A new Lie algebra expansion method: Galilei expansions to Poincaré and Newton–Hooke

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Abstract

We modify a Lie algebra expansion method recently introduced for the (2 + 1)-dimensional kinematical algebras so as to work for higher dimensions. This new improved and geometrical procedure is applied to expanding the (3 + 1)-dimensional Galilei algebra and leads to its physically meaningful ‘expanded’ neighbours. One expansion gives rise to the Poincaré algebra, introducing a curvature $-1/c^2$ in the flat Galilean space of worldlines, while keeping a flat spacetime which changes from absolute to relative time in the process. This formally reverses, at a Lie algebra level, the well known non-relativistic contraction $c \to \infty$ that goes from the Poincaré group to the Galilei one; this expansion is done in an explicit constructive way. The other possible expansion leads to the Newton–Hooke algebras, endowing with a non-zero spacetime curvature $\pm 1/\tau^2$ the spacetime, while keeping a flat space of worldlines.
Introduction

Expansions of Lie algebras can be considered as the opposite processes of Lie algebra contractions. In general, starting from a Lie algebra a contraction gives rise to another more abelian algebra by making some structure constants to vanish, while an expansion goes to another less abelian algebra producing some non-zero structure constants. The idea of contractions of Lie algebras and groups historically appeared in relation with the non-relativistic limit where the relativistic constant $c$ (the speed of light) goes to infinity; that limit brings relativistic mechanics (Poincaré group) to classical mechanics (Galilei group). In the framework of kinematical algebras, the scheme of contractions is well known: starting from de Sitter algebras a sequence of different contractions leads to Poincaré, Newton–Hooke, Galilei, . . . , ending up at the last stage in the so called Static algebra. From the viewpoint of graded contraction theory, the $(3+1)$-dimensional case has been studied in, and kinematical contractions in arbitrary dimension have been obtained in.

Unlike the Lie algebra contractions, the theory of expansions has not been so systematized. As a kind of formal ‘inverse’ of contractions its status is clear, but as yet there is no a general constructive theory where explicit realizations of an ‘expanded’ Lie algebra can be given in terms of an ‘initial’ one. However some specific procedures, valid for certain algebras and certain expansions, have been introduced. Expansions from the inhomogeneous pseudo-orthogonal algebras $\text{iso}(p,q)$ to the semisimple ones $\text{so}(p+1,q)$ with $(p+q = N)$ can be found in; these contain as particular cases those expansions starting from the Euclidean algebra and leading to either the elliptic or hyperbolic ones, and also those expansions from Poincaré to both de Sitter algebras:

\begin{align}
\text{Euclidean expans.:} & \quad \text{iso}(N) \to \text{so}(N+1) \quad \text{iso}(N) \to \text{so}(N,1) \\
\text{Poincaré expans.:} & \quad \text{iso}(N-1,1) \to \text{so}(N,1) \quad \text{iso}(N-1,1) \to \text{so}(N-1,2)
\end{align} (1.1)

A different method enables to perform expansions from $t_{qp}(\text{so}(p) \oplus \text{so}(q)) \to \text{so}(p,q)$. Similar expansions for unitary algebras can be found in the above references.

Recently, another new expansion method was proposed in. Although formulated in algebraic terms, the ideas follow from geometrical considerations. The main trait is to control the expansion by a parameter which is the curvature of some homogeneous spaces associated to both the initial and expanded Lie algebras; the procedure rests on the Casimir operators associated to both the initial and expanded Lie algebras. This method was applied to the expansions within the set of $(2+1)$-dimensional kinematical algebras. In this $(2+1)$D case, known expansions as were recovered and, furthermore, other new expansions were established; amongst them, we remark several algebra expansions which starting from Galilei give rise to either Poincaré or Newton–Hooke, and further expansions going from Newton–Hooke to de Sitter.

In this method the emphasis is put in the structure of homogeneous spaces associated to the two Lie groups involved in the contraction/expansion. Contraction would be associated to vanishing curvature and/or degenerating the metric, while expansion would mean to produce non-vanishing curvature or to make the metric non-degenerate.

A natural and physically interesting frame to discuss these questions is provided by homogeneous models of spacetimes (see e.g.), which are either relativistic or non-relativistic, and have a spacetime curvature which can be either zero or non-zero (this
would correspond to a cosmological constant). It seems interesting to perform a mathematical study of expansion procedures which would work in these \((3+1)D\) physically relevant cases. However, when followed literally the method in \([10]\) do not work in the \((3+1)D\) case.

In this paper we set two objectives. First, in the same geometrical vein, we propose a new expansion procedure which, in principle, can be applied to any Lie algebra in any dimension; the method in \([10]\) can be seen as a particular instance of the new expansion method. The second and main goal of this paper is to apply this procedure to the Galilei algebra and we obtain the Poincaré one within the universal enveloping algebra of the former. The other reverses the ordinary zero-curvature limit in the non-relativistic Newton–Hooke spacetimes, whose Lie algebras are obtained within the universal enveloping algebras of the centrally extended \((3+1)D\) Galilei algebra.

The paper is organized as follows: in the next section we recall the basics of Galilei algebra structure and its main associated symmetric homogeneous spaces (spacetime and space of worldlines) showing two natural ways of expansion to either Poincaré or Newton–Hooke algebras. In section 3 we summarize the steps of the ‘improved’ expansion method that we propose. Its application to the expansions going from Galilei to either Poincaré or Newton–Hooke algebras are explicitly developed in the sections 4 and 5, respectively. Finally, some remarks are pointed out in the last section.

2 The Galilei algebra and associated homogeneous spaces

Let \(H, P_i, K_i\) and \(J_i\) \((i = 1, 2, 3)\) be the usual generators of time translation, space translations, boosts and spatial rotations, respectively. The Lie brackets of the \((3+1)D\) Galilei algebra, \(\mathcal{G}\), are given by

\[
\begin{align*}
[J_i, J_j] &= \varepsilon_{ijk}J_k \\
[P_i, P_j] &= 0 \\
[H, P_i] &= 0 \\
[J_i, P_j] &= \varepsilon_{ijk}P_k \\
[J_i, K_j] &= \varepsilon_{ijk}K_k \\
[J_i, K_j] &= 0 \\
[K_i, K_j] &= 0 \\
[H, J_i] &= 0
\end{align*}
\]  

(2.1)

where \(i, j, k = 1, 2, 3\) and \(\varepsilon_{ijk}\) is the completely skewsymmetric tensor with \(\varepsilon_{123} = 1\). Hereafter any generator or object with three components is denoted as \(X = (X_1, X_2, X_3)\); its ‘square’ and its ‘product’ with other element, say \(Y = (Y_1, Y_2, Y_3)\), are

\[
X^2 = X_1^2 + X_2^2 + X_3^2 \\
XY = X_1Y_1 + X_2Y_2 + X_3Y_3.
\]

(2.2)

The Galilei algebra has two Casimir invariants which read \([16]\):

\[
\begin{align*}
\mathcal{C}_1 &= P^2 = P_1^2 + P_2^2 + P_3^2 \\
\mathcal{C}_2 &= W^2 = W_1^2 + W_2^2 + W_3^2
\end{align*}
\]

(2.3)

where the components of \(W\) are given by

\[
W_1 = P_3K_2 - P_2K_3 \\
W_2 = P_1K_3 - P_3K_1 \\
W_3 = P_2K_1 - P_1K_2.
\]

(2.4)

The second-order Casimir \(\mathcal{C}_1\) (which is related with the Killing–Cartan form) corresponds in the free kinematics of a particle in the Galilean spacetime to the square of the linear
momentum (i.e., to the non-relativistic energy), while the fourth-order invariant $C_2$ can be identified with the square of the angular momentum.

We remark that the Galilei algebra is isomorphic to a twice inhomogeneous orthogonal algebra:

$$G \equiv iiso(3) \equiv t_4 \odot (t_3 \odot so(3))$$

with the two abelian subalgebras $t_4$, $t_3$ and the orthogonal subalgebra $so(3)$ spanned by

$$t_4 = \langle H, P \rangle \quad t_3 = \langle K \rangle \quad so(3) = \langle J \rangle.$$

As any kinematical group, the Galilei group generated by the Lie algebra $G$ has two symmetric homogeneous spaces identified with the spacetime and space of (time-like) wordlines. According to the two involutive automorphisms parity $\Pi$ and the product $\Pi T$ (T is the time-reversal), defined by \[11\]

$$\Pi_T : (H, P, K, J) \rightarrow (-H, -P, K, J) \quad \Pi : (H, P, K, J) \rightarrow (H, -P, -K, J)$$

we find two Cartan Lie algebra decompositions given by

$$\Pi_T : G = p^{(1)} \oplus h^{(1)} \quad p^{(1)} = \langle H, P \rangle \quad h^{(1)} = \langle K, J \rangle$$

$$\Pi : G = p^{(2)} \oplus h^{(2)} \quad p^{(2)} = \langle P, K \rangle \quad h^{(2)} = \langle H, J \rangle = \langle H \rangle \oplus \langle J \rangle$$

fulfilling

$$[h^{(l)}, h^{(l)}] \subset h^{(l)} \quad [h^{(l)}, p^{(l)}] \subset p^{(l)} \quad [p^{(l)}, p^{(l)}] = 0 \quad l = 1, 2. \quad (2.9)$$

Notice that both $h^{(l)}$ and $p^{(l)}$ are Lie subalgebras; the latter is an abelian one. Consequently, the Galilei group $G \equiv IISO(3)$ is the motion group of the following symmetrical homogeneous spaces:

$$S^{(1)} = G/H^{(1)} = IISO(3)/ISO(3) \quad \text{dim} (S^{(1)}) = 3 + 1 \quad \text{curv} (S^{(1)}) = 0$$

$$S^{(2)} = G/H^{(2)} = ISO(3)/\mathbb{R} \otimes SO(3) \quad \text{dim} (S^{(2)}) = 3 + 3 \quad \text{curv} (S^{(2)}) = 0. \quad (2.10)$$

The Galilei subgroups $H^{(1)}$, $H^{(2)}$ (whose Lie algebras are $h^{(1)}$, $h^{(2)}$) are the isotopy subgroups of an event and a time-like line, respectively. Therefore $S^{(1)}$ is identified with the $(3 + 1)$D Galilean spacetime, while $S^{(2)}$ is the 6D space of time-like lines in the Galilean spacetime $S^{(1)}$. Both spaces are of zero curvature: Galilean spacetime is a flat universe with a degenerate metric of absolute time, and the set of time-like lines is a flat rank-two space with a degenerate metric where distance corresponds to relative velocity.

By taking into account the spaces (2.10), two possible Galilei algebra expansions arise in a natural way. First, if we consider the space of wordlines $S^{(2)}$ we can try to obtain a Lie algebra for another kinematics whose corresponding space of time-like lines has a curvature different from zero but keeping a flat spacetime. This expansion allows us to reach the Poincaré algebra by introducing in a suitable way a (negative) curvature in $S^{(2)}$ equal to $-1/c^2$. In this sense, this process is a relativistic expansion as it gives rise to the Minkowskian spacetime which is also a flat universe but of relative time. This is exactly the opposite process to the well known non-relativistic limit or contraction studied by Inonü and Wigner [1], Segal [2] and Saletan [3]. This contraction is also called speed-space contraction [1] and it corresponds to the limit $c \rightarrow \infty$ (i.e. $-1/c^2 \rightarrow 0$) in the Poincaré algebra.
The second possibility is to start with the Galilean spacetime \( S^{(1)} \) and to reach a Lie algebra whose associated spacetime has a non-zero curvature but keeps a flat (rank-two) space of worldlines. In this way we obtain the two Newton–Hooke algebras; these are non-relativistic expansions and provide curved spacetimes whose curvature is \( \kappa = \pm 1/\tau^2 \) (where \( \tau \) is the universe ‘radius’ measured in time units) but are still of absolute time. The opposite process is the so-called spacetime contraction characterized by the limit \( \tau \to \infty \) (i.e. \( \kappa \to 0 \)) in the Newton–Hooke algebras.

3 An ‘improved’ expansion method

In a previous paper we proposed an expansion method, which worked in the kinematical \((2+1)\)D, where the Lie algebras are contractions of \( so(4) \). This is a (only semisimple) rank-two algebra. Furthermore, \( so(n) \) has as many Casimirs as its rank, one quadratic and the others higher order polynomials in the generators with an exceptional behaviour in \( so(4) \), where the additional Casimir is also essentially quadratic. Then it could happen that a method working for \( so(4) \) may not be directly extensible to higher dimensions. Thus this expansion method should be replaced by a more general one. A proposal, which keeps the geometric flavour of the previous method can be described as follows.

Let \( g \) a Lie algebra which is obtained as a contraction from another Lie algebra \( g' \): \( g' \to g \). Suppose that the contraction corresponds to making equal to zero the curvature \( \omega \) of some homogeneous spaces associated to \( g \) and \( g' \) (by taking the quotient associated to the common subalgebra invariant under the contraction). Assume for simplicity that \( g', g \) are rank-two algebras and let \( C_1, C_2 \) the two Casimirs of the initial Lie algebra \( g \) (with \( \omega = 0 \)) and \( C'_1, C'_2 \) those of the final algebra \( g' \) (with \( \omega \neq 0 \)). The main steps of our expansion method are:

(i) Write each expanded Casimir as polynomials on the curvature \( \omega \) we aim to recover:

\[
C'_1 = C_1 + \omega J_1 + \omega^2 M_1 + \ldots \quad C'_2 = C_2 + \omega J_2 + \omega^2 M_2 + \ldots
\]

where \( C_l, J_l, M_l, \ldots (l = 1, 2) \) are independent of \( \omega \). Obviously the terms which are zero-order in \( \omega \) are just the ‘contracted’ Casimirs \( C_l \).

(ii) Assume to work in the universal enveloping algebra of the initial Lie algebra \( g \) within an irreducible representation and consider as ‘expansion seed’ a linear combination formed by the terms linear in the curvature \( \omega \):

\[
\mathcal{J} = \alpha_1 J_1 + \alpha_2 J_2
\]

where \( \alpha_1, \alpha_2 \) are two constants to be determined.

(iii) The expanded generators \( X'_k \) of \( g' \) are the elements in the universal enveloping algebra of \( g \) defined by the following functions of the generators \( X_k \) of \( g \):

\[
X'_k := \begin{cases} 
X_k & \text{if } [\mathcal{J}, X_k] = 0 \\
\frac{[\mathcal{J}, X_k]}{[\mathcal{J}, X_k]} & \text{if } [\mathcal{J}, X_k] \neq 0
\end{cases}
\]

(iv) Impose the new generators \( X'_k \) to close a Lie algebra isomorphic to \( g' \). This gives (if possible at all) some conditions that characterize the constants \( \alpha_1 \) and \( \alpha_2 \).
Some comments are pertinent. First, this is only a proposal, not a full-fledged method, so there is no a priori success guarantee. Yet the method works in the cases we study, with a caveat: in some cases the initial algebra should be taken after being centrally extended. Second, the same procedure can also be applied when both Lie algebras \( g, g' \) are higher rank. In principle, the operator \( J \) will have as many terms as the number of Casimirs (the rank of the algebras). In this sense, we recall that the expansion method introduced in \([10]\) for the \((2+1)D\) kinematical algebras only considered Casimirs linear in the curvature. Hence the extension of the method to higher dimensions consists in keeping the terms which are first-order in the curvature in order to reproduce the whole final algebra \( g' \).

In the next sections we apply this method to the \((3+1)D\) Galilei algebra and hereafter we denote with a prime any element associated to the expanded algebra \( g' \) (either Poincaré or Newton–Hooke), and we drop the prime when we deal with the initial one \( g \) (Galilei).

## 4 Recovering the speed of light: from Galilei to Poincaré

The Lie brackets of the \((3+1)D\) Poincaré algebra \( \mathcal{P} \equiv \text{iso}(3,1) \) read

\[
\begin{align*}
[J'_i, J'_j] &= \varepsilon_{ijk} J'_k \\
[J'_i, P'_j] &= \varepsilon_{ijk} P'_k \\
[P'_i, P'_j] &= 0 \\
[J'_i, K'_j] &= \varepsilon_{ijk} K'_k \\
[K'_i, K'_j] &= -\frac{1}{c^2} \delta_{ij} H' \\
[H', P'_i] &= 0 \\
[H', K'_i] &= -P'_i \\
[H', J'_i] &= 0.
\end{align*}
\]

(4.1)

The two Poincaré Casimir invariants are given by \([14]\):

\[
\begin{align*}
C'_1 &= P'^2 - \frac{1}{c^4} H'^2 = P'_1^2 + P'_2^2 + P'_3^2 - \frac{1}{c^4} H'^2 \\
C'_2 &= W'^2 - \frac{1}{c^2} (J'P')^2 = W'_1^2 + W'_2^2 + W'_3^2 - \frac{1}{c^2} (J'_1 P'_1 + J'_2 P'_2 + J'_3 P'_3)^2.
\end{align*}
\]

(4.2)

where the components of \( W' \) are

\[
\begin{align*}
W'_1 &= -\frac{1}{c^2} H' J'_1 + P'_3 K'_2 - P'_2 K'_3 \\
W'_2 &= -\frac{1}{c^2} H' J'_2 + P'_1 K'_3 - P'_3 K'_1 \\
W'_3 &= -\frac{1}{c^2} H' J'_3 + P'_2 K'_1 - P'_1 K'_2.
\end{align*}
\]

(4.3)

The Casimir \( C'_1 \) is the energy of the particle in the free kinematics in the Minkowskian spacetime, while \( C'_2 \) is the square of the Pauli–Lubanski vector.

The Cartan decompositions \((2.8)\) also hold for the Poincaré algebra although in this case the vector subspaces \( p^{(i)} \) satisfy \([p^{(1)}, p^{(1)}] = 0\) and \([p^{(2)}, p^{(2)}] \subset \mathfrak{h}^{(2)}\) in the relations \((2.3)\). Therefore the Poincaré group \( \mathcal{P} \equiv \text{ISO}(3,1) \) is the motion group of the symmetric homogeneous spaces given by

\[
\begin{align*}
\mathcal{S}^{(1)} &= P/H^{(1)} = \text{ISO}(3,1)/\text{SO}(3,1) & \text{curv} (\mathcal{S}^{(1)}) &= 0 \\
\mathcal{S}^{(2)} &= P/H^{(2)} = \text{ISO}(3,1)/(\mathbb{R} \otimes \text{SO}(3)) & \text{curv} (\mathcal{S}^{(2)}) &= -1/c^2
\end{align*}
\]

(4.4)

where \( \mathcal{S}^{(1)} \) and \( \mathcal{S}^{(2)} \) are identified with the \((3+1)D\) Minkowskian spacetime and the 6D space of time-like lines in Minkowskian spacetime, respectively.
The non-relativistic limit $c \to \infty$ leads to the contraction $\mathcal{P} \to \mathcal{G}$ and makes the curvature of $S^{(2)}$ to vanish so that the commutators $[P_i, K_j], [K_i, K_j]$ are equal to zero in $\mathcal{G}$. Our aim now is to reverse this process, that is, to consider $\mathcal{G}$ as the initial Lie algebra, thus trying to recover $\mathcal{P}$ by introducing the relativistic constant $c$: $\mathcal{G} \to \mathcal{P}$.

According to the first step of the expansion method \([3.1]\) we rewrite the Poincaré algebra (4.1); this leads to two conditions for the constants \(t\)ions are given in the Appendix.

Some commutators between elements of the universal enveloping Galilei algebra which are within an irreducible representation of $G$ are defined in (2.4). Note that in this case, the power series in the curvature $-1/c^2$ are first-order for $C_1'$ and second-order for $C_2'$. The second step of the procedure \([3.2]\) gives the linear combination

\[
\mathcal{J} = \alpha_1 \mathcal{J}_1 + \alpha_2 \mathcal{J}_2 = \alpha_1 H^2 + 2 \alpha_2 H \mathbf{J} \mathbf{W} + \alpha_2 (\mathbf{J} \mathbf{P})^2.
\]

Hence, the new generators that we want to close the Poincaré algebra $X_k'$, are obtained by commuting $\mathcal{J}$ with the Galilei generators $X_k$ following the third step \([3.3]\); they turn out to be

\[
\begin{align*}
H' &= H & J_1' &= J_1 & J_2' &= J_2 & J_3' &= J_3 \\
P_1' &= 2 \alpha_2 H (P_2 W_3 - P_3 W_2) & P_2' &= 2 \alpha_2 H (P_3 W_1 - P_1 W_3) & P_3' &= 2 \alpha_2 H (P_1 W_2 - P_2 W_1) \\
K_1' &= -2 \alpha_1 H P_1 - 2 \alpha_2 \mathbf{J} \mathbf{W} P_1 + 2 \alpha_2 H (K_3 W_3 - K_3 W_2) + 3 \alpha_2 (P_2 W_3 - P_3 W_2) + 2 \alpha_2 \mathbf{J} \mathbf{P} W_1 \\
K_2' &= -2 \alpha_1 H P_2 - 2 \alpha_2 \mathbf{J} \mathbf{W} P_2 + 2 \alpha_2 H (K_3 W_1 - K_1 W_3) + 3 \alpha_2 (P_3 W_1 - P_1 W_3) + 2 \alpha_2 \mathbf{J} \mathbf{P} W_2 \\
K_3' &= -2 \alpha_1 H P_3 - 2 \alpha_2 \mathbf{J} \mathbf{W} P_3 + 2 \alpha_2 H (K_1 W_2 - K_2 W_1) + 3 \alpha_2 (P_1 W_2 - P_2 W_1) + 2 \alpha_2 \mathbf{J} \mathbf{P} W_3.
\end{align*}
\]

Some commutators between elements of the universal enveloping Galilei algebra which are useful in the obtention of the above ‘expanded’ generators as well as in further computations are given in the Appendix.

The last step is to impose the expanded generators \([1.7]\) to fulfil the Lie brackets of the Poincaré algebra \([1.1]\); this leads to two conditions for the constants $\alpha_1, \alpha_2$. The resulting expansion process is summarized by

**Theorem 1.** The generators defined by \([4.7]\) close the \((3+1)D\) Poincaré algebra whenever the constants $\alpha_1, \alpha_2$ satisfy

\[
\alpha_1 C_1 + \alpha_2 C_2 = 0 \quad \alpha_2 = \frac{1}{4 \epsilon^2 C_1 C_2}
\]

where $\epsilon$ is the relativistic constant (the speed of light) and $C_1, C_2$ are the Galilei Casimirs \([2.3]\) which within an irreducible representation of $G$ turn into scalar values.

**Proof.** The generators which are invariant in this expansion span the isotopy subalgebra of a time-like line $h^{(2)} = \{H, J\}$. Hence by taking into account the second Cartan...
decomposition (2.8) it can be checked that the assumptions of the proposition 1 of [10] are automatically fulfilled, which in turn implies that the Lie brackets either between two generators belonging to \( h^{(2)} \), or between one generator of \( h^{(2)} \) and another of \( p^{(2)} \) remain in the same form as in the Galilei algebra, as it should be for Poincaré. Consequently, we have only to compute the 15 commutators that involve two generators of \( p^{(2)} \): \( K'_1, P'_i \). We distinguish four types of Lie brackets: \([P'_i, P'_j], [P'_i, K'_j] \) which must vanish again, and \([P'_i, K'_j], [K'_i, K'_j] \) which according to (4.1) must be now different from zero.

As it is shown in (A.4), \( H, P_i \) and \( W_j \) commute amongst themselves, so that the three commutators \([P'_i, P'_j] \) are directly equal to zero. For the second type, let us compute for instance \([P'_1, K'_2] \); if we consider (A.3) and (A.4) then we obtain that

\[
[P'_1, K'_2] = -4\alpha_2^2 H[P_2 W_3 - P_3 W_2, JW]P_1 + 4\alpha_2^2 H[H, K_3 W_1 - K_1 W_3](P_2 W_3 - P_3 W_2)
+ 4\alpha_2^2 H[P_2 W_3 - P_3 W_2, JP]W_1.
\]  

(4.9)

We introduce (A.7), (A.8) and (2.3) thus finding that this bracket is equal to zero due to the identity (A.1):

\[
[P'_1, K'_2] = -4\alpha_2^2 H \left\{ P_1 P_2 C_2 + (P_3 W_1 - P_1 W_3)(P_2 W_3 - P_3 W_2) + C_1 W_1 W_2 \right\}
= -4\alpha_2^2 H(P_2 W_1 + P_1 W_2)(P_1 W_1 + P_2 W_2 + P_3 W_3) = 0.
\]  

(4.10)

The same happens for the five remaining Lie brackets \([P'_i, K'_j] \). Similar computations allow us to deduce the three commutators \([P'_i, K'_j] \), all of them leading to same condition. Let us choose

\[
[P'_1, K'_1] = -4\alpha_2^2 H[P_2 W_3 - P_3 W_2, JW]P_1 + 4\alpha_2^2 H[H, K_2 W_3 - K_3 W_2](P_2 W_3 - P_3 W_2)
+ 4\alpha_2^2 H[P_2 W_3 - P_3 W_2, JP]W_1
= -4\alpha_2^2 H \left\{ P_1^2 C_2 + (P_2 W_3 - P_3 W_2)^2 + C_1 W_1^2 \right\}.
\]  

(4.11)

By expanding, substituting the term \(-2P_2 P_3 W_3 W_3 \) from (A.3) and imposing the corresponding Poincaré bracket (A.1), we obtain that

\[
[P'_1, K'_1] = -4\alpha_2^2 H C_1 C_2 \equiv -\frac{1}{c^2} H'
\]  

(4.12)

which leads to the second relation of (1.8).

The last step is to compute the three commutators \([K'_1, K'_j] \); each of them gives rise to the two conditions (1.8). Let us consider

\[
[K'_1, K'_2] = 8\alpha_1 \alpha_2 H \left\{ P_3(P_1 W_1 + P_2 W_2) - P_1^2 W_3 - P_2^2 W_3 \right\} - 4\alpha_2^2 H C_2(P_2 K_1 - P_1 K_2)
- 4\alpha_2^2 H \left\{ (P_3 K_2 - P_2 K_3)W_1 + (P_1 K_3 - P_3 K_1)W_2 + (P_2 K_1 - P_1 K_2)W_3 \right\} W_3
+ 4\alpha_2^2 JW \left\{ P_3 P_1 W_1 + P_2 P_3 W_2 - P_1^2 W_3 - P_2^2 W_3 \right\}
- 4\alpha_2^2 JP \left\{ P_3^2 W_1^2 + P_3 W_1 W_3 - P_2 W_3 W_3 \right\}
- 4\alpha_2^2 [JW, JP](P_1 W_2 - P_2 W_1).
\]  

(4.13)

In the terms that multiply \( H \), we use the identity (A.1) as \( P_1 W_1 + P_2 W_2 = -P_3 W_3 \), introduce the components \( W_i \) (2.4) and group terms in order to ‘construct’ the Galilei Casimirs (2.3). Next we write the commutator (A.9) and group the terms multiplying each generator of rotations \( J_i \). It can be checked that those factors associated to \( J_1 \) and \( J_2 \) vanish directly. This gives

\[
[K'_1, K'_2] = -8\alpha_2 H \left\{ \alpha_1 C_1 + \alpha_2 C_2 \right\} W_3 + 4\alpha_2^2 J_3 \left\{ 2P_1 P_2 W_1 W_2 + 2P_1 P_3 W_1 W_3 \right\}
\]  

(4.15)
+2P_2P_3W_2W_3 - P_1^2(W_2^2 + W_3^2) - P_2^2(W_1^2 + W_3^2) - P_3^2(W_1^2 + W_2^2) \) (4.14)

and by applying the identity \([A.2]\) we finally obtain that

\[
\begin{align*}
[K'_1, K'_2] &= -8\alpha_2 H \{ \alpha_1 C_1 + \alpha_2 C_2 \} W_3 - 4\alpha_2^2 J_3 C_1 C_2 \equiv -\frac{1}{c^2} J'_3
\end{align*}
\] (4.15)

so that the theorem 1 is proven.

Consequently, this Galilei Lie algebra expansion reverses the non-relativistic contraction of the Poincaré algebra and allows us to introduce a constant negative curvature \(\omega = -1/c^2\) in the 6D space of worldlines \(S^{(2)}\), thus replacing the flat Galilean space of worldlines by the (curved) space \(P/H^{(2)}\) \([14]\). (Recall however that the space \(S^{(2)}\) is rank-two, so its geometry is very different from a rank-one 6D space of constant curvature, either in the curved or in the flat case). This procedure keeps flat the ordinary spacetime, but the change from zero to non-zero curvature in the space of worldlines is unavoidably linked to the change from the Galilean ‘absolute time’ to the Minkowskian ‘relative time’ nature. At the level of the Cartan decompositions the expansion gives \([p^{(2)}, p^{(2)}] = 0 \to [p^{(2)}, p^{(2)}] \subset h^{(2)}\).

We would like to stress that the same procedure also holds for the expansion going from Galilei to the Euclidean algebra \(E \equiv iso(4)\). If we replace in all the Poincaré expressions the negative curvature \(-1/c^2\) by a positive constant then we obtain the commutation rules, Casimirs, etc. corresponding to \(iso(4)\); note that this is equivalent to set \(c\) equal to a pure imaginary complex number. The expansion \(G \to E\) would lead to similar ‘expanded’ generators and conditions as those characterized by the theorem 1. In this case, the symmetric space \(S^{(1)} = ISO(4)/SO(4)\) is the 4D flat Euclidean space and \(S^{(2)} = ISO(4)/(\mathbb{R} \otimes SO(3))\) is a rank-two and positively curved 6D space of lines in Euclidean space.

5 Recovering the universe time radius: from extended Galilei to Newton–Hooke

Besides the Poincaré and Euclidean algebras, the Galilei algebra has other two physically remarkable neighbours: the oscillating (or anti) and the expanding Newton–Hooke algebras \([1]\), hereafter denoted \(N_+\) and \(N_-\), respectively. Both of them are the Lie algebras of the motion groups of absolute time universes but with non-zero curvature \(\kappa\). The commutation rules of \(N_\pm\) are given by

\[
\begin{align*}
[j'_i, j'_j] &= \varepsilon_{ijk} j'_k \quad & [j'_i, p'_j] &= \varepsilon_{ijk} p'_k \quad & [j'_i, k'_j] &= \varepsilon_{ijk} K'_k \\
p'_i, p'_j] &= 0 \quad & [p'_i, k'_j] &= 0 \quad & [k'_i, k'_j] &= 0 \quad \ \ \ \ \ \ \ \ \ \ \ \ \ (5.1)
\end{align*}
\]

where the curvature \(\kappa\) can be expressed in terms of the universe time radius \(\tau\) either by \(\kappa = 1/\tau^2\) for \(N_+\), or by \(\kappa = -1/\tau^2\) for \(N_-\); notice that \(\tau\) is a characteristic time \([1]\) so that is measured in time units. The Casimirs of \(N_\pm\) turn out to be \([16]\):

\[
\begin{align*}
C'_0 &= p'^2 + \kappa k'^2 = p'^2_1 + p'^2_2 + p'^2_3 + \kappa (k'^2_1 + k'^2_2 + k'^2_3) \\
C'_2 &= w'^2 = w'^2_1 + w'^2_2 + w'^2_3 \quad \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ (5.2)
\end{align*}
\]

where the components of \(W'\) are formally identical to the Galilei ones \([2,4]\).
Due to the non-zero Lie brackets $[H', P'_i] = \kappa K'_i$, the vector subspaces $p^{(i)}$ of the Cartan decompositions (5.3) verify now $[p^{(1)}, p^{(1)}] \subset h^{(1)}$ and $[p^{(2)}, p^{(2)}] = 0$. The Newton–Hooke groups $N_\pm$ are the motion groups of the following symmetric homogeneous spaces

$$
S^{(1)} = N_+/H^{(1)} = T_6(SO(2) \otimes SO(3))/ISO(3) \quad \text{curv}(S^{(1)}) = 1/\tau^2 \\
S^{(2)} = N_+/H^{(2)} = T_6(SO(2) \otimes SO(3))/(SO(2) \otimes SO(3)) \quad \text{curv}(S^{(2)}) = 0 \\
S^{(1)} = N_-/H^{(1)} = T_6(SO(1, 1) \otimes SO(3))/ISO(3) \quad \text{curv}(S^{(1)}) = -1/\tau^2 \\
S^{(2)} = N_-/H^{(2)} = T_6(SO(1, 1) \otimes SO(3))/(SO(1, 1) \otimes SO(3)) \quad \text{curv}(S^{(2)}) = 0
$$

(5.3) (5.4)

The spaces $S^{(1)}$ and $S^{(2)}$ correspond, in this order, to the $(3 + 1)$D non-relativistic curved spacetime and the 6D flat space of worldlines in these spacetimes; we remark that the natural metric in this last flat space is definite positive for $N_+$ ($\kappa > 0$) and indefinite, with signature $(3, 3)$ for $N_-$ ($\kappa < 0$). We recall that a recent study of the metric structure of both Newton–Hooke spacetimes has been carried out in [17].

The limit $\kappa \to 0$, or equivalently $\tau \to \infty$, produces the spacetime contraction $N_\pm \to G$. Now we consider the opposite situation and as in the previous section we take $G$ as the initial Lie algebra analysing the way of obtaining $N_\pm$ by introducing the universe time radius $\tau$ (i.e. $\kappa$): $G \to N_\pm$. We apply our expansion method and decompose the Casimirs (5.2) according to the curvature $\kappa$:

$$
C'_1 = C_1 + \kappa J_1 \quad J_1 = K^2 \quad M_1 = 0 \\
C'_2 = C_2 \quad J_2 = 0 \quad M_2 = 0
$$

(5.5)

where $C_1$, $C_2$ are the Galilei Casimirs (2.3). Therefore the linear combination depending on the linear terms in the curvature (3.2) is simply

$$
J = \alpha_1 J_1 = \alpha_1 K^2
$$

(5.6)

where $\alpha_1$ is a constant to be determined. The relation (5.3) gives the expanded generators; they are $H' = 2\alpha_1 K^2$ and all the remaining ones are unchanged. It can be straightforwardly checked that these new generators do not span the Newton–Hooke algebras; it is necessary to take an initial Lie algebra less abelian in order to be able to perform the expansion. The natural choice is to start from the (centrally) extended Galilei algebra. This was exactly what was required in the $(2 + 1)$D case [10]. Hence we introduce a central extension, with central generator $\Xi$ and parameter $m$ (the mass of the particle). The commutation rules of the centrally extended Galilei algebra $\overline{G}$ are given by (2.1) once the vanishing bracket $[P_i, K_j]$ is replaced by

$$
[P_i, K_j] = \delta_{ij} m \Xi \quad [\Xi, \cdot] = 0.
$$

(5.7)

We apply (3.3) with the element (5.3) for the Lie brackets of $\overline{G}$, finding that the new expanded generators read

$$
J'_i = J_i \quad K'_i = K_i \quad i = 1, 2, 3 \\
H' = 2\alpha_1 K^2 + 3\alpha_1 m \Xi \\
P'_i = -2\alpha_1 m \Xi K_i.
$$

(5.8)

This expansion is characterized by

**Theorem 2.** The generators defined by (5.8) give rise to the $(3 + 1)$D Newton–Hooke algebras provided that the constant $\alpha_1$ fulfils

$$
\alpha_1^2 = -\frac{\kappa}{4m^2 \Xi^2} = \pm \frac{1}{4\tau^2 m^2 \Xi^2} \quad \text{for } N_\pm.
$$

(5.9)
Proof. The generators which are unchanged in the expansion close the isotopy subalgebra of an event (a point in the spacetime) \( h^{(1)} = \{K, J\} \). Thus the proposition 1 of [10] can be applied and we only need to compute the 6 Lie brackets involving the generators of \( p^{(1)} \): \( H' \) and \( P'_i \). By direct computations we obtain

\[
[P'_i, P'_j] = 0 \quad [H', P'_i] = -4\alpha_i^2 m^2 \Xi^2 K_i \equiv \kappa K'_i
\]

and the relation (5.10) is proven.

Therefore this expansion introduces a constant curvature \( \kappa \) in the flat spacetime \( G/H^{(1)} \) (2.10), leading to curved spacetimes \( N_{\pm}/H^{(1)} \) (5.4). At the level of the Cartan decompositions this corresponds to the transition \([p^{(1)}, p^{(1)}] = 0 \to [p^{(1)}, p^{(1)}] \subset h^{(1)}\).

6 Concluding remarks

The expansion method proposed in [10] for the \((2+1)\)D kinematical algebras has been improved in order to be applied to higher dimensions and explicitly tested with the \((3+1)\)D Galilei algebra; the method works in the two physically meaningful ‘expansion directions’ which lead from Galilei spacetime to either the flat relativistic Minkowskian spacetime or to the curved Newton–Hooke non-relativistic spacetimes. In order to summarize, we represent the expansions studied in this paper collectively in the following diagram:

\[
\begin{array}{c}
\mathcal{E} \equiv iso(4) \\
\kappa = 0, \ c \ \text{imaginary} \\
\text{4D Euclidean space} \\
\uparrow \\
\mathcal{N}_{+} \equiv t_6(so(2) \oplus so(3)) \\
\kappa = +1/\tau^2, \ c = \infty \\
\text{Oscillating NH} \\
(3+1)D \text{ spacetime} \\
\mathcal{G} \equiv iiso(3) \text{ or } \mathcal{G} \equiv iiso(3) \\
\kappa = 0, \ c = \infty \\
\text{Galilean} \\
\text{(3+1)D spacetime} \\
\mathcal{N}_{-} \equiv t_6(so(1,1) \oplus so(3)) \\
\kappa = -1/\tau^2, \ c = \infty \\
\text{Expanding NH} \\
(3+1)D \text{ spacetime} \\
\downarrow \\
\mathcal{P} \equiv iso(3,1) \\
\kappa = 0, \ c \ finite \\
\text{Minkowskian} \\
(3+1)D \text{ spacetime}
\end{array}
\]

The method uses as ‘expansion seed’ an operator built out from the ordinary expansion in powers of a parameter —interpreted as a curvature— of the Casimir operators of the expanded algebra. The number of these terms (or the number of Casimirs involved in the expansion) is equal to the rank of the homogeneous space behind the expansion.

In the \((3+1)D\) case we have discussed in detail, there is a fourth-order Casimir (of Pauli–Lubanski type), depending \textit{quadratically} on the curvature. However the terms of the expanded Casimirs required in the seed are only those which are first-order in the curvature to recover. We have done some attempts by using a ‘seed’ which keeps terms which are higher order in the curvatures, but the result seems to coincide with the one obtained by restricting the seed to have only ‘first-order’ terms in the curvature. Why this happens is an intriguing property.
The spacetime $S^{(1)}$ is a rank-one space and when we endow it with a non-zero curvature obtaining the Newton–Hooke algebras, the linear combination $\mathcal{J}$ has a single term (5.6). On the other hand, the space of time-like lines $S^{(2)}$ has rank-two and when we introduce the curvature only in the space of time-like worldlines, reaching the Poincaré algebra, the combination $\mathcal{J}$ has two terms (4.6) so that in this case both Casimirs are essential. In this sense, we remark that all other explicit procedures already known for expanding Poincaré or Euclidean algebras to the simple pseudo-orthogonal algebras (1.1) are rank-one (they introduce curvature in $S^{(1)}$) and involve only the quadratic Casimir [7]. These can also be obtained (in any dimension) by following our expansion method, thus recovering the results given in [7].

Finally, it is worth mentioning that we have only worked at a Lie algebra level. Hence an interesting open problem which naturally arises is to analyze how to implement this kind of processes in the representation theory. Recall that contractions of representations have been already formulated (see, e.g., [4]).

Acknowledgments

This work was partially supported by DGES (Project PB98–0370) from the Ministerio de Educación y Cultura de España and by Junta de Castilla y León (Project CO2/399).

Appendix: some relations in the universal enveloping Galilei algebra

We present some useful Lie brackets between elements of the universal enveloping Galilei algebra which are needed in the proof of the expansion to the Poincaré algebra. First, we write down some identities which are used in the obtention of the commutators listed below as well as in the computations of the expansion:

$$\mathbf{PW} = P_1 W_1 + P_2 W_2 + P_3 W_3 = 0 \quad \mathbf{KW} = K_1 W_1 + K_2 W_2 + K_3 W_3 = 0. \quad (A.1)$$

The first identity leads to other four ones:

$$- 2P_1 P_2 W_1 W_2 - 2P_1 P_3 W_1 W_3 - 2P_2 P_3 W_2 W_3 = P_1^2 W_1^2 + P_2^2 W_2^2 + P_3^2 W_3^2 \quad (A.2)$$

$$P_1^2 W_1^2 - P_2^2 W_2^2 - P_3^2 W_3^2 - 2P_2 P_3 W_2 W_3 = 0$$

$$P_2^2 W_2^2 - P_1^2 W_1^2 - P_3^2 W_3^2 - 2P_1 P_3 W_1 W_3 = 0 \quad (A.3)$$

$$P_3^2 W_3^2 - P_1^2 W_1^2 - P_2^2 W_2^2 - 2P_1 P_2 W_1 W_2 = 0.$$

Likewise, the second relation in (A.1) can be used to obtain four identities analogous to (A.2) and (A.3) but with the generators $K_i$ instead of the $P_i$.

Now we display the Lie brackets between the Galilei generators, the components $W_i$ and the products $JP$, $JW$:

$$[W_i, H] = 0 \quad [W_i, J_j] = \epsilon_{ijk} W_k \quad [W_i, P_j] = 0 \quad [W_i, K_j] = 0 \quad [W_i, W_j] = 0 \quad (A.4)$$
Finally, other necessary commutators are given by:

\[
\begin{align*}
[J\mathbf{P}, J\mathbf{H}] &= 0 & [J\mathbf{P}, J_i] &= 0 & [J\mathbf{P}, P_i] &= 0 & [J\mathbf{P}, K_i] &= W_i \\
[J\mathbf{P}, W_1] &= -(P_2 W_3 - P_3 W_2) \\
[J\mathbf{P}, W_2] &= -(P_3 W_1 - P_1 W_3) \\
[J\mathbf{P}, W_3] &= -(P_1 W_2 - P_2 W_1)
\end{align*}
\]  

(A.5)

\[
\begin{align*}
[J\mathbf{W}, J\mathbf{H}] &= 0 & [J\mathbf{W}, J_i] &= 0 & [J\mathbf{W}, W_i] &= 0 \\
[J\mathbf{W}, P_1] &= P_2 W_3 - P_3 W_2 \\
[J\mathbf{W}, P_2] &= P_3 W_1 - P_1 W_3 \\
[J\mathbf{W}, P_3] &= P_1 W_2 - P_2 W_1 \\
[J\mathbf{W}, K_1] &= K_2 W_3 - K_3 W_2 \\
[J\mathbf{W}, K_2] &= K_3 W_1 - K_1 W_3 \\
[J\mathbf{W}, K_3] &= K_1 W_2 - K_2 W_1.
\end{align*}
\]  

(A.6)

\[
\begin{align*}
[J\mathbf{W}, P_2 W_3 - P_3 W_2] &= C_1 W_1 \\
[J\mathbf{W}, P_3 W_1 - P_1 W_3] &= C_1 W_2 \\
[J\mathbf{W}, P_1 W_2 - P_2 W_1] &= C_1 W_3 \\
[J\mathbf{W}, K_2 W_3 - K_3 W_2] &= K P W_1 \\
[J\mathbf{W}, K_3 W_1 - K_1 W_3] &= K P W_2 \\
[J\mathbf{W}, K_1 W_2 - K_2 W_1] &= K P W_3 \\
[J\mathbf{W}, P_2 W_3 - P_3 W_2] &= -C_2 P_1 \\
[J\mathbf{W}, P_3 W_1 - P_1 W_3] &= -C_2 P_2 \\
[J\mathbf{W}, P_1 W_2 - P_2 W_1] &= -C_2 P_3 \\
[J\mathbf{W}, K_2 W_3 - K_3 W_2] &= -C_2 K_1 \\
[J\mathbf{W}, K_3 W_1 - K_1 W_3] &= -C_2 K_2 \\
[J\mathbf{W}, K_1 W_2 - K_2 W_1] &= -C_2 K_3
\end{align*}
\]  

(A.7)

\[
\begin{align*}
[J\mathbf{W}, J\mathbf{P}] &= J_1 (P_2 W_3 - P_3 W_2) + J_2 (P_3 W_1 - P_1 W_3) + J_3 (P_1 W_2 - P_2 W_1).
\end{align*}
\]  

(A.9)

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