JENSEN POLYNOMIALS ARE NOT A PLAUSIBLE ROUTE TO PROVING THE RiemANN HYPOTHESIS

DAVID W. FARMER

ABSTRACT. Recent work on the Jensen polynomials of the Riemann xi-function and its derivatives found a connection to the Hermite polynomials. Those results have been suggested to give evidence for the Riemann Hypothesis, and furthermore it has been suggested that those results shed light on the random matrix statistics for zeros of the zeta-function. We place that work in the context of prior results, and explain why the appearance of Hermite polynomials is interesting and surprising, and may represent a new type of universal law which refines M. Berry’s “cosine as a universal attractor” principle. However, we find there is no justification for the suggested connection to the Riemann Hypothesis, nor for the suggested connection to the conjectured random matrix statistics for zeros of L-functions. These considerations suggest that Jensen polynomials, as well as a large class of related polynomials, are not useful for attacking the Riemann Hypothesis. We propose general criteria for determining whether an equivalence to the Riemann Hypothesis is likely to be useful.

1. Introduction

Two recent papers \[10, 2\] revisit the classical result of Jensen \[12, 18\] that the Riemann Hypothesis (RH) is true if and only if all of the associated Jensen polynomials, defined in \eqref{2.2} below, have only real zeros. The two recent papers actually concern another version of the Jensen polynomials, which we call the “even” Jensen polynomials, defined in \eqref{3.2}. An interesting connection was found with the Hermite polynomials.

In this paper we examine the recent work on Jensen polynomials in the context of prior work on repeated differentiation of entire functions \[1, 9, 13, 14\], and on differentiation-like operations \[8, 22\]. That perspective explains why the new connection to Hermite polynomials is interesting, but it also suggests why there is no connection to the Riemann Hypothesis nor to the random matrix statistics of zeros of the zeta function. These considerations further suggest that the Jensen polynomials, as well as a large class of related polynomials, are not a useful tool for approaching the Riemann Hypothesis. We introduce terminology which can serve as a guide to deciding whether an equivalence to RH is likely to be useful for resolving the Riemann Hypothesis, or if the equivalence is just a curiosity.

---

Key words and phrases. Jensen polynomial, Riemann Hypothesis, zeta function, xi function, GUE, Hermite polynomial, cosine universality, L-function.

This research was supported by the National Science Foundation.
2. **The classical Jensen polynomials**

Suppose

$$f(z) = \sum_{j=0}^{\infty} \frac{\alpha(j)}{j!} z^j \tag{2.1}$$

is an entire function of order less than two. One can associate the \textit{dth classical Jensen polynomial for the nth derivative of} \( f \), given by

$$J_{f,cl}^{d,n}(z) := \sum_{j=0}^{d} \binom{d}{j} \alpha(j+n) z^j. \tag{2.2}$$

The “\( cl \)” in the subscript refers to these polynomials being “classical” in the sense that (2.2) is the standard definition of the Jensen polynomials. An alternate notation for those polynomials is \( J_{\alpha,cl}^{d,n} \), where the first subscript refers to the Taylor series coefficients instead of to the function. We will also consider the “even” Jensen polynomials, defined in (3.1).

One reason for interest in the classical Jensen polynomials is:

1. \( \lim_{d \to \infty} J_{f,cl}^{d,n}(z/d) = f^{(n)}(z) \), with uniform convergence for \( z \) in a compact set, and
2. \( f \) has only real zeros if and only if \( J_{f,cl}^{d,0} \) has only real zeros for all \( d \).

Note that item (1) directly gives one of the implications in item (2). For real entire functions of order less than two, the property of having only real zeros is preserved under differentiation, so an equivalent reformulation of item (2) is that \( f \) has only real zeros if and only if \( J_{f,cl}^{d,n} \) has only real zeros for all \( d \) and all \( n \).

We will describe results in the literature as they apply to the classical Jensen polynomials \( J_{f,cl}^{d,n} \) as \( n \to \infty \), and then consider the corresponding problem for the even Jensen polynomials considered in [10, 2].

For the functions under consideration here, differentiation preserves real zeros. Much more is true. A beautiful result of Kim [14] asserts that if \( f \) is an entire function of order less than 2, which is real on the real axis, and which has all zeros in a strip \( |\Im(z)| < A \), then for any fixed \( R > 0 \), if \( n \) is sufficiently large then \( f^{(n)} \) has only real zeros in \( |z| < R \). In other words, in any compact region, if you differentiate such functions enough times, all zeros are real. A corollary is that for any \( d \), if \( n \) is large enough then the classical Jensen polynomial \( J_{f,cl}^{d,n} \) has only real zeros.

For a large subset of the functions for which Kim’s theorem applies, even more is conjectured: not only do the zeros move to the real axis, they also approach equal spacing. Since (up to a simple change of variables) the only even, real, entire function of order less than 2 with equal spaced real zeros is the cosine function, it is conjectured that for a large class of functions, repeated differentiation leads to the cosine function, up to a simple rescaling. A precise form of this conjecture was made by Berry [1], who phrased it as

$$\cos(\omega_n t + \delta_n) \text{ is a universal attractor of the derivative map}$$

and by Farmer and Rhoades [9] from a slightly different perspective based on the density of zeros of the function.
A relevant instance of that conjecture was proven by Ki [13]. Let
\[ \Xi(z) = \xi(\frac{1}{2} + iz) \]
be the Riemann \( \Xi \)-function, where
\[ \xi(s) = \frac{1}{2}s(1 - s)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s). \]
The function \( \Xi \) is even and is real on the real axis, and has all zeros in the strip \(-\frac{1}{2} < \Im(z) < \frac{1}{2}\). Thus, it is a theorem that all zeros of \( \Xi^{(n)}(z) \) for \(|z| < T\) are real if \( n \) is sufficiently large, and so for each \( d \), if \( n \) is large enough the classical Jensen polynomial \( J^{d,n}_{\Xi,cl} \) has only real zeros. It was further conjectured [9] that, suitably rescaled, \( \Xi^{(n)}(z) \) approaches \( \cos(z) \). That conjecture was proven by Ki [13]:

**Theorem 2.1 (Ki [13]).** There exist positive decreasing sequences \( A_n \) and \( C_n \) such that
\[ \lim_{n \to \infty} (-1)^n A_n \Xi^{(2n)}(C_n z) = \cos(z), \quad (2.3) \]
uniformly on compact subsets of \( \mathbb{C} \).

That theorem also follows from a proposition of Coffey [5]. The analogous result holds for functions in the extended Selberg class [11]. Functions in that class have a functional equation but not necessarily an Euler product, and so it includes many examples that do not satisfy the analogue of the Riemann Hypothesis. The same result holds for random functions [19], which by construction satisfy the analogue of the Riemann Hypothesis but have Poisson statistics for their zeros. Conrey’s result [9] that \( \Xi^{(n)} \) has \((100 - O(1/n^2))\) percent of its zeros on the real axis is a quantitative version of Kim’s theorem, and can be seen as a foreshadowing of Ki’s result.

Theorem 2.1 implies that the rescaled Taylor series coefficients of \( \Xi^{(n)} \) converge to those of cosine. Since the Taylor coefficients of cosine have a simple form, the Jensen polynomials of cosine can be written explicitly:
\[ J^{d,0}_{\cos,cl}(z) = \frac{1}{2}((1 + iz)^d + (1 - iz)^d). \quad (2.4) \]
By Theorem 2.1 and (2.4) we have

**Corollary 2.2.** There exist positive decreasing sequences \( A_n \) and \( C_n \) such that
\[ \lim_{n \to \infty} (-1)^n A_n J^{d,2n}_{\Xi,cl}(C_n z) = \frac{(1 + iz)^d + (1 - iz)^d}{2} \]
as \( n \to \infty \). In particular, for each \( d \), if \( n \) is sufficiently large then \( J^{d,2n}_{\Xi,cl} \) has only real zeros.

We see that the classical Jensen polynomials \( J^{d,n}_{f,cl} \) having real zeros for large \( n \) is a general phenomenon, following from the fact that, for a large class of entire functions, repeated differentiation leads to the cosine function. In particular, differentiation causes a loss of information about the zeros of the functions considered here, and so in terms of the Riemann Hypothesis there is little revealed by the derivatives of the function. In Section 4 we elaborate with an illustrative example. But first we consider a different form of the Jensen polynomials.
3. The even Jensen polynomials

If \( f \) is an even function, it is natural to write

\[
f(z) = \sum_{j=0}^{\infty} \gamma(j) \frac{z^{2j}}{j!}. \tag{3.1}
\]

From this we define the **even Jensen polynomials**, which are the subject of [10, 2]:

\[
J_{d,n}^{f,\text{ev}}(z) := \sum_{j=0}^{d} \binom{d}{j} \gamma(j + n) z^j. \tag{3.2}
\]

As in the classical case, the first subscript could be the even Taylor coefficients, \( \gamma \), instead of the function.

Note that the even Jensen polynomial of \( f(z) \) is the classical Jensen polynomial of \( f(\sqrt{z}) \). In the case of the Riemann \( \xi \)-function, the Riemann hypothesis is equivalent to the assertion that \( \xi(\frac{1}{2} + \sqrt{z}) \) has zeros only on the negative real axis, or equivalently, \( \Xi(\sqrt{z}) \) has zeros only on the positive real axis.

The terminology of “classical” and “even” Jensen polynomials is not standard, but we felt the terminology was necessary in order to avoid confusion. We write \( J_{d,n}^{f,\text{ev}} \) when we wish to make a statement that applies to either case.

The main results of [10] are precise asymptotics for \( \xi(2n) \left( \frac{1}{2} \right) \) and a new phenomenon relating asymptotic properties of certain sequences to the Hermite polynomials. Those results combine to produce:

**Theorem 3.1** (Griffin, Ono, Rolen, and Zagier [10]). There exist sequences \( A_n, B_n, \) and \( C_n \) such that

\[
\lim_{n \to \infty} A_n J_{d,n}^{f,\text{ev}}(C_n z + B_n) = H_d(z),
\]

uniformly for \( z \) in a compact subset of \( \mathbb{C} \), where \( H_d \) is the \( d \)th Hermite polynomial and the subscript \( \xi \) refers to \( \xi(\frac{1}{2} + z) \).

We compare this to Ki’s theorem quoted above, which implies the following:

**Corollary 3.2.** There exist sequences \( A_n \) and \( C_n \) such that

\[
\lim_{n \to \infty} A_n J_{d,n}^{f,\text{ev}}(C_n z) = (1 + z)^d,
\]

uniformly for \( z \) in a compact subset of \( \mathbb{C} \).

How can we reconcile the fact that the \( d \)th even Jensen polynomials simultaneously converge both to \( (1+z)^d \) and to \( H_d(z) \)? This apparent conundrum is easily resolved by examining a plot of the polynomial. In Figure 3.1 we show graphs of \( A_n J_{d,n}^{f,\text{ev}}(C_n x) \) for \( d = 6, n = 10000 \). That is, the 6th even Jensen polynomial of the 10000th derivative of \( \xi(\frac{1}{2} + z) \). The plot on the left covers the range \(-2 \leq x \leq 0\), and the plot on the right covers \(-1.012 \leq x \leq -0.988\).

Each plot in Figure 3.1 actually contains a superposition with a second graph: \( (1+x)^6 \) on the left, and \( H_6(x) \), shifted and scaled, on the right. In both cases the plots are so close that the two graphs are indistinguishable to the eye.
Figure 3.1. The even Jensen polynomial $J_{6,10000}^{\xi, ev}(x)$, rescaled as described in the text, for $-2 \leq x \leq 0$ on the left, and $-1.012 \leq x \leq -0.988$ on the right. The plot on the left is superimposed with the graph of $(1 + x)^6$, and the plot on the right is superimposed with the Hermite polynomial $H_6(x)$, shifted and scaled.

We see that the main result of [10] contains more information than the theorem of Ki [13] because $H_d(x)$, suitable shifted and scaled, looks just like $(1 + x)^d$, but the converse is not true.

We suggest that the results in [10] can be interpreted as a refinement of the general “cosine universality” of Berry and Farmer-Rhoades. That is:

**Principle 3.3 (‘‘Hermite Universality’’).** For a large class of functions, not only does repeated differentiation lead to the (rescaled) cosine function, but the convergence occurs in a particularly regular and uniform way, characterized by the appearance of the Hermite polynomials within the shifted and rescaled even Jensen polynomials.

We can be more specific about what this principle predicts. Suppose $f(z)$ is an even real entire function for which Cosine Universality should hold. Interpreting Cosine Universality as a statement about Taylor coefficients, we see that (suitably scaled but not shifted), the $n$th derivative of $f(\sqrt{z})$ approaches $e^{-z}$, and so $J_{f, ev}^{d, n}(z)$ (suitably scaled but not shifted) approaches $(1 - z)^d$ as $n \to \infty$. Taken at face value, that limit does not directly imply that $J_{f, ev}^{d, n}(z)$ has only real zeros for sufficiently large $n$ (although one might conclude that from other considerations). Hermite Universality does imply that $J_{f, ev}^{d, n}(z)$ has only real zeros for sufficiently large $n$, and it further implies that those zeros are arranged like the zeros of a Hermite polynomial, shifted and scaled into a small interval around $z = 1$.

Griffin, Ono, Rolen, and Zagier [10] verify this principle in many cases, and also consider it as applied to sequences that are not being viewed as the derivatives of an entire function.

Note that the principle is not restricted to even functions. However the concept of “even” Jensen polynomial has yet to be defined for functions which are not even, but which when repeatedly differentiated and slightly shifted, converge to the cosine function. Presumably there are functions for which Cosine Universality applies but Hermite Universality does not – perhaps functions without sufficient regularity in the spacings of their zeros.

There are a couple of facts that point to Principle 3.3 as an interpretation of the results in [10]. First is that the work of Ki [13], its generalization to the extended Selberg class [11],
the proof of Newman’s conjecture and its generalization \cite{8, 22}, and the work under discussion \cite{10}, all rely on the fact that functions under consideration can be written in a form similar to

\[ \Xi(z) = \int_{-\infty}^{\infty} \varphi(u)e^{izu}du \]  

(3.3)

where \( \varphi \) decreases rapidly. Such an expression is amenable to analyzing derivatives of \( \Xi \), and \cite{10} carries the analysis farther than previous efforts.

The second reason comes from considering some simple examples, which we describe after initial preparations in the next section.

4. NOT ALL EQUIVALENCES TO RH ARE CREATED EQUAL

We have seen that the Jensen polynomials of derivatives, \( J_{d,n} \) for \( n \geq 1 \), do not shed any light on the Riemann Hypothesis, because each increase in the differentiation index loses information about the location of the zeros. In this section we give another example to further illustrate that point, but our main purpose now is to complete the claim in the title of this paper, describing why \( J_{d,0} \), with differentiation index 0, is also not a useful tool for exploring RH.

Our argument is in three parts. First we divide the equivalences to RH into different categories. Then we suggest criteria for deciding, within each category, whether an equivalence is likely to be helpful for proving RH. In particular, we make the point that some equivalences to RH are unlikely to be useful for proving RH.

Given this perspective, we then consider the case of Jensen polynomials and a family of related equivalences.

4.1. TOWARDS A TAXONOMY OF EQUIVALENCES. Many equivalences to RH fall into one or more of the following categories.

(A) A subset or superset of an existing equivalence. (In the case of superset, there are two subcategories, depending on whether or not the additional conditions are logical consequences of the previous conditions.)

(B) A repackaging of an existing equivalence.

(C) A translation into a different language.

In case (A), it is reasonable to interpret a subset equivalence as a promising route to proving RH, because there are fewer conditions to satisfy. And a superset equivalence, if the additional conditions are not logical consequences of the existing conditions, can be interpreted as a promising route to disproving RH, because there are more opportunities to obtain a contradiction. Thus, except in the case where simple logic indicates that the extra conditions provide no additional information, such equivalences cannot be easily ruled out as a plausible route to resolving RH.

In case (B), the potential usefulness of the equivalence hinges on whether the information in the previous equivalence has been concentrated or dispersed. We illustrate the idea with the equivalences of Robin \cite{21} and Lagarias \cite{15}. Those equivalences involve upper bounds of the form

\[ \sigma(n) \leq f(n) \]  

(4.1)
where \( \sigma(n) = \sum_{d \mid n} d \) is the divisor sum function and \( f \) is given explicitly. The proofs of those equivalences start with the RH equivalence involving the error term in the prime number theorem:

\[
\pi(x) = \text{Li}(x) + O(x^{1/2+\varepsilon}). \tag{4.2}
\]

A violation of (4.2) for a particular \( x \) is used to exhibit an integer \( n \) where \( \sigma(n) \) is particularly large: large enough to violate (4.1). The relevance to our discussion here is that the integer \( n \) is enormously larger than \( x \). Thus we say the equivalence has dispersed the information: the new condition requires searching further in order to obtain the same information which was previously available. The dispersal of information is an indication that the equivalence is unlikely to be helpful for resolving RH.

In case (C), the issue is whether the translation could allow the use of new tools. An example which does afford new tools is the equivalence between RH and (4.2). Indeed, the Prime Number Theorem is equivalent to the nonvanishing of the \( \zeta \)-function on the line \( \sigma = 1 \), and both parts of the equivalence have been proven independently.

One could view Robin’s and Lagarias’ equivalences as falling into case (C), since \( \sigma(n) \) does not literally appear in (4.2). However, the use of \( \sigma(n) \) is just convenient packaging, and there are no special properties of the \( \sigma \)-function which are relevant to the proof.

We consider one more example of case (C) before returning to the Jensen polynomials.

**Lemma 4.1.** The following are equivalent:

1. The Riemann Hypothesis is true and all zeros of the \( \zeta \)-function are simple,
2. For all \( R > 0 \), if \( n \geq n(R) \) is sufficiently large, then inside the disc \( |z| < R \) the \( n \)th order Taylor polynomial for \( \Xi(z) \) has only real zeros.

**Proof.** If (1) is true, then (2) follows from Taylor’s theorem, Rouché’s theorem, and the fact that the Taylor coefficients of the \( \Xi \)-function are real.

In the other direction, suppose the \( \Xi \)-function had a multiple zero. By Theorem 2.1, for large \( n \) the signs of \( \Xi^{(2n)}(0) \) alternate, so in a neighborhood of the multiple zero the \( (2n) \)th order Taylor polynomial is alternately larger and smaller than the \( \Xi \)-function. So a double zero of \( \Xi \) would alternately be a pair of real zeros and a pair of complex zeros of its \( (2n) \)th order Taylor polynomial, and a higher odd-order zero would only contribute a single real zero to the Taylor polynomial.

Does that equivalence to (RH + simple zeros) open the possibility of applying new tools to the problem? The answer might not be as definitive as in the previous examples, but (in the author’s opinion) it seems fairly clear that nothing has been gained by translating to Taylor polynomials.

Thus, in each of cases (A), (B), and (C) we have criteria to judge whether or not a given equivalence is a plausible route to resolving RH. We do not claim to “prove” that an equivalence cannot be used to resolve RH, but mathematics is a human endeavor, and human effort is limited, so it is helpful to have reasons for deciding what effort is likely to be fruitful. A similar sentiment was expressed by Poincaré more than 100 years ago [20]:

For a construction to be useful and not mere waste of mental effort, for it to serve as a stepping-stone to higher things, it must first of all possess a
kind of unity enabling us to see something more than the juxtaposition of its elements.

In the next section we explain why the Jensen polynomials are even less useful than the Taylor polynomials as an approach to RH.

4.2. Jensen polynomials disperse the information. There is evidence in the literature that $J_{d, 0}^d$ is not effective at detecting violations of the Riemann Hypothesis. Namely, Chasse [4] proved that if all the zeros $\rho = \beta + i\gamma$ of the zeta-function are on the critical line for $|\gamma| < T$, then $J_{d, 0}^d$ has only real zeros for $d < T^2$. In other words, Jensen polynomials disperse the information about zeros. We illustrate this idea with a simple example.

Consider a function which is entire of order 1, even, real on the real axis, and has all its zeros in a strip $-A < \Im(z) < A$. The analogue of the Riemann Hypothesis is that all of the zeros are real. An example, which presumably has only real zeros, is the Riemann $\Xi$-function. An example which does not have only real zeros is

$$X_{10}(z) = \cos(z) \frac{(z^2 - (10 + i)^2)(z^2 - (10 - i)^2)}{(z^2 - (\frac{5\pi}{2})^2)(z^2 - (\frac{7\pi}{2})^2)}.$$ 

In words, $X_j(z)$ is the function obtained when the pairs of zeros of $\cos(z)$ closest to $\pm j$ are moved to $\pm j \pm i$, and above is a formula for $X_{10}$. Figure 4.1 shows a graph of $X_{10}(x)$.

![Figure 4.1. A graph of the function obtained by moving the zeros at $x = \pm \frac{5\pi}{2}$ and $\pm \frac{7\pi}{2}$ of $\cos(x)$ to $\pm 10 \pm i$.](image)

Examining the graph of $X_{10}$, it can be seen that all zeros of the first derivative, $X'_{10}$, are real, therefore the same is true of all higher derivatives. Thus, the classical Jensen polynomial $J_{X_{10}, cl}^d$ has only real zeros for all $n \geq 1$, as does the even Jensen polynomial $J_{X_{10}, ev}^d$ for all even $n \geq 2$.

But what about $J_{X_{10}, cl}^{d, 0}$? We know that this will have non-real zeros if $d$ is large enough, but how large is large enough? The first two zeros (in magnitude) of $X_{10}$ are real, and then a pair of complex conjugate zeros. Since the “Riemann Hypothesis” fails for $X_{10}$ almost immediately, one might guess that $J_{X_{10}, cl}^{d, 0}$ should have a non-real zero for $d$ quite small. This is not the case. By a direct calculation (we used Mathematica), $J_{X_{10}, cl}^{d, 0}$ has only real zeros for $d \leq 118$, and for all larger $d$ it has non-real zeros.

Table 4.1 shows, for various $X_j$, the maximal $d$ such that $J_{X_j, cl}^{d, 0}$ has only real zeros. The data in that table confirm the impression from Chasse’s theorem, that Jensen polynomials...
JENSEN POLYNOMIALS AND RH

| $j$ | 10 | 20 | 40 | 60 |
|-----|----|----|----|----|
| # first zeros of $X_j$ are real | 2  | 4  | 12 | 18 |
| $J_{X_j,cl}^{d,0}$ has only real zeros for $d \leq$ | 118 | 749 | 1897 | 4242 |
| $d$th Taylor polynomial detects non-real zero for $d \geq$ | 20  | 60  | 118 | 175 |

**Table 4.1.** Tabulating the relative effectiveness of the Jensen polynomials and the Taylor polynomials for detecting violations of the Riemann Hypothesis, using the function $X_j$ as a model.

are inefficient at detecting non-real zeros. Furthermore, the Jensen polynomials, which are defined in terms of the Taylor series coefficients, are not efficient at extracting information from those coefficients. The $d$th order Taylor polynomial of $X_j$ also detects the non-real zero if $d$ is large enough, in accord with Lemma 4.1: this is shown in the bottom row of Table 4.1. We see that the Jensen polynomials require significantly more Taylor coefficients than the Taylor polynomials to detect the non-real zeros.

In the terminology of Section 4.1, Jensen polynomials are a repackaging of the equivalence in Lemma 4.1. And since the Jensen polynomials disperse the information in the Taylor polynomials, we are justified in asserting that the Jensen polynomials are even less useful than the Taylor polynomials as a tool for resolving RH.

A similarity between Jensen and Taylor polynomials is that they approximate $\Xi(z)$ when $|z|$ is small. It is tempting to view the Jensen polynomials as “better” because for larger $z$ the zeros of Jensen polynomials are real, while Taylor polynomials tend to have many complex zeros. But, that apparently nice property is just a distraction. One set of functions has meaningless zeros on a line, and the other has meaningless zeros near a circle. In both cases the extraneous zeros say very little about the function being approximated. The apparently nice property of having extra real zeros comes at the cost of converging to the function more slowly. One must distinguish between elegance in the statement of a proposition, and actually being useful as a tool to prove new results. That criticism also applies to the equivalences due to Robin and to Lagarias.

**4.3. Other Jensen-like polynomials.** There are other polynomials generated from the Taylor coefficients which only have real zeros if and only if the original function has only real zeros. For example, a recent paper of O’Sullivan [17] considers the polynomials

$$P^{d,n}(z) := \sum_{j=0}^{d} \binom{d}{j} \gamma(j + n) H_{d-j}(z).$$

(4.3)

In other words, the Jensen polynomial with $z^j$ replaced by the Hermite polynomial $H_j(z)$.

O’Sullivan shows that these polynomials have the same property that make the Jensen polynomials interesting: $\Xi(z)$ has only real zeros if and only if $P^{d,n}$ has only real zeros for all $d$, $n$. That result is a special case of a more general result whereby any element of the Laguerre-Pólya class produces a sequence of polynomials which can be put in place of the Hermite polynomials in (4.3). Thus, there is a wealth of seemingly different sequences of polynomials, any one of which can detect a violation of the Riemann Hypothesis.
We have argued that the Jensen polynomials are not a useful tool for attacking the Riemann Hypothesis. Might one of those other sequences of polynomials turn out to be more useful? Sadly, no. O’Sullivan goes on to show that if the even Jensen polynomial $J_{d,n}^{\text{ev}}$ has only real zeros, then so does $P_{d,n}$. In other words, $P_{d,n}$ is less useful at detecting violations of the Riemann Hypothesis. The proof in [17] is in the context of $P_{d,n}$, but presumably the analysis extends to all the other sequences of polynomials.

4.4. Other differentiation-like operations. de Bruijn [7] and Newman [16] considered the following operation, which uses the notation of (3.3),

$$\Xi_t(z) = \int_{-\infty}^{\infty} e^{tu^2} \varphi(u)e^{izu} du.$$  \hspace{1cm} (4.4)

The de Bruijn-Newman constant is defined by $\Lambda = \inf \{ t : \Xi_t \text{ has only real zeros} \}$. Since $\Xi_0 = \Xi$, the Riemann Hypothesis is equivalent to $\Lambda \leq 0$. Newman conjectured $\Lambda \geq 0$, which was proven recently by Rodgers and Tao [22]. That result was generalized to the extended Selberg class (most of which does not satisfy the analogue of the Riemann Hypothesis) by Dobner [8]. As Dobner notes, the method “does not require any information about the zeros” of the function. In particular, these results say nothing about Lehmer pairs of zeros, which is somewhat ironic since previously the lower bounds on $\Lambda$ came from Lehmer pairs.

The de Bruijn-Newman operation $\Xi \to \Xi_t$ has several properties in common with differentiation $\Xi \to \Xi^{(j)}$. For example, if $\Xi_{t_0}$ has only real zeros then $\Xi_t$ has only real zeros for all $t > t_0$. Also, if $t > 0$ then as $x \to \infty$ the zeros of $\Xi_t(x)$ approach equal spacing. Thus, the de Bruijn-Newman operation is even more efficient than differentiation at losing information and causing the zeros to approach equal spacing.

In some sense, de Bruijn-Newman operation is like repeated differentiation $\Xi^{(j)}(z)$ where $j$ is an increasing function of $z$. It is possible to be somewhat precise about that remark. As shown in Section 3 of [8], if $z$ is real then the main contribution in (4.4) is concentrated near $u \approx z$. If $n$ is the integer closest to $tz^2$, then the largest term in the Taylor series for $e^{tu^2}$ is approximately

$$\frac{t^n}{n!}u^{2n} \approx \frac{t^n}{n!}u^{2tz^2}. \hspace{1cm} (4.5)$$

In other words, as $z \to \infty$ the zeros of $\Xi_t(z)$ are approaching equal spacing at a rate comparable to the $2tz^2$rd derivative $\Xi^{(2tz^2)}(z)$. That analysis may not be rigorous, but it does explain why the de Bruijn-Newman operation is extremely effective at causing zeros to become equally spaced.

The above discussion was intended to emphasize the point that functions with a representation similar to (3.3) have a nice distribution to their zeros, which becomes nicer under operations similar to differentiation. But those seemingly magical properties are from the realm of analysis, not number theory, and those properties hold whether or not the function satisfies a Riemann Hypothesis.

5. Hermite polynomials and the function $X_{10}$

The function $X_{10}$ in the previous section approaches cosine under repeated differentiation. We now view that function through the lens of the results in [10]. Let $X_{10}(x) = \sum \alpha(n)x^n/n!$. 

Using Cauchy’s theorem, as described in [3], we computed the $\alpha(n)$ for a few $n$ near 100000, shown in Table 5.1. In the notation of (3.1), $\gamma(n) = \alpha(2n)n!(2n)!$, so Table 5.1 is sufficient to approximate $J_{X_{10, ev}}^d$ for $d \leq 6$.

Let

\[ A = 6.2880774766370030074398478301164858 \times 10^{516790} \]
\[ B = 1.600352019320098623551973272940701704 \times 10^{21} \]
\[ C = 7.15586265552087639602840363255312494 \times 10^8. \tag{5.1} \]

Then we find that

\[ AJ_{X_{10, ev}}^6(Cx + B) = -120 + 5.368x + 180.045x^2 - 0.894x^3 - 30.0058x^4 + 0x^5 + x^6. \tag{5.2} \]

For comparison, $H_6(x/2) = -120 + 180x^2 - 30x^4 + x^6$, so we see that the coefficients are close.

As another example, consider $\text{sinc}(x) = \sin(x)/x$. The Taylor coefficients of that function are easy to compute analytically, making it possible to explore high derivatives. For the 10-millionth derivative, with $A$, $B$, and $C$ chosen appropriately, we find

\[ AJ_{\text{sinc, ev}}^6(Cx + B) = -120 - 0.536x + 180.000045x^2 + 0.089x^3 - 30.000058x^4 + 0x^5 + x^6, \tag{5.3} \]

which is even closer to $H_6(x/2)$, and also suggests that the rate of convergence is on the scale of $1/\sqrt{n}$. These examples help support the suggestion that the appearance of the Hermite polynomials in the even Jensen polynomials is a universal phenomenon.

### 6. On the Random Matrix Conjectures for (Derivatives of) L-functions

We end by addressing the claims that the appearance of the Hermite polynomials in the even Jensen polynomials has implications for the random matrix statistics of L-functions.

The first issue is that the Hermite polynomials appear in the even Jensen polynomials for $\xi^{(n)}(\frac{1}{2} + z)$, in other words, in the classical Jensen polynomials for $\xi^{(n)}(\frac{1}{2} + \sqrt{x})$. Since the zeros of $\xi^{(n)}(\frac{1}{2} + \sqrt{x})$ lie in the left half-plane, close to the negative real axis, as $n$ increases those zeros move onto the negative real axis and shift to the left. In the limit, the zeros fall off the negative real edge of the complex plane, and suitably rescaled (but not shifted), the limiting function is $e^z$. In the scaled but unshifted classical Jensen polynomials, all the zeros
accumulate at $z = -1$. Since it is $J_{d,n}^{\xi(\frac{1}{2}+\sqrt{\cdot})}(z/d)$ which converges to $\xi^{(n)}(\frac{1}{2}+\sqrt{\cdot})$ as $d \to \infty$, those zeros do not reveal anything about the limiting function at $z$ in a compact subset.

The second issue is the density of zeros. The main claim for a connection to random matrix statistics was that both the zeros of Hermite polynomials and the eigenvalues of matrices in the Gaussian Unitary Ensemble (GUE) have a density given by the semicircular law. That is true, but when using the GUE to model zeros of L-functions, the semicircle density is a defect, not a feature. One must artificially rescale the eigenvalues of GUE matrices to achieve the flat density of zeros of L-functions. It is the statistics of the spacings, not the density of zeros, which are modeled by random matrices. For $x$ in a bounded interval, the zeros of the Hermite polynomial $H_d(x)$ approach equal spacing as $d \to \infty$. If those were modeling zeros of derivatives, it would merely be a reflection of the limiting cosine function, where all information about the original distribution of zeros has been lost.

REFERENCES

[1] M. Berry, Universal oscillations of high derivatives, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 461 (2005) 1735-1751.
[2] Enrico Bombieri, New progress on the zeta function: from old conjectures to a major breakthrough. Proc. Natl. Acad. Sci. USA 116 (2019), no. 23, 11085-11086.
[3] Folkmar Bornemann, Accuracy and stability of computing high-order derivatives of analytic functions by Cauchy integrals. Found. Comput. Math. 11 (2011), no. 1, 1-63.
[4] Matthew Chasse, Laguerre multiplier sequences and sector properties of entire functions. Complex Var. Elliptic Equ. 58 (2013), no. 7, 875-885.
[5] Mark Coffey. Asymptotic estimation of $\xi^{(2n)}(1/2)$: On a conjecture of Farmer and Rhoades. Math. Comp. 78 (2009) no. 266, 1147-1154.
[6] Brian Conrey, Zeros of derivatives of Riemann’s $\xi$-function on the critical line. J. Number Theory 16 (1983), no. 1, 49-74.
[7] N. G. de Bruijn, The roots of trigonometric integrals, Duke Math. J. 17 (1950), 197-226.
[8] Alexander Dobner, A proof of Newman’s conjecture for the extended Selberg class. Acta Arith. 201 (2021), no. 1, 29–62. arXiv:2005.05142
[9] David W. Farmer, Robert C. Rhoades, Differentiation evens out zero spacings. Trans. Amer. Math. Soc. 357 (2005), no. 9, 3789-3811.
[10] Michael Griffin, Ken Ono, Larry Roelen, Don Zagier, Jensen polynomials for the Riemann zeta function and other sequences. Proc. Natl. Acad. Sci. USA 116 (2019), no. 23, 11103-11110.
[11] Jos Gunn, Christopher Hughes, The effect of repeated differentiation on L-functions. J. Number Theory 194 (2019), 30-43.
[12] J. L. W. V. Jensen, Recherches sur la théorie des équations. Acta Math. 36 (1913), no. 1, 181-195
[13] Haseo Ki, The Riemann $\Xi$-function under repeated differentiation. J. Number Theory 120 (2006), no. 1, 120-131.
[14] Y.-O. Kim, Critical points of real entire functions and a conjecture of Pólya, PAMS, Vol. 124, No. 3, (1996), 819-829.
[15] J.C. Lagarias, An elementary problem equivalent to the Riemann Hypothesis, Amer. Math. Monthly 109, 534-543, 2002.
[16] Charles M. Newman, Fourier transforms with only real zeros, Proc. Amer. Math. Soc. 61 (1976), no. 2, 245-251 (1977).
[17] Cormac O’Sullivan, Zeros of Jensen polynomials and asymptotics for the Riemann xi function, preprint, arXiv:2007.13582
[18] G. Pólya, Über die algebraisch-funktionentheoretischen Untersuchungen von J. L. W. V. Jensen. Kgl Danske Vid Sel Math-Fys Medd 7:3-33.
[19] Robin Pemantle, Sneha Subramanian, Zeros of a random analytic function approach perfect spacing under repeated differentiation. Trans. Amer. Math. Soc. 369 (2017), no. 12, 8743-8764.

[20] H. Poincaré, Science and Hypothesis, The Walter Scott Publishing Co, Ltd., New York (1905). https://gutenberg.org/ebooks/37157

[21] G. Robin, Grandes valeurs de la fonction somme des diviseurs et Hypothèse de Riemann, J. Math. Pures Appl. 63, 2, 187-213, 1984.

[22] Brad Rogers and Terrence Tao, The de Bruijn-Newman constant is non-negative. Forum Math. Pi 8 (2020)

American Institute of Mathematics, 600 East Brokaw Road, San Jose, CA 95112-1006
Email address: farmer@aimath.org