A Microscopical Description of Giant Gravitons II: 

The $AdS_5 \times S^5$ Background

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ABSTRACT

In this article we continue with the microscopical investigation of giant graviton configurations in $AdS_m \times S^n$ spacetimes, initiated in hep-th/0207199. Using dualities and a Matrix theory derivation we propose an action that describes multiple Type IIB gravitons. This action contains multipole moment couplings to the Type IIB background potentials. Using these couplings, we study, from the microscopical point of view, the giant graviton and dual giant graviton configurations in the $AdS_5 \times S^5$ background. In both cases the gravitons expand into a non-commutative 3-sphere, that is defined as an $S^1$-bundle over a fuzzy 2-sphere. When the number of gravitons is large we find perfect agreement with the Abelian, macroscopical description of giant gravitons in this spacetime, given in the literature.

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1 Introduction

Giant gravitons \([1, 2, 3, 4]\) are stable brane configurations with non-zero angular momentum, that are wrapped around \((n - 2)\)- or \((m - 2)\)-spheres in \(AdS_m \times S^n\) spacetimes, and carry a dipole moment with respect to the background gauge potential. They are not topologically stable, but are at a dynamical equilibrium because the contraction due to the tension of the brane is precisely cancelled by the expansion due to the coupling of the angular momentum to the background flux field. It turns out that these spherical brane configurations are massless, conserve the same number of supersymmetries and carry the same quantum numbers of a graviton. The fact that they are extended objects of finite size has lead to the name of giant gravitons.

These configurations were first proposed in \([1]\) as a way to satisfy the stringy exclusion principle implied by the AdS/CFT correspondence. The spherical \((n - 2)\)-brane expands into the \(S^n\) part of the geometry with a radius proportional to its angular momentum. Since this radius is bounded by the radius of the \(S^n\), the configuration has associated a maximum angular momentum. The \((m - 2)\)-brane giant graviton configurations later found in \([2, 3]\) expand, on the other hand, into the \(AdS_m\) component of the spacetime, so that there is no upper bound implied on their angular momentum. They are referred to in the literature \([2]\) as dual giant gravitons. One expects that quantum mechanically these different states will tunnel into each other, forming a unique ground state. In \([2]\) it is speculated that the stringy exclusion principle may still be satisfied if once the quantum mixing is taken into account there is no supersymmetric ground state for angular momenta beyond the exclusion principle bound. Other giant graviton solutions in various backgrounds have been studied elsewhere in the literature \([5, 6]\). Giant gravitons in pp-wave backgrounds have been studied in \([7, 8, 9, 10, 11]\).

The appearance of giant gravitons as blown up massless particles hints to a connection with other examples of expanded brane configurations, more precisely to the dielectric effect, where multiple coinciding D\(_p\)-branes can expand into higher dimensional D-brane configurations. There are two complementary descriptions of this effect. Consider the case in which the D\(_p\)-branes expand into a spherical D\((p + 2)\)-brane. The first one is an Abelian description, describing the spherical D\((p + 2)\)-brane with a large number of D\(_p\)-branes dissolved on its worldvolume \([12]\). The second one is a non-Abelian formulation \([13, 14, 15]\), describing how multiple coinciding D\(_p\)-branes expand into a D\((p + 2)\)-brane with the topology of a fuzzy 2-sphere \([15]\). Both descriptions agree in the limit where the number \(N\) of D\(_p\)-branes is very large.

It has been suggested in the literature (see for example \([14, 17, 7, 18]\)) that it should be possible to describe the giant gravitons of \([1, 2, 3, 4]\) in terms of dielectric gravitational waves. Since massless particles, in particular gravitons, are the source terms for gravitational waves, it is natural to expect that a dielectric effect for these gravitational waves will provide a microscopic picture for the giant graviton configurations. By analogy with the dielectric effect for D-branes, it is believed that in the limit when the number of gravitons, \(N\), is large, this non-Abelian, microscopical description should match the Abelian, macroscopical description of \([1, 2, 3, 4]\), in terms of a spherical brane with angular momentum, where microscopically the angular momentum of the Abelian description is interpreted as the total momentum of the multiple gravitational waves, propagating in the spherical part of the geometry.

In \([15]\) an action describing Type IIB gravitational waves was derived using Matrix string theory in a weakly curved background. This action shows the linear couplings to closed string backgrounds, and contains the now familiar multipole couplings that give rise to the dielectric effect \([15]\). From them one can construct configurations of multiple coinciding gravitational waves expanding into higher dimensional fuzzy surfaces. However, with a non-Abelian action for gravitational waves only known up to linear order in the background fields it is not possible to check whether dielectric gravitational waves really provide a microscopical description for the giant gravitons in \(AdS_m \times S^n\) backgrounds, given that the \(AdS_m \times S^n\) spacetime cannot be taken as a linear perturbation to Minkowski.

In reference \([19]\) we gave one step further in this direction, by providing a closed expression for the worldvolume action for multiple M-theory gravitons, valid beyond the linear approximation. A linear action for M-theory gravitons can easily be constructed by just uplifting the linear
action for multiple Type IIA gravitons constructed in [18] using Matrix string theory in a weakly curved background\(^4\). Demanding consistency with the action for coincident D0-branes when the gravitons propagate along the eleventh direction it is possible to extend this action beyond the linear approximation used in the Matrix theory calculation. In fact, a non-trivial check of this action is that it predicts the polarisation of the gravitons into fuzzy spheres, in such a way that in \(AdS_m \times S^n\) backgrounds the energy and radius of the dielectric configurations exactly match those of the giant graviton solutions of \([1, 2]^5\).

With a closed expression for the action for M-theory gravitational waves it is possible to construct an action for Type IIA gravitational waves valid beyond the linear approximation of [18]. This action was presented in [19]. Given that Type II waves are simply related by a T-duality transformation, it is straightforward to construct an action for multiple Type IIB gravitational waves valid beyond the linear approximation of [18]. With this action it is then possible to construct explicit configurations of multiple Type IIB gravitons in the \(AdS_5 \times S^5\) background, which we can compare with the giant graviton [1] and the dual giant graviton [2, 3] solutions in this spacetime. This is precisely the aim of this paper.

The construction of non-Abelian giant gravitons in \(AdS_5 \times S^5\) turns out to be more involved than the eleven-dimensional non-Abelian giant graviton solutions of [19]. In this background the gravitational waves expand into a spherical D3-brane, and hence a fuzzy 3-sphere Ansatz is needed in the non-Abelian construction, rather than the well-known fuzzy 2-sphere, in terms of \(SU(2)\) matrices. Non-commutative odd spheres have been constructed in [20, 21, 22], as subspaces of fuzzy even spheres. It turns out that these constructions are not applicable to our case. The main reason is that due to our construction of the action for Type IIB gravitational waves via T-duality, this action exhibits an extra isometry direction associated to the coordinate over which the T-duality was performed. This isometry will require a non-manifestly \(SO(4)\)-symmetric solution, which is not of the type presented in [20, 21, 22]. Furthermore, the fact that we are dealing with gravitational waves, which have a one-dimensional worldvolume, allows only non-Abelian couplings similar to the ones one encounters in the case of fuzzy two-spheres. The solution to this apparent paradox is to consider the three-sphere as a \(U(1)\)-bundle over \(S^2\) and choose the extra isometry direction to be the fibre coordinate. The fuzzy version of the \(S^3\) will then consist of a (Abelian) \(U(1)\)-fibre over a fuzzy \(S^2\).

The paper is organised as follows: we start in Section 2 with the construction of the action for Type IIB gravitational waves in the way explained in the previous paragraphs. This action is valid beyond the linear order approximation taken in [18], and can therefore be used to study the \(AdS_5 \times S^5\) background considered in this paper. With this action, we provide, in Section 3, a microscopical description of the giant graviton configuration of [1] in \(AdS_5 \times S^5\). This solution is expected to occur, microscopically, in the form of \(N\) gravitons polarised in a fuzzy 3-sphere contained in \(S^6\). Describing the \(S^3\) as an \(S^1\) bundle over \(S^2\) and making non-commutative the base 2-sphere we find a giant graviton solution which in the large \(N\) limit reproduces exactly the (Abelian) result of [1]. This is a non-trivial check of the validity of our microscopical description and, in particular, of our non-commutative Ansatz. Further support is provided by the calculation in Section 4. In this section we construct microscopically the dual giant graviton solution of [2, 3]. Describing the 3-sphere contained in \(AdS_5\) as an \(S^1\) bundle over \(S^2\) and making this \(S^2\) non-commutative we find a dual giant graviton solution which is also in perfect agreement with the results in [2, 3] when \(N\) is large. Section 5 contains the Discussion, where we comment on the relations between our non-commutative 3-sphere solutions and other constructions of odd non-commutative spheres previously discussed in the literature [20, 21, 22].

\(^{4}\)The derivation in [18] considers Matrix string theory in a weakly curved background in the Sen-Seiberg limit, so that the strings carry spatial momentum, and then takes static gauge. In this way one arrives at an action that describes multiple massless particles carrying momentum. In the Abelian limit, this action is related to the perturbative action for massless particles through a Legendre transformation (we refer the reader to [18] for the details of this construction).

\(^{5}\)The explicit calculation in [19] is for the giant graviton solution in \(AdS_7 \times S^4\) and the dual giant graviton solution in \(AdS_4 \times S^7\), i.e. the cases involving fuzzy 2-spheres.
2 The action for multiple Type IIB gravitational waves

In this section we construct an action for Type IIB gravitational waves suitable for the study of giant graviton configurations in the \( AdS_5 \times S^5 \) background. The starting point is the action for multiple Type IIA gravitational waves presented in \[19\]. This action is simply the weakly coupled (Type IIA) version of the action there proposed for the microscopical study of M-theory gravitons in \( AdS_m \times S^n \) spacetimes. The M-theory action of \[19\] gives the couplings of multiple gravitational waves to M-theory background fields, in the form of a closed expression valid beyond the weak background field limit. The extension beyond the linear order of Matrix theory can be done by demanding agreement with the action for multiple D0-branes when the gravitons propagate along the eleventh direction. T-dualising the action for the Type IIA waves we will obtain an action for Type IIB waves also valid beyond the linear order approximation taken in \[18\].

The Born-Infeld action proposed in \[19\] to describe multiple Type IIA gravitational waves is given by

\[
S_{\text{BI}}^{\text{IA}} = -T_0 \int d\tau \left \{ k^{-1} \sqrt{-P} [E_{00} + E_{0i}(Q^{-1} - \delta)\delta_k E_{00}] \det(Q_j^i) \right \},
\]

where

\[
E_{\mu \nu} = g_{\mu \nu} - k^{-2}k_{\mu}k_{\nu} + k^{-1}e^\phi (i_k C^{(3)})_{\mu \nu},
\]

\[
Q_j^i = \delta_j^i + i[X^i, X^j]e^{-\phi}kE_{kj}, \quad i, j = 1, \ldots, 9.
\]

This action is manifestly invariant under global gauge transformations along the Killing direction: \( \delta X^\mu = \Lambda k^\mu \), since this direction is projected out through the effective metric \( G_{\mu \nu} = g_{\mu \nu} - k^{-2}k_{\mu}k_{\nu} \) and the contraction of the 3-form with \( k^\mu \). The condition that \( k^\mu G_{\mu \nu} = 0 \) implies that the gravitational field is transversal to the direction of propagation of the waves. For simplicity, this action has been calculated for vanishing \( C^{(1)}, B^{(2)} \) and the worldvolume scalar field \( A \).

We now make a T-duality transformation along a transverse direction \( Z \) in order to obtain the action for Type IIB gravitational waves. This transformation was explained in detail in reference \[18\], for the linearised action. The action for Type IIB gravitational waves contains two worldvolume scalars \( A \) and \( Z \), being, respectively, the T-dual of the Type IIA worldvolume scalar \( A \) and the T-dual of the embedding scalar in the direction of the dualisation. Their gauge invariant curvatures are given by \( F = \partial A - P[12]C^{(2)} \) and \( F = \partial Z + P[12]B^{(2)} \) and together they form a doublet under the Type IIB S-duality transformations. Consistently with S-duality and the truncation imposed in the Type IIA action, we will set \( C^{(2)}, B^{(2)}, A \) and \( Z \) equal to zero. Also, for simplicity we take \( k_z = 0 \). This truncation is suitable for the study of gravitational waves in the \( AdS_5 \times S^5 \) background, as we show below. Hence, we obtain the following Born-Infeld action:

\[
S_{\text{BI}}^{\text{IB}} = -T_0 \int d\tau \left \{ k^{-1} \sqrt{-P} [E_{00} + E_{0i}(Q^{-1} - \delta)\Lambda k E_{00}] \det(Q_j^i) \right \},
\]

where now

\[
E_{\mu \nu} = g_{\mu \nu} - k^{-2}k_{\mu}k_{\nu} - l^{-2}l_{\mu}l_{\nu} - k^{-1}l^{-1}e^\phi (i_k l^1 C^{(4)})_{\mu \nu},
\]

\[
Q_j^i = \delta_j^i + i[X^i, X^j]e^{-\phi}kE_{kj}.
\]

Here \( l^\mu \) is a new Killing vector, pointing along the direction in which we performed the T-duality transformation. It is easy to check that this action is invariant under the global isometric transformations generated by \( k^\mu \) and \( l^\mu \): \( \delta X^\mu = \Lambda^{(1)} k^\mu + \Lambda^{(2)} l^\mu \), since these directions are projected

\footnote{We will be dealing throughout the article with only the bosonic part of the worldvolume effective actions. This is enough for the study of the bosonic giant graviton configurations.}

\footnote{This is inferred by the analysis of the monopole term in the Chern-Simons effective action, which shows that the momentum arises as the charge with respect to the background field \( k^{-2}k_{\mu} \). For details see \[13\].}

\footnote{The worldvolume scalar field comes from the reduction of the eleventh scalar, and forms an invariant field strength with the RR 1-form potential.}
out through the effective metric \( G_{\mu\nu} = g_{\mu\nu} - k^{-2}k_{\mu}k_{\nu} - l^{-2}l_{\mu}l_{\nu} \) and the double contraction of the 4-form with the two Killing vectors. It is due to the fact that we have two isometric directions that \( C^{(4)} \) can couple to the Born-Infeld part of the action. This coupling plays a key role in the microscopical description of the \( AdS_5 \times S^5 \) giant graviton solution \( \Pi \), that we will perform in the next section. In this action \( k^\mu \) still points in the direction of propagation of the waves, since under the T-duality transformation and with the truncations above the monopole term is mapped onto itself. The other isometric direction, \( Z \), is however not physical, but just an artifact of the T-duality transformation. We are therefore describing waves which are smeared in the \( Z \)-direction.

In the Abelian limit, (2.4) is still a complicated expression containing two isometric directions. However, together with the (Abelian) Chern-Simons coupling, which is given by \[ S_{\text{CS}} = T_0 \int d\tau k^{-2}k_{\mu}\partial X^\mu, \] this effective action can be related to the (dimensional reduction along the \( Z \) direction of the) usual perturbative action for massless particles

\[ S[\gamma] = -\frac{NT_0}{2} \int d\tau \sqrt{|\gamma|}^{-1}\partial X^\mu\partial X^\nu g_{\mu\nu}, \] by means of a Legendre transformation that restores the dependence on the direction of propagation (see section 2 in [18] for the details). In the non-Abelian case, however, it is not possible to restore this dependence. First of all, this direction is not matrix-valued, so even though we could in principle restore some explicit dependence on its time derivative, the new terms would be Abelian. Second, it is clear that the Legendre transformation cannot give rise to non-Abelian commutators involving this direction, so we would end up in any case with an action with reduced transverse rotational invariance. Thus, we are constrained to work, in the non-Abelian case, with an action with two isometries, and assume the presence of the extra unphysical isometry.

This isometry will however play a key role in our microscopical description of giant graviton configurations in \( AdS_5 \times S^5 \). Indeed, the presence of this compact isometry suggests the representation of the 3-sphere of the giant graviton configurations as a \( U(1) \) bundle over \( S^2 \), with the \( U(1) \) invariance being associated to translations along this direction. We will see that the giant graviton configurations correspond to the polarisation of the gravitons in fuzzy 3-spheres represented as \( U(1) \) bundles over a fuzzy \( S^2 \).

Finally, we turn to the Chern-Simons part of the action for the Type IIB waves. This action was constructed, to linear order in the background fields, in reference [18] (see expression (4.5)). In particular, the linear coupling to the 4-form RR-potential, relevant for the construction of giant gravitons in the \( AdS_5 \times S^5 \) background, was shown to be given by:

\[ S_{\text{CS}} = -i \int d\tau \text{STr}\{ P[(i X_i i X)_{[4} C^{(4)]}] \}. \]

In this action the pull-backs into the worldvolume are defined in terms of gauge covariant derivatives

\[ D X^\mu = \partial X^\mu - A^{(1)} k^\mu - A^{(2)} l^\mu = \partial X^\mu - k^{-2}k_\rho\partial X^\rho k^\mu - l^{-2}l_\rho\partial X^\rho l^\mu \] with respect to the scaling symmetry

\[ \delta X^\mu = \Lambda^{(1)}(\tau)k^\mu + \Lambda^{(2)}(\tau)l^\mu. \]

In this way we ensure (local) invariance under the isometric transformations generated by the two Killing vectors. Using gauge covariant pull-backs it is possible to eliminate the pull-back of the isometric coordinates, and to reproduce the isometric couplings in the action in a manifestly covariant way. For example, the pull-back of the reduced metric in (2.4), \( g_{\mu\nu} \partial X^\mu \partial X^\nu \), can be written as \( g_{\mu\nu} D X^\mu D X^\nu \), in terms of the covariant pull-backs. The action is then given by a gauged sigma model of the type considered in [23, 24, 25].
The coupling in \(2.9\) plays a key role in the microscopic description of the \(AdS_5 \times S^5\) dual giant graviton solution \([2, 3]\), that we perform in Section 4. It is due to the fact that we have a second isometric direction that the \(C^{(4)}\) electric potential of this background can couple in the one-dimensional worldvolume effective action of the gravitational waves.

The effective action that we will use in the next sections to study the giant graviton configurations in the \(AdS_5 \times S^5\) background is then given by:

\[
S_{WB} = S_{WB}^{BI} + S_{WB}^{CS},
\]

with \(S_{WB}^{BI}\) given by \((2.4)\) and \(S_{WB}^{CS}\) by \((2.9)\).

### 3 The giant graviton in \(AdS_5 \times S^5\)

#### 3.1 The macroscopical description

The giant graviton solution in \(AdS_5 \times S^5\) was computed in \([1]\), by looking at stable test brane solutions where a D3-brane with angular momentum in \(S^5\) had expanded to a 3-sphere contained inside the \(S^5\). We briefly review this construction in order to compare it, in the end, with our microscopic description.

Taking the line element for the metric on \(AdS_5 \times S^5\) as

\[
\begin{align*}
    ds^2_{AdS} &= -(1 + \frac{r^2}{L^2})dt^2 + \frac{dr^2}{1 + \frac{r^2}{L^2}} + r^2 d\Omega^2_3, \\
    ds^2_{S^5} &= L^2(d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\Omega^2_3),
\end{align*}
\]

the trial solution considered in \([1]\) has \(\theta = \text{constant}, \phi = \phi(\tau), \) where \(t = \tau\) in static gauge, and \(r = 0\), i.e. it corresponds to a spherical D3-brane with radius \(L \sin \theta\) orbiting the \(S^5\) in the \(\phi\) direction:

\[
ds^2 = -dt^2 + L^2 \cos^2 \theta d\phi^2 + L^2 \sin^2 \theta d\Omega^2_3.
\]

This D3-brane carries a non-vanishing magnetic moment with respect to the RR 4-form potential of the background, which prevents its collapse to zero size. Parametrising the unit 3-sphere in \([3, 5]\) as

\[
d\Omega^2_3 = d\beta_1^2 + \sin^2 \beta_1(d\beta_2^2 + \sin^2 \beta_2 d\beta_3^2),
\]

the RR 4-form potential is given by

\[
C^{(4)}_{\phi \beta_1 \beta_2 \beta_3} = L^4 \sin^4 \theta \sqrt{g_3},
\]

where \(\sqrt{g_3}\) is the volume element on the unit 3-sphere.

Substituting this trial solution into the worldvolume action of the D3-brane, one arrives at the following Hamiltonian:

\[
H = \frac{P_\phi}{L} \sqrt{1 + \tan^2 \theta \left(1 - \frac{\tilde{N}}{P_\phi} \sin^2 \theta \right)^2}.
\]

Here \(\tilde{N}\) is the integer arising through the quantisation condition of the 4-form flux on \(S^5\)

\[
2\pi^2 T_3 = \frac{\tilde{N}}{L^4},
\]

with \(T_3\) the tension of the D3-brane, and \(P_\phi\) is the angular momentum carried by the brane, which is a constant given that \(\phi\) is a cyclic coordinate.

The Hamiltonian \((3.6)\) has two stable minima, one for \(\sin^2 \theta = 0\) and another for

\[
\sin^2 \theta = P_\phi/\tilde{N}.
\]
The value of the Hamiltonian is in both cases \( E = P_\phi / L \), i.e. both solutions represent massless particles with angular momentum \( P_\phi \). So the first solution corresponds to a pointlike graviton, while the second minimum corresponds to a D3-brane with radius \( L(P_\phi / \hat{N})^{1/2} \), which is the giant graviton solution. The giant graviton satisfies the stringy exclusion principle, since the condition \( \sin^2 \theta \leq 1 \) implies an upper bound on the angular momentum: \( P_\phi \leq \hat{N} \).

### 3.2 The microscopical description

In this section we provide a description of the giant graviton solution in terms of coincident gravitons expanding into a D3-brane, which is inside the \( S^3 \)-part of the background geometry. We will check the correctness of this description by looking whether, for a large number of gravitons, it is in agreement with the previous macroscopical description in terms of a test D3-brane.

The similarity between the giant graviton construction in [1] and the Abelian description of the dielectric (or magnetic moment) effect for Dp-branes suggests a microscopical description of the giant graviton in terms of gravitons expanding into a D3-brane with the topology of a fuzzy 3-sphere. In this description the expansion of the gravitons takes place due to their interaction with the RR 4-form potential of the background. At the level of the graviton worldvolume effective action this interaction occurs in the form of a non-Abelian dielectric coupling.

The action that we have proposed in section 2 to describe Type IIB gravitons contains two isometric directions, one of which is identified with the direction of propagation, whereas the other one reflects the fact that the background on which the gravitons propagate contains a compact direction, which is the direction of the T-duality transformation involved in the construction. The point now is to identify these isometries in the background (3.3). It is clear that the first isometry lays in the \( \phi \)-direction. The second isometric direction can, on the other hand, be identified when one considers the 3-sphere as a \( U(1) \)-bundle over \( S^2 \). The natural choice for the second isometry is the direction associated to the \( U(1) \)-fibre.

Representing the \( S^3 \) with radius \( L \sin \theta \) as a submanifold of \( \mathbb{C}^2 \) with coordinates \((z_0, z_1)\) satisfying \( z_0 z_0 + z_1 z_1 = L^2 \sin^2 \theta \), the Hopf fibering, \( p : S^3 \to S^2 \), is given by a stereographic projection of a point \((z_0, z_1)\) of the \( S^3 \) to a point \( z \) of the \( S^2 \) (see for example [26]):

\[
p(z_0, z_1) = \begin{cases} 
  z = z_1 / z_0 & \text{when } z_0 \neq 0 , \\
  1/z = z_0 / z_1 & \text{when } z_1 \neq 0 . 
\end{cases}
\]  

(3.9)

Points on the \( S^3 \) that differ by an overall factor \( \lambda \in U(1) \) get mapped onto the same point \( z \) of \( S^2 \): \( p(z_0, z_1) = p(\lambda z_0, \lambda z_1) \). In this way the Hopf map is dividing out a \( U(1) \) fibre in the \( S^3 \). Inversely the coordinates \( x_i \) \((i = 1, 2, 3)\) on the \( S^2 \) with \( x_1^2 + x_2^2 + x_3^2 = R^2 \) can be obtained from the Hopf map via

\[
x_1 = \frac{2}{a} \Re (z_0 \bar{z}_1), \quad x_2 = \frac{2}{a} \Im (z_0 \bar{z}_1), \quad x_3 = \frac{1}{a} (|z_0|^2 - |z_1|^2),
\]

(3.10)

where \( a \) is an arbitrary parameter with dimension of length, that relates the radius \( L \sin \theta \) of the \( S^3 \) with the radius \( R \) of the \( S^2 \):

\[
R^2 = \sum_{i=1}^{3} (x_i)^2 = \frac{1}{a^2} \left[ z_0 z_0 + \bar{z}_1 z_1 \right]^2 = \frac{L^4 \sin^4 \theta}{a^2} .
\]

(3.11)

Parametrizing the geometry of the \( S^3 \) in terms of Euler angles

\[
z_0 = L \sin \theta e^{i(\chi_3 + \chi_2)/2} \cos \frac{\chi_1}{2}, \quad z_1 = L \sin \theta e^{i(\chi_3 - \chi_2)/2} \sin \frac{\chi_1}{2},
\]

(3.12)

where \( \chi_1 \in [0, \pi] \), \( \chi_2 \in [0, 2\pi] \) and \( \chi_3 \in [0, 4\pi] \), we get the round metric for \( S^3 \):

\[
d_{S^3}^2 = \frac{L^2 \sin^2 \theta}{4} \left( d\chi_1^2 + \sin^2 \chi_1 d\chi_2^2 + (d\chi_3 + \cos \chi_1 d\chi_2)^2 \right).
\]

(3.13)
Here the angles $\chi_1$ and $\chi_2$ parametrise the $S^2$-base manifold and $\chi_3$ the $S^1$-fibre bundle. Note that the metric has the necessary twist in the fibre in order to obtain the $S^3$ as the global space. The coordinates $x_i$ on the $S^2$ are then given by

$$x_1 = R \sin \chi_1 \cos \chi_2, \quad x_2 = R \sin \chi_1 \sin \chi_2, \quad x_3 = R \cos \chi_1,$$

and we can identify the coordinate $\chi_3$ with the isometric direction in the gravitons effective action not associated to the direction of propagation.

Using Euler angles to write the metric of the three-sphere in coordinates adapted to the fibre structure, the background metric (3.3) and four-form gauge field are given by

$$ds^2 = -dt^2 + L^2 \cos^2 \theta d\phi^2 + \frac{L^2 \sin^2 \theta}{4R^2} \left( dx_1^2 + dx_2^2 + dx_3^2 \right),$$

$$C_{\phi \chi_1 \chi_2 \chi_3}^{(4)} = -\frac{1}{8} L^4 \sin \theta \sin \chi_1.$$

In order to make a non-Abelian Ansatz, it is convenient to go to Cartesian coordinates describing the $S^2$-base manifold of the $S^3$, keeping in mind the constraint that $(x_1)^2 + (x_2)^2 + (x_3)^2 = R^2$. In these coordinates, the metric and the four-form RR-field become

$$ds^2 = -dt^2 + L^2 \cos^2 \theta d\phi^2 + \frac{L^2 \sin^2 \theta}{4R^2} \left[ dx_1^2 + dx_2^2 + dx_3^2 \right],$$

$$C_{\chi_1 \chi_2 \phi}^{(4)} = \frac{L^4 \sin^2 \theta}{8R^3} \epsilon_{ijk} x^k, \quad \text{for } i, j, k = 1, 2, 3.$$

We can then make the following non-commutative Ansatz for the 2-sphere that is parametrised by $x^1, x^2, x^3$:

$$X^i = \frac{R}{\sqrt{N^2 - 1}} J^i, \quad i = 1, 2, 3$$

with the $J^i$, forming an $N \times N$ representation of $SU(2)$ (in our conventions $[J^i, J^j] = 2i \epsilon^{ijk} J^k$). Trivially, with this choice

$$(X^1)^2 + (X^2)^2 + (X^3)^2 = R^2 \mathbb{1},$$

so we are dealing with a non-commutative version of the $S^2$ contained in (3.15). Therefore, with this non-commutative Ansatz, the 3-sphere becomes an $S^1$-bundle over a fuzzy $S^2$. This situation is forced by the topology of the space in which the Type IIB gravitons propagate, having an extra compact $S^1$ direction. Thus, the physical picture will correspond to the gravitons expanding into a non-commutative D3-brane, described in coordinates that reflect a $S^2_{\text{fuzzy}} \times S^1$ structure. To see that this is the right microscopical picture we have to check that in the limit in which the number of gravitons is large we recover the description in II, in terms of a D3-brane with the topology of a classical, commutative, $S^3$. We will carry out this calculation in the remaining part of this section.

The action (2.12) for Type IIB waves in the $\text{AdS}_5 \times S^5$ background described by (the non-commutative version of) the coordinates (3.16), contains

$$E_{00} = -1, \quad E_{0i} = 0, \quad E_{ij} = \frac{L^2 \sin^2 \theta}{4R^2} \left[ \delta_{ij} - \frac{\sin \theta}{R \cos \theta} \epsilon_{ijk} X^k \right],$$

$$Q^i_j = \delta^i_j - \frac{L^4 \sin^3 \theta \cos \theta}{4R \sqrt{N^2 - 1}} \epsilon_{ijk} X^k + \frac{L^4 \sin^4 \theta}{4R^2 \sqrt{N^2 - 1}} \left( X^i X_j - \delta^i_j X^2 \right)$$

(3.19)
in the Born-Infeld part, while the contributions to the Chern-Simons part vanish. We then find:

\[
S_{\text{Wb}} = -T_0 \int d\tau \text{STr} \left\{ \frac{1}{L \cos \theta} \sqrt{1 - \frac{L^4 \sin^4 \theta}{2R^2 \sqrt{N^2 - 1}} X^2 + \frac{L^8 \sin^6 \theta \cos^2 \theta}{16R^4(N^2 - 1)} X^2 + \frac{L^8 \sin^8 \theta}{16R^4(N^2 - 1)} X^2 X^2} \right\},
\]

(3.20)

where we have dropped those contributions to \( \det Q \) that will vanish when taking the symmetrised average involved in the symmetrised trace prescription. In our description, since the direction of propagation is isometric, we are effectively dealing with a static configuration, for which we can compute the potential as minus the Lagrangian. In the limit \( L^4 \sin^4 \theta \ll \sqrt{N^2 - 1} \) we can approximate the square root by

\[
1 - \frac{L^4 \sin^4 \theta}{4R^2 \sqrt{N^2 - 1}} X^2 + \frac{L^8 \sin^6 \theta \cos^2 \theta}{32R^4(N^2 - 1)} X^2,
\]

(3.21)

and since \( X^2 = R^2 \), we have for the potential

\[
V_{\text{Wb}}(\theta) = \frac{NT_0}{L \cos \theta} \left( 1 - \frac{L^4 \sin^4 \theta}{4\sqrt{N^2 - 1}} + \frac{L^8 \sin^6 \theta \cos^2 \theta}{32(N^2 - 1)} \right),
\]

(3.22)

which can be seen as the first order expansion of

\[
V_{\text{Wb}}(\theta) = \frac{NT_0}{L} \sqrt{1 + \tan^2 \theta \left( 1 - \frac{L^4 \sin^2 \theta}{4\sqrt{N^2 - 1}} \right)^2}.
\]

(3.23)

in the same limit above. Introducing \( \cos \theta \) inside the square root we have

\[
V_{\text{Wb}}(\theta) = \frac{NT_0}{L} \sqrt{1 + \tan^2 \theta \left( 1 - \frac{L^4 \sin^2 \theta}{4\sqrt{N^2 - 1}} \right)^2}.
\]

(3.24)

This potential has two minima, the point-like graviton at \( \sin \theta = 0 \), and a solution that should correspond to the giant graviton solution at

\[
\sin^2 \theta = \frac{4\sqrt{N^2 - 1}}{L^4}.
\]

(3.25)

Both solutions have an energy \( E = NT_0/L \), and are therefore associated to massless particles with angular momentum \( P_\phi = NT_0 \). There is an upper bound on the angular momentum that comes from the condition \( \sin^2 \theta \leq 1 \), which implies that

\[
P_\phi = NT_0 \leq \frac{T_0}{4} \sqrt{L^8 + 16}.
\]

(3.26)

Comparing the potential (3.21), the radius (3.24) and the upper bound (3.26) with their Abelian counterparts in the macroscopical derivation, we find that there is exact agreement in the large \( N \) limit when

\[
\frac{\tilde{N}}{P_\phi} = \frac{L^4}{4N} \iff P_\phi = 8\pi^2 NT_3.
\]

(3.27)

Making use of the fact that in the microscopical description the total angular momentum is quantised in terms of the tension of the gravitational waves, \( P_\phi = NT_0 \), we obtain the following relation between the tension \( T_0 \) of the waves and the tension \( T_3 \) of the D3-brane:

\[
T_0 = 8\pi^2 T_3,
\]

(3.28)

which is indeed satisfied, given that the isometric transverse direction is an angular variable with period \( 4\pi \).
4 The dual giant graviton in $\text{AdS}_5 \times S^5$

4.1 The macroscopical description

The dual giant graviton solution in $\text{AdS}_5 \times S^5$ was computed in [2,3], by looking at stable test brane solutions where the D3-brane with angular momentum in $S^5$ expands to the 3-sphere contained inside the $\text{AdS}_5$ component of the spacetime. For this giant graviton type of solution there is no upper bound for the angular momentum, because it expands in the non-compact part of the geometry. Let us briefly summarise the construction in these references in order to compare it, in the end, with our microscopical description.

The trial solution in this case is taken with $r = \text{constant}$, $\phi = \phi(\tau)$ and $\theta = 0$, which corresponds to a spherical D3-brane with radius $r$ orbiting the $S^5$ in the $\phi$ direction:

$$ds^2 = -(1 + \frac{r^2}{L^2})dt^2 + r^2 d\Omega_3^2 + L^2 d\phi^2 . \tag{4.1}$$

This D3-brane carries a non-vanishing dipole moment with respect to the RR 4-form potential, which prevents its collapse to zero size. Parametrising the unit 3-sphere in (4.1) as

$$d\Omega_3^2 = d\alpha_1^2 + \sin^2 \alpha_1 (d\alpha_2^2 + \sin^2 \alpha_2 d\alpha_3^2) , \tag{4.2}$$

we have

$$C_{\alpha_1\alpha_2\alpha_3}^{(4)} = -\frac{r^4}{L} \sqrt{g_{\alpha}} . \tag{4.3}$$

Substituting this trial solution into the worldvolume action of the D3-brane, one arrives at the following Hamiltonian [2,3]:

$$H = \frac{1}{L} \left[ \sqrt{\left(1 + \frac{r^2}{L^2}\right)\left(P_\phi^2 + \frac{\tilde{N}^2 r^6}{L^6}\right)} - \frac{\tilde{N} r^4}{L^4} \right] , \tag{4.4}$$

where $\tilde{N}$ is given by (3.7).

The stable solutions correspond to $r = 0$, the point-like graviton, and $r = L\sqrt{P_\phi/\tilde{N}}$, the dual giant graviton, both of which have energy $E = P_\phi/L$, representing massless particles with angular momentum $P_\phi$. Contrary to the giant graviton solution of the previous section, the dual giant graviton solution does not satisfy the stringy exclusion principle, because the absence of an upper bound for its radius implies that $P_\phi$ is neither bounded.

4.2 The microscopical description

We now want to provide a microscopical description of the dual giant graviton solution. This description will be in terms of coincident gravitons expanding into a D3-brane which is now expanded in a fuzzy surface contained inside the $\text{AdS}_5$ part of the geometry. As in the previous section, we represent the $S^3$ in (4.1) as an $S^1$ bundle over $S^2$, take Euler angles such that

$$ds^2_{S^3} = \frac{r^2}{4} \left( d\chi_1^2 + \sin^2 \chi_1 d\chi_2^2 + (d\chi_3 + \cos \chi_1 d\chi_2)^2 \right) \tag{4.5}$$

and identify the coordinate $\chi_3$ parametrising the $S^1$ with the compact isometric direction coming from the T-duality construction. The isometry associated to the propagation direction is again taken to be $\phi$. The background metric (4.1) is then given by

$$ds^2 = -(1 + \frac{r^2}{L^2})dt^2 + \frac{r^2}{4} \left( d\chi_1^2 + \sin^2 \chi_1 d\chi_2^2 + (d\chi_3 + \cos \chi_1 d\chi_2)^2 \right) + L^2 d\phi^2 . \tag{4.6}$$

Taking Cartesian coordinates in the $S^2$ that is parametrised by $\chi_1$ and $\chi_2$

$$x^1 = R \sin \chi_1 \cos \chi_2 , \quad x^2 = R \sin \chi_1 \sin \chi_2 , \quad x^3 = R \cos \chi_1 , \tag{4.7}$$
the metric and 4-form potential take the form
\begin{align*}
    ds^2 &= -(1 + \frac{r^2}{L^2}) dt^2 + L^2 d\phi^2 + \frac{r^2}{4R^2} \left( dx_1^2 + dx_2^2 + dx_3^2 \right) \\
    &+ \frac{r^2}{4} \left[ d\chi_3 + \frac{x_3}{R(x_1^2 + x_2^2)} \left( x_1 dx_2 - x_2 dx_1 \right) \right]^2
\end{align*}

\begin{equation}
    C^{(4)}_{\chi_{0}ij} = -\frac{r^4}{8R^3 L} \epsilon_{ijk} x^k \tag{4.8}
\end{equation}

for \(i, j, k = 1, 2, 3\). Thus, the natural non-commutative Ansatz to make in this case is
\begin{equation}
    X^i = \frac{R}{\sqrt{N^2 - 1}} J^i, \tag{4.9}
\end{equation}

with \(J^i\) forming an \(N \times N\) representation of \(SU(2)\). Our description of the dual giant graviton will again be in terms of gravitons expanding into a non-commutative D3-brane with topology \(S^2_{\text{fuzzy}} \times S^1\), the validity of which should be tested by checking the agreement with the macroscopical description for large number of gravitons.

The action (2.12) for Type IIB waves in the particular background defined by (the non-commutative version of) (4.8) contains:
\begin{align*}
    E_{00} &= -(1 + \frac{r^2}{L^2}), \quad E_{0i} = 0, \quad E_{ij} = \frac{r^2}{4R^2} \delta_{ij}, \tag{4.10} \\
    Q^i_j &= \delta^i_j - \frac{Lr^3}{4R\sqrt{N^2 - 1}} \epsilon^{ijk} X^k. \tag{4.11}
\end{align*}

The Born-Infeld part of the action then takes the form:
\begin{equation}
    S_{W}^{\text{BI}} = -\frac{T_0}{L} \int d\tau \text{Str} \left\{ \sqrt{\left( 1 + \frac{r^2}{L^2} \right) \left( 1 + \frac{L^2 r^6}{16R^2(N^2 - 1)} X^2 \right)} \right\}, \tag{4.12}
\end{equation}

while the Chern-Simons part is, in turn, given by:
\begin{equation}
    S_{W}^{\text{CS}} = -T_0 \int d\tau \text{Str} \{ iP[(iX)i_X i^{(4)}] \} = \int d\tau \frac{N T_0}{4\sqrt{N^2 - 1}} \frac{r^4}{L}. \tag{4.13}
\end{equation}

We then have a potential:
\begin{equation}
    V_{W}(r) = \frac{T_0}{L} \text{Str} \sqrt{\left( 1 + \frac{r^2}{L^2} \right) \left( 1 + \frac{L^2 r^6}{16R^2(N^2 - 1)} X^2 \right)} - \frac{N T_0}{4\sqrt{N^2 - 1}} \frac{r^4}{L}. \tag{4.14}
\end{equation}

In the limit \(Lr^3 \ll \sqrt{N^2 - 1}\) we can approximate the square root by its first order expansion, take the symmetrised trace (which to this order will only produce a factor of \(N\) in front of the action) and regard the remaining expression as the first order expansion of the potential
\begin{equation}
    V_{W}(r) = \frac{N T_0}{L} \sqrt{\left( 1 + \frac{r^2}{L^2} \right) \left( 1 + \frac{L^2 r^6}{16(N^2 - 1)} \right)} - \frac{N T_0}{4\sqrt{N^2 - 1}} \frac{r^4}{L}, \tag{4.15}
\end{equation}

exactly as we did in the previous section.

This potential has two minima, at \(r = 0\), corresponding to the point-like graviton, and at
\begin{equation}
    r^2 = \frac{4\sqrt{N^2 - 1}}{L^2}, \tag{4.16}
\end{equation}

which should correspond to the dual giant graviton solution. Both minima have energy \(E = N T_0/L\). Comparing these results with the Hamiltonian and the radius of the macroscopical derivation, we find that, in the large \(N\) limit, there is exact agreement when the condition (3.27) is fullfilled.
In this paper we have discussed a explicit Matrix action which is solved by a non-commutative 3-sphere. This Matrix model arises as an action for coincident Type IIB gravitational waves. This action is constructed using T-duality from the action for coincident Type IIA gravitational waves of [15, 19], and is such that the T-duality direction appears as a special isometric direction, on which neither the background fields nor the pull-backs depend. The presence of this compact isometric direction suggests the representation of the fuzzy 3-sphere solution as an $S^1$ bundle over a fuzzy 2-sphere base manifold. Accordingly, our solution does not show manifest $SO(4)$ covariance, this invariance being broken down to $SU(2) \times U(1)$. The $SO(4)$ invariance should, still, be present in a non-manifold way, in the same fashion than the $SO(4)$ covariance of the classical 3-sphere is not explicit when the $S^3$ is described as an $S^1$ Hopf fibered. This fuzzy 3-sphere should occur as a BPS solution of the Matrix model, preserving the same supersymmetries as the point-like graviton. We have not checked out this property explicitly, but it is expected from the agreement with the dual macroscopical description, in which the supersymmetry properties of these spherical configurations are demonstrated explicitly [20, 21, 22].

Odd non-commutative spheres have been previously discussed in the literature [20, 21, 22]. We would like to mention some relations between these different constructions.

In references [20, 21, 22] an $SO(4)$-covariant matrix realisation of the condition $\sum_{i=1}^{4} X_{i}^{2} = 1$ is found, in terms of matrices acting on some vector space. As for any fuzzy sphere with more than two dimensions, this matrix algebra contains more representations than is necessary to describe functions on the 3-sphere, so certain projections need to be done to eliminate the excess of degrees of freedom. The non-commutative 3-sphere that results has manifest $SO(4)$ covariance, but its non-commutativity cannot be removed completely in the large $N$ limit. On the contrary, the 3-sphere solution that we have constructed in this paper shows only manifest $SU(2) \times U(1)$ covariance, but approaches neatly the classical $S^3$ in the large $N$ limit, where all the non-commutativity disappears. The fuzzy 3-sphere of [20, 21, 22] can be represented in the large $N$ limit as a fibration of a 2-sphere over an interval$^9$. For finite $N$, it is expected that this representation is in terms of a discrete $S^1$ fibration over a 2-sphere, where both the $S^2$ and the $S^3$ are non-commutative. The reason for this is that the algebra of functions on $S^3$ is infinite dimensional so it cannot fit inside a finite Matrix Algebra as that of the fuzzy $S^3$. It would be interesting to develop a careful description of this $S^1$ fibration structure.

The fuzzy $S^3$ solution with manifest $SO(4)$ covariance is expected to play a role in a plausible description of Type IIB gravitons in terms of an action with no isometry directions (other than the direction of propagation). We should mention at this point that the reason why we have not succeeded in constructing such an action is a technical one, namely the impossibility of restoring the dependence on the T-duality direction in the non-Abelian case. Should the construction of such an action be possible, an $SO(4)$ covariant solution as that of [20, 21, 22] would most likely be the right Ansatz for the study of giant graviton configurations.

In fact, an explicit physical realisation of the fuzzy 3-sphere discussed above is in the context of non-BPS Type IIB D0-branes [20]. The 3-sphere arises as a solution of the action for coincident non-BPS Type IIB D0-branes [27, 28] in a flat background with constant 5-form field strength. In this construction the tachyonic field of the D0-branes plays an essential role in describing the fuzzy $S^3$ as a subspace of a fuzzy $S^4$. This solution cannot however be interpreted in the large $N$ limit as a spherical D3-brane, because its dipole moment vanishes for large $N$ [10]. Our 3-sphere solution, on the other hand, has an interpretation as a spherical D3-brane, since its dipole, or magnetic moment, coupling to the 5-form field strength resembles that of a spherical D3-brane, not only at finite $N$ but also in the large $N$ limit$^{11}$. A dual macroscopical description in terms of a single D3-

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$^9$We thank Sanjaye Ramgoolam for pointing this out to us and for the discussion below for finite $N$.

$^{10}$As discussed in [26], this could be due to the fact that the background they consider is not a consistent supergravity background.

$^{11}$This result seems to confirm the previous observation in [20], since our solution occurs in a consistent supergravity background. One should take into account however that we are dealing with gravitational waves and not with non-BPS D0-branes.
brane with whom to compare the microscopical description in [20] is however not possible, given that it is not known how to dissolve non-BPS D0-branes in the worldvolume of a BPS D3-brane\textsuperscript{12}. Our solution on the contrary arises when studying the polarisation of, BPS, gravitational waves. For this system not only the Lagrangian is defined in a better way, since there is no uncontrolled tachyon dynamics, but, moreover, a dual description in terms of a single D3-brane exists, which can be compared in the large $N$ limit with the microscopical description. The agreement between the two approaches provides in fact the strongest support to our construction.

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