VERTICAL STRUCTURE OF STATIONARY ACCRETION DISKS WITH A LARGE-SCALE MAGNETIC FIELD

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ABSTRACT

In earlier works we pointed out that the disk’s surface layers are non-turbulent and thus highly conducting (or non-diffusive) because the hydrodynamic and/or magnetorotational instabilities are suppressed high in the disk where the magnetic and radiation pressures are larger than the plasma thermal pressure. Here, we calculate the vertical profiles of the stationary accretion flows (with radial and azimuthal components), and the profiles of the large-scale, magnetic field taking into account the turbulent viscosity and diffusivity and the fact that the turbulence vanishes at the surface of the disk. Also, here we require that the radial accretion speed be zero at the disk’s surface and we assume that the ratio of the turbulent viscosity to the turbulent magnetic diffusivity is of order unity. Thus, at the disk’s surface there are three boundary conditions. As a result, for a fixed dimensionless viscosity \( \alpha \)-value, we find that there is a definite relation between the ratio \( R \) of the accretion power going into magnetic disk winds to the viscous power dissipation and the midplane plasma-\( \beta \), which is the ratio of the plasma to magnetic pressure in the disk. For a specific disk model with \( \beta \)-value, we find that there is a definite relation between the ratio \( R \) and of order unity for a stationary solution is \( \beta_0 \approx 2.4 \alpha h / (\alpha h) \), where \( h \) is the disk’s half thickness. For weaker magnetic fields, \( \beta > \beta_0 \), we argue that the poloidal field will advect outward while for \( \beta < \beta_0 \), it will advect inward. Alternatively, if the disk wind is negligible (\( R \ll 1 \)), there are stationary solutions with \( \beta \gg \beta_0 \).

Key words: accretion, accretion disks – galaxies: jets – magnetic fields – magnetohydrodynamics (MHD) – X-rays: binaries

1. INTRODUCTION

Analysis of the diffusion and advection of a large-scale magnetic field in an accretion disk with a turbulent viscosity and magnetic diffusivity arising from the magnetorotational instability (MRI) show that a weak large-scale field diffuses outward rapidly (van Ballegooijen 1989; Lubow et al. 1994). We mention but do not consider here the opposite limit where the magnetic field is sufficiently strong that it suppresses the MRI instability so that the disk is non-turbulent but accretion occurs due to angular momentum outflow to a magnetic disk wind or jet (Lovelace et al. 1994, 1997). Earlier, Bisnovatyi-Kogan & Lovelace (2007) pointed out that the disk’s surface layers are highly conducting because the MRI instability is suppressed in this region where the magnetic and radiative energy densities are larger than the thermal gas energy density. Rothstein & Lovelace (2008) analyzed this problem in further detail and discussed the connections with global and shearing box magnetohydrodynamic (MHD) simulations of the MRI. Lovelace et al. (2009), hereafter LRBK, developed an analytic model for the vertical \( (z) \) profiles of the stationary accretion flows (with radial and azimuthal components), and the profiles of the large-scale, magnetic field taking into account the turbulent viscosity and diffusivity due to the MRI and the fact that the turbulence vanishes at the surface of the disk.

Here, we require that the radial accretion speed be zero at the disk’s surface, and we assume that the ratio of the turbulent viscosity to the turbulent magnetic diffusivity is of order unity as suggested by MHD shearing-box simulations (Guan & Gammie 2009). For a fixed dimensionless viscosity \( \alpha \)-value, we find that there is a definite relation between the ratio \( R \) of the accretion power going into magnetic disk winds to the viscous power dissipation and the midplane plasma-\( \beta \), which is the ratio of the plasma to magnetic pressure in the disk.

Section 2 discusses the model for the flow and ordered magnetic field in a viscous diffusive disk. Section 3 discusses the conclusions.

2. THEORY

Following LRBK we consider the non-ideal MHD of a thin axisymmetric, viscous, resistive disk threaded by a large-scale dipole-symmetry magnetic field \( B \). We use a cylindrical \( (r, \phi, z) \) inertial coordinate system in which the time-averaged magnetic field is \( B = B_r \hat{r} + B_\phi \hat{\phi} + B_z \hat{z} \), and the time-averaged flow velocity is \( v = v_r \hat{r} + v_\phi \hat{\phi} + v_z \hat{z} \). The main equations are

\[
\rho \frac{d v}{d t} = -\nabla p + \rho g + \frac{1}{c^2} J \times B + F^v, \tag{1}
\]

\[
\frac{\partial B}{\partial t} = \nabla \times (v \times B) - \nabla \times (\eta \nabla \times B). \tag{2}
\]

These equations are supplemented by the continuity equation, \( \nabla \cdot (\rho v) = 0 \), by \( \nabla \times B = 4\pi J / c \) and by \( \nabla \cdot B = 0 \). Here, \( \eta \) is the magnetic diffusivity, \( F^v = -\nabla \cdot T^v \) is the viscous force with \( T^v_{jk} = -\rho v_\delta \partial v_j / \partial x_k + \rho v_\delta \partial v_j / \partial x_j - (2/3) \delta_{jk} \nabla \cdot v \) (in Cartesian coordinates), and \( v \) is the kinematic viscosity. For simplicity, in place of an energy equation we consider the adiabatic dependence \( p \propto \rho^{\gamma} \), with \( \gamma \) the adiabatic index.

We assume that both the viscosity and the diffusivity are due to MRI turbulence in the disk so that

\[
v = P \eta = \alpha \frac{c^2_0}{\Omega_k} g(z), \tag{3}
\]

where \( P \) is the magnetic Prandtl number of the turbulence assuming a constant of order unity (Bisnovatyi-Kogan &
Ruzmaikin 1976), \(a \leq 1\) is the dimensionless Shakura & Sunyaev (1973) parameter, \(c_{a0}\) is the midplane isothermal sound speed, \(\Omega_K \equiv (GM/r^3)^{1/2}\) is the Keplerian angular velocity of the disk, and \(M\) is the mass of the central object. The function \(g(z)\) accounts for the absence of turbulence in the surface layer of the disk (Bisnovatyi-Kogan & Lovelace 2007; Rothstein & Lovelace 2008). In the body of the disk \(g \equiv 1\), whereas at the surface of the disk, at say \(z_S\), \(g\) tends over a short distance to a very small value \(\sim 10^{-8}\), effectively zero, which is the ratio of the Spitzer diffusivity of the disk’s surface layer to the turbulent diffusivity of the body of the disk. At the disk’s surface the density is much smaller than its midplane value.

This step-function profile of the viscosity and diffusivity \(g(z)\) is a major idealization of the turbulence in the disk because of the turbulent overshoot which is known to occur in the transition region between an unstable and stable layers of a fluid. A more general treatment is deferred to a future work.

We consider stationary solutions of Equations (1) and (2) for a weak large-scale magnetic field. These can be greatly simplified for thin disks where the disk half-thickness, of the order of \(h \equiv c_{a0}/\Omega_K\), is much less than \(r\). Thus, we have the small parameter

\[
\varepsilon = \frac{h}{r} = \frac{c_{a0}}{v_K} \ll 1. \quad (4)
\]

It is useful in the following to use the dimensionless height \(\xi \equiv z/h\). The midplane plasma beta is taken to be

\[
\beta \equiv \frac{4\pi \rho_0 c_{a0}^2}{B_0^2}, \quad (5)
\]

where \(\beta = c_{a0}^2/v_K^2\) and \(v_{a0} = B_0/(4\pi\rho_0)^{1/2}\) is the midplane Alfvén velocity. Note that the conventional definition of beta is \(2\beta\). The rough condition for the MRI instability and the associated turbulence in the disk is \(\beta \gtrsim 1\) (Balbus & Hawley 1998) and this is assumed here.

The three magnetic field components are assumed to be of comparable magnitude on the disk’s surface, but \(B_z = 0 = B_\phi\) on the midplane. On the other hand, the axial magnetic field changes by only a small amount going from the midplane to the surface, \(\Delta B_z \sim \varepsilon B_z \ll B_z\) (from \(\nabla \cdot \mathbf{B} = 0\)) so that \(B_z \approx\) const inside the disk. As a consequence, the \(\partial B_z/\partial r\) terms in the magnetic force in Equation (1) can all be dropped in favor of the \(\partial B_\phi/\partial z\) terms (with \(j = r, \phi\)). It is important to keep in mind that \(B_z\) is the large-scale field; the approximation does not apply to the small-scale field which gives the viscosity and diffusivity. The three velocity components are assumed to satisfy \(v_z^2 \ll c_{a0}^2\) and \(v_r^2 \ll v_\phi^2\). Consequently, \(v_\phi(r, z)\) is close in value to the Keplerian value \(v_K(r) \equiv (GM/r^2)^{1/2}\), except in the outer disk layers where the radial magnetic force may be comparable with the centrifugal and gravitational forces. We normalize the field components by \(B_0 = B_z(r, z = 0)\), with \(b_z = B_z/B_0, b_\phi = B_\phi/B_0\), and \(b_r = B_r/B_0 \approx 1\). Also, we define \(u_\phi \equiv v_\phi(r, z)/v_K(r)\) and the accretion speed \(u_r \equiv -v_r/(\alpha c_{a0})\). For the assumed dipole field symmetry, \(b_r\) and \(b_\phi\) are odd functions of \(\xi\) whereas \(u_r\) and \(u_\phi\) are even functions.

Integration over the vertical extent of the disk gives the average accretion speed

\[
\overline{u_r} = u_0 - \frac{2b_{\phi S+}}{\alpha \beta \Sigma} \quad (6)
\]

(LRBK) which is the sum of a viscous contribution, \(u_0 \equiv 3\varepsilon k_v\) (with \(k_v\) a numerical constant of order unity) and a magnetic contribution \(\propto b_{\phi S+}\) due to the loss of angular momentum from the surface of the disk where necessarily \(b_{\phi S+} \leq 0\) (Lovelace et al. 1994).

We assume \(p \propto \rho^\gamma\) so that the vertical hydrostatic equilibrium gives

\[
\frac{\rho}{\rho_0} = \left(1 - \frac{(\gamma - 1)\varepsilon^2}{2\gamma}\right)^{1/(\gamma - 1)} , \quad (7)
\]

for \(\beta \gg 1\). The density goes to 0 at \(z_m = [2\gamma/(\gamma - 1)]^{1/2}\). However, before this distance is reached the MRI turbulence is suppressed and \(g(\xi)\) in Equation (3) is effectively zero.

The different components of Equations (1) and (2) can be combined (LRBK) to give the following equation for the radial accretion speed:

\[
\alpha^2 \beta^2 \frac{\partial^2}{\partial \xi^2} \left(\frac{\partial}{\partial \xi} \left(\frac{\rho g}{\partial \xi} \frac{\partial u_r}{\partial \xi} \right) \right) + \frac{\beta}{\rho} \frac{\partial^2}{\partial \xi^2} \left(\rho g (u_r - g u_0)\right) + \frac{\rho^2}{\partial \xi^2} \left(\frac{u_r}{\rho g}\right) + 3 \beta \frac{\rho^2}{\partial \xi^2} = 0. \quad (8)
\]

The equation can be integrated from \(\xi = 0\) out to the surface of the disk \(\xi_S\) where boundary conditions apply.

For specificity we take \(g(\xi) = (1 - \xi^2/\xi_m^2)^5\), where \(\xi_m < \xi_m\) and \(\delta\) is a constant. That is, we neglect the ratio of the Spitzer diffusivity on the surface of the disk to its value in the central part of the disk. An estimate of \(\xi_S\) can be made by noting that \(\beta(\xi) = 4\pi \rho(\xi)/B_0^2 = \beta(\rho)^2 \approx 1\) at \(\xi_S\). This gives \(\xi_S^2/\xi_m^2 = 1 - \beta^{-(\gamma - 1)}\gamma\) and \(\rho_S/\rho_0 = \beta^{-1/\gamma}\).

### 2.1 Boundary Conditions

We consider only the solutions which have net mass accretion,

\[M = 4\pi r h_{\rho_{0,ph}} c_{a0} \Sigma \eta_{\rho} > 0\]

and have \(b_{\phi S} \leq 0\) on the disk’s surface. This condition on \(b_{\phi S}\) corresponds to an efflux of angular momentum and energy (or their absence) from the disk to its corona rather than the reverse. The condition \(b_{\phi S} < 0\) is the same as \(\eta_{\rho} > u_0\), where \(u_0\) is the minimum (viscous) accretion speed. Note that \(\eta = \bar{u}_r/u_0 - 1\) is the ratio of the accretion power which goes into the disk winds to the viscous power dissipation. Clearly, the condition on \(b_{\phi S}\) implies that \(M > 0\) so that there is only one condition. In general, there is a continuum of values of \(b_{\phi S} \leq 0\) for the considered solutions inside the disk. The value of \(b_{\phi S}\) can be determined by matching the calculated fields \(b_{\phi S}\) and \(b_{\phi S}\) onto an external field and flow as discussed in LRBK and here in Section 3.

LRBK showed that there is no jump in \(b_r\) across the conducting surface layer. This implies that \(\partial u_r/\partial \xi_{\xi_S} = 0\), which represents a first boundary condition of the solution of Equation (8). The equations inter-relating \(b_r, b_\phi, u_r, \) and \(\delta u_\phi\) are summarized in the Appendix. A second boundary condition, \(u_r|_{\xi_S} = 0\), follows from Equation (12) of LRBK evaluated just outside of the conducting layer. A third boundary condition, \(\partial b_\phi/\partial \xi_{\xi_S} = 0\), is derived in LRBK.
3. SPECIFIC SOLUTIONS

Here, to simplify the analysis we consider the limit where \( \gamma \to \infty \) in Equation (7) and \( \delta \to 0 \) in Equation (18) of LRBK. Then, \( \zeta_0 \to \zeta_m \) and both \( \beta \) and \( g \) are unit step functions going to 0 at \( \zeta_m = \sqrt{2} \). Also, \( \Pi_r = \langle u_r \rangle \) and \( \Sigma = \sqrt{2} \). Thus, the above physical condition \( \Pi_r \geq 3 \varepsilon k_r = u_0 \) implies that \( b_{rS} \geq u_0 \zeta_0 \beta P \)
from Equation (13) of LRBK. We assume \( k_p = -\partial \ln p/\partial \ln r \) and \( k_0 = 1 \).

The solutions to Equation (8) are \( u_r \propto \exp(i k_j \zeta) \) (with \( j = 1, 2, 3 \)),

\[
\alpha^2 \beta^2 (k_j^2)^3 + 2P \alpha^2 \beta (k_j^2)^2 + (\alpha^2 \beta^2 + P^2)k_j^2 - 3 \beta^2 P^2 = 0 \tag{9}
\]
is a cubic in \( k_j^2 \). The discriminant of the cubic is negative so that there is one real root, \( k_1^2 \), and a complex conjugate pair of roots, \( k_2^2, k_3^2 \). Because \( u_r \) is an even function of \( \zeta \) we can write

\[
u_r = a_1 \cos(k_0 \zeta) + a_2 \cos(k_r \zeta) \cos(k_\zeta),
+ a_3 \sin(k_r \zeta) \sin(k_\zeta), \tag{10}
\]
where \( k_0 = \sqrt{k_1^2} \), \( k_r = \text{Re}(\sqrt{k_2^2}) \), and \( k_\zeta = \text{Im}(\sqrt{k_2^2}) \).

We consider a thin disk, \( \varepsilon = h/r = 0.05 \), and a viscosity parameter, \( \alpha = 0.1 \). Figure 1 shows the dependences of the surface field components on the average accretion speed for \( \beta = 240 \) and \( P = 1 \). The \( b_{rS} \) dependence is given by Equation (6) and is independent of \( P \) while the \( b_{rS} \) is given by \( b_{rS} = P \zeta_0 \Pi_r \) from LRBK (Equation (13)).

There are three boundary conditions at the surface of the disk. Additionally, there is the ratio \( R \) of the accretion power going into magnetic disk winds to the viscous power dissipation. The solution for \( u_r, (\zeta) \) has three unknown coefficients, \( a_1, a_2, \) and \( a_3 \) which are dependent on \( \alpha, P, \) and \( \beta \). If \( \alpha \) and \( P \) are fixed, then there is a definite relation between \( R \) and \( \beta \). Figure 2 shows this relation determined numerically.

Figures 3 shows the vertical profile of the accretion flow for \( \beta = 240 \). The profiles of \( \delta u_r = (\nu_r - v_K)/v_K, b_r, b_\phi \) follow from the equations in the Appendix.

The value of \( b_{\phi S} \leq 0 \) or \( R \) is not fixed by the solution for the field and flow inside the disk. Its value can be determined by matching the calculated surface fields \( b_{rS} \) and \( b_{\phi S} \), onto an external magnetic wind or jet solution. Stability of the wind jet solution to current driven kinking is predicted to limit the ratio of the toroidal to axial magnetic field components at the disk’s surface \( |b_{\phi S}| \) to values \( \lesssim O(2\pi) \) (Hsu & Bellan 2002; Nakamura et al. 2007). Recall that \( R = \Pi_r/u_0 - 1 = 2|b_{\phi S}|/(\alpha \beta \sqrt{2} u_0) \) is the ratio of the accretion power going into the disk wind to the viscous dissipation in the disk. For the mentioned upper limit on \( |b_{\phi S}|, R \lesssim O(2\sqrt{2} \pi/(\alpha \beta u_0)) \approx 2.5 \). This implies a critical value of \( \beta \) for a stationary disk solution (for \( P = 1 \), \( \beta_c \approx 1.2(\alpha \varepsilon)^{-1} \). For \( \alpha = 0.1 \) and \( \varepsilon = 0.05 \), \( \beta_c \approx 240 \).

For \( \beta \lesssim \beta_c \), the disk and large-scale magnetic field are not to be in a stationary state. Equation (6) for the average accretion speed still applies and can be written as \( \Pi_r = u_0 + u_\phi \beta_c \), where \( u_\phi \) is the magnetic contribution to the accretion due to the outflow of angular momentum from the disk’s surfaces. In general, \( u_\phi \) is an increasing function of the magnetic field strength if \( R \sim 1 \) (Lovelace et al. 1994). For this reason, for \( \beta < \beta_c \), the ordered poloidal magnetic field threading the disk will be advected inward in the disk while for \( \beta > \beta_c \), the field will advect outward. Thus, \( \beta_c \) acts as a threshold value for the buildup of a large-scale field in the inner regions of an accretion disk.

Another possibility is that the disk winds are negligible so that \( R \ll 1 \). Figure 2 suggests that in this limit we have a stable stationary solutions with \( \beta \) values much larger than the mentioned critical value.
4. CONCLUSIONS

For a specific disk model with turbulent Prandtl number $P = 1$ and $R \sim 1$, we show that there is a critical value $\beta_c \approx 1.2r/(\alpha h)$ for a stationary disk threaded by a large-scale magnetic field. For $\alpha = 0.1$ and $\varepsilon = 0.05$, $\beta_c \approx 240$. For $\beta < \beta_c$, we argue that the large-scale field will be advected inward and lead to the buildup of the ordered field in the central region of the disk. For $\beta > \beta_c$, the ordered field will advect outward. Alternatively, if the disk wind is negligible so that $R \ll 1$, stationary disk equilibrium with a large-scale field are possible for $\beta$ much larger than the mentioned $\beta_c$.

The three-dimensional MHD simulations of Beckwith et al. (2009) show the inward advection of a significant fraction of the initial unipolar vertical magnetic flux threading an MRI unstable plasma torus around a black hole. The initial average $\beta$ in the torus is 100 which is significantly less than the critical value estimated here. A broader range of simulations for much larger initial values of $\beta$ would be needed to determine if the initial flux diffuses outward. Axisymmetric MHD simulations of Romanova et al. (2011) show episodes of enhanced accretion on the disk surfaces for the case of an MRI unstable disk around a rotating magnetized stars. However, the presence of the stellar magnetic field complicates the analysis of flux transport in the disk.

We have shown that the presence of a large-scale poloidal magnetic field in an accretion disk around a black hole not only plays a decisive role in the jet formation, particle acceleration, and observed hard energy radiation (Bosch-Ramon 2011), but also creates the possibility for a new type of non-stationary behavior in these objects, which could be related to appearance of different spectral and luminosity states and transitions between them as observed in microquasars (Malzac & Belmont 2009).

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APPENDIX

SUBSIDIARY EQUATIONS

Here, we summarize the relations between the field and velocity components. From the radial force balance,

$$\frac{\partial b_r}{\partial \zeta} = \frac{\beta}{\varepsilon} \left(1 - k_p \varepsilon^2 - u_\phi^2\right) + \alpha \beta \frac{\partial}{\partial \zeta} \left(\tilde{\rho} g \frac{\partial u_r}{\partial \zeta}\right),$$

(A1)

where $k_p \equiv -d \ln p/d \ln r$ is assumed positive and of order unity. The $\phi$-component of Equation (2) gives

$$\frac{\partial b_\phi}{\partial \zeta} = \frac{\alpha \beta}{2} (\varepsilon k_r g - u_\phi) - \frac{\alpha \beta}{\varepsilon} \frac{\partial}{\partial \zeta} \left(\tilde{\rho} g \frac{\partial u_\phi}{\partial \zeta}\right),$$

(A2)

where $k_r \equiv d \ln (\rho c_s^2 r^2/h)/d \ln r > 0$ is also assumed to be of order unity.

The toroidal component of Ohm’s law gives

$$\frac{\partial b_r}{\partial \zeta} = \frac{\rho g}{\tilde{\rho} g} u_\phi.$$  

(A3)

The other components of Ohm’s law give

$$\frac{\partial u_\phi}{\partial \zeta} = \frac{3 \varepsilon}{2} b_r - \frac{\alpha \varepsilon}{\rho g} \frac{\partial}{\partial \zeta} \left(\tilde{\rho} g \frac{\partial u_r}{\partial \zeta}\right).$$

(A4)

Combining Equations (A1) and (A3) gives

$$u_r = \frac{\beta \tilde{\rho} g}{\varepsilon \rho g} \left(1 - k_p \varepsilon^2 - u_\phi^2\right) + \frac{\alpha^2 \varepsilon g}{\rho g} \left(\frac{\partial b_r}{\partial \zeta}\right).$$

(A5)

For thin disks, $\varepsilon \ll 1$ and $\beta > 1$, we have $u_\phi = 1 + \delta u_\phi$ with $(\delta u_\phi)^2 \ll 1$. Consequently,

$$\delta u_\phi = -\frac{k_p \varepsilon^2}{2} - \frac{\varepsilon \rho g u_r}{2 \beta \tilde{\rho} g} + \frac{\alpha^2 \varepsilon}{2 \tilde{\rho} g} \left(\frac{\partial b_r}{\partial \zeta}\right),$$

(A6)

to a good approximation.

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