Hypersurfaces in $H^n$ and the space of its horospheres

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Abstract

A classical theorem, mainly due to Aleksandrov [Ale58] and Pogorelov [Pog73], states that any Riemannian metric on $S^2$ with curvature $K > -1$ is induced on a unique convex surface in $H^3$. A similar result holds with the induced metric replaced by the third fundamental form. We show that the same phenomenon happens with yet another metric on immersed surfaces, which we call the horospherical metric.

This result extends in higher dimension, the metrics obtained are then conformally flat. One can also study equivariant immersions of surfaces or the metrics obtained on the boundaries of hyperbolic 3-manifolds. Some statements which are difficult or only conjectured for the induced metric or the third fundamental form become fairly easy when one considers the horospherical metric.

The results concerning the third fundamental form are obtained using a duality between $H^3$ and the de Sitter space $S^3_1$. In the same way, the results concerning the horospherical metric are proved through a duality between $H^n$ and the space of its horospheres, which is naturally endowed with a fairly rich geometrical structure.

Résumé

Un théorème bien connu, dû essentiellement à Aleksandrov [Ale58] et Pogorelov [Pog73], affirme que chaque métrique à courbure $K > -1$ sur $S^2$ est induite sur une unique surface convexe dans $H^3$; un résultat analogue est valable lorsque la métrique induite est remplacée par la troisième forme fondamentale. On montre ici que le même phénomène se produit si on considère une autre métrique sur les surfaces. qu’on appelle métrique horosphérique.

Ce résultat s’étend en dimension plus grande, les métriques obtenues étant alors conformément plates. On peut aussi étudier les immersions équivariantes de surfaces ou les métriques obtenues sur les bords de variétés hyperboliques de dimension 3, et des énoncés difficiles ou seulement conjecturés pour la métrique induite ou la troisième forme fondamentale deviennent faciles pour cette métrique horosphérique.

Les résultats concernant la troisième forme fondamentale sont obtenus en utilisant une dualité connue entre $H^3$ et l’espace de Sitter $S^3_1$; de la même manière, les résultats concernant la métrique horosphérique découlent d’une dualité qu’on décrit entre $H^n$ et l’espace de ses horosphères, qui est muni naturellement d’une structure géométrique assez riche.

Convex surfaces in $H^3$. Let $S$ be a smooth, strictly convex, compact surface in $H^3$. Then $S$ is diffeomorphic to $S^2$, and the Gauss formula indicates that its induced metric has curvature $K > -1$. A well-known theorem, to which several mathematicians have contributed (e.g. Weyl, Nirenberg [Nir53], Aleksandrov [Ale58, AZ67] and Pogorelov [Pog73]; see [Lab89] for a modern approach):

**Theorem 0.1.** Each smooth metric with curvature $K > -1$ on $S^2$ is induced on a unique convex surface in $H^3$.

Note that a similar result holds in $\mathbb{R}^3$, and also in the 3-dimensional sphere $S^3$. The uniqueness here is of course up to global isometries of $H^3$.

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Although the “usual” way of considering this theorem is as a statement on surfaces in $H^3$, it can also be understood as a remarkable statement of existence and uniqueness for a strongly non-linear boundary value problem: finding a hyperbolic metric on the 3-dimensional ball $B^3$ which induces a given metric on the boundary. When considered in this way a basic question is whether the boundary condition chosen here is the only one possible, or indeed the best. One of the goals of this paper is to show that there is an alternative candidate.

The third fundamental form of a surface  This is a fairly classical bilinear form, called $\mathcal{III}$ here, on the tangent of an immersed surface ($\mathcal{III}$ is defined in section 1). When the surface is strictly convex, it is a Riemannian metric; for surfaces in $\mathbb{R}^3$, $\mathcal{III}$ is just the pull-back by the Gauss map of the canonical metric on $S^2$.

An interesting point is that $\mathcal{III}$ provides another good boundary condition for the existence and uniqueness of hyperbolic metrics on $B^3$:

**Theorem** [Sch96]. Let $h$ be a smooth metric on $S^2$. $h$ is the third fundamental form of a convex surface $S$ in $H^3$ if and only if it has curvature $K < 1$. $S$ is then unique up to global isometries.

This result is quite strongly related to analogous polyhedral statements – just like theorem 0.1 was related to the investigation of polyhedra in $H^3$ (see [Ale51]). See [HR93, Sch00] for some related questions.

The de Sitter space  Theorem is strongly related to an interesting duality between $H^3$ and the 3-dimensional de Sitter space, which will be denoted here $S_3^1$. $S_3^1$ is the 3-dimensional, geodesically complete, simply connected Lorentz space with constant curvature 1. It is described more fully in section 1. The duality between $H^3$ and $S_3^1$ – also briefly recalled in section 1 – associates to each point of $H^3$ a totally geodesic, space-like plane in $S_3^1$, and to each totally geodesic (oriented) plane in $H^3$ a point in $S_3^1$. Thus each strictly convex surface $S$ in $H^3$ has a well defined “dual surface” $S^d$ in $S_3^1$, which is space-like and convex. Moreover, the third fundamental of $S$ is the induced metric on $S^d$. Theorem 0.1 can therefore be considered as an isometric embedding theorem in $S_3^1$ – and indeed that is how it is proved.

The space of horospheres in $H^n$  Just like $S_3^1$ is the space of oriented planes in $H^3$ with a natural geometric structure, we can consider the space of horospheres in $H^n$. This is done in some details in section 1; this space – called $C^n_+$ here for reasons that should become clear – is $n$-dimensional, and it has a natural degenerate metric, of signature $(n−1, 0)$. It also has a natural foliation by curves, which we call “vertical lines”, which are tangent to the kernel of the degenerate metric, and come with a canonical parametrization. One of the goals of this paper is to show that one can do some interesting geometry in this space which, in a sense, can be considered as a “degenerate” constant curvature space.

Moreover, there is a natural duality between $H^n$ and $C^n_+$, sending a hypersurface in $H^n$ to the set $S^*$ of points in $C^n_+$ corresponding to the horospheres tangent to $S$; this duality – on which some details are given in section 2 – has some similarities with the $H^3/S^3_1$ duality.

The “horospherical metric” of a surface  When $S$ is a smooth oriented hypersurface in $H^n$, we define the “horospherical metric” on $S$ as the “metric” induced on the dual “hypersurface” $S^*$, and we denote it by $I^*$. In general $I^*$ might be degenerate or otherwise badly behaved, just like the third fundamental form of a non-convex surface in $H^3$. There is a natural class of hypersurfaces in $H^n$, however, for which $I^*$ is a smooth Riemannian metric: the “H-convex” hypersurfaces, which are those which at each point lie on one side of their tangent horosphere. If $S$ is such an H-convex hypersurface, its dual $S^*$ is a smooth, “space-like” hypersurface in $C^n_+$, and moreover it is convex in a natural sense (see section 3).

The metric $I^*$ has a simple expression in terms of the usual extrinsic invariants of a hypersurface: $I^* = I + 2H + \mathcal{III}$, where $H$ is the second fundamental form. The point is that it provides another good boundary condition for the existence and uniqueness of hyperbolic metrics on $B^3$. There is class of metrics on $S^2$, which we call “H-admissible” (resp. “C-admissible”), and which have a rather simple definition (see definition 1.4); those metrics have curvature $K < 1$ (resp. $K \in (−1, 1)$), and are exactly the horospherical metrics of the H-convex (resp. convex) surfaces in $H^3$.
Theorem 5.2. Let \( h \) be a smooth metric on \( S^2 \). It is the horospherical metric \( I^* \) of a H-convex immersed sphere \( S \) in \( H^3 \) if and only if it is H-admissible. It is the horospherical metric of a convex embedded sphere \( S \subset H^3 \) if and only if it is C-admissible. In each case, \( S \) is unique up to the global isometries of \( H^3 \).

In higher dimension, it is not so clear what the metrics induced on e.g. the convex hypersurfaces are. Of course not all metrics are possible, and the conformal flatness of the metrics plays a role \cite{Car10}. It turns out that the situation is much simpler for the horospherical metric, since here again a simple results holds and is easy to prove.

Theorem 5.1. Let \( h \) be smooth metric on \( S^{n-1} \). \( h \) is the horospherical metric \( I^* \) of a H-convex sphere \( S \) in \( H^n \) if and only if:

- \( h \) is conformal to \( \text{can}_{S^{n-1}} \);
- \( h \) is H-admissible, in the sense that it is conformal to \( \text{can}_{S^{n-1}} \) and that \( 2\text{ric}_h - \frac{S}{n-2} - (n-3)h \) is everywhere negative definite.

\( S \) is then unique up to the isometries of \( H^n \). Moreover, \( S \) is convex if and only if all eigenvalues of \( 2(n-2)\text{ric}_h - S_h \) are in \((-n-2)(n-3), (n-2)(n-3))\).

Again, there is also a simple characterization of the metrics which are the horospherical metrics of convex hypersurfaces.

Equivariant surfaces Let \( \Sigma \) be a surface of genus at least two. Although \( \Sigma \) carries many metrics with curvature \( K > -1 \), they can of course not be induced by an embedding in \( H^3 \), since it should then be convex. One needs the slightly refined notion of equivariant embedding. That is a couple \( (\phi, \rho) \), where \( \phi \) is an embedding of the universal cover \( \tilde{\Sigma} \) of \( \Sigma \), and \( \rho \) is a morphism from \( \pi_1(\Sigma) \) into \( \text{Isom}(H^3) \), such that:

\[
\forall x \in \tilde{\Sigma}, \forall \gamma \in \pi_1(\Sigma), \phi(\gamma x) = \rho(\gamma)\phi(x).
\]

One can then search for equivariant embeddings inducing a given metric; it turns out that (because of the index theorem) there are too many of those, so that one can impose an additional condition on \( \rho \).

Theorem 0.2 (Gromov \cite{Gro86}). Let \( \Sigma \) be a surface of genus at least 2, and let \( h \) be a smooth metric on \( \Sigma \) with curvature \( K > -1 \). There is an equivariant isometric embedding \( (\phi, \rho) \) of \( (\Sigma, h) \) into \( H^3 \) such that \( \rho \) fixes a plane.

A remarkable point is that it is still not know whether the uniqueness holds in the theorem above. On the other hand, an analogous results holds with the induced metric replaced by the third fundamental form:

Theorem 0.3 (\cite{LS00}). Let \( \Sigma \) be a surface of genus at least 2, and let \( h \) be a smooth metric on \( \Sigma \) with curvature \( K < 1 \). There is a unique equivariant embedding \( (\phi, \rho) \) of \( \Sigma \) into \( H^3 \) such that the third fundamental form \( \text{III} \) of \( \phi \) is \( h \) and that \( \rho \) fixes plane.

The uniqueness above is of course up to global isometries of \( H^3 \).

Considering the horospherical metric on \( \Sigma \) instead of either the induced metric or the third fundamental form leads to simpler results again. There are simple definitions (see \cite{LS00} of “H-admissible” and “C-admissible” metrics on \( \Sigma \), which are sub-classes of the metrics with curvature \( K < 1 \) and \( K \in (-1, 1) \) respectively. Then:

Theorem 5.4. A smooth metric \( h \) on \( \Sigma \) is the horospherical metric of a H-convex equivariant immersion whose representation fixes a plane if and only if \( h \) is H-admissible. It is the horospherical metric of a convex embedding whose representation fixes a plane if and only if \( h \) is C-admissible. The equivariant immersion/embedding is then unique up to global isometries.

Here again results also hold in higher dimension, and the proof is quite simple.
Manifolds with boundaries As stated above, theorem 0.1 can be considered as a boundary value problem for hyperbolic metrics on the 3-dimensional ball. When considered in this way it should be possible to generalize it to manifolds other than $B^3$. Such a generalization was proposed in the following conjecture.

Conjecture 0.4 (Thurston). Let $M$ be a 3-dimensional manifold with boundary which admits a complete, convex co-compact metric. Then, for any smooth metric $h$ on $\partial M$ with curvature $K > -1$, there is a unique hyperbolic metric $g$ on $M$ which induces $h$ on the boundary, and for which the boundary is convex.

The proof of the existence part of the conjecture was obtained by Labourie [Lab92b, Lab92a], but the uniqueness remains unknown. Theorem 0.3 also suggests that the same kind of result might hold with the induced metric replaced by the third fundamental form; actually the main point of this paper is that the "horospherical metric" works quite well for this kind of results. There are natural classes of "H-admissible" and "C-admissible" metrics, defined in 6.1, which have curvature $K < 1$, such that:

Theorem 6.2. Let $h$ be a smooth metric on $\partial M$.

1. $h$ is the horospherical metric of a H-convex immersion $\phi$ of $\partial M$ in $M$ for a complete hyperbolic metric $g$ on $M$, such that the image of $\phi$ can be deformed through immersions to the boundary at infinity of $M$, if and only if $h$ is H-admissible. $g$ and $\phi$ are then unique.

2. $h$ is the horospherical metric of $\partial M$ for a hyperbolic metric $g$ on $M$, such that $\partial M$ is convex, if and only if $h$ is C-admissible. $g$ is then unique.

In this setting again, the proof is easy, although it uses a deep result, the Ahlfors-Bers theorem (seen here as a bijection between conformal structures on $\partial M$ and hyperbolic metrics on $M$; see [Ahl66]). Actually, theorem 5.4 is a direct consequence of theorem 6.2; it might still be helpful to some readers to have stated it separately.

Hypersurfaces in $S^n$ Note that if $S \subset H^n$ is a convex surface, and if $S^d$ is the dual surface in $S^n$ (which is also a convex surface, and moreover is space-like) then the first, second and third fundamental forms on $S^d$ are $I^d = \mathbb{I}^d$, $\mathbb{II}^d = \mathbb{I}$, and $\mathbb{III}^d = I$ respectively. Therefore:

$$I + 2\mathbb{II} + \mathbb{III} = I^d + 2\mathbb{II}^d + \mathbb{III}^d,$$

so that most of the themes described in this paper for hypersurfaces in $H^n$ are also valid for convex hypersurfaces in $S^n$, and can be proved by considering the dual surface in $H^n$. Presumably a weaker hypothesis than convexity could be used (like the H-convexity condition in $H^n$); it should be possible to repeat some of the arguments below without reference to the dual hypersurface in $H^n$, by replacing the horospheres in $H^n$ by their dual hypersurfaces in $S^n$.

The main point of all this is that some results which are either rather difficult or actually still conjectures for the induced metric or the third fundamental form of (hyper-)surfaces become easy when one considers the horospherical metric instead. Section 7 contains examples of some other areas where this metric might be of interest.

1 The space of horospheres in $H^n$

We will describe in this section the natural geometric structure on the space of horocycles in $H^n$. This structure will be a basic tool in the sequel, so it will be important to understand various basic aspects of it, for instance what the “hyperplanes” or the “umbilical hypersurfaces” are, and how the isometries act.
Horospheres in $H^n$  
To any point at infinity $y \in \partial_{\infty} H^n$, one associates a “Busemann function”, defined, up to the addition of a constant, as:

$$B_y(x) = \lim_{t \to \infty} d(\gamma(t), x_0) - d(\gamma(t), x) ,$$

where $x_0 \in H^n$ is any fixed point and $\gamma$ is any geodesic ray with $\lim_{t \to \infty} \gamma(t) = y$. The level sets of the Busemann functions are called the horospheres of $H^n$. By construction, each horosphere is associated to a unique point at infinity; if two horospheres have the same point at infinity, then they are equidistant.

Let $H$ be a smooth oriented hypersurface in $H^n$, and let $X$ and $Y$ be two vector fields on $H$. Call $D$ the Levi-Civită connection of $H^n$. Then:

$$D_X Y = \overline{D} X Y + \II(X, Y) N ,$$

where $\overline{D}$ is the Levi-Civită connection of the induced metric $I$ on $H$, and $N$ is the unit normal vector field on $H$. $\II$ is called the second fundamental form of $H$, it is a symmetric bilinear form on $TH$. The Weingarten operator $\II$ of $H$ is then defined by:

$$\II(X, Y) = I(-BX, Y) = I(X, -BY) .$$

The sign convention used here is not so standard but will make things easier because we will want to use the exterior normal of e.g. spheres in $H^n$. The third fundamental form of $H$ is:

$$\III(X, Y) = I(BX, BY) .$$

Horospheres in $H^n$ are characterized by the equation:

$$\III = \II = I ,$$

in particular they are umbilical.

The Poincaré model of $H^n$  
There is a convenient model of hyperbolic $n$-space, called the Poincaré model, which is a conformal map from $H^n$ to the Euclidean disc $D^n$ (see e.g. [GHLS]). Moreover, there is also a conformal map from the $n$-sphere $S^n$ minus a point to the Euclidean space $\mathbb{R}^n$. This map can be obtained by stereographic projection. Composing those maps gives us a conformal map from $H^n$ to a geodesic ball in $S^n$, whose radius can be chosen by choosing the right radius for the image of the Poincaré model of $H^n$ in $\mathbb{R}^n$. 

The Klein model of $H^{n+1}$ and the $H^{n+1}/S_q^{n+1}$ duality  
There is also another model of $H^{n+1}$, called the “Klein” or “projective” model. This is a map from $H^{n+1}$ to $D^{n+1}$ which has the striking property that the geodesics of $H^{n+1}$ are mapped to the segments of $D^{n+1}$.

It has a natural extension to a projective model of a “hemisphere” of the $n+1$-dimensional de Sitter space $S_1^{n+1}$ on the complement of $D^{n+1}$ in $\mathbb{R}^{n+1}$. $S_1^{n+1}$ can be also seen as a quadric in Minkowski $n + 2$-space, with the induced metric, as follows:

$$S_1^{n+1} = \{ x \in \mathbb{R}^{n+2}_1 \mid \langle x, x \rangle = 1 \} .$$

There is a natural duality between $H^{n+1}$ and $S_1^{n+1}$, which associates to a totally geodesic hyperplane in $H^{n+1}$ a point in $S_1^{n+1}$. It can be defined in the Minkowski models of $H^{n+1}$ and $S_1^{n+1}$ as follows. Remember that $H^{n+1}$ can be seen as:

$$H^{n+1} = \{ x \in R_1^{n+2} \mid \langle x, x \rangle = -1 \text{ and } x_0 > 0 \} .$$

Given a point $x \in H^{n+1}$, let $D$ be the line going through 0 and $x$ in $R_1^{n+2}$, and let $D^d$ be its orthogonal, which is a space-like hyperplane in $R_1^{n+2}$. The dual of $x$ is the intersection $x^d := D^d \cap S_1^{n+1}$. The same works in the opposite direction, from points in $S_1^{n+1}$ to oriented hyperplanes in $H^{n+1}$.

Given a smooth, oriented, strictly convex hypersurface $S \subset H^{n+1}$, the set of points in $S_1^{n+1}$ which are the duals of the hyperplanes tangent to $S$ is called the dual hypersurface; the notations used here will be $S^d$. It is a space-like, convex hypersurface in $S_1^{n+1}$; its induced metric is $I^d = \III^d = \I^d = I$. See e.g. [Sch93, RH93] for a detailed construction and some additional remarks (in particular concerning polyhedra) on the projective model.
The space of horospheres in $H^n$. The Poincaré model of $H^n$ therefore allows us to consider $H^n$ as the interior of a geodesic sphere $S_0$ in $S^n$. $S^n$ can, through the Klein model of $H^{n+1}$, be considered as the boundary at infinity of $H^{n+1}$. $S_0$ is the boundary of a totally geodesic hyperplane $H_0 \subset H^{n+1}$; let $S_0^*$ be the point in $S_1^{n+1}$ which is dual of $H_0$. The horospheres in $H^n$ are then identified with the spheres in $S^n$ which are interior to and tangent to $S_0$; they are the boundaries of the totally geodesic hyperplanes in $H^{n+1}$ which have a point at infinity in $S_0$. The set of points in $S_1^{n+1}$ which are the duals of those hyperplanes is part of the cone of lines in $\mathbb{R}^{n+1}$ going through $S_0^*$ and tangent to $S^n$; more precisely, it is the set of points of this cone which lie strictly between $S_0^*$ and $S^n$, or, in other terms, the positive light-cone of a point in $S_1^{n+1}$ – whence the notation $C_+^n$.

We already see that $C_+^n$ inherits from this construction a degenerate metric – the one induced on the cone by the de Sitter metric – and a foliation by a family of lines – those going through $S_0^*$. We call those lines "vertical". By construction both the metric and the family of vertical lines are independent of the choices made in the construction. The vertical lines are actually characterized as the curves which are everywhere tangent to the kernel of the (degenerate) metric $g_0$.

Note that $C_+^n$ has, by construction, a very large group of "isometries" which fix both $g_0$ and the vertical lines. This indicates that it is a kind of "degenerate constant curvature space".

![Figure 1: conical model of $C_+^n$](image)

A cylindrical model. A slightly different model, which might sometimes be more convenient, is obtained by taking $H^n$ as a hemisphere in $S^n$; $S_0$ is then an "equatorial" $n-1$-sphere, and its dual point $S_0^*$ is at infinity, so that $C_+^n$ is identified with the union of the lines tangent to $S^n$ at a point of $S_0$, and parallel to the line in $\mathbb{R}^{n+1}$ which is orthogonal to the hyperplane containing $S_0$. 
The induced structure As a submanifold of $S^{n+1}$, $C^n_+$ inherits a degenerate metric $g_0$, i.e., a bilinear form on the tangent space which is at each point of rank $n - 1$. Moreover the kernel of this bilinear form, which at each point is made of a line in the tangent space, integrate as “lines” in $C^n_+$. Those lines are the lines in $\mathbb{R}^{n+1}$ which contain $S^*_0$ and are tangent to $S^n$. They are therefore light-like geodesics of $S^{n+1}_1$, and are naturally equipped with a connection; in other terms they have a parametrization by $\mathbb{R}$ which is defined up to an affine transformation. But those lines actually also have a natural parametrization (up to a constant); namely, it is easy to check that they correspond to the sets of horospheres which have a given focal point at infinity, so that the horospheres corresponding to two points in a given line are equidistant. The distance between them defines the required parametrization.

Note that, from $g_0$ and this canonical parametrization of the vertical lines, one could define a (family of) Riemannian metrics on $C^n_+$. But it does not seem very helpful to do this.

Totally geodesic hyperplanes $C^n_+$ comes equipped with a collection of hypersurfaces which play a special role, and that will be called “totally geodesic hyperplanes”. They are the sets of points duals to the horospheres containing a given point in $H^n$. In the cone model described above, they correspond to the intersections of the cone with the hyperplanes of $\mathbb{R}^{n+1}$ which are tangent to $S^n$ at an interior point of $S_0$. Thus the metric induced on those totally geodesic hyperplanes is isometric to the canonical metric on $S^{n-1}$. Moreover, it should be clear from the description below that they are the only space-like hypersurfaces in $C^n_+$ with an induced metric isometric to $(S^{n-1}, \text{can})$.

By definition, the set of those totally geodesic hyperplanes is an $n$-dimensional manifold – it is parametrized by $H^n$.

Lemma 1.1. Let $x \in C^n_+$, and let $P \subset T_x C^n_+$ be a hyperplane which is transverse to the vertical line at $x$. There is a unique totally geodesic hyperplane $H_0$ in $C^n_+$ which is tangent to $P$ at $x$.

Proof. Consider the cylindrical model of $C^n_+$ described above. $P$ corresponds to an $n - 1$-plane in $\mathbb{R}^{n+1}$ which is disjoint from $S^n$. There are two hyperplanes containing $P$ which are tangent to $S^n$, and one of them is tangent to $C^n_+$ along a line; so there is a unique hyperplane $\overline{P}$ which contains $P$, is transverse to $C^n_+$, and is tangent to $S^n$. $\overline{P}$ intersects $C^n_+$ along an $n - 1$-dimensional manifold which, by construction, is a totally geodesic hyperplane in $C^n_+$. □
Parallel transport along the vertical lines In the cone model above, the tangent space to \( C^n_+ \) is parallel (in \( S^{n+1} \)) along the “vertical lines” (which are the lines in \( C^n_+ \) which are tangent to \( S^n \) at the points of \( S_0 \)). Therefore, the restriction of the Levi-Civitè connection of \( S^{n+1}_1 \) defines a connection along the vertical lines in \( C^n_+ \), and thus also a natural notion of parallel transport along those lines. We call this induced connection \( D^V \).

A kind of connection Now let \( x_0 \in C^n_+ \), and let \( H \) be a hyperplane in \( T_{x_0}C^n_+ \) which is transverse to the vertical direction. We can define a kind of connection, which we call \( D^H \), along the vectors tangent to \( H \) at \( x_0 \). Note that it depends on the choice of \( H \)! It is defined as follows. Call \( H_0 \) the totally geodesic hyperplane tangent to \( H \) at \( x_0 \), let \( X \in H \), and let \( Y \) be a vector field defined in a neighborhood of \( x_0 \), which is tangent to \( H_0 \); then define:

\[
D^H_X Y = D^0_X Y,
\]

where \( D^0 \) is the Levi-Civitè connection of \( H_0 \) for the induced metric. Moreover, if \( T \) is the vector field everywhere parallel to the vertical lines, and with length given by the natural parametrization of those lines, then we decide that, for any function \( f \) on \( C^n_+ \),

\[
D^H_X f T = df(X)T.
\]

This clearly defines \( D^H \) by linearity for any vector field \( Y \) on \( C^n_+ \).

Note that, on the other hand, we do not define a canonical connection on \( C^n_+ \) – and we will not really need one here.

The definition of \( D^H \) can also be obtained in an extrinsic way as follows. For \( x_0 \) and \( H \) chosen as above, there is a unique hyperplane of \( R^{n+1} \) which is transverse to \( C^n_+ \), tangent to \( S^n \), and contains \( H \). This plane contains a unique light-like line \( D' \) containing \( x_0 \). Now choose \( X \in H \), and let \( Y \) be a vector field defined on \( C^n_+ \) in a neighborhood of \( x_0 \). One can project on \( T_{x_0}C^n_+ \) along \( D' \) the vector \( D^S_1 X Y \), where \( D^S_1 \) is the Levi-Civitè connection of \( S^{n+1}_1 \). The reader might want to check that this indeed defines the same vector as \( D^H_X Y \). Of course the point is that the result depends on \( D' \), and therefore on \( H \).

2 H-convex hypersurfaces in \( H^n \)

This section contains some elementary remarks about the dual, in \( C^n_+ \) of some hypersurfaces in \( H^n \). They are then used to give an intrinsic, and quite simple, expression of the metric on \( C^n_+ \).

H-convex hypersurfaces The following notion of convexity is important in our context.

Definition 2.1. Let \( S \) be an oriented hypersurface in \( H^n \), let \( x \in S \), and let \( h \) be a horosphere in \( H^n \). We say that \( h \) is tangent to \( S \) at \( x \) if \( h \) is tangent to \( S \) at \( x \) in the usual sense, and moreover the convex side of \( h \) is on the exterior side of \( S \).

Definition 2.2. Let \( S \) be an oriented hypersurface in \( H^n \). \( S \) is H-convex if, at each point \( x \in S \), \( S \) remains on the concave side of the horosphere tangent to \( S \) at \( x \). \( S \) is strictly H-convex if, moreover, the distance between \( S \) and that horosphere does not vanish up to the second order in any direction at \( x \).

From now on, “H-convex” will be understood as “strictly H-convex” except when otherwise stated. “Convex” will also mean “strictly convex”.

Definition 2.3. Let \( S \) be a hypersurface in \( H^n \). We denote by \( S^* \) the set of points in \( C^n_+ \) which are dual to the horospheres tangent to \( S \).

Definition 2.4. Let \( S \) be a smooth hypersurface in \( C^n_+ \). We say that \( S \) is space-like if \( S \) is everywhere transverse to the vertical lines.
It is not difficult to check that the only compact space-like hypersurfaces in $C^\infty_+$ are spheres.

We will often implicitly identify a hypersurface $S$ with its dual, using the natural map sending a point $x \in S$ to the dual $h^*$ of the horosphere tangent to $S$ at $x$.

**Lemma 2.5.** If $S$ is an $H$-convex hypersurface in $H^n$ such that its principal curvatures are nowhere equal to $-1$, then $S^*$ is an immersed surface in $C^\infty_+$. This happens in particular when $S$ is $H$-convex, and $S^*$ is then space-like. The metric induced by $g_0$ on $S^*$ is:

$$I^* := I + 2\Pi + \Pi$$

where $\Pi$ and $\Pi$ are the second and third fundamental forms of $S$ respectively.

Note that, for instance here, the identification of $I^*$ with $I + 2\Pi + \Pi$ implicitly uses the map from $S$ to $S^*$ sending a point of $S$ to the point of $S^*$ which is dual to the horosphere tangent to $S$ at that point.

The proof of this lemma will use the cylindrical model of $C^\infty_+$ in an explicit way. Consider an $H$-convex hypersurface $H$ in $H^n$, and let $x \in H$. We will use the cylindrical model of $C^\infty_+$, with $x$ located at the "north pole" of $S^n$; the dual of the horosphere $h$ which is tangent to $H$ at $x$ is then a point $h^*$ of the intersection of $C^\infty_+$ (seen as a cylinder) with the hyperplane in $R^{n+1}$ which is tangent to $S^*$ at $x$.

The tangent space to $H$ at $x$ is identified with an affine $n-1$-dimensional subspace $V$ of $R^{n+1}$, and the tangent space to $C^\infty_+$ at $h^*$ can be seen as an $n$-dimensional affine subspace $W$ of $R^{n+1}$ which contain an $n-1$-plane parallel to $V$. We call $\phi$ the duality map from $H$ to $H^*$, sending a point $y$ in $H$ to the dual of the horosphere tangent to $H$ at $y$, and we consider $d\phi$ as a map from $V$ to $W$, where $W \supset V$. Then:

**Proposition 2.6.** The linearized map at $x$ is $T_x\phi = E + B$, where $E$ is the identity map on $V = T_xH$.

**Proof.** Let $v \in T_xH$; call $v^*$ the vector in $W$ corresponding to the variation of the dual point to the horosphere tangent to $H$ at a point which moves in the direction of $V$ on $H$. $v^*$ is the sum of a term $v^*_1$ corresponding to the displacement of $x$ (with a parallel transport of the tangent hyperplane) and a term $v^*_2$ corresponding to the variation of the tangent hyperplane, while $x$ doesn’t move. Using the cylindrical model, one checks that $v^*_1 = v$ (with both terms seen as in $W$) while $v^*_2 = Bv$. □

**Proof of lemma 2.5.** The previous proposition shows that $S^*$ is smooth except maybe when $B$ has $-1$ as one of its eigenvalues.

Moreover, the bilinear form induced on $W$ by $g_0$ (i.e. by the de Sitter metric on the outside of the ball) is a degenerate metric which coincides, on the parallel transport of $V$, with the metric induced on $V$ by $H^n$. Therefore, if $v, v' \in T_xH$, we have that $v^*, v'^* \in V$ and:

$$\langle v + Bv, v' + Bv' \rangle = \langle v, v' \rangle + \langle Bv, v' \rangle + \langle v, Bv' \rangle + \langle Bv, Bv' \rangle,$$

so that:

$$\langle v^*, v'^* \rangle = I(v, v') + 2\Pi(v, v') + \Pi(v, v'),$$

and the result follows. □

**This is a duality** An important point is that the map sending a hypersurface $S$ in $H^n$ to its dual $S^*$ in $C^\infty_+$ is a real duality, in the following sense. First remark that to each totally geodesic hyperplane $H_0$ in $C^\infty_+$ is associated a point in $H^n$, namely the intersection of all the horospheres duals to the point of $H_0$. We call this point the dual of $H_0$, and denote it by $H_0^*$. Then:

**Lemma 2.7.** If $S^*$ is smooth, then $S$ is the set of points in $H^n$ which are duals of a totally geodesic hyperplane in $C^\infty_+$.

**Proof.** This follows again from proposition 2.6, and from the correspondance between vectors on $S$ and on $S^*$. □
An intrinsic definition of the metric $g_0$ The previous lemma can be used to give a simple form of the metric on $C^+_n$; using it will relieve us from the constant use of the cone model, the de Sitter space and so on.

Lemma 2.8. There exists an isometry $\Phi$ from $C^+_n$ to $S^{n-1} \times \mathbb{R}$ with the (degenerate) metric:

$$g_0 \simeq e^{2t} \text{csch}^{2n-2}_n,$$

where $\text{csch}^{2n-2}_n$ is the canonical metric on $S^{n-1}$. Moreover the vertical lines are sent to the lines $\{s\} \times \mathbb{R}$, for $s \in S^{n-1}$, with the same parametrization.

Proof. Let $x_0 \in H^n$. For $t \in \mathbb{R} \setminus \{0\}$, call $S_t$ the geodesic sphere of radius $|t|$ centered at $x_0$, with the normal oriented towards the exterior for $t > 0$ and towards the interior for $t < 0$. Define a map $\Psi$ from $S^{n-1} - \{0\}$ to $C^+_n$ sending $(s, t)$ to the horosphere tangent to $S_t$ at the point $\exp_{x_0}(ts)$, where $s$ is considered as a unit vector in $T_{x_0}H^n$. $\Psi$ can then be extended by continuity to a map from $S^{n-1} \times \mathbb{R}$ to $C^+_n$. $\Phi$ is the inverse of $\Psi$.

By lemma 2.5, the metric induced on $S^*_n$ is:

$$I^*_t = I_t + 2II_t + III_t = \sinh^2(t) \text{csch}^{2n-4}_n(1 + 2 \coth(t) + \coth^2(t)) = (\sinh^2(t) + 2 \sinh(t) \cosh(t) + \cosh^2(t)) \text{csch}^{2n-2}_n = e^{2t} \text{csch}^{2n-2}_n .$$

Now, using e.g. the cylindrical model described above, with $x_0$ as the “north pole” in $S^n$, shows that the surfaces $S^*_t$ are the intersections of $C^+_n$ (seen as a cylinder in $R^{n+1}$) with the horizontal hyperplanes, i.e. the hyperplanes in $R^{n+1}$ which are parallel to the hyperplane containing $S_0$. Therefore the lines $\{s\} \times \mathbb{R}$ are in the kernel of $g_0$, and moreover they correspond to the vertical lines. Finally, by definition of their parametrization (by the distance between equidistant horospheres) it is the same as the one they have in $S^{n-1} \times \mathbb{R}$. \qed

A conformal map Now we remark that all the space-like hypersurfaces in $C^+_n$ can be naturally identified in a conformal way; they are moreover all naturally conformal to the boundary at infinity of $H^n$. Call $\Pi_0$ the map from $C^+_n$ to $\partial_\infty H^n$ sending a horosphere to its point at infinity. Then:

Lemma 2.9. 1. Let $H_1$ and $H_2$ be two compact space-like hypersurfaces in $C^+_n$. The projection from $H_2$ to $H_1$ along the vertical lines is conformal for the induced metrics on $H_1$ and $H_2$.

2. For each space-like hypersurface $H_1 \subset C^+_n$, the restriction of $\Pi_0$ to $H_1$ is conformal for the induced metric on $H_1$ and the usual conformal metric on $\partial_\infty H^n$.

Proof. The first point is a direct consequence of lemma 2.8 above. For the second point remark that, if $x_0$ is the point in $H^n$ which is the intersection of the horospheres in $H_1$, then the map sending a horosphere $h \in H_1$ to its point at infinity is by construction an isometry between $H_1$ with its induced metric and $\partial_\infty H^n$ with the visual metric at $x_0$. It is therefore a conformal map. \qed

Let $H$ be an oriented hypersurface in $H^n$; there is a natural map from $H$ to $\partial_\infty H^n$, which sends a point $x \in H$ to the end point of the ray starting at $x$ in the direction of the oriented normal vector to $H$ at $x$. We call this “Gauss map” $G$ (see e.g. Lab92 for some applications of this map). As a consequence of lemma 2.9 we obtain the following:

Lemma 2.10. If $H$ is an $H$-convex hypersurface in $H^n$, the conformal structure obtained on $H$ as the pull-back by $G$ of the conformal structure on $\partial_\infty H^n$ is the conformal structure of $I^*$.

Umbilical hyperplanes Some hypersurfaces in $C^+_n$ play a special role and have a very simple geometry; they are the surfaces $H^*$, where $H$ is an umbilical hypersurface in $H^n$. By lemma 2.5, $H^*$ is then homothetic to $H$. This is specially interesting when $H$ is a totally geodesic hyperplane in $H^n$, since then $H^*$ is isometric to $H$. We call those surfaces “dual hyperplanes”. It is not difficult to check that the image of a dual hyperplane by the projection on a totally geodesic hyperplane along the vertical lines is a hemisphere.
Isometries  Let $\gamma$ be an isometry of $H^n$. Consider the cone model of $C^n_+$ described in section 1. Then $\gamma$ acts on $S^n$ as a Möbius transformation leaving $S_0$ stable. Therefore it acts as an isometry on $H^{n+1}$, seen as the interior of $S^n$, and therefore also as an isometry on the de Sitter space which lies on the outside of $S^n$, leaving invariant the cone made of the (light-like) lines tangent to $S^n$ along $S_0$ and containing $S^n_0$. So, by construction, $\gamma$ also acts on $C^n_+$ without changing its metric or its vertical lines.

Note that if $\gamma$ has no fixed point in $\partial_\infty H^n$, then it has no fixed point in $C^n_+$ – since an isometry fixing a horosphere should fix its point at infinity. This strongly contrasts with the $H^n/S^n_0$ duality, where all isometries of $H^n$ without fixed point in $H^n \cup \partial_\infty H^n$ have at least one fixed point in $S^n_0$.

The isometries of $C^n_+$ can be characterized in the following simple ways.

Lemma 2.11.  1. Let $H$ be a totally geodesic hyperplane in $C^n_+$. For any isometry $\gamma$ of $H^n$, (the extension to $C^n_+$ of) $\gamma$, composed with the projection on $H$ along the vertical lines is a conformal transformation of $H$.

2. Moreover, any conformal transformation of $H$ corresponds in this way to a unique isometry.

3. Let $D$ be a dual hyperplane in $C^n_+$. Each isometry of $D$ extends in exactly two ways as an isometry of $C^n_+$, one of which preserves orientation.

Proof. Isometries correspond by definition to isometries of $H^n$, which act conformally on $\partial_\infty H^n$, and thus on $H$ by lemma 2.11 point (1) follows. Conversely, any conformal transformation of $H$ defines by lemma 2.8 a conformal transformation of $\partial_\infty H^n$, and therefore an isometry of $H^n$, and also an isometry of $C^n_+$. This proves point (2).

For point (3), let $D^*$ be the dual hyperplane of $D$, i.e. the totally geodesic hyperplane in $H^n$ such that $D$ corresponds to the set of horospheres tangent to $D^*$. Let $\gamma$ be an isometry of $D$. By construction, $D^*$ is isometric to $D$, so that $\gamma$ defines an isometry $\gamma^*$ of $D^*$. Since $D^*$ is an hyperplane in $H^n$, $\gamma^*$ has two extensions as an isometry of $H^n$, one of which preserves orientation. We call this orientation preserving extension $\gamma^+$ again. $\gamma^+$ defines a unique isometry $\gamma$ of $C^n_+$, which leaves $D$ stable by construction. The same works for the other extension of $\gamma^+$.

Group actions  It might be interesting to understand what the quotient of e.g. $C^n_+$ (resp. $C^n_3$) by the action of discrete group acting co-compactly on $H^2$ (resp. $H^3$) is.

3 More on the dual of a hypersurface

Second fundamental forms in $C^n_+$  Let $H$ be a hypersurface in $C^n_+$. Let $x \in H$, and call $H_0 \in T_x C^n_+$ the totally geodesic hyperplane tangent to $H$ at $x$. Let $X$ and $Y$ be vector fields on $H$. Locally (in the neighborhood of $x$) $H$ intersects exactly once each vertical line; therefore, the "vertical connection" $D^v$ defined in section 1 allows us to extend $X$ and $Y$ as vector fields on a neighborhood of $x$ in $C^n_+$ by parallel transport along the vertical lines. We can then unse the kind of connection defined in section 1 to define a "second fundamental form" of $H$ at $x$.

Definition 3.1. The second fundamental form of $H$ at $x$ is defined as:

$$II^*(X,Y) := \Pi(D^v_{H_0} Y),$$

for the extended vector fields, where $\Pi$ is the projection on the vertical direction in $T_x C^n_+$ along the direction of $H_0$.

Lemma 3.2.  1. $II^*$ defines a symmetric bilinear form on $H_0$.

2. If $P_0$ is the (unique) totally geodesic hyperplane in $C^n_+$ which is tangent to $H_0$ at $x$, then $H$ is locally the graph of a function $u$ above $P_0$; $II^*$ is then the hessian of $u$ at $x$ for the metric induced on $P_0$.

3. $II^*$ is also the hessian at $x$ of $u$, seen as a function on $H$, for the induced metric $I^*$ on $H$.  

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In the second part of this lemma, \( u \) is the function such that, at a point \( y \in P_0 \) near \( x \), \( u(y) \) is the "oriented distance" from \( y \) to the intersection of \( H \) with the vertical line through \( y \), for the natural parametrization of that vertical line.

**Proof.** The first point is obviously a consequence of the other. For the second point note that, in the neighborhood of \( x \), the extended vector field \( Y \) is of the form:

\[
Y = Y_0 + du(Y_0)T,
\]

with \( Y_0 \) tangent to \( P_0 \). Therefore the definition of \( D^{H_0} \) shows that:

\[
D^{H_0}_X Y = D^0_X Y_0 + (X. du(Y_0))T,
\]

where \( D^0 \) is the Levi-Civit\`a connection of the induced metric \( g_0 \) on \( P^0 \), and the result follows since \( du = 0 \) at \( x \).

For the third point note that, by lemma 2.8, \( I^* = e^{2u}g_0 \), so that the Levi-Civit\`a connection \( D^* \) of \( I^* \) is given by:

\[
D^*_X Y = D^0_X Y + du(X)Y + du(Y)X - g_0(X, Y)D^0 u,
\]

where vector fields on \( H \) and \( P_0 \) are identified through the projection along the vertical lines. Therefore (by the usual conformal transformation formulas, see e.g. [Bes87], chapter 1):

\[
(D^* du)(X, Y) = (D^0 du)(X, Y) - 2du(X)du(Y) + g_0(X, Y)\|du\|^2_{g_0},
\]

so that \( D^* du = D^0 du \) at \( x \) since \( du = 0 \) at \( x \).

We then use \( I^* \) to define the "Weingarten operator" of a hypersurface \( H \) in \( C^n_+ \):

**Definition 3.3.** If \( H \) is a space-like hypersurface in \( C^n_+ \) and \( x \in H \), the "Weingarten operator" of \( H \) at \( x \) is the linear map \( B^* \) from \( T_x H \) to \( T_x H \), self-adjoint for \( I^* \), defined by:

\[
I^*(X, Y) = I^*(B^* X, B^* Y) = I^*(X, B^* Y).
\]

**An inversion formula** We have already seen in lemma 2.3 that:

\[
I^*(X, Y) = I((E + B)X, (E + B)Y).
\]

Together with the previous lemma, it shows that:

**Lemma 3.4.** If \( S \) is a hypersurface in \( H^n \) with no principal curvature equal to \(-1\) at any point, then:

\[
B^* = (E + B)^{-1}.
\]

**Convex hypersurfaces** Using the previous definition, we can define a convex hypersurface in \( C^n_+ \):

**Definition 3.5.** Let \( H \) be a space-like hypersurface in \( C^n_+ \). We say that \( H \) is convex if \( B^* \) is positive definite at each point of \( H \). \( H \) is tamely convex if all eigenvalues of \( B^* \) are in \((0,1)\) at each point.

The point is that convex hypersurfaces in \( C^n_+ \) have smooth dual hypersurfaces in \( H^n \), and that tamely convex hypersurfaces have convex duals. More precisely:

**Lemma 3.6.** Let \( H \) be a hypersurface in \( C^n_+ \) such that \( B^* \) is nowhere degenerate. Then \( H^* \) is smooth, and its induced metric is:

\[
I(X, Y) = I^*(B^* X, B^* Y).
\]

\( H \) is tamely convex if and only if \( H^* \) is convex.

**Proof.** This follows again from proposition 2.6. 

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4 Isometric embeddings in $C^n_+$

The point of this section is to give an elementary study of the induced metrics on hypersurfaces in $C^n_+$, like the one which can be found in elementary differential geometry books for hypersurfaces in e.g. $\mathbb{R}^n$. The results are a little different, however, due to the degeneracy of the metric.

The Gauss formula. The curvature tensor of the induced metric on a hypersurface in $C^n_+$ is determined by the following analog of the Gauss formula:

**Lemma 4.1.** Let $H$ be a space-like hypersurface in $C^n_+$. Let $x \in H$, call $P_0$ the (unique) totally geodesic hyperplane in $C^n_+$ which is tangent to $H$ at $x$. Let $X, Y, Z$ be three vector fields on $H$. The Riemann curvature tensor $R^*$ of the induced metric $I^*$ on $H$ is given by:

$$R^*_{X,Y}Z = R^0_{X,Y}Z + II^*(X,Z)Y - II^*(Y,Z)X - I^*(Y,Z)B^*X + I^*(X,Z)B^*Y,$$

where $R^0$ is the curvature tensor of $P_0$.

Note that this formula differs from the Euclidean one, in particular because it is linear in $B^*$ instead of quadratic.

**Proof.** We call also $X, Y$ and $Z$ the projections of the vector fields on $P_0$, and $g_{P_0}$ its metric, which has constant curvature 1. The metric on $H$ is then the pull-back of $e^{2u}g_{P_0}$ under the projection of $H$ to $P_0$ along the vertical lines. Therefore, the Levi-Civita connection $\overline{\nabla}$ of $I^*$ is (see e.g. [Bes87], chap. 1):

$$\overline{\nabla}_XY = D_XY + du(X)Y + du(Y)X - g_{P_0}(X,Y)Du,$$

where $D$ is the Levi-Civita connection of $g_{P_0}$. Thus, using the fact that $du = 0$ at $x$, we find that, still at $x$:

$$R^*_{X,Y}Z = \overline{\nabla}_X\overline{\nabla}_YZ - \overline{\nabla}_Y\overline{\nabla}_XZ - \overline{\nabla}_{[X,Y]}Z$$

$$= D_X(D_YZ - D_YD_XZ - D_{[X,Y]}Z)$$

$$= D_X(D_YZ + du(Y)Z + du(Z)Y - g_{P_0}(Y,Z)Du) -$$

$$- D_Y(D_XZ + du(X)Z + du(Z)X - g_{P_0}(X,Z)Du) - D_{[X,Y]}Z$$

$$= R^0_{X,Y}Z + (D_Xdu)(Y)Z + (D_Xdu)(Z)Y -$$

$$- (D_Ydu)(X)Z - (D_Ydu)(Z)X - I^*(Y,Z)D_XDu + I^*(X,Z)D_YDu,$$

and the result follows. \qed

Some consequences. To simplify somewhat the exposition, we concentrate here on surfaces, i.e. the $n = 3$ case. The above formula becomes, for the Gauss curvature of a surface:

$$K^* = 1 - \text{tr}(B^*) .$$

From lemma 4.4, this can be translated as:

$$K^* = 1 - \text{tr}((E + B)^{-1}) = 1 - \frac{\text{tr}(E + B)}{\det(E + B)},$$

so that:

$$K^* = \frac{\det(E + B) - \text{tr}(E + B)}{\det(E + B)} = \frac{\det(B) - 1}{1 + \text{tr}(B) + \det(B)},$$

and, since $\det(B) - 1$ is the Gauss curvature $K$ of the dual surface by the (usual) Gauss formula in $H^3$:

$$K^* = \frac{K}{K + 2H + 2},$$

where $H$ is the mean curvature of the dual surface in $H^3$. Therefore, when $K \neq 0$, we have:

$$K^* = \frac{1}{1 + 2(H + 1)/K}.$$

Thus the constant mean curvature $-1$ surface in $H^3$ are characterized as those whose dual has constant curvature $1$ (of course the minus sign is just a question of orientation).
The Codazzi theorem  Another basic point is that, just as for hypersurfaces in Euclidean space, we have:

**Lemma 4.2.** Let $H$ be a space-like hypersurface in $C^n_+$ with a smooth dual hypersurface, and let $D^*$ be the Levi-Civitā connection of its induced metric. Then, for any vector fields $X,Y$ on $H$:

$$D_X^* Y = B^* D_X (B^{*-1} Y) ,$$

and:

$$(D_X^* B^*) Y = (D_Y^* B^*) X .$$

**Proof.** For the first part of the lemma, we want to show that the connection (again called $D^*$) defined by:

$$D_X^* Y = (E + B)^{-1} D_X ((E + B) Y)$$

is torsion-free and compatible with $I^*$. But it is torsion-free because:

$$D_X^* Y - D_Y^* X = (E + B)^{-1} (D_X ((E + B) Y) - D_Y (E + B) X)$$

$$= (E + B)^{-1} ((E + B) (D_X Y - D_Y X) + (D_X E) Y - (D_Y E) X + (D_X B) Y - (D_Y B) X)$$

$$= D_X Y - D_Y X ,$$

the last step using the Codazzi equation on the dual hypersurface. Therefore $D_X^* Y - D_Y^* X = [X,Y]$, and $D^*$ is torsion-free.

To check that $D^*$ is compatible with $I^*$ is even more direct. If $X,Y,Z$ are vector fields on $H$, then:

$$X I^*(Y,Z) = X I((E + B) Y, (E + B) Z)$$

$$= I(D_X ((E + B) Y), (E + B) Z) + I((E + B) Y, D_X ((E + B) Z))$$

$$= I^*(D_X^* Y, Z) + I^*(Y, D_X^* Z) .$$

The second point of the lemma is easy to prove using the first; if $X$ and $Y$ are vector fields on $H$, then:

$$(D_X^* B^*) Y - (D_Y^* B^*) X = D_X^* (B^* Y) - D_Y^* (B^* X) - B^* (D_X Y - D_Y X)$$

$$= B^* (D_X Y - D_Y X) - B^* (D_X^* Y - D_Y^* X)$$

$$= B^* [X,Y] - B^* [X,Y]$$

$$= 0 .$$

\[\square\]

**Remark**  Lemma 4.2 provides another proof of the formulas given above, relating $K$ and $K^*$ for surfaces in $H^3$ and in $C^n_+$. Indeed, let $(e_1, e_2)$ be an orthonormal frame on a surface $S \subset H^3$; then, by definition of $I^*$, $(\overline{\mathbf{e}_1}, \overline{\mathbf{e}_2}) := ((E + B)^{-1} e_1, (E + B)^{-1} e_2)$ is an orthonormal frame for $I^*$ on $S^*$. Moreover, the connection 1-forms $\omega$ and $\overline{\omega}$ of those frames are the same:

$$\omega(u) := I(D_u e_1, e_2)$$

$$= I^* ((E + B)^{-1} D_u e_1, (E + B)^{-1} e_2)$$

$$= I^* (D_u^* ((E + B)^{-1} e_1), (E + B)^{-1} e_2)$$

$$= I^* (D_u^* \mathbf{\overline{e}_1}, \mathbf{\overline{e}_2})$$

$$=: \overline{\omega}(u) .$$

Therefore, the curvatures on $S$ and $S^*$ differ only by the same factor as the area forms, so that:

$$K^* = \frac{K}{\det(E + B)} .$$
Induced metrics – higher dimensions  Here we take $n \geq 4$, the next paragraph will center on $n = 3$. Let $h$ be a smooth metric on $S^{n-1}$, we have the following elementary characterization of whether $h$ can be obtained as the induced metric on a space-like hypersurface in $C^4_+$. 

**Theorem 4.3.** $(S^{n-1}, h)$ admits a space-like isometric embedding into $C^4_+$ if and only if $h$ is conformal to $\text{can}_{S^{n-1}}$. In this case the embedding is unique up to the isometries of $C^4_+$. 

**Proof.** Let $P_0$ be any totally geodesic hyperplane in $C^4_+$. If $(S^{n-1}, h)$ has a space-like isometric embedding in $C^4_+$, then the projection from the image to $P_0$ is conformal by lemma 2.9. Therefore $h$ is conformal to $\text{can}_{S^{n-1}}$. Conversely, if $h$ is conformal to $\text{can}_{S^{n-1}}$ then there exists a function $u : S^{n-1} \to \mathbb{R}$ such that $h = e^{2u} \text{can}_{S^{n-1}}$; then the graph of $u$ above $P_0$ is, by lemma 2.8, isometric to $h$. 

A more interesting – but still easy – question is to determine when $h$ is induced on a convex or tamely convex hypersurface in $C^4_+$. We call $S_h$ the scalar curvature of $h$. 

**Theorem 4.4.** $h$ is induced on a convex space-like hypersurface $H$ in $C^4_+$ if and only if $h$ is conformal to $\text{can}_{S^{n-1}}$ and $2\text{ric}_h - \frac{h^2}{2} - (n-3) h$ is everywhere negative definite. $H$ is then unique up to isometries of $C^4_+$. $H$ is tamely convex if and only if all eigenvalues of $2(n-2)\text{ric}_h - S_h h$ are in $(-(n-2)(n-3), (n-2)(n-3))$. 

We will say that $h$ is $\textbf{H-admissible}$ if it satisfies the ”positive definite” hypothesis of the theorem, and that $h$ is $\textbf{C-admissible}$ if it satisfies the eigenvalue hypothesis. 

**Proof.** Let $(e_i)_{1 \leq i \leq n-1}$ be an orthonormal frame for $I^*$ which diagonalizes $B^*$, and let $(k_i)_{1 \leq i \leq n-1}$ be the associated eigenvalues of $B^*$. Call $K_{i,j}$ the sectional curvature of $h$ on the 2-plane generated by $e_i$ and $e_j$. Then, by lemma 4.1, 

$$K_{i,j} = 1 - k_i - k_j,$$

so that:

$$\text{ric}_h(e_i, e_i) = \sum_{j \neq i} K_{i,j} = (n - 2) - (n - 3)k_i - \sum_j k_j,$$

and

$$S_h = \sum_i \text{ric}_h(e_i, e_i) = (n - 1)(n - 2) - 2(n - 2) \sum_j k_j,$$

so that:

$$k_i = \frac{S_h + (n - 2)(n - 3) - 2(n - 2)\text{ric}_h(e_i, e_i)}{2(n - 2)(n - 3)} = \frac{S_h - 2(n - 2)\text{ric}_h(e_i, e_i)}{2(n - 2)(n - 3)} + \frac{1}{2},$$

and both results follow. 

**Induced metrics – $n = 3$**  The analog of theorem 4.3 is even simpler in dimension $n = 3$, since in that case all metrics on $S^2$ are conformal to the canonical metric. Therefore, if $h$ is a smooth metric on $S^2$: 

**Theorem 4.5.** $(S^2, h)$ admits a unique (up to the isometries of $C^3_+$) isometric embedding in $C^3_+$. 

To understand the metrics induced on convex surfaces we have to introduce a definition (which is also a lemma). 

**Definition 4.6.** Let $h$ be a smooth metric on $S^2$. Let $x \in S^2$. There is a unique function $u_x$ on $S^2$ such that the metric $e^{-2u_x} h$ has constant curvature 1 and that $u_x(x) = du_x(x) = 0$. We say that $h$ is $\textbf{H-admissible}$ if, for each $x \in S^2$, the hessian of $u_x$ at $x$ is positive definite, and that $h$ is $\textbf{C-admissible}$ if, for each $x$, all eigenvalues of the hessian of $u_x$ at $x$ are in $(0, 1)$. 

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Moreover, by the previous lemma, \( h \) is tamely convex. S

\[ S \text{ only if } h \text{ to } S \]

\[ \text{Proof. Since } h \text{ is } H\text{-admissible.} \]

Now let \( x \in S \). By lemma \[1.1\], there exists a unique totally geodesic plane \( P_0 \) in \( C^2 \) which is tangent to \( S \) at \( x \). \( P_1 \) is the graph above \( S \) of a function \( v \) on \( S \). Then \( e^{-2v}h \) is the metric induced on \( P_1 \), and is isometric to \( \text{can}_{S^2} \), so \( v \) satisfies the conditions set on \( u_x \).

Conversely, if \( w : S \rightarrow \mathbb{R} \) satisfies those conditions, then the graph \( P \) of \( w \) above \( S \) has as induced metric \( \text{can}_{S^2} \), so it is a totally geodesic plane, and moreover it is tangent to \( S \) at \( x \). Thus, by lemma \[1.1\], \( P = P_1 \), and \( w = v \).

Now:

**Theorem 4.7.** Let \( h \) be a smooth metric on \( S^2 \). \( h \) is induced on a convex surface in \( C^3 \) if and only if \( h \) is \( H\)-admissible. \( h \) is induced on a tamely convex surface if and only if \( h \) is \( C\)-admissible.

**Proof.** Since \( h \) is conformal to \( \text{can}_{S^2} \), so there exists a function \( u : S^2 \rightarrow \mathbb{R} \) such that \( e^{2u}\text{can}_{S^2} = h \). Choose a totally geodesic plane \( P_0 \subset C^4 \), and let \( S \) be the graph of \( u \) above \( P_0 \). Then, by lemma \[2.3\] the metric induced on \( S \) is \( h \).

Now let \( x \in S \). By lemma \[1.1\], there exists a unique totally geodesic plane \( P_1 \) in \( C^2 \) which is tangent to \( S \) at \( x \). \( P_1 \) is the graph above \( S \) of a function \( v \) on \( S \). Then \( e^{-2v}h \) is the metric induced on \( P_1 \), and is isometric to \( \text{can}_{S^2} \), so \( v \) satisfies the conditions set on \( u_x \).

Conversely, if \( w : S \rightarrow \mathbb{R} \) satisfies those conditions, then the graph \( P \) of \( w \) above \( S \) has as induced metric \( \text{can}_{S^2} \), so it is a totally geodesic plane, and moreover it is tangent to \( S \) at \( x \). Thus, by lemma \[1.1\], \( P = P_1 \), and \( w = v \).

Now:

**Remark 4.8.** \( H\)-admissible metrics on \( S^2 \) have curvature \( K < 1 \), while \( C\)-admissible metrics on \( S^2 \) have curvature in \((−1, 1)\). The converse, however, is not true.

**Proof.** Theorem \[4.7\] shows that any \( H\)-convex metric is induced on a convex surface in \( C^3 \), and lemma \[4.1\] then indicates that it has curvature strictly below 1. Similarly \( C\)-convex metrics are induced on tamely convex surfaces, which have curvature \( K \in (−1, 1) \) by lemma \[4.1\].

## 5 Surfaces in \( H^3 \)

We will use in this section the results concerning the metrics on convex hypersurfaces to understand the dual metrics on \( H\)-convex spheres in \( H^n \), and then on equivariant hypersurfaces.

**Compact surfaces in \( H^n \), \( n \geq 4 \)** As a consequence of theorems \[4.3\] and \[4.4\], we have for \( n \geq 4 \):

**Theorem 5.1.** Let \( h \) be smooth metric on \( S^{n−1} \). \( h \) is the horospherical metric \( I^* \) of a \( H\)-convex sphere \( S \) in \( H^n \) if and only if:

- \( h \) is conformal to \( \text{can}_{S^{n−1}} \);
- \( h \) is \( H\)-admissible, in the sense that it is conformal to \( \text{can}_{S^{n−1}} \) and \( 2\text{ric}_h − \frac{S_{n−1}}{n−2} − (n−3)h \) is everywhere negative definite.

\( S \) is then unique up to the isometries of \( H^n \). Moreover, \( H \) is tamely convex if and only if, at each point, all eigenvalues of \( 2(n−2)\text{ric}_h − S_{n−1}h \) are in \((−(n−2)(n−3), (n−2)(n−3))\).

**Compact surfaces in \( H^3 \)** The same theorem holds in \( H^3 \) with the adequate notion of \( H\)-convexity; it is a consequence of theorems \[4.3\] and \[4.4\].

**Theorem 5.2.** Let \( h \) be a smooth metric on \( S^2 \). It is the horospherical metric \( I^* \) of a \( H\)-convex immersed sphere \( S \) in \( H^3 \) if and only if it is \( H\)-admissible. It is the horospherical metric of a convex embedded sphere \( S \subset H^3 \) if and only if it is \( C\)-admissible. In each case, \( S \) is unique up to the global isometries of \( H^3 \).
Equivariant surfaces We consider now a surface $\Sigma$ of genus at least 2. First we introduce a class of metrics on $\Sigma$ in the following way – this definition is also a lemma.

**Definition 5.3.** Let $h$ be a smooth metric on $\Sigma$; we also call $h$ the pull-back metric on the universal cover $\tilde{\Sigma}$ of $\Sigma$. For each $x \in \tilde{\Sigma}$, there is a unique function $u_x : \Sigma \to \mathbb{R}$ such that $e^{-2uh}$ is isometric to a hemisphere of $(S^2, \text{can})$, and that $u_x(x) = du_x(x) = 0$. $h$ is H-admissible if, for each $x$, the hessian of $u_x$ is positive definite at $x$. $h$ is C-admissible if, for each $x$, all eigenvalues of the hessian at $x$ of $u_x$ are in $(0, 1)$.

Note that, here again, H-admissible metrics have $K < 1$, and C-admissible metrics have $K \in (-1, 1)$, while the converse is false.

**Proof.** We only have to prove the existence and uniqueness of $u_x$.

It is well known that there exists a unique hyperbolic metric in the conformal class of $h$, i.e. a unique function $u : \Sigma \to \mathbb{R}$ such that $e^{-2uh}$ has constant curvature $-1$. We also call $u$ the induced function on $\tilde{\Sigma}$. Then $(\tilde{\Sigma}, e^{-2uh})$ is isometric to $H^2$, and this defines a function $u$ on $H^2$ which is invariant under an action of $\pi_1(\Sigma)$ by isometries.

Now choose a dual plane $P_0 \subset C^3_+$. Its induced metric is isometric to that of $H^2$; choose an isometry between $P_0$ and $(\tilde{\Sigma}, e^{-2uh})$. This defines a function $u$ on $P_0$, and by construction and lemma 2.8, the graph of $u$ above $P_0$ is isometric to $(\tilde{\Sigma}, h)$. We identify $\tilde{\Sigma}$ with this graph.

Now choose $x \in \tilde{\Sigma}$, and let $P_1 \subset \pi_1(\Sigma)$ be the totally geodesic plane tangent to $\tilde{\Sigma}$ at $x$. $P_0$ is a graph above an hemisphere $P_{1,+}$ of $P_1$, thus $\Sigma$ is also the graph above $P_{1,+}$ of a function $v$; by construction, $v$ satisfies the conditions on $u_x$.

Conversely, if $u$ is a function satisfying those conditions, then the graph of $-w$ above $\Sigma$ is a hemisphere of a totally geodesic plane which is tangent to $\Sigma$ at $x$, so $w = v$. 

This leads to a characterization of the metrics induced on equivariant surfaces in $H^3$ as follows.

**Theorem 5.4.** A smooth metric $h$ on $\Sigma$ is the horospherical metric of a H-convex equivariant immersion whose representation fixes a plane if and only if $h$ is H-admissible. It is the horospherical metric of a convex equivariant embedding whose representation fixes a plane if and only if $h$ is C-admissible. The equivariant immersion/embedding is then unique up to global isometries.

**Proof.** First note that any metric $h$ on $\Sigma$ has an equivariant isometric embedding into $C^3_+$ whose representation fixes a dual plane. Indeed, there is a unique function $u : \Sigma \to \mathbb{R}$ such that $e^{-2uh}$ is hyperbolic; $u$ can then be identified with an equivariant function defined on a dual plane $P_0 \subset C^3_+$, and then $(\Sigma, h)$ is isometric to the graph of $u$ above $P_0$.

The previous proof then indicates that $\Sigma \subset C^3_+$ is convex if and only if $h$ is H-admissible, and tame convex if and only if $h$ is C-admissible. Therefore, the dual immersion in $H^3$ is H-convex if and only if $h$ is H-admissible, and convex if and only if $h$ is C-admissible. In this last case, the convexity implies that the immersion is an embedding.

Some kind of analogous results in higher dimension might hold, but they could be less interesting since the metrics obtained are conformally flat, which is a fairly strong condition. On the other hand they might be used to put special (e.g. hyperbolic) metrics on conformally flat manifolds, through deformations of equivariant sub-manifolds of $H^n$ or $C^3_+$.

6 Hyperbolic manifolds with boundaries

**Why do all this?** As pointed out in the introduction, a natural question along conjecture 0.4 is to find the right boundary condition necessary to obtain a unique hyperbolic metric on a given 3-manifold with boundary. While conjecture 0.4 strongly suggests that one should consider the metric induced on the boundary, theorem 0.3 indicates that the third fundamental form of the boundary could be another choice.

The same question can be asked in higher dimensions, with hyperbolic metrics replaced by Einstein metrics of negative curvature. A basic step is taken in [Sch01], where it is shown that any small deformation of the metric induced on the boundary of a hyperbolic ball can be “followed” by an (essentially
unique) Einstein deformation of the metric in the interior. However, in this case again it is not completely obvious whether the induced metric on the boundary is the right object to consider.

It appears clearly from recent work (see e.g. [GL91, GW, Wit98, And1, Anda]) that, when one considers complete, conformally compact manifolds instead of metrics for which the boundary is at finite distance, then the conformal class of the boundary is what one needs. This does not indicate in any clear way what one should use when the boundary is at finite distance, because, in a conformally compact manifold, the hypersurfaces which are “close” to the boundary in the conformal compact model are “almost umbilical”, so that the conformal class of the induced metric is also (asymptotically) the conformal class of the second or third fundamental forms.

So here again the solution advocated here is that the horospherical metric might be the right thing to consider; the main argument is that, for hyperbolic 3-manifolds, one can then obtain a satisfying existence and uniqueness result in a very simple way. Of course the real challenge will be to obtain similar results in higher dimensions, for Einstein manifolds with boundary, or in other settings.

**H-Admissible metrics** We consider now a geometrically finite 3-manifold with boundary \((M, \partial M)\) which admits a complete convex co-compact hyperbolic metric. Then the universal cover of \((M, \partial M)\) is \((B^3, S^2)\), where \(B^3\) is the 3-dimensional ball. Moreover, if \(h\) is a Riemannian metric on \(\partial M\), then \(h\) defines a complete metric on an open dense subset of \(S^2\), which is invariant under a conformal action of \(\pi_1 M\). Moreover, its conformal structure defines a conformal structure on an open dense set of \(S^2\), which extends to a conformal metric on \(S^2\), and the universal cover \(\tilde{\partial M}\) of \(\partial M\) has a unique conformal embedding into \(S^2\) whose image is an open dense set (see e.g. [Ahl66]).

We can thus define a proper class of metrics on \(\partial M\).

**Definition 6.1.** Let \(h\) be a smooth metric on \(\partial M\), and let \(x \in \tilde{\partial M}\). There exists a unique function \(u_x\) on \(\partial M\) such that \(e^{-2u_x}h\) extends to a constant curvature 1 metric on \(S^2\), and such that \(u_x(x) = du_x(x) = 0\). We say that \(h\) is H-admissible if, for all \(x\), the hessian of \(u_x\) at \(x\) is positive definite. \(h\) is C-admissible if, for all \(x\), the eigenvalues of the hessian of \(u_x\) at \(x\) are in \((0, 1)\).

Note that this definition coincides with definitions 4.4 and 5.3 in the corresponding special cases. Again as above, H-admissible metrics have curvature \(K < 1\), and C-admissible metrics have curvature \(K \in (-1, 1)\).

**Proof.** Again we only have to prove the existence and uniqueness of \(u_x\).

We know that there exists a function \(u\) on the universal cover of \(\partial M\) such that \(e^{-2u}h\) is isometric to an open dense subset of \(S^2\). This defines \((\Sigma, h)\) as the graph of \(u\) above an open dense subset of a totally geodesic plane \(P_0\) (with the induced metric).

The rest of the proof is just like for definition 4.6 and 5.3, and uses the uniqueness of the conformal change of metric.

**Existence and uniqueness** We can now state the analog of conjecture 1.4 for the horospherical metric.

**Theorem 6.2.** Let \(h\) be a smooth metric on \(\partial M\).

1. \(h\) is the horospherical metric of a H-convex immersion \(\phi\) of \(\partial M\) in \(M\) for a complete hyperbolic metric \(g\) on \(M\), such that the image of \(\phi\) can be deformed through immersions to the boundary at infinity of \(M\), if and only if \(h\) is H-admissible. \(g\) and \(\phi\) are then unique.

2. \(h\) is the horospherical metric of \(\partial M\) for a hyperbolic metric \(g\) on \(M\), such that \(\partial M\) is convex, if and only if \(h\) is C-admissible. \(g\) is then unique.

**Proof.** We already know from the proof of definition 6.1 that \((\tilde{\partial M}, h)\) is isometric to the graph of a unique (up to global isometries) graph above a totally geodesic plane \(P_0\).

Taking the dual surface in \(H^3\) gives an immersion \(\phi\) of \(\tilde{\partial M}\) in \(H^3\) which is H-convex if \(h\) is H-admissible, and convex if \(h\) is C-admissible.

Moreover, \(\pi_1 M\) acts by conformal transformations on \(P_0\), so, by lemma 2.11, by isometries on \(H^3\).

By construction, \(\phi(\tilde{\partial M})\) is invariant under those isometries. Thus \((\tilde{\partial M}, h)\) is isometric to the quotient by \(\pi_1 M\) of the image of \(\phi\) with its horospherical metric.
Note that, if \( h \) is only \( H \)-convex, we only obtain a priori an immersion of \( \partial M \) in \( M \), which can be deformed through immersions to an embedding. If \( h \) is \( C \)-admissible, on the other hand, \( \partial M \) is obtained as a convex surface in \( M \), so it is embedded (and it bounds a convex domain in \( M \)).

It should be pointed out that theorem 5.4 is a direct consequence of theorem 5.2. Indeed consider the manifold \( (M, \partial M) = (\Sigma \times [-1, 1], \Sigma \times \{-1, 1\}) \), where \( \Sigma \) is a surface of genus at least 2, and take on \( \partial M \) a metric which is identical on both copies of \( \Sigma \). Then the uniqueness statement in theorem 5.2 implies that the metric \( g \) obtained will have a \( \mathbb{Z}/2\mathbb{Z} \) symmetry, which will exchange the two connected components of \( \partial M \). Therefore \( \partial \tilde{M} \) will be immersed/embedded in \( H^3 \) as two equivariant surfaces, symmetric with respect to a plane which is fixed by both representations.

**Higher dimensions** Similar results might hold in the corresponding cases in higher dimension, with conformally flat metrics on the boundary. This should not be too interesting, however, since conformally flat metrics should be quite rigid in this situation.

### 7 Moreover

**An elementary approach** A large part of what we have described here can be reduced essentially to a simple (but remarkable) property. Let \( H \) be a complete oriented hypersurface in \( H^n \), which is “uniformly \( H \)-convex” in the most natural sense. Let \( u \) be a function on \( h \), with a differential which is ”small”. For each point \( x \in H \), consider the horosphere \( h_x \) tangent to \( H \) at \( x \), and its equidistant horosphere \( h'_x \) at distance \( u(x) \). Then let \( H' \) be the envelope of the horospheres \( h'_x \), and let \( \phi \) be the map sending \( x \in H \) to the point \( \phi(x) \in H' \) where \( h'_x \) is tangent to \( H' \) (this is well defined if \( u \) and \( H \) are well behaved. Then \( \phi \) is an isometry between \( (H, e^{2u} I_H^H) \) and \( (H', I_{H'}) \).

Of course this is basically a translation, in purely \( H^n \) terms, of the basic properties of the metric on \( C^n_+ \), as described in lemma 2.8. Moreover the statement is quite imprecise concerning the precise conditions on \( u \); of course things are clear in \( C^n_+ \), the point is only that \( u \) has to be such that the graph of \( u \) above \( H^n \) (which will be the dual of \( H^n \)) remains convex, so that \( H' \) remains \( H \)-convex. More generally, I guess that some of the results obtained here could be achieved without using \( C^n_+ \), but I doubt whether it could improve the clarity of this matter.

**Symmetric spaces and dualities** Given a symmetric space \( G/K \), there is a quite general way of constructing other spaces (of the form \( G/H \), for various choices of \( H \subset G \)) which are in ”duality” with \( G/K \) – see e.g. [Hel00, Hel94]. The duality between \( H^n \) and \( C^n_+ \) can be seen as a special case of this (with \( G = \text{SO}(n, 1), K = \text{SO}(n) \) and \( H = \text{Isom}((\mathbb{R}^{n+1}),) \), just like the duality between \( H^n \) and \( S^n_1 \) (with \( G = \text{SO}(n, 1), K = \text{SO}(n) \) and \( H = \text{SO}(n-1, 1) \)). In this general setting there is a natural – and well understood – duality between the functions or distributions on a space and on its dual. The duality between the hypersurfaces can be put in this context by replacing a hypersurface by some measure which it defines, the dual hypersurface is then the support of the dual measure.

In the cases which we have described, however, one should not use the measure associated to the area form on the hypersurfaces, since the duality would then act with a factor equal to the Gauss-Kronecker curvature of the hypersurfaces (in the case of the \( H^n/S^n_1 \) duality) or the determinant of \( E + B \) (in the \( H^n/C^n_+ \) duality). Rather one should normalize this area measure by a factor \( 1/\sqrt{\det(B)} \) or \( 1/\sqrt{\det(E + B)} \) in \( H^n \), and \( 1/\sqrt{\det(B^*)} \) in \( S^n_1 \) or \( C^n_+ \).

A natural question is to understand to what extend the duality properties of hypersurfaces in those spaces extend from the cases described above to a more general setting, and what one could get out of it.

**Induced metrics and third fundamental form** One striking feature of the results above is that they are simpler to obtain – and more powerful in some cases – that the corresponding results obtained for convex (hyper-)surfaces when one considers on them the induced metrics or third fundamental forms. This leads to the idea that those results could be used as a tool to obtain results on the induced metrics or third fundamental form; for this one should obtain rigidity results on the way the induced metric (resp. third fundamental form) varies when a deformation changes the horospherical metric.
Einstein manifolds, etc  The most natural framework in which conjecture could be extended is the theory of negatively curved Einstein manifolds with boundaries; indeed, in dimension 3, negatively curved Einstein metrics are the same as hyperbolic metrics.

An elementary (and far too restricted) first step was taken in this direction in [Sch01] (see also [RS99, RS00] for some strikingly related rigidity results obtained by very different methods). The outstanding problem there, however, is that the infinitesimal rigidity result which is needed – stating that an infinitesimal deformation of the interior metric induces a non-trivial deformation of the boundary metric – is only obtained when the boundary is umbilical.

A natural question is therefore whether an analog of the horospherical metric (maybe defined as \( I + 2I_2 + I_3 \)) could lead to some infinitesimal rigidity result for Einstein manifolds with boundary; this would open the door to possible results on the existence and/or uniqueness of Einstein metrics inducing a given horospherical metric on the boundary.

Note that the theory concerning complete metrics is rather more advanced; in that case one only prescribes the conformal structure on the boundary at infinity, and the Einstein metrics are required to be conformally compact. In dimension 3 it is just the classical Ahlfors-Bers theorem, while in higher dimension the theory seems to be advancing (see the previous section for references).

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