TRIANGULATIONS OF SIMPLICIAL COMPLEXES AND THETA POLYNOMIALS

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Abstract. The theta polynomial of a triangulation $\Delta$ of a ball with boundary $\partial \Delta$ is defined as the difference of the $h$-polynomial of $\partial \Delta$ from the $h$-polynomial of $\Delta$. A basic theory for the face enumeration of triangulations of simplicial complexes, parallel to that of Stanley, is developed in which the role of local $h$-polynomials is played by theta polynomials. Positivity and monotonicity properties of theta polynomials are investigated and shown to have strong implications in the study of $h$-polynomials and local $h$-polynomials. In particular, some unimodality and gamma-positivity properties of barycentric subdivisions are extended to larger classes of triangulations, including antiprism triangulations.

1. Introduction

A theory for the face enumeration of triangulations $\Delta'$ of a simplicial complex $\Delta$ was developed by Stanley [23, Part I], in order to study the behavior of the $h$-polynomials of such triangulations (see Section 2 for any undefined terminology) and their monotonicity properties. A major role in this theory is played by the concept of the local $h$-polynomial of the restriction of $\Delta'$ to a face $F \in \Delta$, which appears in Stanley's locality formula [23, Theorem 3.2] (see also Theorem 2.1) as the 'local contribution' to $h(\Delta', x)$ of $\Delta'$ at $F$.

This paper develops the basics of a parallel theory, in which the local $h$-polynomial is replaced by a simpler enumerative invariant, named theta polynomial. There are good reasons for doing so. Most importantly, in the analogue of Stanley's locality formula (see Theorem 3.4), the $h$-polynomials of the links of the faces of $\Delta$ are replaced by the $h$-polynomials of their barycentric subdivisions. The latter have strong unimodality and $\gamma$-positivity properties which, via the formula, can be transferred to the $h$-polynomial of $\Delta'$. One disadvantage of our theory is that theta polynomials do not always have nonnegative coefficients (although they do have symmetric coefficients). However, mild assumptions (see Proposition 3.6) imply nonnegativity and the theory works well if one is willing to focus on triangulations of $\Delta$ which can be roughly thought of as those having at least one interior vertex at every nonempty face of $\Delta$.

The motivation for studying theta polynomials comes from several directions. First, a locality type formula for local $h$-polynomials, in which theta polynomials of barycentric subdivisions appear, is established (see Theorem 3.4).

Date: September 7, 2022.

Mathematics Subject Classifications: Primary: 05E45; Secondary: 13C14, 55U10.

Key words and phrases. Simplicial complex, homology ball, homology sphere, $h$-polynomial, local $h$-polynomial, unimodality, $\gamma$-positivity.
subdivisions appear, was already discovered by Juhnke-Kubitzke, Murai and Sieg [17, Theorem 4.4] (see also Theorem 3.5). The special properties of these theta polynomials are crucial there to deduce that local $h$-polynomials of barycentric subdivisions of regular cell decompositions of the simplex are $\gamma$-positive. Second, theta polynomials play a crucial role in studying the real-rootedness of uniform triangulations of simplicial complexes [6] (a class of triangulations which includes barycentric subdivisions), and antiprism triangulations in particular [7]. Third, a unimodality result for theta polynomials follows from a recent theorem of Adiprasito and Yashfe [3, Theorem 50] (see Theorem 5.1). Moreover, the general problem motivating this paper is to extend the known strong unimodality, $\gamma$-positivity and real-rootedness properties of barycentric subdivisions to more general types of triangulations, as was the case in [6, 7, 8].

This paper is organized as follows. Section 2 includes background material on the face enumeration of simplicial complexes and their triangulations (barycentric subdivisions, in particular). For simplicity, we consider only geometric triangulations of simplicial complexes in this paper, although the theory can be suitably generalized to the quasi-geometric topological (simplicial) subdivisions, considered in [23, Part I]. Section 3 defines the theta polynomial of any homology ball $\Delta$ as

$$\theta(\Delta, x) = h(\Delta, x) - h(\partial \Delta, x)$$

and studies its main properties. The monotonicity of $\theta(\Delta, x)$ under triangulations of $\Delta$ is studied in Section 4. Sections 3 and 4 include several applications of their main results (Theorems 3.4 and 4.2) to the unimodality and $\gamma$-positivity of $h$-polynomials and local $h$-polynomials of triangulations of Cohen–Macaulay simplicial complexes and simplices, respectively. Section 5 discusses the unimodality and $\gamma$-positivity of theta polynomials of homology balls and conjectures natural conditions under which $\gamma$-positivity holds (see Conjecture 5.4). This conjecture is shown to be equivalent to the Link Conjecture for flag homology spheres, already proposed by Chudnovsky and Nevo [13] as a strengthening of Gal’s conjecture [15], a fact which advocates for its validity. As an application of the results of previous sections, Section 6 answers some questions about the $\gamma$-positivity of antiprism triangulations, posed in [7, Section 8].

2. Triangulations and face enumeration

This section includes definitions and background on simplicial complexes, triangulations and their enumerative invariants which are of primary interest in this paper. We denote by $|V|$ the cardinality and by $2^V$ the power set of a finite set $V$.

Polynomials. A polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{R}[x]$ is called

- symmetric, with center of symmetry $n/2$, if $a_i = a_{n-i}$ for all $0 \leq i \leq n$,
- unimodal, with a peak at position $k$, if $a_0 \leq a_1 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_n$,
- $\gamma$-positive, with center of symmetry $n/2$, if $p(x) = \sum_{i=0}^{[\frac{n}{2}]} \gamma_i x^i (1 + x)^{n-2i}$ for some nonnegative real numbers $\gamma_0, \gamma_1, \ldots, \gamma_{[\frac{n}{2}]}$,
- real-rooted, if every root of $p(x)$ is real, or $p(x) \equiv 0$. 
Every $\gamma$-positive polynomial is symmetric and unimodal and every real-rooted and symmetric polynomial with nonnegative coefficients is $\gamma$-positive; see [5, 10, 22] for more information on the connections among these concepts.

Every polynomial $p(x) \in \mathbb{R}[x]$ of degree at most $n$ can be written uniquely in the form $p(x) = a(x) + xb(x)$ for some polynomials $a(x), b(x) \in \mathbb{R}[x]$ of degrees at most $n$ and $n-1$, which are symmetric with centers of symmetry $n/2$ and $(n-1)/2$, respectively. This expression is called the symmetric decomposition of $p(x)$ with respect to $n$; see [8, 11] and references therein. We say that this decomposition is nonnegative, unimodal, $\gamma$-positive or real-rooted if both $a(x)$ and $b(x)$ have the corresponding property. We note that every polynomial which has a nonnegative and unimodal symmetric decomposition with respect to $n$ is unimodal, with a peak at $\lceil (n+1)/2 \rceil$.

**Simplicial complexes.** All simplicial complexes we consider will be abstract and finite. Thus, given a finite set $\Omega$, a simplicial complex on the ground set $\Omega$ is a collection $\Delta$ of subsets of $\Omega$ such that $F \subseteq G \in \Delta \Rightarrow F \in \Delta$. The elements of $\Delta$ are called faces. The dimension of a face $F$ is defined as one less than the cardinality of $F$. The dimension of $\Delta$ is the maximum dimension of a face and is denoted by $\dim(\Delta)$. Faces of $\Delta$ of dimension zero or one are called vertices or edges, respectively, and those which are maximal with respect to inclusion are called facets. The link of the face $F \in \Delta$ is the subcomplex of $\Delta$ defined as $\text{link}_\Delta(F) = \{G \setminus F : G \in \Delta, \ F \subseteq G\}$; in particular, $\text{link}_\Delta(\emptyset) = \Delta$.

All topological properties of $\Delta$ we mention will refer to those of the geometric realization of $\Delta$ [9, Section 9], uniquely defined up to homeomorphism. All homologic properties of $\Delta$ will be considered with respect to a fixed field $k$. Thus, $\Delta$ is said to be Cohen–Macaulay (over $k$) if

$$\widetilde{H}_i \left( \text{link}_\Delta(F), k \right) = 0$$

for every $F \in \Delta$ and every $i < \dim \text{link}_\Delta(F)$, where $\widetilde{H}_*(\Gamma, k)$ denotes reduced simplicial homology of $\Gamma$ (with coefficients in $k$). Moreover, $\Delta$ is called a homology sphere (over $k$) if

$$\widetilde{H}_i \left( \text{link}_\Delta(F), k \right) = \begin{cases} k, & \text{if } i = \dim \text{link}_\Delta(F) \\ 0, & \text{otherwise} \end{cases}$$

for every $F \in \Delta$ and every $i$. An $(n-1)$-dimensional simplicial complex $\Delta$ is called a homology ball (over $k$) if there exists a subcomplex $\partial \Delta$ of $\Delta$, called the boundary of $\Delta$, with the following properties:

- $\partial \Delta$ is an $(n-2)$-dimensional homology sphere (over $k$),
- for every $F \in \Delta$ and every $i$,

$$\widetilde{H}_i \left( \text{link}_\Delta(F), k \right) = \begin{cases} k, & \text{if } F \notin \partial \Delta \text{ and } i = \dim \text{link}_\Delta(F) \\ 0, & \text{otherwise} \end{cases}$$

The interior of $\Delta$ is then defined as $\text{int}(\Delta) = \Delta \setminus \partial \Delta$.

We assume familiarity with basic properties of Cohen–Macaulay simplicial complexes [9, Section 11] [25, Chapter II] and with those of homology balls and spheres as explained, for
instance, in [4, Section 2B]. We note, in particular, that the complex \( \text{link}_\Delta(F) \) is Cohen–Macaulay for every Cohen–Macaulay simplicial complex \( \Delta \) and every \( F \in \Delta \). Moreover, if \( \Delta \) is a homology ball or a homology sphere, then \( \text{link}_\Delta(F) \) is a homology sphere for every \( F \in \text{int}(\Delta) \) and for every \( F \in \Delta \), respectively, and if \( \Delta \) is a homology ball, then \( \text{link}_\Delta(F) \) is a homology ball with boundary \( \partial \text{link}_\Delta(F) = \text{link}_\partial \Delta(F) \) for every \( F \in \partial \Delta \). Every cone over a homology sphere \( \Delta \) (meaning, the simplicial complex \( \Delta \cup \{ F \cup \{ v \} : F \in \Delta \} \), where \( v \) is a new vertex, not in \( \Delta \)) is a homology ball with boundary \( \partial \text{link}_\Delta(F) \).

Following [8], we use the term Cohen–Macaulay* instead of uniformly Cohen–Macaulay for the class of simplicial complexes introduced and studied by Matsuoka and Murai [19]. Thus, a Cohen–Macaulay simplicial complex \( \Delta \) (over \( k \)) is called Cohen–Macaulay* (over \( k \)) if the simplicial complex obtained from \( \Delta \) by removing any of its facets is Cohen–Macaulay (over \( k \)) of the same dimension as \( \Delta \). This class of simplicial complexes includes all doubly Cohen–Macaulay simplicial complexes [25, p. 94] (and, in particular, all homology spheres).

A convenient way to record the face numbers of a simplicial complex \( \Delta \) is the \( h \)-polynomial, defined by the formula

\[
 h(\Delta, x) = \sum_{i=0}^{n} f_{i-1}(\Delta) x^i (1 - x)^{n-i},
\]

where \( f_{i}(\Delta) \) is the number of \( i \)-dimensional faces of \( \Delta \) and \( n-1 \) is its dimension. The sequence \( h(\Delta) = (h_0(\Delta), h_1(\Delta), \ldots, h_n(\Delta)) \) is the \( h \)-vector of \( \Delta \). The polynomial \( h(\Delta, x) \) has nonnegative coefficients for every Cohen–Macaulay complex \( \Delta \) (in particular, for homology balls and spheres). Moreover, it is symmetric, with center of symmetry \( n/2 \), if \( \Delta \) is a homology sphere and has the property that \( x^n h(\Delta, 1/x) = h(\text{int}(\Delta), x) \) if \( \Delta \) is a homology ball, where \( h(\text{int}(\Delta), x) \) is defined by the right-hand side of Equation (2) if \( f_{i-1}(\Delta) \) is replaced by the number of \( (i-1) \)-dimensional faces of \( \text{int}(\Delta) \). In particular, \( h_n(\Delta) = 0 \) for every homology ball \( \Delta \) of dimension \( n-1 \). We refer to [25] for the significance and for more information on \( h \)-vectors of Cohen–Macaulay simplicial complexes.

**Triangulations.** By the term triangulation of a simplicial complex \( \Delta \) we will always mean a geometric triangulation. Thus, a simplicial complex \( \Delta' \) is a triangulation of \( \Delta \) if there exists a geometric realization \( K' \) of \( \Delta' \) which geometrically subdivides a geometric realization \( K \) of \( \Delta \). The carrier of a face \( F' \in \Delta' \) is the smallest face \( F \in \Delta \) for which the face of \( K \) corresponding to \( F \) contains the face of \( K' \) corresponding to \( F' \). The restriction of \( \Delta' \) to a face \( F \in \Delta \) is a triangulation of the simplex \( 2^F \) denoted by \( \Delta_F \).

The local \( h \)-polynomial of a triangulation \( \Gamma \) of a simplex \( 2^V \) was defined by Stanley [23, Definition 2.1] by the formula

\[
 \ell_V(\Gamma, x) = \sum_{F \subseteq V} (-1)^{|V \setminus F|} h(\Gamma_F, x).
\]

Stanley [23] showed that \( \ell_V(\Gamma, x) \) has nonnegative coefficients and that it is symmetric, with center of symmetry \( |V|/2 \). He exploited these properties and the formula of the
following theorem in order to prove nontrivial results about the face enumeration of triangulations of a simplicial complex $\Delta$. We recall that $\Delta$ is called pure if all its facets have the same dimension.

**Theorem 2.1.** ([23, Theorem 3.2]) For every pure simplicial complex $\Delta$ and every triangulation $\Delta'$ of $\Delta$,

$$h(\Delta', x) = \sum_{F \in \Delta} \ell_F(\Delta', x) h(\text{link}_\Delta(F), x).$$

Stellar subdivisions provide a simple way to triangulate a simplicial complex $\Delta$. Given a face $F \in \Delta$, the stellar subdivision of $\Delta$ on $F$ (with new vertex $v$) is the simplicial complex obtained from $\Delta$ by removing all its faces containing $F$ and adding all sets of the form $\{v\} \cup E \cup E'$, where $E \subseteq F$ and $E' \in \text{link}_\Delta(F)$.

**Barycentric subdivisions.** The barycentric subdivision of a simplicial complex $\Delta$ is defined as the simplicial complex $\text{sd}(\Delta)$ on the vertex set $\Delta \setminus \{\emptyset\}$ whose faces are the chains $F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k$ of nonempty faces of $\Delta$. This simplicial complex is naturally a triangulation of $\Delta$; the carrier of the chain $F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k$ is its top element $F_k \in \Delta$. As a result, $\partial(\text{sd}(\Delta)) = \text{sd}(\partial \Delta)$ for every homology ball $\Delta$.

The face enumeration of $\text{sd}(\Delta)$ was studied by Brenti and Welker [12]. We summarize some of their results in the form of the following proposition.

**Proposition 2.2.** ([12]) For integers $0 \leq k \leq n$ there exist polynomials $p_{n,k}(x)$ of degree at most $n$ with nonnegative coefficients, such that

$$h(\text{sd}(\Delta), x) = \sum_{k=0}^{n} h_k(\Delta)p_{n,k}(x)$$

for every $(n-1)$-dimensional simplicial complex $\Delta$. The $p_{n,k}(x)$ satisfy the recurrence

$$p_{n,k}(x) = x \sum_{i=0}^{k-1} p_{n-1,i}(x) + \sum_{i=k}^{n} p_{n-1,i}(x)$$

for every $n \geq 1$ and all $0 \leq k \leq n$, with the initial condition $p_{0,0}(x) = 1$, and have the property that

$$p_{n,n-k}(x) = x^n p_{n,k}(1/x)$$

for $0 \leq k \leq n$.

For example,

$$p_{3,k}(x) = \begin{cases} 1 + 4x + x^2, & \text{if } k = 0 \\ 4x + 2x^2, & \text{if } k = 1 \\ 2x + 4x^2, & \text{if } k = 2 \\ x + 4x^2 + x^3, & \text{if } k = 3 \end{cases} \quad \text{and} \quad p_{4,k}(x) = \begin{cases} 1 + 11x + 11x^2 + x^3, & \text{if } k = 0 \\ 8x + 14x^2 + 2x^3, & \text{if } k = 1 \\ 4x + 16x^2 + 4x^3, & \text{if } k = 2 \\ 2x + 14x^2 + 8x^3, & \text{if } k = 3 \\ x + 11x^2 + 11x^3 + x^4, & \text{if } k = 4. \end{cases}$$
Brenti and Welker [12, Theorem 2] showed that \( h(\text{sd}(\Delta), x) \) is real-rooted, hence unimodal, for every Cohen–Macaulay simplicial complex \( \Delta \) (more generally, for every Boolean cell complex \( \Delta \) with nonnegative \( h \)-vector). The location of the peak of \( h(\text{sd}(\Delta), x) \) was studied by Kubitzke and Nevo [18] using methods of commutative algebra, and by Murai [20] using combinatorial methods. The following statement is a more detailed version of [18, Corollary 4.7] (and applies to Boolean cell complexes with nonnegative \( h \)-vector as well), which we will use in Section 3 to obtain a more general result.

**Proposition 2.3.** Let \( \Delta \) be any \((n-1)\)-dimensional Cohen–Macaulay simplicial complex. Then, \( h(\text{sd}(\Delta), x) \) can be written as a sum of three polynomials with nonnegative, symmetric and unimodal coefficients and centers of symmetry \( (n-1)/2 \), \( n/2 \) and \( (n+1)/2 \), respectively.

In particular, \( h(\text{sd}(\Delta), x) \) is unimodal, with a peak at position \( n/2 \), if \( n \) is even, and at \((n-1)/2\) or \((n+1)/2\), if \( n \) is odd.

The proof uses the following lemma. We recall that a real-rooted polynomial \( p(x) \), with roots \( \alpha_1 \geq \alpha_2 \geq \cdots \), interlaces a real-rooted polynomial \( q(x) \), with roots \( \beta_1 \geq \beta_2 \geq \cdots \), if \( \cdots \leq \alpha_2 \leq \beta_2 \leq \alpha_1 \leq \beta_1 \).

**Lemma 2.4.** The polynomial \( p_{n,k}(x) \) has a nonnegative, real-rooted (in particular, unimodal and \( \gamma \)-positive) symmetric decomposition with respect to \( n \), for all \( n/2 \leq k \leq n \).

**Proof.** By [11, Theorem 2.6], it suffices to show that \( p_{n,k}(x) \) has a nonnegative symmetric decomposition with respect to \( n \) and that it is real-rooted and interlaced by \( x^n p_{n,k}(1/x) \), for \( n/2 \leq k \leq n \). The first claim is a special case of [3, Lemma 4.2], from which explicit formulas for the symmetric parts of \( p_{n,k}(x) \) in terms of the polynomials \( p_{n-1,i}(x) \) can also be deduced. Given that \( x^n p_{n,k}(1/x) = p_{n,n-k}(x) \) (see Proposition 2.2), the second claim follows from the fact that \( (p_{n,0}(x), p_{n,1}(x), \ldots, p_{n,n}(x)) \) is an interlacing sequence of real-rooted polynomials; see, for instance, [10, Example 7.8.8].

**Proof of Proposition 2.3**  By Lemma 2.4 and the symmetry property (7) we know that \( p_{n,k}(x) \) can be written as a sum of two polynomials with nonnegative, symmetric and unimodal coefficients and centers of symmetry \( n/2 \) and \((n+1)/2\) (respectively, \( n/2 \) and \((n-1)/2\)) for all \( n/2 \leq k \leq n \) (respectively, \( 0 \leq k \leq n/2 \)). Given that, the proof follows from Equation (5) and the fact that \( h_k(\Delta) \geq 0 \) for every \( k \).

The local \( h \)-polynomial \( \ell_V(\text{sd}(2^V), x) \) of the barycentric subdivision of the \((n-1)\)-dimensional simplex \( 2^V \) is called the \( n \)th derangement polynomial and is denoted by \( d_n(x) \). A simple combinatorial interpretation in terms of permutation enumeration was given in [23, Proposition 2.4]; see also [5, Section 3.3.1]. The polynomial \( d_n(x) \) is \( \gamma \)-positive, hence unimodal, for every \( n \); see [5, Section 2.1.4] and references therein.

3. Basic properties of theta polynomials

This section discusses basic properties of theta polynomials and some of their immediate consequences for \( h \)-polynomials and local \( h \)-polynomials.
Definition 3.1. The theta polynomial is defined as \( \theta(\Delta, x) = h(\Delta, x) - h(\partial \Delta, x) \) for any homology ball \( \Delta \).

The formula for \( \theta(\Delta, x) \) in the following proposition is equivalent to the one for \( h(\partial \Delta, x) \), given in [24, Lemma 2.3]; see also Section 5. The last statement follows directly from the definitions of \( h(\Delta, x) \) and \( h(\partial \Delta, x) \).

Proposition 3.2. ([24]) For every \( n \geq 1 \) and every \((n-1)\)-dimensional homology ball \( \Delta \),

\[
\theta(\Delta, x) = \sum_{i=1}^{n-1} (h_{n-1}(\Delta) + \cdots + h_{n-i}(\Delta) - h_0(\Delta) - \cdots - h_{i-1}(\Delta)) x^i.
\]

In particular, \( \theta(\Delta, x) \) is symmetric with center of symmetry \( n/2 \), i.e., \( x^n \theta(\Delta, 1/x) = \theta(\Delta, x) \). Moreover, it has zero constant term and the coefficient of \( x \) is equal to one less than the number of interior vertices of \( \Delta \).

Example 3.3. (a) For an \((n-1)\)-dimensional simplex \( \Delta = 2^V \) we have

\[
\frac{1}{2}
\]

since \( h(\partial \Delta, x) = 1 + x + \cdots + x^{n-1} \) for \( n \geq 1 \). Note that \( \Delta = \{ \emptyset \} \), for \( n = 0 \).

(b) By Proposition 3.2 we have

\[
\theta(\Delta, x) = \begin{cases} 
(r-1)x, & \text{if } \dim(\Delta) = 1, \\
(r-1)(x + x^2), & \text{if } \dim(\Delta) = 2,
\end{cases}
\]

where \( r \) is the number of interior vertices of \( \Delta \).

(c) Since the coning operation leaves the \( h \)-polynomial invariant, we have \( \theta(\Delta, x) = 0 \) for every homology ball \( \Delta \) which is the cone over a homology sphere.

(d) The coefficient of \( x^2 \) in \( \theta(\Delta, x) \) is equal to \( f_1^1(\Delta) - f_0^1(\Delta) - (n-2) f_0^0(\Delta) + n - 1 \) for every \((n-1)\)-dimensional homology ball \( \Delta \), where \( f_0^0(\Delta) \) and \( f_1^1(\Delta) \) are the numbers of interior vertices and interior edges of \( \Delta \), respectively. \( \square \)

The significance of theta polynomials stems from the following two theorems, the second of which was discovered and proven in [17] in order to show that the local \( h \)-polynomial \( \ell_V(\text{sd}(C), x) \) is \( \gamma \)-positive for every regular cell decomposition \( C \) of the simplex \( 2^V \).

Theorem 3.4. For every pure simplicial complex \( \Delta \) and every triangulation \( \Delta' \) of \( \Delta \),

\[
h(\Delta', x) = \sum_{F \in \Delta} \theta(\Delta'_F, x) h(\text{sd(link}_\Delta(F)), x).
\]

Theorem 3.5. ([17, Theorem 4.4]) For every triangulation \( \Gamma \) of the simplex \( 2^V \),

\[
\ell_V(\Gamma, x) = \sum_{F \subseteq V} \theta(\Gamma_F, x) d_{|V \setminus F|}(x).
\]
Proof of Theorem 3.4. Applying Theorems 2.1 and 3.5 successively, changing the order of summation and applying Theorem 2.1 once more, with $\Delta$ and $\Delta'$ replaced by $\text{link}_\Delta(F)$ and its barycentric subdivision, respectively, we get

\[
h(\Delta', x) = \sum_{G \subseteq \Delta} \ell_G(\Delta'_G, x) h(\text{link}_\Delta(G), x)
\]

\[
= \sum_{G \subseteq \Delta} \left( \sum_{F \subseteq G} \theta(\Delta'_F, x) d_{[G \setminus F]}(x) \right) h(\text{link}_\Delta(G), x)
\]

\[
= \sum_{F \subseteq \Delta} \theta(\Delta'_F, x) \left( \sum_{F \subseteq G \in \Delta} h(\text{link}_\Delta(G), x) d_{[G \setminus F]}(x) \right)
\]

\[
= \sum_{F \subseteq \Delta} \theta(\Delta'_F, x) h(\text{sd}(\text{link}_\Delta(F)), x)
\]

and the proof follows. \hfill \Box

We will say that a homology ball $\Delta$ has the \textit{interior vertex property} if no facet of $\Delta$ has all its vertices on $\partial \Delta$ (equivalently, if every facet of $\Delta$ has an interior vertex). We will also say that $\Delta$ is

- \textit{theta positive}, if $\theta(\Delta, x)$ has nonnegative coefficients;
- \textit{theta unimodal}, if $\theta(\Delta, x)$ has (nonnegative and) unimodal coefficients;
- \textit{theta $\gamma$-positive}, if $\theta(\Delta, x)$ is $\gamma$-positive.

A triangulation $\Delta'$ of any simplicial complex $\Delta$ will be called \textit{theta positive}, \textit{theta unimodal} or \textit{theta $\gamma$-positive} if the restriction $\Delta'_F$ has the corresponding property for every $F \in \Delta$.

The following statement, which is a special case of Stanley’s monotonicity theorem [24, Theorem 2.1], provides a natural sufficient condition for theta positivity.

**Proposition 3.6.** Let $\Delta$ be a homology ball having the interior vertex property.

(a) \cite{24} $\Delta$ is theta positive.

(b) More generally, $\theta(\text{link}_\Delta(F), x) \geq 0$ for every $F \in \partial \Delta$.

**Proof.** Part (a) is a direct consequence of \cite{24} Theorem 2.1, applied to $\Delta$ and its subcomplex $\partial \Delta$. Part (b) follows from part (a) and the fact that the interior vertex property is inherited by the links in $\Delta$ of faces $F \in \partial \Delta$. Indeed, suppose that $E$ is a facet of $\text{link}_\Delta(F)$ for some $F \in \partial \Delta$. Then, $E \cup F$ is a facet of $\Delta$ and hence it has an interior vertex, say $v$. Clearly, $v \in E$. Since $\partial \text{link}_\Delta(F) = \text{link}_{\partial \Delta}(F)$ and $v \notin \partial \Delta$, $v$ must be an interior vertex of $\text{link}_\Delta(F)$. \hfill \Box

**Corollary 3.7.** We have:

(a) $h(\Delta', x) \geq h(\text{sd}(\Delta), x)$ for every Cohen–Macaulay simplicial complex $\Delta$ and every theta positive triangulation $\Delta'$ of $\Delta$.

(b) $\ell_\nu(\Gamma, x) \geq \ell_\nu(\text{sd}(2^\nu), x)$ for every theta positive triangulation $\Gamma$ of the simplex $2^\nu$. 
In particular, the barycentric subdivisions \( \text{sd}(\Delta') \) and \( \text{sd}(2^V) \) minimize coefficientwise \( h(\Delta', x) \) and \( \ell_V(\Gamma, x) \) among all triangulations \( \Delta' \) of \( \Delta \) and \( \Gamma \) of \( 2^V \), respectively, whose restrictions to all nonempty faces of \( \Delta \) and \( 2^V \) have the interior vertex property.

**Proof.** Since \( \theta(\Delta'_F, \emptyset, x) = 1 \), one may rewrite Equation (9) as

\[
h(\Delta', x) = h(\text{sd}(\Delta'), x) + \sum_{F \in \Delta \setminus \{\emptyset\}} \theta(\Delta'_F, x) h(\text{sd}(\text{link}_{\Delta}(F)), x)
\]

and the proof of (a) follows by the nonnegativity of \( \theta(\Delta'_F, x) \) and \( h(\text{sd}(\text{link}_{\Delta}(F)), x) \). The proof of (b) is similar. \( \square \)

**Corollary 3.8.**
(a) The polynomial \( h(\Delta', x) \) is unimodal, with a peak at position \( n/2 \), if \( n \) is even, and at \((n-1)/2 \) or \((n+1)/2 \), if \( n \) is odd, for every \((n-1)\)-dimensional Cohen–Macaulay simplicial complex \( \Delta \) and every theta unimodal triangulation \( \Delta' \) of \( \Delta \).

(b) The polynomial \( \ell_V(\Gamma, x) \) is unimodal (respectively, \( \gamma \)-positive) for every theta unimodal (respectively, theta \( \gamma \)-positive) triangulation \( \Gamma \) of the simplex \( 2^V \).

**Proof.** Given that products of polynomials with nonnegative, symmetric and unimodal coefficients have the same property [22, Proposition 1], Theorem 3.4 and Proposition 2.3 imply that \( h(\Delta', x) \) can be written as a sum of three polynomials with nonnegative, symmetric and unimodal coefficients and centers of symmetry \((n-1)/2\), \( n/2 \) and \((n+1)/2\), respectively. This implies part (a). For part (b) one uses Theorem 3.5, the unimodality and \( \gamma \)-positivity of derangement polynomials and the fact that products of \( \gamma \)-positive polynomials are \( \gamma \)-positive. \( \square \)

**Corollary 3.9.** Let \( \Delta \) be an \((n-1)\)-dimensional simplicial complex.

(a) If \( \Delta \) is a homology sphere, then \( h(\Delta', x) \) is unimodal (respectively, \( \gamma \)-positive) for every theta unimodal (respectively, theta \( \gamma \)-positive) triangulation \( \Delta' \) of \( \Delta \).

(b) If \( \Delta \) is Cohen–Macaulay*, then \( h(\Delta', x) \) has a unimodal (respectively, \( \gamma \)-positive) symmetric decomposition with respect to \( n \) for every theta unimodal (respectively, theta \( \gamma \)-positive) triangulation \( \Delta' \) of \( \Delta \).

(c) If \( \Delta \) is a homology ball, then \( h(\Delta', x) \) has a unimodal (respectively, \( \gamma \)-positive) symmetric decomposition with respect to \( n-1 \) for every theta unimodal (respectively, theta \( \gamma \)-positive) triangulation \( \Delta' \) of \( \Delta \).

**Proof.** This follows from Theorem 3.4 and known results about barycentric subdivisions, as was the case with Corollary 3.8. For part (a), we note that \( \text{link}_{\Delta}(F) \) is a homology sphere of dimension \( n-|F| \) for every \( F \in \Delta \). As a result, \( h(\text{sd}(\text{link}_{\Delta}(F)), x) \) is symmetric, with nonnegative coefficients and center of symmetry \((n-|F|)/2\), and real-rooted by the main result of [12], thus also unimodal and \( \gamma \)-positive. Under our assumptions on \( \Delta' \), it follows that all terms of the sum on the right-hand side of (9) are symmetric and unimodal (respectively, \( \gamma \)-positive) with center of symmetry \( n/2 \), and hence so is the left-hand side \( h(\Delta', x) \).

For part (b) we use instead the fact that \( h(\text{sd}(\text{link}_{\Delta}(F)), x) \) has a nonnegative and real-rooted symmetric decomposition with respect to \( n-|F| \) for every \( F \in \Delta \). This is
essentially due to \[11, \text{Theorem 5.1}\]; see \[8, \text{Corollary 4.3 (a)}\]. For part (c) we use the fact that either \(h(\text{sd}(\text{link}\, \Delta\, (F), x))\) is symmetric, with nonnegative coefficients and center of symmetry \((n - |F|)/2\), and real-rooted, and \(\theta(\Delta', x)\) has zero constant term (if \(F \in \text{int}(\Delta)\)), or \(h(\text{sd}(\text{link}\, \Delta\, (F), x))\) has a nonnegative and real-rooted symmetric decomposition with respect to \(n - |F| - 1\) (if \(F \in \partial\Delta\)). The latter was shown in \[6, \text{Proposition 8.1}\]. □

Remark 3.10. The proofs of Corollaries \ref{cor:3.8} and \ref{cor:3.9} show that in their statements, \(h(\Delta', x)\) and \(\ell_V(\Gamma, x)\) can be replaced by \(h(\Delta', x) - h(\text{sd}(\Delta), x)\) and \(\ell_V(\Gamma, x) - \ell_V(\text{sd}(2^V), x)\), respectively.

The following question is similar in spirit to \[23, \text{Problem 4.13}\] for local \(h\)-polynomials.

Question 3.11. Which homology balls \(\Delta\) have the property that \(\theta(\Delta, x) = 0\)?

4. Monotonicity of theta polynomials

This section proves the monotonicity of theta polynomials, under mild assumptions, and explores its consequences on their unimodality and \(\gamma\)-positivity. We state two main theorems with similar proofs, based on Theorems \ref{thm:2.1} and \ref{thm:3.4} respectively.

Theorem 4.1. We have \(\theta(\Delta', x) \geq \theta(\Delta, x)\) for every homology ball \(\Delta\) having the interior vertex property and every triangulation \(\Delta'\) of \(\Delta\).

Proof. Applying Equation \ref{eq:4} to the defining equation \ref{eq:1} of \(\theta(\Delta', x)\), we get

\[
\theta(\Delta', x) = h(\Delta', x) - h(\partial\Delta', x)
\]

\[
= \sum_{F \in \Delta} \ell_F(\Delta'_F, x) h(\text{link}\, \Delta(F), x) - \sum_{F \in \partial\Delta} \ell_F(\Delta'_F, x) h(\text{link}\, \partial\Delta(F), x)
\]

\[
= h(\Delta, x) - h(\partial\Delta, x) + \sum_{F \in \text{int}(\Delta)} \ell_F(\Delta'_F, x) h(\text{link}\, \Delta(F), x) + \sum_{F \in \partial\Delta \setminus \{\emptyset\}} \ell_F(\Delta'_F, x) \theta(\text{link}\, \Delta(F), x)
\]

By Proposition \ref{prop:3.6}, we have \(\theta(\text{link}\, \Delta(F), x) \geq 0\) for every \(F \in \partial\Delta\) and the proof follows. □

Theorem 4.2. Let \(\Delta\) be a homology ball and \(\Delta'\) be a theta positive triangulation of \(\Delta\). Then, \(\theta(\Delta', x) \geq \theta(\text{sd}(\Delta), x)\). Moreover:
(a) if $\Delta'$ is a theta unimodal triangulation of $\Delta$, then $\theta(\Delta', x)$ and $\theta(\Delta', x) - \theta(sd(\Delta), x)$ have nonnegative and unimodal coefficients;

(b) if $\Delta'$ is a theta $\gamma$-positive triangulation of $\Delta$, then $\theta(\Delta', x)$ and $\theta(\Delta', x) - \theta(sd(\Delta), x)$ are $\gamma$-positive.

Proof. Applying Equation (9) to the defining equation (1) of $\theta(\Delta', x)$, we get

$$
\theta(\Delta', x) = h(\Delta', x) - h(\partial \Delta', x) = \sum_{F \in \Delta} \theta_F(\Delta'_F, x) h(sd(link(\Delta(F))), x) - \sum_{F \in \partial \Delta} \theta_F(\Delta'_F, x) h(sd(link_{\partial \Delta}(F))), x)
$$

$$
= h(sd(\Delta), x) - h(\partial(\Delta), x) + \sum_{F \in \partial \Delta \setminus \{ \emptyset \}} \theta_F(\Delta'_F, x) (h(sd(link(\Delta(F))), x) - h(sd(\partial(link(\Delta(F))), x)))
$$

$$
= \sum_{F \in int(\Delta')} \theta_F(\Delta'_F, x) h(sd(link(\Delta(F))), x) + \sum_{F \in int(\Delta)} \theta_F(\Delta'_F, x) h(sd(link(\Delta(F))), x) + \sum_{F \in \partial \Delta \setminus \{ \emptyset \}} \theta_F(\Delta'_F, x) h(sd(link(\Delta(F))), x).
$$

All claims in the statement of the proposition follow, since $h$-polynomials and $\theta$-polynomials of barycentric subdivisions of homology spheres and balls, respectively, are known to be $\gamma$-positive (and, in particular, unimodal); see [5, Section 3.1.1] and Section 5. □

Remark 4.3. As expected, we also have $\theta(sd(\Delta), x) \geq \theta(\Delta, x)$ for every homology ball $\Delta$ as a consequence of the formula

$$
\theta(sd(\Delta), x) = \sum_{i=0}^{n-1} (h_n(\Delta) + \cdots + h_{n-i}(\Delta) + x (h_n(\Delta) + \cdots + h_{i+1}(\Delta))) p_{n-1,i}(x),
$$

where $n - 1 = \dim(\Delta)$, and Equation (8). This formula follows from Lemmas 3.5 and 4.2 in [3] (in the special case of barycentric subdivisions).

Corollary 4.4. Given any triangulation $\Gamma$ of the simplex $2^V$, the polynomials $\ell_V(\Gamma', x)$ and $\ell_V(\Gamma', x) - \ell_V(sd(\Gamma), x)$ are unimodal (respectively, $\gamma$-positive) for every theta unimodal (respectively, theta $\gamma$-positive) triangulation $\Gamma'$ of $\Gamma$. 
Proof. Applying Theorem 3.5 to Γ and sd(Γ) gives

\[
\ell_V(\Gamma', x) = \sum_{F \subseteq V} \theta(\Gamma'_{F}, x) d_{|V \setminus F|}(x),
\]

\[
\ell_V(\Gamma', x) - \ell_V(\text{sd}(\Gamma), x) = \sum_{F \subseteq V} (\theta(\Gamma'_{F}, x) - \theta(\text{sd}(\Gamma_{F}), x)) d_{|V \setminus F|}(x).
\]

The result follows from these formulas and Theorem 4.2, since the latter implies the unimodality (respectively, γ-positivity) of \( \theta(\Gamma'_{F}, x) \) and \( \theta(\Gamma'_{F}, x) - \theta(\text{sd}(\Gamma_{F}), x) \) for each \( F \subseteq V \).

We end this section with a discussion of the following general problem.

**Question 4.5.** Let \( \Delta \) and \( \Delta' \) be homology balls such that \( \Delta \) is a subcomplex of \( \Delta' \). Under what conditions does the inequality \( \theta(\Delta', x) \geq \theta(\Delta, x) \) hold? Does it suffice to assume, for instance, that \( \Delta \) and \( \Delta' \) have the interior vertex property and the same dimension?

The following theorem provides a partial answer to this question. The proof follows that of [24, Theorem 2.1] and assumes familiarity with the basics of Stanley–Reisner theory [23, Chapter II].

**Theorem 4.6.** Let \( \Delta \) and \( \Delta' \) be homology balls of the same dimension, such that \( \Delta \) is a subcomplex of \( \Delta' \). If no facet of \( \Delta' \) has all its vertices in \( \partial \Delta \cup \partial \Delta' \), then \( \theta(\Delta', x) \geq \theta(\Delta, x) \).

**Proof.** By extending the field \( k \), if necessary, we may assume it is infinite. We consider the polynomial ring over \( k \) in variables \( x_1, x_2, \ldots, x_m \) corresponding to the vertices of \( \Delta' \), endowed with the standard grading, and the Stanley–Reisner rings \( k[\Delta'] \) and \( k[\Delta] \). Because of our assumption on the facets of \( \Delta' \), just as in the proof of [24, Theorem 2.1] we may choose a linear system of parameters \( (\eta') = (\eta'_1, \eta'_2, \ldots, \eta'_{n}) \) for \( k[\Delta'] \) such that \( \eta'_{n} \) is a linear combination of vertices not in \( \partial \Delta \cup \partial \Delta' \), where \( n-1 \) is the common dimension of \( \Delta' \) and \( \Delta \). Then, \( (\eta'_0) = (\eta'_1, \eta'_2, \ldots, \eta'_{n-1}) \) is a linear system of parameters for \( k[\partial \Delta] \) and the images \( (\eta) \) and \( (\eta_0) \) of \( (\eta') \) and \( (\eta'_0) \) in \( k[\Delta] \) and \( k[\partial \Delta] \) are linear systems of parameters for these rings, respectively. Moreover, we have a diagram of (standard) graded, surjective homomorphisms

\[
k[\Delta']/(\eta') \xrightarrow{\varphi'} k[\partial \Delta']/(\eta'_0)
\]

\[
\downarrow \psi
\]

\[
k[\Delta]/(\eta) \xrightarrow{\varphi} k[\partial \Delta]/(\eta_0)
\]

where \( \varphi', \varphi \) and \( \psi \) are the obvious maps. The kernels, say \( \mathcal{I}' \) and \( \mathcal{I} \), of \( \varphi' \) and \( \varphi \) are graded ideals of \( k[\Delta']/(\eta') \) and \( k[\Delta]/(\eta) \), respectively. Since \( \Delta', \Delta \) and their boundaries are Cohen–Macaulay, \( \mathcal{I}' \) and \( \mathcal{I} \) have Hilbert series equal to \( \theta(\Delta', x) \) and \( \theta(\Delta, x) \), respectively. Thus, it suffices to verify that \( \mathcal{I} \subseteq \psi(\mathcal{I}') \). This is indeed the case, since \( \mathcal{I} \) is generated by
the classes in $k[\Delta]/(\eta)$ of squarefree monomials which correspond to interior faces of $\Delta$ and every such face is also an interior face of $\Delta'$.

\textbf{Remark 4.7.} The assumptions that $\Delta$ and $\Delta'$ have the same dimension and nonnegative theta polynomials do not suffice to guarantee that $\theta(\Delta', x) \geq \theta(\Delta, x)$. For example, let $\Delta'$ be any $(n - 1)$-dimensional homology ball which has at least two facets and a vertex $v$ which belongs to a unique facet of $\Delta'$, where $n \geq 4$, and let $\Delta$ be obtained from $\Delta'$ by removing all faces containing $v$. Then, $\Delta$ is also an $(n - 1)$-dimensional homology ball and $\theta(\Delta, x) = \theta(\Delta', x) + (x^2 + x^3 + \cdots + x^{n-2})$. Note that, in this situation, $\Delta'$ does not have the interior vertex property.

One may pick $\Delta'$ so that $\theta(\Delta', x) \geq 0$, for instance, by letting $\Delta'$ be the 4-fold edgewise subdivision of the three-dimensional simplex, in which case $\theta(\Delta', x) = 0$ (see Section 3.2 and Example 7.2 of [6]).

\section{5. Unimodality and Gamma-Positivity}

This section investigates the unimodality and $\gamma$-positivity of theta polynomials. These questions are naturally raised by the results of the previous sections and are closely related to the unimodality and $\gamma$-positivity of the symmetric decompositions of $h$-polynomials of homology balls. Indeed, given an $(n - 1)$-dimensional homology ball $\Delta$, since $h(\partial \Delta, x)$ and $\theta(\Delta, x)$ are symmetric polynomials with centers of symmetry $\frac{(n - 1)}{2}$ and $\frac{n}{2}$, respectively, and the latter has zero constant term, the expression

\begin{equation}
(11) \quad h(\Delta, x) = h(\partial \Delta, x) + x \theta(\Delta, x)/x
\end{equation}

is the symmetric decomposition of $h(\Delta, x)$ with respect to $n - 1$. Thus, the unimodality (respectively, $\gamma$-positivity) of this symmetric decomposition of $h(\Delta, x)$ is equivalent to the unimodality (respectively, $\gamma$-positivity) of $h(\partial \Delta, x)$ and $\theta(\Delta, x)$.

\textbf{Unimodality.} Because of Equation (8), the unimodality of $\theta(\Delta, x)$ is equivalent to the inequalities

\begin{equation}
(12) \quad h_i(\Delta) \leq h_{n-1-i}(\Delta), \quad 0 \leq i \leq (n - 1)/2,
\end{equation}

where $n - 1 = \dim(\Delta)$. Moreover, since $h(\partial \Delta, x) = h(\Delta, x) - \theta(\Delta, x)$, Equation (8) is equivalent to the formulas

\begin{equation}
(11) \quad h_i(\partial \Delta) = h_0(\Delta) + h_1(\Delta) + \cdots + h_i(\Delta) - h_n(\Delta) - h_{n-1}(\Delta) - \cdots - h_{n-i}(\Delta)
\end{equation}

for $0 \leq i \leq n - 1$, where $h_n(\Delta) = 0$. Hence, the unimodality of $h(\partial \Delta, x)$ is equivalent to the inequalities $h_i(\Delta) \geq h_{n-i}(\Delta)$ for $1 \leq i \leq \lceil n/2 \rceil$ and the unimodality of the symmetric decomposition (11) (meaning, the unimodality of both $h(\partial \Delta, x)$ and $\theta(\Delta, x)$) is equivalent to the inequalities

\begin{equation}
(13) \quad h_0(\Delta) \leq h_{n-1}(\Delta) \leq h_1(\Delta) \leq h_{n-2}(\Delta) \leq \cdots \leq h_{\lceil n/2 \rceil}(\Delta),
\end{equation}

sometimes referred to as the \textit{alternatingly increasing property} for $h(\Delta)$.

Given that the nonnegativity of $\theta(\Delta, x)$ is guaranteed by the interior vertex property (see Proposition 3.6), it seems natural to expect that its unimodality will be valid under the stronger assumption that $\partial \Delta$ is an induced subcomplex of $\Delta$. Indeed, the almost
strong Lefschetz property \cite{18} for $\Delta$ implies the existence of an injective map from a vector space of dimension $h_i(\Delta)$ to a vector space of dimension $h_j(\Delta)$ for all $0 \leq i \leq j \leq n-1-i$. In particular, it implies the inequalities (12) and that
\begin{equation}
\label{eq:14}
h_0(\Delta) \leq h_1(\Delta) \leq \cdots \leq h_{\lfloor (n-1)/2 \rfloor}(\Delta).
\end{equation}
The almost strong Lefschetz property for $\Delta$ follows from a recent result of Adiprasito and Yashfe \cite[Theorem 50]{3}, and the $g$-theorem for homology spheres \cite{1, 2, 21}, under the assumption that $\partial \Delta$ is an induced subcomplex of $\Delta$. Thus, the following statement holds for all fields $k$ for which the $g$-theorem does.

**Theorem 5.1.** (\cite{3}) The polynomial $\theta(\Delta, x)$ is unimodal for every homology ball $\Delta$, such that $\partial \Delta$ is an induced subcomplex of $\Delta$.

The following example shows that the condition that $\partial \Delta$ is an induced subcomplex of $\Delta$ cannot be replaced by the interior vertex property.

**Example 5.2.** Let $\Gamma$ be the triangulation of the three-dimensional ball having two facets $F = \{a,b,c,d\}$ and $G = \{b,c,d,e\}$. Let $\Delta$ be obtained from $\Gamma$ by two stellar subdivisions on these facets, meaning that one adds two new vertices $u, v$ and the unions of the proper subsets of $F$ and $G$ with $\{u\}$ and $\{v\}$, respectively, to obtain $\Delta$ from $\Gamma$. Then,
\begin{align*}
h(\Delta, x) &= 1 + 3x + 2x^2 + 2x^3 \\
h(\partial \Delta) &= h(\partial \Gamma) = 1 + 2x + 2x^2 + x^3,
\end{align*}
so that $\theta(\Delta, x) = x + x^3$ is not unimodal. Note that $\partial \Delta$ is not an induced subcomplex of $\Delta$, since $\{b,c,d\}$ is an interior face of $\Delta$ having all its vertices on $\partial \Delta$. On the other hand, all facets of $\Delta$ have an interior vertex, namely $u$ or $v$. \hfill \Box

**Remark 5.3.** Stanley proved \cite[Theorem 5.2]{23} that the local $h$-polynomial $\ell_V(\Gamma, x)$ is unimodal for every regular triangulation $\Gamma$ of the simplex $2^V$. Moreover, he conjectured \cite[Conjecture 5.4]{23} that the same holds for every triangulation of $2^V$ (see \cite[Question 3.5]{4} for the more general class of topological simplicial subdivisions of the simplex for which this conjecture could be true). Theorem 5.1 combined with Theorem 3.5 shows that $\ell_V(\Gamma, x)$ is unimodal for every triangulation $\Gamma$ of the simplex $2^V$ such that for every $F \subseteq V$, the restriction of $\Gamma$ to $\partial(2^F)$ is an induced subcomplex of $\Gamma$. \hfill \Box

**Gamma-positivity.** A simplicial complex $\Delta$ is said to be flag if every set $F$ of vertices of $\Delta$ such that $\{u, v\} \in \Delta$ for all $u, v \in F$ is a face of $\Delta$. The $h$-polynomials of simplicial complexes seem to behave well with respect to $\gamma$-positivity under the flagness condition. For example, $h$-polynomials of flag homology spheres \cite[Conjecture 2.1.7]{15} and local $h$-polynomials of flag triangulations of simplices \cite[Conjecture 5.4]{4} are conjectured to be $\gamma$-positive. In the spirit of these conjectures, it seems natural to formulate the following flag conjectural analogue of Theorem 5.1.

**Conjecture 5.4.** The polynomial $\theta(\Delta, x)$ is $\gamma$-positive for every flag homology ball $\Delta$, such that $\partial \Delta$ is an induced subcomplex of $\Delta$. 

The following example shows that, once again, the condition that $\partial \Delta$ is an induced subcomplex of $\Delta$ cannot be replaced by the interior vertex property.

**Example 5.5.** Consider the boundary complexes $\Delta_1$ and $\Delta_2$ of two octahedra, which are glued along a common facet $F$, and let $\Delta$ be the union of the cones of $\Delta_1$ and $\Delta_2$ on two vertices $u_1$ and $u_2$. Thus, $\Delta$ is a three-dimensional flag simplicial ball with eleven vertices and sixteen facets which has the interior vertex property (every facet contains either $u_1$ or $u_2$). The boundary $\partial \Delta$ has nine vertices and fourteen facets, and

$$h(\Delta, x) = 1 + 7x + 6x^2 + 2x^3$$

$$h(\partial \Delta) = 1 + 6x + 6x^2 + x^3,$$

so that $\theta(\Delta', x) = x + x^3$ is not $\gamma$-positive (not even unimodal). Since $F$ is an interior face of $\Delta$ having all its vertices on $\partial \Delta$, the latter is not an induced subcomplex of $\Delta$. □

Conjecture [5,4] turns out to be equivalent to a strengthening of Gal’s conjecture [15, Conjecture 2.1.7], proposed recently by Chudnovsky and Nevo [13] in two equivalent forms (named the Link Conjecture and the Equator Conjecture), as we now show. We recall that the $\gamma$-polynomial of an $(n - 1)$-dimensional homology sphere $\Delta$ is defined as $\gamma(\Delta, x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i(\Delta)x^i$, where $h(\Delta, x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i(\Delta)x^i(1 + x)^{n-2i}$.

**Proposition 5.6.** For every positive integer $n$, the following statements are equivalent:

1. Conjecture [5,4] holds for all homology balls of dimension $n - 1$.
2. ([13, Conjecture 1.2]) We have $\gamma(\Delta, x) \geq \gamma(\text{link}_\Delta(v), x)$ for every $(n-1)$-dimensional flag homology sphere $\Delta$ and every vertex $v$ of $\Delta$.
3. ([13, Conjecture 1.3]) We have $\gamma(\Delta, x) \geq \gamma(\Gamma, x)$ for every $(n-1)$-dimensional flag homology sphere $\Delta$ and every $(n-2)$-dimensional flag homology sphere $\Gamma$ which is an induced subcomplex of $\Delta$.

**Proof.** The equivalence (ii) ⇔ (iii) has already been proven in [13, Proposition 3.1]. To prove that (i) ⇒ (ii), we assume that (i) holds and consider an $(n - 1)$-dimensional flag homology sphere $\Delta$, along with a vertex $v$. Then, $\Gamma := \Delta \setminus v$ is an $(n - 1)$-dimensional homology ball with boundary $\partial \Gamma = \text{link}_\Delta(v)$. The flagness of $\Delta$ implies that $\partial \Gamma$ is an induced subcomplex of $\Gamma$ and that $\Gamma$ and $\partial \Gamma$ are both flag. Thus, $\theta(\Gamma(x), x)$ is $\gamma$-positive. Since, as an easy consequence of the definition of the $h$-polynomial, $h(\Delta, x) = h(\Gamma, x) + xh(\text{link}_\Delta(v), x)$, we have

$$\theta(\Gamma, x) = h(\Gamma, x) - h(\partial \Gamma, x) = h(\Gamma, x) - h(\text{link}_\Delta(v), x)$$

$$= h(\Delta, x) - (1 + x)h(\text{link}_\Delta(v), x).$$

Hence, the $\gamma$-positivity of $\theta(\Delta, x)$ exactly means that $\gamma(\Delta, x) \geq \gamma(\text{link}_\Delta(v), x)$.

The proof of (ii) ⇒ (i) is similar. Given an $(n - 1)$-dimensional flag homology ball $\Gamma$ with induced boundary $\partial \Gamma$, one adds the cone of $\partial \Gamma$ over a new vertex $v$ to $\Gamma$ to obtain an $(n - 1)$-dimensional flag homology sphere $\Delta$ with $\text{link}_\Delta(v) = \partial \Gamma$. Because $\partial \Gamma$ is an induced subcomplex of $\Gamma$, the sphere $\Delta$ is also flag and hence $\gamma(\Delta, x) \geq \gamma(\text{link}_\Delta(v), x)$ by (ii). As already shown, the latter inequality is equivalent to the $\gamma$-positivity of $\theta(\Gamma, x)$. □
One may also state Conjecture 5.4 in the language of symmetric decompositions and place it within the phenomena of ‘nonsymmetric γ-positivity’ discussed in [5, Section 5.1].

**Proposition 5.7.** For every positive integer \(d\), the following statements are equivalent:

(i) Conjecture 5.4 holds in all dimensions less than \(d\).

(ii) For all \(1 \leq n \leq d\), the polynomial \(h(\Gamma, x)\) has a γ-positive symmetric decomposition with respect to \(n - 1\) for every \((n - 1)\)-dimensional flag homology ball \(\Gamma\), such that \(\partial \Gamma\) is an induced subcomplex of \(\Gamma\).

**Proof.** As already discussed, the γ-positivity of the symmetric decomposition of \(h(\Gamma, x)\) with respect to \(n - 1\) is equivalent to the γ-positivity of \(\theta(\Gamma, x)\) and \(h(\partial \Gamma, x)\). Thus, we only need to verify that condition (i) implies the γ-positivity of the \(h\)-polynomials of flag homology spheres of dimension less than \(d\). This follows from Proposition 5.6 since, as noted in [13], the validity of condition (ii) there for all \(n \leq d\) implies the validity of Gal’s conjecture in all dimensions less than \(d\) by induction on the dimension. □

**Remark 5.8.** Conjecture 5.4 would imply, in view of Theorem 3.5, that \(\ell_V(\Gamma, x)\) is γ-positive for every flag triangulation \(\Gamma\) of the simplex \(2^V\) such that for every \(F \subseteq V\), the restriction of \(\Gamma\) to \(\partial(2^F)\) is an induced subcomplex of \(\Gamma\) (thus verifying [4, Conjecture 5.4] in this situation). □

Conjecture 5.4 is true, among other special cases, when \(\Delta\) is:

- the barycentric subdivision of any regular cell decomposition of a ball, by results of Karu and Ehrenborg [14] (see [17, Section 4] for a detailed explanation and [5, Theorem 3.9]),
- the \(r\)-fold edgewise subdivision (for \(r \geq n\)) and the \(r\)-colored barycentric subdivision of any \((n - 1)\)-dimensional homology ball; see part (b) of Corollaries 5.1 and 5.4 in [8],
- the antiprism triangulation of any \((n - 1)\)-dimensional homology ball; see Section 6.

The stronger property that \(\theta(\Delta, x)\) has only real roots is proven for edgewise subdivisions and \(r\)-colored barycentric subdivisions in [8]. Further evidence in favor of Conjecture 5.4 can be found in [13].

### 6. Antiprism triangulations

The antiprism triangulation of a simplicial complex \(\Delta\), denoted by \(sd_A(\Delta)\), was first considered by Izmestiev and Joswig [16] in the context of branched coverings of manifolds. As a simplicial complex, it may be defined to have vertices the pointed faces \((F, v)\) of \(\Delta\), meaning pairs of faces \(F \in \Delta\) and vertices \(v \in F\), and faces the sets consisting of the pointed faces \((F_1, v_1), (F_2, v_2), \ldots, (F_m, v_m)\) of \(\Delta\), such that

- \(F_1 \subseteq F_2 \subseteq \cdots \subseteq F_m\) and
- \(F_i \not\subseteq F_j \Rightarrow v_j \in F_j \setminus F_i\), for \(i < j\).

This simplicial complex naturally triangulates \(\Delta\); the carrier of the face consisting of \((F_1, v_1), (F_2, v_2), \ldots, (F_m, v_m)\), as above, is the face \(F_m \in \Delta\).
The combinatorial properties of antiprism triangulations were studied systematically in [7] and shown to have similarities to those of barycentric subdivision, but often to be more challenging to analyze. For example, it is an open problem [7, Conjecture 1.1] to decide whether the $h$-polynomial of $\text{sd}_A(\Delta)$ is real-rooted for every Cohen–Macaulay simplicial complex $\Delta$. The key property that $\theta(\text{sd}_A(2^V), x)$ has only real roots (and hence is unimodal and $\gamma$-positive) for every simplex $2^V$ was proven in [7] (see Theorem 5.2 there). Given that, we can reprove the main result of [7] about the unimodality of $h(\text{sd}_A(\Delta), x)$ and answer in the affirmative some of the questions posed in [7, Section 8] (see Remarks 3 and 4 there). Part (b) of the following corollary verifies Gal’s conjecture in a new special case, namely that of antiprism triangulations of homology spheres.

**Corollary 6.1.** The polynomial $h(\text{sd}_A(\Delta), x)$:

(a) (cf. [7, Theorem 1.3]) is unimodal, with a peak at position $n/2$, if $n$ is even, and at $(n - 1)/2$ or $(n + 1)/2$, if $n$ is odd, for every Cohen–Macaulay $(n - 1)$-dimensional simplicial complex $\Delta$;

(b) is $\gamma$-positive for every $(n - 1)$-dimensional homology sphere $\Delta$;

(c) has a $\gamma$-positive symmetric decomposition with respect to $n$ for every $(n - 1)$-dimensional Cohen–Macaulay* simplicial complex $\Delta$;

(d) has a $\gamma$-positive symmetric decomposition with respect to $n - 1$ for every $(n - 1)$-dimensional homology ball $\Delta$.

**Proof.** This follows from Corollary 3.8 (a), Corollary 3.9 and the theta unimodality and theta $\gamma$-positivity of antiprism triangulations of simplicial complexes, established in [7].

**Corollary 6.2.** The polynomial $\ell_V(\text{sd}_A(\Gamma), x)$ is $\gamma$-positive for every triangulation $\Gamma$ of the simplex $2^V$.

**Proof.** This follows from Corollary 1.3 and the theta $\gamma$-positivity of antiprism triangulations of simplicial complexes, established in [7].

**Question 6.3.** Is $\ell_V(\text{sd}(\Gamma), x)$ real-rooted for every triangulation $\Gamma$ of the simplex $2^V$? Is $\ell_V(\text{sd}_A(\Gamma), x)$ real-rooted for every triangulation $\Gamma$ of $2^V$?

**Acknowledgments.** The author wishes to thank Satoshi Murai and Isabella Novik for useful comments, especially for pointing out that Theorem 5.1 follows from [3, Theorem 50] and the $g$-theorem. This work was supported by the Hellenic Foundation for Research and Innovation (H.F.R.I.) under the ‘2nd Call for H.F.R.I. Research Projects to support Faculty Members & Researchers’.

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