ON THE STRUCTURE OF SUPERSYMMETRIC $T^3$ FIBRATIONS

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Abstract. We formulate some precise conjectures concerning the existence and structure of supersymmetric $T^3$ fibrations of Calabi–Yau threefolds, and describe how these conjectural fibrations would give rise to the Strominger–Yau–Zaslow version of mirror symmetry.

Mirror symmetry between Calabi–Yau manifolds remains, some twenty years after its discovery, one of the biggest mysteries in mathematics. Originally formulated as a physical relationship between certain pairs of Calabi–Yau manifolds [15, 26, 5] (with astonishing mathematical consequences relating enumerative geometry to Hodge theory [14, 61]), the mirror symmetry proposal was refined in 1996 by Strominger, Yau, and Zaslow [78] into a much more geometric statement, again based on physics. But while the main idea of the Strominger–Yau–Zaslow proposal has been clear from the outset, many of the details have remained elusive. One of the reasons for this is that the proposal involves special Lagrangian submanifolds of a Calabi–Yau manifold, and very few tools are available for studying such submanifolds. As a consequence, the initial period of intense study of the original Strominger–Yau–Zaslow proposal has largely ended[1] and much of the recent work on mirror symmetry has shifted to other approaches (including reformulations of the Strominger–Yau–Zaslow proposal), as is recounted in detail elsewhere in this volume.

Our purpose in this paper is to give a quite precise conjectural formulation of the Strominger–Yau–Zaslow version of mirror symmetry for Calabi–Yau threefolds. The conjectures we formulate are modifications of conjectures previously made by Gross, Ruan, and Joyce; we also restate some other conjectures from [78, 41, 55]. Our formulation is unfortunately not directly based on examples, since—as mentioned above—tools for constructing concrete examples are not currently available. The conjectures are, however, motivated on the one hand by qualitative features of the Strominger–Yau–Zaslow proposal which have been discovered by mathematicians,

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[1] A very readable summary of the progress made on the original proposal, and the transition to the more recent approaches, was given recently by Gross [34].
and on the other hand by some suggestive arguments from physics. We focus on the motivation from mathematics in this paper.

For simplicity, we restrict our attention for the most part to Calabi–Yau manifolds of complex dimension one, two, or three. We expect, however, that similar conjectures could be formulated in higher dimension, at the expense of greater combinatorial complexity.

1. Supersymmetric torus fibrations

A Calabi–Yau metric is a Riemannian metric on a manifold $X$ of dimension $2n$ whose Riemannian holonomy is precisely $SU(n)$. The representation theory of the holonomy group gives rise to various geometric structures on $X$: there is a compatible complex structure (unique up to complex conjugation when $n \geq 3$), a 2-form $\omega$ which serves as the Kähler form of the given metric with respect to that complex structure, and a nowhere-vanishing holomorphic $n$-form $\Omega$. Both $\omega$ and $\Omega$ are covariantly constant (as is the almost-complex structure operator $J$).

Thanks to a result conjectured by Calabi [11] and proven by Yau [80], when $X$ is a compact Kähler manifold with a nowhere-vanishing holomorphic $n$-form, there is a unique Ricci-flat metric in each de Rham cohomology class $[\omega] \in H^2(X, \mathbb{R})$ containing a Kähler form. These metrics have holonomy contained in $SU(n)$, so they will be Calabi–Yau under our definition provided that the holonomy does not reduce to a subgroup. As a consequence, the existence of a nowhere-vanishing holomorphic $n$-form on a Kähler manifold $X$ is often taken as a definition of Calabi–Yau manifold.

Given a Calabi–Yau metric on $X$, an associated Kähler form $\omega$, and a holomorphic $n$-form $\Omega$, we say that a submanifold $L$ of (real) dimension $n$ is special Lagrangian if $\omega|_L \equiv 0$ and $\text{Im}(e^{i\theta}\Omega)|_L \equiv 0$ for some $\theta$ called the phase of $L$. This notion was introduced by Harvey and Lawson [45] as a key example of a calibrated geometry: such submanifolds have a local volume-minimizing property. Unfortunately, very few examples of special Lagrangian submanifolds are known in the compact case.

Let $X$ be a compact Calabi–Yau manifold. Physical arguments predict that for most such $X$, there should exist a mirror partner $Y$, which is another compact Calabi–Yau manifold whose physical (but not geometrical) properties are closely related to those of $X$.

However, as has been recognized since the early days of mirror symmetry [13, 48, 49], $X$ is expected to have a mirror partner only when the complex structure of $X$ is sufficiently close to a “large complex structure limit point,” which is a class of boundary points in the compactified moduli space $\mathcal{M}_X$ characterized by having maximally unipotent monodromy [61, 21, 62]. Strominger, Yau, and Zaslow [78] argued on physical grounds that any such Calabi–Yau manifold should have a map $\pi : X \to B$ whose general fiber $\pi^{-1}(b)$ is a special Langrangian $n$-torus $T^n$;

Some authors use the term Calabi–Yau metric when the holonomy is any subgroup of $SU(n)$.

For a brief review of Calabi–Yau manifolds and mirror symmetry, see [66]. More extensive reviews can be found in [20, 65, 47].
such a structure is called a *supersymmetric torus fibration of* \( X \), since the special Lagrangian condition is the geometric counterpart to the preservation of half of the supersymmetry in a physical model. Strominger, Yau, and Zaslow also proposed that the mirror partner should be given, to first approximation, by a compactification of the family of dual tori \( \bigcup (\pi^{-1}(b))^\vee \).

It is worth saying a few words about the physical construction of the mirror partner. Strominger, Yau, and Zaslow argue that the original Calabi–Yau manifold and its mirror partner should be related by a physical construction known as “T-duality on the \( n \)-torus fibers.” In the absence of holomorphic disks with boundaries on the \( n \)-tori, this T-duality simply replaces each nonsingular torus by its dual torus (cf. [58]), while doing something unknown at the singular fibers. This description is expected to be modified when holomorphic disks are present, but the precise effect of the holomorphic disks has not yet been worked out. And as we shall see, the current expectation (at least when \( n = 3 \)) is that such holomorphic disks will be present for at least some of the tori in the torus fibration. This has made it difficult to formulate a mathematical version of the original Strominger–Yau–Zaslow proposal which is both precise and accurate.

Because the arguments used by Strominger, Yau, and Zaslow implicitly assume that the Calabi–Yau metric is uniformly large, we put that hypothesis in the following version of their existence conjecture.

**Conjecture 1** (Existence; cf. [78]). *For any Calabi–Yau metric on a compact complex manifold* \( X \) *of complex dimension* \( n \) *whose complex structure is sufficiently close to a large complex structure limit point and whose Kähler class is sufficiently deep in the Kähler cone, there exists a supersymmetric torus fibration* \( \pi : X \rightarrow B \), *where* \( B \) *is a homology* \( n \)-*sphere.*

There is by now considerable indirect evidence in favor of this conjecture, including an explicit construction in a (slightly degenerate) limiting case [40], as well as two strategies ([69, 70, 71, 72] and [75, 76, 77]) for constructing weak forms of these fibrations for a certain class of compact Calabi–Yau threefolds. However, it has become clear that proving this conjecture will require developing new techniques for studying special Lagrangian submanifolds of a Calabi–Yau manifold. In spite of our lack of tools to prove the conjecture, though, many qualitative features of supersymmetric torus fibrations have been inferred in various ways, and this paper is devoted to explaining our best current (conjectural) understanding of those qualitative features.

We introduce the following terminology and notation. Given a supersymmetric torus fibration \( \pi : X \rightarrow B \), we let \( \Sigma \subset X \) be the set of *singular points of fibers* of \( \pi \), and let \( \Delta = \pi(\Sigma) \) be the *discriminant locus* of the fibration.

## 2. Examples in Low Dimension

In low dimension, supersymmetric torus fibrations of compact Calabi–Yau manifolds are completely understood.
In the case of elliptic curves \((n = 1)\), any Calabi–Yau metric is flat and a special Lagrangian 1-torus is just a closed geodesic. As is well known, if the homology class is fixed then there is a fibration of the elliptic curve over \(B = S^1\) by closed geodesics in the specified class, with no singular fibers.

In the case of K3 surfaces \((n = 2)\), an analysis is possible due to the non-uniqueness of the compatible complex structure. In fact, the original paper of Harvey and Lawson [45] showed that if \(L \subset X\) is special Lagrangian, then there is a different complex structure on \(X\) (compatible with the given Calabi–Yau metric) such that \(L \subset X\) is a complex submanifold. Thus, a special Lagrangian \(T^2\) fibration can be interpreted in another complex structure as a holomorphic elliptic fibration \(\pi : X \rightarrow B\) (with base \(B = \mathbb{C}P^1 = S^2\)), and the structure of these is known in detail. (In fact, a generic Ricci-flat metric on a K3 surface admits such a fibration [64], so the existence conjecture holds in this case.) Thanks to work of Kodaira [54], a complete classification of possible singular fibers of such fibrations is known: they are characterized by the conjugacy class of the monodromy action on \(H^1(T^2, \mathbb{Z})\). The simplest fibers, called semistable, are associated to unipotent monodromy transformations. In an appropriate basis, the monodromy matrix takes the form

\[
M = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.
\]

The topology of a semistable degeneration with \(k = 1\) is very familiar. One of the cycles on the two-torus extends over the degeneration, and the other “vanishing” cycle shrinks to a point; if we follow the torus around a loop encircling the degeneration point in the base, there is a Dehn twist along the vanishing cycle. In spite of this twisting of the topology of the torus, though, the total space of the fibration is non-singular. (In the physics literature, the corresponding geometry is known as the “Taub-NUT metric.”)

For the generic elliptic fibration of a K3 surface, all fibers are semistable, and there are exactly 24 of them, each with \(k = 1\). The monodromy data for such a generic fibration (choosing an arbitrary base point \(b \in B\)) gives a natural homomorphism

\[
\pi_1(B - \{P_1, \ldots, P_{24}\}, b) \rightarrow SL(2, \mathbb{Z})
\]

whose generating loops all map to matrices conjugate to eq. (2.1).

The mirror partner of a given K3 surface (with a fixed Ricci-flat metric) is known to be another K3 surface with a different Ricci-flat metric [6]. When passing to the mirror, the monodromy matrices \(M\) are replaced by \(tM^{-1}\); since \(tM^{-1}\) is conjugate to \(M\), the monodromy data does not change. We will conjecturally extend this kind of “topological” mirror symmetry statement to dimension 3 in the next section. Note that, in any dimension, if we replace all nonsingular tori by their dual tori, the monodromy matrices change as \(M \mapsto tM^{-1}\).
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3. Smooth $T^3$ Fibrations

The first step in studying supersymmetric $T^3$ fibrations of compact Calabi–Yau threefolds is to study more general $T^3$ fibrations, without imposing the “special Lagrangian” condition. For Calabi–Yau hypersurfaces in toric varieties (of arbitrary dimension), Zharkov [81] constructed a (topological) $T^n$ fibration. A general program to understand such fibrations $\pi : X \to B$ (in dimension 3) for which the map $\pi$ is smooth (i.e., $C^\infty$) was initiated by Gross [28, 29, 31], and parallel results were obtained by Ruan [69, 70, 71] in his study of a specific class of $T^3$ fibrations of Calabi–Yau hypersurfaces. The monodromy of such fibrations and the topology of the singular fibers was determined under a suitable assumption of genericity, analogous to the assumption of “generic elliptic fibration” in the case of K3 surfaces which guaranteed that all fibers were semistable with $k = 1$. We can summarize the analysis in a general conjecture (which conjecturally extends their results to the general case).

Conjecture 2 (Topology; cf. [28, 29, 31, 69, 70, 71]). Let $\pi : X \to B$ be a smooth $T^3$ fibration of a compact Calabi–Yau threefold which is generic in a suitable sense. Then

a) The discriminant locus of the fibration is a trivalent graph $\Gamma$.

b) The topology near the edges of $\Gamma$ is modeled by the product of a cylinder with the $k = 1$ semistable degeneration of two-tori. In particular, a Dehn twist along the vanishing cycle and a nonsingular total space are features of this topology.

c) For any loop around an edge of $\Gamma$, the monodromy on either $H^1 \cong H_2$ or $H_1$ of the 3-tori is conjugate to

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (3.1)

In particular, both monodromy actions have a 2-dimensional fixed plane.

d) The vertices of $\Gamma$ come in two types: near a positive vertex, the three monodromy actions on $H^1 \cong H_2$ near the vertex have fixed planes whose intersection is 1-dimensional, while the three monodromy actions on $H_1$ have a common 2-dimensional fixed plane. In an appropriate basis, the monodromy matrices on $H_1$ take the form

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (3.2)

On the other hand, near a negative vertex, the three monodromy actions on $H^1 \cong H_2$ near the vertex have a common 2-dimensional fixed plane, while the three monodromy actions on $H_1$ have fixed planes whose intersection is

\[\text{We stress that it is the map } \pi \text{ which is smooth, not the fibers of the fibration.}\]
1-dimensional. In an appropriate basis, the monodromy matrices on $H_1$ take the form

\begin{align*}
&\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
&\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
&\begin{pmatrix}
1 & -1 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\end{align*}

(e) The fiber of $\pi$ over any point of $B$ other than a vertex of $\Gamma$ has a fixed point free $U(1)$ action and in particular has Euler characteristic 0. The fiber of $\pi$ over each positive vertex has Euler characteristic 1, and the fiber over each negative vertex has Euler characteristic $-1$. (In fact, Gross [31] and Ruan [71] gave explicit descriptions of these singular fibers, but we will not reproduce those descriptions here.)

There is an induced global monodromy action on $H_1$

\begin{equation}
\pi_1(B - \Delta, b) \to SL(3, \mathbb{Z}),
\end{equation}

whose generators satisfy the conditions spelled out in the conjecture. If we have a compact Calabi–Yau threefold and its mirror partner, with smooth $T^3$ fibrations whose nonsingular fibers are dual to each other, then the monodromy transformations will be related as $M \mapsto t M^{-1}$. This implies that the roles of the positive and negative vertices in the fibration are reversed between a Calabi–Yau threefold and its mirror partner. Since part (e) of the conjecture implies that the topological Euler number of $X$ can be calculated via

\begin{equation}
\chi_{top}(X) = \#\{\text{positive vertices}\} - \#\{\text{negative vertices}\},
\end{equation}

the effect of mirror symmetry on monodromy then shows that the Euler number changes sign:

\begin{equation}
\chi_{top}(Y) = \#\{\text{negative vertices}\} - \#\{\text{positive vertices}\} = -\chi_{top}(X),
\end{equation}

as expected from physical mirror symmetry arguments.
In [31], Gross showed how to go further, and use the data from a generic smooth $T^3$ fibration of a given compact Calabi–Yau threefold to construct a manifold which is a candidate mirror partner. The transpose inverse of the original monodromy representation produces a mirror monodromy representation, describing the monodromy on the family of dual tori (with positive and negative vertices reversed). Gross proved a “Reconstruction Theorem:” the family of dual tori can be completed to a compact topological manifold with a smooth $T^3$ fibration, satisfying the properties stated in the conjecture.

4. Combinatorics of $\Gamma$

For smooth $T^3$ fibrations of a compact Calabi–Yau threefold, the combinatorics of the graph $\Gamma$ are beautiful and intricate. For example, in the case of a quintic hypersurface in $\mathbb{CP}^3$, the graph depends on choices of triangulations of the two-dimensional faces of the Newton polytope of the defining equation; one such choice is shown on the left side of Figure 1. One constructs the dual graph of each such triangulation, in which each face of the triangulation gives a vertex of the graph, and each edge of the triangulation is crossed by an edge of the graph (as shown on the right side of Figure 1). That dual graph then becomes a piece of $\Gamma$ (illustrated on the left side of Figure 2) in which each vertex is “negative.” The pieces are assembled according to the combinatorics of the Newton polytope, in which the free ends of the dual graph meet free ends from other faces of the Newton polytope, forming trivalent vertices which are the “positive” vertices of $\Gamma$. The pieces thus attach three at a time; a neighborhood of one such attachment is illustrated on the right side of Figure 2.

A general description of these graphs, for Calabi–Yau complete intersections in toric varieties, was given by Haase and Zharkov [42, 43, 44] and by Gross [33]. Their results show that the classes of mirror pairs described by Batyrev [8] and by Batyrev

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The proofs of some of the results stated by Gross [33] were deferred to another paper which has not yet appeared.
and Borisov [10, 9] admit smooth $T^3$ fibrations satisfying Conjecture 2 and that those $T^3$ fibrations are mirror duals of each other.

This theory of smooth $T^3$ fibrations of compact Calabi–Yau threefolds can also be related to some topological aspects of mirror symmetry which have played important rôles in the physics literature [79, 4, 25]. First, the construction of $\Gamma$ for a Calabi–Yau hypersurface $X$ in a toric fourfold depends on a choice of triangulation of the faces of the Newton polytope, and this choice of triangulation is equivalent to a choice of large complex structure limit point in the moduli space $\mathcal{M}_X$. Mirror symmetry (as developed in the physics literature) offers an alternate interpretation: each large complex structure limit point corresponds to a different birational model of the mirror partner, and the Kähler cones of the birational models fit together into a common space (after complexification), mirroring the complex structure moduli space $\mathcal{M}_X$. In this interpretation, the choice of birational model depends explicitly on a choice of triangulation of the Newton polytope in Batyrev’s construction [8]. The simplest birational change—a “flop”—is realized by the simple change of triangulation illustrated in Figure 3. As Gross pointed out [31, Remark 4.5], the corresponding change in dual graph (also illustrated in the figure) has the correct monodromy properties to be allowed as a new graph $\Gamma'$. One expects that by appropriately varying the complex structure and the Kähler metric, the connecting edge in the original graph will shrink to zero length, giving the 4-valent vertex shown in the intermediate stage; further variation then causes a new connecting edge to grow, changing the topology to that of $\Gamma'$.

The intermediate step illustrated in the middle of Figure 3 is a partial triangulation, which is expected to correspond to a “conifold” singularity on the mirror Calabi–Yau threefold. A second topological feature of mirror symmetry is the “conifold transition” [12, 25] in which that conifold singularity is resolved with a small blowup, rather than being smoothed with a change of complex structure. (The conifold singularity is on the mirror partner, but this transition can also be described on the original Calabi–Yau threefold [65].) The graph $\Gamma$ appears to change as follows, as proposed independently by Gross [30] and Ruan [73]: after shrinking the connecting edge to zero size, leaving a 4-valent vertex, the two arms of the graph crossing at that vertex are separated into different planes, as illustrated in Figure 4. As Gross and Ruan verify, this change is compatible with the monodromies around the edges.

**Figure 3.** The change of triangulation corresponding to a flop.
and produces the expected change in topological Euler characteristic for a conifold transition. However, the relation between this construction and more global versions of the conifold transition remains mysterious and needs further study.

5. Affine structures on the base

Local moduli for a compact special Lagrangian submanifold $L$ of a compact Calabi–Yau manifold $X$ were determined by McLean [59]: the deformation space is smooth, and its tangent space is canonically identified with the space of harmonic 1-forms of $L$. Hitchin [46] used this identification to construct two affine structures on $B - \Delta$ (if $\pi$ is smooth), which geometrize the monodromy transformations that occurred in Conjecture 2.

If $V$ is a normal vector field to $L$ in $X$, then the contraction $\iota(V)\omega$ of $V$ with the Kähler form gives a harmonic 1-form on $L$, and the contraction $\iota(V)\Omega$ of $V$ with the holomorphic $n$-form gives a harmonic $(n - 1)$-form on $L$. These constructions give rise to the affine structures, which can be seen by considering periods of the harmonic forms. If $\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_n}$ are vector fields on the deformation space $\mathcal{M}$ which span the tangent space to $\mathcal{M}$ at $[L]$, and $A_1, \ldots, A_n$ is a basis for $H_1(L, \mathbb{Z})$, then we can form the period matrix

$$\lambda_{ij} = \int_{A_i} \iota\left(\frac{\partial}{\partial t_j}\right)\omega.$$  

The 1-forms $\sum_j \lambda_{ij} dt_j$ on the deformation space $\mathcal{M}$ are closed, and can be integrated to give local coordinates $u_1, \ldots, u_n$ on $\mathcal{M}$ at $[L]$ satisfying $du_i = \sum_j \lambda_{ij} dt_j$. Such coordinate systems provide an affine structure, that is, the transition functions between any two such coordinate systems lie in the affine group $\mathbb{R}^n \rtimes \text{GL}(\mathbb{Z}^n)$. Intrinsically, the lattice $\mathbb{Z}^n$ should be identified with $H_1(L, \mathbb{Z})$ in this case, and this affine structure carries the information about the monodromy on $H_1(L, \mathbb{Z})$.

Similarly, if $B_1, \ldots, B_n$ is a basis for $H_{n-1}(L, \mathbb{Z})$, then we can form the period matrix

$$\mu_{ij} = \int_{B_i} \iota\left(\frac{\partial}{\partial t_j}\right)\Omega.$$
The 1-forms $\sum_j \mu_{ij} dt_j$ on the deformation space $\mathcal{M}$ of $L$ are closed, and can be integrated to give local coordinates $v_1, \ldots, v_n$ on $\mathcal{M}$ satisfying $dv_i = \sum_j \mu_{ij} dt_j$. Such coordinate systems provide the other affine structure, which carries the information about the monodromy on $H_{n-1}(L, \mathbb{Z}) \cong H^1(L, \mathbb{Z})$.

Hitchin shows that these two affine structures are related by a Legendre transform with respect to a suitable locally defined function $K$ (which also determines a canonical metric on the deformation space). This is a version of mirror symmetry which is formulated strictly on the base of the torus fibrations, a notion which was further developed in [41, 55].

6. THE LARGE COMPLEX STRUCTURE LIMIT

When $X$ is a compact Calabi–Yau threefold, it is expected that the map $\pi$ giving a supersymmetric torus fibration $\pi : X \rightarrow B$ will be only piecewise smooth, so the analysis of Sections 3 and 5 does not directly apply. Gross showed [29] that if $\pi$ is smooth, the discriminant locus $\Delta \subset B$ must have codimension two; in the more general “piecewise smooth” case, $\Delta$ may have codimension 1. However, we do expect that $\Delta$ will always have a retraction onto a subset $\Gamma$ of codimension two. And there is a particular limiting situation in which this retraction should become evident: the large complex structure limit.

In fact, Gross–Wilson [41] and Kontsevich–Soibelman [55] have formulated a precise conjecture about the large complex structure limit of a supersymmetric torus fibration.

**Conjecture 3** (Large Complex Structure Limit; cf. [41, 55]). Let $\mathcal{X} \rightarrow S$ be a maximally unipotent degeneration of compact simply-connected Calabi–Yau manifolds of complex dimension $n$, degenerating at $0 \in S$, let $s_i \in S$ be a sequence with $\lim s_i = 0$, and let $g_i$ be a sequence of Ricci-flat metrics on $X_{s_i}$ with diameter bounded above and below. Then there exists a subsequence $(X_{s_{i_j}}, g_{i_j})$ which converges in the sense of Gromov–Hausdorff [27] to a metric space $(X_\infty, d_\infty)$, where $X_\infty$ is homeomorphic to the sphere $S^n$, and $d_\infty$ is induced by a Riemannian metric on $X_\infty - \Gamma_\infty$ for some $\Gamma_\infty \subset X_\infty$ of codimension two.

Following the affine structure to this limit, one expects to find a limiting affine structure, and in fact the discriminants $\Delta_{i_k}$ should have collapsed to $\Gamma_\infty$ in the limit.

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Note there has been substantial additional progress on these limits and on various structures on the base in subsequent work of Kontsevich and Soibelman [56, 57], reviewed elsewhere in this volume.
7. NON-COMPACT EXAMPLES OF SPECIAL LAGRANGIAN FIBRATIONS

Harvey and Lawson’s original paper about calibrations \[45\] gave an explicit example of a special Lagrangian fibration. Define \( f : \mathbb{C}^3 \to \mathbb{R}^3 \) by

\[
(7.1) \quad f(z_1, z_2, z_3) = (\text{Im}(z_1 z_2 z_3), |z_1|^2 - |z_2|^2, |z_1|^2 - |z_3|^2).
\]

Then the fibers of \( f \) are special Lagrangian, and are all invariant under the action of the diagonal torus with determinant 1:

\[
(7.2) \quad \{ \text{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \mid \theta_1 + \theta_2 + \theta_3 = 0 \}.
\]

The singularities of fibers are located where \( z_i = z_j = 0 \) for some pair of indices \( i \) and \( j \); the images of these give three rays within the plane \( \{ x_1 = 0 \} \subset \mathbb{R}^3 \), namely

(i) \( x_2 = 0, x_3 \leq 0 \),

(ii) \( x_2 \leq 0, x_3 = 0 \),

(iii) \( x_2 = x_3 \geq 0 \). This is illustrated in Figure 5.

The nonsingular fibers are all homeomorphic to \( T^2 \times \mathbb{R} \). Note that whenever \( x_1 = 0 \), if we write \( z_j = r_j e^{i\theta_j} \) then either \( \theta_1 + \theta_2 + \theta_3 = 0 \) (in which case \( \text{Re}(z_1 z_2 z_3) \geq 0 \)), or \( \theta_1 + \theta_2 + \theta_3 = \pi \) (in which case \( \text{Re}(z_1 z_2 z_3) \leq 0 \)). Thus, there are two natural subsets \( f^{-1}(0, x_2, x_3)^\pm \) of the fiber \( f^{-1}(0, x_2, x_3) \), distinguished by the sign of \( \text{Re}(z_1 z_2 z_3) \).

These subsets meet along

\[
(7.3) \quad f^{-1}(0, x_2, x_3) \cap \{ z_1 z_2 z_3 = 0 \}.
\]

When the fiber is smooth, each subset is a manifold with boundary, and they meet along their common boundary. However, when the fiber is singular and \( (x_2, x_3) \neq (0, 0) \), each subset is itself a smooth special Lagrangian submanifold, homeomorphic to \( S^1 \times \mathbb{R}^2 \). Note that \( f^{-1}(0, 0, 0) \) is a special case: as Harvey and Lawson pointed out, each subset \( f^{-1}(0, 0, 0)^\pm \) is a cone over \( T^2 \), and those cones meet precisely at the origin in \( \mathbb{C}^3 \).

Further development of examples of this type was made by Goldstein \[24\] and Gross \[30\], and they have been extensively studied in the physics literature (e.g., in \[3, 2, 1\]).
Figure 6. The discriminant locus as a ribbon.

Joyce [49] builds some new special Lagrangian fibrations by carefully combining subsets of Harvey–Lawson fibers. Let

\[ N^+_a = \{ |z_1|^2 - a = |z_2|^2 + a = |z_3|^2 + |a|, \right. \\
\left. \quad \text{Im}(z_1z_2z_3) = 0, \text{Re}(z_1z_2z_3) \geq 0 \} \]

and

\[ N^-_a = \{ |z_1|^2 - a = |z_2|^2 + a = |z_3|^2 + |a|, \right. \\
\left. \quad \text{Im}(z_1z_2z_3) = 0, \text{Re}(z_1z_2z_3) \leq 0 \} \]

Then

\[ N^+_a = \begin{cases} f^{-1}(0, 2a, 2a) & \text{when } a \geq 0, \\ f^{-1}(0, a, 0) & \text{when } a \leq 0. \end{cases} \]

and similarly for \( N^-_a \).

To build a special Lagrangian fibration, Joyce considers translations of these manifolds for \( c \in \mathbb{C} \). Let

\[ N^{\pm}_{a,c} = \{ |z_1|^2 - a = |z_2|^2 + a = |z_3 - c|^2 + |a|, \right. \\
\left. \quad \text{Im}(z_1z_2(z_3 - c)) = 0, \pm \text{Re}(z_1z_2(z_3 - c)) \geq 0. \} \]

These can be made the fibers of special Lagrangian fibrations by defining \( F^\pm : \mathbb{C}^3 \to \mathbb{R} \times \mathbb{C} \) by

\[ F^\pm(z_1, z_2, z_3) = \begin{cases} \left( \frac{1}{2}(|z_1|^2 - |z_2|^2), z_3 \right) & \text{if } |z_1| = |z_2| = 0 \\ \left( \frac{1}{2}(|z_1|^2 - |z_2|^2), z_3 \mp \frac{z_1 \bar{z}_2}{|z_1|^2} \right) & \text{if } |z_2|^2 \leq |z_1|^2 \neq 0 \\ \left( \frac{1}{2}(|z_1|^2 - |z_2|^2), z_3 \mp \frac{\bar{z}_1 z_2}{|z_2|^2} \right) & \text{if } |z_2|^2 > |z_1|^2 \end{cases} \]

With this definition, \( (F^\pm)^{-1}(a, c) = N^{\pm}_{a,c} \).

Notice that the fibrations \( F^\pm \) are only piecewise smooth, and that the discriminant locus in each case is \( \{(0, c)\} \subseteq \mathbb{R} \times \mathbb{C} \), which has codimension 1. Notice also that when \( a > 0 \), both \( N^+_{a,c} \) and \( N^-_{a,c} \) contain the boundary of the holomorphic disk \( \{ |z_1|^2 \leq a \} \) of area \( 2\pi a \), and that when \( a < 0 \), both \( N^+_{a,c} \) and \( N^-_{a,c} \) contain the boundary of the holomorphic disk \( \{ |z_2|^2 \leq -a \} \) of area \( -2\pi a \). In some sense, these shrinking disks are “responsible” for the singularity being created at \( a = 0 \).

To go further, Joyce invokes the extensive theory which he developed in [50, 51, 52] concerning the structure of special Lagrangian 3-manifolds with a \( U(1) \) action. Using
that theory, he is able to construct \cite[Theorem 6.5]{Joyce} a special Lagrangian fibration $\hat{F}: \mathbb{C}^3 \to \mathbb{R}^3$ whose discriminant locus is a ribbon, that is, the locus $\{(0, x_2, x_3) \mid 0 \leq x_2 \leq 1\} \subset \mathbb{R}^3$, as illustrated in Figure 6. There are several important properties of this example of Joyce’s. First, the fiber over an interior point of the ribbon has two singularities—one locally modeled by $F^+$ and the other locally modeled by $F^-$. Second, as in those local models, there are holomorphic disks with boundary in the fiber for $x_1 \neq 0$ (and $0 < x_2 < 1$), whose area approaches 0 as $x_1$ approaches 0; in fact, there is one such holomorphic disk for each of the two singular points.

Third, as we approach the boundary of the strip within the plane $x_1 = 0$, something interesting happens: the two bounding circles approach each other and the holomorphic disks cancel out as the boundary of the strip (either $x_2 = 0$ or $x_2 = 1$) is reached. There are no holomorphic disks when $x_2 < 0$ or $x_2 > 1$. The region in which holomorphic disks are present is illustrated in Figure 7.

Thus, along the boundary of the strip, one has a singularity of multiplicity 2 (in an appropriate sense), which bifurcates into a pair of singularities in the middle of the strip, and those singularities rejoin at the other boundary.

Note that the plane which contains the discriminant locus can be identified intrinsically using local affine coordinates. The cycle $\gamma \in H_1(N_{a,c}^\pm, \mathbb{Z})$ which bounds a holomorphic disk is the vanishing cycle for the family, and it defines a dual subspace in $\gamma^\perp \subset H_c^1(N_{a,c}^\pm, \mathbb{Z})$, which can be locally identified with the plane containing the discriminant locus. This plane can also be characterized as the monodromy-invariant plane in the compactly-supported cohomology of the fiber.

Joyce conjectures that his examples exhibit generic behavior. In fact, even Lagrangian fibrations (not just special Lagrangian fibrations) are expected to exhibit these phenomena, as explained in \cite{Joyce}.

**Conjecture 4** (Singular Fibers; cf. \cite{Joyce}). Let $\pi : X \to B$ be a supersymmetric $T^3$ fibration of a compact Calabi–Yau threefold with respect to a Calabi–Yau metric...
whose compatible complex structure is sufficiently close to a large complex structure limit point, and whose Kähler class is sufficiently deep in the Kähler cone. Then

1. The discriminant locus $\Delta \subset B$ has codimension one. In affine coordinates, $\Delta$ is locally contained in the plane corresponding to the monodromy-invariant subspace of $H^1(\pi^{-1}(b), \mathbb{Z})$ for $b$ near $\Delta$.
2. The fiber over the general point of $\Delta$ has two singular points, one of which is modeled locally by $F^+$ and the other of which is modeled locally by $F^-$. 
3. The fiber over the general point of the boundary of $\Delta$ is modeled locally by $\hat{F}$.
4. Let $\mathcal{H} \subset B$ be the set of fibers which contain the boundary of at least one holomorphic disk in $X$, or are singular. Then the boundary of $\Delta$ is contained in the boundary of $\mathcal{H}$.

The last statement about which fibers contain the boundaries of disks was not conjectured by Joyce, but is consistent with the behavior exhibited by his example $\hat{F}$ (as illustrated in Figure 7).

8. Amoebas

A common feature of the constructions of Zharkov [81] and Ruan [69, 70, 71, 72], which dovetails nicely with the analysis of Joyce described in Section 7, is the description of the discriminant locus of a supersymmetric torus fibration as an amoeba. Amoebas were introduced by Gelfand, Kapranov, and Zelevinsky [23]; we will briefly review the theory, following Mikhalkin [60] (see also [32]).

Let $f = \sum a_I x^I$ be a Laurent polynomial in $n$ complex variables. (Here, $I$ is a multi-index with negative powers allowed, but $f$ has only finitely many non-zero terms.) The amoeba of $f$ is the set $\mathcal{A}_f = \text{Log}(V_f)$, where $V_f = \{ \vec{z} \in (\mathbb{C}^*)^n \mid f(\vec{z}) = 0 \}$, and $\text{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$ is defined by

$$ \text{Log}(z_1, \ldots, z_n) = (\log |z_1|, \ldots, \log |z_n|). $$

A simple example is given by $f(z, w) = z + w + 1$, which is often chosen because it can be graphed exactly, as in Figure 8. More complicated examples have “holes” in the amoeba, as indicated on the left side of Figure 9.
The Laurent polynomial \( f \) has an associated Newton polytope \( \Delta_f \) and toric variety \( T_f \), and there is a moment map \( \mu : T_f \to \Delta_f \subset \mathbb{R}^n \) once a symplectic structure has been chosen on \( T_f \). The closure of \( \mu(V_f) \) is called the \textit{compactified amoeba} of \( f \). Note that \( \text{Log} \) and \( \mu \) are closely related: the interior of the image \( \Delta_f \) of \( \mu \) is mapped homeomorphically by \( \text{Log} \circ \mu^{-1} \) to all of \( \mathbb{R}^n \). An example of a compactified amoeba is illustrated on the right side of Figure 9.

Forsberg, Passare, and Tsikh [22] showed that each component of the complement \( \mathbb{R}^n - A_f \) is convex, and that there is an injective map from the set of components to the lattice points \( \Delta_f \cap \mathbb{Z}^n \) in the Newton polytope, in which the bounded components (the “holes”) map to points in the interior of \( \Delta_f \). For a given polyhedron \( \Delta \), there exist functions \( f \) with \( \Delta_f = \Delta \) whose amoebas have the maximum number of holes, but typically there also exist functions with \( \Delta_f = \Delta \) whose amoebas have fewer holes.

In the case \( n = 2 \), a formula of Baker [7, 53] identifies the genus of a smooth compactification of the affine curve \( V_f \) with the number of interior lattice points in the Newton polytope. Thus, in that case, the maximum number of holes coincides with the genus. There is an associated topological picture when the number of holes is maximal: the map \( \text{Log} \) will in this case be 2-to-1 over the interior of the amoeba, and 1-to-1 on the boundary of the amoeba. It is easy to see that this gives the right answer for the genus.

Additional information about the amoeba can be obtained by considering the \textit{Ronkin function} \( N_f : \mathbb{R}^n \to \mathbb{R} \) defined by

\[
N_f(x_1, \ldots, x_n) = \frac{1}{(2\pi i)^n} \int_{\mu^{-1}(x_1, \ldots, x_n)} \log |f(z_1, \ldots, z_n)| \frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_n}{z_n}.
\]

Ronkin [68] and Passare–Rullgård [67] show that \( N_f \) is well-defined on all of \( \mathbb{R}^n \), is convex over \( A_f \), and is locally linear on the complement of \( A_f \). Let \( \{E\} \) be the set of components of the complement, and let \( N_E \) be the extension of \( N_f|_E \) to a linear function on all of \( \mathbb{R}^n \). Passare and Rullgård define

\[
N_f^\infty = \max_{E} N_E,
\]
which is a piecewise linear function on $\mathbb{R}^n$, and then define the spine of the amoeba $A_f$ to be the set $S_f \subset \mathbb{R}^n$ of points at which the function $N_f^\infty$ is not locally linear. A key theorem of [67] is that the spine $S_f$ is a strong deformation retract of the amoeba $A_f$.

Note that the spine of the amoeba which is shown in Figure 8 is precisely given by Figure 5.

Ruan [74] observed that for a Calabi–Yau hypersurface $X$ in a toric fourfold $T$ which is close to the large complex structure limit, the intersections $C_{jk} = X \cap T_j \cap T_k$ with pairs of toric divisors have amoebas with the maximum number of holes, and these amoebas retract to their spines as the large complex structure limit is approached. The spines in fact form the pieces of the graph $\Gamma$ used to describe a topological $T^3$ fibration, which are dual graphs of appropriate triangulations of the Newton polytopes (as was illustrated in Figure 1).

An amoeba for $C_{jk}$, together with its spine, is shown in the case of the quintic threefold in Figure 10. The spine is precisely the graph which occurred on the left side of Figure 2.

As in the case of $\Gamma$ itself, moving among different large complex structure limit points causes the combinatorics of the triangulation to change (as discussed at the end of Section 3); we expect a corresponding change in the combinatorics of the amoebas.

9. Reconstruction in the Lagrangian case

Gross’s reconstruction theorem produced a topological 6-manifold out of the data describing a smooth $T^3$ fibration, and this provides (in principle) a method for constructing mirror partners when they are not known, provided that one has a smooth $T^3$ fibration.
Castaño-Bernard and Matessi [18] have proved an analogous theorem which produces a compact symplectic 6-manifold with a piecewise smooth Lagrangian $T^3$ fibration, starting from the data describing the smooth fibers of this fibration. The starting data in this case is the affine structure on the base—a refinement of the simple monodromy data which Gross’s theorem needed. The key technique of stitching together smooth $T^3$ fibrations along a common boundary had been developed in earlier work of these authors [17, 16].

A particularly interesting feature of Castaño-Bernard and Matessi’s construction is the behavior of the discriminant locus $\Delta$. Their fibrations have a discriminant locus which is a trivalent graph near the positive vertices, but has codimension 1 near the negative vertices. The discriminant locus retracts onto a trivalent graph; the inverse “thickening” of parts of this graph to a codimension 1 set replaces each neighborhood of a negative vertex with an amoeba-like shape which retracts back to the graph. Moreover, the fibration is smooth outside of a set which retracts to (a subset of) the graph. This result, when combined with the discussion in Section 8, helps to motivate our conjectures in the next two Sections.

10. The geometry of $T^3$ fibrations

Prior to Joyce’s analysis of the structure of special Lagrangian fibrations [49], there had been speculation that supersymmetric $T^3$ fibrations of compact Calabi–Yau threefolds would always be smooth, so that the detailed structure (in the generic case) would be the one given in Conjecture 2. However, Joyce’s analysis prompted many of us to rethink the question, and to try to formulate properties analogous to those of Conjecture 2 which we would expect supersymmetric $T^3$ fibrations to have. Once such formulation appears in Conjecture 5 below.

In Conjecture 2, the discriminant locus is a graph $\Gamma$, but in general we should expect a discriminant locus $\Delta$ which only retracts to a graph $\Gamma$. As Joyce pointed out (and was already mentioned in Conjecture 4), the first thing to expect is that the edges of the graph $\Gamma$ should thicken to ribbons; moreover, one should see two singular points in each fiber over an interior point of the ribbon, with the two points coming together to a single singular point along the edges of the ribbon. The next thing to expect was also proposed by Joyce [49]: since at a “negative” vertex, the local monodromy transformations share a common 2-dimensional fixed plane, $\Delta$ should remain planar, and the negative vertex should be replaced by a “trivalent ribbon” of the sort illustrated in Figure 11. (This is the same structure found by Castaño-Bernard and Matessi [18] in the Lagrangian case.) Near a “positive” vertex, the three planes containing parts of $\Delta$ share a common line but are distinct; Joyce also made a specific proposal for the structure in this case, but we will make a slightly different proposal in our main conjecture below.

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8Note that this is still not a special Lagrangian fibration, but provides an important intermediate step between the cases of topological fibration and special Lagrangian fibration.
Another motivation for our conjecture is the observation by Ruan that in his construction (and also in Zharkov’s construction), the discriminant locus is built out of amoebas, in fact, out of amoebas with the maximum number of holes. Since such amoebas arise from moment maps which are 2-to-1 over the interior and 1-to-1 over the edges, it is natural to identify the set of singular points of $\pi$ with the algebraic curve whose moment map image is the amoeba. This is what we do in our main conjecture.

A third input to our conjecture is the Harvey–Lawson fibration of $\mathbb{C}^3$, which is the standard model of a “positive” vertex with a Lagrangian structure. In that fibration, the set of singular points consists of the coordinate axes in $\mathbb{C}^3$, which meet in a “transverse triple point.” We conjecture that this is a general property of positive vertices.

**Conjecture 5 (Geometry).** Let $\pi : X \to B$ be a supersymmetric $T^3$ fibration of a compact Calabi–Yau threefold with respect to a Calabi–Yau metric whose compatible complex structure is sufficiently close to a large complex structure limit point, and whose Kähler class is sufficiently deep in the Kähler cone. Then $\pi$ is piecewise smooth and

a) The set $\Sigma \subset X$ of singular points of fibers of $\pi$ is a complex subvariety of $X$ of complex dimension 1.

b) All singular points of $\Sigma$ are transverse triple points, locally of the form $\{z_1z_2 = z_1z_3 = z_2z_3 = 0\}$ for local complex coordinates $z_1, z_2, z_3$.

c) For each connected component $\Sigma_\alpha$ of $\Sigma$, $\pi(\Sigma_\alpha)$ is contained in a (real) surface $\mathcal{A}_\alpha \subset B$, and the map $\pi|_{\Sigma_\alpha}$ is generically 2-to-1 onto its image $\pi(\Sigma_\alpha)$, which has the topology of a compactified amoeba with $g(\Sigma_\alpha)$ holes.

d) The discriminant locus $\Delta$ retracts to a trivalent graph $\Gamma$ which is the union of the spines of the (topological) compactified amoebas $\pi(\Sigma_\alpha)$. The graph $\Gamma$ has all of the properties in Conjecture 2.
e) The positive vertices of the graph $\Gamma$ are the points in $\Delta$ at which the spines of the various compactified amoebas meet. The map $\pi$ puts the singular points of $\Sigma$ in one-to-one correspondence with the positive vertices.

f) The set $H \subset B$ of fibers which are either singular or contain the boundary of at least one holomorphic disk retracts onto the discriminant locus $\Delta$. (Although in the local example illustrated in Figure 7 the set $H$ extended far away from $\Delta$, we expect that in global examples $H$ will be confined to a small neighborhood of $\Delta$, as illustrated in Figure 12.) The map $\pi$ is smooth \footnote{The referee points out that due to the smoothness of local moduli for special Lagrangian submanifolds \cite{Joyce}, we should even expect $\pi$ to be smooth on the larger set $X - \pi^{-1}(\Delta)$.} when restricted to $X - \pi^{-1}(H)$.

We have been deliberately vague about the notion of a topological amoeba and its spine, since we don’t know how much of the theory of amoebas should be expected to go through. It would be very interesting to know, for example, if some version of the Ronkin function can be defined for a supersymmetric $T^3$ fibration.

Note that one of the things which could happen if we attempt to deform this structure too far away from a large complex structure limit point is that $\pi|_{\Sigma_\alpha}$ might stop being generically 2-to-1, as happens for moment maps of algebraic curves. It would be very interesting to see what happens to supersymmetric $T^3$ fibrations in that case. Presumably, something more general than Joyce’s phenomenon of two singular points per fiber is going on here (as Joyce briefly discusses in \cite[Section 8.2]{Joyce}).

Among the consequences of our conjecture is a specific prediction for the structure of $\Delta$ near a positive vertex. Using local affine coordinates to identify a neighborhood of the vertex with the first cohomology of the fiber, the three local pieces of $\Delta$ must be contained in the three monodromy-invariant 2-planes, which meet along a common line but are distinct. However, because the corresponding singular point of $\Sigma$ is a transverse triple point, the thickening of each piece of the discriminant locus will need to “thin down” near the positive vertex so that the three pieces of $\Delta$ meet in a single point, leading to a description of the discriminant locus similar to that illustrated in Figure 13. (This “thinning down” was absent from Joyce’s proposal about the positive...
vertices.) Notice that, as Joyce observed, the discriminant locus has a markedly
different local structure near positive and negative vertices and therefore we cannot
hope for a mirror symmetry statement which simply dualizes all nonsingular tori in
the fibration.

The conjecture that $\pi$ is smooth on a region whose complement retracts to $\Delta$
was motivated in part by the properties of the construction of Castaño-Bernard and
Matessi [18]. The conjecture that $\mathcal{H}$ provides such a region is motivated in part
by Joyce’s observation that—at least in examples—the boundary between the set of
tori bounding holomorphic disks and the set not bounding holomorphic disks is a
boundary along which $\pi$ fails to be smooth. An additional motivation for part (f) of
the conjecture is the hope that a proper understanding of the disk contributions to the
physical “T-duality” construction will restore the symmetry between the $T^3$ fibrations
on the original Calabi–Yau manifold and its mirror partner: the fibrations would
consist of dual tori on the complement of $\mathcal{H}$ (where Hitchin’s Legendre transform
will relate the affine structures), with the duality between the tori somehow modified
within $\mathcal{H}$ by the disk contributions.

11. Degenerations

We close with a final conjecture, which is perhaps less well-motivated than Con-
jecture [5] but which proposes an explanation for why the structures we expect from
supersymmetric $T^3$ fibrations are related to the complex structure being near a large
complex structure limit point (as the physics suggests). This final conjecture also
points the way towards a connection between our conjectures and the interesting
program of Gross and Siebert [36, 37, 38, 39], which formulates mirror symmetry in
terms of degenerations of algebraic varieties.

Our final conjecture essentially says that the algebraic curves $\Sigma_\alpha$ should arise from
a large complex structure degeneration of the Calabi–Yau threefolds.
Conjecture 6 (Degeneration). Let $\mathcal{X} \to S$ be a proper flat family of threefolds whose generic point $X_\eta$ is a Calabi–Yau threefold, and whose fiber $X_{s_0}$ at some special point $s_0 \in S$ is a large complex structure degeneration of the form $X_{s_0} = \bigcup X_j$, where the $X_j$ are the components of $X_{s_0}$. Equip $\mathcal{X} \to S$ with a relative Kähler metric $g$ whose Kähler classes are sufficiently deep in the Kähler cone. Then there exist (non-flat) families of subvarieties $C_{jk} \subset \mathcal{X}$ such that $(C_{jk})_{s_0} = X_j \cap X_k$, but $(C_{jk})_s$ is nonsingular of complex dimension 1 when $X_s$ is nonsingular such that for all $s$ sufficiently close to $s_0$ there is a supersymmetric $T^3$ fibration of $X_s$ with respect to $g_s$ whose singular locus $\Sigma_s$ is precisely $\bigcup (C_{jk})_s$.

This is the structure found in the case of Calabi–Yau hypersurfaces in toric fourfolds: in that case, each $X_j \cap X_k$ is an intersection of toric divisors, which meets the nearby nonsingular Calabi–Yau threefolds $X_s$ in a complex curve $(C_{jk})_s$; the union of those curves, in the constructions of Zharkov and of Ruan, forms the set $\Sigma_s$ of singular points of $X_s$.

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