An Empirical Dynamic Programming Algorithm for Continuous MDPs

William B. Haskell, Rahul Jain, Hiteshi Sharma and Pengqian Yu

Abstract—We propose universal randomized function approximation-based empirical value iteration (EVI) algorithms for Markov decision processes. The ‘empirical’ nature comes from each iteration being done empirically from samples available from simulations of the next state. This makes the Bellman operator a random operator. A parametric and a non-parametric method for function approximation using a parametric function space and the Reproducing Kernel Hilbert Space (RKHS) respectively are then combined with EVI. Both function spaces have the universal function approximation property. Basis functions are picked randomly. Convergence analysis is done using a random operator framework with techniques from the theory of stochastic dominance. Finite time sample complexity bounds are derived for both universal approximate dynamic programming algorithms. Numerical experiments support the versatility and effectiveness of this approach.

Index Terms—Continuous state space MDPs; Dynamic programming; Reinforcement Learning.

I. INTRODUCTION

There exist a wide variety of approximate dynamic programming (DP) [1] Chapter 6], [2] and reinforcement learning (RL) algorithms [3] for finite state space Markov decision processes (MDPs). But many real-world problems of interest have either a continuous state space, or large enough that it is best approximated as one. Approximate DP and RL algorithms do exist for continuous state space MDPs but choosing which one to employ is an art form: different techniques (state space aggregation and function approximation [4]) and algorithms work for different problems, and universally applicable algorithms are lacking. For example, fitted value iteration [5] is very effective for some problems but requires the choice of an appropriate basis functions for good approximation. No guidelines are available on how many basis functions to pick to ensure a certain quality of approximation. The strategy then employed is to use a large enough basis but has demanding computational complexity.

In this paper, we propose approximate DP algorithms for continuous state space MDPs that are universal (approximating function space can provide arbitrarily good approximation for any problem), computationally tractable, simple to implement and yet we have non-asymptotic sample complexity bounds. The first is accomplished by picking functions spaces for approximation that are dense in the space of continuous functions. The second goal is achieved by relying on randomized selection of basis functions for approximation and also by ‘empirical’ dynamic programming [6]. The third is enabled because standard python routines can be used for function fitting and the fourth is by analysis in a random operator framework which provides non-asymptotic rate of convergence and sample complexity bounds.

There is a large body of well-known literature on reinforcement learning and approximate dynamic programming for continuous state space MDPs. We discuss the most directly related. In [7], a sampling-based state space aggregation scheme combined with sample average approximation for the expectation in the Bellman operator is proposed. Under some regularity assumptions, the approximate value function can be computed at any state and an estimate of the expected error is given. But the algorithm seems to suffer from poor numerical performance. A linear programming-based constraint-sampling approach was introduced in [8]. Finite sample error guarantees, with respect to this constraint-sampling distribution are provided but the method suffers from issues of feasibility. The closest paper to ours is [5] that does function fitting with a given basis and does ‘empirical’ value iteration in each step. Unfortunately, it is not a universal method as approximation quality depends on the function basis picked. Other papers worth noting are [9] that discusses kernel-based value iteration and the bias-variance tradeoff, and [10] that proposed a kernel-based algorithm with random sampling of the state and action spaces, and proves asymptotic convergence.

This paper is inspired by the ‘random function’ approach that uses randomization to (nearly) solve otherwise intractable problems (see e.g., [11], [12]) and the ‘empirical’ approach that reduces computational complexity of working with expectations [6]. We propose two algorithms. For the first parametric approach, we pick a parametric function family. In each iteration a number of functions are picked randomly for function fitting by sampling the parameters. For the second non-parametric approach, we pick a RKHS for approximation. Both function spaces are dense in the space of continuous functions. In each iteration, we sample a few states from the state space. Empirical value iteration (EVI) is then performed on these states. Each step of EVI involves approximating the Bellman operator with an empirical (random) Bellman operator by plugging a sample average approximation from simulation for the expectation. This is akin to doing stochastic approximations with step size 1. But an elegant random operator framework for convergence analysis using stochastic
dominance was introduced recently in [6] and comes in handy for us as well.

The main contribution of this paper is development of randomized function approximation-based (offline) dynamic programming algorithms that are universally applicable (i.e., do not require appropriate choice of basis functions for good approximation). A secondary contribution is further development of the random operator framework for convergence analysis in the $L_p$-norm that also yields finite time sample complexity bounds.

The paper is organized as follows. Section II presents preliminaries including the continuous state space MDP model and the empirical dynamic programming framework for finite state MDPs introduced in [6]. Section III presents two empirical value iteration algorithms - first, a randomized parametric function fitting method, and second, a non-parametric randomized function fitting in an RKHS space. We also provide statements of main theorems about non-asymptotic error guarantees. Section IV presents a unified analysis of the two algorithms in a random operator framework. Numerical results are reported in Section V. Secondary proofs are relegated to the appendix.

II. PRELIMINARIES

Consider a discrete time discounted MDP given by the 5-tuple, $(S, A, Q, c, \gamma)$. The state space $S$ is a compact subset of $\mathbb{R}^d$ with the Euclidean norm, with corresponding Borel $\sigma$-algebra $\mathcal{B}(S)$. Let $\mathcal{F}(S)$ be the space of all $\mathcal{B}(S)$-measurable bounded functions $f : S \rightarrow \mathbb{R}$ in the supremum norm $\|f\|_\infty := \sup_{s \in S} |f(s)|$. Moreover, let $\mathcal{M}(S)$ be the space of all probability distributions over $S$ and define the $L_2$ norm as $\|f\|_{2, \mu}^2 := \int |f|^2 \mu(ds)$ for given $\mu \in \mathcal{M}(S)$. We assume that the action space $A$ is finite. The transition law $Q$ governs the system evolution. For $B \in \mathcal{B}(S)$, $Q(B | s, a)$ is the probability of next visiting the set $B$ given that action $a \in A$ is chosen in state $s \in S$. The cost function $c : S \times A \rightarrow \mathbb{R}$ is a measurable function that depends on state-action pairs. Finally, $\gamma \in (0, 1)$ is the discount factor.

We will denote by $\Pi$ the class of stationary deterministic Markov policies: mappings $\pi : S \rightarrow A$ which only depend on history through the current state. For a given state $s \in S$, $\pi(s) \in A$ is the action chosen in state $s$ under the policy $\pi$. The state and action at time $t$ are denoted $s_t$ and $a_t$, respectively. Any policy $\pi$ $\in \Pi$ and initial state $s \in S$ determine a probability measure $P^\pi_s$ and a stochastic process $\{(s_t, a_t), t \geq 0\}$ defined on the canonical measurable space of trajectories of state-action pairs. The expectation operator with respect to $P^\pi_s$ is denoted $E^\pi_s[\cdot]$.

We will assume that the cost function $c$ satisfies $|c(s, a)| \leq c_{\text{max}} < \infty$ for all $(s, a) \in S \times A$. Under this assumption, $\|v^\pi\|_{\infty} \leq v_{\text{max}} := c_{\text{max}}/(1 - \gamma)$ where $v^\pi$ is the value function for policy $\pi \in \Pi$ defined as $v^\pi(s) = E^\pi_s[\sum_{i=0}^{\infty} \gamma^i c(s_i, a_i)], \forall s \in S$. For later use, we define $\mathcal{F}(S; v_{\text{max}})$ to be the space of all functions $f \in \mathcal{F}(S)$ such that $\|f\|_{\infty} \leq v_{\text{max}}$.

The optimal value function is $v^* (s) := \inf_{\pi \in \Pi} E^\pi_s[\sum_{i=0}^{\infty} \gamma^i c(s_i, a_i)], \forall s \in S$. To characterize the optimal value function, we define the Bellman operator $T : \mathcal{F}(S) \rightarrow \mathcal{F}(S)$ via

$$[Tv](s) := \min_{a \in A} \{c(s, a) + \gamma \mathbb{E}_{X \sim Q(\cdot | s, a)} [v(X)]\}, \forall s \in S.$$  

It is known that the optimal value function $v^*$ is a fixed point of $T$, i.e. $Tv^* = v^*$. Classical value iteration is based on iterating $T$ to obtain a fixed point, it produces a sequence $(v_k)_{k \geq 0} \subset \mathcal{F}(S)$ given by $v_{k+1} = T v_k, k \geq 0$. Also, we know that $(v_k)_{k \geq 0}$ converges to $v^*$ geometrically in $\|\cdot\|_{\infty}$.

We are interested in approximating the optimal value function $v^*$ within a tractable class of approximating functions $\mathcal{F} \subset \mathcal{F}(S)$. We have the following definitions which we use to measure the approximation power of $\mathcal{F}$ with respect to $T$. We define $d_{p, \mu}(T f, \mathcal{F}) := \inf_{f \in \mathcal{F}} \|f' - T f\|_{p, \mu}$ to be the approximation error for a specific $f$; and

$$d_{p, \mu}(T, \mathcal{F}) := \sup_{f \in \mathcal{F}} d_{p, \mu}(T f, \mathcal{F})$$

to be the inherent $L_p$ Bellman error for the function class $\mathcal{F}$. Similarly, the inherent $L_{\infty}$ Bellman error for an approximating class $\mathcal{F}$ is

$$d_{\infty}(T, \mathcal{F}) \equiv \sup_{g \in \mathcal{F}} \inf_{f \in \mathcal{F}} \|f - T g\|_{\infty}.$$  

We often compare $\mathcal{F}$ to the Lipschitz continuous functions $\text{Lip}(L)$ defined as

$$\{f \in \mathcal{F}(S) : |f(s) - f(s')| \leq L \|s - s'\|, \forall s, s' \in S\}.$$  

In our case, we say that an approximation class $\mathcal{F}$ is universal if $d_{\infty}(\text{Lip}(L), \mathcal{F}) = 0$ for all $L \geq 0$.

One of the difficulties of dynamic programming algorithms like value iteration above is that each iteration of the Bellman operator involves computation of an expectation which may be expensive. Thus, in [13], we proposed replacing the Bellman operator with an empirical (or random) Bellman operator,

$$\hat{T}_n v(s) := \min_{a \in A} \left\{ c(s, a) + \frac{1}{n} \sum_{i=1}^{n} [v(X_i)] \right\},$$

where $X_i$ are samples of the next state from $Q(\cdot | s, a)$ which can be obtained from simulation. Now, we can iterate the empirical Bellman operator,

$$v_{k+1} = \hat{T}_n v_k$$

an algorithm we called Empirical Value Iteration (EVI). The sequence of iterates $\{v_k\}$ is a random process. Since $T$ is a contractive operator, its’ iterates converge to its fixed point $v^*$. The random operator $\hat{T}_n$ may be expected to inherit the contractive property in a probabilistic sense and its’ iterates converge to some sort of a probabilistic fixed point. We introduce $(\epsilon, \delta)$ versions of two such notions introduced in [6].

Definition 1. A function $\hat{v} : S \rightarrow \mathbb{R}$ is an $(\epsilon, \delta)$-strong probabilistic fixed point for a sequence of random operators $\{\hat{T}_n\}$ if there exists an $N$ such that for all $n > N$,

$$\mathbb{P} \{ ||\hat{T}_n \hat{v} - \hat{v}|| > \epsilon \} < \delta.$$
It is called a strong probabilistic fixed point, if the above is true for every positive \( \epsilon \) and \( \delta \).

**Definition 2.** A function \( \hat{v} : S \to \mathbb{R} \) is an \((\epsilon, \delta)\)-weak probabilistic fixed point for a sequence of random operators \( \{\hat{T}_n\} \) if there exist \( N \) and \( K \) such that for all \( n > N \) and all \( k > K \),

\[
P \left( \|\hat{T}_n^k v_0 - \hat{v}\| > \epsilon \right) < \delta, \quad \forall v_0 \in F(S).
\]

It is called a weak probabilistic fixed point, if the above is true for every positive \( \epsilon \) and \( \delta \). Note that the stochastic iterative algorithms such as EVI often find the weak probabilistic fixed point of \( \{\hat{T}_n\} \) whereas what we are looking for is \( v^* \), the fixed point of \( T \). In [13], it was shown that asymptotically the weak probabilistic fixed point of \( \{\hat{T}_n\} \) coincides with its strong probabilistic fixed points which coincide with the fixed point of \( T \) under certain fairly weak assumptions and a natural relationship between \( T \) and \( \{\hat{T}_n\} \)

\[
\lim_{n \to \infty} P \left( \|\hat{T}_n v - T v\| > \epsilon \right) = 0, \quad \forall v \in F(S).
\]

This implies that stochastic iterative algorithms such as EVI will find approximate fixed points of \( T \) with high probability.

**III. The Algorithms and Main Results**

When the state space \( S \) is very large, or even uncountable, exact dynamic programming methods are not practical, or even feasible. Instead, one must use a variety of approximation methods. In particular, function approximation (or fitting the value function with a fixed function basis) is a common technique. The idea is to sample a finite set of states from \( S \), approximate the Bellman update at these states, and then extend to the rest of \( S \) through function fitting similar to [5]. Furthermore, the expectation in the Bellman operator, for example, is also approximated by taking a number of samples of the next state. There are two main difficulties with this approach: First, the function fitting depends on the function basis chosen, making the results problem-dependent. Second, with a large basis (for good approximation), function fitting can be computationally expensive.

In this paper, we aim to address these issues by first picking universal approximating function spaces, and then using randomization to pick a smaller basis and thus reduce computational burden of the function fitting step. We consider two functional families, one is a parametric family \( F(\Theta) \) parameterized over parameter space \( \Theta \) and the other is a non-parametric regularized RKHS. By \( \mu \in M(S) \), we denote a probability distribution from which to sample states in \( S \), and by a \( F \subset F(S) \) we denote a functional family in which to do value function approximation.

Let us denote by \( (\nu_k)_{k \geq 0} \subset F(S; \nu_{max}) \), the iterates of the value functions produced by an algorithm and a sample of size \( N \geq 1 \) from \( S \) is denoted \( s_1^{1,N} = (s_1, \ldots, s_N) \). The empirical \( p \)-norm of \( f \) is defined as \( \|f\|_{p, \nu}^p = \frac{1}{N} \sum_{n=1}^{N} |f(s_n)|^p \) for \( p \in [1, \infty) \) and as \( \|f\|_{\infty, \mu} = \sup_{n=1, \ldots, N} |f(s_n)| \) for \( p = \infty \), where \( \mu \) is the empirical measure corresponding to the samples \( s_1^{1,N} \).

We will make the following technical assumptions for the rest of the paper similar to those made in [5].

**Assumption 1.** (i) For all \((s, a) \in S \times A\), \( Q(\cdot | s, a) \) is absolutely continuous with respect to \( \mu \) and

\[
C_\mu := \sup_{(s, a) \in S \times A} \|dQ(\cdot | s, a) / d\mu\|_\infty < \infty.
\]

(ii) Given any sequence of policies \( \{\pi_m\}_{m \geq 1} \), the future state distribution \( \rho Q^1 \cdots Q^m \) is absolutely continuous with respect to \( \mu \),

\[
c_{p, \mu}(m) := \sup_{\pi_1, \ldots, \pi_m} \|d(\rho Q^1 \cdots Q^m) / d\mu\|_\infty < \infty,
\]

and \( C_{p, \mu} := \sum_{m \geq 0} \gamma^m c_{p, \mu}(m) < \infty \).

The above assumptions are conditions on transition probabilities, the first being a sufficient condition for the second.

**A. Random Parametric Basis Function (RPBF) Approximation**

We introduce an empirical value iteration algorithm with function approximation using random parametrized basis functions (EVI+RPBF). It requires a parametric family \( F \) built from a set of parameters \( \Theta \) with probability distribution \( \nu \) and a feature function \( \phi : S \times \Theta \to \mathbb{R} \) (that depends on both states and parameters) with the assumption that \( \sup_{(s, \theta) \in S \times \Theta} |\phi(s; \theta)| \leq 1 \). This can easily be met in practice by scaling \( \phi \) whenever \( S \) and \( \Theta \) are both compact and \( \phi \) is continuous in \((s, \theta)\). Let \( \alpha : \Theta \to \mathbb{R} \) be a weight function and define \( F(\Theta) := \{f(\cdot) = \int_{\Theta} \phi(\cdot; \theta) \alpha(\theta) d\theta : |\alpha(\theta)| \leq C \nu(\theta), \forall \theta \in \Theta\} \).

We note that the condition \( |\alpha(\theta)| \leq C \nu(\theta) \) for all \( \theta \in \Theta \) is equivalent to requiring that \( \|\alpha\|_\infty, \nu := \sup_{\theta \in \Theta} |\alpha(\theta)| \nu(\theta) \leq C \). We call this the \( \nu \)-weighted supremum norm of \( \alpha \) and \( C \) is a constant.

The function space \( F(\Theta) \) may be chosen to have the ‘universal’ function approximation property in the sense that any Lipschitz continuous function can be approximated arbitrarily closely in this space as shown in [11]. By [11, Theorem 2], many such choices of \( F(\Theta) \) are possible and are developed in [11, Section 5]. For example, \( F(\Theta) \) is universal in the following two cases:

1. \( \phi(s; \theta) = \cos(\langle \omega, s \rangle + b) \) where \( \theta = (\omega, b) \in \mathbb{R}^{d+1} \) and \( \nu(\theta) \) is given by \( \omega \sim \text{Normal}(0, 2 \gamma I) \) and \( b \sim \text{Uniform}[-\pi, \pi] \).
2. \( \phi(s; \theta) = \text{sign}(s_k - t) \) where \( \theta = (t, k) \in \mathbb{R} \times \{1, \ldots, d\} \) and \( \nu(\theta) \) is given by \( k \sim \text{Uniform}\{1, \ldots, d\} \) and \( t \sim \text{Uniform}[-a, a] \).

In this approach, we have a parametric function family \( F(\Theta) \) but instead of optimizing over parameters in \( \Theta \), we randomly sample them first and then do function fitting which involves optimizing over finite weighted combinations \( \sum_{j=1}^{J} \alpha_j \phi(\cdot; \theta_j) \). Unfortunately, this leads to a non-convex optimization problem. Hence, instead of optimizing over \( \theta_{1,J} = (\theta_1, \ldots, \theta_J) \) and \( \alpha_{1,J} = (\alpha_1, \ldots, \alpha_J) \) jointly, we first do randomization over \( \theta_{1,J} \) and then optimization over \( \alpha_{1,J} \), as in [12], to bypass the non-convexity inherent in optimizing over \( \theta_{1,J} \) and \( \alpha_{1,J} \) simultaneously.

This approach allows us to deploy rich parametric families without much additional computational cost. Once we draw a
random sample \( \{ \theta_j \}_{j=1}^J \) from \( \Theta \) according to \( \nu \), we obtain a random function space: \( \tilde{F} (\theta^{1:j}) := \left\{ f (\cdot) = \sum_{j=1}^J \alpha_j \phi (\cdot; \theta_j) : \| (\alpha_1, \ldots, \alpha_j) \|_{\infty} \leq C/J \right\} \).

Step 1 of such an algorithm (Algorithm 1) involves sampling states \( s^{1:N} \) over which to do value iteration and sampling parameters \( \theta^{1:j} \) to pick basis functions \( \phi (\cdot; \theta) \) which are used to do function fitting. Step 2 involves doing an empirical value iteration over states \( s^{1:N} \) by sampling next states \( (X_{m+1}^{s_{n+1}})_{m=1}^M \) according to the transition kernel \( Q \), and using the current iterate of the value function \( v_k \). Note that fresh (i.i.d.) samples of the next state are regenerated in each iteration. Step 3 involves finding the best fit to \( \tilde{v}_k \), the iterate from Step 2, within \( \tilde{F} (\theta^{1:j}) \) wherein randomly sampled parameters \( \theta^{1:j} \) specify the basis functions for function fitting and weights \( \alpha^{1:j} \) are optimized, which is a convex optimization problem.

**Algorithm 1** EVI with random parameterized basis functions (EVI+RPBF)

**Input:** probability distribution \( \mu \) on \( \mathbb{S} \) and \( \nu \) on \( \Theta \);

Sample sizes \( N \geq 1 \), \( M \geq 1 \), \( J \geq 1 \); initial seed \( v_0 \). For \( k = 1, \ldots, K \):

1. Sample \( (s_n)_{n=1}^N \sim \mu^N \) and \( (\theta_j)_{j=1}^J \sim \mu_j^J \).
2. Compute
   \[
   \tilde{v}_k (s_n) = \min_{a \in A} \left\{ c (s_n, a) + \frac{\gamma}{M} \sum_{m=1}^M v_k (X_{m+n}^{s_n,a}) \right\},
   \]
   where \( (X_{m+n}^{s_n,a}) \sim Q (s_n | s_m, a), m = 1, \ldots, M \) are i.i.d.
3. \( \alpha^k = \arg \min_{\alpha} \frac{1}{N} \sum_{n=1}^N \left( \sum_{j=1}^J \alpha_j \phi (s_n; \theta_j) - \tilde{v}_k (s_n) \right)^2 \)
   s.t. \( \| (\alpha_1, \ldots, \alpha_J) \|_{\infty} \leq C/J \), \( v_{k+1} (s) = \sum_{j=1}^J \alpha_j \phi (s; \theta_j) \).
4. Increment \( k \leftarrow k + 1 \) and return to Step 1.

We note that Step 3 of the algorithm can be replaced by another method for function fitting (as we do in the next subsection). The above algorithm differs from Fitted Value Iteration (FVI) algorithm of [5] in how it does function fitting. FVI does function fitting with a deterministic and given set of basis functions which limits its universality while we do function fitting in a much larger space which has a universal function approximation property but are able to reduce computational complexity by exploiting randomization.

In [5] Section 7), it is shown that if the transition kernel and cost are smooth, in the sense that

\[
\| Q (s, a) - Q (s', a) \|_{TV} \leq L_Q \| s - s' \|_2
\]

and

\[
\| c (s, a) - c (s', a) \| \leq L_c \| s - s' \|_2
\]

hold for all \( s, s' \in \mathbb{S} \) and \( a \in A \), then the Bellman operator \( T \) maps bounded functions to Lipschitz continuous functions. In particular, if \( v \) is uniformly bounded by \( v_{\max} \) then \( T v = (L_c + \gamma v_{\max} L_Q) v \) is Lipschitz continuous. Subsequently, the inherent \( L_{\infty} \) Bellman error satisfies \( d_{\infty} (T, F, \mathcal{F}) \leq d_{\infty} (\text{Lip} (L), \mathcal{F}) \) since \( T \mathcal{F} \subseteq \text{Lip} (L) \). So, it only remains to choose an \( \mathcal{F} (\Theta) \) that is dense in \( \text{Lip} (L) \) in the supremum norm, for which many examples exist.

We now provide non-asymptotic sample complexity bounds to establish that Algorithm 1 yields an approximately optimal value function with high probability. We provide guarantees for both the \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) metrics on the error. Denote

\[
\begin{align*}
N_1 (\varepsilon, \delta) &= 275^2 \beta_{\max}^2 \log \left( \frac{40 e (J+1)}{\delta} \right) (10 e \tilde{v}_{\max})^J, \\
M_1 (\varepsilon, \delta') &= \left( \frac{\tilde{v}_{\max}^2}{2} \right) \log \left( \frac{10 N |A|}{\delta'} \right), \\
J_1 (\varepsilon, \delta') &= \left( \frac{5C}{\varepsilon} \right) \left( 1 + \sqrt{2 \log \frac{5}{\delta'}} \right)^2, \\
K_1^* &= \left[ \ln \left( C_{\rho,\mu} \varepsilon - \ln (2 v_{\max}) \right) \right],
\end{align*}
\]

and

where \( C \) is the same constant that appears in the definition of \( \mathcal{F} (\Theta) \) (see [12]) and \( \tilde{v}_{\max} = v_{\max} / \varepsilon \). Set \( \delta' := 1 - (1 - \delta / 2^{1/K_1^* - 1}) \).

**Theorem 1.** Given an \( \varepsilon > 0 \), and a \( \delta \in (0, 1) \), choose \( J \geq J_1 (\varepsilon, \delta') \), \( N \geq N_1 (\varepsilon, \delta') \), \( M \geq M_1 (\varepsilon, \delta') \), Then, for \( K \leq \log \left( 4 / (\delta \mu^* (\delta; K_1^*)) \right) \), we have

\[
\| v_K - v^\star \|_1, \rho \leq 2 C_{\rho,\mu} (d_{1,\mu} (T F (\Theta), F (\Theta)) + 2 \varepsilon)
\]

with probability at least \( 1 - \delta \).

**Remarks.** That is, if we choose enough samples \( N \) of the states, enough samples \( M \) of the next state, and enough random samples \( J \) of the parameter \( \theta \), and then for large enough number of iterations \( K \), the \( \mathcal{L}_1 \) error in the value function is determined by the inherent Bellman error of the function class \( F (\Theta) \). For the function families \( F (\Theta) \) discussed earlier, the inherent Bellman error is indeed zero, and so the value function will have small \( \mathcal{L}_1 \) error with high probability. Next we give a similar guarantee for \( \mathcal{L}_2 \) error for Algorithm 1 by considering approximation in \( L_{2,\mu} (\mathbb{S}) \). We note that \( L_{2,\mu} (\mathbb{S}) \) is a Hilbert space and that many powerful function approximation results exist for this setting because of the favorable properties of a Hilbert space.

Denote

\[
\begin{align*}
N_2 (\varepsilon, \delta') &= 27^2 \beta_{\max}^2 \log \left( \frac{40 e (J+1)}{\delta} \right) (10 e \tilde{v}_{\max})^J, \\
M_2 (\varepsilon, \delta') &= \left( \frac{\tilde{v}_{\max}^2}{2} \right) \log \left( \frac{10 N |A|}{\delta'} \right), \\
J_2 (\varepsilon, \delta') &= \left( \frac{5C}{\varepsilon} \right) \left( 1 + \sqrt{2 \log \frac{5}{\delta'}} \right)^2, \\
K_2^* &= 2 \left[ \ln \left( C_{1/2,\mu} \varepsilon - \ln (2 v_{\max}) \right) \right],
\end{align*}
\]

where again \( \tilde{v}_{\max} = v_{\max} / \varepsilon \). Set \( \delta' := 1 - (1 - \delta / 2^{1/K_2^* - 1}) \).
Theorem 2. Given an \( \varepsilon > 0 \), and a \( \delta \in (0, 1) \), choose \( J \geq J_2(\varepsilon, \delta') \), \( N \geq N_2(\varepsilon, \delta') \), \( M \geq M_2(\varepsilon, \delta') \), Then, for \( K \geq \log(4/\delta \mu^* (\delta_K^*)^2) \), we have

\[
\|v_K - v^*\|_{2, \rho} \leq 2\varepsilon^{1/2}C_{\rho, \mu}^{1/2} (d_{2, \rho}(T \mathcal{F}(\Theta), \mathcal{F}(\Theta)) + 2\varepsilon)
\]

with probability at least \( 1 - \delta \).

Remarks. Again, note that the above result implies that if we choose a function families \( \mathcal{F}(\Theta) \) with inherent Bellman error zero, enough samples \( N \) of the states, enough samples \( M \) of the next state, and enough random samples \( J \) of the parameter \( \theta \), and then for large enough number of iterations \( K \), the value function will have small \( \mathcal{L}_2 \) error with high probability.

B. Non-parametric Function Approximation in RKHS

We now consider non-parametric function approximation combined with EVI. We employ a Reproducing Kernel Hilbert Space (RKHS) for function approximation since for suitably chosen kernels, it is dense in the space of continuous functions and hence has a ‘universal’ function approximation property. In the RKHS setting, we can obtain guarantees directly with respect to the supremum norm.

We will consider a regularized RKHS setting with a continuous, symmetric and positive semidefinite kernel \( K : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R} \) and a regularization constant \( \lambda > 0 \). The RKHS space, \( \mathcal{H}_K \) is defined to be the closure of the linear span of \( \{ K(s, \cdot) \}_{s \in \mathcal{S}} \) endowed with an inner product \( \langle \cdot, \cdot \rangle_{\mathcal{H}_K} \). The inner product \( \langle \cdot, \cdot \rangle_{\mathcal{H}_K} \) for \( \mathcal{H}_K \) is defined such that \( \langle K(x, \cdot), (y, \cdot) \rangle_{\mathcal{H}_K} = K(x, y) \) for all \( x, y \in \mathcal{S} \), i.e., \( \sum_{i,j} \alpha_i \beta_j K(x_i, \cdot) \sum_{i,j} \beta_i \alpha_j K(y_j, \cdot) \rangle_{\mathcal{H}_K} = \sum_{i,j} \alpha_i \beta_j K(x_i, y_j) \). Subsequently, the inner product satisfies the reproducing property: \( (K(s, \cdot), f)_{\mathcal{H}_K} = f(s) \) for all \( s \in \mathcal{S} \) and \( f \in \mathcal{H}_K \).

We assume that our kernel \( K \) is bounded so that \( \lambda \) is necessarily equal to \( \|f\|^{2}_{\mathcal{H}_K} \).

To find the best fit \( f \in \mathcal{H}_K \) to a function with data \( \{ (s_n, \hat{v}(s_n)) \}_{n=1}^{N} \), we solve the regularized least squares problem:

\[
\min_{f \in \mathcal{H}_K} \left\{ \frac{1}{N} \sum_{n=1}^{N} (f(s_n) - \hat{v}(s_n))^2 + \lambda \|f\|_{\mathcal{H}_K}^2 \right\}.
\]

This is a convex optimization problem (the norm squared is convex), and has a closed form solution by the Representer Theorem. In particular, the optimal solution is of the form \( \hat{f}(s) = \sum_{n=1}^{N} \alpha_n K(s_n, s) \) where the weights \( \alpha^{1:N} = (\alpha_1, \alpha_2, \ldots, \alpha_N) \) are the solution to the linear system

\[
(K(s_i, s_j))_{i,j=1}^{N} = \lambda N I \text{ and } (\alpha_n)_{n=1}^{N} = \left( \hat{v}(s_n) \right)_{n=1}^{N}.
\]

This yields EVI algorithm with randomized function fitting in a regularized RKHS (EVI+RKHS) displayed as Algorithm 2.

Algorithm 2 EVI with regularized RKHS (EVI+RKHS)

Input: probability distribution \( \mu \) on \( \mathcal{S} \); sample sizes \( N \geq 1 \), \( \lambda \geq 1 \); penalty \( \lambda \); initial seed \( v_0 \); counter \( k = 0 \).

1) Sample \( \{ s_n \}_{n=1}^{N} \sim \mu \).
2) Compute

\[
\hat{v}_k(s_n) = \min_{a \in \mathcal{A}} \left\{ c(s_n, a) + \frac{\gamma}{M} \sum_{n=1}^{M} v_k(X_m^{s_n, a}) \right\},
\]

where \( \{ X_m^{s_n, a} \}_{m=1}^{M} \sim Q(s_n, a) \) are i.i.d.
3) \( v_{k+1}(\cdot) \) is given by

\[
\arg \min_{f \in \mathcal{H}_K} \left\{ \frac{1}{N} \sum_{n=1}^{N} (f(s_n) - \hat{v}(s_n))^2 + \lambda \|f\|_{\mathcal{H}_K} \right\}.
\]

4) Increment \( k \leftarrow k + 1 \) and return to Step 1.

We define the regression function \( f_M : \mathcal{S} \rightarrow \mathbb{R} \) via

\[
f_M(s) = \min_{\alpha \in \mathcal{A}} \left\{ c(s, a) + \frac{\gamma}{M} \sum_{n=1}^{M} v(X_m^{s, a}) \right\}, \quad \forall s \in \mathcal{S},
\]

it is expected value of our empirical estimator of \( T v \). As expected, \( f_M \rightarrow T v \) as \( M \rightarrow \infty \). We note that \( f_M \) is not necessarily equal to \( T v \) by Jensen’s inequality. We require the following assumption on \( f_M \) to continue.

Assumption 2. For every \( M \geq 1 \), \( f_M(s) = \sum K(s, s') \alpha(y) \mu(dy) \) for some \( \alpha \in \mathcal{L}_{2, \mu}(\mathcal{S}) \).

Regression functions play a key role in statistical learning theory. Assumption 2 states that the regression function lies in the span of the kernel \( K \). Additionally, when \( \mathcal{H}_K \) is dense in the Lipschitz functions, then the inherent Bellman error is zero. For example, \( K(s, s') = \exp(-\gamma \|s - s'\|_2) \), \( K(s, s') = 1 - \frac{1}{\alpha} \|s - s'\|_1 \), and \( K(s, s') = \exp(\gamma \|s - s'\|_1) \) are all universal kernels.

Denote

\[
N_{\infty}(\varepsilon, \delta') = \left( \frac{4 C_K \varepsilon}{\varepsilon (1 - \gamma)} \right)^{\delta} \log \left( \frac{\delta}{\delta'} \right)
\]

\[
M_{\infty}(\varepsilon) = \frac{160 \varepsilon^2 \max_{s} \left\| \gamma (8 \varepsilon v_{\max} - (1 - \gamma)) \right\|_2}{\varepsilon (1 - \gamma)(2 - \gamma)}
\]

\[
K_{\infty} = \frac{\ln(\varepsilon) - \ln(4 \varepsilon v_{\max})}{\ln \gamma}
\]

where \( C_K \) is a constant independent of the dimension of \( \mathcal{S} \) (see [14] for the details on how \( C_K \) depends on \( K \)) and set \( \gamma = 1 - (1 - \delta/2)^{-1/(K_{\infty} - 1)} \).
Theorem 3. Suppose Assumption 2 holds. Given any $\varepsilon > 0$ and $\delta \in (0, 1)$, choose an $N \geq N_{\infty}(\varepsilon, \delta')$ and an $M \geq M_{\infty}(\varepsilon)$. Then, for any $K \geq \log(M/(\delta \mu^*(\delta; K^*)))$,

$$\|v_K - v^*\|_{\infty} \leq \varepsilon$$

with probability at least $1 - \delta$.

Note that we provide guarantees on $L_1$ and $L_2$ error (can be generalized to $L_p$) with the RPBF method and for $L_{\infty}$ error with the RKHS-based randomized function fitting method. Getting guarantees for the $L_p$ error with the RKHS method has proved quite difficult, as has bounds on the $L_{\infty}$ error with the RBPF method.

IV. ANALYSIS IN A RANDOM OPERATOR FRAMEWORK

We will analyze Algorithms 1 and 2 in terms of random operators since this framework is general enough to encompass many such algorithms. The reader can see that Step 2 of both algorithms involves iteration of the empirical Bellman operator while Step 3 involves a randomized function fitting step which is done differently and in different spaces in both algorithms. We use random operator notation to write these algorithms in a compact way, and then derive a clean and to a large extent unified convergence analysis. The key idea is to use the notion of stochastic dominance to bound the error process with an easy to analyze “dominating” Markov chain. Then, we can extend unified convergence analysis. The key idea is to use the notion of stochastic dominance to bound the error process with an easy to analyze “dominating” Markov chain. Then, we can infer the solution quality of our algorithms via the probability distribution of the dominating Markov chain. This analysis idea refines (and in fact, simplifies) the idea we introduced in [13] for MDPs with finite state and action spaces (where there is a function fitting in the supremum norm). In this paper, we develop the technique further, give a stronger convergence rate, account for randomized function approximation, and also generalize the technique to $L_p$ norms.

We introduce a probability space $(\Omega, B(\Omega), P)$ on which to define random operators, where $\Omega$ is a sample space with elements denoted $\omega \in \Omega$, $\mathcal{B}(\Omega)$ is the Borel $\sigma$–algebra on $\Omega$, and $P$ is a probability distribution on $(\Omega, \mathcal{B}(\Omega))$. A random operator is an operator-valued random variable on $(\Omega, \mathcal{B}(\Omega), P)$. We define the first random operator on $\mathcal{F}(\mathbb{S})$ as $\tilde{T}(v) = (s_n, \hat{v}(s_n))_{n=1}^N$ where $(s_n)_{n=1}^N$ is chosen from $\mathbb{S}$ according to a distribution $\mu \in \mathcal{M}(\mathbb{S})$ and

$$\hat{v}(s_n) \equiv \min_{a \in A} \left\{ c(s_n, a) + \frac{\gamma M}{\epsilon} \sum_{m=1}^{M} v(X_{nm}^n, a) \right\},$$

$n = 1, \ldots, N$ is an approximation of $\lfloor T v \rfloor (s_n)$ for all $n = 1, \ldots, N$. In other words, $\tilde{T}$ maps from $v \in \mathcal{F}(\mathbb{S}; v_{\max})$ to a randomly generated sample of $N$ input-output pairs $(s_n, \hat{v}(s_n))_{n=1}^N$ of the function $T v$. Note that $\tilde{T}$ depends on sample sizes $N$ and $M$. Next, we have the function reconstruction operator $\tilde{\Pi}_F$ which maps the data $(s_n, \hat{v}(s_n))_{n=1}^N$ to an element in $\mathcal{F}$. Note that $\tilde{\Pi}_F$ is not necessarily deterministic since Algorithms 1 and 2 use randomized function fitting. We can now write both algorithms succinctly as

$$v_{k+1} = \tilde{G} v_k := \tilde{\Pi}_F \tilde{T} v_k,$$

which can be further written in terms of residual error $\varepsilon_k = \tilde{G} v_k - T v_k$ as

$$v_{k+1} = \tilde{G} v_k = T v_k + \varepsilon_k.$$  

(6)

Iteration of these operators corresponds to repeated samples from $(\Omega, \mathcal{B}(\Omega), P)$, so we define the space of sequences $(\Omega^\infty, \mathcal{B}(\Omega^\infty), P)$ where $\Omega^\infty = \times_{k=0}^\infty \Omega$ with elements denoted $\omega = (\omega_k)_{k=0}^\infty$, $\mathcal{B}(\Omega^\infty) = \times_{k=0}^\infty \mathcal{B}(\Omega)$, and $P$ is the probability measure on $(\Omega^\infty, \mathcal{B}(\Omega^\infty))$ guaranteed by the Kolmogorov extension theorem applied to $P$.

The random sequences $(v_k)_{k=0}^\infty$ in Algorithms 1 and 2 given by

$$v_{k+1} = \tilde{\Pi}_F \tilde{T} (\omega_k) v_k$$

$$= \tilde{\Pi}_F \tilde{T} (\omega_k) \tilde{\Pi}_F \tilde{T} (\omega_{k-1}) \cdots \tilde{\Pi}_F \tilde{T} (\omega_0) v_0,$$

for all $k \geq 0$ is a stochastic process defined on $(\Omega^\infty, \mathcal{B}(\Omega^\infty), P)$. We now analyze error propagation over the iterations.

Let us now investigate how the Bellman residual at each iteration of EVI affects the quality of the resulting policy. There have already been some results which address the error propagation both in $L_\infty$ and $L_p$ ($p \geq 1$) norms [15]. After adapting [5, Lemma 3], we obtain the following point-wise error bounds on $v_K - v^*$ in terms of the errors $(\varepsilon_k)_{k \geq 0}$.

Lemma 4. For any $K \geq 1$, and $\varepsilon > 0$, suppose $||\varepsilon_k||_{p, \mu} \leq \varepsilon$ for all $k = 0, 1, \ldots, K - 1$, then

$$||v_K - v^*||_{p, \mu} \leq 2 \left( \frac{1 - \gamma^{K+1}}{1 - \gamma} \right)^{\frac{1}{p-1}} \left[ C_{\mu}^{1/p} \varepsilon + \gamma^{K/2} (2 v_{\max}) \right],$$

(7)

where $C_{\mu}$ is the discounted-average concentrability coefficient of the future-state distributions as defined in [5, Assumption A2]. Note that Lemma 4 assumes that $||\varepsilon_k||_{2, \mu} \leq \varepsilon$ which we will show in the subsequent section that it is true with high probability.

The second inequality is for the supremum norm.

Lemma 5. For any $K \geq 1$ and $\varepsilon > 0$, suppose $||\varepsilon_k||_{\infty} \leq \varepsilon$ for all $k = 0, 1, \ldots, K - 1$, then

$$||v_K - v^*||_{\infty} \leq \varepsilon / (1 - \gamma) + \gamma^{K/2} (2 v_{\max}).$$

(8)

Inequalities (7) and (8) are the key to analyzing iteration of Equation (6).

A. Convergence analysis using stochastic dominance

We now provide a (unified) convergence analysis for iteration of a sequence of random operators given by (5) and (6). Later, we will show how it can be applied to Algorithms 1 and 2. We will use $\| \cdot \|$ to denote a general norm in the following discussion, since our idea applies to all instances of $p \in [1, \infty)$ and $p = \infty$ simultaneously. The magnitude of the error in iteration $k \geq 0$ is then $||\varepsilon_k||$. We make the following key assumption for a general EVI algorithm.

Assumption 3. For $\varepsilon > 0$, there is a $q \in (0, 1)$ such that

$$Pr \{ ||\varepsilon_k|| \leq \varepsilon \} \geq q \text{ for all } k \geq 0.$$
Assumption 3 states that we can find a lower bound on the probability of the event \( \{ \| \varepsilon_k \| \leq \varepsilon \} \) that is independent of \( k \) and \( (v_k)_{k \geq 0} \) (but does depend on \( \varepsilon \)). Equivalently, we are giving a lower bound on the probability of the event \( \{ \| T v_k - \hat{G} v_k \| \leq \varepsilon \} \). This is possible for all of the algorithms that we proposed earlier. In particular, we can control \( q \) in Assumption 3 through the sample sizes in each iteration of EVI. Naturally, for a given \( \varepsilon, q \) increases as the number of samples grows.

We first choose \( \varepsilon > 0 \) and the number of iterations \( k^* \) for our EVI algorithms to reach a desired accuracy (this choice of \( K^* \) comes from the inequalities (7) and (8)). We call iteration \( k \) “good” if the error \( \| \varepsilon_k \| \) is within our desired tolerance \( \varepsilon \) and “bad” when the error is greater than our desired tolerance. We then construct a stochastic process \( (X_k)_{k \geq 0} \) on \( (\Omega^\infty, \mathcal{B}(\Omega^\infty), \mathcal{P}) \) with state space \( \mathcal{K} := \{ 1, 2, \ldots, K^* \} \) such that

\[
X_{k+1} = \begin{cases} 
\max \{ X_k - 1, 1 \}, & \text{if iteration } k \text{ is "good"}, \\
K^*, & \text{otherwise}.
\end{cases}
\]

The stochastic process \( (X_k)_{k \geq 0} \) is easier to analyze than \( (v_k)_{k \geq 0} \) because it is defined on a finite state space, however \( (X_k)_{k \geq 0} \) is not necessarily a Markov chain.

We next construct a “dominating” Markov chain \( (Y_k)_{k \geq 0} \) to help us analyze the behavior of \( (X_k)_{k \geq 0} \). We construct \( (Y_k)_{k \geq 0} \) on \( (\mathcal{K}^\infty, \mathcal{B}) \), the canonical measurable space of trajectories on \( \mathcal{K} \), so \( \mathcal{Y}_k : \mathcal{K}^\infty \to \mathbb{R} \), and we let \( \mathcal{Q} \) denote the probability measure on \( (\mathcal{K}^\infty, \mathcal{B}) \). Since \((Y_k)_{k \geq 0}\) will be a Markov chain by construction, the probability measure \( \mathcal{Q} \) is completely determined by an initial distribution on \( \mathbb{R} \) and a transition kernel for \((Y_k)_{k \geq 0}\). We always initialize \( Y_0 = K^* \), and then construct the transition kernel as follows

\[
Y_{k+1} = \begin{cases} 
\max \{ Y_k - 1, 1 \}, & \text{w.p. } q, \\
K^*, & \text{w.p. } 1 - q,
\end{cases}
\]

where \( q \) is the probability of a “good” iteration with respect to the corresponding norm. Note that the \( (Y_k)_{k \geq 0} \) we introduce here is different and has much smaller state space than the one we introduced in [6] leading to stronger convergence guarantees.

We now describe a stochastic dominance relationship between the two stochastic processes \((X_k)_{k \geq 0}\) and \((Y_k)_{k \geq 0}\). We will establish that \((Y_k)_{k \geq 0}\) is “larger” than \((X_k)_{k \geq 0}\) in a stochastic sense.

**Definition 3.** Let \( X \) and \( Y \) be two real-valued random variables, then \( X \) is stochastically dominated by \( Y \), written \( X \leq_{st} Y \), when \( \mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)] \) for all increasing functions \( f: \mathbb{R} \to \mathbb{R} \).

Let \( \mathcal{F}_k \) denote the filtration on \( (\Omega^\infty, \mathcal{B}(\Omega^\infty), \mathcal{P}) \) corresponding to the evolution of information about \((X_k)_{k \geq 0}\), and let \([X_{k+1}, \mathcal{F}_k]\) denote the conditional distribution of \( X_{k+1} \) given \( \mathcal{F}_k \). We have the following initial results on the relationship between \((X_k)_{k \geq 0}\) and \((Y_k)_{k \geq 0}\).

The following theorem, our main result for our random operator analysis, compares the marginal distributions of \((X_k)_{k \geq 0}\) and \((Y_k)_{k \geq 0}\) at all times \( k \geq 0 \) when the two stochastic processes \((X_k)_{k \geq 0}\) and \((Y_k)_{k \geq 0}\) start from the same state.

**Theorem 6.** (i) \( X_k \leq_{st} Y_k \) for all \( k \geq 0 \).

(ii) \( \Pr\{ Y_k \leq \eta \} \geq \Pr\{ X_k \leq \eta \} \) for any \( \eta \in \mathbb{R} \) and all \( k \geq 0 \).

**Proof.** First we note that, by [6] Lemma A.1, \([Y_{k+1}, Y_k = \eta]\) is stochastically increasing in \( \eta \) for all \( k \geq 0 \), i.e. \([Y_{k+1}, Y_k = \eta]\) \( \leq_{st} [Y_{k+1}, Y_k = \eta^\prime] \) for all \( \eta \leq \eta^\prime \). Then, by [13] Lemma A.2, \([X_{k+1}, X_k = \eta, \mathcal{F}_k]\) \( \leq_{st} [Y_{k+1}, Y_k = \eta] \) for all \( \eta \in \mathcal{F}_k \) for all \( k \geq 0 \).

(i) Trivially, \( X_0 \leq_{st} Y_0 \) since \( X_0 \leq_{as} Y_0 \). Next, we see that \( X_1 \leq_{st} Y_1 \) by [13] Lemma A.1. We prove the general case by induction. Suppose \( X_k \leq_{st} Y_k \) for \( k \geq 1 \), and for this proof define the random variable

\[
\mathcal{Y}(\theta) = \begin{cases} 
\max \{ \theta - 1, 1 \}, & \text{w.p. } q, \\
K^*, & \text{w.p. } 1 - q.
\end{cases}
\]

to be the conditional distribution of \( Y_k \) conditional on \( \theta \), as a function of \( \theta \). We see that \( Y_{k+1} \) has the same distribution as \( \mathcal{Y}(\theta) \mid \theta = Y_k \) by definition. Since \( \mathcal{Y}(\theta) \) are stochastically increasing by Lemma [13] Lemma A.1, we see that \( \mathcal{Y}(\theta) \mid \theta = Y_k \) \( \geq_{st} \mathcal{Y}(\theta) \mid \theta = X_k \) by [10] Theorem 1.A.6 and our induction hypothesis. Now, \( \mathcal{Y}(\theta) \mid \theta = X_k \) \( \geq_{st} [X_{k+1}, X_k, \mathcal{F}_k] \) by [16] Theorem 1.A.3(d) and Lemma [13] Lemma A.2 for all histories \( \mathcal{F}_k \). It follows that \( Y_{k+1} \geq_{st} X_{k+1} \) by transitivity of \( \geq_{st} \).

(ii) Follows from part (i) by the definition of \( \leq_{st} \). \( \blacksquare \)

By Theorem 6 if \( X_K \leq_{st} Y_K \) and we can make \( \Pr\{ Y_K \leq \eta \} \) large, then we will also obtain a meaningful bound on \( \Pr\{ X_K \leq \eta \} \). Following this observation, the next two corollaries are the main mechanisms for our general sample complexity results for EVI.

**Corollary 7.** For a given \( p \in [1, \infty) \), and any \( \varepsilon > 0 \), and \( \delta \in (0, 1) \), suppose Assumption 3 holds for this \( \varepsilon \), and choose any \( K^* \geq 1 \). Then for \( q \geq (1 + \delta/2)(2K^* + 1) \) and \( K \geq \log(4/(1/2 - \delta/2)(1 - q)q K^{*-1}) \), we have

\[
\| v_K - v^* \|_{p, \rho} \leq 2 \left( \frac{1 - \gamma K^{*+1}}{1 - \gamma} \right)^{\frac{p - 1}{p}} \left[ C_{\rho, \mu^*} + \gamma K^*/p (2v_{\max}) \right]
\]

with probability at least \( \delta \).

**Proof.** Since \((v_k)_{k \geq 0}\) is an irreducible Markov chain on a finite state space, its steady state distribution \( \mu = (\mu(i))_{i=1}^{K^*} \) on \( \mathcal{K} \) exists. By [13] Lemma 4.3, the steady state distribution of \((Y_k)_{k \geq 0}\) is \( \mu = (\mu(i))_{i=1}^{K^*} \) given by:

\[
\mu(1) = q K^{*-1}, \quad \mu(i) = (1 - q) q K^{*-i}, \quad \forall i = 2, \ldots, K^* - 1, \quad \mu(K^*) = 1 - q.
\]

The constant

\[
\mu_{\min}(q; K^*) := \min \left\{ q K^{*-1}, (1 - q) q(K^*-2), (1 - q) \right\},
\]

for all \( q \in (0, 1) \) and \( K^* \geq 1 \), which is the minimum of the steady state probabilities appears shortly in the Markov chain.
mixing time bound for $(Y_k)_{k \geq 0}$. We note that $\mu^*(q; K^*) = (1 - q) q^{K^*-1} \leq \mu_{\text{min}}(q; K^*)$ is a simple lower bound for $\mu_{\text{min}}(q; K^*)$ (we defined $\mu^*(q; K^*) = (1 - q) q^{K^*-1}$ earlier).

Now, recall that $\|\mu - \nu\|_{TV} = \frac{1}{2} \sum_{q=1}^{K^*} |\mu(\eta) - \nu(\eta)|$ is the total variation distance for probability distributions on $K$. Let $Q^k$ be the marginal distribution of $Y_k$ for $k \geq 0$. By a Markov chain mixing time argument, e.g., [17 Theorem 12.3], we have that

$$t_{\text{mix}}(\delta') := \min \left\{ k \geq 0 : \|Q^k - \mu\|_{TV} \leq \delta' \right\} \leq \log \left( \frac{1}{\delta' \mu_{\text{min}}(q; K^*)} \right) \leq \log \left( \frac{1}{\delta'(1 - q) q^{K^*-1}} \right)$$

for any $\delta' \in (0, 1)$. So, for $K \geq \log \left( 1 / (\delta'(1 - q) q^{K^*-1}) \right)$ we have $|\Pr \{Y_K = 1\} - \mu(1)| \leq 2 \|Q^K - \mu\|_{TV} \leq 2 \delta'$, where we use $\mu(1) = q^{K^*-1}$. By Theorem 6, $\Pr \{X_K = 1\} \geq \Pr \{Y_K = 1\}$ and so

$$\Pr \{X_K = 1\} \geq q^{K^*-1} - 2 \delta'.$$

Choose $q$ and $\delta'$ to satisfy $q^{K^*-1} = 1/2 + \delta/2$ and $2 \delta' = q^{K^*-1} - \delta = 1/2 - \delta/2$ to get $q^{K^*-1} - 2 \delta' \geq \delta$, and the desired result follows.

The next result uses the same reasoning for the supremum norm case.

**Corollary 8.** Given any $\epsilon > 0$ and $\delta \in (0, 1)$, suppose Assumption 3 holds for this $\epsilon$, and choose any $K^* \geq 1$. For $q \geq \log \left( \left( \frac{1}{2} + \frac{1}{2} \right)^{1/(K^*-1)} \right)$ and $K \geq \log \left( 4 / \left( (1/2 - \delta/2) (1 - q) q^{K^*-1} \right) \right)$, we have

$$\Pr \left\{ \|v_K - \nu^*\|_{\infty} \leq \epsilon / (1 - \gamma) + q^{K^*} (2 v_{\max}) \right\} \geq \delta.$$

The sample complexity results for both EVI algorithms from Section III follow from Corollaries 7 and 8. This is shown next.

**B. Proofs of Theorems 7, 2 and 3**

We now apply our random operator framework to both EVI algorithms. We will see that it is easy to check the conditions of Corollaries 7 and 8 from which we obtain specific sample complexity results.

We can now give the proof of Theorem 1 by using our Markov chain technique. Based on Theorem 16 in the appendix, we let $p(N, M, J, \epsilon)$ denote the lower bound on the probability of the event $\{\|T v - T v\|_{1, \mu} \leq \epsilon\}$ for $\epsilon > 0$. We also note that $d_{1, \mu}(T v, F(\Theta)) \leq d_{1, \mu}(T F(\Theta), F(\Theta))$ for all $v \in F(\Theta)$.

**Proof.** (of Theorem 1) Starting with inequality (7) for $p = 1$ and using the statement of Theorem 16 we have

$$\|v_K - \nu^*\|_{1, \mu} \leq 2 C_{\rho, \mu} (d_{1, \mu}(T F(\Theta), F(\Theta)) + \epsilon) + 4 v_{\max} \gamma K^*$$

when $\|\varepsilon_k\|_{1, \mu} \leq d_{1, \mu}(T F(\Theta), F(\Theta)) + \epsilon$ for all $k = 0, 1, \ldots, K - 1$. Choose $K^*$ such that

$$4 v_{\max} \gamma K^* \leq 2 C_{\rho, \mu} \epsilon \Rightarrow K^* = \left\lceil \frac{\ln(C_{\rho, \mu} \epsilon)}{\ln \gamma} \right\rceil.$$

Based on Corollary 7, we just need to choose $N, M, J$ such that $p(N, M, J, \epsilon) \geq (1 - \delta/2)^{1/(K^*-1)}$. We then apply the statement of Theorem 16 with probability $1 - (1 - \delta/2)^{1/(K^*-1)}$.

We now give the proof of Theorem 2 along similar lines based on Theorem 17. Again, we let $p(N, M, J, \epsilon)$ denote the lower bound on the probability of the event $\{\|T v - T v\|_{2, \mu} \leq \epsilon\}$.

**Proof.** (of Theorem 2) Starting with inequality (7) for $p = 2$ and using the statement of Theorem 17 we have

$$\|v_K - \nu^*\|_{2, \rho} \leq 2 \left( \frac{1}{1 - \gamma} \right)^{1/2} C_{\rho, \mu} (d_{2, \mu}(T F(\Theta), F(\Theta)) + \epsilon)$$

$$+ 4 \left( \frac{1}{1 - \gamma} \right) v_{\max} \gamma K^* / 2,$$

when $\|\varepsilon_k\|_{2, \mu} \leq d_{2, \mu}(T F(\Theta), F(\Theta)) + \epsilon$ for all $k = 0, 1, \ldots, K - 1$. We choose $K^* \geq 1$ to satisfy

$$4 \left( \frac{1}{1 - \gamma} \right)^{1/2} v_{\max} \gamma K^*/2 \leq 2 \left( \frac{1}{1 - \gamma} \right)^{1/2} C_{\rho, \mu} \epsilon$$

which implies $K^* = 2 \left\lceil \frac{\ln(C_{\rho, \mu} \epsilon)}{\ln \gamma} - \ln(2 v_{\max}) \right\rceil / \ln \gamma$. Based on Corollary 7, we just need to choose $N, M, J$ such that $p(N, M, J, \epsilon) \geq (1 - \delta/2)^{1/(K^*-1)}$. We then apply the statement of Theorem 16 with $p = 1 - (1 - \delta/2)^{1/(K^*-1)}$.

We now provide proof of $L_\infty$ function fitting in RKHS based on Theorem 19 in the appendix. For this proof, we let $p(N, M, \epsilon)$ denote a lower bound on the probability of the event $\{\|T v - T v\|_{\infty} \leq \epsilon\}$.

**Proof.** (of Theorem 3) By inequality (8), we choose $\epsilon$ and $K^* \geq 1$ such that $\epsilon / (1 - \gamma) \leq \epsilon/2$ and $\gamma K^* (2 v_{\max}) \leq \epsilon/2$ by setting

$$K^* \geq \left\lceil \frac{\ln \epsilon - \ln(4 v_{\max})}{\ln \gamma} \right\rceil.$$

Based on Corollary 7, we next choose $N$ and $M$ such that $p(N, M, \epsilon) \geq (1 - \delta/2)^{1/(K^*-1)}$. We then apply the statement of Theorem 19 with error $\epsilon (1 - \gamma)/2$ and probability $1 - (1 - \delta/2)^{1/(K^*-1)}$.

**V. Numerical Experiments**

We now present numerical performance of our algorithm by testing it on the benchmark optimal replacement problem [7], [5]. The setting is that a product (such as a car) becomes more costly to maintain with time/miles, and must be replaced it some point. Here, the state $s_t \in \mathbb{R}_+$ represents the accumulated utilization of the product. Thus, $s_t = 0$ denotes a brand new durable good. Here, $A = \{0, 1\}$, so at each time step, $t$, we can either replace the product ($a_t = 0$) or keep it ($a_t = 1$). Replacement incurs a cost $C$ while keeping the product has
a maintenance cost, $c(s_t)$, associated with it. The transition probabilities are as follows:

$$q(s_{t+1} | s_t, a_t) = \begin{cases} \lambda e^{-\lambda(s_{t+1} - s_t)}, & \text{if } s_{t+1} \geq s_t \text{ and } a_t = 1 \\ \lambda e^{-\lambda s_{t+1}}, & \text{if } s_{t+1} \geq 0 \text{ and } a_t = 0 \\ 0, & \text{otherwise} \end{cases}$$

and the reward function is given by

$$r(s_t, a_t) = \begin{cases} -c(s_t), & \text{if } a_t = 1 \\ -C - c(0), & \text{if } a_t = 0 \end{cases}$$

For our computation, we use $\gamma = 0.6, \lambda = 0.5, C = 30$ and $c(s) = 4s$. The optimal value function and the optimal policy can be computed analytically for this problem. For EVI+RPBF, we use $J$ random parameterized Fourier functions $\{\phi(s, \theta_j) = \cos(\theta_j^T s + b)\}_{j=1}^J$ with $\theta_j \sim \mathcal{N}(0, 0.01)$ and $b \sim \text{Unif}[-\pi, \pi]$. We fix $J = 5$. For EVI+RKHS, we use Gaussian kernel defined as $k(x, y) = \exp(||x - y||^2/(2\sigma^2))$ with $1/\sigma^2 = 0.01$ and $\mathcal{L}_2$ regularization. We fix the regularization coefficient to be $10^{-2}$. The underlying function space for FVI is polynomials of degree 4. The results are plotted after 20 iterations.

The error in each iteration for different algorithms with $N = 100$ states and $M = 5$ is shown in Figure 1. On Y-axis, it shows the relative error computed as $\sup_{s \in S} |v^*(s) - v_k(s)|$ with iterations $k$ on the X-axis. It shows that EVI+RPBF has relative error below 10% after 20 iterations. FVI is close but EVI+RKHS is slower though it may improve with a higher $M$. This is also reflected in the actual run-time performance: EVI+RPBF takes 8,705s, FVI 8,654s and EVI+RKHS takes 42,173s to get within 0.1 relative error.

Note that performance of FVI depends on being able to choose suitable basis functions which for the optimal replacement problem is easy. For other problems, we may expect both EVI algorithms to perform better. So, we tested the algorithms on the cart-pole balancing problem, another benchmark problem but for which the optimal value function is unknown. We formulate it as a continuous 2-dimensional state space with 2 action MDP. The state comprises of the position of the cart and angle of the pole. The actions are to add a force of $-10N$ or $+10N$ to the cart, pushing it left or right. We add $\pm 50\%$ noise to these actions. System dynamics for this system are given in [3]. Rewards are zero except for failure state (if the position of cart reaches beyond $\pm 2.4$, or the pole exceeds an angle of $\pm 12$ degrees), it is $-1$. For our experiments, we choose $N = 100$ and $M = 1$. In case of RPBF, we consider parameterized Fourier basis of the form $\cos(w^T s + b)$ where $w = [w_1, w_2]$, $w_1, w_2 \sim \mathcal{N}(0, 1)$ and $b \sim \text{Unif}[-\pi, \pi]$. We fix $J = 10$ for our EVI+RPBF. For RKHS, we consider Gaussian kernel, $K(s_1, s_2) = \exp(-\sigma ||s_1 - s_2||^2/2)$ with $\sigma = 0.01$. We limit each episode to 1000 time steps. We compute the average length of the episode for which we are able to balance the pole without hitting the failure state. This is the goal in Table I The other columns show run-time needed for the algorithms to learn to achieve such a goal.

From the table, we can see that EVI+RPBF outperforms FVI and EVI+RKHS. Note that guarantees for FVI are only available for $\mathcal{L}_2$-error and for EVI-RPBF for $\mathcal{L}_p$-error. EVI-RKHS is the only algorithm that can provide guarantees on the sup-norm error. Also note that when for problems for which the value functions are not so regular, and good basis functions difficult to guess, the EVI+RKHS method is likely to perform better but as of now we do not have a numerical example to demonstrate this.

**VI. CONCLUSION**

In this paper, we have introduced universally applicable approximate dynamic programming algorithms for continuous state space MDPs. The algorithms introduced are based on using randomization to break the ‘curse of dimensionality’ via the synthesis of the ‘random function approximation’ and ‘empirical’ approaches. Our first algorithm is based on a random parametric function fitting by sampling parameters in each iteration. The second is based on sampling states which then yield a set of basis functions in an RKHS from the kernel. Both function fitting steps involve convex optimization problems and can be implemented with standard packages. Both algorithms can be viewed as iteration of a type of random Bellman operator followed by a random projection operator. This situation can be quite difficult to analyze. But viewed as a random operator, following [6], we can construct Markov chains that stochastically dominates the error sequences which are quite easy to analyze. They yield convergence but also non-asymptotic sample complexity bounds. Numerical experiments on the cart-pole balancing problem suggests good performance in practice. More rigorous numerical analysis will be conducted as part of future work.
A. Rahimi and B. Recht, “Uniform approximation of functions with dimensionality.” John Wiley & Sons, 2007, vol. 703.

3. R. S. Sutton and A. G. Barto, Reinforcement Learning: An introduction. Cambridge Univ Press, 1998, vol. 1, no. 1.

4. R. A. DeVore, “Nonlinear approximation.” Acta numerica, vol. 7, pp. 51–150, 1998.

5. R. Munos and C. Szepesvári, “Finite-time bounds for fitted value iteration.” The Journal of Machine Learning Research, vol. 9, pp. 815–857, 2008.

6. W. B. Haskell, R. Jain, and D. Kalathil, “Empirical dynamic programming.” Mathematics of Operations Research, vol. 41, no. 2, pp. 402–429, 2016.

7. J. Rust, “Using randomization to break the curse of dimensionality.” Econometrica: Journal of the Econometric Society, pp. 487–516, 1997.

8. D. P. De Farias and B. Van Roy, “On constraint sampling in the linear programming approach to approximate dynamic programming.” Mathematics of operations research, vol. 29, no. 3, pp. 462–478, 2004.

9. D. Ormoneit and S. Sen, “Kernel-based reinforcement learning.” Machine learning, vol. 49, no. 2-3, pp. 161–178, 2002.

10. S. Gneuwselder, G. Lever, L. Baldassarre, M. Pontil, and A. Gretton, “Modelling transition dynamics in mdps with rkhs embeddings,” arXiv preprint arXiv:1206.4655, 2012.

11. A. Rahimi and B. Recht, “Uniform approximation of functions with random bases,” in Communication, Control, and Computing, 2008 46th Annual Allerton Conference on. IEEE, 2008, pp. 555–561.

12. “Weighted sums of random kitchen sinks: Replacing minimization with randomization in learning,” in Advances in neural information processing systems, 2009, pp. 1313–1320.

13. W. B. Haskell, R. Jain, and D. Kalathil, “Empirical dynamic programming.” To appear in Mathematics of Operations Research, 2015.

14. S. Smale and D.-X. Zhou, ’Shannon sampling ii: Connections to learning theory,” Applied and Computational Harmonic Analysis, vol. 19, no. 3, pp. 285–302, 2005.

15. R. Munos, “Performance bounds in l,p-norm for approximate value iteration,” SIAM journal on control and optimization, vol. 46, no. 2, pp. 541–561, 2007.

16. M. Shaked and J. G. Shanthikumar, Stochastic Orders. Springer, 2007.

17. D. A. Levin, Y. Peres, and E. L. Wilmer, Markov Chains and Mixing Times. American Mathematical Society, 2008.

18. M. Anthony and P. L. Bartlett, Neural network learning: Theoretical foundations. cambridge university press, 2009.

19. D. Haussler, “Sphere packing numbers for subsets of the boolean n-cube with bounded vapnik-cherovenkis dimension,” Journal of Combinatorial Theory, Series A, vol. 69, no. 2, pp. 217–232, 1995.

APPENDIX

A. Supplement for Section III

The following computation shows that $T$ maps bounded functions to Lipschitz continuous functions when $Q$ and $c$ are both Lipschitz continuous in the sense of $(1)$ and $(2)$. Suppose $\|v\|_\infty \leq v_{\text{max}}$, then $T v$ is Lipschitz continuous with constant $L_c + \gamma v_{\text{max}}L_Q$. We have

$$\| [T v] (s) - [T v] (s') \| \leq L_c \max_{a \in \mathbb{A}} |c (s, a) - c (s', a) | + \gamma \max_{a \in \mathbb{A}} \int v (y) Q (dy | s, a) - v (y) Q (dy | s', a) | \leq 2 \| s - s' \|^2 + \gamma v_{\text{max}} \max_{a \in \mathbb{A}} \int \| Q (dy | s, a) - Q (dy | s', a) \|$$

$$\leq (L_c + \gamma v_{\text{max}}L_Q) \| s - s' \|^2.$$

B. Supplement for Section IV

First, we need to adapt [3] Lemma 3 to obtain point-wise error bounds on $v_{K} - v^*$ in terms of the errors $\{\varepsilon_k\}_{k \geq 0}$. These bounds are especially useful when analyzing the performance of EVI with respect to other norms besides the supremum norm, since $T$ does not have a contractive property with respect to any other norm.

For any $\pi \in \Pi$, we define the operator $Q^\pi : F (\mathbb{S}) \to F (\mathbb{S})$ (which gives the transition mapping as a function of $\pi$) via

$$\left( Q^\pi v \right) (s) \triangleq \int_{\mathbb{S}} v (y) Q (dy | s, \pi (s)) , \forall s \in \mathbb{S}.$$

Then we define the operator $T^\pi : F (\mathbb{S}) \to F (\mathbb{S})$ via

$$\left[ T^\pi v \right] (s) \triangleq c (s, \pi (s)) + \gamma \int_{\mathbb{S}} v (x) Q (dx | s, \pi (s)) , \forall s \in \mathbb{S}.$$

For later use, we let $\pi^* \in \Pi$ be an optimal policy satisfying

$$\pi^* (s) \in \arg \min_{a \in \mathbb{A}} \left\{ c (s, a) + \gamma \int v^* (x) Q (dx | s, a) \right\},$$

$\forall s \in \mathbb{S}$, i.e., it is greedy with respect to $v^*$. More generally, a policy $\pi \in \Pi$ is greedy with respect to $v \in F (\mathbb{S})$ if $T^\pi v = T v$.

For use throughout this section, we let $\pi_k$ be a greedy policy with respect to $v_k$ so that $T^\pi v_k = T v_k$ for all $k \geq 0$. Then, for fixed $K \geq 1$ we define the operators

$$A_K \triangleq \frac{1}{2} \left[ \left( (Q^\pi)^K + Q^{\pi_{K-1}} Q^{\pi_{K-2}} \ldots Q^{\pi_0} \right) , \right.$$

$$A_K \triangleq \frac{1}{2} \left[ \left( (Q^\pi)^K \right) K^{-1} + Q^{\pi_{K-1}} Q^{\pi_{K-2}} \ldots Q^{\pi_{K+1}} \right],$$

for $k = 0, \ldots, K - 1$, formed by composition of transition kernels. We let $\bar{I}$ be the constant function equal to one on $\mathbb{S}$, and we define the constant $\gamma \bar{I} = \frac{2 (1 - K + 1)}{1 - \rho}$ for use shortly.

We note that $\{A_k\}_{k=0}^K$ are all linear operators and $A_k \bar{I} \bar{I} = \bar{I}$ for all $k = 0, \ldots, K$.

Lemma 9. For any $K \geq 1$,

(i) $v_{K} - v^* \leq \sum_{k=0}^{K-1} \gamma K^{-k-1} (Q^\pi)^K (v_0 - v^*) + \gamma^K (Q^\pi^* \bar{I}) (v_0 - v^*)$.\n
(ii) $v_{K} - v^* \leq \sum_{k=0}^{K-1} \gamma K^{-k-1} (Q^\pi_{K-1} Q^{\pi_{K-2}} \ldots Q^{\pi_{K+1}}) (v_0 - v^*) + \gamma^K (Q^\pi_{K-1} Q^{\pi_{K-2}} \ldots Q^{\pi_{K+1}}) (v_0 - v^*)$.\n
(iii) $\|v_{K} - v^*\| \leq 2 \sum_{k=0}^{K-1} \gamma K^{-k-1} A_k \| \varepsilon_k \| + \gamma^K A_K (2 v_{\text{max}})$.\n
Proof. (i) For any $k \geq 1$, we have $T v_k \leq T^\pi v_k$ and $T^\pi v_k - T^\pi v^* = \gamma Q^\pi (v_k - v^*)$, so $v_{k+1} - v^* = T v_k + \varepsilon_k - T^\pi v_k + T^\pi v_k - T^\pi v^* \leq \gamma Q^\pi (v_k - v^*) + \varepsilon_k$. The result then follows by induction.

(ii) Similarly, for any $k \geq 1$, we have $T v^* \leq T^\pi v_k$ and $T v_k - T^\pi v^* = \gamma Q^\pi (v_k - v^*)$, so $v_{k+1} - v^* = T v_k + \varepsilon_k - T^\pi v_k + T^\pi v_k - T^\pi v^* \leq \gamma Q^\pi (v_k - v^*) + \varepsilon_k$. Again, the result follows by induction.

(iii) If $f \leq g \leq h \in F (\mathbb{S})$, then $|g| \leq |f| + |h|$, so combining parts (i) and (ii) gives

$$\|v_{K} - v^*\| \leq 2 \sum_{k=0}^{K-1} \gamma K^{-k-1} A_k \| \varepsilon_k \| + \gamma^K A_K (v_0 - v^*).$$
Then we note that $|v_0 - v^*| \leq 2v_{\max}$. □

Now we use Lemma 9 to derive $p$-norm bounds.

Proof. (of Lemma 4) Using $\sum_{k=0}^{K} \gamma^k = (1 - \gamma^{K+1}) / (1 - \gamma)$, we define the constants

$$
\alpha_k = \frac{(1 - \gamma)\gamma^{K-k-1}}{1 - \gamma^{K+1}}, \quad \forall k = 0, \ldots, K - 1,
$$

$$
\alpha_K = \frac{(1 - \gamma)\gamma^K}{1 - \gamma^{K+1}},
$$

and we note that $\sum_{k=0}^{K} \alpha_k = 1$. Then, we obtain $|v_K - v^*|$

$$
\leq \tilde{\gamma} \sum_{k=0}^{K-1} \alpha_k A_k |\tilde{\varepsilon}_k| + \alpha_K A_K (2v_{\max})
$$

from Lemma 9 iii). Next, we compute $\|v_K - v^*\|_p, \rho =$

$$
\int_{S} |v_K (s) - v^* (s)|^p \rho (ds)
$$

$$
\leq \tilde{\gamma}^p \int_{S} \left[ \sum_{k=0}^{K-1} \alpha_k A_k |\tilde{\varepsilon}_k| + \alpha_K A_K (2v_{\max}) \right]^p (s) \rho (ds)
$$

$$
\leq \tilde{\gamma}^p \int_{S} \left[ \sum_{k=0}^{K-1} \alpha_k A_k |\tilde{\varepsilon}_k| + \alpha_K A_K (2v_{\max}) \right]^p (s) \rho (ds),
$$

using Jensen’s inequality and convexity of $x \rightarrow |x|^p$. Now, we have $\rho A_k \leq c_{\rho, \mu} (K-k-1) \mu$ for $k = 0, \ldots, K - 1$ by Assumption 4 ii) and so for all $k = 0, \ldots, K - 1$,

$$
\int_{S} [A_k |\tilde{\varepsilon}_k|^p] (s) \rho (ds) \leq c_{\rho, \mu} (K-k-1) \|\tilde{\varepsilon}_k\|^p_{p, \rho}.
$$

We arrive at $\|v_K - v^*\|_p, \rho$

$$
\leq \tilde{\gamma}^p \sum_{k=0}^{K-1} \alpha_k c_{\rho, \mu} (K-k-1) \|\tilde{\varepsilon}_k\|^p_{\rho, \mu} + \alpha_K (2v_{\max})^p
$$

$$
= 2^p \tilde{\gamma}^p \sum_{k=0}^{K-1} \gamma^{K-k-1} c_{\rho, \mu} (K-k-1) \|\tilde{\varepsilon}_k\|^p_{\rho, \mu} + \gamma^K (2v_{\max})^p,
$$

where we use $|v_0 - v^*|^p \leq (2v_{\max})^p$. Now, by subadditivity of $x \rightarrow |x|^t$ for $t = 1/p \in (0, 1]$ with $p \in [1, \infty)$, assumption that $\|\tilde{\varepsilon}_k\|_{\rho, \mu} \leq \varepsilon$ for all $k = 0, 1, \ldots, K - 1$, and since $\sum_{k=0}^{K-1} \gamma^{K-k-1} c_{\rho, \mu} (K-k-1) \leq C_{\rho, \mu}$ by Assumption 4 ii), we see

$$
\|v_K - v^*\|_{p, \rho} \leq 2 \left( 1 - \frac{\gamma^{K+1}}{1 - \gamma} \right)^{K+1} \left[ C_{\rho, \mu} \varepsilon + \gamma^K (2v_{\max}) \right],
$$

which gives the desired result. □

Supremum norm error bounds follow more easily from Lemma 9.

Proof. (Proof of Lemma 5) We have

$$
\|v_K - v^*\|_{\infty} \leq \max\{ (\sum_{k=0}^{K-1} \gamma^{K-k-1} (Q^\pi)^{K-k-1} \varepsilon_k + \gamma^K (Q^\pi)^K (v_0 - v^*) \|_{\infty},
$$

$$
\sum_{k=0}^{K-1} \gamma^{K-k-1} (Q^\pi)^{K-k-1} \varepsilon_k + \gamma^K (Q^\pi)^K (v_0 - v^*) \|_{\infty},
$$

by Lemma 9. Now,

$$
\sum_{k=0}^{K-1} \gamma^{K-k-1} (Q^\pi)^{K-k-1} \varepsilon_k + \gamma^K (Q^\pi)^K (v_0 - v^*) \|_{\infty}
$$

and

$$
\sum_{k=0}^{K-1} \gamma^{K-k-1} (Q^\pi)^{K-k-1} \varepsilon_k + \gamma^K (Q^\pi)^K (v_0 - v^*) \|_{\infty}
$$

where we use the triangle inequality, the fact that $(Qf) (s) \leq \int_{S} |f (y)| Q (dy) s \leq \|f\|_{\infty}$ for any transition kernel $Q$ on $S$ and $f \in \mathcal{F}(S)$, and $|v_0 - v^*| \leq 2v_{\max}$. For any $K \geq 1$,

$$
\|v_K - v^*\|_{\infty} \leq \sum_{k=0}^{K-1} \gamma^{K-k-1} \|\tilde{\varepsilon}_k\|_{\infty} + \gamma^K (2v_{\max}),
$$

(9)

Follows immediately since $\sum_{k=0}^{K-1} \gamma^{K-k-1} \varepsilon \leq \varepsilon / (1 - \gamma)$ for all $K \geq 1$. □

We emphasize that Lemma 5 does not require any assumptions on the transition probabilities, in contrast to Lemma 4 which requires Assumption 4 ii).

C. Supplement for Section 11 Function approximation

We record several pertinent results here on the type of function reconstruction used in our EVI algorithms. The first lemma is illustrative of approximation results in Hilbert spaces, it gives an $O\left(1/\sqrt{J}\right)$ convergence rate on the error from using $\tilde{F} (\theta^{1,J})$ compared to $F(\Theta)$ in $L_2, \mu (S)$ in probability.

Lemma 10. [12] Lemma 1] Fix $f^* \in \mathcal{F}(\Theta)$, for any $\delta \in (0, 1)$ there exists a function $\tilde{f} \in \tilde{F}(\theta^{1,J})$ such that

$$
\|f^* - \tilde{f}\|_{2, \mu} \leq C \frac{\delta}{\sqrt{J}} \left( 1 + \sqrt{2 \log \frac{1}{\delta}} \right)
$$

with probability at least $1 - \delta$.

The next result is an easy consequence of [12] Lemma 1] and bounds the error from using $\tilde{F}(\theta^{1,J})$ compared to $F(\Theta)$ in $L_1, \mu (S)$. 

Lemma 11. Fix $f^* \in \mathcal{F}(\Theta)$, for any $\delta \in (0, 1)$ there exists a function $\hat{f} \in \mathcal{F}(\Theta^{1:J})$ such that

$$\|\hat{f} - f^*\|_{1, \mu} \leq \frac{C \sqrt{J}}{\sqrt{\delta}} \left(1 + \sqrt{2 \log \frac{1}{\delta}}\right)$$

with probability at least $1 - \delta$.

Proof. Choose $f, g \in \mathcal{F}(\mathbb{S})$, then by Jensen’s inequality we have

$$\|f - g\|_{1, \mu} = \mathbb{E}_\mu \|f(S) - g(S)\|$$

$$= \mathbb{E}_\mu \left[\left((f(S) - g(S))^2\right)^{1/2}\right] \leq \sqrt{\mathbb{E}_\mu \left[(f(S) - g(S))^2\right]}.$$

The desired result then follows by [12] Lemma 1.

Now we consider function approximation in the supremum norm. Recall the definition of the regression function

$$f_M(s) = \mathbb{E} \left[ \min_{a \in A} \left\{ c(s, a) + \frac{M}{\gamma} \sum_{m=1}^{M} v(X_m^a) \right\} \right],$$

$\forall s \in \mathbb{S}$. Then we have the following approximation result, for which we recall the constant $\kappa := \sup_{s \in \mathbb{S}} \sqrt{K}(s, s)$.

Corollary 12. [14] Corollary 5 For any $\delta \in (0, 1),

$$\|f_M - f\|_{\mathcal{H}_K} \leq \hat{C} \kappa \left(\frac{\log(4/\delta)^2}{N}\right)^{1/6} \text{ for } \lambda = \left(\frac{\log(4/\delta)^2}{N}\right)^{1/3},$$

with probability at least $1 - \delta$.

Proof. Uses the fact that for any $f \in \mathcal{H}_K$, $\|f\|_{\infty} \leq \kappa \|f\|_{\mathcal{H}_K}$. For any $s \in \mathbb{S}$, we have $|f(s)| = |\langle K(s, \cdot), f(\cdot) \rangle_{\mathcal{H}_K}|$ and subsequently

$$|\langle K(s, \cdot), f(\cdot) \rangle_{\mathcal{H}_K}| \leq \|K(s, \cdot)\|_{\mathcal{H}_K} \|f\|_{\mathcal{H}_K} = \sqrt{K(s, \cdot), K(s, \cdot)}_{\mathcal{H}_K} \|f\|_{\mathcal{H}_K} = \sqrt{K(s, s)} \|f\|_{\mathcal{H}_K} \leq \sup_{s, a} \sqrt{K(s, s)} \|f\|_{\mathcal{H}_K},$$

where the first inequality is by Cauchy-Schwarz and the second is by assumption that $K$ is a bounded kernel.

The preceding result is about the error when approximating the regression function $f_M$, but $f_M$ generally is not equal to $T v$. We bound the error between $f_M$ and $T v$ as well in the next subsection.

D. Bellman error

The layout of this subsection is modeled after the arguments in [12], but with the added consideration of randomized function fitting. We use the following easy-to-establish fact.

Fact 13. Let $X$ be a given set, and $f_1 : X \to \mathbb{R}$ and $f_2 : X \to \mathbb{R}$ be two real-valued functions on $X$. Then,

(i) $|\inf_{x \in X} f_1(x) - \inf_{x \in X} f_2(x)| \leq \sup_{x \in X} |f_1(x) - f_2(x)|$, and

(ii) $|\sup_{x \in X} f_1(x) - \sup_{x \in X} f_2(x)| \leq \sup_{x \in X} |f_1(x) - f_2(x)|$.

For example, Fact [13] can be used to show that $T$ is contractive in the supremum norm.

The next result is about $\hat{T}$, it uses Hoeffding’s inequality to bound the estimation error between $\{\hat{v}(s_n)\}_{n=1}^N$ and $\{Tv(s_n)\}_{n=1}^N$ in probability.

Lemma 14. For any $p \in [1, \infty)$, $f, v \in \mathcal{F}(\mathbb{S}; \nu_{\max})$, and $\varepsilon > 0$, $\Pr\{||f - T v||_{p, \mu} - ||f - \hat{v}||_{p, \mu} > \varepsilon\}$

$$\leq 2N |A| \exp\left(-2M \frac{\varepsilon^2}{\nu_{\max}^2}\right).$$

Proof. First we have $||f - T v||_{p, \mu} - ||f - \hat{v}||_{p, \mu} \leq ||Tv - \hat{v}||_{p, \mu}$ by the reverse triangle inequality. Then, for any $s \in \mathbb{S}$ we have $||Tv(s) - \hat{v}(s)||$

$$\leq \gamma \max_{a \in A} \left| \frac{\gamma}{M} \sum_{m=1}^{M} v(X_m^a) \right|$$

by Fact [13]. We may also take $v(s) \in [0, \nu_{\max}]$ for all $s \in \mathbb{S}$ by assumption on the cost function, so by the Hoeffding inequality and the union bound we obtain

$$\Pr\left\{ \max_{n=1, \ldots, N} \left| \frac{1}{N} \sum_{n=1}^{N} v(s_n) - \hat{v}(s_n) \right| \geq \varepsilon \right\} \leq 2N |A| \exp\left(-2M \frac{\varepsilon^2}{\nu_{\max}^2}\right)$$

and thus

$$\Pr\{||Tv - \hat{v}||_{p, \mu} \geq \varepsilon\} = \Pr\left\{ \left(\frac{1}{N} \sum_{n=1}^{N} ||Tv(s_n) - \hat{v}(s_n)||^p \right)^{1/p} \geq \varepsilon \right\} \leq \Pr\left\{ \max_{n=1, \ldots, N} ||Tv(s_n) - \hat{v}(s_n)|| \geq \varepsilon \right\},$$

which gives the desired result.

To continue, we introduce the following additional notation corresponding to a set of functions $\mathcal{F} \subset \mathcal{F}(\mathbb{S})$:

- $\mathcal{F}(s^{1:N}) \triangleq \{(f(s_1), \ldots, f(s_N)) : f \in \mathcal{F}\}$;

- $\mathcal{N}(\varepsilon, \mathcal{F}(s^{1:N}))$ is the $\varepsilon$–covering number of $\mathcal{F}(s^{1:N})$ with respect to the $1$–norm on $\mathbb{R}^N$.

The next lemma uniformly bounds the estimation error between the true expectation and the empirical expectation over the set $\mathcal{F}(\Theta^{1:J})$ (in the following statement, $e$ is Euler’s number).

Lemma 15. For any $\varepsilon > 0$ and $N \geq 1$

$$\Pr\left\{ \sup_{f \in \mathcal{F}(\Theta^{1:J})} \left| \frac{1}{N} \sum_{n=1}^{N} f(s_n) - \mathbb{E}_\mu [f(S)] \right| > \varepsilon \right\} \leq 8e(J + 1) \left(\frac{2J \nu_{\max}}{\varepsilon}\right)^J \exp\left(-\frac{N \varepsilon^2}{128 \nu_{\max}^2}\right).$$

Proof. For any $\mathcal{F} \subset \mathcal{F}(\mathbb{S}; \nu_{\max}), \varepsilon > 0$, and $N \geq 1$, we have
It remains to bound \( \Pr \left\{ \sup_{f \in \mathcal{F}(\theta^{1:J})} \left| \frac{1}{N} \sum_{n=1}^{N} f(S_n) - E_{\mu}[f(S)] \right| > \varepsilon \right\} \)

\[ \leq 8 \mathbb{E} \left[ N \left( \varepsilon/8, \tilde{F}(\theta^{1:J}) (s^{1:N}) \right) \right] \exp \left( \frac{-N \varepsilon^2}{128 v_{\max}^2} \right) . \]

It remains to bound \( \mathbb{E} \left[ N \left( \varepsilon/8, \tilde{F}(\theta^{1:J}) (s^{1:N}) \right) \right] \). We note that \( \tilde{F}(\theta^{1:J}) \) is a subset of

\[ \left\{ f(\cdot) = \sum_{j=1}^{J} \alpha_j \phi(\cdot; \theta_j) \mid (\alpha_1, \ldots, \alpha_J) \in \mathbb{R}^J \right\} , \]

which is a vector space with dimension \( J \). By [18 Corollary 11.5], the pseudo-dimension of \( \tilde{F}(\theta^{1:J}) \) is bounded above by \( J \). Furthermore,

\[ N \left( \varepsilon, \tilde{F}(\theta^{1:J}) (s^{1:N}) \right) \leq e(J + 1) \left( \frac{2 \epsilon v_{\max}}{\varepsilon} \right)^J \]

by [19 Corollary 3] which gives the desired result.

To continue, we let \( v' = v' (v, N, M, J, \mu, \nu) \) denote the (random) output of one iteration of EVI applied to \( v \in \mathcal{F}(S) \) as a function of the parameters \( N, M, J \geq 1 \) and the probability distributions \( \mu \) and \( \nu \). The next theorem bounds the error between \( Tv \) and \( v' \) in one iteration of EVI with respect to \( L_{1,\mu} (S) \), it is a direct adaptation of [5 Lemma 1] modified to account for the randomized function fitting and the effective function space being \( \mathcal{F}(\theta^{1:J}) \).

**Theorem 16.** Choose \( v \in \mathcal{F}(S; v_{\max}), \varepsilon > 0, \) and \( \delta \in (0, 1) \). Also choose \( J \geq \left( \frac{5C}{\varepsilon} \left( 1 + \sqrt{2 \log \frac{5}{\delta}} \right) \right)^2 \), \( N \geq 2^{75^2 v_{\max}^2} \log \left( \frac{40 \epsilon (J + 1)}{\delta} \right) (10 \epsilon v_{\max})^J \), and \( M \geq \left( \frac{v_{\max}^2}{2} \right)^2 \log \left( \frac{10 N |A|}{\delta} \right) \). Then, for \( v' = v'(v, N, M, J, \mu, \nu) \) we have \( \| v' - T v \|_1, \mu \leq d_{1,\mu} (Tv, F(\Theta)) + \varepsilon \) with probability at least \( 1 - \delta \).

**Proof.** Let \( \varepsilon > 0 \) be arbitrary and choose \( f^* \in \mathcal{F}(\Theta) \) such that \( \| f^* - T v \|_1, \mu \leq \inf_{f \in \mathcal{F}(\Theta)} \| f - T v \|_1, \mu + \varepsilon' \). Then, choose \( f \in \tilde{F}(\theta^{1:J}) \) such that \( \| f - T v \|_1, \mu \leq \| f^* - T v \|_1, \mu + \varepsilon'/5 \) with probability at least \( 1 - \delta/5 \) by Lemma [11] by choosing \( J \geq 1 \) to satisfy

\[ \frac{C}{\sqrt{J}} \left( 1 + \sqrt{2 \log \frac{1}{(\delta/5)}} \right) \leq \varepsilon/5 \]

\( \Rightarrow J \geq \left( \frac{5C}{\varepsilon} \left( 1 + \sqrt{2 \log \frac{5}{\delta}} \right) \right)^2 . \)

Now consider the inequalities:

\[ \| v' - T v \|_1, \mu \leq \| v' - T v \|_1, \mu + \varepsilon/5 \]

\[ \leq \| v' - \tilde{v} \|_1, \mu + 2 \varepsilon/5 \]

\[ \leq \| \tilde{f} - \tilde{v} \|_1, \mu + 2 \varepsilon/5 \]

\[ \leq \| \tilde{f} - T v \|_1, \mu + 3 \varepsilon/5 \]

\[ \leq \| f - T v \|_1, \mu + 4 \varepsilon/5 \]

\[ \leq \| f^* - T v \|_1, \mu + \varepsilon \]

\[ \leq d_{1,\mu} (Tv, F(\Theta)) + \varepsilon + \varepsilon' \].

First, note that inequality [12] is immediate since \( \| v' - \tilde{v} \|_1, \mu \leq \| f - \tilde{v} \|_1, \mu \) for all \( f \in \mathcal{F}(\theta^{1:J}) \) by the choice of \( v' \) as the minimizer in Step 3 of Algorithm 1. Second, inequalities [10] and [14] follow from Lemma [15] by choosing \( N \geq 1 \) to satisfy

\[ 8 e (J + 1) \left( \frac{2 \epsilon v_{\max}}{\varepsilon/5} \right)^J \exp \left( \frac{-N \varepsilon^2}{128 v_{\max}^2} \right) \leq \frac{\delta}{5} \]

\[ \Rightarrow N \geq 2^{75^2 v_{\max}^2} \log \left[ \frac{40 e (J + 1)}{\delta} \right] (10 e v_{\max})^J \).

Third, inequality [16] follows from the choice of \( f^* \in \mathcal{F} \). Finally, inequalities [11] and [13] follow from Lemma [14] by choosing \( M \geq 1 \) to satisfy

\[ 2 N |A| \exp \left( -2 M \frac{\varepsilon^2}{v_{\max}^2} \right) \leq \frac{\delta}{5} \]

\[ \Rightarrow M \geq \left( \frac{v_{\max}^2}{2 \varepsilon^2} \right)^2 \log \left[ \frac{10 N |A|}{\delta} \right] . \]

Since \( \varepsilon' \) was arbitrary, the desired result then follows by the union bound.

Using similar steps as Theorem [16] the next theorem bounds the error in one iteration of EVI with respect to \( L_{2,\mu} (S) \).

**Theorem 17.** Choose \( v \in \mathcal{F}(S; v_{\max}), \varepsilon > 0, \) and \( \delta \in (0, 1) \). Also choose \( J \geq \left( \frac{5C}{\varepsilon} \left( 1 + \sqrt{2 \log \frac{5}{\delta}} \right) \right)^2 \), \( N \geq 2^{75^2 v_{\max}^2} \log \left( \frac{40 \epsilon (J + 1)}{\delta} \right) (10 \epsilon v_{\max})^J \), and \( M \geq \left( \frac{v_{\max}^2}{2} \right)^2 \log \left( \frac{10 N |A|}{\delta} \right) \). Then, for \( v' = v'(v, N, M, J, \mu, \nu) \) we have \( \| v' - T v \|_2, \mu \leq d_{2,\mu} (Tv, F(\Theta)) + \varepsilon \) with probability at least \( 1 - \delta \).

In the next lemma we show that we can make the bias between the regression function \( f_M \) and the Bellman update \( T v \) arbitrarily small uniformly over \( s \in S \) through the choice of \( M \geq 1 \).

**Lemma 18.** For any \( \varepsilon > 0 \) and \( M \geq 1 \),

\[ \| f_M - T v \|_\infty \leq \gamma \left[ \varepsilon + 2 |A| \exp \left( -2 M \frac{\varepsilon^2}{v_{\max}^2} \right) (v_{\max} - \varepsilon) \right] . \]
Proof. For any $s \in S$, we compute

$$|f_M(s) - \langle T v \rangle(s)|$$

$$\leq \mathbb{E} \left[ \min_{a \in \mathcal{A}} \left\{ c(s, a) + \frac{\gamma}{M} \sum_{m=1}^{M} v(X_m | s, a) \right\} \right]$$

$$\leq \min_{a \in \mathcal{A}} \left\{ c(s, a) + \frac{\gamma}{M} \mathbb{E}_{X \sim Q(\cdot | s, a)}[v(X)] \right\}$$

$$\leq \gamma \mathbb{E} \left[ \max_{a \in \mathcal{A}} \left\{ \frac{1}{M} \sum_{m=1}^{M} v(X_m | s, a) - \mathbb{E}_{X \sim Q(\cdot | s, a)}[v(X)] \right\} \right]$$

$$\leq \gamma \left[ \varepsilon + 2|\mathcal{A}| \exp \left( -\frac{2M \varepsilon^2}{\nu_{\max}^2} \right) (v_{\max} - \varepsilon) \right],$$

where the second inequality follows from Fact 13 and the third is by the Hoeffding inequality. \hfill \Box

We make use of the following fact for the one step Bellman error in the supremum norm.

**Theorem 19.** Fix $v \in \mathcal{F}(S; v_{\max})$, $\varepsilon > 0$, and $\delta \in (0, 1)$. Also choose $N \geq \left( \frac{2 \gamma K \kappa}{\varepsilon} \right)^{6} \log \left( \frac{(4\delta)^2}{4 - 2\gamma} \right) \frac{\varepsilon^2}{2(\varepsilon/4)^2}$, where $C_K$ is a constant independent of the dimension of $S$. Then for

$$\hat{f}_\lambda \doteq \arg \min_{f \in \mathcal{H}_K} \left\{ \frac{1}{N} \sum_{n=1}^{N} (f(S_n) - Y_n)^2 + \lambda \|f\|_{\mathcal{H}_K}^2 \right\},$$

we have $\|\hat{f}_\lambda - T v\|_{\infty} \leq \varepsilon$ with probability at least $1 - \delta$.

Proof. By the triangle inequality, $\|\hat{f}_\lambda - T v\|_{\infty} \leq \|\hat{f}_\lambda - f_M\|_{\infty} + \|f_M - T v\|_{\infty}$. We choose $N \geq 1$ to satisfy

$$C_K \frac{\log \left( \frac{(4\delta)^2}{N} \right)}{\varepsilon} \leq \frac{\varepsilon}{2} \Rightarrow N \geq \left( \frac{2 \gamma K \kappa}{\varepsilon} \right)^{6} \log \left( \frac{(4\delta)^2}{4 - 2\gamma} \right) \frac{\varepsilon^2}{2(\varepsilon/4)^2},$$

so that $\|\hat{f}_\lambda - f_M\|_{\infty} \leq \varepsilon/2$ with probability at least $1 - \delta$ by [13] Corollary 5 and the fact that $\|f\|_{\infty} \leq \kappa \|f\|_{\mathcal{H}_K}$. Then, we choose $M \geq 1$ to satisfy

$$\gamma \left[ \varepsilon/4 + 2|\mathcal{A}| \exp \left( -\frac{2M \varepsilon^2}{\nu_{\max}^2} \right) (v_{\max} - \varepsilon/4) \right] \leq \frac{\varepsilon}{2}$$

$$\Rightarrow M \geq \frac{\nu_{\max}^2}{2(\varepsilon/4)^2} \log \left( \frac{4|\mathcal{A}| \gamma (v_{\max} - \varepsilon/4)}{(4 - 2\gamma) \varepsilon} \right),$$

so that $\|f_M - T v\|_{\infty} \leq \varepsilon/2$ by Lemma 18. \hfill \Box

**Rahul Jain** is an associate professor and the K. C. Dahlberg Early Career Chair in the EE department at the University of Southern California. He received his PhD in EECS and an MA in Statistics from the University of California, Berkeley, his B.Tech from IIT Kanpur. He is winner of numerous awards including the NSF CAREER award, an IBM Faculty award and the ONR Young Investigator award. His research interests span stochastic systems, statistical learning, queueing systems and game theory with applications in communication networks, power systems, transportation and healthcare.

**Hiteshi Sharma** is a Ph.D. student at USC with particular interests in approximate dynamic programming, reinforcement learning and stochastic optimization. Prior to enrolling at USC, she was at IIT Bombay where she worked on spectrum sharing in cognitive radio networks.

**Pengqian Yu** was born in Yunnan, China, in 1989. He received the B.S. degree in mechanical design, manufacturing and automation from the College of Mechanical Engineering, Chongqing University, Chongqing, China, in 2012, and the Ph.D. degree in the Department of Mechanical Engineering at the National University of Singapore (NUS), in 2016. He is currently a Postdoctoral Research Fellow of the Department of Industrial Systems Engineering and Management at NUS.

**William Haskell** received his B.S. degree in Mathematics and M.S. degree in Econometrics from the University of Massachusetts Amherst in 2006, and his Ph.D. degree in Operation Research from the University of California Berkeley in 2011. Currently, he is an assistant professor in the Department of Industrial Systems Engineering and Management at the National University of Singapore. His research interests include convex optimization, risk-aware decision making, and dynamic programming.