Additive Partial Matchings Induced by Persistence Morphisms

Rocio Gonzalez-Diaz¹, Manuel Soriano-Trigueros²,¹, and Alvaro Torras-Casas³,¹

¹Universidad de Sevilla, Spain
²Institute of Science and Technology Austria, Austria
³Inserm, Center for Research in Epidemiology and Statistics (CRESS) UMR 1153

rogodi@us.es, msoriano@ist.ac.at, atorras@us.es

Abstract

Given a morphism of persistence modules (a.k.a. persistence morphism) \( f : V \to U \), we introduce a novel operator that determines a partial matching between the barcodes of \( V \) and \( U \) induced by \( f \). We show that the proposed operator is additive with respect to the direct sum of persistence morphisms and contains more information than the image persistence module \( fV \) and the rank invariant. We also illustrate some advantages of using our induced partial matching over the Bauer-Lesnick partial matching \( \chi_f \). Lastly, we provide a family of persistence morphisms that contain modules built from Morse filtrations, for which \( \chi_f, fV \), the rank invariant and our proposed induced partial matching are equivalent.

1 Introduction

Persistence modules (PMs) have been extensively used to study persistent homology, one of the main tools of topological data analysis (TDA). Morphisms of PMs (so-called persistence morphisms) appear in different settings, for example, when calculating the persistent homology of a filtered complex coming from two point clouds, one embedded into the other [9].

Informally, a PM is defined as a finite sequence of finite dimensional vector spaces over a fixed field \( k \) and linear maps between them,

\[
V_1 \longrightarrow V_2 \longrightarrow \ldots \longrightarrow V_n.
\]

A persistence morphism \( f : V \to U \) between two PMs \( V \) and \( U \) is a finite set of linear maps \( f_i : V_i \to U_i \) for \( i = 1, 2, \ldots, n \), making the following diagram commutative,

\[
\begin{array}{c}
U \\
\downarrow f_i \downarrow \\
V \\
\end{array}
\begin{array}{c}
U_1 \\
\longrightarrow U_2 \\
\longrightarrow \ldots \\
\longrightarrow U_n \\
\end{array}
\begin{array}{c}
V_1 \\
\longrightarrow V_2 \\
\longrightarrow \ldots \\
\longrightarrow V_n \\
\end{array}
\]

The structure theorem [8,6] states that PMs can always be described using collections of intervals, known as barcodes. Barcodes help researchers to gain insight into the topological structure of data [10]. The usual pipeline is the following: First, a nested sequence of simplicial complexes is obtained from the point cloud using, for example, the Vietoris-Rips filtration. Then, the homology functor is applied to obtain a vector space from each of these complexes and linear maps between them, leading to a PM. The intuition is that the longer an interval is, the more important the homology feature it represents. See Figure 1 for an example.

Now, see Figure 1 where our visual intuition suggests that the large circles sampled by \( X \) and \( X \cup Y \) are the same, although the sets of points differ. If we denote the PMs obtained from
Figure 1: On the top, the point clouds $X$ and $Y$ are displayed. On the bottom, the barcodes obtained from the 1-dimensional homology of the associated Vietoris-Rips filtration of $X$ and $X \cup Y$ are shown. Both barcodes detect two large circles represented by long bars. In addition, the barcode of $X \cup Y$ detects several holes depicted by short bars, which are interpreted as noise in this case.

Figure 2: On the top, the partial matching suggested by our visual intuition, where the intervals representing the left (resp. right) circle are matched. On the bottom, the partial matching that minimizes the bottleneck distance.

the point clouds $X$ and $X \cup Y$ as $V$ and $U$, we can use the inclusion $X \hookrightarrow X \cup Y$ to obtain a persistence morphism $f : V \to U$ [7]. Barcodes are usually compared using partial matching, i.e. a partial biyection between intervals. In practice, barcodes are compared using the bottleneck distance [10, 6], which is defined using a partial matching that minimizes the supremum among the difference between endpoints of the matched intervals as well as the length of the unmatched
intervals. Note that the partial matching used to define the bottleneck distance may not align with our visual intuition, see Figure 2. In this paper, our aim is to provide an algorithm that obtains a partial matching induced by \( f \), identifying, for example in Figure 1, the right (resp. left) circle sampled by \( X \) with the right (resp. left) circle sampled by \( X \cup Y \), as a consequence of the inclusion between \( X \) and \( X \cup Y \).

Besides, in this paper, we study the properties that ensure the proposed induced partial matching behaves as expected in situations such as the one depicted in Figure 1, like being additive with respect to the direct sum of persistence morphisms. We also prove that it is richer than other invariants used to study \( f \), like the rank invariant or the image persistence module \( fV \).

In the rest of this section, we motivate the problem, relate our construction to other TDA concepts, and further explain the aforementioned properties.

1.1 Decomposition of persistence morphisms

Persistence morphisms with the structure displayed in Diagram 1 belong to a more general class of PMs called ladder modules \([11]\). Ladder modules can be uniquely described, up to isomorphism, as direct sums of indecomposable ladder modules \([11]\), as illustrated in the following example.

**Example 1.1.** The following persistence morphism

\[
\begin{array}{c}
U \\
\cap \\
V
\end{array} \xrightarrow{f} \begin{array}{c}
k^2 \xrightarrow{id} k^2 \\
\uparrow \\
0 \\
\uparrow \\
k^2 \xrightarrow{10} k
\end{array}
\]

is isomorphic to the following direct sum of indecomposables:

\[
\begin{array}{c}
k \\
\uparrow \\
0 \\
\uparrow \\
k \\
\uparrow \\
0 \\
\uparrow \\
k \\
\uparrow \\
0 \\
\uparrow \\
k \\
\uparrow \\
0
\end{array} \oplus \begin{array}{c}
k \xrightarrow{id} k \\
0 \xrightarrow{0} 0 \\
k \xrightarrow{id} k \\
0 \xrightarrow{0} 0
\end{array}
\]

In this case, there is not a clear way of inducing a partial matching from \( f \), as there was in Example 1.1.

Note that, when the indecomposables have a simple form as in the previous example, we can deduce directly a partial matching induced by the persistence morphism from its decomposition \([15]\). However, as the following example shows, there are also more complex indecomposables that do not clearly determine a partial matching.

**Example 1.2.** The following persistence morphism is an indecomposable.

\[
\begin{array}{c}
U \\
\cap \\
V
\end{array} \xrightarrow{f} \begin{array}{c}
k \xrightarrow{[1]} k^2 \xrightarrow{[10]} k \\
\uparrow \\
k \xrightarrow{[0]} k^2 \xrightarrow{[10]} k
\end{array}
\]

In this case, there is not a clear way of inducing a partial matching from \( f \), as there was in Example 1.1.

Moreover, the set of indecomposable ladder modules (including persistence morphisms) is wild for \( n > 5 \) \([11]\). Up to our knowledge, there is no computationally efficient way of calculating indecomposables in such case.
1.2 Persistence morphism invariants

Since calculating the decomposition of a persistence morphism \( f \) is generally impractical, we need to find other ways to study \( f \). One possibility is the rank invariant \([5]\), given by the rank of the linear maps between all possible pairs of vector spaces in Diagram 1. Another possibility is to calculate the image persistence module \( fV \) \([7]\), given by the following submodule of \( U \)

\[
f_1V_1 \rightarrow \ldots \rightarrow f_nV_n.
\]

As we will see, our induced partial matching provides in general richer information than these invariants.

There exist other induced partial matchings in the literature. The one that appeared first is the Bauer-Lesnick induced partial matching \( \chi_f \), which combines the barcodes of \( V \), \( fV \) and \( U \) to match intervals from the barcode of \( V \) to the barcode of \( U \). The virtue of \( \chi_f \) is that it is fast to calculate and provides a constructive proof of the stability theorem \([2, 3]\). However, its output sometimes contradicts the intuition. For instance, if \( f \) is the persistence morphism from Example 1.1, then \( \chi_f \) induces the partial matching

\[
\emptyset \mapsto [1, 2], \quad [2, 3] \mapsto [1, 2] \quad \text{and} \quad [2, 2] \mapsto \emptyset
\]

instead of the expected partial matching described in Example 1.1. On the contrary, our proposed operator gives the expected partial matching as explained in Example 5.2.

Some other partial matchings induced by persistence morphisms have been proposed: the cycle registration method \([16]\) and the cohomological cycle matching \([12]\). As mentioned in those papers, those matchings can be seen as variants of \( \chi_f \) applied in different settings.

An alternative construction to relate persistence diagrams is the persistent extension method introduced in \([19]\), although that procedure is based on the witness complex and the Functorial Dowker Theorem, and then have different properties to ours.

Another related concept is the block function \( \mathcal{M}_f \) introduced by the authors of this paper in \([13]\). A block function can be seen as a weaker version of a partial matching. In \([13]\) we showed that \( \mathcal{M}_f \) is additive with respect to the direct sum of persistence morphisms, so, when \( f \) from Example 1.1 is used, the output of \( \mathcal{M}_f \) aligns with the intuition,

\[
\emptyset \mapsto [1, 2], \quad [2, 3] \mapsto \emptyset \quad \text{and} \quad [2, 2] \mapsto [1, 2].
\]

However, the block function \( \mathcal{M}_f \) does not determine a partial matching in general. For instance, if \( f \) is the persistence morphism given in Example 1.2 then from \( \mathcal{M}_f \) we deduce the following assignment

\[
[1, 4] \mapsto [1, 3] \quad \text{and} \quad [2, 3] \mapsto [1, 3],
\]

that is not a partial matching. We will see later that the method proposed in this paper always induces a unique partial matching for each \( f \). Further, in \([B]\) we show that the new method is equal to \( \mathcal{M}_f \) when the latter induces a partial matching.

All of these constructions are strongly related to persistence basis calculation and ladder module decomposition, see \([18, 14, 11, 21]\). An exhaustive study of the relation between all these constructions, including ours, is beyond the scope of this work and is left for future study. We explain in \([A] \) and \([B]\) the relation with the partial matching \( \chi_f \), and with the block function \( \mathcal{M}_f \).

1.3 Results presented in this paper

The paper is structured as follows. We introduce the necessary background in Section 2 and important operators used in the rest of the paper in Section 3. We propose a newly defined block function
\(\widetilde{M}_f\) in Section 4 and explain how to calculate a partial matching from it. In Section 5, we show some properties of \(\widetilde{M}_f\), including how \(fV\) or the rank invariant can be obtained from it, and its additivity. We also introduce here the concept of ULRE persistence morphisms, which include the persistence morphisms built from Morse filtrations, and for which \(fV\), the rank invariant, \(\chi_f\), and \(\widetilde{M}_f\) coincide. The algorithm for calculating \(\widetilde{M}_f\) by means of Gaussian reductions is given in Section 6. In Section 7, we conclude with an overall analysis of the results and some comments about future directions. The relations with some previous constructions are left as appendices. The interested reader can find a concise comparison of \(\widetilde{M}_f\) with \(\chi_f\) and \(M_f\) in A and B, respectively. Finally, some technical lemmas can be found in C, D and E. The notation used in this paper is provided in Table 1.

2 Background

In this section, we provide the notions and preliminary results needed to understand the rest of the paper. For a more general introduction to persistence modules, please consult [6].

2.1 Persistence modules

All vector spaces considered in this paper are defined over a fixed field \(\mathbb{F}\). Given \(n \in \mathbb{N}\), the expression \([n]\) denotes the set of integers \(\{1, 2, \ldots, n\}\). A persistence module (PM) \(V\) indexed by the set \([n]\) consists of a finite set of vector spaces \(V_p\) for \(p \in [n]\), and a set of linear maps \(V_p \xrightarrow{\rho_{pq}} V_q\) for \(p, q \in [n]\) and \(p \leq q\) such that \(\rho_{q1}\rho_{pq} = \rho_{p1}\) if \(p \leq q \leq l\) and \(\rho_{pp}\) is the identity map. The maps \(\rho_{pq}\) are known as the structure maps of \(V\) and are denoted simply by \(\rho\) when no confusion arises. To simplify notation, we add, to each PM, the trivial vector spaces \(V_0 = V_{n+1} = 0\) as well as the respective trivial structure maps \(\rho_{0p}\) and \(\rho_{p(n+1)}\) for \(0 \leq p \leq n + 1\).

Given \(a\) and \(b\) in \([n]\) such that \(a \leq b\), we write \(I = [a, b]\) to denote the interval set \([a, a + 1, \ldots, b]\), and \(I(n)\) to denote all the intervals that are subsets of \([n]\). The interval module \(k_I\) is composed by \(k_I = k\) for all \(t \in I\) and \(k_I = 0\) otherwise, while the structure maps are given by the identity map whenever possible and the zero map otherwise.

In the following, we always denote the structure maps of the PMs \(V\) and \(U\) as \(\rho\) and \(\phi\), respectively. The persistence module \(V\) is a submodule of \(U\) if \(V_p \subseteq U_p\) for all \(p \in [n]\) and \(\rho_{pq} = \phi_{pq}\big|_{V_p}\) for any pair \(p \leq q\). The direct sum of \(V\) and \(U\), \(V \oplus U\), is defined using the vector spaces \(V_p \oplus U_p\) and the structure maps \(\rho_{pq} \oplus \phi_{pq}\). If \(V\) is a submodule of \(U\), the quotient \(U/V\) is the persistence module whose vector spaces are \(U_i/V_i\) for all \(i \in [n]\) and whose structure maps are induced by the structure maps of \(U\).

As anticipated in Section 1 a persistence morphism \(f : V \rightarrow U\) is a set of linear maps \(\{f_p\}_{p \in [n]}\), making the following diagram commutative,

\[
\begin{array}{cccccc}
0 & \rightarrow & U_1 & \xrightarrow{\phi_{12}} & U_2 & \xrightarrow{\phi_{23}} \cdots & \xrightarrow{\phi_{n-1n}} & U_n & \rightarrow & 0 \\
& & f \downarrow & & f \downarrow & & f \downarrow & & f \downarrow & & \\
0 & \rightarrow & V_1 & \xrightarrow{\rho_{12}} & V_2 & \xrightarrow{\rho_{23}} \cdots & \xrightarrow{\rho_{n-1n}} & V_n & \rightarrow & 0
\end{array}
\]

The persistence morphism \(f\) is an isomorphism/surjection/injection of PMs when all \(f_p\) are isomorphisms/surjections/injections of vector spaces for \(p \in [n]\).

In the following, we omit the subindices of the linear map when these are clear from the context, for example, we might write \(fV_p\) instead of \(f_pV_p\). In addition, we can obtain a persistence module
| Section | Table 1: Notation used in this paper. |
|----------|--------------------------------------|
|          | Persistence modules (PMs)            |
|          | Structure maps of $V$, $U$            |
|          | Persistence morphism                  |
|          | Intervals                              |
|          | Set of intervals that are subsets of $[n]$ |
|          | Interval modules                       |
|          | Persistence diagram (PD)               |
|          | Barcode                                |
|          | Persistence basis                      |
|          | Persistence generator                   |
|          | Image/kernel operators                  |
|          | PMs associated to $V$ and $I$           |
|          | Persistence bases for $V_I^\pm$, $V_I$ |
|          | Partial matching                        |
|          | Block Function                          |
|          | Sections of $V_I$                      |
|          | Persistence bases for $\hat{V}_I^\pm$, $\hat{V}_I$ |
|          | Block function induced by $f$ defined in [13] |
|          | Block function induced by $f$ defined here |
|          | PM used to define $\hat{M}_f(I,J)$      |
|          | Sections of $U_{Jd}$, $V_{Ia}$         |
|          | Total orders on the intervals           |
|          | Sequence of filtered complexes          |
|          | Matrix associated to $f$                |
|          | Gaussian column reduction of $F$        |
|          | Minors of $R$ used to calculate $\hat{M}_f(I,J)$ |

| $V$, $U$ | $V_p \to V_q, U_p \to U_q$ |
| $f : V \to U$ | $I = [a, b], J = [c, d]$ |
| $I(n)$ | $k_{[a,b]}$ |
| $\text{PD}(V) = \{(I, m^V_I)\}$ | $\mathbf{B}(V) = \{(I, i) \mid i \in \{m^V_I\}\}$ |
| $\alpha : \bigoplus_{I \in \text{PD}(V)} k_{[a,b]} \to V$ | $a_i : k_{[a,b]} \to V$ |
| $\text{Im}^\pm, \text{Ker}^\pm$ | $V_I^\pm, V_I$ |
| $\mathcal{A}_f^\pm, \mathcal{A}_I$ | $\sigma : \mathbf{B}(V) \to \mathbf{B}(U)$ |
| $\mathcal{M} : I(n) \times I(n) \to \mathbb{Z}_{\geq 0}$ | $\hat{M}_f$ |
| $\mathcal{M}_f$ | $\hat{M}_f$ |
| $Z_{IJ}^\pm$ | $Z_{IJ}^\pm$ |
| $X_i$ | $\leq \cdot, \leq^*$ |
| $\{X_i\}_{i=1}^n$ | $F$ |
| $R$ | $\hat{R}_{IJ}^\pm, \hat{R}_{IJJ}$ |
from $f$ from its image vector spaces,
\[ fV := fV_1 \xrightarrow{\phi_1|_{V_1}} fV_2 \xrightarrow{\phi_2|_{V_2}} \cdots \xrightarrow{\phi_{n-1}|_{V_{n-1}}} fV_n, \]
known as the image persistence module and introduced in [7].

2.2 Persistence diagrams and barcodes

A multiset can be seen as a set where each element can be repeated more than once. Formally, a multiset is defined as $\mathcal{S} = \{ (s, m_s) \mid s \in S, m_s \in \mathbb{N} \}$ where $S$ is a set of elements and $m_s$ represents the number of copies of $s$ in $S$, i.e. its multiplicity. The representation of a multiset, $\text{Rep}(\mathcal{S})$ is the set defined by
\[ \{ (s, i) \in S \times \mathbb{N} \mid i \in [m_s] \}. \]
The size of $\mathcal{S}$ coincides with the size of its representation as calculated using the following formula,
\[ \#\mathcal{S} = \#\text{Rep}(\mathcal{S}) = \sum_{s \in S} m_s. \]

The following theorem states that any PM $V$ can be expressed uniquely, up to isomorphism, as a direct sum of interval modules.

**Theorem 2.1** ([20], [8]). If $V$ is a PM then there exists a multiset $\{ (I, m_I) \}$ of intervals $I \in I(V)$ and integers $m_I \in \mathbb{N}$ such that
\[ V \cong \bigoplus_{I \in I(V)} \left( \bigoplus_{\ell=1}^{m_I} k_I \right). \]

The multiset given by Theorem 2.1 is called the persistence diagram (PD) of $V$, and is denoted as $\text{PD}(V)$. We denote the representation of $\text{PD}(V)$ as the barcode of $V$, $\text{B}(V) := \text{Rep}(\text{PD}(V))$.

PDs are plotted depicting the intervals as points in the plane with an index representing their multiplicity [10]. The multiplicity index might be dropped when it is not important for the representation, as in Figure 1. Be aware that, although persistence diagrams and barcodes are often considered equivalent, this paper uses persistence diagrams for multisets and barcodes for their representations. In the following, we write the elements of $\text{B}(V)$ as $I_i$ instead of $(I, i)$. Lets us see an example.

**Example 2.2.** The PD of $V$ and $U$ in Example 1.1 are $\text{PD}(V) = \{ ([2, 2], 1), ([2, 3], 1) \}$ and $\text{PD}(U) = \{ ([1, 2], 2) \}$ respectively. Their barcodes are $\text{B}(V) = \{ [2, 2], [2, 3] \}$ and $\text{B}(U) = \{ [1, 2], [1, 2, 2] \}$.

If $(I, 1)$ is the only copy of the interval $I$ in the barcode, we might just write $I$ instead of $I_1$. We also might write $m_I^V$ to specify that the multiplicity of $I$ corresponds to the decomposition of $V$.

2.3 Persistence bases

A persistence basis for $V$ [18, 6] is an isomorphism
\[ \alpha : \bigoplus_{\ell \in [\#\text{PD}(V)]} k_{[a_{\ell}, b_{\ell}]} \to V. \]

Such isomorphism always exists by Theorem 2.1. The persistence generator $\alpha_{\ell} : k_{[a_{\ell}, b_{\ell}]} \to V$ is defined as the persistence morphism $\alpha$ restricted to $k_{[a_{\ell}, b_{\ell}]}$ for $\ell \in [\#\text{PD}(V)]$. When we write $\alpha_{\ell} \sim [a_{\ell}, b_{\ell}]$, we mean that $k_{[a_{\ell}, b_{\ell}]}$ is the domain of $\alpha_{\ell}$. We also specify a persistence basis $\alpha$ by its set of persistence generators $\mathcal{A} = \{ \alpha_{\ell} \}_{\ell \in [\#\text{PD}(V)]}$. 

7


**Example 2.3.** Consider the PM $V$ given in Example 2.2. Since $V \cong k_{[1,4]} \oplus k_{[2,3]}$ then $\alpha : k_{[1,4]} \oplus k_{[2,3]} \rightarrow V$ is a persistence basis for $V$ and the persistence generators $\alpha_1$ and $\alpha_2$ are given by the following commutative diagrams:

\[
\begin{align*}
V & \xrightarrow{\alpha_1} k_{[1,4]} \xrightarrow{k} k_{[2,3]} \xrightarrow{k} V \\
& \quad \uparrow{\text{id}} \quad \uparrow{\begin{bmatrix} 1 & 0 \end{bmatrix}} \quad \uparrow{\begin{bmatrix} 1 & 0 \end{bmatrix}} \quad \uparrow{\text{id}} \\
\end{align*}
\]

and

\[
\begin{align*}
V & \xrightarrow{\alpha_2} k_{[2,3]} \xrightarrow{k} k_{[1,4]} \xrightarrow{k} V \\
& \quad \uparrow{0} \quad \uparrow{\begin{bmatrix} 1 & 0 \end{bmatrix}} \quad \uparrow{\begin{bmatrix} 1 & 0 \end{bmatrix}} \quad \uparrow{0} \\
\end{align*}
\]

**Definition 2.4.** Given a subset $S = \{a_i\}_{i \in \Lambda}$ of a persistence basis $A$ with indexes $\Lambda \subseteq [\#A]$ we define the span of $S$, denoted by $\langle S \rangle$, as the image of the sum of persistence generators in $S$, that is

\[
\langle S \rangle = \text{Im}\left( \bigoplus_{i \in \Lambda} \alpha_i : \bigoplus_{i \in \Lambda} k_{[a_i, b_i]} \rightarrow V \right).
\]

For $t \in [n]$, we define $S_t^j := \{a_i^j \mid i \in \Lambda$ and $t \in \{a_i, b_i\}\}$, where $a_i^j$ stands for $\alpha_i(1_k)$. In particular, $V_t \cong \langle A_t \rangle$ and $\langle S_t^j \rangle \cong \langle S_t \rangle$, where $A_t$ and $S_t$ are sets of linearly independent vectors in $V_t$.

We use the following subsets of $A$ to characterize some spaces in the next subsection,

- $I_c^+(A) = \{ a_i \in A \mid a_i \sim [a_i, b_i] \text{ such that } a_i \leq c \}$,
- $I_c^-(A) = \{ a_i \in A \mid a_i \sim [a_i, b_i] \text{ such that } a_i < c \}$,
- $K_c^+(A) = \{ a_i \in A \mid a_i \sim [a_i, b_i] \text{ such that } b_i \leq c \}$,
- $K_c^-(A) = \{ a_i \in A \mid a_i \sim [a_i, b_i] \text{ such that } b_i < c \}$,

and write $I^\pm_c(A)$ and $K^\pm_c(A)$ instead of $(I^\pm_c(A))_t$ and $(K^\pm_c(A))_t$.

**Example 2.5.** Consider the persistence basis with persistence generators $\alpha_1 : k_{[1,4]} \rightarrow V$ and $\alpha_2 : k_{[2,3]} \rightarrow V$ obtained in Example 2.3. Then, for $c = 2$, we have

\[
\begin{align*}
I_2^+(A) &= \{ a_1, a_2 \}, \quad I_2^-(A) = \{ a_1 \}, \quad I_2^+(A) = \{ a_2^1, a_2^2 \}, \quad \text{and} \quad I_2^-(A) = \{ a_2^1 \}.
\end{align*}
\]

### 2.4 Image and kernel operators of PMs

The operators $\text{Im}^+_a$, $\text{Ker}^+_a$, defined for fixed $a, b \in [n]$, were used in [8] to prove Theorem 2.1 and in [13] to define the block function $M_f$. Since we need these operators to introduce $\tilde{M}_f$, we recall their definitions:

\[
\begin{align*}
\text{Im}^+_a(V) &:= \text{im} \rho_{ait}, \quad \text{Ker}^+_b(V) := \ker \rho_{iti(b+1)}, \\
\text{Im}^-_a(V) &:= \text{im} \rho_{a(i-1)t}, \quad \text{Ker}^-_b(V) := \ker \rho_{iti}.
\end{align*}
\]

Observe that when $a$ (resp. $b$) is fixed, $\text{Im}^+_a(V)$ (resp. $\text{Ker}^+_b(V)$) is a PM, since $\rho_{its} (\text{Im}^+_a(V)) = \text{Im}^+_a(V)$ (resp. $\rho_{its} (\text{Ker}^+_b(V)) \subseteq \text{Ker}^+_b(V)$) for $t < s$.

The following lemma gives an intuitive idea of their structure:

**Lemma 2.6 ([13]).** For any $a, b \in [n]$, we have the equalities:

- $\text{Im}^+_a(V) = \langle I^+_a(A) \rangle$ for all $t \in [a, n]$,
Remark 2.9. Example 2.10. compute the decomposition of implies that Lemma 2.8 of

Theorem 2.11. Let $A$ be a vector space and let $\{(F^-, F^+): \lambda \in \Lambda\}$ be a disjoint set of sections of $A$. Then

$$\bigoplus_{\lambda \in \Lambda} (F^+ / F^-) \subseteq A.$$ 

Particularly, if the set of sections is disjoint and covers $A$ then

$$\bigoplus_{\lambda \in \Lambda} (F^+ / F^-) \cong A.$$ 

2.5 Sections

Sections are the main algebraic tools used in [8] to prove Theorem 2.1. They will be used in some of the proofs in the upcoming sections. We recall their definition and properties here and recall a sketch of their role in [8] to illustrate their utility. A section of a vector space $A$ is a pair of vector spaces $(F^-, F^+)$ such that $F^- \subseteq F^+ \subseteq A$. We say that a set $\{(F^\lambda, F^\mu): \lambda, \mu \in \Lambda\}$ of sections of $A$ with index set $\Lambda$ is disjoint if, for all $\lambda \neq \mu$, either $F^\lambda \not\subseteq F^\mu$ or $F^\mu \not\subseteq F^\lambda$. It is said that it covers $A$ provided that for any subspace $B \subseteq A$ there is some $\lambda \in \Lambda$ satisfying that $B + F^\lambda \neq B + F^\mu$. 

Theorem 2.11. Let $A$ be a vector space and let $\{(F^\lambda, F^\mu): \lambda \in \Lambda\}$ be a disjoint set of sections of $A$. Then

$$\bigoplus_{\lambda \in \Lambda} (F^+ / F^-) \subseteq A.$$ 

In particular, note that these operators respect direct sum. The following simple property will be important in further sections.

Lemma 2.7. If $a < b$ then, $\text{Im}_a^+(V) \subseteq \text{Im}_b^+(V)$ and $\text{Ker}_a^+(V) \subseteq \text{Ker}_b^+(V)$.

We combine these PMs to create new ones associated with intervals $I = [a, b] \in I(n)$.

$$V_I^+ := \text{Im}_a^+(V) \cap \text{Ker}_b^+(V) = \text{im} \rho_{al} \cap \ker \rho_{tl(b+1)},$$

$$V_I^- := \text{Im}_a^-(V) \cap \text{Ker}_b^+(V) + \text{Im}_a^+(V) \cap \text{Ker}_b^-(V)$$

$$= \text{im} \rho_{al} \cap \ker \rho_{tl(b+1)} + \text{im} \rho_{al} \cap \ker \rho_{tb},$$

$$V_I := \frac{V_I^+}{V_I^-}.$$ 

Note that they are PMs since

$$\rho_{ls} (\text{Im}_a^+(V) \cap \text{Ker}_b^+(V)) \subseteq \rho_{ls} (\text{Im}_a^+(V)) \cap (\text{Ker}_b^+) \subseteq \text{Im}_a^+(V) \cap \text{Ker}_b^+.$$ 

Furthermore, the previous operators can be described in terms of subsets of $A$:

$$A_I^+ = \{ a_i \in A | a_i \leq a \text{ and } b_i \leq b \},$$

$$A_I^- = \{ a_i \in A | (a_i < a \text{ and } b_i \leq b) \text{ or } (a_i \leq a \text{ and } b_i < b) \},$$

$$A_I = \{ a_i \in A | (a_i = a \text{ and } b_i = b) \}.$$ 

Lemma 2.8 ([13]). We have that $V_I^\pm \cong \langle A_I^\pm \rangle$ and $V_I \cong \langle A_I \rangle$ for any $t \in I$.

Remark 2.9. Since $\#A_I = \#A_I^$ for $t \in I$ and $\#A_I = m_I^V$ by definition then the previous lemma implies that $m_I^V = \text{dim} V_I$ for any $t \in I$. Then, $V_I^\pm$ provide all the information necessary to compute the decomposition of $V$. An additional consequence is that $V_I = 0$ for $I \notin I(V)$.

Example 2.10. Consider the PM $V$ from Example 2.2

$$V \cong k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} k^2 \xrightarrow{\text{Id}} k^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} k.$$ 

We have $V_{[1,4]}^+ \cong k_{[1,4]}$, $V_{[1,4]}^- = 0$, $V_{[2,3]}^- \cong k_{[2,3]}$ and $V_{[1,4]}^+ = 0$. Altogether we obtain that $V_{[1,4]} \cong k_{[1,4]}$ and $V_{[2,3]} \cong k_{[2,3]}$. Additionally, one might check that $V_I = 0$ for all $I \neq [1, 4], [2, 3]$. Then, $V \cong k_{[1,4]} \oplus k_{[2,3]}$.
Proof. The first statement is a direct consequence of the arguments used in the proof of Theorem 6.1 of [8]. The second statement is precisely Theorem 6.1 of [8]. □

In order to prove Theorem 2.1, the author of [8] defined $F^+_{\{a,b\}} = \mathrm{Im}_{ul}^+(V) \cap \ker_{bl}^+(V) + \mathrm{Im}_{ul}^+(V)$ for each $\ell \in [n]$ and $[a, b] \in I(n)$, and proved that

- $\{(F^-_I, F^+_I)\}_{I \in I(n)}$ is a disjoint set of sections that covers $V_I$, and
- $F^-_I / F^+_I \cong \bigoplus_{\ell=1}^{m^+_I} k_{I \ell}$ for each $I \in I(n)$ and some $m^+_I \in \mathbb{Z}_{\geq 0}$.

Then, from Theorem 2.11 we deduce the isomorphisms

$$
\bigoplus_{I \in I(n)} (F^+_I / F^-_I) \cong V_I,
$$

and the fact these structure maps of $F^+_I$ commute with these isomorphisms concludes the proof.

To prove that $F^+_I$ compose a set of sections that covers $V_I$, an additional concept was used in [8] which we introduce next. A set of sections $\{(F^-_\lambda, F^+_\lambda) : \lambda \in \Lambda\}$ strongly covers a vector space $A$ provided that for all subspaces $B, C \subseteq A$ with $C \nsubseteq B$ there is some $\lambda \in \Lambda$ with

$$
B + (F^-_\lambda \cap C) \neq B + (F^+_\lambda \cap C).
$$

We can combine a set that strongly covers $A$ with one that covers $A$ to produce a new set that covers $A$, see Lemma [D.1]. Lastly, we have added some technical lemmas about sections to [14].

### 2.6 Partial matchings and block functions

A block function [13] between $V$ and $U$ is a function $\mathcal{M} : I(n) \times I(n) \to \mathbb{Z}_{\geq 0}$ satisfying that $\mathcal{M}(I, J) = 0$ if $I \notin I(V)$ or $J \notin I(U)$ and

$$
\sum_{J \in I(U)} \mathcal{M}(I, J) \leq m^+_I. \quad (4)
$$

A partial matching between two barcodes $B(V)$ and $B(U)$ is a bijection $\sigma : R \to R'$ where $R \subseteq B(V)$ and $R' \subseteq B(U)$. Given two partial matchings between $B(V)$ and $B(U)$, $\sigma_1 : R_1 \to R'_1$ and $\sigma_2 : R_2 \to R'_2$, we say that $\sigma_1$ and $\sigma_2$ are isomorphic if there exist bijections $\gamma, \gamma'$ making the following diagram commutative

$$
\begin{array}{ccc}
R_1 & \xrightarrow{\sigma_1} & R'_1 \\
\gamma \downarrow & & \downarrow \gamma' \\
R_2 & \xrightarrow{\sigma_2} & R'_2,
\end{array}
$$

and such that if $\gamma(I_i) = L_{\ell}$ then $I = L$ and if $\gamma'(J_j) = L_{\ell}$ then $J = L$. By abuse of notation, we may write $\sigma : B(V) \to B(U)$ instead of $\sigma : R \to R'$ and $I_i \mapsto J_j$ instead of $\sigma(I_i) = J_j$. Besides, we may write $\sigma(I_i) = \emptyset$ or $I_i \mapsto \emptyset$ when we mean $I_i \notin R$.

Notice that, for each partial matching, we can define a block function

$$
\mathcal{M}(I, J) := \#\{i \mid \exists j, \text{ such that } \sigma(I_i) = J_j\},
$$

that satisfies the additional property

$$
\sum_{I \in I(V)} \mathcal{M}(I, J) \leq m^+_J. \quad (5)
$$

And vice versa, a block function satisfying inequalities (4) and (5) determines a unique (up to isomorphism) partial matching between two barcodes, assigning $\mathcal{M}(I, J)$ copies of the interval $I$ to copies of the interval $J$. 

10
Example 2.12. Given the barcodes

\[ B(V) = \{[2, 2], [2, 3]\} \quad \text{and} \quad B(U) = \{[1, 2], [1, 2]\} \]

the partial matchings

\[ [2, 2] \mapsto [1, 2] \quad \text{and} \quad [2, 3] \mapsto [1, 2], \quad \text{and} \]

\[ [2, 2] \mapsto [1, 2] \quad \text{and} \quad [2, 3] \mapsto [1, 2], \]

are isomorphic and define the same block function,

\[ \mathcal{M}([2, 2], [1, 2]) = 1, \quad \mathcal{M}([2, 3], [1, 2]) = 1 \]

and \( \mathcal{M}(I, J) = 0 \) for any other \( I, J \in I(n) \).

3 Interval blocks constructed using disjoint covering sets of sections

At the end of Section 1, we mentioned that the block function \( \mathcal{M}_f \) introduced in [13] does not determine a unique partial matching in general. As pointed out by Corollary 5.6 of the same paper, this only happens if there are nested intervals in \( B(V) \); this is the case of Example 1.2. The root of this problem lies in the fact that, whenever there are nested intervals in \( B(V) \), the sections \( \{ (V_{I_t}, V_{I_t}^+) \}_{t \in I(n)} \) may fail to be disjoint, as it is shown in the following example.

Example 3.1. Consider \( V \) from Examples 1.2 and 2.10 where \( B(V) = \{[1, 4], [2, 3]\} \). In this case we have that \( V_{(1, 4)}^+ \not\subset V_{(2, 3)}^- \) while also \( V_{(2, 3)}^- \subset V_{(1, 4)}^- \) for all \( t \in I \cap J \). Thus, \( \{ (V_{I_t}, V_{I_t}^+) \}_{t \in I(4)} \) is not a disjoint set of sections of \( V_t \).

To obtain a disjoint set of sections, we use the following definitions based on the \( F_{I_t}^\pm \) terms defined in Section 7 of [8].

Definition 3.2. Let \( I = [a, b] \in I(n) \). Let us define the following PMs,

\[
\hat{V}_{I_t}^+ = V_{I_t}^+ + \text{Im}_{I_t}^-(V), \quad \hat{V}_{I_t}^- = V_{I_t}^- + \text{Im}_{I_t}^+(V), \quad \text{when } t \in [a, n],
\]

\[
\text{and } 0 \text{ otherwise;}
\]

\[
\hat{V}_{I_t}^+ = V_{I_t}^+ + \text{Ker}_{I_t}^-(V), \quad \hat{V}_{I_t}^- = V_{I_t}^- + \text{Ker}_{I_t}^+(V), \quad \text{when } t \in [1, b],
\]

\[
\text{and } 0 \text{ otherwise.}
\]

In order to give an intuition of the structure of these spaces, we define the following subsets

\[ \hat{A}_I^+ = (I_a^+(A) \cap \mathcal{K}_b^+(A)) \cup I_a^-(A) = \{ a_t \in A \mid (a_t < a) \text{ or } (a_t = a \text{ and } b_t \leq b) \} \]

\[ \hat{A}_I^- = (I_a^+(A) \cap \mathcal{K}_b^-(A)) \cup I_a^-(A) = \{ a_t \in A \mid (a_t < a) \text{ or } (a_t = a \text{ and } b_t < b) \} \].

Analogously, we define

\[ \hat{A}_I^+ = (I_a^+(A) \cap \mathcal{K}_b^+(A)) \cup \mathcal{K}_b^-(A) = \{ a_t \in A \mid (b_t < b) \text{ or } (b_t = b \text{ and } a_t \leq a) \} \]

\[ \hat{A}_I^- = (I_a^-(A) \cap \mathcal{K}_b^+(A)) \cup \mathcal{K}_b^-(A) = \{ a_t \in A \mid (b_t < b) \text{ or } (b_t = b \text{ and } a_t < a) \} \].

These subsets of \( A \) are used to describe \( \hat{V}_{I_t}^\pm \) and \( \hat{V}_{I_t}^\pm \) in terms of persistence bases.

Proposition 3.3. \( \hat{V}_{I_t}^\pm = \langle \hat{A}_{I_t}^\pm \rangle \) and \( \hat{V}_{I_t}^\pm = \langle \hat{A}_{I_t}^\pm \rangle \) for all \( t \in I \).
Consequently, as these sections are disjoint and cover $\mathcal{K}_B^+(A) \cup \mathcal{K}_B^-(A) = \{ \mathcal{A}_{I_t}^+ \}$, implying the first case. The second case can be proven in a similar way. 

**Example 3.4.** Recall $V$ from Example 3.2 given by

$$V \cong k \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} k^2 \xrightarrow{\text{Id}} k^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} k.$$ 

We have

$$\tilde{V}_{I_t}^+ = \text{Im}^+_V(V) \cap \text{Ker}^+_V(V) + \text{Im}^-_V(V) = \langle \mathcal{I}^+_V(A) \cap \mathcal{K}_B^+(A) \rangle + \langle \mathcal{I}^-_V(A) \cap \mathcal{K}_B^-(A) \rangle = \langle \mathcal{A}_{I_t}^+ \rangle,$$

implying the second case can be proven in a similar way.

**Proposition 3.5.** \{\{(\tilde{V}_{I_t}^+, \tilde{V}_{I_t}^-)\}_{t \in I(n)}\} are disjoint sets of sections that cover $V_t$.

**Proof.** Fix $t \in [n]$, by Lemma 7.1 from [8], \{(\text{Im}^-_V, \text{Im}^+_V)\}_{t \in I(n)} and \{(\text{Ker}^+_V, \text{Ker}^-_V)\}_{t \in I(n)} are disjoint and strongly cover $V_t$. Then, we can apply Lemma B.1 to obtain that \{(\tilde{V}_{[a,b]}^+, \tilde{V}_{[a,b]}^-)\}_{a \leq b} and \{(\tilde{V}_{[a,b]}^-, \tilde{V}_{[a,b]}^+)\}_{a \leq b} are disjoint sets that cover $V_t$. Using that these spaces are trivial if $t \notin I$ for $I \in I(n)$, we can extend the index set to the whole $I(n)$ and obtain the desired result.

**Example 3.6.** Let $V$ be the domain of the persistence morphism given in Examples 3.2 and 3.4. In light of Proposition 3.3 we can repeat the computations from Example 3.4 in a more straightforward manner. From Example 3.3 we know that $a_1 \sim [1,4]$ and $a_2 \sim [2,3]$ constitute a persistence basis for $V$. Thus, we deduce the equalities

$$\tilde{V}_{[2,3]}^+ = \langle \mathcal{A}_{[2,3]}^+ \rangle = \langle a_{1,2}, a_{2,2} \rangle = \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle \text{ and } \tilde{V}_{[2,3]}^- = \langle \mathcal{A}_{[2,3]}^- \rangle = \langle a_{1,2} \rangle = \langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle.$$

Similarly one can obtain $\tilde{V}_{[1,4]}^+ = \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$ and $\tilde{V}_{[1,4]}^- = \langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle$. In this case $\tilde{V}_{[1,4]}^+ \subseteq \tilde{V}_{[2,3]}^+$ and so the sections are disjoint. Also, we might check that the sections cover $V_2$. Let $B \subseteq V_2$, if $B + \tilde{V}_{[1,4]}^+ = B + \tilde{V}_{[1,4]}^+$ then the only possibility is that $B = \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$; however this implies $B + \tilde{V}_{[2,3]}^+ \neq B + \tilde{V}_{[2,3]}^-$. Consequently, as the sections are disjoint and cover $V_2$, we can see that the following holds in line with Theorem 2.7:

$$V_2 = \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle \oplus \langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle \cong \langle \tilde{V}_{[1,4]}^+ / \tilde{V}_{[1,4]}^- \rangle \oplus \langle \tilde{V}_{[2,3]}^+ / \tilde{V}_{[2,3]}^- \rangle.$$

We have seen in Proposition 3.5 that $\tilde{V}_{I_t}^+$ and $\tilde{V}_{I_t}^-$ lead to sets of disjoint sections covering $V_t$. In the next proposition, we show that from these sections we can recover $V_t$. 

12
Proposition 3.7. For each \( I \in \mathcal{I}(n) \) and \( t \in [n] \) we have that,

\[
\frac{\hat{V}_{1t}^-}{\hat{V}_{1t}^+} \cong \frac{\hat{V}_{I}^-}{\hat{V}_{I}^+} \cong \langle A_{I} \rangle \cong V_{I}.
\]

Proof. The isomorphisms \( \hat{V}_{1t}^+/\hat{V}_{1t}^- \cong \hat{V}_{I}^+/\hat{V}_{I}^- \cong \langle A_{I} \rangle \) are a direct consequence of Proposition 3.3 and, by Lemma 2.8, the three vector spaces are generated by the same basis. Note that if \( t \notin I \) or \( I \notin \mathcal{I}(V) \) all the spaces are trivial.

In particular, since generators commute with the structure maps, we also have that

\[
\frac{\hat{V}_{I}^+}{\hat{V}_{I}^-} \cong \frac{\hat{V}_{I}^+}{\hat{V}_{I}^-} \cong V_{I} \cong \bigoplus_{\ell=1}^{m_{I}} k_{I}.
\]

That is, we have recovered the interval block summand of \( V \) associated to \( I \) by means of disjoint covering sets of sections. This is key to proving in Section 4.1 that \( \hat{M}_{f} \) induces a unique partial matching.

4 Definition of \( \hat{M}_{f} \)

Given \( f : V \to U \), we introduce the operator \( \hat{M}_{f} : \mathcal{I}(n) \times \mathcal{I}(n) \to \mathbb{Z}_{\geq 0} \). Later, we prove that \( \hat{M}_{f} \) is actually a block function that induces a unique partial matching between \( B(V) \) and \( B(U) \).

First, recall that the spaces \( \hat{V}_{I}^+ \) and \( \hat{U}_{I}^+ \) contain the information on the barcode decomposition. Then, if we find how these spaces are related via \( f \), we can obtain a relation for their respective barcodes. For this purpose, we define the following vector spaces,

\[
Z_{IJ,t} := \frac{f(\hat{V}_{I}^+ \cap \hat{U}_{J}^+) - f(\hat{V}_{I}^- \cap \hat{U}_{J}^+) + f(\hat{V}_{I}^+ \cap \hat{U}_{J}^-)}{f(\hat{V}_{I}^+ \cap \hat{U}_{J}^-) + f(\hat{V}_{I}^- \cap \hat{U}_{J}^-)}
\]

when \( t \in I \cap J \), and \( Z_{IJ,t} = 0 \) otherwise. Note that \( Z_{IJ} \) is a PM with the structure maps inherited from \( U \). An expression of the space \( Z_{IJ,t} \) in terms of persistence bases can be found in Section 6.

We also use the following order for closed intervals. Given \( I = [a, b] \) and \( J = [c, d] \), we say that \( J \leq I \) if \( c \leq a \leq d \leq b \). Note that when \( J \notin I \), the only possible morphism of interval modules \( g : k_{I} \to k_{J} \) is \( g_{t} = 0 \) for all \( t \in [n] \). The following lemma is an analogous result for \( Z_{IJ,t} \).

Lemma 4.1. If \( J \notin I \) then \( Z_{IJ,t} = 0 \) for all \( t \in [n] \).

Proof. Since \( J \notin I \) at least one of the following strict inequalities must hold: either \( a < c \) or \( d < a \) or \( b < d \). Suppose first that \( a < c \). Note that, by the commutativity of \( f \), we have that

\[
f_{t}(\hat{V}_{I}^+) \subseteq f_{t}(\text{Im}_{al}(V)) = f_{t}(\rho_{al}(V_{a})) = \phi_{al}(f_{a}V_{a}) \subseteq \text{Im}^{+}_{al}(U).
\]

As \( a < c \), \( \text{Im}^{+}_{al}(U) \subseteq \text{Im}^{-}_{al}(U) \) by Lemma 2.7 and \( f(\hat{V}_{I}^+) \cap \hat{U}_{J}^+ \subseteq \text{Im}^{+}_{al}(U) \cap \hat{U}_{J}^+ \subseteq \hat{U}_{J}^+ \), where the last inclusion follows by the sequence

\[
\text{Im}^{-}_{al}(U) \cap \hat{U}_{J}^+ = \text{Im}^{-}_{al}(U) \cap (U_{J}^+ + \ker_{al}^{-}(U)) \subseteq \text{Im}^{-}_{al}(U) \cap \ker_{al}^{+}(U) \subseteq \hat{U}_{J}^+.
\]

Then, \( f(\hat{V}_{I}^+) \cap \hat{U}_{J}^+ \subseteq f(\hat{V}_{I}^+) \cap \hat{U}_{J}^- \) and the quotient \( Z_{IJ,t} \) is zero.

Next, suppose that \( d < a \). If \( t < a \) then \( \hat{V}_{I}^+ = 0 \) and if \( t \geq a > d \) then \( \hat{U}_{J}^+ = 0 \). Both cases imply
In this case, we have used Lemma E.2. Then, using Lemma E.2 together with Lemma C.4, we obtain

The structure maps of $\mathcal{Z}$ and so

Altogether, using again Lemma C.2, we obtain

First, since $f$ commutes with the structure maps, we get

Note that we have used the inclusions $\text{Ker}^+_b(V) \subseteq \text{Ker}^-_{d_l}(V)$, as $b < d$, and $\text{Im}^-_a(V) \subseteq \widetilde{V}_I^-$. Next, using the linearity of $f$, we obtain the first equality

In this case, we have used $f(\text{Ker}^-_{d_l}(V)) \subseteq \text{Ker}^-_{d_l}(U)$, which follows by commutativity of $f$ with the structure maps of $V$ and $U$. Finally, using Lemma C.2 we obtain

 Altogether, using again Lemma C.2 we obtain

and so $Z_{IJ_t}$ vanishes.

As the following lemma states, $Z_{IJ_t}$ only reaches non-null values for intervals in the PDs of $V$ and $U$.

**Lemma 4.2.** If $I \not\in I(V)$ or $J \not\in I(U)$, then $Z_{IJ_t} = 0$ for all $t \in [n]$.

**Proof.** If either condition is satisfied, we have that $\widetilde{V}_I^- = \widetilde{V}_I^+$ or $\widetilde{U}_J^- = \widetilde{U}_J^+$ respectively, implying that $f\widetilde{V}_I^- \cap \widetilde{U}_J^+ = f\widetilde{V}_I^+ \cap \widetilde{U}_J^+$ and the result follows.

As we will see in Definition 4.4, the value of $\widetilde{M}_f(I, J)$ is given by the dimension of the vector spaces $Z_{IJ_t}$. This value is constant for $t \in I \cap J$.

**Lemma 4.3.** Given any pair $s \leq t$ from $I \cap J$, we have that $\phi_{st}$ induces an isomorphism $Z_{IJS} \cong \widetilde{Z}_{IJ_t}$.

**Proof.** First, since $f$ commutes with the structure maps, we get $\phi_{st} f\tilde{V}_I^\pm = f \phi_{st} \tilde{V}_I^\pm = f \tilde{V}_I^\pm$, where we have used Lemma E.2. Then, using Lemma E.2 together with Lemma C.4, we obtain

Thus, we define $\widetilde{M}_f(I, J)$ as follows.

**Definition 4.4.** Let $f : V \to U$ be persistence morphism. For any $I = [a, b]$ and $J = [c, d]$ in $I(n)$, we define

We will see in the next subsection that $\widetilde{M}_f(I, J)$ is indeed a block function. Note that we evaluated the dimension of $Z_{IJ}$ at $d$, but we could have evaluated it for any $t \in I \cap J$ due to Lemma 4.3.

**Remark 4.5.** There are different situations for which $\widetilde{M}_f(I, J)$ is null:
• In particular, $\widetilde{M}_f(I, J) = 0$ if $I \cap J = \emptyset$ by the definition of $Z_{IJ}$.

• More generally, $\widetilde{M}_f(I, J) = 0$ if $I \not\subseteq J$ by Lemma 4.1.

• $\widetilde{M}_f(I, J) = 0$ if $I \notin I(V)$ or $J \notin I(U)$ by Lemma 4.2.

**Remark 4.6.** By Lemma 4.3, $Z_{IJ}$ is isomorphic to the direct sum of $\widetilde{M}_f(I, J)$ copies of $k_{IJ}$, since its structure maps are isomorphisms inside $I \cap J$, and by definition $Z_{IJ} = 0$ for $t \not\subseteq I \cap J$.

### 4.1 $\widetilde{M}_f$ induces a unique partial matching

The aim of this subsection is to prove the following statement.

**Theorem 4.7.** The operator $\widetilde{M}_f$ is a block function and always induces a unique partial matching between $\mathcal{B}(V)$ and $\mathcal{B}(U)$.

**Example 4.8.** Consider the persistence morphism $f : V \rightarrow U$ from Example 1.2. Also, recall from Example 3.4 that $V_{[2,3]}^+ \simeq \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$ and $V_{[2,3]}^- \simeq \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle$, which implies that $fV_{[2,3]}^+ = \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$ and $fV_{[2,3]}^- = \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle$. Similarly, one can obtain that

$$\hat{U}_{[1,3]}^+ = \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle$$

and $\hat{U}_{[1,3]}^- = \langle 0 \rangle$ as well as $\hat{U}_{[2,2]}^+ = \langle 0 \rangle$ and $\hat{U}_{[2,2]}^- = 0$.

Altogether, we obtain

$$Z_{[2,3][1,3]}^2 = \frac{\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle \cap \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle}{\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle} = \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle \cong 0$$

and

$$Z_{[2,3][2,2]}^2 = \frac{\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle \cap \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle \cap \langle 0 \rangle}{\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle} = \langle 0 \rangle \cong k.$$ 

And then,

$$\widetilde{M}_f([2, 3], [2, 2]) = \dim Z_{[2,3][2,2]}^2 = 1 \quad \text{and} \quad \widetilde{M}_f([2, 3], [1, 3]) = \dim Z_{[2,3][1,3]}^2 = \dim Z_{[2,3][1,3]}^2 = 0.$$ 

It can also be checked that $\widetilde{M}_f([1, 4], [1, 3]) = 1$ and $\widetilde{M}_f([1, 4], [2, 2]) = 0$. Then, $\widetilde{M}_f$ induces the partial matching

$$[2, 3] \mapsto [2, 2] \quad \text{and} \quad [1, 4] \mapsto [1, 3],$$

in opposition to $M_f$ that does not determine a unique partial matching (recall Eq. (2) in Section 1.2).

As explained in Section 2.6, $\widetilde{M}_f$ is a block function if it satisfies Inequality (4) and it induces a unique partial matching if it also satisfies Inequality (5). Our strategy consists of showing that $Z_{IJ}$ can be injected in both $V_{IJ}$ and $U_{IJ}$, and using this fact, to compare $\widetilde{M}_f$ and the multiplicities $m^V_f$ and $m^U_f$. To achieve this, we define the following subspaces.

**Definition 4.9.** Given $I, J \in I(n)$, we define the following subspaces of $U_{IJ}$ as

$$Z_{JI}^- := \frac{f\hat{V}_{JI}^- \cap \hat{U}_{JI}^-}{\hat{U}_{JI}^-} \quad \text{and} \quad Z_{JI}^+ := \frac{f\hat{V}_{JI}^+ \cap \hat{U}_{JI}^+}{\hat{U}_{JI}^+}.$$
if \( t \in J \) and \( Z_{1J}^+ = 0 \) otherwise.

Similarly, we define the following subspaces of \( V_I \), as

\[
Y_{1J}^- := \frac{\bar{V}_I + f^{-1}_t \bar{U}_J}{\bar{V}_I} \quad \text{and} \quad Y_{1J}^+ := \frac{\bar{V}_I + f^{-1}_t \bar{U}_J + \bar{V}_I}{\bar{V}_I}
\]

if \( t \in I \) and \( Y_{1J}^\pm = 0 \) otherwise.

The quotient of these spaces generates \( Z_{1J}^+ \).

**Proposition 4.10.** \( Z_{1J} \cong Z_{1J}^+ / Z_{1J}^- \cong Y_{1J}^+ / Y_{1J}^- \).

**Proof.** In order to prove the first isomorphism, denote

\[
B = f \bar{V}_I^- \cap \bar{U}_J^+, \quad C = f \bar{V}_I^- \cap \bar{U}_J^+ \quad \text{and} \quad D = \bar{U}_J^+ \text{, where } C \subseteq B.
\]

Then, one has that \( Z_{1J} = B/(C + B \cap D) \), since \( B \cap D = f \bar{V}_I^- \cap \bar{U}_J^+ \). In addition, one can check that \( Z_{1J}^+ / Z_{1J}^- \cong (B + D)/(C + D) \). Then, by Lemma C.1 the first isomorphism follows. The isomorphism \( Z_{1J} \cong Y_{1J}^+ / Y_{1J}^- \) is obtained using first Lemma C.1 and then Lemma C.4.

We also have the following result.

**Lemma 4.11.** Consider \( f : V \to U \). For \( J = [c, d] \in I(U) \),

\[
\left\{ \begin{array}{c}
(\bar{Z}_{1J}^+, \bar{Z}_{1J}^-) \\
(\bar{Y}_{1J}^+, \bar{Y}_{1J}^-)
\end{array} \right\} \in I(U) \quad \text{is a disjoint set of sections of } U_{Jd}.
\]

Analogously, for \( I = [a, b] \in I(V) \),

\[
\left\{ \begin{array}{c}
(\bar{Y}_{1J}^+, \bar{Y}_{1J}^-)
\end{array} \right\} \in I(U) \quad \text{is a disjoint set of sections of } V_{Ja}.
\]

**Proof.** Notice that \( \left\{ \begin{array}{c}
(\bar{Z}_{1J}^+, \bar{Z}_{1J}^-) \\
(\bar{Y}_{1J}^+, \bar{Y}_{1J}^-)
\end{array} \right\} \in I(U) \) is a set of sections of \( U_{Jd} \cong \bar{U}_d^- / \bar{U}_d^+ \). Let \( I, I' \in I(U) \). Since, by Proposition 3.5 \( \left\{ \begin{array}{c}
(\bar{V}_d^- / \bar{V}_d^+), \bar{V}_d^+ \end{array} \right\} \in I(U) \) is a disjoint set of sections of \( V_d \), we might assume without loss of generality that \( \bar{V}_d^+ \preceq \bar{V}_d^- \). In turn this implies \( f(\bar{V}_d^+ \preceq f(\bar{V}_d^-) \) and necessarily \( Z_{1J} \preceq Z_{1J}^- \). A similar reasoning can be made for \( \left\{ \begin{array}{c}
(\bar{Y}_{1J}^+, \bar{Y}_{1J}^-)
\end{array} \right\} \in I(U) \).

**Proof of Theorem 4.7.** All we need to show is that \( \tilde{M}_f \) satisfies (4) and (5). Using Lemma 4.11 and Theorem 2.11 we obtain the following injections

\[
\bigoplus_{I \in I(U)} Z_{1J} \hookrightarrow U_{Jd} \quad \text{and} \quad \bigoplus_{J \in I(U)} Y_{1J} \hookrightarrow V_{Ja}.
\]

For a fixed \( J = [c, d] \in I(U) \), we can deduce the following inequality using the first injection in (6)

\[
\sum_{I \in I(U)} \tilde{M}_f(I, J) = \sum_{I \in I(U)} \dim Z_{1Jd} = \dim \bigoplus_{I \in I(U)} Z_{1Jd} \leq \dim U_{Jd} = m_J^U,
\]

where the last equality follows by Remark 2.9. Thus, \( \tilde{M}_f \) satisfies (4).

Next, we proceed to show that \( \tilde{M}_f \) satisfies (4), i.e. \( \sum_{J \in I(U)} \tilde{M}_f(I, J) \leq m_I^V \). For this, we fix \( I = [a, b] \in I(U) \) and, as we need to vary \( J \), we denote max(\( J \)) as \( r(J) \). Recall that if \( \dim Z_{1Jd} \neq 0 \) then \( d \in I \cap J, I \in I(V), J \in I(U) \), and \( J \preceq I \) by Remark 4.3. In this case, \( I \cap J = [a, r(J)] \) if \( \tilde{M}_f(I, J) \neq 0 \). Then, by Lemma 4.3 and Proposition 4.10 we have that \( Y_{1Ja} \cong Z_{1Ja} \cong Z_{1Jr(J)} \). Using again the second injection of (6), we have that

\[
\sum_{J \in I(U)} \tilde{M}_f(I, J) = \sum_{J \in I(U)} \dim Z_{1J} = \dim \bigoplus_{J \in I(U)} Y_{1Ja} \leq \dim V_{Ja} = m_J^V,
\]

where we have used Remark 2.9 for the last equality.
5 Properties of \( \widetilde{M}_f \)

The definition of \( \widetilde{M}_f \) is quite formal and makes it difficult to understand the intuition behind it. In this section, we show some properties that may give a better understanding of its nature.

First, note that the block function \( \widetilde{M}_f \) is an invariant under persistence morphisms since it is defined using the dimension, intersection and quotient of vector spaces that are invariant themselves. The following result justifies that the block function \( \widetilde{M}_f \) is compatible with the inner structure of the persistence morphism \( f \).

**Theorem 5.1.** Given two persistence morphisms, \( h : V \to U \) and \( g : V' \to U' \), and intervals \( I, J \in I(n) \), we have that

\[
\widetilde{M}_{h \oplus g}(I, J) = \widetilde{M}_h(I, J) + \widetilde{M}_g(I, J).
\]

**Proof.** The result follows since direct sums commute with quotients, finite intersections and sums of vector spaces, and \( \dim(V_t \oplus U_t) = \dim V_t + \dim U_t \).

As a consequence, if the persistence morphism \( f \) decomposes in a simple way, the partial matching induced by \( \widetilde{M}_f \) will behave as expected.

**Example 5.2.** Recall the morphism \( f \) from Example 1.1. It could be expressed as

\[
\begin{array}{ccccccc}
V & \cong & k & \overset{Id}{\rightarrow} & k & \rightarrow & 0 & 0 & \rightarrow & 0 & \rightarrow & 0 & k & \overset{Id}{\rightarrow} & k & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \oplus & & \uparrow & & \oplus & & \uparrow & & \text{Id} & & \uparrow & & \uparrow \\
U & & 0 & \rightarrow & 0 & \rightarrow & 0 & 0 & \rightarrow & 0 & \rightarrow & k & \overset{Id}{\rightarrow} & k & \rightarrow & 0 & \rightarrow & 0.
\end{array}
\]

Using Theorem 5.1 the only not-null value of \( \widetilde{M}_f \) is \( \widetilde{M}_f([2,2], [1,2]) = 1 \). Then, the partial matching induced by \( \widetilde{M}_f \) is

\[
\emptyset \mapsto [1,2], \quad [2,3] \mapsto \emptyset \quad \text{and} \quad [2,2] \mapsto [1,2].
\]

Example 5.2 is the expected partial matching mentioned in Example 1.1, which is different from the one given by \( \chi_f \), see Example A.2. In fact, additivity is one of the main differences compared to the partial matching \( \chi_f \). See A for a more in-depth comparison.

### 5.1 Obtaining \( fV \) and the rank invariant from \( \widetilde{M}_f \)

This subsection aims to prove that \( \widetilde{M}_f \) describes \( fV \) up to isomorphism. As a consequence, we also show that the partial matching induced by \( \widetilde{M}_f \) pairs \( \#\PD(fV) \) intervals in each barcode, and that it can be used to calculate the rank invariant. The main result of the section is the following.

**Theorem 5.3.** Given \( f : V \to U \), we have the isomorphism

\[
fV \cong \bigoplus_{I \in I(V)} Z_{IJ}.
\]

In order to prove Theorem 5.3, we define a set of sections \( W_{IJ}^\pm \) that allow to write \( fV \) in terms of \( Z_{IJ} \). To achieve this, we improve Proposition 3.5 to show that \( \widehat{V}_{IJ}^\pm \) and \( \widehat{V}_{IJ}^- \) strongly cover \( V \), and prove that \( \{(f\widehat{V}_{IJ}^+, f\widehat{V}_{IJ}^-)\}_{I \in I(n)} \) strongly covers \( fV \) as well.

17
**Proposition 5.4.** For each \( t \in [n] \), we have that the sets of sections

\[
\{ (\tilde{V}_{I_t}^-, \tilde{V}_{I_t}^+) \}_{t \in I(n)} \quad \text{and} \quad \{ (\tilde{V}_{I_t^+}, \tilde{V}_{I_t^-}) \}_{t \in I(n)}
\]

strongly cover \( V_t \).

Before proving this proposition, we need to define two different total orders in \( I(n) \).

**Definition 5.5.** We say that \( J \leq^* I \) if \( d < b \) or whenever \( d = b \) then \( c \leq a \) and, that \( J \leq \) \( I \) if \( c < a \) or whenever \( c = a \) then \( d \leq b \).

Note that \( J \leq \) \( I \) implies both \( J \leq^* I \) and \( J \leq \) \( I \).

**Proof of Proposition 5.4.** For a fixed \( t \), we define the set \( I(n, t) = \{ I \in I(n) \mid t \in I \} \) and claim that the subset \( \{ (\tilde{V}_{I_t}^-, \tilde{V}_{I_t}^+) \}_{t \in I(n, t)} \) strongly covers \( V_t \). Then, the result follows by Remark D.3. Let us now proceed to prove our claim by using Lemma D.2. First, we can use \( \leq^* \) to fix a total order on the intervals \( I \in I(n, t) \),

\[
[1, t] \leq [1, t+1] \leq \ldots \leq [1, n] \leq [2, t] \leq [2, t+1] \leq \ldots \leq [t, n-1] \leq [t, n].
\]

Then, since we are considering that \( V_0 = V_{n+1} = 0 \), we have that \( \tilde{V}_{[1,t]}^- = 0 \) and \( \tilde{V}_{[t,n]}^+ \supseteq \text{Im}^+_n(V) \cap \text{Ker}^+_n(V) = V_t \). This is the second requirement of Lemma D.2. For the first requirement, consider a pair of consecutive intervals in the order \( \leq^* \); there are two cases:

(i) \([a, b-1] \leq [a, b]\) for some \( a, b \in [n] \) such that \( t \in [a, b] \), or

(ii) \([a-1, t] \leq [a, t]\) for some \( a \in [n] \) with \( a \leq t \).

In addition, observe that by definition \( \text{Im}^+_n(V) = \text{Im}^{-}_m(V) \), \( \text{Ker}^+_m(V) = \text{Ker}^{-}_n(V) \). Then, in case (i) it is straightforward to check that \( \tilde{V}^-_{[a,b-1]} = \tilde{V}^-_{[a,b]} \). Now, in case (ii) we obtain

\[
\tilde{V}^+_{[a-1,n]} = \text{Im}^+_m(V) \cap \text{Ker}^+_m(V) + \text{Im}^-_{(a-1)n}(V) = \rho_{(a-1)n}(V_{a-1}) \cap V_t + \rho_{(a-2)n}(V_{a-2})
\]

where we have used that \( \text{Ker}^+_m(V) = V_t \) and \( \text{Ker}^+_m(V) = 0 \). Then, the two conditions of Lemma D.2 are satisfied and \( \{ (\tilde{V}_{I_t^-}, \tilde{V}_{I_t^+}) \}_{t \in I(n, t)} \) strongly covers \( V_t \). A similar reasoning can be applied to \( \{ (\tilde{V}_{I_t^-}, \tilde{V}_{I_t^+}) \}_{t \in I(n, t)} \). In this case, the total order is induced by \( \leq^* \),

\[
[1, t] \leq^* [2, t] \leq^* \ldots \leq^* [t, t] \leq^* [1, t+1] \leq^* [2, t+1] \leq^* \ldots \leq^* [t-1, n] \leq^* [t, n]
\]

and then the two conditions of Lemma D.2 can be proved analogously.

We also have a similar result for \( fV_t \).

**Proposition 5.6.** \( \{ (f\tilde{V}_{I_t^-}, f\tilde{V}_{I_t^+}) \}_{t \in I(n)} \) is a strong disjoint set of sections that covers \( fV_t \).

**Proof.** Fix \( t \in [n] \). From the proof of Proposition 5.4, we know that \( \tilde{V}_{[1,t]}^- = 0, \tilde{V}_{[t,n]}^+ = V_t \) and given two consecutive intervals \( I \leq I' \) from the set \( I(n, t) \), it is true that \( \tilde{V}_{I'}^+ = \tilde{V}_{I_t}^+ \). This implies \( f\tilde{V}_{[1,t]}^- = 0, f\tilde{V}_{[t,n]}^+ = fV_t \) and \( f\tilde{V}_{I_t^+} = f\tilde{V}_{I_t^-} \), so we can apply Lemma D.2 and Remark D.3 again.

Now we have all the requisites to complete the proof of Theorem 5.3.
Proof of Theorem 5.3. First, by Proposition 5.4, \((\hat{U}_{J_1}, \hat{U}_{J_1}^+)\) strongly covers \(U_V\). Now, using the definition of strongly covering sections, we consider any \(B 
subseteq fV_t\) and take \(C = fV_t\), so that there exists \(J \in I(n)\) such that \(B + \hat{U}_{J_1}^- \cap fV_t \neq B + \hat{U}_{J_1}^+ \cap fV_t\). In particular, we deduce that \((\hat{U}_{J_1}^- \cap fV_t, \hat{U}_{J_1}^+ \cap fV_t)\) covers \(fV_t\). Second, using Proposition 5.6 and Lemma D.1, \(W_{I,J_1}^\pm = f\hat{V}_{I,J_1}^\pm \cap \hat{U}_{J_1}^\pm + \hat{U}_{J_1}^- \cap fV_t\) form a disjoint set of sections that covers \(fV_t\), and

\[
\bigoplus_{I \in I(n)} \bigoplus_{J \in I(n)} \frac{W_{I,J_1}^+}{W_{I,J_1}} \cong fV_t
\]

by Theorem 2.11. In addition, since \(f\hat{V}_{I,J_1} \subset fV_t\), we can use Lemma C.2 to obtain

\[
W_{I,J_1}^+ = (f\hat{V}_{I,J_1}^+ \cap \hat{U}_{J_1}^+ + \hat{U}_{J_1}^-) \cap fV_t.
\]

Denoting \(A = fV_t, B = f\hat{V}_{I,J_1} \cap \hat{U}_{J_1}^+ + \hat{U}_{J_1}^-\) and \(C = f\hat{V}_{I,J_1} \cap \hat{U}_{J_1}^+ + \hat{U}_{J_1}^-\), it can be easily seen that \(B \subset A + C\). Then, we can use Lemma C.3 to obtain

\[
\frac{W_{I,J_1}^+}{W_{I,J_1}^-} = \frac{(f\hat{V}_{I,J_1}^+ \cap \hat{U}_{J_1}^+ + \hat{U}_{J_1}^-) \cap fV_t}{(f\hat{V}_{I,J_1}^- \cap \hat{U}_{J_1}^+ + \hat{U}_{J_1}^-) \cap fV_t} \cong \frac{f\hat{V}_{I,J_1}^+ \cap \hat{U}_{J_1}^+ + \hat{U}_{J_1}^-}{f\hat{V}_{I,J_1}^- \cap \hat{U}_{J_1}^+ + \hat{U}_{J_1}^-},
\]

which is isomorphic to \(Z_{I,J_1}\) by Definition 4.9 and Proposition 4.10. Putting all together

\[
fV_t \cong \bigoplus_{I \in I(n)} \bigoplus_{J \in I(n)} \frac{W_{I,J_1}^+}{W_{I,J_1}} \cong \bigoplus_{I \in I(n)} \bigoplus_{J \in I(n)} Z_{I,J_1}.
\]

Now, combining Remark 4.5 with Remark 4.6, we obtain that if either \(I \notin I(V)\) or \(J \notin I(U)\), then \(Z_{I,J_1} = 0\). Consequently, the first direct sum of the previous expression can be simplified to

\[
\bigoplus_{I \in I(V)} \bigoplus_{J \in I(U)} Z_{I,J_1} \cong fV_t.
\]

Notice that the structure maps of \(Z_{I,J_1}\) and \(fV_t\) are both induced by the structure maps of \(U\), so it can be checked that these maps commute with the isomorphisms and can be assembled to form the claimed isomorphism. Then,

\[
\bigoplus_{I \in I(V)} \bigoplus_{J \in I(U)} Z_{I,J} \cong fV.
\]

A consequence of the previous result is that \(fV\) can be obtained from \(\tilde{M}_f\).

Corollary 5.7. Given \(L \in I(fV)\), we have that

\[
m_L^V = \sum_{I \cap J = L} \tilde{M}_f(I, J).
\]

In particular, \(PD(fV)\) can be calculated from \(\tilde{M}_f(I, J)\).

Proof. Due to Theorem 5.3, each summand \(Z_{I,J}\) contributes \(\tilde{M}_f(I, J)\) copies of \(k_{I \cap J}\) in the decomposition of \(fV\). Thus, the number of copies of \(k_L\) is determined by the sum of \(\tilde{M}_f(I, J)\) such that \(I \cap J = L\).
When considering barcodes instead of PDs, the previous corollary takes a stronger form.

**Corollary 5.8.** Let \( \sigma : \mathcal{B}(V) \rightarrow \mathcal{B}(U) \) be the partial matching induced by \( \widehat{M}_f \) as defined in Section 2.6. Then, there exist a bijection between matched pairs, \( \sigma(I_i) = J_i \), and intervals \( L_e \in \mathcal{B}(fV) \), such that \( I \cap J = L \). In particular, the number of pairs matched by \( \sigma \) equals \( \# \mathcal{B}(fV) \).

**Example 5.9.** Recall that the barcodes of \( V \) and \( U \) in Example 1.2 are \( \mathcal{B}(V) = \{[1, 4], [2, 3]\} \) and \( \mathcal{B}(U) = \{[1, 3], [2, 2]\} \). Moreover, the barcode of the image persistence module is \( \mathcal{B}(fV) = \{[1, 3], [2, 2]\} \). Then, the matching induced by \( \widehat{M}_f(I, J) \) is

\[
[1, 4] \mapsto [1, 3] \quad \text{and} \quad [2, 3] \mapsto [2, 2],
\]

since it is the only matching for which the set obtained from the intersection of the paired intervals coincides with \( \mathcal{B}(fV) \).

As mentioned in the introduction, there is no computationally efficient way to calculate the decomposition of persistence morphisms, so other methods must be used to analyze \( f \). The most simple one is the rank invariant \([5]\).

It is not difficult to see that, for persistence morphisms, the rank invariant can be obtained from \( fV \). The following result uses this fact to provide an expression of the rank invariant in terms of \( \widehat{M}_f \).

**Proposition 5.10.** Given \( f : V \rightarrow U \) and \( s \leq t \), the ranks of the linear maps between \( V_s, V_t \) and \( U_s, U_t \) are given, respectively, by

\[
\text{rank } (\rho_{st}) = \sum_{I \in \mathcal{I}(V)} m^V_I \quad \text{and} \quad \text{rank } (\phi_{st}) = \sum_{J \in \mathcal{J}(U)} m^U_J.
\]

Moreover, the ranks of the linear maps between \( V_s \) and \( U_t \) are given by

\[
\text{rank } (f_t \rho_{st}) = \text{rank } (\phi_{st} f_s) = \sum_{I \in \mathcal{I}(V)} \sum_{J \in \mathcal{J}(U)} \widehat{M}_f(I, J).
\]

**Proof.** The first two equalities are the persistent Betti numbers \( \beta^{ij}_p \), see Section VII.1 of \([10]\). The last equality follows from the isomorphism in Theorem 5.3 where the structure map \( \phi_{st} \) restricted to \( f(V_s) \) is only nonzero on those addends \( k_{I \cap J} \) such that \( s, t \in I \cap J \).

However, the converse of Corollary 5.7 and Proposition 5.10 are not true, that is, we cannot obtain \( \widehat{M}_f \) from the rank invariant or \( fV \), as the following example shows. In particular, this implies that \( \widehat{M}_f \) is a richer invariant than \( fV \) and the rank invariant.

**Example 5.11.** Consider the following two persistence morphisms

\[
\begin{array}{cccccccc}
U & \cong & k & \xrightarrow{\text{Id}} & k & \xrightarrow{\text{Id}} & k & \xrightarrow{\text{Id}} & k
\end{array}
\]

\[
\begin{array}{cccccccc}
V & \cong & 0 & \xrightarrow{\text{Id}} & k & \xrightarrow{\text{Id}} & 0 & \rightarrow & 0
\end{array}
\]

\[
\begin{array}{cccccccc}
U & \cong & k & \xrightarrow{\text{Id}} & k & \xrightarrow{\text{Id}} & k & \xrightarrow{\text{Id}} & k
\end{array}
\]

\[
\begin{array}{cccccccc}
V & \cong & 0 & \xrightarrow{\text{Id}} & 0 & \xrightarrow{\text{Id}} & 0 & \rightarrow & k & \xrightarrow{\text{Id}} & k
\end{array}
\]

20
Clearly, the partial matching induced by $\widetilde{M}_f$ is $[2, 3] \mapsto [1, 3]$, while the one induced by $\widetilde{M}_g$ is $[2, 3] \mapsto [2, 3]$. On the other hand, the image modules coincide,

$$fV \cong gV \cong 0 \longrightarrow k \longrightarrow k,$$

and so do their ranks,

$$\text{rank}(f_2) = \text{rank}(g_2) = 1, \quad \text{rank}(\phi_{23}f_2) = \text{rank}(\phi_{23}g_2) = 1 \quad \text{and} \quad \text{rank}(f_3) = \text{rank}(g_3) = 1.$$

All the other ranks are either those from the structure maps of $V$ and $U$ or those from linear maps starting at $V$. Thus, $f$ and $g$ cannot be distinguished by their image modules or ranks, and yet $\widetilde{M}_f \neq \widetilde{M}_g$.

Interestingly, there exists a family of persistence morphisms, including those built on Morse filtrations, for which $\widetilde{M}_f$ and the rank invariant are equivalent, along with other invariants like $fV$ and $\chi_f$. This is the topic of the next subsection.

### 5.2 ULRE persistence morphisms

In this section we introduce a particular case of persistence morphisms for which the rank invariant, $fV$ and the matching induced by $\widetilde{M}_f$ are equivalent. This also applies to the matching $\chi_f$, see [16].

We assume that $f : V \rightarrow U$ is such that all intervals from $\text{PD}(V)$ have multiplicity 1 and distinct left endpoints, while all intervals from $\text{PD}(U)$ have multiplicity 1 and distinct right endpoints. We call this an **ULRE persistence morphism**, where ULRE stands for Unique Left and Right Endpoint. A first observation is that since all intervals from $I(V)$ and $I(U)$ have multiplicity one, then all intervals $I \in I(V)$ and $J \in I(U)$ have a single representative in the barcodes $I \in \mathcal{B}(V)$ and $J \in \mathcal{B}(U)$. Another consequence is that $\widetilde{M}_f(I, J) \leq 1$ for all intervals $I, J$. The interest in ULRE persistence morphisms arises from the fact that they include persistence morphisms coming from a Morse filtrations. Morse filtrations appear frequently in TDA, for example, they are used to define the cycle registration in [16].

Now, let us denote the homology functor of rank $m$ over the field $k$ as $H_m(-)$. Consider a filtered complex $\{X_i\}_{i=1}^{n}$ which is Morse; i.e. for each $j \in [n-1]$ the inclusion $i_j : X_j \hookrightarrow X_{j+1}$ induces a morphism $i_j^* : H_m(X_j) \rightarrow H_m(X_{j+1})$ which is either

1. an isomorphism, or
2. injective and $\dim \left(H_m(X_{j+1})\right) = \dim \left(H_m(X_j)\right) + 1$, or
3. surjective and $\dim \left(H_m(X_{j+1})\right) = \dim \left(H_m(X_j)\right) - 1$.

We denote by $PH_m(X)$ the persistence module given by $H_m(X_i)$ for $i \in [n]$ and the structure maps $i_j^*$ for $j \in [n-1]$. Notice that the barcode decomposition of $PH_m(X)$ is such that at each index $i \in [n]$ only one of the following is true: (1) no bar is born, or (2) one bar is born or (3) one bar dies. Now, consider another Morse filtration $\{Y_i\}_{i=1}^{n}$ with inclusions $'i'$ and consider cellular morphisms $l_i : X_i \rightarrow Y_i$ for all $i \in [n]$ such that $l_{i+1} \circ i_j = i'_j \circ l_j$ for all $j \in [n-1]$. Hence, there exist induced linear maps $(l_i)_* : H_m(X_i) \rightarrow H_m(Y_i)$ for all $i \in [n]$ and such that $(l_{i+1})_* \circ (i_j)_* = (l'_j)_* \circ (i_j)_*$ for all $j \in [n-1]$. This implies that $l_s = \{(l_i)_*\}_{i \in [n]}$ is a persistence morphism. By the aforementioned observation on the barcode decomposition of $PH_m(X)$, we also have that $l_s$ is an ULRE persistence morphism.

As we saw in Proposition 5.10 the rank invariant can be obtained from $\widetilde{M}$. Now, we show that the converse holds for ULRE persistence morphisms.
Proposition 5.12. Let $f : V \to U$ be an ULRE persistence morphism. Given intervals $I = [a, b]$ and $J = [c, d]$ and assuming that $a \leq d$, the following relation holds

$$\widetilde{M}_f(I, J) = \text{rank } (\phi_{ad}f_a) - \text{rank } (\phi_{(a-1)d}f_{a-1})$$

$$- \text{rank } (\phi_{(a-1)(d+1)}f_a) + \text{rank } (\phi_{(a-1)(d+1)}f_{a-1}).$$

Proof. By using the last equality from Proposition 5.10, we obtain

$$\text{rank } (\phi_{ad}f_a) = \sum_{I' \in I(V)} \sum_{J' \in I(U)} \widetilde{M}_f(I', J')$$

$$= \sum_{I' \in I(V)} \sum_{a'} \sum_{J' \in I(U)} \widetilde{M}_f(I', J') + \sum_{I' \in I(V)} \sum_{a-1 \notin I'} \sum_{J' \in I(U)} \widetilde{M}_f(I', J').$$

However, note that since $f$ is a ULRE persistence morphism, $I$ is the only interval in $B(V)$ that is born at $a$. Hence,

$$\sum_{I' \in I(V)} \sum_{a-1 \notin I'} \sum_{J' \in I(U)} \widetilde{M}_f(I', J') = \sum_{J' \in I(U)} \widetilde{M}_f(I, J')$$

and

$$\text{rank } (\phi_{ad}f_a) = \text{rank } (\phi_{(a-1)d}f_{a-1}) + \sum_{J' \in I(U)} \widetilde{M}_f(I, J').$$

We can repeat the same computation, swapping $d$ by $(d+1)$, to obtain

$$\text{rank } (\phi_{ad+1}f_a) = \text{rank } (\phi_{(a-1)(d+1)}f_{a-1}) + \sum_{J' \in I(U)} \widetilde{M}_f(I, J').$$

The result follows by using the obtained equalities together with the following equation:

$$\widetilde{M}_f(I, J) = \sum_{J' \in I(U)} \widetilde{M}_f(I, J') - \sum_{J' \in I(U)} \widetilde{M}_f(I, J') - \sum_{J' \in I(U)} \widetilde{M}_f(I, J').$$

Next, we show that if $f$ is ULRE, then $\widetilde{M}_f$ can be obtained directly from $B(fV)$.

Proposition 5.13. Let $f : V \to U$ be a ULRE persistence morphism. We have that

$$\widetilde{M}_f(I, J) = \begin{cases} 1 & \text{if } I \cap J \in B(fV), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Corollary 5.8, for each interval $L$ in $B(fV)$, $\widetilde{M}_f$ is pairing an interval $I$ in $B(V)$ with the same left endpoint that $L$ with an interval $J$ in $B(U)$ with the same right endpoint that $L$. Since $f$ is ULRE, there is only one possible $I$ and one possible $J$ with such endpoints.

Example 5.14. Consider Example 1.2 and notice that it is a ULRE persistence morphism. The image persistence module is $B(fV) = \{(1,3), (2,2)\}$, and so, we obtain $\widetilde{M}_f$ directly from $B(fV)$:

$$\widetilde{M}_f([1,4], [1,3]) = 1 \text{ and } \widetilde{M}_f([2,3], [2,2]) = 1.$$

In Proposition A.4 of [A], we prove a similar result for $\chi_f$. Putting all these results together, we obtain the following theorem.

Theorem 5.15. Given an ULRE persistence morphism, $f : V \to U$, then $fV$, the rank invariant, $\widetilde{M}_f$ and $\chi_f$ are equivalent.
6 A matrix algorithm to compute $\widetilde{M}_f$

One of the strengths of $\widetilde{M}_f$ is that, despite being sensitive to the decomposition of $f$, it can be calculated efficiently. In this section, we provide an algorithm to calculate it using Gaussian column reductions.

We fix two persistence bases $A = \{a_i\}_{i \in \#PD(V)}$ for $V$ and $B = \{\beta_i\}_{i \in \#PD(U)}$ for $U$. Recall the inequalities from Definition 5.5. We assume that the elements of $A$ are ordered in such a way that if $i \leq j$, then $[a_i, b_j] \leq [a_j, b_j]$. Similarly, if $[c_i, d_i],[c_j, d_j] \in B$ and $i \leq j$, then $[c_i, d_i] \leq^* [c_j, d_j]$.

Now, to define a matrix method for computing $\widetilde{M}_f$, we need to work with the matrix associated to a persistence morphism $f : V \rightarrow U$. This concept has already been introduced in the literature; see for example [13]. However, we provide a brief and precise definition of what we mean by such a matrix.

**Definition 6.1.** Let $f : V \rightarrow U$ be a persistence morphism and consider persistence bases $A$ for $V$ and $B$ for $U$. Further, suppose that $A$ and $B$ have their generators sorted in such a way that the respective orders $\leq$ and $\leq^*$ are respected by the intervals associated to them. The matrix $F$ associated to $f$ in the persistence bases $A$ and $B$ is a $\#PD(U) \times \#PD(V)$ matrix with coefficients in $k$ such that, for each generator $a_j \in A$,

- $f(a^1_{ja_j}) = \sum_{\beta_i \in B_{a_j}} F_{ij} \beta^1_{ia_j}$ and
- $F_{ij} = 0$ for all $\beta_i \in B$ such that $a_j \not\in [a_i, b_i]$.

where the entries of $F$ are denoted by $F_{ij}$ with $i \in \#PD(U)$ and $j \in \#PD(V)$.

As $B_{a_j}$ gives a basis for $U_{a_j}$ for all $j \in \#PD(V)$, it follows that $F$ is uniquely defined.

**Remark 6.2.** For $f : V \rightarrow U$ and a fixed $t \in [n]$, the minor of $F$, $F_t$, given by the columns associated to $A_t$, and the rows to $B_t$, is precisely the matrix associated to the linear map $f_t : V_t \rightarrow U_t$.

Let $F$ be the matrix associated to $f$ in the persistence bases $A$ and $B$, which are ordered in a compatible way with $\leq$ and $\leq^*$ respectively. Note that this order is also followed on the minors $F_t$ for all $t \in [n]$. Our work now focuses on obtaining a matrix method on $F$ to compute the induced matching $\widetilde{M}_f$.

We denote by $R$ the reduced matrix that one obtains after applying a Gaussian column elimination on $F$. Now we define $\widetilde{R}_{IJ}$ to be the minor of $R$ with rows from $B_J$ and the columns from $A_I$ with pivots in $B_J$. Let $J = [c, d]$ and denote by $\langle \widetilde{R}_{IJ} \rangle$ the subspace of $\langle B_{Jd} \rangle$ spanned by the columns from $\widetilde{R}_{IJ}$ (the columns from $\widetilde{R}_{IJ}$ can be embedded as vectors in $\langle B_{Jd} \rangle$). Similarly, we denote by $R_t$ the reduced matrix of $F_t$. Given $t \in I \cap J$, we define the following minors:

- $\widetilde{R}^\pm_{IJ} :\equiv$ minor of $R_t$ with rows from $B_{Jt}$ and the columns from $\widetilde{A}^\pm_{Jt}$ with pivots in $B_{Jt}$,
- $\widetilde{R}^\pm_{IJt} :\equiv$ minor of $R_t$ with rows from $B_{Jt}$ and the columns from $A_{It}$ with pivots in $B_{Jt}$.

In particular, notice that $\widetilde{R}_{IJ} = \widetilde{R}^+_{IJt} \setminus \widetilde{R}^-_{IJt}$; i.e. the minor of $\widetilde{R}^+_{IJt}$ obtained taking the columns that are not contained in $\widetilde{R}^-_{IJt}$. We define by $\langle \widetilde{R}^\pm_{IJt} \rangle$ the subspace of $\langle B_{Jt} \rangle$ spanned by the columns from $\widetilde{R}^\pm_{IJt}$ and $\langle \widetilde{R}^\pm_{IJ} \rangle$ the subspace of $\langle B_{J} \rangle$ spanned by the columns from $\widetilde{R}_{IJ}$.

Next, we would like to relate the Gaussian elimination $R$ of $F$ with $\widetilde{M}_f$. The following lemma is a key step.

**Lemma 6.3.** $Z^\pm_{IJt} \equiv \langle \widetilde{R}^\pm_{IJt} \rangle$ for all $t \in I \cap J$. 

23
\textbf{Proof.} If $J \nsubseteq I$, $Z_{IJ_t}^\pm$ is zero and $\tilde{R}_{IJ_t}$ is empty, so the isomorphism is trivial. Thus, we assume that $J \subseteq I$.

\[ Z_{IJ_t}^\pm = \frac{f \tilde{V}_{IJ_t}^\pm \cap \tilde{U}_{IJ_t}^\pm + \tilde{U}_{IJ_t}^-}{\tilde{U}_{IJ_t}^-} \cong \frac{f \langle \tilde{A}_{IJ_t}^\pm \rangle \cap \langle \tilde{B}_{IJ_t}^- \rangle + \langle \tilde{B}_{IJ_t}^- \rangle}{\langle \tilde{B}_{IJ_t}^- \rangle} \subseteq \langle B_{IJ_t} \rangle, \]

and note that the last quotient is generated by the columns from the minor of $R_t$ given by the rows from $B_{IJ_t}$ and the columns from $A_{IJ_t}^\pm$ with pivots in $B_{IJ_t}$, that is, we obtain the claimed isomorphism.

\[ \square \]

Now, the following result relates minors from $R$ with $\tilde{M}_f$.

\textbf{Theorem 6.4.} $Z_{IJ_t} \cong \langle \tilde{R}_{IJ_t} \rangle \cong \langle \tilde{R}_{IJ} \rangle$ for all $t \in I \cap J$. Thus, \( \tilde{M}_f(I, J) = \text{rank} \langle \tilde{R}_{IJ} \rangle \).

\textbf{Proof.} First, recall that, by Lemma 6.3, $Z_{IJ_t}^\pm \cong \langle \tilde{R}_{IJ_t}^\pm \rangle$. Second, since $\tilde{R}_{IJ_t}^- \subseteq \tilde{R}_{IJ_t}^+$, using Lemma B.2. from [13] we obtain $\langle R_{IJ_t} \rangle = \langle \tilde{R}_{IJ_t}^+ \setminus \tilde{R}_{IJ_t}^- \rangle \cong \langle \tilde{R}_{IJ_t}^+ \rangle / \langle \tilde{R}_{IJ_t}^- \rangle$. Altogether, there are isomorphisms

\[ Z_{IJ_t} \cong \frac{Z_{IJ_t}^+ \cong \langle R_{IJ_t}^+ \rangle}{Z_{IJ_t}^- \cong \langle R_{IJ_t}^- \rangle} \cong \langle \tilde{R}_{IJ_t} \rangle. \]

Finally, using the fact that $t \in I \cap J$, it follows that $\langle \tilde{R}_{IJ_t} \rangle \cong \langle \tilde{R}_{IJ} \rangle$.

\[ \square \]

Theorem 6.4 implies the following straightforward procedure to compute the block function $\tilde{M}_f$ from $f$. To start, we fix the persistence bases $A$ and $B$ for $V$ and $U$ respectively. Next, we compute the associated matrix $F$ and compute its column reduced form $R$. Then, by Theorem 6.4

\[ \tilde{M}_f(I, J) = \# \{ \text{columns from } R \text{ associated to } A_I \text{ with pivots in } B_J \}. \]

Thus, we obtain $\tilde{M}_f(I, J)$ directly from the pivots in $R$. Notice that, with a little different column order, the same Gaussian reduction on $F$ was used in [18] to obtain a persistence basis for $fV$.

\textbf{Example 6.5.} Let us consider again Example 1.2. In this case, $V$ has a persistence basis given by two generators $\alpha_1 \sim [1, 4]$ and $\alpha_2 \sim [2, 3]$ while $U$ has a persistence basis formed by $\beta_1 \sim [2, 2]$ and $\beta_2 \sim [1, 3]$. The matrix $F$ associated with $f : V \to U$ on this choice of persistence bases as well as its Gaussian reduction $R$ are

\[ F = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \longrightarrow R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

Reading the pivots from $R$, we see that the column of $\alpha_1$ has $\beta_2$ as pivot while the column of $\alpha_2$ has $\beta_1$ as pivot. Hence,

\[ \tilde{M}_f([1, 4], [1, 3]) = \tilde{M}_f([2, 3], [2, 2]) = 1, \text{ and } \tilde{M}_f([1, 4], [2, 2]) = \tilde{M}_f([2, 3], [1, 3]) = 0. \]

Then, the induced partial matching is $[1, 4] \to [1, 3]$ and $[2, 3] \to [2, 2]$. Notice that we have obtained the same result as in Example 5.14. However, this method applies generally to persistence morphisms, independently of being ULRE or not.
7 Conclusions and future work

We have defined a new way of inducing a partial matching from a persistence morphism that satisfies the following properties:

- it is an invariant of persistence morphisms,
- it contains more information than $fV$ and the rank invariant,
- it is additive with respect to the direct sum of persistence morphisms, and
- it can be calculated in a single matrix column reduction.

We have also introduced the family of ULRE persistence modules, for which the rank invariant, $fV$ and $\tilde{\mathcal{M}}_f$ are equivalent.

As a future work direction, we plan to develop an implementation to compute the induced partial matching efficiently, adapting the algorithm to the concrete case of persistent homology of simplicial complexes. These results can also be generalized to persistence modules over $\mathbb{R}$ as in [13]. The use of sections to prove most of the results may simplify this generalization. Another pending task is to find an explicit relation between the proposed matching and the other works mentioned in Section 1.2.

Moreover, we explained in the introduction that one can obtain a morphism $f : V \rightarrow U$ from two point clouds when one is contained within the other. When the two point clouds share some points, but there is no inclusion relation between them, we obtain a pair of persistence morphisms, $V \leftarrow U \rightarrow W$.

It would be interesting to study how the concepts introduced in this paper can be adapted to this situation as well as their relation with [16, 12]. In addition, we plan to study the stability properties of the partial matching. This is key to understand how $\tilde{\mathcal{M}}_f$ can be used in topological data analysis.

Lastly, in [4] we explain that a combination of $\tilde{\mathcal{M}}_f$ and the block function $\mathcal{M}_f$ can offer valuable insights into the decomposition of persistence morphisms. We also plan to explore results in that direction in future works.

Acknowledgments. Partially funded by the European Union under grant agreement no. 101070028-2 REXASI-PRO, and by MCIN/AEI and the NextGenerationEU/PRTR, under project TED2021-129438B-I00. Álvaro Torras-Casas was also partially funded by EPSRC grant EP/W522405/1.

References

[1] H. Asashiba, E.G. Escolar, Y. Hiraoka, and H. Takeuchi. Matrix method for persistence modules on commutative ladders of finite type. *Jpn. J. Ind. Appl. Math.*, 2019.

[2] U. Bauer and M. Lesnick. Induced matchings and the algebraic stability of persistence barcodes. *J. Comput. Geom.*, 2015.

[3] U. Bauer and M. Lesnick. Persistence diagrams as diagrams: A categorification of the stability theorem. In *Topological Data Analysis*. Springer Int. Publ., 2020.

[4] G. Carlsson, A. Dwaraknath, and B. J. Nelson. Persistent and zigzag homology: A matrix factorization viewpoint, 2021.

[5] G. Carlsson and A. Zomorodian. The theory of multidimensional persistence. *Discrete Comput. Geom.*, 2009.
[6] F. Chazal, V. de Silva, M. Glisse, and S. Oudot. *The Structure and Stability of Persistence Modules*. Briefs in Mathematics. Springer Int. Publ., 2016.

[7] D. Cohen-Steiner, H. Edelsbrunner, J. Harer, and D. Morozov. Persistent homology for kernels, images, and cokernels. In *Proc. 20th Annu. ACM-SIAM Symp. Disc. Algorithms*, SODA’09, 2009.

[8] W. Crawley-Boevey. Decomposition of pointwise finite-dimensional persistence modules. *J. Algebra Appl.*, 2015.

[9] Sebastiano Cultrera di Montesano, Ondřej Draganov, Herbert Edelsbrunner, and Morteza Saghafian. Persistent homology of chromatic alpha complexes, 2024.

[10] H. Edelsbrunner and J. Harer. *Computational Topology: An Introduction*. American Mathematical Society, 2009.

[11] E.G. Escolar and Y. Hiraoka. Persistence modules on commutative ladders of finite type. *Discrete Comput. Geom.*, 2014.

[12] I. García-Redondo, A. Monod, and A. Song. Fast topological signal identification and persistent cohomological cycle matching, 2022.

[13] R. Gonzalez-Diaz, M. Soriano-Trigueros, and A. Torras-Casas. Partial matchings induced by morphisms between persistence modules. *Comput. Geom.*, 2023.

[14] A. De Gregorio, M. Guerra, S. Scaramuccia, and F. Vaccarino. Parallel decomposition of persistence modules through interval bases, 2021.

[15] E. Jacquard, V. Nanda, and U. Tillmann. The space of barcode bases for persistence modules. *J. Appl. Comput. Topol.*, 2023.

[16] Yohai Reani and Omer Bobrowski. Cycle registration in persistent homology with applications in topological bootstrap. *IEEE Trans. Pattern Anal. Mach. Intell.*, 2023.

[17] Nico Stucki, Johannes C. Paetzold, Suprosanna Shit, Bjoern Menze, and Ulrich Bauer. Topologically faithful image segmentation via induced matching of persistence barcodes. In *Proc. 40th Int. Conf. Mach. Learn.*, volume 202, pages 32698–32727. PMLR, 2023.

[18] A. Torras-Casas. Distributing persistent homology via spectral sequences. *Discrete Comput. Geom.*, 2023.

[19] H.R. Yoon, R. Ghrist, and C. Giusti. Persistent extensions and analogous bars: data-induced relations between persistence barcodes. *J. Appl. Comput. Topol.*, 2023.

[20] A. Zomorodian and G. Carlsson. Computing persistent homology. *Discret. & Comput. Geom.*, 2004.

[21] Živa Urbančič and Jeffrey Giansiracusa. Ladder decomposition for morphisms of persistence modules, 2023.
A Comparison between \( \widetilde{\mathcal{M}}_f \) and \( \chi_f \)

Here, we compare \( \widetilde{\mathcal{M}}_f \) with the induced partial matching \( \chi_f \). This matching has been used not only in the proof of the stability theorem \([2, 3]\) but also in various TDA applications \([17]\).

As explained in Section \([1,2]\), \( \chi_f \) may produce outputs that do not align with the decomposition of the persistence morphism \( f \). However, we could construct an additive operator \( \chi'_f \) considering the unique decomposition of \( f \) into indecomposables \( f_1 \oplus \cdots \oplus f_n \), and then combining each \( \chi_{f_i} \) to obtain the partial matching. The main obstacle to this construction is that, as mentioned in the introduction, there is currently no efficient way to compute indecomposables. Since we have provided a matrix algorithm to calculate \( \widetilde{\mathcal{M}}_f \), it is natural to ask if \( \widetilde{\mathcal{M}}_f \) is equivalent to \( \chi'_f \), as this would offer a way to compute an additive version of \( \chi_f \). However, as explained in Example \([A.7]\), this is not the case.

We now recall the procedure to obtain the partial matching \( \chi_f \) from a persistence morphism \( f \) \([2, 3]\).

Firstly, given \( e \in [n] \), denote by \( \mathcal{B}(V)_e \) the subset of \( \mathcal{B}(V) \) formed by the elements \( I_e \in \mathcal{B}(V) \) such that \( I \) has \( e \) as its right endpoint, and similarly, \( \mathcal{B}(V)e \) contains \( I_e \in \mathcal{B}(V) \) if \( e \) is the left endpoint of \( I \). Observe that we can fix an order in each of these sets, for \( \mathcal{B}(V)_e \), \( [a, e] \leq_e [b, e] \) if \( a < b \) or \( a = b \) and \( i < j \); and for \( \mathcal{B}(V)e \), \( [e, a] \leq_e [e, b] \) if \( a > b \) or \( a = b \) and \( i < j \). The definition of \( \chi_f \) (presented later) is based on the following result.

**Theorem A.1** (Theorem 4.2 from \([2]\)). If \( f : V \to U \) is injective, then for each \( e \in [n] \) we have that

\[
\#\mathcal{B}(V)_e \leq \#\mathcal{B}(U)_e
\]

and if \( f : V \to U \) is surjective, then

\[
\#\mathcal{B}(V)e \geq \#\mathcal{B}(U)e.
\]

Now, observe that, given \( f : V \to U \), there exists a unique decomposition \( V \xrightarrow{f'} fV \subseteq U \) where \( f' \) is surjective and \( f'' \) is injective \([2, 3]\). Applying Theorem \([A.1]\) we have that \( \#\mathcal{B}(V)_e \geq \#\mathcal{B}(fV)_e \) for each \( e \in [n] \). Using the defined order for these sets, we can build a partial matching \( \sigma_1 : \mathcal{B}(V) \to \mathcal{B}(fV) \) that maps the \( j \)-th element of \( \mathcal{B}(V)_e \) to the \( j \)-th element of \( \mathcal{B}(fV)_e \) for all \( e \in [n] \). This way, all the bars in \( \mathcal{B}(fV) \) are matched. Analogously, we can match the \( j \)-th element of \( \mathcal{B}(fV)e \) to the \( j \)-th element of \( \mathcal{B}(U)e \), obtaining an injection \( \sigma_2 : \mathcal{B}(fV) \to \mathcal{B}(U) \). Then, a partial matching \( \sigma \) between \( \mathcal{B}(V) \) and \( \mathcal{B}(U) \) can be built concatenating \( \sigma_1 \) and \( \sigma_2 \).

**Example A.2.** Going back to Example \([I.7]\) where

\[
\begin{array}{cccccccc}
U & k & \longrightarrow & k & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & & & & & & & & \\
V & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & k & \longrightarrow & 0 \\
\end{array}
\]

It can be checked that \( \mathcal{B}(fV) = \{ [2, 2]_1 \} \). Note now that the order in \( \mathcal{B}(V)_2 \) establishes that \( [2, 3]_1 \leq_2 [2, 2]_1 \), and the one in \( \mathcal{B}(U)_2 \) establishes that \( [1, 2]_1 \leq^2 [1, 2]_2 \). Then, the partial matching \( \chi_f \) is given by

\[
\begin{array}{c}
[2, 3]_1 \mapsto [1, 2]_1, \quad \emptyset \mapsto [1, 2]_2 \quad \text{and} \quad [2, 2]_1 \mapsto \emptyset.
\end{array}
\]

Note that this partial matching differs from the one induced by the decomposition of \( f \), as illustrated in Example \([I.7]\).
Remark A.3. A consequence of how \( \chi_f \) is constructed is that, for each element \( L \) in \( \mathbf{B}(fV) \), there must be another element \( I \) in \( \mathbf{B}(V) \) with the same left endpoint, and another element \( J \) in \( \mathbf{B}(U) \) with the same right endpoint, so that \( \chi_f(I) = J \) and \( I \cap J = L \).

Note the similarity between Remark A.3 and Corollary 5.8. This allows us to prove that \( \chi_f \) and \( \mathcal{M}_f \) are actually equivalent for ULRE persistence morphisms.

**Proposition A.4.** Let \( f : V \to U \) be a ULRE persistence morphism. We have that

\[
\mathcal{M}_f(I, J) = \begin{cases} 1 & \text{if } \chi_f(I) = J, \\ 0 & \text{otherwise}. \end{cases}
\]

**Proof.** By Remark A.3, for each interval \( L \) in \( \mathbf{B}(fV) \), \( \chi_f \) is pairing an interval \( I \) in \( \mathbf{B}(V) \) with the same left endpoint that \( L \) with an interval \( J \) in \( \mathbf{B}(U) \) with the same right endpoint that \( L \). Since \( f \) is ULRE, there is only one possible \( I \) and one possible \( J \) with these endpoints. Moreover, from Corollary 5.7 we deduce that \( I \cap J \) is in \( I(fV) \) if and only if \( \mathcal{M}_f(I, J) = 1 \). This implies the result. \( \square \)

Given two partial matching \( \sigma_1 : B_1 \to B'_1 \) and \( \sigma_2 : B_2 \to B'_2 \), we denote by \( \sigma_1 \cup \sigma_2 : B_1 \cup B_2 \to B'_1 \cup B'_2 \) the new partial matching given by the disjoint union of the barcodes \( B_1 \) and \( B_2 \), and their relations. Using this notation, and combining Proposition A.4 and Theorem 5.1, we obtain that

**Proposition A.5.** If \( f : V \to U \) is an ULRE persistence morphism such that \( f \cong f_1 \oplus f_2 \), then \( \chi_f = \chi_{f_1} \cup \chi_{f_2} \).

**Example A.6.** Consider Example 1.2 noticing that \( f \) is ULRE. In this case, \( \mathcal{M}_f \) coincides with \( \chi_f \). The barcode of \( fV \) is \( \mathbf{B}(fV) = \{[1, 3], [2, 2] \} \), and so, by the definition of \( \chi_f \), we obtain

\[
\mathcal{M}_f([1, 4], [1, 3]) = 1 \text{ and } \mathcal{M}_f([2, 3], [2, 2]) = 1
\]

that coincides with the result obtained in Example 4.8. Se dice lo mismo arriba y abajo. Solo habría que ponerlo una vez.

Lastly, we provide an example to illustrate that \( \mathcal{M} \) does not coincide with \( \chi \) when applied to the indecomposables of \( f \). In particular, we show that \( \mathcal{M}_f \) cannot be seen as applying \( \chi \) to the indecomposables of \( f \).

**Example A.7.** Consider the following indecomposable:

\[
\begin{array}{ccccccc}
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\mathbf{U} & \cong & \mathbf{k}^2 & [01] & \mathbf{k} & [01] & \mathbf{k} \\
\mathbf{V} & \cong & 0 & \mathbf{k}^2 & \mathbf{k} & \mathbf{k} \\
\end{array}
\]

We have that \( \mathbf{B}(V) = \{[2, 3], [2, 4] \} \), \( \mathbf{B}(U) = \{[2, 2], [1, 3] \} \) and \( \mathbf{B}(fV) = \{[2, 2], [2, 3] \} \). Hence, the partial matching given by \( \chi_f \) is

\[
[2, 3] \mapsto [2, 2] \quad \text{and} \quad [2, 4] \mapsto [1, 3].
\]

However, if we order the generators of \( V \) as \( \alpha_1 \sim [2, 3] \) and \( \alpha_2 \sim [2, 4] \), and the ones of \( U \) as \( \beta_1 \sim [2, 2] \) and \( \beta_2 \sim [2, 3] \). The matrix \( F \) associated with \( f : V \to U \) on this choice of persistence bases as well as its Gaussian reduction \( R \) are

\[
F = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.
\]
and the partial matching given by $\widetilde{M}_f$ is,

$$[2, 3] \mapsto [1, 3] \quad \text{and} \quad [2, 4] \mapsto [2, 2].$$

From these results, two interesting open questions arise: (1) Is it possible to modify $\widetilde{M}_f$ in such a way that it coincides with applying $\chi$ to the indecomposables of $f$? Note also that $fV$ can be obtained from $M_f$, and $\chi_f$ can be obtained from $fV$. Thus, $\chi_f$ is completely determined by $\widetilde{M}_f$. (2) Can this fact be used to derive new properties of the relation between $\chi_f$ and $\widetilde{M}_f$?

**B Comparison between $\widetilde{M}_f$ and $M_f$**

In [13], we defined a block function $M_f$ induced by a persistence morphism $f : V \to U$ that is the precedent of $\widetilde{M}_f$, presented here. More precisely, $M_f$ is built from the PM $X_{IJ}$, defined for each $I, J \in I(n)$ using the vector spaces

$$X_{IJ} := \frac{fV_I^+ \cap U_J^+}{fV_I^- \cap U_J^- + fV_I^+ \cap U_J^+}$$

if $t \in I \cap J$ and 0 otherwise, with structure maps induced by the quotient.

Since the PMs we consider in this paper are indexed by sets $[n]$, the definition of the induced block function $M_f : I(n) \times I(n) \to \mathbb{Z}_{\geq 0}$ is simplified as follows:

$$M_f(I, J) = \dim X_{IJd}$$

when $J \leq I$, and 0 otherwise. Note that, originally, $M_f$ was defined using a direct limit, but, since we only consider PMs indexed by finite sets, here, it is defined just by evaluating $\dim X_{IJd}$ on the right endpoint of $I \cap J$, that we denote $d$.

In [13 Section 6], we provided a matrix method for calculating $M_f$. Specifically, when evaluating $\dim X_{IJd}$ on $d$, we consider minors of $F$. More precisely, using the persistence bases in [5] from Section 2.4, we consider the matrices

$$L_{IJd} := \begin{pmatrix} A_{I,d} & A_{J,d} \\ B_{J,d} & * & * \\ B_d \setminus B_{J,d} & * & * \end{pmatrix}.$$ 

Then, a Gaussian column reduction is performed similarly to Section 6 to obtain a reduced matrix $N_{IJd}$. We then considered its minor $R_{IJd}$, given by the rows of $N_{IJd}$ associated to $B_{J,d}$ and the columns from $A_{I,d}$ which are zero on the rows associated with $B_d \setminus B_{J,d}$. By [13 Theorem 6.2], $\langle R_{IJd} \rangle \cong X_{IJd}$, and then, $M_f$ can be calculated counting the pivots of $R_{IJd}$.

**Example B.1.** Consider again Example [12] and recall that $V \cong k_{[1,4]} \oplus k_{[2,3]}$ and $U \cong k_{[1,3]} \oplus k_{[2,2]}$. Using the matrix method for calculating $M_f$ explained above, we have that $M_f([1,4], [1,3]) = 1$, $M_f([2,3], [1,3]) = 1$ and $M_f(I, J) = 0$ otherwise, as anticipated in equation (2) of Subsection 1.2.

**Lemma B.2.** Let $f : V \to U$ and let $I \in I(V)$ and $J = [c, d] \in I(U)$ with $J \leq I$. Consider the following set of intervals,

$$S(I, d) = \{I' \in I(V) \mid I' \not\subseteq I, I' \subseteq I \text{ and } d \in I'\},$$

$$T(J) = \{J' \in I(U) \mid d \in J' \text{ and } J \subseteq J'\}.$$ 

Then, if $\widetilde{M}_f(I', J') = 0$ for all $I' \in S(I, d)$ and $J' \in T(J)$, we have that $M_f(I, J) = \widetilde{M}_f(I, J)$.
Proof. First, observe that, the generators from \( B_d \) have right endpoint equal or greater than \( d \), while generators in \( B^+_d \) have right endpoint \( d \). In addition, those generators in \( B_d \setminus B^+_d \) with endpoint \( d \) must have left endpoint greater to \( c \). Hence, we can sort generators of \( B_d \cup B^+_d \) in a compatible way with \( \leq^* \); keeping the generators of \( B_d \) smaller than the generators from \( B_d \setminus B^+_d \). A similar reasoning can be applied to columns of \( A^+_{1d} \) and \( A_{1d} \) with respect to the order \( \leq^* \). Consequently, the rows and columns from \( L_{1d} \), can be sorted following the interval orders \( \leq^* \) and \( \leq \), respectively without breaking the block matrix structure. Now, consider the following minor from \( F_d \), which results by restricting to the corresponding rows and columns indicated by the generator sets on the margins,

\[
F_{1d} := \begin{pmatrix}
\tilde{A}^-_{1d} & A_{1d} \\
B^-_{1d} & B_{1d} \setminus B^+_d
\end{pmatrix}.
\]

It follows that \( L_{1d} \) is a minor of \( F_{1d} \) since \( B^+_d = \hat{B}^+_d \) and also \( A^-_{1d} \subseteq \tilde{A}^-_{1d} \). Denote \( \hat{N}_{1d} \) and \( R_{1d} \) to the reduced matrices of \( L_{1d} \) and \( F_{1d} \). Observe that, in general, the pivots of \( \hat{N}_{1d} \) may be different to the ones of \( R_{1d} \) since the column reduced in the first case are \( A^-_{1d} \cup A_{1d} \), and in the second \( \tilde{A}^-_{1d} \cup A_{1d} \). However, using Theorem 6.4, our hypotheses on \( \hat{M}_f \) implies that \( \hat{R}_{1d} \) are trivial for all \( I' \leq I \) such that \( I' \notin I \) and \( J \leq J' \). Consequently, all columns of \( R_{1d} \) associated to \( \tilde{A}^-_{1d} \setminus A_{1d}^- \) must be zero. In other words, all columns from \( R_{1d} \) which are nonzero can be considered as columns from \( \hat{N}_{1d} \), and so \( \langle \tilde{R}_{1d} \rangle = \langle R_{1d} \rangle \), and the result follows. \( \square \)

Remark B.3. Note that the previous lemma also holds if \( S(I, d) \) is empty. Observe as well that \( T(J) \) cannot be empty, since \( J \) is always in it.

Example B.4. Consider Example 1.2 where \( \mathbf{B}(V) = \{(1, 4), [2, 3]\} \) and \( \mathbf{B}(U) = \{(1, 3), [2, 2]\} \). We know from Example 4.8 that \( \hat{M}_f([1, 4], [1, 3]) = 1 \). In addition, \( S([1, 4], 3) = \emptyset \). Hence, by Remark B.3, we can apply Lemma B.2 and obtain:

\[ \hat{M}_f([1, 4], [1, 3]) = \hat{M}_f([1, 4], [1, 3]) = 1. \]

However, since \( S([2, 3], 2) = \{(1, 4)\} \), \( T([2, 2]) = \{(2, 2), [1, 3]\} \) and \( \hat{M}_f([1, 4], [1, 3]) \neq 0 \), we cannot apply Lemma B.2 to obtain \( \hat{M}_f([2, 3], [2, 2]). \)

Proposition B.5. If there are no nested intervals in \( I(V) \), then \( \hat{M}_f = \hat{M}_f \).

Proof. If there are no nested bars in \( I(V) \), we have that \( I' \leq I \) implies \( I' \leq I \). Then, \( S(I, d) = \emptyset \) and the requisites of Lemma B.2 are satisfied by Remark B.3. \( \square \)

Proposition B.5 provides an alternative proof of Corollary 5.6 from 13, which states that \( \mathcal{M}_f \) is a partial matching when there are no nested intervals in \( I(V) \). In addition, this result facilitates the computation of \( \mathcal{M}_f \), as the matrix algorithm presented in 13 for computing \( \mathcal{M}_f \) requires a Gaussian column reduction for each \( I \in I(V) \) whereas \( \hat{M}_f \) can be computed with a single Gaussian reduction.

Note that the differences between \( \hat{M}_f \) and \( \hat{M}_f \) can provide information about the indecomposables of \( f \), as demonstrated by the following example.

Example B.6. Consider again Example 1.2 and recall that \( \mathbf{V} = k_{[1,4]} \oplus k_{[2,2]} \) and \( \mathbf{U} = k_{[1,3]} \oplus k_{[2,2]} \). We show now that \( f \) is indecomposable by using the outputs of \( \hat{M}_f \) and \( \hat{M}_f \). Suppose that \( f \) decomposes as two nontrivial persistence morphisms \( f = f' \oplus f'' : V' \oplus V'' \rightarrow U' \oplus U'' \). Without loss of generality, we assume that \( V' \neq 0 \) and that \( k_{[1,4]} \) is a summand in the decomposition of \( V' \). Now, since \( \hat{M}_f \) preserves direct sums, it follows from \( \hat{M}_f([1, 4], [1, 3]) = 1 \) that \( k_{[1,3]} \) must be a
summand in the decomposition of $U'$. On the other hand, $M_f([2, 3], [1, 3]) = 1$ implies that $k_{[2,3]}$ is also a summand in the decomposition of $V'$, and so $V'' = 0$. Now, since $\widetilde{M}_f([2, 3], [2, 2]) = 1$ and $\widetilde{M}_f$ also commutes with direct sums, this implies that $k_{[2,2]}$ is also a summand in the decomposition of $U'$, and so $U'' = 0$. This implies that $f'' : V'' \to U''$ is a trivial persistence morphism, leading to a contradiction. Hence, $f$ is indecomposable.

C Lemmas concerning vector space quotients

Lemma C.1. Let $A$ be a vector space, and consider subspaces $B, C, D \subseteq A$ such that $C \subseteq B$. Then, there is an isomorphism

$$\frac{B + D}{C + D} \cong \frac{B}{C + B \cap D}.$$  

Proof. Consider the inclusion $\iota : B \hookrightarrow B + D$, which induces a linear map

$$\tilde{\iota} : \frac{B}{C + B \cap D} \longrightarrow \frac{B + D}{C + D}.$$ 

First, notice that $\tilde{\iota}$ is well-defined since $\iota(C + B \cap D) \subseteq C + D$. We claim that $\tilde{\iota}$ is an isomorphism. To prove injectivity, let $b \in B$ and assume that $\tilde{\iota}(b)$ is trivial; that is, $b \in C + D$. Hence, there exists some vector $c \in C$ such that $b - c \in D$. However, $b - c \in B$ by the hypothesis $C \subseteq B$. Altogether, $b = c + (b - c) \in C + B \cap D$ and injectivity follows. Surjectivity follows since, for any class $b + d$ in the codomain of $\tilde{\iota}$, we have the equality $\tilde{\iota}(b + d) = \tilde{\iota}(b)$.  

Lemma C.2. Let $C, D, E$ be subspaces of a vector space $A$. If $D \subseteq E$ then

$$(C + D) \cap E = C \cap E + D.$$ 

Proof. First, notice we have the inclusion $C \cap E + D = C \cap E \cap D \subseteq (C + D) \cap E$. On the other hand, let $w \in (C + D) \cap E$. Then, there exist $c \in C$ and $d \in D$ such that $w = c + d$ and so $c = w - d \in E$, since $D \subseteq E$. Thus, $w = c + d \in C \cap E + D$ as claimed.  

Lemma C.3. Consider a linear map $\zeta : A \rightarrow B$ together with $C \subseteq A$ and $D \subseteq B$. Then, $\zeta(C \cap \zeta^{-1}D) = \zeta C \cap D$.

Proof. The inclusion $\subseteq$ follows by

$$\zeta(C \cap \zeta^{-1}D) \subseteq (\zeta C) \cap (\zeta \zeta^{-1}D) \subseteq \zeta C \cap D,$$

while the complementary inclusion $\supseteq$ follows by showing that all elements $v \in \zeta C \cap D$ are such that $v \in \zeta(C \cap \zeta^{-1}D)$, which we proceed to prove. Since $v \in \zeta C$, there exists $w \in C$ such that $\zeta(w) = v$ and since $v \in D$ we have that $w \in \zeta^{-1}D$. Altogether $w \in C \cap \zeta^{-1}D$ which implies $v = \zeta(w) \in \zeta(C \cap \zeta^{-1}D)$ and the claim follows.  

Lemma C.4. Consider a linear map $\zeta : A \rightarrow B$ together with a pair of sections $(F^-, F^+)$ and $(G^-, G^+)$ for $A$ and $B$ respectively. Then, $\zeta$ induces the following isomorphism of quotients:

$$\zeta : \frac{F^+ \cap \zeta^{-1}G^+}{F^- \cap \zeta^{-1}G^+ + F^+ \cap \zeta^{-1}G^-} \cong \frac{\zeta F^+ \cap G^+}{\zeta F^- \cap G^+ + \zeta F^+ \cap G^-}.$$ 

Proof. The map $\zeta$ is surjective and well defined by the following equalities

(i) $\zeta(F^+ \cap \zeta^{-1}G^+) = \zeta F^+ \cap G^+$ and
we have that sequence of spaces

\[ \zeta(F^\ast \cap \zeta^{-1}G^+ + F^+ \cap \zeta^{-1}G^-) = \zeta F^\ast + \zeta F^+ \cap G^- \]

which we proceed to show next.

First, notice that equality \ref{1} follows by using Lemma C.3 with \( C = F^+ \) and \( D = G^+ \). Next, equation \ref{2} follows by

\[
\zeta(F^\ast \cap \zeta^{-1}G^+ + F^+ \cap \zeta^{-1}G^-) = \zeta F^\ast + \zeta F^+ \cap G^- \]

where we have used again Lemma C.3 on second equality.

Now we prove injectivity. Suppose that \( w \in F^+ \cap \zeta^{-1}G^+ \) and \( \zeta(w) \in F^\ast \cap \zeta^{-1}G^+ + \zeta F^+ \cap G^- \). Then, by equality \ref{2} there exists \( h \in F^\ast \cap \zeta^{-1}G^+ + \zeta F^+ \cap G^- \) such that \( \zeta(h) = \zeta(w) \), and so \( h - w \in \text{Ker} \zeta \). In particular, \( h - w \in \zeta^{-1}G^- \), and since \( h, w \in F^+ \), it follows that \( w - h \in F^+ \cap \zeta^{-1}G^- \) so that \( w = (w - h) + h \) has trivial class in the domain of \( \zeta \).

Lemma C.5. Let \( C \subset B \) and \( A \) be vector spaces such that \( B \subset A + C \), then we have the isomorphism

\[
\frac{B \cap A}{C \cap A} \cong \frac{B}{C}
\]

Proof. The injectivity is clear since \( x \in B \cap A \) and \( x \in C \) implies \( x \in C \cap A \). The surjectivity follows from the fact that every \( b \in B \) can be written as \( a + c \) with \( a \in A \) and \( c \in C \), then \( b - c = a \) is in \( B \) and \( a \in B \cap A \). This implies that \( a \) has the same class as \( b \).

D Lemmas concerning sections of vector spaces

Lemma D.1 (Lemma 6.2 in \[8\]). If \( \{(F^{-}_\lambda, F^+_\lambda) : \lambda \in \Lambda \} \) is a disjoint set of sections that covers a vector space \( A \), and \( \{(G^{-}_\sigma, G^+_\sigma) : \sigma \in \Sigma \} \) is a disjoint set of sections that strongly covers \( A \), then the set

\[
\{(G^{-}_\sigma \cdot F^+_\lambda, G^+_\sigma \cdot F^+_\lambda + F^-_\lambda) : (\lambda, \sigma) \in \Lambda \times \Sigma \},
\]

is a disjoint set of sections that covers \( A \).

Lemma D.2. Consider a vector space \( A \), a finite totally ordered set \( \{\theta_i\}_{i \in [m]} \) and a set of sections of \( A \), \( \{ F^{-}_{\theta_i}, F^+_{\theta_i} \}_{i \in [m]} \). Then, if

\[
\begin{align*}
F^+_\theta &= F^-_{\theta_{i+1}} \text{ for all } 1 \leq i < m, \\
F^-_{\theta_i} &= 0 \text{ and } F^+_{\theta_m} = A,
\end{align*}
\]

we have that \( \{ F^-_{\theta_i}, F^+_{\theta_i} \}_{i \in [m]} \) is a disjoint set of sections that strongly covers \( A \).

Proof. We start proving the disjoint property. Let \( i < j \), by the first condition, there exists a sequence of spaces \( F^+_\theta = F^-_{\theta_{i+1}} \subseteq F^+_{\theta_{i+1}} = \ldots = F^-_{\theta_j} \) and \( F^+_{\theta_j} \subseteq F^+_{\theta_j} \). Now, let us prove that \( \{ F^-_{\theta_i}, F^+_{\theta_i} \}_{i \in [m]} \) strongly covers \( A \). For any \( C \supseteq B \) with \( C, B \subseteq A \), define the sets

\[
\Theta^-_{B,C} = \{ \theta_i \mid F^-_{\theta_i} \cap C \subseteq B \}, \quad \Theta^+_{B,C} = \{ \theta_i \mid F^+_{\theta_i} \cap C \not\subseteq B \}.
\]

and note that due to the second condition in the lemma, neither of these sets are empty. In addition, if \( \theta_i = \max(\Theta^-_{B,C}) \), then \( \theta_{i+1} \notin \Theta^-_{B,C} \). Since \( F^-_{\theta_i} = F^-_{\theta_{i+1}} \), \( \theta_i \) must be in \( \Theta^+_{B,C} \) and

\[
B + F^-_{\theta_i} \cap C = B \neq B + F^+_{\theta_i} \cap C.
\]

Note that this result also holds when \( \theta_i = \theta_m \). Hence, \( \{ F^-_{\theta_i}, F^+_{\theta_i} \}_{i \in [m]} \) strongly covers \( A \).
**Remark D.3.** Notice that if \( \{ F_{A}^{\pm} \}_{A \in \Lambda} \) is a disjoint set of sections of \( \Omega \subseteq \Lambda \), and \( \{ F_{A}^{\pm} \}_{A \in \Omega} \) (strongly) covers \( \Lambda \), then so does \( \{ F_{A}^{\pm} \}_{A \in \Lambda} \).

### E  An extra property of the sections of \( V \)

**Lemma E.1.** Consider a linear map \( \zeta : A \to B \) together with subsets \( C, D \subseteq B \) such that 
\[
D = \zeta^{-1}(D), \quad \zeta^{-1}(C) + \zeta^{-1}(D) = \zeta^{-1}(C + D).
\]

**Proof.** The inclusion \( \zeta^{-1}(C) + \zeta^{-1}(D) \subseteq \zeta^{-1}(C + D) \) follows directly. Consider \( v \in \zeta^{-1}(C + D) \) so that \( \zeta(v) \in C + D \). We can take \( d \in D \) such that \( \zeta(v) - d \in C \); since \( \zeta^{-1}(D) = D \), there must exist \( q \in \zeta^{-1}(D) \) such that \( \zeta(q) = d \). Hence, \( \zeta(v - q) = \zeta(v) - \zeta(q) = \zeta(v) - d \in C \), and so \( v - q \in \zeta^{-1}(C) \). Altogether \( v = (v - q) + q \in \zeta^{-1}(C) + \zeta^{-1}(D) \) and the second claim follows.

**Lemma E.2.** Given \( s \leq t \) in \( I \), \( \rho_{st}(\hat{V}_{I}^{\pm}) = \hat{V}_{I}^{\pm} \) and also \( \rho_{st}^{-1}(\hat{V}_{I}^{\pm}) = \hat{V}_{I}^{\pm} \).

**Proof.** Let \( I = [a, b] \). Recall that Lemma 2.2 from [8] states that \( \rho_{st} \Im_{as}^{\mp}(V) = \Im_{as}^{\mp}(V) \) as well as \( \rho_{st}^{-1} \Ker_{bs}^{\pm}(V) = \Ker_{bs}^{\pm}(V) \). Then, the first equality follows by

\[
\rho_{st}(\hat{V}_{I}^{\pm}) = \rho_{st}(V_{I}^{\pm} + \Im_{as}^{\mp}(V)) = \rho_{st}V_{I}^{\pm} + \rho_{st} \Im_{as}^{\mp}(V) = V_{I}^{\pm} + \Im_{as}^{\mp}(V) = \hat{V}_{I}^{\pm},
\]

where we have also used Lemma 3.1 from [8] that implies \( \rho_{st}(V_{I}^{\pm}) = V_{I}^{\pm} \).

For the second equality, first note that \( \rho_{st}^{-1}(\hat{V}_{I}^{\pm}) = \rho_{st}^{-1}(V_{I}^{\pm} + \Ker_{bs}^{\pm}(V)) \). We would like to use Lemma E.1 with \( \zeta = \rho_{st} \), \( C = \Ker_{bs}^{\pm}(V) \) and \( D = V_{I}^{\pm} \). In order to do it, we need to check that \( D = \eta^{-1}(D) \).

\[
\rho_{st}\rho_{st}^{-1}(V_{I}^{\pm}) = \rho_{st}(\rho_{st}^{-1}V_{I}^{\pm}) = \rho_{st}(V_{I}^{\pm} + \ker \rho_{st}) = \rho_{st}(V_{I}^{\pm}) + \rho_{st}(\ker \rho_{st}) = V_{I}^{\pm},
\]

where we have used \( \rho_{st}(V_{I}^{\pm}) = V_{I}^{\pm} \) in the last equality. Then, applying Lemma E.1 we obtain

\[
\rho_{st}^{-1}(\hat{V}_{I}^{\pm}) = \rho_{st}^{-1}(V_{I}^{\pm} + \Ker_{bs}^{\pm}(V)) = \rho_{st}^{-1}(V_{I}^{\pm}) + \rho_{st}^{-1} \Ker_{bs}^{\pm}(V) = \rho_{st}^{-1}(V_{I}^{\pm} + \ker \rho_{st}) + \ker \rho_{st},
\]

where we have used \( \ker \rho_{st} \subseteq \ker \rho_{sb} = \Ker_{bs}^{\pm}(V) \) in the second to last equality.

\[\Box\]