Set theory with a proper class of indiscernibles

Ali Enayat

Dedicated to the memory of Ken Kunen

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Abstract

We investigate an extension of ZFC set theory, denoted ZFI$_<$, which is equipped with a well-ordering $<$ of the universe $V$ of set theory, and a proper class $I$ of indiscernibles over the structure $(V,\in,\vartriangleleft)$.

Our main results are Theorems A, B, and C below. Note that the equivalence of condition (ii) and (iii) in Theorem A was established in an earlier (2004) published work of the author. In what follows GBC is the Gödel-Bernays theory of classes with global choice. In Theorem C the symbol $\rightarrow$ is the usual Erdős-arrow notation for partition calculus.

**Theorem A.** The following are equivalent for a sentence $\varphi$ in the language $\{=,\in\}$ of set theory:

- $(i)$ ZFI$_<$ $\vdash \varphi$.
- $(ii)$ ZFC + $\Lambda$ $\vdash \varphi$, where $\Lambda = \{\lambda_n : n \in \omega\}$, and $\lambda_n$ is the sentence asserting the existence of an $n$-Mahlo cardinal $\kappa$ such that $V(\kappa)$ is a $\Sigma_n$-elementary submodel of the universe $V$.
- $(iii)$ GBC + “Ord is weakly compact” $\vdash \varphi$.

**Theorem B.** Every $\omega$-model of ZFI$_<$ satisfies $V \neq L$.

**Theorem C.** The sentence expressing $\forall m,n \in \omega (\text{Ord} \rightarrow (\text{Ord})^n_m)$ is not provable in the theory $T = \text{GBC} + \text{“Ord is weakly compact”}$, assuming $T$ is consistent.

The paper also includes results about the interpretability relationship between the theories ZFC + $\Lambda$, ZFI$_<$, and GBC + “Ord is weakly compact”.

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**Key Words.** Zermelo-Fraenkel set theory, Gödel-Bernays class theory, indiscernibles, Mahlo cardinal, weakly compact cardinal, satisfaction class.

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1. INTRODUCTION

The principal focus of this paper is on an extension $\text{ZFI}_<$ of Zermelo-Fraenkel set theory $\text{ZF}$ that is equipped with a global well-ordering $<$ and a proper class $I$ of ordinals such that $(I, \in)$ is a collection of order indiscernibles over the structure $(V, \in, <)$. Moreover, the axioms of $\text{ZFI}_<$ stipulate that the expanded universe $(V, \in, <, I)$ satisfies the axioms of $\text{ZF}$ in the extended language incorporating $<$ and $I$. Thus $\text{ZFI}_<$ is a system of set theory that can be described as strongly ‘anti-Leibnizian’: The Leibniz dictum on the identity of indiscernibles bars the existence of a single pair of distinct indiscernibles in the universe $(V, \in)$ of sets, but models of $\text{ZFI}_<$ are endowed, intuitively speaking, with an unnameable number of such objects that are grouped into a proper class $I$ that can be used in set-theoretical reasoning.

The precise definition of $\text{ZFI}_<$ is given in Section 3. The definition makes it clear that (a) if $\kappa$ is a weakly compact cardinal, then $(V(\kappa), \in)$ has an expansion that satisfies any prescribed finitely axiomatized subtheory of $\text{ZFI}_<$, and (b) if $\kappa$ is a Ramsey cardinal, then $(V(\kappa), \in)$ has an expansion to a model of $\text{ZFI}_<$. One of our main results is Theorem 4.1 (a refinement of Theorem A of the abstract) that shows that the purely set-theoretical consequences of $\text{ZFI}_<$ coincide with the theorems of the theory obtained by augmenting $\text{ZFC}$ with the Levy scheme $\Lambda$, a scheme that ensures that the class of ordinals behaves like an $\omega$-Mahlo cardinal (the precise definition of $\Lambda$ is given in Definition 2.4.10). Theorem 4.1 complements the main results in [E-2] and [E-3] that exhibit the surprising ways in which $\text{ZFC}+\Lambda$ manifests itself as a canonical theory, especially in the context where the model theory of $\text{ZF}$ is compared with the model theory of $\text{PA}$ (Peano Arithmetic). In contrast, parts (e) and (f) of Theorem 3.8 (which refine Theorem B of the abstract) show that an $\omega$-model of $\text{ZFI}_<$ (i.e., a model of $\text{ZFI}_<$ whose $\omega$ is well-founded in the real world) satisfies large cardinal hypotheses significantly stronger than the existence of $\omega$-Mahlo cardinals. Our third main result is Theorem 4.9 (Theorem C of the abstract), which should be contrasted with the fact that $\text{GBC}+\text{ "Ord is weakly compact"}$ can prove sentences of the form $\forall \kappa (\text{Ord} \rightarrow (\text{Ord})^n_\kappa)$, where $n$ ranges over nonzero natural numbers in the real world. We also include some interpretability-theoretic results concerning $\text{ZFI}_<$ and variants of $\text{ZFI}_<$.

There is a notable series of papers investigating combinatorial features of $n$-Mahlo cardinals, beginning with the groundbreaking work of Schmerl [S], which eventually culminated in the Hajnal-Kanamori-Shelah paper [HKS]. The relationship between $n$-Mahlo cardinals and various types of sets of indiscernibles has also been extensively studied by many researchers including McAloon, Ressayre, Friedman, Finkel and Todorcević (see, e.g., [FR] and [FT]). However, the proofs of our results dominantly employ techniques from the model theory of set theory together with classical combinatorial ideas, thus they do not rely on the machinery developed in the above body of work. Of course it would be interesting to work out the relationship between our results and the aforementioned literature.

The organization of the paper is as follows. Section 2 contains a mix of preliminary material employed in the paper; the reader is advised to pay special attention to Subsections 2.3 and 2.4. Section 3 introduces $\text{ZFI}$ and $\text{ZFI}_<$ and mostly focuses on their model theory. Section 4 is devoted to the calibration of the purely set-theoretical consequences of $\text{ZFI}_<$, and Section 5 studies $\text{ZFI}_<$ from an interpretability-theoretic point of view. In Section 6 we discuss four systems that are closely related to $\text{ZFI}_<$. Finally, we close the paper by presenting a few open questions in Section 7.

History and Acknowledgments. This paper might appear as a natural sequel to [E-3], but the work reported here arose in a highly indirect way as a result of an engagement with certain potent

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1. The impact of Leibnizian motifs in set theory and its model theory is explored in [E-4] and [E-5].
2. The Levy scheme $\Lambda$ was denoted $\Phi$ in earlier work of the author, and in particular in [E-3]. The new notation is occasioned by the author’s appreciation of the role played by Azriel Levy in the investigations of the Mahlo hierarchy and reflection phenomena, masterfully overviewed in Kanamori’s portraitue [Kan-3]. In Subsection 2.4 we review the basic features of the Levy Scheme.
ideas proposed by Jan Mycielski concerning Leibnizian motifs in set theory, an engagement that culminated in the trilogy of papers [E-4], [E-5], and [E-6]. Informed by Bohr’s aphorism “The opposite of a correct statement is a false statement. But the opposite of a profound truth may well be another profound truth”, and as if to maintain a cognitive balance, upon the completion of the aforementioned trilogy my attention and curiosity took an opposite turn towards the highly ‘anti-Leibnizian’ systems of set theory studied here. The protoforms of the results of this paper were first presented at the New York Logic Conference (2005), IPM Logic Conference (2007, Tehran, Iran), the Kunen Fest Meeting (2009, Madison, Wisconsin), and most recently at the Oxford Set Theory Seminar (2020).

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2. PRELIMINARIES

In this section we collect the basic definitions, notations, conventions, and results that will be used in the remaining sections.

2.1. Models of set theory

2.1.1. Definitions and basic facts. (Models, languages, and theories) Let $\mathcal{L}_{\text{Set}} = \{=, \in\}$ be the usual language of set theory. In what follows we make the blanket assumption that $\mathcal{M}, \mathcal{N}$, etc. are $\mathcal{L}$-structures, where $\mathcal{L} \supseteq \mathcal{L}_{\text{Set}}$. By a model of set theory, we mean an $\mathcal{L}$-structure that satisfies enough of ZF set theory to support a decent theory of ordinals, and of the von Neumann levels $V(\alpha)$ of the universe $V$ of ZF.

(a) We follow the convention of using $\mathcal{M}, \mathcal{M}^*, \mathcal{M}_0$, etc. to denote (respectively) the universes of discourse of structures $\mathcal{M}, \mathcal{M}^*, \mathcal{M}_0$, etc. Given a structure $\mathcal{M}$, we write $\mathcal{L}(\mathcal{M})$ for the language of $\mathcal{M}$. Given some relation symbol $R \in \mathcal{L}(\mathcal{M})$, we often write $R^\mathcal{M}$ for the $\mathcal{M}$-interpretation of $R$. In particular, we denote the membership relation of $\mathcal{M}$ by $\in^\mathcal{M}$; thus an $\mathcal{L}_{\text{Set}}$-structure $\mathcal{M}$ is of the form $(\mathcal{M}, \in^\mathcal{M})$.

(b) For $c \in \mathcal{M}$, $\text{Ext}_\mathcal{M}(c)$ is the $\mathcal{M}$-extension of $c$, i.e.,

$$\text{Ext}_\mathcal{M}(c) := \{m \in \mathcal{M} : m \in^\mathcal{M} c\}.$$ 

We say that a subset $X$ of $\mathcal{M}$ is coded in $\mathcal{M}$ if there is some $c \in \mathcal{M}$ such that $\text{Ext}_\mathcal{M}(c) = X$. $X$ is piecewise coded in $\mathcal{M}$ if $X \cap \text{Ext}_\mathcal{M}(m)$ is coded for each $m \in \mathcal{M}$. For $A \subseteq \mathcal{M}$, $\text{Cod}_\mathcal{M}(A)$ is the collection of sets of the form $A \cap \text{Ext}_\mathcal{M}(c)$, where $c \in \mathcal{M}$.

(c) $\text{Ord}^\mathcal{M}$ is the class of “ordinals” of $\mathcal{M}$, i.e., $\text{Ord}^\mathcal{M} := \{m \in \mathcal{M} : \mathcal{M} \models \text{Ord}(m)\}$, where $\text{Ord}(x)$ expresses “$x$ is transitive and is well-ordered by $\in$”. More generally, for a formula $\varphi(\overline{x})$, where $\overline{x} = (x_1, \cdots, x_k)$, we write $\varphi^\mathcal{M}$ for $\{\overline{m} \in \mathcal{M}^k : \mathcal{M} \models \varphi(m_1, \cdots, m_k)\}$. We write $\omega^\mathcal{M}$ for the set of finite ordinals (i.e., natural numbers) of $\mathcal{M}$, and $\omega$ for the set of finite ordinals in the real world, whose members we refer to as metatheoretic natural numbers. $\mathcal{M}$ is said to be $\omega$-standard if $(\omega, \in)^\mathcal{M} \cong (\omega, \in)$. For $\alpha \in \text{Ord}^\mathcal{M}$ we often use $\mathcal{M}(\alpha)$ to denote the substructure of $\mathcal{M}$ whose universe is $(V(\alpha))^\mathcal{M}$. 


(d) \( \mathcal{N} \) is said to end extend \( \mathcal{M} \) (equivalently: \( \mathcal{M} \) is an initial submodel of \( \mathcal{N} \)), written \( \mathcal{M} \subseteq_{\text{end}} \mathcal{N} \), if \( \mathcal{M} \) is a submodel of \( \mathcal{N} \) and for every \( a \in \mathcal{M}, \text{Ext}_\mathcal{M}(a) = \text{Ext}_\mathcal{N}(a) \). We often write “e.e.e.” instead of “elementary end extension”. It is easy to see that if \( \mathcal{N} \) is an e.e.e. of a model \( \mathcal{M} \) of ZF, then \( \mathcal{N} \) is a rank extension of \( \mathcal{M} \), i.e., whenever \( a \in \mathcal{M} \) and \( b \in \mathcal{N} \setminus \mathcal{M} \), then \( \mathcal{N} = \text{Ext}(a) \in \mathcal{M}(b) \), where \( \rho \) is the usual ordinal-valued rank function defined by \( \rho(x) = \sup\{\rho(y) + 1 : y \in x\} \).

(e) We treat ZF as being axiomatized as usual, except that instead of including the scheme of replacement among the axioms of ZF, we include the schemes of separation and collection, as in [CK, Appendix A]. Thus, in our set-up the axioms of Zermelo set theory Z are obtained by removing the scheme of collection from the axioms of ZF. More generally, we construe ZF(\( \mathcal{L} \)) to be the natural extension of ZF in which the schemes of separation and collection are extended to \( \mathcal{L} \)-formulae, and we will denote Z(\( \mathcal{L} \)) (Zermelo set theory over \( \mathcal{L} \)) as the result of extending Z with the \( \mathcal{L} \)-separation scheme Sep(\( \mathcal{L} \)), which consists of the universal closures of \( \mathcal{L} \)-formulae of the form:

\[ \forall u \exists v \forall x (x \in w \iff x \in v \land \varphi(x, \overline{v})) \]

Thus ZF(\( \mathcal{L} \)) is the result of augmenting Z(\( \mathcal{L} \)) with the \( \mathcal{L} \)-collection scheme Coll(\( \mathcal{L} \)), which consists of the universal closures of \( \mathcal{L} \)-formulae of the form:

\[ (\forall x \in v \exists y \varphi(x, y, \overline{y})) \rightarrow (\exists w \forall x \exists y \in w \varphi(x, y, \overline{y})) \]

When \( \mathcal{L} = \mathcal{L}_{\text{Set}} \cup \{X\} \), we will write Sep(\( X \)), Coll(\( X \)), etc. instead of Sep(\( \mathcal{L} \)), Coll(\( \mathcal{L} \)), etc. (respectively).

(f) Suppose \( n \in \omega \). \( \Sigma_n(\mathcal{L}) \) is the natural extension to \( \mathcal{L} \)-formulae of the usual Levy hierarchy. Thus \( \Sigma_0(\mathcal{L}) \) is the smallest family of \( \mathcal{L} \)-formulae that contains all atomic \( \mathcal{L} \)-formulae and is closed under Boolean operations and bounded quantification. We write \( \mathcal{M} \prec_{\Sigma_n(\mathcal{L})} \mathcal{N} \) to indicate that \( \mathcal{M} \) is a proper \( \Sigma_n(\mathcal{L}) \)-elementary submodel of \( \mathcal{N} \), i.e., \( \mathcal{M} \) is a proper submodel of \( \mathcal{N} \), and for each \( k \)-ary \( \varphi(\overline{v}) \in \Sigma_n(\mathcal{L}) \) and each \( k \)-tuple \( \overline{m} \) from \( \mathcal{M} \), \( \mathcal{M} \models \varphi(\overline{m}) \) iff \( \mathcal{N} \models \varphi(\overline{m}) \).

(g) Given a language \( \mathcal{L} \) and a predicate symbol \( X \), we often write \( \mathcal{L}(X) \) instead of \( \mathcal{L} \cup \{X\} \). Similarly, we write \( \Sigma_n(X) \) instead of \( \Sigma_n(\mathcal{L}(X)) \), and \( ZF(X) \) instead of \( ZF(\mathcal{L}(X)) \). Given \( \mathcal{M} \models ZF(\mathcal{L}) \), we say that a subset \( X_{\mathcal{M}} \) of \( \mathcal{M} \) is \( \mathcal{M} \)-amenable if \( (\mathcal{M}, X_{\mathcal{M}}) \models ZF(\mathcal{L}(X)) \). It is well-known that if \( \mathcal{M} \models ZF \) and \( X_{\mathcal{M}} \subseteq \mathcal{M} \), then \( (\mathcal{M}, X_{\mathcal{M}}) \models ZF(\mathcal{L}) \) iff \( X_{\mathcal{M}} \) is piecewise coded in \( \mathcal{M} \) and \( (\mathcal{M}, X_{\mathcal{M}}) \models \text{Coll}(\mathcal{L}) \).

(h) Suppose \( X \subseteq M^k \), for \( 1 \leq k \in \omega \), \( X \) is \( \mathcal{M} \)-definable if \( X = \varphi^{\mathcal{M}} \) for some \( \mathcal{L}(\mathcal{M}) \)-formula. \( X \) is parametrically \( \mathcal{M} \)-definable if \( X = \varphi^{\mathcal{M}^+} \) for some \( \mathcal{L}(\mathcal{M}^+) \)-formula, where \( \mathcal{M}^+ \) is the expansion \( (\mathcal{M}, m)_{m \in M} \) of \( \mathcal{M} \). A parametrically \( \mathcal{M} \)-definable function is a function \( f : M^k \rightarrow M \) (where \( 1 \leq k \in \omega \)) such that the graph of \( f \) is parametrically \( \mathcal{M} \)-definable. If \( M^* \) is an elementary extension of \( \mathcal{M} \), then any such \( f \) extends naturally to a parametrically \( \mathcal{M}^* \)-definable function according to the same definition; we may also denote this extension as \( f \).

(i) \( \mathcal{M} \) has definable Skolem functions if for every \( \mathcal{L}(\mathcal{M}) \)-formula \( \varphi(x, y_1, \ldots, y_k) \), whose free variable(s) include a distinguished free variable \( x \) and whose other free variables (if any) are \( y_1, \ldots, y_k \), there is an \( \mathcal{M} \)-definable function \( f \) such that (abusing notation slightly):

\[ \mathcal{M} \models \forall y_1 \ldots \forall y_k \exists x \varphi(x, y_1, \ldots, y_k) \rightarrow \varphi(f(y_1, \ldots, y_k), y_1, \ldots, y_k) \]

(j) If \( \mathcal{M} \) has definable Skolem functions, then given any \( X \subseteq M \), there is a least elementary substructure \( \mathcal{M}_X \) of \( \mathcal{M} \) that contains \( X \), whose universe is the set of all applications of \( \mathcal{M} \)-definable functions to tuples from \( X \). We will refer to \( \mathcal{M}_X \) as the submodel of \( \mathcal{M} \) generated by \( X \).

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3We will often conflate \( X \) and \( X_{\mathcal{M}} \) to lighten the notation. Also note that some authors use the expression ‘\( X \) is a class of \( \mathcal{M} \)’ instead of ‘\( X \) is amenable over \( \mathcal{M} \)’.

4This fact is essentially due to Keisler, its proof is implicit in the proof of Theorem C of [Kei].
(k) Given a distinguished binary relation symbol $<$, the global well-ordering axiom, denoted GW, is the conjunction of the sentences “$<$ is a linear order” and “every nonempty set has a $<$-least element”. It is well-known that within $ZF(<) + GW$ there is global well-ordering $<^*$ that is set-like and thus is of order-type $\text{Ord}$ (by defining $x <^* y$ iff $[(\nu(x) = \nu(y) \text{ and } x < y) \lor (\nu(x) \in \nu(y))]$, where $\nu$ is the usual ordinal-valued rank function).

(l) Given a distinguished unary function symbol $f$, the global choice axiom, denoted GC, is the axiom $\forall x (x \neq \varnothing \rightarrow f(x) \in x)$.

The following two theorems are well-known. A proof of Theorem 2.1.2 can be found in [L, Section V.4]; for Theorem 2.1.3 see [Fe].

2.1.2. Theorem. The theories $ZF(\mathcal{L}(<)) + GW$ and $ZF(\mathcal{L}(f)) + GC$ are definitionally equivalent for every language $\mathcal{L} \supseteq \mathcal{L}_{\text{Set}}$

2.1.3. Theorem. Suppose $\mathcal{M} \models ZFC(\mathcal{L})$ for some countable language $\mathcal{L}$, and $\text{Ord}^\mathcal{M}$ has countable cofinality. Then $\mathcal{M}$ has an expansion $(\mathcal{M}, <_M) \models ZF(\mathcal{L}) + GW$.

The following proposition provides us with a large class of models of set theory that have definable Skolem functions.

2.1.4. Proposition. For any language $\mathcal{L}$ that includes $<$, every model of $ZF(\mathcal{L}) + GW$ has definable Skolem functions.

Proof. Given $\varphi = \varphi(x, y_1, \ldots, y_k)$, we can define a Skolem function $f$ for $\varphi$ by first choosing $\alpha$ to be the first ordinal such that $\exists x \in V(\alpha) \varphi(x, y_1, \ldots, y_k)$, if $\exists x \varphi(x, y_1, \ldots, y_k)$, and then defining $f(y_1, \ldots, y_k)$ to be the $<$-first element of:

$$\{x : x \in V(\alpha) \land \varphi(x, y_1, \ldots, y_k)\}.$$

We define $f(y_1, \ldots, y_k) = 0$ if $\neg \exists x \varphi(x, y_1, \ldots, y_k)$.

For models of ZF, the $\mathcal{L}_{\text{Set}}$-sentence $\exists p (V = \text{HOD}(p))$ expresses: “there is some $p$ such that every set is first order definable in some structure of the form $(V(\alpha), \in, p, \beta)_{\beta < \alpha}$ with $p \in V(\alpha)$”. The following theorem is well-known; the equivalence of (a) and (b) will be revisited in Remark 4.3.

2.1.5. Theorem. The following statements are equivalent for $\mathcal{M} \models ZF$:

(a) $\mathcal{M} \models \exists p (V = \text{HOD}(p))$.

(b) For some $p \in M$ and some set-theoretic formula $\varphi(x, y, z)$, $\mathcal{M}$ satisfies “$\varphi(x, y, p)$ well-orders the universe”.

(c) For some $p \in M$ and some set-theoretic formula $\psi(x, y, z)$, $\mathcal{M}$ satisfies “$\psi(x, y, p)$ is the graph of a global choice function”.

2.2. Indiscernibles

This subsection includes the basic notation and facts about indiscernibles that will be used in later sections.

- Given a linear order $(X, <)$, and nonzero $n \in \omega$, we use $[X]^n$ to denote the set of all increasing sequences $x_1 < \cdots < x_n$ from $X$.\footnote{Two theories $T_1$ and $T_2$ are said to be definitionally equivalent if they have a common definitional extension. Definitional equivalence is also commonly referred to as synonymy, see [M].}
2.2.1. Definition. Given a structure $\mathcal{M}$ and some linear order $(I, <)$ where $I \subseteq M$, we say that $(I, <)$ is a set of order indiscernibles in $\mathcal{M}$ if for any $\mathcal{L}(\mathcal{M})$-formula $\varphi(x_1, \cdots, x_n)$, and any two $n$-tuples $\overline{i}$ and $\overline{j}$ from $[I]^n$, we have:

$$\mathcal{M} \models \varphi(i_1, \cdots, i_n) \leftrightarrow \varphi(j_1, \cdots, j_n).$$

The following classical result is due to Ehrenfeucht and Mostowski; see, e.g., Theorem 3.3.11 of [CK]. In what follows we use the notation $\mathcal{M}_I$ introduced in part (j) of Definition 2.1.1 to denote the elementary submodel of $\mathcal{M}$ generated by $I$.

2.2.2. Theorem. (Fundamental Theorem of Indiscernibles) Suppose $\mathcal{M}$ is a structure with definable Skolem functions, $(I, <)$ is a set of order indiscernibles in $\mathcal{M}$, and $\mathcal{L} = \mathcal{L}(\mathcal{M})$.

(a) (Subset Theorem) For each subset $I_0$ of $I$, $\mathcal{M}_{I_0} \preceq \mathcal{M}_I \preceq \mathcal{M}$. Moreover, if $I$ is infinite and $I_0 \neq I$, then $\mathcal{M}_{I_0} \prec \mathcal{M}_I$.

(b) (Stretching Theorem) If $I$ is infinite and $(K, <_K)$ is a linear order, then there is a model $\mathcal{M}_K \equiv \mathcal{M}$ in which $(K, <_K)$ forms a set of indiscernibles, $K$ generates $\mathcal{M}_K$, and for any $\mathcal{L}$-formula $\varphi(x_1, \cdots, x_n)$ we have:

$$\forall \overline{i} \in [I]^n \forall \overline{k} \in [K]^n \mathcal{M} \models \varphi(i_1, \cdots, i_n) \iff \mathcal{M}_K \models \varphi(k_1, \cdots, k_n).$$

(c) (Elementary Embedding Theorem) Let $\mathcal{M}_K$ be as in (b). Then each injective order-preserving embedding $e$ of $(I, <)$ into $(K, <_K)$ induces an elementary embedding $\hat{e}$ of $\mathcal{M}_I$ into $\mathcal{M}_K$, defined by

$$\hat{e}(f(i_1, \cdots, i_n)) = f(e(i_1), \cdots, e(i_n)),$$

where $f$ is an $\mathcal{M}$-definable function. Moreover, if $e$ is surjective, then so is $\hat{e}$.

2.2.3. Remark. Since the moreover clause of part (a) of Theorem 2.2.2 is not included in Theorem 3.3.11 of [CK], we outline its proof here. It suffices to show that if $f(x_1, \cdots, x_n)$ is an $\mathcal{M}$-definable function, $(i_1, \cdots, i_n) \in [I]^n$, and $j$ is an element of $I$ such that $j \notin \{i_1, \cdots, i_n\}$, then $f(i_1, \cdots, i_n) \neq j$. By part (b) of Theorem 2.2.2 we may assume that the order-type of $I$ is $\mathbb{Q}$ (the rationals). There are three cases to consider:

Case A: $j$ is below $i_1$.

Case B: $j$ is between $i_k$ and $i_{k+1}$, where $1 \leq k \leq n - 1$.

Case C: $j$ is above $i_n$.

We only consider Case B and leave the other two cases (which are handled similarly) to the reader. Suppose to the contrary that for $1 \leq k \leq n - 1$:

$$(*) \quad i_k < j < i_{k+1}, \text{ and } f(i_1, \cdots, i_n) = j.$$

Since the order-type of $I$ is assumed to be $\mathbb{Q}$, there is an element $j' \neq j$ in $I$ such that $i_k < j' < i_{k+1}$. By indiscernibility, $(*)$ implies that $f(i_1, \cdots, i_n) = j'$, which is impossible since $f$ is a function.

2.3. Satisfaction classes

Satisfaction classes are generalizations of the familiar model-theoretic notion of ‘elementary diagram’. They play an important role in this paper; the material below is the bare minimum that we will need.

2.3.1. Definition. Reasoning within $\text{ZF}$, for each object $a$ in the universe of sets, let $c_a$ be a constant symbol denoting $a$ (where the map $a \mapsto c_a$ is $\Delta_1$). For each finite extension $\mathcal{L}$ of $\mathcal{L}_{\text{Set}}$, let $\text{Sent}_{\mathcal{L}}(x)$ be the $\mathcal{L}_{\text{Set}}$-formula that defines the class $\text{Sent}_{\mathcal{L}}$ of sentences in the language $\mathcal{L}^+ = \mathcal{L} \cup \{c_a : a \in V\}$, and let $\text{Sent}_{\mathcal{L}^+}(i, x)$ be the $\mathcal{L}_{\text{Set}}$-formula that expresses “$i \in \omega, x \in \text{Sent}_{\mathcal{L}^+}$, and $x$ is a $\Sigma_i(\mathcal{L}^+)$-sentence”.
2.3.2. Definition. Suppose $\mathcal{L}$ is a finite extension of $\mathcal{L}_{\text{set}}, \mathcal{M} \models \text{ZF}(\mathcal{L}), S \subseteq M$, and $k \in \omega^M$.

(a) $S$ is a $\Sigma_k$-satisfaction class for $\mathcal{M}$ if $(\mathcal{M}, S) \models \text{Sat}(k, S)$, where $\text{Sat}(k, S)$ is the universal generalization of the conjunction of the axioms $(I)$ through $(IV)$ below. We assume that first order logic is formulated using only the logical constants $\{\neg, \lor, \exists\}$.

(I) $[(S(c_x = c_y) \iff x = y) \land (S(c_x \in c_y) \iff x \in y)]$.

(II) $[\text{Sent}_{\mathcal{L}^+(k, \varphi)} \land (\varphi = \lnot \psi)] \rightarrow [S(\varphi) \iff \lnot S(\psi)]$.

(III) $[\text{Sent}_{\mathcal{L}^+(k, \varphi)} \land (\varphi = \psi_1 \lor \psi_2)] \rightarrow [S(\varphi) \iff (S(\psi_1) \lor S(\psi_2))]$.

(IV) $[\text{Sent}_{\mathcal{L}^+(k, \varphi)} \land (\varphi = \exists v \, \psi(v))] \rightarrow [S(\varphi) \iff \exists x \, S(\psi(c_x))]$.

(b) $S$ is a $\Sigma_\omega$-satisfaction class for $\mathcal{M}$ if for each $k \in \omega$, $S$ is a $\Sigma_k$-satisfaction class over $\mathcal{M}$. In other words, $S$ is a $\Sigma_\omega$-satisfaction class for $\mathcal{M}$ if $S$ agrees with the usual Tarskian satisfaction class for $\mathcal{M}$ on all standard $\mathcal{L}$-formulae. Note that if $\mathcal{M}$ is not $\omega$-standard, then such a satisfaction class $S$ does not necessarily satisfy Tarski’s compositional clauses for formulae of nonstandard length in $\mathcal{M}$. However, using a routine overspill argument, it can be readily checked that if $S$ is a $\Sigma_\omega$-satisfaction class for $\mathcal{M}$ and $S$ is $\mathcal{M}$-amenable, then there is a nonstandard $c \in \omega^M$ such that $S$ is a $\Sigma_c$-satisfaction class over $\mathcal{M}$; indeed, all that is needed for the overspill argument is for $(\mathcal{M}, S)$ to satisfy the scheme of induction over $\omega^M$, a scheme that holds in $(\mathcal{M}, S)$ since $(\mathcal{M}, S)$ satisfies the separation scheme Sep($\mathcal{L}$).

(c) $S$ is a full satisfaction class for $\mathcal{M}$ if for each $k \in \omega^M$, $S$ is a $\Sigma_k$-satisfaction class over $\mathcal{M}$. In other words, $S$ is a full satisfaction class for $\mathcal{M}$ if $S$ satisfies $(I)$, and the strengthened versions of $(II)$, $(III)$, and $(IV)$ from part (a) in which the conjunct Sent$_{\mathcal{L}^+(k, \varphi)}$ is replaced by Sent$_{\mathcal{L}^+(\varphi)}$. Thus, in contrast with $\Sigma_\omega$-satisfaction classes which are only guaranteed to satisfy Tarski’s compositional clauses for standard formulae, full satisfaction classes satisfy Tarski’s compositional clauses for all formulae in $\mathcal{M}$ (including the nonstandard ones, if any).

(d) Recall that given any language $\mathcal{L}$, $\mathcal{L}_{\infty, \infty}$ is the union of logics $\mathcal{L}_{\kappa, \lambda}$, where $\kappa$ and $\lambda$ are infinite cardinals and the logic $\mathcal{L}_{\kappa, \lambda}$ is the extension of first order logic that allows conjunctions and disjunctions of sets of formulae of cardinality less than $\kappa$ and blocks of existential quantifiers and blocks of universal quantifiers of length less than $\lambda$. Thus $\mathcal{L}_{\omega, \omega}$ is none other than the usual first order logic based on the language $\mathcal{L}$. $S$ is an $\mathcal{L}_{\infty, \infty}$-satisfaction class for $\mathcal{M}$ if $S$ satisfies $(I)$, the strengthened version of $(II)$ from part (a) in which the conjunct Sent$_{\mathcal{L}^+(k, \varphi)}$ is replaced by the formula Sent$_{\mathcal{L}^+_{\infty, \infty}}(\varphi)$ that expresses “$\varphi$ is a sentence of $\mathcal{L}^+_{\infty, \infty}$”, as well as the following stronger variants of $(III)$ and $(IV)$. Note that in $(III)^*$ below $\Psi$ ranges over sets of formulae of $\mathcal{L}_{\infty, \infty}$

$(III)^*$ $[\text{Sent}_{\mathcal{L}^+_{\infty, \infty}}(\varphi) \land (\varphi = \forall \Psi)] \rightarrow [S(\varphi) \iff \exists \psi \in \Psi \, S(\psi)]$.

$(IV)^*$ $[\text{Sent}_{\mathcal{L}^+_{\infty, \infty}}(\varphi) \land (\varphi = \exists x_\alpha: \alpha < \lambda) \, \psi(x_\alpha: \alpha < \lambda)] \rightarrow$

$[S(\varphi) \iff (\exists x_\alpha: \alpha < \lambda) \, S(\psi(c_{x_\alpha}: \alpha < \lambda))]$.

- Given a satisfaction class $S$, in the interest of a lighter notation, we will often write $\varphi(a_1, \cdots, a_n) \in S$ instead of $\varphi(c_{a_1}, \cdots, c_{a_n}) \in S$.

2.3.3. Remark. It is a well-known result of Levy that if $\mathcal{M} \models \text{ZF}$, then there is a $\Sigma_0$-satisfaction class for $\mathcal{M}$ that is definable in $\mathcal{M}$ by a $\Sigma_1$-formula (see [3] p. 186 for a proof). This makes it clear that for each $n \geq 1$, there is a $\Sigma_n$-satisfaction class for $\mathcal{M}$ that is definable in $\mathcal{M}$ by a $\Sigma_n$-formula. Levy’s result extends to models of $\text{ZF}(\mathcal{L})$ if $\mathcal{L}$ is finite. We use Sat$_{\Delta_0}$ to refer to the canonical $\Sigma_0$-satisfaction class (recall that by definition $\Delta_0 = \Sigma_0$ in the Levy Hierarchy).
2.4. The theory GBC + “Ord is weakly compact”

The theory GBC + “Ord is weakly compact” was first studied by McAloon and Ressayre [MR], and then later, using different methods and motivations, by the author [R-3]. Here we bring together a number of results about this theory that are not only of intrinsic foundational interest, but also play an essential role in the proofs of the results in later sections.

2.4.1. Definition. GBC is the Gödel-Bernays theory of classes GB with global choice. Our set-up for GB is the standard one in which models of GB are viewed as two-sorted structures of the form \((\mathcal{M}, x)\), where \(\mathcal{M} \models ZF\), and \(x \subseteq \mathcal{P}(M)\). Thus, the language appropriate to GB (referred to as the language of class theory) is a two-sorted language: a sort for sets (represented by lower case letters), a sort for classes (represented by upper case letters), and a special membership relation symbol \(\in\) for indicating that a set \(x\) is a member of a class \(X\), written \(x \in X\). In the interest of a lighter notation, we use \(\in\) both as the formal symbol indicating membership between sets, and also for the membership relation between sets and classes (since we use upper case letters to symbolize classes, there is no risk of confusion). Also, since coding of sequences is available in GB, we shall use expressions such as “\(F \in x\)”, where \(F\) is a function, as a substitute for the precise but lengthier expression “there is a class in \(x\) that canonically codes \(F\)”. We will say \(X \in x\) is a proper class if there is no \(c \in M\) such that \(\text{Ext}_M(c) = X\), otherwise we say that \(X\) is coded as a set in \(M\).

2.4.2. Remark. It is well-known that for \(x \subseteq \mathcal{P}(M)\), and \(\mathcal{M} \models ZF\), \((\mathcal{M}, x) \models GB\) iff the following two conditions hold:

(a) If \(X_1, \ldots, X_n \in x\), then \((\mathcal{M}, X_1, \ldots, X_n) \models ZF(X_1, \ldots, X_n)\).

(b) If \(X_1, \ldots, X_n \in x\), and \(Y\) is parametrically definable in \((\mathcal{M}, X_1, \ldots, X_n)\), then \(Y \in x\).

2.4.3. Definition. “Ord is weakly compact” is the statement in the language of class theory asserting that every Ord-tree has a branch, where Ord-trees are defined in analogy with the familiar notion of \(\kappa\)-trees in infinite combinatorics: \((\tau, <_\tau)\) is an Ord-tree, if \((\tau, <_\tau)\) is a well-founded tree of height Ord such that the collection of nodes of any prescribed ordinal rank is a set (as opposed to a proper class).

The following result is the GBC-adaptation of the ZFC-formulation of the classical Erdös-Hajnal-Rado Ramification Lemma. The Ramification Lemma is a ZFC-theorem with a parameter \(\kappa\) that ranges over infinite cardinals \(\kappa\); in the GBC-adaptation below the class of ordinals Ord plays the role typically played by \(\kappa\). Lemma 2.4.4 shows that within each model \((\mathcal{M}, x)\) of GBC, one can canonically associate an Ord-tree \(\tau_F\) to each coloring of \(F\) of \([\text{Ord}]^{n+1}\), where \(1 \leq n \in \omega^\mathcal{M}\) into set-many colors such that the color associated by \(F\) to each increasing chain of length \(n + 1\) in \(\tau_F\) is independent of the maximum element of the chain.

2.4.4. Lemma. Suppose \((\mathcal{M}, x) \models GBC\), \(1 \leq n \in \omega^\mathcal{M}\), and \(F: [\text{Ord}]^{n+1} \rightarrow \lambda\), where \(F \in x\) and \(\lambda\) is a cardinal in \(\mathcal{M}\). There is a structure \(\tau_F = (\text{Ord}^\mathcal{M}, <_F)\) coded in \(x\) such that the following hold in \((\mathcal{M}, F, \tau_F)\):

(a) For all ordinals \(\alpha\) and \(\beta\), if \(\alpha <_F \beta\), then \(\alpha \in \beta\). In particular, \(\tau_F\) is a well-founded tree.

(b) \(F(\alpha_1, \alpha_2, \cdots, \alpha_n, \alpha_{n+1}) = F(\alpha_1, \alpha_2, \cdots, \alpha_n, \beta)\) whenever

\[\alpha_1 <_F \alpha_2 <_F \cdots <_F \alpha_{n-1} <_F \alpha_n <_F \alpha_{n+1}, \text{ and } \alpha_n <_F \beta.\]
(c) For each ordinal $\alpha$, the $\alpha$-th level of the tree $\tau_F$ has cardinality at most $\lambda^{[\omega+n]}$; in particular $\tau_F$ is an Ord-tree.

Proof. The ZFC-proof presented in [Kan-1, Lemma 7.2] can be readily adapted to the GBC context by replacing the cardinal $\sigma$ in that proof with the proper class $\text{Ord}$.

The above Lemma lies at the heart of the proof of the theorem below. In parts (b) and (c) of the theorem, $\text{Ord} \to (\text{Ord})^n$ stands for the sentence in the language of class theory that asserts that for every class function $F : [\text{Ord}]^n \to \kappa$ (where $0 < n \in \omega$, $\kappa$ is a finite or infinite cardinal, and $[\text{Ord}]^n$ is the class of all increasing sequences of ordinals of length $n$) there is an unbounded $H \subseteq \text{Ord}$ such that $H$ is $F$-homogeneous, i.e., for any two increasing $n$-tuples $\bar{x}$ and $\bar{y}$ from $H$, $F(\bar{x}) = F(\bar{y})$.

2.4.5. Theorem. Suppose $(\mathcal{M}, \mathcal{X}) \models \text{GBC} + \text{“Ord is weakly compact”}$. Then:

(a) If $1 \leq n \in \omega^\mathcal{M}$, $F \in \mathcal{X}$, $\kappa \in \text{Ord}^\mathcal{M}$ and $(\mathcal{M}, F) \models F : [\text{Ord}]^{n+1} \to \kappa$, then there is some proper class $H \in \mathcal{X}$ that is ‘end-homogeneous’, i.e., $(\mathcal{M}, F, H)$ satisfies:

\[ \forall \bar{x} \in [H]^{n+2} \ F(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) = F(\alpha_1, \ldots, \alpha_n, \alpha_{n+2}). \]

(b) For every cardinal $\kappa$ in $\mathcal{M}$, $(\mathcal{M}, \mathcal{X}) \models \forall n \in \omega \setminus \{0\} \ (\varphi(n, \kappa) \to \varphi(n+1, \kappa))$, where:

\[ \varphi(n, \kappa) := (\text{Ord} \to (\text{Ord})^\kappa)^n. \]

(c) If $1 \leq n \in \omega$, and $\kappa$ is a cardinal of $\mathcal{M}$, then $(\mathcal{M}, \mathcal{X}) \models \text{Ord} \to (\text{Ord})^\kappa$.

Proof. To verify (a), we argue in $(\mathcal{M}, \mathcal{X})$. Suppose $F : [\text{Ord}]^{n+1} \to \kappa$, where $1 \leq n \in \omega^\mathcal{M}$ and $F \in \mathcal{X}$, and let $\tau_F$ be as in Lemma 2.4.4. By weak compactness of Ord, there is some proper class $H \subseteq \text{Ord}$ that is a cofinal branch of $\tau_F$. Lemma 2.4.4 assures us that $F(\alpha_1, \ldots, \alpha_n, \beta) = F(\alpha_1, \ldots, \alpha_n, \beta')$ if $\bar{\sigma} \in [H]^n$ and $\beta$ and $\beta'$ are any two elements of $H$ that are above $\alpha_n$. Thus $H$ is end-homogeneous, as desired.

To see that (b) holds, suppose $(\mathcal{M}, \mathcal{X}) \models \varphi(n, \kappa)$ for some nonzero $n \in \omega^\mathcal{M}$ and some cardinal $\kappa$ of $\mathcal{M}$. To verify that $(\mathcal{M}, \mathcal{X}) \models \varphi(n+1, \kappa)$, suppose that for some $F \in \mathcal{X}$, $(\mathcal{M}, F) \models F : [\text{Ord}]^{n+1} \to \kappa$. By (a) we can get hold of an end-homogeneous $H$ for $F$. Consider the function $G : [H]^n \to \kappa$ defined in $(\mathcal{M}, F)$ by:

\[ G(\alpha_1, \ldots, \alpha_n) := F(\alpha_1, \ldots, \alpha_n, \beta), \text{ where } \beta \in H \text{ and } \beta > \alpha_n. \]

The end-homogeneity of $H$ assures us that $G$ is well-defined. Hence by the assumption that $\varphi(n, \kappa)$ holds in $(\mathcal{M}, \mathcal{X})$, there is a proper class $H' \subseteq H$ that is $G$-homogeneous. This makes it evident that $H'$ is $F$-homogeneous, thus completing the proof of (b).

(c) follows immediately from (b) by induction on metatheoretic natural numbers $n$. □

Next we will describe a minor extension of another tree construction, first introduced in [E-2, Section 3], and later simplified in [EH, Definition 2.2], where it was used to prove that models of GBC of the form $(\mathcal{M}, \mathcal{X})$, where $\mathcal{X}$ is the collection of parametrically $\mathcal{M}$-definable subsets of $\mathcal{M} \models \text{ZFC}$, never satisfy the axiom “Ord is weakly compact”.

2.4.6. Definition. Suppose $(\mathcal{M}, \mathcal{X}) \models \text{GBC}$. Fix some ordering $<_\mathcal{M}$ of $\mathcal{M}$ in $\mathcal{X}$ such that $(\mathcal{M}, <_\mathcal{M}) \models \text{GW}$. Within $(\mathcal{M}, <_\mathcal{M})$, given ordinals $\alpha \in \beta$ let:

\footnote{As shown in Theorem 4.9 the statement $\theta = \forall m, n \in \omega \ (\text{Ord} \to (\text{Ord})^m_n)$ is not provable in $\text{GBC} + \text{“Ord is weakly compact”}$, but part (b) of Theorem 2.4.5 shows that $\theta$ is provable in the theory obtained by augmenting $\text{GBC} + \text{“Ord is weakly compact”}$ with $\Pi^1_2$-induction (over the ambient $\omega$).}
\[ V_{\beta,\alpha} = (V(\beta),\in,<,a)_{a \in V(\alpha)} \, . \]

Thus \( V_{\beta,\alpha} \) is an \( \mathcal{L}_\alpha \)-structure, where \( \mathcal{L}_\alpha \) is the result of augmenting \( \mathcal{L}_{\text{Set}}(\in) \) with constant symbols \( c_a \) for each \( a \in V(\alpha) \). Given \( X \in \mathcal{X} \) and \( n \in \omega \), within \( (\mathcal{M},<,M,X) \), let \( \tau_n(X) \) be the tree whose elements are of the form:

\[ T(X,\beta,\alpha,s) := \text{Th}(V_{\beta,\alpha},X \cap V(\beta),s) \, , \]

where \( s \in V(\beta) \setminus V(\alpha) \), with the additional requirement that:

\[ (V(\beta),\in,<,X \cap V(\beta)) \prec_{\Sigma_n(X)} (V,\in,<,X) \, . \]

Note that \( T(X,\beta,\alpha,s) \) consists of \( \mathcal{L}_\alpha(X,c) \)-sentences that hold in \( (V_{\beta,\alpha},X \cap V(\beta),s) \), where:

\[ \mathcal{L}_\alpha(X,c) := \{<,X,c\} \cup \{c_a : a \in V(\alpha)\} \, . \]

In the above, \( X \) is a unary predicate (that is conflated with its denotation), and \( c \) is a new constant symbol whose denotation is \( s \). The ordering relation on \( \tau_n(X) \) is set-inclusion.

2.4.7. Theorem. Suppose \( (\mathcal{M},\mathcal{X}) \models \text{GBC} \), and let \( <_M \) be a member of \( \mathcal{X} \) such that \( (\mathcal{M},<_M) \models \text{GW} \). Then:

(a) For each \( X \in \mathcal{X} \), \( (\mathcal{M},\mathcal{X}) \models "\tau_n(X) \text{ is an Ord-tree}" \).

(b) If \( n \geq 1 \) and the tree \( \tau_n(X) \) as computed in \( \mathcal{M}^+ := (\mathcal{M},<_M,X) \) has a branch \( B \in \mathcal{X} \), then there is an \( \mathcal{L}_{\text{Set}}(\in,X) \)-structure \( \mathcal{N}^+ := (\mathcal{N},<_N,X_N) \) and a proper \( \Sigma_n(\in,X) \)-elementary end embedding

\[ j : \mathcal{M}^+ \to \mathcal{N}^+ \, . \]

Moreover, \( \mathcal{X} \) contains both the embedding \( j \), and a full satisfaction class for the structure \( \mathcal{N}^+ \).

(c) If \( (\mathcal{M},\mathcal{X}) \models "\text{Ord is weakly compact}" \), then for every \( X \in \mathcal{X} \) and every \( n \in \omega \), there is a \( \Sigma_n(\in,X) \)-e.e.e. \( \mathcal{N}^+ \) of \( \mathcal{M}^+ := (\mathcal{M},<_M,X) \) that has a minimum element. Consequently, there is some \( S_X \in \mathcal{X} \) that is a full satisfaction class for \( \mathcal{M}^+ \); indeed there is even some \( S_{X,\infty} \in \mathcal{X} \) such that \( S_{X,\infty} \) is an \( \mathcal{L}_{\infty,\infty} \)-satisfaction predicate for every \( \mathcal{L} \)-set-structure. [EH, III.2].

Proof. The proofs of (a) and (b) are minor variants of Lemmas 2.3 and 2.5 of [EH], so we do not present them here. To prove (c), given \( X \in \mathcal{X} \) and \( n \in \omega \), we first use (b) and the assumption that \( (\mathcal{M},\mathcal{X}) \models "\text{Ord is weakly compact}" \) to construct a \( \Sigma_n(\in,X) \)-e.e.e. \( \mathcal{N}^+ \) of \( \mathcal{M}^+ \). Then we use the following result to arrange for \( \text{Ord}^\mathcal{N} \setminus \mathcal{M} \) to have a minimum element. Note that this immediately implies the existence of the satisfaction classes \( S_X \) and \( S_{X,\infty} \) as in the second assertion in (c) since if \( n \geq 2 \), then \( \mathcal{N}^+ \) is a model of a substantial fragment of ZF, including KP (Kripke-Platek set theory), and already KP is sufficient for defining the \( \mathcal{L}_{\infty,\infty} \)-satisfaction predicate for every \( \mathcal{L} \)-set-structure. [EH, III.2].

2.4.8. Theorem. Suppose \( (\mathcal{M},\mathcal{X}) \models \text{GBC} \), \( X \in \mathcal{X} \), and \( (\mathcal{M},<_M) \models \text{GW} \), where \( <_M \) is in \( \mathcal{X} \). Suppose furthermore that \( \mathcal{X} \) contains a full satisfaction class for \( \mathcal{M}^+ := (\mathcal{M},<_M,X_M) \) and also a full satisfaction class for some \( \Sigma_{n+3}(\in,X) \)-e.e.e. \( \mathcal{N}^+ := (\mathcal{N},<_N,X_N) \) of \( \mathcal{M}^+ \), where \( n \geq 1 \). Then there is some \( \mathcal{K}^+ := (\mathcal{K},<_K,X_K) \) such that:

(a) \( \mathcal{K}^+ \) is a \( \Sigma_{n+1}(\in,X) \)-e.e.e. of \( \mathcal{M}^+ \).

(b) \( \mathcal{X} \) contains a full satisfaction class for \( \mathcal{K}^+ \), and

(c) \( \text{Ord}^\mathcal{K} \setminus \mathcal{M} \) has a minimum element.

Proof. The proof is similar to the proofs of [EH, Theorem 3.3] and [EH, Theorem 2.1]. Choose \( S \in \mathcal{X} \) such that \( S \) is a full satisfaction class for \( \mathcal{M}^+ \), where \( \mathcal{M}^+ \prec_{\Sigma_{n+3}(\in,X)} \mathcal{N}^+ \). For \( n \in \omega \) consider the statement \( \varphi_n \) that expresses the following instance of the reflection theorem:
Note that $\varphi_\alpha$ is a $\Pi_{n+3}(<,X)$-statement for each $1 \leq k \in \omega$ since the satisfaction predicate for $\Sigma_k(<,X)$-formulae is $\Sigma_k(<,X)$-definable. Therefore $\mathcal{N}^+ \models \varphi$ since $\varphi$ holds in $\mathcal{M}^+$ by the reflection theorem. So we can fix some $\lambda \in \text{Ord}^\mathcal{N}\setminus\text{Ord}^\mathcal{M}$ and some $\mathcal{N}$-ordinal $\beta > \lambda$ such that:

$$\mathcal{N}^+(\beta) \prec_{\Sigma_{n+1}(<,X)} \mathcal{N}^+, \text{ and}$$

where $\mathcal{N}^+(\beta) := (V(\beta),<,\in,\mathcal{M}^+(\beta))$. Note that this implies that $\mathcal{N}^+(\beta)$ can meaningfully define the satisfaction predicate for every set-structure ‘living in’ $\mathcal{N}^+(\beta)$. For any $\alpha \in \text{Ord}^\mathcal{M}$ with $\alpha < \beta$, within $\mathcal{N}^+$ one can define the submodel $K^+_{\alpha} := (K_\alpha,\prec_{K_\alpha},X_{K_\alpha})$ of $(V(\beta),<,X \cap V(\beta))$ whose universe $K_\alpha$ is defined via:

$$K_\alpha := \{ a \in V(\beta) : a \text{ is first order definable in } (V(\beta),<,X \cap V(\beta),\lambda,m)_{m \in V(\alpha)} \}.$$  

Clearly $M(\alpha) \cup \{ \lambda \} \subset K_\alpha$ and $K^+_{\alpha} \prec_{\Sigma_{n+1}(<,X)} \mathcal{N}^+$, and of course $K^+_{\alpha}$ is coded in $\mathcal{N}^+$. Next let:

$$K := \bigcup_{\alpha \in \text{Ord}^\mathcal{M}} K_\alpha, \text{ and } X_K := X_{\mathcal{N}} \cap K,$$

and let the $\mathcal{L}_{\text{Set}}(<,X)$-structure $K^+$ be the submodel of $\mathcal{N}^+(\beta)$ whose universe is $K$. Note that:

$$\mathcal{M}^+ \prec_{\text{end},\Sigma_{n+1}(<,X)} K^+,$$

since:

$$\mathcal{M}^+ \prec_{\text{end}} K^+ \preceq \mathcal{N}^+(\beta) \prec_{\Sigma_{n+1}(<,X)} \mathcal{N}^+.$$

- Observe that if $S$ is a full satisfaction class for $\mathcal{N}^+$ such that $S \in \mathcal{X}$, then there are full satisfaction classes in $\mathcal{X}$ for the structures $\mathcal{N}^+(\beta)$ and $K^+$.

To prove that $\text{Ord}^\mathcal{K}\setminus\text{Ord}^\mathcal{M}$ has a least element, suppose to the contrary that $\text{Ord}^\mathcal{K}\setminus\text{Ord}^\mathcal{M}$ has no least element. Within $\mathcal{N}^+$ let $S_\alpha$ be the full satisfaction class for the structure

$$(V(\beta),\in,<,X \cap V(\beta),\lambda,m)_{m \in V(\alpha)}.$$  

Now let $\Phi := \bigcup_{\alpha \in \text{Ord}^\mathcal{M}} \Phi_\alpha$, where:

$$\Phi_\alpha := \{ \varphi(c,m) \in M : \mathcal{N}^+ \models \varphi(c,m) \in S_\alpha \},$$

Note that $\Phi \in \mathcal{X}$ since $S_\alpha \in \mathcal{X}$, in particular $\Phi$ is $\mathcal{M}^+$-amenable. Also observe that in the above definition of $\Phi_\alpha$, $\varphi(c,m)$ ranges over formulae of the language $\mathcal{L}_\alpha(X,c)$ (in the sense of $\mathcal{M}$), where $\mathcal{L}_\alpha(X,c)$ is as in Definition 2.4.6. Also note that the constant $c$ is interpreted as $\lambda$ in the right-hand-side of the above definition of $\Phi_\alpha$. Thus $\Phi$ can be thought of as the type of the element $\lambda$ in the structure $\mathcal{N}^+(\beta)$ over the parameter set $M$ (with the important provision that $\Phi$ includes nonstandard formulae if $\mathcal{M}$ is $\omega$-nonstandard). Now let:

$$\Gamma := \{ t(c,m) \in M : t(c,m) \in \Phi \text{ and } \forall \theta \in \text{Ord} \ (t(c,m) > c_\theta) \in \Phi \},$$

$$\forall \lambda \in \text{Ord} \exists \beta \in \text{Ord} \left( \lambda \in \beta \land (V(\beta),<,\in,\mathcal{M}(\beta)) \prec_{\Sigma_{n+1}(X)} (V,\in,\mathcal{X}) \right).$$
where \( t \) is a definable function in the language \( \mathcal{L}^+ = \bigcup_{\alpha \in \text{Ord}^\mathcal{M}} \mathcal{L}_\alpha (X, c) \). So, officially speaking, \( \Gamma \) consists of syntactic objects \( \varphi(c, c_m, x) \) in \( \mathcal{M} \) that satisfy the following three conditions in \((\mathcal{M}, \Phi)\):

1. \([\exists! x \varphi(c, c_m, x)] \in \Phi\).
2. \([\forall x (\varphi(c, c_m, x) \rightarrow x \in \text{Ord})] \in \Phi\).
3. \([\forall \theta \in \text{Ord} [\forall x (\varphi(c, c_m, x) \rightarrow c_\theta \in x)] \in \Phi\).

Note that \( \Gamma \) is definable in \((\mathcal{M}^+, \Phi)\). Since we assumed that \( \text{Ord}^\mathcal{K} \setminus \text{Ord}^\mathcal{M} \) has no minimum element, \((\mathcal{M}^+, \Phi) \models \psi\), where:

\[
\psi := \forall t (t \in \Gamma \rightarrow (\exists t' \in \Gamma \land [t' \in t] \in \Phi)).
\]

Choose \( k \in \omega \) such that \( \psi \) is a \( \Sigma_k(\beta, X, \Phi) \)-statement, and use the reflection theorem in \((\mathcal{M}^+, \Phi)\) to pick \( \mu \in \text{Ord}^\mathcal{M} \) such that:

\[
(\mathcal{M}^+(\mu), \Phi \cap M(\mu)) \models \Sigma_k(\beta, X, \Phi) (\mathcal{M}^+, \Phi).
\]

Then \( \psi \) holds in \((\mathcal{M}^+(\mu), \Phi \cap M(\mu))\), so by DC (dependent choice, which holds in \( \mathcal{M} \) since \( \text{AC} \) holds in \( \mathcal{M} \)), there is some function \( f_c \) in \( \mathcal{M} \) such that:

\[
(\mathcal{M}^+, \Phi) \models \forall n \in \omega [f_c(n + 1) \in f_c(n)] \in \Phi.
\]

Let \( \alpha \in \text{Ord}^\mathcal{M} \) be large enough so that \( M(\alpha) \) contains all constants \( c_m \) that occur in any of the terms in the range of \( f \); let \( f_\lambda(n) \) be defined in \( \mathcal{N} \) as the result of replacing all occurrences of the constant \( c \) with \( c_\lambda \) in \( f_c(n) \); and let \( g(n) \) be defined in \( \mathcal{N}^+ \) as the interpretation of \( f_\lambda(n) \) in:

\[
(V(\beta), \in, X \cup V(\beta), \lambda, m)_{m \in V_\alpha}.
\]

Then \( \mathcal{N}^+ \) satisfies:

\[
\forall n \in \omega (g(n) \in g(n + 1)),
\]

which contradicts the foundation axiom in \( \mathcal{N}^+ \). This completes the proof of Theorem 2.4.8, which in turn concludes the proof of part (c) of Theorem 2.4.7.

2.4.9. Theorem. (Different faces of weak compactness of \( \text{Ord} \)) The following are equivalent for any model \((\mathcal{M}, X)\) of GBC:

(i) (Tree property) \((\mathcal{M}, X) \models \psi_1\), where \( \psi_1 \) expresses: Every \( \text{Ord} \)-tree has a branch.

(ii) (Weak compactness) \((\mathcal{M}, X) \models \psi_2\), where \( \psi_2 \) expresses: For any language \( L \), if \( T \) is an \( L_{\infty, \infty} \)-theory of cardinality \( \text{Ord} \) such that every set-sized subtheory of \( T \) has a model, then there is a full satisfaction class for a model of \( T \).

(iii) (Ramsey property for an arbitrary set of colors in \( \mathcal{M} \) and an arbitrary metatheoretic exponent \( n \geq 2 \)) \((\mathcal{M}, X) \models \psi_{3,n}\), where \( \psi_{3,n} \) expresses: \( \forall \kappa (\text{Ord} \rightarrow (\text{Ord})^n_{\kappa}) \).

(iv) (Ramsey property for exponent 2 and 2 colors) \((\mathcal{M}, X) \models \psi_4\), where \( \psi_4 \) expresses: \( \text{Ord} \rightarrow (\text{Ord})^2 \).

(v) (Keisler property) \((\mathcal{M}, X) \models \psi_5\), where \( \psi_5 \) expresses: For all \( X \) there is some \( S \) such that \( S \) is an \( L_{\infty, \infty} \)-satisfaction class for an \( L_{\infty, \infty} \)-e.e. of \((V, \in, X)\).

(vi) (\( \Pi^1 \)-Reflection) For every \( L_{\text{Set}}(X, Y) \)-formula \( \varphi(X, Y, x) \), and for each \( m \in \mathcal{M} \) and \( A \in X \), \((\mathcal{M}, X)\) satisfies the following sentence in which \( A_\alpha := A \cap V(\alpha)\):

As shown in Theorem 4.9, this result cannot be strengthened by quantifying over \( n \) within the theory GBC + “\( \text{Ord} \) is weakly compact”.

12
\[
\forall X \varphi(X, A, m) \implies \\
[\exists \alpha \forall X \subseteq V(\alpha) (V(\alpha), \epsilon, X, A_\alpha) \models \varphi(X, A_\alpha, m)].
\]

**Proof.** With the help of Theorem 2.4.5, the equivalence of (i), (ii), (iii), and (iv) can be verified with the same strategy as in the usual ZFC-proofs (e.g., as in [Kan-1, Theorem 7.8]) of the equivalence of various formulations of weak compactness of a cardinal. It is easy to see that \((v) \implies (i)\). To show the equivalence of \((v)\) with any of \((i)\) through \((iv)\), however, takes much more effort in contrast to the ZFC-setting, e.g., in order to show that \((ii) \implies (iv)\) one first needs to know that if \((M, X)\) is a model of GBC in which \((ii)\) holds, and \(X \in X\), then the \(L_{\infty, \infty}\)-elementary diagram of \((M, X)\) is available as a model of \(X\) (where \(L = L_{\text{Set}}(X)\)). More officially, we need to know that \(X\) contains an \(L_{\infty, \infty}\)-satisfaction class for \((M, X)\) (as defined in Definition 2.3.2(d)). This is precisely where part \((c)\) of Theorem 2.4.7 comes to the rescue. With the equivalence of \((v)\) with each of \((i)\) through \((iv)\) at hand, the proof will be complete once we show that \((v) \implies (vi) \implies (i)\). To see that \((v) \implies (vi)\), suppose \((M, X)\) is a model of GBC in which \((v)\) holds, and suppose \((M, X) \models \forall X \varphi(X, A, m)\) for some \(m \in M\) and \(A \in X\). Let \((N, B)\) be an \(L_{\infty, \infty}\)-elementary end extension of \((M, A)\), where for some \(S \in X\), \(S\) is a \(L_{\infty, \infty}\)-satisfaction class for \((M, A)\). Recall that in ZFC the well-foundedness of \(\epsilon\) is expressible in \(L_{\omega_1, \omega_1}\) via the sentence \(\psi\) below:

\[
\psi := \neg \exists x \in X : x \in \omega \wedge x_{n+1} \in x_n.
\]

Therefore, since \(M\) satisfies \(\psi\) and \(N\) is an \(L_{\infty, \infty}\)-elementary extension of \(M\), \(\text{Ord}^N \setminus \text{Ord}^M\) has a minimum element \(\kappa\), and thus \((N(\kappa), B \cap N(\kappa)) = (M, A)\). Hence:

\[
(N, B) \models \forall X \subseteq V(\kappa) \varphi^{V(\kappa)}(X, A, m),
\]

where \(\varphi^{V(\kappa)}\) is the result of restricting the (set) quantifiers of \(\varphi\) to \(V(\kappa)\). Therefore since \((M, A) \prec (N, B)\), we conclude:

\[
(M, A) \models \exists \alpha \forall X \subseteq V(\alpha) \varphi^{V(\alpha)}(X, A, m),
\]

thus completing the proof of \((v) \implies (vi)\). The proof of \((vi) \implies (i)\) is routine and uses the standard strategy of showing within ZFC that the \(\Pi_1^1\)-Reflection property of an inaccessible cardinal \(\kappa\) implies that \(\kappa\) has tree property.

Recall that the notion “\(\kappa\) is \(\alpha\)-Mahlo” is defined recursively by decreeing that “\(\kappa\) is 0-Mahlo” means that \(\kappa\) is strongly inaccessible, and for an ordinal \(\alpha > 0\) “\(\kappa\) is \(\alpha\)-Mahlo” means that for all \(\beta < \alpha\) the collection of cardinals that are \(\beta\)-Mahlo are stationary in \(\kappa\). It is a classical fact that, provably in ZFC, every weakly compact cardinal \(\kappa\) is \(\kappa\)-Mahlo.

Theorem 2.4.12 below summarizes some well-known facts about the Levy scheme \(\Lambda\); the statement of the theorem uses the following definition.

**2.4.10. Definition.** In what follows \(X\) is a unary predicate symbol (which will be conflated with its interpretation in a given structure).

(a) \(\Lambda = \{\lambda_n : n \in \omega\}\), where \(\lambda_n\) is the \(L_{\text{Set}}\)-sentence asserting the existence of an \(n\)-Mahlo cardinal \(\kappa\) such that \((V(\kappa), \epsilon) \prec_{\Sigma_n} (V, \epsilon)\). More generally, \(\Lambda(X) = \{\lambda_n(X) : n \in \omega\}\), and \(\lambda_n(X)\) is the \(L_{\text{Set}}(X)\)-sentence asserting the existence of an \(n\)-Mahlo cardinal \(\kappa\) such that \((V(\kappa), \epsilon, X \cap V(\kappa)) \prec_{\Sigma_n(X)} (V, \epsilon, X)\).

(b) For \(n \in \omega\), \(A_n(X) = \{\lambda_{n,i}(X) : i \in \omega\}\), and \(\lambda_{n,i}(X)\) is the sentence asserting the existence of an \(n\)-Mahlo cardinal \(\kappa\) such that \((V(\kappa), \epsilon, X \cap V(\kappa)) \prec_{\Sigma_i(X)} (V, \epsilon, X)\).

(c) \(\Lambda^-\) is the fragment of \(\Lambda\) consisting of statements of the form “there is an \(n\)-Mahlo cardinal”, for \(n \in \omega\).
2.4.11. Theorem. (Folklore).

(a) For \( n \in \omega, \kappa \) is \((n+1)\)-Mahlo iff for every \( X \subseteq V(\kappa), (V(\kappa), \in, X) \models \Lambda_n(X) \).

(b) \( \kappa \) is \( \omega \)-Mahlo iff for every \( X \subseteq V(\kappa), (V(\kappa), \in, X) \models \Lambda(X) \).

(c) ZFC + \( \Lambda^- \) is mutually interpretable with ZFC + \( \Lambda \).

(d) Assuming the consistency of ZFC + “there is an \( \omega \)-Mahlo cardinal”, ZFC + \( \Lambda^- \) \( \not\models \Lambda \).

(e) If \( \mathcal{M} \models \text{ZFC +} \mathcal{L} \), then \( L^\mathcal{M} \models \Lambda \) (where \( L^\mathcal{M} \) is the constructible universe of \( \mathcal{M} \)).

(f) If \( \mathcal{M} \models \text{ZFC +} \mathcal{L}, \mathbb{P} \) is a set notion of forcing \( \mathbb{P} \) in \( \mathcal{M} \), and \( G \) is \( \mathbb{P} \)-generic over \( \mathcal{M} \), then \( \mathcal{M}[G] \models \Lambda \).

Proof. Suppose \( \kappa \) is a strongly inaccessible cardinal and \( X \subseteq V(\kappa) \). Let

\[ C := \{ \lambda \in \kappa : (V(\lambda), \in, X \cap V(\lambda)) \prec (V(\kappa), \in, X) \} \cdot \]

A routine Skolem hull argument shows that \( C \) is closed and unbounded in \( \kappa \). This fact lies at the heart of the proofs of (a) through (c); note that the proof of (c) uses Orey's Compactness Theorem 5.3. To verify (d), work in a model of ZFC + “there is an \( \omega \)-Mahlo cardinal”, and for each \( n \in \omega \) let \( \kappa_n \) be the first \( n \)-Mahlo cardinal, and \( \kappa_\omega := \sup_{n \in \omega} \kappa_n \). Choose the first strongly inaccessible cardinal \( \lambda > \kappa_\omega \). Then ZFC + \( \Lambda^- \) clearly holds in \( (V(\lambda), \in) \). We will show that \( \Lambda \) fails in \( (V(\lambda), \in) \). To see this, we first note:

(*) \((V(\lambda), \in) \models \text{“the collection of Mahlo cardinals is bounded in Ord”}\).

On the other hand, “there are unboundedly many \( n \)-Mahlo cardinals in the universe” holds in \((V(\kappa), \in)\) for any \((n+1)\)-Mahlo cardinal \( \kappa \), and therefore if \((V(\kappa), \in) \prec_{\Sigma_n} (V, \in), \) where the statement \( \varphi = \text{“there are unboundedly many} \ n\text{-Mahlo cardinals}” \) is a \( \Sigma_n \)-sentence, then \( \varphi \) holds in the universe. Together with (*), this makes it clear that \( \Lambda \) fails in \((V(\lambda), \in)\). Part (e) follows from routine absoluteness considerations, and part (f) is a consequence of the preservation of both (1) the \( n \)-Mahlo property of a cardinal \( \kappa \), and (2) the property \( V(\kappa) \prec_{\Sigma_n} V \), in \( \mathbb{P} \)-generic extensions satisfying \( \mathbb{P} \in V(\kappa) \). (1) is established along the lines of the proof of [Kan-1, Proposition 10.13]; (2) follows from a standard truth-and-forcing argument.

The theorem below reveals the close relationship between the class theory GBC+ “Ord is weakly compact”, and the set theory ZFC + \( \Lambda \).

2.4.12. Theorem. [E-3, Corollary 2.1.1] Let \( \varphi \) be an \( \mathcal{L}_{\text{Set}} \)-sentence. The following are equivalent:

(i) GBC + “Ord is weakly compact” \( \vdash \varphi \).

(ii) ZFC + \( \Lambda \vdash \varphi \).

3. BASIC FEATURES OF ZFI AND ZFI

In this section we officially meet the principal characters of our paper, namely the theory ZFI, and its extension ZFI. We establish two useful schemes (apartness and diagonal indiscernibility) within ZFI. These schemes are then used to demonstrate some basic model-theoretic facts about ZFI and ZFI. In particular, we show that \( \omega \)-nonstandard models of ZF that have an expansion to ZFI are recursively saturated, and \( \omega \)-standard models of ZF that have an expansion to ZFI satisfy “\( 0^# \) exists”.

3.1. Definition. ZFI is the theory formulated in the language \( \mathcal{L}_{\text{Set}}(I) \), where \( I \) is a unary predicate, whose axioms consist of the three groups below.
• Note that we often write \( x \in I \) instead of \( I(x) \).

(1) \( \text{ZF}(I) \). Recall from part (e) of Definition 2.1.1 that \( \text{ZF}(I) \) includes the separation scheme \( \text{Sep}(I) \) and the collection scheme \( \text{Coll}(I) \).

(2) The sentence \( \text{Cof}(I) \) expressing “\( I \) is a cofinal subclass \( \text{Ord} \)”.

(3) The scheme \( \text{Indis}(I) = \{ \text{Indis}_\varphi(I) : \varphi \) is a formula of \( \mathcal{L}_{\text{Set}} \} \) ensuring that \( I \) forms a class of order indiscernibles for the ambient model \( (V, \in) \) of set theory. More explicitly, for each \( n \)-ary formula \( \varphi(v_1, \cdots, v_n) \) in the language \( \{=, \in\} \), \( \text{Indis}_\varphi(I) \) is the sentence:

\[
\forall x_1 \in I \cdots \forall x_n \in I \forall y_1 \in I \cdots \forall y_n \in I
\]

\[\left[ (x_1 \in \cdots \in x_n) \land (y_1 \in \cdots \in y_n) \rightarrow (\varphi(x_1, \cdots, x_n) \leftrightarrow \varphi(y_1, \cdots, y_n)) \right].\]

The theory \( \text{ZF}_I \) is an extension of \( \text{ZF} \); it is formulated in the language \( \mathcal{L}_{\text{Set}(I, <)} \), whose axioms consist of \( \text{Cof}(I) \) above, together with the following strengthenings of the axioms in (1) and (3) above:

\( (1^+) \) \( \text{ZF}(I, <) \) + \( \text{GW} \).

\( (3^+) \) The scheme \( \text{Indis}_\varphi(I) = \{ \text{Indis}_\varphi(I) : \varphi \) is a formula of \( \mathcal{L}_{\text{Set}(<)} \} \) ensuring that \( I \) forms a class of order indiscernibles for \( (V, \in, <) \).

• The above definition can be model-theoretically recast as follows: \( \mathcal{M} \models \text{ZF} \) has an expansion (\( \mathcal{M}, I \models \text{ZF} \)) if there is an \( \mathcal{M} \)-amenable cofinal subset \( I \) of \( \text{Ord}^{\mathcal{M}} \) such that \( (I, \in_I) \) forms a class of indiscernibles over \( \mathcal{M} \). Similarly, a model \( (\mathcal{M}, <_M, I) \models \text{ZF}(<) \) + \( \text{GW} \) has an expansion \( (\mathcal{M}, <_M, I) \models \text{ZF}_I \) if there is an \( (\mathcal{M}, <_M) \)-amenable cofinal subset \( I \) of \( \text{Ord}^{\mathcal{M}} \) such that \( (I, \in_I) \) forms a class of indiscernibles over \( (\mathcal{M}, <_M) \). Therefore by Theorem 2.1.5 if \( \mathcal{M} \models \text{ZF} + V = \text{HOD} \), and \( \mathcal{M} \) has an expansion to a model of \( \text{ZFI} \), then \( \mathcal{M} \) is also expandable to a model of \( \text{ZFI}_< \). Moreover, by Theorem 3.2(b) below, the assumption that \( \mathcal{M} \models V = \text{HOD} \) can be weakened to the assumption that \( \mathcal{M} \models \exists p(V = \text{HOD}(p)) \).

3.2. Theorem. Let \( \text{ZF}^* \) be the subsystem of \( \text{ZF} \) axiomatized by \( \text{ZF} + \text{Coll}(I) + \text{Cof}(I) + \text{Indis}(I) \). The following schemes are provable in \( \text{ZF}^* \):

(a) The **apartness** scheme for \( \mathcal{L}_{\text{Set}} \)-formulae:

\[
\text{Apart}(I) = \{ \text{Apart}_\varphi(I) : \varphi \in \text{Form}_{n+1}(\mathcal{L}_{\text{Set}}), n \in \omega \},
\]

where \( \text{Form}_{n}(\mathcal{L}_{\text{Set}}) \) is the collection of \( \mathcal{L}_{\text{Set}} \)-formulae whose free variables are \( x_1, \cdots, x_n \), and \( \text{Apart}_\varphi(I) \) is the following formula:

\[
\forall i \in I \forall j \in I [i < j \rightarrow \forall \overline{x} \in (V(i))^n (\exists y \varphi(\overline{x}, y) \rightarrow \exists y \in V(j) \varphi(\overline{x}, y))].
\]

(b) The **diagonal indiscernibility** scheme for \( \mathcal{L}_{\text{Set}} \)-formulae:

\[
\text{Indis}^+_\varphi(I) = \{ \text{Indis}^+_\varphi(I) : \varphi \in \text{Form}_{n+1+r}(\mathcal{L}_{\text{Set}}), n, r \in \omega, r \geq 1 \},
\]

where \( \text{Indis}^+_\varphi(I) \) is the following formula:

\[
\forall i \in I \forall j \in [I]^r \forall \overline{\mathcal{I}} \in [I]^r \left[ (i < j_1) \land (i < k_1) \right] \rightarrow
\]

\[
[\forall \overline{\mathcal{x}} \in (V(i))^n (\varphi(\overline{x}, i, j_1, \cdots, j_r) \leftrightarrow \varphi(\overline{x}, i, k_1, \cdots, k_r))].
\]
Similarly, let $ZFI^*_<$ be the subsystem of $ZFI_<$ axiomatized by $ZF(<) + GW + Coll(I) + Cof(I) + \text{Indis}_<(I)$. The following schemes are provable in $ZFI^*_<$:

(c) The **apartness** scheme for $\mathcal{L}_{\text{Set}}(\langle \rangle)$-formulae:

$$\text{Apart}_<(I) = \{ \text{Apart}_\varphi(I) : \varphi \in \text{Form}_{n+1}(\mathcal{L}_{\text{Set}}(<)), \, n \in \omega \},$$

where $\text{Form}_n(\mathcal{L}_{\text{Set}}(<))$ is the collection of $\mathcal{L}_{\text{Set}}(<)$-formulae whose free variables are $x_1, \ldots, x_n$, and $\text{Apart}_\varphi(I)$ is the following formula:

$$\forall i \in I \forall j \in I [i < j \rightarrow \forall \langle \rangle \in (V(i))^n (\exists y \varphi(\langle \rangle, y) \rightarrow \exists y \in V(j) \varphi(\langle \rangle, y))] .$$

(d) The **diagonal indiscernibility** scheme for $\mathcal{L}_{\text{Set}}(\langle \rangle)$-formulae:

$$\text{Indis}^+_<(I) = \{ \text{Indis}^+_\varphi(I) : \varphi \in \text{Form}_{n+1+r}(\mathcal{L}_{\text{Set}}(<)), \, n, r \in \omega, \, r \geq 1 \},$$

where $\text{Indis}^+_\varphi(I)$ is the following formula:

$$\forall i \in I \forall j \in [I]^r \forall \langle \rangle \in [I]^r [(i < j_1) \land (i < k_1)] \rightarrow [\forall \langle \rangle \in (V(i))^n (\varphi(\langle \rangle, i, j, j_1, \ldots, j_r) \leftrightarrow \varphi(\langle \rangle, i, k, k_1, \ldots, k_r))].$$

**Proof.** We will only establish (a) and (b) since the proof of (c) is similar to the proof of (a) and the proof of (d) is similar to the proof of (b). Let $(\mathcal{M}, I) \models ZFI^*$. To verify that the apartness scheme holds in $(\mathcal{M}, I)$, fix some $i_0 \in I$ and some $\varphi(\langle \rangle, y) \in \text{Form}_{n+1}(\mathcal{L}_{\text{Set}})$. Then, since the collection scheme $\text{Coll}(I)$ holds in $(\mathcal{M}, I)$, and $I$ is cofinal in $\text{Ord}^\mathcal{M}$, there is some $j_0 \in I$ with $i_0 < j_0$ such that:

$$(\mathcal{M}, I) \models \forall \langle \rangle \in (V(i_0))^n (\exists y \varphi(\langle \rangle, y) \rightarrow \exists y \in V(j_0) \varphi(\langle \rangle, y)).$$

The above, together with the indiscernibility of $I$ in $\mathcal{M}$, makes it evident that $(\mathcal{M}, I) \models \text{Apart}_\varphi.$

To verify that $\text{Indis}^+_\varphi(I)$ holds in $(\mathcal{M}, I)$, we will first establish a weaker form of diagonal indiscernibility of $I$ in which all $j_n < k_1$ (thus all the elements of $\langle \rangle$ are less than all the elements of $\bar{\gamma}$). Fix some $\varphi \in \text{Form}_{n+1+r}(\mathcal{L}_{\text{Set}})$ and $i_0 \in I$. Within $\mathcal{M}$ consider the function $f : [\text{Ord}]^r \rightarrow \mathcal{P}(V(i_0)^n)$ by:

$$f(\bar{\gamma}) := \{ \langle \rangle \in (V(i_0))^n : \varphi(\langle \rangle, i_0, \bar{\gamma}) \}. $$

Since $(\mathcal{M}, I)$ satisfies the collection scheme $\text{Coll}(I)$ and $I$ is cofinal in $\text{Ord}^\mathcal{M}$, there are $\mathcal{M}$-ordinals $\gamma_1 < \cdots < \gamma_{2r}$ in $I$ such that:

$$f(\gamma_1, \cdots, \gamma_r) = f(\gamma_{r+1}, \cdots, \gamma_{2r}).$$

Thus we have:

$$(\mathcal{M}, I) \models [\forall \langle \rangle \in (V(i_0))^n (\varphi(\langle \rangle, i_0, \gamma_1, \ldots, \gamma_r) \leftrightarrow \varphi(\langle \rangle, i_0, \gamma_{r+1}, \ldots, \gamma_{2r}))].$$

By indiscernibility of $I$ in $\mathcal{M}$, the above implies the following weaker form of $\text{Indis}^+_\varphi(I)$:

$$\forall i \in I \forall j \in [I]^n \forall k \in [I]^n [(i < j_1) \land (j_n < k_1)] \rightarrow [\forall \langle \rangle \in (V(i))^n (\varphi(\langle \rangle, i, j_1, \ldots, j_r) \leftrightarrow \varphi(\langle \rangle, i, k_1, \ldots, k_r))].$$
We will now show that the above weaker form of Indis$_2(I)$ implies Indis$_3(I)$. Given $i \in I$, $\tau \in [I]^r$ and $\beta \in [I]^r$, with $i < \alpha_1$ and $i < \beta_1$, choose $\gamma \in [I]^r$ with $\gamma_1 > \max\{\alpha_n, \beta_n\}$. Then by the above we have:

$$\mathcal{M} \models [\forall \tau \in (V(i))^n (\varphi(\tau, i, \alpha_1, \cdots, \alpha_r) \iff \varphi(\tau, i, \gamma_1, \cdots, \gamma_r))],$$

and

$$\mathcal{M} \models [\forall \tau \in (V(i))^n (\varphi(\tau, i, \beta_1, \cdots, \beta_r) \iff \varphi(\tau, i, \gamma_1, \cdots, \gamma_r))],$$

which together imply:

$$\mathcal{M} \models [\forall \tau \in (V(i))^n (\varphi(\tau, i, \alpha_1, \cdots, \alpha_r) \iff \varphi(\tau, i, \beta_1, \cdots, \beta_r))].$$

\[\Box\]

- Note that the diagonal indiscernibility scheme for $\mathcal{L}_{\text{Set}}$-formulae ensures that if $(\mathcal{M}, I) \models ZFI$ and $i \in I$, then $I^{\geq i}$ is a set of indiscernibles over the expanded structure $(\mathcal{M}, m)_{m \in V(i)}$, where $I^{\geq i} = \{j \in I : j \geq i\}$. Similarly, the diagonal indiscernibility scheme for $\mathcal{L}_{\text{Set}}(<)$-formulae ensures that if $(\mathcal{M}, <, I) \models ZF(\sigma)$ and $i \in I$, then $I^{\geq i}$ is a set of indiscernibles over the expanded structure $(\mathcal{M}, <, m)_{m \in V(i)}$.

The fact that the apartness scheme holds in ZF and ZF$_{<}$ will be employed in the following theorem to show that ZF is able to define a $\Sigma_\omega$-satisfaction predicate over the ambient model of ZF, and ZF$_{<}$ is able to define a $\Sigma_\omega$-satisfaction predicate over the ambient model of ZF($\sigma$) + GW (in the sense of part (b) of Definition 2.3.2).

3.3. Theorem. There is a formula $\sigma(x)$ in the language $\mathcal{L}_{\text{Set}}(I)$ such that for all models $(\mathcal{M}, I)$ of ZF, $\sigma^\mathcal{M}$ is a $\Sigma_\omega$-satisfaction class for $\mathcal{M}$. In particular:

(a) If $(\mathcal{M}, I) \models ZFI$, then $\sigma^\mathcal{M}$ is an amenable $\Sigma_\omega$-satisfaction class for $\mathcal{M}$.

(b) If $(\mathcal{M}, I) \models ZFI$, and $\mathcal{M}$ is $\omega$-standard, then $\sigma^\mathcal{M}$ is an amenable full satisfaction class for $\mathcal{M}$.

Similarly, there is a formula $\sigma_{<}(x)$ in the language $\mathcal{L}_{\text{Set}}(<, I)$ such that for all models $(\mathcal{M}, <, I)$ of ZF$_{<}$, $\sigma_{<}^\mathcal{M}$ is a $\Sigma_\omega$-satisfaction class for $(\mathcal{M}, <, I)$. In particular:

(c) If $(\mathcal{M}, <, I) \models ZFI$, then $\sigma_{<}^\mathcal{M}$ is an amenable $\Sigma_\omega$-satisfaction class for $(\mathcal{M}, <, I)$.

(d) If $(\mathcal{M}, <, I) \models ZFI$, and $\mathcal{M}$ is $\omega$-standard, then $\sigma^\mathcal{M}_{<}$ is an amenable full satisfaction class for $(\mathcal{M}, <, I)$.

Proof. (a) and (b) are immediate consequences of the first assertion of the theorem, which we will establish. The proofs of (c) and (d) are similar and will not be presented. The following definition takes place in $(\mathcal{M}, I)$: Given any $\varphi(\tau) \in \text{Form}_k(\mathcal{L}_{\text{Set}})$ and any $k$-tuple $\tau$, let $i_0$ be the first element of $I$ such that $\tau \in V(i_0)$, and for each $n \in \omega$, let $i_{n+1}$ be the first element of $I$ that exceeds $i_n$. Then let $\alpha := \sup_{n \in \omega} i_n$.

It is easy to see, by Tarski’s test (for elementarity) and the veracity of the Apartness scheme in $(\mathcal{M}, I)$ that $\mathcal{M}(\alpha) \prec \mathcal{M}$. Therefore, if $S$ is defined in $\mathcal{M}$ by:

$$\varphi(\tau) \in S \iff (V(\alpha), \epsilon) \models \varphi(\tau),$$

then $S$ is a $\Sigma_\omega$-satisfaction class for $\mathcal{M}$. Our description of $S$ makes it clear that $S$ is definable in $\mathcal{M}$ by a parameter-free formula $\sigma(x)$ in the language $\mathcal{L}_{\text{Set}}(I)$.

3.4. Remark. Theorem 2.4.7(c) together with the proof of the $(iii) \Rightarrow (i)$ direction of Theorem 4.1 shows that if $(\mathcal{M}, I) \models ZFI_{<}$, then $\mathcal{M}_I$ carries an amenable full satisfaction class $S$ (but $S$ need not be
definable in \( \mathcal{M}_I \)). On the other hand, it is known [EKM, Theorem 6.3] that if \( \mathcal{M} \) and \( \mathcal{N} \) are models of ZFC such that \( \mathcal{M} \) is a cofinal elementary submodel of \( \mathcal{N} \), then for any \( \mathcal{M} \)-amenable subset \( X_M \) of \( M \), there is a (unique) subset \( X_N \) of \( N \) such that \((\mathcal{M}, X_M) < (\mathcal{N}, X_N)\). Thus, if \((\mathcal{M}, I) \models \text{ZFI}_<\), then \( \mathcal{M} \) carries an amenable full satisfaction class.

### 3.5. Corollary
Suppose \( \mathcal{M} \models \text{ZF} \). There is no parametrically \( \mathcal{M} \)-definable subset \( I \) of \( \text{Ord}^\mathcal{M} \) such that \((\mathcal{M}, I) \models \text{ZFI}_<\). Similarly, if \( \mathcal{M} \) has an expansion \((\mathcal{M}, <_M) \models \text{ZF}(\langle \rangle) + \text{GW} \), then there is no parametrically \((\mathcal{M}, <_M)\)-definable subset \( I \) of \( \text{Ord}^\mathcal{M} \) such that \((\mathcal{M}, <_M, I) \models \text{ZFI}_<\).

**Proof.** Put Theorem 3.3 together with Tarski’s theorem on undefinability of truth. Alternatively, one can take advantage of diagonal indiscernibility. \( \square \)

### 3.6. Corollary
If \((\mathcal{M}, I) \models \text{ZFI}_<\), and \( \mathcal{M} \) is \( \omega \)-nonstandard, then \( \mathcal{M} \) is recursively saturated. Similarly, if \((\mathcal{M}, <_M, I) \models \text{ZFI}_<\), and \( \mathcal{M} \) is \( \omega \)-nonstandard, then \((\mathcal{M}, <_M) \) is recursively saturated.

**Proof.** We will only verify the ZFI case; a similar strategy works for \( \text{ZFI}_<\). This is established using a well-known overspill argument using the fact that induction over \( \omega^\mathcal{M} \) holds in \((\mathcal{M}, S)\), where \( S \) is a \( \Sigma^\omega \)-satisfaction class given by Theorem 3.3. More specifically, since \( S \) satisfies Tarski’s compositional conditions for each \( \Sigma_n \)-formula (where \( n \in \omega \)), by overspill we can fix some nonstandard \( c \in \omega^\mathcal{M} \) such that \( S \) satisfies Tarski conditions for \( \Sigma_c \)-formulae. Next let \( \langle \varphi_i(x) : i \in \omega \rangle \) be a recursive enumeration in the real world of the formulae of a recursive type \( p(x) \) (involving finitely many parameters from \( \mathcal{M} \)), where \( p(x) \) is finitely realizable in \( \mathcal{M} \). This enumeration can be extended to some enumeration \( \langle \varphi_i(x) : i \in \omega^\mathcal{M} \rangle \) in \( \mathcal{M} \). For each \( i \in \omega^\mathcal{M} \) let

\[
\psi_i := \exists x \bigwedge_{j \leq i} \varphi_j(x).
\]

Then for every \( n \in \omega \), \((\mathcal{M}, S) \models \theta(n)\), where \( \theta(i) := (\psi_i \in \Sigma_c) \land S(\psi_i) \), and therefore by overspill, there is some nonstandard \( d \in \omega^\mathcal{M} \) such that \((\mathcal{M}, S) \models \theta(d)\). It is now easy to see (using the fact that \( S \) satisfies Tarski’s compositional clauses for all \( \Sigma_c \)-formulae) that \( p(x) \) is realized in \( \mathcal{M} \). \( \square \)

### 3.7. Corollary
A countable \( \omega \)-nonstandard model \( \mathcal{M} \models \text{ZFC} \) has an expansion to a model of \( \text{ZFI}_<\) iff \( \mathcal{M} \) is recursively saturated and \( \mathcal{M} \models \Lambda \).

**Proof.** The left-to-right direction follows from Corollary 3.6 and Theorem 4.1. The right-to-left direction follows from Theorem 4.1 and the resplendence property of countable recursively saturated models [Kay, Theorem 15.7]. \( \square \)

- In what follows \( M_X \) is the elementary submodel of \((\mathcal{M}, <_M)\) generated by \( X \), as in part \( j \) of Definition 2.1.1, thus the universe \( M_X \) of \( M_X \) consists of the elements of \( M \) that are definable in \((\mathcal{M}, <_M)\) with parameters from \( X \).

### 3.8. Theorem
Suppose \((\mathcal{M}, <_M, I) \) is an \( \omega \)-standard model of \( \text{ZFI}_<\). Then:

- \( \textbf{(a)} \) For each subset \( X \) of \( M \) that is definable in \((\mathcal{M}, <_M, I)\), \( M_X \) is definable in \((\mathcal{M}, <_M, I)\).

- \( \textbf{(b)} \) \( M_{I_1} \cong M_{I_2} \) for any cofinal subsets \( I_1 \) and \( I_2 \) of \( I \) that are definable in \((\mathcal{M}, <_M, I)\). Moreover, the isomorphism between \( M_{I_1} \) and \( M_{I_2} \) is definable in \((\mathcal{M}, <_M, I)\).

- \( \textbf{(c)} \) There is a nontrivial elementary embedding \( j : M_I \rightarrow M_I \) such that \( j \) is definable in \((\mathcal{M}, <_M, I)\).

- \( \textbf{(d)} \) \( M_I \) is a proper subset of \( M \).

- \( \textbf{(e)} \) \( \mathcal{M} \models \text{“}\theta^\# \text{ exists”}, \) in particular \( \mathcal{M} \models V \neq L \).

\( \text{Remarks.} \) Recall that if \( \mathcal{M} \) is resplendent, and \( \mathcal{M} \) has an elementary extension to a recursive (computable) theory \( T \) (such as \( \text{ZFI}_<\)) formulated in a language extending the language of \( \mathcal{M} \), then \( \mathcal{M} \) has an expansion to a model of \( T \).
(f) The core model $K^M$ of $\mathcal{M}$ satisfies “there is a proper class of almost Ramsey cardinals” (in the sense of \cite{VW}).

**Proof.** (a) can be easily verified with the help of Theorem 3.3.

To prove (b), first we observe that within $ZF(\mathcal{L})$ (for any $\mathcal{L}$) one can prove that if $I_1$ and $I_2$ are definable cofinal subsets of the class of ordinals, then there is a definable isomorphism $g : I_1 \rightarrow I_2$. By Theorem 2.2.2, $g$ lifts to an isomorphism $\hat{g} : \mathcal{M}_{I_1} \rightarrow \mathcal{M}_{I_2}$. Let $\hat{g} : \mathcal{M}_{I_1} \rightarrow \mathcal{M}_{I_2}$ be given by

$$\hat{g}(f(i_1, \cdots, i_n)) = f(g(i_1), \cdots, g(i_n)),$$

where $f$ is an $\mathcal{M}$-definable function. On the other hand, by Theorem 3.3(d), there is a full satisfaction predicate over $(\mathcal{M}, <_M)$ that is definable in $(\mathcal{M}, <_M, I)$, which together with (a) makes it clear that the proof of the fact that $\hat{g}$ is an isomorphism of $\mathcal{M}_{I_1}$ and $\mathcal{M}_{I_2}$ can be carried out within $(\mathcal{M}, <_M, I)$.

To see that (c) holds, we first observe that, reasoning in ZFI, there is a definable order-embedding $f : \text{Ord} \rightarrow I$, and thus there exists a definable enumeration $\langle i_\xi : \xi \in \text{Ord} \rangle$ of $I$, where $f(\xi) = i_\xi$. Therefore the map $h : I \rightarrow I$ given by $h(i_\xi) = i_{\xi+1}$ is an $(\mathcal{M}, I)$-definable order-preserving map whose range $I_0$ is a proper subset of $I$. By part (a) of Theorem 2.2.2, $h$ induces an elementary embedding $\hat{h}$ of $\mathcal{M}_I$ onto $\mathcal{M}_{I_0}$, where $\mathcal{M}_{I_0}$ is a proper elementary submodel of $\mathcal{M}_I$. Note that by Theorem 3.3, $\hat{h}$ is definable in $(\mathcal{M}, <_M, I)$. Thus $\hat{h}$ is the desired nontrivial elementary self-embedding $j$ of $\mathcal{M}_I$.

To verify (d), suppose $M_I = M$. Then by (c) there is a nontrivial elementary embedding $j : \mathcal{M} \rightarrow \mathcal{M}$ such that $j$ is $\mathcal{M}$-amenable. But Kunen’s venerable theorem \cite[Theorem 17.7]{Je} bars the existence of such an embedding $j$. Thus $M_I \subseteq M$.

Next we establish (e). The fact that there is an $\mathcal{M}$-amenable satisfaction class over $\mathcal{M}$ makes it clear that there is a cofinal subset $X \subseteq \text{Ord}^M$ such that $\mathcal{M}(\alpha) < \mathcal{M}$ for each $\alpha \in X$. Therefore for each $\alpha \in X$ the statement:

$$“I \cap L(\alpha) is a set of indiscernibles over (L(\alpha), \in)”$$

holds in $\mathcal{M}$. So by picking an element $\alpha \in X$ such that $\mathcal{M}$ satisfies “$I \cap L(\alpha)$ is uncountable”, we can deduce that $\mathcal{M}$ satisfies that $0^\#$ exists by a classical theorem of Silver \cite[Corollary 18.18]{Je}. Alternatively, one can put (c) together with Kunen’s theorem \cite[Theorem 18.20]{Je} that says that $0^\#$ exists iff the constructible universe admits a nontrivial elementary self-embedding. This is because within $(\mathcal{M}, <_M, I)$, there is an isomorphism between $L^M$ and $L^{M_I}$, and therefore if $j : M_I \rightarrow M_I$ is a nontrivial elementary embedding such that $j$ is definable in $(\mathcal{M}, <_M, I)$, then $j$ induces a nontrivial elementary self-embedding of $L^M$ that is $\mathcal{M}$-amenable.

The proof of (f) is based on a key result of Vickers and Welch \cite{VW}, which states that if there is an inner model $\mathcal{M}_0$ of a model $\mathcal{M}$ of ZFC, and an $\mathcal{M}$-amenable nontrivial elementary embedding $j : \mathcal{M}_0 \rightarrow \mathcal{M}$, then the core model $K^M$ of $\mathcal{M}$ satisfies “there is a proper class of almost Ramsey cardinals”. Note that by (d), $M_I$ is a proper elementary submodel of $\mathcal{M}$, and by (a) its universe $M_I$ is definable in $(\mathcal{M}, <_M, I)$, therefore if $c : M_I \rightarrow M_0$ is the collapsing map of $M_I$ onto an inner model $\mathcal{M}_0$ of $\mathcal{M}$, then $c^{-1} : M_0 \rightarrow M$ is a nontrivial elementary embedding that is clearly $\mathcal{M}$-amenable.

**3.9. Remark.** Parts (d), (e), and (f) of Theorem 3.8 can be strengthened, as explained below.

Part (d) holds also when $\mathcal{M}$ is $\omega$-nonstandard. To see this, suppose $M_I = M$ for $(\mathcal{M}, <_M, I) \models ZFI_<$, where $\mathcal{M}$ is $\omega$-nonstandard. By Theorem 3.3 there is a $\Sigma_\omega$-satisfaction class $S$ on $\mathcal{M}$ that is definable in $(\mathcal{M}, I)$. Consider the function

$$h : M \rightarrow \text{Ext}_M(\omega^M),$$

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where \( h \) is defined in \((\mathcal{M}, I)\) by \( h(m) := \) the (Gödel number of) the least \( L_{\text{Set}}\)-formula \( \varphi(x, \bar{y}) \) such that, as deemed by \( S \), \( m \) is defined by \( \varphi(x, \bar{7}) \) for some tuple \( \bar{7} \) of parameters from \( I \), i.e., \( S \) contains the sentences \( \varphi(m, \bar{7}) \) and \( \exists x \varphi(x, \bar{7}) \). Note that the set of standard elements of \( \omega^\mathcal{M} \) are definable in \((\mathcal{M}, I)\) as the set of \( i \in \omega \) such that \( i < j \in \omega \) for some \( j \) in the range of \( h \). Thus \((\mathcal{M}, I)\) is an \( \omega \)-nonstandard model of \( ZF_{<} \), in which the set of standard elements of \( \omega^\mathcal{M} \) is definable, which is impossible.

A straightforward modification of the proof of part (e) shows that the statement “\( r^\# \) exists for all \( r \subseteq \omega \)” holds in every \( \omega \)-standard model of \( ZF_{<} \).

Finally, by taking advantage of the diagonal indiscernibility property of \( I \), the proof of part (f) can be modified to show that if \((\mathcal{M}, <_M, I)\) is an \( \omega \)-standard model of \( ZF_{<} \), then for any \( m \in M \) there is an inner model \( M_0 \) of \( M \) such that \( m \in M_0 \), and for some \( M \)-amenable nontrivial elementary embedding \( j : M_0 \to M \), \( j(m) = m \). This shows that if \((\mathcal{M}, <_M, I)\) is an \( \omega \)-standard model of \( ZF_{<} \), then \( M \) exhibits “inner model reflection” in the sense of \( [BCFHRS] \), i.e., any first order property of \( M \) (parameters allowed) reflects to a proper inner model of \( M \). This result is a variant of a theorem of Vickers and Welch \( [VW\text{ Theorem 2.3}(i)] \) that derives inner model reflection from the existence of a proper class \( I \) of “good indiscernibles” for \((V, \in)\)\(^{10}\).

3.10. Corollary. No well-founded model \( \mathcal{M} \) of \( ZF \) that satisfies any of the conditions below has an expansion to a model of \( ZF_{<} \).

(a) \( \mathcal{M} \models V = L \).

(b) \( \mathcal{M} = (V(\kappa), \in) \), where \( \kappa \) is the first cardinal satisfying \( P(\kappa) \), and \( P(\kappa) \) is a large cardinal property consistent with \( V = L \), e.g., \( P(\kappa) = \text{“}\kappa \text{ is inaccessible/Mahlo/weakly compact/ineffable”} \).

4. WHAT ZF\(_{<} \) KNOWS ABOUT SET THEORY

In contrast to the previous section whose main focus was on the model-theoretic behavior of the theories \( ZF \) and \( ZF_{<} \), the main focus of this section is to use model-theoretic methods to gauge the proof-theoretic strength of these theories. As mentioned in the introduction, a simple compactness argument shows that \( ZF_{<} \) is consistent if there is a weakly compact cardinal. The main result of this section is Theorem 4.1, which pinpoints the set-theoretical strength of \( ZF_{<} \). Note that Theorem 4.1 shows that the consistency strength of \( ZF_{<} \) is roughly the consistency strength of the existence of an \( \omega \)-Mahlo cardinal, which is considerably below the consistency strength of the existence of a weakly compact cardinal. This calibration of the consistency strength of \( ZF_{<} \) also follows from part (b) of Theorem 5.5. As explained in Remark 4.8, Theorem 4.1 can be strengthened by adding two additional equivalent conditions to the five equivalent conditions of the theorem.

4.1. Theorem. The following are equivalent for an \( L_{\text{Set}}\)-sentence \( \varphi \):

(i) \( ZF_{<} \models \varphi \), where \( ZF_{<} \) is the subsystem of \( ZF_{<} \) axiomatized by:

\[
ZF(\langle) + GW + \text{Coll}(\langle, I) + \text{Cof}(I) + \text{Indis}(I).
\]

(ii) \( ZF_{<} \models \varphi \).

\(^{10}\) \( I \) is a set of good indiscernibles over a model \((\mathcal{M}, <_M) \models ZF(\langle) + GW \) if (1) \( I \) is a cofinal subset of \( \text{Ord}^\mathcal{M} \) that is \( M \)-amenable, (2) for each \( \alpha \in I \), \((\mathcal{M}(\alpha), <_{\mathcal{M}(\alpha)}) \prec (\mathcal{M}, <_M) \), and (3) \( I \) satisfies the diagonal indiscernibility scheme. In light of Theorem 3.2, \( I \) is a set of good indiscernibles over \((\mathcal{M}, <_M) \models ZF(\langle) + GW \text{ iff } (\mathcal{M}, <_M, I) \models ZF_{<} \) and (2) holds. Thus \( I \) is a set of good indiscernibles over \((\mathcal{M}, <_M) \models ZF(\langle) + GW \text{ iff } (\mathcal{M}, <_M, I) \models ZF_{<}^{\text{good}} \), where \( ZF_{<}^{\text{good}} \) is as in part (iii) of Theorem 4.1.
(iii) \( \text{ZFI}^\text{Good}_< \vdash \varphi \), where \( \text{ZFI}^\text{Good}_< := \text{ZFI}_< + \psi \), where \( \psi \) is the single sentence expressing:

\[
\forall \alpha, \beta \in I [\alpha \in \beta \rightarrow (V(\alpha), \in, <) \prec (V(\beta), \in, <)].
\]

(iv) \( \text{ZFC} + \Lambda \vdash \varphi \).

(v) \( \text{GBC} + \text{“Ord is weakly compact”} \vdash \varphi \).

**Proof.** Recall that Theorem 2.4.12 assures us of the equivalence of (iv) and (v). Since (i) \( \Rightarrow \) (ii) and (ii) \( \Rightarrow \) (iii) are both trivial, the proof of the theorem will be complete once we establish:

\[ (iii) \Rightarrow (iv) \Rightarrow (i). \]

To prove (iii) \( \Rightarrow \) (iv), suppose that for some \( \mathcal{L}_\text{Set} \)-sentence \( \varphi \) we have:

1. \( \text{ZFI}^\text{Good}_< \vdash \varphi \).

Assume on the contrary that \( \text{ZFC} + \Lambda + \neg \varphi \) is consistent. By Theorem 2.4.12 and the completeness theorem for first order logic, there is a model \( (\mathcal{M}_0, \mathfrak{x}_0) \models \text{GBC} + \text{“Ord is weakly compact”} \) such that:

2. \( \mathcal{M}_0 \models \neg \varphi \).

Since by Theorem 2.4.9 for each metatheoretic natural number \( n \geq 2 \),

\[
(\mathcal{M}_0, \mathfrak{x}_0) \models \text{Ord} \rightarrow (\text{Ord})^n_{2^c},
\]

there is an elementary extension \( (\mathcal{M}, \mathfrak{x}) \) of \( (\mathcal{M}_0, \mathfrak{x}_0) \) such that for some nonstandard \( c \in \omega^M \) we have:

3. \( (\mathcal{M}, \mathfrak{x}) \models \text{Ord} \rightarrow (\text{Ord})^n_{2^c} \).

Let \( <_M \) be a member of \( \mathfrak{x}_0 \) such that \( (\mathcal{M}, <_M) \models \text{GW} \). By Theorem 2.4.7(c) we can get hold of a full satisfaction class \( S \in \mathfrak{x} \) for \( (\mathcal{M}, <_M) \). Since \( S \) is \( \mathcal{M} \)-amenable, by using the reflection theorem within \( (\mathcal{M}, S) \), there is an \( (\mathcal{M}, S) \)-definable unbounded subset \( I_0 \) of \( \text{Ord}^M \) (in particular, \( I_0 \in \mathfrak{x} \)) such that:

4. For each \( \alpha \) in \( I_0 \) \( (\mathcal{M}(\alpha), <_{M_\alpha}, S_\alpha) \prec_{\Sigma_2(<, S)} (\mathcal{M}, <_M, S) \), where \( S_\alpha := S \cap M(\alpha) \).

Since the predicate “\( S \) is a full satisfaction class for \((V, \in, <)\)” is \( \Pi_2(<, S) \), by (4) \( S_\alpha \) is a full satisfaction class for \( (\mathcal{M}(\alpha), <_{M_\alpha}) \) for each \( \alpha \) in \( I_0 \). This fact, in turn, readily implies:

5. \( (\mathcal{M}, I_0) \models \forall \alpha, \beta \in I_0 [\alpha \in \beta \rightarrow (V(\alpha), \in, <) \prec (V(\beta), \in, <)] \).

On the other hand, it is easy to construct a recursive list of \( \mathcal{L}_\text{Set}(<) \)-formulae \( \langle \varphi_i(x_1, \ldots, x_i) : i < \omega \rangle \) such that the free variables of \( \varphi_i \) are among \( x_1, \ldots, x_i \). This enumeration can be naturally prolonged within \( \mathcal{M} \) so as to obtain an enumeration \( \langle \varphi_i(x_1, \ldots, x_i) : i < c \rangle \) of \( \mathcal{L}_\text{Set}(<) \)-formulae in the sense of \( \mathcal{M} \).

Next we define the following evaluation function \( e_S : [I_0]^c \rightarrow \{0, 1\}^c \) within \( (\mathcal{M}, S) \) by:

\[
e_S(\alpha_1, \ldots, \alpha_c) = \langle \| \varphi_i(\alpha_1, \ldots, \alpha_i) \|_S : i < c \rangle,
\]

where \( \alpha_1 < \cdots < \alpha_i \) and for all \( i < c \)

\[
\| \varphi_i(\alpha_1, \ldots, \alpha_i) \|_S = 1 \text{ iff } \varphi_i(\alpha_1, \ldots, \alpha_i) \in S.
\]

By (3) there is some \( I \in \mathfrak{x} \), such that \( I \subseteq I_0 \), \( I \) is homogeneous for \( e_S \) and \( I \) is unbounded in \( \text{Ord}^M \).

It is evident that \( I \) is a cofinal set of indiscernibles over \( (\mathcal{M}, <_M) \) that is \( \mathcal{M} \)-amenable. Thus, in light of (5) and the fact that \( I_0 \subseteq I \), \( (\mathcal{M}, <_M, I) \models \text{ZFI}^\text{Good}_< \), so by (1) \( \mathcal{M} \models \varphi \), which contradicts (2). This contradiction concludes the proof of (iii) \( \Rightarrow \) (iv).
The proof of (iv) ⇒ (i) of Theorem 4.1 relies on the following lemma, in which \( \mathcal{M} \) is a model of ZF(\(<\)) + GW, \( \mathcal{M}_J \) is the elementary submodel of \( \mathcal{M} \) generated by \( \mathcal{M} \)-definable functions (as in part (j) of Definition 2.1.1) and \( \mathcal{M}_{I+J} \) is the elementary extension of \( \mathcal{M}_I \) resulting from stretching \( I \) to the linear order \( I + J \), as in Theorem 2.2.2(b). Here \( I + J \) is the linear order on \( I \cup J \) in which the elements of \( J \) all exceed the elements of \( I \) (where \( I \) and \( J \) are disjoint). Thus, \( \mathcal{M}, \mathcal{M}_I, \) and \( \mathcal{M}_{I+J} \) are \( \mathcal{L}_{\operatorname{Set}}(\langle \cdot \rangle \rangle \)-structures that satisfy ZF(\(<\)) + GW.

4.2. Lemma. Suppose \( \mathcal{M} \) is a model of ZF(\(<\)) + GW that has an expansion \( (\mathcal{M}, I) \models ZF_{<\omega}^I \). Let \( (I, \langle, J) \) be a linear order without a minimum element that is disjoint from \( I \), and let \( \mathfrak{X} : = \operatorname{Cod}_{\mathcal{M}_I}(\mathcal{M}_{I+J}) \). Then the following hold:

(a) \( \mathcal{M}_I \prec_{\text{end}} \mathcal{M}_{I+J} \).

(b) \( I \) is downward cofinal in \( \mathcal{M}_{I+J} \backslash \mathcal{M}_I \), i.e., \( \forall x \in \mathcal{M}_{I+J} \setminus \mathcal{M}_I \exists j \in J (j <_{\mathcal{M}_{I+J}} x) \).

(c) \((\mathcal{M}_I, \mathfrak{X}) \models \text{GBC} \).

(d) \((\mathcal{M}_I, \mathfrak{X}) \models \text{“Ord is weakly compact”} \).

Proof. To prove (a), we note that by the Stretching Theorem 2.2.2(b), \( \mathcal{M}_{I+J} \) is an elementary extension of \( \mathcal{M}_I \), so the proof of (a) is complete once we verify that \( \mathcal{M}_{I+J} \) end extends \( \mathcal{M}_I \). For this purpose, since \( I \) is cofinal in the ordinals of \( \mathcal{M}_I \) it suffices to show that if \( f \) is an \( \mathcal{M} \)-definable function, where \( f \) is \((n+s)\)-ary, \( \overline{t} \in [I]^n \), and \( \overline{j} \in [J]^s \), then the following statement \((\nabla)\) holds for any \( i \in I \):

\[
(\nabla) \quad [\mathcal{M}_{I+J} \models f(\overline{t}, \overline{j}) \in V(i)] \implies f(\overline{t}, \overline{j}) \in \mathcal{M}_I.
\]

To establish \((\nabla)\), suppose:

(1) \( \mathcal{M}_{I+J} \models f(\overline{t}, \overline{j}) \in V(i) \) for some \( \overline{t} \in [I]^n \), \( \overline{j} \in [J]^s \), and \( i \in I \).

Let \( i_n = \max(\overline{t}) \). Putting (1) together with the assumption that \( \mathcal{M}_{I+J} \) is obtained by stretching \( I \) to \( I + J \) implies:

(2) \( \mathcal{M}, I \models \forall \overline{x} \in [I]^s \exists i_n < x_1 < \cdots < x_s \implies f(\overline{t}, x_1, \cdots, x_s) \in V(i) \).

By \( \mathcal{M} \)-amenability of \( I \), the Collection Scheme Coll(\( I \)) holds in \((\mathcal{M}, I)\), which coupled with (2) yields:

(3) \( \mathcal{M}, I \models \exists y \in V(i) \forall \alpha \in \text{Ord} \exists \overline{x} \in [I]^s (\alpha < x_1 < \cdots < x_s) \land f(\overline{t}, x_1, \cdots, x_s) = y \).

Since \( \mathcal{M}_I \prec \mathcal{M} \), by (3), we can find \( \overline{r}, \overline{t} \in [I]^2 \) with \( i_n < M k_1 < M \cdots < M k_s < M l_1 < M \cdots < M l_s \) such that:

(4) \( \mathcal{M}_I \models f(\overline{r}, \overline{t}) = f(\overline{t}, \overline{r}) \).

By combining (4) with the assumption that \( \mathcal{M}_{I+J} \) is obtained by stretching \( I \) to \( I + J \) we can conclude that \( f(\overline{t}, \overline{j}) = f(\overline{r}, \overline{t}) \in \mathcal{M}_I \), which shows that \((\nabla)\) holds, thus completing the proof of (a). Note that the assumption that \( J \) has no minimum element was not invoked in the proof of (a).

We next establish (b). In light of (a) it is sufficient to show:

(\(\nabla\)) If \( c \in M_{I+J} \) and \( c <_{M_{I+J}} j \) for each \( j \in J \), then \( c \in M_I \).

We will establish the following stronger form \((\nabla^+)\) of \((\nabla)\). In what follows \( f \) is an \((n+s)\)-ary \( \mathcal{M} \)-definable function, \( \overline{t} \in [I]^n \), \( \overline{j} \in [J]^s \) and \( i_n = \max(\overline{t}) \).

(\(\nabla^+)\) If \( f(\overline{t}, \overline{j}) \in M_{I+J} \), and for all \( j \in J M_{I+J} \models f(\overline{t}, \overline{j}) < j \), then \( f(\overline{t}, \overline{j}) < k' \) for any \( k' \in I \) such that \( i_n < M k' \).

To establish \((\nabla^+)\), suppose that for all \( j \in J \) \( M_{I+J} \models f(\overline{t}, \overline{j}) < j \). Since \( J \) has no minimum element, there is some \( j' \in J \) that is below \( j_1 = \min(\overline{j}) \), therefore:
(5) For all \( j \in J \), \( \mathcal{M}_{I+j} \models f(\overline{i}, j) < j' < j_1 \).

The fact that \( \mathcal{M}_{I+j} \) is the elementary extension of \( \mathcal{M}_I \) resulting from stretching \( I \) to \( I + J \) assures us that if we choose \( k' > i_a \) with \( k' \in I \) and some \( \overline{k} \in [I]^s \) with \( k' < k_1 = \min(\overline{k}) \), then:

(6) \( \mathcal{M}_I \models f(\overline{i}, \overline{k}) < k' \).

Thanks to (6), we can conclude that \( \mathcal{M}_{I+j} \models f(\overline{i}, \overline{j}) < k' \), thus \((\diamondsuit^+)\) holds. This concludes our verification of (b). Note that (b) implies that \( \mathcal{M}_{I+j} \backslash \mathcal{M}_I \) has no \(<_{\mathcal{M}_{I+j}}\)-minimum element.

To establish (c), we first claim:

\((\diamondsuit)\) If \( X \in \mathfrak{X} \), then \( X = D \cap M_I \) for some \( D \subseteq M \) such that \( D \) is parametrically \((\mathcal{M}, I)\)-definable.

To demonstrate \((\diamondsuit)\) let \( X \in \mathfrak{X} \), and choose \( a \in M_{I+j} \) such that \( X = \text{Ext}_{\mathcal{M}_{I+j}}(a) \). Then \( a = f(\overline{i}, \overline{j}) \) for some \( \mathcal{M} \)-definable \((n + s)\)-ary function \( f \), where \( \overline{i} \in [I]^n \), and \( \overline{j} \in [J]^s \). Thus

\[ X = \{ m \in M_I : \mathcal{M}_{I+j} \models f(\overline{i}, \overline{j}) \}. \]

Note that the veracity of the diagonal indiscernibility scheme in \( \mathcal{M}_{I+j} \) implies:

(7) For \( m \in M \), \( \mathcal{M}_{I+j} \models m \in f(\overline{i}, \overline{j}) \) iff there is some “sufficiently large” \( \overline{k} \in [I]^s \), \( \mathcal{M} \models m \in f(\overline{i}, \overline{k}) \),

where “sufficiently large” means that there is some \( u \in I \) such that \( \mathcal{M}(u) \) (i.e., \( V^\mathcal{M}(u) \)) contains \( \overline{i} \) and \( m \) and \( u < k_1 \) (recall that \( k_1 = \min(\overline{k}) \)). Thus (7) makes it clear that \( X = D \cap M_I \), where:

\[ D := \{ m \in M : (M, I) \models \exists u \in I \exists \overline{k} \in [I]^s \ u < k_1 \land \{ \overline{i}, m \} \subseteq V^\mathcal{M}(u) \land m \in f(\overline{i}, \overline{k}) \}. \]

This concludes the verification of \((\diamondsuit)\). Note that \((\diamondsuit)\) readily implies that for each \( X \in \mathfrak{X} \) \((\mathcal{M}_I, X)\) satisfies the following weak form of Coll(X):

(8) \((\mathcal{M}_I, X) \models \forall v \left( \forall x \in v \exists y \left( \langle x, y \rangle \in X \right) \rightarrow \left( \exists w \forall x \in v \exists y \in w \left( \langle x, y \rangle \in X \right) \right) \].

We will next verify \((\bigtriangleup)\) below, which together with (8) will allow us to conclude that \((\mathcal{M}_I, X)\) satisfies the full scheme Coll(X).

\((\bigtriangleup)\) If \( X \in \mathfrak{X} \) and \( Y = \{ m \in M_I : \mathcal{M}_{I+j} \models \varphi(m, p, X) \} \) for some \( \mathcal{L}_{\text{Set}}(\langle, X)\)-formula \( \varphi(x, y, X) \) and some parameter \( p \in M \), then \( Y \in \mathfrak{X} \).

The proof of \((\bigtriangleup)\) is carried out by induction on the complexity of \( \varphi \). We may assume that the logical connectives consist of \( \{ -, \lor, \exists \} \). The atomic case and the Boolean cases go through smoothly (since \( \mathfrak{X} \) is readily seen to be closed under complements and unions), but the existential case requires a nontrivial argument. To handle the existential case, we need to show:

\((*)\) If \( Y := \{ x \in M_I : \exists y \in M_I \langle x, y \rangle \in X \} \), then \( Y \in \mathfrak{X} \).

Let \( r \in M_{I+j} \) such that \( X = \text{Ext}_{\mathcal{M}_{I+j}}(r) \cap M_I \). To verify \((*)\), it is sufficient to show \((**\ast)\) below:

\((**\ast)\) There is some \( c \in M_{I+j} \) such that \( \forall x, y \in M_{I+j} \mathcal{M}_{I+j} \models \langle x, y \rangle \in r \land y < c \) iff \( y \in M_I \).

To see that \((**\ast) \Rightarrow (*)\), choose \( d \) in \( \mathcal{M}_{I+j} \) such that \( \mathcal{M}_{I+j} \models d = \{ x : \exists y < c \langle x, y \rangle \in r \} \), thus:

\[ Y = \text{Ext}_{\mathcal{M}_{I+j}}(d) \cap M_I, \]

which makes it clear that \( Y \in \mathfrak{X} \). In order to establish \((**\ast)\), choose a function \( g \) in \( M_{I+j} \) such that \( \mathcal{M}_{I+j} \) thinks that the domain of \( g \) is the same as the domain \( \text{Dom}(r) \) of \( r \), where \( \text{Dom}(r) = \{ x : \exists y \langle x, y \rangle \in r \} \) and
\[ M_{I+J} \models \forall x \in \text{Dom}(r) \ [g(x) = \triangleleft \text{least } y \text{ such that } \langle x, y \rangle \in r]. \]

Choose an \( M \)-definable function \( f \), where \( \vec{t} \in [I]^n \), and \( \vec{J} \in [J]^s \) such that \( g = f(\vec{t}, \vec{J}) \), and let \( G := \text{Ext}_{M_{I+J}}(g) \cap M_I \).

Note that \( Y = \{ x \in M_I : g(x) \in M_I \} \). We will establish (***) by showing that there is a lower bound \( c \in M_{I+J} \) for \( \{ g(x) : x \in M_I, \ g(x) \notin M_I \} \) (in the sense of \( <_{M_{I+J}} \)). We may assume that \( I \) is cofinal in \( (M_I, <_M) \) (by replacing \( <_M \), if necessary, with \( <_{M_I} \), as in part (k) of Definition 2.1.1). Coupled with (8), we may conclude:

(9) \( \forall k \in I \ \exists k' \in I \) such that \( (M_I, G) \models \forall x < k \ (x \in \text{Dom}(G) \rightarrow G(x) < k') \).

Given elements \( k \) and \( k' \) of \( I \), let

\[ Z_{k,k'} := \{ y \in M_{I+J} : M_{I+J} \models \exists x < k, \ y = g(x) > k' \} . \]

Choose any \( k \in I \). By (9) there is some \( k' \in I \) such that \( Z_{k,k'} \subseteq M_{I+J} \setminus M_I \). Reasoning in \( M_{I+J} \), let \( u \in M_{I+J} \) be the \( <_M \)-least element of \( Z_{k,k'} \). By part(b) of Lemma 4.2, there is some \( j' \in J \) such that \( j' < u \) and therefore \( j' \) is a strict lower bound for \( Z_{k,k'} \). Thus:

(10) \( M_{I+J} \models \forall x < k \ \forall y \ [(y = g(x) > k') \rightarrow j' < y] \).

Recall that \( g = f(\vec{t}, \vec{J}) \). So (10) states that \( M_{I+J} \) satisfies a particular first order statement with parameters \( k, k', j', \vec{t}, \vec{J} \), and \( \vec{T} \) (all of which are in \( J \)), which coupled with the indiscernibility property of \( I \), shows that (11) holds for any \( k < k' \) in \( I \), as long as \( k \) is above \( \vec{t}_n \). This shows that \( j' \) serves as the element \( c \) in (**), thus concluding the verification of (\( \triangle \)).

Thanks to (8) and (\( \triangle \)) we have:

(11) \( (M_I, X) \models \text{Coll}(X) \).

On the other hand, each member of \( X \) is clearly piecewise coded in \( M_I \) since \( M_I \) is a rank-extension of \( M_{I+J} \) (thanks to (a) and the fact that elementary end extensions of models of ZF are rank extensions). As pointed out in 2.2.2(g), the piecewise codability of \( X \) together with (11) allows us to conclude that \( (M_I, X) \models \text{ZF}(X) \). Therefore thanks to the fact that finitely many members of \( X \) can be coded by a single member of \( X \), we have:

(12) \( (M_I, X_1, ..., X_n) \models \text{ZF}(X_1, ..., X_n) \) for any finite subset \( \{ X_1, ..., X_n \} \) of \( X \).

In light of Remark 2.4.2, (\( \triangle \)) and (12) make it clear that \( (M_I, X) \models \text{GBC} \), thus concluding the proof of (c).

Finally, we turn to establishing (d). Suppose \( \tau \) is an Ord-tree (in the sense of \( M_I \)) coded in \( X \). Thanks to the existence of the global well-ordering \( <_M \) in \( X \) we may assume without loss of generality that \( \tau = (M_I, \triangleleft) \) for some tree-ordering relation \( \triangleleft \) coded in \( X \). Fix \( r \in M_{I+J} \) such that

\[ \text{Ext}_{M_{I+J}}(r) \cap M = \{ (x, y) \in M_I : x \triangleleft y \} . \]

Within \( M_{I+J} \) let \( k \) be the field of \( r \), i.e., the set of elements that occur as the first or second coordinates of an ordered pair in \( r \). Without loss of generality we may assume that every element of \( r \) is an ordered pair from the point of view of \( M_{I+J} \). Consider the relational structure \( \tau^* := (k, r) \in M_{I+J} \). Within \( M_{I+J} \) for each ordinal \( \alpha \), let \( \tau^*(\alpha) \) be the initial segment of \( \tau \) consisting of elements of \( \tau^* \) whose rank (in the tree \( \tau^* \)) is at most \( \alpha \), and let \( \theta(x) \) be the following formula that expresses:

\[ \text{"} x \in \text{Ord and } \tau^*(x) \text{ is a well-founded tree"}. \]
The assumption that \((M_\tau, \tau) \models \text{"\(\tau\) is a well-founded tree of height \(\text{Ord}\)"} implies:

(12) \(M_{I+J} \models \theta(\alpha)\) for all \(\alpha \in \text{Ord}^{M_I}\).

Recall that (c) implies that \(\text{Ord}^{M_{I+J}} \setminus M_I\) has no least element. Therefore (12) assures us via a simple overspill argument there is some \(\beta \in \text{Ord}^{M_{I+J}}\) such that \(M_{I+J} \models \theta(\beta)\). This shows that the initial segment \(\tau(\beta)\) of \(\tau^*\) in \(M_{I+J}\) properly end extends \(\tau\), i.e., \(\tau(\beta)\) does not contain any new elements \(<\)-below the elements of \(\tau\). So we can construct a branch \(B\) of \(\tau\) such that \(B \in \mathcal{X}\) by considering the elements below a member of \(\tau(\beta)\) whose height is above \(\text{Ord}^M\). More specifically, choose \(t \in \text{Ext}_{M_{I+J}}(\tau(\beta)) \setminus M_I\), and define the desired branch \(B \in \mathcal{X}\) of \(\tau\) by

\[
B := \{m \in M_I : (m, t) \in r\}.
\]

\[\square\ (\text{Lemma 4.2})\]

With Lemma 4.2 at hand, we are now in a position to smoothly verify the direction (iv) \(\Rightarrow\) (i) of Theorem 4.1. Suppose \(\text{ZFC} + \Lambda \vdash \varphi\), and assume on the contrary that \(\text{ZFI}^< + \neg \varphi\) is consistent, and therefore there is a countable \((M, I)\) of \(\text{ZFI}^<_p\) such that \(M \models \neg \varphi\). Let \(J\) be any linear order with no minimum element that is disjoint from \(M\), and let \(M_{I+J}\) be the elementary extension of \(M_I\) resulting from stretching \(I\) to the linear order \(I + J\). By Lemma 4.2 \(M_{I+J}\) is an elementary extension of \(M_I\) and \((M_I, \mathcal{X}) \models \text{GBC} + \text{"Ord is weakly compact"}\). So by Theorem 2.4.12 \(M_I\) satisfies \(\text{ZFC} + \Lambda\), which in light of the fact that \(M_I < M\) implies that \(\varphi\) holds in \(M\), contradiction. \(\square\ (\text{Theorem 4.1})\)

4.3. Remark. It is not clear whether the scheme \(\Lambda\) is provable in \(\text{ZFCI}\) (i.e., \(\text{ZFI}\) plus the axiom of choice). However, note that by part (b) of Theorem 3.2 for any \((M, I) \models \text{ZFI}\), and any \(p \in M\), a tail of \(I\) is indiscernible in \(\text{HOD}^M(p)\). Together with the fact that there is a well-ordering of \(\text{HOD}^M(p)\) that is parametrically definable in \(M\), one can use the strategy of the (iv) \(\Rightarrow\) (i) direction of the proof of Theorem 4.1 so as to show that if \((M, I) \models \text{ZFI}\), and \(m \in M\), then \(\text{HOD}^M(m) \models \Lambda\).

4.4. Remark. The proof of Theorem 4.1 makes it clear that the following hold:

(a) If \((M, \mathcal{X}) \models \text{GBC} + \text{"Ord is weakly compact"}, and \(M\) is \(\omega\)-nonstandard, then \(M\) has an expansion to a model of \(\text{ZFI}^<_p\).

(b) If \((M, I) \models \text{ZFI}^<_p\), then the elementary submodel \(M_I\) of \(M\) has an expansion to a model of \(\text{GBC} + \text{"Ord is weakly compact"}\).

Next we define the extensions \(\text{ZFI}^<_p\) and \(\text{ZFI}^<_\omega\) of \(\text{ZFI}^<_p\), which despite their powerful appearance, turn out to be rather mild extensions of \(\text{ZFI}^<_p\).

4.5. Definition. The theory \(\text{ZFI}^<_\omega\) is the union of the theories \(\text{ZFI}^<_k\) for \(1 \leq k \leq \omega\), where \(\text{ZFI}^<_k\) is formulated in the language \(\mathcal{L}_k = \mathcal{L}_{\text{Set}} \cup \{I_j(x) : j < k\}\), and each \(I_j(x)\) is a unary predicate. The axioms of \(\text{ZFI}^<_k\) are obtained from the axioms of \(\text{ZFI}^<_p\) simply by renaming \(I\) as \(I_0\). The axioms of \(\text{ZFI}^<_k\) consist of the union of the axioms of \(\text{ZFI}^<_k\) with the following four groups of sentences:

(1\(_{k+1}\)) \(\text{ZFC}(\mathcal{L}_{k+1})\);

(2\(_{k+1}\)) The sentence \(\text{Cof}(I_k)\) expressing "\(I_k\) is a cofinal subclass of the class of ordinals"; and

(3\(_{k+1}\)) The scheme \(\text{Indis}_\varphi(I_k) = \{\text{Indis}_\varphi(I_k) : \varphi\) is a formula of \(\mathcal{L}_k\}\) ensuring that \(I_k\) is a class of order indiscernibles for the structure \((\mathcal{V}, \in, <)^j_{j < k}\). More explicitly, for each \(n\)-ary formula \(\varphi(v_1, \ldots, v_n)\) in the language \(\mathcal{L}_k\), \(\text{Indis}_\varphi(I_k)\) is the following sentence:

\[
\forall x_1 \in I_k \cdots \forall x_n \in I_k \forall y_1 \in I_k \cdots \forall y_n \in I_k
\]

\[
[(x_1 < \cdots < x_n) \land (y_1 < \cdots < y_n) \rightarrow (\varphi(x_1, \ldots, x_n) \leftrightarrow \varphi(y_1, \ldots, y_n))].
\]
The sentence asserting that $I_k$ is subclass of $I_{k-1}$ (for $k \geq 1$).

• Thus $\text{ZF}^k_\prec$ bears the same relation to $\text{ZF}^k$ that $\text{ZF}^k_\prec$ bears to $\text{ZF} + \text{G}W$, i.e., for $1 \leq k < \omega$, a model $(M, <_M)$ of $\text{ZF} + \text{G}W$ has an expansion to a model of $\text{ZF}^k_\prec$ if and only if there is a nested sequence $I_0 \supseteq \cdots \supseteq I_k$ of cofinal subsets of $\text{Ord}^M$ such that $(I_0, \in)$ is indiscernible over $(M, <_M)$, $(I_1, \in)$ is indiscernible over $(M, <_M, I_0)$, \ldots, and $(I_k, \in)$ is indiscernible over $(M, <_M, I_0, \cdots, I_{k-1})$.

4.6. Theorem. Suppose $\varphi$ is a sentence in the language $L_{\text{Set}}$, then:

$$\text{ZF}^k_\prec \models \varphi \iff \text{GBC+ "Ord is weakly compact" \models \varphi}.$$  

Proof. Note that the right-to-left direction of the above equivalence is an immediate consequence of $(iii) \Rightarrow (i)$ of Theorem 4.1. The left-to-right direction of the above equivalence is an elaboration of the proof of $(ii) \Rightarrow (iii)$ of Theorem 4.1. More explicitly, it suffices to show that for any nonzero $k \in \omega$, if $(M, \mathfrak{X}) \models \text{GBC+ "Ord is weakly compact"}$, then a sufficient condition for $M$ to have an expansion to a model of $\text{ZF}^k_\prec$ is that there is a nonstandard $c \in \omega^M$ such that:

1. $(M, \mathfrak{X}) \models \text{Ord} \rightarrow (\text{Ord})^c.$

By the reasoning of the proof of $(iii) \Rightarrow (iv)$ of Theorem 4.1 using (1) we can find some $<_M$ in $\mathfrak{X}$ and $I_0 \in \mathfrak{X}$ such that $(M, <_M, I_0) \models \text{ZF}^k_\prec$. Let $(\varphi_i(x_1, \cdots, x_i))_{i < \omega}$ be a recursive list of $L_{\text{Set}}(\prec, I_0)$-formulae $(\varphi_i(x_1, \cdots, x_i))_{i < \omega}$ such that the free variables of $\varphi_i$ are among $x_1, \cdots, x_i$, and let $(\varphi_i(x_1, \cdots, x_i))_{i < c}$ be an extension of this enumeration in $M$. Fix a full satisfaction class $S \in \mathfrak{X}$ for $(M, <_M, I_0)$ and let $e_S : \{0, 1\}^c \rightarrow [\text{Ord}]^c$ within $(M, S)$ by:

$$e_S(a_1, \cdots, a_c) = (\|\varphi_i(a_1, \cdots, a_i)\|_S : i < c),$$

where $a_1 < \cdots < a_i$ and for all $i < c$

$$\|\varphi_i(a_1, \cdots, a_i)\|_S = 1 \iff \varphi_i(a_1, \cdots, a_i) \in S.$$  

By (1) there is some $I_1 \in \mathfrak{X}$ with $I_1 \subseteq I_0$ such that $I_1$ is homogeneous for $e$ and unbounded in $\text{Ord}^M$. It is evident that $(M, <_M, I_0, I_1) \models \text{ZF}^2_\prec$. By repeating this argument we can thus obtain an expansion of $M$ that satisfies $\text{ZF}^k_\prec$ for any desired nonzero $k \in \omega$.

4.7. Theorem. If $\widehat{M} = (M, <_M, I_k)_{1 \leq k < \omega}$ is a model of $\text{ZF}^\omega_\prec$ and $\mathfrak{X}$ is the collection of parametrically $\widehat{M}$-definable subsets of $M$, then $(M, \mathfrak{X}) \models \text{GBC+ "Ord is weakly compact"}.$

Proof. It should be clear that $(M, \mathfrak{X}) \models \text{GBC}$. To verify that “Ord is weakly compact” holds in $(M, \mathfrak{X})$, by Theorem 2.4.9 it suffices to verify that the partition relation $\text{Ord} \rightarrow (\text{Ord})^2_\prec$ holds in $(M, \mathfrak{X})$. But this is easy, since if for some $F \in \mathfrak{X}$ we have:

$$(M, F) \models F : [\text{Ord}]^2 \rightarrow \{0, 1\},$$

then $F$ is definable in $(M, <_M, I_k)_{1 \leq k \leq m}$ for some $m \in \omega$, and therefore $I_{m+1}$ is proper class that is an unbounded $F$-homogeneous member of $\mathfrak{X}$, as desired.

4.8. Remark. If the model $\widehat{M}$ as in Theorem 4.7 is recursively saturated, then the proof of Theorem 4.1 of $L_{\Sigma_2}$ shows that the model $(M, \mathfrak{X})$ in the statement of Theorem 4.7 also satisfies the scheme $\Sigma_1^1$-AC (and therefore the scheme $\Delta_1^1$-CA). Together with Theorem 4.6, this shows that Theorem 4.1 can be strengthened by asserting that the following two conditions $(vi)$ and $(vii)$ on an $L_{\text{Set}}$-sentence $\varphi$ are also equivalent to conditions $(i)$ through $(v)$ of that theorem.
(vi) $\text{ZFI}^\kappa_\prec \vdash \varphi$

(vii) GBC+ “Ord is weakly compact” $+ \Sigma^1_1$-AC $\vdash \varphi$

The next result shows that a statement that one might expect to be provable in the theory GBC+ “Ord is weakly compact” is actually unprovable in that theory.

4.9. Theorem. If the theory GBC+ “Ord is weakly compact” is consistent, then it does not prove the statement $\theta = \forall m, n \in \omega (\text{Ord} \rightarrow (\text{Ord})^\omega_m)$.

Proof. Let $\langle \varphi_i(x_1, \ldots, x_i) : i < \omega \rangle$ be a recursive list of $\mathcal{L}_{\text{Set}}(\prec)$-formulae such that the free variables of $\varphi_i$ are among $x_1, \ldots, x_i$. For each $n \in \omega$ let $T_n$ be the fragment of ZF$_\prec$ whose axioms consist of ZF($\mathcal{L}$) + GW for $\mathcal{L} = \mathcal{L}_{\text{Set}}(I, \prec, \cdot)$ and Cof($I$) and sentences Indisc$_{\varphi_i}$ for $i \leq n$. We next prove a key lemma.

4.10. Lemma. GBC + $\theta \vdash \forall n \in \omega \text{Con}(T_n)$.

Proof. Let $(\mathcal{M}, \mathfrak{X}) \models \text{GBC+ } \theta$. Then in particular $(\mathcal{M}, \mathfrak{X})$ satisfies $\forall n \in \omega (\text{Ord} \rightarrow (\text{Ord})^\omega_m)$ . Given any fixed $n \in \omega^\mathcal{M}$ and arguing in $(\mathcal{M}, \mathfrak{X})$, we will show the consistency of $T_n$. By Theorem 2.4.7(c) there is a full satisfaction class $S \in \mathfrak{X}$ for $\mathcal{M}$, which can use together with $\text{Ord} \rightarrow (\text{Ord})^\omega_m$ to get hold of an unbounded homogeneous set $I \in \mathfrak{X}$ for the map $e^n_S : [\text{Ord}]^\omega_n \rightarrow \{0, 1\}^n$ that is defined within $(\mathcal{M}, S)$ by:

$$e^n_S(\alpha_1, \ldots, \alpha_n) = (\llbracket \varphi_i(\alpha_1, \ldots, \alpha_i) \rrbracket_S : i < n),$$

where $\alpha_1 < \cdots < \alpha_i$ and for all $i < n$

$$\llbracket \varphi_i(\alpha_1, \ldots, \alpha_i) \rrbracket_S = 1 \text{ iff } \varphi_i(\alpha_1, \ldots, \alpha_i) \in S.$$ 

Clearly $(\mathcal{M}, I) \models T_n$. By Theorem 2.4.7(c) there is a full satisfaction predicate $S$ for $(\mathcal{M}, I)$ such that $S \in \mathfrak{X}$, which shows that $\text{Con}(T_n)$ holds in $\mathcal{M}$. \hfill {\flushright $\square$} (Lemma 4.10)

By Lemma 4.10 and compactness, $\text{Con}(\text{ZFI}_\prec)$ is provable in GBC + $\theta$. Since Theorem 4.1 is readily verifiable in ZFC, the formal consistency of GBC + “Ord is weakly compact” is provable in GBC + $\theta$. In light of Gödel’s second incompleteness theorem, the proof is complete. \hfill {\flushright $\square$} (Theorem 4.9)

5. INTERPRETABILITY ANALYSIS OF ZFI$_\prec$

In this section we study ZFI and ZFI$_\prec$ through the lens of interpretability theory, a lens that brings both the semantic and syntactic features of the theories under its scope into a finer focus. We review some relevant interpretability-theoretic preliminaries before presenting our results.

5.1. Definitions. Suppose $U$ and $V$ are first order theories, and for the sake of notational simplicity, let us assume that $U$ and $V$ are theories that support a definable pairing function. We use $\mathcal{L}_U$ and $\mathcal{L}_V$ to respectively designate the languages of $U$ and $V$.

(a) An interpretation $\mathcal{I}$ of $U$ in $V$, written:

$$\mathcal{I} : U \rightarrow V,$$

is given by a translation $\tau$ of each $\mathcal{L}_U$-formula $\varphi$ into an $\mathcal{L}_V$-formula $\varphi^\tau$ with the requirement that $V \vdash \varphi^\tau$ for each $\varphi \in U$, where $\tau$ is determined by an $\mathcal{L}_V$-formula $\delta(x)$ (referred to as a domain formula), and a mapping $P \mapsto A_P$ that translates each $n$-ary $\mathcal{L}_U$-predicate $P$ into some $n$-ary $\mathcal{L}_V$-formula $A_P$. The translation is then lifted to the full first order language in the obvious way by making it commute with propositional connectives, and subject to:
\[(\forall x\varphi)^\mathcal{I} = \forall x(\delta(x) \rightarrow \varphi^\tau) \text{ and } (\exists x\varphi)^\mathcal{I} = \exists x(\delta(x) \wedge \varphi^\tau).\]

- Note that each interpretation \(\mathcal{I} : U \rightarrow V\) gives rise to an inner model construction that **uniformly builds a model** \(\mathcal{M}^\mathcal{I} \models \forall \) for any \(\mathcal{M} \models \forall\).

(b) \(U\) is interpretable in \(V\) (equivalently: \(V\) interprets \(U\)), written \(U \preceq V\), iff there is an interpretation \(\mathcal{I} : U \rightarrow V\). \(U\) is locally interpretable in \(V\), written \(U \preceq_{\text{loc}} V\) if \(U_0 \preceq V\) for every finitely axiomatizable subtheory \(U_0\) of \(U\).

(c) \(U\) and \(V\) are mutually interpretable when \(U \preceq V\) and \(V \preceq U\).

(d) \(U\) is a retract of \(V\) iff there are interpretations \(\mathcal{I}\) and \(\mathcal{J}\) with \(\mathcal{I} : U \rightarrow V\) and \(\mathcal{J} : V \rightarrow U\), and a binary \(U\)-formula \(F\) such that \(F\) is, \(U\)-verifiably, an isomorphism between \(\text{id}_U\) (the identity interpretation on \(U\)) and \(\mathcal{J} \circ \mathcal{I}\). In model-theoretic terms, this translates to the requirement that the following holds for every \(\mathcal{M} \models \forall\):

\[F^\mathcal{M} : \mathcal{M} \xrightarrow{\cong} \mathcal{M}^* := (\mathcal{M}^\mathcal{J})^\mathcal{I}.\]

(e) \(U\) and \(V\) are bi-interpretable iff there are interpretations \(\mathcal{I}\) and \(\mathcal{J}\) as above that witness that \(U\) is a retract of \(V\), and additionally, there is a \(V\)-formula \(G\), such that \(G\) is, \(V\)-verifiably, an isomorphism between \(\text{id}_V\) and \(\mathcal{J} \circ \mathcal{I}\). In particular, if \(U\) and \(V\) are bi-interpretable, then given \(\mathcal{M} \models \forall\) and \(\mathcal{N} \models \forall\), we have

\[F^\mathcal{M} : \mathcal{M} \xrightarrow{\cong} \mathcal{M}^* := (\mathcal{M}^\mathcal{J})^\mathcal{I} \text{ and } G^\mathcal{N} : \mathcal{N} \xrightarrow{\cong} \mathcal{N}^* := (\mathcal{N}^\mathcal{I})^\mathcal{J}.\]

(f) The above notions can also be localized at a pair of models. Suppose \(\mathcal{N}\) is an \(L_U\)-structure and \(\mathcal{M}\) is an \(L_V\)-structure. We say that \(\mathcal{N}\) is parametrically interpretable in \(\mathcal{M}\), written \(\mathcal{N} \preceq_{\text{par}} \mathcal{M}\) (equivalently: \(\mathcal{M} \supseteq_{\text{par}} \mathcal{N}\)) iff the universe of discourse of \(\mathcal{N}\), as well as all the \(\mathcal{N}\)-interpretations of \(L_U\)-predicates are \(\mathcal{M}\)-definable. Similarly, we say that \(\mathcal{M}\) and \(\mathcal{N}\) are parametrically bi-interpretable if there are parametric interpretations \(\mathcal{I}\) and \(\mathcal{J}\), together with an \(\mathcal{M}\)-definable \(F\) and an \(\mathcal{N}\)-definable map \(G\) such that:

\[F^\mathcal{M} : \mathcal{M} \xrightarrow{\cong} \mathcal{M}^* := (\mathcal{M}^\mathcal{J})^\mathcal{I} \text{ and } G^\mathcal{N} : \mathcal{N} \xrightarrow{\cong} \mathcal{N}^* := (\mathcal{N}^\mathcal{I})^\mathcal{J}.\]

(g) A **sequential theory** is a theory equipped with a ‘\(\beta\)-function’ for handling finite sequences of objects in the domain of discourse.

The following theorems are classical. Theorem 5.2 was first proved for PA by Mostowski. His argument was later generalized by Montague as in Theorem 5.2 below. In part (b) of the theorem, \(L_{\text{Arith}}\) is the usual language of arithmetic \(+, -, \cdot, 0, 1\), and for \(L \supseteq L_{\text{Arith}}\), \(\text{PA}(L)\) is the natural extension of PA in which \(L\)-formulae can appear in the scheme of induction.

5.2. **Theorem.** (Montague) If \(T\) is a sequential theory and \(T\) can prove the induction scheme over its ambient set of natural numbers, then \(T\) is a reflexive theory, i.e., \(T\) proves the formal consistency of each of its finite subtheories. In particular:

(a) For all \(L \supseteq L_{\text{Arith}}\), every extension (in the same language) of \(\text{PA}(L)\) is reflexive.

(b) For all \(L \supseteq L_{\text{Set}}\), every extension (in the same language) of \(\text{Z}(L)\) is reflexive, where \(\text{Z}(L)\) is Zermelo set theory, as in Definition 2.1.1(e).

5.3. **Theorem.** (Orey’s Compactness Theorem) If \(U\) is a reflexive sequential theory, and \(V \preceq_{\text{loc}} U\) for some recursively enumerable theory \(V\), then \(V \preceq U\).

5.4. **Theorem.** \(\text{GB} \not\subseteq \text{ZF}\). Similarly, \(\text{GBC}\) is not interpretable in \(\text{ZF}(<) + \text{GW}\).
We are now ready to present the new results of this section. In part (b) of Theorem 5.5, $\Lambda^-$ is the subset of $\Lambda$ consisting of sentences of the form “there is an $n$-Mahlo cardinal” for each metatheoretic $n \in \omega$, as in part (c) of Definition 2.4.12.

5.5. Theorem. (Relative interpretability results)

(a) GBC + “Ord is weakly compact” is not interpretable in $\text{ZFI}_\prec$.

(b) The theories $\text{ZFC} + \Lambda^-$, $\text{ZFC} + \Lambda$ and $\text{ZFI}_\prec$ are pairwise mutually interpretable.

(c) $\text{ZFI}_\prec$ is interpretable in GBC + “Ord is weakly compact”.

Proof. The proof of (a) combines Theorem 4.1 together with the strategy that proves Theorem 5.4. More specifically, since $\text{ZFI}_\prec$ is a reflexive theory (by Theorem 5.2(a)), and the theory GBC + “Ord is weakly compact” is finitely axiomatizable, the interpretability of GBC + “Ord is weakly compact” in $\text{ZFI}_\prec$ would imply that $\text{ZFI}_\prec$ proves the $\text{L}_{\text{set}}$-sentence expressing the formal consistency of GBC + “Ord is weakly compact”, which in light of Theorem 4.1 contradicts Gödel’s second incompleteness theorem.

To prove (b), first recall that by part (c) of Theorem 2.4.11 $\text{ZFC} + \Lambda^-$ and $\text{ZFC} + \Lambda$ are mutually interpretable. Also note that since Theorem 4.1 assures us that $\text{ZFI}_\prec \vdash \Lambda$, the identity interpretation serves as a witness to the interpretability of $\text{ZFC} + \Lambda$ within $\text{ZFI}_\prec$. So the proof of (b) will be complete once we establish the interpretability of $\text{ZFI}_\prec$ within $\text{ZFC} + \Lambda$. Towards this goal, thanks to Orey’s Compactness Theorem 5.3, it will suffice to show that every finite subtheory of $\text{ZFI}_\prec$ is interpretable in $\text{ZFC} + \Lambda$. Indeed we will show that for each $n \in \omega$, $\text{ZFC} + \Lambda$ can interpret the subtheory $T_n$ of $\text{ZFI}_\prec$, where $T_n$ is the same theory as in the proof of Theorem 4.9. Fix some nonzero $n \in \omega$, and reasoning within $\text{ZFC} + \Lambda$, we consider the class function $F : [\text{Ord}]^n \to \{0,1\}^n$, where

$$F(\alpha_1, \cdots, \alpha_n) = (\parallel \varphi_i(\alpha_1, \cdots, \alpha_i) \parallel : 1 \leq i \leq n),$$

where for all $i < n$

$$\parallel \varphi_i(\alpha_1, \cdots, \alpha_i) \parallel = 1 \text{ iff } \varphi_i(\alpha_1, \cdots, \alpha_i).$$

Note that if $F(\alpha_1, \cdots, \alpha_n) = F(\beta_1, \cdots, \beta_n)$, then $\varphi_i(\alpha_1, \cdots, \alpha_i) \leftrightarrow \varphi_i(\beta_1, \cdots, \beta_i)$ whenever $1 \leq i \leq n$. Let $\tau_F$ be the Ord-tree as in Lemma 2.4.4 whose cofinal branches are end-homogeneous proper classes for $F$, i.e., not dependent on the $n$-th coordinate of any increasing chain of length $n$. Let $\kappa_1$ be an $m$-Mahlo cardinal such that $V(\kappa_1)$ is a $\Sigma_m$-elementary submodel of the universe, where $m \geq n$, each $\varphi_i$ is in $\Sigma_m$, and also $m$ is large enough so that the following statement is in $\Sigma_m$:

“$\tau_F$ is an Ord-tree, and the value of $F$ on any increasing chain in $\tau_F$ of length $n$ is independent of its $n$-th component”.

Choose any ordinal $\lambda$ above $\kappa_1$ and let $H_1$ be the intersection of $\kappa_1$ with the collection of ordinals that are below $\lambda$ in the sense of the ordering $<_F$ of $\tau_F$. Then by the choice of $\kappa_1$, $H_1$ is a cofinal branch of the tree computed in $(V(\kappa_1), \in)$ via the defining formula of $\tau_F$. Hence:

$$(V(\kappa_1), \in, H_1) \models H_1 \text{ is end-homogeneous for } F,$$

i.e., $F(\alpha_1, \cdots, \alpha_{n-1}, \alpha_n) = F(\alpha_1, \cdots, \alpha_{n-1}, \beta)$ for any increasing elements $\alpha_1 < \cdots < \alpha_{n-1} < \alpha_n$ from $H_1$, and any $\beta \in H_1$ that is greater than $\alpha_n$.

In the next step we consider the restriction of the function $F$ to $[H_1]$ within $(V(\kappa_1), \in, H_1)$ and obtain an $(m-1)$-Mahlo cardinal $\kappa_2 < \kappa_1$ that satisfies the following two properties:

(1) $(V(\kappa_2), \in, H_1 \cap V(\kappa_2)) \prec (V(\kappa_1), \in, H_1)$.

(2) There is some cofinal subset $H_2$ of $\kappa_2$ such that the value of $F$ on any increasing chain of length $n$ from $H_2$ is independent of the choices of the $(n-1)$-th and the $n$-th components of the chain.
Recall that \( m \geq n \), so by the \( m \)-Mahlo property of \( \kappa_1 \), we can repeat this process so as to obtain an \((m - n)\)-Mahlo cardinal \( \kappa_n \) and for some cofinal subset \( H_{n-1} \) of \( \kappa_n \) such that \( (V(\kappa_n), \epsilon) \prec (V(\kappa_1), \epsilon) \) such that for all increasing \( n \)-tuples \( \bar{a} = (\alpha_1, \cdots, \alpha_n) \) and \( \bar{b} = (\beta_1, \cdots, \beta_n) \) from \( H_{n-1} \), \( \alpha_1 = \beta_1 \) implies that \( F(\bar{a}) = F(\bar{b}) \). With such a set \( H_{n-1} \) at hand, it is then easy to build \( H_n \subseteq H_{n-1} \) (as in the proof of Theorem 2.4.5(b)) such that:

\[
H_n \text{ is cofinal in } \kappa_n \text{ and } (V(\kappa_n), \epsilon, H_n) \models H_n \text{ is homogeneous for } F.
\]

This makes it clear that \( (V(\kappa_n), \epsilon, H_n) \) is our desired model of the subtheory \( T_n \) of \( ZF\). This concludes the proof of (b).

Finally, to demonstrate (c), we can simply put part (b) together with Theorem 2.4.12 that assures us that \( ZFC + \Lambda \) is provable in the theory \( GBC + \text{"Ord is weakly compact"} \).

5.6. Remark. By a slight modification of the proof strategy of part (b) of Theorem 5.5, one could also show that \( ZFC + \Lambda \) is mutually interpretable with the extension \( ZF_\prec \) of \( ZF\) studied in the previous section. This modified proof can be combined with Theorem 4.7 to give a new proof of (i) \( \Rightarrow \) (ii) of Theorem 2.4.12.

The following definition is motivated by the work of Albert Visser [V]; it was introduced in [E-7].

5.7. Definition. Suppose \( T \) is a first order theory. \( T \) is solid iff the following property (\( \nabla \)) holds for all models \( M, M^* \), and \( N \) of \( T \):

\[
(\nabla) \quad \text{If } M \supseteq_{\text{par}} N \supseteq_{\text{par}} M^* \text{ and there is a parametrically } M\text{-definable isomorphism } i_0 : M \to M^*, \text{ then there is a parametrically } M\text{-definable isomorphism } i : M \to N.
\]

Visser showed that PA is a solid theory, a result that was extended to ZF and Kelley-Morse theory of classes in [E-7]. An examination of the proof of solidity of ZF presented in [E-7] shows a slightly more general result that plays a crucial role in the proof of Theorem 5.9 below, namely:

5.8. Theorem. Suppose \( M \) and \( M^* \) are models of \( ZF \), and \( (N, X) \models ZF(X) \). Then \( (\nabla^+) \) below holds:

\[
(\nabla^+) \quad \text{If } M \supseteq_{\text{par}} (N, X) \supseteq_{\text{par}} M^* \text{ and there is a parametrically } M\text{-definable isomorphism } i_0 : M \to M^*, \text{ then there is a parametrically } M\text{-definable isomorphism } i : M \to N.
\]

The following general result shows that in contrast with Theorem 5.5(b), the theories \( ZFC + \Lambda \) and \( ZF_\prec \) are not bi-interpretable.

5.9. Theorem. No model of \( ZF \) is parametrically bi-interpretable with a model of \( ZF\).

Proof. Suppose to the contrary that there are interpretations \( I \) and \( J \) that witness that some model \( M \) of \( ZFC \) is parametrically bi-interpretable with a model of \( ZF\). Then by Theorem 5.8, \( M \) can parametrically define a class \( I \) of indiscernibles for itself. But this contradicts Corollary 3.5. \( \square \)

6. SOME VARIANTS OF \( ZF_\prec \)

In this section we discuss four variants of \( ZF_\prec \). We begin with presenting two of these variants that turn out to be conservative over \( ZFC \). The first such system \( ZFI_\prec \) below can be intuitively thought of as weakening the stipulation in \( ZFC_\prec \) that there is a proper class of indiscernibles over the universe to the stipulation that there are arbitrarily large sets of indiscernibles over the universe.

6.1. Definition. \( ZFI_\prec \) is a theory formulated in the language \( L_{\text{Set}} \cup \{\prec, I(x,y)\} \), where \( I(x,y) \) is a binary predicate, whose axioms consist of the following three groups of axioms:

- We will write \( I(x, \alpha) \) as \( x \in I_\alpha \) for better readability.
(1) ZF($<, I$) + GW.

(2) The conjunction of $\forall \alpha \in \text{Ord} \ \forall x (x \in I_\alpha \rightarrow (x \in \text{Ord} \land \alpha \in \text{Ord}))$ with $\forall \alpha \in \text{Ord} \ | \{x : x \in I_\alpha\}| \geq \aleph_\alpha$.

(3) A scheme consisting of sentences of the form $\forall \alpha \in \text{Ord} (\text{Indis}_\varphi(I_\alpha))$, for each formula $\varphi$ in the language $\mathcal{L}_{\text{Set}}(\prec)$. This scheme ensures that $(I_\alpha, \in)$ is a set of order indiscernibles for the ambient model $(V, \in)$ of set theory for each ordinal $\alpha$. More explicitly, if $\varphi = \varphi(v_1, \cdots, v_n)$, then Indis$_\varphi(I_\alpha)$ is the formula below:

$$\forall x_1 \in I_\alpha \cdots \forall x_n \in I_\alpha \ \forall y_1 \in I_\alpha \cdots \forall y_n \in I_\alpha$$

$$[(x_1 \in \cdots \in x_n) \land (y_1 \in \cdots \in y_n) \rightarrow (\varphi(x_1, \cdots, x_n) \Leftrightarrow \varphi(y_1, \cdots, y_n))].$$

- Thus ZFI$^\prec_\prec$ is a theory that ensures that for each ambient infinite cardinal $\kappa_\alpha$, there is a set of indiscernibles for $(V, \in, \prec)$ of size at least $\kappa_\alpha$.

6.2. Theorem. ZFI$^\prec_\prec$ is a conservative extension of ZFC.

Proof. To show the conservativity of ZFI$^\prec_\prec$ over ZFC, it suffices to show that every countable model $\mathcal{M}$ of ZFC has an elementary extension to a model $\mathcal{M}^*$ which has an expansion to ZFI$^\prec_\prec$. So let $\mathcal{M}$ be a countable model of ZFC. By Theorem 2.1.3, there is an expansion $(\mathcal{M}, <, I)$ of $\mathcal{M}$ that satisfies ZF($<, I$) + GW. By compactness, to show the existence of the desired elementary extension $\mathcal{M}^*$ of $\mathcal{M}$, it suffices to show that the elementary diagram of $\mathcal{M}$ is consistent with ZFC$^\prec_\prec$. Towards this goal, fix some list $\langle \varphi_i(x_1, \cdots, x_i) : i < \omega \rangle$ of $\mathcal{L}_{\text{Set}}(\prec)$-formulae such that the free variables of $\varphi_i$ are among $x_1, \cdots, x_i$, and let:

$$T := \text{Th}(\mathcal{M}, <, m)_{m \in \mathcal{M}} \cup \text{ZFI}^\prec_\prec,$$

and let $T_0$ be a finite subset of $T$. Then there is some $j \in \omega$ such that if an axiom of the form $\forall \alpha \in \text{Ord} (\text{Indis}_\varphi(I_\alpha))$ is included in $T_0$, then $\varphi$ is among $\{\varphi_0, \cdots, \varphi_j\}$. Recall that by the classical Erdős-Rado theorem [Kan-1] Theorem 7.3, ZFC proves:

$$\exists \kappa (\kappa) \to (\kappa^+)_{\kappa+1} \text{ for every infinite cardinal } \kappa \text{ and every } n \in \omega,$$

where $\exists \kappa (\kappa)$ is the Beth function, defined by: $\exists \kappa (\kappa) = \kappa$ and $\exists \kappa+1 = 2^{\exists \kappa (\kappa)}$. The Erdős-Rado theorem, together with a global well-ordering $<, I$, then allows us to define within $(\mathcal{M}, <, I)$ a function $F : \text{Ord} \rightarrow V$ such that for each $\alpha \in \text{Ord}$, $F(\alpha)$ is a set of ordinals of cardinality at least $\aleph_\alpha$, and $(F(\alpha), \in)$ is homogeneous for $\{\varphi_0, \cdots, \varphi_j\}$. This makes it clear that if $I_\mathcal{M}(x, y)$ is defined in $(\mathcal{M}, <, I)$ as $[y \in \text{Ord} \land x \in F(y)]$, then:

$$(\mathcal{M}, <, I_\mathcal{M}) \models T_0.$$

Thus every finite subset of $T$, and therefore $T$ itself, is consistent, as promised.

The second variant of ZFI$^\prec_\prec$ we consider, denoted ZFI$^\prec_\prec$ is obtained from ZFI$^\prec_\prec$ by weakening the demand that $I$ is amenable to the demand that it satisfies Sep($\mathcal{L}_{\text{Set}}(\prec, I)$). Note that ZFI$^\prec_\prec$ does include Coll($\mathcal{L}_{\text{Set}}$).

6.3. Definition. Let ZFI$^\prec_\prec$ be the subsystem of ZFI$^\prec_\prec$ whose axioms consist of the following:

(1) ZF($<, I$) + GW + Sep($<, I$).

(2) The sentence expressing that $I$ is cofinal in Ord.
(3) The scheme Indis$_{\mathcal{L}_{\text{Set}}}(I)$ (as in Definition 3.1).

**6.4. Theorem.** $\text{ZFI}_<^<$ is a conservative extension of $\text{ZFC}$.

**Proof.** It suffices to show that every countable model $\mathcal{M}$ of $\text{ZFC}$ has an elementary extension to a model $\mathcal{M}^*$ which has an expansion to $\text{ZFI}_<$.

Let $\mathcal{M}$ be a countable model of $\text{ZFC}$, and $(\mathcal{M}, <_M)$ be an expansion of $\mathcal{M}$ that satisfies $\text{ZF}(\prec) + \text{GW}$. Then let:

$$T := \text{Th}(\mathcal{M}, <_M, m)_{m \in M} \cup \text{ZFI}_<^<,$$

and let $T_0$ be a finite subset of $T$. Let $n \in \omega$ be large enough so that any sentence in $T_0$ that belongs to the elementary diagram of $\mathcal{M}$ is $\Sigma_n$, and let $j \in \omega$ be large enough so that if the sentence $\text{Indis}_\varphi(I)$ is in $T_0$, then $\varphi$ is among $\{\varphi_0, \cdots, \varphi_j\}$. By the reflection theorem for $\text{ZF}(\prec)$, there is some $\alpha \in \text{Ord}_M$ such that:

$$(\mathcal{M}(\alpha), <_{M(\alpha)}) \prec \Sigma_n (\mathcal{M}, <_M) \text{ and } \mathcal{M} \models \text{cf}(\alpha) = \omega,$$

where $<_{M(\alpha)}$ is the restriction of $<_M$ to $M(\alpha)$. Since $\alpha$ has countable cofinality in $\mathcal{M}$, by the infinite Ramsey theorem applied within $\mathcal{M}$, there is a cofinal subset $I$ of $\alpha$ that is $\varphi_i$-indiscernible for each $i \leq j$. Since any expansion of $\mathcal{M}(\alpha)$ within $\mathcal{M}$ satisfies the remarkability condition of $\text{ZFI}_<$, this makes it clear that:

$$(\mathcal{M}(\alpha), <_{M(\alpha)}, I) \models T_0,$$

which completes the proof of consistency of $T$.

Finally, in the remarks below, we briefly discuss two natural strengthenings of $\text{ZFI}_<$ whose purely set-theoretical consequences go beyond $\text{ZFC} + \Lambda$.

**6.5. Remark.** Recall the classical fact of large cardinal theory that the Silver indiscernibles (of the constructible universe) are closed and unbounded in the ordinals, and satisfy the so-called remarkability condition [Kan-1, Lemma 9.10]. A moment’s reflection reveals that the axiom “$I$ is closed and unbounded in $\text{Ord}$” is inconsistent with $\text{ZFI}$ based on cofinality considerations and indiscernibility. More specifically, $\text{ZFI}$ implies that either all limit ordinals in $I$ have cofinality $\omega$, or they are all of uncountable cofinality; each of which is inconsistent with $I$ being closed and unbounded. On the other hand, if $(\mathcal{M}, <, I) \models \text{ZFI}_<$, using a class-theoretic adaptation of Baumgartner’s characterization of $n$-ineffable cardinals in terms of regressive partition relations [Ban], the remarkability condition of $I$ can be recast as asserting that $I$ is “definably stationary” in $\mathcal{M}$, i.e., $I$ intersects every closed unbounded subset of $\text{Ord}_M$ that is parametrically definable in $(\mathcal{M}, <)$. Using this equivalence one can readily show that the remarkability condition can be consistently added to $\text{ZFI}_<$, assuming that $\text{ZFC} + \text{“there is a cardinal } \kappa \text{ that is } n\text{-ineffable for each } n \in \omega \text{”}$ is consistent. Moreover, the techniques of this paper can be extended to show that $\mathcal{L}_{\text{Set}}$-consequences of the strengthening of $\text{ZFI}_<$ by an axiom scheme expressing the remarkability of $I$ turn out to coincide with the theorems of $\text{ZFC} + \Theta$ where $\Theta = \{\theta_n : n \in \omega\}$ and $\theta_n$ is the $\mathcal{L}_{\text{Set}}$-sentence asserting that there is an $n$-ineffable cardinal $\kappa$ such that $\text{V}(\kappa)$ is a $\Sigma_n$-elementary submodel of the universe $V$. Another axiomatization for $\Theta$, in the presence of $\text{ZFC}$, is $\Theta' = \{\theta'_n : n \in \omega\}$, where $\theta'_n$ is the $\mathcal{L}_{\text{Set}}$-sentence asserting that there is an $n$-subtle cardinal $\kappa$ such that $\text{V}(\kappa)$ is a $\Sigma_n$-elementary submodel of the universe $V$. It is worth mentioning that the proof of 2.4.12 can be modified to show that $\text{ZFC} + \Theta$ axiomatizes the purely set-theoretical consequences of the class theory $\text{GBC} + \{\text{Ord is } n\text{-ineffable: } n < \omega\}$; this class theory can also be axiomatized by $\text{GBC} + \{\text{Ord is } n\text{-subtle: } n < \omega\}$.  

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6.6. Remark. Recall from part (iii) of Theorem 4.1 that $ZFI^\text{Good}_<$ is the result of augmenting the theory $ZFI_<$ with the sentence $\psi$ that expresses:

$$\forall \alpha, \beta \in I[\alpha \in \beta \rightarrow (V(\alpha), \in, <) \prec (V(\beta), \in, <)].$$

Note that thanks to $\psi$ within $ZFI^\text{Good}_<$ there is a definable full satisfaction predicate $S$ for $(V, \in, <)$, since $S$ can simply be defined as the union of the Tarskian satisfaction predicates for structures of the form $(V(\alpha), \in, <)$ as $\alpha$ ranges in $I$. Using $S$ we can formulate the following axiom $\sigma$ that expresses that elements of $I$ are order indiscernible in the sense of $S$ for the structure $(V, \in, <)$:

$$\forall \alpha, \beta \in I[\alpha \in \beta \rightarrow (V(\alpha), \in, <) \prec (V(\beta), \in, <)].$$

It is easy to see that if $\kappa$ is a Ramsey cardinal, then $(V(\kappa), \in)$ has an expansion to $ZFI^\text{Good}_+\sigma$. In contrast to the consistency of $ZFI^\text{Good}_+\sigma$ the existence of $0^\#$, as well as the existence of a proper class of almost Ramsey cardinals in the core model. This is because the proof strategy of Theorem 3.8(c,d,e) can be carried out within $ZFI^\text{Good}_+\sigma$ to obtain a nontrivial elementary self-embedding of $(V_I, \in, <)$ by shifting the indiscernibles, where $V_I$ is the proper class consisting of sets that are definable (in the sense of $S$) in $(V, \in, <)$ with parameters from $I$.

It is not clear whether the purely set-theoretical consequences of $ZFI^\text{Good}_+\sigma$ can be axiomatized by a ‘natural’ extension of $ZFC + \Lambda$. However, it is noteworthy that the purely set-theoretical consequences of $ZFI^\text{Good}_+\sigma$ coincide with the purely set-theoretical consequences of $GBC + \text{"Ord is Ramsey}^\text{"}$, where “$\text{Ord is Ramsey}^\text{"}$” expresses: $\text{Ord} \rightarrow (\text{Ord})^{\omega}_2$. This follows from the fact that a model $M$ of $ZFC$ has an expansion to $ZFI^\text{Good}_+\sigma$ iff $M$ has an expansion to $GBC + \text{"Ord is Ramsey}^\text{"}$. The right-to-left direction of this equivalence is handled by a routine argument; the left-to-right direction is established by noting that if $(M, <_M, I)$ is a model of $ZFI^\text{Good}_+\sigma$, and $\mathcal{X}$ is chosen as the collection of subsets of $M$ that are parametrically definable in $(M, <_M)$ in the sense of the aforementioned $(M, <_M)$-definable full satisfaction predicate $S$, then $(M, \mathcal{X})$ satisfies $GBC + \text{"Ord is Ramsey}^\text{"}$.

7. OPEN QUESTIONS

Here we draw attention to some natural questions that arise from the results of the paper.

7.1. Question. Does $ZFCI \vdash \Lambda$?

- One would expect that by the use of a generic global well-ordering one could show that $ZFI_<$ is a conservative extension of $ZFCI$, but our attempts in this direction have been unsuccessful. See also Remark 4.3.

7.2. Question. Can Theorem 4.9 be improved by weakening the statement $\theta$ of that theorem to the statement $\theta^- = \forall n \in \omega (\text{Ord} \rightarrow (\text{Ord})^n_\omega)$?

- We conjecture that the answer to Question 7.2 is in the positive, in analogy with the well-known unprovability of the statement $\psi = \forall n \in \omega (\omega \rightarrow (\omega)^n_\omega)$ in ACA$_0$.

7.3. Question. Let $T \vdash_\pi \varphi$ indicate that $\pi$ is the (binary code of) a proof of $\varphi$ from axioms in the theory $T$. Is there a polynomial-time computable function $f$ such that for all $L_{\text{Set}}$-sentences $\varphi$, the following holds:

$$ZFI_<> \vdash_\pi \varphi \Rightarrow ZFC + \Lambda \vdash f(\pi) \varphi?$$

- We suspect that Question 7.3 has a positive answer.
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Department of Philosophy, Linguistics, and the Theory of Science
University of Gothenburg, Gothenburg, Sweden

email: ali.enayat@gu.se