Local discontinuous Galerkin method for the Backward Feynman-Kac Equation

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Abstract Anomalous diffusions are ubiquitous in nature, whose functional distributions are governed by the backward Feynman-Kac equation. In this paper, the local discontinuous Galerkin (LDG) method is used to solve the 2D backward Feynman-Kac equation in a rectangular domain. The spatial semi-discrete LDG scheme of the equivalent form (obtained by Laplace transform) of the original equation is established. After discussing the properties of the fractional substantial calculus, the stability and optimal convergence rates $O(h^{k+1})$ of the semi-discrete scheme are proved by choosing an appropriate generalized numerical flux. The $L1$ scheme on the graded meshes is used to deal with the weak singularity of the solution near the initial time. Based on the theoretical results of a semi-discrete scheme, we investigate the stability and convergence of the fully discrete scheme, which shows the optimal convergence rates $O(h^{k+1} + \tau^{\min\{2-\alpha, \gamma\delta\}})$. Numerical experiments are carried out to show the efficiency and accuracy of the proposed scheme. In addition, we also verify the effect of the central numerical flux on the convergence rates and the condition number of the coefficient matrix.

Keywords Backward Feynman-Kac equation · Fractional substantial calculus · LDG method · Generalized numerical flux · Graded meshes · $L1$ scheme

Mathematics Subject Classification (2020) 65D15 · 35R11

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1 Introduction

The origin of Feynman-Kac transform can be traced back to Richard Feynman’s research on “path integrals” in the 1940s. Later, Mark Kac realized that the solution of Schrödinger equation (heat equation with external potential term) describing the functional distribution of diffusion motion can be obtained by this transformation [16]. The functional of anomalous diffusion has attracted extensive attention of physicists with the in-depth study of non-Brownian motion and anomalous diffusion. Similar to the functional of Brownian motion, the functional of anomalous diffusion can be defined as

\[ A_t = \int_0^t \kappa(Y_{E_s}) ds, \]

where \( \kappa(x) \) is a bounded function on the state space of the stochastic process \( Y_{E_t} \) with \( E_t \) being the inverse of the driftless subordinator with Lévy measure \( \mu(dx) := -d\omega(x) \) that is independent of the Markov process \( Y_t \).

The distribution of the functional defined by (1) is governed by the 2D backward equation for Feynman-Kac transform with the abstract form [5]

\[ \partial_t \omega, \kappa(x) u(t, x) = \mathcal{L} u(t, x) - \kappa(x) I^{\omega, \kappa(x)} u(t, x) \quad \text{on} \quad (0, T] \times \Omega, \]

where \( \omega(t) \) is an unbounded right continuous decreasing function on \((0, \infty)\) that is integrable on \((0, 1]\) and has \( \lim_{t \to +\infty} \omega(t) = 0 \), \( \mathcal{L} \) is the generator of the stochastic process \( Y_t \), \( \partial_t \omega, \kappa(x) u(t, x) \) denotes the generalized time fractional derivative, defined by

\[ \partial_t^{\omega, \kappa(x)} u(t, x) = \frac{\partial}{\partial t} I^{\omega, \kappa(x)}(u(t, x) - u(0, x)) \]

with \( I^{\omega, \kappa(x)} \) being the generalized time fractional integral

\[ I^{\omega, \kappa(x)} u(t, x) := \int_0^t e^{-\kappa(x)(t-\tau)} \omega(t-\tau) u(\tau, x) d\tau. \]

The boundary condition of (2) is periodic and the initial condition is

\[ u(0, x) = u_0(x), \quad x \in \Omega, \]

where the rectangular region \( \Omega \subset \mathbb{R}^2 \).

The main challenges for analyzing and solving (2) come from the spatiotemporal coupling and nonlocality of the generalized time fractional derivative (3); it is harder to get analytical solution. So, finding an effective numerical method to solve (2) seems to be urgent. Currently, there are many discussions for the numerical algorithms of fractional partial differential equations, such as discontinuous Galerkin method (DG) [11,24,30], finite difference method [10,29], finite element method [9], and spectral method [31], etc. To the best of our knowledge, the DG method for (2) has not been discussed, and the main challenge lies in that \( \kappa(x) \) may be negative. In the paper, this problem has been
overcome by using some special techniques given in Lemma 2 and Lemma 3 (see below).

In 1973, the DG method was used to solve the neutron transport equation by Reed and Hill and it allows the basis functions in each element to be relatively independent and requires information exchange by defining the numerical fluxes at the boundaries of adjacent elements. Since the 1990s, the Runge-Kutta discontinuous Galerkin method proposed by Cockburn and Shu [38,39] has been widely used. The DG method can flexibly deal with many problems that are not easy to deal with by continuous finite element method and it’s conducive to the formation of adaptive mesh [33] and parallel computing. Nowadays, the DG method is widely used to solve the equations in physics, chemistry, biology, and atmosphere science, generally having the property of strong convection. Initially, the DG method was mainly used to solve and analyze first-order problems. Due to the convenience of solving the first-order problem by DG method, Bassi and Rebay transformed the high-order equation into a first-order system by introducing auxiliary variables to solve the Navier-Stokes equation [1], and finally they solved the high-order problem conveniently by using the DG method. This method is called “local discontinuous Galerkin method” (LDG) with the reason that auxiliary variables can be solved in local elements. Later, Cockburn and Shu [8] established the theoretical framework of LDG method.

Initially, the LDG method was used by Cockburn and Shu [7,8] to solve the convection diffusion problem. Deng and Hesthaven [11] establish a theoretical framework for LDG method to solve the spatial fractional diffusion equation in 2013. Since then, the LDG method has been widely used to solve the fractional diffusion equation with integral fractional Laplacian [24], the 2D fractional diffusion equation [25], the fractional telegraph equation [34], the time tempered fractional diffusion equation [30], the fractional convection diffusion equation [37], the fractional Burgers equation [19], and the fractional Allen-Cahn equation [36], etc. This method inherits the flexibility of DG method and it can better retain the physical properties of the model when dealing with models with poor regularity. The combination of the LDG method with the adaptive strategy can better reflect its advantages.

Although the DG method can be used to discretize the time derivatives [22], more often it is used to approximate spatial operators. The existing discussions for the semi-discrete LDG scheme of spatial fractional partial differential equations are usually with the time classical derivatives [11,24,36,37]. Because of the nonlocal property of fractional derivative, the theoretical analysis of LDG method for time fractional partial differential equation (TFPDE) is often directly based on its fully discrete scheme [14,25,34,35]. As far as we know, there are few studies on LDG semi-discrete schemes for TFPDE. Although the properties of continuous Riemann-Liouville time fractional derivatives provide ideas for studying the properties of other types of time fractional derivatives, there are still great challenges when $\kappa(x)$ is negative. To overcome this problem, we will use Fourier transform and Cauchy integral theorem.
If the solution of the equation to be solved is sufficiently regular, the convergence order of $L_1$ scheme of Caputo derivative on uniform meshes can reach $(2-\alpha)$ in theory [18]. Since the solution of Caputo fractional derivative problem has weak singularity near the initial value [20,27], i.e., $|u^{(l)}(t)| \leq C_{u}(1+t^{\alpha-l})$, $l=0,1,2$, the $L_1$ scheme cannot reach the convergence order of $(2-\alpha)$ on uniform meshes. Fortunately, the non-uniform mesh method can better overcome this problem. Stynes et al. [28] proposed the $L_1$ scheme based on graded meshes in order to overcome the weak singularity of the solution at the initial time in the time fractional reaction-diffusion equation. Huang and Stynes [15] proposed a finite element scheme based on graded meshes for the time fractional initial boundary value problem. Li et al. [17] established the finite difference scheme on non-uniform meshes for nonlinear fractional differential equations. In this paper, we also use graded meshes to overcome the weak singularity of the solution of (2) near the initial value.

In the following, we take $\Delta = \Delta$ and $\omega(\tau) = \tau^{1-\alpha} \Gamma(1-\alpha)$, $0 < \alpha < 1$, where $\Delta$ is the Laplacian operator, i.e., the generator of Brownian motion, and $\Gamma(\cdot)$ is the Gamma function. Then Eq. (2) is called the backward Feynman-Kac equation [2,3,10,32]. In fact, according to the definition of the generalized time fractional derivative in (3), $\partial_t^{\omega,\kappa}(x)u(t,x)$ can be rewritten as

$$\partial_t^{\omega,\kappa}(x)u(t,x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t e^{-\kappa(x)(t-\tau)}(t-\tau)^{-\alpha}(u(\tau,x) - u(0,x))d\tau. \quad (6)$$

Then we can further get the equivalent form of Eq. (2) (see the Appendix)

$$\Delta u(t,x) = e^{-\kappa(x)t} D_t^\alpha \left(e^{\kappa(x)t}u(t,x)\right) := C_0 D_t^{\alpha,\kappa}(x)u(t,x), \quad (7)$$

where $C_0 D_t^\alpha u(t,x)$ is the Caputo fractional derivative of order $\alpha \in (0,1)$, defined by

$$C_0 D_t^\alpha u(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \partial_\tau u(\tau,x)d\tau.$$

This paper will focus on the construction and analysis of the numerical scheme for Eq. (7) with periodic boundary and initial conditions (5).

Since $\kappa(x)$ is a bounded function on $\Omega$, naturally the following hold:

(i) There exists a positive constant $C_\kappa$ such that $|\kappa(x)| \leq C_\kappa$, $x \in \Omega$; \quad (8)

(ii) There exist two positive constants $C_{\min}$ and $C_{\max}$ such that

$$C_{\min} \leq e^{-\kappa(x)t} \leq C_{\max}, \quad (t,x) \in [0,T] \times \Omega. \quad (9)$$

The rest of this paper is organized as follows. The properties of fractional substantial calculus are proved in Section 2. In Section 3 we present the spatial
semi-discrete LDG scheme of the equivalent form of the original equation using the
generalized alternating numerical flux for two-dimensional space; and the
$L^2$-stability and a priori error estimate are also proposed in Theorem 1 and
Theorem 2 respectively. In Section 3, the fully discrete scheme is established
by the L1 scheme of fractional substantial derivative on graded mesh. Based
on the theoretical results of semi-discrete scheme, the fully discrete scheme
is theoretically analyzed. Section 5 contains some numerical results. The pa-
per concludes with some discussions in the last section. In the Appendix, we
provide the proof for the equivalent form of the original equation.

2 Notations and some preliminaries

In this section, we introduce the fractional substantial calculuses, analyze their
properties, and also introduce the used function spaces.

2.1 Properties of the time fractional substantial calculus

First, we introduce the time fractional substantial calculus [4,10,33].

**Definition 1** For any $\alpha > 0$, the time fractional substantial integral of the
function $u(t)$ defined on $[0, \infty)$ is given by

$$0\mathcal{I}^{\alpha, \kappa(x)}_t u(t) = e^{-\kappa(x)t}0\mathcal{I}^{\alpha}_t [e^{\kappa(x)t}u(t)],$$

where $\kappa(x)$ is a prescribed function in (1). Here $0\mathcal{I}^{\alpha}_t u(t)$ is the Riemann-
Liouville fractional integral of order $\alpha$, which is defined by

$$0\mathcal{I}^{\alpha}_t u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}u(\tau)d\tau.$$

**Definition 2** For any $\alpha \in (0, 1)$, the time fractional substantial derivative of
the function $u(t)$ defined on $[0, \infty)$ is given by

$$R_0^\alpha D^{\alpha, \kappa(x)}_t u(t) = e^{-\kappa(x)t}R_0^\alpha [e^{\kappa(x)t}u(t)],$$

where $\kappa(x)$ is a prescribed function in (1). Here $R_0^\alpha D^{\alpha}_t u(t)$ is the Riemann-
Liouville fractional derivative of order $\alpha \in (0, 1)$, which is defined by

$$R_0^\alpha D^{\alpha}_t u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha}u(\tau)d\tau.$$

**Lemma 1** For $u(t) \in L^2(\mathbb{R})$ and $\alpha > 0, \kappa(x)$ defined on $\Omega$, it holds that

$$\mathcal{F}[e^{\kappa(x)t}u(t)](\omega) = \tilde{u}(\kappa(x) + i\omega),$$

$$\mathcal{F}[-\infty I^{\alpha, \kappa(x)}_t u(t)](\omega) = (\kappa(x) + i\omega)^{-\alpha} \tilde{u}(\omega).$$
If \( u(t) \in H^\alpha(\mathbb{R}) \) further, then
\[
\mathcal{F}[\int_{-\infty}^{t} D_t^{\alpha,\kappa(x)} u(t)](\omega) = (\kappa(x) + i\omega)^\alpha \tilde{u}(\omega),
\]
where \( i = \sqrt{-1} \) and \( H^\alpha(\mathbb{R}) \) is the fractional Sobolev space with the norm
\[
\|u(t)\|^2_{H^\alpha(\mathbb{R})} := \int_{-\infty}^{\infty} (1 + |\omega|^{2\alpha}) |\tilde{u}(\omega)|^2 d\omega.
\]
Here \( \mathcal{F} \) denotes Fourier transform operator, i.e.,
\[
\tilde{u}(\omega) := \mathcal{F}[u(t)](\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} u(t) dt.
\]

To keep the technical details in the forthcoming numerical analyses at a moderate level, similar to [21, 22, 23], the properties of the time fractional substantial calculus can be first proved. To be convenient, \( u(t) \) and \( v(t) \) are used instead of \( u(t, x) \) and \( v(t, x) \) in this section, respectively. Next, we introduce two Lemmas.

**Lemma 2** Let \( u(t) \) be a piecewise \( C^1 \) function on \([0, T]\) and \( \kappa(x) \) be a given function on \( \Omega \). Then
\[
\int_0^T u(t) \cdot R_0^\alpha \kappa(x) t^{-\alpha} v(t) dt \geq C_\alpha T^{-\alpha} \int_0^T u^2(t) dt, \quad 0 < \alpha < 1,
\]
where \( C_\alpha = \frac{1}{\alpha + 1} \left( \frac{\alpha \pi}{\alpha + 1} \right)^\alpha \cdot \min \{ C_{\max}^{-4+4\alpha}, 1 \} \). Here \( C_{\max} \) is defined in (9).

**Proof** First, we assume that there exists a set \( \Omega_1 \subset \Omega \) such that
\[
\begin{cases}
\kappa(x) \geq 0 & \text{for } x \in \Omega_1, \\
\kappa(x) < 0 & \text{for } x \in \Omega \setminus \Omega_1.
\end{cases}
\]

Let \( v(t) \) be a piecewise \( C^1 \) function on \([0, 1]\). Thus, \( R_0^\alpha (e^{\kappa(x)T} v(t)) \) is continuous except for weak singularities at the breakpoints of \( e^{\kappa(x)T} v(t) \). We extend \( v(t) \) by zero outside the interval \([0, 1]\). Then, one can get
\[
\tilde{v}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iy t} v(t) dt = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-iy t} v(t) dt = \frac{1}{\sqrt{2\pi}} \hat{v}(iy),
\]
where \( \hat{v} \) is the Laplace transform of \( v \) and \( i = \sqrt{-1} \).

By Plancherel’s theorem, the fact that \( \hat{v}(iy) = \hat{v}(-iy) \) (\( v \) is a real-valued function), we find that
\[
\int_0^1 v(t) \cdot R_0^\alpha \kappa(x) T v(t) dt = \int_{-\infty}^{\infty} v(t) \cdot R_0^\alpha \kappa(x) T v(t) dt
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} (iy + \kappa(x) T)^\alpha |\hat{v}(iy)|^2 dy,
\]
(10)
where Lemma \text[1] is used. Here \( \hat{v}(iy) \) represents the complex conjugate of \( \hat{v}(iy) \).

By convention, denote the left hand side of (10) by \( I_1 \). In what follows, we only need to consider two cases for \( \kappa(x) \):

Case I: \( \kappa(x) \geq 0 \) for \( x \in \Omega_1 \). By (10), one can obtain

\[
I_1 = \frac{1}{\pi} \int_0^\infty \text{Re}[\hat{v}(iy + \kappa(x)T)^\alpha] |\hat{v}(iy)|^2 dy \\
= \frac{1}{\pi} \int_0^\infty |iy + \kappa(x)T|^\alpha \cos \theta |\hat{v}(iy)|^2 dy \\
\geq \frac{1}{\pi} \cos \left( \frac{\pi \alpha}{2} \right) \int_0^\infty y^\alpha : |\hat{v}(iy)|^2 dy,
\]

(11)

where \( \theta \leq \frac{\pi \alpha}{2} \) is the argument of a complex number \( (iy + \kappa(x)T)^\alpha \) and \( \text{Re}[z] \) is the real part of a complex number \( z \).

For any \( \varepsilon > 0 \), using the Cauchy-Schwarz inequality, one can get

\[
\int_0^\varepsilon |\hat{v}(iy)|^2 dy = \int_0^\varepsilon \left| \int_0^1 e^{-iyt} v(t) dt \right|^2 dy \leq \varepsilon \int_0^1 v^2(t) dt.
\]

(12)

By Plancherel’s theorem for \( v(t) \), there is

\[
\int_0^1 v^2(t) dt \leq \frac{\varepsilon}{\pi} \int_0^1 v^2(t) dt + \frac{1}{\pi} \int_\varepsilon^\infty |\hat{v}(iy)|^2 dy,
\]

where the property (12) is used in the last step.

For \( 0 < \varepsilon < \pi \), it follows that

\[
\left(1 - \frac{\varepsilon}{\pi}\right) \int_0^1 v^2(t) dt = \frac{1}{\pi} \int_\varepsilon^\infty |\hat{v}(iy)|^2 dy \\
\leq \frac{1}{\pi} \int_\varepsilon^\infty \left( \frac{y}{\varepsilon} \right)^\alpha |\hat{v}(iy)|^2 dy \\
\leq \frac{1}{\pi \varepsilon^\alpha} \int_0^\infty y^\alpha |\hat{v}(iy)|^2 dy.
\]

(13)

Combining (11) and (13) leads to

\[
\int_0^1 v(t) \cdot R_{\alpha,\kappa(x)T} v(t) dt \geq \left( \varepsilon^\alpha - \frac{\varepsilon^{\alpha+1}}{\pi} \right) \cos \left( \frac{\pi \alpha}{2} \right) \int_0^1 v^2(t) dt.
\]

Hence

\[
\int_0^1 v(\tau) \cdot R_{\alpha,\kappa(x)T} v(\tau) d\tau \geq C_{\alpha 1} \int_0^1 v^2(\tau) d\tau,
\]

where \( C_{\alpha 1} = \frac{1}{\alpha+1} \left( \frac{\alpha \pi}{\alpha+1} \right)^\alpha \cos \left( \frac{\pi \alpha}{2} \right) \).

Case II: \( \kappa(x) < 0 \) for \( x \in \Omega \setminus \Omega_1 \). If \( y > \kappa(x)T \cdot \tan \left( \frac{\pi}{2\alpha} \right) \), there exists

\[
\text{Re}[(iy + \kappa(x)T)^\alpha] > 0.
\]
Hence, by (11), it holds that
\[
\lim_{y \to +\infty} Re[(iy + \kappa(x)T)^\alpha] |\hat{v}(iy)|^2 = 0.
\] (14)

In complex plane, we define
\[
\Gamma_i := \{iy + \kappa(x)T| -\infty < y < +\infty\},
\Gamma_2 := \{iy - \kappa(x)T| -\infty < y < +\infty\},
\Gamma_3 := \{i(y_0 + x)| \kappa(x)T \leq x \leq -\kappa(x)T, \ y_0 > 0\},
\Gamma_4 := \{-iy_0 + x| \kappa(x)T \leq x \leq -\kappa(x)T, \ y_0 > 0\},
\Sigma := \{z| \Re{z} \leq -\kappa(x)T\},
\] (15)
where \(\Gamma_i (i = 1, 2, 3, 4)\) are with the directions being the same as the ones of the corresponding coordinate axes.

By using (10), it has
\[
I_1 = \frac{1}{2\pi} \int_{\Gamma_1} z^\alpha \cdot |\hat{v}(z - \kappa(x)T)|^2 dz.
\]
Since \(v(t)\) has a compact support in \(\mathbb{R}\), i.e., \(v(t)\) is zero outside of \([0, 1]\), we know that \(z^\alpha \cdot |\hat{v}(z - \kappa(x)T)|^2\) is analytic in the strip shape area \(\Sigma\). By Cauchy integral theorem, there exists
\[
\int_{\Gamma_1 - \Gamma_2} z^\alpha \cdot |\hat{v}(z - \kappa(x)T)|^2 dz + \lim_{y_0 \to +\infty} \int_{\Gamma_3 - \Gamma_4} z^\alpha \cdot |\hat{v}(z - \kappa(x)T)|^2 dz = 0,
\]
where \(-\Gamma_2\) means the direction is opposite to \(\Gamma_2\). By (14) and (15), we have
\[
\lim_{y_0 \to +\infty} \int_{\Gamma_3 - \Gamma_4} z^\alpha \cdot |\hat{v}(z - \kappa(x)T)|^2 dz = 2 \int_{\Gamma_3 y_0 \to +\infty} \lim_{y_0 \to +\infty} Re[z^\alpha \cdot |\hat{v}(z - \kappa(x)T)|^2 dz = 0.
\]
Thus,
\[
I_1 = \frac{1}{2\pi} \int_{\Gamma_1} z^\alpha \cdot |\hat{v}(z - \kappa(x)T)|^2 dz
\]
\[= \frac{1}{2\pi} \int_{-\infty}^{\infty} (iy - \kappa(x)T)^\alpha \cdot |\hat{v}(iy - 2\kappa(x)T)|^2 dy.
\] (16)

Similar to (11), there exists
\[
I_1 \geq \frac{1}{\pi} \cos \left(\frac{\pi \alpha}{2}\right) \int_{0}^{\infty} y^\alpha \cdot |\hat{v}(iy - 2\kappa(x)T)|^2 dy.
\] (17)

For any \(\varepsilon > 0\), using the Cauchy-Schwarz inequality leads to
\[
\int_{0}^{\varepsilon} |\hat{v}(iy - 2\kappa(x)T)|^2 dy = \int_{0}^{\varepsilon} \left| \int_{0}^{1} e^{-(iy - 2\kappa(x)T)t} v(t) dt \right|^2 dy
\]
\[\leq \varepsilon \int_{0}^{1} v^2(t) dt,
\] (18)
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where (11) is used. Next, invoking Plancherel’s theorem for $e^{2\kappa(x) T} v(t)$ followed by an application of (18), it holds that

$$\frac{1}{C_{\max}^4} \int_0^1 v^2(t) dt \leq \int_0^1 e^{4\kappa(x) T} v^2(t) dt$$

$$= \frac{1}{\pi} \int_0^\infty |\hat{v}(iy - 2\kappa(x) T)|^2 dy$$

$$\leq \frac{1}{\pi} \int_0^1 v^2(t) dt + \frac{1}{\pi} \int_\varepsilon^\infty |\hat{v}(iy - 2\kappa(x) T)|^2 dy.$$ 

For $0 < \varepsilon < \frac{\pi}{C_{\max}^4}$, it follows that

$$\frac{1}{\pi} \int_\varepsilon^1 v^2(t) dt = \frac{1}{\pi} \int_\varepsilon^\infty |\hat{v}(iy - 2\kappa(x) T)|^2 dy$$

$$\leq \frac{1}{\pi \varepsilon^\alpha} \int_0^\infty y^\alpha |\hat{v}(iy - 2\kappa(x) T)|^2 dy. \quad (19)$$

By (17) and (19), we obtain

$$\int_0^1 v(t) \cdot R_0^\alpha,\kappa(x) T v(t) dt \geq \left( \varepsilon^{\alpha} - \varepsilon^{\alpha+1} \right) \frac{C_{\max}^4 - \varepsilon}{2 \pi} \int_0^1 v^2(t) dt.$$ 

Then one can easily see that

$$\int_0^1 v(\tau) \cdot R_0^\alpha,\kappa(\kappa(x) T \tau) v(\tau) d\tau \geq C_{\alpha 2} \int_0^1 v^2(\tau) d\tau,$$

where $C_{\alpha 2} = \frac{1}{(\alpha + 1) C_{\max}^4} \left( \frac{\alpha \pi}{\alpha + 1} \right) \cos \left( \frac{\pi \alpha}{2} \right).$

Combining the results of Case I and Case II lead to

$$\int_0^1 v(\tau) \cdot R_0^\alpha,\kappa(x) T v(\tau) d\tau \geq C_{\alpha} \int_0^1 v^2(\tau) d\tau. \quad (20)$$

Finally, by the scaling argument, we arrive at

$$\int_0^T u(t) \cdot R_0^\alpha,\kappa(x) u(t) dt \geq C_{\alpha} T^{-\alpha} \int_0^T u^2(t) dt,$$

where $u(t) = v(t/T)$. Thus the proof is completed.

Lemma 3 Let $u(t)$ and $v(t)$ be piecewise continuous on $[0, T]$ and $\phi(x)$ be a given function on $\Omega$. Then

$$\left| \int_0^T e^{-\phi(x)t} u(t) \cdot v(t) dt \right|^2 \leq \sec^2 \left( \frac{\pi \alpha}{2} \right) \int_0^T v(t) \cdot R_0^\alpha,\phi(x) v(t) dt$$

$$\cdot \int_0^T e^{2[\phi(x)] + 2\phi(x)t} u(t) \cdot R_0^\alpha,2[\phi(x)] - \phi(x) u(t) dt.$$
Proof First, we assume that there exists a set \( \Omega_1 \subset \Omega \) such that
\[
\begin{cases}
\phi(x) \geq 0 & \text{for } x \in \Omega_1, \\
\phi(x) < 0 & \text{for } x \in \Omega \setminus \Omega_1.
\end{cases}
\]

In what follows, we analyze two cases.

**Case I:** \( \phi(x) \geq 0 \) for \( x \in \Omega_1 \). The similar analysis can be found in (10).

By Plancherel’s theorem, there exists
\[
\int_0^T e^{-\phi(x)t} u(t) \cdot v(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(z_1) \cdot \hat{v}(iy) dy
\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{u}(z_1)| \cdot |\hat{v}(iy)| dy,
\]
where \( z_1 = iy + \phi(x) \).

For any \( \varepsilon > 0 \), using the Cauchy-Schwarz inequality in the bounded interval implies that
\[
\Phi^2(\varepsilon) = \left( \int_{-\varepsilon}^{\varepsilon} |\hat{u}(z_1) z_1^{-\alpha/2} \cdot \hat{v}(iy)| dy \right)^2
\leq \int_{-\varepsilon}^{\varepsilon} |z_1^{-\alpha/2} \cdot \hat{v}(iy)|^2 dy \cdot \int_{-\varepsilon}^{\varepsilon} |\hat{u}(z_1)|^2 dy,
\]
where \( \Phi(\varepsilon) \) is defined by
\[
\Phi(\varepsilon) = \int_{-\varepsilon}^{\varepsilon} |\hat{u}(z_1)| \cdot |\hat{v}(iy)| dy.
\]

Since \( \text{Re}[z_1^\alpha] = |z_1|^\alpha \cos \theta \), we have
\[
\Phi^2(\varepsilon) \leq 4 \int_0^\infty \text{Re}[z_1^\alpha] \cdot |\hat{v}(iy)|^2 dy \int_0^\infty \text{Re}[z_1^{-\alpha}] \cdot |\hat{u}(z_1)|^2 dy
\leq 4 \sec^2 \left( \frac{\pi \alpha}{2} \right) \int_0^\infty \text{Re}[z_1^\alpha] \cdot |\hat{v}(iy)|^2 dy \int_0^\infty \text{Re}[z_1^{-\alpha}] \cdot |\hat{u}(z_1)|^2 dy
= \sec^2 \left( \frac{\pi \alpha}{2} \right) \int_{-\infty}^{\infty} z_1^\alpha \cdot |\hat{v}(iy)|^2 dy \int_{-\infty}^{\infty} z_1^{-\alpha} \cdot |\hat{u}(z_1)|^2 dy,
\]
where \( \theta \leq \frac{\pi \alpha}{2} \) is the argument of a complex number \( z_1^\alpha \) and \( \text{Re}[z_1] \) is the real part of a complex number \( z_1 \). Next, applying Plancherel’s theorem and Lemma [1] we have
\[
\Phi^2(\varepsilon) \leq (2\pi)^2 \sec^2 \left( \frac{\pi \alpha}{2} \right) \int_0^T v(t) \cdot R D_t^{\alpha, \phi(x)} v(t) dt
\cdot \int_0^T e^{-2\phi(x)t} u(t) \cdot 0 I_t^{\alpha, 0} u(t) dt.
\]
As a result of this estimate and (21), we have
\[
\left| \int_0^T e^{-\phi(x) t} u(t) \cdot v(t) dt \right|^2 \leq \sec^2 \left( \frac{\pi \alpha}{2} \right) \int_0^T v(t) \cdot R D^{\alpha,0}_t u(t) v(t) dt \
\cdot \int_0^T e^{-2\phi(x) t} u(t) \cdot I^\alpha,0_0 u(t) dt.
\]

**Case II**: \(\phi(x) < 0\) for \(x \in \Omega \setminus \Omega_1\). By Plancherel’s theorem, there exists
\[
\int_0^T e^{-\phi(x) t} u(t) \cdot v(t) dt = \int_0^T e^{2\phi(x) t} v(t) \cdot e^{-3\phi(x) t} u(t) dt
\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{v}(iy - 2\phi(x))| \cdot |\hat{u}(iy + 3\phi(x))| dy.
\]

Similar to (2.1), for any \(\varepsilon > 0\), we have
\[
\Psi^2(\varepsilon) \leq \sec^2 \left( \frac{\pi \alpha}{2} \right) \int_{-\infty}^{\infty} z_1^\alpha \cdot |\hat{v}(z_2 - \phi(x))|^2 dy \cdot \int_{-\infty}^{\infty} z_2^{-\alpha} \cdot |\hat{u}(iy + 3\phi(x))|^2 dy \\
= \sec^2 \left( \frac{\pi \alpha}{2} \right) \int_{-\infty}^{\infty} z_1^\alpha \cdot |\hat{v}(iy)|^2 dy \cdot \int_{-\infty}^{\infty} z_2^{-\alpha} \cdot |\hat{u}(iy + 3\phi(x))|^2 dy,
\]
where
\[
z_1 = iy + \phi(x), \quad z_2 = iy - \phi(x),
\]
\[
\Psi(\varepsilon) = \int_{-\varepsilon}^{\varepsilon} |\hat{v}(iy - 2\phi(x))| \cdot |\hat{u}(iy + 3\phi(x))| dy,
\]
and similar arguments to prove (10) are used. Further applying Plancherel’s theorem, we have
\[
\Psi^2(\varepsilon) \leq (2\pi)^2 \sec^2 \left( \frac{\pi \alpha}{2} \right) \int_0^T v(t) \cdot R D^{\alpha,0}_t v(t) dt \\
\cdot \int_0^T e^{-3\phi(x) t} u(t) \cdot e^{-3\phi(x) t} I^\alpha,0_0 u(t) dt.
\]

As a result of this estimate and (22), we have
\[
\left| \int_0^T e^{-\phi(x) t} u(t) \cdot v(t) dt \right|^2 \leq \sec^2 \left( \frac{\pi \alpha}{2} \right) \int_0^T v(t) \cdot R D^{\alpha,0}_t v(t) dt \\
\cdot \int_0^T e^{-6\phi(x) t} u(t) \cdot I^\alpha,0_0 u(t) dt.
\]
Thus the proof is completed.
2.2 Notations and function spaces

Now, let’s introduce the symbols to be used later. Let \( \Omega_h = \{ \Omega_{ij} \}_{i=1, \ldots, N} \) denote a tessellation of \( \Omega \) with rectangular elements \( \Omega_{ij} = I_i \times J_j \), where \( I_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \) and \( J_j = (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}) \) with the length \( h_i^x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \) and width \( h_j^y = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}} \). Set \( h_{ij} = \max(h_i^x, h_j^y) \) and denote \( h = \max_{\Omega_{ij} \in \Omega_h} h_{ij} \).

We also assume that \( \Omega_h \) is quasi-uniform in this paper. Moreover, we define

\[
L^2(\Omega) := \{ v : \Omega \to \mathbb{R} \mid v|_{\Omega_{ij}} \in L^2(\Omega_{ij}), \forall \Omega_{ij} \in \Omega_h \}
\]

and the finite element space consisting of piecewise polynomials

\[
V_h^k = \{ v \in L^2(\Omega_h) \mid v|_{\Omega_{ij}} \in Q_h^k(\Omega_{ij}), \forall \Omega_{ij} \in \Omega_h \},
\]

\[
V_h^k = \{(v, w) \mid v, w \in V_h^k\},
\]

where \( Q_h^k(\Omega_{ij}) = P_h(I_i) \times P_h(J_j) \) with \( \otimes \) being the tensor product. Here \( P_h(I_i) \) and \( P_h(J_j) \) denote the set of all polynomials of degrees at most \( k \) on edges \( I_i \) and \( J_j \), respectively.

The broken Sobolev space \( H^s(\Omega_h) \), for any given integer \( s \geq 0 \), is defined as

\[
H^s(\Omega_h) := \{ v : \Omega \to \mathbb{R} \mid v|_{\Omega_{ij}} \in H^s(\Omega_{ij}), \forall \Omega_{ij} \in \Omega_h \},
\]

\[
H^s(\Omega_h) = \{(v, w) \mid v, w \in H^s(\Omega_h)\},
\]

equipped with the broken Sobolev norm

\[
\| v \|_{H^s(\Omega_h)} = \left( \sum_{\Omega_{ij} \in \Omega_h} \| v \|^2_{H^s(\Omega_{ij})} \right)^{1/2},
\]

\[
\| v \|_{H^s(\Omega_h)} = \left( \sum_{\Omega_{ij} \in \Omega_h} \| v \|^2_{H^s(\Omega_{ij})} \right)^{1/2}.
\]

For any \( \Omega_{ij} \in \Omega_h \), we write \((\cdot, \cdot)_{\Omega_{ij}}\) and \(\| \cdot \|_{\Omega_{ij}}\) as the inner product and norm associated with \( L^2(\Omega_{ij}) \). To simplify symbols, summing over all the elements, we denote

\[
(v, r) = \sum_{\Omega_{ij} \in \Omega_h} (v, r)_{\Omega_{ij}},
\]

\[
\| v \| = \left( \sum_{\Omega_{ij} \in \Omega_h} \| v \|^2_{\Omega_{ij}} \right)^{1/2},
\]

\[
(v, r) = \sum_{\Omega_{ij} \in \Omega_h} (v, r)_{\Omega_{ij}},
\]

\[
\| v \| = \left( \sum_{\Omega_{ij} \in \Omega_h} \| v \|^2_{\Omega_{ij}} \right)^{1/2},
\]

\[
\langle v, v \cdot n \rangle = \sum_{\Omega_{ij} \in \Omega_h} \langle v, v \cdot n \rangle_{\partial \Omega_{ij}},
\]

\[
\langle v, v \cdot n \rangle_{\partial \Omega_{ij}} = \int_{\partial \Omega_{ij}} v(s) v(s) \cdot n ds,
\]

where \( n \) is the outward unit normal vector to \( \partial \Omega_{ij} \).

As usual, we refer to the interior information of the element by a superscript “-” and to the exterior information by a superscript “+”. Let \( v_{+,i+1/2,y} \) and \( v_{-,i+1/2,y} \) represent the limit values of the function \( v(x) \) at \((x_{i+1/2}, y)\) and
Let us introduce the auxiliary variable $p$. The proof details of the system in Section 4.

In the section, we present the semi-discrete LDG scheme for Eq. (7). The numerical fluxes for $f$ are defined as single valued functions at the cell interfaces, $u_{i+1/2}$, $v_{i+1/2}$, and $w_{i+1/2}$. Moreover, the weighted averages are denoted by

$$
i_{i+1/2} = \sigma_1 u_{i+1/2} + (1 - \sigma_1) v_{i+1/2},$$

$$v_{i+1/2} = \sigma_2 v_{i+1/2} + (1 - \sigma_2) w_{i+1/2},$$

where $\sigma_1, \sigma_2$ are the weights.

3 The spatial semi-discrete LDG scheme

In the section, we present the semi-discrete LDG scheme for Eq. (7). The proof details of the $L^2$-stability and the optimal convergence results for the semi-discrete scheme are provided. These results are helpful to the numerical analyses in Section 4.

3.1 Variational formulation and numerical scheme

Let us introduce the auxiliary variable $p$, and rewrite Eq. (7) as a first-order system

$$
\begin{align*}
\begin{cases}
C_0 D_t^{p, \alpha}(\mathbf{x}) u(t, \mathbf{x}) - \nabla \cdot \mathbf{p}(t, \mathbf{x}) = 0, & (t, \mathbf{x}) \in (0, T] \times \Omega, \\
\mathbf{p}(t, \mathbf{x}) - \nabla u(t, \mathbf{x}) = 0, & (t, \mathbf{x}) \in (0, T] \times \Omega, \\
u(0, \mathbf{x}) = u_0(\mathbf{x}), & \mathbf{x} \in \Omega.
\end{cases}
\end{align*}
$$

Assume that $(u, p)$ as the exact solution of (23) belongs to

$$H^1(0, T; H^1(\Omega)) \times L^2(0, T; H^1(\Omega)).$$

Taking the inner product of test functions $(v, w)$ and doing integration by part, from (24) we get

$$
\begin{align*}
\begin{cases}
(C_0 D_t^{p, \alpha}(\mathbf{x}) u(t, \mathbf{x}), v)|_{\Omega_{ij}} + (\mathbf{p}(t, \mathbf{x}), \nabla v)|_{\Omega_{ij}} - (\mathbf{p}(t, \mathbf{x}) \cdot \mathbf{n}, v)|_{\partial \Omega_{ij}} = 0, \\
(\mathbf{p}(t, \mathbf{x}), \mathbf{w})|_{\Omega_{ij}} + (u(t, \mathbf{x}), \nabla \cdot \mathbf{w})|_{\Omega_{ij}} - (u(t, \mathbf{x}), \mathbf{w} \cdot \mathbf{n})|_{\partial \Omega_{ij}} = 0,
\end{cases}
\end{align*}
$$

for $\Omega_{ij} \in \Omega_h$, where $(v, w) \in H^1(\Omega_h) \times H^1(\Omega_h)$.

To obtain the spatial semi-discrete scheme for Eq. (7), we present the numerical fluxes $\tilde{u}_h$ and $\tilde{p}_h$ as single valued functions defined at the cell interfaces, in general depending on the values of the numerical solution $u_h$ and $p_h$ from both sides of the interfaces

$$
\begin{align*}
\tilde{u}_h^{x,i+1/2,y} = \tilde{u}_h^{x,i+1/2,y} = \tilde{u}_h^{x,i+1/2,y} = \tilde{u}_h^{x,i+1/2,y}, \\
\tilde{u}_h^{y,i+1/2,x} = \tilde{u}_h^{y,i+1/2,x} = \tilde{u}_h^{y,i+1/2,x} = \tilde{u}_h^{y,i+1/2,x},
\end{align*}
$$

$$
\begin{align*}
\tilde{p}_h^{x,i+1/2,y} = \tilde{p}_h^{x,i+1/2,y} = \tilde{p}_h^{x,i+1/2,y} = \tilde{p}_h^{x,i+1/2,y}.
\end{align*}
$$
where \( p_h = [p_h \quad q_h]^T \).

Now, we construct the spatial semi-discrete LDG scheme of Eq. (7). Find \((u_h(t, x), p_h(t, x)) \in H^1(0, T; V_h^k) \times L^2(0, T; V_h^k)\), which is the approximation of \((u(t, x), p(t, x))\), such that

\[
\begin{cases}
(D_{0}^{n, \alpha}(x) u_h, v)_{\Omega_{ij}} + (p_h, \nabla v)_{\Omega_{ij}} - \langle \widehat{p}_h \cdot n, v \rangle_{\partial \Omega_{ij}} = 0, \\
(p_h, w)_{\Omega_{ij}} + (u_h, \nabla \cdot w)_{\Omega_{ij}} - \langle \widehat{u}_h, w \cdot n \rangle_{\partial \Omega_{ij}} = 0, (26)
\end{cases}
\]

where \((v, w) \in V_h^k \times V_h^k\) for \(\Omega_{ij} \in \Omega_h\).

In the following sections, we choose the generalized alternating numerical fluxes

\[
\begin{aligned}
\hat{u}_h^{i+1/2,y} &= (u_h)^{i+1/2,y}, \\
\hat{p}_h^{i+1/2,y} &= (p_h)^{1-\sigma_1,y}, \\
\hat{u}_h^{i,y+j+1/2} &= (u_h)^{i,y+j+1/2}, \\
\hat{p}_h^{i,y+j+1/2} &= (p_h)^{x,1-\sigma_2},
\end{aligned}
\]

where \(\sigma_1 \neq 1/2, \sigma_2 \neq 1/2, i = 1, \cdots, N_x, j = 1, \cdots, N_y\).

Remark 1 When \(\sigma_1 = \sigma_2 = 1/2\), the generalized alternating numerical fluxes are called central numerical fluxes, which can not guarantee the optimal error estimates of the numerical scheme in theory. The numerical results for central numerical fluxes are given in Section 5.

3.2 Stability analysis of the semi-discrete scheme

To ensure the validity of the numerical scheme, we need to prove the \(L^2\)-stability and convergence of the semi-discrete LDG scheme (26). First, we consider the bilinear form

\[
B(u_h, v; p_h, w) = (u_h, \nabla \cdot w) - \langle \hat{u}_h, w \cdot n \rangle + (p_h, \nabla v) - \langle \hat{p}_h \cdot n, v \rangle. (28)
\]

Lemma 4 Assume that \(u_h\) and \(p_h\) are defined in the rectangular region \(\Omega_h\) with periodic boundary conditions and the numerical fluxes \(\hat{u}_h\) and \(\hat{p}_h\) are given in (27). Then

\[
B(u_h, u_h; p_h, p_h) = 0. (29)
\]

Proof The formula (29) can be directly calculated by substituting (27) into (28) and using the periodic boundary conditions of \(u_h(t, x)\) and \(p_h(t, x)\). Therefore, we omit the details.

Theorem 1 (\(L^2\)-stability) The semi-discrete LDG scheme (26) is unconditionally stable, i.e.,

\[
\int_0^T \|u_h(t, x)\|^2 dt \leq C \|u_h(0, x)\|^2, (30)
\]

where \(C = \sec^2 \left( \frac{\pi \alpha}{2} \right) \cdot T \max\left\{ C_\alpha, C_\alpha F(2 - \alpha) \right\} \). Here \(C_\alpha\) and \(C_\alpha\) are defined in (9) and Lemma (2) respectively.
Proof Setting the test functions \( v = u_h \) and \( w = p_h \), and summing up over all \( u \) yield
\[
\sum_{0}^{C} D_{t}^{\alpha,\kappa}(x) u_h, u_h) + B(u_h, u_h; p_h, p_h) + \|p_h\|^2 = 0,
\]
where the bilinear form \( B(\cdot, \cdot; \cdot, \cdot) \) is defined in \((28)\). Employing Lemma\(4\) leads to
\[
\sum_{0}^{C} D_{t}^{\alpha,\kappa}(x) u_h, u_h) + \|p_h\|^2 = 0. \quad (31)
\]
Since \( \int_{0}^{T} D_{t}^{\alpha,\kappa}(x) u_h, u_h) dt \), one has
\[
\int_{0}^{T} D_{t}^{\alpha,\kappa}(x) u_h, u_h) dt = \int_{0}^{T} \int_{0}^{R} D_{t}^{\alpha,\kappa}(x) u_h, u_h) dt
\]
\[
- \int_{0}^{T} (\omega_{1-\alpha}(t)e^{-\alpha(x)}u_h(0, x), u_h) dt \leq 0,
\]
where \( \omega_{1-\alpha}(t) = t^{-\alpha}/\Gamma(1-\alpha) \). Denote the first and second line of the right hand side of \((32)\) by \( I \) and \( II \), respectively. The analyses to \( I \) and \( II \) rely on Lemma\(2\) and Lemma\(3\). Taking the functions \( v(t, x) = u_h \) and \( u(t, x) = \omega_{1-\alpha}(t)u_h(0, x) \) in Lemma\(3\) results in
\[
\begin{align*}
II & \leq \sec \left( \frac{\pi \alpha}{2} \right) T^{1/2} \cdot \left( \int_{0}^{T} \int_{\Omega} e^{-\alpha(x)}t\omega_{1-\alpha}(t)u_h(0, x) \right)^{1/2} \\
& \leq \sec \left( \frac{\pi \alpha}{2} \right) T^{1/2} \cdot \max \{C_{\max}^{3}, 1\} \left( \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \right)^{1/2} \|u_h(0, x)\| \\
& \leq \frac{1}{2} I + \frac{1}{2} \sec^{2} \left( \frac{\pi \alpha}{2} \right) \cdot \max \{C_{\max}^{6}, 1\} \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \|u_h(0, x)\|^2,
\end{align*}
\]
where \( C_{\kappa}, C_{\max} \) are defined by \((33)\) and the fact that \( \int_{0}^{T} (\omega_{1-\alpha}(t)) = 1 \) is used.

Combining the estimates \((32)\) and \((33)\) leads to
\[
I \leq \sec^{2} \left( \frac{\pi \alpha}{2} \right) \cdot \max \{C_{\max}^{6}, 1\} \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \|u_h(0, x)\|^2.
\]
In addition, taking the function \( u(t, x) = u_h(t, x) \) in Lemma\(2\) there exists
\[
I \geq C_{\alpha} T^{-\alpha} \int_{0}^{T} \|u_h(t, x)\|^2 dt.
\]
As a result of these estimates, one can obtain
\[
\int_{0}^{T} \|u_h(t, x)\|^2 dt \leq C \|u_h(0, x)\|^2,
\]
which finishes the proof of the stability result.
3.3 Error estimates of the semi-discrete scheme

In the following, we will present error estimates for the LDG semi-discrete scheme \([20]\). The optimal convergence results of order \(k+1\) are lost if the projection error at all element boundaries cannot be treated in a nice way. In light of this point, the optimal error estimates are obtained in the forthcoming error analysis by using the approximation properties of the so-called Gauss-Radau projection. The generalized Gauss-Radau projection \([6]\) \(P_{\sigma_1, \sigma_2} : L^2(\Omega) \rightarrow V_h^k\) of the scalar function is defined by

\[
(P_{\sigma_1, \sigma_2} u, v)_{\Omega_j} = (u, v)_{\Omega_j}, \quad \forall v \in Q^{k-1}(\Omega_j),
\]

\[
(P_{\sigma_1, \sigma_2} u, v)_{i+1/2, y} = (u_{i+1/2, y}, v)_{J_j}, \quad \forall v \in P^{k-1}(J_j),
\]

\[
(P_{\sigma_1, \sigma_2} u, v)_{x, j+1/2} = (u_{x, j+1/2}, v)_{I_i}, \quad \forall v \in P^{k-1}(I_i),
\]

\[
(P_{\sigma_1, \sigma_2} u)_{i+1/2, j+1/2} = u_{i+1/2, j+1/2},
\]

for any \(i = 1, \cdots, N_x\) and \(j = 1, \cdots, N_y\). Here and in what follows, we use the notation defined in \([20]\) to represent the weighted average on each edge and use

\[
u_{i+1/2, j+1/2}(\sigma_1, \sigma_2) = \sigma_1 \sigma_2 u(x_{i+1/2}, y_{j+1/2}) + (1-\sigma_1)(1-\sigma_2) u(x_{i+1/2}, y_{j+1/2}) + \sigma_1 (1-\sigma_2) u(x_{i+1/2}, y_{j+1/2}) + (1-\sigma_1) \sigma_2 u(x_{i+1/2}, y_{j+1/2}),
\]

to represent the weighted average at the corner point.

In order to deal with auxiliary variable, we propose the following generalized Gauss-Radau projection \([6]\) \(Q_{\sigma_1, \sigma_2} : L^2(\Omega) \rightarrow V_h^k\) of scalar function

\[
(Q_{\sigma_1, \sigma_2} u, v)_{\Omega_j} = (u, v)_{\Omega_j}, \quad \forall v \in P^{k-1}(I_i) \otimes P^k(J_j),
\]

\[
(Q_{\sigma_1, \sigma_2} u, v)_{i+1/2, y} = (u_{i+1/2, y}, v)_{J_j}, \quad \forall v \in P^k(J_j),
\]

\[
(Q_{\sigma_2} u)_{x, j+1/2} = (u_{x, j+1/2}, v)_{I_i}, \quad \forall v \in P^k(I_i).
\]

for any \(i = 1, \cdots, N_x\) and \(j = 1, \cdots, N_y\). Moreover, the generalized Gauss-Radau projection \(Q_{\sigma_1, \sigma_2} : L^2(\Omega) \rightarrow V_h^k\) of vector function can be defined as \(Q_{\sigma_1, \sigma_2} u = [Q_{\sigma_1} u_1 \ Q_{\sigma_2} u_2]^T\), where \(u = [u_1 \ u_2]^T\). Then, the following three approximation properties hold \([6]\).

**Lemma 5** Assume that \(u(x) \in H^{s+1}(\Omega) \cap H^2(\Omega)\) with \(\Omega \subset \mathbb{R}^2\) and \(s \geq 0\). If \(\sigma_1 \neq 1/2\) and \(\sigma_2 \neq 1/2\), the projections \(P_{\sigma_1, \sigma_2}\) satisfy

\[
\|\eta\| + h^{1/2} \|\eta\|_{L^2(\Gamma_h)} \leq C h^{\min(k+1,s+1)} \|u\|_{H^{s+1}(\Omega)}.
\]

where \(\eta = P_{\sigma_1, \sigma_2} u(x) - u(x)\). The positive constant \(C\) is independent of \(u(x)\) and \(h\). Here \(\Gamma_h\) denotes the set of boundary points of all elements \(\Omega_i \in \Omega_h\).
Lemma 6 Assume that \( u(x) \in H^{s+1}(\Omega_h) \) with \( \Omega_h \subset \mathbb{R}^2 \) and \( s \geq 0 \). If \( \sigma_1 \neq 1/2 \), the projections \( Q_{\sigma_1} \) satisfy

\[
\| \eta \| + h^{1/2} \| \eta \|_{L^2(\Gamma_h)} \leq C h^{\min(k+1,s+1)} \| u \|_{H^{s+1}(\Omega_h)}, \quad l = 1, 2,
\]

where \( \eta = Q_{\sigma_1} u(x) - u(x) \). The positive constant \( C \) is independent of \( u(x) \) and \( h \). Here \( \Gamma_h \) denotes the set of boundary points of all elements \( \Omega_i \in \Omega_h \).

Lemma 7 Assume that \( u(x) \in H^{k+2}(\Omega_h) \) with \( \Omega_h \subset \mathbb{R}^2 \). If \( \sigma_1 \neq 1/2 \) and \( \sigma_2 \neq 1/2 \), the projections \( P_{\sigma_1, \sigma_2} \) satisfy

\[
\| (\eta, \nabla \cdot v) - (\tilde{\eta}, v \cdot n) \| \leq C h^{k+1} \| u \|_{H^{k+2}(\Omega_h)} \| v \|, \quad \forall v \in V_h^k, \quad (35)
\]

where \( \eta = P_{\sigma_1, \sigma_2} u(x) - u(x) \) and \( n \) is the outward unit normal vector to \( \partial \Omega_i \). The positive constant \( C \) is independent of \( u(x) \) and \( h \). The definition of \( \tilde{\eta} \) is the same as that of \( \tilde{u} \) in (27).

Proof Following [6, Lemma 3.6], it is easy to get (35).

With the above preliminary knowledge, we next numerically analyze the convergence of the semi-discrete scheme.

Theorem 2 Let \( u_h \) be the numerical solution of the semi-discrete LDG scheme (26) and \( u(t, x) \in H^1((0, T]; H^{s+1}(\Omega_h) \cap H^2(\Omega_h)) \) the exact solution of Eq. (7) with \( s \geq k \). Then there exists a positive constant \( C \), being independent of \( h \) and \( u_h \), such that

\[
\int_0^T \| u(t, x) - u_h(t, x) \|^2 dt \leq C h^{2k+2}.
\]

Proof Let \( (u, p) \) be the exact solution of Eq. (25) and \( (u_h, p_h) \) the numerical solution of the scheme (26). Summing up (25) and (26) over all \( j \) leads to

\[
\begin{align*}
\left( \frac{C}{0} D_t^{\alpha, \kappa}(x) u_h, v \right) + B(u_h, v; p_h, w) + (p_h, w) &= 0, \quad (36) \\
\left( \frac{C}{0} D_t^{\alpha, \kappa}(x) u, v \right) + B(u, v; p, w) + (p, w) &= 0, \quad (37)
\end{align*}
\]

where the bilinear form \( B(\cdot, \cdot; \cdot, \cdot) \) is defined in (28). Subtracting (36) from (37) gets the error equation

\[
\left( \frac{C}{0} D_t^{\alpha, \kappa}(x) e_u, v \right) + B(e_u, v; e_p, w) + (e_p, w) = 0,
\]

where \( e_u = u - u_h \), \( e_p = p - p_h \).

By convention, let

\[
\begin{align*}
\zeta_u &= P_{\sigma_1, \sigma_2} e_u, \quad \eta_u = P_{\sigma_1, \sigma_2} u - u, \\
\zeta_p &= Q_{1-\sigma_1, 1-\sigma_2} e_p, \quad \eta_p = Q_{1-\sigma_1, 1-\sigma_2} p - p.
\end{align*}
\]
Then the errors can be divided into \( e_u = \zeta_u - \eta_u \) and \( e_p = \zeta_p - \eta_p \), which implies that

\[
\left( \int_0^T D_t^{o,\kappa}(x) \zeta_u, v \right) + \left( \zeta_p, w \right) = \left( \int_0^T D_t^{o,\kappa}(x) \eta_u, v \right) + \left( \eta_p, w \right) + B(\eta_u, v; \eta_p, w) - B(\zeta_u, v; \zeta_p, w).
\]

Setting the test functions \( v = \zeta_u \), \( w = \zeta_p \) in the above equation leads to

\[
\left( \int_0^T D_t^{o,\kappa}(x) \zeta_u, \zeta_u \right) + \left( \zeta_p, \zeta_p \right) = \left( \int_0^T D_t^{o,\kappa}(x) \eta_u, \zeta_u \right) + \left( \eta_p, \zeta_p \right) + B(\eta_u, \zeta_u; \eta_p, \zeta_p) - B(\zeta_u, \zeta_u; \zeta_p, \zeta_p).
\]

Next, we analyze the two bilinear forms in the above equation. By using (27), (34), and Lemma 7, we have

\[
\int T \int (\hat{\eta}_p \cdot n, \zeta_u = 0, \quad \int (\hat{\eta}_u \cdot \nabla \cdot \zeta_p - (\hat{\eta}_u \cdot \zeta_p) n | \leq Ch^{k+1} || \zeta_p ||,
\]

and as a result,

\[
|B(\eta_u, \zeta_u; \eta_p, \zeta_p)| \leq Ch^{k+1} || \zeta_p ||.
\]

From Lemma 4 there exists \( B(\zeta_u, \zeta_u; \zeta_p, \zeta_p) = 0 \). Then, one can get

\[
\left( \int_0^T D_t^{o,\kappa}(x) \zeta_u, \zeta_u \right) + || \zeta_p ||^2 \leq \left( \int_0^T D_t^{o,\kappa}(x) \eta_u, \zeta_u \right) + \left( \eta_p, \zeta_p \right) + Ch^{k+1} || \zeta_p ||. \quad (38)
\]

Now, multiplying \( e^{-2C_o t} \) on both sides of the inequality (38), we can do the estimate further for the right-hand side of (38) by employing the Cauchy-Schwarz inequality

\[
\left( \int_0^T D_t^{o,\kappa}(x) \zeta_u, e^{-2C_o t} \zeta_u \right) \leq \left( \int_0^T D_t^{o,\kappa}(x) \eta_u, e^{-2C_o t} \zeta_u \right) + \frac{1}{2} || \eta_p ||^2 + Ch^{2k+2}. \quad (39)
\]

Similar to (32), there exist

\[
\int_0^T \left( \int_0^T D_t^{o,\kappa}(x) \zeta_u, e^{-2C_o t} \zeta_u \right) dt = \int_0^T \left( \int_0^T D_t^{o,\kappa}(x) \eta_u, e^{-2C_o t} \zeta_u \right) dt
\]

and

\[
\int_0^T \left( \int_0^T D_t^{o,\kappa}(x) \eta_u, e^{-2C_o t} \zeta_u \right) dt = \int_0^T \left( \int_0^T D_t^{o,\kappa}(x) \eta_u, e^{-2C_o t} \zeta_u \right) dt
\]

\[
- \int_0^T \left( \omega_{1-\alpha}(t) \eta_u(0, x), \zeta_u e^{-\alpha(x) 2C_o t} \right) dt,
\]

where \( \zeta_u(0, x) = 0 \) is used.

Denote each line of the right-hand sides of (41) and (11) by III, IV, and V, respectively. The analyses of III–V mainly depend on Lemma 2 and
Lemma 3 Setting functions \( v(t, x) = e^{-C_\alpha t} \zeta_u \), \( u(t, x) = e^{\kappa(x)t}D_t^\alpha D_t^\alpha \eta_u \), and \( \phi(x) = \kappa(x) + C_\alpha > 0 \) in Lemma 3 one can get

\[ IV \leq \sec \left( \frac{\pi \alpha}{2} \right) III^{1/2} \cdot VI \leq \frac{1}{4} III + \sec^2 \left( \frac{\pi \alpha}{2} \right) \cdot VI, \]  

(42)

where

\[ VI = \int_0^T \left( \int_0^t D_t^\alpha \kappa(x) \eta_u, e^{-2C_\alpha t} \eta_u \right) dt. \]

Similarly, taking functions \( v(t, x) = e^{-C_\alpha t} \zeta_u \), \( u(t, x) = \omega_{1-\alpha}(t) \eta_u(0, x) \), and \( \phi(x) = \kappa(x) + C_\alpha > 0 \) in Lemma 3 one can obtain

\[ V \leq \sec \left( \frac{\pi \alpha}{2} \right) III^{1/2} \cdot \left( \int_0^T \omega_{1-\alpha}(t) \cdot \| e^{-(\kappa(x)+C_\alpha)t} \eta_u(0, x) \|^2 dt \right)^{1/2} \]

\[ \leq \frac{1}{4} III + \sec^2 \left( \frac{\pi \alpha}{2} \right) \cdot \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \| \eta_u(0, x) \|^2. \]

(43)

Substituting (40)–(43) into (39) leads to

\[ III \leq 2 \sec^2 \left( \frac{\pi \alpha}{2} \right) \cdot \left( VI + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \| \eta_u(0, x) \|^2 \right) + \int_0^T \| \eta \|^2 dt + Ch^{2k+2}. \]  

(44)

In addition, using the properties of projections in Lemma 3 and Lemma 5 yields

\[ III \leq 2 \sec^2 \left( \frac{\pi \alpha}{2} \right) VI + Ch^{2k+2}. \]

In what follows, we will analyze \( VI \). By integrating \( VI \) by parts with respect to \( t \), one can get

\[ VI = \int_0^T \left( \frac{T - \tau}{T} \right)^{-\alpha} \left( e^{-(\kappa(x)+2C_\alpha)T} \eta_u(T, x), e^{\kappa(x)T} \eta_u(T, x) \right) d\tau \]

\[ - \int_0^T \int_0^\tau \left( \frac{T - \tau}{T} \right)^{-\alpha} \left( e^{\kappa(x)T} \eta_u(T, x), \partial_t \left( e^{-(\kappa(x)+2C_\alpha)T} \eta_u(T, x) \right) \right) \right) d\tau dt. \]

By using the Cauchy-Schwarz inequality, the boundedness of \( \kappa(x) \), and Lemma 5 one can obtain \( VI \leq Ch^{2k+2} \). Hence \( III \leq Ch^{2k+2} \). Moreover, taking function \( u(t, x) = e^{-C_\alpha t} \zeta_u(t, x) \) in Lemma 2 there exists

\[ III \geq C_\alpha T^{-\alpha} \int_0^T \| e^{-C_\alpha t} \zeta_u(t, x) \|^2 dt \]

\[ \geq C_\alpha C_{\min}^2 T^{-\alpha} \int_0^T \| \zeta_u(t, x) \|^2 dt. \]

Thus

\[ \int_0^T \| \zeta_u(t, x) \|^2 dt \leq Ch^{2k+2}. \]

Then the proof can be completed after using the triangle inequality.
4 The fully discrete LDG scheme

In this section, we first introduce the fully discrete scheme for Eq. (7). Then, the stability analysis of the fully discrete scheme is proposed. Based on the obtained error estimates of semi-discrete scheme, we provide the error estimates of the fully discrete scheme.

4.1 Construction of the fully discrete scheme

Let $M$ be a positive integer. Assume that $t_n = (n/M)T$ for $n = 0, 1, \cdots, M$, is the mesh point of the interval $[0, T]$, where the constant mesh grading $\gamma > 1$. In particular, when $\gamma = 1$, graded mesh is uniform one. Set step size $\tau_n = t_n - t_{n-1}$ and the maximum step size $\tau := \max_{1 \leq i \leq M} \tau_i$. An approximation to the time fractional derivative $\frac{D_t^{\alpha,\kappa}}{\partial t^{\alpha,\kappa}} u(t, x)$ in Eq. (7), called the L1 scheme, can be obtained by a simple quadrature formula as

$$
\frac{D_t^{\alpha,\kappa}}{\partial t^{\alpha,\kappa}} u(t_n, x) = \frac{e^{-\kappa(x)t_n}}{\Gamma(1 - \alpha)} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{\partial}{\partial s} [e^{\kappa(x)s} u(s, x)] \cdot \frac{ds}{(t_n - s)^{\alpha}}
$$

$$
= \frac{e^{-\kappa(x)\tau_n}}{\Gamma(1 - \alpha)} \sum_{i=1}^{n} \frac{e^{\kappa(x)\tau_i} u(t_i, x) - e^{\kappa(x)\tau_{i-1}} u(t_{i-1}, x)}{\tau} \cdot \int_{t_{i-1}}^{t_i} \frac{ds}{(t_n - s)^{\alpha}} + \mathcal{T}^n(x)
$$

$$
= \frac{e^{-\kappa(x)\tau_n}}{\Gamma(2 - \alpha)} \sum_{i=1}^{n} \frac{e^{\kappa(x)\tau_i} u(t_i, x) - e^{\kappa(x)\tau_{i-1}} u(t_{i-1}, x)}{\tau} \cdot [(t_n - t_{i-1})^{1-\alpha} - (t_n - t_i)^{1-\alpha}] + \mathcal{T}^n(x),
$$

$$
= C_0 \frac{D_t^{\alpha,\kappa}}{\partial t^{\alpha,\kappa}} u(t_n, x) + \mathcal{T}^n(x)
$$

where $\mathcal{T}^n(x)$ is temporal truncation error of the L1 scheme of $\frac{D_t^{\alpha,\kappa}}{\partial t^{\alpha,\kappa}} u(t, x)$ at $t = t_n$. By simple calculations, one can get

$$
\frac{D_t^{\alpha,\kappa}}{\partial t^{\alpha,\kappa}} u(t_n, x) := A_n^0 u(t_n, x) - A_{n-1}^0 e^{-\kappa(x)\tau_n} u(t_0, x)
$$

$$
= A_n^0 u(t_n, x) - A_{n-1}^0 e^{-\kappa(x)\tau_n} u(t_0, x) + \sum_{i=1}^{n-1} (A_i^0 - A_{i-1}^0) e^{-\kappa(x)(t_n - t_{i-1})} u(t_{n-i}, x),
$$

where

$$
A_n^0 = \frac{(t_n - t_{n-1})^{1-\alpha} - (t_n - t_{n-i+1})^{1-\alpha}}{\Gamma(2 - \alpha)\tau_{n-i+1}}, \quad i = 1, 2, \cdots, n.
$$

Using the mean value theorem, one can easily prove that

$$
A_n^0 \geq A_n^1 \geq \cdots \geq A_n^n > 0, \quad 1 \leq n \leq M.
$$
Based on the spatial semi-discrete LDG scheme (29) and (46), one can obtain

\[
\begin{align*}
&\left\{\begin{array}{l}
(C D_{D}^{\alpha,\kappa}(x) u(t_n, x), v)_{\Omega_{ij}} = -(T^{n}_{w}(x), v)_{\Omega_{ij}} - (p_{h}(t_n, x), \nabla v)_{\Omega_{ij}} \\
(p_{h}(t_n, x), w)_{\Omega_{ij}} + (u_{h}(t_n, x), \nabla \cdot w)_{\Omega_{ij}} - (\hat{u}_{h}(t_n, x), w \cdot n)_{\partial \Omega_{ij}} = 0,
\end{array}\right.
\end{align*}
\]

where (v, w) ∈ \(V_{h}^{k} \times V_{h}^{k}\) for \(\Omega_{ij} \subset \Omega_{h}\). Denote the approximation of \((u(t_n, x), p(t_n, x))\) by \((u_{h}^{n}, p_{h}^{n})\) ∈ \(V_{h}^{k} \times V_{h}^{k}\). With the semi-discrete scheme (29), there exists the fully discrete LDG scheme:

\[
\begin{align*}
&\left\{\begin{array}{l}
(C D_{D}^{\alpha,\kappa}(x) u_{h}^{n}, v)_{\Omega_{ij}} + (p_{h}^{n}, \nabla v)_{\Omega_{ij}} - (\hat{p}_{h}^{n} \cdot n, v)_{\partial \Omega_{ij}} = 0, \\
(p_{h}^{n}, w)_{\Omega_{ij}} + (u_{h}^{n}, \nabla \cdot w)_{\Omega_{ij}} - (\hat{u}_{h}^{n}, w \cdot n)_{\partial \Omega_{ij}} = 0, \\
(u_{h}^{0}, v)_{\Omega_{ij}} = (u_{h}(0), v)_{\Omega_{ij}},
\end{array}\right.
\end{align*}
\]

where (v, w) ∈ \(V_{h}^{k} \times V_{h}^{k}\) for \(\Omega_{ij} \subset \Omega_{h}\), and \(\hat{u}_{h}^{n}\) and \(\hat{p}_{h}^{n}\) are defined in (27).

### 4.2 Stability analysis of the fully discrete scheme

Similar to the previous numerical analysis of the spatial semi-discrete scheme, we will analyze the effectiveness of the fully discrete scheme (50) in theory. We prove that the fully discrete scheme is stable, which is the key to ensure that the numerical implementation can be effectively performed.

**Theorem 3** \((L^{2}\text{-stability})\) For periodic boundary condition, the fully discrete LDG scheme (50) is unconditionally stable. There exists a positive constant \(C_{\max}\) depending on \(\kappa(x)\), such that the numerical solution \(u_{h}^{n}\) satisfies

\[
\|u_{h}^{n}\| \leq C_{\max} \|u_{h}^{0}\|, \quad n \geq 1,
\]

where \(C_{\max}\) is defined by (4).

**Proof** Taking test functions \(v = u_{h}^{n}, w = p_{h}^{n}\), and summing all terms of (50), we have

\[
-B(u_{h}^{n}, u_{h}^{n}; p_{h}^{n}, p_{h}^{n}) = (C D_{D}^{\alpha,\kappa}(x) u_{h}^{n}, u_{h}^{n})_{\Omega} + \|p_{h}^{n}\|^{2},
\]

where \(B(u_{h}^{n}, u_{h}^{n}; p_{h}^{n}, p_{h}^{n})\) is defined by (25). By using Lemma 4, one can get

\[
B(u_{h}^{n}, u_{h}^{n}; p_{h}^{n}, p_{h}^{n}) = 0.
\]

After a simple calculation, there exists

\[
A_{n}^{0}\|u_{h}^{0}\|^{2} \leq \sum_{i=1}^{n-1} (A_{n-1}^{0} - A_{n}^{0})(e^{-\kappa(x)(t_{n}-t_{n-i})} u_{h}^{n-i}, u_{h}^{n-i}) + A_{n-1}^{0}(e^{-\kappa(x)t_{n}} u_{h}^{0}, u_{h}^{0})
\]

\[
\leq \sum_{i=1}^{n-1} (A_{n-1}^{0} - A_{n}^{0})\|e^{-\kappa(x)(t_{n}-t_{n-i})} u_{h}^{n-i}\| \cdot \|u_{h}^{n-i}\|
\]

\[
+ A_{n-1}^{0}\|e^{-\kappa(x)t_{n}} u_{h}^{0}\| \cdot \|u_{h}^{0}\|,
\]

(53)
where (40), (42), (44) and (45) are used.

Next, we use mathematical induction to prove the stability of the numerical scheme. For $n = 1$, there is only the second term on the right side of (53). By using (8), one can get
\[ \|u_h^1\| \leq e^{C_\alpha t_1} \|u_h^0\|. \]

Now, we suppose the following inequalities hold
\[ \|u_h^m\| \leq e^{C_\alpha t_m} \|u_h^0\|, \quad m = 1, \ldots, n - 1. \] (54)

For $m = n$, plugging (54) into (53) leads to
\[
A_n^0 \|u_h^n\| \leq \sum_{i=1}^{n-1} (A_{i-1}^n - A_i^n) e^{C_\alpha (t_n - t_{n-1})} \|u_h^{n-i}\| + A_{n-1}^n e^{C_\alpha t_n} \|u_h^0\|
\]
\[
\leq \sum_{i=1}^{n-1} (A_{i-1}^n - A_i^n) e^{C_\alpha (t_n - t_{n-1})} e^{C_\alpha t_{n-i}} \|u_h^0\| + A_{n-1}^n e^{C_\alpha t_n} \|u_h^0\|
\]
\[
= A_n^0 e^{C_\alpha t_n} \|u_h^0\|. \]

Thus,
\[ \|u_h^n\| \leq e^{C_\alpha t_n} \|u_h^0\| \leq C_{\text{max}} \|u_h^0\|, \]
where $C_{\text{max}}$ is defined by (9). This finishes the proof of the stability result.

4.3 Error estimates of the fully discrete scheme

Now, we consider the numerical convergence of the fully discrete scheme (50). The coefficient (47) is used to define a convolutional coefficient recursively [26].

For $n \geq 1$,
\[
P_n^0 = \frac{1}{A_0^n}, \quad P_{n-j}^n = \frac{1}{A_0^n} \sum_{i=j+1}^{n} (A_{i-j}^n - A_{i-j}^i) P_{n-i}^n, \quad 1 \leq j \leq n - 1. \] (55)

The discrete coefficient $P_{n-j}^n$ is defined to simulate the convolution kernel of the Riemann-Liouville fractional integral of order $\alpha$. According to (55), it is not difficult to find that the discrete convolution kernel $P_{n-j}^n$ satisfies the following properties
\[
\sum_{j=m}^{n} P_{n-j}^j A_{j-m}^j \equiv 1, \quad 1 \leq m \leq n \leq M. \] (56)

In fact, one can also derive (55) based on (54) and both of them can be used as the definition of discrete convolution kernel.

**Lemma 8** Let $P_{n-j}^n$ and $A_{i-j}^i$ be defined by (55) and (47). Then
\[
P_0^n \sum_{i=1}^{n-j} (A_{i-1}^n - A_i^n) P_{n-i}^{n-j} = P_{n-j}^n, \quad i, j = 1, 2, \ldots, n - 1. \] (57)
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Proof It is straightforward to evaluate. Denoting the left-hand side of (57) by \( L \), for \( 1 \leq m \leq n - 1 \), one can obtain

\[
\sum_{j=m}^{n-1} L \cdot A_j^m = \sum_{j=m}^{n-1} \left[ P^m_0 \cdot \sum_{i=1}^{n-j} (A_{i-1}^n - A^n_i) P^{n-i}_{n-i-j} \right] \cdot A_j^m
\]

\[
= \sum_{i=1}^{n-m} P^m_0 \cdot \left[ \sum_{j=m}^{n-i} P^{n-i}_{n-i-j} A_j^m \right] \cdot (A_i^n - A^n_i)
\]

\[
= \sum_{i=1}^{n-m} P^m_0 \cdot (A_i^n - A^n_i)
\]

\[
= 1 - P^m_0 \cdot A_{n-m}^n.
\]

On the other hand, according to (56), there exists

\[
\sum_{j=m}^{n-1} P^n_{n-j} \cdot A_j^m = 1 - P^n_0 \cdot A_{n-m}^n,
\]

which concludes the proof.

Next Lemma suggests that the convergence order of global truncation error of L1 scheme is closely related to the regularity of the solution; even using small mesh parameter \( \gamma \) the optimal time accuracy of order \( O(\tau^{2-\alpha}) \) can be achieved under the condition that the regularity of the solution is well. This result can be clearly observed in Table 7 of Section 5.

Lemma 9 For \( u \in C^2_\delta((0, T]; H^2(\Omega_h)) \), denoting \( \Upsilon^n(x) \) as the temporal truncation error of the L1 scheme of \( \hat{C}_0^{\alpha,\kappa}(x,t) u(t, x) \) at \( t = t_n \), there exists a constant \( C > 0 \), such that

\[
\sum_{j=1}^{n} P^n_{n-j} \| \Upsilon^n(x) \| \leq \frac{C}{\delta(1-\alpha)} \tau^{\min(2-\alpha, \gamma \delta)}, \quad 1 \leq n \leq M,
\]

(58)

where \( C^2_\delta((0, T]) = \{ u(t) \mid u \in C^2((0, T]), |u^{(l)}| \leq C_n(1 + t^{\delta-l}), \; l = 1, 2, \; 0 < t \leq T \} \) and the regularity parameter \( \delta \in (0, 1) \cup (1, 2) \).

Proof The estimate (58) can be obtained by combining (9) and the proof process of [26, Lemma 3.2].

We will use the convergence results of semi-discrete scheme obtained in the previous section to prove the convergence of fully discrete scheme (50). Next, we only discuss the relationship between the solution of the spatial semi-discrete scheme and the solution of the fully discrete scheme, that is, the error estimates of temporal discretization.
Theorem 4 Let \( u_h \in C^2([0, T]; V_h^k) \) and \( u_h^n \in V_h^k \) be the numerical solutions of the spatial semi-discrete scheme (26) and the fully discrete scheme (50), respectively. Then the following estimate holds

\[
\|u_h(t_n, \mathbf{x}) - u_h^n(\mathbf{x})\| \leq C\tau^{\text{min}(\alpha, \gamma)}, \quad n = 1, \ldots, M,
\]

where \( C \) is a constant independent of \( \tau \).

Proof Suppose that \((u_h, p_h)\) and \((u_h^n, p_h^n)\) are the numerical solutions of the scheme (25) and (50), respectively. Then \((u_h(t_n, \mathbf{x}), p_h(t_n, \mathbf{x}))\) satisfies Eq. (49).

Combining (50) with (49) implies that

\[
-B(e_u^n; v; e_p^n, w) = (\mathcal{C}D_M^{\alpha, \kappa}(x)e_u^n, v) + (e_p^n, w) + (\mathcal{T}^n(\mathbf{x}), v),
\]

where \( e_u^n = u_h(t_n, \mathbf{x}) - u_h^n \), \( e_p^n = p_h(t_n, \mathbf{x}) - p_h^n \).

Taking the test functions \( v = e_u^n \in V_h^k \), \( w = e_p^n \in V_h^k \) in (59) leads to

\[
(\mathcal{C}D_M^{\alpha, \kappa}(x)e_u^n, e_p^n) + \|e_p^n\|^2 = -(\mathcal{T}^n(\mathbf{x}), e_u^n),
\]

where we use the fact that \( B(e_u^n; e_u^n, e_p^n, e_p^n) = 0 \). By using (46) and the fact that \( e_u^0 = 0 \), one can obtain

\[
A_0^n \|e_u^n\|^2 \leq \sum_{i=1}^{n-1} (A_i^n - A_{i-1}^n) \left( e^{-\kappa(x)(t_n-t_{n-i})}e_u^{n-i}, e_u^n \right) + \|\mathcal{T}^n(\mathbf{x})\| \cdot \|e_u^n\|.
\]

Further, there exists

\[
A_0^n \|e_u^n\| \leq \sum_{i=1}^{n-1} (A_i^n - A_{i-1}^n) e^{C_\kappa(t_n-t_{n-i})} \|e_u^{n-i}\| + \|\mathcal{T}^n(\mathbf{x})\|,
\]

where \( C_\kappa \) is defined by (5).

Let \( \theta^{n-i} = e^{C_\kappa(t_n-t_{n-i})} \|e_u^{n-i}\| \). Then \( \theta^n = \|e_u^n\| \) and one can get

\[
A_0^n \theta^n \leq \sum_{i=1}^{n-1} (A_i^n - A_{i-1}^n) \theta^{n-i} + \|\mathcal{T}^n(\mathbf{x})\|.
\]

Next, we prove \( \theta^n \leq \sum_{j=1}^n P^{n-j}_{m-j} \cdot \|\mathcal{T}^j(\mathbf{x})\| \) by mathematical induction. For \( n = 1 \), there is only the second term on the right side of (60). By using (55), one can get \( \theta^1 \leq P^0_0 \cdot \|\mathcal{T}^1(\mathbf{x})\| \). Now, we suppose

\[
\theta^m \leq \sum_{j=1}^m P^m_{m-j} \cdot \|\mathcal{T}^j(\mathbf{x})\|, \quad m = 1, \ldots, n - 1.
\]
For \( m = n \), plugging (51) into (60) leads to
\[
A^n_0 \theta^n \leq \sum_{i=1}^{n-1} (A^n_{i-1} - A^n_i) \sum_{j=1}^{n-i} P^{n-i}_{n-i-j} \cdot \|T^j(x)\| + \|T^n(x)\|
\]
\[
= \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} (A^n_{i-1} - A^n_i) \cdot P^{n-i}_{n-i-j} \cdot \|T^j(x)\| + \|T^n(x)\|
\]

According to (55) and Lemma 8, there exists
\[
\theta^n \leq \sum_{j=1}^{n} P^n_{n-j} \cdot \|T^j(x)\|, \quad n = 1, \ldots, M.
\]

Finally, by using Lemma 9 it holds
\[
\|e^n_i\| \leq C_{\min}^{\tau \min(2-\alpha, \gamma \delta)}, \quad n = 1, \ldots, M,
\]
which completes the proof.

**Theorem 5** Let \( u(t, x) \in C^2((0, T); H^{s+1}(\Omega_h) \cap H^2(\Omega_h)) \) and \( u^n_i \) be the exact solution of Eq. (7) and the numerical solution of the fully discrete scheme (60), respectively. Then for \( s \geq k \), the following estimate holds
\[
\sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|H_{1,n}u(t, x) - \bar{u}_h(t, x)\| dt \leq C(h^{k+1} + \tau^{\min(2-\alpha, \gamma \delta)}), \quad (62)
\]

where \( C > 0 \) is a constant independent of \( \tau, h, \) and \( H_{1,n}u(t, x) \) and \( \bar{u}_h(t, x) \) are linear interpolation functions on the time interval \((t_{n-1}, t_n)\) with \( H_{1,n}u(t, x) = u(t, x), \bar{u}_h(t, x) = u^n_i, i = n-1, n.\)

**Proof** Let \( u_h \) be the numerical solution of the spatial semi-discrete scheme (20) and \( H_{1,n}u_h(t, x) \) be linear interpolation function of \( u_h(t, x) \) on the time interval \((t_{n-1}, t_n)\). By using Theorem 2 and the Cauchy-Schwarz inequality, one can get
\[
\sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|H_{1,n}(u(t, x) - u_h(t, x))\| dt \leq \int_{0}^{T} \|u(t, x) - u_h(t, x)\| dt + C_3 \tau^2
\]
\[
\leq C(h^{k+1} + \tau^2).
\]

According to Theorem 4 there exists
\[
\sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|H_{1,n}u_h(t, x) - \bar{u}_h(t, x)\| dt \leq T \cdot \max_{0 \leq n \leq M} \|u_h(t_n, x) - u^n_i(x)\|
\]
\[
\leq C_{\tau \min}^{\tau \min(2-\alpha, \gamma \delta)}.
\]

Finally, the proof is completed by using the triangle inequality.
5 Numerical experiments

To justify the theoretical results, we perform extensive numerical experiments. We define the $L^2$ error between the numerical solution $u_h^n$ of the fully discrete scheme (50) and the exact solution $u(t_n, x)$ of Eq. (7)

$$E(h, M) = \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \| \Pi_{1,n} u(t, x) - \bar{u}_h(t, x) \| dt,$$

where $\Pi_{1,n} u$ and $\bar{u}_h$ are defined by Theorem 5.

In the following examples, the $L^2$ errors and the spatial convergence orders of piecewise $Q^0$, $Q^1$, $Q^2$ polynomials to approximate the exact solution are calculated, respectively. In order to verify the effectiveness of the generalized numerical fluxes, the fully discrete LDG scheme (50) with the generalized numerical fluxes with different parameters are used.

In particular, the generalized numerical flux is called the central numerical fluxes when $\sigma_1 = 1/2$, $\sigma_2 = 1/2$. According to the previous theoretical analyses, for the central numerical fluxes the optimal spatial convergence orders cannot be obtained. However, we still numerically calculate the $L^2$ errors and the spatial convergence rates of the central numerical fluxes in this section and the numerical results show the effect of the fluxes on the spatial convergence rates. In addition, the influence of the change of the generalized numerical fluxes on the condition number of coefficient matrix is also observed.

**Example 1** As the first example, we consider the homogeneous Feynman-Kac backward equation

$$\frac{\partial^{\alpha-2}}{\partial t^\alpha} u(t, x) = \Delta u(t, x), \quad (t, x) \in (0, T] \times \Omega,$$

where $\Omega = [0, 1] \times [0, 1]$. The boundary condition is periodic and the initial value condition is

$$u_0(x) = \sin(2\pi y) \cos(2\pi x).$$

The exact solution of Eq. (63) with the above initial boundary value conditions is

$$u(t, x) = e^{2t} E_\alpha(-8\pi^2 t^\alpha) \cos(2\pi x) \sin(2\pi y),$$

where $E_\alpha(t) = \sum_{l=0}^{\infty} \frac{t^l}{l!(\alpha + 1)}$, called Mittag-Leffler function.

In order to weaken the influence of temporal errors, we divide the time interval into sufficiently small parts when verifying the convergence rates of spatial errors. In Table 1 we take $M = 20$, $\gamma = 2$, $T = 0.1$, and approximate the exact solution using piecewise $Q^0$ polynomial. In Table 2 we take $M = 20$, $\gamma = 3$, $T = 1$, and approximate the exact solution by piecewise $Q^1$ polynomial. In Table 3 we take $M = 100$, $\gamma = 6$, $T = 0.1$, and approximate the exact solution using $Q^2$ polynomial. The numerical simulation results show that the spatial convergence rates in Tables 1-3 are $O(h)$, $O(h^2)$, and $O(h^3)$, respectively, which are in good agreement with the theoretical results (62).
Local discontinuous Galerkin method for the Backward Feynman-Kac Equation

### Table 1

The $L^2$ errors and spatial convergence orders using piecewise $Q^0$ polynomial with the generalized alternating numerical fluxes in Example 1.

| $(\sigma_1, \sigma_2, \alpha)$ \(\backslash\) $h$ | 1/12     | 1/14     | 1/16     | 1/18     |
|-----------------------------------------------|----------|----------|----------|----------|
| (0, 1, 0.7)                                  | 2.9071e-04 | 2.4957e-04 | 2.1866e-04 | 1.9459e-04 |
| Rates                                        | 0.9898   | 0.9903   | 0.9901   |          |
| (0.6, 0.3, 0.7)                              | 3.1971e-04 | 2.6880e-04 | 2.3232e-04 | 2.0483e-04 |
| Rates                                        | 1.1253   | 1.0921   | 1.0694   |          |
| (0.4, 0.5, 0.7)                              | 3.2375e-04 | 2.7139e-04 | 2.3412e-04 | 2.0614e-04 |
| Rates                                        | 1.1444   | 1.1065   | 1.0806   |          |

### Table 2

The $L^2$ errors and spatial convergence orders using piecewise $Q^1$ polynomial with the generalized alternating numerical fluxes in Example 1.

| $(\sigma_1, \sigma_2, \alpha)$ \(\backslash\) $h$ | 1/12     | 1/14     | 1/16     | 1/18     |
|-----------------------------------------------|----------|----------|----------|----------|
| (0, 1, 0.5)                                  | 2.2780e-04 | 1.6766e-04 | 1.2852e-04 | 1.0165e-04 |
| Rates                                        | 1.9886   | 1.9908   | 1.9917   |          |
| ---                                          | 1/20     | 1/22     | 1/24     | 1/26     |
| (0.1, 0.8, 0.5)                              | 1.0563e-04 | 8.7725e-05 | 7.4005e-05 | 6.3265e-05 |
| Rates                                        | 1.9484   | 1.9547   | 1.9589   |          |
| ---                                          | 1/32     | 1/36     | 1/40     | 1/44     |
| (0.3, 0.7, 0.5)                              | 6.3955e-05 | 5.1027e-05 | 4.1653e-05 | 3.4652e-05 |
| Rates                                        | 1.9173   | 1.9266   | 1.9306   |          |

### Table 3

The $L^2$ errors and spatial convergence orders using piecewise $Q^2$ polynomial with the generalized alternating numerical fluxes in Example 1.

| $(\sigma_1, \sigma_2, \alpha)$ \(\backslash\) $h$ | 1/10     | 1/12     | 1/14     | 1/16     |
|-----------------------------------------------|----------|----------|----------|----------|
| (1, 0, 0.3)                                  | 1.9815e-06 | 1.1514e-06 | 7.2730e-07 | 4.8878e-07 |
| Rates                                        | 2.9778   | 2.9800   | 2.9762   |          |
| (0.8, 0.3, 0.3)                              | 1.5636e-06 | 8.9614e-07 | 5.6167e-07 | 3.7590e-07 |
| Rates                                        | 3.0532   | 3.0306   | 3.0076   |          |
| (0.6, 0.5, 0.3)                              | 1.3772e-06 | 7.8569e-07 | 4.9134e-07 | 3.2858e-07 |
| Rates                                        | 3.0782   | 3.0452   | 3.0132   |          |

Comparing Table 1 with Table 3, it is not difficult to find that the change of parameters of numerical fluxes has little effect on the spatial convergence rates using piecewise $Q^0$ and $Q^2$ polynomials, which seems to be contrary to the results by piecewise $Q^1$ polynomial in Table 2.

### Example 2

Let us further consider the inhomogeneous Feynman-Kac backward equation.

\[
\mathcal{C}t^0 D_t^{\cos(2\pi x)} u(t, x) = \Delta u(t, x) + f(t, x), \quad (t, x) \in (0, 0.1] \times \Omega, \quad (65)
\]
where $\Omega = [0, 1] \times [0, 1]$. The boundary condition is periodic and the initial value condition is $u_0(x) = 0$. Let the exact solution of Eq. (65) be $u(t, x) = e^{-t \cos(2\pi x)} \delta \cos(2\pi x) \sin(2\pi y)$. Then one can choose the source term
\[
f(t, x) = e^{-t \cos(2\pi x)} \left( \frac{\Gamma(1 + \delta)}{\Gamma(\delta + 1 - \alpha)} t^{\delta - \alpha} \right) \cos(2\pi x) \sin(2\pi y) + \left( 2 - t \cos(2\pi x) \right) g(t, x) + \left( 2 + t \cos(2\pi x) \right) h(t, x),
\]
where
\[
g(t, x) = 4t^\delta \pi^2 e^{-t \cos(2\pi x)} \cos(2\pi x) \sin(2\pi y),
\]
\[
h(t, x) = 4t^{\delta + 1} \pi^2 e^{-t \cos(2\pi x)} \sin^2(2\pi x) \sin(2\pi y),
\]
and the regularity parameter $\delta \in (0, 1) \cup (1, 2)$.

First, we take $\delta = \alpha$ to verify the spatial convergence orders. In Table 4, take $M = 20$, $\gamma = 2$, and approximate the exact solution using piecewise constant function. In Table 5, take $M = 20$, $\gamma = 3$, and approximate the exact solution by piecewise $Q^2$ polynomial. In Table 6, take $M = 100$, $\gamma = 6$, and approximate the exact solution using piecewise $Q^2$ polynomial. The numerical simulation results show that the spatial convergence orders in Tables 4-6 are $O(h)$, $O(h^2)$, and $O(h^3)$, respectively, which are in good agreement with the theoretical results (62).

In Example 2, we verify the influence of central fluxes on the orders of convergence in detail. By choosing different parameters of numerical fluxes in Table 5, it can be noted that although the spatial orders of convergence can reach order $O(h^2)$ stably when using piecewise $Q^1$ polynomial, the closer the numerical fluxes are to the central ones the finer the subdivision required for the spatial convergence orders to reach stability. More than that, the numerical results in Table 5 also show that the spatial convergence orders are only the first order when the numerical fluxes are taken as the central ones, which is consistent with the existing conclusion [13].

Theorem 5 shows that the temporal convergence orders depend on the regularity of the solution. For the solution with well regularity ($\delta$ large) even if the small mesh parameter $\gamma$ is used, we can still achieve an optimal time accuracy of order $O(\tau^2 - \alpha)$, that is, when $\gamma \delta \geq 2 - \alpha$, the temporal convergence orders $O(\tau^2 - \alpha)$ can be got. Therefore, we take $\gamma \geq \max \{ (2 - \alpha) / \delta, 1 \}$ in order to observe the optimal temporal convergence orders. Now, we set the space stepsize as $h = M^{(\alpha - 2) / (k + 1)}$ in Table 7 based on the optimal spatial convergence orders. Here, we take parameters of numerical flux $\sigma_1 = 0$, $\sigma_2 = 1$, and the degree $k = 0$. The numerical results in Table 7 show that the temporal convergence rates can reach the optimal accuracy of order $O(\tau^2 - \alpha)$, which is consistent with the theoretical result (62).

In the last part of this section, we reveal the relationship between the condition number of the coefficient matrix of the fully discrete scheme (50) and the numerical flux and degree of polynomials.
### Table 4 The $L^2$ errors and spatial convergence orders using piecewise $Q^0$ polynomial with the generalized alternating numerical fluxes in Example 2

| $(\sigma_1, \sigma_2, \alpha) \setminus \Delta h$ | 1/12    | 1/14    | 1/16    | 1/18    |
|---------------------------------------------|--------|--------|--------|--------|
| $(1, 0.3)$                                  | 5.3743e-03 | 4.6124e-03 | 4.0391e-03 | 3.5923e-03 |
| Rates                                      | 0.9918  | 0.9938  | 0.9952  |        |
| $(0.7, 0.2, 0.3)$                           | 5.6634e-03 | 4.7934e-03 | 4.1600e-03 | 3.6770e-03 |
| Rates                                      | 1.0819  | 1.0615  | 1.0478  |        |
| $(0.3, 0.6, 0.3)$                           | 5.7668e-03 | 4.8576e-03 | 4.2026e-03 | 3.7067e-03 |
| Rates                                      | 1.1120  | 1.0848  | 1.0660  |        |
| $(0, 1, 0.7)$                               | 2.1386e-03 | 1.8358e-03 | 1.6075e-03 | 1.4298e-03 |
| Rates                                      | 0.9911  | 0.9953  | 0.9948  |        |
| $(0.3, 0.8, 0.7)$                           | 2.2450e-03 | 1.9022e-03 | 1.6520e-03 | 1.4699e-03 |
| Rates                                      | 1.0748  | 1.0562  | 1.0438  |        |
| $(0.4, 0.5, 0.7)$                           | 2.3053e-03 | 1.9396e-03 | 1.6768e-03 | 1.4781e-03 |
| Rates                                      | 1.1205  | 1.0904  | 1.0704  |        |

### Table 5 The $L^2$ errors and spatial convergence orders using piecewise $Q^1$ polynomial with the generalized alternating numerical fluxes in Example 2

| $(\sigma_1, \sigma_2, \alpha) \setminus \Delta h$ | 1/12    | 1/14    | 1/16    | 1/18    |
|---------------------------------------------|--------|--------|--------|--------|
| $(1, 0.3)$                                  | 5.9690e-04 | 4.3915e-04 | 3.3660e-04 | 2.6616e-04 |
| Rates                                      | 1.9888  | 1.9917  | 1.9935  |        |
| $(0, 1, 0.7)$                               | 2.3747e-04 | 1.7479e-04 | 1.3399e-04 | 1.0596e-04 |
| Rates                                      | 1.9879  | 1.9908  | 1.9925  |        |
| $(0.5, 0.5, 0.3)$                           | 1.1287e-03 | 9.5578e-04 | 8.2960e-04 | 7.3330e-04 |
| Rates                                      | 1.0786  | 1.0603  | 1.0476  |        |
| $(0.5, 0.5, 0.7)$                           | 4.4600e-04 | 3.7780e-04 | 3.2800e-04 | 2.8997e-04 |
| Rates                                      | 1.0765  | 1.0586  | 1.0462  |        |

| $(\sigma_1, \sigma_2, \alpha) \setminus \Delta h$ | 1/20    | 1/22    | 1/24    | 1/26    |
|---------------------------------------------|--------|--------|--------|--------|
| $(0.9, 0.2, 0.3)$                           | 2.7614e-04 | 2.2929e-04 | 1.9337e-04 | 1.6524e-04 |
| Rates                                      | 1.9806  | 1.9583  | 1.9643  |        |
| $(0.1, 0.8, 0.7)$                           | 1.0930e-03 | 9.1293e-04 | 7.7003e-04 | 6.5812e-05 |
| Rates                                      | 1.9491  | 1.9564  | 1.9620  |        |

| $(\sigma_1, \sigma_2, \alpha) \setminus \Delta h$ | 1/32    | 1/36    | 1/40    | 1/44    |
|---------------------------------------------|--------|--------|--------|--------|
| $(0.7, 0.3, 0.3)$                           | 1.4712e-04 | 1.3325e-04 | 1.0865e-04 | 9.0246e-05 |
| Rates                                      | 1.9228  | 1.9370  | 1.9475  |        |
| $(0.2, 0.7, 0.7)$                           | 5.7992e-05 | 4.6161e-05 | 3.7604e-05 | 3.1222e-05 |
| Rates                                      | 1.9371  | 1.9461  | 1.9514  |        |
Table 6 The $L^2$ errors and spatial convergence orders using piecewise $Q^2$ polynomial with the generalized alternating numerical fluxes in Example 2.

| $(\sigma_1, \sigma_2, \alpha) \setminus h$ | 1/10  | 1/12  | 1/14  | 1/16  |
|----------------------------------|-------|-------|-------|-------|
| (1, 0, 0.3)                      | 4.4230e-05 | 2.5701e-05 | 1.6225e-05 | 1.0888e-05 |
| Rates                           | 2.9775  | 2.9838  | 2.9876  |        |
| (0.8, 0.3, 0.3)                  | 3.5359e-05 | 2.0197e-05 | 1.2621e-05 | 8.4140e-06 |
| Rates                           | 3.0716  | 3.0500  | 3.0366  |        |
| (0.6, 0.5, 0.3)                  | 3.1299e-05 | 1.7729e-05 | 1.1032e-05 | 7.3361e-06 |
| Rates                           | 3.1175  | 3.0776  | 3.0551  |        |
| (0, 1, 0.7)                      | 1.7602e-05 | 1.0232e-05 | 6.4639e-06 | 4.3426e-06 |
| Rates                           | 2.9754  | 2.9797  | 2.9787  |        |
| (0.2, 0.7, 0.7)                  | 1.4075e-05 | 8.0437e-05 | 5.0315e-06 | 3.3698e-06 |
| Rates                           | 3.0689  | 3.0436  | 3.0221  |        |
| (0.4, 0.6, 0.7)                  | 1.2600e-05 | 7.1447e-06 | 4.5236e-06 | 2.9686e-06 |
| Rates                           | 3.1118  | 3.0681  | 3.0355  |        |

Taking $\{\phi^r_i(x)\phi^s_j(y)\}_{r,s=0}^k$ as the basis of $Q^k$, on the element $\Omega_{ij}$, our implementation uses the orthogonal Legendre polynomials, and express $u^h_k(x) \in V^k_h$ as

$$u^h_k|_{\Omega_{ij}} = \sum_{r,s=0}^k \sum_{r=0}^k u^r_{ij}(t_n) \phi^r_i(x) \phi^s_j(y), \quad (66)$$
Table 8 Condition numbers of the coefficient matrix $F_M$ for the fully scheme (50).

| $(Q^k, \alpha)$ | $(\sigma_1, \sigma_2)$ | (1,0)      | (0.9,0.1) | (0.8,0.2) | (0.7,0.3) |
|-----------------|----------------------|------------|------------|------------|------------|
| $(Q^0, 0.3)$    |                      | 2.2489e+02 | 1.4429e+02 | 8.8317e+01 | 6.5928e+01 |
| $(Q^1, 0.3)$    |                      | 9.3401e+02 | 8.4305e+02 | 7.7026e+02 | 7.1675e+02 |
| $(Q^2, 0.3)$    |                      | 2.5750e+03 | 1.8470e+03 | 1.2961e+03 | 9.5453e+02 |
| $(Q^0, 0.7)$    |                      | 1.4294e+01 | 9.5084e+00 | 6.1848e+00 | 4.8554e+00 |
| $(Q^1, 0.7)$    |                      | 5.6369e+01 | 5.0972e+01 | 4.6659e+01 | 4.3490e+01 |
| $(Q^2, 0.7)$    |                      | 1.5365e+02 | 1.1046e+02 | 7.7802e+01 | 5.7456e+01 |

with $u^{n+1}_h(t_n)$ being the unknown coefficients of numerical solution $u^n_h$. Introducing $u^n_h$ (66) into the fully discrete scheme (50) leads to its matrix form

$$F_n U^n = \sum_{l=0}^{n-1} G_l U^l, \quad n = 1, 2, \cdots, M, \quad (67)$$

where $U^n$ is the coefficient vector of $u^n_h$.

We take $M = 100, h = 1/12, T = 0.1$, and calculate the condition number of $F_M$. In Table 8 we find that the condition number of $F_M$ decreases as the numerical fluxes approach the central ones. On the other hand, the condition number of $F_M$ increases gradually with the increase of polynomial degree. The condition numbers in Table 8 are not very large, implying that the equation still can be effectively solved using the matrix form (67).

6 Conclusion

The non-Brownian functional is an important class of statistical observables, the distribution of which is governed by Feynman-Kac equation [12]. In this paper, we first derive the equivalent form of the backward Feynman-Kac equation. Then the spatial semi-discrete scheme is constructed by LDG method. Following the properties of the fractional substantial calculus, the stability and optimal spatial convergence orders $O(h^{k+1})$ of the semi-discrete scheme with the generalized alternating numerical fluxes are obtained. Based on the theoretical results of the semi-discrete scheme, we obtain the optimal convergence orders $O(h^{k+1} + \tau \min(1, \gamma \delta))$ of the fully discrete scheme. Finally, extensive numerical experiments are carried out to demonstrate the validity of the numerical scheme and justify the theoretical findings. The closer the numerical fluxes are to the central ones the finer the subdivision are required for the spatial convergence orders to reach stability by using piecewise odd polynomial. In particular, if using piecewise odd polynomial and central numerical fluxes, the spatial convergence orders are just $O(h^k)$. In addition, as the numerical fluxes approach the central ones and the degree of polynomial
decreases, the condition number of the coefficient matrix of the fully discrete scheme decreases gradually.

Appendix

Equivalent form of Eq. (2). Let \( 0 < \alpha < 1 \) and \( u(t, x) \in C_{2, \delta}((0, T]; H^2(\Omega)) \). Then

\[
\partial_t^{\alpha, \kappa(x)} u(t, x) = \Delta u(t, x) - \kappa(x) L_t^{\alpha, \kappa(x)} u(t, x),
\]

(68)
is equivalent to

\[
e^{-\kappa(x)t} C_0 \int_0^t e^{\kappa(x)s} u(t, x) = \Delta u(t, x),
\]

where \( C_{2, \delta}((0, T]) = \{ u(t) \mid |u^{(l)}(t)| \leq C_u (1 + t^{\delta-l}), \ l = 0, 1, 2, \ 0 < t \leq T \} \) and the regularity parameter \( \delta \in (0, 1) \cup (1, 2) \).

Proof In the following, we define Laplace transform of \( u(t, x) \) about the time variable as

\[
\mathcal{L}\{u(t, x); s\} = \int_0^\infty e^{-st} u(t, x) dt.
\]

Let \( \partial_t^{\alpha, \kappa(x)} u(t, x) = \frac{\partial}{\partial t} g(t, x) \) in (2). According to the properties of Laplace transform, there exist

\[
\mathcal{L}\{\partial_t^{\alpha, \kappa(x)} u(t, x); s\} = s\hat{g}(s, x) - g(0, x)
\]

and

\[
\hat{g}(s, x) = \frac{1}{(s + \kappa(x))^{1-\alpha}} (\hat{u}(s, x) - \frac{1}{s} u(0, x)).
\]

Since \( u(t, x) \in C_{2, \delta}((0, T]; H^2(\Omega)) \), one can get

\[
\|g(0, x)\| = \lim_{t \to 0^+} \frac{1}{\Gamma(1-\alpha)} \left\| \int_0^t e^{-\kappa(x)(t-\tau)} (t-\tau)^{-\alpha} (u(\tau, x) - u(0, x)) d\tau \right\|
\]

\[
\leq \lim_{t \to 0^+} \frac{C_{\max}}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \|u(\tau, x) - u(0, x)\| d\tau
\]

\[
\leq \lim_{t \to 0^+} \frac{2C_{\max} C_u}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} (1 + \tau^\delta) d\tau
\]

\[
= 0,
\]

where we denote \( \| \cdot \| \) as the norms associated with \( L^2(\Omega) \) and \( C_{\max} \) is defined by (2). One can further get \( g(0, x) = 0 \). Therefore

\[
\mathcal{L}\{\partial_t^{\alpha, \kappa(x)} u(t, x); s\} = \frac{1}{(s + \kappa(x))^{1-\alpha}} (s\hat{u}(s, x) - u(0, x)).
\]

(69)

Using inverse Laplace transform for (69) results in

\[
\partial_t^{\alpha, \kappa(x)} u(t, x) = \int_0^t (t-\tau)^{-\alpha} \frac{1}{\Gamma(1-\alpha)} \int_0^\infty e^{\kappa(x)\tau} \frac{\partial}{\partial \tau} u(\tau, x) d\tau.
\]

(70)
Substituting (70) and (4) into (68) leads to
\[
\begin{align*}
\Delta u(t, x) &= \nabla \cdot \left( \omega, \kappa(x) \nabla u(t, x) \right) + \kappa(x) \int_0^t \left( \frac{\partial}{\partial \tau} u(\tau, x) + \kappa(x) u(\tau, x) \right) d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t e^{-\kappa(x)(t-\tau)/(t-\tau)} (t-\tau)^{-\alpha} \left[ \frac{\partial}{\partial \tau} u(\tau, x) + \kappa(x) u(\tau, x) \right] d\tau \\
&= e^{-\kappa(x)t} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{D_\tau^\alpha (e^{\kappa(x)t} u(t, x))}{D_\tau^\alpha (\kappa(x))} d\tau
\end{align*}
\]
which concludes the proof.

References

1. Bassi, F., Rebay, S.: A high-order accurate discontinuous finite element method for the numerical solution of the compressible Navier-Stokes equations. J. Comput. Phys. 131, 267–279 (1997)
2. Carmi, S., Barkai, E.: Fractional Feynman-Kac equation for weak ergodicity breaking. Phys. Rev. E 84, 061104 (2011)
3. Carmi, S., Turgeman, L., Barkai, E.: On distributions of functionals of anomalous diffusion paths. J. Stat. Phys. 141, 1071–1092 (2010)
4. Chen, M.H., Deng, W.H.: High order algorithms for the fractional substantial diffusion equation with truncated Lévy flights. SIAM J. Sci. Comput. 37, A890–A917 (2015)
5. Chen, Z.Q., Deng, W.H., Xu, P.B.: Feynman-Kac transform for anomalous processes. SIAM J. Math. Anal. 53, 6017–6047 (2021)
6. Cheng, Y., Meng, X., Zhang, Q.: Application of generalized Gauss-Radau projections for the local discontinuous Galerkin method for linear convection-diffusion equations. Math. Comp. 86, 1233–1267 (2017)
7. Cockburn, B., Dong, B.: An analysis of the minimal dissipation local discontinuous Galerkin method for convection-diffusion problems. J. Sci. Comput. 32, 233–262 (2007)
8. Cockburn, B., Shu, C.W.: The local discontinuous Galerkin method for time-dependent convection-diffusion systems. SIAM J. Numer. Anal. 35, 2440–2463 (1998)
9. Deng, W.H.: Finite element method for the space and time fractional Fokker-Planck equation. SIAM J. Numer. Anal. 47, 612–626 (2008)
10. Deng, W.H., Chen, M.H., Barkai, E.: Numerical algorithms for the forward and backward fractional Feynman-Kac equations. J. Sci. Comput. 62, 718–746 (2015)
11. Deng, W.H., Hesthaven, J.S.: Local discontinuous Galerkin methods for fractional diffusion equations. ESAIM Math. Model. Numer. Anal. 47, 1845–1864 (2013)
12. Deng, W.H., Wang, X.D., Nie, D.X.: Functional Distribution of Anomalous and Non-ergodic Diffusion: From Stochastic Processes to PDEs. World Scientific Publishing Co. Pte. Ltd., Singapore (2022)
13. Hesthaven, J.S., Warburton, T.: Nodal Discontinuous Galerkin Methods: Algorithms, Analysis, and Applications. Springer, New York, USA (2008)
14. Huang, C.B., An, N., Yu, X.J.: A local discontinuous Galerkin method for time-fractional diffusion equation with discontinuous coefficient. Appl. Numer. Math. 151, 367–379 (2020)
15. Huang, C.B., Stynes, M.: Optimal spatial H1-norm analysis of a finite element method for a time-fractional diffusion equation. J. Comput. Appl. Math. 367, 112435 (2020)
16. Kac, M.: On distributions of certain wiener functionals. Trans. Amer. Math. Soc. 65, 1–13 (1949)
17. Li, C.P., Yi, Q., Chen, A.: Finite difference methods with non-uniform meshes for non-linear fractional differential equations. J. Comput. Phys. 316, 614–631 (2016)
18. Liu, Y.M., Xu, C.J.: Finite difference/spectral approximations for the time-fractional diffusion equation. J. Comput. Phys. 225, 1533–1552 (2007)
19. Mao, Z.P., Karniadakis, G.E.: Fractional Burgers equation with nonlinear non-locality: spectral vanishing viscosity and local discontinuous Galerkin methods. J. Comput. Phys. 336, 145–163 (2017)
20. McLean, W.: Regularity of solutions to a time-fractional diffusion equation. ANZIAM J. 52, 123–138 (2010)
21. McLean, W.: Fast summation by interval clustering for an evolution equation with memory. SIAM J. Sci. Comput. 34, A3039–A3056 (2012)
22. Mustapha, K., Abdallah, B., Furati, K.M., Nour, M.: A discontinuous Galerkin method for time fractional diffusion equations with variable coefficients. Numer. Algorithms 73, 517–534 (2016)
23. Mustapha, K., Schötzau, D.: Well-posedness of hp-version discontinuous Galerkin methods for fractional diffusion wave equations. IMA J. Numer. Anal. 34, 1426–1446 (2014)
24. Nie, D.X., Deng, W.H.: Local discontinuous Galerkin method for the fractional diffusion equation with integral fractional Laplacian. Comput. Math. Appl. 104, 44–49 (2021)
25. Qiu, L.L., Deng, W.H., Hesthaven, J.S.: Nodal discontinuous Galerkin methods for fractional diffusion equations on 2D domain with triangular meshes. J. Comput. Phys. 298, 678–694 (2015)
26. Ren, J.C., Liao, H.L., Zhang, J.W., Zhang, Z.M.: Sharp H1-norm error estimates of two time-stepping schemes for reaction-subdiffusion problems. J. Comput. Appl. Math. 389, 113352 (2021)
27. Stynes, M.: Too much regularity may force too much uniqueness. Fract. Calc. Appl. Anal. 19, 1554–1562 (2016)
28. Stynes, M., O’Riordan, E., Gracia, J.L.: Error analysis of a finite difference method on graded meshes for a time-fractional diffusion equation. SIAM J. Numer. Anal. 55, 1075–1079 (2017)
29. Sun, J., Nie, D.X., Deng, W.H.: High-order BDF fully discrete scheme for backward fractional Feynman-Kac equation with nonsmooth data. Appl. Numer. Math. 161, 82–100 (2021)
30. Sun, X.R., Li, C., Zhao, F.Q.: Local discontinuous Galerkin methods for the time tempered fractional diffusion equation. Appl. Math. Comput. 365, 1–16 (2020)
31. Tian, W.Y., Deng, W.H., Wu, Y.J.: Polynomial spectral collocation method for space fractional advection-diffusion equation. Numer. Methods for Partial Differential Equations 30, 514–535 (2014)
32. Turgeman, L., Carmi, S., Barkai, E.: Fractional Feynman-Kac equation for non-Brownian functionals. Phys. Rev. Lett. 103, 190201 (2009)
33. Wang, X.D., Deng, W.H.: Discontinuous Galerkin methods and their adaptivity for the tempered fractional (convection) diffusion equations. J. Comput. Math. 38, 841–869 (2020)
34. Wei, L.L., Dai, H.Y., Zhang, D.L., Si, Z.Y.: Fully discrete local discontinuous Galerkin method for solving the fractional telegraph equation. Calcolo 51, 175–192 (2014)
35. Wei, L.L., He, Y.N.: Analysis of a fully discrete local discontinuous Galerkin method for time-fractional fourth-order problems. Appl. Math. Model. 38, 1511–1522 (2014)
36. Xia, Y.H., Xu, Y., Shu, C.W.: Application of the local discontinuous Galerkin method for the Allen-Cahn/Cahn-Hilliard system. Commun. Comput. Phys. 5, 821–835 (2009)
37. Xu, Q.W., Hesthaven, J.S.: Discontinuous Galerkin method for fractional convection-diffusion equations. SIAM J. Numer. Anal. 52, 405–423 (2014)
38. Xu, Z.L., Chen, X.Y., Liu, Y.J.: A new Runge-Kutta discontinuous Galerkin method with conservation constraint to improve CFL condition for solving conservation laws. J. Comput. Phys. 278, 348–377 (2014)
39. Zhang, Q., Shu, C.W.: Error estimates to smooth solutions of Runge-Kutta discontinuous Galerkin methods for scalar conservation laws. SIAM J. Numer. Anal. 42, 641–666 (2004)