A MAPPING DEFINED BY THE SCHUR-SZEGÓ COMPOSITION

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Abstract. Each degree \( n + k \) polynomial of the form \((x + 1)^k(x^n + c_1x^{n-1} + \cdots + c_n)\), \(k \in \mathbb{N}\), is representable as Schur-Szegő composition of \( n \) polynomials of the form \((x + 1)^{n+k-1}(x + a_j)\). We study properties of the affine mapping \( \Phi_{n,k} : (c_1, \ldots, c_n) \mapsto (\sigma_1, \ldots, \sigma_n) \), where \( \sigma_i \) are the elementary symmetric polynomials of the numbers \( a_j \). We also study properties of a similar mapping for functions of the form \( e^x P \), where \( P \) is a polynomial, \( P(0) = 1 \), and we extend the Descartes rule to them.

The Schur-Szegő composition (SSC) of two polynomials \( A := \sum_{j=0}^{n} \binom{n}{j} \alpha_j x^j \) and \( B := \sum_{j=0}^{n} \binom{n}{j} \beta_j x^j \) is defined by the formula
\[ A \ast B := \sum_{j=0}^{n} \binom{n}{j} \alpha_j \beta_j x^j. \]
The SSC is commutative and associative. The above formula can be generalized in a self-evident way to the case of composition of more than two polynomials.

Obviously, \((x + 1)^n \ast A = A\) for any degree \( n \) polynomial \( A\); that is, in the case of the SSC the polynomial \((x + 1)^n\) plays the role of unity. If the polynomials \( A \) and \( B \) are considered as degree \( n + k \) ones, their first \( k \) coefficients being equal to 0, then the formula for \( A \ast B \) will be a different one. To avoid such an ambiguity we assume throughout this paper that the leading coefficient of at least one of the composed polynomials is non-zero. See more about the SSC in [8] and [9].

In this paper we study the affine mappings \( \Phi_{n,k} \) (connected with the SSC and defined after the proof of Lemma 1) and their generalization \( \Phi \) for entire functions (defined before Remarks 1). We also generalize the Descartes rule, see Theorem 3.

The following formulae are proved in [2] (\( S \) is a degree \( n - 1 \) polynomial):
\[
(0.1) \quad (A \ast B)' = (1/n)(A' \ast B') \quad \text{and} \quad (xS \ast B) = (x/n)(S \ast B').
\]

**Proposition 1.** (Proposition 1.4 in [6].) If the polynomials \( A \) and \( B \) have roots \( x_A \neq 0 \) and \( x_B \neq 0 \) of multiplicities \( m_A \) and \( m_B \) respectively, where \( m_A + m_B \geq n \), then \(-x_Ax_B\) is a root of \( A \ast B \) of multiplicity \( m_A + m_B - n \).

The following proposition is used to define below the mappings \( \Phi_{n,k}, k \geq 1 \):

**Proposition 2.** Each polynomial \( P := (x + 1)^k(x^n + c_1x^{n-1} + \cdots + c_n) \) is representable as SSC
\[
(0.2) \quad P = K_{n,k;a_1} \ast \cdots K_{n,k;a_n} \quad \text{with} \quad K_{n,k;a_i} := (x + 1)^{n+k-1}(x + a_i),
\]
where the complex numbers \( a_i \) are unique up to permutation.

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Proof:

For $k = 1$ the proposition is announced in Remark 7 of [2] and is proved in [1]. For $k > 1$ it can be deduced from there as follows: write $P$ in the form $(x + 1)(x + 1)\cdots(x + 1)(x^n + c_1x^{n-1} + \cdots + c_n))$. The second factor is a polynomial of degree $n + k - 1$ to which one can apply the result from [1] with $n$ replaced by $n + k - 1$. Hence $P$ is SSC of $n + k - 1$ composition factors. One can deduce from Proposition [1] that $k - 1$ of these composition factors equal $K_{n,k;1}$ (because −1 is a $(k - 1)$-fold root of the second factor) and hence can be skipped. □

Lemma 1. The coefficient of $x^s$ in $P$ is zero if and only if one of the numbers $a_i$ equals $-s/(n + k - s)$.

This follows from the formula

$$K_{n,k;a_i} = \sum_{s=0}^{n+k} \binom{n+k}{s} \left( \frac{n+k-s}{n+k} a_i + \frac{s}{n+k} \right) x^s.$$ 

Indeed, the coefficient of $x^s$ in at least one polynomial $K_{n,k;a_i}$ must equal 0. □

With $c_i$ and $a_i$ as in Proposition 2, the mapping $\Phi_{n,k}$ is defined like this:

$$\Phi_{n,k} : (c_1, \ldots, c_n) \mapsto (\sigma_1, \ldots, \sigma_n),$$

where $\sigma_j := \sum_{1 \leq i_1 < \cdots < i_j \leq n} a_{i_1} \cdots a_{i_j}$.

The mapping $\Phi_{n,k}$ is affine. For $k = 1$ this is proved in [3]. For any $k$ it follows from there (the coefficients of the polynomial $P/(x + 1)$ are affine functions of the variables $c_i$). Properties of $\Phi_{n,1}$ are studied in [3], [4], [5] and [7]. In this paper we continue the study of paper [5] and extend it to the case of entire functions.

The SSC of the entire functions $f := \sum_{j=0}^{\infty} \gamma_j x^j / j!$ and $g := \sum_{j=0}^{\infty} \delta_j x^j / j!$ is defined by the formula $f * g = \sum_{j=0}^{\infty} \gamma_j \delta_j x^j / j!$. Set $P_m := 1 + c_1 x + \cdots + c_m x^m$, $\tilde{\sigma}_k := \sum_{i \leq j_1 < \cdots < j_k \leq m} 1/(a_{i_1} \cdots a_{i_k})$. The following proposition allows to define an analog of the mappings $\Phi_{n,k}$:

Proposition 3. Each function $e^x P_m$, where $P_m$ is a degree $m$ polynomial such that $P_m(0) = 1$, is representable in the form

$$e^x P_m = \kappa_{a_1} \cdots \kappa_{a_m},$$

where $\kappa_{a_j} = e^x (1 + x/a_j)$. The numbers $a_j$ are unique up to permutation.

Indeed, it is easy to show by induction on $m$ (the proof is left for the reader) that the SSC of $m$ composition factors $\kappa_{a_j}$ is of the form $(1 + \sum_{i=1}^{m} b_i x^i) e^x$, where $b_i = \sum_{l=1}^{m} \zeta_{i,l} \tilde{\sigma}_l$, $\zeta_{i,l} \in \mathbb{N}$, $\zeta_{i,i} = 1$. The mapping $(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_m) \mapsto (b_1, \ldots, b_m)$ is linear upper-triangular and non-degenerate from where the proposition follows. □

Define the mapping $\Phi$ as follows: $\Phi : (c_1, \ldots, c_m) \mapsto (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_m)$.

Remarks 1. 1) The mapping $\Phi$ is a limit of mappings $\Phi_{n,k}$ as $k \to \infty$: each polynomial $k^k(x/k + 1)^k(x^\kappa + c_1 x^{\kappa-1} + \cdots + c_n)$ can be represented as SSC of $n$ composition factors of the form $k^{n+k-1}(x/k + 1)^{n+k-1}(x + a_j)$. The proof of this is completely analogous to the proof of Proposition 2. There remains to observe that $\lim_{k \to \infty} (x/k + 1)^k = e^x$. To avoid the constant factors $k^k$ and $k^{n+k-1}$ which tend to
infinity as \( k \to \infty \), one can consider instead polynomials \((x/k+1)^k(c_0x^n+c_1x^{n-1}+\cdots+c_{n-1}x+1)\) and composition factors of the form \((x/k+1)^{n+k-1}(x/a_i+1)\) or \((x/k+1)^{n+k-1}x\) in which case no constant factors are necessary.

2) For the composition factors \( \kappa_{a_j} \) one has the formula

\[
\kappa_{a_j} = \sum_{j=0}^{\infty} (1/j!)(1+j/a_j)x^j.
\]

3) If \( P \) (resp. \( P_m \)) is a real polynomial, then part of the numbers \( a_j \) in formula (0.2) (resp. (0.4)) are real and the rest form complex conjugate couples. Indeed, otherwise conjugation of the two sides of (0.2) or (0.4) would produce a new set of numbers \( a_j \) which contradicts their uniqueness.

**Notation 1.** We denote by \( U_n \subset \mathbb{R}^n \cong Oc_1\cdots c_n \) the subset defined by the conditions \((-1)^i c_i \geq 0\). By \( \Pi_n \) we denote the hyperbolicity domain of the family of polynomials \( P \), i.e. the set of values of the coefficients \( c_i \) for which \( P \) is hyperbolic. We write \( V_n \subset \mathbb{R}^n \) for the set of values of the coefficients of \( P \) for which the real parts of all roots are non-negative. It is easy to show that \((\Pi_n \cap U_n) \subset V_n \subset U_n\).

By \( T[f] \) we denote the Taylor series at 0 of the entire function \( f \).

**Theorem 1.** For each \( n \geq 1 \) and for each \( k \geq 1 \) one has \( \Phi_{n,k}(U_n) \subset U_n \).

**Corollary 1.** For the mapping \( \Phi \) one has \( \Phi(U_n) \subset U_n \).

To obtain the corollary consider \( \Phi \) as a limit of \( \Phi_{n,k} \) as \( k \to \infty \), see Remarks [1].

**Proof of Theorem 1:**

We prove the theorem by induction on \( n \). For \( n = 1 \) the mapping \( \Phi_{n,k} \) is the identity mapping and there is nothing to prove. Further we use the same reasoning as the one used in [5] (for \( k = 1 \) the theorem coincides with part (2) of Theorem 1.4 in [5]). Set \( P := xQ + R \), where \( R := c_n(x+1)^k \). For the polynomial \( xQ \) one of the numbers \( a_i \) defined in Proposition [2] equals 0. Set

\[
(x+1)^k xQ := (x+1)^{n+k-1}x \ast (x+1)^{n+k-1}(x+h_2) \ast \ldots \ast (x+1)^{n+k-1}(x+h_{n-1}).
\]

Apply formulae (0.1). The right-hand side of the last equality is representable as

\[
x((x+1)^{n+k-2}(x+g_2)\ast\ldots\ast(x+1)^{n+k-2}(x+g_{n-1}))\text{, where } g_i = \frac{(n+k-1)h_i+1}{n+k}.
\]

The last composition (excluding the factor \( x \)) is the representation of the polynomial \((x+1)^k Q\) in the form (1.2). By inductive assumption, if \( \sigma_0^j \) (resp. \( \sigma_1^j \) or \( \sigma_2^j \)) stands for the \( j \)th elementary symmetric polynomial of the quantities \( g_i \) (resp. \( l_j := (n+k)g_i/(n+k-1) \) or \( h_i \)), then \((-1)^j \sigma_0^j \geq 0\) (resp. \((-1)^j \sigma_1^j \geq 0\)). We set \( \sigma_0^0 = \sigma_0^1 = \sigma_2^0 = 1 \). Having in mind that \( h_i = l_i - 1/(n+k-1) \) and that the signs of \( \sigma_1^j \) alternate, one sees that \((-1)^j \sigma_2^j \geq 0 \). Indeed, one has \( \sigma_2^j = \sum_{v=0}^{n} (-1)^v r_v \sigma_v^j \).

We show for the half-axis \( Oc_n \) (positive for odd and negative for even \( n \)) that \( \Phi_{n,k}(Oc_n) \subset U_n \). As \( \Phi_{n,k} \) is affine, this implies \( \Phi(U_n) \subset U_n \).
The first $n$ coefficients of $R$ are 0, therefore $n$ of the numbers $a_i$ defined for $\Phi_{n,k}(R)$ equal $\infty$ and $-s/(n+k-s)$, $s = n + k - 1, \ldots, k + 1$, see Lemma 1. By Proposition 1.1 the remaining $k - 1$ of them equal 1. Therefore the numbers $a_i$ define a polynomial of the form $(x + 1)^k(c_0 x^{n-1} + \cdots + c_0)$ with $(-1)^{\nu} c_0 > 0$. □

**Remark 1.** One can deduce from the proof of Theorem 1 that if $A \in \partial U_n$ (the boundary of $U_n$), then $\Phi_{n,k}(A) \in \partial U_n$ if and only if $A \in \{c_n = 0\}$.

**Theorem 2.** If $P$ is real and with $\nu$ positive roots, then at least $\nu$ of the numbers $a_i$ defined by formula (9.2) are negative and belonging to different intervals of the kind $I_{n,k,s} := \{-s/(n+k-1-s), -s/(n+k-s)\}$.

**Proof:**

The polynomial $P$ has $\nu$ positive roots. By the Descartes rule, there are at least $\nu$ sign changes in the sequence $\Sigma$ of its coefficients. On the other hand, when the polynomial $K_{n,k,a_i}$ is real (i.e. when $a_i$ is real), there is at most one sign change in the sequence of its coefficients. This follows from formula (9.3—the numbers $((n + k - s)/(n + k))a_i + (s/(n + k))$ for $s = 0, \ldots, n + k$ form an arithmetic progression. For a couple of polynomials $K_{n,k,a_i}, K_{n,k,a_j}$, their SSC is a polynomial with all coefficients positive. The same is true for couples of polynomials $K_{n,k,a_i}, K_{n,k,a_j}$ with $a_i$ and $a_j$ belonging to one and the same interval $I_{n,k,s}$, and for polynomials $K_{n,k,a_i}$, with $a_i > 0$. Hence the $\nu$ sign changes in the sequence $\Sigma$ are due only to numbers $a_i$ belonging to different intervals $I_{n,k,s}$. □

**Remarks 2.** When $P$ or $P_n$ is hyperbolic (i.e. with all roots real), the mapping $\Phi_{n,k}$ (resp. $\Phi$) exhibits different properties in the cases when all roots are positive and when they are all negative. For instance, if all quantities $a_i$ are positive, then the composition $K_{n,k,a_1} \cdots \cdot K_{n,k,a_n}$ is a polynomial with all roots negative; this follows from Proposition 1.5 in [6]. But it is not true that when $P$ has all roots negative, then all quantities $a_i$ are real positive. Example:

\[(x + 1)^{k+1} x \ast (x + 1)^{k+1} x = (x + 1)^{k} x(x + 1/(k + 2)) .\]

Perturb the composition factors in the left-hand side into $(x + 1)^{k+1}(x \pm \varepsilon i)$. The polynomial to the right will have all roots negative (one of which by Proposition 1 is a $k$-fold root at $-1$). This follows from the comparison of the signs of the constant terms to the left and right. A similar example can be given about the mapping $\Phi$:

\[e^\varepsilon (x + 1) \ast e^\varepsilon (x + 1) = e^\varepsilon (x^2 + 3x + 1) .\]

Here $x^2 + 3x + 1$ has two negative roots. After this perturb the two composition factors to the left into $e^\varepsilon (x + 1 \pm \varepsilon i)$. For $\varepsilon > 0$ small enough the polynomial multiplying $e^\varepsilon$ in the right-hand side still has two negative roots.

When all roots of $P$ are positive, then all quantities $a_i$ are negative, see Theorem 2.2. But when all quantities $a_i$ are negative, then all roots of $P$ are not necessarily positive. E.g. the following polynomial has two complex conjugate roots:

\[(x + 1)^k ((x^2 - (2kx)/(k + 2)x + 1) = (x + 1)^{k+1}(x - 1) \ast (x + 1)^{k+1}(x - 1) .\]

In the case of the mapping $\Phi$ an analogous example is given by the equality

\[e^\varepsilon (x - 1) \ast e^\varepsilon (x - 1) = e^\varepsilon (x^2 - x + 1) .\]
and the analog of Theorem 2 in the case of the mapping $\Phi$ is Corollary 2 below.

**Notation 2.** For a polynomial $P = x^n + c_1x^{n-1} + \cdots + c_n$ we set

$$\Xi[P] := x(x-1) \cdots (x-n+1) + c_1x(x-1) \cdots (x-n+2) + \cdots + c_{n-1}x + c_n.$$ 

**Remark 2.** One checks directly that $e^x P(x) = \sum_{j=0}^{\infty} \Xi[P](j)x^j/j!$. It is easy to show that the numbers $-a_j$ defined by (0.4) are roots of the polynomial $\Xi[P]$.

Set $\Xi'[P] := x^n + c_{1,\nu}x^{n-1} + \cdots + c_{n-1,\nu}x + c_{n,\nu}$, $c_{0,\nu} := 1$. It is clear that $c_{n,\nu} = c_{n,0}$ for all $\nu$.

**Proposition 4.** 1) For each real polynomial $P$ as in Notation 2 there exists $\nu_0 \in \mathbb{N}$ such that for $\nu \geq \nu_0$ the signs of all summands to the right are the same. Hence the coefficient $c_{1,\nu}$ alternate.

2) One has $\lim_{\nu \to \infty} |e_{s,k}/e_{s-1,\nu}| = \infty$ for $s = 1, \ldots, n-1$.

3) For $\nu$ large enough the signs of the first $n$ coefficients of $T[e^xP]$ alternate.

**Proof:**

Observe first that $c_{0,\nu} = 1$ and $c_{n,\nu} = c_{n,0}$ for all $\nu$. The coefficient $c_{1,\nu}$ equals $c_1 - \nu n(n-1)/2$. Hence for $\nu$ sufficiently large this coefficient is $< 0$. Moreover, after its sign stabilizes, its absolute value increases with each new iteration of $\Xi$ and tends to $\infty$. Hence $\lim_{\nu \to \infty} |c_{1,\nu}/c_{0,\nu}| = \infty$.

Suppose that each of the coefficients $c_{j,\nu}$, $j = 1, \ldots, l - 1$ of $\Xi'[P]$ has the Property A: For $\nu$ large enough its sign is the same as the one of $(-1)^j$; moreover, after its sign stabilizes, its absolute value increases with each new iteration of $\Xi$.

Set $x(x-1) \cdots (x-n+1+l) := \sum_{j=0}^{n-l-1} r_{j,l} x^{n-l-j}$. Hence $r_{j,l} = 0$ for $j > n-l-1$ and $(-1)^j r_{j,l} > 0$. In particular, $r_{0,1} = 1$. The constants $r_{j,l}$ depend on $n$, $l$ and $j$, but not on $\nu$. One has $c_{l+1,\nu} = c_{l,\nu} + \sum_{j=1}^{n-l} r_{j,l} c_{-j,\nu}$ (**). For $\nu$ sufficiently large the signs of all summands to the right are the same. Hence the coefficient $c_{l,\nu}$ also has the Property A if $l < n$. This implies part 1).

Notice that $|r_{j,l}| \geq 1$ with equality only for $j = 0$ and for $l = n-2$. Therefore part 2) of the proposition follows from (**). Part 3) results from part 2). □

**Proposition 5.** If the real polynomial $P$ is with all roots real positive, then the polynomial $\Xi[P]$ is with all roots real positive and distinct.

**Proof:**

The non-degenerate affine mapping $\Phi$ is the limit as $k \to \infty$ of the non-degenerate affine mappings $\Phi_{n,k}$, see Remark 1. For each $(n,k)$ fixed the numbers $a_i$ defined for the polynomial $P$ from the composition product $[0.2]$, are negative, see Theorem 2 Therefore their limits are nonpositive. The limits are $\neq 0$, otherwise one should have $P(0) = 0$. By Remark 2 the roots of $\Xi(P)$ are all positive.

Set $\kappa_1 \cdots \kappa_{\alpha_n} := e^x P_j(x)$. Hence $e^x P_{m-1}(x) * e^x (1 + x/a_m) = e^x P_m(x)$ and

$$P_m(x) = (1 + x/a_m)P_{m-1}(x) + (x/a_m)P'_{m-1}(x).$$

By inductive assumption the polynomial $P_{m-1}$ is with distinct positive roots. Hence the term $(x/a_m)P_{m-1}(x)$ changes sign at the consecutive roots of $P_{m-1}$; that is, there is a root of $P_m$ between any two consecutive roots of $P_{m-1}$. This makes $m-2$ distinct positive roots of $P_m$. One has $\operatorname{sgn}(P_j(\infty)) = (-1)^j$, $j = m - 1, m$ (because the quantities $a_j$ are negative) and $\operatorname{sgn} P_j(0) = 1$ (see (0.6)). This means that there
is a root of $P_m$ in $(0, \lambda)$ and there is a root in $(\gamma, \infty)$, where $\lambda$ is the smallest and $\gamma$ is the largest of the roots of $P_{m-1}$. Thus $P_m$ has $m$ distinct positive roots. □

The following theorem (proved at the end of the paper) extends the Descartes rule to functions which are products of exponential functions and polynomials.

**Theorem 3.** If the real degree $m$ polynomial $P$ has $k$ positive roots, $1 \leq k \leq m$, then there are at least $k$ sign changes in the sequence of the coefficients of $T[e^x P]$.

**Corollary 2.** If there are $k$ sign changes in the sequence of coefficients of $T[e^x P]$, then at least $k$ of the numbers $a_i$ in the composition formula \( \bigcup_{i=1}^{k} \) are negative, distinct and belonging to different intervals of the kind $[-l, -l]$, $l \in \mathbb{N} \cup \{0\}$. The conclusion is true in particular when the real polynomial $P$ has $k$ positive roots.

**Proof:**

Formula (1.3) implies that there is at most one change of sign in the sequence of coefficients of the Taylor series $T[\kappa_a]$. This change occurs only if $a_i < 0$. By Theorem 3 there are at least $k$ sign changes in the sequence of coefficients of $T[e^x P_m]$. Composition factors $\kappa_a$, with complex $a_j$ present in (1.4) only in complex conjugate couples (see part 3) of Remarks 1), and for each composition of the kind $\kappa_a * \kappa_b$ all coefficients of $T[\kappa_a * \kappa_b]$ are positive. The same is true for two composition factors whose numbers $a_{j_1}, a_{j_2}$ belong to one and the same interval $[-l, -l]$. Hence the sign changes can come only from composition factors with negative numbers $a_j$ which belong to different intervals $[-l, -l]$. □

**Corollary 3.** For $P$ as in Notation 2 there exists $\nu_0 \in \mathbb{N}$ such that for $\nu \geq \nu_0$ the polynomial $\Xi^\nu [P]$ is with real and distinct roots, $n - 1$ or all of them being positive.

The corollary follows from part 3) of Proposition 4 and from Corollary 2. Whether all roots or all but one are positive depends on the sign of the constant term of the polynomial. Indeed, the mapping $P \mapsto \Xi^\nu [P]$ preserves the constant term.

**Remark 3.** By analogy with the proof of part (5) of Theorem 1.4 in [5] one can prove that for each $(n, k)$ fixed there exists $\nu(n, k)$ such that for $\nu_0 \geq \nu(n, k)$ the mapping $\Phi_{\nu_0}^{n,k}$ sends each point of $U_n$ into $U_n \cap \Pi_n$. In [5] this is proved for $k = 1$. The following example shows that this is not true for the mapping $\Phi$.

**Example 1.** Represent $f := e^x(1+ax+bx^2) \quad (a < 0, b > 0)$ in the form $e^x(1+x/\alpha) \ast e^x(1+x/\beta)$. Then $1/\alpha + 1/\beta = a - b$, $1/(\alpha \beta) = b$. Hence $\Phi[f] = e^x(1+(a-b)x+bx^2)$ and $\Phi^\nu[f] = e^x(1+(a-b)x+bx^2)$. For every $a_0 \in \mathbb{N}$ one can find $a < 0$ and $b > 0$ such that $1+(a-b)x+bx^2$ is hyperbolic for $s \geq a_0$ and not hyperbolic for $s < a_0$.

**Proposition 6.** For $m = 3$ the mapping $\Phi$ does not send the set $V_m$ into itself.

**Proof:**

Consider the functions of the kind $e^x(x^3 + ax^2 + bx + c)$, $a \leq 0$, $b \geq 0$, $c \leq 0$. Their subset whose roots have non-negative real parts is bounded by the hyperbolic paraboloid $P : c = ab$ and the hyperplanes $H_1 : a = 0$, $H_2 : b = 0$ and $H_3 : c = 0$. It is defined by the system $c \geq ab, a \leq 0, b \geq 0, c \leq 0$. Its boundary is $A \cup B$, where $A := \{c = 0, a \leq 0, b \geq 0\}$, $B := \{c = ab, a \leq 0, b \geq 0\}$.
The polynomials corresponding to the set $A$ have a root at 0, the ones from $B$ are of the form $R := (x - d)(x^2 + \Lambda) = x^3 - dx^2 + \Lambda x - d\Lambda$, $d \geq 0$, $\Lambda \geq 0$. Set

\begin{equation}
 e^x(x^3 - dx^2 + \Lambda x - d\Lambda) = e^x(x + \alpha) \ast e^x(x + \beta) \ast e^x(x + \gamma),
 \end{equation}

$\sigma_i := \alpha + \beta + \gamma$, $\sigma_j := \alpha\beta + \alpha\gamma + \beta\gamma$, $\sigma_3 := \alpha\beta\gamma$. Comparing the coefficients of 1, $x$ and $x^2$ in the two sides of (0.7) one obtains the system

$$
\sigma_3 = -d\Lambda, \ 1 + \sigma_1 + \sigma_2 + \sigma_3 = \Lambda - d\Lambda, \ 8 + 4\sigma_1 + 2\sigma_2 + \sigma_3 = -2d + 2\Lambda - d\Lambda.
$$

This means that $\Phi[e^x R] = e^x(x^3 + (-d-3)x^2 + (\Lambda + d + 2)x - d\Lambda)$. The coefficients $a, b, c$ of the last polynomial factor satisfy the condition $c = (a + 3)(b + a + 1)$. This defines a hypersurface $Y \subset \mathbb{R}^3$. Consider the intersection $B \cap Y$. It is defined by the conditions $a \leq 0$, $b \geq 0$, $c = ab = (a + 3)(b + a + 1)$. The point $W := (a, b, c) = (-2, 1/3, -2/3)$ belongs to this intersection. Fix $a = -2$ and vary $b$. Close to the point $W$ there are points of $Y$ which are inside and points which are outside the domain $V_3$. This proves the proposition. □

**Proof of Theorem 3.**

10. Assume (which is not restrictive) that $P$ is monic and that $P(0) \neq 0$. Set $P := x^m + d_1x^{m-1} + \cdots + d_m$, $d := |d_1| + \cdots + |d_m|$. There exists $N \in \mathbb{N}$ such that the coefficient of $x^j$ in $T[e^xP]$ is positive for $j \geq N$. Indeed, this coefficient equals

$$
1/(j-m)! + d_1/(j-m+1)! + \cdots + d_m/j!
$$

which is $> 0$ for $d < j - m + 1$ (i.e. for $j > d + m - 1$).

20. Suppose that $P$ has a root $x_0 > 0$ of multiplicity $\mu > 1$ (if all positive roots are simple, then go directly to 30). Denote by $x_1, \ldots, x_s$ its other positive roots and by $m_1, \ldots, m_s$ their multiplicities. For $\varepsilon > 0$ small enough the coefficients of $T[e^xP + \varepsilon x^N(\prod_{i=1}^s(x-x_i)^{m_i})(x-x_0)^{\mu-1}]$ have the same signs as the coefficients of $T[e^xP]$. The root $x_0$ bifurcates into a root of multiplicity $\mu - 1$ and a simple root close to it, both positive. The other positive roots and their multiplicities remain the same. In the same way one can change the function $e^xP$ to a nearby one $e^xP + g$ ($g$ is a polynomial) with the same number of positive roots (counted with multiplicity, but which are all simple) and the same signs of its Taylor coefficients.

30. Fix an interval $I := [\delta_1, \delta_2]$ ($0 < \delta_1 < \delta_2$) containing in its interior all positive roots of $e^xP + g$. The series $T[e^xP]$ converges absolutely for all real $x$ and its coefficients except finitely many are positive. Therefore one can find a partial sum $S$ of $T[e^xP + g]$ with the same number of roots in $I$ as $e^xP + g$ (all of them being simple) and with the same number of sign changes in the sequence of its coefficients. Hence $S$ has $\geq k$ positive roots and the number of sign changes in the sequence of the Taylor coefficients (which is the same for $T[e^xP]$) is $\geq k$. □

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