The strong 3-rainbow index of edge-comb product of a path and a connected graph

Zata Yumni Awanis\textsuperscript{a}, A.N.M. Salman\textsuperscript{a}, Suhadi Wido Saputro\textsuperscript{a}

\textsuperscript{a}Combinatorial Mathematics Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesa 10, Bandung 40132, Indonesia

\texttt{zata.yumni@s.itb.ac.id, msalman@math.itb.ac.id, suhadi@math.itb.ac.id}

Abstract

A tree in an edge-colored connected graph $G$ is a rainbow tree if all of its edges have different colors. Let $k$ be an integer with $2 \leq k \leq n$ and $S$ be a $k$-subset of $V(G)$. The strong $k$-rainbow index $sr_xk(G)$ of $G$ is the smallest number of colors required in an edge-coloring of $G$ such that every set $S$ in $G$ is connected by a rainbow tree with minimum size. In this paper, we investigate the $sr_x3$ of edge-comb product of a path and a connected graph, denoted by $P_n \triangleright_e H$. It is obvious that the natural upper bound for $sr_x3(P_n \triangleright_e H)$ is $|E(P_n \triangleright_e H)|$. Hence, we first provide graphs $H$ with $sr_x3(P_n \triangleright_e H) = |E(P_n \triangleright_e H)|$, then provide a sharper upper bound for $sr_x3(P_n \triangleright_e H)$ where $sr_x3(P_n \triangleright_e H) \neq |E(P_n \triangleright_e H)|$. We also provide the exact values of $sr_x3(P_n \triangleright_e H)$ for some graphs $H$.

Keywords: edge-comb product, rainbow coloring, rainbow Steiner tree, strong 3-rainbow index

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1. Introduction

Throughout this paper, all graphs are finite, simple, and connected. The terminology and notation refer to Diestel [11]. For simplifying, we define a set $[a, b] = \{x : a \leq x \leq b\}$. Let $G(V, E)$ be an edge-colored graph of order $n \geq 3$. A tree in $G$ is a rainbow tree if all of its edges have different colors. Let $k$ be an integer with $k \in [2, n]$. The smallest number of colors required in an edge-coloring of $G$ such that every $k$-subset $S$ of $V(G)$ is connected by a rainbow tree is
called the $k$-rainbow index $rx_k(G)$ of $G$. These concepts were first proposed by Chartrand et al. in 2010 [9]. If $k = 2$, then $rx_2(G) = rc(G)$, where $rc(G)$ denotes the rainbow connection number of $G$ [8]. Hence, $rc(G) = rx_2(G) \leq rx_3(G) \leq \cdots \leq rx_n(G)$. Caro et al. [6] conjectured that determining the rainbow connection number of graphs is an NP-Hard problem. Chakraborty et al. in [7] then confirmed this conjecture. Therefore, the determination of rainbow connection number is mostly done by limiting the study to certain classes of graphs. The readers can see [8, 12, 14, 15, 16, 18, 19, 20] for more results about the rainbow connection number of graphs.

The concept of $k$-rainbow index has useful and interesting applications in the security of a communication network. Suppose that every $k$ people are expected to communicate and exchange information securely. To achieve this, we can assign passwords to the line which connects them (which may have other people as intermediaries) so that no passwords are repeated. Since the economic aspect is taken into consideration, the number of passwords that being used are expected to be as minimum as possible. The $k$-rainbow index represents the smallest number of these distinct passwords.

For two vertices $x, y \in V(G)$, the length of a shortest $x - y$ path in $G$ is called the distance between $x$ and $y$, denoted by $d(x, y)$. The largest distance between two vertices of $G$ is called the diameter of $G$, denoted by $diam(G)$. The Steiner distance $d(S)$ of $S$ is the minimum size of a tree containing $S$. The $k$-Steiner diameter of $G$, denoted by $sdiam_k(G)$, is the maximum Steiner distance of $S$ among all sets $S$ in $G$. If $S = \{x, y\}$, then $d(S) = d(x, y)$ and $sdiam_2(G) = diam(G)$. Chartrand et al. [9] stated that for any graph $G$ of order $n \geq 3$ and each integer $k \in [3, n]$, $k - 1 \leq sdiam_k(G) \leq rx_k(G) \leq n - 1$. They also determined the $rx_k$ of trees and cycles, where the $rx_k$ of trees is equal to the upper bound for $rx_k(G)$. The first and second authors [4] investigated the $rx_3$ of amalgamation of some graphs of diameter 2, meanwhile Liu and Hu [17] investigated the $rx_3$ of three graph product operations, which are strong product, Cartesian product, and lexicographic product. Some other results about $rx_k$ of graphs can be found in [9, 10, 13, 15, 16].

In [3], we proposed a new concept called a strong $k$-rainbow index. An edge-coloring of $G$ is called a strong $k$-rainbow coloring if every set $S$ in $G$ is connected by rainbow tree of size $d(S)$. Such a tree is called a rainbow Steiner $S$-tree. A rainbow Steiner $S$-tree is called a rainbow $x - y$ geodesic if $S = \{x, y\}$ [8]. The strong $k$-rainbow index of $G$, denoted by $srx_k(G)$, is the smallest number of colors required such that $G$ admits a strong $k$-rainbow coloring. Following the definition, $rx_k(G) \leq srx_k(G)$. If $k = 2$, then $srx_2(G) = src(G)$, where $src(G)$ denotes the strong rainbow connection number of $G$ [8]. Therefore, $src(G) = srx_2(G) \leq srx_3(G) \leq \cdots \leq srx_n(G)$ for any graph $G$ of order $n \geq 2$. Chartrand et al. [8] stated that $diam(G) \leq rc(G) \leq src(G) \leq |E(G)|$.

It is clearly that for any connected graph $G$, the strong $k$-rainbow index is defined for $G$ since every edge-coloring that assigns different colors to the edges of $G$ is a strong $k$-rainbow coloring. Thus, we have

$$sdiam_k(G) \leq rx_k(G) \leq srx_k(G) \leq |E(G)|.$$  \hspace{1cm} (1)

Graph operations have an important rule in making a larger and complex communication network. Hence, we investigated the $srx_3$ of amalgamation and comb product of some graphs. We
also investigated the $srx_3$ of some certain graphs (see [1, 2, 3]). The following theorems are needed.

**Theorem 1.1.** [3] Let $T_n$ be a tree of order $n \geq 3$. Then $srx_3(T_n) = |E(T_n)| = n - 1$.

**Theorem 1.2.** [3] Let $L_n$ be a ladder graph of order $2n$ ($n \geq 3$). Then $srx_3(L_n) = n$.

**Theorem 1.3.** [3] Let $K_{n,n}$ be a regular complete bipartite graph of order $2n$ ($n \geq 3$). Then $srx_3(K_{n,n}) = n$.

**Theorem 1.4.** [3] Let $C_n$ be a cycle of order $n \geq 3$. Then
\[
srx_3(C_n) = \begin{cases} 
2, & \text{for } n = 3; \\
n - 2, & \text{for } n \in \{4, 5, 6, 8\}; \\
n, & \text{otherwise.}
\end{cases}
\]

Figure 1 illustrates the strong 3-rainbow colorings of $C_n$ for $n \in [3, 6]$ and $n = 8$.

**Theorem 1.5.** [1] Let $F_n$ be a fan of order $n + 1$ ($n \geq 3$). Then
\[
srx_3(F_n) = \begin{cases} 
3, & \text{for } n = 4; \\
\left\lceil \frac{n}{2} \right\rceil, & \text{otherwise.}
\end{cases}
\]

The following definition of edge-comb product of two graphs is referred to [5]. Given an undirected graph $G$, an **orientation** of $G$ is an assignment of a direction to every edge of $G$. Let $G$ and $H$ be two connected graphs. Let $O$ be an orientation of $G$ and $\vec{e}$ be an oriented edge of $H$. The **edge-comb product** of $G$ and $H$ on $\vec{e}$ (under the orientation $O$), denoted by $G^o \searrow_{\vec{e}} H$, is a graph formed by taking one copy of $G$ and $|E(G)|$ copies of $H$ and identifying the $i$-th copy of $H$ at the edge $\vec{e}$ to the $i$-th edge of $G$, where the two edges have the same orientation.

In this paper, we investigate the strong 3-rainbow index of $P_n^o \searrow_{\vec{e}} H$. In Section 2, we first provide graphs $H$ with $srx_3(P_n^o \searrow_{\vec{e}} H) = |E(P_n^o \searrow_{\vec{e}} H)|$, then we provide a sharper upper bound for $srx_3(P_n^o \searrow_{\vec{e}} H)$. In Section 3, we determine the exact value of $srx_3(P_n^o \searrow_{\vec{e}} H)$ for some graphs $H$. In Section 4, we give concluding remarks and some open problems for further investigation.
2. Sharp upper bound for $sr x_3(P_n^o \triangleright_e H)$

For two integers $n, m \geq 3$, let $P_n^o$ be a path $P_n = v_1v_2 \ldots v_n$ of order $n$ with orientation $O$, where every edge of $P_n$ has an orientation from $v_i$ to $v_{i+1}$ for each $i \in [1, n-1]$, and $H$ be a connected graph of order $m$ with $V(H) = \{w_1, w_2, \ldots, w_m\}$ and $\overrightarrow{e} = w_aw_b$ be an oriented edge of $H$ which has an orientation from $w_a$ to $w_b$. Now, we consider graphs $P_n^o \triangleright_e H$. For $i \in [1, n-1]$, let the $i$-th copy of $H$ is denoted by $H^i$ with $V(H^i) = \{v_1^i, v_2^i, \ldots, v_m^i\}$ and $E(H^i) = \{v_i^pv_i^q: p, q \in [1, m] \text{ and } w_pw_q \in E(H)\}$. We define $V(P_n^o \triangleright_e H) = \bigcup_{i=1}^{n-1} V(H^i)$ and $E(P_n^o \triangleright_e H) = \bigcup_{i=1}^{n-1} E(H^i)$, where $v_i^p = v_i$ and $v_i^q = v_{i+1}$ for each $i \in [1, n-1]$.

Let $X \subseteq E(P_n^o \triangleright_e H)$. For further discussion, if $c$ is a strong 3-rainbow coloring of $P_n^o \triangleright_e H$, then the set of colors assigned to the edges in $X$ is denoted by $c(X)$. By considering any three vertices of $P_n$ and using Theorem 1.1, we have

$$|c(E(P_n))| = n - 1. \quad (2)$$

According to (1), the natural upper bound for $sr x_3(P_n^o \triangleright_e H)$ is $|E(P_n^o \triangleright_e H)|$. The following theorem shows that $sr x_3(P_n^o \triangleright_e T_m) = |E(P_n^o \triangleright_e T_m)|$.

**Theorem 2.1.** For two integers $n, m \geq 3$, let $P_n$ and $T_m$ be a path of order $n$ and a tree of order $m$, respectively. Let $\overrightarrow{e}$ be any oriented edge of $T_m$. Then $sr x_3(P_n^o \triangleright_e T_m) = (m - 1)(n - 1)$.

**Proof.** Note that $P_n^o \triangleright_e T_m$ is a tree with $|E(P_n^o \triangleright_e T_m)| = |E(T_m)|(n - 1)$, thus $sr x_3(P_n^o \triangleright_e T_m) = |E(P_n^o \triangleright_e T_m)| = |E(T_m)|(n - 1) = (m - 1)(n - 1)$ by Theorem 1.1. $\square$

Following theorem above, a natural thought arises: Is there any nontrivial graph $H$ of order $m$ besides a tree with $sr x_3((P_n^o \triangleright_e H) = |E(P_n^o \triangleright_e H)|$? The next theorem provides the characterization of connected graphs $H$ with $sr x_3((P_n^o \triangleright_e H) = |E(P_n^o \triangleright_e H)|$.

**Theorem 2.2.** For two integers $n, m \geq 3$, let $P_n$ and $H$ be a path of order $n$ and a connected graph of order $m$, respectively. Let $\overrightarrow{e}$ be any oriented edge of $H$. Then $H$ is a tree if and only if $sr x_3(P_n^o \triangleright_e H) = |E(P_n^o \triangleright_e H)|$.

**Proof.** Let $H$ be a tree. Then by using Theorem 2.1, $sr x_3(P_n^o \triangleright_e T_m) = |E(P_n^o \triangleright_e T_m)|$.

Conversely, let $H$ be a connected graph with $sr x_3(P_n^o \triangleright_e T_m) = |E(P_n^o \triangleright_e T_m)|$ and not a tree. Hence, graph $H$ contains cycles. Let $g \geq 3$ be the girth of $H$. For $i \in [1, n-1]$, let $C_g^i$ be a cycle of length $g$ in $H^i$. Since $n \geq 3$, consider graphs $H^1$ and $H^2$. For each $i \in [1, 2]$, relabeling vertices of $H^i$ such that $V(C_g^i) = \{v_1^i, v_2^i, \ldots, v_g^i\}$, $E(C_g^i) = \{v_p^iv_{p+1}^i: p \in [1, g] \text{ and } v_{p+1}^i = v_1^i\}$, and $d_{H^i}(v_2, v_1^i) \leq d_{H^i}(v_2, v_p^i)$ for all $p \in [2, g]$. For further discussion, let $d_{H^i}(v_2, v_p^i) = l_p^i$ for each $i \in [1, 2]$ and $p \in [1, g]$. Thus, by assumption, we have $l_p^1 \in [l_1^1, l_1^1 + p - 1]$ for $p \in [1, \lfloor \frac{g}{2} \rfloor + 1]$ and $l_p^2 \in [l_1^2, l_1^2 + g - p + 1]$ for $p \in [\lfloor \frac{g}{2} \rfloor + 2, g]$.

The following two cases show that there is an edge of $C_g^i$ for each $i \in [1, 2]$ which is not contained in a $v_2 - v_{t}^i$ geodesic for any $t \in [1, g]$.

**Case 1.** $v_2 \in V(C_g^i)$ for some $i \in [1, 2]$ 

It means \( v_2 = v_1^1 \). Thus, we have \( l_i^p = p - 1 \) for \( p \in [1, \lfloor \frac{g}{2} \rfloor + 1] \) and \( l_i^{p} = g - p + 1 \) for \( p \in [\lceil \frac{g}{2} \rceil + 2, g] \). If \( g \) is odd, then \( v_i^{\lfloor \frac{g}{2} \rfloor + 1}, v_i^{\lfloor \frac{g}{2} \rfloor + 2} \) is not contained in a \( v_2 - v_i^t \) geodesic for any \( t \in [1, g] \). If \( g \) is even, then there are two \( v_2 - v_i^{\lfloor \frac{g}{2} \rfloor + 1} \) geodesics, one path contains \( v_i^{\lfloor \frac{g}{2} \rfloor + 1}, v_i^{\lfloor \frac{g}{2} \rfloor + 2} \) and another path contains \( v_i^{\lfloor \frac{g}{2} \rfloor + 1}, v_i^{\lfloor \frac{g}{2} \rfloor + 2} \). Hence, we can choose \( v_i^{\lfloor \frac{g}{2} \rfloor + 1}, v_i^{\lfloor \frac{g}{2} \rfloor + 2} \) to be an edge that is not contained in a \( v_2 - v_i^t \) geodesic for any \( t \in [1, g] \).

**Case 2.** \( v_2 \notin V(C_g^i) \) for some \( i \in [1, 2] \)

We define several sets as follows.

- For odd \( g \), let \( V_i^{1,1} \) be a set of \((v_i^p, v_i^q)\) such that \( v_i^p, v_i^q \in V(C_g^i) \) and \( l_i^p = l_i^q \) for distinct \( p, q \in [1, \lceil \frac{g}{2} \rceil + 1] \), and \( V_i^{1,2} \) be a set of \((v_i^p, v_i^q)\) such that \( v_i^p, v_i^q \in V(C_g^i) \) and \( l_i^p = l_i^q \) for distinct \( p, q \in \{1\} \cup [\lceil \frac{g}{2} \rceil + 2, g] \).

- For even \( g \), let \( V_i^{2,1} \) be a set of \((v_i^p, v_i^q)\) such that \( v_i^p, v_i^q \in V(C_g^i) \) and \( l_i^p = l_i^q \) for distinct \( p, q \in [1, \lceil \frac{g}{2} \rceil + 1] \), and \( V_i^{2,2} \) be a set of \((v_i^p, v_i^q)\) such that \( v_i^p, v_i^q \in V(C_g^i) \) and \( l_i^p = l_i^q \) for distinct \( p, q \in \{1\} \cup [\lceil \frac{g}{2} \rceil + 1, g] \).

Note that regardless the parity of \( g \), we have either Subcase 2.1 or 2.2 as follows.

**Subcase 2.1.** \( |V_i^{r,s}| \geq 1 \) for some \( s \in [1, 2] \)

Choose a pair \((v_i^p, v_i^q)\) \( \in V_i^{r,s} \) so that \( d_{C_g}(v_i^p, v_i^q) \) has the smallest value. Thus, we have \( d_{C_g}(v_i^p, v_i^q) = 1 \) or 2, since there is another pair \((v_i^p, v_i^q)\) \( \in V_i^{r,s} \) such that \( d_{C_g}(v_i^p, v_i^q) < d_{C_g}(v_i^p, v_i^q) \) if \( d_{C_g}(v_i^p, v_i^q) \geq 3 \), contradicts the assumption. If \( d_{C_g}(v_i^p, v_i^q) = 1 \), then \( v_i^p v_i^q \) is not contained in a \( v_2 - v_i^p \) geodesic and a \( v_2 - v_i^q \) geodesic. This implies \( v_i^p v_i^q \) is also not contained in a \( v_2 - v_i^t \) geodesic for any \( t \in [1, g] \). If \( d_{C_g}(v_i^p, v_i^q) = 2 \), then there is \( v_i^k \in V(C_g^i) \) such that \( v_i^p v_i^k, v_i^q v_i^k \in E(C_g^i) \) and \( l_i^k = l_i^k + 1 \). Hence, there are two \( v_2 - v_i^k \) geodesics, one path contains \( v_i^p v_i^k \) and another path contains \( v_i^q v_i^k \). Similar to Case 1 for even \( g \), edge \( v_i^p v_i^k \) can be chosen to be an edge that is not contained in a \( v_2 - v_i^t \) geodesic for any \( t \in [1, g] \).

**Subcase 2.2.** \( |V_i^{r,s}| = 0 \) for all \( s \in [1, 2] \)

Since \( |V_i^{r,s}| = 0 \) for all \( s \in [1, 2] \), we have \( l_i^p = l_i^1 + p - 1 \) for \( p \in [1, \lfloor \frac{g}{2} \rfloor + 1] \) and \( l_i^p = l_i^1 + g - p + 1 \) for \( p \in [\lceil \frac{g}{2} \rceil + 2, g] \). Thus, similar to Case 1, we obtain that \( v_i^{\lfloor \frac{g}{2} \rfloor + 1}, v_i^{\lfloor \frac{g}{2} \rfloor + 2} \) is not contained in a \( v_2 - v_i^t \) geodesic for any \( t \in [1, g] \).

Let \( S \) be a 3-subset of \( V(P_n^o \triangleright_e H) \). According to Cases 1 and 2, there is an edge \( e_i \in E(C_g^i) \) for each \( i \in [1, 2] \) such that \( e_i \) is not contained in a \( v_2 - v_i^t \) geodesic for any \( t \in [1, g] \). Therefore, by assigning the color 1 to the edges \( e_1 \) and \( e_2 \) and the colors 2, 3, \ldots, \( |E(P_n^o \triangleright_e H)| - 1 \) to the remaining \( |E(P_n^o \triangleright_e H)| - 2 \) edges of \( P_n^o \triangleright_e H \), there is a rainbow Steiner S-tree in \( P_n^o \triangleright_e H \). Hence, \( sx_3(P_n^o \triangleright_e H) \leq |E(P_n^o \triangleright_e H)| - 1 \), contradicts the assumption.

According to Theorem 2.2, graph \( P_n^o \triangleright_T m \) is the only graph whose \( sx_3 \) is equal to its size. The following theorem provides a sharper upper bound for \( sx_3(P_n^o \triangleright_T H) \).
Theorem 2.3. For two integers \( n, m \geq 3 \), let \( P_n \) and \( H \) be a path of order \( n \) and a connected graph of order \( m \), respectively. Let \( \vec{e} \) be any oriented edge of \( H \). Then

\[
\text{srx}_3(P_n^o \triangleright_{\vec{e}} H) \leq \text{srx}_3(H)(n-1).
\]

Proof. Let \( \vec{e} = w_iw_b \). For \( i \in [1, n-1] \), we color all edges of \( H^i \) with \( \text{srx}_3(H) \) colors so that each \( H^i \) admits a strong 3-rainbow coloring where \( c(E(H^i)) \cap c(E(H^j)) = \emptyset \) for all \( j \in [1, n-1] \) with \( j \neq i \). According to the definition, graph \( P_n^o \triangleright_{\vec{e}} H \) can be formed by identifying vertices \( v_i^b \) and \( v_{i+1}^b \) for each \( i \in [1, n-1] \). Since each \( H^i \) admits a strong 3-rainbow coloring with \( c(E(H^i)) \cap c(E(H^j)) = \emptyset \) for distinct \( i, j \in [1, n-1] \), there is a rainbow Steiner \( S \)-tree for every 3-subset \( S \) of \( V(P_n^o \triangleright_{\vec{e}} H) \). Thus, \( \text{srx}_3(P_n^o \triangleright_{\vec{e}} H) \leq \text{srx}_3(H)(n-1) \). \( \Box \)

Since \( \text{srx}_3(T_m) = m-1 \) by Theorem 1.1, it follows by Theorem 2.1 that \( \text{srx}_3(P_n^o \triangleright_{\vec{e}} T_m) \) is also equal to the upper bound given in Theorem 2.3. Thus, the upper bound is sharp. There are other graphs \( H \) such that \( \text{srx}_3(P_n^o \triangleright_{\vec{e}} H) = \text{srx}_3(H)(n-1) \). These results are given in Section 3.

3. The strong 3-rainbow index of \( P_n^o \triangleright_{\vec{e}} H \) for some connected graphs \( H \)

Our first two results show that there are two connected graphs \( H \) such that \( \text{srx}_3(P_n^o \triangleright_{\vec{e}} H) = \text{srx}_3(H)(n-1) \).

For a ladder graph \( L_m \) of order \( 2m \) \((m \geq 3)\), we define \( V(L_m) = \{w_i : i \in [1, 2m]\} \) and \( E(L_m) = \{w_iw_{i+1} : i \in [1, m-1] \cup [m+1, 2m-1] \} \cup \{w_iw_{i+m} : i \in [1, m]\} \). The following theorem shows that \( \text{srx}_3(P_n^o \triangleright_{\vec{e}} L_m) = \text{srx}_3(L_m)(n-1) \).

Theorem 3.1. For two integers \( n, m \geq 3 \), let \( P_n \) and \( L_m \) be a path of order \( n \) and a ladder of order \( 2m \), respectively. Let \( \vec{e} \) be an oriented edge of \( L_m \) where \( \vec{e} = w_1w_{m+1} \). Then \( \text{srx}_3(P_n^o \triangleright_{\vec{e}} L_m) = m(n-1) \).

Proof. Since \( \text{srx}_3(L_m) = m \) by Theorem 1.2, it follows by Theorem 2.3 that \( \text{srx}_3(P_n^o \triangleright_{\vec{e}} L_m) \leq m(n-1) \). Now, let \( c \) be a strong 3-rainbow coloring of \( P_n^o \triangleright_{\vec{e}} L_m \). For \( i \in [1, n-1] \), let \( X_i = \{v_i^2, v_i^p, v_i^{p+1} : p \in [2, m-1]\} \). We first verify two properties as follows.

(A1) \( c(X_i) \cap c(E(P_n)) = \emptyset \) for each \( i \in [1, n-1] \)

Suppose that there are \( e \in X_i \) for some \( i \in [1, n-1] \) and \( f \in E(P_n) \) such that \( c(e) = c(f) \).

Let \( e = uv \) and \( f = xy \), and assume that \( d(v_i, x) < d(v_i, y) \). Observe that edges \( e \) and \( f \) should be contained in any rainbow Steiner \( \{u, v, y\} \)-tree, but \( c(e) = c(f) \), a contradiction.

(A2) \( c(X_i) \cap c(X_j) = \emptyset \) for \( i, j \in [1, n-1] \) with \( i \neq j \)

Suppose that there are \( e \in X_i \) and \( f \in X_j \) for distinct \( i, j \in [1, n-1] \) such that \( c(e) = c(f) \).

Let \( e = uv \) and \( f = xy \), and assume that \( d(v_j, x) < d(v_j, y) \). By considering \( \{u, v, y\} \), we will obtain a contradiction.

Since \( |c(X_i)| \geq m - 1 \) for each \( i \in [1, n-1] \), by using (2), (A1), and (A2), \( \text{srx}_3(P_n^o \triangleright_{\vec{e}} L_m) \geq m(n-1) \). \( \Box \)
According to (1), the natural lower bound for \( \text{sr}X_3(P_n^o \triangleright_e H) \) is \( \text{sdiam}_3(P_n^o \triangleright_e H) \). Consider graphs \( P_n^o \triangleright_e L_m \) where \( \bar{e} \in E(L_m) \) with \( \bar{e} = w_1w_{m+1} \). It is easy to check that \( \text{sdiam}_3(P_3^o \triangleright_e L_m) = 2m \) and \( \text{sdiam}_3(P_n^o \triangleright_e L_m) = n + 3m - 4 \) for \( n \geq 4 \). Hence, by Theorem 3.1, \( \text{sr}X_3(P_n^o \triangleright_e L_m) = \text{sdiam}_3(P_n^o \triangleright_e L_m) \) for \( n \in [3, 4] \).

For further discussion, we define path \( \nu_p \nu_q \nu_r \nu_t = \nu_p \nu_q \nu_r \). For \( m \geq 3 \), let \( K_{m,m} \) be a regular complete bipartite graph of order \( 2m \) with \( V(K_{m,m}) = \{ w_i: \ i \in [1, 2m] \} \) and \( E(K_{m,m}) = \{ w_iw_j: \ i \in [1, m], j \in [m+1, 2m] \} \). The next theorem shows that \( \text{sr}X_3(P_n^o \triangleright_e K_{m,m}) = \text{sr}X_3(K_{m,m}) \).

**Theorem 3.2.** For two integers \( n, m \geq 3 \), let \( P_n \) and \( K_{m,m} \) be a path of order \( n \) and a regular complete bipartite graph of order \( 2m \), respectively. Let \( \bar{e} \) be any oriented edge of \( K_{m,m} \). Then \( \text{sr}X_3(P_n^o \triangleright_e K_{m,m}) = m(n-1) \).

**Proof.** Without loss of generality, let \( \bar{e} = w_1w_{m+1} \) such that \( v_i^1 = v_i \) and \( v_i^{m+1} = v_{i+1} \) for each \( i \in [1, n-1] \). By using Theorems 1.3 and 2.3, we have \( \text{sr}X_3(P_n^o \triangleright_e K_{m,m}) \leq m(n-1) \). Now, let \( c \) be a strong 3-rainbow coloring of \( P_n^o \triangleright_e K_{m,m} \). We first verify two properties as follows.

(B1) \( c(v_i v_i^p) \notin c(E(P_n)) \) for each \( i \in [1, n-1] \) and \( p \in [m+2, 2m] \)

Suppose that there are \( i \in [1, n-1] \) and \( p \in [m+2, 2m] \) such that \( c(v_i v_i^p) \in c(E(P_n)) \). Let \( e = uv \in E(P_n) \) with \( c(v_i v_i^p) = c(e) \), and assume that \( d(v_i, u) < d(v_i, v) \). Observe that edges \( v_i v_i^p \) and \( e \) should be contained in any rainbow Steiner \( \{v_i, v_i^p, v\} \)-tree, but \( c(v_i v_i^p) = c(e) \), a contradiction.

(B2) \( c(v_i v_i^p) \neq c(v_j v_j^q) \) for \( i, j \in [1, n-1] \) with \( i \neq j \) and \( p, q \in [m+2, 2m] \)

Let \( i < j \). By considering \( \{v_i, v_i^p, v_j^q\} \) for \( p, q \in [m+2, 2m] \), it is clearly that \( c(v_i v_i^p) \neq c(v_j v_j^q) \).

Note that \( d_{K_{m,m}}(v_i) = m \) for \( i \in [1, n-1] \), thus \( \text{sr}X_3(P_n^o \triangleright_e K_{m,m}) \geq m(n-1) \) by (2), (B1), and (B2).

According to Theorems 2.1, 3.1, and 3.2, the values of \( \text{sr}X_3(P_n^o \triangleright_e H) \) for some graphs \( H \) does not depend on the order of \( H \). Nevertheless, there are some graphs \( H \) so that the values of \( \text{sr}X_3(P_n^o \triangleright_e H) \) depends on the order of \( H \). These results are given in Theorems 3.3 and 3.4.

A fan graph \( F_m \) of order \( m+1 \) \((m \geq 3)\) is a graph formed by joining a vertex to each vertex of \( P_m \). We define \( V(F_m) = \{ w_i: \ i \in [1, m+1] \} \) and \( E(F_m) = \{ w_iw_i: \ i \in [2, m+1] \} \cup \{ w_iw_{i+1}: \ i \in [2, m] \} \). For \( i \in [2, m+1] \), vertex \( w_i \) and edge \( w_iw_i \) are called the center vertex and the spoke of \( F_m \), respectively. In [1], we obtained the following lemma which will be used to prove Theorem 3.3.

**Lemma 3.1.** \([1]\) Let \( F_m \) be a fan of order \( m+1 \) \((m \geq 3)\) which has a strong 3-rainbow coloring. Then each color is assigned to at most two spokes \( w_iw_i \) and \( w_iw_j \) where \( w_iw_j \in E(F_m) \).

**Theorem 3.3.** For two integers \( n, m \geq 3 \), let \( P_n \) and \( F_m \) be a path of order \( n \) and a fan of order \( m+1 \), respectively. Let \( \bar{e} \) be an oriented edge of \( F_m \) where \( \bar{e} = w_1w_2 \). Then

\[
\text{sr}X_3(P_n^o \triangleright_e F_m) = \left\{ \begin{array}{ll}
2n - 1, & \text{for } m = 4; \\
\left\lceil \frac{m}{2} \right\rceil (n-1), & \text{otherwise}.
\end{array} \right.
\]
Proof. Let $c$ be a strong 3-rainbow coloring of $P_n^o \triangleright_{\varepsilon} F_m$. Similar to the proof of Theorem 3.2, we have two properties as follows.

(C1) $c(v_i v^p_i) \notin c(E(P_n))$ for each $i \in [1, n-1]$ and $p \in [4, m+1]$
(C2) $c(v_i v^p_i) \neq c(v_j v^q_j)$ for $i, j \in [1, n-1]$ with $i \neq j$ and $p, q \in [4, m+1]$

Now, we distinguish two cases.

Case 1. $m = 4$

Suppose that $srx_3(P_n^o \triangleright_{\varepsilon} F_4) \leq 2n-2$. Let $c : E(P_n^o \triangleright_{\varepsilon} F_4) \rightarrow [1, 2n-2]$ be a strong 3-rainbow coloring of $P_n^o \triangleright_{\varepsilon} F_4$. By using (2), Lemma 3.1, (C1), and (C2), we need at least $2n - 2$ different colors assigned to the edges of $P_n$ and spokes of $F_4$, for all $i \in [1, n-1]$. Without loss of generality, let $c(v_i v_{i+1}) = c(v_i v^3_i) = i$ and $c(v_i v^1_i) = c(v_i v^2_i) = c(v_i v^4_i) = i + n - 1$ for $i \in [1, n-1]$. Now, observe that identifying vertex $v_2$ in a rainbow Steiner $\{v_2, v^4_1, v^5_1\}$-tree and a rainbow $v_2 - v_n$ geodesic will obtain a rainbow Steiner $\{v^5_1, v^5_n\}$-tree. Similarly, identifying vertex $v_2$ in a rainbow Steiner $\{v_2, v^4_1, v^5_1\}$-tree and a rainbow $v_2 - v^5_n$ geodesic for all $i \in [2, n-1]$ will obtain a rainbow Steiner $\{v^1_i, v^5_i, v^5_n\}$-tree. Since these rainbow Steiner trees must contain edge $v^4_i v^5_i$, we have $c(v^4_i v^5_i) \notin \{c(v_i v_{i+1}), c(v_i v^5_i)\}$ for all $i \in [2, n-1]$. This means we have two colors, which are 1 and $n$, to be assigned to the three edges in a Steiner tree containing $\{v_2, v^4_1, v^5_1\}$, which is impossible. Thus, $srx_3(P_n^o \triangleright_{\varepsilon} F_4) \geq 2n - 1$.

Next, we show that $srx_3(P_n^o \triangleright_{\varepsilon} F_4) \leq 2n - 1$. For each $i \in [1, n-1]$, define $c(v_i v_{i+1}) = c(v_i v^3_i) = c(v^3_i v^i_1) = i$, $c(v_i v^1_i) = c(v_i v^2_i) = c(v_i v^4_i) = i + n - 1$, and $c(v_i v^5_i) = 2n - 1$. Let $S$ be a 3-subset of $V(P_n^o \triangleright_{\varepsilon} F_4)$. If $S \subseteq V(F^i_4)$ for some $i \in [1, n-1]$, then it is not hard to find a rainbow Steiner $S$-tree. Hence, there are two possible sets $S$ as follows. First, we consider case when both vertices of $S$ belong to the same fan $F^i_4$ for some $i \in [1, n-1]$. Let $y \in V(F^j_4)$ for $j \in [1, n-1]$ with $j \neq i$. For $i < j$, let $P$ be a $v_{i+1} - v_j$ geodesic. Then there is a rainbow Steiner $S$-tree as given in Table 1. The proof for $i > j$ is similar to the case for $i < j$.

| Set $S$ | Condition | A rainbow Steiner $S$-tree |
|---------|------------|----------------------------|
| $\{v_i, v_{i+1}, y\}$ | $p = 1, q = 2$ | $v_i v_{i+1} \cup P \cup v_j y$ |
| $\{v_i, v^q_i, y\}$ | $p = 1, q = 3$ | $v^q_i v_{i+1} \cup P \cup v_j y$ |
| $\{v_{i+1}, v^q_i, y\}$ | $p = 2, q = 3$ | $v^q_i v_{i+1} \cup P \cup v_j y$ |
| $\{v^p_i, v^q_i, y\}$ | $p, q \in [3, 5], p < q$ | $v^p_i v^{p-1}_i v^q_i v_{i+1} \cup P \cup v_j y$ |

Next, we consider case when each vertex of $S$ belongs to three different fans $F^i_4, F^j_4$, and $F^k_4$ for $i, j, k \in [1, n-1]$. Without loss of generality, let $i < j < k$. Let $S = \{v^p_i, v^q_j, z\}$ where $z \in V(F^k_4)$. Let $P$ be a $v_{i+1} - v_k$ geodesic, $P^1_i = v_{i+1} v^3_i v^p_i$ if $p \in [3, 4], P^2_i = v_{i+1} v^3_i v^5_i$ if $p = 5$, $P^1_j = v_{j+1} v^3_j$ if $q = 3, P^2_j = v^3_j v^q_j$ if $q \in [4, 5]$, and $P_k = v_k z$. Then the tree $T = P \cup P^a_i \cup P^b_j \cup P_k$.
The strong 3-rainbow index of edge-comb product of a path and a connected graph

with \( a, b \in [1, 2] \) is a rainbow Steiner \( S \)-tree, where the values of \( a \) and \( b \) depend on the values of \( p \) and \( q \), respectively. Note that the case when \( S \) contains at least one vertex of \( P_n \) has been proven.

An illustration of a strong 3-rainbow coloring of \( P_n \oplus_F \vec{e} F_4 \) is given in Figure 2.

![Figure 2. A strong 3-rainbow coloring of \( P_n \oplus_F \vec{e} F_4 \).](image)

Case 2. \( m = 3 \) or \( m \geq 5 \)

Since \( srx_3(F_m) = \lceil \frac{m}{2} \rceil \) by Theorem 1.5, it follows by Theorem 2.3 that \( srx_3(P_n \oplus_F \vec{e} F_m) \leq \lceil \frac{m}{2} \rceil (n - 1) \). Now, let \( e \) be a strong 3-rainbow coloring of \( P_n \oplus_F \vec{e} F_m \). By using (2), Lemma 3.1, (C1), and (C2), we have \( srx_3(P_n \oplus_F \vec{e} F_m) \geq \lceil \frac{m}{2} \rceil (n - 1) \).

Following Theorem 3.3, we obtain that \( srx_3(P_n \oplus_F \vec{e} F_m) \) for \( m = 3 \) or \( m \geq 5 \) is equal to the upper bound given in Theorem 2.3.

Now, we consider graphs \( P_n \oplus_F C_m \) where \( \vec{e} \) is any oriented edge of \( C_m \). For \( m \geq 3 \), let \( V(C_m) = \{w_i : i \in [1, m]\} \) and \( E(C_m) = \{w_iw_{i+1} : i \in [1, m] \text{ and } w_{m+1} = w_1\} \). Our next result provides the exact value of \( srx_3(P_n \oplus_F \vec{e} C_m) \).

**Theorem 3.4.** For two integers \( n \geq 3 \) and \( m \geq 4 \), let \( P_n \) and \( C_m \) be a path of order \( n \) and a cycle of order \( m \), respectively. Let \( \vec{e} \) be any oriented edge of \( C_m \). Then

\[
srx_3(P_n \oplus_F \vec{e} C_m) = \begin{cases} 
2n - 2, & \text{for } m = 4; \\
2n + \lfloor \frac{n}{2} \rfloor - 2, & \text{for } m = 5; \\
(m - 3)(n - 1) + 1, & \text{for } m \in \{6, 8\}; \\
(m - 1)(n - 1) + 1, & \text{for odd } m \geq 7; \\
(m - 2)(n - 1) + 3, & \text{for even } m \geq 10.
\end{cases}
\]

Proof. Without loss of generality, let \( \vec{e} = w_1w_2 \) such that \( v_i^1 = v_i \) and \( v_i^2 = v_{i+1} \) for each \( i \in [1, n - 1] \). We consider several cases.

**Case 1.** \( m \) is odd

We distinguish two subcases.

Subcase 1.1. \( m = 5 \)
Suppose that $srx_3(P_n \bowtie C_5) \leq 2n + \lfloor \frac{n}{2} \rfloor - 3$. Let $c : E(P_n \bowtie C_5) \rightarrow [1, 2n + \lfloor \frac{n}{2} \rfloor - 3]$ be a strong 3-rainbow coloring of $P_n \bowtie C_5$. Observe that $c(v_iv_{i+1}) \notin c(E(P_n))$ and $c(v_iv_j) \neq c(v_jv_{j+1})$ for $i, j \in [1, n - 1]$ with $i \neq j$. Hence, by using (2), we need at least $2n - 2$ different colors assigned to all edges $v_iv_{i+1}$ and $v_jv_{j+1}$ for $i \in [1, n - 1]$, implying that we have at most $\lfloor \frac{n}{2} \rfloor - 1$ colors left. Let $X$ be the set of these $\lfloor \frac{n}{2} \rfloor - 1$ colors. Now, we consider edges $v_{i+1}v_i^3, v_i^4v_i^5$, and $v_jv_{j+1}$ for all $i \in [1, n - 1]$. By considering $\{v_{i+1}, v_i^3, v_j\}$ and $\{v_i^4, v_{i+1}^p, v_j\}$ for all $j \in [1, n - 1]$, $j \neq i, j \neq i + 1$, and $p \in [3, 4]$, we obtain that these three edges cannot be assigned with colors from $c(E(P_n) \setminus \{v_1v_{i+1}\})$. Furthermore, by considering $\{v_i^3, v_j^3, v_j^4\}$ and $\{v_i^4, v_{i+1}^p, v_{i+1}^p\}$ for all $j \in [1, n - 1], j \neq i$, and $p \in [3, 4]$, these three edges also cannot be assigned with $c(v_jv_{j+1})$. This implies $\{c(v_{i+1}v_i^3), c(v_i^3v_j^4), c(v_i^4v_{j+1})\} \subseteq \{c(v_{i+1}v_i^3), c(v_i^4v_{j+1})\} \cup X$ for all $i \in [1, n - 1]$. Since every two adjacent edges in $C_5$ must have different colors, this forces

$$c(v_{i+1}v_i^3) \in \{c(v_i^4v_{j+1})\} \cup X, \ c(v_i^3v_j^4) \in \{c(v_{i+1}v_i^3), c(v_i^4v_{j+1})\} \cup X, \text{ and} \ c(v_i^4v_{j+1}) \in \{c(v_{i+1}v_i^3)\} \cup X,$$

and at least one edge of edges $v_{i+1}v_i^3, v_i^3v_j^4$, or $v_i^4v_{j+1}$ should be assigned with colors from $X$. This condition implies there are two possible proofs that might happen. Before we proceed further, we consider the following two properties.

(D1) $\{c(v_{i+1}v_i^3), c(v_i^3v_j^4)\} \cap \{c(v_jv_{j+1}v_i), c(v_jv_{j+1}v_i)\} = \emptyset$ for $i, j \in [1, n - 1]$ with $i \neq j$.

Without loss of generality, let $i < j$. By considering $\{v_i^4, v_{j+1}^3, v_j^3\}$ and $\{v_i^4, v_{j+1}^3, v_j^3\}$, we have $\{c(v_i^4, v_{j+1}^3), c(v_{j+1}^3, v_j^3)\} \cap \{c(v_{j+1}^3, v_j^3)\} = \emptyset$.

(D2) $c(v_i^4v_{j+1}) \neq c(v_j^3v_i^3)$ for $i, j \in [1, n - 1]$ with $i \neq j$.

Without loss of generality, let $i < j$. By considering $\{v_i^4, v_j^3, v_j^4\}$, we have $c(v_i^4v_{j+1}) \neq c(v_j^3v_i^3)$.

Now, we consider these two possible proofs, which are: (i) $\{c(v_{i+1}v_i^3), c(v_i^4v_{j+1})\} \subseteq X$ for some $i \in [1, n - 1]$; and (ii) $c(v_i^4v_{j+1}) \in X$ for some $i \in [1, n - 1]$. If the first case happens, then observe that there are at most $\lfloor \frac{n}{2} \rfloor - 1$ pairs of two edges $\{v_{i+1}v_i^3, v_i^4v_{j+1}\}$ for some $i \in [1, n - 1]$ such that $\{c(v_{i+1}v_i^3), c(v_i^4v_{j+1})\} \subseteq X$. Hence, by using (D1), there are at least $n - \lfloor \frac{n}{2} \rfloor$ pairs of two edges $\{v_{i+1}v_i^3, v_i^4v_{j+1}\}$ for all $j \in [1, n - 1]$ with $j \neq i$ such that $\{c(v_{i+1}v_i^3), c(v_i^4v_{j+1})\} \not\subseteq X$. This forces $c(v_{i+1}v_i^3) = c(v_{i+1}v_i^3), c(v_i^4v_{j+1}) = c(v_{i+1}v_i^3), \text{ and } c(v_i^4v_{j+1}) \in X$ by (3). However, by using (D2), we need at least $n - \lfloor \frac{n}{2} \rfloor$ different colors assigned to the edges $v_i^4v_{j+1}$ for all $j \in [1, n - 1]$ with $j \neq i$, which is impossible since $|X| \leq \lfloor \frac{n}{2} \rfloor - 1$. A similar argument applies if the second case happens.

For the upper bound, we first define an edge-coloring $c$ of $P_n \bowtie C_5$ using $2n + \lfloor \frac{n}{2} \rfloor - 2$ colors as follows.

1. For each $i \in [1, n - 1]$, define $c(v_{i+1}v_i) = i$ and $c(v_{i+1}v_i^3) = c(v_i^4v_{j+1}) = i + n - 1$.
2. For each $i \in [1, \lfloor \frac{n}{2} \rfloor]$, define $c(v_i^4v_{j+1}) = i + 2(n - 1)$ and $c(v_i^3v_{j+1}) = c(v_{i+1}v_i)$.
3. For each $i \in [\lfloor \frac{n}{2} \rfloor + 1, n - 1]$, define $c(v_i^4v_{j+1}) = i - \lfloor \frac{n}{2} \rfloor + 2(n - 1)$ and $c(v_i^4v_{j+1}) = c(v_{i+1}v_i)$.

Now, let $S$ be a 3-subset of $V(P_n \bowtie C_5)$. Since the edge-coloring $c$ assigns 3 different colors to all edges of $C_5$ and has the same coloring pattern as given in Figure 1, there is a rainbow Steiner $S$-tree if $S \subseteq V(C_5)$ for some $i \in [1, n - 1]$. Hence, we distinguish two cases.
First, we consider \( S = \{v_i^p, v_j^q, v_j^r\} \) for distinct \( i, j \in [1, n - 1] \). For \( i < j \), let \( P' \) be a \( v_{i+1} - v_j \) geodesic, \( P_1^j = v_{j+1}v_j^3 \) if \( r = 3 \), and \( P_2^j = v_jv_j^3v_j^3 \) if \( r \in [4, 5] \). Let \( P_1 = P_1^j \cup P_2^b \) with \( b \in [1, 2] \). Then there is a rainbow Steiner \( S \)-tree as given in Table 2. Meanwhile, for \( i > j \), let \( P'' \) be a \( v_{j+1} - v_i \) geodesic, \( P_3^j = v_jv_j^3v_{j+1} \) if \( r \in [3, 4] \), and \( P_4^j = v_j^5v_jv_{j+1} \) if \( r = 5 \). Let \( P_2 = P_2^b \cup P'' \) with \( b \in [3, 4] \). Then there is a rainbow Steiner \( S \)-tree as given in Table 2. Note that the value of \( b \) for each case depends on the value of \( r \in [3, 5] \).

| Set \( S \) | Condition | A rainbow Steiner \( S \)-tree |
|---|---|---|
| \( \{v_i, v_{i+1}, v_j^r\} \) | \( i < j, p = 1, q = 2 \) | \( v_i, v_{i+1} \cup P_1 \) |
| \( \{v_i, v_{i+1}^q, v_j^r\} \) | \( i < j, p = 1, q = 3 \) | \( v_i, v_{i+1}^q \cup P_1 \) |
| \( \{v_i, v_{i+1}^q, v_{i+1}^q\} \) | \( i < j, p = 1, q \in [4, 5] \) | \( v_i, v_{i+1}^q \cup P_1 \) |
| \( \{v_i, v_{i+1}^q, v_j^r\} \) | \( i > j, p = 1, q = 3 \) | \( v_i, v_{i+1}^q \cup P_1 \) |
| \( \{v_i, v_{i+1}^q, v_{i+1}^q\} \) | \( i > j, p = 1, q \in [4, 5] \) | \( v_i, v_{i+1}^q \cup P_1 \) |
| \( \{v_i, v_{i+1}^q, v_j^r\} \) | \( i < j, p = 2, q \in [3, 4] \) | \( v_i, v_{i+1}^q \cup P_1 \) |
| \( \{v_i, v_{i+1}^q, v_j^r\} \) | \( i < j, p = 2, q = 5 \) | \( v_i, v_{i+1}^q \cup P_1 \) |
| \( \{v_i, v_{i+1}^q, v_j^r\} \) | \( i > j, p = 2, q \in [3, 4] \) | \( v_i, v_{i+1}^q \cup P_1 \) |
| \( \{v_i, v_{i+1}^q, v_j^r\} \) | \( i > j, p = 2, q = 5 \) | \( v_i, v_{i+1}^q \cup P_1 \) |
| \( \{v_i, v_{i+1}^q, v_{i+1}^q\} \) | \( i < j, p, q \in [3, 5], p < q \) | \( v_i, v_{i+1}^q \cup P_1 \) |
| \( \{v_i, v_{i+1}^q, v_{i+1}^q\} \) | \( i < j, p, q \in [3, 5], p < q \) | \( v_i, v_{i+1}^q \cup P_1 \) |

Next, we consider \( S = \{v_i^p, v_j^q, v_k^r\} \) for distinct \( i, j, k \in [1, n - 1] \). Without loss of generality, let \( i < j < k \). Let \( P \) be a \( v_{i+1} - v_k \) geodesic, \( P_1^i = v_{i+1}^q v_k^r \) if \( p \in [3, 4] \), \( P_2^i = v_{i+1}^q v_k^r \) if \( p = 5 \), \( P_1^j = v_{j+1}^q v_k^r \) if \( j \in [2, \frac{n}{2}] \) and \( q = 3 \), \( P_2^j = v_{j+1}^q v_k^r \) if \( j \in [2, \frac{n}{2}] \) and \( q \in [4, 5] \), \( P_3^j = v_{j+1}^q v_k^r \) if \( j \in [\frac{n}{2} + 1, n - 2] \) and \( q \in [3, 4] \), \( P_4^j = v_{j+1}^q v_k^r \) if \( j \in [\frac{n}{2} + 1, n - 2] \) and \( q = 5 \), \( P_5^j = v_{k+1}^q v_k^r \) if \( k = 3 \), and \( P_6^j = v_{k+1}^q v_k^r \) if \( r \in [4, 5] \). Then the tree \( T = P \cup P_1^i \cup P_2^i \cup P_3^i \cup P_4^i \cup P_5^i \cup P_6^i \) with \( a, c \in [1, 2] \) and \( b \in [1, 4] \) is a rainbow Steiner \( S \)-tree, where the values of \( a \) and \( c \) depend on the values of \( p \) and \( r \), respectively, and the value of \( b \) depends on the values of \( j \) and \( q \).

Note that the case when \( S \) contains at least one vertex of \( P_n \) has been proven for each of the above cases. Figure 3 illustrates a strong 3-rainbow coloring of \( P_n^o \geq C_5 \).

**Subcase 1.2.** \( m \geq 7 \)

Suppose that \( \operatorname{sr}(P_n^o \geq C_m) \leq (m - 1)(n - 1) \). Let \( c : E(P_n^o \geq C_m) \to [1, (m - 1)(n - 1)] \) be a strong 3-rainbow coloring of \( P_n^o \geq C_m \). According to Theorem 1.4, all edges of \( C_m \) should be assigned with different colors. Hence, by considering \( \{v_i^p, v_j, v_j^q\} \) and \( \{v_i^p, v_j^q, v_j^r\} \) for \( i, j \in [1, n - 1] \) with \( i < j \), \( p \in \lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 2, \) \( q \in \lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 1 \), we need at least \((m - 1)(n - 1)\) different colors assigned to the edges of \( P_n^o \geq C_m \) except edges \( v_i^p v_j, v_j^q v_j^r \) for all \( i \in [1, n - 1] \). This means we have used all colors. Now, we consider edge \( v_1^p v_4^q + 1 \). By using
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Theorem 1.4 and considering \( \{v_1^{\lceil \frac{m}{2} \rceil}, v_1^{\lceil \frac{m}{2} \rceil + 1}, v_i^p \} \) for all \( i \in [2, n - 1] \) and \( p \in \{\lceil \frac{m}{2} \rceil, \lceil \frac{m}{2} \rceil + 1\} \), we need one new different color assigned to the edge \( v_1^{\lceil \frac{m}{2} \rceil}v_1^{\lceil \frac{m}{2} \rceil + 1} \), which is impossible.

For the upper bound, we first define an edge-coloring \( c \) of \( P^o_n \bowtie C_m \) using \((m-1)(n-1)+1\) colors as follows.

1. Assign the colors \( 1, 2, \ldots, (m-1)(n-1) \) to all edges of \( P^o_n \bowtie C_m \) except edges \( v_i^{\lceil \frac{m}{2} \rceil}v_i^{\lceil \frac{m}{2} \rceil + 1} \) for all \( i \in [1, n-1] \).

2. Define \( c(v_1^{\lceil \frac{m}{2} \rceil}v_1^{\lceil \frac{m}{2} \rceil + 1}) = (m-1)(n-1) + 1 \) and \( c(v_i^{\lceil \frac{m}{2} \rceil}v_i^{\lceil \frac{m}{2} \rceil + 1}) = c(v_i^{-1}v_i^{-1} + 1 v_i^{-1} + 2) \) for each \( i \in [2, n-1] \).

Now, let \( S \) be a 3-subset of \( V(P^o_n \bowtie C_m) \). Observe that edges of \( P^o_n \bowtie C_m \) have different colors except edges \( v_i^{\lceil \frac{m}{2} \rceil + 1}v_i^{\lceil \frac{m}{2} \rceil + 2} \) for \( i \in [1, n-2] \) and \( v_i^{\lceil \frac{m}{2} \rceil + 1}v_i^{\lceil \frac{m}{2} \rceil + 1} \) for \( i \in [2, n-1] \), that is \( c(v_i^{\lceil \frac{m}{2} \rceil}v_i^{\lceil \frac{m}{2} \rceil + 1}) = c(v_i^{-1}v_i^{-1} + 1 v_i^{-1} + 2) \) for all \( i \in [2, n-1] \). This means if \( S \subseteq V(C_m^i) \) for some \( i \in [1, n-1] \), then there is a rainbow Steiner \( S \)-tree since all edges of \( C_m^i \) have different colors. Hence, without loss of generality, we distinguish two cases.

First, we consider \( S = \{v_i^p, v_j^q, v_j^p\} \) for distinct \( i, j \in [1, n-1] \). For \( i < j \), observe that there is a rainbow \( v_{i+1} - v_j \) geodesic \( T_1 \) in \( P_n \), a rainbow Steiner \( \{v_{i+1}, v_i^p, v_j^q\} \)-tree \( T_2 \) in \( C_m^i \), and a rainbow \( v_j - v_j^q \) geodesic \( T_3 \) in \( C_m^j \), so that \( c(E(T_a)) \cap c(E(T_b)) = \emptyset \) for all distinct \( a, b \in [1, 3] \). Then the tree \( T = T_1 \cup T_2 \cup T_3 \) is a rainbow Steiner \( S \)-tree. The proof for \( i > j \) is similar to the case for \( i < j \).

Next, we consider \( S = \{v_i^p, v_j^q, v_k^p\} \) for distinct \( i, j, k \in [1, n-1] \). Without loss of generality, let \( i < j < k \). Note that there are a rainbow \( v_{i+1} - v_k \) geodesic \( T_1 \) in \( P_n \), a rainbow \( v_{i+1} - v_k^p \) geodesic \( T_2 \) in \( C_m^i \), a rainbow \( v_j - v_j^q \) geodesic \( T_3 \) in \( C_m^j \), a rainbow \( v_j^q - v_j^q \) geodesic \( T_4 \) in \( C_m^i \), and a rainbow \( v_k - v_k^p \) geodesic \( T_5 \) in \( C_m^k \), so that \( c(E(T_a)) \cap c(E(T_b)) = \emptyset \) for all distinct \( a, b \in [1, 5] \). Then the tree \( T = T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5 \) is a rainbow Steiner \( S \)-tree.

Note that the case when \( S \) contains at least one vertex of \( P_n \) has been proven for each of the above cases. Figure 4 illustrates a strong 3-rainbow coloring of \( P^o_n \bowtie C_7 \).

**Case 2.** \( m \) is even
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An illustration of a strong 3-rainbow coloring of $P_6^m \bowtie_w C_4$.

Figure 4. A strong 3-rainbow coloring of $P_6^m \bowtie_w C_7$.

We first consider the following properties. Let $c$ be a strong 3-rainbow coloring of $P_6^m \bowtie_w C_m$. For $i \in [1, n-1]$, let $X_i = E(C_{m}^{i}) \setminus \{v_i v_{i+1}, v_i^m v_i^{m+1}, v_i^{m+1} v_i^{m+2}\}$.

(E1) $c(X_i) \cap c(E(P_n)) = \emptyset$ for each $i \in [1, n-1]$
By considering $\{v_i, v_i^p, v_j\}$ for $j \in [1, n-1], j \not= i$, and $p \in \{\frac{m}{2}, \frac{m}{2} + 2\}$, it is clearly that $c(X_i) \cap c(E(P_n)) = \emptyset$

(E2) $c(X_i) \cap c(X_j) = \emptyset$ for $i, j \in [1, n-1]$ with $i \neq j$
By considering $\{v_i, v_i^p, v_j^q\}$ for $i < j$ and $p, q \in \{\frac{m}{2}, \frac{m}{2} + 2\}$, we obtain that $c(X_i) \cap c(X_j) = \emptyset$.

(E3) For each $i \in [1, n-1]$, $|c(X_i)| \geq 1$ for $m = 4$, $|c(X_i)| \geq m - 4$ for $m \in \{6, 8\}$, and $|c(X_i)| \geq m - 3$ for $m \geq 10$
For $m = 4$, it is clearly that $|c(X_i)| \geq 1$. For $m = 6$, since $c(v_i^6 v_i^6) \not= (v_i^6 v_i^6)$ for each $i \in [1, n-1]$, we have $|c(X_i)| \geq 2$. For $m = 8$, suppose that $|c(X_i)| \leq 3$. Observe that edges $v_i^6 v_i^7, v_i^7 v_i^8, v_i v_i^8$ should be assigned with different colors, which means every color in $c(X_i)$ should be used to color these edges. Next, we consider edges $v_i^7 v_i^7, v_i^3 v_i^3, v_i v_i^8$ for each $i \in [1, n-1]$, we have $|c(X_i)| \geq 2$. For $m = 10$, suppose that $|c(X_i)| \leq 3$. Observe that edges $v_i^6 v_i^7, v_i^7 v_i^8, v_i v_i^8$ should be assigned with different colors, which means every color in $c(X_i)$ should be used to color these edges. Next, we consider edges $v_i^7 v_i^7, v_i^3 v_i^3, v_i v_i^8$ for each $i \in [1, n-1]$, we have $|c(X_i)| \geq 2$.

For $m \geq 10$, it is clearly that $|c(X_i)| \geq m - 3$ by Theorem 1.4.

Now, we distinguish three subcases.

**Subcase 2.1. $m = 4$**

By using (2), (E1), (E2), and (E3), we have $sr x_3(P_6^m \bowtie_w C_4) \geq 2n - 2$. Furthermore, since $sr x_3(C_4) = 2$ by Theorem 1.4. it follows by Theorem 2.3 that $sr x_3(P_6^m \bowtie_w C_4) \leq 2(n - 1) - 2n - 2$. An illustration of a strong 3-rainbow coloring of $P_6^m \bowtie_w C_4$ is given in Figure 5.

**Subcase 2.2. $m \in \{6, 8\}$**

Suppose that $sr x_3(P_6^m \bowtie_w C_m) \leq (m - 3)(n - 1)$. Let $c : E(P_6^m \bowtie_w C_m) \rightarrow [1, (m - 3)(n - 1)]$ be a strong 3-rainbow coloring of $P_6^m \bowtie_w C_m$. By using (2), (E1), (E2), and (E3), we need at least $(m - 3)(n - 1)$ different colors assigned to all of the edges of $P_6^m \bowtie_w C_m$ except edges $v_i^m v_i^{m+1}$ and $v_i^{m+1} v_i^{m+2}$ for $i \in [1, n-1]$. Now, we consider $\{v_i^p, v_i^{p+1}, v_i^q\}$ for all $i \in [2, n-1], p \in \{\frac{m}{2}, \frac{m}{2} + 1\}$, $p \in \{\frac{m}{2}, \frac{m}{2} + 1\}$, and $q \in \{\frac{m}{2}, \frac{m}{2} + 1\}$.
A rainbow Steiner $P_n \Rightarrow C_3$.

Figure 5. A strong 3-rainbow coloring of $P_n \Rightarrow C_4$.

and $q \in \{\frac{m}{2}, \frac{m}{2} + 2\}$. This forces \( \{c(v_1^m v_1^0), c(v_1^{m+1} v_1^{1+2})\} \subseteq \{c(v_1v_2)\} \cup c(X_1) \), implying that all edges of $C_3^n$ are assigned with $m - 3$ different colors, contradicts Theorem 1.4.

For the upper bound, we first define an edge-coloring $c$ of $P_n \Rightarrow C_m$ using $(m - 3)(n - 1) + 1$ colors as follows. For $m = 6$ and $i \in [1, n - 1]$, define $c(v_iv_{i+1}) = i$, $c(v_i v_{i+1} v_{i+1} v_1 v_2) = i + n - 1$, $c(v_i v_j) = i + 2(n - 1)$, and $c(v_i v_j) = 3(n - 1) + 1$. Meanwhile for $m = 8$ and $i \in [1, n - 1]$, define $c(v_i v_{i+1}) = i$, $c(v_i v_{i+1} v_{i+1} v_1 v_2) = i + n - 1$, $c(v_i v_j) = i + 2(n - 1)$, $c(v_i v_j) = 3(n - 1) + 1$, and assign the colors $3(n - 1) + 2, 3(n - 1) + 3, \ldots, 5(n - 1) + 1$ to the remaining $2(n - 1)$ edges of $P_n \Rightarrow C_8$.

Now, let $S$ be a 3-subset of $V(P_n \Rightarrow C_m)$. Similar to the proof of Subcase 1.1, we distinguish two cases. First, we consider $S = \{v_i^p, v_i^q, v_j^r\}$ for distinct $i, j \in [1, n - 1]$. For $i < j$, let $P$ be a $v_{i+1} - v_j$ geodesic, $P_i = v_j v_{j+1} v_j v_j v_j^r$ or $P_j = v_j v_j v_j^r v_j v_j$ if $r \in [\frac{m}{2}, \frac{m}{2} + 1]$, and $P_i = v_j v_j v_j^r v_j v_j$ if $r \in [\frac{m}{2} + 2, m]$. Let $P_i = P \cup P_j$ with $b \in [1, 2]$. Then there is a rainbow Steiner $S$-tree as given in Table 3, where the value of $b$ depends on the value of $r \in [3, m]$. The proof for $i > j$ is similar to the case for $i < j$.

| Set $S$ | Condition | A rainbow Steiner $S$-tree |
|--------|-----------|---------------------------|
| $\{v_i, v_{i+1}, v_j^r\}$ | $p = 1, q = 2$ | $v_i v_{i+1} \cup P_1$ |
| $\{v_i, v_i^q, v_j^r\}$ | $p = 1, q \in [\frac{m}{2}, \frac{m}{2}] + 1$ | $v_i^q v_i v_i v_i v_{i+1} \cup P_1$ |
| $\{v_{i+1}, v_i^q, v_j^r\}$ | $p = 2, q \in [\frac{m}{2}, \frac{m}{2}] + 1$ | $v_i^q v_i v_i v_i v_{i+1} \cup P_1$ |
| | $p = 2, q \in [\frac{m}{2} + 2, m]$ | $v_i^q v_i v_i v_i v_{i+1} \cup P_1$ |
| | $p < q$ | $v_i^q v_i v_i v_i v_{i+1} \cup P_1$ |
| | $m = 3, p = 3, q = 6$ | $v_i v_i v_i v_i v_{i+1} \cup P_1$ |
| | $m = 6, p = 4, q = 6$ | $v_i^q v_i v_i v_i v_{i+1} \cup P_1$ |
| | $m = 8, p \in [3, 4], q \in [7, 8]$ | $v_i^q v_i v_i v_i v_{i+1} \cup P_1$ |
| | $m = 8, p = 5, q \in [7, 8]$ | $v_i^q v_i v_i v_i v_{i+1} \cup P_1$ |
Next, without loss of generality, we consider \( S = \{v^p_i, v^q_j, v^r_k\} \) for \( i, j, k \in [1, n - 1] \) with \( i < j < k \). Let \( P \) be a \( v_i+1 - v_k \) geodesic, \( P^1_i = v_i+1v^p_i v^{p-1}_i v^p_i \) if \( p \in [3, \frac{m}{2} + 1] \), \( P^2_i = v_i+1v^m_i v^{m+1}_i v^p_i \) if \( p \in [\frac{m}{2} + 2, m] \), \( P^1_j = v_j+1v^q_j v^{q-1}_j v^q_j \) if \( q \in [3, \frac{m}{2} + 1] \), \( P^2_j = v_jv^{m+1}_j v^q_j \) if \( q \in [\frac{m}{2} + 2, m] \), \( P^1_k = v_kv_{k+1}v^r_k v^{r-1}_k v^r_k \) if \( r \in [3, \frac{m}{2} + 1] \), and \( P^2_k = v_kv^m_k v^r_k \) if \( r \in [\frac{m}{2} + 2, m] \). Then the tree \( T = P \cup P^1_i \cup P^2_i \cup P^1_j \cup P^2_j \cup P^1_k \cup P^2_k \) is a rainbow Steiner \( S \)-tree, where the values of \( a, b, c \) depend on the values of \( p, q, r \), respectively.

Note that the case when \( S \) contains at least one vertex of \( P_n \) has been proven for each of the above cases. Figure 6 illustrates the strong 3-rainbow colorings of \( P^n_8 \triangleright_\varepsilon C_m \) for \( m \in \{6, 8\} \).

![Figure 6. Strong 3-rainbow colorings of \( P^n_8 \triangleright_\varepsilon C_m \) for \( m \in \{6, 8\} \).](image)

**Subcase 2.3. \( m \geq 10 \)**

Suppose that \( srx_3(P^n_m \triangleright_\varepsilon C_m) \leq (m - 2)(n - 1) + 2 \). Let \( c : E(P^n_m \triangleright_\varepsilon C_m) \to [1, (m - 2)(n - 1) + 2] \) be a strong 3-rainbow coloring of \( P^n_m \triangleright_\varepsilon C_m \). By using (2), (E1), (E2), and (E3), we need at least \( (m - 2)(n - 1) \) different colors assigned to the edges of \( P^n_m \triangleright_\varepsilon C_m \) except edges \( v^m_i v^{m+1}_i \) and \( v^m_i v^{m+2}_i \) for \( i \in [1, n - 1] \). This means we have at most two colors left, say 1 and 2. Next, we consider edges \( v^m_i v^{m+1}_i \) and \( v^m_i v^{m+2}_i \). Note that by Theorem 1.4, edges of \( C_m \) should be assigned with different colors. Hence, by considering \( \{v^p_i, v^{p+1}_i, v^q_i\} \) for all \( i \in [2, n - 1] \), \( p \in [\frac{m}{2}, \frac{m}{2} + 1] \), and \( q \in [\frac{m}{2}, \frac{m}{2} + 2] \), we obtain that edges \( v^m_i v^{m+1}_i \) and \( v^m_i v^{m+2}_i \) cannot be assigned with colors from \( c(E(P_n)) \) and \( c(X_i) \) for all \( i \in [1, n - 1] \). This
forces \(\{c(v_1^{\frac{m}{2}}, v_1^{\frac{m}{2}+1}), c(v_1^{\frac{m}{2}+1}, v_1^{\frac{m}{2}+2}\}\} \subseteq \{1, 2\}\). Without loss of generality, let \(c(v_1^{\frac{m}{2}}, v_1^{\frac{m}{2}+1}) = 1\) and \(c(v_1^{\frac{m}{2}+1}, v_1^{\frac{m}{2}+2}) = 2\). Similarly, we obtain that edges \(v_2^{\frac{m}{2}}, v_2^{\frac{m}{2}+1}\) and \(v_2^{\frac{m}{2}+1}, v_2^{\frac{m}{2}+2}\) can not be assigned with colors from \(c(E(P_n) \setminus \{v_1v_2\})\) and \(c(X_i)\) for all \(i \in [2, n-1]\), implying that \(\{c(v_1v_2), c(v_2^{\frac{m}{2}+1}, v_2^{\frac{m}{2}+2}\}\} \subseteq \{c(v_1v_2), 1, 2\} \cup c(X_1)\). However, by considering \(\{v_2^{\frac{m}{2}}, v_2^{\frac{m}{2}+2}, v_1^p\}\) for \(p \in \{\frac{m}{2} + 1, \frac{m}{2} + 3\}\), this forces \(\{c(v_2^{\frac{m}{2}}, v_2^{\frac{m}{2}+1}), c(v_2^{\frac{m}{2}+1}, v_2^{\frac{m}{2}+2}\}\} \subseteq \{2, c(v_1^{\frac{m}{2}+2}, v_1^{\frac{m}{2}+3}\}\}. But, there is no rainbow Steiner \(\{v_2^{\frac{m}{2}}, v_2^{\frac{m}{2}+2}, v_1^m\}\)-tree, a contradiction.

For the upper bound, we first define an edge-coloring \(c\) of \(P_n^{o} \triangleright_{\varepsilon} C_m\) using \((m-2)(n-1) + 3\) colors as follows.

1. Assign the colors \(1, 2, \ldots, (m-2)(n-1)\) to the edges of \(P_n^{o} \triangleright_{\varepsilon} C_m\) except edges \(v_i^{\frac{m}{2}}, v_i^{\frac{m}{2}+1}\) and \(v_i^{\frac{m}{2}+1}, v_i^{\frac{m}{2}+2}\) for all \(i \in [1, n-1]\).

2. Define \(c(v_i^{\frac{m}{2}}, v_i^{\frac{m}{2}+1}) = (m-2)(n-1) + 1\), \(c(v_i^{\frac{m}{2}+1}, v_i^{\frac{m}{2}+2}) = (m-2)(n-1) + 2\), and \(c(v_i^{\frac{m}{2}}, v_i^{\frac{m}{2}+1}) = (m-2)(n-1) + 3\).

3. For each \(i \in [2, n-1]\), define \(c(v_i^{\frac{m}{2}+1}, v_i^{\frac{m}{2}+2}) = (v_i^{\frac{m}{2}+1}, v_i^{\frac{m}{2}+2})\).

4. For each \(i \in [3, n-1]\), define \(c(v_i^{\frac{m}{2}}, v_i^{\frac{m}{2}+1}) = (v_i^{\frac{m}{2}}, v_i^{\frac{m}{2}+1})\).

Let \(S\) be a 3-subset of \(V(P_n^{o} \triangleright_{\varepsilon} C_m)\). Similar to the proof of Subcase 1.2, we can find a rainbow Steiner \(S\)-tree in \(P_n^{o} \triangleright_{\varepsilon} C_m\). Figure 7 illustrates a strong 3-rainbow coloring of \(P_5^{o} \triangleright_{\varepsilon} C_{10}\).

![Figure 7. A strong 3-rainbow coloring of \(P_5^{o} \triangleright_{\varepsilon} C_{10}\).](image)

Following Theorem 3.4, we obtain that \(srx_3(P_n^{o} \triangleright_{\varepsilon} C_4)\) is equal to the upper bound in Theorem 2.3, meanwhile for other values of \(m\), the \(srx_3(P_n^{o} \triangleright_{\varepsilon} C_{m})\) is not equal to the upper bound.
4. Conclusion

We have shown that $H$ is a tree if and only if $srx_3(P_n^o \triangleright e H) = |E(P_n^o \triangleright e H)|$. Further, we have also provided a sharper upper bound for $srx_3(P_n^o \triangleright e H)$, that is $srx_3(P_n^o \triangleright e H) \leq srx_3(H)(n-1)$, and have determined the exact values of $srx_3(P_n^o \triangleright e H)$ for some connected graphs $H$.

There are many classes of connected graphs $H$ for which the $srx_3(P_n^o \triangleright e H)$ is not known. Hence, it is interesting to continue the study by determining the exact value of $srx_3(P_n^o \triangleright e H)$ for other connected graphs $H$. These results are expected to help characterize the connected graphs $H$ with $srx_3(P_n^o \triangleright e H) = srx_3(H)(n-1)$. Since a path is one of classes of trees, it is also interesting to study the $srx_3$ of edge-comb product of a tree and a connected graph.

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