VANISHING THEOREMS FOR THE DE RHAM COMPLEX OF UNITARY LOCAL SYSTEM

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Abstract. We will prove a Kodaira-Nakano type of vanishing theorem for the logarithmic de Rham complex of unitary local system. We will then study the weight filtration on the logarithmic de Rham complex, and prove a stronger statement for the associated graded complex.

Introduction

Let $X$ be a smooth projective variety of dimension $n$ over $\mathbb{C}$ and $D \subset X$ a simple normal crossing divisor. In [2], Deligne constructed the canonical extension $E$ for any local system $L$ defined on $U$ (over complex topology). $E$ is equipped with a flat connection,

$$\nabla : E \to E \otimes \Omega_X^{(\log D)}$$

and it is characterized by the following two properties

1. The flat sections of $\nabla$ coincide with $L$ on $U$.
2. The eigenvalues of the residue of $\nabla$ lie in the strip

$$\{z \in \mathbb{C} | 0 \leq \text{Re}(z) < 1\}$$

Let $V$ be a unitary local system on $U := X - D$ (over complex topology). Let $(E, \nabla)$ be the canonical extension of $V$. Write $\text{DR}_X(D, E)$ for the de Rham complex

$$E \xrightarrow{\nabla} E \otimes \Omega_X^{(\log D)} \to \cdots \xrightarrow{\nabla} E \otimes \Omega_X^{(\log D)}$$

First, we will prove a Kodaira-Nakano type of vanishing theorem

Theorem 1. Let $L$ be an ample line bundle on $X$, then

$$H^q(X, E \otimes \Omega_X^{p}(\log D) \otimes L) = 0$$

for $p + q > \dim X$.

The de Rham complex $\text{DR}_X(D, E)$ comes with an decreasing filtration $F$ (Hodge filtration or "naive" filtration) and a increasing filtration $W$ (weight filtration). The weight filtration $W$ will be defined in Section [2] $F$ and $W$ together will define a mixed Hodge structure on $\text{DR}_X(D, E)$.

Then, we will prove a more refined version of the above theorem

Theorem 2. Let $L$ be an ample line bundle on $X$, then

$$H^q(X, \text{Gr}^W(\Omega_X^{p} \otimes E) \otimes L) = 0$$

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1. Residue Map

In this section we will define a residue map \( \text{Res}(E) \) on the complex \( \text{DR}_X(D, E) \). Similar to the usual residue map on the holomorphic de Rham complex, \( \Omega_X, \text{Res}(E) \) will define a weight filtration on \( \text{DR}_X(D, E) \). \( \text{Res}(E) \) has been defined and studied in [8].

For \( m = 1, \ldots, n \), let \( D_m \) be the union of \( m \)-fold intersection of components of \( D \); Let \( \tilde{D}_m \) be the disjoint union of components of \( D_m \); Let \( v_m : \tilde{D}_m \to X \) be the composition of the projection map onto \( D_m \) and the inclusion map. \( \tilde{C}_m := v_m^* D_{m+1} \) is either empty or a normal

**Theorem 3.** [8]

1. \( V_m := j_* V|_{D_m - D_{m+1}} \) is a unitary local system on \( D_m - D_{m+1} \).
2. There exist a unique subvectorbundle \( E_m \) of \( v_m^* E \) and a unique holomorphic integrable connection \( \nabla_m \) on \( E_m \) with logarithmic poles along \( C_m \) such that
\[
\ker \nabla_m|_{D_m - \tilde{C}_m} = v_m^{-1} V_m
\]
3. There exists a unique subvectorbundle \( E_m^* \) of \( v_m^* E \) with
\[
E_m \oplus E_m^* = v_m^* E
\]

**Proof.** All of the statements above are local. Therefore, we can assume \( X \) is a polydisk. Write \( X = \Delta_1 \times \cdots \times \Delta_n \), and let \( z_i \) be the coordinate on \( \Delta_i \). Suppose \( D \) is defined by
\[
z_1 \times \cdots \times z_s = 0
\]

1. The local system \( V \) on \( U \) is equivalent to an unitary representation
\[
T : \pi_1(U) \to \text{GL}(r, \mathbb{C})
\]
As \( \pi_1(U) \) is abelian and \( T \) is unitary, we can simultaneously diagonalize all \( T(\gamma_i) \), where \( \gamma_i \)'s form a generating set of \( \pi_1(U) \) (see Appendix 1). Therefore, we can assume \( V \) is a direct sum of rank 1 unitary local systems. Write
\[
V = V^1 \oplus \cdots \oplus V^r
\]
For each \( V^i \), let \( \lambda_{i,j} \) be its monodromy around \( D^j \). So \( V^i \) extends to \( D^j \) if and only if \( \lambda_{i,j} = 1 \). Now let \( D^{j_1} \cap \cdots \cap D^{j_m} \) be one component of \( D_m \), and let \( x \in D^{j_1} \cap \cdots \cap D^{j_m} \). Then, near \( x \) \( V_m \) is
\[
\bigoplus_{\lambda_{i,j_1} = \cdots = \lambda_{i,j_m} = 1} V^i
\]
This shows that \( V_m \) is a unitary local system.

2. The uniqueness of the subvectorbundle \( E_m \) follows from the uniqueness of canonical connection. Therefore, we only need to show the existence part. Use the notation from part 1, and assume \( V \) decomposes as direct sum of rank 1 unitary local system \( V^i \). Let \( E^i \) be the canonical connection of \( V^i \). Then, it is clear that
\[
E_m = \bigoplus_{\lambda_{i,j_1} = \cdots = \lambda_{i,j_m} = 1} v_m^* E^i
\]
3. $E$ inheits a flat Hermitian form from $V$. Define $E^*_m$ as the complement of $E_m$ with respect to this metric. On $\Delta$, $E^*_m$ is the direct sum of $v_i^* E_i$ not appearing in the definition of $E_m$.

\[ \square \]

**Remark 1.** $E_m$ could have different ranks on different component of $\tilde{D}_m$.

For each $m \leq p \leq \dim D_m$, there exists a residue map crossing divisor in $\tilde{D}_m$.

\[ \text{Res}_m : \Omega^p_X(\log D) \to v_m*(\Omega^{p-m}_{\tilde{D}_m}) \]

Res$_m$ commutes with exterior derivative $d$, making it a homomorphism of complexes

\[ \text{Res}_m : \Omega_X(\log D) \to v_m*\Omega^p_{\tilde{D}_m}(\log \tilde{C}_m)[-m] \]

Consider the following variation of the residue map Res$_m$

\[ \text{Res}_m(E) : \Omega^p_X(\log D) \otimes E \xrightarrow{\Res_m \otimes \text{id}} v_m*(\Omega^{p-m}_{\tilde{D}_m}(\log \tilde{C}_m)) \otimes E \]

\[ = v_m*(\Omega^{p-m}_{\tilde{D}_m}(\log \tilde{C}_m) \otimes v_m^* E) \]

\[ \xrightarrow{\text{id} \otimes p_m} v_m*(\Omega^{p-m}_{\tilde{D}_m}(\log \tilde{C}_m) \otimes E_m) \]

where $p_m : v_m^* E \to E_m$ is the projection onto the $E_m$ component.

**Lemma 1.** [8] $\text{Res}_m(E) \circ \nabla = \nabla_m \circ \text{Res}_m(E)$, i.e. $\text{Res}_m(E)$ is homomorphism of complexes

\[ DR_X(D, E) \to v_m*DR_{\tilde{D}_m}(\tilde{C}_m, E_m)[-m] \]

**2. Weight filtration on the de Rham Complex**

The residue map

\[ \text{Res}_m(E) : DR_X(D, E) \to v_m*DR(\tilde{C}_m, E_m, \nabla_m)[-m] \]

can be used to define a weight filtration $W$ on $DR_X(D, E)$ [8]

\[ W_m(DR_X(D, E)) = \ker \text{Res}_{m+1}(E) \quad \text{if } m \geq 0 \]

\[ W_m(DR_X(D, E)) = 0 \quad \text{if } m < 0 \]

Local descriptions of $W_m(DR_X(D, E))$ have been given in [8]. We will review them here:

Let $\Delta = \Delta_1 \times \cdots \times \Delta_n$ be a polydisk of $X$ with coordinate $z_1, \cdots, z_n$. Suppose $D$ is defined as

\[ z_1 \times \cdots \times z_s = 0 \]

As in part 1 of Theorem [3] we assume $V$ is the direct sum of rank 1 unitary local systems on $\Delta$, and write

\[ V = V^1 \oplus \cdots \oplus V^r \]

**Definition 1.** We say $\frac{\partial}{\partial z_j}$ acts on $V^i$ by identity if $\lambda_{ij} = 1$, i.e. the monodromy of $V^i$ by a small circle around $D_j$ is the identity.
Let $E^i$ be be canonical extension of $V^i$ on $\Delta$;
Let $\mu_i$ be a generator of $E^i$, then
$$dz_{j_1} \wedge \cdots \wedge dz_{j_k} \wedge dz_{j_{k+1}} \cdots \wedge dz_{j_p} \otimes \mu_i$$
is in $W_m(\text{DR}_X(D, E))$ if and only if there are at most $m$ log forms acting on $V^i$ by identity.

**Proposition 1.** [8]

1. $W_m(\text{DR}(D, E, \nabla))$ is an increasing filtration.
2. $\text{Res}_m(E)$ induces an isomorphism
$$\text{Gr}^W_m(\text{DR}(D, \nabla, E)) \to v_m(W_0(\text{DR}_{\tilde{D}_m}(\tilde{C}_m, \tilde{E}_m))[-m])$$

**Proof.** The statements are local. We can assume $X$ is a polydisk and $V$ is a unitary local system of rank 1.

1. From the local description of $W_m(\text{DR}_X(D, E))$, it is clear that $W_m$ is an increasing filtration.

2. Let $s$ be a section $W_m(\text{DR}_X(D, E))$. Use the local description above, $s$ is of the form
$$\omega \otimes \mu$$
where
$$\omega = dz_{j_1} \wedge \cdots \wedge dz_{j_k} \wedge dz_{j_{k+1}} \cdots \wedge dz_{j_p}$$
and $\omega$ has at most $m$ log 1-forms acting on $V$ by identity. $\mu$ is a generating section of $E$.

First, we show $\text{Res}_m(E)(s) \in W_0(\Omega^{p-m}_{\tilde{D}_m}(\log \tilde{C}_m) \otimes \tilde{E}_m)$.

$$\text{Res}_m(E)(s) = \text{Res}_m(\omega) \otimes \mu_m$$
By the construction of $\omega$, $\text{Res}_m(\omega)$ does not have log form $dz_{j_i}$ acting on $V$ by identity. This shows that
$$\text{Res}_m(E)(s) \in W_0(\Omega^{p-m}_{\tilde{D}_m}(\log \tilde{C}_m) \otimes \tilde{E}_m)$$
If $\omega_0 \otimes \mu_m \in W_0(\Omega^{p-m}_{\tilde{D}_m}(\log \tilde{C}_m) \otimes \tilde{E}_m)$, to get a preimage in $W_m(\Omega^p_X(\log D) \otimes E)$, simply take
$$\omega_m \wedge \omega_0 \otimes \mu$$
where $\omega_m$ is any $m$-form. And $\omega_m \wedge \omega_0 \otimes \mu \in W_m(\Omega^p_X(\log D) \otimes E)$ by the construction of $\omega_0$. This shows that
$$\text{Res}_m(E) : W_m(\text{DR}(D, E, \nabla)) \to W_0(\text{DR}(\tilde{C}_m, \tilde{E}_m, \tilde{\nabla}_m))$$
is surjective.
If $\text{Res}_m(E)(s) = 0$, that means in $\omega$, there are at most $m - 1$ log forms acting on $V$ by identity. This is precisely the local description of $W_{m-1}(\Omega^p_X(\log D) \otimes E)$.
3. Mixed Hodge Structure on the de Rham Complex

The framework for studying the mixed Hodge structure on $\text{DR}_X(D, E)$ has been worked out by Deligne in [3] and [4]. The analysis of the mixed Hodge structure on $\text{DR}_X(D, E)$ was given by Timmerscheidt in [8]. We will give an overview about the results from both authors. The vanishing theorem in the following section is a consequence of the mixed Hodge structure on $\text{DR}_X(D, E)$.

Let $A$ denote $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ and $A \otimes \mathbb{Q}$ the field $\mathbb{Q}$ or $\mathbb{R}$.

Assume $V$ has an $A$-lattice throughout this section, i.e. there exists a unitary local system $V_A$ defined over $A$ such that

$$V = V_A \otimes_A \mathbb{C}$$

Let $D^+(A)$ (resp. $D^+(C)$) denote the derived category of $A$-modules (resp. $C$-vector spaces).

The main result of this section is

**Theorem 4.** [3](Proposition 6.4)

$$(\mathbb{R}j_* V_A, (\mathbb{R}j_* V_A \otimes \mathbb{Q}), (\text{DR}_X(D, E), F, W))$$

is an $A$-cohomological mixed Hodge complex.

For readers’ sake, we included all relevant definitions involved in the above theorem here. They can be found in [4] or [5] (Section 5).

**Definition 2.** (Hodge Structure (HS)) A Hodge structure of weight $n$ is defined by the data:

1. A finitely generated abelian group $H_{\mathbb{Z}}$;
2. A decomposition by complex subspaces:

$$H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=n} H^{p,q}$$

satisfying

$$H^{p,q} = \overline{H^{q,p}}$$

**Definition 3.** (Hodge Complex (HC)) A Hodge $A$-complex $K$ of weight $n$ consists of

1. A complex $K_A$ of $A$-modules, such that $H^k(K_A)$ is an $A$-module of finite type for all $k$;
2. A filtered complex $(K_{\mathbb{C}}, F)$ of $\mathbb{C}$-vector spaces;
3. An isomorphism

$$\alpha : K_A \otimes \mathbb{C} \to K_{\mathbb{C}}$$

in $D^+(\mathbb{C})$;

The following axioms must be satisfied

1. The spectral sequence defined by $(K_{\mathbb{C}}, F)$ degenerates at $E_1$;
2. for all $k$, the filtration $F$ on $H^k(K_{\mathbb{C}}) \cong H^k(K_A) \otimes \mathbb{C}$ defines an $A$-Hodge structure of weight $n + k$ on $H^k(K_A)$

**Definition 4.** Let $X$ be a topological space. An $A$-Cohomological Hodge Complex (CHC) $K$ of weight $n$ on $X$, consists of:

1. A complex of sheaves $K_A$ of $A$-modules on $X$;
2. A filtered complex of sheaves $(K_{\mathbb{C}}, F)$ of $\mathbb{C}$-vector spaces on $X$;
(3) an isomorphism
\[ \alpha : K_A \otimes \mathbb{C} \to K_C \]
in \( D^+(X, \mathbb{C}) \)
Moreover, the triple \((R\Gamma(K_A), R\Gamma(K_C, F), R\Gamma(\alpha))\) is a Hodge Complex of weight \( n \)

**Definition 5.** (Mixed Hodge Complex) An A-Mixed Hodge Complex (MHC) \( K \) consists of:

1. A complex \( K_A \) of \( A \)-modules such that \( H^k(K_A) \) is an \( A \)-module of finite type for all \( k \);
2. A filtered complex \( (K_A \otimes \mathbb{Q}, W) \) of \( A \otimes \mathbb{Q} \)-vector spaces with an increasing filtration \( W \);
3. An isomorphism \( K_A \otimes \mathbb{Q} \cong K_A \otimes \mathbb{Q} \) in \( D^+(A \otimes \mathbb{Q}) \);
4. A bi-filtered complex \( (K_C, W, F) \) of \( \mathbb{C} \)-vector spaces with an increasing (resp. decreasing) filtration \( W \) (resp. \( F \)) and an isomorphism:
\[ \alpha : (K_A \otimes \mathbb{Q}, W) \otimes \mathbb{C} \cong (K_C, W) \]
in \( D^+ F(\mathbb{C}) \).
Moreover, the following axiom needs to be satisfied: For all \( n \), the system consisting of:
- the complex \( \text{Gr}^W_n(K_A \otimes \mathbb{Q}) \) of \( A \otimes \mathbb{Q} \)-vector spaces,
- the complex \( \text{Gr}^W_n(K_C, F) \) of \( \mathbb{C} \)-vector spaces with induced \( F \) filtration,
- the isomorphism \( \text{Gr}^W_n(\alpha) : \text{Gr}^W_n(K_A \otimes \mathbb{Q}) \otimes \mathbb{C} \to \text{Gr}^W_n(K_C) \)

is an \( A \otimes \mathbb{Q} \)-Hodge Complex of weight \( n \).

**Definition 6.** (Cohomological Mixed Hodge Complex (CMHC)) An A-Cohomological Mixed Hodge Complex (CMHC) \( K \) on a topological space \( X \) consists of:

1. A complex of sheaves \( K_A \) of sheaves of \( A \)-modules on \( X \) such that \( H^k(X, K_A) \) are \( A \)-modules of finite type;
2. A filtered complex \( (K_A \otimes \mathbb{Q}, W) \) of sheaves of \( A \otimes \mathbb{Q} \)-vector spaces on \( X \) with an increasing filtration \( W \) and an isomorphism
\[ K_A \otimes \mathbb{Q} \cong K_A \otimes \mathbb{Q} \]
in \( D^+(X, A \otimes \mathbb{Q}) \);
3. A bi-filtered complex of sheaves \( (K_C, W, F) \) of \( \mathbb{C} \)-vector spaces on \( X \) with an increasing (resp. decreasing) filtration \( W \) (resp. \( F \)) and an isomorphism:
\[ \alpha : (K_A \otimes \mathbb{Q}, W) \otimes \mathbb{C} \to (K_C, W) \]
in \( D^+ F(X, \mathbb{C}) \).
Moreover, the following axiom needs to be satisfied: For all \( n \), the system consisting of:
- the complex \( \text{Gr}^W_n(K_A \otimes \mathbb{Q}) \) of sheaves of \( A \otimes \mathbb{Q} \)-vector spaces on \( X \),
- the complex \( \text{Gr}^W_n(K_C, F) \) of \( \mathbb{C} \)-vector spaces with induced \( F \) filtration,
- the isomorphism \( \text{Gr}^W_n(\alpha) : \text{Gr}^W_n(K_A \otimes \mathbb{Q}) \otimes \mathbb{C} \to \text{Gr}^W_n(K_C) \)
is an \( A \otimes \mathbb{Q} \)-Cohomological Hodge Complex of weight \( n \).

The following example of Cohomological Mixed Hodge Complex can be found in [4] and [5].
Example 1. Let $X$ be a smooth projective variety over $\mathbb{C}$, $D \subset X$ a simple normal crossing divisor. Let $U := X - D$ and let

$$j : U \to X$$

be the inclusion map.

Let $\mathbb{Q}_U$ be the constant sheaf with $\mathbb{Q}$-coefficient on $U$. $\mathbb{R}j_*\mathbb{Q}_U \otimes \mathbb{C} = \mathbb{R}\mathbb{C}_U$ is quasi-isomorphic to the logarithmic de Rham complex

$$O_X \xrightarrow{d} \Omega_X (\log D) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_X (\log D)$$

For any complex $K$ of sheaves on $X$, let $\tau$ be the canonical increasing filtration

$$\tau_m K^q = \begin{cases} K^q & \text{if } q < m \\ \ker d^q \subset K^q & \text{if } q = m \\ 0 & \text{if } q > m \end{cases}$$

See [5] (Corollary 6.4) for the following result:

The system consisting of

1. $(\mathbb{R}j_*\mathbb{Q}_U, \tau)$;
2. $(\Omega_X (\log D), W, F)$ with usual weight and Hodge filtration $W$ and $F$;
3. The quasi-isomorphism

$$(\mathbb{R}j_*\mathbb{Q}_U, \tau) \otimes \mathbb{C} \cong (\Omega_X (\log D), W)$$

is a Cohomological Mixed Hodge Complex on $X$.

4. Vanishing Theorem for the de Rham Complex

we have seen in the previous section that if $V$ has a real lattice, then

$$(\mathbb{R}j_*V_A, (\mathbb{R}j_*V_{A\otimes\mathbb{Q}}), (\text{DR}_X(D, E), F, W))$$

is an $A$-cohomological mixed Hodge complex. As a result of the general theory developed in [4], we have

Theorem 5. Assume there is a real-valued unitary local system $V_\mathbb{R}$ defined on $U$ such that $V = V_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$

Let $V$ and $\text{DR}_X(D, E)$ be as above. The spectral sequence associated to the Hodge filtration on $\text{DR}_X(D, E)$,

$$E_1^{p,q} = H^q(X, \Omega^p_X (\log D) \otimes E) \Rightarrow \mathbb{H}^{p+q}(X, \text{DR}_X(D, E))$$

degenerates at $E_1$.

If $V$ does not have an $A$-lattice with $A \subset \mathbb{R}$, then we cannot expect $\text{DR}_X(D, E)$ to carry a mixed Hodge structure. However, the degeneration of Hodge spectral sequence still holds true. One can prove of this in [8]. We will give a simpler proof here:

Let $\bar{V}$ denote the conjugate of $V$, i.e. the monodromy representation of $\bar{V}$ is the complex conjugate of the monodromy representation of $V$.

Lemma 2. There exists a real unitary local system $W_\mathbb{R}$ of rank $2r$ such that $V \oplus \bar{V} \cong W_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$.
Proof. We will construct $W^j$ locally, and show it is canonically determined by $V$. Over a polydisk, we can assume $V$ is diagonal, and we write

$$V = \bigoplus_{j=1}^{r} V^j$$

where $V^j$ is a unitary local system of rank 1 with monodromy

$$\lambda_j = \cos \theta_j + i \sin \theta_j$$

We will construct $W^j$ for each $j$. The monodromy of $\bar{V}^j$ is $\bar{\lambda}_j$ and the monodromy of $V^j \oplus \bar{V}^j$ is

$$\begin{bmatrix} \cos \theta_j + i \sin \theta_j & 0 \\ 0 & \cos \theta_j - i \sin \theta_j \end{bmatrix}$$

Since

$$\begin{bmatrix} \cos \theta_j + i \sin \theta_j & 0 \\ 0 & \cos \theta_j - i \sin \theta_j \end{bmatrix} \text{ and } \begin{bmatrix} \cos \theta_j & \sin \theta_j \\ -\sin \theta_j & \cos \theta_j \end{bmatrix}$$

have the same characteristic polynomial over $\mathbb{C}$, they must be conjugate over $\mathbb{C}$. Therefore, we can take $W^j$ to be

$$\begin{bmatrix} \cos \theta_j & \sin \theta_j \\ -\sin \theta_j & \cos \theta_j \end{bmatrix}$$

Then,

$$W^j = \bigoplus_{j=1}^{r} W^j$$

Now, let $V$ be any unitary local system on $X - D$, and let $\text{DR}_X(D, E)$ be its de Rham complex.

**Corollary 1.** The Hodge spectral sequence

$$E_1^{p,q} := H^q(X, \Omega^p_X(\log D) \otimes E) \Rightarrow H^{p+q}(X, \text{DR}_X(D, E)) = H^{p+q}(X, \mathbb{R}, V)$$

degenerates at $E_1$.

**Proof.** Direct sum and taking cohomology commutes.

**Theorem 6.** (Corollary 3.5) Suppose $U$ is an affine variety of complex dimension $n$. Then, for any constructible sheaf $L$ on $U$

$$H^k(U, L) = 0$$

for $k > n$

**Corollary 2.** Let $V$ and $\text{DR}_X(D, E)$ be as above. Suppose $U$ is affine, then

$$H^q(X, \Omega^p_X(\log D) \otimes E) = 0$$

for $p + q > \dim X$

**Lemma 3.** Suppose $B$ is a smooth divisor transversal to $D$. Then, there is short exact sequence

$$0 \rightarrow \Omega^p_X(\log D + B) \otimes O_X(-B) \rightarrow \Omega^p_X(\log D) \rightarrow \Omega^p_B(\log D \cap B) \rightarrow 0$$

where $i$ is the inclusion map, and $r$ is the restriction map.
Proof. For simplicity, we prove the case for $p = 1$. We may also assume $X$ is affine. Let $X = \text{Spec} A$, and let $f_1, \cdots, f_s$ be the regular sequence corresponding to $D$, and let $b$ be the defining equation of $B$.

The basis of $\Omega^1_X(\log D + B) \otimes O_X(-B)$ as an $A$-module is
\[
\frac{df_1}{f_1} \otimes b, \cdots, \frac{df_s}{f_s} \otimes b, \frac{db}{b} \otimes b
\]

The basis of $\Omega_X(\log D)$ as an $A$-module is
\[
\frac{df_1}{f_1}, \cdots, \frac{df_s}{f_s}
\]

The basis of $\Omega_B(\log D \cap B)$ as an $A^D$-module is
\[
\frac{df_1}{f_1}, \cdots, \frac{df_s}{f_s}
\]
where by abuse of notation $f_i$ are regarded as their image in $A^D$.

Then, it is clear how to define $i$ and $r$ show that the above sequence is exact. □

Lemma 4. Suppose $B$ is a smooth divisor transversal to $D$. Then, $E_B := E \otimes O_B$ is the canonical extension of $V_B := V|_{B - B \cap D}$.

Proof. The statement is local, therefore we may assume $X$ is a polydisk
\[
\Delta_1 \times \cdots \times \Delta_n
\]
such that the analytic coordinate of $\Delta_i$, for $i = 1, \cdots, s$, are defining equation of $D_i$, and the analytic coordinate of $\Delta_n$ is the defining equation of $B$.

First, we study $V_B$ by computing its monodromy representation:

Let $T : \pi_1(X - D, x) \rightarrow \text{GL}(r, \mathbb{C})$ be the monodromy representation of $V$. For each generator $\gamma_i$ of $\pi_1(X - D, x)$, let $\Gamma_i = T(\gamma_i)$. As $\Gamma_i$ are commuting and unitary, we can use one matrix to diagonalize all of them. Therefore, we can assume all $\Gamma_i$ are diagonal matrices. Moreover, as $V$ is undefined only on $D$, so for each $i$, $\Gamma_i^{s^j} = 1$, for $j = s + 1, \cdots, n$.

Now, $B = \Delta_1 \times \cdots \times \Delta_{n-1}$, and the monodromy representation of $V|_{B - B \cap D}$ is given by
\[
\pi_1(B - B \cap D) \overset{i}{\rightarrow} \pi_1(X - D) \overset{T}{\rightarrow} \text{GL}(r, \mathbb{C})
\]

where $i$ is the natural inclusion map. It is clear that one can choose the basis of $\pi_1(B - B \cap D)$ and $\pi_1(X - D)$ such that $i$ can be realized as the identity map. Therefore, the monodromy representations of $V_B$ are also $\Gamma_i$, for $i = 1, \cdots, s$.

To show $E_B$ is the canonical extension of $V_B$, we compute the connection matrix of $E|_B$ and relate it to the monodromy representations of $V|_{B - B \cap D}$.

One can assume $E$ is trivial over $X$. Choose a local frame of $V$ on $X$, and use it as a trivialization of $E$. With respect to this trivialization, the connection $\nabla$ can be realized as
\[
d + N_1 \frac{dz_1}{z_1} + \cdots + N_s \frac{dz_s}{z_s}
\]
where $N_1, \cdots, N_s$ are commuting matrices with eigenvalues in the stripe
\[
\{ z \in \mathbb{C} | 0 \leq \text{Re} z < 1 \}
\]
such that $e^{-2\pi i N_i} = \Gamma_i$. 


Now, restrict $E$ to $B$, we see that the connection $\nabla|_B$ can still be realized as
\[
d + N_1 \frac{dz_1}{z_1} + \cdots + N_n \frac{dz_n}{z_n}
\]
As monodromy representations of $V_{B-B \cap D}$ are $\Gamma_i$, it follows that $E|_B$ is the canonical extension of $V_{B-B \cap D}$.

**Theorem 7.** Suppose $L$ is very ample on $X$. Then
\[
H^q(X, E \otimes \Omega_X^p (\log D) \otimes L) = 0
\]
for $p + q > \dim X$.

**Proof.** Let $B$ be a smooth divisor transversal to $D$ such that $L \cong O_X(B)$. By Lemma 3, we have the following exact sequence
\[
0 \to \Omega_X^p (\log D + B) \overset{i}{\to} \Omega_X^p (\log D) \otimes O_X(B) \overset{r}{\to} \Omega_B^p (\log D \cap B) \otimes O_X(B) \to 0
\]
Tensor it by $E$ and take the cohomology sequence, we get:
\[
\cdots \to H^q(X, \Omega_X^p (\log D + B) \otimes E) \to H^q(X, \Omega_X^p (\log D) \otimes O_X(B) \otimes E) \\
\to H^q(X, \Omega_B^p (\log D \cap B) \otimes O_X(B) \otimes E) \to \cdots
\]
Therefore, to prove the theorem, it is enough to show
**Claim 1:** $H^q(X, \Omega_X^p (\log D + B) \otimes E) = 0$
**Claim 2:** $H^q(X, \Omega_B^p (\log D \cap B) \otimes O_X(B) \otimes E) = 0$ for $p + q > \dim X$.

**Proof of claim 1:** Consider the maps
\[
X - (B + D) \xrightarrow{i} X - B \overset{h}{\to} X
\]
Let $V^\alpha$ be the restriction of $V$ on $X - (B + D)$. The complex $\text{DR}(D + B, E, \nabla)$ is quasi-isomorphic to $\mathbb{R}(h \circ f)_* V^\alpha$. Therefore,
\[
H^k(X - (B + D), V^\alpha) = \mathbb{H}^k(X, \text{DR}(D + B, E, \nabla))
\]
The claim then follows from Corollary 2.

**End of Proof**

Claim 2 follows from induction on the dimension of the variety.

Now to finish the proof, it remains to show the base case of Claim 2. One may assume now that $X$ is a smooth projective curve over $\mathbb{C}$, we need to show that
\[
H^1(X, \Omega_X (\log D) \otimes E \otimes L) = 0
\]
But for the curve case, $\Omega_X (\log D) \otimes O_X(B) = \Omega_X (\log D + B)$. So the result follows again from Theorem 2.

Now suppose $L$ is any ample line bundle. Let $m$ be an integer such that $L^\otimes m$ is very ample. Take a smooth divisor $B$ transversal to $D$ such that $L^\otimes m \cong O_X(B)$. Let $\varphi$ be the local equation of $B$ on some affine open set, and let $\pi : X' \to X$ be the normalization of $X$ in $\mathbb{C}(X)(\sqrt[\varphi^{-1}})$.

**Proposition 2.** Let $\pi : X' \to X$, $B$ and $L$ be as above
1. $X'$ is smooth.
2. $\pi^* B = m \tilde{B}$, where $\tilde{B} = (\pi^* B)_{\text{red}}$. 

(3) $D' := \pi^* D$ is a normal crossing divisor on $X'$.
(4) $\tilde{B}$ is transversal to $\pi^* D$.
(5) $\pi^* \Omega^n_X(\log D) = \Omega^n_{X'}(\log D')$.
(6) $\pi^* E$ is the canonical extension of $\pi^{-1} V$.

Proof. 1. We will construct $X'$ by constructing its affine cover and specifying the gluing morphisms. Let $U_i = \text{Spec} A_i$ be an affine cover of $X$, and let $f_i$ be the defining equation of $D$ in $A_i$.

For each $A_i$, $\frac{A_i[Y]}{(Y^m - f_i)}$ is integrally closed in $\mathbb{C}(X)(f_i^{1/m})$. Therefore,

$$U'_i := \text{Spec} \frac{A_i[Y]}{(Y^m - f_i)}$$

is the normalization of $U_i$ in $\mathbb{C}(X)(f_i^{1/m})$.

The same morphisms used to glue $U_i$ into $X$ can be used to glue $U'_i$ into $X'$. Therefore, to show $X'$ is smooth, it is enough to show $\frac{A_i[Y]}{(Y^m - f_i)}$ is a regular ring.

2. The local defining equation of $\tilde{B}$ is $Y$, and $\pi^*(f_1) = Y^m$.

3. To see this, we describe $\pi^* D$ in $\pi^* U$ for any polydisk $U = \Delta_1 \times \cdots \times \Delta_n$. If $B \cap U \neq \emptyset$, then construct $\Delta_i$ such that defining equation of $D_i$, for $i = 1, \cdots, s$, are coordinates of $D_i$, for $i = 1, \cdots, s$; and the defining equation of $B$ is the coordinate of $D_n$. Then,

$$\pi^* U = \Delta_1 \times \Delta_1 \cdots \Delta_{n-1} \times \Sigma^m$$

where $\Sigma^m$ is the $m$-sheeted cover over a complex disk branched over the origin. In this case, $\pi^* D$ is still defined by $z_1 \times z_2 \times \cdots z_s$.

If $B \cap U = \emptyset$, then $\pi^* U$ is etale over $U$. Therefore, $\pi^* D$ is etale over $D$. So $\pi^* D$ is again a simple normal crossing divisor.

4.This is clear from the case 1 of part 3.

5. Straightforward computation. 6. We compute the monodromy representation of $\pi^{-1} V$ first:

Let $T : \pi_1(U - D, x) \to \text{GL}(r, \mathbb{C})$ be the representation corresponding to the local system $V$.

**Case 1:** Suppose $x \notin B$, then $\pi^{-1}(U)$ is etale over $U$. Let $U'$ be an component of $\pi^{-1}(U)$, and let $x' \in U'$ be a preimage of $x$. Then,

$$T' : \pi_1(U' - D', x') \xrightarrow{\pi_*} \pi_1(U - D, x) \xrightarrow{T} \text{GL}(r, \mathbb{C})$$

is the representation corresponding to $\pi^{-1} V$.

**Case 2:** Suppose $x \in B$, then use the description from part 3, we know that

$$\pi^{-1} U = \Delta_1 \times \Delta_2 \times \cdots \times \Sigma^m$$

In both cases, $\pi^{-1} U - D'$ is homotopic to $S_1 \times S_2 \times \cdots \times S_s$ So we can define generators of $\pi_1(U' - D', x')$ and $\pi_1(U - D, x)$ such that $\pi_*$ is the identity map.

To show $\pi^* E$ is the canonical extension of $\pi^{-1} V$, we only need to compute the connection matrix of $\pi^* E$ and relate it to the monodromies of $\pi^{-1} V$.

Let $\gamma_i$ be a small circle around $D_i$, and let $\Gamma_i$ be the monodromy $T(\gamma_i)$. As

$$\pi_* : \pi_1(U' - D', x') \to \pi_1(U - D, x)$$


is the identity map, $\Gamma_i$ are also the monodromy representations of $\pi^{-1}V$. Next, we compute the connection matrix of $E$. Let $U$ be small enough so that $E$ is trivial over it. Choose a local frame of $V$, and use it as a trivialization of $E$. With respect to this trivialization, the connection $\nabla$ can be realized as

$$d + N_1 \frac{dz_1}{z_1} + \cdots + N_s \frac{dz_s}{z_s}$$

where $N_1, \cdots, N_s$ are commuting matrices with eigenvalue in the stripe

$$\{ z \in \mathbb{C} | 0 \leq \Re z < 1 \}$$

such that $e^{-2\pi i N_i} = \Gamma_i$.

As $\pi^*z_i = z_i$, for $i = 1, \cdots, s$, we see that the $\pi^*\nabla$ over $\pi^{-1}U$ can be realized as:

$$d + N_1 \frac{dz_1}{z_1} + \cdots + N_s \frac{dz_s}{z_s}$$

This shows that $\pi^*E$ is the canonical extension of $\pi^{-1}V$. 

\[ \square \]

**Corollary 3.** For any ample line bundle $L$ on $X$,

$$H^q(X, E \otimes \Omega^p_X(\log D) \otimes L) = 0$$

for $p + q > \text{dim } X$

**Proof.** Let $m, B$ and $\pi : X' \to X$ be as above. By Theorem 7

$$H^q(X', \pi^*(E \otimes \Omega^p_X(\log D) \otimes L)) = 0$$

for $p + q > \text{dim } X' = \text{dim } X$.

$\pi : X' \to X$ is a finite morphism, so for $i > 0$, $R^i\pi_*\mathcal{F} = 0$ for any coherent sheaf $\mathcal{F}$ on $X'$. This implies

$$H^q(X', \pi^*(E \otimes \Omega^p_X(\log D) \otimes L)) = H^q(X, \pi_*(\pi^*(E \otimes \Omega^p_X(\log D) \otimes L)))$$

$$= H^q(X, \pi_*(\mathcal{O}_Y) \otimes E \otimes \Omega^p_X(\log D) \otimes L)$$

$$= 0$$

for $p + q > \text{dim } X$. The second equality follows from the projection formula. As $\pi_*(\mathcal{O}_Y) \cong \bigoplus_{i=0}^{m-1} \mathcal{O}_X(-L^\otimes i)$, the result follows. \[ \square \]

5. **PARTIAL WEIGHT FILTRATION**

In the previous section, we proved the vanishing theorem for the complex

$$\text{DR}_X(D, E) \otimes \mathcal{O}_X(B)$$

where $B$ is a smooth very ample divisor transversal to $D$. The intermediate step for the proof is the vanishing theorem for the complex

$$\text{DR}(D + B, E)$$

In this section, we define a partial weight filtration on the complex

$$\text{DR}_X(D + B, E)$$

It is a more refined weight filtration than the one defined in Section 2 and it will be used to prove the vanishing theorem for the graded complex

$$\text{Gr}^W \text{DR}_X(D, E)$$
For simplicity, suppose $V$ is a rank 1 unitary local system. We will define partial weight filtration by giving local description of forms. Then, we will show it is a well-defined global notion after Theorem 8. Let $\omega$ be a section of $E$. Recall that $W_m \Omega^p_X(\log D) \otimes E$ consists of sections of the form

$$\omega \otimes \mu$$

where $\omega$ can be written as

$$\frac{dz_{j_1}}{z_{j_1}} \wedge \frac{dz_{j_2}}{z_{j_2}} \wedge \cdots \frac{dz_{j_k}}{z_{j_k}} \wedge \frac{dz_{j_{k+1}}}{z_{j_{k+1}}} \wedge \cdots \wedge \frac{dz_j}{z_j}$$

Moreover, $\omega$ has at most $m$ log forms acting on $V$ by identity. Now let $W^{D^1} \wedge \cdots \wedge D^l \Omega_{\log D}(D, E)$ be the set of forms that can be written as $\omega \otimes \mu$, where $\omega$ can be written as

$$\frac{dz_{j_1}}{z_{j_1}} \wedge \frac{dz_{j_2}}{z_{j_2}} \wedge \cdots \frac{dz_{j_k}}{z_{j_k}} \wedge \frac{dz_{j_{k+1}}}{z_{j_{k+1}}} \wedge \cdots \wedge \frac{dz_j}{z_j}$$

Moreover, let $g$ be the cardinality of the following set

$$\{ \text{log forms in } \omega \text{ acting on } V \text{ by identity} \} \cap \left\{ \frac{dz_1}{z_1}, \cdots, \frac{dz_s}{z_s} \right\}$$

Then $g \leq m$.

To generalize, $W^{D^1 + \cdots + D^l} \Omega_{\log D}(D, E)$ is the set of forms that can be written as $\omega \otimes \mu$, where $\omega$ can be written as

$$\frac{dz_{i_1}}{z_{i_1}} \wedge \frac{dz_{i_2}}{z_{i_2}} \wedge \cdots \frac{dz_{i_k}}{z_{i_k}} \wedge \frac{dz_{i_{k+1}}}{z_{i_{k+1}}} \wedge \cdots \wedge \frac{dz_j}{z_j}$$

Moreover, let $g$ be the cardinality of the following set

$$\{ \text{log forms in } \omega \text{ acting on } V \text{ by identity} \} \cap \left( \left\{ \frac{dz_1}{z_1}, \cdots, \frac{dz_s}{z_s} \right\} - \left\{ \frac{dz_{i_1}}{z_{i_1}}, \cdots, \frac{dz_{i_l}}{z_{i_l}} \right\} \right)$$

Write $T$ for $D + B$. Let $T_2$ be the union of 2-fold intersections of components of $T$. Let $v_1: D_1 \to D$ be the normalization map, i.e. $D_1$ is the disjoint union of components of $D$. Let $F_1 = v_1^* T_2$. Then, $F_1$ is a normal crossing divisor in $D_1$. We have seen in Section 1 that the restriction of $j_* V$ on $D_1 - T_2$ is a unitary local system, denote it by $V_1$; and let $E_1$ be the subbundle of $v_1^* E$ which is the canonical extension of $V_1$.

**Proposition 3.** There is an exact sequence

$$0 \to W_0^B \Omega^p_{\log D + B}(D + B, E, \nabla) \xrightarrow{i} \Omega^p_{\log D + B}(D + B, E, \nabla) \xrightarrow{\text{res}} v_{1*} \Omega^p_{F_1, E_1} \xrightarrow{\text{res}} 0$$

where $i$ is the inclusion map, and res is the residue map.

**Proof.** Suppose for simplicity $D$ is smooth, i.e. $D$ has only one component. Also, suppose $V$ is a unitary local system of rank 1. Let $\mu$ be a local section of $E$.

Let $z_1$ be the local equation of $D$. Suppose $\frac{dz_1}{z_1}$ acts on $V$ by identity, then $V$ extends to a unitary local system on $D - D \cap B$. In this case, $D_1 = D$, and $F_1 = D \cap B$. Let $z_n$ be the local equation for $B$. Then, locally over a polydisk

1. $W_0^B \Omega^p_{\log D + B}(D + B) \otimes E$ is generated by sections of the form

$$\frac{dz_n}{z_n} \wedge \omega \otimes \mu$$

where $\omega \in \Omega^{p-1}_X$. 

2. $\Omega^p_X(\log D + B)$ is generated by sections of the form
\[
\frac{dz_n}{z_n} \wedge \omega \otimes \mu
\]
where $\omega \in \Omega_X^{p-1}(\log D)$.

3. $\Omega^{p-1}_{D_1}(\log F_1) \otimes E_1$ is generated by sections of the form
\[
\frac{dz_n}{z_n} \wedge \omega \otimes \mu_1
\]
where $\omega \in \Omega^{p-2}_{D_1}(\log F_1)$.

Use the local description, it is clear that the sequence is exact.

\[\square\]

**Theorem 8.** Let $(E_B, \nabla_B)$ be the restriction of $(E, \nabla)$ on $B$, and let $\text{DR}(B \cap D, E_B, \nabla_B)$ be the complex
\[
0 \to E_B \to \Omega^1_B(B \cap D) \otimes E_B \to \cdots
\]
then there is an exact sequence of complexes
\[
0 \to W^B_m \text{DR}_X(D + B, E) \overset{i}{\to} W_m \text{DR}_X(D, E) \otimes \Omega_X(B) \overset{r}{\to} W_m \text{DR}_B(D \cap B, E_B) \otimes \Omega_X(B) \to 0
\]
i is the inclusion map, and $r$ is the restriction map.

**Proof.** For simplicity, we assume $E$ has rank 1. The statement is local, so we work on a polydisk, and we use the notation from above. Let $\mu$ be a generating section of $E$, then

1. $W^B_m \text{DR}_X(D + B, E) \otimes \Omega_X(\log D)$ is generated by
\[
\omega \otimes \mu \otimes z_n
\]
where $\omega \in \Omega^p_X(\log D + B)$ is a $p$-form that has at most $m \log$ forms coming from
\[
\left\{ \frac{dz_1}{z_1}, \ldots, \frac{dz_s}{z_s} \right\}
\]
acting on $E$ by identity.

2. $W_m \text{DR}_X(D, E)$ is generated by
\[
\omega \otimes \mu
\]
where $\omega \in \Omega^p_X(\log D)$ is a $p$-form that has at most $m \log$ forms acting on $E$ by identity.

3. $W_m \text{DR}_B(D \cap B, E_B)$ is generated by
\[
\omega \otimes \mu
\]
where $\omega \in \Omega^p_B(\log B \cap D)$ is a $p$-form that has at most $m \log$ forms acting on $\mu_B$ by identity.

The map $i$ is the natural inclusion map, i.e. $\frac{dz_n}{z_n} \otimes z_n \mapsto dz_n$; The map $r$ is the restriction on $B$.  

\[\square\]
The above theorem also gives a description of
\[ W^B_m \text{DR}_X(D + B, E) \]
as the kernel of the restriction map
\[ r : \text{DR}_X(D + B, E) \otimes O_X(B) \to W_m \text{DR}_B(B \cap D, E_B) \otimes O_X(B) \]
It means that \( W^B_m \text{DR}_X(D + B, E) \) is indeed globally well-defined.

6. Mixed Hodge Structure on the Complex \( W^B_0 \text{DR}(D + B, E, \nabla) \)
Throughout this section, we assume the unitary local system \( V \) has a real lattice \( V_R \) such that
\[ V = V_R \otimes \mathbb{C} \]
We will study the mixed Hodge structure on the complex
\[ W^B_0 \text{DR}_X(D + B, E) \]
Consider the maps
\[ X - (D + B) \xrightarrow{L} X - B \xrightarrow{h} X \]
Write \( V^o \) (resp. \( V^o_R \)) for the restriction of \( V \) (resp. \( V_R \)) on \( X - (D + B) \).
Let \( \tau \) be the canonical filtration on \( h_* f_* V^o_R \); let \( W \) be the increasing filtration on \( W^B_0 \text{DR}_X(D + B, E) \) defined as
\[ W^B_m W^B_0 \text{DR}_X(D + B, E) = \begin{cases} 0 & \text{if } m < 0 \\ W^B_0 \text{DR}_X(D + B, E) & \text{if } m = 0 \\ W^B_0 \text{DR}_X(D + B, E) & \text{if } m > 0 \end{cases} \]
The main result of this section is
**Theorem 9.**
\[ (\mathbb{R}h_* f_* V^o_R, (\mathbb{R}h_* f_* V^o_R, \tau), (W^B_0 \text{DR}_X(D + B, E), F^\cdot, W)) \]
is a \( \mathbb{R} \)-cohomological mixed Hodge complex.

**Proposition 4.** \( \mathbb{R}h_* (f_* V^o) \) is quasi-isomorphic to
\[ W^B_0 \text{DR}_X(D + B, E) \]
**Proof.** The statement is local, so we can assume \( X \) is a polydisk. For the basic case, one can assume \( V \) is of rank 1, \( D \) has two components \( D_1 \) and \( D_2 \) such that the monodromy of \( V \) around \( D_1 \) is trivial, and the monodromy of \( V \) around \( D_2 \) is nontrivial.
Let \( Y = X - B \). Then, \( \Omega_Y(\log D^2) \otimes h^* E \) is a resolution of \( f_* V^o \) (see [8]).
Let \( g : Y - D^2 \to Y \) be the inclusion map. By a theorem of Griffith[?] and Deligne[3], the inclusion map
\[ i : \Omega_Y(\log D^2) \to g_* \mathcal{A}_Y - D^2 \]
is a quasi-isomorphism. Therefore, \( f_* V \) is quasi-isomorphic to
\[ g_* \mathcal{A}_Y - D^2 \]
As \( g_* \mathcal{A}_Y - D^2 \) is a complex of flasque sheaves, \( \mathbb{R}h_* f_* V \) is quasi-isomorphic to
\[ h_* g_* \mathcal{A}_Y - D^2 \]
Now,
\[ W_0^B \Omega_X^2(\log D + B) \otimes E = \Omega_X^2(\log D^2 + B) \otimes E \]
But as we have seen the complex \( \Omega_X^2(\log D^2 + B) \) is quasi-isomorphic to
\[(h \circ g)_{\ast} \mathcal{A}D - D^2\]
So the result for the basic case follows.
Now, let \( V \) be of rank \( r \). For each \( i = 1, 2 \), let \( \Gamma_i \) be the monodromy of \( V \) around \( D^i \). As \( V \) is unitary, we can simultaneously diagonalize all \( \Gamma_1 \) and \( \Gamma_2 \). Therefore, we can assume \( V \) is the direct sum of two rank 1 unitary local systems. As \( RH_{\ast} \) and \( f_{\ast} \) commutes with direct sum. The result follows.
Now, let \( V \) be of rank 1 and let \( D^1, \ldots, D^s \) be components of \( D \). Now let \( D_1 \) be the subdvisor of \( D \) over which \( V \) has identity monodromy; and let \( D_2 \) be the subdvisor of \( D \) over which \( V \) has nontrivial monodromy. Then, the result follows after the same steps in the basic case. \( \square \)

**Proposition 5.** The inclusion map
\[ i : (W_0^B DR_X(D + B, E), \tau) \to (W_0^B DR_X(D + B, E), W) \]
is a quasi-isomorphism of filtered complexes.

**Proof.** This is again a local statement, so we can assume \( X \) is a polydisk and \( V \) is of rank 1. We need to show that the induced maps of \( i \)
\[ H^k(i) : H^k(Gr_m^W W_0^B DR_X(D + B, E)) \to H^k(Gr_m^W W_0^B DR_X(D + B, E)) \]
are isomorphisms.

\[ H^k(Gr_m^W W_0^B DR_X(D + B, E)) = \begin{cases} H^m(W_0^B DR_X(D + B, E)) & \text{if } i = m \\ 0 & \text{otherwise} \end{cases} \]

**Claim 1** If \( m > 1 \), then \( H^m(W_0^B DR_X(D + B, E)) = 0 \).

**Proof of Claim 1** We have a short exact sequence of complexes
\[ 0 \to W_0^B DR_X(D + B, E) \to W_0^B DR_X(D + B, E) \xrightarrow{res} W_0^B DR_B(B \cap D, E_B)[-1] \to 0 \]
where the \( DR(D \cap B, E_B, \nabla_B) \) is the complex
\[ \cdots \to \Omega_B^m(\log B \cap D) \otimes E_B \xrightarrow{\nabla_B} \Omega_B^{m+1}(\log B \cap D) \otimes E_B \to \cdots \]
and the map \( res \) is the residue map.
Taking cohomology, we get
\[ \cdots \to H^k(W_0^B DR_X(D + B, E)) \to H^k(W_0^B DR_X(D + B, E)) \]
\[ \to H^{k-1}(W_0^B DR_B(B \cap D, E_B)) \to \cdots \]
Timmerscheidt proved in the Appendix D of [7] that \( W_0^B DR_X(D + B, E) \) is a resolution of \( (h \circ f)_{\ast} V \). Therefore, \( W_0^B DR_X(D + B, E) \) is exact. Likewise,
\[ W_0^B DR_B(B \cap D, E_B) \]
is also exact.
So the conclusion follows.

**End of Proof**
The above proof also shows that
\[H^k(\text{Gr}^W_0 W^B_0 \text{DR}_X(D + B, E)) = \begin{cases} H^1(\text{Gr}^W_1 W^B_0 \text{DR}_X(D + B, E)) & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases}\]

\[H^k(\text{Gr}^W_0 W^B_0 \text{DR}_X(D + B, E)) = \begin{cases} H^0(W^0_0 \text{DR}_X(D + B, E)) & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{cases}\]

Therefore, to prove
\[i : (W^B_0 \text{DR}_X(D + B, E), \tau) \to (W^B_0 \text{DR}_X(D + B, E), W)\]
is a quasi-isomorphism of filtered complexes, it remains to prove that both
\[H^0(i) : H^0(\text{Gr}^W_0 W^B_0 \text{DR}_X(D + B, E)) \to H^0(\text{Gr}^W_0 W^B_0 \text{DR}_X(D + B, E))\]
and
\[H^1(i) : H^1(\text{Gr}^W_0 W^B_0 \text{DR}_X(D + B, E)) \to H^1(\text{Gr}^W_0 W^B_0 \text{DR}_X(D + B, E))\]
are isomorphisms.

Now,
\[H^0(\text{Gr}^W_0 W^B_0 \text{DR}_X(D + B, E)) = \ker(E \xrightarrow{\nabla} W^B_0(\Omega^1_X\log(D + B) \otimes E))\]
and
\[H^0(\text{Gr}^W_0 W^B_0 \text{DR}_X(D + B, E)) = \ker(E \xrightarrow{\nabla} W_0(\Omega^1_X\log(D + B) \otimes E))\]
It is clear that the map \(H^0(i)\) is an isomorphism.

To simplify notations, write \(K^\cdot\) for \(W^B_0 \text{DR}_X(D + B, E)\), from the proof of Claim 1, we have a commutative diagram

\[
\begin{array}{ccc}
H^1(\text{Gr}^W_1 K^\cdot) & \xrightarrow{H^1(i)} & H^1(\text{Gr}^W_1 K^\cdot) \\
\downarrow & & \downarrow \\
H^1(K^\cdot) & \xrightarrow{\text{res}} & H^1(W^0_0 \text{DR}_B(B \cap D, E_B)[-1])
\end{array}
\]

and the residue map on the second row is an isomorphism. As the residue map on the first row is an isomorphism (even on the complex level), we see that the map \(H^1(i)\) is an isomorphism.

\[\square\]

For reader’s sake, we restate the main theorem of this Section:

**Theorem 10.**

\((\mathbb{R} h_* f_* V^\omega, \mathbb{R} h_* f_* V^\omega, \tau), (W^B_0 \text{DR}_X(D + B, E), F^\cdot, W)\)
is a cohomological mixed \(\mathbb{R}\)-Hodge complex

**Proof.** The quasi-isomorphism

\((\mathbb{R} h_* f_* V^\omega, \tau) \otimes \mathbb{C} \to (W^B_0 \text{DR}_X(D + B, E), W)\)

was proved in the previous proposition.

It remains to show

\((\text{Gr}^m \mathbb{R} h_* f_* V^\omega, (\text{Gr}^m W^B_0 \text{DR}_X(D + B, E), F))\)
is a cohomological $\mathbb{R}$-complex of weight $m$, i.e. the Hodge spectral sequence of $(\text{Gr}_m^W W_0^B \text{DR}_X(D + B, E), F)$ degenerates at $E_1$, and the induced filtration on

$$H^k(X, \text{Gr}_m^W W_0^B \text{DR}_X(D + B, E)) = H^k(X, \text{Gr}_m^W \mathbb{R} f_* V_R^D) \otimes \mathbb{C}$$

defines a pure $\mathbb{R}$-Hodge structure of weight $k + m$ on

$$H^k(X, \text{Gr}_m^W \mathbb{R} f_* V_R^D)$$

i.e. the induced filtration $F$ on $H^k(X, \text{Gr}_m^W W_0^B \text{DR}_X(D + B, E))$ is $m + k$ opposed to its conjugate.

For $m > 1$, all $\text{Gr}_m^W W_0^B \text{DR}_X(D + B, E)$ are 0, so we only need to show the case for $m = 0, 1$.

For $m = 0$,

$$(\text{Gr}_m^W W_0^B \text{DR}_X(D + B, E), F) = (W_0 \text{DR}_X(D + B, E), F)$$

Timmerscheidt showed that it is a cohomological $\mathbb{R}$-complex of weight 0 in [7] (Appendix D).

For $m = 1$, we have seen that

$$\text{Gr}_1^W W_0^B \text{DR}_X(D + B, E) \cong W_0 \text{DR}(B \cap D, E_B, \nabla_B)[-1]$$

Let $F$ be the induced Hodge filtration on $\text{Gr}_1^W W_0^B \text{DR}_X(D + B, E)$, and let $F_B$ be the usual Hodge filtration on $W_0 \text{DR}(B \cap D, E_B, \nabla_B)$. Let $\bar{F}$ and $\bar{F}_B$ be their conjugates.

To show $F$ and $\bar{F}$ are $k + 1$ opposed on $H^k(X, \text{Gr}_1^W W_0^B \text{DR}_X(D + B, E))$, we show that

$$\text{Gr}_q^F \text{Gr}_p^F H^k(X, \text{Gr}_1^W W_0^B \text{DR}_X(D + B, E)) = 0 \text{ if } p + q \neq k + 1$$

As $\text{Gr}_1^W W_0^B \text{DR}_X(D + B, E) \cong W_0 \text{DR}_B(B \cap D, E_B)[-1],$

$$\text{Gr}_q^F \text{Gr}_p^F H^k(X, \text{Gr}_1^W W_0^B \text{DR}_X(D + B, E)) = \text{Gr}_q^{F_B} \text{Gr}_p^{F_B} H^{k-1}(B, W_0 \text{DR}_B(B \cap D, E_B))$$

$$\text{Gr}_q^F \text{Gr}_p^F H^k(X, \text{Gr}_1^W W_0^B \text{DR}_X(D + B, E)) = \text{Gr}_q^{\bar{F}_B} \text{Gr}_p^{\bar{F}_B} H^{k-1}(B, W_0 \text{DR}_B(B \cap D, E_B))$$

Therefore, $\text{Gr}_q^F \text{Gr}_p^F H^k(X, \text{Gr}_1^W W_0^B \text{DR}_X(D + B, E)) = 0$ if $p - 1 + q - 1 \neq k - 1$.

The $E_1$-degeneration of $(\text{Gr}_1^W W_0^B \text{DR}_X(D + B, E), F)$ follows from the $E_1$-degeneration of $(W_0 \text{DR}(B \cap D, E_B, \nabla_B), F_B)$.

$\square$

So far, we have shown that if $V$ has a real lattice $V_R$, i.e.

$$V = V_R \otimes_{\mathbb{R}} \mathbb{C}$$

Then, the Hodge spectral sequence

$$E_1^{p,q} = H^q(X, W_0^B \Omega^p_X(\log D + B) \otimes E) \Rightarrow H^{p+q}(X, W_0^B \text{DR}_X(D + B, E))$$

degenerates at $E_1$.

Now, consider the case when $V$ does not have a real-lattice.
7. Vanishing Theorem for the complex $\text{Gr}^W_{DR_X}(D, E)$

Now, let $V$ be any unitary local system over $\mathbb{C}$. We have seen in Section 4 that even if $V$ does not have a real lattice, the spectral sequence of $(DR_X(D, E), F)$ still have $E_1$-degeneration. Similarly, we have

**Lemma 5.** Let $B$ be a smooth divisor transversal to $D$, then the spectral sequence of $(W_0^B DR(D + B, E), F)$:

$$E_1^{p, q} = H^q(X, W_0^B(\Omega_X^p(\log D + B))) \Rightarrow H^{p+q}(X, W_0^B(DR_X(D + B, E)))$$

degenerates at $E_1$.

**Theorem 11.** Let $B$ be a smooth very ample divisor transversal to $D$, then for $m = 0, \cdots, n - 1$

$$H^q(X, \text{Gr}^W_m DR^p(D, E, \nabla) \otimes O_X(B)) = 0$$

for $p + q > n + 1$.

**Proof.** We show first that

$$H^q(X, W_0 DR^p(D, E, \nabla) \otimes O_X(B)) = 0$$

for $p + q > n + 1$.

By Theorem 8 we have the exact sequence

$$0 \to W_0^B DR_X(D + B, E) \to W_0 DR_X(D, E) \otimes O_X(B) \to W_0 DR_B(B \cap D, E_B) \otimes O_X(B) \to 0$$

Take cohomology sequence, we get

$$\cdots \to H^q(X, W_0^B(\Omega_X^p(\log D + B) \otimes E)) \to H^q(X, W_0(\Omega_X^p(\log D) \otimes E) \otimes O_X(B)) \to$$

$$\to H^q(B, W_0(\Omega_B^p(\log B \cap D) \otimes E) \otimes O_X(B)) \to \cdots$$

Therefore, it is enough to show

**Claim 1:** $H^q(X, W_0^B(\Omega_X^p(\log D + B) \otimes E)) = 0$ for $p + q > n$.

**Claim 2:** $H^q(B, W_0(\Omega_B^p(\log B \cap D) \otimes E_B)) \otimes O_X(B)) = 0$ for $p + q > n$.

**Proof of Claim 1:** Consider the maps

$$X - (B + D) \xrightarrow{\tilde{L}} X - B \xrightarrow{\tilde{h}} X$$

Write $V^\circ$ for the restriction of $V$ on $X - B$. We have seen in Lemma 5 that the spectral sequence

$$E_1^{p, q} = H^q(X, W_0^B(\Omega_X^p(\log D + B) \otimes E)) \Rightarrow H^{p+q}(X, W_0^B DR_X(D + B, E))$$

$$= H^{p+q}(X, \mathbb{R} h_* f_* V^\circ)$$

$$= H^{p+q}(X - B, f_* V^\circ)$$

As $X - B$ is affine, it follows from Theorem 6 that

$$H^q(X, W_0^B(\Omega_X^p(\log D + B) \otimes E)) = 0$$

for $p + q > n$.

**End**

**Proof of Claim 2:** Induct on dimension of $X$.

Therefore, it remains to show that if $X$ is a smooth projective curve, then

$$H^1(X, W_0(\Omega_X(\log D) \otimes E) \otimes O_X(B)) = 0$$

But for the curve case,

$$W_0(\Omega_X(\log D) \otimes E) \otimes O_X(B) = W_0^B(\Omega_X(\log D + B) \otimes E)$$
Therefore the result follows again from Theorem 6.
To finish the rest of the proof, we use the identification from proposition
\[(W_m/W_m^{-1})DR_X(D, E) \cong W_0DR(\tilde{C}_m, E_m, \nabla_m)[-m]\]
and then apply the above argument to \(\tilde{D}_m\).

**Corollary 4.** For any \(m \in \mathbb{Z}\),
\[H^q(X, W_m(\Omega_X^p(\log D) \otimes E) \otimes O_X(B)) = 0\]
for \(p + q > \dim X\).

**Corollary 5.** Let \(L\) be an ample line bundle on \(X\), then
\[H^q(X, Gr^W(\Omega_X^p(\log D) \otimes E) \otimes L) = 0\]
for \(p + q > n\).

**Proof.** Like in Theorem 11, it is enough to show
\[H^q(X, W_0DR^p(D, E, \nabla) \otimes L) = 0\]
for \(p + q > n\).

Let \(m\) be a large enough integer such that \(L \otimes m\) is very ample. Let \(B\) be a smooth hyperplane divisor transversal to \(D\) so that
\[L \cong O_X(B)\]
Use the same idea from Corollary 3 we construct a cyclic cover of degree \(m\) branched over \(B\)
\[\pi : X' \rightarrow X\]
To finish the proof, it remains to show
\[\pi^*W_0DR_X(D, E) = W_0DR(\tilde{D}, \tilde{E}, \tilde{\nabla})\]
But this is clear from the local description of \(W_0\). □

8. **Appendix**

8.1. **Linear algebra.**

**Theorem 12.** Let \(U\) be an unitary matrix over \(\mathbb{C}\), then \(U\) is diagonalizable.

**Theorem 13.** Let \(A\) and \(B\) be commuting diagonalizable \(n \times n\) matrices over any field \(k\), then \(A\) and \(B\) can be simultaneously diagonalized.

**Proof.** Let \(V\) be the vector space \(k^n\). It is enough to show that \(A\) and \(B\) share the same eigenvectors.

**Claim 1:** \(A\) and \(B\) share at least one eigenvector.

**Proof of Claim 1:** Let \(v\) be an eigenvector of \(A\) with eigenvalue \(\lambda\), then
\[ABv = BAv = B\lambda v = \lambda Bv\]
i.e. \(Bv\) is also an eigenvector of \(A\) with eigenvalue \(\lambda\).

Let \(W\) be the subspace spanned by
\[v, Bv, \cdots, B^n v\]

Then, \(W\) is invariant under \(B\). As \(V\) has a basis by eigenvectors of \(B\), one can choose a vector \(w \in W\) which is an eigenvector of \(B\). Then, from the construction of \(W\), \(w\) is also an eigenvector of \(A\). End
Let $w$ be as above, with $Bw = \mu w$; Let $e_1, \cdots, e_n$ be the standard basis of $V$; Let $V'$ be the subspace spanned by $e_1, \cdots, e_{n-1}$; Let $\phi : V \to V$ be the linear map such that $\phi(e_n) = w$.

\[
\begin{align*}
\phi^{-1} \circ A \circ \phi &= A' \oplus \text{Diag}(\lambda) \\
\phi^{-1} \circ B \circ \phi &= B' \oplus \text{Diag}(\mu)
\end{align*}
\]

where $A'$ and $B'$ are $n-1 \times n-1$ submatrices of $A$ and $B$, representing the restriction of $A$ and $B$ on $V'$.

Now, $A'$ and $B'$ are diagonalizable, and they commute, therefore, by inducting on the size of the matrix, we are done. \qed
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