On the classification of unstable $H^*V - A$-modules

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Abstract

In this work, we begin studying the classification, up to isomorphism, of unstable $H^*V - A$-modules $E$ such that $F_2 \otimes_{H^*V} E$ is isomorphic to a given unstable $A$-module $M$. In fact this classification depends on the structure of $M$ as unstable $A$-module. In this paper, we are interested in the case $M$ a nil-closed unstable $A$-module and the case $M$ is isomorphic to $\sum^n F_2$. We also study, for $V = \mathbb{Z}/2\mathbb{Z}$, the case $M$ is the Brown-Gitler module $J(2)$.

1 Introduction

Let $V$ be an elementary abelian 2-group of rank $d$, that is a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^d$, $d \in \mathbb{N}$, $BV$ be a classifying space for the group $V$ and $H^*V = H^*(BV; F_2)$. We recall that $H^*V$ is an $F_2$-polynomial algebra $F_2[t_1, \ldots, t_d]$ on $d$ generators $t_i$, $1 \leq i \leq d$, of degree one.

Let $A$ be the mod.2 Steenrod algebra and $\mathcal{U}$ the category of unstable

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$A$-modules. We recall that $H^*V - U$ is the category whose objects are unstable $H^*V - A$-modules and morphisms are $H^*V$-linear and $A$-linear maps of degree zero. For example, the mod.2 equivariant cohomology of a $V$-CW-complex, which is the cohomology of the Borel construction, is an unstable $H^*V - A$-module.

Let $E$ be an unstable $H^*V - A$-module, we denote by $\overline{E}$ the unstable $A$-module $\mathbb{F}_2 \otimes_{H^*V} E = E/\tilde{H}^*V.E$, where $\tilde{H}^*V$ denotes the augmentation ideal of $H^*V$.

We have the following problem:

\[(P): \text{Let } M \text{ be an unstable A-module.} \]

Classify, up to isomorphism, unstable $H^*V - A$-modules such that $\overline{E} \cong M$ (as unstable $A$-modules).

It is clear that, for every subgroup $W$ of $V$, the unstable $H^*V - A$-module:

$$H^*W \otimes M$$

is a solution for the problem $(P)$.

For $W = 0$, a solution of $(P)$ is given by the unstable $H^*V - A$-module $M$ which is trivial as an $H^*V$-module.

For $W = V$, a solution of $(P)$ is given by the unstable $H^*V - A$-module $H^*V \otimes M$ which is free as an $H^*V$-module.

If $V = \mathbb{Z}/2\mathbb{Z}$ and $M = \Sigma N$ a suspension of an unstable $A$-module $N$, then we have, at least, the following two solutions of the problem $(P)$ which are free as $H^*(\mathbb{Z}/2\mathbb{Z})$-modules:

1. $\Sigma(H^*(\mathbb{Z}/2\mathbb{Z}) \otimes N)$.

2. $((H^*(\mathbb{Z}/2\mathbb{Z})^{\geq 1}) \otimes N$. 

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These two solutions are different as unstable $A$-modules (here $H^*(\mathbb{Z}/2\mathbb{Z})_{\geq 1}$ is the sub-algebra of $H^*(\mathbb{Z}/2\mathbb{Z})$ of elements of degree bigger than or equal to one). This shows that the solutions of the problem $(P)$ i.e. the classification, up to isomorphism, of unstable $H^*V - A$-modules such that $\overline{E} \cong M$ (as unstable $A$-modules), depends on the structure of $E$ as an $H^*V$-module and on the structure of $M$ as unstable $A$-module.

In this paper we will discuss the solutions of $(P)$ if $M$ is a nil-closed unstable $A$-module and $E$ is free as an $H^*V$-module and the solutions of $(P)$ if $M$ is isomorphic to $\sum F_2$ or to $J(2)$ and $E$ is free as an $H^*V$-module.

We begin by proving the following result (which is solution of $(P)$ when $M$ is a nil-closed unstable $A$-module).

**Theorem 1.1.** Let $E$ be unstable $H^*V - A$-module which is free as an $H^*V$-module. If $\overline{E}$ is a nil-closed unstable $A$-module, then there exists two reduced $U$-injectives $I_0$, $I_1$ and an $H^*V - A$-linear map $\varphi : H^*V \otimes I_0 \to H^*V \otimes I_1$ such that:

1. $E \cong \ker \varphi$
2. $\overline{E} \cong \ker \overline{\varphi}$

The proof of this result is based on the classification of $H^*V - U$-injectives and on some properties of the injective hull in the category $H^*V - U$.

Our work is naturally motivated by topology as shown in the study of homotopy fixed points of a $\mathbb{Z}/2$-action (see [L1]). Let $X$ be a space equipped with an action of $\mathbb{Z}/2$ and $X^{h\mathbb{Z}/2}$ denote the space of homotopy fixed points of this action. The problem of determining the mod.2 cohomology of $X^{h\mathbb{Z}/2}$ (we ignore deliberately the questions of 2-completion) involves two steps:
- determining the mod. 2 equivariant cohomology \( H^*_{Z/2}X \);
- determining \( \text{Fix}_{Z/2} H^*_{Z/2}X \) (for the definition of the functor \( \text{Fix}_{Z/2} \) see section 2).

For the first step, see for example [DL], the main information one has about the \( Z/2 \)-space \( X \) is that the Serre spectral sequence, for mod. 2 cohomology, associated to the fibration

\[
X \to X_{hZ/2} \to BZ/2
\]
collapses (\( X_{hZ/2} \) denotes the Borel construction \( EZ/2 \times_{Z/2} X \)). This collapsing implies that \( H^*_{Z/2}X \) is H-free and that \( H^*_{Z/2}X \) is canonically isomorphic to \( H^*X \). This gives clearly a topological application of problem (P).

We then prove the following results (related to the case \( E \) is \( \sum^n \mathbb{F}_2 \) and \( J(2) \)).

**Theorem 1.2.** Let \( E \) be unstable \( H^*V - A \)-module which is free as an \( H^*V \)-module. If \( E \) is isomorphic to \( \sum^n \mathbb{F}_2 \), then there exists an element \( u \) in \( H^*V \) such that:

1. \( u = \prod_i \theta_i^{\alpha_i} \), where \( \theta_i \in (H^1V) \setminus \{0\} \) and \( \alpha_i \in \mathbb{N} \)
2. \( E \cong \sum^d uH^*V \) with \( d + \sum_i \alpha_i = n \)

**Proposition 1.3.** Let \( E \) be an \( H - A \)-module which is \( H \)-free and such that \( E \) is isomorphic to \( J(2) \) then:
\( E \cong H \otimes J(2) \)
or
\( E \) is the sub-\( H - A \)-module of \( H \oplus \sum H \) generated by \((t, \Sigma 1)\) and \((t^2, 0)\).
The proofs of these two results are based on Smith theory, some properties of the functor Fix and on a result of J.P. Serre.

The paper is structured as follows. In section 2, we introduce the definitions of reduced and nil-closed unstable $A$-modules. We give the classification of injective modules in the category $\mathcal{U}$ and in the category $H^*V - \mathcal{U}$. We also recall the algebraic Smith theory. In section 3, we establish some properties of $E$ when $\overline{E}$ is a reduced unstable $A$-module. The results will be useful in section 4, where we give the solutions of the problem $(\mathcal{P})$ when $E$ is free as an $H^*V$-module and $\overline{E}$ is nil-closed. In section 5, we give some topological applications. In section 6, we give the solutions of the problem $(\mathcal{P})$ when $E$ is free as an $H^*V$-module and $\overline{E}$ is isomorphic to $\sum^n \mathbb{F}_2$, we also give a topological application. In section 7, we solve the problem $(\mathcal{P})$ when $\overline{E}$ is the Brown-Gitler module $J(2)$ and $V$ is $\mathbb{Z}/2\mathbb{Z}$.

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2 Preliminaries on the categories $\mathcal{U}$ and $H^*V - \mathcal{U}$

In this section, we will fix some notations, recall some definitions and results about the categories $\mathcal{U}$ and $H^*V - \mathcal{U}$.
2.1 Nilpotent unstable $A$-modules

Let $N$ be an unstable $A$-module. We denote by $Sq_0$ the $\mathbb{Z}/2\mathbb{Z}$-linear map:

$$Sq_0 : N \to N, \ x \mapsto Sq_0(x) = Sq^{|x|}x.$$ 

An unstable $A$-module $N$ is called nilpotent if:

$$\forall \ x \in N, \ \exists \ n \in \mathbb{N}; \ Sq_0^n x = 0.$$ 

For example, finite unstable $A$-modules and suspension of unstable $A$-modules are nilpotent. Let $Tor_{H^V}^1(F_2, N)$ be the first derived functor of the functor $F_2 \otimes_{H^V} - : H^V - U \to U$, we have the following useful result.

**Proposition 2.1.1.** ([S] page 150) Let $N$ be an unstable $H^V - A$-module, then the unstable $A$-module $Tor_{H^V}^1(F_2, N)$ is nilpotent.

2.2 Reduced unstable $A$-modules

An unstable $A$-module $M$ is called reduced if the $\mathbb{Z}/2\mathbb{Z}$-linear map:

$$Sq_0 : M \to M, \ x \mapsto Sq_0(x) = Sq^{|x|}x,$$

is an injection.

Another characterization of reduced unstable $A$-module in terms of nilpotent modules is the following.

**Lemma 2.2.1.** ([LZ1]) An unstable $A$-module is reduced if it does not contain a non-trivial nilpotent module.

In particular, any $A$-linear map from a nilpotent $A$-module to a reduced one is trivial.
2.3 Nil-closed unstable A-modules

Let $M$ be an unstable $A$-module. We denote by $Sq_1$ the $\mathbb{Z}/2\mathbb{Z}$-linear map:

$$Sq_1 : N \to N, \ x \mapsto Sq_1(x) = Sq^{|x|-1}x.$$

**Definition 2.3.1.** ([EP]) An unstable $A$-module $M$ is called nil-closed if:

1. $M$ is reduced.
2. $\text{Ker}(Sq_1) = \text{Im}(Sq_0)$.

We have the following two characterizations of unstable nil-closed $A$-modules.

**Lemma 2.3.2.** ([LZ1]) Let $M$ be an unstable $A$-module and $\mathcal{E}(M)$ be its injective hull. The unstable $A$-module $M$ is nil-closed if and only if $M$ and the quotient $\mathcal{E}(M)/M$ are reduced.

Let $Ext^s_U(-, M)$ be the $s$-th derived functor of the functor $\text{Hom}_U(-, M)$.

**Lemma 2.3.3.** ([LZ1]) An unstable $A$-module $M$ is nil-closed if and only if $Ext^s_U(N, M) = 0$ for any nilpotent unstable $A$-module $N$ and $s = 0, 1$.

2.4 Injectives in the category $\mathcal{U}$

Let $I$ be an unstable $A$-module, $I$ is called an injective in the category $\mathcal{U}$ or $\mathcal{U}$-injective for short, if the functor $\text{Hom}_\mathcal{U}(-, I)$ is exact.

The classification of $\mathcal{U}$-injectives (see [LZ1], [LS]) is the following.

Let $J(n), \ n \in \mathbb{N},$ be the $n$-th Brown-Gitler module, characterized up to isomorphism, by the functorial bijection on the unstable $A$-module $M$:

$$\text{Hom}_\mathcal{U}(M, J(n)) \cong \text{Hom}_{\mathbb{F}_2}(M^n, \mathbb{F}_2)$$
Clearly $J(n)$ is an $\mathcal{U}$-injective and it is a finite module.

Let $\mathcal{L}$ be a set of representatives for $\mathcal{U}$-isomorphism classes of indecomposable direct factors of $H^*(\mathbb{Z}/2\mathbb{Z})^m$, $m \in \mathbb{N}$ (each class is represented in $\mathcal{L}$ only once).

We have:

**Theorem 2.4.1.** Let $I$ be an $\mathcal{U}$-injective module. Then there exists a set of cardinals $a_{L,n}$, $(L, n) \in \mathcal{L} \times \mathbb{N}$, such that $I \cong \bigoplus_{(L, n)} (L \otimes J(n))^\oplus a_{L,n}$.

Conversely, any unstable $A$-module of that form is $\mathcal{U}$-injective.

Let’s remark that $H^*V$ is an $\mathcal{U}$-injective.

### 2.5 The injectives of the category $H^*V - \mathcal{U}$

The classification of injectives of the category $H^*V - \mathcal{U}$ ($H^*V - \mathcal{U}$-injectives for short) is given by Lannes-Zarati [LZ2] as follows.

Let $J_V(n)$, $n \in \mathbb{N}$, be the unstable $H^*V - A$-module characterized, up to isomorphism, by the functorial bijection on the unstable $H^*V - A$-module $M$:

$$\text{Hom}_{H^*V - \mathcal{U}}(M, J_V(n)) \cong \text{Hom}_{F_2}(M^n, F_2)$$

Clearly $J_V(n)$ is an $H^*V - \mathcal{U}$-injective.

Let $\mathcal{W}$ be the set of subgroups of $V$ and let $(W, n) \in \mathcal{W} \times \mathbb{N}$, we write

$$E(V, W, n) = H^*V \otimes_{H^*V/W} J_{V/W}(n)$$

(in this formula $H^*V$ is an $H^*V/W$-module via the map induced in mod.2 cohomology by the canonical projection $V \rightarrow V/W$).

**Theorem 2.5.1.** ([LZ2]) If $I$ is an injective of the category of $H^*V - \mathcal{U}$, then $I \cong \bigoplus_{(L,W,n) \in \mathcal{L} \times \mathcal{W} \times \mathbb{N}} (E(V, W, n) \otimes_{F_2} L)^\oplus a_{L,W,n}$.

Conversely, each $H^*V - A$-module of this form is an $H^*V - \mathcal{U}$-injective.
Clearly $H^*V$ is an $H^*V - \mathcal{U}$-injective.

## 2.6 Algebraic Smith theory

### 2.6.1 The functors $\text{Fix}$

We introduce the functors $\text{Fix}$ ([L1], [L2]). We denote by 

$$\text{Fix}_V : H^*V - \mathcal{U} \to \mathcal{U}$$

the left adjoint of the functor 

$$H^*V \otimes - : \mathcal{U} \to H^*V - \mathcal{U}$$

We have the functorial bijection:

$$\text{Hom}_{H^*V - \mathcal{U}}(N, H^*V \otimes P) \cong \text{Hom}_\mathcal{U} (\text{Fix}_V N, P)$$

for every unstable $H^*V - A$-module $N$ and every unstable $A$-module $P$.

The functor $\text{Fix}_V$ has the following properties.

1. The functor $\text{Fix}_V$ is an exact functor.
2. Let $N$ be an unstable $H^*V - A$-module and $\mathcal{E}(N)$ be its injective hull. Then, the module $\text{Fix}_V \mathcal{E}(N)$ is the injective hull of $\text{Fix}_V N$.

### 2.6.2

Let $N$ be an unstable $H^*V - A$-module, we denote by 

$$\eta_V : N \to H^*V \otimes \text{Fix}_V N$$

the adjoint of the identity of $\text{Fix}_V N$. We denote by $c_V = \prod_{u \in H^*V - \{0\}} u$ the top Dickson invariant, we have the following result (see [L2] corollary 2.3).
Proposition 2.6.1. Let $N$ be an unstable $H^*V - A$-module. The localization of the map $\eta_V$

$$\eta_V[c^{-1}_V] : N[c^{-1}_V] \rightarrow H^*V[c^{-1}_V] \otimes \text{Fix}_V N$$

is an injection.

This shows in particular, that if $N$ is torsion-free then the map $\eta_V$ is an injection.

The proposition 2.6.1 can be reformulated as follows.

Proposition 2.6.2. Let $N$ be an unstable $H^*V - A$-module. If $N$ is torsion-free then its injective hull in $H^*V - \mathcal{U}$ is free as an $H^*V$-module and is isomorphic to \( \bigoplus_{(L,n) \in \mathcal{L} \times N} (H^*V \otimes J(n)) \otimes L \)

Proof. Since the module is torsion-free then the map $\eta_V : N \rightarrow H^*V \otimes \text{Fix}_V N$ adjoint of the identity of $\text{Fix}_V N$ is an injection. So $N$ is a sub-$H^*V - A$-module of $H^*V \otimes \text{Fix}_V N$. By 2.6.1.1 and 2.6.1.2, we have that the injective hull of $N$ is isomorphic to $H^*V \otimes I$, where $I$ is an $\mathcal{U}$-injective. \qed

Remark 2.6.3. As a consequence of proposition 2.6.2, we have that if $E$ is an unstable $H^*V - A$-module which is free as an $H^*V$-module then its injective hull (in the category $H^*V - \mathcal{U}$) is also free as an $H^*V$-module.

Proposition 2.6.4. [LZ2]. Let $N$ be an unstable $H^*V - A$-module which is of finite type as an $H^*V$-module. The localization of the map $\eta_V$

$$\eta_V[c^{-1}_V] : N[c^{-1}_V] \rightarrow H^*V[c^{-1}_V] \otimes \text{Fix}_V N$$

is an isomorphism.

In particular, the previous result shows that:
1. If \( N \) is free as an \( H^*V \)-module, then the map \( \eta_V \) is an injection.

2. The isomorphism of the proposition proves that \( \text{dim} \overline{E} = \text{dim} \text{Fix}_V E \) where \( \text{dim} \) is the total dimension (see [LZ2]).

## 3 Some properties of \( E \) when \( \overline{E} \) is reduced

In this section we will prove some algebraic results which will be useful for section 4. In fact, we will analyze the relation between an unstable \( H^*V - A \)-module \( E \) and its (associated) unstable \( A \)-module \( \overline{E} \). For this, we will begin by giving some technical results.

### 3.1 Technical results

**Lemma 3.1.1.** Let \( P \) and \( Q \) be unstable \( H^*V - A \)-modules, free as \( H^*V \)-modules and \( f : P \to Q \) an \( H^*V - A \)-linear map. If the induced map \( \overline{f} : \overline{P} \to \overline{Q} \) is an injection then \( f \) is also an injection.

**Proof.** Let’s denote by \( \text{Im} f \) the image of \( f \), by \( \overline{f} : \overline{P} \to \overline{Q} \) the natural surjection and by \( i : \text{Im} f \hookrightarrow Q \) the inclusion of \( \text{Im} f \) in \( Q \). Since \( \overline{f} \) is an injection so the induced map \( (\overline{f}) \) is an isomorphism of unstable \( A \)-modules and then the induced map \( \overline{i} \) is an injection. This shows that \( \overline{\text{Im} f} \) is the image of \( \overline{f} \). Since the module \( \text{Im} f \) is a sub-\( H^*V \)-module of the \( H^*V \)-free module \( Q \) and \( \overline{i} : \overline{\text{Im} f} \hookrightarrow \overline{Q} \) is an injection, so \( \text{Im} f \) is free as an \( H^*V \)-module. In particular, we have that \( \text{Tor}_1^{H^*V}(\mathbb{F}_2, \text{Im} f) = 0 \) (see for example [R]). Let’s denote by \( N \) the kernel of the map \( \overline{f} \), so we have the following short exact sequence in \( H^*V - \mathcal{U} \):

\[
0 \longrightarrow N \longrightarrow P \overset{\overline{f}}\longrightarrow \text{Im} f \longrightarrow 0 .
\]
By applying the functor \((\mathbb{F}_2 \otimes_{H^*V} -)\) to the previous sequence, we prove that \(\overline{N}\) is trivial (since the map \(\overline{f}\) is an isomorphism and \(Imf\) is free as an \(H^*V - A\)-module). Hence the module \(N\) is trivial and the map \(f\) is an injection.

\(\square\)

The converse of this lemma is not true in general, but we have the following result:

**Lemma 3.1.2.** Let \(P\) and \(Q\) be unstable \(H^*V - A\)-modules, free as \(H^*V\)-modules and \(f: P \rightarrow Q\) an \(H^*V - A\)-linear map which is an injection. If \(\overline{P}\) is a reduced unstable \(A\)-module, then the induced map \(\overline{f}: \overline{P} \rightarrow \overline{Q}\) is an injection.

**Proof.** We denote by \(C\) the quotient of \(Q\) by \(P\), we have the following short exact sequence in \(H^*V - U\):

\[
0 \rightarrow P \xrightarrow{f} Q \xrightarrow{} C \xrightarrow{} 0 .
\]

By applying the functor \((\mathbb{F}_2 \otimes_{H^*V} -)\) to the previous sequence, we obtain an exact sequence in \(U\):

\[
0 \rightarrow Tor_1^{H^*V}(\mathbb{F}_2, C) \xrightarrow{} \overline{P} \xrightarrow{\overline{f}} \overline{Q} \xrightarrow{} \overline{C} \xrightarrow{} 0 .
\]

Since \(\overline{P}\) is reduced as unstable \(A\)-module and \(Tor_1^{H^*V}(\mathbb{F}_2, C)\) is nilpotent (see proposition 2.1.1), then the map \(\overline{f}\) is an injection.

\(\square\)
3.2 Statement of some properties of $E$ when $\overline{E}$ is reduced

The first result of this paragraph concerns the relation between the injective hull of $E$ and the induced module $\overline{E}$.

**Theorem 3.2.1.** Let $E$ be an unstable $H^*V - A$-module which is free as an $H^*V$-module and let $\mathcal{E}(E)$ be its injective hull (in the category $H^*V - U$). We suppose that $\overline{E}$ is reduced and let $I$ be its injective hull in the category $U$.

Then $\mathcal{E}(E)$ is isomorphic, as an unstable $H^*V - A$-module, to $H^*V \otimes I$.

**Proof.** Since $E$ is free as an $H^*V$-module, then $\mathcal{E}(E)$ is isomorphic, in the category $H^*V - U$, to $H^*V \otimes J$, where $J$ is an $U$-injective (see proposition 2.6.2).

Let’s denote by $i$ the inclusion of $E$ in $\mathcal{E}(E)$, we have, by lemma 3.1.2, that the induced map $\tilde{i}$ is an injection. We will prove, by using the definition, that $J$ is the injective hull of $\overline{E}$, in the category $U$. Let $P$ be a sub-$A$-module of $J$ such that the $A$-module $(\tilde{i})^{-1}(P)$ is trivial, we have to show that the unstable $A$-module $P$ is trivial.

Since $(\tilde{i})^{-1}(P)$ is trivial then the composition: $\pi \circ \tilde{i}: E \xrightarrow{\tilde{i}} J \xrightarrow{\pi} J/P$ is an injection. By lemma 3.1.1, the following composition

$E \xrightarrow{i} H^*V \otimes J \xrightarrow{\pi} H^*V \otimes (J/P)$

is an injection, which proves that the unstable $H^*V - A$-module $i^{-1}(H^*V \otimes P)$ is trivial. Since $H^*V \otimes J$ is the injective hull of $E$ so the unstable $H^*V - A$-module $H^*V \otimes P$ is trivial.

**Corollary 3.2.2.** Let $E$ be an unstable $H^*V - A$-module such that:

1. $E$ is free as an $H^*V$-module.

2. $\overline{E}$ is reduced as unstable $A$-module.
Then $E$ is reduced as unstable $A$-module.

Proof. We have, by theorem 3.2.1, that the injective hull of $E$ is $\text{H}^*V \otimes I$, where $I$ is the injective hull of $\overline{E}$ in $\mathcal{U}$. Since $\overline{E}$ is reduced, then $I$ is a reduced $\mathcal{U}$-injective. This shows that $E$ is reduced as an unstable $A$-module because its injective hull (in the category $\text{H}^*V - \mathcal{U}$) is $\text{H}^*V \otimes I$ which is reduced as unstable $A$-module.

Remark 3.2.3. In the previous result the condition (1): $E$ is free as an $\text{H}^*V$-module is necessary. In fact, the finite $\text{H} - A$-module $J_{\mathbb{Z}/2\mathbb{Z}}(1)$ is not free as an $\text{H}$-module and not reduced as an unstable $A$-module, however $J_{\mathbb{Z}/2\mathbb{Z}}(1) = F_2$ is a reduced unstable $A$-module. Observe that $J_{\mathbb{Z}/2\mathbb{Z}}(1)$ is isomorphic, as unstable $A$-module, to $F_2 \oplus \sum F_2$, the structure of $\text{H}$-module is given by: $t.\iota = \Sigma \iota$, where $\iota$ is the generator of $F_2$ and $t$ the generator of $\text{H}$.

Observe that the converse of corollary 3.2.2 is false. In fact, the $\text{H} - A$-module $E = H^{\geq 1}$ is reduced as unstable $A$-module however the unstable $A$-module $\overline{E} \cong \sum F_2$ is not reduced.

4 Description of $E$ when $\overline{E}$ is nil-closed

The main result of this paragraph concerns the relation between the two first terms of a (minimal) injective resolution of $E$ and $\overline{E}$.

Theorem 4.1. Let $E$ be an unstable $\text{H}^*V - A$-module which is free as an $\text{H}^*V$-module. We suppose that:

1. $\overline{E}$ is nil-closed.

2. $0 \to \overline{E} \to I_0 \to I_1 \to \cdots$ is the beginning of a (minimal) $\mathcal{U}$-injective resolution of $\overline{E}$. 

\[ \text{E} \quad \text{I}_1 \quad \text{I}_2 \quad \cdots \text{I}_n \quad \overline{\text{I}}_1 \quad \overline{\text{I}}_2 \quad \cdots \]
Then there exists an \( H^*V - A \)-linear map \( \varphi : H^*V \otimes I_0 \to H^*V \otimes I_1 \) such that:

1. \( 0 \to E \to H^*V \otimes I_0 \xrightarrow{\varphi} H^*V \otimes I_1 \to \cdots \) is the beginning of a (minimal) injective resolution of \( E \) (in the category \( H^*V - U \)).

2. \( \varphi = i_1 \)

Proof. The unstable \( A \)-module \( \overline{E} \) is nil-closed so is reduced, we have then, by theorem 3.2.1, that the injective hull of \( E \) is \( H^*V \otimes I_0 \). We denote by \( C_0 \) the quotient of \( H^*V \otimes I_0 \) by \( E \). We have the following short exact sequence in \( H^*V - U \):

\[
0 \to E \xrightarrow{i_0} H^*V \otimes I_0 \to C_0 \to 0.
\]

Since the induced map \( i_0 \) is an injection (see lemma 3.1.2), then the unstable \( A \)-module \( Tor^H^*V_1(\mathbb{F}_2, C_0) \) is trivial; this shows that the module \( C_0 \) is free as an \( H^*V \)-module (see for example [NS], proposition A.1.5).

We verify that the \( U \)-injective hull of \( C_0 \) is \( I_1 \) and that \( C_0 \) is reduced since \( \overline{C_0} \) is reduced (see corollary 3.2.2). This implies, by theorem 3.2.1, that the \( H^*V - U \)-injective hull of \( C_0 \) is isomorphic to \( H^*V \otimes I_1 \). \( \square \)

Remark 4.2. let \( M \) be a nil-closed unstable \( A \)-module and \( 0 \to M \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1 \to \cdots \) be the beginning of a (minimal) \( U \)-injective resolution of \( M \). We denote by \( (\text{Hom}_{H^*V - U}(H^*V \otimes I_0, H^*V \otimes I_1))_{i_1} \)

the set of \( H^*V - A \)-linear map \( \varphi : H^*V \otimes I_0 \to H^*V \otimes I_1 \) such that \( \varphi = i_1 \).

Using Lannes T-functor (see [LI]) we have:

\[
(\text{Hom}_{H^*V - U}(H^*V \otimes I_0, H^*V \otimes I_1))_{i_1} \cong (\text{Hom}_U(T_V I_0, I_1))_{i_1}
\]

where \( (\text{Hom}_U(T_V I_0, I_1))_{i_1} \) is the set of \( A \)-linear map \( \psi : T_V I_0 \to I_1 \) such that \( \psi \circ i = i_1 \), where \( i : I_0 \hookrightarrow T_V I_0 \) denotes the natural inclusion.
The kernel of any element $\psi \in (\text{Hom}_U(T_V I_0, I_1))_{i_1}$, which is free as an $H^*V$-module, is an unstable $H^*V - A$-module such that $\overline{\ker \psi} \cong M$.

**Remark 4.3.** If $E$ is an $U$-injective then the only unstable free $H^*V - A$-module, up to isomorphism, solution of the problem $(P)$ is $H^*V \otimes \overline{E}$.

Let $n$ be an even integer. The unstable free $H - A$-modules, up to isomorphism, solution of the problem $(P)$ when $M$ is $H^*BSO(n)$ are $H^*BO(n)$ and $H \otimes H^*BSO(n)$. We verify that these two $H - A$-modules are not isomorphic in the category $H - U$ (since it does not exist an $A$-linear section of the projection $H^*BO(n) \to H^*BSO(n)$).

## 5 Applications

### 5.1

Our first application concerns the determination of the mod. 2 cohomology of the mapping space $\text{hom}(B(\mathbb{Z}/2^n), Y)$ whose domain is a classifying space for the group $\mathbb{Z}/2^n$ and whose range is a space $Y$ such that $H^*Y$ is concentrated in even degrees.

We will just recall some facts, ignoring the $p$-completion problems. For further details see [DL].

One proceeds by induction on the integer $n$. Let us set

$$X = \text{hom}(E(\mathbb{Z}/2^n)/(\mathbb{Z}/2^{n-1}), Y).$$

The space $X$ has the homotopy type of $\text{hom}(B(\mathbb{Z}/2^{n-1}), Y)$ and is equipped of an action $\mathbb{Z}/2$ such that one has a homotopy equivalence

$$\text{hom}(B(\mathbb{Z}/2^n), Y) \cong X^{h\mathbb{Z}/2},$$
$X^{h\mathbb{Z}/2}$ denoting the homotopy fixed point space: $\text{hom}_{\mathbb{Z}/2}(E\mathbb{Z}/2, X)$. Using $\text{Fix}_{\mathbb{Z}/2}$-theory \[L1\], one gets:

$$H^*\text{hom}(B(\mathbb{Z}/2^n), Y) \cong \text{Fix}_{\mathbb{Z}/2} H^*_{\mathbb{Z}/2}X.$$ 

Since the computation of the functor $\text{Fix}_{\mathbb{Z}/2}$ on an unstable $H - A$-module is not difficult in general, the determination of the mod. 2 cohomology of the mapping space $\text{hom}(B(\mathbb{Z}/2^n), Y)$ is reduced to the determination of the unstable $H - A$-module $H^*_{\mathbb{Z}/2}X$. As we are going to explain, this last point is closely related to problem $(P)$.

One knows by induction on $n$ that the mod. 2 cohomology of the space $X$ as the one of the space $Y$ is concentrated in even degrees and one checks that the action of $\mathbb{Z}/2$ on $H^*(Y; \mathbb{Z})$ is trivial. These two facts imply that the Serre spectral sequence, for mod. 2 cohomology, associated to the fibration

$$X \to X^{h\mathbb{Z}/2} \to B\mathbb{Z}/2$$

collapses ($X^{h\mathbb{Z}/2}$ denotes the Borel construction $E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X$). This collapsing implies in turn that $H^*_{\mathbb{Z}/2}X$ is H-free and that $H^*_{\mathbb{Z}/2}X$ is isomorphic to $H^*X$. So the determination of $H^*\text{hom}(B(\mathbb{Z}/2^n), Y)$ is indeed reduced to the resolution of a problem $(P)$.

We conclude this subsection by a concrete example (we follow \[De\], section 6); we take $n = 2$ and $Y = \text{BSU}(2)$. Using $T_{\mathbb{Z}/2}$-computations one sees that $X$ has the homotopy type of $\text{BSU}(2) \coprod \text{BSU}(2)$; one checks also that the $\mathbb{Z}/2$-action preserves the connected components. The $(P)$-problem associated to the determination of the unstable $H - A$-module $H^*_{\mathbb{Z}/2}X$ is the following one:

Find the unstable $H - A$-modules $E$ such that

- $E$ is H-free;
the unstable A-module $E$ is isomorphic to $H^{*}{\text{BSU}}(2)$.

Using the fact that the injective hull, in the category $H\mathcal{U}$, of $E$ is $H \otimes H$ (see theorem 3.2), one checks that one has two possibilities:

- $E \cong H \otimes H^{*}{\text{BSU}}(2);$

- $E \cong H \otimes_{H^{*}{\text{BU}}(1)} H^{*}{\text{BU}}(2)$ (the structures of unstable $H^{*}{\text{BU}}(1) - A$-modules on $H = H^{*}{\text{BO}}(1)$ and $H^{*}{\text{BU}}(2)$ are respectively induced by the inclusion of $O(1)$ in $U(1)$ and the determinant homomorphism from $U(2)$ to $U(1)$).

5.2

The theorem 4.1 can be illustrated, topologically, as follows:

**Proposition 5.2.1.** Let $X$ be a CW-complex on which acts an elementary abelian group 2-group $V$. Suppose that:

1. $H^{*}X$ is nil-closed
2. $0 \rightarrow H^{*}X \rightarrow I_{0} \xrightarrow{\alpha} I_{1} \rightarrow \cdots$ is the beginning of a (minimal) $U$-injective resolution of $H^{*}X$
3. $H^{*}_{V}X$ is free as an $H^{*}V$-module.

Then there exists an $H^{*}V - A$-linear map $\varphi : H^{*}V \otimes I_{0} \rightarrow H^{*}V \otimes I_{1}$ such that:

1. $H^{*}_{V}X \cong Ker(\varphi)$. 
2. $0 \longrightarrow H^*_V X \longrightarrow H^* V \otimes I_0 \overset{\varphi}{\longrightarrow} H^* V \otimes I_1 \longrightarrow \cdots$ is the beginning of a (minimal) injective resolution of $H^*_V X$ (in the category $H^* V - \mathcal{U}$).

3. $\varphi = \alpha : I_0 \rightarrow I_1$.

In particular, we have:

**Corollary 5.2.2.** Let $X$ be a CW-complex on which acts an elementary abelian group 2-group $V$. Suppose that:

1. $H^* X$ is a reduced $\mathcal{U}$-injective,
2. $H^*_V X$ is free as an $H^* V$-module.

Then $H^*_V X \cong H^* V \otimes H^* X$.

### 6 Description of $E$ when $\overline{E}$ is isomorphic to $\sum^n \mathbb{F}_2$

In this section, we prove the following result.

**Theorem 6.1.** Let $E$ be unstable $H^* V - A$-module which is free as an $H^* V$-module. If $\overline{E}$ is isomorphic to $\sum^n \mathbb{F}_2$, then there exists an element $u$ in $H^* V$ such that:

1. $u = \prod_i \theta_i^{\alpha_i}$, where $\theta_i \in (H^1 V) \setminus \{0\}$ and $\alpha_i \in \mathbb{N}$
2. $E \cong \sum^d u H^* V$ with $d + \sum_i \alpha_i = n$. 
Proof. Let $N$ be an unstable $A$-module, we denote by $\dim N$ the total dimension of $N$ that is $\dim N = \sum_i \dim N^i$. We have the equality $\dim E = 1 = \dim Fix_\nu E$ (see [LZ3]), so we deduce that $Fix_\nu E = \sum^l \mathbb{F}_2$, where $l \in \mathbb{N}$. Let $\eta_\nu : E \to H^*V \otimes Fix_\nu E$ be the adjoint of the identity of $Fix_\nu E$ (see [LZ2]). Since the map $\eta_\nu$ is an injection, then the module $E$ is a sub-$H^*V - A$-module of $\sum^l H^*V$. Let’s write $E = \sum^l E'$, where $E'$ is sub-$H^*V - A$-module of $H^*V$. By a result of J-P. Serre (see [Se]), there exists $N$ such that: $c_N^V H^*V \subset E' \subset H^*V$. Since $E'$ is free as an $H^*V$-module and of dimension one, then there exists $u \in H^*V$ such that $E' = uH^*V$. The inclusion $c_N^V H^*V \subset uH^*V$ proves that $u = \prod_i \theta^\alpha_i$, where $\theta_i \in (H^1V) \setminus \{0\}$ and $\alpha_i \in \mathbb{N}$.

Remark 6.2. We remark that by the previous result, we can determine $E$ when $\overline{E}$ is isomorphic to $\mathbb{F}_2 \oplus \sum^n \mathbb{F}_2$. In this case, we verify that $E \cong H^*V \oplus \sum^d uH^*V$, where $u = \prod_i \theta^\alpha_i$, $\theta_i \in H^*V \setminus \{0\}$, $\alpha_i \in \mathbb{N}$ and $d + \sum_i \alpha_i = n$. In fact, since the $H^*V - U$-injective module $H^*V$ is a sub-$H^*V$-module of $E$, then $E \cong H^*V \oplus E'$, where $E'$ is an unstable $H^*V - A$-module, free as an $H^*V$-module and such that $\overline{E'} \cong \sum^n \mathbb{F}_2$. The result holds from theorem 6.1.

6.3 Example

We give an example showing how to realize topologically the cases of theorem 6.1 and remark 6.2.

Let $\rho : V \to O(d)$ be a group homomorphism. $\rho$ gives both an action of $V$ on $D^d$, $S^{d-1}$ and a $d$-dimensional orthogonal bundle whose mod.2 Euler class is denoted by $e(\rho)$.

The long exact sequence of the pair $(D^d, S^{d-1})$ and the Thom isomorphism give the long (Gysin) exact sequence (see for example [Hu]):

$$
\cdots \longrightarrow H^{* - 1}V \longrightarrow H^{* - 1}S^{d - 1} \longrightarrow \sum^d H^*V \overset{\cdot e(\rho)}{\longrightarrow} H^*V \longrightarrow H^*_V S^{d - 1} \longrightarrow \cdots
$$
The decomposition \( \rho \cong \oplus_{i=1}^{d} \rho_i \) of the representation \( \rho \) into orthogonal representations of dimension 1 gives \( e(\rho) = \prod_i e(\rho_i) \). We have now two cases.

- If none of the representations \( \rho_i \) is trivial then \( e(\rho) \) is non zero and \( H^*_V(\mathcal{D}^d, S^{d-1}) \) is isomorphic to \( e(\rho)H^*V \) as an \( H^*V - A \)-module. This illustrates theorem 6.1.

- Otherwise, let’s write \( \rho = \sigma \oplus \tau \), \( \sigma \) (resp. \( \tau \)) being the direct sum of the non trivial (resp. trivial) representations \( \rho_i \). Then \( H^*_V S^{d-1} \cong H^*V \oplus \Sigma \dim \tau e(\sigma)H^*V \) and \( H^*_V(S^{d-1}) \) is an illustration of the remark 6.2.

### 7 Determination of \( E \) when \( V \) is \( \mathbb{Z}/2\mathbb{Z} \) and \( \overline{E} \) is \( J(2) \)

In this section, we assume that \( V \) is \( \mathbb{Z}/2\mathbb{Z} \) and \( \overline{E} \) is the Brown-Gitler module \( J(2) \).

We denote by \( H = \mathbb{F}_2[t] \) the cohomology of \( \mathbb{Z}/2\mathbb{Z} \), where \( t \) is an element of \( H \) of degree one. We have the following result.

**Proposition 7.1.** Let \( E \) be an \( H - A \)-module which is \( H \)-free and such that \( \overline{E} \) is isomorphic to \( J(2) \) then:

\( E \cong H \otimes J(2) \)

or

\( E \) is the sub-\( H - A \)-module of \( H \oplus \sum H \) generated by \((t, \Sigma 1)\) and \((t^2, 0)\).

**Proof.** This proof uses the Smith theory (see [DW], [LZ2] theorem 2.1) which gives us an exact sequence \((*)\) in \( H - \mathcal{U} \):

\[
(*) \quad 0 \longrightarrow E \overset{\eta}{\longrightarrow} H \otimes \text{Fix}E \longrightarrow C \longrightarrow 0
\]
where $C$ the quotient of $H \otimes \text{Fix}E$ is finite and also $\text{Fix}E$ is finite.

If the module $C$ is trivial then $E$ is isomorphic to $H \otimes J(2)$.

When $C$ is a non trivial module. By applying the functor $\mathbb{F}_2 \otimes H$ to the exact sequence (*), we obtain:

$$0 \longrightarrow \sum \tau C \longrightarrow \overline{E} = J(2) \longrightarrow \text{Fix}E \longrightarrow C \longrightarrow 0$$

where $\tau C$ is the trivial part of $C$ (see [BHZ]).

Let’s denote by $Q$ the quotient of $E$ by $\sum \tau C$. By properties of the module $J(2)$, we have that $\sum \tau C = \sum^2 \mathbb{F}_2$ and $Q = \sum \mathbb{F}_2$. The exact sequence:

$$0 \longrightarrow \sum \mathbb{F}_2 \longrightarrow \text{Fix}E \longrightarrow C \longrightarrow \sum \mathbb{F}_2$$

gives that $\text{Fix}E \cong \sum \mathbb{F}_2 \oplus C$. One checks that the module $C$ is either isomorphic to $\mathbb{F}_2$ or $\sum \mathbb{F}_2$. If $C = \sum \mathbb{F}_2$ then $\text{Fix}E \cong \sum \mathbb{F}_2 \oplus \sum \mathbb{F}_2$ as an unstable $A$-module, which implies that the module $E$ is a suspension which is impossible because $E = J(2)$ is not a suspension. We conclude that $C = \mathbb{F}_2$. Since $\tau C = \sum \mathbb{F}_2$ then we get $C$ is isomorphic to $H^\leq 1$, where $H^\leq 1$ denotes the sub-$H - A$-module of $H$ consisting of elements of degree less or equal than 1. We have the following exact sequence in $H - \mathcal{U}$:

$$0 \longrightarrow \sum \mathbb{F}_2 \longrightarrow H \oplus \sum H \longrightarrow H^\leq 1 \longrightarrow 0.$$ 

The module $E$, we are searching for, is the kernel of $\varphi$ and we check that it is the sub-$H - A$-module of $H \oplus \sum H$ generated by the elements $(t, \Sigma 1)$ and $(t^2, 0)$.

Remark 7.2. Let be $\mathbb{Z}/2\mathbb{Z}$ act on a real projective space $\mathbb{R}P^2$; let $x_0$ be a fixed point of this action (the set of fixed point is not empty for example by an argument of Lefschetz number). We have:

- The Serre spectral sequence collapses to give that: $H^*_V(\mathbb{R}P^2, x_0)$ is $H$-free and $H^*_V(\mathbb{R}P^2, x_0)$ is isomorphic to $J(2)$.

- In [DW], Dwyer and Wilkerson have shown that $H^*_V \mathbb{R}P^2 = \mathbb{F}_2[t, y]/(f)$
where \( y \) restricts to \( x \) and \( f = y^i(y + t)^j \) for \( i + j = 3 \). It is easy to check that this computation agrees with theorem 7.1.

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