Towards understanding the probability of $0^+$ ground states in even-even many-body systems

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Abstract

For single-$j$ shells with $j = \frac{7}{2}, \frac{9}{2}$ and $\frac{11}{2}$, we relate the large probability of $I^+$ ground states to the largest (smallest) coefficients $\alpha^J_{I(v)\beta} = \langle n\nu\beta I | A^J \cdot A^I | n\nu\beta I \rangle$, where $n$ is the particle number, $\nu$ is the seniority, $\beta$ is an additional quantum number, and $I$ is the angular momentum of the state. Interesting regularities of the probabilities of $I^+$ ground states are noticed and discussed for 4-particle systems. Several counter examples of the $0^+$ ground state (0GS) predominance are noticed for the first time.

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The shell structure and pairing are “universals” which play important roles in many-body systems such as metal clusters, atoms, and nuclei etc. Therefore, a study which relates intrinsic properties of the shell model with the origin of the pairing phenomenon in many-body systems is extremely important. Recent work by Johnson et al [1–3] was the first effort along these lines. In that work, the low-lying spectra of many-body systems with even number of particles were examined using random two-body interactions (TBRE), and its results showed a preponderance of $I^\pi = 0^+$ ground states (0GS). This phenomenon also occurs in $sd$-boson systems, and was referred to as “a robust property” of the valence space [1–3]. All these studies demonstrate that the features of pairing arise from a very large ensemble of two-body interactions and seem to be independent of the specific character of the force. These observations, however, seem to be contrary to the traditional assumption in nuclear physics, whereby the 0GS dominance in even-even nuclei is a reflection of a strong pairing interaction associated with a strong short-range attraction between identical nucleons.

It is therefore very important and interesting to study why the 0GS are favored in even-even nuclei when using the TBRE [1–3]. There have been several efforts to understand this phenomenon. In Ref. [1] an analysis was carried out on systems of identical particles occupying orbits with $j = \frac{1}{2}, \frac{3}{2},$ and $\frac{5}{2}$, and it was indicated that there is a correspondence between a larger distribution width of $0^+$ states and the 0GS dominance. In another paper, it was proposed [2] that for a system of interacting bosons the probability that the ground state has a certain value of the angular momentum is not really fixed by the full distribution of eigenvalues, but rather by that of the lowest one. In [3], Muhall et al. discussed the 0GS dominance within single-$j$ shells by using geometric chaoticity and uniformly changed random interactions. In [4], Kusnesov discussed the $sp$-boson case. In [5], Kaplan et al. studied the correlation between the eigenvalues and spins of the states in some simple cases.

All these studies [1–10] on the 0GS dominance are interesting and important, and have potentially impacted our understanding on the origin of one of the most characteristic features of nuclear spectra. An essential understanding of the 0GS predominance, however, has not yet been achieved.

Towards that goal, we present in this Rapid Communication a new view on the origin of the 0GS dominance. We are able to achieve this new understanding because we focus on the simplest non-trivial problem, that of identical nucleons in a single-$j$ shell. This problem has the advantage that it is simply solvable, thus permitting useful insight to be obtained which can hopefully then be applied to more general problems. In the process, some previously unrecognized features, such as counter examples of the 0GS predominance, and regularities of probability of $I^+$ ground states, have been discerned.

We first discuss the properties of linear combinations of random numbers. Let $G_J$ be a set of random numbers with a distribution function

$$
\rho(G_J) = \frac{1}{\sqrt{\pi}} exp(-G_J^2), \quad J = 0, 2, \cdots, 2(j - 1), \quad (0.1)
$$

and $F(k)$ be a set of linear combinations of $G_J$: 

\[ F(k) = \sum_j \alpha^j_k G_j, \quad k = 1, 2, \cdots N, \quad (0.2) \]

where \( j \) labels the single-\( j \) shell, and \( N \) is the total number of states (to be introduced later). It can be shown that distribution functions of random \( F(k) \) are

\[ \rho(F(k)) = \frac{1}{\sqrt{\pi}} \exp(- (F(k))^2 / g_k^2), \quad g_k^2 = \sum_j (\alpha^j_k)^2. \quad (0.3) \]

If \( \alpha^j_m \) in Eq. (0.2) is the largest (or the smallest) among all the \( \alpha^j_k \) \( (k = 1, \cdots N) \), the probability of \( F(m) \) being both the smallest and the largest number is large. To show this, let us look at

\[ \mathcal{F}(k) = F(k) - F(m) = c^j_k G_j + \left( \sum_{j \neq j'} (\alpha^j_k - \alpha^j_m) G_j \right) = c^j_k G_j + \mathcal{F}'(k), \quad (0.4) \]

where \( k \neq m \), and \( c^j_k = (\alpha^j_k - \alpha^j_m) \), \( \mathcal{F}'(k) = \sum_{j \neq j'} (\alpha^j_k - \alpha^j_m) G_j \). The right hand side of Eq. (0.4) has two terms, both of which are Gaussian type random numbers. The coefficient \( c^j_k \) is negative or positive, and thus effectively produces a shift in \( \mathcal{F}(k) \), as is evident from Eq. (0.4). Therefore, all of the functions \( \mathcal{F}(k) \) have large probabilities to be both negative and positive, depending on the sign of the shift, i.e., \( F(m) \) has a large probability to be both the smallest and the largest.

Roughly speaking, one may use \( d = \frac{\alpha^j_m}{c^j_k} \), which is the smallest among all the \( |c^j_k| (k \neq m) \), as an estimate of the shifts, and use the maximum of \( D = \sqrt{\sum_{j \neq j'} (\alpha^j_k - \alpha^j_m)^2} \) as an estimate of the distribution widths of the functions \( \mathcal{F}'(k) \). The probability of \( F(m) \) being the smallest (largest) is very large when \( d \) is comparably large (\( d \sim D \)). This occurs when there are a very few \((3 \sim 6)\) parameters in the Hamiltonian. The probability of \( F(m) \) being the smallest (largest) is roughly determined by \( g_k \), the widths of the quantities \( F(k) \), when \( d << D \). The states with large eigenvalue widths have large probabilities to be both the smallest and the largest. This situation occurs if there are many (e.g., more than 20) TBRE parameters. In Ref. [1], the 0GS probability are determined by the total widths of the energy eigenvalues because there are 30 parametrized T=1 interactions.

If there are two or more coefficients \( \alpha^j_m \) \( (j' = 0, 2, \cdots 2j - 1) \) which are the largest or smallest for different functions \( F(k) \), the probability of finding \( F(m) \) as both the smallest and the largest increases.

Keeping the above ideas in mind, we now return to the shell model calculation for single-\( j \) shells. The Hamiltonian for a single-\( j \) shell is

\[ H = \sum_j \sqrt{2j + 1} G^j J^j \times \tilde{A}^j = \left( j^2 J |V| j^2 J \right). \]

The parameters \( G^j \) are distributed according to Eq. (0.1).
Consider the simplest example, a $j = 7/2$ shell with 4 particles, where all the states are labeled by their total angular momenta $I$ and their seniority quantum numbers $(v)$. The eigenvalues $E_{I(v)}$ are as follows \cite{11}:

$$E_{0(0)} = \frac{3}{2} G_0 + \frac{5}{6} G_2 + \frac{3}{2} G_4 + \frac{13}{6} G_6,$$

$$E_{2(2)} = G_2 + \frac{42}{11} G_4 + \frac{13}{11} G_6,$$

$$E_{4(4)} = \frac{7}{3} G_2 + 1 \cdot G_4 + \frac{8}{3} G_6,$$

$$E_{6(2)} = \frac{1}{2} G_0 + \frac{5}{6} G_2 + \frac{3}{2} G_4 + \frac{19}{6} G_6,$$

where bold font is used for the largest and italic for the smallest amplitudes in an expansion in terms of $G_J$,

$$E_{I(v)} = \sum_J \alpha^J_{I(v)} G_J. \hspace{1cm} (0.6)$$

Using Eq. $(0.5)$, we can predict the probability of $I^+$ ground states without carrying out energy calculations with the TBRE. For example, the 0GS probability is determined by

$$\int dG_0 \int dG_2 \int dG_4 \int dG_6 \int dE_{0(0)} \int dE_{2(2)} \cdots \int dE_{8(4)} \delta \left( E_{0(0)} - \sum_J \alpha^J_{0(0)} G_J \right) \cdots \delta \left( E_{8(4)} - \sum_J \alpha^J_{8(4)} G_J \right) \rho(G_0) \rho(G_2) \rho(G_4) \rho(G_6). \hspace{1cm} (0.7)$$

The probabilities of finding $I$ as the spin of ground states for a 4-particle system with $j = 7/2$ are listed in Table I. The row “test” corresponds to results obtained by using Eq. $(0.5)$ and 1000 sets of the TBRE. The row “cal.” corresponds to the probabilities predicted by an integral for each $I^+$ state similar to Eq. $(0.7)$ for the $0^+$ state. The probabilities calculated using the TBRE and those predicted by using integrals like Eq. $(0.7)$ are all consistent with one another.

Using Eq. $(0.5)$ and Eq. $(0.3)$, $g_{I(v)}$—the distribution width of $E_{I(v)}$—is obtained and listed in the last row of Table I. It is found that there is no correspondence between the width $g_{I(v)}$ and the probability of finding $I^+ (v)$ to be the ground state.

Because there are only 4 parameters in Eq. $(0.3)$, we can use the shift argument to predict which states have large probabilities to be the ground. They are the states with the largest (smallest) $\alpha^J_{I(v)}$: $I(v)=0(0), 2(4), 4(4), 8(4)$.

Our success in explaining which states can have large probabilities to be the ground state for the $j = 7/2$ shell encouraged us to go to higher $j$ shells. The study of 4 particles in the $j = 9/2$ shell is very interesting because the 0GS is predominant in this case (See Fig. 1). This is therefore a good case in which to check whether the above assumptions continue to apply.
Unfortunately, the eigenstates of 4 particles in this shell are complicated by mixings between states with the same and different seniorities. In such a case there is no simple relation between the eigenvalues and the two-body interaction parameters, \( G_J \).

To simplify the analysis for the \( j = \frac{9}{2} \) case, we assume that the mixings are negligible, and
\[
\langle \nu \beta I | \sum_{J} \sqrt{2J + 1} \left[ A^I_d \times \tilde{A}^J \right]^0 \nu \beta I \rangle
\]
are the ”eigenvalues”. This assumption is supported by the fact that in the basis \( | \nu \beta I \rangle \) mixings are observed to be small for \( I > 0 \) states in even-\( N \) systems in the \( j = \frac{9}{2} \) shell. It is also supported by a recent claim \([2]\) that a many-body system picks out seniority as an approximate quantum number, even though it is not obviously implicit in the Hamiltonian. If we could take those mixings into account in a simple way, those off-diagonal matrix elements would presumably pull down the lower ”eigenstates” even lower, and push up the higher ones even higher. For \( I = 0 \) states, the off-diagonal matrix elements are comparable in magnitude to the diagonal matrix elements, and the pulling-down and pushing-up effects should be more important than for \( I \neq 0 \) states. Therefore, this assumption leads to an underestimate of the 0GS probability.

The coefficients \( \alpha_J^I \) for each \( G_J \) in the “eigenvalues” are listed in Table II. There are three largest \( \alpha_J^I \) \((J = 0, 4, 6)\) in the \( I = 0 \) states. Therefore, the 0GS probability must be very large. In addition, there are one largest \( \alpha_J^I \) \((J = 2)\) and one smallest \( \alpha_J^I \) \((J = 6)\) in one of the 4^+ states. Finally, there are one largest \( \alpha_J^I \) \((J = 8)\) and two smallest \( \alpha_J^I \) \((J = 2, 4)\) in the highest angular momentum state \((I = 12)\). This indicates large probabilities of 4^+ ground states and 12^+ ground states. In one of the 2^+ states \((v = 2)\) the coefficient of \( G_6 \) is very near to the smallest. Thus, there should be a sizable percentage of 2^+ ground states as well.

Shell model calculations based on the TBRE are consistent with the above discussion: the probability of 0^+ states to be the ground state is 66.4\%, that of 4^+ is 12.2\% and that of 12^+ is 17.4\%. The probability of finding 2^+ states as the ground state is 3.4\%. There are very small probabilities \((0.6\% in total)\) for all other \( I^+ \) ground states. The predicted 0GS probability obtained using an integral similar to Eq. (D.7), where no mixing is assumed \((which is exact for j = \frac{9}{2})\), is 45.11\%. As expected, this is smaller than the probability \((66.4\%)\) found when the mixings are taken into account. We note without further details that the above argument is also applicable to 4 and 6 particles in the \( j = \frac{11}{2} \) shell, and to the general features of odd-A fermion system in small single-\( j \) shells and \( sd \)-boson systems \([12]\).

We have also carried out several other sets of calculations, both for larger single-\( j \) shells and in some cases for two-\( j \) shells. Some of the principal outcomes are:

1) The probability of ground states with odd angular momenta is much smaller than that of their neighboring even angular momenta even if the number of states with the same angular momentum \( I \) found in \( j^n \) configuration is comparably large.

2) The unique \( I_{max} \) state has a large probability to be the ground state, although this probability decreases with \( j \). According to ref. \([3]\), by using random interactions which distribute uniformly between \(-1\) and \(1\), the probability of \( I_{max}^+ \) ground states staggers rapidly
and becomes 0 for several single $j$.

3) The $I = 2^+$ and $4^+$ states have large probabilities to be the ground state. This indicates that small and even angular momentum states are favored as the ground state in single-$j$ shells.

4) The 0GS predominance obtained using the TBRE is not a “rule” without exceptions. There are counter examples: $j = \frac{7}{2}$ and $j = \frac{13}{2}$ shells with 4 particles (refer to Fig. 1); two-$j$ shell ($j = \frac{7}{2}$ plus $\frac{5}{2}$) with 4 particles, where the probability of the 0GS is 24% while that of the $2^+$ GS is 33%.

A summary of the above regularities is shown in Fig. 1, from which we see that when the 0GS probability is a maximum, the probability of $2^+$ ground states is a minimum, and vice versa. Fig. 1 indicates that the fluctuation of the 0GS probability decreases with $j$, the 0GS probability saturates around 35%, and the GS probabilities of $I = 2, 4$ and $I_{\text{max}}$ states decrease and converge around 9%. These regularities are worthy of further study.

To summarize, we have presented in this Rapid Communication an interpretation of the 0GS probability in terms of the largest and smallest coefficients $\alpha_J^{I(v)}$ for several single-$j$ shells. The interpretation applies when seniority is either a conserved or an approximately conserved quantum number. Further study showed that this method is not only applicable to even number of fermions in a small single-$j$ shell, but also is a good benchmark to explain the IGS probability of odd-$A$ fermions in a small single-$j$ shell and general features of sd-boson systems [12]. Therefore, for the first time, the IGS probabilities of a variety of systems can be discussed on the same footing using the method proposed in this paper.

The 0GS probabilities have been calculated using integrals like Eq. (0.7), the predicted probabilities being reasonably consistent with those obtained using the TBRE. We showed that the width of the distribution of $0^+$ eigenvalues is not a manifestation of the 0GS preponderance in small single-$j$ shells.

Two other conclusions not yet mentioned are as follows: 1) If $I^+$ states have a very large probability to be the ground state, there must be, on average, a large energy gap between the $I^+$ ground states and other states. This gap comes from a large shift $d$ in small $j$ shells, where there are very few parameters, or from the difference between the width of eigenvalues for the $I^+$ states and those for all other angular momentum states in very large single-$j$ shells and multi-$j$ shells. 2) If certain $I^+$ states are favored to be the ground state, they are also favored to be the highest state with the same probability.

As $j$ increases, the approximations made in this study, i.e., the omission of mixings between states with different seniorities and those among states with the same seniority deteriorate. In principle, the relationship between the eigenvalues and the two-body interactions is not linear. As a consequence, more general or more complicated cases may not be studied in this simple way. Nevertheless, our analysis should provide helpful clues for what is going on with regards to 0GS predominance in more general even-even many-body systems.

Finally, it is not yet understood why certain coefficients $\alpha_J^{I(v)}$ turn out to be the largest
and/or the smallest among all of them. Since this observation seems to be critical to ex-
plaining 0GS dominance, at least in small single-\textit{j} shells, further consideration of this issue
is warranted.

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Caption:
Fig. 1. Probabilities of $I^+$ ground states for different $j$ shells with 4 particles. All probabilities are obtained from 1000 runs of the TBRE.
TABLE I. The probability of each state to be the ground state and its distribution width for 4 particles in a $j = 7/2$ shell. All eigenstates are uniquely labeled by their seniorities and angular momenta. The first line specifies the angular momentum and seniority for each state. The probabilities labeled by “test” are obtained from 1000 runs of the TBRE, and those labeled by “cal.” are obtained by integrals like Eq. (0.7) for $0^+$ state (see the text for further details). The distribution width, $g_{I(v)}$, of each energy level is listed in the 3rd row. The shift, $d$, of each level is given in the last row. The $d$ inside a bracket corresponds to a shift coming from the smallest $\alpha_{I(v)}^J$.

| $I(v)$ | 0(0) | 2(2) | 2(4) | 4(2) | 4(4) | 5(4) | 6(2) | 8(4) |
|--------|------|------|------|------|------|------|------|------|
| test   | 19.9%| 1.2% | 31.7%| 0.0% | 25.0%| 0.0% | 0.0% | 22.2%|
| cal.   | 18.19%| 0.89%| 33.25%| 0.00%| 22.96%| 0.00%| 0.02%| 24.15%|
| $g_{I(v)}$ | 3.14 | 3.25 | 4.12 | 3.45 | 3.68 | 3.62 | 3.64 | 4.22 |
| $d$    | 1.00 | -    | 1.32 (0.98) | -    | 0.50(0.50) | -    | -    | 0.68(0.35) |

TABLE II. The coefficients $\alpha_{I(v)\beta}^J$ for 4 particles in a $j = 9/2$ shell. Bold font is used for the largest $\alpha_{I(v)}^J$ are the largest and italic for the smallest $\alpha_{I(v)}^J$.

| $I(v)$ | $G_0$ | $G_2$ | $G_4$ | $G_6$ | $G_8$ |
|--------|-------|-------|-------|-------|-------|
| 0      | 0.00  | 0.20  | 2.57  | 2.91  | 0.32  |
| 2      | 0.60  | 1.43  | 1.22  | 0.893 | 1.86  |
| 2      | 0.00  | 1.35  | 1.69  | 1.70  | 1.26  |
| 3      | 0.00  | 0.36  | 2.28  | 2.63  | 0.71  |
| 4      | 0.00  | 0.50  | 2.08  | 2.43  | 0.99  |
| 4      | 0.00  | 2.04  | 1.02  | 0.890 | 2.06  |
| 4      | 0.60  | 0.68  | 1.04  | 2.40  | 1.28  |
| 5      | 4.00  | 1.00  | 1.59  | 1.84  | 1.57  |
| 6      | 4.00  | 1.64  | 0.98  | 1.08  | 2.29  |
| 6      | 4.00  | 0.39  | 1.85  | 2.34  | 1.43  |
| 6      | 2.60  | 0.34  | 1.66  | 1.33  | 2.07  |
| 7      | 4.00  | 1.20  | 1.09  | 1.40  | 2.31  |
| 8      | 4.00  | 0.41  | 1.42  | 2.05  | 2.13  |
| 8      | 2.60  | 0.55  | 0.68  | 1.58  | 2.59  |
| 9      | 4.00  | 0.17  | 1.33  | 2.12  | 2.38  |
| 10     | 4.00  | 0.70  | 0.69  | 1.41  | 3.21  |
| 12     | 4.00  | 0.00  | 0.52  | 1.69  | 3.78  |