DIRAC STRUCTURES AND DIXMIER-DOUADY BUNDLES

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ABSTRACT. A Dirac structure on a vector bundle $V$ is a maximal isotropic subbundle $E$ of the direct sum $V \oplus V^*$. We show how to associate to any Dirac structure a Dixmier-Douady bundle $A_E$, that is, a $\mathbb{Z}_2$-graded bundle of $C^*$-algebras with typical fiber the compact operators on a Hilbert space. The construction has good functorial properties, relative to Morita morphisms of Dixmier-Douady bundles. As applications, we show that the Dixmier-Douady bundle $A_{\text{Spin}^c_G}$ over a compact, connected Lie group (as constructed by Atiyah-Segal) is multiplicative, and we obtain a canonical ‘twisted Spin$_c$-structure’ on spaces with group valued moment maps.

Dedicated to Richard Melrose on the occasion of his 60th birthday.

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1. Introduction

A classical result of Dixmier and Douady [11] states that the degree three cohomology group $H^3(M, \mathbb{Z})$ classifies Morita isomorphism classes of $C^*$-algebra bundles $\mathcal{A} \to M$, with typical fiber $\mathbb{K}(\mathcal{H})$ the compact operators on a Hilbert space. Here a Morita isomorphism $\mathcal{E}: \mathcal{A}_1 \to \mathcal{A}_2$ is a bundle $\mathcal{E} \to M$ of bimodules, locally modeled on the $\mathbb{K}(\mathcal{H}_2) - \mathbb{K}(\mathcal{H}_1)$ bimodule $\mathbb{K}(\mathcal{H}_1, \mathcal{H}_2)$. Dixmier-Douady bundles $\mathcal{A} \to M$ may be regarded as higher
analagous of line bundles, with Morita isomorphisms replacing line bundle isomorphisms. An important example of a Dixmier-Douady bundle is the Clifford algebra bundle of a Euclidean vector bundle of even rank; a Morita isomorphism $\mathbb{C}l(V) \to \mathbb{C}$ amounts to a $\text{Spin}_c$-structure on $V$.

Given a Dixmier-Douady bundle $A \to M$, one has the twisted $K$-homology group $K_0(M, A)$, defined as the $K$-homology of the $C^*$-algebra of sections of $A$ (see Rosenberg [28]). Twisted $K$-homology is a covariant functor relative to morphisms 

$$(\Phi, E): A_1 \to A_2,$$

given by a proper map $\Phi: M_1 \to M_2$ and a Morita isomorphism $E: A_1 \to \Phi^* A_2$. For example, if $M$ is an even-dimensional Riemannian manifold, the twisted $K$-group $K_0(M, \mathbb{C}l(TM))$ contains a distinguished Kasparov fundamental class $[M]$, and in order to push this class forward under the map $\Phi: M \to \text{pt}$ one needs a Morita morphism $\mathbb{C}l(TM) \to \mathbb{C}$, i.e. a $\text{Spin}_c$-structure on $M$. The push-forward $\Phi_* [M] \in K_0(\text{pt}) = \mathbb{Z}$ is then the index of the associated $\text{Spin}_c$-Dirac operator. Similarly, if $A \to G$ is a Dixmier-Douady bundle over a Lie group, the definition of a ‘convolution product’ on $K_0(G, A)$ as a push-forward under group multiplication $\text{mult}: G \times G \to G$ requires an associative Morita morphism $(\text{mult}, E)$: $\text{pr}_1^* A \otimes \text{pr}_2^* A \to A$.

In this paper, we will relate the Dixmier-Douady theory to Dirac geometry. A (linear) Dirac structure $(V, E)$ over $M$ is a vector bundle $V \to M$ together with a subbundle

$$E \subset V := V \oplus V^*,$$

such that $E$ is maximal isotropic relative to the natural symmetric bilinear form on $V$. Obvious examples of Dirac structures are $(V, V)$ and $(V, V^*)$.

One of the main results of this paper is the construction of a Dirac-Dixmier-Douady functor, associating to any Dirac structure $(V, E)$ a Dixmier-Douady bundle $A_E$, and to every ‘strong’ morphism of Dirac structures $(V, E) \to (V', E')$ a Morita morphism $A_E \to A_{E'}$.

The Dixmier-Douady bundle $A_V$ is canonically Morita trivial, while $A_V$ (for $V$ of even rank) is canonically Morita isomorphic to $\mathbb{C}l(V)$. An interesting example of a Dirac structure is the Cartan-Dirac structure $(\mathbb{T}G, E)$ for a compact Lie group $G$. The Cartan-Dirac structures is multiplicative, in the sense that there exists a distinguished Dirac morphism

$$(1) \quad (\mathbb{T}G, E) \times (\mathbb{T}G, E) \to (\mathbb{T}G, E)$$

(with underlying map the group multiplication). The associated Dixmier-Douady bundle $A_E =: A_G^{\text{spin}}$ is related to the spin representation of the loop group $LG$. This bundle (or equivalently the corresponding bundle of projective Hilbert spaces) was described by Atiyah-Segal [6, Section 5], and plays a role in the work of Freed-Hopkins-Teleman [14]. As an immediate consequence of our theory, the Dirac morphism (1) gives rise to a Morita morphism

$$(2) \quad (\text{mult}, E): \text{pr}_1^* A_G^{\text{spin}} \otimes \text{pr}_2^* A_G^{\text{spin}} \to A_G^{\text{spin}}.$$
Another class of examples comes from the theory of quasi-Hamiltonian $G$-spaces, that is, spaces with $G$-valued moment maps $\Phi : M \rightarrow G$ \cite{2}. Typical examples of such spaces are products of conjugacy classes in $G$. As observed by Bursztyn-Crainic \cite{7}, the structure of a quasi-Hamiltonian space on $M$ defines a strong Dirac morphism $(TM, T^*M) \rightarrow (TG, E)$ to the Cartan-Dirac structure. Therefore, our theory gives a Morita morphism $A_{TM} \rightarrow A^\text{Spin}_G$. On the other hand, as remarked above $A_{TM}$ is canonically Morita isomorphic to the Clifford bundle $\mathbb{C}l(TM)$, provided $\dim M$ is even (this is automatic if $G$ is connected). One may think of the resulting Morita morphism

\begin{equation}
\mathbb{C}l(TM) \rightarrow A^\text{Spin}_G
\end{equation}

(with underlying map $\Phi$) as a ‘twisted Spin$_c$-structure’ on $M$ (following the terminology of Bai-Lin Wang \cite{33} and Douglas \cite{12}). In a forthcoming paper \cite{19}, we will define a pre-quantization of $M$ \cite{31, 34} in terms of a $G$-equivariant Morita morphism $(\Phi, E) : \mathbb{C} \rightarrow A^\text{preq}_G$. Tensoring with \cite{3}, one obtains a push-forward map in equivariant twisted $K$-homology

$$
\Phi_* : K^G_0(M, \mathbb{C}l(TM)) \rightarrow K^G_0(G, A^\text{preq}_G \otimes A^\text{Spin}_G).
$$

For $G$ compact, simple and simply connected, the Freed-Hopkins-Teleman theorem \cite{13, 14} identifies the target of this map as the fusion ring (Verlinde algebra) $R_k(G)$, where $k$ is the given level. The element $Q(M) = \Phi_*[M]$ of the fusion ring will be called the quantization of the quasi-Hamiltonian space. We will see in \cite{19} that its properties are similar to the geometric quantization of Hamiltonian $G$-spaces.

The organization of this paper is as follows. In Section 2 we consider Dirac structures and morphisms on vector bundles, and some of their basic examples. We observe that any Dirac morphism defines a path of Dirac structures inside a larger bundle. We introduce the ‘tautological’ Dirac structure over the orthogonal group and show that group multiplication lifts to a Dirac morphism. Section 3 gives a quick review of some Dixmier-Douady theory. In Section 4 we give a detailed construction of Dixmier-Douady bundles from families of skew-adjoint real Fredholm operators. In Section 5 we observe that any Dirac structure on a Euclidean vector bundle gives such a family of skew-adjoint real Fredholm operators, by defining a family of boundary conditions for the operator $\frac{\partial}{\partial t}$ on the interval $[0, 1]$. Furthermore, to any Dirac morphism we associate a Morita morphism of the Dixmier-Douady bundles, and we show that this construction has good functorial properties. In Section 7 we describe the construction of twisted Spin$_c$-structures for quasi-Hamiltonian $G$-spaces. In Section 8 we show that the associated Hamiltonian loop group space carries a distinguished ‘canonical line bundle’, generalizing constructions from \cite{13} and \cite{21}.

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2. Dirac structures and Dirac morphisms

We begin with a review of linear Dirac structures on vector spaces and on vector bundles \([1, 8]\). In this paper, we will not consider any notions of integrability.

2.1. Dirac structures. For any vector space \(V\), the direct sum \(V = V \oplus V^*\) carries a non-degenerate symmetric bilinear form extending the pairing between \(V\) and \(V^*\),

\[
\langle x_1, x_2 \rangle = \mu_1(v_2) + \mu_2(v_1), \quad x_i = (v_i, \mu_i).
\]

A morphism \((\Theta, \omega) : V \rightarrow V'\) is a linear map \(\Theta : V \rightarrow V'\) together with a 2-form \(\omega \in \bigwedge^2 V^*\). The composition of two morphisms \((\Theta, \omega) : V \rightarrow V'\) and \((\Theta', \omega') : V' \rightarrow V''\) is defined as follows:

\[
(\Theta', \omega') \circ (\Theta, \omega) = (\Theta' \circ \Theta, \omega + \Theta^* \omega').
\]

Any morphism \((\Theta, \omega) : V \rightarrow V'\) defines a relation between elements of \(V, V'\) as follows:

\[
(v, \alpha) \sim_{(\Theta, \omega)} (v', \alpha') \iff v' = \Theta(v), \quad \alpha = v \omega + \Theta^* \alpha'.
\]

Given a subspace \(E \subset V\), we define its forward image to be the set of all \(x' \in V'\) such that \(x \sim_{(\Theta, \omega)} x'\) for some \(x \in E\). For instance, \(V^*\) has forward image equal to \((V')^*\). Similarly, the backward image of a subspace \(E' \subset V'\) is the set of all \(x \in V\) such that \(x \sim_{(\Theta, \omega)} x'\) for some \(x' \in E'\). The backward image of \(\{0\} \subset V'\) is denoted \(\ker(\Theta, \omega)\), and the forward image of \(V\) is denoted \(\text{ran}(\Theta, \omega)\).

A subspace \(E\) is called Lagrangian if it is maximal isotropic, i.e. \(E^\perp = E\). Examples are \(V, V^* \subset V\). The forward image of a Lagrangian subspace \(E \subset U\) under a Dirac morphism \((\Theta, \omega)\) is again Lagrangian. On the set of Lagrangian subspaces with \(E \cap \ker(\Theta, \omega) = 0\), the forward image depends continuously on \(E\). The choice of a Lagrangian subspace \(E \subset V\) defines a (linear) Dirac structure, denoted \((V, E)\). We say that \((\Theta, \omega)\) defines a Dirac morphism

\[
(\Theta, \omega) : (V, E) \rightarrow (V', E')
\]

if \(E'\) is the forward image of \(E\), and a strong Dirac morphism if furthermore \(E \cap \ker(\Theta, \omega) = 0\). The composition of strong Dirac morphisms is again a strong Dirac morphism.

Examples 2.1. (a) Every morphism \((\Theta, \omega) : V \rightarrow V'\) defines a strong Dirac morphism \((V, V^*) \rightarrow (V', (V')^*)\).

(b) The zero Dirac morphism \((0, 0) : (V, E) \rightarrow (0, 0)\) is strong if and only if \(E \cap V = 0\).
(c) Given vector spaces $V, V'$, any 2-form $\omega \in \wedge^2 V^*$ defines a Dirac morphism $(0, \omega): (V, V) \to (V', (V')^*)$. It is a strong Dirac morphism if and only if $\omega$ is non-degenerate. (This is true in particular if $V' = 0$.)

(d) If $E = V$, a Dirac morphism $(\Theta, \omega): (V, V) \to (V', E')$ is strong if and only if $\ker(\omega) \cap \ker(\Theta) = 0$.

2.2. Paths of Lagrangian subspaces. The following observation will be used later on. Suppose $\boxed{4}$ is a strong Dirac morphism. Then there is a distinguished path connecting the subspaces

\[
E_0 = E \oplus (V')^*, \quad E_1 = V^* \oplus E',
\]

of $V \oplus V'$, as follows. Define a family of morphisms $(j_t, \omega_t): V \to V \oplus V'$ interpolating between $(\text{id} \oplus 0, 0)$ and $(0 \oplus \Theta, \omega)$:

\[
 j_t(v) = ((1 - t)v, t\Theta(v)), \quad \omega_t = t\omega.
\]

Then

\[
\ker(j_t, \omega_t) = \begin{cases} 
0 & t \neq 1, \\
\ker(\Theta, \omega) & t = 0.
\end{cases}
\]

Since $(\Theta, \omega)$ is a strong Dirac morphism, it follows that $E$ is transverse to $\ker(j_t, \omega_t)$ for all $t$. Hence the forward images $E_t \subset V \oplus V'$ under $(j_t, \omega_t)$ are a continuous path of Lagrangian subspaces, taking on the values $\boxed{5}$ for $t = 0, 1$. We will refer to $E_t$ as the standard path defined by the Dirac morphism $\boxed{4}$.

Given another strong Dirac morphism $(\Theta', \omega'): (V', E') \to (V'', E'')$, define a 2-parameter family of morphisms $(j_{tt'}, \omega_{tt'}): V \to V \oplus V' \oplus V''$ by

\[
 j_{tt'}(v) = ((1 - t - t')v, t\Theta(v), t'\Theta'((\Theta(v))))), \quad \omega_{tt'} = t\omega + t'(\omega + \Theta^* \omega')
\]

Then

\[
\ker(j_{tt'}, \omega_{tt'}) = \begin{cases} 
0 & t + t' \neq 1 \\
\ker(\Theta, \omega) & t + t' = 1, t \neq 0, \\
\ker((\Theta', \omega') \circ (\Theta, \omega)) & t = 0, t' = 1
\end{cases}
\]

In all cases, $\ker(j_{tt'}, \omega_{tt'}) \cap E = 0$, hence we obtain a continuous 2-parameter family of Lagrangian subspaces $E_{tt'} \subset V \oplus V' \oplus V''$ by taking the forward images of $E$. We have,

\[
E_{00} = E \oplus (V')^* \oplus (V'')^*, \quad E_{10} = V^* \oplus E' \oplus (V'')^*, \quad E_{01} = V^* \oplus (V')^* \oplus E''
\]

Furthermore, the path $E_{a0}$ (resp. $E_{0s}$, $E_{1-s,s}$) is the direct sum of $(V'')^*$ (resp. of $(V')^*$, $V^*$) with the standard path defined by $(\Theta, \omega)$ (resp. by $(\Theta', \omega') \circ (\Theta, \omega)$, $(\Theta', \omega')$).
2.3. The parity of a Lagrangian subspace. Let Lag($\mathbb{V}$) be the Lagrangian Grassmannian of $\mathbb{V}$, i.e. the set of Lagrangian subspaces $E \subset \mathbb{V}$. It is a submanifold of the Grassmannian of subspaces of dimension dim $\mathbb{V}$. Lag($\mathbb{V}$) has two connected components, which are distinguished by the mod 2 dimension of the intersection $E \cap \mathbb{V}$. We will say that $E$ has even or odd parity, depending on whether dim($E \cap \mathbb{V}$) is even or odd. The parity is preserved under strong Dirac morphisms:

**Proposition 2.2.** Let $(\Theta, \omega): (\mathbb{V}, E) \rightarrow (\mathbb{V}', E')$ be a strong Dirac morphism. Then the parity of $E'$ coincides with that of $E$.

**Proof.** Clearly, $E$ has the same parity as $E_0 = E \oplus (V')^*$, while $E'$ has the same parity as $E_1 = V^* \oplus E'$. But the Lagrangian subspaces $E_0, E_1 \subset \mathbb{V} \oplus \mathbb{V}'$ have the same parity since they are in the same path component of Lag($\mathbb{V} \oplus \mathbb{V}'$).

2.4. Orthogonal transformations. Suppose $V$ is a Euclidean vector space, with inner product $B$. Then the Lagrangian Grassmannian Lag($\mathbb{V}$) is isomorphic to the orthogonal group of $V$, by the map associating to $A \in O(V)$ the Lagrangian subspace

$$E_A = \{((I - A^{-1})v, (I + A^{-1})\frac{v}{2}) \mid v \in V\}.$$ 

Here $B$ is used to identify $V^* \cong V$, and the factor of $\frac{1}{2}$ in the second component is introduced to make our conventions consistent with [1]. For instance,

$$E_{-I} = V, \quad E_I = V^*, \quad E_{A^{-1}} = (E_A)^{op}$$

where we denote $E^{op} = \{(v, -\alpha) \mid (v, \alpha) \in E\}$. It is easy to see that the Lagrangian subspaces corresponding to $A_1, A_2$ are transverse if and only if $A_1 - A_2$ is invertible; more generally one has $E_{A_1} \cap E_{A_2} \cong \ker(A_1 - A_2)$. As a special case, taking $A_1 = A$, $A_2 = -I$ it follows that the parity of a Lagrangian subspace $E = E_A$ is determined by $\det(A) = \pm 1$.

**Remark 2.3.** The definition of $E_A$ may also be understood as follows. Let $V^-$ denote $V$ with the opposite bilinear form $-B$. Then $V \oplus V^-$ with split bilinear form $B \oplus (-B)$ is isometric to $\mathbb{V} = V \oplus V^*$ by the map $(a, b) \mapsto (a-b, (a+b)/2)$. This defines an inclusion $\kappa: O(V) \hookrightarrow O(V \oplus V^-) \cong O(\mathbb{V})$. The group $O(\mathbb{V})$ acts on Lagrangian subspaces, and one has $E_A = \kappa(A) \cdot V^*$.

2.5. Dirac structures on vector bundles. The theory developed above extends to (continuous) vector bundles $V \to M$ in a straightforward way. Thus, Dirac structures $(\mathbb{V}, E)$ are now given in terms of Lagrangian sub-bundles $E \subset \mathbb{V} = V \oplus V^*$. Given a Euclidean metric on $V$, the Lagrangian sub-bundles are identified with sections $A \in \Gamma(O(V))$. A Dirac morphism $(\Theta, \omega): (\mathbb{V}, E) \rightarrow (\mathbb{V}', E')$ is a vector bundle map $\Theta: V \to V'$ together with a 2-form $\omega \in \Gamma(\wedge^2 V^*)$, such that the fiberwise maps and 2-forms define Dirac morphisms $(\Theta_m, \omega_m): (\mathbb{V}_m, E_m) \rightarrow (\mathbb{V}'_m, E'_m)$. Here $\Phi$ is the map on the base underlying the bundle map $\Theta$. 


Example 2.4. For any Dirac structure $(\mathcal{V}, E)$, let $U := \text{ran}(E) \subset V$ be the projection of $E$ along $V^*$. If $U$ is a sub-bundle of $V$, then the inclusion $U \hookrightarrow V$ defines a strong Dirac morphism, $(U, U) \longrightarrow (V, E)$. More generally, if $\Phi: N \to M$ is such that $U := \Phi^* \text{ran}(E) \subset \Phi^*V$ is a sub-bundle, then $\Phi$ together with fiberwise inclusion defines a strong Dirac morphism $(U, U) \longrightarrow (V, E)$. For instance, if $(V, E)$ is invariant under the action of a Lie group, one may take $\Phi$ to be the inclusion of an orbit.

2.6. The Dirac structure over the orthogonal group. Let $X$ be a vector space, and put $\mathcal{X} = X \oplus X^*$. The trivial bundle $V_{\text{Lag}(\mathcal{X})} = \text{Lag}(\mathcal{X}) \times X$ carries a tautological Dirac structure $(V_{\text{Lag}(\mathcal{X})}, E_{\text{Lag}(\mathcal{X})})$, with fiber $(E_{\text{Lag}(\mathcal{X})})_m$ at $m \in \text{Lag}(\mathcal{X})$ the Lagrangian subspace labeled by $m$. Given a Euclidean metric $B$ on $X$, we may identify $\text{Lag}(\mathcal{X}) = O(X)$; the tautological Dirac structure will be denoted by $(V_{O(X)}, E_{O(X)})$. It is equivariant for the conjugation action of $O(X)$. We will now show that the tautological Dirac structure over $O(X)$ is multiplicative, in the sense that group multiplication lifts to a strong Dirac morphism. Let $\Sigma: V_{O(X)} \times V_{O(X)} \to V_{O(X)}$ be the bundle map, given by the group multiplication on $V_{O(X)}$ viewed as a semi-direct product $O(X) \ltimes X$. That is,

\begin{equation}
\Sigma((A_1, \xi_1), (A_2, \xi_2)) = (A_1 A_2, A_2^{-1} \xi_1 + \xi_2).
\end{equation}

Let $\sigma$ be the 2-form on $V_{O(X)} \times V_{O(X)}$, given at $(A_1, A_2) \in O(X) \times O(X)$ as follows,

\begin{equation}
\sigma_{(A_1,A_2)}((\xi_1, \xi_2),(\zeta_1, \zeta_2)) = \frac{1}{2}(B(\xi_1, A_2 \zeta_2) - B(A_2 \xi_2, \xi_1)).
\end{equation}

Similar to [1, Section 3.4] we have:

Proposition 2.5. The map $\Sigma$ and 2-form $\sigma$ define a strong Dirac morphism

\[(\Sigma, \sigma): (V_{O(X)}, E_{O(X)}) \times (V_{O(X)}, E_{O(X)}) \longrightarrow (V_{O(X)}, E_{O(X)})\]

This morphism is associative in the sense that

\[(\Sigma, \sigma) \circ (\Sigma \times \text{id}, \sigma \times 0) = (\Sigma, \sigma) \circ (\text{id} \times \Sigma, 0 \times \sigma)\]

as morphisms $(\mathcal{V}, E) \times (\mathcal{V}, E) \times (\mathcal{V}, E) \longrightarrow (\mathcal{V}, E)$.

Outline of Proof. Given $A_1, A_2 \in O(X)$ let $A = A_1 A_2$, and put

\begin{equation}
e(\xi) = ((I - A^{-1})\xi, (I + A^{-1})\xi), \quad \xi \in X.
\end{equation}

Define $e_1(\xi)$ similarly for $A_1, A_2$. One checks that

\[e_1(\xi_1) \times e_2(\xi_2) \sim (\Sigma, \sigma) e(\xi)\]

if and only if $\xi_1 = \xi_2 = \xi$. The straightforward calculation is left to the reader. It follows that every element in $E_{O(X)}|_A$ is related to a unique element in $E_{O(X)}|_{A_1} \times E_{O(X)}|_{A_2}$. \qed
2.7. Cayley transform and exponential map. The trivial bundle $V_{\wedge 2}X = \wedge^2 X \times X$ carries a Dirac structure $(\mathcal{V}_{\wedge 2}X, E_{\wedge 2}X)$, with fiber at $a \in \wedge^2 X$ the graph $\text{Gr}_a = \{(\mu a, \mu) \mid \mu \in X^*\}$. It may be viewed as the restriction of the tautological Dirac structure under the inclusion $\wedge^2 X \hookrightarrow \text{Lag}(X)$, $a \mapsto \text{Gr}_a$. Use a Euclidean metric $B$ on $X$ to identify $\wedge^2 X = \mathfrak{o}(X)$, and write $(\mathcal{V}_{\mathfrak{o}(X)}, E_{\mathfrak{o}(X)})$. The orthogonal transformation corresponding to the Lagrangian subspace $\text{Gr}_a$ is given by the Cayley transform $I_{a/2}$. Hence, the bundle map

$$\Theta: V_{\mathfrak{o}(X)} \to V_{\mathcal{O}(X)}, \ (a, \xi) \mapsto \left(\frac{I_{a/2}}{I_{a/2}}, \xi\right)$$

together with the zero 2-form define a strong Dirac morphism

$$(\Theta, 0): (\mathcal{V}_{\mathfrak{o}(X)}, E_{\mathfrak{o}(X)}) \rightarrow (\mathcal{V}_{\mathcal{O}(X)}, E_{\mathcal{O}(X)}),$$

with underlying map the Cayley transform. On the other hand, we may also try to lift the exponential map $\exp: \mathfrak{o}(X) \to \mathcal{O}(X)$. Let

$$(9) \quad \Pi: V_{\mathfrak{o}(X)} \to V_{\mathcal{O}(X)}, \ (a, \xi) \mapsto (\exp(a), \frac{I_{-a} - a}{a} \xi),$$

the exponential map for the semi-direct product $\mathfrak{o}(X) \ltimes X \rightarrow \mathcal{O}(X) \ltimes X$. Define a 2-form $\varpi$ on $V_{\mathfrak{o}(X)}$ by

$$(10) \quad \varpi_a(\xi_1, \xi_2) = -B(\frac{a - \sinh(a)}{a^2}, \xi_1, \xi_2).$$

The following is parallel to [11, Section 3.5].

**Proposition 2.6.** The map $\Pi$ and the 2-form $\varpi$ define a Dirac morphism

$$(\Pi, -\varpi): (\mathcal{V}_{\mathfrak{o}(X)}, E_{\mathfrak{o}(X)}) \rightarrow (\mathcal{V}_{\mathcal{O}(X)}, E_{\mathcal{O}(X)}).$$

It is a strong Dirac morphism over the open subset $\mathfrak{o}(V)_2$ where the exponential map has maximal rank.

**Outline of Proof.** Let $a \in \mathfrak{o}(X)$ and $A = \exp(a)$ be given. Let $e(\xi)$ be as in [3] and define $e_0(\xi) = (a\xi, \xi)$. One checks by straightforward calculation that

$$e_0(\xi) \sim_{(\Pi, -\varpi)} e(\xi)$$

proving that $\ (\Pi, -\varpi): (\mathcal{V}_{\mathfrak{o}(X)}, E_{\mathfrak{o}(X)}) \rightarrow (\mathcal{V}_{\mathcal{O}(X)}, E_{\mathcal{O}(X)})$ is a Dirac morphism. Suppose now that the exponential map is regular at $a$. By the well-known formula for the differential of the exponential map, this is equivalent to invertibility of $\Pi_a$. An element of the form $(a\xi, \xi)$ lies in $\ker(\Theta, \omega)$ if and only if $\Pi_a(a\xi) = 0$ and $\xi = a\xi \varpi_a$. The first condition shows $a\xi = 0$, and then the second condition gives $\xi = 0$. Hence $e_0(\xi) \sim_{(\Pi, -\varpi)} 0 \Rightarrow \xi = 0$. Conversely, if $\Pi_a$ is not invertible, and $\xi \neq 0$ is an element in the kernel, then $(a\xi, \xi) \sim_{(\Pi, -\varpi)} 0.$

3. Dixmier-Douady bundles and Morita morphisms

We give a quick review of Dixmier-Douady bundles, geared towards applications in twisted $K$-theory. For more information we refer to the articles [11, 13, 18] and the monograph [26]. Dixmier-Douady bundles are also known as Azumaya bundles.
3.1. Dixmier-Douady bundles. A Dixmier-Douady bundle is a locally trivial bundle $A \to M$ of $\mathbb{Z}_2$-graded $C^*$-algebras, with typical fiber $\mathbb{K}(\mathcal{H})$ the compact operators on a $\mathbb{Z}_2$-graded (separable) complex Hilbert space, and with structure group $\text{Aut}(\mathbb{K}(\mathcal{H})) = \text{PU}(\mathcal{H})$, using the strong operator topology. The tensor product of two such bundles $A_1, A_2 \to M$ modeled on $\mathbb{K}(\mathcal{H}_1), \mathbb{K}(\mathcal{H}_2)$ is a Dixmier-Douady bundle $A_1 \otimes A_2$ modeled on $\mathbb{K}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. For any Dixmier-Douady bundle $A \to M$ modeled on $\mathbb{K}(\mathcal{H})$, the bundle of opposite $C^*$-algebras $A^{\text{op}} \to M$ is a Dixmier-Douady bundle modeled on $\mathbb{K}(\mathcal{H}^{\text{op}})$, where $\mathcal{H}^{\text{op}}$ denotes the opposite (or conjugate) Hilbert space.

3.2. Morita isomorphisms. A Morita isomorphism $\mathcal{E} : A_1 \to A_2$ between two Dixmier-Douady bundles over $M$ is a $\mathbb{Z}_2$-graded bundle $\mathcal{E} \to M$ of Banach spaces, with a fiberwise $A_2 - A_1$ bimodule structure

$$A_2 \otimes \mathcal{E} \otimes A_1$$

that is locally modeled on $\mathbb{K}(\mathcal{H}_2) \otimes \mathbb{K}(\mathcal{H}_1, \mathcal{H}_2) \otimes \mathbb{K}(\mathcal{H}_1)$. Here $\mathbb{K}(\mathcal{H}_1, \mathcal{H}_2)$ denotes the $\mathbb{Z}_2$-graded Banach space of compact operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. In terms of the associated principal bundles, a Morita isomorphism is given by a lift of the structure group $\text{PU}(\mathcal{H}_1) \times \text{PU}(\mathcal{H}_1^{\text{op}})$ of $A_2 \otimes A_1^{\text{op}}$ to $\text{PU}(\mathcal{H}_2 \otimes \mathcal{H}_1^{\text{op}})$. The composition of two Morita isomorphisms $\mathcal{E} : A_1 \to A_2$ and $\mathcal{E}' : A_2 \to A_3$ is given by $\mathcal{E}' \circ \mathcal{E} = \mathcal{E}' \otimes A_2 \mathcal{E}$, the fiberwise completion of the algebraic tensor product over $A_2$. In local trivializations, it is given by the composition $\mathbb{K}(\mathcal{H}_2, \mathcal{H}_3) \times \mathbb{K}(\mathcal{H}_1, \mathcal{H}_2) \to \mathbb{K}(\mathcal{H}_1, \mathcal{H}_3)$.

Examples 3.1. (a) A Morita isomorphism $\mathcal{E} : C \to A$ is called a Morita trivialization of $A$, and amounts to a Hilbert space bundle $\mathcal{E}$ with an isomorphism $A = \mathbb{K}(\mathcal{E})$.

(b) Any $*$-bundle isomorphism $\phi : A_1 \to A_2$ may be viewed as a Morita isomorphism $A_1 \to A_2$, by taking $\mathcal{E} = A_2$ with the $A_2 - A_1$-bimodule action $x_2 \cdot y \cdot x_1 = x_2 y \phi(x_1)$.

(c) For any Morita isomorphism $\mathcal{E} : A_1 \to A_2$ there is an opposite Morita isomorphism $\mathcal{E}^{\text{op}} : A_2 \to A_1$, where $\mathcal{E}^{\text{op}}$ is equal to $\mathcal{E}$ as a real vector bundle, but with the opposite scalar multiplication. Denoting by $\chi : \mathcal{E} \to \mathcal{E}^{\text{op}}$ the anti-linear map given by the identity map of the underlying real bundle, the $A_1 - A_2$-bimodule action reads $x_1 \cdot \chi(e) \cdot x_2 = \chi(x_1^2 \cdot e \cdot x_1^2)$. The Morita isomorphism $\mathcal{E}^{\text{op}}$ is ‘inverse’ to $\mathcal{E}$, in the sense that there are canonical bimodule isomorphisms

$$\mathcal{E}^{\text{op}} \circ \mathcal{E} \cong A_1, \quad \mathcal{E} \circ \mathcal{E}^{\text{op}} \cong A_2.$$

3.3. Dixmier-Douady theorem. The Dixmier-Douady theorem (in its $\mathbb{Z}_2$-graded version) states that the Morita isomorphism classes of Dixmier-Douady bundles $A \to M$ are classified by elements

$$\text{DD}(A) \in H^3(M, \mathbb{Z}) \times H^1(M, \mathbb{Z}_2),$$

called the Dixmier-Douady class of $A$. Write $\text{DD}(A) = (x, y)$. Letting $\hat{A}$ be the Dixmier-Douady-bundle obtained from $A$ by forgetting the $\mathbb{Z}_2$-grading,
the element $x$ is the obstruction to the existence of an (ungraded) Morita trivialization $\hat{E} : \mathbb{C} \rightarrow \hat{A}$. The class $y$ corresponds to the obstruction of introducing a compatible $\mathbb{Z}_2$-grading on $\hat{E}$. In more detail, given a loop $\gamma : S^1 \rightarrow M$ representing a homology class $[\gamma] \in H_1(M, \mathbb{Z})$, choose a Morita trivialization $(\gamma, \hat{F}) : \mathbb{C} \rightarrow \hat{A}$. Then $y([\gamma]) = \pm 1$, depending on whether or not $\hat{F}$ admits a compatible $\mathbb{Z}_2$-grading.

(a) The opposite Dixmier-Douady bundle $A^{\text{op}}$ has class $\text{DD}(A^{\text{op}}) = -\text{DD}(A)$.

(b) If $\text{DD}(A_i) = (x_i, y_i), i = 1, 2,$ are the classes corresponding to two Dixmier-Douady bundles $A_1, A_2$ over $M$, then \cite{[6], Proposition 2.3}

$$\text{DD}(A_1 \otimes A_2) = (x_1 + x_2 + \beta(y_1 \cup y_2), y_1 + y_2)$$

where $y_1 \cup y_2 \in H^2(M, \mathbb{Z}_2)$ is the cup product, and $\beta : H^2(M, \mathbb{Z}_2) \rightarrow H^3(M, \mathbb{Z})$ is the Bockstein homomorphism.

3.4. 2-isomorphisms. Let $A_1, A_2$ be given Dixmier-Douady bundles over $M$.

**Definition 3.2.** A 2-isomorphism between two Morita isomorphisms $\mathcal{E}, \mathcal{E}' : A_1 \rightarrow A_2$ is a continuous bundle isomorphism $\mathcal{E} \rightarrow \mathcal{E}'$, intertwining the norms, the $\mathbb{Z}_2$-gradings and the $A_2 - A_1$-bimodule structures.

Equivalently, a 2-isomorphism may be viewed as a trivialization of the $\mathbb{Z}_2$-graded Hermitian line bundle

$$L = \text{Hom}_{A_2 - A_1}(\mathcal{E}, \mathcal{E}')$$

given by the fiberwise bimodule homomorphisms. Any two Morita bimodules are related by \eqref{eq:11} as $\mathcal{E}' = \mathcal{E} \otimes L$. It follows that the set of 2-isomorphism classes of Morita isomorphisms $A_1 \rightarrow A_2$ is either empty, or is a principal homogeneous space (torsor) for the group $H^2(M, \mathbb{Z}_2) \times H^0(M, \mathbb{Z}_2)$ of $\mathbb{Z}_2$-graded line bundles.

**Example 3.3.** Suppose the Morita isomorphisms $\mathcal{E}, \mathcal{E}'$ are connected by a continuous path $\mathcal{E}_s$ of Morita isomorphisms, with $\mathcal{E}_0 = \mathcal{E}, \mathcal{E}_1 = \mathcal{E}'$. Then they are 2-isomorphic, in fact $L_s = \text{Hom}_{A_2 - A_1}(\mathcal{E}, \mathcal{E}_s)$ is a path connecting \eqref{eq:11} to the trivial line bundle.

**Example 3.4.** Suppose $A_s, s \in [0, 1]$ is a continuous family of Dixmier-Douady-bundles over $M$, i.e. their union defines a Dixmier-Douady bundle $A \rightarrow [0, 1] \times M$. Then there exists a continuous family of isomorphisms $\phi_s : A_0 \rightarrow A_s$, i.e. an isomorphism $\text{pr}_2^* A_0 \cong A$ of bundles over $[0, 1] \times M$. (The existence of such an isomorphism is clear in terms of the associated principal $\text{PU}(H)$-bundles.) By composing with $\phi_0^{-1}$ if necessary, we may assume $\phi_0 = \text{id}$. Any other such family of isomorphisms $\phi'_s : A_0 \rightarrow A_s$, $\phi'_0 = \text{id}$ is related to $\phi_s$ by a family $L_s$ of line bundles, with $L_0$ the trivial line
bundle. We conclude that the homotopy of Dixmier-Douady bundles \( A_s \) gives a distinguished 2-isomorphism class of isomorphisms \( A_0 \to A_1 \).

3.5. Clifford algebra bundles. Suppose that \( V \to M \) is a Euclidean vector bundle of rank \( n \). A Spin\(_c\)-structure on \( V \) is given by an orientation on \( V \) together with a lift of the structure group of \( V \) from SO\((n)\) to Spin\(_c\)(\(n\)), where \( n = \text{rk}(V) \). According to Connes \[10\] and Plymen \[23\], this is equivalent to Definition 3.5 below in terms of Dixmier-Douady bundles.

Recall that if \( n \) is even, then the associated bundle of complex Clifford algebras \( \text{Cl}(V) \) is a Dixmier-Douady bundle, modeled on \( \text{Cl}(\mathbb{R}^n) = \text{End}(\wedge^n \mathbb{R}^2) \). In this case, a Spin\(_c\)-structure may be defined to be a Morita trivialization \( \tilde{S} : \mathbb{C} \to \text{Cl}(V) \), with \( \tilde{S} \) is the associated spinor bundle. To include the case of odd rank, it is convenient to introduce \( \tilde{\mathcal{V}} = V \oplus \mathbb{R}^n \), \( \tilde{\text{Cl}(V)} = \text{Cl}(\tilde{\mathcal{V}}) \).

**Definition 3.5.** A Spin\(_c\)-structure on a Euclidean vector bundle \( V \) is a Morita trivialization

\[
\tilde{S} : \mathbb{C} \to \tilde{\text{Cl}(V)}
\]

The bundle \( \tilde{S} \) is called the corresponding spinor bundle. An isomorphism of two Spin\(_c\)-structures is a 2-isomorphism of the defining Morita trivializations.

If \( n \) is even, one recovers \( S \) by composing with the Morita isomorphism \( \tilde{\text{Cl}(V)} \to \text{Cl}(V) \). The Dixmier-Douady class \((x, y)\) of \( \text{Cl}(V) \) is the obstruction to the existence of a Spin\(_c\)-structure: In fact \( x \) is the third integral Stiefel-Whitney class \( \tilde{\beta}(w_2(V)) \in H^3(M, \mathbb{Z}) \), while \( y \) is the first Stiefel-Whitney class \( w_1(V) \in H^1(M, \mathbb{Z}_2) \), i.e. the obstruction to orientability of \( V \).

Any two Spin\(_c\)-structures on \( V \) differ by a \( \mathbb{Z}_2 \)-graded Hermitian line bundle, and an isomorphism of Spin\(_c\)-structures amounts to a trivialization of this line bundle. Observe that there is a Morita trivialization

\[
\wedge \tilde{\mathcal{V}} : \mathbb{C} \to \tilde{\text{Cl}(V \oplus V)} = \tilde{\text{Cl}(V)} \otimes \tilde{\text{Cl}(V)}
\]

defined by the complex structure on \( \tilde{\mathcal{V}} \oplus \tilde{\mathcal{V}} \cong \tilde{\mathcal{V}} \otimes \mathbb{R}^2 \). Hence, given a Spin\(_c\)-structure, we can define the Hermitian line bundle

\[
K_{\tilde{S}} = \text{Hom}_{\tilde{\text{Cl}(V \oplus V)}}(\tilde{S} \otimes \tilde{S}, \wedge \tilde{\mathcal{V}}).
\]

(If \( n \) is even, one may omit the \( \sim \)'s.) This is the canonical line bundle of the Spin\(_c\)-structure. If the Spin\(_c\)-structure on \( V \) is defined by a complex structure \( J \), then the canonical bundle coincides with \( \text{det}(V_-) = \wedge^{n/2} V_- \), where \( V_- \subset V \) is the \(-i\) eigenspace of \( J \).

3.6. Morita morphisms. It is convenient to extend the notion of Morita isomorphisms of Dixmier-Douady bundles, allowing non-trivial maps on the base. A Morita morphism

\[
(\Phi, \mathcal{E}) : A_1 \to A_2
\]
of bundles $A_i \to M_i$, $i = 1, 2$ is a continuous map $\Phi: M_1 \to M_2$ together with a Morita isomorphism $\mathcal{E}: A_1 \to \Phi^* A_2$. A given map $\Phi$ lifts to such a Morita morphism if and only if $\text{DD}(A_1) = \Phi^* \text{DD}(A_2)$. Composition of Morita morphisms is defined as $(\Phi', \mathcal{E}') \circ (\Phi, \mathcal{E}) = (\Phi' \circ \Phi, \Phi^* \mathcal{E}' \circ \mathcal{E})$. If $\mathcal{E}: \mathbb{C} \to A$ is a Morita trivialization, we can think of $\mathcal{E}^\text{op}: A \to \mathbb{C}$ as a Morita morphism covering the map $M \to \text{pt}$. As mentioned in the introduction, a Morita morphism such that $\Phi$ is proper induces a push-forward map in twisted K-homology.

3.7. Equivariance. The Dixmier-Douady theory generalizes to the $G$-equivariant setting, where $G$ is a compact Lie group. $G$-equivariant Dixmier-Douady bundles over a $G$-space $M$ are classified by $H^3_G(M, \mathbb{Z}) \times H^1_G(M, \mathbb{Z}/2)$. If $M$ is a point, a $G$-equivariant Dixmier-Douady bundle $A \to \text{pt}$ is of the form $A = \mathbb{K}(H)$ where $H$ is a $\mathbb{Z}/2$-graded Hilbert space with an action of a central extension $\tilde{G}$ of $G$ by $U(1)$. (It is a well-known fact that $H^3_G(\text{pt}, \mathbb{Z}) = H^3(BG, \mathbb{Z})$ classifies such central extensions.) The definition of Spin$_c$-structures in terms of Morita morphisms extends to the $G$-equivariant in the obvious way.

4. Families of skew-adjoint real Fredholm operators

In this Section, we will explain how a continuous family of skew-adjoint Fredholm operators on a bundle of real Hilbert spaces defines a Dixmier-Douady bundle. The construction is inspired by ideas in Atiyah-Segal [6], Carey-Mickelsson-Murray [9, 22], and Freed-Hopkins-Teleman [14, Section 3].

4.1. Infinite dimensional Clifford algebras. We briefly review the spin representation for infinite dimensional Clifford algebras. Excellent sources for this material are the book [24] by Plymen and Robinson and the article [5] by Araki.

Let $\mathcal{V}$ be an infinite dimensional real Hilbert space, and $\mathcal{V}^\mathbb{C}$ its complexification. The Hermitian inner product on $\mathcal{V}^\mathbb{C}$ will be denoted $\langle \cdot, \cdot \rangle$, and the complex conjugation map by $v \mapsto v^*$. Just as in the finite-dimensional case, one defines the Clifford algebra $\mathbb{C}l(\mathcal{V})$ as the $\mathbb{Z}/2$-graded unital complex algebra with odd generators $v \in \mathcal{V}^\mathbb{C}$ and relations, $vv = \langle v, v \rangle$. The Clifford algebra carries a unique anti-linear anti-involution $x \mapsto x^*$ extending the complex conjugation on $\mathcal{V}^\mathbb{C}$, and a unique norm $\| \cdot \|$ satisfying the $C^*$-condition $\|x^* x\| = \|x\|^2$. Thus $\mathbb{C}l(\mathcal{V})$ is a $\mathbb{Z}/2$-graded pre-$C^*$-algebra.

A (unitary) module over $\mathbb{C}l(\mathcal{V})$ is a complex $\mathbb{Z}/2$-graded Hilbert space $\mathcal{E}$ together with a $\ast$-homomorphism $\varrho: \mathbb{C}l(\mathcal{V}) \to \mathcal{L}(\mathcal{E})$ preserving $\mathbb{Z}/2$-gradings. Here $\mathcal{L}(\mathcal{E})$ is the $\ast$-algebra of bounded linear operators, and the condition on the grading means that $\varrho(v)$ acts as an odd operator for each $v \in \mathcal{V}^\mathbb{C}$. 
We will view $\mathcal{L}(\mathcal{V})$ (the bounded $\mathbb{R}$-linear operators on $\mathcal{V}$) as an $\mathbb{R}$-linear subspace of $\mathcal{L}(\mathcal{V}^C)$. Operators in $\mathcal{L}(\mathcal{V})$ will be called real. A real skew-adjoint operator $J \in \mathcal{L}(\mathcal{V})$ is called an orthogonal complex structure on $\mathcal{V}$ if it satisfies $J^2 = -I$. Note $J^* = -J = J^{-1}$, so that $J \in O(\mathcal{V})$.

The orthogonal complex structure defines a decomposition $\mathcal{V}^C = \mathcal{V}_+ \oplus \mathcal{V}_-$ into maximal isotropic subspaces $\mathcal{V}_\pm = \ker(J \mp i) \subset \mathcal{V}^C$. Note $v \in \mathcal{V}_+ \iff v^* \in \mathcal{V}_-$. Define a Clifford action of $\mathbb{C}l(\mathcal{V})$ on $\wedge \mathcal{V}_+$ by the formula

$$\rho(v) = \sqrt{2} (\epsilon(v_+) + \iota(v_-)),$$

writing $v = v_+ + v_-$ with $v_+ \in \mathcal{V}_+$. Here $\epsilon(v_+)$ denotes exterior multiplication by $v_+$, while the contraction $\iota(v_-)$ is defined as the unique derivation such that $\iota(v_-)w = \langle v_+, w \rangle$ for $w \in \mathcal{V}^C \subset \wedge \mathcal{V}^C$. Passing to the Hilbert space completion one obtains a unitary $\mathbb{Z}_2$-graded Clifford module

$$S_J = \wedge \mathcal{V}_+,$$

called the spinor module or Fock representation defined by $J$.

The equivalence problem for Fock representations was solved by Shale and Stinespring [32]. See also [24, Theorem 3.5.2].

**Theorem 4.1** (Shale-Stinespring). The $\mathbb{C}l(\mathcal{V})$-modules $S_1, S_2$ defined by orthogonal complex structures $J_1, J_2$ are unitarily isomorphic (up to possible reversal of the $\mathbb{Z}_2$-grading) if and only if $J_1 - J_2 \in \mathcal{L}_{HS}(\mathcal{V})$. In this case, the unitary operator implementing the isomorphism is unique up to a scalar $z \in U(1)$. The implementer has even or odd parity, according to the parity of $\frac{1}{2} \dim \ker(J_1 + J_2) \in \mathbb{Z}$.

**Definition 4.2.** [29, p. 193], [14] Two orthogonal complex structures $J_1, J_2$ on a real Hilbert space $\mathcal{V}$ are called equivalent (written $J_1 \sim J_2$) if their difference is Hilbert-Schmidt. An equivalence class of complex structures on $\mathcal{V}$ (resp. on $\mathcal{V} \oplus \mathbb{R}$) is called an even (resp. odd) polarization of $\mathcal{V}$.

By Theorem 4.1 the $\mathbb{Z}_2$-graded $C^*$-algebra $\mathbb{K}(S_J)$ depends only on the equivalence class of $J$, in the sense that there exists a canonical identification $\mathbb{K}(S_{J_1}) \equiv \mathbb{K}(S_{J_2})$ whenever $J_1 \sim J_2$. That is, any polarization of $\mathcal{V}$ determines a Dixmier-Douady algebra.

4.2. **Skew-adjoint Fredholm operators.** Suppose $D$ is a real skew-adjoint (possibly unbounded) Fredholm operator on $\mathcal{V}$, with dense domain $\text{dom}(D) \subset \mathcal{V}$. In particular $D$ has a finite-dimensional kernel, and 0 is an isolated point of the spectrum. Let $J_D$ denote the real skew-adjoint operator,

$$J_D = i \text{sign}(\frac{1}{2}D)$$

(using functional calculus for the self-adjoint operator $\frac{1}{2}D$). Thus $J_D$ is an orthogonal complex structure on $\ker(D) \perp$, and vanishes on $\ker(D)$. If $\ker(D) = 0$, we may also write $J_D = \frac{D}{|D|}$. The same definition of $J_D$ also applies to complex skew-adjoint Fredholm operators. We have:
Proposition 4.3. Let $D$ be a (real or complex) skew-adjoint Fredholm operator, and $Q$ a skew-adjoint Hilbert-Schmidt operator. Then $J_{D+Q} - J_D$ is Hilbert-Schmidt.

The following simple proof was shown to us by Gian-Michele Graf.

Proof. Choose $\epsilon > 0$ so that the spectrum of $D, D + Q$ intersects the set $|z| < 2\epsilon$ only in $\{0\}$. Replacing $D$ with $D + i\epsilon$ if necessary, and noting that $J_{D+i\epsilon} - J_D$ has finite rank, we may thus assume that $0$ is not in the spectrum of $D$ or of $D + Q$. One then has the following presentation of $J_D$ as a Riemannian integral of the resolvent $R(z) = (D - z)^{-1}$,

$$J_D = \frac{-1}{\pi} \int_{-\infty}^{\infty} R_t(D) dt,$$

convergent in the strong topology. Using a similar expression for $J_{D+Q}$ and the second resolvent identity $R_t(D + Q) - R_t(D) = -R_t(D + Q) Q R_t(D)$, we obtain

$$J_{D+Q} - J_D = \frac{1}{\pi} \int_{-\infty}^{\infty} R_t(D + Q) Q R_t(D) dt.$$

Let $a > 0$ be such that the spectrum of $D, D + Q$ does not meet the disk $|z| \leq a$. Then $||R_t(D)||, ||R_t(D + Q)|| \leq (t^2 + a^2)^{-1/2}$ for all $t \in \mathbb{R}$. Hence

$$||R_t(D + Q) Q R_t(D)||_{HS} \leq \frac{1}{t^2 + a^2} ||Q||_{HS},$$

using $||AB||_{HS} \leq ||A|| ||B||_{HS}$. Since $\int (t^2 + a^2)^{-1} dt = \pi/a$, we obtain the estimate

(14) $||J_{D+Q} - J_D||_{HS} \leq \frac{1}{a} ||Q||_{HS}. \quad \Box$

A real skew-adjoint Fredholm operator $D$ on $\mathcal{V}$ will be called of even (resp. odd) type if ker($D$) has even (resp. odd) dimension. As in [14], Section 3.1, we associate to any $D$ of even type the even polarization defined by the orthogonal complex structures $J \in \text{O}(\mathcal{V})$ such that $J - J_D$ is Hilbert-Schmidt. For $D$ of odd type, we similarly obtain an odd polarization by viewing $J_D$ as an operator on $\mathcal{V} \oplus \mathbb{R}$ (equal to 0 on $\mathbb{R}$).

Two skew-adjoint real Fredholm operators $D_1, D_2$ on $\mathcal{V}$ will be called equivalent (written $D_1 \sim D_2$) if they define the same polarization of $\mathcal{V}$, and hence the same Dixmier-Douady algebra $\mathcal{A}$. Equivalently, $D_i$ have the same parity and $J_{D_1} - J_{D_2}$ is Hilbert-Schmidt. In particular, $D \sim D + Q$ whenever $Q$ is a skew-adjoint Hilbert-Schmidt operator. In the even case, we can always choose $Q$ so that $D + Q$ is invertible, while in the odd case we can choose such a $Q$ after passing to $\mathcal{V} \oplus \mathbb{R}$.

Remark 4.4. The estimate (14) show that for fixed $D$ (such that $D, D + Q$ have trivial kernel), the difference $J_{D+Q} - J_D \in \mathcal{L}_{HS}(\mathcal{X})$ depends continuously on $Q$ in the Hilbert-Schmidt norm. On the other hand, it also depends continuously on $D$ relative to the norm resolvent topology [27, page 284].
This follows from the integral representation of $J_{D+Q} - J_D$, together with resolvent identities such as
\[ R_t(D') - R_t(D) = R_t(D')R_1(D')^{-1}(R_1(D') - R_1(D))R_1(D)^{-1}R_t(D). \]
giving estimates $||R_t(D') - R_t(D)|| \leq (t^2 + a^2)^{-1}||R_1(D') - R_1(D)||$ for $a > 0$ such that the spectrum of $D, D'$ does not meet the disk of radius $a$.

4.3. Polarizations of bundles of real Hilbert spaces. Let $\mathcal{V} \rightarrow M$ be a bundle of real Hilbert spaces, with typical fiber $\mathcal{X}$ and with structure group $O(\mathcal{X})$ (using the norm topology). A polarization on $\mathcal{V}$ is a family of polarizations on $\mathcal{V}_m$, depending continuously on $m$. To make this precise, fix an orthogonal complex structure $J_0 \in O(\mathcal{X})$, and let $L_{res}(\mathcal{X})$ be the Banach space of bounded linear operators $S$ such that $[S, J_0]$ is Hilbert-Schmidt, with norm $||S|| + ||[S, J_0]||_HS$. Define the restricted orthogonal group $O_{res}(\mathcal{X}) = O(\mathcal{X}) \cap L_{res}(\mathcal{X})$, with the subspace topology. It is a Banach Lie group, with Lie algebra $\mathfrak{o}_{res}(\mathcal{X}) = \mathfrak{o}(\mathcal{X}) \cap L_{res}(\mathcal{X})$. The unitary group $U(\mathcal{X}) = U(\mathcal{X}, J_0)$ relative to $J_0$, equipped with the norm topology is a Banach subgroup of $O_{res}(\mathcal{X})$. For more details on the restricted orthogonal group, we refer to Araki [5] or Pressley-Segal [25].

Definition 4.5. An even polarization of the real Hilbert space bundle $\mathcal{V} \rightarrow M$ is a reduction of the structure group $O(\mathcal{X})$ to the restricted orthogonal group $O_{res}(\mathcal{X})$. An odd polarization of $\mathcal{V}$ is an even polarization of $\mathcal{V} \oplus \mathbb{R}$.

Thus, a polarization is described by a system of local trivializations of $\mathcal{V}$ whose transition functions are continuous maps into $O_{res}(\mathcal{X})$. Any global complex structure on $\mathcal{V}$ defines a polarization, but not all polarizations arise in this way.

Proposition 4.6. Suppose $\mathcal{V} \rightarrow M$ comes equipped with a polarization. For $m \in M$ let $A_m$ be the Dixmier-Douady algebra defined by the polarization on $\mathcal{V}_m$. Then $A = \bigcup_{m \in M} A_m$ is a Dixmier-Douady bundle.

Proof. We consider the case of an even polarization (for the odd case, replace $\mathcal{V}$ with $\mathcal{V} \oplus \mathbb{R}$). By assumption, the bundle $\mathcal{V}$ has a system of local trivializations with transition functions in $O_{res}(\mathcal{X})$. Let $S_0$ be the spinor module over $\mathcal{C}I(\mathcal{X})$ defined by $J_0$, and $PU(S_0)$ the projective unitary group with the strong operator topology. A version of the Shale-Stinespring theorem [21] Theorem 3.3.5] says that an orthogonal transformation is implemented as a unitary transformation of $S_0$ if and only if it lies in $O_{res}(\mathcal{X})$, and in this case the implementer is unique up to scalar. According to Araki [5] Theorem 6.10(7)], the resulting group homomorphism $O_{res}(\mathcal{X}) \rightarrow PU(S_0)$ is continuous. That is, $A$ admits the structure group $PU(S_0)$ with the strong topology.

In terms of the principal $O_{res}(\mathcal{X})$-bundle $\mathcal{P} \rightarrow M$ defined by the polarization of $\mathcal{V}$, the Dixmier-Douady bundle is an associated bundle
\[ A = \mathcal{P} \times O_{res}(\mathcal{X}) \mathbb{K}(S_0). \]
4.4. Families of skew-adjoint Fredholm operators. Suppose now that
\( D = \{ D_m \} \) is a family of (possibly unbounded) real skew-adjoint Fredholm
operators on \( \mathcal{V}_m \), depending continuously on \( m \in M \) in the norm resolvent
sense \[27\], page 284]. That is, the bounded operators \( (D_m - I)^{-1} \in \mathcal{L}(\mathcal{V}_m) \)
define a continuous section of the bundle \( \mathcal{L}(\mathcal{V}) \) with the norm topology. The
map \( m \mapsto \text{dim ker}(D_m) \) is locally constant \( \mod \ 2 \). The family \( D \) will be
called of even (resp. odd) type if all \( \dim \ker(D_m) \) are even (resp. odd). Each
\( D_m \) defines an even (resp. odd) polarization of \( \mathcal{V}_m \), given by the complex
structures on \( \mathcal{V}_m \) or \( \mathcal{V}_m \oplus \mathbb{R} \) whose difference with \( J_{D_m} \) is Hilbert-Schmidt.

**Proposition 4.7.** Let \( D = \{ D_m \} \) be a family of (possibly unbounded) real
skew-adjoint Fredholm operators on \( \mathcal{V}_m \), depending continuously on \( m \in M \)
in the norm resolvent sense. Then the corresponding family of polarizations
on \( \mathcal{V}_m \) depends continuously on \( m \) in the sense of Definition 4.4. That is,
\( D \) determines a polarization of \( \mathcal{V} \).

**Proof.** We assume that the family \( D \) is of even type. (The odd case is dealt
with by adding a copy of \( \mathbb{R} \).) We will show the existence of a system of local
trivializations
\[
\phi_\alpha : \mathcal{V}|_{U_\alpha} = U_\alpha \times \mathcal{X}
\]
and skew-adjoint Hilbert-Schmidt perturbations \( Q_\alpha \in \Gamma(\mathcal{L}_{\text{HS}}(\mathcal{V}|_{U_\alpha})) \) of
\( D|_{U_\alpha} \), continuous in the Hilbert-Schmidt norm\(^1\), so that
\begin{enumerate}[(i)]  
\item \( \ker(D_m + Q_\alpha|_{m}) = 0 \) for all \( m \in U_\alpha \), and
\item \( \phi_\alpha \circ J_{D+Q_\alpha} \circ \phi_\alpha^{-1} = J_0 \).
\end{enumerate}
The transition functions \( \chi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : U_\alpha \cap U_\beta \to O(\mathcal{X}) \) will
then take values in \( O_{\text{res}}(\mathcal{X}) \): Indeed, by Proposition 4.3 the difference \( J_{D+Q_\beta} - J_{D+Q_\alpha} \)
is Hilbert-Schmidt, and (using (14) and Remark 4.4) it is a continuous section of \( \mathcal{L}_{\text{HS}}(\mathcal{V}) \) over \( U_\alpha \cap U_\beta \). Conjugating by \( \phi_\alpha \), and using (ii) it follows
that
\[
(15) \quad \chi_{\alpha\beta}^{-1} \circ J_0 \circ \chi_{\alpha\beta} - J_0 : U_\alpha \cap U_\beta \to \mathcal{L}(\mathcal{X})
\]
takes values in Hilbert-Schmidt operators, and is continuous in the Hilbert-Schmidt
norm. Hence the \( \chi_{\alpha\beta} \) are continuous functions into \( O_{\text{res}}(\mathcal{X}) \).

It remains to construct the desired system of local trivializations. It
suffices to construct such a trivialization near any given \( m_0 \in M \). Pick
a continuous family of skew-adjoint Hilbert-Schmidt operators \( Q \) so that
\( \ker(D_{m_0} + Q_{m_0}) = 0 \). (We may even take \( Q \) of finite rank.) Hence \( J_{D_{m_0}+Q_{m_0}} \)
is a complex structure. Choose an isomorphism \( \phi_{m_0} : \mathcal{V}_{m_0} \to \mathcal{X} \) intertwining
\( J_{D_{m_0}+Q_{m_0}} \) with \( J_0 \), and extend to a local trivialization \( \phi : \mathcal{V}|_U \to U \times \mathcal{X} \)
over a neighborhood \( U \) of \( m_0 \). We may assume that \( \ker(D_m + Q_m) = 0 \)
for \( m \in U \), defining complex structures \( J_m = \phi_m \circ J_{D_m+Q_m} \circ \phi_m^{-1} \). By
construction \( J_{m_0} = J_0 \), and hence \( ||J_m - J_0|| < 2 \) after \( U \) is replaced by a

\(^1\)The sub-bundle \( \mathcal{L}_{\text{HS}}(\mathcal{V}) \subset \mathcal{L}(\mathcal{V}) \) carries a topology, where a sections is continuous
at \( m \in M \) if its expression in a local trivialization of \( \mathcal{V} \) near \( m \) is continuous. (This is
independent of the choice of trivialization.)
smaller neighborhood if necessary. By [24, Theorem 3.2.4], Condition (ii) guarantees that
\[ g_m = (I - J_m J_0) |I - J_m J_0|^{-1} \]
gives a well-defined continuous map \( g: U \to O(X) \) with \( J_m = g_m J_0 g_m^{-1} \). Hence, replacing \( \phi \) with \( g \circ \phi \) we obtain a local trivialization satisfying (i), (ii).

To summarize: A continuous family \( D = \{ D_m \} \) of skew-adjoint real Fredholm operators on \( V \) determines a polarization of \( V \). The fibers \( P_m \) of the associated principal \( O_{\text{res}}(X) \)-bundle \( P \to M \) defining the polarization are given as the set of isomorphisms of real Hilbert spaces \( \phi_m: V_m \to X \) such that \( J_0 - \phi_m J_{D_m} \phi_m^{-1} \) is Hilbert-Schmidt. In turn, the polarization determines a Dixmier-Douady bundle \( A \to M \).

We list some elementary properties of this construction:

(a) Suppose \( V \) has finite rank. Then \( A = \text{Cl}(V) \) if the rank is even, and \( A = \text{Cl}(V \oplus \mathbb{R}) \) if the rank is odd. In both cases, \( A \) is canonically Morita isomorphic to \( \tilde{\text{Cl}}(V) \).

(b) If \( \ker(D) = 0 \) everywhere, the complex structure \( J = D|D|^{-1} \) gives a global a spinor module \( S \), defining a Morita trivialization
\[ S: \mathbb{C} \to A. \]

(c) If \( V = V' \oplus V'' \) and \( D = D' \oplus D'' \), the corresponding Dixmier-Douady algebras satisfy \( A \cong A' \otimes A'' \), provided the kernels of \( D' \) or \( D'' \) are even-dimensional. If both \( D' \), \( D'' \) have odd-dimensional kernels, we obtain \( A \otimes \text{Cl}(\mathbb{R}^2) \cong A' \otimes A'' \). In any case, \( A \) is canonically Morita isomorphic to \( A' \otimes A'' \).

(d) Combining the three items above, it follows that if \( V' = \ker(D) \) is a sub-bundle of \( V \), then there is a canonical Morita isomorphism
\[ \tilde{\text{Cl}}(V') \to A. \]

(e) Given a \( G \)-equivariant family of skew-adjoint Fredholm operators (with \( G \) a compact Lie group) one obtains a \( G \)-Dixmier-Douady bundle.

Suppose \( D_1, D_2 \) are two families of skew-adjoint Fredholm operators as in Proposition 4.7. We will call these families equivalent and write \( D_1 \sim D_2 \) if they define the same polarization of \( V \), and therefore the same Dixmier-Douady bundle \( A \to M \). We stress that different polarizations can induce isomorphic Dixmier-Douady bundles, however, the isomorphism is usually not canonical.

5. FROM DIRAC STRUCTURES TO DIXMIER-DOUADY BUNDLES

We will now use the constructions from the last Section to associate to every Dirac structure \((V, E)\) over \( M \) a Dixmier-Douady bundle \( A_E \to M \), and to every strong Dirac morphism \((\Theta, \omega): (V, E) \to (V', E')\) a Morita morphism. The construction is functorial 'up to 2-isomorphisms'.
5.1. The Dixmier-Douady algebra associated to a Dirac structure.

Let \((\mathcal{V}, E)\) be a Dirac structure over \(M\). Pick a Euclidean metric on \(V\), and let \(\mathcal{V} \to M\) be the bundle of real Hilbert spaces with fibers \(\mathcal{V}_m = L^2([0,1], V_m)\).

Let \(A \in \Gamma(O(V))\) be the orthogonal section corresponding to \(E\), as in Section 2.4. Define a family \(D_E = \{(D_E)_m, \ m \in M\}\) of operators on \(\mathcal{V}\), where \((D_E)_m = \frac{\partial}{\partial t}\) with domain (16) \(\text{dom}((D_E)_m) = \{f \in V_m | \dot{f} \in V_m, f(1) = -A_m f(0)\}\).

The condition that the distributional derivative \(\dot{f}\) lies in \(L^2 \subset L^1\) implies that \(f\) is absolutely continuous; hence the boundary condition \(f(1) = -A_m f(0)\) makes sense. The unbounded operators \((D_E)_m\) are closed and skew-adjoint (see e.g. [27, Chapter VIII]). By Proposition A.4 in the Appendix, the family \(D_E\) is continuous in the norm resolvent sense, hence it defines a Dixmier-Douady bundle \(A_E\) by Proposition 4.7.

The kernel of the operator \((D_E)_m\) is the intersection of \(V_m \subset V_m\) (embedded as constant functions) with the domain (16). That is, \(\ker((D_E)_m) = \ker(A_m + I) = V_m \cap E_m\).

**Proposition 5.1.** Suppose \(E \cap V\) is a sub-bundle of \(V\). Then there is a canonical Morita isomorphism \(\widetilde{\text{Cl}}(E \cap V) \to A_E\).

In particular there are canonical Morita isomorphisms \(\text{C} \to A_{V^*}, \widetilde{\text{Cl}}(V) \to A_V\).

**Proof.** Since \(\ker(D_E) \cong E \cap V\) is then a sub-bundle of \(V\), the assertion follows from item (d) in Section 4.4. \(\square\)

**Remark 5.2.** The definition of \(A_E\) depends on the choice of a Euclidean metric on \(V\). However, since the space of Euclidean metrics is contractible, the bundles corresponding to two choices are related by a canonical 2-isomorphism class of isomorphisms. See Example 3.4.

**Remark 5.3.** The Dixmier-Douady class \(\text{DD}(A_E) = (x,y)\) is an invariant of the Dirac structure \((\mathcal{V}, E)\). It may be constructed more directly as follows: Choose \(V'\) such that \(V \oplus V' \cong X \times \mathbb{R}^N\) is trivial. Then \(E \oplus (V')^*\) corresponds to a section of the orthogonal bundle, or equivalently to a map \(f: X \to O(N)\). The class \(\text{DD}(A_E)\) is the pull-back under \(f\) of the class over \(O(N)\) whose restriction to each component is a generator of \(H^3(\cdot, \mathbb{Z})\) respectively \(H^1(\cdot, \mathbb{Z}_2)\). (See Proposition 6.2 below.) However, not all classes in \(H^3(X, \mathbb{Z}) \times H^1(X, \mathbb{Z}_2)\) are realized as such pull-backs.

The following Proposition shows that the polarization defined by \(D_E\) depends very much on the choice of \(E\), while it is not affected by perturbations of \(D_E\) by skew-adjoint multiplication operators \(M_\mu\). Let \(L^\infty([0,1], \sigma(V))\)
denote the Banach bundle with fibers $L^\infty([0,1], \sigma(V_m))$. Its continuous sections $\mu$ are given in local trivialization of $V$ by continuous maps to $L^\infty([0,1], \sigma(X))$. Fiberwise multiplication by $\mu$ defines a continuous homomorphism

$$L^\infty([0,1], \sigma(V)) \to \mathcal{L}(V), \; \mu \mapsto M_\mu.$$  

**Proposition 5.4.**

(a) Let $E, E'$ be two Lagrangian sub-bundles of $V$. Then $D_E \sim D_{E'}$ if and only if $E = E'$.

(b) Let $\mu \in \Gamma(L^\infty([0,1], \sigma(V)))$, defining a continuous family of skew-adjoint multiplication operators $M_\mu \in \Gamma(\mathcal{L}(V))$. For any Lagrangian sub-bundle $E \subset V$ one has

$$D_E + M_\mu \sim D_E.$$

The proof is given in the Appendix, see Propositions A.2 and A.3.

5.2. Paths of Lagrangian sub-bundles. Suppose $E_s, \; s \in [0,1]$ is a path of Lagrangian sub-bundles of $V$, and $A_s \in \Gamma(O(V))$ the resulting path of orthogonal transformations. In Example 3.4 we remarked that there is a path of isomorphisms $\phi_s : A_{E_0} \to A_{E_s}$ with $\phi_0 = \text{id}$, and the 2-isomorphism class of the resulting isomorphism $\phi_1 : A_{E_0} \to A_{E_1}$ does not depend on the choice of $\phi_s$. It is also clear from the discussion in Example 3.4 that the isomorphism defined by a concatenation of two paths is 2-isomorphic to the composition of the isomorphisms defined by the two paths.

If the family $E_s$ is differentiable, there is a distinguished choice of the isomorphism $A_{E_0} \to A_{E_1}$, as follows.

**Proposition 5.5.**

Suppose that $\mu_s := -\frac{\partial A_t}{\partial s} A_s^{-1}$ defines a continuous section of $L^\infty([0,1], \sigma(V))$. Let $M_\gamma \in \Gamma(O(V))$ be the orthogonal transformation given fiberwise by pointwise multiplication by $\gamma_t = A_t A_0^{-1}$. Then

$$M_\gamma \circ D_{E_0} \circ M_\gamma^{-1} = D_{E_1} + M_\mu \sim D_{E_1}.$$

Thus $M_\gamma$ induces an isomorphism $A_{E_0} \to A_{E_1}$.

**Proof.** We have

$$f(1) = -A_0 f(0) \iff (M_\gamma f)(1) = -A_1 (M_\gamma f)(0),$$

which shows $M_\gamma(\text{dom}(D_{E_0})) = \text{dom}(D_{E_1})$, and

$$A_t A_0^{-1} \frac{\partial}{\partial t} (A_0 A_t^{-1} f) = \frac{\partial f}{\partial t} + \mu_t f.$$

□

Examples 5.6.

(a) Suppose $E$ corresponds to $A = \exp(a)$ with $a \in \Gamma(\sigma(V))$. Then $A_s = \exp(sa)$ defines a path from $A_0 = I$ and $A_1 = A$. Hence we obtain an isomorphism $A_{V^*} \to A_E$. (The 2-isomorphism class of this isomorphism may depend on the choice of $a$.)
(b) Any 2-form \( \omega \in \Gamma(\wedge^2 V^*) \) defines an orthogonal transformation of \( V \),
given by \((v, \alpha) \mapsto (v, \alpha - \iota_v \omega)\). Let \( E^\omega \) be the image of the Lagrangian subbundle \( E \subset V \) under this transformation. The corresponding orthogonal transformations \( A, A^\omega \) are related by
\[
A^\omega = (A - \omega(A - I))(I - \omega(A - I))^{-1}
\]
where we identified the 2-form \( \omega \) with the corresponding skew-adjoint map \( \omega \in \Gamma(\sigma(V)) \). Replacing \( \omega \) with \( s\omega \), one obtains a path \( E_s \) from \( E_0 = E \) to \( E_1 = E^\omega \), defining an isomorphism \( A_E \to A_{E^\omega} \).

5.3. The Dirac-Dixmier-Douady functor. Having assigned a Dixmier-Douady bundle to every Dirac structure on a Euclidean vector bundle \( V \),
\[
(V, E) \leadsto A_E
\]
we will now associate a Morita morphism to every strong Dirac morphism:
\[
(\Theta, \omega): (V, E) \to (V', E') \leadsto (\Phi, \mathcal{E}): A_E \to A_{E'}.
\]
Here \( \Phi: M \to M' \) is underlying the map on the base. Theorem 5.7 below states that (18) is compatible with compositions ‘up to 2-isomorphism’. Thus, if we take the morphisms for the category of Dixmier-Douady bundles to be the 2-isomorphism classes of Morita morphisms, and if we include the Euclidean metric on \( V \) as part of a Dirac structure, the construction (17), (18) defines a functor. We will call this the Dirac-Dixmier-Douady functor.

The Morita isomorphism \( \mathcal{E}: A_E \to \Phi^* A_{E'} = A_{\Phi^* E'} \) in (18) is defined as a composition
\[
A_E \to A_{E \oplus \Phi^*(V')} \cong A_{V^* \oplus \Phi^* E'} \to A_{\Phi^* E'}
\]
where the middle map is induced by the path \( E_s \) from \( E_0 = E \oplus \Phi^*(V')^* \) to \( E_1 = V^* \oplus \Phi^* E' \), constructed as in Subsection 2.2. By composing with the Morita isomorphisms \( A_E \to A_{E \oplus \Phi^*(V')} \) and \( A_{V^* \oplus \Phi^* E'} \to A_{E'} \) this gives the desired Morita morphism \( A_E \to A_{E'} \).

Theorem 5.7. i) The composition of the Morita morphisms \( A_E \to A_{E'} \) and \( A_{E'} \to A_{E''} \) defined by two strong Dirac morphisms \((\Theta, \omega)\) and \((\Theta', \omega')\) is 2-isomorphic to the Morita morphism \( A_E \to A_{E''} \) defined by \((\Theta', \omega') \circ (\Theta, \omega)\). ii) The Morita morphism \( A_E \to A_E \) defined by the Dirac morphism \((\text{id}_V, 0): (V, E) \to (V, E)\) is 2-isomorphic to the identity.

Proof. i) By pulling everything back to \( M \), we may assume that \( M = M' = M'' \) and that \( \Theta, \Theta' \) induce the identity map on the base. As in Section 2.2 consider the three Lagrangian subbundles
\[
E_{00} = E \oplus (V')^* \oplus W^*, \ E_{10} = V^* \oplus E' \oplus W^*, \ E_{01} = V^* \oplus (V')^* \oplus E''
\]
of \( V \oplus V' \oplus V'' \). We have canonical Morita isomorphisms
\[
A_E \to A_{E_{00}}, \ A_{E'} \to A_{E_{10}}, \ A_{E''} \to A_{E_{01}}.
\]
The morphism \([19]\) may be equivalently described as a composition
\[
A_E \rightarrow A_{E_{01}} \cong A_{E_{10}} \rightarrow A_{E'},
\]
since the path from \(E_{00}\) to \(E_{10}\) (constructed as in Subsection \([2,2]\)) is just the direct sum of \(W^*\) with the standard path from \(E \oplus (V')^*\) to \(V^* \oplus E'\). Similarly, one describes the morphism \(A_{E'} \rightarrow A_{E''}\) as
\[
A_{E'} \rightarrow A_{E_{10}} \cong A_{E_{01}} \rightarrow A_{E''}.
\]
The composition of the Morita morphisms \(A_E \rightarrow A_{E'} \rightarrow A_{E''}\) defined by \((\Theta, \omega)\), \((\Theta', \omega')\) is hence given by
\[
A_E \rightarrow A_{E_{10}} \cong A_{E_{01}} \cong A_{E_{01}} \rightarrow A_{E''}.
\]
The composition \(A_{E_{10}} \cong A_{E_{01}} \cong A_{E_{01}}\) is 2-isomorphic to the isomorphism defined by the concatenation of standard paths from \(E_{00}\) to \(E_{10}\) to \(E_{01}\). As observed in Section \([2,2]\) this concatenation is homotopic to the standard path from \(E_{00}\) to \(E_{01}\), which defines the morphism \(A_E \rightarrow A_{E''}\) corresponding to \((\Theta', \omega') \circ (\Theta, \omega)\).

ii) We will show that the Morita morphism \(A_E \rightarrow A_{E_0} \cong A_{E_1} \rightarrow A_E\) defined by \((\text{id}_V, 0)\) is homotopic to the identity. Here \(E_0 = E \oplus V^*,\ E_1 = V^* \oplus E,\) and the isomorphism \(A_{E_0} \cong A_{E_1}\) is defined by the standard path \(E_t\) connecting \(E_0, E_1\). By definition, \(E_t\) is the forward image of \(E\) under the morphism \((j_t, 0): V \rightarrow V \oplus V\) where
\[
j_t: V \rightarrow V \oplus V,\ y \mapsto ((1 - t)y, ty).
\]

It is convenient to replace \(j_t\) by the isometry,
\[
\tilde{j}_t = (t^2 + (1 - t)^2)^{-1/2} j_t.
\]
This is homotopic to \(j_t\) (e.g. by linear interpolation), hence the resulting path \(\tilde{E}_t\) defines the same 2-isomorphism class of isomorphisms \(A_{E_0} \rightarrow A_{E_1}\).

The splitting of \(V \oplus V\) into \(V_t := \text{ran}(\tilde{j}_t)\) and \(V_t^\perp\) defines a corresponding orthogonal splitting of \(V \oplus V\). The subspace \(\tilde{E}_t\) is the direct sum of the intersections
\[
\tilde{E}_t \cap V_t^\perp = \text{ann}(V_t) = (V_t^\perp)^*,\ \tilde{E}_t \cap V_t =: \tilde{E}_t'.
\]
This defines a Morita isomorphism
\[
A_{\tilde{E}_t} \rightarrow A_{\tilde{E}_t'}
\]
On the other hand, the isometric isomorphism \(V \rightarrow V_t\) given by \(\tilde{j}_t\) extends to an isomorphism \(V \rightarrow V_t\), taking \(E\) to \(\tilde{E}_t'\). Hence \(A_{\tilde{E}_t'} \cong A_E\) canonically. In summary, we obtain a family of Morita isomorphisms
\[
A_E \rightarrow A_{E_0} \cong A_{\tilde{E}_t} \rightarrow A_{\tilde{E}_t'} \cong A_E.
\]
For \(t = 1\) this is the Morita isomorphism defined by \((\text{id}_V, 0)\), while for \(t = 0\) it is the identity map \(A_E \rightarrow A_E\).
5.4. Symplectic vector bundles. Suppose $V \to M$ is a vector bundle, equipped with a fiberwise symplectic form $\omega \in \Gamma(\wedge^2 V^*)$. Given a Euclidean metric $B$ on $V$, the 2-form $\omega$ is identified with a skew-adjoint operator $R_\omega$, defining a complex structure $J_\omega = R_\omega/[R_\omega]$ and a resulting spinor module $S_\omega: \mathbb{C} \to \mathbb{C}l(V)$. (We may work with $\mathbb{C}l(V)$ rather than $\mathbb{C}l(V)$, since $V$ has even rank.)

**Proposition 5.8.** The Morita isomorphism

$$S_\omega^*: \mathbb{C}l(V) \to \mathbb{C}$$

defined by the $\text{Spin}_\omega$-structure $S_\omega$ is 2-isomorphic to the Morita isomorphism $\mathbb{C}l(V) \to \mathcal{A}_V$, followed by the Morita isomorphism $\mathcal{A}_V \to \mathbb{C}$ defined by the strong Dirac morphism $(0, \omega): (\mathbb{V}, V) \to (0, 0)$ (cf. Example 2.1).

**Proof.** Consider the standard path for the Dirac morphism $(0, \omega): (\mathbb{V}, E) \to (0, 0)$,

$$E_t = \{(1-t)v, \alpha) \mid tv \omega + (1-t)\alpha = 0\} \subset \mathbb{V},$$

defining $\mathcal{A}_V = \mathcal{A}_{E_0} \cong \mathcal{A}_{E_1} = \mathcal{A}_V^\text{op} \to \mathbb{C}$. The path of orthogonal transformations defined by $E_t$ is

$$A_t = \frac{t R_\omega - \frac{1}{2}(1-t)^2}{t R_\omega + \frac{1}{2}(1-t)^2}.$$

We will replace $A_t$ with a more convenient path $\tilde{A}_t$,

$$\tilde{A}_t = -\exp(t\pi J_\omega).$$

We claim that this is homotopic to $A_t$ with the same endpoints. Clearly $A_0 = -I = -\tilde{A}_0$ and $A_1 = I = \tilde{A}_1$. By considering the action on any eigenspace of $R_\omega$, one checks that the spectrum of both $J_\omega A_t$ and $J_\omega \tilde{A}_t$ is contained in the half space $\text{Re}(z) \geq 0$, for all $t \in [0, 1]$. Hence

$$J_\omega A_t + I, \quad J_\omega \tilde{A}_t + I$$

are invertible for all $t \in [0, 1]$. The Cayley transform $C \mapsto (C - I)/(C + I)$ gives a diffeomorphism from the set of all $C \in \text{O}(V)$ such that $C + I$ is invertible onto the vector space $\mathfrak{o}(V)$. By using the linear interpolation of the Cayley transforms one obtains a homotopy between $J_\omega A_t$, $J_\omega \tilde{A}_t$, and hence of $A_t, \tilde{A}_t$.

By Proposition 5.5, the path $\tilde{A}_t$ defines an orthogonal transformation $M_\gamma \in \text{O}(V)$, taking the complex structure $J_0$ for $E_0 = V^\text{op}$ to a complex structure $J_1 = M_\gamma \circ J_0 \circ M_\gamma^{-1}$ in the equivalence class defined by $D_{E_1}$. Consider the orthogonal decomposition $V = V' \oplus V''$ with $V' = \ker(D_V) \cong V$. Let $J''$ be the complex structure on $V''$ defined by $D_V$, and put $J' = J_\omega$. Since

$$M_\gamma \circ D_{V'} \circ M_\gamma^{-1} = D_V + \pi J_\omega,$$

we see that $J_1 = J' \oplus J''$, hence $S_1 = S' \otimes S'' = S_\omega \otimes S''$. The Morita isomorphism $\mathbb{C}l(V) \to \mathcal{A}_V$ is given by the bimodule $\mathcal{E} = S'' \otimes \mathbb{C}l(V)$.
Since $\mathcal{C}l(V) = S_\omega \otimes S_\omega^{op}$, it follows that $E = S'' \otimes \mathcal{C}l(V) = S_1 \otimes S_\omega^{op}$, and

$$S_1^{op} \otimes A_{\mathcal{C}l} E = S_\omega^{op}.$$  

\[\square\]

6. The Dixmier-Douady bundle over the orthogonal group

6.1. The bundle $A_{O(X)}$. As a special case of our construction, let us consider the tautological Dirac structure $(\mathbb{V}_{O(X)}, E_{O(X)})$ for a Euclidean vector space $X$. Let $A_{O(X)}$ be the corresponding Dixmier-Douady bundle; its restriction to $SO(X)$ will be denoted $A_{SO(X)}$. The Dirac morphism $(\mathbb{V}_{O(X)}, E_{O(X)}) \times (\mathbb{V}_{O(X)}, E_{O(X)}) \rightarrow (\mathbb{V}_{O(X)}, E_{O(X)})$ gives rise to a Morita morphism

$$pr_1^* A_{O(X)} \otimes pr_2^* A_{O(X)} \rightarrow A_{O(X)},$$

which is associative up to 2-isomorphisms.

**Proposition 6.1.**

(a) There is a canonical Morita morphism $C \rightarrow A_{O(X)}$ with underlying map the inclusion of the group unit, $\{I\} \hookrightarrow O(X)$.

(b) For any orthogonal decomposition $X = X' \oplus X''$, there is a canonical Morita morphism

$$pr_1^* A_{O(X')} \otimes pr_2^* A_{O(X'')} \rightarrow A_{O(X)}$$

with underlying map the inclusion $O(X') \times O(X'') \hookrightarrow O(X)$.

**Proof.** The Proposition follows since the restriction of $E_{O(X)}$ to $I$ is $X^*$, while the restriction to $O(X') \times O(X'')$ is $E_{O(X')} \times E_{O(X'')}$. \[\square\]

The action of $O(X)$ by conjugation lifts to an action on the bundle $\mathbb{V}_{O(X)}$, preserving the Dirac structure $E_{O(X)}$. Hence $A_{O(X)}$ is an $O(X)$-equivariant Dixmier-Douady bundle.

The construction of $A_{O(X)}$, using the family of boundary conditions given by orthogonal transformations, is closely related to a construction given by Atiyah-Segal in [6], who also identify the resulting Dixmier-Douady class. The result is most nicely stated for the restriction to $SO(X)$; for the general case use an inclusion $O(X) \hookrightarrow SO(X \oplus \mathbb{R})$.

**Proposition 6.2.** [6, Proposition 5.4] Let $(x, y) = DD(A_{SO(X)})$ be the Dixmier-Douady class.

(a) For dim $X \geq 3$, dim $X \neq 4$ the class $x$ generates $H^2(SO(X), \mathbb{Z}) = \mathbb{Z}$.

(b) For dim $X \geq 2$ the class $y$ generates $H^1(SO(X), \mathbb{Z}_2) = \mathbb{Z}_2$.

Atiyah-Segal’s proof uses an alternative construction $A_{SO(X)}$ in terms of loop groups (see below). Another argument is sketched in Appendix [B].
6.2. Pull-back under exponential map. Let \( \mathcal{V}_{\varnothing(X)}, E_{\varnothing(X)} \) be as in Section 2.7 and let \( \mathcal{A}_{\varnothing(X)} \) be the resulting \( O(X) \)-equivariant Dixmier-Douady bundle. Since \( E_{\varnothing(X)}|_{\varnothing} = \text{Gr}_{\varnothing} \), its intersection with \( X \subset \mathbb{X} \) is trivial, and so \( \mathcal{A}_{\varnothing(X)} \) is Morita trivial. Recall the Dirac morphism \( (\Pi, -\varpi) : (\mathcal{V}_{\varnothing(X)}, E_{\varnothing(X)}) \rightarrow (\mathcal{V}_{\varnothing(X)}^*, E_{\varnothing(X)}^*) \), with underlying map \( \exp : \varnothing(X) \rightarrow O(X) \). We had shown that it is a strong Dirac morphism over the subset \( \varnothing(X)_\sharp \) where the exponential map has maximal rank, or equivalently where \( \Pi_a = (I - e^{-a})/a \) is invertible. One hence obtains a Morita morphism

\[
\mathcal{A}_{\varnothing(X)}|_{\varnothing(X)_\sharp} \rightarrow \mathcal{A}_{O(X)}. 
\]

Together with the Morita trivialization \( \mathbb{C} \rightarrow \mathcal{A}_{\varnothing(X)} \), this gives a Morita trivialization of \( \exp^* \mathcal{A}_{O(X)} \) over \( \varnothing(X)_\sharp \).

On the other hand, \( \exp^* E_{\varnothing(X)} \) is the Lagrangian sub-bundle of \( \varnothing(X) \times \mathbb{X} \) defined by the map \( a \mapsto \exp(a) \in O(X) \). Replacing \( \exp(a) \) with \( \exp(sa) \), one obtains a homotopy \( E_s \) between \( E_1 = \exp^* E_{\varnothing(X)} \) and \( E_0 = X^* \), hence another Morita trivialization of \( \exp^* \mathcal{A}_{O(X)} \) (defined over all of \( \varnothing(X) \)). Let \( L \rightarrow \varnothing(X)_\sharp \) be the \( O(X) \)-equivariant line bundle relating these two Morita trivializations.

**Proposition 6.3.** Over the component containing 0, the line bundle \( L \rightarrow \varnothing(X)_\sharp \) is \( O(X) \)-equivariantly trivial. In other words, the two Morita trivializations of \( \exp^* \mathcal{A}_{O(X)}|_{\varnothing(X)_\sharp} \) are 2-isomorphic over the component of \( \varnothing(X)_\sharp \) containing 0.

**Proof.** The linear retraction of \( \varnothing(X) \) onto the origin preserves the component of \( \varnothing(X)_\sharp \) containing 0. Hence it suffices to show that the \( O(X) \)-action on the fiber of \( L \) at 0 is trivial. But this is immediate since both Morita trivializations of \( \exp^* \mathcal{A}_{O(X)} \) at \( 0 \in \varnothing(X)_\sharp \) coincide with the obvious Morita trivialization of \( \mathcal{A}_{O(X)}|_{\varnothing} \). \( \square \)

6.3. Construction via loop groups. The bundle \( \mathcal{A}_{SO(X)} \) has the following description in terms of loop groups (cf. [9]). Fix a Sobolev level \( s > 1/2 \), and let \( \mathcal{P} \text{SO}(X) \) denote the Banach manifold of paths \( \gamma : \mathbb{R} \rightarrow SO(X) \) of Sobolev class \( s + 1/2 \) such that \( \pi(\gamma) := \gamma(t + 1)\gamma(t)^{-1} \) is constant. (Recall that for manifolds \( Q, P \), the maps \( Q \rightarrow P \) of Sobolev class greater than \( k + \dim Q/2 \) are of class \( C^k \).) The map

\[
\pi : \mathcal{P} \text{SO}(X) \rightarrow SO(X), \ \gamma \mapsto \pi(\gamma)
\]

is an \( \text{SO}(X) \)-equivariant principal bundle, with structure group the loop group \( LSO(X) = \pi^{-1}(e) \). Here elements of \( \text{SO}(X) \) acts by multiplication from the left, while loops \( \lambda \in LSO(X) \) acts by \( \gamma \mapsto \gamma \lambda^{-1} \). Let \( \mathcal{X} = L^2([0, 1], X) \) carry the complex structure \( J_0 \) defined by \( \frac{d}{dt} \) with anti-periodic boundary conditions, and let \( S_0 \) be the resulting spinor module. The action of the group \( LSO(X) \) on \( \mathcal{X} \) preserves the polarization defined by \( J_0 \), and hence carries a continuous map \( LSO(X) \rightarrow O_{\text{res}}(\mathcal{X}) \). Using its composition with the map \( O_{\text{res}}(\mathcal{X}) \rightarrow \text{PU}(S_0) \), we have:
Proposition 6.4. The Dixmier-Douady bundle $\mathcal{A}_{SO(X)}$ is an associated bundle $\mathcal{P}SO(X) \times_{LSO(X)} \mathbb{K}(S_0)$.

Proof. Given $\gamma \in \mathcal{P}SO(X)$, consider the operator $M_\gamma$ on $X = L^2([0, 1], X)$ of pointwise multiplication by $\gamma$. As in Proposition 5.5, we see that $M_\gamma$ takes the boundary conditions $f(1) = -f(0)$ to $(M_\gamma f)(1) = -\pi(\gamma)(M_\gamma f)(0)$, and induces an isomorphism $\mathbb{K}(S_0) = A_I \rightarrow A_{\pi(\gamma)}$. This defines a map $P_{SO(X)} \times \mathbb{K}(S_0) \rightarrow A_{SO(X)}$ with underlying map $\pi: \mathcal{P}SO(X) \rightarrow SO(X)$. This map is equivariant relative to the action of $LSO(X)$, and descends to the desired bundle isomorphism. □

In particular $\pi^* A_{SO(X)} = \mathcal{P}SO(X) \times \mathbb{K}(S_0)$ has a Morita trivialization defined by the trivial bundle $\mathcal{E}_0 = \mathcal{P}SO(X) \times S_0$. The Morita trivialization is $LSO(X) \times SO(X)$-equivariant, using the central extension of the loop group obtained by pull-back of the central extension $U(S_0) \rightarrow PU(S_0)$.

7. q-Hamiltonian G-spaces

In this Section, we will apply the correspondence between Dirac structures and Dixmier-Douady bundles to the theory of group-valued moment maps [2]. Most results will be immediate consequences of the functoriality properties of this correspondence. Throughout this Section, $G$ denotes a Lie group, with Lie algebra $\mathfrak{g}$. We denote by $\xi^L, \xi^R \in \mathfrak{X}(G)$ the left, right invariant vector fields defined by the Lie algebra element $\xi \in \mathfrak{g}$, and by $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ the Maurer-Cartan forms, defined by $\iota(\xi^L)\theta^L = \iota(\xi^R)\theta^R = \xi$. For sake of comparison, we begin with a quick review of ordinary Hamiltonian $G$-spaces from the Dirac geometry perspective.

7.1. Hamiltonian G-spaces. A Hamiltonian $G$-space is a triple $(M, \omega_0, \Phi_0)$ consisting of a $G$-manifold $M$, an invariant 2-form $\omega_0$ and an equivariant moment map $\Phi_0: M \rightarrow \mathfrak{g}^*$ such that

(i) $d\omega_0 = 0$

(ii) $\iota(\xi_M)\omega_0 = -d\langle \Phi_0, \xi \rangle$, $\xi \in \mathfrak{g}$

(iii) $\ker(\omega_0) = 0$.

Conditions (ii) and (iii) may be rephrased in terms of Dirac morphisms. Let $E_{\mathfrak{g}^*} \subset T\mathfrak{g}^*$ be the Dirac structure spanned by the sections $\epsilon_0(\xi) = \langle \xi^L, (d\mu, \xi) \rangle$, $\xi \in \mathfrak{g}$.

Here $\xi^L \in \mathfrak{X}(\mathfrak{g}^*)$ is the vector field generating the co-adjoint action (i.e. $\xi^L|_\mu = (\text{ad}\xi)^*\mu$), and $\langle d\mu, \xi \rangle \in \Omega^1(\mathfrak{g}^*)$ denotes the 1-form defined by $\xi$. Then Conditions (ii), (iii) hold if and only if

$$(d\Phi_0, \omega_0): (TM, TM) \rightarrow (T\mathfrak{g}^*, E_{\mathfrak{g}^*})$$
is a strong Dirac morphism. Using the Morita isomorphism \( \widetilde{\mathcal{C}}l(TM) \to A_{TM} \), and putting \( A^\Spin_g := A_{E_g^*} \) we obtain a \( G \)-equivariant Morita morphism

\[
(\Phi_0, \mathcal{E}_0): \widetilde{\mathcal{C}}l(TM) \to A^\Spin_g.
\]

Since \( E_g^* \cap Tg^* = 0 \), the zero Dirac morphism \( (Tg^*, E_g^*) \to (0, 0) \) is strong, hence it defines a Morita trivialization \( A^\Spin_g \to C \). From Proposition 5.8, we see that the resulting equivariant Spin\(_c\)-structure \( \widetilde{\mathcal{C}}l(TM) \to C \) is 2-isomorphic to the Spin\(_c\)-structure defined by the symplectic form \( \omega_0 \). (Since symplectic manifolds are even-dimensional, we may work with \( \mathcal{C}l(TM) \) in place of \( \tilde{\mathcal{C}}l(TM) \).)

### 7.2. q-Hamiltonian \( G \)-spaces

An Ad\((G)\)-invariant inner product \( B \) on \( g \) defines a closed bi-invariant 3-form

\[
\eta = \frac{1}{12} B(\theta^L, [\theta^L, \theta^L]) \in \Omega^3(G).
\]

A q-Hamiltonian \( G \)-manifold \( M \) is a \( G \)-manifold, together with an an invariant 2-form \( \omega \), and an equivariant moment map \( \Phi: M \to G \) such that

1. \( d\omega = -\Phi^*\eta \),
2. \( i(\xi_M)\omega = -\frac{1}{2} \Phi^* B((\theta^L + \theta^R), \xi) \)
3. \( \ker(\omega) \cap \ker(d\Phi) = 0 \) everywhere.

The simplest examples of q-Hamiltonian \( G \)-spaces are the conjugacy classes in \( G \), with moment map the inclusion \( \Phi: C \hookrightarrow G \). Again, the definition can be re-phrased in terms of Dirac structures. Let \( E_G \subset TG \) be the Lagrangian sub-bundle spanned by the sections

\[
e(\xi) = (\xi^L, \frac{1}{2} B(\theta^L + \theta^R, \xi)), \quad \xi \in g.
\]

Here \( \xi^L = \xi^L - \xi^R \in \mathfrak{X}(G) \) is the vector field generating the conjugation action. \( E_G \) is the Cartan-Dirac structure introduced by Alekseev, Ševera and Strobl \[7, 30\]. As shown by Bursztyn-Crainic \[7\], Conditions (ii) and (iii) above hold if and only if

\[
(\Phi, \omega): (TM, TM) \to (TG, E_G)
\]

is a strong Dirac morphism. Let

\[
A^\Spin_G := A_{E_G}
\]

be the \( G \)-equivariant Dixmier-Douady bundle over \( G \) defined by the Cartan-Dirac structure. The strong Dirac morphism \( (\Phi, \omega) \) determines a Morita morphism \( A_{TM} \to A^\Spin_G \). Since \( A_{TM} \) is naturally Morita isomorphic to \( \mathcal{C}l(TM) \) we obtain a distinguished 2-isomorphism class of \( G \)-equivariant Morita morphisms

\[
(\Phi, \mathcal{E}): \mathcal{C}l(TM) \to A^\Spin_G.
\]

**Definition 7.1.** The Morita morphism \( (\Phi, \mathcal{E}) \) is called the **canonical twisted Spin\(_c\)-structure** for the q-Hamiltonian \( G \)-space \((M, \omega, \Phi)\).
Remarks 7.2. (a) Equation (21) generalizes the usual Spin$\_c$-structure for a symplectic manifold. Indeed, if $G = \{e\}$ we have $A^{\text{Spin}}_G = \mathbb{C}$, and a q-Hamiltonian $G$-space is just a symplectic manifold. Proposition 5.8 shows that the composition $\widetilde{Cl}(TM) \longrightarrow A_T M \longrightarrow \mathbb{C}$ in that case is 2-isomorphic to the Morita trivialization defined by an $\omega$-compatible almost complex structure.

(b) The tensor product $\widetilde{Cl}(TM) \otimes \widetilde{Cl}(TM) = \widetilde{Cl}(TM \oplus TM)$ is canonically Morita trivial (see Section 3.5). Hence, the twisted Spin$\_c$-structure on a q-Hamiltonian $G$-space defines a $G$-equivariant Morita trivialization

$$\mathbb{C} \longrightarrow \Phi^*(A^{\text{Spin}}_G) \otimes \mathbb{C}.$$  

One may think of (22) as the counterpart to the canonical line bundle. Indeed, for $G = \{e\}$, (22) is a Morita isomorphism from the trivial bundle over $M$ to itself. It is thus given by a Hermitian line bundle, and from (a) above one sees that this is the canonical line bundle associated to the Spin$\_c$-structure of $(M, \omega)$.

Remark 7.3. In terms of the trivialization $TG = G \times g$ given by the left-invariant vector fields $\xi^L$, the Cartan-Dirac structure $(\mathbb{T}G, E_G)$ is just the pull-back of the tautological Dirac structure $(\mathbb{V}_{O(g)}, E_{O(g)})$ under the adjoint action $\text{Ad}: G \rightarrow O(g)$. Similarly, $A^{\text{Spin}}_G$ is simply the pull-back of $A_{O(g)} \rightarrow O(g)$ under the map $\text{Ad}: G \rightarrow O(g)$.

In many cases q-Hamiltonian $G$-spaces have even dimension, so that we may use the usual Clifford algebra bundle $Cl(TM)$ in (21):

Proposition 7.4. Let $(M, \omega, \Phi)$ be a connected q-Hamiltonian $G$-manifold. Then $\dim M$ is even if and only if $\text{Ad}_{\Phi(m)} \in SO(g)$ for all $m \in M$. In particular, this is the case if $G$ is connected.

Proof. This is proved in [4], but follows much more easily from the following Dirac-geometric argument. The parity of the Lagrangian sub-bundle $TM \subset \mathbb{T}M$ is given by $(-1)^{\dim M} = \pm 1$. By Proposition 2.2, the parity is preserved under strong Dirac morphisms. Hence it coincides with the parity of $E_G$ over $\Phi(M)$, and by Remark 7.3 this is the same as the parity of the tautological Dirac structure $E_{O(g)}$ over $\text{Ad}(\Phi(M)) \subset O(g)$. The latter is given by $\det(\text{Ad}_\Phi) = \pm 1$. This shows $\det(\text{Ad}_\Phi) = (-1)^{\dim M}$. □

As a noteworthy special case, we have:

Corollary 7.5. A conjugacy class $C = \text{Ad}(G)g \subset G$ of a compact Lie group $G$ is even-dimensional if and only if $\det(\text{Ad}_g) = 1$.

7.3. Stiefel-Whitney classes. The existence of a Spin$\_c$-structure on a symplectic manifold implies the vanishing of the third integral Stiefel-Whitney class $W^3(M) = \beta(w_2(M))$, while of course $w_1(M) = 0$ by orientability. For q-Hamiltonian spaces we have the following statement:
Corollary 7.6. For any q-Hamiltonian $G$-space,
\[ W^3(M) \equiv \tilde{\beta}(w_2(M)) = \Phi^* x, \ w_1(M) = \Phi^* y. \]
where $(x, y) \in H^3(G, \mathbb{Z}) \times H^1(G, \mathbb{Z}_2)$ is the Dixmier-Douady class of $\mathcal{A}_G^{\text{Spin}}$. A similar statement holds for the $G$-equivariant Stiefel-Whitney classes.

Remarks 7.7.
(a) The result gives in particular a description of $w_1(C)$ and $\tilde{\beta}(w_2(C))$ for all conjugacy classes $C \subset G$ of a compact Lie group.
(b) If $G$ is simply connected, so that $H^1(G, \mathbb{Z}_2) = 0$, it follows that $w_1(M) = 0$. Hence q-Hamiltonian spaces for simply connected groups are orientable. In fact, there is a canonical orientation [4].
(c) Suppose $G$ is simple and simply connected. Then $x$ is $h^\vee$ times the generator of $H^3(G, \mathbb{Z}) = \mathbb{Z}$, where $h^\vee$ is the dual Coxeter number of $G$. This follows from Remark 7.3, since $\operatorname{Ad}^*: H^3(\text{SO}(g), \mathbb{Z}) = \mathbb{Z} \rightarrow H^3(G, \mathbb{Z}) = \mathbb{Z}$ is multiplication by $h^\vee$. We see that a conjugacy class $C$ of $G$ admits a Spin$_c$-structure if and only if the pull-back of the generator of $H^3(G, \mathbb{Z})$ is $h^\vee$-torsion. Examples of conjugacy classes not admitting a Spin$_c$-structure may be found in [20].

7.4. Fusion. Let $\mu: G \times G \rightarrow G$ be the group multiplication, and denote by $\sigma \in \Omega^2(G \times G)$ the 2-form
\[ (23) \quad \sigma = -\frac{1}{2} B(\text{pr}_1^* \theta^L, \text{pr}_2^* \theta^R) \]
where $\text{pr}_j: G \times G \rightarrow G$ are the two projections. By [1, Theorem 3.9] the pair $(d \mu, \sigma)$ define a strong $G$-equivariant Dirac morphism
\[ (d \mu, \sigma): (TG, E_G) \times (TG, E_G) \rightarrow (TG, E_G). \]
This can also be seen using Remark 7.3 and Proposition 2.5, since left trivialization of $TG$ intertwines $d \mu$ with the map $\Sigma$ from (6), taking (23) to the 2-form $\sigma$ on $V_{O(g)} \times V_{O(g)}$. It induces a Morita morphism
\[ (24) \quad (\mu, E): \text{pr}_1^* \mathcal{A}_G^{\text{Spin}} \otimes \text{pr}_2^* \mathcal{A}_G^{\text{Spin}} \rightarrow \mathcal{A}_G^{\text{Spin}}. \]
If $(M, \omega, \Phi)$ is a q-Hamiltonian $G \times G$-space, then $M$ with diagonal $G$-action, 2-form $\omega_{\text{fus}} = \omega + \Phi^* \sigma$, and moment map $\Phi_{\text{fus}} = \mu \circ \Phi: M \rightarrow G$ defines a q-Hamiltonian $G$-space
\[ (25) \quad (M, \omega_{\text{fus}}, \Phi_{\text{fus}}). \]
The space (25) is called the fusion of $(M, \omega, \Phi)$. Conditions (ii), (iii) hold since
\[ (26) \quad (d \Phi_{\text{fus}}, \omega_{\text{fus}}) = (d \mu, \sigma) \circ (d \Phi, \omega) \]
is a composition of strong Dirac morphisms, while (i) follows from $d \sigma = \mu^* \eta - \text{pr}_1^* \eta - \text{pr}_2^* \eta$. The Dirac-Dixmier-Douady functor (Theorem 5.7) shows that the twisted Spin$_c$-structures are compatible with fusion, in the following sense:
Proposition 7.8. The Morita morphism $\widetilde{Cl}(TM) \longrightarrow A_G^{\text{Spin}}$ for the $q$-Hamiltonian $G$-space $(M, \omega_\text{Hus}, \Phi_\text{Hus})$ is equivariantly $2$-isomorphic to the composition of Morita morphisms

$$\widetilde{Cl}(TM) \longrightarrow \text{pr}_1^* A_G^{\text{Spin}} \otimes \text{pr}_2^* A_G^{\text{Spin}} \longrightarrow A_G^{\text{Spin}}$$

defined by the twisted Spin$_c$-structure for $(M, \omega, \Phi)$, followed by [24].

7.5. Exponentials. Let $\exp : g \rightarrow G$ be the exponential map. The pull-back $\exp^* \eta$ is equivariantly exact, and admits a canonical primitive $\varpi \in \Omega^2(g)$ defined by the homotopy operator for the linear retraction onto the origin.

Remark 7.9. Explicit calculation shows [3] that $\varpi$ is the pull-back of the $2$-form (denoted by the same letter) $\varpi \in \Gamma(\wedge^2 V_0^*) \cong C^\infty(\mathfrak{o}(g), \wedge^2 \mathfrak{g}^*)$ from Section 2.7 under the adjoint map, $\text{ad} : g \rightarrow \mathfrak{o}(g)$. Using the inner product to identify $\mathfrak{g}^* \cong g$, the Dirac structure $E_\varpi \equiv E_g$ is the pull-back of the Dirac structure $E_{o(\varpi)}$ by the map $\text{ad} : g \rightarrow \mathfrak{o}(g)$.

The differential of the exponential map together with the $2$-form $\varpi$ define a Dirac morphism

$$(d\exp, -\varpi) : (Tg, E_g) \longrightarrow (TG, E_G)$$

which is a strong Dirac morphism over the open subset $g$, where $\exp$ has maximal rank. See [11, Proposition 3.12], or Proposition 2.6 above.

Let $(M, \Phi_0, \omega_0)$ be a Hamiltonian $G$-space with $\Phi_0(M) \subset g_0$, and $\Phi = \exp \Phi_0$, $\omega = \omega_0 - \Phi_0^* \varpi$. Then $(d\Phi, \omega) = (d\exp, -\varpi) \circ (d\Phi_0, \omega_0)$ is a strong Dirac morphism, hence $(M, \omega, \Phi)$ is a $q$-Hamiltonian $G$-space. It is called the exponential of the Hamiltonian $G$-space $(M, \omega_0, \Phi_0)$.

The canonical twisted Spin$_c$-structure for $(M, \omega, \Phi)$ can be composed with the Morita trivialization $\Phi_* A_G^{\text{Spin}} = \Phi_0^* \exp^* A_G^{\text{Spin}} \longrightarrow \mathbb{C}$ defined by the Morita trivialization of $\exp^* A_G^{\text{Spin}}$, to produce an ordinary equivariant Spin$_c$-structure. On the other hand, we have the equivariant Spin$_c$-structure defined by the symplectic form $\omega_0$.

Proposition 7.10. Suppose $(M, \omega_0, \Phi_0)$ is a Hamiltonian $G$-space, such that $\Phi_0$ takes values in the zero component of $g_0 \subset g$. Let $(M, \omega, \Phi)$ be its exponential. Then the composition

$$\widetilde{Cl}(TM) \longrightarrow \Phi_* A_G^{\text{Spin}} \longrightarrow \mathbb{C}$$

is $2$-isomorphic to the Morita morphism $\widetilde{Cl}(TM) \longrightarrow \mathbb{C}$ given by the canonical Spin$_c$-structure for $\omega_0$.

Proof. Proposition 7.9 shows that over the zero component of $g_0$, the Morita trivialization of $\exp^* A_G^{\text{Spin}}$ is $2$-isomorphic to the composition of the Morita

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2We could also write $\widetilde{Cl}(TM)$ in place of $\widetilde{Cl}(TM)$ since dim $M$ is even.
isomorphism $\mathcal{A}^{\text{Spin}}_g \to \mathcal{A}^{\text{Spin}}_G^1$ induced by $(\text{d exp}, -\omega)$, with the Morita trivialization of $\mathcal{A}^{\text{Spin}}_g$ (induced by the Dirac morphism $(Tg^*, E_g) \to (0, 0)$).

The result now follows from Theorem 5.7. □

7.6. Reduction. In this Section, we will show that the canonical twisted $\text{Spin}_c$-structure is well-behaved under reduction. Let $(M, \omega, \Phi)$ be a q-Hamiltonian $K \times G$-space. Thus $\Phi$ has two components $\Phi_K, \Phi_G$, taking values in $K, G$ respectively. Suppose $e \in G$ a regular value of $\Phi_G$, so that $Z = \Phi_G^{-1}(e)$ is a smooth $K \times G$-invariant submanifold. Let $\iota: Z \to M$ be the inclusion. The moment map condition shows that the $G$-action is locally free on $Z$, and that $\iota^* \omega$ is $G$-basic. Let us assume for simplicity that the $G$-action on $Z$ is actually free. Then

$$M_{\text{red}} = Z/G$$

is a smooth $K$-manifold, the $G$-basic 2-form $\iota^* \omega$ descends to a 2-form $\omega_{\text{red}}$ on $M_{\text{red}}$, and the restriction $\Phi|_Z$ descends to a smooth $K$-equivariant map $\Phi_{\text{red}}: M_{\text{red}} \to K$.

**Proposition 7.11.** [2] The triple $(M_{\text{red}}, \omega_{\text{red}}, \Phi_{\text{red}})$ is a q-Hamiltonian $K$-space. In particular, if $K = \{e\}$ it is a symplectic manifold.

We wish to relate the canonical twisted $\text{Spin}_c$-structures for $M_{\text{red}}$ to that for $M$. We need:

**Lemma 7.12.** There is a $G \times K$-equivariant Morita morphism

$$\widetilde{\mathcal{C}}l(TM)|_Z \to \widetilde{\mathcal{C}}l(TM_{\text{red}}),$$

with underlying map the quotient map $\pi: Z \to M_{\text{red}}$.

**Proof.** Consider the exact sequences of vector bundles over $Z$,

$$0 \to Z \times g \to TZ \to \pi^*TM_{\text{red}} \to 0,$$

where the first map is inclusion of the generating vector fields, and

$$0 \to TZ \to TM|_Z \to Z \times g^* \to 0,$$

where the map $TM|_Z \to g^* \cong g = T_eG$ is the restriction $(d\Phi)|_Z$. (We are writing $g^*$ in (29) to avoid confusion with the copy of $g$ in (28).) The Euclidean metric on $TM$ gives orthogonal splittings of both exact sequences, hence it gives a $K \times G$-equivariant direct sum decomposition

$$TM|_Z = \pi^*TM_{\text{red}} \oplus Z \times (g \oplus g^*).$$

The standard symplectic structure

$$\omega_{g \oplus g^*}(v_1, \mu_1, v_2, \mu_2) = \mu_1(v_2) - \mu_2(v_1)$$

defines a $K \times G$-equivariant $\text{Spin}_c$-structure on $Z \times (g \oplus g^*)$, and gives the desired equivariant Morita isomorphism. □
Note that the restriction of the Morita morphism $\widetilde{\mathcal{C}}l(TM) \rightarrow \mathcal{A}_{K \times G}^{\text{Spin}}$, to $Z \subset M$ takes values in $\mathcal{A}_{K \times G}^{\text{Spin}}|K \times \{e\}$. Let

$$\mathcal{A}_{K \times G}^{\text{Spin}}|K \times \{e\} \rightarrow \mathcal{A}_{K}^{\text{Spin}}$$

be the Morita isomorphism defined by the Morita trivialization of $\mathcal{A}_{G}^{\text{Spin}}|\{e\}$. The twisted Spin$_c$-structure for $(M, \omega, \Phi)$ descends to the twisted Spin$_c$-structure for the $G$-reduced space $(M_{\text{red}}, \omega_{\text{red}}, \Phi_{\text{red}})$, in the following sense.

**Theorem 7.13 (Reduction).** Suppose $(M, \omega, \Phi)$ is a q-Hamiltonian $K \times G$-manifold, such that $e$ is a regular value of $\Phi^{-1}(e)$ and such that $G$ acts freely on $\Phi^{-1}(e)$. The diagram of $K \times G$-equivariant Morita morphisms

$$\begin{array}{ccc}
\widetilde{\mathcal{C}}l(TM)|Z & \rightarrow & \mathcal{A}_{K \times G}^{\text{Spin}}|K \times \{e\} \\
\downarrow & & \downarrow \\
\widetilde{\mathcal{C}}l(TM_{\text{red}}) & \rightarrow & \mathcal{A}_{K}^{\text{Spin}}
\end{array}$$

commutes up to equivariant 2-isomorphism. Here the vertical maps are given by (27) and (32).

The proof uses the following normal form result for $TM|Z$.

**Lemma 7.14.** For a suitable choice of invariant Euclidean metric on $TM$, the decomposition $TM|Z = \pi^*TM_{\text{red}} \oplus Z \times (g \oplus g^*)$ from (30) is compatible with the 2-forms. That is,

$$\omega|Z = \pi^*\omega_{\text{red}} \oplus \omega_{g \oplus g^*}.$$

**Proof.** We will construct $K \times G$-equivariant splittings of the exact sequences (28) and (29) so that (30) is compatible with the 2-forms. (One may then take an invariant Euclidean metric on $TM|Z$ for which these splittings are orthogonal, and extend to $TM$.) Begin with an arbitrary $K \times G$-invariant splitting

$$TM|Z = TZ \oplus F.$$

Since $F \cap \ker(\omega) = 0$, the sub-bundle $F^\omega \subset TM|Z$ (the set of vectors $\omega$-orthogonal to all vectors in $F$) has codimension $\text{codim}(F^\omega) = \dim F = \dim g$. The moment map condition shows that $\omega$ is non-degenerate on $F \oplus Z \times g$. Hence $(Z \times g) \cap F^\omega = 0$, and therefore

$$TM|Z = (Z \times g) \oplus F^\omega.$$

Let $\phi: TM|Z \rightarrow Z \times g$ be the projection along $F^\omega$. The subspace

$$F' = \{v - \frac{1}{2}\phi(v) \mid v \in F\}$$

is again an invariant complement to $TZ$ in $TM|Z$, and it is isotropic for $\omega$. Indeed, if $v_1, v_2 \in F$,

$$\omega(v_1 - \frac{1}{2}\phi(v_1), v_2 - \frac{1}{2}\phi(v_2)) = \frac{1}{2}\omega(v_1, v_2 - \phi(v_2)) + \frac{1}{2}\omega(v_1 - \phi(v_1), v_2)$$
vanishes since \( v_i - \phi(v_i) \in F^\omega \). The restriction of \((d\Phi_G)|_Z: TM|_Z \to g^*\) to \(F'\) identifies \( E' = Z \times g^* \). We have hence shown the existence of an invariant decomposition \( TM|_Z = TZ \oplus Z \times g^* \) where the second summand is embedded as an \( \omega \)-isotropic subspace, and such that \((d\Phi_G)|_Z\) is projection to the second summand. From the \(G\)-moment map condition

\[
\iota(\xi_M)\omega|_Z = -\frac{1}{2} \Phi_G^*B((\theta^L + \theta^R)|_Z, \xi) = -B((d\Phi_G)|_Z, \xi), \quad \xi \in g,
\]

we see that the induced 2-form on the sub-bundle \( Z \times (g \oplus g^*) \) is just the standard one, \( \omega_{g \oplus g^*} \). The \( \omega \)-orthogonal space \( Z \times (g \oplus g^*)^{\omega} \) defines a complement to \( Z \times g \subset TZ \), and is hence identified with \( \pi^*TM_{\text{red}} \).

**Proof of Theorem 7.13.** Let \( \Theta: TM|_Z \to TM_{\text{red}} \) be the bundle morphism given by projection to the first summand in (30), followed by the quotient map. Then

\[
(\Theta, \omega_{g \oplus g^*}): (TM|_Z, TM|_Z) \to (TM_{\text{red}}, TM_{\text{red}}),
\]

is a strong Dirac morphism, and the resulting Morita morphism \( \mathcal{A}_{TM}|_Z \to \mathcal{A}_{TM_{\text{red}}} \) fits into a commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}l(TM)|_Z & \to & \mathcal{A}_{TM}|_Z \\
\downarrow & & \downarrow \\
\mathcal{C}l(TM_{\text{red}}) & \to & \mathcal{A}_{TM_{\text{red}}}
\end{array}
\]

On the other hand, letting \( \text{pr}_1: T(K \times G)|_{K \times \{e\}} \to TK \) be projection to the first summand, we have

\[
(\text{pr}_1, 0) \circ (d\Phi|_Z, \omega|_Z) = (d\Phi_{\text{red}}, \omega_{\text{red}}) \circ (\Theta, \omega_{g \oplus g^*}),
\]

so that the resulting diagram of Morita morphisms

\[
\begin{array}{ccc}
\mathcal{A}_{TM}|_Z & \to & \mathcal{A}_{K \times G}|_{K \times \{e\}} \\
\downarrow & & \downarrow \\
\mathcal{A}_{TM_{\text{red}}} & \to & \mathcal{A}_{K}^{\text{Spin}}
\end{array}
\]

commutes up to 2-isomorphism. Placing (33) next to (34), the Theorem follows.

**Remark 7.15.** If \( e \) is a regular value of \( \Phi_G \), but the action of \( G \) on \( Z \) is not free, the reduced space \( M_{\text{red}} \) is usually an orbifold. The Theorem extends to this situation with obvious modifications.

**Remark 7.16.** Reduction at more general values \( g \in G \) may be expressed in terms of reduction at \( e \), using the shifting trick: Let \( G_g \subset G \) be the centralizer of \( g \), and \( \text{Ad}(G)g^{-1} \cong G/G_g^{-1} \) the conjugacy class of \( g^{-1} \). Then

\[
M/\!/gG := \Phi_G^{-1}(g)/G_g = (M \times \text{Ad}(G)g^{-1})/G
\]
where $M \times \text{Ad}(G).g^{-1}$ is the fusion product. Again, one finds that $g$ is a regular value of $\Phi_G$ if and only if the $G_g$-action on $\Phi^{-1}(g)$ is locally free, and if the action is free then $M//gG$ is a $q$-Hamiltonian $K$-space.

8. Hamiltonian $LG$-spaces

In his 1988 paper, Freed [15] argued that for a compact, simple and simply connected Lie group $G$, the canonical line bundle over the Kähler manifold $LG/G$ (and over the other coadjoint orbits of the loop group) is a $\hat{L}G$-equivariant Hermitian line bundle $K \to LG/G$, where the central circle of $\hat{L}G$ acts with a weight $-2h^\vee$, where $h^\vee$ is the dual Coxeter number. In [21], this was extended to more general Hamiltonian $LG$-spaces.

In this Section we will use the correspondence between Hamiltonian $LG$-spaces and $q$-Hamiltonian $G$-spaces to give a new construction of the canonical line bundle, in which it is no longer necessary to assume $G$ simply connected. We begin by recalling the definition of a Hamiltonian $LG$-space.

Let $G$ be a compact Lie group, with a given invariant inner product $B$ on its Lie algebra. We fix $s > 1/2$, and take the loop group $LG$ to be the Banach Lie group of maps $S^1 \to G$ of Sobolev class $s + 1/2$. Its Lie algebra $Lg$ consists of maps $S^1 \to g$ of Sobolev class $s + 1/2$. We denote by $Lg^*$ the $g$-valued 1-forms on $S^1$ of Sobolev class $s - 1/2$, with the gauge action $g \cdot \mu = \text{Ad}_g(\mu) - g^* \theta^R$. A Hamiltonian $LG$-manifold is a Banach manifold $N$ with an action of $LG$, an invariant (weakly) symplectic 2-form $\sigma \in \Omega^2(N)$, and a smooth $LG$-equivariant map $\Psi : N \to Lg^*$ satisfying the moment map condition

$$\iota(\xi^2)\sigma = -d\langle \Psi, \xi \rangle, \quad \xi \in Lg.$$

Here the pairing between elements of $Lg^*$ and of $Lg$ is given by the inner product $B$ followed by integration over $S^1$.

Suppose now that $G$ is connected, and let $PG$ be the space of paths $\gamma : \mathbb{R} \to G$ of Sobolev class $s + 1/2$ such that $\pi(\gamma) = \gamma(t + 1)\gamma(t)^{-1}$ is constant. The map $\pi : PG \to G$ taking $\gamma$ to this constant is a $G$-equivariant principal $L\gamma$-bundle, where $a \in G$ acts by $\gamma \mapsto a\gamma$ and $\Lambda \in LG$ acts by $\gamma \mapsto \gamma\Lambda^{-1}$. One has $PG/G \cong Lg^*$ with quotient map $\gamma \mapsto \gamma^{-1}\gamma dt$. Let $\tilde{N} \to N$ be the principal $G$-bundle obtained by pull-back of the bundle $PG \to Lg^*$, and $\tilde{\Psi} : \tilde{N} \to PG$ the lifted moment map. Then $\tilde{\Psi}$ is $L\gamma \times G$-equivariant. Since the $L\gamma$-action on $PG$ is a principal action, the same is true for the action on $\tilde{N}$. Assuming that $\Psi$ (hence $\tilde{\Psi}$) is proper, one obtains a smooth compact manifold $M = \tilde{N}/L\gamma$ with an induced $G$-map $\Phi : M \to G = PG/LG$. 

$$\begin{array}{ccc}
\tilde{N} & \xrightarrow{\tilde{\Psi}} & PG \\
\pi_M \downarrow & & \downarrow \pi_G \\
M & \xrightarrow{\Phi} & G
\end{array}$$
In [2], it was shown how to obtain an invariant 2-form \( \omega \) on \( M \), making \( (M, \omega, \Phi) \) into a q-Hamiltonian \( G \)-spaces. This construction sets up a 1-1 correspondence between Hamiltonian \( LG \)-spaces with proper moment maps and q-Hamiltonian spaces.

As noted in Remark 7.2, the canonical twisted \( \text{Spin}_c \)-structure for \( (M, \omega, \Phi) \) defines a \( G \)-equivariant Morita trivialization of the bundle \( \mathcal{E} : C \rightarrow \Phi^*\mathcal{A}_G^\text{Spin} \otimes^2 \) over \( M \). On the other hand, let \( \widetilde{LG}\text{Spin} \) be the pull-back of the basic central extension \( \widetilde{L}SO(\mathfrak{g}) \) under the adjoint action. By the discussion in Section 6.3, the pull-back bundle \( \mathcal{A}_G^\text{Spin} \) to \( P_G \) has a canonical \( \widetilde{LG}\text{Spin} \times G \)-equivariant Morita trivialization, \( S_0 : C \rightarrow \pi^*\mathcal{A}_G^\text{Spin} \), where the central circle of \( \widetilde{LG}\text{Spin} \) acts with weight 1. Tensoring \( S_0 \) with itself, and pulling everything back to \( \tilde{N} \) we obtain two Morita trivializations \( \pi^*_M \mathcal{E} \) and \( \Psi^*(S_0 \otimes S_0) \) of the Dixmier-Douady bundle \( \mathcal{C} \) over \( \tilde{N} \), given by the pull-back of \( \mathcal{A}_G^\text{Spin} \otimes^2 \) under \( \Phi \circ \pi_M = \pi_G \circ \tilde{\Psi} \). Let

\[ \tilde{K} := \text{Hom}_C(\tilde{\Psi}^*(S_0 \otimes S_0), \pi^*_M \mathcal{E}) \]

Then \( \tilde{K} \) is a \( \widetilde{LG}\text{Spin} \times G \)-equivariant Hermitian line bundle, where the central circle in \( \widetilde{LG}\text{Spin} \) acts with weight \(-2\). Its quotient \( K = \tilde{K}/G \) is the desired canonical bundle for the Hamiltonian \( LG \)-manifold \( N \).

**Remark 8.1.** For \( G \) simple and simply connected, the central extension \( \widetilde{LG}\text{Spin} \) is the \( h^\vee \)-th power of the ‘basic central’ extension \( \widetilde{LG} \). We may thus also think of \( K_N \) as a \( \widetilde{LG} \)-equivariant line bundle where the central circle acts with weight \(-2h^\vee\).

The canonical line bundle is well-behaved under symplectic reduction. That is, if \( e \) is a regular value of \( \Phi \) then \( 0 \in L\mathfrak{g}^* \) is also a regular value of \( \Psi \), and \( \Phi^{-1}(e) \cong \Psi^{-1}(0) \) as \( G \)-spaces. Assume that \( G \) acts freely on these level sets, so that \( M/G = N/G \) is a symplectic manifold. The canonical line bundle for \( M/G = K_{M/G} = K_N|_{\psi^{-1}(0)}/G \). As in [21], one can sometimes use this fact to compute the canonical line bundle over moduli spaces of flat \( G \)-bundles over surfaces.

**Appendix A. Boundary conditions**

In this Section, we will prove several facts about the operator \( \partial/\partial t \) on the complex Hilbert-space \( L^2([0, 1], \mathbb{C}^n) \), with boundary conditions defined by \( A \in U(n) \),

\[ \text{dom}(D_A) = \{ f \in L^2([0, 1], \mathbb{C}^n) | \dot{f} \in L^2([0, 1], \mathbb{C}^n), f(1) = -Af(0) \} \]

Let \( e^{2\pi i \lambda(1)}, \ldots, e^{2\pi i \lambda(n)} \) be the eigenvalues of \( A \), with corresponding normalized eigenvectors \( v^{(1)}, \ldots, v^{(n)} \in \mathbb{C}^n \). Then the spectrum of \( D_A \) is given by
the eigenvalues $2\pi i(\lambda^{(r)} + k - \frac{1}{2})$, \( k \in \mathbb{Z}, r = 1, \ldots, n \) with eigenfunctions

$$
\phi_k^{(r)}(t) = \exp(2\pi i(\lambda^{(r)} + k - \frac{1}{2}) t) v^{(r)}.
$$

We define \( J_A = i \text{sign}(-iD_A) \); this coincides with \( J_A = D_A/|D_A| \) if \( D_A \) has trivial kernel.

**Proposition A.1.** Let \( A, A' \in \mathbb{U}(n) \). Then \( J_{A'} - J_A \) is Hilbert-Schmidt if and only if \( A' = A \).

**Proof.** Suppose \( A' \neq A \). Let \( \Pi, \Pi' \) be the orthogonal projection operators onto \( \ker(J_A - i), \ker(J_{A'} - i) \). It suffices to show that \( \Pi' - \Pi \) is not Hilbert-Schmidt, i.e. that \( (\Pi' - \Pi)^2 \) is not trace class. Since

$$
(\Pi - \Pi')^2 = \Pi(I - \Pi')\Pi + (I - \Pi)\Pi'(I - \Pi)
$$

is a sum of two positive operators, it suffices to show that \( \Pi(I - \Pi')\Pi \) is not trace class. Let \( \phi_l^{(s)} \) be the eigenfunctions of \( D_{A'} \), defined similar to those for \( D_A \), with eigenvalues \( 2\pi i(\lambda^{(s)} + l - \frac{1}{2}) \). Indicating the eigenvalues and eigenfunctions for \( A' \) by a prime \( ' \), we have

$$
\text{tr}(\Pi(I - \Pi')\Pi) = \sum \left| \langle \phi_k^{(r)}, \phi_l^{(s)} \rangle \right|^2.
$$

where the sum is over all \( k, r, l, s \) satisfying \( \lambda^{(r)} + k - \frac{1}{2} > 0 \) and \( \lambda^{(s)} + l - \frac{1}{2} \leq 0 \). But

$$
\left| \langle \phi_k^{(r)}, \phi_l^{(s)} \rangle \right|^2 = \left| \langle v^{(r)}, v^{(s)} \rangle (e^{2\pi i(\lambda^{(s)} - \lambda^{(r)}) - 1}) \right|^2.
$$

Since \( A' \neq A \), we can choose \( r, s \) such that

$$
e^{2\pi i\lambda^{(r)}} \neq e^{2\pi i\lambda^{(s)}} \quad \text{and} \quad \langle v^{(r)}, v^{(s)} \rangle \neq 0.
$$

For such \( r, s \), the enumerator is a non-zero constant, and the sum over \( k, l \) is divergent. \( \square \)

**Proposition A.2.** Given \( A, A' \in \mathbb{U}(n) \), let

$$
\gamma: [0, 1] \to \text{Mat}_n(\mathbb{C})
$$

be a continuous map with

$$
A'\gamma(0) = \gamma(1)A,
$$

and such that \( \dot{\gamma} \in L^\infty([0, 1], \text{Mat}_n(\mathbb{C})) \). Let \( M_\gamma \) be the bounded operator on \( L^2([0, 1], \mathbb{C}^n) \) given as multiplication by \( \gamma \). Then

$$
M_\gamma J_A - J_{A'} M_\gamma
$$

is Hilbert-Schmidt.

**Proof.** This is a mild extension of Proposition(6.3.1) in Pressley-Segal \[25\], page 82], and we will follow their line of argument. Using the notation from the proof of Proposition A.1, it suffices to show that \( M_\gamma \Pi - \Pi'M_\gamma \) is Hilbert-Schmidt, or equivalently that both \( (I - \Pi')M_\gamma \Pi \) and \( \Pi'M_\gamma(I - \Pi) \) are
Hence \( M_{\gamma} - \gamma \)\( J \)

Proof. Let \( D \) to \( J \)

Proposition A.3. Let \( A \in U(n) \), and let \( \mu \in L^\infty([0,1],u(n)) \). Consider \( D_{\lambda,\mu} = D_{\lambda} + M_{\mu} \) with domain equal to that of \( D_{\lambda} \), and define \( J_{\lambda,\mu} \) similar to \( J_{\lambda} \). Then \( J_{\lambda,\mu} - J_{\lambda} \) is Hilbert-Schmidt.

Proof. Let \( \gamma \in C([0,1], U(n)) \) be the solution of the initial value problem \( \dot{\gamma} \gamma^{-1} = -\mu \) with \( \gamma(0) = I \). Let \( A = \gamma(1)A' \). The operator \( M_{\gamma} \) of multiplication by \( \gamma \) takes \( \text{dom}(D_{A'}) \) to \( \text{dom}(D_{A}) \), and

\[
M_{\gamma} D_{A'} M_{\gamma}^{-1} = D_{A} - \gamma \gamma^{-1} = D_{\lambda,\mu}.
\]

Hence \( M_{\gamma} J_{\lambda} M_{\gamma}^{-1} = J_{\lambda,\mu} \). By Proposition A.2, \( M_{\gamma} J_{\lambda} M_{\gamma}^{-1} \) differs from \( J_{\lambda} \) by a Hilbert-Schmidt operator.

Let us finally consider the continuity properties of the family of operators \( D_{\lambda}, A \in U(n) \). Recall [27] Chapter VIII that the norm resolvent topology on the set of unbounded skewadjoint operators on a Hilbert space is defined by declaring that a net \( D_i \) converges to \( D \) if and only if \( R_1(D_i) = (D_i -
$I)^{-1} \rightarrow R_1(D) = (D - I)^{-1}$ in norm. This then implies that $R_z(D_i) \rightarrow R_z(D)$ in norm, for any $z$ with non-zero real part, and in fact $f(D_i) \rightarrow f(D)$ in norm for any bounded continuous function $f$. For bounded operators, convergence in the norm resolvent topology is equivalent to convergence in the norm topology.

**Proposition A.4.** The map $A \mapsto D_A$ is continuous in the norm resolvent topology.

**Proof.** We will use that $||R_1(D)|| = ||(D - I)^{-1}|| < 1$ for any skew-adjoint operator $D$. Let us check continuity at any given $A \in U(n)$. Given $a \in u(n)$, let us write $D_a = D_{\exp(a)A}$. We will prove continuity at $A$ by showing that

$$||R_1(D_a) - R_1(D_0)|| \leq 3||a||.$$

Let $U_a \in U(V)$ be the operator of pointwise multiplication by $\exp(ta) \in U(V)$. Then

$$||U_a - U_0|| = \sup_{t\in[0,1]} ||\exp(ta) - I|| \leq ||a||.$$

The operator $U_a$ takes the domain of $D_0$ to that of $D_a$, since $f(1) = -Af(0)$ implies $(U_a f)(1) = \exp(a)f(1) = -\exp(a)Af(0)$. Furthermore,

$$D_a = U_a(D_0 + M_a)U_a^{-1}$$

Hence

$$R_1(D_a) = U_a R_1(D_0 + M_a) U_a^{-1}.$$ 

The second resolvent identity $R_1(D_0+M_a) - R_1(D_0) = R_1(D_0+M_a)M_a R_1(D_0)$ shows

$$||R_1(D_0 + M_a) - R_1(D_0)|| \leq ||M_a|| = ||a||.$$ 

Hence

$$||R_1(D_a) - R_1(D_0)|| = ||U_a R_1(D_0 + M_a) U_a^{-1} - U_0 R_1(D_0) U_0^{-1}||$$

$$\leq ||(U_a - U_0) R_1(D_0 + M_a) U_a^{-1}|| + ||U_0 R_1(D_0 + M_a) (U_a^{-1} - U_0^{-1})||$$

$$+ ||U_0 (R_1(D_0 + M_a) - R_1(D_0)) U_0^{-1}||$$

$$\leq 2||a|| ||R_1(D_0 + M_a)|| + ||R_1(D_0 + M_a) - R_1(D_0)|| < 3||a||. \square$$

**Appendix B. The Dixmier-Douady bundle over $S^1$**

Let $S^1 = \mathbb{R}/\mathbb{Z}$ carry the trivial action of $S^1$. The Morita isomorphism classes of $S^1$-equivariant Dixmier-Douady bundles $A \to S^1$ are labeled by their class

$$DD_{S^1}(A) \in H^3_{S^1}(S^1, \mathbb{Z}) \times H^1(S^1, \mathbb{Z}).$$

The bundle corresponding to $x \in H^3_{S^1}(S^1, \mathbb{Z}) = H^2_{S^1}(pt) = \mathbb{Z}$ and $y \in H^1(S^1, \mathbb{Z}) = H^0(pt, \mathbb{Z}) = \mathbb{Z}_2$ may be described as follows. Let $L_{(x, y)} \cong \mathbb{C}$ be the $\mathbb{Z}_2$-graded $S^1$-representation, of parity given by the parity of $y$, and with $S^1$-weight given by $x$. Choose a $\mathbb{Z}_2$-graded $S^1$-equivariant Hilbert space
\( \mathcal{H} \) with an equivariant isomorphism \( \tau: \mathcal{H} \to \mathcal{H} \otimes L \) preserving \( \mathbb{Z}_2 \)-gradings. Then \( \tau \) induces an \( S^1 \)-equivariant \(*\)-homomorphism
\[
\tau: \mathbb{K}(\mathcal{H}) \to \mathbb{K}(\mathcal{H} \otimes L) = \mathbb{K}(\mathcal{H}),
\]
preserving \( \mathbb{Z}_2 \)-gradings. The bundle \( \mathcal{A} \to S^1 \) with Dixmier-Douady class \( (x, y) \) is obtained from the trivial bundle \([0, 1] \times \mathbb{K}(\mathcal{H})\), using \( \tau \) to glue \( \{0\} \times \mathbb{K}(\mathcal{H}) \) and \( \{1\} \times \mathbb{K}(\mathcal{H}) \). Given another choice \( \mathcal{H}', \tau' \), one obtains a Morita isomorphism \( \mathcal{E}: \mathcal{A} \to \mathcal{A}' \), where \( \mathcal{E} \) is obtained from a similar boundary identification for \([0, 1] \times \mathbb{K}(\mathcal{H}', \mathcal{H})\).

A convenient choice of \( H, \tau \) defining the bundle with \( x = 1, y = 1 \) is as follows. Let \( \mathcal{H} \) be a Hilbert space with orthonormal basis of the form \( s_k \), indexed by the subsets \( K = \{k_1, k_2, \ldots\} \subset \mathbb{Z} \) such that \( k_1 > k_2 > \cdots \) and \( k_l = k_{l+1} + 1 \) for \( l \) sufficiently large. Let
\[
m_K = \#\{k \in K| k > 0\} - \#\{k \in \mathbb{Z} - K| k \leq 0\}.
\]
Let \( \mathcal{H} \) carry the \( S^1 \)-action such that \( s_K^n \) is a weight vector of weight \( m_K \), and a \( \mathbb{Z}_2 \)- grading, defined by the weight spaces of even/odd weight. Let \( \tau(K) = \{k + 1| k \in K\} \). Then \( m_{\tau(K)} = m_K + 1 \), hence the automorphism \( \tau: \mathcal{H} \to \mathcal{H} \) taking \( s_K \) to \( s_{\tau(K)} \) has the desired properties.

The Hilbert space \( \mathcal{H} \) can also be viewed as a spinor module. Let \( \mathcal{V} \) be a real Hilbert space, with complexification \( \mathcal{V}^\mathbb{C} \), and let \( f_k, k \in \mathbb{Z} \) be vectors such that \( f_k \) together with \( f_k^* \) are an orthonormal basis. The elements \( s_K \) for \( K = \{k_1, k_2, \cdots\} \) with \( k_1 > k_2 > \cdots \) are written as formal infinite wedge products
\[
s_K = f_{k_1} \wedge f_{k_2} \wedge \cdots
\]
suggesting the action of the Clifford algebra: \( g(f_k) \) acts by exterior multiplication, while \( g(f_k^*) \) acts by contraction. The automorphism \( \tau \in U(\mathcal{H}) \) is an implementer of the orthogonal transformation \( T \in O(\mathcal{V}) \),
\[
(T f_k) = f_{k+1}, \quad (T f_k^*) = f_{k+1}^*.
\]
Let us denote the resulting Dixmier-Douady bundle by \( \mathcal{A}_{(1,1)} \).

**Proposition B.1.** The Dixmier-Douady bundle \( \mathcal{A}_{(1,1)} \to S^1 \) is equivariantly isomorphic to the Dixmier-Douady bundle \( \mathcal{A} \to SO(2) \cong S^1 \), constructed as in Section 6.

**Proof.** For \( s \in \mathbb{R} \), let \( A_s \in SO(2) \) be the matrix of rotation by \( 2\pi s \), and let \( D_s \) be the skew-adjoint operator \( \frac{\partial}{\partial s} \) on \( L^2([0, 1], \mathbb{R}^2) \) with boundary conditions \( f(1) = -A_s f(0) \). The operator \( D_0 \) has an orthonormal system of eigenvectors \( f_k, f_k^* \), \( k \in \mathbb{Z} \) given by
\[
f_k(t) = e^{2\pi i (k - \frac{1}{2})} u_t,
\]
with \( u = \frac{1}{\sqrt{2}}(1, i) \). The eigenvalues for \( f_k, f_k^* \) are \( \pm 2\pi i (k - \frac{1}{2}) \). We see that the \( +i \) eigenspace of \( J = D_0/|D_0| \) is given by
\[
\mathcal{V}_+ = \text{span}\{f_3, f_2, f_1, f_0^*, f_{-1}^*, \cdots\}.
\]
There is a unique isomorphism of $\mathbb{C} l(\mathcal{V})$-modules $S_f \to \mathcal{H}$ taking the ‘vacuum vector’ $1 \in S_f = \wedge \mathcal{V}_+$ to the ‘vacuum vector’ $f_0 \wedge f_{-1} \wedge \cdots$.

For $s \in \mathbb{R}$, define orthogonal transformations $U_s \in O(\mathcal{V})$, where $U_s$ is pointwise multiplication by $t \mapsto A_{st}$. On $f_k$ the operator $U_s$ acts as multiplication by $e^{2\pi i st}$, and on $f_k^*$ as multiplication by $e^{-2\pi i st}$. Hence

$$f_k^{(s)} = U_s f_k, \quad (f_k^{(s)})^* = U_s f_k^*$$

are the eigenvectors of $D_s$, with shifted eigenvalues $\pm 2\pi i(k - \frac{1}{2} + s)$. The complex structure

$$J_s = U_s J U_s^{-1}$$

differs from $J_{D_s} = i \text{sign}(-iD_s)$ by a finite rank operator. Hence, letting $S_s$ denote the $\mathbb{C} l(\mathcal{V})$-module defined by $J_s$, the fiber of $A \to SO(2)$ at $A(s)$ may be described as $\mathbb{K}(S_s)$. The orthogonal transformation $U_s$ extends to an orthogonal transformation of $\wedge \mathcal{V}$, taking $S = \wedge \mathcal{V}_+$ to $S_s = \wedge \mathcal{V}_{+,s}$, where $\mathcal{V}_{+,s} = U_s \mathcal{V}_{+,s}$. Hence each $S_s$ is identified with $S \cong \mathcal{H}$ as a Hilbert space (not as a $\mathbb{C} l(\mathcal{V})$-module). The identification $\mathbb{K}(S_0) \cong \mathbb{K}(S_1)$ is given by the choice of any isomorphism of $\mathbb{C} l(\mathcal{V})$-modules $S_0 \to S_1$. In terms of the identifications with $\mathcal{H}$, such an isomorphism is given by an implementer of the orthogonal transformation $U_1$. The proof is completed by the observation that $U_1 = T$ (cf. (36)), which is implemented by $\tau$.

We are now in position to outline an alternative argument for the computation of the Dixmier-Douady class of $A_{SO(n)}$, Proposition 6.2. Note that $A_{SO(n)}$ is equivariant under the conjugation action of $SO(n)$. One has $H^3_{SO(n)}(SO(n), \mathbb{Z}) = \mathbb{Z}$ for $n \geq 2$, $n \neq 4$, and the natural maps to ordinary cohomology are isomorphisms for $n \geq 3, n \neq 4$. Similarly $H^1_{SO(n)}(SO(n), \mathbb{Z}_2) = \mathbb{Z}_2$ for $n \geq 2$, and the natural map to $H^1(SO(n), \mathbb{Z}_2)$ is an isomorphism. On the other hand, the map $H^3_{SO(n)}(SO(n), \mathbb{Z}) \to H^3_{SO(2)}(SO(2), \mathbb{Z})$ (defined by the inclusion $SO(2) \hookrightarrow SO(n)$ as the upper left corner) is an isomorphism for $n \geq 2, n \neq 4$, and likewise for $H^1(\cdot, \mathbb{Z}_2)$. It hence suffices to check that the bundle over $SO(2)$ has equivariant Dixmier-Douady class $(1, 1) \in \mathbb{Z} \times \mathbb{Z}_2$. But this is clear from our very explicit description of $A_{SO(2)}$.

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