Fractional-order correlation imaging with thermal light

De-Zhong Cao,1 Qing-Chen Li, Xu-Cai Zhuang, Cheng Ren,1 Su-Heng Zhang,2 and Xin-Bing Song†

1Department of Physics, Yantai University, Yantai 264005, Shandong Province, China
2College of Physics Science & Technology, Hebei University, Baoding 071002, China
3Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel

In thermal light ghost imaging, the correlation orders were usually positive integers in previous studies. In this paper, we examine the fractional-order moments, whose correlation order are fractional numbers, between the bucket and reference relation signals in the ghost imaging system. The crucial step in theory is to determine the precise relation between the bucket signals and reference signals. We deduce the joint probability density function between the bucket and reference signals by regarding the reference signals as an array of independent stochastic variables. In calculating the fractional-order moments, the correlation order for the reference signals must be positive to avoid infinity. While the correlation order for the bucket signals can be positive or negative numbers. Negative (positive) ghost images are obtained with negative (positive) orders of the bucket signals. The visibility degree and signal-to-noise ratio of ghost images from the fractional-order moments are analysed. The experimental results and numerical simulations meet our analysis based on probability theory.

I. INTRODUCTION

Measuring high-order intensity correlation function is the key tool to reconstruct the object information in ghost imaging with thermal light (GITL) [1, 2]. The correlation orders were natural numbers in all present scenarios. The most favorite order was 2, and the second-order correlation functions in GITL were widely investigated both in theory and in practice. Since the source plays a role of a conjugate mirror [7], ghost imaging with thermal light can be implemented without lenses [8, 9]. The ghost images in computational GITL can be formed with bucket signals measured by only one single pixel-detector [10]. Now GITL were applied in remote sensing [11], lidar [12], imaging encryption [13], and biomedical imaging [14]. Multi-color GITL has been investigated to discriminate wavelength information [15] and to reconstruct RGB information of the color object [16]. Higher-order correlation functions were used to enhance the visibility degree [17] and improve contrast-to-noise ratios [18, 19] of the ghost images. Third-order GITL were also applied to construct two ghost images [20]. Recent investigations showed that ghost images can be formed in first-order correlation measurements with thermal light [21].

Nevertheless, the orders of natural numbers are relative rough parameters in application. Besides integer-order moments, the fractional-order moments made great success in such processes as truncated Lévy flights [22] or atmospheric laser scintillations [23]. In this paper, we report a GITL experiment in which the fractional-order moments of the stochastic bucket and reference intensity signals are calculated. That is, the object information is reconstructed by measuring the fractional-order moments of the bucket and reference signals. In calculating the fractional-order moments, the positive orders of the reference signals are set to avoid infinity, while the orders of the bucket signals can be positive or negative numbers. We find that negative (positive) ghost images can be obtained with negative (positive) orders of the bucket signals.

In theory, an elaborate analysis based on probability theory is provided. The crucial step is to determine the precise relation between the bucket signals and reference signals. The reference signals can be regarded as an array of independent stochastic variables, each of which meets negative exponential distribution [13]. So the probability density function of the bucket signals, as well as the joint probability function between the bucket and reference signals, can be obtained since the bucket signals can be regarded as a linear sum of the reference signals. Also the visibility degrees and signal-to-noise ratios of the ghost images are analyzed according to our theory. The experimental results and numerical simulations are in good agreement with our theory.

Our paper is organised as follows. Section II gives the theory of the joint probability density function between the bucket and reference signals. Section III shows the theoretical analysis and experimental results of the fractional-order moments for binary objects. Section IV shows the numerical simulations of the fractional-order moments for a complicated object. The conclusions and discussions are shown in Sec. V.

II. JOINT PROBABILITY DENSITY FUNCTION BETWEEN THE BUCKET AND REFERENCE SIGNALS

Figure 1 shows the sketch of our experimental setup of GITL. The two sets of correlated random speckles are the thermal light fields in the object plane and reference plane. In the object arm, the bucket detector D_B converts the total optical intensity out of the object, depicted by the letter “A”, into bucket signal I_B. While the reference detector D_f scans and converts local intensity into reference signal I_r. The bucket detector and the reference detector are two charge coupled devices (CCDs) in experiment. The correlator, which is a computer in-
deed, is used to measure fractional-order moment functions \( \langle P_B^m I_r^n \rangle \), where \( \mu \) and \( \nu \) are fractional numbers. The fraction-order moment function retrieves the object information which is shown in the screen.

![Image 1](image.png)

**FIG. 1:** Sketch of the experimental setup. The random speckles are two identical pseudo-thermal light beams. The upper one is the object beam and the lower is the reference beam. \( D_B \) is the bucket detector, and the pixel detector \( D_I \) registers the reference beam. The fractional moments \( \langle P_B^m I_r^n \rangle \) is performed by the correlator.

The resolving power of GITL is inversely proportional to the coherence length of the optical fields in the object and reference planes. Throughout the paper we consider the case of perfect GITL that the GITL system has the ability to completely reconstruct the object information. For simplicity of mathematics, as shown in Fig. 1, we synchronously divide the object and reference planes into such \( n \) small units that (i) the details of the object are maintained, and (ii) the thermal fields in all the units are statistically independent from each other.

From the viewpoint of probability theory, the reference signals, i.e., the thermal light intensities in all the units, can be regarded as a set of stochastic variables \( I_r = \{I_1, I_2, \ldots, I_n\} \). Each element of the reference signal \( I_r \) \( (i = 1, 2, \ldots, n) \) meets the negative exponential probability distribution \( p(I_r) = \frac{1}{I_0} e^{-\frac{I}{I_0}} \), where the constant \( I_0 \) represents the intensity average. The fractional moment of the reference signal is \( \langle I_r^{\nu} \rangle = \Gamma(1 + \nu) I_0^{\nu} \) for any fractional number \( \nu \), where the Gamma function is \( \Gamma(1 + \nu) = \int_0^\infty t^\nu \exp(-t) dt \). Due to the fact that the largest probability of the reference signal is \( p(0) \) that \( p(0) \geq p(I_r) \), we usually set \( \nu \) positive \( (\nu > 0) \) in experiment to avoid infinity.

The bucket detector \( D_B \), which collects the variables scattered from the object, output the bucket signals

\[
I_B = I_r T = \sum_{i=1}^n I_i t_i, \quad (1)
\]

where \( 0 \leq t_i \leq 1 \) is the transmittance or reflectivity of the \( i \)-th unit in the object \( T = \{t_1, t_2, \ldots, t_n\} \prime \) (the prime denotes matrix transposition). Obviously, the bucket signal \( I_B \) in Eq. (1) is a linear sum of \( n \) independent variables, each of which fulfills the negative exponential probability density function with weight \( t_i \). The probability density function for the variable from the \( i \)-th object unit becomes

\[
p(I_i/t_i) = \frac{1}{t_i} e^{-\frac{I_i}{t_i}}, \quad \text{and its Laplace transformation is}
\]

\[
\tilde{p}_i(s) = \int_0^\infty \frac{p(I_i/t_i)}{t_i} e^{-st} dI = \frac{1}{1 + s \times I_0 t_i}. \quad (2)
\]

We can see that each variable from the object unit also fulfill the negative exponential distribution, with a modified average \( I_0 t_i \). After some algebra, the probability density function of the bucket signal is calculated out

\[
P_B(I_B) = \mathcal{L}^{-1}_{s \rightarrow I_B} \left[ \prod_{i=1}^n \tilde{p}_i(s) \right], \quad (3)
\]

where \( \mathcal{L}^{-1} \) denotes the inverse Laplace transformation from variable \( s \) to \( I_B \).

We note that \( P_B(I_B) \) in Eq. (3) does not completely depend on the object structure. The probability density function \( P_B(I_B) \) can be calculated out as long as the histogram of the object is known. A remarkable feature of the bucket probability density function is that \( P_B(0) = 0 \), if the object is composed of at least two nonzero units. So the fractional moment of the bucket signal \( \langle I_B^{\nu} \rangle = \int_0^\infty I_B^{\nu} P_B(I_B) dI_B \), where \( \mu \) is a fractional number, is tenable for both \( \mu > 0 \) and \( \mu < 0 \).

Since the reference signal \( I_r \) is one constituent of the bucket signal \( I_B \), we combine Eqs. (1) and (3), and write the joint probability density function between the bucket signal \( I_B \) and the reference signal \( I_i \) as

\[
P_2(I_B, I_i) = P_B'(I_B - I_i t_i) \times p(I_i), \quad (4)
\]

where \( P_B'(x) = \mathcal{L}^{-1}_{s \rightarrow x} \left[ \prod_{j=1,j\neq i}^n \tilde{p}_j(s) \right], \quad \text{and} \quad I_B \geq I_i t_i.
\]

The object information can be reconstructed by calculating out the fractional-order moments \( \langle I_B^{\nu} I_r^\mu \rangle \) in a computer in experiment. According to the probability theory, the function of the fractional-order moment is

\[
\langle I_B^{\nu} I_r^\mu \rangle = \int_0^\infty dI_B \int_0^\infty dI_i \times I_B^{\nu} I_i^\mu P_2(I_B, I_i). \quad (5)
\]

As mentioned above, the fractional numbers in Eq. (5) meet \( \nu > 0 \) and \( \mu \neq 0 \). What we should pay more attention to is the fractional-order moments \( \langle I_B^{\nu} I_r^\mu \rangle_{t_i = 0} \) and \( \langle I_B^{\nu} I_r^\mu \rangle_{t_i = 1} \) for \( t_i = 0 \) and \( 1 \), respectively. The former defines the background of the ghost images \( \langle I_B^{\nu} I_r^\mu \rangle_{t_i = 0} = \langle I_B^{\nu} \rangle \langle I_r^\mu \rangle \). However, the latter meets

\[
\langle I_B^{\nu} I_r^\mu \rangle_{t_i = 1} > \langle I_B^{\nu} I_r^\mu \rangle_{t_i = 0}, \quad (\mu > 0, \nu > 0)
\]

\[
\langle I_B^{\nu} I_r^\mu \rangle_{t_i = 1} < \langle I_B^{\nu} I_r^\mu \rangle_{t_i = 0}, \quad (\mu < 0, \nu > 0)
\]

It is clear that the ghost image is above its background when \( \mu > 0 \), and is below its background when \( \mu < 0 \). That is, negative ghost images can be obtained for negative fractional order \( \mu \). Consequently, the visibility degree and peak SNR of the ghost images in fractional-order moments are defined as

\[
V = \frac{\langle (I_B^{\nu} I_r^\mu)_{t_i = 1} - (I_B^{\nu} I_r^\mu)_{t_i = 0}\rangle^2}{\langle (I_B^{\nu} I_r^\mu)_{t_i = 1} \rangle^2 - \langle (I_B^{\nu} I_r^\mu)_{t_i = 0} \rangle^2},
\]

\[
R_p = \sqrt{\frac{\langle (I_B^{\nu} I_r^\mu)_{t_i = 1} \rangle^2 - \langle (I_B^{\nu} I_r^\mu)_{t_i = 0} \rangle^2}{\langle (I_B^{\nu} I_r^\mu)_{t_i = 1} \rangle^2 + \langle (I_B^{\nu} I_r^\mu)_{t_i = 0} \rangle^2}} \quad (7)
\]

where \( N \) is the number of sampling in experiment.

In the following, we show the experimental results of the fractional-order moments for binary objects in ghost imaging, and then show the numerical simulations of the fractional moments for a complicated object in ghost imaging.
III. EXPERIMENT RESULTS OF BINARY OBJECTS

To explicitly exploit the probability theory method and to illustrate the characteristics of ghost images from fractional-order moments in GITL, we consider the case of binary objects that the values of the object units are \( t_i = 0 \) or 1. From Eq. (3), the bucket signals meet Gamma distribution with probability density function as

\[
P_B(I_B) = \frac{I_B^{m-1}}{(m-1)!I_0^m} \exp[-\frac{I_B}{I_0}],
\]

where \( m \) is the number of the effective object units (whose values are \( t_i = 1 \)). The subplot in Fig. 1 shows probability density functions of Eq. (8) for \( p(I_r) \) = 1 with solid black line, \( P_B(I_B) \) = 2 with dashed blue line, and \( P_B(I_B) \) = 5 with dotted red line, respectively. We can find that \( p(0) \geq p(I_r) \) for single-unit object and \( P_B(0) \leq P_B(I_B) \) for complex object. The joint probability density function between the reference and bucket signals is

\[
P_2(I_B, I_r) = \begin{cases} 
\frac{(I_r-t_i)^{m-2}}{(m-2)!I_0^{m-2}} \exp[-\frac{I_r}{I_0}], (t_i = 1) \\
\frac{I_0^{m-1}}{(m-1)!I_0^m} \exp[-\frac{I_r}{I_0}], (t_i = 0)
\end{cases}
\]

where \( I_i \leq I_B \) must be considered.

The ensemble average of the reference signals is \( \langle I_r \rangle = I_0 \), and the ensemble average of the bucket signals is \( \langle I_B \rangle = mI_0 \). The fractional-order moments is calculated out

\[
\langle I_B^{\mu+\nu} \rangle_0 = \frac{\Gamma(m+\mu)\Gamma(1+\nu)}{\Gamma(m)} I_0^{\mu+\nu},
\]

\[
\langle I_B^{\mu+\nu} \rangle_1 = \frac{\Gamma(m+\mu+\nu)\Gamma(1+\nu)}{\Gamma(m+\nu)} I_0^{\mu+\nu},
\]

for \( t_i = 0 \) and \( t_i = 1 \), respectively. The visibility degree and peak SNR of the ghost images are

\[
V = \frac{\left| \Gamma(m+\mu+\nu)\Gamma(m) - \Gamma(m+\mu)\Gamma(m+\nu) \right|}{\Gamma(m+\mu+\nu)\Gamma(m+\mu+\nu)}
\]

\[
R_p = \frac{\left| \Gamma(m+\mu+\nu)\Gamma(m+\nu) - \Gamma(m)\Gamma(m+\mu+\nu) \right|}{\sqrt{\frac{\Gamma(m+\mu+\nu+2)\Gamma(m+\nu+2)}{N^2\Gamma(m+\nu+2)} - \frac{\Gamma(m+\mu+\nu+2)\Gamma(m+\nu+2)}{N^2\Gamma(m+\nu+2)}}}
\]

respectively for binary objects.

In experiment, the pseudo-thermal light source is obtained by projecting a laser beam (laser diode: \( \lambda = 650nm \)) onto a rotating ground glass plate [17] (which is not shown in Fig. 1). We set the diameter of the laser spot on the glass plate \( d = 4.50mm \), and the distance between the object and the glass plate \( L = 8.50cm \). The coherent length of the random laser speckles in the object plane is about \( 1.22\frac{\lambda L}{d} \approx 14.98\mu m \), which is smaller than the pixel pitch of the CCD 20.0\mu m. This ensures well-performed GITL, and fulfills the two assumptions proposed above.

Figure 2 shows our experimental results of measuring the fractional-order moments in GITL with binary objects over \( N = 120,000 \) samplings. The object is depicted in Fig. 3(a). Figures 3(b-f) show the normalized fractional-order moments \( \langle I_r^{\mu+\nu} \rangle_0 \). The ghost images from the fractional-order-moments are depicted in Figs. 2(b), 2(c), 2(d), 2(e), 2(f), and 2(g), for \( \mu = -2.7183, -1.414, -0.618, 0.618, 1.414, 2.7183 \) respectively. The parameter \( \nu = 0.5 \) is fixed in all the ghost images. We can see that the negative ghost images are obtained for negative orders \( \mu < 0 \) in Figs. 2(b), 2(c), and 2(d). While the positive ghost images are obtained for positive orders \( \mu > 0 \) in Figs. 2(e), 2(f), and 2(g). We find that the greater the absolute order \( |\mu| \), the higher visibility degree of the ghost image becomes. In general, the negative ghost images has better visibility than the positive ones. But the behaviors of the signal-to-noise ratio (SNR) differs greatly from that of the visibility degree. The SNRs decrease when \( |\mu| \) increases. The Visibility degrees are \( V = 6.77 \times 10^{-3}, 3.45 \times 10^{-3}, 1.49 \times 10^{-3}, 1.47 \times 10^{-3}, 3.32 \times 10^{-3}, 6.26 \times 10^{-3} \), and the peak SNRs (defined in Eq. (12)) are \( R_p = 2.118, 2.126, 2.130, 2.132, 2.133, 2.131 \) for Figs. 2(b), 2(c), 2(d), 2(e), 2(f), and 2(g), respectively.

The orders \( \mu \) and \( \nu \) determine the quality of ghost images. Therefore, fractional orders provide more detailed parameters to exquisitely adjust or modify the quality of the ghost images in practice than integer orders. Below is our theory of the fractional-order moments in GITL.

The visibility \( V \) in Eq. (12) and the relative peak SNR \( R_p/\sqrt{N} \) in Eq. (13) versus the fractional orders are plotted in Fig. 3. The number of effective object unit is \( m = 20 \) in Figs. 3(a) and 3(b), and \( m = 30 \) in Figs. 3(c) and 3(d), respectively. We can see from figs. 3(a) and 3(c) that the visibility degree increases as the absolute values of the correlation orders \( \mu \) and \( \nu \) increase. An evident feature is that the visibility degree of the negative image \( \mu < 0 \) increase faster than that of the positive image \( \mu > 0 \). Also, the greater number of effective object unit can degrade the visibility degree.

The relative peak SNR is plotted in Figs. 3(b) and 3(d). The peak SNR first increases and then decreases as the correlation orders \( |\mu| \) and \( \nu \) increase. We can also find that the maximum value of peak SNR of the neg-
The visibility degree is greater than that of the positive image ($\mu > 0$). We can conclude that the image quality of the negative ghost images can be better than that of the positive ghost images for the opposite fractional orders. Furthermore, the visibility degree and SNR of the ghost image vary continuously with fractional (continuous) orders. So we can adjust the visibility degree and SNR at will by choosing appropriate experimental parameters.

![Figure 3](image3.png)  
**FIG. 3:** Visibility degrees $V$ (a,c) and relative peak SNR $R_\mu/\sqrt{N}$ (b,d) of binary objects. The object unit number (for $t_x = 1$) is $m = 20$ in (a,b), and $m = 30$ in (c,d).

### IV. NUMERICAL SIMULATIONS OF COMPLEX OBJECTS

So far we have presented the theory of the fractional ghost imaging with thermal light, and have illustrated ghost images in fractional-order moments in experiment with binary objects. Our proposal of course is suitable for more complicated objects. In this section we show the numerical simulations of the fractional-order moments in GITL with a more complicated object.

The object is an image of the part of cameraman (size: $64 \times 64$ pixels). Figure 4 shows the results of numerical simulations by calculating the normalized fractional-order moments $\langle I^\mu \rangle / \langle I \rangle$. The reference signals are the stochastic numbers created by a computer. The number of sampling is $N = 200,000$. Other parameters are the same as in Fig. 2. We again obtain negative ghost images for $\mu < 0$ in Figs. 4(a), 4(b), and 4(c). The positive ghost images for $\mu > 0$ in Figs. 4(d), 4(e), and 4(f) are also obtained. The visibility degrees are $V = 7.84 \times 10^{-4}, 3.97 \times 10^{-4}, 1.76 \times 10^{-4}, 1.76 \times 10^{-4}, 4.06 \times 10^{-4}, 7.72 \times 10^{-4}$, for Figs. 4(a), 4(b), 4(c), 4(d), 4(e), and 4(f), respectively. Again the visibility increases when the absolute value of the order increases. The corresponding peak SNRs for all the ghost images are $R_\mu = 5.27, 5.57, 5.44, 6.95, 5.65,$ and $6.18$, respectively. The peak SNRs of the ghost images vary with the fraction orders.

![Figure 4](image4.png)  
**FIG. 4:** Numerical simulations of ghost images from fractional moments with a gray object ($64 \times 64$ pixels). The number of sampling is $N = 200,000$. Other parameters are the same in Fig. 2.

### V. CONCLUSION

In conclusion, we have investigated the fractional-order moments in GITL experiment. The reference signals have been regarded as an array of stochastic variables, while the bucket signals have been regarded as a linear sum of these stochastic variables. We then have deduced the joint probability density function between the bucket signals and reference signals according probability theory. The object information can be retrieved through fractional-order moments. We have found that negative (positive) ghost images can be obtained if the orders of the bucket signals are smaller (greater) than zero. Ghost imaging with fractional-order moments has been implemented in experiments with a binary object and in numerical simulations with a more complicated object. The visibility degrees and peak SNRs of the ghost images vary with the correlation fractional-orders. So we have the chance to carefully adjust the image quality by choosing appropriate fractional orders in GITL. Our technique can provide abundant ghost images, and has potential to work in complex environments.

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