THE ASYMPTOTIC BEHAVIOR OF GLOBALLY SMOOTH SOLUTIONS OF BIPOLAR NON-ISENTROPIC COMPRESSIBLE EULER-MAXWELL SYSTEM FOR PLASMA

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Abstract. The bipolar non-isentropic compressible Euler-Maxwell system is investigated in $R^3$ in the present paper, and the $L^9$ time decay rate for the global smooth solution is established. It is shown that the total densities, total temperatures and magnetic field of two carriers converge to the equilibrium states at the same rate $(1 + t)^{-\frac{1}{2} + \frac{3}{2q}}$ in $L^9$ norm. But, both the difference of densities and the difference of temperatures of two carriers decay at the rate $(1 + t)^{-2 - \frac{1}{q}}$, and the velocity and electric field decay at the rate $(1 + t)^{-\frac{3}{2} + \frac{1}{q}}$. This phenomenon on the charge transport shows the essential difference between the non-isentropic unipolar Euler-Maxwell and the bipolar isentropic Euler-Maxwell system.

Keywords: Bipolar non-isentropic Euler-Maxwell equations, Plasma, Globally smooth solution, Asymptotic behavior

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1. Introduction and main results

The Euler-Maxwell system is used to model and simulate the transport of charged particles in plasma[1,3,8,9,20]. Usually, it takes the form of compressible non-isentropic Euler equations forced by the electromagnetic field, which is governed by the self-consistent Maxwell equation. In present paper, we consider the Cauchy problem for the bipolar non-isentropic Euler-Maxwell
For asymptotic limits with small parameters, see [14, 15] and references therein. For the three
is established by Peng et al in [16] and Ueda et al in [19] respectively, and the decay rate of
amplitude to the Cauchy problem in the whole space and to the periodic problem in the torus
solutions has been given in [2] by the compensated compactness method. For the three di-
In a simplified one dimensional isentropic Euler-Maxwell system, the global existence of entropy
the Cauchy problem (1.2)-(1.3) has a local smooth solution when the initial data are smooth.
where the unknowns are the density \( n_\mu > 0 \), the velocity \( u_\mu = (u_\mu^1, u_\mu^2, u_\mu^3) \), the absolute temperature \( \theta_\mu > 0 \), the total energy \( \mathcal{E}_\mu = \frac{1}{2} |u_\mu|^2 + C_\nu \theta_\mu \), the pressure function \( p_\mu = R_\nu n_\mu \theta_\mu \) for \( \mu = e, i \), the electronic field \( E \) and magnetic field \( B \). Furthermore, the constants \( C_\nu > 0 \), \( R_\nu > 0 \)
are the heat capacity at constant volume and the coefficient of heat conductivity respectively. Throughout
this paper, we set \( C_\nu = R_\nu = 1 \) without loss of generality. Then, the system (1.1)
is equivalent to
\[
\begin{align*}
\partial_t n_e + \nabla \cdot (n_e u_e) &= 0, \\
\partial_t u_e + (u_e \cdot \nabla) u_e + \frac{\theta_\mu}{n_e} \nabla n_e + \nabla \theta_e = -(E + u_e \times B) - u_e, \\
\partial_t \theta_e + \nabla \cdot (\theta_e u_e) + (\theta_e - 1) &= 0, \\
\partial_t n_i + \nabla \cdot (n_i u_i) &= 0, \\
\partial_t u_i + (u_i \cdot \nabla) u_i + \frac{\theta_i}{n_i} \nabla n_i + \nabla \theta_i = (E + u_i \times B) - u_i, \\
\partial_t \theta_i + \nabla \cdot (\theta_i u_i) + (\theta_i - 1) &= 0, \\
\partial_t E - \nabla \times B &= n_e u_e - n_i u_i, \\
\partial_t B + \nabla \times E &= 0, \\
\nabla \cdot E &= n_i - n_e, \\
\nabla \cdot B &= 0, \\
\end{align*}
\]
(1.2)

Initial data is given as
\[
(n_\mu, u_\mu, \theta_\mu, E, B)|_{t=0} = (n_\mu(0), u_\mu(0), \theta_\mu(0), E(0, B(0)), x \in \mathbb{R}^3, \\
\text{with the compatible condition}
\]
\[
\nabla \cdot E_0 = n_\mu(0) - n_e(0), \\
\nabla \cdot B_0 = 0, \\
x \in \mathbb{R}^3.
\]

The Euler-Maxwell system (1.2) is a symmetrizable hyperbolic system for \( n_\mu, \theta_\mu > 0 \). Then the Cauchy problem (1.2)-(1.3) has a local smooth solution when the initial data are smooth. In a simplified one dimensional isentropic Euler-Maxwell system, the global existence of entropy solutions has been given in [2] by the compensated compactness method. For the three dimensional isentropic Euler-Maxwell system, the existence of global smooth solutions with small amplitude to the Cauchy problem in the whole space and to the periodic problem in the torus is established by Peng et al in [16] and Ueda et al in [19] respectively, and the decay rate of the smooth solution when \( t \) goes to infinity is obtained by Duan in [4] and Ueda et al in [18]. For asymptotic limits with small parameters, see [14, 15] and references therein. For the three
dimensional bipolar isentropic Euler-Maxwell system, the global existence and the asymptotic behavior of the smooth solution is also obtained by Duan et al in [5]. Recently, Yang et al in [21] consider the diffusive relaxation limit of the three dimensional unipolar non-isentropic Euler-Maxwell system, and Wang et al asymptotics and global existence in [6].

However, there is no analysis on the asymptotics and global existence for the bipolar non-isentropic Euler-Maxwell system in three space dimensions yet. Therefore, the goal of the present paper is to establish the global existence of smooth solutions around a equilibrium solution of system (1.2) and the decay rate of the smooth solution as \( t \to \infty \).

The main result of this paper can be stated as follows.

**Theorem 1.1.** Assume (1.3) hold. If \( \| [n_{0 \mu} - 1, u_{0 \mu}, \theta_{0 \mu} - 1, E_{0}, B_{0}] \|_{s} \leq \delta_{0} \) for \( s \geq 4 \). Then, there is a unique global solution \( [n_{\mu}(t, x), u_{\mu}(t, x), \theta_{\mu}(t, x), E(t, x), B(t, x)] \) to the initial value problem (1.2) - (1.3) which satisfies

\[
[n_{\mu} - 1, u_{\mu}, \theta_{\mu} - 1, E, B] \in C^{1}([0, T); H^{s-1}(\mathbb{R}^{3})) \cap C([0, T); H^{s}(\mathbb{R}^{3}))
\]

and

\[
\sup_{t \geq 0} \| [n_{\mu}(t) - 1, u_{\mu}(t), \theta_{\mu}(t) - 1, E(t), B(t)] \|_{s} \leq C_{0} \| [n_{0 \mu} - 1, u_{0 \mu}, \theta_{0 \mu} - 1, E_{0}, B_{0}] \|_{s},
\]

where \( \delta_{0}, C_{0} > 0 \) are constants independent of time.

Moreover, if \( \| [n_{0 \mu} - 1, u_{0 \mu}, \theta_{0 \mu} - 1, E_{0}, B_{0}] \|_{L^{1+H^{13}}} \leq \delta_{1} \), then the solution \( [n_{\mu}(t, x), u_{\mu}(t, x), \theta_{\mu}(t, x), E(t, x), B(t, x)] \) satisfies

\[
\| [n_{e}(t) - n_{i}(t), \theta_{e}(t) - \theta_{i}(t)] \|_{L^{q}} \leq C_{1} (1 + t)^{-\frac{1}{2} - \frac{1}{q}},
\]

\[
\| [n_{e}(t) + n_{i}(t) - 2, \theta_{e}(t) + \theta_{i}(t) - 2] \|_{L^{q}} \leq C_{1} (1 + t)^{-\frac{3}{2} + \frac{1}{q}},
\]

\[
\| u_{e}(t) \pm u_{i}(t), E(t) \|_{L^{q}} \leq C_{1} (1 + t)^{-\frac{2}{2} + \frac{1}{q}},
\]

\[
\| B(t) \|_{L^{q}} \leq C_{1} (1 + t)^{-\frac{3}{2} + \frac{1}{2q}},
\]

for any \( t \geq 0 \) and \( 2 \leq q \leq \infty \). Where, constants \( \delta_{1}, C_{1} > 0 \) are also independent of time.

**Remark 1.1.** It should be emphasized that both the velocity and temperature relaxation term of the bipolar non-isentropic Euler-Maxwell system (1.2) plays a key role in the proof of Theorem 1.1.

**Notations.** In this paper, \( f \sim g \) means \( \gamma a \leq b \leq \gamma a \) for a constant \( 0 < \gamma < 1 \). \( H^{s} \) denotes the standard Sobolev space \( W^{s,2}(\mathbb{R}^{3}) \). We use \( H^{s} \) to denote the corresponding s-order homogeneous Sobolev space. Set \( L^{2} = H^{0} \). The norm of \( H^{s} \) is denoted by \( \| \cdot \|_{s} \) with \( \| \cdot \| = \| \cdot \|_{0} \), and \( \langle \cdot, \cdot \rangle \) denotes the inner product over \( L^{2}(\mathbb{R}^{3}) \). For the multi-index \( \alpha = (\alpha_{1}, \alpha_{2}, \alpha_{3}) \), we denote \( \partial^{\alpha} = \partial_{x_{1}^{\alpha_{1}}} \partial_{x_{2}^{\alpha_{2}}} \partial_{x_{3}^{\alpha_{3}}} = \partial_{x_{1}^{\alpha_{1}}} \partial_{x_{2}^{\alpha_{2}}} \partial_{x_{3}^{\alpha_{3}}} \) and \( |\alpha| = \alpha_{1} + \alpha_{2} + \alpha_{3} \). For an integrable function \( f : \mathbb{R}^{3} \to \mathbb{R} \), its Fourier transform is defined by

\[
\hat{f}(k) = \int_{\mathbb{R}^{3}} e^{-i x \cdot k} f(x) dx, \quad x \cdot k := \sum_{j=1}^{3} x_{j} k_{j}, \quad k \in \mathbb{R}^{3},
\]

where \( i = \sqrt{-1} \in \mathbb{C} \) is the imaginary unit.

The rest of the paper is arranged as follows. In Section 2, the transformation of the initial value problem and the proof of the global existence and uniqueness of solutions are presented. In Section 3, we study the linearized homogeneous equations to get the \( L^{p} - L^{q} \) decay property.
and the explicit representation of solutions. In the last Section 4, we investigate the decay rates of solutions to the transformed nonlinear equations and complete the proof of Theorem [1.1]

2. Global solutions for equations (1.2)

2.1. Preliminary. Suppose \([n_{\mu}(t,x), u_{\mu}(t,x), \theta_{\mu}(t,x), E(t,x), B(t,x)]\) be a smooth solution of the initial value problem for the bipolar non-isentropic Euler-Maxwell equations (1.2) with initial data (1.3) which satisfies (1.4). Set

\[
\begin{align*}
n_{\mu}(t,x) &= 1 + \rho_{\mu}(t,x), \theta_{\mu}(t,x) = 1 + \Theta_{\mu}(t,x).
\end{align*}
\]

Thus, we can rewrite the system (1.2)-(1.4) as

\[
\begin{align*}
\partial_t \rho_e + \nabla \cdot ((1 + \rho_e) u_e) &= 0, \\
\partial_t u_e + (u_e \cdot \nabla) u_e + \frac{1 + \Theta_e}{1 + \rho_e} \nabla \rho_e + \nabla \Theta_e &= -(E + u_e \times B) - u_e, \\
\partial_t \Theta_e + \nabla \cdot ((1 + \Theta_e) u_e) + \Theta_e &= 0, \\
\partial_t \rho_i + \nabla \cdot ((1 + \rho_i) u_i) &= 0, \\
\partial_t u_i + (u_i \cdot \nabla) u_i + \frac{1 + \Theta_i}{1 + \rho_i} \nabla \rho_i + \nabla \Theta_i &= (E + u_i \times B) - u_i, \\
\partial_t \Theta_i + \nabla \cdot ((1 + \Theta_i) u_i) + \Theta_i &= 0, \\
\partial_t E - \nabla \times B - u_e + u_i &= \rho_e u_e - \rho_i u_i, \\
\partial_t B + \nabla \times E &= 0,
\end{align*}
\]

with initial data

\[
U_{\mid t=0} = U_0 := [\rho_{\mu 0}, u_{\mu 0}, \Theta_{\mu 0}, E_0, B_0], \ x \in \mathbb{R}^3,
\]

which satisfies the compatible condition

\[
\nabla \cdot E_0 = \rho_{\mu 0} - \rho_{\mu 0}, \ \nabla \cdot B_0 = 0, \ x \in \mathbb{R}^3.
\]

Here, \(\rho_{\mu 0} = n_{\mu 0} - 1\).

In the following, we usually assume \(s \geq 4\). Moreover, for \(U = [\rho_{\mu}, u_{\mu}, \Theta_{\mu}, E, B]\), we use \(E_s(U(t)), \mathcal{E}_s^h(U(t)), \mathcal{D}_s(U(t))\) and \(\mathcal{D}_s^h(U(t))\) to define the energy functional, the high-order energy functional, the dissipation rate and the high-order dissipation rate as

\[
\begin{align*}
\mathcal{E}_s(U(t)) &\sim ||[\rho_{\mu}, u_{\mu}, \Theta_{\mu}, E, B]||_{s}^2, \\
\mathcal{E}_s^h(U(t)) &\sim ||\nabla[\rho_{\mu}, u_{\mu}, \Theta_{\mu}, E, B]||_{s-1}^2, \\
\mathcal{D}_s(U(t)) &\sim ||\nabla[\rho_{\mu}, \rho_i]||_{s-1}^2 + ||[u_{e}, u_{i}, \Theta_{e}, \Theta_{i}]||_{s}^2 \\
&+ ||E||_{s}^2 + ||\nabla B||_{s-2}^2 + ||\rho_e - \rho_i||_{s}^2
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{D}_s^h(U(t)) &\sim ||\nabla^2[\rho_{e}, \rho_i]||_{s-2}^2 + ||\nabla[u_{e}, u_{i}, \Theta_{e}, \Theta_{i}]||_{s-1}^2 \\
&+ ||\nabla E||_{s-2}^2 + ||\nabla^2 B||_{s-3}^2 + ||\nabla[\rho_e - \rho_i]||_{s}^2.
\end{align*}
\]

respectively. Now, concerning the transformed initial value problem (2.2)-(2.3), we have the global existence result as follows.
Proposition 2.1. Assume that $U_0 = [\rho_0, u_0, \Theta_0, E_0, B_0]$ satisfies the compatible condition (2.3). If $\mathcal{E}_s(U_0)$ is small enough, then, for any $t \geq 0$, the initial value problem (2.2) has a unique global nonzero solution $U = [\rho_\mu, u_\mu, \Theta_\mu, E, B]$ which satisfies

$$
(2.9) \quad U \in C^1([0, T); H^{s-1}(\mathbb{R}^3)) \cap C([0, T); H^s(\mathbb{R}^3)),
$$

and

$$
(2.10) \quad \mathcal{E}_s(U(t)) + \gamma \int_0^t \mathcal{D}_s(U(s))ds \leq \mathcal{E}_s(U_0).
$$

Obviously, from the Proposition 2.1, it is straightforward to get the existence result of Theorem 1.1. Furthermore, solutions of Proposition 2.1 really decay under some extra conditions on $U_0 = [\rho_0, u_0, \Theta_0, E_0, B_0]$. For this purpose, we define $\omega_s(U_0)$ as

$$
(2.11) \quad \omega_s(U_0) = \|U_0\|_s + \|\rho_0, u_0, \Theta_0, E_0, B_0\|_{L_1}^s
$$

for $s \geq 4$. Then, we obtain the following decay results.

Proposition 2.2. Assume that $U_0 = [\rho_0, u_0, \Theta_0, E_0, B_0]$ satisfies (2.3). If $\omega_{s+2}(U_0)$ is sufficiently small, then system (2.2) has a solution $U = [\rho_\mu, u_\mu, \Theta_\mu, E, B]$ satisfying

$$
(2.12) \quad \|U(t)\|_s \leq C\omega_{s+2}(U_0)(1 + t)^{-\frac{s}{2}}
$$

for any $t \geq 0$. Moreover, if $\omega_{s+6}(U_0)$ is sufficiently small, then, for any $t \geq 0$, the solution also satisfies

$$
(2.13) \quad \|\nabla U(t)\|_{s-1} \leq C\omega_{s+6}(U_0)(1 + t)^{-\frac{s}{3}}.
$$

Thus, one can obtain the decay rates (1.5)-(1.8) through the method of bootstrap and the Proposition stated above.

2.2. Weighted energy estimates. In this subsection, we shall give the proof of Proposition 2.1 for the global existence and uniqueness of solutions to the initial value problem (2.2)-(2.3). Since hyperbolic equations (2.2) is quasi-linear symmetrizable, thus one has the local existence of smooth solutions to (2.2) as follows.

Lemma 2.1. (Local existence of smooth solutions, see [11, 12]) Let $s > \frac{5}{2}$ and $(\rho_0, u_0, \Theta_0, E_0, B_0) \in H^s(\mathbb{R}^3)$. Then there exist $T > 0$ and a unique smooth solution $(u_\mu, \Theta_\mu, E, B)$ to the Cauchy problem (1.2)-(1.3) satisfying $(\rho_\mu, u_\mu, \Theta_\mu, E, B) \in C^1([0, T); H^{s-1}(\mathbb{R}^3)) \cap C([0, T); H^s(\mathbb{R}^3))$.

Then, with the help of the continuity argument, the global existence of solutions satisfying (2.9) and (2.10) follows by combing Lemma 2.1 and a priori estimate as follows.

Theorem 2.1. Assume that $U = [\rho_\mu, u_\mu, \Theta_\mu, E, B] \in C^1([0, T); H^{s-1}(\mathbb{R}^3)) \cap C([0, T); H^s(\mathbb{R}^3))$ is smooth for $T > 0$ with

$$
(2.14) \quad \sup_{0 \leq t \leq T} \|U(t)\|_s \leq \delta
$$

for $\delta \leq \delta_0$ with $\delta_0$ sufficiently small and suppose $U$ to be the solution of the equations (2.2) for $t \in (0, T)$. Then, for a constant $0 < \gamma < 1$ and any $0 \leq t \leq T$, it holds that

$$
(2.15) \quad \frac{d}{dt}\mathcal{E}_s(U(t)) + \gamma \mathcal{D}_s(U(t)) \leq C[\mathcal{E}_s(U(t))^{\frac{1}{2}} + \mathcal{E}_s(U(t))] \mathcal{D}_s(U(t)).
$$
Proof. We will use five steps to finish the proof as follows. In step 1, we establish the estimate of Euler part and Maxwell part of the system (2.2) by using weighted energy estimate method. In the following steps 2–4, we utilize the skew-symmetric structure of the system (2.2) to get the dissipative estimates for $\rho_\mu$, $E$ and $B$.

**Step 1.** It holds that

\[
\frac{d}{dt} \|U\|^2_s + \|u_e, u_i, \Theta_e, \Theta_i\|^2_s \leq C \|U\|_s \left( \|u_e, u_i, \Theta_e, \Theta_i\|^2_s + \|\nabla [\rho_e, \rho_i]\|^2_{s-1} \right).
\]

In fact, from the first six equations of (2.2), weighted energy estimate on $\partial^\alpha \rho_\mu, \partial^\alpha u_\mu$ and $\partial^\alpha \Theta_\mu$ with $|\alpha| \leq s$ imply

\[
\frac{1}{2} \frac{d}{dt} \sum_{\mu=e,i} \left( \langle 1 + \frac{\Theta_\mu}{1 + \rho_\mu}, |\partial^\alpha \rho_\mu|^2 \rangle + \langle 1 + \rho_\mu, |\partial^\alpha u_\mu|^2 \rangle + \langle 1 + \rho_\mu, |\partial^\alpha \Theta_\mu|^2 \rangle \right)
\]

\[
+ \sum_{\mu=e,i} \left( \langle 1 + \rho_\mu, |\partial^\alpha u_\mu|^2 \rangle + \langle 1 + \rho_\mu, |\partial^\alpha \Theta_\mu|^2 \rangle \right) + \langle (1 + \rho_e) \partial^\alpha E, \partial^\alpha u_e \rangle
\]

\[
- \langle (1 + \rho_i) \partial^\alpha E, \partial^\alpha u_i \rangle = - \sum_{\beta < \alpha} C^\alpha_{\beta} I_{\alpha, \beta}(t) + I_1(t).
\]

Where, $I_{\alpha, \beta}(t) = I^e_{\alpha, \beta}(t) + I^i_{\alpha, \beta}(t)$, $I_1(t) = I^e_1(t) + I^i_1(t)$ with

\begin{align*}
I^e_{\alpha, \beta}(t) &= \langle 1 + \frac{\Theta_e}{1 + \rho_e}, \partial^\alpha \rho_e \nabla \partial^\beta u_e, \partial^\alpha \rho_e \rangle + \langle 1 + \frac{\Theta_e}{1 + \rho_e}, \partial^\alpha u_e \nabla \partial^\beta \rho_e, \partial^\alpha \rho_e \rangle \\
&\quad + \langle 1 + \frac{\rho_e}{1 + \Theta_e}, \partial^\alpha \rho_e \nabla \partial^\beta \Theta_e, \partial^\alpha \rho_e \rangle + \langle 1 + \frac{\rho_e}{1 + \Theta_e}, \partial^\alpha \Theta_e \nabla \partial^\beta u_e, \partial^\alpha \Theta_e \rangle \\
&\quad + \langle (1 + \rho_e) \partial^\alpha u_e \nabla \partial^\beta \rho_e, \partial^\alpha u_e \rangle + \langle (1 + \rho_e) \partial^\alpha \Theta_e \nabla \partial^\beta \rho_e, \partial^\alpha u_e \rangle \\
&\quad + \langle (1 + \rho_e) \partial^\alpha \rho_e \nabla \partial^\beta \rho_e, \partial^\alpha \rho_e \rangle,
\end{align*}

\begin{align*}
I^i_{\alpha, \beta}(t) &= \langle 1 + \frac{\Theta_i}{1 + \rho_i}, \partial^\alpha \rho_i \nabla \partial^\beta u_i, \partial^\alpha \rho_i \rangle + \langle 1 + \frac{\Theta_i}{1 + \rho_i}, \partial^\alpha u_i \nabla \partial^\beta \rho_i, \partial^\alpha \rho_i \rangle \\
&\quad + \langle 1 + \frac{\rho_i}{1 + \Theta_i}, \partial^\alpha \rho_i \nabla \partial^\beta \Theta_i, \partial^\alpha \rho_i \rangle + \langle 1 + \frac{\rho_i}{1 + \Theta_i}, \partial^\alpha \Theta_i \nabla \partial^\beta u_i, \partial^\alpha \Theta_i \rangle \\
&\quad + \langle (1 + \rho_i) \partial^\alpha u_i \nabla \partial^\beta \rho_i, \partial^\alpha u_i \rangle + \langle (1 + \rho_i) \partial^\alpha \Theta_i \nabla \partial^\beta \rho_i, \partial^\alpha u_i \rangle \\
&\quad - \langle (1 + \rho_i) \partial^\alpha \rho_i \nabla \partial^\beta \rho_i, \partial^\alpha \rho_i \rangle,
\end{align*}

and

\[
I_1^e(t) = \frac{1}{2} \left\langle \partial_t \left( 1 + \frac{\Theta_e}{1 + \rho_e} \right), |\partial^\alpha \rho_e|^2 \right\rangle + \langle \nabla \Theta_e \partial^\alpha u_e, \partial^\alpha \rho_e \rangle + \langle \nabla \rho_e \partial^\alpha u_e, \partial^\alpha \Theta_e \rangle
\]

\[
+ \frac{1}{2} \left\langle \nabla \cdot \left( 1 + \frac{\Theta_e}{1 + \rho_e} u_e \right), |\partial^\alpha \rho_e|^2 \right\rangle + \frac{1}{2} \left\langle \partial_t \left( 1 + \frac{\rho_e}{1 + \Theta_e} \right), |\partial^\alpha \Theta_e|^2 \right\rangle
\]

\[
+ \frac{1}{2} \left\langle \nabla \cdot \left( 1 + \frac{\rho_e}{1 + \Theta_e} u_e \right), |\partial^\alpha \Theta_e|^2 \right\rangle - \langle (1 + \rho_e) u_e \times \partial^\alpha B, \partial^\alpha u_e \rangle
\]
\[ I_1(t) = \frac{1}{2} \left\langle \partial_t \left( \frac{1 + \Theta}{1 + \rho_i} \right), |\partial^\alpha \rho_i|^2 \right\rangle + \left\langle \nabla \Theta_i \partial^\alpha u_i, \partial^\alpha \rho_i \right\rangle + \left\langle \nabla \rho_i \partial^\alpha u_i, \partial^\alpha \Theta_i \right\rangle \\
+ \frac{1}{2} \left\langle \nabla \cdot \left( \frac{1 + \Theta}{1 + \rho_i} u_i \right), |\partial^\alpha \rho_i|^2 \right\rangle + \frac{1}{2} \left\langle \partial_t \left( \frac{1 + \rho_i}{1 + \Theta_i} \right), |\partial^\alpha \Theta_i|^2 \right\rangle \\
+ \frac{1}{2} \left\langle \nabla \cdot \left( \frac{1 + \rho_i}{1 + \Theta_i} u_i \right), |\partial^\alpha \Theta_i|^2 \right\rangle + \left\langle (1 + \rho_i) u_i \times \partial^\alpha B, \partial^\alpha u_i \right\rangle,
\]

where we have used integration by parts. When $|\alpha| = 0$, one has

\[ I_1(t) = I_1^i(t) + I_1^\alpha(t) \]

\[ = \sum \left\{ \frac{1}{2} \left\langle \partial_{\Theta} \left( \frac{1 + \Theta}{1 + \rho} \right) \partial_t \Theta + \partial_{\rho} \left( \frac{1 + \Theta}{1 + \rho} \right) \partial_t \rho, |\Theta|^2 \right\rangle + \left\langle \nabla \Theta \partial u, \rho \right\rangle \right\} \\
+ \frac{1}{2} \left\langle \partial_{\Theta} \left( \frac{1 + \Theta}{1 + \rho} \right) \partial_t \Theta + \partial_{\rho} \left( \frac{1 + \Theta}{1 + \rho} \right) \partial_t \rho, |\Theta|^2 \right\rangle + \left\langle \nabla \rho, |\Theta|^2 \right\rangle \\
+ \frac{1}{2} \left\langle \partial_{\Theta} \left( \frac{1 + \Theta}{1 + \rho} \right) \partial_t \Theta + \partial_{\rho} \left( \frac{1 + \Theta}{1 + \rho} \right) \partial_t \rho, |\Theta|^2 \right\rangle + \left\langle \nabla \rho, |\Theta|^2 \right\rangle \\
- \left\langle (1 + \rho) u_e \times B, u_e \right\rangle + \left\langle (1 + \rho) u_i \times B, u_i \right\rangle \]

\[ \leq C \left\| \rho \right\|_L \left\| \rho \right\|_L \left\{ \left\| \partial_{\Theta} \left( \frac{1 + \Theta}{1 + \rho} \right) \right\|_L \left( \left\| 1 + \Theta \right\|_L \left\| \nabla \cdot u \right\| \right) + \left\| \nabla \Theta \right\|_L \left\| \nabla \cdot u \right\| \right\} \]

\[ + \left\| \partial_{\rho} \left( \frac{1 + \rho}{1 + \Theta} \right) \right\|_L \left( \left\| 1 + \rho \right\|_L \left\| \nabla \rho \right\|_L \left\| u \right\| \right) + \left\| \nabla \Theta \right\|_L \left\| \nabla \cdot u \right\| \right\} \]

\[ + C \left\| \Theta \right\|_L \left\| \Theta \right\|_L \left\{ \left\| \partial_{\Theta} \left( \frac{1 + \rho}{1 + \Theta} \right) \right\|_L \left( \left\| 1 + \Theta \right\|_L \left\| \nabla \cdot u \right\| \right) + \left\| \nabla \Theta \right\|_L \left\| \nabla \cdot u \right\| \right\} \]

\[ + \left\| \partial_{\rho} \left( \frac{1 + \rho}{1 + \Theta} \right) \right\|_L \left( \left\| 1 + \rho \right\|_L \left\| \nabla \rho \right\|_L \left\| u \right\| \right) + \left\| \nabla \Theta \right\|_L \left\| \nabla \cdot u \right\| \right\} \]

\[ + C \left\| \nabla \rho \right\|_L \left\| u \right\|_L \left\| \Theta \right\|_L \left\| \rho \right\|_L \left\| B \right\| \left\| u \right\|_L \left\| L \right\| \left\| L \right\| \]
\[ \leq C \left( \| \nabla u_\mu \| + \| u_\mu \| + \| \nabla \Theta_\mu \| + \| \nabla \rho_\mu \| \right) \left( \| \rho_\mu \| \| \nabla \rho_\mu \| \right) \| \Theta_\mu \| \| \nabla \Theta_\mu \| \right) \] 
\[ + C \| \nabla \rho_\mu \| \| u_\mu \| \| \nabla \Theta_\mu \| + \| \nabla \Theta_\mu \| \| u_\mu \| \| \nabla \rho_\mu \| \right) + C \| u_\mu \| \| B \| \| \nabla u_\mu \| \] 
\[ \leq C \| \rho_\mu, u_\mu, \Theta_\mu, B \| \left( \| \nabla \rho_\mu \|^2 + \| u_\mu \|^2 + \| \nabla \Theta_\mu \|^2 \right), \]

which will further be bounded by the right hand side term of (2.16), and where we have used (2.14). When \(|\alpha| \geq 1\), similarly as before, one has

\[ I_{\alpha, \beta}(t) + I_1(t) \leq C \| \rho_\mu, u_\mu, \Theta_\mu, B \|_N \left( \| \nabla \rho_\mu \|_{s-1}^2 + \| u_\mu, \Theta_\mu \|_{s-1}^2 \right), \]

which will also be bounded by the right hand side term of (2.16).

Besides, for \(|\alpha| \leq s\), standard energy estimates on \( \partial^\alpha E \) and \( \partial^\alpha B \) from (2.2) yield

\[ \frac{1}{2} \frac{d}{dt} \left( \| \partial^\alpha E \|^2 + \| \partial^\alpha B \|^2 \right) - 2 \left( 1 + \rho_e \right) \partial^\alpha u_e - (1 + \rho_i) \partial^\alpha u_i, \partial^\alpha E \right) \]
\[ = \left( \partial^{\alpha-\beta} \partial^\alpha u_e - \partial^{\alpha-\beta} \partial^\alpha u_i, \partial^\alpha E \right) \]
\[ \leq C \| E \|_s \left( \| u_\mu \|_s^2 + \| \nabla \rho_\mu \|_{s-1}^2 \right), \]

which will be bounded by the right hand side term of (2.16). Then, with the help of (2.14), the summation (2.17) and (2.18) over \(|\alpha| \leq s\), one has (2.16).

**Step 2.** It holds that

\[ \frac{d}{dt} \sum_{|\alpha| \leq s-1, \mu=e,i} \langle \partial^\alpha u_\mu, \nabla \partial^\alpha \rho_\mu \rangle + \gamma \left( \| \nabla [\rho_e, \rho_i] \|^2_{s-1} + \| \rho_e - \rho_i \|^2 \right) \]
\[ \leq C \left( \| u_\mu \|_s^2 + \| \rho_\mu, u_\mu, \Theta_\mu, B \|_s^2 \left( \| \nabla \rho_\mu \|_{s-1}^2 + \| u_\mu, \Theta_\mu \|_s^2 \right) \right). \]

In fact, we can rewrite the equations (2.2) as

\[
\begin{align*}
\partial_t \rho_e + \nabla \cdot u_e &= g_{1e}, \\
\partial_t u_e + \nabla \rho_e + \nabla \Theta_e + u_e + E &= g_{2e}, \\
\partial_t \Theta_e + \nabla \cdot u_e + \Theta_e &= g_{3e}, \\
\partial_t \rho_i + \nabla \cdot u_i &= g_{1i}, \\
\partial_t u_i + \nabla \rho_i + \nabla \Theta_i + u_i - E &= g_{2i}, \\
\partial_t \Theta_i + \nabla \cdot u_i + \Theta_i &= g_{3i}, \\
\partial_t E - \nabla \times B - u_e + u_i &= g_{4e} - g_{4i}, \\
\partial_t B + \nabla \times E &= 0, \\
\nabla \cdot E &= \rho_i - \rho_e, \\
\nabla \cdot B &= 0, \\
(t, x) &\in (0, \infty) \times \mathbb{R}^3,
\end{align*}
\]
where

\[
\begin{align*}
    g_{1e} &= -\rho_e \nabla \cdot u_e - u_e \nabla \rho_e, \\
    g_{2e} &= -(u_e \cdot \nabla) u_e - \left( \frac{\Theta_e + 1}{1 + \rho_e} - 1 \right) \nabla \rho_e - u_e \times B, \\
    g_{3e} &= -\Theta_e \nabla \cdot u_e - u_e \nabla \rho_e, \\
    g_{4e} &= \rho_e u_e, \\
    g_{1i} &= -\rho_i \nabla \cdot u_i - u_i \nabla \rho_i, \\
    g_{2i} &= -(u_i \cdot \nabla) u_i - \left( \frac{\Theta_i + 1}{1 + \rho_i} - 1 \right) \nabla \rho_i + u_i \times B, \\
    g_{3i} &= -\Theta_i \nabla \cdot u_i - u_i \nabla \Theta_i, \\
    g_{4i} &= \rho_i u_i.
\end{align*}
\]

(2.21)

Let $|\alpha| \leq s - 1$. Utilizing $\partial^\alpha$ to the second equation of (2.20), multiplying it by $\nabla \partial^\alpha \rho_e$, integrating over $\mathbb{R}^3$ and using the last equation in (2.22), replacing $\partial_t \rho_e$ from the first equation of (2.20) implies

\[
\frac{d}{dt} \left( \partial^\alpha u_e, \nabla \partial^\alpha \rho_e \right) + \| \nabla \partial^\alpha \rho_e \|^2 + \| \partial^\alpha \rho_e \|^2 - \left( \partial^\alpha \rho_i, \partial^\alpha \rho_e \right) + \left( \nabla \partial^\alpha \Theta_e, \nabla \partial^\alpha \rho_e \right) = \| \partial^\alpha \nabla \cdot u_e \|^2 + \left( \partial^\alpha \nabla \rho_i, \partial^\alpha \rho_i \right) - \left( \partial^\alpha u_i, \nabla \partial^\alpha \rho_i \right) - \left( \partial^\alpha \nabla \cdot u_i, \partial^\alpha g_{1i} \right).
\]

Similarly as before, from the fourth and fifth equations of (2.20), we have

\[
\frac{d}{dt} \left( \partial^\alpha u_i, \nabla \partial^\alpha \rho_i \right) + \| \nabla \partial^\alpha \rho_i \|^2 + \| \partial^\alpha \rho_i \|^2 - \left( \partial^\alpha \rho_i, \partial^\alpha \rho_i \right) + \left( \nabla \partial^\alpha \Theta_i, \nabla \partial^\alpha \rho_i \right) = \| \partial^\alpha \nabla \cdot u_i \|^2 + \left( \partial^\alpha \nabla \rho_i, \partial^\alpha \rho_i \right) - \left( \partial^\alpha u_i, \nabla \partial^\alpha \rho_i \right) - \left( \partial^\alpha \nabla \cdot u_i, \partial^\alpha g_{1i} \right).
\]

Furthermore, the summation of the two equations above gives

\[
\frac{d}{dt} \left( \langle \partial^\alpha u_e, \nabla \partial^\alpha \rho_e \rangle + \langle \partial^\alpha u_i, \nabla \partial^\alpha \rho_i \rangle \right) + \| \nabla \partial^\alpha \rho_e \|^2 + \| \partial^\alpha \rho_e \|^2 + \| \partial^\alpha (\rho_e - \rho_i) \|^2 = \| \partial^\alpha \nabla \cdot u_e \|^2 + \left( \partial^\alpha \nabla \rho_i, \nabla \partial^\alpha \rho_i \right) - \left( \partial^\alpha \nabla \cdot u_i, \partial^\alpha g_{1i} \right)
\]

(2.22)

Therefore, after using Cauchy-Schwarz inequality, one has

\[
\frac{d}{dt} \left( \langle \partial^\alpha u_e, \nabla \partial^\alpha \rho_e \rangle + \langle \partial^\alpha u_i, \nabla \partial^\alpha \rho_i \rangle \right) + \lambda \left( \| \nabla \partial^\alpha \rho_e \| \|^2 + \| \nabla \partial^\alpha \rho_i \| \|^2 + \| \partial^\alpha (\rho_e - \rho_i) \|^2 \right) \leq C \left( \| \partial^\alpha \nabla \cdot u_e \|^2 + \| \partial^\alpha \rho_i \|^2 + \| \partial^\alpha \nabla \Theta_i \|^2 + \| \partial^\alpha g_{1i} \|^2 + \| \partial^\alpha \rho_i \|^2 \right)
\]

(2.22)

From the definition of $g_{j;\mu}$, $(j = 1, 2)$, one can check that

\[
\| \partial^\alpha g_{1;\mu} \|^2 + \| \partial^\alpha g_{2;\mu} \|^2 \leq C \left( \| \rho_{\mu, \Theta_{\mu}} \| \| \Theta_{\mu} \| + \| B \| \left( \| \nabla \rho_{\mu} \|^2_{s-1} + \| u_{\mu} \|^2 + \| \Theta_{\mu} \|^2 \right) \right)
\]

Putting this into (2.22), then, (2.19) follows by taking summation over $|\alpha| \leq s - 1$. 

Step 3. It holds that

\[
\frac{d}{dt} \sum_{|\alpha| \leq s-1} \langle \partial^\alpha (u_e - u_i), \partial^\alpha E \rangle + \gamma \| E \|^2_{s-1} \leq C \left( \| u_{\mu}, \Theta_{\mu} \| \|^2 + C \| \nabla \rho_{\mu} \|^2_{s-1} + C \| u_{\mu} \|^2 \right)
\]

(2.23)
In fact, for $|\alpha| \leq s - 1$, from the second and fifth equation of (2.20), one has
\[\partial_t (u_e - u_i) + \nabla (\rho_e - \rho_i) + \nabla (\Theta_e - \Theta_i) + 2E = g_{2e} - g_{2i} - (u_e - u_i).\]

Utilizing $\partial^\alpha$ to (2.24), multiplying it by $\partial^\alpha E$, integrating over $\mathbb{R}^3$ and replacing $\partial_t E$ from the seventh equation of (2.2) implies
\[
\frac{d}{dt} \langle \partial^\alpha (u_e - u_i), \partial^\alpha E \rangle + \| \partial^\alpha (\rho_e - \rho_i) \|^2 + 2\| \partial^\alpha E \|^2 \\
= -\langle \partial^\alpha (\Theta_e - \Theta_i), \partial^\alpha (\rho_e - \rho_i) \rangle + \langle \partial^\alpha (u_e - u_i), \partial^\alpha E \rangle + \langle \partial^\alpha (u_e - u_i), \nabla \times \partial^\alpha B \rangle \\
+ \| \partial^\alpha (u_e - u_i) \|^2 + \langle \partial^\alpha (u_e - u_i), \partial^\alpha (\rho_e u_e - \rho_i u_i) \rangle + \langle \partial^\alpha (g_{2e} - g_{2i}), \partial^\alpha E \rangle,
\]

Therefore, after using Cauchy-Schwarz inequality, one has
\[
\frac{d}{dt} \langle \partial^\alpha (u_e - u_i), \partial^\alpha E \rangle + \| \partial^\alpha E \|^2 \\
\leq C \left( \| \partial^\alpha u_{\mu} \|^2 + \| \partial^\alpha \Theta_{\mu} \|^2 + \| \partial^\alpha \nabla \rho_{\mu} \|^2 \right) + C \| [u_e, u_i] \|_{\mu} \| \nabla B \|_{s-2} \\
+ C \| [\rho_{\mu}, u_{\mu}, \Theta_{\mu}, B] \|^2 \left( \| \nabla \rho_{\mu} \|_{s-1}^2 + \| [u_{\mu}, \Theta_{\mu}] \|_{s}^2 \right).
\]

Thus, with help of the summation of the previous estimate over $|\alpha| \leq s - 1$, one can obtain (2.23).

**Step 4.** It holds that
\[\langle \partial^\alpha E, -\nabla \times \partial^\alpha B \rangle + \gamma \| \nabla B \|_{s-2}^2 \leq C(\| [u_{\mu}, E] \|_{s-1}^2 + \| \nabla \rho_{\mu} \|_{s-1}^2 \| u_{\mu} \|_{s}^2).
\]

In fact, for $|\alpha| \leq s - 2$, applying $\partial^\alpha$ to the seventh equation of (2.2), multiplying it by $-\partial^\alpha \nabla \times B$, integrating over $\mathbb{R}^3$ and then utilizing the eighth equation of (2.2) gives
\[
\frac{d}{dt} \sum_{|\alpha| \leq s - 2} \langle \partial^\alpha E, -\nabla \times \partial^\alpha B \rangle + \| \nabla \times \partial^\alpha B \|^2 \\
= \| \nabla \times \partial^\alpha E \|^2 - \langle \partial^\alpha (u_e - u_i), \nabla \times \partial^\alpha B \rangle + \langle \partial^\alpha (\rho_e u_e - \rho_i u_i), -\nabla \times \partial^\alpha B \rangle
\]

Furthermore, with the help of Cauchy-Schwarz inequality and the summation over $|\alpha| \leq s - 2$, we yield (2.25). Where we have used
\[\| \partial^\alpha \partial_i B \| = \| \partial_i \triangle^{-1} \nabla \times (\nabla \times \partial^\alpha B) \| \leq C \| \nabla \times \partial^\alpha B \|\]
for $1 \leq i \leq 3$, due to $\nabla \cdot B = 0$ and the fact that $\partial_i \triangle^{-1} \nabla$ is bounded from $L^p$ to $L^p$ with $1 < p < \infty$, see [17].

**Step 5.** Now, based on the four previous steps, we will search (2.15). We define the energy functional as
\[
\mathcal{E}_e(U(t)) = \| U \|_{\mu}^2 + K_1 \sum_{|\alpha| \leq s-1} \sum_{\alpha=\mu, i} \langle \partial^\alpha u_{\mu}, \nabla \partial^\alpha \rho_{\mu} \rangle \\
+ K_2 \sum_{|\alpha| \leq s-1} \langle \partial^\alpha (u_e - u_i), \partial^\alpha E \rangle + K_3 \sum_{|\alpha| \leq s-2} \langle \partial^\alpha E, -\nabla \times \partial^\alpha B \rangle,
\]
for constants $0 < K_3 \ll K_2 \ll K_1 \ll 1$ to be chosen later. Notice that as soon as $0 < K_j \ll 1$ $(1 \leq j \leq 3)$ is sufficiently small, then $\mathcal{E}_e(U(t)) \sim \| U \|_{\mu}^2$ holds true. Furthermore, the summation of
\[ (2.16), (2.19) \times K_1, \quad (2.23) \times K_2 \quad \text{and} \quad (2.25) \times K_3 \] implies that there is \( 0 < \gamma < 1 \) such that
\[
\frac{d}{dt} E_s (t) + \left( [u_e, u_i, \Theta_e, \Theta_i] \right)_{+}^{2} + \gamma K_1 (\| \nabla [\rho_e, \rho_i] \|^2_{s-1} + \| \rho_e - \rho_i \|^2_{s-1}) \\
+ \gamma K_2 \| E \|^2_{s-1} + \gamma K_3 \| \nabla B \|^2_{s-2}
\leq C [E_s (t)]^2 + E_s (t) D_s (t) + CK_1 \| u_{\mu} \|^2_{s} + CK_2 (\| [u_{\mu}, \Theta_{\mu}] \|^2_{s} + \| \nabla \rho_{\mu} \|^2_{s-1})
\leq C [E_s (U(t))]^{\frac{1}{2}} + E_s (U(t)) D_s (U(t)) + CK_1 \| u_{\mu} \|^2_{s} + CK_2 (\| [u_{\mu}, \Theta_{\mu}] \|^2_{s} + \| \nabla \rho_{\mu} \|^2_{s-1})
\leq C \left( K_{2}^3 \| u_{\mu} \|^2_{s} + K_{2}^3 \| \nabla B \|^2_{s-2} \right) + CK_3 \| [u_{\mu}, E] \|^2_{s-1}.
\]

By letting \( 0 < K_3 \ll K_2 \ll K_1 \ll 1 \) be sufficiently small with \( K_{2}^3 \ll K_3 \), we obtain (2.15). Now, we complete the proof of the Theorem 2.1. \( \square \)

3. Linearized homogeneous equations

In this section, for searching the time-decay property of solutions to the nonlinear equations (2.2) in the last section, we have to consider the decay properties of the linearized equations (2.20). Let us introduce the transformation
\[
(3.1) \quad \rho_1 = \frac{\rho_e - \rho_i}{2}, \quad u_1 = \frac{u_e - u_i}{2}, \quad \Theta_1 = \frac{\Theta_e - \Theta_i}{2}.
\]

Then, from system (2.2), \( U_1 = [\rho_1, u_1, \Theta_1, E, B] \) satisfies
\[
\begin{align*}
\partial_t \rho_1 + \nabla \cdot u_1 &= \frac{1}{2} (g_{1e} - g_{1i}), \\
\partial_t u_1 + \nabla \rho_1 + \nabla \Theta_1 + E + u_1 &= \frac{1}{2} (g_{2e} - g_{2i}), \\
\partial_t \Theta_1 + \nabla \cdot u_1 + \Theta_1 &= \frac{1}{2} (g_{3e} - g_{3i}), \\
\partial_t E - \nabla \times B - 2u_1 &= g_{4e} - g_{4i}, \\
\partial_t B + \nabla \times E &= 0, \\
\frac{1}{2} \nabla \cdot E &= -\rho_1, \quad (t, x) \in (0, \infty) \times \mathbb{R}^3,
\end{align*}
\]

with initial value \( U_1 |_{t=0} = U_{1,0} := [\rho_{1,0}, u_{1,0}, \Theta_{1,0}, E_0, B_0] \), \( x \in \mathbb{R}^3 \) which satisfies the compatibility conditions \( \frac{1}{2} \nabla \cdot E_0 = -\rho_{1,0}, \nabla \cdot B_0 = 0 \). Where, \( [\rho_{1,0}, u_{1,0}, \Theta_{1,0}] \) is given from \( [\rho_{\mu0}, u_{\mu0}, \Theta_{\mu0}] \) from the transformation (3.1). Moreover, we introduce another transformation
\[
(3.3) \quad \rho_2 = \frac{\rho_e + \rho_i}{2}, \quad u_2 = \frac{u_e + u_i}{2}, \quad \Theta_2 = \frac{\Theta_e + \Theta_i}{2}.
\]

Then \( U_2 = [\rho_2, u_2, \Theta_2] \) satisfies
\[
\begin{align*}
\partial_t \rho_2 + \nabla \cdot u_2 &= \frac{1}{2} (g_{1e} + g_{1i}), \\
\partial_t u_2 + \nabla \rho_2 + \nabla \Theta_2 + u_2 &= \frac{1}{2} (g_{2e} + g_{2i}), \\
\partial_t \Theta_2 + \nabla \cdot u_2 + \Theta_2 &= \frac{1}{2} (g_{3e} + g_{3i}), \quad (t, x) \in (0, \infty) \times \mathbb{R}^3,
\end{align*}
\]
with initial value $U_2|_{t=0} = U_{2,0} := [\rho_{2,0}, u_{2,0}, \Theta_{2,0}], x \in \mathbb{R}^3$, where $[\rho_{2,0}, u_{2,0}, \Theta_{2,0}]$ is from the transformation (3.3). Therefore, one can define the solution $U_1 = [\rho_1, u_1, \Theta_1, E, B]$ and $U_2 = [\rho_2, u_2, \Theta_2]$, respectively, as follows.

\begin{align}
U_1(t) &= e^{tL_1}U_{1,0} + \frac{1}{2} \int_0^t e^{(t-y)L_1} [g_{1e} - g_{1i}, g_{2e} - g_{2i}, g_{3e} - g_{3i}, 2(g_{4e} - g_{4i})](y)dy,
\end{align}

and

\begin{align}
U_2(t) &= e^{tL_2}U_{2,0} + \frac{1}{2} \int_0^t e^{(t-y)L_2} [g_{1e} + g_{1i}, g_{2e} + g_{2i}, g_{3e} + g_{3i}](y)dy,
\end{align}

where $e^{tL_1}U_{1,0}$ and $e^{tL_2}U_{2,0}$, respectively, denote the solution of the following homogeneous initial value problems (3.7)-(3.8) and (3.10)-(3.11), which will be given as follows:

The linearized homogeneous equations corresponding to (3.2) is

\begin{align}
\begin{aligned}
\partial_t \rho_1 + \nabla \cdot u_1 &= 0, \\
\partial_t u_1 + \nabla \rho_1 + \nabla \Theta_1 + E + u_1 &= 0, \\
\partial_t \Theta_1 + \nabla \cdot u_1 + \Theta_1 &= 0, \\
\partial_t E - \nabla \times B - 2u_1 &= 0, \\
\partial_t B + \nabla \times E &= 0,
\end{aligned}
\end{align}

with initial value

\begin{align}
U_1|_{t=0} = U_{1,0} := [\rho_{1,0}, u_{1,0}, \Theta_{1,0}, E_0, B_0], x \in \mathbb{R}^3
\end{align}

which satisfies the compatible conditions

\begin{align}
\frac{1}{2} \nabla \cdot E_0 = -\rho_{1,0}, \nabla \cdot B_0 = 0.
\end{align}

And the linearized homogeneous equations corresponding to (3.6) is

\begin{align}
\begin{aligned}
\partial_t \rho_2 + \nabla \cdot u_2 &= 0, \\
\partial_t u_2 + \nabla \rho_2 + \nabla \Theta_2 + u_2 &= 0, \\
\partial_t \Theta_2 + \nabla \cdot u_2 + \Theta_2 &= 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^3,
\end{aligned}
\end{align}

with initial value

\begin{align}
U_2|_{t=0} = U_{2,0} := [\rho_{2,0}, u_{2,0}, \Theta_{2,0}], x \in \mathbb{R}^3.
\end{align}

Here $[\rho_{2,0}, u_{2,0}, \Theta_{2,0}]$ is from the transform (3.6). In the sequel, we usually denote $U_1 = [\rho_1, u_1, \Theta_1, E, B]$ as the solution of the linearized homogeneous equations (3.7), and $U_2 = [\rho_2, u_2, \Theta_2]$ as the one of (3.10).

Firstly, for the linearized homogeneous system (3.7)-(3.8), similarly as [6], we obtain the $L^p - L^q$ decay property as follows

**Proposition 3.1.** Assume $U_1(t) = e^{tL_1}U_{1,0}$ is the solution to the initial value problem (3.7)-(3.8) with $U_{1,0} = [\rho_{1,0}, u_{1,0}, \Theta_{1,0}, E_0, B_0]$ which satisfies (3.8). Then, for any $t \geq 0$, $U_1 = [\rho_1,
in this paper, we set \( \tilde{E} \) satisfies

\[
\begin{align*}
\|\rho_1(t), & \Theta_1(t)\| \leq Ce^{-\frac{t}{2}} \|\rho_{1,0}, u_{1,0}, \Theta_{1,0}\|, \\
\|u_1(t)\| & \leq Ce^{-\frac{t}{2}} \|\rho_{1,0}, \Theta_{1,0}\| + C(1 + t)^{-\frac{3}{2}} \|[u_{1,0}, E_0, B_0]\|_{L^1 \cap H^2}, \\
\|E(t)\| & \leq C(1 + t)^{-\frac{1}{2}} \|[u_{1,0}, \Theta_{1,0}, E_0, B_0]\|_{L^1 \cap H^3}, \\
\|B(t)\| & \leq C(1 + t)^{-\frac{1}{2}} \|[u_{1,0}, E_0, B_0]\|_{L^1 \cap H^2},
\end{align*}
\]

(3.12)

\[
\begin{align*}
\|\rho_1(t), & \Theta_1(t)\|_{L^\infty} \leq Ce^{-\frac{t}{2}} \|\rho_{1,0}, u_{1,0}, \Theta_{1,0}\|_{L^2 \cap H^2}, \\
\|u_1(t)\|_{L^\infty} & \leq Ce^{-\frac{t}{2}} \|\rho_{1,0}, \Theta_{1,0}\|_{L^1 \cap H^2} + C(1 + t)^{-2} \|[u_{1,0}, E_0, B_0]\|_{L^1 \cap H^5}, \\
\|E(t)\|_{L^\infty} & \leq C(1 + t)^{-2} \|[u_{1,0}, \Theta_{1,0}, E_0, B_0]\|_{L^1 \cap H^6}, \\
\|B(t)\|_{L^\infty} & \leq C(1 + t)^{-2} \|[u_{1,0}, E_0, B_0]\|_{L^1 \cap H^5},
\end{align*}
\]

(3.13)

and

\[
\begin{align*}
\|\nabla B(t)\| & \leq C(1 + t)^{-\frac{3}{2}} \|[u_{1,0}, E_0, B_0]\|_{L^1 \cap H^4}, \\
\|\nabla^s [E(t), B(t)]\| & \leq C(1 + t)^{-\frac{3}{2}} \|[u_{1,0}, \Theta_{1,0}, E_0, B_0]\|_{L^2 \cap H^{s+3}}.
\end{align*}
\]

(3.14)

3.1. Explicit solutions of (3.10)-(3.11). Firstly, let us search the explicit Fourier transform solution \( U_2 = [\rho_2, u_2, \Theta_2] \) of the initial value problem (3.10)-(3.11).

From the three equations of (3.10), one has

\[
\partial_{tt} \rho_2 + 2 \partial_t \rho_2 - 2 \Delta \partial_t \rho_2 + \partial_t \rho_2 - \Delta \rho_2 = 0,
\]

with initial value

\[
\begin{align*}
\rho_2|_{t=0} &= \rho_{2,0}, \\
\partial_t \rho_2|_{t=0} &= -\nabla \cdot u_{2,0}, \\
\partial_{tt} \rho_2|_{t=0} &= \Delta \rho_{2,0} + \nabla \cdot u_{2,0} + \Delta \Theta_{2,0}.
\end{align*}
\]

(3.15)

(3.16)

After taking the Fourier transform on (3.15) and (3.16), it follows that

\[
\partial_{tt} \hat{\rho}_2 + 2 \partial_t \hat{\rho}_2 + (1 + 2|k|^2) \partial_t \hat{\rho}_2 + |k|^2 \hat{\rho}_2 = 0,
\]

with initial value

\[
\begin{align*}
\hat{\rho}_2|_{t=0} &= \hat{\rho}_{2,0}, \\
\partial_t \hat{\rho}_2|_{t=0} &= -i|k|\tilde{k} \cdot \hat{u}_{2,0}, \\
\partial_{tt} \hat{\rho}_2|_{t=0} &= -|k|^2 \hat{\rho}_{2,0} + i|k|\tilde{k} \cdot \hat{u}_{2,0} - |k|^2 \hat{\Theta}_{2,0},
\end{align*}
\]

(3.17)

(3.18)

in this paper, we set \( \tilde{k} = \frac{k}{|k|} \). The characteristic equation of (3.17) is

\[
F(\chi) := \chi^3 + 2\chi^2 + (1 + 2|k|^2) \chi + |k|^2 = 0.
\]

For the roots of the previous characteristic equation and their properties, we obtain

**Lemma 3.1.** Assume \( |k| \neq 0 \). Then, \( F(\chi) = 0 \), \( \chi \in \mathbb{C} \) has a real root \( \sigma = \sigma(|k|) \in (-\frac{1}{2}, 0) \) and two conjugate complex roots \( \chi_{\pm} = \beta \pm i\omega \) with \( \beta = \beta(|k|) \in (-1, -\frac{3}{4}) \) and \( \omega = \omega(|k|) \in (0, +\infty) \) which satisfy the following properties:

\[
\beta = -1 - \frac{\sigma}{2}, \quad \omega = \frac{1}{2} \sqrt{3\sigma^2 + 4\sigma + 8|k|^2}.
\]

(3.19)
σ, β, ω are smooth in |k| > 0, and σ(|k|) is strictly decreasing over |k| > 0, with

\[
\lim_{|k| \to 0} \sigma(|k|) = 0, \quad \lim_{|k| \to \infty} \sigma(|k|) = -\frac{1}{2}.
\]

Furthermore, the asymptotic behavior as follows hold true:

\[
\sigma(|k|) = -O(1)|k|^2, \quad \beta(|k|) = -1 + O(1)|k|^2, \quad \omega(|k|) = O(1)|k|
\]

whenever |k| ≤ 1 is sufficiently small, and

\[
\sigma(|k|) = -\frac{1}{2} + O(1)|k|^{-2}, \quad \beta(|k|) = -\frac{3}{4} - O(1)|k|^{-2}, \quad \omega(|k|) = O(1)|k|
\]

whenever |k| ≥ 1 is sufficiently large. Here and in the sequel O(1) means strictly positive constant.

Proof. Assume |k| ≠ 0. First of all, we look for the possibly existing real root for \(F(\chi) = 0\) over \(\chi \in \mathbb{R}\). Since

\[
F'(\chi) = 3\chi^2 + 4\chi + 1 + 2|k|^2 > 0,
\]

and \(F(-\frac{1}{2}) = -\frac{1}{8} < 0, \quad F(0) = |k|^2 > 0\), then equation \(F(\chi) = 0\) really has one and only one real root defined as \(\sigma = \sigma(|k|)\) which satisfies \(-\frac{1}{2} < \sigma < 0\). After taking derivative of \(F(\sigma(|k|)) = 0\) in |k|, one has

\[
\sigma'(|k|) = -\frac{|k|(2 + 4\sigma)}{3\sigma^2 + 4\sigma + 1 + 2|k|^2} < 0,
\]

so that \(\sigma(\cdot)\) is strictly decreasing in |k| > 0. Since \(F(\sigma) = 0\) can be re-written as

\[
\sigma \left[ \frac{\sigma(\sigma + 2)}{1 + 2|k|^2} + 1 \right] = -\frac{|k|^2}{1 + 2|k|^2},
\]

then \(\sigma\) has limits 0 and \(-\frac{1}{2}\) as \(|k| \to 0\) and \(|k| \to \infty\), respectively.

\(F(\sigma(|k|)) = 0\) is also equivalent with

\[
\sigma + \frac{1}{2} = \frac{\frac{1}{2}(\sigma + 1)^2}{(\sigma + 1)^2 + 2|k|^2}
\]

Therefore, it follows that \(\sigma(|k|) = -O(1)|k|^2\) whenever \(|k| < 1\) is small enough and \(\sigma(|k|) = -\frac{1}{2} + O(1)|k|^{-2}\) whenever \(|k| \geq 1\) is large enough. Next, let us search roots of \(F(\chi) = 0\) on \(\chi \in \mathbb{C}\). Since \(F(\sigma) = 0\) with \(\sigma \in \mathbb{R}\), \(F(\chi) = 0\) can be split up into

\[
F(\chi) = (\chi - \sigma) \left[ \left(\chi + 1 + \frac{\sigma}{2}\right)^2 + \frac{3}{4}\sigma^2 + \sigma + 2|k|^2 \right] = 0.
\]

Therefore, there are two conjugate complex roots \(\chi_\pm = \beta \pm i\omega\) which satisfy

\[
\left(\chi + 1 + \frac{\sigma}{2}\right)^2 + \frac{3}{4}\sigma^2 + \sigma + 2|k|^2 = 0.
\]

After solving the above equation, one can get that \(\beta = \beta(|k|), \quad \omega = \omega(|k|)\) take the form of (3.19). From the asymptotic behavior of \(\sigma(|k|)\) at \(|k| = 0\) and \(\infty\), one can directly acquire that of \(\beta(|k|), \omega(|k|)\). Now, we complete the proof of Lemma 3.1.\(\square\)

Based on Lemma 3.1, one can define the solution of (3.17) as

\[
(3.20) \quad \hat{\rho}_2(t, k) = c_1(k)e^{\alpha t} + e^{\beta t} (c_2(k) \cos \omega t + c_3(k) \sin \omega t),
\]
where $c_i(k), 1 \leq i \leq 3$, is to be ascertained by (3.18) later. In fact, (3.18) implies

$$(3.21) \quad \begin{bmatrix} 1 & 1 & 0 \\ \beta & \omega & 0 \end{bmatrix} = A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & \beta & \omega \\ \sigma & 2 \beta - \omega \end{bmatrix}.$$ 

It is directly to check that

$$\det A = \omega \left[ \omega^2 + (\sigma - \beta)^2 \right] = \omega \left( 3\sigma^2 + 4\sigma + 1 + 2|k|^2 \right) > 0$$

and

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} (\beta^2 + \omega^2) \omega & -2\beta \omega & \omega \\ \sigma (\sigma - 2\beta) \omega & 2\beta \omega & -\omega \\ \sigma (\beta^2 - \omega^2 - \sigma \beta) \omega^2 + \sigma^2 - \beta^2 & \beta - \sigma \end{bmatrix}.$$ 

Notice that (3.21) together with (3.18) gives

$$[c_1, c_2, c_3]^T = \frac{1}{3\sigma^2 + 4\sigma + 1 + 2|k|^2} \begin{bmatrix} \beta^2 + \omega^2 - |k|^2 \\ \sigma^2 - 2\beta \beta + |k|^2 \\ \sigma(\beta^2 - \omega^2 - \sigma \beta) - (\beta - \sigma)|k|^2 \end{bmatrix} - i|k|(2\beta + 1) - |k|^2 i|k|(\sigma + 1) + \frac{1 + \delta}{\omega} |k|^2.$$ 

Similarly, from the three equations of (3.10), one has

$$(3.23) \quad \partial_{tt}\hat{\Theta}_2 + 2\partial_t \hat{\Theta}_2 + (1 + 2|k|^2) \partial_t \hat{\Theta}_2 + |k|^2 \hat{\Theta}_2 = 0,$$

with initial value

$$(3.24) \quad \begin{cases} \hat{\Theta}_2|_{t=0} = \hat{\Theta}_{2,0}, \\ \partial_t \hat{\Theta}_2|_{t=0} = -i|k|k \cdot \hat{u}_{2,0}, \\ \partial_t \hat{\Theta}_2|_{t=0} = -|k|^2 \hat{\rho}_{2,0} + 2i|k|k \cdot \hat{u}_{2,0} + (1 - |k|^2) \hat{\Theta}_{2,0}. \end{cases}$$ 

Based on Lemma 3.1, one can also set the solution of (3.23) as

$$(3.25) \quad \hat{\Theta}_2(t, k) = c_4(k)e^{\sigma t} + e^{\beta t} (c_5(k) \cos \omega t + c_6(k) \sin \omega t),$$

where $c_i(k), 4 \leq i \leq 6,$ is to be ascertained by (3.24) later. In fact, after tenuous computation, (3.24) implies

$$[c_4, c_5, c_6]^T = \frac{1}{3\sigma^2 + 4\sigma + 1 + 2|k|^2} \begin{bmatrix} -|k|^2 - i|k|(1 + \sigma) \\ |k|^2 i|k|(1 + \sigma) \\ -\frac{1}{\omega}(\frac{3}{2}\sigma(\sigma + 2) + 1 + 2|k|^2) \end{bmatrix} \begin{bmatrix} \hat{\rho}_{2,0} \\ \hat{\Theta}_{2,0} \end{bmatrix}.$$
Similarly, again from the three equations of (3.10), we also have

\[
\partial_{ttt}(\hat{k} \cdot \hat{u}_2) + 2\partial_{tt}(\hat{k} \cdot \hat{u}_2) + (1 + 2|k|^2)\partial_t(\hat{k} \cdot \hat{u}_2) + |k|^2(\hat{k} \cdot \hat{u}_2) = 0,
\]

with initial value

\[
\begin{align*}
\hat{k} \cdot \hat{u}_2|_{t=0} &= \hat{k} \cdot \hat{u}_{2,0}, \\
\partial_t(\hat{k} \cdot \hat{u}_2)|_{t=0} &= -i|k|\hat{\rho}_{2,0} - \hat{k} \cdot \hat{u}_{2,0} - i|k|\hat{\Theta}_{2,0}, \\
\partial_{tt}(\hat{k} \cdot \hat{u}_2)|_{t=0} &= i|k|\hat{\rho}_{2,0} + (1 - 2|k|^2)\hat{k} \cdot \hat{u}_{2,0} + 2i|k|\hat{\Theta}_{2,0}.
\end{align*}
\]

From Lemma 3.1, one can also check that the solution of (3.27) has the form

\[
\hat{k} \cdot \hat{u}_2(t, k) = c_7(k)e^{\alpha t} + e^{\beta t}(c_8(k)\cos \omega t + c_9(k)\sin \omega t),
\]

with

\[
[c_7, c_8, c_9]^T = \frac{1}{3\sigma^2 + 4\sigma + 1 + 2|k|^2}
\[
\begin{bmatrix}
-\frac{i|k|}{\omega}(\frac{3\sigma}{2}\sigma + 1 - 2|k|^2) & -\frac{i|k|}{\omega}(1 + \sigma) & i|k| \\
\frac{i|k|}{\omega}(1 + \sigma) & \frac{1}{2}(1 + \sigma + 2|k|^2) & \frac{1}{2}\sigma \\
-i|k|\sigma & -\frac{i}{\omega}(1 + \sigma - 2|k|^2) & i|k|\sigma
\end{bmatrix}
\begin{bmatrix}
\hat{\rho}_{2,0} \\
\hat{k} \cdot \hat{u}_{2,0}
\end{bmatrix}.
\]

Furthermore, after taking the curl for the second equation of (3.10) and making the Fourier transform in \(x\), we have

\[
\partial_t \left( \hat{k} \times (\hat{k} \times \hat{u}_2) \right) + \hat{k} \times (\hat{k} \times \hat{u}_2) = 0,
\]

with initial value

\[
\hat{k} \times (\hat{k} \times \hat{u}_2)|_{t=0} = \hat{k} \times (\hat{k} \times \hat{u}_{2,0}).
\]

After solving (3.31)-(3.32), we have

\[
\hat{k} \times (\hat{k} \times \hat{u}_2) = e^{-t} \left( \hat{k} \times (\hat{k} \times \hat{u}_{2,0}) \right).
\]

Now, we can obtain the explicit Fourier transform solution \(U_2 = [\rho_2, u_2, \Theta_2]\) as follows from the above computations.

**Theorem 3.1.** Assume \(U_2 = [\rho_2, u_2, \Theta_2]\) be the solution of the initial value problem (3.10)-(3.11) on the linearized homogeneous equations. For \((t, k) \in (0, \infty) \times \mathbb{R}^3\) with \(|k| \neq 0\), we obtain

\[
\begin{bmatrix}
\hat{\rho}_{2}(t, k) \\
\hat{u}_{2||}(t, k) \\
\hat{\Theta}_{2}(t, k)
\end{bmatrix} =
\begin{bmatrix}
\hat{\rho}_{2}(t, k) \\
\hat{u}_{2||}(t, k) \\
\hat{\Theta}_{2}(t, k)
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}.
\]

Here \(\hat{u}_{2||}, \hat{u}_{2\perp}\) are defined by

\[
\hat{u}_{2||} = \hat{k}\hat{k} \cdot \hat{u}_2, \quad \hat{u}_{2\perp} = -\hat{k} \times (\hat{k} \times \hat{u}_2) = (I_3 - \hat{k} \otimes \hat{k})\hat{u}_2.
\]

Then, there exit matrices \(G_{5 \times 5}^I(t, k)\) and \(G_{3 \times 3}^{II}(t, k)\) such that

\[
\begin{bmatrix}
\hat{\rho}_{2}(t, k) \\
\hat{u}_{2||}(t, k) \\
\hat{\Theta}_{2}(t, k)
\end{bmatrix} = G_{5 \times 5}^I(t, k)
\begin{bmatrix}
\hat{\rho}_{2,0}(k) \\
\hat{u}_{2||,0}(k) \\
\hat{\Theta}_{2,0}(k)
\end{bmatrix}.
\]
and
\[ (3.36) \quad \hat{u}_{2\perp}(t,k) = G^H_{3 \times 3}(t,k) \hat{u}_{2\perp,0}(k), \]
where \( G^H_{5 \times 5} \) is explicitly ascertained by representations \( (3.20), (3.29), (3.25) \) for \( \hat{\rho}_2(t,k), \hat{\omega}_2(t,k), \hat{\Theta}_2(t,k) \) with \( c_i(k), (1 \leq i \leq 9) \) are defined as \( (3.22), (3.30), (3.26) \) in terms of \( \hat{\rho}_{20}(k), \hat{\omega}_{2\perp,0}(k), \hat{\Theta}_{2,0}(k) \); and \( G^H_{3 \times 3} \) is chosen by the representations \( (3.33) \) for \( \hat{u}_{2\perp}(t,k) \) in terms of \( \hat{u}_{2\perp,0}(k) \).

3.2. \( L^p - L^q \) decay property. In this subsection, we use Theorem 3.1 to obtain \( L^p - L^q \) decay property for every component of the solution \( U_2 = [\rho_2, u_2, \Theta_2] \). For this aim, we first search the rigorous time-frequency estimates on \( \hat{U}_2 = [\hat{\rho}_2, \hat{u}_2, \hat{\Theta}_2] \) as follows

**Lemma 3.2.** Assume \( U_2 = [\rho_2, u_2, \Theta_2] \) be the solution to the initial value problem \( (3.10)-(3.11) \) on the linearized homogeneous equations. Then, there are constants \( \gamma > 0, C > 0 \) such that for all \( (t,k) \in (0, \infty) \times \mathbb{R}^3 \),

\[ (3.37) \quad |\hat{\rho}_2(t,k)| \leq C \left| \begin{bmatrix} \hat{\rho}_{2,0}(t,k), \hat{u}_{2,0}(t,k), \hat{\Theta}_{2,0}(t,k) \end{bmatrix} \right| \cdot \left\{ \begin{array}{ll} e^{-\gamma t} + e^{-\gamma |k|^2 t} & \text{if } |k| \leq 1, \\ e^{-\gamma t} + e^{-\gamma |k|^2 t} & \text{if } |k| > 1, \end{array} \right. \]

\[ (3.38) \quad |\hat{u}_2(t,k)| \leq C \left| \begin{bmatrix} \hat{\rho}_{2,0}(t,k), \hat{u}_{2,0}(t,k), \hat{\Theta}_{2,0}(t,k) \end{bmatrix} \right| \cdot \left\{ \begin{array}{ll} e^{-\gamma t} + |k| e^{-\gamma |k|^2 t} & \text{if } |k| \leq 1, \\ |k|^{-1} e^{-\gamma t} + e^{-\gamma |k|^2 t} & \text{if } |k| > 1, \end{array} \right. \]

and

\[ (3.39) \quad |\hat{\Theta}_2(t,k)| \leq C \left| \begin{bmatrix} \hat{\rho}_{2,0}(t,k), \hat{u}_{2,0}(t,k), \hat{\Theta}_{2,0}(t,k) \end{bmatrix} \right| \cdot \left\{ \begin{array}{ll} e^{-\gamma t} + e^{-\gamma |k|^2 t} & \text{if } |k| \leq 1, \\ e^{-\gamma t} + e^{-\gamma |k|^2 t} & \text{if } |k| > 1, \end{array} \right. \]

**Proof.** Firstly, let us look for the upper bound of \( \hat{\rho}_2 \) defined as \( (3.37) \). In fact, from Lemma 3.1 it is directly to check \( (3.22) \) to get

\[
\begin{bmatrix}
  c_1 \\
  c_2 \\
  c_3
\end{bmatrix} =
\begin{bmatrix}
  O(1) & -O(1)|k|i & -O(1)|k|^2 \\
  O(1)|k| & -O(1)|k|^2 & O(1)|k| \\
  O(1)|k|^{-1} & -O(1)|k|^2 & O(1)|k|^{-1}
\end{bmatrix}
\begin{bmatrix}
  \hat{\rho}_{2,0} \\
  \hat{u}_{2,0} \\
  \hat{\Theta}_{2,0}
\end{bmatrix}
\]

as \( |k| \to 0 \), and

\[
\begin{bmatrix}
  c_1 \\
  c_2 \\
  c_3
\end{bmatrix} =
\begin{bmatrix}
  O(1) & -O(1)|k|^{-1}i & -O(1)|k|^{-1} \\
  O(1)|k|^2 & -O(1)|k|^2 & O(1)|k|^2 \\
  O(1)|k|^{-1} & -O(1)|k|^2 & O(1)|k|^{-1}
\end{bmatrix}
\begin{bmatrix}
  \hat{\rho}_{2,0} \\
  \hat{u}_{2,0} \\
  \hat{\Theta}_{2,0}
\end{bmatrix}
\]

as \( |k| \to \infty \).

Therefore, after putting the previous computations into \( (3.20) \), it holds that

\[
\hat{\rho}_2(t,k) = \left( O(1)\hat{\rho}_{2,0} - O(1)|k|i\hat{k} \cdot \hat{u}_{2,0} - O(1)|k|^2\hat{\Theta}_{2,0} \right) e^{\alpha t}
\]

\[
+ \left( O(1)\hat{\rho}_{2,0} + O(1)|k|i\hat{k} \cdot \hat{u}_{2,0} + O(1)|k|^2\hat{\Theta}_{2,0} \right) e^{\beta t}\cos \omega t
\]

\[
+ \left( O(1)|k|\hat{\rho}_{2,0} - O(1)|k|^2 i\hat{k} \cdot \hat{u}_{2,0} + O(1)|k|\hat{\Theta}_{2,0} \right) e^{\beta t}\sin \omega t,
\]

as \( |k| \to 0 \), and

\[
\hat{\rho}_2(t,k) = \left( O(1)\hat{\rho}_{2,0} - O(1)|k|^{-1}i\hat{k} \cdot \hat{u}_{2,0} - O(1)\hat{\Theta}_{2,0} \right) e^{\alpha t}
\]

\[
+ \left( O(1)\hat{\rho}_{2,0} + O(1)|k|^{-1}i\hat{k} \cdot \hat{u}_{2,0} + O(1)\hat{\Theta}_{2,0} \right) e^{\beta t}\cos \omega t
\]

\[
+ \left( O(1)|k|^{-1}\hat{\rho}_{2,0} - O(1)i\hat{k} \cdot \hat{u}_{2,0} + O(1)|k|^{-1}\hat{\Theta}_{2,0} \right) e^{\beta t}\sin \omega t,
\]
as $|k| \to \infty$.

Based on Lemma 3.1, we find that there is $\gamma > 0$ such that
\[
\begin{cases}
\sigma(k) \leq -\gamma|k|^2, & \beta(k) = -1 - \frac{\sigma}{2} \leq -\gamma \quad \text{over } |k| \leq 1, \\
\sigma(k) \leq -\gamma, & \beta(k) = -1 - \frac{\sigma}{2} \leq -\frac{\gamma}{|k|^2} \quad \text{over } |k| \geq 1.
\end{cases}
\]

Thus, one can obtain, for $|k| \leq 1$
\[
|\hat{\rho}_2(t,k)| \leq C \left( e^{-\gamma t} + e^{-\gamma|k|^2t} \right) \left| \left[ \hat{\rho}_{2,0}, \hat{k}, \hat{u}_{2,0}, \hat{\Theta}_{2,0} \right] \right|,
\]
and for $|k| \geq 1$
\[
|\hat{\rho}_2(t,k)| \leq C \left( e^{-\gamma t} + e^{-\gamma|k|^2t} \right) \left| \left[ \hat{\rho}_{2,0}, \hat{k}, \hat{u}_{2,0}, \hat{\Theta}_{2,0} \right] \right|.
\]

Furthermore, one has
\[
|\hat{\rho}_2(t,k)| \leq C \left| \left[ \hat{\rho}_{2,0}, \hat{u}_{2,0}, \hat{\Theta}_{2,0} \right] \right| \cdot \begin{cases}
(e^{-\gamma t} + e^{-\gamma|k|^2t}) & \text{if } |k| \leq 1, \\
(e^{-\gamma t} + e^{-\gamma t}) & \text{if } |k| \geq 1.
\end{cases}
\]

Similarly, we obtain (3.38) and (3.39). Now, we complete the proof of Lemma 3.2. \hfill \square

From Lemma 3.2, it is straightforward to acquire the decay property for every component of the solution $U_2 = [\rho_2, u_2, \Theta_2]$. So that we omitted the details of proof for briefness. See for instance [6].

**Theorem 3.2.** Assume $1 \leq p, r \leq 2 \leq q \leq \infty, l \geq 0$ and an integer $m \geq 0$. Suppose $U_2(t) = e^{tL_2}U_{2,0}$ be the solution of the initial value problem (3.10)-(3.11). Then, for any $t \geq 0$, $U_2 = [\rho_2, u_2, \Theta_2]$ satisfies decay property as follows

\[
\|\nabla^m \rho_2(t)\|_{L^q} \leq C(1 + t)^{-\frac{4}{2} \left( \frac{1}{r} - \frac{1}{q} \right) - \frac{m}{2} \left( l + 3 \left( \frac{1}{r} - \frac{1}{q} \right) \right)} \|[\rho_{2,0}, u_{2,0}, \Theta_{2,0}]\|_{L^p}
\]
\[+ C(1 + t)^{-\frac{1}{2}} \left\| \nabla^{m+\left[ l+3 \left( \frac{1}{r} - \frac{1}{q} \right) \right]} \left[ \rho_{2,0}, u_{2,0}, \Theta_{2,0} \right] \right\|_{L^r},
\]
\[
(3.40)
\]

\[
\|\nabla^m u_2(t)\|_{L^q} \leq C(1 + t)^{-\frac{4}{2} \left( \frac{1}{r} - \frac{1}{q} \right) - \frac{m+1}{2} \left( l + 3 \left( \frac{1}{r} - \frac{1}{q} \right) \right)} \|[\rho_{2,0}, u_{2,0}, \Theta_{2,0}]\|_{L^p}
\]
\[+ C(1 + t)^{-\frac{1}{2}} \left\| \nabla^{m+\left[ l+3 \left( \frac{1}{r} - \frac{1}{q} \right) \right]} \left[ \rho_{2,0}, u_{2,0}, \Theta_{2,0} \right] \right\|_{L^r},
\]
\[
(3.41)
\]

\[
\|\nabla^m \Theta_2(t)\|_{L^q} \leq C(1 + t)^{-\frac{4}{2} \left( \frac{1}{r} - \frac{1}{q} \right) - \frac{m}{2} \left( l + 3 \left( \frac{1}{r} - \frac{1}{q} \right) \right)} \|[\rho_{2,0}, u_{2,0}, \Theta_{2,0}]\|_{L^p}
\]
\[+ C(1 + t)^{-\frac{1}{2}} \left\| \nabla^{m+\left[ l+3 \left( \frac{1}{r} - \frac{1}{q} \right) \right]} \left[ \rho_{2,0}, u_{2,0}, \Theta_{2,0} \right] \right\|_{L^r},
\]
\[
(3.42)
\]

where
\[
[l + 3 \left( \frac{1}{r} - \frac{1}{q} \right)] = \begin{cases}
[l + 3 \left( \frac{1}{r} - \frac{1}{q} \right)]_+ + 1 & \text{if } r = q = 2 \text{ and } l \text{ is an integer,} \\
[l + 3 \left( \frac{1}{r} - \frac{1}{q} \right)]_+ & \text{otherwise,}
\end{cases}
\]

where, we use $[\cdot]_+$ to denote the integer part of the argument.

From Theorem 3.2 let us list some particular cases as follows for later use.
Corollary 3.1. Let $U_2(t) = e^{tL_2}U_{2,0}$ be the solution of the initial value problem (3.10)-(3.11). Then, for any $t \geq 0$, $U_2 = [\rho_2, u_2, \Theta_2]$ satisfies

\begin{align}
\|\rho_2(t)\| \leq C(1+t)^{-\frac{3}{2}}\|\rho_{2,0}, u_{2,0}, \Theta_{2,0}\|_{L^1 \cap H^2}, \\
\|u_2(t)\| \leq C(1+t)^{-\frac{1}{2}}\|\rho_{2,0}, u_{2,0}, \Theta_{2,0}\|_{L^1 \cap H^3}, \\
\|\Theta_2(t)\| \leq C(1+t)^{-\frac{1}{2}}\|\rho_{2,0}, u_{2,0}, \Theta_{2,0}\|_{L^1 \cap H^2},
\end{align}

(3.43)

and

\begin{align}
\|\nabla \rho_2(t)\| \leq C(1+t)^{-\frac{3}{2}}\|\rho_{2,0}, u_{2,0}, \Theta_{2,0}\|_{L^1 \cap H^4}, \\
\|\nabla u_2(t)\| \leq C(1+t)^{-\frac{5}{2}}\|\rho_{2,0}, u_{2,0}, \Theta_{2,0}\|_{L^1 \cap H^5}, \\
\|\nabla \Theta_2(t)\| \leq C(1+t)^{-\frac{7}{2}}\|\rho_{2,0}, u_{2,0}, \Theta_{2,0}\|_{L^1 \cap H^4},
\end{align}

(3.44)

and

\begin{align}
\|\rho_2(t)\|_{L^\infty} \leq C(1+t)^{-\frac{3}{2}}\|\rho_{2,0}, u_{2,0}, \Theta_{2,0}\|_{L^1 \cap H^5}, \\
\|u_2(t)\|_{L^\infty} \leq C(1+t)^{-2}\|\rho_{2,0}, u_{2,0}, \Theta_{2,0}\|_{L^1 \cap H^6}, \\
\|\Theta_2(t)\|_{L^\infty} \leq C(1+t)^{-\frac{3}{2}}\|\rho_{2,0}, u_{2,0}, \Theta_{2,0}\|_{L^1 \cap H^5}.
\end{align}

(3.45)

4. Decay rates for system (2.2)

4.1. Decay rates for the energy functional. In this subsection, we will prove the decay rate (2.12) in Proposition 2.2 for the energy $\|U(t)\|^2$. We begin with the Lemma as follows which can be seen directly from the proof of Theorem 2.1.

Lemma 4.1. Assume that $U = [\rho_\mu, u_\mu, \Theta_\mu, E, B]$ is the solution of the initial value problem (2.2)-(2.3) with $U_0 = [\rho_{0,0}, u_{0,0}, \Theta_{0,0}, E_0, B_0]$ which satisfies (2.4). If $E_s(U_0)$ is small enough, then, for any $t \geq 0$

\begin{align}
\frac{d}{dt}E_s(U(t)) + \lambda D_s(U(t)) \leq 0.
\end{align}

(4.1)

From Lemma 4.1, we can check that

\begin{align}
(1+t)^l E_s(U(t)) + \gamma \int_0^t (1+y)^l D_s(U(y))dy \\
\leq E_s(U_0) + l \int_0^t (1+y)^{l-1} E_s(U(y))dy \\
\leq E_s(U_0) + C l \int_0^t (1+y)^{l-1} \left(\|B(y)\|^2 + \|\rho_c(y)\|^2 + D_{s+1}(U(y))\right)dy,
\end{align}

where we have used $E_s(U(t)) \leq \|B(t)\|^2 + \|\rho_c + \rho_i(t)\|^2 + D_{s+1}(U(t))$. Using (4.1) again, we have

\begin{align}
E_{s+2}(U(t)) + \gamma \int_0^t D_{s+2}(U(y))dy \leq E_{s+2}(U_0)
\end{align}

and

\begin{align}
(1+t)^{l-1} E_{s+1}(U(t)) + \gamma \int_0^t (1+y)^{l-1} D_{s+1}(U(y))dy \\
\leq E_{s+1}(U_0) + C (l-1) \int_0^t (1+y)^{l-2} \left(\|B(y)\|^2 + \|\rho_c + \rho_i(y)\|^2 + D_{s+2}(U(y))\right)dy.
\end{align}
Then, by iterating the previous estimates, we obtain

\[
(1 + t)^l \mathcal{E}_s(U(t)) + \gamma \int_0^t (1 + y)^{l-1} D_s(U(y)) \, dy \leq C \mathcal{E}_{s+2}(U_0) + C \int_0^t (1 + y)^{l-1} \left( \|B(y)\|^2 + \|(\rho_e + \rho_i)(y)\|^2 \right) \, dy
\]

for \(1 < l < 2\).

Now, let us establish the estimate on the integral term on the right hand side of (4.2). Applying the estimate on \(B\) in (3.12) and the estimate on \(\rho_2\) in (3.43) to (3.5) and (3.6), respectively, we have

\[
\|B(t)\| \leq C (1 + t)^{-\frac{3}{2}} \|[u_{1,0}, E_0, B_0]\|_{L^1 \cap H^2} + C \int_0^t (1 + t - y)^{-\frac{3}{2}} \|[g_{2e}(y) - g_{2i}(y), g_{4e}(y) - g_{4i}(y)]\|_{L^1 \cap H^2} \, dy,
\]

(4.3)

\[
\|(\rho_e + \rho_i)(t)\| \leq C \|\rho_2(t)\| \leq C (1 + t)^{-\frac{3}{2}} \|[\rho_{\mu_0}, u_{\mu_0}, \Theta_{\mu_0}]\|_{L^1 \cap H^2} + C \int_0^t (1 + t - y)^{-\frac{3}{2}} \|[g_{1e} + g_{1i}, g_{2e} + g_{2i}, g_{3e} + g_{3i}] (y)\|_{L^1 \cap H^2} \, dy.
\]

(4.4)

It is directly to check that for any \(0 \leq y \leq t\),

\[
\|[g_{2e}(y) - g_{2i}(y), g_{4e}(y) - g_{4i}(y)]\|_{L^1 \cap H^2} \leq C \mathcal{E}_s(U(y)) \leq C (1 + y)^{-\frac{3}{2}} \mathcal{E}_{s,\infty}(U(t)),
\]

\[
\|[g_{1e} + g_{1i}, g_{2e} + g_{2i}, g_{3e} + g_{3i}] (y)\|_{L^1 \cap H^2} \leq C \mathcal{E}_s(U(y)) \leq C (1 + y)^{-\frac{3}{2}} \mathcal{E}_{s,\infty}(U(t)),
\]

where \(\mathcal{E}_{s,\infty}(U(t)) := \sup_{0 \leq y \leq t} (1 + y)^{\frac{3}{2}} \mathcal{E}_s(U(y))\). Plugging the two previous inequalities into (4.3) and (4.4) respectively implies

\[
\|B(t)\| \leq C (1 + t)^{-\frac{3}{2}} \left( \|[u_{\mu_0}, E_0, B_0]\|_{L^1 \cap H^2} + \mathcal{E}_{s,\infty}(U(t)) \right)
\]

and

\[
\|(\rho_e + \rho_i)(t)\| \leq C (1 + t)^{-\frac{3}{2}} \left( \|[\rho_{\mu_0}, u_{\mu_0}, \Theta_{\mu_0}]\|_{L^1 \cap H^2} + \mathcal{E}_{s,\infty}(U(t)) \right).
\]

Next, we search the uniform bound of \(\mathcal{E}_{s,\infty}(U(t))\) which will imply the decay rates of \(\mathcal{E}_s(U(t))\) or equivalent to \(\|U(t)\|_s^2\). In fact, after choosing \(l = \frac{3}{2} + \varepsilon\) in (4.2) with \(\varepsilon > 0\) sufficiently small and utilizing (1.5) and (4.6), one obtain

\[
(1 + t)^{\frac{3}{2} + \varepsilon} \mathcal{E}_s(U(t)) + \gamma \int_0^t (1 + y)^{\frac{3}{2} + \varepsilon} D_s(U(y)) \, dy \leq C \mathcal{E}_{s+2}(U_0) + C (1 + t)^{\varepsilon} \left( \|[\rho_{\mu_0}, u_{\mu_0}, \Theta_{\mu_0}, E_0, B_0]\|_{L^1 \cap H^2}^2 + [\mathcal{E}_{s,\infty}(U(t))]^2 \right),
\]

which implies

\[
(1 + t)^{\frac{3}{2}} \mathcal{E}_s(U(t)) \leq C \left( \mathcal{E}_{s+2}(U_0) + \|[\rho_{\mu_0}, u_{\mu_0}, \Theta_{\mu_0}, E_0, B_0]\|_{L^1 \cap H^2}^2 + [\mathcal{E}_{s,\infty}(U(t))]^2 \right),
\]

and therefore,

\[
\mathcal{E}_{s,\infty}(U(t)) \leq C \left( \omega_{s+2}(U_0)^2 + [\mathcal{E}_{s,\infty}(U(t))]^2 \right),
\]

since \(\omega_{s+2}(U_0) > 0\) is small enough, it holds that \(\mathcal{E}_{s,\infty}(U(t)) \leq C \omega_{s+2}(U_0)^2\) for any \(t \geq 0\), which implies \(\|U(t)\|_s \leq C \mathcal{E}_s(U(t))^{\frac{1}{2}} \leq C \omega_{s+2}(U_0) (1 + t)^{-\frac{3}{4}}\), that is (2.12).
4.2. Decay rate for high-order energy functional. In this subsection, we will look for the decay estimate of the high-order energy \( \| \nabla U(t) \|^2_{s-1} \), that is \((2.13)\) of Proposition [2.2]. We begin with the following Lemma.

**Lemma 4.2.** Assume \( U = [\rho, u, \Theta, E, B] \) is the solution of the initial value problem \((2.2)-(2.3)\) with \( U_0 = [\rho_0, u_0, \Theta_0, E_0, B_0] \) which satisfies \((2.1)\) in the sense of Proposition [2.1]. If \( E_s(U_0) \) is small enough, then, there exist the high-order energy function \( E_s^h(\cdot) \) and the high-order dissipative rate \( D_s^h(\cdot) \) such that

\[
\frac{d}{dt} E_s^h(U(t)) + \gamma D_s^h(U(t)) \leq 0,
\]

holds for any \( t \geq 0 \).

**Proof.** The proof is very similar to the proof of Theorem [2.1]. In fact, by letting \( |\alpha| \geq 1 \), then corresponding to \((2.16), (2.19), (2.23)\) and \((2.25)\), it can also be checked that

\[
\frac{d}{dt} \| \nabla U \|^2_{s-1} + \| \nabla[u_e, u_i, \Theta_e, \Theta_i] \|^2_{s-1} \leq C \| U \|_s \| \nabla[\rho_e, \rho_i, u_e, u_i, \Theta_e, \Theta_i] \|^2_{s-1},
\]

\[
\frac{d}{dt} \sum_{1 \leq |\alpha| \leq s-1} \langle \partial^\alpha u, \nabla \partial^\alpha \rho \rangle + \gamma \left( \| \nabla^2 [\rho_e, \rho_i] \|^2_{s-2} + \| \nabla [\rho_e - \rho_i] \|^2 \right) \leq C \left( \| \nabla u \|^2_{s-1} + \| U \|^2_s \| \nabla[\rho, u, \Theta] \|^2_{s-1} \right),
\]

and

\[
\frac{d}{dt} \sum_{1 \leq |\alpha| \leq s-2} \langle \partial^\alpha E, -\nabla \times \partial^\alpha B \rangle + \gamma \| \nabla^2 B \|^2_{s-3} \leq C \left( \| \nabla E \|^2_{s-2} + \| \nabla u \|^2_{s-1} + \| \nabla[\rho, u, \Theta] \|^2_{s-1} \| U \|^2_s \right).
\]

Now, similarly done as that in Step 5 of Theorem [2.1]. Let us define the high-order energy functional as

\[
E_s(U(t)) = \| \nabla U \|^2_{s-1} + K_1 \sum_{1 \leq |\alpha| \leq s-1} \sum_{\mu=e,i} \langle \partial^\alpha u, \nabla \partial^\alpha \rho \rangle
\]

\[
+ K_2 \sum_{1 \leq |\alpha| \leq s-1} \langle \partial^\alpha (u_e - u_i), \partial^\alpha E \rangle + K_3 \sum_{1 \leq |\alpha| \leq s-2} \langle \partial^\alpha E, -\nabla \times \partial^\alpha B \rangle,
\]

Similarly, one can take \( 0 < K_3 \ll K_2 \ll K_1 \ll 1 \) be sufficiently small with \( K_3^3 \ll K_3 \), such that \( E_s^h(U(t)) \sim \| \nabla U(t) \|^2_{s-1} \), that is \( E_s^h(\cdot) \) is really a high-order energy functional which satisfies \((2.6)\), and moreover, the sum of the four previously estimates with coefficients corresponding to \((4.8)\) gives \((4.7)\). Now, we complete the proof of Lemma [4.2].

Based on Lemma [4.2], one can check that

\[
\frac{d}{dt} E_s^h(U(t)) + \gamma E_s^h(U(t)) \leq C \left( \| \nabla B \|^2 + \| \nabla [E, B] \|^2 + \| \nabla (\rho_e + \rho_i) \|^2 \right),
\]
which implies

\[ \mathcal{E}_s^h(U(t)) \leq e^{-\gamma t} \mathcal{E}_s^h(U_0) \]  
(4.9)

\[ + C \int_0^t e^{-\gamma(t-y)} \left( \| \nabla B(y) \|^2 + \| \nabla^s [E, B](y) \|^2 + \| \nabla (\rho_e + \rho_i)(y) \|^2 \right) dy. \]

Now, let us estimate the time integral term on the right hand side of the previous inequality. Noting that the equations of \( E \) and \( B \) in bipolar non-isentropic Euler-Maxwell system are the same as that in bipolar isentropic Euler-Maxwell system, similarly as that in [3], we obtain

**Lemma 4.3.** Assume \( U = [\rho_{\mu}, u_{\mu}, \Theta_{\mu}, E, B] \) is the solution of the initial value problem (2.2) - (2.3) with \( U_0 = [\rho_{\mu 0}, u_{\mu 0}, \Theta_{\mu 0}, E_0, B_0] \) which satisfies (2.4) in the sense of Proposition 2.1 If \( \omega_{s+6}(U_0) \) is small enough, then, for any \( t \geq 0 \)

\[ \| \nabla B(t) \|^2 + \| \nabla^s [E(t), B(t)] \|^2 + \| \nabla (\rho_e + \rho_i)(t) \|^2 \leq C \omega_{s+6}(U_0)^2 (1 + t)^{-\frac{5}{2}}. \]

**Proof.** Utilize the estimate (3.14) to (3.3) of the solution \( U_1(t) \) so that

\[ \| \nabla B(t) \| \leq C(1 + t)^{-\frac{5}{2}} \|[\rho_{\mu 0}, E_0, B_0]\|_{L^1 \cap \dot{H}^4} \]

\[ + C \int_0^t (1 + t - y)^{-\frac{5}{2}} \|[g_{2e}(y) - g_{2i}(y), g_{4e}(y) - g_{4i}(y)]\|_{L^1 \cap \dot{H}^4} dy \]

\[ \leq C(1 + t)^{-\frac{5}{2}} \|[\rho_{\mu 0}, E_0, B_0]\|_{L^1 \cap \dot{H}^4} + C \int_0^t (1 + t - y)^{-\frac{5}{2}} \|U(y)\|^2_{\max\{s, 4\}} dy \]

\[ \leq C(1 + t)^{-\frac{5}{2}} \|[\rho_{\mu 0}, E_0, B_0]\|_{L^1 \cap \dot{H}^4} + C \int_0^t (1 + t - y)^{-\frac{5}{2}} \omega_{s+6}(U_0)^2 (1 + y)^{-\frac{5}{2}} dy \]

\[ \leq C \omega_{s+6}(U_0) (1 + t)^{-\frac{5}{2}} \]

and

\[ \| \nabla^s [E(t), B(t)] \| \]

\[ \leq C(1 + t)^{-\frac{5}{2}} \|[\rho_{\mu 0}, \Theta_{\mu 0}, E_0, B_0]\|_{L^2 \cap \dot{H}^{s+3}} \]

\[ + C \int_0^t (1 + t - y)^{-\frac{5}{2}} \|[g_{2e}(y) - g_{2i}(y), g_{3e}(y) - g_{3i}(y), g_{4e}(y) - g_{4i}(y)]\|_{L^2 \cap \dot{H}^{s+3}} dy \]

\[ \leq C(1 + t)^{-\frac{5}{2}} \|[\rho_{\mu 0}, \Theta_{\mu 0}, E_0, B_0]\|_{L^2 \cap \dot{H}^{s+3}} + C \int_0^t (1 + t - y)^{-\frac{5}{2}} \|U(y)\|^2_{s+4} dy \]

\[ \leq C(1 + t)^{-\frac{5}{2}} \|[\rho_{\mu 0}, \Theta_{\mu 0}, E_0, B_0]\|_{L^2 \cap \dot{H}^{s+3}} + C \int_0^t (1 + t - y)^{-\frac{5}{2}} \omega_{s+6}(U_0)^2 (1 + y)^{-\frac{5}{2}} dy \]

\[ \leq C \omega_{s+6}(U_0) (1 + t)^{-\frac{5}{2}}. \]
Similarly, utilizing the estimate on $\rho_2$ in (3.31) to (3.36) of the solution $U_2(t)$, we obtain
\[
\|\nabla (\rho_e + \rho_i) (t)\| \\
\leq C(1 + t)^{-\frac{\gamma}{2}} \|[\rho_{\mu 0}, u_{\mu 0}, \Theta_{\mu 0}]\|_{L^{1+H^5}} \\
+ C \int_0^t (1 + t - y)^{-\frac{\gamma}{2}} \|[g_{1e}(y) + g_{1i}(y), g_{2e}(y) + g_{2i}(y), g_{3e}(y) + g_{3i}(y)]\|_{L^{1+H^5}} dy \\
\leq C(1 + t)^{-\frac{\gamma}{2}} \|[\rho_{\mu 0}, u_{\mu 0}, \Theta_{\mu 0}]\|_{L^{1+H^5}} + C \int_0^t (1 + t - y)^{-\frac{\gamma}{2}} \|U(y)\|_{\max\{5,s\}}^2 dy \\
\leq C(1 + t)^{-\frac{\gamma}{2}} \|[\rho_{\mu 0}, u_{\mu 0}, \Theta_{\mu 0}]\|_{L^{2+H^3}} + C \int_0^t (1 + t - y)^{-\frac{\gamma}{2}} \omega s+6(U_0)^2 (1 + y)^{-\frac{\gamma}{4}} dy \\
\leq C\omega s+6(U_0) (1 + t)^{-\frac{\gamma}{4}}.
\]
Where we have used (2.12) and the smallness of $\omega s+6(U_0)$. Now, we complete the proof of Lemma 4.3.

Then, after plugging (4.10) into (4.9), we have
\[
\mathcal{E}_s^h(U(t)) \leq e^{-\gamma t} \mathcal{E}_s^h(U_0) + C\omega s+6(U_0)^2 (1 + t)^{-\frac{\gamma}{4}}.
\]
Since $\mathcal{E}_s^h(U(t)) \sim \|\nabla U(t)\|_{s-1}^2$ holds true for any $t \geq 0$, (2.13) follows. Now, we finish the proof of Proposition 2.2.

4.3. Decay rate in $L^q$. In this subsection, we are to look for the decay rates of solutions $U = [\rho_{\mu}, u_{\mu}, \Theta_{\mu}, E, B]$ in $L^q$ $(2 \leq q \leq +\infty)$ of the initial value problem (2.22) by proving the second part of Theorem 1.1. Throughout this subsection, we usually suppose that $\omega_{13}(U_0)$ is small enough. Firstly, for $s \geq 4$, Proposition 2.2 shows that if $\omega s+2(U_0)$ is small enough,
\[
\|U(t)\|_{s} \leq C\omega s+2(U_0)(1 + t)^{-\frac{\gamma}{4}},
\]
and if $\omega s+6(U_0)$ is small enough,$$
(4.11)\|\nabla U(t)\|_{s-1} \leq C\omega s+6(U_0)(1 + t)^{-\frac{\gamma}{4}}.
$$
Now, let us establish the estimates on $B, [u_e-u_i, E], u_e+u_i, [\rho_e-\rho_i, \Theta_e-\Theta_i]$ and $[\rho_e+\rho_i, \Theta_e+\Theta_i]$ as follows.

Estimate on $\|B\|_{L^q}$. For $L^2$ rate, it is directly from (4.11) to get
\[
\|B(t)\| \leq C\omega_{0}(U_0)(1 + t)^{-\frac{\gamma}{4}}.
\]
For $L^\infty$ rate, by applying $L^\infty$ estimate on $B$ of (3.13) to (3.35), we obtain
\[
\|B(t)\|_{L^\infty} \leq C(1 + t)^{-\frac{\gamma}{2}} \|[u_{\mu 0}, E_0, B_0]\|_{L^{1+H^5}} \\
+ C \int_0^t (1 + t - y)^{-\frac{\gamma}{2}} \|[g_{2e} - g_{2i}, g_{4e} + g_{4i}]\|_{L^{1+H^5}} dy.
\]
Because of (4.11),
\[
\|[g_{2e} - g_{2i}, g_{4e} + g_{4i}]\|_{L^{1+H^5}} \leq C \|U(t)\|_{6}^2 \leq C\omega_{8}(U_0)^2 (1 + t)^{-\frac{\gamma}{4}},
\]
we have
\[
\|B(t)\|_{L^\infty} \leq C\omega_{8}(U_0)(1 + t)^{-\frac{\gamma}{4}}.
\]
Therefore, by $L^2 - L^\infty$ interpolation
\[
(4.13)\|B(t)\|_{L^q} \leq C\omega_{8}(U_0)(1 + t)^{-\frac{\gamma}{4} + \frac{q}{2}}.
\]
for $2 \leq q \leq \infty$.

**Estimate on $||[u_e - u_i, E]||_{L^q}$**. For $L^2$ rate, applying the $L^2$ estimate on $u_e - u_i$ and $E$ in (3.12) to (3.5), one has

$$
|| (u_e - u_i) (t) || \leq C (1 + t)^{-\frac{q}{4}} (\| \rho_{\mu_0}, \Theta \mu_0 \|_{L^1 \cap H^2} + \| [u_{\mu_0}, E_0, B_0] ||_{L^1 \cap H^2}) \\
+ C \int_0^t (1 + t - y)^{-\frac{q}{2}} \| [g_{1e} - g_{1i}, g_{3e} - g_{3i}] (y) \| dy \\
+ C \int_0^t (1 + t - y)^{-\frac{q}{2}} \| [g_{2e} - g_{2i}, g_{4e} - g_{4i}] (y) \|_{L^1 \cap H^2} dy
$$

and

$$
|| E (t) || \leq C (1 + t)^{-\frac{q}{4}} \| [u_{\mu_0}, \Theta \mu_0, E_0, B_0] ||_{L^1 \cap H^3} \\
+ C \int_0^t (1 + t - y)^{-\frac{q}{2}} \| [g_{1e} - g_{1i}, g_{3e} - g_{3i}, g_{4e} - g_{4i}] (y) \|_{L^1 \cap H^3} dy.
$$

Since by (4.11),

$$
\| [g_{1e} - g_{1i}, g_{3e} - g_{3i}] (t) \| + \| [g_{2e} - g_{2i}, g_{3e} - g_{3i}, g_{4e} - g_{4i}] (t) \|_{L^1 \cap H^3} \\
\leq C \| U (t) \|^2 \leq C \omega_0 (U_0)^2 (1 + t)^{-\frac{q}{2}},
$$

which implies that

(4.14) \hspace{1cm} || [u_e - u_i, E] (t) || \leq C \omega_0 (U_0) (1 + t)^{-\frac{q}{2}}.

For $L^\infty$ rate, utilize the $L^\infty$ estimates on $u_e - u_i$ and $E$ in (3.13) to (3.5), we have

$$
|| (u_e - u_i) (t) ||_{L^\infty} \leq C (1 + t)^{-2} (\| \rho_{\mu_0}, \Theta \mu_0 \|_{L^1 \cap H^2} + \| [u_{\mu_0}, E_0, B_0] ||_{L^1 \cap H^2}) \\
+ C \int_0^t (1 + t - y)^{-2} \| [g_{1e} - g_{1i}, g_{3e} - g_{3i}] (y) \|_{L^1 \cap H^2} dy \\
+ C \int_0^t (1 + t - y)^{-2} \| [g_{2e} - g_{2i}, g_{4e} - g_{4i}] (y) \|_{L^1 \cap H^2} dy
$$

and

$$
|| E (t) ||_{L^\infty} \leq C (1 + t)^{-2} \| [u_{\mu_0}, \Theta \mu_0, E_0, B_0] ||_{L^1 \cap H^6} \\
+ C \int_0^t (1 + t - y)^{-2} \| [g_{2e} - g_{2i}, g_{3e} - g_{3i}, g_{4e} - g_{4i}] (y) \|_{L^1 \cap H^6} dy.
$$

Since

$$
\| [g_{1e} - g_{1i}, g_{2e} - g_{2i}, g_{3e} - g_{3i}, g_{4e} - g_{4i}] (t) \|_{L^1} \\
\leq C \| U (t) \| \| (u_e - u_i) (t) \| + \| U (t) \| + \| \nabla U (t) \| \\
\leq \omega_{10} (U_0)^2 (1 + t)^{-\frac{3}{2}},
$$

and

$$
\| [g_{1e} - g_{1i}, g_{2e} - g_{2i}, g_{3e} - g_{3i}, g_{4e} - g_{4i}] (t) \|_{H^5 \cap H^6} \leq C \| \nabla U (t) \|^2 \leq \omega_{13} (U_0)^2 (1 + t)^{-\frac{3}{2}},
$$

then, one has

$$
\| [u_e (t) - u_i (t), E (t)] \|_{L^\infty} \leq C \omega_{13} (U_0)^2 (1 + t)^{-\frac{3}{2}}.
$$

Therefore, by $L^2 - L^\infty$ interpolation

(4.15) \hspace{1cm} \| [u_e (t) - u_i (t), E (t)] \|_{L^q} \leq C \omega_{13} (U_0) (1 + t)^{-\frac{q}{2} + \frac{1}{2q}},
for $2 \leq q \leq \infty$.

Estimate on $\|u_e + u_i\|_{L^q}$. For $L^2$ rate, utilizing the $L^2$ estimates on $u_e + u_i$ in (3.33) to (3.36), we have

\[
\|u_e + u_i\|_t \leq C (1 + t)^{-\frac{1}{2}} \|\rho \mu_0, u\mu_0, \Theta \mu_0\|_{L^1 \cap H^3} + C \int_0^t (1 + t - y)^{-\frac{3}{4}} \|\rho e + g_{i1} + g_{22} + g_{22}, g_{33} + g_{33}\|_t \|L^1 \cap H^3 \ dy.
\]

Since by (4.11),

\[
\|\rho e + g_{i1} + g_{22} + g_{22}, g_{33} + g_{33}\|_t \leq C \|U(t)\|^2_1 \leq \omega_0(U_0)^2 (1 + t)^{-\frac{3}{2}},
\]

it follows that

\[
\|u_e + u_i\|_t \leq C \omega_0(U_0)(1 + t)^{-\frac{3}{2}}.
\]

For $L^\infty$ rate, utilizing the $L^\infty$ estimates on $u_e + u_i$ in (3.45) to (3.36), we have

\[
\|u_e + u_i\|_t \leq C (1 + t)^{2} \|\rho \mu_0, u\mu_0, \Theta \mu_0\|_{L^1 \cap H^6} + C \int_0^t (1 + t - y)^{-2} \|\rho e + g_{i1} + g_{22} + g_{22}, g_{33} + g_{33}\|_t \|L^1 \cap H^6 \ dy
\]

Since by (4.11),

\[
\|\rho e + g_{i1} + g_{22} + g_{22}, g_{33} + g_{33}\|_t \leq C \|U(t)\|^2_1 \leq \omega_0(U_0)^2 (1 + t)^{-\frac{3}{2}},
\]

it follows that

\[
\|u_e(t) + u_i(t)\|_t \leq C \omega_0(U_0)(1 + t)^{-\frac{3}{2}}.
\]

Therefore, by $L^2 - L^\infty$ interpolation

\[
\|u_e(t) + u_i(t)\|_t \leq C \omega_0(U_0)(1 + t)^{-\frac{3}{2} + \frac{1}{2q}},
\]

for $2 \leq q \leq \infty$.

Then from (4.15) and (4.16) we have

\[
\|u_{\mu}(t)\|_t \leq C \omega_1(U_0)(1 + t)^{-\frac{3}{2} + \frac{1}{2q}},
\]

for $2 \leq q \leq \infty$.

Estimate on $\|\rho_e - \rho_i, \Theta - \Theta_i\|_t$ and $\|\rho_e + \rho_i, \Theta - \Theta_i\|_t$. For $L^2$ rate, utilizing the $L^2$ estimates on $\rho_e - \rho_i$ and $\Theta - \Theta_i$ in (3.32) to (3.35), we have

\[
\|\rho_e - \rho_i, \Theta - \Theta_i\|_t \leq C e^{-\frac{t}{2}} \|\rho \mu_0, u\mu_0, \Theta \mu_0\|_t + C \int_0^t e^{-\frac{1}{2}u} \|\rho e - g_{i1} + g_{22} - g_{22}, g_{33} - g_{33}\|_t \|B(t)\|_t \|L^\infty \ dy.
\]

Because of

\[
\|\rho e - g_{i1} + g_{22} - g_{22}, g_{33} - g_{33}\|_t \leq C \left( \|\nabla U(t)\|^2_1 + \|u_e + u_i(t)\|_t \|B(t)\|_t \right) \leq C \omega_{10}(U_0)^2 (1 + t)^{-\frac{5}{2}},
\]

where (4.12), (4.13) and (4.16) were used. Then (4.18) yields the decay estimate

\[
\|\rho_e - \rho_i, \Theta - \Theta_i\|_t \leq C \omega_{10}(U_0)(1 + t)^{-\frac{5}{2}}.
\]

Similarly for $\|\rho_e - \rho_i, \Theta - \Theta_i\|_t$, by utilizing the $L^2$ estimate on $\|\rho_e + \rho_i, \Theta - \Theta_i\|_t$ in (3.43) to (3.36), we obtain the decay estimate

\[
\|\rho_e + \rho_i, \Theta - \Theta_i\|_t \leq C \omega_6(U_0)(1 + t)^{-\frac{5}{2}}.
\]
Combining (4.19) and (4.20), we obtain
\begin{equation}
\| [\rho_\mu, \Theta_\mu] (t) \| \leq C \omega_10(U_0)(1 + t)^{-\frac{3}{4}},
\end{equation}
For \( L^\infty \) rate, by utilizing the \( L^\infty \) estimate on \([\rho_e - \rho_i, \Theta_e - \Theta_i] \) in (3.13) to (3.5), we have the decay estimate
\begin{equation}
\| [\rho_e - \rho_i, \Theta_e - \Theta_i] (t) \|_{L^\infty} \leq C e^{-\frac{t}{2}} \| [\rho_{i0}, u_{i0}, \Theta_{i0}] \|_{L^2 \cap \dot{H}^2} + C \int_0^t e^{-\frac{1-t}{2}} \| [g_{1e} - g_{1i}, g_{2e} - g_{2i}, g_{3e} - g_{3i}] (y) \|_{L^2 \cap \dot{H}^2} dy.
\end{equation}
Notice that one can check
\begin{equation}
\| [g_{1e} - g_{1i}, g_{2e} - g_{2i}, g_{3e} - g_{3i}] (t) \|_{L^2 \cap \dot{H}^2} \leq C \| \nabla U(t) \|_4 (\| [\rho_\mu(t), \Theta_\mu(t)] \| + \| u_\mu(t) \| + \| [u_\mu(t), B(t)] \|_{L^\infty}) \\
\leq C \omega_13(U_0)^2 (1 + t)^{-2},
\end{equation}
where we have used (4.12), (4.13), (4.17) and (4.21). Which implies from (4.22) that
\begin{equation}
\| [\rho_e - \rho_i, \Theta_e - \Theta_i] (t) \|_{L^\infty} \leq C \omega_13(U_0)(1 + t)^{-2}.
\end{equation}
Therefore, by \( L^2 - L^\infty \) interpolation
\begin{equation}
\| [\rho_e - \rho_i, \Theta_e - \Theta_i] \|_{L^q} \leq C \omega_13(U_0)(1 + t)^{-2} - \frac{3}{4},
\end{equation}
for \( 2 \leq q \leq \infty \).
For \( \| [\rho_e + \rho_i, \Theta_e + \Theta_i] \|_{L^\infty} \), by utilizing the \( L^\infty \) estimate on \([\rho_e + \rho_i, \Theta_e + \Theta_i] \) in (3.15) to (3.6), we have the decay estimate
\begin{equation}
\| [\rho_e + \rho_i, \Theta_e + \Theta_i] (t) \|_{L^\infty} \leq C \omega_8(U_0)(1 + t)^{-\frac{3}{4}}.
\end{equation}
Then from (4.20) and (4.25) we have
\begin{equation}
\| [\rho_e + \rho_i, \Theta_e + \Theta_i] (t) \|_{L^q} \leq C \omega_8(U_0)(1 + t)^{-\frac{3}{4} + \frac{3}{2q}}.
\end{equation}
Thus, (4.24), (4.26), (4.15)-(4.16) and (4.13) give (1.5), (1.6), (1.7) and (1.8), respectively. Now, we complete the proof of Theorem 1.1.

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