Hierarchy of Non-Markovianity and $k$-divisibility phase diagram of Quantum Processes in Open Systems

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In recent years, much effort has been devoted to the construction of a proper measure of quantum non-Markovianity. However, those proposed measures are shown to be at variance with different situations. In this work, we utilize the theory of $k$-positive maps to generalize a hierarchy of $k$-divisibility and develop a powerful tool, called $k$-divisibility phase diagram, which can provide a further insight into the nature of quantum non-Markovianity. By exploring the phase diagram with several paradigms, we can explain the origin of the discrepancy between two frequently used measures and find the condition under which the two measures coincide with each other.

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Introduction. Open quantum systems have attracted increasing attention due to their fundamental importance and applications in various fields. In these systems, one has to deal with the interactions between the system and its environments, which may induce dissipation, decoherence, or rephasing. The environment usually consists of a huge amount of degrees of freedom, and keeping track of them exactly is impossible. When dealing with the interactions, a typical approach is to employ the Born-Markov approximation [1], leading to the celebrated Lindblad master equation [2, 3]. Its solutions form a family of quantum dynamical semigroups. In an authentic physical system, however, the dynamics is expected to deviate from the idealized Markovian evolution, in which the memory effects are essentially ignored. In order to take the memory effects into account, many improved techniques had been developed, such as path-integral formalisms [4–7], Monte Carlo algorithms [8], hierarchy equations of motion (HEOM) [9–11], the reaction-coordinate method [12, 13], and non-Markovian quantum master equations [14–16].

Although the notion of non-Markovianity has been used extensively, there is no unique definition nowadays. Recently, much effort has been devoted to the construction of an appropriate measure of quantum non-Markovianity [17]. Most of them are based on continuously monitoring the time variation of certain quantities of interest, such as trace-distance [18], entanglement [19], mutual information [20], channel capacity [21], and the set of accessible states [22]. When these quantities decrease monotonically in time, the system is said to undergo a Markovian process. On the other hand, whenever the revival of these quantities in a time period is detected, the corresponding measure of non-Markovianity can be constructed according to an optimal amount of the revival.

In the non-Markovianity measure proposed by Breuer, Lane, and Piilo (BLP) [18], they consider the time-varying rate $\sigma$ defined as

$$\sigma (\rho_1, \rho_2; t) = \frac{\partial}{\partial t} \| E_{t,0} (\rho_1 - \rho_2) \|_1, \quad (1)$$

for a pair of arbitrary initial states $\rho_1$ and $\rho_2$, where $E_{t,0}$ is a completely positive (CP) and trace-preserving (TP) quantum process and $\| A \|_1 = \text{Tr}\sqrt{A^\dagger A}$ denotes the trace norm of a matrix $A$. The BLP measure interprets $\sigma (\rho_1, \rho_2; t)$ as an information flow, and $\sigma (\rho_1, \rho_2; t) > 0$ witnesses a back-flow of information from the environment into the system which increases the distinguishability of $\rho_1$ and $\rho_2$ and indicates the non-Markovian character of the process $E_{t,0}$.

The underlying origin of the revival of these quantities is the divisibility of the processes [23, 24]. A CPTP quantum process $E_{t,0}$ is said to be CP-divisible if, $\forall t, \tau > 0$, there exist a complement process $\Lambda_{t+\tau,t}$ which is also CPTP and satisfies the composition law

$$E_{t+\tau,t} = \Lambda_{t+\tau,t} \circ E_{t,0}. \quad (2)$$

A process $E_{t,0}$ is said to be Markovian if and only if it is CP-divisible. Hence the measure of non-Markovianity proposed by Rivas, Huelga, and Plenio (RHP) [19] is the degree of a process deviating from being CP-divisible, i.e.,

$$g(t) = \lim_{\epsilon \to 0^+} \frac{\| (I \otimes \Lambda_{t+\tau,t}) (\Psi) (\Psi) \|_1 - 1}{\epsilon}, \quad (3)$$

where $|\Psi\rangle$ denotes a maximally-entangled state between the system and a copy of well-isolated ancilla possessing the same degrees of freedom of the system and $I$ is the identity process acting on the ancillary degree of freedoms. Due to the Choi-Jamiołkowski isomorphism [25, 26], the complement process $\Lambda_{t+\tau,t}$ is CPTP if and only if $g(t) = 0$, $\forall t > 0$, namely, $E_{t,0}$ is CP-divisible and Markovian.

Apart from some special cases in which only a single decoherence channel is present [27, 28], many compara-

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tive studies [29–31] showed that these measures are essentially at variance, especially when the process consists of multiple decoherence channels.

In view of the discrepancies among different measures, the following questions naturally arise: What is the underlying reason that makes these measures at variance and under what circumstance these measures coincide with each other? Is that possible to construct a better measure when all the measures fail to work? To address these questions, we adopt the concept of $k$-divisibility [32], a natural generalization of CP-divisibility in Eq. (2), to develop the $k$-divisibility phase diagram, which can provide a further insight into the nature of quantum non-Markovianity.

$k$-positive maps. Suppose that $\mathcal{A}$ is a $C^\ast$-algebra of linear operators on the $n$-dimensional Hilbert space $\mathcal{H}_n$, and $\mathcal{A}^+$ is the subset of positive elements. A TP map $\mathcal{E}$ is said to be positivity-preserving (PP) if $\mathcal{E}(\mathcal{A}^+) \subseteq \mathcal{A}^+$. However, a quantum system may have nonclassical correlations (e.g. entanglement) with some other ancillary degrees of freedom. Hence the notion of PP must be generalized to a series of $k$-positivity: a TP map $\mathcal{E}$ is said to be $k$-positive if $\mathcal{E}_k \otimes \mathcal{E}: \mathcal{M}_k \otimes \mathcal{A} \to \mathcal{M}_k \otimes \mathcal{A}$ is PP, and CP if $\mathcal{E}$ is $k$-positive for all positive integers $k$, where $\mathcal{E}_k$ is the identity map acting on the $k \times k$ matrix algebra $\mathcal{M}_k$. Although the structure of CP maps has been studied thoroughly [25, 26], there is still no efficient criterion for determining whether a map is $k$-positive or not [33, 34].

$k$-divisibility and divisibility phase diagram. Having the notion of $k$-positive maps, we can generalize CP-divisibility to a hierarchy of $k$-divisibility: A CPTP quantum process $\mathcal{E}_{t,0}$ is said to be $k$-divisible if, $\forall t, \tau > 0$, the complement process $\Lambda_{t+\tau,t}$ in Eq. (2) is $k$-positive. Consequently, $n$-divisibility is equivalent to CP-divisibility and $\mathcal{E}_{t,0}$ is 0-divisible if $\Lambda_{t+\tau,t}$ is not a positivity-preserving map. Introducing a family of sets $\mathcal{D}_k$ containing processes $\mathcal{E}_{t,0}$ with divisibility less than $k$, one has a chain of inclusions

$$\mathcal{D}_0 \subset \mathcal{D}_1 \subset \cdots \subset \mathcal{D}_{n-1} \subset \mathcal{D}_n, \tag{4}$$

where $\mathcal{D}_n$ consists of all quantum processes, no matter Markovian or non-Markovian, and $\mathcal{D}_0$ consists of 0-divisible processes, which is called to be essentially non-Markovian by Chruściński et al. [32]. Now we propose to define the set of proper $k$-divisibility $\mathcal{PD}_k = \mathcal{D}_k - \mathcal{D}_{k-1}$, then $\mathcal{PD}_n = \mathcal{D}_n - \mathcal{D}_{n-1}$ consists of processes which are exactly $n$-divisible, i.e., Markovian processes. Thus the inclusion chain in Eq. (4) can be rewritten into a partition of $\mathcal{D}_n$ in terms of $\mathcal{PD}_k$

$$\mathcal{D}_n = \bigcup_{k=0}^n \mathcal{PD}_k. \tag{5}$$

To make the partition in Eq. (5) intuitive, in the following, we consider several paradigms and show the explicit visualization by means of $k$-divisibility phase diagram which can provide us further insight into the nature of quantum non-Markovianity. The algorithm to the partition in Eq. (5) based on the criteria for $k$-positivity and all the calculations for the following paradigms are described in the Supplementary [35]. Additionally, a similar partition has been reported recently in an all optical setup [36].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Fig1.png}
\caption{(Color online) The $k$-divisibility phase diagram for the Pauli dephasing channel in 3-dimensional $\gamma_j(t)$ space with (a) $\gamma_3(t) \geq 0$ and (b) $\gamma_3(t) < 0$. The $\mathcal{PD}_2$ (gray) is for the Markovian processes with nonnegative rates $\gamma_j(t) \geq 0$ confined in the first octant, $\mathcal{PD}_1$ (blue) is for the non-Markovian processes with rates satisfying conditions Eqs. (8), and $\mathcal{PD}_0$ (red) is for the essentially non-Markovian processes away from the above regions. The upper bound of the BLP measure coincides with the border between $\mathcal{PD}_1$ and $\mathcal{PD}_0$.}
\end{figure}

Examples. Firstly, we consider a master equation for a qubit in the standard Lindblad form

$$\frac{\partial}{\partial t} \rho = \mathcal{L}_t \rho = \frac{1}{2} \sum_{j=1}^3 \gamma_j(t) (\bar{\sigma}_j \rho \sigma_j - \rho), \tag{6}$$

where $\bar{\sigma}_j$ are the Pauli matrices. It is well-known that the necessary and sufficient conditions for its corresponding process $\mathcal{E}_t = \mathcal{T} \exp \left[ \int_0^t \mathcal{L}_\tau d\tau \right] \in \mathcal{PD}_2$ (being Markovian) are

$$\gamma_1(t) \geq 0, \quad \gamma_2(t) \geq 0, \quad \gamma_3(t) \geq 0. \tag{7}$$

On the other hand, the conditions for $\mathcal{E}_t \in \mathcal{PD}_1$ are weaker than that of Eqs. (7) [32], i.e.,

$$\gamma_1(t) + \gamma_2(t) \geq 0, \quad \gamma_2(t) + \gamma_3(t) \geq 0, \quad \gamma_3(t) + \gamma_1(t) \geq 0. \tag{8}$$

Although the region of Eqs. (7) is enclosed by that of Eqs. (8), $\mathcal{PD}_2$ should be excluded from $\mathcal{PD}_1$ by definition. The above conditions can be depicted in a 3-dimensional $\gamma_j(t)$ space for a clear comprehension. In Fig. 1, we show the half space with (a) $\gamma_3(t) \geq 0$ and (b) $\gamma_3(t) < 0$. The first octant consists of all Markovian processes satisfying Eqs. (7) and therefore lies in the region of $\mathcal{PD}_2$ (gray). The $\mathcal{PD}_1$ (blue) consists of non-Markovian processes satisfying Eqs. (8) surrounding the region of $\mathcal{PD}_2$, and the $\mathcal{PD}_0$ (red) consists of essentially non-Markovian ones which get rid of the above regions.
Furthermore, it has been shown that whenever $E_t \in \mathcal{P}D_1$, $\sigma(\rho_1, \rho_2; t) \leq 0$ is always fulfilled \([32, 37]\). Namely, BLP measure can only detect the non-Markovianity in the region of $\mathcal{P}D_0$ and be blind to the non-Markovianity in the region of $\mathcal{P}D_1$. The underlying reason of this weakness lies in that the definition of $\sigma(\rho_1, \rho_2; t)$ does not take the full advantage of CP-divisibility.

To further show the powerfulness of the divisibility phase diagrams, we consider the eternal non-Markovian process proposed by Hall et al. \([38]\) with decay rates

$$\gamma_1(t) = \gamma_2(t) = 1, \quad \gamma_3(t) = -\tanh t. \quad (9)$$

It is clear that these decay rates lie exactly in the region of $\mathcal{P}D_1$, hence the corresponding process is non-Markovian and can never be detected by BLP measure. Along the same line, one can easily construct more eternal non-Markovian processes such as

$$\gamma_1(t) = 1, \quad \gamma_2(t) = -\gamma_3(t) = \sin t. \quad (10)$$

The second example considered is a pair of qubits coupled with each other via a C-NOT gate. Besides the C-NOT gate, the T-qubit undergoes an isotropic depolarizing channel. The C-qubit is a mixture and has population $a$ on $|1\rangle\langle 1|$. For $a = 1$ or $0$, the C-qubit is pure and has no information that can flow into T-qubit. Hence the T-qubit is Markovian ($\mathcal{P}D_2$). As $a$ approaches 0.5, the T-qubit becomes more and more uncertain and contains more information. The information-flow into the T-qubit resultantly dominates and more uncertain and contains more information. Consequently, the T-qubit can only decohere Markovianly. As a result, the dynamics of T-qubit shows a transition from $\mathcal{P}D_2$ to $\mathcal{P}D_0$ with $a$ approaching 0.5 in Fig. 2.

Moreover, we find out that the upper bound of RHP measure coincides into the border of $\mathcal{P}D_1$ and $\mathcal{P}D_2$. This means that, in this case, the RHP measure is a perfect measure which can thoroughly detect the non-Markovianity in both region of $\mathcal{P}D_0$ and $\mathcal{P}D_1$. On the other hand, the upper bound of BLP measure also coincides with the border of $\mathcal{P}D_0$ and $\mathcal{P}D_1$, and the reason is similar to that in the previous example.

In the following, we proceed to address the question that under what circumstance can BLP measure be perfect as shown in Ref. \([27, 39]\). We consider the qubit dynamics subjected to a amplitude-damping channel

$$\frac{\partial}{\partial t} \rho = \gamma(t) \left( \hat{\sigma}_- \rho \hat{\sigma}_+ - \frac{1}{2} (\hat{\sigma}_+ \hat{\sigma}_- \rho) \right). \quad (12)$$

The time-varying rate $\gamma(t)$ is determined by the spectral density function $J(\omega)$ of the environment. In this example, we employ the Lorentzian spectral density function $J(\omega) = \gamma_0 \lambda^2 / [2\pi (\omega^2 + \lambda^2)]$. In Fig. 3, we depict the $k$-divisibility phase diagram in $C - \lambda$ plane, and the criteria for $\mathcal{P}D_2$ and $\mathcal{P}D_1$ are exactly the same. Consequently,
the $\gamma_0 - \lambda$ plane is divided into only two part, $\mathcal{PD}_2$ and $\mathcal{PD}_0$, and the degeneracy of $\mathcal{PD}_1$ can be shown explicitly by the divisibility phase diagram for such dynamics possessing only one decoherence channel. As expected, the BLP measure can detect all non-Markovianity in the region of $\mathcal{PD}_0$ with $\gamma_0 > \lambda/2$. Due to the degeneracy, the BLP measure can detect all non-Markovianity in $\gamma_0 - \lambda$ plane and is equivalent to the RHP measure for this model. This is in line with the conclusions of Ref. [27, 39].

Additionally, similar conclusion can be drawn in a special case of the first example, where the qubit system is subject to only one of the three dephasing channels, e.g., $\gamma_1(t) = \gamma_2(t) = 0$ and $\gamma_3(t) \neq 0$. The border of $\mathcal{PD}_1$ is merged with that of $\mathcal{PD}_2$, and the degeneracy occurs. Then the BLP measure again works perfectly as RHP measure does, as shown in Ref. [27].

![Graph](image)

FIG. 4. (Color online) The $k$-divisibility phase diagram for a pair of atoms with the superradiant process, where $x$ is defined as the inter-atomic distance divided by the wave number $q$. The environment atom has an initial population $a$ in its excited state, which is related to the rate of energy feedback to the system atom. For $a = 0$ or $x = n\pi$, the system is Markovian ($\mathcal{PD}_2$). The region surrounding $\mathcal{PD}_2$ with small $a$ or $x$ closed to $n\pi$ is dominated by $\mathcal{PD}_1$. When $a$ increases, the system atom becomes more non-Markovian and is dominated by $\mathcal{PD}_0$. The non-Markovianity reduces to Markovianity and appears periodically when increasing $x$. In this case, the BLP measure can only detect the non-Markovianity in a small region of $\mathcal{PD}_0$ (the region above the dashed curve).

Apart from these theoretical models described by the standard Lindblad form, we finally consider the superradiant phenomenon in a two-atom system which can be implemented experimentally and attracts much interests recently [40, 41]. In this system, the two atoms are coupled with each other via a common photon reservoir with the wave number $q$. Although the dynamics for the two-atom system can be described by a standard Lindblad form and is considered to be Markovian [40], it is not the case if we pay attention to only one of the atoms by tracing out the other one. Assume that the two atoms are separated by a distance $d$ with no initial correlation. Similar to the second example, the traced-out atom plays the role of non-Markovian environment, and its initial population $a$ of the excited state is related to the rate of energy feedback to the system atom.

In Fig. 4, we show the divisibility phase diagram for the system atom. When $a = 0$, the environment atom is in its ground state and no energy feedback into the system atom can occur. Thus the system atom undergoes a purely Markovian dissipation and the dynamics belongs to $\mathcal{PD}_2$. When $x = qd$ is a multiple of $\pi$, both the atoms are effectively blind to each other due to the destructive interference. This inhibits the inter-atom energy exchange and also leads to the Markovian behavior of the system atom, regardless of the value of $a$. In general, the system atom is more non-Markovian with increasing $a$, due to the stronger feedback of energy from the environment atom. Hence $\mathcal{PD}_0$ distributes over the most of the region with large $a$, whereas $\mathcal{PD}_1$ can only occupy the region with small $a$ and the narrow regions surrounding $x = n\pi$.

In contrast to the previous examples, the BLP measure can only detect the non-Markovianity in a small region of $\mathcal{PD}_0$, which is above the dashed curve in Fig. 4. This is because the information from the environment to the system atom is reduced when increasing the inter-atom distance $d$. However, the non-Markovianity should be kept even for a large distance, due to the inter-atom coupling mediated by photon modes [40]. With the peculiar excitation-transfer rate in the form of $\sin(qd)/qd$, the transition between Markovian and non-Markovian behavior should possess periodicity of $qd = n\pi$, and the measure of non-Markovianity keeps finite but decreasing as increasing the distance.

**Conclusions.** In summary, we have utilized the theory of $k$-positivity maps to generalize the notion of CP-divisibility to a hierarchy of $k$-divisibility. This results in a refinement of quantum non-Markovianity and allows us to classify quantum processes into the partition of $\mathcal{D}_k$ in terms of $\mathcal{PD}_k$. Further visualization of the $\mathcal{PD}_k$ partition leads to a useful tool, referred as $k$-divisibility phase diagrams. These phase diagrams show the landscape of non-Markovianity and allow one to study different measures in a unified framework. We can acquire a deeper insight into the nature of quantum non-Markovianity, such as an intuitive way to realize the cause of the eternal non-Markovian dynamics, the reason for the perfection of RHP measure, the clue for the weakness of BLP measure, and the circumstance when the measures coincide with each other. Finally, we consider the superradiant phenomenon of a two-atom system which can be implemented experimentally. The Markovian region is reduced to several straight lines instead of an area due to the destructive interference or the lack of feedback energy. The distribution of non-Markovian regions is related to the rate of energy feedback to the system atom and possesses periodicity due to the peculiar form of photon-mediated interaction. In this case, the BLP measure can only de-
tect a small part of the non-Markovian regions. This weakness is due to the increasing the inter-atom distance may reduce the back flow of information and thus the distinguishability can hardly increase with time.

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[1] H. P. Breuer and F. Petruccione, The Theory of Open Quantum Systems (Oxford University Press, New York, 2002).
[2] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. 17, 821 (1976).
[3] G. Lindblad, Comm. Math. Phys. 48, 119 (1976).
[4] A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, A. Garg, and W. Zwerger, Rev. Mod. Phys. 59, 1 (1987).
[5] U. Weiss, Quantum Dissipative Systems, 4th ed. (World Scientific, Singapore, 2012).
[6] H. Grabert, P. Schramm, and G.-L. Ingold, Phys. Rep. 168, 115 (1988).
[7] M. Grifoni and P. Hänggi, Phys. Rep. 304, 229 (1999).
[8] R. Egger, L. Mühlbacher, and C. H. Mak, Phys. Rev. E 61, 5961 (2000).
[9] Y. Tanimura, Phys. Rev. A 41, 6676 (1990).
[10] Y. Tanimura and R. Kubo, J. Phys. Soc. Jpn. 58, 101 (1989).
[11] Y. Tanimura, J. Phys. Soc. Jpn. 75, 082001 (2006).
[12] J. Ilies-Smith, N. Lamberti, and A. Nazir, Phys. Rev. A 90, 032114 (2014).
[13] A. Garg, J. N. Onuchic, and V. Ambegaokar, J. Chem. Phys. 83, 4491 (1985).
[14] F. Shiba and T. Arimitsu, J. Phys. Soc. Jpn. 49, 891 (1980).
[15] S. Mukamel, I. Oppenheim, and J. Ross, Phys. Rev. A 17, 1988 (1978).
[16] M. A. Palenberg, R. J. Silbey, C. Warus, and P. Reineker, J. Chem. Phys. 114, 4386 (2001).
[17] A. Rivas, S. F. Huelga, and M. B. Plenio, Rep. Prog. Phys. 77, 094001 (2014).
[18] H.-P. Breuer, E.-M. Laine, and J. Piilo, Phys. Rev. Lett. 103, 210401 (2009).
[19] A. Rivas, S. F. Huelga, and M. B. Plenio, Phys. Rev. Lett. 105, 050403 (2010).
[20] S. Luo, S. Fu, and H. Song, Phys. Rev. A 86, 044101 (2012).
[21] B. Bylicka, D. Chruściński, and S. Maniscalco, Sci. Rep. 4, 5720 (2014).
[22] S. Lorenzo, F. Plastina, and M. Paternostro, Phys. Rev. A 88, 020102 (2013).
[23] M. M. Wolf and J. I. Cirac, Comm. Math. Phys. 279, 147 (2008).
[24] M. M. Wolf, J. Eisert, T. S. Cubitt, and J. I. Cirac, Phys. Rev. Lett. 101, 150402 (2008).
[25] A. Jamiołkowski, Rep. Math. Phys. 3, 275 (1972).
[26] M.-D. Choi, Linear Alg. Appl. 10, 285 (1975).
[27] H.-S. Zeng, N. Tang, Y.-P. Zheng, and G.-Y. Wang, Phys. Rev. A 84, 032118 (2011).
[28] P. Haikka, J. Goold, S. McEntoo, F. Plastina, and S. Maniscalco, Phys. Rev. A 85, 060101 (2012).
[29] D. Chruściński, A. Kossakowski, and A. Rivas, Phys. Rev. A 83, 052128 (2011).
[30] T. J. G. Apollaro, S. Lorenzo, C. Di Franco, F. Plastina, and M. Paternostro, Phys. Rev. A 90, 012310 (2014).
[31] C. Addis, B. Bylicka, D. Chruściński, and S. Maniscalco, Phys. Rev. A 90, 052103 (2014).
[32] D. Chruściński and S. Maniscalco, Phys. Rev. Lett. 112, 120404 (2014).
[33] K. S. Ranade and M. Ali, Open Sys. Info. Dyn. 14, 371 (2007).
[34] L. Skowronek, E. Størmer, and K. Życzkowski, J. Math. Phys. 50, 062106 (2009).
[35] See Supplemental Material at [URL will be inserted by publisher] for detailed information of the criteria and algorithm for $k$-positivity.
[36] N. K. Bernardes, A. Cuevas, A. Orieux, C. H. Monken, P. Mataloni, F. Sciarrino, and M. F. Santos, arXiv:1504.01602 .
[37] D. Chruśc?ki?ski and F. A. Wudarski, Phys. Lett. A 377, 1425 (2013).
[38] M. J. W. Hall, J. D. Cresser, L. Li, and E. Andersson, Phys. Rev. A 89, 042120 (2014).
[39] E.-M. Laine, J. Piilo, and H.-P. Breuer, Phys. Rev. A 81, 062115 (2010).
[40] G.-Y. Chen, S.-L. Chen, C.-M. Li, and Y.-N. Chen, Sci. Rep. 3, 2514 (2013).
[41] J. A. Mlynek, A. A. Abdumalikov, C. Eichler, and A. Wallraff, Nat. Commun. 5, 5186 (2014).