THE INDEX OF FAMILIES OF PROJECTIVE OPERATORS

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Abstract. Let $1 \to \Gamma \to \tilde{G} \to G \to 1$ be a central extension by an abelian finite group. In this paper, we compute the index of families of $\tilde{G}$-transversally elliptic operators on a $G$-principal bundle $P$. We then introduce the notion of families of projective operators on fibrations equipped with an Azumaya bundle $\mathcal{A}$. We define and compute the index of such families using the cohomological index formula for families of $SU(N)$-transversally elliptic operators. More precisely, a family $A$ of projective operators can be pulled back in a family $\tilde{A}$ of $SU(N)$-transversally elliptic operators on the $PU(N)$-principal bundle of trivialisations of $\mathcal{A}$. Through the distributional index of $\tilde{A}$, we can define an index for the family $A$ of projective operators and using the index formula in equivariant cohomology for families of $SU(N)$-transversally elliptic operators, we derive an explicit cohomological index formula in de Rham cohomology. Once this is done, we define and compute the index of families of projective Dirac operators. As a second application of our computation of the index of families of $\tilde{G}$-transversally elliptic operators on a $G$-principal bundle $P$, we consider the special case of a family of $Spin(2n)$-transversally elliptic Dirac operators over the bundle of oriented orthonormal frames of an oriented fibration and we relate its distributional index with the index of the corresponding family of projective Dirac operators.

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Introduction

This paper is devoted to an application of the cohomological index theorem shown in [7] using equivariant cohomology. In particular, using the main result of [7], we define a cohomological index for families of projective operators following [54, 55, 62]. Let us recall that in the standard case introduced in [54], Mathai, Melrose and Singer associated with an elliptic projective operator an analytical index and then computed...
this index by a cohomological formula à la Atiyah-Singer \cite{2,3,4}. This setting allows them to introduce a projective Dirac operator $\hat{\theta}_M^\gamma$ for any oriented manifold and as expected they obtained
\[
\text{Ind}_\gamma(\hat{\theta}_M^\gamma) = (2i\pi)^{-n} \int_M \hat{A}(TM),
\]
see \cite{54}. In \cite{55}, the same authors showed that a projective operator $A$ can be represented by a $SU(N)$-transversally elliptic operator $\hat{A}$ and they showed that the analytical index of the projective operator $A$ can be computed as the pairing of the distributional index of $\hat{A}$ with any smooth function on $SU(N)$ equal to 1 on a neighborhood of $\text{Id} \in SU(N)$. Recall that the operator $\hat{A}$ is obtained by pulling back the operator $A$ to the $\text{PU}(N)$-principal bundle associated with the Azumaya bundle $A \to M$ considered in the definition of the projective operator $A$, see Section 4 and \cite{54,55,53} for more details. Notice that here we have a central extension by an abelian finite group
\[
1 \longrightarrow \mathbb{Z}_N \longrightarrow SU(N) \longrightarrow \text{PU}(N) \longrightarrow 1.
\]
In \cite{62}, Paradan considered the general case of a central extension by an abelian finite group $\Gamma$ of a compact group $G$
\[
1 \longrightarrow \Gamma \longrightarrow \hat{G} \overset{\zeta}{\longrightarrow} G \longrightarrow 1,
\]
and computed the distributional index of any $\hat{G}$-transversally elliptic operator acting on a $G$-principal bundle $\mathcal{P}$. As shown by Atiyah in \cite{1}, this distributional character is supported in the subset $S$ of $\hat{G}$ of elements $\gamma \in \hat{G}$ such that $\mathcal{P}^\gamma \neq \emptyset$. Since $\hat{G}$ acts on $\mathcal{P}$ through the morphism $\zeta$ it follows that $S \subset \Gamma$. This allows Paradan to recover the index formula shown in \cite{54} for projective operators using the Berline-Paradan-Vergne index theorem for transversally elliptic operators, see \cite{17,05}. In particular, around any point of the support of the distributional index character, the index is given by a Atiyah-Singer formula, see \cite{62} Theorem 4.1. Here we point out that this result is completely similar to the results obtained in \cite{70} and that the result of Paradan \cite{62} generalises the result obtained for projective Dirac operators in \cite{75}.

In this paper, we follow Paradan’s approach and generalise it to the case of families. More precisely, we consider a central extension by an abelian finite group as in Equation (2) and a $G$-principal bundle $\mathcal{P} \to M$ where $M \to B$ is a fibration of compact manifolds. In this context, we compute the index of a family of $\hat{G}$-transversally elliptic operators along the fibres of $\mathcal{P} \to B$. We obtain the following generalisation of \cite{62} Theorem 4.1 to families using the index theorem à la Berline-Paradan-Vergne shown in \cite{7} for families of transversally elliptic operators.

**Theorem.** Let $\sigma \in K_G^\gamma(T_G(\mathcal{P}|B))$, we have $\text{Ind}^{\mathcal{P}|B}_{-\infty}(\sigma) = \sum_{\gamma \in \Gamma} T_\gamma(\sigma) * \delta_\gamma$, where
\[
T_\gamma(\sigma) = (-2i\pi)^{-\dim M + \dim B} \exp_\gamma\left( \int_{T(M|B)} \text{Ch}_\gamma(\sigma) \wedge \hat{A}(T(M|B))^2 \wedge e^\Theta \right).
\]
Here $\text{Ch}_\gamma(\sigma)$ is the twisted Chern character, see Definition 3.5 and $e^\Theta$ is the Chern-Weil morphism, see Section 5.1.

We then introduce the notion of families of projective operators by considering the special case given by the extension of Equation (1). Following \cite{55}, we define the analytical index of such families using the corresponding pairing with a smooth function on $SU(N)$ equal to 1 around $\text{Id} \in SU(N)$ with the distributional index defined in \cite{7}, see also Equation (5). Once this is done, we show using the previous theorem that the index of a projective family can be computed with a cohomological formula à la Atiyah-Singer with values in the de Rham cohomology of the base $B$.

The paper is divided as follows. We start by recalling standard results about functions and distributions on compact Lie groups. We then recall briefly the definitions of the equivariant cohomologies used in our computations. In Section 2 we recall the materials from \cite{6} regarding the index of families of transversally elliptic operators, see also \cite{8}. In Section 3 we prove the main result of this paper. In Section 4 we introduce the notion of families of projective operators and show the corresponding cohomological
We would like to mention that other directions have been investigated in [52, 53, 11, 12, 21, 22] and the references therein. In [62, 63], the authors deal with projective families of operators. In this case, the twist comes from the base space of the fibration and they obtained an index theorem in twisted $K$-theory and then deduced a cohomological formula. In [11], Benameur and Gorokhovsky showed a local index formula for projective families of Dirac operators using Bismut’s superconnection approach [18], see also [65]. In [22], Carrillo Rouse and Wang extended the setting from [63] to the case of foliations and showed a twisted index theorem in $K$-theory. In [21], Carrillo Rouse defined the pseudodifferential calculus that corresponds to the twisted $K$-theory for Lie groupoids. Independently, in [12] Benameur, Gorokhovsky and Leichtnam defined the corresponding pseudodifferential calculus in the special case of foliation, i.e. for the holonomy groupoid and showed higher index formulae using Bismut’s superconnection approach and extended the result of [11]. We point out that none of this results encompass our setting of families of projective operators and therefore in particular the setting of [54, 55]. We refer to [21] Section 7.2 for a discussion on this subject.

For interesting results concerning index theory, Lie groups and more generally groupoids, we refer the reader to [9, 10, 13, 23, 24, 26, 36, 37, 38, 46, 57, 77] and the references therein. In particular, we point out the similar setting of gauge-invariant operators investigated in [58, 59, 60].

1. Preliminaries

In this section we gather some well known facts about compact Lie groups that we will use in the sequel.
1.1. **Standard applications of Poincaré–Birkhoff–Witt theorem.** This subsection is devoted to standard results related with Poincaré–Birkhoff–Witt theorem, see for example [19]. Let $H$ be a compact connected Lie group and $\mathfrak{h}$ its Lie algebra. Recall that $H$ acts on itself on the right by $R_g(x) = xg^{-1}$, on the left by $L_g(x) = gx$ and therefore by conjugation $Ad(g)x = R_g L_g(x) = L_g R_g(x)$. The action by conjugation is called the adjoint action. We denote the induced action of an element $s \in H$ on $\mathfrak{h}$ again by $Ad(s)$.

Let $\mathcal{U}(\mathfrak{h})$ denote the universal enveloping algebra and $\mathcal{Z}(\mathfrak{h}) := \mathcal{Z}(\mathcal{U}(\mathfrak{h}))$ its center. Denote by $\mathcal{U}(\mathfrak{h})^b := \{u \in \mathcal{U}(\mathfrak{h}) \mid uX = Xu, \forall X \in \mathfrak{h}\}$. We clearly have $\mathcal{Z}(\mathfrak{h}) \subset \mathcal{U}(\mathfrak{h})^b$ and similarly if $v \in \mathcal{U}(\mathfrak{h})^b$ then for any $Y_1, \ldots, Y_k \in \mathfrak{h}$ we have $[v, Y_1 \cdots Y_k] = 0$ and such products $Y_1 \cdots Y_k$ generate $\mathcal{U}(\mathfrak{h})$. In other words, $\mathcal{Z}(\mathfrak{h}) = \mathcal{U}(\mathfrak{h})^b$. We denote by $C^{-\infty}_c(H)$ the set of distributions on $H$ supported in $\gamma$. Let $S(\mathfrak{h})$ be the symmetric algebra. The following results are well known, we will only give the main ideas of the proofs for the convenience of the reader.

**Proposition 1.1.**

1. The enveloping algebra $\mathcal{U}(\mathfrak{h})$ can be canonically identified with the algebra $C^{-\infty}_c(H)$ of distributions on $H$ supported at the identity.

2. The center $\mathcal{Z}(\mathfrak{h})$ corresponds to the set $C^{-\infty}_c(H)^{Ad(H)}$ of $Ad(H)$-invariant distributions on $H$ supported at the identity.

3. Let $\gamma \in \mathcal{Z}(H) := \{h \in H \mid \forall t \in H, \, ht = th\}$. The map $\mathcal{Z}(\mathfrak{h}) \to C^{-\infty}_c(H)^{Ad(H)}$ given by $T \mapsto T \ast \delta_\gamma$, where $\delta_\gamma$ is the Dirac delta function in $\gamma$, is an isomorphism.

4. The exponential map $exp : \mathfrak{h} \to H$ defines a linear isomorphism (but not of algebras) $exp_\ast : S(\mathfrak{h})^{Ad(H)} \to \mathcal{Z}(\mathfrak{h})$.

where $S(\mathfrak{h})^{Ad(H)}$ is viewed as the algebra of $Ad$-invariant distributions on $\mathfrak{h}$ supported at $0$.

**Proof.** Recall that $C^{-\infty}_c(H)$ is an algebra for the convolution defined by $T \ast T'(f) = T \otimes T'(\mu^* f)$, where $\mu : H \times H \to H$ is the product on $H$, i.e. $\mu^*$ is the comultiplication. In other words, $T \ast T'(f) = \langle T_{h_1}, (T_{h_2}(R_{h_2}(f))(h_1)) \rangle$.

1. Denote by $\delta_1$ the Dirac delta function in $1 \in H$. Let $D_1 : \mathfrak{h} \to C^{-\infty}_c(H)$ be the map given by $X \mapsto D_1(X) := X^\mu_1 \delta_1$, where $X^\mu_1 \delta_1(f) = -X^\mu_1(f)(1)$ is the derivative of $\delta_1$ along $X \in \mathfrak{h}$. Clearly, $D_1([X, Y]) = D_1(X) + D_1(Y) - D_1(Y \ast D_1(X)$. Therefore, the universal property of $\mathcal{U}(\mathfrak{h})$ implies that $D_1$ can be extended to the universal enveloping algebra. The map $D_1$ is injective since, by Poincaré–Birkhoff-Witt theorem, a basis of $\mathcal{U}(\mathfrak{h})$ is given by products $X_{j_1}^{a_1} \cdots X_{j_n}^{a_n}$, where $X_i$ is a basis of $\mathfrak{h}$ and $j_i \geq 0$. Moreover, the images are linearly independent differential operators composed with the Dirac delta function. The surjectivity follows from [21] Theorem XXXV p 100].

2. Since $H$ is compact and connected every element $h \in H$ is in the image of the exponential map. Therefore, $u \in \mathcal{Z}(\mathfrak{h})$ if and only if it commutes with every $X \in \mathfrak{h}$. But this is equivalent to $Ad(e^X)u^* \delta_1 = u^* \delta_1$ for any $X \in \mathfrak{h}$. In other words, $u^* \delta_1$ is $Ad(H)$-invariant.

3. Let $T \in \mathcal{Z}(\mathfrak{h})$ be identified with its corresponding element in $C^{-\infty}_c(H)^{Ad(H)}$. Clearly the convolution by $\delta_\gamma$ as support $\gamma$ since $\delta_1 \ast \delta_\gamma$ as support $\gamma$. Moreover, the convolution by $\delta_\gamma$ is an isomorphism since the convolution by $\delta_{\gamma^{-1}}$ is an inverse. Now since $\gamma$ is central we have $Ad(h)\gamma = \gamma$. Therefore we get $T \ast \delta_\gamma(Ad(h)f) = T \ast \delta_\gamma(\mu^*(Ad(h)f)) = T(Ad(h)R_{Ad(h)}(f)) = T(R_{\gamma}(f)) = T \ast \delta_\gamma(f)$.

4. Let $v_1, \ldots, v_n$ be a basis of $\mathfrak{h}$. We can see $S(\mathfrak{h})$ as the algebra of distributions on $\mathfrak{h}$ supported in $0$ using the map $\sum a_\alpha v^\alpha \mapsto \sum (-1)^{\alpha} a_\alpha (v^*)^{\alpha} \delta_0$, where $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $v^\alpha = v_1^{\alpha_1} \cdots v_n^{\alpha_k}$. The isomorphism is given on monomial by $\exp_\ast(X_{i_1} \cdots X_{i_p}) = \frac{1}{p!} \sum_{\sigma \in S_p} X_{i_{\sigma(1)}} \cdots X_{i_{\sigma(p)}}$ and clearly if $T \in S(\mathfrak{h})$ is $Ad(H)$-invariant then its image sits in $\mathcal{Z}(\mathfrak{h})$. Indeed, $Ad(e^X)\exp_\ast(T) = \exp_\ast(Ad(e^X)T) = \exp_\ast(T)$ and therefore $\exp_\ast(T)$ commutes with every $X \in \mathfrak{h}$, i.e. $\exp_\ast(T) \in \mathcal{Z}(\mathfrak{h})$. Notice that the convolution on $S(\mathfrak{h})$ is commutative since it comes from the additive structure on $\mathfrak{h}$. 


Recall the identifications of $S(h)_{Ad(H)}$ with the algebra of $Ad(H)$-invariant distributions on $h$ supported in 0 and of $Z(h)$ with the algebra of $Ad(H)$-invariant distributions on $H$ supported in 1. The map $\exp_s$ is the usual pushforward of distributions, i.e. if $T \in C^\infty_c(U)$, $\phi : U \to V$ is a smooth map such that $f_{|\text{supp}(T)}$ is proper, and $f \in C^\infty_c(V)$ then $\phi_* T(f) = T(\phi^* f)$.

\[ \begin{array}{c}
\text{Lemma 1.3.}\quad \text{results, see [31, Section 2.2 & 2.3] for more details.}
\end{array} \]

Recall that $H(s) := \{ h \in H, hs = sh \}$ can be seen as the closed subgroup of $H$ given by the stabilizer $\text{Stab}_{Ad(H)}(s) = \{ h \in H, Ad(h)s = s \}$ in $H$ of $s$ for the adjoint action but also as the submanifold of $H$ of fixed points $H^{Ad(s)} = \{ h \in H, Ad(h)h = h \}$ by $Ad(s)$. Denote by $h(s) := \{ Y \in h, Ad(s)Y = Y \}$ the Lie algebra of $H(s)$. If we chose a $Ad(H)$-invariant scalar product on $h$ then we get a bi-invariant metric on $H$, i.e. a metric which is both left invariant and right invariant. Using this metric $Ad(H)s \times h(s)$ can be identified with the orthogonal to the $Ad(H)$ orbit of $s$. By the slice theorem we obtain that there is an open set $\mathcal{U}_s(0) \subset h(s)$ such that $H \times H(s) \mathcal{U}_s(0)$ identifies with an open neighbourhood $W(s, 0)$ of $Ad(H)s$. The identification $\Phi$ is given by $[k, Y] \mapsto Ad(k)\exp_s(Y) = ks^\gamma k^{-1}$ since the exponential map for a bi-invariant Riemannian metric coincides with the Lie group exponential map.

When $V$ is a finite dimensional vector space, we denote by $\det_V(J)$ the determinant of $J \in \text{End}(V)$. Since $h(s)$ is $Ad(s)$-invariant, we can restrict $Ad(s)$ to $q(s) = h(s) \supset h(s)$ and $\det_X(J)$ is a smooth map such that $\det_X(J) = s^\gamma s^{-1}$ (since the exponential map for a bi-invariant Riemannian metric coincides with the Lie group exponential map.

\[ \begin{array}{c}
\text{Lemma 1.3.}\quad \text{results, see [31, Section 2.2 & 2.3] for more details.}
\end{array} \]

\[ \begin{array}{c}
(1) \quad \text{We have } h(s) = \text{im}(id - Ad(s)) \text{ and } \det_{q(s)}(id - Ad(s)) > 0.
(2) \quad \text{If } \mathcal{U}_s(0) \text{ is a small enough neighbourhood of } 0 \in h(s) \text{ then } \det_{q(s)}(id - Ad(se^Y)) > 0, \forall Y \in \mathcal{U}_s(0).
(3) \quad \text{The differential of } \Phi : H \times H(s) \mathcal{U}_s(0) \to H, \text{ i.e. the differential } d_{[k, Y]}\Phi \text{ is given modulo composition with } dL_%{k^{-1}s^{-1}} \text{ by }
\quad \quad \quad D(X, Z) = Ad(k)(e^{-Y}\exp_X Y + (Ad(se^Y)^{-1} - id)X)
\text{ for any } (X, Z) \in q(s) \times h(s) \cong T_{[k, Y]} H \times H(s) \mathcal{U}_s(0).
(4) \quad \text{We have }
\quad \quad \quad |\det(d_{[k, Y]}\Phi)| = |\det_{q(s)}(id - Ad(se^Y))|\det_{h(s)}(e^{-Y}\exp_X Y)|.
\end{array} \]

\[ \begin{array}{c}
\text{Proof.} \quad 1. \quad \text{Let } v \in h \text{ then } v - Ad(s)v \in h(s)^\perp. \text{ Indeed, let } w \in h(s) \text{ then } Ad(s)w = Ad(s^{-1})w = w \text{ therefore } \langle v - Ad(s)v, w \rangle = \langle v, w \rangle - \langle v, Ad(s)^{-1}w \rangle = \langle v, w \rangle - \langle v, w \rangle = 0.
\text{But } h(s) = \text{ker}(id - Ad(s)) \text{ and dim } h(s)^\perp = \text{dim } h - \text{dim } h(s) = \text{dim im(id - Ad(s))). \text{ Recall that } t \mapsto \det_t(id - tAd(s)) \text{ is a real polynomial. Now, since } Ad(s) \text{ is orthogonal, every eigenvalue has modulus 1 and on } q(s) = h/s \text{ every eigenvalue is different from 1. If } -1 \text{ is an eigenvalue then } 1 - (-1)^2 = 2 \text{. Moreover, every complex eigenvalue } \mu \text{ is pair conjugate and } (1 - \bar{\mu})(1 - \mu) = |1 - \mu|^2 > 0.
\text{2. The map } Y \mapsto f(Y) = \det_{q(s)}(id - Ad(se^Y)) \text{ is continuous and } \det_{q(s)}(id - Ad(s)) > 0 \text{ therefore } f^{-1}([0, +\infty]) \text{ is open in } h(s) \text{ and contains 0.}
\text{3. Notice that the tangent space } T_{(0, Y)}q(s) \times h(s) \text{ to } q(s) \times h(s) \text{ in } (0, Y) \text{ identifies with the tangent space } T_{[k, Y]} H \times H(s) \mathcal{U}_s(0) \text{ to } H \times H(s) \mathcal{U}_s(0) \text{ in } [k, Y]. \text{ Indeed, use the differential of the map } j : q(s) \times
\[ D(X, Z) = \frac{d}{dt} |_{t=0} ke^{-Y}s^{-1}k^{-1}ke^{tX}se^{Y} e^{TZ}e^{-tX}k^{-1} \]
\[ = Ad(k) \frac{d}{dt} |_{t=0} e^{-Y}s^{-1}ke^{tX}se^{Y} e^{TZ}e^{-tX} \]
\[ = Ad(k)(Ad(e^{-Y}s^{-1}X + e^{-Y}Adp(Y)Z - X) \]
\[ = Ad(k)(e^{-Y}Adp(Y)Z + (Ad(se^{Y})^{-1} - id)X). \]

4. We have \( e^{-Y}Adp(Y)Z \in \mathfrak{h} \) since \( e^{Y+TZ} \in H(s) \) and similarly \( (id - Ad(se^{Y}))X \in \mathfrak{q}(s) \) because \( (id - Ad(se^{Y}))q(s) \subset \mathfrak{q}(s) \). Since the metric on \( \mathfrak{h} \) is \( Ad(H) \)-invariant, we get that \( |det_{\mathfrak{h}}(Ad(k))| = |det_{\mathfrak{q}(s)}(Ad(se^{Y}))| = 1 \). This gives the result using 3.

Recall that the differential \( Adp(Y) \) of the exponential map \( h(s) \rightarrow H(s) \) is given in \( Y \in \mathfrak{h}(s) \) by
\[ Adp(Y) = e^{Y} \frac{1 - e^{-ad(Y)}}{ad(Y)}, \]
where \( ad \) is the differential of the adjoint action \( Ad \) on \( \mathfrak{h}(s) \). We denote as usual by \( j_{\mathfrak{h}}(Y) = det_{\mathfrak{h}}(1 - e^{-ad(Y)}) \) the Jacobian determinant of \( Adp(Y) \) which is positive on \( U_{\mathfrak{h}}(0) \) if \( U_{\mathfrak{h}}(0) \) is small enough.

Let \( \mu_{s} \) and \( \mu \) denote respectively the normalized Haar measures on \( H(s) \) and \( H \). Recall that there is a unique \( H \)-invariant measure \( \mu_{H/H(s)} \) on \( H/H(s) \) such that \( d\mu = d\mu_{H/H(s)} \) defined by the linear functional \( f \in C(H/H(s)) \rightarrow \int_{H/H(s)} f d\mu_{H/H(s)} := \int_{H} f \circ \pi(h) d\mu(h), \) where \( \pi : H \rightarrow H/H(s) \) is the canonical fibration. We denote \( d\mu_{H/H(s)} \) by \( dq \) and \( d\mu_{h} \) by \( dy \). Denote by \( dy \) the tangent Lesbeq measure on \( \mathfrak{h} \) and respectively by \( dY \) and \( dQ \) Lebesgue measures on \( h(s) \) and \( q(s) \) tangent to \( dy \) and \( dq \) such that \( dX = dYdQ \), see [31].

**Theorem 1.4.** [31] Let \( \alpha \in C^{\infty}(H)^{AdH} \) and \( \varphi \in C^{\infty}(H) \) be functions supported in \( W(s, 0) \cong H \times H(s)U_{\mathfrak{h}}(0) \). Then
\[ \int_{W(s, 0)} \alpha(h)\varphi(h)d\mu(h) = \int_{H/H(s)} \int_{U_{\mathfrak{h}}(0)} \alpha(se^{Y})\varphi(qse^{Y}q^{-1})det_{\mathfrak{q}(s)}(id - se^{Y})j_{\mathfrak{h}}(Y)dYdq. \]

Let \( \theta \in C^{\infty}(H)^{AdH} \) then there is a unique \( \psi \in C^{\infty}(U_{\mathfrak{h}}(0))^{H(s)} \) such that \( \forall \varphi \in C^{\infty}(H)^{AdH} \) supported in \( W(s, 0) \)
\[ \int_{W(s, 0)} \theta(h)\varphi(h)d\mu(h) = \int_{H/H(s)} \int_{U_{\mathfrak{h}}(0)} \psi(se^{Y})\varphi(qse^{Y}q^{-1})det_{\mathfrak{q}(s)}(id - se^{Y})j_{\mathfrak{h}}(Y)dYdq. \]

This means that \( \theta \in C^{\infty}(H)^{H} \) defines by restriction an element \( \psi \in C^{\infty}(U_{\mathfrak{h}}(0))^{H(s)} \). We will denote the restricted element by \( \theta \|_{\mathfrak{h}} \). For details on restrictions of invariant generalized functions see for instance [31] [65].

1.3. **Equivariant cohomology.** Here we recall the definition of equivariant cohomologies used in the sequel, see [15] [16] [31]. Let again \( H \) be a compact Lie group and \( \mathfrak{h} \) its Lie algebra. Assume that \( H \) acts smoothly on a manifold \( W \) (we say that \( W \) is a \( H \)-manifold). Let \( X \in \mathfrak{h} \) and denote by \( X_{W} \) the vector field generated by \( X \) on \( W \) that is \( X_{W}(f)(w) = \frac{1}{\mu_{W}} \int_{W} f(e^{-tX} \cdot w, w \in W) \). Let \( d \) be the de Rham differential and let \( \iota(Y) \) denote the contraction by a vector field \( Y \). Let \( \mathcal{A}(W) \) be the space of differential forms on \( W \). Recall that the group \( H \) acts on \( \mathcal{A}(W) \) and consider the tensored product \( C^{\infty}(\mathfrak{h}) \otimes \mathcal{A}(W) \) equipped with the tensored action given by \( (s \cdot \alpha)(X) = s(\alpha(Ad(s^{-1}X))), \) for any \( \alpha \in C^{\infty}(\mathfrak{h}) \otimes \mathcal{A}(W) \). Let \( \mathcal{A}^{\infty}_{H}(\mathfrak{h}, W) \) denote the algebra \( (C^{\infty}(\mathfrak{h}) \otimes \mathcal{A}(W))^{H} \) of \( H \)-invariant smooth functions on \( \mathfrak{h} \) with values in \( \mathcal{A}(W) \). Let \( D \) be the equivariant differential on \( \mathcal{A}^{\infty}_{H}(\mathfrak{h}, W) \) given by
\[ (D\alpha)(X) = d(\alpha(X)) - \iota(X_{W})(\alpha(X)). \]
We have \( (D^{2}\alpha)(X) = -\mathcal{L}(\alpha)(X) \) so \( D^{2} \) is zero on \( \mathcal{A}^{\infty}_{H}(\mathfrak{h}, W) \) because any element of \( \mathcal{A}^{\infty}_{H}(\mathfrak{h}, W) \) is \( H \)-invariant.
Definition 1.5. The equivariant cohomology \( \mathcal{H}_H^\infty(h, W) \) with smooth coefficients is the cohomology of the complex \( (\mathcal{A}_H^\infty(h, W), D) \).

We now recall the definition of the equivariant cohomology with generalised coefficients \[30\], see also \[47\]. Let \( C^{-\infty}(h, \mathcal{A}(W)) \) be the space of generalised functions on \( h \) with values in \( \mathcal{A}(W) \). By definition, this is the space of continuous linear maps from the space \( \mathcal{D}(h) \) of \( C^\infty \) densities with compact support on \( h \) to \( \mathcal{A}(W) \), where \( \mathcal{D}(h) \) and \( \mathcal{A}(W) \) are equipped with the \( C^\infty \) topologies. Therefore, if \( \alpha \in C^{-\infty}(h, \mathcal{A}(W)) \) and if \( \phi \in \mathcal{D}(h) \) then \( \langle \alpha, \phi \rangle \) is a differential form on \( W \) denoted by \( \int_h \alpha(X)\phi(X)dX \). A \( C^\infty \) density with compact support on \( h \) is also called a test density, and a \( C^\infty \) function with compact support on \( h \) is called a test function. Denote by \( E^i \) a basis of \( h \) and \( E_i \) its dual basis. Let \( d \) be the operator on \( C^{-\infty}(h, \mathcal{A}(W)) \) defined by
\[
\langle d\alpha, \phi \rangle = d\langle \alpha, \phi \rangle, \quad \forall \phi \in \mathcal{D}(h).
\]

Let \( \iota \) be the operator defined by
\[
\langle \iota \alpha, \phi \rangle = \sum_i \iota(E^i)\langle \alpha, E_i \otimes \phi \rangle,
\]
where \( E^i_W \) means as before the vector field generated by \( E^i \in h \) on \( W \) and \( (E_i \otimes \phi)(X) = E_i(X)\phi(X) = X_i\phi(X) \), for any \( X = \sum X_i E^i \in h \). Let then \( d_h \) be the operator on \( C^{-\infty}(h, \mathcal{A}(W)) \) defined by
\[
d_h\alpha = d\alpha - \iota\alpha.
\]
The operator \( d_h \) coincides with the equivariant differential on \( C^\infty(h, \mathcal{A}(W)) \subset C^{-\infty}(h, \mathcal{A}(W)) \). The group \( H \) acts naturally on \( C^{-\infty}(h, \mathcal{A}(W)) \) by \( \langle g\cdot \alpha, \phi \rangle = g \cdot \langle \alpha, g^{-1} \cdot \phi \rangle \) and this action commutes with the operators \( d \) and \( \iota \). The space of \( H \)-invariant generalised functions on \( h \) with values in \( \mathcal{A}(W) \) is denoted by
\[
\mathcal{A}_H^{-\infty}(h, \mathcal{A}(W)) = C^{-\infty}(h, \mathcal{A}(W))^H.
\]
The operator \( d_h \) preserves \( \mathcal{A}_H^{-\infty}(h, W) \) and satisfies \( d_h^2 = 0 \). Similarly, if we replace \( \mathcal{A}(W) \) with \( \mathcal{A}_c(W) \) the space of compactly supported forms then we can define \( \mathcal{A}_c,H^{-\infty}(h, W) = C^{-\infty}(h, \mathcal{A}_c(W))^H \).

We also need to consider \( H \)-equivariant generalised forms which are defined on an open neighbourhood of the origin in \( h \). If \( O \) is an \( H \)-invariant open subset of \( h \), we denote by \( \mathcal{A}_H^{-\infty}(O, W) \) and \( \mathcal{A}_c,H^{-\infty}(O, W) \) the spaces obtained similarly. Let \( U \) be a \( H \)-invariant open subset of \( W \). The space of forms with generalised coefficients and with support in \( U \) is denoted by \( \mathcal{A}_H^{-\infty}(O, W) \). This is the space of differential forms with generalized coefficients such that there is a \( H \)-invariant closed subspace \( C_\alpha \subset U \) such that \( \int \alpha(X)\phi(X)dX \) is supported in \( C_\alpha \) for any test density \( \phi \).

Notation 1.5.1. The cohomology of the complex \( (\mathcal{A}_H^{-\infty}(h, W), d_h) \) is denoted by \( \mathcal{H}_H^{-\infty}(h, W) \).
The cohomology of the complex \( (\mathcal{A}_c,H^{-\infty}(h, W), d_h) \) is denoted by \( \mathcal{H}_c,H^{-\infty}(h, W) \).
The cohomology of the complex \( (\mathcal{A}_H^{-\infty}(O, W), d_h) \) is denoted by \( \mathcal{H}_H^{-\infty}(O, W) \).
The cohomology of the complex \( (\mathcal{A}_c,H^{-\infty}(O, W), d_h) \) is denoted by \( \mathcal{H}_c,H^{-\infty}(O, W) \).
Let \( F \subset W \) be a closed subspace and let \( \mathcal{H}_F^{-\infty}(h(s), W^s) \) be the projective limit of the projective system \( (\mathcal{H}_U^{-\infty}(h(s), W^s))_{F \subset U} \).

There is a natural map
\[
\mathcal{H}_H^{-\infty}(h, W) \to \mathcal{H}_H^{-\infty}(h, W)
\]
induced by the inclusion \( \mathcal{A}_H^{-\infty}(h, \mathcal{A}(W)) \to \mathcal{A}_H^{-\infty}(h, \mathcal{A}(W)) \). If \( p : M \to B \) is an oriented \( H \)-equivariant fibration, then integration along the fibres \( \int_M \) defines a map from \( \mathcal{A}_c,H^{-\infty}(h, M) \) to \( \mathcal{A}_c,H^{-\infty}(h, B) \):
\[
\langle \int_M \alpha, \phi \rangle := \int_M \langle \alpha, \phi \rangle, \quad \forall \phi \in \mathcal{D}(h),
\]
and induces a well defined map:
\[
\int_M : \mathcal{H}_c,H^{-\infty}(h, M) \to \mathcal{H}_c,H^{-\infty}(h, B).
\]
Finally note that if $\alpha \in \mathcal{H}_{c,H}^\infty (\mathfrak{h}, M)$, and $\beta \in \mathcal{H}_{c,H}^\infty (\mathfrak{h}, B)$ then $\alpha \wedge p^*\beta \in \mathcal{H}_{c,H}^\infty (\mathfrak{h}, M)$ and

$$\int_{M|B} \alpha \wedge p^*\beta = (\int_{M|B} \alpha) \wedge \beta.$$ 

2. The index of transversally elliptic families

In this section, we first recall the setting of [8] and refer to it for details. Then we describe the support of the distributional index of families of $H$-transversally elliptic operators introduced in [7]. Let $H$ be a compact Lie group and let $p : Z \to B$ be a compact $H$-fibration with trivial action on $B$. We denote by $Z_b = p^{-1}(b)$ the fibre sitting above $b \in B$. We denote by $T(Z|B) := \ker(dp)$ the vertical subbundle of $TZ$. Let $\mathfrak{h}$ be the Lie algebra of $H$ and recall that an element $X \in \mathfrak{h}$ defines a vector field $X_Z(x) := \frac{d}{dt}|_{t=0} e^{-tX} x$ and that $X_Z(x) \in T_x(Z|B)$ is vertical. Using a $H$-invariant Riemannian metric on $Z$, we identify $T^*Z$ and $TZ$.

Let us recall the definition of the vertical $H$-transversal cotangent space $T^*_H(Z|B)$. Following [8], we denote by $T^*_HZ := \{(x, \alpha) \in T^*Z, \alpha(X_Z(x)) = 0, \forall X \in \mathfrak{h}\}$ and identify it with the set $T_HZ$ of vectors orthogonal to the orbits with the help of the $H$-invariant Riemannian metric. Similarly, we can consider $T^*_H(Z|B) := \{(x, \alpha) \in T^*(Z|B), \alpha(X_Z(x)) = 0, \forall X \in \mathfrak{h}\}$ and we can identify it with the set $T_H(Z|B)$ of vertical tangent vectors orthogonal to the orbits using the $H$-invariant Riemannian metric. We then call $T_H(Z|B) := T(Z|B) \cap T_HZ$ the vertical $H$-transversal tangent space.

Let $E = E^+ \oplus E^-$ be a $\mathbb{Z}_2$-graded hermitian vector bundle. In the sequel, we shall denote by $\Psi^m(Z|B, E)$ the set of smooth families of order $m$ pseudodifferential operators on $Z$ and by $\Psi^{-\infty}(Z|B, E)$ the smoothing families, see [5].

We shall say that a $H$-invariant smooth family $A_0 := (A_{0,b} : C^\infty(Z_b, E^+_b) \to C^\infty(Z_b, E^-_b))_{b \in B}$ of pseudodifferential operators is $H$-transversally elliptic if its principal symbol $\sigma(A_0)(\xi)$ is invertible for any non zero vector $\xi \in T_H(Z|B)$, see [9]. Recall that every element $a \in K_H(T_H(Z|B))$ of the compactly supported $H$-equivariant $K$-theory group of $T_H(Z|B)$ can be represented by the principal symbol $\sigma(A_0)|_{T_H(Z|B)}$ of a $H$-invariant family $A_0$ of $H$-transversally elliptic operators. Let $A_0^*$ be the formal adjoint of $A_0$ and denote by $A := \begin{pmatrix} 0 & A_0^* \\ A_0 & 0 \end{pmatrix}$. We denote by $E = E^+ \oplus E^-$ the Hilbert $C(B)$-module associated with the continuous field $(L^2(Z_b, E_b; \mu_b))_{b \in B}$ of square integrable sections along the fibres with respect to a $H$-invariant continuous family of Borel measures $(\mu_b)_{b \in B}$ in the Lebesgue class. When $A_0$ is a family of order $0$ pseudodifferential operators, $A$ extends to an adjointable operator in $L^2(C(B, E))$. Let $C^*H$ be the $C^*$-algebra of the compact group $H$. Recall the representation $\pi : C^*H \to \mathcal{L}(C(B, E))^\prime$ of $C^*H$ as adjointable operators on $E$ given by $\pi(\varphi)s(x) = \int_H \varphi(h)\hat{s}(s(h^{-1}x))dh$, where $\varphi \in C(H)$, $s \in C(Z, E)$ and the integration is with respect to the Haar measure on $H$. We shall denote the Kasparov’s bivariant $K$-theory group of the pair of $C^*$-algebras $(C^*H, C(B))$ by $KK(C^*H, C(B))$, see [11, 13].

**Definition 2.1.** [8] The analytical index map

$$\text{Ind}^{Z|B} : K_H(T_H(Z|B)) \to KK(C^*H, C(B))$$

is defined by

$$\text{Ind}^{Z|B}([\sigma(A_0)|_{T_H(Z|B)}]) := [E, \pi, A].$$

Denote by $\hat{H}$ the set of isomorphism classes of unitary irreducible representations of $H$.

**Proposition 2.2.** Let $H$ be a compact Lie group and $Z \to B$ be a compact $H$-fibration with trivial action on $B$. Then the analytical index map is a $\hat{H}(H)$-morphism in the following sense

$$\text{Ind}^{Z|B}(a \otimes [V]) = j^H[V] \otimes_{C^*H} \text{Ind}^{Z|B}(a),$$

where $V \in \hat{H}$ and $[V] \in K_H(\mathbb{C})$ is the corresponding element.
Proof. This is exactly the multiplicative property shown in [6] with \( Z' = \{ * \} \to B' = \{ * \} \) and \( H' = \{ 1 \} \), see also [8]. Let us recall briefly the proof in this simpler case for the benefit of the reader. The index class \( \text{Ind}_Z \) is defined because \((V \otimes \pi, \rho) \). Let us recall briefly the proof in this simpler case for the benefit of the reader. The index class \( \text{Ind}_Z \) is defined because \( \pi(C^* H), \pi \mid \eta \in \pi \in C(H), \eta \in \pi \). The K-multiplicity is the completion of \( C(H, V) \) with respect to the \( C^* H \)-valued scalar product given by
\[
\langle v_1, v_2 \rangle(h) := \int_H \langle v_1(k), v_2(\chi(k)) \rangle dk, \quad \forall v_1, v_2 \in C(H, V),
\]
and \( \rho(\phi)(v) = \int_H \phi(k)v(k^{-1}h)dk, \forall \phi \in C(H) \) and \( v \in C(H, V) \). Notice that the operator \( 1 \otimes \pi A \) is well defined because \( \pi(C^* H), \pi \mid \eta \in \pi \in C(H), \eta \in \pi \). Furthermore, for \( v_1, v_2 \in C(H, V) \) and \( \eta_1, \eta_2 \in \pi \), the identity
\[
\langle \pi(v_1 \otimes \eta_1), \pi(v_2 \otimes \eta_2) \rangle = \langle \eta_1, \pi(v_1, v_2) \rangle \eta_2
\]
can be checked as follows. Using the \( \pi \)-invariance of the scalar product on \( \pi \), we have
\[
\langle \pi(v_1 \otimes \eta_1), \pi(v_2 \otimes \eta_2) \rangle = \int_{H^2} \langle v_1(k), v_2(h) \rangle(k \eta_1, h \eta_2) dkd\eta
\]
\[
= \int_{H^2} \langle \eta_1, v_1(k), v_2(h) \rangle k^{-1} dkd\eta.
\]
The substitution \( u = k^{-1}h \) gives directly
\[
\langle \pi(v_1 \otimes \eta_1), \pi(v_2 \otimes \eta_2) \rangle = \int_{H^2} \langle \eta_1, v_1(k), v_2(ku) \rangle(\eta_2) dkd\eta
\]
\[
= \int_{H^2} \langle \eta_1, v_1(k), \eta_2 \rangle(\eta_2) dkd\eta
\]
\[
= \langle \eta_1, \pi(v_1, v_2) \rangle \eta_2.
\]
To show that \( \pi(C(H, V) \otimes \pi \pi) \) is dense in \( \pi \otimes \pi \), consider an approximate identity \( (e_i) \) of \( C^* H \), composed of continuous functions on \( H \) which are supported as close as we please to the neutral element of \( H \). Then \( \pi((v \otimes e_i) \otimes \eta) = v \otimes \pi(e_i) \eta \) converges to \( v \otimes \eta \) for any \( v \in V \) and \( \eta \in \pi \). Similar computations also imply that \( U \) intertwines operators and representations. \( \square \)

Recall that Green-Julg isomorphism \( K_H(\pi) \cong KK_C(C^* H) \) [11] is given by \( \theta \in \hat{H} \mapsto \chi_\theta = \chi_0 \otimes \pi \), where \( \chi_0 = [\pi, 0] \in KK_C(C^* H) \) and the Hilbert \( C^* H \)-module structure is given by
\[
\langle \lambda, \lambda' \rangle(g) = \lambda(\lambda')g, \quad \lambda \cdot \phi = \lambda \int_H \phi(h) d\lambda
\]
where \( \lambda, \lambda' \in C \) and \( \phi \in C(H) \).

Let \( V \in \hat{H} \) and consider the Hilbert \( C(B) \)-module \( C_V := (V \otimes \pi) \) and the operator \( A_V := (id_V \otimes A) E^H C \). We can now introduce the definition of K-multiplicity of an irreducible unitary representation of \( H \) from [6].

**Definition 2.3.** [6] The K-multiplicity \( m_A(V) \) of an irreducible unitary representation \( V \) of \( H \) in the index class \( \text{Ind}_Z \) is the image of the class \([E^H_V, A^H_V]) \in KK_C(C(B)) \) under the isomorphism \( KK_C(C(B)) \cong K(B) \). So \( m_A(V) \) is the class of a virtual vector bundle over \( B \), an element of the topological K-theory group \( K(B) \). The class \([E^H_V, A^H_V]) \) coincides (as expected) with the Kasparov product
\[
\chi_V \otimes \text{Ind}_Z \in KK_C(C(B)),
\]
where \( \chi_V = \chi_0 \otimes j^H[V] \in KK_C(C^* H) \) is the element image of \([V] \in K_H(C) \) by the Green-Julg isomorphism.
Since $KK(C^* H, C(B)) \cong \text{Hom}(R(H), K(B))$ (see for instance [70]), we have the following description of the index map:

**Proposition 2.4.** [7] The index class of a $H$-invariant family $A_0$ of $H$-transversally elliptic operators is totally determined by its multiplicities and we have:

$$\text{Ind}_{C^* B}(A_0) = \sum_{V \in H} m_A(V) \chi_V.$$

The next proposition explains that the index map is a $R(H)$-module homomorphism, using the description of the index map from the previous proposition.

**Proposition 2.5.** For any $a \in K_H(T_H(Z|B))$, we have

$$\text{Ind}_{Z|B}(a \cdot [V]) = \sum_{W \in \hat{H}} m_a(W) \chi_V \chi_W.$$

**Proof.** Let $\theta \in \hat{H}$. We have $\langle \text{Ind}_{Z|B}(a \cdot [V]), \chi_\theta \rangle = m_a \otimes (\chi_\theta \otimes j^H([V]) \otimes \text{Ind}_{Z|B}(a))$. Using Green-Julg isomorphism, it follows

$$\langle \text{Ind}_{Z|B}(a \cdot [V]), \chi_\theta \rangle = \chi_0 \otimes j^H([\theta]) \otimes j^H([V]) \otimes \text{Ind}_{Z|B}(a)$$

$$= \chi_0 \otimes j^H((\theta \otimes [V]) \otimes \text{Ind}_{Z|B}(a)$$

$$= \chi_0 \otimes \chi_V \otimes \text{Ind}_{Z|B}(a)$$

$$= m_a(\theta \otimes V)$$

$$= \langle \text{Ind}_{Z|B}(a), \chi_\theta \rangle.$$

The last equality follows from the relations

$$\theta \otimes V = \sum_{W \in \hat{H}} \dim \left( (W^* \otimes (\theta \otimes V))^H \right) W,$$

$$m_a(\theta \otimes V) = \sum_{W \in \hat{H}} \dim \left( (W^* \otimes (\theta \otimes V))^H \right) m_a(W),$$

$$\langle \chi_W, \chi_V \chi_\theta \rangle = \dim \left( (W^* \otimes (\theta \otimes V))^H \right).$$

\[\square\]

Let $C^{-\infty}(H)^{Ad(H)}$ be the set of $Ad(H)$-invariant distributions on $H$ and $\mathcal{H}_{dR}^{ev}(B)$ be the even part of the de Rham cohomology. Assume $B$ oriented. It is shown in [7] that there is a well defined map

$$\text{Ind}_{\chi}^{Z|B} : K_H(T_H(Z|B)) \rightarrow C^{-\infty}(H)^{Ad(H)} \otimes \mathcal{H}_{dR}^{ev}(B) \cong \mathcal{H}_{H}^{-\infty, ev}(h, B)$$

called the distributional index map given by

$$\text{Ind}_{\chi}^{Z|B}(\sigma(A_0)|T_H(Z|B)) = \sum_{V \in H} \text{Ch}(m_A(V)) \chi_V,$$

where $\text{Ch}(m_A(V)) \in \mathcal{H}_{dR}^{ev}(B)$ is the usual Chern character of $m_A(V)$ and $\chi_V$ is the character of $V \in \hat{H}$.

We have the following generalisation of [11 Theorem 4.6].

**Lemma 2.6** (localisation). Let $H$ be a compact Lie group and $Z \rightarrow B$ be a compact $H$-fibration with $B$ a $H$-trivial oriented manifold. If $A_0$ is a family of $H$-transversally elliptic operators on $Z \rightarrow B$ then

$$\text{supp}(\text{Ind}_{\chi}^{Z|B}(A_0)) \subset \{ h \in H, Z^h \neq \emptyset \}.$$
Proof. The proof follows exactly the same line than Atiyah’s proof [11 Theorem 4.6]. Let \( \text{Stab}_H(Z) \) be the finite set of conjugacy classes of isometry subgroup of \( H \) for the action on \( Z \). Let \( h \in H \). If \( Z^h = \emptyset \) then \( h \) is not conjugate to any element belonging in \( K \in \text{Stab}_H(Z) \). Therefore by [11 Lemma 4.5], there is \([V] \in K_H(\mathbb{C}) \) such that \( \chi(h) \neq 0 \) and \( \chi_K = 0 \), for any \( K \in \text{Stab}_H(Z) \). Using [11 Lemma 4.4], we obtain \([V]^N K_H(Z) = 0 \) but \( K_H(T_H(Z|B)) \) is a unitary module on \( K_H(Z) \) therefore \([V]^N K_H(T_H(Z|B)) = 0 \). Since \( \text{Ind}^{Z|B}_h \) is a \( R(H) = K_H(\mathbb{C}) \)-homomorphism, it follows that \( 0 = \text{Ind}^{Z|B}_h(\chi^N. a) = [V]^N \cdot \text{Ind}^{Z|B}_h(a) \) and the same is true for the distributional index. Since \( \chi_N(h) \neq 0 \) this implies \( h \notin \text{supp} (\text{Ind}^{Z|B}_h(A)) \). \( \square \)

2.1. The Berline-Paradan-Vergne form of the index map for families. Here we recall the main result of [7]. We will not insist on the construction of the Chern character used in [7] to proved the index theorem. This is justified by the fact that in the sequel the vertical transversal space will define a vector bundle.

Let us denote by \( r : T^*(Z|B) \to T^*Z \) the inclusion induced by the Riemannian metric. The Liouville 1-form \( \omega_Z \) on \( T^*Z \) defines by restriction a 1-form \( r^*\omega_Z \) on \( T^*(Z|B) \), see [7] for more details. Assume \( B \) oriented and \( H \)-trivial. It can be shown that the 1-form \( r^*\omega \) is \( H \)-invariant and that the subspace \( C_{r^*\omega} = \{ \xi \in T^*(Z|B) \mid r^*\omega(\xi) = 0, \forall X \in \mathfrak{h} \} \) of \( T^*(Z|B) \) is equal to \( T_H^*(Z|B) \), see [7] for instance.

Let \( \sigma \) be a \( H \)-transversally elliptic symbol along the fibres of \( p : Z \to B \). We recalled above the definition of the distributional index \( \text{Ind}^{Z|B}_h(\sigma) \in C^{-\infty}(H, \mathcal{H}^{cr}_{dR}(B))^{Ad(H)} \). We can restrict such element through its associated generalized function because such element belongs to \( C^{-\infty}(H, \mathcal{H}^{cr}_{dR}(B)) \).

In the next theorem, we shall denote by \( \text{Ch}_c(\sigma, s^*\omega, s)(Y) \in \mathcal{H}^{-\infty}_c(\mathbb{h}, T(Z|B)) \) the \( s \)-equivariant Chern character of a \( H \)-transversally elliptic morphism along the fibres, see [7] and [64, 65] when \( B = * \). We denote by \( \hat{A}(T(Z|B), Y) \in \mathcal{H}^H_0(\mathbb{h}, B) \) the equivariant \( \hat{A} \)-genus of \( T(Z|B) \), see [15].

The main result of [7] is the following theorem.

Theorem 2.7. [7] Let \( \sigma \) be a \( H \)-transversally elliptic symbol along the fibres of a compact \( H \)-equivariant fibration \( p : Z \to B \) with oriented and \( H \)-trivial. Denote by \( N^s \) the normal vector bundle to \( Z^s \) in \( Z \).

1. There is a unique generalized function with values in the cohomology of \( B \) denoted

\[
\text{Ind}^{H,Z|B}_{coh} : K_H(T_H(Z|B)) \to C^{-\infty}(H, \mathcal{H}^{cr}_{dR}(B))^{Ad(H)}
\]
satisfying the following local relations:

\[
\text{Ind}^{H,Z|B}_{coh}([\sigma])_s(Y) = (2i\pi)^{-\dim(Z^s|B)} \int_{T(Z^s|B)} \text{Ch}_c(\sigma, s^*\omega, s)(Y) \wedge \hat{A}^2(T(Z^s|B), Y),
\]

\( \forall s \in H, \forall Y \in \mathfrak{h}(s) \) small enough such that the equivariant classes \( \hat{A}^2(T(Z^s|B), Y) \) and \( D(N^s, Y) \) are defined. 2. Furthermore, we have the following index formula:

\[
\text{Ind}^{H,Z|B}_{coh}([\sigma]) = \text{Ind}^{H,Z|B}_{coh}([\sigma]) = C^{-\infty}(H, \mathcal{H}^{cr}_{dR}(B, \mathbb{C}))^{Ad(H)}.
\]

Remark 2.8. The definition of the form \( D_s(N^s, Y) \in \mathcal{H}^{\infty}_c(\mathbb{h}, Z^s) \) can be found in [10, 17, 65] but will not be needed in the sequel since under the assumptions of the next sections \( N^s \) will be reduced to \( Z \times 0 \).

Outside of \( T_H^*Z \), the \( H \)-equivariant form \( \beta(\omega) = -i\omega \int_0^\infty e^{itD\omega} dt \) is well defined as a \( H \)-equivariant form with generalized coefficients, and we have \( D\beta(\omega) = 1 \) outside \( T_H^*Z \), see [65, Equation (15)]. Let \( U \) be a \( H \)-invariant open neighborhood of \( T_H^*Z \) and let \( \chi \) be a smooth \( H \)-invariant function on \( T^*Z \) with support in \( U \) and equal to 1 in a neighborhood of \( T_H^*Z \). Recall [65, Proposition 3.11] that this allows to define a closed equivariant differential form on \( T^*Z \), with generalized coefficients, and supported in \( U \)

\[
\text{One}(\omega, \chi) = \chi + d\chi \beta(\omega) \in \mathcal{A}_U^{-\infty}(\mathfrak{g}, T^*Z).
\]

Moreover, its cohomology class \( \text{One}_U(\omega) \in \mathcal{H}^{\infty}_c(\mathfrak{g}, T^*Z) \) does not depend on \( \chi \).

Definition 2.1. [65] The collection \( \text{One}_U(\omega) \) defines an element \( \text{One}(\omega) \in \mathcal{H}^{\infty}_c(T_H^*Z, \mathbb{h}, T^*Z) \).
Remark 2.9. If $H = \{ e \}$ then $T_H^*Z = T^*Z$ and $\text{One}(\omega) = 1$.

We denote by $\text{Ch}_{s}(\sigma, s) = (\gamma) = \mathcal{H}_{s, \text{supp}(\sigma), \text{hor}}(\mathfrak{g}, T(Z^*|B))$ the $s$-equivariant Chern character of a vertical symbol $\sigma$ defined as in [63] Definition 3.7, see also [7].

Proposition 2.10. Let $\sigma$ be a symbol which is $H$-transversally elliptic along the fibres of $Z \to B$. We have

$$\text{Ch}_c(\sigma, r^*\omega, s)(Y) = \text{Ch}_{s}(\sigma, s)(Y) \wedge r^* \text{One}(\omega_s) \in \mathcal{H}_{c, H}^- (\mathfrak{g}, T(Z^*|B)),$$

where $\text{Ch}_c(\sigma, r^*\omega, s)(Y)$ is the $s$-equivariant Chern character defined in [7] using [64].

Proof. This follows directly from [64] Theorem 3.22, see also [65].

3. Transversal index for central extension by finite groups

In this section, we generalize the setting from [62] to the context of fibration. We recall that $B$ is assumed to be oriented. Let $p : M \to B$ be a compact fibration. Let $G$ be a compact connected Lie group and $\pi : P \to M$ be a $G$-principal fibration. In particular, we get a compact fibration $p \circ \pi : P \to B$ and $G$ acts trivially on $B$ as in the previous section. As in [62], we consider a central extension $1 \xrightarrow{\iota} \Gamma \xrightarrow{\zeta} \tilde{G} \xrightarrow{\gamma} G \xrightarrow{\iota} 1$ by a finite group $\Gamma$. In this context, $P \to B$ becomes a $\tilde{G}$-fibration when equipped with the action given by $\tilde{g} \cdot x = \zeta(\tilde{g}) \cdot x$, for any $x \in P$ and $\tilde{g} \in \tilde{G}$. We denote simply by $\tilde{g} \cdot x = \tilde{g}x$ and $g \cdot x = gx$ the actions of $\tilde{G}$ and $G$.

We denote by $\mathfrak{g}$ the Lie algebra of $G$ and similarly by $\tilde{\mathfrak{g}}$ the Lie algebra of $\tilde{G}$. Notice that $\mathfrak{g} = \tilde{\mathfrak{g}}$ because $\Gamma$ is discrete. Since the action of $G$ is free on $P$, the map $P \times \mathfrak{g} \to TP$ is an isomorphism on its image. This implies that $T_G P = T_{\tilde{G}} P$ and $T_G (P|B) = T_{\tilde{G}} (P|B)$ are vector subbundles of $TP$. Clearly, the quotient maps by the $G$-action induce isomorphisms $T_G P/G \cong TM$ and $T_G (P|B)/G \cong T(M|B)$.

We are interested in families of $\tilde{G}$-transversally elliptic operators on $P \to B$. Let $\sigma \in K_{\tilde{G}} (T_G (P|B))$. Using Lemma 2.6 we know that $\text{supp}(\text{Ind}_{P|B}^\mathfrak{g}(\sigma)) \subset \{ \tilde{g} \in \tilde{G}, P\tilde{g} \neq \emptyset \} = \Gamma$. It follows that we can write

$$\text{Ind}_{P|B}^\mathfrak{g}(\sigma) = \sum_{\gamma \in \Gamma} Q_\gamma(\sigma),$$

where $Q_\gamma(\sigma) \in C^{-\infty}(\tilde{G}) \otimes \mathcal{H}_{s, \text{hor}}^\mathfrak{g}(B)$ is supported in $\gamma \in \Gamma$. Using Proposition 1.1 we obtain that there is $T_\gamma(\sigma) \in Z(\mathfrak{g}) \otimes \mathcal{H}_{s, \text{hor}}(B)$ such that $Q_\gamma(\sigma) = T_\gamma(\sigma) \ast \delta_\gamma$, compare with [76] [29]. With this in mind, our next goal is to determine $\exp^{-1} \otimes \text{id}_{\mathcal{H}_{s, \text{hor}}(B)}(T_\gamma(\sigma))$.

3.1. Vertical twisted Chern character. Let $E_1, \ldots, E_r$ be an orthonormal basis of $\mathfrak{g}$ and let $\theta = \sum \theta_i \otimes E_i \in (A^1(P) \otimes \mathfrak{g})^G$ be a connection 1-form on $P \to M$. We denote by $\Theta = \sum \Theta_i \otimes E_i \in (A^\mathfrak{g}_{\text{hor}}(P) \otimes \mathfrak{g})^G$ its curvature, where $A^\mathfrak{g}_{\text{hor}}(P)$ is the algebra of horizontal forms of even degree on $P$. We shall denote by $X_1, \ldots, X_r$ coordinates in the basis $(E_i)$. Recall that the Chern-Weil morphism $CW : S(\mathfrak{g})^G \to A(\mathfrak{g})_{\text{hor}}^\mathfrak{g} \cong \mathcal{A}^\mathfrak{g}(M)$ is given by $CW(P)(\Theta) = P(\Theta_1, \ldots, \Theta_r)$ and that this can be extended to $C^\infty(\mathfrak{g})^G$ using a Taylor expansion at 0. Let us recall what this means. Denote as before $(X^\alpha)^* = (X_1)^{\alpha_1} \cdots (X_r)^{\alpha_r}$ the induced differential operator on $G$ by the monomial $X^\alpha = X_1^{\alpha_1} \cdots X_r^{\alpha_r}$, where $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ is a multi-index. Let $\varphi \in C^\infty(\mathfrak{g})^G$ and write $\varphi(X_1, \ldots, X_r) = \sum_{|\alpha| \leq \text{dim } P} (X^\alpha)^*(\varphi)(0) X^\alpha + o(|X|^\text{dim } P)$, with $|\alpha| = \sum \alpha_i$ and $\alpha! = \alpha_1! \cdots \alpha_r!$ then $\varphi(\Theta) = \sum_{|\alpha| \leq \text{dim } P} (X^\alpha)^*(\varphi)(0) \Theta^\alpha \in \mathcal{H}_{s, \text{hor}}(M)$.

Using the identification of $S(\mathfrak{g})$ with $C_0^\infty(\mathfrak{g})$, see Proposition 1.1 the Chern-Weil morphism can be written $e^{\Theta} \ast \delta_\theta$, i.e. $\varphi(\Theta) = (e^{\Theta} \ast \delta_\theta)(\varphi)$. In the sequel, we will denote simply the Chern-Weil morphism by $e^\Theta$ using the previous convention.

Remark 3.1. If $\varphi = 1$ on a neighbourhood of 0 then $\varphi(\Theta) = 1$. 


**Definition 3.2.** \[\text{[62]}\] For any closed form \(\alpha \in \mathcal{A}_c(T(M|B))\) with compact support, the expression \(\alpha \wedge e^\Theta\) defines an element in \(C_0^\infty(g) \otimes \mathcal{A}_c(T(M|B))\). Denote by \(\overline{\nabla}(X) = \int_X \varphi(Ad(g)X)dg\) the average of \(\varphi \in C^\infty(g)\) with respect to the Haar measure on \(G\). Then \(\int_{T(M|B)} \alpha \wedge e^\Theta\) defines an element in \(C_0^\infty(g) \otimes \mathcal{A}(B)\) by

\[
\left\langle \int_{T(M|B)} \alpha \wedge e^\Theta, \varphi \right\rangle_g := \int_{T(M|B)} \alpha \wedge \overline{\nabla}(\Theta),
\]

for any \(\varphi \in C^\infty(g)\).

Following \[\text{[62]}\] we now introduce the twisted Chern character \(Ch_\gamma(\sigma)\) of a \(\tilde{G}\)-transversally elliptic symbol along the fibres of \(P \to B\). Since \(\sigma\) is \(G\)-transversally elliptic along the fibres, the intersection of its support and \(T_G(P|B)\) is compact. Seen as a morphism over the manifold \(T_G(P|B)\), \(\sigma\) is then compactly supported therefore the Chern character \(Ch_\gamma(\sigma, \gamma) \in H^\infty_G(g, T_G(P|B))\) is well defined, see \[\text{[14]} \text{[65]} \text{[62]}\]. Since the finite subgroup \(\Gamma\) acts trivially on \(P\), we have a canonical isomorphism between \(H^\infty_{c,G}(g, T_G(P|B))\) and \(H^\infty_{c,G}(g, T_G(P|B))\).

**Definition 3.3.** \[\text{[62]}\] Let \(H_{dR,c}(T(M|B))\) denote the de Rham cohomology of \(T(M|B)\) with compact support. The twisted Chern character \(Ch_\gamma(\sigma) \in H_{dR,c}(T(M|B))\) is defined as the image of \(Ch_\gamma(\sigma, \gamma)\) under the Chern–Weil isomorphism \(\mathcal{H}^\infty_{c,G}(g, T_G(P|B)) \to H_{dR,c}(T(M|B))\) that is associated with the principal \(G\)-bundle \(T_G(P|B) \to T(M|B)\).

Let us recall an explicit construction for this Chern character \[\text{[62]}\].

**Remark 3.4.** Let \(\Pi : T(P|B) \to P\) be the projection and \(\sigma : \Pi^*E^+ \to \Pi^*E^-\) be a given \(G\)-transversally elliptic symbol along the fibres. Let \(\nabla^+\) be a \(\tilde{G}\)-equivariant connection on the vector bundle \(E^+ \to P\). The pull-back \(\nabla^{\Pi^*E^+} := \Pi^*\nabla^+\) is then a connection on \(\Pi^*E^+\) viewed as a vector bundle on the manifold \(T_G(P|B)\). Since \(\text{supp}(\sigma) \cap T_G(P|B)\) is compact, we can define on the vector bundle \(\Pi^*E^- \to T_G(P|B)\) a connection \(\nabla^{\Pi^*E^-}\) such that the relation \(\nabla^{\Pi^*E^+} = \sigma \circ \nabla^{\Pi^*E^-} \circ \sigma^{-1}\) holds outside a compact subset of \(T_G(P|B)\). We consider the equivariant Chern character, twisted by the central element \(\gamma \in \Gamma\):

\[
Ch_\tilde{G}(\sigma) := Ch_\gamma(\nabla^{\Pi^*E^+} \oplus \nabla^{\Pi^*E^-}),
\]

see \[\text{[13]} \text{[65]} \text{[7]}\] and the references therein for more details.

3.2. **The index formula for central extensions by finite groups.** Let \(\theta\) be a connection 1-form on \(\pi : P \to M\) and assume that the metric on \(P\) is compatible with the decomposition \(TP = T_GP \oplus P \times g\) induced by the connection \(\theta\). We denote by \(\pi_1\) and \(\pi_2\) the projections corresponding to the first and second factor in the decomposition \(TP = T_GP \oplus P \times g\). The differential map \(d\pi\) restricted to the subbundle \(T_GP\) coincides with the quotient map \(q : T_GP \to TM\) by the \(G\)-action. Let \(\nu \in \mathcal{A}(P \times g)^G\) be given by \(\nu(x, X)(v, Y) = (\theta(x)v, X)_g\), where \((x, X) \in P \times g\), \((v, Y) \in T_xP \times TXg = T_xP \times g\) and \((\cdot, \cdot)_g\) is our metric on \(g\) compatible with the connection and the metric on \(P\). Let \(\omega_P\) and \(\omega_M\) be respectively the Liouville 1-form on \(P\) and \(M\). Recall that if \(Z\) is a manifold then the Liouville 1-form on \(T^*Z\) is given in local coordinates \((q, p)\) by \(\omega = -\sum p_i dq_i\). In other words \(\langle \omega(x, \xi), W \rangle = -\xi(d_x\pi(W))\) for any \((x, \xi) \in T^*Z\) and \(W \in T_xT^*Z\). With the notations \(d\pi, \pi_2\) and the decomposition \(T(P|B) = T_G(P|B) \oplus P \times g\), from the previous section, we have the following result.

**Proposition 3.5.** Assume that the metric on \(P\) is compatible with the metrics on \(B\) and \(M\). Denote by \(r : T(P|B) \to TP\) the inclusion. Then

\[
r^*\omega_P = r^*(d\pi)^*\omega_M - r^*\pi_2^*\nu.
\]

Furthermore,

\[
r^*\text{One}(\omega_P) = r^*(d\pi)^*\text{One}(\omega_M) \wedge r^*\pi_2^*\text{One}(\nu) \in H^\infty_{G,c}(g, T(P|B))
\]

where \(\text{One}(\omega_M) \in H_{dR}(TM)\) and \(\text{One}(\nu) \in H^\infty_{G,c}(g, P \times g) = H^\infty_{G,c}(g, P \times g)\).
Proof. From [64] Theorem 4.5, we have
\[ \omega_P = (d\pi)^* \omega_M - \pi^*_2 \nu, \]
and
\[ \text{One}(\omega_P) = (d\pi)^* \text{One}(\omega_M) \wedge \pi^*_2 \text{One}(-\nu), \]
see also [65, Section 4.1]. The result follows applying the restriction \( r^* \).

Lemma 3.6. We have
\[ \hat{A}(T(P|B))^2(X) = (d\pi)^* \hat{A}(T(M|B))^2 j_\theta(X)^{-1}. \]

Proof. Indeed, take on \( T(P|B) = q^*(T(M|B)) \oplus P \times \mathfrak{g} \) the connection given by \( \nabla^{T(P|B)} = q^* \nabla^{T(M|B)} \oplus d \otimes \text{id}_\mathfrak{g} \) where \( \nabla^{T(M|B)} \) is a connection on \( T(M|B) \) and \( d \) is the de Rham differential on \( P \). Then we have
\[ \mu^{T(P|B)}(X) = L^{T(P|B)}(X) - \nabla_{X^P}^{T(P|B)} = L^{P \times \mathfrak{g}}(X) - \iota(X) d \otimes \text{id}_\mathfrak{g} = \text{id}_P \otimes d \theta(X), \]
and the curvature of \( \nabla^{T(P|B)} \) is \( R^{T(P|B)} = q^* R^{T(M|B)} \) where \( R^{T(M|B)} \) is the curvature of \( \nabla^{T(M|B)} \). Denoting by \( R\hat{g}(X) = R^{T(P|B)} + \mu^{T(P|B)}(X) \), we have by definition
\[ \hat{A}(T(P|B))^2(X) = \det \left( \frac{R\hat{g}(X)}{e^{R\hat{g}(X)/2} - e^{-R\hat{g}(X)/2}} \right), \]
see [15] Section 7.1. The result follows then easily from the relation \( R\hat{g}(X) = q^* R^{T(M|B)} \oplus \text{id}_P \otimes d \theta(X) \) and the fact that the adjoint action is orthogonal.

We shall denote by \( \text{dim}(M|B) := \text{dim} M - \text{dim} B \) and \( \text{dim}(P|B) := \text{dim} P - \text{dim} B \).

Theorem 3.7. Let \( \sigma \in K\tilde{G}(T_G(P|B)) \), we have \( \text{Ind}_{-\infty}^{P|B}(\sigma) = \sum_{\gamma \in \Gamma} T_\gamma(\sigma) * \delta_\gamma \), where
\[ T_\gamma(\sigma) = (2i\pi)^{-\text{dim}(M|B)} \exp \left( \int_{T(M|B)|B} \text{Ch}_\gamma(\sigma) \wedge \hat{A}(T(M|B))^2 \wedge e^\Theta \right), \]
Here \( \text{Ch}_\gamma(\sigma) \) is the twisted Chern character, see Definition 3.3.

Proof. Recall that we consider a central extension \( 1 \longrightarrow \Gamma \longrightarrow \tilde{G} \xrightarrow{\zeta} G \longrightarrow 1 \) by a finite group \( \Gamma \) and therefore \( \gamma \in \Gamma \) acts trivially on \( P \) since \( \tilde{G} \) acts by \( \tilde{g} \cdot p = \zeta(\tilde{g})p \). In particular, we have \( P^\gamma = P \), \( N^\gamma = P \times \{0\} \) and thus \( D_\gamma(N^\gamma, X) = 1 \). We know that \( \text{Ind}_{-\infty}^{P|B}(\sigma) \) is supported in \( \Gamma \). Let \( \gamma \in \Gamma \). Using Theorem 2.7 we have
\[ \text{Ind}_{-\infty}^{P|B}(\sigma)|_{\gamma}(X) = (2i\pi)^{-\text{dim}(M|B)} \int_{T(P|B)|B} \text{Ch}_\gamma(\sigma, X) \wedge r^* \text{One}(\omega_P) \wedge \hat{A}(T(P|B))^2(X). \]
Since \( \text{Ch}_\gamma(\sigma, X) \) is supported in \( T_G(P|B) \) we have \( \text{Ch}_\gamma(\sigma, X) = \pi^*_2 \text{Ch}_\tilde{G}(\sigma)(X) \) and \( r^*(d\pi)^* \text{One}(\omega_M) = 1 \) because \( C_{r^* \omega_M} = T(M|B) \), see Remark 2.9. Therefore applying Corollary 3.3 we get
\[ \text{Ind}_{-\infty}^{P|B}(\sigma)|_{\gamma}(X) = (2i\pi)^{-\text{dim}(P|B)} \int_{T(P|B)|B} \text{Ch}_\gamma(\sigma, X) \wedge r^* \pi^*_2 \text{One}(-\nu) \wedge (d\pi)^* \hat{A}(T(M|B))^2 j_\theta(X)^{-1} \]
\[ = (2i\pi)^{-\text{dim}(P|B)} j_\theta(X)^{-1} \int_{T_G(P|B)|B} \text{Ch}_\tilde{G}(\sigma) \wedge (d\pi)^* \hat{A}(T(M|B))^2 \int_{\mathfrak{g}} \text{One}(-\nu). \]
But using [65] Lemma 4.5, \( \int_{\mathfrak{g}} \text{One}(-\nu) = (2i\pi)^{\text{dim} \tilde{G}} e^{\Theta} * \delta_0 \theta_t \cdots \theta_1 \). Therefore, we obtain
\[ \text{Ind}_{-\infty}^{P|B}(\sigma)|_{\gamma}(X) = (2i\pi)^{-\text{dim}(M|B)} j_\theta(X)^{-1} \int_{T(M|B)|B} \text{Ch}_\gamma(\sigma) \wedge \hat{A}(T(M|B))^2 e^{\Theta} * \delta_0. \]
Since \( \tilde{G}(\gamma) = \tilde{G} \), the result follows from Theorem 1.4. \( \square \)
Corollary 3.8. Let $\gamma \in \Gamma$ and $\varphi \in C^{\infty}(\hat{G})$ be a function equal to 1 on a neighbourhood of $\gamma$ with small enough support. Then

$$\langle \text{Ind}_{B}^{P}(\sigma), \varphi \rangle_{\hat{G}} = (2i\pi)^{-\dim(M|B)} \int_{T(M|B)} \text{Ch}_{\gamma}(\sigma) \wedge \hat{A}(T(M|B))^{2}. $$

Proof. If the support of $\varphi$ is small enough then the only element of $\Gamma$ contained in the support of $\varphi$ is $\gamma$. Therefore, Theorem 3.7 gives

$$\langle \text{Ind}_{B}^{P}(\sigma), \varphi \rangle_{\hat{G}} = \langle T_{\gamma}(\sigma) \ast \delta_{\gamma}, \varphi \rangle_{\hat{G}},$$

where $T_{\gamma}(\sigma) = (-2i\pi)^{-\dim(M|B)} \exp_{\gamma}(\int_{T(M|B)} \text{Ch}_{\gamma}(\sigma) \wedge \hat{A}(T(M|B))^{2} \wedge e^{\Theta}).$ Since $\varphi$ is equal to 1 around $\gamma$, we get the result because $(e^{\Theta} \ast \delta_{0}(X), \varphi(\gamma e^{X}))/\theta$ is equal to 1 in cohomology. \qed

Following [62], we consider the group $\hat{\Gamma}$ of characters of the finite abelian group $\Gamma$ and we decompose any $\hat{G}$-transversally elliptic symbol along the fibres $\sigma \in C^{\infty}(T(P|B), \text{Hom}(\Pi^{*}E^{+}, \Pi^{*}E^{-}))$ as $\sigma = \bigoplus_{\chi \in \Gamma} \sigma_{\chi},$ where $\sigma_{\chi} \in C^{\infty}(T(P|B), \text{Hom}(\Pi^{*}E^{+}_{\chi}, \Pi^{*}E^{-}_{\chi}))$ is a $\hat{G}$-transversally elliptic symbol along the fibres on $P$. Here $E^{\pm}_{\chi}$ is the subbundle of $E^{\pm}$ where $\Gamma$ acts through the character $\chi$. From Definition 3.3 it is obvious that the twisted Chern character $\text{Ch}_{\gamma}(\sigma)$ admits the decomposition

$$\text{Ch}_{\gamma}(\sigma) = \sum_{\chi \in \Gamma} \chi(\gamma)\text{Ch}_{e}(\sigma_{\chi}),$$

see also [62]. We then obtain the following theorem, see again [62] Theorem 4.3 for the case $B = \ast$.

Theorem 3.9. Let $\sigma \in K_{\hat{G}}(T_{G}(P|B))$ with decomposition $\sigma = \bigoplus_{\chi \in \Gamma} \sigma_{\chi}$. We have

$$\text{Ind}_{B}^{P}(\sigma) = \sum_{(\chi, \gamma) \in \Gamma \times \Gamma} \chi(\gamma)T_{e}(\sigma_{\chi}) \ast \delta_{\gamma},$$

where $T_{e}(\sigma_{\chi}) = (2i\pi)^{-\dim(M|B)} \exp_{\chi}(\int_{T(M|B)} \text{Ch}_{e}(\sigma_{\chi}) \hat{A}(T(M|B))^{2} e^{\Theta}).$

Proof. This follows using linearity and Theorem 3.7. \qed

Let us give now an example.

Example 3.10. Let $S^{3} \rightarrow \mathbb{CP}^{1} = S^{2}$ be the Hopf $S^{1}$-principal fibration. Let $\theta = \frac{i}{2} \sum_{i=0}^{1} z_{i} d\bar{z}_{i} - \bar{z}_{i} dz_{i}$ be the standard connection associated with the Fubini-Study metric on $\mathbb{CP}^{1}$. Using the connection we can identify $T^{*}(S^{3}|S^{2}) = S^{3} \times \mathbb{R}^{*}$ with the orthogonal to the horizontal bundle $\mathcal{H} = \text{ker} \theta$, i.e. $T^{*}(S^{3}|S^{2}) \cong \{ \xi \in T^{*}P, \xi|_{\mathcal{H}} = 0 \}$. Recall that the identification is given by $\eta \rightarrow \eta \circ \theta$ for $\zeta \in \mathbb{R}^{*}$. Denote by $r : T^{*}(S^{3}|S^{2}) \rightarrow T^{*}S^{3}$ the induced inclusion and by $\omega_{S^{3}}$ the Liouville 1-form. Then if $\xi$ denote the coordinate in $\mathbb{R}^{*}$ and $\pi^{S^{3}|S^{2}} : T^{*}(S^{3}|S^{2}) \rightarrow S^{3}$ the bundle projection, we get $-r^{*}(\omega_{S^{3}}) = \xi(\pi^{S^{3}|S^{2}}) \ast \theta$ which will be denoted simply by $\wedge \theta$ since $T^{*}(S^{3}|S^{2}) = S^{3} \times \mathbb{R}^{*}$ is trivial. Indeed, let $(x, \xi) \in T^{*}(S^{3}|S^{2}) = S^{3} \times \mathbb{R}^{*}$ and $W \in T_{(x,\xi)}(T^{*}(S^{3}|S^{2}))$ then

$$-r^{*}(\omega_{S^{3}})(x, \xi)(W) = \langle r(\xi), d\pi_{S^{3}}(dr(W)) \rangle.$$ 

Since $\pi \circ r$ is the bundle projection of $\pi^{S^{3}|S^{2}} : T^{*}(S^{3}|S^{2}) \rightarrow S^{3},$ we get

$$-r^{*}(\omega_{S^{3}})(x, \xi)(W) = \langle r(\xi), d\pi_{S^{3}|S^{2}}(W) \rangle$$

$$= \xi \circ \theta(d\pi_{S^{3}|S^{2}}(W))$$

$$= \xi(1)\theta(d\pi_{S^{3}|S^{2}}(W)).$$

Consider now the fibrations $\pi : P = S^{1} \times S^{3} \rightarrow S^{2} \times S^{1} = M$ and $p : M = S^{3} \times S^{1} \rightarrow S^{3} = B.$ The Liouville 1-form on $P$ is given by $p_{1}^{*}\omega_{S^{3}} + p_{2}^{*}\omega_{S^{1}},$ where $p_{1} : S^{3} \times S^{1} \rightarrow S^{3}$ and $p_{2} : S^{3} \times S^{1} \rightarrow S^{1}$ are the projections. We assume that $S^{1}$ acts on $P$ by its action on $S^{3}$. Let $\sigma \in K_{S^{3}}(T_{S^{1}}(P|B))$ and let us compute $\text{Ind}_{B}^{P}(\sigma).$ Notice that since we consider the free action of $S^{1}$ the index is supported in $e.$ We follow [65]. We have that
where the change of sign is due to 

$$\text{Ind}_{-\infty}^{P|B}(\sigma) = (2i\pi)^{-2} \int_{T^*(P|B)_B} \text{Ch}_{c}(\sigma, X) \wedge \text{One}(r^*\omega_P)$$

$$= (2i\pi)^{-2} \int_{S^3 \times T^* S^1 | S^2} \text{Ch}_{c}^{S^1}(\sigma) \int_{\mathbb{R}^*} \text{One}(r^*\omega_{S^3}).$$

Let us compute \( \int_{\mathbb{R}^*} \text{One}(r^*\omega_P) \). Let \( g \in C_c^\infty(\mathbb{R}) \) be equal to 1 on a neighborhood of 0. Consider the function \( \chi(\zeta) = g(\zeta^2) \) on \( P \times \mathbb{R}^* \) and represent \( \text{One}(r^*\omega_{S^3}) \) by

$$\text{One}(r^*\omega_{S^3}) = \chi + d\chi \wedge (-ir^*\omega_{S^3}) \int_0^\infty e^{itD(r^*\omega_{S^3})} dt.$$ 

Let us denote by \( \hat{\phi}(\zeta) = \int_{\mathbb{R}} e^{i\zeta X} \phi(X) dX \) the Fourier transform of any smooth compactly supported function \( \phi \) on \( \mathbb{R} \). We now compute \( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^*} \text{One}(r^*\omega_{S^3}) \right) \phi(X) dX = \int_{\mathbb{R}^*} \int_{\mathbb{R}} d\chi(i\zeta\theta) \int_0^\infty e^{-it(d\zeta \wedge \theta + \zeta\Theta - \zeta X)} dt \phi(X) dX $$

$$= \int_{\mathbb{R}^*} \int_{\mathbb{R}} d\chi(i\zeta\theta) \int_0^\infty (1 - it\zeta\Theta)e^{itX} \phi(X) dtdX,$$

because \( \theta \wedge \theta = 0 \) and \( \Theta \) is a 2-form on the 2 dimensional manifold \( S^2 \). It follows

$$\int_{\mathbb{R}} \left( \int_{\mathbb{R}^*} \text{One}(r^*\omega_{S^3}) \right) \phi(X) dX = \int_{\mathbb{R}^*} \int_{\mathbb{R}} d\chi(i\zeta\theta) \int_0^\infty (1 - it\zeta\Theta) \hat{\phi}(t\zeta) dt$$

$$= i\theta \int_{\mathbb{R}^*} \int_0^\infty -2g'((\zeta^2)\zeta(1 - it\zeta\Theta)) \hat{\phi}(t\zeta) dt d\zeta,$$

where the change of sign is due to \( -d\zeta \wedge \theta = \theta \wedge d\zeta \) which give the orientation on the fibers of T* \((S^3|S^2) \rightarrow S^2 \) induced from the Liouville 1-forms of \( S^3 \) and \( S^2 \). We now use the substitution \( \zeta_1 = t\zeta \) to obtain

$$\int_{\mathbb{R}} \left( \int_{\mathbb{R}^*} \text{One}(r^*\omega_{S^3}) \right) \phi(X) dX = i\theta \int_{\mathbb{R}^*} \int_0^\infty -2g'(\frac{\zeta_1^2}{t^2}) \frac{\zeta_1}{t^2} (1 - i\zeta_1\Theta) \hat{\phi}(\zeta_1) dt \frac{d\zeta_1}{t}$$

$$= i\theta \int_{\mathbb{R}^*} \int_0^\infty \frac{dt}{d\zeta_1}(g'(\frac{\zeta_1^2}{t^2}))(1 - i\zeta_1\Theta) \hat{\phi}(\zeta_1) dtd\zeta_1.$$

Since \( g \) is compactly supported and equal to 1 on a neighborhood of 0 it follows

$$\int_{\mathbb{R}} \left( \int_{\mathbb{R}^*} \text{One}(r^*\omega_{S^3}) \right) \phi(X) dX = i\theta \int_{\mathbb{R}^*} (1 - i\zeta_1\Theta) \hat{\phi}(\zeta_1) d\zeta_1,$$

$$= 2\pi i\theta(\hat{\phi}(0) + \hat{\phi}'(0)\Theta)$$

$$= 2\pi i\theta(\Theta),$$

where we have used that \( \int_{\mathbb{R}^*} \hat{\phi}(\zeta) d\zeta = 2\pi \hat{\phi}(0) \) and \( \int_{\mathbb{R}^*} -i\hat{\phi}'(\zeta) d\zeta = 2\pi \hat{\phi}'(0) \). Notice that \( \theta \) can be integrated along the fibers of \( S^3 \rightarrow S^2 = \mathbb{C}P^1 \) using the trivialisations

$$t_0 : \mathbb{C} \times S^1 \rightarrow S^3|_{U_0}, \ (\alpha, \beta) \rightarrow \frac{\beta}{1 + |\alpha|^2}(1, \alpha),$$

and

$$t_1 : \mathbb{C} \times S^1 \rightarrow S^3|_{U_1}, \ (\alpha, \beta) \rightarrow \frac{\beta}{1 + |\alpha|^2}(\alpha, 1),$$
where $U_i$ are the standard affine charts of $\mathbb{C}P^1$. More precisely, we have above $U_0$

$$
\int_{U_0 \times S^1 | U_0} t^*_0 (z_0 d\bar{z}_0 - \bar{z}_0 dz_0) = \int_0^{2\pi} \left( \frac{e^{i\lambda}}{\sqrt{1 + |\alpha|^2}} \frac{-ie^{-i\lambda}}{\sqrt{1 + |\alpha|^2}} d\lambda - \frac{e^{-i\lambda}}{\sqrt{1 + |\alpha|^2}} \frac{ie^{i\lambda}}{\sqrt{1 + |\alpha|^2}} d\lambda \right)
$$

$$
= -\frac{4i\pi}{1 + |\alpha|^2},
$$

$$
\int_{U_0 \times S^1 | U_0} t^*_0 (z_1 d\bar{z}_1 - \bar{z}_1 dz_1) = \int_0^{2\pi} \left( \frac{e^{i\lambda} \alpha}{\sqrt{1 + |\alpha|^2}} \frac{-i\alpha e^{-i\lambda}}{\sqrt{1 + |\alpha|^2}} d\lambda - \frac{\alpha e^{-i\lambda}}{\sqrt{1 + |\alpha|^2}} \frac{i\alpha e^{i\lambda}}{\sqrt{1 + |\alpha|^2}} d\lambda \right)
$$

$$
= -\frac{4i\pi|\alpha|^2}{1 + |\alpha|^2},
$$

and therefore as expected

$$
\int_{U_0 \times S^1 | U_0} \theta = \frac{i}{2} \left( \frac{-4i\pi}{1 + |\alpha|^2} + \frac{-4i\pi|\alpha|^2}{1 + |\alpha|^2} \right) = 2\pi.
$$

Notice that $\tilde{\phi}(\Theta) = \int_{S^1} \phi(Ad_z(X)) dz = \phi(\Theta)$ because $S^1$ is commutative. All together this gives the result.

**Remark 3.11.**

(1) To get concrete example, we can take any elliptic operator of positive order on $S^1$ and consider it as a constant family of $S^1$-transversally elliptic operators on $S^1 \times S^1$.

(2) The same computation for the fibration $q : S^3 \to S^2$ computes the distributional index of the zero family of $S^1$-transversally elliptic operators $0 : C^\infty(S^3) \to 0$. In this case, we have that

$$
\text{Ind}_{S^1 \times S^2}^S(0) = \sum_{n \in \mathbb{Z}} m_0(C_n) \chi_n,
$$

where $m_0(C_n) = S^3 \times S^1$, $V$ and $\chi_n(z) = z^n$ is the character of the irreducible representation $C_n$. Therefore, using Chern-Weil isomorphism we get that

$$
\text{Ch}(m_0(C_n)) = \text{Ch}(P \times C_n)(\Theta) = e^{\mu n(1)\Theta},
$$

where $\mu_n(X) = n i = \mathcal{L}_{c^0}(X) - X_{S^3}$ is the moment of the trivial connection $d$ on $P \times C_n$. Since $\mu_n(1) = in$, we get $\text{Ch}(m_0(C_n)) = 1 + in\Theta$ and therefore

$$
\text{Ind}_{S^1 \times S^2}^S(0) = \sum_{n \in \mathbb{Z}} (1 + in\Theta) \chi_n.
$$

In other words, $\forall \phi \in C_c^\infty(\mathbb{R})$ supported close enough from $0$, we have

$$
\langle \text{Ind}_{S^1 \times S^2}^S(0)(e^{iX}), \phi(X) \rangle = \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} (1 + in\Theta) e^{inX} \phi(X) dX,
$$

$$
= 2\pi \sum_{n \in \mathbb{Z}} (\hat{\phi}(n) + \Theta \hat{\phi}'(n)),
$$

$$
= 2\pi (\phi(0) + \Theta \phi'(0)),
$$

$$
= \langle \text{Ind}_{\text{coh}}^{S^1 \times S^2}(0)(e^{iX}), \phi(X) \rangle.
$$

(3) We can consider a central extension of $S^1$ by $S^1$ given by $z \to z^n$ with kernel $\mathbb{Z}_n$. In this case, the computation done in the last example can be reproduced mutatis mutandis to get the distributional coefficients at $\gamma \in \mathbb{Z}_n$.

4. THE INDEX OF FAMILIES OF PROJECTIVE OPERATORS

In this section, we extend the setting of [54, 55] to the case of families. Let $\mathcal{H}$ be a Hilbert space and denote by $K(\mathcal{H})$ the $C^*$-algebra of compact operators. Let us first recall the definitions of Azumaya bundles and projective bundles.

**Definition 4.1.** [54, 53] An Azumaya bundle $\mathcal{A}$ over a manifold $M$ is a vector bundle with fibres which are Azumaya algebras and which has local trivializations reducing these algebras to $M_N(\mathbb{C})$. A projective vector bundle $E$ over $M$ is a projection valued section of $\mathcal{A} \otimes K(\mathcal{H})$. 
Recall that the transpose Azumaya bundle $A^t$ is $A$ with multiplication reversed. Since the structure group of $A \otimes A^t$ acts by the adjoint representation $PU(N) \to PU(N^2)$ which lift canonically to a $U(N)$ action, the bundle $A \otimes A^t$ is trivial as an Azumaya bundle, see [54].

**Lemma 4.2.** [54] Let $E_1$ and $E_2$ be projective bundles associated to $A$. Then the bundle $\text{hom}(E_1, E_2)$ with fibres $\text{hom}(E_{1x}, E_{2x})$ at $x \in M$ is a vector bundle.

Let $\pi : P_A \to M$ be the $PU(N)$-principal bundle of trivialisations of $A \to M$. Then the lift $\pi^*A$ of $A$ to $P_A$ is trivial, i.e. it is a homomorphism bundle. Let $E_1$ be a projective vector bundle. Then $E_1 = \pi^*E_1$ is a finite dimensional vector bundle such that $E_1 \subset \mathbb{C}^N \otimes \mathcal{H}$ which is equivariant for the standard action of $U(N)$ on $\mathbb{C}^N$ interpreted as covering the action of $PU(N)$ on $P_A$. Let $E_2$ be another projective vector bundle associated with $A$. Recall that the action of $U(N)$ on $\text{hom}(\hat{E}_1, \hat{E}_2)$ is by conjugation. Therefore, $\text{hom}(\hat{E}_1, \hat{E}_2)$ defines a $PU(N)$-equivariant vector bundle over $P_A$ which descends to a well defined vector bundle $\text{hom}(E_1, E_2)$ on $M$.

Unfortunately, the “big” homomorphism bundle $\text{Hom}(\hat{E}_1, \hat{E}_2)$ is only a projective vector bundle over $M^2 = M \times M$ since it is associated with $A \otimes A^t$ over $M^2$. By the previous discussion, $\text{Hom}(\hat{E}_1, \hat{E}_2)$ restricts to the diagonal in a vector bundle, reducing there to $\text{hom}(E_1, E_2)$.

Denote by $d$ the distance function associated with the Riemannian metric on $M$. Let $N_\varepsilon := \{(x, x') \in M^2, d(x, x') < \varepsilon\}$.

Let $p : M \to B$ be a compact fibration as before. Let us recall the following fundamental result [54].

**Proposition 4.3.** Given two projective bundles, $E_1$ and $E_2$, associated to a fixed Azumaya bundle and $\varepsilon > 0$ sufficiently small, the exterior homomorphism bundle $\text{Hom}(\hat{E}_1, \hat{E}_2)$ over $M^2 := M \times_B M = \{(x, x') \in M \times M, p(x) = p(x')\}$, descends from a neighborhood of the diagonal in $P_A \times_B P_A = \{(z, z') \in P_A \times P_A, p(z) = p(z')\}$ to a vector bundle, $\text{Hom}^A(E_1, E_2)$, over $N_{\varepsilon,B} := N_{\varepsilon} \cap M^2$ extending $\text{hom}(E_1, E_2)$.

For any three such bundles there is a natural associative composition law

$$\text{Hom}^A(x', x'')(E_2, E_3) \times \text{Hom}^A(x, x')(E_1, E_2) \to \text{Hom}^A(x, x''')(E_1, E_3),$$

given by $(a, a') \mapsto a \circ a'$ for any $(x'', x')$, $(x, x') \in N_{\varepsilon/2,B}$ which is consistent with the composition over the units in $M^2$.

**Proof.** It is shown in [54] Proposition 1] that for $\varepsilon > 0$ sufficiently small, the exterior homomorphism bundle $\text{Hom}(\hat{E}_1, \hat{E}_2)$, descends from a neighborhood of the diagonal in $P_A \times P_A$ to a vector bundle, $\text{Hom}^A(E_1, E_2)$, over $N_{\varepsilon}$ extending $\text{hom}(E_1, E_2)$ with the associative composition law. The result follows then by restriction to $N_{\varepsilon,B}$. □

Let $F_1$ and $F_2$ be vector bundles over $M$. Denote by $|\lambda(M; B)|$ the vector bundle of vertical densities over $M$ and by $|\lambda(M; B)|$ its pullback to $M^2$ through the first projection. Recall that families of smoothing operators $\Psi^{-\infty}(M; B, F_1, F_2)$ can be defined as operators associated with smooth kernels $C^\infty(M^2, \text{Hom}(F_1, F_2) \otimes |\Lambda(M; B)|)$ over $M^2 = M \times_B M$, i.e. $A \in \Psi^{-\infty}(M; B, F_1, F_2)$ is given by a smooth section $A(x, x') \in C^\infty(M^2, \text{Hom}(F_1, F_2) \otimes |\Lambda(M; B)|)$ by the formula

$$A_2(x) = \int_{M^2} A(x, x') s(x'), \quad s \in C^\infty(M, F_1).$$

Furthermore, if $F_3$ is another vector bundle over $M$ then the composition

$$\Psi^{-\infty}(M; B, F_2, F_3) \circ \Psi^{-\infty}(M; B, F_1, F_2) \subset \Psi^{-\infty}(M; B, F_1, F_3)$$

is given by

$$A \circ B(x, x') = \int_{M^2} A(x, x'') \circ B(x'', x').$$

Following [54], we now define the linear space of families of smoothing operators and families of pseudodifferential operators with kernels supported in $N_{\varepsilon,B}$ for any pair $E_1, E_2$ of projective bundles associated to a fixed Azumaya bundle.
Definition 4.4. Let $E_1$, $E_2$ be projective bundles associated to a fixed Azumaya bundle $\mathcal{A}$. The linear space of families of smoothing operators with kernel supported in $N_{\varepsilon, B}$ is
\[ \Psi_{\varepsilon}^{\infty}(M|B, E_1, E_2) := C_c^\infty(N_{\varepsilon, B}, \Hom^A(E_1, E_2) \otimes |\Lambda(M|B)|). \]

Proposition 4.5. \cite{54} Let $E_1$, $E_2$ and $E_3$ be projective bundles associated to a fixed Azumaya bundle $\mathcal{A}$. The composition law of usual families of smoothing operators can be extended directly to define
\[ \Psi_{\varepsilon/2}^{-\infty}(M|B; E_2, E_3) \circ \Psi_{\varepsilon/2}^{-\infty}(M|B; E_1, E_2) \subset \Psi_{\varepsilon}^{-\infty}(M|B; E_1, E_3). \]
For $A \in \Psi_{\varepsilon/4}^{-\infty}(M|B, E_4, E_3)$, $B \in \Psi_{\varepsilon/4}^{-\infty}(M|B, E_3, E_2)$ and $C \in \Psi_{\varepsilon/4}^{-\infty}(M|B, E_2, E_1)$ this product is associative, i.e.
\[ A \circ (B \circ C) = (A \circ B) \circ C. \]

Proof. As in \cite{54}, this follows directly from the composition law of Proposition 4.3 \hfill \Box

Definition 4.6. Let $E_1$, $E_2$ be projective bundles associated to a fixed Azumaya bundle $\mathcal{A}$. The space of families of order $m$ pseudodifferential operators with kernel supported in $N_{\varepsilon, B}$ is
\[ \Psi^m_{\varepsilon}(M|B, E_1, E_2) := \left\{ \begin{array}{c} I_m^{\varepsilon - \dim B/4}(N_{\varepsilon, B}, M) \\
C_c^\infty(N_{\varepsilon, B}, \Hom^A(E_1, E_2)) \end{array} \right\}, \]
where $I_m^{\varepsilon - \dim B/4}(N_{\varepsilon, B}, M)$ is the set of compactly supported order $m - \frac{\dim B}{4}$ conormal distributions to $M$ on $N_{\varepsilon, B}$, see \cite{40, 53}.

We have the following standard results, see for example \cite{39, 40, 50, 54, 72}. See also \cite{27, 48, 50, 61, 74}.

Theorem 4.7. \cite{54} Let $E_1$, $E_2$ and $E_3$ be projective bundles associated to a fixed Azumaya bundle $\mathcal{A}$.

1. Then\[ 0 \longrightarrow \Psi^{-1}(M|B; E_1, E_2) \longrightarrow \Psi^m(M|B; E_1, E_2) \longrightarrow_{\sigma_m} C^\infty(S^*(M|B), \hom(E_1, E_2) \otimes N_m) \longrightarrow 0, \]
where $N_m$ is the line bundle over $S^*(M|B)$ of smooth functions on $T(M|B) \setminus 0$ which are homogeneous of degree $m$.

2. The composition law of usual families of smoothing operators can be extended directly to define\[ \Psi_{\varepsilon/2}^{m}(M|B; E_2, E_3) \circ \Psi_{\varepsilon/2}^{m}(M|B; E_1, E_2) \subset \Psi_{\varepsilon/2}^{m+m'}(M|B; E_1, E_3). \]

3. For $A \in \Psi_{\varepsilon/4}^{m}(M|B, E_4, E_3)$, $B \in \Psi_{\varepsilon/4}^{m}(M|B, E_3, E_2)$ and $C \in \Psi_{\varepsilon/4}^{m}(M|B, E_2, E_1)$ we have\[ A \circ (B \circ C) = (A \circ B) \circ C. \]

4. Furthermore, the symbol map satisfies\[ \sigma_{m+m'}(AB) = \sigma_m(A)\sigma_{m'}(B). \]

5. If $A \in \Psi_{\varepsilon/2}^{m}(M|B; E_1, E_2)$ is elliptic, i.e. $\sigma_m(A)$ is pointwise invertible on $T(M|B) \setminus 0$, then there exists $Q \in \Psi_{\varepsilon/2}^{-m}(M|B; E_2, E_1)$ such that $Q \circ A = \Id - E_R$, $A \circ Q = \Id - E_L$, where $E_R \subset \Psi_{\varepsilon}^{-\infty}(M|B; E_1, E_2)$ and $E_L \subset \Psi_{\varepsilon}^{-\infty}(M|B; E_2, E_1)$. Furthermore, any two such choices $Q'$ and $Q$ satisfy $Q - Q' \subset \Psi_{\varepsilon}^{-\infty}(M|B; E_2, E_1)$.

Recall the central extension \[ 1 \longrightarrow \mathbb{Z}_N \longrightarrow SU(N) \longrightarrow PU(N) \longrightarrow 1. \]

The following result is shown in \cite{55} Proposition 4 in the case $B = \{ \ast \}$.

Theorem 4.8. \cite{55} Let $\Omega \subset \mathcal{P}_{\mathcal{A}, \pi_{\text{top}}}$ be a sufficiently small neighborhood of $\mathcal{P}_{\mathcal{A}}$ invariant under the diagonal $PU(N)$-action. Then there is a well defined push-forward map into the families of projective pseudodifferential operators\[ \pi_* : \Psi^m_{\varepsilon}(\mathcal{P}_{\mathcal{A}}|B; \tilde{E}_1, \tilde{E}_2)^{SU(N)} := \{ A \in \Psi^m(\mathcal{P}_{\mathcal{A}}|B; \tilde{E}_1, \tilde{E}_2)^{SU(N)}, \ \text{supp}(A) \subset \Omega \} \to \Psi^m_{\varepsilon}(M|B; E_1, E_2) \]
which preserves composition of elements with support in $\Omega'$ such that $\Omega' \circ \Omega' \subset \Omega$. 


Theorem 4.10. Let \( \Psi^m_\Omega(\mathcal{P}_A|B; \hat{E}_1, \hat{E}_2)^{SU(N)} \) and denote by \( A_b(p, p') \) the family of Schwartz kernels over \( \mathcal{P}_{A, \pi_\varphi} \). We then define the map \( \pi_* \) as in [55] by the formula
\[
\pi_* A_b(x, x') = \int_{\pi^{-1}(x) \times \pi^{-1}(x')} A_b(p, p').
\]

The rest of the proof is completely similar to [55 Proposition 4] and is omitted here. \( \square \)

Let \( \pi_M : T(M|B) \to M \) be the projection. As in [54, 55], the symbol \( \sigma(A) \) of an elliptic family of projective operators \( A \) defines an element \( [\sigma(A)] \in K(T(M|B), \pi_M^*A) \) of the compactly supported twisted K-theory, see [20, 28, 72, 68, 73] and the references therein. Denoting by \( \pi_{\mathcal{P}_A} : T_{SU(N)}(\mathcal{P}_A|B) \to \mathcal{P}_A \) the projection, we obtain as in [55] a map in K-theory
\[
i : K(T(M|B), \pi_M^*A) \to K_{SU(N)}(T_{SU(N)}(\mathcal{P}_A|B)),
\]
given by \( \iota([\sigma(A)]) = [\tilde{\sigma(A)}, \pi_{\mathcal{P}_A}^*\hat{E}_1, \pi_{\mathcal{P}_A}^*\hat{E}_2] \) where \( \tilde{\sigma(A)} \in C(T_{SU(N)}(\mathcal{P}_A|B) \setminus \mathcal{P}_A, \text{hom}(\hat{E}_1, \hat{E}_2))^{PU(N)} \) is the \( PU(2^n) \)-invariant section corresponding to \( \sigma(A) \in C(T(M|B) \setminus M, \text{hom}(E_1, E_2)) \) with respect to the \( PU(N) \)-principal bundle \( T_{SU(N)}(\mathcal{P}_A|B) \to T(M|B) \). This is \( SU(N) \)-invariant with respect to the action covering the \( PU(N) \)-action since the action by conjugation does not depend on the representative of a lift of an element of \( PU(N) \).

Definition 4.9. Let \( A \in \Psi^m_\Omega(M|B; E_1, E_2) \) be an elliptic family of projective operators. Denote by \( \hat{A} \) the pullback family to \( \mathcal{P}_A \). Let \( \phi \in C^\infty(SU(N)) \) be a function equal to 1 in a small enough neighbourhood of the identity. Then we define the analytical index of \( A \) by
\[
\text{Ind}^{|M|B}_u(A) = \sum_{V \in SU(N)} \text{Ch}(m_{\hat{A}}(V))(\chi_V, \varphi) \in \mathcal{H}^\text{ev}_{dR}(B).
\]

We have the following Atiyah-Singer index formula.

Theorem 4.10. Let \( A \in \Psi^m_\Omega(M|B; E_1, E_2) \) be an elliptic family of projective operators. Then
\[
\text{Ind}^{|M|B}_u(A) = (2\pi i)^{-\dim(M|B)} \int_{T(M|B)|B} \text{Ch}._c(\iota([\sigma(A)]) \wedge \hat{A}(T(M|B))^2) \in \mathcal{H}^\text{ev}_{dR}(B).
\]

Proof. We apply Corollary K.8 to the central extension
\[
1 \longrightarrow \mathbb{Z}_N \longrightarrow SU(N) \longrightarrow PU(N) \longrightarrow 1.
\]
\( \square \)

4.1. Families of projective Dirac operators. Assume that \( T(M|B) \) is oriented and that \( \dim(M|B) = \dim M - \dim B = 2n \). Recall that \( B \) is also assumed to be oriented. Consider the special case where the Azumaya bundle \( A = \mathcal{C}l(M|B) \) is the complexified Clifford bundle of \( T(M|B) \). Denote as before by \( \pi : \mathcal{P} \to M \) the \( PU(2^n) = \text{Aut}(\mathcal{C}l(2n)) \)-principal bundle of trivializations associated with \( A \). We assume that the metric \( g_M \) on \( M \) is constructed from the pull back of a metric \( g_B \) on \( B \) and a metric on \( T(M|B) \), i.e. \( g_M = g_M|B \oplus p^*g_B \). Notice that this can be achieved by picking a random metric on \( M \) and replacing the metric on the orthogonal to \( T(M|B) \) by \( p^*g_B \). Similarly, we assume that the metric \( g_P \) on \( \mathcal{P} \) is given by \( g_P = (\cdot, \cdot)_\text{su} \oplus \pi^*g_M \), where \((\cdot, \cdot)_\text{su} \) is a metric on \( T(\mathcal{P}|M) \).

Let \( \mathcal{F} : = F^{SO}(M|B) \) be the bundle of oriented orthonormal frames of \( T(M|B) \). We have the identification \( \mathcal{P} = \mathcal{F} \times_{SO(2n)} PU(2^n) \), where \( SO(2n) \hookrightarrow PU(2^n) = \text{Aut}(\mathcal{C}l(2n)) \) is the standard embedding, see [49].

Following [54], we proceed now to the construction of the family of projective Dirac operators. Let us fix from now on a \( PU(2^n) \)-equivariant \( * \)-isomorphism
\[
\Phi : \mathcal{C}l(2n) \to M_{2^n}(\mathbb{C}).
\]
Lemma 4.11. (1) We have the following $PU(2^n)$-equivariant trivialisation $T : \pi^*Cl(M|B) \to \mathcal{P} \times Cl(2^n)$ given by
\[ T(f, \varphi) = (f, f^{-1}(\varphi)), \]
where $f \in \mathcal{P}$ is seen as an $*$-isomorphism $f : Cl(2n) \to Cl(M|B)_{\pi(f)}$.
(2) Since $\mathcal{P} = \mathcal{F} \times SO(2n)$ $PU(2^n)$ and $Cl(M|B) = \mathcal{F} \times SO(2n) Cl(2n)$, the previous trivialisation can be rewritten
\[ ([\mathcal{E}, A], [\mathcal{E}, \mathcal{F}]) \mapsto ([\mathcal{E}, A], \hat{A}^{-1} \hat{\varphi} \hat{A}), \]
where $\hat{A} \in SU(2^n)$ is any lift of $A \in PU(2^n)$.
(3) The previous trivialisation induces using $\Phi$ the trivialisation
\[ \pi^*Cl(M|B) \to \mathcal{P} \times M_{2n}(\mathbb{C}). \]
(4) We can then define
\[ c : \pi^*T(M|B) \to \mathcal{P} \times M_{2n}(\mathbb{C}). \]

Proof. The action of $PU(2^n)$ on $\mathcal{P}$ is given by $f \cdot A = f \circ A$ therefore we clearly have
\[ T((f, \varphi) \cdot A) = T(f \circ A, \varphi) = (f \circ A, A^{-1} \circ f^{-1}(\varphi)). \]
Since any automorphism of $Cl(2n) \cong M_{2n}(\mathbb{C})$ is inner, we can write $T((f, \varphi) \cdot A) = (f \circ A, \hat{A}^{-1}(f^{-1}(\varphi)) \hat{A})$, where $\hat{A} \in SU(2^n)$ is any lift of $A \in PU(2^n)$ through the central extension $SU(2^n) \to PU(2^n)$ by $\mathbb{Z}_2$.

The three last items are similar and therefore we shall only explain the last item. Recall that $T_{SU(2^n)}(\mathcal{P}|B) = \pi^*TM$ and that any element of $\pi^*TM$ is of the form $([\mathcal{E}, A], [\mathcal{E}, \mathcal{V}]) \in (\mathcal{F} \times SO(2n) PU(2^n)) \times_M (\mathcal{F} \times SO(2n) \mathbb{R}^{2n})$. Using the isomorphism $\Phi$ and the trivialisation $\pi^*Cl(M|B) \to \mathcal{P} \times Cl(2^n)$, we can define the map
\[ c : \pi^*T(M|B) \to \mathcal{P} \times M_{2n}(\mathbb{C}) \]
by $c([\mathcal{E}, A], [\mathcal{E}, \mathcal{V}]) = ([\mathcal{E}, A], \Phi(\hat{A}^{-1} \hat{\mathcal{V}} \hat{A}))$, where $\hat{\mathcal{V}} : \mathbb{R}^{2n} \to Cl(2n)$ is the standard map and $\hat{A} \in SU(2^n)$ is any lift of $A \in PU(2^n)$. The map $c$ does not depend on the choices. Indeed, $c$ clearly does not depend on the choice of the lift in $SU(2^n)$ because of conjugation and for any lift $\hat{R} \in Spin(2n)$ of $R \in SO(2n)$, we have $\hat{\mathcal{V}}(R^{-1} \mathcal{V}) = \hat{R}^{-1} \hat{\mathcal{V}} \hat{R}$ therefore $c([\mathcal{E}, \mathcal{V}, R, R^{-1} A], [\mathcal{E}, \mathcal{V}, R, R^{-1} \mathcal{V}]) = ([\mathcal{E}, A], f(\hat{A}^{-1} \hat{R} \hat{\mathcal{V}} \hat{R}^{-1} \hat{A})) = c([\mathcal{E}, A], [\mathcal{E}, \mathcal{V}]).$ 

Denote by $\omega_C = \iota^n \mathcal{E}(e_1) \cdots \mathcal{E}(e_{2n})$ the chirality element in $Cl(2n)$ and let $\tilde{\omega}_C([\mathcal{E}, A]) = \Phi(\hat{A}^{-1} \omega_C \hat{A})$ be the corresponding section of $\mathcal{P} \times M_{2n}(\mathbb{C})$. Notice that this is the equivariant section obtained from the chirality global section $\omega_M|B : M \to Cl(M|B)$ which at $x \in M$ is given by $\omega_M|B(x) = \iota^n v_1 \cdots v_{2n}$ for any local oriented orthonormal basis $(v_i)$ of $T_x(M|B)$.

Following \cite{59}, let us introduce the projective vertical $Spin$ bundle.

Definition 4.12. We call the $SU(2^n)$-equivariant trivial bundle $\tilde{\mathcal{S}} := \mathcal{P} \times \mathbb{C}^{2^n}$ the vertical projective $Spin$ bundle. The vertical projective half spinor bundle $\mathcal{S}^\pm$ are the projective bundles associated with the projections $\frac{1 + \omega_M|B}{2}$. They can be represented by the $SU(2^n)$-equivariant vector bundles $\mathcal{S}^\pm := \left( \frac{1 + \mathcal{S}}{2} \right) \tilde{\mathcal{S}}$.

Remark 4.13. The previous definition coincides with the definition of the projective $Spin$ bundle introduced in \cite{57} when $B$ is reduced to a point.

As in \cite{55, 57}, the Levi-Civita connection induces partial connections $\nabla^\pm$ on $\mathcal{S}^\pm$. More precisely, on $T(M|B)$ we consider the connection $\nabla^{LCV} = \pi^* \nabla^{LC}$, where $\pi^* : TM \to T(M|B)$ is the projection induced by the metric on $M$ and $\nabla^{LC}$ is the Levi-Civita connection on $M$. Then $\nabla^{LCV}$ induces the Levi-Civita connection on each fibre $M_b$ and the pullback of $\nabla^{LCV}$ is a metric connection on $T_{SU}(\mathcal{P}|B) = \pi^*T(M|B)$. This in turn defines $SU(2^n)$-equivariant connections $\nabla^\pm$ on $\mathcal{S}^\pm$ using the group embedding $\rho : SO(2n) \to PU(2^n)$, see \cite{49} for instance.

Lemma 4.14. Let $\theta$ be the connection one form on $\mathcal{F}$ induced by $\nabla^{LCV}$.
\(1\) The 1-form \( \tilde{\theta} : TF \times TPU(2^n) \to \mathfrak{pu}(2^n) \cong \mathfrak{su}(2^n) \) given by
\[
\tilde{\theta}(\xi, v, A, X) = Ad_{A^{-1}}(\rho_s(\theta(v))) + X, \quad (\xi, v, A, X) \in TF \times PU(2^n) \times \mathfrak{pu}(2^n)
\]
induces a \(SU(2^n)\)-invariant connection \( \tilde{\theta} : TP \to \mathfrak{pu}(2^n) \).

\(2\) The \(SU(2^n)\)-equivariant connection \( \nabla \) on \( \hat{S} \) is given by \( \nabla = d + U_*(\tilde{\theta}) \), where \( U_* : \mathfrak{pu}(2^n) \to M_{2^n}(\mathbb{C}) \) is the composition of the isomorphism \( \mathfrak{pu}(2^n) \to \mathfrak{su}(2^n) \) and the inclusion \( \mathfrak{su}(2^n) \to M_{2^n}(\mathbb{C}) \). Then the connections \( \nabla^\pm \) on \( \hat{S}^\pm \) are given by \( \nabla^\pm = \frac{1}{2} [\tilde{\theta}, \nabla] \).

\(3\) The connection \( \nabla \) is a Clifford connection, i.e. if \( \nabla^{\pi \times M_{2^n}(\mathbb{C})} = \Phi \circ T \nabla^{\pi \times \text{Cl}(M[B])} T^{-1} \circ \Phi^{-1} \) is the connection induced by the pullback connection of \( \nabla^{LCV} \) modulo the isomorphism \( \Phi \circ T \) then
\[
[\nabla, C] = \nabla^{\pi \times M_{2^n}(\mathbb{C})}(C), \quad \forall C \in C^\infty(P, M_{2^n}(\mathbb{C})).
\]

Proof. 1. Denote by \( r : SO(2n) \times F \times PU(2^n) \to F \times PU(2^n) \) the action, i.e. \( r(R, \xi, A) = (\xi \circ R, R^{-1}A) \). We have
\[
\tilde{\theta} \left( dr((R,Y), (\xi, v, A, X)) \right) = \tilde{\theta} \left( (\xi \circ R, d_{\xi}r_{R}(v + Y_{\xi}(\xi))) \right) = \tilde{\theta} \left( (\xi, v), (A, X) \right),
\]
where we have used the identification of \( TPU(2^n) \cong PU(2^n) \times \mathfrak{pu}(2^n) \) given by left translation. Furthermore, for any \( g \in SU(2^n) \) we have
\[
\tilde{\theta} \left( (\xi, v), (A, X) \right) = Ad_{g^{-1}} \left( Ad_{A^{-1}}(\rho_s(\theta(v))) \right) + dL_{g^{-1}A^{-1}}(AXg)
\]
\[
\tilde{\theta} \left( (\xi, v), (A, X) \right) = Ad_{g^{-1}} \left( \tilde{\theta} \left( (\xi, v), (A, X) \right) \right).
\]
Therefore, \( \tilde{\theta} \) induces a \(SU(2^n)\)-invariant connection \( \tilde{\theta} \) on \( P \).

2. This follows directly from the \(SU(2^n)\)-invariance of \( \tilde{\theta} \). Indeed, let \( s \in \mathbb{C}^{2^n}, g \in SU(2^n), v \in C^\infty(P, TP) \) and denote by \( r : P \times SU(2^n) \to P \) the action, i.e. \( r(f, g) = f \circ \beta(g) \) where \( \beta : SU(2^n) \to PU(2^n) \) is the quotient map, then
\[
g \left( \nabla_{d_{\xi}r_{\xi}(v)}(f \circ g) \right) = g \left( d_{d_{\xi}r_{\xi}(v)}(f \circ g) \right) = g \left( d_{\rho_s(\theta(v))} \circ d_{\tilde{\theta}(v)}(f \circ g) \right) + gU_*(\mathfrak{su}(2^n)) s(f \circ \beta(g))
\]
\[
= d_f(g \cdot s)(v) + gU_*(Ad_{g^{-1}}B \tilde{\theta}(v)) s(f \circ \beta(g))
\]
\[
= d_f(g \cdot s)(v) + U_*(\tilde{\theta}(v))(g \cdot s)(f),
\]
where we recall that \( g \cdot s(f) = g(s(f \circ \beta(g))) \).

3. The last statement is clear because the connection on \( \pi^*\text{Cl}(M[B]) \cong P \times \text{Cl}(2n) \) is given by \( d + \text{Ad}_*(\tilde{\theta}) \), where \( \text{Ad} : PU(2^n) \to \text{Aut}(\text{Cl}(2n)) \) is the representation given through the isomorphism \( \Phi \) by conjugation by \( SU(2^n) \). Therefore by definition, we get \( \Phi \left( \text{Ad}_*(\tilde{\theta}) \left( \Phi^{-1}(C) \right) \right) = U_*(\tilde{\theta}) C - CU_*(\tilde{\theta}) \).

As in [54], the Levi-Civita connection on \( M \) also induces similarly connections \( \nabla^{S^\pm} \) on \( S^\pm \). Furthermore, the homomorphism bundle of the vertical \( \text{Spin} \) bundle \( S \) can be identified with \( \text{Cl}(M[B]) \) and recall that it has an extension to \( \hat{\text{Cl}}(M[B]) = \hat{A} \) in a neighborhood of the diagonal, and this extended bundle also has
an induced connection, see Proposition 4.3 and [54]. We can then define the associated family \( \hat{\theta}_{M|B} \) of projective Dirac operators with kernel

\[
\hat{\theta}_{M|B} := c \cdot \nabla^S_L (\kappa_{id}), \quad \kappa_{id} = \delta(z - z') \text{Id}_S.
\]

Here, as in [54], \( \kappa_{id} \) is the kernel of the identity operator seen as a family of projective differential operators on \( S \) (i.e. a family of projective pseudodifferential operators with support in the diagonal) and \( \nabla^S_L \) is the connection restricted to the left variables with \( c \) the contraction given by the Clifford action of \( T(M|B) \) on the left. The operator \( \hat{\theta}_{M|B} \) is then odd with respect to the graduation and elliptic with symbol \( \sigma(\hat{\theta}_{M|B})(\xi) = cl(\xi) \) the Clifford multiplication.

We now represent the previous family of projective operators by a family \( \hat{\theta}_{P|B} \) of \( SU(2^n) \)-transversally elliptic operators. Let \( \nabla^\pm \) be the \( SU(2^n) \)-equivariant connections induced by the Levi-Civita on the \( SU(2^n) \)-equivariant vector bundles \( \tilde{S}^\pm \). We then obtain the corresponding family \( \hat{\theta}_{P|B} \) of \( SU(2^n) \)-transversally elliptic operators on \( P \) which is given by

\[
\hat{\theta}_{P|B} = \sum c(e_i) \nabla^+_e, \quad \text{where} \quad c(e_i) \text{ is the Clifford multiplication introduced above and} \quad (e_i) \text{ is any local orthonormal basis of} \quad T(M|B), \text{ see also [55, 75].} \quad \text{Since the principal symbol of} \quad \hat{\theta}_{P|B} \text{ is given by} \quad \sigma(\hat{\theta}_{P|B})(\xi) = c(\xi) \text{ for any} \quad \xi \in T_{SU(2^n)}(P|B), \text{ we get:}
\]

**Corollary 4.15.** The index of the family of projective Dirac operators is given by

\[
\text{Ind}_{su}^{M|B}(\hat{\theta}_{M|B}) = (2\pi i)^{-n} \int_{M|B} \hat{A}(T(M|B)) \in \mathcal{H}_{4R}^\ast(B).
\]

**Proof.** Using Theorem 4.10 we see that we need to show that

\[
\text{Ch}_{su}(\sigma(\hat{\theta}_{P|B})) = (2\pi i)^n \hat{A}(T(M|B))^{-1} \wedge \text{Thom}(T(M|B)).
\]

This follows from the fact that the curvature of \( \hat{\theta} \) (respectively its image by \( U_* \)) corresponds to the curvature of \( \theta \) (respectively to its image by \( \mathfrak{so}(2n) \to \mathfrak{spin}(2n) \)) and the standard computation of the Chern character of the symbol of families of Dirac operators. First notice that the moment map \( \mu^\nabla(X) \) vanishes because \( \mu^\nabla(X) = \mathcal{L}_X^S(X) - \nabla_X = U_*(X) - U_*(\hat{\theta}(X_P)) = 0. \) It follows that \( \mu^\nabla^\pm(X) = 0 \) because \( \hat{\omega}_\mathcal{C} \) is \( SU(2^n) \)-invariant. Now let \( (U_i) \in I \) be a finite cover of \( M \) of trivialisations \( \delta_i : U_i \times SO(2n) \to F = F_{SO(2n)}(M|B) \) of the bundle of oriented orthonormal frames of \( T(M|B) \) and let \( \phi_i : U_i \times PU(2^n) \to P \) be the induced trivialisations of \( P \). We shall denote again by \( \delta_i : U_i \to F \) the section given by \( \delta_i(x) = \delta_i(x, \text{Id}) \) and write \( \delta_i(x, R) = \delta_i(x) \circ R = \delta_i(x) R \). Let \( (f_i) \in I \) be a partition of unity subordinate to \( (U_i) \). We can then write

\[
\text{Ch}_{su}(\sigma(\hat{\theta}_{P|B})) = \sum f_i (d\delta^{-1}_i) \cdot (d\delta^{-1}_i)^* \text{Ch}_{su}^{PU(2^n)}(\sigma(\hat{\theta}_{P|B}))(X),
\]

which does not depend on \( X \in \mathfrak{su}(2^n) \) since the moment maps vanish. Denote by \( \delta_i : U_i \times \mathbb{R}^{2n} \to TU_i \) the trivialisation given by \( \delta_i(x, v) = \delta_i(x)(v) \). We have that

\[
(d\delta_i)^* (\sigma(\hat{\theta}_{P|B}))(x, v, A) = \hat{A}^{-1} \hat{c}(\delta_i^{-1}(x)v)A = \hat{A}^{-1}((\delta_i^{-1})^* \hat{c})(x, v) \hat{A},
\]

for any \( (x, v, A) \in T(U_i|B) \times PU(2^n) \). Now \( (\delta_i^{-1})^* \hat{c} \) is just the Clifford multiplication for the trivial bundle \( T(U_i|B) \) acting on the trivial half \( Spin \) bundle \( U_i \times S^k \) and the curvature of \( \hat{\theta} \) (respectively its image by \( U_* \)) corresponds to the curvature of \( \theta \) (respectively to its image by \( \mathfrak{so}(2n) \to \mathfrak{spin}(2n) \)) therefore using the Chern-Weil isomorphism \( CW : \mathcal{H}_{s\mathfrak{u}(2n)}^\infty(T(P|B)) \to \mathcal{H}_{s\mathfrak{u}(2n)}(T(U_i|B)), \) we obtain that

\[
f_i \text{ Ch}_{su}(d\delta_i)^* (\sigma(\hat{\theta}_{P|B})) = f_i \text{ Ch}_{su}((\delta_i^{-1})^* \hat{c}) = f_i (2\pi i)^n \hat{A}(T(U_i|B))^{-1} \wedge \text{Thom}(T(U_i|B)),
\]

where \( \text{Thom}(T(U_i|B)) \) is the Thom form of the bundle \( T(M|B)|_{U_i} \to U_i \). Finally, we get

\[
\text{Ch}_{su}(\sigma(\hat{\theta}_{P|B})) = \sum f_i (2\pi i)^n \hat{A}(T(U_i|B))^{-1} \wedge \text{Thom}(T(U_i|B)) = (2\pi i)^n \sum f_i (\hat{A}(T(M|B))^{-1} \wedge \text{Thom}(T(M|B)))|_{(T(M|B)|_{U_i})} \quad \text{Thom}(T(M|B)).
\]
Remark 4.16. The equality \( \text{Ch}_{\text{id}}(\sigma(\mathcal{P}_B)) = (2\pi)^n \hat{A}(T(M|B))^{-1} \wedge \text{Thom}(T(M|B)) \) can also be shown by twisting the class \( \sigma(\mathcal{P}_B) \) by the \( SU(2n) \)-equivariant vector bundle \( \mathcal{P} \times \mathbb{S}^\ast \) and looking at the standard formula since the twisted symbol corresponds to the \( SU(2n) \)-equivariant symbol of the family of signature operators.

5. Families of Spin(2n)-transversally elliptic Dirac operators

In this section, we discuss the application of Theorem 3.7 given by a family of Spin(2n)-transversally elliptic operators over the bundle of oriented orthonormal frames of an oriented fibration with even dimensional fibres. The motivation for the study of the index of families of Spin(2n)-transversally elliptic Dirac operators comes from the index of families of projective Dirac operators. Indeed, we will see that the index of families of projective Dirac operators is captured by the index of families of Spin(2n)-transversally elliptic Dirac operators. This was already noticed in [35, 62].

Let \( p : M \to B \) be a fibration of compact manifolds as before and assume that \( T(M|B) \) is oriented and that \( \dim(M|B) = \dim M - \dim B = 2n \). Recall that \( B \) is also assumed to be oriented. Let \( \mathcal{C}(M|B) \) be the complexified Clifford bundle of \( T(M|B) \) as before. Assume that the metric \( g_M \) on \( M \) is constructed from the pull back of a metric \( g_B \) on \( B \) and a metric on \( T(M|B) \), i.e. \( g_M = g_{M|B} \oplus p^*g_B \).

Let \( q : \mathcal{F} = FSO(M|B) \to M \) be the bundle of oriented orthonormal frames of \( T(M|B) \). Let \( \nabla^{LC} \) be the Levi-Civita connection on \( M \) and consider the connection \( \nabla^{LCV} = \pi^V \circ \nabla^{LC} \) on \( T(M|B) \), where \( \pi^V : TM \to T(M|B) \) is the projection induced by the metric on \( M \). Then \( \nabla^{LCV} \) induces the Levi-Civita connection on each fibre \( M_b \). Denote by \( \theta \) the induced connection 1-form on \( \mathcal{F} = FSO(M|B) \) and equip \( \mathcal{F} \) with the metric \( g_F = q^*g_M + \langle \theta, \theta \rangle_{SO(2n)} \), where \( \langle \cdot, \cdot \rangle_{SO(2n)} \) is an \( Ad \)-invariant metric on the Lie algebra \( so(2n) \) of \( SO(2n) \).

We have the following trivialisation of \( T_{SO(2n)}^*(\mathcal{F}|B) \), see [62] for the case \( B = \{ b \} \) (i.e. when \( B \) is reduced to a point).

Lemma 5.1. (1) The map \( \alpha : T_{SO(2n)}^*(\mathcal{F}|B) \to \mathcal{F} \times (\mathbb{R}^{2n})^\ast \) given by \( \alpha(\mathcal{E}, \xi) = (\mathcal{E}, \xi \circ (\mathcal{E} q)^{-1} \circ \mathcal{E}) \) is an \( SO(2n) \)-equivariant isomorphism. We shall denote by \( \alpha_\mathcal{E} : (T_{SO(2n)}^*(\mathcal{F}|B))_\mathcal{E} \to (\mathbb{R}^{2n})^\ast \) the induced map.

(2) The map \( \alpha \) induces an isomorphism \( \tilde{\alpha} \) between the bundle of oriented orthonormal frames of \( T_{SO(2n)}(\mathcal{F}|B) \) and \( \mathcal{F} \times SO(2n) \).

(3) The bundle \( T_{SO(2n)}(\mathcal{F}|B) \) has a spin structure.

Proof. (i) Recall that the map \( d_\mathcal{E} q : T_{SO(2n)}(\mathcal{F}|B)_\mathcal{E} \to T_{q(\mathcal{E})}(M|B) \) is an isomorphism for any \( \mathcal{E} \in \mathcal{F} \). By definition \( \mathcal{E} \) gives an isomorphism \( \mathcal{E}^* : \mathbb{R}^{2n} \to T_{q(\mathcal{E})}(M|B) \). Furthermore, \( \alpha \) is \( SO(2n) \)-equivariant because for any \( R \in SO(2n) \) we have,

\[
\alpha(\mathcal{E} \circ R, \xi) = (\mathcal{E} \circ R, \xi \circ (d_\mathcal{E} q)^{-1} \circ (d_\mathcal{E} q)^{-1} \circ \mathcal{E} \circ R)
\]

where \( r_R \) denotes the right action of \( R \).

(ii) This follows directly from (i). Indeed, if \( \langle \mathcal{E}, \mathcal{W} \rangle \) is a frame of \( T_{SO(2n)}(\mathcal{F}|B) \) then \( \tilde{\alpha}(\mathcal{E}, \mathcal{W}) = (\mathcal{E}, \mathcal{W}^{-1} \circ (d_\mathcal{E} q)^{-1} \circ \mathcal{E}) \in \mathcal{F} \times SO(2n) \).

(iii) The map \( \mathcal{F} \times Spin(2n) \to \mathcal{F} \times SO(2n) \) defines a spin structure. \( \square \)

Let us denote by \( S^\pm \) the half spinor associated with \( Spin(2n) \) and by \( F^{Spin}(\mathcal{F}|B) = \mathcal{F} \times Spin(2n) \) the bundle of spin frames.

We consider the spin vector bundles \( S^\pm(\mathcal{F}|B) = F^{spin}(\mathcal{F}|B) \times_{Spin(2n)} S^\pm = \mathcal{F} \times S^\pm \to \mathcal{F} \) associated with the spin structure. The Clifford multiplication \( c : T_{SO(2n)}^*(\mathcal{F}|B) \to \text{Hom}(S^+, S^-) \) is then given by
where $\Theta$ is the curvature of the $SO(2n)$-principal bundle $F \to M$. In particular, if $\varphi \in C^\infty(Spin(2n))^{Ad(Spin(2n))}$ is a function equal to 1 around $\text{Id}$ and 0 around $-\text{Id}$ then

$$\langle \text{Ind}_{-\infty}^{\mathcal{F}}(\hat{\varphi}^+_{F|B}), \varphi \rangle = (2\pi i)^{-n} \int_{M|B} \hat{A}(T(M|B)) \in \mathcal{H}_{	ext{Ad}}(\mathcal{B}).$$

**Proof.** Recall the central extension

$$1 \to \mathbf{Z}_2 \to Spin(2n) \to SO(2n) \to 1.$$ 

Applying Theorem 3.7 we get that $\text{Ind}_{-\infty}^{\mathcal{F}}(\hat{\varphi}^+_{F|B}) = T_{M|B} \cdot \delta_{\text{Id}} - T_{M|B} \cdot \delta_{-\text{Id}}$, where

$$T_{\varphi}(\text{Id} \cdot (\hat{\varphi}^+_{F|B})) = (2\pi i)^{-n} \text{Ch}_{\varphi}(\text{Id} \cdot (\hat{\varphi}^+_{F|B})) \exp\left(\int_{T(M|B)\text{Id}} \text{Ch}_{\gamma}(\text{Id} \cdot (\hat{\varphi}^+_{F|B})) \right)$$

and where $\text{Ch}_{\gamma}(\text{Id} \cdot (\hat{\varphi}^+_{F|B}))$ is the twisted Chern character, see Definition 5.3.

Since $\text{Ch}_{\text{Id}}(\text{Id} \cdot (\hat{\varphi}^+_{F|B})) = -\text{Ch}_{-\text{Id}}(\text{Id} \cdot (\hat{\varphi}^+_{F|B}))$, we only need to show that

$$\text{Ch}_{\text{Id}}(\text{Id} \cdot (\hat{\varphi}^+_{F|B})) = (2\pi i)^{n} \hat{A}(T(M|B))^{-1} \wedge \text{Thom}(T(M|B)).$$

This can be obtained as in the proof of Corollary 6.15. Let us give an other proof based on the $Spin$ structure, see [15, 63].

Recall that the vector bundle $T_{SO(2n)}(\mathcal{F}|B) = T_{Spin(2n)}(\mathcal{F}|B)$ is spin therefore using [16, Proposition 7.43]

$$(2\pi i)^{-n} \text{Ch}_{\text{Id}}^{Spin(2n)}(\text{Id} \cdot (\hat{\varphi}^+_{F|B}))(X) = \hat{A}(T_{SO(2n)}(\mathcal{F}|B))(X)^{-1} \wedge \text{Thom}(T_{SO(2n)}(\mathcal{F}|B))(X),$$

where $\text{Thom}(T_{SO(2n)}(\mathcal{F}|B))(X)$ denotes the Thom form in equivariant cohomology, see also [63]. Now recalling the identification $T_{SO(2n)}(\mathcal{F}|B) = q^*T(M|B)$, we get

$$\hat{A}(T_{SO(2n)}(\mathcal{F}|B))(X) = q^* \hat{A}(T(M|B)) \otimes \phi(X),$$

and

$$\text{Thom}(T_{SO(2n)}(\mathcal{F}|B))(X) = q^* \text{Thom}(T(M|B)) \otimes \phi(X),$$

where $\text{Thom}(T(M|B))$ is the Thom form and $\phi$ is equal to 1 on a small neighbourhood of $0 \in \text{spin}(2n)$. In fact, $\phi$ can be taken constant equal to 1 because if we equipped $T_{SO(2n)}(\mathcal{F}|B)$ with the pull-back connection $\pi^*\nabla$ of a connection $\nabla$ on $T(M|B)$ then the moment $\mu_{T_{SO(2n)}(\mathcal{F}|B)}(X) = 0$. Indeed, here $\mathcal{L}^{T_{SO(2n)}(\mathcal{F}|B)}(X)$ coincides with $X_F$ and $\pi^*\nabla_X$ coincides also with $X_F$ therefore $\mu_{T_{SO(2n)}(\mathcal{F}|B)}(X) = \mu_{\mathcal{L}^{T_{SO(2n)}(\mathcal{F}|B)}}(X)$.
Recall that the image of $\phi$ through the Chern-Weil morphism gives 1 in cohomology. Applying the Chern-Weil isomorphism, it follows
\[
\text{Ch}_{\text{Id}}(\sigma(\mathcal{P}_{\mathcal{F}|B})) = (2\pi i)^n \hat{A}(T(M|B))^{-1} \wedge \text{Thom}(T(M|B)).
\]
This complete the proof. \hfill \Box

**Remark 5.3.** As a corollary, we obtain that the index of a family of projective Dirac operators can be computed from the index of the corresponding family of $\text{Spin}(2n)$-transversally elliptic Dirac operators on $\mathcal{F} = \text{FSO}(M|B)$. More precisely, if $\varphi \in C^\infty(\text{Spin}(2n))^{\text{Ad}(\text{Spin}(2n))}$ is a function equal to 1 around $\text{Id}$ and 0 around $-\text{Id}$ then
\[
\text{Ind}^M_{M|B}(\mathcal{P}^+_M|B) = (\text{Ind}^B_{\mathcal{F}|B}(\mathcal{P}^+_B), \varphi) = (2\pi i)^{-n} \int_{M|B} \hat{A}(T(M|B)) \in H^*_M(B).
\]

The family $\mathcal{P}^+_M|B$ of $\text{Spin}(2n)$-transversally elliptic operators induces in some sense the family of $SU(2^n)$-transversally elliptic operators. Indeed, the connection on an orthonormal frame. This allows to define a family $\text{Id}$ $\text{Spin}$ of differential operators along the fibers $\mathcal{F} \rightarrow M$ by $q : \mathcal{F} \rightarrow M$ and the central extension of $\text{PU}(2^n)$ by the central extension of $\text{SO}(2n)$, then we recover the family $\mathcal{P}^+_M|B$ of projective Dirac operators from the family $\mathcal{P}^+_M|B$ of $\text{Spin}(2n)$-transversally elliptic operators. In other words, the pull back to $\mathcal{F}$ of the family $\mathcal{P}^+_M|B$ of projective Dirac operators is the family $\mathcal{P}^+_M|B$ of $\text{Spin}(2n)$-transversally elliptic operators, see also [35, 62].

**5.1. Example.** We now discuss an example. Let us consider the complex projective plan $\mathbb{CP}^2$ equipped with the Fubini-Study metric $g^{FS}$. Recall that the Fubini-Study metric is given in affine charts using complex coordinates $(z_1, z_2, z_3)$ by
\[
g^{FS}_{ij} = g^{FS}(\partial_{z_i}, \partial_{z_j}) = \frac{(1 + |z|^2)\delta_{ij} - z_i z_j}{(1 + |z|^2)^2},
\]
and $g^{FS}_{ij} = g^{FS}(\partial_{\bar{z}_i}, \partial_{\bar{z}_j}) = 0$. We let $S^1$ act isometrically on $\mathbb{CP}^2$ by $\gamma \cdot [z_0, z_1, z_2] = [z_0, z_1, \gamma z_2]$. Let $S^5 \subset \mathbb{CP}^2$ be the unit sphere equipped with the induced metric $g^{S^5}$ from $\mathbb{CP}^2$. Recall the principal $S^1$-bundle $q : S^5 \rightarrow \mathbb{CP}^2$ given by the orbits of the diagonal action of $S^1$ on $S^5 \subset \mathbb{CP}^2$. By letting $S^1$ act diagonally on $S^5 \times \mathbb{CP}^2$ by $\gamma \cdot ((w_0, w_1, w_2), [z_0, z_1, z_2]) = ((\gamma w_0, \gamma w_1, \gamma w_2), [z_0, z_1, \gamma z_2])$, we obtain a principal $S^1$-bundle $\rho : S^5 \times \mathbb{CP}^2 \rightarrow M := S^5 \times S^1 \mathbb{CP}^2$. The metric $g^{S^5 \times \mathbb{CP}^2} = g^{S^5} + g^{\mathbb{CP}^2}$ on $S^5 \times \mathbb{CP}^2$ is $S^1$ invariant, we obtain a metric on $M$. Let us now consider the fibration $p : M := S^5 \times S^1 \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ given by $p([x, [z]]) = q(x)$. It is then clear that $p$ is also a Riemannian fibration since it is obtained from the Riemannian fibration $S^5 \times \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ with projection $(x, [z]) \mapsto q(x) = q \circ p_1(x, [z]).$

The vertical tangent bundle $T(M|B) = \ker dp$ is given by $S^5 \times_{S^1} T\mathbb{CP}^2$ and the bundle $F^{SO}(M|B)$ of oriented orthonormal frames of $T(M|B)$ is given by $\mathcal{F} := F^{SO}(M|B) = S^5 \times_{S^1} F^{SO}(\mathbb{CP}^2)$, where $F^{SO}(\mathbb{CP}^2)$ is the bundle of oriented orthonormal frames of $\mathbb{CP}^2$. Let $\mathcal{C}(\mathbb{CP}^2)$ be the complexified Clifford bundle of $\mathbb{CP}^2$. Similarly, let $\mathcal{C}(M|B)$ be the Clifford bundle of $T(M|B)$ then $\mathcal{C}(M|B) = S^5 \times_{S^1} \mathcal{C}(\mathbb{CP}^2)$.

Notice that $\mathbb{CP}^2$ can be seen as a fibration of oriented Riemannian manifolds of dimension $2 \times 2$ over the point. Denote the $\text{Spin}(4)$-transversally elliptic Dirac $\mathcal{P}^{FS}(\mathbb{CP}^2)$ associated over the bundle of oriented orthonormal frame. This allows to define a family $\text{Id}_{S^5} \otimes \mathcal{P}^{FS}(\mathbb{CP}^2)$ of differential operators along the fibers of $S^5 \times \mathbb{CP}^2 \rightarrow S^5$, see [35].

**Lemma 5.4.** The family $\text{Id}_{S^5} \otimes \mathcal{P}^{FS}(\mathbb{CP}^2)$ on $S^5 \times F^{SO}(\mathbb{CP}^2) \rightarrow S^5$ is a family of $\text{Spin}(4)$-transversally elliptic operators which is $S^1$-invariant with respect to the diagonal $S^1$-action introduced above.

**Proof.** Indeed, recall the trivialisation $\alpha : T_{SO(2n)}(F^{SO(2n)}(\mathbb{CP}^2) \rightarrow F^{SO(2n)}(\mathbb{CP}^2) \times \mathbb{R}^4$ given by $\alpha(\mathcal{E}, w) = (\mathcal{E}, \mathcal{E}^{-1} \circ dw \circ q(w))$. This induces the trivialisation $\bar{\alpha} : F^{SO}(T_{SO(2n)}(F^{SO}(\mathbb{CP}^2))) \rightarrow F^{SO}(\mathbb{CP}^2) \times SO(4)$ given by $\bar{\alpha}(\mathcal{E}, \mathcal{W}) = (\mathcal{E}, \mathcal{E}^{-1} \circ dw \circ q \circ \mathcal{W})$. Let us look at the action of $S^1$ through the trivialisations $\alpha$ and $\bar{\alpha}$. The
action on $T_{SO(2n)}(F^{SO(2n)}(\mathbb{CP}^2))$ is given by $e^{i\theta} \cdot (\mathcal{E}, w) = (d_q(\mathcal{E})e^{i\theta} \circ \mathcal{E}, d_q(e^{i\theta})dw)$. We compute then

$$\alpha(e^{i\theta} \cdot (\mathcal{E}, w)) = \alpha(d_q(\mathcal{E})e^{i\theta} \circ \mathcal{E}, d_q(e^{i\theta})dw)$$

$$= (d_q(\mathcal{E})e^{i\theta} \circ \mathcal{E}, d_q(e^{i\theta})dw - i \circ [d_q(\mathcal{E})e^{i\theta}] q \circ d_q(e^{i\theta})dw)$$

$$= (d_q(\mathcal{E})e^{i\theta} \circ \mathcal{E}, d_q(e^{i\theta})dw - i \circ d_q(e^{i\theta})d_q(w))$$.

Since $q \circ d_q(e^{i\theta}) = e^{i\theta} \cdot q$, in other words since $q$ is $S^1$-equivariant, we obtain

$$\alpha(e^{i\theta} \cdot (\mathcal{E}, w)) = (d_q(\mathcal{E})e^{i\theta} \circ \mathcal{E}, d_q(e^{i\theta})dw - i \circ d_q(e^{i\theta})d_q(w))$$.

Therefore, the induced action on $F^{SO(\mathbb{CP}^2)} \times SO(4)$ is given by

$$e^{i\theta} \cdot (\mathcal{E}, \mathcal{W}) = (d_q(\mathcal{E})e^{i\theta} \circ \mathcal{E}, d_q(e^{i\theta} \circ \mathcal{E})dw - i \circ d_q(e^{i\theta})d_q(w))$$

and consequently the actions on $F^{SO(\mathbb{CP}^2)} \times Spin(2n)$ and $F^{SO(\mathbb{CP}^2)} \times S^\pm$ are given by the same formula, that is the action is through the action on the first component $F^{SO}(\mathbb{CP}^2)$. It follows then also that $\forall (\mathcal{E}, w) \in T_{SO(4)}F^{SO}(\mathbb{CP}^2)$, we have

$$c(e^{i\theta} \cdot (\mathcal{E}, w)) = c(\mathcal{E}, d_q(e^{i\theta})dw - i \circ d_q(e^{i\theta})d_q(w)).$$

This shows that the operator is $S^1$-invariant because $\nabla_+^{\varepsilon_q(\mathcal{E})}(s \circ d_q(e^{i\theta})) = \nabla d_q(e^{i\theta})d_q(w)(s)(d_q(e^{i\theta} \circ \mathcal{E})).$

We then get the result because the vertical transversal symbol of $\text{Id}_{S^5} \otimes \vartheta_{F^{SO}(\mathbb{CP}^2)}$ is given $\forall (x, \mathcal{E}, w) \in S^5 \times T_{SO(2n)}F^{SO}(\mathbb{CP}^2)$ by

$$\sigma(\text{Id}_{S^5} \otimes \vartheta_{F^{SO}(\mathbb{CP}^2)})(\mathcal{E}, w) = c(\mathcal{E}, w)$$

is invertible for non zero $w$. $\square$

By restriction to $S^1$-invariant functions, we obtain a family over $B = S^5/S^1 = \mathbb{CP}^2$ of $Spin(4)$-transversally elliptic operators

$$\vartheta_{\mathcal{F}|B}^+ \colon C^\infty(\mathcal{F}, S^+) \to C^\infty(\mathcal{F}, \mathcal{S}^-).$$

Let $v = \frac{i}{2} \sum_{i=0}^2 Z_k dZ_k - Z_k dZ_k$ be the standard $S^1$-connection on the principal bundle $S^5 \to \mathbb{CP}^2$ associated with the Fubini-Study metric on $\mathbb{CP}^2$. Here $(Z_0, Z_1, Z_2) \in S^5 \subset \mathbb{C}^3$. Denote by $\nabla$ the curvature of $v$. Recall that $\nabla$ is given in trivialisation corresponding to affine charts $U_i = \{ Z_i \neq 0 \}$ of $\mathbb{CP}^2$ with coordinates $z = (z_1, z_2)$ by $\nabla = \frac{i}{2} \left( \frac{1}{1+|z|^2} \sum dz_k \wedge d\bar{z}_k - \sum dz_k \wedge dz_{k\wedge} \sum z_k d\bar{z}_k \wedge d\bar{z}_k \right)$ and is therefore a $2$-form with real coefficients.

**Proposition 5.5.** Let $\vartheta_{\mathcal{F}|B}^+$ be the family of projective Dirac operators corresponding to $\vartheta_{\mathcal{F}|B}^+$, see Section 4.1 and Remark 5.3. We have

$$\text{Ind}_{\alpha}^M(\vartheta_{\mathcal{F}|B}^+) = \langle \text{Ind}_{\alpha}^{F^\infty}(\vartheta_{\mathcal{F}|B}^+), \alpha \rangle = -\frac{1}{2} \cdot \frac{\bar{Y}}{27} \cdot \frac{133}{3^2} \frac{1}{15}$$

for any $\varphi \in C^\infty(Spin(4))^{Ad(Spin(4))}$ equal to 1 around $\text{Id}$ and 0 around $-\text{Id}$.

**Proof.** Recall that the $\hat{A}$-genus $\hat{A}(E, \nabla) = \det^{1/2}(\frac{F/2}{\sinh(F/2)})$ of a real vector bundle $E \to Z$ with connection $\nabla$ and curvature $F$ over a manifold $Z$ belongs to $\hat{A}(Z, \mathbb{R})$. Furthermore, it is the form associated with the power series $h(t) = \frac{1}{2} \ln(\frac{F/2}{\sinh(F/2)})$, that is

$$\hat{A}(E, \nabla) = \exp(\frac{1}{2} \ln(\frac{F/2}{\sinh(F/2)})),$$

see [13] Section 1.5. Recall that $M = S^5 \times S^1 \subset \mathbb{CP}^2$ is 8-dimensional and therefore its $\hat{A}$-genus only has non vanishing terms of degree 0, 4 and 8. We then have using the expansion series of $f(t)$ that

$$\hat{A}(T(M|B)) = 1 - \frac{1}{12} \cdot \text{Tr}((F^T(M|B))^2) + \frac{1}{24 \cdot 360} \cdot \text{Tr}((F^T(M|B))^4) + \frac{1}{24 \cdot 288} \cdot \text{Tr}((F^T(M|B))^2)^2,$$

since higher coefficients vanish because their degree is bigger than $\dim M = 8$. 

\[\square\]
Using Theorem 5.2, we get that

\[ \langle \text{Ind}_{-\infty}^{\infty}(\theta^+_{F|B}), \varphi \rangle = -(2\pi)^{-2} \int_{M|B} \hat{A}(T(M|B)). \]

Now recall that integration along the fibers commutes with the Chern-Weil morphism, see [15] Proposition 7.35 for instance. Let then \( \nabla^{LC} \) be the Levi-Civita connection on \( T\mathbb{CP}^2 \) and denote by \( \nabla^{\alpha} \) for instance. Let then \( \nabla^{\alpha} \delta \) be the vanishing of the torsion of \( \nabla^{\alpha} \) as given by the diagonal matrix \( \nabla^{\alpha} = \text{diag} \quad \nabla^{\alpha} \delta \). It is shown in [15] Lemma 7.37 that this data defines a curvature 2-form \( F_{C\overline{C}} \) on \( T(M|B) \) given by \( F_{C\overline{C}} = \delta + \Theta \delta \), where 1 is the basis of Lie\( (S^1) \).

We shall denote by \([f \otimes \alpha]_{\max} = f \otimes [\alpha]_{\max}\) the component of \( f \otimes \alpha \in \mathcal{C}^\infty(\mathbb{R}) \otimes \mathcal{A}(\mathbb{CP}^2) \) with maximal degree (with respect to the form degree on \( \mathcal{A}(\mathbb{CP}^2) \)). We have

\[ [\text{Tr}((F(X))^2)]_{\max} = [\text{Tr}((F + X \mu(1))^2)]_{\max} = \text{Tr}(F^2) \]

and

\[ [\text{Tr}(F(X)^4)]_{\max} = [\text{Tr}((F^2 + X F \mu(1) + X \mu(1) F + X^2 \mu(2))]_{\max} \]
\[ = X^2 \text{Tr}(F^2 \mu(1)^2) + X^2 \text{Tr}((F \mu(1))^2) + X^2 \text{Tr}(F \mu(1)^2 F^2) \]
\[ + X^2 \text{Tr}((\mu(1)F)^2) + X^2 \text{Tr}(\mu(1)F^2 \mu(1)) + X^2 \text{Tr}(\mu(1)^2 F^2) \]
\[ = X^2 \left( 4 \text{Tr}(F^2 \mu(1)^2) + 2 \text{Tr}((F \mu(1)) \right) \]

and

\[ [\text{Tr}(F(X)^2)^2]_{\max} = X^2 \left( 2 \text{Tr}(F^2) \text{Tr}(\mu(1)^2) + 4 \text{Tr}(F \mu(1))^2 \right). \]

We did the computations with the help of the free software Sagemath in the coordinate chart \( U_0 = \{ z_0 \neq 0 \} \) with coordinates \((z_1, z_2) \in \mathbb{C}^2 \). In the sequel \( |z| = |z_1|^2 + |z_2|^2 \). We obtained that the square of the curvature \( F \) is given by the diagonal matrix

\[ F^2 = \text{diag} \left( \frac{3}{1 + |z|^2} \frac{3}{1 + |z|^2} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \right), \]

and therefore \( \text{Tr}(F^2) = \frac{12}{(1 + |z|^2)^3} \). We can then compute

\[ \frac{(2\pi)^{-2}}{-48} \int_{\mathbb{CP}^2} \text{Tr}(F^2) = \int_{U_0} \frac{12}{(1 + |z|^2)^3} \]
\[ = \frac{1}{48} \int_{\mathbb{R}^4} \frac{4}{(1 + |z|^2)^3} \]
\[ = \frac{12}{48} \int_{\mathbb{R}^2} dx_1 dy_1 dx_2 dy_2. \]

We chose to compute this integral using polar coordinates for \((x_i, y_i) = r_i (\cos(\theta_i), \sin(\theta_i))\) and then again polar coordinates with respect to \((r_1, r_2)\) because the other integrals seem to be easier to compute using this.
choice. We get

\[
\frac{(2i\pi)^{-2}}{-48} \int_{\mathbb{C}P^2} \text{Tr}(F^2) = - \int_{\mathbb{R}^+} \frac{r_1 r_2}{(1 + r_1^2 + r_2^2)^3} dr_1 dr_2
\]

\[
= - \int_{\mathbb{R}^+} \int_0^{\pi/2} r^3 \cos(\theta) \sin(\theta) \frac{d\theta dr}{(1 + r^2)^3}
\]

\[
= - \left[ \frac{\sin(\theta)^2}{2} \right]_0^{\pi/2} \left[ \frac{r^4}{4(1 + r^2)^2} \right]_0^{\infty}
\]

\[
= -\frac{1}{8}.
\]

Since \( F^2 \) is diagonal, we obtain that

\[
\text{Tr}(F^2 \mu(1)^2) = \frac{3\text{Tr}(\mu(1)^2)}{(1 + |z|^2)^3} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2.
\]

We have

\[
\text{Tr}(\mu(1)^2) = \frac{2|z_1|^4 + 2|z_2|^4 - 2|z_1|^2|z_2|^2 - 2|z_2|^2 + 2|z_1|^2 + 1}{(1 + |z|^2)^2}.
\]

We can then compute as we did before

\[
\int_{\mathbb{C}P^2} \text{Tr}(F^2 \mu(1)^2) = \int_{U_0} \frac{3\text{Tr}(\mu(1)^2)}{(1 + |z|^2)^3} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2
\]

\[
= \frac{(2\pi)^2}{2} \frac{3}{2}
\]

\[
= \frac{1}{4} \int_{\mathbb{C}P^2} \text{Tr}(F^2) \text{Tr}(\mu(1)^2).
\]

We have

\[
\text{Tr}(F\mu(1)) = \frac{-8|z_1|^4 + 9|z_2|^4 - 9(|z_1|^2 + 1)|z_2|^2 + 4|z_1|^2 + 2}{(1 + |z|^2)^5} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2.
\]

Therefore, we get

\[
\int_{\mathbb{C}P^2} \text{Tr}(F\mu(1))^2 = 4(2\pi)^2.
\]

For the last needed term, we obtain

\[
\text{Tr}\left((F\mu(1))^2\right) = \frac{-8|z_1|^4 + 9|z_2|^4 - 13|z_1|^2|z_2|^2 - 13|z_2|^2 + 16|z_1|^2 + 8}{(1 + |z|^2)^5}
\]

and therefore

\[
\int_{\mathbb{C}P^2} \text{Tr}\left((F\mu(1))^2\right) = \frac{5(2\pi)^2}{3}.
\]

Putting everything together, we finally get

\[
\text{Ind}^M_{\mathcal{B}}(\phi^+_M|B) = \frac{1}{8} + \frac{(2i\pi)^{-2}}{24 \cdot 23 \cdot 32} \left[ \frac{4}{5} \int_{\mathbb{C}P^2} \text{Tr}(F^2 \mu(1)^2) + \frac{2}{5} \int_{\mathbb{C}P^2} \text{Tr}(\mu(1)^2) \right]
\]

\[
+ \frac{1}{2} \int_{\mathbb{C}P^2} \text{Tr}(F^2) \text{Tr}(\mu(1)^2) + \int_{\mathbb{C}P^2} \left( \text{Tr}(F\mu(1)) \right)^2
\]

\[
= \frac{1}{8} - \frac{1}{24 \cdot 23 \cdot 32} \left[ \frac{6}{5} + \frac{10}{15} + 3 + 4 \right]
\]

\[
= \frac{1}{8} - \frac{1}{27 \cdot 32 \cdot 15}.
\]
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