Performing Quantum Measurement in Suitably Entangled States
Originates the Quantum Computation Speed Up

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Abstract

We introduce a local concept of speed-up applicable to intermediate stages of a quantum algorithm. We use it to analyse the complementary roles played by quantum parallel computation and quantum measurement in yielding the speed-up. A severe conflict between there being a speed-up and the many worlds interpretation is highlighted.

I. INTRODUCTION

Why quantum computation can be more efficient than its classical counterpart is a fundamental problem that has already attracted significant attention (Ekert and Jozsa, 1997; Kitaev, 1997). The reason has naturally been sought in the special features of quantum mechanics exploited in quantum computation, like state superposition, entanglement and quantum interference. For what concerns quantum measurement, until now it has been thought to perform the following functions:

• of course, accessing the outcome of the quantum computation process;

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• selecting that outcome in a random way. In fact, in the case of some algorithms, a certain number of different outcomes – obtained through repetition of the overall algorithm – is needed to identify the problem solution. Randomness assures that, by repeating the algorithm, we do not always obtain the same outcome.

We will show that quantum measurement also plays a well defined role in yielding the speed-up. This is due to the non-causal principle that the measurement outcome is always a single eigenvalue of the measurement basis. Given a suitable condition, this principle imposes a system of algebraic equations representing the problem to be solved (or the hard part thereof), whereas the measurement outcome, by satisfying that system, yields the problem solution.

The condition is that the state before measurement is (suitably) entangled with respect to the contents of two computer registers\(^1\). Measuring either content, through the single outcome principle, imposes and solves the above mentioned algebraic system.

The time required to measure the content of either register is linear in the number of the register qubits, and it is not affected by registers entanglement – entanglement is interaction free. All the problem complexity is represented in this entanglement and does not affect measurement time: this is also essential to achieve the speed-up.

We shall examine all quantum algorithms found so far under the above perspectives.

\(^1\)In the current formulation of some quantum algorithms (e.g. the seminal 1985 Deutsch’s algorithm and 1996 Grover’s algorithm), this entanglement does not appear. Likely, this has prevented understanding the role played by the interplay between quantum parallel computation and quantum measurement in yielding the speed-up. We will reformulate such algorithms by physically representing both the problem and the solution algorithm: in this way the required entanglement appears.
II. OVERVIEW

For the sake of clarity, we shall provide an overview of our explanation of the speed-up, applied to a simplified version of Simon’s algorithm. All details are deferred to the subsequent Sections.

The problem is as follows. Given \( B \sim \{0,1\} \) and \( B^n \sim \{0,1,...,N-1\} \sim \mathbb{Z}_N \) with \( N = 2^n \), we consider a function \( f(x) \) from \( B^n \) to \( B^n \). The argument \( x \) ranges over \( 0,1,..., N-1; \) \( n \) is said to be the size of the problem. \( f(x) \) has the following properties:

- it is a 2-to-1 function, namely for any \( x \in B^n \) there is one and only one second argument \( x' \in B^n \) such that \( x \neq x' \) and \( f(x) = f(x') \);
- such \( x \) and \( x' \) are evenly spaced by a constant value \( r \), namely: \( |x - x'| = r \);
- given a value \( x \) of the argument, computing the corresponding value of \( f(x) \) requires a time polynomial in \( n \) [i.e. \( \text{poly}(n) \)]; whereas, given a value \( f \) of the function, finding an \( x \) such that \( f(x) = f \), requires a time exponential in \( n \) [i.e. \( \text{exp}(n) \)] with any known classical algorithm; the function is “hard to reverse”.

Besides knowing the above properties, the problem solver can access a quantum computer that, given any input \( x \), produces the output \( f(x) \) in \( \text{poly}(n) \) time. The problem is to find \( r \) in an efficient way, which turns out to require \( \text{poly}(n) \) time rather than the \( \text{exp}(n) \) time required by classical computation.

The computer operates on two registers \( X \) and \( F \), each of \( n \) qubits; \( X \) contains the argument \( x \) and \( F \) – initially set at zero – will contain the result of computing \( f(x) \). We denote by \( \mathcal{H}_{XF} \equiv \text{span}\{|x\rangle_X,|y\rangle_F\} \), with \( (x,y) \) running over \( B^n \times B^n \), the Hilbert space of the two registers. The initial state of these registers is

\[
|\varphi,t_0\rangle_{XF} = |0\rangle_X |0\rangle_F
\]

They are, so to speak, blank. By using standard operations like the Hadamard transform,
and the quantum computer to perform function evaluation (see IV for details), we obtain in poly($n$) time the following registers state (indexes are as in IV):

$$|\varphi, t_2\rangle_{XF} = \frac{1}{\sqrt{N}} \sum_x |x\rangle_X |f(x)\rangle_F,$$  \hspace{1cm} (2)

with $x$ running over $0, 1, ..., N - 1$. This is of course the result of “parallel quantum computation”.

Let us designate by $[F]$ the number stored in register $F$ – by $[X]$ the number stored in $X$. We measure $[F]$ in state (2) \[F\]. Given the character of $f(x)$, the measurement outcome has the form:

$$|\varphi, t_3\rangle_{XF} = \frac{1}{\sqrt{2}} (|\overline{x}\rangle_X + |\overline{x} + r\rangle_X) |\overline{f}\rangle_F,$$  \hspace{1cm} (3)

where $\overline{f}$ is the value of the measured observable, and $f(\overline{x}) = f(\overline{x} + r) = \overline{f}$. The remaining part of the algorithm serves to extract $r$ out of the above superposition, by using quantum interference within register $X$, measurement of $[X]$ and repetition of the overall process for a sufficient number of times (see Section IV.A).

However, we will see that, under a reasonable criterion, the speed-up has already been achieved by preparing state (3), thus by performing the two stages of quantum parallel computation and quantum measurement. How the two stages interplay will be shown afterwards.

Since the speed-up is referred to an efficient classical computation that yields the same result, the quantum character of state (3) constitutes a difficulty. This is avoided by introducing a local definition of speed-up. We should explicitly distinguish between a quantum

\[2\]This intermediate measurement can be skipped, but we will show (Section IV.A) that either performing or skipping it is mathematically equivalent. It has been introduced by Ekert and Jozsa (1998) to clarify the way Shor’s and Simon’s algorithms operate; it also serves to clarify the speed-up.
state as a physical thing and its symbolic description. By this we mean that, for exam-
ple, the quantum state described by (2) should be considered a preparation in a lab, while
expression (2), namely the “string” – in the acception of formal languages –

\[ \frac{1}{\sqrt{N}} \sum_x |x\rangle_X |f(x)\rangle_F , \]

once all \( x \) and \( f(x) \) appearing in the sum are substituted with the proper numerical values,
is the symbolic description of the preparation.

Let us consider the quantum computation cost (as a function of problem size) of trans-
forming a quantum state \( A \) into a quantum state \( B \). This is benchmarked with the classical
computation cost of transforming the symbolic description of \( A \) into the symbolic description
of \( B \). The speed-up is the difference between the two. Cost is usually expressed as computa-
tion time versus problem size, provided that all other computing resources are comparable.
This definition can be applied to intermediate stages of a quantum algorithm, and coincides
with the usual one if applied to the whole algorithm.

First, we will use it to focus on stage (1) through (3) of Simon’s algorithm, in order to
check that the speed-up, as defined above, originates here. This will also serve to clarify the
algebraic character of quantum computation (Section III).

On the one hand, we should assess the time required to produce by classical computation
the symbolic description (3) starting from the symbolic description (1) (i.e. from scratch).
Naturally, \( \bar{x} \), \( \bar{x} + r \), and \( \overline{T} \) should be considered proper numerical values. Finding them
requires solving the following system of numerical algebraic equations:

\[ f(x_1) = f(x_2) , \]

\[ x_1 \neq x_2 . \]

Fig. 1

It is useful to resort to the network representation of system (4) – fig. 1. The gate
c \( (x_1, x_2) \) imposes the condition that if \( x_1 \neq x_2 \) then the output is 1, if \( x_1 = x_2 \) then the
output is 0, and vice-versa. To impose \( x_1 \neq x_2 \), the gate output must be set at 1. Note that this network is just a way of representing a system of algebraic equations (useful to highlight a topological feature, as we will see): time is not involved and gates are simply logical constraints. Each of the two gates \( f(x) \) imposes that if the input is \( x \) then the output is \( f(x) \) or, conversely, if the output is \( f \) then the input is an \( x \) such that \( f(x) = f \).

This network is hard to satisfy by classical means. Because of the looped network topology, finding a valuation of \( x_1, x_2 \) and \( f \) which satisfies the network requires reversing \( f(x) \) at least once, which takes by assumption \( \exp(n) \) time.

On the other hand, the time required to produce the quantum state (3) with Simon’s algorithm, is the sum of the \( \text{poly}(n) \) time required to produce state (2) by means of parallel quantum computation, and the time required to measure \([F]\). This latter is independent of state (2) entanglement and is simply linear in \( n \), the number of qubits of register \( F \). The overall time is \( \text{poly}(n) \). The speed-up – as defined above – has already been achieved (in Section IV we will verify from another standpoint that the remaining part of the algorithm cannot possibly host any speed-up).

Let us see how the speed-up is originated by the interplay between quantum parallel computation and quantum measurement. This will be shown at a conceptual level herebelow, and formally in Section III.

We shall focus on stage (2) through (3). Analyzing the classical computation cost of deriving the symbolic description (3) from (2), provides an appropriate background for showing how quantum measurement operates.

Description (2) can be conveniently visualized as the print-out of the sum of \( 2^n \) tensor products (see eq. 2). Loosely speaking, two values of \( x \) such that \( f(x_1) = f(x_2) \), must be \( \exp(n) \) spaced. Otherwise such a pair of values could be found in \( \text{poly}(n) \) time by classically computing \( f(x) \) on a \( \text{poly}(n) \) number of consecutive arguments.
The point is that the print-out would create a Library of Babel\footnote{From the story “The Library of Babel” by J.L. Borges.} effect. Even for a small $n$, it would fill the entire galaxy with, say, $\ldots |x_1\rangle_X |f(x_1)\rangle_F \ldots$ here, and $\ldots |x_2\rangle_X |f(x_2)\rangle_F \ldots$ [such that $f(x_1) = f(x_2)$] in Alpha Centauri. Finding such a pair of print-outs would still require $\exp(n)$ time.

By the way, this shows that a capability of directly accessing the so-called “exponential wealth” of parallel quantum computation, namely the above print-out (the fact this is impossible is sometimes “regretted”) would be nullified by its “exponential dilution”.

Let us see the way quantum measurement operates on the result of quantum parallel computation. It \textit{distills} the desired pair of arguments in a time linear in $n$ (the number of qubits of register $F$). In fact, it does more than randomly selecting one measurement outcome; by selecting a single outcome – a definite value of $f$ – it \textit{performs a logical operation crucial for solving the problem} (i.e. selecting the two values of $x$ associated with the value of that outcome). So to speak, quantum measurement would perform as an “exponentially efficient” librarian. Its computational role, complementary to the production of the parallel computation outputs, appears to be self-evident (given a proper understanding of the context).

Noticeably, this way of operating of quantum measurement “strongly” violates the principle of causality (i.e. that all events have an antecedent cause), in the following sense. It is not only the case that a measurement outcome occurs at random (this would be the usual violation), it is also computed – and physically determined – in a non-causal way.

This is due to the fact that a quantum evolution undergoing selective measurement is affected by \textit{both} the initial conditions and the final condition that there is a single measurement outcome. This outcome is therefore caused by both ends, which naturally violates the principle of antecedent causes. This appears to be an interesting and completely new physical notion, highlighted by a computational context. It will be further expounded in
Sections III and IV.

**III. QUANTUM ALGEBRAIC COMPUTATION**

We will show that quantum computation is an entirely new paradigm where there is isomorphism between the *algebraic definition* of a solution and its *physical determination*. This is unlike the usual notion of algorithm, where the isomorphism is between the definition of a sequential computational procedure and its dynamical implementation. As we will see, in quantum computation the definition of a solution is non-sequential and the corresponding physical determination is non-dynamical in character.

Let us show that the state after measurement (3) (which we consider for short to be the problem solution) solves a system of algebraic equations imposed by measuring \([F]\) in state (2). This is parallel to solving, by classical computation, the system of algebraic equations (4).

It is useful to apply von Neumann’s quantum measurement model. Besides registers \(X\) and \(F\), we must consider the pointer of a measurement apparatus that will give the result of measuring \([F]\). Let \(|\psi, t_2\rangle_{XF} = |\varphi, t_2\rangle_{XF} |0\rangle_P\) be the state before measurement of the whole system; \(|0\rangle_P\) denotes the pointer initial state.

von Neumann’s model has two steps. The first, corresponding to the measurement interaction, is a unitary evolution \(U\) leading from the state before measurement to a “provisional description” of the state after measurement:

\[
|\psi, t_2\rangle_{XF} = \frac{1}{\sqrt{N}} \sum_x |x\rangle_X |f(x)\rangle_F |0\rangle_P \xrightarrow{U} \]

\[
|\psi, t_3\rangle_{XF} = \frac{1}{\sqrt{N}} \sum_i (|x_i\rangle_X + |x_i + r\rangle_X) |f_i\rangle_F |f_i\rangle_P ,
\]

with \(f_i = f(x_i) = f(x_i + r)\) running over all the values taken by \(f(x)\). As stated before, measurement time \(t_3 - t_2\) is linear in \(n\). As well known, description (5) yields the appropriate correlation between \([F]\) and the indication of the pointer, but it conflicts with the empirical evidence that both (i.e. \(f_i\)) must have a definite measured value.
The second step of von Neumann’s model can be seen as a *reinterpretation* of description (5): the tensor products appearing in the right hand of (5) should be considered mutually exclusive measurement outcomes (still at the same time \( t_3 \)) with probability distribution the square modules of the respective probability amplitudes\(^4\). This yields a measurement outcome of the form:

\[
\frac{1}{\sqrt{2}} (|\vec{x}\rangle_X + |\vec{x} + r\rangle_X) \langle \vec{f} | \langle \vec{f} | \phi_{XF} ,
\]

where \( \vec{f} \) is a definite value. In the following, we will disregard the mathematically redundant factor \( |\vec{f}\rangle_F \).

Note that this reinterpretation, as it is, does not involve the notion of time and is “transparent” to dynamics, in the sense it does not affect measurement time \( t_3 - t_2 \). Interestingly, the selection of \( |\vec{x}\rangle_X + |\vec{x} + r\rangle_X \), essential to achieve the speed-up, stems from the reinterpretation, i.e. by the non-causal principle that there must be a single measurement outcome.

This principle is usually applied in a procedural way: by performing a sequence of typographic operations (in the acceptation of formal languages) on the symbolic description of the state before measurement, we obtain the symbolic description of a possible state after measurement.

This procedural approach conceals the algebraic character of the reinterpretational step. In fact, by applying a system of algebraic equations to an “unknown” vector \( |\varphi\rangle_{XF} \) of the Hilbert space of the two registers \( \mathcal{H}_{XF} \), we obtain the same result. The form of this vector is naturally \( |\varphi\rangle_{XF} = \sum_{x,y} \alpha_{x,y} |x\rangle_X |y\rangle_F \), where \( (x, y) \) runs over \( B^n \times B^n \), and \( \alpha_{x,y} \) are complex variables independent of each other up to normalization: \( \sum_{x,y} |\alpha_{x,y}|^2 = 1 \).

Let \( \{|f\rangle_F\} \) be the set of the eigenstates of register \( F \) and \( |\vec{f}\rangle_F \in \{|f\rangle_F\} \) be the eigenstate selected by the quantum measurement of \([F]\). The first equation is

\[
|\vec{f}\rangle_F \langle \vec{f} | |\varphi\rangle_{XF} = |\varphi\rangle_{XF}, \tag{6}
\]

\(^4\)It is the same in decoherence theory, where the terms of a mixture become mutually exclusive measurement outcomes.
$|\overline{f}\rangle_F \langle \overline{f}|_F$ is naturally the projector on the Hilbert subspace $\mathcal{H}_{XF}^\overline{f} = \text{span}\{ |x\rangle_X, |\overline{f}\rangle_F \}$ with $x$ running over $B^n$ and $|\overline{f}\rangle_F$ being fixed; a $|\varphi\rangle_{XF}$ satisfying eq. (6) is a free linear combination of all the tensor products of $\mathcal{H}_{XF}$ containing $|\overline{f}\rangle_F$:

$$|\langle \varphi|_{XF} |\varphi,t_2\rangle_{XF}| \text{ must be maximum; (7)}$$

$|\varphi\rangle_{XF}$ satisfying equations (6) and (7) becomes the projection of $|\varphi,t_2\rangle_{XF}$ on $\mathcal{H}_{XF}^\overline{f}$:

$$|\varphi\rangle_{XF} = \sqrt{\frac{N}{2}} |\overline{f}\rangle_F \langle \overline{f}|_F |\varphi,t_2\rangle_{XF};$$

this means that $|\overline{f}\rangle_F$ has “dragged” all the tensor products of $|\varphi,t_2\rangle_{XF}$ containing it. The solution of the two equations is indeed the state after measurement $|\varphi,t_3\rangle_{XF}$ given by eq. (3), formerly derived in the procedural way.

To sum up, solving the system of algebraic equations (6-7) is equivalent to performing the reinterpretational step of von Neumann’s model. This does not affect the first step. In other words, performing the first step gives “for free” (without incurring any further dynamical cost) the solution of (6-7). Under the current definition of speed-up, solving equations (6-7) is equivalent to solving the system of algebraic equations (4), namely the classically hard part of the problem.

This isomorphism between algebraic definition and physical determination of a solution, blurs a long-standing distinction of mathematical logic between the notions of “implicit definition” and “computation”.

In problem solving, a problem implicitly defines its solution. For example, let us consider factorization: given the known product $c$ of two unknown prime numbers $x$ and $y$, the

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5In any way, the process of satisfying equations (6-7) must be comprised within the measurement interaction, namely in the time interval $[t_2, t_3]$, which is linear in $n$.

6If the problem admits no solution, we should consider the meta-problem whether the problem admits a solution (which always admits a solution).
numerical algebraic equation \( x \cdot y = c \) implicitly defines the values of \( x \) and \( y \) that satisfy it. The system of algebraic equations (4) constitutes a similar example.

In the classical framework, an implicit or algebraic definition of a solution does not prescribe how to find it. In order to find the solution, the implicit definition must be changed into an equivalent constructive definition, namely into an algorithm (which is always possible with the problems we are dealing with). An algorithm is an abstraction of the way things can be constructed in reality – inevitably in a model thereof – and prescribes a computation process that builds the solution.

The current notion of algorithm still reflects the way things can be constructed in the traditional classical reality – namely through a sequential (dynamical/causal) process. Turing machine computation and the Boolean network representation of computation are examples of sequential (dynamical/causal) computation. In fact, an algorithm specifies a causal propagation of logical implication from a completely defined input to a completely defined output which contains the solution. It is thus meant to be executable through a dynamical/causal process\(^7\).

We can see that a feature of quantum computation essential to achieve the speed-up, namely selective measurement, is extraneous to the causal notions of both algorithm and dynamics. In particular, quantum computation is not “quantum Turing machine computation”. Out of habit, we will keep the name of algorithm even in the case of a quantum computation yielding a speed-up.

**IV. FOUR TYPES OF QUANTUM ALGORITHMS**

\(^7\)Classical analog computation and classical nondeterministic computation are not considered here to be fundamentally different, being still performed through causal processes (classical randomness can be considered to be pseudo-randomness). The important thing is that both differ from quantum computation in their inability to host the non-causal effects we are dealing with.
A. Modified Simon’s algorithm

Without significant loss of generality, we will follow the simplified version (Cleeve et al., 1997) of Simon’s algorithm (1994). For the sake of illustration, the following table gives a trivial example.

| x | f(x) |
|---|-----|
| 0 | 0   |
| 1 | 1   |
| 2 | 0   |
| 3 | 1   |

Table I

The block diagram of the simplified algorithm is given in Fig. 2 – we should disregard /F for the time being.

Fig. 2

The blocks in Fig. 2 represent the following transformations.

- The $f(x)$ transform (a reversible Boolean gate in the time-diagram of computation) leaves the content of register $X$ unaltered, so that an input $x$ is repeated in the corresponding output, and computes $f(x)$ adding it to the former content of register $F$ (which was set to zero). If the input state is not sharp but is a quantum superposition, the same transformation applies to any tensor product appearing in it.

- $H$ is the Hadamard transform defined as follows: $|0\rangle_i \xrightarrow{H} \frac{1}{\sqrt{2}} (|0\rangle_i + |1\rangle_i)$, $|1\rangle_i \xrightarrow{H} \frac{1}{\sqrt{2}} (|0\rangle_i - |1\rangle_i)$. In the case of a register of $n$ qubits, containing the number $\overline{x}$, it yields $|\overline{x}\rangle_X \xrightarrow{H} \frac{1}{\sqrt{N}} \sum_x (-1)^{\overline{x} \cdot \overline{x}} |x\rangle_X$, where $N = 2^n$, $x$ ranges over $0, 1, ..., N - 1$, and $\overline{x} \cdot \overline{x}$ denotes the module 2 inner product of the two numbers in binary notation (seen as row matrices).
• $M$ represents the action of measuring the numerical content of a register.

We shall give some significant computational states (also for the example). The preparation is $|\varphi, t_0\rangle_{XF} = |0\rangle_X |0\rangle_F$. After performing Hadamard on $X$ and function evaluation, we have:

$$|\varphi, t_2\rangle_{XF} = \frac{1}{\sqrt{N}} \sum_x |x\rangle_X |f(x)\rangle_F = \frac{1}{2} (|0\rangle_X |0\rangle_F + |1\rangle_X |1\rangle_F + |2\rangle_X |0\rangle_F + |3\rangle_X |1\rangle_F);$$

this is the state before the intermediate measurement of $[F]$.

Measuring $[F]$ yields, say, $\overline{f} = 1$; the state after measurement is consequently:

$$|\varphi, t_3\rangle_{XF} = \frac{1}{\sqrt{2}} (|\overline{x}\rangle_X + |\overline{x} + r\rangle_X) |\overline{f}\rangle_F = \frac{1}{\sqrt{2}} (|1\rangle_X + |3\rangle_X) |1\rangle_F,$$

Ekert and Jozsa (1998) have shown that quantum entanglement between qubits is essential for providing a speed-up, in terms of time or other computing resources, in the class of quantum algorithms we are dealing with. After measuring $f(x)$, the state of the two registers becomes factorizable, and all entanglement is destroyed. The remaining actions, performed on register $X$, use interference (which generates no entanglement) to “extract” $r$ out of the superposition $\frac{1}{\sqrt{2}} (|\overline{x}\rangle_a + |\overline{x} + r\rangle_a)$. We must conclude from another standpoint that the speed-up has been achieved by preparing $|\varphi, t_3\rangle_{XF}$.

Performing again $H$ on register $X$ yields:

$$|\varphi, t_4\rangle_{XF} = \frac{1}{\sqrt{2N}} \sum_z (-1)^{\overline{z} \cdot \overline{x}} [1 + (-1)^{r \cdot z}] |z\rangle_X |\overline{f}\rangle_F.$$
satisfy this system (r is “easily” extracted from this superposition). The overall procedure requires poly(n) time, whereas solving equations (4) by classical computation would require \(\exp(n)\) time.

Finally, let us show that performing or skipping the “intermediate measurement” of \([F]\) in \(|\varphi, t_2\rangle_{XF}\), is equivalent. If we skip it (in fig. 2, the symbol \(M\) on \(F\) should be erased), the state of registers \(X\) and \(F\) at time \(t_4\) remains entangled. In the case of the example, this is

\[
|\varphi, t_4\rangle_{XF} = \frac{1}{2} \left[ (|00\rangle_X + |01\rangle_X) |0\rangle_F + (|00\rangle_X - |01\rangle_X) |1\rangle_F \right]
\]  

(8)

A direct measurement of register \(X\) qubits is equivalent to measuring first the content of \(X\) with respect to the following basis:

\[
\{|00\rangle + |01\rangle, |00\rangle - |01\rangle, |10\rangle + |11\rangle, |10\rangle - |11\rangle\}
\]

In fact, one can see that introducing this previous measurement is a redundant operation with respect to measuring the individual qubits. Because of state (8) entanglement, this measurement is equivalent to measuring \([F]\) instead.

The result is, say:

\[
\frac{1}{\sqrt{2}} (|00\rangle_X - |01\rangle_X) |1\rangle_F.
\]  

(9)

It is useful to resort to the notion of wave function collapse – by the way without having to adhere to it. We are just discussing a mathematical equivalence, and the notion of collapse is a mathematically legitimate one. According to von Neumann, collapse on state (9) can be backdated in time provided that its result (9) undergoes in an inverted way the same transformation undergone by the time-forward evolution (the usual one). It can readily be seen that backdating this collapse at time \(t_2\) is equivalent to having performed the intermediate measurement of \([F]\) in state (2).

Finally, it should be noted that \([F]\) does not actually need to be measured. This measurement is a virtual consequence of measuring only \([X]\) at the end.
B. Shor’s algorithm

The problem of factoring an integer \( L \) – the product of two unknown primes – is transformed into the problem of finding the period of the function \( f(x) = a^x \mod L \), where \( a \) is an integer between 0 and \( L - 1 \), and is coprime with \( L \) (Shor, 1994; Cleve et al., 1997). Figure 2 can also represent Shor’s algorithm, provided that \( f(x) \) is as defined above and the second Hadamard transform \( H \) is substituted by the discrete Fourier transform \( F \). The state before measurement has the form \( |\varphi, t_2\rangle_{XF} = \frac{1}{\sqrt{L}} \sum_x |x\rangle_X |f(x)\rangle_F \). Measuring or not measuring \( f(x) \) in \( |\varphi, t_2\rangle_{XF} \) is still equivalent. By measuring it, the above quantum state changes into the superposition

\[
\overline{k} (|\varphi\rangle_X + |\varphi + r\rangle_X + |\varphi + 2r\rangle_X + ...) |f\rangle_F,
\]

where \( f(\varphi) = f(\varphi + r) = ... = \overline{f} \), and \( \overline{k} \) is a normalization factor.

The second part of the algorithm generates no entanglement and serves to “extract” \( r \) in polynomial time, by using Fourier-transform interference and auxiliary, off line, mathematical considerations. Under the current assumptions, the quantum speed-up has been achieved by preparing state (10): the discussion is completely similar to that of the previous algorithm.

C. Deutsch’s 1985 algorithm

The seminal 1985 Deutsch’s algorithm has been the first demonstration of a quantum speed-up. In its usual form, this algorithm yields a deterministic output, apparently ruling out the role of selective measurement. A more careful examination will show that this is not the case.

Deutsch’s algorithm, and more in general quantum oracle computing, is better seen as a competition between two players. One – the oracle – produces the problem, the other should produce the solution. Sticking to Greek tradition, we shall call the former player Sphinx, the latter Oedipus.
The competition is formalized as follows. Both players have complete knowledge of a set of functions $f_k : B^n \to B^n$ ($k$ labels the elements of the set). They can also access a computer that, set in its $k$-th mode, computes $f_k(x)$ given any input $x \in B^n$. The Sphinx chooses $k$ at random, sets the computer in its $k$-th mode and passes it on to Oedipus. Oedipus knows nothing of the Sphinx’ choice and must efficiently find $k$ by testing the computer input-output behaviour. He is naturally forbidden to inspect the computer mode. If the computer is quantum, then we speak of “quantum oracle computing”.

Deutsch’s (1985) algorithm, as modified in (Cleve et al., 1997), is as follows. Let $\{f_k\}$ be the set of all possible functions $f_k : B \to B$, namely:

| $x$ | $f_{00}(x)$ | $x$ | $f_{01}(x)$ | $x$ | $f_{10}(x)$ | $x$ | $f_{11}(x)$ |
|-----|-------------|-----|-------------|-----|-------------|-----|-------------|
| 0   | 0           | 0   | 0           | 0   | 1           | 0   | 1           |
| 1   | 1           | 1   | 1           | 1   | 0           | 1   | 1           |

$\{f_k\}$ is divided into a couple of subsets: the balanced functions, characterized by an even number of zero and one function values, thus labeled by $k = 01, 10$, and the unbalanced ones, labeled by $k = 00, 11$. Oedipus must find, with a minimum number of computer runs, whether the computer (whose mode has been randomly set by the Sphinx) computes a balanced or an unbalanced function. The algorithm is illustrated in Fig. 3(a). The computation of $f_k(x)$ is represented as a reversible Boolean gate, like in the previous algorithms but for the fact that the result of the computation is now module 2 added to the former content of register $F$.

Fig. 3(a),(b)

Let us consider the significant computational states. The preparation is the same for all $k$: $|\varphi_k, t_0\rangle_{av} = \frac{1}{\sqrt{2}} |0\rangle_a (|0\rangle_v - |1\rangle_v)$. The state at time $t_3$ before measurement, after performing function evaluation by running the computer only once (Fig. 3a), depends on the Sphinx’ choice:
if $k = 00$ then $|\psi_{00}, t_3 \rangle_{XF} = \frac{1}{\sqrt{2}} |0 \rangle_X (|0 \rangle_F - |1 \rangle_F)$

if $k = 01$ then $|\psi_{01}, t_3 \rangle_{XF} = \frac{1}{\sqrt{2}} |1 \rangle_X (|0 \rangle_F - |1 \rangle_F)$

if $k = 10$ then $|\psi_{10}, t_3 \rangle_{XF} = -\frac{1}{\sqrt{2}} |1 \rangle_X (|0 \rangle_F - |1 \rangle_F)$

if $k = 11$ then $|\psi_{11}, t_3 \rangle_{XF} = -\frac{1}{\sqrt{2}} |0 \rangle_X (|0 \rangle_F - |1 \rangle_F)$

It can be seen that measuring $[X]$ gives Oedipus’ answer (1 for balanced, 0 for unbalanced).

This algorithm is more efficient than any classical algorithm, where two runs of the computer would be required to establish the answer. However, the result is apparently reached in a deterministic way, without quantum measurement performing any selection.

This must be ascribed to the incomplete physical representation of the situation. In Sections IV.A and IV.B, we had a problem that implicitly defined its solution. The physical determination of the solution was obtained by measuring the content of a computer register in an entangled state representing the problem to be solved (or the hard part thereof). This obviously requires that the problem is physically represented\footnote{In Sections IV.A and IV.B, all knowledge of the function and ignorance about $r$ were physically represented in a superposition of the form (2).}, whereas presently an essential part of it, the Sphinx choosing the oracle mode, is not. For example, the logical if-then conditions above are external to the physical representation.

In order to complete the physical representation, we introduce the extended gate $F(k, x)$ which computes the function $F(k, x) = f_k(x)$ for all $k$ and $x$. This gate has an ancillary input register $K$ which contains $k$, namely the oracle mode [Figure 3(b) gives the extended algorithm]. This input is identically repeated in a corresponding output – to keep gate reversibility. Of course, Oedipus is forbidden to inspect register $K$. The preparation becomes

$$|\varphi, t_0 \rangle_{KXF} = \frac{1}{2\sqrt{2}} \left( (|00 \rangle_K + e^{i\delta_1} |01 \rangle_K + e^{i\delta_2} |10 \rangle_K + e^{i\delta_3} |11 \rangle_K) |0 \rangle_X (|0 \rangle_F - |1 \rangle_F) \right).$$

where $\delta_1$, $\delta_2$ and $\delta_3$ are independent random phases. The superposition $|00 \rangle_K + e^{i\delta_1} |01 \rangle_K +$
$e^{i\delta_2} |10\rangle_K + e^{i\delta_3} |11\rangle_K$ represents the fact that the Sphinx gives Oedipus the computer in a mode $k$ randomly chosen among $k = 00, 01, 10, 11$. To Oedipus, this is indistinguishable from a mixture – the above superposition is in fact a mixture represented with the method of random phases (Finkelstein, 1996).

Let us go directly to the state before measurement – see Fig. 3(b):

$$|\varphi, t_3\rangle_{KXF} = \frac{1}{2\sqrt{2}} \left[ (|00\rangle_K - e^{i\delta_3} |11\rangle_K) |0\rangle_X + (e^{i\delta_1} |01\rangle_K - e^{i\delta_2} |10\rangle_K) |1\rangle_X \right] (|0\rangle_F - |1\rangle_F).$$

(11)

It can be seen that this entangled state represents the mutual definition between the Sphinx’ choice of the computer mode $k$ and Oedipus’ answer. The problem implicitly defining its solution appears here in the form of the mutual definition of the moves of the two players. Reaching state (11) still requires one oracle run.

The action of measuring $[K]$ in state (11), equivalent to the Sphinx’ choice of the oracle mode, by imposing equations (6-7) transforms mutual definition into correlation between individual measurement outcomes (like in an EPR situation). In other words, the Sphinx’ choice of $k$ simultaneously determines the problem solution, namely Oedipus’ answer – retrievable by measuring $[X]$. In the classical framework instead, the Sphinx’ choice should necessarily be propagated to Oedipus’ answer by means of a causal process (requiring two computer runs).

Achieving the speed-up still involves measurement of an entangled state, namely selective measurement.

D. An instance of Grover’s algorithm

The rules of the game are the same as before. This time we have the set of the $2^n$ functions $f_k : B^n \rightarrow B$ such that $f_k (x) = \delta_{k,x}$, where $\delta$ is the Kronecker symbol. We shall consider

$^9$Of course, $|\varphi, t_2\rangle_{XF}$ of Section III must be changed into $|\varphi, t_3\rangle_{KXF}$ and $\mathcal{H}_{XF}$ into $\mathcal{H}_{KXF}$.
the simplest instance $n = 2$. This yields four functions $f_k(x)$, labeled $k = 0, 1, 2, 3$. Figure 4(a) gives Grover’s algorithm (1996) in the standard version provided in (Cleve et al., 1997) for $n = 2$. Without entering into detail, the complete physical representation of the state before measurement becomes (Fig. 4b):

$$\frac{1}{2\sqrt{2}} \left( |0\rangle_K |0\rangle_X + e^{i\delta_1} |1\rangle_K |1\rangle_X + e^{i\delta_2} |2\rangle_K |2\rangle_X + e^{i\delta_3} |3\rangle_K |3\rangle_X \right) (|0\rangle_F - |1\rangle_F),$$

where $\delta_1, \delta_2$ and $\delta_3$ are independent random phases.

Again, we have the mutual definition of the Sphinx’ choice and Oedipus’ answer. This is transformed into physical determination by measuring the content of either register, as in the previous oracle problem.

V. CONCLUSIONS

This work should provide a better understanding of what quantum computation is and is not at a fundamental level. It is not, as often believed, the quantum transposition of sequential-causal computation (e.g. of Turing machine computation). This is only the first stage of the quantum algorithm. In the measurement stage, there is isomorphism between the implicit definition of the problem solution and its (non-causal) physical determination. The speed-up essentially relies on the non-causal principle that the measurement outcome is a single eigenvalue of the measurement basis.

Detaching quantum computation from the notion of causal algorithm – a classical vestige – should be a precondition for pursuing further developments at a fundamental level.

For example, let us consider the possibility of exploiting particle statistics symmetrizations to achieve a quantum speed-up. Such symmetrizations can be seen as projections on symmetric (constrained) Hilbert subspaces. There is no relation between a projection and reversible Turing machine computation, namely a unitary evolution. If instead quantum computation is (properly) seen as a projection on a constrained Hilbert subspace, which
amounts to solving a problem, then we have an analogy with particle statistics symmetrizations to work with. Refs. (Castagnoli, 1998; Castagnoli et al., 1998) provide still abstract attempts in this direction.

More generally, this work highlights the essential role played by non-dynamical effects in quantum computation. Let us mention in passing that a form of quantum computation which is of geometric rather than dynamic origin has recently been provided (Jones et al., 2000). This concretely shows that there are ways of getting out of the usual quantum computation paradigm.

This work conflicts (Castagnoli et al., 2000) with the many worlds interpretation. As well known, this interpretation aims to restore the principle of causality by denying the objectivity of the (non-causal) principle that there is always a single measurement outcome (denoted by $S$ in the following). The fact we experience $S$, would be subjective in character – ascribable to a limitation of our perception.

Thus, from the one hand, $S$ would be subjective in character; from the other, it yields the speed-up, an objective consequence in a most obvious way. To avoid this contradiction, we are obliged to accept the objectivity of $S$. It can be argued that the “strong” form of causality violation highlighted in this work (Section II) is too strong to be denied by the many worlds interpretation.

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Fig. 1

Fig. 2

Fig. 3(a)

Fig. 3(b)

Fig. 4(a)

Fig. 4(b)