On the Ihara expression of graph zeta functions

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Abstract

We consider the generalized weighted zeta function for a finite digraph, and show that it has the Ihara expression, a determinant expression of graph zeta functions, with a certain specified definition for inverse arcs. A finite digraph in this paper allows multi-arcs or multi-loops.

1 Introduction

A graph zeta function is a formal power series associated with a finite graph. It enumerates the closed paths of given length, exposes the primes, or depicts the cycles in a finite graph. The prototype of the graph zeta functions was introduced by Y. Ihara [6] in 1966 from the number theoretical point of view. Ihara’s zeta was subsequently pointed out by J. -P. Serre [17] that it can be formulated in terms of finite graphs, and is now called the Ihara zeta function for a finite graph [5, 12, 19]. In a paper by H. Bass [2], as was implicitly implied by Ihara [6], the Ihara zeta is provided the determinant expression described by the adjacency matrix and the degree matrix of the corresponding graph. This determinant expression is now called the Ihara expression [7, 10] (see also [14]), and the theorem is called the Bass-Ihara theorem. The Ihara expression is one of the main interests in the study of graph zeta functions, and many researches have pursued on this subject [1, 3, 4, 7, 10, 13, 15, 16, 18].

It is also necessary to mention that the Ihara expression was recently provided a new point of view from quantum walk theory [11], and thus the significance of the Ihara expression is now increasing in related areas.

The subject of the present paper is the Ihara expression for the generalized weighted zeta function. The generalized weighted zeta function was introduced in [14] as a single scheme which unifies the graph zetas appeared in previous studies, for instance, the Ihara zeta [6], the Bartholdi zeta [1], the Mizuno-Sato zeta [13] and the Sato zeta [16]. Graph zetas may have in general four expressions called the exponential expression, the Euler expression, the Hashimoto expression and the Ihara expression. It is verified in [14] that the first three expressions are equivalent for graph zeta functions. The last two expressions are both
determinant expressions, where the size of the matrix used in the last one, the edge matrix, is smaller in general than the other ones, the adjacency matrix and the degree matrix.

For known examples of graph zetas, the Ihara expression is obtained by transforming the Hashimoto expression. In this paper, we will show that the existence of the Ihara expression for the generalized weighted zeta function by reformulating its Hashimoto expression. In particular, we verify the main theorem for the case where the underlying graph is a finite digraph, which generalizes the developments in previous researches. Graph zetas are usually defined via the symmetric digraph of a given finite graph, so it is natural to define for finite digraphs rather than finite graphs. In addition, a finite digraph in this paper allows multi-arcs and multi-loops, and one will see in the procedure that it is an unavoidable issue how one defines the inverses for each arc of a digraph. For this, we can consider two extreme ways. One is the case where all the arcs with inverse direction to an arc $a$ are defined to be the inverses of $a$, and the other one is the case where a single arc with inverse direction, if exists, defined to be the inverse of $a$. In [10], we treat the former case. In the present article, we treat the latter case. These two cases are natural generalization for the case where the underlying graph is a finite graph.

Throughout this paper, we use the following notation. The ring of integers is denoted by $\mathbb{Z}$. The field of rational numbers and complex numbers are denoted by $\mathbb{Q}$ and $\mathbb{C}$ respectively. For a set $X$, the cardinality of $X$ is denoted by $|X|$. The Kronecker delta is denoted by $\delta_{xy}$, which returns 1 if $x = y$, 0 otherwise. The symbol $I$ stands for the identity matrix.

2 Preliminaries

2.1 Graphs and Digraphs

A digraph is a pair $\Delta = (V, A)$ of a set $V$ and a multi-set $A$ consisting of ordered pairs $(u, v)$ of elements $u, v$ in $V$. If the cardinalities of $V$ and $A$ are finite, then $\Delta$ is called a finite digraph. An element of $V$ (resp. $A$) is called a vertex (resp. an arc) of $\Delta$. An arc $a = (u, v)$ is depicted by an arrow from $u$ to $v$. An arc $l$ of the form $(u, u)$ is called a loop, and the vertex $u$ is called the nest of $l$. The set of loops is denoted by $L$. The vertex $u$ is called the tail of $a$, and $v$ the head of $a$, which are denoted by $t(a)$ and $h(a)$ respectively. Let $u, v \in V$. We denote by $A_{uv}$ the set

$$\{a \in A \mid t(a) = u, h(a) = v\}$$

of arcs with the tail $u$ and the head $v$. Hence $L = \sqcup_{u \in V} A_{uu}$. Note that $|A_{uv}| \geq 1$ may occur in general. Thus an arc $a \in A_{uv}$ sometimes called a multi-arc, and the cardinality $|A_{uv}|$ is called the multiplicity of $a$. Similarly, a loop $l \in A_{uu}$ may be called a multi-loop. The cardinality $|A_{uu}|$ is called the multiplicity of $l$. Set $A_{u*} = \sqcup_{v \in V} A_{uv}$ and $A_{v*} = \sqcup_{u \in V} A_{uv}$. A digraph $\Delta = (V, A)$ is called simple if $A_{uu} = \emptyset$ for any $u \in V$ and $|A_{uv}| = 1$ if $A_{uv} \neq \emptyset$. Let $A(u, v) = A_{uv} \cup A_{vu}$ denote the set of arcs lying between vertices $u$ and $v$. A digraph $\Delta$ is called connected if $A(u, v) \neq \emptyset$ for any distinct $u, v$. A digraph in this paper is always assumed to be connect otherwise stated.
Let \( \Delta = (V, \mathcal{A}) \) be a finite digraph, and \( u, v \) two distinct vertices. We may assume that \( |\mathcal{A}_{uv}| \leq |\mathcal{A}_{vu}| \). If \( \mathcal{A}_{uv} \) and \( \mathcal{A}_{vu} \) are both not empty, then one can fix an injection
\[
\iota_{uv} : \mathcal{A}_{uv} \to \mathcal{A}_{vu}.
\]
If \( u = v \), then we agree that \( \iota_{uv} \) is the identity map. In this case, we say that an arc \( a \in \mathcal{A}_{uv} \) has inverse, and the arc \( \iota_{uv}(a) \in \mathcal{A}_{vu} \) is the inverse arc, or simple the inverse of \( a \), denoted by \( a^{-1} \), and vice versa, \( a \) is the inverse of \( a^{-1} \). We also note that, by this definition, a loop \( l \in \mathcal{A}_{uv} \) satisfies \( l^{-1} = l \), that is, each loop is self-inverse. An arc \( a' \in \mathcal{A}_{vu} \) not lying in the image of \( \iota_{uv} \) has no inverse. In the case where \( \mathcal{A}_{uv} = \emptyset \), any arc \( a' \in \mathcal{A}_{vu} \) is defined to have no inverse. Alternatively, one can also define any arc belonging to \( \mathcal{A}_{vu} \) to be inverse of an arc of \( \mathcal{A}_{uv} \). This alternate definition also works, and the development with this definition will be found in \cite{10}.

A graph is a pair \( \Gamma = (V, E) \) of a set \( V \) and a multi-set \( E \) consisting of 2-subsets \( \{u, v\} \) of \( V \). If \( V \) and \( E \) are finite (multi-) sets, then the graph \( \Gamma \) is called finite. An element \( \{u, v\} \in E \) is called an edge. In particular, an edge of the form \( l = \{u, u\} \) is called an loop. The vertex \( u \) is called the nest of \( l \). The set of loops is denoted by \( L \). Obviously, \( \{u, v\} \in E \setminus L \) implies \( u \neq v \). We also suppose that an edge or a loop has multiplicity. Hence these are sometimes called a multi-edge and multi-loop. In other words, if we denote by \( E(u, v) \) the set of multi-arcs lying between vertices \( u, v \in V \), then we assume that \( |E(u, v)| \geq 1 \) for any \( u, v \in V \). Note that \( E(u, u) \) denotes the set of loops with nest \( u \). The cardinality \( |E(u, v)| \) is called the multiplicity of an edge \( \{u, v\} \). A graph is called simple if it has no loops and the multiplicity of any edge is at most one. The matrix
\[
A_\Gamma = (|E(u, v)|)_{u, v \in V}
\]
is called the adjacency matrix of \( \Gamma \). For a vertex \( u \in V \), the number of edges \( \{u, v\} \ (v \in V) \) is called the degree of \( u \), denoted by \( d_u \). Thus we have \( d_u = \sum_{v \in V} |E(u, v)| \) for \( u \in V \). The diagonal matrix
\[
D_\Gamma = (\delta_{uv}d_u)_{u, v \in V}
\]
is called the degree matrix of \( \Gamma \).

Let \( \Gamma = (V, E) \) be a finite graph. We recall the definition of the symmetric digraph of \( \Gamma \). We assign for each edge \( \{u, v\} \in E \setminus L \), two arc \( (u, v) \) and \( (v, u) \) in mutually reverse direction. For a loop \( \{u, u\} \in L \), we assign a single directed loop \( (u, u) \). Then we have an set of arcs
\[
\mathcal{A} = \{(u, v), (v, u) \mid \{u, v\} \in E \setminus L \} \cup \{(u, u) \mid \{u, u\} \in L \}.
\]
The finite digraph constructed in this manner is called the symmetric digraph of a finite graph \( \Gamma \). An arc \( a' = (v, u) \) is called the inverse of \( a = (u, v) \), and denote it by \( a' = a^{-1} \). Any loop \( l = (u, u) \) is defined to be self-inverse, i.e., \( l^{-1} = l \). Therefore, the notion of inverse arcs is straightforwardly defined for the symmetric digraph of a finite graph, and one can see that the preceding definitions of inverse arcs, including the alternating one, are natural generalization of the case for the symmetric digraph. It can readily be confirmed that the symmetric digraph of a simple graph is simple.
2.2 Graph zeta functions

Let $\Delta = (V, A)$ be a finite digraph, $A^\mathbb{Z}$ the set of two-sided infinite sequence. Let $\varphi$ be the left shift operator on $A^\mathbb{Z}$, and $\Xi$ a $\varphi$-stable subset of $A^\mathbb{Z}$, i.e., a subset of $A^\mathbb{Z}$ satisfying $\varphi(\Xi) \subset \Xi$. An example of $\Xi$ is the set

$$\Pi_\Delta = \{(a_i)_{i \in \mathbb{Z}} \in A^\mathbb{Z} \mid h(a_i) = t(a_{i+1}), \forall i \in \mathbb{Z}\}$$

of two-sided infinite path of $\Delta$. If we denote the restriction $\varphi|_\Xi$ by $\lambda$, then we have a quasi-finite dynamical system $(\Xi, \lambda)$, that is, the set $X_m = \{x \in \Xi \mid \lambda^m(x) = x\}$ of $m$-periodic points in $(\Xi, \lambda)$ is a finite set for each $m \geq 1$, since we have $|X_1| = |A|^m$. For $x \in X_m$, the integer $m$ is called a period of $x$. Thus the union $X = \bigcup_{m \geq 1} X_m$ consists of all periodic points in $(\Xi, \lambda)$. Note that the union is not disjoint, since any multiple of a period of $x \in X$ is again a period of it. Let $x = (a_i) \in X$, and let a period $m$ of $x$ be fixed. A consecutive $m$-section $(a_k, a_{k+1}, \ldots, a_{k+m-1})$ is called a fundamental section of $x \in X_m$. In the case where $\Xi = \Pi_\Delta$, then an element $x \in X_m$ is called a closed path of $\Delta$ of length $m$. Thus the closed paths of $\Delta$ are in one-to-one correspondence with the fundamental sections of $X_m$. If we consider the $\varphi$-stable subset $\Pi_\Delta^0 = \{(a_i) \in \Pi_\Delta \mid a_i^{-1} \neq a_{i+1}, \forall i\}$, then an element of $X_m$ is called a reduced closed path of length $m$. Let $\varpi(x)$ denote the minimum period of $x$. Hence it is obvious that $x \in X_{\varpi(x)}$ for any $x \in X$. If we consider $x \in X$ as an element of $X_{\varpi(x)}$, then $x$ is called a prime element. If $x \in X$ is prime, then we denote it by $\pi(x)$. Obviously we have $\pi(x) \in X_{\varpi(x)}$. In the case where $\Xi = \Pi_\Delta$ (resp. $\Pi_\Delta^0$), a prime element is called a prime (resp. prime reduced) closed path of $\Delta$.

Let $\Delta = (V, A)$ be a finite digraph, $R$ a commutative $\mathbb{Q}$-algebra, and $\theta : A \times A \to R$ a map. A finite sequence $w = a_0a_1 \cdots a_{m-1}$ on $A$ is called a word with alphabet $A$. The set of words with alphabet $A$ is denoted by $A^*$. A word $w = a_0a_1 \cdots a_{m-1}$ is called prime if there exists no word $u \in A^*$ satisfying $w = u^k$ for some positive integer $k$. Given a word $w = a_0a_1 \cdots a_{m-1} \in A^*$, we consider the two-sided infinite sequence $x = (a_i) \in A^\mathbb{Z}$ defined by $a_i = a_j$ if $i$ is congruent to $j$ modulo $m$. We denote this sequence by $w^\Delta$. The following product

$$\theta(a_0, a_1)\theta(a_1, a_2) \cdots \theta(a_{m-2}, a_{m-1})\theta(a_{m-1}, a_0)$$

is denoted by circ$_\theta(w)$, called the circular product of $\theta$ along with $w$. Let $\Xi$ be a $\varphi$-stable subset of $A^\mathbb{Z}$. If $\text{circ}_\theta(w) \neq 0 \Rightarrow w^\Delta \in \Xi$

for any prime word $w \in A^*$, then we say that $\Xi$ satisfies the path condition on $\theta$, or sometime we say simply that $(\Xi, \theta)$ closed path of $\Delta$.

Let $\Delta = (V, A)$ be a finite digraph, $R$ a commutative $\mathbb{Q}$-algebra, and $\theta : A \times A \to R$ a map. Let $x = (a_i) \in X_m$ and $w = a_ka_{k+1} \cdots a_{k+m}$ any fundamental section regarded as a word on $A^\mathbb{Z}$. Obviously the circular product circ$_\theta(w)$ does not depend on the choice of fundamental section, and we denote it by circ$_\theta(x)$ for $x \in X_m$. Let $N_m(\theta)$ denote the sum
\[ \sum_{x \in X_m} \circ \rho(x), \text{ and consider the following formal power series with a variable } t: \]

\[ Z_{\Xi}(t; \theta) = \exp \left[ \sum_{m \geq 1} \frac{N_m(\theta)}{m} t^m \right]. \]

**Definition 1 (Graph zeta function)** Let \( \Delta = (V, A) \) be a finite digraph, \( R \) a commutative \( \mathbb{Q} \)-algebra, and \( \theta: A \times A \to R \) a map. Let \( \Xi = \Pi_\Delta \), and suppose that \( (\Xi, \theta) \) satisfies the path condition. Then the formal power series \( Z_{\Xi}(t; \theta) \) is called the graph zeta function for \( \Delta \) with a weight map \( \theta \), which is denoted by \( Z_\Delta(t; \theta) \).

**Example 2 (Ihara zeta function)** Given a map \( \theta^1: A \times A \to R \) by

\[ \theta^1(a, a') = \delta_{b(a)l(a')} - \delta_{a^{-1}a'}, \]

one can see that \( (\Pi_\Delta, \theta^1) \) satisfies the path condition as follows. Suppose that a prime word \( w = \alpha_0 \cdots \alpha_{m-1} \in A^* \) satisfies the condition \( \circ \rho(w) \neq 0 \). This implies \( \theta^1(\alpha_{i-1}, \alpha_i) \neq 0 \), hence \( b(\alpha_{i-1}) = t(\alpha_i) \) for all \( i = 1, 2, \ldots, m \), where \( \alpha_m = \alpha_0 \). Therefore we have \( w^* \in \Pi_\Delta \), and the graph zeta \( Z_\Delta(t; \theta^1) \) is called the Ihara zeta function \([2, 5, 6, 12, 17]\).

**Remark 3** If a digraph \( \Delta \) consists of several connected components \( \Delta = \bigcup_{i=1}^n \Delta_i \), then one can see easily that \( Z_\Delta(t; \theta) = \prod_{i=1}^n Z_{\Delta_i}(t; \theta) \).

### 2.3 Three expressions

Graph zeta functions have two other expressions. Let \( \Delta = (V, A) \) be a finite digraph, \( R \) a commutative \( \mathbb{Q} \)-algebra, and \( \theta: A \times A \to R \) a map. Two elements \( x, y \in \Pi_\Delta \) is called equivalent iff there exists an integer \( k \) satisfying \( y = \lambda^k(x) \). We denote by \( x \sim y \) this equivalence relation on \( \Pi_\Delta \). Note that the relation \( \sim \) also affords an equivalence relation on \( X \). An equivalence class with representative \( x \in \Xi \) is denoted by \([x]\). An element of the coset \( X = X/\sim \) is called a cycle of \( \Delta \). Since the relation \( \sim \) affords an equivalence relation on each \( X_m \), we have \( X = \bigcup_{m \geq 1} X_m \), where \( X_m = X_m/\sim \). If \([x] \in X_m \), then positive integer \( m \) is called the period of the cycle \([x]\). A cycle \([x]\) with reduced (resp. prime) \( x \) is called a reduced (resp. prime) cycle of \( \Delta \). If \([x]\) is prime, then we denote it by \( \pi([x]) \), which belongs to \( X_{\pi(x)} \). In other words, we have \( \pi([x]) = \pi(x) \). Let \( M_\Delta(\theta) = (\theta(a, a'))_{a, a' \in A} \), which is a square matrix of degree \(|A| \). We consider the following two formal power series:

\[ E_\Delta(t; \theta) = \prod_{[x] \in X} \frac{1}{1 - \circ \rho(\pi([x]))t\pi([x])}, \quad H_\Delta(t; \theta) = \frac{1}{\det(I - tM_\Delta(\theta))}. \]

**Proposition 4** For a finite digraph \( \Delta \), it follows that \( Z_\Delta(t; \theta) = E_\Delta(t; \theta) = H_\Delta(t; \theta) \).

For a graph zeta, these three expressions are equivalent to each other. The first identity follows only from the definitions of both sides. The second identity actually needs the path condition. See [14] for precise information. These three expressions are called the exponential expression, the Euler expression and the Hashimoto expression, respectively. The existence of the Hashimoto expression is significant for our development. We will construct the Ihara expression by reformulating the Hashimoto expression (c.f., [20]).
3 Main result

Let \( \Delta = (V, A) \) be a finite digraph and \( R \) a commutative \( \mathbb{Q} \)-algebra. Given two functions \( \tau, \nu : A \to R \), we consider the weight map \( \theta^G : A \times A \to R \) defined by

\[
\theta^G(a, a') = \tau(a')\delta_{h(a)\cup(a')} - \nu(a')\delta_{a^{-1}a'},
\]

for \( a, a' \in A \). The graph zeta function \( Z_{\Delta}(t; \theta^G) \) is called the generalized weighted zeta function [14].

Remark 5 Let \( \Delta \) be the symmetric digraph of a finite graph. If \( \tau = \nu = 1 \), then \( Z_{\Delta}(t; \theta^G) \) is nothing but the Ihara zeta function [2, 5, 6, 12, 17]. If \( \tau = \nu = 1 \), it is the graph zeta function defined in [1]. The edge zeta function and the path zeta function [18] also come out from this framework [14].

One can easily verify that the map \( \theta^G \) satisfies the adjacency condition, and hence \( (\Pi_\Delta, \theta^G) \) satisfies the path condition. Therefore the generalized weighted zeta \( Z_{\Delta}(t; \theta^G) \) has the three expressions. See [14] for precise information.

Proposition 6 We have the identities \( Z_{\Delta}(t; \theta^G) = E_{\Delta}(t; \theta^G) = H_{\Delta}(t; \theta^G) \).

Thus we are in position to construct the Ihara expression for the generalized weighted zeta \( Z_{\Delta}(t; \theta^G) \) with the definition of inverse arc in this paper. Let \( \Delta = (V, A) \) be a finite digraph which allows multi-arcs and multi-loops. For \( u, v \in V \), recall that \( \Phi_\Delta = \{(u, v) \in \Phi_\Delta \mid A(u, v) \neq \emptyset\} \). Thus it follows that \( A = \bigcup_{(u,v) \in \Phi_\Delta} A(u, v) \). For vertices \( u, v \in V \), we write \( u \leq v \) if \( u \neq v \) and \( |A_{uv}| \leq |A_{vu}| \). If \( u \neq v \) and \( |A_{uv}| < |A_{vu}| \), we write \( u < v \). Thus, any \( (u, v) \in V \times V \) satisfies \( u \leq v, v < u \) or \( u = v \). If \( u = v \) or \( u \leq v \), then we fix a injection \( \nu_{uv} : A_{uv} \to A_{vu} \) as in section 2.1. In particular, the map \( \nu_{uv} \) is assume to be the identity map on \( A_{uv} \). Let

\[
\begin{align*}
\Phi^{(1)}_\Delta &= \{(u, v) \in \Phi_\Delta \mid u \leq v, A_{uv} \neq \emptyset\}, \\
\Phi^{(2)}_\Delta &= \{(u, u) \in \Phi_\Delta \mid A_{uu} \neq \emptyset\}, \\
\Phi^{(3)}_\Delta &= \{(u, v) \in \Phi_\Delta \mid u < v, A_{uv} = \emptyset\},
\end{align*}
\]

and let

\[
\begin{align*}
A^{(1)} &= \bigcup_{(u,v) \in \Phi^{(1)}_\Delta} A_{uv}, \quad A^{(-1)} = \bigcup_{(u,v) \in \Phi^{(1)}_\Delta} A_{uv}^{-1}, \quad \overline{A^{(1)}} = \bigcup_{(u,v) \in \Phi^{(1)}_\Delta} A_{uv} \setminus A_{uv}^{-1}, \\
A^{(2)} &= \bigcup_{(u,u) \in \Phi^{(2)}_\Delta} A_{uu}, \quad A^{(3)} = \bigcup_{(u,v) \in \Phi^{(3)}_\Delta} A_{vu},
\end{align*}
\]

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where $\mathcal{A}_u^{-1} = \{\iota_{uv}(a) \mid a \in \mathcal{A}_{uv}\}$. Thus, it follows that $\mathcal{A} = \mathcal{A}^{(1)} \cup \mathcal{A}^{(-1)} \cup \overline{\mathcal{A}}^{(1)} \cup \mathcal{A}^{(2)} \cup \mathcal{A}^{(3)}$. The set of arcs with inverse is given by $\mathcal{A}^{(1)} \cup \mathcal{A}^{(2)}$, and $\overline{\mathcal{A}}^{(1)} \cup \mathcal{A}^{(3)}$ give the set of arcs without inverse. We set $\mathcal{A}^\times = \overline{\mathcal{A}}^{(1)} \cup \mathcal{A}^{(3)}$. For an arc $a \in \mathcal{A}$, let

$$\mathcal{E}(a) = \begin{cases} \{a, a^{-1}\}, & \text{if } a \in \mathcal{A}^{(1)}, \\ \{a\}, & \text{otherwise.} \end{cases}$$

and let $c_a(t) = c_a(t; \theta^G) = 1 - \prod_{a \in \mathcal{E}(a)} (-v(a)t)$. For $u, v \in V$, define

$$a_{uv} = \sum_{a \in \mathcal{A}_{uv}} \frac{\tau(a)}{c_a(t)} \in R[t], \quad b_{uv} = \delta_{uv} \sum_{a \in \mathcal{A}^{(1)} \cap \mathcal{A}_{uv}} \frac{\tau(a) \tau(a^{-1})}{c_a(t)} \in R[t].$$

**Definition 7** Let $\Delta = (V, \mathcal{A})$ be a finite digraph. The following $|V| \times |V|$ matrices

$$A_\Delta(\theta^G) = (a_{uv})_{u,v \in V}, \quad B_\Delta(\theta^G) = (b_{uv})_{u,v \in V}$$

are called the *weighted adjacency matrix* and the *weighted backtrack matrix* for $\Delta$ respectively.

**Example 8** Let $V = \{1, 2, 3\}$ and $\mathcal{A}_{12} = \{a_1\}$, $\mathcal{A}_{21} = \{a_2, a_3\}$, $\mathcal{A}_{23} = \{a_4\}$, $\mathcal{A}_{32} = \{a_5\}$, $\mathcal{A}_{13} = \emptyset$, $\mathcal{A}_{31} = \{a_6\}$, $\mathcal{A}_{11} = \{a_7, a_8\}$, say $\iota_{12}(a_1) = a_2$, $\iota_{23}(a_4) = a_5$, $\iota_{11}(a_7) = a_7$, $\iota_{11}(a_8) = a_8$, i.e., $a_1^{-1} = a_2$, $a_4^{-1} = a_5$, $a_7^{-1} = a_7$, $a_8^{-1} = a_8$, and $a_3, a_6$ have no inverse. In this case, we have: $\Phi^{(1)}_\Delta = \{(1, 2), (2, 3)\}$, $\Phi^{(2)}_\Delta = \{(1, 1)\}$, $\Phi^{(3)}_\Delta = \{(1, 3)\}$; $\mathcal{A}^{(1)} = \{a_1, a_4\}$, $\mathcal{A}^{(-1)} = \{a_2, a_5\}$, $\mathcal{A}^{(1)} = \{a_3\}$, $\mathcal{A}^{(2)} = \{a_7, a_8\}$, $\mathcal{A}^{(3)} = \{a_6\}$; $c_{a_1}(t) = c_{a_2}(t) = 1 - v(a_1)v(a_2)t^2$, $c_{a_4}(t) = c_{a_5}(t) = 1 - v(a_4)v(a_5)t^2$, $\tau_{a_7}(t) = 1 + v(a_7)t$, $\tau_{a_8}(t) = 1 + v(a_8)t$, $c_{a_3}(t) = c_{a_6}(t) = 1$; and

$$A_\Delta(\theta^G) = \begin{bmatrix} \frac{\tau(a_7)}{1-v(a_1)v(a_2)t^2} + \frac{\tau(a_8)}{1+v(a_1)v(a_2)t^2} & 0 \\ 0 & \frac{\tau(a_1)}{1-v(a_1)v(a_2)t^2} \end{bmatrix},$$

$$B_\Delta(\theta^G) = \begin{bmatrix} 0 & \frac{\tau(a_7)}{1-v(a_1)v(a_2)t^2} + \frac{\tau(a_8)}{1+v(a_1)v(a_2)t^2} \\ \frac{\tau(a_1)v(a_2)}{1-v(a_1)v(a_2)t^2} & 0 \\ 0 & \frac{\tau(a_7)}{1-v(a_1)v(a_2)t^2} + \frac{\tau(a_8)}{1+v(a_1)v(a_2)t^2} \end{bmatrix}. $$

**Remark 9** Let $\Gamma = (V, E)$ be a finite simple graph and $\Delta = \Delta(\Gamma)$ the symmetric digraph. Note that by definition $\Gamma$ has no loops. Then one can see that $A$ and $B$ are natural generalization of the adjacency matrix $A_\Gamma$ and the degree matrix $D_\Gamma$ of $\Gamma$ respectively (c.f., [7]). In this case, we have $|A_{uv}| = |A_{vu}| = 1$ for non-empty $A(u, v)$. In addition, if we consider the case where $\theta^G = \theta^1$, i.e., $\tau = v = 1$, then it follows that

$$c_a(t) = 1 - t^2$$

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for all $a \in A$, since $|A_{uv}| = 1$ if $A_{uv} \neq \emptyset$ and the inclusions $\iota_{uv} : A_{uv} \to A_{vu}$ are bijective. A simple observation shows that, for $u, v \in V$,

$$a_{uv} = \frac{|A_{uv}|}{1 - t^2}, \quad b_{uu} = \frac{|A^{(1)} \cap A_{us}|}{1 - t^2}.$$

One can easily see that $|A_{uv}| = 1$ iff $\{u, v\} \in E$ (otherwise zero), and $|A^{(1)} \cap A_{us}|$ gives the number of edges in $\Gamma$ satisfying $\{u, v\} \in E$ for some $v \in V$. This shows that

$$A_\Delta(\theta^G) = \frac{1}{1 - t^2} A_\Gamma, \quad B_\Delta(\theta^G) = \frac{1}{1 - t^2} D_\Gamma.$$

Thus we can regard that the weighted adjacency matrix and the weighted backtrack matrix are, respectively, natural generalization of the adjacency matrix and the degree matrix.

Let $\Delta = (V, A)$ be a finite digraph. Recall that $M = M_\Delta(\theta^G) = (\theta^G(a, a'))_{a, a' \in A}$. Let

$$H = H_\Delta(\theta^G) = (\tau(a')\delta_{\theta(a)}(a'))_{a, a' \in A},$$
$$J = J_\Delta(\theta^G) = (\nu(a')\delta_{a^{-1}}(a'))_{a, a' \in A},$$
$$K = K_\Delta(\theta^G) = (\delta_{\theta(a)}(v))_{a \in A, v \in V},$$
$$L = L_\Delta(\theta^G) = (\tau(a')\delta_{ut(a')})_{u \in V, a' \in A}.$$

For each arc $a \in A$, we consider the following restrictions

$$J(a) = (\nu(a')\delta_{a^{-1}}(a'))_{a, a' \in E\langle a \rangle},$$
$$K(a) = (\delta_{\theta(a)}(v))_{a \in E\langle a \rangle, v \in V},$$
$$L(a) = (\tau(a')\delta_{ut(a')})_{u \in V, a' \in E\langle a \rangle}$$

for the matrices $J$, $K$, and $L$. Note that $J(a)$ is $2 \times 2$-matrix if $a \in A^{(1)}$, $1 \times 1$ otherwise. Hence we can arrange the arcs so as to the matrix $J$ is a direct sum

$$J = \bigoplus_{a \in A^{(1)}} J(a) \oplus \bigoplus_{a \in A \setminus (A^{(1)} \cup A^{(-1)})} J(a).$$
of $2 \times 2$ blocks and $1 \times 1$ blocks. We fix such a total order on $\mathcal{A}$. If we denote by $I(a) \ (a \in \mathcal{A})$ the identity matrix of degree $|\mathcal{E}(a)|$, then the matrix $I + tJ$ is a direct sum of the matrices $\oplus_{a \in \mathcal{A}\setminus \mathcal{A}^{-1}}(I(a) + tJ(a))$, where the direct summands are all invertible on $R[[t]]$.

**Lemma 10** The matrix $I + tJ$ is invertible.

For $\Delta$ and $\theta^G$, we denote by $I_{\Delta}(t; \theta^G)$ the following formal power series with indeterminate $t$:

$$
\frac{1}{\det(I + tJ) \det(I - tA_{\Delta}(\theta^G) + t^2 B_{\Delta}(\theta^G))}.
$$

**Theorem 11** (Main theorem) Let $\Delta$, $R$ and $\theta^G$ be as above. We have

$$Z_{\Delta}(t; \theta^G) = I_{\Delta}(t; \theta^G).$$

**Proof.** Since $(\Pi_{\Delta}, \theta^G)$ satisfies the path condition (c.f., [14]), we have the identity

$$Z_{\Delta}(t; \theta^G) = \frac{1}{\det(I - tM)},$$

where $M = M_{\Delta}(\theta^G)$. Let $H, J, K$ and $L$ be as above. By definition, it follows that $M = H - J$. It also follows that $H = KL$, thus $M = KL - J$. Hence we have

$$
\det(I - tM) = \det(I - t(KL - J)) = \det((I + tJ) - tKL) = \det(I + tJ)\det(I - t(I + tJ)^{-1}KL) = \det(I + tJ)\det(I - tL(I + tJ)^{-1}K),
$$

where the final identity follows from the well-known identity $\det(I - AB) = \det(I - BA)$ in linear algebra. Since each direct summand of

$$I + tJ = \bigoplus_{a \in \mathcal{A}\setminus \mathcal{A}^{-1}} I(a) + tJ(a)$$

is invertible, we have $(I + tJ)^{-1} = \bigoplus_{a \in \mathcal{A}\setminus \mathcal{A}^{-1}}(I(a) + tJ(a))^{-1}$, and it follows that

$$L(I + tJ)^{-1}K = \sum_{a \in \mathcal{A}\setminus \mathcal{A}^{-1}} L(a)(I(a) + tJ(a))^{-1}K(a).$$

Note that $\det(I(a) + tJ(a)) = c_a(t)$ for $a \in \mathcal{A} \setminus \mathcal{A}^{-1}$, and we have

$$(I(a) + tJ(a))^{-1} = \begin{cases} c_a(t)^{-1}(I(a) - tJ(a)), & \text{if } a \in \mathcal{A}^{(1)} \\ c_a(t)^{-1}I(a), & \text{if } a \in \mathcal{A}^{(2)} \cup \mathcal{A}^\times. \end{cases}$$
Hence it follows that
\[
\sum_{a \in \mathcal{A} \setminus \mathcal{A}^{(-1)}} \frac{L(a)(I(a)+tJ(a))^{-1}K(a)}{a \in \mathcal{A} \setminus \mathcal{A}^{(-1)}} = \sum_{a \in \mathcal{A} \setminus \mathcal{A}^{(-1)}} c_a(t)^{-1} L(a)K(a) - t \sum_{a \in \mathcal{A}^{(1)}} c_a(t)^{-1} L(a)J(a)K(a).
\]

The \((u, v)\)-entry \(r_{uv}\) of the matrix \(\sum_{a \in \mathcal{A} \setminus \mathcal{A}^{(-1)}} c_a(t)^{-1} L(a)K(a)\) is given by
\[
\quad r_{uv} = \sum_{a \in \mathcal{A} \setminus \mathcal{A}^{(-1)}} c_a(t)^{-1} \sum_{\alpha \in \mathcal{E}(a)} \tau(\alpha)\delta_{ut(\alpha)}\delta_{h(\alpha)v}.
\]

Note that \(\delta_{ut(\alpha)}\delta_{h(\alpha)v} \neq 0\) is equivalent to \(a \in \mathcal{A}_{uv}\). It follows that
\[
\quad r_{uv} = \sum_{a \in \mathcal{A} \setminus \mathcal{A}^{(-1)}} c_a(t)^{-1} \sum_{\alpha \in \mathcal{E}(a) \cap \mathcal{A}_{uv}} \tau(\alpha).
\]

We verify \(r_{uv} = a_{uv}\) for all \((u, v) \in V \times V\). Suppose that \(u \leq v\). If \(a \in \mathcal{A}^{(1)}\), then \(\mathcal{E}(a) \cap \mathcal{A}_{uv} = \{a\}\). Otherwise, we have \(\mathcal{E}(a) \cap \mathcal{A}_{uv} = \emptyset\). This implies that \(r_{uv} = \sum_{a \in \mathcal{A} \setminus \mathcal{A}^{(-1)}} c_a(t)^{-1} \tau(a)\). In the case where \(v < u\), \(\mathcal{E}(a) \cap \mathcal{A}_{uv} \neq \emptyset\) implies \(a \in \mathcal{A}^x\), and we have \(\mathcal{E}(a) \cap \mathcal{A}_{uv} = \{a\}\) for \(a \in \mathcal{A}^x\). Suppose that \(u = v\). In this case, \(\mathcal{E}(a) \cap \mathcal{A}_{uu} \neq \emptyset\) implies \(a \in \mathcal{A}^{(2)}\), and we have \(\mathcal{E}(a) \cap \mathcal{A}_{uu} = \{a\}\). Therefore, putting all these together, it follows that \(r_{uv} = a_{uv}\) for all \((u, v) \in V \times V\).

The \((u, v)\)-entry \(s_{uv}\) of the matrix \(\sum_{a \in \mathcal{A}^{(1)}} c_a(t)^{-1} L(a)J(a)K(a)\) is given by
\[
\quad s_{uv} = \sum_{a \in \mathcal{A}^{(1)}} c_a(t)^{-1} \sum_{\alpha, \beta \in \mathcal{E}(a)} \tau(\alpha)v(\beta)\delta_{ut(\alpha)}\delta_{a^{-1}\beta}\delta_{h(\beta)v}.
\]

We verify \(s_{uv} = b_{uv}\) for any \((u, v) \in V \times V\). Let \(a \in \mathcal{A}^{(1)}\). We have \(\mathcal{E}(a) = \{a, a^{-1}\}\) with \(a \neq a^{-1}\). Thus it follows that
\[
\quad s_{uv} = \sum_{a \in \mathcal{A}^{(1)}} c_a(t)^{-1} \tau(a)v(a^{-1})\delta_{ut(a)}\delta_{h(a^{-1})u},
\]
which equals \(\sum_{a \in \mathcal{A}^{(1)} \cap \mathcal{A}_{uv}} c_a(t)^{-1} \tau(a)v(a^{-1})\). Now we have show that \(s_{uv} = b_{uv}\). \(\square\)

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