COFINITENESS OF WEAKLY LASKERIAN LOCAL COHOMOLOGY MODULES

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Abstract. Let $I$ be an ideal of a Noetherian ring $R$ and $M$ be a finitely generated $R$-module. We introduce the class of extension modules of finitely generated modules by the class of all modules $T$ with $\dim T \leq n$ and we show it by $\text{FD} \leq n$ where $n \geq -1$ is an integer. We prove that for any $\text{FD} \leq 0$ (or minimax) submodule $N$ of $H^t_I(M)$ the $R$-modules $\text{Hom}_R(R/I, H^j_I(M)/N)$ and $\text{Ext}_R^1(R/I, H^j_I(M)/N)$ are finitely generated, whenever the modules $H^0_I(M)$, $H^1_I(M)$, ..., $H^{t-1}_I(M)$ are $\text{FD} \leq 1$ (or weakly Laskerian). As a consequence, it follows that the associated primes of $H^t_I(M)/N$ are finite. This generalizes the main results of Bahmanpour and Naghipour [4] and [5], Brodmann and Lashgari [7], Khashyarmanesh and Salarian [21] and Hong Quy [18]. We also show that the category $\mathcal{FD}_{\leq 1}(R, I)$ of $I$-cofinite $\text{FD} \leq 1 R$-modules forms an Abelian subcategory of the category of all $R$-modules.

1. Introduction

The following conjecture was made by Grothendieck in [15]:

Conjecture: For any ideal $I$ of a Noetherian ring $R$ and any finite $R$-module $M$, the module $\text{Hom}_R(R/I, H^j_I(M))$ is finitely generated for all $j \geq 0$.

Here, $H^j_I(M)$ denotes the $j^{th}$ local cohomology module of $M$ with support in $I$. Although the conjecture is not true in general as was shown by Hartshorne in [16], there are some attempts to show that under some conditions, for some number $t$, the module $\text{Hom}_R(R/I, H^j_I(M))$ is finite, see [2, Theorem 3.3], [11, Theorem 6.3.9], [13, Theorem 2.1], [4, Theorem 2.6] and [5, Theorem 2.3].

In [16], Hartshorne defined an $R$-module $L$ to be $I$-cofinite, if $\text{Supp}(L) \subseteq V(I)$ and $\text{Ext}_R^i(R/I, L)$ is finitely generated module for all $i$. He asked:

If $I$ is an ideal of $R$ and $M$ is a finitely generated $R$-module, when is $H^i_I(M)$ $I$-cofinite for all $i$ ?

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In this direction in section 3 we generalize [2, Theorem 3.3], [4, Theorem 2.6] and [5, Theorem 2.3] to the class of extension modules of finitely generated modules by the class of all modules $T$ with $ \dim T \leq 1$ (FD $\leq 1$). Note that the class of weakly Laskerian modules is contained in the class of FD $\leq 1$ modules. More precisely, we shall show that:

**Theorem 1.1.** Let $R$ be a Noetherian ring and $I$ an ideal of $R$. Let $M$ be a finitely generated $R$-module and $t \geq 1$ be a positive integer such that the $R$-modules $H^i_I(M)$ are FD $\leq 1$ modules (or weakly Laskerian) for all $i < t$. Then, the following conditions hold:

(i) The $R$-modules $H^i_I(M)$ are $I$-cofinite for all $i < t$.

(ii) For all FD $\leq 0$ (or minimax) submodule $N$ of $H^t_I(M)$, the $R$-modules

$$\text{Hom}_R(R/I, H^t_I(M)/N) \text{ and } \text{Ext}^1_R(R/I, H^t_I(M)/N)$$

are finitely generated.

As an immediate consequence we prove the following corollary that is a generalization of Bahmanpour-Naghipour’s results in [4] and also the Delfino-Marley’s result in [10] and Yoshida’s result in [27] for an arbitrary Noetherian ring.

**Corollary 1.2.** Let $R$ be a Noetherian ring and $I$ an ideal of $R$. Let $M$ be a finitely generated $R$-module such that the $R$-modules $H^i_I(M)$ are FD $\leq 1$ (or weakly Laskerian) $R$-modules for all $i$. Then,

(i) the $R$-modules $H^i_I(M)$ are $I$-cofinite for all $i$.

(ii) For any $i \geq 0$ and for any FD $\leq 0$ (or minimax) submodule $N$ of $H^i_I(M)$, the $R$-module $H^i_I(M)/N$ is $I$-cofinite.

Abazari and Bahmanpour in [1] studied cofiniteness of extension functors of cofinite modules as a generalization of Huneke-Koh’s results in [17]. In Corollary 3.8 we generalise the results of Abazari and Bahmanpour.

Hartshorn also posed the following question:

**Whether the category $\mathcal{M}(R, I)_{cof}$ of $I$-cofinite modules forms an Abelian subcategory of the category of all $R$-modules?** That is, if $f : M \to N$ is an $R$-module homomorphism of $I$-cofinite modules, are $\ker f$ and $\coker f$ $I$-cofinite?

Hartshorne proved that if $I$ is a prime ideal of dimension one in a complete regular local ring $R$, then the answer to his question is yes. On the other hand, in [10], Delfino and Marley extended this result to arbitrary complete local rings. Recently, Kawasaki [20] generalized the Delfino and Marley’s result for an arbitrary ideal $I$ of dimension one in a local ring $R$. Finally, more recently, Sedghi, Bahmanpour and Naghipour in [6] completely have removed local assumption on $R$. One of the main results of this section is to prove that the class of $I$-cofinite FD $\leq 1$ modules compose an Abelian category (see Theorem 3.7).

Let $R$ denote a commutative Noetherian ring, and let $I$ be an ideal of $R$. Throughout this paper, $R$ will always be a commutative Noetherian ring with non-zero identity and $I$ will be an ideal of $R$. We denote $\{p \in \text{Spec } R : p \supseteq a\}$ by $V(a)$. For any unexplained notation and terminology we refer the reader to [9] and [24].
2. Preliminaries

Yoshizawa in [28, Definition 2.1] defined classes of extension modules of Serre subcategory by another one as below.

**Definition 2.1.** Let $S_1$ and $S_2$ be Serre subcategories of the category of all $R$-modules. We denote by $(S_1, S_2)$ the class of all $R$-modules $M$ with some $R$-modules $S_1 \in S_1$ and $S_2 \in S_2$ such that a sequence $0 \rightarrow S_1 \rightarrow M \rightarrow S_2 \rightarrow 0$ is exact.

We will denote the class of all modules $M$ with $\text{dim} M \le n$ by $D_{\le n}$ and the class of extension modules of finitely generated modules by the class of $D_{\le n}$ modules by $FD_{\le n}$ where $n \ge -1$ is an integer. Note that the class of $FD_{\le -1}$ is the same as finitely generated $R$-modules. Recall that a module $M$ is a minimax module if there is a finitely generated submodule $N$ of $M$ such that the quotient module $M/N$ is artinian. Thus the class of minimax modules is the class of extension modules of finitely generated modules by the class of Artinian modules. Minimax modules have been studied by Zink in [29] and Zöschinger in [30, 31]. See also [26]. Recall too that an $R$-module $M$ is called weakly Laskerian if $\text{Ass}(M/N)$ is a finite set for each submodule $N$ of $M$. The class of weakly Laskerian modules introduced in [14], by Divaani-Aazar and Mafi. Recently, Hung Quy [18], introduced the class of extension modules of finitely generated modules by the class of all modules of finite support and named it FSF modules. By the following theorem over a Noetherian ring $R$ an $R$-module $M$ is weakly Laskerian if and only if is FSF.

**Theorem 2.2.** Let $R$ be a Noetherian ring and $M$ a nonzero $R$-module. The following statements are equivalent:

1. $M$ is a weakly Laskerian module;
2. $M$ is an FSF module.

**Proof.** See [3, Theorem 3.3]. □

**Lemma 2.3.** Let $R$ be a Noetherian ring. Then the following conditions hold:

(i) Any finitely generated $R$-module and any $D_{\le n}$ $R$-module are $FD_{\le n}$.
(ii) The class of $FD_{\le n-1}$ modules is contained in the class of $FD_{\le n}$ modules for all $n \ge 0$.
(iii) The class of minimax modules is contained in the class of $FD_{\le 0}$ that is the class of extension modules of finitely generated modules by semiar tinian modules.
(iv) The class of weakly Laskerian modules is contained in the class of $FD_{\le 1}$.
(v) The class of $FD_{\le n}$ $R$-modules forms a Serre subcategory of the category of all $R$-modules.

**Proof.** (i), (ii), (iii) are trivial.
(iv) Use Theorem [2, 2].
(v) See [28, Corollary 4.3 or 4.5]. □
Lemma 3.2. Let $f$ be finitely generated. Therefore, in view of Proposition 2.5, the $R$-module $N$ is $\ast$-cofinite. Now it follows from the exact sequence $(\text{Ext})$ that $\dim R/N = 0$.

Now it follows from the exact sequence $(\text{Ext})$ that $\dim R/N = 0$.

Proposition 2.5. Let $I$ be an ideal of a Noetherian ring $R$ and $M$ be an $D_{\leq 1}$ module such that $\hom R(R/I, M)$ and $\text{Ext}_1^R(R/I, M)$ are finitely generated.

Proof. See [13, Theorem 2.1] and [12, Theorem A].

3. Cofiniteness of local cohomology

In what follows the next theorem plays an important role.

Theorem 3.1. Let $I$ be an ideal of a Noetherian ring $R$ and $M$ be an $FD_{\leq 1}$ $R$-module such that $\text{Supp } M \subseteq V(I)$. Then the following statements are equivalent:

(i) $M$ is $I$-cofinite,

(ii) The $R$-modules $\hom R(R/I, M)$ and $\text{Ext}_1^R(R/I, M)$ are finitely generated.

Proof. $(i) \Rightarrow (ii)$ is clear. In order to prove $(ii) \Rightarrow (i)$, by Definition there is a finitely generated submodule $N$ of $M$ such that the $R$-module $\dim (M/N) \leq 1$ and $\text{Supp } M/N \subseteq V(I)$. Also, the exact sequence

$$0 \to N \to M \to M/N \to 0, \quad (*)$$

induces the following exact sequence

$$0 \to \hom R(R/I, N) \to \hom R(R/I, M) \to \hom R(R/I, M/N) \to \text{Ext}_1^R(R/I, N) \to \text{Ext}_1^R(R/I, M) \to \text{Ext}_1^R(R/I, M/N) \to \text{Ext}_2^R(R/I, N).$$

Whence, it follows that the $R$-modules $\hom R(R/I, M/N)$ and $\text{Ext}_1^R(R/I, M/N)$ are finitely generated. Therefore, in view of Proposition 2.5, the $R$-module $M/N$ is $I$-cofinite. Now it follows from the exact sequence $(*)$ that $M$ is $I$-cofinite.

Lemma 3.2. Let $I$ be an ideal of Noetherian ring $R$, $M$ a non-zero $R$-module and $t \in \mathbb{N}_0$. Suppose that the $R$-module $H^i_I(M)$ is $I$-cofinite for all $i = 0, ..., t - 1$, and the $R$-modules $\text{Ext}_1^R(R/I, M)$ and $\text{Ext}_1^{t+1}_R(R/I, M)$ are finitely generated. Then the $R$-modules $\hom R(R/I, H^i_I(M))$ and $\text{Ext}_1^R(R/I, H^i_I(M))$ are finitely generated.

Proof. See [13, Theorem 2.1] and [12, Theorem A].

Lemma 3.3. Let $I$ be an ideal of a Noetherian ring $R$ and $M$ be an $FD_{\leq 0}$ $R$-module such that $\text{Supp } M \subseteq V(I)$. Then the following statements are equivalent:

(i) $M$ is $I$-cofinite,

(ii) The $R$-module $\hom R(R/I, M)$ is finitely generated.
Proof. The proof is similar to the proof of [25, Proposition 4.3].

We are now ready to state and prove the following main results (Theorem 3.4 and the Corollaries 3.5 and 3.6) which are extension of Bahmanpour-Naghipour’s results in [4] and [5], Brodmann-Lashgari’s result in [7], Khashyarmanesh-Salarian’s result in [21], Hong Quy’s result in [18], and also the Delfno-Marley’s result in [10] and Yoshida’s result in [27] for an arbitrary Noetherian ring.

**Theorem 3.4.** Let $R$ be a Noetherian ring and $I$ an ideal of $R$. Let $M$ be a finitely generated $R$-module and $t \geq 1$ be a positive integer such that the $R$-modules $H^i_I(M)$ are FD$_{\leq 1}$ $R$-modules for all $i < t$. Then, the following conditions hold:

(i) The $R$-modules $H^i_I(M)$ are $I$-cofinite for all $i < t$.

(ii) For all FD$_{\leq 0}$ (or minimax) submodule $N$ of $H^i_I(M)$, the $R$-modules

$$\text{Hom}_R(R/I, H^i_I(M)/N)$$

are finitely generated. In particular the set $\text{Ass}_R(H^i_I(M)/N)$ is a finite set.

**Proof.** (i) We proceed by induction on $t$. By Lemma 3.2 the case $t = 1$ is obvious since $H^0_I(M)$ is finitely generated. So, let $t > 1$ and the result has been proved for smaller values of $t$. By the inductive assumption, $H^i_I(M)$ is $I$-cofinite for $i = 0, 1, ..., t - 2$. Hence by Lemma 3.2 and assumption, $\text{Hom}_R(R/I, H^{t-1}_I(M))$ and $\text{Ext}^1_R(R/I, H^{t-1}_I(M))$ are finitely generated. Therefore by Corollary 3.1, $H^i_I(M)$ is $I$-cofinite for all $i < t$. This completes the inductive step.

(ii) In view of (i) and lemma 3.2, $\text{Hom}_R(R/I, H^i_I(M))$ and $\text{Ext}^1_R(R/I, H^i_I(M))$ are finitely generated. On the other hand, according to Lemma 3.3 or Melkersson’s result [25, Proposition 4.3], $N$ is $I$-cofinite. Now, the exact sequence

$$0 \longrightarrow N \longrightarrow H^i_I(M) \longrightarrow H^i_I(M)/N \longrightarrow 0$$

induces the following exact sequence,

$$\text{Hom}_R(R/I, H^i_I(M)) \longrightarrow \text{Hom}_R(R/I, H^i_I(M)/N) \longrightarrow \text{Ext}^1_R(R/I, N) \longrightarrow$$

$$\text{Ext}^1_R(R/I, H^i_I(M)) \longrightarrow \text{Ext}^2_R(R/I, H^i_I(M)/N) \longrightarrow \text{Ext}^2_R(R/I, N).$$

Consequently

$$\text{Hom}_R(R/I, H^i_I(M)/N)$$

are finitely generated, as required. □

**Corollary 3.5.** Let $R$ be a Noetherian ring and $I$ an ideal of $R$. Let $M$ be a finitely generated $R$-module such that the $R$-modules $H^i_I(M)$ are FD$_{\leq 1}$ (or weakly Laskerian) $R$-modules for all $i$. Then, the

(i) The $R$-modules $H^i_I(M)$ are $I$-cofinite for all $i$.

(ii) For any $i \geq 0$ and for any FD$_{\leq 0}$ (or minimax) submodule $N$ of $H^i_I(M)$, the $R$-module $H^i_I(M)/N$ is $I$-cofinite.
Proof. (i) Clear.
(ii) In view of (i) the R-module $H^i_I(M)$ is $I$-cofinite for all $i$. Hence the R-module $\text{Hom}_R(R/I, N)$ is finitely generated, and so it follows from Lemma 3.3 or [25, Proposition 4.3] that $N$ is $I$-cofinite. Now, the exact sequence

$$0 \to N \to H^i_I(M) \to H^i_I(M)/N \to 0,$$

implies that the R-module $H^i_I(M)/N$ is $I$-cofinite. \qed

Corollary 3.6. Let $R$ be a Noetherian ring and $I$ an ideal of $R$. Let $M$ be a finitely generated $R$-module and $t \geq 1$ be a positive integer such that the $R$-modules $H^i_I(M)$ are weakly Laskerian for all $i < t$. Then, the following conditions hold:

(i) The $R$-modules $H^i_I(M)$ are $I$-cofinite for all $i < t$.
(ii) For all $\text{FD}_{\leq 0}$ (or minimax) submodule $N$ of $H^t_I(M)$, the $R$-modules $\text{Hom}_R(R/I, H^t_I(M)/N)$ and $\text{Ext}^1_R(R/I, H^t_I(M)/N)$ are finitely generated. In particular the set $\text{Ass}_R(H^t_I(M)/N)$ is a finite set.

Proof. Use Theorem 2.2 and note that the category of weakly Laskerian modules is contained in the category of $\text{FD}_{\leq 1}$ modules. \qed

One of the main result of this section is to prove that for an arbitrary ideal $I$ of a Noetherian ring $R$, the Category of $I$-cofinite $\text{FD}_{\leq 1}$ modules compose an Abelian category.

Theorem 3.7. Let $I$ be an ideal of a Noetherian ring $R$. Let $\mathcal{F} \mathcal{D}^1(R, I)_{\text{cof}}$ denote the category of $I$-cofinite $\text{FD}_{\leq 1}$ $R$-modules. Then $\mathcal{F} \mathcal{D}^1(R, I)_{\text{cof}}$ is an Abelian category.

Proof. Let $M, N \in \mathcal{F} \mathcal{D}^1(R, I)_{\text{cof}}$ and let $f : M \to N$ be an $R$-homomorphism. It is enough that to show that the $R$-modules $\ker f$ and $\coker f$ are $I$-cofinite.

To this end, the exact sequence

$$0 \to \ker f \to M \to \text{im} f \to 0,$$

induces an exact sequence

$$0 \to \text{Hom}_R(R/I, \ker f) \to \text{Hom}_R(R/I, M) \to \text{Hom}_R(R/I, \text{im} f) \to \text{Ext}^1_R(R/I, \ker f) \to \text{Ext}^1_R(R/I, M),$$

that implies the $R$-modules $\text{Hom}_R(R/I, \ker f)$ and $\text{Ext}^1_R(R/I, \ker f)$ are finitely generated. Therefore it follows from Theorem 3.1 that $\ker f$ is $I$-cofinite. Now, the assertion follows from the following exact sequences

$$0 \to \ker f \to M \to \text{im} f \to 0,$$

and

$$0 \to \text{im} f \to N \to \coker f \to 0.$$

\qed

The following corollary is a generalization of [1, Theorem 2.7].
Corollary 3.8. Let $I$ be an ideal of a Noetherian ring $R$. Let $M$ be an $\text{FD}_1$ $I$-cofinite $R$-module. Then, the $R$-modules $\text{Ext}_R^i(N, M)$ and $\text{Tor}_R^i(N, M)$ are $I$-cofinite and $\text{FD}_1$ modules, for all finitely generated $R$-modules $N$ and all integers $i \geq 0$.

Proof. Since $N$ is finitely generated it follows that $N$ has a free resolution of finitely generated free modules. Now the assertion follows using Theorem 3.7 and computing the modules $\text{Tor}_R^i(N, M)$ and $\text{Ext}_R^i(N, M)$, by this free resolution. □

Corollary 3.9. Let $I$ be an ideal of a Noetherian ring $R$, $M$ a non-zero finite $R$-module such that $\dim M/IM \leq 1$ (e.g., $\dim R/I \leq 1$). Then for each finite $R$-module $N$, the $R$-modules $\text{Ext}_R^i(N, H^j_I(M))$ and $\text{Tor}_R^j(N, H^i_I(M))$ are $I$-cofinite for all $i \geq 0$ and $j \geq 0$.

Proof. Note that $\dim \text{Supp} H^i_I(M) \leq \dim M/IM \leq 1$ thus it is an $\text{FD}_1$ module for all $i \geq 0$, now use Corollary 3.8. □

Lemma 3.10. Let $R$ be a Noetherian ring, $I$ a proper ideal of $R$ and $M$ a non-zero $\text{D}_{\leq 1}$ and $I$-cofinite $R$-module. Then for each non-zero finitely generated $R$-module $N$ with support in $V(I)$, the $R$-modules $\text{Ext}_R^i(M, N)$ are finitely generated, for all integers $i \geq 0$.

Proof. See [19, Theorem 2.8]. □

Corollary 3.11. Let $R$ be a Noetherian ring and $I$ be an ideal of $R$. Let $M$ be an $\text{FD}_{\leq 1}$ and $I$-cofinite $R$-module. Then, the $R$-modules $\text{Ext}_R^i(M, N)$ and $\text{Tor}_R^i(M, N)$ are finitely generated, for all finitely generated $R$-modules $N$ with $\text{Supp}(N) \subseteq V(I)$ and all integers $i \geq 0$.

Proof. The assertion follows from the definition using Lemma 3.9 and [25, Theorem 2.1]. □

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