Orbital period derivative of a binary system using an exact orbital energy equation

Vikram H. Zaver
B-4/6, Avanti Apt., Harbanslal Marg, Sion, Mumbai 400022 INDIA
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It is proposed that the equations of motion in periodic relativity which yielded major predictions of general relativity are exact in nature and can be applied to pulsars and inspiraling compact binaries for analyzing orbital period derivative and two polarization gravitational wave forms. Exactness of these equations eliminates the need for higher order xPN corrections to the orbital energy part of the balance equation. This is mainly due to the introduction of dynamic WEP which states that the gravitational mass is equal to the relativistic mass.

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1. INTRODUCTION

The famous quadrupole formalism of Einstein [1] is the lowest order wave generation formalism in the Newtonian limit $1/c \rightarrow 0$. Using this formalism, Peters and Mathews [2] calculated the time average of the energy radiated from a system of two gravitating masses. From these results, secular decays of the semi major axis, and eccentricity were found by Peters [3]. Based on this work, Esposito and Harrison [4], and Wagoner [5] heuristically formulated the orbital period derivative for the binary pulsar system in light of the discovery of a radio pulsar PSR 1913+16 by Hulse and Taylor [6]. Timing measurement of PSR 1913+16 by Weisberg and Taylor [7] provided confirmation for the rate of orbital period decay due to gravitational radiation damping as predicted by general relativity. Subsequently both the theoretical model and the measurement accuracies were improved over a period of time [8-17].

In periodic relativity theory (PR) [15,19], we extended the concept of space-time symmetry to include heavier de Broglie particles, which enabled us to introduce energy-momentum invariant into the space-time invariant resulting in an invariant relationship between the force and the energy. We utilized this relation to address two-body problem in gravitational field by proposing the following equation of orbital motion

$$\frac{d}{dt}(E - m_0c^2) = -\frac{\mu m_1}{r^2} \frac{dr}{dt} (\cos \psi + \sin \psi), \quad (1.1)$$

where $m = \gamma m_0$ is the relativistic mass, $E$ total energy of the particle, $\psi$ is the angle between the radial vector $r$ and the velocity vector $\mathbf{v}$. Factor $(\cos \psi + \sin \psi)$ is responsible for giving us geodesic like trajectories. Appearance of this factor is due to the introduction of the Newtonian gravitational potential in the form of summation of two components due to acceleration $(\mu m/r^2) \cos \psi$ and $(\mu m/r^2) \sin \psi$ acting on the gravitating body. With this set-up for the equation of motion, we successfully obtained exactly the same lowest order expressions for gravitational redshift and deflection of light as in general relativity. We also obtained second order non-homogeneous non-linear differential equation of motion equivalent to general relativity which yielded same values of the perihelion precession for the planets of the solar system. These are reasons enough to conclude that Eq. (1.1) is an exact equation of motion [20] and can provide us exact orbital energy for two body systems. If we indiscriminately introduce xPN corrections to these equations, there is a good chance that we may not be able to derive correct formula for gravitational redshift and bending of light from such corrected equations. This means that we have locked on to an orbital energy equation which we cannot alter. This is mainly because the present theory is energy based theory. In comparison, general relativity uses Newtonian orbital energy equations and adds higher order post-Newtonian corrections [8,21-27] to get desired relativistic effects for computing center of mass binding energy and gravitational wave energy flux. In PR, these corrections are fundamentally built into the orbital energy equation. The basis of this equation is the introduction of dynamic WEP(weak equivalence principle) in PR. Dynamic WEP states that the gravitational mass is equal to the relativistic mass. For more details see [18,19]. This situation provides an ideal testing ground for proving this theory in the area of gravitational radiation.

2. ORBITAL ENERGY OF A BINARY SYSTEM

For a binary system, we can rewrite Eq. (1.1) as

$$\frac{d}{dt}(E - m_1c^2) = -\frac{\mu_1 m_2 \gamma}{r^2} \mathbf{v} (\cos \psi + \sin \psi) \hat{r}, \quad (2.1)$$

where $m_1$ is the mass of the pulsar and $m_2$ that of the companion star. For Binary, $\mu_1 = Gm_1$, $r$ is the separation. Total orbital energy $E_0$ of the system can be given
by
\[
E^0 = (E - m_2c^2) + \int \frac{\mu_1 m_2 \gamma}{r^2} \frac{dr}{dt} (\cos \psi + \sin \psi) \ dt.
\] (2.2)

If we evaluate \( E^0 \) at periastron, we can introduce following approximations. \( \psi = \pi/2 \) and \( m_2 \gamma = \text{constant} \). Therefore,
\[
E^0 = (E - m_2c^2) + \mu_1 m_2 \gamma \left( -\frac{1}{r} \right), \quad (2.3)
\]
\[
E^0 = m_2c^2 \left[ (\gamma - 1) - \frac{\mu_1 \gamma}{r c^2} \right], \quad (2.4)
\]
\[
E^0 = m_2c^2 \left[ 1 - \frac{v_p^2}{c^2} \right] \left\{ 1 - \frac{\mu_1}{r c^2} \right\} - 1, \quad (2.5)
\]

where \( v_p \) is the velocity at periastron. One can observe that the xPN corrections are naturally and fundamentally built into the orbital energy equation Eq. (2.5). Unlike general relativity, there is no limit to which this equation can be expanded. However, here we will expand Eq. (2.5) equivalent to 3.5PN order in general relativity.
\[
E^0 = m_2c^2 \left[ \left( 1 + \frac{v_p^2}{2c^2} + \frac{3v_p^4}{8c^4} + \frac{5v_p^6}{16c^6} + \frac{35v_p^8}{128c^8} + \ldots \right) \times \left\{ 1 - \frac{\mu_1}{r c^2} \right\} - 1 \right].
\] (2.6)

where \( v_p = h/r_p \), \( h^2 = \mu_1 a(1 - e^2) \) and \( r_p = a(1 - e) \) at periastron. Variation of \( E^0 \) with time is then given by Eq. (2.8). It is to be noted here that introduction of the deviation to flat Minkowski metric \([18, 19]\) in Eq. (2.2) and Eq. (2.8) does not alter the formulation because the effect gets cancelled out.

\[
\begin{aligned}
\frac{dE^0}{dt} &= \left[ \frac{\mu_1 m_2}{2a^2} - \frac{1}{4} \frac{\mu_1^2 m_2(1 + e)(3e - 1)}{c^2 a^3(1 - e)^2} - \frac{3}{16} \frac{\mu_1^3 m_2(1 + e)^2(5e - 1)}{c^2 a^4(1 - e)^3} - \frac{5}{32} \frac{\mu_1^4 m_2(1 + e)^3(7e - 1)}{c^2 a^5(1 - e)^4} \right] \frac{da}{dt} \\
&\quad + \left[ \frac{\mu_1^2 m_2 e}{c^2 a^2(1 - e)^3} + \frac{1}{8} \frac{\mu_1^3 m_2(7e^2 + 4e + 1)}{c^2 a^3(1 - e)^4} + \frac{15}{8} \frac{\mu_1^4 m_2 e(1 + e)^2}{c^2 a^4(1 - e)^5} \right] \frac{de}{dt} + \mathcal{O} \left( \frac{1}{c^5} \right). \\
\end{aligned}
\] (2.8)

3. ORBITAL PERIOD DERIVATIVE

Laser interferometric observations of gravitational waves demands energy balance equation computed to an extraordinarily high degree of accuracy of order 1/c^6. Equations of variation with time of the orbital frequency and orbital phase of an inspiralling compact binary are derived from this energy balance equation and the theoretical templates of the compact inspiral binary is obtained by introducing these highly accurate parameters into binary’s two polarization wave-forms \( h_+ \) and \( h_{\times} \). In case of binary pulsars, the energy balance equation is given by
\[
\frac{dE^0}{dt} = -\mathcal{L},
\] (3.1)

where \( E^0 \) is the orbital energy and \( \mathcal{L} \) the total gravitational luminosity (or wave flux) of the source. The time average value \( \langle \mathcal{L} \rangle \) has been computed to 1PN order by Blanchet and Sch"{o}fer\(^\text{[28]}\) and can be given in our terminology by
\[
\langle \mathcal{L} \rangle = \frac{1024}{5Gc^3 \mu^5(1 - e^2)^7} \left\{ 1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right\}

+ \frac{-E^0}{1664\mu^2(1 - e^2)^5} \left[ 13 - 6414e^2 - \frac{27405}{4} e^4 - \frac{5377}{16} e^6 \right]

+ \left( -840 - \frac{6419}{2} e^2 - \frac{5103}{8} e^4 - \frac{259}{8} e^6 \right) \nu \right\},
\] (3.2)

where \( \mu = (m_1 m_2)/m_0 \), \( m = (m_1 + m_2) \), \( \nu = \mu/m \) and \( E^0 \) to 1PN accuracy can be obtained from Eq. (2.7) and given by
\[
E^0 = \frac{-\mu_1 m_2}{2a} \left[ 1 + \frac{\mu_1(1 + e)(3e - 1)}{ac^2(1 - e)^2} \right].
\] (3.3)

We can truncate Eq. (2.8) to 1PN accuracy as follows.
\[
\begin{aligned}
\frac{dE^0}{dt} &= \left[ \frac{\mu_1 m_2}{2a^2} - \frac{1}{4} \frac{\mu_1^2 m_2(1 + e)(3e - 1)}{c^2 a^3(1 - e)^2} \right] \frac{da}{dt} \\
&\quad + \left[ \frac{\mu_1^2 m_2 e}{c^2 a^2(1 - e)^3} \right] \frac{de}{dt}.
\end{aligned}
\] (3.4)
We will utilize following lowest order expression for \( \langle de/dt \rangle \) given by Peters \[3\]. 1PN expression could be more consistent but it may not affect the result significantly.

\[
\langle de/dt \rangle = \frac{19c(1-e^2)(1+(121/304)e^2)}{12a(1+(73/24)e^2+(37/96)e^4)} \langle da/dt \rangle.
\] (3.5)

We get following relation from Kepler’s third law,

\[
d \frac{da}{dt} = \frac{2a \dot{P}_b}{3 \dot{P}_b},
\] (3.6)

where \( \dot{P}_b \) is the orbital period and \( \dot{P}_b \) orbital period derivative. Substitution of Eqs. (3.5) and (3.6) in Eq. (3.4) yields,

\[
\langle dE/\dot{E}_b \rangle = \frac{\mu_1 m_2}{3a} \left[ 1 - \frac{\mu_1 (\sigma_1 + \sigma_2)}{\sigma_3} \right] \frac{\dot{P}_b}{\dot{P}_b},
\] (3.7)

where \( \sigma_1 = (-3 + 6e + (151/8)e^2 + (149/4)e^3) \),
\( \sigma_2 = ((1081/32)e^4 + (79/8)e^5 + (111/32)e^6) \),
\( \sigma_3 = 6ac^2(1-e)^2(1 + (73/24)e^2 + (37/96)e^4) \).

Introducing Eqs. (3.2) and (3.7) into the balance Eq. (3.1) and replacing \( a \) with Kepler’s third law in the form

\[
\frac{1}{a} = \left( \frac{2\pi}{\dot{P}_b} \right)^2 \mu_2^{-\frac{3}{2}}, \quad \text{where} \quad \mu_2 = G(m_1 + m_2),
\] (3.8)

we get, \( \dot{P}_b = k_1(1 + k_2) \), where, \( k_1 = -\frac{192\pi}{5e^5} \left( \frac{2\pi G}{\dot{P}_b} \right)^{\frac{3}{2}} \frac{m_1 m_2 (m_1 + m_2)^{-\frac{3}{2}}}{(1-e^2)^{\frac{3}{2}}} \)

\[
\times \left\{ 1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right\},
\] (3.9)

\[
k_2 = \left[ (k_6 - 1) + \left( \frac{k_4 k_5 k_6}{k_1} \right) \right],
\] (3.11)

\[
k_3 = -\frac{4\pi}{35} \left[ \frac{m_1 m_2 (m_1 + m_2)^{\frac{5}{2}}}{c^2 (1-e^2)^{\frac{5}{2}}} \left( \frac{2\pi G}{P_b} \right)^{\frac{3}{2}} \right],
\] (3.13)

\[
k_4 = \left[ 1 + \frac{1}{4} \frac{(1 + e)(1 - 3e)}{c^2 (1-e^2)^{\frac{3}{2}}} \left( \frac{2\pi G}{P_b} \right)^{\frac{3}{2}} m_1 (m_1 + m_2)^{-\frac{3}{2}} \right],
\] (3.14)

\[
k_5 = \left[ 13 - \frac{6414 e^2}{4} - \frac{27405}{8} e^2 - \frac{5103}{8} e^4 + \frac{259}{8} e^6 \right] \nu.
\] (3.15)

For binary pulsar PSR 1913+16, factor \( k_2 \) given by Eq. (3.11) in this theory turns out to be \( k_2 = -1.865 \times 10^{-5} \) compared to \( +2.15 \times 10^{-5} \) given by Blanchet and Schäfer \[28\], which is \(-1.1528 \) times factor \( k_2 \) in our theory and \(-60 \) times the result of another closest rival theory of Spyrou and Papadopoulos \[29\]. Therefore, in a way this theory is in remarkable agreement with that of Blanchet and Schäfer. As of today this value remains below the accuracy in the measurement of \( \dot{P}_b \).

In this section, we will simply relate the orbital energy Eq. (2.7) to the 3.5PN accurate gravitational wave-form model discussed by Blanchet et al. \[32\]. Since the orbits of the inspiralling compact binaries are circularized, for
We have derived precise value for the orbital period derivative which is very close to the currently accepted theoretical value derived by Blanchet and Schäfer [28]. This needs to be experimentally verified for establishing the correct theoretical approach. It would be impossible to derive the expression for the orbital period derivative in this theory without the use of relativistic mass for the orbiting body in the inverse square law of gravitation. In PR [18, 19] static WEP is modified to dynamic WEP which states that the gravitational mass is equal to the relativistic mass. It is the use of relativistic mass that eliminates the need for higher order $x$PN corrections to the orbital energy part of the balance equation. Further justification for using the relativistic mass is discussed at length in [18, 19].

5. Conclusion

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