LINEAR AND MULTIPLICATIVE 2-FORMS

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ABSTRACT. We study the relationship between multiplicative 2-forms on Lie groupoids and linear 2-forms on Lie algebroids, which leads to a new approach to the infinitesimal description of multiplicative 2-forms and to the integration of twisted Dirac manifolds.

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1. Introduction

The main purpose of this paper is to offer an alternative viewpoint to the study of multiplicative 2-forms on Lie groupoids and their infinitesimal counterparts carried out in [3]. This study turns out to be closely related to topics such as equivariant cohomology and generalized moment map theories, see, e.g., [2, 3, 19]. A particularly important case is that of the symplectic multiplicative 2-forms of symplectic groupoids [6], whose infinitesimal counterparts are Poisson structures. As shown in [3], infinitesimal versions of more general multiplicative 2-forms include twisted Dirac structures in the sense of [17].

Let $\mathcal{G}$ be a Lie groupoid over $M$, with source and target maps $s, t : \mathcal{G} \rightarrow M$, and multiplication $m : \mathcal{G}^{(2)} \rightarrow \mathcal{G}$. Let $A$ be the Lie algebroid of $\mathcal{G}$, with Lie bracket $[\cdot, \cdot]$ on $\Gamma(A)$ and anchor $\rho : A \rightarrow TM$. A 2-form $\omega \in \Omega^2(\mathcal{G})$ is called multiplicative if

$$m^* \omega = p_1^* \omega + p_2^* \omega,$$
where \(p_1, p_2 : G^{(2)} \to G\) are the natural projections. Given a closed 3-form \(\phi \in \Omega^3(M)\), we say that \(\omega\) is \textit{relatively} \(\phi\)-\textit{closed} if \(d\omega = s^*\phi - t^*\phi\). The main result in [3] asserts that, if \(G\) is \(s\)-simply-connected, then there exists a one-to-one correspondence between multiplicative 2-forms \(\omega \in \Omega^2(G)\) and vector-bundle maps \(\sigma : A \to T^*M\) satisfying

\[
(\sigma(u), \rho(v)) = -(\sigma(v), \rho(u))
\]

\[
\sigma([u, v]) = \mathcal{L}_{\rho(u)}\sigma(v) - i_{\rho(v)}^*d\sigma(u) + i_{\rho(v)}i_{\rho(u)}^*\phi,
\]

for all \(u, v \in \Gamma(A)\). We refer to such maps \(\sigma\) as IM 2-forms relative to \(\phi\) (IM stands for \textit{infinitesimal multiplicative}). If \(L \subset TM \oplus T^*M\) is a \(\phi\)-twisted Dirac structure [7, 17], then the projection \(L \to T^*M\) is naturally an IM 2-form, so the correspondence above includes the integration of twisted Dirac structures as a special case.

The IM 2-form associated with a multiplicative 2-form \(\omega \in \Omega^2(G)\) is simply

\[
(1.1) \quad \sigma(u) = i_u\omega|_{TM}, \quad u \in A,
\]

where \(A\) and \(TM\) are naturally viewed as subbundles of \(TG|_M\). The construction of \(\omega\) from a given \(\sigma : A \to T^*M\) in [3, Sec. 5] relies on the identification of \(G\) with \(A\)-homotopy classes of \(A\)-paths (in the sense of [9], cf. [16]), in such a way that \(\omega\) is obtained by a variation of the infinite dimensional reduction procedure of [5]. A different, more general, viewpoint to this problem has been recently studied in [1], where this correspondence is seen as part of a general Van Est isomorphism.

In this paper, we avoid the use of path spaces by noticing that the construction of a multiplicative \(\omega \in \Omega^2(G)\) out of an IM 2-form \(\sigma\) can be phrased as the integration of a suitable Lie algebroid morphism, similar in spirit to the approach of Mackenzie and Xu [13, 14] to the problem of integrating Lie bialgebroids to Poisson groupoids, which served as our main source of inspiration.

We notice that any multiplicative 2-form \(\omega \in \Omega^2(G)\) naturally induces a 2-form \(\Lambda \in \Omega^2(A)\) on the total space of \(A\), which is \textit{linear} in a suitable sense. We show that, when \(\omega\) is relatively \(\phi\)-closed, the 2-form \(\Lambda\) is totally determined by the map \(\sigma\) (1.1) and \(\phi\) via the formula

\[
(1.2) \quad \Lambda = -(\sigma^*\omega_{\text{can}} + \rho^*\tau(\phi)),
\]

where \(\omega_{\text{can}}\) is the canonical symplectic form on \(T^*M\), and \(\tau(\phi) \in \Omega^2(TM)\) is the 2-form defined, at each point \(X \in TM\), by \(\tau(\phi)|_X = p^*_M(i_X \phi)\), where \(p_M : TM \to M\) denotes the natural projection.

As a key step to reconstructing multiplicative 2-forms from infinitesimal data, consider an arbitrary Lie algebroid \(A \to M\), together with a vector-bundle map \(\sigma : A \to T^*M\) and a closed form \(\phi \in \Omega^2(M)\). Let us use \(\sigma\) and \(\phi\) to define \(\Lambda \in \Omega^2(A)\) by (1.2). Our main observation is that the bundle map

\[
\Lambda^2 : TA \to T^*A, \quad U \mapsto i_U\Lambda,
\]

is a morphism between tangent and cotangent Lie algebroids (see [13]) if and only if \(\sigma\) is an IM 2-form relative to \(\phi\). This result can be immediately applied to the integration of IM 2-forms: the morphism of groupoids \(TG \to T^*G\) obtained by integrating the morphism \(\Lambda^2 : TA \to T^*A\) determines the desired multiplicative 2-form. Our approach to multiplicative 2-forms can be naturally extended in different directions, e.g., to forms of higher degree or forms with no prescription on
their exterior derivatives, as recently done in [1] from a different perspective. These extensions and a comparison with [1] will be discussed in a separate paper.

The paper is organized as follows. In Section 2 we briefly recall the definitions and main properties of tangent and cotangent Lie algebroids and groupoids. In Section 3, we discuss the construction of linear 2-forms on Lie algebroids associated with multiplicative 2-forms on Lie groupoids. In Section 4, we relate IM 2-forms with linear 2-forms defining algebroid morphisms $T\mathcal{A} \rightarrow T^*\mathcal{A}$, and apply our results to the integration of IM 2-forms.

1.1. Notations and conventions. For a Lie groupoid $\mathcal{G}$ over $M$, its source and target maps are denoted by $s, t$. Composable pairs $(g, h) \in \mathcal{G}^{(2)} = \mathcal{G} \times_M \mathcal{G}$ are such that $s(g) = t(h)$, and the multiplication map is denoted by $m : \mathcal{G}^{(2)} \rightarrow \mathcal{G}$, $m(g, h) = gh$. Its Lie algebroid is $A\mathcal{G} = \ker(Ts)|_M$, with anchor $Tt|_A : A \rightarrow TM$, and bracket induced by right-invariant vector fields. For a general Lie algebroid $A \rightarrow M$, we denote its anchor by $\rho_A$ and bracket by $[,]_A$ (or simply $\rho$ and $[,]$ if there is no risk of confusion). Given vector bundles $A \rightarrow M$ and $B \rightarrow M$, vector-bundle maps $A \rightarrow B$ in this paper are assumed to cover the identity map, unless otherwise stated. Einstein’s summation convention is consistently used throughout the paper.

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2. Tangent and cotangent structures

In this section, we briefly recall tangent and cotangent algebroids and groupoids, following [12, 13], where readers can find more details.

2.1. Tangent and cotangent Lie groupoids. Let $\mathcal{G}$ be a Lie groupoid over $M$, with Lie algebroid $A\mathcal{G}$ (if there is no risk of confusion, we may denote $A\mathcal{G}$ simply by $A$). The tangent bundle $T\mathcal{G}$ has a natural Lie groupoid structure over $TM$, with source (resp., target) map given by $Ts : T\mathcal{G} \rightarrow TM$ (resp., $Tt : T\mathcal{G} \rightarrow TM$). The multiplication on $T\mathcal{G}$ is defined by $Tm : T\mathcal{G}^{(2)} = (T\mathcal{G})^{(2)} \rightarrow T\mathcal{G}$. We refer to this groupoid as the tangent groupoid of $\mathcal{G}$.

The cotangent bundle $T^*\mathcal{G}$ has a Lie groupoid structure over $A^*$, known as the cotangent groupoid of $\mathcal{G}$. The source and target maps are given by

$$\tilde{s}(\alpha_g)u = \alpha_g(Tl_g(u - Tt(u))), \quad \tilde{t}(\beta_g)v = \beta_g(Tr_g(v)),$$

where $\alpha_g, \beta_g \in T^*_g \mathcal{G}$, $u \in A_{s(g)}$, and $v \in A_{t(g)}$. Here $l_g : t^{-1}(s(g)) \rightarrow t^{-1}(t(g))$ and $r_g : s^{-1}(t(g)) \rightarrow s^{-1}(s(g))$ denote the left and right multiplications by $g \in \mathcal{G}$, respectively. The multiplication on $T^*\mathcal{G}$, denoted by $\circ$, is defined by

$$\alpha_g \circ \beta_h(Tm(X_g, Y_h)) = \alpha_g(X_g) + \beta_h(Y_h),$$

for $(X_g, Y_h) \in T_{(g, h)}\mathcal{G}^{(2)}$. 

In this section, we briefly recall tangent and cotangent algebroids and groupoids, following [12, 13], where readers can find more details.
2.2. **Tangent double vector bundles and duals.** Let $q_A : A \rightarrow M$ be a vector bundle. There is a natural *double vector bundle* [12, 15] associated with it, referred to as the **tangent double vector bundle** of $A$, and defined by the following diagram:

\[(2.4)\]

\[
\begin{array}{ccc}
TA & \overset{Tq_A}{\longrightarrow} & TM \\
\downarrow{p_A} & & \downarrow{p_M} \\
A & \overset{q_A}{\longrightarrow} & M.
\end{array}
\]

Here the vertical arrows are the usual tangent bundle structures. Similarly, one can consider the tangent double vector bundle of $q_{A^*} : A^* \rightarrow M$, which defines a double vector bundle $TA^*$:

\[(2.5)\]

\[
\begin{array}{ccc}
TA^* & \overset{Tq_{A^*}}{\longrightarrow} & TM \\
\downarrow{p_{A^*}} & & \downarrow{p_M} \\
A^* & \overset{q_{A^*}}{\longrightarrow} & M.
\end{array}
\]

It will be useful to consider coordinates on these bundles. If $(x^j)$, $j = 1, \ldots, \dim(M)$, are local coordinates on $M$ and $\{e_d\}$, $d = 1, \ldots, \text{rank}(A)$, is a basis of local sections of $A$, we write the corresponding coordinates on $A$ as $(x^j, u^d)$ and tangent coordinates on $TA$ as $(x^j, u^d, \dot{x}^j, \dot{u}^d)$. For each $x = (x^j)$, note that $(u^d)$ specifies a point in $A_x$, $(\dot{x}^j)$ gives a point in $T_xM$, whereas $(\dot{u}^d)$ determines a point on a second copy of $A_x$, tangent to the fibres of $A \rightarrow M$, known as the **core** of $TA$ (defined by $\ker(p_A) \cap \ker(Tq_A)$, see [12, 15]). Similarly, we have local coordinates $(x^j, \xi_d)$ on $A^*$ (relative to the basis $\{e_d\}$, dual to $\{e_d\}$), and tangent coordinates $(x^j, \xi_d, \dot{x}^j, \dot{\xi}_d)$, where now the coordinates $(\dot{\xi}_d)$ represent the core directions.

Let $T^*A \rightarrow TM$ be the vector bundle defined by dualizing the fibres of $Tq_A : TA \rightarrow TM$, $(x^j, u^d, \dot{x}^j, \dot{u}^d) \mapsto (x^j, \dot{x}^j)$. This fits into the double vector bundle

\[(2.6)\]

\[
\begin{array}{ccc}
T^*A & \overset{T_{q^*}}{\longrightarrow} & TM \\
\downarrow{p_M} & & \downarrow{p_M} \\
A^* & \overset{q_{A^*}}{\longrightarrow} & M.
\end{array}
\]

Here the vertical map $T^*A \rightarrow A^*$ is defined by $(x^j, \zeta_d, \dot{x}^j, \eta_d) \mapsto (x^j, \eta_d)$, where $T^*A$ is locally written as $(x^j, \zeta_d, \dot{x}^j, \eta_d)$, with $(\zeta_d)$ dual to $(u^d)$, and $(\eta_d)$ dual to $(\dot{u}^d)$.

The double vector bundles (2.5) and (2.6) turn out to be isomorphic: as shown in [13, Proposition 5.3], by applying the tangent functor to the natural pairing $A^* \times_M A \rightarrow \mathbb{R}$ (followed by the fibre projection $T\mathbb{R} \rightarrow \mathbb{R}$) one obtains a nondegenerate pairing $TA^* \times TM A \rightarrow \mathbb{R}$, which induces an isomorphism of double vector bundles

\[(2.7)\]

\[I : TA^* \rightarrow T^*A.\]

Locally, this identification amounts to the flip

\[(x^j, \zeta_d, \dot{x}^j, \dot{\xi}_d) \mapsto (x^j, \dot{\xi}_d, \dot{x}^j, \zeta_d).\]

The cotangent bundle $T^*A$ can be locally written in coordinates $(x^j, u^d, p_j, \zeta_d)$, where $(p_j)$ determines a point in $T^*_xM$ and $\zeta_d$ in $A^*_x$ (dual to the direction tangent
to the fibres $A \rightarrow M$). If $c_A : T^*A \rightarrow A$, $c_A(x^j, u^d, p_j, \zeta_d) = (x^j, u^d)$ denotes the natural projection, we see that $T^*A$ fits into the following double vector bundle:

$$
T^*A \xrightarrow{r} A^* \xleftarrow{c_A} A \xrightarrow{q_A} M,
$$

where the bundle projection $r : T^*A \rightarrow A^*$ is given locally by $r(x^j, u^d, p_j, \zeta_d) = (x^j, \zeta_d)$. The same construction can be applied to the vector bundle $A^* \rightarrow M$, yielding a double vector bundle structure for $T^*A^*$. These double vector bundles can be identified by a Legendre type transform $[13, \text{Thm. 5.5}]$ (cf. [18]):

$$
R : T^*A^* \rightarrow T^*A,
$$
given locally by $(x^j, \xi_d, p_j, u^d) \mapsto (x^j, u^d, -p_j, \xi_d)$.

There are two other identifications involving tangent and cotangent double vector bundles that we need to recall. For an arbitrary manifold $M$, we first have the canonical involution

$$
J_M : T^*M \rightarrow T^*M
$$

defined in local coordinates by $J_M(x^j, \dot{x}^j, \delta x^j, \delta \dot{x}^j) = (x^j, \delta \dot{x}^j, \dot{x}^j, \delta \dot{x}^j)$.

There is also an isomorphism of double vector bundles (also restricting to the identity on side bundles and cores),

$$
\Theta_M : TT^*M \rightarrow T^*TM
$$
defined in local coordinates by $\Theta_M(x^j, p_j, \dot{x}^j, \dot{p}_j) = (x^j, \dot{x}^j, \dot{p}_j, p_j)$.

Here $(x^j, p_j)$ are cotangent coordinates on $T^*M$. Equivalently, $\Theta_M = J_M^* \circ I_M$, where $J_M^* : T^*TM \rightarrow T^*TM$ is the dual of (2.10), and

$$
I_M : TT^*M \rightarrow T^*TM
$$
is as in (2.7) (with $A = TM$).

2.3. **Tangent and cotangent Lie algebroids.** Suppose that the vector bundle $A \rightarrow M$ carries a Lie algebroid structure, which can be equivalently described by a fibrewise linear Poisson structure on $A^*$ (see, e.g., [4, Sec. 16.5]). Since any Poisson structure on a manifold defines a Lie algebroid structure on its cotangent bundle (see, e.g., [4, Sec. 17.3]), we obtain a Lie algebroid structure on $T^*A^*$; it follows that $TA^*$ inherits a Poisson structure, which turns out to be linear with respect to both vector bundle structures on $TA^*$ (2.5). Hence the vector bundle $T^*A^* \rightarrow TM$, dual to $TA^* \rightarrow TM$, is a Lie algebroid. Using the identification $T^*A^* \cong TA$ as in (2.7),
we obtain a Lie algebroid structure on $TA \to TM$, referred to as the **tangent Lie algebroid** of $A$.

To describe this algebroid structure more explicitly, we recall that any section $u \in \Gamma(A)$ gives rise to two types of sections on $TA$: the first one is just $Tu : TM \to TA$, and the second one, denoted by $\tilde{u}$, identifies $u$ at each point with a core element in $TA$; locally, using coordinates $(x^j, u^d)$ for $A$ and $(x^j, \dot{x}^j, \dot{u}^d)$ for $TA$, $\tilde{u} : TM \to TA$ is defined by

\[(2.13) \quad \tilde{u}(x^j, \dot{x}^j) = (x^j, 0, \dot{x}^j, u^d(x)).\]

These two types of sections generate the space of sections of $TA \to TM$. The Lie algebroid structure on $TA$ is completely described in terms of these sections by the relations [13]:

\[(2.14) \quad \langle \tilde{u}, \tilde{v} \rangle_{TA} = 0, \quad [Tu, \tilde{v}]_{TA} = [\tilde{u}, [Tu, T]\tilde{v}]_{\hat{A}}, \quad [Tu, \tilde{v}]_{TA} = T[u, v]_{\hat{A}},\]

for $u, v \in \Gamma(A)$; the anchor map is $\rho_{TA} = J_M \circ T\rho_A$, where $J_M : T(TM) \to T(TM)$ is as in (2.10).

On the other hand, since $T^*A^* \to A^*$ is a Lie algebroid (defined by the linear Poisson structure on $A^*$), one can induce a Lie algebroid structure on $r : T^*A \to A^*$ using the identification (2.9). This is known as the **cotangent Lie algebroid** of $A$.

Explicit formulas for its bracket and anchor will be recalled in Section 4.3.

Suppose that $A = AG$ is the Lie algebroid of a Lie groupoid $G$, and consider the natural inclusion $\iota_{AG} : AG \to T\hat{G}$, which is a bundle map over the unit map $M \to G$. Then the canonical involution $J_G : T(T\hat{G}) \to T(T\hat{G})$ (2.10) restricts to a Lie algebroid isomorphism

\[(2.15) \quad j_G : T(AG) \to A(T\hat{G}).\]

In other words, we have a commutative diagram

\[(2.16) \quad T(\hat{G}) \xrightarrow{j_G} A(T\hat{G}) \xrightarrow{\iota_{A(T\hat{G})}} T(\hat{G}). \]

The canonical pairing $T^*G \times G T\hat{G} \to \mathbb{R}$ is a morphism of groupoids, and applying the Lie functor one obtains a nondegenerate pairing $A(T^*G) \times_{AG} A(T\hat{G}) \to \mathbb{R}$, explicitly given by

\[\langle U, V \rangle = \langle I_G(\iota_{AG}(U)), \iota_{A(T\hat{G})}(V) \rangle,\]

where $U \in A(T^*G)$, $V \in A(T\hat{G})$, and $I_G$ is as in (2.12). This induces an isomorphism $A(T^*G) \to A^*(T\hat{G})$, where $A^*(T\hat{G})$ is obtained by dualizing the fibres of $A(T\hat{G}) \to A(G)$, and the composition of this map with $j_G^* : A^*(T\hat{G}) \to T^*(AG)$ defines a Lie algebroid isomorphism

\[(2.17) \quad \theta_G : A(T^*G) \to T^*(AG).\]

Alternatively, one can check that $\theta_G = (T\iota_G)^t \circ \Theta_G \circ \iota_{A(T^*G)}$, where $(T\iota_AG)^t : \iota_{AG}^*T^*(T\hat{G}) \to T^*(AG)$ is dual to the tangent map $T\iota_AG : T(AG) \to \iota_{AG}^*T(T\hat{G})$. 
3. Tangent lifts and the Lie functor

We now discuss how multiplicative forms on Lie groupoids relate to differential forms on Lie algebroids. As a first step, we need to recall a natural operation that lifts differential forms on a manifold to its tangent bundle,

\[(3.18) \quad \Omega^k(M) \rightarrow \Omega^k(TM), \quad \alpha \mapsto \alpha_T,\]

known as the tangent (or complete) lift, see [10, 20].

3.1. Tangent lifts of differential forms. The properties of tangent lifts recalled in this subsection can be found (often in more generality) in [10]; we included the proofs of some key facts for the sake of completeness.

Given the tangent bundle \(p_M : TM \rightarrow M, (x^j, \dot{x}^j) \mapsto (x^j)\), consider the two vector bundle structures associated with \(T(TM)\):

\[(3.19) \quad T(TM)_{\text{TPM}} \rightarrow TM, \quad T(TM)_{\text{TPM}} \rightarrow TM, \]

where \(p_{TM}(x^j, \dot{x}^j, \delta x^j, \delta \dot{x}^j) = (x^j, \dot{x}^j)\) and \(T_{p_M}(x^j, \dot{x}^j, \delta x^j, \delta \dot{x}^j) = (x^j, \delta \dot{x}^j)\). We use the notation

\[T(TM) \times_{TPM} T(TM), \quad T(TM) \times_{TPM} T(TM), \]

to specify the vector bundle structure used for fibre products over \(TM\); more general \(k\)-fold fibre products over \(TM\) are denoted by

\[\prod_{1}^{k} T(TM)_{\text{TPM}}, \quad \prod_{1}^{k} T(TM)_{\text{TPM}}.\]

Using the involution \((2.10)\), given by \(J_M(x^j, \dot{x}^j, \delta x^j, \delta \dot{x}^j) = (x^j, \delta \dot{x}^j, \dot{x}^j, \delta x^j)\) in local coordinates, we obtain a natural isomorphism

\[(3.20) \quad J_M^{(k)} : \prod_{1}^{k} T(TM)_{\text{TPM}} \rightarrow \prod_{1}^{k} T(TM).\]

Given a \(k\)-form \(\alpha \in \Omega^k(M)\), \(k \geq 1\), consider the bundle map

\[(3.21) \quad \alpha^\sharp : \prod_{1}^{k-1} T(TM)_{\text{TPM}} \rightarrow T^*M, \quad \alpha^\sharp(X_1, \ldots, X_{k-1}) = i_{X_{k-1}} \ldots i_{X_1} \alpha.\]

(For \(k = 1\), \(\alpha^\sharp : M \rightarrow T^*M\) is just \(\alpha\) viewed as a section of \(T^*M\).) Using the natural identification \(T(\prod_{1}^{k} TM_{\text{TPM}}) = \prod_{1}^{k} T_{TPM}(TM)\), we consider the tangent map

\[T^\alpha^\sharp : \prod_{1}^{k-1} T(TM)_{\text{TPM}} \rightarrow T(T^*M).\]

The tangent (or complete) lift of a \(k\)-form on \(M\) is defined as follows (cf. [20]):

- If \(f \in \Omega^0(M) = C^\infty(M)\), then \(f_T \in C^\infty(TM)\) is the fibrewise linear function on \(TM\) defined by \(df\),

\[f_T(X) = (df)_{p_M(X)}(X), \quad X \in TM.\]
If $\alpha \in \Omega^k(M)$, $k \geq 1$, we define
\[ (\alpha_T)^{\sharp} : \prod_{p \in TM} T(TM) \to T^*(TM), \quad (\alpha_T)^{\sharp} := \Theta_M \circ T\alpha^{\sharp} \circ J_{TM}^{(k-1)}, \]
and then $\alpha_T \in \Omega^k(TM)$ is given by
\[ \alpha_T(U_1, \ldots, U_k) := \left< \alpha^{\sharp}_T(U_1, \ldots, U_{k-1}), U_k \right>. \]

One can directly verify that $\alpha_T$ is multilinear. The fact that it is indeed a $k$-form on $TM$ follows from the next lemma (cf. [10, 20]).

**Lemma 3.1.** The following holds:

(i) For $f \in C^\infty(M)$, $df_T = (df)_T$.
(ii) For $f \in C^\infty(M)$, $\alpha \in \Omega^k(M)$,
\[ (f\alpha)_T = f_T \alpha^\vee + f^\vee \alpha_T, \]
where $\beta^\vee = p^*_M \beta$ for any $\beta \in \Omega^l(M)$.
(iii) For $k \geq 2$, the tangent lift $(dx^1 \wedge \ldots \wedge dx^k)_T$ equals
\[ \sum_{m=1}^k (dx^1)^\vee \wedge \ldots \wedge (dx^m)^\vee \wedge (dx^{m+1})_T \wedge (dx^{m+1})^\vee \wedge \ldots \wedge (dx^k)^\vee. \]

(Whenever there is no risk of confusion, we write $(dx^j)^\vee$ simply as $dx^j$.)

**Proof.** To verify (i), let us consider $X \in TM$ and $U \in T_X(TM)$. In local coordinates, we write $X = (x^j, \dot{x}^j)$ and $U = (x^j, \dot{x}^j, \delta x^j, \delta \dot{x}^j)$. Then $f_T(X) = \frac{\partial f}{\partial x^j} \dot{x}^j$, and
\[ (\alpha_T)^{\sharp} : \prod_{p \in TM} T(TM) \to T^*(TM), \quad (\alpha_T)^{\sharp} := \Theta_M \circ T\alpha^{\sharp} \circ J_{TM}^{(k-1)}, \]
and, as a consequence,
\[ \Theta_M(T(df)^{\sharp}(x^j, \dot{x}^j)) = (x^j, \dot{x}^j, \frac{\partial f}{\partial x^j} \dot{x}^j, \frac{\partial f}{\partial \dot{x}^j}). \]

It immediately follows that $(df)_T(X) \alpha^\vee(U_1, \ldots, U_{k-1})$ agrees with (3.22).

Let us show that (ii) holds for $k > 1$ (the cases $k = 0, 1$ are simpler). One can directly check that $(f\alpha)^{\sharp} = f\alpha^{\sharp}$ and
\[ T(f\alpha)^{\sharp}(U_1, \ldots, U_{k-1}) = \alpha^\sharp(U_1, \ldots, X_{k-1})(df)_x(Y) + f(x)T\alpha^\sharp(U_1, \ldots, U_{k-1}), \]
where $X_i = p_{TM}(U_i) \in T_xM$, and $Y = (p_M)_*(U_1) = \ldots = (p_M)_*(U_{k-1})$. In the last formula, addition and multiplication by scalars are with respect to the vector bundle structure $T(T^*M) \to TM$ (in the fibre over $Y \in T_xM$), and $\alpha^\sharp(U_1, \ldots, X_{k-1}) \in T^*_xM$ is viewed inside $T(T^*M)$ as the core (i.e., tangent to $T^*M$-fibres). Since
\[ \Theta_M : T(T^*M) \longrightarrow T^*(TM) \] is a double vector bundle isomorphism restricting to the identity on side bundles and cores, we have

\[
\Theta_M^T(f_{\alpha})^\sharp(U_1, \ldots, U_{k-1}) = \alpha^\sharp(X_1, \ldots, X_{k-1})(df)_x(Y) + f(x)\Theta_MT\alpha^\sharp(U_1, \ldots, U_{k-1}),
\]

where now the addition and scalar multiplication operations are relative to the vector bundle \( T^*(TM) \rightarrow TM \), \( \alpha^\sharp(X_1, \ldots, X_{k-1}) \) belongs to the core fibre in \( T^*(TM) \) (i.e., cotangent to \( M \)). Writing \( (U_1, \ldots, U_{k-1}) = J_{\alpha}^{(k-1)}(V_1, \ldots, V_{k-1}) \), then \( X_i = (p_M)_*(V_i) \) and \( Y = p_T(M(V_i)) \), so (3.23) yields

\[
(f_{\alpha})^\sharp_x = (f_T)^\vee + f^\vee\theta_T^\sharp.
\]

Let us now prove (iii). Note that

\[
(d\sigma x^1 \land \ldots \land d\sigma x^k)^\sharp_x(B_1, \ldots, B_{k-1}) = \sum_{\sigma \in S_k} (-1)^\sigma x_{\sigma(1)}^1 \cdots x_{\sigma(k-1)}^k(d\sigma x)^\sharp_x(B_1, \ldots, B_{k-1}),
\]

where \( B_i = (x^i, \delta x_i, \delta j x_i) \) and \( V_k = (x^i, (\delta x_i)^\sharp_x, \delta j x_i^\sharp_x) \). Since \( (dx^i)_T = d\delta x^i \) (by (i)), one checks that

\[
\sum_{n=1}^k (d\delta x^i)^\vee_x \land \ldots \land (d\delta x^{i_{n-1}})^\vee_x \land (d\delta x^{i_n})_T \land (d\delta x^{i_{n+1}})^\vee_x \land \ldots \land (d\delta x^k)^\vee_x(V_1, \ldots, V_k),
\]

where \( V_i = (x^i, (\delta x_i)^\sharp_x, \delta j x_i^\sharp_x) \) (so that \( J_M(V_i) = U_i \)), equals

\[
\sum_{\sigma \in S_k} (-1)^\sigma x_{\sigma(1)}^1 \cdots x_{\sigma(n-1)}^n (\delta \sigma x)^\sharp_x \sigma_{\sigma(n)} x_{\sigma(n+1)}^1 \cdots x_{\sigma(k)}^k,
\]

which agrees with (3.23) after reshuffling indices.

Let us now consider the operation

\[
\tau : \Omega^k(M) \longrightarrow \Omega^{k-1}(TM), \quad \tau(\alpha) = p_M^*(i_X\alpha),
\]

where \( X \in TM \) and \( k \geq 1 \). In other words, given \( U_1, \ldots, U_{k-1} \in T_X(TM), \)

\[
\tau(\alpha)(U_1, \ldots, U_{k-1}) = \alpha(X, (p_M)_*(U_1), \ldots, (p_M)_*(U_{k-1})).
\]

In coordinates, writing \( \alpha = \frac{1}{k!} \alpha_{i_1 \ldots i_k}(x)dx^{i_1} \land \ldots \land dx^{i_k} \) (with \( \alpha_{i_1 \ldots i_k} \) totally anti-symmetric), we have

\[
\tau(\alpha)_X = \frac{1}{(k-1)!} \alpha_{i_1 \ldots i_k}(x)X^{i_1}dx^{i_2} \land \ldots \land dx^{i_k}.
\]
Example 3.2. Consider the map $\omega^2: TM \longrightarrow T^*M$, $\omega^2(X) = i_X \omega$, associated with a 2-form $\omega \in \Omega^2(M)$. A direct computation shows that
\[ \tau(\omega) = (\omega^2)^{\ast} \theta_{\text{can}}, \]
where $\theta_{\text{can}} \in \Omega^1(T^*M)$ is the canonical 1-form, $\theta_{\text{can}} = p_i dx^i$.

The tangent lift can be computed by the following Cartan-like formula (cf. [10]).

**Proposition 3.3.** For $\alpha \in \Omega^k(M)$, the tangent lift of $\alpha$ is given by the formula
\[ \alpha_T = d\tau(\alpha) + \tau(d\alpha). \]

**Proof.** It suffices to check (3.26) locally, so we replace $M$ by a neighborhood with coordinates $(x^j)$, so that $TM$ has coordinates $(x^j, \dot{x}^j)$. Let us consider the vector field $V$ on $TM$ defined by
\[ V_X := \dot{x}^j \frac{\partial}{\partial x^j} \in T_X(TM), \]
where $X = (x^j, \dot{x}^j) \in TM$. This vector field has the property that $Tp_M(V_X) = X$.

One can directly check that
\[ f_T = \mathcal{L}_V(p_M^*f), \quad \text{and} \quad (dx^j)_T = d\dot{x}^j = \mathcal{L}_V(p_M^*dx^j), \]
where $f \in C^\infty(M)$. From the definition of $\tau$, it immediately follows that
\[ \tau(\beta) = i_V p_M^* \beta, \quad \beta \in \Omega^k(M). \]

Given $\alpha = \frac{1}{k!} \alpha_{i_1...i_k}(x) dx^{i_1} \wedge \ldots \wedge dx^{i_k}$, using Lemma 3.1 we obtain
\[ \alpha_T = \frac{1}{k!} (\alpha_{i_1...i_k})_T p_M^* (dx^{i_1} \wedge \ldots \wedge dx^{i_k}) + \frac{1}{k!} p_M^* \alpha_{i_1...i_k} (dx^{i_1} \wedge \ldots \wedge dx^{i_k})_T \]
\[ = \frac{1}{k!} (\alpha_{i_1...i_k})_T p_M^* (dx^{i_1} \wedge \ldots \wedge dx^{i_k}) + \]
\[ \frac{1}{k!} p_M^* \alpha_{i_1...i_k} \sum_{n=1}^k dx^{i_1} \wedge \ldots \wedge (dx^{i_n})_T \wedge \ldots \wedge dx^{i_k}. \]

It then follows from (3.27) that $\alpha_T = \mathcal{L}_V p_M^* \alpha$. Using (3.28) and Cartan’s formula, we have
\[ \alpha_T = d(i_V p_M^* \alpha) + i_V p_M^* d\alpha = d\tau(\alpha) + \tau(d\alpha). \]

\[ \square \]

**Example 3.4.** From Example 3.2, it follows that if $\omega \in \Omega^2(M)$, then
\[ \omega_T = -(\omega^2)^{\ast} \omega_{\text{can}} + \tau(d\omega). \]

Here $\omega_{\text{can}} = -d\theta_{\text{can}} = dx^i \wedge dp_i$ is the canonical symplectic form on $T^*M$. (For the tangent lift of closed 2-forms, see also [8, Sec. 3]).

An immediate consequence of (3.26) is the fact that tangent lifts and exterior derivatives commute.

**Corollary 3.5.** For $\alpha \in \Omega^k(M)$, $d(\alpha_T) = (d\alpha)_T$. 
3.2. Lie functor on multiplicative differential forms. Let $\mathcal{G}$ be a Lie groupoid over $M$, $A = A\mathcal{G}$ its Lie algebroid, and let $\alpha \in \Omega^k(\mathcal{G})$. We can define an induced $k$-form on $A$ by pulling back the tangent lift $\alpha_T \in \Omega^k(T\mathcal{G})$ via the inclusion $\iota_A : A \to T\mathcal{G}$. In this section we discuss this operation when $\alpha$ is multiplicative.

Recall that a $k$-form $\alpha \in \Omega^k(\mathcal{G})$ is multiplicative if

\begin{equation}
\alpha \in \Omega^k(\mathcal{G}) \quad \text{is multiplicative if and only if} \quad (3.32)
\end{equation}

and (3.33), we see that if $X$ is multiplicative, then by (3.29) we have

\begin{equation}
m^*\alpha = p_1^*\alpha + p_2^*\alpha,
\end{equation}

where $p_1, p_2 : G(2) \to G$ are the natural projections, and $m$ is the groupoid multiplication. We denote the associated $k$-form on $A$ by

\begin{equation}
\text{Lie}(\alpha) := \iota_A\alpha_T.
\end{equation}

Note that it follows from Corollary 3.5 that

\begin{equation}
d\text{Lie}(\alpha) = \text{Lie}(d\alpha).
\end{equation}

In order to explain in which sense $\text{Lie}(\alpha)$ is the infinitesimal counterpart of $\alpha$, we will need a known alternative characterization of multiplicative forms.

The tangent groupoid structure on the tangent bundle $p_G : T\mathcal{G} \to \mathcal{G}$ over $TM$ induces a groupoid structure on the direct sum

\begin{equation}
\prod_{p_G}^n T\mathcal{G} = T\mathcal{G} \oplus \ldots \oplus T\mathcal{G}
\end{equation}

over the base $\prod_{p_M}^n TM = TM \oplus \ldots \oplus TM$ in a canonical way.

**Lemma 3.6.** A $k$-form $\alpha \in \Omega^k(\mathcal{G})$ ($k \geq 1$) is multiplicative if and only if the bundle map $\alpha^2 : \prod_{p_G}^{k-1} T\mathcal{G} \to T^*\mathcal{G}$ (see (3.21)) is a groupoid morphism.

**Proof.** Let us consider the following identities, obtained by differentiating basic identities on any Lie groupoid (see [3, Lem. 3.1]):

\begin{equation}
(Tm)_{(t(g),g)}(Tt(X),X) = X = (Tm)_{(g,s(g))}(X,Ts(X)), \quad \forall X \in T_g\mathcal{G},
\end{equation}

\begin{equation}
(Tr_g)_{(t(g))}(u) = (Tm)_{(t(g),g)}(u,0), \quad (Tl_g)_{s(g)}(v) = (Tm)_{(g,s(g))}(0,v)
\end{equation}

where $u \in A_{t(g)} = \text{Ker}(Ts)_{t(g)}$ and $v \in \text{Ker}(Tl)_{s(g)}$. Using the first identities in (3.32) and (3.33), we see that if $\alpha$ is multiplicative, then by (3.29) we have

\begin{equation}
\alpha(Tt(X_1),\ldots,Tt(X_{k-1}),u) = \alpha(X_1,\ldots,X_{k-1},Tr_g(u)),
\end{equation}

where $X_i \in T_g\mathcal{G}, u \in A_{t(g)}$. This is precisely the compatibility of $\alpha^2$ with the target maps on $\prod_{p_G}^{k-1} T\mathcal{G}$ and $T^*\mathcal{G}$. Similarly, note that (3.32) and (3.29) imply that, if $Z_1, \ldots, Z_k \in TM$, then $\alpha(Z_1,\ldots,Z_k) = 0$. Using this fact, along with (3.29) and the second identities in (3.32) and (3.33), we obtain the compatibility of $\alpha^2$ and the source maps:

\begin{equation}
\alpha(Ts(X_1),\ldots,Ts(X_{k-1}),u) = \alpha(X_1,\ldots,X_{k-1},Tl_g(u-Tt(u))),
\end{equation}

where $X_i \in T_g\mathcal{G}, u \in A_{t(g)}$.

Assuming that $\alpha^2$ is compatible with the source and target maps, we see that it is a groupoid morphism if and only if

\begin{equation}
\alpha^2(TM(X_1,Y_1),\ldots,Tm(X_{k-1},Y_{k-1})) = \alpha^2(X_1,\ldots,X_{k-1}) \circ \alpha^2(Y_1,\ldots,Y_{k-1}).
\end{equation}
By evaluating each side of the last equation on $Tm(X_k, Y_k)$, we see that this condition is equivalent to

$$\alpha(Tm(X_1, Y_1), \ldots, Tm(X_k, Y_k)) = \alpha(X_1, \ldots, X_k) + \alpha(Y_1, \ldots, Y_k),$$

which is precisely the multiplicativity condition (3.29). \qed

Given a groupoid morphism $\psi : \mathcal{G}_1 \to \mathcal{G}_2$, we denote the associated morphism of Lie algebroids (given by the restriction of $T\psi : T\mathcal{G}_1 \to T\mathcal{G}_2$ to $A\mathcal{G}_1 \subset T\mathcal{G}_1$) by

$$\text{Lie}(\psi) : A\mathcal{G}_1 \to A\mathcal{G}_2.$$

The natural projection $p_\mathcal{G} : T\mathcal{G} \to \mathcal{G}$ is a groupoid morphism, and one can directly verify that there is a canonical identification

$$A(\prod_{p_\mathcal{G}} T\mathcal{G}) = \prod_{\text{Lie}(p_\mathcal{G})} A(T\mathcal{G}).$$

Using this identification we get, for any given multiplicative $k$-form $\alpha \in \Omega^k(\mathcal{G})$, a Lie algebroid morphism

$$\text{Lie}(\alpha^\sharp) : \prod_{\text{Lie}(p_\mathcal{G})} A(T\mathcal{G}) \to A(T^*\mathcal{G}). \quad (3.34)$$

The isomorphism $j_\mathcal{G} : T(A\mathcal{G}) \to A(T\mathcal{G})$, see (2.15), induces an identification

$$j_\mathcal{G}^{(k)} : \prod_{p_A} T(A\mathcal{G}) \to \prod_{\text{Lie}(p_\mathcal{G})} A(T\mathcal{G}). \quad (3.35)$$

Recall the isomorphism $\theta_\mathcal{G} : A(T^*\mathcal{G}) \to T^*(A\mathcal{G})$ defined in (2.17).

**Proposition 3.7.** For a multiplicative $k$-form $\alpha \in \Omega^k(\mathcal{G})$, $\text{Lie}(\alpha)$ and $\text{Lie}(\alpha^\sharp)$ are related by

$$\text{Lie}(\alpha)^\sharp = \theta_\mathcal{G} \circ \text{Lie}(\alpha^\sharp) \circ j_\mathcal{G}^{(k-1)} : \prod_{p_A} T(A\mathcal{G}) \to T^*(A\mathcal{G}).$$

**Proof.** Recall that $\theta_\mathcal{G} = (T\iota_{A\mathcal{G}})^* \circ \Theta_\mathcal{G} \circ \iota_{T^*(A\mathcal{G})}$ and $J_\mathcal{G} \circ T\iota_{A\mathcal{G}} = \iota_{A(T\mathcal{G})} \circ j_\mathcal{G}$. This last identity immediately implies that

$$J_\mathcal{G}^{(k)} \circ (\prod_{p_A} T\iota_{A\mathcal{G}}) = (\prod_{p_A} \iota_{A(T\mathcal{G})}) \circ j_\mathcal{G}^{(k)}.$$

Since $\iota_{A(T^*\mathcal{G})} \circ \text{Lie}(\alpha^\sharp) = T\alpha^\sharp \circ \prod_{p_A} \iota_{A(T\mathcal{G})}$, it follows that

$$\theta_\mathcal{G} \circ \text{Lie}(\alpha^\sharp) \circ J_\mathcal{G}^{(k-1)} = (T\iota_{A\mathcal{G}})^* \circ \Theta_\mathcal{G} \circ T\alpha^\sharp \circ \prod_{p_A} \iota_{A(T\mathcal{G})} \circ J_\mathcal{G}^{(k-1)} = (T\iota_{A\mathcal{G}})^* \circ \alpha^\sharp_{T} \circ (\prod_{p_A} T\iota_{A\mathcal{G}}),$$

and this last term is $(\iota_{T}^* \alpha_T)^\sharp = (\text{Lie}(\alpha))^\sharp$. \qed

**Corollary 3.8.** If $\alpha \in \Omega^k(\mathcal{G})$ is multiplicative and $\mathcal{G}$ is s-connected, then $\alpha = 0$ if and only if $\text{Lie}(\alpha) = 0$. 
proof. If $G$ is $s$-connected, then $\prod_{p \in G}^k T G$ also has connected source-fibres. We now use the fact that if two groupoid morphisms $G_1 \to G_2$ induce the same Lie algebroid morphism and $G_1$ has source-connected fibres, then they must coincide. Hence $\alpha^s = 0$ if and only if $\text{Lie}(\alpha^s) = 0$. The conclusion now follows since $\alpha^s = 0$ (resp., $\text{Lie}(\alpha^s) = 0$) is equivalent to $\alpha^s = 0$ (resp., $\text{Lie}(\alpha^s) = 0$), and $\text{Lie}(\alpha^s) = 0$ if only if $\text{Lie}(\alpha^s) = 0$ by Proposition 3.7.

4. Multiplicative 2-forms and their infinitesimal counterparts

4.1. Linear 2-forms on vector bundles. Let $q : A \to M$ be a vector bundle, and consider the double vector bundles $TA$ and $T^*A$, as in Section 2.2. A 2-form $\Lambda \in \Omega^2(A)$ is called linear if $\Lambda^s : TA \to T^*A$ is a morphism of double vector bundles (cf. [11, Sec. 7.3]). In particular, there is a vector bundle map $\lambda : TM \to A^*$ (over the identity) such that the following diagram is commutative:

\[
\begin{array}{ccc}
TA & \xrightarrow{\Lambda^s} & T^*A \\
\downarrow{Tq} & & \downarrow{r} \\
TM & \xrightarrow{\lambda} & A^*
\end{array}
\]

In this case we say that $\Lambda$ covers $\lambda$.

Remark 4.1. The fact that a bivector field $\pi$ on a vector bundle $A$ is linear is equivalent [11, 13] to the bundle map $\pi^s : T^*A \to TA$ being a morphism of double vector bundles. Hence linear 2-forms are just their dual analogues.

It is simple to check from the definition that a linear 2-form has a local expression of the form:

\[
\Lambda = \frac{1}{2} \Lambda_{ij}(x,u)dx^i \wedge dx^j + \Lambda_{jd}(x,u)dx^j \wedge du^d
\]

(4.37)

\[
= \frac{1}{2} \Lambda_{ij,d}(x,u)dx^i \wedge dx^j + \lambda_{jd}(x)dx^j \wedge du^d.
\]

where $(x,u) = (x^i, u^d)$ are local coordinates in $A$ (relative to a local basis $\{e_d\}$), and $\lambda_{jd} = \langle \lambda(\frac{\partial}{\partial x^j}), e_d \rangle$.

Example 4.2. The canonical symplectic form $\omega_{\text{can}} = dx^i \wedge dp_j$ on the cotangent bundle $T^*M$ is linear. Any vector bundle map $\sigma : A \to T^*M$, locally written as $\sigma(e_d) = \sigma_{jd}dx^j$, defines a linear 2-form on $A$ by pullback,

\[
\sigma^*\omega_{\text{can}} = u^d \frac{\partial \sigma_{id}}{\partial x^k} dx^i \wedge dx^k + \sigma_{id}dx^i \wedge du^d,
\]

covering the map $\lambda = \sigma^t : TM \to A^*$, where $\sigma^t$ is the fibrewise transpose of $\sigma$.

From the local expression (4.37), one can directly verify that Example 4.2 completely characterizes linear closed 2-forms:

Proposition 4.3. A linear 2-form $\Lambda \in \Omega^2(A)$ is closed if and only if it is of the form

\[
\Lambda = (\lambda^s)^*\omega_{\text{can}},
\]
where $\Lambda^t : A \to T^*M$ is the fibrewise transpose of the vector-bundle map $\lambda : TM \to A^*$ (see (4.36)).

A proof of this result can be found in [11, Sec. 7.3].

**Example 4.4.** If $\omega \in \Omega^2(M)$, then its tangent lift $\omega_T \in \Omega^2(TM)$ is linear and covers the map $\lambda = \omega^\sharp : TM \to T^*M$. If $\omega$ is closed, then so is $\omega_T$ (it is in fact exact, by Proposition 3.3). It follows from Proposition 4.3 and the fact that $(\omega^\sharp)^* = -\omega^\sharp$ that

$$\omega_T = -(\omega^\sharp)^*\omega_{can},$$

in agreement with Example 3.4.

**Example 4.5.** Let $\phi \in \Omega^3(M)$ be a 3-form on $M$. Then the 2-form $\tau(\phi)$ on $TM$ defined by (3.25) is linear; it covers the bundle map $\lambda : A \to T^*M$ that is zero on each fibre.

4.2. **Linear 2-forms on Lie algebroids.** Let $A \to M$ be a Lie algebroid. We will discuss two natural ways to obtain linear 2-forms on $A$.

First, given any 3-form $\phi \in \Omega^3(M)$, we can use the anchor $\rho : A \to TM$ to pull-back the linear 2-form $\tau(\phi)$ to $A$. The resulting 2-form

$$\rho^*(\tau(\phi)) \in \Omega^2(A)$$

is linear, covering the map $\lambda : TM \to A^*$ that is zero on each fibre.

On the other hand, if $A = AG$ is the Lie algebroid of a Lie groupoid $G$, then one obtains linear 2-forms on $A$ as infinitesimal versions of multiplicative 2-forms on $G$.

**Proposition 4.6.** Let $\omega \in \Omega^2(G)$ be a multiplicative 2-form, and let $\lambda : TM \to A^*$ be defined by $\lambda(X)(u) = \omega(X,u)$, for $X \in TM$ and $u \in A$. Then

1. $\Lambda = \text{Lie}(\omega) \in \Omega^2(A)$ is linear and covers $\lambda$.
2. Given $\phi \in \Omega^3(M)$ closed and if $G$ is s-connected, then $d\omega = s^*\phi - t^*\phi$ if and only if

$$\Lambda = (\lambda^t)^*\omega_{can} - \rho^*(\tau(\phi)).$$

**Proof.** Let us prove (1). Note that $\text{Lie}(\omega) = \iota_A^*\omega_T$ is linear since $\omega_T \in \Omega^2(TG)$ is linear, and the pull back of a linear 2-form to a vector subbundle is again linear.

From Lemma 3.6, we know that $\omega^\sharp : TG \to T^*G$ is a groupoid morphism, which restricts to the map $\lambda : TM \to A^*$ on identity sections. As a result, $\text{Lie}(\omega^\sharp)$ fits into the following commutative diagram:

$$\begin{array}{ccc}
A(TG) & \xrightarrow{\text{Lie}(\omega^\sharp)} & A(T^*G) \\
\downarrow & & \downarrow \\
TM & \xrightarrow{\lambda} & A^*,
\end{array}$$

and it follows from Proposition 3.7 that $\Lambda = \text{Lie}(\omega)$ covers $\lambda$.

For part (2), note that

$$\text{Lie}(s^*\phi - t^*\phi) = \iota_A^*(s^*\phi)_T - \iota_A^*(t^*\phi)_T.$$
(\(A\) is tangent to the \(s\)-fibres) and \(T_t \circ \iota_A = \rho\), we obtain \(\text{Lie}(s^*\phi - t^*\phi) = -d\rho^*\tau(\phi)\).

By Corollary 3.8, we know that
\[
d\omega - (s^*\phi - t^*\phi) = 0 \iff \text{Lie}(d\omega - (s^*\phi - t^*\phi)) = 0.
\]

But \(\text{Lie}(d\omega - (s^*\phi - t^*\phi)) = d(\Lambda + \rho^*\tau(\phi))\). Since the linear 2-form \(\Lambda + \rho^*\tau(\phi)\) covers \(\lambda\), it follows from Proposition 4.3 that
\[
d(\Lambda + \rho^*\tau(\phi)) = 0 \iff \Lambda + \rho^*\tau(\phi) = (\lambda^t)^*\omega_{\text{can}},
\]
as desired. \(\Box\)

To make the connection between this paper and the results in [3] more transparent, it will be convenient to consider the map \(\sigma : A \to T^*M\) induced by \(\omega \in \Omega^2(G)\) via
\[
\sigma_\omega(u)(X) = \omega(u, X), \quad u \in A, X \in TM.
\]
In the notation of Proposition 4.6, we have \(\sigma_\omega = -\lambda^t\), so under the assumptions in part (2), \(\Lambda = \text{Lie}(\omega)\) and \(\sigma_\omega\) are related by
\[
\Lambda = -(\sigma_\omega^*\omega_{\text{can}} + \rho^*\tau(\phi)),
\]
in such a way that \(\Lambda\) covers \(-\sigma_\omega^t : TM \to A^*\). The following result describes when such a 2-form induces a morphism between the tangent and cotangent algebroid structures.

**Theorem 4.7.** Let \(\Lambda \in \Omega^2(A)\) be as in (4.40). The following are equivalent:

(i) The map \(\Lambda^t : TA \to T^*A\) is a Lie algebroid morphism.
(ii) The map \(\sigma : A \to T^*M\) satisfies
\[
\langle \sigma(u), \rho(v) \rangle = -\langle \sigma(v), \rho(u) \rangle
\]
\[
\sigma([u, v]) = \mathcal{L}_{\rho(u)}\sigma(v) - i_{\rho(v)}d\sigma(u) + i_{\rho(v)}i_{\rho(u)}\phi,
\]
for all \(u, v \in \Gamma(A)\).

Vector bundle maps \(\sigma : A \to T^*M\) satisfying conditions (4.41) and (4.42) were introduced in [3] and are referred to as **IM 2-forms** on \(A\) (relative to \(\phi\)). We also recall that a **morphism** between Lie algebroids \(A \to M\) and \(B \to N\) (see, e.g., [12]) is a vector bundle map \(\Psi : A \to B\), covering \(\psi : M \to N\), which is compatible with anchors, meaning that
\[
\rho_B \circ \Psi = T\psi \circ \rho_A,
\]
and compatible with brackets in the following sense. Consider the pull-back bundle \(\psi^*B \to M\), and let us keep denoting by \(\Psi\) the induced map \(\Gamma(A) \to \Gamma(\psi^*B)\) at the level of sections. Given sections \(u, v \in \Gamma(A)\) such that \(\Psi(u) = f_j\psi^*u_j\) and \(\Psi(v) = g_i\psi^*v_i\), where \(f_j, g_i \in C^\infty(M)\) and \(u_j, v_i \in \Gamma(B)\), the following condition should be valid:
\[
\Psi([u, v]_A) = f_jg_i\psi^*[u_j, v_i]_B + \mathcal{L}_{\rho_A(u)}g_i\psi^*v_i - \mathcal{L}_{\rho_A(v)}f_j\psi^*u_j.
\]

**4.3. IM 2-forms and Lie algebroid morphisms.** This subsection presents the key step to the integration of IM 2-forms.

Let \(A \to M\) be a Lie algebroid, with bracket \([- , -]\) and anchor \(\rho\). Let \(\sigma : A \to T^*M\) be a vector bundle map (over the identity) and \(\phi \in \Omega^3(M)\) a closed 3-form. Motivated by (4.39), let us consider the linear 2-form \(\Lambda \in \Omega^2(A)\) defined by
\[
\Lambda = -(\sigma_\omega^*\omega_{\text{can}} + \rho^*\tau(\phi)),
\]
covering \(-\sigma_\omega^t : TM \to A^*\). The following result describes when such a 2-form induces a morphism between the tangent and cotangent algebroid structures.
We will need explicit local formulas for the tangent and cotangent Lie algebroids. For a basis of local sections \( \{ e_d \} \) of \( A \), we denote the corresponding Lie algebroid structure functions by \( \rho_A \) and \( C_{ab}^c \):

\[
\rho_A(e_a) = \rho_a^j \frac{\partial}{\partial x^j}, \quad [e_a, e_b] = C_{ab}^c e_c.
\]

Recall from Section 2.3 that any section \( u : M \rightarrow A \) defines two types of sections of \( TA \rightarrow TM \), denoted by \( Tu \) and \( \tilde{u} \). From (2.14), the tangent Lie algebroid structure can be written as follows:

\[
\rho_{TA}(e_a) = \rho_a^j \frac{\partial}{\partial x^j} + d\rho_a^j \frac{\partial}{\partial \xi^j}, \quad \rho_{TA}(e_a) = \rho_a^j \frac{\partial}{\partial x^j}.
\]

To describe the Lie algebroid structure on \( T^*A \rightarrow A^* \) explicitly, we also consider two types of sections that generate the space of sections of \( T^*A \) over \( A^* \). The first type is induced from a section \( u \in \Gamma(A) \), and denoted by \( u^L \). In local coordinates \( (x^j, \xi_d) \) on \( A^* \) (relative to the basis of local sections \( \{ e^d \} \) of \( A^* \), dual to \( \{ e_d \} \)), it is given by

\[
u^L(x^j, \xi_d) = (x^j, u^d(x), 0, \xi_d),
\]

where \( T^*A \) is written locally in coordinates \( (x^j, u^d, p_j, \xi_d) \) as in Section 2.2. The second type are core sections: locally, for each \( \alpha = \alpha_j dx^j \in \Gamma(T^*M) \), we define the section \( \tilde{\alpha} \) of \( T^*A \rightarrow A^* \) by

\[
\tilde{\alpha}(x^j, \xi_d) = (x^j, 0, \alpha_j(x), \xi_d).
\]

The cotangent Lie algebroid is defined by the relations:

\[
\rho_{\tilde{T^*A}}(dx^j) = \rho_a^j \frac{\partial}{\partial x^j} - dC_{ab}^c \xi_c + C_{ab}^c d\xi_c, \quad \rho_{\tilde{T^*A}}(dx^j) = \rho_a^j \frac{\partial}{\partial \xi^j} + C_{ab}^c \xi_c \frac{\partial}{\partial \xi^j}.
\]

We now turn to the proof of Theorem 4.7.

**Proof.** We work locally, so we assume that \( M \) has coordinates \( (x^j) \). Then \( A \) has coordinates \( (x^j, u^d) \) (relative to a basis of local sections \( \{ e_d \} \)), \( TA \) has tangent coordinates \( (x^j, u^d, \dot{x}^j, \dot{u}^d) \), while induced coordinates on \( T^*A \) are denoted by \( (x^j, u^d, p_j, \xi_d) \). Similarly, \( A^* \) has dual coordinates \( (x^j, \xi_d) \), inducing coordinates \( (x^j, \xi_d, \dot{x}^j, \dot{\xi}_d) \) on \( TA^* \).

We start by discussing when \( A^2 \) is compatible with the anchors, i.e.,

\[
T(-\sigma^i) \circ \rho_{TA} = \rho_{\tilde{T^*A}} \circ A^2.
\]

Let us consider local expressions of the relevant maps. We write \( \sigma : A \rightarrow T^*M \) and \( \sigma^i : TM \rightarrow A^* \) locally as

\[
\sigma(x^j, u^d) = (x^j, u^d \sigma_{jd}(x)), \quad \sigma^i(x^j, \dot{x}^j) = (x^j, \dot{x}^j |_{\sigma_{jd}(x)}).
\]

Denoting coordinates on \( TM \) by \( (x^j, \dot{x}^j) \), and on \( T(TM) \) by \( (x^j, \dot{x}^j, \delta x^j, \delta \dot{x}^j) \), we get

\[
T(-\sigma^i)(x^j, \dot{x}^j, \delta x^j, \delta \dot{x}^j) = (x^j, -\delta x^j, \sigma_{lj} \delta x^j, -\delta \dot{x}^j |_{\sigma_{lj}(x)} \delta x^k - \sigma_{jd} \delta \dot{x}^j) \in TA^*.
\]
One can directly verify that the map $\Lambda^\sharp$ can be locally written as follows:

\begin{equation}
\Lambda^\sharp(x^j, u^d, \dot{x}^j, \dot{u}^d) = (x^j, u^d, p_j, \zeta_d),
\end{equation}

where

\[ p_j = \dot{x}^j u^d \left( \frac{\partial\sigma_{jd}}{\partial x^j} - \frac{\partial\sigma_{ld}}{\partial x^i} \right) + \dot{u}^d \sigma_{jd} - \phi_{ijk} u^d \rho_d^b \dot{x}^i, \quad \zeta_d = -\dot{x}^i \sigma_{ld}. \]

The space of sections of $TA \longrightarrow TM$ is generated by sections of types $Te_a$ and $\widehat{e}_b$. We have

\begin{equation}
\Lambda^\sharp(Te_a|_{(x,\dot{x})}) = \left( x^j, \delta_{ad}, \dot{x}^i \left( \frac{\partial\sigma_{ja}}{\partial x^i} - \frac{\partial\sigma_{ta}}{\partial x^i} \right) - \phi_{ijk} \rho_d^b \dot{x}^i, -\dot{x}^l \sigma_{ld} \right),
\end{equation}

\begin{equation}
\Lambda^\sharp(\widehat{e}_b|_{(x,\dot{x})}) = (x^j, 0, \sigma_{jb}, -\dot{x}^l \sigma_{ld}).
\end{equation}

Using (4.45) and (4.49), one can directly check that

\[ T(-\sigma^t)(\rho_{TA}(\widehat{e}_b|_{(x,\dot{x})})) = (x^j, -\dot{x}^l \sigma_{ld}, 0, -\sigma_{ld} \rho_d^b) \in (-\sigma^t)^*TA^*. \]

On the other hand, using the local expression (4.51), we have

\[ \rho_{TA}(\Lambda^\sharp(\widehat{e}_b|_{(x,\dot{x})})) = (x^j, -\dot{x}^l \sigma_{ld}, 0, \rho_d^l \sigma_{lb}). \]

It follows that the compatibility (4.50) for core sections amounts to

\[ \langle \rho(e_b), \sigma(e_d) \rangle = -\langle \rho(e_d), \sigma(e_b) \rangle, \]

which is equivalent to (4.41).

For sections of type $Te_b$, again using (4.45) and (4.49), we get

\[ T(-\sigma^t)(\rho_{TA}(Te_b|_{(x,\dot{x})})) = (x^j, -\dot{x}^l \sigma_{ld}, \rho_d^l \sigma_{lb}, \zeta_d) \in (-\sigma^t)^*TA^*, \]

where

\begin{equation}
\zeta_d = -\dot{x}^l \left( \frac{\partial\sigma_{ld}}{\partial x^l} \rho_d^b + \sigma_{ld} \rho_d^l \right) = -\langle \mathcal{L}_{\rho(e_b)} \sigma(e_d), \dot{x} \rangle.
\end{equation}

Similarly, we compute

\[ \rho_{TA}(\Lambda^\sharp(Te_b|_{(x,\dot{x})})) = (x^j, -\dot{x}^l \sigma_{ld}, \rho_d^l \sigma_{lb}, \zeta_d'), \]

where

\begin{equation}
\zeta_d' = \dot{x}^j \rho_d^k \left( \frac{\partial\sigma_{kb}}{\partial x^j} - \frac{\partial\sigma_{db}}{\partial x^k} \right) - \phi_{ij} \rho_d^b \dot{x}^i \rho_d^l \sigma_{lc} + C_{db}^c \dot{x}^l \sigma_{lc}
\end{equation}

\[ = \langle -i_{\rho(e_d)}(d\sigma(e_b)) + i_{\rho(e_d)}i_{\rho(e_b)} \sigma([e_d, e_b]), \dot{x} \rangle. \]

Comparing (4.54) and (4.55), it follows that the compatibility (4.50) for sections of the type $Te_b$ is verified if and only if (4.42) holds.

Let us now check the bracket-preserving condition (4.43), that in our case reads

\begin{equation}
\Lambda^\sharp([U, V]_{TA}|_{(x,\dot{x})}) = f_j g_i [U, V_i]_{TA} \rho_{TA}(U) g_i V_i |_{-\sigma^t} + \mathcal{L}_{\rho_{TA}(U)} g_i V_i |_{-\sigma^t} - \mathcal{L}_{\rho_{TA}(V)} f_j U_j |_{-\sigma^t},
\end{equation}

where $U, V \in \Gamma(TA)$, and $f_j, g_i \in C^\infty(TM)$, $U_j, V_i \in \Gamma(T^*A)$ are such that $\Lambda^\sharp(U) = f_j(-\sigma^t)^*U_j$ and $\Lambda^\sharp(V) = g_i(-\sigma^t)^*V_i$.

From (4.52), (4.53), we can write

\begin{equation}
\Lambda^\sharp(Te_a|_{(x,\dot{x})}) = \rho_a^l |_{-\sigma^t} + f_j^a \dot{x}^l |_{-\sigma^t},
\end{equation}

\begin{equation}
\Lambda^\sharp(\widehat{e}_a|_{(x,\dot{x})}) = \sigma(e_a) |_{-\sigma^t} = g_i^a \dot{x}^i |_{-\sigma^t}. \]
where
\[ f_j^a = \dot{x}^l \left( \frac{\partial \sigma_{ja}}{\partial x^l} - \frac{\partial \sigma_{ia}}{\partial x^j} \right) - \phi_{ijk} \rho_a^k \dot{x}^i, \quad g_i^a = \sigma_{ia}, \]
so we can express the images in terms of sections of types (4.46) and (4.47) on \( T^*A \).
It will be useful to note that the functions \( f_j^a = f_j^a(x, \dot{x}) \) satisfy
\[ f_j^a \, dx^j = i_{\dot{x}} d\sigma(e_a) - i_{\dot{x}} i_{\rho(e_a)} \phi, \]
viewed as an equality of horizontal 1-forms on \( TM \), i.e. 1-forms of type \( \alpha_j(x, \dot{x}) \, dx^j \) (in this formula, \( \dot{x} \) is seen as the vector field \( \dot{x}^l \frac{\partial}{\partial x^l} \) on \( TM \)). In fact, locally, there is an identification of the space of horizontal 1-forms on \( TM \) with a subspace of sections of \( (\sigma^t)^*T^*A \) via
\[ \Omega^1_{hor}(TM) \longrightarrow \Gamma((\sigma^t)^*T^*A), \quad \alpha_j(x, \dot{x}) \, dx^j \mapsto \alpha_j(x, \dot{x}) \, dx^j |_{\sigma^t(x, \dot{x})}. \]
In the remainder of this section, we will use this identification to view horizontal 1-forms on \( TM \) as sections of the bundle \( (\sigma^t)^*T^*A \). In particular, in order to simplify our notation, we will write \( d\dot{x}^j |_{\sigma^t(x, \dot{x})} \) just as \( dx^j \).
Since it suffices to verify condition (4.56) for sections of types \( T e_a \) (linear) and \( \widehat{e}_a \) (core), we have three cases to analyze.

**Core-core sections**

If \( U = \widehat{e}_a \) and \( V = \widehat{e}_b \) are core sections, then by (4.44) we know that \( [\widehat{e}_a, \widehat{e}_b]_{T^*A} = 0 \), so the left-hand side of (4.56) vanishes. On the other hand, from (4.45), the Lie derivatives on the right-hand side of (4.56) are only with respect to the variable \( \dot{x} \).
Since the functions \( g_j \) in (4.58) do not depend on \( \dot{x} \) and \( [\widehat{e}_a, \widehat{e}_b]_{T^*A} = 0 \), it follows that, for a pair of core sections, the right-hand side of (4.56) vanishes as well.

**Core-linear sections**

Let us consider (4.56) when \( U = T e_a \) and \( V = \widehat{e}_b \). Since \( [T e_a, \widehat{e}_b]_{T^*A} = C_{ab}^c \widehat{e}_c \), it follows from (4.58) that the left-hand side of (4.56) is
\[ \Lambda^i([T e_a, \widehat{e}_b]_{T^*A}) = \sigma([e_a, e_b]). \]
Using the bracket relations (4.48), one directly sees that the first term on the right-hand side of (4.56) is just \( \sigma_{ib} \rho^i_a \). For the second term, we have
\[
(L_{\rho(T^*A(T e_a))} \sigma_{ib}) \, dx^i = L_{\rho_b} \left( \frac{\partial \sigma_{ja}}{\partial x^l} - \frac{\partial \sigma_{ia}}{\partial x^j} \right) - \sigma_{ib} L_{\rho^i_a} \, dx^i \\
= L_{\rho_b} \sigma(e_b) - \sigma_{ib} \rho^i_a.
\]
The third term on the right-hand side of (4.56) is given by
\[
(L_{T A(\widehat{e}_b)} f_j^a) \, dx^j = \left( \rho_b \left( \frac{\partial \sigma_{ja}}{\partial x^l} - \frac{\partial \sigma_{ia}}{\partial x^j} \right) - \phi_{ijk} \rho_a^k \chi^i \right) \, dx^j \\
= i_{\rho(\widehat{e}_b)} d\sigma(e_a) - i_{\rho(\widehat{e}_b)} i_{\rho(e_a)} \phi.
\]
As a result, in this case, (4.56) is equivalent to
\[ \sigma([e_a, e_b]) = L_{\rho(e_a)} \sigma(e_b) - i_{\rho(e_b)} d\sigma(e_a) + i_{\rho(e_a)} i_{\rho(e_a)} \phi, \]
which agrees with condition (4.42).

**Linear-linear sections**
We finally consider (4.56) when \( U = Te_a \) and \( V = Te_b \). From (4.44), (4.57) and (4.58), and using (4.60), we see that the left-hand side of (4.56) is

\[
\Lambda^2([Te_a, Te_b]_{TA}) = C^c_{ab} c^L_{-} \sigma'^{i}(x, \hat{x}) + C^c_{ab} f^c_{J} \sigma^J(e_c) + dC^c_{ab}(\hat{x}) \sigma(e_c)
\]

\[
= [e_a, e_b] L_{-} \sigma'^{i}(x, \hat{x}) + C^c_{ab}(i \hat{x} \sigma(e_c) - i \hat{x} i \rho(e_c) \phi) + dC^c_{ab}(\hat{x}) \sigma(e_c)
\]

\[
= [e_a, e_b] L_{-} \sigma'^{i}(x, \hat{x}) + i \hat{x} \sigma(e_c) + dC^c_{ab}(\sigma(e_c), \hat{x}) - i \hat{x} i \rho(e_c) \phi.
\]

As in (4.60), we abuse notation and use \( \hat{x} \) also to represent the vector field \( \hat{x}^I \frac{\partial}{\partial x^I} \).

By (4.57), we can write \( \Lambda^2(Te_a) = e^L_a + f^b_J \sigma^J \) and \( \Lambda^2(Te_b) = e^L_b + f^b_J \sigma^J \). Using (4.48), we see that the first term on the right-hand side of (4.56) is

\[
[e_a, e_b] L_{-} \sigma'(x, \hat{x}) + dC^c_{ab}(\sigma'(\hat{x}), e_c) - f^a_V d\rho^b_i + f^b_V d\rho^a_i.
\]

Note that the second and third terms on the right-hand side of (4.56) are \( L_{\rho TA}(Te_a) f^a_V d\sigma^J \) and \( L_{\rho TA}(Te_b) f^a_V d\sigma^J \), respectively. Let us find a more explicit expression for the latter (the former is clearly completely analogous). Since

\[
L_{\rho TA}(Te_b) f^a_V d\sigma^J = L_{\rho TA}(Te_b) (f^a_V d\sigma^J) - \int f^a_J L_{\rho TA}(Te_b) d\sigma^J,
\]

it follows that

\[
L_{\rho TA}(Te_b) f^a_V d\sigma^J = L_{\rho TA}(Te_b) i \hat{x} \sigma(e_a) - L_{\rho TA}(Te_b) i \hat{x} \rho(e_a) \phi - f^a_J d\rho^i.
\]

Let us consider the (local) vector fields on \( TM \) given by \( V_b = d\rho^b_i (\hat{x}) \frac{\partial}{\partial x^i} \) and \( V'_b = d\rho^b_i (\hat{x}) \frac{\partial}{\partial x^i} \), so that \( \rho TA(Te_b) = \rho(e_b) + V'_b \). It is simple to check that \( [\rho(e_a), \hat{x}] = -V'_b \) and \( L_{V'_b} i \hat{x} \alpha = i \hat{x} \alpha \) for any \( \alpha = \frac{1}{2} \alpha_{ij}(x) dx^i \wedge dx^j \). Using Cartan calculus, we find

\[
L_{\rho TA}(Te_b) i \hat{x} \sigma(e_a) = L_{\rho(e_b)} i \hat{x} \sigma(e_a) + L_{V'_b} i \hat{x} \sigma(e_a)
\]

\[
= -i V'_b \sigma(e_a) + i \hat{x} L_{\rho(e_b)} d\sigma(e_a) + i \hat{x} \rho(e_a).
\]

Similarly,

\[
L_{\rho TA}(Te_b) i \hat{x} \rho(e_a) \phi = i \hat{x} L_{\rho(e_b)} i \rho(e_a) \phi.
\]

As a result, we obtain

\[
L_{\rho TA}(Te_b) f^a_V d\sigma^J = i \hat{x} \rho(e_b) d\sigma(e_a) - i \hat{x} L_{\rho(e_b)} i \rho(e_a) \phi - f^a_V d\rho^i.
\]

Analogously, we have

\[
L_{\rho TA}(Te_a) f^b_V d\sigma^J = i \hat{x} \rho(e_b) d\sigma(e_b) - i \hat{x} L_{\rho(e_b)} i \rho(e_a) \phi - f^b_V d\rho^i.
\]

Hence (4.56) amounts to the identity

\[
(4.62) \quad i \hat{x} \sigma([e_a, e_b]) - i \hat{x} i \rho(e_a, e_b) \phi = i \hat{x} d\rho(e_b) d\sigma(e_a) = i \hat{x} L_{\rho(e_b)} i \rho(e_a) \phi - i \hat{x} d\rho(e_b) d\sigma(e_a) = i \hat{x} \rho(e_b) d\sigma(e_a) + i \hat{x} L_{\rho(e_b)} i \rho(e_a) \phi.
\]

By basic Cartan calculus of forms, we have the identity

\[
i \rho(e_a, e_b) \phi - L_{\rho(e_a)} i \rho(e_b) \phi + L_{\rho(e_b)} i \rho(e_a) \phi = d i \rho(e_b) i \rho(e_a) \phi.
\]

It follows that (4.62) is equivalent to

\[
d\sigma([e_a, e_b]) = d(i \rho(e_a) d\sigma(e_b) - i \rho(e_b) d\sigma(e_a) + i \rho(e_b) i \rho(e_a) \phi)
\]

\[
= d(L_{\rho(e_a)} \sigma(e_b) - d i \rho(e_a) \sigma(e_b) - i \rho(e_b) d\sigma(e_a) + i \rho(e_b) i \rho(e_a) \phi)
\]

\[
= d(L_{\rho(e_a)} \sigma(e_b) - i \rho(e_b) d\sigma(e_a) + i \rho(e_b) i \rho(e_a) \phi).
\]
which holds by (4.42).

4.4. Applications to integration. In this section we present an alternative proof of the main result in [3], which describes IM 2-forms as infinitesimal versions of multiplicative 2-forms ([3, Theorem 2.5]).

Let $\mathcal{G}$ be a Lie groupoid over $M$, with Lie algebroid $A$. Let us denote the space of multiplicative 2-forms on $\mathcal{G}$ by $\Omega^2_{mult}(\mathcal{G})$, and the space of linear 2-forms on $A$ by $\Omega^2_{lin}(A)$. We also consider the subspace $\Omega^2_{alg}(A) \subset \Omega^2_{lin}(A)$ of linear 2-forms $\Lambda$ for which $\Lambda^2 : TA \longrightarrow T^*A$ is a Lie algebroid morphism.

A direct consequence of Proposition 3.7 is that $\text{Lie}(\alpha)^{\sharp}$ is a Lie algebroid morphism for any multiplicative $k$-form $\alpha$ on $\mathcal{G}$. Using Proposition 4.6, part (1), we conclude that the Lie functor on multiplicative forms gives rise to a well-defined map

$$\text{Lie} : \Omega^2_{mult}(\mathcal{G}) \longrightarrow \Omega^2_{alg}(A), \ \omega \mapsto \Lambda = \text{Lie}(\omega).$$

**Proposition 4.8.** If $\mathcal{G}$ is s-simply-connected, then (4.63) is a bijection.

**Proof.** We will show that (4.63) has an inverse map. If $\Lambda \in \Omega^2_{alg}(A)$, then $\Lambda^2 : TA \longrightarrow T^*A$ is a morphism of algebroids. So

$$\theta^{-1}_G \circ \Lambda^2 \circ j^{-1}_G : A(TG) \longrightarrow A(T^*G)$$

is a Lie algebroid morphism. Since $\mathcal{G}$ is s-simply-connected, so is $TG$. By Lie’s second theorem for algebroids (see, e.g., [12]), there exists a unique Lie groupoid morphism $\omega^\sharp : TG \longrightarrow T^*G$ with $\text{Lie}(\omega^\sharp) = \theta^{-1}_G \circ \Lambda^2 \circ j^{-1}_G$, or $\Lambda^2 = \text{Lie}(\omega^\sharp) \circ j_G$.

It remains to check that $\omega^\sharp$ is indeed the bundle map associated with a 2-form $\omega \in \Omega^2(\mathcal{G})$, i.e., that it is a vector-bundle map (covering the identity) with respect to the bundle structures $TG \longrightarrow \mathcal{G}$ and $T^*G \longrightarrow \mathcal{G}$, and that $(\omega^\sharp)^t = -\omega^\sharp$. A proof of this fact can be given just as in [14]: the key point is that the bundle projections $p_G : TG \longrightarrow \mathcal{G}$, $c_G : T^*G \longrightarrow \mathcal{G}$, the vector bundle sums $TG \times_{p_G} TG \longrightarrow TG$, $T^*G \times_{c_G} T^*G \longrightarrow T^*G$, and scalar multiplications $TG \times \mathbb{R} \longrightarrow TG$, $T^*G \times \mathbb{R} \longrightarrow T^*G$, as well as the natural pairing $TG \times_{(p_G, c_G)} T^*G \longrightarrow \mathbb{R}$ are all groupoid morphisms. The corresponding maps for Lie algebroids (after the identifications (2.15) and (2.17)) are precisely the vector bundle structure maps and pairing for $p_A : TA \longrightarrow A$ and $c_A : T^*A \longrightarrow A$, see, e.g., [13]. For example, to prove that $c_G \circ \omega^\sharp = p_G$, it suffices to verify this condition (by the connectivity of the source-fibres) at the level of algebroids. But then we have

$$\text{Lie}(c_G \circ \omega^\sharp) = c_A \circ \Lambda^2 = p_A = \text{Lie}(p_G).$$

The other properties of $\omega^\sharp$ are derived from those of $\Lambda^2$ similarly, as in [14, Theorem 4.1].

**Corollary 4.9 ([3]).** If $\mathcal{G}$ is s-simply-connected and $\phi \in \Omega^3(M)$ is closed, there is a one-to-one correspondence between multiplicative 2-forms on $\mathcal{G}$ satisfying $d\omega = s^\star \phi - t^\star \phi$ and IM 2-forms $\sigma : A \longrightarrow T^*M$ relative to $\phi$.

**Proof.** We know that $\text{Lie} : \Omega^2_{mult}(\mathcal{G}) \longrightarrow \Omega^2_{alg}(A), \ \omega \mapsto \Lambda = \text{Lie}(\omega)$ is a bijection, and by Proposition 4.6, part (2), $d\omega = s^\star \phi - t^\star \phi$ if and only if $\Lambda = -(\sigma^\star \omega_{can} + p^\star \tau(\phi))$. The conclusion now follows from Theorem 4.7.
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