No star is good news: a unified look at rerandomization based on $p$-values from covariate balance tests

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Abstract

Randomized experiments balance all covariates on average and provide the gold standard for estimating treatment effects. Chance imbalances nevertheless exist more or less in realized treatment allocations, complicating the interpretation of experimental results. To inform readers of the comparability of treatment groups at baseline, modern scientific publications often report covariate balance tables with not only covariate means by treatment group but also the associated $p$-values from significance tests of their differences. The practical need to avoid small $p$-values as indicators of poor balance motivates balance check and rerandomization based on these $p$-values from covariate balance tests (ReP) as an attractive tool for improving covariate balance in randomized experiments. Despite the intuitiveness of such strategy and its possibly already widespread use in practice, the existing literature lacks results about its implications on subsequent inference, subjecting many effectively rerandomized experiments to possibly inefficient analyses. To fill this gap, we examine a variety of potentially useful schemes for ReP and quantify their impact on subsequent inference. Specifically, we focus on three estimators of the average treatment effect from the unadjusted, additive, and fully interacted linear regressions of the outcome on treatment, respectively, and derive their asymptotic sampling properties under ReP. The main findings are threefold. First, ReP improves covariate balance between treatment groups, thereby strengthening the causal conclusions that can be drawn from experimental data. In addition to increasing comparability of treatment groups, the improved balance also reduces the asymptotic conditional biases of the estimators and ensures more coherent inferences between covariate-adjusted and unadjusted analyses. Second, the estimator from the fully interacted regression is asymptotically the most efficient under all ReP schemes examined, and permits convenient regression-assisted inference identical to that under complete randomization. Third, ReP improves the asymptotic efficiency of the estimators from the unadjusted and additive regressions. The corresponding standard regression analyses are accordingly still valid but in general overconservative. As a result, the combination of ReP for treatment allocation and fully interacted regression for analysis ensures both covariate balance and convenient and efficient inference. Importantly, our theory is design-based and holds regardless of how well the models involved in both the rerandomization and analysis stages represent the true data-generating processes.

Keywords: Constrained randomization; design-based inference; design of experiment; Gaussian correlation inequality; potential outcomes; regression adjustment

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1. Introduction

1.1. Rerandomization based on $p$-values

Covariate balance increases comparability of units under different treatment conditions, strengthening the causal conclusions that can be drawn from data. Randomized experiments balance all observed and unobserved covariates on average, providing the gold standard for estimating treatment effects. Chance imbalances nevertheless exist more or less in realized allocations, complicating the interpretation of experimental results. Rerandomization, termed by Cox (1982) and Morgan and Rubin (2012), enforces covariate balance by discarding randomizations that do not satisfy a prespecified balance criterion. Bruhn and McKenzie (2009) conducted a survey of leading experimental researchers in development economics, and suggested that rerandomization is commonly used yet often poorly documented.

To inform readers of the comparability of treatment groups at baseline, modern social and biomedical scientific publications are often encouraged to report baseline covariate balance tables with not only covariate means by treatment group but also the associated $p$-values from significance tests of their differences. The practical need to avoid small $p$-values as indicators of poor balance motivates conducting rerandomization directly based on these $p$-values from balance tests (Bruhn and McKenzie 2009; Ashraf et al. 2010). Formally, rerandomization based on $p$-values (ReP) runs one or more statistical tests to check the covariate balance of a realized randomization, and accepts the allocation if and only if the $p$-values of interest all exceed some prespecified thresholds. In their popular textbook on modern field experiments, Gerber and Green (2012, Chapter 4.5) made this recommendation as a way to “quickly approximate blocking”.

Despite its decade-long existence and close relevance to randomized experiments in practice, the theory of ReP has not been addressed in the literature. Hansen and Bowers (2008) discussed two hypothesis testing-based techniques for balance check in randomized experiments but left the issue of rerandomization and corresponding inference untouched. Gerber and Green (2012) gave the practical recommendation yet did not discuss its theoretical implications. The existing discussion on rerandomization, on the other hand, focused mostly on balance criteria based on the Mahalanobis distance between covariate means by treatment group (Morgan and Rubin 2012, 2015; Branson et al. 2016; Li et al. 2018, 2020; Li and Ding 2020; Branson and Shao 2021; Zhao and Ding 2021, 2023; Johansson et al. 2021; Johansson and Schultzberg 2022). The resulting procedure, though convenient in theory, is in general not a straightforward choice in practice. Another related literature is that on restricted randomization, also known as constrained randomization, which improves
covariate balance by blocking, stratification, matched pairing, covariate-adaptive adjustment, etc. See, e.g., Bailey (1987), Imai et al. (2009), Bruhn and McKenzie (2009), Miratrix et al. (2013), Higgins et al. (2016), Bugni et al. (2018, 2019), Fogarty (2018), Liu and Yang (2020), Wang et al. (2021), Pashley and Miratrix (2021), Bai et al. (2022), and Ye et al. (2022). See also Johansson and Schultzberg (2022) for a discussion on the connection and comparison between stratification and rerandomization. The existing work in this literature however concerns restrictions distinct from those based on covariate balance tests. The gap between theory and practice results in many effectively ReP-based experiments being analyzed as if they were completely randomized, risking overconservative inferences that hinder the detection of statistically significant findings (Bruhn and McKenzie 2009). The goal of this paper is to fill this gap and clarify the theoretical implications of ReP.

1.2. Our contributions

First, we formalize ReP as a powerful tool for improving covariate balance in randomized treatment-control experiments, and propose a variety of potentially useful schemes based on standard statistical tests. The proposed ReP schemes use $p$-values from two-sample $t$-tests, linear regression, and logistic regression to form the covariate balance criteria, respectively, allowing for easy implementation via standard software packages.

Next, we quantify for the first time the impact of the proposed ReP schemes on subsequent inference, filling the gap in the literature. Specifically, we focus on three estimators of the average treatment effect from the ordinary least squares (OLS) fits of the unadjusted, additive, and fully interacted linear regressions of the outcome on treatment, respectively, and evaluate their sampling properties under the proposed ReP schemes from the design-based perspective. In short, the design-based perspective takes the physical act of randomization as the sole source of randomness in evaluating the sampling properties of quantities of interest, yielding results that are robust to model misspecification and unverifiable assumptions (Neyman 1923; Freedman 2008b; Lin 2013; Imbens and Rubin 2015). The main findings are threefold. First, ReP improves the covariate balance between treatment groups, which in turn (i) simplifies the interpretation of experimental results, (ii) reduces the conditional biases of the estimators (c.f. Ding 2021, Section 4.3), and (iii) promotes more coherent inferences between covariate-adjusted and unadjusted analyses (c.f. Zhao and Ding 2023, Section S5.3). Second, the estimator from the fully interacted regression is asymptotically the most efficient under all ReP schemes considered, with the asymptotic sampling distribution unaffected by the rerandomization. It is thus our recommendation for subsequent anal-
ysis under the proposed ReP schemes, allowing for convenient regression-based inferences identical to that under complete randomization. Specifically, we can use the coefficient of the treatment in the OLS fit of the outcome on the treatment, covariates, and their interactions as the point estimator of the average treatment effect, and adopt the associated Eicker–Huber–White (EHW) standard error as a convenient approximation to the true standard error. Third, ReP improves the asymptotic efficiency of the estimators from the unadjusted and additive regressions, rendering inference based on the usual normal approximation overconservative. This highlights the importance of rerandomization-specific inference under ReP when the unadjusted or additive regression is used and, by contrast, demonstrates the advantage of the fully interacted regression for efficient and straightforward inference by normal approximation. We thus recommend the combination of ReP and fully interacted adjustment to ensure both covariate balance and efficient inference.

Lastly, we extend the theory of ReP to experiments with more than two treatment arms and establish analogous results for ReP based on F-tests and multinomial logistic regression.

As a novel technical contribution, we establish the asymptotic equivalence of the likelihood ratio test (LRT) and the Wald test for logistic and multinomial logistic regressions from the design-based perspective, complementing the recent discussions in Freedman (2008b), Hansen and Bowers (2008), and Guo and Basse (2023) on design-based inference from nonlinear regressions. See Theorem S1 in the Supplementary Material. Without invoking any assumption of the logistic or multinomial logistic model, we view logistic and multinomial logistic regressions as purely numeric procedures based on maximum likelihood estimation (MLE), and evaluate the sampling properties of the test statistics over the distribution of the treatment assignments.

1.3. Notation and definitions

For a set of tuples \( \{(u_i, v_i) : u_i \in \mathbb{R}, v_i \in \mathbb{R}^K, i = 1, \ldots, N\} \), denote by \( \text{lm}(u_i \sim v_i) \) the OLS fit of the linear regression of \( u_i \) on \( v_i \), and by \( \text{logit}(u_i \sim v_i) \) the MLE fit of the logistic or multinomial logistic regression of \( u_i \) on \( v_i \). We will clarify the regression specification in the context, and focus on the numeric outputs of OLS and MLE without invoking any of the underlying modeling assumptions. Assume default tests and p-values from standard software packages throughout unless specified otherwise.

For two \( K \times 1 \) vectors \( T = (t_1, \ldots, t_K)^T \) and \( a = (a_1, \ldots, a_K)^T \), denote by \( |T| \leq a \) if \( |t_k| \leq a_k \) for all \( k = 1, \ldots, K \). Let \( \text{diag}(u_k)_{k=1}^K = \text{diag}(u_1, \ldots, u_K) \) denote the \( K \times K \) diagonal matrix with \( u_k \)'s on the diagonal. For a \( K \times K \) symmetric matrix \( V = (V_{kk'})_{k,k'=1,\ldots,K} \) with \( V_{kk} > 0 \) for all \( k = 1, \ldots, K \), let \( \sigma(V) = \text{diag}(V_{kk}^{1/2})_{k=1}^K \) and \( D(V) = \{\sigma(V)\}^{-1}V\{\sigma(V)\}^{-1} \). Intuitively, \( D(V) \) gives
the corresponding correlation matrix when \( V \) is a covariance matrix. Let \( \|e\|_M = e^T \{ \text{cov}(e) \}^{-1} e \) denote the Mahalanobis distance of a random vector \( e \) from the origin. Let \( \sim \) denote convergence in distribution, and let \( E_a \) denote asymptotic expectation.

Lastly, we use the following notion of peakedness \( \text{(Sherman 1955)} \) to quantify the relative efficiency between different estimators \( \text{(Li et al. 2020)} \).

**Definition 1.** For two symmetric random vectors \( A \) and \( B \) in \( \mathbb{R}^K \), we say \( A \) is more peaked than \( B \) if \( \mathbb{P}(A \in C) \geq \mathbb{P}(B \in C) \) for all symmetric convex sets \( C \) in \( \mathbb{R}^K \), denoted by \( A \succeq B \).

For \( K = 1 \), a more peaked random variable has narrower central quantile ranges. For \( A \) and \( B \) with finite second moments, \( A \succeq B \) implies \( \text{cov}(A) \leq \text{cov}(B) \) \( \text{(Li et al. 2020, Proposition 4)} \). For \( A \) and \( B \) that are both normal with zero means, \( A \succeq B \) is equivalent to \( \text{cov}(A) \leq \text{cov}(B) \). This suggests peakedness as a more refined measure for comparing relative efficiency of estimators than the commonly used covariance. We formalize the intuition in Definition 2 below.

**Definition 2.** Assume that \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \) are two consistent estimators for parameter \( \theta \in \mathbb{R}^K \) as the sample size \( N \) tends to infinity, with \( \sqrt{N}(\hat{\theta}_1 - \theta) \sim A_1 \) and \( \sqrt{N}(\hat{\theta}_2 - \theta) \sim A_2 \) for some symmetric random vectors \( A_1 \) and \( A_2 \). We say

(i) \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \) are asymptotically equally efficient if \( A_1 \) and \( A_2 \) have the same distribution, denoted by \( \hat{\theta}_1 \sim \hat{\theta}_2 \);

(ii) \( \hat{\theta}_1 \) is asymptotically more efficient than \( \hat{\theta}_2 \) if \( A_1 \succeq A_2 \), denoted by \( \hat{\theta}_1 \succeq_\infty \hat{\theta}_2 \).

By Definition 2 an asymptotically more efficient estimator has not only a smaller asymptotic covariance but also narrower central quantile ranges.

## 2. Basic setting of the treatment-control experiment

### 2.1. Regression-based inference under complete randomization

Consider an intervention of two levels, indexed by \( q = 0, 1 \), and a finite population of \( N \) units, indexed by \( i = 1, \ldots, N \). Let \( Y_i(q) \in \mathbb{R} \) be the potential outcome of unit \( i \) under treatment level \( q \in \{0, 1\} \) \( \text{(Neyman 1923, Imbens and Rubin 2015)} \). The individual treatment effect is \( \tau_i = Y_i(1) - Y_i(0) \) for unit \( i \), and the finite-population average treatment effect is \( \tau = N^{-1} \sum_{i=1}^N \tau_i = \bar{Y}(1) - \bar{Y}(0) \), where \( \bar{Y}(q) = N^{-1} \sum_{i=1}^N Y_i(q) \).

For some prespecified, fixed integer \( N_1 > 0 \), complete randomization draws completely at random \( N_1 \) units to receive level 1 of the intervention and then assigns the remaining \( N_0 = N - N_1 > 0 \)
units to receive level 0. Let \( e_q = N_q / N \) denote the proportion of units under treatment level \( q \in \{0, 1\} \).

Let \( Z_i \in \{0, 1\} \) indicate the treatment level received by unit \( i \). The observed outcome equals \( Y_i = Z_i Y_i(1) + (1 - Z_i) Y_i(0) \). Let \( \hat{Y}(q) = N_q^{-1} \sum_{i : Z_i = q} Y_i \) denote the average observed outcome under treatment level \( q \in \{0, 1\} \). The difference in means \( \hat{\tau}_N = \hat{Y}(1) - \hat{Y}(0) \) is unbiased for \( \tau \) under complete randomization (Neyman 1923), and can be computed as the coefficient of \( Z_i \) from the simple, unadjusted linear regression \( \text{lm}(Y_i \sim 1 + Z_i) \) over \( i = 1, \ldots, N \).

The presence of covariates promises the opportunity to improve estimation efficiency. Let \( x_i = (x_{i1}, \ldots, x_{ij})^T \) denote the \( J \) pretreatment covariates for unit \( i \), centered at \( \bar{x} = N^{-1} \sum_{i=1}^N x_i = 0_J \). Fisher (1935) suggested an estimator \( \hat{\tau}_f \) for \( \tau \), which equals the coefficient of \( Z_i \) from the additive linear regression \( \text{lm}(Y_i \sim 1 + Z_i + x_i) \) over \( i = 1, \ldots, N \). Freedman (2008a) criticized the possible efficiency loss by \( \hat{\tau}_f \) compared to \( \hat{\tau}_N \). Lin (2013) recommended an improved estimator, denoted by \( \hat{\tau}_l \), as the coefficient of \( Z_i \) from the fully interacted linear regression \( \text{lm}(Y_i \sim 1 + Z_i + x_i + Z_i x_i) \) over \( i = 1, \ldots, N \), and showed its asymptotic efficiency over \( \hat{\tau}_N \) and \( \hat{\tau}_f \). In addition, Lin (2013) also showed that the corresponding EHW standard errors are asymptotically appropriate for estimating the true standard errors of \( \hat{\tau}_f \) and \( \hat{\tau}_l \). This justifies large-sample Wald-type inference of \( \tau \) based on OLS.

We adopt the design-based perspective for all theoretical statements in this article, which views the physical act of randomization, as represented by \((Z_i)_{i=1}^N\), as the sole source of randomness in evaluating the sampling properties of quantities of interest. Accordingly, despite the estimators \( \hat{\tau}_N \), \( \hat{\tau}_f \), and \( \hat{\tau}_l \) are all outputs from the unadjusted, additive, and fully interacted linear regressions, respectively, we invoke no assumption of the underlying linear models but evaluate the sampling properties of \( \hat{\tau}_i \)’s over the distribution of \((Z_i)_{i=1}^N\) conditioning on the potential outcomes and covariates. All theoretical guarantees of the \( \hat{\tau}_i \)’s are therefore design-based and hold even when the linear models are misspecified.

### 2.2. Covariate balance and rerandomization

The regression adjustment by Fisher (1935) and Lin (2013), as it turns out, can be viewed as adjusting for imbalances in covariate means. Let \( \hat{\tau}_x = \hat{x}(1) - \hat{x}(0) \) denote the difference in covariate means between treatment groups, with \( \hat{x}(q) = N_q^{-1} \sum_{i : Z_i = q} x_i \) representing the sample covariate mean under treatment level \( q \in \{0, 1\} \). Let \( \hat{\gamma}_f \) be the coefficient vector of \( x_i \) from the additive regression \( \text{lm}(Y_i \sim 1 + Z_i + x_i) \) over \( i = 1, \ldots, N \), and let \( \hat{\gamma}_l = e_0 \hat{\gamma}_{l,1} + e_1 \hat{\gamma}_{l,0} \), where \( \hat{\gamma}_{l,q} \) denotes the coefficient vector of \( x_i \) from the treatment group-specific regression \( \text{lm}(Y_i \sim 1 + x_i) \) over \( \{i : Z_i = q\} \).
Zhao and Ding (2021, Proposition 1) showed that

\[ \hat{\tau}_* = \hat{\tau}_N - \hat{\tau}_x^T \hat{\gamma}_* \quad \text{for} \quad * = n, f. \]

The two covariate-adjusted estimators are thus variants of \( \hat{\tau}_N \) with corrections for the difference in covariate means. A small \( \hat{\tau}_x \) intuitively promotes more coherent inferences between \( \hat{\tau}_* \) \((* = n, f)\) and the asymptotically most efficient \( \hat{\tau}_L \), thereby enhancing the asymptotic efficiency of \( \hat{\tau}_N \) and \( \hat{\tau}_f \).

Rerandomization, on the other hand, enforces covariate balance in the design stage (Cox 1982; Morgan and Rubin 2012), and accepts an allocation if and only if it satisfies some prespecified covariate balance criterion. Assume complete randomization for the initial allocations throughout this article. Morgan and Rubin (2012) and Li et al. (2018) studied a special type of rerandomization, known as ReM, that uses the Mahalanobis distance of \( \hat{\tau}_x \) as the balance criterion, and accepts a randomization if and only if \( \|\hat{\tau}_x\|_M \leq a_0 \) for some prespecified threshold \( a_0 \). The practical need to avoid small \( p \)-values in baseline covariate balance tables instead motivates ReP that accepts a randomization if and only if the \( p \)-values from relevant balance tests all exceed some prespecified thresholds. To fill the gap in the literature regarding the theoretical properties of ReP, we examine nine hypothesis testing-based covariate balance criteria for conducting ReP under the treatment-control experiment, and quantify their respective impact on subsequent inference from the design-based perspective. We start with three two-sample \( t \)-test-based criteria in Section 3 given their direct connections with the balance tables in practice, and extend the discussion to six regression-based alternatives in Section 4. The results provide the basis for generalizations to experiments with more than two treatment arms, which we formalize in Section 5.

3. ReP based on two-sample \( t \)-tests

3.1. Marginal, joint, and consensus rules

The difference in covariate means provides an intuitive measure of covariate balance under the treatment-control experiment. Depending on whether we examine the \( J \) covariates separately or together, this motivates three two-sample \( t \)-test-based criteria for ReP.

To begin with, recall \( x_{ij} \) as the \( j \)th covariate of unit \( i \). A common approach to balance check is to run one two-sample \( t \)-test for each covariate \( j \in \{1, \ldots, J\} \) based on \( (x_{ij}, Z_i)_{i=1}^N \), and use the resulting two-sided \( p \)-value, denoted by \( p_{j,t} \), to measure its balance between treatment groups. This yields \( J \) marginal \( p \)-values, \( \{p_{j,t} : j = 1, \ldots, J\} \), that occupy the last column of the covariate balance tables. An intuitive, and possibly already widely-used, criterion for ReP is then to accept a
randomization if and only if \( p_{j,t} \geq \alpha_j \) for all \( j = 1, \ldots, J \) for some prespecified thresholds \( \alpha_j \in (0, 1) \) (Bruhn and McKenzie 2009). We call this the marginal rule based on \( J \) marginal tests of individual covariates.

Alternatively, we can test the difference in means of all \( J \) covariates together by a multivariate version of the two-sample \( t \)-test, and accept a randomization if and only if the \( p \)-value from this joint test exceeds some prespecified threshold. Let \( \hat{\Omega} \) be the pooled estimated covariance of \( \hat{\tau}_x \). The two-sample Hotelling’s \( T^2 \) test takes \( W_t = \hat{\tau}_x^T \hat{\Omega}^{-1} \hat{\tau}_x \) as the test statistic, and computes a one-sided \( p \)-value, denoted by \( p_{0,t} \), by comparing \( W_t \) against the Hotelling’s \( T^2 \) distribution. Alternatively, we can replace the Hotelling’s \( T^2 \) distribution with the asymptotically equivalent \( \chi^2 \) distribution and compute the \( p \)-value based on the joint Wald test. A joint rule then accepts a randomization if and only if \( p_{0,t} \geq \alpha_0 \) for some prespecified threshold \( \alpha_0 \in (0, 1) \).

In situations where both marginal and joint balances are desired, we can adopt a consensus rule that accepts a randomization if and only if it is acceptable under both the marginal and joint rules with \( p_{j,t} \geq \alpha_j \) for all \( j = 0, 1, \ldots, J \).

Index by “mg”, “jt”, and “cs” the marginal, joint, and consensus rules, respectively. This defines three ReP schemes by two-sample \( t \)-tests, summarized in Definition 3 below. Of interest are their implications on the subsequent inference based on \( \hat{\tau}_s \) (* = N, F, L). We address this question in Sections 3.2 and 3.3 below.

**Definition 3.** Assume ReP by two-sample \( t \)-tests. Let \( \mathcal{A}_{t,\text{mg}} = \{ p_{j,t} \geq \alpha_j \} \) for all \( j = 1, \ldots, J \}, \mathcal{A}_{t,\text{jt}} = \{ p_{0,t} \geq \alpha_0 \} \}, \text{ and } \mathcal{A}_{t,\text{cs}} = \mathcal{A}_{t,\text{mg}} \cap \mathcal{A}_{t,\text{jt}} = \{ p_{j,t} \geq \alpha_j \} \text{ for all } j = 0, 1, \ldots, J \} \text{ denote the acceptance criteria under the marginal, joint, and consensus rules, respectively.}

### 3.2. Asymptotic theory

We derive in this subsection the asymptotic sampling properties of \( \hat{\tau}_s \) (* = N, F, L) under the three ReP schemes in Definition 3. The results demonstrate the multiple benefits of ReP in strengthening causal conclusions from experimental data, and elucidate the advantage of Lin (2013)’s method for convenient and efficient inference under ReP.

Let \( S^2_x = (N - 1)^{-1} \sum_{i=1}^{N} x_i x_i^T \) denote the finite-population covariance of the centered \( (x_i)_{i=1}^{N} \). Condition [1] below gives the standard regularity conditions for design-based finite-population asymptotic analysis, which embeds the study population into an infinite sequence of finite populations with increasing sizes and defines the asymptotic distribution of any sample quantity as its limiting distribution along this hypothetical sequence. See Li and Ding (2017) for a review. We assume this framework for all asymptotic statements in this article.
Condition 1. As \( N \to \infty \), (i) \( e_q = N_q/N \) has a limit in \((0,1)\) for \( q = 0, 1 \), (ii) the first two finite-population moments of \( \{Y_i(0), Y_i(1), x_i\}_{i=1}^N \) have finite limits; \( S_x^2 \) and its limit are both nonsingular, and (iii) \( N^{-1} \sum_{i=1}^N Y_i^4(q) = O(1) \) for \( q = 0, 1 \); \( N^{-1} \sum_{i=1}^N \|x_i\|^4 = O(1) \).

Let \( \gamma_q \) be the coefficient vector of \( x_i \) from \( \mathbb{1}{\{Y_i(q) \sim 1 + x_i\}} \) over \( i = 1, \ldots, N \). This is a theoretical fit with \( \{Y_i(q)\}_{i=1}^N \) only partially observable depending on the treatment assignment. Condition 1 ensures that \( e_q, \gamma_q, \) and \( S_x^2 \) all have finite limits as \( N \) tends to infinity. For notational simplicity, we will use the same symbols to denote their respective limits when no confusion would arise. Lemma 1 below follows from Zhao and Ding (2021) and specifies the asymptotic distributions of \( \hat{\tau}_s \) \((s = N, F, L)\) under complete randomization. This provides the baseline for evaluating the efficiency gains by ReP.

Lemma 1. Under complete randomization and Condition 1, we have
\[
\sqrt{N} \begin{pmatrix} \hat{\tau}_s - \tau \\ \hat{\tau}_s \end{pmatrix} \sim \mathcal{N} \left( 0_{J+1}, \begin{pmatrix} v_s & c_s^T \\ c_s & v_x \end{pmatrix} \right) \quad (s = N, F, L)
\]
with \( v_x = (e_0 e_1)^{-1} S_x^2 \),
\[
c_N = S_x^2 (e_0^{-1} \gamma_0 + e_1^{-1} \gamma_1), \quad c_F = S_x^2 (e_1^{-1} - e_0^{-1}) (\gamma_1 - \gamma_0), \quad c_L = 0_J,
\]
and \( v_s - v_l = c_s^T v_x^{-1} c_s \geq 0 \) for \( s = N, F, L \). We give the explicit expressions of \( v_s \) \((s = N, F, L)\) in the Supplementary Material.

Recall \( \alpha_j \) \((j = 1, \ldots, J)\) and \( \alpha_0 \) as the thresholds for the marginal and joint rules. Let \( a_0 \) be the \((1 - \alpha_0)\)th quantile of the \( \chi^2_J \) distribution. Let \( a_j \) be the \((1 - \alpha_j/2)\)th quantile of the standard normal distribution, vectorized as \( a = (a_1, \ldots, a_J)^T \). Let \( \epsilon \sim \mathcal{N}(0, 1) \) be a standard normal random variable. Let
\[ \mathcal{L} \sim \epsilon_0 \mid \{\|\epsilon_0\|^2_2 \leq a_0\}, \quad \mathcal{T}_t \sim \epsilon_t \mid \{\|\epsilon_t\| \leq a\}, \quad \mathcal{T}'_t \sim \epsilon_t \mid \{\|\epsilon_t\|_M \leq a_0\} \quad (1) \]
be three truncated normal random vectors independent of \( \epsilon \), with \( \epsilon_0 \sim \mathcal{N}(0_J, I_J) \) and \( \epsilon_t \sim \mathcal{N}\{0_J, D(v_x)\} \). Proposition 1 below gives the asymptotic sampling distributions of \( \hat{\tau}_s \) \((s = N, F, L)\) under the three ReP schemes in Definition 3. For comparison, we also include the results under ReM to highlight the connection (Zhao and Ding 2021). Let \( \mathcal{A}_{\text{rem}} = \{\|\hat{\tau}_s\|_M \leq a_0\} \) denote the acceptance criterion under ReM with threshold \( a_0 \). Let \( \hat{\theta} \mid \mathcal{A} \) represent the distribution of \( \hat{\theta} \) under rerandomization with acceptance criterion \( \mathcal{A} \).
Assume Condition 1. Then

\[
\sqrt{N}(\hat{\tau}_* - \tau) \mid \mathcal{A}_{\text{rem}} \leadsto v_0^{1/2} + c_*\bar{v}_x^{-1/2}L,
\]
\[
\sqrt{N}(\hat{\tau}_* - \tau) \mid \mathcal{A}_{\text{jt}} \leadsto v_0^{1/2} + c_*\bar{v}_x^{-1/2}L,
\]
\[
\sqrt{N}(\hat{\tau}_* - \tau) \mid \mathcal{A}_{\text{mg}} \leadsto v_0^{1/2} + c_*\bar{v}_x^{-1/2}\sigma(v_x)\mathcal{T},
\]
\[
\sqrt{N}(\hat{\tau}_* - \tau) \mid \mathcal{A}_{\text{cs}} \leadsto v_0^{1/2} + c_*\bar{v}_x^{-1/2}\sigma(v_x)\mathcal{T}',
\]

for \( * = N, F \), whereas \( \sqrt{N}(\hat{\tau}_L - \tau) \mid \mathcal{A} \sim \mathcal{N}(0, v_L) \) for all \( \mathcal{A} \in \{\mathcal{A}_{\text{rem}}, \mathcal{A}_{\text{jt}}, \mathcal{A}_{\text{mg}}, \mathcal{A}_{\text{cs}}\} \).

Proposition 1 has two implications. First, all three estimators remain consistent under all four rerandomization schemes, with the joint rule being asymptotically equivalent to ReM. Second, the asymptotic distributions of \( \hat{\tau}_L \) remain the same as that under complete randomization in Lemma 1 whereas those of \( \hat{\tau}_N \) and \( \hat{\tau}_F \) change to convolutions of normal and truncated normal when their respective \( c_* \neq 0, J \). We show in Lemma [S4] in the Supplementary Material that \( L \gtrless \epsilon_0 \) and \( \mathcal{T}, \mathcal{T}' \gtrless \epsilon_t \) by the Gaussian correlation inequality due to Royen [2014]. This provides the basis for quantifying the impact of ReP on the asymptotic efficiency of each \( \hat{\tau}_* \), as well as the asymptotic relative efficiency of \( \hat{\tau}_* \) across \( * = N, F, L \). We formalize the intuition in Theorem 1 below.

\[ \rho(J, a_0) = P(\chi_{J+2} \leq a_0)/P(\chi_J \leq a_0) < 1. \]

Theorem 1. Assume Condition 1. For all \( \mathcal{A} \in \{\mathcal{A}_{\text{rem}}, \mathcal{A}_{\text{jt}}, \mathcal{A}_{\text{mg}}, \mathcal{A}_{\text{cs}}\} \), we have

(i)

\[
(\hat{\tau}_x \mid \mathcal{A}) \succeq_{\infty} \hat{\tau}_x
\]

with

\[
\frac{E_a(\|\hat{\tau}_x\|_2^2 \mid \mathcal{A}_{\text{jt}})}{E_a(\|\hat{\tau}_x\|_2^2)} = \frac{E_a(\|\hat{\tau}_x\|_2^2 \mid \mathcal{A}_{\text{rem}})}{E_a(\|\hat{\tau}_x\|_2^2)} = \rho(J, a_0);
\]

(ii)

\[
(\hat{\tau}_N \mid \mathcal{A}) \succeq_{\infty} \hat{\tau}_N, \quad (\hat{\tau}_F \mid \mathcal{A}) \succeq_{\infty} \hat{\tau}_F, \quad (\hat{\tau}_L \mid \mathcal{A}) \sim \hat{\tau}_L,
\]

\[
(\hat{\tau}_L \mid \mathcal{A}) \succeq_{\infty} (\hat{\tau}_* \mid \mathcal{A}) \quad \text{for} \quad * = N, F,
\]

with \( (\hat{\tau}_N \mid \mathcal{A}) \sim \hat{\tau}_N \sim \hat{\tau}_L \) if and only if \( c_N = 0, J \) and \( (\hat{\tau}_F \mid \mathcal{A}) \sim \hat{\tau}_F \sim \hat{\tau}_L \) if and only if \( c_F = 0, J \);

(iii) for \( * \in \{N, F, L\} \), the asymptotic conditional bias of \( \hat{\tau}_* \) given \( \hat{\tau}_x \) satisfies

\[
\frac{E_a \left[ \frac{E_a(\hat{\tau}_* - \tau \mid \hat{\tau}_x, \mathcal{A})}{E_a(\hat{\tau}_x)} \right]^2}{E_a \left[ \frac{E_a(\hat{\tau}_* - \tau \mid \hat{\tau}_x)^2}{E_a(\hat{\tau}_x)} \right]} \leq 1
\]
particular, we use utility of ReP in reducing conditional biases and improving coherence across \( \hat{\tau} \) Section 3.3. Third, Theorem 1(iii)–(iv) are direct consequences of Theorem 1(i) and illustrate the asymptotic efficiency of \( \hat{\tau} \). Second, Theorem 1(ii) shows the utility of ReP in improving peakedness of \( \hat{\tau} \), and use of \( \hat{\tau} \). These implications together illustrate the value of ReP: despite having no effect on the asymptotic efficiency of \( \hat{\tau}_L \), ReP promotes not only covariate balance between treatment groups but also more coherent inferences across different estimators. We thus recommend the combination of ReP and \( \hat{\tau}_L \) for subsequent inference with details given in Section 3.3. Third, Theorem 1(iii)–(iv) are direct consequences of Theorem 1(i) and illustrate the utility of ReP in reducing conditional biases and improving coherence across \( \hat{\tau}_N \) and \( \hat{\tau}_F \). In particular, we use \( \mathbb{E}_{a}(\hat{\tau}_s - \tau \mid \hat{\tau}_x, A) \) and \( \mathbb{E}_{a}(\hat{\tau}_s - \tau \mid \hat{\tau}_x) \) to measure the asymptotic conditional biases of \( \hat{\tau}_s \) given \( \hat{\tau}_x \) under rerandomization and complete randomization, respectively, in Theorem 1(iii), and use \( \mathbb{E}_{a}(\hat{\tau}_s - \hat{\tau}_{ss}^*)^2 \mid A \) and \( \mathbb{E}_{a}(\hat{\tau}_s - \hat{\tau}_{ss}^* )^2 \) to measure the coherence between estimators \( \hat{\tau}_s \) and \( \hat{\tau}_{ss}^* \) in Theorem 1(iv).

For a random quantity \( \hat{\theta}, (\hat{\theta} \mid A) \Rightarrow_n \hat{\theta} \) implies that rerandomization increases the asymptotic peakedness of \( \hat{\theta} \), whereas \( (\hat{\theta} \mid A) \sim \hat{\theta} \) implies that rerandomization has no effect asymptotically. The implications of Theorem 1 are hence threefold. First, Theorem 1(i) establishes the utility of the two-sample \( t \)-test-based ReP to improve covariate balance in terms of the asymptotic distribution of \( \hat{\tau}_x \). This gives another measure of improved covariate balance in addition to the self-evident improvement in the realized allocation. Second, Theorem 1(ii) shows the utility of ReP in improving the asymptotic efficiency of \( \hat{\tau}_N \) and \( \hat{\tau}_F \), and ensures the asymptotic efficiency of \( \hat{\tau}_L \) over \( \hat{\tau}_N \) and \( \hat{\tau}_F \) under all three ReP schemes with the asymptotic efficiency unaffected by the rerandomization. We thus recommend the combination of ReP and \( \hat{\tau}_L \) for subsequent inference with details given in Section 3.3. Third, Theorem 1(iii)–(iv) are direct consequences of Theorem 1(i) and illustrate the utility of ReP in reducing conditional biases and improving coherence across \( \hat{\tau}_N \) and \( \hat{\tau}_F \). In particular, we use \( \mathbb{E}_{a}(\hat{\tau}_s - \tau \mid \hat{\tau}_x, A) \) and \( \mathbb{E}_{a}(\hat{\tau}_s - \tau \mid \hat{\tau}_x) \) to measure the asymptotic conditional biases of \( \hat{\tau}_s \) given \( \hat{\tau}_x \) under rerandomization and complete randomization, respectively, in Theorem 1(iii), and use \( \mathbb{E}_{a}(\hat{\tau}_s - \hat{\tau}_{ss}^*)^2 \mid A \) and \( \mathbb{E}_{a}(\hat{\tau}_s - \hat{\tau}_{ss}^* )^2 \) to measure the coherence between estimators \( \hat{\tau}_s \) and \( \hat{\tau}_{ss}^* \) in Theorem 1(iv).

These implications together illustrate the value of ReP: despite having no effect on the asymptotic efficiency of \( \hat{\tau}_L \), ReP promotes not only covariate balance between treatment groups but also more coherent inferences across different estimators. We thus recommend the combination of ReP and fully interacted adjustment to ensure both covariate balance and efficient inference. The theory of most previous papers on rerandomization focused on the efficiency gain by ReM analogous to Theorem 1(ii). Here we give more results on the multiple benefits of ReP.

The asymptotic equivalence of \( (\hat{\tau}_L \mid A) \) and \( \hat{\tau}_L \) for \( A \in \{A_{rem}, A_{t, jt}, A_{t, mg}, A_{t, cs}\} \) in Proposition 1 and Theorem 1(ii) is no coincidence but the consequence of \( \hat{\tau}_L \) being asymptotically independent of \( \hat{\tau}_x \) under complete randomization from Lemma 1. Balance criteria based on \( \hat{\tau}_x \) thus have no effect on \( \hat{\tau}_L \) asymptotically, with \( A_{rem}, A_{t, jt}, A_{t, mg}, \) and \( A_{t, cs} \) all being special cases. The same argument
underpins the asymptotic equivalence of \( \hat{\tau}_* \) and \( (\hat{\tau}_* \mid A) \) for \( * \in \{ N, F \} \) when \( c_* = 0_J \) under special configurations of the potential outcomes; an example is that \( c_F = 0_J \) when the individual treatment effects \( \tau_i \) are constant across all \( i \)’s. The resulting \( \hat{\tau}_* \) \( (\equiv N, F) \) is asymptotically identically distributed as \( \hat{\tau}_L \) under complete randomization by Lemma 1 with the asymptotic sampling distribution unaffected by rerandomization.

More generally, the linear projection of \( \hat{\tau}_* \) on \( \hat{\tau}_x \) equals \( \text{proj}(\hat{\tau}_* \mid \hat{\tau}_x) = \tau + c_*^T v_x^{-1} \hat{\tau}_x \) with regard to the asymptotic distribution under complete randomization in Lemma 1, and is asymptotically independent of the corresponding residual, denoted by \( \text{res}(\hat{\tau}_* \mid \hat{\tau}_x) = \hat{\tau}_* - \text{proj}(\hat{\tau}_* \mid \hat{\tau}_x) = \hat{\tau}_* - \tau - c_*^T v_x^{-1} \hat{\tau}_x \). This ensures

\[
\hat{\tau}_* = \text{proj}(\hat{\tau}_* \mid \hat{\tau}_x) + \text{res}(\hat{\tau}_* \mid \hat{\tau}_x) = \tau + c_*^T v_x^{-1} \hat{\tau}_x + \text{res}(\hat{\tau}_* \mid \hat{\tau}_x),
\]

where the residual \( \text{res}(\hat{\tau}_* \mid \hat{\tau}_x) \) satisfies \( \sqrt{N} \text{res}(\hat{\tau}_* \mid \hat{\tau}_x) \sim N(0, v_L) \) and is asymptotically independent of \( \hat{\tau}_x \). Balance criteria based on \( \hat{\tau}_x \) can thus only affect the \( c_*^T v_x^{-1} \hat{\tau}_x \) part in (6) asymptotically, and turn it into a truncated normal with greater peakedness when \( c_* \neq 0_J \). This gives the intuition behind Proposition 1 and Theorem 1.

3.3. Wald-type inference

Proposition 1 and Theorem 1 together establish the asymptotic distributions and relative efficiency of \( \hat{\tau}_* \) \( (\equiv N, F, L) \) under the two-sample \( t \)-test-based ReP. The results provide two guidelines on subsequent Wald-type inference of the average treatment effect.

First, the Wald-type inference based on \( \hat{\tau}_L \) is asymptotically the most efficient and can be conducted using the same normal approximation as under complete randomization. Specifically, let \( \hat{s}_L \) denote the EHW standard error of \( \hat{\tau}_L \) from the same OLS fit. Lin (2013) and Li et al. (2018, Lemma A16) ensured that it is asymptotically appropriate for estimating the true standard error of \( \hat{\tau}_L \) under both complete randomization and the three ReP schemes in Definition 3, justifying the Wald-type inference based on \( (\hat{\tau}_L, \hat{s}_L) \) and normal approximation. This illustrates the advantage of ReP for allowing for convenient regression-assisted analysis by the fully interacted regression.

Second, ReP narrows the asymptotic sampling distributions of \( \hat{\tau}_N \) and \( \hat{\tau}_F \) in general, rendering the usual normal approximation based on \( \hat{\tau}_* \) \( (\equiv N, F) \) and their associated EHW standard errors overconservative for estimating \( \tau \). Rerandomization-specific sampling distributions are thus necessary for better-calibrated inference based on \( \hat{\tau}_* \) \( (\equiv N, F) \). Recall the definition of \( (\hat{\gamma}_{L,0}, \hat{\gamma}_{L,1}) \) from Section 2.2 as the sample analogs of \( (\gamma_0, \gamma_1) \). With \( v_L \) and \( c_* \) being the only unknowns in the asymptotic distributions of \( \hat{\tau}_* \) \( (\equiv N, F) \) in Proposition 1 we can estimate them using \( N\hat{s}_L^2 \) and
the sample analogs \( \hat{c}_S = S_t^2(\epsilon_0^{-1}\hat{c}_{1,0} + \epsilon_1^{-1}\hat{c}_{1,1}) \) and \( \hat{c}_F = S_t^2(\epsilon_0^{-1} - \epsilon_0^{-1})(\hat{c}_{1,1} - \hat{c}_{1,0}) \), respectively, and conduct inference based on the resulting plug-in distributions \( \text{(Li et al., 2018)} \). This modification mitigates the overconservativeness of the Wald-type inference based on \( \hat{\tau}_* \) \( *(= N, F) \) under ReP and, by contrast, illustrates the convenience of \( \text{Lin (2013)} \)'s method for efficient and well-calibrated inference.

4. ReP based on linear and logistic regressions

4.1. Linear and logistic regressions for assessing covariate balance

The two-sample \( t \)-tests measure covariate balance by the difference in covariate means and are numerically equivalent to a component-wise regression of \( x_i \) on \( (1, Z_i) \), assessing how \( x_i \) varies with different values of \( Z_i \). The idea of the propensity score \( \text{(Rosenbaum and Rubin, 1983)} \), on the other hand, motivates an alternative measure of covariate balance by assessing how \( Z_i \) varies with \( x_i \).

Consider the linear regression of \( Z_i \) on \( (1, x_i) \), denoted by \( \text{lm}(Z_i \sim 1 + x_{i1} + \cdots + x_{iJ}) \). Let \( \hat{\beta}_j \) denote the coefficient of the \( j \)th covariate \( x_{ij} \) for \( j = 1, \ldots, J \). The magnitude of \( \hat{\beta}_j \) gives an intuitive measure of the influence of covariate \( j \) on the final assignment, with a well-balanced assignment expected to have all \( \hat{\beta}_j \)'s close to zero; see, e.g., \( \text{de Mel et al. (2009, Table 1)} \) and \( \text{Kuziemko et al. (2015, Table 3)} \) for balance tables based on these regression outputs. This motivates three linear regression-based ReP schemes under the marginal, joint, and consensus rules, respectively.

To begin with, denote by \( p_{j,\text{lm}} \) the \( p \)-value associated with \( \hat{\beta}_j \) from standard software packages. The marginal rule accepts a randomization if and only if \( p_{j,\text{lm}} \geq \alpha_j \) for all \( j = 1, \ldots, J \) for some prespecified thresholds \( \alpha_j \in (0, 1) \).

Alternatively, let \( p_{0,\text{lm}} \) be the \( p \)-value from the \( F \)-test of \( \text{lm}(Z_i \sim 1 + x_{i1} + \cdots + x_{iJ}) \) against the empty model \( \text{lm}(Z_i \sim 1) \). It is a standard output of linear regression by most software packages and provides a summary measure of the magnitudes of all \( \hat{\beta}_j \)'s. The joint rule then accepts a randomization if and only if \( p_{0,\text{lm}} \geq \alpha_0 \) for some prespecified threshold \( \alpha_0 \in (0, 1) \). This is the recommendation by \( \text{Gerber and Green (2012)} \).

The consensus rule, accordingly, accepts a randomization if and only if it is acceptable under both marginal and joint rules with \( p_{j,\text{lm}} \geq \alpha_j \) for all \( j = 0, 1, \ldots, J \). This extends the three two-sample \( t \)-test-based criteria in Definition 3 to the linear regression of \( Z_i \) on \( (1, x_i) \).

One concern with the above approach based on \( \text{lm}(Z_i \sim 1 + x_{i1} + \cdots + x_{iJ}) \) is that linear regression is in general not intended for binary responses like \( Z_i \). An immediate alternative is to consider the logistic regression of \( Z_i \) on \( (1, x_i) \), denoted by \( \text{logit}(Z_i \sim 1 + x_{i1} + \cdots + x_{iJ}) \), instead.
Table 1: Nine ReP schemes under the treatment-control experiment

| Rule          | Model option: $\dagger = t, \text{lm, logit}$ |
|---------------|-----------------------------------------------|
| marginal (mg) | $p_{j,\dagger} \geq \alpha_j$ ($j = 1, \ldots, J$) |
| joint (jt)    | $p_{0,\dagger} \geq \alpha_0$                 |
| consensus (cs)| $p_{j,\dagger} \geq \alpha_j$ ($j = 0, 1, \ldots, J$) |

and form the acceptance criteria based on $p$-values from its MLE fit \cite{Hansen2008}.

Specifically, let $p_{j, \text{logit}}$ be the $p$-value associated with the coefficient of $x_{ij}$ from \text{logit}$($Z_i \sim 1 + x_{i1} + \cdots + x_{ij}$) for $j = 1, \ldots, J$, and let $p_{0, \text{logit}}$ be the $p$-value from the likelihood ratio test (LRT) of \text{logit}$($Z_i \sim 1 + x_{i1} + \cdots + x_{ij}$) against the empty model \text{logit}$($Z_i \sim 1$). They are all standard outputs of logistic regression by most software packages, and allow us to form the marginal, joint, and consensus criteria in identical ways as those based on $\{p_{j,\dagger} : j = 0, 1, \ldots, J\}$ for $\dagger = t, \text{lm}$.

This defines in total nine ReP schemes, as the combinations of three \textit{model options} — the two-sample $t$-tests of $x_i$ (“t”), the linear regression of $Z_i$ on $(1, x_i)$ (“lm”), and the logistic regression of $Z_i$ on $(1, x_i)$ (“logit”) — and the marginal (“mg”), joint (“jt”), and consensus (“cs”) rules, summarized in Table 1. We extend below the results under the two-sample $t$-test-based schemes to the six regression-based variants, respectively.

4.2. Asymptotic theory

We derive in this subsection the asymptotic sampling properties of $\hat{\tau}_s$ ($s = N, F, L$) under the six linear or logistic regression-based ReP schemes in Table 1. Echoing the comments after Theorem 1, the result illustrates the utility of the six regression-based ReP schemes in improving covariate balance, reducing conditional biases, and promoting coherence across different estimators, and establishes the asymptotic efficiency of $\hat{\tau}_L$ under all six criteria with the asymptotic sampling distribution unaffected by ReP.

To this end, we first introduce an additional regularity condition that underpins the design-based properties of logistic regression.

\textbf{Condition 2.} Let

$$\pi(\tilde{x}_i, \theta) = \frac{\exp(\tilde{x}_i^T \theta)}{1 + \exp(\tilde{x}_i^T \theta)} \quad \text{for} \quad \tilde{x}_i = (1, x_i^T)^T \text{ and } \theta \in \mathbb{R}^{J+1}.$$

As $N \to \infty$, $H(\theta) = -N^{-1} \sum_{i=1}^N \pi(\tilde{x}_i, \theta) (1 - \pi(\tilde{x}_i, \theta)) \tilde{x}_i \tilde{x}_i^T$ converges to a negative-definite matrix
H_\infty(\theta) < 0 \text{ for all } \theta \in \mathbb{R}^{J+1}, \text{ and the convergence is uniform on any compact set } \Theta \subset \mathbb{R}^{J+1}.

The H(\theta) in Condition 2 gives the Hessian matrix of the log-likelihood function under the logistic model scaled by N^{-1}. To gain intuition about the uniform convergence in Condition 2, consider a superpopulation working model where the x_i’s are independent and identically distributed with finite second moment. The uniform law of large numbers holds for H(\theta) and ensures that it converges to some H_\infty(\theta) almost surely on \mathbb{R}^{J+1}, with the convergence being uniform on any compact set \Theta \subset \mathbb{R}^{J+1} [Newey and McFadden 1994; Ferguson 1996, Chapter 16]. This suggests the mildness of the requirement on uniform convergence. In addition, we can show that H(\theta) \leq 0 for all \theta \in \mathbb{R}^{J+1}. This suggests the mildness of the requirement on the negative definiteness of H_\infty(\theta).

Let \mathcal{A}_{t,jt} = \{p_{0,t} \geq \alpha_0\}, \mathcal{A}_{t,mg} = \{p_{j,t} \geq \alpha_j \text{ for all } j = 1, \ldots, J\}, \text{ and } \mathcal{A}_{t,cs} = \mathcal{A}_{t,jt} \cap \mathcal{A}_{t,mg} = \{p_{j,t} \geq \alpha_j \text{ for all } j = 0, 1, \ldots, J\} \text{ denote the acceptance criteria under the six regression-based ReP schemes for } t = \text{lm, logit.}

Recall the definitions of v_x, \epsilon, \mathcal{L}, a_0, \text{ and } a = (a_1, \ldots, a_J)^T \text{ from Section 3.2. Let}

\begin{align*}
T_{\text{lm}} &\sim \epsilon_{\text{lm}} \mid \{|\epsilon_{\text{lm}}| \leq a\}, \\
T'_{\text{lm}} &\sim \epsilon_{\text{lm}} \mid \{|\epsilon_{\text{lm}}| \leq a, \|\epsilon_{\text{lm}}\|_M \leq a_0\}
\end{align*}

be two truncated normal random vectors independent of \epsilon, with \epsilon_{\text{lm}} \sim \mathcal{N}\{0, J(D(v_x^{-1}))\}. The Gaussian correlation inequality ensures that T_{\text{lm}}, T'_{\text{lm}} \geq \epsilon_{\text{lm}}; \text{ see Lemma S4 in the Supplementary Material. This underlies the improved asymptotic efficiency of } \hat{\tau}_n \text{ and } \hat{\tau}_f \text{ under regression-based ReP. We state the details in Proposition 2 and Theorem 2 below.}

**Proposition 2.** Assume Condition 1 for \dag = \text{lm} \text{ and Conditions 1,2 for } \dag = \text{logit. For } \dag \in \{\text{lm, logit}\}, \text{ we have}

\begin{align*}
\sqrt{N}(\hat{\tau}_* - \tau) \mid \mathcal{A}_{t,jt} &\rightsquigarrow v_L^{1/2} \epsilon + e_x^T v_x^{-1/2} \mathcal{L}, \\
\sqrt{N}(\hat{\tau}_* - \tau) \mid \mathcal{A}_{t,mg} &\rightsquigarrow v_L^{1/2} \epsilon + e_x^T \sigma(v_x^{-1}) T_{\text{lm}}, \\
\sqrt{N}(\hat{\tau}_* - \tau) \mid \mathcal{A}_{t,cs} &\rightsquigarrow v_L^{1/2} \epsilon + e_x^T \sigma(v_x^{-1}) T'_{\text{lm}},
\end{align*}

for * = n, f, whereas \sqrt{N}(\hat{\tau}_L - \tau) \mid \mathcal{A} \rightsquigarrow \mathcal{N}(0, v_L) \text{ for all } \mathcal{A} \in \{\mathcal{A}_{t,\circ} : \dag = \text{lm, logit}; \circ = jt, mg, cs\}.

**Theorem 2.** Theorem 1 holds for all \mathcal{A} \in \{\mathcal{A}_{t,\circ} : \dag = \text{lm, logit}; \circ = jt, mg, cs\}.

All comments after Proposition 1 and Theorem 1 extend here with no need of modification. Proposition 2 gives the asymptotic sampling distributions of \hat{\tau}_* (* = n, f, L) under the six regression-based ReP schemes, and establishes the asymptotic equivalence of the linear and logistic regression model options under all three rules. As a direct implication of Proposition 2, Theorem 2 highlights
the utility of the regression-based ReP in improving covariate balance, reducing conditional biases, and promoting coherence between adjusted and unadjusted analyses, and ensures the asymptotic efficiency of $\hat{\tau}_L$ under all six criteria. The fully interacted regression is thus our recommendation for subsequent inference under the linear and logistic regression-based ReP as well, with all discussion in Section 3.3 extending here verbatim.

Juxtapose Proposition 2 with Proposition 1. The three joint criteria are asymptotically equivalent to not only each other but also ReM with threshold $a_0$. This is no coincidence but the consequence of the test statistics used by these criteria all being asymptotically equivalent to $\|\hat{\tau}_x\|_M$; see Remark S2 in the Supplementary Material for details. The marginal criteria based on the linear and logistic regressions, on the other hand, differ from that based on the two-sample $t$-tests even asymptotically. The difference is nevertheless immaterial based on simulation evidence.

Echoing the comments at the end of Section 2.1, we view the linear and logistic regressions as purely numeric procedures based on OLS or MLE for computing the $p$-values and estimators, and invoke none of the underlying modeling assumptions in evaluating their outputs. The results in Proposition 2 and Theorem 2 are therefore all design-based, and hold regardless of how well the underlying linear or logistic models represent the true data-generating processes. This concludes our discussion on ReP under the treatment-control experiment. The criteria based on two-sample $t$-tests are arguably the most straightforward, making the discussion on the regression-based variants seem to be of theoretical interest only. The logistic regression nevertheless provides a key stepping stone for extending the current results to experiments with more than two treatment arms. We give the details in the next section.

5. ReP in multi-armed experiments

5.1. Basic setting and covariate balance criteria

Multi-armed experiments enable comparisons of more than two treatment levels simultaneously, and are intrinsic to applications with multiple factors of interest. To conduct rerandomization in such settings, a straightforward option is to check balance for all pairs of treatment arms, and accept a randomization if and only if all pairwise comparisons pass the balance check (Morgan 2011). Depending on the number of treatment arms in question, however, this may result in a large number of pairwise comparisons and become unwieldy in practice. A more practical alternative is to use a test that directly measures the balance across all treatment arms.

To this end, the covariate-wise $F$-test provides a natural way of extending the marginal two-
sample t-test to more than two treatment arms, measuring the balance of individual covariates across all treatment arms simultaneously. See, e.g., de Mel et al. (2013) and Dupas and Robinson (2013) for balance tables based on such F-tests. The multinomial logistic regression, on the other hand, is a straightforward extension of the logistic regression and provides a way to measure both covariate-wise and overall balances across all treatment arms by the idea of the propensity score. See Gerber et al. (2009) for an example of balance check based on the multinomial logistic regression. We formalize below their extensions to ReP.

Consider a multi-armed experiment with \( Q > 2 \) treatment levels, indexed by \( q \in Q = \{1, \ldots, Q\} \), and a study population of \( N \) units, indexed by \( i = 1, \ldots, N \). Renew \( x_i = (x_{i1}, \ldots, x_{ij})^T \) as the centered covariate vector and \( Z_i \in Q = \{1, \ldots, Q\} \) as the initial treatment assignment of unit \( i \). For \( j = 1, \ldots, J \), let \( p_{j,t} \) denote the p-value from the marginal F-test on covariate \( j \) based on \( (x_{ij}, Z_i)_{i=1}^N \). The marginal F-test-based criterion for ReP accepts a randomization if and only if \( p_{j,t} \geq \alpha_j \) for all \( j = 1, \ldots, J \) for some prespecified thresholds \( \alpha_j \in (0, 1) \). Let \( I_{iq} = 1(Z_i = q) \) denote the indicator of treatment level \( q \). The F-test on covariate \( j \) can be implemented via \( \text{lm}(x_{ij} \sim 1 + I_{i1} + \cdots + I_{i,Q-1}) \).

The multinomial logistic regression, on the other hand, accommodates the marginal, joint, and consensus rules for ReP together via one mle fit. Renew \( \logit(Z_i \sim 1 + x_{i1} + \cdots + x_{iJ}) \) as the multinomial logistic regression of \( Z_i \in Q \) on \((1, x_i)\) over \( i = 1, \ldots, N \). Assume without loss of generality level \( Q \) as the reference level. The mle fit of \( \logit(Z_i \sim 1 + x_{i1} + \cdots + x_{iJ}) \) yields one coefficient of \( x_{ij} \) for each non-reference level \( q \in Q_+ = \{1, \ldots, Q - 1\} \), denoted by \( \tilde{\beta}_{qj} \). We use the subscript + to signify quantities associated with the non-reference levels. Let \( p_{qj,\logit} \) be the p-value associated with \( \tilde{\beta}_{qj} \) from standard software packages. The marginal rule accepts a randomization if and only if \( p_{qj,\logit} \geq \alpha_{qj} \) for all \( q \in Q_+ \) and \( j = 1, \ldots, J \) for some prespecified thresholds \( \alpha_{qj} \in (0, 1) \).

Alternatively, let \( p_{0,\logit} \) be the p-value from the LRT of \( \logit(Z_i \sim 1 + x_{i1} + \cdots + x_{iJ}) \) against the empty model \( \logit(Z_i \sim 1) \). It is a standard output of multinomial logistic regression from most software packages and gives a summary measure of the magnitudes of \( \tilde{\beta}_{qj} \)'s as a whole. The joint rule then accepts a randomization if and only if \( p_{0,\logit} \geq \alpha_0 \) for some prespecified threshold \( \alpha_0 \in (0, 1) \). The consensus rule, accordingly, accepts a randomization if and only if it is acceptable under both the marginal and joint rules. This defines three additional criteria for conducting ReP under multi-armed experiments. We summarize the definitions in Table 2.

Other criteria can be formed based on tests for multivariate analysis of variance (Morgan 2011) or linear regression of \( I_{iq} = 1(Z_i = q) \) on \((1, x_i)\). These alternatives nevertheless involve more technical subtleties and can be unwieldy in practice. To save space, we give in this section the
Table 2: Four ReP schemes under multi-armed experiments

| Rule    | $F$-test                  | Multinomial logistic regression |
|---------|---------------------------|--------------------------------|
| marginal| $p_{j,f} \geq \alpha_j$ for all $j = 1, \ldots, J$ | $p_{qj,\text{logit}} \geq \alpha_{qj}$ for all $qj$ |
| joint   | n.a.                      | $p_{0,\text{logit}} \geq \alpha_0$ |
| consensus| n.a.                     | $p_{qj,\text{logit}} \geq \alpha_{qj}$ for all $qj$, $p_{0,\text{logit}} \geq \alpha_0$ |

theory of ReP for multi-armed experiments based on marginal $F$-tests and multinomial logistic regression due to their practical convenience, and relegate details on the alternative criteria to the Supplementary Material.

5.2. Treatment effects and regression estimators

We next define the average treatment effect and regression estimators under multi-armed experiments, extending the notation and definitions in Section 2. Renew $Y_i(q) \in \mathbb{R}$ as the potential outcome of unit $i$ if assigned to treatment level $q \in Q = \{1, \ldots, Q\}$. The observed outcome equals $Y_i = \sum_{q \in Q} I_{iq} Y_i(q)$ with $I_{iq} = 1(Z_i = q)$. Renew $\bar{Y}(q) = N^{-1} \sum_{i=1}^N Y_i(q)$ as the average potential outcome under treatment level $q \in Q$, vectorized as $\bar{Y} = (\bar{Y}(1), \ldots, \bar{Y}(Q))^T \in \mathbb{R}^Q$. The goal is to estimate the finite-population average causal effect

$$\tau = G\bar{Y}$$ (7)

for some prespecified contrast matrix $G$ with rows orthogonal to $1_Q$. The $\tau = \bar{Y}(1) - \bar{Y}(0)$ under the treatment-control experiment is a special case with $\bar{Y} = (\bar{Y}(1), \bar{Y}(0))^T$ and $G = (1, -1)$.

Consider

$$N: \quad \mathbf{1m}(Y_i \sim I_{i1} + \cdots + I_{iQ}),$$
$$F: \quad \mathbf{1m}(Y_i \sim I_{i1} + \cdots + I_{iQ} + x_i),$$
$$L: \quad \mathbf{1m}(Y_i \sim I_{i1} + \cdots + I_{iQ} + I_{i1}x_i + \cdots + I_{iQ}x_i)$$

as the unadjusted, additive, and fully interacted linear regressions of $Y_i$ on $\{I_{iq} : q \in Q\}$ and $x_i$, respectively, indexed by $* = N$ (unadjusted), $F$ (additive), and $L$ (fully interacted). Let

$$\hat{Y}_* = (\hat{Y}_*(1), \ldots, \hat{Y}_*(Q))^T \quad (* = N, F, L)$$
denote the coefficient vectors of \((\mathcal{I}_1, \ldots, \mathcal{I}_Q)^T\) from these three regressions, respectively. They are consistent for estimating \(\bar{Y}\) under complete randomization \([\text{Lu 2016} \text{ Zhao and Ding 2023}]\), and allow us to estimate \(\tau = G \bar{Y}\) by

\[
\hat{\tau}_s = G \hat{Y}_s \quad (s = N, F, L).
\]

Of interest are the validity and relative efficiency of \(\hat{\tau}_s\)'s under ReP. We give the details in the next subsection.

### 5.3. Asymptotic theory

We present in this subsection the asymptotic theory of ReP under multi-armed experiments. Assume throughout that the initial allocation is obtained by complete randomization. The experimenter assigns completely at random \(N_q > 0\) units to level \(q \in Q\) with \(\sum_{q \in Q} N_q = N\), and accepts the allocation if and only if the assignments satisfy the prespecified covariate balance criterion.

#### 5.3.1. Baseline efficiency under complete randomization

Recall the definitions of \(\hat{x}(q) = N_q^{-1} \sum_{i: Z_i = q} x_i\), \(e_q = N_q/N\), \(\gamma_q\), and Condition 1 in Sections 2–3 under the treatment-control experiment. Renew them for multi-armed experiments with \(q \in Q = \{1, \ldots, Q\}\). Let \(\hat{x} = (\hat{x}(1)^T, \ldots, \hat{x}(Q)^T)^T \in \mathbb{R}^{JQ}\) and \(\gamma^F = \sum_{q \in Q} e_q \gamma_q\). Lemma 2 below follows from \([\text{Zhao and Ding 2023}]\) and states the asymptotic distributions of \(\hat{Y}_s\) \((s = N, F, L)\) under complete randomization. The results ensure \(\hat{\tau}_L \geq \hat{\tau}_N, \hat{\tau}_F\) under complete randomization and provide the baseline for evaluating the efficiency gains under ReP.

**Lemma 2.** Under complete randomization and the multi-armed version of Condition 1 we have

\[
\sqrt{N} \left( \begin{pmatrix} \hat{Y}_s - \bar{Y} \\ \hat{x} \end{pmatrix} \right) \rightsquigarrow \mathcal{N} \left\{ \begin{pmatrix} 0_{Q+JQ} \\ V_x \begin{pmatrix} V_x & \Gamma_s V_x \\ V_x \Gamma_s^T & V_x \end{pmatrix} \end{pmatrix} \right\} \quad (s = N, F, L)
\]

with \(V_x = N \text{cov}(\hat{x}) = \{\text{diag}(e_q^{-1})_{q \in Q} - 1_{Q \times Q}\} \otimes S_x^2\),

\[
\Gamma_N = \text{diag}(\gamma^T_q)_{q \in Q}, \quad \Gamma_F = \text{diag}\{(\gamma_q - \gamma^F)\}_{q \in Q}, \quad \Gamma_L = 0_{Q \times JQ},
\]

and \(V_s = V_L + \Gamma_s V_x \Gamma_s^T \geq V_L\) for \(s = N, F, L\). We give the explicit expressions of \(V_s\)'s in the Supplementary Material.
5.3.2. ReP based on the marginal $F$-tests

Let $A_f = \{p_{j,f} \geq \alpha_j \text{ for all } j = 1, \ldots, J\}$ denote the acceptance criterion under ReP based on the marginal $F$-tests. Renew $\epsilon \sim \mathcal{N}(0, I_Q)$ as a $Q \times 1$ standard normal random vector. Let $T_f \sim \epsilon_f | \{i \in Q \mid e_{iq} \leq a_j^2 S_{x,j}^2 \text{ for all } j = 1, \ldots, J\}$ be a truncated normal random vector independent of $\epsilon$, where $\epsilon_f = (\epsilon_{f,q})_{q \in Q, j = 1, \ldots, J} \sim \mathcal{N}(0, JQ, V_{x,j})$, and $a_j^2$ denotes the $(1 - \alpha_j)^{th}$ quantile of the $\chi^2_{Q-1}$ distribution. Lemma S4 in the Supplementary Material ensures $T_f \succeq \epsilon_f$ by the Gaussian correlation inequality.

**Proposition 3.** Assume the multi-armed version of Condition 1. Then

\[
\sqrt{N}(\hat{Y}_* - \bar{Y}) | A_f \xrightarrow{\mathcal{D}} \mathcal{N}(0, V_{f}^{1/2} + \Gamma_f T_f), \quad (* = N, F),
\]

\[
\sqrt{N}(\hat{Y}_l - \bar{Y}) | A_f \xrightarrow{\mathcal{D}} \mathcal{N}(0, V_l).
\]

Renew $\hat{\tau}_x = (G_x \otimes I_J) \hat{x}$, where $G_x$ is some prespecified contrast matrix with rows orthogonal to $1_Q$. It defines a general measure of the difference in $\{\hat{x}(q) : q \in Q\}$, extending $\hat{\tau}_x = \hat{x}(1) - \hat{x}(0)$ under the treatment-control experiment to multi-armed experiments (Zhao and Ding 2023).

**Theorem 3.** Assume Condition 1.

(i) Equations (2)–(3) hold for $\hat{\tau}_x = (G_x \otimes I_J) \hat{x}, \hat{\tau}_s = G \hat{Y}_s$ (* = N, F, L), and $A = A_f$ for arbitrary $G$ and $G_x$. In particular, $(\hat{\tau}_N | A_f) \sim \hat{\tau}_N \sim \hat{\tau}_L$ if $\Gamma_N = 0_{Q \times JQ}$ and $(\hat{\tau}_F | A_f) \sim \hat{\tau}_F \sim \hat{\tau}_L$ if $\Gamma_F = 0_{Q \times JQ}$.

(ii) Analogous to equations (4)–(5), for $* \neq ** \in \{N, F, L\}$, we have

\[
\frac{\mathbb{E}_a \left[ \| \mathbb{E}_a(\hat{\tau}_s - \tau | \hat{\tau}_x, A_f) \|^2 \right]}{\mathbb{E}_a \left[ \| \mathbb{E}_a(\hat{\tau}_s - \tau | \hat{\tau}_x) \|^2 \right]} \leq 1, \quad \frac{\mathbb{E}_a \left\{ \| \hat{\tau}_s - \tau_{**} \|_2^2 | A_f \right\}}{\mathbb{E}_a \left\{ \| \hat{\tau}_s - \tau_{**} \|_2^2 \right\}} \leq 1.
\]

Echoing the comments after Theorems 1 and 2, Theorem 3 illustrates the utility of the marginal $F$-test-based ReP for improving covariate balance, reducing conditional biases, and promoting coherence across different estimators under multi-armed experiments, and ensures the asymptotic efficiency of $\hat{\tau}_L$ with identical asymptotic sampling distribution as under complete randomization. Subsequent inference can thus be conducted based on $\hat{\tau}_L$ and its EHW covariance in full parallel with the discussion in Section 3.3. We relegate the details to the Supplementary Material.
5.3.3. ReP based on the multinomial logistic regression

We now address the ReP schemes based on the multinomial logistic regression of $Z_i$ on $(1, x_i)$. Condition 3 below extends Condition 2 to multi-armed experiments and underpins the design-based properties of multinomial logistic regression.

**Condition 3.** For $\tilde{x}_i = (1, x_i^T)^T$ and $\theta = (\theta_1^T, \ldots, \theta_{J-1}^T)^T \in \mathbb{R}^{(J+1)(Q-1)}$ with $\theta_q \in \mathbb{R}^{J+1}$, let

$$
\pi_q(\tilde{x}_i, \theta) = \frac{\exp(\tilde{x}_i^T \theta_q)}{1 + \sum_{q' \in Q_+} \exp(\tilde{x}_i^T \theta_{q'})} \quad (q \in Q_+),
$$

and let $H(\theta) = (H_{qq'}(\theta))_{q,q' \in Q_+}$ with $H_{qq'}(\theta) = N^{-1} \sum_{i=1}^N \pi_q(\tilde{x}_i, \theta)\{\pi_{q'}(\tilde{x}_i, \theta) - 1(q = q')\} \tilde{x}_i \tilde{x}_i^T$. As $N \to \infty$, $H(\theta)$ converges to a negative-definite matrix $H_\infty(\theta) < 0$ for all $\theta \in \mathbb{R}^{(J+1)(Q-1)}$, and the convergence is uniform on any compact set $\Theta \subset \mathbb{R}^{(J+1)(Q-1)}$.


The same intuition about Condition 2 extends here, with $H(\theta) \preceq 0$ giving the Hessian matrix of the scaled log-likelihood function under the multinomial logistic model.

Let $A_{\text{logit},jt} = \{p_{0,\text{logit}} \geq \alpha_j\}$, $A_{\text{logit},mg} = \{p_{qj,\text{logit}} \geq \alpha_q \}$ for all $qj$, and $A_{\text{logit},cs} = A_{\text{logit},jt} \cap A_{\text{logit},mg} = \{p_{0,\text{logit}} \geq \alpha_0; p_{qj,\text{logit}} \geq \alpha_q \}$ denote the acceptance criteria under the joint, marginal, and consensus rules, respectively.

Let $a_{qj}$ be the $(1 - \alpha_{qj}/2)\text{th}$ quantile of the standard normal distribution for $q \in Q_+$ and $j = 1, \ldots, J$. Without introducing new notation, renew $a_0$ as the $(1 - \alpha_0)\text{th}$ quantile of the $\chi^2_{J(Q-1)}$ distribution, and renew $a = (a_{qj})_{q \in Q_+; j = 1, \ldots, J}$ as the vectorization of $a_{qj}$ in lexicographical order. The definitions of $a_0$ and $a$ reduce to those in [1] with $(\alpha_j, a_j) = (\alpha_1j, a_1j)$ when $Q = 2$.

Recall that $\epsilon \sim \mathcal{N}(0, I_Q)$. Let

$$
\mathcal{L} \sim \epsilon_0 \mid \{|\epsilon_0|^2 \leq a_0\}, \quad \mathcal{T}_{\text{logit}} \sim \epsilon_{\text{logit}} \mid \{|\epsilon_{\text{logit}}| \leq a\}, \quad \mathcal{T}'_{\text{logit}} \sim \epsilon'_{\text{logit}} \mid \{|\epsilon'_{\text{logit}}| \leq a, \|\epsilon'_{\text{logit}}\|_M \leq a_0\}
$$

be three truncated normal random vectors independent of $\epsilon$, with $\epsilon_0 \sim \mathcal{N}(0, I_{J(Q-1)}, I_{J(Q-1)})$ and $\epsilon_{\text{logit}} \sim \mathcal{N}(0, I_{J(Q-1)}, D(V_\Psi))$. We have $\mathcal{L} \geq \epsilon_0$ and $\mathcal{T}_{\text{logit}}, \mathcal{T}'_{\text{logit}} \succeq \epsilon_{\text{logit}}$ by the Gaussian correlation inequality; see Lemma S4 in the Supplementary Material.

Recall the definition of $\Gamma^*_s$ ($s = N, F, L$) from Lemma 2. Let $\Gamma'_s = \Gamma^*_s\{(I_{Q-1} - e_Q^T e_+^T) \otimes I_J\}$ for $s = N, F, L$ with $e_+ = (e_1, \ldots, e_{Q-1})^T$. Let $\Psi = \{(\Phi^{-1} \text{diag}(e_+)) \otimes (S_2^2)^{-1}\}$ with $\text{diag}(e_+) = \text{diag}(e_{qj})_{q \in Q_+}$ and $\Phi = \text{diag}(e_+) - e_+ e_+^T$. We use the subscript + to signify quantities associated with the non-reference levels, and give the intuition behind $\Gamma^*_s$ and $\Psi$ in the Supplementary Material.
Let

\[ V_x = N \text{cov}(\hat{x}_+) = (R^{-1} - 1_{(Q-1) \times (Q-1)}) \otimes S_x^2, \quad V_\Psi = N \text{cov}(\Psi \hat{x}_+) = \Psi V_x, \Psi^T, \]

with \( \hat{x}_+ = (\hat{x}^{(1)}_T, \ldots, \hat{x}^{(Q-1)}_T)^T \) and \( \text{cov}(\cdot) \) denoting covariance under complete randomization.

**Proposition 4.** Assume the multi-armed version of Condition 1 and Condition 3. Then

\[ \sqrt{N}(\hat{Y} - \bar{Y}) | A_{\text{logit},jt} \rightsquigarrow V_1^{1/2} \epsilon + \Gamma' V_1^{1/2} L, \]
\[ \sqrt{N}(\hat{Y} - \bar{Y}) | A_{\text{logit},mg} \rightsquigarrow V_1^{1/2} \epsilon + \Gamma' \Psi^{-1} \sigma(V_\Psi) T_{\text{logit}}, \]
\[ \sqrt{N}(\hat{Y} - \bar{Y}) | A_{\text{logit},cs} \rightsquigarrow V_1^{1/2} \epsilon + \Gamma' \Psi^{-1} \sigma(V_\Psi) T_{\text{logit}}', \]

for \( * = N, F \), whereas \( \sqrt{N}(\hat{Y} - \bar{Y}) | A_{\text{logit},o} \rightsquigarrow \mathcal{N}(0_Q, V_L) \) for \( \diamond = jt, mg, cs. \)

**Theorem 4.** Theorem 3 holds if we replace all \( A_f \) with \( A_{\text{logit},o} \) for \( \diamond = jt, mg, cs. \)

All comments after Theorem 3 extend here after changing the “marginal F-test-based” to “multinomial logistic regression-based”. We omit the details to avoid repetition. Echoing the comments after Theorem 2, the results in Proposition 4 and Theorem 4 are all design-based, and hold regardless of how well the models underlying the multinomial logistic and linear regressions for rerandomization and analysis represent the true data-generating processes. The proof of Proposition 4 further introduces a novel technical result on the asymptotic equivalence of the LRT and the Wald test for logistic and multinomial logistic regressions from the design-based perspective. We relegate the details to Theorem S1 in the Supplementary Material.

### 6. Numerical examples

We now illustrate the finite-sample properties of ReP by simulation. Due to space limitations, we focus on the treatment-control experiment in the main paper, and relegate additional examples based on a real-world multi-armed experiment to the Supplementary Material. The results are coherent with the asymptotic theory in Sections 3–5, featuring considerably improved covariate balances and the overall efficiency of \( \hat{\tau}_L \) over \( \hat{\tau}_N \) and \( \hat{\tau}_F \).

Consider a treatment-control experiment with \( N = 500 \) units, indexed by \( i = 1, \ldots, N \), and treatment arm sizes \((N_0, N_1) = (400, 100)\). For each \( i \), we draw a \( J = 5 \) dimensional covariate vector \( x_i = (x_{i1}, \ldots, x_{i5})^T \) with \( x_{ij} \) as independent Uniform\((-1, 1)\), and generate the potential outcomes as \( Y_i(0) \sim \mathcal{N}(-\sum_{j=1}^5 x_{ij}^3, 0.1^2) \) and \( Y_i(1) \sim \mathcal{N}(\sum_{j=1}^5 x_{ij}^3, 0.4^2) \). We center the \( Y_i(0) \)’s and \( Y_i(1) \)’s
respectively to ensure $\tau = \bar{Y}(1) - \bar{Y}(0) = 0$, and fix $\{Y_i(0), Y_i(1), x_i\}_{i=1}^N$ in the simulation. For each iteration, we draw a random permutation of $N_1$ 1’s and $N_0$ 0’s to obtain the initial allocation under complete randomization.

Figure 1 shows the distributions of $\|\hat{\tau}_x\|_2 = \|\hat{x}(1) - \hat{x}(0)\|_2$, $\hat{\tau}_N - \hat{\tau}_F$, $\hat{\tau}_N - \hat{\tau}_L$, and $\hat{\tau}_F - \hat{\tau}_L$ under complete randomization and the three two-sample t-test-based ReP schemes over 50000 independent initial allocations. The results under complete randomization are summarized over all 50000 allocations, whereas those under ReP are summarized over the subsets of allocations that satisfy the respective balance criteria. We vary the thresholds for the marginal rule from $\alpha_j = 0.15$ to $\alpha_j = 0.5$ for $j = 1, \ldots, J$, and choose $\alpha_0$ accordingly to ensure that the joint rule has approximately the same acceptance rate as the marginal rule. The message is coherent across different rules and thresholds: ReP considerably reduces the difference in covariate means and the differences across different estimators, both in line with the theoretical results in Theorem 1. Compare sub-plots (a) and (b) under different values of $\alpha_j$’s. More stringent thresholds result in greater reduction in the differences when everything else stays the same.

Figure 2 shows the distributions of $\hat{\tau}_s$ ($* = N, F, L$). The message is coherent across different rules and thresholds: ReP considerably improves the efficiency of $\hat{\tau}_N$ and $\hat{\tau}_F$ but leaves that of $\hat{\tau}_L$ unchanged, both in line with Theorem 1. Compare sub-plots (a) and (b) under different values of $\alpha_j$’s. Increasing the thresholds improves the efficiency of $\hat{\tau}_s$ ($* = N, F$) when everything else stays the same.

7. Discussion

ReP provides a powerful tool for improving covariate balance in randomized experiments. We examined thirteen rerandomization schemes under this category and quantified their theoretical properties from the design-based perspective. The theory clarifies three important issues regarding the impact of ReP on subsequent inference. First, ReP improves covariate balance, which in turns reduces conditional biases and ensures more coherent inferences across different estimators. Second, the estimator from the fully interacted regression is asymptotically the most efficient under all schemes examined, with the asymptotic sampling distribution remaining unchanged by ReP. We can thus conduct subsequent inference based on this estimator and its EHW standard error or covariance via identical procedure as that under complete randomization. Third, ReP improves the efficiency of the estimators from the unadjusted and additive regressions asymptotically, necessitating rerandomization-specific inference to avoid overconservativeness. These results illustrate the value of ReP for strengthening the causal conclusions that can be drawn from experimental data,
Figure 1: Distributions of $\|\hat{\tau}_x\|_2 = \|\hat{x}(1) - \hat{x}(0)\|_2$, $\hat{\tau}_N - \hat{\tau}_F$, $\hat{\tau}_N - \hat{\tau}_L$, and $\hat{\tau}_F - \hat{\tau}_L$ under complete randomization and the three two-sample $t$-test-based ReP schemes over 50000 independent initial allocations. The results under complete randomization, labeled as “CRE”, are summarized over all 50000 allocations, whereas those under ReP, labeled as “t_marginal”, “t_joint”, and “t_consensus”, respectively, are summarized over the subsets of allocations that satisfy the respective balance criteria. The vertical lines correspond to the 0.025 and 0.975 empirical quantiles, respectively.

(a) $\alpha_j = 0.15$ for $j = 1, \ldots, J$, and $\alpha_0 = 0.55$.

(b) $\alpha_j = 0.50$ for $j = 1, \ldots, J$, and $\alpha_0 = 0.95$. 
Figure 2: Distributions of $\hat{\tau}_* (\ast = N, F, L)$ under complete randomization and the three two-sample $t$-test-based ReP schemes over 50000 independent initial allocations. The results under complete randomization, labeled as “CRE”, are summarized over all 50000 allocations, whereas those under ReP, labeled as “t_marginal”, “t_joint”, and “t_consensus”, respectively, are summarized over the subsets of allocations that satisfy the respective balance criteria. The true $\tau$ is 0. The vertical lines correspond to the 0.025 and 0.975 empirical quantiles, respectively.

(a) $\alpha_j = 0.15$ for $j = 1, \ldots, J$, and $\alpha_0 = 0.55$.

(b) $\alpha_j = 0.50$ for $j = 1, \ldots, J$, and $\alpha_0 = 0.95$. 
and highlight the value of the fully interacted adjustment for efficient and well-calibrated inference under ReP.

We focused on the thirteen criteria in Tables 1 and 2 because of their already widespread use for balance checks in practice. The variety of other test options for balance check promises a whole spectrum of alternative schemes for conducting ReP, catering to the needs of different studies. We give the details in the Supplementary Material.

We focused on asymptotic results in establishing the theoretical properties of ReP. Finite-sample exact unbiasedness is in general lost after rerandomization or regression adjustment. An exception is that the unadjusted difference in means remains unbiased under ReP when the treatment groups are of equal sizes; see Morgan and Rubin (2012). Alternatively, the Fisher randomization test provides a way to conduct finite-sample exact inference under ReP. Zhao and Ding (2021) established the theory of the Fisher randomization test under ReM. All results therein extend to ReP with minimal modifications.

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Supplementary Material

Section S1 gives the additional results for ReP in multi-armed experiments.
Section S2 presents extensions to alternative covariate balance criteria for ReP.
Section S3 reviews the test statistics that underlie the p-values we use to form ReP.
Section S4 states the key lemmas for proving the results in the main paper. In particular, Theorem S1 is a novel technical result, and formalizes the design-based properties of the mle outputs from logistic and multinomial logistic regressions. The result establishes the asymptotic equivalence of the lrt and the Wald test for logistic and multinomial logistic regressions from the design-based perspective.
Section S5 gives the proofs of the results in the main paper.
Section S6 gives the proof of Theorem S1.

Assume centered covariates with \( \bar{x} = N^{-1} \sum_{i=1}^{N} x_i = 0 \) throughout to simplify the presentation. For two sequences of random vectors \( \{A_N\}_{N=1}^{\infty} \) and \( \{B_N\}_{N=1}^{\infty} \) with \( A_N \rightsquigarrow A \) and \( B_N \rightsquigarrow B \) in \( \mathbb{R}^m \), write \( A_N \succeq \infty B_N \) if \( A \succeq B \), and write \( A_N \sim B_N \) if \( A \) and \( B \) have the same distribution. Definition 2 in the main paper is a special case of this definition of \( \succeq \infty \) and \( \sim \), with \( \hat{\theta}_1 \sim \hat{\theta}_2 \) and \( \hat{\theta}_1 \succeq \infty \hat{\theta}_2 \) being abbreviations of \( \sqrt{N}(\hat{\theta}_1 - \theta) \sim \sqrt{N}(\hat{\theta}_2 - \theta) \) and \( \sqrt{N}(\hat{\theta}_1 - \theta) \succeq \infty \sqrt{N}(\hat{\theta}_2 - \theta) \), respectively, when the meaning of \( \theta \) is clear from the context.

S1. Additional results for ReP with multiple arms

S1.1. Simulation based on a multi-armed field experiment

We now illustrate our theory for ReP in general experiments using data from a multi-armed experiment studying the effect of public recognition on giving (Karlan and McConnell 2014).

Karlan and McConnell (2014) conducted a field experiment in collaboration with Dwight Hall, a service club at Yale University, as part of their annual fundraising campaign from October 2007 to May 2008. The sample frame consisted of 4,168 alumni in Dwight Hall’s prior-donor database who had a valid phone number and had not already made a donation between January 2007 and October 2007. Campaign staffs made calls to these potential donors and randomly assigned them to \( Q = 4 \) treatment scripts with equal probability:

1. Control: We are hoping you will continue your support to Dwight Hall with a gift of $100.
2. 100 circle: We are hoping you will continue your support to Dwight Hall with a gift of $100. With a donation of at least $100, you will become a member of our Friend donor circle. Friends
will be listed by name in the Dwight Hall Fall 2008 newsletter.

(3) 500 circle: We are hoping you will continue your support to Dwight Hall with a gift of $100. With a donation of at least $500, you will become a member of our Benefactor donor circle. Benefactors will be listed by name in the Dwight Hall Fall 2008 newsletter.

(4) 100 circle and 500 circle: We are hoping you will continue your support to Dwight Hall with a gift of $100. With a donation of at least $100, you will become a member of our Friends donor circle. With a donation of at least $500, you will become a member of our Benefactor donor circle. Both Friends and Benefactors will be listed by name in the Dwight Hall Fall 2008 newsletter.

The three active treatment scripts, indexed by \( q = 2, 3, 4 \), contain information on the donor circles, whereas the control treatment script, indexed by \( q = 1 \), does not.

One outcome of interest is the probability of a gift large enough to publish in the Friends circle, represented by \( Y_i = 1 \) (unit \( i \) made a donation of at least $100). For arbitrary \( \tau = G\bar{Y} \) of interest, we can compute \( \hat{Y}_* \) \((* = N,F,L) \) by the linear regressions in Section 5.2 of the main paper, and form the point estimators as \( \hat{\tau}_* = G\hat{Y}_* \). For concreteness, we consider the inference of \( \tau = 3^{-1}\{\bar{Y}(2) + \bar{Y}(3) + \bar{Y}(4)\} - \bar{Y}(1) \), as the average treatment effect of the three active treatment scripts relative to the control. The results for other treatment effects are similar and thus omitted.

We include \( J = 7 \) pretreatment covariates: marital status, gender, age, average prior gift, and indicators of whether the largest last gift falls between \( I \) for \( I = (0, 100), [100, 500), \) and \( [500, +\infty) \), respectively. We exclude units with incomplete covariate information, resulting in \( N = 2,298 \) units in \( Q = 4 \) treatment groups with \( (N_1, N_2, N_3, N_4) = (526, 610, 584, 578) \). The sampling distributions of \( \hat{\tau}_* \) \((* = N,F,L) \) depend on all potential outcomes including the unobserved ones. To make the simulation more realistic, we impute the unobserved potential outcomes by simple model fitting. Specifically, we fit a logistic regression of the observed outcome \( Y_i \) on the indicators of the treatment levels, all covariates, and their interactions, and then impute the unobserved potential outcomes, namely \( Y_i(q) \) for \( q \in \{1, 2, 3, 4\} \backslash \{Z_i\} \), based on the fitted probability of \( Y_i(q) = 1 \) with a cutoff at 0.5. For the simulated data set, the true value of \( \tau \) equals 0.0203.

Figure S1 shows the distributions of \( \hat{\tau}_* \) \((* = N,F,L) \) under complete randomization and the four ReP schemes from Table 2 over 50000 independent initial allocations. We set \( \alpha_{qj} = 0.2 \) for the marginal rule based on the multinomial logistic regression, \( \alpha_0 = 0.96 \) for the joint rule based on the multinomial logistic regression, and \( \alpha_j = 0.55 \) for the marginal rule based on the \( F \)-tests to ensure that the resulting ReP procedures have approximately the same acceptance rate.
Figure S1: Distributions of $\hat{\tau}_n$ ($\ast = N, \ F, \ L$) under complete randomization and the four ReP schemes in Table 2 over 50000 independent initial allocations. The results under complete randomization, labeled as CRE, are summarized over all 50000 allocations, whereas those under ReP, labeled as logit marginal, logit joint, logit consensus, and F mg, respectively, are summarized over the subsets of allocations that satisfy the respective balance criteria. The true $\tau$ is 0.0203. The vertical lines correspond to the 0.025 and 0.975 empirical quantiles, respectively.

The results are coherent with the asymptotic theory in Theorems 3 and 4 and illustrate the duality between ReP and regression adjustment in improving the efficiency of $\hat{\tau}_N$. The similarity between $\hat{\tau}_F$ and $\hat{\tau}_L$, on the other hand, is likely the result of the $\gamma_q$'s being relatively homogeneous across treatment groups such that $\Gamma_F \approx 0$. This is not surprising given the binary nature of the outcome, which inevitably restricts the variability in the $\gamma_q$'s. We have observed substantial efficiency gain of $\hat{\tau}_L$ over $\hat{\tau}_F$ in simulation with heterogeneous $\gamma_q$'s but omit the details to save space. Importantly, recall that $\hat{\tau}_n$ ($\ast = N, F, L$) are based on outputs from three linear regressions that are in general considered suboptimal for fitting binary outcomes. The results here highlight the validity of the proposed inferential procedure regardless of how well the linear models involved approximate the true data-generating process.
S1.2. Wald-type inference

Recall from Theorems 3 and 4 that $\hat{\tau}_l$ is asymptotically the most efficient under ReP with multiple arms, with $\sqrt{N}(\hat{Y}_l - \bar{Y}) \mid A \sim \mathcal{N}(0, V_l)$ for all $A \in \{A_f, A_{\text{logit}}, \cdot \cdot = \text{jt, mg, cs}\}$. This suggests subsequent inference based on $\hat{\tau}_l$ and its ehw covariance by normal approximation.

Specifically, let $\hat{V}'_l$ be the ehw covariance of $\hat{Y}_l$ from the same OLS fit. Zhao and Ding (2023) showed that it is asymptotically appropriate for estimating the true sampling covariance under complete randomization; see Lemma S2 in Section S4.1. The same reasoning as in Li et al. (2018, Lemma A16) ensures that the asymptotic appropriateness extends to ReP based on the marginal $F$-tests as well. This, together with the asymptotic normality of $\hat{\tau}_l$ from Lemma 2 and Proposition 3 in the main paper, justifies the Wald-type inference of $\tau$ based on $(\hat{\tau}_l, G\hat{V}'_l G^T)$ under both complete randomization and ReP. The Fisher randomization test can be conducted similarly using $\hat{\tau}^*_l (G\hat{V}'_l G^T)^{-1}\hat{\tau}_l$ as the test statistic for both the strong and weak null hypotheses (Wu and Ding 2020). This illustrates the convenience of the fully interacted regression for inference under general experiments.

The asymptotic sampling distribution of $\hat{\tau}_* (\cdot = N, F)$, on the other hand, is altered by ReP into a convolution of independent normal and truncated normal when $\Gamma_* \neq 0$, resulting in greater peakedness than that under complete randomization. Inference based on the usual normal approximation, as a result, is overly conservative, deterring statistically significant findings. This, again, illustrates the value of the fully interacted regression for efficient and well-calibrated inference under ReP for general experiments.

S1.3. Extensions of the joint $t$-test and linear regression

The marginal $F$-tests give an immediate extension of the marginal two-sample $t$-tests to more than two treatment arms. The range of tests commonly used in multivariate analysis of variance, on the other hand, provide reasonable substitutes to the Hotelling’s $T^2$ test under the joint rule (Morgan 2011). Common choices of test statistics include Wilks’ $\Lambda$, the Lawley–Hotelling trace, the Pillai–Bartlett trace, and Roy’s largest root. One complication is that the distributions of these test statistics are not well studied under the design-based inference. Morgan (2011) recommended using the Fisher randomization test to generate an empirical distribution for the test statistic of choice. This is sound in theory but can become unwieldy in practice.

Generalization of the linear regression-based criteria can be accomplished by a dichotomization of the treatment assignment variable. Recall that $I_{iq} = 1(Z_i = q)$ denotes the indicator for assignment to treatment arm $q \in Q = \{1, \ldots, Q\}$ in a general experiment. We can conduct balance
checks based on $Q$ separate linear regressions as $\text{lm}(I_q \sim 1 + x_{i1} + \cdots + x_{ij})$ over $i = 1, \ldots, N$ for $q = 1, \ldots, Q$. The process yields one joint and $J$ marginal $p$-values for each $q \in Q$, measuring the influence of $x_{ij}$’s on assignment to treatment level $q$. We can form acceptance criteria accordingly based on whether some or all of them exceed some prespecified thresholds. Despite the conceptual straightforwardness of this approach, however, it requires additional data transformations, and gives only measures of covariate balance for the $Q$ treatment levels separately.

S1.4. Covariate-wise and treatment-wise $p$-values

The multinomial logistic regression of $Z_i$ on $(1, x_i)$ provides a powerful tool for conducting ReP under general experiments. We formed the marginal rule based on $(p_{qj, \text{logit}})_{q \in Q_+; j=1,\ldots,J}$, with one $p$-value for each estimated coefficient $\tilde{\beta}_{qj}$, corresponding to the correlation between covariate $j$ and assignment to treatment level $q \in Q_+$. Alternatively, most standard software packages also allow us to test the overall effect of a given covariate $j \in \{1, \ldots, J\}$ across all treatment levels; see, e.g., the `test` command in stata. Denote by $p_{\cdot, \text{logit}}$ the resulting $p$-value associated with the overall effect of covariate $j$. We can also form the balance criterion using the $p_{\cdot, \text{logit}}$’s, and accept a randomization if and only if $p_{\cdot, \text{logit}} \geq \alpha_j$ for all $j = 1, \ldots, J$. The resulting criterion parallels the marginal rules under the treatment-control experiment, and yields analogous results with $\hat{\tau}_l$ being our recommendation.

More generally, we can arrange the $\tilde{\beta}_{qj}$’s into a matrix, with rows corresponding to the $Q - 1$ non-reference treatment levels and columns corresponding to the $J$ covariates:

| Non-reference treatment level | Covariate | Treatment-wise |
|------------------------------|-----------|----------------|
| 1                            | ... $j$   | ... $J$        |
| $\tilde{\beta}_{11}$        | ... $\tilde{\beta}_{1j}$ | $\tilde{\beta}_{1J}$ |
| ...                          | ...       | ...            |
| $\tilde{\beta}_{q1}$        | ... $\tilde{\beta}_{qj}$ | $\tilde{\beta}_{qJ}$ |
| ...                          | ...       | ...            |
| $\tilde{\beta}_{Q-1,1}$     | ... $\tilde{\beta}_{Q-1,j}$ | $\tilde{\beta}_{Q-1,J}$ |
| Covariate-wise               | $p_{1, \text{logit}}$ | ... $p_{J, \text{logit}}$ |
| $p_{0, \text{logit}}$       |           |                |

The $p_{qj, \text{logit}}$’s, $p_{j, \text{logit}}$’s, and $p_{0, \text{logit}}$ then correspond to the cells, columns, and the entire matrix, respectively, measuring the deviations of the corresponding $\tilde{\beta}_{qj}$’s from 0.

By symmetry, we can also conduct one Wald test for each row of the matrix, $\tilde{\beta}_q = (\tilde{\beta}_{q1}, \ldots, \tilde{\beta}_{qJ})^T$, and accept a randomization if and only if the resulting treatment-wise $p$-values, denoted by $p_{q, \text{logit}}$.
for \( q \in Q_+ \), satisfy some prespecified criterion. The magnitude of \( p_{q, \text{logit}} \) intuitively reflects the covariate balance between the treatment level \( q \in Q_+ \) and the reference level \( Q \). The acceptance rule based on \( \{ p_{q, \text{logit}} : q \in Q_+ \} \) hence involves \( Q - 1 \) pairwise comparisons of the non-reference levels to the reference level, simplifying the approach that consults all pairwise comparisons.

This yields four types of \( p \)-values, namely \( p_{qj, \text{logit}}, p_{j, \text{logit}}, p_{q, \text{logit}}, \) and \( p_{0, \text{logit}}, \) summarized in the first row of Table S1. They measure the covariate balance at the treatment-covariate pair, covariate, treatment, and overall levels, respectively, and allow us to form balance criteria like (i) the marginal \( p \)-value \( p_{qj, \text{logit}} \), (ii) the treatment-wise \( p \)-value \( p_{j, \text{logit}} \), (iii) the two-sample \( t \)-test of \( \{ x_{ij} : Z_i = q \} \) and \( \{ x_{ij} : Z_i = Q \} \) for each pair of \( (q, j) \in Q_+ \times \{ 1, \ldots, J \} \), respectively.

The same discussion extends to the \( p \)-values from (i) the treatment-wise regressions \( \mathbf{1}_i(\mathcal{I}_{iq} \sim 1 + x_{i1} + \cdots + x_{ij}) \) over \( i = 1, \ldots, N \) for \( q \in Q \); (ii) the covariate-wise regressions \( \mathbf{1}_i(x_{ij} \sim 1 + \mathcal{I}_{i1} + \cdots + \mathcal{I}_{i,Q-1}) \) over \( i = 1, \ldots, N \) for \( j \in \{ 1, \ldots, J \} \); and (iii) the two-sample \( t \)-test of \( \{ x_{ij} : Z_i = q \} \) and \( \{ x_{ij} : Z_i = Q \} \) for each pair of \( (q, j) \in Q_+ \times \{ 1, \ldots, J \} \), respectively.

Specifically, recall that the treatment-wise regression \( \mathbf{1}_i(\mathcal{I}_{iq} \sim 1 + x_{i1} + \cdots + x_{ij}) \) extends the linear regression of \( Z_i \) on \( (1, x_i) \) under the treatment-control experiment to general experiments, measuring the influence of covariates on assignment to treatment level \( q \in Q \). Denote by \( \hat{\beta}_q = (\hat{\beta}_{q1}, \ldots, \hat{\beta}_{qJ})^T \) the coefficient vector of \( (x_{i1}, \ldots, x_{ij})^T \) from the OLS fit. It yields two types of \( p \)-values, namely

(i) the marginal \( p \)-value associated with each individual \( \hat{\beta}_{qj} \), denoted by \( p_{qj, \text{lm}} \); and
(ii) the treatment-wise \( p \)-value from the \( F \)-test of \( \mathbf{1}_i(\mathcal{I}_{iq} \sim 1 + x_{i1} + \cdots + x_{ij}) \) against \( \mathbf{1}_i(\mathcal{I}_{iq} \sim 1) \), denoted by \( p_{q, \text{lm}} \).

They are analogous to the \( p_{qj, \text{logit}} \)’s and \( p_{q, \text{logit}} \)’s from the multinomial logistic regression, respectively, and allow us to form balance criteria like (i) \( p_{qj, \text{lm}} \geq \alpha_{qj} \) for all \( q, j \), or (ii) \( p_{q, \text{lm}} \geq \alpha_q \) for all \( q \in Q \). See the second row of Table S1.

Next, recall \( \mathbf{1}_i(x_{ij} \sim 1 + \mathcal{I}_{i1} + \cdots + \mathcal{I}_{i,Q-1}) \) as a regression formulation for computing the \( p_{j,t} \) from the marginal \( F \)-test of \( (x_{ij}, Z_i)_{i=1}^N \). The resulting fit, in addition to yielding \( p_{j,t} \) as the \( p \)-value from
Table S1: Four strategies for forming the treatment-covariate-wise, covariate-wise, treatment-wise, and joint $p$-values under general experiments. Let $J = \{1, \ldots, J\}$

|                         | Treatment-covariate-wise $(q,j)$ | Covariate-wise $(j)$ | Treatment-wise $(q)$ | Joint |
|-------------------------|----------------------------------|----------------------|----------------------|-------|
| $\logit(Z_i \sim 1 + x_{i1} + \cdots + x_{ij})$ over $i = 1, \ldots, N$ | $p_{qj,\text{logit}}$ | $p_{j,\text{logit}}$ | $p_{q,\text{logit}}$ | $p_{0,\text{logit}}$ |
| $\text{lm}(I_{iq} \sim 1 + x_{i1} + \cdots + x_{ij})$ over $i = 1, \ldots, N$ for $q \in Q$ | $p_{qj,\text{lm}}$ | n.a. | $p_{q,\text{lm}}$ | n.a. |
| $\text{lm}(x_{ij} \sim 1 + I_{i1} + \cdots + I_{iQ-1})$ over $i = 1, \ldots, N$ for $j \in J$ | $p_{qj,\text{f}}$ | $p_{j,\text{f}}$ | n.a. | n.a. |
| Two-sample $t$-test of $\{x_{ij} : Z_i = q\}$ and $\{x_{ij} : Z_i = Q\}$ for $j \in J$ and $q \in Q_+$ | $p_{qj,\text{t}}$ | n.a. | n.a. | n.a. |

the $F$-test of $\text{lm}(x_{ij} \sim 1 + I_{i1} + \cdots + I_{i,Q-1})$ against $\text{lm}(x_{ij} \sim 1)$, also yields one marginal $p$-value for the coefficient of each $I_{iq}$ ($q \in Q_+$), denoted by $p_{qj,\text{f}}$. The two types of $p$-values are analogous to the $p_{qj,\text{logit}}$’s and $p_{j,\text{logit}}$’s, respectively, and allow us to form balance criteria accordingly. The $p_{j,\text{f}}$’s under the treatment-control experiment are a special case with $Q = 2$ and $I_{i1} = Z_i$. See also Barrera-Osorio et al. (2011, Table 2) and Dupas and Robinson (2013, Tables 1 and A1) for applications that use the $p_{qj,\text{f}}$’s for balance check.

Last but not least, we can conduct one two-sample $t$-test of $\{x_{ij} : Z_i = q\}$ and $\{x_{ij} : Z_i = Q\}$ for each pair of $(q,j) \in Q_+ \times \{1, \ldots, J\}$, comparing the balance of covariate $j$ between treatment groups $q \in Q_+$ and $Q$. Denote by $p_{qj,\text{t}}$ the resulting two-sided $p$-value. It can be implemented by fitting $\text{lm}(x_{ij} \sim 1 + I_{iq})$ over $\{i : Z_i \in \{q, Q\}\}$, and allows us to form balance criteria like $p_{qj,\text{t}} \geq \alpha_{qj}$ for all $qj$ for some prespecified thresholds $\alpha_{qj} \in (0,1)$.

This yields four strategies for forming treatment-covariate-wise, covariate-wise, treatment-wise, and joint $p$-values under general experiments, summarized in Table S1. The multinomial logistic regression accommodates all four types of $p$-values via one MLE fit, and is hence our recommendation in general.

S2. Alternative covariate balance criteria

S2.1. Rerandomization with tiers of covariates

When covariates vary in a priori importance, Morgan and Rubin (2015) proposed rerandomizing based on Mahalanobis distance within tiers of covariate importance, imposing more stringent cri-
criteria for covariates that are thought to be more important. Extension of ReP to such settings is straightforward under the marginal rules by setting the covariate-wise thresholds according to the importance. To construct joint rules for ReP with tiers of covariates, we can conduct one joint test for each tier of covariates, and set the tier-wise thresholds according to the importance. The consensus rules then follow as the intersections of the corresponding marginal and joint rules. These ReP schemes complicate the asymptotic distributions of $\hat{\tau}_\ast (\ast = N,F)$ but keep that of $\hat{\tau}_L$ unchanged. We recommend the same analysis based on $\hat{\tau}_L$.

S2.2. Alternative joint tests from regression

The discussion so far assumed default tests from standard software packages. As a result, we conducted an $F$-test to construct the joint criterion under the linear regression model option, namely $\mathcal{A}_{lm,jt} = \{p_{0,lm} \geq \alpha_0\}$, and conducted an LRT to construct the joint criterion under the logistic and multinomial logistic regression model options, namely $\mathcal{A}_{logit,jt} = \{p_{0,logit} \geq \alpha_0\}$, respectively. The Wald test, on the other hand, enables definition of the joint criteria in a unified way. Specifically, recall $\ln(Z_i \sim 1 + x_i)$ and $\logit(Z_i \sim 1 + x_i)$ as the linear and logistic regressions under the treatment-control experiment. Let $\hat{\beta} = (\hat{\beta}_1, \ldots, \hat{\beta}_J)^T$ and $\hat{\beta} = (\hat{\beta}_1, \ldots, \hat{\beta}_J)^T$ be the coefficient vectors of $x_i$ from the OLS and MLE fits, respectively, with $\hat{V}$ and $\tilde{V}$ as the corresponding estimated covariances. The Wald tests of $\hat{\beta}$ and $\tilde{\beta}$ compare $W_{lm} = \hat{\beta}^T \hat{V}^{-1} \hat{\beta}$ and $W_{logit} = \tilde{\beta}^T \tilde{V}^{-1} \tilde{\beta}$ against the $\chi^2_J$ distribution, and define two alternatives to the $F$-test and LRT, respectively, measuring the magnitudes of $(\hat{\beta}_j)_{j=1}^J$ and $(\tilde{\beta}_j)_{j=1}^J$ as a whole. All results in Proposition 2 and Theorem 2 on $\mathcal{A}_{lm,jt}$ and $\mathcal{A}_{logit,jt}$ extend verbatim to the resulting ReP schemes, with $\hat{\tau}_L$ being our recommendation.

Likewise for all results in Proposition 4 and Theorem 4 on $\mathcal{A}_{logit,jt}$ to extend verbatim to the ReP based on the Wald test of $\hat{\beta} = (\hat{\beta}_q)_{q \in \mathcal{Q}_+; j=1,\ldots,J}$ under the general experiment, with $\hat{\tau}_L$ being our recommendation.

See Lemmas S1, S6, S9 and Theorem S1 in Section S4 for the proof of the asymptotic equivalence of the Wald test to the $F$-test and LRT, respectively. See also de Mel et al. (2009, Table 1) for an application of the Wald test to balance check.

S2.3. EHW standard errors for balance test

We assumed the default $p$-values from standard software packages for forming the balance criteria. The standard error and covariance involved in their computation are hence in general the classic standard errors and covariances derived under homoskedasticity. Alternatively, we can form the
test statistics using the EHW robust standard errors and covariances as in the analysis stage, and compute the $p$-values accordingly. We give the explicit forms of the resulting test statistics in Remark [S1] in Section [S3.5] and show in Remark [S2] in Section [S4.4] their equivalence with the classic counterparts as $N$ tends to infinity. All results thus extend to ReP based on the robustly studentized test statistics, with $\hat{\tau}$ being our recommendation.

S2.4. $p$-values from other standard statistical tests

The discussion so far concerned $p$-values from the linear, logistic, and multinomial logistic regressions in standard software packages. Alternatively, the probit and multinomial probit regressions provide two intuitive variants to the logistic and multinomial logistic regressions of $Z_i$ on $(1, x_i)$, respectively, accommodating marginal, joint, and consensus rules via one MLE fit. We conjecture that the results are analogous, and leave the technical details to future work.

In addition, the two-sample Kolmogorov–Smirnov test and the chi-square test of independence exemplify alternative test choices for forming covariate balance criteria under the treatment-control and general experiments, respectively. See Gerber et al. (2009) and Chen et al. (2010) for their applications to covariate balance check. We leave the theory on their properties for rerandomization to future work.

S3. Test statistics and equivalent forms for acceptance criteria in Tables 1 and 2

We review in this section the test statistics that underlie the $p$-values we use to form the ReP schemes in Tables 1 and 2, respectively. To avoid repetition, we treat the logistic regression for treatment-control experiments as a special case of the multinomial logistic regression with $Q = 2$ and level 2 relabeled as level 0.

Assume $\alpha_j$ $(j = 1, \ldots, J)$ and $\alpha_0$ as the thresholds for the marginal and joint criteria under the treatment-control experiment, respectively.

Assume $\alpha_j$ $(j = 1, \ldots, J)$, $\alpha_{qj}$ $(q \in Q_+; j = 1, \ldots, J)$, and $\alpha_0$ as the thresholds for the marginal $F$-tests and the marginal and joint tests based on the multinomial logistic regression, respectively, under the general experiment.
S3.1. Two-sample \( t \)-tests

**Marginal tests.** Let \( \hat{\tau}_{x,j} = \hat{x}_j(1) - \hat{x}_j(0) \), where \( \hat{x}_j(q) = N_q^{-1} \sum_{i:Z_i=q} x_{ij} \), be the difference in means of the \( j \)th covariate, equaling the \( j \)th component of \( \hat{\tau}_x \). The pooled standard error for \( \hat{\tau}_{x,j} \) equals

\[
\hat{\text{se}}_j = \sqrt{\frac{(N_1 - 1)\hat{S}_{x,j}^2(1) + (N_0 - 1)\hat{S}_{x,j}^2(0)}{N - 2} \left( \frac{1}{N_1} + \frac{1}{N_0} \right)},
\]

with \( \hat{S}_{x,j}^2(q) = (N_q - 1)^{-1} \sum_{i:Z_i=q} (x_{ij} - \hat{x}_j(q))^2 \) for \( q = 0, 1 \). The two-sample \( t \)-test uses

\[
t_{j,t} = \frac{\hat{\tau}_{x,j}}{\hat{\text{se}}_j}
\]

as the test statistic, and computes \( p_{j,t} \) based on the \( t_{N-2} \) distribution as

\[
p_{j,t} = P(|A| \geq |t_{j,t}|), \quad \text{where } A \sim t_{N-2}.
\]

Let \( T_t = (t_{1,t}, \ldots, t_{J,t})^T \). The marginal criterion equals

\[
\mathcal{A}_{t,mg} = \{ p_{j,t} \geq \alpha_j \text{ for all } j = 1, \ldots, J \}
\]

\[
= \{ |t_{j,t}| \leq a_{j,t} \text{ for all } j = 1, \ldots, J \}
\]

\[
= \{|T_t| \leq a_t \},
\]

with \( a_t = (a_{1,t}, \ldots, a_{J,t})^T \) and \( a_{j,t} \) denoting the \((1 - \alpha_j/2)\)th quantile of the \( t_{N-2} \) distribution. Numerically, \( \hat{\tau}_{x,j}, \hat{\text{se}}_j, t_{j,t}, \) and \( p_{j,t} \) equal the coefficient, classic standard error, \( t \)-value, and \( p \)-value associated with \( Z_i \) from the OLS fit of \( 1m(x_{ij} \sim 1 + Z_i) \) over \( i = 1, \ldots, N \), respectively. This gives an alternative implementation of the marginal \( t \)-tests via OLS.

**Joint test.** Recall \( W_t = \hat{\tau}_x^T \hat{\Omega}^{-1} \hat{\tau}_x \) as the test statistic for the joint two-sample \( t \)-test. The pooled estimated covariance equals

\[
\hat{\Omega} = \frac{(N_1 - 1)\hat{S}_x^2(1) + (N_0 - 1)\hat{S}_x^2(0)}{N - 2} \left( \frac{1}{N_1} + \frac{1}{N_0} \right),
\]

with \( \hat{S}_x^2(q) = (N_q - 1)^{-1} \sum_{i:Z_i=q} (x_i - \hat{x}(q))^T (x_i - \hat{x}(q)) \) for \( q = 0, 1 \). Assume the joint Wald test for computing \( p_{0,t} \), with \( \chi^2_J \) as the reference distribution. The acceptance criterion equals

\[
\mathcal{A}_{t,jt} = \{ p_{0,t} \geq \alpha_0 \} = \{ W_t \leq a_0 \},
\]
where \( a_0 \) denotes the \((1 - \alpha_0)\)th quantile of the \( \chi^2_J \) distribution.

### S3.2. Linear regression

**Marginal tests.** Let \( \hat{\beta} = (\hat{\beta}_1, \ldots, \hat{\beta}_J)^T \) denote the coefficient vector of \( x_i = (x_{i1}, \ldots, x_{iJ})^T \) from \( \text{lm}(Z_i \sim 1 + x_{i1} + \cdots + x_{iJ}) \). Let \( \hat{V} \) be the associated estimated covariance, with \( \hat{V}_{jj} \) as the \((j, j)\)th element for \( j = 1, \ldots, J \). The marginal test of \( \hat{\beta}_j \) takes

\[
t_{j,\text{lm}} = \frac{\hat{\beta}_j}{\hat{V}_{jj}^{1/2}}
\]

as the test-statistic, and computes \( p_{j,\text{lm}} \) based on the \( t_{N-1,J} \) distribution as

\[
p_{j,\text{lm}} = P(|A| \geq |t_{j,\text{lm}}|), \quad \text{where} \ A \sim t_{N-1,J}.
\]

Let \( T_{\text{lm}} = (t_{1,\text{lm}}, \ldots, t_{J,\text{lm}})^T \). The marginal criterion equals

\[
\mathcal{A}_{\text{lm},\text{mg}} = \{ p_{j,\text{lm}} \geq \alpha_j \text{ for all } j = 1, \ldots, J \}
\]

\[
= \{ |t_{j,\text{lm}}| \leq a_{j,\text{lm}} \text{ for all } j = 1, \ldots, J \}
\]

\[
= \{ |T_{\text{lm}}| \leq a_{\text{lm}} \},
\]

with \( a_{\text{lm}} = (a_{1,\text{lm}}, \ldots, a_{J,\text{lm}})^T \) and \( a_{j,\text{lm}} \) denoting the \((1 - \alpha_j/2)\)th quantile of the \( t_{N-1,J} \) distribution.

**Joint test.** The \( F \)-test for linear regression compares \( \text{lm}(Z_i \sim 1 + x_{i1} + \cdots + x_{iJ}) \) against the empty model \( \text{lm}(Z_i \sim 1) \). Let \( \text{RSS}_0 \) and \( \text{RSS}_1 \) denote the residual sums of squares from the null and full regressions, respectively. The test statistic equals

\[
F = \frac{(\text{RSS}_0 - \text{RSS}_1)/J}{\text{RSS}_1/(N - 1 - J)},
\]

and is compared against the \( F_{J,N-1,J} \) distribution to compute \( p_{0,\text{lm}} \) as

\[
p_{0,\text{lm}} = P(A \geq F), \quad \text{where} \ A \sim F_{J,N-1,J}.
\]

The acceptance criterion equals

\[
\mathcal{A}_{\text{lm},\text{jt}} = \{ p_{0,\text{lm}} \geq \alpha_0 \} = \{ F \leq f_{J,N-1,J} \},
\]

where \( f_{J,N-1,J} \) denotes the \((1 - \alpha_0)\)th quantile of the \( F_{J,N-1,J} \) distribution.
Marginal tests. The marginal test of $\tilde{x}$ algorithm assumes that $(\tilde{Q})$

Recall level $S3.3$. Multinomial logistic regression

where $a = 2$, $Q$ simplifies to $\tilde{Q}$ with $\tilde{a}$ asymptotic equivalence between $p_0$ and $\tilde{p}_0$ for constructing the joint criterion from linear regression; see Lemma S6. For two sequences of events ($A_N$) and ($B_N$) write $A_N \equiv B_N$ if $P(A_N | B_N) = o(1)$ and $P(B_N | A_N) = o(1)$.

Lemma S1. $F = J^{-1}W_{lm}$ and $f_{j,N-1-J} = J^{-1}a_0 + o(1)$ such that

$A_{lm,jt} = \{p_{0,lm} \geq a_0\} = \{F \leq f_{j,N-1-J}\} = \{W_{lm} \leq a_t\} = \{W_{lm} \leq a_0\} = \{p'_{0,lm} \geq a_0\}$,

where $a_t = J \cdot f_{j,N-1-J} = a_0 + o(1)$.

S3.3. Multinomial logistic regression

Recall level $Q$ as the reference level when fitting $\logit(Z_i \sim 1 + x_{i1} + \cdots + x_{ij})$ by MLE. The fitting algorithm assumes that $(x_i, Z_i)_{i=1}^N$ are independent samples from

$Z_i \mid x_i \sim \text{multinomial}\{1; (\pi_1(x_i), \ldots, \pi_Q(x_i))\}$,

with

$$\log \frac{\pi_q(x_i)}{\pi_Q(x_i)} = \beta_{q0} + x_i^T \beta_q$$

for $q \in Q_+ = \{1, \ldots, Q-1\}$ and $\beta_q = (\beta_{q1}, \ldots, \beta_{qJ})^T$. (S4)

Let $\tilde{\beta}_q = (\tilde{\beta}_{q1}, \ldots, \tilde{\beta}_{qJ})^T$ be the MLE of $\beta_q$ for $q \in Q_+$. Let $\tilde{\beta} = (\tilde{\beta}_1^T, \ldots, \tilde{\beta}_{Q-1}^T)^T = (\tilde{\beta}_q)_{q \in Q_+ ; j=1,\ldots,J} \in \mathbb{R}^{J(Q-1)}$, with $\tilde{V} = (\tilde{V}_{j,q,q'})$ as the estimated covariance from the same MLE fit. The notation simplifies to $\tilde{\beta} = (\tilde{\beta}_1^T, \ldots, \tilde{\beta}_{J-1}^T)^T$ and $\tilde{V} = (\tilde{V}_{j,j'})_{j,j'=1,\ldots,J}$ under the treatment-control experiment with $Q = 2$, $Q_+ = \{1\}$, $\tilde{\beta} = \tilde{\beta}_1$, and $(\tilde{\beta}_j, \tilde{V}_{j,j'}) = (\tilde{\beta}_{1j}, \tilde{V}_{1j,1j'})$.

Marginal tests. The marginal test of $\beta_{qj}$ uses

$$t_{qj,\logit} = \frac{\tilde{\beta}_{qj}}{\sqrt{V_{qj,qj}}}$$
as the test statistic, and computes $p_{qj, \text{logit}}$ as

$$p_{qj, \text{logit}} = \mathbb{P}(\lvert A \rvert \geq \lvert t_{qj, \text{logit}} \rvert), \quad \text{where } A \sim \mathcal{N}(0, 1).$$

Let $T_{\text{logit}} = (t_{qj, \text{logit}})_{q \in Q_+; j = 1, \ldots, J } \in \mathbb{R}^{J(Q-1)}$ in lexicographical order of $qj$. The marginal criterion equals

$$A_{\text{logit, mg}} = \{p_{qj, \text{logit}} \geq \alpha_{qj} \text{ for all } q \in Q_+ \text{ and } j = 1, \ldots, J\}$$
$$= \{|t_{qj, \text{logit}}| \leq a_{qj} \text{ for all } q \in Q_+ \text{ and } j = 1, \ldots, J\}$$
$$= \{|T_{\text{logit}}| \leq a\},$$

with $a = (a_{qj})_{q \in Q_+; j = 1, \ldots, J} \in \mathbb{R}^{J(Q-1)}$ and $a_{qj}$ denoting the $(1 - \alpha_{qj}/2)$th quantile of the standard normal distribution.

When $Q = 2$, we have $(\tilde{\beta}_j, \tilde{V}_{jj}, p_{j, \text{logit}}, \alpha_j) = (\tilde{\beta}_{1j}, \tilde{V}_{1j, 1j}, p_{1j, \text{logit}}, \alpha_{1j})$ for $j = 1, \ldots, J$. The definition of $a$ simplifies to $a = (a_1, \ldots, a_J)^T \in \mathbb{R}^J$, with $a_j$ equaling the $(1 - \alpha_j/2)$th quantile of the standard normal distribution. The definition of $T_{\text{logit}}$ simplifies to $T_{\text{logit}} = (t_{1, \text{logit}}, \ldots, t_{J, \text{logit}})^T$ with

$$t_{j, \text{logit}} = \tilde{\beta}_j / \tilde{V}_{jj}^{1/2}.$$

The marginal criterion equals

$$A_{\text{logit, mg}} = \{p_{j, \text{logit}} \geq \alpha_j \text{ for all } j = 1, \ldots, J\} \quad \text{(S5)}$$
$$= \{|t_{j, \text{logit}}| \leq a_j \text{ for all } j = 1, \ldots, J\}$$
$$= \{|T_{\text{logit}}| \leq a\}.$$ 

**Joint test.** The LRT for multinomial logistic regression tests $\text{logit}(Z_i \sim 1 + x_i)$ against the empty model $\text{logit}(Z_i \sim 1)$. Denote by $\lambda_{\text{LRT}}$ the resulting test statistic, with the explicit form given in (S28). Then $p_{0, \text{logit}}$ is computed based on the $\chi^2_{J(Q-1)}$ distribution as

$$p_{0, \text{logit}} = \mathbb{P}(A \geq \lambda_{\text{LRT}}), \quad \text{where } A \sim \chi^2_{J(Q-1)}.$$

The joint criterion equals

$$A_{\text{logit,jt}} = \{p_{0, \text{logit}} \geq \alpha_0\} = \{\lambda_{\text{LRT}} \leq a_0\},$$

S13
where \( a_0 \) denotes the \((1 - \alpha_0)\)th quantile of the \( \chi^2_{J(Q-1)} \) distribution. The definition of \( a_0 \) reduces to that under the treatment-control experiment, namely the \((1 - \alpha_0)\)th quantile of the \( \chi^2_J \) distribution, with \( Q = 2 \).

Whereas the LRT is the default joint test for multinomial logistic regression returned by most software packages, we can also compute the joint \( p \)-value via a joint Wald test with test statistic \( W_{\text{logit}} = \tilde{\beta}^T \tilde{V}^{-1} \tilde{\beta} \). The resulting \( p \)-value equals

\[
p'_{0,\text{logit}} = \mathbb{P}(A \geq W_{\text{logit}}), \quad \text{where} \quad A \sim \chi^2_{J(Q-1)},
\]

with \( \{p'_{0,\text{logit}} \geq \alpha_0\} = \{W_{\text{logit}} \leq a_0\} \). Theorem [1] in Section [4] ensures \( \lambda_{\text{lrt}} - W_{\text{logit}} = o_P(1) \). This underpins the asymptotic equivalence between the LRT and the Wald test for constructing the joint criterion from multinomial logistic regression; see Lemma [6].

**S3.4. Marginal F-tests under the general experiment**

Renew \( \hat{x}_j(q) = N_q^{-1} \sum_{i:Z_i=q} x_{ij} \) for \( q \in Q = \{1, \ldots, Q\} \). The \( F \)-test of covariate \( j \) uses

\[
F_j = \frac{\sum_{q \in Q} N_q \hat{x}^2_j(q)/(Q-1)}{\sum_{q \in Q} \sum_{i:Z_i=q} (x_{ij} - \hat{x}_j(q))^2/(N-Q)}
\]

as the test statistic, and compares it against the \( F_{Q-1,N-Q} \) distribution to compute \( p_{j,f} \) as

\[
p_{j,f} = \mathbb{P}(A \geq F_j), \quad \text{where} \quad A \sim F_{Q-1,N-Q}.
\]

The marginal criterion equals

\[
A_{f,mg} = \{p_{j,f} \geq \alpha_j \text{ for all } j = 1, \ldots, J\} = \{F_j \leq a_{j,f} \text{ for all } j = 1, \ldots, J\}, \quad (S7)
\]

where \( a_{j,f} \) denotes the \((1 - \alpha_j)\)th quantile of the \( F_{Q-1,N-Q} \) distribution.

**S3.5. Unification**

Table [2] summarizes the regression realizations, test statistics, and reference distributions for the thirteen ReP schemes in Tables [1] and [2]. Table [3] summarizes the alternative expressions of the nine criteria in Table [1] for treatment-control experiments in terms of the test statistics.

In particular, recall \( \tilde{T}_\uparrow = (t_1, \ldots, t_{J,\uparrow})^T \) as the vector of the marginal \( t \)-statistics for \( \uparrow = t, \text{lm, logit} \) under the treatment-control experiment. Direct comparison shows that \( \hat{\Omega}_{jj} = s_e^2_j \) for
Table S2: Regression realizations, test statistics, and reference distributions for the thirteen ReP schemes in Tables 1 and 2. Let \( I_{i,+} = (I_{i1}, \ldots, I_{i,Q-1})^T \) for \( i = 1, \ldots, N \). 

\[
\begin{array}{cccccc}
Q & Model option & Regression realization & Test statistics and reference distributions & Marginal & Joint (default) & Joint (Wald) \\
\hline
2 & t & \text{lm}(x_{ij} \sim 1 + Z_i) & \hat{\tau}_{x,j}/\hat{se}_j & t_{N-2} & \hat{\tau}_x^T \hat{\Omega}^{-1} \hat{\tau}_x & \chi^2_J \\
   & \text{lm} & \text{lm}(Z_i \sim 1 + x_i) & \hat{\beta}_j/\sqrt{\hat{V}_{jj}} & t_{N-1-J} & F_{J,N-1-J} & \hat{\beta}_j \hat{T}^{-1} \hat{\beta}_j & \chi^2_J \\
   & \text{logit} & \text{logit}(Z_i \sim 1 + x_i) & \hat{\beta}_j/\sqrt{\hat{V}_{jj}} & \mathcal{N}(0,1) & \lambda_{\text{LRT}} & \chi^2_J & \hat{\beta}_j \hat{T}^{-1} \hat{\beta}_j & \chi^2_J \\
\geq 2 & f & \text{lm}(x_{ij} \sim 1 + I_{i,+}) & F_j & F_{Q-1,N-Q} & \mathcal{N}(0,1) & \lambda_{\text{LRT}} & \chi^2_{J(Q-1)} & \hat{\beta}_j \hat{T}^{-1} \hat{\beta}_j & \chi^2_{J(Q-1)} \\
   & \text{logit} & \text{logit}(Z_i \sim 1 + x_i) & \hat{\beta}_{qj}/\sqrt{V_{qj,qj}} & \mathcal{N}(0,1) & \lambda_{\text{LRT}} & \chi^2_{J(Q-1)} & \hat{\beta}_j \hat{T}^{-1} \hat{\beta}_j & \chi^2_{J(Q-1)} \\
\end{array}
\]

\( j = 1, \ldots, J \), and allows us to unify the marginal criteria in \([S1], [S3] \), and \([S5] \) as 

\[
\mathcal{A}_{\text{t,mg}} = \{|T_\hat{\tau}| \leq a_t\} \quad (\hat{\tau} = \text{t, lm, logit}),
\]

with \( a_{\text{logit}} = a \) and 

\[
\begin{align*}
T_t &= \text{diag}(\hat{se}_j^{-1})_{j=1}^J \hat{\tau}_x = \sigma(\hat{\Omega})^{-1} \hat{\tau}_x, \\
T_{\text{lm}} &= \text{diag}(\hat{V}_{jj}^{-1/2})_{j=1}^J \hat{\beta} = \sigma(\hat{V})^{-1} \hat{\beta}, \\
T_{\text{logit}} &= \text{diag}(\hat{V}_{jj}^{-1/2})_{j=1}^J \hat{\beta} = \sigma(\hat{V})^{-1} \hat{\beta}.
\end{align*}
\]

The expressions in the last column of Table \([S3] \) also extend to the criteria based on the multinomial logistic regression for general experiments with renewed definitions of \( T_{\text{logit}} = (t_{qj,\text{logit}})_{q \in \mathcal{Q} +; \ j=1,\ldots,J} \), 

\( a = (a_{qj})_{q \in \mathcal{Q} +; \ j=1,\ldots,J} \), and \( a_0 \).

Table S3: Acceptance criteria for \( \hat{\tau} = \text{t, lm, logit} \) under the treatment-control experiment. 

\[
\begin{array}{cccccc}
\text{by p-values} & t & \text{lm} & \text{logit} \\
\hline
\mathcal{A}_{\text{t,mg}} & p_{j,\text{t}} \geq \alpha_j (j = 1, \ldots, J) & |T_t| \leq a_t & |T_{\text{lm}}| \leq a_{\text{lm}} & |T_{\text{logit}}| \leq a \\\n\mathcal{A}_{\text{t,lt}} & p_{0,\text{t}} \geq \alpha_0 & W_t \leq a_0 & W_{\text{lm}} \leq a_t & \lambda_{\text{LRT}} \leq a_0 \\
\mathcal{A}_{\text{t,cs}} & p_{j,\text{t}} \geq \alpha_j (j = 0, 1, \ldots, J) & |T_t| \leq a_t, W_t \leq a_0 & |T_{\text{lm}}| \leq a_{\text{lm}}, W_{\text{lm}} \leq a_t & |T_{\text{logit}}| \leq a, \lambda_{\text{LRT}} \leq a_0 \\
\end{array}
\]

The test statistics satisfy 

\[
\begin{align*}
T_t &= \sigma(\hat{\Omega})^{-1} \hat{\tau}_x, \quad T_{\text{lm}} = \sigma(\hat{V})^{-1} \hat{\beta}, \quad T_{\text{logit}} = \sigma(\hat{V})^{-1} \hat{\beta}, \\
W_t &= \hat{\tau}_x^T \hat{\Omega}^{-1} \hat{\tau}_x, \quad W_{\text{lm}} = \hat{\beta}^T \hat{T}^{-1} \hat{\beta}, \quad W_{\text{logit}} = \hat{\beta}^T \hat{T}^{-1} \hat{\beta},
\end{align*}
\]

with \( a_{\text{t}} = a + o(1) \) (\( \hat{\tau} = \text{t, lm} \)) and \( a_{\text{t}} = a_0 + o(1) \).

\textbf{Remark S1.} Recall \( \hat{se}_j \) as the classic standard error of \( \hat{\tau}_{x,j} \) from \( \text{lm}(x_{ij} \sim 1 + Z_i) \). Echoing the discussion in Section \([S2.3] \) we can replace it with the EHW standard error from the same OLS fit,
denoted by $\hat{s}e_j'$, and conduct the marginal two-sample $t$-test based on the robustly studentized $t$-statistic $t_{j,t} = \hat{\tau}_{x,j}/\hat{se}_j'$ for $j = 1, \ldots, J$.

Extensions to other criteria are straightforward by replacing the $\hat{\Omega}$, $\hat{V}$, and $\hat{V}$ in Tables S2 and S3 with their respective heteroskedasticity-robust counterparts. In particular, the $\text{EHW}$ counterparts of $\hat{V}$ and $\tilde{V}$ can be obtained as direct outputs from the same linear, logistic, and multinomial logistic regressions, respectively. The robust counterpart of $\hat{\Omega}$ can be computed as $\hat{\Omega}' = (N-1)^{-1} \sum_{i=1}^N \{Y_{i,*}(q) - \bar{Y}(q)\} \{Y_{i,*}(q') - \bar{Y}(q')\}$. Let $V_* = \text{diag}(S*_{*qq}/e_q)_{q \in Q} - S_*$ ($* = N,F,L$).

Condition [●] ensures that $e_q$, $\gamma_q$, $\gamma_*$, $S_*$, and $V_*$ all have finite limits as $N$ tends to infinity. For notational simplicity, we will use the same symbols to denote their respective limits when no confusion would arise. Recall that $\hat{x} = (\hat{x}(1)^T, \ldots, \hat{x}(Q)^T)^T$, with $\hat{x}(q) = N_q^{-1} \sum_{i:Z_i=q} x_i$ for $q \in Q$. Let $\hat{V}'_*$ be the $\text{EHW}$ covariance of $\hat{Y}_*$ from the same $\text{OLS}$ fit. Lemma S2 follows from Zhao and Ding (2023), and clarifies the design-based properties of $\hat{Y}_*$ and $\hat{V}'_*$.

**Lemma S2.** Assume a completely randomized general experiment and Condition [●] For $* = N,F,L$,
(i) \[
\sqrt{N} \left( \frac{\hat{Y}_* - \bar{Y}}{\hat{x}} \right) \rightsquigarrow \mathcal{N} \left\{ 0_{Q \times JQ}, \begin{pmatrix} V_* & \Gamma_* V_x \\ V_x \Gamma^T_* & V_x \end{pmatrix} \right\},
\]

with \( V_x = N\text{cov}(\hat{x}) = \{\text{diag}(e^{-1}_q)_{q \in Q} - 1_{Q \times Q}\} \otimes S^2_x \), \( V_* = V_L + \Gamma_* V_x \Gamma^T_* \geq V_L \) for

\[
\Gamma_N = \text{diag}(\gamma^T_q)_{q \in Q}, \quad \Gamma_F = \text{diag}\{(\gamma_q - \gamma_F)^T\}_{q \in Q}, \quad \Gamma_L = 0_{Q \times JQ},
\]

and hence \( \hat{Y}_L \succeq \infty \hat{Y}_N, \hat{Y}_F \); (ii) \( N\hat{V}'_* - V_* = S_* + o_p(1) \) with \( S_* \geq 0 \).

Lemma \([S2, i]\) ensures the consistency and asymptotic normality of \( \hat{\tau}_* = G\hat{Y}_* \) for estimating \( \tau = GY \) under complete randomization. Lemma \([S2, ii]\) ensures the asymptotic appropriateness of the EHW covariance for estimating the true sampling covariance, and thereby justifies the Wald-type inference of \( \tau \) based on \( (\hat{\tau}_*, G\hat{V}'_* G^T) \) and normal approximation.

The theory under the treatment-control experiment then follows as a special case with \( \hat{\tau}_* = \hat{Y}_*(1) - \hat{Y}_*(0) \) by the invariance of OLS to non-degenerate transformation of the regressor vector. Let \( \hat{se}_* \) be the EHW standard error of \( \hat{\tau}_* \) from the same OLS fit.

**Corollary S1.** Assume a completely randomized treatment-control experiment and Condition \([\text{II}]\). For \( * = N, F, L, \)

(i)

\[
\sqrt{N} \left( \frac{\hat{\tau}_* - \tau}{\hat{\tau}_x} \right) \rightsquigarrow \mathcal{N} \left\{ 0_{J+1}, \begin{pmatrix} v_* & c^T_* \\ c_* & v_x \end{pmatrix} \right\},
\]

with \( v_* = (-1,1)V_x(-1,1)^T \), \( v_x = (e_0 e_1)^{-1} S^2_x \), and

\[
c_N = S^2_x(e_0^{-1} \gamma_0 + e_1^{-1} \gamma_1), \quad c_F = S^2_x(e_1^{-1} - e_0^{-1})(\gamma_1 - \gamma_0), \quad c_L = 0_J
\]

satisfying \( v_* - v_L = c^T_* v_x^{-1} c_* \geq 0 \);

(ii) \( N(\hat{se}_*)^2 - v_* = (-1,1)S_*(-1,1)^T + o_p(1) \) with \( (-1,1)S_*(-1,1)^T \geq 0 \).

Parallel to the comments after Lemma \([S2, i]\), Corollary \([S1, i]\) states the consistency and asymptotic normality of \( \hat{\tau}_* \) for estimating \( \tau = \bar{Y}(1) - \bar{Y}(0) \) under complete randomization, and ensures the
asymptotic efficiency of $\hat{\tau}_t$ over $\hat{\tau}_n$ and $\hat{\tau}_f$. Corollary S1.3 justifies the Wald-type inference based on $(\hat{\tau}_n, \hat{se}_n)$ and normal approximation.

### S4.2. Peakedness

Lemma S3 below states the celebrated Gaussian correlation inequality, with the recent breakthrough proof due to Royen (2014); see also Lataša and Matlak (2017).

**Lemma S3** (Gaussian correlation inequality). Let $\mu$ be an $m$-dimensional Gaussian probability measure on $\mathbb{R}^m$, that is, $\mu$ is a multivariate normal distribution, centered at the origin. Then $\mu(C_1 \cap C_2) \geq \mu(C_1)\mu(C_2)$ for all convex sets $C_1, C_2 \subset \mathbb{R}^m$ that are symmetric about the origin.

Lemma S3 immediately implies Corollary S2 below, which states that a mean-zero Gaussian measure restricted to a symmetric convex set is more peaked than the original unrestricted measure.

**Corollary S2.** Let $\epsilon \sim \mathcal{N}(0,m,\Sigma)$. Then $\epsilon | \{\epsilon \in C\} \geq \epsilon$ for arbitrary convex set $C \subset \mathbb{R}^m$ that is symmetric about the origin.

**Proof of Corollary S2.** Let $\mathcal{T} \sim \epsilon | \{\epsilon \in C\}$. The result follows from

$$P(\mathcal{T} \in C_1) = P(\epsilon \in C_1 | \epsilon \in C) = \frac{P(\epsilon \in C_1, \epsilon \in C)}{P(\epsilon \in C)} \geq P(\epsilon \in C_1)$$

for arbitrary symmetric convex set $C_1 \subset \mathbb{R}^m$ by Lemma S3. □

Recall that

$$\mathcal{T}_t \sim \epsilon_t | \{\epsilon_t \leq a\}, \quad \mathcal{T}'_t \sim \epsilon_t | \{\epsilon_t \leq a, \|\epsilon_t\|_M \leq a_0\},$$

$$\mathcal{T}_{im} \sim \epsilon_{im} | \{\epsilon_{im} \leq a\}, \quad \mathcal{T}'_{im} \sim \epsilon_{im} | \{\epsilon_{im} \leq a, \|\epsilon_{im}\|_M \leq a_0\},$$

$$\mathcal{T}_{logit} \sim \epsilon_{logit} | \{\epsilon_{logit} \leq a\}, \quad \mathcal{T}'_{logit} \sim \epsilon_{logit} | \{\epsilon_{logit} \leq a, \|\epsilon_{logit}\|_M \leq a_0\},$$

$$\mathcal{L} \sim \epsilon_0 | \{\|\epsilon_0\|_2^2 \leq a_0\}, \quad \mathcal{T}_f \sim \epsilon_f | \{\sum_{q \in Q} e_q \epsilon_{qj}^2 \leq a_j' S_{x,j}^2 \text{ for all } j = 1, \ldots, J\},$$

with $\epsilon_t \sim \mathcal{N}\{0,J, D(V_x)\}, \epsilon_{im} \sim \mathcal{N}\{0,J, D(V_x^{-1})\}, \epsilon_{logit} \sim \mathcal{N}\{0,J(Q-1), D(V_{\Phi})\}, \epsilon_0 \sim \mathcal{N}\{0,J(Q-1), I_{J(Q-1)}\}$, and $\epsilon_f \sim \mathcal{N}\{0,J_Q, V_x\}$.

**Lemma S4.** $\mathcal{T}_s, \mathcal{T}'_s \geq \epsilon_s$ for $s = t, im, logit, \mathcal{L} \geq \epsilon_0$, and $\mathcal{T}_f \geq \epsilon_f$.

**Proof of Lemma S4.** The results follow from Corollary S2 and the convexity of $\{u \in \mathbb{R}^m : |u| \leq a\}$, $\{u \in \mathbb{R}^m : \|u\|_M \leq a_0\}$, $\{u \in \mathbb{R}^m : \|u\|_2^2 \leq a_0\}$, and $\{u = (u_{qj})_{q \in \mathcal{Q}}; j = 1, \ldots, J \in \mathbb{R}^{JQ} : \sum_{q \in \mathcal{Q}} e_q u_{qj}^2 \leq a_j' S_{x,j}^2 \text{ for all } j = 1, \ldots, J\}$. □
Lemma S5 below reviews two classical results in probability for comparing peakedness between random vectors. Li et al. (2020) used them before. The proofs follow from Dharmadhikari and Joag-Dev (1988, Lemma 7.2 and Theorem 7.5).

Lemma S5. (i) If two $m \times 1$ symmetric random vectors $A$ and $B$ satisfy $A \succeq B$, then $CA \succeq CB$ for any matrix $C$ with compatible dimensions.

(ii) Let $A$, $B_1$, and $B_2$ be three independent $m \times 1$ symmetric random vectors. If $A$ is normal and $B_1 \succeq B_2$, then $A + B_1 \succeq A + B_2$.

S4.3. Design-based properties of multinomial logistic regression

Theorem S1 below is a novel technical result, and clarifies the design-based properties of the MLE outputs from the logistic and multinomial logistic regressions, respectively. The result ensures the asymptotic equivalence between the LRT and the Wald test in terms of both the test statistics and the corresponding $p$-values. We relegate the proof to Section S6.

Recall that $\hat{x}_+ = (\hat{x}(1)^T, \ldots, \hat{x}(Q-1)^T)^T$, with $V_{x+} = N\operatorname{cov}(\hat{x}_+)$ denoting its scaled sampling covariance under complete randomization. Recall that $V_\Psi = N\operatorname{cov}(\Psi \hat{x}_+) = \Psi V_{x+} \Psi^T$, with $\Psi = \{\Phi^{-1} \operatorname{diag}(e_+)\} \otimes (S_\hat{x}^2)^{-1}$, $e_+ = (e_1, \ldots, e_{Q-1})^T$, and $\Phi = \operatorname{diag}(e_+) - e_+e_+^T$.

Theorem S1. Consider a completely randomized experiment with $Q \geq 2$ treatment arms. Under Conditions 1 and 3, we have

$$\sqrt{N}(\hat{\beta} - \Psi \hat{x}_+) = o_P(1), \quad \sqrt{N}\hat{\beta} \sim N(0_{J(Q-1)}, V_\Psi), \quad N\tilde{V} = V_\Psi + o_P(1),$$

$$\lambda_{LRT} - N\hat{\beta}^T V_{x+}^{-1} \hat{x}_+ = o_P(1), \quad W_{\logit} - N\hat{\beta}^T V_{x+}^{-1} \hat{x}_+ = o_P(1), \quad \lambda_{LRT} - W_{\logit} = o_P(1),$$

$$\lambda_{LRT} \sim \chi^2_{J(Q-1)}, \quad W_{\logit} \sim \chi^2_{J(Q-1)}.$$

For a treatment-control experiment with $Q = 2$ and the reference level $q = 2$ relabeled as 0, the results simplify to

$$\sqrt{N}\{\hat{\beta} - (S_\hat{x}^2)^{-1} \tau_x\} = o_P(1), \quad \sqrt{N}\hat{\beta} \sim N(0_J, (e_0e_1)^{-1}(S_\hat{x}^2)^{-1}),$$

$$N\tilde{V} = (e_0e_1)^{-1}(S_\hat{x}^2)^{-1} + o_P(1).$$

S4.4. Weak convergence

Lemma S6 below underpins the asymptotic equivalence between balance criteria based on asymptotically equivalent thresholds or test statistics. The proof follows from standard probability calculation and is thus omitted.
Lemma S6. Let \((A_N)_{N=1}^{\infty}\) be a sequence of \(m \times 1\) random vectors. Let \((B_N)_{N=1}^{\infty}\) and \((B_N')_{N=1}^{\infty}\) be two sequences of random variables with \(B_N - B_N' = o(1)\) and \((A_N, B_N)_{N=1}^{\infty}\) having a continuous limiting distribution, represented by \((A, B)\). Let \((b_N)_{N=1}^{\infty}\) be a sequence of constants with a finite limit, \(b_\infty = \lim_{N \to \infty} b_N < \infty\). Then

(i) \(A_N | \{B_N \leq b_N\} \sim A_N | \{B_N \leq b_\infty\}\) provided \(P(B \leq b_\infty) > 0\);

(ii) \(A_N | \{B_N \leq b\} \sim A_N | \{B'_N \leq b\}\) for arbitrary fixed \(b \in \mathbb{R}\) that satisfies \(P(B \leq b) > 0\).

Definition S1 below extends the notion of rerandomization with general covariate balance criterion (ReG) from [Li et al. (2018)].

Definition S1. Let \(\phi(B, C)\) be a binary covariate balance indicator function, where \(\phi(\cdot, \cdot)\) is a binary indicator function and \((B, C)\) are two statistics computed from the data. An ReG accepts a randomization if \(\phi(B, C) = 1\).

The definition of ReG is general and includes all nine criteria in Table S3 as special cases. Table S4 below summarizes the covariate balance indicator functions for ReM and the nine ReP schemes in Table S3 under the treatment-control experiment, respectively. As an illustration, \((B, C) = (\sqrt{N\hat{\tau}_x}, N\hat{\Omega})\) under the two-sample t-test model option, with \(\phi(\cdot, \cdot)\) equaling

\[
\phi(u, v) = \begin{cases} 
1\{|\sigma(v)^{-1}u| \leq a_1\} & \text{under the marginal rule;} \\
1\{u^Tv^{-1}u \leq a_0\} & \text{under the joint rule;} \\
1\{|\sigma(v)^{-1}u| \leq a_1, \ u^Tv^{-1}u \leq a_0\} & \text{under the consensus rule.} 
\end{cases}
\]

The resulting covariate balance indicator functions equal

\[
\phi(B, C) = \begin{cases} 
1\{|\sigma(\hat{\Omega})^{-1}\hat{\tau}_x| \leq a_1\} & \text{under the marginal rule;} \\
1\{\hat{\tau}_x^T\hat{\Omega}^{-1}\hat{\tau}_x \leq a_0\} & \text{under the joint rule;} \\
1\{|\sigma(\hat{\Omega})^{-1}\hat{\tau}_x| \leq a_1, \ \hat{\tau}_x^T\hat{\Omega}^{-1}\hat{\tau}_x \leq a_0\} & \text{under the consensus rule} 
\end{cases}
\]

given \(\phi(u, v) = \phi(u/\sqrt{N}, v/N)\) in all three cases.

Lemma S7 below is a generalization of [Li et al. (2018) Proposition A1], and gives the asymptotic distribution of arbitrary random elements under ReG. To this end, Condition S1 below imposes some smoothness constraints on the associated \(\phi\) to prevent the acceptance region from being a set of measure zero. All covariate balance criteria in Table S4 satisfy Condition S1.

Condition S1. The binary indicator function \(\phi(\cdot, \cdot)\) satisfies: (i) \(\phi(\cdot, \cdot)\) is almost surely continuous; (ii) for \(u \sim \mathcal{N}(0, I, v_0)\), we have \(P\{\phi(u, v_0) = 1\} > 0\) for all \(v_0 > 0\), and \(\text{cov}\{u | \phi(u, v_0) = 1\}\) is a
Table S4: The covariate balance indicator functions and the corresponding \((B, C)\) for ReM and the nine ReP schemes in Table S3 under the treatment-control experiment. The joint criterion under the “logit” model option is given in terms of the asymptotically equivalent Wald statistic to highlight the analogy across different model options.

| \(B, C\) | \(\sqrt{N\hat{\tau}_x, Ncov(\hat{\tau}_x)}\) | \(\sqrt{N\hat{\tau}_x, N\Omega}\) |
| --- | --- | --- |
| Marginal | n.a. | \(1\{|\sigma(\hat{\Omega})^{-1}\hat{\tau}_x| \leq a_t\}\) |
| Joint | \(1\{|\hat{\tau}_x^T\text{cov}(\hat{\tau}_x)^{-1}\hat{\tau}_x| \leq a_0\}\) | \(1\{|\hat{\tau}_x^T\hat{\Omega}^{-1}\hat{\tau}_x| \leq a_0\}\) |
| Consensus | n.a. | \(1\{|\sigma(\hat{\Omega})^{-1}\hat{\tau}_x| \leq a_t, \hat{\tau}_x^T\hat{\Omega}^{-1}\hat{\tau}_x| \leq a_0\}\) |

\[ \text{ReM} \hspace{1cm} \text{t} \]

| \(B, C\) | \(\sqrt{N\hat{\beta}, NV}\) | \(\sqrt{N\hat{\beta}, NV}\) |
| --- | --- | --- |
| Marginal | \(1\{|\sigma(\hat{V})^{-1}\hat{\beta}| \leq a_{\text{im}}\}\) | \(1\{|\sigma(\hat{V})^{-1}\hat{\beta}| \leq a\}\) |
| Joint | \(1\{|\hat{\beta}^T\hat{V}^{-1}\hat{\beta}| \leq a_t\}\) | \(1\{|\hat{\beta}^T\hat{V}^{-1}\hat{\beta}| \leq a_0\}\) |
| Consensus | \(1\{|\sigma(\hat{V})^{-1}\hat{\beta}| \leq a_{\text{im}}, \hat{\beta}^T\hat{V}^{-1}\hat{\beta}| \leq a_t\}\) | \(1\{|\sigma(\hat{V})^{-1}\hat{\beta}| \leq a, \hat{\beta}^T\hat{V}^{-1}\hat{\beta}| \leq a_0\}\) |

The continuous function of \(v_0\).

**Lemma S7** (Weak convergence under ReG). Assume \((\phi_N)_{N=1}^{\infty}\) as a sequence of binary indicator functions under Condition S1 that converges to \(\phi\). For a sequence of random elements \((A_N, B_N, C_N)_{N=1}^{\infty}\) that satisfies \((A_N, B_N, C_N) \sim (A, B, C)\) as \(N \to \infty\), we have

\[
(A_N, B_N) \mid \{\phi_N(B_N, C_N) = 1\} \sim (A, B) \mid \{\phi(B, C) = 1\}
\]

in the sense that, for any continuity set \(S\) of \((A, B) \mid \{\phi(B, C) = 1\}\),

\[
\mathbb{P}\{(A_N, B_N) \in S \mid \phi_N(B_N, C_N) = 1\} = \mathbb{P}\{(A, B) \in S \mid \phi(B, C) = 1\} + o(1).
\]

Lemma S8 below gives the asymptotic joint distributions of \(\hat{\tau}_x\) and the elements that determine the covariate balance measures in Table S4. The result provides the basis for verifying Propositions 1 and 2 in a unified way.

**Lemma S8.** Assume a completely randomized treatment-control experiment and Conditions 1 and 2. For \(* = N, F, L\), we have

\[
\begin{align*}
(\sqrt{N}(\hat{\tau}_x - \tau), \sqrt{N}\hat{\tau}_x, N\text{cov}(\hat{\tau}_x)) & \sim (A_*, B, v_x), \\
(\sqrt{N}(\hat{\tau}_x - \tau), \sqrt{N}\hat{\tau}_x, N\Omega) & \sim (A_*, B, v_x), \\
(\sqrt{N}(\hat{\tau}_x - \tau), \sqrt{N}\hat{\beta}, NV) & \sim (A_*, v_x^{-1}B, v_x^{-1}), \\
(\sqrt{N}(\hat{\tau}_x - \tau), \sqrt{N}\hat{\beta}, N\hat{\Omega}) & \sim (A_*, (S_x^2)^{-1}B, (\epsilon_0e_1)^{-1}(S_x^2)^{-1})
\end{align*}
\]
with
\[
\begin{pmatrix}
  A_s \\
  B
\end{pmatrix}
\sim \mathcal{N}\left\{ 0_{J+1}, \begin{pmatrix} v_s & c_s^T \\ c_s & v_x \end{pmatrix} \right\}.
\]

**Proof of Lemma S8.** The result on ReM follows from Corollary S1. The result on the “logit” model option follows from Theorem S1. We verify below the results on the two-sample \(t\) ("t") and linear regression ("lm") model options, respectively.

**Result on the two-sample \(t\)-test model option.** The joint distribution of \(\sqrt{N}(\hat{\tau}_*-\tau, \hat{\tau}_x)\) follows from Corollary S1. The probability limit of \(N\hat{\Omega}\) follows from \(N\hat{\Omega} = (e_0e_1)^{-1}S_x^2 + o_P(1) = v_x + o_P(1)\) by S2 and \(\hat{S}_x^2(q) = S_x^2 + o_P(1)\) under complete randomization and Condition 1.

**Result on the linear regression model option.** The Frisch–Waugh–Lovell theorem ensures that
\[
\hat{\beta} = \frac{N_1}{N-1}(S_x^2)^{-1}\hat{x}(1) = \frac{N_0N_1}{(N-1)N}(S_x^2)^{-1}\hat{\tau}_x = \frac{N}{N-1}v_x^{-1}\hat{\tau}_x
\]
\[\text{(S8)}\]
is a non-degenerate linear transformation of \(\hat{\tau}_x\). The joint distribution of \(\sqrt{N}(\hat{\tau}_*-\tau, \hat{\beta})\) then follows from Corollary S1.

Further let
\[
S_Z^2 = \frac{1}{N-1}\sum_{i=1}^N (Z_i - \bar{Z})^2 = \frac{N}{N-1}e_0e_1, \quad S_{xZ} = \frac{1}{N-1}\sum_{i=1}^N x_i(Z_i - \bar{Z}) = \frac{N}{N-1}e_1\hat{x}(1),
\]
with \(\bar{Z} = N^{-1}\sum_{i=1}^N Z_i = e_1\). The probability limit of \(\hat{V}\) follows from
\[
N\hat{V} = \sigma^2\left(\sum_{i=1}^N x_ix_i^T\right)^{-1} = \frac{N}{N-1}\hat{\sigma}^2(S_x^2)^{-1}
\]
with
\[
\hat{\sigma}^2 = \frac{1}{N-1-J}\sum_{i=1}^N (Z_i - \bar{Z} - x_i^T\hat{\beta})^2 = \frac{N-1}{N-1-J}(S_Z^2 + \hat{\beta}^TS_x^2\hat{\beta} - 2\hat{\beta}^TS_xZ) = e_0e_1 + o_P(1).
\]

\[\square\]

Recall that \(\Gamma'_s = \Gamma_s\kappa\), with \(\kappa = (I_Q^{-1}, -e_Q^{-1}e_+)^T \otimes I_J, \Gamma_N = \text{diag}(\gamma_q^T)_{q \in Q}, \Gamma_v = \text{diag}(\gamma_q -
\( \gamma_{\ell}^{(T)} \}_{\ell \in \mathcal{Q}} \), and \( \Gamma_L = 0_{Q \times JQ} \). It follows from \( 0_J = \bar{x} = \sum_{q \in \mathcal{Q}} c_q \hat{x}(q) \) that

\[
\hat{x} = \kappa \hat{x}, \quad \Gamma_\ast \hat{x} = \Gamma_\ast \hat{x}_+ \quad (\ast = N, F, L).
\] (S9)

This gives the intuition behind the definition of \( \Gamma'_\ast \). Lemma S9 gives the asymptotic joint distributions of \( (\hat{Y}_\ast, \hat{x}_+) \) and \( (\hat{Y}_\ast, \hat{\beta}, \hat{V}) \) under the general experiment, respectively, analogous to Lemma S8.

**Lemma S9.** Assume a completely randomized general experiment and Conditions 1 and 3. Let \( A \sim \mathcal{N}(0_Q, V_L) \), \( B \sim \mathcal{N}(0_{JQ}, Q_x) \), and \( B' \sim \mathcal{N}(0_{J(Q-1)}, V_{x+}) \) be independent normal random vectors. For \( \ast = N, F, L \), we have

(i) \( \sqrt{N}(\hat{Y}_\ast - \hat{Y}, \hat{x}) \rightsquigarrow (A + \Gamma_\ast B, B) \), \( \sqrt{N}(\hat{Y}_\ast - \hat{Y}, \hat{x}_+) \rightsquigarrow (A + \Gamma'_\ast B', B') \);

(ii) \( (\sqrt{N}(\hat{Y}_\ast - \hat{Y}), \sqrt{N}\hat{\beta}, N\hat{V}) \rightsquigarrow (A + \Gamma'_\ast B', \Psi B', V_\Psi) \);

(iii) \( \hat{Y}_\ast | \{ \lambda_{\text{LRT}} \leq a_0 \} \rightsquigarrow \hat{Y}_\ast | \{ W_{\text{logit}} \leq a_0 \} \).

**Proof of Lemma S9.** The result on \( \sqrt{N}(\hat{Y}_\ast - \hat{Y}, \hat{x}) \) follows from Lemma S2. This, together with Lemma S9, further ensures \( \sqrt{N}(\hat{Y}_\ast - \hat{Y} - \Gamma_\ast \hat{x}_+, \hat{x}_+) \rightsquigarrow (A, B') \), and hence \( \sqrt{N}(\hat{Y}_\ast - \hat{Y}, \hat{x}_+) \rightsquigarrow (A + \Gamma'_\ast B', B') \) for \( \ast = N, F, L \).

The result on \( (\sqrt{N}(\hat{Y}_\ast - \hat{Y}), \sqrt{N}\hat{\beta}, N\hat{V}) \) then follows from \( \sqrt{N}(\hat{\beta} - \Psi \hat{x}_+) = o_{\mathbb{P}}(1) \) and \( N\hat{V} = V_\Psi + o_{\mathbb{P}}(1) \) by Theorem S1.

The asymptotic equivalence between \( \hat{Y}_\ast | \{ \lambda_{\text{LRT}} \leq a_0 \} \) and \( \hat{Y}_\ast | \{ W_{\text{logit}} \leq a_0 \} \) follows from \( \lambda_{\text{LRT}} - N\hat{x}_+^T V_{x+}^{-1} \hat{x}_+ = o_{\mathbb{P}}(1) \) and \( W_{\text{logit}} - N\hat{x}_+^T V_{x+}^{-1} \hat{x}_+ = o_{\mathbb{P}}(1) \) by Theorem S1 and Lemma S6.

**Remark S2.** Lemma S8 and its proof also imply some of the comments we made in the main text.

First, Lemma S8 together with Lemma S8 and \( \sqrt{N}\{ \hat{\beta} - (S_2^T)^{-1} \hat{\tau}_x \} = o_{\mathbb{P}}(1) \) from Theorem S1 ensures the asymptotic equivalence between \( \| \hat{\tau}_x \|_M = \hat{\tau}_x^T \text{cov}(\hat{\tau}_x)^{-1} \hat{\tau}_x \), \( W_t = \tau_x^T \hat{\Omega}^{-1} \hat{\tau}_x \), \( W_{\text{lm}} = \hat{\beta}^T \hat{V}^{-1} \hat{\beta} \), and \( W_{\text{logit}} = \hat{\beta}^T \hat{V}^{-1} \hat{\beta} \) in the sense that \( W_t - \| \hat{\tau}_x \|_M = o_{\mathbb{P}}(1) \) for \( t = \text{t}, \text{lm}, \text{logit} \). This elucidates the asymptotic equivalence between ReM and the joint criteria under the treatment-control experiment by Lemma S6.

Next, recall \( \hat{s}_{e,j} \) and \( \hat{s}_{e,j}' \) as the classic and eHW standard errors of \( \hat{\tau}_{x,j} \), respectively, from Remark S1 It follows from the Frisch–Waugh–Lovell theorem and Zhao and Ding (2021, Lemma S1) that

\[
\hat{s}_{e,j}^2 = \frac{1}{N - 2} \left( \frac{S_{2,j}^2}{S_Z^2} - \hat{\tau}_{x,j}^2 \right), \quad (\hat{s}_{e,j}')^2 = \frac{(Z - 1_N \hat{Z})^T \text{diag}(\hat{\tau}_{x,j}^2)^N (Z - 1_N \hat{Z})}{\| Z - 1_N Z \|_2^2}
\]
with \( Z = (Z_1, \ldots, Z_N)^T \), \( S_{x,j}^2 = (N - 1)^{-1} \sum_{i=1}^{N} x_{ij}^2 \), \( \epsilon_{ij} = x_{ij} - Z_i \hat{\tau}_{x,j} \), and

\[
N \hat{\sigma}_j^2 = (e_0 e_1)^{-1} S_{x,j}^2 + o_P(1), \quad N (\hat{\sigma}_j')^2 = (e_0 e_1)^{-1} S_{x,j}^2 + o_P(1).
\]

This ensures \( t_{j,t}' = \hat{\tau}_{x,j}/\hat{\sigma}_j' = t_{j,t} + o_P(1) \), and hence the asymptotic equivalence of the classic and EHW standard errors for constructing the marginal criterion under the two-sample \( t \)-tests. The results for other criteria are similar and thus omitted.

Importantly, the asymptotic equivalence between the classic and EHW standard errors does not hold for \( \hat{\sigma}_j' (\ast = n, f, l) \) and their classic counterparts based on the default outputs of \( \text{lm}(Y_i \sim 1 + Z_i) \), \( \text{lm}(Y_i \sim 1 + Z_i + x_i) \), and \( \text{lm}(Y_i \sim 1 + Z_i + x_i + Z_i x_i) \) in general. Specifically, the classic standard errors of \( \hat{\tau}_j' (\ast = n, f, l) \) are not necessarily asymptotically conservative for estimating the true sampling variances, and can thus lead to invalid inferences. As a result, the use of EHW standard errors is immaterial for rerandomization yet crucial for analysis.

S5. Proofs of the main results

S5.1. Asymptotic distributions in Propositions 1–4

Proof of Propositions 1 and 2. Let

\[
\begin{pmatrix}
A_s \\
B
\end{pmatrix} \sim \mathcal{N} \left\{ 0_{J+1}, \begin{pmatrix} v_s & c_s^T \\ c_s & v_x \end{pmatrix} \right\}
\]

with \( \sqrt{N} (\hat{\tau}_s - \tau, \hat{\tau}_x) \sim (A_s, B^T) \) for \( \ast = n, f, l \). Recall the definitions of \( A_{t,\phi} \) for \( t = t, \text{lm}, \text{logit} \) and \( \phi = jt, \text{mg}, \text{cs} \) in terms of the test statistics from Table S3.

Two-sample \( t \)-tests. For \( \ast = n, f, l \), let \( (A_N, B_N, C_N) = (\sqrt{N} (\hat{\tau}_s - \tau), \sqrt{N} \hat{\tau}_x, N \hat{\Omega}) \) with \( (A_N, B_N, C_N) \sim (A_s, B, v_x) \) by Lemma S8.

- Recall that \( A_{t,jt} = \{ p_{0,t} \geq \alpha_0 \} = \{ W_t \leq a_0 \} \) under the joint rule. We have

\[
\sqrt{N} (\hat{\tau}_s - \tau) \mid A_{t,jt} = \sqrt{N} (\hat{\tau}_s - \tau) \mid \{ \hat{\tau}_x^T \hat{\Omega}^{-1} \hat{\tau}_x \leq a_0 \} \\
\sim A_s \mid \{ B^T v_x^{-1} B \leq a_0 \} \\
\sim A_s \mid \{ \| B \|_M \leq a_0 \}
\]

by applying Lemma S7 to \( (A_N, B_N, C_N) \) and \( \phi_N(u, v) = \phi(u, v) = 1(u^T v^{-1} u \leq a_0) \).
• Recall that $A_{t,\text{mg}} = \{p_{j,t} \geq \alpha_j, j = 1, \ldots, J\} = \{|T_i| \leq a_t\}$ under the marginal rule. We have

$$
\sqrt{N}(\hat{\tau}_* - \tau) | A_{t,\text{mg}} = \sqrt{N}(\hat{\tau}_* - \tau) | \{|\sigma(\hat{\Omega})^{-1}\hat{\tau}_x| \leq a_t\} \\
\sim A_* | \{|\sigma(v_x)^{-1}B| \leq a\}
$$

by applying Lemma S7 to $(A_N, B_N, C_N)$, $\phi_N(u, v) = 1\{|\sigma(v)^{-1}u| \leq a_t\}$, and $\phi(u, v) = 1\{|\sigma(v)^{-1}u| \leq a\}$.

• Recall that $A_{t,\text{cs}} = \{p_{j,t} \geq \alpha_j, j = 0, 1, \ldots, J\} = \{W_t \leq a_0, \ |T_i| \leq a_t\}$ under the consensus rule. We have

$$
\sqrt{N}(\hat{\tau}_* - \tau) | A_{t,\text{cs}} = \sqrt{N}(\hat{\tau}_* - \tau) | \{\hat{\tau}_x^\top\hat{\Omega}^{-1}\hat{\tau}_x \leq a_0, \ |\sigma(\hat{\Omega})^{-1}\hat{\tau}_x| \leq a_t\} \\
\sim A_* | \{\|B\| \leq a_0, \ |\sigma(v_x)^{-1}B| \leq a\}
$$

by applying Lemma S7 to $(A_N, B_N, C_N)$, $\phi_N(u, v) = 1\{|u^Tv^{-1}u \leq a_0\} \cdot 1\{|\sigma(v)^{-1}u| \leq a_t\}$, and $\phi(u, v) = 1\{|u^Tv^{-1}u \leq a_0\} \cdot 1\{|\sigma(v)^{-1}u| \leq a\}$.

It thus suffices to compute

$$
A_* | \{\|B\| \leq a_0\}, \quad A_* | \{|\sigma(v_x)^{-1}B| \leq a\}, \quad A_* | \{\|B\| \leq a_0, \ |\sigma(v_x)^{-1}B| \leq a\},
$$

respectively.

To this end, write

$$
A_* = (A_* - c_x^Tv_x^{-1}B) + c_x^Tv_x^{-1}B, \quad (S10)
$$

with $A_* - c_x^Tv_x^{-1}B \sim \mathcal{N}(0, v_{L})$ and independent of $B$. This ensures

$$
A_* | \{\|B\| \leq a_0\} \sim (A_* - c_x^Tv_x^{-1}B) + c_x^Tv_x^{-1}B | \{\|B\| \leq a_0\} \\
\sim v_1^{1/2}T + c_x^Tv_x^{-1}(v_1^{1/2}L). \quad (S11)
$$

Likewise for the results under the marginal and consensus rules. In particular, let $\epsilon_t = \sigma(v_x)^{-1}B \sim \mathcal{N}\{0, D(v_x)\}$ to write

$$
B = \sigma(v_x)\epsilon_t, \quad \{|\sigma(v_x)^{-1}B| \leq a\} = \{\epsilon_t \leq a\}.
$$
This ensures

$$B \mid \{ |\sigma(v_x)^{-1}B| \leq a \} = \sigma(v_x)\epsilon_t \mid \{ |\epsilon_t| \leq a \}$$

$$\sim \sigma(v_x)\mathcal{T}_t,$$

$$B \mid \{ \|B\|_M \leq a_0, |\sigma(v_x)^{-1}B| \leq a \} = \sigma(v_x)\epsilon_t \mid \{ \|\epsilon_t\|_M \leq a_0, |\epsilon_t| \leq a \}$$

$$\sim \sigma(v_x)\mathcal{T}'_t,$$

and thus

$$A_* \mid \{ |\sigma(v_x)^{-1}B| \leq a \} \sim v_L^{1/2} \epsilon + c_s v_x^{-1} \sigma(v_x)\mathcal{T}_t,$$

$$A_* \mid \{ \|B\|_M \leq a_0, |\sigma(v_x)^{-1}B| \leq a \} \sim v_L^{1/2} \epsilon + c_s v_x^{-1} \sigma(v_x)\mathcal{T}'_t$$

by [S10].

**Linear regression.** For $* = N, F, \lambda$, let $(A_N, B_N, C_N) = (\sqrt{N}(\hat{\tau}_\lambda - \tau), \sqrt{N}\beta, N\hat{V})$ with $(A_N, B_N, C_N) \sim (A_*, v_x^{-1}B, v_x^{-1})$ by Lemma [S8]

- Recall that $A_{\text{lm}, \text{jt}} = \{ p_{0, \text{lm}} \geq \alpha_0 \} = \{ W_{\text{lm}} \leq a_t \}$ under the joint rule by Lemma [S1]. We have

  $$\sqrt{N}(\hat{\tau}_\lambda - \tau) \mid A_{\text{lm}, \text{jt}} = \sqrt{N}(\hat{\tau}_\lambda - \tau) \mid \{ \hat{\beta}^T\hat{V}^{-1}\hat{\beta} \leq a_t \}$$

  $$\sim A_* \mid \{ (v_x^{-1}B)^T v_x (v_x^{-1}B) \leq a_0 \}$$

  $$\sim A_* \mid \{ \|B\|_M \leq a_0 \}$$

  by applying Lemma [S7] to $(A_N, B_N, C_N), \phi_N(u, v) = 1(u^T v^{-1}u \leq a_t), \text{and } \phi(u, v) = 1(u^T v^{-1}u \leq a_0)$.

- Recall that $A_{\text{lm}, \text{mg}} = \{ p_{j, \text{lm}} \geq \alpha_j, j = 1, \ldots, J \} = \{ |T_{\text{lm}}| \leq a_{\text{lm}} \}$ under the marginal rule. We have

  $$\sqrt{N}(\hat{\tau}_\lambda - \tau) \mid A_{\text{lm}, \text{mg}} = \sqrt{N}(\hat{\tau}_\lambda - \tau) \mid \{ |\sigma(\hat{V})^{-1}\hat{\beta}| \leq a_{\text{lm}} \}$$

  $$\sim A_* \mid \{ |\sigma(v_x^{-1})v_x^{-1}B| \leq a \}$$

  by applying Lemma [S7] to $(A_N, B_N, C_N), \phi_N(u, v) = 1\{ |\sigma(v)^{-1}u| \leq a_{\text{lm}} \}, \text{and } \phi(u, v) = 1\{ |\sigma(v)^{-1}u| \leq a \}$.

- Recall that $A_{\text{lm}, \text{cs}} = \{ p_{j, \text{lm}} \geq \alpha_j, j = 0, 1, \ldots, J \} = \{ W_{\text{lm}} \leq a_t, |T_{\text{lm}}| \leq a_{\text{lm}} \}$ under the
consensus rule. We have

\[ \sqrt{N}(\hat{\tau}_s - \tau) | A_{\text{im,cs}} = \sqrt{N}(\hat{\tau}_s - \tau) | \{\hat{\beta}^T \hat{\beta} \leq a, \ |\sigma(\hat{\beta})| \leq a_{\text{im}} \} \]

\[ \sim A_s | \{(v_x^{-1} B)^T v_x (v_x^{-1} B) \leq a_0, \ |\sigma(v_x^{-1} B) \leq a \} \]

\[ \sim A_s | \{\|B\|_M \leq a_0, \ |\sigma(v_x^{-1} B) \leq a \} \]

by applying Lemma 7\textsuperscript{7} to \((A_N, B_N, C_N), \phi_N(u, v) = 1(u^T v^{-1} u \leq a) \cdot 1\{|\sigma(v)^{-1} u| \leq a_{\text{im}}\}, \) and \(\phi(u, v) = 1(u^T v^{-1} u \leq a_0) \cdot 1\{|\sigma(v)^{-1} u| \leq a\} \).

It thus suffices to compute \(A_s | \{\|B\|_M \leq a_0\}, \ A_s | \{|\sigma(v_x^{-1} B) \leq a\}, \) and \(A_s | \{\|B\|_M \leq a_0, \ |\sigma(v_x^{-1} B) \leq a\} \)

The distribution of \(A_s | \{\|B\|_M \leq a_0\} \) is given by (11). For \(A_s | \{|\sigma(v_x^{-1} B) \leq a\} \) and \(A_s | \{\|B\|_M \leq a_0, \ |\sigma(v_x^{-1} B) \leq a\} \), let \(\epsilon_{\text{im}} = \sigma(v_x^{-1} B) \sim N\{0, D(v_x^{-1})\} \)

\[ B = v_x \sigma(v_x^{-1}) \epsilon_{\text{im}}, \quad \{|\sigma(v_x^{-1} B) \leq a\} = \{\epsilon_{\text{im}} \leq a\}. \]

This ensures

\[ B | \{|\sigma(v_x^{-1} B) \leq a\} = v_x \sigma(v_x^{-1}) \epsilon_{\text{im}} | \{|\epsilon_{\text{im}}| \leq a\} \]

\[ \sim v_x \sigma(v_x^{-1}) T_{\text{im}} \]

\[ B | \{|\|B\|_M \leq a_0, \ |\sigma(v_x^{-1} B) \leq a\} = v_x \sigma(v_x^{-1}) \epsilon_{\text{im}} | \{|\epsilon_{\text{im}}| \leq a_0, \ |\epsilon_{\text{im}}| \leq a\} \]

\[ \sim v_x \sigma(v_x^{-1}) T'_{\text{im}} \]

and thus

\[ A_s | \{|\sigma(v_x^{-1} B) \leq a\} \sim v_l^{1/2} \epsilon + c_{v_{\text{im}}} \sigma(v_x^{-1}) T_{\text{im}} \]

\[ A_s | \{\|B\|_M \leq a_0, \ |\sigma(v_x^{-1} B) \leq a\} \sim v_l^{1/2} \epsilon + c_{v_{\text{im}}} \sigma(v_x^{-1}) T'_{\text{im}} \]

by 10\textsuperscript{7}.

**Logistic regression.** For \(* = N, F, L, \) let \((A_N, B_N, C_N) = (\sqrt{N}(\hat{\tau}_s - \tau), \sqrt{N} \hat{\beta}, N \hat{V}) \) with \((A_N, B_N, C_N) \leadsto (A_s, (S_x^2)^{-1} B, (e_0 e_1)^{-1} (S_x^2)^{-1}) \) by Lemma 8. We verify below that \((\hat{\tau}_s | A_{\text{logit, } \circ}) \sim (\hat{\tau}_s | A_{\text{im, } \circ}) \) for \(\circ = \text{jt, mg, cs}. \)

- Recall that \(A_{\text{logit, } \circ} = \{p_{0, \text{logit}} \geq a_0\} = \{\lambda_{\text{LRT}} \leq a_0\} \) under the joint rule. By \(\hat{Y}_s | \{\lambda_{\text{LRT}} \leq a_0\} \)
Recall that \( a_0 \sim \hat{Y}_* \mid \{ W_{\logit} \leq a_0 \} \) from Lemma \( \text{S9} \) we have

\[
\begin{align*}
\hat{\tau}_s \mid A_{\logit,jt} &= \hat{\tau}_s \mid \{ \lambda_{\text{LRT}} \leq a_0 \} \\
&\sim \hat{\tau}_s \mid \{ W_{\logit} \leq a_0 \} = \hat{\tau}_s \mid \{ \hat{\beta}^T \hat{V}^{-1} \hat{\beta} \leq a_0 \},
\end{align*}
\]

with

\[
\sqrt{N}(\hat{\tau}_s - \tau) \mid \{ \hat{\beta}^T \hat{V}^{-1} \hat{\beta} \leq a_0 \} \sim A_s \mid \{ B^T (S_x^2)^{-1} (e_0 e_1 S_x^2) (S_x^2)^{-1} B \leq a_0 \}
\]

by applying Lemma \( \text{S7} \) to \((A_N, B_N, C_N)\) and \( \phi_N(u, v) = \phi(u, v) = 1(u^T v^{-1} u \leq a_0) \). This ensures

\[
\sqrt{N}(\hat{\tau}_s - \tau) \mid A_{\logit,jt} \sim A_s \mid \{ \| B \|_M \leq a_0 \},
\]

identical to the limiting distribution of \( \sqrt{N}(\hat{\tau}_s - \tau) \mid A_{\text{lim,jt}}. \)

- Recall that \( A_{\logit,mg} = \{ p_j, \logit \geq \alpha_j,\ j = 1, \ldots, J \} = \{ |T_{\logit}| \leq a \} \) under the marginal rule. We have

\[
\sqrt{N}(\hat{\tau}_s - \tau) \mid A_{\logit,mg} = \sqrt{N}(\hat{\tau}_s - \tau) \mid \{ |\sigma(\hat{V})^{-1} \hat{\beta}| \leq a \}
\]

\[
\sim A_s \mid \{ |\sigma(v_x^{-1})^{-1} v_x^{-1} B| \leq a \}
\]

by applying Lemma \( \text{S7} \) to \((A_N, B_N, C_N)\) and \( \phi_N(u, v) = \phi(u, v) = 1\{ |\sigma(v)^{-1} u| \leq a \} \); in particular, the limit of \( \{ |\sigma(\hat{V})^{-1} \hat{\beta}| \leq a \} \) follows from

\[
\sigma(\hat{V})^{-1} \hat{\beta} \sim \sigma((e_0 e_1)^{-1} (S_x^2)^{-1}) (S_x^2)^{-1} B
\]

\[
= \sigma((e_0 e_1)^{-2} v_x^{-1}) (e_0 e_1) v_x^{-1} B = \sigma(v_x^{-1}) v_x^{-1} B
\]

given \( v_x = (e_0 e_1)^{-1} S_x^2 \) and \( \sigma((e_0 e_1)^{-2} v_x^{-1}) = (e_0 e_1)^{-1} \sigma(v_x^{-1}) \). This is identical to the limiting distribution of \( \sqrt{N}(\hat{\tau}_s - \tau) \mid A_{\text{lim,mg}}. \)

- Recall that \( A_{\logit,cs} = \{ p_j, \logit \geq \alpha_j,\ j = 0, 1, \ldots, J \} = \{ \lambda_{\text{LRT}} \leq a_0,\ |T_{\logit}| \leq a \} \) under the consensus rule. The same reasoning as in \( \text{(S12)} \) ensures

\[
\hat{\tau}_s - \tau \mid A_{\logit,cs} = \hat{\tau}_s - \tau \mid \{ \lambda_{\text{LRT}} \leq a_0,\ |\sigma(\hat{V})^{-1} \hat{\beta}| \leq a \}
\]

\[S28\]
\[ \sim \, \hat{\tau}_s - \tau \mid \{ \hat{\beta}^T \hat{V}^{-1} \hat{\beta} \leq a_0, \| \sigma(\hat{V})^{-1} \hat{\beta} \| \leq a_0 \} \]

with

\[ \sqrt{N}(\hat{\tau}_s - \tau) \mid \{ \hat{\beta}^T \hat{V}^{-1} \hat{\beta} \leq a_0, \| \sigma(\hat{V})^{-1} \hat{\beta} \| \leq a_0 \} \sim A_\ast \mid \{ \| B \|_M \leq a_0, \| \sigma(v^{-1}v^{-1}B) \| \leq a_0 \} \]

by applying Lemma S7 to \((A_N, B_N, C_N)\) and \(\phi_N(u, v) = \phi(u, v) = 1(u^T v u \leq a_0)\). This ensures

\[ \sqrt{N}(\hat{\tau}_s - \tau) \mid A_{\logit, cs} \sim A_\ast \mid \{ \| B \|_M \leq a_0, \| \sigma(v^{-1}v^{-1}B) \| \leq a_0 \} \]

identical to the limiting distribution of \(\sqrt{N}(\hat{\tau}_s - \tau) \mid A_{\lm, cs} \).

Proof of Proposition

Recall \(A_{f, mg} = \{ F_j \leq a_{j,t} \text{ for all } j = 1, \ldots, J \} \) from S7 with the explicit forms of \(F_j\) in S6. With \((N - 1)S_{x,j}^2 = \sum_{i=1}^N x_{ij}^2 = \sum_{q \in Q} \sum_{z_i: z_i = q} \{ x_{ij} - \hat{x}_j(q) \}^2 + \sum_{q \in Q} N_q \hat{x}_j^2(q)\) by direct algebra, the expression of \(A_{f, mg}\) can be simplified to

\[ A_{f, mg} = \{ F_j \leq a_{j,t} \text{ for all } j \} = \left\{ \sum_{q \in Q} e_q \{ \sqrt{N} \hat{x}_j(q) \}^2 \leq a''_j \text{ for all } j \right\} , \]

where

\[ a''_j = \frac{(N - 1)S_{x,j}^2}{(N - Q)/\{(Q - 1)a_{j,t}\} + 1} = a'_j S_{x,j}^2 + o(1) \]

by \(a_{j,t} = (Q - 1)^{-1}a'_j + o(1)\).

Observe that \(\sqrt{N} \hat{x} \sim \epsilon_f\) and \(\sqrt{N} \hat{x}_j(q) \sim \epsilon_{f,j}\) by Lemma S2. The result follows from Lemmas S7 and S9 as

\[ \sqrt{N}(\hat{Y}_s - \bar{Y}) \mid A_{f, mg} \]

\[ = \sqrt{N}(\hat{Y}_s - \bar{Y} - \Gamma_s \hat{x}) + \Gamma_s \sqrt{N} \hat{x} \mid \left\{ \sum_{q \in Q} e_q \{ \sqrt{N} \hat{x}_j(q) \}^2 \leq a''_j \text{ for all } j \right\} \]

\[ \sim V_1^{1/2} \epsilon + \Gamma_s \epsilon_f \mid \left\{ \sum_{q \in Q} e_q \epsilon_{f,j}^2 \leq a'_j S_{x,j}^2 \text{ for all } j \right\} . \]
Proof of Proposition 4. Renew \( a_0 \) as the \((1 - \alpha_0)\)th quantile of the \( \chi^2_{(Q-1)} \) distribution. Renew \( a = (a_{qj})_{q \in Q_+; j=1,...,J} \), where \( a_{qj} \) denotes the \((1 - \alpha_{qj}/2)\)th quantile of the standard normal distribution.

The marginal, joint, and consensus criteria based on the multinomial logistic regression equal

\[
\begin{align*}
A_{\logit, mg} &= \{|T_{\logit}| \leq a\}, \\
A_{\logit, jt} &= \{\lambda_{\text{LRT}} \leq a_0\}, \\
A_{\logit, cs} &= \{\lambda_{\text{LRT}} \leq a_0, |T_{\logit}| \leq a\},
\end{align*}
\]  

(S13)

respectively, with \( T_{\logit} = \text{diag}(\tilde{V}_q^{-1/2})\tilde{\beta} = \sigma(\tilde{V})^{-1}\tilde{\beta} \) and \( \lambda_{\text{LRT}} - \tilde{\beta}^T\tilde{V}^{-1}\tilde{\beta} = o_p(1) \).

Let \( A \sim \mathcal{N}(0, V_L) \) and \( B \sim \mathcal{N}(0, J_{(Q-1)}, V_x) \) be two independent normal random vectors with \( \text{cov}(\Psi B) = V_\Psi \). For \(* = \text{n, f, l} \), let \((A_N, B_N, C_N) = (\sqrt{N}(\tilde{Y}_* - \bar{Y}), \sqrt{N}\tilde{\beta}, N\tilde{V}) \) with \((A_N, B_N, C_N) \sim (A + \Gamma_* B, \Psi B, V_\Psi) \) by Lemma S9.

- Under the joint rule, Lemma S9 ensures

\[
\sqrt{N}(\tilde{Y}_* - \bar{Y}) | A_{\logit, jt} = \tilde{Y}_* | \{\lambda_{\text{LRT}} \leq a_0\} \sim \tilde{Y}_* | \{\tilde{\beta}^T\tilde{V}^{-1}\tilde{\beta} \leq a_0\}
\]  

(S14)

with

\[
\sqrt{N}(\tilde{Y}_* - \bar{Y}) | \{\tilde{\beta}^T\tilde{V}^{-1}\tilde{\beta} \leq a_0\} \sim A + \Gamma_* B | \{(\Psi B)^T V_\Psi^{-1} \Psi B \leq a_0\}
\]

\sim 
A + \Gamma_* B | \{||B||_M \leq a_0\}

by applying Lemma S7 to \((A_N, B_N, C_N)\) and \( \phi_N(u, v) = \phi(u, v) = 1(u^T v^{-1} u \leq a_0) \). This ensures

\[
\sqrt{N}(\tilde{Y}_* - \bar{Y}) | A_{\logit, jt} \sim A + \Gamma_* B | \{||B||_M \leq a_0\}.
\]

- Under the marginal rule, we have

\[
\sqrt{N}(\tilde{Y}_* - \bar{Y}) | A_{\logit, mg} = \sqrt{N}(\tilde{Y}_* - \bar{Y}) | \{||\sigma(\tilde{V})^{-1}\tilde{\beta}|| \leq a\}
\]

\sim 
A + \Gamma_* B | \{||\sigma(\tilde{V})^{-1}\Psi B|| \leq a\}

by applying Lemma S7 to \((A_N, B_N, C_N)\) and \( \phi_N(u, v) = \phi(u, v) = 1\{||\sigma(v)^{-1}|| \leq a\} \).
• Under the consensus rule, we have

\[ \hat{Y}_s \mid A_{\logit,cs} = \hat{Y}_s \mid \{\lambda_{\text{lrt}} \leq a_0, |\sigma(\hat{V})^{-1}\hat{\beta}| \leq a\} \sim \hat{Y}_s \mid \{\hat{\beta}^T \hat{V}^{-1} \hat{\beta} \leq a_0, |\sigma(\hat{V})^{-1}\hat{\beta}| \leq a\} \]

by (S14) with

\[ \sqrt{N}(\hat{Y}_s - \bar{Y}) \mid \{\hat{\beta}^T \hat{V}^{-1} \hat{\beta} \leq a_0, |\sigma(\hat{V})^{-1}\hat{\beta}| \leq a\} \sim A + \Gamma_s' B \mid \|B\|_M \leq a_0, |\sigma(V_\phi)^{-1}\Psi B| \leq a \]

by applying Lemma S7 to \((A_N, B_N, C_N)\) and \(\phi_N(u, v) = \phi(u, v) = 1(u^T v - 1 u \leq a_0) \cdot 1(|\sigma(v)^{-1} u| \leq a)\). This ensures

\[ \sqrt{N}(\hat{Y}_s - \bar{Y}) \mid A_{\logit,cs} \sim A + \Gamma_s' B \mid \|B\|_M \leq a_0, |\sigma(V_\phi)^{-1}\Psi B| \leq a \].

With \(A \sim V_L^{1/2} \epsilon\) and independent of \(B\), it suffices to compute \(B \mid \{\|B\|_M \leq a_0\}, B \mid \{|\sigma(V_\phi)^{-1}\Psi B| \leq a\}, \) and \(B \mid \{\|B\|_M \leq a_0, |\sigma(V_\phi)^{-1}\Psi B| \leq a\}, \) respectively.

For the joint criterion, we have \(B \mid \{\|B\|_M \leq a_0\} \sim V_L^{1/2} L\), and thus

\[ A + \Gamma_s' B \mid \{\|B\|_M \leq a_0\} \sim V_L^{1/2} \epsilon + \Gamma_s' V_L^{1/2} L. \]

For the marginal and consensus criteria, let \(\epsilon_{\logit} = \sigma(V_\phi)^{-1}\Psi B \sim \mathcal{N}(0, D(V_\phi))\) to write

\[ B = \Psi^{-1}\sigma(V_\phi)\epsilon_{\logit}, \quad \{|\sigma(V_\phi)^{-1}\Psi B| \leq a\} = \{ |\epsilon_{\logit}| \leq a\}. \]

This ensures

\[ B \mid \{|\sigma(V_\phi)^{-1}\Psi B| \leq a\} = \Psi^{-1}\sigma(V_\phi)\epsilon_{\logit} \mid \{|\epsilon_{\logit}| \leq a\} \sim \Psi^{-1}\sigma(V_\phi) T_{\logit}, \]

\[ B \mid \{\|B\|_M \leq a_0, |\sigma(V_\phi)^{-1}\Psi B| \leq a\} = \Psi^{-1}\sigma(V_\phi)\epsilon_{\logit} \mid \{|\epsilon_{\logit}| \leq a, \|\epsilon_{\logit}\|_M \leq a_0\} \sim \Psi^{-1}\sigma(V_\phi) T'_{\logit}, \]

and thus

\[ A + \Gamma_s' B \mid \{|\sigma(V_\phi)^{-1}\Psi B| \leq a\} \sim V_L^{1/2} \epsilon + \Gamma_s' \Psi^{-1}\sigma(V_\phi) T_{\logit}, \]
\[ A + \Gamma' B \mid \{ \| B \|_{\mathcal{M}} \leq a_0, |\sigma(V_\Psi)^{-1} \Psi B| \leq a \} \sim V_{L_1}^{1/2} + \Gamma' \Psi^{-1} \sigma(V_\Psi) T_{logit}'. \]

\[ \square \]

S5.2. Covariate balance and asymptotic relative efficiency in Theorems 1–4

**Proof of Theorems 1–4.** The asymptotic relative efficiency of \( \hat{\tau}_* \) \((* = N, F, L)\) follows immediately from Propositions 1–4 and Lemmas S4–S5. We verify below the improved covariate balance under the ReP schemes based on the multinomial logistic regression. The results under the two-sample \( t \)-test-based, linear or logistic regression-based, and marginal \( F \)-test-based criteria are analogous and thus omitted. Note that the asymptotic conditional bias and the difference between different estimators are all functions of \( \hat{\tau}_* \). The results associated with the covariance reduction factor \( \rho(J, a_0) \) under the joint rules then follow from the results on ReM from Morgan and Rubin (2012) and Li et al. (2018).

Recall the acceptance criteria by test statistics from (S13). Recall from (S9) that \( \hat{x} = \kappa \hat{x}_+ \) and hence \( \hat{\tau}_* = G_x \hat{x} = G_x \kappa \hat{x}_+ \). Let \( A \sim N(0, (Q-1)_x, V_{x+}) \) with \( \sqrt{N} \hat{x}_+ \sim A \) and \( \sqrt{N} \hat{\tau}_* \sim G_x \kappa A \) by Lemma S9. The result follows from

\[
\sqrt{N} \hat{\tau}_* \mid A_{logit,jt} = G_x \kappa (\sqrt{N} \hat{x}_+) \mid \{ \lambda_{LRT} \leq a_0 \}
\sim G_x \kappa (\sqrt{N} \hat{x}_+) \mid \{ \sqrt{N} x^T V_{x+}^{-1} \hat{x}_+ \leq a_0 \}
\sim G_x \kappa A \mid \{ A^T V_{x+}^{-1} A \leq a_0 \}
\geq G_x \kappa A,
\]

\[
\sqrt{N} \hat{\tau}_* \mid A_{logit,mg} = G_x \kappa (\sqrt{N} \hat{x}_+) \mid \{ |\sigma(\bar{V})^{-1} \bar{\beta}| \leq a \}
\sim G_x \kappa (\sqrt{N} \hat{x}_+) \mid \{ |\sigma(V_\Psi)^{-1} \sqrt{N} \Psi \hat{x}_+| \leq a \}
\sim G_x \kappa A \mid \{ |\sigma(V_\Psi)^{-1} \Psi A| \leq a \}
\geq G_x \kappa A,
\]

\[
\sqrt{N} \hat{\tau}_* \mid A_{logit,cs} = G_x \kappa (\sqrt{N} \hat{x}_+) \mid \{ \lambda_{LRT} \leq a_0, |\sigma(\bar{V})^{-1} \bar{\beta}| \leq a \}
\sim G_x \kappa (\sqrt{N} \hat{x}_+) \mid \{ \sqrt{N} x^T V_{x+}^{-1} \hat{x}_+ \leq a_0, |\sigma(V_\Psi)^{-1} \sqrt{N} \Psi \hat{x}_+| \leq a \}
\sim G_x \kappa A \mid \{ A^T V_{x+}^{-1} A \leq a_0, |\sigma(V_\Psi)^{-1} \Psi A| \leq a \}
\geq G_x \kappa A.
\]

In particular, the three \( \sim \) follow from \( \lambda_{LRT} - \sqrt{N} x^T V_{x+}^{-1} \hat{x}_+ = o_P(1) \), \( \sqrt{N}(\bar{\beta} - \Psi \hat{x}_+) = o_P(1) \), and \( N \bar{V} - V_\Psi = o_P(1) \); the three \( \geq \) follow from Corollary S7.
S6. Proof of Theorem S1

We verify in this section the properties of the logistic and multinomial logistic regressions in Theorem S1. The results ensure the asymptotic equivalence of the LRT and the Wald test for logistic and multinomial logistic regressions from the design-based perspective. To this end, we first review some useful numeric facts about the multinomial logistic regression in Section S6.1, and then give the proof in Section S6.2.

Recall that $\hat{x}_+ = (e_1, \ldots, e_{Q-1})^T$, $\text{diag}(e_+)_q \in \mathcal{Q}_+$, and $\Psi = \{\Phi^{-1} \text{diag}(e_+)\} \otimes (S_x^2)^{-1}$, with $\Phi = \text{diag}(e_+) - e_+ e_+^T$. Let $R_+ = \text{diag}(e_+)$ be a shorthand for $\text{diag}(e_+)$ to write $\Psi = (\Phi^{-1} R_+) \otimes (S_x^2)^{-1}$ with

$$
\Phi = R_+ - e_+ e_+^T = 
\begin{pmatrix}
  e_1(1-e_1) & -e_1 e_2 & \ldots & -e_1 e_{Q-1} \\
  -e_2 e_1 & e_2(1-e_2) & \ldots & -e_2 e_{Q-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  -e_{Q-1} e_1 & -e_{Q-1} e_2 & \ldots & e_{Q-1}(1-e_{Q-1})
\end{pmatrix}.
$$

Recall that $\hat{x}_+ = (\hat{x}(1)^T, \ldots, \hat{x}(Q-1)^T)^T$, with $V_{x_+} = N \text{cov}(\hat{x}_+)$. Then $V_{x_+} = (R_+^{-1} - 1_{(Q-1)\times(Q-1)}) \otimes S_x^2$ equals the upper $J(Q-1) \times J(Q-1)$ submatrix of $V_x = \{\text{diag}(e_q^{-1})_q \in \mathcal{Q} - 1_{Q\times Q}\} \otimes S_x^2$. This ensures

$$
V_{\Psi} = N \text{cov}(\Psi \hat{x}_+) = \Psi V_{x_+} \Psi^T
\quad \text{(S15)}
$$

$$
= \{\Phi^{-1} R_+\} \otimes (S_x^2)^{-1} \setminus \{R_+^{-1} - 1_{(Q-1)\times(Q-1)}\} \otimes S_x^2 \setminus \{\Phi^{-1} R_+^T \otimes (S_x^2)^{-1}\}
= \{\Phi^{-1} R_+ (R_+^{-1} - 1_{(Q-1)\times(Q-1)}) R_+ \Phi^{-1}\} \otimes (S_x^2)^{-1}
= \Phi^{-1} \otimes (S_x^2)^{-1}
$$

given $\Phi = R_+ (R_+^{-1} - 1_{(Q-1)\times(Q-1)}) R_+$. As a result, we have $\Psi = V_{\Psi} (R_+ \otimes I_J)$ and hence

$$
(R_+ \otimes I_J) V_{\Psi} (R_+ \otimes I_J) = \Psi^T V_{\Psi}^{-1} \Psi = V_{x_+}^{-1}.
\quad \text{(S16)}
$$
S6.1. Numeric facts about the multinomial logistic regression

Recall the multinomial logistic model from (S4). We have

\[ \pi_q(x_i) = \pi_q(\tilde{x}_i, \theta) = \frac{\exp(\tilde{x}_i^T \theta_q)}{1 + \sum_{q' \in Q_+} \exp(\tilde{x}_i^T \theta_{q'})} \quad \text{for} \quad q = 1, \ldots, Q, \tag{S17} \]

with \( \tilde{x}_i = (1, x_i^T)^T \), \( \theta_Q = 0_{J+1} \), \( \theta_q = (\beta_{q0}, \beta_q^T) \) for \( q \in Q_+ \), and \( \theta = (\theta_1^T, \ldots, \theta_{Q-1}^T)^T \). The scaled log-likelihood function of \((x_i, Z_i)^N_{i=1}\) equals

\[ \bar{L}(\theta) = N^{-1} \sum_{i=1}^N \log \{ \pi_{Z_i}(x_i) \} \]

\[ = N^{-1} \sum_{i=1}^N \left[ \sum_{q \in Q_+} I_{iq} \tilde{x}_i^T \theta_q - \log \left\{ 1 + \sum_{q' \in Q_+} \exp(\tilde{x}_i^T \theta_{q'}) \right\} \right]. \tag{S18} \]

The score function of \( \bar{L}(\theta) \) equals

\[ U(\theta) = \frac{\partial \bar{L}(\theta)}{\partial \theta} = \begin{pmatrix} U_1(\theta) \\ \vdots \\ U_{Q-1}(\theta) \end{pmatrix} \tag{S19} \]

with

\[ U_q(\theta) = \frac{\partial \bar{L}(\theta)}{\partial \theta_q} = N^{-1} \sum_{i=1}^N \tilde{x}_i \{ I_{iq} - \pi_q(\tilde{x}_i, \theta) \} \quad (q \in Q_+). \]

The Hessian matrix of \( \bar{L}(\theta) \) equals

\[ H(\theta) = \frac{\partial^2 \bar{L}(\theta)}{\partial \theta \partial \theta^T}, \tag{S20} \]

with the explicit form given by Condition 3.

S6.2. The proof

Lemmas [S10] and [S11] provide the basis for proving Theorem [S1].

Lemma S10. (Rudin 1976, Theorem 7.17) Suppose \( \{ f_N(x) \}_{N=1}^\infty \) is a sequence of functions, differentiable on \([a, b]\) and such that \( \{ f_N(x_0) \}_{N=1}^\infty \) converges for some point \( x_0 \) on \([a, b]\). If the sequence of derivatives, \( \{ f'_N(x_0) \}_{N=1}^\infty \), converges uniformly on \([a, b]\), then \( \{ f_N(x) \}_{N=1}^\infty \) converges uniformly on

S34
\([a, b]\), to a function \(f(x)\), and

\[
\lim_{N \to \infty} f_N'(x) = f'(x) \quad (a \leq x \leq b).
\]

**Lemma S11.** (Newey and McFadden [1994] Theorem 2.7) If there is a function \(Q_0(\theta)\) such that (i) \(Q_0(\theta)\) is uniquely maximized at \(\theta_0\); (ii) \(\theta_0\) is an element of the interior of a convex set \(\Theta\) and \(Q_N(\theta)\) is concave; and (iii) \(Q_N(\theta) = Q_0(\theta) + o_P(1)\) for all \(\theta \in \Theta\), then the maximizer of \(Q_N(\theta)\), denoted by \(\hat{\theta}_N\), exists with probability approaching one and \(\hat{\theta}_N = \theta_0 + o_P(1)\).

**Proof of Theorem S1.** Zhao and Ding (2021, Lemma S5) ensures that \(\hat{x}(q) = o(1)\) for all \(q \in Q\) almost surely under Condition 1. For notational simplicity, we assume that (S21) is true in the following proof. The simplification does not affect the validity of the proof given all results concern either convergence in probability or convergence in distribution.

Let \(\hat{\beta}_q\) and \(\hat{\theta}_q = (\hat{\beta}_{q0}, \hat{\beta}_q^T)^T\) be the MLEs of \(\beta_q\) and \(\theta_q = (\beta_{q0}, \beta_q^T)^T\) for \(q \in Q\), respectively, concatenated as \(\hat{\beta} = (\hat{\beta}_1^T, \ldots, \hat{\beta}_{Q-1}^T)^T\) and \(\hat{\theta} = (\hat{\theta}_1^T, \ldots, \hat{\theta}_{Q-1}^T)^T\).

**Convergence of \(\hat{\theta}\).** As a key intermediate result, we first verify

\[
\tilde{\theta} = \theta^* + o_P(1) \tag{S22}
\]

for

\[
\theta^* = ((\theta_1^*)^T, \ldots, (\theta_{Q-1}^*)^T)^T, \quad \text{where} \quad \theta_q^* = (\beta_{q0}^*, 0_T^T) \quad \text{with} \quad \beta_{q0}^* = \log(e_q/e_Q).
\]

By Lemma S11 it suffices to show that there exists a function \(\bar{L}_\infty(\theta)\) such that (i) \(\bar{L}_\infty(\theta)\) is uniquely maximized at \(\theta^*\); (ii) \(\bar{L}(\theta)\) is concave on \(\mathbb{R}^{(J+1)(Q-1)}\); and (iii) \(\bar{L}(\theta) = \bar{L}_\infty(\theta) + o(1)\) for all \(\theta \in \mathbb{R}^{(J+1)(Q-1)}\). We verify below these three sufficient conditions in the order of (iii) to (i) to (ii).

First, it follows from \(\bar{x}_i^T \theta_q^* = \beta_{q0}^* = \log(e_q/e_Q)\) that \(\pi_q(\bar{x}_i, \theta^*) = e_q\) for all \(q \in Q\) and \(i = 1, \ldots, N\) by (S17). Plug this in the expressions of \(\bar{L}(\theta)\), \(U(\theta)\), and \(H(\theta)\) from (S18), (S19), and Condition 3 to see

\[
\bar{L}(\theta^*) = \sum_{q \in Q} e_q \log(e_q),
\]
\[ U(\theta^*) = \begin{pmatrix} U_1(\theta^*) \\ \vdots \\ U_{Q-1}(\theta^*) \end{pmatrix}, \quad \text{where} \quad U_q(\theta^*) = \begin{pmatrix} 0 \\ e_q \hat{x}(q) \end{pmatrix}, \quad (S23) \]

\[ H(\theta^*) = -\Phi \otimes (\delta S^2_x) \quad \text{with} \quad H_{qq}(\theta^*) = e_q(e_q - 1(q = q'))(\delta S^2_x), \]

where \( \delta = 1 - N^{-1} \) and \( S^2_x = (N - 1)^{-1} \sum_{i=1}^N \tilde{x}_i\tilde{x}_i^T = \text{diag}(\delta^{-1}, S^2_x) \). Under (S21), this ensures that \( \bar{L}(\theta), U(\theta), \) and \( H(\theta) \) all converge pointwise at \( \theta = \theta^* \), with

\[ H^* = H_\infty(\theta^*) = -\Phi \otimes \text{diag}(1, S^2_x) < 0. \quad (S24) \]

In addition, Condition 3 ensures that \( \partial U(\theta)/\partial \theta = H(\theta) \) converges uniformly to \( H_\infty(\theta) \) on any compact set in \( \mathbb{R}^{(J+1)(Q-1)} \). Let \( U_{qj}(\theta) = \partial \bar{L}(\theta)/\partial \beta_{qj} \) be the \((q,j)\)th element of \( U(\theta) \) for \( q \in Q_+ \) and \( j = 0, 1, \ldots, J \). Applying Lemma S10 component-wise to \( f_N = U_{qj} \) ensures that there exists a function, denoted by \( U_\infty(\theta) \), such that

\[ U(\theta) = U_\infty(\theta) + o(1), \quad \partial U_\infty(\theta)/\partial \theta = H_\infty(\theta) \]

for all \( \theta \in \mathbb{R}^{(J+1)(Q-1)} \), and the convergence is uniform on any compact set in \( \mathbb{R}^{(J+1)(Q-1)} \).

Sufficient condition (iii) then follows from applying Lemma S10 component-wise to \( f_N = \bar{L} \), which ensures that there exists a function, denoted by \( \bar{L}_\infty(\theta) \), such that

\[ \bar{L}(\theta) = \bar{L}_\infty(\theta) + o(1), \quad \partial \bar{L}_\infty(\theta)/\partial \theta = U_\infty(\theta) \]

for all \( \theta \in \mathbb{R}^{(J+1)(Q-1)} \).

Sufficient condition (i) then follows from \( \partial^2 \bar{L}_\infty(\theta)/\partial \theta \partial \theta^T = H_\infty(\theta) < 0 \) by Condition 3 and \( U_\infty(\theta^*) = \lim_{N \to \infty} U(\theta^*) = 0 \) by (S23).

For sufficient condition (ii), let \( H_i(\theta) = (H_{i,qq'}(\theta))_{q,q' \in Q_+} \) with

\[ H_{i,qq'}(\theta) = \pi_q(\tilde{x}_i, \theta)(\pi_{q'}(\tilde{x}_i, \theta) - 1(q = q'))\tilde{x}_i\tilde{x}_i^T. \]

Then \( H(\theta) = N^{-1} \sum_{i=1}^N H_i(\theta) \) by the explicit form of \( H(\theta) \) in Condition 3. Observe that \( H_i(\theta) = -\Phi_i \otimes (\tilde{x}_i\tilde{x}_i^T) \), with \( \Phi_i = (\Phi_{i,qq'})_{q,q' \in Q_+} \) where \( \Phi_{i,qq'} = \pi_q(\tilde{x}_i, \theta)\{1(q = q') - \pi_{q'}(\tilde{x}_i, \theta)\} \). It follows from \( \Phi_1 \geq 0 \) that \( H_i(\theta) \leq 0 \) and hence \( H(\theta) \leq 0 \).

**Asymptotic equivalence of \( \hat{\beta} \) and \( \Psi \hat{x}_+ \).** We next verify \( \sqrt{N}(\hat{\beta} - \Psi \hat{x}_+) = o_p(1) \). The proof follows from the same reasoning as that of [Newey and McFadden (1994)](1994) Theorem 3.1) for independent and
identically distributed samples.

Recall $U_{qj}(\theta) = \partial \bar{L}(\theta)/\partial \beta_{qj}$ as the $(qj)$th element of $U(\theta)$ for $q \in Q_+$ and $j = 0, 1, \ldots, J$. Let $H_{qj}(\theta) = \partial U_{qj}(\theta)/\partial \theta \in \mathbb{R}^{(J+1)(Q-1)}$, with

$$H(\theta) = \partial U(\theta)/\partial \theta^T = (H_{1,0}(\theta), H_{1,1}(\theta), \ldots, H_{Q-1,J}(\theta))^T.$$ 

Expanding $U_{qj}(\theta)$ at $\theta^*$ yields

$$0 = U_{qj}(\tilde{\theta}) = U_{qj}(\theta^*) + \{H_{qj}(\theta^*)\}^T(\tilde{\theta} - \theta^*), \quad (S25)$$

where $\theta^* \in \mathbb{R}^{(J+1)(Q-1)}$ is a point on the line segment between $\tilde{\theta}$ and $\theta^*$. That $\tilde{\theta} = \theta^* + o_P(1)$ from (S22) ensures $\theta_{qj}' = \theta^* + o_P(1)$ and hence $H_{qj}(\theta^*) = H_{qj}(\theta^*) + o_P(1)$ for all $q \in Q_+$ and $j = 0, 1, \ldots, J$.

Let $H'$ be the matrix with rows $H_{qj}(\theta_{qj}')$ in lexicographical order of $(qj)$. Then $H' = H^* + o_P(1)$ with $H^* < 0$ and hence

$$1_{|H'|} = 1(H' \text{ is nonsingular}) = 1 + o_P(1)$$

by Condition 3. Stacking (S25) in lexicographical order of $(qj)$ yields $0 = U(\theta^*) + H'(\tilde{\theta} - \theta^*)$ and hence

$$1_{|H'|}\sqrt{N}(\tilde{\theta} - \theta^*) = -1_{|H'|}(H')^{-1}\sqrt{N}U(\theta^*).$$

This, together with $\sqrt{N}U(\theta^*)$ being asymptotically normal by (S23) and Lemma S2, ensures

$$\sqrt{N}(\tilde{\theta} - \theta^*) = O_P(1) \quad (S26)$$

and hence

$$\sqrt{N}(\tilde{\theta} - \theta^*) = 1_{|H'|}\sqrt{N}(\tilde{\theta} - \theta^*) + (1 - 1_{|H'|})\sqrt{N}(\tilde{\theta} - \theta^*) \quad (S27)$$

$$= -1_{|H'|}(H')^{-1}\sqrt{N}U(\theta^*) + (1 - 1_{|H'|})\sqrt{N}(\tilde{\theta} - \theta^*)$$

$$= -(H^*)^{-1}\sqrt{N}U(\theta^*) + o_P(1)$$

by Slutsky’s theorem. Observe that $(H^*)^{-1} = -\Phi^{-1} \otimes \text{diag}\{1, (S^2_\delta)^{-1}\}$ from (S24). Removing the
dimensions corresponding to \( \{ \beta_q : q \in \mathcal{Q} \} \) in (S28) yields

\[
\sqrt{N} \hat{\beta} = \{ \Phi^{-1} \otimes (S_2^q)^{-1} \} \sqrt{N}(R_\pi \otimes I_J)\hat{x}_+ + o_P(1) = \sqrt{N} \Psi \hat{x}_+ + o_P(1)
\]

by the explicit form of \( U(\theta^*) \) from (S23).

**Asymptotic normality of \( \hat{\beta} \).** That \( \sqrt{N} \hat{\beta} \sim \mathcal{N}(0, M_{Q-1}V_\psi) \) then follows from the asymptotic normality of \( \hat{x}_+ \) by Lemma [S9] and Slutsky’s theorem.

**Convergence of \( N \bar{V} \).** Let \( \bar{H} = H(\hat{\theta}) \) be the value of \( H(\theta) \) evaluated at the MLE. Then \( \bar{V} \) equals the submatrix of \((-N\bar{H})^{-1}\) after removing the rows and columns corresponding to the \( Q-1 \) intercepts, namely \( \{ \beta_q : q \in \mathcal{Q}_+ \} \). That \( N \bar{V} = V_\psi + o_P(1) \) follows from

\[
\bar{H}^{-1} = (H^*)^{-1} + o_P(1) = -\Phi^{-1} \otimes \text{diag}\{1, (S_2^q)^{-1}\} + o_P(1)
\]

by (S22) and (S24), and the definition of the Kronecker product.

**Asymptotic equivalence of \( \lambda_{LRT} \) and \( N\hat{\bar{x}}_+^T V_\psi^{-1} \hat{x}_+ \).** The LRT tests \( \logit(Z_i \sim 1 + x_i) \) against \( H_0 : \logit(Z_i \sim 1) \). Let \( \Theta_0 = \{ \theta = (\theta_1^T, \ldots, \theta_{Q-1}^T)^T : \theta_q = (\beta_q, 0_J)^T \} \) be the restricted parameter space under \( H_0 \), with \( \bar{\theta}_0 \in \Theta_0 \) as the MLE. The test statistic equals

\[
\lambda_{LRT} = -2N \left\{ \sup_{\theta \in \Theta_0} \bar{L}(\theta) - \sup_{\theta \in \mathbb{R}^{(j+1)(Q-1)}} \bar{L}(\theta) \right\}
\]

For \( \theta \in \Theta_0 \), we have \( \hat{x}_i^T \theta_q = \beta_q \) such that (S18) reduces to \( \bar{L}(\theta) = \sum_{q \in \mathcal{Q}} e_q \log(\pi_q) \), with

\[
\pi_q = \frac{\exp(\beta_q)}{1 + \sum_{q' \in \mathcal{Q}_+} \exp(\beta_{q'})}
\]

denoting the identical value of \( \pi_q(x_i) \) across \( i = 1, \ldots, N \); see (S17). The invariance of MLE to non-degenerate transformation of the parameters ensures that the MLEs of \( \pi_q \) and \( \beta_q \) equal \( e_q \) and \( \log(e_q/e_Q) = \beta_q^* \), respectively, for \( q \in \mathcal{Q}_+ \). This ensures \( \bar{\theta}_0 = \theta^* \) and hence

\[
\lambda_{LRT} = 2N \{ \bar{L}(\bar{\theta}) - \bar{L}(\theta^*) \}. \tag{S28}
\]

We verify below \( \lambda_{LRT} - N\hat{\bar{x}}_+^T V_\psi^{-1} \hat{x}_+ = o_P(1) \).

First, \( \bar{L}(\theta^*) = \bar{L}(\bar{\theta}) + 2^{-1}(\theta^* - \bar{\theta})^T H(\theta'')(\theta^* - \bar{\theta}) \) for some \( \theta' \) on the line segment of \( \bar{\theta} \) and \( \theta^* \). This, together with (S28), ensures

\[
\lambda_{LRT} = -N(\bar{\theta} - \theta^*)^T H(\theta')(\bar{\theta} - \theta^*) = -N(\bar{\theta} - \theta^*)^T H^*(\bar{\theta} - \theta^*) + o_P(1)
\]

S38
given \( H(\theta') = H^* + o_P(1) \) by \([S22]\) and \( \sqrt{N}(\hat{\theta} - \theta^*) = O_P(1) \) by \([S26]\).

Next, it follows from \([S28]\) that

\[-N(\tilde{\theta} - \theta^*)^T H^*(\tilde{\theta} - \theta^*) = -N\{U(\theta^*)\}^T (H^*)^{-1} U(\theta^*) + o_P(1).\]

The result then follows from

\[-N\{U(\theta^*)\}^T (H^*)^{-1} U(\theta^*)
= N(e_1 \hat{x}(1)^T, \ldots, e_{Q-1} \hat{x}(Q-1)^T)\{\Phi^{-1} \otimes (S^2_x)^{-1}\}
= N\{\hat{x}_+^T (R_+ \otimes I_J)\} V_{\Psi}\{(R_+ \otimes I_J)\hat{x}_+\}
= N\hat{x}_+^T V_{x+}^{-1} \hat{x}_+\]

by \([S23]–[S24]\) and \([S15]–[S16]\).

**Asymptotic equivalence of \( W_{\text{logit}} \) and \( N\hat{x}_+^T V_{x+}^{-1} \hat{x}_+ \).** The result follows from

\[W_{\text{logit}} = \tilde{\beta}^T \tilde{V}^{-1} \tilde{\beta} = N(\Psi \hat{x}_+)^T V_{\Psi}^{-1} (\Psi \hat{x}_+) + o_P(1) = N\hat{x}_+^T V_{x+}^{-1} \hat{x}_+ + o_P(1)\]

with \( \sqrt{N}(\tilde{\beta} - \Psi \tilde{x}_+) = o_P(1) \), \( N\tilde{V} - V_{\Psi} = o_P(1) \) as we just proved and \( \Psi^T V_{\Psi}^{-1} \Psi = V_{x+}^{-1} \) by \([S16]\).

**Asymptotic distributions of \( \lambda_{\text{LRT}} \) and \( W_{\text{logit}} \).** The result follows from \( N\hat{x}_+^T V_{x+}^{-1} \hat{x}_+ \sim \chi^2_{J(Q-1)} \) by Lemma \([S9]\) and Slutsky’s theorem.

**Simplification under the treatment-control experiment.** The result follows from \( \hat{x}_+ = \hat{x}(1) = e_0 \tilde{x}_x \) and \( \Psi = e_0^{-1}(S^2_x)^{-1} \) such that \( \Psi \hat{x}_+ = e_0^{-1}(S^2_x)^{-1} \hat{x}(1) = (S^2_x)^{-1} \hat{x}_x \) and \( V_{\Psi} = (e_0 e_1)^{-1}(S^2_x)^{-1} \). \( \square \)