The Frustration in being Odd: area law violation in local systems

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We demonstrate the existence of a new quantum phase of matter that arises in antiferromagnetic spin chains with a \textit{weak frustration} –just one bond in a large chain. This is the case, for instance, of systems with an odd number of spins with periodic boundary conditions. Such new phase is extended, gapless, but not relativistic: the low-energy excitations have a quadratic (Galilean) spectrum. Locally, the correlation functions on the ground state do not show significant deviations compared to the non-frustrated case, but correlators involving a number of sites (or distances) scaling like the system size display new behaviors. In particular, the von Neumann entanglement entropy is found to follow new rules, for which neither area law applies, nor one has a divergence of the entropy with the system size. Such very long-range correlations are novel and of potential technological interest. We display such new phase in a few prototypical chains using numerical simulations and we study analytically the paradigmatic example of the Ising chain. Through these examples we argue that this phase emerges generally in (weakly) frustrated systems with discrete symmetries.

Frustration is the result of competing interactions, each favoring a different type of ordering, so that not all terms in the Hamiltonian can be minimized simultaneously. Notice that any genuine quantum phase includes some amount of frustration, since non-commuting terms in the Hamiltonian clearly promote contrasting local arrangements\cite{1,2}. Frustration and entanglement are strictly related: from one side, monogamy of entanglement\cite{3,4} implies that entanglement cannot be maximized between every component of a system and thus that a trade-off has to be established, and from the other side the superposition of states resulting from the trade-off due to frustration may generate entanglement. However, with the term frustration one usually refers to the so-called “\textit{geometrical frustration}”, which emerged first in classical systems. Prototypical are models characterized by antiferromagnetic/repulsive (AFM) interactions with closed loops of odd lengths or by a competition between nearest and next-to-nearest neighbor AFM interactions. In a frustrated quantum system, geometrical and quantum frustration are in general intertwined and it is not easy to discern between the different effects of the two sources.

The simplest toy model to visualize geometrical frustration are three spins arranged on the vertices of a triangle, with AFM couplings along the bonds. In a classical system with Ising variables as magnetic moments, the interaction cannot be minimized simultaneously on all bonds, resulting into a six-fold degenerate ground state corresponding to the three positions of the defect spin times the two orientations it can take. It is easy to generalize these considerations for longer spin loops with nearest-neighbor AFM bonds: while on even chains one of the two Néel states of alternating spins minimizes all local interactions (and thus the whole Hamiltonian), for loops of odd lengths \( N = 2M + 1 \), one bond avoid minimization, resulting into a \( 2N \) degenerate ground state. Promoting the magnetic moments from Ising variables to three-dimensional spins does not alleviate the frustration (for sufficiently large systems), still resulting into a ground-state degeneracy scaling like the system length\cite{5}. It is worth noticing that adding a single site to an AFM loop changes the system dramatically, turning a double degeneracy into a massive one and vice-versa, thus demonstrating that the effect of frustration is non-perturbative in nature.

A certain degree of frustration is a common phenomenon in most realistic systems and can give rise to peculiar properties, such as the aforementioned massive ground-states degeneracies. Nonetheless, one expects that an extensive amount of frustration is necessary to produce a non-trivial phase with a non-trivial behavior. This is the case, for instance, both for regular (lattice) systems, such as the ANNNI model\cite{6} or spin ices\cite{7}, and for disordered systems, such as the Sherrington-Kirkpatrick model\cite{8} and general spin glasses\cite{9}. All these examples show unique behaviors different from those of unfrustrated systems, such as algebraic decay of correlation functions without criticality\cite{10,11}, local zero-modes\cite{12,13} and residual entropy at near-zero temperature\cite{14,15}, and give rise to peculiar emergent properties, such as artificial electromagnetism\cite{10,11} and light with parameter depended velocity, monopoles and Dirac strings\cite{18}.
It is possible to quantify the amount of frustration both for classical and quantum systems, and all the aforementioned examples show extensive frustration, meaning that the amount of conflicting terms in the Hamiltonian grow together with the dimension of the lattice, or that the number of AFM loops contributing to frustration keeps growing \[19, 20\].

In this work we concentrate on a class of systems with nonextensive frustration, whose effect has been largely ignored until now. We show that even such weak frustration can lead to a new gapless phase, with a quadratic spectrum of excitations and very long-range correlations. We characterize these behaviors also using (by now traditional) quantum information theory techniques, revealing a peculiar violation of the area law \[21\] for the entanglement entropy, which yet does not result into its divergence for large systems. This is because the peculiarities of these systems emerge in a scaling limit in which distances are scaled with the total system size. Our results prove the existence of a new phase, robust against perturbation and with universal behavior, characterized by a finite amount of entanglement, but with correlations spread throughout the whole system.

I. WEAKLY FRUSTRATED SPIN CHAINS

Let us introduce a generic nearest-neighbor one-dimensional spin-$\frac{1}{2}$ spin chain with $N$ spins in a magnetic field:

$$
H = \frac{J}{2} \sum_{l=1}^{N-1} \left[ \left( \frac{1+\gamma}{2} \right) \sigma_l^x \sigma_{l+1}^x + \left( \frac{1-\gamma}{2} \right) \sigma_l^y \sigma_{l+1}^y \right] + \frac{\Delta}{2} \sum_{l=1}^{N-1} \sigma_l^z \sigma_{l+1}^z - \sum_{l=1}^{N} h \sigma_l^z
$$

(1)

where $\sigma_l^\alpha$, with $\alpha = x, y, z$, are Pauli matrices which describe spin-$1/2$ operators on the $l$-th lattice site of the chain. The nearest neighbor interactions and the external magnetic field are non-commuting terms that favor different kinds of orders, thus providing a quantum nature to this model. Setting $J_N = J$ restores translational invariance with periodic boundary conditions. Choosing $J = 1$ (up to an energy scale) favors AFM order. On an odd periodic lattice $N = 2M + 1$, this order shows both classical and quantum sources of frustration. To prove this, we can set $h = 0$ and see that the system does not satisfy the quantum Toulouse conditions \[1, 2\]. For a system with a continuous $U(1)$ symmetry (for instance, the $XXZ$ chain obtained by setting $\gamma = 0$ in \[1\]), the effect of such frustration is trivial: while for even lengths $N = 2M$ the ground state can achieve zero total magnetization $S_T^z = 0$, in the frustrated case $N = 2M + 1$ there are two equivalent ground states with $S_T^z = \pm \frac{N}{2}$ (whose degeneracy is immediately lifted for a nonzero $h$).

In \[23\], Campostrini et al. considered the odd length ($N = 2M + 1$), ferromagnetic Ising chain, obtained by settings $J = -1$, $\gamma = 1$, and $\Delta = 0$ in \[1\]. When the defect $J_N$ differs from $J$, it breaks translational invariance and for $J_N > 0$ favors AFM order along the $x$-direction between the first and last spins of the chain. If $J_N < 0$, they found that, for $|h| < 1$, the point $J_N = 1$ represents a quantum critical point separating two different phases for $J_N \lesssim 1$. Notice that on this point $J_N = 1$ the model can be mapped into the translational invariant AFM Ising chain using local rotations on the set of even spins. The authors connect this critical behavior to the metastability of this model under the perturbation provided by a longitudinal magnetic field $\delta H = h_x \sum_{l=1}^{N} \sigma_l^x$. In fact, it is known that the point $h_x = 0$ corresponds to a first order phase transition \[23\, 26\].

The algebraic decay of the correlation functions at $J_N = 1$ derived in \[25\] was reexamined in \[27\] where Dong et al. focused on the translational invariant version of the same model. In this way, the defect is not localized at the “end” of the chain, but it is rather a frustration due to a AFM loop of odd length. We will discuss the solution of this model below, but what was observed in \[27\] is that this weak frustration is sufficient to significantly scramble the energy spectrum. In this case, for $|h| < J$, the ground-state is unique with a band of $2N - 1$ levels above it, forming a gapless continuum in the thermodynamic limit. This model can be mapped exactly into a system of free fermions also in this frustrated phase, so that various calculations can be carried out analytically.

Quite interestingly, we notice that there are two types of correlation functions: some, which we deem “quasi-local”, characterized by a dependence of the distance $R$ scaling like $N$, and some, “local”, without this.
interplay. These two families give rise to an intriguing interplay of correlations decaying algebraically and exponentially. These behaviors are exemplified by the two-point spin correlation functions. Extending the results of [27] we have

\[ C_{\text{xx}}(R) \equiv \langle \sigma_{i}^{x} \sigma_{i+R}^{x} \rangle = (-1)^{R} \left( 1 - \frac{h^2}{J^2} \right)^{1/4} \left[ 1 + \frac{c^{2}(h)}{R^2} \left( \frac{h^2}{J^2} \right)^{R} \right] \left( 1 - \frac{2R}{N} \right), \]  

(2)

\[ C_{\text{zz}}(R) \equiv \langle \sigma_{i}^{z} \sigma_{i+R}^{z} \rangle = m_{z}^{2} - \frac{c_{1}^{2}(h)}{R^2} \left( \frac{h^2}{J^2} \right)^{R} + \frac{4m_{z}}{N} \left[ 1 + \frac{c_{2}^{2}(h)(-1)^{R}}{R} \right] \left( \frac{h}{J} \right)^{R}, \]  

(3)

where the exact form of \( c^{2}(h) \) and \( c_{1,2}^{2}(h) \) is not relevant for our considerations. \( C_{\text{xx}}(R) \) belongs to the class of quasi-local correlation functions, while \( C_{\text{zz}}(R) \) is local. If one first takes the thermodynamic limit \( N \to \infty \), these functions reduce to the standard ones of the Ising chain [28], which decay exponentially to saturation, with correlation length \( \xi = -\frac{1}{\ln(|h|/j^2)} \).

However, this procedure does not allow for a correct evaluation of the spontaneous longitudinal magnetization. In fact, the proper prescription [28] is to first evaluate (2) at antipodal points \( (R \approx N/2) \) and only at the limit \( N \to \infty \) is performed. Doing so, we find \( \lim_{N \to 0} C_{\text{xx}} \left( \frac{N-1}{2} \right) = 0 \) because of the slow algebraic decay in (2), implying \( \langle \sigma^{x} \rangle = 0 \). This is a surprising result, since a nonvanishing longitudinal magnetization is the hallmark of the \( Z_{2} \) spontaneous symmetry breaking, for which the Ising model is the poster-child. While a finite magnetization along the \( x \)-direction is observed for ferromagnetic and AFM chains with even length, the frustrated AFM chain with odd length shows a striking different behavior. Indeed, this is consistent with the aforementioned fact that the ground state remains unique even for \( |h| < 1 \), unlike in the standard case.

Thus, while locally the correlation functions of the frustrated AFM Ising chain are indistinguishable from those of the unfrustrated version, at large distances important differences emerge. To capture this diversity one should consider an improved thermodynamic limit, in which distances are measured in terms of the chain length: \( r \equiv \frac{R}{N} \), which is kept fixed as \( N \to \infty \). Under this improved limit, quasi-local correlation functions such as (2) are characterized by an algebraic decay, as if \( \xi = \infty \).

From a technical point of view, the difference between local and quasi-local correlators is that the latter involve in their computation a number of quasi-particle excitations growing with the distance, while for the former this number stays constant. However, this explanation depends on the specific features of the Ising chain and one naturally wonders how robust the behaviors we discussed are under generic perturbations. In order to address this question we explore a wide range of parameters for the generic spin chain in (1) using DMRG simulations.

II. THE ENTANGLEMENT ENTROPY

Also, to better understand this non-trivial frustrated phase, we look at the entanglement entropy (EE), as a probe into the ground state structure in this phase and as a representative on an explicitly non-local correlation functions. To do so, we divide the system into two parts: a subsystem \( A \) consisting on \( R \) contiguous sites and its complement \( B \) with \( N-R \) spins. We extract the reduced density matrix \( \rho_{A}(R) = \text{tr}_{N-R}(GS) \langle GS \rangle \) of subsystem \( A \), which is a mixed density matrix because of the severed correlations between \( A \) and \( B \). One of the standard measures of entanglement we will use is the Von Neumann entropy [29] [30], defined as

\[ S_{A}(R) = -\text{tr}_{A} \left[ \rho_{A}(R) \ln \rho_{A}(R) \right]. \]  

(4)

Nowadays, von Neumann entropy is a ubiquitous tools in different areas of physics which provides fundamental information on a quantum phase [31]. It typically follows some universal behaviors for sufficiently large subsystems: while for high energy states it is proportional to the volume of \( A \) for ground-states of systems with local interactions it is expected to satisfy an area law, with possible logarithmic violations for critical phases (with Fermi surfaces). Intuitively, the area law stems from the fact that entanglement reflects the correlations shared between the subsystems and these are localized in a shell of the order of a few correlation
FIG. 1. Comparison between the EE of standard phases (gapped and CFT critical) and that of the weakly frustrated case, showing the distinct different behavior of the latter with a violation of the area law. The EE \( S_A(R) \) for the reduced density matrix evaluated on a block of \( R \) adjacent spins is plotted as function of \( R \) for total chain length \( N = 501 \) and different sets of Hamiltonian (1) parameters. In considering finite-size systems, it is customary to plot the entropy as a function of \( x \equiv \frac{N}{\pi} \sin \frac{\pi R}{N} \), in order to account for the periodic boundary condition and the symmetry of the entropy around \( R = 2/N \), but here we prefer to show the raw data.

lengths around the boundaries. For gapless systems, correlations extend with an algebraic decay, resulting into a dependence on the subsystem size. Thus, in one dimension, the EE of the ground state should either saturate to a constant [21,22] (since the boundary area is just two points) or show the characteristic universal behavior \( S(R) \approx \frac{c}{6} \ln R \) of conformal field theories (CFTs) with central charge \( c \) [32].

As we mentioned, the frustrated Ising chain for \(|h| < J\) is found to be gapless: this fact and the algebraic decay of some correlation functions point against an area-law behavior. On the other hand the spectrum of low energy excitations is quadratic (Galilean) and thus violates relativistic invariance of CFT and hence we have no reason to expect the presence of a logarithmic divergence of the EE [33]. We present in Fig. 1 the typical behavior we observe for the frustrated phase, together with the area-law saturation of the corresponding ferromagnetic system and the logarithmic divergence at CFT criticality. For comparison:

1. For small \( R \), compared to the correlation length of the correspondent ferromagnetic model, (i.e. the model obtained changing \( J \) from 1 to \(-1\)), the EE of the ferromagnetic and the antiferromagnetic systems almost coincide.

2. Increasing \( R \) in the unfrustrated case the EE indeed saturates quickly. In contrast, the frustrated chains still show a growth which is well fitted, in the bulk, by an empirical \( S_A(R) \approx a(N)R^{b(N)} \) where the fitting parameters depend on \( N \) as well as on the Hamiltonian ones (Fig. 2). We will come back to this point later. Notice that the dependence on \( N \) of the EE prevents it from diverging in the thermodynamic limit.

3. In fact, the saturation of the EEs in the limit of large \( N \) can be appreciated in Fig. 3. In the spirit of the improved thermodynamic limit introduced before, we keep the size of the subsystem \( A \) equal to a fixed ratio \( r = R/N \) of the total length of the chain and plot the EE as \( N \) is increased. In such a limit, we observe, that the frustrated models show a behavior of the EE proportional to the inverse of the system size: \( S_A(N) \approx a_r + \frac{b_r}{N} \), which saturates for large sizes.
a) Ising: $J=1, \gamma=1, \Delta=0., h=0.2$

b) $XZ + h_z$: $J=1, \gamma=1, \Delta=0.1, h=0.8$

c) $XYZ + h_z$: $J=1, \gamma=0.9, \Delta=0.1, h=0.8$

$\log S_A(R)$

FIG. 2. Area Law violation in the weakly frustrated phase. The dependence of the $S_A(R)$ on $N$ is plotted in log-log plot to show that in the bulk it follows a power-law of the type $S_A(R) \simeq a(N) R^{b(N)}$, shown as a dashed grey line.

$\log[1/N] S_A(R)$

FIG. 3. Dependence of the EE $S_A(R)$ on $N$ while keeping the ratio $r = R/N$ constant, in the weakly frustrated phase, for different Hamiltonian parameters. In all the panels the black squares stands for the results obtained keeping $r = 1/5$, the red points for $r = 1/3$ and the blue diamonds for $r = 9/20$ while the lines represent for the best fit obtained with a function of the form $a_r + \frac{b_r}{N}$.
In these plots, we collected data from different point in the phase-space of the generic spin system \[ \mathbf{1} \], including the Ising chain, the XY-chain in a longitudinal magnetic field, and the XYZ-chain in an external magnetic field. While the Ising chain is akin to a free model, the last two are not even integrable (and have been obtained through extensive DMRG simulations). The qualitatively similar behaviors in all these different models is evident. All the numerical results for the nonintegrable models were obtained by ordinary DMRG\[34\]. In our DMRG computations, we have considered up to 300 kept states to represent the truncated Hilbert space of each DMRG block. Typically, the truncation error is smaller than \(10^{-12}\).

This agreement can also be made quantitative. Collecting all entropy saturation points in the \(N \to \infty\) limit for the different values of the parameters in the same plot, we observe in Fig. 4, that they fall on the same universal curve, for which we empirically extract two equivalent analytical representations\[\] \[\]

\[
S_A(r) = a_{1/2} \left[ \frac{1}{2} + \sqrt{2} \left( r(1-r) \right)^{3/4} \right] = a_{1/2} \left[ \frac{2}{\pi} E \left(1 - 2r\right) \right]^{3/2}. \tag{5}
\]

Here \(a_{1/2}\) is a constant that depends on the Hamiltonian parameters (corresponding to the saturation point determined in Fig. 3), \(r = R/N\), and \(E(x)\) is the complete elliptic integral of the second kind

\[
E(x) \equiv \int_0^1 dt \frac{\sqrt{1-x^2t^2}}{\sqrt{1-t^2}}. \tag{6}
\]

This result is in strong contrast both with the divergence shown by standard (CFT) critical models and with the exponential convergence to a constant value that is found in systems satisfying the area law. Note the first equality in (5) provides an algebraic dependence of the EE on the system size, which is a simple

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1The two functions \(E(\sqrt{1-x^2})\) and \(\frac{\pi}{2} \left[ \frac{1 + x^{1/2}}{x} \right]^{2/3} \) are almost indistinguishable and relate the 2nd complete elliptic integral to the arc length of an ellipse\[35\] \[36\].
FIG. 5. Frustration as function of the size of the system, for the two phases of the Ising model. The blue(red) points/lines represent the amount of frustration of a single interaction in the AFM(Ferromagnetic) case, respectively, while the black curve is their difference \( g_F \), representing the amount of geometrical frustration. The quantum phase \( h < 1 \) is the one which spontaneously break the \( Z_2 \) symmetry for \( J = -1 \) and generates the frustrated phase for \( J = 1 \) and is the only one showing a finite amount of frustration. Similar results hold for the generic XYZ chain.

A power-law only for small values of \( R/N \), while trying a power-law fitting for larger values of \( R/N \) results on a varying exponent. Thus, the weakly frustrated chains present a peculiar violation of the area law which yet does not result into a divergence. While at the moment we do not have a general theory describing this frustrated case, these results, characterized by a very unusual and universal behavior, show that its existence is quite general robust and not related to specific, fine-tuned models. Indeed, to the best of our knowledge, the only property shared by all the models we considered is a discrete global \( Z_2 \) symmetry. Thus, we conclude that this non-trivial phase is the result of its combination with a weak frustration.

To further analyze the role of the weak frustration, we present in Fig. 5 the behavior of the frustration measure \( F(J, \gamma, \Delta, h) \) defined in \cite{1} for each single interaction. As in completely unfrustrated systems each term in the Hamiltonian can be minimized independently, this measure of frustration coincides with the Hilbert-Schmidt distance between the projector in the local ground space (i.e. the subspace in which every single interaction would take the system if all the other terms of the Hamiltonian were turned off) and the ground-state that is actually realized for the whole system. As the distance increases, the frustration of an interaction term grows. Notice that, due to its definition, such a measure of frustration cannot discern between quantum and geometrical frustration. Since the ferromagnetic model presents only the former, to distillate the contribution of the latter we may use the following quantity:

\[
g_F = N \left[ F(1, \gamma, \Delta, h) - F(-1, \gamma, \Delta, h) \right]. \tag{7}
\]

In other words we estimate the weight of the geometrical frustration as the extra amount of frustration in the antiferromagnetic system with respect to the ferromagnetic case. As we can observe in Fig. 5 for large \( N \), while in the paramagnetic phase \( g_F \) vanishes, in the weakly ordered phase it goes to a constant value. This is in perfect agreement with the naive observation that the amount of geometrical frustration does not increase with the length of the chain.

**THE FRUSTRATED ISING CHAIN**

Although our results for the XYZ chain show that this weakly frustrated phase is quite general and non limited to the odd AFM Ising chains, it is instructive to look in details at this last model to see how these
unusual behaviors emerge.

Let us specialize (1) to $J = J_N + 1, \gamma = 1$, and $\Delta = 0$:

$$H_{\text{Ising}} = \frac{1}{2} \sum_{l=1}^{N} (\sigma_{l}^{x} \sigma_{l+1}^{x} - h \sigma_{l}^{z}) , \quad (8)$$

where periodic boundary conditions $\sigma_{l+N} = \sigma_{l}^{x}$ are assumed. The $\mathbb{Z}_2$ symmetry of the model is implemented by the parity operator $P = \prod_{l=1}^{N} \sigma_{l}^{z}$ which measures the parity $\pm 1$ of the magnetization along the $z$-axis and which commutes with the Hamiltonian $[H_{\text{Ising}}, P] = 0$.

To study this chain, the standard procedure is to first apply the Jordan-Wigner transformation (JWT) which maps spin-1/2 variables into spinless fermions[37]:

$$\sigma_{l}^{+} = e^{i \pi \sum_{j<l} \psi_{j}^{\dagger} \psi_{l}} , \quad \sigma_{l}^{z} = 1 - 2 \psi_{l}^{\dagger} \psi_{l} , \quad (9)$$

so that an empty fermionic site corresponds to a spin up, with further phase decoration due the non-local string in [37]. Although the JWT solves the difficult problem of dealing with spins, it explicitly breaks translational invariance, by selecting a first site from which the string starts. Because of this, the Hamiltonian written in terms of fermions presents a defect, set by the parity operator $P = \pm 1$, in the coupling between the first and last spin. One way to deal with this issue is to separate from the start the Hilbert space into two subspaces of different parities. Then, the defect is removed by imposing periodic or anti-periodic boundary conditions to the fermionic system depending on the parity, which, in turn, is reflected in the choice of integer/half-integer quantization for the Fourier momenta. Finally, the Hamiltonian in Fourier space is quadratic and can be diagonalized by means of a Bogoliubov rotation[38]. After this sequence of non-local mapping, the Ising chain (8) is transformed exactly into the free fermionic Hamiltonian

$$H = \frac{1+P}{2} H^{+} + \frac{1-P}{2} H^{-} , \quad H^{\pm} = \sum_{q \in \Gamma_{\pm}} \varepsilon \left( \frac{2\pi}{N} q \right) \left\{ \chi_{q}^{\dagger} \chi_{q} - \frac{1}{2} \right\} , \quad (10)$$

with spectrum

$$\varepsilon(\alpha) \equiv \sqrt{(h + \cos \alpha)^2 + \sin^2 \alpha} , \quad (11)$$

with the possible exception of momenta $\frac{2\pi}{N} q = 0, \pi$, since these modes have energy $h \pm 1$ respectively. The set of allowed momenta is given by $\Gamma_{P} = \{ n + \frac{1+P}{4} \}^{N-1}_{n=0}$. The 0- and $\pi$-modes are special: in the unfrustrated cases they are responsible for the double degeneracy in the symmetry broken phase, while for the frustrated chain they close the gap. Let us discuss only the latter case here[38].

The ground state of (10) for $N = 2M + 1$ is always the vacuum of Bogoliubov fermions $\chi_{q} | GS \rangle = 0$ for $q = \frac{1}{2}, \ldots, N - \frac{1}{2}$. For $|h| < 1$ it has energy

$$E_{0} = -\frac{1}{2} \sum_{q=0}^{2M} \varepsilon \left[ \frac{2\pi}{N} \left( q + \frac{1}{2} \right) \right] + 1 - h . \quad (12)$$

The $\pi$-mode (corresponding to $q = M$) has negative energy and so its absence costs energy. However, it cannot be occupied alone, because such state would have odd parity and does not belong to the Hilbert space. Note that in the odd parity sector an exact $\pi$-mode is not allowed because of the quantization condition and thus the odd parity sector does not have a negative energy mode. Therefore, the lowest energy excited states in the even parity sector are of the type $\chi_{M+1/2}^{\dagger} \chi_{p+1/2}^{\dagger} | GS \rangle$ and have energies

$$E_{p} = -\frac{1}{2} \sum_{q=0}^{2M} \varepsilon \left[ \frac{2\pi}{N} \left( q + \frac{1}{2} \right) \right] + \varepsilon \left[ \frac{2\pi}{N} \left( p + \frac{1}{2} \right) \right] ,$$

which lie arbitrarily close to $E_{0}$, with a quadratic dispersion: $E(k) \approx E_{0} + \frac{1}{2} \left( \frac{\lambda}{1 - \eta} \right) (k - \pi)^2 + \ldots$. In the thermodynamic limit this set of states form a continuum above the ground state. In the odd parity sector, the lowest energy state has energy greater than $E_{0}$ and also lies at the bottom of a quadratic gapless band of $N$ states $\chi_{p}^{\dagger} | GS' \rangle$ (where $| GS' \rangle$ is the state annihilated
by all the $\chi_q$, for $q = 0, \ldots, N - 1$, where $p = M, M + 1$ has the lowest energy. As $N \to \infty$, the bands in the even and odd sector mix, with the energy difference between the lowest energy states in the two sectors vanishing polynomially. In total, the ground state is part of a band of doubly – and in some points foursomes– degenerate $2N$ states.

A special role in this construction is played by the negative-energy mode, whose occupation reduces the total energy of the system. The crucial difference between the frustrated and the non-frustrated case is that in the former this mode appears in the even parity sector and cannot be occupied alone, while in the latter belongs to the odd parity sector and thus lowers the energy of the lowest energy state, while not closing the gap with the rest of the band [35]. Also, as we mentioned, the energy difference between the lowest energy states in the two sectors closes polynomially in $N$ in the weakly-frustrated phase and exponentially in the ferromagnetic phase of the non-frustrated models.

One can visualize what happens in the frustrated phase starting from the classical point $h = 0$. In this case, for $N = 2M$, the ground state would be given by the one of the two Néel states. However, moving from even to odd $N$, since these states do not satisfy anymore the AFM condition for a pair of neighbor spins, they are degenerate with the additional $2N - 2$ states with one domain wall. Turning on a finite $h$ splits this degeneracy, but, unlike what happens to other very symmetric points under perturbations, in this case the gap between the states is not proportional to the strength of the perturbation $h$ and thus these $2N$ state fan out into the band discussed above [24].

Having the ground state representation in the free fermionic language allows for the calculation of the physical spin correlation functions, by inverting the transformations sketched above [35]. Even more striking, from the fundamental two-point functions one can construct the correlation matrix, whose eigenvalues provide the diagonal form of the reduced density matrix needed for the EE, as explained in [1]. Defining the (Majorana) fermionic operators $A_l \equiv c_l + \bar{c}_l$ and $B_l \equiv \iota(\bar{c}_l - c_l)$, both the spin correlation functions and the correlation matrix can be expressed in terms of three kind of expectation values, i.e. $\langle A_l \bar{A}_m \rangle$, $\langle B_l \bar{B}_m \rangle$ and $\langle A_l \bar{B}_m \rangle$. The first two of them, for both the frustrated and the unfrustrated Ising model are $\langle A_l \bar{A}_m \rangle = \delta_{l,m}$ and $\langle A_l \bar{B}_m \rangle \equiv lG(r)$, both the spin correlation functions and physical spin correlation functions hold an expression, in terms of the fermionic ones, in which the number of terms increase with the distance, typically because of the Jordan-Wigner string in [10]. In such cases, the role played by the contribution $\nu(h,r)$ must be taken into account also in the (improved) thermodynamic limit and leads to an algebraic decay, as for [2].

A fortiori, in agreement with the aforementioned picture, the EE, which can be evaluated in terms of the eigenvalues of the correlation matrix, can be considered as a correlator involving a number of two-point functions $G(r, J, h)$ growing with the subsystem size. This fact is consistent with the common-sense knowledge that the EE is a non-local quantity.

**DISCUSSION AND CONCLUSIONS**

We have shown that a series of quantum spin chains with a weak frustration (one frustrated bond over $N$) develop a new quantum phase of matter, which presents a mixture of some correlation functions decaying exponentially and some decaying algebraically. The power-law correlations are very slowly decaying, since
the relevant parameter is \( r = \frac{\alpha}{\sqrt{N}} \). We also characterized this phase using the Von Neumann entanglement entropy, which shows a violation of the area law with an algebraic growth with subsystem sizes, which still does not lead to a divergence of the EE with large systems. Such behavior supports the idea that, as in gapped chains, the total amount of entanglement in the system is finite, but, similarly to critical systems, the entanglement is distributed through the whole chain, with the possibility of distilling Bell-pairs with arbitrary distance.

Although, to the best of our knowledge, such behavior has not been reported in any system before, this is not the first class of local, translational invariant systems which presents a violation of the area law. Recently, two such examples have been introduced and are known as the Motzkin\(^{40}\) and the Fredkin chain\(^{41}\). These are frustration-free (FF) systems, in the sense that the Hamiltonian can be decomposed as a sum of local terms, all sharing the same ground state. This feature also allows for a direct evaluation of the entanglement entropy, which scales either logarithmically with the subsystem size for low-spin chain, or as a square-root for higher spin-variable lengths. These models share similarities and profound difference with the class of frustrated systems we considered in this work: for instance, both are related to a massive degeneracy of the ground state manifold, but in a very different way. For the FF systems, a massive degeneracy exists for periodic boundary conditions, but the area law violation requires an open chain with certain conditions at the borders, which selects from the manifold a unique, highly entangled, ground state. In the frustrated case, the massive degeneracy is lifted by a perturbation (namely, the external magnetic field) and periodic boundary conditions are crucially needed to enforce frustration and observe the area law violation. Also, in the FF models, the area law violation is accompanied by a divergence of the EE for large systems, which is not the case for the frustrated cases. Most of all, the FF systems are somewhat artificial in their construction, especially so for the cases of square-root violation of the area law. To the contrary, the frustrated systems we considered are very natural and, as we show, robust against perturbations.

As a matter of fact, these systems are so common that it is rather surprising that the effects of weak frustrations have been overlooked so far. In particular for the Ising chain we discussed in detail our analysis could have been carried out several decades ago, while the effect of a single defect was deemed to be negligible. Indeed, for local observables this is the case, and one has to consider very long distances or non-local correlators (such as the EE) to reveal the non-negligible contribution of such a single defect, but the sensibility to look at such signatures has matured only recently.

From a more technical angle, we notice that, at least for the Ising chain, the ground state of the frustrated chain has the same correlation functions of certain low-lying states of the non-frustrated case. From this point of view, the frustrations can be considered as a way to render stable against decay an otherwise low energy state. It is clear that such an effect, once understood better, could be improved to help in state engineering for quantum technologies. Another intriguing perspective in this direction is the fact that the ground states we analyzed for the frustrated systems show very long-range correlations, scaling like the system size, which could be harvested for quantum information processing or transmission. As we mentioned, these states seem to have a finite amount of entanglement, but spread in a peculiar way. And it is known that it is not really important the total amount of entanglement in a system, but how it is distributed and how you can use it\(^{42}\).

Finally, we have not discussed what are the ingredients necessary to produce this frustrated phase, because we cannot provide a definitive answer at the moment. The collapse of the EE data for the all the models we considered on a single scaling function (see eq. 5 and Fig. 4) points toward a universal nature for this frustrated phase. In this regard, it would be really important to develop a field theory capable to capture this phenomenology in its scaling limit, but we are not aware on any already existing in the literature. Since boundary conditions seem so important in establishing the frustration, it is tempting to speculate that such field theory should contain some topological contribution. Also, since we have examples of similarly frustrated systems with \( U(1) \) symmetry, such as the XXZ chain, which do not develop a non-trivial phase, we put forward the conjecture that what we discussed in this work pertains to systems with discrete symmetries and we wonder about the ramification of considering more complex groups, compared to the \( \mathbb{Z}_2 \) we focused on. Also, preliminary results on a natural extension of the Ising chain (namely, the \( XY \) chain) show that the frustrated phenomenology we discussed can actually comprise different phases with different behavior, such as persistent currents and even spontaneous breaking of translational invariance\(^{43}\).
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