Diagonalization of the Hamiltonian in a Chromomagnetic Field

Sh. Mamedov*

Institute for Physical Problems, Baku State University,
Z.Khalilov st. 23, AZ-1148, Baku, Azerbaijan

Abstract

We demonstrate how to find both the color diagonal form of the squared Dirac equation in the axial color background and the transformation of the color space, which makes this equation diagonal.

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*Email: shahin@theory.ipm.ac.ir & Sh-Mamedov@bsu.az
**Introduction**

Classical and Quantum Mechanics of non-abelian charged particles are studied during a long period. In Refs. [1] and [2] the motion of a colored particle in the constant color fields in the framework of Quantum Mechanics was considered. The squared Dirac equation for this motion was solved in the finite space and the spectrum and wave functions for the considered cases were found. This equation, in addition to non-diagonal spin matrices, contains the non-diagonal color matrices as well. Obviously, this non-diagonality does not permit us to write an eigenvalue equation separately for the each color state. To this end we need in the diagonal form of the Hamiltonian in the color space. Having the explicit form of the Hamiltonian and knowing its hermicity, its diagonal form of it and the transformation of the color space, which diagonalizes Hamiltonian in this space can be found. We are going to realize this idea for the case of the axial chromomagnetic background, which was applied for the study of motion of the colored particle [1] and supersymmetry of the Dirac equation [9, 3, 7].

**1 Dirac equation in an axial field**

The Dirac equation for a colored particle in an external color field has the form:

\[
(\gamma^\mu P_\mu - M) \psi = 0, \tag{1.1}
\]

where \(P_\mu = p_\mu + gA_\mu = p_\mu + gA^a_\mu \lambda^a / 2;\) the \(\lambda^a\) are Gell-Mann matrices and the color index \(a\) runs \(a = \overline{1,8}\). \(g\) is the color interaction constant. In terms of the Majorana spinors \(\phi\) and \(\chi\) the equation (1.1) has the following form

\[
(\sigma^i P_i)^2 \psi = - \left( \frac{\partial^2}{\partial t^2} + M^2 \right) \psi, \tag{1.2}
\]

where the Pauli matrices \(\sigma^i\) describe the particle’s spin. In Eq. (1.2) and afterwards \(\psi\) means \(\phi\) or \(\chi\). The spinors \(\phi\) and \(\chi\) have two components corresponding to the two spin states of a particle \(\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}\). Each component of \(\psi\) transforms under the fundamental representation of the color group \(SU_c(3)\) and has three color components describing color states of the particle and corresponding to three eigenvalues of the color spin \(\lambda^3\)

\[
\psi_\pm = \begin{pmatrix} \psi_+ (\lambda^3 = +1) \\ \psi_+ (\lambda^3 = -1) \\ \psi_+ (\lambda^3 = 0) \end{pmatrix} = \begin{pmatrix} \psi^{(1)}_{\pm} \\ \psi^{(2)}_{\pm} \\ \psi^{(3)}_{\pm} \end{pmatrix}. \tag{1.3}
\]

We are going to diagonalize Eq. (1.2) for the case of chromomagnetic background given by the constant vector potentials \(A^a_\mu\) introduced in [4]. Remind, for giving the axial chromomagnetic field the components of \(A^a_\mu\) are choosen as below

\[
A^1_\mu = \sqrt{\tau} \delta_{1a}, \ A^2_\mu = \sqrt{\tau} \delta_{2a}, \ A^3_\mu = 0, \ A^0_\mu = 0, \tag{1.4}
\]
where $\tau$ is a constant and $\delta_{\mu\alpha}$ is the Kroneker symbol.

For this $A^a_\mu$ the field strength tensor $F^c_{\mu\nu} = g f^{abc} A^a_\mu A^b_\nu$ has only one non-zero component

$$F^3_{12} = g\tau = H^3_z,$$

and Eq. (1.4) gives a constant field directed along the third axes of the ordinary and color spaces. Here $f^{abc}$ are the structure constants of the $SU_c(3)$ group.

In this field Eq. (1.2) splits into the two equations for the $\psi_\pm$, which have following forms:

$$\left[ p^2 + \frac{1}{2} g^2 \tau I_2 + g\tau^{1/2} \left(p_1 \lambda^1 + p_2 \lambda^2 \mp \frac{1}{2} g\tau^{1/2} \lambda^3 \right) \right] \psi_\pm = \left( E^2 - M^2 \right) \psi_\pm. \quad (1.6)$$

Here we have set $i\partial\psi/\partial t = E\psi$ and introduced the color matrix $I_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Since field (1.5) is directed along the third axis, $\psi_\pm$ describe the spin states with the up and down projections of $\sigma^3$ and the corresponding Hamiltonians are defined as $H_\pm \psi_\pm = E^2 \psi_\pm$. $H_\pm$ have not diagonal color structure because of the non-diagonal $\lambda^1$ and $\lambda^2$ matrices and so, the eigenvalue equation

$$H_\pm \psi_\pm^{(i)} = E^2 \psi_\pm^{(i)}$$

cannot be written for pure color states $\psi_\pm^{(i)}$. The explicit matrix form of the general Hamiltonian in the combined color and spin spaces is:

$$H = \begin{pmatrix} p^2 & \mathcal{G}_{p_-} & 0 & 0 & 0 & 0 \\ \mathcal{G}_{p_+} & p^2 + \mathcal{G}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & p^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & p^2 + \mathcal{G}^2 & \mathcal{G}_{p_-} & 0 \\ 0 & 0 & 0 & \mathcal{G}_{p_+} & p^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & p^2 \end{pmatrix}. \quad (1.7)$$

In Eq. (1.7) we have introduced notations $p^2 = p^2 + M^2$, $\mathcal{G} = g\tau^{1/2}$, $p_\pm = p_1 \mp ip_2$. In order to deal with the matrices with less dimensionality, instead of one with dimension $6 \times 6$ we can consider two Hamiltonians having dimensions $3 \times 3$:

$$H_+ = \begin{pmatrix} p^2 & \mathcal{G}_{p_-} & 0 \\ \mathcal{G}_{p_+} & p^2 + \mathcal{G}^2 & 0 \\ 0 & 0 & p^2 \end{pmatrix}, \quad H_- = \begin{pmatrix} p^2 + \mathcal{G}^2 & \mathcal{G}_{p_-} & 0 \\ \mathcal{G}_{p_+} & p^2 & 0 \\ 0 & 0 & p^2 \end{pmatrix}.$$

These two Hamiltonians correspond to the spin up and spin down states $\psi_+$ and $\psi_-$ respectively. The eigenvalue equation, obviously, can be written only for the diagonal matrix form of the Hamiltonian. For this purpose we need in a diagonal form of the Hamiltonian $H$ in the combined spin-color spin space, i.e. in diagonal forms of $H_\pm$. Eigenfunctions of the diagonalized Hamiltonian will describe the states of the colored particle, which have definite value of energy. Because the Hamiltonian is the hermitian

*In Ref. [3] we call these states the energy states.
matrix, it has diagonal form in the basis of its eigenfunctions and this diagonal form is unique. We can find diagonal form of $H'$ for the Hamiltonian (1.7) and then its diagonal elements will correspond to the energy spectrum of the particle. By $H'$ we will able to write an eigenvalue equation with the eigenvalues $E_k^2$:

$$H'\Psi' = E_k^2 \Psi'. \quad (1.8)$$

Here $\Psi'$ is the eigenfunction of $H'$, which is different from $\psi^{(1),(2)}_\pm$. In order to get the diagonal $H'$ from the non-diagonal Hamiltonian $H$ (1.7), we should make some transformation $U$ in the combined spin-color spin space. The wave functions $\psi^{(i)}_\pm$ will be transformed on this transformation as well. More precisely, the Hamiltonian (1.7) will get the diagonal form under some $U$ transformation of the basis vectors of the combined spin-color spin space. The basis vectors in the combined space are the functions $\psi^{(i)}_\pm$, since these functions are the eigenfunctions of the spin and color spin operators $\sigma^i$ and $\lambda^a$. Under $U$ transformation of the combined space, the basis vectors transform according to the rule:

$$\Psi' = U\Psi, \quad \Psi = \begin{pmatrix} \psi^{(1)}_+ \\ \psi^{(2)}_+ \\ \psi^{(3)}_+ \\ \psi^{(1)}_- \\ \psi^{(2)}_- \\ \psi^{(3)}_- \end{pmatrix}. \quad (1.9)$$

Components of the new basis vector $\Psi'$ will be some superposition of $\psi^{(i)}_\pm$ and will not be the states with the definite value of projection of color spin. As we have stated above, if $H'$ is diagonal, then basis vectors of this space are the eigenvectors of this Hamiltonian. So, these components are the states with the definite value of the energy. Since $H'$ is unique, the transformation matrix $U$ and the basis vector $\Psi'$ are unique as well.

## 2 Diagonalization of the Hamiltonian

Let us consider how much information can be obtained from the hermicity of $H$ for the finding $U$ transformation and $H'$. Under transformation (1.9) the Hamiltonian (1.7), as any matrix in this space, is transformed as:

$$U^{-1}HU = H'. \quad (2.1)$$

The difficulty of determining of $H'$ is that, two of three matrices in (2.1) are unknown; we have no explicit form of either $U$ or $H'$. It turns out possible to find both of these matrices, relying on their properties. From the hermicity of $H'$ we immediately conclude, that the matrix $U$ should be unitary, in addition to its uniqueness:

$$H'^\dagger = U^\dagger H^\dagger U^{-1\dagger} = H' = U^{-1}HU,$$

\[\dagger\text{Expressions of } \psi^{(i)}_\pm \text{ and } E_k \text{ can be found in Refs. [3, 1, 10].}\]
\[ U^{-1\dagger} = U \Rightarrow UU^\dagger = 1. \]

Eq. (2.1) is the basic relation between two Hamiltonian matrices \( H \) and \( H' \). Having multiplied from the right hand side by \( U \) matrix it gets form:

\[ HU = UH', \tag{2.2} \]

which contains the linear relations between the elements \( u_{ij} \) of \( U \) matrix. Another important property of the unitary transformation is that, it does not change determinant and trace of a matrix. In our case this means that \( H \) and \( H' \), due to unitarity of \( U \), have the same trace and the same determinant:

\[ \det H' = \det H, \quad \text{Tr} H = \text{Tr} H'. \tag{2.3} \]

The matrices \( H, H' \) and \( U \) are matrices of dimensionality \( 6 \times 6 \). It will be easier to apply Eqs. (2.2) and (2.3) for \( H' \pm \) and for \( H' \) separately, than for \( H \). To this end we have to divide the \( U \) matrix into \( U' \) and \( U' \) parts, which transform \( H' \) and \( H' \), then after solving Eq. (2.2) construct the \( U \) matrix by means of \( U' \) and \( U' \) matrices. Let us remark, that since \( H' \) and \( H' \) differ from each other and each matrix has unique transformation diagonalizing it, the \( U' \) and \( U' \) matrices should be different as well.

According to (2.3) the determinants of \( H' \) and \( H' \) equal to determinants of diagonalized ones, i.e. \( H' \) and \( H' \). Besides, they have the same values:

\[ \det H' = \det H' = \left( \mathcal{P}^2 + G^2 / 2 \right) \left( \mathcal{P}^2 - G^2 / 2 \right) \mathcal{P}^2 = f_1 f_2 f_3, \tag{2.4} \]

where \( f_i \) denote \( f_{1,2} = \mathcal{P}^2 \pm G^2 / 2, \ f_3 = \mathcal{P}^2 \) and \( \mathcal{P}^2 = \sqrt{p_{\perp}^2 + G^2 / 4}, \ p_{\perp}^2 = p_1^2 + p_2^2 \). Let us construct firstly the \( H' \) by help of (2.4). Determinant of \( H' \) is the product of the diagonal elements:

\[ \det H' = \det \begin{pmatrix} h'_{11} & 0 & 0 \\ 0 & h'_{22} & 0 \\ 0 & 0 & h'_{33} \end{pmatrix} = h'_{11} h'_{22} h'_{33}. \tag{2.5} \]

We have two products: the product of three factors \( f_i \) in (2.4) and the product of three diagonal elements \( h'_{ii} \) in (2.5), which are equal:

\[ f_1 f_2 f_3 = h'_{11} h'_{22} h'_{33}. \]

Also sum of factors \( f_i \) in (2.4) is equal to the sum of diagonal elements of \( H' \) and to the sum of \( h'_{jj} \) in accordance with (2.3):

\[ \sum_i f_i = 3\mathcal{P}^2 + G^2 = \sum_j h'_{jj} = \sum_j h'_{jj}. \]

Equalities of the product and sum allows us to assume that factors \( f_i \) in (2.4) are the same with the diagonal elements \( h'_{jj} \) in (2.5), since we know invariance of \( \text{Tr} H' \) and \( \det H' \) under \( U' \) transformation. But, we encounter the question of place of \( f_i \) along the diagonal of \( H' \), i.e. which factor \( f_i \) corresponds to the which diagonal element \( h'_{ii} \), since
we have $3! = 6$ different variants for this corresponding. Each variant can be offered as a candidate for $H'_+$, while we have only one $H'_+$. Due to uniqueness of the diagonal form of the hermitian matrix, only one of the constructed variants will give us the correct $H'_+$. We may choose some variant for $f_i = h'_{ij}$ identification and then verify this choice by means of (2.2). According to the uniqueness of $H'_+$ and $U$ the only correct variant of identification will satisfy Eq. (2.2), i.e. will be solved for the $u_{ij}$ without mathematical nonsense. We intuitively make the identification below:

$$h'_{11} = f_2 = \mathcal{P}^2 - \mathcal{G}\mathcal{P}_p + \mathcal{G}^2/2, \quad h'_{22} = f_1 = \mathcal{P}^2 + \mathcal{G}\mathcal{P}_p + \mathcal{G}^2/2, \quad h'_{33} = f_3 = \mathcal{P}^2$$ (2.6)

and Eq. (2.2) for this choice has got the following explicit form:

$$
\begin{pmatrix}
\mathcal{P}^2 & \mathcal{G}\mathcal{P}_p & 0 \\
\mathcal{G}\mathcal{P}_p & \mathcal{P}^2 + \mathcal{G}^2 & 0 \\
0 & 0 & \mathcal{P}^2
\end{pmatrix}
\begin{pmatrix}
u_{11} & \nu_{12} & \nu_{13} \\
\nu_{21} & \nu_{22} & \nu_{23} \\
\nu_{31} & \nu_{32} & \nu_{33}
\end{pmatrix} =
\begin{pmatrix}
\mathcal{P}^2 - \mathcal{G}\mathcal{P}_p + \mathcal{G}^2/2 & 0 & 0 \\
0 & \mathcal{P}^2 + \mathcal{G}\mathcal{P}_p + \mathcal{G}^2/2 & 0 \\
0 & 0 & \mathcal{P}^2
\end{pmatrix}.
$$

From this equality we get the following system of equations:

$$
\begin{cases}
p_- u_{21} = -(\mathcal{P}_p - \mathcal{G}/2) u_{11} \\
p_- u_{22} = (\mathcal{P}_p + \mathcal{G}/2) u_{12} \\
p_+ u_{11} = -(\mathcal{P}_p + \mathcal{G}/2) u_{21} \\
p_+ u_{12} = (\mathcal{P}_p - \mathcal{G}/2) u_{22} \\
u_{13} = u_{23} = u_{31} = u_{32} = 0
\end{cases}
$$ (2.7)

The first and second equations in (2.7) are equivalent to the third and fourth ones, respectively. Thus in (2.7) we have four equations for the eight real unknowns $Re u_{ij}$ and $Im u_{ij}$, i.e. the system (2.7) is unsolvable. But it can be supplemented by the relations between $u_{ij}$ following from the unitarity of $U_+$ matrix:

$$
\begin{cases}
u_{11} u_{11}^* + \nu_{12} u_{12}^* = 1 \\
u_{11} u_{21}^* + \nu_{12} u_{22}^* = 0 \\
u_{21} u_{11}^* + \nu_{22} u_{12}^* = 0 \\
u_{21} u_{21}^* + \nu_{22} u_{22}^* = 1 \\
u_{33} u_{33}^* = 1
\end{cases}
$$ (2.8)

Besides, the color symmetry group of the Hamiltonian (1.7) is $SU(3)$ and so, $U_+$ transformation belongs to this symmetry group. One more relation following from the unimodularity condition $\det U_+ = 1$ can be added to the equations of (2.8)\footnote{We solve (1.19) for the choice $u_{33} = 1.$}

$$
u_{11} u_{22} - \nu_{12} u_{21} = 1.$$

Equations in (2.8) are non-linear relative to these unknowns and solving of the systems (2.7) and (2.8) together enables us to find only the module of $u_{ij}$:

$$
u_{11} u_{11}^* = \frac{1}{2} + \frac{\mathcal{G}}{4\mathcal{P}_p}, \quad \nu_{12} u_{12}^* = \frac{1}{2} - \frac{\mathcal{G}}{4\mathcal{P}_p}, \quad \nu_{21} u_{21}^* = \frac{1}{2} - \frac{\mathcal{G}}{4\mathcal{P}_p}, \quad \nu_{22} u_{22}^* = \frac{1}{2} + \frac{\mathcal{G}}{4\mathcal{P}_p}.$$ (2.9)
We can parametrize $u_{ij}$ introducing free angle parameters $\alpha_i$ and $\beta_i$:

$$u_{11} = \left(\frac{1}{2} + \frac{G}{4P_\perp}\right)^{1/2} e^{i\alpha_1}, \quad u_{22} = \left(\frac{1}{2} + \frac{G}{4P_\perp}\right)^{1/2} e^{i\alpha_2},$$

$$u_{12} = \left(\frac{1}{2} - \frac{G}{4P_\perp}\right)^{1/2} e^{i\beta_1}, \quad u_{21} = \left(\frac{1}{2} - \frac{G}{4P_\perp}\right)^{1/2} e^{i\beta_2}.$$  \hspace{1cm} (2.10)

It is useful to write the first equation of the system (2.7) in the terms of $Re \, u_{ij}$ and $Im \, u_{ij}$:

$$\left\{ \begin{array}{l}
p_1 \cos \beta_2 + p_2 \sin \beta_2 = -p_\perp \cos \alpha_1 \\
p_1 \sin \beta_2 - p_2 \cos \beta_2 = -p_\perp \sin \alpha_1
\end{array} \right.,$$

which, in the angle parametrization (2.10), gives simple relation between $\alpha_1$ and $\beta_1$:

$$\left\{ \begin{array}{l}
p_\perp \cos \beta_2 = -(p_1 \cos \alpha_1 - p_2 \sin \alpha_1) \\
p_\perp \sin \beta_2 = -(p_2 \cos \alpha_1 + p_1 \sin \alpha_1)
\end{array} \right..$$  \hspace{1cm} (2.11)

Similarly, the second equation of (2.7) relates the parameter $\alpha_2$ with $\beta_1$:

$$\left\{ \begin{array}{l}
p_\perp \cos \beta_1 = p_1 \cos \alpha_2 + p_2 \sin \alpha_2 \\
p_\perp \sin \beta_1 = p_1 \sin \alpha_2 - p_2 \cos \alpha_2
\end{array} \right..$$  \hspace{1cm} (2.12)

Having took these relations into account in (2.10) we eliminate $\beta_i$ parameters and thus cut down the number of free parameters up to two. As a result, $u_{12}$ and $u_{21}$ are expressed in the $\alpha_j$ angles as well:

$$u_{12} = \frac{1}{\sqrt{2}} \left(\frac{1}{P_\perp (P_\perp + G/2)}\right)^{1/2} p_- e^{i\alpha_2}, \quad u_{21} = -\frac{1}{\sqrt{2}} \left(\frac{1}{P_\perp (P_\perp + G/2)}\right)^{1/2} p_+ e^{i\alpha_1}.$$  \hspace{1cm} (2.13)

Unimodularity condition relate $\alpha_1$ and $\alpha_2$ parameteres: $\alpha_2 = \alpha_1 \equiv \alpha$, and then the $U_+$ matrix depends on only one free parameter $\alpha$:

$$U_+ = \left(\frac{1}{2P_\perp}\right)^{1/2} \left(\begin{array}{ccc}
(P_\perp + G/2)^{1/2} e^{i\alpha} & p_- (P_\perp + G/2)^{-1/2} e^{-i\alpha} & 0 \\
-p_+ (P_\perp + G/2)^{-1/2} e^{i\alpha} & (P_\perp + G/2)^{1/2} e^{-i\alpha} & 0 \\
0 & 0 & (2P_\perp)^{1/2}
\end{array}\right).$$  \hspace{1cm} (2.14)

As is seen from (2.14), the $U_+$ matrix does not transform the $\psi^{(3)}_\perp$. It can be easily checked, that the transformation (2.14) reduces $H_+$ to the $H'_+$ with the diagonal elements (2.6). Let us remark, that any other than (2.6) choice of diagonal elements leads to mathematical nonsense on solving (2.7) and (2.8) jointly. So, we can state that (2.6) is the unique diagonal form of the $H'_+$ and (2.14) is the unique matrix $U_+$ diagonalazing $H_+$.

Now we can diagonalize the $H_+$. Hamiltonian repeating the scheme used for the $H_\perp$. According to (2.3) and (2.4) we can suppose equality of the diagonal elements of $H'_\perp$ and the factors $f_i$. We make the following identification of diagonal elements $h''_{ij}$ with the factors $f_i$:

$$h''_{11} \equiv f_1 = P^2 + GP_\perp + G^2/2, \quad h''_{22} \equiv f_2 = P^2 - GP_\perp + G^2/2, \quad h''_{33} \equiv f_3 = P^2.$$  \hspace{1cm} (2.15)
For the choice (2.15) the relation $H_- U_- = U_- H'_-$ leads to the following system of equations:

$$
\begin{align*}
p_- u_{21} &= (P_\perp - G/2) u_{11} \\
p_- u_{22} &= - (P_\perp + G/2) u_{12} \\
p_+ u_{11} &= (P_\perp + G/2) u_{11} \\
p_+ u_{12} &= - (P_\perp - G/2) u_{22} \\
u_{13} &= u_{23} = u_{31} = u_{32} = 0
\end{align*}
$$

(2.16)

Having solved this system together with (2.8) we find the $U_-$ matrix diagonalizing $H_-:

$$
U_- = \left( \frac{1}{2P_\perp} \right)^{1/2} \begin{pmatrix}
(P_\perp + G/2)^{1/2} e^{i\beta} & -p_- (P_\perp + G/2)^{-1/2} e^{-i\beta} & 0 \\
p_+ (P_\perp + G/2)^{-1/2} e^{i\beta} & (P_\perp + G/2)^{1/2} e^{-i\beta} & 0 \\
0 & 0 & (2P_\perp)^{1/2}
\end{pmatrix}.
$$

(2.17)

According to (2.1) the matrix

$$
\mathcal{U} = \begin{pmatrix}
U_+ & 0 \\
0 & U_-
\end{pmatrix}
$$

will reduce the Hamiltonian (1.7) to its diagonal form:

$$
H' = \begin{pmatrix}
h_{11}' & 0 & 0 & 0 & 0 & 0 \\
0 & h_{22}' & 0 & 0 & 0 & 0 \\
0 & 0 & h_{33}' & 0 & 0 & 0 \\
0 & 0 & 0 & h_{11}'' & 0 & 0 \\
0 & 0 & 0 & 0 & h_{22}'' & 0 \\
0 & 0 & 0 & 0 & 0 & h_{33}''
\end{pmatrix}.
$$

(2.17)

In spite of $\mathcal{U}$ matrix is the transformation of the 6-dimensional spin-color spin space, actually it transforms the color space, because has quasidiagonal form and does not mix spin indices.

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\footnote{We do not denote the elements of $U_-$ matrix differently from ones of $U_+$, though they differs.}
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