Article

On Some New Simpson’s Formula Type Inequalities for Convex Functions in Post-Quantum Calculus

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Abstract: In this work, we prove a new \((p, q)\)-integral identity involving a \((p, q)\)-derivative and \((p, q)\)-integral. The newly established identity is then used to show some new Simpson’s formula type inequalities for \((p, q)\)-differentiable convex functions. Finally, the newly discovered results are shown to be refinements of comparable results in the literature. Analytic inequalities of this type, as well as the techniques used to solve them, have applications in a variety of fields where symmetry is important.

Keywords: Simpson’s inequalities; post-quantum calculus; convex functions

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1. Introduction

During his lifetime, Thomas Simpson created important approaches for numerical integration and the estimation of definite integrals, which became known as Simpson’s rule (1710–1761). J. Kepler, who made a comparable calculation roughly a century before Newton, is the inspiration for Kepler’s rule. Estimations based exclusively on a three-step quadratic kernel are commonly referred to as Newton-type results because Simpson’s technique incorporates the three-point Newton–Cotes quadrature rule.

(1) Simpson’s quadrature formula (Simpson’s 1/3 rule)

\[ \int_{\pi_1}^{\pi_2} F(x) \, dx \approx \frac{\pi_2 - \pi_1}{6} \left[ F(\pi_1) + 4F\left( \frac{\pi_1 + \pi_2}{2} \right) + F(\pi_2) \right]. \]

(2) Simpson’s second formula or Newton–Cotes quadrature formula (Simpson’s 3/8 rule)

\[ \int_{\pi_1}^{\pi_2} F(x) \, dx \approx \frac{\pi_2 - \pi_1}{8} \left[ F(\pi_1) + 3F\left( \frac{2\pi_1 + \pi_2}{3} \right) + 3F\left( \frac{\pi_1 + 2\pi_2}{3} \right) + F(\pi_2) \right]. \]

The following estimation, known as Simpson’s inequality, is one of many linked with these quadrature rules in the literature:
Theorem 1. Suppose that $\mathcal{F} : [\pi_1, \pi_2] \to \mathbb{R}$ is a four-times continuously differentiable mapping on $(\pi_1, \pi_2)$, and let $\|\mathcal{F}^{(4)}\|_\infty = \sup_{x \in [\pi_1, \pi_2]} |\mathcal{F}^{(4)}(x)| < \infty$. Then, one has the inequality

$$
\left| \frac{1}{3} \left[ \frac{\mathcal{F}(\pi_1) + \mathcal{F}(\pi_2)}{2} + 2\mathcal{F} \left( \frac{\pi_1 + \pi_2}{2} \right) \right] - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \mathcal{F}(x) dx \right| \leq \frac{1}{2880} \|\mathcal{F}^{(4)}\|_\infty (\pi_2 - \pi_1)^4.
$$

Many researchers have focused on Simpson-type inequality in various categories of mappings in recent years. Because convexity theory is an effective and powerful technique to solve a huge number of problems from various disciplines of pure and applied mathematics, some mathematicians have worked on the results of Simpson’s and Newton’s type in obtaining a convex map. The novel Simpson’s inequalities and their applications in numerical integration quadrature formulations were presented by Dragomir et al. [1]. Furthermore, Alomari et al. [2] discovered a number of inequalities in Simpson’s kind of s-convex functions. The variance of Simpson-type inequality as a function of convexity was then observed by Sarikaya et al. in [3]. Refs. [4–6] can be consulted for further research on this subject.

On the other hand, quantum and post-quantum integrals for many types of functions have been used to study many integral inequalities. The authors of [7–21] employed left–right $q$-derivatives and integrals to prove HH integral inequalities and associated left–right estimates for convex and coordinated convex functions. Noor et al. proposed a generalized version of quantum integral inequalities in their paper [22]. In [23], the authors demonstrated some parameterized quantum integral inequalities for generalized quasi-convex functions. In [24], Khan et al. used the green function to prove quantum HH inequality. For convex and coordinated convex functions, the authors of [25–30] constructed new quantum Simpson’s and quantum Newton’s type inequalities. Consult [31–33] for quantum Ostrowski’s inequality for convex and co-ordinated convex functions. Using the left $(p, q)$-difference operator and integral, the authors of [34] expanded the results of [9] and demonstrated HH-type inequalities and associated left estimates. In [16], the authors discovered the right estimates of HH-type inequalities, as demonstrated in [34]. Vivas-Cortez et al. [35] recently generalized the results of [11] and used the right $(p, q)$-difference operator and integral to prove HH-type inequalities and associated left estimates.

We use the $(p, q)$-integral to establish some new post-quantum Simpson’s type inequalities for $(p, q)$-differentiable convex functions, as inspired by recent research. The newly revealed inequalities are also shown to be extensions of previously discovered inequalities.

The structure of this article is as follows. The principles of $q$-calculus, as well as other relevant topics in this subject, are briefly discussed in Section 2. The basics of $(p, q)$-calculus, as well as some recent research in this topic, are covered in Section 3. In Section 4, we prove a new $(p, q)$-integral identity involving a $(p, q)$-derivative. Section 5 describes the Simpson’s type inequalities for $(p, q)$-differentiable functions via $(p, q)$-integrals. It is also taken into account the relationship between the findings given here and similar findings in the literature. Section 6 finishes with some research suggestions for the future.

2. Preliminaries of $q$-Calculus and Some Inequalities

In this section, we revisit several previously regarded ideas. In addition, we utilize the following notation here and elsewhere (see [36]):

$$(n)_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \ldots + q^{n-1}, \quad q \in (0, 1).$$

In [37], Jackson gave the $q$-Jackson integral from 0 to $\pi_2$ for $0 < q < 1$ as follows:

$$
\int_0^{\pi_2} \mathcal{F}(x) \ dq x = (1 - q) \pi_2 \sum_{n=0}^{\infty} q^n \mathcal{F}(\pi_2 q^n) \quad (1)
$$
provided that the sum converges absolutely.

**Definition 1 ([38]).** For a function \( F : [\pi_1, \pi_2] \to \mathbb{R} \), the left \( q \)-derivative of \( F \) at \( x \in [\pi_1, \pi_2] \) is characterized by the expression

\[
\pi_1 D_q F(x) = \frac{F(x) - F(qx + (1 - q)\pi_1)}{(1 - q)(x - \pi_1)}, \quad x \neq \pi_1.
\]  

(2)

If \( x = \pi_1 \), we define \( \pi_1 D_q F(\pi_1) = \lim_{x \to \pi_1^-} \pi_1 D_q F(x) \) if it exists and it is finite.

**Definition 2 ([11]).** For a function \( F : [\pi_1, \pi_2] \to \mathbb{R} \), the right \( q \)-derivative of \( F \) at \( x \in [\pi_1, \pi_2] \) is characterized by the expression

\[
\pi_2 D_q F(x) = \frac{F(qx + (1 - q)\pi_2) - F(x)}{(1 - q)(\pi_2 - x)}, \quad x \neq \pi_2.
\]  

(3)

If \( x = \pi_2 \), we define \( \pi_2 D_q F(\pi_2) = \lim_{x \to \pi_2^+} \pi_2 D_q F(x) \) if it exists and it is finite.

**Definition 3 ([38]).** Let \( F : [\pi_1, \pi_2] \to \mathbb{R} \) be a function. Then, the left \( q \)-definite integral on \([\pi_1, \pi_2]\) is defined as

\[
\int_{\pi_1}^{\pi_2} F(x) \, d_q x = (1 - q)(\pi_2 - \pi_1) \sum_{n=0}^{\infty} q^n F(q^n \pi_2 + (1 - q^n)\pi_1) \]  

(4)

\[
= (\pi_2 - \pi_1) \int_0^1 F((1 - t)\pi_1 + t\pi_2) \, dq t.
\]

**Definition 4 ([11]).** Let \( F : [\pi_1, \pi_2] \to \mathbb{R} \) be a function. Then, the right \( q \)-definite integral on \([\pi_1, \pi_2]\) is defined as

\[
\int_{\pi_1}^{\pi_2} F(x) \, d_q x = (1 - q)(\pi_2 - \pi_1) \sum_{n=0}^{\infty} q^n F(q^n \pi_1 + (1 - q^n)\pi_2) \]

(5)

\[
= (\pi_2 - \pi_1) \int_0^1 F(t\pi_1 + (1 - t)\pi_2) \, dq t.
\]

Alp et al. [9] proved the following Hermite–Hadamard-type inequalities for convex functions via \( q \)-integral.

**Theorem 2.** For the convex mapping \( F : [\pi_1, \pi_2] \to \mathbb{R} \), the following inequality holds

\[
F\left(\frac{q\pi_1 + \pi_2}{2}, q\right) \leq \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} F(x) \, d_q x \leq \frac{qF(\pi_1) + F(\pi_2)}{2}.
\]

In [11], Bermudo et al. established the following quantum Hermite–Hadamard-type inequalities:

**Theorem 3.** For the convex mapping \( F : [\pi_1, \pi_2] \to \mathbb{R} \), the following inequality holds

\[
F\left(\frac{\pi_1 + q\pi_2}{2}, q\right) \leq \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} F(x) \, d_q x \leq \frac{F(\pi_1) + qF(\pi_2)}{2}.
\]
\[ \mathcal{F}\left(\frac{\pi_1 + \pi_2}{2}\right) \leq \frac{1}{2(\pi_2 - \pi_1)} \left[ \int_{\pi_1}^{\pi_2} \mathcal{F}(x) \, n_1 d_q x + \int_{\pi_1}^{\pi_2} \mathcal{F}(x) \, n_2 d_q x \right] \leq \frac{\mathcal{F}(\pi_1) + \mathcal{F}(\pi_2)}{2}. \]

Recently, Siricharuanun et al. [29] proved the following Simpson’s formula type inequality for convex functions.

**Theorem 4.** Let \( \mathcal{F} : [\pi_1, \pi_2] \to \mathbb{R} \) be a \( q^{\pi_2} \)-differentiable function on \((\pi_1, \pi_2)\) such that \( q^{\pi_2} D_q \mathcal{F} \) is continuous and integrable on \([\pi_1, \pi_2]\). If \( q^{\pi_2} D_q \mathcal{F} \) is convex on \([\pi_1, \pi_2]\), then we have the following inequality for \( q^{\pi_2} \)-integrals:

\[
\left| \frac{1}{(\pi_2 - \pi_1)} \int_{\pi_1}^{\pi_2} \mathcal{F}(s) \, n_2 d_q s \right| - 1\cdot 6^q \left[ \mathcal{F}(\pi_1) + q^2 [4^q \mathcal{F}\left(\frac{\pi_1 + \pi_2}{2}\right) + q \mathcal{F}(\pi_2) \right] \leq q(\pi_2 - \pi_1) \{ | q^{\pi_2} D_q \mathcal{F}(\pi_1)| [A_1(q) + A_2(q)] + | q^{\pi_2} D_q \mathcal{F}(\pi_2)| [B_1(q) + B_2(q)] \},
\]

where \( 0 < q < 1 \) and

\[
A_1(q) = \frac{2q^2[2]^2 + [6]^2 q \left( [6]_q - [3]_q \right)}{[2]^3 [3]_q [6]^3_q},
\]

\[
B_1(q) = \frac{2q^2 [3]_q [6]^2_q - q^2}{[2]^3 [3]_q [6]^3_q} + \frac{1}{[2]^3} \left( q + q^2 - q^2 + 2q \right),
\]

\[
A_2(q) = \frac{2q^2 [5]^3_q}{[2]^3 [3]_q [6]^3_q} + \frac{[6]_q \left( 1 + [2]^2_q \right) - [3]_q [5]_q \left( 1 + [2]^2_q \right)}{[2]^3 [3]_q [6]^3_q},
\]

\[
B_2(q) = \frac{2q^2 [5]^3_q [6]^2_q [3]_q - q^2 [5]^3_q}{[2]^3 [3]_q [6]^3_q} \frac{1}{[2]^3} \left( q + q^2 \right) - \frac{q^2}{[2]^3 [3]_q [6]^3_q} - \frac{q^2}{[2]^3 [3]_q [6]^3_q} - \frac{q^2}{[3]_q}.
\]

3. Post-Quantum Calculus and Some Inequalities

In this section, we review some fundamental notions and notations of \((p, q)\)-calculus. The \([n]_{p,q}\) is said to be \((p, q)\)-integers and expressed as:

\[
[n]_{p,q} = \frac{p^n - q^n}{p - q}
\]

with \( 0 < q < p \leq 1 \). The \([n]_{p,q}\)! and \( \left[ \begin{array}{c} n \\ k \end{array} \right] \) are called \((p, q)\)-factorial and \((p, q)\)-binomial, respectively, and expressed as:

\[
[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}, \quad n \geq 1, \quad [0]_{p,q}! = 1,
\]

\[
\left[ \begin{array}{c} n \\ k \end{array} \right] ! = \frac{[n]_{p,q}!}{[n-k]_{p,q}! [k]_{p,q}!}.
\]

**Definition 5 ([39])**. The \((p, q)\)-derivative of mapping \( \mathcal{F} : [\pi_1, \pi_2] \to \mathbb{R} \) is given as:

\[
D_{p,q} \mathcal{F}(x) = \frac{\mathcal{F}(px) - \mathcal{F}(qx)}{(p - q)x}, \quad x \neq 0
\]
with \(0 < q < p \leq 1\).

**Definition 6** ([40]). The left \(p,q\)-derivative of mapping \(F : [\pi_1, \pi_2] \to \mathbb{R}\) is given as:

\[
\pi_1 D_{p,q} F(x) = \frac{F(px + (1 - p)\pi_1) - F(qx + (1 - q)\pi_1)}{(p - q)(x - \pi_1)}, \ x \neq \pi_1
\]  

(7)

with \(0 < q < p \leq 1\). For \(x = \pi_1\), we state that \(\pi_1 D_{p,q} F(\pi_1) = \lim_{x \to \pi_1} \pi_1 D_{p,q} F(x)\) if it exists and it is finite.

**Definition 7** ([35]). The right \((p,q)\)-derivative of mapping \(F : [\pi_1, \pi_2] \to \mathbb{R}\) is given as:

\[
\pi_2 D_{p,q} F(x) = \frac{F(qx + (1 - q)\pi_2) - F(px + (1 - p)\pi_2)}{(p - q)(\pi_2 - x)}, \ x \neq \pi_2
\]  

(8)

with \(0 < q < p \leq 1\). For \(x = \pi_2\), we state that \(\pi_2 D_{p,q} F(\pi_2) = \lim_{x \to \pi_2} \pi_2 D_{p,q} F(x)\) if it exists and it is finite.

**Remark 1.** It is clear that if we use \(p = 1\) in (7) and (8), then the equalities (7) and (8) reduce to (2) and (3), respectively.

**Definition 8** ([40]). The left \((p,q)\)-integral of mapping \(F : [\pi_1, \pi_2] \to \mathbb{R}\) on \([\pi_1, \pi_2]\) is stated as:

\[
\int_{\pi_1}^{x} F(\tau) \ \pi_1 d_{p,q} \tau = (p - q)(x - \pi_1) \sum_{n=0}^{\infty} q^n \frac{q^n}{p^{n+1}} \mathcal{F} \left( \frac{q^n}{p^{n+1}} x + \left(1 - \frac{q^n}{p^{n+1}}\right) \pi_1 \right)
\]  

(9)

with \(0 < q < p \leq 1\).

**Definition 9** ([35]). The right \((p,q)\)-integral of mapping \(F : [\pi_1, \pi_2] \to \mathbb{R}\) on \([\pi_1, \pi_2]\) is stated as:

\[
\int_{x}^{\pi_2} F(\tau) \ \pi_2 d_{p,q} \tau = (p - q)(\pi_2 - x) \sum_{n=0}^{\infty} q^n \frac{q^n}{p^{n+1}} \mathcal{F} \left( \frac{q^n}{p^{n+1}} x + \left(1 - \frac{q^n}{p^{n+1}}\right) \pi_2 \right)
\]  

(10)

with \(0 < q < p \leq 1\).

**Remark 2.** It is evident that if we select \(p = 1\) in (9) and (10), then the equalities (9) and (10) change into (4) and (5), respectively.

**Remark 3.** If we take \(\pi_1 = 0\) and \(x = \pi_2 = 1\) in (9), then we have

\[
\int_{0}^{1} F(\tau) \ \pi_1 d_{p,q} \tau = (p - q) \sum_{n=0}^{\infty} q^n \frac{q^n}{p^{n+1}} \mathcal{F} \left( \frac{q^n}{p^{n+1}} \right).
\]

Similarly, by taking \(x = \pi_1 = 0\) and \(\pi_2 = 1\) in (10), then we obtain that

\[
\int_{0}^{1} F(\tau) \ \pi_2 d_{p,q} \tau = (p - q) \sum_{n=0}^{\infty} q^n \frac{q^n}{p^{n+1}} \mathcal{F} \left( 1 - \frac{q^n}{p^{n+1}} \right).
\]

**Remark 4.** If \(f\) is a symmetric function—i.e., \(F(s) = F(\pi_2 + \pi_1 - s), for s \in [\pi_1, \pi_2]\)—then we have

\[
\int_{\pi_1}^{\pi_2 (1-p)\pi_1} F(s) \ \pi_1 d_{p,q} s = \int_{\pi_1 (1-p)\pi_1}^{\pi_2} F(s) \ \pi_2 d_{p,q} s.
\]
Lemma 1 ([35]). We have the following equalities
\[
\int_{\pi_1}^{\pi_2} (\pi_2 - x)^{\pi_1} \pi_2 d_{p,q}x = \frac{(\pi_2 - \pi_1)^{\pi_1+1}}{[\pi_1 + 1]_{p,q}} \quad (11)
\]
\[
\int_{\pi_1}^{\pi_2} (x - \pi_1)^{\pi_1} \pi_1 d_{p,q}x = \frac{(\pi_2 - \pi_1)^{\pi_1+1}}{[\pi_1 + 1]_{p,q}} \quad (12)
\]
where \(\pi_1 \in \mathbb{R} - \{ -1 \} \).

Recently, M. Vivas-Cortez et al. [35] proved the following HH-type inequalities for convex functions using the \((p, q)^{\pi_2}\)-integral.

Theorem 5 ([35]). For a convex mapping \(F : [\pi_1, \pi_2] \to \mathbb{R}\), which is differentiable on \([\pi_1, \pi_2]\), the following inequalities hold for the \((p, q)^{\pi_2}\)-integral:
\[
F\left(\frac{p\pi_1 + q\pi_2}{2\eta_{p,q}}\right) \leq \frac{1}{p(\pi_2 - \pi_1)} \int_{\pi_1}^{\pi_2} F(x) \pi_2 d_{p,q}x \leq \frac{pF(\pi_1) + qF(\pi_2)}{2\eta_{p,q}},
\]
where \(0 < q < p \leq 1\).

Theorem 6 ([35]). For a convex function \(F : [\pi_1, \pi_2] \to \mathbb{R}\), the following inequality holds:
\[
F\left(\frac{\pi_1 + \pi_2}{2}\right) \leq \frac{1}{2p(\pi_2 - \pi_1)} \left[ \int_{\pi_1}^{p(\pi_2 - \pi_1)} F(x) \pi_2 d_{p,q}x + \int_{\pi_1}^{\pi_2} F(x) \pi_1 d_{p,q}x \right] \leq \frac{F(\pi_1) + F(\pi_2)}{2},
\]
where \(0 < q < p \leq 1\).

4. An Identity

In this section, we deal with an identity that is required to reach our major estimates. In the following lemma, we first build an identity based on a two-stage kernel.

Lemma 2. Let \(F : [\pi_1, \pi_2] \to \mathbb{R}\) be a differentiable function on \([\pi_1, \pi_2]\). If \(\pi_1 \>_{p,q} F\) is continuous and integrable on \([\pi_1, \pi_2]\), then one has the identity
\[
\frac{p}{[b]_{p,q}}\left[ p^2 F(\pi_2) + q\left(\frac{[b]_{p,q}}{p+q} - 1\right) F\left(\frac{p\pi_2 + q\pi_1}{p+q}\right) + qF(\pi_1) \right] \quad (15)
\]
\[- \frac{1}{(\pi_2 - \pi_1)} \int_{\pi_1}^{p(\pi_2 - \pi_1)} F(s) \pi_1 d_{p,q}s \]
\[= pq(\pi_2 - \pi_1) \int_{0}^{1} \Lambda(s) \pi_1 \>_{p,q} F(s\pi_2 + (1-s)\pi_1) \>_{p,q} s,
\]
where
\[
\Lambda(s) = \begin{cases} 
1 - \frac{1}{[b]_{p,q}}, & s \in \left[0, \frac{p}{[b]_{p,q}}\right) \\
\frac{[b]_{p,q}}{[b]_{p,q}} - 1, & s \in \left[\frac{p}{[b]_{p,q}}, 1\right].
\end{cases}
\]
**Proof.** Using the fundamental properties of \((p, q)\)-integrals and the definition of function \(\Lambda(s)\), we find that

\[
\int_0^1 \Lambda(s) \pi_1 D_{\rho,q} F(s\pi_2 + (1 - s)\pi_1) \; d_{\rho,q}s
\]

(16)

\[
= \frac{[5]_{\rho,q} - 1}{[6]_{\rho,q}} \int_0^p \pi_1 D_{\rho,q} F(s\pi_2 + (1 - s)\pi_1) \; d_{\rho,q}s
\]

\[
+ \int_0^1 \left( s - \frac{[5]_{\rho,q}}{[6]_{\rho,q}} \right) \pi_1 D_{\rho,q} F(s\pi_2 + (1 - s)\pi_1) \; d_{\rho,q}s.
\]

According to Definition 6, one must also have

\[
\pi_1 D_{\rho,q} F(s\pi_2 + (1 - s)\pi_1) = \frac{F(ps\pi_2 + (1 - ps)\pi_1) - F(qs\pi_2 + (1 - qs)\pi_1)}{(p - q)(\pi_2 - \pi_1)s}.
\]

Now, if we substitute the above equation into (16), we obtain

\[
\int_0^1 \Lambda(s) \pi_1 D_{\rho,q} F(s\pi_2 + (1 - s)\pi_1) \; d_{\rho,q}s
\]

(17)

\[
= \frac{[5]_{\rho,q} - 1}{[6]_{\rho,q}} \int_0^p \frac{F(ps\pi_2 + (1 - ps)\pi_1) - F(qs\pi_2 + (1 - qs)\pi_1)}{(p - q)(\pi_2 - \pi_1)s} \; d_{\rho,q}s
\]

\[
+ \int_0^1 \frac{F(ps\pi_2 + (1 - ps)\pi_1) - F(qs\pi_2 + (1 - qs)\pi_1)}{(p - q)(\pi_2 - \pi_1)s} \; d_{\rho,q}s.
\]

When the first integral on the right-hand side of (17) is calculated using Definition 8, it is discovered that

\[
\frac{[5]_{\rho,q}}{[6]_{\rho,q}} \int_0^p \frac{F(ps\pi_2 + (1 - ps)\pi_1) - F(qs\pi_2 + (1 - qs)\pi_1)}{(p - q)(\pi_2 - \pi_1)s} \; d_{\rho,q}s
\]

(18)

\[
= \frac{1}{(\pi_2 - \pi_1)} \left\{ \sum_{k=0}^{\infty} F \left( \frac{q^k}{p^{k+1}} \frac{p^2}{2} \pi_2 + \left( 1 - \frac{q^k}{p^{k+1}} \frac{p^2}{2} \right) \pi_1 \right) - \sum_{k=0}^{\infty} F \left( \frac{q^{k+1}}{p^{k+1}} \frac{p}{2} \pi_2 + \left( 1 - \frac{q^{k+1}}{p^{k+1}} \frac{p}{2} \right) \pi_1 \right) \right\}
\]

\[
= \frac{1}{(\pi_2 - \pi_1)} \left\{ F \left( \frac{p\pi_2 + q\pi_1}{2} \right) - F(\pi_1) \right\}.
\]
5. Main Results

For \((p, q)\)-differentiable convex functions, we prove some new Simpson’s formula type inequalities in this section. For the sake of brevity, we start this section with certain notations that will be utilized in our new results.

\[
A_1(p, q) = \frac{2}{p + q} \left( \frac{3}{p + q} - \frac{2}{p \cdot q} \right) + p^2 \left( \frac{6}{p + q} - \frac{3}{p \cdot q} \right),
\]

\[
B_1(p, q) = \frac{2}{p + q} \left( \frac{3}{p + q} - \frac{6}{p \cdot q} - \frac{3}{p \cdot q} + \frac{2}{p \cdot q} \right)
+ \left( \frac{p^2}{p + q} - \frac{2}{p \cdot q} + \frac{3}{p \cdot q} + p^2 \frac{3}{p \cdot q} \right),
\]

\[
A_2(p, q) = \frac{3}{p + q} \left( \frac{5}{p + q} - \frac{2}{p \cdot q} \right) + \frac{p^3}{p + q} - \frac{2}{p \cdot q} \left( \frac{5}{p + q} - \frac{3}{p \cdot q} \right) + \frac{2}{p \cdot q} \left( \frac{6}{p + q} - \frac{2}{p \cdot q} \right),
\]

\[
B_2(p, q) = \frac{5}{p + q} \left( \frac{2}{p \cdot q} \right) - \frac{5}{p + q} \left( \frac{3}{p \cdot q} \right) + \frac{5}{p + q} \left( \frac{2}{p \cdot q} \right)
- \left( \frac{p^2}{p + q} - \frac{2}{p \cdot q} \right) + \frac{2}{p \cdot q} \left( \frac{3}{p \cdot q} \right) + \frac{5}{p \cdot q} \left( \frac{2}{p \cdot q} \right).
\]
Theorem 7. Assume that the conditions of Lemma 2 hold. If $|\pi_1 D_{p,q} F|$ is convex on $[\pi_1, \pi_2]$, then we have following inequality for $(p,q)_{\pi_2}$-integrals:

$$\left| \frac{p}{[6]_{p,q}} \int_{\pi_2}^{\pi_1} 1_{p,q} + q \left( [5]_{p,q} - 1 \right) F \left( \frac{p [\pi_2 + q \pi_1]}{p + q} \right) + q F(\pi_1) \right|$$

$$\leq \frac{1}{(\pi_2 - \pi_1)} \int_{\pi_1}^{\pi_2 - (1-p)\pi_1} F(s) \, d_{p,q} s$$

Proof. We observe that when we take the modulus in Lemma 2, because of the modulus' characteristics, we have

$$\left| \frac{1}{[6]_{p,q}} \int_{\pi_2}^{\pi_1} 1_{p,q} + q \left( [5]_{p,q} - 1 \right) F \left( \frac{p [\pi_2 + q \pi_1]}{p + q} \right) + q F(\pi_1) \right|$$

$$\leq \frac{1}{(\pi_2 - \pi_1)} \int_{\pi_1}^{\pi_2 - (1-p)\pi_1} F(s) \, d_{p,q} s$$

Using the convexity of $|\pi_1 D_{p,q} F|$, we may calculate integrals on the right-hand side of (26) as follows:

$$\left| \frac{p}{[6]_{p,q}} \int_{\pi_2}^{\pi_1} 1_{p,q} + q \left( [5]_{p,q} - 1 \right) F \left( \frac{p [\pi_2 + q \pi_1]}{p + q} \right) + q F(\pi_1) \right|$$

$$\leq \frac{1}{(\pi_2 - \pi_1)} \int_{\pi_1}^{\pi_2 - (1-p)\pi_1} F(s) \, d_{p,q} s$$

When we apply the equality (12) idea to the aforementioned post-quantum integrals, we obtain

$$\left| \frac{p}{[6]_{p,q}} \int_{\pi_2}^{\pi_1} 1_{p,q} + q \left( [5]_{p,q} - 1 \right) F \left( \frac{p [\pi_2 + q \pi_1]}{p + q} \right) + q F(\pi_1) \right|$$

$$\leq \frac{1}{(\pi_2 - \pi_1)} \int_{\pi_1}^{\pi_2 - (1-p)\pi_1} F(s) \, d_{p,q} s$$

We may calculate integrals on the right-hand side of (26) as follows:

$$\left| \frac{p}{[6]_{p,q}} \int_{\pi_2}^{\pi_1} 1_{p,q} + q \left( [5]_{p,q} - 1 \right) F \left( \frac{p [\pi_2 + q \pi_1]}{p + q} \right) + q F(\pi_1) \right|$$

$$\leq \frac{1}{(\pi_2 - \pi_1)} \int_{\pi_1}^{\pi_2 - (1-p)\pi_1} F(s) \, d_{p,q} s$$

We may calculate integrals on the right-hand side of (26) as follows:

$$\left| \frac{p}{[6]_{p,q}} \int_{\pi_2}^{\pi_1} 1_{p,q} + q \left( [5]_{p,q} - 1 \right) F \left( \frac{p [\pi_2 + q \pi_1]}{p + q} \right) + q F(\pi_1) \right|$$

$$\leq \frac{1}{(\pi_2 - \pi_1)} \int_{\pi_1}^{\pi_2 - (1-p)\pi_1} F(s) \, d_{p,q} s$$

We may calculate integrals on the right-hand side of (26) as follows:

$$\left| \frac{p}{[6]_{p,q}} \int_{\pi_2}^{\pi_1} 1_{p,q} + q \left( [5]_{p,q} - 1 \right) F \left( \frac{p [\pi_2 + q \pi_1]}{p + q} \right) + q F(\pi_1) \right|$$

$$\leq \frac{1}{(\pi_2 - \pi_1)} \int_{\pi_1}^{\pi_2 - (1-p)\pi_1} F(s) \, d_{p,q} s$$
and
\[
\int_0^1 \frac{\psi'}{p_{\alpha\beta}} (1 - s) \left| s - \frac{1}{6} \right|_{p_{\alpha\beta}} \left| s - \frac{1}{6} \right|_{p_{\alpha\beta}} \, dp_{\alpha\beta} s
\]
\[
= \int_0^1 \frac{\psi'}{p_{\alpha\beta}} (1 - s) \left( \frac{1}{6} \right)_{p_{\alpha\beta}} \, dp_{\alpha\beta} s + \int_0^1 \frac{\psi'}{p_{\alpha\beta}} (1 - s) \left( s - \frac{1}{6} \right)_{p_{\alpha\beta}} \, dp_{\alpha\beta} s
\]
\[
= 2 \int_0^1 \frac{\psi'}{p_{\alpha\beta}} (1 - s) \left( \frac{1}{6} \right)_{p_{\alpha\beta}} \, dp_{\alpha\beta} s + \int_0^1 \frac{\psi'}{p_{\alpha\beta}} (1 - s) \left( s - \frac{1}{6} \right)_{p_{\alpha\beta}} \, dp_{\alpha\beta} s
\]
\[
= 2 \int_0^1 \frac{\psi'}{p_{\alpha\beta}} (1 - s) \left( \frac{1}{6} \right)_{p_{\alpha\beta}} \, dp_{\alpha\beta} s + \int_0^1 \frac{\psi'}{p_{\alpha\beta}} (1 - s) \left( s - \frac{1}{6} \right)_{p_{\alpha\beta}} \, dp_{\alpha\beta} s
\]
\[
= \left[ \frac{2}{p_{\alpha\beta}} \frac{3}{p_{\alpha\beta}} \frac{6}{p_{\alpha\beta}} \right]_{p_{\alpha\beta}} + \left( \frac{p^2 \frac{3}{p_{\alpha\beta}} \frac{6}{p_{\alpha\beta}} - p \frac{2}{p_{\alpha\beta}} \frac{3}{p_{\alpha\beta}} - p^3 \frac{6}{p_{\alpha\beta}} + p^2 \frac{6}{p_{\alpha\beta}}}{2 \frac{3}{p_{\alpha\beta}} \frac{3}{p_{\alpha\beta}} \frac{6}{p_{\alpha\beta}}} \right).
\]

Thus, we obtain
\[
\int_0^1 \frac{\psi'}{p_{\alpha\beta}} \left| s - \frac{1}{6} \right|_{p_{\alpha\beta}} \left| s - \frac{1}{6} \right|_{p_{\alpha\beta}} \, dp_{\alpha\beta} s = \left[ \frac{2}{p_{\alpha\beta}} \frac{3}{p_{\alpha\beta}} \frac{6}{p_{\alpha\beta}} \right]_{p_{\alpha\beta}} + \left( \frac{p^2 \frac{3}{p_{\alpha\beta}} \frac{6}{p_{\alpha\beta}} - p \frac{2}{p_{\alpha\beta}} \frac{3}{p_{\alpha\beta}} - p^3 \frac{6}{p_{\alpha\beta}} + p^2 \frac{6}{p_{\alpha\beta}}}{2 \frac{3}{p_{\alpha\beta}} \frac{3}{p_{\alpha\beta}} \frac{6}{p_{\alpha\beta}}} \right). \tag{27}
\]

Similarly, we have
\[
\int_0^1 \frac{\psi'}{p_{\alpha\beta}} \left| s - \frac{5}{6} \right|_{p_{\alpha\beta}} \left| s - \frac{5}{6} \right|_{p_{\alpha\beta}} \, dp_{\alpha\beta} s \leq \left[ \frac{2}{p_{\alpha\beta}} \frac{3}{p_{\alpha\beta}} \frac{6}{p_{\alpha\beta}} \right]_{p_{\alpha\beta}} + \left( \frac{p^3 \frac{6}{p_{\alpha\beta}} - p^2 \frac{5}{p_{\alpha\beta}} \frac{3}{p_{\alpha\beta}} + [2] \frac{3}{p_{\alpha\beta}} \frac{6}{p_{\alpha\beta}}}{2 \frac{3}{p_{\alpha\beta}} \frac{3}{p_{\alpha\beta}} \frac{6}{p_{\alpha\beta}}} \right) \tag{28}
\]

We obtain the inequality (25) by placing (27) and (28) in (26). This completes the proof. \(\Box\)
Corollary 1. In Theorem 7, if we set \( p = 1 \), then we have the following new Simpson’s type inequality for \( q \)-integrals:

\[
\left| \frac{1}{[6]_q} \left[ F(\pi_2) + q \left( [5]_q - 1 \right) F \left( \frac{\pi_2 + q \pi_1}{1 + q} \right) + q F(\pi_1) \right] - \frac{1}{(\pi_2 - \pi_1)} \int_{\pi_1}^{\pi_2} F(s) \, \pi_1 d_q s \right|
\leq q(\pi_2 - \pi_1) \left\{ \sum_{i=1}^{3} A_i(1, q) + A_2(1, q) \right\}
\]

Remark 5. In Theorem 7, if we assume \( p = 1 \) and later take the limit as \( q \to 1^- \), then we obtain the following Simpson’s type inequality:

\[
\left| \frac{1}{6} \left[ F(\pi_1) + 4 F \left( \frac{\pi_1 + \pi_2}{2} \right) + F(\pi_2) \right] - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} F(s) \, ds \right|
\leq \frac{5(\pi_2 - \pi_1)}{72} \left[ |F'(\pi_1)| + |F'(\pi_2)| \right].
\]

This is proven by Alomari et al. in [2].

Now, we can see how the inequalities appear when we utilize maps with convex \( q \)-derivative powers in an absolute value.

Theorem 8. Assume that the conditions of Lemma 2 hold. If \( |\pi_1 D_{p,q} F|^{p_1} \) is convex on \([\pi_1, \pi_2]\) for some \( p_1 > 1 \), then we have following inequality for \( (p, q) \)-integrals:

\[
\left| \frac{p}{[6]_{p,q}} \left[ p^\frac{q}{p} F(\pi_2) + q \left( [5]_{p,q} - 1 \right) F \left( \frac{p \pi_2 + q \pi_1}{p + q} \right) + q F(\pi_1) \right] \right|
\leq \frac{pq(\pi_2 - \pi_1)}{\theta(p, q)} \theta(p, q)
\]

\[
\left( \frac{p}{[2]_{p,q}^3} | \pi_1 D_{p,q} F(\pi_2) |^{p_1} + \frac{p[2]_{p,q}^3 - p^2}{[2]_{p,q}^3} | \pi_1 D_{p,q} F(\pi_1) |^{p_1} \right) \right)^{\frac{1}{p_1}}
\]

\[
+ \theta(p, q)
\]

\[
\left( \frac{[2]_{p,q}^3 - p^2}{[2]_{p,q}^3} | \pi_1 D_{p,q} F(\pi_2) |^{p_1} + \frac{[2]_{p,q}^3 - [2]_{p,q}^2 - p[2]_{p,q}^2 + p^2}{[2]_{p,q}^3} | \pi_1 D_{p,q} F(\pi_1) |^{p_1} \right) \right)^{\frac{1}{p_1}},
\]

where \( 0 < q < q \leq 1, \frac{1}{p_1} + \frac{1}{p_1} = 1 \) and

\[
\theta(p, q) = \int_{0}^{\pi_1} s - \frac{1}{[6]_{p,q}} \left| d_{p,q} s \right|^{p_1}
\]

\[
\theta(p, q) = \int_{\pi_1}^{1} s - \frac{[5]_{q}}{[6]_{q}} \left| d_{p,q} s \right|^{p_1}
\]

Proof. When the integrals on the right-hand side of (26) are subjected to the well-known Hölder’s inequality for post-quantum integrals, it is discovered that
\[ \left| \frac{p}{[6]_{p,q}} \left[ p^5 F(\tau_2) - q \left( [5]_{p,q} - 1 \right) F \left( \frac{p\tau_2 + q\tau_1}{p + q} \right) + q F(\tau_1) \right] - \frac{1}{(\tau_2 - \tau_1)} \int_{\tau_2}^{\tau_2+(1-p)\tau_1} F(s) \left( s, \tau_1, \pi, d_{p,q} s \right) \right| \]

\[
\leq p(q(\tau_2 - \tau_1) \left[ \left( \int_0^{p/6} |s - 1|^{\tau_1} d_{p,q} s \right)^{1/\tau_1} \right] \\
\times \left( \int_0^{p/6} |s - 1|^{\tau_1} d_{p,q} s \right)^{1/\tau_1} \\
+ pq(\tau_2 - \tau_1) \left[ \left( \int_0^{p/6} |s - 1|^{\tau_1} d_{p,q} s \right)^{1/\tau_1} \right] \\
\times \left( \int_0^{1} \left| \frac{p}{\pi} \right| d_{p,q} s \right)^{1/\tau_1} \\
\times \left( \int_0^{1} \left| \frac{p}{\pi} \right| d_{p,q} s \right)^{1/\tau_1} \\
\times \left( \int_0^{1} \left| \frac{p}{\pi} \right| d_{p,q} s \right)^{1/\tau_1} \right].
\]

By using the convexity of \(|\n_1 D_{p,q} F|^{p_1}\), we obtain

\[
\left| \frac{p}{[6]_{p,q}} \left[ p^5 F(\tau_2) - q \left( [5]_{p,q} - 1 \right) F \left( \frac{p\tau_2 + q\tau_1}{p + q} \right) + q F(\tau_1) \right] - \frac{1}{(\tau_2 - \tau_1)} \int_{\tau_2}^{\tau_2+(1-p)\tau_1} F(s) \left( s, \tau_1, \pi, d_{p,q} s \right) \right| \]

\[
\leq p(q(\tau_2 - \tau_1) \left[ \left( \int_0^{p/6} |s - 1|^{\tau_1} d_{p,q} s \right)^{1/\tau_1} \right] \\
\times \left( \int_0^{p/6} |s - 1|^{\tau_1} d_{p,q} s \right)^{1/\tau_1} \\
+ pq(\tau_2 - \tau_1) \left[ \left( \int_0^{p/6} |s - 1|^{\tau_1} d_{p,q} s \right)^{1/\tau_1} \right] \\
\times \left( \int_0^{1} \left| \frac{p}{\pi} \right| d_{p,q} s \right)^{1/\tau_1} \\
\times \left( \int_0^{1} \left| \frac{p}{\pi} \right| d_{p,q} s \right)^{1/\tau_1} \\
\times \left( \int_0^{1} \left| \frac{p}{\pi} \right| d_{p,q} s \right)^{1/\tau_1} \right].
\]

Using equality (12), we see that, for the other integrals on the right-hand side of (30),

\[
\int_0^{p/\pi} s d_{p,q} s = \frac{\pi p^2}{[2]_{p,q}^3},
\]

\[
\int_0^{p/\pi} (1-s) d_{p,q} s = \frac{\pi p^2}{[2]_{p,q}^3} - p^2.
\]
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Similarly, we obtain
\[
\int_{\frac{1}{2} p_{p,q}}^{1} sd_{p,q}s = \frac{|2|_{p,q}^2 - p^2}{\left(\frac{1}{p_{p,q}}\right)^3} \tag{33}
\]
\[
\int_{\frac{1}{2} p_{p,q}}^{1} (1-s)d_{p,q}s = \frac{|2|_{p,q}^3 - |2|_{p,q}^2 - p|2|_{p,q}^2 + p^2}{\left(\frac{1}{p_{p,q}}\right)^3}. \tag{34}
\]

We obtain the desired inequality (29) by inserting (31)–(34) into (30), which completes the proof. \(\square\)

**Corollary 2.** In Theorem 8, if we set \(p = 1\), then we obtain the following new Simpson’s type inequality for \(q\)-integrals:

\[
\left| \frac{1}{6}_p \left[ F(\pi_2) + q\left(\frac{5}{q} - 1\right) F\left(\frac{\pi_2 + q\pi_1}{1 + q}\right) + qF(\pi_1) \right] - \frac{1}{(\pi_2 - \pi_1)} \int_{\pi_1}^{\pi_2} F(s) \pi_1 dq_s \right| \leq q(\pi_2 - \pi_1) \left[ \Theta_1^\frac{1}{\pi} (1,q) \right.
\]
\[
\times \left( \frac{1}{[2]_q^3} |_{\pi_1} D_q F(\pi_2) |_{p_1} + \frac{|2|_q^2 - 1}{[2]_q^3} |_{\pi_1} D_q F(\pi_1) |_{p_1} \right)^\frac{1}{\pi_1} 
\]
\[
+ \Theta_2^\frac{1}{\pi} (1,q) 
\times \left( \frac{|2|_q^2 - 1}{[2]_q^3} |_{\pi_1} D_q F(\pi_2) |_{p_1} + \frac{|2|_q^3 - |2|_q^2 |2|_q^2 + 1}{[2]_q^3} \right) 
\]
\[
\left. \times \left( \frac{1}{(\pi_2 - \pi_1)} \int_{\pi_1}^{\pi_2} F(s) \pi_1 dq_s \right) \right].
\]

**Theorem 9.** Assume that the conditions of Lemma 2 hold. If \(|_{\pi_1} D_{p,q} F |_{p_1}\) is convex on \([\pi_1, \pi_2]\) for some \(p_1 \geq 1\), then we have following inequality for \((p,q)_{\pi_1}\)-integrals:

\[
\left| \frac{p}{6_{p,q}} \left[ F(\pi_2) - q\left(\frac{5}{q} - 1\right) F\left(\frac{p\pi_2 + q\pi_1}{p + q}\right) + qF(\pi_1) \right] \right| 
\]
\[
- \frac{1}{(\pi_2 - \pi_1)} \int_{\pi_1}^{\pi_2} F(s) \pi_1 dq_s \right| \leq pq(\pi_2 - \pi_1) \left[ \left( \frac{2}{[2]_{p,q}^2} - \frac{1}{[2]_{p,q}^3} \right) \left( \frac{2}{[2]_{p,q}} \left(\frac{2}{[2]_{p,q}} - \frac{1}{[2]_{p,q}^3} \right) \right)^{-\frac{1}{\pi_1}} 
\]
\[
\times \left( A_1(p,q) |_{\pi_1} D_{p,q} F(\pi_2) |_{p_1} + B_1(p,q) |_{\pi_1} D_{p,q} F(\pi_1) |_{p_1} \right)^\frac{1}{\pi_1} 
\]
\[
+ \left( \frac{2}{[2]_{p,q}^2} - \frac{1}{[2]_{p,q}^3} \right) \left( \frac{2}{[2]_{p,q}} \left(\frac{2}{[2]_{p,q}} - \frac{1}{[2]_{p,q}^3} \right) \right)^{-\frac{1}{\pi_1}} 
\]
\[
\times \left( A_2(p,q) |_{\pi_1} D_{p,q} F(\pi_2) |_{p_1} + B_2(p,q) |_{\pi_1} D_{p,q} F(\pi_1) |_{p_1} \right)^\frac{1}{\pi_1},
\]

where \(0 < q < p \leq 1\) and \(A_1(p,q), A_2(p,q), B_1(p,q), B_2(p,q)\) are given as in (21)–(24), respectively.
Proof. Using the conclusions obtained in the proof of Theorem 7 after applying the well-known power mean inequality to the integrals on the right-hand side of (26), we discover that, due to the convexity of \( |\tau_{1} D_{p,q} F|^{p_1} \),

\[
\left| \frac{p}{6}_{p,q} \left[ p^5 F(\pi_2) - q \left( \frac{[5]_{p,q}}{p} - 1 \right) F \left( \frac{p \tau_{1} + q \tau_{1}}{p + q} \right) + q F(\pi_1) \right] \right| \leq \frac{1}{(\pi_2 - \pi_1)} \int_{\pi_1}^{\pi_2} \mathcal{F}(s) \left| d_{p,q} s \right| \left| d_{p,q} s \right|
\]

\[
\times \left( \left| \int_0^{\frac{p}{6}_{p,q}} s - \frac{1}{[6]_{p,q}} d_{p,q} s \right| \left| \int_0^{\frac{p}{6}_{p,q}} s - \frac{1}{[6]_{p,q}} d_{p,q} s \right| \right)\]

\[
\left( A_1(p,q) | \tau_{1} D_{p,q} F(\pi_2) |^{p_1} + B_1(p,q) | \tau_{1} D_{p,q} F(\pi_1) |^{p_1} \right) \frac{1}{p_1}
\]

\[
+ p \left( \int_{\pi_1}^{\pi_2} s - \frac{[5]_{p,q}}{[6]_{p,q}} d_{p,q} s \right) \left( \int_{\pi_1}^{\pi_2} s - \frac{[5]_{p,q}}{[6]_{p,q}} d_{p,q} s \right) \frac{1}{p_1}
\]

\[
\times \left( A_2(p,q) | \tau_{1} D_{p,q} F(\pi_2) |^{p_1} + B_2(p,q) | \tau_{1} D_{p,q} F(\pi_1) |^{p_1} \right) \frac{1}{p_1} \].

We also observe that

\[
\int_{\pi_1}^{\pi_2} s - \frac{[5]_{p,q}}{[6]_{p,q}} d_{p,q} s = 2 \int_0^{\frac{[5]_{p,q}}{[6]_{p,q}}} \left( \frac{1}{[6]_{p,q}} - s \right) d_{p,q} s + \int_0^{\frac{p}{6}_{p,q}} \left( s - \frac{1}{[6]_{p,q}} \right) d_{p,q} s
\]

\[
= 2 \left( \frac{[2]_{p,q} - 1}{[2]_{p,q} [6]_{p,q}} \right) + \frac{p^2 [6]_{p,q} - p [2]_{p,q}^2}{[6]_{p,q} [2]_{p,q}^2},
\]

and by using similar operations, we have

\[
\int_{\pi_1}^{\pi_2} s - \frac{[5]_{p,q}}{[6]_{p,q}} d_{p,q} s = 2 \frac{[2]_{p,q} [5]_{p,q}}{[2]_{p,q} [6]_{p,q}} - \frac{[5]_{p,q}}{[2]_{p,q} [6]_{p,q}} + \frac{[5]_{p,q}}{[6]_{p,q}} - \frac{p [5]_{p,q} [2]_{p,q}^2 - p^2 [6]_{p,q}^2}{[6]_{p,q} [2]_{p,q}^2}.
\]
We obtain the needed inequality (35) by swapping (37) and (38) in (36). As a result, the proof is complete. □

Corollary 3. In Theorem 9, if we set \( p = 1 \), then we obtain the following new Simpson’s type inequality for the \( q \)-integral:

\[
\frac{1}{[6]_q} \left[ F(\pi_2) - q \left( [5]_q - 1 \right) F\left( \frac{\pi_2 + q \pi_1}{1 + q} \right) + q F(\pi_1) \right] - \frac{1}{(\pi_2 - \pi_1)} \int_{\pi_1}^{\pi_2} F(s) \, d_q s \leq q(\pi_2 - \pi_1) \left[ \frac{\left( 2 \left[ 2 \right]_q - 1 \right)}{\left[ 2 \right]_q \left[ 6 \right]_q} + \frac{\left[ 2 \right]_q \left[ 2 \right]_q - \left[ 2 \right]_q^2}{\left[ 6 \right]_q \left[ 2 \right]_q^3} \right]^{1 - \frac{1}{\pi_1}}
\]

\[
\times \left( A_1(1, q) \left| \pi_1 D_q F(\pi_2) \right|^{\frac{1}{\pi_1}} + B_1(1, q) \pi_1 D_q F(\pi_1) \right)^{\frac{1}{\pi_1}}
\]

\[
+ \frac{\left( 2 \left[ 2 \right]_q \left[ 5 \right]_q^2 - \left[ 5 \right]_q^2 \right)}{\left[ 2 \right]_q \left[ 6 \right]_q} + \frac{1}{\left[ 2 \right]_q} - \frac{\left[ 5 \right]_q - [6]_q}{\left[ 6 \right]_q \left[ 2 \right]_q^3} \right]^{1 - \frac{1}{\pi_1}}
\]

\[
\times \left( A_2(1, q) \left| \pi_1 D_q F(\pi_2) \right|^{\frac{1}{\pi_1}} + B_2(1, q) \pi_1 D_q F(\pi_1) \right)^{\frac{1}{\pi_1}}.
\]

Remark 6. In Theorem 9, if we set \( p = 1 \) and later take the limit as \( q \to 1^- \), then we have the following Simpson’s type inequality:

\[
\frac{1}{6} \left[ F(\pi_1) + 4 F\left( \frac{\pi_1 + \pi_2}{2} \right) + F(\pi_2) \right] - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} F(s) \, ds \leq \frac{1}{(1296)^{\frac{1}{\pi_1}}} \left( \frac{5}{52} \right)^{1 - \frac{1}{\pi_1}} (\pi_2 - \pi_1)
\]

\[
\times \left[ 61 \left| F'(\pi_1) \right|^{\frac{1}{\pi_1}} + 29 \left| F'(\pi_2) \right|^{\frac{1}{\pi_1}} \right]^{\frac{1}{\pi_1}} + \left[ 29 \left| F'(\pi_1) \right|^{\frac{1}{\pi_1}} + 61 \left| F'(\pi_2) \right|^{\frac{1}{\pi_1}} \right]^{\frac{1}{\pi_1}}.
\]

This is given by Alomari et al. in [2].

6. Conclusions

In this investigation, we have proven different variants of Simpson’s formula type inequalities for \((p, q)\)-differentiable convex functions via post-quantum calculus. We conclude that the findings of this research are universal in nature and contribute to inequality theory, as well as applications in quantum boundary value problems, quantum mechanics, and special relativity theory for determining solution uniqueness. The findings of this study can be utilized in symmetry. Results for the case of symmetric functions can be obtained by applying the concept in Remark 4, which will be studied in future work. Future researchers will be able to obtain similar inequalities for different types of convexity and co-ordinated convexity in their future work, which is a new and important problem.

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