SU($N$) group-theory constraints on color-ordered five-point amplitudes at all loop orders

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Abstract

Color-ordered amplitudes for the scattering of $n$ particles in the adjoint representation of SU($N$) gauge theory satisfy constraints arising solely from group theory. We derive these constraints for $n = 5$ at all loop orders using an iterative approach. These constraints generalize well-known tree-level and one-loop group theory relations.

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1 Introduction

The discovery of the color-kinematic (BCJ) duality of gauge theory and the double-copy property of gravity [1,2] constitutes only the most recent chapter of the tremendous advances that have occurred in our understanding of perturbative gauge and gravity amplitudes over the last decade. Tree-level relations implied by the BCJ conjecture have been verified in refs. [3–6], and the BCJ conjecture has also been tested at loop level for four- [2,7] and five-point [8,9] amplitudes of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. This subject has been reviewed in refs. [7,10], which also contain references to other work on the subject.

The color structure of a gauge theory amplitude may be expressed by decomposing the amplitude in either a trace basis [11] or a basis of color factors [12,13]. An advantage of the trace basis decomposition is that its coefficients, called color-ordered amplitudes, are individually gauge-invariant. The trace basis is also conducive to exhibiting the $1/N$ expansion of the gauge theory, and moreover has a close connection to the string theory expansion of the scattering amplitudes. Color-kinematic duality implies the existence of linear constraints among tree-level color-ordered amplitudes, which were proven in refs. [3–6].

Even before BCJ duality is imposed, however, the color-ordered gauge theory amplitudes are known to obey various constraints solely as a consequence of SU($N$) group theory. At tree level, these include the U(1) decoupling [14,15] and Kleiss-Kuijf relations [16]. These and similar group-theory relations for one-loop amplitudes [11,17] can be elegantly derived using the alternative color decomposition of the amplitude [12,13]. Four-point color-ordered amplitudes are also known to obey group-theory relations at two loops [18], and these were recently generalized to all loop orders, where it was shown that there exist four relations among color-ordered four-point amplitudes for each $L \geq 2$ [19]. Other recent work on constraints among loop-level amplitudes includes refs. [20–23].

The purpose of this paper is to derive all SU($N$) group theory relations satisfied by five-point color-ordered amplitudes at two and higher loops, generalizing the known relations at tree level and one loop. We employ a recursive approach [19] to derive the constraints satisfied by any $L$-loop diagram (containing only adjoint fields) that can be obtained by attaching a rung between two external legs of an $(L-1)$-loop diagram. We assume that the most general $L$-loop color factor can be obtained from this subset using Jacobi relations. Then, by seeding the recursion relation with the six known constraints on tree-level five-point amplitudes, we show that there are ten constraints among color-ordered five-point amplitudes at each odd loop order, and twelve constraints at each even loop order.

In order to state our results up front, we define $A^{(L,k)}$ as the part of the $L$-loop five-point amplitude that is suppressed by $0 \leq k \leq L$ powers of $N$ relative to the leading planar amplitude, and $A^{(L,k)}_{\lambda}$ as the coefficients of this amplitude in a trace basis consisting of single-trace terms for $\lambda = 1, \cdots, 12$ and double-trace terms for $\lambda = 13, \cdots, 22$. (A precise definition of this trace basis is given in the main body of the paper.) The six tree-level U(1) decoupling relations [15] can be expressed as

$$\sum_{\lambda=1}^{12} A^{(0)}_{\lambda} x^{(0)}_{\lambda j} = 0, \quad j = 1, \cdots, 6 \quad (1.1)$$
where $x^{(0)}$ are constants defined in eq. (3.11). The ten one-loop U(1) decoupling relations \[11\] can be expressed as

$$A^{(1,1)}_{\lambda} = \sum_{\kappa=1}^{12} A^{(1,0)}_{\kappa} m^{(1)}_{\kappa,\lambda-12}, \quad \lambda = 13, \cdots, 22 \tag{1.2}$$

where $m^{(1)}$ are constants defined in eq. (4.11). In this paper, we show that, for all odd loop orders, the most-subleading-color five-point amplitudes are given by

$$A^{(L,L)}_{\lambda} = \sum_{\kappa=1}^{12} A^{(L,L-1)}_{\kappa} m^{(1)}_{\kappa,\lambda-12}, \quad \lambda = 13, \cdots, 22, \quad \text{odd } L \tag{1.3}$$

and for all even loop orders, the most-subleading-color amplitudes obey the six constraints\[2\]

$$\sum_{\lambda=1}^{12} A^{(L,L)}_{\lambda} x^{(0)}_{\lambda j} = 0, \quad j = 1, \cdots, 6, \quad \text{even } L \geq 2. \tag{1.4}$$

We show that five-point amplitudes obey six additional constraints at all even loop orders\[3\]

$$\sum_{\lambda=1}^{12} \left(10 A^{(L,L-2)}_{\lambda} x^{(0)}_{\lambda j} + A^{(L,L)}_{\lambda} x^{(2)}_{\lambda j}\right) + \sum_{\lambda=13}^{22} A^{(L,L-1)}_{\lambda} x^{(1)}_{\lambda-12,j} = 0, \quad j = 1, \cdots, 6, \quad \text{even } L \geq 2 \tag{1.5}$$

where $x^{(1)}$ and $x^{(2)}$ are constants defined in eqs. (1.9) and (1.16). Eqs. (1.4) and (1.5) may be combined to express each of the most-subleading-color amplitudes $A^{(L,L)}$ as linear combinations of $A^{(L,L-1)}$ and $A^{(L,L-2)}$, using the constants $m^{(2)}$ defined in eq. (4.20).

We have obtained the relations (1.3)–(1.5) by using the connection between the color basis \[12\] and \[13\] and the trace basis of gauge theory amplitudes. Since the independent color basis at $L$ loops is smaller than the trace basis, the null eigenvectors of the transformation matrix from one basis to the other imply constraints among the trace basis coefficients. One can alternatively obtain some, but not all, of these constraints by expanding the amplitude in a U($N$) trace basis, and observing that an amplitude containing one or more photons vanishes since the U(1) structure constants are zero \[11\] [21]. Such U(1) decoupling relations can be used to derive eqs. (1.3) and (1.4), but not eq. (1.5). The constraints (1.3)–(1.5) reduce the number of independent $L$-loop color-ordered five-point amplitudes from $d(L)$ to $d(L-1)$ where $d(L) = 10L + 2\left\lfloor \frac{L}{2} \right\rfloor + 12$. No further constraints on $L$-loop color-ordered five-point amplitudes arise from group theory alone.

The remainder of this paper is organized as follows. In sec. 2 we review the trace basis for five-point amplitudes at arbitrary loop order. In sec. 3 we describe the color basis and explain how each null eigenvector of the transformation matrix from the color basis to the trace basis implies a constraint among color-ordered amplitudes. Finally, in sec. 4 we utilize a recursive approach to derive the all-loop group-theory constraints on five-point color-ordered amplitudes.

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2This relation was previously observed for $L = 2$ in ref. [21].

3The $L = 2$ constraints have been independently obtained by C. Boucher-Veronneau and L. Dixon [24].
2 The trace basis for all-loop five-point amplitudes

In this section, we review the trace and $1/N$ decomposition of five-point amplitudes. Five-point amplitudes of an SU($N$) gauge theory can be expressed in terms of a basis $\{T_\lambda\}$, $\lambda = 1, \cdots, 22$, of single and double traces. We choose an explicit basis given by

$$T_1 = \left[ \text{Tr}(12345) - \text{Tr}(15432) \right], \quad T_7 = \left[ \text{Tr}(12543) - \text{Tr}(13254) \right],$$

$$T_2 = \left[ \text{Tr}(14325) - \text{Tr}(15234) \right], \quad T_8 = \left[ \text{Tr}(14523) - \text{Tr}(13254) \right],$$

$$T_3 = \left[ \text{Tr}(13425) - \text{Tr}(15243) \right], \quad T_9 = \left[ \text{Tr}(13524) - \text{Tr}(14253) \right],$$

$$T_4 = \left[ \text{Tr}(12435) - \text{Tr}(15342) \right], \quad T_{10} = \left[ \text{Tr}(12534) - \text{Tr}(14352) \right],$$

$$T_5 = \left[ \text{Tr}(14235) - \text{Tr}(15324) \right], \quad T_{11} = \left[ \text{Tr}(14532) - \text{Tr}(12354) \right],$$

$$T_6 = \left[ \text{Tr}(13245) - \text{Tr}(15423) \right], \quad T_{12} = \left[ \text{Tr}(13542) - \text{Tr}(12453) \right],$$

$$T_{13} = \text{Tr}(12) \left[ \text{Tr}(345) - \text{Tr}(543) \right], \quad T_{18} = \text{Tr}(13) \left[ \text{Tr}(245) - \text{Tr}(542) \right],$$

$$T_{14} = \text{Tr}(23) \left[ \text{Tr}(451) - \text{Tr}(154) \right], \quad T_{19} = \text{Tr}(24) \left[ \text{Tr}(351) - \text{Tr}(153) \right],$$

$$T_{15} = \text{Tr}(34) \left[ \text{Tr}(512) - \text{Tr}(215) \right], \quad T_{20} = \text{Tr}(35) \left[ \text{Tr}(412) - \text{Tr}(214) \right],$$

$$T_{16} = \text{Tr}(45) \left[ \text{Tr}(123) - \text{Tr}(321) \right], \quad T_{21} = \text{Tr}(41) \left[ \text{Tr}(523) - \text{Tr}(325) \right],$$

$$T_{17} = \text{Tr}(51) \left[ \text{Tr}(234) - \text{Tr}(432) \right], \quad T_{22} = \text{Tr}(52) \left[ \text{Tr}(134) - \text{Tr}(431) \right],$$

where $\text{Tr}(12 \cdots) \equiv \text{Tr}(T^a_1 T^a_2 \cdots)$, and the matrices $T^a$ are the generators in the defining representation of SU($N$), normalized according to $\text{Tr}(T^a T^b) = \delta^{ab}$. All other possible trace terms vanish in SU($N$) since $\text{Tr}(T^a) = 0$.

The $L$-loop amplitude may be further decomposed in powers of $N$ as

$$A^{(L)} = \sum_{\lambda=1}^{12} \left( \sum_{k=0}^{\lfloor L/2 \rfloor} N^{L-2k} A^{(L,2k)}_\lambda \right) T_\lambda + \sum_{\lambda=13}^{22} \left( \sum_{k=0}^{\lfloor (L-1)/2 \rfloor} N^{L-2k-1} A^{(L,2k+1)}_\lambda \right) T_\lambda$$

$$= \sum_{\lambda=1}^{12} \left( \sum_{k=0}^{\lfloor L/2 \rfloor} N^{L-2k} t^{(L)}_{\lambda+22k} \right) T_\lambda + \sum_{\lambda=13}^{22} \left( \sum_{k=0}^{\lfloor (L-1)/2 \rfloor} N^{L-2k-1} t^{(L)}_{\lambda+22k} \right) T_\lambda$$

The $1/N$ expansion suggests enlarging the 22-dimensional basis to a $d(L)$-dimensional basis which takes into account powers of $N$:

$$d(L) = \begin{cases} 11L + 12, & L \text{ even,} \\ 11L + 11, & L \text{ odd.} \end{cases}$$
We then write

\[ A^{(L)}(L) = d(L) \sum_{\lambda=1}^{\lambda_{22k}} A^{(L)}(L) \lambda_t(L) \lambda, \quad \text{where} \quad A^{(L)}_{\lambda + 22k} = \begin{cases} A^{(L,2k)}_{\lambda}, & \lambda = 1, \ldots, 12, \\ A^{(L,2k+1)}_{\lambda}, & \lambda = 13, \ldots, 22. \end{cases} \tag{2.6} \]

In the remainder of the paper, we demonstrate that these \( d(L) \) color-ordered amplitudes \( A^{(L)}_{\lambda} \) obey a set of group-theory constraints

\[ \sum_{\lambda=1}^{\lambda_{22k}} A^{(L)}(L) \lambda_t(L) \lambda = 0, \quad j = 1, \ldots, \begin{cases} 12, & \text{even } L \geq 2, \\ 10, & \text{odd } L. \end{cases} \tag{2.7} \]

In particular, these constraints can be used to express the most-subleading color amplitudes \( A^{(L,L)}(L) \) at each loop order as linear combinations of \( A^{(L,L-1)}(L) \) for \( L \) odd, and in terms of \( A^{(L,L-1)}(L) \) and \( A^{(L,L-2)}(L) \) for \( L \) even.

### 3 The color basis for five-point amplitudes

In this section, we review the decomposition of the amplitude in a basis of color factors \[12, 13\]. The \( n \)-point amplitude in a gauge theory containing only fields in the adjoint representation of \( SU(N) \) (such as pure Yang-Mills or supersymmetric Yang-Mills theory) can be written in a loop expansion, with the \( L \)-loop amplitude given by a “parent-graph” decomposition \[27\]

\[ A^{(L)} = \sum_i a^{(L)}_i c^{(L)}_i. \tag{3.1} \]

Here \( \{c^{(L)}_i\} \) represents a complete set of \( L \)-loop \( n \)-point diagrams built from cubic vertices with a factor of the \( SU(N) \) structure constants \( \tilde{f}^{abc} \) at each vertex; we have suppressed all momentum and spin dependence. (Contributions from Feynman diagrams containing quartic vertices with factors of \( \tilde{f}^{abc} \tilde{f}^{cde} \), \( \tilde{f}^{ace} \tilde{f}^{bde} \), and \( \tilde{f}^{ade} \tilde{f}^{bce} \) can be parcelled out among other diagrams containing only cubic vertices.)

The two decompositions (2.6) and (3.1) may be related by expressing the color factors as linear combinations of the trace basis (2.5)

\[ c^{(L)}_i = \sum_{\lambda=1}^{\lambda_{22k}} M^{(L)}_{i\lambda} f^{(L)}_\lambda. \tag{3.2} \]

Combining eqs. (3.1) and (3.2) then yields

\[ A^{(L)}_{\lambda} = \sum_i a^{(L)}_i M^{(L)}_{i\lambda}. \tag{3.3} \]

\[ \text{This set of color factors need not be independent but may satisfy constraints } \sum_i \ell_i c_i = 0. \text{ Such an overcomplete basis is usually required to make color-kinematic duality manifest } [14, 18]. \]
The number of independent color factors $n_{\text{color}}^{(L)}$ is less than the dimension of the trace basis $d(L)$, so the transformation matrix $M_{i\lambda}^{(L)}$ will have a set of $n_{\text{con}}^{(L)}$ right null eigenvectors.

$$\sum_{\lambda=1}^{d(L)} M_{i\lambda}^{(L)} r_{\lambda j}^{(L)} = 0, \quad j = 1, \ldots, n_{\text{con}}^{(L)},$$

where $n_{\text{con}}^{(L)} = d(L) - n_{\text{color}}^{(L)}$. (3.4)

In other words, the vectors $r_{\lambda j}^{(L)}$, $j = 1, \ldots, n_{\text{con}}^{(L)}$ span the kernel of the transformation matrix. Equation (3.3) then implies $n_{\text{con}}^{(L)}$ constraints on the color-ordered amplitudes

$$\sum_{\lambda=1}^{d(L)} A_{\lambda}^{(L)} T_{\lambda j}^{(L)} = 0, \quad j = 1, \ldots, n_{\text{con}}^{(L)}.$$ (3.5)

Hence, the kernel of the transformation matrix determines the group-theory constraints on the color-ordered amplitudes.

At tree level, we may choose an independent basis of color factors to be $[12, 13]\$ of color factors to be $[12, 13]$

$$c_1^{(0)} = f_{a_1 a_2 b} f_{b a_3 c} f_{c a_4 a_5}, \quad c_2^{(0)} = f_{a_1 a_2 b} f_{b a_3 c} f_{c o_{2}a_5}, \quad c_3^{(0)} = f_{a_1 a_2 b} f_{b a_3 c} f_{c a_2 a_5},$$

$$c_4^{(0)} = f_{a_1 a_2 b} f_{b a_4 c} f_{c o_{2}a_5}, \quad c_5^{(0)} = f_{a_1 a_4 b} f_{b a_2 c} f_{c a_3 a_5}, \quad c_6^{(0)} = f_{a_1 a_3 b} f_{b a_2 c} f_{c a_4 a_5}. \quad (3.6)$$

By writing

$$f_{abc} = i\sqrt{2} f_{abc} = \text{Tr}([T^a, T^b] T^c)$$

and using the SU($N$) identities

$$\text{Tr}(PT^a) \text{Tr}(QT^a) = \text{Tr}(PQ) - \frac{1}{N} \text{Tr}(P) \text{Tr}(Q)$$

$$\text{Tr}(PT^a Q T^a) = \text{Tr}(P) \text{Tr}(Q) - \frac{1}{N} \text{Tr}(PQ) \quad (3.8)$$

we find that

$$c_i^{(0)} = \sum_{\lambda=1}^{12} M_{i\lambda}^{(0)} t_{\lambda}^{(0)} \quad \text{with} \quad M^{(0)} = \left( \begin{array}{cccccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right)$$

where

$$m^{(0)} = \left( \begin{array}{cccccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right). \quad (3.10)$$

Constraints $\sum_i \ell_i c_i = 0$ among the color factors correspond to left null eigenvectors of the transformation matrix: $\sum_i \ell_i M_{i\lambda} = 0$.

A larger, fifteen-dimensional basis is required to manifest color-kinematic duality [1].
We denote the six right null eigenvectors of $M^{(0)}$ by $x^{(0)}_{\lambda j}$, $j = 1, \cdots, 6$, where
\[
x^{(0)} = \left( -I_{6\times6}^{(0)} \right) = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix}.
\tag{3.11}
\]
The six constraints on color-ordered tree amplitudes implied by these null vectors
\[
\sum_{\lambda=1}^{12} A^{(0)}_{\lambda} x^{(0)}_{\lambda j} = 0, \quad j = 1, \cdots, 6
\tag{3.12}
\]
are precisely the tree-level U(1) decoupling relations for five-point amplitudes \cite{15}. In the next section, we will use eqs. (3.9) and (3.10) as the starting point of a recursive procedure to derive higher-loop relations among five-point amplitudes.

4 Relations among $L$-loop five-point amplitudes

In this section, we employ a recursive procedure \cite{19} to obtain the set of right null eigenvectors at arbitrary loop level, which then determine the constraints on the color-ordered amplitudes. An $(L+1)$-loop diagram may be obtained from an $L$-loop diagram by attaching a rung between two of its external legs, $i$ and $j$. This corresponds to contracting its color factor with $\epsilon_{ij} \tilde{f}^{a_i a'_i} \tilde{f}^{a_j a'_j}$. If $i$ and $j$ are not adjacent, this will convert a planar diagram into a nonplanar diagram.

First consider the effect of this procedure on the trace basis (2.1) -(2.2)
\[
T_\lambda \rightarrow \sum_{\kappa=1}^{22} G_{\lambda \kappa} T_\kappa, \quad \text{with} \quad G = \begin{pmatrix} NA & B \\
C & ND \end{pmatrix}
\tag{4.1}
\]
where explicit expressions for the submatrices $A, B, C,$ and $D$ are given in the appendix. On the expanded basis (2.3), the same procedure yields
\[
t^{(L)}_\lambda \rightarrow \sum_{\kappa=1}^{d(L+1)} g_{\lambda \kappa} t^{(L+1)}_\kappa
\tag{4.2}
\]
where \( g^{(L)} \) is the \( d(L) \times d(L+1) \) matrix

\[
g^{(L)} = \begin{pmatrix}
A & B & 0 & 0 & 0 & \ldots \\
0 & D & C & 0 & 0 & \ldots \\
0 & 0 & A & B & 0 & \ldots \\
0 & 0 & 0 & D & C & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]  

(4.3)

Now consider a complete set \( \{c_i^{(L)}\} \) of \( L \)-loop color factors, expressed in terms of the trace basis via eq. (3.2). Attaching a rung between two external legs of \( c_i^{(L)} \) yields

\[
c_i^{(L)} \rightarrow \sum_{\lambda=1}^{d(L)} \sum_{\kappa=1}^{d(L+1)} M_{i\lambda}^{(L)} g_{\lambda\kappa}^{(L)} t_{\kappa}^{(L+1)}.
\]  

(4.4)

If we assume that this procedure generates a complete set of \( (L+1) \)-loop color factors \( \{c_i^{(L+1)}\} \), then the kernel of \( M_{i\kappa}^{(L+1)} \) is the kernel of \( \sum_{\lambda=1}^{d(L)} M_{i\lambda}^{(L)} g_{\lambda\kappa}^{(L)} \); in other words,

\[
\sum_{\lambda=1}^{d(L)} \sum_{\kappa=1}^{d(L+1)} M_{i\lambda}^{(L)} g_{\lambda\kappa}^{(L)} r_{\kappa j}^{(L+1)} = 0.
\]  

(4.5)

The solutions \( r_{\kappa j}^{(L+1)} \) of this equation are given by the solutions of

\[
\sum_{\kappa=1}^{d(L+1)} g_{\lambda\kappa}^{(L)} r_{\kappa j}^{(L+1)} = \text{linear combination of} \ \{r_{\lambda j'}^{(L)}\}
\]  

(4.6)

by virtue of eq. (3.4). We stress that eqs. (4.5) and (4.6) must hold for arbitrary values of the parameters \( e_{ij} \) in the matrices (A.1).

Our task is now to solve eq. (4.6) recursively. We begin with the tree-level transformation matrix \( M^{(0)} \) given in eqs. (3.9) and (3.10). At one loop, there are ten solutions of

\[
\sum_{\lambda=1}^{12} \sum_{\kappa=1}^{22} M_{i\lambda}^{(0)} g_{\lambda\kappa}^{(0)} r_{\kappa j}^{(1)} = 0
\]  

(4.7)

where \( g^{(0)} = (A \ B) \); these null eigenvectors are given by

\[
r^{(1)} = \begin{pmatrix}
10x^{(0)} & 0 \\
x^{(1)} & y^{(1)}
\end{pmatrix}
\]  

(4.8)
where \( x^{(0)} \) was defined in eq. (3.11), and \( x^{(1)} \) and \( y^{(1)} \) are given by\(^7\)

\[
x^{(1)} = \begin{pmatrix}
-2 & 0 & 0 & 2 & 1 & -1 \\
-2 & -1 & -2 & -2 & -1 & -2 \\
-1 & -2 & -2 & -1 & -2 & -2 \\
1 & 2 & 0 & 0 & -2 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 \\
-1 & 1 & 2 & 0 & 0 & -2 \\
-2 & -2 & -1 & -2 & -2 & -1 \\
0 & 0 & 2 & 1 & -1 & -2 \\
-2 & -1 & 1 & 2 & 0 & 0
\end{pmatrix}, \quad
y^{(1)} = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
-1 & -1 & 0 & -1 \\
1 & 0 & 0 & 1 \\
-1 & 0 & -1 & -1 \\
-1 & -1 & -1 & -1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\tag{4.9}
\]

The last four eigenvectors of eq. (4.8) imply constraints among the one-loop double-trace coefficients

\[
\sum_{\lambda=13}^{22} A^{(1,1)}_{\lambda} y^{(1)}_{\lambda-12,j} = 0, \quad j = 1, \cdots 4
\tag{4.10}
\]

while the other six eigenvectors imply relations between the one-loop single-trace and double-trace coefficients. Equivalently we can write the ten one-loop null eigenvectors eq. (4.8) as

\[
r^{(1)} = \left( n^{(1)} - I_{10 \times 10} \right), \quad \text{where} \quad m^{(1)} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.
\tag{4.11}
\]

These eigenvectors allow us to express each of \( A^{(1)}_{13} \) through \( A^{(1)}_{22} \) (which correspond to double-trace coefficients) in terms of a linear combination of \( A^{(1)}_{1} \) through \( A^{(1)}_{12} \) (which correspond to single-trace coefficients) namely

\[
A^{(1,1)}_{\lambda} = \sum_{\kappa=1}^{12} A^{(1,0)}_{\kappa} m^{(1)}_{\kappa,\lambda-12}, \quad \lambda = 13, \cdots, 22.
\tag{4.12}
\]

\(^7\)Each \( x^{(1)}_j \) is defined only up to the addition of arbitrary linear combinations of the \( y^{(1)}_j \). This freedom could be used, for example, to set the bottom four entries of \( x^{(1)}_j \) to zero. We have chosen the \( x^{(1)}_j \) to be consistent with the two-loop eigenvectors given below.
These are precisely the one-loop U(1) decoupling relations \[11\] as previously observed \[25\].

Next we turn to the two-loop case. The two-loop null eigenvectors \(r^{(2)}\) satisfy
\[
\sum_{\lambda=1}^{22} \sum_{\kappa=1}^{34} M^{(1)}_{\lambda \kappa} g^{(1)}_{\lambda \kappa} r^{(2)}_{\kappa j} = 0,
\]
where \(g^{(1)} = \begin{pmatrix} A & B & 0 \\ 0 & D & C \end{pmatrix}\).

In ref. \[25\], the form of \(M^{(1)}\) was obtained by expressing the independent twelve-dimensional basis of pentagon color factors in terms of the one-loop trace basis. In this paper, we instead (and equivalently) construct \(M^{(1)}\) as the matrix that annihilates the set of known one-loop null eigenvectors \(4.11\), namely,
\[
M^{(1)} = \begin{pmatrix} \mathbb{1}_{12 \times 12} & m^{(1)} \end{pmatrix}.
\]

Then one finds that eq. \(4.13\) has twelve solutions
\[
r^{(2)} = \begin{pmatrix} 10 x^{(0)} & 0 \\ x^{(1)} & 0 \\ x^{(2)} & x^{(0)} \end{pmatrix},
\]
where \(x^{(0)}\) and \(x^{(1)}\) were given in eqs. \(3.11\) and \(4.9\), and \(x^{(2)}\) is given by
\[
x^{(2)} = \begin{pmatrix} 1 & 2 & 4 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 & 2 & 4 \\ 1 & 0 & 1 & 2 & 4 & 2 \\ 2 & 4 & 2 & 1 & 0 & 1 \\ 4 & 2 & 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 4 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\]

We observe that the last six eigenvectors of eq. \(4.15\) imply constraints among \(A_{23}^{(L)}\) through \(A_{34}^{(L)}\), that is, among the two-loop subleading-color single-trace coefficients:
\[
\sum_{\lambda=1}^{12} A_{\lambda j}^{(2,2)} x^{(0)}_{\lambda j} = 0, \quad j = 1, \ldots, 6.
\]

These are simply the two-loop U(1) decoupling relations, analogous to the tree-level U(1) decoupling relations \(3.12\), and can be obtained by an extension of the arguments of ref. \[11\]. Equation \(4.17\) was previously observed in ref. \[21\]. In that paper, it was also shown that analogs of
the Kleiss-Kuijf relations hold for the two-loop subleading-color single-trace coefficients through seven points, but are modified for eight-point functions.

The first six eigenvectors of eq. (4.15) correspond to additional relations among two-loop leading-color and subleading-color five-point amplitudes. These constraints, however, cannot be obtained by U(1) decoupling arguments.

The twelve two-loop null eigenvectors (4.15) can equivalently be expressed as

\[
\sum_{\lambda=1}^{12} \left( 10 A^{(2,0)}_{\lambda} x^{(0)}_{\lambda j} + A^{(2,2)}_{\lambda} x^{(2)}_{\lambda j} \right) + \sum_{\lambda=13}^{22} A^{(2,1)}_{\lambda} x^{(1)}_{\lambda-12,j} = 0, \quad j = 1, \cdots 6. \tag{4.18}
\]

These constraints, however, cannot be obtained by U(1) decoupling arguments.

The twelve two-loop null eigenvectors (4.15) can equivalently be expressed as

\[
r^{(2)} = \begin{pmatrix} m^{(2)}_1 \\ -2 \times I_{12 \times 12} \end{pmatrix} \tag{4.19}
\]

where

\[
m^{(2)} = \begin{pmatrix} 0 & -2 & -4 & 2 & -4 & 2 & -2 & -4 & -10 & -4 & -2 & 4 \\ -2 & 0 & 2 & -4 & 2 & -4 & -4 & -2 & 4 & -2 & -4 & -10 \\ -4 & 2 & 0 & -2 & -4 & 2 & -2 & 4 & -2 & -4 & -10 & -4 \\ 2 & -4 & -2 & 0 & 2 & -4 & -4 & -10 & -4 & -2 & 4 & -2 \\ -4 & 2 & -4 & 2 & 0 & -2 & -10 & -4 & -2 & 4 & -2 & -4 \\ 2 & -4 & 2 & -4 & -2 & 0 & 4 & -2 & -4 & -10 & -4 & -2 \\ -2 & -4 & -2 & -4 & 10 & 4 & 0 & -2 & 4 & 2 & 4 & -2 \\ -4 & 2 & 4 & 10 & -4 & -2 & -2 & 0 & 4 & 2 & 4 & -2 \\ 10 & 4 & -2 & -4 & -2 & -4 & 4 & -2 & 0 & -2 & 4 & 2 \\ -4 & -2 & -4 & -2 & 4 & 10 & 2 & 4 & -2 & 0 & -2 & 4 \\ -2 & -4 & 10 & 4 & -2 & -4 & 4 & 2 & 4 & -2 & 0 & -2 \\ 4 & 10 & -4 & -2 & -4 & -2 & -2 & 4 & 2 & 4 & -2 & 0 \end{pmatrix}. \tag{4.20}
\]

These allow one to express all of the most-subleading-color amplitudes \( A^{(2,2)} \) in terms of linear combinations of \( A^{(2,0)} \) and \( A^{(2,1)} \).

\(^8\)C. Boucher-Veronneau and L. Dixon have independently obtained these relations by recasting an independent two-loop color basis into the trace basis [24].
Three-loop eigenvectors are the solutions of
\[ \sum_{\lambda=1}^{34} \sum_{\kappa=1}^{44} M_{i\lambda}^{(2)} g_{\lambda \kappa}^{(2) (3)} r_{\kappa j}^{(3)} = 0. \] (4.21)

One could construct \( M^{(2)} \) from an independent basis of two-loop color factors, but instead we simply write it as the matrix whose kernel is given by eq. (4.19), namely
\[ M_{i\lambda}^{(2)} = (2 \times 22 \times 22 \ m^{(2)}) \] (4.22)

One then finds that eq. (4.21) has precisely ten solutions,
\[ r^{(3)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 10x^{(0)} & 0 \\ x^{(1)} & y^{(1)} \end{pmatrix} \] (4.23)
where \( x^{(0)} \), \( x^{(1)} \), and \( y^{(1)} \) were given in eqs. (3.11) and (4.9). These are exactly analogous to the one-loop null eigenvectors (4.8).

The fact that there are no three-loop null eigenvectors with non-zero entries in the top two blocks makes it possible to solve the recursion relations to all loop orders. This is based on the following observation. We have written the \( L \)-loop null eigenvectors in block form, with \( r^{(L)} \) having \((L + 1)\) blocks of alternating height 12 and 10. By setting \( e_2 = e_3 = e_4 = e_5 = 1 \) in the matrices (A.1), one obtains \( A = D = I \) and \( B = C = 0 \). Then eq. (4.6) implies that the top \( L \) blocks of \( r^{(L)} \) must be a linear combination of \((L - 1)\)-loop null eigenvectors. Then, since the top two blocks of \( r^{(3)} \) vanish, the top two blocks of \( r^{(4)} \) must also vanish, so that eq. (4.6) for \( r^{(4)} \) is precisely equivalent to eq. (4.6) for \( r^{(2)} \). Similarly, eq. (4.6) for \( r^{(5)} \) is precisely equivalent to eq. (4.6) for \( r^{(3)} \), and so forth. Thus, there are ten null eigenvectors at each odd-loop order and twelve null eigenvectors at each even-loop order (except at tree level, where there are six), which are given explicitly by
\[ r^{(2\ell+1)} = \begin{pmatrix} \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \\ 10x^{(0)} & 0 \\ x^{(1)} & y^{(1)} \end{pmatrix}, \quad r^{(2\ell+2)} = \begin{pmatrix} \vdots & \vdots \\ 0 & 0 \\ 10x^{(0)} & 0 \\ x^{(1)} & 0 \\ x^{(2)} & x^{(0)} \end{pmatrix}. \] (4.24)

Finally, the \( L \)-loop relations among color-ordered five-point amplitudes implied by these null eigenvectors are given in eqs. (1.3)–(1.5).

5 Conclusions

In this paper, we have derived group-theory identities for color-ordered five-point amplitudes in SU(\( N \)) gauge theories at all loop orders. We used a recursive procedure to derive the constraints
on $L$-loop color factors that are generated by attaching a rung between two external legs of an $(L-1)$-loop color factor. Assuming that all $L$-loop color factors are linear combinations of those just described (i.e., via Jacobi relations), the constraints derived apply to all $L$-loop color-ordered amplitudes.

At odd-loop levels, we obtained ten relations, analogous to the one-loop U(1) decoupling relations, which allow one to express the most-subleading-color amplitudes $A^{(L,L)}$ in terms of the second-most-subleading-color amplitudes $A^{(L,L-1)}$. At even-loop levels, we obtained six relations that relate the most-subleading-color amplitudes $A^{(L,L)}$ to one another, analogous to the tree-level U(1) decoupling relations. For even $L \geq 2$, we obtained six additional relations relating $A^{(L,L)}$ to other amplitudes, which cannot be obtained by U(1) decoupling. All twelve of the even-loop relations can be combined to express $A^{(L,L)}$ as linear combinations of $A^{(L,L-1)}$ and $A^{(L,L-2)}$.

It would clearly be of interest to extend these results to six-point amplitudes and beyond.

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A Appendix

The submatrices that comprise $G$ in eq. (4.1) are given by

$$A = \begin{pmatrix}
  e_2+e_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & e_4+e_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & e_3+e_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & e_2+e_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & e_4+e_5 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & e_3+e_5 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & e_2+e_5 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & e_3+e_5 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_2+e_5 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_2+e_3 \\
\end{pmatrix},$$

$$B = \begin{pmatrix}
  e_2-e_3 & -e_3+e_4 & 0 & e_3-e_4 & -e_4+e_5 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & e_2-e_3 & 0 & e_2-e_5 & 0 & 0 & 0 & e_3-e_4 & e_2-e_3 \\
  0 & 0 & e_2-e_4 & 0 & e_2+e_5 & e_3-e_4 & e_3-e_5 & e_3-e_4 & -e_4-e_4 & e_4-e_4 \\
  -e_2+e_4 & 0 & 0 & 0 & e_3-e_5 & e_3-e_4 & e_3-e_4 & e_3-e_4 & e_3-e_4 & e_3-e_4 \\
  0 & 0 & 0 & 0 & e_3-e_5 & e_3-e_4 & e_3-e_4 & e_3-e_4 & e_3-e_4 & e_3-e_4 \\
  0 & 0 & 0 & 0 & e_3-e_5 & e_3-e_4 & e_3-e_4 & e_3-e_4 & e_3-e_4 & e_3-e_4 \\
  -e_2+e_4 & 0 & 0 & 0 & e_3-e_5 & e_3-e_4 & e_3-e_4 & e_3-e_4 & e_3-e_4 & e_3-e_4 \\
  e_2-e_4 & 0 & 0 & 0 & e_3-e_5 & e_3-e_4 & e_3-e_4 & e_3-e_4 & e_3-e_4 & e_3-e_4 \\
  e_2-e_3 & -e_3+e_5 & 0 & e_3-e_5 & 0 & 0 & 0 & 0 & -e_4-e_5 & e_4-e_5 \\
  -e_2+e_3 & 0 & 0 & 0 & e_3-e_5 & e_3-e_4 & e_3-e_4 & e_3-e_4 & e_3-e_4 & e_3-e_4 \\
\end{pmatrix},$$

$$C = \begin{pmatrix}
  -e_3+e_5 & 0 & 0 & e_4-e_5 & 0 & 0 & -e_3+e_5 & 0 & 0 & e_4-e_5 & -e_3+e_4 & -e_3+e_4 \\
  e_3-e_4 & 0 & 0 & 0 & e_4-e_5 & 0 & e_4-e_5 & 0 & 0 & e_4-e_5 & 0 & e_4-e_5 \\
  e_3-e_4 & 0 & 0 & 0 & e_3-e_4 & 0 & e_3-e_4 & 0 & 0 & e_3-e_4 & 0 & e_3-e_4 \\
  -e_3+e_5 & 0 & 0 & 0 & e_3-e_4 & 0 & e_3-e_4 & 0 & 0 & e_3-e_4 & 0 & e_3-e_4 \\
  e_2-e_5 & 0 & 0 & e_2-e_5 & 0 & 0 & e_2-e_5 & 0 & 0 & e_2-e_5 & 0 & e_2-e_5 \\
  0 & 0 & 0 & 0 & e_3-e_4 & 0 & e_3-e_4 & 0 & 0 & e_3-e_4 & 0 & e_3-e_4 \\
  0 & 0 & 0 & e_3-e_4 & 0 & 0 & e_3-e_4 & 0 & 0 & e_3-e_4 & 0 & e_3-e_4 \\
  0 & e_3-e_4 & 0 & 0 & e_3-e_4 & 0 & 0 & e_3-e_4 & 0 & 0 & e_3-e_4 & 0 \\
  0 & e_2-e_5 & 0 & e_2-e_5 & 0 & 0 & e_2-e_5 & 0 & 0 & e_2-e_5 & 0 & e_2-e_5 \\
\end{pmatrix},$$

$$D = \begin{pmatrix}
  2e_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & e_4+e_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & e_2+e_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & e_2+e_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 2e_5 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 2e_3 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & e_3+e_5 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & e_3+e_5 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_2+e_4 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_3+e_4 \\
\end{pmatrix} \quad (A.1)$$

where the coefficient of $e_i \equiv e_{1i}$ denotes the matrix obtained by attaching a rung between external legs 1 and $i$ of the trace basis [2.1], [2.2]. We have omitted for the sake of readability terms in these matrices that correspond to connecting other pairs of external legs. They may be obtained from these matrices via permutations of rows and columns.
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