Spin Dynamics of a Fermi Liquid in an Electric Field

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Abstract

The influence of an external electric field on the spin dynamics of an electrically neutral Fermi liquid is considered, the mechanism of such an influence being the relativistic spin-orbital interaction. As a result, Leggett’s equations for the spin dynamics of weakly polarized Fermi liquids are generalized to the case of non-zero external electric field. In addition, we obtained the transverse spin dynamics equation for strongly spin-polarized liquids in an electric field at zero temperature. In both situations covariant derivatives depending on the electric field are shown to be substituted for spatial gradients in line with the SU(2) gauge invariance of the microscopic Hamiltonian. The new equations are applied to the study of spin flow along a channel, where an electric field is found to bring about an additional phase shift of the order of magnitude of the phase shift in superfluid $^3$He-$B$ but growing with time.

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1 Introduction

The advances in experimental technique in recent years made feasible the observation of effects due to electric field on the motion of electrically neutral superfluid $^3$He [1]. Experiments on superfluid $^3$He-$B$ in an electric field, using the spin current Josephson effect, are under way at the moment [2]. The influence of an external electric field on superfluid $^3$He has also been treated theoretically in scientific literature [3, 4, 5]. However, a similar investigation for a normal Fermi liquid is lacking. It is this question which is attempted to be answered by the present paper.

There have been predicted two competitive mechanisms of the influence of an electric field on the spin dynamics of the superfluid $^3$He [5], viz., (i) slight deformation of the electronic shells of $^3$He atoms by the gradient of the order parameter and (ii) spin-orbital coupling $(\mu \times p)E/mc$ of the electric field with the moving magnetic moments of the $^3$He nuclei, where $\mu$ is the magnetic moment of $^3$He atom. In the present paper this latter relativistic effect is incorporated into spin dynamics equations of a normal Fermi liquid. The result complies with the gauge invariance arguments developed in Ref. [5].

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In superfluid $^3$He-\text{B} in an experiment with fixed spin supercurrent along a channel of length $L$, connecting two reservoirs with homogeneously precessing magnetizations, the supercurrent, induced by an electric field, leads to an additional phase shift of the order of

$$\delta \alpha \sim \frac{2\mu E}{\hbar c} L \sim 10^{-4} \text{ rad} \quad (1)$$

for $E \approx 3 \cdot 10^4 \text{ V/cm}$ and $L \approx 1 \text{ cm}$. According to [1], this is an experimentally detectable quantity.

In a normal Fermi liquid a phase shift of the same order is demonstrated to arise and grow with time in a similar one-dimensional spin flow geometry.

The set-up of the paper is as follows. The second section is dedicated to introducing the SU(2) gauge invariant theory of the interaction of electromagnetic field with a Fermi liquid. In the third section the weakly polarized Fermi liquid is considered on the basis of the modifications to the Landau theory resulting from the appearance of an electric field. Although this section does not overtly leans on the gauge invariance, the outcome agrees with what one expects from Sec. II. The next section is given over to the study of one-dimensional spin flow as an example of application of the new equations.

The invariance properties of the microscopic Hamiltonian are employed in treating strongly spin-polarized liquids at zero temperature in Sec. V. The last section sums up the main results.

2 Gauge Invariance

In the SU(2)-gauge-invariant wording of Ref. [5] electric and magnetic fields are introduced uniformly as components of a gauge field $A_\mu$, where $\mu = 0, 1, 2, 3$:

$$gA_0 = gH \equiv \omega_L, \quad gA^\alpha_i = -\frac{g}{c} \epsilon_{\alpha ik} E_k. \quad (2)$$

Here $g$ is the gyromagnetic ratio for $^3$He nuclei, $\omega_L$ is the Larmor frequency.

The Hamiltonian of particles with the mass $m$ and the magnetic moment $\mu = \hbar g/2$ in an external electromagnetic field in the second quantization representation is

$$\mathcal{H} = \int \psi^+(r) \left[ \frac{\hat{p}^2}{2m} - \hat{\mu} \left( H - \frac{1}{mc} \hat{p} \times E \right) \right] \psi(r) \, d^3r, \quad (3)$$

where $\hat{\mu} = \mu \hat{\sigma}$ is the magnetic moment operator. The Hamiltonian can be rewritten equivalently using the gauge field $A_\mu$:

$$\mathcal{H} = \int \psi^+(r) \left[ \frac{(\hat{p} - \hat{\mu} A_i)^2}{2m} - \hat{\mu} A_0 \right] \psi(r) \, d^3r. \quad (4)$$

The last expression is invariant against rotations of the spin space if only they are accompanied by the corresponding change of the gauge field:

$$\psi \rightarrow \mathcal{R} \psi, \quad gA_\mu \rightarrow g\mathcal{R}A_\mu + \omega_\mu, \quad (5)$$
The rotation \( R \) and \( K(\theta) \) is a 3D rotation matrix corresponding to \( \mathcal{R} \). The frequency tensor of this rotation \( \omega_{\mu} \) is defined as 
\[-i\omega_{\mu}\hat{\sigma}/2 = \mathcal{R}^+ \partial_{\mu} \mathcal{R}, \]
and the gradients are designated as \( \partial_{\mu} \), where \( \partial_0 = \partial_t, \partial_i = \nabla_i \).

This symmetry allows one to refer to \( A_{\mu} \) as the gauge field. Quite generally, gauge fields appear when the invariance of a Hamiltonian against some local gauge symmetry is stipulated. Nevertheless, it might be worth mentioning that this symmetry is somewhat formal since it is valid only for the Hamiltonian of \(^3\)He atoms and not for the electromagnetic field.

The SU(2) gauge invariance of the Hamiltonian calls for all observables to be gauge invariant as well. Hence all space-time derivatives can enter spin dynamics equations only in combinations with the gauge field, i.e., in the form of covariant derivatives:
\[ D_{\mu}X = \partial_{\mu}X + gA_{\mu} \times X. \tag{6} \]

As we will see later, this conjecture is correct.

### 3 Weak Polarizations

Spin dynamics equations in the case of weak polarization have been formulated by Silin \([6]\) and Leggett \([7]\) in the framework of the Landau theory \([8]\). Leggett showed that the first two harmonics of the distribution function, viz., the spin and the spin current densities, decouple from the rest for slow enough spatial variations. The purpose of this section is to generalize the Leggett equations to the case of an electric field present.

The Landau theory of Fermi liquids operates with phenomenologically defined quantities. So it is not convenient to literally use the gauge invariance approach discussed above. We will set this approach aside until Sec. V and formulate the theory in the usual terms of the quasiparticle distribution matrix \( \hat{n}_k \) and the quasiparticle energy matrix \( \hat{\epsilon}_k \).

In the absence of fields the equilibrium values of \( \hat{n}_k \) and \( \hat{\epsilon}_k \) are
\[ \hat{\epsilon}_0^k = \xi_k \hat{1}, \quad \hat{n}_0^k = f_0(\hat{\epsilon}_0^k). \tag{7} \]

Here \( f_0(\xi) = [\exp(\xi/T)+1]^{-1} \) is the Fermi distribution function and \( \xi_k = \hbar^2 k^2 / 2m^* - \epsilon_F \), where \( m^* \) is the quasiparticle mass.

According to Landau’s ideas the departure from equilibrium distribution \( \delta \hat{n}_k \) changes the quasiparticle energy along with external fields:
\[ \delta \hat{\epsilon}_k = \delta \hat{\epsilon}_{\text{ext}} + \sum_{k'} \text{Sp}'[f_{kk'}\delta \hat{n}_{k'}], \tag{8} \]
where \( \delta \hat{\epsilon}_{\text{ext}} \) is the energy of a quasiparticle in external fields, \( f_{kk'} = N_F^{-1} (F_{kk'}^a \hat{1} + F_{kk'}^a \hat{\sigma} \hat{\sigma}') \) is the quasiparticle interaction and \( N_F = m^* k_F / \pi^2 \hbar^2 \) is the density of states on the Fermi surface.
As the microscopic Hamiltonian (4) indicates, an electric field alters the equilibrium (kinetic) part $\hat{\varepsilon}_k^0$ of the energy matrix $\hat{\varepsilon}_k$ by shifting all the momenta of quasiparticles:

$$\hat{\varepsilon}_k^0 \rightarrow \hat{\varepsilon}_k^0 + \delta \hat{\varepsilon}_k^0, \quad \text{where } \delta \hat{\varepsilon}_k^0 = -v_{F, i} \hat{\mu} A_i.$$  

(10)

As a consequence, the equilibrium distribution $\hat{n}_k^0 \equiv f_0(\hat{\varepsilon}_k^0)$ will also change: $\hat{n}_k^0 \rightarrow \hat{n}_k^0 + (df_0/d\xi)\delta \hat{\varepsilon}_k^0$, which, in its turn, will alter the departure from equilibrium $\delta \hat{n}_k$ defined as $\delta \hat{n}_k = \hat{n}_k - \hat{n}_k^0$:

$$\delta \hat{n}_k \rightarrow \delta \hat{n}_k - \frac{df_0}{d\xi} \delta \hat{\varepsilon}_k^0.$$  

(11)

The amendment in $\delta \hat{n}_k$ through relation (8) will change $\delta \hat{\varepsilon}_k$:

$$\delta \hat{\varepsilon}_k \rightarrow \delta \hat{\varepsilon}_k + \sum_{k'} \text{Sp}'[f_{kk'} \left( -\frac{df'_0}{d\xi} \right) \delta \hat{\varepsilon}_{k'}^0] = \delta \hat{\varepsilon}_k + \frac{F_a}{3} \delta \hat{\varepsilon}_k^0.$$  

(12)

Now let introduce macroscopic quantities. Spin and spin current densities are defined according to the formulas:

$$S = \frac{\hbar}{2} \sum_k \text{Sp}[\hat{\sigma} \delta \hat{n}_k], \quad J_i = \frac{1}{2} \sum_k \text{Sp}[\hat{\sigma} \delta \hat{n}_k \nabla_{k_i} \hat{\varepsilon}_k].$$  

(13)

Both of these expressions in equilibrium in the absence of external field are equal to zero. Expanding them to within the first order in departures from equilibrium yields

$$S = \frac{\hbar}{2} \sum_k \text{Sp}[\hat{\sigma} \delta \hat{n}_k],$$  

$$J_i = \frac{1}{2} \sum_k \text{Sp}[\hat{\sigma} \left( \delta \hat{n}_k \nabla_{k_i} \hat{\varepsilon}_k^0 + \hat{n}_k^0 \nabla_{k_i} \delta \hat{\varepsilon}_k \right)] = \frac{\hbar}{2} \sum_k v_{F, i} \text{Sp}[\hat{\sigma} \left( \delta \hat{n}_k - \frac{df_0}{d\xi} \delta \hat{\varepsilon}_k \right)].$$  

(14, 15)

where the Fermi velocity $v_{F, i} = \hbar k_i/m^*$ and we took the second term in (15) by parts. This term describes backflow (cf. [9]). The expression in parentheses in (13) can be easily seen to represent the departure from the local equilibrium $\hat{n}_k^{loc} = f_0(\varepsilon_k^0 + \delta \hat{\varepsilon}_k) \approx f_0(\varepsilon_k^0) + (df_0/d\xi)\delta \hat{\varepsilon}_k$.

When an electric field is switched on both spin and spin current densities in equilibrium will still be zero. This follows from the fact that the amendments to their magnitudes brought about by $\delta \hat{n}_k^0$ and $\delta \hat{\varepsilon}_k^0$ are described by eqs. (14) and (15) respectively. But these expressions vanish when $\delta \hat{n}_k^0$ and $\delta \hat{\varepsilon}_k^0$ are substituted for $\delta \hat{n}_k$ and $\delta \hat{\varepsilon}_k$. Thus an external electric field produces no spin current in equilibrium. We will exploit this fact later when writing out the collision integral in the kinetic equation in the $\tau$-approximation.

Substituting the expression (8) along with the renewed $\delta \hat{n}_k$ (11) and $\delta \hat{\varepsilon}_k$ (12) into (13) yields

$$J_i = \frac{\hbar N_F}{4} \langle v_{F, i} \text{Sp}[\hat{\sigma} \delta \hat{\varepsilon}^{ext, eff.}] \rangle_k + \frac{\hbar}{2} \left( 1 + \frac{F_a}{3} \right) \sum_k v_{F, i} \text{Sp}[\hat{\sigma} \delta \hat{n}_k],$$  

(16)
where the corrections due to electric field to \( \delta \hat{\sigma}^n_k \) and \( \delta \hat{\epsilon}_k \) can be conveniently incorporated into an effective external-fields energy

\[
\delta \hat{\epsilon}^{\text{ext eff.}}_k = -\frac{\hbar \hat{\sigma}}{2} \left( \omega_L + v_{Fi} \left( 1 + \frac{F_1^a}{3} \right) g A_i \right)
\]  

(17)

On plugging this into (16), we obtain the following expression for the spin current:

\[
\mathbf{J}_i = \mathbf{J}_i^0 + \hbar \left( 1 + \frac{F_1^a}{3} \right) \sum_k v_{Fi} \sigma_k,
\]

(18)

where \( \sigma_k = \frac{1}{2} \text{Sp}[\hat{\sigma} \delta \hat{n}_k] \). As compared to the case of no electric field [7], the expression for the spin current contains an additional term

\[
\mathbf{J}_i^0 = \frac{\mu \rho \hbar}{2m^* c} e_{aik} E_k \left( 1 + \frac{F_1^a}{3} \right) = -\frac{\chi_n}{g^2} \frac{w^2}{3} g A_i,
\]

(19)

where \( \rho = k_F^3 / 3\pi^2 \) is the liquid’s density, \( w^2 = v_F^2 (1 + F_0^a) (1 + F_1^a / 3) \) and \( \chi_n = g^2 \hbar^2 N_F / 4 (1 + F_1^a) \) is the normal Fermi-liquid susceptibility. This term appeared previously in the spin supercurrent in superfluid Fermi liquid [3].

We now proceed to the derivation of the dynamics equations themselves.

Let expand \( \hat{n}_k \) and \( \hat{\epsilon}_k \) in the basis of the identity and the Pauli matrices:

\[
\hat{n}_k = f_k \hat{1} + \sigma_k \hat{\sigma}, \quad \hat{\epsilon}_k = \epsilon_k \hat{1} + \epsilon_k \hat{\sigma}.
\]

(20)

In the weak-polarization case, where spin polarization is a very small fraction of the total number of spins in the liquid, \( f_k \) and \( \epsilon_k \) may be put equal to their equilibrium values (7) (see [7]). Then from eqs. (8), (9), (10) and (12) we obtain

\[
\epsilon_k = -\frac{\hbar}{2} \left( \omega_L + v_{Fi} \left( 1 + \frac{F_1^a}{3} \right) g A_i \right) + \frac{2}{N_F} \sum_{k'} F_{kk'}^a \sigma_{k'}.
\]

(21)

Note that here the electrical-field-dependent corrections to \( \epsilon_k^0 \) and \( \delta \hat{\epsilon}_k \) also appear in effect as if it is the external energy \( \delta \hat{\epsilon}^{\text{ext}}_k \) which contained them. What we mean is that the above expression could be obtained from the expression (8) with the effective external energy (17) without allowing for the corrections (10) and (12). It is tempting therefore to attribute the term in (17) that depends on electric field to an external spin-orbital energy. Nevertheless, since it contains Fermi-liquid constants this term is not, strictly speaking, an energy in an external field. Quite the reverse, it appeared from corrections to the kinetic energy. Thus we conclude that the parallel drawn is formal.

The Silin-Leggett [7] kinetic equation for the spin part \( \sigma_k \) of \( \hat{n}_k \) is (for brevity we omit \( k \)-indices)

\[
\partial_t \sigma - \frac{2}{\hbar} \epsilon \times \sigma + v_{Fi} \nabla_i \left( \sigma - \frac{df_0}{d\xi} \epsilon \right) = (\partial_t \sigma)_{\text{coll}}.
\]

(22)

Note the wrong sign before the second term in [3, 4].
we can further reduce the kinetic equation to
\[ D_t \mathbf{m}_k - \frac{4}{\hbar N_F} \langle F^a(\mathbf{k}\mathbf{k}') \mathbf{m}_{k'} \rangle_{k'} \times \mathbf{m}_k + v_{Fi} \nabla_i \left( \mathbf{m}_k + \langle F^a(\mathbf{k}\mathbf{k}') \mathbf{m}_{k'} \rangle_{k'} \right) + v_{Fi} \left( 1 + \frac{F_i^a}{3} \right) g \mathbf{A}_i \times \mathbf{m}_k - \frac{\hbar N_F}{4} v_{Fi} \nabla_i \left( \mathbf{\omega}_L + v_{Fj} \left( 1 + \frac{F_j^a}{3} \right) g \mathbf{A}_j \right) = (\partial_t \mathbf{m}_k) \tag{24} \]

Following Leggett, we seek after a solution in the form of the sum of the zeroth and first harmonics of \( \mathbf{m}_k \):
\[ \hbar \mathbf{m}_k = \mathbf{S} + \frac{3}{v_F} \left( 1 + \frac{F_i^a}{3} \right)^{-1} \hat{\mathbf{k}}_i (\mathbf{J}_i - \mathbf{J}_i^0). \tag{25} \]

The higher harmonics can be proved to decouple from the first two if only the characteristic spatial scale \( \lambda \) is greater than the mean free path or the molecular field length \([7]\):
\[ \lambda > \min \left\{ v_F \tau, \frac{v_F}{\kappa \omega_L} \right\}. \tag{26} \]
Here \( \kappa = (F_i^a/3 - F_0^a)/(1 + F_0^a) \). This constraint coincides with that for the case of zero electric field because the “electric length” \( l_E = (gE/c)^{-1} \) appearing as a new scale in the problem is much greater than all the above mentioned scales (for an electric field as strong as \( E = 3 \cdot 10^4 \text{V/cm} \) the “electric length” shortens only to \( l_E = 10^6 \text{cm} \)).

When the mean free path is shorter than the molecular field length (which obviously is equivalent to the inequality \( \kappa \omega_L \tau < 1 \)) we deal with the hydrodynamic regime. The opposite case corresponds to the collisionless regime.

After some algebra we obtain a set of two equations:
\[ D_t \mathbf{S} + D_i \mathbf{J}_i = 0, \tag{27} \]
\[ D_t \mathbf{J}_i - \partial_t \mathbf{J}_i^0 + \frac{w^2}{3} D_i (\mathbf{S} - \mathbf{S}^{\text{eq}}) + \frac{\kappa g^2}{\chi_n} \mathbf{S} \times \mathbf{J}_i = -\frac{\mathbf{J}_i}{\tau_1}. \tag{28} \]
Here \( \mathbf{S}^{\text{eq}} = \chi_n \mathbf{\omega}_L / g^2 \) is the equilibrium magnetization, \( \tau_1 = \tau/(1 + F_i^a/3) \). As compared to the Leggett equations, all the spatial derivatives in \((27), (28)\) are replaced with the covariant ones. The term \( \partial_t \mathbf{J}_i^0 = (\chi_n / g^2)(w^2/3)(\nabla_i \mathbf{\omega}_L - \nabla \mathbf{\omega}_L) \) is non-zero only for an electric field varying in time. We will assume this is not the case in what follows.

First, from now on we make use of the units wherein \( \chi_n = g^2 \). Next, we transform to the frame, rotating with the local Larmor frequency \( \mathbf{\omega}_L (r) \) that we assume to be parallel to \( \mathbf{z} \). To that end we substitute \( \mathbf{S}(t) \) with \( \mathbf{R}(t) \mathbf{S}(t) \) and similarly for \( \mathbf{J}_i \), where \( \mathbf{R} = \mathbf{R}(-\mathbf{\omega}_L t) \) is a 3D rotation matrix. Then we get from \((27), (28)\)
\[ \partial_t \mathbf{S} + \bar{D}_i \mathbf{J}_i = 0, \tag{29} \]
\[ \partial_t \mathbf{J}_i + \frac{w^2}{3} \bar{D}_i (\mathbf{S} - \mathbf{\omega}_L) + \kappa \mathbf{S} \times \mathbf{J}_i = -\frac{\mathbf{J}_i}{\tau_1}, \tag{30} \]
where \( \tilde{D}_i = (\nabla_i + g \tilde{A}_i \times J_i) \) and \( \tilde{A}_i = R^{-1}A_i \).

We here used the fact that \( e_{\alpha ij}R_{i\beta}R_{j\gamma} = R_{\alpha k}e_{k\beta\gamma} \), which follows from the general equality holding for an arbitrary matrix: \( e_{\alpha ij}R_{\alpha k}R_{i\beta}R_{j\gamma} = e_{k\beta\gamma} \det R \).

Eq. (30) is an inhomogeneous linear differential equation on \( J_i \). Its solution can be represented as the sum of the general solution of the corresponding homogeneous equation and a particular solution of the inhomogeneous one. The general solution of the homogeneous equation contains the factor \( \exp(-t/\tau_1) \) and dies out in a time \( \tau_1 \). Thus for the most of the time of an experiment we may take into account only the particular-solution part of the spin current.

In the absence of electric fields \( \tilde{D}_i = \nabla_i \) and both the inhomogeneous term and the coefficients in Eq. (30) are independent of time. In this case Eqs. (29), (30) allow exactly stationary solutions, wherein \( \partial_t S \) and \( \partial_t J_i \) are exactly zero in the Larmor frame.

In the presence of an electric field, the inhomogeneous term in (30) contains, besides the part constant in time, a fast contribution from the rotation matrix that oscillates with the Larmor frequency \( \omega_L \). Hence the solution will be the sum of constant and oscillating terms. The oscillating term in the Larmor frame will appear from the point of view of the laboratory frame as a doubled frequency motion and must induce a doubled frequency harmonic in the NMR signal.

However the amplitude of this harmonic as well as the correction to the stationary motion due to electric field will be of the order of smallness of \( gEL/c \), where \( L \) is the spatial scale of the problem. Therefore we can expand \( S \) and \( J_i \) into degrees of electric field to within the first order: \( S(E) = S + \delta S \), and similarly for \( J_i \), where \( S \) and \( J_i \) now designate the corresponding quantities in the absence of electric field that meet the conventional Leggett equations. For the departures \( \delta S \) and \( \delta J_i \) proportional to the electric field we obtain from (29), (30)

\[
\begin{align*}
\partial_t \delta S &+ \nabla_i \delta J_i + g \tilde{A}_i \times J_i = 0, \quad (31) \\
\partial_t \delta J_i &+ \frac{w^2}{3} \left( \nabla_i \delta S + g \tilde{A}_i \times (S - \omega_L) \right) + \kappa (\delta S \times J_i + S \times \delta J_i) = -\frac{\delta J_i}{\tau_1}. \quad (32)
\end{align*}
\]

Thus \( \delta S \) and \( \delta J_i \) (deemed as components of one unknown quantity) satisfy a system of inhomogeneous linear differential equations with coefficients depending on coordinates but independent of time. The electric field enters the equations only through inhomogeneous terms and hence the corresponding homogeneous system describes the time evolution of small perturbations to a known unperturbed zero-electric-field solution. The general solution of the homogeneous system is the sum of particular solutions in which \( \delta S \) and \( \delta J_i \) depend on time through the factors \( e^{-i\omega t} \). The perturbation frequencies \( \omega \) are to be determined by solution of the homogeneous system with the corresponding boundary conditions. For the unperturbed distribution to be stable the imaginary parts of all the possible frequencies \( \omega \) need to be negative. Then the arisen perturbations will die out exponentially.

To find the quasistationary part of the solution (i.e. the one that is constant in time
in the Larmor frame), we should average the inhomogeneous terms over the period of precession $1/\omega_L$, which yields $\langle \hat{A}_i \rangle = A_i^z \hat{z}$.

We remind that $A_i^z = -c^{-1} e_z k E_k$. Thus the $\hat{z}$-component of the electric field have no effect on the quasistationary dynamics. If we choose $E \parallel \hat{x}$ for definiteness, then $A_i^z = c^{-1} E \hat{y}_i$ and the electric field falls out from the equations for the $\hat{x}_i$- and $\hat{z}_i$-orbital components of the spin current, i.e. the electric field affects the quasistationary motion only if the $\hat{y}_i$-components of the spin current is non-zero.

4 One-Dimensional Flow

As an example we consider spin flow through a thin channel directed along the $\hat{y}_i$-axis. But first we examine the spin distribution in such a geometry in the absence of an electric field. The situation is reminiscent of the flow along the $\hat{z}_i$-directed channel studied experimentally [10] and theoretically [11]. We may extend the results obtained in Ref. [11] to our situation since in the derivation of distributions and in the stability analysis in Ref. [11] only the fact that the flow is one-dimensional is used and the name of the coordinate along which the distribution changes makes no difference.

4.1 In the Absence of Electric Field

In the idealized geometry of Ref. [11] two reservoirs of a Fermi-liquid polarized by an external magnetic field directed along $\hat{z}$ are connected via a long thin tube of length $2L$ and cross-sectional area $A \ll L^2$. After a $\pi$-pulse of a rf magnetic field, inverting the spins, is applied to one of the volumes, diffusion tends to diminish the large longitudinal polarization gradient along the channel and eventually equalize polarization in the two chambers (see Fig. 1).

Since the time $L^2/D$ to diffuse through the tube is much longer than the time $A/D$ to diffuse away from the tube entrances, we may, following [11], assume fixed boundary conditions $S(y = \pm L) = \pm S_0 \hat{z}$, where $S_0$ is the magnetization density in the reservoirs. For convenience we transform to the dimensionless units: $s = y/L$, $\tau = D_0 t/L^2$, $m = S/S_0$, $j = L J_y/D_0 S_0$, and introduce the parameter $v = -\kappa S_0 \tau_1$.  

Figure 1: Geometry of idealized spin-diffusion experiment. The circles are turns of wire of the rf coil used to invert the magnetization in the left reservoir.
Here $D_0 = w^2 \tau_1 / 3$ is the diffusion coefficient. As a matter of fact, $\kappa$ is positive for real liquids and thus $v$ is negative.

Now we rederive the results of [11] generalized to the case of the parameter $\hat{m}_0 \neq \hat{x}$ (see below).

The Leggett equations in the Larmor frame, which are just eqs. (29), (30) with $\tilde{D}_i = \nabla_i$, for stationary solutions depending on only one spatial coordinate yield

$$\partial_s j = 0, \quad (\partial_s + v j \times) m + j = 0. \quad (33)$$

We get

$$j = \text{const}, \quad m(s) = R(-svj)m_0 - sj, \quad (34)$$

where $m_0 \equiv m(s = 0)$ is the magnetization in the center of the channel. Its absolute value is $\sqrt{1 - j^2}$ and its direction remains an arbitrary parameter. This uncertainty in the direction of $\hat{m}_0$ corresponds to the indefiniteness of the initial phase of the precession. Actually the result in [11] is written out (with some slips) in the coordinate representation corresponding to the choice of $m_0 \parallel \hat{x}$. This is why we dwelt on the derivation.

Then the boundary conditions $m|_{s=\pm1} = \pm \hat{z}$ require $R(2vj)m_0 = -m_0$. Hence either $m_0 = 0$ and we get

$$j = -\hat{z}, \quad m(s) = s\hat{z}. \quad (35)$$

Or $j \perp m_0 \neq 0$ and $vj = \pi/2 + \pi n, \ n \in \mathbb{Z}$. The stability analysis and numerical simulations [11] show that only the solutions with $n = 0$ and $n = -1$ are stable. For liquids with $v$ negative $n = -1$ is relevant, while $n = 0$ corresponds to $v$ positive. Using the overt form of the rotation matrix $R(\theta) = (\delta_{\alpha \beta} - \hat{\theta}_\alpha \hat{\theta}_\beta) \cos \theta + \hat{\theta}_\alpha \hat{\theta}_\beta - e_{\alpha \beta \gamma} \hat{\theta}_\gamma \sin \theta$, we obtain for $n = -1$:

$$j = -\frac{\pi}{2v}, \quad \dot{j} = -j\hat{z} + m_0 \times \dot{\hat{z}}, \quad m(s) = -m_0 \times j \sin(\pi s/2) + m_0 \cos(\pi s/2) - sj\dot{j}. \quad (36)$$

In such a solution the polar angle $\pi - \vartheta$ between $\dot{j}$ and $\dot{\hat{z}}$ is fixed by the condition: $\dot{j} = -(\dot{j}\hat{z}) = \cos \vartheta$. (Here we see that of necessity $\dot{j} \leq 1$ and $|v| \geq \pi/2$.) The arbitrary azimuthal angle of $\dot{j}$ is parameterized by a constant unit vector $\hat{m}_0 = \hat{z} \times j/m_0$.

The three unit vectors $\hat{m}_0 \times \dot{j}$, $\hat{m}_0$, $\dot{j}$ form a right-handed basis (see Fig. 2).

Subject to the value of $|v|$ either of the above stable stationary solutions takes place. For $|v| \leq \pi/2$ the solution is the longitudinal diffusion (35). For $|v| \geq \pi/2$ this configuration becomes unstable (Castaing instability) and (36) is realized.

Properly speaking, arbitrary boundary conditions can be considered in the same manner but only the ones involved, i.e. of the diametrically opposed magnetizations in the two reservoirs, appear to have a solution of more or less simple form.

In fact, the diffusion under consideration is only quasi-stationary. That is, it is independent of time on the short scale $L^2/D_0$. But on long times owing to the finite size of the reservoirs, slowly as it will, the current will change the magnetization densities in the reservoirs, thus changing the boundary conditions. Thanks to this slowness we may deem the situation at any given moment as stationary on times $\sim \tau$.
and simply substitute $\hat{S}_0$ for $\hat{z}$ in the boundary conditions, so that now they read $\mathbf{m}(s = \pm 1) = \pm \hat{S}_0$.

For the long-time dependence of the magnetization in the right reservoir we then get

$$T \frac{dS_0}{dt} = S_0 j,$$

(37)

where $T = VL/D_0A$ is the long-time scale. Thus the quasistationary approximation is valid as long as $T \gg L^2/D_0$, or, equivalently, $V \gg LA$. For experimental conditions of $[10]$ $T \sim 100$ sec.

For $|v| \leq \pi/2$ the current $j = -\hat{S}_0$ and the magnetization decays exponentially:

$$S_0(t) = S_0(0) \exp(-t/T).$$

(38)

For $|v| \geq \pi/2$ the magnetization density diminishes algebraically until it reaches the critical value $\pi/2$. It is convenient to introduce a constant $\zeta = \pi/2\kappa_1$ so that $j = \zeta/S_0$. Upon substituting $j = j(-j\hat{S}_0 + \sqrt{1 - j^2}\mathbf{m}_0 \times \hat{S}_0)$ eq. (37) yields

$$T \frac{dS_0}{dt} = -\frac{\zeta^2}{S_0}, \quad T \frac{d\hat{S}_0}{dt} = \frac{\zeta}{S_0} \sqrt{1 - \frac{\zeta^2}{S_0^2}} \mathbf{m}_0 \times \hat{S}_0.$$

(39)

Then while $v(t)$ exceeds the critical value, i.e. for times $t \leq t^* - T/2$, where $t^* = TS_0^2(0)/2\zeta^2$, we get

$$S_0(t) = \zeta \sqrt{2 \frac{t^* - t}{T}}; \quad \hat{S}_0(t) = \mathcal{R} \left( \hat{m}_0 \int_0^t \frac{\zeta}{S_0} \sqrt{1 - \frac{\zeta^2}{S_0^2} dt} \frac{1}{S_0^2/T} \right) = \mathcal{R}(\hat{m}_0 \beta(t))\hat{z},$$

(40)
where
\[ \beta(t) = \left[ \vartheta(t) - \sqrt{\frac{2t^* - t}{T} - 1} \right]_0^t \approx j(0)\sqrt{1 - j^2(0)} \frac{t}{T} + O\left(\frac{t^2}{T^2}\right). \quad (41) \]

The absolute value of magnetization decays as a square root of time and its direction rotates around \( \hat{m}_0 \), i.e. in the \( \hat{j} - \hat{z} \) plane (see Fig. (3a)). The angle \( \beta \) increases monotonically with time. As numerical evaluation shows, it reaches \( \pi \) at the critical point for the first time when the duration of the non-longitudinal part of diffusion constitutes \( t^* \approx 10T \).

Note that the time evolution of the magnetizations in the two reservoirs is symmetric as a result of the conservation of current. And at any moment the directions of the magnetizations in the vessels remain opposite to ensure the diametrically opposed conditions involved.

For \( t \geq t^* - T/2 \) the magnetization will proceed by exponential slump (see Fig. (4)) with the angle \( \beta_f = \beta(t^* - T/2) \neq 0 \) between \( \hat{z} \) and \( \hat{S}_0 \) (see Fig. (3b)).

4.2 In the Presence of an Electric Field

Let now an electric field \( \mathbf{E} = E\hat{x} \) be applied to the liquid in the channel at some time \( t_E > 0 \). In accordance with what was said above, all quantities will acquaint increments proportional to the electric field: the magnetization and current in the channel \( \mathbf{m} \rightarrow \mathbf{m} + \delta\mathbf{m}, \mathbf{j} \rightarrow \mathbf{j} + \delta\mathbf{j} \). And the magnetization in the right reservoir \( S_0 \rightarrow S_0 + \delta S_0 \).

The boundary condition for \( \delta\mathbf{m} \) will be \( \delta\mathbf{m}(s = \pm 1) = \pm \delta S_0 / S_0 \).

Since the quasistationary inhomogeneous terms \( g \langle \hat{A}_i \rangle \times \) are independent of time, we seek for a quasistationary solution \( \partial_t \delta\mathbf{m} = 0, \partial_t \delta\mathbf{j} = 0 \). The oscillating terms will be discussed later. Of course, \( S_0, \mathbf{m} \) and \( \mathbf{j} \) all depend on the “long” time \( t/T \).
After introducing the additional parameter $\epsilon = gEL/c$ we get

$$\partial_s \delta j + \epsilon \hat{z} \times j = 0, \quad (42)$$

$$(\partial_s + vj \times)\delta m + \epsilon \hat{z} \times m + \delta j + v\delta j \times m = 0. \quad (43)$$

The first of the equations yields

$$\delta j(s) = \delta j_\epsilon(s) + \delta j_0, \quad (44)$$

where

$$\delta j_\epsilon(s) = -\epsilon \hat{z} \times js \quad (45)$$

and $\delta j_0$ is a constant to be determined from eq. (43) with the boundary conditions imposed. We can make a substitution $\delta m(s) = R(-vsj)\delta \tilde{m}(s)$ into eq. (43), after which it can be easily integrated

$$\delta m(s) = R(-vsj)\int_{-1}^{s} R(vsj)(\epsilon \hat{z} \times m + \delta j + v\delta j \times m)ds - \frac{\delta S_0}{S_0}. \quad (46)$$

And then the boundary conditions reduce to

$$\int_{-1}^{1} R(vsj)(\epsilon \hat{z} \times m + \delta j + v\delta j \times m)ds = 2R(vj)\frac{\delta S_0}{S_0}. \quad (47)$$

As a matter of fact, $\delta j_0$ should be determined from this equality. At the moment $t_E$ of switching on of the electric field $\delta S_0 = 0$ and the above equation can be solved analytically. $\delta j_0$ then turns out to be zero. Nonetheless, although $\delta j_0$ is, generally speaking, non-zero at arbitrary times, this part of $\delta j$ is constant in space and does not change the symmetric picture of the diffusion.
It is the part $\delta j_\epsilon$ of $\delta j$ that depends on $s$ and brings about an observable effect of interest to us. This part has opposite signs at the two tube entrances, thus changing the magnetizations in the two reservoirs by the same amount, i.e. \emph{not symmetrically}. Unless $j$ is parallel to $\hat{z}$, the term $\delta j_\epsilon$ is directed along $\hat{m}_0$ perpendicular to the $\hat{j} - \hat{z}$ plane. Therefore, as diffusion goes on, the non-symmetric change of magnetization in the reservoirs will lead to an \emph{additional phase shift} between the two vessels.

To find the change in the time evolution of $S_0$ due to electric field we substitute $S_0$ with $S_0 + \delta S_0$, where $S_0(t)$, as previously, belongs to the case of the absence of electric field and the part $\delta S_0$ is proportional to the electric field. This part obeys an equation, which can be obtained by taking variations of eq. (37) using (44):

$$T \frac{d\delta S_0}{dt} = S_0 [\delta j_0 + \delta j_\epsilon(s = 1)] + \delta S_0 j.$$  \hfill (48)

Next we expand $\delta S_0$ onto the components in and perpendicular to the $\hat{j} - \hat{z}$ plane:

$$\delta S_0 = \delta S_0^m + \delta S_0^{j - z}.$$  For the component $\delta S_0^m$ perpendicular to the $\hat{j} - \hat{z}$ plane we get

$$T d\delta S_0^m / dt = S_0 [\delta j_0^m + \delta j_\epsilon(1)].$$

The phase shift is described by the term $\delta j_\epsilon$. Hence the phase shift increases with time as

$$\delta \alpha(t) \approx -2 \frac{\delta S_0^m(t)}{S_0(t) \sin \beta(t)} = -2 \int_{t_E}^{t} S_0(t) \delta j_\epsilon(1; t) \frac{dt}{T},$$

where we have neglected the presumably small component $\delta S_0^{j - z}$ compared to $S_0$ in the denominator.

For the non-longitudinal stable solution (35) we have $\delta j_\epsilon(1) = \epsilon j \sin[\beta(t) - \vartheta(t)]$, and the corresponding integral can be found only numerically. For times close to $t_E$ we have ($t > t_E$)

$$\delta \alpha \approx 2 \epsilon \frac{t - t_E}{T} j(t_E) \frac{\sin[\beta(t_E) - \vartheta(t_E)]}{\sin \beta(t_E)}.$$  \hfill (50)

When $t_E \ll T$, the last expression simplifies

$$\delta \alpha \approx -2 \epsilon \frac{t - t_E}{t_E}.$$  \hfill (51)

However, this formula cannot be applied to $t_E$ extremely close to zero. Namely, in the case that the transverse component of the magnetization in the reservoir $S_0(t) \sin \beta(t_E)$ is of the order of magnitude of $\delta S_0^m$, the up-to-the-first-order extension of the transverse component of $S_0$ in electric field is not valid.

For the longitudinal stable solution (35) eq. (49) can be easily integrated and yields

$$\delta \alpha(t) = -2 \epsilon \left[ \exp((t - t_0) / T) - 1 \right]$$

for all values of $\beta_f \equiv \beta(t^* - T/2)$ except for $\beta_f = \pi n$, $n \in \mathbb{Z}$, when the phase shift is undefined. In particular, if there were no non-longitudinal regime at all (due to
insufficient initial polarization), and hence the magnetization is parallel to \(\hat{z}\), there would be no additional phase shift.

The time \(t_0\) in (52) is either \(t^* - T/2\), when the longitudinal solution follows the non-longitudinal regime, or \(t_E\) if an electric field is turned on already at the longitudinal part of the diffusion. In the former case, eq. (52) should be added up with the phase shift accumulated during the non-longitudinal part.

Exp. (52) formally tends to infinity with time, but owing to the small coefficient \(\epsilon\) it still remains very small for all of the time of an experiment.

The appearance of an additional phase shift in an electric field is quite understandable. The diffusion process may be outlined as a transfer of down spins from the left reservoir to the right one. Naturally, the current along \(\hat{y}_i\) of down spins is equivalent to the current of up spins in the opposite direction. In accordance with the boundary conditions the spins of \(^3\text{He}\) atoms at the left end of the tube are directed preferably down. The “surplus” of down spins drifts to the right and interacts with the electric field \(E = E\hat{x}\) as if there was an increase to the magnetic field: \(- (E v_F/c) \hat{\mu} \hat{z}\).

On the contrary, the up spins at the right end float to the left, and their spin-orbital energy \((E v_F/c) \hat{\mu} \hat{z}\) amounts to an effective decrease of the magnetic field. The spins at the opposite ends thus precess at slightly different frequencies, unless, of course, they are not confined to the \(\hat{z}\) direction like in the purely longitudinal solution. Hence in the second solution (for \(|v| \geq \pi/2\)) there develops a phase shift between the two entrances of the tube.

The conclusion about the existence of an additional phase shift was drawn for a situation with diametrically opposed magnetization directions at the tube ends as the boundary condition. But the inference in itself remains valid for other boundary conditions as well, irrespective of the fact that we are ignorant of zero-electric-field distribution in that case and of exact expression for the phase shift. This follows, first, from the current conservation in the absence of electric field, which secures the self-similar boundary conditions in the quasistationary diffusion process. And secondly, from eq. (54), which holds for all stationary solutions in one-dimensional geometry and leads to non-symmetric change of the magnetization in the reservoirs when an electric field is applied.

As for the oscillating part of the gauge field, it does not contribute to the current along the channel. Indeed, the oscillating part of the gauge field is

\[
g\hat{\mathbf{A}}_i = (gE/c)\hat{z}_i(\hat{x}\sin \omega_L t - \hat{y}\cos \omega_L t)
\]

if the electric field is parallel to \(\hat{x}\) as before, that is, \(g\hat{\mathbf{A}}_i\) is proportional to \(\hat{z}_i\). Taking into account that different \(i\)-components of \(\delta \mathbf{J}_i\) do not enter each other’s evolution equation (52), it means that in the particular solution of the inhomogeneous system only \(\delta \mathbf{S}\) and \(\delta \mathbf{J}_z\) will be non-zero and \(\delta \mathbf{J}_x\) and \(\delta \mathbf{J}_y\) will figure only in the general solution of the homogeneous equation. According to what was said above, if only the unperturbed solution is stable, the latter dies out exponentially with the rate \((\text{Im} \omega)^{-1}\), where \(\omega\) is an eigenvalue of (51), (52).
To be exact, the oscillating part of the gauge field brings about the appearance of a fine structure across the channel, which vary on a spatial scale of $\sqrt{A}$ and oscillates with the Larmor frequency in the Larmor frame. Though in general it is rather complicated, the main features of such a structure may be seen on the easier example of the longitudinal diffusion, which is considered in the Appendix.

5 Strongly-Polarized Fermi-Liquid

Landau theory cannot be literally applied to strongly spin-polarized Fermi liquids [12]. However for $T = 0$ Fomin derived microscopically transverse spin dynamics equations [12], the form of which, in fact, coincides with the collisionless limit of the Leggett equations.

In this section we will consider the application of the SU(2) gauge invariance directly to a microscopic Hamiltonian of a liquid at $T = 0$. The Hamiltonian of the liquid involved in the absence of external fields consists of the kinetic energy

$$H_K = -\frac{1}{2m} \int \psi^+(r) \nabla^2_i \psi(r) \, d^3r$$

and an interaction $H_{\text{int}}$ between particles. We neglect the very small spin-orbital interaction between particles.

To include electric and magnetic fields into consideration we must use (4) instead of $H_K$. Such a replacement can be interpreted as a result of a gauge transformation $\partial_\mu \rightarrow D_\mu = \partial_\mu - ig\hat{\sigma}_A \mu$. It is convenient to write the Hamiltonian (4) in the form:

$$\delta H = - \int gA_0(r)S(r) \, d^3r - \int gA_i(r)J_i(-gA_i, r) \, d^3r,$$

where the spin and the spin-current operators are defined as follows

$$S(r) = \psi^+(r)\frac{\hat{\sigma}}{2}\psi(r),$$

$$J_i(X_i, r) = \frac{1}{2m} \left[(i\nabla_i \psi^+(r))\frac{\hat{\sigma}}{2}\psi(r) + \psi^+(r)\frac{\hat{\sigma}}{2}(-i\nabla_i \psi(r)) + \frac{1}{4}\psi^+(r)\psi(r)X_i \right].$$

Following Fomin [12] we now transform in each space point to a frame precessing at such a frequency $\Omega$, that the magnetization in the rotating frame is equilibrium. $\Omega$ is supposed to vary slowly enough in space and time so that we could leave only first terms in $\omega$ and $k$, where $\omega$ and $k$ are respectively the characteristic frequency and wave number of $\Omega$ in the Larmor reference system.

As a result we get the following expression for the Hamiltonian in the rotating frame

$$H = H_K + H_{\text{int}} + \delta H + \int S(r)\Omega \, d^3r.$$

Now we introduce $\alpha$ and $\beta$ as the spherical coordinates of $-\Omega$: $\Omega = -\Omega(\sin \beta \cos \alpha, \sin \beta \sin \alpha, \cos \beta)$ and rotate the spin space $\psi \rightarrow R\psi$, where

$$R = \exp(-i\alpha\sigma^z/2)\exp(-i\beta\sigma^y/2)\exp(-i\gamma\sigma^z/2)$$

We leave out spin indices of the field operators of $^3$He atoms $\psi$ for brevity.
they constitute the unperturbed Hamiltonian of a Fermi liquid in a uniform field. The ground state of this Hamiltonian is

\[ \psi \rightarrow R \psi + (\partial_\mu R) \psi = RD_\mu \psi, \]  

with the covariant derivative \( D_\mu \psi = \partial_\mu \psi + (R^\dagger \partial_\mu R) \psi = \partial_\mu \psi - i \frac{e}{2} \omega_\mu \psi. \) This must be regarded as the definition of \( \omega_\mu. \) It can be written out equivalently as

\[ \omega_\mu = \frac{1}{2} e_{\alpha\beta\gamma} R_{\beta j} \partial_\mu R_{\gamma j} = \begin{pmatrix}
- \sin \beta \cos \gamma \partial_\mu \alpha & \sin \gamma \partial_\mu \beta \\
\sin \beta \sin \gamma \partial_\mu \alpha & \cos \gamma \partial_\mu \beta \\
\cos \beta \partial_\mu \alpha & \partial_\mu \gamma
\end{pmatrix}, \]  

where \( R_{\alpha i} = R_z(-\gamma) R_y(-\beta) R_z(-\alpha) \) is a 3D rotation matrix.

Under this transformation

\[ H \rightarrow H_K + H_{\text{int}} + \int S(r) R \Omega \, d^3r + \delta H|_{gA_\mu \rightarrow gRA_\mu + \omega_\mu} = \]

\[ = H_K + H_{\text{int}} + \int S(r) R \Omega \, d^3r - \int (\omega_0 + \omega_L \hat{R} \hat{z}) S(r) \, d^3r - \]

\[ - \int (\omega_i + gR A_i) J_i (\omega_i - gR A_i, r) \, d^3r. \]  

As can be easily seen, \( R \hat{z} = (- \sin \beta \cos \gamma, \sin \beta \sin \gamma, \cos \beta), \) \( R \Omega = -\Omega \hat{z}. \)

In case that the absolute value of \( \Omega \) is greater than its variations in the reference system that precedes at the Larmor frequency (i.e., greater than \( \dot{\alpha} + \omega_L, \dot{\beta}, \dot{\gamma} \) and \( \nabla^2 \alpha/m, \nabla^2 \beta/m, \nabla^2 \gamma/m \)), then the first three terms in (62) are the greatest. Together they constitute the unperturbed Hamiltonian of a Fermi liquid in a uniform field \( -\Omega \hat{z}. \) The ground state of this Hamiltonian is \( \langle S \rangle = (0, 0, S) \) and \( \langle J_i \rangle = 0. \) The former is extremely important because shows that \( \alpha \) and \( \beta \) serve as spherical coordinates of \( S \) in the laboratory frame.

The other terms in (62) are the perturbation. Its magnitude in the ground state equals

\[ - (\omega_0 + \omega_L \hat{R} \hat{z}) \langle S \rangle = -S(\dot{\gamma} + \cos \beta(\dot{\alpha} + \omega_L)). \]  

The expression above contains only time derivatives. To include space derivatives one should allow for the second order corrections, which can be evaluated as in Maki’s paper [13]:

\[ \langle \Delta H \rangle = \Delta F = -\frac{\chi^J}{2} (\omega_i + gR A_i)^2 \]

\[ = -\frac{\chi^J}{2} [(\nabla \alpha)^2 + (\nabla \beta)^2 + (\nabla \gamma)^2 + 2 \cos \beta \nabla \alpha \nabla \gamma] - \chi^J \frac{g^2}{c^2} E^2 + \chi^J e_{\alpha i k} \bar{\omega}_{\alpha i} \bar{g}(64) \]

Note the wrong sign in Ref. [13]. A new definition is

\[ \bar{\omega}_i = R^{-1} \omega_i = \begin{pmatrix}
\sin \beta \cos \alpha \nabla_i \gamma - \sin \alpha \nabla_i \beta \\
\sin \beta \sin \alpha \nabla_i \gamma + \cos \alpha \nabla_i \beta \\
\cos \beta \nabla_i \gamma + \nabla_i \alpha
\end{pmatrix}. \]  

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Thus the effective Lagrangian
\[ L_{\text{eff}} = -S(\dot{\gamma} + \cos \beta \dot{\alpha} + \hat{S} \times \frac{gE}{c}) + \langle \Delta H \rangle. \]
Varying this with respect to \( \gamma \), we obtain
\[ \dot{S} + \chi J \nabla (\nabla \gamma + \cos \beta \nabla \alpha + \hat{S} \times \frac{gE}{c}) = 0. \quad (66) \]

In the end we could set a constraint on \( \gamma \) guided by the reasons of rationality, i.e. so that \( \dot{S} = 0 \), which requires \( \nabla \gamma + \cos \beta \nabla \alpha = -\hat{S} \times gE/c \). Substituting this condition into the equations obtained by varying \( L_{\text{eff}} \) with respect to \( \alpha \) and \( \beta \), we obtain a set of two equations for \( \alpha \) and \( \beta \) which are equivalent to
\[ D_t \hat{S} = \frac{\chi J}{S} D_i (D_i \hat{S} \times \hat{S}). \quad (67) \]

It is interesting to trace the formal transition from Eqs. (27), (28) to Eq. (67) though the modified Leggett equations, definitely, cannot be applied to a strongly-polarized liquid.

First of all, for a strongly-polarized liquid \( S \) is much greater than \( \omega_L \), hence, \( S_{\text{eq}} \) could be neglected in (28). Moreover, the collision integral \( J_i/\tau_1 \) vanishes as compared to the molecular-field term \( \kappa S \times J_i \) by virtue of the parameter \( \kappa S \tau_1 \gg 1 \). Thus we can get to the limit \( \kappa S \tau_1 \gg 1 \) in an expression for the quasistationary current, which can be obtained by resolving Eq. (28) with respect to \( J_i \) provided that \( D_t J_i = 0 \):
\[ J_i = -\frac{D_0}{1 + (\kappa S \tau_1)^2} [ D_i S + \kappa \tau_1 S \times D_i S + (\kappa \tau_1)^2 S (S \nabla_i S) ]. \quad (68) \]

As was discussed concerning the weak-polarization case, the transition to the quasistationary current is self-consistent only for the quasistationary part of the solution because only then both sides of (68) are constant in time in the Larmor frame. The senior (the third) term in (68) identically vanishes for homogeneous spatial distributions of the absolute value of magnetization. We assume this condition to fulfil, for the collisionless regime of the weak-polarization case it can be proved.

Substituting then the expression for the quasistationary current into (27), we obtain (67) with \( \chi_J = w^2/3\kappa \), or, in ordinary units, \( \chi_J = (\chi_n/g^2)(w^2/3\kappa) \). It is this value of \( \chi_J \) that the Fermi-liquid theory gives in the limit of weak polarizations [12]. Still \( \chi_J \) cannot be obtained in the general case.

From the fact that the modified Leggett equations (27), (28) yield (67) in the deep collisionless limit in the quasistationary approximation it follows immediately that all conclusions about the quasistationary part of spin flow through a channel and, in particular, of the additional phase shift remain valid in the strong-polarization case.

6 Conclusion

In this paper we considered how an external electric field can influence spin dynamics of an electrically neutral Fermi liquid through the spin-orbital interaction with nuclear magnetic moments. In the framework of Landau’s theory of Fermi liquids attributing an additional energy due to spin-orbital interaction to each quasiparticle leads
to generalization of Leggett’s equations. The amendments are consistent with the requirements imposed by intrinsic SU(2) gauge invariance of the interaction of a magnetic moment with electromagnetic field. The corrections caused by electric field in the Leggett’s equations are responsible for an additional phase shift in a one-dimensional spin flow experiment. This phase shift, which is proportional to the electric field, coincides in the order of magnitude with that in superfluid $^3$He-$B$ but grows with time. Its magnitude is observable experimentally.

For strongly spin-polarized Fermi liquids, electric field is incorporated into Fomin’s equations for transverse spin dynamics of a strongly spin-polarized Fermi liquid at zero temperature. The formal correspondence between these equations and the collisionless limit of Leggett’s equations spreads to the case of an electric field present as well.

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**Appendix**

In the case of longitudinal diffusion the inhomogeneous term in (31) is identically zero, but not in (32). Thus the solution of the coupled system (31), (32) should as usual be sought in the form of the sum of the two terms depending on time through the factors $\sin \omega_L t$ and $\cos \omega_L t$.

According to what was said in the body of the paper, we seek for a solution of the system

$$\partial_t \delta S + \partial_z J_z = 0, \quad \partial_t \delta J_z + \frac{w^2}{3} (\partial_z S + \frac{gE}{c} (\hat{x} \sin \omega_L t - \hat{y} \cos \omega_L t) \times (S - \omega_L)) + \kappa \delta S \times J_z = -\frac{\delta J_z}{\tau_1}$$

in the form $\delta J_z = (D_0 S_0 / L) (a \sin \omega_L t + b \cos \omega_L t)$. From the first equation we get $\delta S = (D_0 S_0 / L) (-b' / \omega_L) \sin \omega_L t + (a' / \omega_L) \cos \omega_L t$, where $a' = \partial_z a$.

Next we should utilize the boundary condition of the absence of spin transfer through the walls of the channel: $\delta J_z = 0$ at the walls. Hence the solution can be expanded into harmonic modes, in which $\delta J_z$ depend on $z$ through the factors of the form $\sin k z$. Here $k = \pi Z / d$ and $d$ is the thickness of the channel, which in general is a function of $x$. 

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\[
\begin{align*}
\mathbf{a} &= \frac{\epsilon(s - \omega_L/S_0)}{C^2 + 1} (C \hat{x} + \hat{y}), \\
\mathbf{b} &= \frac{\epsilon(s - \omega_L/S_0)}{C^2 + 1} (\hat{x} - C \hat{y}),
\end{align*}
\]
where \( C = \omega_L \tau_1 - D_0 k^2/\omega_L - vs \). We here allowed for the fact that due to diffusion after some finite time after the beginning of the experiment \( S_0 \) is not equal to \( \omega_L \).

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