Algebraic non-integrability of magnetic billiards

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Abstract
We consider billiard ball motion in a convex domain of the Euclidean plane bounded by a piece-wise smooth curve under the action of a constant magnetic field. We show that if there exists a first integral polynomial in the velocities of the magnetic billiard flow, then every smooth piece $\gamma$ of the boundary must be algebraic, and either is a circle or satisfies very strong restrictions. In particular, it follows that any non-circular magnetic Birkhoff billiard is not algebraically integrable for all but finitely many values of the magnitude of the magnetic field. Moreover, a magnetic billiard in ellipse is not algebraically integrable for all values of the magnitude of the magnetic field. We conjecture that the circle is the only integrable magnetic billiard, not only in the algebraic sense, but also for a broader meaning of integrability. We also introduce what we call outer magnetic billiards. As an application of our method, we prove analogous results on algebraically integrable outer magnetic billiards.

Keywords: magnetic billiards, polynomial integrals, Birkhoff conjecture

1. Introduction and main results

1.1. Magnetic Birkhoff billiards

In this paper we consider a magnetic billiard inside a convex domain $\Omega \subset \mathbb{R}^2$ bounded by a simple piece-wise smooth closed curve. We consider the influence of a magnetic field of constant magnitude $\beta > 0$ on the billiard motion, so that the particle moves inside $\Omega$ with unit speed along a Larmor circle of constant radius $r = \frac{1}{\beta}$ in counterclockwise direction. Upon hitting the boundary, the particle is reflected according to the law of geometric optics. We call such a model a magnetic Birkhoff billiard.

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We shall assume that every smooth piece $\gamma$ of the boundary of $\Omega$ satisfies
\[ \beta < \min_{\gamma} k, \]
where $k$ is the curvature. In other words, we assume that the magnetic field is relatively weak with respect to the curvature. It is an exercise in the differential geometry of curves that in this case the boundary of the domain $\Omega$ is strictly convex with respect to circles of radius $r = \frac{1}{2}$. This means, in particular, that the intersection of any circle of radius $r$ with $\Omega$ consists of at most one arc. Moreover, under this assumption, if a circle of radius $r$ oriented in the same direction as the boundary is tangent to $\partial \Omega$ (with the agreed orientation), then it contains the domain $\Omega$ inside. Magnetic Birkhoff billiards were studied in many papers; see, e.g., [1, 2, 8, 11, 13, 15, 18, 20]. The motivation for the present paper comes from [15], where computer evidence of chaotic behavior of a magnetic billiard inside an ellipse is demonstrated for all magnitudes $\beta$ of the magnetic field. For $\beta$ positive, unlike the case $\beta = 0$, the pictures show that the billiard is not integrable. We examine this problem in an algebraic setting, using ideas from our recent papers on ordinary Birkhoff billiards [3, 4], extending previous results of [5, 19]. It seems plausible that other approaches to integrability are applicable. For example, it is interesting to test for meromorphic integrals via variational equations along a 2-periodic orbit (see figure 1) in the spirit of [14].

### 1.2. Polynomial integrals

We are concerned with the existence of first integrals polynomial in the velocities for magnetic billiards.

**Definition 1.1.** Let $\Phi : T\Omega \to \mathbb{R}$ be a function on the unit tangent bundle, $\Phi = \sum_{k+l=0}^{N} a_{kl}(x)v_1^k v_2^l$, which is a polynomial in the components $v_1, v_2$ of $v$ with coefficients continuous up to the boundary, $a_{kl} \in C(\Omega)$. We call $\Phi$ a polynomial integral of the magnetic billiard if the following conditions hold.

1. $\Phi$ is an integral of the magnetic flow $g^t$ inside $\Omega$,
   \[ \Phi(g^t(x, v)) = \Phi(x, v); \]
(2) $\Phi$ is preserved under the reflections at smooth points of the boundary $\partial \Omega$: for any smooth point $x \in \partial \Omega$,

$$\Phi(x, v) = \Phi(x, v - 2\langle n, v \rangle n),$$

for any $v \in T_x \Omega$, $|v| = 1$, where $n$ is the unit normal to $\partial \Omega$ at $x$.

**Remark 1.** It turns out that the condition of convexity with respect to the circles of radius $r$ which we introduced above can be relaxed if one adds in the definition 1.1 of the integral the following requirement: for any given circle of radius $r$ intersecting the domain $\Omega$ in several arcs, the integral $\Phi$ must have the same value on all those arcs. Our method strongly relies on this additional requirement and it is not clear to us whether it can be discarded or not.

**Example 1.** Let $\gamma$ be a circle with the center at the origin. Then the function which measures the distance of the center of Larmor circle to the origin is invariant under reflections and hence is an integral $h$ of the billiard flow. Specifically, $h$ has the form:

$$h(x, v) = x_1^2 + x_2^2 + \frac{2}{3}(v_1 x_2 - v_2 x_1).$$

In fact, we are not aware of any other piece-wise smooth example of integrable magnetic billiard. Similarly to Birkhoff' conjecture for the ordinary billiards, we ask if the only integrable magnetic billiard is the circular one. As usual, the integrability can be understood in various ways. In this paper we study integrals polynomial in the velocities for magnetic billiards. Another approach, that of the so-called total integrability, was considered in [2].

### 1.3. Phase space of magnetic billiard

Throughout this paper we shall use the following construction. Denote by $J$ the standard complex structure on $\mathbb{R}^2$ and introduce the mapping

$$\mathcal{L} : T_x \Omega \to \mathbb{R}^2, \quad \mathcal{L}(x, v) = x + rJv,$$

which assigns to every unit tangent vector $v \in T_x \Omega$ the center of the corresponding Larmor circle. Varying the unit vector $v$ in $T_x \Omega$, for a fixed point $x \in \Omega$, the corresponding Larmor centers form a circle of radius $r$ centered at $x$. The domain swept by all these circles when $x$ runs over $\Omega$ will be denoted by $\Omega_r$. Vice versa, one can prove that for any circle of radius $r$ lying in $\Omega_r$, its center necessarily belongs to $\Omega$.

For a magnetic Birkhoff billiard we always choose counterclockwise orientation of $\partial \Omega$. Moreover, for any smooth piece $\gamma$ of the boundary $\partial \Omega$ we define two curves as follows. Fixing an arc-length parameter $s$ of positive orientation, we set

$$\gamma_{+r}(s) = \mathcal{L}(\gamma(s), \tau(s)), \quad \gamma_{-r}(s) = \mathcal{L}(\gamma(s), -\tau(s)), \quad (2)$$

where $\tau(s) = \dot{\gamma}(s)$. It is easy to see that $\Omega_r \subset \mathbb{R}^2$ is a bounded domain in the plane homeomorphic to an annulus and the curves $\gamma_{\pm r}$, called parallel curves to $\gamma$, lie on the boundary $\partial \Omega_r$. Here $\gamma_{+ r}$ lies on the outer boundary of the annulus, and $\gamma_{- r}$ lies on the inner boundary. Other pieces of the boundary of $\partial \Omega_r$ are circular arcs of radius $r$ with the centers at the corners of the boundary $\partial \Omega$, but they will not play any role in the sequel.
Remark 2. The curves $\gamma_{\pm r}$ are also called equidistant curves, or fronts, in singularity theory, or offset curves in computer aided geometric design, see [16, 17].

One easily computes the curvature of the parallel curves to be

$$k_{+r} = \frac{k(\gamma)}{r \cdot k(\gamma) - 1}, \quad k_{-r} = \frac{k(\gamma)}{r \cdot k(\gamma) + 1}.$$ 

So the curvature of the inner boundary $k_{+r}$ and that of the outer boundary $k_{-r}$ always satisfy the bounds

$$k_{+r} > \beta, \quad 0 < k_{-r} < \beta. \quad (3)$$

This shows, in particular, that any circle of radius $r$ with the center at $\gamma(s)$ is tangent to the outer boundary from inside at $\gamma(s) - rJ\gamma(s)$, and to the inner boundary from outside at the point $\gamma(s) + rJ\gamma(s)$. Moreover, apart from these tangencies, this circle remains entirely inside $\Omega_r$ (see figure 2).

By the definition, $\mathcal{L}$ has constant value on every Larmor circle, so the components of $\mathcal{L}$ are integrals of the magnetic flow $g_t$ inside $\Omega$.

Moreover, we introduce the mapping $\mathcal{M} : \Omega_r \to \Omega_r$ by the following rule: let $C_-$ and $C_+$ be two Larmor circles centered at $P_-$ and $P_+$, respectively. We define

$$\mathcal{M}(P_-) = P_+ \iff C_- \text{ is transformed to } C_+$$

after billiard reflection at the boundary $\partial \Omega$.

With this definition, $\mathcal{M} : \Omega_r \to \Omega_r$ preserves the standard symplectic form in the plane (see section 3, proposition 3.1 for the proof), and thus $\Omega_r$ naturally becomes the phase space of the magnetic Birkhoff billiard. We shall call $\mathcal{M}$ the magnetic billiard map. Notice that on the boundaries $\gamma_{\pm r}$, the map $\mathcal{M}$ acts as the identity map, while on the connecting circles of the boundary corresponding to the corners of $\Omega$, $\mathcal{M}$ is not defined.

Given a polynomial integral $\Phi = \sum_{k \geq 1} a_k(x)v_1^k v_2^k$ of the magnetic billiard, we define the function $F : \Omega_r \to \Omega_r$ by the requirement

Figure 2. Circle of radius $r$ centered at $\gamma(s)$ is tangent to $\partial \Omega_r = \gamma_{+r} \cup \gamma_{-r}$. 

\[\begin{array}{c}
\Omega_r \\
\gamma_{+r} \\
\Omega \\
\gamma(s) \\
\gamma_{-r}
\end{array}\]
This is a well-defined construction, since \( \Phi \) is an integral of the magnetic flow, and therefore takes constant values on any Larmor circle. Moreover, since \( \Phi \) is invariant under the billiard flow, \( F \) is invariant under the billiard map \( \mathcal{M} \):

\[
F \circ \mathcal{M} = F.
\]

In coordinates, the definition (4) reads

\[
F(x_1 - rv_2, x_2 + rv_1) = \Phi(x, v) = \sum_{0 \leq k + l \leq N} a_{kl}(x)v_1^k v_2^l.
\]

Notice that since \( \Phi \) is a polynomial in \( v \), the function \( F \) satisfies the following property: \( F \) restricted to any circle of radius \( r \) lying in \( \Omega_r \) is a trigonometric polynomial of degree at most \( N \). The next theorem claims that in such a case \( F \) itself is a polynomial function.

**Theorem 1.2.** Let \( \Omega_r \) be a domain in \( \mathbb{R}^2 \) which is the union of all circles of radius \( r \) whose centers run over a domain \( \Omega \) (for example, the whole \( \mathbb{R}^2 \)). Let \( F : \Omega_r \to \mathbb{R} \) be a continuous function such that the restriction of \( F \) to any circle of radius \( r \) of \( \Omega_r \) is a trigonometric polynomial of degree at most \( N \). Then \( F \) is a polynomial in \( (x, y) \) of degree at most \( 2N \).

The proof of this theorem uses lemma 2.1, whose elegant proof was communicated to us by S Tabachnikov. The proofs of the lemma and theorem 1.2 are given in section 2.1. Notice that if one allows \( r \) to be arbitrary in theorem 1.2, then the statement is obvious; however, the result holds true when \( r \) is fixed.

The next statement is an immediate consequence of theorem 1.2 and (5):

**Corollary 1.3.** The coefficients of the integral \( \Phi \) are polynomials in \( (x, y) \) of degree at most \( 2N - (k + l) \).

Recall that the coefficients of the integral \( \Phi \) are assumed to be continuous on the closure of \( \Omega \), therefore it follows from the very construction of the function \( F \) that it extends continuously to the boundary \( \partial \Omega_r \), and hence by theorem 1.2, \( F \) coincides with a polynomial on \( \Pi_r \).

Moreover, we will prove the following:

**Proposition 1.4.** Suppose that the magnetic billiard in \( \Omega \) admits a polynomial integral \( \Phi \) and let \( F \) be the corresponding polynomial on \( \Pi_r \). Then for every smooth piece \( \gamma \) of the boundary \( \partial \Omega \) it holds that

\[
F|_{\gamma \pm r} = \text{const}.
\]

**Remark 3.** Given a smooth piece \( \gamma \) of the boundary one can assume that the polynomial integral \( F \) of \( \mathcal{M} \) is such that the constant in proposition 1.4 is 0, for both parallel curves \( \gamma_{\pm r} \). Indeed, if \( F|_{\gamma - r} = c_1 \) and \( F|_{\gamma + r} = c_2 \), one can replace \( F \) by \( F^2 - (c_1 + c_2)F + c_1 \cdot c_2 \) to annihilate both constants \( c_1, c_2 \). In terms of the integral \( \Phi \), this means that on \( \gamma \) one can assume that

\[
\Phi(x, \pm \tau) = 0,
\]

for every point \( x \in \gamma \) and unit tangent vector \( \tau \) to \( \gamma \) at \( x \).
We shall prove proposition 1.4 in section 3.

Proposition 1.4 and the remark 3 imply that

\[ \gamma_{\pm r} \subset \{ F = 0 \}, \]

and thus \( \gamma_{\pm r} \) is contained in the algebraic curve \( \{ F = 0 \} \). This fact implies then that \( \gamma \) itself is algebraic. In the sequel we shall denote by \( f_{\pm r} \) the minimal defining polynomials of the irreducible component in \( \mathbb{C}^2 \) containing \( \gamma_{\pm r} \), respectively. Since the curves \( \gamma_{\pm r} \) are real, \( f_{\pm r} \) have real coefficients. Notice that it may happen that both \( \gamma_{\pm r} \) belong to the same component, so that \( f_{\pm r} = f_{\mp r} \). For instance, this is the case for parallel curves to \( \gamma \) when \( \gamma \) is an ellipse \([6, 16, 17]\). In this case \( f_{\pm r} = f_{\mp r} \) is an irreducible polynomial of degree 8.

1.4. Main result and corollaries

We now turn to the formulation of our main result:

**Theorem 1.5.** Let \( \Omega \) be a convex bounded domain with piece-wise smooth boundary \( \partial \Omega \), such that every smooth piece of \( \partial \Omega \) has curvature at least \( \beta \). Suppose that the magnetic billiard in \( \Omega \) admits a polynomial integral \( \Phi \). Then the following alternative holds: either \( \partial \Omega \) is a circle, or every smooth piece \( \gamma \) of \( \partial \Omega \) is not circular and has the property that the affine curves \( \{ f_{\pm r} = 0 \} \) are smooth in \( \mathbb{C}^2 \). Moreover, any non-singular point of intersection of the projective curve \( \{ f_{\pm r} = 0 \} \) in \( \mathbb{C}P^2 \) with the infinite line \( \{ z = 0 \} \) away from the isotropic points \( (1 : \pm i : 0) \) must be a tangency point with the infinite line. Here \( f_{\pm r} \) is a homogenization of \( f_{\pm r} \).

**Corollary 1.6.** For any non-circular domain \( \Omega \) in the plane, the magnetic billiard inside \( \Omega \) is not algebraically integrable for all but finitely many values of \( \beta \).

**Proof.** Indeed, \( f_{\pm r} \) depends on \( r \) as a polynomial function, so \( f_{\pm r} \) is a polynomial in \( x, y, \) and \( r \). Moreover, since every piece \( \gamma \) has positive curvature bounded from below by \( \beta \) there is an open interval \( \frac{1}{r} \in (k_{\min}, k_{\max}) \) where one can claim, using a differential geometry argument, that \( \gamma_{\pm r} \) does have singularities. Hence, the system of equations

\[
\partial_x f_{\pm r} = \partial_y f_{\pm r} = f_{\pm r} = 0
\]

defines an algebraic curve in \( \mathbb{C}^3 \) and its projection on the \( r \)-line is a Zariski open set. It then follows that singularities persist for all but finitely many \( r \).

One can hope that refining our method below one can prove that the only magnetic billiard admitting a polynomial integral is circular. But this is out of reach at the present moment.

**Remark 4.** Our main result implies, in particular, that if one of the arcs of the magnetic billiard is circular, then only the circle is algebraically integrable. This is in contrast to the case of zero magnetic field, where there exist polygons for which the billiard flows admit polynomial integrals (see \([12]\)).

**Corollary 1.7.** Let \( \Omega \) be the interior of the standard ellipse

\[ \partial \Omega = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}, \quad 0 < b < a. \]
Then for any magnitude of the magnetic field \(0 < \beta < k_{\text{min}} = \frac{b}{a}\), the magnetic billiard in the ellipse is not algebraically integrable.

**Proof.** The equation of the parallel curves for an ellipse reads (see, e.g., [6]):

\[
a^8(b^4 + (r^2 - y^2)^2 - 2b^2(r^2 + x^2)) + b^4(r^2 - x^2)^2(b^4 - 2b^2(r^2 - x^2 + y^2)) + (x^2 + y^2 - r^2)^2 - 2a^6(b^6 + (r^2 - y^2)^2(r^2 + x^2 - y^2) - b^4(r^2 - 2x^2 + 3y^2)) - b^2(r^4 + 3y^2(x^2 - y^2) + r^2(3x^2 + 2y^2)) + 2a^2b^2(-b^6(r^2 + x^2) - (-r^2 + x^2 + y^2)^2)
\]

\[
\times (r^4 - x^2y^2 - r^2(x^2 + y^2)) + b^4(r^4 - 3x^4 + 3x^2y^2 + r^2(2x^2 + 3y^2)) + b^2(r^6 - 2x^6 + x^4y^2 - 3x^2y^4 + r^4(-4x^2 + 2y^2) + r^2(5x^4 - 3x^2y^2 - 3y^4)) + a^4(b^6 + 2b^6(r^2 + 3x^2 - 2y^2) + (r^2 - y^2)^2(-r^2 + x^2 + y^2)^2)
\]

\[
- 2b^4(3r^4 - 3x^4 + 5x^2y^2 - 3y^4 + 4r^2(x^2 + y^2)) + 2b^2(r^6 - 3x^4y^2 + x^2y^4 - 2y^6 + 2r^4(x^2 - 2y^2) + r^2(-3x^4 - 3x^2y^2 + 5y^4))) = 0.
\]

It is known to be irreducible (see [6]). Moreover, the parallel curves \(\gamma_{\pm r}\) have singularities in the complex plane for every \(r > \frac{1}{k_{\text{min}}} = \frac{a^2}{a^2} > b\), namely

\[
\left(0, \pm \frac{\sqrt{b^2 - a^2} \sqrt{a^2 - r^2}}{a}, \frac{\pm \sqrt{a^2 - b^2} \sqrt{b^2 - r^2}}{b}, 0\right).
\]

Therefore the result follows from theorem 1.5.

\[\square\]

### 1.5. Outer magnetic billiards

It is remarkable that the action of the billiard map \(M\) coincides with what we call an outer magnetic billiard. In addition, the result which we get by our method provides the extension of the result by Tabachnikov [19] to the case of outer magnetic billiards.

Let us introduce outer magnetic billiards in a natural way. Let \(\Gamma\) be a smooth convex curve in the plane with a fixed orientation (not necessarily counterclockwise). Let \(\beta > 0\) be the magnitude of the magnetic field. Given a point \(P\) outside \(\Gamma\) we define \(T(P)\) as follows:

Consider the Larmor circle of radius \(r = \frac{1}{\beta}\) starting from \(P\) tangent to \(\Gamma\) at \(\Gamma(s)\) with the agreed orientation at \(\Gamma(s)\), and define \(T(P)\) to lie on the same Larmor circle so that the arcs \((P, \Gamma(s))\) and \((\Gamma(s), T(P))\) have the same angular measure.

Notice that two different cases arise:

(1) If the orientation on \(\Gamma\) is clockwise, then \(T\) is well defined for any \(\beta > 0\) (see figure 3).
However, if the orientation on $\Gamma$ is counterclockwise, then $T$ is well defined for $b_0 \min$ (see figure 4).

Remark 5. In fact, in the second case one can allow $\Gamma$ to be a $C^1$-smooth curve which is piece-wise $C^2$, having arcs of radius $r$ interlaced between non-circular $C^2$ pieces.

In both cases (1) and (2) the domain where the outer billiard map $T$ is defined is the annulus $A$ bounded by $\Gamma$ and $\Gamma_{+2r}$ (see figures 3 and 4). We call the dynamical system $T$ the...
outer magnetic billiard. It is not hard to check that $T$ is a symplectic map of the annulus $A$. The next theorem shows that the outer magnetic billiard $T$ in the case $(2)$ is in fact isomorphic to the magnetic Birkhoff billiard $\mathcal{M}$:

**Theorem 1.8.** The magnetic billiard map $\mathcal{M}: \Omega_r \to \Omega_r$ (defined in section 1.3) coincides with the outer billiard $T$ determined by the inner boundary $\gamma_+^r$ of $\Omega_r$, endowed with the counterclockwise orientation.

Indeed, let $P_-, P_+$ be the centers of two Larmor circles $C_-, C_+$, such that $C_-$ is reflected to $C_+$ at the point $Q$ of the smooth part $\gamma$ of the boundary $\partial \Omega$. Then it follows from the definition of the reflection law that the circle of radius $r$ with the center in $Q$, oriented counterclockwise, passes from $P_-$ to $P_+$ and is tangent to $\gamma_+^r$ at the point $P = \mathcal{L}(Q, \tau(Q))$ (see figures 5 and 6 for two different configurations of $C_-$ and $C_+$).

**Remark 6.** Let us remark that the map $T$ in the case $(1)$ (see figure 3) is not isomorphic to any magnetic Birkhoff billiard globally, for topological reasons. Indeed, the difference between the rotation numbers of the two boundaries $\Gamma$ and $\Gamma_2^r$ equals $0$ in case $(1)$ and equals $2\pi$ in case $(2)$. Nevertheless, since our method below is concentrated near the boundaries, it applies for both cases $(1)$ and $(2)$.

Let $F$ be a polynomial which is invariant under $T$. As before, we shall call $F$ a polynomial integral. Then similarly to the magnetic Birkhoff billiard, we have that $\Gamma$ and $\Gamma_2^r$ lie in $\{F = \text{const}\}$ and therefore are algebraic (similarly to proposition 1.4). Our result for outer magnetic billiards reads:

**Theorem 1.9.** Assume that there exists a non-constant polynomial $F$ such that $F$ is invariant under the outer billiard map $T$. Let $f$ and $f_2^r$ be irreducible defining polynomials of $\Gamma$ and $\Gamma_2^r$. Then the following alternative holds: either $\Gamma$ is a circle, or the curves $\{f = 0\}$ and $\{f_2^r = 0\}$ in $\mathbb{C}^2$ are smooth with the property that any non-singular intersection point of the...
projective curves \( \tilde{f} = 0 \) and \( \tilde{f}_{1,2} = 0 \) in \( \mathbb{CP}^2 \) with the infinite line \( \{z = 0\} \) which is not an isotropic point \((1 : \pm i : 0)\), must be a point of tangency (here \( \tilde{f} \) is a homogenization of \( f \)).

**Corollary 1.10.** The outer magnetic billiard for an ellipse is not algebraically integrable.

**Proof.** In the case of an ellipse the curve \( \tilde{f} = 0 \) is smooth everywhere in \( \mathbb{CP}^2 \) and intersects transversally the infinite line in two points away from the isotropic points. Thus theorem 1.9 implies that there exists no polynomial integral.

Exactly as in corollary 1.6, we have the following:

**Corollary 1.11.** For all but finitely many values of the magnitude \( \beta \) of the magnetic field, the outer magnetic billiard of \( \Gamma \) is not algebraically integrable unless \( \Gamma \) is a circle.

The rest of the paper is organized as follows. In section 2 we prove theorem 1.2 and lemma 2.1. In section 3 we deal with the boundary values of the integral \( F \) and prove that the map \( \mathcal{M} \) is symplectic. In section 4 we derive a remarkable equation for \( F \). Finally, in section 5 we prove theorem 1.5. The proofs of theorem 1.9 and corollary 1.11 are completely analogous and therefore are omitted.

2. Proof of theorem 1.2

We shall use the following lemma.

**Lemma 2.1.** Let \( F : A \rightarrow \mathbb{R} \) be a \( C^\infty \) function, where

\[
A = \{ (x, y) : (r - \delta)^2 \leq x^2 + y^2 \leq (r + \delta)^2 \},
\]

is an annulus in \( \mathbb{R}^2 \). Suppose that the restriction of \( F \) to any circle of radius \( r \) lying in \( A \) is a trigonometric polynomial of degree at most \( N \). Then \( F \) is a polynomial in \( x \) and \( y \) of degree at most \( 2N \).

**Proof.** (After S Tabachnikov). We shall say that \( F \) has property \( P_N \) if the restriction of \( F \) to any circle of radius \( r \) lying in \( A \) is a trigonometric polynomial of degree at most \( N \). The proof of the lemma goes by induction on the degree \( N \).

1. For \( N = 0 \), the lemma obviously holds, since if \( F \) has property \( P_0 \), then \( F \) is a constant on any circle of radius \( r \) and hence must be a constant on the whole \( A \), because any two points of \( A \) can be connected by a union of a finite number of circular arcs of radius \( r \).

2. Assume now that any function satisfying property \( P_{N-1} \) is a polynomial of degree at most \( 2(N-1) \).

Let \( F \) be any smooth function on \( A \) with property \( P_N \). Denote by \( C_0 \) be the core circle of \( A \), i.e., \( C_0 = \{ x^2 + y^2 = r^2 \} \), and let \( F_0 \) be a polynomial in \( (x, y) \) of degree \( N \) satisfying \( F|_{C_0} = F_0|_{C_0} \). Then one can find a \( C^\infty \) function \( G : A \rightarrow \mathbb{R} \) such that

\[
F(x, y) - F_0(x, y) = (x^2 + y^2 - r^2)G(x, y), \quad \forall (x, y) \in A.
\]
(This can be proved with the help of ‘polar’ coordinates on $A$
\[
(x, y) \to (u, v); \quad u = x^2 + y^2 - r^2, \quad v = \arg(x + iy),
\]
applying Hadamard’s lemma to the function $F - F_0$ with respect to the variable $u$ and $v$ regarded as a parameter.) Let us show now that $G$ has property $P_{N-1}$. Then by induction we will have that $G$ is a polynomial of degree $2(N - 1)$ and thus by (6), $F$ is a polynomial of degree at most $2N$. We need to show that the function $g := G|_C$ is a trigonometric polynomial of degree at most $N - 1$, for any circle $C$ of radius $r$ in $A$. With no loss of generality we may assume that the circle $C$ is centered on the $x$-axis (otherwise one applies a suitable rotation of the plane). Then
\[
C = \{(x, y) \in A : (x - a)^2 + y^2 = r^2\}, \quad |a| < \delta.
\]
Substituting $x = a + r \cos t$, $y = r \sin t$ into (6), we have
\[
(F - F_0)|_C = (a^2 + 2ar \cos t) \cdot g.
\]
Expanding the left- and the right-hand sides in Fourier series we get
\[
\sum_{k=-\infty}^{+\infty} f_k e^{ikt} = a(a + re^{it} + re^{-it}) \sum_{k=-\infty}^{+\infty} g_k e^{ikt},
\]
where $f_k$ are Fourier coefficients of $(F - F_0)|_C$. Moreover, we have
\[
f_k = 0, \quad |k| > N,
\]
since both $F$ and $F_0$ have property $P_N$. Thus we obtain a linear recurrence relation for the coefficients $g_k$:
\[
r g_{k+1} + a g_k + r g_{k-1} = 0, \quad |k| > N.
\]
The characteristic polynomial of this difference equation
\[
\lambda^2 + \frac{a}{r} \lambda + 1 = 0,
\]
has two complex conjugate roots $\lambda_{1,2} = e^{\pm i\alpha}$ and therefore we can write
\[
g_{N+l} = c_1 e^{il\alpha} + c_2 e^{-il\alpha}, \quad l \geq 2,
\]
where
\[
c_1 + c_2 = g_N, \quad c_1 e^{i\alpha} + c_2 e^{-i\alpha} = g_{N+1}.
\]
It is obvious now that if at least one of the coefficients $g_N$ or $g_{N+1}$ does not vanish, then at least one of the constants $c_1$, $c_2$ does not vanish, and therefore the sequence $\{g_{N+l}\}$ does not converge to 0 when $l \to +\infty$. Thus contradicts the continuity of $g$. Therefore, both $g_N$ and $g_{N+1}$ must vanish, and so $g$ is a trigonometric polynomial of degree at most $N - 1$, proving that $G$ has property $P_{N-1}$. This completes the proof of theorem 1.2.

Next we give the proof of theorem 1.2.

**Proof.** Take any circle of radius $r$ lying in $\Omega$, and let $A$ be the annulus which is the closure of its $\delta$-neighborhood. Using the convolution with a $C^\infty$ mollifier $\rho_\epsilon$ compactly supported in a small disc of radius $\epsilon$, we get a $C^\infty$ function
It is easy to see that if \( F \) has property \( P_N \), then also \( F' \) has property \( P_N \) on the chosen annulus \( A \), for all \( \epsilon \) small enough, \( 0 < \epsilon < \epsilon_0 \). Then by lemma 2.1, \( F' \) must be a polynomial on \( A \) of degree at most \( 2N \), for \( 0 < \epsilon < \epsilon_0 \). Recall that \( F' \) converge to \( F \) uniformly on \( A \) as \( \epsilon \to 0 \). Therefore, since the space of polynomials of degree at most \( N^2 \) is finite-dimensional, it follows that \( F \) is also a polynomial on \( A \) of degree at most \( N^2 \). The set \( W_r \) can be covered by annuli like \( A \), therefore \( F \) must be a polynomial of degree at most \( N^2 \) on the whole \( \Omega_r \). This completes the proof of theorem 1.2.

3. Boundary values of the integral. Proof that \( \mathcal{M} \) is symplectic

In this section we prove proposition 1.4 and the fact that the map \( \mathcal{M} \) is symplectic.

Take a point \( Q \) on a smooth piece \( \gamma \) of the boundary \( \partial \Omega \). Let \( \tau(Q) \) be a positive unit tangent vector to \( \gamma \). Let \( C_- \) and \( C_+ \) be the incoming and outgoing circles with the unit tangent vectors \( v_- \) and \( v_+ \) at the impact point \( Q \). We are interested in the two cases when the reflection angle between \( \tau \) and \( v_- \) or between \( -v_+ \) and \( \tau \) is close to zero. These two possibilities correspond (see figures 5 and 6) to the following cases:

(a) \( v_- = R_{-\epsilon} \tau, \quad v_+ = R_\epsilon \tau, \)

(b) \( v_- = R_\epsilon (-\tau), \quad v_+ = R_{-\epsilon} (-\tau), \)

where \( R_\epsilon \) is the counterclockwise rotation of the plane by a small angle \( \epsilon \). In figures 5 and 6 \((q;Q)\) \(\) and \((Q;p)\) are arcs of the circles \( C_- \) \(\) and \( C_+ \), respectively.

We define

\[
P_- (\epsilon) = \mathcal{L}(Q, v_-) = Q + rJ(v_-), \quad P_+ (\epsilon) = \mathcal{L}(Q, v_+) = Q + rJ(v_+).
\]

In the case (a) we have

\[
P_- (\epsilon) = Q + rJ(R_{-\epsilon} \tau), \quad P_+ (\epsilon) = Q + rJ(R_\epsilon \tau).
\]  

(7)

In the case (b),

\[
P_- (\epsilon) = Q - rJ(R_\epsilon \tau), \quad P_+ (\epsilon) = Q - rJ(R_{-\epsilon} \tau).
\]  

(8)

In the figures 5 and 6 we abbreviate \( P_- := P_- (\epsilon), P_+ := P_+ (\epsilon), P_0 := P_+ (0) = P_- (0) \).

Notice that in case (a) the middle point of the short arc that connects the points \( P_-(\epsilon) \) and \( P_+(\epsilon) \) is \( P_0 = P = (Q + rJ\tau) \in \gamma_{-\epsilon}, \) while for the case (b) the middle point is \( P_0 = (Q - rJ\tau) \in \gamma_+ \) (see figures 5 and 6).

Proof of proposition 1.4. The condition 2. of definition 1.1 reads in terms of \( F \)

\[
F(P_+(\epsilon)) = F(P_-(\epsilon)).
\]  

(9)

Differentiating this equality with respect to \( \epsilon \) at \( \epsilon = 0 \) and using the fact that \( \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} R_\epsilon = J \), we compute in the case (a):

\[
\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} F(P_+(\epsilon)) = dF|_{P_0}(-r \cdot \tau),
\]

\[
\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} F(P_-(\epsilon)) = dF|_{P_0}(r \cdot \tau).
\]
Here $dF|_{P_0}(w)$ is the differential of the function $F$ at the point $P_0$ applied to the vector $w$. Thus (9) implies that

$$dF|_{P_0}(\tau) = 0,$$

in which $\tau$ should be understood as the unit tangent vector to $\gamma_{\pm r}$ at the point $P_0$, proving the claim in the case (a). The case (b) is completely analogous. This proves that

$$F|_{\gamma_{\pm r}} = \text{const.}$$

Next we prove that the magnetic billiard map $M$ is symplectic.

**Proposition 3.1.** The map $M$ preserves the standard symplectic form $\omega$ in the plane.

**Proof.** $M$ maps $P_{-}$ to $P_{+}$. We shall stick to the case (a) for concreteness. Let $s$ be the arc-length parameter along $\gamma$. Let $(\tau, n)$, where $\tau = \dot{\gamma}$, $n = J\tau$, be the Frenet orthonormal frame along $\gamma$.

In view of (7), it is natural to introduce the local coordinates $(s, \epsilon)$ in $\Omega_r$ by

$$(s, \epsilon) \mapsto P = \gamma(s) + rJR_{\epsilon} \tau(s).$$

Notice that according to (7), the points $P_{-}$, $P_{+}$ correspond the opposite signs of the coordinate $\epsilon$. Differentiating with respect to $s$ and $\epsilon$ we get

$$\frac{\partial P}{\partial s} = \tau(s) + krJR_{\epsilon} n(s) = \tau(s) - kr\epsilon \tau(s),$$

$$\frac{\partial P}{\partial \epsilon} = rJR_{\epsilon} \tau(s) = -r\epsilon \tau(s) - r \cos \epsilon \tau(s) - r \sin \epsilon n(s).$$

Using these formulas we compute immediately the symplectic form $\omega$:

$$\omega\left(\frac{\partial P}{\partial s}, \frac{\partial P}{\partial \epsilon}\right) = \omega(\tau(s), -r\epsilon \tau(s)) = -r \sin \epsilon.$$

This means that in the coordinates $(s, \epsilon)$ $\omega$ has the form:

$$\omega = -r \sin \epsilon \, ds \wedge d\epsilon.$$

Notice that the sign change of $\epsilon$ leaves this expression invariant. This proves the invariance of $\omega$ under the map $M$. Let us remark that the last expression for the form $\omega$ coincides with the familiar invariant 2-form for the ordinary Birkhoff billiard. \qed

**4. A remarkable equation**

For any function $F$ which is invariant under $M$, we can rewrite equation (9) at any non-critical point $P_0 \in \gamma_{\pm r}$ as follows.

Denote by $n = J\tau$ the unit normal vector to $\gamma$. Then $n$ is also normal to $\gamma_{\pm r}$ and $\gamma_{-r}$ at the corresponding points. It is useful to rewrite formulas (7) and (8) in a slightly more convenient form. For the case (a) we have

$$P_{\pm}(\epsilon) = Q + rJ (R_{\pm \epsilon} \tau) = Q + rJR_{\pm \epsilon} n$$

$$= Q + rm - r(n - R_{\pm \epsilon} n) = P_0 - r(I - R_{\pm \epsilon}) n.$$  (10)
Analogously, for the case (b) we get from (8)
\[ P_\delta(\epsilon) = Q - r J(R_\delta \tau) = Q - r R_\delta n \]
\[ = Q - m + r(n - R_\delta n) = P_0 + r(I - R_\delta n). \]  
(11)

Notice that for the unit normal to the curves \( \gamma_\delta \) and \( \gamma_r \) at \( P_0 \) one has \( n = \pm \frac{\nabla F}{|\nabla F|} \), where the sign is irrelevant since we can change the sign of \( F \). Using this remark we can rewrite equation (9) with the help of (10) and (11) in the two cases (a) and (b) simultaneously:
\[ F \left( P_0 + r(I - R_\delta) \left( \frac{\nabla F}{|\nabla F|} \right)(P_0) \right) \]
\[ - F \left( P_0 + r(I - R_r) \left( \frac{\nabla F}{|\nabla F|} \right)(P_0) \right) = 0, \quad P_0 \in \gamma_{\delta,r}. \]  
(12)

This can be written for \( P_0 = (x, y) \in \gamma_{\delta,r} \) explicitly:
\[ F \left( x + r \frac{F_x(1 - \cos \epsilon) + F_y \sin \epsilon}{|\nabla F|}, y + r \frac{F_y(1 - \cos \epsilon) - F_x \sin \epsilon}{|\nabla F|} \right) \]
\[ - F \left( x + r \frac{F_y(1 - \cos \epsilon) - F_x \sin \epsilon}{|\nabla F|}, y + r \frac{F_x(1 - \cos \epsilon) + F_y \sin \epsilon}{|\nabla F|} \right) = 0. \]  
(13)

The next step is to expand equation (13) in power series in \( \epsilon \). The coefficient at \( \epsilon^3 \) reads
\[ (F_{xx} F_y^2 - 3F_{xxy} F_y F_x + 3F_{xyy} F_y F_x^2 - F_{yyy} F_x^3) \]
\[ + 3\beta (F_x^2 + F_y^2) + (F_{xx} F_y + F_{xy} (F_{yy} - F_{yx}) - F_{yy} F_x F_x) = 0, \quad (x, y) \in \gamma_{\delta,r}. \]  
(14)

Remarkably, the left-hand side of (14) is the complete derivative along the tangent vector field \( \nu \) to \( \gamma_{\delta,r} \), \( \nu = (F_x, -F_y) \), of the following expression, which therefore must be constant:
\[ H(F) + \beta |\nabla F|^3 = \text{const}, \quad (x, y) \in \gamma_{\delta,r}; \]  
(15)

here we used the notation
\[ H(F) = F_{xx} F_y^2 - 2F_{xy} F_x F_y + F_{yy} F_x^2. \]

Let us remark that (13) and therefore also (15) are valid only for those points where \( \nabla F \) does not vanish. However, very often this requirement is not satisfied if the polynomial \( F \) is reducible. Therefore, we proceed as follows. Let us denote by \( f_{\gamma_{\delta,r}} \), irreducible defining polynomial of \( \gamma_{\delta,r} \) (the proof for the curve \( \gamma_r \) is identical). Then we have
\[ F = f_{\gamma_{\delta,r}}^k \cdot g, \]
for some integer \( k \geq 1 \), where the polynomial \( g \) does not vanish identically on \( \gamma_{\delta,r} \). Given an arc of \( \gamma_{\delta,r} \) where \( g \) does not vanish, we may assume that \( g \) is positive on the arc (otherwise we change the sign of \( F \)). Moreover, since \( f_{\gamma_{\delta,r}} \) is an irreducible polynomial, we may assume that \( \nabla f_{\gamma_{\delta,r}} \) does not vanish on the arc. Therefore equation (15) can be derived in the same manner for the function \( F^k = f_{\gamma_{\delta,r}} \cdot g^k \), which obviously is invariant under the map \( M \) exactly as \( F \) is. Thus we have
\[ H(f_{\gamma_{\delta,r}} \cdot g^k) + \beta \left| \nabla (f_{\gamma_{\delta,r}} \cdot g^k) \right|^3 = \text{const}, \quad (x, y) \in \{f_{\gamma_{\delta,r}} = 0\}. \]  
(16)
Using the identities

\[ H(f_{+r} \cdot g^2) = g^2 H(f_{+r}), \quad \nabla (f_{+r} \cdot g^2) = g^2 \nabla (f_{+r}), \]

which are valid for all \((x, y) \in \{f_{+r} = 0\}\), we obtain from (16) that

\[ g^2 (H(f_{+r}) + \beta |\nabla f_{+r}|^k) = \text{const}, \quad (x, y) \in \gamma_{+r}. \tag{17} \]

Raising back to the power \(k\) we get

\[ g^3 (H(f_{+r}) + \beta |\nabla f_{+r}|^k)^k = \text{const}, \quad (x, y) \in \gamma_{+r}. \tag{18} \]

Next we establish the following

**Proposition 4.1.** The constant in equation (18) cannot be 0.

**Proof.** Recall the formulas for the curvature \(k\) of the curve defined implicitly by \(f_{+r} = 0\):

\[ \text{div} \left( \frac{\nabla f_{+r}}{|\nabla f_{+r}|} \right) = \frac{H(f_{+r})}{|\nabla f_{+r}|^3} = \pm k_{+r}. \tag{19} \]

Now we take any point on \(\gamma_{+r}\) and substitute into (18). This gives that the constant must be non-zero. Indeed, if the constant is zero, then

\[ \frac{H(f_{+r})}{|\nabla f_{+r}|^3} = -\beta. \]

Then by formulas (19) we have

\[ k_{+r} = \pm \beta. \]

But this is not possible, because of the bounds on the curvature of the parallel curves (3). \(\square\)
5. Proof of the main theorem 1.5

In this section we finish the proof of theorem 1.5. We start with the following:

**Theorem 5.1.** Suppose that the magnetic billiard in $\Omega$ admits a non-constant polynomial integral $\Phi$. If at least one piece of the boundary $\partial \Omega$ is a circular arc, then $\partial \Omega$ is a circle.

**Proof.** Recall that for circular magnetic billiard there exists a simple integral, given by example 1. It is very convenient to pass from $\Phi$ to $F$ defined as above:

$$ F \circ \mathcal{L} = \Phi. $$

Recall that the mapping $\mathcal{L}$ sends a unit vector $v$ at $x$ to the center of the Larmor circle passing through $x$ in the direction of $v$. Hence if $(Q, v)$ is reflected into $(Q, v')$, then the points $P_\pm = \mathcal{L}(v_\pm)$ lie on the circle of radius $r$ which is tangent to the curves $\gamma_{+r}$ and $\gamma_{-r}$ at the points $\mathcal{L}(Q, \tau)$ and $\mathcal{L}(Q, -\tau)$, which are the middle points of the two arcs of the circle connecting $P_-$ and $P_+$ (see figures 5 and 6).

Consider two pieces of $\partial \Omega$: $\gamma^{(1)}$ is an arc of the circle $C$ of radius $d$, and $\gamma^{(2)}$ is the adjacent piece. Consider also the annulus bounded by the two concentric circles, $C_r$ of radius $d + r$ and $C_{-r}$ of radius $r - d$ (see figure 7). Let us consider together with the given magnetic billiard another one acting inside the circle $C$. So the annulus between the two concentric circles of $C_r$ and $C_{-r}$ is the phase space of the magnetic billiard inside the circle $C$.

We claim that the polynomial function $F$ must have a constant value on every circle concentric with $C$. To show this, we denote by $(\rho, \phi)$ the polar coordinates centered at the center of these circles. In these coordinates the mapping $P_- \to P_+$ corresponding to the circular billiard is given by

$$ \rho(P_\pm) = \rho(P_\pm), \quad \phi(P_\pm) = \phi(P_\pm) + \alpha(\rho), $$

where

$$ \alpha(\rho) = 2 \arccos \left( \frac{\rho^2 + d^2 - r^2}{2\rho d} \right). $$

It follows from (21) that the function $\alpha$ is analytic in the annulus bounded by the two circles, $r - d < \rho < d + r$. Consider now the function

$$ \Delta(x, y) = F(P_\pm) - F(P_\pm), $$

where $P_\pm$ has coordinates $(x, y)$ and the coordinates of $P_\pm$ are determined according to (20) and (21). It follows from the analyticity of $\alpha$ and polynomiality of $F$ that the function $\Delta(x, y)$ is analytic on the open annulus $r - d < \rho < d + r$. Furthermore, since $F$ is built via the integral $\Phi$ for the billiard inside $\Omega$ and $\gamma^{(1)}_{+r} \subset C_{-r}$, it follows that $\Delta$ vanishes on an open subset of the annulus, and therefore must vanish identically on the annulus. This fact together with the denseness of the invariant circles with irrational rotation numbers for the map (20), yields that the polynomial $F$ has constant values on every concentric circle lying inside the annulus, and therefore on every concentric circle in the plane (not necessarily inside the annulus). This proves the claim.

Suppose now that the adjacent piece $\gamma^{(2)}$ does not lie on the circle $C$. This implies that $\gamma^{(2)}_{+r}$ necessarily intersects an open set of concentric circles. On every circle $F$ has a constant value by the claim above, and $F$ is also a constant on $\gamma^{(2)}_{+r}$, by proposition 1.4. Therefore, $F$
must be a constant on an open set and hence everywhere, contrary to the assumptions. This completes the proof.

Now we are in position to finish the proof of theorem 1.5.

Proof. Consider the equation (18) in $S^2$. It follows from (18) and proposition 4.1 that the curve $\{f_{r,0} = 0\}$ has no singular points in $S^2$, since at singular points both $H(f_{r,0})$ and $\nabla(f_{r,0})$ vanish. Now consider now in $P^2$ with homogeneous coordinates $(x : y : z)$ the projective curve $[\tilde{f}_{r,0} = 0]$. We shall denote the homogeneous polynomials corresponding to $f$ and $g$ by $\tilde{f}$ and $\tilde{g}$, respectively. Then the homogeneous version of (18) for $(x : y : z) \in [\tilde{f}_{r,0} = 0]$ reads

$$\tilde{g}^3 (z \cdot H(\tilde{f}_{r,0}) + \beta ((\tilde{f}_{r,0})^2 + (\tilde{f}_{r,0})^2)^2)^k = \text{const} \cdot z^p. \quad (22)$$

Here the power $p = 3 \deg g + 3k(\deg f_{r,0} - 1)$ must be positive unless the degree of the polynomial $f_{r,0}$ and that of $F$ are equal to one. But this is impossible, due to our convexity assumptions. Let $Z$ be any point of intersection of $[\tilde{f}_{r,0} = 0]$ with the infinite line $[z = 0]$. Then by (22) for such a point we have the two relations

$$(\tilde{f}_{r,0})^2 + (\tilde{f}_{r,0})^2 = 0, \quad x(\tilde{f}_{r,0})_x + y(\tilde{f}_{r,0})_y + z(\tilde{f}_{r,0})_z = x(\tilde{f}_{r,0})_x + y(\tilde{f}_{r,0})_y = 0.$$ 

But these two relations are compatible only in two cases: either

$$x^2 + y^2 = z = 0,$$

or

$$(\tilde{f}_{r,0})_x = (\tilde{f}_{r,0})_y = 0.$$ 

This completes the proof.

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