B \cdot p + C p^0) \sqrt{\mathcal{J}}$ satisfies

(211)

Also, we have that $L(f) = L\{I - P\}f$. Together with (208), (209), (210), and (211), we finally obtain the local conservation laws satisfied by $(A, B, C)$:

$$
\partial_t A + \lambda_0 \partial_t C + \lambda_{10} \nabla_x \cdot B = -\nabla_x \cdot \{\{I - P\}f, \sqrt{\mathcal{J}} \frac{p}{p^0} \},
$$

(212)

$$
\lambda_1 \partial_t B + \lambda_{10} \nabla_x A + \lambda_1 \nabla_x C = -\nabla_x \cdot \{\{I - P\}f, \sqrt{\mathcal{J}} \frac{p \otimes p}{p^0} \},
$$

$$
\lambda_0 \partial_t A + \lambda_{00} \partial_t C + \lambda_{10} \nabla_x \cdot B = 0.
$$

Comparing the first and the third conservation laws, we obtain

$$
\partial_t A \left( 1 - \frac{\lambda_0^2}{\lambda_{00}} \right) + \nabla_x \cdot B \left( \frac{\lambda_0}{\lambda_{00}} - \frac{\lambda_{00} \lambda_1}{\lambda_{00}} \right) = -\nabla_x \cdot \{\{I - P\}f, \sqrt{\mathcal{J}} \frac{p}{p^0} \},
$$

(213)

$$
\lambda_1 \partial_t B + \lambda_{10} \nabla_x A + \lambda_1 \nabla_x C = -\nabla_x \cdot \{\{I - P\}f, \sqrt{\mathcal{J}} \frac{p \otimes p}{p^0} \},
$$

$$
\lambda_0 \partial_t A + \lambda_{00} \partial_t C + \lambda_{10} \nabla_x \cdot B = -\nabla_x \cdot \{\{I - P\}f, \sqrt{\mathcal{J}} \frac{p}{p^0} \}.
$$

We also mention that we have the following lemma on the coefficients $A, B, C$ by the conservation of mass, momentum, and energy:

**Lemma 11.** Let $f(t, x, p)$ be the local solution to the linearized relativistic Boltzmann equation (21) which is shown to exist in Theorem 10 which satisfies the mass, momentum, and
energy conservation laws (20). Then we have

\[ (214) \quad \int_{\mathbb{T}^3} A(t, x) dx = \int_{\mathbb{T}^3} B_i(t, x) dx = \int_{\mathbb{T}^3} C(t, x) dx = 0, \]

where \( i \in 1, 2, 3 \).

We also list two lemmas that helps us to control the coefficients in the linear microscopic term \( l \) and the non-linear higher-order term \( \Gamma \).

**Lemma 12.** For any coefficient \( l_\mu \) for the microscopic term \( l \), we have

\[ (215) \quad \sum_{\mu \in D} ||l_\mu||_{H^N_x} \lesssim \sum_{|\alpha| \leq N} ||\{ I - P \} \partial^\alpha f \|^2_{L^2_{x,p} (\mathbb{T}^3 \times \mathbb{R}^3)}. \]

**Proof.** In order to estimate the size for \( H^N_x \) norm, we take

\[ (216) \quad \langle \partial^\alpha l(\{ I - P \} f), e_k \rangle = -\langle \hat{p} \cdot \nabla_x (\{ I - P \} \partial^\alpha f), e_k \rangle - \langle L(\{ I - P \} \partial^\alpha f), e_k \rangle. \]

For any \(|\alpha| \leq N - 1 \), the \( L^2 \)-norm of the first part of the right-hand side is

\[ (217) \quad ||\langle \hat{p} \cdot \nabla_x (\{ I - P \} \partial^\alpha f), e_k \rangle||_{L^2_x}^2 \lesssim \int_{\mathbb{T}^3 \times \mathbb{R}^3} dx dp |e_k| ||\{ I - P \} \nabla_x \partial^\alpha f|^2 \]

\[ \lesssim ||\{ I - P \} \nabla_x \partial^\alpha f||^2_{L^2_{x,p} (\mathbb{T}^3 \times \mathbb{R}^3)}. \]

Similarly, we have

\[ (218) \quad ||\langle L(\{ I - P \) \partial^\alpha f), e_k \rangle||_{L^2_x}^2 \lesssim \||\{ I - P \} \partial^\alpha f||^2_{L^2_{x,p} (\mathbb{T}^3 \times \mathbb{R}^3)}. \]

This completes the proof. \( \square \)

**Lemma 13.** Let \( ||f||_{H^N_x}^2 \leq M \) for some \( M > 0 \). Then, we have

\[ (219) \quad \sum_{\mu \in D} ||G_\mu||_{H^N_x} \lesssim \sqrt{M} \sum_{|\alpha| \leq N} ||\partial^\alpha f||^2_{L^2_{x,p} (\mathbb{T}^3 \times \mathbb{R}^3)}. \]

**Proof.** In order to estimate the size for \( H^N_x \) norm, we consider \( \langle \Gamma(f, f), e_k \rangle \). By (157), for any \( m \geq 0 \),

\[ (220) \quad ||\langle \Gamma(f, f), e_k \rangle||_{H^N_x} \lesssim \sum_{|\alpha| \leq N - 1} \sum_{|\alpha_1| \leq \alpha} \left\| \partial^{\alpha - \alpha_1} f \right\|_{L^2_{x,p} (\mathbb{T}^3 \times \mathbb{R}^3)} \left\| \partial^{\alpha_1} f \right\|_{L^2_{x,p} (\mathbb{T}^3 \times \mathbb{R}^3)} \]

\[ \lesssim \| f \|^2_{L^2_{x,p} (\mathbb{T}^3 \times \mathbb{R}^3)} \sum_{|\alpha| \leq N} ||\partial^\alpha f||^2_{L^2_{x,p} (\mathbb{T}^3 \times \mathbb{R}^3)}. \]

This completes the proof. \( \square \)

These two lemmas above, the macroscopic equations, and the local conservation laws will together prove the following theorem on the coercivity estimate for the microscopic term \( \{ I - P \} f \) which is crucial for the energy inequality which will imply the global existence of the solution with the continuity argument.
Theorem 14. Given the initial condition \( f_0 \in H \) which satisfies the mass, momentum, and energy conservation laws (20) and the assumptions in Theorem 10, we can consider the local solution \( f(t,x,p) \) to the linearized relativistic Boltzmann equation (21). Then, there is a constant \( M > 0 \) such that if

\[
||f(t)||^2_H \leq M_0,
\]

then there are universal constants \( \delta > 0 \) and \( C > 0 \) such that

\[
\sum_{|\alpha| \leq N} ||\{I - P\} \partial^\alpha f||^2_{L^2_{x,\gamma}}(t) \geq \delta \sum_{|\alpha| \leq N} ||P \partial^\alpha f||^2_{L^2_{x,\gamma}}(t) - C \frac{dI(t)}{dt},
\]

where \( I(t) \) is the interaction potential defined as

\[
I(t) = \sum_{|\alpha| \leq N-1} \{I^a_\alpha(t) + I^b_\alpha(t) + I^c_\alpha(t)\}
\]

and each of the sub-potentials \( I^a_\alpha(t) \), \( I^b_\alpha(t) \), and \( I^c_\alpha(t) \) is defined as

\[
I^a_\alpha(t) = \sum_{i=1}^{3} \int_{T^3} \partial_i \partial^\alpha m_{ia} \partial^\alpha A(t,x) dx,
\]

\[
I^b_\alpha(t) = -\sum_{i=1}^{3} \sum_{j \neq i} \int_{T^3} \partial_j \partial^\alpha m_{ij} \partial^\alpha B_i dx,
\]

\[
I^c_\alpha(t) = \int_{T^3} (\nabla \cdot \partial^\alpha B) \partial^\alpha C(t,x) dx + \sum_{i=1}^{3} \int_{T^3} \partial_i \partial^\alpha m_{ia} \partial^\alpha C(t,x) dx.
\]

Proof. Since \( Pf = A + B \cdot p + Cp^0 \), we have that

\[
||P \partial^\alpha f(t)||^2_{L^2_{x,\gamma}} \lesssim ||\partial^\alpha A(t)||^2_{L^2_{\gamma-x}} + ||\partial^\alpha B(t)||^2_{L^2_{\gamma-x}} + ||\partial^\alpha C(t)||^2_{L^2_{\gamma-x}}.
\]

Thus, it suffices to prove the following estimate:

\[
||\partial^\alpha A(t)||^2_{H^\gamma_{\alpha-x}} + ||\partial^\alpha B(t)||^2_{H^\gamma_{\alpha-x}} + ||\partial^\alpha C(t)||^2_{H^\gamma_{\alpha-x}}
\]

\[
\lesssim \sum_{|\alpha| \leq N} ||\{I - P\} \partial^\alpha f(t)||^2_{L^2_{\gamma-x}} + C \sum_{|\alpha| \leq N} ||\partial^\alpha f(t)||^2_{L^2_{\gamma-x}} + \frac{dI(t)}{dt}.
\]

Note that the term \( M \sum_{|\alpha| \leq N} ||\partial^\alpha f(t)||^2_{L^2_{\gamma-x}} \) can be ignored because we have

\[
\sum_{|\alpha| \leq N} ||\partial^\alpha f(t)||^2_{L^2_{\gamma-x}} \lesssim \sum_{|\alpha| \leq N} ||P \partial^\alpha f(t)||^2_{L^2_{\gamma-x}} + \sum_{|\alpha| \leq N} ||\{I - P\} \partial^\alpha f(t)||^2_{L^2_{\gamma-x}}
\]

\[
\lesssim ||\partial^\alpha A(t)||^2_{H^\gamma_x} + ||\partial^\alpha B(t)||^2_{H^\gamma_x} + ||\partial^\alpha C(t)||^2_{H^\gamma_x} + \sum_{|\alpha| \leq N} ||\{I - P\} \partial^\alpha f(t)||^2_{L^2_{\gamma-x}}.
\]

Therefore, with sufficiently small \( M > 0 \), (226) will imply Theorem 14.

In order to prove (226), we will estimate each of the \( \partial^\alpha \) derivatives of \( A, B, C \) for \( 0 < |\alpha| \leq N \) separately. Later, we will use Poincaré inequality to estimate the \( L^2 \)-norm of \( A, B, C \) to finish the proof.

For the estimate for \( A \), we use the second equation in the system of macroscopic equations 207 which tells

\[
\partial_t A = -\partial_t m_{ia} + l_{ia} + G_{ia}.
\]

We take \( \partial_t \partial^\alpha \) onto this equation for \( |\alpha| \leq N - 1 \)
and sum over $i$ and obtain that

$$
(228) \quad - \Delta \partial^\alpha A = \sum_{i=1}^{3} (\partial_t \partial_i \partial^\alpha m_{ia} - \partial_i \partial^\alpha (l_{ia} + G_{ia})).
$$

We now multiply $\partial^\alpha A$ and integrate over $T^3$ to obtain

$$
(229) \quad ||\nabla \partial^\alpha A||_{L^2_T}^2 \leq ||\partial^\alpha (l_{ia} + G_{ia})||_{L^2_T}||\nabla \partial^\alpha A||_{L^2_T} + \frac{d}{dt} \sum_{i=1}^{3} \int_{T^3} \partial_i \partial^\alpha m_{ia} \partial^\alpha A(t, x) dx - \sum_{i=1}^{3} \int_{T^3} \partial_i \partial^\alpha m_{ia} \partial_i \partial^\alpha A(t, x) dx.
$$

We define the interaction functional

$$
(230) \quad I^\alpha_a(t) = \sum_{i=1}^{3} \int_{T^3} \partial_i \partial^\alpha m_{ia} \partial_i \partial^\alpha A(t, x) dx.
$$

For the last term, we use the first equation of the local conservation laws (213) to obtain that

$$
(231) \quad \int_{T^3} \sum_{i=1}^{3} |\partial_t \partial^\alpha m_{ia} \partial_i \partial^\alpha A(t, x)| dx \leq \zeta ||\nabla \cdot \partial^\alpha B||_{L^2_T}^2 + C_\zeta \sum_{|\alpha| \leq N} ||\{I - P\} \nabla \partial^\alpha f||_{L^2_T}^2,
$$

for any $\zeta > 0$. Together with Lemma 12 and Lemma 13, we obtain that

$$
(232) \quad ||\nabla \partial^\alpha A||_{L^2_T}^2 - \zeta ||\nabla \cdot \partial^\alpha B||_{L^2_T}^2 \lesssim C_\zeta \sum_{|\alpha| \leq N} ||\{I - P\} \nabla \partial^\alpha f||_{L^2_T}^2 + \frac{dI^\alpha_a}{dt} + M \sum_{|\alpha| \leq N} ||\partial^\alpha f||_{L^2_T}^2.
$$

For the estimate for $C$, we use the fourth equation in the system of macroscopic equations 207 which tells $\partial_t C + \partial_i B_i = -\partial_t m_{ic} + l_{ic} + G_{ic}$. We take $\partial_t \partial^\alpha$ onto this equation for $|\alpha| \leq N - 1$ and sum over $i$ and obtain that

$$
(233) \quad - \Delta \partial^\alpha C = \frac{d}{dt} (\nabla \cdot \partial^\alpha B) + \sum_{i=1}^{3} (\partial_t \partial_i \partial^\alpha m_{ic} - \partial_i \partial^\alpha (l_{ic} + G_{ic})).
$$

We now multiply $\partial^\alpha C$ and integrate over $T^3$ to obtain

$$
(234) \quad ||\nabla \partial^\alpha C||_{L^2_T}^2 \leq \frac{d}{dt} \int_{T^3} (\nabla \cdot \partial^\alpha B) \partial^\alpha C(t, x) dx - \int_{T^3} (\nabla \cdot \partial^\alpha B) \partial_i \partial^\alpha C(t, x) dx + \sum_{i=1}^{3} \int_{T^3} \partial_i \partial^\alpha m_{ic} \partial_i \partial^\alpha C(t, x) dx + \frac{d}{dt} \sum_{i=1}^{3} \int_{T^3} \partial_i \partial^\alpha m_{ia} \partial_i \partial^\alpha C(t, x) dx - \sum_{i=1}^{3} \int_{T^3} \partial_i \partial^\alpha m_{ia} \partial_i \partial^\alpha C(t, x) dx.
$$

We define the interaction functional

$$
(235) \quad I^\alpha_c(t) = \int_{T^3} (\nabla \cdot \partial^\alpha B) \partial^\alpha C(t, x) dx + \sum_{i=1}^{3} \int_{T^3} \partial_i \partial^\alpha m_{ia} \partial^\alpha C(t, x) dx.
$$
We also use the third equation of the local conservation laws (213) to obtain that

\[
\int_{T^3} \sum_{i=1}^{3} |\partial_t \partial^\alpha m_{ij} \partial_i \partial^\alpha C(t, x)| dx \leq \zeta \|\nabla \cdot \partial^\alpha B\|^2_{L^2_x} + C_\zeta \|\{I - P\} \nabla \partial^\alpha f\|^2_{L^2_{x,t}^{\frac{2+\gamma}{2}}},
\]

for any \( \zeta > 0 \). Together with Lemma 12 and Lemma 13, we obtain that

\[
||\nabla \partial^\alpha C||^2_{L^2_x} - \zeta ||\nabla \cdot \partial^\alpha B\||^2_{L^2_x} \lesssim C_\zeta \sum_{|\alpha| \leq N} ||\{I - P\} \partial^\alpha f\||^2_{L^2_{x,t}^{\frac{2+\gamma}{2}}} + \frac{dI^\alpha}{dt} + M \sum_{|\alpha| \leq N} ||\partial^\alpha f||^2_{L^2_{x,t}^{\frac{2+\gamma}{2}}}.
\]

For the estimate for \( B \), we use the last equation in the system of macroscopic equations (207) which tells \((1 - \delta_{ij}) \partial_i B_j + \partial_j B_i = -\partial_t m_{ij} + l_{ij} + G_{ij} \). Note that when \( i = j \), we have

\[
\partial_t B_i = -\partial_t m_{ii} + l_{ii} + G_{ii}.
\]

Also, if \( i \neq j \), we have

\[
\partial_i B_j + \partial_j B_i = -\partial_t m_{ij} + l_{ij} + G_{ij}.
\]

We take \( \partial_j \partial^\alpha \) on both equations above for \(|\alpha| \leq N - 1\) and sum on \( j \) to obtain

\[
\Delta \partial^\alpha B_i = -\partial_i \partial_j \partial^\alpha B_i + 2\partial_j \partial^\alpha l_{ii} + 2\partial_i \partial^\alpha G_{ii} + \sum_{j \neq i} (-\partial_i \partial^\alpha l_{jj} - \partial_j \partial^\alpha G_{jj} + \partial_i \partial^\alpha l_{ij} + \partial_j \partial^\alpha G_{ij} - \partial_t \partial^\alpha m_{ij}).
\]

We now multiply \( \partial^\alpha B_i \) and integrate over \( T^3 \) to obtain

\[
||\nabla \partial^\alpha B_i||^2_{L^2_x} \leq -\frac{d}{dt} \sum_{j \neq i} \int_{T^3} \partial_j \partial^\alpha m_{ij} \partial^\alpha B_j dx + \sum_{j \neq i} \int_{T^3} \partial_j \partial^\alpha m_{ij} \partial_t \partial^\alpha B_i dx
\]

\[
+ \sum_{\mu \in D} ||\partial^\alpha (l_\mu + G_\mu)||^2_{L^2_x}.
\]

We define the interaction functional

\[
I^\alpha_0(t) = -\sum_{i=1}^{3} \sum_{j \neq i} \int_{T^3} \partial_j \partial^\alpha m_{ij} \partial^\alpha B_i dx.
\]

We also use the second equation of the local conservation laws (213) to obtain that

\[
\sum_{i=1}^{3} \sum_{j \neq i} \int_{T^3} |\partial_j \partial^\alpha m_{ij} \partial_t \partial^\alpha B_i(t, x)| dx
\]

\[
\leq \zeta (||\nabla \cdot \partial^\alpha A||^2_{L^2_x} + ||\nabla \cdot \partial^\alpha C||^2_{L^2_x}) + C_\zeta ||\{I - P\} \nabla \partial^\alpha f||^2_{L^2_{x,t}^{\frac{2+\gamma}{2}}},
\]

for any \( \zeta > 0 \). Together with Lemma 12 and Lemma 13, we obtain that

\[
||\nabla \partial^\alpha B||^2_{L^2_x} - \zeta (||\nabla \cdot \partial^\alpha A||^2_{L^2_x} + ||\nabla \cdot \partial^\alpha C||^2_{L^2_x}) \lesssim C_\zeta \sum_{|\alpha| \leq N} ||\{I - P\} \partial^\alpha f||^2_{L^2_{x,t}^{\frac{2+\gamma}{2}}} + \frac{dI^\alpha_0}{dt} + M \sum_{|\alpha| \leq N} ||\partial^\alpha f||^2_{L^2_{x,t}^{\frac{2+\gamma}{2}}}.
\]
Choose sufficiently small \( \zeta > 0 \). Then, (232), (237), and (244) implies that

\[
||\nabla A||^2_{H^{N-1}} + ||\nabla B||^2_{H^{N-1}} + ||\nabla C||^2_{H^{N-1}} \lesssim \sum_{|\alpha| \leq N} \|\{I - P\}\partial^\alpha f\|^2_{L^2} + \frac{dI}{dt} + M \sum_{|\alpha| \leq N} ||\partial^\alpha f||^2_{L^{2+2}}.
\]

(245)

On the other hand, with the Poincaré inequality and Lemma 11, we obtain that

\[
||A||^2 \lesssim (||\nabla A|| + \left| \int_{\mathbb{T}^3} A(t, x) dx \right|)^2 = ||\nabla A||^2 \lesssim \sum_{|\alpha| \leq N} ||\partial^\alpha f||^2_{L^{2+2}}.
\]

(246)

This same estimate holds for \( b \) and \( c \). Therefore, the inequality (226) holds and this finishes the proof for the theorem.

We now use this coercive estimate to prove that the local solutions from the theorem 10 should be global-in-time solutions by standard continuity argument. We will also prove that the solutions have rapid exponential time decay.

Before we go into the proof for the global existence, we would like to mention a coercive lower bound for the linearized Boltzmann collision operator \( L \) which also gives the positivity of the operator:

**Theorem 15.** There is a constant \( \delta > 0 \) such that

\[
\langle Lf, f \rangle \geq \delta \|\{I - P\}f\|^2_{L^{2+\gamma}}.
\]

(247)

**Proof.** By following the proof for Theorem 1.1 of [42] with relativistic collision kernel and using that \( g \geq \frac{|p - q|}{\sqrt{p_0 q_0}} \), we can obtain that

\[
\langle Lf, f \rangle \geq \delta_1 \|f\|^2_{L^{2+\gamma}}
\]

(248)

where \( \delta_1 > 0 \) is a constant. Also, by Lemma 7, we have

\[
\langle Lf, f \rangle \geq |f|^2_{L^{2+\gamma}} - C \|f\|^2_{L_1^2(B_C)}
\]

(249)

for some \( C > 0 \). Then, for any \( \delta_2 \in (0, 1) \),

\[
\langle Lf, f \rangle = \delta_2 \langle Lf, f \rangle + (1 - \delta_2) \langle Lf, f \rangle \geq \delta_2 |f|_{L^{2+\gamma}} - C \delta_2 \|f\|^2_{L_1^2(B_C)} + (1 - \delta_2) \delta_1 |f|_{L^{2+\gamma}}.
\]

(250)

Since \( C > 0 \) is finite, we have \( |f|^2_{L^{2+\gamma}} \gtrsim |f|_{L_1^2(B_C)}^2 \). By choosing \( \delta_2 > 0 \) sufficiently small and supposing \( f = \{I - P\}f \), we obtain the theorem.

Now, we define the dissipation rate \( D \) as

\[
D = \sum_{|\alpha| \leq N} ||\partial^\alpha f(t)||^2_{L^{2+\gamma}}.
\]

(251)

We will use the energy functional \( \mathcal{E}(t) \) to be a high-order norm which satisfies

\[
\mathcal{E}(t) \approx \sum_{|\alpha| \leq N} ||\partial^\alpha f(t)||^2_{L^2(T^3 \times \mathbb{R}^3)}.
\]

(252)

This functional will be precisely defined during the proof. Then, we would like to set up the following energy inequality:

\[
\frac{d}{dt} \mathcal{E}(t) + D(t) \leq C \sqrt{\mathcal{E}(t)D(t)}.
\]

(253)
We will prove this energy inequality and use this to show the global existence.

**Proof.** (Proof for Theorem 1) We denote \( D \overset{\mathrm{def}}{=} D_0 \) and \( \mathcal{E} \overset{\mathrm{def}}{=} \mathcal{E}_0 \). By the definitions on interaction functionals, there is a sufficiently large constant \( C'' > 0 \) for any \( C' > 0 \) such that
\[
||f(t)||^2_{L^2 H^N_x} \leq (C'' + 1)||f(t)||^2_{L^2 H^N_x} - C'I(t) \lesssim ||f(t)||^2_{L^2 H^N_x}.
\]

Note that \( C'' \) doesn’t depend on \( f(t, x, p) \) but only on \( C' \) and \( I \). Here we define the energy functional \( \mathcal{E}(t) \) as
\[
\mathcal{E}(t) = (C'' + 1)||f(t)||^2_{L^2 H^N_x} - C'I(t).
\]

Then, the above inequalities show that the definition of \( \mathcal{E} \) satisfies (252).

Recall the local existence theorem 10 and Theorem 14 and choose \( M_0 \leq 1 \) so that both theorems hold. We choose \( M_1 \leq \frac{M_0}{2} \) and consider initial data \( \mathcal{E}(0) \) so that
\[
\mathcal{E}(0) \leq M_1 < M_0.
\]

From the local existence theorem, we define \( T > 0 \) so that
\[
T = \sup\{t \geq 0|\mathcal{E}(t) \leq 2M_1\}.
\]

By taking the spatial derivative \( \partial \alpha \) onto the linearized relativistic Boltzmann equation (21), integrating over \((x, p)\), and summing over \( \alpha \), we obtain
\[
\frac{1}{2} \frac{d}{dt} ||f(t)||^2_{L^2 H^N_x} + \sum_{|\alpha| \leq N} (L\partial \alpha f, \partial \alpha f) = \sum_{|\alpha| \leq N} (\partial \alpha \Gamma(f, f), \partial \alpha f).
\]

By the estimates from Lemma 3, we have
\[
\sum_{|\alpha| \leq N} (\partial \alpha \Gamma(f, f), \partial \alpha f) \lesssim \sqrt{\mathcal{E}} D.
\]

Since our choice of \( M_1 \) satisfies \( \mathcal{E}(t) \leq 2M_1 \leq M_0 \), we see that the assumption for Theorem 14 is satisfied. Then, Theorem 14 and Theorem 15 tells us that
\[
\sum_{|\alpha| \leq N} (L\partial \alpha f, \partial \alpha f) \geq \delta ||\{I - P\}f||^2_{L^2_{t, \alpha, \gamma}}
\]
\[
\geq \frac{\delta}{2} ||\{I - P\}f||^2_{L^2_{t, \alpha, \gamma}} + \frac{\delta \delta'}{2} \sum_{|\alpha| \leq N} ||P\partial \alpha f||^2_{L^2_{t, \alpha, \gamma}}(t) - \frac{\delta C}{2} \frac{dI(t)}{dt}.
\]

Let \( \delta'' = \min\{\frac{\delta}{2}, \frac{\delta \delta'}{2}\} \) and let \( C' = \delta C \). Then, we have
\[
\frac{1}{2} \frac{d}{dt} ||f(t)||^2_{L^2 H^N_x} - C'I(t) + \delta'' D \lesssim \sqrt{\mathcal{E}} D.
\]

We multiply (258) by \( \frac{\mathcal{E}}{2} \) and add this onto this inequality above and use the positivity of \( L \) from Theorem 15 to conclude that
\[
\frac{d\mathcal{E}(t)}{dt} + \delta'' D(t) \leq C\sqrt{\mathcal{E}(t)}D(t),
\]
for some \( C > 0 \). Suppose \( M_1 = \min\{\frac{\delta''}{4C}, \frac{M_0}{2}\} \). Then, we have
\[
\frac{d\mathcal{E}(t)}{dt} + \delta'' D(t) \leq C\sqrt{\mathcal{E}(t)}D(t) \leq C\sqrt{2M_1 D(t)} \leq \frac{\delta''}{2} D(t).
\]
Now, we integrate over $0 \leq t \leq \tau < T$ and obtain
\begin{equation}
\mathcal{E}(\tau) + \frac{\delta''}{2} \int_0^\tau \mathcal{D}(t) dt \leq \mathcal{E}(0) \leq M_1 < 2M_1.
\end{equation}
Since $\mathcal{E}(\tau)$ is continuous in $\tau$, $\mathcal{E}(\tau) \leq M_1$ if $T < \infty$. This contradicts the definition of $T$ and hence $T = \infty$. This proves the global existence. Also, notice that $\mathcal{E}(t) \lesssim \mathcal{D}(t)$. This and the equation (263) show the exponential time decay. \hfill \square

9. Appendix

9.1. Relativistic collision geometry. Consider the center-of-momentum expression for the collision operator. Under the expression, note that

\begin{equation}
p' - q' = g \omega + g(\gamma - 1)(p + q) \frac{(p + q) \cdot \omega}{|p + q|^2}
\end{equation}

Thus, $\omega$ can be represented as

\begin{equation}
\omega = \frac{1}{g} (p' - q' - \sqrt{s}(p^0 - q^0)(\gamma - 1)(p + q) \frac{1}{|p + q|^2})
\end{equation}

On the other hand,

\begin{equation}
\cos \theta := \frac{(p' - q' \cdot (p' - q'))}{g^2}
\end{equation}

\begin{align*}
\frac{1}{g^2} \left( - (p^0 - q^0)(p^0 - q^0) + (p - q) \cdot (p' - q') \right) \\
= \frac{1}{g^2} \left( - (p^0 - q^0)(p^0 + q^0 + \sqrt{s}) \omega \cdot (p + q) + (p - q) \cdot (g \omega + g(\gamma - 1)(p + q) \frac{(p + q) \cdot \omega}{|p + q|^2}) \right) \\
= \frac{1}{g^2} \left( - (p^0 - q^0)(p^0 + q^0 + \sqrt{s}) \omega \cdot (p + q) + \sqrt{s}(p - q) \cdot (p^0 + q^0 + \sqrt{s}) \cdot (p - q) \cdot \omega \cdot (p + q)(p^0 - q^0) \right) \\
= \frac{1}{g^2} \left( (\sqrt{s} + 2q^0) \frac{p - (\sqrt{s} + 2p^0)q}{g(p^0 + q^0 + \sqrt{s})} \cdot \omega \right) \\
= k \cdot \omega.
\end{align*}
Note that \(|k| = 1\). This expression on \(\cos \theta\) gives us the intuition on the relationship between \(\cos \theta\) expressed as the Lorentzian inner product of 4-vectors and that expressed as the usual Euclidean inner product of 3-vectors. Thus, we can see that even in the relativistic collisional kinetics, the geometry can be expressed by using the usual 3-vectors and the usual Euclidean inner product with the above translation.

9.2. **Dual representation.** In this section, we develop the Carleman representation of the relativistic gain and loss terms which arise many times throughout this paper represented as an integral over \(E^p_{q-p'}\) where the set is defined as:

\[
E^p_{q-p'} \overset{\text{def}}{=} \{ P \in \mathbb{R}^3 | (p^\mu - q^\mu)(q_\mu - p'_\mu) = 0 \}.
\]

We first derive the Carleman dual representation of the relativistic gain term. The relativistic gain term part of the inner product \(\langle \Gamma(f, h), \eta \rangle\) is written as

\[
\langle \Gamma^+(f, h), \eta \rangle = c \int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dq}{q^0} \delta(p^\mu q_\mu + 1) \delta(q'^\mu q'_\mu + 1) u(p^0) u(q^0)
\]

where \(u(x) = 1\) if \(x \geq 1\) and \(= 0\) otherwise. Then, we obtain that

\[
\langle \Gamma^+(f, h), \eta \rangle = \frac{c}{4 \pi} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \eta(p') \int_{\mathbb{R}^3} \frac{dq}{q^0} f(q) \int_{\mathbb{R}^3} dp^\mu h(p) \int_{\mathbb{R}^3} dq'^\mu e^{\frac{\eta g}{2 g^2}} u(p^0) u(q^0)
\]

\[
\times \delta(p'^\mu p_\mu + 1) \delta(q'^\mu q'_\mu + 1) s \sigma(g, \omega) \delta^4(p'^\mu + q'^\mu - p'^\mu - q'^\mu).
\]

We reduce the integral \(\int_{\mathbb{R}^3} dq'^\mu\) by evaluating the last delta function and obtain

\[
\langle \Gamma^+(f, h), \eta \rangle = \frac{c}{4 \pi} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \eta(p') \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dp^\mu h(p) e^{-\frac{\eta g}{2 g^2}} u(p^0)
\]

\[
\times u(q^0 - p^0 + p^0) \sigma(p^\mu p_\mu + 1) \delta((q^\mu - p'^\mu + p^\mu)(q_\mu - p'_\mu + p_\mu) + 1) s \sigma(g, \omega)
\]

The terms in the second delta function can be rewritten as

\[
(q^\mu - p'^\mu + p^\mu)(q_\mu - p'_\mu + p_\mu) + 1 = (q^\mu - p'^\mu)(q_\mu - p'_\mu) + 2(q^\mu - p'^\mu)p_\mu = \tilde{g}^2 + 2p^\mu(q_\mu - p'_\mu).
\]

Therefore, by evaluating the first delta function, we finally obtain the dual representation of the gain term as

\[
\langle \Gamma^+(f, h), \eta \rangle = \frac{c}{4 \pi} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \eta(p') \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dp^\mu h(p) e^{-\frac{\eta g}{2 g^2}}
\]

\[
\times u(q^0 - p^0 + p^0) \delta(\frac{\tilde{g}^2 + 2p^\mu(q_\mu - p'_\mu)}{2 \tilde{g}}).
\]

where the measure \(d \pi_p\) is defined as

\[
d \pi_p = u(q^0 - p^0 + p^0) \delta(\frac{\tilde{g}^2 + 2p^\mu(q_\mu - p'_\mu)}{2 \tilde{g}}).
\]
We also want to compute the dual representation for the loss term. We start from the following.

\[
\langle \Gamma(f, h), \eta \rangle = \int_{R^3} dp \int_{R^3} dq \int_{S^2} d\omega \ v_\phi f(q) h(p) \sigma(g, \omega)(\sqrt{J(q')\eta(q') - \sqrt{J(q)\eta(p)}} - \sqrt{J(q)\eta(p)})
\]

(276)

Initially, suppose that \( \int_{S^2} d\omega |\sigma_0(\theta)| < \infty \) and that \( \int_{S^2} d\omega \sigma_0(\theta) = 0 \). Then,

\[
\langle \Gamma(f, h), \eta \rangle = \int_{R^3} dp \int_{R^3} dq \int_{S^2} d\omega \ v_\phi f(q) h(p) \Phi(g) \sigma_0(\theta)(\sqrt{J(q')\eta(q') - \sqrt{J(q)\eta(p)}}).
\]

(277)

This is the relativistic Boltzmann gain term and its dual representation is shown above to be the following:

\[
\frac{c}{2} \int_{R^3} dp' \int_{R^3} \frac{dq'}{q'^2} \int_{E_{q-p'}^p} d\pi_p \frac{s\sigma(g, \theta)}{g} f(q) h(p) \sqrt{J(q')\eta(p')}. \]

(278)

On the geometry \( E_{q-p'}^p \), \( (p'^\mu - p^\mu)(q_\mu - p'_\mu) = 0 \) Thus we have \( g^2 + \tilde{g}^2 = g^2 \).

Note that

\[
(p'^\mu - q'^\mu)(p_\mu - q_\mu) = (2p'^\mu - p^\mu - q'^\mu)(p_\mu - q_\mu)
\]

(279)

\[
= (p'^\mu - p^\mu + p'^\mu - q'^\mu)(p_\mu - p'_\mu + p'_\mu - q_\mu)
\]

\[
= (p'^\mu - p^\mu)(p_\mu - p'_\mu) + (p'^\mu - q'^\mu)(p'_\mu - q_\mu)
\]

\[
= -g^2 + \tilde{g}^2.
\]

Since \( \cos \theta \overset{\text{def}}{=} \frac{(p'^\mu - q'^\mu)(p_\mu - q_\mu)}{g^2} \), we have that

\[
\cos \theta \overset{\text{def}}{=} -\frac{\tilde{g}^2 + g^2}{g^2 + \tilde{g}^2}.
\]

(280)

Define \( t = \frac{-\tilde{g}^2 + g^2}{g^2 + \tilde{g}^2} \). Then, we obtain \( dt = dq \frac{-4g\tilde{g}}{g^2 + \tilde{g}^2} \).

Since \( \int_{-1}^{1} dt \sigma_0(t) = 0 \), we have

\[
\int_{0}^{\infty} \frac{4g\tilde{g}^2}{(g^2 + \tilde{g}^2)^2} \sigma_0 \left( \frac{-\tilde{g}^2 + g^2}{g^2 + \tilde{g}^2} \right) d\tilde{g} = 0.
\]

(281)

From the estimation part for the inequality on the set \( E_{q-p'}^p \), we may find a proper variable \( \omega' \in H^2 \) such that \( \mathbb{R}^+ \times H^2 = E_{q-p'}^p \). Then, the integral is now

\[
\int_{H^2} d\omega' \int_{0}^{\infty} dq' \frac{4g\tilde{g}^2}{(g^2 + \tilde{g}^2)^2} \sigma_0 \left( \frac{-\tilde{g}^2 + g^2}{g^2 + \tilde{g}^2} \right) = 0.
\]

(282)

Then, we obtain

\[
\int_{E_{q-p'}^p} \frac{\tilde{g}^2}{(g^2 + \tilde{g}^2)^2} \sigma_0 \left( \frac{-\tilde{g}^2 + g^2}{g^2 + \tilde{g}^2} \right) = 0.
\]

(283)

Therefore by multiplying constant terms with respect to \( p \), we have

\[
\int_{E_{q-p'}^p} \frac{d\pi_p \sigma(g, \theta) \tilde{g}^2 \Phi(\tilde{g})}{p^3} \frac{f(q)h(p')\eta(p')}{\tilde{g}^4 \Phi(\tilde{g})} = 0.
\]

(284)
Now we subtract this expression from the Carleman representation just written for $\langle \Gamma(f, h), \eta \rangle$ must equal the usual representation. This will be called the relativistic dual representation. Thus,

$$\langle \Gamma(f, h), \eta \rangle$$

$$= \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \ \nu \phi f(q) h(p) \sigma(g, \omega) (\sqrt{J(q')} \eta(p') - \sqrt{J(q)} \eta(p))$$

$$= \frac{e}{2} \int_{\mathbb{R}^3} dp' \int_{\mathbb{R}^3} dq' \int_{\mathbb{E}^p_{q' - p'}} \frac{d\sigma_{p' \sigma(g, \theta)}}{\sqrt{v}} f(q) \eta(p') \{ h(p) \sqrt{J(q')} - \frac{s g' \Phi(\tilde{g})}{sg' \Phi(g)} h(p') \sqrt{J(q)} \}.$$ 

We claim that this representation holds even when the mean value of $\sigma_0$ is not zero. Suppose that $\int_{\mathbb{S}^2} d\omega |\sigma_0(\theta)| < \infty$ and that $\int_{\mathbb{S}^2} d\omega \sigma_0(\theta) \neq 0$. Define

$$\sigma_0'(t) = \sigma_0(t) - 1_{\{1 - \epsilon\}}(t) \int_{-\epsilon}^1 dt' \sigma_0(t').$$

Then, we have $\int_{-1}^1 \sigma_0'(t) dt = 0$ vanishing on $\mathbb{S}^2$.

Now, define

$$\langle \Gamma_\epsilon(f, h), \eta \rangle$$

$$= \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \ \nu \phi f(q) h(p) \sigma_0'(\cos \theta) (\sqrt{J(q')} \eta(p') - \sqrt{J(q)} \eta(p)).$$

Note that $t = \cos \theta$. Then,

$$|\langle \Gamma(f, h), \eta \rangle - \langle \Gamma_\epsilon(f, h), \eta \rangle|$$

$$= \left| \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \ \nu \phi f(q) h(p) \Phi(g)$$

$$\cdot (\sqrt{J(q')} \eta(p') - \sqrt{J(q)} \eta(p)) 1_{\{1 - \epsilon, 1\}}(\cos \theta) \frac{1}{\epsilon} \int_{-\epsilon}^1 \sigma_0(t') dt' \right|.$$ 

Here, we briefly discuss some properties under the condition $\cos \theta = 1$. By the definition, we have

$$\cos \theta = \frac{(p'^\mu - q'^\mu) (p'_\mu - q'_\mu)}{g^2}.$$ 

Thus, if $\cos \theta = 1$,

$$(p'^\mu - q'^\mu) (p'_\mu - q'_\mu) = g^2$$

$$= (p'^\mu - q'^\mu) (p_\mu - q_\mu).$$

Then we have

$$(p'^\mu - q'^\mu) (p'_\mu - p_\mu) = 0.$$ 

By the collision geometry $(p'^\mu - p'^\mu) (p'_\mu - q_\mu) = 0$, we have

$$(p'^\mu - p'^\mu) (p_\mu - p'_\mu) = g^2 = 0.$$ 

Thus, we get $\tilde{g} = 0$.

Equivalently, this means that

$$(p'^0 - p'^0)^2 = |p' - p|^2.$$
And this implies that $p^0 = p'^0$ and $p = p'$ because

\[
|p'^0 - p^0| = \frac{|p'|^2 - |p|^2}{p'^0 + p^0} < |p' - p|.
\]

(293)

Therefore, if $\cos \theta = 1$, we have $p'^\mu = p^\mu$ and $q'^\mu = q^\mu$.

Thus, as $\epsilon \to 0$, the norm in (287) $\to 0$ because the integrand vanishes on the set $\cos \theta = 1$.

Therefore, we can call (284) as the dual representation because if we define

\[
\langle \Gamma(f, h), \eta \rangle = \langle T_f \eta, h \rangle = \langle \eta, T^*_f h \rangle.
\]

(294)

9.3. Representations in other variables. The collision integral below can be represented in some other ways:

\[
\int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dq}{q^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \frac{dp'}{p'^0} \sigma(g, \theta) \delta^{(4)}(p'^\mu + q'^\mu - p^\mu - q^\mu) A(p, q, p')
\]

(295)

where $A$ is some Schwartz function.

The 12-fold integral can be written as an 8-fold integral in the center-of-momentum system by getting rid of the delta function and we obtain:

\[
\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \ v_\phi \sigma(g, \omega) A(p, q, p')
\]

where $v_\phi = \frac{\sqrt{2}}{p'^0}.$

9.3.1. Here we look for another expression as an integration on the set $\mathbb{R}^3 \times \mathbb{R}^3 \times E_{p'_{p+q}}$ where $E_{p'_{p+q}}$ is the hyperplane

\[
E_{p'_{p+q}} = \{ p' \in \mathbb{R}^3 : (p'^\mu - p^\mu)(p_\mu + q_\mu) = 0 \}.
\]

We rewrite (295) as

\[
\int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dq}{q^0} B(p, q, p')
\]

where $B = B(p, q, p')$ is defined as

\[
B = \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \int_{\mathbb{R}^3} \frac{dq'}{q'^0} \sigma(g, \theta) \delta^{(4)}(p'^\mu + q'^\mu - p^\mu - q^\mu) A(p, q, p')
\]

(296)

\[
d\Theta(p'^\mu, q'^\mu) = dp'^\mu dq'^\mu u(q'^\mu) u(p'^0) \delta(s - q^2 - 4) \delta((p'^\mu - q'^\mu)(p^\mu + q^\mu)) \text{ and } u(r) = 0 \text{ if } r < 0 \text{ and } u(r) = 1 \text{ if } r \geq 0.
\]

Now we apply the change of variable

\[
\tilde{q}^\mu = q'^\mu - p'^\mu.
\]
Then with this change of variable the integral becomes

\[ B = \int_{\mathbb{R}^4 \times \mathbb{R}^4} d\Theta(\vec{q}^\mu, q_0^\mu) s\sigma(g, \theta) \delta^{(4)}(2p^\mu + \vec{q}^\mu - q^\mu) A(p^\mu, q^\mu, q'^\mu) \]

where \( d\Theta(\vec{q}^\mu, q_0^\mu) \overset{\text{def}}{=} dp^\mu dq^\mu u(q^0 + p^0)u(p^0)\delta(s - g^2 - 4)\delta(\vec{q}^\mu(p^\mu + q'^\mu)) \). This change of variables gives us the Jacobian \( = 1 \).

Finally we evaluate the delta function to obtain

\[ B = \int_{\mathbb{R}^4} d\Theta(p'^\mu)s\sigma(g, \theta)A(p^\mu, q^\mu, p'^\mu) \]

where we are now integrating over the four vector \( p'^\mu \) and \( d\Theta(p'^\mu) = dp'^\mu u(q^0 + p^0 - p'^0)u(p'^0)\delta(s - g^2 - 4)\delta((p'^\mu + q'^\mu)(p_\mu + q_\mu - 2p'_\mu)) \). We conclude that the integral is given by

\[ B = \int_{E^0_{p\to q}} \frac{d\pi_{p'}}{2\sqrt{s^0}} s\sigma(g, \theta)A(p, q, p') \]

where \( d\pi_{p'} = dp' u(p^0 + q^0 - p'^0)\delta \left( -\frac{s}{\sqrt{s'}} - \frac{p'^\mu(p_\mu + q_\mu)}{\sqrt{s'}} \right) \). This is an 2 dimensional surface measure on the hypersurface \( E^0_{p\to q} \) in \( \mathbb{R}^3 \).

9.3.2. Similarly, we can also look for another expression as an integration on the set \( \mathbb{R}^3 \times \mathbb{R}^3 \times E^0_{p\to q} \) where \( E^0_{p\to q} \) is the hyperplane

\[ E^0_{p\to q} = \{ q \in \mathbb{R}^3 : (p'^\mu - p^\mu)(p_\mu + q_\mu) = 0 \} \]

We rewrite (295) as

\[ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} B(p, q, p') \]

where \( B = B(p, q, p') \) is defined as

\[ B = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} d\Theta(q^\mu, q'^\mu)s\sigma(g, \theta)\delta^{(4)}(p'^\mu + q'^\mu - p^\mu - q^\mu)A(p, q, p') \]

(300)

where \( d\Theta(q^\mu, q'^\mu) \overset{\text{def}}{=} dq^\mu dq'^\mu u(q^0)u(q'^0)\delta(s - g^2 - 4)\delta((q^\mu - q'^\mu)(q'^\mu + q'^\mu)) \) and \( u(r) = 0 \) if \( r < 0 \) and \( u(r) = 1 \) if \( r \geq 0 \).

Now we apply the change of variable

\[ q'^\mu = q'^\mu + q^\mu. \]

Then with this change of variable the integral becomes

\[ B = \int_{\mathbb{R}^4 \times \mathbb{R}^4} d\Theta(\vec{q}^\mu, q_0^\mu)s\sigma(g, \theta)\delta^{(4)}(p'^\mu + \vec{q}^\mu - p^\mu)A(p^\mu, q^\mu, q'^\mu) \]

where \( d\Theta(\vec{q}^\mu, q_0^\mu) \overset{\text{def}}{=} dq^\mu dq'^\mu u(q^0 + \vec{q}^0)u(q'^0)\delta(s - g^2 - 4)\delta(\vec{q}^\mu(2q^\mu + \vec{q}^\mu)) \). This change of variables gives us the Jacobian \( = 1 \).

Finally we evaluate the delta function to obtain

\[ B = \int_{\mathbb{R}^4} d\Theta(q^\mu)s\sigma(g, \theta)A(p^\mu, q^\mu, p'^\mu) \]
where we are now integrating over the four vector \( q^\mu \) and 
\[
d\Theta(q^\mu) = dq^\mu u(p^0 - p'^0 + q^0)u(q^0)\delta(s - g^2 - 4)\delta((p^\mu - p'^\mu)(2q_\mu + p_\mu - p'_\mu))
\]
We conclude that the integral is given by
\[
B = \int_{E_{p'-p}^\delta} \frac{d\pi_q}{2q^0} s\sigma(g, \theta)A(p, q, p')
\]
where
\[
d\pi_q = dq u(p^0 + q^0 - p'^0)\delta \left( \frac{q^0}{2} + \frac{g^2(p_\mu - p'_\mu)}{g} \right).
\]
This is a 2-dimensional surface measure on the hypersurface \( E_{p'-p}^\delta \) in \( \mathbb{R}^3 \).

9.4. Alternative form of the collision operator. Here, we also want to introduce an alternative way of writing the collision operator. The 12-fold integral (295) will be written in 9-fold integral in this subsection in \( (p, p', \bar{q}) \) where we define \( \bar{q} \) as below.

We write (295) using Fubini as follows
\[
(304) \quad I \overset{\text{def}}{=} \int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \int_{\mathbb{R}^4} dq \int_{\mathbb{R}^3} dq' \int_{\mathbb{R}^3} \frac{dp''}{p''^0} \int_{\mathbb{R}^4} dq'' \int_{\mathbb{R}^4} dq''' \frac{d\pi_q}{2q^0} s\sigma(g, \theta)\delta^{(4)}(p'^\mu + q''\mu - p''\mu - q'''\mu)A(p, q, q', q'')A(p, q, q', q'').
\]
By adding two delta functions and two step functions, we can express the integral above as follows
\[
(305) \quad I = \int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \int_{\mathbb{R}^4} dq' \int_{\mathbb{R}^4} dq'' \int_{\mathbb{R}^4} dq''' \frac{d\pi_q}{2q^0} s\sigma(g, \theta)\delta^{(4)}(p'^\mu + q''\mu - p''\mu - q'''\mu)A(p, q, q', q'').
\]
where we are now integrating over the 14-vector \( (p, p', q^\mu, q'^\mu) \), \( u \) is defined by \( u(r) = 0 \) if \( r < 0 \) and \( u(r) = 1 \) if \( r \geq 0 \), and we let \( \bar{q} \overset{\text{def}}{=} q(q^0, q'^0) \) and \( \bar{s} \overset{\text{def}}{=} s(q^0, q'^0) \). We will convert the integral over \( (q^\mu, q'^\mu) \) into the integral over \( q''\mu - q'''\mu \) and \( q''\mu + q'''\mu \).

Now we apply the change of variables
\[
q''_\mu \overset{\text{def}}{=} q^\mu + q''\mu, \quad q'''_\mu \overset{\text{def}}{=} q''\mu - q'''\mu.
\]
This will do the change \( (q^\mu, q'^\mu) \rightarrow (q''_\mu, q'''_\mu) \) with Jacobian = 16. With this change, the integral \( I \) becomes
\[
(306) \quad I = \int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \int_{\mathbb{R}^4} dq' \int_{\mathbb{R}^4} dq'' \frac{d\pi_q}{2q^0} u(q^0)u(-q''_s q_s - 4)\delta(-q''_q q_s - q''_q q_s - 4)\delta(q''_s q_s)\delta^{(4)}(p''\mu - p''\mu - q'''_\mu)A(p, \frac{q_\mu + q_\mu}{2}, p').
\]
Then we evaluate the third delta function to obtain
\[
(307) \quad I = \int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \int_{\mathbb{R}^4} dq' \frac{d\pi_q}{2q^0} u(q^0)u(-q''_s q_s - 4)\delta(-q''_q q_s - \bar{g}^2 - 4)\delta(q''_s (p''_\mu - p''_\mu))s\sigma(g, \theta)A(p, \frac{q_\mu + q_\mu}{2}, p').
\]
Note that $-q_s^μq_{sμ} - 4 = \tilde{g}^2 \geq 0$ by the first delta function, and thus we always have $u(-q_s^μq_{sμ} - 4) = 1$. Also, since $\tilde{s} = \tilde{g}^2 + 4$, we have

$$\begin{align*}
u(k^0) \delta(-q_s^μq_{sμ} - \tilde{g}^2 - 4) &= \nu(k^0) \delta(-q_s^μq_{sμ} - \tilde{s}) \\
&= \nu(k^0) \delta((k^0_s)^2 - |q_s|^2 - \tilde{s}) \\
&= \frac{\delta((k^0_s)^2 - \sqrt{|q_s|^2 + \tilde{s}})}{2\sqrt{|q_s|^2 + \tilde{s}}}. \\
\end{align*}$$

(308)

Then we finally carry out an integration using the first delta function and obtain

$$I = \int_{\mathbb{R}^3} \frac{dp}{p^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \int_{\mathbb{R}^3} \frac{dq_s}{2\sqrt{|q_s|^2 + \tilde{s}}} \delta(q_s^μ(p'_μ - p_μ)) sσ(g, θ) A(p, \frac{q_s + q_θ}{2}, p').$$

(309)

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