CONSTRUCTIONS OF LINDELÖF SCATTERED P-SPACES

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Abstract. We construct locally Lindelöf scattered P-spaces (LLSP spaces, in short) with prescribed widths and heights under different set-theoretic assumptions.

We prove that there is an LLSP space of width $\omega_1$ and height $\omega_2$ and that it is relatively consistent with ZFC that there is an LLSP space of width $\omega_1$ and height $\omega_3$. Also, we prove a stepping up theorem that, for every cardinal $\lambda \geq \omega_2$, permits us to construct from an LLSP space of width $\omega_1$ and height $\lambda$ satisfying certain additional properties an LLSP space of width $\omega_1$ and height $\alpha$ for every ordinal $\alpha < \lambda^+$. Then, we obtain as consequences of the above results the following theorems:

1. For every ordinal $\alpha < \omega_3$ there is an LLSP space of width $\omega_1$ and height $\alpha$.
2. It is relatively consistent with ZFC that there is an LLSP space of width $\omega_1$ and height $\alpha$ for every ordinal $\alpha < \omega_4$.

1. Introduction

The cardinal sequence of a scattered space is the sequence of the cardinalities of its Cantor-Bendixson levels. The investigation of the cardinal sequences of different classes of topological spaces is a classical problem of set theoretic topology. Many important results were proved in connection with the cardinal sequences of locally compact scattered (LCS, in short) spaces, see e.g. 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16. In 5 a complete characterization of the cardinal sequences of the 0-dimensional, of the regular, and of the Hausdorff spaces was given.

Recall that a topological space $X$ is a $P$-space, if the intersection of every countable family of open sets in $X$ is open in $X$. The aim of this paper is to start the systematic investigation of cardinal sequences of locally Lindelöf scattered P-spaces. We will see that several methods applied to LCS spaces can be applied here, but typically we should face more serious technical problems.

If $X$ is a topological space and $\alpha$ is an ordinal, we denote by $X^\alpha$ the $\alpha$-th Cantor-Bendixson derivative of $X$. Then, $X$ is scattered if $X^\alpha = \emptyset$ for some $\alpha$.

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Assume that \( X \) is a scattered space. We define the \textit{height} of \( X \) by

\[ \text{ht}(X) = \text{the least ordinal } \alpha \text{ such that } X^\alpha = \emptyset. \]

For \( \alpha < \text{ht}(X) \), we write \( I_\alpha(X) = X^\alpha \setminus X^{\alpha+1} \). If \( x \in I_\alpha(X) \), we say that \( \alpha \) is the \textit{level} of \( x \) and we write \( \rho(x, X) = \alpha \), or simply \( \rho(x) = \alpha \) if no confusion can occur. Note that \( \rho(x) = \alpha \) means that \( x \) is an accumulation point of \( I_\beta(X) \) for \( \beta < \alpha \) but \( x \) is not an accumulation point of \( X^\alpha = \bigcup \{ I_\beta(X) : \beta \geq \alpha \} \). We define the \textit{width} of \( X \) as

\[ \text{wd}(X) = \sup\{|I_\alpha(X)| : \alpha < \text{ht}(X)\}. \]

If \( X \) is a scattered space, \( x \in X \) and \( U \) is a neighbourhood of \( x \), we say that \( U \) is a \textit{cone on} \( x \), if \( x \) is the only point in \( U \) of level \( \geq \rho(x, X) \).

By an \textit{LLSP space}, we mean a locally Lindelöf, scattered, Hausdorff P-space.

**Proposition 1.1.** An LLSP space is 0-dimensional.

**Proof.** By [13, Proposition 4.2(b)], a Lindelöf Hausdorff P-space \( X \) is normal, so a locally Lindelöf Hausdorff P-space is regular. Thus, by [13, Corollary 3.3], \( X \) is 0-dimensional. \( \square \)

So, by Proposition 1.1 above, if \( X \) is an LLSP space, \( x \in X \) and \( B_x \) is a neighbourhood basis of \( x \), we may assume that every \( U \in B_x \) is a Lindelöf clopen cone on \( x \).

It was proved by Juhász and Weiss in [6] that for every ordinal \( \alpha < \omega_2 \) there is an LCS space of height \( \alpha \) and width \( \omega \). Then, we will transfer this theorem to the setting of LLSP spaces, showing that for every ordinal \( \alpha < \omega_3 \) there is an LLSP space of height \( \alpha \) and width \( \omega_1 \).

To obtain an LCS space of height \( \omega_1 \) and width \( \omega \), in [6] Juhász and Weiss, using transfinite recursion, constructed a sequence \( \langle X_\alpha : \alpha \leq \omega_1 \rangle \) of LCS spaces such that \( X_\alpha \) had height \( \alpha \) and width \( \omega \), and for \( \alpha < \beta \), the space \( X_\alpha \) was just the first \( \alpha \) Cantor-Bendixson levels of \( X_\beta \).

Since \( X_\alpha \) is dense in \( X_{\alpha+1} \), Juhász and Weiss had to guarantee that \( X_\alpha \) is not compact. But it was automatic, because if \( \alpha = \gamma + 1 \), then \( X_\alpha \) had a top infinite Cantor-Bendixson level, so \( X_\alpha \) was not compact. If \( \alpha \) is a limit ordinal, then the open cover \( \{ X_\xi : \xi < \alpha \} \) witnessed that \( X_\alpha \) is not compact.

What happens if we try to adopt that approach for LLSP spaces? To obtain an LLSP space of height \( \omega_2 \) and width \( \omega_1 \), we can try, using transfinite recursion, to construct a sequence \( \langle X_\alpha : \alpha \leq \omega_2 \rangle \) of LLSP spaces such that \( X_\alpha \) has height \( \alpha \) and width \( \omega_1 \), and for \( \alpha < \beta \), the space \( X_\alpha \) is just the first \( \alpha \) levels of \( X_\beta \).

Since \( X_\alpha \) is dense in \( X_{\alpha+1} \), we have to guarantee that \( X_\alpha \) is not closed in \( X_{\alpha+1} \), in particular, \( X_\alpha \) is not Lindelöf. (Since in a P-space, Lindelöf subspaces are closed.) However, in our case it is not automatic in limit steps,
because the increasing countable union of open non-Lindelöf subspaces can be Lindelöf.

So some extra effort is needed to guarantee the non-Lindelöfness in limit steps.

Assume that \( \kappa \) is an uncountable cardinal and \( \alpha \) is a non-zero ordinal. If \( X \) is an LLSP space such that \( \text{ht}(X) = \alpha \) and \( \text{wd}(X) = \kappa \), we say that \( X \) is a \((\kappa, \alpha)\)-LLSP space.

Then, we will also transfer the results proved in [3] and [11] on thin-tall spaces to the context of locally Lindelöf P-spaces, showing that Con(ZFC) implies Con(ZFC + “there is an \((\omega_1, \alpha)\)-LLSP space for every ordinal \( \alpha < \omega_4 \).”

2. Construction of an LLSP space of width \( \omega_1 \) and height \( \omega_2 \)

By a decomposition of a set \( A \) of size \( \omega_1 \), we mean a partition of \( A \) into subsets of size \( \omega_1 \). In this section we will prove the following result.

**Theorem 2.1.** There is an \((\omega_1, \omega_2)\)-LLSP space.

**Proof.** We construct an \((\omega_1, \omega_2)\)-LLSP space whose underlying set is \( \omega_2 \). For every \( \alpha < \omega_2 \), we put \( I_\alpha = (\omega_1 \cdot (\alpha + 1)) \setminus (\omega_1 \cdot \alpha) \), and for every ordinal \( \xi < \omega_1 \), we define the “column” \( N_\xi = \{ \omega_1 \cdot \mu + \xi : \mu < \omega_2 \} \). Write \( \xi \in N_{\eta(\xi)} \).

Our aim is to construct, by transfinite induction on \( \alpha < \omega_2 \) an LLSP space \( X_\alpha \) satisfying the following:

1. \( X_\alpha \) is an \((\omega_1, \alpha + 1)\)-LLSP space such that \( I_\beta(X_\alpha) = I_\beta \) for every \( \beta \leq \alpha \).
2. For every \( \xi < \omega_1 \), \( N_\xi \cap X_\alpha \) is a closed discrete subset of \( X_\alpha \).
3. If \( \beta < \alpha \) and \( x \in X_\beta \), then a neighbourhood basis of \( x \) in \( X_\beta \) is also a neighbourhood basis of \( x \) in \( X_\alpha \).

For every \( \alpha < \omega_2 \) and \( x \in I_\alpha \), in order to define the required neighbourhood basis \( B_x \) of \( x \) in \( X_\alpha \), we will also fix a Lindelöf cone \( V_x \) of \( x \) in \( X_\alpha \) such that the following holds:

4. \( V_x \cap I_\alpha = \{ x \} \).
5. \( V_x = \bigcup B_x \).
6. There is a club subset \( C_x \) of \( \omega_1 \) such that \( \omega_1 \setminus C_x \) is unbounded in \( \omega_1 \) and \( V_x \cap \bigcup \{ N_\nu : \nu \in C_x \} = \emptyset \).

We define \( X_0 \) as the set \( I_0 = \omega_1 \) with the discrete topology, and for \( x \in I_0 \) we put \( V_x = \{ x \} \) and \( C_x = \{ y \in \omega_1 : y \text{ is a limit ordinal > } x \} \). So, assume that \( \alpha > 0 \). If \( \alpha = \beta + 1 \) is a successor ordinal, we put \( Z = X_\beta \). And if \( \alpha \) is a limit ordinal, we define \( Z \) as the direct union of \( \{ X_\beta : \beta < \alpha \} \). So, the underlying set of the required space \( X_\alpha \) is \( Z \cup I_\alpha \). If \( x \in Z \), then a basic neighbourhood of \( x \) in \( X_\alpha \) is a neighbourhood of \( x \) in \( Z \). Our purpose is to define a neighbourhood basis of each element of \( I_\alpha \). Let \( \{ x_\nu : \nu < \omega_1 \} \) be an enumeration without repetitions of \( Z \). By the induction hypothesis, for
every $\xi < \omega_1$ there is a club subset $C_\xi$ of $\omega_1$ such that $\omega_1 \setminus C_\xi$ is unbounded in $\omega_1$ and $V_{x_\xi} \cap \bigcup \{ N_\nu : \nu \in C_\xi \} = \emptyset$. Let $C = \Delta \{ C_\xi : \xi < \omega_1 \}$, the diagonal intersection of the family $\{ C_\xi : \xi < \omega_1 \}$. As $V_{x_\xi} \cap \bigcup \{ N_\nu : \nu \in C_\xi \} = \emptyset$, by the definition of $C$, for every $\xi < \omega_1$, $V_{x_\xi} \cap \bigcup \{ N_\nu : \nu \in C \} \subset \bigcup \{ N_\nu : \nu \leq \xi \}$, and clearly $\omega_1 \setminus C$ is unbounded in $\omega_1$. Then, we will define for every element $y \in I_\alpha$, a neighbourhood basis of $y$ from a set $V_y$ in such a way that for some final segment $C'$ of $C$ we will have that $V_y \cap \bigcup \{ N_\nu : \nu \in C' \} = \emptyset$. We distinguish the following three cases:

**Case 1.** $\alpha = \beta + 1$ is a successor ordinal.

For each $\xi < \omega_1$ we take a Lindelöf clopen cone $U_\xi$ on some $u_\xi$ in $Z$ as follows. We take $U_0 \subseteq V_{x_0}$ as a Lindelöf clopen cone on $x_0$ such that $(U_0 \setminus \{ x_0 \}) \cap N_0 = \emptyset$. Suppose that $\xi > 0$. Let $u_\xi$ be the first element $x_\eta$ in the enumeration $\{ x_\nu : \nu < \omega_1 \}$ of $Z$ such that $u_\xi \not\in \bigcup \{ U_\mu : \mu < \xi \}$. Since $I_\beta \cap \bigcup \{ U_\mu : \mu < \xi \} \subset \{ \eta_\mu : \mu < \xi \}$, the element $u_\xi$ is defined. Then, we choose $U_\xi \subset V_{x_\eta}$ as a Lindelöf clopen cone on $u_\xi$ such that $U_\xi \cap \bigcup \{ U_\mu : \mu < \xi \} = \emptyset$ and $(U_\xi \setminus \{ u_\xi \}) \cap \bigcup \{ N_\nu : \nu \leq \eta \} = \emptyset$. So, as $V_{x_\eta} \cap \bigcup \{ N_\nu : \nu \in C \} \subset \bigcup \{ N_\nu : \nu \leq \eta \}$, we deduce that $(U_\xi \setminus \{ u_\xi \}) \cap \bigcup \{ N_\nu : \nu \in C \} = \emptyset$. And clearly, $\{ U_\xi : \xi < \omega_1 \}$ is a partition of $Z$. Let

$$A = \{ \xi \in \omega_1 : u_\xi \in I_\beta \cap N_\rho \ \text{for some} \ \rho \in \omega_1 \setminus C \}.$$ 

Since $I_\beta \subset \{ u_\xi : \xi < \omega_1 \}$, we have $|A| = \omega_1$. Let $\{ A_\xi : \xi < \omega_1 \}$ be a decomposition of $A$. Fix $\xi < \omega_1$. Let $y_\xi = \omega_1 \cdot \alpha + \xi$. Then, we define

$$V_{y_\xi} = \{ y_\xi \} \cup \bigcup \{ U_\nu : \nu \in A_\xi \}.$$ 

Note that since $\bigcup \{ U_\nu : \nu \in A_\xi \} \cap \bigcup \{ N_\nu : \nu \in C \} = \emptyset$, we infer that $V_{y_\xi} \cap \bigcup \{ N_\nu : \nu \in C \}$ and $\nu > \xi = \emptyset$. Now, we define a basic neighbourhood of $y_\xi$ in $X_\alpha$ as a set of the form

$$\{ y_\xi \} \cup \bigcup \{ U_\nu : \nu \in A_\xi, \nu \geq \zeta \}$$

where $\zeta < \omega_1$. Then, it is easy to check that conditions $(1)$ – $(6)$ hold.

**Case 2.** $\alpha$ is a limit ordinal of cofinality $\omega_1$.

Let $\langle \alpha_\nu : \nu < \omega_1 \rangle$ be a strictly increasing sequence of ordinals cofinal in $\alpha$. For every $\xi < \omega_1$, we choose a Lindelöf clopen cone $U_\xi$ on some point $u_\xi$ in $Z$ as follows. If $\xi$ is not a limit ordinal, let $u_\xi$ be the first element $x_\eta$ in the enumeration $\{ x_\nu : \nu < \omega_1 \}$ of $Z$ such that $u_\xi \not\in \bigcup \{ U_\mu : \mu < \xi \}$ and let $U_\xi \subset V_{x_\eta}$ be a Lindelöf clopen cone on $u_\xi$ such that $U_\xi \cap \bigcup \{ U_\mu : \mu < \xi \} = \emptyset$. Now, assume that $\xi$ is a limit ordinal. Let $\nu < \omega_1$ be such that $\alpha_\nu > \sup \{ \rho(u_\mu, Z) : \mu < \xi \}$. Then, we pick $u_\xi$ as the first element $x_\eta$ in the enumeration $\{ x_\nu : \nu < \omega_1 \}$ of $Z$ such that $u_\xi \in I_{\alpha_\nu}(Z) \cap N_\delta$ for some $\delta \in \omega_1 \setminus C$ with $\delta > \xi$. Note that by the election of $\alpha_\nu$, we have that
Then, we choose \( U_\xi \subset V_{x_\eta} \) as a Lindelöf clopen cone on \( u_\xi \) such that

\[
U_\xi \cap \bigcup \{ U_\mu : \mu < \xi \} = \emptyset
\]

and

\[
(U_\xi \setminus \{ u_\xi \}) \cap \bigcup \{ N_\nu : \nu \leq \eta \} = \emptyset.
\]

Then since \( V_{x_\eta} \cap \bigcup \{ N_\nu : \nu \in C \} \subset \bigcup \{ N_\nu : \nu \leq \eta \} \) and \( \delta \not\in C \), we infer that \( U_\xi \cap \bigcup \{ N_\nu : \nu \in C \} = \emptyset \).

Now, let \( \{ A_\xi : \xi < \omega_1 \} \) be a decomposition of the set of limit ordinals of \( \omega_1 \). Fix \( \xi < \omega_1 \). Let \( y_\xi = \omega_1 \cdot \alpha + \xi \). Then, we define

\[
V_{y_\xi} = \{ y_\xi \} \cup \bigcup \{ U_\mu : \mu \in A_\xi \}.
\]

Clearly, \( V_{y_\xi} \cap \bigcup \{ N_\nu : \nu \in C, \nu > \xi \} = \emptyset \). Now, we define a basic neighbourhood of \( y_\xi \) in \( X_\alpha \) as a set of the form

\[
V_{y_\xi} \setminus \bigcup \{ U_\nu : \nu \in A_\xi, \nu < \zeta \}
\]

where \( \zeta < \omega_1 \).

Note that the condition that \( \delta > \xi \) in the election of \( u_\xi \) for \( \xi \) a limit ordinal is needed to assure that \( N_\xi \cap X_\alpha \) is a closed discrete subset of \( X_\alpha \) for \( \xi < \omega_1 \). So, conditions (1) – (6) hold.

**Case 3.** \( \alpha \) is a limit ordinal of cofinality \( \omega \).

Let \( \langle \alpha_n : n < \omega \rangle \) be a strictly increasing sequence of ordinals converging to \( \alpha \). Proceeding by transfinite induction on \( \xi < \omega_1 \), we construct a sequence \( \langle u_n^\xi : n < \omega \rangle \) of points in \( Z \) and a sequence \( \langle U_n^\xi : n < \omega \rangle \) such that each \( U_n^\xi \subset V_{u_n^\xi} \) is a Lindelöf clopen cone on \( u_n^\xi \) as follows. Fix \( \xi < \omega_1 \), and assume that for \( \mu < \xi \) the sequences \( \langle u_n^\mu : n < \omega \rangle \) and \( \langle U_n^\mu : n < \omega \rangle \) have been constructed. Let \( C^* = \bigcap \{ C_n^\mu : \mu < \xi, n < \omega \} \). Note that \( C^* \) is a club subset of \( \omega_1 \), because it is a countable intersection of club subsets of \( \omega_1 \). Now, since for every \( \mu < \xi \) and \( n < \omega \), we have that \( V_{u_n^\mu} \cap \bigcup \{ N_\nu : \nu \in C_n^\mu \} = \emptyset \), we infer that

\[
\bigcup \{ V_{u_n^\mu} : \mu < \xi, n < \omega \} \cap \bigcup \{ N_\nu : \nu \in C^* \} = \emptyset.
\]

Hence, for every ordinal \( \beta < \alpha \),

\[
|I_\beta \setminus \bigcup \{ V_{u_n^\mu} : \mu < \xi, n < \omega \}| = \omega_1.
\]

Now, we construct the sequences \( \langle u_n^\xi : n < \omega \rangle \) and \( \langle U_n^\xi : n < \omega \rangle \) by induction on \( n \). If \( n \) is even, let \( u_n^\xi \) be the first element \( x_n \) in the enumeration \( \{ x_\nu : \nu < \omega_1 \} \) of \( Z \) such that \( u_n^\xi \not\in \bigcup \{ U_k^\mu : \mu < \xi, k < \omega \} \cup \bigcup \{ U_n^\xi : k < n \} \), and let \( U_n^\xi \subset V_{x_n} \) be a Lindelöf clopen cone on \( u_n^\xi \) such that
for \( \alpha < \omega \), we have that

\[
U_\xi^\xi \cap \left( \bigcup \{ U_\mu^\mu : \mu < \xi, k < \omega \} \cup \bigcup \{ U_\xi^\xi : k < n \} \right) = \emptyset.
\]

Now, suppose that \( n \) is odd. Let \( k \in \omega \) be such that \( \alpha_k > \sup \{ \rho(u_n^\xi, Z) : m < n \} \). First, we pick \( \tilde{u}_n^\xi \) as the first element \( x_n \) in the enumeration \( \{ x_\nu : \nu < \omega_1 \} \) of \( Z \) such that \( \tilde{u}_n^\xi \in I_{\alpha_k + 1}(Z) \cap N_\zeta \), for some \( \zeta \in C^* \). So, \( \tilde{u}_n^\xi \notin \bigcup \{ U_\mu^\mu : \mu < \xi, m < \omega \} \cup \bigcup \{ U_\xi^\xi : m < n \} \). Now, we choose \( \tilde{U}_n^\xi \subset V_{x_n} \)
as a Lindelöf clopen cone on \( \tilde{u}_n^\xi \) such that

\[
\tilde{U}_n^\xi \cap \left( \bigcup \{ U_\mu^\mu : \mu < \xi, m < \omega \} \cup \bigcup \{ U_\xi^\xi : m < n \} \right) = \emptyset.
\]

and

\[
(\tilde{U}_n^\xi \setminus \{ \tilde{u}_n^\xi \}) \cap \bigcup \{ N_\nu : \nu \leq \eta \} = \emptyset.
\]

Then as \( \tilde{u}_n^\xi = x_n \) and \( V_{x_\rho} \cap \bigcup \{ N_\nu : \nu \in C \} \subset \bigcup \{ N_\nu : \nu \leq \eta \} \), we infer that \( (\tilde{U}_n^\xi \setminus \{ \tilde{u}_n^\xi \}) \cap \bigcup \{ N_\nu : \nu \in C \} = \emptyset \). However, note that if \( \zeta \) is the ordinal such that \( \tilde{u}_n^\xi \in N_\zeta \), it may happen that \( \zeta \in C \). Then, we pick \( u_n^\xi \) as the first element \( x_\rho \) in the enumeration \( \{ x_\nu : \nu < \omega_1 \} \) of \( Z \) such that \( u_n^\xi \in \tilde{U}_n^\xi \cap I_{\alpha_k}(Z) \cap N_\delta \) for some \( \delta > \xi \). Note that \( \delta \notin C \), because \( (\tilde{U}_n^\xi \setminus \{ \tilde{u}_n^\xi \}) \cap \bigcup \{ N_\nu : \nu \in C \} = \emptyset \). Now, we choose \( U_n^\xi \subset \tilde{U}_n^\xi \cap V_{x_\rho} \) as a Lindelöf clopen cone on \( u_n^\xi \) such that

\[
( U_n^\xi \setminus \{ u_n^\xi \} ) \cap \bigcup \{ N_\nu : \nu \leq \rho \} = \emptyset.
\]

Hence as \( V_{x_\rho} \cap \bigcup \{ N_\nu : \nu \in C \} \subset \bigcup \{ N_\nu : \nu \leq \rho \} \) and \( \delta \notin C \), we infer that

\[
U_n^\xi \cap \bigcup \{ N_\nu : \nu \in C \} = \emptyset.
\]

Now, let \( \{ A_\xi : \xi < \omega_1 \} \) be a decomposition of \( \omega_1 \). Fix \( \xi < \omega_1 \). Let \( y_\xi = \omega_1 \cdot \alpha + \xi \). Then, we define

\[
V_{y_\xi} = \{ y_\xi \} \cup \bigcup \{ U_\mu^\mu : \mu \in A_\xi, n \text{ odd} \}.
\]

As \( \bigcup \{ U_\mu^\mu : \mu \in A_\xi, n \text{ odd} \} \cap \bigcup \{ N_\nu : \nu \in C \} = \emptyset \), we deduce that \( V_{y_\xi} \cap \bigcup \{ N_\nu : \nu \in C \text{ and } \nu > \xi \} = \emptyset \). Then, we define a basic neighbourhood of \( y_\xi \) in \( X_\alpha \) as a set of the form

\[
\{ y_\xi \} \cup \bigcup \{ U_\mu^\mu : \mu \in A_\xi, \mu \geq \zeta, n \text{ odd} \}
\]

where \( \zeta < \omega_1 \). Now, it is easy to see that conditions (1) – (6) hold.

Then, we define the desired space \( X \) as the direct union of the spaces \( X_\alpha \) for \( \alpha < \omega_2 \).

\( \square \)

\textbf{Remark 2.2.} Note that by the construction carried out in the proof of Theorem 2.1, we have that

if \( U \subset X \) is Lindelöf then \( \{ \xi : N_\xi \cap U \neq \emptyset \} \in NS(\omega_1) \).
3. A stepping up theorem

In this section, for every cardinal \( \lambda \geq \omega_2 \) we will construct from an \((\omega_1, \lambda)\)-LLSP space satisfying certain additional properties an \((\omega_1, \alpha)\)-LLSP space for every ordinal \( \alpha < \lambda^+ \). As a consequence of this construction, we will be able to extend Theorem 2.1 from \( \omega_2 \) to any ordinal \( \alpha < \omega_3 \). We need some preparation.

Definitions 3.1. (a) Assume that \( X \) is an LLSP space, \( \beta + 1 < \text{ht}(X) \), \( x \in I_{\beta+1}(X) \) and \( \mathbb{B}_x \) is a neighbourhood basis for \( x \). We say that \( \mathbb{B}_x \) is admissible, if there is a pairwise disjoint family \( \{U_\nu : \nu < \omega_1\} \) such that for every \( \nu < \omega_1 \), \( U_\nu \) is a Lindelöf clopen cone on some point \( x_\nu \in I_\beta(X) \) in such a way that \( \mathbb{B}_x \) is the collection of sets of the form

\[
\{x\} \cup \bigcup\{U_\nu : \nu \geq \xi\},
\]

where \( \xi < \omega_1 \). Then, we will say that \( \mathbb{B}_x \) is the admissible basis for \( x \) given by \( \{U_\nu : \nu < \omega_1\} \).

(b) Now, we say that \( X \) is an admissible space if for every \( x \in X \) there is a neighbourhood basis \( \mathbb{B}_x \) such that for every successor ordinal \( \beta + 1 < \text{ht}(X) \) the following holds:

1. \( \mathbb{B}_x \) is an admissible basis for every point \( x \in I_{\beta+1}(X) \),
2. if \( x, y \in I_{\beta+1}(X) \) with \( x \neq y \) and \( \rho(x) = \rho(y) \), \( \mathbb{B}_x \) is given by \( \{U_\nu : \nu < \omega_1\} \) and \( \mathbb{B}_y \) is given by \( \{U'_\nu : \nu < \omega_1\} \), then for every \( \nu, \mu < \omega_1 \) we have \( U_\nu \cap U'_\mu = \emptyset \).

Note that the space \( X \) constructed in the proof of Theorem 2.1 is admissible.

Definition 3.2. We say that an LLSP space \( X \) is good, if for every ordinal \( \alpha < \text{ht}(X) \) and every set \( \{U_n : n \in \omega\} \) of Lindelöf clopen cones on points of \( X \), the set \( I_\alpha(X) \setminus \bigcup\{U_n : n \in \omega\} \) is uncountable.

Note that the space \( X \) constructed in the proof of Theorem 2.1 is good.

Assume that \( X \) is a good LLSP space. Then, we define the space \( X^* \) as follows. Its underlying set is \( X \cup \{z\} \) where \( z \notin X \). If \( x \in X \), a basic neighbourhood of \( x \) in \( X^* \) is a neighbourhood of \( x \) in \( X \). And a basic neighbourhood of \( z \) in \( X^* \) is a set of the form

\[
X^* \setminus \bigcup\{U_n : n \in \omega\}
\]

where each \( U_n \) is a Lindelöf clopen cone on some point of \( X \). Clearly, \( X^* \) is a Lindelöf scattered Hausdorff P-space with \( \text{ht}(X^*) = \text{ht}(X) + 1 \).

Theorem 3.3. Let \( \lambda \geq \omega_2 \) be a cardinal. Assume that there is a good \((\omega_1, \lambda)\)-LLSP space that is admissible. Then, for every ordinal \( \alpha < \lambda^+ \) there is a good \((\omega_1, \alpha)\)-LLSP space.
So, we obtain the following consequence of Theorems 2.1 and 3.3.

**Corollary 3.4.** For every ordinal $\alpha < \omega_3$ there is a good $(\omega_1, \alpha)$-LLSP space.

**Proof of Theorem 3.3.** We may assume that $\lambda \leq \alpha < \lambda^+$. We proceed by transfinite induction on $\alpha$. If $\alpha = \lambda$, the case is obvious. Assume that $\alpha = \beta + 1$ is a successor ordinal. Let $Y$ be a good $(\omega_1, \beta)$-LLSP space. For every $\nu < \omega_1$ let $Y_\nu$ be a P-space homeomorphic to $Y^*$ in such a way that $Y_\nu \cap Y_\mu = \emptyset$ for $\nu < \mu < \omega_1$. Clearly, the topological sum of the spaces $Y_\nu$ ($\nu < \omega_1$) is a good $(\omega_1, \alpha)$-LLSP space.

Now, assume that $\alpha > \lambda$ is a limit ordinal. Let $\theta = \text{cf}(\alpha)$. Note that since there is a good admissible $(\omega_1, \lambda)$-LLSP space and $\theta \leq \lambda$, there is a good admissible LLSP space $T$ of width $\omega_1$ and height $\theta$. Let $\{\alpha_\xi : \xi < \theta\}$ be a closed strictly increasing sequence of ordinals cofinal in $\alpha$ with $\alpha_0 = 0$. For every ordinal $\xi < \theta$, we put $J_\xi = \{\alpha_\xi\} \times \omega_1$. We may assume that the underlying set of $T$ is $\bigcup\{J_\xi : \xi < \theta\}$, $I_\xi(T) = J_\xi$ for every $\xi < \theta$ and $I_\theta(T) = \emptyset$.

Fix a system of neighbourhood bases, $\{B_x : x \in T\}$, which witnesses that $T$ is admissible. Write $V_s = \bigcup B_s$ for $s \in T$.

So, writing

$$T' = \{s \in T : \rho(s, T) \text{ is a successor ordinal}\},$$

for each $s \in T$ with $\rho(s, T) = \xi + 1$, there is $D_s = \{d_\xi^s : \xi < \omega_1\} \subseteq [I_\xi(T)]^{\omega_1}$ and for each $d \in D_s$ there is a Lindelöf cone $U_d$ on $d$ such that

$$B_s = \{s\} \cup \bigcup_{\eta \leq \xi} U_{d_\eta^s} : \eta < \omega_1\}.$$

In order to carry out the desired construction, we will insert an adequate LLSP space between $I_\xi(T)$ and $I_{\xi+1}(T)$ for every $\xi < \theta$. If $\xi < \theta$, we define $\delta_\xi = \text{o.t.}(\alpha_{\xi+1} \setminus \alpha_\xi)$. We put $y_{\delta_\xi + 1}^\xi = \langle \alpha_{\xi+1}, \nu \rangle$ for $\xi < \theta$ and $\nu < \omega_1$, and we put $D_\delta^\xi = \{x \in T : \rho(x, T) = \xi$ and $x \in V_{y_{\delta_\xi + 1}^\xi}\} = D_{y_{\delta_\xi + 1}^\xi}$. Since $T$ is admissible, $D_{y_\nu} \cap D_{y_\mu} = \emptyset$ for $\nu \neq \mu$.

Now, by the induction hypothesis, for every point $y = y_{\delta_\xi + 1}^\xi$ where $\xi < \theta$ and $\nu < \omega_1$ there is a Lindelöf scattered Hausdorff P-space $Z_y$ of height $\delta_\xi + 1$ such that $I_\delta(Z_y) = D_\delta^\xi$, $|I_\nu(Z_y)| = \omega_1$ for $\nu < \delta_\xi$, $I_\delta(Z_y) = \{y\}$ and $Z_y \cap T = D_\delta^\xi \cup \{y\}$. Also, we assume that $Z_{y_{\delta_\xi + 1}^\xi} \cap Z_{y_{\nu}^\eta + 1} = \emptyset$ for $\nu \neq \mu$ and $\eta < \theta$ and $\mu < \omega_1$.

Now, our aim is to define the desired $(\omega_1, \alpha)$-LLSP space $Z$. Its underlying set is

$$Z = T \cup \bigcup \{Z_y : y \in T'\}.$$

If $V$ is a Lindelöf clopen cone on a point $z \in T$, we define
$V^* = V \cup \{(Z_y \setminus T) : y \in V \cap T^\prime\}$. 

Observe that if $y \in V \cap T^\prime$, then $Z_y \setminus V^* = D_y \setminus V$ and $D_y \setminus V$ is countable because $T$ is admissible. So $Z_y \cap V^*$ is open in $Z_y$ because $Z_y$ is a P-space.

Now, assume that $x \in Z_s$ for some $s \in T^\prime$. Then, if $U$ is a Lindelöf clopen cone on $x$ in $Z_s$, we define

$$U^\sim = U \cup \{(U_y)^* : y \in D_s \cap U\}.$$ 

Note that for every $s \in T^\prime$ we have $(V_s)^* = (Z_s)^\sim$.

After that preparation we can define the bases of the points of $Z$. Suppose that $x \in Z = T \cup \bigcup \{Z_s : y \in T^\prime\}$. If $x \in T \setminus T^\prime$, then let

$$B_x^Z = \{V^* : V \text{ is a Lindelöf clopen cone on } x \text{ in } T\}.$$ 

If $x \in (Z \setminus T) \cup T^\prime$, then pick first the unique $s \in T^\prime$ such that $x \in Z_s \setminus I_0(Z_s)$, and let

$$B_x^Z = \{U^\sim : U \text{ is a Lindelöf clopen cone on } x \text{ in } Z_s\}.$$ 

**Claim 1.** \(\{B_x^Z : x \in Z\}\) is a system of neighbourhood bases of a topology \(\tau_Z\).

**Proof.** Assume that $y \in W \in B_x^Z$. We should show that $B_x^Z \cap P(W) \neq \emptyset$.

Assume first that $x \in T \setminus T^\prime$, and so $W = V^*$ for some Lindelöf clopen cone $V$ on $x$ in $T$.

If $y \in T \setminus T^\prime$, then $y \in V$ and so $S \subset V$ for some Lindelöf clopen clone $S$ on $y$ in $T$. Thus $y \in S^* \subset V^*$ and $S^* \in B_y^Z$.

If $y \in (Z \setminus T) \cup T^\prime$ then pick first the unique $s \in T^\prime$ such that $y \in Z_s \setminus I_0(Z_s)$. Then $s \in V$ because otherwise $y \in V^*$ is not possible. So as we observed, $V^* \cap Z_s$ is open in $Z_s$. So let $S$ be a Lindelöf clopen cone on $y$ in $Z_s$ with $S \subset V^* \cap Z_s$. Then $y \in S^\sim \subset V^*$ and $S^\sim \in B_y^Z$.

Assume now that $x \in (Z \setminus T) \cup T^\prime$, then pick first the unique $s \in T^\prime$ such that $x \in Z_s \setminus I_0(Z_s)$. Then $W = U^\sim$ for some Lindelöf clopen cone $U$ on $x$ in $Z_s$.

If $y \in Z_s \setminus I_0(Z_s)$, then $S \subset U$ for some Lindelöf clopen cone $S$ on $y$ in $Z_s$, and so $S^\sim \in B_y^Z$ and $S^\sim \subset U^\sim$.

If $y \notin Z_s \setminus I_0(Z_s)$, then $y \in (U_d)^*$ for some $d \in I_0(Z_s) \cap U$, and so there is $S \in B_y^Z$ with $S \subset (U_d)^*$ using what we proved so far. Thus $S \subset U^\sim$ as well. 

**Claim 2.** \(\tau_Z\) is Hausdorff.

**Proof.** Assume that \(\{x, y\} \in |Z|^2\). Let $s$ and $t$ be elements of $T$ such that $x \in Z_s \setminus I_0(Z_s)$ if $x \notin T \setminus T^\prime$ and $s = x$ otherwise, and $y \in Z_t \setminus I_0(Z_t)$ if $y \notin T \setminus T^\prime$ and $t = y$ otherwise.
If $s \neq t$, consider disjoint Lindel"of clopen cones $U$ and $V$ on $s$ and $t$ in $T$ respectively. Note that if $w \in U \cap T'$, then $Z_w \setminus T \subset U^*$ because $w \in U$, but $(Z_w \setminus T) \cap V^* = \emptyset$ because $w \notin V$, and analogously if $w \in V \cap T'$ then $Z_w \setminus T \subset V^*$ but $(Z_w \setminus T) \cap U^* = \emptyset$. So, $U^*$ and $V^*$ are disjoint open sets containing $x$ and $y$ respectively.

If $s = t$, then there are disjoint cones in $Z_s$ on $x$ and $y$, $U$ and $V$, respectively. Then $U^*$ and $V^*$ are disjoint open sets containing $x$ and $y$, respectively. □

It is trivial from the definition that $Z$ is a $P$-space because $T$ is a $P$-space and the $Z_s$ are $P$-spaces.

By transfinite induction on $\delta < \alpha$ it is easy to check that

$$I_\delta(Z) = \begin{cases} J_\xi & \text{if } \delta = \alpha_\xi, \\ \cup \{I_\eta(Z_s) : s \in I_{\xi+1}(T)\} & \text{if } \alpha_\eta < \delta = \alpha_\xi + \eta < \alpha_{\xi+1}, \end{cases}$$

so $Z$ is scattered with height $\alpha$ and width $\omega_1$.

**Claim 3.** $Z$ is locally Lindel"of.

**Proof.** Note that if $x \in T \setminus T'$ and $U^* \in \mathbb{B}_x^\infty$, then for every $V^* \in \mathbb{B}_x^\infty$ with $V^* \subset U^*$ we have that $U^* \setminus V^* = \cup\{W^*_n : n \in \nu\}$ where $\nu \leq \omega$ in such a way that each $W_n$ is a Lindel"of clopen cone on some point $v_n \in T \cap U$ in $T$ with $\rho(v_n, T) < \rho(x, T)$.

Also, if $x \in T' \cup (Z \setminus T)$ and $U^* \in \mathbb{B}_x^\infty$ then for every $V^* \in \mathbb{B}_x^\infty$ with $V^* \subset U^*$, if $s$ is the element of $T'$ with $x \in Z_s \setminus I_0(Z_s)$, we have that $U^* \setminus V^* = \cup\{U'_n : n \in \nu\}$ where $\nu \leq \omega$ in such a way that for every $n \in \nu$, either $U'_n = U_n^\sim$ where $U_n$ is a Lindel"of clopen cone on some point $u_n \in Z_s \cap U$ in $Z_s$ with $0 < \rho(u_n, Z_s) < \rho(x, Z_s)$ or $U'_n = U^*_n$ where $U_n$ is a Lindel"of clopen cone on some point $u_n \in D_s \cap U$ in $T$.

Now, proceeding by transfinite induction on $\rho(x, Z)$, we can verify that if $x \in T \setminus T'$ and $U$ is a Lindel"of clopen cone on $x$ in $T$, then $U^*$ is a Lindel"of clopen cone on $x$ in $Z$, and that if $x \in Z_s \setminus I_0(Z_s)$ for some $s \in T'$ and $U$ is a Lindel"of clopen cone on $x$ in $Z_s$, then $U^*$ is a Lindel"of clopen cone on $x$ in $Z$. Therefore, $Z$ is locally Lindel"of. □

**Claim 4.** $Z$ is good.

**Proof.** Let $\delta < \alpha = ht(Z)$ and let $\{W_n : n \in \omega\}$ be a family of Lindel"of cones in $Z$. Since every $W_n$ is covered by countably many Lindel"of cones from the basis, we can assume that $W_n \in \mathbb{B}_{x_n}^\infty$ for some $x_n \in Z$ for each $n \in \omega$. For each $n$ pick $y_n \in T$ such that $y_n = x_n$ if $x_n \in T$ and $x_n \in Z_{y_n}$ otherwise.

Then $W_n \subset W'_n$ for some $W'_n \in \mathbb{B}_{y_n}^\infty$, so we can assume that $\{x_n : n \in \omega\} \subset T$. We can also assume that if $x_n \in T'$, then $W_n$ is as large as possible, i.e. $W_n = Z_{x_n}^\sim = (V_{x_n})^\ast$.

If $x_n \in T \setminus T'$, then $W_n = S_n^\ast$ for some Lindel"of cone $S_n$ on $x_n$ in $T$. 

If \( \delta = \alpha \xi \) for some \( \xi \), then \( I_\delta(Z) \cap W_\alpha = I_\delta(Z) \cap V_{x_\alpha} \) if \( x_\alpha \in T' \) and \( I_\delta(Z) \cap W_\alpha = I_\delta(Z) \cap S_\alpha \) if \( x_\alpha \in T \setminus T' \).

So \( I_\delta(Z) \setminus \bigcup_{n \in \omega} W_n \) is uncountable because \( T \) is good.

Assume that \( \alpha \xi < \delta < \alpha \xi + 1 \) and let \( \delta = \alpha \xi + \eta \).

Pick \( s \in I_{\alpha \xi+1}(Z) \setminus \bigcup_{n \in \omega} W_n \). Then \( Z_s \setminus \bigcup_{n \in \omega} W_n \supset I_\eta(Z_s) \), and so \( I_\delta(Z) \setminus \bigcup_{n \in \omega} W_n \supset I_\eta(Z_s) \), and hence \( I_\delta(Z) \setminus \bigcup_{n \in \omega} W_n \) is uncountable.

Thus, the space \( Z \) is as required.

\[ \square \]

4. Cardinal sequences of length \(< \omega_4\)

In this section, we will show the following result.

**Theorem 4.1.** If \( V=L \), then there is a cardinal-preserving partial order \( \mathbb{P} \) such that in \( V^\mathbb{P} \) there is an \((\omega_1, \alpha)\)-LLSP space for every ordinal \( \alpha < \omega_4 \).

If \( S = \bigcup \{ \{ \alpha \} \times A_\alpha : \alpha < \eta \} \) where \( \eta \) is a non-zero ordinal and each \( A_\alpha \) is a non-empty set of ordinals, then for every \( s = (\langle \alpha, \xi \rangle) \in S \) we write \( \pi(s) = \alpha \) and \( \zeta(s) = \xi \).

The following notion is a refinement of a notion used implicitly in [3].

**Definition 4.2.** We say that \( S = \langle S, \preceq, i \rangle \) is an LLSP poset, if the following conditions hold:

(P1) \( \langle S, \preceq \rangle \) is a partial order with \( S = \bigcup \{ S_\alpha : \alpha < \eta \} \) for some non-zero ordinal \( \eta \) such that each \( S_\alpha = \{ \alpha \} \times A_\alpha \) where \( A_\alpha \) is a non-empty set of ordinals.

(P2) If \( s < t \) then \( \pi(s) < \pi(t) \).

(P3) If \( \alpha < \beta < \eta \) and \( t \in S_\beta \), then \( \{ s \in S_\alpha : s < t \} \) is uncountable.

(P4) If \( \gamma < \eta \) with \( cf(\gamma) = \omega \), \( t \in S_\gamma \) and \( \{ t_n : n \in \omega \} \) is a sequence of elements of \( S \) such that \( t_n < t \) for every \( n \in \omega \), then for every ordinal \( \beta < \gamma \) the set \( \{ s \in S_\beta : s < t \) and \( s \neq t_n \) for \( n \in \omega \} \) is uncountable.

(P5) \( i : [S]^2 \to [S]^\omega \) such that for every \( \{ s, t \} \in [S]^2 \) the following holds:

(a) If \( v \in i\{s, t\} \) then \( v \preceq s, t \).

(b) If \( u \preceq s, t \), then there is \( v \in i\{s, t\} \) such that \( u \preceq v \).

If there is an uncountable cardinal \( \lambda \) such that \( |S_\alpha| = \lambda \) for \( \alpha < \eta \), we will say that \( \langle S, \preceq, i \rangle \) is a \((\lambda, \eta)\)-LLSP poset.

If \( S = \langle S, \preceq, i \rangle \) is an LLSP poset with \( S = \bigcup \{ S_\alpha : \alpha < \eta \} \), we define its associated LLSP space \( X = X(S) \) as follows. The underlying set of \( X(S) \) is \( S \). If \( x \in S \) we write \( U(x) = \{ y \in S : y \preceq x \} \). Then, for every \( x \in S \) we define a basic neighbourhood of \( x \) in \( X \) as a set of the form \( U(x) \setminus \bigcup \{ U(x_n) : n \in \omega \} \) where each \( x_n < x \). It is easy to check that \( X \) is a locally Lindelöf scattered Hausdorff P-space (see [4] for a parallel proof). And by conditions (P3) and (P4) in Definition 4.2, we infer that \( \text{ht}(X) = \eta \) and \( I_\alpha(X) = S_\alpha \) for every \( \alpha < \eta \).
In order to prove Theorem 4.1, first we will construct an \((\omega_1, \omega_3)\)-LLSP space \(X\) in a generic extension by means of an \(\omega_1\)-closed \(\omega_2\)-c.c. forcing, by using an argument similar to the one given by Baumgartner and Shelah in [3].

Recall that a function \(F : [\omega_3]^2 \rightarrow [\omega_3]^{<\omega_1}\) has property \(\Delta\), if \(F\{\alpha, \beta\} \subset \min\{\alpha, \beta\}\) for every \(\{\alpha, \beta\} \in [\omega_3]^2\) and for every set \(D\) of countable subsets of \(\omega_3\) with \(|D| = \omega_2\) there are \(a, b \in D\) with \(a \neq b\) such that for every \(\alpha \in a \setminus b, \beta \in b \setminus a\) and \(\tau \in a \cap b\) the following holds:

(a) if \(\tau < \alpha, \beta\) then \(\tau \in F\{\alpha, \beta\}\),
(b) if \(\tau < \beta\) then \(F\{\alpha, \tau\} \subset F\{\alpha, \beta\}\),
(c) if \(\tau < \alpha\) then \(F\{\tau, \beta\} \subset F\{\alpha, \beta\}\).

By a result due to Velickovic, it is known that \(\square_{\omega_2}\) implies the existence of a function \(F : [\omega_3]^2 \rightarrow [\omega_3]^{<\omega_1}\) satisfying property \(\Delta\) (see [17], Chapter 7 and Lemma 7.4.9., for a proof).

**Proof of Theorem 4.1.** Let \(F : [\omega_3]^2 \rightarrow [\omega_3]^{<\omega_1}\) be a function with property \(\Delta\). First, we construct by forcing an \((\omega_1, \omega_3)\)-LLSP poset. Let \(S = \bigcup\{S_\alpha : \alpha < \omega_3\}\) where \(S_\alpha = \{\alpha\} \times \omega_1\) for each \(\alpha < \omega_3\). \(S\) will be the underlying set of the required poset. We define \(P\) as the set of all \(p = \langle x_p, \preceq_p, i_p \rangle\) satisfying the following conditions:

1. \(x_p\) is a countable subset of \(S\).
2. \(\preceq_p\) is a partial order on \(x_p\) such that:
   (a) if \(s \prec_p t\) then \(\pi(s) < \pi(t)\),
   (b) if \(s \prec_p t\) and \(\pi(t)\) is a successor ordinal \(\beta + 1\), then there is \(v \in S_\beta\) such that \(s \preceq_p v \prec_p t\).
3. \(i_p : [x_p]^2 \rightarrow [x_p]^{<\omega}\) satisfying the following conditions:
   (a) if \(s \prec_p t\) then \(i_p\{s, t\} = \{s\}\),
   (b) if \(s \not\prec_p t\) and \(\pi(s) < \pi(t)\), then \(i_p\{s, t\} \subset \bigcup\{S_\alpha : \alpha \in F\{\pi(s), \pi(t)\}\}\),
   (c) if \(s, t \in x_p\) with \(s \neq t\) and \(\pi(s) = \pi(t)\) then \(i_p\{s, t\} = \emptyset\),
   (d) \(v \preceq_p s, t\) for all \(v \in i_p\{s, t\}\),
   (e) for every \(u \preceq_p s, t\) there is \(v \in i_p\{s, t\}\) such that \(u \preceq_p v\).

If \(p, q \in P\), we write \(p \leq q\) iff \(x_q \subset x_p, \preceq_p \upharpoonright x_q = \preceq_q\) and \(i_p \upharpoonright [x_q]^2 = i_q\). We put \(\mathbb{P} = \langle P, \leq\rangle\).

Clearly, \(\mathbb{P}\) is \(\omega_1\)-closed. And since the function \(F\) has property \(\Delta\), it is easy to check that \(\mathbb{P}\) has the \(\omega_2\)-c.c., and so \(\mathbb{P}\) preserves cardinals.

Now, let \(G\) be a \(\mathbb{P}\)-generic filter. We write \(\preceq = \bigcup\{\preceq_p : p \in G\}\) and \(i = \bigcup\{i_p : p \in G\}\). It is easy to see that \(S = \bigcup\{x_p : p \in G\}\) and \(\preceq\) is a partial order on \(S\). Then, we have that \(\langle S, \preceq, i\rangle\) is an \((\omega_1, \omega_3)\)-LLSP poset. For this, note that conditions \((P1), (P2), (P5)\) in Definition 4.2 are obvious, and condition \((P3)\) follows from a basic density argument. So, we verify condition \((P4)\). For every \(t \in S\) such that \(\gamma = \pi(t)\) has cofinality \(\omega\), for every sequence \(\langle t_n : n \in \omega\rangle\) of elements of \(S\), for every ordinal \(\beta < \gamma\) and for every ordinal \(\xi < \omega_1\) let
If every open cover of $X$ for some $\kappa$ cardinal-preserving partial order above theorems, we can show the following more general results:

1. We mean a scattered Hausdorff $P$ than $\kappa$ for every ordinal $\alpha < \omega_4$.

Since $P$ is $\omega_1$-closed, we have that $D_{t,\{t_n \in \omega\}, \beta, \xi} \in V$. Then, consider $p = \langle x_p, \leq_p, i_p \rangle \in P$. We define a $q \in D_{t,\{t_n \in \omega\}, \beta, \xi}$ such that $q \leq p$.

Without loss of generality, we may assume that $t \in x_p$. We distinguish the following cases.

Case 1. $t_n \notin x_p$ for some $n \in \omega$.

We define $q = \langle x_q, \leq_q, i_q \rangle$ as follows:

a. $x_q = x_p \cup \{t_n : n \in \omega\}$,

b. $\leq_q = \leq_p$,

c. $i_q \{x, y\} = i_p \{x, y\}$ if $\{x, y\} \notin [x_p]^2$, $i_q \{x, y\} = \emptyset$ otherwise.

Case 2. $t_n \in x_p$ for every $n \in \omega$.

If $t_n \notin x_p$ for some $n \in \omega$, we put $q = p$. So, assume that $t_n \leq_p t$ for all $n \in \omega$. Let $u \in S_\beta \setminus x_p$ be such that $\zeta(u) > \xi$. We define $q = \langle x_q, \leq_q, i_q \rangle$ as follows:

a. $x_q = x_p \cup \{u\}$,

b. $\leq_q = \leq_p \cup \{(u, v) : t \leq_p v\}$,

c. $i_q \{x, y\} = i_p \{x, y\}$ if $\{x, y\} \in [x_p]^2$, $i_q \{x, y\} = \{x\}$ if $\zeta_q x, i_q \{x, y\} = \{y\}$ if $y \leq_q x, i_q \{x, y\} = \emptyset$ otherwise.

So, $D_t,\{t_n \in \omega\}, \beta, \xi$ is dense in $P$, and hence condition (P4) holds. Let $X = X((S, \leq, i))$. For every $x \in S$, we write $U(x) = \{y \in S : y \leq x\}$. By conditions (2)(b) and (3)(c) in the definition of $P$, we see that if $x \in S_{\beta+1}$ for some $\beta < \omega_3$, then $x$ has an admissible basis in $X$ given by $\{U(y) : y \leq x, \pi(y) = \beta\}$. Thus, $X$ is an admissible space. And clearly, $X$ is good. So, by Theorem 3.3, we can construct from the space $X$ an $\omega_1, \alpha$-LLSP space for every ordinal $\omega_3 \leq \alpha < \omega_4$.

Now, assume that $\kappa$ is an uncountable regular cardinal. Recall that a topological space $X$ is a $P_\kappa$-space, if the intersection of any family of less than $\kappa$ open subsets of $X$ is open in $X$. And we say that $X$ is $\kappa$-compact, if every open cover of $X$ has a subcover of size less $< \kappa$. By an $SP_\kappa$ space we mean a scattered Hausdorff $P_\kappa$-space. Then, we want to remark that by using arguments that are parallel to the ones given in the proofs of the above theorems, we can show the following more general results:

1. For every uncountable regular cardinal $\kappa$ and every ordinal $\alpha < \kappa^{+++}$, there is a locally $\kappa$-compact $SP_\kappa$ space $X$ such that $ht(X) = \alpha$ and $wd(X) = \kappa$.

2. If $V=L$ and $\kappa$ is an uncountable regular cardinal, then there is a cardinal-preserving partial order $P$ such that in $V^P$ we have that for every
ordinal $\alpha < \kappa^{++}$ there is a locally $\kappa$-compact $SP_\kappa$ space $X$ such that $\text{ht}(X) = \alpha$ and $\text{wd}(X) = \kappa$.

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