Asymptotic behavior of dispersive electromagnetic waves in bounded domains

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Abstract. We analyze the stability of Maxwell equations in bounded domains taking into account electric and magnetization effects. Well-posedness of the model is obtained by means of semigroup theory. A passivity assumption guarantees the boundedness of the associated semigroup. Further the exponential or polynomial decay of the energy is proved under suitable sufficient conditions. Finally, several illustrative examples are presented.

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1. Introduction

In this paper, we analyze the stability of Maxwell equations for a general class of dispersion law in a bounded domain \( \Omega \) of \( \mathbb{R}^3 \) with a Lipschitz boundary \( \Gamma \). More precisely, the Maxwell equations in \( \Omega \) are given by

\[
\begin{align*}
D_t - \text{curl} \, H &= 0 \quad \text{in} \quad Q := \Omega \times (0, +\infty), \\
B_t + \text{curl} \, E &= 0 \quad \text{in} \quad Q,
\end{align*}
\]

where \( E \) and \( H \) are, respectively, the electric and magnetic fields, while \( D \) and \( B \) are, respectively, the electric and magnetic flux densities. In case of electric and magnetization effects, these latter ones take the form

\[
\begin{align*}
D(x, t) &= \varepsilon(x) E(x, t) + P(x, t), \\
B(x, t) &= \mu(x) H(x, t) + M(x, t),
\end{align*}
\]

where \( \varepsilon \) (resp. \( \mu \)) is the permittivity (resp. permeability) of the medium, while \( P \) (resp. \( M \)) is the retarded electric polarization (resp. magnetization) that in most applications (see [7,15],[24, Chapter 11]) are of integral form

\[
\begin{align*}
P(x, t) &= \int_0^t \nu_E(t - s, x) E(x, s) \, ds, \\
M(x, t) &= \int_0^t \nu_H(t - s, x) H(x, s) \, ds,
\end{align*}
\]

where \( \nu_E(t, x) \) (resp. \( \nu_H(t, x) \)) is the electric (resp. magnetic) susceptibility kernel. Some particular models (corresponding to certain kernels), like Debye, Lorentz or Drude models, can be reduced to a system coupling Maxwell’s equations to a finite number of differential equations, see [22]. In such a case, semigroup
theory can be applied to obtain existence and decay behavior of the solutions. Our goal is to analyze the general system (1.1) supplemented with the electric boundary conditions

\[ E \times \mathbf{n} = 0, \ H \cdot \mathbf{n} = 0 \text{ on } \Gamma = \partial \Omega, \]  

and initial conditions

\[ E(\cdot, 0) = E_0, \ H(\cdot, 0) = H_0 \text{ in } \Omega, \]  

and find sufficient conditions that guarantee exponential or polynomial decay (on infinite time horizons) of the solutions.

In [13], existence and uniqueness of solutions for problem (1.1)–(1.7) are studied by transforming the system to a Volterra integral equation. We use here a different approach based on semigroup theory as in [8–10,19].

In [5], the propagation of waves in unbounded dispersive media is studied by using the so-called Perfectly Matched Layers technique in order to realize artificial absorbing conditions. For dispersive isotropic Maxwell equations, necessary and sufficient conditions for stability of the PML are given.

In this paper, we restrict ourselves to the case when the permittivity and the permeability are positive constants, while the kernels are real valued and do not depend on the space variable, namely we assume that \( \nu_E(t, x) = \nu_E(t) \) and \( \nu_H(t, x) = \nu_H(t) \), this already corresponds to a large class of physical examples, see, for instance, [15,24]. We further assume that \( \nu_E, \nu_H \in K \), where \( K \) is the set of kernels \( \nu \in C^2([0, \infty)) \), that satisfy

\[ \lim_{t \to \infty} \nu'(t) = 0, \]  

and that there exists two positive constants \( C \) and \( \delta \) (depending on \( \nu \)) such that

\[ |\nu''(t)| \leq C e^{-\delta t}, \forall t \geq 0. \]  

Again these assumptions cover a large class of physical models, see section 6 for some illustrative examples.

For brevity, we define the function \( w \) by

\[ w(t) = C e^{-\delta t}, \forall t \geq 0. \]

The paper is organized as follows. In Sect. 2, we study the well-posedness of the model in an appropriate Hilbert setting by means of semigroup theory. In Sect. 3, we show, under the passivity assumption (see (3.1) below), that the semigroup associated with the model is bounded. Sections 4 and 5 are devoted to the exponential or polynomial decay of the energy under appropriate sufficient conditions. Finally, in Sect. 6 we give several illustrative examples.

Let us conclude this introduction with some notation used in the paper:

The \( L^2(\Omega) \)-inner product (resp. norm) will be denoted by \( \langle \cdot, \cdot \rangle \) (resp. \( \| \cdot \| \)). The usual norm and semi-norm of \( H^s(\Omega) \) \( (s \geq 0) \) are denoted by \( \| \cdot \|_{s, \Omega} \) and \( | \cdot |_{s, \Omega} \), respectively. For \( s = 0 \), we drop the index \( s \). By \( a(t) \lesssim b(t) \), we mean that there exists a constant \( C > 0 \) independent of the functions \( a, b \) and the time \( t \), such that \( a(t) \leq Cb(t) \).

2. Well-posedness result

Even if existence result for problem (1.1)–(1.7) can be obtained using Volterra integral equation method (see, for instance, [13]), we here prefer to use a first-order past history framework (see [8,10,19] for second-order framework and [9] for first-order one) in order to translate our system into a semigroup context (useful for the stability analysis). First, we notice that combining expressions (1.2) to (1.5) with (1.1), we obtain the integro-differential system
\[
\begin{align*}
\varepsilon E_t + \nu_E(0)E + \int_0^t \nu'_E(t-s)E(\cdot, s) \, ds - \text{curl} \, H &= 0 \text{ in } Q, \\
\mu H_t + \nu_H(0)H + \int_0^t \nu'_H(t-s)H(\cdot, s) \, ds + \text{curl} \, E &= 0 \text{ in } Q.
\end{align*}
\] (2.1)

Assuming for the moment that the solution \((E, H)\) of (2.1) with boundary conditions (1.6) and initial conditions (1.7) exists, then for all \((t, s) \in [0, \infty) \times (0, \infty)\) we introduce the cumulative past histories

\[
\eta^t_E(\cdot, s) = \min_{s \leq t} \int_0^s E(\cdot, t-y) \, dy,
\]

\[
\eta^t_H(\cdot, s) = \min_{s \leq t} \int_0^s H(\cdot, t-y) \, dy,
\]

that, respectively, satisfy the transport equation

\[
\begin{align*}
\partial_t \eta^t_E(\cdot, s) &= -\partial_s \eta^t_E(\cdot, s) + E(\cdot, t), \\
\partial_t \eta^t_H(\cdot, s) &= -\partial_s \eta^t_H(\cdot, s) + H(\cdot, t),
\end{align*}
\] (2.4, 2.5)

the boundary condition

\[
\lim_{s \to 0} \eta^t_E(\cdot, s) = \lim_{s \to 0} \eta^t_H(\cdot, s) = 0,
\] (2.6)

and the initial condition

\[
\eta^0_E(\cdot, s) = \eta^0_H(\cdot, s) = 0.
\] (2.7)

Since formal integration by parts yields the identities

\[
\begin{align*}
\int_0^t \nu'_E(t-s)E(\cdot, s) \, ds &= -\int_0^\infty \nu''_E(s)\eta^t_E(\cdot, s) \, ds, \\
\int_0^t \nu'_H(t-s)H(\cdot, s) \, ds &= -\int_0^\infty \nu''_H(s)\eta^t_H(\cdot, s) \, ds,
\end{align*}
\]

system (2.1) is (formally) equivalent to

\[
\begin{align*}
\varepsilon E_t + \nu_E(0)E - \int_0^\infty \nu''_E(s)\eta^t_E(\cdot, s) \, ds - \text{curl} \, H &= 0 \text{ in } Q, \\
\mu H_t + \nu_H(0)H - \int_0^\infty \nu''_H(s)\eta^t_H(\cdot, s) \, ds + \text{curl} \, E &= 0 \text{ in } Q.
\end{align*}
\] (2.8)

All together by setting

\[
U = \begin{pmatrix} E \\ H \\ \eta_E \\ \eta_H \end{pmatrix},
\]

we obtain the abstract Cauchy problem

\[
\begin{align*}
U_t &= AU, \\
U(0) &= U_0,
\end{align*}
\] (2.9)
where

\[
A \begin{pmatrix} E \\ H \\ \eta_E \\ \eta_H \end{pmatrix} = \begin{pmatrix} \varepsilon^{-1}(-\nu_E(0)E + \int_0^\infty \nu'_E(s)\eta_E(\cdot, s) \, ds + \text{curl } H) \\ \mu^{-1}(-\nu_H(0)H + \int_0^\infty \nu'_H(s)\eta_H(\cdot, s) \, ds - \text{curl } E) \\ -\partial_s \eta_E(\cdot, s) + E \\ -\partial_s \eta_H(\cdot, s) + H \end{pmatrix},
\]

(2.10)

and

\[U_0 = (E_0, H_0, 0, 0)^T.\]

The existence of a solution to (2.9) is obtained by using semigroup theory in the appropriate Hilbert setting described here below: First, we introduce the Hilbert spaces

\[J(\Omega) = \{ \chi \in L^2(\Omega)^3 | \text{div} \chi = 0 \},\]

\[\hat{J}(\Omega) = \{ \chi \in J(\Omega) | \chi \cdot n = 0 \text{ on } \Gamma \},\]

recalling that for a field \( \chi \in J(\Omega) \), \( \chi \cdot n \) has a meaning as an element of \( H^{-\frac{1}{2}}(\Gamma) \), see [11, Theorem I.2.5].

Observe that \( w \in L^\infty((0, \infty)) \) and recall that for a Hilbert space \( X \) with inner product \( (\cdot, \cdot)_X \) and induced norm \( \| \cdot \|_X \), \( L^2_w((0, \infty); X) \) is the Hilbert space comprised of Bochner measurable functions \( \eta \) defined on \((0, \infty)\) with values in \( X \) such that

\[
\int_0^\infty \| \eta(s) \|^2_X w(s) \, ds < \infty,
\]

with the natural inner product

\[
\int_0^\infty (\eta(s), \eta'(s))_X w(s) \, ds, \forall \eta, \eta' \in L^2_w((0, \infty); X).
\]

Let us notice that \( L^2_w((0, \infty); X) \) is quite large since it contains all polynomials with coefficients in \( X \), namely for any nonnegative integer \( n \) and any \( a_i \in X, i = 0, \ldots, n \), the polynomial \( p \) defined by

\[p(s) = \sum_{i=0}^n a_i s^i, \forall s \in (0, \infty),\]

belongs to \( L^2_w((0, \infty); X) \).

Now, we introduce the Hilbert space

\[\mathcal{H} = J(\Omega) \times \hat{J}(\Omega) \times L^2_w((0, \infty); J(\Omega)) \times L^2_w((0, \infty); \hat{J}(\Omega)),\]

with the inner product

\[
((E, H, \eta_E, \eta_H)^\top, (E', H', \eta'_E, \eta'_H)^\top)_{\mathcal{H}} := \int_\Omega (\varepsilon E \cdot \tilde{E}' + \mu H \cdot \tilde{H}') \, dx + \int_0^\infty \int_\Omega (\eta_E(x, s) \cdot \tilde{\eta}_E(x, s) + \eta_H(x, s) \cdot \tilde{\eta}_H(x, s)) \, dx w(s) \, ds,
\]

for all \((E, H, \eta_E, \eta_H)^\top, (E', H', \eta'_E, \eta'_H)^\top \in \mathcal{H}\).
We then define the operator \( A \) as follows:

\[
\mathcal{D}(A) = \{ (E, H, \eta_E, \eta_H)^\top \in \mathcal{H} | \text{curl } E, \text{curl } H \in L^2(\Omega)^3, E \times n = 0 \text{ on } \Gamma, \\
\partial_s \eta_E \in L^2(0, \infty); J(\Omega), \partial_s \eta_H \in L^2(0, \infty); \hat{J}(\Omega) \\
\text{and } \eta_E(0) = \eta_H(0) = 0 \},
\]

(2.11)

and for all \( U = (E, H, \eta_E, \eta_H)^\top \in \mathcal{D}(A), AU \) is given by (2.10). Note that for a field \( E \in H(\text{curl}; \Omega) = \{ E \in L^2(\Omega)^3 : \text{curl } E \in L^2(\Omega)^3 \}, E \times n \) has a meaning as an element of \( H^{-\frac{1}{2}}(\Gamma)^3 \), see [11, Theorem I.2.11].

We now check that \( A \) generates a \( C_0 \)-semigroup on \( \mathcal{H} \).

**Theorem 2.1.** The operator \( A \) defined by (2.10) with domain (2.11) generates a \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \) on \( \mathcal{H} \). Therefore for all \( U_0 \in \mathcal{H} \), problem (2.9) has a mild solution \( U \in \mathcal{C}([0, \infty), \mathcal{H}) \) given by \( U(t) = T(t)U_0 \), for all \( t \geq 0 \). If moreover \( U_0 \in \mathcal{D}(A^k) \), with \( k \in \mathbb{N}^* \), problem (2.9) has a classical solution \( U \in \bigcap_{j=0}^k \mathcal{C}^j([0, \infty), \mathcal{D}(A^{k-j})) \).

**Proof.** It suffices to show that \( A - \kappa I \) is a maximal dissipative operator for some \( \kappa \geq 0 \); then by Lumer–Phillips’ theorem it generates a \( C_0 \)-semigroup of contractions on \( \mathcal{H} \) and consequently \( A \) generates a \( C_0 \)-semigroup on \( \mathcal{H} \).

Let us first show the dissipativeness. Let \( U = (E, H, \eta_E, \eta_H)^\top \in \mathcal{D}(A) \) be fixed. Then by the definition of \( A \), we have

\[
(AU, U)_{\mathcal{H}} = \int_{\Omega} \left( (-\nu_E(0)E + \int_0^\infty \nu'_E(s)\eta_E(\cdot, s) \, ds + \text{curl } H) \cdot \bar{E} \\
+ (-\nu_H(0)H + \int_0^\infty \nu'_H(s)\eta_H(\cdot, s) \, ds - \text{curl } E) \cdot \bar{H} \right) \, dx \\
+ \int_0^\infty \int_{\Omega} \left( (-\partial_s \eta_E(\cdot, s) + E) \cdot \bar{\eta}_E + (-\partial_s \eta_H(\cdot, s) + H) \cdot \bar{\eta}_H \right) \, dx \, w(s) \, ds.
\]

Note that by the density of \( \mathcal{D}(\Omega) \) into \( \{ E \in H(\text{curl}; \Omega) : E \times n = 0 \text{ on } \Gamma \} \), the following Green’s formula holds

\[
\int_{\Omega} (\text{curl } E \cdot \bar{H} - E \cdot \text{curl } \bar{H}) \, dx = 0;
\]

(2.12)

furthermore, integrating by parts, we have

\[
\int_0^\infty \int_{\Omega} \partial_s \eta_E(\cdot, s) \cdot \bar{\eta}_E \, dx \, w(s) \, ds = -\int_0^\infty \int_{\Omega} \eta_E(\cdot, s) \cdot \partial_s \bar{\eta}_E \, dx - \int_0^\infty \int_{\Omega} |\eta_E(\cdot, s)|^2 \, dx \, w'(s) \, ds.
\]

Using these identities, we find

\[
\Re(AU, U)_{\mathcal{H}} = -\int_{\Omega} \left( \nu_E(0)|E|^2 + \nu_H(0)|H|^2 \right) \, dx \\
+ \Re \int_{\Omega} \left( \int_0^\infty \nu'_E(s)\eta_E(\cdot, s) \, ds \cdot \bar{E} + \int_0^\infty \nu'_H(s)\eta_H(\cdot, s) \, ds \cdot \bar{H} \right) \, dx
\]

as the real part of the energy.
As \( w'(s) \leq 0 \), we deduce that
\[
\Re(AU, U)_H \leq \int_\Omega (\nu_E(0)|E|^2 + \nu_H(0)|H|^2) \, dx + \int_\Omega \left( \int_0^s \nu''_E(s) \eta_E(\cdot, s) \, ds \cdot \bar{E} + \int_0^s \nu''_H(s) \eta_H(\cdot, s) \, ds \cdot \bar{H} \right) \, dx
\]
\[+ \Re \int_\Omega \int_0^\infty (E \cdot \bar{\eta}_E + H \cdot \bar{\eta}_H) \, dx \, w(s) \, ds.\]

Now using assumption (1.9), the definition of \( w \) and Cauchy–Schwarz’s inequality, we find that there exists a positive constant \( \kappa \) such that
\[
\Re(AU, U)_H \leq \kappa \|U\|^2_H.
\]

This shows that \( A - \kappa I \) is dissipative.

We proceed with the maximality. Let \( \lambda > 0 \) be fixed. For \((F, G, R, S)^\top \in H\), we look for \( U = (E, H, \eta_E, \eta_H)^\top \in D(A) \) such that
\[
(\lambda I - A)U = (F, G, R, S)^\top. \tag{2.13}
\]

According to (2.10), this is equivalent to
\[
\varepsilon \lambda E + \nu_E(0)E - \int_0^\infty \nu''_E(s) \eta_E(\cdot, s) \, ds - \text{curl } H = \epsilon F, \tag{2.14}
\]
\[
\mu \lambda H + \nu_H(0)E - \int_0^\infty \nu''_H(s) \eta_H(\cdot, s) \, ds + \text{curl } E = \mu G, \tag{2.15}
\]
\[
\lambda \eta_E + \partial_s \eta_E(\cdot, s) - E = R, \tag{2.16}
\]
\[
\lambda \eta_H + \partial_s \eta_H(\cdot, s) - H = S. \tag{2.17}
\]

Assume for the moment that \( U \) exists. Then the last two equations allow to eliminate \( \eta_E \) and \( \eta_H \) since they are equivalent to
\[
\eta_E(s) = \frac{1 - e^{-\lambda s}}{\lambda} E + \int_0^s e^{-\lambda (s-y)} R(y) \, dy, \tag{2.18}
\]
\[
\eta_H(s) = \frac{1 - e^{-\lambda s}}{\lambda} H + \int_0^s e^{-\lambda (s-y)} S(y) \, dy. \tag{2.19}
\]

Thus, plugging these expressions into (2.14) and (2.15), we find that
\[
\varepsilon \lambda E + \left( \nu_E(0) - \frac{1}{\lambda} \int_0^\infty \nu''_E(s)(1 - e^{-\lambda s}) \, ds \right) E - \text{curl } H = \epsilon F + r(\lambda), \tag{2.20}
\]

where
\[
r(\lambda) = \int_0^\infty \left( \frac{1 - e^{-\lambda s}}{\lambda} \int_0^s e^{-\lambda (s-y)} R(y) \, dy \right) \, ds.
\]
\mu \lambda H + \left( \nu_H(0) - \frac{1}{\lambda} \int_0^\infty \nu_H''(s)(1 - e^{-\lambda s}) \, ds \right) H + \text{curl} \, E = \mu G + s(\lambda),  \tag{2.21}

where

\begin{align*}
    r(\lambda) &= \int_0^\infty \nu_E''(s) \int_0^s e^{-\lambda(s-y)} R(y) \, dy \, ds, \\
    s(\lambda) &= \int_0^\infty \nu_H''(s) \int_0^s e^{-\lambda(s-y)} S(y) \, dy \, ds,
\end{align*}

with the regularity \( r(\lambda) \in J(\Omega) \) and \( s(\lambda) \in J(\Omega) \). But two integrations by parts allow to show that (see Appendix A)

\begin{align*}
    \nu_E(0) - \frac{1}{\lambda} \int_0^\infty \nu_E''(s)(1 - e^{-\lambda s}) \, ds &= \nu_E(0) + \frac{1}{\lambda} (\nu_E'(0) + \mathcal{L} \nu_E''(\lambda)) = \lambda \mathcal{L} \nu_E(\lambda),
\end{align*}

where we recall that \( \mathcal{L} \nu_E \) is the Laplace transform of \( \nu_E \), see (A.5). Hence, the previous identities (2.20) and (2.21) may be equivalently written as

\begin{align*}
    \lambda (\epsilon + \mathcal{L} \nu_E(\lambda)) E - \text{curl} \, H &= \epsilon F + r(\lambda),  \\
    \lambda (\mu + \mathcal{L} \nu_H(\lambda)) H + \text{curl} \, E &= \mu G + s(\lambda).  \tag{2.24}
\end{align*}

Owing to (A.8), for \( \lambda \) large enough, we have

\( \epsilon + \mathcal{L} \nu_E(\lambda) > 0 \) as well as \( \mu + \mathcal{L} \nu_H(\lambda) > 0 \).

Therefore, for \( \lambda \) sufficiently large system (2.24)–(2.25) fits into a standard framework (see, for instance, [21, Lemma 3.1]) and a unique solution \((E, H)\) exists with the regularity

\begin{align*}
    E &\in X_N(\Omega) = \{ U \in J(\Omega) : \text{curl} \, U \in L^2(\Omega)^3 \text{ and } U \times n = 0 \text{ on } \Gamma \},  \\
    H &\in X_T(\Omega) = \{ U \in J(\Omega) : \text{curl} \, U \in L^2(\Omega)^3 \},
\end{align*}

because \( \epsilon F + r \) (resp. \( \mu G + s \)) belongs to \( J(\Omega) \) (resp. \( J(\Omega) \)).

The surjectivity of \( \lambda I - A \) for \( \lambda \) large enough finally holds because once \( E \) and \( H \) are given, we obtain \( \eta_E \) and \( \eta_H \) with the help of (2.18) and (2.19), respectively, and easily verify the required regularity.  \( \square \)

3. Boundedness of the semigroup

In order to apply standard results on the decay of semigroups (see Lemmas 4.1, 5.1 and 5.2 below), the first step is to show that the semigroup \((T(t))_{t \geq 0}\) generated by \( A \) is bounded. This property is based on the passitivity assumption (or equivalently the assumption that the material is passive, see [7, Definition 2.5] and [20, (2.15)]), that says that (see (A.10))

\begin{align*}
    \Re (i \omega \mathcal{L} \nu_E(i \omega)) \geq 0, \quad \Re (i \omega \mathcal{L} \nu_H(i \omega)) \geq 0, \quad \forall \omega \in \mathbb{R}. \tag{3.1}
\end{align*}

Note that this property is equivalent to

\begin{align*}
    \omega \Im \mathcal{L} \nu_E(i \omega) \leq 0, \quad \omega \Im \mathcal{L} \nu_H(i \omega) \leq 0, \quad \forall \omega \in \mathbb{R}. \tag{3.2}
\end{align*}
Lemma 3.1. Under the additional assumption (3.1), there exists a positive constant $M$ such that

$$
\|T(t)\| \leq M, \quad \forall t \geq 0.
$$

(3.3)

Proof. Take $U_0 = (E_0, H_0, \eta^0_E, \eta^0_H) \in D(A)$ and let $U(t) = (E(t), H(t), \eta^t_E, \eta^t_H) = T(t)U_0$, for all $t \geq 0$. Then by Theorem 2.1, $U \in C([0, \infty), D(A)) \cap C^1([0, \infty), \mathcal{H})$ is a classical solution of problem (2.9), which means that (2.4)–(2.5) and (2.8) hold for all $t > 0$.

But we notice that

$$
\eta^t_E(\cdot, s) = \tilde{\eta}^0_E(\cdot, s - t) + \int_0^{\min\{s,t\}} E(\cdot, t - y) dy,
$$

(3.4)

$$
\eta^t_H(\cdot, s) = \tilde{\eta}^0_H(\cdot, s - t) + \int_0^{\min\{s,t\}} H(\cdot, t - y) dy,
$$

(3.5)

where $\tilde{\eta}^0_E$ is the extension of $\eta^0_E$ by zero on $(-\infty, 0)$. Plugging these expressions in (2.8), we find that

$$
\begin{align*}
\begin{cases}
\varepsilon E_t + \nu_E(0) E + \int_0^t \nu'_E(t - s) E(\cdot, s) \, ds - \text{curl} \, H = F(t) \text{ in } Q, \\
\mu H_t + \nu_H(0) H + \int_0^\infty \nu'_H(s) H(\cdot, s) \, ds + \text{curl} \, E = G(t) \text{ in } Q,
\end{cases}
\end{align*}
$$

(3.6)

where

$$
\begin{align*}
F(t) &:= \int_0^\infty \nu''(s) \tilde{\eta}^0_E(\cdot, s - t) \, ds = \int_0^\infty \nu''(s) \eta^0_E(\cdot, s - t) \, ds, \\
G(t) &:= \int_0^\infty \nu''(s) \tilde{\eta}^0_H(\cdot, s - t) \, ds = \int_0^\infty \nu''(s) \eta^0_H(\cdot, s - t) \, ds.
\end{align*}
$$

Now, we note that

$$
\nu_E(0) E + \int_0^t \nu'_E(t - s) E(\cdot, s) \, ds = \frac{d}{dt} \left( \int_0^t \nu_E(t - s) E(\cdot, s) \, ds \right),
$$

and therefore system (3.6) is equivalent to

$$
\begin{align*}
\begin{cases}
\varepsilon E_t + \frac{d}{dt} \left( \int_0^t \nu_E(t - s) E(\cdot, s) \, ds \right) - \text{curl} \, H = F(t) \text{ in } Q, \\
\mu H_t + \frac{d}{dt} \left( \int_0^t \nu_H(t - s) H(\cdot, s) \, ds \right) + \text{curl} \, E = G(t) \text{ in } Q.
\end{cases}
\end{align*}
$$

(3.7)

Now, we adapt an argument used in the proof of [20, Theorem 3.1]. For a fixed $T > 0$, we multiply the first identity by $\tilde{E}(t)$ and the second one by $\tilde{H}(t)$, integrate both in $\Omega \times (0,T)$, take the sum and find
\[
\int_0^T \int_\Omega \left( \left( \varepsilon E_t + \frac{d}{dt} \left( \int_0^t \nu_E(t-s)E(\cdot,s) \, ds \right) - \text{curl} \, H(t) \right) \cdot \bar{E}(t) \right.
\]
\[
\left. + \left( \mu H_t + \frac{d}{dt} \left( \int_0^t \nu_H(t-s)H(\cdot,s) \, ds \right) + \text{curl} \, E(t) \right) \cdot \bar{H}(t) \right) \, dx dt
\]
\[
= \int_0^T \int_\Omega \left( F(t) \cdot \bar{E}(t) + G(t) \cdot \bar{H}(t) \right) \, dx dt.
\]
Taking the real part of this identity and applying Green's formula (2.12), we get
\[
\Re \int_0^T \int_\Omega \left( \left( \varepsilon E_t + \frac{d}{dt} \left( \int_0^t \nu_E(t-s)E(\cdot,s) \, ds \right) \right) \cdot \bar{E}(t) \right.
\]
\[
\left. + \left( \mu H_t + \frac{d}{dt} \left( \int_0^t \nu_H(t-s)H(\cdot,s) \, ds \right) \right) \cdot \bar{H}(t) \right) \, dx dt
\]
\[
= \Re \int_0^T \int_\Omega \left( F(t) \cdot \bar{E}(t) + G(t) \cdot \bar{H}(t) \right) \, dx dt.
\]
Now, if we define \( \tilde{E}_T \) (and similarly for \( \tilde{H}_T \)) by
\[
\tilde{E}_T(\cdot,t) = \begin{cases} 
E(\cdot,t) & \text{if } t \in (0,T), \\
0 & \text{else},
\end{cases}
\]
the previous identity can be written as
\[
\Re \int_0^T \int_\Omega \left( \varepsilon E_t \cdot \bar{E}(t) + \mu H_t \cdot \bar{H}(t) \right) \, dx dt
\]
\[
= -\Re \int_\mathbb{R} \int_\Omega \left( \frac{d}{dt}(\tilde{v}_E * \tilde{E}_T)(t) \cdot \overline{\tilde{E}_T(t)} + \frac{d}{dt}(\tilde{v}_H * \tilde{H}_T)(t) \cdot \overline{\tilde{H}_T(t)} \right) \, dx dt
\]
\[
+ \Re \int_0^T \int_\Omega \left( F(t) \cdot \bar{E}(t) + G(t) \cdot \bar{H}(t) \right) \, dx dt,
\]
where \( f * \) g means the convolution in \( \mathbb{R} \), namely
\[
(f * g)(t) = \int_\mathbb{R} f(t-s)g(s) \, ds, \forall t \in \mathbb{R}.
\]
Now by Parseval's identity, we have
\[
\int_\mathbb{R} \int_\Omega \left( \frac{d}{dt}(\tilde{v}_E * \tilde{E}_T)(t) \cdot \bar{E}_T(t) + \frac{d}{dt}(\tilde{v}_H * \tilde{H}_T)(t) \cdot \bar{H}_T(t) \right) \, dx dt
\]
\[
= \int_\Omega \int_\mathbb{R} i\omega \left( \mathcal{F}(\tilde{v}_E(i\omega))|\mathcal{F}(\tilde{E}_T(i\omega)|^2 + \mathcal{F}(\tilde{v}_H(i\omega))|\mathcal{F}(\tilde{H}_T(i\omega)|^2 \right) \, d\omega dx.
\]
From our passivity assumption (3.1), we deduce that
\[
\Re \int_{\mathbb{R}} \int_{\Omega} \left( \frac{d}{dt}(\tilde{\nu}_E \ast_t \tilde{E}_T(t)) \cdot \tilde{E}_T(t) + \frac{d}{dt}(\tilde{\nu}_H \ast_t \tilde{H}_T(t)) \cdot \tilde{H}_T(t) \right) dx dt \geq 0.
\]
This estimate in the identity (3.8) leads to
\[
\int_{\Omega} (\varepsilon |E(x, T)|^2 + \mu |H(x, T)|^2) dx \leq \int_{\Omega} (\varepsilon |E_0(x)|^2 + \mu |H_0(x)|^2) dx + 2\Re \int_{0}^{T} \int_{\Omega} (F(t) \cdot \tilde{E}(t) + G(t) \cdot \tilde{H}(t)) dx dt.
\]
(3.9)

By setting
\[
E(t) = \frac{1}{2} \int_{\Omega} (\varepsilon |E(x, t)|^2 + \mu |H(x, t)|^2) dx, \forall t \geq 0
\]
and using Cauchy–Schwarz inequality in the last estimate, we obtain
\[
E(T) \leq E(0) + \sqrt{2} \int_{0}^{T} \left( \int_{\Omega} (|F(x, t)|^2 + |G(x, t)|^2) dx \right)^{\frac{1}{2}} E^{\frac{1}{2}}(t) dt, \forall T > 0.
\]
By Gronwall’s inequality (see, for instance, [20, Lemma 3.1]), we deduce that
\[
E(t) \leq \left( E(0)^{\frac{1}{2}} + \frac{\sqrt{2}}{2} \int_{0}^{t} \left( \int_{\Omega} (|F(x, s)|^2 + |G(x, s)|^2) dx \right)^{\frac{1}{2}} ds \right)^{2}, \forall t > 0.
\]
(3.10)

Now, we need to estimate the term
\[
\int_{0}^{t} \left( \int_{\Omega} (|F(x, s)|^2 + |G(x, s)|^2) dx \right)^{\frac{1}{2}} ds.
\]
But using the definition of \( F \) and \( G \), the assumption (1.9) and Cauchy–Schwarz inequality, we see that
\[
\int_{0}^{t} \left( \int_{\Omega} (|F(x, s)|^2 + |G(x, s)|^2) dx \right)^{\frac{1}{2}} ds \lesssim \left( \int_{0}^{t} e^{-\delta s} ds \right) \left( \| \eta^0_E \|_{L^2((0, \infty); J(\Omega))} + \| \eta^0_H \|_{L^2((0, \infty); J(\Omega))} \right),
\]
and therefore
\[
\int_{0}^{t} \left( \int_{\Omega} (|F(x, s)|^2 + |G(x, s)|^2) dx \right)^{\frac{1}{2}} ds \lesssim \| \eta^0_E \|_{L^2((0, \infty); J(\Omega))} + \| \eta^0_H \|_{L^2((0, \infty); J(\Omega))}.
\]

Plugging this estimate into (3.10), we have obtained that
\[
E(t) \lesssim \| U_0 \|^2_{H_T}.
\]
(3.11)
Now, we return to (3.4) and (3.5) to estimate the norm of $\eta_t^E$ and $\eta_t^H$. Let us perform the estimation for $\eta_t^E$. By (3.4), we have

$$\|\eta_t^E(\cdot,s)\|^2_{L^2_w((0,\infty);J(\Omega))} \leq 2 \int_0^\infty w(s) \int_\Omega |\eta_t^0(x,s-t)|^2 \, dx \, ds + 2 \int_0^\infty w(s) \left( \int_0^{\min\{s,t\}} \int_0 E(x,t-y) \, dy \right)^2 \, dx \, ds.$$ 

The first term is easily estimated via a change in variable and the property $w(s+t) = e^{-\delta t}w(s)$, valid for all $s, t \geq 0$, yielding

$$\int_0^\infty w(s) \int_\Omega |\tilde{\eta}_0^E(x,s-t)|^2 \, dx \, ds = \int_0^\infty w(s) \int_\Omega |\eta_t^0(x,s-t)|^2 \, dx \, ds = \int_0^\infty w(s+t) \int_\Omega |\eta_t^0(x,s)|^2 \, dx \, ds \leq e^{-\delta t} \int_0^\infty w(s) \int_\Omega |\eta_t^0(x,s)|^2 \, dx \, ds.$$ 

This means that

$$\|\eta_t^0(x,\cdot-t)\|_{L^2_w((0,\infty);J(\Omega))} \leq e^{-\delta t} \|\eta_t^0\|_{L^2_w((0,\infty);J(\Omega))}.$$  

For the second term, by Cauchy–Schwarz inequality and Fubini’s theorem, we have

$$\int_0^\infty w(s) \left( \int_0^{\min\{s,t\}} \int_0 E(x,t-y) \, dy \right)^2 \, dx \, ds \leq \int_0^\infty w(s) \int_\Omega s \min\{s,t\} \int_0 |E(x,t-y)|^2 \, dy \, dx \, ds \leq 2 \int_0^\infty sw(s) \int_0 \mathcal{E}(t-y) \, dy \, ds.$$ 

Hence using the estimate (3.11), we find

$$\int_0^\infty w(s) \left( \int_0^{\min\{s,t\}} \int_0 E(x,t-y) \, dy \right)^2 \, dx \, ds \lesssim \|U_0\|^2_{\mathcal{H}} \int_0^\infty w(s)s^2 \, ds \lesssim \|U_0\|^2_{\mathcal{H}}.$$ 

These last estimates show that

$$\|\eta_t^E(\cdot,s)\|^2_{L^2_w((0,\infty);J(\Omega))} \lesssim \|U_0\|^2_{\mathcal{H}}.$$ 

Since the same arguments yield

$$\|\eta_t^H(\cdot,s)\|^2_{L^2_w((0,\infty);J(\Omega))} \lesssim \|U_0\|^2_{\mathcal{H}},$$

...
the combination of these two estimates with (3.11) leads to
\[ \|U(t)\|^2_{L^2} \lesssim \|U_0\|^2_{L^2}, \quad \forall t > 0. \]
Since \( D(A) \) is dense in \( H \), we conclude that
\[ \|T(t)U_0\|^2_{L^2} \lesssim \|U_0\|^2_{L^2}, \quad \forall t > 0, \quad U_0 \in H, \]
which was the claim. \( \square \)

**Corollary 3.2.** Under the additional assumption (3.1), the resolvent set \( \rho(A) \) of \( A \) contains the right-half plane, namely
\[ \{ \lambda \in \mathbb{C} : \Re \lambda > 0 \} \subset \rho(A). \]

**Proof.** Direct consequence of Lemma 3.1 and of Theorem 5.2.1 of [3]. \( \square \)

### 4. Strong stability

One simple way to prove the strong stability of (2.9) is to use the following theorem due to Arendt & Batty and Lyubich & Vû (see [2,18]).

**Theorem 4.1.** (Arendt & Batty/Lyubich & Vû) Let \( X \) be a reflexive Banach space and \( (T(t))_{t \geq 0} \) be a bounded \( C_0 \) semigroup generated by \( A \) on \( X \). Assume that \( (T(t))_{t \geq 0} \) is bounded and no eigenvalues of \( A \) lie on the imaginary axis. If \( \sigma(A) \cap i\mathbb{R} \) is countable, then \( (T(t))_{t \geq 0} \) is stable.

We now want to take advantage of this Theorem. Since the resolvent of our operator is not compact, we have to analyze the full spectrum of \( A \) on the imaginary axis. For that purpose, we actually need a stronger assumption than the passivity, namely in addition to (3.1), we need that
\[ \Re(i\omega L\nu_E(i\omega)) + \Re(i\omega L\nu_H(i\omega)) > 0, \quad \forall \omega \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}. \] (4.1)

As before, this property is equivalent to
\[ \omega \Im L\nu_E(i\omega) + \omega \Im L\nu_H(i\omega) < 0, \quad \forall \omega \in \mathbb{R}^*. \] (4.2)

We first prove a preliminary result related to a family of operators defined in
\[ H_1 := J(\Omega) \times \hat{J}(\Omega). \]
First, let us consider the unbounded operator \( B \) from \( H_1 := J(\Omega) \times \hat{J}(\Omega) \) into itself with the domain
\[ D(B) := \{(E,H) \in H_1 | \text{curl } E, \text{curl } H \in L^2(\Omega)^3, \text{ and } E \times n = 0 \text{ on } \Gamma \}, \]
defined by
\[ B((E,H)) = (\epsilon E - \text{curl } H, \mu H + \text{curl } E). \]
As noted before, \( B \) is an isomorphism from \( D(B) \) into \( H_1 \), with a compact resolvent. Consequently for any \( \omega \in \mathbb{R} \), the operator
\[ B_\omega((E,H)) = (i\omega (\epsilon + L\nu_E(i\omega)) E - \text{curl } H, i\omega (\mu + L\nu_H(i\omega)) H + \text{curl } E), \]
with the same domain as \( B \) is a compact perturbation of \( B \). Therefore, for all \( \omega \), \( B_\omega \) is a Fredholm operator of index 0. Hence, it will be an isomorphism if and only if it is injective. This is proved in the next lemma.

**Lemma 4.2.** Under the additional assumptions (3.1) and (4.1), and if \( \Omega \) is simply connected with a connected boundary, then the operator \( B_\omega \) is an isomorphism from \( D(B) \) into \( H_1 \).
Proof. Let \((E, H) \in \ker \mathbf{B}_\omega\), then we have
\[
0 = (\mathbf{B}_\omega(E, H), (E, H))_{\mathcal{H}_1}
= \int_\Omega \left( (i\omega (\epsilon + L\nu_E(i\omega)) E - \text{curl} \, H) \cdot \bar{E} 
+ (i\omega (\mu + L\nu_H(i\omega)) H + \text{curl} \, E) \cdot \bar{H} \right) \, dx.
\]
Hence applying Green’s formula and taking the real part of the identity, we find that
\[
0 = \int_\Omega \left( \Re (i\omega (\epsilon + L\nu_E(i\omega)) |E|^2 + \Re (i\omega (\mu + L\nu_H(i\omega)) |H|^2) \right) \, dx.
\]
By our assumptions (3.1) and (4.1), we may distinguish between the three cases:

1. If \(\Re (i\omega (\epsilon + L\nu_E(i\omega))) > 0\), we deduce that \(E = 0\) and by the definition of \(\mathbf{B}_\omega\), we deduce that \(\text{curl} \, H = 0\). This property added to the fact that \(H \in \hat{J}(\Omega)\) allows to conclude that \(H = 0\) owing to Proposition 3.14 of [1].

2. If \(\Re (i\omega (\mu + L\nu_H(i\omega))) > 0\), we deduce that \(H = 0\) and by the definition of \(\mathbf{B}_\omega\), we deduce that \(\text{curl} \, E = 0\). This property added to the fact that \(E\) is divergence free and satisfies \(E \times n = 0\) on \(\Gamma\), allows to conclude that \(E = 0\) owing to Proposition 3.18 of [1].

3. If \(\Re (i\omega (L\nu_E(i\omega) + L\nu_H(i\omega))) = 0\) at \(\omega = 0\), then we directly deduce that \(\text{curl} \, E = 0 = \text{curl} \, H = 0\), and we conclude that \(E = H = 0\) with the help of Propositions 3.14 and 3.18 of [1]. \(\square\)

Remark 4.3. Obviously the assumption that \(\Omega\) is simply connected and that its boundary is connected can be weakened if (4.1) can be replaced by a stronger assumption.

Lemma 4.4. Under the assumptions of Lemma 4.2,
\[
i\mathbb{R} \equiv \{ i\beta \mid \beta \in \mathbb{R} \} \subset \rho(\mathcal{A}).
\]
Proof. The proof is similar to the proof of the maximality of \(\mathcal{A}\). Indeed fix \(\omega \in \mathbb{R}\) and let \((F, G, R, S)^\top \in \mathcal{H}\). Then we look for \(U = (E, H, \eta_E, \eta_H)^\top \in D(\mathcal{A})\) such that
\[
(i\omega I - \mathcal{A}) U = (F, G, R, S)^\top.
\]
Arguing as in the proof of the maximality, this means that we first look for \((E, H)\) solution of (2.24)–(2.25) with \(\lambda = i\omega\), or equivalently solution of
\[
\mathbf{B}_\omega(E, H) = (\epsilon F + r(i\omega), \mu G + s(i\omega)).
\]
Note that \(r(i\omega)\) (resp. \(s(i\omega)\)) belongs to \(J(\Omega)\) (resp. \(\hat{J}(\Omega)\)) because by Fubini’s theorem and Cauchy–Schwarz inequality we have
\[
\|r(i\omega)\|_{\Omega} \lesssim \int_0^\infty |\nu''_E(s)| \int_0^s \|R(\cdot, y)\|_{\Omega} \, dy \, ds \\
\lesssim \int_0^\infty w(s) \int_0^s \|R(\cdot, y)\|_{\Omega} \, dy \, ds \\
\lesssim \int_0^\infty \|R(\cdot, y)\|_{\Omega} w(y) \, dy
\]
Lemma 4.5. Under the assumptions of Lemma 4.2, energy we use the following result (see [23] or [12]):

Our stability results are based on a frequency domain approach, namely for the exponential decay of the energy we use the following result (see (2.18) and (2.19), respectively (with \( \lambda = i\omega \)) and easily check their right requested regularity.

As a direct consequence of this Lemma and Theorem 4.1, we obtain the following result.

Lemma 5.1. Let \((e^{t\mathcal{L}})_{t \geq 0}\) be a bounded \(C_0\) semigroup on a Hilbert space \(H\). Then it is exponentially stable, i.e., it satisfies

\[
\| e^{t\mathcal{L}}U_0 \| \leq C e^{-\omega t}\|U_0\|_H, \quad \forall U_0 \in H, \quad \forall t \geq 0,
\]

for some positive constants \(C\) and \(\omega\) if and only if

\[
i\mathbb{R} \subset \rho(\mathcal{L}), \quad (5.1)
\]

and

\[
\sup_{\beta \in \mathbb{R}} \| (i\beta - \mathcal{L})^{-1} \| < \infty. \quad (5.2)
\]

In contrast, the polynomial decay of the energy is based on the following result stated in Theorem 2.4 of [6] (see also [4,17] for weaker variants).

Lemma 5.2. Let \((e^{t\mathcal{L}})_{t \geq 0}\) be a bounded \(C_0\) semigroup on a Hilbert space \(H\) such that its generator \(\mathcal{L}\) satisfies (5.1) and let \(\ell\) be a fixed positive real number. Then the following properties are equivalent

\[
\| e^{t\mathcal{L}}U_0 \| \leq C t^{-\frac{1}{2}}\|U_0\|_{\mathcal{D}(\mathcal{L})}, \quad \forall U_0 \in \mathcal{D}(\mathcal{L}), \quad \forall t > 1,
\]

\[
\| e^{t\mathcal{L}}U_0 \| \leq C t^{-1}\|U_0\|_{\mathcal{D}(\mathcal{L}^{\ell})}, \quad \forall U_0 \in \mathcal{D}(\mathcal{L}^{\ell}), \quad \forall t > 1,
\]

\[
\sup_{\beta \in \mathbb{R}} \frac{1}{1 + |\beta|^\ell} \| (i\beta - \mathcal{L})^{-1} \| < \infty. \quad (5.3)
\]

As Lemma 4.4 guarantees the validness of assumption (5.1), it remains to check whether (5.2) or (5.3) is valid. This is possible by improving assumption (4.1) with a precise behavior of \(\Re(i\omega\mathcal{L}U_E(i\omega))\) and of \(\Re(i\omega\mathcal{L}U_H(i\omega))\) at infinity. More precisely, we suppose that there exist four nonnegative constants \(\sigma_E\), \(\sigma_H\), \(\omega_0\), and \(m\) with \(\sigma_E + \sigma_H > 0\) such that

\[
\Re(i\omega\mathcal{L}U_E(i\omega)) |X|^2 + \Re(i\omega\mathcal{L}U_H(i\omega)) |Y|^2 
\geq |\omega|^{-m}(\sigma_E |X|^2 + \sigma_H |Y|^2), \quad \forall X, Y \in \mathbb{C}^3, \quad \omega \in \mathbb{R} : |\omega| \geq \omega_0. \quad (5.4)
\]
**Lemma 5.3.** In addition to the assumptions of Lemma 4.2, assume that (5.4) holds. Then the operator $A$ satisfies (5.3) with $\ell = m$.

**Proof.** We use a contradiction argument, namely suppose that (5.3) is false. Then there exist a sequence of real numbers $\beta_n \to +\infty$ and a sequence of vectors $z_n = (E_n, H_n, \eta_{E,n}, \eta_{H,n})^T$ in $D(A)$ with

$$
\|z_n\|_H = 1,
$$

(5.5)
satisfying

$$
\beta_n \left( e i \beta_n E_n + \nu_E(0)E_n \right.
\left. - \int_0^\infty \nu_E''(s) \eta_{E,n}(\cdot, s) \, ds - \text{curl} \, H_n \right) = \epsilon F_n \to 0 \text{ in } J(\Omega),
$$

(5.6)

$$
\beta_n \left( \mu i \beta_n H_n + \nu_H(0)E \right.
\left. - \int_0^\infty \nu_H''(s) \eta_{H,n}(\cdot, s) \, ds + \text{curl} \, E_n \right) = \mu G_n \to 0 \text{ in } \hat{J}(\Omega),
$$

(5.7)

$$
\beta_n \left( i \beta_n \eta_{E,n} + \partial_s \eta_{E,n}(\cdot, s) - E_n \right) = R_n \to 0 \text{ in } L^2_w((0, \infty); J(\Omega)),
$$

(5.8)

$$
\beta_n \left( i \beta_n \eta_{H,n} + \partial_s \eta_{H,n}(\cdot, s) - H_n \right) = S_n \to 0 \text{ in } L^2_w((0, \infty); \hat{J}(\Omega)).
$$

(5.9)

By these two last identities, $\eta_{E,n}$ and $\eta_{H,n}$ are given by (see (2.18)–(2.19))

$$
\eta_{E,n}(s) = \frac{1 - e^{-i \beta_n s}}{i \beta_n} E_n + \beta_n^{-\ell} \int_0^s e^{-i \beta_n (s-y)} R_n(y) \, dy,
$$

(5.10)

$$
\eta_{H,n}(s) = \frac{1 - e^{-i \beta_n s}}{i \beta_n} H_n + \beta_n^{-\ell} \int_0^s e^{-i \beta_n (s-y)} S_n(y) \, dy.
$$

(5.11)

Thus plugging these expressions in (5.6) and (5.7), we find that (compare with (2.20)–(2.21) and (2.24)–(2.25))

$$
\beta_n \left( i \beta_n (\epsilon + L \nu_E(i \beta_n)) E_n - \text{curl} \, H_n \right) = \epsilon F_n + r_n(i \beta_n),
$$

(5.12)

$$
\beta_n \left( i \beta_n (\mu + L \nu_H(i \beta_n)) H_n + \text{curl} \, E_n \right) = \mu G_n + s_n(i \beta_n),
$$

(5.13)

where

$$
r_n(i \beta_n) = \int_0^\infty \nu_E''(s) \int_0^s e^{-i \beta_n (s-y)} R_n(y) \, dy \, ds,
$$

(5.14)

$$
s_n(i \beta_n) = \int_0^\infty \nu_H''(s) \int_0^s e^{-i \beta_n (s-y)} S_n(y) \, dy \, ds,
$$

(5.15)

that have the regularity $r_n \in J(\Omega)$ and $s_n \in \hat{J}(\Omega)$ with

$$
\|r_n(i \beta_n)\|_\Omega + \|s_n(i \beta_n)\|_\Omega \lesssim \|R_n\|_{L^2_w((0, \infty); J(\Omega))} + \|S_n\|_{L^2_w((0, \infty); \hat{J}(\Omega))} = o(1).
$$

(5.16)
Now multiplying (5.12) (resp. (5.13)) by $E_n$ (resp. $H_n$), integrating over $\Omega$, and summing the two identities, we get
\[
\beta_n^\ell \int_{\Omega} \left( (i\beta_n (\epsilon + \mathcal{L}\nu_E(i\beta_n)) E_n - \text{curl} \, H_n) \cdot \bar{E}_n \right. \\
+ (i\beta_n (\mu + \mathcal{L}\nu_H(i\beta_n)) H_n + \text{curl} \, E_n) \cdot \bar{H}_n \bigg) \, dx \\
= \int_{\Omega} \left( (\epsilon F_n + r_n(i\beta_n)) \cdot \bar{E}_n + (\mu G_n + s_n(i\beta_n)) \cdot \bar{H}_n \right) \, dx.
\]

Again by Green’s formula (2.12), and taking the real part, we find
\[
\beta_n^\ell \Re \int_{\Omega} \left( (i\beta_n (\epsilon + \mathcal{L}\nu_E(i\beta_n)) |E_n|^2 + i\beta_n (\mu + \mathcal{L}\nu_H(i\beta_n)) |H_n|^2) \right) \, dx
\]
\[
= \Re \int_{\Omega} \left( (\epsilon F_n + r_n(i\beta_n)) \cdot \bar{E}_n + (\mu G_n + s_n(i\beta_n)) \cdot \bar{H}_n \right) \, dx.
\]

Owing to (5.5), (5.6), (5.7), and (5.16), this right-hand side tends to zero as $n$ goes to infinity, i.e., we have
\[
\beta_n^\ell \Re \int_{\Omega} \left( (i\beta_n (\epsilon + \mathcal{L}\nu_E(i\beta_n)) |E_n|^2 + i\beta_n (\mu + \mathcal{L}\nu_H(i\beta_n)) |H_n|^2) \right) \, dx = o(1).
\]

Taking into account our assumption (5.4), for $n$ large enough, the previous property implies that
\[
\beta_n^\ell \Re \int_{\Omega} \left( (\epsilon F_n + r_n(i\beta_n)) \cdot \bar{E}_n + (\mu G_n + s_n(i\beta_n)) \cdot \bar{H}_n \right) \, dx = o(1).
\]

Hence taking $\ell = m$, we find that
\[
\int_{\Omega} \left( \sigma_E |E_n|^2 + \sigma_H |H_n|^2 \right) \, dx = o(1).
\]

We then distinguish between three cases:
1) If $\sigma_E$ and $\sigma_H$ are both positive, then (5.18) directly guarantees that
\[
\|E_n\|_\Omega + \|H_n\|_\Omega = o(1).
\]

Once this property holds, we return to (5.10) and (5.11) to get a contradiction with (5.5), since we will show that
\[
\|\eta_{E,n}\|_{L^2_w((0,\infty);J(\Omega))} + \|\eta_{H,n}\|_{L^2_w((0,\infty);\hat{J}(\Omega))} = o(1).
\]

Let us check this property for $\eta_{E,n}$ (the treatment of $\eta_{H,n}$ is completely similar and is omitted), namely we will show that
\[
\|\eta_{E,n}\|_{L^2_w((0,\infty);J(\Omega))} \lesssim \|R_n\|_{L^2_w((0,\infty);J(\Omega))} + \|E_n\|_\Omega
\]

which by (5.8) and (5.19) leads to
\[
\|\eta_{E,n}\|_{L^2_w((0,\infty);J(\Omega))} = o(1).
\]
The first step is to show that $\eta_{E,n}$ belongs to $L^2_w((0, \infty); J(\Omega))$. Indeed the first term of the right-hand side of (5.10) clearly belongs to $L^2_w((0, \infty); J(\Omega))$, so let us concentrate on the second term. Namely let us set

$$
\Psi_n(s, \cdot) = \int_0^s e^{-i\beta_n(s-y)} R_n(y, \cdot) \, dy, \ \forall s \geq 0.
$$

Then we easily see that $\Psi_n(0, \cdot) = 0$ and $\Psi_n$ satisfies the transport equation

$$
\partial_s \Psi_n(s, \cdot) + i\beta_n \Psi_n(s, \cdot) = R_n(s, \cdot), \ \forall s > 0.
$$

Hence multiplying this identity by $\bar{\Psi}_n w(s)$, and integrating in $\Omega$ and in $s \in (0, y)$ for any $y > 0$, we find that

$$
\int_\Omega \int_0^y \left( \partial_s \Psi_n(s, x) + i\beta_n \Psi_n(s, x) \right) \bar{\Psi}_n(s, x) w(s) \, ds \, dx
$$

$$
= \int_\Omega \int_0^y R_n(s, x) \cdot \bar{\Psi}_n(s, x) w(s) \, ds \, dx.
$$

Taking the real part of this identity, we find

$$
\frac{1}{2} \int_\Omega \int_0^y \partial_s (|\Psi_n(s, x)|^2) w(s) \, ds \, dx = \Re \int_\Omega \int_0^y R_n(s, x) \cdot \bar{\Psi}_n(s, x) w(s) \, ds \, dx.
$$

Integrating by parts on the left-hand side, we obtain

$$
\delta \int_\Omega \int_0^y |\Psi_n(s, x)|^2 w(s) \, ds \, dx + \int_\Omega |\Psi_n(y, x)|^2 w(y) \, dx
$$

$$
= 2\Re \int_\Omega \int_0^y R_n(s, x) \cdot \bar{\Psi}_n(s, x) w(s) \, ds \, dx.
$$

Hence, Cauchy–Schwarz inequality leads to

$$
\delta \left( \int_\Omega \int_0^y |\Psi_n(s, x)|^2 w(s) \, ds \, dx \right)^{\frac{1}{2}} \leq 2 \left( \int_\Omega \int_0^y |R_n(s, x)|^2 w(s) \, ds \, dx \right)^{\frac{1}{2}}.
$$

Passing to the limit as $y$ tends to infinity, we deduce that $\Psi_n \in L^2_w((0, \infty); J(\Omega))$ with

$$
||\Psi_n||_{L^2_w((0, \infty); J(\Omega))} \lesssim ||R_n||_{L^2_w((0, \infty); J(\Omega))}.
$$

(5.21)

Coming back to (5.10), we then have

$$
||\eta_{E,n}||_{L^2_w((0, \infty); J(\Omega))} \lesssim \frac{1 - e^{-i\beta_n}}{i\beta_n} ||E_n||_{L^2_w((0, \infty); J(\Omega))}
$$

$$
+ \beta_n^{-\ell} ||\Psi_n||_{L^2_w((0, \infty); J(\Omega))}.
$$

(5.22)

Let us then estimate the first term of this right-hand side. First, we observe that
\[
\left\| \frac{1 - e^{-i\beta_n t}}{i\beta_n} E_n \right\|_{L^2_w((0,\infty);J(\Omega))}^2 = (\int_{\Omega} |E_n(x)|^2 \, dx) \left( \int_{0}^{\infty} \left| \frac{1 - e^{-i\beta_n s}}{i\beta_n} \right|^2 w(s) \, ds \right) \\
\leq \frac{4}{\beta_n^2} \left( \int_{0}^{\infty} w(s) \, ds \right) \|E_n\|_{\Omega}^2.
\]

Hence for \( n \) large enough, we have
\[
\left\| \frac{1 - e^{-i\beta_n t}}{i\beta_n} E_n \right\|_{L^2_w((0,\infty);J(\Omega))} \lesssim \|E_n\|_{\Omega}.
\]

Using this estimate and (5.21) combined with (5.22) leads to (5.20).

2) If \( \sigma_E \) is positive, then (5.18) only yields
\[
\|E_n\|_{\Omega} = o(1). \tag{5.23}
\]

Hence, to obtain a contradiction, it remains to show that
\[
\|H_n\|_{\Omega} = o(1). \tag{5.24}
\]

To do so, we first multiply (5.13) by \( \overline{H}_n \) and integrate in \( \Omega \) to get
\[
(\mu + \mathcal{L}_H(i\beta_n)) \int_{\Omega} |H_n|^2 \, dx + \frac{1}{i\beta_n} \int_{\Omega} \text{curl} \, E_n \cdot \overline{H}_n \, dx = o(1).
\]

Then using Green’s formula (2.12), we get
\[
(\mu + \mathcal{L}_H(i\beta_n)) \int_{\Omega} |H_n|^2 \, dx + \frac{1}{i\beta_n} \int_{\Omega} E_n \cdot \text{curl} \, \overline{H}_n \, dx = o(1).
\]

Now, we use (5.12) to get
\[
(\mu + \mathcal{L}_H(i\beta_n)) \int_{\Omega} |H_n|^2 \, dx - \left( \epsilon + \mathcal{L}_E(i\beta_n) \right) \int_{\Omega} |E_n|^2 \, dx = o(1). \tag{5.25}
\]

But we notice that (A.10) guarantees that
\[
|\mathcal{L}_E(i\beta_n)| + |\mathcal{L}_H(i\beta_n)| = o(1).
\]

This property combined with (5.5) allows to transform (5.25) into
\[
\mu \int_{\Omega} |H_n|^2 \, dx - \epsilon \int_{\Omega} |E_n|^2 \, dx = o(1).
\]

Therefore, (5.24) holds owing to (5.23).

3) If \( \sigma_H \) is positive, then (5.18) only yields (5.24) but the previous argument shows that then (5.23) holds.

The proof is then complete. \qed

**Remark 5.4.** Estimate (5.21) is in accordance with (3.12) because this last one combined with Lemma 5.1 shows that the resolvent of the transport operator is bounded (in the \( L^2_w \)-norm) in the imaginary axis.

This Lemma and Lemma 5.1 (resp. 5.2) directly yield the

**Corollary 5.5.** In addition to the assumptions of Lemma 4.2, assume that (5.4) holds with \( m = 0 \). Then the semigroup \( (e^{tA})_{t \geq 0} \) is exponentially stable, in particular the solution \( (E(t), H(t)) \) of (2.1), (1.6) and (1.7) tends exponentially to zero in \( \mathcal{H}_1 \).
Corollary 5.6. In addition to the assumptions of Lemma 4.2, assume that (5.4) holds with \( m > 0 \). Then the semigroup \( (e^{tA})_{t \geq 0} \) is polynomially stable, i.e.,
\[
\|e^{tA}U_0\| \lesssim t^{-\frac{m}{2}} \|U_0\|_{\mathcal{D}(A)}, \quad \forall U_0 \in \mathcal{D}(A), \quad \forall t > 1.
\]
In particular, the solution \((E(t), H(t))\) of (2.1), (1.6) and (1.7) satisfies
\[
\| (E(t), H(t)) \|_{\mathcal{H}_1} \lesssim t^{-\frac{m}{2}} \| (E_0, H_0) \|_{\mathcal{D}(A)}, \quad \forall (E_0, H_0) \in \mathcal{D}(B), \quad \forall t > 1.
\]

6. Some illustrative examples

6.1. Some dispersive models

All physical examples of dispersive models that we found in the literature (see [14], [16], [24, §11.2], [7], [5], and [20]) are summarized in the following example.

Let \( J \) be a positive integer and for all \( j \in \{1, \cdots, J\} \), let \( p_j, q_j \) be real-valued polynomials (of one variable). Let \( z_j \) be a complex number with \( \Re z_j = x_j < 0 \) and define
\[
\nu_E(t) = \sum_{j=1}^{J} (p_j(t) \cos(y_j t) + q_j(t) \sin(y_j t)) e^{x_j t},
\]
where \( y_j = \Im z_j \). Define similarly \( \nu_H \) by taking other polynomials \( p_j, q_j \) and other complex numbers \( z_j \) with negative real parts. For simplicity, we only examinate the case of \( \nu_E \), when necessary we will add the index \( E \) or \( H \) to distinguish polynomials related to \( \nu_E \) or \( \nu_H \).

First, it is easy to check that \( \nu_E \) satisfies (1.8) and (1.9). Furthermore by rewriting \( \nu_E \) in the equivalent form
\[
\nu_E(t) = \sum_{j=1}^{J} P_j(t) e^{x_j t},
\]
where \( P_j \) is a (complex-valued) polynomial of degree \( d_j \), we see that
\[
\mathcal{L} \nu_E(\lambda) = \sum_{j=1}^{J} \frac{d_j}{\lambda} \frac{P_j^{(\ell)}(0)}{(\lambda-z_j)^{\ell+1}},
\]
where \( P_j^{(\ell)} \) denotes the derivative of \( P_j \) of order \( \ell \). This means that \( i\omega \mathcal{L} \nu_E(i\omega) \) is a rational fraction in \( \omega \), more precisely
\[
\begin{align*}
& i\omega \mathcal{L} \nu_E(i\omega) = \frac{P_r(\omega)}{Q_r(\omega)} + \frac{P_t(\omega)}{Q_t(\omega)}, \\
& \text{where } P_r, Q_r, P_t, Q_t \text{ are real-valued polynomials such that } \deg P_r \leq \deg Q_r \text{ and } \deg P_t \leq \deg Q_t.
\end{align*}
\]
This means that (3.1) holds if and only if
\[
\frac{P_{E,r}(\omega)}{Q_{E,r}(\omega)} \geq 0 \quad \text{and} \quad \frac{P_{H,r}(\omega)}{Q_{H,r}(\omega)} \geq 0, \quad \forall \omega \in \mathbb{R}.
\]
Similarly, (4.1) is valid if and only if
\[
\begin{align*}
R(\omega) = \frac{P_{E,r}(\omega)}{Q_{E,r}(\omega)} + \frac{P_{H,r}(\omega)}{Q_{H,r}(\omega)} \quad &\text{satisfies} \\
& R(\omega) > 0, \forall \omega \in \mathbb{R}^*.
\end{align*}
\]
By writing

\[ R(\omega) = \sum_{n=0}^{N_1} a_n \omega^n - \sum_{n=0}^{N_2} b_n \omega^n, \]  

with \( N_1 \leq N_2, a_{N_1} \neq 0 \) and \( a_{N_2} \neq 0 \), we notice that two necessary conditions for (6.5) are

\[ N_2 - N_1 \text{ even and } \frac{a_{N_1}}{b_{N_2}} > 0. \]  

Finally, the last passivity assumption (5.4) is obviously related to the behavior at infinity of \( R(\omega) \). Using (6.6), we deduce that (5.4) holds with \( m = N_2 - N_1 \) if and only if (6.7) holds.

Let us finish this subsection by some particular cases.

**Example 6.1.** The Debye model [24, §11.2.1] corresponds to the choice \( \nu_H(t) = 0 \) and \( \nu_E(t) = \beta e^{-\frac{t}{\tau}} \), with \( \beta \) and \( \tau \) two positive real numbers. Hence

\[ \mathcal{L}\nu_E(\lambda) = \frac{\beta \tau}{\tau \lambda + 1}, \]

and we find

\[ R(\omega) = \frac{\beta \tau^2 \omega^2}{1 + \tau^2 \omega^2}. \]

This means that (3.1) and (4.1) hold and that (5.4) is valid with \( m = 0 \). Hence, by Corollary 5.5 we deduce the exponential decay of the energy if \( \Omega \) is simply connected with connected boundary (see [22, Theorem 4.12], where the first assumption is missing).

**Example 6.2.** The Lorentz model [24, §11.2.2] corresponds to the choice \( \nu_H(t) = 0 \) and

\[ \nu_E(t) = \beta \sin(\nu_0 t)e^{-\frac{t}{\lambda}}, \]

with \( \beta, \nu \) and \( \nu_0 \) three positive real numbers. Hence

\[ \mathcal{L}\nu_E(\lambda) = \frac{\beta \nu_0}{\nu_0^2 + \lambda^2 + \nu \lambda}, \]

with \( \omega_0^2 = \nu_0^2 + \nu^2/4 \). Then we easily check that (3.1) and (4.1) hold and that (5.4) is valid with \( m = 2 \). Hence, by Corollary 5.6 we deduce a decay of the energy as \( t^{-1} \) if \( \Omega \) is simply connected with connected boundary (see [22, Theorem 4.12], where the first assumption is missing).

**Example 6.3.** The Drude model [24, §11.2.3] (also called lossy Drude model) corresponds to the choice \( \nu_H(t) = 0 \) and

\[ \nu_E(t) = \beta (1 - e^{-\nu t}), \]

with \( \beta \) and \( \nu \) two positive real numbers. Hence,

\[ \mathcal{L}\nu_E(\lambda) = \frac{\beta \nu}{\nu \lambda + \lambda^2}. \]

Then we easily check that (3.1) and (4.1) hold and that (5.4) is valid with \( m = 2 \). Again we deduce a decay of the energy in \( t^{-1} \) if \( \Omega \) is simply connected and its boundary connected.

Other examples from [24, §11.2] also fall under our framework.
6.2. An academic example

For all \( j \in \mathbb{N}^* \), let \( z_j = x_j < 0 \) and let \( a_j, b_j \) be real-valued numbers such that
\[
\sum_{j=1}^{\infty} (|a_j| + |b_j|) < \infty.
\]

Then we can define
\[
\nu_E(t) = \sum_{j=1}^{\infty} (a_j \cos(y_j t) + b_j \sin(y_j t))e^{x_j t},
\]
(6.8)

where \( y_j = \Im z_j \). For simplicity, take \( \nu_H = 0 \).

Assuming that there exists \( \xi > 0 \) such that \( x_j \leq -\xi, \forall j \in \mathbb{N}^* \)
then we directly check that (1.8) and (1.9) hold.

Furthermore by rewriting \( \nu_E \) in the equivalent form
\[
\nu_E(t) = \sum_{j=1}^{\infty} A_j e^{z_j t},
\]
(6.9)

where \( A_j \) is a complex number such that
\[
\sum_{j=1}^{\infty} |A_j| < \infty,
\]
we see that
\[
L \nu_E(\lambda) = \sum_{j=1}^{\infty} \frac{A_j}{\lambda - z_j}.
\]

Now simple calculations show that for all \( \omega \in \mathbb{R}^* \), we have
\[
\Re(i\omega L \nu_E(i\omega)) = \omega^2 \sum_{j=1}^{\infty} \frac{\alpha_j}{x_j^2 + (\omega - y_j)^2} + \omega \sum_{j=1}^{\infty} \frac{x_j \beta_j - y_j \alpha_j}{x_j^2 + (\omega - y_j)^2},
\]
when \( A_j = \alpha_j + i\beta_j \), with \( \alpha_j, \beta_j \in \mathbb{R} \).

For the sake of simplicity, we now treat two different particular cases for which the second term of this right-hand side is zero.

1. Assume that \( y_j = \beta_j = 0 \), for all \( j \), then
\[
\Re(i\omega L \nu_E(i\omega)) = \omega^2 \sum_{j=1}^{\infty} \frac{\alpha_j}{x_j^2 + \omega^2}.
\]

Hence assuming further that
\[
x_j \geq -\Xi, \forall j \in \mathbb{N},
\]
(6.10)

for some positive real number \( \Xi \), we find that
\[
\xi^2 + \omega^2 \leq x_j^2 + \omega^2 \leq \Xi^2 + \omega^2, \forall j \in \mathbb{N},
\]
and consequently
\[
\Re(i\omega \mathcal{L} \nu_E(i\omega)) \geq \omega^2 \frac{a}{\xi^2 + \omega^2} + \frac{b}{\xi^2 + \omega^2} + \omega^2 \frac{((a+b)\omega^2 + a\xi^2 + b\Xi^2)}{(\Xi^2 + \omega^2)(\xi^2 + \omega^2)},
\]
where
\[
a = \sum_{j: \alpha_j > 0} \alpha_j, \quad b = \sum_{j: \alpha_j < 0} \alpha_j.
\]
This means that the assumptions
\[
a + b = \sum_{j=1}^{\infty} \alpha_j > 0 \quad \text{and} \quad a\xi^2 + b\Xi^2 \geq 0
\]
guarantee that (5.4) holds with \( m = 0 \) and hence an exponential decay of the energy (under the same assumptions on \( \Omega \) and its boundary as before). In contrast, if we assume that
\[
a + b = \sum_{j=1}^{\infty} \alpha_j = 0 \quad \text{and} \quad a\xi^2 + b\Xi^2 > 0,
\]
then (5.4) is valid with \( m = 2 \) and again we deduce a decay of the energy as \( t^{-1} \).

2. Assume that \( x_j \beta_j - y_j \alpha_j = 0 \), for all \( j \); then
\[
\Re(i\omega \mathcal{L} \nu_E(i\omega)) = \omega^2 \sum_{j: \alpha_j \neq 0} \frac{\alpha_j}{x_j^2 + \left(\omega - \frac{x_j \beta_j}{\alpha_j}\right)^2}.
\]
As before assuming further that (6.10) holds as well as
\[
\frac{\beta_j^2}{\alpha_j} \leq \Lambda, \forall j : \alpha_j \neq 0,
\]
for some positive real number \( \Lambda \), one can show that there exist four positive constants \( c, C, \theta, \Theta \), with \( c < 1 < C \), such that
\[
c(\theta^2 + \omega^2) \leq x_j^2 + \left(\omega - \frac{x_j \beta_j}{\alpha_j}\right)^2 \leq C(\Theta^2 + \omega^2), \quad \forall j \in \mathbb{N} : \alpha_j \neq 0,
\]
Therefore,
\[
\Re(i\omega \mathcal{L} \nu_E(i\omega)) \geq \omega^2 \frac{a}{C(\Theta^2 + \omega^2)} + \frac{b}{c(\theta^2 + \omega^2)} \geq \omega^2 \frac{((ac + bC)\omega^2 + ac\theta^2 + bC\Theta^2)}{cC(\Xi^2 + \omega^2)(\xi^2 + \omega^2)}.
\]
Thus, the assumptions
\[
ac + bC > 0 \quad \text{and} \quad ac\theta^2 + bC\Theta^2 \geq 0
\]
guarantee an exponential decay of the energy, while the conditions
\[
ac + bC = 0 \quad \text{and} \quad ac\theta^2 + bC\Theta^2 > 0,
\]
yield a decay of the energy as \( t^{-1} \).
6.3. Another academic example

Take $\nu_H = 0$ and

$$\nu_E(t) = e^{-t^2}, \forall t \geq 0.$$  

Then we easily check that (1.8) and (1.9) hold. Furthermore by Cauchy’s theorem, one sees that

$$i\omega \mathcal{L}(\nu_E)(i\omega) = e^{-\frac{\omega^2}{4}}(i\sqrt{\frac{\pi}{2}}\omega + |\omega|I_{|\omega|}), \forall \omega \in \mathbb{R}^*, $$

and

$$\int_0^\infty e^{-t^2} dt = \sqrt{\frac{\pi}{2}}$$

Hence

$$\Re(i\omega \mathcal{L}(\nu_E)(i\omega)) = e^{-\frac{\omega^2}{4}}|\omega|I_{|\omega|},$$

which means that (3.1) and (4.1) hold. On the other hand, as

$$I_{|\omega|} \to \infty \quad \text{as} \quad |\omega| \to \infty,$$

by L'Hôpital’s rule, we have

$$\lim_{\omega \to \infty} \frac{I_\omega}{\omega^{-1}e^{\frac{\omega^2}{4}}} = \lim_{\omega \to \infty} \frac{1}{\frac{1}{2} - \frac{1}{\omega^2}} = 2$$

and we deduce that

$$I_{|\omega|} \sim |\omega|^{-1}e^{\frac{\omega^2}{4}}, \forall |\omega| \text{ large }.$$ 

Hence for $\omega$ large enough, one concludes that

$$\Re(i\omega \mathcal{L}(\nu_E)(i\omega)) \gtrsim 1,$$

which means that again (5.4) is valid with $m = 0$ and by Corollary 5.5 we deduce the exponential decay of the energy.

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A. Some properties of the Laplace transform

In this section, we state some results for the Laplace transform of kernels in $K$

**Lemma A.1.** Let $\nu \in K$, then we have

$$\nu'(t) = -\int_t^\infty \nu''(y) dy.$$  

(A.1)
Proof. First we notice that (1.8) is equivalent to the identity
\[ \nu'(0) + \int_0^\infty \nu''(y) \, dy = 0, \] (A.2)
simply because Lebesgue’s bounded convergence theorem guarantees that
\[ \int_0^\infty \nu''(y) \, dy = \lim_{R \to \infty} \int_0^R \nu''(y) \, dy. \]
As \( \nu \) is twice differentiable, we may write
\[ \nu'(t) = \nu'(0) + \int_0^t \nu''(y) \, dy \]
and by (A.2), we get (A.1). \( \square \)

Corollary A.2. Let \( \nu \in K \), then we have
\[ |\nu'(t)| \lesssim e^{-\delta t}, \quad \forall t \geq 0, \] (A.3)
as well as
\[ |\nu(t)| \lesssim 1 + e^{-\delta t}, \quad \forall t \geq 0. \] (A.4)

Proof. Estimate (A.3) directly follows from (A.1) and assumption (1.9).
For the second estimate, we again may write
\[ \nu(t) = \nu(0) + \int_0^t \nu'(y) \, dy, \]
and we conclude by (A.3). \( \square \)

The previous results allow to give a meaning to the Fourier–Laplace transform of \( \nu \in K \) defined by
\[ \mathcal{L}\nu(\lambda) = \int_0^\infty e^{-\lambda s} \nu(s) \, ds, \] (A.5)
for all \( \lambda \in \mathbb{C} \) such that \( \Re \lambda > 0 \). Furthermore, the following identities will be valid
\[ \lambda \mathcal{L}\nu(\lambda) = \nu(0) + \mathcal{L}\nu'(\lambda), \] (A.6)
\[ \lambda \mathcal{L}\nu(\lambda) = \nu(0) + \frac{1}{\lambda} (\nu'(0) + \mathcal{L}\nu''(\lambda)), \] (A.7)
for all \( \lambda \in \mathbb{C} \) such that \( \Re \lambda > 0 \).
As estimate (A.3) guarantees that \( \nu' \) is integrable, by Lebesgue’s bounded convergence theorem we deduce that
\[ \mathcal{L}\nu'(\lambda) \to 0 \text{ as } \Re \lambda \to \infty; \]
by (A.6), we then deduce that
\[ \mathcal{L}\nu(\lambda) \to 0 \text{ as } \Re \lambda \to \infty. \] (A.8)
Finally, since for \( \nu \in K \) its derivative is exponentially decaying at infinity (see (A.3)), the Fourier–Laplace transform of \( \nu' \) is also well-defined on the imaginary axis and the mapping
\[ \Re \to \mathbb{C} : \omega \to \mathcal{L}\nu'(i\omega) \]
is continuous and bounded. In view to (A.6), we then have (in the distributional sense)

\[ i\omega L\nu(i\omega) = \nu(0) + L\nu'(i\omega), \forall \omega \in \mathbb{R}, \]  
(A.9)

and consequently the mapping

\[ \omega \rightarrow i\omega L\nu(i\omega) \]

is continuous on \( \mathbb{R} \) and bounded. (A.10)

Note also that for \( \nu \in K \), and any \( \omega \in \mathbb{R} \), \( L\nu'(i\omega) \) corresponds to the Fourier transform of \( \tilde{\nu}' \), the extension by zero of \( \nu' \) in \((-\infty, 0)\), as

\[ L\nu'(i\omega) = \int_{-\infty}^{\infty} e^{-i\omega s} \nu'(s) \, ds = \int_{-\infty}^{\infty} e^{-i\omega s} \tilde{\nu}'(s) \, ds. \]

References

[1] Amrouche, C., Bernardi, C., Dauge, M., Girault, V.: Vector potentials in three-dimensional nonsmooth domains. Math. Methods Appl. Sci. 21, 823–864 (1998)
[2] Arendt, W., Batty, C.J.K.: Tauberian theorems and stability of one-parameter semigroups. Trans. Am. Math. Soc. 306(2), 837–852 (1988)
[3] Arendt, W., Batty, C.J.K., Hieber, M., Neubrander, F.: Vector-Valued Laplace Transforms and Cauchy Problems, Volume of 96 Monographs in Mathematics, vol. of. Birkhäuser, Basel (2001)
[4] Bátkai, A., Engel, K.-J., Prüss, J., Schnaubelt, R.: Polynomial stability of operator semigroups. Math. Nachr. 279(13–14), 1425–1440 (2006)
[5] Bécache, E., Joly, P., Vinoles, V.: On the analysis of perfectly matched layers for a class of dispersive media and application to negative index metamaterials. Math. Comput. 87(314), 2775–2810 (2018)
[6] Borichev, A., Tomilov, Y.: Optimal polynomial decay of functions and operator semigroups. Math. Ann. 347(2), 455–478 (2010)
[7] Cassier, M., Joly, P., Kachanovska, M.: Mathematical models for dispersive electromagnetic waves: an overview. Comput. Math. Appl. 74(11), 2792–2830 (2017)
[8] Conti, M., Gatti, S., Pata, V.: Uniform decay properties of linear Volterra integro-differential equations. Math. Models Methods Appl. Sci. 18(1), 21–45 (2008)
[9] Danese, V., Geredeli, P.G., Pata, V.: Exponential attractors for abstract equations with memory and applications to viscoelasticity. Discrete Contin. Dyn. Syst. 35(7), 2881–2904 (2015)
[10] Giorgi, C., Naso, M.G., Pata, V.: Energy decay of electromagnetic systems with memory. Math. Models Methods Appl. Sci. 15(10), 1489–1502 (2005)
[11] Girault, V., Raviart, P.-A.: Finite Element Methods for Navier–Stokes Equations Theory and Algorithms, Volume 5 of Springer Series in Computational. Springer, Berlin (1986)
[12] Huang, F.L.: Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces. Ann. Differ. Equ. 1(1), 43–56 (1985)
[13] Ioannidis, A.D., Kristensson, G., Stratis, I.G.: On the well-posedness of the Maxwell system for linear bianisotropic media. SIAM J. Math. Anal. 44(4), 2459–2473 (2012)
[14] Jackson, J.D.: Classical Electrodynamics, 1st edn. Wiley, New York (1962)
[15] Kristensson, G., Karlsson, A., Rikte, S.: Electromagnetic wave propagation in dispersive and complex material with time domain techniques. Inverse Probl. 1, 277–294 (2002)
[16] Kristensson, G., Rikte, S., Siivola, A.: Mixing formulas in the time domain. J. Opt. Soc. Am. A 15(5), 1411–1422 (1998)
[17] Liu, Z., Rao, B.: Characterization of polynomial decay rate for the solution of linear evolution equation. Z. Angew. Math. Phys. 56(4), 630–644 (2005)
[18] Lyubich, Y.I., Vü, Q.P.: Asymptotic stability of linear differential equations in Banach spaces. Stud. Math. 88(1), 37–42 (1988)
[19] Muñoz Rivera, J.E., Naso, M.G., Vuk, E.: Asymptotic behaviour of the energy for electromagnetic systems with memory. Math. Methods Appl. Sci. 27(7), 819–841 (2004)
[20] Nguyen, H.-M., Vinoles, V.: Electromagnetic wave propagation in media consisting of dispersive metamaterials. C. R. Math. Acad. Sci. Paris 356(7), 757–775 (2018)
[21] Nicaise, S.: Exact boundary controllability of Maxwell’s equations in heterogeneous media and an application to an inverse source problem. SIAM J. Control Optim. 38(4), 1145–1170 (2000). (electronic)
[22] Nicaise, S.: Stabilization and asymptotic behavior of dispersive medium models. Syst. Control Lett. 61(5), 638–648 (2012)
[23] Prüss, J.: On the spectrum of $C_0$-semigroups. Trans. Am. Math. Soc. \textbf{284}(2), 847–857 (1984)
[24] Sihvola, A.: Electromagnetic Mixing Formulas and Applications. Number 47 in Electromagnetic Waves Series. IET, London (1999)

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