Quantum mechanics for three versions of the Dirac equation in a curved spacetime

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1 Quantum Mechanics in a gravitational field

- Quantum effects in the classical gravitational field are observed, e.g. in neutron interferometry \[1, 2\]. All of the effects observed until now had been previously predicted by using the non-relativistic Schrödinger equation in the Newtonian gravity potential \[2, 3, 4\], and are still described by this same non-relativistic approximation.

- However, gravity is currently described by relativistic theories with curved spacetime.

- The usual way to write the wave equations of quantum mechanics (QM) in a curved spacetime is by covariantization (this is connected with the equivalence principle): at any given event $X$, the sought-for equation in a curved spacetime should coincide with the flat-spacetime version in coordinates where the connection vanishes at $X$ and $g_{\mu\nu}(X) = \eta_{\mu\nu}$. 
For the Dirac equation with standard (spinor) transformation, this leads to the standard equation (Dirac-Fock-Weyl or DFW), which does not obey the equivalence principle, at least not the genuine one [5].

In a previous work, two alternative equations were got by applying directly the classical-quantum correspondence [5]. Thus, we have indeed three different a priori inequivalent versions of the Dirac equation in a curved spacetime.

2 Three Dirac equations in a curved spacetime

The three gravitational Dirac equations have the same form:

\[ \gamma^\mu D_\mu \psi = -im\psi, \]  

where \( \gamma^\mu = \gamma^\mu(X) \) (\( \mu = 0, ..., 3 \)) is a field of \( 4 \times 4 \) complex matrices defined on the spacetime \( V \) [endowed with a Lorentzian metric \( g_{\mu\nu} \), with inverse matrix \( (g_{\mu\nu})^{-1} \)], such that

\[ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} 1_4, \quad \mu, \nu \in \{0, ..., 3\} \quad (1_4 \equiv \text{diag}(1, 1, 1, 1)); \]

and where \( \psi \) is a bispinor field for the standard, DFW equation, but is a four-vector field for the two alternative equations [5], based on the tensor representation of the Dirac field (TRD) [6, 7];

and where \( D_\mu \) is a covariant derivative, associated with a specific connection: one for each of the three equations.

2.1 Dirac equation with vector wave function

For Dirac’s original equation, the wave function is a (bi)spinor. This is due to the Dirac matrices \( \gamma^\mu \) being assumed Lorentz-invariant. However, a matrix usually does not remain invariant after a coordinate change.
In TRD, the wave function $\psi$ is a (4-)vector instead, and the set of the four Dirac matrices $\gamma^\mu$ builds a third-order tensor: after a coordinate change,

$$\psi'^\mu = \frac{\partial x'^\mu}{\partial x^\sigma} \psi^\sigma, \quad \gamma'^\mu_{\rho\nu} \equiv (\gamma'^\mu)^\rho_{\nu} = \frac{\partial x'^\mu}{\partial x^\sigma} \frac{\partial x^\rho}{\partial x'^\sigma} \frac{\partial x^\nu}{\partial x'^\tau} \gamma^\tau_{\sigma\tau}.$$  

(3)

This too leaves the usual Dirac equation covariant [6]. Moreover, the associated QM predictions in a flat spacetime are left unchanged, for:

1) The explicit (coordinate) expression of the Dirac equation is the same as with the standard (spinor) transformation behaviour, and

2) There is no influence of the possible set of Dirac matrices [7].

2.2 The three different connections

- For the two alternative equations (TRD), this is an affine connection:

$$ (D_\mu \psi)^\nu = \partial_\mu \psi^\nu + \Delta_\mu^\nu \psi^\rho. $$

(4)

- For one of the two TRD equations (TRD-1), this is the Levi-Civita connection. I.e., $\Delta_\mu^\nu = \{^\nu_\mu\}$, the Christoffel symbols associated with the spacetime metric $g_{\mu\nu}$. The corresponding gravitational Dirac equation obeys the equivalence principle [5].

- For TRD-2, the connection $\Delta$ is defined from the spatial Levi-Civita connection in an assumed preferred reference frame [5].

- For the standard equation (DFW), one uses the “spin connection”, which depends on the $\gamma^\mu$ matrices and is generally complex [8].

3 A common tool: the hermitizing matrices

For TRD, the set $(\gamma^\mu)$ builds a tensor, hence cannot be fixed. This is true even for the “flat” matrices $\gamma^{\flat\alpha}$ if one defines

$$\gamma^\mu = a^\mu_{\alpha} \gamma^{\flat\alpha}$$

(5)
with \( a^\mu_\alpha \) an orthonormal tetrad. We must be able to use any possible set \((\gamma^\mu)\) of Dirac matrices. Note that, also for DFW, in which the gamma matrices are always defined through a tetrad by Eq. (5), one should study the influence of the choice of the “flat” matrices \( \gamma^{2\alpha} \) and the tetrad \( a^\mu_\alpha \).

The solution is to use the hermitizing matrix: this is a \( 4 \times 4 \) matrix \( A \) such that
\[
A^\dagger = A, \quad (A\gamma^\mu)^\dagger = A\gamma^\mu \quad \mu = 0, \ldots, 3,
\]
where \( M^\dagger \equiv M^{*T} \) = Hermitian conjugate of matrix \( M \). For the usual sets \((\gamma^{2\alpha})\) (Dirac’s, “chiral”, Majorana), \( A = \gamma^{20} \).

The matrix \( A \), introduced by Bargmann, was studied mainly by Pauli [9]. The existence of \( A \) (and \( B \): for the \( \alpha^\mu \) matrices) has been proved by us for any set \((\gamma^\mu)\) in a general metric [7].

### 4 Definition of the probability current

In a flat spacetime, the current is unambiguously defined as
\[
J^\mu = \bar{\psi} A\gamma^\mu \psi.
\]

The definition (7) is generally-covariant, the current being indeed a four-vector, for TRD and for DFW as well. Thus it holds true in a curved spacetime \( V \). (Then \( \gamma^\mu, A \) depend on \( X \in V \).)

The current (7) is independent of the choice of the Dirac matrices: if one changes one set \((\gamma^\mu)\) for another one \((\tilde{\gamma}^\mu)\), the second set can be obtained from the first one by a point-dependent similarity transformation:
\[
\exists S = S(X) \in \text{GL}(4, \mathbb{C}) : \quad \tilde{\gamma}^\mu(X) = S\gamma^\mu(X)S^{-1}, \quad \mu = 0, \ldots, 3,
\]
With the change \( \tilde{\psi} = S\psi \), this leaves the current unchanged [7, 10].

### 5 Condition for current conservation

This is specified by the following result [10]:
Theorem 1. Consider the general Dirac equation in a curved spacetime (1), thus either DFW or any of the two TRD equations. In order that any $\psi$ solution of (1) satisfy the current conservation

$$D_\mu J^\mu = 0,$$

(9)

it is necessary and sufficient that

$$D_\mu (A \gamma^\mu) = 0.$$ 

(10)

Corollary 1. For DFW theory, the hermitizing matrix field $A(X)$ can be imposed to be the constant matrix $A^2$, i.e., a hermitizing matrix for the "flat" matrices $\gamma^{\flat \alpha}$ of Eq. (4). Then the current conservation applies to any solution of the DFW equation.

6 Admissible coefficient fields

Theorem 1 means that not all possible coefficient fields $(\gamma^\mu, A)$ of the Dirac equation are physically admissible, but merely the ones which, in addition to the anticommutation relation (2), satisfy the field equation (10) ensuring current conservation. Such systems we call "admissible."

Example: in a flat spacetime, relevant fields $\gamma^\mu$ are ones which are constant in Cartesian coordinates (and hence also the field $A$). Then the condition for current conservation (10) is satisfied.

If one selects the gamma field [satisfying (2)] "at random," the condition (10) and the current conservation do not generally apply to the solutions of the Dirac equation (1) even in a flat spacetime—except for DFW.

7 The Hamiltonian is frame-dependent

The Dirac equation (1) can be put into Schrödinger form:

$$i \frac{\partial \psi}{\partial t} = H \psi, \quad (t \equiv x^0),$$

(11)
with
\[ H \equiv m\alpha^0 - i\alpha^j D_j - i(D_0 - \partial_0), \quad (12) \]
where
\[ \alpha^0 \equiv \gamma^0/g^{00}, \quad \alpha^j \equiv \gamma^0 \gamma^j / g^{00}. \quad (13) \]
In order that the Hamiltonians \( H \) and \( H' \) before and after a coordinate change be equivalent operators, the coordinate change must be a spatial change:
\[ x'^0 = x^0, \quad x'^j = f^j((x^k)). \quad (14) \]
Then, both sides of the Schrödinger equation (11) behave as a scalar for DFW, and as a vector for TRD. Thus \( H \) depends on the reference frame, i.e., on the three-dimensional congruence of world lines (observers) which is considered. This frame dependence of the Hamiltonian and the resulting necessity of restricting oneself to spatial coordinate changes is a general result that holds true for other wave equations. In Ref. [10], we give a precise definition of a reference frame in this context, and we study it mathematically.

8 Hermiticity condition for the Hamiltonian

The Hilbert scalar product is fixed by the following result [10]:

**Theorem 5.** A necessary condition for the scalar product of time-independent wave functions to be time independent and for the Hamiltonian \( H \) to be a Hermitian operator, is that the scalar product should be
\[ (\psi | \varphi) \equiv \int_{\mathbb{R}^3} \psi^\dagger A\gamma^0 \varphi \sqrt{-g} \, d^3x. \quad (15) \]

The hermiticity condition for the Hamiltonian, w.r.t. this scalar product, is then given by [10]:

**Theorem 6.** Assume that the coefficient fields \((\gamma^\mu, A)\) satisfy the two admissibility conditions (2) and (10). In order that the Dirac Hamiltonian (12) be Hermitian for the scalar product (15), it is necessary and sufficient that
\[ \partial_0 (\sqrt{-g} A\gamma^0) = 0. \quad (16) \]
9 For DFW, hermiticity is not stable under a local similarity transformation

For DFW, all local similarity transformations are admissible, since condition \(10\) is always satisfied [with the choice \(A(X) = A^\sharp\); see Corollary 1]. In contrast, for TRD, condition \(10\) is quite demanding.

For DFW, in very general coordinates, the tetrad \((a^\mu_\alpha)\) may be chosen to satisfy \(a^0_j = 0\). Taking for “flat” matrices the standard Dirac matrices \(\gamma^\sharp_\alpha\), the hermiticity condition \(16\) then reduces to Leclerc’s [11]:

\[
\partial_0 (\sqrt{-g} g^{00}) = 0.
\]

But, after a local similarity \(S\), the condition \(16\) becomes

\[
\partial_0 (\sqrt{-g} g^{00} S^\dagger S) = 0,
\]

which obviously cannot be satisfied if \(17\) is, and if moreover \(S^\dagger S = F(t)\).

10 Conclusion

- Two new gravitational Dirac equations were previously derived from wave mechanics [5]. One obeys the equivalence principle, the other one has a preferred reference frame. Both see the wave function as a vector. This leaves QM associated with the Dirac equation of special relativity unchanged, i.e., for Minkowski spacetime in Cartesian coordinates [7].

- The three gravitational Dirac equations have been studied in a common framework, using the hermitizing matrix (field) \(A\) [10]:
  - The current conservation asks for the matrix equation \(D_\mu (A\gamma^\mu) = 0\). Thus, not all coefficient fields of the Dirac equation are admissible.
  - The hermiticity condition is \(\partial_0 (\sqrt{-g} A\gamma^0) = 0\). This is frame dependent, as is natural indeed. We gave a formal definition of a reference frame.

• For DFW, this condition is not stable under admissible similarity transformations: the standard version of the gravitational Dirac equation seems to have a uniqueness problem. We are currently pursuing the study to clarify this.

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