SOME RESULTS IN GENERALIZED ŠERSTNEV SPACES

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Abstract. In this paper, we show that D-compactness in Generalized Šerstnev spaces implies D-boundedness and as in the classical case, a D-bounded and closed subset of a characteristic Generalized Šerstnev is not D-compact in general. Finally, in the finite dimensional Generalized Šerstnev spaces a subset is D-compact if and only if it is D-bounded and closed.

1. Introduction and Preliminaries

Probabilistic normed spaces (PN spaces henceforth) were introduced by Šerstnev in 1963 [3]. In the sequel, we adopt the new definition of Generalized Šerstnev PN spaces given in the paper by LaFuera-Guillén and Rodríguez [6]. The notations and concepts used are those of [2, 3, 4, 6] and [9].

In the sequel, the space of probability distribution functions is denoted by \( \Delta^+ \) and \( D^+ \subseteq \Delta^+ \) is defined as follows:

\[
D^+ = \{ F \in \Delta^+ : I^- F(+\infty) = 1 \}.
\]

The space \( \Delta^+ \) is partially ordered by the usual point-wise ordering of functions i.e., \( F \leq G \) if and only if \( F(x) \leq G(x) \) for all \( x \) in \( \mathbb{R} \). The maximal element for \( \Delta^+ \) in this order is \( \varepsilon_0 \), a distributive defined by

\[
\varepsilon_0 = \begin{cases} 
0 & \text{if } x \leq 0 \\
1 & \text{if } x > 0.
\end{cases}
\]

A triangle function is a binary operation on \( \Delta^+ \), namely a function \( \tau : \Delta^+ \times \Delta^+ \rightarrow \Delta^+ \) that is associative, commutative, nondecreasing and which has \( \varepsilon_0 \) as unit, viz.

For all \( F, G, H \in \Delta^+ \), we have

\[
\begin{align*}
\tau(\tau(F, G), H) &= \tau(F, \tau(G, H)), \\
\tau(F, G) &= \tau(G, F), \\
F \leq G &\implies \tau(F, H) \leq \tau(G, H), \\
\tau(F, \varepsilon_0) &= F.
\end{align*}
\]

Continuity of a triangle functions means continuity with respect to the topology of weak convergence in \( \Delta^+ \).

Typical continuous triangle functions are \( \tau_T(F, G)(x) = \sup_{s+t=x} T(F(s), G(t)) \) and \( \tau_{T^*}(F, G) = \inf_{s+t=x} T^*(F(s), G(t)) \).

Here, \( T \) is a continuous t-norm, i.e. a continuous binary operation on \([0, 1]\) that is commutative, associative, nondecreasing in each variable and has 1 as identity.

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and $T^*$ is a continuous t-conorm, namely a continuous binary operation on $[0, 1]$ which is related to the continuous t-norm $T$ through $T^*(x, y) = 1 - T(1 - x, 1 - y)$.

**Definition 1.1.** A probabilistic normed (briefly PN) space is a quadruple $(V, \nu, \tau, \tau^*)$, where $V$ is a real vector space, $\tau$ and $\tau^*$ are continuous triangle functions, and $\nu$ is a mapping from $V$ into $\Delta^+$ such that, for all $p, q \in V$, the following conditions hold:

1. $\nu_p = \varepsilon_0$ if and only if, $p = \theta$, where $\theta$ is the null vector in $V$;
2. $\nu_{-p} = \nu_p$, for each $p \in V$;
3. $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$, for all $p, q \in V$;
4. $\nu_p \leq \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)p})$, for all $\alpha \in [0, 1]$.

If the inequality (N4) is replaced by the equality

$$\nu_p = \tau_M(\nu_{\alpha p}, \nu_{(1-\alpha)p}),$$

then the PN space is called Šerstnev space and, as a consequence, a condition stronger than (N2) holds, namely

$$\nu_{\lambda p}(x) = \nu_p\left(\frac{x}{|\lambda|}\right),$$

for all $p \in V$, $\lambda \neq 0$ and $x \in \mathbb{R}$.

Following [1][3], for $0 < b \leq +\infty$, let $M_b$ be the set of $m$-transforms consist on all continuous and strictly increasing functions from $[0, b]$ onto $[0, +\infty]$. More generally, let $\tilde{M}$ be the set of non decreasing left-continuous functions $\phi : [0, +\infty] \rightarrow [0, +\infty]$, $\phi(0) = 0, \phi(+\infty) = +\infty$ and $\phi(x) > 0$, for $x > 0$. Then $M_b \subseteq \tilde{M}$ once $m$ is extended to $[0, +\infty]$ by $m(x) = +\infty$ for all $x \geq b$. Note that a function $\phi \in \tilde{M}$ is bijective if and only if $\phi \in M_{+\infty}$. Sometimes, the probabilistic norms $\nu$ and $\nu'$ of two given PN spaces satisfy $\nu' = \nu\phi$ for some $\phi \in M_{+\infty}$, non necessarily bijective. Let $\phi$ be the (unique) quasi-inverse of $\phi$ which is left-continuous. Recall from [2] page 49, that $\phi$ is defined by $\phi(0) = 0, \phi(+\infty) = +\infty$ and $\phi(t) = \sup\{u : \phi(u) < t\}$ for all $0 < t < +\infty$. It follows that $\phi(\phi(x)) \leq x$ and $\phi(\phi(y)) \leq y$ for all $x$ and $y$.

**Definition 1.2.** [3]. A quadruple $(V, \nu, \tau, \tau^*)$ satisfying the $\phi$-Šerstnev condition

$$\nu_{\lambda p}(x) = \nu_p(\hat{\phi}\left(\frac{\phi(x)}{|\lambda|}\right)),$$

for all $x \in \mathbb{R}^+, p \in V$ and $\lambda \in \mathbb{R} - \{0\}$ is called a $\phi$-Šerstnev PN space (Generalized Šerstnev spaces).

**Lemma 1.3.** If $|\alpha| \leq |\beta|$, then $\nu_{\beta p} \leq \nu_{\alpha p}$ for every $p \in V$.

**Definition 1.4.** Let $(V, \nu, \tau, \tau^*)$ be a PN space. For each $p \in V$ and $\lambda > 0$, the strong $\lambda$-neighborhood of $p$ is the set

$$N_p(\lambda) = \{q \in V : \nu_{p-q}(\lambda) > 1 - \lambda\},$$

and the strong neighborhood system for $V$ is the union $\bigcup_{p \in V} N_p$, where $N_p = \{N_p(\lambda) : \lambda > 0\}$.

The strong neighborhood system for $V$ determines a Hausdorff topology for $V$. 
**Definition 1.5.** Let \((V, \nu, \tau, \tau^*)\) be a PN space, a sequence \(\{p_n\}_n\) in \(V\) is said to be strongly convergent to \(p\) in \(V\) if for each \(\lambda > 0\), there exists a positive integer \(N\) such that \(\nu_{p_n - p_m}(\lambda) > 1 - \lambda\), whenever \(m, n > N\). A PN space \((V, \nu, \tau, \tau^*)\) is said to be strongly complete in the strong topology if and only if every strongly Cauchy sequence in \(V\) is strongly convergent to a point in \(V\).

**Definition 1.6.** \([4]\). Let \((V, \nu, \tau, \tau^*)\) be a PN space and \(\lambda > 0\). A nonempty set \(A\) in \((V, \nu, \tau, \tau^*)\) is said to be distributionally bounded, or simply D-bounded if either

(a) certainly bounded, if \(R_A(x_0) = 1\) for some \(x_0 \in (0, +\infty)\);

(b) perhaps bounded, if one has \(R_A(x) < 1\), for every \(x \in (0, +\infty)\) and \(l^- R_A(+\infty) = 1\);

(c) perhaps unbounded, if \(R_A(x_0) > 0\) for some \(x_0 \in (0, +\infty)\) and \(l^- R_A(+\infty) \in (0, 1)\);

(d) certainly unbounded, if \(l^- R_A(+\infty) = 0\) i.e., if \(R_A = \varepsilon_\infty\).

Moreover, \(A\) is said to be distributionally bounded, or simply D-bounded if either (a) or (b) holds, i.e. \(R_A \in D^+\). If \(R_A \in \Delta^+ - D^+\), \(A\) is called D-unbounded.

**Theorem 1.8.** \([5, 6]\). A subset \(A\) in the PN space \((V, \nu, \tau, \tau^*)\) is D-bounded if and only if there exists a d.f. \(G \in D^+\) such that \(\nu_p \geq G\) for every \(p \in A\).

**Definition 1.9.** \([5, 6]\). A subset \(A\) of TVS \(V\) is said to be topologically bounded if for every sequence \(\{\alpha_n\}\) of real numbers that converges to zero as \(n \to +\infty\) and for every \(\{p_n\}\) of elements of \(A\), one has \(\alpha_n p_n \to \theta\), in the strong topology.

**Theorem 1.10.** \([6]\). Suppose \((V, \nu, \tau, \tau^*)\) is a PN space, when it is endowed with the strong topology induced by the probabilistic norm \(\nu\). Then it is a topological vector space if and only if for every \(p \in V\) the map from \(\mathbb{R}\) into \(V\) defined by

\[
\alpha \mapsto \alpha p
\]

is continuous.

The PN space \((V, \nu, \tau, \tau^*)\) is called characteristic whenever \(\nu(V) \subseteq D^+\).

**Theorem 1.11.** \([5]\). Let \(\phi \in \tilde{M}\) such that \(\lim_{x \to -\infty} \phi(x) = \infty\). Then a \(\phi\)-\v{S}erstnev PN space \((V, \nu, \tau, \tau^*)\) is a TVS if and only if it is characteristic.

**Lemma 1.12.** \([5, 6]\). Let \(\phi \in \tilde{M}\) such that \(\lim_{x \to -\infty} \phi(x) = \infty\). Let \((V, \nu, \tau, \tau^*)\) be a characteristic \(\phi\)-\v{S}erstnev PN space. Then for a subset \(A\) of \(V\) the following are equivalent

(a) For every \(n \in \mathbb{N}\), there is a \(k \in \mathbb{N}\) such that \(A \subseteq kN_\phi(1/n)\).

(b) \(A\) is D-bounded.

(c) \(A\) is topologically bounded.
2. D-bounded and D-compact sets in φ-Šerstnev spaces

**Theorem 2.1.** Let \( \phi \in \tilde{M} \) such that \( \lim_{x \to \infty} \phi(x) = \infty \). Then in a characteristic φ-Šerstnev PN space \((V, \nu, \tau, \tau^*)\) if \( p_m \to p \) in \( V \) and \( A = \{p_m : m \in \mathbb{N}\} \), then \( A \) is a D-bounded subset of \( V \).

**Proof.** Let \( p_m \to p_0 \) and \( a_m \to 0 \). Then there exists \( m_0 \in \mathbb{N} \) such that for every \( m \geq m_0 \), \( 0 < a_m < 1 \), then

\[
\nu_{a_m p_m} \geq \tau(\nu_{a_m(p_m - p_0)}, \nu_{a_m p_0}) > \tau(\nu_{p_m - p_0}, \nu_{a_m p_0}) \to \tau(\varepsilon_0, \varepsilon_0) = \varepsilon_0,
\]

as \( m \) tend to infinity. \( \square \)

**Example 2.2.** The 4-tuple \((\mathbb{R}, \nu, \tau_\pi, \tau_M)\), where \( \nu : \mathbb{R} \to \Delta^+ \) is defined by

\[
\nu_p(x) = \begin{cases} 
0 & \text{if } x = 0 \\
\frac{a\pi}{x + |x|} & \text{if } 0 < x < +\infty \\
1 & \text{if } x = +\infty
\end{cases}
\]

\( a \in (0, 1) \), \( \nu_0 = \varepsilon_0 \) and \( \phi(x) = x \), is a φ-Šerstnev space (see [6]). The sequence \( \{\frac{1}{m}\} \) is convergent but \( A = \{\frac{1}{m} : m \in \mathbb{N}\} \) is not D-bounded set. The only D-bounded set in this space is \( \{0\} \).

**Definition 2.3.** The characteristic φ-Šerstnev PN space \((V, \nu, \tau, \tau^*)\) is said to be distributionally compact (simply D-compact) if every sequence \( \{p_m\}_m \) in \( V \) has a convergent subsequence \( \{p_{m_k}\}_k \). A subset \( A \) of a characteristic φ-Šerstnev PN space \((V, \nu, \tau, \tau^*)\) is said to be D-compact if every sequence \( \{p_m\}_m \) in \( A \) has a subsequence \( \{p_{m_k}\}_k \) convergent to a element \( p \in A \).

**Lemma 2.4.** A D-compact subset of a characteristic φ-Šerstnev PN space \((V, \nu, \tau, \tau^*)\), is D-bounded and closed.

**Proof.** Suppose that \( A \subseteq V \) is D-compact. From Lemma 1.12 it is enough show that \( A \) is topologically bounded. On the contrary there is a sequence \( \{p_m\}_m \subseteq A \) and a real sequence \( a_m \to 0 \) such that \( \{a_m p_m\} \) doesn’t tend to the origin in \( V \). Then there is an infinite set \( J \subseteq \mathbb{N} \) such that the sequence \( \{a_m p_m\}_{m \in J} \) lies in the complement of a neighborhood of the origin. Now \( \{p_m\} \) is a subset of D-compact set \( A \), so it has a convergent subsequence \( \{p_m\}_{m \in J'} \). From Lemmas 1.12 and 2.1 \( \{p_m\}_{m \in J'} \) is topologically bounded and so \( \{a_m p_m\}_{m \in J'} \) tends to origin which is a contradiction. The closedness of \( A \) is trivial. \( \square \)

As in the classical case, a D-bounded and closed subset of a characteristic φ-Šerstnev is not D-compact in general, as one can see from the next example.

**Example 2.5.** We consider quadruple \((\mathbb{Q}, \nu, \tau_\pi, \tau_M)\), where \( \pi(x, y) = xy \), for every \( x, y \in [0, 1] \) and probabilistic norm \( \nu_p(t) = \frac{t}{t + |p|} \). It is straightforward to check that \((\mathbb{Q}, \nu, \tau_\pi, \tau_M)\) is a characteristic φ-Šerstnev PN space. In this space, convergence of a sequence is equivalent to its convergence in \( \mathbb{R} \). We consider the subset \( A = [a, b] \cap \mathbb{Q} \) which \( a, b \in \mathbb{R} - \mathbb{Q} \). Since \( R_A(t) = \frac{t}{t + \max(|a|, |b|)} \), then \( A \) is D-bounded set and since \( A \) is closed in \( \mathbb{Q} \) classically, and so is closed in \((\mathbb{Q}, \nu, \tau_\pi, \tau_M)\). We
know $A$ is not classically compact in $\mathbb{Q}$, i.e. there exists a sequence in $\mathbb{Q}$ with no convergent subsequence in classical sense and so in $(\mathbb{Q}, \nu, \tau_\pi, \tau_M)$. Hence $A$ is not $D$-compact.

**Theorem 2.6.** Consider a finite dimensional characteristic $\phi$-$\check{\text{Serstnev}}$ PN space $(V, \nu, \tau, \tau^*)$ on real field $(\mathbb{R}, \nu', \tau', \tau'^*)$. Every subset $A$ of $V$ is $D$-compact if and only if $A$ is $D$-bounded and closed.

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