Ergodic averaging with and without invariant measures

Michael Blank

Institute for Information Transmission Problems RAS (Kharkevich Institute), Moscow, Russia
National Research University Higher School of Economics, Moscow, Russia

E-mail: blank@iitp.ru

Received 13 February 2017, revised 19 September 2017
Accepted for publication 29 September 2017
Published 16 November 2017

Abstract
The classical Birkhoff ergodic theorem in its most popular version says that the time average along a single typical trajectory of a dynamical system is equal to the space average with respect to the ergodic invariant distribution. This result is one of the cornerstones of the entire ergodic theory and its numerous applications. Two questions related to this subject will be addressed: how large is the set of typical trajectories, in particular in the case when there are no invariant distributions, and how the answer is connected to properties of the so called natural measures (limits of images of ’good’ measures under the action of the system).

Keywords: ergodic theorem, invariant measure, typical trajectory
Mathematics Subject Classification numbers: 28D05, 37A30, 37A50
(Some figures may appear in colour only in the online journal)

1. Introduction

A conventional point-wise ergodic ‘Theorem’ says that: For a given invariant measure $\mu$ an ergodic average along a trajectory starting from a certain initial point converges $\mu$-almost everywhere. Examples: celebrated Birkhoff et al theorems. Those results are well known but have two serious disadvantages. First, the ergodic invariant measure may have a small support in which case the behavior of trajectories starting from points outside of the support is not described by such statements. Second, as we will see there are simple examples of low dimensional dynamical systems having no invariant measures and thus formally having nothing to do with such claims. Our aim is to try to overcome these difficulties, namely to find conditions under which claims of ergodic averaging type may be obtained for reasonably general
dynamical systems for almost all initial points with respect to a ‘good’ reference measure (e.g. Lebesgue measure).

Let \((X, \rho)\) be a compact metric space equipped with a \(\sigma\)-algebra of measurable sets \(\mathcal{B}\) and a probability reference measure \(m\), and let \(T : X \to X\) be a measurable map from this space into itself. In this paper we restrict ourselves to the questions related to the generalization of one of the most known and widely used results in ergodic theory of dynamical systems – the classical Birkhoff ergodic theorem, which claims that

\[
\frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) \xrightarrow{\text{a.s.}} \int f \, d\mu
\]

(1.1)

for an ergodic \(T\)-invariant measure \(\mu\), each integrable function \(f \in L^1_\mu\) and \(\mu\)-a.e. \(x \in X\).

In a number of cases there is a special invariant measure called Sinai–Ruelle–Bowen (SRB) measure \(\tilde{\mu}\) (exact definitions will be given in section 2) which represents averages along trajectories starting from a set of positive \(m\)-measure. For a given probability measure \(\mu\) consider the set of \(\mu\)-typical points

\[
Z_\mu := \{ x \in X : \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) \xrightarrow{\text{a.s.}} \int f \, d\mu \} \quad \forall f \in C^0(X),
\]

(1.2)

where \(C^0(X)\) stands for the set of continuous functions on \(X\). Our aim is to find conditions under which \(Z_\mu\) is the set of full \(m\)-measure, i.e. \(m(Z_\mu) = 1\). Moreover, we will show that the set of typical points may be large not only for ergodic invariant measures (which is not surprising), but also for some non-invariant measures (see section 4).

To start with, let us demonstrate that the situation when \(m(\text{supp}(\tilde{\mu}))\) is much smaller than \(m(Z_\mu)\) appears very naturally in the simplest examples of dynamical systems.

**Example 1.1.** Let \(T\) be a map from the unit disc \(X := \{(\phi, R) : 0 \leq \phi < 2\pi, \ 0 \leq R \leq 1\}\) into itself defined in the polar coordinates \((\phi, R)\) by the relation:

\[
T(\phi, R) := (\phi + 2\pi\alpha + \beta(R - r) \mod 2\pi, \gamma(R - r) + r)
\]

(1.3)

with parameters \(\alpha, \beta, \gamma, r \in (0, 1)\).

The circle \(\{R = r\}\) is the only attractor of this map with the basin of attraction consisting of all the points in \(X\) except the unstable fixed point located at the origin. See section 3.2 for the detailed discussion of ergodic properties of this system.

This example in fact is based on a trivial observation that a uniformly contractive dynamical system governed by the map \(Tx := x/2\) on the unit interval is uniquely ergodic (the Dirac measure \(\delta_0\) at the origin is invariant) while \(m(\text{supp}(\delta_0)) = 0 < 1 = m(Z_\delta)\).

Without a kind of attraction property for points outside of the support of the measure \(\tilde{\mu}\) one cannot expect the generalization of the Birkhoff theorem, e.g. a presence of another attractor of positive reference measure obviously would contradict it. As we will see the contraction alone is not enough for this.

In what follows we give necessary and sufficient conditions for the property \(m(Z_\mu) = 1\) in three different cases: regular (theorem 3.1 in section 3), irregular (theorem 4.1 in section 4) and self-consistent (theorem 5.1 in section 5). Definitions of these cases will be given in the corresponding sections. Roughly speaking the regular case corresponds to the situation when the map \(T\) is smooth enough, in the irregular case due to discontinuities of the map \(T\) there are no invariant measures, while the self-consistent case is a deterministic version of a nonlinear Markov chain. Technically the hard part of the proofs of all subsequent results is described
is section 3.3 which is devoted to the regular case only, while the proofs of corresponding claims made for more complicated situations discussed in sections 4 and 5 are reduced to the those in section 3.3 by means of a special trick: introduction of the notion of weakly ergodic measures. In section 2 we discuss basic definitions and constructions as well as some known results about ergodic averaging.

2. Basic definitions and constructions

We start with basic definitions and constructions used throughout the paper.

**Definition 2.1.** Let \( (X, \mathcal{B}) \) be a compact measurable space and let \( T : X \to X \) be a measurable map from this space into itself. The map induces the transfer-operator \( T_\mu \) acting in the space of probability measures \( \mathcal{M} \) on \( X \) according to the formula \( T_\mu(A) := \mu(T^{-1}A) \) for each measurable set \( A \subseteq X \). A measure \( \mu \in \mathcal{M} \) is called \( T \)-invariant if \( T_\mu \mu = \mu \).

Throughout the paper we always assume that \( \mathcal{B} \) is a Borel \( \sigma \)-algebra of measurable sets and only probability measures will be taken into consideration. Additionally we fix a certain reference measure \( m \in \mathcal{M} \) on this \( \sigma \)-algebra being positive on each open measurable subset and call it a reference measure.

**Definition 2.2.** A measurable map \( T : X \to X \) is called nonsingular with respect to the reference measure \( m \) if for any \( A \in \mathcal{B} \), \( T_*m(A) = 0 \) if and only if \( m(A) = 0 \). \( T \) is conservative if for each set \( A \in \mathcal{B} \) of positive \( m \)-measure there exists \( n \in \mathbb{N} \) such that \( m(T^{-n}A) > 0 \). A measurable set is called wandering, if all its images under the action of the map \( T \) are disjoint. A probability measure \( \mu \) is called ergodic if \( \mu(A) \in \{0, 1\} \) for each \( T \)-invariant set. Recall that a set \( A \subseteq \mathcal{B} \) is \( T \)-invariant if \( T^{-1}A = A \).

Observe that the ergodicity of a probability measure does not necessarily imply its invariance, for example for each point \( x \in X \) the \( \delta \)-measure \( \delta_x \) is ergodic for any map \( T : X \to X \).

**Definition 2.3.** For a given probability measure \( \nu \) denote by \( \mathcal{M}(\nu) \subseteq \mathcal{M} \) the set of probability measures \( \mu \) on \( X \) absolutely continuous with respect to \( \nu \) (notation \( \mu \ll \nu \)).

Recall that \( \mu \ll \nu \) means that \( \nu(A) = 0 \) implies \( \mu(A) = 0 \). The convergence of measures throughout the paper is considered always in the weak* sense.

**Definition 2.4.** A measure \( \mu \) is said to be wandering if all its images under the action of the transfer-operator \( T_\mu \) are mutually singular.

Recall that the mutual singularity of measures \( \mu, \nu \) (notation \( \mu \perp \nu \)) means that there exists a set \( A \in \mathcal{B} \) such that \( \mu(A) = \nu(X \setminus A) = 0 \). Observe also that the presence of an wandering measure \( \mu \in \mathcal{M}(\nu) \) is equivalent the existence of an wandering set of positive \( \nu \) measure.

**Definition 2.5.** Fix a reference measure \( m \in \mathcal{M} \). A measure \( \tilde{\mu}^{\text{nat}} \in \mathcal{M} \) is called natural if there is an open set \( U \subseteq X \) with \( m(U) > 0 \) such that

\[
\frac{1}{n} \sum_{k=0}^{n-1} T_\mu^k \mu \xrightarrow{n \to \infty} \tilde{\mu}^{\text{nat}} \quad \forall \mu \in \mathcal{M}(m), \quad \mu(U) = 1.
\]  

A measure \( \tilde{\mu}^{\text{obs}} \in \mathcal{M} \) is called observable if there is an open set \( U \subseteq X \) with \( m(U) > 0 \) such that

\[
\frac{1}{n} \sum_{k=0}^{n-1} T_\delta^k \delta \xrightarrow{n \to \infty} \tilde{\mu}^{\text{obs}} \quad \forall \delta \in U', \quad \mu(U) = 1.
\]
where \( \delta_x \) is the Dirac measure at point \( x \) and \( m(U \setminus U') = 0 \). The set \( U \) in these constructions is the basin of attraction of the corresponding measure.

The natural and observable measures are different instances of the so-called Sinai–Ruelle–Bowen (SRB) measures. For the discussion of the connections between them we refer a reader to [5], where the question when these objects coincide has been raised in the first time, and to [12, 16] where further clarifications were obtained. A number of nontrivial examples demonstrating not only the difference between these objects but that the existence of one of them does not guarantee the existence of another were constructed and studied in [5, 12, 16]. Curiously, despite that the authors of [12] claimed that they ‘give a complete description of relations between observable and natural measures’, a number of situations were not taken into account there. The most important among those omitted are the case when the limit measure \( \tilde{\mu}_{\text{nat}} \) is singular with respect to the reference measure and the case when \( \tilde{\mu}_{\text{nat}} \) does not exist.

We shall study both these situations in the present paper.

It is worth noting that the so-called operator approach (see e.g. [4]) provides very effective tools for the analysis of the natural measures which makes them more preferable from the applied point of view than the observable ones.

**Definition 2.6.** Support of a measure \( \mu \), which is denoted by \( \text{supp}(\mu) \), is a union of all points \( x \in X \) satisfying the property that each open neighborhood of the point \( x \) has strictly positive \( \mu \)-measure. For a given natural measure \( \tilde{\mu} \) we denote by \( S \) its support and by \( m_S \) the conditional probability measure constructed from the reference measure \( m \) on the set \( S \).

To simplify the presentation we assume that if a sequence of points \( \{x_n\} \), which belongs either to the set \( S \) or to its complement, converges to a point \( x \) as \( n \to \infty \), then the sequence of their images \( \{T_{x_n}\} \) has a limit which in general may differ from \( Tx \).

The condition \( m(S) > 0 \) implies the existence of the conditional measure \( m_S \), namely \( m_S(A) := m(A \cap S)/m(S) \), otherwise if \( m(S) = 0 \) one needs additional assumptions for the existence of the conditional measure (see a general construction related to measurable partitions, e.g. in [18]). Therefore to avoid these difficulties we assume that in the situations under consideration the measure \( m_S \) is always well defined and that the map \( T \) is nonsingular with respect to this conditional measure. In the examples which we shall consider this is indeed the case.

Combining the results obtained in [5, 11, 12, 16] we get the following information about the connections between natural and observable measures.

**Theorem 2.1** ([5, theorem 2.1, [12, theorem 2.4, [11, theorem 1]]].) Let \( T \) be a measurable map, having no wandering sets of positive \( m \)-measure, and let there exist \( \tilde{\mu}^{\text{obs}} \) with the open basin of attraction \( U \) of positive \( m \)-measure. Then there exists the natural measure \( \tilde{\mu}^{\text{nat}} \) with the same basin of attraction and \( \tilde{\mu}^{\text{nat}} = \tilde{\mu}^{\text{obs}} \). Conversely, let there exist a natural \( T \)-invariant measure \( \tilde{\mu}^{\text{nat}} \) with the open basin of attraction \( U \) of positive \( m \)-measure. Then \( m_U \ll \tilde{\mu}^{\text{nat}} \) implies that \( \tilde{\mu}^{\text{nat}} \) is ergodic and observable (i.e. \( \tilde{\mu}^{\text{obs}} = \tilde{\mu}^{\text{nat}} \)).

The importance of the nonexistence of wandering sets of positive \( m \)-measure to the construction of invariant measures was first observed in [11] and later it has been shown that conditions of this sort are necessary for the presence of invariant measures absolutely continuous with respect to \( m \)-measure (see e.g. [10]). In our setup it is simpler to consider a slight generalization of this notion – wandering measures absolutely continuous with respect to the reference measure.

It turns out that the assumption of continuity of the map \( T \) simplifies a lot the construction of the natural measure.
Theorem 2.2 ([16, theorem 2.1]). Let $T$ be a continuous map and let the limit measure
\[ \tilde{\mu} := \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} T_k^* \mu \]
eexist and be ergodic with respect to $T$. Then $\tilde{\mu}$ is the natural measure with the basin of attraction $\text{supp}(m)$.

An alternative approach to is based on the idea of genericity. For example, the ergodicity assumption used in all ergodic theorems mentioned in section 1 may be justified by the well known Oxtoby–Ulam result [17], according to which a generic volume preserving homeomorphism of a compact manifold is ergodic. This explains to some extent reasons why this is exactly the case considered in a vast majority of textbooks on dynamical systems theory. In turn ergodicity by the Birkhoff ergodic theorem means that almost all points (with respect to the volume measure) are typical in this case. In a recent paper [1] this reasoning was extended to generic continuous maps without the assumption of the volume preservation and it has been shown that for a generic map the Birkhoff average converges almost everywhere, but the limit value may depend sensitively on the initial point. This disproves a conjecture by Ruelle [19] who expected that generically those averages should diverge, which he called historical behavior. A different approach to this question together with a comprehensive review of corresponding results may be found in [2], see also [7–9]. Anyway, this discussion demonstrates that without some specific assumptions about the dynamics there is no hope to obtain positive results in this direction.

3. Regular case

By the ‘regular case’ we mean that the map $T$ is good enough and the natural measure $\tilde{\mu}$ is ‘finer’ than the reference measure $m$. To be precise we make the following standing assumption in this section: the transfer-operator $T_\ast$ is continuous at the point $\tilde{\mu}$ and $S := \text{supp}(\tilde{\mu}) \subseteq \text{supp}(m)$. Observe that this property does not require the map $T$ to be continuous.

3.1. Main results

Throughout the paper we fix a compact metric phase space $(X, \rho)$ equipped with a Borel $\sigma$-algebra of measurable sets $\mathcal{B}$ and a probabilistic reference measure $m$ on this $\sigma$-algebra. To have a simple picture in mind a reader may think that we deal with the unit interval $X := [0, 1]$ with the uniform metric, the standard Borel $\sigma$-algebra and the Lebesgue measure $m$ on it. Let us formulate our main results in the classical setting when invariant measures of a dynamical system under study are present. Namely, we assume that the natural measure $\tilde{\mu}$ exists. Additionally we assume that the map $T$ is nonsingular not only with respect to the reference measure $m$, but with respect to the conditional reference measure $m_S$.

Theorem 3.1. (Necessary and sufficient conditions) The property $m(Z_{\tilde{\mu}}) \cdot m_S(Z_{\tilde{\mu}}) = 1$ is equivalent to the following three assumptions

(i) $\frac{1}{n} \sum_{k=0}^{n-1} T_k^* \mu \to \tilde{\mu}$ $\forall \mu \in \mathcal{M}(m) \cup \mathcal{M}(m_S)$,
(ii) the limit measure $\tilde{\mu}$ is ergodic,
(iii) there are no wandering measures in $\mathcal{M}(m_S)$.

This result is in fact stronger than what we have discussed in the Introduction, since we claim additionally that the conditional reference measure $m_S(Z_{\tilde{\mu}}) = 1$.

A somewhat unusual assumption (iii) of theorem 3.1 may be replaced by another one looking much simpler, but in fact being stronger.
Remark 3.1. The assumption (iii) holds if

(iii') \( \tilde{\mu} \in \mathcal{M}(m_S) \).

Indeed, if (iii') holds true then the presence of an wandering measure absolutely continuous with respect to the measure \( m_S \) would contradict to the definition of the invariant measure \( \tilde{\mu} \). This new assumption is much stronger and does not allow to study situations with singular invariant measures, but it is easier to control, which we shall use in section 5 dedicated to self-consistent dynamical systems. Note that the last claim made in theorem 2.1 is very similar to this assumption.

In the case of continuous maps above conditions may be considerably simplified.

Theorem 3.2 (Case of continuous maps). Let \( T : X \to X \) be a continuous map. Then the \( m \)-a.e. convergence in (1.1) is equivalent to the following three conditions:

(i) \( \frac{1}{n} \sum_{k=0}^{n-1} T_k^* m \to_{n \to \infty} \tilde{\mu} \),

(ii) the limit measure \( \tilde{\mu} \) is ergodic,

(iii) there are no wandering measures in \( \mathcal{M}(m_S) \).

Moreover, if \( \tilde{\mu} \ll m \), then the conditions (ii), (iii) may be omitted altogether.

3.2. Analysis of the examples

The main aim of this section is to discuss in detail ergodic properties of the dynamical systems introduced as examples in the Introduction and their simple modifications.

Since the situation with the one-dimensional contracting map \( Tx := x/2 \) (mentioned in section 1) is trivial, we start with the example 1.1 (the corresponding phase portrait is sketched in figure 1 (left)). Recall that in this example we consider a family of maps from the unit disc \( X := \{ (\phi, R) : 0 \leq \phi < 2\pi, 0 \leq R \leq 1 \} \) into itself defined in the polar coordinates \( (\phi, R) \) by the relation:

\[
T(\phi, R) := (\phi + 2\pi \alpha + \beta(R - r) \mod 2\pi, \gamma(R - r) + r).
\]

This simple example is very instructive since choosing a different quadruple of admissible parameters \( (\alpha, \beta, r, \gamma) \in (0, 1)^4 \) or making a slight modification of this system we are able to construct illustrations and counter-examples to our main results formulated in the previous section.

Let the reference measure \( m \) be chosen as the two-dimensional Lebesgue measure on \( X \) normalized by \( \pi \), \( C := \{ R = r \} \)—the centered circle of radius \( r \), and let \( m_C \) be the one-dimensional Lebesgue measure on \( C \), normalized by \( 2\pi r \), i.e. the normalized conditional two-dimensional Lebesgue measure on the one-dimensional set \( C \).

Obiously for each admissible quadruple \( (\alpha, \beta, r, \gamma) \) the system \( T, X \) possesses a single attractive set \( C \) (with respect to the Euclidean metric \( \rho(\cdot, \cdot) \) on \( X \)) and the measure \( m_C \) is \( T \)-invariant.

Proposition 3.2.

(a) \( \rho(T^n x, C) \to 0 \) for all \( x \in X \setminus \{0\} \).

(b) \( (T, C, m_C) \) is ergodic if and only if \( \alpha \) is irrational.

(c) \( m(Z_{m_C}) = 1 \) whenever \( \alpha \) is irrational.

(d) The measure \( m_C \) is natural for each admissible \( \alpha \).

All these claims are more or less straightforward and we leave the proof for the reader.

Observe that by (d) even for rational values of the parameter \( \alpha \) each \( m \)-smooth measure converges to \( m_C \) in Cesaro means. Moreover, making an arbitrary \( C^\infty \)-small perturbation to
the maps from this family (slowing down the rotation around $S$ in its small neighborhood) one gets a system, for which $(T, C, m_C)$ is non-ergodic, i.e. $\alpha$ is rational, but the property $m(Z_{m_C}) = 1$ remains valid. This explains, why the properties $m(Z_{m_C}) = 1$ and $m_C(Z_{m_C}) = 1$ should be considered separately, despite that they look very similar.

Let us show that eliminating the rotation outside of the attractor $S$ in the example 1.1 we come to the situation when $m(Z_{m_C}) = 0$ while $m_C(Z_{m_C}) = 1$.

**Example 3.1.**

$$T(\phi, R) := \begin{cases} 
\phi + 2\pi \alpha & \text{if } R = r, \\
\phi & \text{if } R \neq r 
\end{cases} \cdot \gamma(R - r)$$

The corresponding phase portrait is sketched in figure 1 (right). It is easy to see that here we have the uniform global convergence to the support of the measure $m_C$ (albeit without the property (i)), but for any point $x := (\phi, R)$ with $R \neq r$ the ergodic average $\frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)$ converges to $f(\phi, r)$ rather than to $\int f \, dm_C$.

3.3. Proofs

We start with the connections between the point-wise convergence described by the relation (1.2) and the convergence of images of measures under the action of the dynamical system, which is formulated as the property (i) in above theorems. Observe that the property (1.2) is equivalent to the existence (and hence uniqueness) of the observable measure $\tilde{\mu}_{obs}$ having the entire space $X$ as the basin of attraction.

Fix a measurable map $T$ from a compact measurable metric space $(X, B)$ into itself.

**Definition 3.1.** A point $x \in X$ is called **typical** for a probability measure $\mu$ if

$$\frac{1}{n} \sum_{k=0}^{n-1} T_k^* \delta_x \rightharpoonup \mu \quad \text{in the weak sense.}$$

Recall that the set of typical points for the measure $\mu \in \mathcal{M}$ is denoted by $Z_\mu$.

In these terms an observable measure is characterized by the property that there is an open set of typical points of positive $m$-measure.
Lemma 3.3. Let $\mu$ be a limit point for the sequence of measures $\frac{1}{n} \sum_{k=0}^{n-1} T^k \delta_x$ for some point $x \in X$ and let the map $T_* : \mathcal{M}(X) \to \mathcal{M}(X)$ be continuous at the measure $\mu$. Then $T_* \mu = \mu$.

Proof. By the assumption of lemma there exists a sequence of indices $\{n_i\} \to \infty$ such that

$$\mu_{n,i} := \frac{1}{n_i} \sum_{k=0}^{n_i-1} T^k \delta_x \to \mu.$$ 

On the other hand, for each continuous function $\phi$ we have

$$|\delta_x - T^\star \delta_x)(\phi)| \leq 2|\phi|_\infty.$$ 

Since the map $T_*$ is continuous at the measure $\mu$ one can interchange the action of the transfer-operator $T_*$ and the operation of passing to the limit. Therefore

$$T_* \mu = \lim_{i \to \infty} T_* \mu_{n,i}$$

$$= \lim_{i \to \infty} \left( \mu_{n,i} - \frac{1}{n_i} (\delta_x - T^\star \delta_x) \right)$$

$$= \lim_{i \to \infty} \mu_{n,i} = \mu.$$ 

Observe that without the continuity assumption one cannot make a conclusion about the invariance of the limit measure. □

One of the key points of our argument is the following result describing metric properties of basins of attraction of general non necessarily invariant measures.

Lemma 3.4. If $\mu$ is an ergodic $T$-invariant measure, then $\mu(Z_\mu) = 1$. Otherwise, if $\mu$ is non-ergodic or ergodic but non-invariant and the map $T_* : \mathcal{M}(X) \to \mathcal{M}(X)$ is continuous at the measure $\mu$, then $\mu(Z_\mu) = 0$.

Proof. If the measure $\mu$ is an ergodic $T$-invariant measure this claim is a trivial consequence of the Birkhoff ergodic theorem. Indeed, it is enough to apply ergodic theorem for the test-function defined as the indicator function of the set $Z_\mu$.

It remains to prove that in the opposite situation the set of typical points is of zero $\mu$-measure. By lemma 3.3 a measure having even a single typical point at which the transfer-operator is continuous inevitably has to be an invariant measure. Therefore by the Ergodic Decomposition theorem (see e.g. [15]) there is a probability measure $\eta$ on the set $\mathcal{M}_T^{\text{erg}}$ of all ergodic $T$-invariant measures $\nu$, such that for each measurable set $B \in \mathcal{B}$ we have

$$\mu(B) = \int \nu(B) \operatorname{d}\eta(\nu).$$

(3.1)

By the first part of the proof, $\nu(Z_\nu) = 1$ for each ergodic measure $\nu$. On the other hand, by the definition of the set of typical points $Z_\nu \cap Z_{\nu'} = \emptyset$ for any two measures $\nu, \nu'$, which implies that

$$Z_\mu \subseteq X \setminus \bigcup_{\nu \in \mathcal{M}_T^{\text{erg}}} Z_\nu.$$
Applying (3.1), we get
\[ \mu(\cup_{\nu \in \mathcal{M}}^\nu Z_\nu) = 1. \]

Therefore
\[ \mu(Z_\mu) \leq \mu(X \setminus \cup_{\nu \in \mathcal{M}}^\nu Z_\nu) = 0, \]
which finishes the proof.

\[ \square \]

Remark 3.5. If \( T \in C^0(X, X) \) then \( T_* \) is continuous at any measure \( \mu \), but otherwise \( T_* \) might be discontinuous even at the invariant measure.

Indeed, consider

\[ \text{Example 3.2. } X := [0, 1] \text{ and } T_x := \begin{cases} \frac{x}{2} + \frac{1}{4} & \text{if } 0 \leq x \leq 1/2, \\ 2x - 1 & \text{otherwise}. \end{cases} \]

Then the transfer-operator \( T_* \) is discontinuous at the \( T \)-invariant measure \( \delta_{1/2} \).

Lemma 3.6. The property \( m(Z_{\tilde{\mu}}) = 1 \) implies (i).

Proof. By definition for a \( m \)-a.a. point \( y \in X \) the weak convergence of the sequence of measures \( \frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^k y} \to \tilde{\mu}^{\text{obs}} \) takes place, i.e. for any continuous function \( \phi : X \to \mathbb{R} \) and \( m \)-a.a. \( y \in X \) we, using the Lebesgue dominated convergence theorem, get
\[
\int \phi(x) \, d \left( \frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^k y} \right) = \frac{1}{n} \sum_{k=0}^{n-1} \int \phi(T^k y) \to \int \phi \, d \tilde{\mu}^{\text{obs}}.
\]

Choose an absolutely continuous (with respect to \( m \)) measure \( \mu \in \mathcal{M}(X) \) and consider Cesaro averages of its images: \( \mu_n := \frac{1}{n} \sum_{k=1}^{n} T^{k-1} \mu \). Then using the above convergence and the absolute continuity of the measure \( \mu \) we get
\[
\int \phi \, d \mu_n = \int \phi \, d \left( \frac{1}{n} \sum_{k=0}^{n-1} T^k \mu \right) = \int \frac{1}{n} \sum_{k=0}^{n-1} \phi(T^k x) \, d \mu \to \int \left( \int \phi \, d \tilde{\mu}^{\text{obs}} \right) \, d \mu = \int \phi \, d \tilde{\mu}^{\text{obs}},
\]
which proves the assertion.

Consider now the questions about the necessity of ergodicity and the absence of wandering sets.

Lemma 3.7. The property \( m(Z_{\mu}) = 1 \) implies ergodicity, i.e. (ii).

Proof. By lemma 3.4 a non-ergodic invariant measure cannot have a full \( m \)-measure set of typical points, which implies the claim.

Despite of this result even the uniqueness of the measure \( \tilde{\mu}^{\text{obs}} \) does not imply that it is ergodic. Indeed, consider the following example (see [5] for details):
\[
T_x := \begin{cases} (1 - \sin(\pi x - \pi/2))/2 & \text{if } 0 < x < 1, \\ x & \text{if } x \in \{0, 1\} \end{cases}.
\]

(3.2)
One can easily show that the locally maximal attractor in this example consists of two fixed points at 0 and 1 and that
\[
\frac{1}{n} \sum_{k=0}^{n-1} T_k \delta_x \xrightarrow{n \to \infty} \frac{1}{2} (\delta_0 + \delta_1) = \tilde{\mu}^{\text{obs}} \text{ for any initial point } x \in (0, 1).
\]
On the other hand, this measure is nonergodic, since the points 0 and 1 are fixed points.

According to lemma 3.4 for the analysis of the regular case we need to study only properties of ergodic invariant measures with a nontrivial support \(S\). Thus the set \(S\) is forward invariant.

**Lemma 3.8.** Let \(\tilde{\mu}\) be an ergodic invariant measure. Then the property \(m_S(Z_{\tilde{\mu}}) = 1\) implies the absence of wandering measures in \(\mathcal{M}(m_S)\).

**Proof.** Assume from the contrary that there exists a wandering measure \(\mu \in \mathcal{M}(m_S)\). Denote \(\tilde{S} := Z_{\tilde{\mu}} \cap S\). Then by the assumption the measure \(\mu_{\tilde{S}}\) also belongs to the set \(\mathcal{M}(m_S)\).

Hence
\[
\frac{1}{n} \sum_{k=0}^{n-1} T_k \mu_{\tilde{S}} \xrightarrow{n \to \infty} \tilde{\mu}.
\]

By the wandering property the measures \(T_k \mu_{\tilde{S}}\) are mutually singular for different \(k\), which together with the forward invariance of \(S\) contradicts to the fact that the support of the limit measure \(\tilde{\mu}\) coincides with the set \(S\). □

It is worth noting that this result is very similar to the analysis of transformations without wandering sets of positive measure in [11] (section 10).

**Proof of theorem 3.1.** Collecting together the results of lemmas 3.6–3.8 we get the first part of the theorem related to the necessity of the assumptions (i)–(iii).

Now we turn to the second part of the proof of theorem 3.1 and consider in detail the set of points converging under dynamics to the support \(S\) of the measure \(\tilde{\mu}\):
\[
Y := \{ x \in X : \lim_{t \to \infty} \rho(T^t x, S) = 0 \}.
\]

**Lemma 3.9.** Let the condition 3.1 (i) hold true. Then \(m(Y) = 1\).

**Proof.** Assume from the contrary that this claim does not hold. Then there exists a subset \(A := \{ x \in X : \liminf_{t \to \infty} \rho(T^t x, S) > 0 \}\) of positive \(m\)-measure. Denote by \(m_A\) the conditional measure induced by the reference measure \(m\) on the set \(A\). Since \(m(A) > 0\) this measure is absolutely continuous with respect to \(m\) and hence by 3.1(i) we have
\[
\frac{1}{n} \sum_{k=0}^{n-1} T_k m_A \xrightarrow{n \to \infty} \tilde{\mu},
\]
which contradicts to the definition of the set \(A\). □

**Definition 3.2.** We say that a point \(y_x \in S\) is weakly tracing a point \(x \in X\) if the following limits (in the weak sense) exist and coincide
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} T_k \delta_x = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} T_k \delta_{y_x}.
\]
In other words, trajectories starting from the points \( x \in X \) and \( y_x \in S \) have the same statistics. Obviously the map \( x \to y_x \) is not injective, however, applying the same argument as in the proof of Lemma 3.9, we deduce that

**Corollary 3.10.** For \( m \)-a.e. point \( x \in Y \) there is a weakly tracing point \( y_x \in S \).

Observe that one cannot use here the stronger point-wise tracing property

\[
\rho(T^n x, T^n y_x) \xrightarrow{n \to \infty} 0.
\]

Indeed, consider a contractive system whose trajectories are wrapping around the limit circle. The dynamics on the circle is supposed to be a pure irrational rotation like in the example 1.1. Then if the rate of convergence to the attractor is slow enough, e.g. of order \( 1/n \), then the outer points do not possess the point-wise tracing counterparts on the attractor.

**Definition 3.3.** Denote by \( \hat{Y} \) the subset of points from the set \( Y \) weakly traced by \( \tilde{\mu} \)-"typical" trajectories on the support of the measure \( \tilde{\mu}^{nat} \).

Our aim is to check that this set of full \( m \)-measure. To this end one is tempted to extend the argument used in the previous proof along the following lines.

Assume from the contrary that there exists a subset \( B \subseteq Y \) of positive \( m \)-measure such that the trajectories starting from this set are traced by non-typical points from \( S \). By the ergodicity assumption 3.1(ii) according to lemma 3.4 we have \( \tilde{\mu}(Z_{\tilde{\mu}}) = 1 \). Hence the \( \tilde{\mu} \)-measure of the set of non-typical points \( S \setminus Z_{\tilde{\mu}} \) is zero. Denote by \( m_B \) the conditional measure constructed from \( m \) on the set \( B \). Since \( m(B) > 0 \) this measure is absolutely continuous with respect to \( m \) and hence by 3.1(i) we have

\[
\frac{1}{n} \sum_{k=0}^{n-1} T^k m_B \xrightarrow{n \to \infty} \tilde{\mu}.
\]

On the other hand, since by construction all points from the set \( B \) are traced by \( \tilde{\mu} \)-non-typical points, the limit measure should be supported by the set \( S \setminus Z_{\tilde{\mu}} \), which seems to be a contradiction.

Unfortunately the set of non-typical points being \( T \)-invariant needs not to be a compact set. Therefore the limit measure may be supported by a larger set of positive \( \mu \)-measure. This explains that some additional assumptions are necessary in order to overcome this difficulty.

One of the possibilities is to assume that the measures \( m_S \) and \( \tilde{\mu} \) are equivalent (see remark 3.1). However we prefer to use a less restrictive assumption about the absence of wandering measures.

**Lemma 3.11.** Let the conditions 3.1(i)–(iii) hold true and let \( m(S) > 0 \). Then \( m(\hat{Y}) = 1 \).

**Proof.** Denote \( \hat{Z} := Z_{\hat{\mu}} \cap S \). From the previous results we have \( \tilde{\mu}(\hat{Z}) = 1 \), but this fact alone does not contradict to \( m(\hat{Z}) < 1 \). If the latter inequality takes place, then the set \( B := S \setminus \hat{Z} \) is of positive \( m \)-measure. By definition only \( \tilde{\mu} \)-non-typical points belong to the set \( B \). Then this together with the assumption (i) and the ergodicity of \( \tilde{\mu} \) imply that the measure \( m_B \) is wandering, which contradicts to the assumption (iii).

Consider now what happens outside of the set \( S \). Denote \( Z := Z_{\tilde{\mu}} \setminus S \). Assume from the contrary that there exists a subset \( C \subseteq (X \setminus S) \cap Y \) of positive \( m \)-measure consisting only of \( \tilde{\mu} \)-non-typical points. Since \( m(Y) = 1 \) by lemma 3.9, the intersection with \( Y \) does not change the measure of the set \( C \). By corollary 3.10 for each point \( x \in C \) there is a tracing point \( y_x \in S \). Decompose the set \( C \) into two parts: \( C_0 \) consisting of points starting from which a trajectory hits the set \( S \) in a finite number of steps, and \( C_1 \) for which trajectories only converge
to $S$. By the 1st part of the proof $m(C_0) = 0$ (as a union of pre-images of the set of $m$-measure zero), while to prove that $m(C_1) = 0$ one uses additionally the property (i).

**Lemma 3.12.** Let the conditions 3.1(i)-(iii) hold true, and let $m(S) = 0$, but $m_S$ be well defined. Then $m(Y) = 1$.

**Proof.** The only difference with the proof of lemma 3.11 is that one needs to use additionally the assumption that the property (3.1(i)) holds not only with respect to the reference measure $m$, but also with respect to the conditional measure $m_S$, which is singular with respect to $m$ in this case. Therefore we omit details.

**Final part of the proof of theorem 3.1.** Collecting the results obtained in lemmas 3.9–3.12 we get that $m(Z_{\tilde{\mu}}) = 1$. To obtain the claim that $m_S(Z_{\tilde{\mu}}) = 1$ one needs to consider the restriction of the dynamical system to the forward invariant set $S$ (similarly to the proof of the necessary conditions).

It remains to prove theorem 3.2. Here the item (i) follows from theorem 2.2. The consideration of items (ii) and (iii) is exactly the same as in the proof of the previous result.

### 4. Irregular case

This section is dedicated to the situation when the transfer-operator $T_*$ is discontinuous at the limit measure $\tilde{\mu} := \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} T_*^k \mu$. Normally this happens when the dynamical system has no invariant measure. A typical example of this sort may be obtained by the following simple modification of the example (1.3):

**Example 4.1.**

$$T(\phi, R) := \begin{cases} (\phi + 2\pi \alpha + \beta(R - r) \mod 2\pi, \gamma(R - r) + r) & \text{if } r(R - r) \neq 0 \\ (\phi + 2\pi \alpha \mod 2\pi, (1 + r)/2) & \text{otherwise.} \end{cases} \quad (4.1)$$

This example and its modifications are very instructive, but to have a simpler picture in mind let us consider the following one-dimensional map:

**Example 4.2.** $X := [0, 1)$ and $T x := \begin{cases} 1 - c & \text{if } x = 0 \\ x^2 & \text{otherwise.} \end{cases}$

If $0 < c < 1$, then this map has no invariant measures, but each absolutely continuous measure converges weakly to $\delta_0$; otherwise if $c = 0$ the situation changes drastically: the limit measure is invariant.

One can show that in the examples above for any smooth probability measure $\mu$ the sequence of measures $\frac{1}{n} \sum_{k=0}^{n-1} T_*^k \mu$ converges weakly to a certain limit measure $\bar{\mu}$ on the circle $\{R = r\}$ or at the origin, but this measure is no longer invariant (if $c \neq 0$ in the example 4.2). Instead of the convergence to the limit circle or to a fixed point one can consider situations when the only attractor of the dynamical system $(T, X)$ is a Cantor set $K$. Then modifying the map $T$ only on the set $K$ in the same way as above (e.g. $\tilde{T}|_{X \setminus K} \equiv T|_{X \setminus K}$, $\tilde{T}K \neq K$) one gets nontrivial examples of the discontinuity of the transfer-operator $\tilde{T}_*$ at the limit measure.

In order to apply arguments similar to those elaborated in section 3 we need to clarify and extend the notion of the invariant measure. To this end we make use of the equivalence of the ergodicity and the fulfillment of the Birkhoff theorem (see e.g. [15]).
Definition 4.1. Motivated by the Birkhoff ergodic theorem, we say that a measure $\mu \in \mathcal{M}$ is weakly ergodic if $\mu(Z_\mu) = 1$.

Indeed, this property obviously holds in the case of a conventional ergodic invariant measure. To demonstrate the reason for this extension, consider the example 4.2. When $0 < c < 1$ this map has no invariant measures, but the $\delta$-measure at the origin is weakly ergodic. More sophisticated situations, concerning the so called nonlinear Markov chains, will be discussed in section 5.

The assumption about the discontinuity of the transfer-operator $T_\mu$ at the measure $\bar{\mu}$ implies obviously the discontinuity of the map $T$. Nevertheless making some additional assumptions about local properties of the map $T$ in the neighborhood of the support of the measure $\bar{\mu}$ in principle one can get an analogue of the necessary part of theorem 3.1. However those assumptions look somewhat clumsy and therefore we restrict ourselves only to sufficient conditions.

Theorem 4.1. The assumptions

(i) $\frac{1}{n} \sum_{k=0}^{n-1} T^k_\mu \to \bar{\mu}$, $\forall \mu \in \mathcal{M}(m) \cup \mathcal{M}(mS)$.
(ii) the limit measure $\bar{\mu}$ is weakly ergodic,
(iii) there are no wandering measures in $\mathcal{M}(mS)$,

imply that $m(Z_\bar{\mu}) \cdot mS(Z_\bar{\mu}) = 1$.

The proof follows basically the same scheme as in the regular case with the necessary usage of the weak ergodicity property instead of references to the ergodic theorem, e.g.:

Lemma 4.1. If $\mu$ is weakly ergodic $T$-invariant measure, then $\mu(Z_\mu) = 1$.

Proof. The claim is an immediate consequence of the definition of the weakly ergodic measure.

This result is an extension of lemma 3.4. Further on the statements of lemmas 3.9–3.12 made in the regular case do not make use of the continuity of the map $T$ and thus remain valid in the discontinuous setting. Collecting the corresponding results we get the proof of theorem 4.1.

5. Self-consistent dynamical systems

In this section we discuss a more complicated and not well studied case when the dynamics depends not only on the current point in the space $X$ but on the current statistics of the system as well. Let $\{T_\mu\}_{\mu \in \mathcal{M}(X)}$ be a family of maps from a compact measurable space $X$ into itself, parametrized by probability measures $\mu \in \mathcal{M}(X)$.

Definition 5.1. By a self-consistent dynamical system we mean a skew product map $T(x, \mu) := (T_\mu x, (T_\mu)^\ast \mu)$ acting in the direct product space $X \times \mathcal{M}(X)$.

The idea of deterministic self-consistent dynamical systems was introduced by Kaneko [13] in order to approximate the dynamics of large systems in terms of a mean-field type perturbation of an isolated sub-system. Despite a large number of attempts to study such systems there are only a few situations when a complete mathematical treatment was successful. See [3, 14] and further references to known numerical results therein. Discussion of similar questions in true random setting may be found e.g. in [6].

Let us give a couple of seemingly trivial examples. Let $X := [0, 1]$ and denote by $E_\mu := \int x \, d\mu$ the mathematical expectation over a probabilistic measure $\mu$ on $X$. 

4661
Example 5.1 (Additive perturbation). \( T_{\varepsilon, \mu} x := Tx + \varepsilon E_{\mu} \mod 1 \), where \( \varepsilon \in [0, 1] \).

Example 5.2 (Multiplicative perturbation).

(a) \( T_{\mu} x := x \cdot E_{\mu} \mod 1 \),
(b) \( T_{\mu} x := x / E_{\mu} \mod 1 \) (here we assume that \( 1/0 \mod 1 = 0 \)).

The example 5.1 with \( T_{\mu} x := 1 - 2|x - 1/2| \) was studied in [14], where it has been shown that images of any probability measure absolutely continuous with respect to the Lebesgue measure \( m \) on \( X \) converge in Cesaro means to \( m \) provided the parameter \( \varepsilon \) be small enough. This is one of a very few cases where the stability of the SRB measure with respect to mean-field type perturbations has been proven.

The example 5.2 represents the situation of multiplicative perturbations which was not studied earlier. In distinction to the case (a), where the dynamics is trivial: there are only two invariant measures stable one \( \delta_0 \) and unstable \( \delta_1 \), the dynamics in the case (b) is much more complicated: there are infinitely many mutually singular probabilistic invariant measures, including the Lebesgue measure. Here the invariance means that the measure is preserved under dynamics (this explanation is necessary since the system itself is defined in the product space \( X \times \mathcal{M}(X) \)).

In the present setting, even under the assumption that all maps \( T_{\mu} \) are continuous, one cannot apply the ergodic theorem directly, since at each time step a different map is chosen. Additionally the conventional definition of ergodicity does no make much sense here. To overcome this difficulty we adapt the notion of the weak ergodicity to the setting of self-consistent dynamical systems.

**Definition 5.2.** A measure \( \mu \) is said to be weakly ergodic for the self-consistent system if the property

\[
\frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x, \mu)) \overset{n \to \infty}{\longrightarrow} \int f \, d\mu
\]

holds for each continuous test function \( f(\cdot, \cdot) \), depending only on the first argument \( x \), and for \( \mu \)-a.e. \( x \in X \).

Note that the Lebesgue measure \( m \) is weakly ergodic in the example 5.2(b), however it is not clear whether smooth probabilistic measures converge to \( m \) even in a certain weak sense under the action of this self-consistent dynamical system.

Let the transfer-operator \( (T_{\bar{\mu}}) \) be continuous at the limit measure \( \bar{\mu} \) (see the assumption (i) below) and which is finer than the given reference measure \( m \), i.e. \( S := \text{supp}(\bar{\mu}) \subseteq \text{supp}(m) \). The definition of \( \bar{\mu} \)-typical points also needs to be modified due to the more complex structure of the phase space \( (X \times \mathcal{M}(X)) \) instead of \( X \):

\[
Z_{\bar{\mu}} := \{ x \in X : \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x, \mu)) \overset{n \to \infty}{\longrightarrow} \int f \, d\bar{\mu} \quad \forall \mu \in \mathcal{M}(m), \forall f \in C^0(X) \}.
\]

The following result gives sufficient conditions for the application of the ergodic averaging in the entire space. Since in the case under consideration the point-wise dynamics depends sensitively on the choice of the initial measure, the construction of necessary conditions is not clear at the moment.
**Theorem 5.1.** Let

(i) \( \frac{1}{n} \sum_{k=0}^{n-1} T^k \mu \overset{n \to \infty}{\rightarrow} \tilde{\mu} \quad \forall \mu \in \mathcal{M}(m) \cup \mathcal{M}(m_S) \),

(ii) the limit measure \( \tilde{\mu} \) be weakly ergodic,

(iii) \( \tilde{\mu} \in \mathcal{M}(m_S) \).

Then \( m(Z_{\tilde{\mu}}) \cdot m_S(Z_{\tilde{\mu}}) = 1 \).

Again as in the previous section the usage of the notion of weakly ergodic measures allows to study the dynamics outside of the set \( S := \text{supp}(\tilde{\mu}) \) and to prove the convergence to \( S \) for \( m \)-a.a. initial points outside of \( S \), while the convergence of statistics of trajectories starting from the points inside of \( S \) follows from the assumption (iii').

As we already discussed in section 3.1 (see remark 3.1) the assumption (iii') is somewhat restrictive and one would prefer a weaker assumption:

(iii) there are no wandering measures \( \mu \in \mathcal{M}(m_S) \).

Unfortunately there are two obstacles here. First, at each time step one applies a different map from the family \( T_\mu \) and thus the notion of the wandering measure needs to be modified.

**Definition 5.3.** A probability measure \( \mu \) is said to be wandering for the self-consistent system \( T \) if the measures \( T_n^* \mu \) and \( T_k^* \mu \) are mutually singular for all \( n \neq k \in \mathbb{Z}^+ \).

A much more delicate point is that due to the same reason (different maps are applied at different time steps) the support of the limit measure \( \tilde{\mu} \) needs not to be an invariant set for the maps from the family \( T_\mu \). Therefore the situation with singular with respect to the conditional measure \( m_S \) limit measures is out of control under the present approach. Nevertheless we expect that the assumption (iii) should be sufficient here.

**Acknowledgments**

The author is grateful to anonymous referees for helpful comments and suggestions which improved the quality of this paper.

This work has been carried out at IITP RAS and we gratefully acknowledge the support of Russian Foundation for Sciences (project No. 14-50-00150).

**References**

[1] Abdenur F and Andersson M 2013 Ergodic theory of generic continuous maps Commun. Math. Phys. 318 831–55

[2] Araujo V and Pinheiro V 2016 Abundance of wild historic behavior, ergodic decomposition and generalized physical measures (arXiv:1609.05356v1)

[3] Bardet J-B, Keller G and Zweimuller R 2009 Stochastically stable globally coupled maps with bistable thermodynamic limit Commun. Math. Phys. 292 237–70

[4] Blank M 1997 Discreteness and Continuity in Problems of Chaotic Dynamics (Providence, RI: American Mathematical Society)

[5] Blank M and Bunimovich L 2003 Multicomponent dynamical systems: SRB measures and phase transitions Nonlinearity 16 387–401

[6] Butkovsky O A 2014 On Ergodic properties of nonlinear markov chains and stochastic McKean–Vlasov equations Theory Probab. Appl. 58 661–74
[7] Catsigeras E 2012 Ergodic theorems with respect to Lebesgue Trans. Math. World Sci. Eng. Acad. Soc. 10 463–79
[8] Chaika J and Masur H 2015 There exists an interval exchange with a non-ergodic generic measure J. Mod. Dyn. 9 289–304
[9] Gelfert K and Kwietnia D 2014 On density of ergodic measures and generic points (arXiv:1404.0456v2)
[10] Hajian A B and Kakutani S 1964 Weakly wandering sets and invariant measures Trans. AMS 110 136–51
[11] Hurewicz W 1944 Ergodic theorem without invariant measure Ann. Math. 45 192–206
[12] Jarvenpaa E and Tolonen T 2005 Relations between natural and observable measures Nonlinearity 18 897–912
[13] Kaneko K 1990 Globally coupled chaos violates the law of large numbers but not the central limit theorem Phys. Rev. Lett. 65 1391–4
[14] Keller G 2000 An ergodic theoretic approach to mean field coupled maps Prog. Probab. 46 183–208
[15] Kornfeld I P, Fomin S V and Sinai Ya G 1982 Ergodic Theory (New York: Springer)
[16] Misiurewicz M 2005 Ergodic natural measures Contemp. Math. 385 1–6
[17] Oxtoby J C and Ulam S M 1941 Measure-preserving homeomorphisms and metrical transitivity Ann. Math. 42 874–920
[18] Rohlin V A 1952 On the fundamental ideas of measure theory Am. Math. Soc. 71 55 (translation)
Rohlin V A 1949 On the fundamental ideas of measure theory Mat. Sb. 25 107–50 (Russian)
[19] Ruelle D 2001 Historical behaviour in smooth dynamical systems Global Analysis of Dynamical Systems ed H Broer et al (London: Institute of Physics) pp 63–6