Some properties of F-harmonic maps

Mohammed Benalili and Hafida Benallal

Abstract. In this note, we investigate estimates of the Morse index for F-harmonic maps into spheres, our results extend partially those obtained in ([14]) and ([15]) for harmonic and p-harmonic maps.

1. Introduction

Harmonic maps have been studied first by J. Eells and J.H. Sampson in the sixties and since then many works were done (see [4], [9], [13], [16], [17], [21]) to cite a few of them. Extensions to notions of p-harmonic, biharmonic, F-harmonic and f-harmonic maps were introduced and similar research has been carried out (see [1], [2], [3], [5], [12], [15], [18], [20]). Harmonic maps were applied to broad areas in sciences and engineering including the robot mechanics (see [6], [8]).

The Morse index for harmonic maps, p-harmonic maps, as well as biharmonic maps, into a standard unit Euclidean sphere $S^n$ has been widely considered (see [12], [14], [15]).

In this paper for a $C^2$-function $F : [0, +\infty[ \rightarrow [0, +\infty[$ such that $F'(t) > 0$ on $t \in ]0, +\infty[$, we consider the Morse index for F-harmonic maps into spheres. Our results generalize partial estimates of the Morse index obtained in ([14]) and ([15]) for harmonic and p-harmonic maps.

Let $(M, g)$ be a compact Riemannian manifold of dimension $m \geq 2$, $S^n$ the unit $n$-dimensional Euclidean sphere with $n \geq 2$ endowed with the canonical metric can induced by the inner product of $R^{n+1}$.

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For a $C^1$- application $\phi : (M, g) \longrightarrow (S^n, can)$, we define the $F$-energy functional by,
\[
E_F(\phi) = \int_M F \left( \frac{|d\phi|^2}{2} \right) dv_g
\]
where $\frac{|d\phi|^2}{2}$ denotes the energy density given by
\[
\frac{|d\phi|^2}{2} = \frac{1}{2} \sum_{i=1}^m |d\phi(e_i)|^2
\]
and where $\{e_i\}$ is an orthonormal basis on $T_x M$ and $dv_g$ is the Riemannian measure associated to $g$ on $M$.

Let $\phi^{-1}TS^n$ be the pullback vector fiber bundle of $TS^n$, $\Gamma (\phi^{-1}TS^n)$ the space of sections on $\phi^{-1}TS^n$ and denote by $\nabla^M$, $\nabla^{S^n}$ and $\tilde{\nabla}$ Levi-Civita connections on $TM$, $T S^n$ and $\phi^{-1}TS^n$ respectively. $\tilde{\nabla}$ is defined by
\[
\tilde{\nabla}_X Y = \nabla^{S^n}_{\phi_* X} Y
\]
where $X \in TM$ and $Y \in \Gamma (\phi^{-1}TS^n)$.

Let $v$ be a vector field on $S^n$ and $(\phi^v_t)$ the flow of diffeomorphisms induced by $v$ on $S^n$ i.e.
\[
\phi^v_0 = \phi, \quad \frac{d}{dt} \phi^v_t \big|_{t=0} = v.
\]

The first variation formula of $E_F(\phi)$ is given by
\[
\frac{d}{dt} E_F(\phi_t) \big|_{t=0} = \int_M F' \left( \frac{|d\phi_t|^2}{2} \right) \langle \nabla_{\partial_t d\phi_t} d\phi_t, d\phi_t \rangle \big|_{t=0} dv_g
\]
\[
= - \int_M \langle v, \tau_F(\phi) \rangle dv_g
\]
where $\tau_F(\phi) = \text{trace}_g \nabla \left( F' \left( \frac{|d\phi|^2}{2} \right) d\phi \right)$ denotes the Euler-Lagrange equation of the $F$-energy functional $E_F$. Remark that if $|d\phi|_{\phi^{-1}TN}$ is constant then $\phi$ is harmonic if and only if $\phi$ is $F$-harmonic.

**Definition 1.** $\phi$ is called $F$-harmonic if and only if $\tau_F(\phi) = 0$ i.e. $\phi$ is a critical point of $F$-energy functional $E_F$.

The second variation of $E_F$ is given as
\[
\frac{d^2}{dt^2} E_F(\phi_t) \big|_{t=0} = \frac{d}{dt} \int_M \frac{d}{dt} F \left( \frac{|d\phi_t|^2}{2} \right) \big|_{t=0} dv_g
\]
\[\begin{align*}
&= \int_M \left[ F'' \left( \frac{|d\phi|^2}{2} \right) (\nabla v, d\phi)^2 + F' \left( \frac{|d\phi|^2}{2} \right) |\nabla v|^2 \right] dv_g \\
&\quad - \int_M \left\langle \nabla_{\partial_t} \frac{\partial \phi_i}{\partial t} |_{t=0}, \text{trace}_g \nabla \left( F' \left( \frac{|d\phi|^2}{2} \right) d\phi \right) \right\rangle dv_g \\
&\quad - \int_M F' \left( \frac{|d\phi|^2}{2} \right) \sum_{i=1}^m \langle R^{S^n} (v, d\phi(e_i)) d\phi(e_i), v \rangle dv_g
\end{align*}\]

and since \( \phi \) is \( F \)-harmonic, \( \tau_F(\phi) = 0 \), then
\[
\frac{d^2}{dt^2} E_F(\phi_t) |_{t=0} = \int_M F'' \left( \frac{|d\phi|^2}{2} \right) (\nabla v, d\phi)^2 dv_g +
\]

\[
(1.1) \quad \int_M F' \left( \frac{|d\phi|^2}{2} \right) \left[ |\nabla v|^2 - \sum_{i=1}^m \langle R^{S^n} (v, d\phi(e_i)) d\phi(e_i), v \rangle \right] dv_g.
\]

Along this paper we consider variation in directions of vector fields of the subspace \( \mathcal{L}(\phi) \) of \( \Gamma(\phi^{-1}TS^n) \) defined by
\[
\mathcal{L}(\phi) = \{ \bar{v} \circ \phi, v \in \mathbb{R}^{n+1} \}
\]
where \( \bar{v} \) is a vector field on \( S^n \) given by \( \bar{v}(y) = v - \langle v, y \rangle y \) for any \( y \in S^n \); it is known that \( \bar{v} \) is a conformal vector field on \( S^n \). Obviously, if \( \phi \) is not constant, \( \mathcal{L}(\phi) \) is of dimension \( n+1 \).

2. Morse index for \( F \)-harmonic application

For any vector field \( v \) on \( S^n \) along \( \phi \), we associate the quadratic form
\[
Q^F_{\phi}(v) = \frac{d^2}{dt^2} E_F(\phi_t) |_{t=0}.
\]

The Morse index of the \( F \)-harmonic map is defined as the positive integer
\[
\text{Ind}_F(\phi) = \sup \{ \dim N, N \subset \Gamma(\phi) \text{ such that } Q^F_{\phi}(v) \text{ negative defined on } N \}
\]
where \( N \) is a subspace of \( \Gamma(\phi) \). The Morse index measures the degree of the instability of \( \phi \) which is called \( F \)-stable if \( \text{Ind}_F(\phi) = 0 \). Let also \( S^g_F(\phi) \) be the \( F \)-stress-energy tensor defined by
\[
(2.1) \quad S^g_F(\phi) = F' \left( \frac{|d\phi|^2}{2} \right) |d\phi|^2 g - 2 \left( F' \left( \frac{|d\phi|^2}{2} \right) + F'' \left( \frac{|d\phi|^2}{2} \right) \right) \phi^{\text{can}}.
\]
For $x \in M$, we put

$$S^o_F(\phi) = \inf \{S^o_F(\phi)(X, X), X \in T_xM \text{ such that } g(X, X) = 1 \}.$$ 

The tensor $S^o_F(\phi)$ will be called positive (resp. positive defined) at $x$ if $S^o_F(\phi) \geq 0$ (resp. $S^o_F(\phi) > 0$).

**Remark.** $F(t) = \frac{1}{p}(2t)^\frac{2}{p}$, with $p \in [2, +\infty]$, $S^p_g(\phi)$ is the stress-energy tensor introduced by Eells and Lemaire for $p = 2$ (\cite{11}) or El Soufi for $p \geq 4$, (\cite{13}).

In this note we state the following result

**Theorem.** Let $\phi$ be an $F$-harmonic map from a compact $m$–Riemannian manifold $(M, g)$ $(m \geq 2)$ into the Euclidean sphere $S^n$ $(n \geq 2)$. Suppose that the $F$-stress-energy tensor $S^F_g(\phi)$ of $\phi$ is positive defined. Then the Morse index of $\phi$, $\text{Ind}_F(\phi) \geq n + 1$.

**Proof.** Let $w = \tilde{v} \circ \phi \in \mathcal{L}(\phi)$ and put $\langle v, \phi \rangle = \phi_0$. For any point $x \in M$, we denote respectively by $w^T(x)$ and $w^\perp(x)$ the tangential and normal components of the vector $w(x)$ on the spaces $d\phi(T_xM)$ and $d\phi(T_xM)^\perp$. Let also $\{e_1, ..., e_m\}$ an orthonormal basis of $T_xM$ which diagonalizes $\phi^*\text{can}$ and such that $\{d\phi(e_1), d\phi(e_2), ..., d\phi(e_l)\}$ forms a basis of $d\phi(T_xM)$.

If $\left( F \left( \frac{|d\phi|^2}{2} \right) + \frac{|d\phi|^2}{2} F'' \left( \frac{|d\phi|^2}{2} \right) \right) \neq 0$ at the point $x$, then

$$|\tilde{v}^T(x)|^2 = \sum_{i=1}^l |d\phi(e_i)|^{-2} \langle \tilde{v}(x), d\phi(e_i) \rangle^2$$

on the other hand, for any $i \leq l$, we have

$$2 \left( F \left( \frac{|d\phi|^2}{2} \right) + \frac{|d\phi|^2}{2} F'' \left( \frac{|d\phi|^2}{2} \right) \right) |d\phi(e_i)|^2 = |d\phi|^2 \left( \frac{|d\phi(x)|^2}{2} \right)$$

(2.2)

$$- S^F_g(\phi)(x)(e_i, e_i) \leq |d\phi|^2 \left( \frac{|d\phi(x)|^2}{2} \right) - S^o_F(\phi)(x)$$

so

$$2 \left( F \left( \frac{|d\phi|^2}{2} \right) + \frac{|d\phi|^2}{2} F'' \left( \frac{|d\phi|^2}{2} \right) \right) \sum_{i=1}^l \langle \tilde{v}(x), d\phi(e_i) \rangle^2$$

and since,

$$\langle \tilde{v}(x), d\phi(e_i) \rangle^2 = \langle v - \langle v, \phi \rangle \phi, d\phi(e_i) \rangle^2$$
$$= \langle v, d\phi(e_i) \rangle^2 = |d\phi_v(e_i)|^2$$

we get

$$\left( |d\phi|^2 F'(\frac{|d\phi(x)|^2}{2}) - S_{g,F}^0(\phi)(x) \right) |w^T(x)|^2 \geq 2 \left( F' \left( \frac{|\phi|^2}{2} \right) + \frac{|d\phi|^2}{2} F'' \left( \frac{|d\phi|^2}{2} \right) \right) |d\phi_v(x)|^2.$$ 

Now, taking into account (2.2), we infer that

$$2 \left( F' \left( \frac{|\phi|^2}{2} \right) + \frac{|d\phi|^2}{2} F'' \left( \frac{|d\phi|^2}{2} \right) \right) |d\phi_v(x)|^2 - |d\phi|^2 F'(\frac{|d\phi(x)|^2}{2}) |\nabla| \leq - |d\phi|^2 F'(\frac{|d\phi(x)|^2}{2}) |\nabla| - S_{g,F}^0(\phi)(x) |T^N(x)| - S_{g,F}^0(\phi)(x) |T^T(x)|^2$$

(2.3)

Now the second variation writes as

$$\frac{d^2}{dt^2} E_F(\phi_t) |_{t=0} = \int_M F'' \left( \frac{|\phi|^2}{2} \right) \langle \nabla v, d\phi \rangle^2 dv_g$$

$$+ \int_M F' \left( \frac{|\phi|^2}{2} \right) \left[ |\nabla v|^2 - |d\phi|^2 |\nabla| + |d\phi_v|^2 \right] dv_g$$

Consequently, we have

$$Q_F^\phi(v) = 2 \int_M \left( \frac{|\phi|^2}{2} F'' \left( \frac{|\phi|^2}{2} \right) + F' \left( \frac{|\phi|^2}{2} \right) \right) |d\phi_v|^2 dv_g$$

$$- \int_M F' \left( \frac{|\phi|^2}{2} \right) |d\phi|^2 |\nabla v|^2 dv_g$$

and taking account of the inequality (2.3), we get that

$$Q_F^\phi(v) \leq -2 \int_M S_{g,F}^0(\phi) |\nabla|^2 dv_g.$$ 

Finally since $S_{g,F}^0(\phi)$ is positive defined, it follows that $Q_F$ is negative defined on $\mathcal{L}(\phi)$. Hence

$$\text{Ind}_F(\phi) \geq n + 1.$$
3. Morse index of particular $F$-harmonic maps

3.1. Stability of the identity map. In this section we borrow ideas from [12] to show the stability of the identity map. Let $(M, g)$ be a compact manifold and consider the identity $I$ on $M$ which is obviously $F$-harmonic, the second variation formula of $I$ writes as

$$Q^F_I(v) = F'' \left( \frac{m}{2} \right) \sum_{i=1}^m \int_M \langle \nabla_{e_i} v, e_i \rangle^2 \, dv_g +$$

$$F' \left( \frac{m}{2} \right) \int_M \left[ |\nabla v|^2 - Ric_M(v, v) \right] \, dv_g.$$  

(3.1)

If $L_v$ denotes the Lie derivative in the direction of $v$, the Yano’s formula [22] leads to

$$\int_M \left[ |\nabla v|^2 - Ric_M(v, v) \right] \, dv_g = \int_M \left[ \frac{1}{2} |L_v g|^2 - (div(v))^2 \right] \, dv_g.$$  

(3.2)

Now if $(e_i)_i$ is an orthonormal basis on $M$ which diagonalizes $L_v g$ we obtain as in [12] that

$$|L_v g|^2 \geq \frac{4}{m} (div(v))^2$$  

(3.3)

therefore by (3.1), (3.2) and (3.3) we infer that

$$Q^F_I(v) \geq \frac{1}{m} \left( F'' \left( \frac{m}{2} \right) + (2 - m) F' \left( \frac{m}{2} \right) \right) \int_M (div(v))^2 \, dv_g.$$  

(3.4)

We deduce the following proposition:

**PROPOSITION 1.** Let $(M, g)$ be a compact Riemannian manifold of dimension $m \geq 3$. Suppose that

$$F'' \left( \frac{m}{2} \right) + (2 - m) F' \left( \frac{m}{2} \right) \geq 0.$$  

(3.5)

The identity map $I$ on $M$ is $F$-stable.

**REMARK 2.** $F(t) = \frac{1}{2-m} e^{(2-m)t} + C$, where $C \geq \frac{1}{m-2}$ is a constant, fulfills the condition (3.3).

3.2. Morse index of the identity map. Now we are interested by the identity map $I$ on $M$. Let $C$ and $K$ denote the space of conformal vector fields and the space of Killing vector fields on $M$ respectively.

**PROPOSITION 2.** Let $(M, g)$ be a compact $m$-dimensional manifold $(m \geq 3)$. Suppose that

$$\frac{m - 2}{m} F' \left( \frac{m}{2} \right) - F'' \left( \frac{m}{2} \right) > 0.$$  

(3.6)
then $\text{Ind}_F (I) \geq \dim (C/K)$.

**Proof.** Plugging (3.1) in (3.2), we get

$$Q^F_I (v) = F'' \left( \frac{m}{2} \right) \int_M \text{div}(v)^2 \, dv_g +$$

$$F' \left( \frac{m}{2} \right) \int_M \left[ \frac{1}{2} |L_v g|^2 - \text{div}(v)^2 \right] \, dv_g$$

(3.7)

and if $v$ is a conformal vector field on $M$ then (see the proof of Theorem 2 in [15])

(3.8)

$$L_v g = - \frac{2}{m} \text{div}(v) g$$

where $m = \dim(M)$. So (3.7) becomes

$$Q^F_I (v) = \left( F'' \left( \frac{m}{2} \right) + \frac{2 - m}{m} F' \left( \frac{m}{2} \right) \right) \int_M \text{div}(v)^2 \, dv_g.$$  

If $\frac{m - 2}{m} F' \left( \frac{m}{2} \right) - F'' \left( \frac{m}{2} \right) > 0$, then

$$Q^F_I (v) \leq 0.$$  

The equality holds if $\text{div}(v) = 0$ which means by (3.8) that $v$ is a Killing vector field. Then on the quotient space $C/K$, we have

$$Q^F_I (v) < 0$$

i.e.

$$\text{Ind}_F (I) \geq \dim(C/K).$$

□

**Remark 3.** $F(t) = \frac{m}{m-2} e^{\frac{m-2}{m-2}t} + Ct$, where $C > 0$ is a constant, fulfills the condition (3.6).

**3.3. Morse index of the homothetic map.** Let $\phi : (M, g) \to (N, h)$ be a homothetic map i.e. $\phi^* h = k^2 g$ where $k \in R$. Clearly $|d\phi|^2_h = mk^2$, where $m = \dim(M)$, in that case the $F$-tension $\tau_F (\phi)$ is proportional to the mean curvature of $\phi$ so $\phi$ is $F$-harmonic if and only if $\phi$ is minimal immersion.

**Proposition 3.** Let $\phi : (M, g) \to (N, h)$ be an $F$-harmonic homothetic map. Then we have

$$\text{Ind}_F (\phi) \geq \text{Ind}_F (I)$$

where $I$ is the identity map of $M$. 

Proof. The second variation of $\phi$ in direction of a vector field $v$ reduces to

$$Q^F_\phi (v) = F'' \left( \frac{mk^2}{2} \right) \int_M \langle \nabla v, d\phi \rangle^2 \phi^{-1} \mathcal{F}_N \, dv_g$$

(3.9) \quad + F' \left( \frac{mk^2}{2} \right) \int_M \left[ |\nabla v|^2 - \sum_{i=1}^{m} \langle R^N (v, d\phi (e_i)) d\phi (e_i), v \rangle \right] \, dv_g

where $\{e_i\}_{1 \leq i \leq m}$ is an orthonormal basis on $M$. Let $\Gamma^T (\phi)$ the subspace of $\Gamma (\phi^{-1}TN)$, consisting of vector fields on $N$ of the form $d\phi (X)$ where $X$ is a vector field on $M$. The restriction of $Q^F_\phi$ to $\Gamma^T (\phi)$, where $I$ is the identity map on $M$, is given by (see Lemma 2.5 [15])

$$Q^I_\phi (d\phi (X)) = k^2 Q^I_\phi (X).$$

(3.10)

As in [15] and since $\nabla d\phi$ takes its value in the normal fiber bundle of $N$, we get

$$\langle \nabla_X d\phi (Y), d\phi (Z) \rangle = \langle (\nabla d\phi) (X, Y), Z \rangle + \langle d\phi (\nabla_X Y), d\phi (Z) \rangle$$

(3.11)

$$= k^2 \langle \nabla_X Y, Z \rangle.$$

Replacing (3.11) and (3.10) in (3.9) we deduce that

$$Q^F_\phi (d\phi (X)) = F'' \left( \frac{mk^2}{2} \right) k^2 \int_M \langle \nabla e_i X, e_i \rangle^2 \, dv_g + F' \left( \frac{mk^2}{2} \right) k^2 Q^I_\phi (X)

= k^2 Q^F_\phi (X).$$

\[\square\]

Propositions (2) and (3) lead to

Corollary 1. Let $\phi : (M, g) \rightarrow (N, h)$ be an $F$-harmonic homothetic map. Suppose that

$$m - 2 \cdot \frac{m}{m F' \left( \frac{m}{2} \right) - F'' \left( \frac{m}{2} \right)} > 0$$

where $m = \dim (M) \geq 3$.

Then

$$\text{Ind}_F (\phi) \geq \dim (C/K).$$

We can deduce an estimation to the $F$-index of an homothetic $F$-harmonic from Theorem [1].

Consider $\phi : (M, g) \rightarrow (S^n, \text{can})$ an homothetic map i.e. $\phi^*\text{can} = k^2 g, k \in R$; where $S^n$ denotes the unit Euclidean $n$-dimensional sphere.
endowed with the canonical metric $\text{can}$. The $F$-stress-energy tensor given by (2.1) writes

$$S_{g}^{F}(\phi) = F'(\frac{|d\phi|^{2}}{2})|d\phi|^{2}g - 2\left(F'(\frac{|d\phi|^{2}}{2}) + F''\left(\frac{|d\phi|^{2}}{2}\right)\right)|d\phi|^{2}\frac{m}{2}g$$

$$= \left(1 - \frac{2}{m}\right)F'(\frac{|d\phi|^{2}}{2}) - \frac{|d\phi|^{2}}{m}F''\left(\frac{|d\phi|^{2}}{2}\right)|d\phi|^{2}g.$$ 

So $S_{g}^{F}(\phi)$ will be positive defined if

$$\left(1 - \frac{2}{m}\right)F'(\frac{|d\phi|^{2}}{2}) - \frac{|d\phi|^{2}}{m}F''\left(\frac{|d\phi|^{2}}{2}\right) > 0.$$ 

As a consequence of Theorem 1, we have

**Proposition 4.** Let $\phi$ be an homothetic $F$-harmonic map from a compact $m$-Riemannian manifold $(M, g)$ $(m \geq 3)$ into the Euclidean sphere $S^{n}$. Suppose that

$$\left(1 - \frac{2}{m}\right)F'(\frac{|d\phi|^{2}}{2}) - \frac{|d\phi|^{2}}{m}F''\left(\frac{|d\phi|^{2}}{2}\right) > 0.$$ 

Then the Morse index of $\phi$, $\text{Ind}_{F}(\phi) \geq n + 1$.

**Remark 4.** The function $F(t) = \frac{m^{2}}{m-2}e^{\frac{m-2}{2}t}$, with $m \geq 3$ fulfills the condition (3.12) for homothetic maps $\phi : (M, g) \to (S^{n}, \text{can})$ i.e. $\phi^{\ast}\text{can} = k^{2}g$ provided that $k^{2} < m$.

**Remark 5.** The space $C$ of conformal vector fields on the unit Euclidean sphere $S^{n}$ is of dimension $\frac{1}{2}(n+1)(n+2)$ and that of Killing vector fields $K$ is of dimension $\frac{1}{2}n(n+1)$. Then $\dim(C/K) = n + 1$. So we recover the result given by Corollary 7.

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DEPT. OF MATHEMATICS, FACULTÉ DES SCIENCES, UNIVERSITÉ ABOU-BELKAÏD, TLEMÇEN
E-mail address: m_benalili@mail.univ-tlemcen.dz
E-mail address: h_benallal@mail.univ-tlemcen.dz