On constant curvature submanifolds
of space forms

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Abstract
We prove a converse to well-known results by E. Cartan and J. D. Moore. Let $f: M^n_c \to \mathbb{Q}^{n+p}_{\tilde{c}}$ be an isometric immersion of a Riemannian manifold with constant sectional curvature $c$ into a space form of curvature $\tilde{c}$, and free of weak-umbilic points if $c > \tilde{c}$. We show that the substantial codimension of $f$ is $p = n - 1$ if, as shown by Cartan and Moore, the first normal bundle possesses the lowest possible rank $n - 1$. These submanifolds are of a class that has been extensively studied due to their many properties. For instance, they are holonomic and admit Bäcklund and Ribaucour transformations.

E. Cartan [1] in 1919 initiated the systematic study of isometric immersions $f: M^n_c \to \mathbb{Q}^{n+p}_{\tilde{c}}$ of an $n$-dimensional connected Riemannian manifold $M^n_c$ of constant sectional curvature $c$ into a simply connected space form. Thus $\mathbb{Q}^{n+p}_{\tilde{c}}$ denotes the Euclidean space $\mathbb{R}^m$, the Euclidean sphere $\mathbb{S}^m_{\tilde{c}}$ or the hyperbolic space $\mathbb{H}^m_{\tilde{c}}$ according to whether $\tilde{c} = 0$, $\tilde{c} > 0$ or $\tilde{c} < 0$, respectively.

Cartan proved for $n \geq 3$ and $c < \tilde{c}$, that the codimension of $f$ satisfies $p \geq n - 1$, and if $p = n - 1$ the normal bundle has to be flat. The dual case of isometric immersions $f: M^n_c \to \mathbb{Q}^{n+p}_{\tilde{c}}$, $n \geq 3$ and $c > \tilde{c}$, has been considered by J. D. Moore [4]. Under the additional assumption that $f$ is free of weak-umbilic points, he again obtained that $p \geq n - 1$, and if $p = n - 1$ that the normal bundle is flat. We recall that $x \in M^n_c$ is called a weak-umbilic for $f$ if there exists a unit normal vector $\zeta \in N_f M(x)$ such that the corresponding shape operator is $A_\zeta = \sqrt{c - \tilde{c}} I$.

The way to argue that $p$ satisfies the lower bound given above is to show that the dimension of the first normal space has to satisfy $\dim N^f_1 \geq n - 1$ at any point. Recall that the first normal space $N^f_1(x)$ of $f$ at $x \in M^n$ is the
subspace of the normal space $N_f M(x)$ spanned by the image of its second fundamental form $\alpha_f : TM \times TM \to N_f M$ at $x \in M^n$, that is,

$$N^f_1(x) = \text{span}\{\alpha_f(X, Y) : X, Y \in T_x M\}.$$
such that all the curvatures with respect to a Frenet frame never vanish. Then let $f: \mathbb{R}^n \to S_{1}^{2n+m-2} \subset \mathbb{R}^{2n+m-1}$ be the isometric immersion given by

$$f(t) = (c(t_1), a_2 \sin(t_2/a_2), a_2 \cos(t_2/a_2), \ldots, a_n \sin(t_n/a_n), a_n \cos(t_n/a_n))$$

where $t = (t_1, \ldots, t_n)$. It is easy to verify that $\dim N^f_1 = n$ and that $f$ has flat normal bundle and is substantial.

## 1 The proof

We first recall some basic facts about submanifolds with flat normal bundle which can be seen in [3].

If an isometric immersion $f: M^n \to \mathbb{Q}^{n+p}_c$ has flat normal bundle, that is, if the curvature tensor of the normal connection vanishes, it is a standard fact that at any point $x \in M^n$ there exists a set of unique pairwise distinct vectors $\eta_i(x) \in N_f M(x)$, $1 \leq i \leq s(x)$, called the principal normals of $f$ at $x$ and an associate orthogonal splitting of the tangent space as

$$T_x M = E_{\eta_1}(x) \oplus \cdots \oplus E_{\eta_s}(x),$$

where

$$E_{\eta_i}(x) = \{ X \in T_x M : \alpha_f(X, Y) = \langle X, Y \rangle_{\eta_i} \text{ for all } Y \in T_x M \}.$$  

Hence the second fundamental form of $f$ has the simple representation

$$\alpha_f(X, Y) = \sum_{i=1}^{s} \langle X^i, Y^i \rangle_{\eta_i}$$  \hspace{1cm} (1)$$

where $X \mapsto X^i$ is the orthogonal projection onto $E_{\eta_i}$.

The dimension of $E_{\eta_i}(x)$ is called the multiplicity of $\eta_i$ of $f$ at $x \in M^n$. If $s(x) = k$ is constant on $M^n$, the maps $x \in M^n \mapsto \eta_i(x)$, $1 \leq i \leq k$, are smooth vector fields, called the principal normal vector fields of $f$, and the distributions $x \in M^n \mapsto E_{\eta_i}(x)$, $1 \leq i \leq k$, are also smooth.

**Proof of Theorem 1**: We first observe that the fact that $f$ has to have flat normal bundle is somehow inside the arguments by Cartan and Moore. For a proof of this fact we refer to Theorem 5.5 in [3], where the statement is for codimension $n-1$. But it is straightforward to verify that the same
conclusion holds under the weaker assumption that dim $N_1^f = n - 1$ since the proof reduces to analyze the algebraic structure of the second fundamental form as a map $\alpha_f: TM \times TM \to N_1^f$.

We show next that at any point there are exactly $n$ distinct principal normals $\eta_1, \ldots, \eta_n$. If $c < \tilde{c}$ this is trivial since if a principal normal $\eta_j$ has multiplicity $m > 1$, we would have from the Gauss equation that $\|\eta_j\|^2 = c - \tilde{c} < 0$. Hence we may assume $c > \tilde{c}$. First observe that there is at most one principal normal of multiplicity $m > 1$. In fact, if $\eta_1 \neq \eta_2$ have both this property, using that $\|\eta_1\|^2 = c - \tilde{c} = \|\eta_2\|^2$ we would conclude that $\eta_1 = \eta_2$.

Suppose that there is one principal normal $\eta_1$ of multiplicity $m > 1$. Using (1) we have

$$\langle \alpha_f(X, Y), \eta_1 \rangle = \sum_{i=1}^{n-m+1} \langle X^i, Y^i \rangle \langle \eta_i, \eta_1 \rangle = (c - \tilde{c})\langle X, Y \rangle,$$

and this is in contradiction with the assumption on weak-umbilics points.

We claim that

1. $\dim N_1^f \geq n - 1$ and
2. $\|\eta_i\|^2 \neq c - \tilde{c}$ for all $1 \leq i \leq n$.

If $c < \tilde{c}$ then (ii) is trivial and (i) follows as an application of either Otsuki’s lemma (see Corollary 6.2 of [3]) or of the theory of flat bilinear forms (see Theorem 5.1 of [3]). If $c > \tilde{c}$ suppose that

$$\|\eta_1\|^2 = c - \tilde{c}. \tag{2}$$

Then

$$\|\eta_j\|^2 \neq c - \tilde{c} \text{ for all } 2 \leq j \leq n \tag{3}$$

since if $\|\eta_j\|^2 = c - \tilde{c}$ for some $2 \leq j \leq n$, then that $\langle \eta_1, \eta_j \rangle = c - \tilde{c}$ easily gives $\eta_1 = \eta_j$, a contradiction. Hence, we always have principal normals $\eta_2, \ldots, \eta_n$ such that (3) holds. These vectors are linearly independent, which proves (i). In fact, if $\sum_{i \geq 2} k_i \eta_i = 0$, then taking the inner product with $\eta_1$ gives $\sum_{i \geq 2} k_i = 0$ whereas the inner product with $\eta_j$, $2 \leq j \leq n$, yields

$$k_j(\|\eta_j\|^2 + \tilde{c} - c) = (\tilde{c} - c) \sum_{i \geq 2} k_i = 0.$$

It follows using (3) that $k_j = 0$ for all $2 \leq j \leq n$. 

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Since $\dim N^f_1 = n - 1$, set $\eta_1 = \sum_{i \geq 2} k_i \eta_i$. Taking the inner product with $\eta_1$ and using (2) gives
\[
\sum_{i \geq 2} k_i = 1. \tag{4}
\]
Taking the inner product with $\eta_i$, $2 \leq i \leq n$, and using (3) and (4) we obtain $k_i = 0$ in contradiction with (4), and the proof of the claim is complete.

Next we show that the subbundle $N^f_1$ is parallel with respect to the normal connection. To see this, we first observe that the Codazzi equation gives
\[
\nabla^\perp_{X_j} \eta_i = \langle \nabla_{X_i} X_j, X_i \rangle (\eta_i - \eta_j), \quad i \neq j \tag{5}
\]
where the $X_i \in \Gamma(E_{\eta_i})$, $1 \leq i \leq n$, are unit local vector fields.

Assume first that $n = 2$. Then the principal normals $\eta_1, \eta_2$ are linearly dependent, in fact, they satisfy
\[
(c - \tilde{c}) \eta_i - \|\eta_i\|^2 \eta_j = 0, \quad i \neq j,
\]
and the parallelism follows using (5).

Next let $n \geq 3$. By the claim, there is a set of principal normals $\eta_1, \ldots, \eta_{n-1}$ that are linearly independent and
\[
\eta_n = \sum_{i=1}^{n-1} \rho_i \eta_i. \tag{6}
\]
Taking the inner product with $\eta_n$ gives
\[
\|\eta_n\|^2 = (c - \tilde{c}) \sum_{i=1}^{n-1} \rho_i
\]
whereas with $\eta_j$, $1 \leq j \leq n - 1$, yields
\[
\rho_j \|\eta_j\|^2 = (\tilde{c} - c) \left( \sum_{i \neq j} \rho_i - 1 \right), \quad 1 \leq j \leq n - 1.
\]
Then
\[
\rho_j = -\frac{\|\eta_n\|^2 + \tilde{c} - c}{\|\eta_j\|^2 + \tilde{c} - c}, \quad 1 \leq j \leq n - 1.
\]
It now follows from (6) that
\[ \sum_{i=1}^{n} \frac{\eta_i}{\|\eta_i\|^2 + \tilde{c} - c} = 0. \] (7)

Taking the inner product with some \( \eta_j \) yields
\[ \sum_{i=1}^{n} \frac{1}{\|\eta_i\|^2 + \tilde{c} - c} = \frac{1}{\tilde{c} - c}. \] (8)

It follows from (7) and (8) that
\[ \eta_i = \sum_{j \neq i}^{n} \frac{\tilde{c} - c}{\|\eta_j\|^2 + \tilde{c} - c}(\eta_i - \eta_j), \quad 1 \leq i \leq n. \] (9)

We obtain using (9) and (5) that
\[ \left(1 - \sum_{j \neq i}^{n} \frac{\tilde{c} - c}{\|\eta_j\|^2 + \tilde{c} - c}\right) \nabla_{X_i}^\perp \eta_i \in N_1^f. \]

Since (8) gives
\[ 1 - \sum_{j \neq i}^{n} \frac{\tilde{c} - c}{\|\eta_j\|^2 + \tilde{c} - c} = \frac{\tilde{c} - c}{\|\eta_i\|^2 + \tilde{c} - c} \neq 0, \]
then \( \nabla_{X_i}^\perp \eta_i \in N_1^f \), and using (5) it follows that \( N_1^f \) is parallel in the normal connection.

To conclude the proof, we recall an elementary fact from the theory of isometric immersions (cf. Proposition 2.1 of [3]). If the first normal spaces form a parallel normal subbundle then the codimension reduces to the rank of \( N_1^f \). Hence, in our case, we obtain that \( p = n - 1 \).

References

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