Dynamical Stability of Coreless Vortex States in $F = 1$ Spinor Bose-Einstein Condensates

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Abstract. We investigate the dynamical stability of the coreless vortex state in $F = 1$ spinor Bose-Einstein condensates, by numerically solving the Gross-Pitaevskii and Bogoliubov-de Gennes equations. We cover both ferromagnetic and antiferromagnetic interaction regions and the magnetizations per particle $M$ from $-1$ to $+1$. It is found that the coreless vortex state is dynamically stable in large parameter regions of magnetization $M$. At $M = -1$, the hyperfine spin component with vortex winding 2 is dominant, and hence the coreless vortex can spontaneously decay even without dissipation. However, it is found that in the ferromagnetic case, the spin-spin interactions tend to stabilize the state. In the antiferromagnetic case, we find new dynamical instability modes which are not obtained in the ferromagnetic interaction region. In the antiferromagnetic interaction region, the spin-spin interactions can stimulate dynamical instabilities, while there still remains a large stability area in the parameter $M$. This is because of certain restrictions in inducing dynamical instability.

1. Introduction
Since the realization of Bose-Einstein condensates (BECs) in cold atom system, a huge amount of studies have been done to understand the macroscopic quantum phenomena of these systems. One of the main topics of this field has been the so-called spinor BEC whose order parameter has multiple components due to the hyperfine spin $F$ [1, 2]. This system has brought up new paradigms. For example, many possibilities of the stationary vortex states arise in spinor BECs with $F = 1, 2$ [3, 4, 5, 6]. The coreless vortex state, which we discuss in this paper, is the one of the interesting phenomena which cannot appear in the scalar BEC. Due to the spin degrees of freedom, the vortex state realized in the spinor BEC becomes fundamentally different from the singular vortex state in scalar BECs [4]. Experimentally, the coreless vortex state can be realized in $^{23}$Na atoms with $F = 1$ by a phase imprinting method [7].

In this paper, we focus on the dynamical stability of the coreless vortex state. In the previous works, the ground state phase diagram of this system in a plane of the external rotation frequency $\Omega$ and the magnetization $M$ has been established by minimizing the Gross-Pitaevskii (GP) energy functional [4]. Our method is based on the Bogoliubov-de Gennes (BdG) equation which describes the stability of the solution of the GP equation.

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Let us clarify the difference between the dynamical stability and instability. A complex eigenvalue in the excitation spectrum makes the stationary state unstable which is called dynamical instability [8]. In the dynamical instability region, the complex-frequency excitation modes either grow or damp exponentially in time even in the absence of dissipation. This evolution satisfies the conservation of both, the total energy and the angular momentum. Without the complex eigenvalue, the state is dynamically stable, because the system does not break spontaneously.

The order parameter of the $F=1$ spinor BEC has three components, which are described by fields $\phi_1$, $\phi_0$, and $\phi_{-1}$. In a stationary state, the coreless vortex is described by a combination of winding numbers $\langle w_1, w_0, w_{-1}\rangle = (0, 1, 2)$, where $w_i$ is the winding number of $\phi_i$ [4, 9]. Since the coreless vortex state accompanies a component with winding 2, which is dynamically unstable in scalar BECs [10], the coreless vortex can easily disintegrate at the $M = -1$ limit. However, it is found that in the case of the ferromagnetic interaction, the repulsive interaction between components suppress its dynamical instabilities. Namely the $m_F = 1$ component, which fills in the vortex core of $\phi_{-1}$, can keep the winding 2 vortex dynamically stable. On the other hand, it is demonstrated that the antiferromagnetic interaction can stimulate the dynamical instability. In this work, we find a new dynamical instability driving the condensate modes towards phase separation, which does not exist in the ferromagnetic interaction region.

2. System

The time-dependent GP equation for the condensate wave function $\Psi_i(\mathbf{r}, t)$ with spin $i$ is derived from the $F = 1$ spinor BEC Hamiltonian [1, 2] to be

$$\frac{i}{\hbar} \frac{\partial \Psi_i(\mathbf{r}, t)}{\partial t} = \left[ -\frac{1}{2} \nabla^2 + V_{\text{trap}}(\mathbf{r}) - \mu_i + g_n \sum_j |\Psi_j(\mathbf{r})|^2 \right] \Psi_i(\mathbf{r}, t) + g_s \sum_{j,k,l} \sum_{\alpha=x,y,z} (F_{\alpha})_{ij}(F_{\alpha})_{ik} \Psi_j^*(\mathbf{r}, t) \Psi_k(\mathbf{r}, t) \Psi_l(\mathbf{r}, t).$$

We first solve the above GP equation to get a stationary state. Then we solve the BdG equation which describes linear response to the small perturbation about the GP solution [1, 2],

$$\Psi_i(\mathbf{r}, t) = \phi_i(\mathbf{r}) + \lambda \left( u_{q_i}(\mathbf{r}) e^{iE_q t} - v_{q_i}(\mathbf{r}) e^{-iE_q t} \right).$$

Here we normalize the length by $d \equiv \sqrt{1/m\omega}$ and energy by the trap frequency $\omega$, where $m$ is the mass of the atom. We also set $\hbar$ to unity. Hereafter this notation is used ubiquitously. And $(F_{\alpha})_{ij}$ is the $(i, j)$ component of the spin matrix $F_{\alpha}$ ($\alpha = x, y, z$) for hyperfine spin $F=1$ system. The suffixes $\{i, j, k, l\}$ take values 1, 0, $-1$. The BEC is confined in a two-dimensional plane with the harmonic trap potential in the radial direction $V_{\text{trap}}(r) = 1/2 r^2$. We consider a cylindrically symmetric vortex state.

In our calculation, we choose the coupling constant of the spin-spin interactions $g_s = \pm 0.001, \pm 0.01$, and the density-density interaction coupling constant is fixed to $g_n = 0.113$. The values of $g_s = 0.001$ and $-0.001$ are comparable to that in the actual system of $^{23}\text{Na}$ and $^{85}\text{Rb}$ atomic BECs, respectively. We also fix the total number of the atoms per unit length along $z$-axis to $N = 1.5 \times 10^6$.

In our calculation, the magnetization $M \equiv \int d\mathbf{r} \left( |\phi_1|^2 - |\phi_{-1}|^2 \right) / \sum_i |\phi_i|^2$ is varied from $-1$ to 1. In $M = -1$ limit, the $(0, 1, 2)$ coreless vortex state has a finite amplitude only in component $\phi_{-1}$ with $w_{-1} = 2$. In this limit, the coreless vortex state coincides with the winding 2 vortex state in scalar BEC and is dynamically unstable in the parameter space we employ [8, 10]. In the opposite limit $M = 1$, the coreless vortex is the nonrotating scalar BEC, which is the ground state. We focus on how the excitation spectrum — especially the part which causes dynamical instability — changes as a function of the magnetization $M$ and the spin-spin interaction.
3. Results

First, we explain how the dynamical instabilities appear in the system. Figure 1 shows the excitation spectrum at \( g_s = -0.01 \) and \( M = -0.90 \). The filled triangle at \( (q_\theta = 0, E_\mathbf{q} = 0) \) corresponds to the GP solution. The complex modes appear at \( (q_\theta = -2, -|\text{Re}[E_\mathbf{q}]|) \) and \( (q_\theta = 2, +|\text{Re}[E_\mathbf{q}]|) \), which are also labeled with filled triangles. This is because the total energy \( E \) and the angular momenta \( q_\theta \) are conserved. These complex modes cause the dynamical instabilities. Here, the two condensed bosons with \( q_\theta = 0 \) and \( E_\mathbf{q} = 0 \) are able to spontaneously break into two quasiparticles with \( (q_\theta = -2, -|\text{Re}[E_\mathbf{q}]|) \) and \( (q_\theta = 2, +|\text{Re}[E_\mathbf{q}]|) \). The threshold of the difference of the energy \( |\text{Re}[E]| \pm \epsilon \) for resonance can be calculated using the same analysis as in Ref. [8].

![Figure 1](image1.png)

**Figure 1.** The low energy excitation spectrum at the ferromagnetic interaction \( g_s = -0.01 \) and \( M = -0.90 \). This figure shows the real part of the eigenvalues depending on the angular momentum \( q_\theta \). The filled triangles correspond to the GP solution \( (q_\theta = 0, E_\mathbf{q} = 0) \) and complex modes \( (q_\theta = \pm 2, \text{Re}[E_\mathbf{q}] = \pm 1.75) \). The stationary state derived by GP equation can decay to different two states with the total energy and the angular momentum conservation.

![Figure 2](image2.png)

**Figure 2.** Imaginary part of the complex eigenvalues as a function of magnetization \( M \). Each graph corresponds to a different coupling constant: (a) \( g_s = -0.001 \), (b) \( g_s = -0.01 \), (c) \( g_s = 0.001 \), and (d) \( g_s = 0.01 \). For the ferromagnetic interactions, there are no imaginary frequencies outside the range of \( M \) shown in panels (a) and (b).

Figure 2 shows the imaginary part of the eigenenergy \( E_\mathbf{q} \) with \( q_\theta = \pm 1, \pm 2 \). For both the ferromagnetic and antiferromagnetic interactions, complex eigenvalues appear near \( M = -1 \), which causes dynamical instabilities towards the splitting of the doubly quantized vortex core in \( \phi_{-1} \). At the \( M = -1 \) limit, the results agree with the winding \( w = 2 \) vortex state in the scalar BEC [8]. The complex modes disappear with increasing \( M \). The eigenmode with \( (q_\theta = -2, -|\text{Re}[E_\mathbf{q}]|) \) and with \( (q_\theta = 2, |\text{Re}[E_\mathbf{q}]|) \) behave differently as a function of \( M \). The intuitive interpretation in the ferromagnetic interaction is that the \( m_F = 1 \) component \( \phi_1 \) fills the vortex core, which acts as a pinning potential for the doubly quantized vortex of \( \phi_{-1} \). Due to the repulsive interaction between \( m_F = 1 \) and \( m_F = -1 \) components, the state becomes dynamically stable. In increasing \( M \) further, complex modes appear again. At these regions, the complex modes arise with the same mechanism as explained above.
In the ferromagnetic case, only the states with \( q_\theta = \pm 2 \) causes dynamical instability at near \( M = -1 \). On the other hand, in the antiferromagnetic case shown in Fig. 2 (c) and (d), additional complex excitations appear with \( q_\theta = \pm 1 \). These modes also satisfy the total energy and the angular momentum conservations.

In Fig. 3, we show the density profiles of the coreless vortex states excited by \( q_\theta = -2 \) and \( q_\theta = -1 \) complex modes. The \( q_\theta = -2 \) mode in Fig. 3 (a) and (b) breaks the winding 2 vortex into a pair of winding 1 vortices. On the other hand, \( q_\theta = -1 \) complex mode leads to the phase separation between \( \phi_{\pm 1} \) and \( \phi_0 \) domains. As seen in Fig. 3 (c) and (e), \( m_F = 1 \) and \( m_F = -1 \) components tend to spatially overlap with each other, which is attributed to the attractive interaction between \( m_F = 1 \) and \( m_F = -1 \) components due to the antiferromagnetic interaction. By this characteristics of the antiferromagnetic coupling constant, dynamical instabilities are stimulated in the extensive region of \( M \) in contrast to the case of the ferromagnetic region in which the dynamical instabilities tend to be suppressed.

![Figure 3. The density profiles of the complex modes.](image)

**Figure 3.** The density profiles of the complex modes. The top row corresponds to \( M = -0.80 \) and \( q_\theta = -2 \) mode, and the bottom row corresponds to \( M = 0.40 \) and \( q_\theta = -1 \) mode. From the left column, \( |\Psi_1|^2, |\Psi_0|^2, |\Psi_{-1}|^2 \) are shown. \( |\Psi_0|^2 \) of the top row is neglected because the amplitude is vanishingly small. Both modes are derived in \( g_s = 0.01 \) antiferromagnetic region.

4. Conclusion

We have studied the stability of the coreless vortex state for \( F = 1 \) spinor Bose-Einstein condensates. By numerically solving Gross-Pitaevskii and Bogoliubov-de Gennes equations, the low energy excitation spectrum has been derived in \(-1 \leq M \leq 1\) and in both ferromagnetic and antiferromagnetic interactions. It has been found that complex eigenmodes appear in the coreless vortex state, which cause dynamical instability near \( M = -1 \) in both interactions and in a broad \( M \) region in the antiferromagnetic interaction with increasing \( M \). The dynamical instability is suppressed for ferromagnetic interactions, while it is stimulated in the antiferromagnetic interaction due to the spin-spin interaction characteristic. In the antiferromagnetic region, we find additional complex modes with \( q_\theta = \pm 1 \) which do not exist in ferromagnetic region. However, it is concluded that the coreless vortex state is dynamically stable in a broad region of \( M \) for the both interactions due to the restrictions in the appearance of the complex modes.

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