Notes on matrix of Fibonacci numbers

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Abstract. For integer m ≥ 3, we discuss m by m matrices of Fibonacci numbers that resemble
the simplicity of Binet formula and maintain the calculation using only integer computation.
We show that the m by m matrices of Fibonacci numbers will eventually store m + 1 distinct
consecutive Fibonacci numbers.

1. Introduction
One of the most well known sequence is the Fibonacci sequence (Fibonacci numbers). The
Fibonacci sequence \( \{f_n\}_{n \geq 0} \) is the sequence of numbers of the form
\[
\begin{align*}
&f_0 \quad f_1 \quad f_2 \quad f_3 \quad f_4 \quad f_5 \quad f_6 \quad f_7 \quad \ldots \\
&0 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad \ldots
\end{align*}
\]
Interesting problem regarding Fibonacci numbers is to find the \( n \)th term of the sequences.
Around 1875 Binet proposed a formula that can calculate the \( n \)th term Fibonacci numbers
known as Binet formula [1, 4]. For \( n \geq 2 \), Binet formula can be formulated as follows
\[
f_n = \frac{1}{\sqrt{5}} (\lambda_1^n - \lambda_2^n) = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n
\]  
(1)
with \( \lambda_1 = \frac{1+\sqrt{5}}{2}, \lambda_2 = \frac{1-\sqrt{5}}{2} \). One difficulty in determining the \( n \)th term \( f_n \) of Fibonacci numbers
is the fact that the formula (1) dealing with calculation of irrational number.

One way to find the Fibonacci numbers, without dealing with irrational number, is using a
recurrence relation that based on the property that for \( n \geq 2 \) each term the Fibonacci numbers
is the sum of the previous two terms [4]. So the Fibonacci numbers can be found using recurrence
relation
\[
f_n = f_{n-1} + f_{n-2}, \quad \text{for } n \geq 2.
\]  
(2)
We note that the recurrence relation (2) will store three consecutive terms of the Fibonacci
numbers.

Based on the recurrence relation (2) an attempt has been made to store three terms of
Fibonacci numbers using a 2 by 2 matrix [2–4]. In this paper, for \( m \geq 3 \) we establish an \( m \) by
\( m \) matrix that will eventually store \( m + 1 \) terms of the Fibonacci numbers. We organize the
paper as follows. In Section 2, we review the 2 by 2 matrix for Fibonacci numbers. In Section 3,
based on the 2 by 2 matrix of Fibonacci numbers, for \( m \geq 3 \) we discuss an \( m \) by \( m \) matrix of
Fibonacci numbers.
2. Case of 2 by 2 matrix

We review the 2 by 2 matrix of Fibonacci numbers as in [2–4]. We start with the special 2 by 2 matrix \( F_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \). Let \( U_n = \begin{pmatrix} u_{n,1} \\ u_{n,2} \end{pmatrix} \) and define a recurrence relation

\[
U_{n+1} = \begin{pmatrix} u_{n+1,1} \\ u_{n+1,2} \end{pmatrix} := F_2 U_n = \begin{pmatrix} u_{n+1,1} + u_{n,2} \\ u_{n+1,2} \end{pmatrix}.
\]

Notice that \( u_{n+1,1} = u_{n,1} + u_{n,2} \) resembles a Fibonacci like recurrences as in (2). Hence if we choose the vector \( U_n \) to be \( U_n = \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix} \) for some two consecutive terms of Fibonacci numbers, then the vector \( U_{n+1} \) will store the two consecutive terms \( f_{n+1} \) and \( f_n \) of the Fibonacci numbers.

Fortunately, the entries of each column of

\[
F_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} f_2 & f_1 \\ f_1 & f_0 \end{pmatrix}
\]

are two consecutive Fibonacci numbers. This implies the power of \( F_2 \) for some positive integer \( n \geq 1 \) will store three consecutive Fibonacci numbers as stated in the following theorem.

**Theorem 1.** [4] Let \( \{f_n\}_{n \geq 0} \) be the sequence of Fibonacci numbers. If \( F_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \), then \( (F_2)^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} \) for \( n \geq 1 \).

**Proof.** We prove by mathematical induction on \( n \). If \( n = 1 \), we have \( (F_2)^1 = \begin{pmatrix} f_{1+1} & f_1 \\ f_1 & f_{1-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \) as in (3).

Assume that for some positive integer \( k \geq 1 \), \( (F_2)^k = \begin{pmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{pmatrix} \). This implies

\[
(F_2)^{k+1} = (F_2)(F_2)^k = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{pmatrix} = \begin{pmatrix} f_{k+1} + f_k & f_k + f_{k-1} \\ f_{k+1} & f_k \end{pmatrix} = \begin{pmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{pmatrix}.
\]

Therefore, we conclude that \( (F_2)^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} \) for \( n \geq 1 \).

We note that the statement of Theorem 1 constitutes the beauty of Binet formula (1) of finding Fibonacci numbers using a simple formula and the beauty of the recurrence relation (2) that finds the Fibonacci numbers using only integer computation.

As a direct consequence of Theorem 1 we have the following result.

**Corollary 2.** Let \( \{f_n\}_{n \geq 0} \) be the sequence of Fibonacci numbers. If \( F = \begin{pmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{pmatrix} \) for some positive integer \( k \), then \( F^n = \begin{pmatrix} f_{(k+1)n+1} & f_{(k+1)n} \\ f_{(k+1)n} & f_{(k+1)n-1} \end{pmatrix} \).
We will generalize the result in Theorem 1 for $m$ by $m$ matrix for some $m \geq 3$. We start with $m = 3$. Let $U_{n+1}$ and $U_n$ be 3 by 1 matrices and consider the recurrence

$$
\begin{bmatrix}
  u_{n+1,1} \\
  u_{n+1,2} \\
  u_{n+1,3}
\end{bmatrix} :=
\begin{bmatrix}
  1 & 1 & 0 \\
  1 & 0 & 0 \\
  0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
  u_{n,1} \\
  u_{n,2} \\
  u_{n,3}
\end{bmatrix}.
$$

Then $u_{n+1,1} = u_{n,1} + u_{n,2}$ and $u_{n+1,j} = u_{n,j-1}$ for $j = 2, 3$. As in the case of 2 by 2 matrix, if we replace $U_n$ with vector with entries consist of three consecutive Fibonacci numbers then $FU_n$ will result in three consecutive Fibonacci numbers.

**Proposition 3.** Let $\{f_n\}_{n \geq 0}$ be the sequence of Fibonacci numbers. If $F_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, then

$$(F_3)^n = \begin{bmatrix} f_{n+1} & f_n & 0 \\ f_n & f_{n-1} & 0 \\ f_{n-1} & f_{n-2} & 0 \end{bmatrix}$$

for $n \geq 2$.

**Proof.** We proof by induction on $n$. If $n = 2$, we have

$$(F_3)^2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} f_3 & f_2 & 0 \\ f_2 & f_1 & 0 \\ f_1 & f_0 & 0 \end{bmatrix}.$$

Assume that $(F_3)^k = \begin{bmatrix} f_{k+1} & f_k & 0 \\ f_k & f_{k-1} & 0 \\ f_{k-1} & f_{k-2} & 0 \end{bmatrix}$ for some $k \geq 2$. Hence we have

$$(F_3)^{k+1} = F_3(F_3)^k = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} f_{k+1} & f_k & 0 \\ f_k & f_{k-1} & 0 \\ f_{k-1} & f_{k-2} & 0 \end{bmatrix} = \begin{bmatrix} f_{k+1} + f_k & f_k + f_{k-1} & 0 \\ f_{k+1} & f_k & 0 \\ f_{k+1} & f_k & 0 \end{bmatrix} = \begin{bmatrix} f_{k+2} & f_{k+1} & 0 \\ f_{k+1} & f_k & 0 \\ f_{k+1} & f_k & 0 \end{bmatrix}.$$
Therefore, 

$$(F_3)^n = \begin{pmatrix} f_{n+1} & f_n & 0 \\ f_n & f_{n-1} & 0 \\ f_{n-1} & f_{n-2} & 0 \end{pmatrix}$$

for integer $n \geq 2$.

Let 

$$F_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} f_2 & f_1 & 0 & 0 \\ f_1 & f_0 & 0 & 0 \\ f_0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$  \tag{4}

Using the same argument as in the case of $3$ by $3$, we have the following result.

**Proposition 4.** Let $\{f_n\}_{n \geq 0}$ be the sequence of Fibonacci numbers. If $F_4$ is defined as in (4), then 

$$\begin{pmatrix} f_{n+1} & f_n & 0 \\ f_n & f_{n-1} & 0 \\ f_{n-1} & f_{n-2} & 0 \\ f_{n-2} & f_{n-3} & 0 \end{pmatrix}$$

for $n \geq 3$.

We now consider, for some $m \geq 5$, the $m$ by $m$ matrix 

$$F_m = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} = \begin{pmatrix} f_2 & f_1 & 0 & 0 & \cdots & 0 & 0 \\ f_1 & f_0 & 0 & 0 & \cdots & 0 & 0 \\ f_0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$  \tag{5}

Then the recurrence relation $U_{n+1} := F_m U_n$ will result in $u_{n+1,1} = u_{n,1} + u_{n,2}$ and for $j = 2, 3, \ldots, m$ we have $u_{n+1,j} = u_{n,j-1}$. Let $F_m(:, k)$ be the $k^{th}$ column of $F_m$, then 

$$(F_m)^{m-2}F_m(:, 1) = \begin{pmatrix} f_m \\ f_m \end{pmatrix}, (F_m)^{m-2}F_m(:, 2) = \begin{pmatrix} f_{m-1} \\ f_{m-2} \\ f_{m-3} \\ \vdots \\ f_0 \end{pmatrix}$$

and 

$$(F_m)^{m-2}F(:, k) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

for $k = 3, \ldots, m$. This implies 

$$(F_m)^{m-1} = \begin{pmatrix} f_m & f_{m-1} & 0 & 0 & \cdots & 0 & 0 \\ f_{m-1} & f_{m-2} & 0 & 0 & \cdots & 0 & 0 \\ f_{m-2} & f_{m-3} & 0 & 0 & \cdots & 0 & 0 \\ f_{m-3} & f_{m-4} & 0 & 0 & \cdots & 0 & 0 \\ f_{m-4} & f_{m-5} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_1 & f_0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$  \tag{6}

We now have the following result.
Theorem 5. Let \( \{ f_n \}_{n \geq 0} \) be the sequence of Fibonacci numbers. If \( F_m \), for some \( m \geq 3 \), is the \( m \) by \( m \) matrices defined in (5), then

\[
(F_m)^n = \begin{pmatrix} f_{n+1} & f_n & 0 & 0 & \cdots & 0 & 0 \\ f_n & f_{n-1} & 0 & 0 & \cdots & 0 & 0 \\ f_{n-1} & f_{n-2} & 0 & 0 & \cdots & 0 & 0 \\ f_{n-2} & f_{n-3} & 0 & 0 & \cdots & 0 & 0 \\ f_{n-3} & f_{n-4} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{n-(m-2)} & f_{n-(m-1)} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}
\]

for \( n \geq m - 1 \).

Proof. We prove by induction on \( n \). Equation (6) guarantees that theorem is true for \( n = m - 1 \).

Assume now that \((F_m)^k = \begin{pmatrix} f_{k+1} & f_k & 0 & 0 & \cdots & 0 & 0 \\ f_k & f_{k-1} & 0 & 0 & \cdots & 0 & 0 \\ f_{k-1} & f_{k-2} & 0 & 0 & \cdots & 0 & 0 \\ f_{k-2} & f_{k-3} & 0 & 0 & \cdots & 0 & 0 \\ f_{k-3} & f_{k-4} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{k-(m-2)} & f_{k-(m-1)} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}\) for \( k \geq m - 1 \). Then

\[
(F_m)^{k+1} = F_m(F_m)^k = \begin{pmatrix} f_{k+1} + f_k & f_k + f_{k-1} & 0 & 0 & \cdots & 0 & 0 \\ f_{k+1} & f_k & 0 & 0 & \cdots & 0 & 0 \\ f_k & f_{k-1} & 0 & 0 & \cdots & 0 & 0 \\ f_{k-1} & f_{k-2} & 0 & 0 & \cdots & 0 & 0 \\ f_{k-2} & f_{k-3} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{k-(m-3)} & f_{k-(m-2)} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}
\]

and hence

\[
(F_m)^{k+1} = \begin{pmatrix} f_{(k+1)+1} & f_{k+1} & 0 & 0 & \cdots & 0 & 0 \\ f_{(k+1)} & f_{(k+1)-1} & 0 & 0 & \cdots & 0 & 0 \\ f_{(k+1)-1} & f_{(k+1)-2} & 0 & 0 & \cdots & 0 & 0 \\ f_{(k+1)-2} & f_{(k+1)-3} & 0 & 0 & \cdots & 0 & 0 \\ f_{(k+1)-3} & f_{(k+1)-4} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{(k+1)-(m-2)} & f_{(k+1)-(m-1)} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.
\]

Therefore,

\[
(F_m)^n = \begin{pmatrix} f_{n+1} & f_n & 0 & 0 & \cdots & 0 & 0 \\ f_n & f_{n-1} & 0 & 0 & \cdots & 0 & 0 \\ f_{n-1} & f_{n-2} & 0 & 0 & \cdots & 0 & 0 \\ f_{n-2} & f_{n-3} & 0 & 0 & \cdots & 0 & 0 \\ f_{n-3} & f_{n-4} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{n-(m-2)} & f_{n-(m-1)} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}
\]

for \( n \geq m - 1 \).

We note that first two columns the matrix \((F_m)^n\) for some \( n \geq m - 1 \) store \( m + 1 \) distinct consecutive terms of the Fibonacci numbers.
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