Topological chiral interface states beyond insulators

Lavi K. Upreti and P. Delplace

Univ Lyon, ENS de Lyon, Univ Claude Bernard, CNRS, Laboratoire de Physique, F-69342 Lyon, France

(Dated: February 24, 2020)

We show how to engineer a chiral spectral flow of interface states in a gapless system. Although the bulk bands touch – thus making the topological indices of the bands ill-defined – this spectral flow has a topological origin that makes it robust even when it coexists with the bulk bands. This phenomenon is illustrated with an adiabatically modulated photonic quantum walk.

I. INTRODUCTION

Topological phases of matter involve both insulators and semimetals. However, their topological nature is revealed through different properties – while insulators are classified according to the global topology of their spectrally isolated bands over the Brillouin zone, semimetals are locally characterized by the topology of their nodal points or lines, that are somehow analogous to topological defects in reciprocal space. Interestingly, these two aspects are related. For instance, a Weyl semimetal can be seen as a stable phase when continuously deforming a 3D topological insulator into a trivial one, provided inversion symmetry is broken. In two dimensions, single Dirac points symptomatically appear at the transition between two distinct topological insulating phases, as in the celebrated Haldane model for Chern insulators.

The existence of boundary states is a universal manifestation of the underlying topology. For insulators, they are well defined in a spectral gap, that they bridge when varying their quasimomentum, which make them propagative. This property is sometimes referred to as the spectral flow of the boundary modes. It gives rise, for instance, to chiral edge states in Chern insulators. The absence of a spectral gap makes the situation quite different for semimetals. Indeed, the nodal points act as topological transition points in momentum space, at which the boundary states are therefore doomed to arise and end. This is the case of Fermi arcs surface states connecting the Weyl points when crossed by the Fermi energy in 3D. This is again the case of edge states in 2D graphenic-like structures that own Dirac points in their energy spectrum. Obviously, the stability of the these Dirac points prevent the edge states from being chiral, so that the only way to get a spectral flow in 2D structures is to impose a gap, even an indirect one, as it is indeed the case for Chern insulators. Moreover, the Chern numbers of the bands of an insulator consistently account for the number of chiral edge states in each gap, according to the celebrated bulk-edge correspondence.

In this paper, we present a scenario to engineer a spectral flow of edge states in 2D semimetallic phases with stable Dirac points. In particular, we show how such edge states can spectrally coexist with bulk modes and still have a well defined topological origin although the Chern numbers of the bands are ill-defined. Such unusual properties are actually made possible by managing different gap inversions mechanisms at the same energy, so that all the gaps never open simultaneously. Even though there is no gap in the Brillouin zone, the topological properties can be captured by associating a topological charge as the quantized Berry flux emanating from degeneracy points in 3D parameter space, similarly to Weyl nodes in synthetic dimensions. This quantity is also a Chern number, but must be distinguished from that of the full bands over the Brillouin zone which are ill-defined here. Remarkably, the resulting topological spectral flows consist in chiral interface states in between 2D semi-metals rather than insulators. All along the paper, we illustrate this idea with the quasienergy bands of a scattering network model that generalizes different experimental setups recently used in the context of topological photonic quantum walks. In that context, the parameter that tunes the spectral flow is an adiabatic phase driving the quantum walk, rather than the usual quasimomentum.

The paper is organized as follows. In Section II, we introduce the scattering network model of an adiabatically modulated 1D quantum walk. That shows how the short time modulation of a dynamical phase allows one to selectively manipulate Dirac points at the same quasienergy. The topological properties of these degeneracy points are given in section III, and their associated spectral flows of uncoupled interface states with bulk modes are computed accordingly. These interface states are found to have no edge state counterpart in finite geometry with open boundary conditions as discussed in section IV.

II. SCATTERING NETWORK MODEL

A. Generalities

Topological properties of networks models have been investigated in details recently in the context of photonics. Here we consider the discrete time-evolution of a state (e.g. quantum state or wavepacket) through the generic network depicted in figure 1. Such evolution decomposes into a staggered sequence of free propagations along the oriented links and scattering events at the vertices encoded into the unitary matrices

$$S_{i,j} = \begin{pmatrix} \cos \theta_{i,j} & i \sin \theta_{i,j} \\ i \sin \theta_{i,j} & \cos \theta_{i,j} \end{pmatrix}$$
where the parameters $\theta_{l,j}$ may depend both on the position $l$ in the lattice and on the time step $j$. Let us impose a periodicity of the network after $N$ time steps so that the scattering nodes satisfy $S_{l,j} = S_{l,j+N}$. Such network can thus model e.g. a one-dimensional Floquet quantum walk. In the following, it will be indeed useful to introduce the Floquet operator $U_F$, that is the unitary evolution operator after $N$ time steps. This operator will depend on the set of parameters $\{\theta_j\}$ where $j$ runs from 1 to $N$ and where we have dropped the position index $l$, assuming invariance along the $x$ direction for now.

![Figure 1. Two-dimensional oriented scattering lattice where the $y$ axis plays the role of time. A time period consists in $N$ successive steps. A phase $\phi_{l,j}$ is added for the states scattered out the node $j$ and propagating leftwards (blue arrows). The unit cell of this lattice is emphasized by a black dashed rectangle.](image)

In addition, we introduce a phase shift carried along by the states when propagating along each link. Importantly, this phase shift evolves according to two time scales with respect to the period of $N$ time steps. At short time scales, the phase shift $\phi$ may also depend on both the discrete position $l$ and the time step $j$ within a period. Each unit cell of the network is thus decorated with a pattern of phase shifts $\phi_{l,j} = Q_{l,j} \phi$ that preserves the periodicity of the network, where $Q_{l,j}$ is some rational number that will be specified later, and $\phi$ is a phase shift of reference. The second time scale appears through the small variation of $\phi$ from one period to the next one. Strictly speaking, the network is therefore not periodic in time anymore, but we shall consider the case where $\phi$ evolves slowly enough so that the long time stroboscopic dynamics can be described by the adiabatically modulated Floquet operator when continuously varying the phase parameter $\phi \in [0, 2\pi]$. Exploiting translation invariance along the $x$ direction, we are finally interested in the Bloch-Floquet parametrized evolution operator $U_F(k_x, \phi, \{\theta_j\})$ where $k_x$ is the quasi-momentum in the $x$ direction. Note that this operator depends in a periodic fashion on its parameters. Its eigenstates are thus parametrized over a $N + 2$-torus that can be seen as a synthetic Brillouin zone. This model could thus provide an interesting framework to investigate physical phenomena in higher dimensions ($> 3$), such as analogs of topological phases provided a spectral gap exists.

This network generalizes previous models whose topological properties have been investigated experimentally in photonics setups. For instance, when $N = 2$ and in the absence of a phase shift ($\phi = 0$), the model describes one-dimensional photonic quantum walk\cite{10} and one-dimensional laser-written modulated photonic lattices in silica,\cite{17} in which boundary modes have been observed. Still when $N = 2$ but now for $\phi_1 = +\phi, \phi_2 = -\phi$ together with fixed coupling parameters $\theta_{l,j} = \pi/4$, it describes pairs of coupled optical fibre loops in which the Berry curvature was measured using wavepacket dynamics.\cite{18}

Interestingly, as we show below, this model can also exhibit spectral flows in the absence of a spectral gap. A particular striking case even consists of the uncoupled superposition of a spectral flow with the bulk bands. The topological nature of this spectral flow can be understood from the existence of topological charges (Berry monopoles) in synthetic dimensions ($k_x, \phi, \Delta$) where $\Delta$ depends on the scattering parameters $\theta_j$. The key feature that yields the coexistence of bulk and boundary modes is the selective manipulation of distinct gaps opening/closure mechanisms that are made possible by specific patterns of $\phi_{l,j}$.

### B. Gapless states in four-steps networks

Our starting point is the two-step network ($N = 2$) for which $\phi_1 = -\phi_2 = \phi$ and where the nodes act as identical 50/50 beam splitters.\cite{18} The quasi-energy spectrum of the Bloch-Floquet operator $U_F(k_x, \phi, \{\theta_1 = \pi/4\})$, shown in Figure 2, is fully gapless and its two bands touch linearly along $k_x$ and $\phi$ at both quasi-energies $0$ and $\pi$. Tuning the coupling parameters away from $\theta = \pi/4$ opens a gap at quasi-energies $0$ and $\pi$, and may lead to a topological regime characterized by gap-valued topological invariants $W \in \mathbb{Z}_3(U(N))$.\cite{24,28} Accordingly, in the presence of boundaries in the $x$ direction, the system exhibits a spectral flow of edge states that bridges each gap when $\phi$ is tuned.\cite{26} Here we show how to implement a similar spectral flow in the absence of such a gap. For that purpose, one needs additional Dirac points at the same quasi-energies, and whose stability depends on different parameters than the preexisting ones. A possible strategy to obtain more Dirac points in the synthetic Brillouin zone ($k_x, \phi$) consists in enlarging the unit cell by allowing more distinct couplings along either the transverse or the propagative direction. Here we choose the second option by considering a period of $N = 4$ steps for the fast dy-
natures, and fixing the distinct phase shifts inside a unit cell as \( \theta_1 = +\phi, \theta_2 = -\phi, \theta_3 = +\phi, \) and \( \theta_4 = -\phi. \) Note that the net phase inside the unit cell is zero, thus conserving inversion symmetry along synthetic dimension \( \phi \), which avoids any winding of the quasienergy bands.\(^{12,26}\)

Using the spatial periodicity along \( x, \) i.e. \( \theta_{1,j} = \theta_j \) the Floquet-Bloch evolution operator can be written as the succession of translation-like operations and local scattering processes as

\[
U_F(\phi, k_x; \{\theta_j\}) = T_-S_4T_3S_2T_2S_1 \tag{2}
\]

where

\[
T_\pm = \begin{pmatrix} e^{i(\pm k_x \phi)/2} & 0 \\ 0 & e^{-i(\pm k_x \phi)/2} \end{pmatrix} \tag{3a}
\]

\[
S_j = \begin{pmatrix} \cos \theta_j & i \sin \theta_j \\ i\sin \theta_j & \cos \theta_j \end{pmatrix} \tag{3b}
\]

As shown in figure 3, the quasienergy spectrum \( \varepsilon \) is fully gapless for the critical value of parameters \( \{\theta_j = \pi/4\} \), as expected. But now, there exists two Dirac points \( A_0, B_0 \) at \( \varepsilon = 0 \), and the two bands also touch at \( \varepsilon = \pi \) along four lines (instead of points) satisfying \( k_x \pm \phi = 0 \) and \( k_x \pm \phi = 2\pi \) (dashed black and green lines in Fig 3). By deviating from this critical point, it is then possible to lift some of these degeneracies leaving untouched the other ones. The different conditions required to lift specific band touching points or lines are inferred by expanding in \( \theta_j \) the Floquet operator around these degeneracies away from \( \pi/4 \).

At the two Dirac points sitting at \( \varepsilon = 0 \), namely \( A_0, B_0 \), the Floquet operator must satisfy \( U_F = 1 \). Substituting their coordinates \( (\phi, k_x) \) – respectively \((0, \pi)\) and \((\pi, 0)\) – in equation (2), an expansion in scattering parameters around the critical point \( \theta_j = \pi/4 \rightarrow \pi/4 + \delta \theta_j \), yields the constrain

\[
S(-\delta \theta_1 + \delta \theta_2 - \delta \theta_3 + \delta \theta_4) = 1 \quad \text{at} \ A_0 \quad \text{and} \ B_0 \tag{4}
\]

(\( S(\theta_j) = S_j \)), which is only satisfied for \(-\delta \theta_1 + \delta \theta_2 - \delta \theta_3 + \delta \theta_4 = 0 \). Conversely, a gap opens at \( A_0 \) and \( B_0 \) when this condition is not fulfilled. An interesting twist comes at \( \varepsilon = \pi \), where now the Floquet operator must satisfy \( U_F = -1 \). Expanding the Floquet operator in scattering parameters for the four degeneracy lines \( k_x \) satisfy

\[
S(\delta \theta_1 + \delta \theta_2 + \delta \theta_3 + \delta \theta_4) = 1 \tag{5}
\]

that differs from the condition (4) for the two degeneracy points at \( \varepsilon = 0 \). Furthermore, there are two special points, namely \( A_\pi \) at \( (\phi = \pi/2, k_x = \pi/2) \) and \( B_\pi \) at \( (\phi = 3\pi/2, k_x = -\pi/2) \), shown with blue dots in Fig 3, where this expansion does not apply. There, one finds a third condition that reads

\[
S(\delta \theta_1 - \delta \theta_2 - \delta \theta_3 + \delta \theta_4) = 1 \quad \text{at} \ A_\pi \quad \text{and} \ B_\pi . \tag{6}
\]

Finally, the different gap opening terms \( \delta \theta_j \) follow from

\[
\nu_1 \delta \theta_1 + \nu_2 \delta \theta_2 + \nu_3 \delta \theta_3 + \nu_4 \delta \theta_4 \neq 0 \tag{7}
\]

with \( \nu_j = \pm 1 \) as summarized in Table I. Thus, doubling the time period of the network indeed brings new degeneracies, namely, \( A_{0,\pi} \) and \( B_{0,\pi} \). However, degeneracies at a fixed quasienergy, 0 or \( \pi \), are (un)stable under the same perturbations \( \delta \theta_j \) . The only exception being at \( \varepsilon = \pi \) where the degeneracy lines (in eq(5)) and degeneracy points (in eq(6)) do not share the same stability, and hence can be gapped separately. A way to get rid of the degeneracy lines and end up with touchings points that can be manipulated separately at the same quasienergy, by changing the phase shift pattern \( \theta_j \) within a period.
C. Selective manipulation of degeneracies

We propose now the following phase shift pattern that decorates the four-step period: \( \phi_1 = 2\phi, \phi_2 = -\phi, \phi_3 = 0, \) and \( \phi_4 = -\phi \). This choice still preserves \( \sum \phi_j = 0 \) and thus prevents windings of the quasi-energy bands.\(^{26}\)

The quasienergy spectrum of the Bloch-Floquet operator is depicted in figure 4 at the critical point \( \{ \theta_j = \pi/4 \} \). This spectrum is still fully gapless, but now the two bands touch at \( \varepsilon = 0 \) and \( \varepsilon = \pi \) only at points, either linearly in both directions (Dirac points \( A_0, C_0, A_\pi \) and \( C_\pi \)) or linearly in one direction and quadratically in the other one (semi-Dirac points \( B_0 \) and \( B_\pi \)).\(^{29-34}\)

![Quasienergy spectrum for the four-step Floquet operator with \( \phi_1 = 2\phi, \phi_2 = -\phi, \phi_3 = 0, \) and \( \phi_4 = -\phi \). It shows Dirac points at \( A_{0/\pi}, C_{0/\pi} \) and semi-Dirac points at \( B_{0/\pi} \).](image)

**Figure 4.** Quasienergy spectrum for the four-step Floquet operator with \( \phi_1 = 2\phi, \phi_2 = -\phi, \phi_3 = 0, \) and \( \phi_4 = -\phi \). It shows Dirac points at \( A_{0/\pi}, C_{0/\pi} \) and semi-Dirac points at \( B_{0/\pi} \).

Applying the same reasoning as in section II B regarding the stability of these six band touching points with respect to a perturbation in scattering parameters \( \delta \theta_j \), one ends up with a new classification shown in Table II. It reveals four distinct gap opening processes. In particular, \( C_0, A_\pi \) and \( C_\pi \) behave similarly under any perturbation \( \delta \theta_j \), but differently than the points \( A_0, B_0 \) and \( B_\pi \). In other words, these different degeneracy points are stable against distinct perturbations (or mass terms). It is thus now possible to lift a specific degeneracy at \( \varepsilon = 0 \) or \( \varepsilon = \pi \) without opening a bulk gap. For the sake of simplicity, instead of considering any combination of all the \( \delta \theta_j \)’s, we fix \( \delta \theta_1 = \delta \theta_3 = 0 \), and focus only on the effect of \( \delta \theta_3 \) and \( \delta \theta_4 \). The stability of the degeneracies under the perturbations \( \{ \theta_3, \theta_4 \} \rightarrow \{ \pi/4 + \nu_3 \delta \theta_3, \pi/4 + \nu_4 \delta \theta_4 \} \) can then be characterized by the sign of the product of \( \nu_3 \nu_4 \) only, that leaves us two possibilities. Therefore, one needs to distinguish two distinct gap opening processes driven by two independent mass terms

\[
m_{\pm} \equiv (\delta \theta_3 \pm \delta \theta_4)/2
\] (8)
as summarized in Table II (in blue). These two mass terms allow us to generate topological spectral flows of boundary modes in a gapless (semi-metallic) regime, where both the Chern numbers and the Floquet winding number of the evolution operator are ill-defined, as we show in the next section.

| Quasienergy | Degeneracy points | \( \nu_1 \) | \( \nu_2 \) | \( \nu_3 \) | \( \nu_4 \) | mass term |
|-------------|-------------------|---------|---------|---------|---------|-----------|
| \( \varepsilon = 0 \) | \( A_0 \) | - | + | - | + | \( m_+ \) |
|             | \( B_0 \) | + | - | - | + | \( -m_- \) |
|             | \( C_0 \) | - | + | + | - | \( m_+ \) |
| \( \varepsilon = \pi \) | \( A_\pi \) | + | - | - | + | \( m_- \) |
|             | \( B_\pi \) | - | + | + | - | \( m_- \) |
|             | \( C_\pi \) | - | - | + | + | \( m_- \) |

**Table II.** Stability of the different band touching points of figure 4, under a perturbation \( \nu_j \delta \theta_j \). The mass terms \( m_{\pm} \) encode this stability when considering \( \nu_3 \delta \theta_3 \) and \( \nu_4 \delta \theta_4 \) only.

III. TOPOLOGICAL SPECTRAL FLOW THROUGH BULK MODES

A. Topological charge of degeneracy points

In the vicinity of each band touching point \( X \), one can expand the (dimensionless) effective Hamiltonian defined via the Floquet operator as

\[
U_F = e^{-iH_{\text{eff}}^X}
\] (9)
at the lowest order terms in couplings \( \delta \theta_j \), phase shift \( \delta \phi \) and quasimomentum \( \delta k_x \). Such Hamiltonians have the generic form

\[
H_{\text{eff}}^X(\delta \phi, \delta k_x, m) = \mathbf{h}^X \cdot \mathbf{\sigma}
\] (10)

where \( \mathbf{\sigma} \) is the vector of Pauli matrices and \( \mathbf{h}^X \) defines a family of continuous maps from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \). Therefore, \( \mathbf{h}^X/|\mathbf{h}^X| \) defines continuous maps from parameter space \( \mathbb{R}^3 \setminus \{ X \} \) to target space \( S^2 \) that are classified by the homotopy group \( \pi_2(S^2) = \mathbb{Z} \). The elements of this group are integer numbers that tell how many times \( \mathbf{h}^X/|\mathbf{h}^X| \) warps the sphere. They are given by the degree of \( \mathbf{h}^X \).
defined as
\[
\text{deg}(h^X) = \sum_{p^{(n)}} \text{sgn} \left[ \det \left( \frac{\partial h^X_j}{\partial \lambda_i} \right) |_{\mu^{(0)}} \right]
\]  
(11)
where the pre-images \( p^{(0)}_j = (\delta \phi_j^{(0)}, \delta k_j^{(0)}, m_i^{(0)}) \) satisfy \( h(p^{(0)}_j) = h^{(0)} \), with \( h^{(0)} \) an arbitrary vector in \( \mathbb{R}^3 \), and where \( \{\lambda_i\} \) stands for \( \{\delta \phi, \delta k_x, m_{\pm}\} \).

For a two band Hamiltonian, this degree is directly related to the Chern number \( C_{\pm} \) of the continuous family of normalized eigenstates \( \psi_{\pm} (\delta \phi, \delta k_x, m) \) of \( H_{\text{eff}}^X \) as
\[
C_{\pm} = \mp \text{deg}(h^X) .
\]
(12)
Importantly, a non vanishing value of \( C_n \) is known to guarantee the existence of a spectral flow towards bands \( n \) when the mass term \((m_{\pm} \text{ here})\) is varied in space and changes sign.\(^{35-40}\) This spectral flow usually consists in a unidirectional mode, localized where the mass term vanishes, and whose (quasi-)energy bridges a spectral gap when a parameter (here \( \phi \)) is tuned.

In the following, we compute this topological index (via the degree formula (11)) for different band touching phases, and whose (quasi-)energy bridges a spectral gap, even in the absence of a gap.

1. Spectral flow induced by a spatial variation of \( m_+ \)

According to Table II, the degeneracy points \( A_0 \) and \( B_\pi \) are both lifted when \( m_+ \neq 0 \). One can thus assign them a topological charge (in the sense of section III A) by computing the degree of their respective expanded effective Hamiltonian, with the parameter (base) space being \( (\delta \phi, \delta k_x, m_{\pm}) \).

Let us detail the calculation for \( A_0 \) whose coordinates are \( (\phi, k_x) = (\pi, \pi) [2\pi] \). At lowest order in each parameter, the effective Hamiltonian \( H_{\text{eff}}^{A_0} = h^{A_0} \cdot \sigma \) yields
\[
h^{A_0}(\delta \phi, \delta k_x, m_+) = \begin{pmatrix} -2m_+ & \delta \phi + \delta k_x \\ \delta \phi & 0 \end{pmatrix} .
\]
(13)
The spectrum of \( H_{\text{eff}}^{A_0} \) simply consists in the two branches \( \varepsilon_{\pm} = \mp |h^{A_0}| \) that touch linearly when \( m_+ = 0 \), as expected (see Fig 5). Since \( h^{A_0} \) is linear with respect to each parameter, there is only one pre-image, so that the degree can be straightforwardly computed as
\[
\text{deg}(h^{A_0}) = \text{sgn det} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = +1
\]
(14)
Likewise, the topological charge for \( B_\pi \) is \( \text{deg}(h^{B_\pi}) = +1 \).

Accordingly, a spectral flow appears in the spectrum when considering now a spatial dependence of \( m_+ (x) \) that changes sign. This anisotropy is taken into account in the network model by considering a variation of \( \delta \theta = \delta \theta_{1,4} = m_+ (l) \) along \( x \), thus breaking translation invariance. For numerical convenience, we consider periodic boundary conditions along \( x \), so that the mass term \( m_+ (l) \) changes sign twice, giving rise to two opposite spectral flows (instead of one) visible in figure 6a. This is a typical situation where chiral interface states bridge a spectral gap. Note that the situation is however different from what is currently encountered in topological insulating phases, since the two bands actually touch at \( \varepsilon = \pi \) at two other points of the Brillouin zone, \( A_\pi \) and \( C_\pi \), that are stable against the perturbation in \( m_+ \). Therefore, these chiral states cannot be interpreted as the interface modes between two distinct topological (e.g. Chern) insulators, as they appear at the interface between two gapless semi-metals. The situation is maybe even more unusual with \( A_0 \), since its \( \phi \) coordinate matches that of \( B_0 \) (see figure 4) which remains stable under the perturbation in \( m_+ \), according to Table II. It follows that the spectral flow coexists with bulk modes and does not bridge a gap, as shown in figure 6b. Notice that the direction of the spectral flow is the same for \( B_\pi \) and \( A_0 \), in agreement with the common value of their topological charge.

2. Spectral flow induced by a spatial variation of \( m_- \)

Similarly, a spatial variation of the mass term \( m_- \) leads to a topological spectral flow for \( B_0, C_0, A_\pi \) and \( C_\pi \) when \( \phi \) is varied, provided \( m_- \) changes sign. Let us focus on \( B_0 \) which is a semi-Dirac point, since their topological charge and their associated spectral flow is overlooked in the literature in comparison to Dirac points. An expansion
The eigenvalues $\varepsilon_B$ which exist along with the gapless $A_\pi$ geometry. For degeneracy point (a) $A_0$ at quasienergy 0, which exists along with the gapless $B_0$ and another for (b) $B_\pi$ at quasienergy $\pi$.

of the effective Hamiltonian $H_{\text{eff}}^{B_0} = h^{B_0} \cdot \sigma$ in coupling parameters and quasimomenta gives

$$h^{B_0}(\delta \phi, \delta k_x, m_-) = \begin{pmatrix} -2m_- - \delta k_x (\delta \phi + \delta k_x) \\ 2\delta \phi m_- \\ \delta \phi - \delta k_x \end{pmatrix}$$  \quad (15)

The eigenvalues $\varepsilon \pm = \pm |h^{B_0}|$ yields a semi-Dirac behavior when $m_- = 0$, as announced (see Fig. 7). The introduction of $m_-$ opens a gap, and allows us to define the topological charge of this degeneracy point as

$$\text{deg}(h^{B_0}) = \sum_{p_1^{(0)}} \text{sgn} \det \begin{pmatrix} -\delta k_x & 2m_- & 1 \\ -\delta k_x - \delta \phi/2 & 0 & -1 \\ -2 & 2\delta \phi & 0 \end{pmatrix}$$

$$= \sum_{p_1^{(0)}} \text{sgn} [4m_- - (3\delta k_x + \delta \phi)\delta \phi]$$  \quad (16)

Three pre-images ($k_x, m_i$) are found to satisfy these conditions: $p_1^{(0)} = (-k_1, k_1, 0)$ with $k_1 > 0$, $p_2^{(0)} = (0, k_2, -k_2^2/4)$ with $k_2 < 0$ and $p_3^{(0)} = (\phi_3, 0, 0)$ with $\phi_3 > 0$. The pre-image $p_1^{(0)}$ yields a positive contribution to the sum (16) while both $p_2^{(0)}$ and $p_3^{(0)}$ contribute negatively, so that finally $\text{deg} h^{B_0} = -1$. A similar calculation leads to $\text{deg} h^{C_\pi} = +1$, $\text{deg} h^{A_\pi} = +1$ and $\text{deg} h^{C_{\pi 0}} = +1$.

Accordingly, a numerical calculation is performed in a periodic geometry where $m_- (l)$ changes sign twice when varying with the discrete position index $l$ on the network. Spectral flows are found in agreement with the value of the topological charge. The cases of the semi-Dirac points $C_\pi$ and $B_0$ are depicted in figure III A 2, where the spectral flow for $B_0$ is indeed the opposite to that of $C_\pi$. They both illustrate the same phenomenology as that discussed in the previous section III A 1. In particular, there is a spectral flow around $B_0$ that coexists with bulk states, since $A_0$, that has the same $\phi$ coordinate, is stable under a perturbation of $m_-$. type.

One can evaluate the pre-images by fixing a direction for $h^{B_0}$, say along $z$. This imposes the three following conditions

$$-4m_- = \delta k_x (\delta \phi + \delta k_x)$$  \quad (17a)

$$\delta \phi m_- = 0$$  \quad (17b)

$$\delta \phi > \delta k_x$$  \quad (17c)

Figure 7. Semi-Dirac point $B_0$ at quasienergy $\varepsilon = 0$ located at $(\phi, k_x) = (\pi, 0)$. This degeneracy point is robust against a perturbation $m_+$ but a gap opens due to the introduction of $m_-$. 

Figure 8. Existence of a spectral flow of modes localized at the two interfaces of the cylindrical geometry (in blue and red), where the mass term $m_-$ changes sign twice, in the vicinity of (a) $C_\pi$ at quasienergy $\pi$, and (b) $B_0$ at quasienergy 0. In the second case, the spectral flow crosses the Dirac point $A_0$.

IV. CHIRAL EDGE STATES IN GAPLESS SYSTEMS

Here is another situation where the bulk topological invariants of the bands are ill-defined, because these bands precisely always touch somewhere in the Brillouin zone. Since the topological spectral flow consists in confined modes at the frontier between two domains of $m_\pm$ with opposite signs, a natural question to ask is thus whether chiral states may exist at the boundary of a finite network with a fixed mass term $m_\pm$, that is specifically when...
for $m_+ < 0$ and $B_0$ for $m_- < 0$, where the gap remains close.

The first remark is that the sign of $m_{\pm}$ that gives rise to edge states does not seem obviously related to the topological charge computed above. Moreover, while these edge states look very similar to what can be found in gapped systems for $A_\pi$, $C_\pi$, $C_0$, and $B_\pi$, as they bridge a local gap, the situation is different for $A_0$ and $B_0$ that are affected by the bulk modes. The inset figures show that these edge states actually do not connect the two bands, but eventually couple to the bulk modes and disappear (figures 9 (a1) and (e1)). This is in sharp contrast with the continuous interfaces in $m_{\pm}$ that revealed a continuous spectral flow through the bulk modes (figures 6b and 8b).

The existence of a spectral flow when a mass term is continuously varied and changes sign, is traditionally understood as a mode emerging at the interface between two topologically nonequivalent systems. This is of course meaningful provided that each system’s topology is well defined in itself, when the mass term is fixed, like in topological insulators. This is however not always the case. In particular, in continuous media, the Chern numbers $C$ of the bands are only well defined when the projectors are regularized at infinity. Otherwise, the topological charge approach used here remains a powerful valid strategy.

V. CONCLUSION

We have reported a model where the gapless spectrum prevents the definition of a Chern number for the bands and a Floquet winding number for the gaps. Still, we have shown that topological chiral spectral flows of interface states can be engineered by a suitable anisotropic perturbation that changes sign. Such spectral flows can even coexist with delocalized (bulk) modes from which they remain uncoupled, in contrast to edge states in finite geometry with open boundaries. These spectral flows can be interpreted as robust chiral states at the interface between gapless semimetallic phases.

\begin{thebibliography}{99}
\bibitem{1} M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
\bibitem{2} N. P. Armitage, E. J. Mele, and A. Vishwanath, Rev. Mod. Phys. 90, 015001 (2018).
\bibitem{3} S. Murakami, New Journal of Physics 9, 356 (2007).
\bibitem{4} S. Murakami, S. Iso, Y. Avishai, M. Onoda, and N. Nagaosa, Phys. Rev. B 76, 205304 (2007).
\bibitem{5} F. D. M. Haldane, Phys. Rev. Lett. 61, 2015 (1988).
\bibitem{6} A. A. Burkov and L. Balents, Phys. Rev. Lett. 107, 127205 (2011).
\bibitem{7} X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, Phys. Rev. B 83, 205101 (2011).
\bibitem{8} P. Delplace, J. Li, and D. Carpentier, EPL (Europhysics Letters) 97, 67004 (2012).
\bibitem{9} S.-Y. Xu, I. Belopolski, N. Alidoust, M. Neupane, G. Bian, C. Zhang, R. Sankar, G. Chang, Z. Yuan, C.-C. Lee, S.-M. Huang, H. Zheng, J. Ma, D. S. Sanchez, B. Wang, A. Bansil, F. Chou, P. P. Shibayev, H. Lin, S. Jia, and M. Z. Hasan, Science 349, 613 (2015).
\bibitem{10} M. Fujita, K. Wakabayashi, K. Nakada, and K. Kusakabe, Journal of the Physical Society of Japan 65, 1920 (1996), https://doi.org/10.1143/JPSJ.65.1920.
\bibitem{11} K. Nakada, M. Fujita, G. Dresselhaus, and M. S. Dresselhaus, Phys. Rev. B 54, 17954 (1996).
\end{thebibliography}
L. Zhou, C. Chen, and J. Gong, Phys. Rev. B 94, 075443 (2016).
X. Ying and A. Kamenev, Phys. Rev. Lett. 121, 086810 (2018).
Y. Hatsugai, Phys. Rev. Lett. 71, 3697 (1993).
G. M. Graf and M. Porta, Communications in Mathematical Physics 324, 851 (2013).
T. Kitagawa, M. A. Broome, A. Fedrizzi, M. S. Rudner, E. Berg, I. Kassal, A. Aspuru-Guzik, E. Demler, and A. G. White, Nat. Commun. 3, 882 (2012).
M. Bellec, C. Michel, H. Zhang, S. Tzortzakis, and P. Delplace, EPL (Europhysics Letters) 119, 14003 (2017).
G. Montambaux, F. Piéchon, J.-N. Fuchs, and M. O. Goerbig, Phys. Rev. B 80, 153412 (2009).
H. Huang, Z. Liu, H. Zhang, W. Duan, and D. Vanderbilt, Phys. Rev. B 92, 161115 (2015).
S. Banerjee, “Anderson localizaion for semi-dirac semi-weyl semi-metal,” (2015), arXiv:1508.05145 [cond-mat.str-el].
C. Zhong, Y. Chen, X. Xie, Y.-Y. Sun, and S. Zhang, Phys. Chem. Chem. Phys. 19, 3820 (2017).
M. Nakahara, Geometry, topology and physics (CRC Press, 2003).
G. E. Volovik, Technology (Oxford University Press, 2009).
P. Delplace, J. B. Marston, and A. Venaille, Science 358, 1075 (2017).
F. Faure, arXiv preprint arXiv:1901.10592 (2019).
M. Perrot, P. Delplace, and A. Venaille, Nature Physics 15, 781 (2019).
M. Marciani and P. Delplace, arXiv preprint arXiv:1906.09057 (2019).
G. Volovik, Zhurnal Ehksperimental’noj i Teoreticheskoj Fiziki 20 (1988).
M. G. Silveirinha, Phys. Rev. B 92, 125153 (2015).
C. Tauber, P. Delplace, and A. Venaille, Journal of Fluid Mechanics 868, R2 (2019).
A. Souslov, K. Dasbiswas, M. Fruchart, S. Vaikuntanathan, and V. Vitelli, Phys. Rev. Lett. 122, 128001 (2019).
C. Tauber, P. Delplace, and A. Venaille, Phys. Rev. Research 2, 013147 (2020).