The Importance of Clipping in Neurocontrol by Direct Gradient Descent on the Cost-to-Go Function and in Adaptive Dynamic Programming

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Abstract—In adaptive dynamic programming, neurocontrol and reinforcement learning, the objective is for an agent to learn to choose actions so as to minimise a total cost function. In this paper we show that when discretized time is used to model the motion of the agent, it can be very important to do “clipping” on the motion of the agent in the final time step of the trajectory. By clipping we mean that the final time step of the trajectory is to be truncated such that the agent stops exactly at the first terminal state reached, and no distance further. We demonstrate that when clipping is omitted, learning performance can fail to reach the optimum; and when clipping is done properly, learning performance can improve significantly.

The clipping problem we describe affects algorithms which use explicit derivatives of the model functions of the environment to calculate a learning gradient. These include Backpropagation Through Time for Control, and methods based on Dual Heuristic Dynamic Programming. However the clipping problem does not significantly affect methods based on Heuristic Dynamic Programming, Temporal Differences or Policy Gradient Learning algorithms. Similarly, the clipping problem does not affect fixed-length finite-horizon problems.

I. INTRODUCTION

In Adaptive Dynamic Programming (ADP) \cite{1}, Neurocontrol \cite{2}, and Reinforcement Learning (RL) \cite{3}, an agent moves in a state space $S \subset \mathbb{R}^n$, such that at integer time step $t$, it has state vector $\vec{x}_t \in S$. $T$ is a fixed set of terminal states, with $T \subset S$. At each time $t$, the agent chooses an action $\vec{u}_t$, which takes it to the next state according to the environment’s model function

$$\vec{x}_{t+1} = f(\vec{x}_t, \vec{u}_t),$$

thus the agent passes through a trajectory of states $(\vec{x}_0, \vec{x}_1, \vec{x}_2, \ldots)$, terminating only when (and if) a terminal state is reached, as illustrated in Fig. 1. As shown in this figure, clipping is the concept of calculating the exact fraction in the final time step at which a boundary of terminal states is reached, and stopping the agent exactly at this boundary. The name clipping is taken by analogy to the concept in computer graphics. Without clipping, the discretization of time would cause the agent to penetrate slightly beyond the terminal boundary, as shown in the figure.

On transitioning from each state $\vec{x}_t$ to the next, the agent receives an immediate scalar cost $U_t$ from the environment according to the function

$$U_t := U(\vec{x}_t, \vec{u}_t).$$

In addition, if the agent reaches a terminal state $\vec{x} \in T$, then an additional terminal cost is given by the scalar function $\Phi(\vec{x})$.

Throughout this paper, subscripts on variables will be used to indicate the time step of a trajectory. And from now on in the paper, we will only consider episodic, or finite horizon, environments; that is environments where all trajectories are guaranteed to meet a terminal state eventually.

The ADP problem is for the agent to learn to choose actions so as to minimise the expectation of the total long-term cost received from any given start state $\vec{x}_0$. Specifically, the problem is to find an action network $A(\vec{x}, \vec{z})$, where $\vec{z}$ is the parameter vector of a function approximator, which calculates an action

$$\vec{u}_t = A(\vec{x}_t, \vec{z})$$

for any given state $\vec{x}_t$, such that the following long-term cost is minimised:

$$J(\vec{x}_0, \vec{z}) := \langle \sum_{t=0}^{T-1} \gamma^t U_t + \gamma^T \Phi(\vec{x}_T) \rangle$$

subject to (1), (2) and (3); where $T$ is the time step at which the first terminal state is reached (which in general will be dependent on $\vec{x}_0$ and $\vec{z}$), where $\gamma \in [0, 1]$ is a constant discount factor that specifies the relative importance of long-term costs over short term ones, and where $\langle \cdot \rangle$ denotes expectation.

The function $J(\vec{x}_0, \vec{z})$ is called the cost-to-go function from state $\vec{x}_0$, or the value function.

In this paper we show that when a large final impulse of cost $\Phi(\vec{x})$ is given at a terminal state $\vec{x} \in T$, then failure to do clipping in the final timestep of the trajectory can

\[\text{Fig. 1: A trajectory reaching a terminal state. The thick curved line indicates a boundary of terminal states. In this diagram, clipping does not take place, and the trajectory penetrates beyond the terminal boundary. When clipping is used correctly, we intend to stop the agent exactly at the point of intersection between the trajectory and the terminal boundary.}\]
very significantly distort the direction of the learning gradient used by certain ADP algorithms, and thus prevent successful
solution of the ADP problem. We also show that this problem is not lessened by sampling the time steps of the underlying
continuous-time process at a higher rate. This problem affects
the commonly used ADP algorithms Dual Heuristic Dynamic
Programming (DHP) [4], [5], and Backpropagation through
time (BPTT) [6], both of which are described in Section
II plus algorithms based on DHP such as Value-Gradient
Learning [7], [8], [9]. These algorithms are all very closely
related to each other [10], [11], and for purposes of explaining
clipping as clearly as possible, we will use BPTT as the
example.

BPTT works by calculating the quantity $\frac{\partial J}{\partial \theta}$ directly and
very efficiently for each trajectory sampled, enabling gradient
descent to be performed on $J$ with respect to $\theta$. However
without clipping being done correctly, the gradient that BPTT
calculates can be distorted enough to prevent learning. Fig. 2
illustrates the problems that arise without clipping.

In general, increasing the sampling rate of the discretization
time will not solve the problem, since that would simply
make the dotted arcs in Fig. 2a squeeze closer together, and
will make the teeth of the saw-tooth blade shape in Fig. 3 finer.
The gradients in Figs. 2b and 3 would still not be helpful for
learning.

We show how to solve the problem by incorporating clipping
in to the model and cost functions, $f(\vec{x}, \vec{u})$ and $U(\vec{x}, \vec{u})$,
when terminal states are reached. BPTT and DHP make
intensive use of the derivatives of these two functions, and
hence we must carefully differentiate through the clipped
versions of these functions. This is the important step that
we derive in this paper, and this step corrects the gradient
$\frac{\partial J}{\partial \theta}$ to make it suitable for learning, and solves the problems
explained by Fig. 2 and Fig. 3.

As well as terminal boundaries in state space that deliver
impulses of cost, similar corrections would need making in
environments where the model and cost functions change
their behaviour discontinuously as the agent traverses a given
continuous boundary in state space. These boundaries would
act as refraction layers do to photons. As the agent crosses
them, the learning gradient $\frac{\partial J}{\partial \theta}$ would get twisted. The solution
to this problem is similar to the one we propose for terminal
boundaries, but we do not consider these non-terminal refrac-
tion layers any more in this paper.

The necessity for clipping affects any algorithm that calcul-
ates the derivatives of the model function, i.e. $\frac{\partial J}{\partial \theta}$ directly, and
when terminal states that deliver impulses of cost are present.
For example the RL method of [12], which implements a
continuous-time numerical differentiation to evaluate $\frac{\partial J}{\partial \theta}$, will
also be affected by this clipping problem. Likewise, the ADP
methods of BPTT, DHP, GDHP [13] and Value-Gradient
Learning are also affected by the requirement for clipping.

Clipping is not necessary for any problem where the ter-
nination condition is simply when a fixed integer number of
time steps is reached, as we discuss further in Section
II-D. Also our experiments in this paper show that the ADP
algorithm called Heuristic Dual Programming (HDP, [4], [11],
[5]) does not need clipping, since this algorithm does not make
significant use of the derivatives of the model function. The
policy-gradient learning methods of [14], [15] do not require
clipping either, since they do not use the derivatives of the
model function.

In the rest of this paper, in Section II we describe the af-
ected ADP algorithms for control problems. In Section III we
describe how to do the clipping and differentiate through the
modified model functions, as is required for effective gradient

(a) Spurious zigzag gradients can occur when clipping is not used.

(b) The graph of $R$ versus $\theta$ yields no useful local gradient informa-
tion. Hence minimising $R$ with respect to $\theta$ using only $dR/d\theta$ would be
impossible.

Fig. 2: An example of the problems that can occur when
clipping is not used.

Fig. 3: A pathological example: Local gradient is opposite to
global gradient.
II. THE ADP/RL LEARNING ALGORITHMS

We describe three main ADP/RL algorithms first in their forms without clipping.

A. Backpropagation Through Time For Control

Backpropagation through time (BPTT) can be applied to control problems, as described by \(6\). In this section we derive the algorithm. This is an algorithm that requires clipping in the environments we consider in this paper.

BPTT is an efficient algorithm to calculate \(\frac{\partial J}{\partial z} \) for a given trajectory. The combination of the BPTT gradient calculation with a gradient descent weight update can be used to solve control problems, i.e. by the weight update \(\Delta \vec{x} = -\alpha \frac{\partial J}{\partial x} \) for some small positive learning rate \(\alpha\).

Throughout this paper we make a notational convention that all vectors are columns, and differentiation of a scalar by a vector gives a column vector (e.g. \(\frac{\partial f}{\partial \vec{u}}\) is a column). We define differentiation of a vector function by a vector argument as the transpose of the usual Jacobian notation. For example, \(\frac{\partial A(\vec{x}, \vec{z})}{\partial \vec{x}}\) is a matrix with element \((i, j)\) equal to \(\frac{\partial A^T}{\partial x_i}\). Similarly, \(\frac{\partial f}{\partial \vec{z}}\) is the matrix with element \((i, j\} = \frac{\partial f^T}{\partial z_j}\).

Parentheses subscripted with a “\(t\)” are what we call trajectory-shorthand notation, which we define to indicate that a quantity is evaluated at time step \(t\) of a trajectory. For example \((\frac{\partial U}{\partial \vec{u}})_{t=0}\) is shorthand for the function \(\frac{\partial U(\vec{x}, \vec{u}, \vec{z})}{\partial \vec{u}}\) evaluated at \((\vec{x}_t, \vec{u}_t)\). Similarly, \((\frac{\partial A}{\partial \vec{z}})_{t=0} := \frac{\partial A(\vec{x}, \vec{z})}{\partial \vec{z}}\big|_{(\vec{x}_t, \vec{z})}\), and \((\frac{\partial A}{\partial \vec{x}})_{t=1} := \frac{\partial A(\vec{x}, \vec{z})}{\partial \vec{x}}\big|_{(\vec{x}_{t+1}, \vec{z})}\).

For any given trajectory starting at state \(\vec{x}_0\), the function \(J(\vec{x}_0, \vec{z})\) given by \(4\) can be written recursively using equations \(1\)–\(3\), as:

\[
J(\vec{x}, \vec{z}) := U(\vec{x}, A(\vec{x}, \vec{z})), \gamma J(f(\vec{x}, A(\vec{x}, \vec{z})), \vec{z})
\]

with \(J(\vec{x}_T, \vec{z}) := \Phi(\vec{x}_T)\) at the trajectory’s terminal state, \(\vec{x}_T \in \mathbb{T}\).

Differentiating \(5\) with the chain rule gives:

\[
\frac{\partial J}{\partial \vec{z}} = \frac{\partial J}{\partial \vec{x}} \bigg(\frac{\partial U}{\partial \vec{u}} + \gamma J(f(\vec{x}, A(\vec{x}, \vec{z})), \vec{z})\bigg) + \gamma \frac{\partial J}{\partial \vec{z}} \bigg(\frac{\partial U}{\partial \vec{u}} + \gamma J(f(\vec{x}, A(\vec{x}, \vec{z})), \vec{z})\bigg)
\]

where we used the chain rule, equations \(1\)–\(3\) and trajectory-shorthand notation. In this equation there are implied matrix-vector products that make use of the matrix notation defined above.

Expanding this recursion gives:

\[
\frac{\partial J}{\partial \vec{z}} = \sum_{t=0}^{T} \gamma^t \left(\frac{\partial A}{\partial \vec{z}}\right)_t \left(\frac{\partial U}{\partial \vec{u}} + \gamma \frac{\partial f}{\partial \vec{u}} \left(\frac{\partial J}{\partial \vec{z}}\right)_t + \gamma \frac{\partial J}{\partial \vec{z}} \left(\frac{\partial U}{\partial \vec{u}} + \gamma \frac{\partial f}{\partial \vec{u}} \left(\frac{\partial J}{\partial \vec{z}}\right)_t + \gamma \frac{\partial J}{\partial \vec{z}}\right)_{t+1}\right)
\]

This equation refers to the quantity \(\frac{\partial J}{\partial \vec{z}}\) which can be found recursively by differentiating \(5\) and using the chain rule, giving

\[
\frac{\partial J}{\partial \vec{z}} = \frac{\partial U}{\partial \vec{u}} + \gamma \frac{\partial f}{\partial \vec{u}} \frac{\partial J}{\partial \vec{z}} + \gamma \frac{\partial J}{\partial \vec{z}} \frac{\partial U}{\partial \vec{u}} + \gamma \frac{\partial f}{\partial \vec{u}} \frac{\partial J}{\partial \vec{z}} + \gamma \frac{\partial J}{\partial \vec{z}} \frac{\partial J}{\partial \vec{z}}\]

with

\[
\frac{\partial J}{\partial \vec{z}} \bigg|_{T} = \frac{\partial \Phi}{\partial \vec{z}}
\]

at the terminal state, \(\vec{x}_T \in \mathbb{T}\).

Equation \(7\) can be understood to be backpropagating the quantity \((\frac{\partial J}{\partial \vec{z}})_{t+1}\) through the action network, model and cost functions to obtain \((\frac{\partial J}{\partial \vec{z}})_{t}\), and giving the algorithm its name. Pseudocode for the whole BPTT algorithm is given in Alg. \(1\) where lines \(2\) to \(7\) of the algorithm come from equations \(5\), \(6\) and \(7\) respectively. In the algorithm, the vector \(\vec{p}\) holds the backpropagated value for \(\frac{\partial J}{\partial \vec{z}}\). \(Q_x\) and \(Q_u\) are the derivatives of the Q-function with respect to \(\vec{x}\) and \(\vec{u}\) respectively, where the Q-function is defined by

\[
Q(\vec{x}, \vec{u}, \vec{z}) = U(\vec{x}, \vec{u}) + \gamma J(f(\vec{x}, \vec{u}), \vec{z})
\]

The Q-function is a model based version of the Q-function defined in Q-learning \(16\). It is similar to the cost-to-go function’s recursive definition \( \frac{\partial J}{\partial \vec{z}}\), but it differs in that it allows the first action chosen to be independent of the action network. This will be useful in deriving the clipping equations in Section \(III\) but for now \(Q_x\) and \(Q_u\) can just be treated as internal variables in Alg. \(1\). The BPTT algorithm runs in time \(O(\dim(\vec{z}))\) per trajectory step.

\[\text{Algorithm 1 Backpropagation Through Time for Control.}\]

\begin{enumerate}
\item \(\frac{\partial f}{\partial \vec{z}} \leftarrow 0\)
\item \(\vec{p} \leftarrow (\frac{\partial f}{\partial \vec{z}})_{t}\)
\item \textbf{for} \(t = T \leftarrow 0 \) \textbf{step} \(-1\) \textbf{do}
\item \(Q_x \leftarrow \left(\frac{\partial U}{\partial \vec{u}}\right)_t + \gamma \left(\frac{\partial f}{\partial \vec{u}}\right)_t \vec{p}\)
\item \(Q_u \leftarrow \left(\frac{\partial U}{\partial \vec{u}}\right)_t + \gamma \left(\frac{\partial f}{\partial \vec{u}}\right)_t \vec{p}\)
\item \(\frac{\partial J}{\partial \vec{z}} \leftarrow \frac{\partial J}{\partial \vec{z}} + \gamma \left(\frac{\partial A}{\partial \vec{z}}\right)_t \vec{p}\)
\item \(\vec{p} \leftarrow Q_x + \left(\frac{\partial J}{\partial \vec{z}}\right)_t Q_u\)
\item \textbf{end for}
\item \(\vec{z} \leftarrow \vec{z} - \alpha \frac{\partial J}{\partial \vec{z}}\)
\end{enumerate}
B. Dual Heuristic Dynamic Programming (DHP) and Heuristic Dynamic Programming (HDP)

Dual Heuristic Dynamic Programming (DHP) and Heuristic Dynamic Programming (HDP) are ADP algorithms which use a critic function, and can require clipping in the environments we consider in this paper. Both of these algorithms were originally by Werbos [4] and are described more recently by [3], [17], [1], and we define them briefly here.

The use of critic functions allows these two algorithms to apply their learning rule on-line, unlike the previously described BPTT which needed to wait until a trajectory was completed before it could apply the learning weight update. DHP makes use of a vector-critic function $\hat{G}(\mathbf{x}, \mathbf{w})$ which produces a vector output of dimension $\mathbb{R}^{\text{dim}(\mathbf{x})}$. This could be the output of a neural network with weight vector $\mathbf{w}$ and dim($\mathbf{x}$) inputs and outputs. The DHP weight update attempts to make the function $\hat{G}(\mathbf{x}, \mathbf{w})$ learn to output the gradient $\frac{\partial \hat{G}}{\partial \mathbf{w}}$.

HDP uses a scalar-critic function $\hat{V}(\mathbf{x}, \mathbf{w})$ which produces a scalar output. This could be the output of a neural network with weight vector $\mathbf{w}$ and dim($\mathbf{x}$) inputs, and just one output node. The HDP weight update attempts to make the function $\hat{V}(\mathbf{x}, \mathbf{w})$ learn to output the function $J(\mathbf{x}, \mathbf{z})$ for all $\mathbf{x} \in \mathcal{S}$. HDP is equivalent to the algorithm “TD(0)” from the RL literature [18].

Pseudocode for DHP is given in Alg. 2. Line 9 of the algorithm trains the critic with a learning rate $\beta > 0$, and line 10 implements a commonly used actor weight update described by [3] (using a learning rate $\alpha > 0$). The algorithm uses the same matrix notation for Jacobians and trajectory-shorthand notation as described in Section II-A, so that for example $\frac{\partial \hat{G}}{\partial \mathbf{w}}$ is the function $\frac{\partial \hat{G}}{\partial \mathbf{w}}$ evaluated at $(\mathbf{x}, \mathbf{w})$.

Pseudocode for HDP is given in Alg. 3. Lines 8 and 9 give the critic and action-network weight updates, respectively. Again the action-network weight update is the one described by [3], but model-free alternatives which don’t require knowledge of the derivatives of $f$ are also possible (e.g. [3] ch.6.6], or [19] sec 4.2)).

Backpropagation [20], [21]) can be used to efficiently calculate $\frac{\partial \hat{G}}{\partial \mathbf{x}}$, $\frac{\partial \hat{V}}{\partial \mathbf{x}}$ and the products involving $\frac{\partial \hat{A}}{\partial \mathbf{x}}$ and $\frac{\partial \hat{A}}{\partial \mathbf{z}}$. Using this method, both DHP and HDP can be implemented in a running time of $O(n)$ operations per time step of the trajectory, where $n = \max(\text{dim}(\mathbf{w}), \text{dim}(\mathbf{z}))$.

The pseudocode gives explicit details of how the function $\Phi(\hat{x})$ is to be used instead of the critic at the final time step of a trajectory. This is an important detail that is necessary to implement clipping and finite-horizon problems correctly.

III. Using and Differentiating Clipping in Learning

In this section we derive the formulae for the clipped model and cost functions, and their derivatives. We will denote the clipped versions of the original functions with a superscripted $C$, so that $f^C$, $U^C$ and $J^C$ will be the function names we use for the clipped versions of the model, cost and cost-to-go functions, respectively. The functions $f^C$ and $U^C$ are only defined for any state $\mathbf{x}_t$ that occurs immediately before a terminal state is reached, i.e. for which $\mathbf{x}_t \notin \mathcal{T}$ and for which $f(\mathbf{x}_t, \mathbf{u}_t) \in \mathcal{T}$.

These three clipped functions, $f^C$, $U^C$ and $J^C$, are key concepts in this paper, because defining them clearly allows us to differentiate them carefully, and hence calculate the learning gradients correctly. This is what allows us to solve the clipping problem. Hence this section is the main contribution of this paper, in terms of implementation details for solving the clipping problem.

Algorithm 2 DHP with a critic network $\hat{G}(\mathbf{x}, \mathbf{w})$ and action network $A(\mathbf{x}, \mathbf{z})$.

\begin{algorithmic}[1]
\State $t \leftarrow 0$
\While {$\mathbf{x}_t \notin \mathcal{T}$}
\State $\mathbf{u}_t \leftarrow A(\mathbf{x}_t, \mathbf{z})$
\EndWhile
\end{algorithmic}

Algorithm 3 HDP with a critic network $\hat{V}(\mathbf{x}, \mathbf{w})$ and action network $A(\mathbf{x}, \mathbf{z})$.

\begin{algorithmic}[1]
\State $t \leftarrow 0$
\While {$\mathbf{x}_t \notin \mathcal{T}$}
\State $\mathbf{u}_t \leftarrow A(\mathbf{x}_t, \mathbf{z})$
\State $\mathbf{x}_{t+1} \leftarrow f(\mathbf{x}_t, \mathbf{u}_t)$
\EndWhile
\end{algorithmic}

A. Calculation of the Clipped Model and Cost Functions

Suppose the agent is transitioning between states $\mathbf{x}_t$ and $f(\mathbf{x}_t, \mathbf{u}_t)$, and the state $f(\mathbf{x}_t, \mathbf{u}_t)$ would be beyond the terminal boundary unless clipping was applied. To calculate the clipping correctly, we imagine this state transition as occurring along the straight line segment from $\mathbf{x}_t$ to $f(\mathbf{x}_t, \mathbf{u}_t)$, i.e. the
\[ \vec{r} = \vec{x}_t + \lambda \vec{u}, \]  
(9)

where
\[ \vec{u} = f(\vec{x}_t, \vec{u}_t) - \vec{x}_t, \]  
(10)

and \( \lambda \in [0, 1] \) is a real parameter. This is illustrated in Fig. 4.

This straight line must intersect a boundary of terminal states. At the point of intersection, the tangent plane of the terminal boundary is given by \( (\vec{r} - \vec{P}) \cdot \vec{n} = 0 \) (i.e. where \( \vec{r} \) is an arbitrary position vector that lies on a plane which has normal \( \vec{n} \) and passes through a point with position vector \( \vec{P} \), and where \( \cdot \) denotes the inner product between two vectors), as illustrated in Fig. 4. The constants \( \vec{P} \) and \( \vec{n} \) should be available from either the physical environment or from the collision-detection routine of the simulated environment.

At the intersection of the line and the plane, we have

\[ (\vec{x}_t + \lambda \vec{u} - \vec{P}) \cdot \vec{n} = 0 \]

\[ \Rightarrow \lambda = \frac{(\vec{P} - \vec{x}_t) \cdot \vec{n}}{\vec{r} \cdot \vec{n}}. \]  
(11)

This value of \( \lambda \) is a real number between 0 and 1 which indicates the transition along the transition line from \( \vec{x}_t \) to \( f(\vec{x}_t, \vec{u}_t) \) at which the terminal boundary was encountered. We will refer to the value \( \lambda \) as the “clipping fraction”, and since it depends on \( \vec{x}_t, \vec{u}_t, \vec{P} \) and \( \vec{n} \), it is defined by the function:

\[ \lambda := \Lambda(\vec{x}_t, \vec{u}_t, \vec{P}, \vec{n}) := \frac{(\vec{P} - \vec{x}_t) \cdot \vec{n}}{(\vec{r} \cdot \vec{n})}. \]  
(12)

Hence the clipped value of the final state is \( \vec{x}_{t+1} = \vec{x}_t + \Lambda(\vec{x}_t, \vec{u}_t, \vec{P}, \vec{n}) f(\vec{x}_t, \vec{u}_t) - \vec{x}_t \), which is found by combining equations (9), (10) and (11). This gives the function for the clipped model function as

\[ f^C(\vec{x}, \vec{u}, \vec{P}, \vec{n}) := \vec{x} + \Lambda(\vec{x}, \vec{u}, \vec{P}, \vec{n}) f(\vec{x}, \vec{u}) - \vec{x}. \]  
(12)

Assuming that “cost” is delivered at a uniform rate during the final state transition, the total clipped cost would be proportional to the clipping fraction, giving:

\[ U^C(\vec{x}, \vec{u}, \vec{P}, \vec{n}) := \Lambda(\vec{x}, \vec{u}, \vec{P}, \vec{n}) U(\vec{x}, \vec{u}). \]  
(13)

Since the final clipped timestep has duration \( \lambda \in [0, 1] \), the terminal cost \( \Phi(\vec{x}_T) \) should only receive a discount of \( \gamma^\lambda \) instead of the full discount \( \gamma \). Hence, at the penultimate time step, \( \vec{x}_{T-1} \), the total cost-to-go is

\[ J^C(\vec{x}_{T-1}, \vec{z}) := U^C(\vec{x}_{T-1}, \vec{u}_{T-1}, \vec{P}, \vec{n}) + \gamma^\lambda \Phi(\vec{x}_T). \]  
(14)

This possibly seems like a pedantic detail, but it is this detail which allows us to solve a version of the cart-pole benchmark problem, which would otherwise be impossible for DHP, in Section IV-B.

Alg. 4 illustrates how equations (11)-(13) and (14) would be used to evaluate a trajectory with clipping.

**Algorithm 4 Unrolling a trajectory with clipping.**

1:  \( t \leftarrow 0, J^C \leftarrow 0 \)
2:  \( \text{while } \vec{x}_t \notin T \text{ do} \)
3:      \( \vec{u}_t \leftarrow A(\vec{x}_t, \vec{z}) \)
4:      \( \vec{x}_{t+1} \leftarrow f(\vec{x}_t, \vec{u}_t) \)
5:      \( \text{if } \vec{x}_{t+1} \in T \text{ then} \)
6:          \( \text{Identify } \vec{P} \text{ and } \vec{n} \text{ by inspection of the intersection} \)
7:              \( \text{with the terminal boundary, } T. \)
8:      \( \lambda \leftarrow \Lambda(\vec{x}_t, \vec{u}_t, \vec{P}, \vec{n}) \)
9:      \( T \leftarrow t + 1 \)
10:     \( \vec{x}_T \leftarrow \vec{x}_t + \lambda (\vec{x}_T - \vec{x}_t) \)
11: \( J^C \leftarrow J^C + (\gamma^t) (\lambda U(\vec{x}_t, \vec{u}_t) + \gamma^\lambda \Phi(\vec{x}_T)) \)
12: \( J^C \leftarrow J^C + (\gamma^t) U(\vec{x}_t, \vec{u}_t) \)
13: \( \text{end if} \)
14: \( t \leftarrow t + 1 \)
15: \( \text{end while} \)
16: \( T \leftarrow t \)

Note that \( \vec{P} \) and \( \vec{n} \) are required by equations (11)-(13). These would be found during the collision-detection routine (i.e. line 6 of Alg. 4), from knowledge of the terminal-boundary orientation, together with knowledge of \( \vec{x}_{T-1} \) and \( f(\vec{x}_{T-1}, \vec{u}_{T-1}) \). Knowledge of the orientation of the terminal boundary could come from a model of the physical environment’s boundary; or if this model was not available, then a physical inspection of the actual boundary would need to take place. Examples of how these two vectors were found in our experiments are given in Sec. IV-A and IV-B.

**B. Calculation of the Derivatives of the Clipped Model and Cost Functions**

The ADP algorithms described in Section II require the derivatives of the model function, and hence they will require the derivatives of the clipped model function \( f^C(\vec{x}, \vec{u}, \vec{P}, \vec{n}) \) too. Fig. 5 shows how different the derivative of \( f \) can be from the derivative of \( f^C \), and hence how important it is to get this correct in ADP/RL. This figure clarifies why algorithms that are dependent on \( \frac{\partial f^C}{\partial \vec{x}} \) are critically affected by the need for clipping, and also that just reducing the duration of each time step tracking or simulating the motion will not solve the problem at all.

Differentiating the formula for \( \Lambda(\vec{x}, \vec{u}, \vec{P}, \vec{n}) \) in (11) gives:

\[ \frac{\partial \Lambda(\vec{x}, \vec{u}, \vec{P}, \vec{n})}{\partial \vec{x}} = \frac{\partial}{\partial \vec{x}} \left( \frac{(\vec{P} - \vec{x}) \cdot \vec{n}}{(\vec{r} \cdot \vec{n})} \right) \]  
by (11)
The cost-to-go function for the penultimate time step, equation (14), can be rewritten as a Q-function of both \( \tilde{x} \) and \( \tilde{u} \), to give

\[
Q(\tilde{x}_{T-1}, \tilde{u}_{T-1}) := U^C(\tilde{x}_{T-1}, \tilde{u}_{T-1}, \tilde{P}, \tilde{n}) + \gamma^L \Phi(f^C(\tilde{x}_{T-1}, \tilde{u}_{T-1}, \tilde{P}, \tilde{n})).
\] (21)

Differentiating this with respect to \( \tilde{u}_{T-1} \) or \( \tilde{x}_{T-1} \) gives:

\[
\left( \frac{\partial Q}{\partial \bullet} \right)_{T-1} = \left( \frac{\partial U^C}{\partial \bullet} \right)_{T-1} + \gamma^L \left( \frac{\partial f^C}{\partial \bullet} \right)_{T-1} \left( \frac{\partial \Phi}{\partial f} \right)_T + (\ln \gamma) \left( \frac{\partial \Lambda}{\partial \bullet} \right)_{T-1} \Phi(\tilde{x}_T)
\] (22)

where \( \bullet \) represents either \( \tilde{u} \) or \( \tilde{x} \).

This equation, which relies upon the derivatives of \( f^C(\tilde{x}, \tilde{u}, \tilde{P}, \tilde{n}) \) and \( U^C(\tilde{x}, \tilde{u}, \tilde{P}, \tilde{n}) \) (as defined in equations (15) to (19)), can be used to modify BPTT from Alg. 4 into its corresponding “with clipping” version given in Alg. 5. Equation (22) appears in the algorithm directly in lines 8-9.

The DHP and HDP algorithms need similar modifications to convert them to include clipping. Clipping needs applying to the final time step of the trajectory unroll, which can be implemented by replacing line 4 of both algorithms by lines 10-13 of Alg. 4. Also, in the case of DHP (Alg. 2), the lines that calculate \( Q_x \) and \( Q_u \) need replacing by lines 10-13 of Alg. 5 and similarly the line that calculates \( Q_u \) in Alg. 4 (HDP) needs the same modification.

**Algorithm 5** Backpropagation Through Time for Control, with Clipping.

**Require:** Trajectory calculated by Alg. 4

1. \( \frac{\partial f}{\partial x} \leftarrow \tilde{q} \)
2. \( \tilde{P} \leftarrow (\frac{\partial f}{\partial x})_t \)
3. for \( t = T - 1 \) to 0 step -1 do
4. if \( \tilde{x}_{t+1} \in \mathbb{T} \) then
5. Calculate \( \left( \frac{\partial f}{\partial x} \right)_t \) and \( \left( \frac{\partial f}{\partial u} \right)_t \) by (15) and (16).
6. Calculate \( \left( \frac{\partial f^C}{\partial x} \right)_t \) and \( \left( \frac{\partial f^C}{\partial u} \right)_t \) by (17) and (18).
7. Calculate \( \left( \frac{\partial U^C}{\partial x} \right)_t \) and \( \left( \frac{\partial U^C}{\partial u} \right)_t \) by (19) and (20).
8. \( Q_x \leftarrow \left( \frac{\partial U^C}{\partial x} \right)_t + \gamma^L \left( \frac{\partial f^C}{\partial x} \right)_t \tilde{p} + (\ln \gamma) \left( \frac{\partial \Lambda}{\partial x} \right)_t \Phi(\tilde{x}_T) \)
9. \( Q_u \leftarrow \left( \frac{\partial U^C}{\partial u} \right)_t + \gamma^L \left( \frac{\partial f^C}{\partial u} \right)_t \tilde{p} + (\ln \gamma) \left( \frac{\partial \Lambda}{\partial u} \right)_t \Phi(\tilde{x}_T) \)
10. else
11. \( Q_x \leftarrow \left( \frac{\partial U}{\partial x} \right)_t + \gamma \left( \frac{\partial f}{\partial x} \right)_t \tilde{p} \)
12. \( Q_u \leftarrow \left( \frac{\partial U}{\partial u} \right)_t + \gamma \left( \frac{\partial f}{\partial u} \right)_t \tilde{p} \)
13. end if
14. \( \frac{\partial f}{\partial x} \leftarrow \frac{\partial f}{\partial x} + \gamma^t \left( \frac{\partial \Lambda}{\partial x} \right)_t Q_u \)
15. \( \tilde{P} \leftarrow Q_x + \left( \frac{\partial \Lambda}{\partial P} \right)_t Q_u \)
16. end for
17. \( \tilde{x} \leftarrow \tilde{x} - \alpha \frac{\partial f}{\partial x} \)
C. Implementing Clipping Efficiently and Correctly

To demonstrate how clipping would be correctly implemented with an ADP/RL algorithm, we use the BPTT algorithm for illustration. In an implementation of BPTT with clipping, we would first evaluate a trajectory by Alg. 4. During this stage, we would record the full trajectory \((\vec{x}_0, \vec{x}_1, \ldots, \vec{x}_T)\) and actions \((\vec{u}_0, \vec{u}_1, \ldots, \vec{u}_{T-1})\) and also, during the collision with the terminal boundary, we would record \(\vec{P}\) and \(\vec{n}\) and the clipping fraction, \(\lambda\). We then have enough information to be able to run the BPTT algorithm with clipping (Alg. 5).

To ensure the correctness of our implementations in each experiment and environment which we tackled, we first verified all of the derivatives of \(\Lambda(\vec{x}, \vec{u}, \vec{P}, \vec{n})\), \(f^C(\vec{x}, \vec{u}, \vec{P}, \vec{n})\) and \(U^C(\vec{x}, \vec{u}, \vec{P}, \vec{n})\) numerically, with respect to both \(\vec{x}\) and \(\vec{u}\), at least a few times. When all of these derivatives were all verified all of the derivatives of our experiments, below). At the start of each experimental trial, the action and critic networks used were multi-layer perceptrons (MLPs, see [23] for details). Each MLP had \(\dim(\vec{x})\) input nodes, 2 hidden layers of 6 nodes each, and one output layer, with short-cut connections connecting all pairs of layers. The output layers were dimensioned as follows: Each action network had \(\dim(\vec{u})\) output nodes; each HDP critic network had 1 output node; and each DHP critic had \(\dim(\vec{d})\) output nodes. All network nodes had bias weights, as is usual in MLP architectures. The activation functions used were hyperbolic tangent functions, except for the critic network’s output layer which was always a linear activation function (with linear slope as specified in the individual experiments, below). At the start of each experimental trial, neural weights were initialised randomly in the range \([-1, 1]\), with uniform probability distribution.

D. Clipping with Trajectories of Fixed or Variable Finite Length

In situations where trajectories are fixed finite length (commonly referred to as a fixed-length finite-horizon problem), clipping is not necessary. This is in contrast to the problems we considered in the introduction, which were variable finite-length problems, since the trajectory lengths were determined by the environment (e.g. a trajectory terminates only when the agent crashes into a wall). In this section we will distinguish between these two situations by referring to them as “fixed finite-length” and “variable finite-length” problems, respectively. Only in variable finite-length problems is clipping necessary.

In the fixed finite-length problem, the clipping fraction defined by \([21]\) is always \(\lambda = 1\), and therefore \(\frac{\partial \Lambda}{\partial \vec{x}} = 0\), \(\frac{\partial \Lambda}{\partial \vec{u}} = 0\) and \(\gamma^\lambda = \gamma\). Hence the clipped model and cost functions are identical to their unclipped counterparts, and therefore it is not necessary to implement any program code specifically to handle clipping. This might be one reason why the need for clipping has not previously been noted in the research literature, since most finite-horizon problems considered have been fixed-finite length.

However the fixed finite-length problem does have one minor different complication, in that it is often necessary to include the time step into the state vector. This is because the optimal actions and cost-to-go function will often be dependent upon the number of incomplete steps in a trajectory.

Of course for both fixed-length and variable-length finite-horizon problems, it is important to ensure the terminal cost function \(\Phi(\vec{x})\) is learnt correctly by the learning algorithm. The pseudocode shows explicitly how to do this (e.g. for BPTT, see line 2 of Algs. 1 and 5. For DHP, see line 5 of Alg. 2. And for HDP, see lines 5 and 6 of Alg. 3).

IV. Experimental Results

This section describes two neural-network based ADP/RL experiments which require clipping to be solved well.

In all experiments the action and critic networks used were multi-layer perceptrons (MLPs, see [23] for details). Each MLP had \(\dim(\vec{x})\) input nodes, 2 hidden layers of 6 nodes each, and one output layer, with short-cut connections connecting all pairs of layers. The output layers were dimensioned as follows: Each action network had \(\dim(\vec{u})\) output nodes; each HDP critic network had 1 output node; and each DHP critic had \(\dim(\vec{d})\) output nodes. All network nodes had bias weights, as is usual in MLP architectures. The activation functions used were hyperbolic tangent functions, except for the critic network’s output layer which was always a linear activation function (with linear slope as specified in the individual experiments, below). At the start of each experimental trial, neural weights were initialised randomly in the range \([-1, 1]\), with uniform probability distribution.

A. Vertical Lander problem

A spacecraft is dropped in a uniform gravitational field, and its objective is to make a fuel-efficient gentle landing. The spacecraft is constrained to move in a vertical line, and a single thruster is available to make upward accelerations. The state vector \(\vec{x} = (h, v, u)^T\) has three components: height \(h\), velocity \(v\), and fuel remaining \(u\). The action vector, \(a\), is one-dimensional (so that \(\vec{u} = a \in \mathbb{R}\)) producing accelerations \(a \in [0, 1]\). The Euler method with time-step \(\Delta t\) is used to integrate the motion, giving model functions:

\[
f((h, v, u)^T, a) = (h + v \Delta t, v + (a - k_g) \Delta t, (k_u)u - a \Delta t)^T
\]

\[
U(((h, v, u)^T, a) = (k_f)a \Delta t
\] (23)

Here, \(k_g = 0.2\) is a constant giving the acceleration due to gravity; the spacecraft can produce greater acceleration than that due to gravity. \(k_f = 4\) is a constant giving fuel penalty. \(k_u = 1\) is a unit conversion constant. We used \(\Delta t = 1\) in our main experiments here.
Trajectories terminate as soon as the spacecraft hits the ground \((h = 0)\) or runs out of fuel \((u = 0)\). These two conditions define \(T\). This is a variable finite-length problem, and there is no need to use a discount factor, so we fixed \(\gamma = 1\). On termination, the algorithms need to choose values for \(\bar{P}\), and \(\bar{v}\) which describe the orientation of the terminal-boundary tangent plane. These choices are given for this experiment in Table I. In the case that the final un-clipped state transition crosses both terminal planes, then the one that is crossed first (i.e. the one that produces a smaller clipping fraction by \((11)\)) is to be used.

In addition to the cost function \(U(\bar{x}, a)\) defined above, a final impulse of cost defined by \(\bar{\Phi}(\bar{x}_T) := \frac{1}{2}mv^2 + m(k_g)h\) is given as soon as the lander reaches a terminal state, where \(m = 2\) is the mass of the spacecraft. The two terms in the final impulse of cost are the kinetic and potential energy, respectively. The first cost term penalises landing too quickly. The second term is a cost term equivalent to the kinetic energy that the spacecraft would acquire by crashing to the ground under free fall (i.e. with \(a = 0\)), so to minimise this cost the spacecraft must learn to not run out of fuel.

The input vector to the action and critic networks was \(\bar{x} = (h/100, v/10, u/50)^T\), and the model and cost functions were redefined to act on this rescaled input vector directly. The action network’s output \(y\) was rescaled to give the action by \(A(\bar{x}, \bar{z}) := (y + 1)/2\) directly. We tested each algorithm in batch mode, operating on five trajectories simultaneously. Those five trajectories had fixed start points, which had been randomly chosen in the region \(h \in (0, 100), v \in (-10, 10)\) and \(u = 30\).

Fig. 6 shows learning performance of the BPTT, DHP and HDP algorithms, both with and without clipping. Each graph shows five curves, and each curve shows the learning performance from a different random weight initialisation. The learning rates for the three algorithms were: BPTT (\(\alpha = 0.01\)); DHP (\(\alpha = 0.001, \beta = 0.00001\)); and HDP (\(\alpha = 0.00001, \beta = 0.00001\)). The critic-network’s output layer’s activation function had a linear slope of 20 in the DHP experiment and 10 in the HDP experiment.

Because HDP is an algorithm which requires stochastic exploration to optimise the ADP/RL problem effectively [24], in the HDP experiment we had to modify \((3)\) to choose exploratory actions. Hence for the HDP experiment we used

\[
\bar{v}_t = A(\bar{x}_t, \bar{z}) + X_{\sigma},
\]

where \(X_{\sigma}\) is a normally distributed random variable with mean zero and standard deviation \(\sigma = 0.1\).

These graphs show the clear stability and performance advantages of using clipping correctly for the BPTT and DHP algorithms. The graphs also confirm that the HDP algorithm is not significantly affected by the need for clipping.

Fig. 7 shows that the need for clipping is not made arbitrarily small by just using a smaller \(\Delta t\) value.

B. Cart Pole Experiment

We investigated the effects of clipping in the well known cart-pole benchmark problem described in Fig. 8. We considered the version of this problem used by [25], where the total trajectory cost is a function of the duration that the pole could be balanced for. Clearly, unless clipping is used properly, the duration will be an integer number of time steps, and since this is not smooth and differentiable, it will cause problems (become impossible) for DHP and BPTT. Hence traditionally when DHP or BPTT are used for the cart-pole problem, a different cost function would be used, one that is differentiable and proportional to the deviation from the balanced position (e.g. see [26]). However, in this section we show that by using clipping, DHP and BPTT can be successful with the duration-based reward. Since it is not possible to do this without clipping, we assume this is the first published version of this solution by DHP/BPTT.

The equation of motion for the frictionless cart-pole system [25, 27, 26] is:

\[
\dot{\theta} = \frac{g \sin \theta - \cos \theta \left[ \frac{E + m g \dot{\theta}^2 \sin \theta}{m l^2 + m l^2 \cos^2 \theta} \right]}{l \left[ \frac{3}{3} - \frac{m g \cos^2 \theta}{m l^2 + m l^2} \right]}
\]

(24)

TABLE I: Terminal Boundary Planes used in vertical-lander experiment. The state vector used here is \(\bar{x} = (h, v, u)^T\).

| Termination Condition Breached | Position vector of Plane, \(P^T\) | Normal Vector to Plane, \(\bar{n}^T\) |
|-------------------------------|---------------------------------|---------------------------------|
| \(h \leq 0\) (this ground)   | \((0, 0, 0)\)                   | \((1, 0, 0)\)                   |
| \(u \leq 0\) (no fuel)       | \((0, 0, 0)\)                   | \((0, 0, 1)\)                   |
Actor-critic architectures in the cart-pole problem. The output state-space scaling can be critical to successful convergence of the algorithms. As noted by [26], choosing an appropriate \( \alpha = 0.01, \alpha = 0.0001 \). The DHP critic used a final-layer activation-function slope of 0.1.

Learning took place on five trajectories simultaneously, with fixed start points randomly chosen from the region \(|x| < 2.4, |\theta| < \frac{\pi}{15}, \dot{x} = 0, \dot{\theta} = 0\). This is similar to the starting conditions used by [25]. The exact derivatives of the model and cost functions were made available to the algorithms.

The performance of the two algorithms, both with and without clipping, are shown in Fig. 9. Each graph shows how the average balancing duration over all five trajectories, versus the training iteration. Each graph shows an ensemble of five different curves, with each curve represents a training run from a different random weight initialisation.

The results show that using clipping correctly enables both the DHP and BPTT algorithms to solve this problem consistently, and without clipping it is impossible for both algorithms.

V. CONCLUSIONS

The problem of clipping for ADP/RL and neurocontrol algorithms has been demonstrated and motivated. Without clipping, algorithms which rely on the derivatives of the model and cost functions can fail to work. The solution is to apply clipping, and then to correctly differentiate the model and cost functions in the final time step. This solution has been given in the form of the equations, plus in the form of clear pseudocode for the two major affected ADP algorithms: DHP and BPTT.

Two neural network experiments have confirmed the importance of applying clipping correctly. These included a cart-pole experiment, where clipping was found to be essential, and in a vertical-lander experiment, where clipping produced a significant improvement of performance.

The situations in which clipping are needed have been made clear, and those situation where it can be ignored have also been specified.

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