Standard Model bundles
of the heterotic string

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Abstract

We show how to construct supersymmetric three-generation models with gauge group and matter content of the Standard Model in the framework of non-simply-connected elliptically fibered Calabi-Yau manifolds $Z$. The elliptic fibration on a cover Calabi-Yau, where the model has 6 generations of $SU(5)$ and the bundle is given via the spectral cover description, has a second section leading to the needed free involution. The relevant involution on the defining spectral data of the bundle is identified for a general Calabi-Yau space of this type and invariant bundles are generally constructible.
1 Introduction

A search for the gauge group $G_{SM}$ and chiral matter content of the Standard model within the framework of string theory may be done under different perspectives. We adopt as guidelines here the following principles: the number and respective ranks of the gauge group factors should not be put in just by hand (like collecting different gauge groups just in the manner needed) but rather come in one package (as from a broken GUT group); gravity should be on the same level as the gauge field theory (and not in a different sector but rather should come united with the gauge forces like in a string compactification model); with an eye on the stabilization of moduli (which has to be achieved later when discussing the continuous parameters of the Standard model) the discussion should not be restricted from the outset to having important geometric moduli fixed (by working in some geometric limit of the space) but rather should be carried out at a generic point in moduli space so that it retains some flexibility when including further requirements; finally the construction should be as simple as possible in the given framework (so that especially it remains manageable when proceeding further).

Efforts to get a (supersymmetric) phenomenological spectrum from the $E_8 \times E_8$ heterotic string on a Calabi-Yau $Z$ started with embedding the spin connection in the gauge connection which gave an unbroken $E_6$ (times a hidden $E_8$ which couples only gravitationally). Then a further breaking of the gauge group can be achieved by turning on Wilson lines using a non-trivial $\pi_1(Z)$. Though the simplest constructions of Calabi-Yau spaces have a trivial fundamental group one can still produce a non-trivial $\pi_1$ by dividing $Z$ by a freely acting group $G = \pi_1$, provided such an operation exists. This leads at the same time to a reduction by $|G|$ of the often large number of generations $\chi(Z)/2$.

This approach was generalised [1] to the case of embedding instead of the tangent bundle an $SU(n)$ bundle for $n = 4$ or 5, leading to unbroken $SO(10)$ resp. $SU(5)$ of even greater phenomenological interest than $E_6$. This subject was revived when the bundle construction was made much more explicit for the case of elliptically fibered Calabi-Yau $\pi : Z \rightarrow B$ [2]. In subsequent work this extended ansatz showed among other things to have a much greater flexibility in providing one with three-generation models of the corresponding unbroken GUT group [3].

There remains to go to the Standard Model gauge group in this framework. For the Hirzebruch surfaces $F_m$, $m = 0, 1, 2$ as bases $B$ the special elliptic fibrations $\pi : Z \rightarrow B$ we will have to consider are smooth. However none of them has non-trivial $\pi_1(Z)$ (the Enriques surface as base $B$ has $\pi_1(B) = \mathbb{Z}_2$ leading to a non-trivial $\pi_1(Z)$ but, as pointed out by E. Witten, does not lead to a three generation model, cf. [10]).
The elliptic framework can nevertheless give a three generation model of Standard Model gauge group and matter content by working with an $SU(5)$ bundle leading to a $SU(5)$ gauge group on a space admitting a free involution $\iota$ which after modding it out gives a smooth Calabi-Yau $Z' = Z/\mathbb{Z}_2$

$$Z \xrightarrow{\rho} Z'$$

$$\pi \downarrow \quad \downarrow \pi'$$

(1.1)

$$B \rightarrow B'$$

On $Z'$ one can turn on a $\mathbb{Z}_2$ Wilson line with generator $1_3 \oplus -1_2$ breaking $SU(5)$ to $G_{SM}$

$$SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$$

(1.2)

On the other hand to actually compute the generation number one has to work ‘upstairs’ as $Z'$ does not have a section but only a bi-section (left over from the two sections of $Z$) and so one can not use the spectral cover method there directly$^2$.

$$V \rightarrow V'$$

$$\downarrow \quad \downarrow$$

$$Z \xrightarrow{\rho} Z'$$

(1.3)

If $V'$ over $Z'$ is the searched for 3-generation bundle then the bundle $V = \rho^*V'$ on $Z$ has 6 generations and is ‘moddable’ by construction. Conversely, having constructed a bundle above on $Z$ with 6 generations, one assures that it can be modded out by $\iota$ (to get the searched for bundle on $Z'$) by demanding that $V$ should be $\iota$-invariant. So one needs to specify on $Z$ an $\iota$-invariant $SU(5)$ bundle that leads to 6 generations and fulfills some further requirements of the spectral cover construction, essentially a bound on the $\eta$ class expressing the five-brane effectivity required from anomaly cancellation.

As it will be our goal to ‘mod’ not just the Calabi-Yau spaces but also the geometric data describing the bundle (and this transformation of bundle data into geometric data uses in an essential way the elliptic fibration structure) we will search only for actions which preserve the fibration structure, i.e. $\zeta \cdot \pi = \pi \cdot \iota$ with $\zeta$ an action on the base.

$$Z \xrightarrow{\iota} Z$$

$$\pi \downarrow \quad \downarrow \pi$$

(1.4)

$$B \xrightarrow{\iota} B$$

$^2$The bundle is described fibrewise (where it decomposes into a sum of line bundles) by a collection of points, for which a choice of reference point $p$ (the "zero" in the group law) has to be made (to represent a degree zero line bundle by the divisor $Q_i - p$); to have the reference point globally one needs a section.
This has the consequence that our elliptically fibered Calabi-Yau spaces will actually have 2 sections\(^3\) \(\sigma_1\) and \(\sigma_2 = \iota \sigma_1\) (\(B\)-model spaces).

Turning this around we will look for an elliptically fibered Calabi-Yau space \(Z\) with a changed type of elliptic fibre so that the global fibration has then besides the usually assumed single section (\(A\)-model) a second one (\(B\)-model); this will then lead to a free involution \(\iota\) on \(Z\). The elliptic Calabi-Yau spaces over Hirzebruch surfaces we will actually consider are close cousins of the "\(STU\)"-models of Hodge numbers \((3,243)\) prominent in the \(N = 2\) string duality (there are versions of this space over the Hirzebruch surface \(F_m\) for \(m = 0, 1, 2\) with the same Hodge numbers). For example over \(F_2\) the exchange of the \(A\)-fibre \(P_{2,3,1}(6)\) with the \(B\)-fibre \(P_{1,2,1}(4)\) leads one from the \(P_{1,1,2,8,12}(24)\) to the \(P_{1,1,2,4,8}(16)\) of \(h^1 = 4\) and \(h^2 = 148\) (one has again three versions of this space for \(m = 0, 1, 2\); we stick to these \(m\) as the only ones for which the global \(B\)-model is smooth).

Demanding the existence of an appropriate involution (fibrewise the translation in the group law with \(\sigma_2\), i.e. \(\sigma_2\) has to be at a 1/2-division point) leaves only 75 deformations.

In section 2 the \(B\)-model spaces along with their cohomological data are introduced; for \(F_0, F_2\) we are writing down a free involution. Then the spectral cover construction of bundles is recalled and the Chern classes of \(V\) and the \(\iota\)-action for our Calabi-Yau are described. Requiring \(c_1(V) = 0\) fixes \(V\) in terms of an \(\eta \in H^{1,1}(B)\) and a pair \((\mu, \lambda)\) of half-integers occurring in the twisting bundle \(L\) on the spectral cover \(C = n\sigma_1 + \eta\)

\[
c_1(L) = \left(\frac{(n\sigma_1 + \eta + c_1)/2 + \lambda(n\sigma_1 - \eta + nc_1) + \mu \epsilon}{C}\right)
\]

(1.5)

(with \(\alpha \in H^2(B)\)). Here the new invariant \(\mu\) occurs as one can use now the new class \(\delta = \mu \epsilon|_C = \mu(\sigma_1 - \sigma_2 + c_1)|_C\) in the \(B\)-model, to get a broader class of bundles for which we identify the \(\iota\)-action; then \(c_2(V)\) is given with its \(\mu\)-dependence

\[
c_2(V) = (1 - \mu)\eta\sigma_1 + \mu\eta\sigma_2 - \mu(1 - \mu)\eta c_1 + kF\]

(1.6)

(with classes of the form \(aF\) which contain the parameter \(\mu\) kept separate) and examples of a 6 generation model ("above") are given (the effectivity bound on \(\eta\) is discussed in appendix C). Finally the action of \(i\) on \(c_2(V)\) is read off which amounts just to the parameter substitution \(\phi_\mu : \mu \rightarrow 1 - \mu\); this suggests already (what is formally demonstrated in appendix A and constitutes the main result of this paper) that \(\phi_\mu\) gives the action \(I\) of the involution also on the bundle itself and not just on the second Chern-class of \(V\) (the relevant bundle \(L\) on \(C\) comes from restriction to \(C\) of a line bundle \(l = \mathcal{O}(D)\) on \(Z\))

\(^3\)We will use the same notation for a section, its image and its cohomology class.
Here we suppressed that one actually starts with \(l\) and first builds \(j_*j^*l = j_*L\), where \(j : C \hookrightarrow Z\), before applying the Fourier-Mukai transform (actually we will prove (1.7) just restricted to \(C\), i.e. only for \(L\) what is sufficient for our purposes); building \(F(j_*L)\) is the spectral cover construction. For \(F_m\) with \(m\) even the invariant element \(\delta_{\mu=1/2} = \epsilon/2|_C\), i.e. \(C\) has to be tuned to be an integral divisor class so that the construction is well-defined.

As the focus of the present paper is on how to construct invariant bundles over a general elliptically fibered Calabi-Yau and not so much on specific examples we do not scan exhaustively all possible bases nor do we search specifically for stable \(\eta\) because the invariance question, as is apparent from (1.7), is independent of specific \(\eta\)-classes.

Note also that as the \(10\) and the \(\bar{5}\) will come in the same number of families by anomaly considerations it is enough to adjust \(c_3(V)/2\) to get all the Standard model fermions.

Finally some remarks on technical points and relation to [4] and [14], [15]. Using the \(B\)-models our discussion will rely on [4] whose main points we will recall. From the generation formula found in [3] it is immediate to write down the \(\eta, \lambda\) parameters for the 6-generations models above on \(Z\). There remain the issues of the \(\eta\)-bound (i.e. effectivity of the five-brane class) and \(\iota\)-invariance. The mentioned models were only partially given in [4] as the ones over \(F_2\) violated an upper bound on the \(\eta\)-class. That actually a less restrictive version of the bound can be used and that when demanding \(\iota\)-invariance the bounds are completely unproblematical is described here in subsect. C.1.

In appendix A the \(\iota\)-action is identified already on the bundle level (not just for \(c_2(V)\)). When forcing two sections in the \(A\)-model and resolving the singularities occurring by the moduli specialisation (leading to a model similar but different from [4]) one has to consider individual resolution classes whose behaviour under the Fourier-Mukai transform can be difficult to treat (one alternative then is to consider more involved constructions on a special \(Z\) given by the fibre product of two elliptic surfaces). A uniform treatment of all these classes is essentially corresponding to \(\delta_{\mu=1/2}\) which is actually to be tuned integral so that here a construction just with \(L\)-classes coming from the full \(Z\) is possible. The mentioned features lead to a relatively simple construction.
2 The Calabi-Yau spaces with free \( \mathbb{Z}_2 \) action

2.1 Change of fibre type

To have the extra structure, which allows for free involution on the elliptically fibered Calabi-Yau we will use a different elliptic curve than the fibre usually taken in the Weierstrass \( A \)-model which we recall first.

\( A \)-model

The mentioned standard description (\( A \)-model) has the form

\[
y^2 + x^3 + z^6 + sxz^4 = 0 \tag{2.8}
\]

We will be interested in a global version of these descriptions over a complex surface \( B \) so that our Calabi-Yau \( Z \) can be described by a generalized Weierstrass equation in a \( \mathbb{P}^2 \) bundle \( W \) over \( B \) (note that the fibre is not \( \mathbb{P}^2_{2,3,1}(6) \)). This gives for the \( A \)-fibre (cf. [2])

\[
zy^2 + x^3 + a_4xz^2 + b_6z^3 = 0 \tag{2.9}
\]

where the given variables \( x, y, z \) and coefficient functions \( a, b \) are sections of \( L^i \), with \( i = 2, 3, 0 \) and \( i = 4, 6 \), respectively, with the line bundle \( L = K_B^{-1} \) over \( B \). One has an obvious projection \( \pi : Z \to B \) and a section \( \sigma : B \to Z \) given by \( z = 0 \). The Chern classes of \( Z \) are given by (cf. [2]) (\( c_1, c_2, \alpha \) will always denote \( c_1(B), c_2(B), \pi^*\alpha \) for \( \alpha \in H^2(B) \))

\[
c_2(Z) = 12 \sigma c_1 + c_2 + 11c_1^2, \quad c_3(Z) = -60c_1^2 \tag{2.10}
\]

From the weights of \( a_4 \) and \( b_6 \) one gets \( 9^2 + 13^2 - 3 - 3 - 1 = 243 \) deformations over \( F_0 \).

\( B \)-model

We will be interested in the so-called \( B \)-fibre \( \mathbb{P}^2_{1,2,1}(4) \) (cf. [4])

\[
y^2 + x^4 + sx^2z^2 + z^4 = 0 \tag{2.11}
\]

In the case of the global \( B \)-model the fibre \( \mathbb{P}^2 \) is actually a weighted \( \mathbb{P}^2_{1,2,1} \) with \( y \) being a section of \( \mathcal{O}(2) \). Then the variables \( x, y, z \) and coefficient functions \( a, b, c \) are sections of \( \mathcal{O}^i \) with \( i = 1, 2, 0 \) and \( i = 2, 3, 4 \), respectively and one has a well-defined equation

\[
y^2 + x^4 + a_2x^2z^2 + b_3xz^3 + c_4z^4 = 0 \tag{2.12}
\]

From the weights of \( a_2, b_3 \) and \( c_4 \) one gets \( 5^2 + 7^2 + 9^2 - 3 - 3 - 1 = 148 \) deformations over \( F_0 \). The generic form of the global \( B \)-model is smooth [4] over a Hirzebruch base
For $m = 0, 1, 2$, the cases we restrict ourselves to from now on. Now a second, cohomologically inequivalent section $σ_2$ occurs: considering the equation at the locus $z = 0$, i.e. $y^2 = x^4$ (after $y \to iy$) one finds 8 solutions which constitute two equivalence classes in $\mathbb{P}(1, 2, 1)$: $(x, y, z) = (1, \pm 1, 0)$. $y = +1$ corresponds to the zero section (in the group law) $σ_1$, while the other one has rank 1 in the Mordell-Weil group, i.e. generates infinitely many sections. For special points in the moduli space we can bring it to a 2-torsion point (in the group law) leading to the shift-involution, cf. subsect 2.2.

Let us keep some relations on record: one has the adjunction relations ($i = 1, 2$)

$$σ_i(σ_i + c_1) = 0 \quad (2.13)$$

(with $σ_i := σ_i(B)$) and, as $σ_1$ and $σ_2$ are disjoint, $σ_1σ_2 = 0$, the important class

$$ε := σ_1 - σ_2 + c_1 \quad \text{(with } σ_1ε = 0) \quad (2.14)$$

turns out to be trivial on $σ_1$. Finally one finds for the Chern classes

$$c_2(Z) = 6 (σ_1 + σ_2) c_1 + c_2 + 5c_1^2 \quad , \quad c_3(Z) = -36c_1^2 \quad (2.15)$$

(Over $\mathbf{F}_m$ cf. $h^{2,1} = 148$ above and $h^{1,1} = 4$.) For a relation between the $A$-model, when enforced to have also a second section, and the $B$-model cf. appendix B.

**Examples over Hirzebruch bases**

Usually, in the $A$-model, one has $k + 1$ $H^2(Z)$-cohomology classes from the divisors ($k := h^{1,1}(B)$), namely $σ$ and the $π^*α$; similarly one finds $k + 1$ $H^4(Z)$-cohomology classes from the curves $F$ (the elliptic fibre) and $σα$ (denoting $σ_α$ resp. equivalently $σ \cdot π^*α$).

In the $B$-model one has $k + 2$ divisor cohomology classes (including $σ_2$) and $2k + 1$ curve cohomology classes (including $σ_2α$). So all but one of the new curve classes $σ_2α$ must be dependent on the other classes. So one has one relation over $B = \mathbf{F}_m$ whose $H^{1,1}$ is generated by the class $b$ of the base-$\mathbb{P}^1$ (of $b^2 = -m$) and $f$ of the fibre-$\mathbb{P}^1$.

From Poincare duality and the intersection products (with $c_1(\mathbf{F}_m) = 2b + (2 + m)f$)

|       | $σ_1$ | $σ_2$ | $π^*b$ | $π^*f$ |
|-------|-------|-------|--------|--------|
| $F$   | 1     | 1     | 0      | 0      |
| $σ_1b$ | $m - 2$ | 0    | $-m$  | 1      |
| $σ_1f$ | $-2$   | 0     | 1      | 0      |
| $σ_2b$ | 0     | $m - 2$ | $-m$  | 1      |
| $σ_2f$ | 0     | $-2$  | 1      | 0      |
one finds, with $X := \frac{1}{2}(\sigma_2 - \sigma_1)f$ as new independent element besides the $F$ and the $\sigma_1 \alpha$,

$$(\sigma_2 - \sigma_1) \alpha = (\alpha \cdot c_1) X \tag{2.16}$$

(as is checked directly for $\alpha = xb + yf$; here $X$ is integral). Note that despite appearance to the contrary the term $\epsilon/2$ (like $X$) shows here integral intersection pairings: for the class $c_1/2$ as well as the class $(\sigma_1 - \sigma_2)/2$ have for $F_m$ with $m$ even, integral intersection pairings with the displayed classes as can be seen from inspecting the intersection table.

$(\sigma_1, \sigma_2)$ or $(\sigma_1, \epsilon)$ build a basis for $H^2(Z)/H^2(B)$ with $D = x\sigma_1 + y\sigma_2 + \alpha = x\sigma_1 - \mu\sigma_2 + \alpha = \bar{x}\sigma_1 + \mu\epsilon + \bar{\alpha}$ where $\alpha \in H^2(B)$, $\mu = -y$, $\bar{x} = x - \mu$ and $\bar{\alpha} = \alpha - \mu c_1$. With

$$\theta := \frac{1}{2c_1^2} \epsilon c_1 = \frac{1}{2} \left( \frac{1}{c_1^2} (\sigma_1 - \sigma_2) c_1 + F \right) = \frac{1}{2} (-X + F) \tag{2.17}$$

one has $y = D \cdot \theta$, i.e. taking intersection with the class $\theta$ is defined in $H^2(Z)/H^2(B)$ and $\sigma_k \cdot \theta = \delta_{k2}$ resp. $\epsilon \cdot \theta = 1$. (For a one-dimensional base one would have to use $\theta := \epsilon/2$.)

An involution

We define an involution $I_D : D_{\mu} \to D_{1-\mu}$ on the space $H^2(Z)$ in the form

$$I_D : \hspace{1cm} \bar{x}\sigma_1 + \mu\epsilon + \bar{\alpha} \longrightarrow \bar{x}\sigma_1 + (1 - \mu)\epsilon + \bar{\alpha} \tag{2.18}$$

In the other basis this amounts to

$$I_D : \hspace{1cm} x\sigma_1 + y\sigma_2 + \alpha \longrightarrow x\sigma_1 + y\sigma_2 + \epsilon + 2y\epsilon + \alpha \tag{2.19}$$

Therefore the affine transformation given by $I_D$ decomposes as follows

$$I_D = T_{\epsilon} \circ R_{\epsilon}, \hspace{1cm} R_{\epsilon} : D \longrightarrow D + (D \cdot \frac{\epsilon c_1}{c_1^2}) \epsilon \tag{2.20}$$

where $T_{\epsilon}$ is the translation by $\epsilon$ and the linear part is given by the reflection $R_{\epsilon}$ in the $\mu = 1/2$ axis in the $(\mu, \bar{x})$-plane (note $R_{\epsilon}(\epsilon) = -\epsilon$ so that $R_{\epsilon} \neq \iota$ as $\iota \epsilon = -\epsilon + 2c_1$).

Remark: Later it will be important to understand the classes $(k = 1, 2)$

$$D_k := R_{\epsilon}(\sigma_k) = \sigma_k + 2\delta_{2k}\epsilon \tag{2.21}$$

Note that the class $D_2 := \sigma_2 + 2\epsilon$ does not$^4$ come from a smooth irreducible effective divisor or else it would be a section (as $F \cdot \epsilon = 0$) and $(D_2 + c_1)D_2 = 6c_1\epsilon = 48(F - X) \neq 0$ would have to vanish, cf. (2.13) (for the situation in the (specialised) $A$-model cf. appendix B). But this product vanishes on (e.g. spectral) surfaces (classes) like $x\sigma_1 + \alpha$ what will be enough for our purposes (cf. remark after (1.7) and the reasoning in appendix A).

$^4$showing that $R_{\epsilon}$ does not come from a map on the space (unlike $\iota$ which keeps irreducibility)
2.2 Free $Z_2$ action on the global $B$-model $Z$

We recall how a smooth $Z$ with (fibration-compatible) free involution $\iota$ leaving the holomorphic $(3,0)$-form invariant is found [4], so that $Z' = Z/\iota$ is a smooth Calabi-Yau.

In the global $B$-model one has a involutive shift symmetry which is free at least on the generic fibre. The existence of the shift symmetry means that the second section is at a globally specified order-2 point; this in turn means specialization $b = 0$ of the complex parameters (the spaces then turn out to be singular; so this has still to be modified)

shift-symmetry \[(x, y, z) \xrightarrow{\iota} (-x, -y, z) \quad \text{for} \quad b = 0 \quad (2.22)\]

As $\iota$ is compatible with the fibration we have an involution $\mathcal{I}$ on $B$ with $\mathcal{I} \cdot \pi = \pi \cdot \iota$. $Z' = Z/\iota$ is again an elliptic fibration (of $B$-type elliptic fibre) over a base $B' = B/\mathcal{I}$. To get $\iota$ one starts with the elliptic fibration $Z$ with 2 sections and uses the operation (2.22) together with an in general non-free involution $\mathcal{I}$ on $B$. As $\iota$ identifies the two sections their image downstairs in $Z'$ (where thereby an independent divisor class in $H^{1,1}$ is lost) will be an irreducible surface $\sigma$ (still isomorphic to $B$) which is only a bisection: from $pr^*\sigma \cdot pr^*f = 2 \sigma \cdot f$ where $f$ denotes the fibre downstairs, lying over a generic point $b'$ in the base $B'$ of $Z'$, one finds $\sigma \cdot f = 2$, the left hand side being 4 as each of the two sections $\sigma_i$ over $\sigma$ intersects each of the two fibers in $Z$ 'above' (lying over the preimages of $b'$ in $B$) twice.

Over $\mathbf{F}_m$ the involution $\mathcal{I}$ will be chosen as a combination of the non-trivial involutions on the two individual $\mathbb{P}^1$'s, so $\mathcal{I} : (z_1, z_2) \to (-z_1, -z_2)$ in local coordinates. The fix-point locus in the base is generically disjoint from the discriminant of the elliptic fibration. The operation over the fix-locus in the base must be free in the fibre, i.e. a shift by a torsion point. So one has to use actually a fibration where such a shift exists globally (even if only on a sublocus of moduli space). The fibre shift must be combined with an operation in the base because at the singular fibers the pure fibre operation alone ceases to be free; also using just the fibre shift would restrict one to the locus $b = 0$ where $Z$ is singular.

One has [4] a free involution for a specialised $Z$ over $F_0$ and $F_2$ ($\Omega_3^{\text{holo}}$, in its explicit coordinate expression, is invariant as $\iota$ involves an even number of minus signs)

\[(z_1, z_2 ; x, y, z) \xrightarrow{\iota} (-z_1, -z_2 ; -x, -y, z) \quad (2.23)\]

which does not necessarily restrict one to the $b = 0$ locus (for $b = 0$ it is identical to the shift provided by the second section, which is then an involution). The involution does not exist on the fibre as such, but can exist, when combined with a base involution, on a subspace of the moduli space (not as 'small' as the locus $b = 0$) where the generic
member is still smooth. From (2.12) the coefficient functions should transform under \( \ell \) as \( a_2^+, b_3^+, c_4^+ \), i.e. over \( F_0 \), say, only monomials \( z_1^p z_2^q \) within \( b_{6,6} \) with \( p + q \) even are forbidden (note that even away from \( b = 0 \) the coordinate involution still maps the two sections on another); similarly in \( a_{4,4} \) and \( c_{8,8} \) \( p + q \) odd is forbidden. Therefore the number of deformations drops to \( h^{2,1} = (5^2 + 1)/2 + (7^2 - 1)/2 + (9^2 + 1)/2 - 1 - 1 - 1 = 75 \). The discriminant of (2.12) remains generic since enough terms in \( a, b, c \) deformations drops to \( h \).

3 The bundles

3.1 The spectral cover description

In the spectral cover description of an \( SU(n) \) bundle [2] one considers the bundle \( V \) first over an elliptic fibre \( E \) and then pastes together these descriptions using global data in the base \( B \). Over \( E \) an \( SU(n) \) bundle \( V \) over \( Z \) (assumed to be fibrewise semistable) decomposes as a direct sum \( \oplus_{i=1}^n \mathcal{L}_{q_i} \) of line bundles \( \mathcal{L}_{q_i} = \mathcal{O}_E(q_i - p) \) of degree zero (\( p \) the zero element); this is described as a set of \( n \) points \( q_i \) which sum to zero. Variation over \( B \) gives a hypersurface \( C \subset Z \) which is a ramified \( n \)-fold cover of \( B \) given as a locus \( w = 0 \) with \( w \) a section of \( \mathcal{O}(\sigma)^n \otimes \mathcal{M} \) (with a line bundle \( \mathcal{M} \) over \( B \) of class \( \eta \in H^{1,1}(B) \))

\[
C = n\sigma_1 + \eta \quad (3.1)
\]

The idea is then to trade in the \( SU(n) \) bundle \( V \) over \( Z \), which is in a sense essentially a datum over \( B \), for a line bundle \( L \) over the \( n \)-fold (ramified) cover \( C \) of \( B \): one has

\[
V = p_*(p_C^*\mathcal{L} \otimes \mathcal{P}) \quad (3.2)
\]

with \( p : Z \times_B C \to Z \) and \( p_C : Z \times_B C \to C \) the projections and \( \mathcal{P} \) the global version of the Poincare line bundle over \( E_I \times E_{II} \) (actually one uses a symmetrized version of this), i.e. the universal bundle which realizes \( E_{II} \) as moduli space of degree zero line bundles over \( E_I \). \( \mathcal{P} = \mathcal{O}(\Delta - \sigma_{I,1} \times_B Z_{II} - Z_I \times_B \sigma_{II,1} - c_1) \) (with \( c_1 \) denoting here \( K_B^{-1} \)) becomes trivial on \( \sigma_{I,1} \times_B Z_{II} \) and \( Z_I \times_B \sigma_{II,1} \) and furthermore \((k = 1,2)\)

\[
\mathcal{P}|_{Z \times_B \sigma_k = Z} = \mathcal{O}(-\delta_k \epsilon) \quad (3.3)
\]

A second parameter in the description of \( V \) is given by a half-integral number \( \lambda \) as the condition \( c_1(V) = \pi_* \left( c_1(L) + \frac{c_1(C) - c_1}{2} \right) = 0 \) gives (with \( \gamma \in \ker \pi_* : H^{1,1}(C) \to H^{1,1}(B) \))

\[
c_1(L) = -\frac{1}{2}(c_1(C) - c_1) + \gamma = \frac{n\sigma_1 + \eta + c_1}{2} + \gamma \quad (3.4)
\]
(for other degrees of freedom cf. [11]) where \( \gamma \) is, in the \( A \)-model, being given by \( (\lambda \in \frac{1}{2}\mathbb{Z}) \)

\[
\gamma_A = \lambda(n\sigma_1 - \eta + nc_1)|_C
\]

(3.5)
as the latter is (in the \( A \)-model) the only generically existent class which projects to zero. In the \( B \)-model one has actually a further possibility (see below).

The most natural requirement (and generically the only one) to assure integrality of

\[
c_1(L) = n\left(\frac{1}{2} + \lambda\right)\sigma + \left(\frac{1}{2} - \lambda\right)\eta + \left(\frac{1}{2} + n\lambda\right)c_1
\]

(3.6)
is\(^5\) \( \lambda \in \frac{1}{2} + \mathbb{Z} \) for \( n \) odd resp. \( \lambda \in \mathbb{Z}, \eta \equiv c_1 \mod 2 \) for \( n \) even (then \( c_2(V) \in H^4(Z, \mathbb{Z}) \)).

genuine \( B \) type bundles

Up to now the influence of choosing a \( B \)-model Calabi-Yau \( Z \) had a rather restricted impact. Essentially the influence of this alteration was restricted to the change in \( c_2(Z) \) whose consequences for the upper bound of \( \eta \) will be discussed below. A much more interesting new freedom arises because one has more divisors (the second section) and so a new option arises to build up a line bundle on the spectral cover from a class \( \delta = \gamma_B \)

\[
\delta : = \mu \epsilon|_C = \mu(\sigma_1 + c_1 - \sigma_2)|_C
\]

(3.7)
(note that \( p_*(\delta) = 0 \); when considered in \( Z \) one would have \( \delta = \mu \epsilon C = \mu \epsilon \eta = \mu (\eta \cdot c_1) (F - X) \) from (2.16)).

One finds for the most general combination \( \gamma = \gamma_A + \delta \) to be used in building up \( L \) that \( \mu \) can be integral with \( \lambda \) restrictions unchanged; \( \mu \) can be half-integral if \( \frac{1}{2}(\sigma_1 - \sigma_2 + c_1)|_C \) belongs to the integral cohomology of \( C \), i.e. if \( C \) can be chosen that way (as \( \epsilon/2 \) was not integral already on \( Z \), i.e. before pulled-back (restricted) to \( C \) by \( j : C \hookrightarrow Z \)).

3.2 The cohomology of the bundles

Grothendieck-Riemann-Roch gives the cohomological data of \( V \) from those of \( L \)

\[
ch(V)Td(Z) = p_* \left( e^{c_1(p_*L \otimes P)} Td(Z \times_B C) \right)
\]

(3.8)
One has from (3.2) and with the relations \( p_*(c_1^2(P)) = -2\eta(\sigma_1 + c_1), p_*(\gamma_A c_1(P)) = 0 \), \( p_*(\delta c_1(P)) = \mu \eta \epsilon, p_*(\gamma_A \delta) = 0 \) by (2.14), \( \pi_*(\gamma_A^2) = -\lambda^2 m\eta(\eta - nc_1) \) and \( \pi_*(\delta^2) = -2\mu^2 \eta c_1 \)

\[5\] More exotic possibilities include \( \lambda = \frac{1}{2n}, \eta \equiv 0 \mod n \) for \( n \) odd or \( \lambda = \frac{1}{4}, \eta = 2c_1 \mod 4 \) for \( n = 4 \) [20].
that
\[
c_2(V) = p_* \left[ \frac{(n\sigma_1 + \eta + c_1)^2}{24} - c_2(Z \times_B C) - c_2(Z) - \frac{1}{2} \left( \gamma + c_1(P) \right)^2 \right]
\]
\[= -\frac{1}{2} p_* \left( c_1^2(P) \right) - p_* \left( \delta c_1(P) \right) - \frac{1}{2} p_* (\delta^2)
\]
\[-\frac{n}{24} \left( (n^2 - 1)c_1^2 + 3\eta(\eta - nc_1) \right) - \eta c_1 - \frac{1}{2} p_* (\gamma_A^2)
\]
\[= \eta \sigma_1 - \mu \eta \epsilon + \mu^2 \eta c_1 - \frac{n^3 - n}{24} c_1^2 + \frac{1}{2} (\lambda^2 - \frac{1}{4}) \eta (\eta - nc_1)
\]
\[= (1 - \mu) \eta \sigma_1 + \mu \eta \sigma_2 - \mu (1 - \mu) \eta c_1 + kF
\] (3.9)

where we have kept separate the classes of the form \( aF \) which contain the parameter \( \mu \).

The generation number

Note that the generation number \( c_3(V)/2 \) is unchanged\(^6\) because a newly occurring term \(-\frac{3}{2} \mu \eta c_1 (\sigma_1 - \sigma_2) \) in \( c_3(V)/2 \) does not contribute (because it vanishes after integration over \( Z \) as both \( \sigma_i \) are sections leading after integration over the fibre to an integral over \( B \) times \( (1 - 1) \)). Concerning chiral matter one finds as number of net generations \([3]\)
\[
\frac{1}{2} c_3(V) = \lambda \eta (\eta - nc_1)
\] (3.10)

Note that in the case of an \( SU(5) \) bundle one has because of the decomposition
\[
248 = (5, 10) \oplus (10, \bar{5}) \oplus (\bar{5}, 10) \oplus (\bar{10}, 5) \oplus (1, 24) \oplus (24, 1)
\] (3.11)
to consider also the \( \Lambda^2 V = 10 \) (unlike in the case of an \( SU(3) \) or \( SU(4) \) bundle) to get the \( \bar{5} \) part of the fermions \( 10 \oplus \bar{5} \); but the \( 10 \) and the \( \bar{5} \) will come in the same number of families by anomaly considerations.

When searching for a 3-generation model 'below', i.e. a 6-generation model 'above', (3.10) restricts one to \( \lambda = \pm 1/2 \) or \( \pm 3/2 \) where one has then to construct on the Calabi-Yau 'above' (before the modding) an \( SU(5) \) model with \( \eta (\eta - 5c_1) = \pm 12 \) or \( \pm 4 \).

Examples, over \( F_2 \), of 6 generations bundles are immediately found from formula (3.10)
\[
\eta = 14b + 22f , \quad \lambda = +3/2 \quad ; \quad \eta = 24b + 30f \quad \lambda = -1/2
\] (3.12)

As the focus in this paper is on the involution-invariance condition we have relegated further discussion of the \( \eta \)-sector to appendix C where further conditions on the \( \eta \) class are described (such as effectivity of the five-brane class occurring in the anomaly cancellation condition).

\(^6\)That \( c_3(V) \) is independent of \( \mu \) is further explained in [4] where also the fact that \( c_2(V) \) is for \( \mu = 0 \) the same as in the \( A \)-model, in principle obvious but not manifest from (3.8), is discussed further.
3.3 Invariance of the bundles under the involution

As conditions on \( \eta \) and \((\mu, \lambda)\) from \( \iota^* V = V \) one finds first \( \underline{\eta}(\eta_1) = \eta_1 \); as in our case \( B = F_0 \) or \( F_2 \) the involution induced on \( H^{1,1}(B) \) is trivial anyway, this does not represent any restriction: \( b \) and \( f \) build a basis of \( H^{1,1}(B) \) and the top \((1,1)\) form of these two \( P_1 \) are given in local coordinates by, say, \( dz d\bar{z} \); but these classes are invariant under the operations (actually \((-1)\)'s) on the \( P_1 \).

Actually also the actual spectral cover surface \( C \) should and can easily be chosen to be invariant under the appropriate induced operation \( \iota' \). As \( C \) sits actually in the dual Calabi-Yau where the shift-symmetry (2.22) is ineffective one has here to use (cf. [15]) a \( \iota' \) where \( sh \) is split off\(^7\).

Finally and most importantly we describe how the operation of the involution on \( V \) is expressed on the level of \( L \). This operation can be discussed separately and at the end as it is independent of the \( \eta \)-sector. The relation (3.9) for the second Chern class

\[
c_2(V) = (1 - \mu) \eta \sigma_1 + \mu \eta \sigma_2 - \mu (1 - \mu) \eta c_1 + kF
\]

suggests that this operation is given not by any non-trivial map on the classes but just by a parameter substitution

\[
V \rightarrow \iota^* V \quad \Leftrightarrow \quad (\mu, \lambda) \rightarrow (1 - \mu, \lambda)
\]

which correctly also leaves invariant the \( aF \) term as \( a \) is invariant under the substitution in \( \mu \). From (3.14) one notices that as this operation, i.e. (1.7) resp. (2.18), acts only in the \( \epsilon \)-sector it keeps the general form of \( c_1(L) \) (which was fixed by \( c_1(V) = 0 \)) in contrast to a possible action \( \sigma_1 \leftrightarrow \sigma_2 \) (in \( L \) !), i.e. again just \( \iota \), which would have been also compatible with (3.9).

Therefore one finds the critical line \( \mu = 1/2 \) as invariance locus. Note that this not only makes the cohomology class \( c_2(V) \) invariant; rather, this being just a parameter substitution, it suggests (3.14), i.e. that the substitution in \( \mu \) already on the level of the (\( V \)-defining) line bundle \( L \) itself corresponds to the involution on the bundle \( V \)

\[
\mathcal{IO}\left(x \sigma_1 + \alpha + \mu \epsilon\right) = \mathcal{O}\left(x \sigma_1 + \alpha + (1 - \mu) \epsilon\right)
\]

(at least when restricted to \( C \) what is the relevant case in the application to \( L \); this relation is checked formally in the appendix A). Note further that \( C \) has to be chosen such that the term \( \frac{1}{2}(\sigma_1 - \sigma_2 + c_1)|_C \) belongs to the integral cohomology of \( C \).

I thank B. Andreas and A. Klemm for discussion.

\(^7\)For \( b_3 = 0 \) this would be \( \iota' : (z_1, z_2; (x, y, z)) \rightarrow (-z_1, -z_2; (-x, y, z)) \); note that then \( \iota = sh \circ \iota' \).
A Appendix: the involution on the bundle data

We identify the action $I$ of the involution directly on the bundle data, specifically on the bundle $L$ on $C$. For this we work on $Z \times_B Z$ rather than $Z \times_B C$ and use the language of the Fourier-Mukai transform $F$ such that $V = p_*(p^*_C L \otimes \mathcal{P}) = F(j_* L)$ where $j : C \to Z$ is the inclusion. As the line bundles $\mathcal{O}(\sigma_i), \mathcal{O}(\eta), \mathcal{O}(c_1)$ (with the notation $\mathcal{O}(c_1) = \mathcal{O}(K_B^{-1})$) from which $L$ is built exist already on $Z$ one has $L = j^* l$ for a line bundle $l$ on $Z$; in the application $l = \mathcal{O}(x\sigma_1 + \alpha + \mu \varepsilon)$ where $c_1(L) = (x\sigma_1 + \alpha + \mu \varepsilon)|_C$. Let us now assume that we have already taken $i'$ $C = C$ (cf. sect. 3.3). Then, taking into account that $I_{j_*} j^* l = j_* j^* I l$, we will identify $I l$ (cf. the analogous case in [15]) where $F I l = i F l$ in general (cf. (1.7))$^8$; more precisely we want to show $I l = l - l_{1-\mu}$, i.e. that $I$ here operates as $I_D$, cf. (2.19), so that $I$ keeps the class of the relevant $l$’s in (1.5). As a locally free sheaf $\mathcal{O}(\alpha)$ coming from the base can be carried through everywhere as a tensor factor we just show (2.19) in the form $(k = 1, 2)$

$$I \mathcal{O}(z\sigma_k) = \mathcal{O}(z\sigma_k + \varepsilon + 2z\delta_{k2}\varepsilon) \quad (A.1)$$

Now note that one has $F \mathcal{G} = Rp_{1*}(p^*_2 \mathcal{G} \otimes \mathcal{P})$ with $^9 \hat{F} = Rp_{1*}(p^*_2 \mathcal{G} \otimes \mathcal{P} \otimes \mathcal{O}(c_1))$ as inverse transform (when also taking into account an exchange of the roles of $Z_1$ and $Z_2$; further a (-1)-shift in the grading is involved here). So $\hat{F} = DFD$ ($D$ is taking the dual sheaf) and we can evaluate $I$ in the form $I = DFD i F$.

Before we do the actual calculation we define inductively (the isomorphism class of) a rank $a$ vector bundle $V_a$ (for $a \geq 1$) by the non-split short exact sequence (SES)

$$0 \longrightarrow \mathcal{O}(ac_1) \longrightarrow V_{a+1} \longrightarrow V_a \longrightarrow 0 \quad (A.2)$$

starting with $V_1 = \mathcal{O}(\sigma_1)$ (and then also $V_0 = \mathcal{O}(\sigma_1^* \mathcal{O}(\sigma_1) = \mathcal{O}(\sigma_1 \mathcal{O}_B(-c_1) = \mathcal{O}(\mathcal{O}_B(-c_1))$ and fulfilling $\pi_* V_a^* = 0$ (i.e. $\mathcal{O}_B$) and $R^1\pi_* V_a^* = \mathcal{O}_B(-ac_1)$. Concerning the uniqueness note that from the Leray spectral sequence of the fibration one has a SES

$$0 \to H^1(B, \pi_* V_a^* \otimes \mathcal{O}_B(ac_1)) \to H^1(Z, V_a^* \otimes \mathcal{O}_Z(ac_1)) \to H^0(B, (R^1\pi_* V_a^*) \otimes \mathcal{O}_B(ac_1)) \to 0$$

and$^{10}$ the group of extensions $\text{Ext}^1(V_a, \mathcal{O}(ac_1)) = H^1(Z, V_a^* \otimes \mathcal{O}(ac_1)) = H^0(B, \mathcal{O}_B) = C$.

Then one finds indeed (we focus on positive $\mu$, i.e. negative $z$; arguments for positive $z$ are analogous) with the auxiliary relations (A.11) and (A.22) presented below that $(k = 1, 2; a > 0$; for $a = 0$ we will find $F^1 \mathcal{O} = V_0)$

---

$^8$Nevertheless we actually prove this only on the $L$-level, i.e. restricted to $C$ what is sufficient.

$^9$In general (cf. (1.7))

$^10$Using $H^1(B, \mathcal{O}_B(ac_1)) \cong H^1(b, \pi_F \mathcal{O}_B(ac_1)) = 0$ with $\pi_F : \mathcal{F}_m \to \mathcal{P}_m^1$, cf. [18] (although $c_1$ is not ample for $\mathcal{F}_2$ so that the Kodaira vanishing theorem does not apply)
\[ F \mathcal{O}(-a\sigma_k) = V_a \otimes \mathcal{O}(-\delta_k \varepsilon)[-1] \quad (A.3) \]
\[ iF \mathcal{O}(-a\sigma_k) = iV_a \otimes \mathcal{O}(-\delta_k \varepsilon)[1] \quad (A.4) \]
\[ D_i F \mathcal{O}(-a\sigma_k) = iV_a^* \otimes \mathcal{O}(\delta_k \varepsilon)[1] \quad (A.5) \]
\[ F D_i F \mathcal{O}(-a\sigma_k) = \mathcal{O}(-\varepsilon + a(2\delta_k \varepsilon + \sigma_k)) \quad (A.6) \]
\[ I \mathcal{O}(-a\sigma_k) = DF D_i F \mathcal{O}(-a\sigma_k) = \mathcal{O}(\varepsilon - 2a\delta_k \varepsilon - a\sigma_k) \quad (A.7) \]

**First auxiliary relation**\(^{11}\)
\[ F \mathcal{O}(\alpha) = \mathcal{O}_{\sigma_1}(\alpha - c_1)[-1] \quad (A.8) \]
\[ F \mathcal{O}_{\sigma_k} = \mathcal{O}(-\delta_k \varepsilon) \quad (A.9) \]

For (A.8) (cf. also eqn. (2.48) of [17]) note that \( F \pi^* \mathcal{O}_B(\alpha) = Rp_1(p_2^*\pi^* \mathcal{O}_B(\alpha) \otimes \mathcal{P}) = \pi^* \mathcal{O}_B(\alpha) \otimes Rp_1 \mathcal{P} \) where the second factor is \( Rp_1 \mathcal{P} = R^1 p_1 \mathcal{P}[-1] \); the latter restricts on its support \( \sigma_1(B) \) to \( \sigma_1^* R^1 p_1 \mathcal{P} = R^1 \pi_* \mathcal{P} \mid_{\sigma_1 \times_B \mathcal{Z}} = R^1 \pi_* \mathcal{O}_Z = (\pi_* \omega_{Z/B})^* = \mathcal{O}(-c_1) \), and the assertion follows (so \( F^1 \mathcal{O} = V_0 \) as mentioned). For (A.9) note \( F \mathcal{O}_{\sigma_k} = Rp_1(p_2^*\sigma_k \mathcal{O}_B \otimes \mathcal{P}) \) and that \( p_2^*\sigma_k \mathcal{O}_B \) has support on \( Z \times_B \sigma_k = Z \) (actually the extension by zero of \( \mathcal{O}_Z = \pi^* \mathcal{O}_B \) there) where (3.3) applies.

**Second auxiliary relation**
\[ F \mathcal{O}(a\sigma_k) = V_a^* \otimes \mathcal{O}(-\delta_k \varepsilon - c_1) \quad (A.10) \]
\[ F \mathcal{O}(-a\sigma_k) = V_a \otimes \mathcal{O}(-\delta_k \varepsilon)[-1] \quad (A.11) \]

For (A.11) one gets for \( a = 1 \) from the SES
\[ 0 \longrightarrow \mathcal{O}(-\sigma_k) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{\sigma_k} \longrightarrow 0 \quad (A.12) \]
the following SES (coming from the LES of (A.12) and using \( F^0 \mathcal{O}_Z = 0, F^1 \mathcal{O}_{\sigma_k} = 0 \))
\[ 0 \longrightarrow \mathcal{O}(-\delta_k \varepsilon) \longrightarrow F^1 \mathcal{O}(-\sigma_k) \longrightarrow \mathcal{O}_{\sigma_1}(-c_1) \longrightarrow 0 \quad (A.13) \]
or, after tensoring with \( \mathcal{O}(-\sigma_1 + \delta_k \varepsilon) \) and noting (2.13), (2.14),
\[ 0 \longrightarrow \mathcal{O}(-\sigma_1) \longrightarrow F^1 \mathcal{O}(-\sigma_k) \otimes \mathcal{O}(-\sigma_1 + \delta_k \varepsilon) \longrightarrow \mathcal{O}_{\sigma_1} \longrightarrow 0 \quad (A.14) \]
identifying the middle term as \( \mathcal{O} \), i.e. giving indeed \( F^1 \mathcal{O}(-\sigma_k) = V_1 \otimes \mathcal{O}(-\delta_k \varepsilon) \). Similarly one finds for the induction step that one gets from the SES
\[ 0 \longrightarrow \mathcal{O}(-(a + 1)\sigma_k) \longrightarrow \mathcal{O}(-a\sigma_k) \longrightarrow \mathcal{O}_{\sigma_k}(ac_1) \longrightarrow 0 \quad (A.15) \]
\(^{11}\)here \( \mathcal{O}_{\sigma_k} \) are actually the sheafs extended by zero \( \sigma_k \mathcal{O}_B \)
and the associated SES (coming from the LES with (A.9))

\[ 0 \rightarrow \mathcal{O}(ac_1 - \delta_{k2}\epsilon) \rightarrow F^1 \mathcal{O}(-(a + 1)\sigma_k) \rightarrow F^1 \mathcal{O}(-a\sigma_k) \rightarrow 0 \quad (A.16) \]

(\text{using the induction hypothesis } F^1 \mathcal{O}(-a\sigma_k) = V_a \otimes \mathcal{O}(-\delta_{k2}\epsilon)) \text{ that}

\[ 0 \rightarrow \mathcal{O}(ac_1) \rightarrow F^1 \mathcal{O}(-(a + 1)\sigma_k) \otimes \mathcal{O}(\delta_{k2}\epsilon) \rightarrow V_a \rightarrow 0 \quad (A.17) \]

identifying indeed the middle term as \( V_{a+1} \). For (A.10) one has in an analogous induction from (A.12) the SES \( 0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(\sigma_k) \rightarrow \mathcal{O}_{\sigma_k}(-c_1) \rightarrow 0 \) and from its LES

\[ 0 \rightarrow F^0 \mathcal{O}(\sigma_k) \rightarrow \mathcal{O}(-\delta_{k2}\epsilon - c_1) \rightarrow \mathcal{O}_{\sigma_1}(-c_1) \rightarrow 0 \quad (A.18) \]

(tensored with \( \mathcal{O}(\delta_{k2}\epsilon + c_1) \)) the \( a = 1 \) case \( F^0 \mathcal{O}(\sigma_k) = \mathcal{O}(-\sigma_1 - \delta_{k2}\epsilon - c_1) \). By the LES of the SES \( 0 \rightarrow \mathcal{O}(a\sigma_k) \rightarrow \mathcal{O}((a + 1)\sigma_k) \rightarrow \mathcal{O}_{\sigma_k}(-c_1 - ac_1) \rightarrow 0 \) one concludes with (A.9).

**Third auxiliary relation**

\[ F^1 \mathcal{O}(\sigma_2 - \sigma_1) = \mathcal{O}_{D_2} \quad (D_k := 2\delta_{k2}\epsilon + \sigma_k) \quad (A.19) \]

with \( D_2 \) an effective divisor of the indicated class. For this consider the SES

\[ 0 \rightarrow \mathcal{O}(\sigma_2 - \sigma_1) \rightarrow \mathcal{O}(\sigma_2) \rightarrow \mathcal{O}_{\sigma_1} \rightarrow 0 \quad (A.20) \]

which gives, from the LES, with \( F^0 \mathcal{O}(\sigma_2 - \sigma_1) = 0 \), \( F^0 \mathcal{O}(\sigma_2) = V^*_1 \otimes \mathcal{O}(-\epsilon - c_1) = \mathcal{O}(\sigma_2 - 2\sigma_1 - 2c_1) = \mathcal{O}(-2\epsilon - \sigma_2) = \mathcal{O}(-D_2) \), \( F^0 \mathcal{O}_{\sigma_1} = \mathcal{O}, F^1 \mathcal{O}(\sigma_2) = 0 \) the SES

\[ 0 \rightarrow \mathcal{O}(-D_2) \rightarrow \mathcal{O} \rightarrow F^1 \mathcal{O}(\sigma_2 - \sigma_1) \rightarrow 0 \quad (A.21) \]

**Fourth auxiliary relation**

\[ X_a := F^1 (iV_a^* \otimes \mathcal{O}(\delta_{k2}\epsilon)) = \mathcal{O}(-\epsilon + a(2\delta_{k2}\epsilon + \sigma_k)) \quad (A.22) \]

The \( a = 1 \) case follows with \( iV_1 = \mathcal{O}(\sigma_2) \) and (A.11) from \( X_1 = F^1 (\mathcal{O}(-\sigma_2 + \delta_{k2}(\sigma_2 - \sigma_1 + c_1)) = \mathcal{O}(\sigma_k - \epsilon + 2\delta_{k2}\epsilon). \) Then one gets from the \( i \)-transform of (A.2) the SES

\[ 0 \rightarrow iV_a^* \otimes \mathcal{O}(\delta_{k2}\epsilon) \rightarrow iV_{a+1}^* \otimes \mathcal{O}(\delta_{k2}\epsilon) \rightarrow \mathcal{O}(\delta_{k2}\epsilon - ac_1) \rightarrow 0 \quad (A.23) \]

and then, from the LES, with (A.8) and (A.19) for \( k = 1 \) and \( 2 \), the SES

\[ 0 \rightarrow X_a \rightarrow X_{a+1} \rightarrow \mathcal{O}_{D_k}((-1)^k c_1 - ac_1) \rightarrow 0 \quad (A.24) \]

After tensoring with \( X_a^{-1} \), one gets a SES from the inductive hypothesis

\[ 0 \rightarrow \mathcal{O} \rightarrow X_{a+1} \otimes X_a^{-1} \rightarrow \mathcal{O}_{D_k}(D_k) \rightarrow 0 \quad (A.25) \]

identifying the middle term as \( \mathcal{O}(D_k) \) and concluding the induction (here we used \( \mathcal{O}_{D_k}((-1)^k c_1 - ac_1 + \epsilon - a(\sigma_k + 2\delta_{k2}\epsilon)) = \mathcal{O}_{D_k}(D_k - a(c_1 + D_k)) \) from \( \epsilon = D_2 - \sigma_1 - c_1 \) and \( \sigma_1|_{D_2} = 0 \) for \( k = 2 \)). Note here that \( (D_2 + c_1)D_2 = 6c_1\epsilon \) vanishes on \( C = n\sigma_1 + \eta. \)

**Remark:** Using restriction to \( C \) we prove (1.7) actually for \( L = l|_C \) what is enough.
Appendix: Relation between A- and B-model

Transitions

We mention the following relation between the global A- and B-models. First one has, as described above, the generic A-model with fibre $\mathbf{P}^2 = \mathbf{P}_{111}$

$$A_{111}^{gen}$$

$$zy^2 + x^3 + \alpha_4xz^2 + \beta_6z^3 = 0 \quad (B.26)$$

This has a specialisation $\alpha_4 = b_4 - a_2^2, \beta_6 = -a_2b_4$ so that one gets (with $w := x - az$ of well-defined $L$-weight)

$$A_{111}^{spec}$$

$$zy^2 + x^3 + (b_4 - a_2^2)z^2x - a_2b_4z^3 =$$

$$zy^2 + (x - a_2z)(x^2 + a_2xz + b_4z^2) = \quad (B.27)$$

$$zy^2 + w\left(w^2 + 3a_2zw + (2a_2^2 + b_4)z^2\right) = 0$$

By this specialisation a second section $\sigma_2$ is enforced at $(w, y, z) = (0, 0, 1)$. From $\Delta = (4b_4 - a_2^2)(b_4 + 2a_2^2)^2$ one sees using Kodaira’s classifications of singular fibers that it acquires an $A_1$ fiber over a divisor $D$ of class $4c_1$, more precisely at $(w, y, z) = (0, 0, 1)$ over $2a_2^2+b_4 = 0$ (cf. last line of (B.27)). If one resolves (by one blow-up) the singularities over $D$ one gets an extremal transition from $Z$ to a model $\hat{Z}$ (considered in type IIA in [7,8]), with the Euler number changed [5] by $\delta = -2e_D = 24c_1^2$, i.e. $e_{\hat{Z}} = -36c_1^2$. This model differs from the B-model over the same base by a birational transformation.

To shed more light on the relation of the A- and B-model we point to the following coordinate transformations. The affine form $y^2 + x^3 + \alpha_4x + \beta_6 = 0$ of $A_{111}^{gen}$ can be completed also in a $\mathbf{P}_{231}(6)$

$$A_{231}^{gen}$$

$$y^2 + x^3 + h_4xz^4 + B_6z^6 = 0 \quad (B.28)$$

Besides the former specialisation $A_4 = b_4 - a_2^2$ and $B_6 = -a_2b_4$ which gives here

$$A_{231}^{spec}$$

$$y^2 + x^3 + (b_4 - a_2^2)z^4 - a_2b_4z^6 = 0 \quad (B.29)$$

one can also consider the specialisation $A_4 = h_4 + 3f_2^2$ and $B_6 = -(1/4g_3^2 + 1/27f_2^3)$, i.e.

$$A_{231}^{spec'}$$

$$y^2 + x^3 + (h_4 + 3f_2^2)z^4 - \left(\frac{1}{4}g_3^2 + \frac{1}{27}f_2^3\right)z^6 = 0 \quad (B.30)$$

which arises also after doing $y \rightarrow y - \frac{1}{2}g_3z^3, x \rightarrow x - \frac{1}{3}f_2z^2$ in

$$y^2 + x^3 + h_4xz^4 + g_3y^3 + f_2x^2z^2 = 0 \quad (B.31)$$
The blow-up \( x_{(2)} \to u_{(2)}, y_{(3)} \to u_{(2)} y_{(1)} \) (with projective weights indicated) of the point \((x_{(2)}, y_{(3)}) = (0, 0)\) in the affine chart \( z \neq 0 \), with \( y_{(1)} \) the new slope-coordinate along the exceptional \( \mathbb{P}^1 \) given (locally and then also globally in the plane) by \( u = 0 \), which replaces the embedding plane \( \mathbb{P}^{231} \) by the new one \( \mathbb{P}^{121} \ni (y, u, z) \), gives the global curve

\[
u^2 + (f_2 z^2 + y^2)u + g_3 y z^3 + h_4 z^4 = 0
\]

resp., after redefining \( u \to u - (f z^2 + y^2)/2 \), the generic \( B \)-model of (2.12)

\[
B_{121}^{\text{gen}}
\]

\[
u^2 - \frac{1}{4} y^4 - \frac{1}{2} f_2 y^2 z^2 + g_3 y z^3 + (h_4 - \frac{1}{4} f_2^2) z^4 = 0
\]

(B.33)

Finally we relate the specialisations \( A_{111}^{\text{spec}} \) (or equivalently \( A_{231}^{\text{spec}} \)) and \( A_{231}^{\text{spec}'} \). For this note that the respective blow-up’s take place for \( A_{111}^{\text{spec}} \) and still so for \( A_{231}^{\text{spec}'} \) at \( (x, y, z) = (a, 0, 1) \) over the curve \( 2a_2^2 + b_4 = 0 \) in \( B \) resp. for \( A_{231}^{\text{spec}'} \) at \( (x, y, z) = (0, 0, 1) \) even in general. Therefore a direct relation arises if \( 2a_2^2 + b \) vanishes not just on a curve in \( B \) but generically; using this further specialisation \( b = -2a_2^2 \) one gets from \( A_{231}^{\text{spec}'} \)

\[
y^2 + x^3 - 3a_2^2 x z^4 + 2a_2^3 z^6 = 0
\]

(B.34)

i.e. essentially just the form of \( A_{231}^{\text{spec}'} \) when that is also further specialised to \( h = g = 0 \) (such that just the identification between \( a_2 \) and \( f_2 \) remains; this corresponds to the doubly specialised \( B \)-model (cf. [4]) with \( b = 0 \) and \( c = d^2 \) in (2.12)).

Alternatively one might also transform (the affine form of)

\[
B_{121}^{\text{gen}}
\]

\[
y^2 + x^4 + A_2 x^2 z^2 + B_3 x z^3 + C_4 z^4 = 0
\]

(B.35)

to the (affine) form of \( A_{231}^{\text{gen}} \) in (B.28) with

\[
A_4 = -\frac{1}{4} \left( \frac{1}{12} (A^2 + 12 C) \right) \quad , \quad B_6 = -\frac{1}{4} \left( \frac{1}{216} A (36 C - A^2) - \frac{1}{16} B^2 \right)
\]

(B.36)

For the special case of \( B_3 = 0 \) one gets the special form \( A_{231}^{\text{spec}'} \) of (B.29) with

\[
a_2 = -\frac{1}{6} A_2 \quad , \quad b_4 = \frac{1}{144} (A_2^2 - 36 C_4)
\]

(B.37)

Specific divisors

As mentioned it is important to understand the geometry of the classes \( D_k = R_\epsilon (\sigma_k) = \sigma_k + 2 \delta_{2k} \epsilon \), e.g. \( D_2 = \sigma_2 + 2 \epsilon \) did not come from a smooth irreducible effective divisor.

Actually, in the (resolved specialised) \( A \)-model (cf. also [14]) \( \sigma_2 \) and \( 2 \epsilon \) can be represented individually by smooth irreducible surfaces for \( 2 \epsilon \) is represented by a surface \( \mathcal{E} \),
rationally ruled by the fibre $\theta$ over the curve $M = \sigma_2 \cdot \mathcal{E}$ in $\sigma_2$ of class $4c_1\sigma_2$

\[ \mathcal{E} \]

\[ D_2 = \sigma_2 + \mathcal{E} \] where \[ \pi_{\mathcal{E}} \downarrow \text{fibre } \theta \]

\[ \sigma_2 \cdot \mathcal{E} = M \]

Concerning the fibration of $\mathcal{E}$ over $M$ by $\theta$ note that the surfaces $\mathcal{E}$ and $4c_1$ cut out in $\sigma_2$ the same curve $M$; if $\mathcal{E} = M$ ” $\times” \theta$ the intersection of $4c_1$ and $\mathcal{E}$ itself is $M^2|_{\sigma_2} = 16c_1^2$ times the fibre but this is now indeed $\theta$ because $\theta = \frac{1}{2c_1}c_1\epsilon = \frac{1}{16c_1}4c_1 \cdot \mathcal{E}$; furthermore $\theta^2|_{\mathcal{E}} = 0$ (that is a fibre in $\mathcal{E}$) as $\theta = \frac{1}{4c_1}c_1 \cdot \mathcal{E}$ gives $\theta^2|_{\mathcal{E}} = \frac{1}{(4c_1)^2}c_1^2 \cdot \mathcal{E} = \frac{1}{16c_1^2}F \cdot \mathcal{E} = 0$; finally $\theta \cdot \mathcal{E} = \frac{1}{2c_1}c_12\epsilon^2 = \frac{1}{2c_1}(\sigma_2 + c_1\epsilon) = -2$ so $\theta$ is just a $\mathbb{P}^1$ (cf. the analogous case of $\theta$-curves in a $K3$). Here $TZ|_{\theta} = T\theta \oplus N\theta \oplus N\mathcal{E}|_{\theta}$ where the middle term is trivial because of the fibration. Note further that the self-intersection number $-c := M^2|_{\mathcal{E}}$ of the base of the fibration is $\sigma_2^2\mathcal{E} = -4c_1^2$. The class $c_1(\mathcal{E}) = 2M + (e_M + e)\theta$ (cf. [18]) is then identified as $c_1(\mathcal{E}) = 2M + (-12c_1^2 + 4c_1^2)\theta = 2M - 8c_1^2\theta$ verifying the adjunction relation $\mathcal{E}(c_1(\mathcal{E})) = 0$ for the irreducible surface $\mathcal{E}$ in the Calabi-Yau $Z$ as $\mathcal{E}|_{\mathcal{E}} = (\sigma_2 = 2c_1)|_{\mathcal{E}} = -2M + 8c_1^2\theta$.

There is a corresponding surface $\iota\mathcal{E}$ rationally ruled by $F - \theta = \iota\theta$ (as $\iota\epsilon = -\epsilon + 2c_1$) over the base curve $\iota M$ in $\sigma_1$. One has $\mathcal{E} \cdot \iota\mathcal{E} = 4\epsilon(-\epsilon + 2c_1) = 4c_1(\sigma_1 + \sigma_2 + c_1) = 4\epsilon_1(2\sigma_2 + \epsilon) = 8(\sigma_2 + c_1^2)\theta$, the proper transform of the curve of singularities in the $A$-model. Fibrewise the affine $A_1$-tree (two $\mathbb{P}^1$ intersecting twice) is built by $\theta$ and $\iota\theta$ as $\theta \cdot \iota\mathcal{E} = \frac{1}{16c_1^2}4c_1\mathcal{E} \cdot \iota\mathcal{E} = F(\sigma_1 + \sigma_2 + c_1) = 2$.

Note that in the $B$-model not only the term $\mathcal{E}$ $(D_2 + c_1)D_2 = 6c_1\epsilon$ vanishes on surfaces of class $n\sigma_1 + \alpha$ but that even $D_2$ itself is, on such a surface, given (as a class) just by its part on the elliptic surface $14\alpha$ as $D_2|_{n\sigma_1 + \alpha} = D_2|_{\alpha}$. What concerns the self-intersection number of $D_2|_{\alpha}$ in $\alpha$ note that only the $\sigma_2$ part of $D_2$ contributes as $(D_2|_{\alpha})^2 = D_2^2\alpha = 4c_1\alpha\epsilon - c_1\alpha\sigma_2 = -c_1\alpha\sigma_2 = \alpha\sigma_2^2 = (\sigma_2|_{\alpha})^2$. Inside $\alpha$ now $D_2$ can be $15$ an irreducible section: one has $(D_2 + c_1)D_2|_{\alpha} = 0$ and its $K_{D_2}|_{\alpha} = K_{\alpha} \cdot D_2|_{\alpha} + (D_2|_{\alpha})^2 = \alpha \cdot D_2 \cdot \alpha - c_1\alpha\sigma_2 = $

---

12 the class $x\sigma_1 + y\sigma_2 + zc_1$ of the blow-up surface $\mathcal{E}$ is identified as $2\epsilon$ from the relations $\mathcal{E}F = 0$, $\mathcal{E}\sigma_1 = 0$ (this constrains also already the base curve in the ansatz) and $\mathcal{E}\theta = -2$ (the last relation from the $A_1$ resolution; for an a posteriori check cf. below)

13 whose non-vanishing implies that $D_2$ (being numerically effective) is not irreducible effective (it would be then a section)

14 which is, for example, just $b \times F$ resp. $K3$ for $\alpha = b = b$ resp. $f$ ($\alpha$ denotes both $\alpha \in H^2(B)$ and $\pi^*\alpha$)

15 Contrast this also with the fact (cf. for example [19]) that, inside the elliptic $K3$ given by $\pi^*f$, the numerical section $D := \sigma_2 + pF$ can not be represented by an irreducible curve or else it would be a section, so rational but one has $D^2f = -c_1\sigma_2|_f + 2p = -2 + 2p \neq -2$ for $p \neq 0$. 

19
\[(\alpha - c_1)\alpha \sigma_2\] equals the the \(K\) of the base curve as \(K_\alpha = K_B \alpha + \alpha^2 = \alpha^2 - c_1 \alpha\) (computed in \(B\); here \(\alpha\) refers to the class in \(H^2(B)\) whereas usually it denotes \(\pi^* \alpha\)).

\section*{C Restrictions on the bundle parameters}

Here we describe conditions on the bundles which appear besides the generation-tuning and the involution-invariance like effectivity of the five-brane class occurring in the anomaly cancellation condition.

\subsection*{C.1 The upper bound on \(\eta\)}

We restrict our attention to the visible sector concerning \(V_1\) (which we simply call \(V\)) and put in the hidden sector \(V_2 = 0\) when embedding the bundle \((V_1, V_2)\) in \(E_8 \times E_8\). Essential restrictions on \(V\) come from bounds on the \(\eta\) class. The upper bound comes from the anomaly cancellation condition \(c_2(Z) = c_2(V_1) + c_2(V_2) + W\) giving the effectiveness restriction \(c_2(V) \leq c_2(Z)\) on the five-brane class \(W = W_B + a_f F\) (as the five-brane must wrap an actual curve). We want to make sure that \(W_B\) is effective and \(a_f\) is non-negative.

For the \(B\)-model the decomposition \((\oplus_i \sigma_i H^2(B)) \oplus H^4(B)\) (with suitable pull-backs understood) gives (with \(\eta_2 = 0\)) for the parts not coming from \(H^4(B)\)

\[
\eta \sigma_1 - \mu \eta (\sigma_1 - \sigma_2) + W_B = 6c_1 \sigma_1 + 6c_1 \sigma_2 \tag{C.39}
\]

\begin{itemize}
\item **The \(\mu = 0\) sector**
\end{itemize}

In the final result for \(c_2(V) = \eta \sigma_1 + \omega\) (where \(\omega \in H^4(B)\), pulled back to \(Z\)) the ‘number’ \(k\) of the model \((1, 2\) for \(A, B)\) cancelled out leaving the \(A\) model result unchanged (cf. previous footn.). On the other hand note that we have a corresponding decomposition \(c_2(Z) = \frac{12}{k} \Sigma c_1 + (c_2 + (\frac{12}{k} - 1)c_1^2)\). This is the term responsible for the upper bound \(\eta \sigma_1 \leq \frac{12}{k} \Sigma c_1\), which when interpreted as the sufficient condition \(\eta \leq \frac{12}{k} c_1\) is thus for \(k = 2\) in the \(B\)-model much sharper than for \(k = 1\) in the \(A\)-model, excluding for example \((x, y) = (14, 22)\) and \((24, 30)\) over \(F_2\).

But actually the situation is slightly better because we established a sufficient condition only. The base part (C.39) of the condition gives for \(\mu = 0\) certainly the sufficient condition \(\eta_1 = xb + yf \leq 6c_1 = 12b + 24f\). But because the curve classes are not all independent there is actually more flexibility. For this let us restrict our attention to the phenomenologically relevant case of the base \(F_2\). Then \(6c_1 \sigma_1 + 6c_1 \sigma_2 = (24b + 24f) \sigma_1 + 24f \sigma_2\) and the condition \(\eta = xb + yf \leq 24b + 24f\) ensues allowing \((x, y) = (14, 22)\).
The $\mu \neq 0$ sector

With $\mu = 1/2$ to get invariant bundles (cf. below) (C.39) amounts to

$$\frac{1}{2} \eta \sigma_1 + \frac{1}{2} \eta \sigma_2 \leq 6c_1 \sigma_1 + 6c_1 \sigma_2$$  \hspace{1cm} (C.40)

Even without exploiting the less restrictive version of the bound as above, one is back to the $A$-model bound $\eta \leq 12c_1$ as sufficient, allowing $(x, y) = (14, 22)$ and $(24, 30)$.

C.2 The lower bound on $\eta$

This is a bound on 'how much instanton number has to be turned on to generate/fill out a certain $SU(n)$ bundle, i.e. to have no greater unbroken gauge group than a certain $G$'.

Recall that in the six-dimensionsional duality [9] between the heterotic string on $K3$ with instanton numbers $(12 \mp m)$ (no five-branes) and $F$-theory on $F_m$ the gauge group is described by the singularities of the $F$-theory fibration; a perturbative heterotic gauge group corresponds to a degeneration over the common base-$\mathbb{P}^1 B_1 = b$ of the heterotic $K3$ resp. the $F_m$, i.e. the discriminant divisor $\Delta = 12c_1(F_m)$ has a component $\delta(G)b$ (with $\delta(G)$ the vanishing order of the discriminant, equivalently the Euler number of the affine resolution tree of the singularity). This gives the relation $m \leq 24 \frac{\delta(G)}{12 - \delta(G)} - \delta(G)b$ for having not a singularity worse than $G$ [13] (as $\Delta' = \Delta - \delta(G)b$ has transversal intersection with $b$ and so $\Delta' \cdot b \geq 0$), i.e. $12 - m \geq 12 - \frac{24}{12 - \delta(G)} = (6 - \frac{12}{12 - \delta(G)})c_1(B_1)$. From this was induced [13] (cf. [12]) the bound in four dimensions

$$\eta_1 \geq (6 - \frac{12}{12 - \delta(G)})c_1$$  \hspace{1cm} (C.41)

(the $(12 \mp m)$ structure generalizes in four dimensions to $\eta_1/2 = 6c_1 \mp t$ from the component part $\eta_1 \sigma + \eta_2 \sigma = 12c_1 \sigma$ of the anomaly cancellation condition $c_2(V_1) + c_2(V_2) + a_f F = c_2(Z)$ for the case of an $A$-model with $W_B = 0$). In our case ($B$-model at the invariant point $\mu = 1/2$) (C.39), including an $\eta_2 \neq 0$, similarly reads $(\eta_1 + \eta_2)\Sigma = 12c_1 \Sigma$ (for the case with $W_B = 0$; here $\Sigma = \sigma_1 + \sigma_2$).

One gets\(^{16}\) the lower bound $\eta \geq \frac{30}{7}c_1$ for an $SU(5)$ unbroken gauge group (with $\delta = 5$).

C.3 Examples

So one has to find an $\eta$ in the strip $\frac{30}{7}c_1 \leq \eta \leq 6c_1$ with $\eta(\eta - 5c_1) = \pm 12$ (for $\lambda = \pm \frac{1}{2}$) resp. $\eta(\eta - 5c_1) = \pm 4$ (for $\lambda = \pm \frac{3}{2}$) with $a_f$ non-negative (we will be considering only

\(^{16}\)As our focus in this paper is on the invariance condition rather than on specific $\eta$-classes we just mention the similar criterion $\eta \geq 5c_1$ (and $\eta b \geq 0$) [16].
where $a_f = c_2 + 10c_1^2$ for $\lambda = \pm \frac{1}{2}$ resp. $a_f = c_2 + 10c_1^2 \mp 20$ for $\lambda = \pm \frac{3}{2}$.

6 generations bundles over $F_2$ are (violating the (naive) upper bound $x \leq 12, y \leq 24$)

$$
\eta = 14b + 22f, \quad \lambda = +3/2 \quad ; \quad \eta = 24b + 30f, \quad \lambda = -1/2 \quad (C.42)
$$

with $W = (5b + 13f)(\sigma_1 + \sigma_2) + 75F$ and $W = 9f(\sigma_1 + \sigma_2) + 99F$, respectively (from $c_2(V_{\mu=1/2}) = \frac{1}{2}\eta(\sigma_1 + \sigma_2) - \frac{1}{2}\eta c_1 - 40 + \text{sign}(\lambda)(|\lambda| - 1/2)\eta$). Note that in both cases $W_B$ is not only effective (the coefficients of $b$ and $f$ are nonnegative) but the class will be represented also by a (smooth) irreducible curve (so that one will not have to worry about intersections of its components) as $W_B \cdot b \geq 0$ (cf. [18]). (But note also $\eta b < 0$.)

References

1. E. Witten, New issues in manifolds of SU(3) holonomy, Nucl. Phys. B268 (1986) 79.

2. R. Friedman, J. Morgan and E. Witten, Vector bundles and F-theory, Comm. Math. Phys. 187 (1997) 679, hep-th/9701162.

3. G. Curio, Chiral matter and transitions in heterotic string models, Phys. Lett. B435 (1998) 39, hep-th/9803224.

4. B. Andreas, G. Curio and A. Klemm, Towards the Standard Model from elliptic Calabi-Yau, Int.J.Mod.Phys. A19 (2004) 1987, hep-th/9903052.

5. A. Klemm, B. Lian, S.-S. Roan and S.-T. Yau, Calabi-Yau fourfolds for M- and F-Theory compactification, Nucl. Phys. B518 (1998) 515, hep-th/9701023.

6. P. Berglund, A. Klemm, P. Mayr and S. Theisen, On Type IIB Vacua With Varying Coupling Constant, Nucl.Phys. B558 (1999) 178, hep-th/9805189.

7. A. Klemm and P. Mayr, Strong Coupling Singularities and Non-abelian Gauge Symmetries in N = 2 String Theory, Nucl.Phys. B469 (1996) 37-50.

8. S. Katz, D. Morrison and R. Plesser Enhanced Gauge Symmetry in Type II String Theory, Nucl.Phys. B477 (1996) 105-140.

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\(^{17}\)cf. also [14]; data which specify a 6 generations bundle over $F_1$ (where no involution was found) and fulfill the bounds mentioned above are given by $\eta = 11b + 15f , \lambda = +3/2$. A 6 generations bundle over $F_0$ with $\eta = 8b + 14f, \lambda = +3/2$ violates the lower bound $x, y \geq 9$ (resp. 10).
9. D. Morrison and C. Vafa, *Compactifications of F-Theory on Calabi–Yau Threefolds – II*, Nucl.Phys. **B476** (1996) 437-469.

10. R. Donagi, A. Lukas, B.A. Ovrut and D. Waldram, *Holomorphic vector bundles and non-perturbative vacua in M-theory*, JHEP **9906** (1999) 034, hep-th/9901009.

11. G. Curio and R. Donagi, Nucl. Phys. **B518** (1998) 603, hep-th/9801057.

12. P. Berglund and P. Mayr, *Heterotic String/F-theory Duality from Mirror Symmetry*, Adv.Theor.Math.Phys. **2** (1999) 1307, hep-th/9811217.

13. G. Rajesh, *Toric Geometry and F-theory/heterotic duality in four dimensions*, hep-th/9811240.

14. R. Donagi, B.A. Ovrut, T. Pantev and D. Waldram, *Standard Models from Heterotic M-theory*, Adv.Theor.Math.Phys. **5** (2002) 93, hep-th/9912208.

15. R. Donagi, B.A. Ovrut, T. Pantev and D. Waldram, *Standard-Model Bundles on Non-Simply Connected Calabi–Yau Threefolds*, JHEP **0108** (2001) 053, hep-th/0008008; *Standard-model bundles*, Adv.Theor.Math.Phys. **5** (2002) 563, math.AG/0008010; *Spectral involutions on rational elliptic surfaces*, Adv.Theor.Math.Phys. **5** (2002) 499, math.AG/0008011.

16. B.A. Ovrut, T. Pantev and J. Park, *Small Instanton Transitions in Heterotic M-Theory*, JHEP **0005** (2000) 045, hep-th/0001133.

17. B. Andreas, G. Curio, D. Hernandez Ruiperez, S.-T. Yau *Fourier-Mukai Transform and Mirror Symmetry for D-Branes on Elliptic Calabi-Yau*, math.AG/0012196; *Fibrewise T-Duality for D-Branes on Elliptic Calabi-Yau*, hep-th/0101129, JHEP **0103** (2001) 020.

18. R. Hartshorne *Algebraic Geometry*, Graduate Texts in Mathematics (1977) Springer-Verlag.

19. B. Andreas and G. Curio, *Horizontal and Vertical Five-Branes in Heterotic/F-Theory Duality*, hep-th/9912025, JHEP **0001** (2000) 013.

20. B. Andreas, *On Vector Bundles and Chiral Matter in N=1 Heterotic Compactifications*, hep-th/9802202, JHEP **9901** (1999) 011.