The analysis of the soliton-type solutions of conformable equations by using generalized Kudryashov method

Melike Kaplan1 · Arzu Akbulut2

Received: 9 March 2021 / Accepted: 10 July 2021 / Published online: 14 August 2021
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2021

Abstract
In this study, we applied the generalized Kudryashov method to two different conformable fractional differential equations and one system namely Burgers’ equation with conformable derivative, Wu-Zhang system with conformable derivative, and the conformable sinh-Gordon equation. Obtained solutions are soliton-type solutions. All obtained solutions have been verified and considered correct by placing them in the original equation with the help of the Maple software program. Also, we have plotted three-dimensional (3D) graphs of some of these solutions with the help of Maple.

Keywords Exact solutions · Generalized Kudryashov method · Symbolic computation · Conformable differential equations

Mathematics Subject Classification 00A69 · 26A33 · 68W30

1 Introduction
In the literature, it is well known that a broad range of problems in many research fields, such as wave propagation phenomenon, dynamical systems, mechanical engineering, fluid mechanics, biology, hydrodynamics, plasma physics, image processing, chemistry, optics, finance, and other fields of engineering and science. Nonlinear fractional partial differential equations (NFPDEs) have been proposed and searched by several scientists (Zubair et al. 2018; Raza et al. 2019; Hosseini et al. 2020). Many proposed different definitions have been introduced in the literature (Yang et al. 2019; Park et al. 2020). Recently, researchers started to see incompleteness in most of the fractional derivative definitions (Samko et al. 1993; Kilbas et al. 2006). Since fractional differential equations have been a significant area in mathematical physics, nonlinear optics, mathematical biology, plasma physics,
quantum field theory, and applied mathematics, solutions of these equations, including soliton-type solutions, have become more important.

Moreover, some functions do not have Taylor power series representation, or their Laplace transform can not calculate. Therefore, a new impressive definition called “the conformable derivative” was propounded by Khalil et al. (2014). This depends just on the basic limit definition of the derivative. The conformable partial differential equations (CPDEs) are simply PDEs in sense of conformable partial fractional derivatives. Namely, conformable derivatives are easy to use while comparing to the other fractional derivatives, as its derivative definition does not include an integral term (Kurt et al. 2015).

There are a lot of studies have been done for the definition and properties of the conformable derivative. Conformable forms of the Taylor power series expansions, Gronwalls inequality, chain rule, exponential functions, Laplace transform, and integration by parts have been presented by Abdeljawad (2015). Benkhettou et al. (2016) have been expressed the calculus of the conformable time-scale. Also, Chung (2015a) has used the conformable derivative and integral to work the fractional Newtonian mechanics. Moreover, deterministic conformable partial differential equations (CPDEs) became a significant topic in the area. So, a lot of scientists paid more elaboration to their approximate and analytical solutions. The stochastic travelling wave solutions for the fractional coupled KdV and 2D KdV equations are acquired by a modified fractional sub-equation method in Ghany et al. (2013) and Ghany and Hyder (2014), respectively.

The generalized Kudryashov method is one of the most useful procedures to procure exact solutions of nonlinear partial differential equations (Kudryashov 2019; Demiray et al. 2014b, 2015; Demiray and Bulut 2016). The basic idea was generalized to a so-called transformed rational function method (Lee and Ma 2009; Ma and Fuchssteiner 1996). Recently, another kind of important solutions, called lump solutions, have been studied extensively and could be formulated by combining different nonlinearities (Ma et al. 2020; Yang et al. 2020; Ma et al. 2021).

This manuscript is organized as follows. Firstly, the methodology of the generalized Kudryashov method has been given. Then, we implement the adopted method to the conformable Burgers’ equation, Wu-Zhang system with conformable derivative, and the conformable sinh-Gordon equation for finding new exact traveling wave solutions and soliton-type solutions. We have categorized the obtained solutions. Also, we plotted the graphics of the founded solutions that have been given by setting some special values for the parameters. Finally, a conclusion is given.

### 2 Summary of conformable fractional derivative

Since the orientation of study in modeling real-world problems is shifting toward the use of fractional order derivative, there are various definitions of fractional derivatives in the literature for example Riemann-Liouville fractional derivative, Caputo fractional derivative, modified Riemann-Liouville fractional derivative, conformable derivative, beta derivative (Demiray 2020a, b) and so on. In this study, we have used the conformable fractional derivative, which is introduced by Khalil et al. The advantages of the conformable derivative are that the constant function is zero and the conformable fractional derivative behaves well in the product rule and chain rule.

The definition of conformable fractional derivative is given as follows:
Assume that \( f : [0, \infty) \to \mathbb{R} \) be a function. The conformable derivative of \( f \) of order \( \alpha, 0 < \alpha \leq 1 \), is defined as
\[
(T_\alpha f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.
\]
for all \( t > 0 \). We can list some helpful features as follows:

\[
T_\alpha(af + bg) = a(T_\alpha f) + b(T_\alpha g), \quad \text{for all } a, b \in \mathbb{R}
\]

Chain rule: Let \( f : (0, \infty) \to \mathbb{R} \) be a differentiable and \( \alpha \)-differentiable function, \( g \) be a differentiable function defined in the range of \( f \).
\[
T_\alpha(fg)(t) = t^{1-\alpha} g'(t)f'(g(t)).
\]

Moreover, the following rules are hold.

\[
T_\alpha(p^\alpha) = \alpha pt^{\alpha-1}, \quad \text{for all } p \in \mathbb{R}
\]

\[
T_\alpha(\lambda) = 0, \quad \text{for all constant functions } f(t) = \lambda
\]

\[
T_\alpha(f/g) = \frac{\alpha g'(t)f(t) - f'(g(t))}{g(t)^2}.
\]

Additively, if \( f \) is differentiable, then \( T_\alpha(f)(t) = t^{1-\alpha} \frac{df}{dt}(t) \) (Khalil et al. 2014; Chung 2015b).

### 3 The generalized Kudryashov method

We take into consideration a general CPDE of the formula (Kudryashov 2012; Bulut et al. 2014):
\[
F(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial x^2}, \ldots) = 0. \tag{1}
\]

The polynomial and derivates of \( u = u(x, t) \) are represented by \( F \), where the nonlinear terms and the highest order derivatives are comprised. In a suitable manner of the following travelling wave transformation:
\[
u(x, t) = u(\xi), \quad \xi = kx - \frac{t^\alpha}{\alpha}, \tag{2}
\]
where \( l \) represents the velocity of the wave, Eq. (1) is reduced thereby forming an ordinary differential equation (ODE) in the form
\[
G(u, u', u'', \ldots, ) = 0. \tag{3}
\]

We note that, in Eq. (3) the differentiation of \( u \) with respect to \( \xi \) is represented by prime. We will integrate all the terms in Eq. (3). Conforming to this technique, the desired solution for the reduced equation is formed by a polynomial in \( R(\xi) \) as
\[
u(\xi) = \sum_{i=0}^{N} a_i R^i(\xi) \quad \text{or} \quad \sum_{j=0}^{M} b_j R^j(\xi), \tag{4}
\]
where \( a_i (i = 0, 1, \ldots, n) \), \( b_j (j = 0, 1, \ldots, m) \) are constants to be found \((a_N \neq 0, b_M \neq 0)\) and \( Q = Q(\xi) \) is the solution of

\[
\frac{dR}{d\xi} = R^2(\xi) - R(\xi). \tag{5}
\]

The solution of Eq. (5) written as

\[
R(\xi) = \frac{1}{1 + C_1 e^{\xi}}, \quad C_1 \text{ is integration constant}.
\]

Based on the homogeneous balance principle, one can find the positive integers \( N \) and \( M \) in Eq. (4) with the use of the Eq. (3). Namely, they can be determined by using homogeneous balance between the nonlinear terms and the highest order derivatives appearing in Eq. (3). Finally, we can obtain a polynomial of \( R \) by subrogating Eqs. (4) into (3) along with Eq. (5). Here, we equate all the coefficients of polynomial \( R \) to zero to obtain an algebraic equation system. Solutions of this system using the assistance of the computer software gives the values of \( a_i (i = 0, 1, \ldots, n) \), \( b_j (j = 0, 1, \ldots, m) \). Lastly, we find the soliton-type solutions of the reduced Eq. (3) by subrogating these acquired values and Eqs. (5) into (4).

There are different types of Kudryashov methods in the literature such as the Kudryashov method, modified Kudryashov method, generalized Kudryashov method, extended Kudryashov method. There are differences among these methods, such as auxiliary equations and the form of the desired solution. We can give a comparison of the methods as follows.

(i) for the classical Kudryashov method, the solution of the auxiliary equation is

\[
R(\xi) = \frac{1}{1 + e^{\xi}},
\]

(ii) for the modified Kudryashov method, the solution of the auxiliary equation is

\[
R(\xi) = \frac{1}{1 \pm a^{\xi}},
\]

and

\[
\frac{dR}{d\xi} = (R^2(\xi) - R(\xi)) \ln a.
\]

(iii) for the extended Kudryashov method

\[
R(\xi) = R^3(\xi) - R(\xi).
\]

For these methods, we assume the solutions as

\[
u(\xi) = \sum_{i=0}^{M} a_i R^i(\xi),
\]

where \( a_i (i = 0, 1, 2, \ldots, M), (a_M \neq 0) \) are constants to be determined (Ray 2016; Ege and Misirli 2018).

We have used the generalized Kudryashov method in this paper, since we seek the solution in a more comprehensive form.
4 Implementations

In the following section, three equations are implemented to show how the method works.

4.1 Burgers’ equation with conformable derivative

Burgers’ equation

\[ u_t + uu_x - vu_{xx} = 0, \]

was firstly presented in 1918 by Bateman has been used as a mathematical model in several fields such as gas dynamics, number theory, elasticity theory, heat conduction, hydrodynamic waves, shock wave theory, elastic waves, termaviscous fluids, and turbulence theory (Bateman 1915). Many scientists have been worked on obtaining numerical and exact solutions for not only Burgers’ equation (Burger 1939, 1948).

Then, the fractional Burgers’ equation

\[ \frac{\partial^{\alpha} u}{\partial t^{\alpha}} + uu_x - vu_{xx} = 0, \quad 0 < x < 1, t > 0, \]

where \( \alpha \in (0, 1) \) and \( \frac{\partial^{\alpha} u}{\partial t^{\alpha}} \) means conformable derivative of \( u(x, t) \) has attracted many scientists’ attention. Using the Cole-Hopf transformation

\[ u = -2v \frac{\delta_x}{\delta}, \]

the equation Eq. (6) transforms into a time fractional heat equation,

\[ \frac{\partial^{\alpha} \delta}{\partial t^{\alpha}} = v \frac{\partial^2 \delta}{\partial x^2} \]

where \( \delta(x, t) \) the solution of the heat equation Eq. (7) and the derivative is an \( \alpha \)-order conformable fractional derivative.

The numerical solution of fractional Burgers’ equation has been acquired (Esen and Tasbozan 2016; Esen et al. 2013) implemented HAM to procure the approximate analytical solution of fractional Burgers’ equation. Also, exact solutions to this equation have been founded by using the Ricatti expansion method (Abdel-Salam et al. 2014). Kurt et al. (2015) applied the homotopy analysis method to this equation and they utilized the Hopf–Cole transform (Kurt et al. 2016). Auto-Bäcklund transform and exact solutions to local conformable time-fractional viscous Burgers system have been found (Huang and Yang 2019). Some new travelling wave solutions for the one-dimensional Burgers equation were obtained (Cenesiz et al. 2017). Also, Demiray et al. applied the generalized Kudryashov method to time-fractional Burgers equation with Riemann-Liouville sense (Demiray et al. 2014a).

Using the following travelling wave transformation

\[ u(x, t) = \xi, \xi = x - c \frac{t}{\alpha}, \]

we can reduce Eq. (6) to the following ODE

\[ -cu' + uu'' - vu''' = 0. \]
Integrating Eq. (9) with respect to ξ once, and taking the integration constant as zero, we find the following equation

\[-cu + \frac{u^2}{2} - vu' = 0.\]  \hspace{1cm} (10)

According to the homogeneous balance principle, balancing the nonlinear term \(u^2\) with the highest order nonlinear term, it can be founded as

\[N - M + 1 = 2N - 2M \Rightarrow N = M + 1\]

By setting \(M = 1\), we get \(N = 2\). So the solution can be expressed as

\[u(\xi) = \frac{a_0 + a_1R + a_2R^2}{b_0 + b_1R},\]  \hspace{1cm} (11)

where \(R = R(\xi)\) is the solution of the Eq. (5). Based on this, we substitute Eqs. (11) into (10) and use Eq. (5). Afterwards, we equate all coefficients of the functions \(R^k\) to zero. Therefore the following equation system can be obtained. Here \(a_0, a_1, a_2, b_0, \) and \(b_1\) are parameters.

\[
\begin{align*}
R^4 & : -va_2b_1 + \frac{a_2^2}{2} = 0, \\
R^3 & : a_1a_2 - ca_2b_1 + va_2b_1 - 2va_2b_0 = 0, \\
R^2 & : -ca_2b_0 + \frac{a_1^2}{2} + a_0a_2 + 2va_2b_0 - va_1b_0 + vb_1a_0 - ca_1b_1 = 0, \\
R^1 & : -ca_0b_1 + va_1b_0 - ca_1b_0 + a_0a_1 - vb_1a_0 = 0, \\
R^0 & : -ca_0b_0 + \frac{a_0^2}{2} = 0.
\end{align*}
\]

From the solution of this algebraic equation set, we find different cases which are discussed as follows (Figs. 1, 2 and 3, 4, 5, 6, 7, 8 and 9).

Case 1:

![Fig. 1](image_url) The kinky periodic solitary wave solution equation with conformable of the Burgers’ derivative within the interval \(-10 \leq x \leq 10, 0 \leq t \leq 2\), when \(C_1 = 4, b_1 = 2, v = 4, \alpha = 0.9\)
Fig. 2 The antikink soliton solution of the Burgers’ equation with conformable derivative within the interval $-10 \leq x \leq 10, 0 \leq t \leq 2$, when $C_1 = 4, b_0 = 1, b_1 = 2, v = 4, \alpha = 0.9$

Fig. 3 The antikink soliton solution of the Burgers’ equation with conformable derivative within the interval $-10 \leq x \leq 10, 0 \leq t \leq 2$, when $C_1 = 4, b_0 = 1, b_1 = 2, v = 4, \alpha = 0.9$
Then, by subrogating the acquire values into Eqs. (11) with (8), we get the solitary wave solution of the conformable Burgers’ equation as follows

\[ a_0 = 0, a_1 = 0, a_2 = 2vb_1, b_0 = -\frac{b_1}{2}, c = 2v. \]

Then, by subrogating the acquire values into Eqs. (11) with (8), we get the solitary wave solution of the conformable Burgers’ equation as follows

\[ u_1(x, t) = \frac{2vb_1}{\left(1+C_1(\cosh(x-\frac{2vt}{a})+\sinh(x-\frac{2vt}{a}))\right)^2} - \frac{b_1}{2} + b_1 \left(\frac{1}{1+C_1(\cosh(x-\frac{2vt}{a})+\sinh(x-\frac{2vt}{a}))}\right). \]  

**Case 2:**

\[ a_0 = 0, a_1 = 2vb_0, a_2 = 2vb_1, c = v. \]

Then, by subrogating the acquire values into Eq. (11) with Eq. (8), we get the soliton-type solution of the conformable Burgers’ equation as follows
Case 3:

\[ a_0 = -2v b_0, \quad a_1 = -2v b_1 + 2v b_0, \quad a_2 = 2v b_1, \quad c = -v. \]

Then, by subrogating the acquisite values into Eq. (11) with Eq. (8), we get the soliton-type solution of the conformable Burgers’ equation as follows

\[
\begin{align*}
  u_2(x, t) &= \frac{2v b_0 \left( \frac{1}{1 + C_1 (\cosh(x - \frac{vt}{\alpha}) + \sinh(x - \frac{vt}{\alpha}))} \right) + 2v b_1 \left( \frac{1}{1 + C_1 (\cosh(x - \frac{vt}{\alpha}) + \sinh(x - \frac{vt}{\alpha}))} \right)^2}{b_0 + b_1 \left( \frac{1}{1 + C_1 (\cosh(x + \frac{vt}{\alpha}) + \sinh(x + \frac{vt}{\alpha}))} \right)}. \quad (13)
\end{align*}
\]

\[
\begin{align*}
  u_3(x, t) &= \frac{-2v b_0 + (-2v b_1 + 2v b_0) \left( \frac{1}{1 + C_1 (\cosh(x + \frac{vt}{\alpha}) + \sinh(x + \frac{vt}{\alpha}))} \right) + 2a b_1 \left( \frac{1}{1 + C_1 (\cosh(x + \frac{vt}{\alpha}) + \sinh(x + \frac{vt}{\alpha}))} \right)^2}{b_0 + b_1 \left( \frac{1}{1 + C_1 (\cosh(x + \frac{vt}{\alpha}) + \sinh(x + \frac{vt}{\alpha}))} \right)}. \\
\end{align*}
\]

(14)
Case 4:

\[ a_0 = 2v b_1, \quad a_1 = -4v b_1, \quad a_2 = 2v b_1, \quad b_0 = -\frac{b_1}{2}, \quad c = -2v. \]
Then, by subrogating the acquisite values into Eqs. (11) with (8), we get the solitary wave solution of the conformable Burgers’ equation as follows

\[
    u_4(x, t) = \frac{2\nu b_1 - 4\nu b_1 \left( \frac{1}{1+C_1(\cosh(x+\frac{2\nu t}{\alpha})+\sinh(x+\frac{2\nu t}{\alpha}))} \right) + 2\nu b_1 \left( \frac{1}{1+C_1(\cosh(x+\frac{2\nu t}{\alpha})+\sinh(x+\frac{2\nu t}{\alpha}))} \right)^2}{\frac{-b_1}{2} + b_1 \left( \frac{1}{1+C_1(\cosh(x+\frac{2\nu t}{\alpha})+\sinh(x+\frac{2\nu t}{\alpha}))} \right)}.
\]

(15)

4.2 Wu-Zhang system with conformable derivative

Wu-Zhang system demonstrates dispersive long waves in two horizontal directions on shallow waters which means that the pure plane has several speeds propagation that makes some of the waves spread outwards in space. A proper comprehension of all
Fig. 10  The kinky periodic solitary wave solution of the Wu-Zhang system with conformable derivative $v_{5,6}$ within the interval $-10 \leq x \leq 10, 0 \leq t \leq 2$, when $C_1 = 4, \alpha = 0.9$

Fig. 11  The kinky periodic solitary wave solution of the Wu-Zhang system with conformable derivative $u_{7,8}$ within the interval $-10 \leq x \leq 10, 0 \leq t \leq 2$, when $C_1 = 4, \alpha = 0.9$

Fig. 12  The kinky periodic solitary wave solution of the Wu-Zhang system with conformable derivative $v_{7,8}$ within the interval $-10 \leq x \leq 10, 0 \leq t \leq 2$, when $C_1 = 4, \alpha = 0.9$

solutions is useful for civil and coastal engineers in order to implement the nonlinear water wave model in the coastal and harbor design.

The $(1+1)$-dimensional Wu-Zhang system with conformable derivative is in the form (Esami and Rezazadeh 2016):
Here $u$ and $v$ represent the elevation and the surface velocity of water, respectively.

In the year 1996, Wu and Zhang (1996) proposed three sets of equations to model long, nonlinear, scattered gravitational waves that describe waves travelling in two horizontal directions over uniform shallow water depth. After doing some transformations and reductions, a dimensional long wave splitter $(1 + 1)$ equation, known as the Wu-Zhang system has been acquired. These equations (Zheng et al. 2003) are advantageous for harbor and coastal designs in civil and coastal engineering.

While there is a gap in the Wu-Zhang system’s harmonious time zone literature, the classical Wu-Zhang system is thought to have achieved soliton solutions by many researchers (Du et al. 2018; Mirzazadeh et al. 2017; Li et al. 2000). In recent years, this system of equations has been discussed by many researchers to find exact solutions of different types. For example, Eslami and Rezazadeh are the first integral method applied to acquire exact solutions of Eq. (16) (Eslami and Rezazadeh 2016). Khater et al. (2019) found numerical solutions through three different numerical schemes and they found exact solutions through a modified auxiliary equation method applied for this system (Khater et al. 2019). Yel and Baskonus (2019) implemented the modified $exp(-\phi(\xi))$- expansion function method.

Let us solve the system above based on the method implemented. We firstly take a travelling wave transformation as follows:

$$u(x, t) = u(\xi), v(x, t) = v(\xi), \xi = x - c \frac{t}{\alpha}. \quad (17)$$

The system Eqs. (16) is reduced the following ODE with use of Eq. (17):

$$-cu' + uu' + v' = 0, \quad (18)$$

Integrating first equation of Eq. (18) with respect to $\xi$ once and setting the constant of the integration to zero, we get the following equation:

$$v = cu - \frac{1}{2}u^2. \quad (19)$$

The following equation can be founded by subrogating Eq. (19) into the second equation of Eq. (18):

$$-c^2u' + 3cuu' - \frac{3}{2}u^2u' + \frac{1}{3}u''' = 0. \quad (20)$$

If we integrate Eq. (20) with respect to $\xi$ once, we find

$$u'' = 3c^2u - \frac{9}{2}cu^2 + \frac{3}{2}u^3. \quad (21)$$

Then we balance the highest order derivative term $u''$ with the nonlinear term $u^3$, we find

$$N - M + 2 = 3N - 3M \Rightarrow N = M + 1$$
By setting $M = 1$, we find $N = 2$. Then, the desired exact solution becomes

$$u(\xi) = \frac{a_0 + a_1 R + a_2 R^2}{b_0 + b_1 R},$$

(22)

where $R = R(\xi)$ is the solution of the Eq. (5). Based on this, we get the following equation system by subrogating Eqs. (22) into (21) along with Eq. (5) and then equating all coefficients of the functions $R_k$ to zero:

$$R^6 : 2a_2 b_1 - \frac{3}{2}a_3 = 0,$$

$$R^5 : -3a_2 b_1^2 + 6a_2 b_0 b_1 - \frac{9}{2}a_1 a_2 + \frac{9}{2}ca^2 b_1 = 0,$$

$$R^4 : -\frac{9}{2}a_0 a_2 + 9ca_1 a_2 b_1 + \frac{9}{2}ca^2 b_0 - 3c^2 a_2 b_1^2 + 6a_2 b_0^2 + a_2 b_1 - 9a_2 b_0 b_1 - \frac{9}{2}a_1^2 a_2 = 0,$$

$$R^3 : -2b_1 a_0 b_0 + 3a_2 b_0 b_1 - 3c^2 a_1 b_1^2 + 9ca_0 a_1 b_2 - 9a_0 a_1 a_2 + a_1 b_1 b_1 - b_1^2 a_0 - 6c^2 a_2 b_0 b_1 - \frac{3}{2}a_3^2 - 10a_2 b_0^2 + \frac{9}{2}ca^2 b_1 + 9ca_1 a_2 b_0 + 2a_1 b_0^2 = 0,$$

$$R^2 : b_1 a_0 - 3c^2 a_2 b_0^2 - \frac{9}{2}a_0 a_2^2 - 3c^2 a_0 b_1^2 - \frac{9}{2}a_2 b_0^2 - 3a_1 b_0^2 - 6c^2 a_1 b_0 b_1 + 9ca_0 a_1 b_1 + 3b_1 a_0 b_0 + 9ca_0 a_2 b_0 + \frac{9}{2}ca^2 b_0 - a_1 b_0 b_1 + 4a_2 b_0^2 = 0,$$

$$R^1 : -6c^2 a_0 b_1 b_0 - b_1 a_0 b_0 - \frac{9}{2}a_0^2 a_1 + 9ca_0 a_1 b_0 + a_1 b_0^2 - 3c^2 a_1 b_0^2 + \frac{9}{2}ca^2 b_1 = 0,$$

$$R^0 : \frac{9}{2}ca^2 b_0 - 3c^2 a_0 b_0^2 - \frac{3}{2}a_3^3 = 0.$$

and $a_0, a_1, a_2, b_0, b_1$ are parameters to be found. Solving these algebraic equations with the assistance of Maple, we attain the following different cases (Figs. 5, 6, 7, 8, 9, 10, 11, and 12):

**Case 1:**

$$a_0 = 0, a_1 = \frac{2\sqrt{3}}{3} b_1, a_2 = \pm \frac{2\sqrt{3}}{3} b_1, b_0 = 0, c = \frac{\sqrt{3}}{3}.$$

By subrogating the acquisite values into Eq. (22), we attain the soliton-type solutions of the system as

$$u_{1,2}(x, t) = \frac{\mp \frac{2\sqrt{3}}{3} C_1 \left( \sinh \left( x \pm \frac{\sqrt{3}}{3a} t^a \right) + \cosh \left( x \pm \frac{\sqrt{3}}{3a} t^a \right) \right) + C_1}{C_1^2 + 1 + 2C_1 \cosh \left( x \pm \frac{\sqrt{3}}{3a} t^a \right)},$$

(23)

and

$$v_{1,2}(x, t) = \frac{2C_1 \left( \sinh \left( x \pm \frac{\sqrt{3}}{3a} t^a \right) + \cosh \left( x \pm \frac{\sqrt{3}}{3a} t^a \right) \right) \left( 6c_1 \sinh \left( x \pm \frac{\sqrt{3}}{3a} t^a \right) \cosh \left( x \pm \frac{\sqrt{3}}{3a} t^a \right) + 6C_1 \sinh \left( x \pm \frac{\sqrt{3}}{3a} t^a \right) + 3 - 3C_1^2 + 6C_1 \cosh \left( x \pm \frac{\sqrt{3}}{3a} t^a \right) + 6C_1 \cosh \left( x \pm \frac{\sqrt{3}}{3a} t^a \right) \right)}{6C_1^2 \sinh \left( x \pm \frac{\sqrt{3}}{3a} t^a \right) \cosh \left( x \pm \frac{\sqrt{3}}{3a} t^a \right) + 6C_1 \sinh \left( x \pm \frac{\sqrt{3}}{3a} t^a \right) + 3 - 3C_1^2 + 6C_1 \cosh \left( x \pm \frac{\sqrt{3}}{3a} t^a \right) + 6C_1 \cosh \left( x \pm \frac{\sqrt{3}}{3a} t^a \right)}.$$

(24)

**Case 2:**
Therefore, if we subrogate the acquire values into Eq. (22), we acquire the soliton-type solutions of the Wu-Zhang system with conformable derivative as follows

\[ u_{3,4}(x, t) = \frac{\pm 2\sqrt{3}}{3} \left( 1 + C_1 \left( \cosh \left( x \mp \frac{\sqrt{3}}{3} r^a \right) + \sinh \left( x \mp \frac{\sqrt{3}}{3} r^a \right) \right) \right), \]  

(25)

and

\[ v_{3,4}(x, t) = \frac{2C_1 \left( \sinh \left( x \pm \frac{\sqrt{3}}{3} r^a \right) + \cosh \left( x \pm \frac{\sqrt{3}}{3} r^a \right) \right)}{6C_1 \sinh \left( x \pm \frac{\sqrt{3}}{3} r^a \right) \cosh \left( x \pm \frac{\sqrt{3}}{3} r^a \right) + 3 - 3C_1 + 6C_1 \cosh \left( x \pm \frac{\sqrt{3}}{3} r^a \right) + 6C_1 \cosh \left( x \pm \frac{\sqrt{3}}{3} r^a \right)}, \]  

(26)

**Case 3:**

\[ a_0 = 0, a_1 = 0, a_2 = \pm \frac{2\sqrt{3}}{3} b_1, b_0 = -\frac{b_1}{2}, c = \pm \frac{2\sqrt{3}}{3}. \]

By subrogating the acquire values into Eq. (22), we acquire the solitary wave solutions of the Wu-Zhang system with conformable derivative as follows

\[ u_{5,6}(x, t) = \pm \frac{4\sqrt{3}C_1 \left( \sinh \left( x \pm \frac{2\sqrt{3}}{3} r^a \right) + \cosh \left( x \pm \frac{2\sqrt{3}}{3} r^a \right) \right)}{6C_1 \cosh \left( x \pm \frac{2\sqrt{3}}{3} r^a \right) \sinh \left( x \pm \frac{2\sqrt{3}}{3} r^a \right) - 3 + 6C_1 \cosh^2 \left( x \pm \frac{2\sqrt{3}}{3} r^a \right) - 3C_1^2}, \]  

(27)

and

\[ v_{5,6}(x, t) = \frac{4}{3} \left( 1 + C_1 \left( \cosh \left( x \pm \frac{2\sqrt{3}}{3} r^a \right) + \sinh \left( x \pm \frac{2\sqrt{3}}{3} r^a \right) \right) \right)^2 \left( \frac{1}{C_1 \left( \cosh \left( x \pm \frac{2\sqrt{3}}{3} r^a \right) + \sinh \left( x \pm \frac{2\sqrt{3}}{3} r^a \right) \right)} - \frac{1}{3} + \frac{1}{C_1 \left( \cosh \left( x \pm \frac{2\sqrt{3}}{3} r^a \right) + \sinh \left( x \pm \frac{2\sqrt{3}}{3} r^a \right) \right) c^2} \right)^{-\frac{1}{2}} - \frac{1}{2} \left( 1 + C_1 \left( \cosh \left( x \pm \frac{2\sqrt{3}}{3} r^a \right) \right) \left( \sinh \left( x \pm \frac{2\sqrt{3}}{3} r^a \right) \right) \right)^2 \left( \frac{1}{C_1 \left( \cosh \left( x \pm \frac{2\sqrt{3}}{3} r^a \right) + \sinh \left( x \pm \frac{2\sqrt{3}}{3} r^a \right) \right) c^2} \right)^{-\frac{1}{2}} \right). \]  

(28)

**Case 4:**

\[ a_0 = \pm \frac{2\sqrt{3}}{3} b_0, a_1 = \pm \frac{2\sqrt{3}}{3} \left( -b_1 + b_0 \right), a_2 = \pm \frac{2\sqrt{3}}{3} b_1, c = \pm \frac{\sqrt{3}}{3}. \]

By subrogating the acquire values into Eq. (22), we acquire the same soliton-type solutions of the Wu-Zhang system with conformable derivative in Case 1.

**Case 5:**
By subrogating the acquire values into Eq. (22), we acquire the solitary wave solutions of the Wu-Zhang system with conformable derivative as follows

\[ a_0 = \pm \frac{2\sqrt{3}}{3} b_1, a_1 = \pm \frac{4\sqrt{3}}{3} b_1, a_2 = \pm \frac{2\sqrt{3}}{3} b_1, b_0 = -\frac{b_1}{2}, c = \pm \frac{2\sqrt{3}}{3}. \]

For all the cases above \( C_1 \) is an integration constant.

### 4.3 The conformable sinh-Gordon equation

In this subsection, we are going to consider the conformable sinh-Gordon equation, which plays a crucial role in many scientific applications such as nonlinear optics, fluid dynamics, solid-state physics, nonlinear optics, kink dynamics, mathematical biology, plasma physics, kink dynamics, quantum field theory, and chemical kinetics (Tajadodi et al. 2021).

\[ D_t^\alpha (u_x) + e^u - e^{-u} = 0, \tag{31} \]

Using the following travelling wave transformation

\[ u(x, t) = \xi, \xi = x - c \frac{\alpha}{\alpha}, v = e^u \tag{32} \]

For all the cases above \( C_1 \) is an integration constant.
we can reduce Eq. (31) to the following ODE

\[-c(vv'' - (v')^2) + v^3 - v = 0\]  \hspace{1cm} (33)

According to the homogeneous balance principle, balancing the nonlinear terms $vv''$ with $v^3$, it can be founded as

\[2N - 2M + 2 = 3N - 3M \Rightarrow N = M + 2\]

By setting $M = 1$, we get $N = 3$. So the solution can be expressed as

\[u(\xi) = \frac{a_0 + a_1R + a_2R^2 + a_3R^3}{b_0 + b_1R},\]  \hspace{1cm} (34)

where $R = R(\xi)$ is the solution of the Eq. (5). Based on this, we substitute Eq. (34) into Eq. (31) and use Eq. (5). Afterwards, we equate all coefficients of the functions $R^k$ to zero. Therefore the following equation system can be obtained. Here $a_0, a_1, a_2, a_3, b_0$, and $b_1$ are parameters.
From the solution of this algebraic equation set, we find different cases which are discussed as follows (Figs. 13 and 14).

Case 1:

\[ a_0 = b_0, a_1 = -4b_0 + b_1, a_2 = -4b_1 + 4b_0, a_3 = 4b_1, c = 2. \]

Then, by subrogating the acquire values into Eqs. (34) with (32), we get the solitary wave solution of the conformable sinh-Gordon equation as follows

\[
\begin{align*}
\int \frac{1 - 2C_1 \left( \cosh \left( x - \frac{2c}{a} \right) - \sinh \left( x - \frac{2c}{a} \right) \right) + C_1 \left( \cosh \left( 2x - \frac{4c}{a} \right) - \sinh \left( 2x - \frac{4c}{a} \right) \right) - (1 + C_1 \left( \cosh \left( x - \frac{2c}{a} \right) + \sinh \left( x - \frac{2c}{a} \right) \right))^2}{1 - 2C_1 \left( \cosh \left( x - \frac{2c}{a} \right) - \sinh \left( x - \frac{2c}{a} \right) \right) + C_1 \left( \cosh \left( 2x - \frac{4c}{a} \right) - \sinh \left( 2x - \frac{4c}{a} \right) \right)}
\end{align*}
\]

(35)

Case 2:

\[ a_0 = -b_0, a_1 = 4b_0 - b_1, a_2 = 4b_1 - 4b_0, a_3 = -4b_1, c = -2. \]
Then, by subrogating the acquire values into Eqs. (34) with (32), we get the solitary wave solution of the conformable sinh-Gordon equation as follows

$$u_2(x, t) = \ln \left( \frac{-1 + 2C_1(\cosh \left( x + \frac{2u}{a} \right) + \sinh \left( x + \frac{2u}{a} \right) ) - C_2 \left( \cosh \left( 2x + \frac{2u}{a} \right) + \sinh \left( 2x + \frac{4u}{a} \right) \right)}{\left( 1 + C_1 \left( \cosh \left( x + \frac{2u}{a} \right) + \sinh \left( x + \frac{2u}{a} \right) \right) \right)^2} \right)$$

(36)

For all the cases above $C_1$ is an integration constant.

5 Conclusions

In this manuscript, the generalized Kudryashov method has been applied to acquire new soliton-type solutions of the conformable Burgers’ equation, Wu-Zhang system with conformable derivative, and the conformable sinh-Gordon equation. By using the conformable fractional derivative definition in the traveling wave transformation, the equations are reduced to nonlinear ordinary differential equations. The considered equations are important in mathematical physics and engineering. The acquired solutions represented by the exponential, hyperbolic trigonometric, and rational functions may be appropriate to understand the mechanism of the complex nonlinear physical phenomena in wave propagation. The obtained results are soliton-type results. We have categorized the obtained solutions. Also, we have plotted the graphs of the acquired solutions under suitable values of constants that have been given. By comparing our results with the existing ones in the literature, we saw that they are different. According to our knowledge, the obtained solutions in this article have the potential to shed light on the area of engineering. We used the Maple software program for checking all obtained solutions and drawing the graphs.

The obtained solutions consisted of hyperbolic functions. These functions are circular functions that have arisen in both mathematics and physics. For instance, the hyperbolic secant functions arise in the profile of a laminar jet, the hyperbolic cosine functions have catenary shape, the hyperbolic tangent functions arise during computations of magnetic moment and rapidity of special relativity, and the hyperbolic cotangent functions arise in the Langevin function for magnetic polarisation.

The generalized Kudryashov method is an effective and powerful technique for acquiring soliton-type solutions of CPDEs. Also, it is deserving to study the Burgers’ equation, Wu-Zhang system, and sinh-Gordon equation with other senses of fractional derivatives.

References

Abdeljawad, T.: On conformable fractional calculus. J. Comput. Appl. Math. 279, 57–66 (2015)
Abdel-Salam, E.A.-B., Yousif, E.A., Arko, Y.A.S., Gumma, E.A.E.: Solution of moving boundary space-time fractional Burger’s equation. J. App. Math. 2014, 218092 (2014)
Bateman, H. Some recent researches on the motion of fluids. Month. Weather Rev. 43, 163–170 (1915)
Benkhettou, N., Hassani, S., Torres, D.F.M.: A conformable fractional calculus on arbitrary time scales. J. King Saud Univ. Sci. 28, 93–98 (2016)
Bulut, H., Pandir, Y., Demiray, S.T.: Exact solutions of time-fractional KdV equations by using generalized Kudryashov method. Int. J. Model. Optim. 4(4), 315–320 (2014)
Burger, J. M.: Mathematical examples illustrating the relations occurring in the theory of turbulent fluid motion, Trans. Roy. Neth. Acad. Sci. Amsterdam 17, 1–53 (1939)
Burger, J.M.: A mathematical model illustrating the theory of turbulence. Adv. Appl. Mech. 1, 171–199 (1948)

Cenesiz, Y., Baleanu, D., Kurt, A., Tasbozan, O.: New exact solutions of Burgers type equations with conformable derivative. Waves Random Complex Media 27, 103–116 (2017)

Chung, W.S.: Fractional Newton mechanics with conformable fractional derivative. J. Comput. Appl. Math. 290, 150–158 (2015a)

Chung, W.S.: Fractional Newton mechanics with conformable fractional derivative. J. Comput. Appl. Math. 290, 150–158 (2015b)

Demiray, S.T.: New solutions of Biswas–Arshed equation with beta time derivative. Optik Int. J. Light Electr. Opt. 222(165405), 1–5 (2020a)

Demiray, S.T.: New soliton solutions of optical pulse envelope E(Z, τ) with beta time derivative. Optik Int. J. Light Electr. Opt. 223(165453), 1–6 (2020b)

Demiray, S.T., Bulut, H.: Generalized Kudryashov method for nonlinear fractional double sinh-Poisson equation. J. Nonlinear Sci. Appl. 9, 1349–1355 (2016)

Demiray, S.T., Pandir, Y., Bulut, H.: Generalized Kudryashov method for time-fractional differential equations. Abs. Appl. Anal. (2014a) Article ID 901540

Demiray, S.T., Pandir, Y., Bulut, H.: The investigation of exact solutions of nonlinear time-fractional Klein-Gordon equation by using generalized Kudryashov method. AIP Conf. Proc. 1637, 283–289 (2014b)

Demiray, S.T., Pandir, Y., Bulut, H.: The analysis of the exact solutions of the space fractional coupled KD equations. AIP Conf. Proc. 1648, 370013–1–5 (2015)

Du, Z., Tian, B., Xie, X.Y., Chai, J., Wu, X.Y.: Backlund transformation and soliton solutions in terms of the Wronskian for the Kadomtsev-Petviashvili-based system in fluid dynamics. Pramana J. Phys. 90, 45 (2018)

Ege, S.M., Misirli, E.: Extended Kudryashov method for fractional nonlinear differential equations. Math. Sci. Appl. e-notes 6(1), 19–28 (2018)

Esen, A., Tasbozan, O.: Numerical solution of time fractional Burgers equation by cubic B-spline finite elements. Mediterr. J. Math. 13, 1325–1337 (2016)

Esen, A., Yagmurlu, N.M., Tasbozan, O.: Approximate analytical solution to time-fractional damped burger and Cahn–Allen equations. Appl. Math. Inf. Sci. 7, 1951–1956 (2013)

Eslami, M., Rezazadeh, H.: The first integral method for Wu–Zhang system with conformable time-fractional derivative. Calcolo 53, 475–485 (2016)

Ghany, H.A., Hyder, A.: Abundant solutions of Wick-type stochastic fractional 2D KdV equations. Chin. Phys. B 23, 0605031–7 (2014)

Ghany, H.A., Okb El Babb, A.S., Zabel, A.M., Hyder, A.: The fractional coupled KdV equations: exact solutions and white noise functional approach. Chin. Phys. B 22, 0805011–7 (2013)

Hosseini, K., Osman, M.S., Mirzazadeh, M., Rabie, F.: Investigation of different wave structures to the generalized third-order nonlinear Schrödinger equation. Optik 206, 164259 (2020)

Huang, S.J., Yang, N.J.: Auto-Bäcklund transform and exact solutions to local conformable time-fractional viscous Burgers system. EPL 125, 15003 (2019)

Khaliril, R., Al Horani, M., Youssef, A., Sababheh, M.: A new definition of fractional derivative. J. Comput. Appl. Math. 264, 65–70 (2014)

Khatner, M.M.A., Attia, R.A.M., Lu, D.: Numerical solutions of nonlinear fractional Wu–Zhang system for water surface versus three approximate schemes. J. Ocean Eng. Sci. 4, 144–148 (2019a)

Khatner, M.M.A., Lu, D., Attia, R.A.M.: Dispersive long wave of nonlinear fractional Wu–Zhang system via a modified auxiliary equation method. AIP Adv. 9, 025003 (2019b)

Kilbas, A., Srivastava, M.H., Trujillo, J.J.: Theory and application of fractional differential equations. In: North Holland Mathematics Studies, Vol. 204 (2006)

Kudryashov, N.A.: One method for finding exact solutions of nonlinear differential equations. Commun. Nonlinear Sci. Numer. Simul. 17(6), 2248–2253 (2012)

Kudryashov, N.A.: Solitary and periodic waves of the hierarchy for propagation pulse in optical fiber. Optik 194, 163060 (2019)

Kurt, A., Çenesiz, Y., Tasbozan, O.: On the solution of Burgers equation with the new fractional derivative. Open Phys. 13, 355–360 (2015)

Kurt, A., Çenesiz, Y., Tasbozan, O.: Cankaya Univ. J. Sci. Eng. 13, 018 (2016)

Lee, J.H., Ma, W.X.: A transformed rational function method and exact solutions to the dimensional Jimbo–Miwa equation. Chaos Solitons Fractals 42(3), 1356–1363 (2009)

Li, Y.S., Ma, W.X., Zhang, J.E.: Darboux transformations of classical Boussinesq system and its new solutions. Phys. Lett. A 275, 60–66 (2000)
Ma, W.X., Fuchssteiner, B.: Explicit and exact solutions to a Kolmogorov-Petrovskii-Piskunov equation. Int. J. Nonlinear Mech. 31, 329–338 (1996)
Ma, W.X., Zhang, Y., Tang, Y.: Symbolic computation of lump solutions to a combined equation involving three types of nonlinear terms. East Asian J. Appl. Math. 10(4), 732–745 (2020)
Ma, W.X., Bai, Y., Adjiri, A.: Nonlinearity-managed lump waves in a spatial symmetric HSI model. Eur Phys. J. Plus 136, 240 (2021)
Mirzazadeh, M., Ekici, M., Eslami, M., Krishnan, E.V., Kumar, S., Biswas, A.: Solitons and other solutions to Wu-Zhang system. Nonlinear Anal. Model. 22, 441–458 (2017)
Park, C., Khatet, M.M.A., Abdel-Aty, A.-H., Attifa, R.A.M., Rezazadeh, H., Zidan, A.M., Mohamed, A.B.A.: Dynamical analysis of the nonlinear complex fractional emerging telecommunication model with higher-order dispersive cubic-quintic. Alex. Eng. J. 59, 1425–1433 (2020)
Ray, S.S.: New analytical exact solutions of time fractional KdV–KZK equation by Kudryashov methods. Chin. Phys. B 25(4), 040204 (2016)
Raza, N., Aslam, M.R., Rezazadeh, H.: Analytical study of resonant optical solitons with variable coefficients in Kerr and non-Kerr law media. Opt. Quant. Electron. 51, 59 (2019)
Samko, G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, Yverdon (1993)
Tajadodi, H., Khan, Z.A., Irshad, A.R., Aguilar, J.M.G., Khan, A., Khan, H.: Exact solutions of conformable fractional differential equations. Results Phys. 22, 103916 (2021)
Wu, T.Y., Zhang, J.E.: On modeling nonlinear long wave. In: Cook, L.P., Roytbhurd, V., Tulin, M. (eds.) Mathematical is for Solving Problems, p. 233. SIAM, Philadelphia (1996)
Yang, H.W., Guo, M., He, H.: Conservation laws of space-time fractional mZK equation for Rossby solitary waves with complete coriolis force. Int. J. Nonlinear Sci. Numer. Simul. 20, 1–16 (2019)
Yang, J.Y., Ma, W.X., Khalique, C.M.: Determining lump solutions for a combined soliton equation in (2+1)-dimensions. Eur. Phys. J. Plus 135(6), 494 (2020)
Yel, G., Baskonus, H.M.: Solitons in conformable time-fractional Wu–Zhang system arising in coastal design. Pramana J. Phys. 93, 57 (2019)
Zheng, X., Chen, Y., Zhang, H.: Generalized extended tanh-function method and its application to (1+1)-dimensional dispersive long wave equation. Phys. Lett. A 311, 145–157 (2003)
Zubair, A., Raza, N., Mirzazadeh, M., Liu, W., Zhou, Q.: Analytic study on optical solitons in parity-time-symmetric mixed linear and nonlinear modulation lattices with non-Kerr nonlinearities. Optik 173, 249–262 (2018)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.