IDEMPOTENT CONVEXITY AND ALGEBRAS FOR THE
CAPACITY MONAD AND ITS SUBMONADS

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Abstract. Idempotent analogues of convexity are introduced. It is proved
that the category of algebras for the capacity monad in the category of com-
pacts is isomorphic to the category of (max, min)-idempotent biconvex com-
pacts and their biaffine maps. It is also shown that the category of algebras for
the monad of sup-measures ((max, min)-idempotent measures) is isomorphic
to the category of (max, min)-idempotent convex compacta and their affine
maps.

INTRODUCTION

Monads (also called triples, [2, 8]) in topological categories and algebras for these
monads are closely related to important objects of analysis and topological algebra.
Swirszcz [17] proved that algebras and their morphisms for the probability mea-
sure monad are precisely convex compact maps of locally convex vector topological
spaces and continuous affine maps.

By a result of Day (cf. Theorem 3.3 of [6]), the category of algebras for the
filter monad in the category of sets is the category of continuous lattices and their
mappings that preserve directed joins and arbitrary meets. Due to Wyler [19]
algebras for the hyperspace monad are compact Lawson semilattices. Zarichnyi [20]
has shown that the category of algebras for the superextension monad is isomorphic
to the category of compacta with (fixed) almost normal T_2-subbase and their convex
maps. We will use a result of Radul [15] who introduced the inclusion hyperspace
triple and proved that its algebras and their morphisms are in fact compact Lawson
lattices and their complete homomorphisms.

Unlike probability (normed additive) measures which are a traditional object of
investigation by means of categorical topology, their non-additive analogues were
paid less attention from this point of view. Meanwhile capacities (normed non-
additive measures) that were introduced by Choquet [4] and rediscovered by Sugeno
under the name fuzzy measures have found numerous applications, e.g. in decision
making under uncertainty [7, 15]. One of the most promising classes of non-additive
measures is one of idempotent measures [1]. For other important classes of capac-
ities and their topological properties see [3]. Upper semicontinuous capacities on
compact spaces were systematically studied in [14].

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Therefore it seems natural to use methods of categorical topology to study non-additive measures. Nykyforchyn and Zarichnyi [21] defined the capacity functor and the capacity monad in the category of compacta, and proved basic topological properties of capacities on metrizable and non-metrizable compacta. Two important dual subfunctors of the capacity functor, namely of $\cup$-capacities (possibility measures) and of $\cap$-capacities (necessity measures) were introduced in [9], and it was shown that they lead to submonads of the capacity monad. The aim of this paper is to describe categories of algebras for the capacity monad, for the monads of $\cup$-capacities and of $\cap$-capacities, and to present internal relations of the capacity monad and its submonads with idempotent mathematics and generalizations of convexity (in the form of join geometry).

1. Preliminaries

A compactum is a compact Hausdorff topological space. We regard the unit segment $I = [0; 1]$ as a subspace of the real line with the natural topology. We write $A \subseteq B$ (resp. $A \subset B$) if $A$ is a closed (resp. open) subset of a space $B$. For a set $X$ the identity mapping $X \rightarrow X$ is denoted by $1_X$. For a compactum $X$ we denote by $\exp X$ the set of all nonempty closed subsets of $X$ with the Vietoris topology. A base of this topology consists of all sets of the form

$\langle U_1, U_2, \ldots, U_n \rangle = \{ F \in \exp X \mid F \subseteq U_1 \cup U_2 \cup \cdots \cup U_n, F \cap U_i \neq \emptyset \text{ for all } 1 \leq i \leq n \}$,

where $n \in \mathbb{N}$ and all $U_i \subseteq X$ are open. The space $\exp X$ for a compactum $X$ is a compactum as well. A nonempty closed subset $F \subseteq \exp X$ is called an inclusion hyperspace if for all $A, B \in \exp X$ an inclusion $A \subset B$ and $A \in F$ imply $B \in F$. The set $GX$ of all inclusion hyperspaces is closed in $\exp(exp X)$. For more on $\exp X$ and $GX$ see [18].

We regard any set $S$ with an idempotent, commutative and associative binary operation $\oplus : S \times S \rightarrow S$ (with an additive denotation) as an upper semilattice with the partial order $x \leq y \iff y \geq x \iff x \oplus y = y$ and the pairwise supremum $x \oplus y$ for $x, y \in S$. Similarly, given an idempotent, commutative and associative operation $\otimes : S \times S \rightarrow S$ (with a multiplicative denotation), we regard $S$ as a lower semilattice with the partial order $x \leq y \iff y \geq x \iff x \otimes y = x$ and $x \otimes y$ being the infimum of $x, y \in S$.

If two operations $\oplus, \otimes : L \times L \rightarrow L$ are idempotent, commutative and associative, and the distributive laws and the laws of absorption are valid, then $L$ is a distributive lattice w.r.t. the partial order $x \leq y \iff y \geq x \iff x \oplus y = y \iff x \otimes y = x$, and $x \oplus y$ and $x \otimes y$ are the pairwise supremum and the pairwise infimum of $x, y \in L$.

If $f, g$ are functions with the same domain and values in a poset, then by $f \lor g$ and $f \land g$ we also define their pointwise supremum and infimum. If $f$ is a function with values in a set $L$ with an operation “$\oplus$” (or “$\otimes$”), and $a \in L$, then $(a \oplus f)(x) = a \oplus f(x)$ (resp. $(a \otimes f)(x) = a \otimes f(x)$) for any valid argument $x$.

An idempotent semiring is a set $R$ with binary operations $\oplus, \otimes : R \times R \rightarrow R$ such that $(R, \oplus)$ is an abelian monoid with a neutral element 0, “$\oplus$” is idempotent, i.e., $a \oplus a = a$ for all $a \in R$, $(R, \otimes)$ is a monoid with a neutral element 1, the operation “$\otimes$” is distributive over “$\oplus$” : $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$ for all $a, b, c \in R$, and $0 \oplus a = a \otimes 0 = 0$ for all $a \in R$. The most popular idempotent semiring is the tropical semiring $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$, where $x \oplus y = \max\{x, y\}, x \otimes y = x + y$, which is the
basis of tropical mathematics [10]. A little less extensively studied is the idempotent semiring \((\mathbb{R} \cup \{-\infty, \emptyset, \odot, \oplus\})\), where \(x \odot y = \max\{x, y\}\), \(x \odot y = \min\{x, y\}\). We will use a semiring which is algebraically and topologically isomorphic to it, but more convenient for our purposes, namely \((I, \oplus, \odot)\) with \(x \oplus y = \max\{x, y\}\), \(x \otimes y = \min\{x, y\}\). In general, any distributive lattice \((L, \oplus, \odot)\) with top and bottom elements is an idempotent semiring.

We denote by \(1_C\) we denote the identity functor in a category \(C\). Recall that all \(\mathbb{F}\)-algebras for a fixed monad \(F\) and all their morphisms form a category of \(\mathbb{F}\)-algebras.

See [21][12] for the definitions of category, morphism, functor, natural transformation, monad, algebra for a monad, morphism of algebras, tripleability and related facts. By \(I\) we denote the identity functor in a category \(C\). Recall that all \(\mathbb{F}\)-algebras for a fixed monad \(F\) and all their morphisms form a category of \(\mathbb{F}\)-algebras.

It is proved in [18] that constructions \(\exp\) and \(G\) can be extended to functors in \(\text{Comp}\) that are functorial parts of monads. For a continuous map of compacta \(f : X \to Y\) the maps \(\exp f : \exp X \to \exp Y\) and \(Gf : GX \to GY\) are defined by the formulae \(\exp f(F) = \{f(x) \mid x \in F\}\), \(F \in \exp X\) and \(Gf(F) = \{B \subseteq Y \mid B \supseteq f(A) \text{ for some } A \in F\}, F \in GX\). For the inclusion hyperspace monad \(\mathcal{G} = (G, \eta_G, \mu_G)\) the components \(\eta_G X : X \to GX\) and \(\mu_G X : G^2X \to GX\) of the unit and the multiplication are defined as follows: \(\eta_G(x) = \{F \in \exp X \mid x \in F\}\), \(x \in X\) and \(\mu_G X(F) = \bigcup \{\bigcap A \mid A \in F\}, F \in G^2X\).

We denote by \(\text{Comp}\) the category of compacta that consists of all compacta and their continuous mappings. If there is a natural transformation of one functor in \(\text{Comp}\) to another with all components being topological embeddings, then the first functor is called a subfunctor of the latter [18]. Similarly an embedding of monads in \(\text{Comp}\) is a morphism of monads with all components being topological embeddings. If there exists an embedding of one monad in \(\text{Comp}\) into another one, then the first monad is called a submonad of the latter.

Now we present the main notions and results of [21] that concern capacities on compacta, the capacity functor and the capacity monad. We call a function \(c : \exp X \cup \{\emptyset\} \to I\) a capacity on a compactum \(X\) if the following three properties hold for all closed subsets \(F, G\) of \(X\):

1. \(c(\emptyset) = 0, c(X) = 1\);
2. if \(F \subseteq G\), then \(c(F) \leq c(G)\) (monotonicity);
3. if \(c(F) < a\), then there exists an open set \(U \supseteq F\) such that \(G \subseteq U\) implies \(c(G) < a\) (upper semicontinuity).

We extend a capacity \(c\) to all open subsets in \(\text{Comp}\) by the formula:

\[
c(U) = \sup\{c(F) \mid F \subseteq X, F \subseteq U\}, U \subseteq X.
\]

It is proved in [21] that the set \(M X\) of all capacities on a compactum \(X\) is a compactum as well, if a topology on \(M X\) is determined by a subbase that consists of all sets of the form

\[
O_-(F, a) = \{c \in M X \mid c(F) < a\},
\]

where \(F \subseteq X, a \in \mathbb{R}\), and

\[
O_+(U, a) = \{c \in M X \mid c(U) > a\} = \{c \in M X \mid \text{there exists a compactum } F \subseteq U, c(F) > a\},
\]

where \(U \subseteq X, a \in \mathbb{R}\).
The assignment $M$ extends to the capacity functor $M$ in the category of compacta, if the map $Mf : MX \to MY$ for a continuous map of compacta $f : X \to Y$ is defined by the formula

$$Mf(c)(F) = c(f^{-1}(F)),$$

where $c \in MX$, $F \subseteq Y$. This functor is the functorial part of the capacity monad $\mathcal{M} = (M, \eta, \mu)$ that was described in [21]. Its unit and multiplication are defined by the formulae

$$\eta X(x) = \delta_x \text{ where } \delta_x(F) = \begin{cases} 1, & \text{if } x \in F, \\ 0, & \text{if } x \notin F, \end{cases} \quad \text{(a Dirac measure concentrated in } x)$$

$$\mu X(C)(F) = \sup\{ \alpha \in I \mid C(\{c \in MX \mid c(F) \geq \alpha\}) \geq \alpha\},$$

where $x \in X$, $C \in M^2X$, $F \subseteq X$.

We call a capacity $c \in MX$ a $\cup$-capacity (also called sup-measure or possibility measure) if $c(A \cup B) = \max\{c(A), c(B)\}$ for all $A, B \subseteq X$. A capacity $c \in MX$ a $\cap$-capacity (or necessity measure) [9] if $c(A \cap B) = \min\{c(A), c(B)\}$ for all $A, B \subseteq X$. The sets of all $\cup$-capacities and of all $\cap$-capacities on a compactum $X$ are denoted by $M_uX$ and $M_cX$. It is proved in [9] that $M_uX$ and $M_cX$ are closed in $MX$, $Mf(M_cX) \subseteq M_cY$ and $Mf(M_uX) \subseteq M_uY$ for any continuous map of compacta $f : X \to Y$, thus we obtain subfunctors $M_u, M_c$ of the capacity functor $M$. Moreover, we get submonads $\mathbb{M}_u$ and $\mathbb{M}_c$ of the capacity monad $\mathbb{M}$.

Observe that for a $\cup$-capacity $c$ and a closed set $F \subseteq X$ we have $c(F) = \max\{c(x) \mid x \in F\}$, and $c$ is completely determined by its values on singletons. Therefore we often identify $c$ with the upper semicontinuous function $X \to I$ that sends each $x \in X$ to $c(\{x\})$, and write $c(x)$ instead of $c(\{x\})$. Conversely, each upper semicontinuous function $c : X \to I$ with max $c = 1$ determines a $\cap$-capacity by the formula $c(F) = \max\{c(x) \mid x \in F\}$, $F \subseteq X$. A similar, but a little more complicated observation is valid for $\cap$-capacities.

2. ALGEBRAS FOR THE MONADS OF $\cup$-CAPACITIES AND $\cap$-CAPACITIES

Let an operation $ic : X \times I \times X \to X$ be given for a set $X$. In the sequel we denote $ic(x, \alpha, y)$ by $x \oplus (\alpha \odot y)$ or simply by $x \oplus ay$ for the sake of shortness. We call $ic$ an idempotent convex combination of two points in $X$ if the following equalities are valid for all $x, y, z \in X$, $\alpha, \beta \in I$:

1) $x \oplus \alpha x = x$;
2) $(x \oplus ay) \oplus \beta z = (x \oplus \beta z) \oplus ay$;
3) $x \oplus \alpha(y \oplus \beta z) = (x \oplus ay) \oplus (\alpha \odot \beta)z$;
4) $x \oplus 1y = y \oplus 1x$;
5) $x \oplus 0y = x$.

We also call the set

$$\Delta^n_\oplus = \{(\alpha_0, \alpha_1, \ldots, \alpha_n) \in I^{n+1} \mid \alpha_0 \odot \alpha_1 \odot \ldots \odot \alpha_n = 1\}$$

the (idempotent) $n$-dimensional $\oplus$-simplex. Now, assuming 1)–5), for any coefficients $(\alpha_0, \alpha_1, \ldots, \alpha_n) \in \Delta^n_\oplus$ and elements $x_0, x_1, \ldots, x_n \in X$ we define the idempotent convex combination of $n + 1$ points as follows (assume that $\alpha_k = 1$ for some
Let \( \xi \) be an idempotent convex combination. Conditions 2), 4) assure that the combination is well defined and does not depend on the order of summands. Obviously \( 1x \oplus ay = x \oplus ay \). By 5) summands with zero coefficients can be dropped, and by 1) and 3), if two summands contain the same point, then a summand with a greater coefficient absorbs a summand with a lesser coefficient. Conditions 2), 3) also imply a "big associative law":

\[
\alpha_0(\beta_0^0 x_0^0 \oplus \ldots \oplus \beta_0^n x_0^n) + \alpha_1(\beta_1^1 x_1^1 \oplus \ldots \oplus \beta_1^n x_1^n) + \ldots + \alpha_n(\beta_n^n x_n^0 \oplus \ldots \oplus \beta_n^n x_n^n) =
\]

\[
(\alpha_0 \oplus \beta_0^0 x_0^0 \oplus \ldots \oplus (\alpha_0 \otimes \beta_0^n) x_0^n + \ldots + \alpha_n \otimes \beta_1^1) x_1^1 + \ldots + (\alpha_n \otimes \beta_n^n) x_n^0 \oplus \ldots \oplus (\alpha_n \otimes \beta_n^n) x_n^n,
\]

where \( x_i^j \in X, (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \Delta^n_0, (\beta_0^0, \beta_1^1, \ldots, \beta_n^n) \in \Delta^n_0 \) for \( i = 0, 1, \ldots, n \).

Properties 1)–4) imply that the operation \( \vee : X \times X \to X, x \vee y = x \oplus ly \) for all \( x, y \in X \), is commutative, associative and idempotent, thus \((X, \vee)\) is an upper semilattice with a partial order \( x \leq y \iff x \vee y = y \) for which \( x \vee y \) is a pairwise supremum of \( x \) and \( y \). If \( X \) is a compactum such that

6) for a neighborhood \( U \) of any element \( x \in X \) there is a neighborhood \( V \) of \( x, V \subseteq U \), such that \( y \oplus 1z \in V \) for all \( y, z \in V \);

then each point of \( X \) has a local base consisting of subsemilattices, and \((X, \vee)\) is a compact Lawson upper semilattice \( \mathbb{M}_\leq \). We will call a pair \((X, ic)\) of a compactum \( X \) with idempotent convex combination \( ic \) that satisfies the property 6) a \((\max, \min)-idempotent convex compactum\).

**Theorem 2.1.** Let \( X \) be a compactum. There is a one-to-one correspondence between continuous maps \( \xi : M_i X \to X \) such that the pair \((X, \xi)\) is an \( \mathbb{M}_\leq \)-algebra, and continuous idempotent convex combinations \( ic : X \times I \times X \to X \) such that \((X, ic)\) is a \((\max, \min)-idempotent convex compactum\).

If for a continuous \( ic : X \times I \times X \to X \) conditions 1)–5) are valid, then 6) implies a stronger property:

6+) for a neighborhood \( U \) of any element \( x \in X \) there is a neighborhood \( V \) of \( x, V \subseteq U \), such that \( y \oplus \alpha z \in V \) for all \( y, z \in V, \alpha \in I \).

**Proof.** Let \((X, \xi)\) be an \( \mathbb{M}_\leq \)-algebra. Define the operation \( ic : X \times I \to X \) by the formula \( ic(x, \alpha, y) = \xi(\delta x \oplus \alpha \delta y) \). It is obvious that \( \xi \) is well-defined, continuous and satisfies 1), 4), 5). To prove 2), observe that by the definition of an algebra for a monad we obtain

\[
(x \oplus ay) \oplus \beta z = \xi(\delta x \oplus \alpha \delta y) \oplus \beta \delta z = \xi \circ M_i \xi(\delta x \oplus \alpha \delta y \oplus \beta \delta z) = \xi(\delta x \oplus \alpha \delta y \oplus \beta \delta z) = \xi(\delta x \oplus \beta \delta z \oplus \alpha \delta y) = (x \oplus \beta z) \oplus \alpha y.
\]

Proof of 3) is quite analogous. Thus the map \( ic \) is an idempotent convex combination of two points, and we consider idempotent convex combinations of arbitrary finite number of points to be defined as described above.

Let \( U \) be a neighborhood of \( x \in X \). By continuity of \( \xi \) and the equality \( \xi(\delta x) = x \) there is a neighborhood \( \tilde{U} \subseteq M_i X \) of \( \delta x \) such that for all \( c \in \tilde{U} \) we have \( \xi(c) \in U \). There also exists a neighborhood \( \tilde{V} \ni x \) such that for all \( y_0, y_1, \ldots, y_n \in \tilde{V}, \)
$(\alpha_0, \alpha_1, \ldots, \alpha_n) \in \Delta^n$ we have $\alpha_0 \delta_{y_0} \oplus \alpha_1 \delta_{y_1} \oplus \cdots \oplus \alpha_n \delta_{y_n} \in \tilde{U}$. It is straightforward to verify that the set

$$V = \{ \alpha_0 y_0 \oplus \alpha_1 y_1 \oplus \cdots \oplus \alpha_n y_n \mid n \in \{0, 1, \ldots\}, (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \Delta^n, y_0, y_1, \ldots, y_n \in \tilde{V} \}$$

is a neighborhood of $x$ requested by 6+), which implies 6). Thus it is proved that an $M_\cdot$-algebra $(X, \xi)$ determines a continuous operation $\iota c$ that satisfies conditions 1)–6).

Now assume that we are given a compactum $X$ and a continuous operation $\iota c : X \times I \times X \to X$ that satisfies conditions 1)–6). Recall that $X$ with the operation $\vee : X \times X \to X$, defined by the formula $x \vee y = x \oplus 1 y$, is a compact Lawson upper semilattice, therefore for all nonempty closed $F \subset X$ there is sup $F$ that depends on $F$ continuously w.r.t. Vietoris topology [13]. Let $c \in M_\cdot X$ and $c(x_0) = 1$ for some $x_0 \in X$. We put $\xi(c) = \sup\{ x_0 \oplus \alpha x \mid x \in X, \alpha \leq c(x) \}$. We will prove that $\xi : M_\cdot X \to X$ is well defined (i.e. does not depend on the choice of $x_0$) and continuous.

For each $x \in X$ let $gr(x)$ be the collection $(x \vee y)_{y \in X} \in X^X$. Then the map of compacta $gr : X \to X^X$ is continuous and injective, therefore is an embedding.

The equality

$$\xi(c) \vee y = \sup\{ x_0 \oplus \alpha x \mid x \in X, \alpha \leq c(x) \} \vee y = \sup\{ y \oplus 1 x_0 \oplus \alpha x \mid x \in X, \alpha \leq c(x) \} = \sup\{ (y \oplus 1 x_0) \vee (y \oplus \alpha x) \mid x \in X, \alpha \leq c(x) \} = \sup\{ y \oplus \alpha x \mid x \in X, \alpha \leq c(x) \}$$

holds for each $y \in X$, and the latter expression does not depend on $x_0$. This implies that $gr(\xi(c))$ and thus $\xi(c)$ are uniquely determined. Moreover, $pr_y \circ gr(\xi(c))$ is the supremum of the image of the closed set $\{ (x, \alpha) \mid x \in X, \alpha \in I, \alpha \leq c(x) \} \subset X \times I$ under the continuous map that sends $(x, \alpha)$ to $y \oplus \alpha x \in X$. Taking into account that this set (the hypograph of the function $c : X \to I$) depends on $c \in M_\cdot X$ continuously, we obtain that the correspondence $c \mapsto gr(\xi(c))$ is continuous, which implies continuity of $\xi : M_\cdot X \to X$.

To show that $(X, \xi)$ is an $M_\cdot$-algebra, we again assume $c(x_0) = 1$ for a capacity $c \in M_\cdot X$. Then

$$y \oplus \alpha \xi(c) = y \oplus \alpha \sup\{ x_0 \oplus \beta x \mid x \in X, \beta \leq c(x) \} = \sup\{ y \oplus \alpha x_0 \oplus (\alpha \oplus \beta) x \mid x \in X, \beta \leq c(x) \} = \sup\{ (y \oplus \alpha x_0) \vee (y \oplus (\alpha \oplus \beta) x) \mid x \in X, \beta \leq c(x) \} = (y \oplus \alpha x_0) \vee \sup\{ y \oplus (\alpha \oplus \beta) x \mid x \in X, \beta \leq c(x) \} = \sup\{ y \oplus (\alpha \oplus \beta) x \mid x \in X, \beta \leq c(x) \}$$

holds for each $y \in X$, $\alpha \in I$. The second equality sign follows from an “infinite distributive law" $y \oplus \alpha \sup F = \sup\{ y \oplus \alpha x \mid x \in F \}$, with $F$ a nonempty subset of $X$. This law is first proved for finite $F$ and then extended to infinite case by continuity of lower upper bounds.

It is obvious that $\xi(\delta_x) = x$ for a point $x \in X$, i.e. $\xi \circ \eta_x X = 1_X$. We choose a capacity $\mathcal{C} \in M^2_\cdot X$ and compare $\xi \circ M_\cdot \xi(\mathcal{C})$ and $\xi \circ \mu_\cdot X(\mathcal{C})$. For a point $y \in X$ we
have
\[
y \lor (\xi \circ M_c \xi (\mathcal{C})) = \sup \{(y \oplus \alpha x) | x \in X, \alpha \leq M_c \xi (\mathcal{C})(x)\} = \min\{(y \oplus \alpha \xi (c) | c \in M_c X, \alpha \leq \mathcal{C}(c)\} = \sup \{\sup\{(y \oplus (\alpha \otimes \beta) x) | x \in X, \beta \leq c(x)\}, c \in M_c X, \alpha \leq \mathcal{C}(c)\} = \sup\{(y \oplus \alpha x) | x \in X, c \in M_c X, \alpha \leq \min\{\mathcal{C}(c), c(x)\}\} = y \lor (\xi \circ \mu_c X (\mathcal{C})).
\]
This implies \( \xi \circ M_c \xi = \xi \circ \mu_c X, \) i.e. \((X, \xi)\) is a \( \mathbb{M}_c \)-algebra.

To prove that the correspondence "\( \mathbb{M}_c \)-algebra \( \leftrightarrow \) idempotent convex combination that satisfies 1)–6)" is one-to-one, assume that for some continuous idc : \( X \times I \times X \) satisfying 1)–6) there is a continuous map \( \xi' : M_c X \rightarrow X \) such that \((X, \xi')\) is a \( \mathbb{M}_c \)-algebra and \( \text{idc}(x, \alpha, y) = \xi'(\delta_x \oplus \alpha \delta_y)\) for all \( x, y \in X, \alpha \in I \). Therefore \( \xi'(\delta_x \oplus \alpha \delta_y) = \xi(\delta_x \oplus \alpha \delta_y) \) for the constructed above map \( \xi \). Let \( 1 \geq \alpha_1 \geq \alpha_2 \geq 0, x_0, x_1, x_2 \in X \), then
\[
\xi(\delta_{x_0} \oplus \alpha_1 \delta_{x_1} \oplus \alpha_2 \delta_{x_2}) = \xi(\delta_{x_0} \oplus \alpha_1 \delta_{x_1} \oplus \alpha_2 \delta_{x_2}) = \xi(\delta_{x_0} \oplus \alpha_1 \delta_{x_1} \oplus \alpha_2 \delta_{x_2}) = \ldots = \xi'(\delta_{x_0} \oplus \alpha_1 \delta_{x_1} \oplus \alpha_2 \delta_{x_2}).
\]
By induction in a similar manner we prove that
\[
\xi(\delta_{x_0} \oplus \alpha_1 \delta_{x_1} \oplus \alpha_2 \delta_{x_2} \oplus \ldots \oplus \alpha_n \delta_{x_n}) = \xi'(\delta_{x_0} \oplus \alpha_1 \delta_{x_1} \oplus \alpha_2 \delta_{x_2} \oplus \ldots \oplus \alpha_n \delta_{x_n})
\]
for arbitrary integer \( n \geq 0 \). By continuity we deduce that \( \xi(c) = \xi'(c) \) for all \( c \in M_c X \).

Let \( \text{idc} : X \times I \times X \rightarrow X \) and \( \text{idc}' : X' \times I \times X' \rightarrow X' \) be idempotent convex combinations. We say that a map \( f : (X, \text{idc} \rightarrow (X', \text{idc}') \) is affine if it preserves idempotent convex combination, i.e. \( f(\text{idc}(x, \alpha, y)) = \text{idc}'(f(x), \alpha, f(y)) \) for all \( x, y \in X, \alpha \in I \).

**Theorem 2.2.** Let \((X, \xi), (X', \xi')\) be \( \mathbb{M}_c \)-algebras, \( \text{idc} : X \times I \times X \rightarrow X \) and \( \text{idc}' : X' \times I \times X' \rightarrow X' \) be the respective idempotent convex combinations. Then a continuous map \( f : X \rightarrow Y \) is a morphism of \( \mathbb{M}_c \)-algebras \((X, \xi) \rightarrow (X', \xi')\) if and only if \( f : (X, \text{idc}) \rightarrow (X', \text{idc}') \) is affine.

**Proof.** Necessity. Let \( f : (X, \xi) \rightarrow (X', \xi') \) be a morphism of \( \mathbb{M}_c \)-algebras, \( x, y \in X, \alpha \in I \). Then
\[
f(\text{idc}(x, \alpha, y)) = f(\xi(\delta_x \oplus \alpha \delta_y)) = \xi'(f(\delta_x \oplus \alpha \delta_y)) = \xi'(\delta_{f(x)} \oplus \alpha \delta_{f(y)}) = \text{idc}'(f(x), \alpha, f(y)).
\]

Sufficiency. Let \( f : (X, \text{idc}) \rightarrow (X', \text{idc}') \) be affine, then \( f(x \lor y) = f(x) \lor f(y) \) for all \( x, y \in X \). Continuity of \( f \) implies that \( f \) preserves suprema of closed sets. For \( c \in M_c X \) we choose a point \( x_0 \in X \) such that \( c(x_0) = 1 \), then \( M_c f(x)(f(x_0)) = 1 \). Therefore :
\[
\xi'(M_c f(c)) = \sup\{(f(x_0) \oplus \alpha x', x' \in X', \alpha \leq M_c f(c)(x')\} = \sup\{f(x_0) \oplus \alpha f(x) | x \in X, \alpha \leq c(x)\} = \sup\{f(x_0) \oplus \alpha c(x) | x \in X, \alpha \leq c(x)\} = f(\sup\{x_0 \oplus \alpha x | x \in X, \alpha \leq c(x)\}) = f(\xi(c)),
\]
and \( f \) is a morphism of \( \mathbb{M}_c \)-algebras.
Theorem 2.4. A pair of a compactum $X$ and a continuous map $\text{ic}: X \times I \times X \to X$ is a (max, min)-idempotent convex compactum if and only if $X$ is a closed convex subset of a compact Lawson (max, min)-idempotent semimodule $(L, \oplus, \otimes)$ such that $\text{ic}(x, \alpha, y) \equiv x \oplus \alpha y$.

Proof. Sufficiency is obvious. To prove necessity, assume that $X$ is a compactum and a continuous map $\text{ic}: X \times I \times X \to X$ satisfies conditions 1)–6). We define an equivalence relation “$\sim$” on $X \times I$ as follows: $(x_1, a_1) \sim (x_2, a_2)$ if $y \oplus a_1 x_1 = y \oplus a_2 x_2$ for all $y \in L$. Then we call $(L, \oplus, \otimes)$ a compact Lawson (max, min)-idempotent semimodule $(L, \oplus, \otimes)$ such that $\text{ic}(x, \alpha, y) \equiv x \oplus \alpha y$. 

Remark 2.3. It is easy to see that (max, min)-idempotent convex compacta and their affine continuous maps constitute a category $\text{Conv}_{\text{max}, \text{min}}$ of (max, min)-idempotent convex compacta that by the latter theorem is monadic (=tripleable) [17] over the category of compacta.

Convex compacta are usually defined as compact closed subsets of locally convex topological vector spaces. To obtain a similar description for (max, min)-idempotent convex compacta, we need some extra definitions and facts. For an idempotent semiring $S = (S, \oplus, \otimes, 0, 1)$ a (left idempotent) $S$-semimodule is a set $L$ with operations $\alpha \ominus y = x \oplus (y \ominus x)$, for all $x, y \in L, \alpha \in S$:

1) $x \oplus y = y \oplus x$;
2) $(x \oplus y) \ominus z = x \oplus (y \ominus z)$;
3) there is an (obviously unique) element $0 \in L$ such that $x \ominus 0 = x$ for all $x$;
4) $0 = \ominus 1$ = $\beta x$;
5) $x \ominus \beta y = x = \alpha \ominus (\beta \ominus x)$;
6) $\Delta_x \ominus y = \ominus x$;
7) $0 \ominus x = 0$.

We adopt the usual convention and write $\alpha x$ instead of $\alpha \ominus x$. Observe that these axioms imply $\alpha 0 = 0$, $x \ominus x = x$. Informally speaking, an idempotent semimodule is a vector space over an idempotent semiring.

If $S = (I, \text{max}, \text{min}, 0, 1)$, we will talk about a (max, min)-idempotent semimodule. In this case we define an operation $\text{ic}: L \times I \times L \to L$ by the formula $\text{ic}(L, x, \alpha, y) = x \oplus (\alpha \ominus y)$, for all $x, y \in L$. It is easy to see that $\text{ic}$ satisfies 1)–5). The combination $\alpha_0 x_0 \oplus \alpha_1 x_0 \oplus \ldots \oplus \alpha_n x_n$ of points $x_0, x_1, \ldots, x_n$ is defined in an obvious way and coincides with the described above operation if $(\alpha_0, \alpha_1, \ldots, \alpha_n) \in \Delta^n_x$. A subset $A$ of a (max, min)-idempotent semimodule $L$ is called convex if $x \oplus \alpha y \in A$ whenever $x, y \in A$, $\alpha \in I$. A convex subset $A \subset L$ contains all idempotent convex combinations of its elements.

Let a (max, min)-idempotent semimodule $L$ be a compactum, the operations $\oplus$ and $\otimes$ be continuous, and the topology on $L$ satisfy an additional condition:

8) for a neighborhood $U$ of any element $x \in L$ there is a neighborhood $V$ of $x$, $V \subset U$, such that $y \ominus z \in V$ for all $y, z \in V$.

Then we call $(L, \oplus, \otimes)$ a compact Lawson (max, min)-idempotent semimodule.

By the above theorem $L$ is a $\mathcal{M}_L$-algebra, which implies

8+) for a neighborhood $U$ of any element $x \in L$ there is a neighborhood $V$ of $x$, $V \subset U$, such that $y \ominus \alpha z \in V$ for all $y, z \in V$, $\alpha \in I$.

Thus for every point of $L$ there is a local base that consists of convex neighborhoods, and we say that $L$ is locally convex.

The nature of a compactum $X$ with an idempotent convex combination that satisfies 1)–6) is clarified by the following
This relation is closed in \((X \times I) \times (X \times I)\), therefore the quotient space \(X \times I/\sim\), which we denote by \(\bar{X}\), is a compact Hausdorff space. We also denote by \([\langle x, a \rangle]\) the equivalence class of the pair \((x, a)\). The map \(i : X \to \bar{X}\) that sends a point \(x \in X\) to \([\langle x, 1 \rangle]\) is an embedding because \((x_1, 1) \sim (x_2, 1)\) is possible only if \(x_1 = x_2\).

We define operations \(\otimes : I \times \bar{X} \to \bar{X}\) and \(\oplus : \bar{X} \times \bar{X} \to \bar{X}\) by the formulae:

\[
[x, a] \oplus [y, b] = \begin{cases} 
[x \oplus by, a \geq b], & a \geq b \\
[y \oplus ay, b], & a \leq b.
\end{cases}
\]

The element \(\bar{0} = [(x, 0)]\) does not depend on \(x\) and satisfies 3). Properties 5), 6), 7) are obvious. Verification that \(\oplus, \otimes\) are well defined, continuous and satisfy 1), 2), 4), 8), is more convenient with a generalization of the mapping \(gr : X \to X^X\) that was defined in the proof of the latter theorem. To avoid introducing extra denotations, we denote by \(gr(x, a)\), where \(x \in X, a \in I\), the collection \((t \oplus ax)_{t \in X}\). Then the map \(gr : X \times I \to X^X\) is continuous (but, as can be shown, not injective).

It is obvious that \((x_1, a_1) \sim (x_2, a_2)\) if and only if \(gr(x_1, a_1) = gr(x_2, a_2)\), thus we will identify the image of the map \(gr\) with the quotient space \(\bar{X} = X \times I/\sim\), and \(gr\) with the quotient map.

Let \(\bar{x}, \bar{y}, \bar{z}\) be points in \(\bar{X}\), and \(\bar{x} = gr(x, a) = (x_t)_{t \in X}, \bar{y} = gr(y, b) = (y_t)_{t \in X}, \bar{z} = gr(z, c) = (z_t)_{t \in X}\). Observe that \(x \oplus y = (x_t \lor y_t)_{t \in X}, a \otimes x = (t \oplus ax_t)_{t \in X}\), therefore \(\bar{x} \oplus \bar{y}\) and \(a \otimes \bar{x}\) are uniquely determined and continuous w.r.t. \(\bar{x}, \bar{y}\) and \(a, \bar{x}\) resp. Similar expressions can be written for \(x \ominus y\) and \(y \ominus z\), and 1), 2) are easily seen. Next, \(a \otimes \bar{x} = (t \oplus ax_t)_{t \in X}\), \(a \ominus \bar{y} = (t \ominus ay_t)_{t \in X}\), thus

\[
(a \otimes \bar{x}) \ominus (a \ominus \bar{y}) = ((t \ominus ax_t) \lor (t \oplus ay_t))_{t \in X} = ((t \oplus ax_t \lor ax_t))_{t \in X} = (a \ominus \bar{x} \ominus \bar{y}).
\]

Similarly,

\[
(a \otimes \bar{x}) \ominus (b \otimes \bar{x}) = ((t \ominus ax_t) \lor (t \oplus bx_t))_{t \in X} = ((t \oplus ax_t \lor bx_t))_{t \in X} = (a \ominus \bar{x} \ominus \bar{y}).
\]

and condition 4) holds.

Let \(G \subset \bar{X}\) be a closed nonempty set, then \(G = gr(F)\) for some closed \(F \subset X \times I\). There is \((x_0, a_0) \in F\) such that \(a_0 = \max\{a \; | \; (x, a) \in F\}\). It is easy to show that sup \(G\) in \(\bar{X}\) is equal to \([\langle x', a_0 \rangle]\) where \(x' = \sup\{x_0 \ominus ax \; | \; (x, a) \in F\}\), thus the upper semilattice \(\bar{X}\) is complete. It is also clearly seen that

\[
gr(x', a_0) = (\sup\{t \ominus ax' \; | \; (x, a) \in F\})_{t \in X} = (\sup\{x_t \; | \; (x_t)_{t \in X} \in G\})_{t \in X},
\]

therefore sup \(G\) depends on \(G\) continuously w.r.t. Vietoris topology. It is a statement equivalent to 8). \(\Box\)

As triples \(M_\Delta\) and \(M_{\otimes}\) are isomorphic through a natural transformation \(\simeq\) defined in [9], and the map \(I \to I\) that sends each \(t\) to \(1 - t\) is an isomorphism of the idempotent semirings \((I, \oplus, \otimes, 0, 1)\) and \((I, \lor, \otimes, 1, 0)\), by duality we immediately can state an analogue of Theorem 2.51. Its proof can be obtained by replacing \(M_\Delta\) by \(M_{\otimes}\), by \(\oplus\), by \(\otimes\), by \(0\), upper semilattices by lower ones, \(\lor\) by \(\land\), sup by inf, \(\Delta_{\otimes}\) by the (idempotent) \(n\)-dimensional \(\otimes\)-simplex:

\[
\Delta^n_{\otimes} = \{(\alpha_0, \alpha_1, \ldots, \alpha_n) \in I^{n+1} \mid \alpha_0 \otimes \alpha_1 \otimes \ldots \otimes \alpha_n = 0\}
\]
and vice versa, where it is necessary. Thus we define dual idempotent convex combinations and (min, max)-idempotent convex compacta that are precisely $\mathbb{M}_-$-algebras. We omit obvious details. Observe that for a given $\mathbb{M}_-$-algebra $(X, \xi)$ the respective dual idempotent convex combination $\xi : X \times I \times X \to X$ is determined by the equality $\xi(x, \alpha, y) = (\delta_d \wedge (\alpha \vee \delta_y))$. Conversely, the value $\xi(c)$ for a capacity $c \in \mathcal{M}_C X$ (assuming that $c(X \setminus \{x_0\}) = 0$) is equal to $\xi(c) = \inf \{\xi(c(x_0, \alpha, x)) \mid x \in X, \alpha \geq c(X \setminus \{x\})\}$.

It is easy also to formulate analogues of Theorems 2.2, 2.4.

3. Algebras for the capacity monad

In the sequel a (min, max)-idempotent biconvex compactum is a compactum $X$ with four operations $\oplus : X \times X \to X$, $\odot : I \times X \to X$, $\odot : X \times X \to X$, $\oplus : I \times X \to X$ such that $(X, \oplus, \odot)$ is a Lawson lattice, $(X, \odot, \odot)$ is an $(I, \odot, \odot)$-semimodule, $(X, \odot, \odot)$ is an $(I, \odot, \odot)$-semimodule, the associative laws $(\alpha \odot x) \odot y = \alpha \odot (x \odot y)$, $(\alpha \odot x) \odot y = \alpha \odot (x \odot y)$ and the distributive laws $\alpha \odot (\beta \odot x) = (\alpha \odot \beta) \odot (\alpha \odot x)$, $\alpha \odot (\beta \odot x) = (\alpha \odot \beta) \odot (\alpha \odot x)$ are valid for all $x, y \in X$, $\alpha, \beta \in I$.

**Theorem 3.1.** Let $X$ be a compactum. There is a one-to-one correspondence between:

1) continuous maps $\xi : MX \to X$ such that the pair $(X, \xi)$ is an $\mathbb{M}$-algebra;
2) quadruples $(\oplus, \odot, \odot, \odot)$ of continuous operations $\oplus : X \times X \to X$, $\odot : I \times X \to X$, $\odot : X \times X \to X$, $\oplus : I \times X \to X$ such that $(X, \odot, \odot, \odot, \odot)$ is a (min, max)-idempotent biconvex compactum;
3) quadruples $(\odot, \odot, p, m)$ of continuous maps $\odot : X \times X \to X$, $\odot : X \times X \to X$, $p : I \to X$ such that

a) $(X, \odot, \odot)$ is a Lawson lattice;
b) $p : (I, \odot) \to (X, \odot)$ is a morphism of upper semilattices that preserves a top element;
c) $m : (I, \odot) \to (X, \odot)$ is a morphism of lower semilattices that preserves a bottom element;
d) for all $\alpha, \beta \in I$ we have $m(\alpha) \odot p(\beta) = p(\alpha \odot \beta)$, $m(\alpha) \odot p(\beta) = m(\alpha \odot \beta)$.

In the case 2) the following property of local biconvexity holds: for a neighborhood $U$ of any element $x \in X$ there is a neighborhood $V$ of $x$, $V \subset U$, such that $y \odot (\alpha \odot z) \in V$, $y \odot (\alpha \odot z) \in V$ for all $y, z \in V$, $\alpha \in I$.

**Proof.** 1)→3). Let $(X, \xi)$ be an $\mathbb{M}$-algebra. We use the fact that $\mathbb{G}$ is a submonad of the capacity monad $\mathbb{M}$. The components of an embedding $i_G : \mathbb{G} \hookrightarrow \mathbb{M}$ are of the form

$$i_G X(A)(F) = \begin{cases} 1, & \text{if } F \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $(X, \xi \circ i_G X)$ is a $\mathbb{G}$-algebra. Theorem 2 [13] states that for a $\mathbb{G}$-algebra $(X, \theta)$ the operations $\odot : X \times X \to X$, $\odot : I \times X \to X$ defined by the formulae $x \odot y = \theta(y \odot (x \cap x \odot y))$ and $x \odot y = \theta(y \odot (x \cap x \odot y))$ are such that $(X, \odot, \odot)$ is a Lawson lattice. We apply this theorem to $\theta = \xi \circ i_G X$ and obtain that $X$ with the operations $x \odot y = \xi(\delta_x \vee \delta_y)$ and $x \odot y = \xi(\delta_x \wedge \delta_y)$ is a Lawson lattice. We denote by $0$ and $1$ its least and greatest elements. Now we put $p(\alpha) = \xi(\delta_0 \vee \alpha \odot \delta_1)$, $m(\alpha) = \xi(\delta_1 \wedge \alpha \odot \delta_0)$. It is obvious that $p, m$ are continuous and
we can show that $x$ above, then distributive laws in ($\alpha \cup \beta$) as well as ($\alpha \cup \beta$) ($\alpha \cup \beta$). It is sufficient to put $X \rightarrow \sigma$.

Similarly $m(\alpha \oplus \beta) = m(\alpha) \oplus m(\beta)$ for all $\alpha, \beta \in I$. We also have

$m(\alpha) \otimes p(\beta) = \xi(\delta(\alpha \cup \beta) \wedge \delta_1) = \xi \circ M \xi(\delta(\alpha \cup \beta) \wedge \delta_1) = $ \[ \xi \circ M \xi(\delta(\alpha \cup \beta) \wedge \delta_1) \]

as well as $m(\alpha) \otimes p(\beta) = m(\alpha \oplus \beta)$. 

3$\rightarrow$2). It is sufficient to put $\alpha \otimes x = m(\alpha) \oplus x$, $\alpha \otimes x = p(\alpha) \oplus x$, and it is clear that all conditions of 2) are satisfied due to the commutative, associative and distributive laws in ($X, \oplus, \ominus$).

Observe also that, if $m, p$ are determined by an $A$-algebra $(X, \xi)$ as described above, then

$\alpha \otimes x = \xi(\delta(\alpha \cup \beta) \wedge \delta_2) = \xi(\delta(\alpha \cup \beta) \wedge \delta_2) = \xi \circ M(\xi(\delta(\alpha \cup \beta) \wedge \delta_1)) = $ \[ \xi \circ M(\xi(\delta(\alpha \cup \beta) \wedge \delta_1)) \]

and similarly $\alpha \oplus x = \xi(\delta_1 \wedge \delta_2)$ for all $x \in X, \alpha \in I$. In the same manner we can show that $x \oplus(\alpha \otimes y) = \xi(\delta_1 \wedge \delta_0 \wedge \delta_2)$, $x \ominus(\alpha \otimes y) = \xi(\delta_1 \wedge \alpha \wedge \delta_0)$ for all $x, y \in X, \alpha \in I$. These formulae are the same that were used to define idempotent semiconvex combinations and dual idempotent semiconvex combinations in the proofs of Theorem 2.1 and the dual theorem.

2$\rightarrow$1).

Now let $(X, \oplus, \ominus, \ominus)$ be a (min, max)-idempotent biconvex compactum. If $ic(x, \alpha, y) = x \ominus(\alpha \otimes y)$, $ci(x, \alpha, y) = x \otimes(\alpha \otimes y)$, then it is obvious that $(X, ic)$ is a (min, max)-idempotent convex compactum and $(X, ci)$ is a (max, min)-idempotent convex compactum. Thus by Theorem 2.1 and the dual theorem, if mappings $\xi, \xi : M \rightarrow X$ and $\xi, \xi : M \rightarrow X$ are defined by the formulæ

$\xi(c) = \sup\{x \cup(\alpha \times x) \mid x \in X, \alpha \leq c\}, \ c \in M, x_0 \in X, c(x_0) = 1, \ and$

$\xi(c) = \inf\{x \cup(\alpha \times x) \mid x \in X, \alpha \geq c(x)\}, \ c \in M, x_0 \in X, c(X \setminus x_0) = 0$, then the pairs $(X, \xi, \xi)$ are resp. an $\mathbb{M}_0$-algebra and an $\mathbb{M}_0$-algebra. In our case we can define $\xi, \xi$ by simpler but equivalent formulæ (the second “$\pm$” sign in each equality is due to complete distributivity of a compact Lawson lattice) :

$\xi(c) = \sup\{c(x) \otimes x \mid x \in X\} = \inf\{c(X \setminus A) \oplus sup A \mid A \subseteq X\}, \ c \in M, X,$

and

$\xi(c) = \inf\{c(X \setminus \{x\}) \oplus x \mid x \in X\} = \sup\{c(X \setminus A) \ominus inf A \mid A \subseteq X\}, \ c \in M, X.$

If $\xi, \xi : M \rightarrow X$ are continuous maps such that the pairs $(X, \xi), (X, \xi)$ are $\mathbb{M}$-algebras and $\xi|_{M_0} = \xi|_{M_0}, \xi|_{M_0} = \xi_0, \xi|_{M_0} = \xi_0$, then the two following
diagrams have to be commutative (we omit explicit notations for restrictions):

\[
\begin{array}{ccc}
M_\mu M_\nu X & \xrightarrow{\mu X} & MX \\
M_\mu \xi \downarrow & & \downarrow (\ast) \\
M_\mu & \xrightarrow{\xi} & X
\end{array}
\quad \begin{array}{ccc}
M_\rho M_\nu X & \xrightarrow{\mu X} & MX \\
M_\rho \xi \downarrow & & \downarrow (\ast) \\
M_\rho & \xrightarrow{\xi'} & X
\end{array}
\]

We show that if \( C, C' \subseteq M_\nu X \) are such that \( \mu X(C) = \mu X(C') \), then \( \xi \circ M_\nu \xi(C) = \xi \circ M_\nu \xi(C') \). Observe that \( \mu X(C) = \mu X(C') \) implies that for all \( A \subseteq X \) and \( \alpha \in I \) the existence of \( c \in M_\nu X \) such that \( C(c) \geq \alpha \) and \( c(A) \geq \alpha \) is equivalent to the existence of \( c' \in M_\nu X \) such that \( C'(c') \geq \alpha \) and \( c'(A) \geq \alpha \). It is also obvious that the same statement is valid for any open \( A \subseteq X \). Thus:

\[
\xi \circ M_\nu \xi(C) = \sup \{ M_\nu \xi(C)(x) \otimes x \mid x \in X \} = \\
\sup \{ \xi \circ \xi(C)(c) \mid c \in M_\nu X \} = \\
\sup \{ C(c) \otimes \sup \{ c(X \setminus A) \otimes \inf A \mid A \subseteq X \} \mid c \in M_\nu X \} = \\
\sup \{ C(c) \otimes c(X \setminus A) \otimes \inf A \mid A \subseteq X, c \in M_\nu X \} = \\
\sup \{ \alpha \otimes \inf A \mid A \subseteq X, c \in M_\nu X, \alpha \otimes \inf A \} = \\
\sup \{ \alpha \otimes \inf A \mid A \subseteq X, c' \in M_\nu X, \alpha \otimes \inf A \} = \\
\cdots = \xi \circ M_\nu \xi(C').
\]

An obvious dual statement is also valid. Taking into account that by Theorem 8 [9] for a compactum \( X \) the equality \( \mu(M_\nu M_\mu X) = \mu(M_\mu M_\nu X) = MX \) is valid, and \( \mu X(M_\nu M_\mu X) : M_\mu M_\nu X \to MX \) and \( \mu X(M_\mu M_\nu X) : M_\nu M_\mu X \to MX \) are quotient maps as continuous surjective maps of compacta, we obtain that the diagrams \((\ast)\) and \((\ast\ast)\) uniquely determine continuous maps \( \xi, \xi' : MX \to X \).

In the diagram

\[
\begin{array}{ccc}
M^2_\mu M_\nu X & \xrightarrow{M_\mu M_\nu X} & M_\mu M_\nu X \\
M_\mu \mu X \downarrow & & \downarrow M_\mu \xi \\
M^2_\mu X & \xrightarrow{\mu X} & M_\mu X \\
M_\mu \xi \downarrow & & \downarrow \xi \\
M_\mu X & \xrightarrow{\mu X} & MX \\
\mu X \downarrow & & \downarrow \xi \\
M_\mu X & \xrightarrow{\mu X} & MX \\
\xi \downarrow & & \downarrow \xi \\
M_\mu X & \xrightarrow{\mu X} & MX \\
\xi \downarrow & & \downarrow \xi \\
M_\mu X & \xrightarrow{\mu X} & MX
\end{array}
\]

the top square and the side squares are commutative, and the leftmost vertical arrow is epimorphic, therefore the bottom square commutes as well. Using also dual arguments, we show that the two following diagrams are commutative:

\[
\begin{array}{ccc}
M_\mu MX & \xrightarrow{\mu X} & MX \\
M_\mu \xi \downarrow & & \downarrow \xi \\
M_\mu X & \xrightarrow{\mu X} & MX
\end{array}
\quad \begin{array}{ccc}
M_\mu MX & \xrightarrow{\mu X} & MX \\
M_\mu \xi \downarrow & & \downarrow \xi \\
M_\mu X & \xrightarrow{\mu X} & MX
\end{array}
\]

\[
\begin{array}{ccc}
M_\rho M_\nu X & \xrightarrow{\mu X} & MX \\
M_\rho \xi \downarrow & & \downarrow \xi \\
M_\rho X & \xrightarrow{\mu X} & MX
\end{array}
\quad \begin{array}{ccc}
M_\rho M_\nu X & \xrightarrow{\mu X} & MX \\
M_\rho \xi \downarrow & & \downarrow \xi \\
M_\rho X & \xrightarrow{\mu X} & MX
\end{array}
\]
We apply the functor $M_\circ$ to the left diagram and combine it with (**):

\[
\begin{array}{ccc}
M_\circ M_\circ M_\circ & \xrightarrow{\mu M_X} & M_\circ M_\circ X \xrightarrow{M_\circ \xi} M_\circ X \\
M_\circ M_\circ X & \xrightarrow{M_\circ \mu X} & M_\circ M_\circ X \xrightarrow{\mu X} M_\circ X \\
M_\circ M_\circ X & \xrightarrow{\mu X} & M_\circ M_\circ X \xrightarrow{\mu X} M_\circ X \\
M_\circ M_\circ X & \xrightarrow{\mu X} & M_\circ M_\circ X \xrightarrow{\mu X} M_\circ X \\
M_\circ M_\circ X & \xrightarrow{\mu X} & M_\circ M_\circ X \xrightarrow{\mu X} M_\circ X
\end{array}
\]

The restriction of $\mu M_X$ to $M_\circ M_\circ M_\circ X$ is an epimorphism, therefore the commutativity of the outer contour imply that the left of the two following diagrams commutes. The right diagram is commutative by dual arguments.

\[
\begin{array}{ccc}
M_\circ M_\circ X & \xrightarrow{\mu X} & M_\circ X \\
M_\circ X & \xrightarrow{\mu X} & M_\circ X \\
X & \xrightarrow{\xi} & X
\end{array}
\]

\[
\begin{array}{ccc}
M_\circ M_\circ X & \xrightarrow{\mu X} & M_\circ X \\
M_\circ X & \xrightarrow{\mu X} & M_\circ X \\
X & \xrightarrow{\xi} & X
\end{array}
\]

Therefore in the diagram

\[
\begin{array}{ccc}
M_\circ M_\circ M_\circ X & \xrightarrow{\mu M_X} & M_\circ M_\circ X \\
M_\circ M_\circ X & \xrightarrow{\mu X} & M_\circ M_\circ X \\
M_\circ M_\circ X & \xrightarrow{\mu X} & M_\circ M_\circ X \\
M_\circ M_\circ X & \xrightarrow{\mu X} & M_\circ M_\circ X \\
M_\circ M_\circ X & \xrightarrow{\mu X} & M_\circ M_\circ X
\end{array}
\]

the front square is commutative, which is the “harder part” of the definition of $M_\circ$-algebra. A proof of the “easier part” $\xi \circ \eta X = 1_X$ is straightforward. Thus $(X, \xi)$ is a unique $M_\circ$-algebra such that $x \oplus (\alpha \otimes y) = \xi(\delta_x \vee (\alpha \otimes \delta_y))$ and $x \otimes (\alpha \oplus y) = \xi(\delta_x \wedge (\alpha \oplus \delta_y))$ for all $x, y \in X$, $\alpha \in I$. As a by-product we obtain that $\xi = \xi'$, i.e. definitions of $\xi$ by the diagrams (*) and (**) are equivalent.

To prove local biconvexity, for a given neighborhood $U$ of a point $x$ by continuity of $\xi$ and the equality $\xi(\delta_x) = x$ we choose a neighborhood $\tilde{U} \subset M_\circ X$ of $\delta_x$ such that for all $c \in \tilde{U}$ we have $\xi(c) \in U$. There exists a neighborhood $\tilde{U} \ni x$ such that for all $x_0, x_1, \ldots, x_n \in \tilde{U}$, $(\alpha_0, \alpha_1, \ldots, \alpha_n) \in \Delta^n_\alpha$ we have $\alpha_0 x_0 \cup \alpha_1 x_1 \cup \cdots \cup \alpha_n x_n \in \tilde{U}$.

Now we choose a neighborhood $\tilde{U} \subset M_\circ X$ of $\delta_x$ such that for all $c \in \tilde{U}$ we have $\xi(c) \in \tilde{U}$. There is also a neighborhood $\tilde{U} \ni x$ such that $y_0, y_1, \ldots, y_n \in \tilde{U}$,
\((a_0, a_1, \ldots, a_n) \in \Delta^n\) imply \(a_0 \delta y_0 \wedge a_1 \delta y_1 \wedge \cdots \wedge a_n \delta y_n \in \hat{U}\). Now we put
\[
\hat{V} = \{(a_0 \circ y_0) \oplus (a_1 \circ y_1) \oplus \cdots \oplus (a_n \circ y_n) \mid n \in \{0, 1, \ldots\}, (a_0, a_1, \ldots, a_n) \in \Delta^n, y_0, y_1, \ldots, y_n \in \hat{U}\},
\]
and the set
\[
V = \{(a_0 \circ y_0) \oplus (a_1 \circ y_1) \oplus \cdots \oplus (a_n \circ y_n) \mid n \in \{0, 1, \ldots\}, (a_0, a_1, \ldots, a_n) \in \Delta^n, x_0, x_1, \ldots, x_n \in \hat{V}\}
\]
is a neighborhood of \(x\) requested by local bicommutativity.

For \((\max, \min)\)-idempotent biconvex compacta \((X, \oplus, \odot, \odot, \odot)\) and \((X', \oplus, \odot, \odot, \odot)\) we say that a map \(f : X \to X'\) is biaffine if it preserves idempotent convex combination and the dual idempotent convex combination, i.e. \(f(x \oplus (\alpha \circ y)) = f(x) \oplus (\alpha \circ f(y))\) whenever \(x, y \in X, \alpha \in I\).

**Theorem 3.2.** Let \((X, \xi), (X', \xi')\) be \(M\)-algebras and quadruples \((\oplus, \odot, \odot, \odot)\) of continuous operations be determined on \(X\) and \(X'\) by \(\xi\) and \(\xi'\) resp. (in the sense of Theorem \[2.7\]). Then a continuous map \(f : X \to Y\) is a morphism of \(M\)-algebras \((X, \xi) \to (X', \xi')\) if and only if \((X, \oplus, \odot, \odot, \odot) \to (X', \oplus, \odot, \odot, \odot)\) is biaffine.

**Proof.** Necessity. Let \(f\) be a morphism of algebras. It was shown in the proof of the previous theorem that the idempotent convex combination and the dual idempotent convex combination of points \(x, y \in X\) are determined by the formulae \(x \oplus (\alpha \circ y) = \xi(\delta_x \vee \alpha \circ \delta_y), x \odot (\alpha \circ y) = \xi(\delta_x \wedge \alpha \odot \delta_y)\) (in \(X'\) the same but \(\xi\) replaced with \(\xi'\)). Then we follow the line of the proof of Theorem \[2.2\].

Sufficiency. Let \(f\) be biaffine. Then by Theorem \[2.2\] and a dual theorem \(f\) is a morphism of \(M\)-algebras \((X, \xi|_{M(X)} \to (X', \xi'|_{M(X')})\) and a morphism of \(M\)-algebras \((X, \xi|_{M(X)} \to (X', \xi'|_{M(X')}\), i.e. the diagrams

\[
\begin{array}{ccc}
M, X & \xrightarrow{Mf} & M, X' \\
\xi|_{M(X)} & \downarrow & \xi'|_{M(X')} \\
X & \xrightarrow{f} & X'
\end{array}
\]

are commutative. Therefore the top face and the side faces of the diagram

\[
\begin{array}{ccc}
M, M, M, X & \xrightarrow{M, M, Mf} & M, M, X' \\
\mu X & \downarrow & \mu X' \\
M, X & \xrightarrow{Mf} & M, X'
\end{array}
\]

\[
\begin{array}{ccc}
M, M, X & \xrightarrow{Mf} & M, M, X' \\
\xi|_{M(X)} & \downarrow & \xi'|_{M(X')} \\
MX & \xrightarrow{f} & MX'
\end{array}
\]

\[
\begin{array}{ccc}
MX & \xrightarrow{f} & MX'
\end{array}
\]

commute. The leftmost arrow \(\mu X : M, M, X \to MX\) is an epimorphism, thus the bottom face commutes as well, i.e. \(f\) is a morphism of \(M\)-algebras. \(\square\)
Remark 3.3. The latter theorem implies that the category BiConv_{max,min} of (max, min)-idempotent biconvex compacta and their continuous biaffine maps is monadic over the category of compacta.

Remark 3.4. Note that a biaffine map \( f : (X, \oplus, \ominus, \odot, \ominus) \to (X', \oplus, \ominus, \odot, \ominus) \) not necessarily preserves operations \( \odot \) and \( \ominus \) (although it preserves \( \oplus \) and \( \ominus \)). E.g., let \( X = X' = I, \oplus = \ominus = \max, \ominus = \ominus = \min \), \( f(x) = \max\{x, \frac{1}{2}\} \). Then \( f \) is biaffine, but \( f(0 \otimes 1) = \frac{1}{2} \neq 0 \otimes f(1) = 0 \). It is easy to show that a biaffine continuous map \( f : (X, \oplus, \ominus, \odot, \ominus) \to (X', \oplus, \ominus, \odot, \ominus) \) preserves \( \odot \) iff it preserves a bottom element, and it preserves \( \ominus \) iff it preserves a top element.

We present an example of (max, min)-idempotent biconvex compacta. Let \( A \) be a set and for each \( a \in A \) a non-decreasing surjective map \( \varphi_a : I \to I \) is fixed. For \( x, y \in I^A \), \( x = (x_a)_{a \in A}, y = (y_a)_{a \in A}, \alpha \in I \) we put \( x \otimes y = (\max\{x_a, y_a\})_{a \in A}, x \ominus y = (\min\{x_a, y_a\})_{a \in A}, \alpha \ominus x = (\max\{\varphi_a(\alpha), x_a\})_{a \in A}, \alpha \ominus x = (\max\{\varphi_a(\alpha), x_a\})_{a \in A} \). Then \( (X, \oplus, \ominus, \odot, \ominus) \) obviously satisfies the definition. In communication with M. Zarichnyi a question arose:

Question 3.5. Does every (max, min)-idempotent biconvex compactum biaffinely embed into some \( I^A \) with the defined above operations?

Provided the answer is positive, any biaffine map \( f : (X, \oplus, \ominus, \odot, \ominus) \to (X', \oplus, \ominus, \odot, \ominus) \) algebraically (with preservation of idempotent and dual idempotent convex combinations) and topologically embeds into a biaffine map that is a projection of some \( I^A \) onto \( I^B, B \subset A \) (operations on \( I^A \) and \( I^B \) are defined as above).

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