1D-Disordered Conductor with Loops Immersed in a Magnetic Field

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Abstract

We investigate the conductance of a 1-D disordered conducting loop with two contacts, immersed in a magnetic flux. We show the appearance in this model of the Al’tshuler-Aronov-Spivak behaviour. We also investigate the case of a chain of loops distributed with finite density: in this case we show that the interference effects due to the presence of the loops can lead to the delocalization of the wave function.

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1 Introduction

The Aharonov-Bohm effect arises combining a non-trivial topology with the presence of the magnetic field \( \phi \). When disorder is added to this picture it has been shown [2] that usually observed quantities acquire new properties. For instance, the conductivity of a thin cylindrical conductor immersed in a magnetic flux \( \phi \) is a periodic function having period \( \phi_0 \), the quantum of magnetic flux, but in the presence of disorder one gets a period \( \phi_0/2 \) (see Ref. [2]). This effect has been observed in experiments [3]. In Ref. [2] computations were carried on for the 2D-case in the framework of weak localization theory [4] and essential use was made of perturbation theory.

The papers [5, 6, 7] have arisen interest toward 1D-models presenting the Al’tshuler-Aronov-Spivak behaviour. In these works the study of the conductivity of a ring immersed in a magnetic flux and having two contacts on opposite sides was carried on. The case of the ordered ring was studied there thoroughly. In [8] it was considered a disordered ring, but localization effects were completely ignored.

In the present paper we consider a loop of one-dimensional disordered conductor immersed in a magnetic flux \( \phi \) (measured in units of \( \phi_0 \)). The geometry of the contact that we are considering differs from the one analyzed in Refs. [3]-[8] (see section 2 and Fig. 1). This allows us to describe the contact in terms of glueing matrices for solutions of the Schrödinger equation rather than in terms of the \( S \)-matrices used in Refs. [3]-[8]. This point may turn useful in generalizations of the model.

We compute the conductivity of the sample in the framework of the Landauer-Büttiker approach [9, 6] and show that it is a periodic function of \( 2\phi \), so that the frequency of oscillations is doubled. This result is in agreement with the considerations put forth in Ref. [2] (see also [10] for a review).

2 Conductance of a loop immersed in a magnetic flux

In order to compute the conductance of the sample depicted in Fig. 1 we resort to Landauer’s formula [9, 6]:

\[
G = \frac{e^2}{2\pi \hbar} \frac{\tau}{1 - \tau},
\]

where \( \tau \) is the transmission probability (calculated at the Fermi level). \( \tau \) can be obtained from the \( 2 \times 2 \) transfer matrix \( T \) which maps the space of solutions \( \psi(x) \) of the Schrödinger equation from one space point to another:

\[
v(x_2) = T v(x_1), \quad v(x) = \left( \begin{array}{c} \psi'(x) + ik\psi(x) \\ \psi'(x) - ik\psi(x) \end{array} \right), \quad \tau = 1/|T_{22}|^2;
\]

here \( k \) denotes the one-dimensional momentum of the electron and the Schrödinger equation has the form

\[- \psi'' + U(x)\psi = k^2\psi.\]
Let us determine the transfer matrix $T$ for a generic loop. The one-dimensional description is adequate only asymptotically, far away from the cross-junction domain. Inside this domain we should use the three-dimensional Schrödinger equation. However, if the domain is small enough (i.e., of size $\sim 1/k$), we just need to get some linear relations among the effective asymptotic one-dimensional wave functions. Since the Schrödinger equation is a second order PDE, the fixing of the boundary values of the wave function determines the solution uniquely with all its derivatives.

This means that among the eight quantities $\psi(a), \psi'(a), \psi(b), \psi'(b), \psi(c), \psi'(c), \psi(d), \psi'(d)$ (see Fig. 2) there should exist four linear relations, which can be written for a symmetric junction as

$$
\begin{pmatrix}
  v_a \\
  v_b \\
\end{pmatrix} =
\begin{pmatrix}
  A & B \\
  B & A \\
\end{pmatrix}
\begin{pmatrix}
  v_c \\
  v_d \\
\end{pmatrix},
$$

with $A$ and $B$ being $2 \times 2$ complex matrices. If we now connect $b$ and $c$ through a generic transfer matrix $T_0$ (or $e^{i\phi}T_0$ in the presence of a magnetic flux $\phi$) we get $v_a$ in terms of $v_d$ as follows:

$$
v_a = T v_d, \quad T = B + e^{i\phi}AT_0(1 - e^{i\phi}BT_0)^{-1}A.\quad (5)
$$

Since $T$ is a transfer matrix, it must satisfy the current conservation condition:

$$
T^*T = 1, \quad \text{with} \quad T^* = \sigma_3 T^\dagger \sigma_3, \quad (6)
$$

and the time-inversion invariance:

$$
\bar{T} = T \quad \text{if} \quad \phi = 0, \quad \text{with} \quad \bar{T} = \sigma_1 T^\dagger \sigma_1. \quad (7)
$$

The same equations hold for $T_0$. Eqs. (6) and (7) constrain the form of the matrices $A$ and $B$. Eq. (6) simply tells us that $\bar{A} = A$ and $\bar{B} = B$. Notice that $2 \times 2$ matrices such that $\bar{X} = X$
have the form $X = x_0 + \vec{x} \cdot \vec{\sigma}$, where $\vec{\sigma}$ is the vector of Pauli matrices, $\vec{x} = (x_1, x_2, ix_3)$ and all the $x_j$ are real. For matrices of this kind the $*$-conjugation just amounts to a "space" reflection $X^* = x_0 - \vec{x} \cdot \vec{\sigma}$. Using these properties, and the fact that (6) and (7) hold for $T_0$, we get the following inversion formula:

$$(1 - e^{i\phi} BT_0)^{-1} = \frac{1}{\Delta} (1 - e^{i\phi} T_0^* B^*),$$

(8)

where

$$\Delta = \det(1 - e^{i\phi} BT_0) = 1 - e^{i\phi} \text{Tr} (BT_0) + e^{2i\phi} \det B;$$

(9)

thus (8) can be rewritten as

$$T = \frac{1}{\Delta} \left( \Delta B + e^{i\phi} AT_0 A - e^{2i\phi} AB^* A \right).$$

(10)

We now plug (10) into the unitarity condition (6), which must be satisfied by arbitrary values of $\phi$ and $T_0$, and is thus equivalent to the vanishing of a matrix-valued trigonometric polynomial. Vanishing of the $\sin 2\phi$-term gives

$$A^* B = -B^* A;$$

(11)

for $\phi = 0$ and from the arbitrariness of $T_0$ we get

$$\det A = 1 - \det B.$$

(12)

The remaining terms give no additional conditions.

We remark that (6) and (7) are also satisfied if we impose on $A$ and $B$ the equations which are obtained from (11) and (12) by reversing the signs of the right hand sides. However, only the positive choice of the sign respect invariance with regard to the interchange of $A$ and $B$. This invariance has to be satisfied for the following reason: if we connect $b$ and $d$ instead of $b$ and $c$ we would obtain (11) and (12) with $A$ and $B$ interchanged. It is easy to verify that the new equations would not be compatible with the previous ones if we had chosen the negative signs.

Using (11) and (12) we can rewrite $T$ in the simpler form

$$T = \frac{e^{i\phi}}{\Delta} [(2 \cos \phi - \text{Tr} BT_0) \cdot B + AT_0 A].$$

(13)

One can easily verify that the general solution of (11) and (12) depends upon six real parameters and is given by

$$B = kSA$$

(14)

with

$$S = i \begin{pmatrix} q & z \\ -\bar{z} & -q \end{pmatrix}, \quad \pm 1 = q^2 - |z|^2, \quad k^2 = \pm (1/\det A - 1),$$

(15)

5 Eq. (12) can also be rewritten as $B^* B + A^* A = 1$. 

3
where \( A = a_0 + \vec{a} \cdot \vec{\sigma} \) is an arbitrary matrix satisfying \( \tilde{A} = A \) and the positive signs in (13) have to be chosen when \( 0 < \det A < 1 \).

Landauer’s formula (1), using (13), now reads

\[
G = \frac{e^2}{2\pi\hbar} \frac{|\Delta|^2}{|Q|^2 - |\Delta|^2},
\]

with

\[
Q = (2\cos \phi - \text{Tr} B T_0) \cdot B_{22} + (A T_0 A)_{22}.
\]

### 3 Appearance of the AAS effect

We shall consider the geometry depicted in Fig. 1, where the loop of length \( L \) is constituted by a 1-dimensional disordered conductor. In this case the transfer matrix describing the propagation through the disordered loop can be parametrized as follows [11]:

\[
T_0 = \begin{pmatrix}
\alpha & \beta \\
\beta^* & \alpha^*
\end{pmatrix} = \begin{pmatrix}
e^{i(\alpha_s + kL) \cosh \Gamma} & e^{i(\beta_s + kL) \sinh \Gamma} \\
e^{-i(\beta_s + kL) \sinh \Gamma} & e^{-i(\alpha_s + kL) \cosh \Gamma}
\end{pmatrix},
\]

where \( \alpha_s, \beta_s \) and \( \Gamma \) are functionals of the impurities potential and can be considered as slow variables (see [12] for details).

The main point in the approach of Refs. [2, 4, 10, 11] is the averaging over an ensemble of impurities potentials. However, in the case of a given sample we also have to separate fast-phase averaging and ensemble averaging. Fast-phase averaging can be performed e.g. in the following way:

\[
G_L = \frac{1}{\Delta L} \int_L^{L+\Delta L} G \, dL' \quad (1/k \ll \Delta L \ll \ell),
\]

where \( \ell \) is the localization length. This average must be considered since the length \( L \) is defined with a precision \( \Delta L \) of the order of the size of the intermediate domain where the wave function is essentially three-dimensional. From (18) it follows that only functions of \( |\alpha|^2 \) and \( |\beta|^2 \) are non-vanishing after fast-phase averaging.

We are now in a position to prove the appearance of the Al’tshuler-Aronov-Spivak effect in our model. From (18) it follows that the coefficients of \( \cos \phi \)-terms appearing in the r.h.s. of (16) are linear combinations of \( \alpha, \alpha^*, \beta \) and \( \beta^* \). Therefore after performing the average (18) only even powers of \( \cos \phi \) survive and \( G \) becomes a function of period \( \pi \) instead of \( 2\pi \). An explicit example is given in the following section. This shows that the conclusion of Ref. [8] that at \( |\beta| \neq 0 \) the \( \cos \phi \)-oscillations of the conductance survive after fast-phase averaging is wrong [1].

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6 We must stress here that the phases of the non-diagonal elements of (18) are fast variables.
4 Explicit computation

Let us first of all consider the simple case of the following 1-parameter solution of (11) and (12):

\[ A = \sqrt{1 - \kappa^2}, \quad B = i\kappa\sigma_3. \]  (20)

From (16) we get

\[ G = e^{2\frac{\kappa^2 \Im^2 \alpha + 4\kappa(1 + \kappa^2) \cos \phi \Im \alpha + 1 + 2\kappa^2 \cos 2\phi + \kappa^4}{(1 - \kappa^2)^2|\beta|^2}}. \]  (21)

After the fast-phase averaging we obtain

\[ G_L = G_\infty + e^{2\frac{\kappa^2 \cos^2 \phi + \kappa^4}{(1 - \kappa^2)^2}} \frac{1}{|\beta|^2}, \]  (22)

\[ G_\infty = e^{2\frac{\kappa^2}{2\pi\hbar (1 - \kappa^2)^2}}. \]  (23)

In the general case the expressions of \( G_L \) in terms of \(|\alpha|^2\), \(|\beta|^2\) and \(\cos^2 \phi\) is rather cumbersome. If \(L/\ell \gg 1\) we have

\[ |\beta|^2 = \exp \log |\beta|^2 \sim \exp \left\langle \log |\beta|^2 \right\rangle \]  (24)

since the logarithm of \(|\alpha|^2\) and \(|\beta|^2\) in the limit \(L/\ell \gg 1\) is additive \([13, 14]\). For large \(L/\ell\) the behaviour of \(\left\langle \log |\beta|^2 \right\rangle\) is well known:

\[ \left\langle \log |\beta|^2 \right\rangle \sim 2 \frac{L}{\ell}. \]  (25)

Thus, the values of \(|\alpha|^2\) and \(|\beta|^2\) are large enough and \(G_L\) can be represented as a series expansion in \(|\beta|^{-2}\). At the first order we get \(G_L = G_\infty + C + D \cos^2 \phi\) where \(C\) and \(D\) are of the order \(\exp(-2L/\ell)\). This is in accordance with estimations \([13]\) made in the framework of weak-localization theory.

5 Chain of loops

In this section we want to show that the interference effects due to the presence of loops can in principle lead to the delocalization of the wave function.

The transfer matrix of a conducting loop with coefficient of internal scattering of order \(\delta x\) \(\text{(i.e. with } A = 1 + A' \delta x, B = B' \delta x\text{)}\) is obtained by linearizing \([13]\):

\[ R = 1 + i \Theta \delta x \]  (26)

where \(\Theta\) is a matrix verifying \(\Theta^* = -\Theta, \tilde{\Theta} = \Theta\). Let us consider a series of \(N\) such loops connected by short wires of disordered conductor of length \(\delta x\), and let us denote by \(x_i\) the position of the \(i\)-th loop. The transfer matrix corresponding to the \(i\)-th loop is

\[ T_i = 1 + (i \varphi(x_i) s_3 + \zeta_+(x_i) s_+ + \zeta_-(x_i) s_-) \delta x = 1 + Q_i \delta x, \]
where $i\varphi$ and $\zeta_\pm$ are local random fields, which in terms of the random potential $V(x)$ have the form $\varphi(x) = -V(x)/k$, $\zeta_\pm(x) = \pm iV(x) e^{\pm 2i k x}/2k$ (see [12] for details). Introducing the unitary matrices $U_i = (R)^i$ we can write the transfer matrix for the loop chain in the following form:

\[
T = R^N \prod_{i+1}^N U_i^* T_i U_i
= R^N \prod_{i=1}^N (1 + U_i^* Q_i U_i \delta x)
= \exp(i\Theta L) \cdot \mathcal{P} \exp \int dx u^*(x) Q(x) U(x),
\]

where $\mathcal{P}$ denotes the operator ordering of the exponential along the integration line. Let us impose the resonant condition that $\Theta$ be diagonal. Choose $k$ so that

\[
i\Theta = -ik\sigma_3,
\]

thus giving

\[
U(x) = e^{-ikx\sigma_3}
\]

and

\[
U^*(x)Q(x)U(x) = \frac{V(x)}{2ik} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.
\]

The path-ordered exponential is easily computed since the matrix (30) is factorized:

\[
T = e^{-ikL\sigma_3} \begin{pmatrix} 1 + \frac{1}{2ik} \int_{-L}^L V(x) dx & -\frac{1}{2ik} \int_{-L}^L V(x) dx \\ \frac{1}{2ik} \int_{-L}^L V(x) dx & 1 - \frac{1}{2ik} \int_{-L}^L V(x) dx \end{pmatrix}
\]

It is worth noting that the non-exponential dependence of the matrix $T$ on $\int_{-L}^L V(t) dt$ follows from the nilpotency of the matrix (30). The resistance of the loop chain is then

\[
T_{12}T_{21} = \frac{1}{4k^2} \int dx dx' V(x)V(x').
\]

This result is exact and does not depend on taking any average (it refers of course to a very particular case). For a random function $V(x)$ with space-homogeneous statistics [13] the right-hand side of (32) is a self-averaging quantity. Therefore the bulk resistivity of the chain is equal to $1/\ell$, where $\ell = 4k^2/D$ is the localization length in the absence of loops. We remember that in the absence of loops the resistance grows exponentially with $L$. The linear behaviour of the resistance (Ohm’s law) obtained in place of the exponential one shows that the localization length diverges.

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