Introduction to Doubly Special Relativity

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Abstract

In these notes, based on the lectures given at 40th Winter School on Theoretical Physics, I review some aspects of Doubly Special Relativity (DSR). In particular, I discuss relation between DSR and quantum gravity, the formal structure of DSR proposal based on $\kappa$-Poincaré algebra and non-commutative $\kappa$-Minkowski space-time, as well as some results and puzzles related to DSR phenomenology.

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1 Introduction

What is the fate of Lorentz symmetry at Planck scale? This question was the main theme of the Winter School and, as the reader could see from the proceedings, there are many possible answers. Here I would like to describe one possibility, whose central postulate is that in spite of the fact that departures from Special Relativity are introduced at scales close to Planck scale, one keeps unchanged the central physical message of the theory of relativity, namely the equivalence of all (inertial) observers. This justifies the term Relativity in the title.

To be more specific, let us start with the set of postulates of Doubly Special Relativity\(^1\) (I will use the acronym DSR in what follows) or Special Relativity with Two Observer Independent Scales, as proposed in [1], [2] (see also [3], [4].) These postulates can be formulate as follows.

- One assumes that the relativity principle holds, i.e., equivalence of all inertial observers in the sense of Galilean Relativity and Special Relativity is postulated.

- There are two observer independent scales: one of velocity \(c\), identified with the speed of light \(^2\), and second of dimension of mass \(\kappa\) (or length \(\lambda = \kappa^{-1}\)), identified with the Planck mass. Of course, it is assumed that in the limit \(\kappa \to \infty\) DSR becomes the standard Special Relativity. This postulate is the reason for the term “Doubly”. Since it turns out that the action of symmetry generators must be deformed in DSR, one may talk about “Deformed Special Relativity”.

It is a quite nontrivial problem, though, how these postulates can be realized in practice, given the fact that at the Planck scale we are to have to do with two scales of length and/or mass. Indeed, we know both from the theory and numerous experiments that in Special Relativity different observers do attribute different lengths and masses to the same measurements: as it

\(^1\)Some authors prefer to use the name Deformed Special Relativity, fortunately leading to the same acronym

\(^2\)Some readers may be confused already at this point since it is often claimed that DSR predicts dependence of the speed of massless particles on energy they carry, so that the speed of light is energy (and wavelength) dependent. Then the question arises to which speed this postulate refers to. As I will show below there are, arguably, good reasons to believe that in DSR the speed of light equals 1, independently of the energy.
is well known, we have to do with Lorentz-FitzGerald contraction and relativistic corrections to mass. How is it then possible to have at the same time relativity principle and the observer-independent scale of length or mass? It turns out that it is possible, but the price to pay is quite high, as one presumably must describe space-time in terms of non-commutative geometry, and to talk about space-time symmetries, one should use the language of quantum groups.

It should be noted also that as an immediate consequence of the postulates DSR theory should possess (like Galilean and Special Relativity theories) a ten dimensional group of symmetries, corresponding to rotations, boosts, and translations, which however, as a result of the presence of the second scale, cannot be just the linear Poincaré group. This immediately poses a problem. Namely, if we have a theory with observer independent scale of mass, then it follows that it should be expected that the standard Special Relativistic Casimir \( E^2 - p^2 = m^2 \) is to be replaced by some nonlinear mass-shell relation, between energy and three-momentum (which would involve the parameter \( \kappa^3 \).) Thus the second scale \( \kappa \) must be encoded into the mass-shell condition so that it is kept invariant by symmetry transformations. But then it follows that the speed of massless particles defined as \( \partial E/\partial p \) would be dependent on the energy they carry, which makes it hard to understand what would be the operational meaning of the observer-independent speed of light. Below I will suggest ways out of this dilemma.

I should warn the reader that the construction of the theory of Doubly Special relativity is not completed yet; in fact we do not even have a single DSR candidate, which would satisfy all the requirements of internal and conceptual self-consistency. Nevertheless during the last three years many results have been obtained, and for example we now control pretty well the one particle sector of the theory, both technically and conceptually. However, many problems remain, for example, we still do not understand the multi-particle sector of DSR theory.

The structure of this notes corresponds to the structure of the lectures I gave at the Winter School. The next section corresponding to the first lecture is devoted to the questions whether and how DSR could emerge as an appropriate limit of quantum gravity. The complete answer to these questions is still unknown but we have some number of evidences suggesting

\[3\text{Note however that there exists a class of models of DSR, in which the dispersion relation between energy and momentum is not deformed (see below.)}\]
that indeed DSR may be rooted in quantum gravity. The third section of these notes is devoted to describing techniques used in a particular, best developed approach to DSR, based on the so-called $\kappa$-Poincaré algebra and $\kappa$-Minkowski space-time. In section 4 I would like to describe main results obtained in the DSR framework, as well as bunch of open problems, mainly related to the multi-particle processes.

2 DSR from quantum gravity?

If the DSR idea is correct, it is quite natural to expect that Doubly Special Relativity emerges somehow as a limit of quantum gravity. It is rather clear why it must be so. In the standard Special Relativity we have only one scale, and there is no natural way in which another scale of mass and/or length could be introduced purely in Special Relativistic setting. On the other hand, in quantum gravity we have, in addition to the velocity scale $c$, three additional dimensionful constants, $G$, $\hbar$ (which I often set equal 1 in what follows), and (sometimes) the cosmological constant $\Lambda$. The immediate idea is that in the limiting procedure, in which the gravitational interactions as well as quantum effects become negligible, and the space-time becomes effectively flat (at least locally), some trace of these constants remains, giving rise to new observer-independent scale $\kappa$. In this section I will try to convince the reader that such scenario may indeed result from quantum theory of gravity.

Usually we take for granted that the $G \to 0$, (and possibly $\Lambda \to 0$ if we start with non-zero cosmological constant) limit of (quantum) gravity is just the Minkowski space-time. But perhaps this is not correct, and we are forced to take the limit (especially in the case in which point particles are present) such that either

1. $\lim_{G, \Lambda \to 0} \sqrt{G/\Lambda} = \kappa^{-1} \neq 0$, or alternatively,

2. $\lim_{G, \hbar \to 0} \sqrt{\hbar/G} = \kappa \neq 0$.

It is not clear which of these scenarios (if any) is realized in Nature, but there are some indirect evidences in favor of the claim that indeed it might be so.

Let us try to investigate the first scenario following the ideas presented in [5]. To this end let us consider first the three-dimensional quantum gravity with positive cosmological constant $\Lambda$. Then it is well known [6] that
the excitations of 3d quantum gravity with cosmological constant transform under representations of the quantum deformed deSitter algebra $SO_q(3,1)$, with $z = \ln q$ behaving in the limit of small $\Lambda \hbar^2/\kappa^2$ as $z \approx \sqrt{\Lambda \hbar}/\kappa$.

I will not discuss at this point the notion of quantum deformed algebras (Hopf algebras) in much details (the book [7] would be a good references for the reader who wants to study this exciting branch of mathematics.) It will suffice to say that quantum algebras consist of several structures, the most important for our current purposes would be the universal enveloping algebra, which could be understand as an algebra of brackets among generators, which are equal to some analytic functions of them. Thus the quantum algebra is a generalization of a Lie algebra, and it is worth observing that the former reduces to the latter in an appropriate limit. Quantum algebras start playing an important role in various branches of theoretical physics; in particular, in some cases, they can play a role of relativistic symmetries in some field theoretical models (see an excellent, pedagogical exposition in [8].) In the case of quantum algebra $SO_q(3,1)$ the algebraic part looks as follows (the parameter $z$ used below is related to $q$ by $z = \ln q$)

$$\begin{align*}
[M_{2,3}, M_{1,3}] &= \frac{1}{z} \sinh(z M_{1,2}) \cosh(z M_{0,3}) \\
[M_{2,3}, M_{1,2}] &= M_{1,3} \\
[M_{2,3}, M_{0,3}] &= M_{0,2} \\
[M_{2,3}, M_{0,2}] &= \frac{1}{z} \sinh(z M_{0,3}) \cosh(z M_{1,2}) \\
[M_{1,3}, M_{1,2}] &= -M_{2,3} \\
[M_{1,3}, M_{0,3}] &= M_{0,1} \\
[M_{1,3}, M_{0,1}] &= \frac{1}{z} \sinh(z M_{0,3}) \cosh(z M_{1,2}) \\
[M_{1,2}, M_{0,2}] &= -M_{0,1} \\
[M_{1,2}, M_{0,1}] &= M_{0,2} \\
[M_{0,3}, M_{0,2}] &= M_{2,3} \\
[M_{0,3}, M_{0,1}] &= M_{1,3} \\
[M_{0,2}, M_{0,1}] &= \frac{1}{z} \sinh(z M_{1,2}) \cosh(z M_{0,3})
\end{align*}$$

(1)

Since this is our first encounter with quantum algebra let us pause for a moment to discuss its main features. First of all, we observe that on the right

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4Since in 3d, the dimension of the gravitational constant is $1/kg$, we write $G = \kappa^{-1}$. 

5
hand sides we do not have linear functions generators, as in the Lie algebra
case, but some (analytic) functions of them. However we still assume that
the brackets are antisymmetric and that Jacobi identity holds.

**Exercise 1.** Convince yourself by direct inspection that for the algebra (1) Jacobi identities indeed hold.

It follows from this observation that contrary to the Lie algebras case, we
are now entitled to use any analytic functions of the initial set of generators
as a basis of the quantum algebra (in the Lie algebra case we can only take
linear combinations of them.) It should be stressed already at this point that
quantum algebras possess more structures than just the enveloping algebra
structure (for more details see [7]); some of them will be relevant in what
follows. Note that in the limit $z \to 0$ the algebra (1) becomes the standard
algebra $SO(3,1)$, and this is the reason for using the term $SO_q(3,1)$.

**Exercise 2.** Denote by $M^z_{\mu\nu}$ the generators of the algebra (1) and by
$M_{\mu\nu}$ the generators of the standard $SO(3,1)$ algebra (obviously the equation
$\lim_{z \to 0} M^z_{\mu\nu} = M_{\mu\nu}$ should hold.) Find explicit expressions for $M^z_{\mu\nu}$ as func-
tions of $M_{\mu\nu}$ and $z$. (If this exercise happens to be too hard do that only up
to the next-to-leading order in $z$.)

The $SO(3,1)$ Lie algebra is the three dimensional de Sitter algebra and
it is well known how to obtain the three dimensional Poincaré algebra from
it. First of all one has to single out the energy and momentum genera-
tors of right physical dimension (note that the generators $M_{\mu\nu}$ of (1) are dimen-
sionless): one identifies three-momenta $P_\mu \equiv (E, P_i)$ ($\mu = 1, 2, 3, i = 1, 2$) as
appropriately rescaled generators $M_{0,\mu}$ and then one takes the Inômï–Wigner
contraction limit (see, for example, [9].)

Let us try therefore to proceed in an analogous way and contract the
algebra (1). To this aim we must first rescale some of the generators by an
appropriate scale, provided by combination of dimensionful constants present
in definition of the parameter $z$

\begin{align*}
E &= \sqrt{\Lambda} h \ M_{0,3} \\
P_i &= \sqrt{\Lambda} h \ M_{0,i} \\
M &= M_{1,2}
\end{align*}
Taking now into account the relation $z \approx \sqrt{\Lambda h}/\kappa$, which holds for small $\Lambda$, from
\[
[M_{2,3}, M_{1,3}] = \frac{1}{z} \sinh(zM_{1,2}) \cosh(zM_{0,3})
\]
we find
\[
[N_2, N_1] = \frac{\kappa}{h\sqrt{\Lambda}} \sinh(\hbar\sqrt{\Lambda}/\kappa M) \cosh(E/\kappa)
\] (3)
Similarly from
\[
[M_{0,2}, M_{0,1}] = \frac{1}{z} \sinh(zM_{1,2}) \cosh(zM_{0,3})
\]
we get
\[
[P_2, P_1] = \sqrt{\Lambda h} \kappa \sinh(\sqrt{\Lambda h}/\kappa M) \cosh(E/\kappa)
\] (4)
Similar substitutions can be made in other commutators of (1). Now going to the contraction limit $\Lambda \to 0$, while keeping $\kappa$ constant we obtain the following algebra
\[
[N_i, N_j] = -M \epsilon_{ij} \cosh(E/\kappa)
\]
\[
[M, N_i] = \epsilon_{ij} N^j
\]
\[
[N_i, E] = P_i
\]
\[
[N_i, P_j] = \delta_{ij} \kappa \sinh(E/\kappa)
\]
\[
[M, P_i] = \epsilon_{ij} P^j
\]
\[
[E, P_i] = 0
\]
\[
[P_2, P_1] = 0
\] (5)
This algebra is called the three dimensional $\kappa$-Poincaré algebra (in the standard basis.)

It turns out that this contracted algebra is again a quantum algebra, i.e., after the contraction all the additional structures of $SO_q(3,1)$ became the analogous structures of the new algebra (which is not obvious a priori because, in principle, it may happen that during the contraction procedure additional structures of the quantum algebra may become not well defined). This really nontrivial and remarkable result has been obtained in early nineties in [10], [11].

Let us pause for a moment here to make couple of comments. First of all, one easily sees that in the limit $\kappa \to \infty$ from the $\kappa$-Poincaré algebra
algebra (5) one gets the standard Poincaré algebra. Second, we see that in this algebra both the Lorentz and translation sectors are deformed. However, as I have been stressing already, in the case of quantum algebras one is free to change the basis of generators in arbitrary, analytic way. It turns out that there exists such a change of basis that the Lorentz part of the algebra becomes classical (i.e., undeformed.) Such a basis, derived in \[12\], is called bicrossproduct (because of the remarkable bicrossproduct structure of the full quantum algebra, see \[7\]), and the Doubly Special Relativity model (both in 3 and 4 dimensions) based on such an algebra is called DSR1. In this basis the algebra looks as follows

\[
\begin{align*}
[N_i, N_j] &= -\epsilon_{ij} M \\
[M, N_i] &= \epsilon_{ij} N^j \\
[N_i, E] &= P_i \\
[N_i, P_j] &= \delta_{ij} \frac{\kappa}{2} \left(1 - e^{-2E/\kappa} + \frac{\vec{P}^2}{\kappa^2}\right) - \frac{1}{\kappa} P_i P_j \\
[M, P_i] &= \epsilon_{ij} P^j \\
[E, P_i] &= 0 \\
[P_1, P_2] &= 0.
\end{align*}
\]

Exercise 3. Derive explicit transformations from variables in (5) to variables in (6) (solution can be found in \[12\].)

Note now that the algebra (6) is exactly of the form needed in Doubly Special Relativity. By construction this is the algebra of symmetries of flat space, being an appropriate limit of the algebra of symmetries of states of quantum gravity. Moreover it manifestly contains the observer-independent scale of dimension of mass \(\kappa\).

Exercise 4. Check that \(\kappa\) is the observer-independent scale in the sense that if \(|\vec{P}| = \kappa\), then \(\delta|\vec{P}| = 0\), where \(\delta\) denotes the change under infinitesimal action of boosts (solution can be found in \[3\].)

This shows that, at least in principle, one can try to construct a theory, which satisfies principles of DSR, and that such a theory may be neither inconsistent, nor trivial. Of course to construct a theory of particle kinematics, with symmetries defined by (5), (6) much more is needed; for example we
must know how to compose momenta for multiparticle systems, what is the
form of conservation laws, etc. I will discuss these issues in the following
sections below.

The algebras (5), (6) has been derived from the limit of the algebra of
symmetries of three dimensional gravity, which, as it is well known, has some
remarkable features, namely it is a topological field theory with no dynamical
degrees of freedom. The question arises as to if it is possible to repeat this
analysis in the most interesting, four dimensional case. One can expect that
this latter case would be much more complex: to go to the appropriate
limit reminding the Special Relativistic setting one should first switch off the
dynamical degrees of freedom of gravity. The good news is that in the limit,
in which the gravitational constant goes to zero, four dimensional gravity
becomes a topological field theory again, reminding the three-dimensional
situation. However, I must admit that it is not known if there exists a limit
of four dimensional quantum gravity, which results in DSR theory. There
are, however, some circumstantial evidences in favor of such a claim.

In the four-dimensional case the excitations of ground state\(^5\) of a qua-
ntum gravity theory are conjectured to transform under represen-
tations of the quantum deformed de Sitter algebra \(SO_q(3,2)\), with \(z = \ln q\) behaving in the
limit of small \(\Lambda \kappa^{-2}\) as, \(z \approx \Lambda \kappa^{-2}\) \(^{13},^{14},^{15},^{16}\). Then (see \(^3\) for more
details) one again takes the limit, which this time is much more involved,
since one must not only rescale variables, as it was done above, but also to
renormalize them (see also \(^10\)), in order to get finite result. It turns out
that now we have to do with one parameter family of contractions, labelled
by real, positive parameter \(r\): for \(0 < r < 1\) as a result of contraction one
obtains the standard Poincaré algebra, for \(r > 1\) the contraction does not
exists and only for a single value \(r = 1\) the contraction gives the desired
four dimensional \(\kappa\)-Poincaré algebra. It remains therefore an open problem
whether and how quantum gravity singles out the value for \(r\) and is this value
1?

We see therefore that it is possible to obtain the DSR1 algebra by con-
tracting the algebras of symmetries of quantum gravity, in dimensions 3 and
4. This strongly suggests that indeed this algebra would be an algebra of
symmetries of particle kinematics taking part in the flat space. It is interest-

\(^5\)We restrict our attention to the ground state, because we are interested only in the
limit in which all local degrees of freedom of quantum gravity are switched off. After all
our goal is to formulate a theory which is to replace Special Relativity!

\(^6\)From now on I put \(\hbar\) equal 1.
ing therefore that, in some cases at least, there are traces of quantum gravity in this algebra. I must stress, however that it remains to prove rigorously that the algebra $SO_q(3, 2)$ indeed plays the conjectured role in quantum gravity.

And now something completely different. In Special Relativity the Poincaré algebra plays dual role: it is an algebra of symmetries of space–time and at the same time it labels momenta and spin of a particle. Deformed Poincaré algebra should also play such a dual role, so now let us investigate the algebras of charges carried by point particles coupled to quantum gravity. As I will show in section 3 below, in the DSR framework it turns out that the four momentum of a particle is not a point in the flat Minkowski space, as in Special Relativity, but instead, the manifold of momenta is a curved manifold of constant curvature, $\kappa^{-2}$ [18], [20]. But then, by the same token, positions, which are identified with “translations” of momenta, cannot commute, so that the space-time of DSR should necessarily be a non-commutative manifold, called $\kappa$-Minkowski space-time [12], [21]. Let us see therefore, how this picture emerges from quantum gravity, this time coupled to point particles, and without cosmological constant.

In what follows I will review the results obtained in [22]. Let us start with the case of three-dimensional quantum gravity now coupled to a point particle. Then it is well known (see the detailed and clear exposition in [23] and references therein) that since in 3d gravity does not have any dynamical degrees of freedom, the theory is fully characterized by Poincaré charges carried by the particle. In other words the theory reduces to a theory of the phase space of the particle, which is different from the phase space of free particles, as a result of the modifications induced by topological degrees of freedom of gravity. This phase space is characterized by the following properties [23]

- The coordinates of the particle (understood as variables on the phase space, which are canonically conjugated to momenta) do not commute and instead
  \[ [x_0, x_i] = -\frac{1}{\kappa} x_i, \quad [x_i, x_j] = 0. \] (7)
  (The bracket above is either the Poisson bracket or the commutator.) Such a non-commutative space-time is called $\kappa$-Minkowski.

- The space of (three-) momenta is not the flat $R^3$ manifold, but the
maximally symmetric space of constant curvature $-\kappa$ (anti de Sitter space of momenta).

- Last but not least it has been shown in numerous works on 3d quantum gravity that the full Hopf $\kappa$-Poincaré algebra with all the quantum group structures plays the role (see e.g., [17] and references therein.)

But as I will show in Chapter 3, these are exactly the properties of phase space of a particle in DSR (in the case of both 3 and 4 dimensional space-time.) Note in passing an interesting duality between curvature and non-commutativity

\[
\text{Curvature of momentum space} \leftrightarrow \text{Non-commutativity of position space}
\]

As I will show below this duality can be understood as a consequence of the co-product structure of quantum Poincaré algebra.

Thus we see again that kinematics of particles in three dimensions is described by the DSR-like structure with observer independent scale. The question arises as to if something similar can happen in four space-time dimensions. I have only circumstantial evidences in favor of such claim, and the argument goes as follows [22].

The main idea is to construct an experimental situation that forces a dimensional reduction from the four dimensional to the $2+1$ dimensional theory. It is interesting that this can be done in quantum theory, using the uncertainty principle as an essential element of the argument. Let us consider a free elementary particle in $3+1$ dimensions, whose mass is less than $G^{-1} = \kappa$. The motion of the particle will be linear, at least in some classes of coordinates systems, not accelerating with respect to the natural inertial coordinates at infinity. Let us consider the particle as described by an inertial observer who travels perpendicular to the plane of its motion, which I will call the $z$ direction. From the point of view of that observer, the particle is in an eigenstate of longitudinal momentum, $\hat{P}_{z}^{\text{total}}$, with some eigenvalue $P_z$. Since the particle is in an eigenstate of $\hat{P}_{z}^{\text{total}}$ its wavefunction will be uniform in $z$, with wavelength $L$ where (note that I assume here that

\[\footnote{See the insightful discussion in [7], in which Shahn Majid argues that this duality indicates a deep relation between non-commutativity and quantization of gravity.} \]
$L$ is so large that I can trust the standard uncertainty relation; besides this uncertainty relation is not being modified in some formulations of DSR)

$$L = \frac{1}{P_{total}}$$     (8)

At the same time, we assume that the uncertainties in the transverse positions are bounded a scale $r$, such that $r \ll 2L$. Then the wavefunction for the the particle has support on a narrow cylinder of radius $r$ which extend uniformly in the $z$ direction. Finally, we assume that the state of the gravitational field is semiclassical, so that to a good approximation, within $C$ the semiclassical Einstein equations hold.

$$G_{ab} = 8\pi G < \hat{T}_{ab}>$$     (9)

Note that we do not have to assume that the semiclassical approximation holds for all states. We assume something much weaker, which is that there are subspaces of states in which it holds. This assumption is, in a sense, analogous to the assumption above that we are interested only in the analysis of ground state of quantum gravity.

Since the wavefunction is uniform in $z$, this implies that the gravitational field seen by our observer will have a spacelike Killing field $k^a = (\partial/\partial z)^a$.

Thus, if there are no forces other than the gravitational field, the particle described semiclassically by (9) must be described by an equivalent 2 + 1 dimensional problem in which the gravitational field is dimensionally reduced along the $z$ direction so that the particle, which is the source of the gravitational field, is replaced by a punctures.

The dimensional reduction is governed by a length $d$, which is the extent in $z$ that the system extends. We cannot take $d < L$ without violating the uncertainty principle. It is then convenient to take $d = L$. Further, since the system consists of the particle, with no intrinsic extent, there is no other scale associated with their extent in the $z$ direction. We can then identify $z = 0$ and $z = L$ to make an equivalent toroidal system, and then dimensionally reduce along $z$. The relationship between the four dimensional Newton’s constant $G^4$ and the three dimensional Newton’s constant $G^3 = G$ is given by

$$G^3 = \frac{G^4}{\bar{P}^z} = \frac{G^4 P_{total}^z}{\hbar}$$     (10)
Thus, in the analogous 3 dimensional system, which is equivalent to the original system as seen from the point of view of the boosted observer, the Newton’s constant depends on the longitudinal momentum.

Of course, in general there will be an additional scalar field, corresponding to the dynamical degrees of freedom of the gravitational field. We will for the moment assume that these are unexcited, but exciting them will not affect the analysis so long as the gravitational excitations are invariant also under the Killing field and are of compact support.

Now we note that, if there are no other particles or excited degrees of freedom, the energy of the system can to a good approximation be described by the hamiltonian $H$ of the two dimensional dimensionally reduced system. This is described by a boundary integral, which may be taken over any circle that encloses the particle. But it is well known that in 3d gravity $H$ is bounded from above. This may seem strange, but it is easy to see that it has a natural four dimensional interpretation.

The bound is given by

$$M < \frac{1}{4G^3} = \frac{L}{4G^4}$$

where $M$ is the value of the ADM hamiltonian, $H$. But this just implies that

$$L > 4G^4 M = 2R_{Sch}$$

i.e. this has to be true, otherwise the dynamics of the gravitational field in $3 + 1$ dimensions would have collapsed the system to a black hole! Thus, we see that the total bound from above of the energy in $2 + 1$ dimensions is necessary so that one cannot violate the condition in $3 + 1$ dimensions that a system be larger than its Schwarzschild radius.

Note that we also must have

$$M > P_{z}^{tot} = \frac{\hbar}{L}$$

Together with (12) this implies $L > l_{Planck}$, which is of course necessary if the semiclassical argument we are giving is to hold.

Now, we have put no restriction on any components of momentum or position in the transverse directions. So the system still has symmetries in the transverse directions. Furthermore, the argument extends to any number of particles, so long as their relative momenta are coplanar. Thus, we learn the following.
Let $\mathcal{H}^{QG}$ be the full Hilbert space of the quantum theory of gravity, coupled to some appropriate matter fields, with $\Lambda = 0$. Let us consider a subspace of states $\mathcal{H}^{\text{weak}}$ which are relevant in the low energy limit in which all energies are small in Planck units. We expect that this will have a symmetry algebra which is related to the Poincaré algebra $\mathcal{P}^4$ in 4 dimensions, by some possible small deformations parameterized by $G^4$ and $\hbar$. Let us call this low energy symmetry group $\mathcal{P}_G^4$.

Let us now consider the subspace of $\mathcal{H}^{\text{weak}}$ which is described by the system we have just constructed. It contains the particle, and is an eigenstate of $\hat{P}_z$ with large $\hat{P}_z$ and vanishing longitudinal momentum. Let us call this subspace of Hilbert space $\mathcal{H}_{P_z}$.

The conditions that define this subspace break the generators of the (possibly modified) Poincaré algebra that involve the $z$ direction. But they leave unbroken the symmetry in the $2 + 1$ dimensional transverse space. Thus, a subgroup of $\mathcal{P}_G^{3+1}$ acts on this space, which we will call $\mathcal{P}_G^{2+1}$.

We have argued that the physics in $\mathcal{H}_{P_z}$ is to good approximation described by an analogue system in of a particle in $2 + 1$ gravity. However, we know from the results mentioned above that the symmetry algebra acting there is not the ordinary 3 dimensional Poincaré algebra, but the $\kappa$-Poincaré algebra in 3 dimensions, with

$$\kappa^{-1} = \frac{4G^4 P^{tot}_z}{\hbar} \quad (14)$$

Now we can note the following. Whatever $\mathcal{P}_G^4$ is, it must have the following properties:

- It depends on $G^4$ and $\hbar$, so that it’s action on each subspace $\mathcal{H}_{P_z}$, for each choice of $P_z$, is the $\kappa$ deformed 3d Poincaré algebra, with $\kappa$ as above.

- It does not satisfy the rule that momenta and energy add, on all states in $\mathcal{H}$, since they are not satisfied in these subspaces.

- Therefore, whatever $\mathcal{P}_G^4$ is, it is not the classical Poincaré group.

Thus the theory of particle kinematics at ultra high energies is not Special Relativity, and the arguments presented above suggest that it might be Doubly Special Relativity. So it is good time now to start discussing the structures of this theory.
3 Doubly Special Relativity and the $\kappa$-Poincaré algebra

Soon after pioneering papers of Amelino-Camelia [1], [2] it was realized in [3] and [4] that the $\kappa$-Poincaré algebra [10], [11], [12] is a perfect mathematical setting to describe one particle kinematics in DSR. Let us recall from the preceding section that in particular, in the bicrossproduct basis the brackets of rotations $M_i$, boosts $N_i$, and the components of momenta $P^\mu$ read

\[
[M_i, M_j] = i \epsilon_{ijk} M_k, \quad [M_i, N_j] = i \epsilon_{ijk} N_k, \quad [N_i, N_j] = -i \epsilon_{ijk} M_k, \quad [M_i, P_j] = i \epsilon_{ijk} P_k, \quad [M_i, P_0] = 0, \quad [N_i, P_j] = i \delta_{ij} \left( \frac{\kappa}{2} \left( 1 - e^{-2P_0/\kappa} \right) + \frac{1}{2\kappa} \vec{P}^2 \right) - i \frac{1}{\kappa} P_i P_j \quad [N_i, P_0] = i P_i
\]

It is important to note that the algebra of $M_i, N_i$ is just the standard Lorentz algebra, so one of the first conclusions is that the Lorentz sector of $\kappa$-Poincaré algebra is not deformed. Therefore in DSR theories, in accordance with the first postulate above, the Lorentz symmetry is not broken but merely nonlinearly realized in its action on momenta. This simple fact has lead some authors (see e.g., [21], [25]) to the claim that DSR is nothing but the standard Special Relativity in non-linear disguise. As we will see this view is clearly wrong, simply because the algebra (15)–(18) describes only half of the phase space of the particle, and the full phase space algebra cannot be reduced to the one of Special Relativity.

As one can easily check, the Casimir of the $\kappa$-Poincaré algebra reads

\[
\kappa^2 \cosh \frac{P_0}{\kappa} - \frac{\vec{P}^2}{2} e^{P_0/\kappa} = M^2.
\]

**Exercise 5.** Check that (19) is indeed the Casimir of the algebra (15)–(18) i.e., its commutators with all the generators of $\kappa$-Poincaré algebra vanish. Is it the only possible Casimir of this algebra? Compute the velocity...
\( v = \partial P_0/\partial |\vec{P}| \). How the behavior of this velocity depends on the sign of \( \kappa \)?

It follows from (19) that the value of three-momentum \( |\vec{P}| = \kappa \) corresponds to infinite energy \( P_0 = \infty \). One can check easily (see Exercise 4 above) that in this particular realization of DSR \( \kappa \) is indeed observer independent \([3, 4]\) (i.e., if a particle has momentum \( |\vec{P}| = \kappa \) for some observer, it has the same momentum for all, Lorentz related, observers.) One also sees that the speed of massless particles, naively defined as derivative of energy over momentum, increases monotonically with momentum and diverges for the maximal momentum \( |\vec{P}| = \kappa \), if \( \kappa \) is positive. As I mentioned already in the DSR terminology, the theory based on the algebra (15)–(18) with Casimir (19) is sometimes called DSR1.

One should note at this point that the bicrossproduct algebra above is not the only possible realization of DSR. For example, in \([26, 27]\) Magueijo and Smolin proposed and carefully analyzed another DSR proposal, called sometimes DSR2. In DSR2 the Lorentz algebra is still not deformed and there are no deformations in the brackets of rotations and momenta. The boosts–momenta generators have now the form

\[
[N_i, p_j] = i \left( \delta_{ij} p_0 - \frac{1}{\kappa} p_i p_j \right),
\]

and

\[
[N_i, p_0] = i \left( 1 - \frac{p_0}{\kappa} \right) p_i.
\]

It is easy to check that the Casimir for this algebra has the form

\[
M^2 = \frac{p_0^2 - \vec{p}^2}{\left( 1 - \frac{p_0}{\kappa} \right)^2}.
\]

**Exercise 6.** Check that (22) is indeed the Casimir of the DSR2 algebra \([20, 21]\). Compute the velocity \( v = \partial P_0/\partial |\vec{P}| \). Find relations between DSR1 and DSR2 momentum variables (the answer can be found in \([19, 21]\).)

Moreover there is a basis of DSR, closely related to the famous Snyder theory \([28]\), in which the energy-momentum space algebra is purely classical (it was first found in \([29]\) and further analyzed in \([19, 21]\).)
Exercise 7. Find explicit transformation from DSR1 to the classical basis, in which all the brackets are identical to those of the standard Poincaré algebra. (See [19], [21], where the relation of the DSR algebra in classical basis and Snyder’s theory is analyzed in details.)

3.1 Space-time of DSR

The formulation of DSR in the energy-momentum space is clearly incomplete, as it lacks any description of the structure of space-time. DSR has been formulated in a somehow unusual way: one started with the energy–momentum space and only then the problem of construction of space-time had been considered. Usually we do the opposite, for example in the standard formulation of Special Relativity one starts with clear operational definition of space-time notions (distance, time interval) and only then the energy-momentum space and phase space is being constructed.

Exercise 7. (Difficult.) Formulate Special Relativity in the operational way, taking as a starting point the space of energy and momenta.

There are in principle many ways how the phase space can be constructed. For example in [30] one constructs the position space along the same lines as the energy-momentum space has been constructed in [26], [27]. Here, following [21], I take another route. As I have been stressing in the preceding section, one of the distinctive features of the \( \kappa \)-Poincaré algebra is that it possesses additional structures that make it a Hopf algebra. Namely one can construct the so called co-products for the rotation, boosts, and momentum generators, which, in turn, can be used to provide a procedure to construct the phase space in a unique way.

The co-product is the mapping from the algebra \( \mathcal{A} \) to the tensor product \( \mathcal{A} \otimes \mathcal{A} \) satisfying some requirements that make it in a sense dual to algebra multiplication (see [7] for details), which essentially provides a rule how the algebra acts on products (of functions, and, in physical applications, on multiparticle states.) For the bicrossproduct \( \kappa \)-Poincaré algebra (15)–(18) the co-products read

\[
\Delta(P_0) = 1 \otimes P_0 + P_0 \otimes 1 
\]

(23)

\(^9\)By this I mean that I do not quite know how to solve it (as a matter of fact I believe nobody does)!
\[
\Delta(P_k) = P_k \otimes e^{-P_0/\kappa} + \mathbb{1} \otimes P_k \\
\Delta(M_i) = M_i \otimes \mathbb{1} + \mathbb{1} \otimes M_i \\
\Delta(N_i) = \mathbb{1} \otimes N_i + N_i \otimes e^{-P_0/\kappa} - \frac{1}{\kappa} \epsilon_{ijk} M_j \otimes P_k
\] (24) (25) (26)

In order to construct the one-particle phase space we must first introduce objects that are dual to \( M_i, N_i, \) and \( P_\mu \). These are the matrix \( \Lambda^{\mu\nu} \) and the vector \( X^\mu \). Let us briefly interpret their physical meaning. \( X^\mu \) are to be dual to momenta \( P_\mu \), which clearly indicates that they should be interpreted as translation of momenta, in other words the positions. The duality between \( \Lambda^{\mu\nu} \) and \( M_{\mu\nu} = (M_i, N_i) \) is a bit more tricky. However if one interprets \( M_{\mu\nu} \) in analogy to the interpretation of momenta, i.e., as Lorentz charge carried by the particle, that is its angular momentum, then the dual object \( \Lambda^{\mu\nu} \) has clear interpretation of Lorentz transformation. Thus we have the structure of the form \( G \times MP \), where \( G \) is the Poincaré group acting on the space of Poincaré charges of the particle \( MP \). We see therefore that we can make use of the powerful mathematical theory of Lie-Poisson groups and co-adjoint orbits (see, for example, [31], [32]) and their quantum deformations.

Following [33] and [34] we assume the following form of the co-product on the group

\[
\Delta(X^\mu) = \Lambda^{\mu\nu} \otimes X^\nu + X^\mu \otimes \mathbb{1}
\]
(27)
and

\[
\Delta(\Lambda^{\mu\nu}) = \Lambda^{\mu\rho} \otimes \Lambda^{\rho\nu}
\]
(28)

The next step is to define the pairing between elements of the algebra and of the group in a canonical way that establish the duality between these two structures.

\[
\langle P_\mu, X^\nu \rangle = i \delta^\nu_\mu
\]
(29)
\[
\langle M^{\alpha\beta}, \Lambda^{\mu\nu} \rangle = i \left( g^{\alpha\mu} \delta^\beta_\nu - g^{\beta\mu} \delta^\alpha_\nu \right)
\]
(30)
\[
\langle \Lambda^{\mu\nu}, 1 \rangle = \delta^\mu_\nu
\]
(31)

In (30) \( g^{\alpha\mu} \) is the Minkowski space-time metric. This pairing must be consistent with the co-product structure in the following sense

\[
\langle A, XY \rangle = \langle A^{(1)}, X \rangle \langle A^{(2)}, Y \rangle,
\]
(32)
\[
\langle AB, X \rangle = \langle A, X^{(1)} \rangle \langle B, X^{(2)} \rangle,
\]
(33)
The rules (29)–(33) make it possible to construct the commutator algebra of the phase space. To this end one makes use of the Heisenberg double procedure [32], [34], that defines the brackets in terms of the pairings as follows (no summation over repeated indices here!)

\[
[X^\mu, P^\nu] = P^\nu \langle X^\mu_{(1)}, P^\nu_{(2)} \rangle X^\mu_{(2)} - P^\nu X^\mu,
\]
(34)

\[
[X^\mu, M^{\rho\sigma}] = M^{\rho\sigma}_{(1)} \langle X^\mu_{(1)}, M^{\rho\sigma}_{(2)} \rangle X^\mu_{(2)} - M^{\rho\sigma} X^\mu,
\]
(35)

and analogously for \(\Lambda^\mu_{\nu}\) commutators, where on the right hand side we make use of the standard ("Sweedler") notation for co-product

\[
\Delta T = \sum T_{(1)} \otimes T_{(2)}.
\]

As an example let us perform these steps in the case of the bicrossproduct \(\kappa\)-Poincaré algebra of DSR1. It follows from (24), and (29), and (33) that

\[
< P_i, X_0 X_j > = -\frac{1}{\kappa} \delta_{ij}, \quad < P_i, X_j X_0 > = 0,
\]

from which one gets

\[
[X_0, X_i] \equiv X_0 X_i - X_i X_0 = -\frac{i}{\kappa} X_i.
\]
(36)

Similarly, using (34) we get the standard relations

\[
[P_0, X_0] = -i, \quad [P_i, X_j] = i \delta_{ij}.
\]
(37)

It turns out that the phase space algebra contains one more non-vanishing commutator (which can be, of course, also obtained from Jacobi identity), namely

\[
[P_i, X_0] = -\frac{i}{\kappa} P_i.
\]
(38)

Thus we have constructed the phase space of the bicrossproduct \(\kappa\)-Poincaré algebra of DSR1. Let us stress that this construction relies heavily on the form of co-product. However, as it will turn out below, some of the commutators are sensitive to the particular form of the DSR, while the others are not. In particular we will see that the non-commutativity of positions \ref{eq:36} is to large extend universal for a whole class of DSR theories. The non-commutative space-time with such Lie-like type of non-commutativity
is called $\kappa$-Minkowski space-time.

**Exercise 8.** Using Jacobi identity derive the brackets of boosts and positions, assuming that they form a Lie algebra. Which algebra is it? (The answer can be found below.)

### 3.2 From DSR theory to DSR theories

The introduction of invariant momentum (or mass) scale $\kappa$ has immediate consequences. The most important is that there is nothing sacred about the bicrossproduct DSR presented above, as one can simply use $\kappa$ to define new energy and momentum (new basis of DSR) as analytic functions of the old ones, to wit

$$ P_i = \mathcal{F}_i(P_i, P_0; \kappa), \quad P_0 = \mathcal{F}_0(P_i, P_0; \kappa), \quad \text{(39)} $$

the only restrictions being that the equations in (39) transform covariantly under rotations and that in the $\kappa \to \infty$ limit $P_\mu = P_\mu$, because we insist on the right low energy limit in all the bases. Observe that such a change of energy and momentum is not possible in a theory without any mass scale, like special relativity and Newtonian mechanics, in which the energy momentum spaces are linear, and the mass shell conditions are expressed by quadratic form.

Then a natural question arises: which momenta are the “right” ones? The hope is that the theory of quantum gravity or some other fundamental theory, from which DSR is descending will tell what is the correct physical choice. One can also contemplate the possibility that in the final, complete formulation of DSR one will have to do with some kind of “energy-momentum general covariance”, i.e., that physical observables do not depend on a particular realization of eq. (39), like observables in general relativity do not depend on coordinate system. Then a natural question arises: is it possible to understand transformations (39) as coordinate transformations on some (energy-momentum) space?

Surprisingly enough the answer to this question is in the positive: indeed the transformations between DSR theories, described by (39) are nothing but coordinate transformation of the constant curvature manifold, on which momenta live. To reach this conclusion one observes first [19], [21] that it follows from the Heisenberg double construction that both the $\kappa$-Minkowski commutator (36) and the commutators between Lorentz charges $M_{\mu\nu}$ and
positions $X_\mu$ are left invariant by the transformations (39). This follows from the fact that the transformations (39) are severely constrained by assumed rotational invariance and the fact that in the $\kappa \to \infty$ limit the new energies and momenta must be the same as in the standard Special Relativity. Since the bicrossproduct DSR variables satisfy this requirement it follows that the new variables cannot differ from the DSR1 ones in the $\kappa^0$ order. Therefore, in the leading order, they must be of the form

$$\mathcal{P}_i \approx P_i + \alpha \frac{1}{\kappa} P_0 + O\left(\frac{1}{\kappa^2}\right), \quad \mathcal{P}_0 = P_0 + \beta \frac{1}{\kappa} P_0^2 + O\left(\frac{1}{\kappa^2}\right)$$

(40)

where $\alpha$ and $\beta$ are numerical parameters. It turns out that in computing the brackets of positions $X$ and the ones of positions with boosts Heisenberg double procedure picks up only the first terms in this expansion, and thus the form of the commutators remains unchanged. Of course, the position-momenta commutators are changed by the transformations (39), (40).

Exercise 9. Using expansion (40) derive the brackets of positions and four-momenta $\mathcal{P}_\mu$. It would help to notice that co-product is a homomorphism and thus $\Delta(ab) = \Delta(a)\Delta(b)$.

Next it was realized in [18], [20] that the algebra of positions and Lorentz charges is nothing but de Sitter $SO(4,1)$ algebra. The positions and Lorentz transformations are, in turn, nothing but the transformations of the manifold, whose points are energy and momenta (energy-momentum manifold.) On this manifold positions are generators of translational symmetry, while boosts and rotations generate Lorentz transformations. Thus the energy–momentum manifold is a four-dimensional manifold with ten-parameter group of symmetries and thus it must be a maximally symmetric space of constant curvature. It follows from the well known theorem of differential geometry that such a manifold must be locally diffeomorphic to one of the three spaces of constant curvature, and since the group of symmetries is $SO(4,1)$, this manifold must be de Sitter space$^{10}$. Then it follows that the algebra of positions and Lorentz transformations is just an algebra of symmetries of de Sitter space, and therefore it is, of course, independent of a coordinate system we use to describe this space.

$^{10}$It turns out that all other spaces of constant curvature are also possible, if one generalizes somehow the definition of $\kappa$-Poincaré algebra, i.e., the phase space associated with $\kappa$-Poincaré algebra can have positive, zero, and negative curvature (see [35] for details.)
De Sitter space of momenta can be constructed as a four dimensional surface of constant curvature $\kappa$ in the five dimensional Minkowski space with coordinates $\eta_A$, $A = 0, \ldots, 4$, to wit
\[- \eta_0^2 + \eta_1^2 + \cdots + \eta_4^2 = \kappa^2.\] (41)

The $SO(4,1)$ generators can be decomposed into positions $X_\mu$ and Lorentz charges $M_{\mu\nu}$, which act on $\eta_A$ variables as follows
\[
\begin{align*}
[X_0, \eta_4] &= \frac{i}{\kappa} \eta_0, \quad [X_0, \eta_0] = \frac{i}{\kappa} \eta_4, \quad [X_0, \eta_i] = 0, \quad (42) \\
[X_i, \eta_4] &= [X_i, \eta_0] = \frac{i}{\kappa} \eta_i, \quad [X_i, \eta_j] = \frac{i}{\kappa} \delta_{ij} (\eta_0 - \eta_4), \quad (43) \\
\end{align*}
\]

and
\[
\begin{align*}
[M_i, \eta_j] &= i \epsilon_{ijk} \eta_k, \quad [N_i, \eta_j] = i \delta_{ij} \eta_0, \quad [N_i, \eta_0] = i \eta_i, \quad (44)
\end{align*}
\]

It should be noted that there is another decomposition of $SO(4,1)$ generators $[18], [20]$, in which the resulting algebra is exactly the one considered by Snyder $[28]$.

On the space (41) one can built various co-ordinate systems, each related to some DSR theory. In particular, one recovers the bicroosproduct DSR1 with the following coordinates (which are, accidentally, the standard “cosmological” coordinates on de Sitter space)
\[
\begin{align*}
\eta_0 &= -\kappa \sinh \frac{P_0}{\kappa} - \frac{\tilde{P}^2}{2\kappa} e^{P_0/\kappa} \\
\eta_i &= -P_i e^{P_0/\kappa} \\
\eta_4 &= \kappa \cosh \frac{P_0}{\kappa} - \frac{\tilde{P}^2}{2\kappa} e^{P_0/\kappa}. \quad (45)
\end{align*}
\]

Using (45), (43), and the Leibnitz rule, one easily recovers the commutators $[15], [18]$.

**Exercise 10.** Check this explicitly.

Other coordinates systems, are possible, of course.

**Exercise 11.** Find the coordinates on de Sitter space of momenta, corresponding to DSR2.
In particular one can choose the “standard basis” in which
\[ P_\mu = \eta_\mu/\eta_4. \] (46)

Note that in this basis (or classical DSR) the commutators of all Poincaré charges, \( P_\mu \) and \( M_{\mu\nu} \) are purely classical. However, the positions brackets, as well as the momenta/positions cross-relations are still non-trivial.

**Exercise 12.** Compute the bracket of positions with energy and momenta in the classical basis.

This means that in the classical bases of DSR the (observer-independent) scale \( \kappa \) disappears completely from the Lorentz sector, but is still present in the translational one. Thus such a theory fully deserves the name DSR.

De Sitter space setting reveals the geometrical structure of DSR theories. As we saw the energy momentum space of DSR is a four dimensional manifold of positive constant curvature, and the curvature radius equals the scale \( \kappa \). The Lorentz charges and positions are identified with the set of ten tangent vectors to the de Sitter energy-momentum space, and as an immediate consequence of this their algebra is independent of any particular coordinate system on this space. However the latter seems to be, at least naively, physically relevant. Each such coordinate system defines for us (up to the redundancy discussed in [20]) the physical energy and momentum. In one-particle sector the particular choice may not be relevant, but it seems that it would be of central importance for the proper understanding of many particles phase spaces, in particular in analysis of the phenomenologically important issue of particles scattering and conservation laws.

Having obtained the one-particle phase space of DSR, it is natural to proceed with construction of the field theory. Here two approaches are possible. One can try to construct field theory on the non-commutative \( \kappa \)-Minkowski space-time. Attempts to construct such a theory has been reported, for example, in [36] and references therein, as well as in [37], [38]. This line of research is, however, far from being able to give any definite results, though some partial results, like an interesting, nontrivial vertex structure reported in [37], [38] may shed some light on physics of the scattering processes. The major obstacle seems to be lack of the understanding of functional analysis on the spaces with Lie-type of non-commutativity, which is most likely a deep and hard mathematical problem (already the definition of appropriate differential and integral calculi is a mater of discussion.) Therefore it seems simpler
(and in fact more along the line of the DSR proposal, where the energy momentum space is more fundamental than the space-time structures) to try to build (quantum) field theory in energy-momentum space directly. This would amount to understand how to define (quantum) fields on the curved energy-momentum space, but, in principle, for spaces of constant curvature at least functional analysis is well understood. It should be noted that such an idea has been contemplated for a long time, and in fact it was one of the main motivations of [28]. Field theories with curved energy-momentum manifold has been intensively investigated by Kadyshevsky and others [39], without any conclusive results, though.

4 Physics with Doubly Special Relativity

Till now I have been discussing formal aspects of Doubly Special Relativity in a particular formulation, in which quantum algebras and non-commutative space-time played the fundamental role. Now it is time to try to turn to more physical questions, related with possible experimental signatures of quantum gravity. In other contributions to this volume, the reader can find much more detailed discussion of the “quantum gravity phenomenology”, here I would like to concentrate on those physical aspects and problems that are directly related to a particular formulation of DSR in terms of $\kappa$-Poincaré algebras.

4.1 Time-of-flight experiments and the issue of velocity in DSR

One of the simplest experimental tests of quantum gravity phenomenology is the time-of-flight experiment. In this experiment which is to be performed in a near future with good accuracy by GLAST satellite (see e.g., [40] and references therein) one measures the energy-dependence of velocity of light coming from a distant source. Naively, most DSR models predicts positive signal in such an experiment (for details see [41].) Indeed, in DSR $\partial E(p)/\partial p$ does, with an exception of the classical bases, depend on energy, which suggest that velocity of massless particles may depend on the energy they carry. This is the case, for example, both in the bicrossproduct DSR1 and in the Magueijo-Smolin DSR2 model.

Of course, the velocity formula should be derived from the first principles. In the careful analysis reported in [44] (based on the calculations presented
some time ago in [42] the authors construct the wave packet from plane waves moving on the $\kappa$-Minkowski space-time, and then calculate the group velocity of such a packet, which, they claim, turns-out to be exactly $v^{(g)} = \frac{\partial E(p)}{\partial p}$\footnote{Notice however that similar analysis presented in [43] resulted in different conclusion. I will discuss below the reason for this discrepancy.}. This result is puzzling in view of the phase space calculation of velocity, which I will present below. Therefore let us analyze this calculation in more details.

The authors of [44] consider the wave packet built of waves moving in non-commutative $\kappa$-Minkowski space-time, centered at $(\omega_0, \vec{k}_0)$, to wit

$$\Psi(\omega_0, \vec{k}_0) = \int e^{i\vec{k} \cdot \vec{x}} e^{-i\omega t} d\mu(47)$$

Here the plane waves have been ordered so that the time variable appear on the right, and $d\mu$ is an appropriate measure on the space of three-momenta, whose detailed form will be irrelevant to what follows. We assume that the plane waves in the integral satisfy appropriate field equations so that $\omega$ is a given function of $\vec{k}$ such that for the pair $(\vec{k}, \omega(\vec{k}))$ the Casimir vanishes identically. Let us assume that the integral in (47) has support on small neighborhood

$$\omega_0 - \Delta \omega \leq \omega \leq \omega_0 + \Delta \omega \quad \vec{k}_0 - \Delta \vec{k} \leq \vec{k} \leq \vec{k}_0 + \Delta \vec{k}$$

Factoring out the phases $e^{i\omega_0 x}$ to the left and $e^{-i\omega_0 t}$ to the right one gets

$$\Psi^m(\omega_0, \vec{k}_0) = e^{i\vec{k}_0 \cdot \vec{x}} \left[ \int e^{i\Delta \vec{k} \cdot \vec{x}} e^{-i\Delta \omega t} d\mu \right] e^{-i\omega_0 t}(48)$$

Now the integral in the middle carries the information about the group velocity of the wave packet. Indeed it follows that the group velocity equals (in deriving the expression above one should make use of the fact that in the limit $\Delta \omega, \Delta \vec{k} \to 0$, the commutator $[e^{-i\Delta \vec{k} \cdot \vec{x}}, e^{i\Delta \omega t}] = 0$)

$$v^{(g)} = \lim_{\Delta \vec{k} \to 0} \frac{\Delta \omega}{|\Delta \vec{k}|} = \frac{d\omega}{d|\vec{k}|} = \frac{dE}{d|P|}.$$ (49)

The expression (48) is, however, ambiguous because the middle, amplitude term does not commute with the exponents on the left and on the right as a result of the identity

$$e^{i\vec{k} \cdot \vec{x}} e^{-i\omega t} = e^{-i\omega t} e^{i\omega/\kappa \vec{k} \cdot \vec{x}}$$
Thus instead of (48) we can use
\[ \Psi_r(\omega_0, \vec{k}_0)(\vec{x}, t) = e^{i\vec{k}_0 \cdot \vec{x}} e^{-i\omega_0 t} \left[ \int e^{i\Delta \vec{k} \cdot \vec{x}} e^{-i\Delta \omega t} d\mu \right] \] (50)
or
\[ \Psi_l(\omega_0, \vec{k}_0)(\vec{x}, t) = \left[ \int e^{i\Delta \vec{k} \cdot \vec{x}} e^{-i\Delta \omega t} d\mu \right] e^{i\vec{k}_0 \cdot \vec{x}} e^{-i\omega_0 t} \] (51)
where in the last expression, we neglected the \( e^{-\Delta \omega/\kappa} \) term in the exponent (it goes to zero in the relevant limit.)

We see therefore that the group velocity depends on the ordering of the wave packet (48), (50), (51) and equals
\[ v^{(g)} = \begin{cases} 
\frac{d\omega}{d|\vec{k}|} & \text{in the cases } m, l \\
\frac{d\omega}{d|\vec{k}|} e^{-\omega/\kappa} & \text{in the case } r 
\end{cases} \] (52)
Using the fact that for massless particles \( \omega \) and \( \vec{k} \) are related by (see (19)
\[ \kappa^2 \cosh \frac{\omega}{\kappa} - \frac{\vec{k}^2}{2} e^{\omega/\kappa} = \kappa^2. \] (53)
we find easily
\[ v^{(g)} = \begin{cases} 
\frac{\kappa}{\kappa - |\vec{k}|} & \text{in the cases } m, l \\
1 & \text{in the case } r 
\end{cases} \] (54)
Thus we see that the ordering ambiguity in the derivation leads to the ambiguity in the prediction of DSR1 concerning one of the few effects that might be in principle observed. In particular, for one ordering we have velocity of massless particles growing with the energy, while for other we have constant speed of light, as in Special Relativity. The only way out, therefore, is to compute the velocity in a different, though physically equally appealing framework.

To this aim let us try to compute the velocity starting from the phase space of DSR theories. This computation has been presented in [45] (see also [46] and [47].)

The idea is to start with the commutators (42)–(44). Note first that since the for the variable \( \eta_4 \), \( [M, \eta_4] = [N, \eta_4] = 0, \kappa \eta_4 \) is a Casimir (cf. (15)) and can be therefore naturally identified with the relativistic Hamiltonian \( \mathcal{H} \) for free particle in any DSR basis as it is by construction Lorentz-invariant, and reduces to the standard relativistic particle hamiltonian in the large \( \kappa \) limit.
Indeed, using the fact that for $P$ small compared to $\kappa$, in any DSR theory $\eta_\mu \sim P_\mu + O(1/\kappa)$ we have

$$\kappa \eta_4 = \kappa^2 \sqrt{1 + \frac{P_0^2 - \vec{P}^2}{\kappa^2}} \sim \kappa^2 + \frac{1}{2} \left( P_0^2 - \vec{P}^2 \right) + O\left( \frac{1}{\kappa^2} \right)$$

(55)

Then it follows from eq. (43) that

$$\eta_\mu = [x_\mu, \kappa \eta_4] = [x_\mu, \mathcal{H}] \equiv \dot{x}_\mu$$

(56)

can be identified with four velocities $u_\mu$. The Lorentz transformations of four velocities are then given by eq. (44) and are identical with those of Special Relativity. Moreover, since

$$u_0^2 - \vec{u}^2 \equiv C = M^2$$

(57)

by the standard argument the three velocity equals $v_i = u_i / u_0$ and the speed of massless particle equals 1. Let me stress here once again that this result is DSR model independent, though, of course, the relation between three velocity of massive particles and energy they carry depends on a particular DSR model one uses.

**Exercise 12.** Compute the velocity of massless particles for DSR1 directly. Use $\kappa \eta_4$ as the hamiltonian and explicit expressions for $\eta_\mu$ as functions of energy and momenta (45). (The answer can be found in [15].)

Thus this calculation indicates that GLAST should not see any signal of energy dependent speed of light, at least if it is correct to think of photons as of point massless classical particles, as I have implicitly assumed in the derivation above.

It should be stressed that the issue of velocity of physical particles is not completely settled on the theoretical ground, and thus any experimental input would be extremely valuable.

### 4.2 Remarks on multi-particle systems

Having obtained the one-particle phase space of DSR it is natural to try to generalize this result to find the two- and multi-particles phase spaces. It turns out however that such a generalization is very difficult, and in spite
of many attempts not much about multi-particles kinematics is known. On the other hand the control over particle scattering processes is of utmost relevance in the analysis of seemingly one of the most important windows to quantum gravity phenomenology, provided by Ultra High Energy Cosmic Rays and possible violations of predictions of Special Relativity in UHECR physics (see e.g., [40], [41] for more detailed discussion and the list of relevant references.)

Ironically, we have in our disposal the mathematical structure that seem to provide a tool to solve multi-particle the problem directly. This structure is co-product. Recall that the co-product is a mapping from the algebra to the tensor product

$$\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$$

and thus it provides the rule how the algebra acts on tensor products of its representations. We know that in ordinary quantum mechanics two-particles states are described as a tensor product of single-particle ones\(^\text{12}\). Note that this is a very strong physical assumption: in making it we claim that any two-particle system is nothing but two particles in a black box, i.e., that the particles preserve their identities even in multi-particle states. But it is well possible that multi-particle states differ qualitatively from the single-particle ones, for example as a result of non-local interactions. Let us, however, assume that in also DSR to obtain the multi-particle states one should only tensor the single-particle ones, and let us try to proceed.

In the case of classical algebras the co-product is trivial: $$\Delta G = G \otimes 1 + 1 \otimes G$$ which means that the group action on two particle states just respects Leibnitz rule. For example the total momentum of two particles in Special Relativity is just the sum of their momenta:

$$\Delta(P^\mu) |1 + 2> = \Delta(P^\mu) |P^{(1)}> \otimes |P^{(2)}> =$$

$$\left(P^\mu \otimes 1 + 1 \otimes P^\mu\right) |P^{(1)}> \otimes |P^{(2)}> = \left(P^{\mu(1)} + P^{\mu(2)}\right) |P^{(1)}> \otimes |P^{(2)}>$$

\(^{12}\)In fact there is more to the description of multi-particles states than just the tensor product, namely one should impose somehow the statistics by symmetrizing or anti-symmetrizing the product. It is well known that in 4 dimensions these are the only possibilities, but the proof relies heavily on the assumption of Poincaré invariance. It is not known if relaxing this assumption by replacing the Poincaré with \(\kappa\)-Poincaré invariance can result in some other, braided statistics.
In the case of quantum algebras the co-product is non-trivial and non-symmetric by definition (if the co-product was symmetric we would have to do instead with just a classical Lie algebra in nonlinear disguise). This immediately leads to the problem, as I will argue below.

Before turning to this problem let us point out yet another one, relevant for DSR1 as well as for DSR2. Namely the co-product has been constructed so that two-particle states transform as the single-particle ones (for example in Special Relativity total momentum is Lorentz vector.) Indeed if we calculate the total energy and momentum of two-particles system using the co-product addition rule of DSR1 from

\[ \Delta(P_0) = \mathbb{1} \otimes P_0 + P_0 \otimes \mathbb{1} \]  
\[ \Delta(P_k) = P_k \otimes e^{-P_0/\kappa} + \mathbb{1} \otimes P_k \]

we find

\[ P_{0+}^{1+2} = P_0^{(1)} + P_0^{(2)} \], \[ P_{k+}^{1+2} = P_k^{(1)} e^{-P_0^{(2)}/\kappa} + P_k^{(2)} \].

But then it follows that total momentum must satisfy the same mass shell relation as the single particle does.

**Exercise 13.** Check that \( P_{0+}^{1+2} \) and \( P_{k+}^{1+2} \) satisfy the dispersion relation of DSR1 if \( P_0^{(1/2)} \), \( P_k^{(1/2)} \) do.

We know however that in the case of the DSR1 we have to do with maximal momentum for particles, of order of Planck mass. While acceptable for Planck scale elementary particles, this is certainly violated for macroscopic bodies. To prove this, the reader can perform a nice quantum gravity phenomenology experiment just by kicking a soccer ball! So we know that there is an experimental proof that either our procedure of attributing momentum to composite system by tensoring and applying co-product, or the bicrossproduct DSR, or both are wrong.

To investigate things further let us turn to the DSR theory, which does not suffer from the “soccer ball problem” namely to the classical basis DSR with standard dispersion relation \( \mathcal{P}_0^2 - \mathcal{P}_i^2 = m^2 \), for which de Sitter coordinates are given by \( (46) \). The co-product for this basis has been calculated in \( \[21\] \).
and up to the leading terms in $1/\kappa$ expansion read

$$\Delta(P_0) = 1 \otimes P_0 + P_0 \otimes 1 + \frac{1}{\kappa} P_i \otimes P_i + \ldots \quad (63)$$

$$\Delta(P_i) = 1 \otimes P_i + P_i \otimes 1 + \frac{1}{\kappa} P_0 \otimes P_i + \ldots \quad (64)$$

Using this we see that according to the co-product addition rule the total momentum of two-particles system is

$$P_{(1+2)}^0 = P_{(1)}^0 + P_{(2)}^0 + \frac{1}{\kappa} P_{(1)}^i P_{(2)}^i \quad (65)$$

$$P_{(1+2)}^i = P_{(1)}^i + P_{(2)}^i + \frac{1}{\kappa} P_{(1)}^0 P_{(2)}^i \quad (66)$$

As it stands, the formulas (65, 66) suffer from two problems: first of all, recalling that $P_\mu$ transforms as a Lorentz vector for single particle, these expressions look terribly non-covariant. Second, even though (65) is symmetric in exchanging particles labels $1 \leftrightarrow 2$, (66) is not. How do we know which particle is first and which is second? Let us try to resolve these puzzles in turn.

That the first puzzle is just an apparent paradox follows immediately from the consistency of the quantum algebra. As I said above the action of boosts on two-particle state is such that total momentum transforms exactly as the single-particle momentum does. This is in fact the very reason of the “soccer ball problem” in the DSR1. In fact the boosts do not only act on $P_{(1)}^\mu$ and $P_{(2)}^\mu$ independently; they also mix them in a special way. This feature was to be expected, since the co-product addition rule mixes single-particle states in a non-trivial way. More specifically, note that boosts must act on two-particle states by co-product as well, therefore in order to find out how a two-particle state changes when we boost it we must compute the commutator $[\Delta(N), \Delta(P)]$. Recall now that the co-product of boosts reads (again up to the leading terms in $1/\kappa$ expansion)

$$\Delta(N_i) = 1 \otimes N_i + N_i \otimes 1 - \frac{1}{\kappa} N_i \otimes P_0 - \frac{1}{\kappa} \epsilon_{ijk} M_j \otimes P_k \quad (67)$$

Using this one easily checks explicitly that

$$[\Delta(N_i), \Delta(P_j)] = \delta_{ij} \Delta(P_0), \quad [\Delta(N_i), \Delta(P_0)] = \Delta(P_i) \quad (68)$$
from which it follows that $P_{t0}^{(1+2)}$ and $P_{t0}^{(1+2)}$ do transform covariantly, as they should\textsuperscript{13}. Of course equation (68) holds to all orders, as it just reflects the defining property of the co-product.

Let us now turn to the second puzzle, the apparent dependence of the total energy/momentum on physically arbitrary labelling of particles. Here I have much less to say, as this paradox has not been yet solved. One should however mention an interesting result obtained in the case of the analogous problem in deformed, non-relativistic model. In the paper \cite{48} the authors find that even though there is an apparent asymmetry in particle labels due to the asymmetry of the co-product, the representations with flipped labels are related to the original ones by unitary transformation, and are therefore physically completely equivalent. In the similar spirit in \cite{49} one uses the fact of such an equivalence in 1+1 dimensions to demand that the action of generators on two particles (bosonic) states is through symmetrized co-product.

During this Winter School Aurelio Grillo and Fernando Mendez produced another interesting puzzle concerning the validity of co-product based momenta addition rule. This puzzle reminds somehow the entanglement problem in quantum mechanics and it can be described as follows.

Suppose we use (62) to formulate conservation rule for two-to-two particles scattering, which would therefore take the following form

\begin{equation}
P_0^{(1)} + P_0^{(2)} = P_0^{(3)} + P_0^{(4)}
\end{equation}

\begin{equation}
P_k^{(1)} e^{-P_0^{(2)}/\kappa} + P_k^{(2)} = P_k^{(1)} e^{-P_0^{(3)}/\kappa} + P_k^{(4)}.
\end{equation}

But what about all other particles in the Universe (spectators)? In principle, their presence would contribute non-trivially to the conservation laws (69), (70), to wit

\begin{equation}
P_0^{(1)} + P_0^{(2)} + P_0^{(\text{univ})} = P_0^{(3)} + P_0^{(4)} + P_0^{(\text{univ})}
\end{equation}

\begin{equation}
\left( P_k^{(1)} e^{-P_0^{(2)}/\kappa} + P_k^{(2)} \right) e^{-P_0^{(\text{univ})}/\kappa} + P_k^{(\text{univ})} = \left( P_k^{(1)} e^{-P_0^{(3)}/\kappa} + P_k^{(4)} \right) e^{-P_0^{(\text{univ})}/\kappa} + P_k^{(\text{univ})}.
\end{equation}

In the standard, Special Relativistic case we neglect the influence of the rest of the Universe, because we believe that the processes are (at least

\textsuperscript{13}This holds, of course, for a DSR theory in any basis, not just in the classical one.
approximately) local, but here we have non-local influence of one particle on another all the time, independently of their separation (in the formulas (71), (72) there is no information concerning separation of particles in space and time.) Thus, the final construction of DSR theory must necessarily solve this spectator problem as well!

5 Conclusion

There is a growing hope that some form of DSR theory indeed describes Nature in the kinematical regime, where the energies of the particles became close to the Planck energy scale and at the same time one could neglect local degrees of gravity, described by (still to be constructed) Quantum Theory of Gravity. This hope is based on the analogy between the ground state of 4d quantum gravity and 3d quantum gravity, both being described by topological quantum field theory.

As we saw, we seem to know some of the ingredients of the DSR theory, and we can even predict some (testable, in principle) DSR phenomenology. It seems however that there would be very hard to derive the complete form of DSR just from the first principles, the hope being that soon we will be able to derive DSR as an appropriate limit of (Loop) Quantum Gravity.

Four years ago during the Winter School entitled “Towards Quantum Gravity” Giovanni Amelino-Camelia asked the insightful question “Are we at the dawn of Quantum Gravity Phenomenology?”. This year we devoted the whole Winter School to discuss possible observable signals of Quantum Gravity. I hope that in four years we will meet to discuss numbers coming from Quantum Gravity experiments that would be already running and producing data. I also hope that it will turn out that these data would agree with the final form of Doubly Special Relativity.

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