STRINGS BELOW THE PLANCK SCALE

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We show that, for a class of critical strings in $\mathbb{R} \times S^1$-target space, the description of string theory given by its field content (analog model) breaks down when the radius of $S^1$ decreases below $R_0 = \sqrt{\alpha'}$, the self-dual point of the partition function $Z(R)$. We find that $Z(R)$ has a soft singularity at $R_0$ (a finite jump in the first derivative of $Z$).
1 Introduction

The study of $c = 1$ conformal matter coupled to the Liouville mode is equivalent to that of a bosonic string in two dimensions. Closed bosonic string theory in two target-space dimensions can be formulated provided we include non-trivial backgrounds [1]. The only propagating field in this string theory is the scalar tachyon mode which actually is massless in two dimensions.

The computation of the genus-one path integral [2, 3] further supports the fact that the $c = 1$ theory can be described by a massless field in two dimensions. However a subtlety must be taken into account. The zero mode of the free scalar field compactified on a circle produces a one loop vacuum energy for the theory (as the result of going from $S^1$ to $\mathbb{R}$ taking $R \to \infty$) which does not correspond to the ultraviolet divergent vacuum energy of a quantum field. $R$-duality actually implies that this finite vacuum energy equals the coefficient which multiplies the $R^{-2}$ dependence in the free energy for a single boson in two dimensions [4].

This letter will be devoted to the study of a class of critical strings in $\mathbb{R} \times S^1$-target space. In this case we will show that the description of string theory by its field content breaks down when the radius of the compactified dimension decreases below the self-dual point ($R = \sqrt{\alpha'}$, we will take $\alpha' = \frac{1}{2}$ throughout). More generally, we will see that the transition across the self-dual point physically corresponds to the fact that, below the Planck scale, the partition function of the string measures degrees of freedom which does not have a quantum-field theoretical counterpart.

In the concluding Sec. 4 a mathematical relation between the vacuum energy of the $c = 1$ non-critical string and the partition function of the states with the symmetry encoded in the Monster group $F_1$ will be given.

2 Two Dimensional Heterotic Strings on $\mathbb{R} \times S^1$

There is a class of two-dimensional heterotic strings whose left-moving sector can be constructed taking the left-moving part of the bosonic string compactified on a 24 dimensional torus defined by a Niemeier lattice [5]. The partition function for the left-moving states in the internal torus reads

$$Z_L = \frac{\Theta_T}{\eta^{24}} = j(\tau) + r_T(1) - 720$$

(1)

where $j(\tau)$ is the modular invariant function of the modular parameter $\tau = \tau_1 + i\tau_2$ normalized in such a way that the coefficient of $q^{-1}$ ($q = e^{2\pi i \tau}$) is 1 [8] and $r_T(1)$, which parametrizes 24 models, is the number of lattice vectors with $(\text{length})^2 = 2$.

To get the right-moving part contribution to the partition function, we first compactify the eight bosonic right-moving coordinates of the supersymmetric string on the only eight-
dimensional self-dual even lattice, $\Gamma_8$; then we break supersymmetry along the lines of [9]. To be more concrete, we mod out the theory by $(-1)^F\vartheta$ where $\vartheta$ corresponds to shifting $\Gamma_8$ by half of a lattice vector. We take $\vartheta = e^{2\pi i P} \delta$ with $\delta = (\frac{1}{2}, 0)$. Now we have a theory in which there are contributions from the four $\text{SO}(10)$ conjugacy classes. The result is

$$Z_R = 48 + \frac{1}{447} \mathbb{I}[\theta_2^8 + \bar{\theta}_3^8 + \bar{\theta}_4^4 - \bar{\theta}_4^4 - \theta_2^4]$$

(2)

From this expression we see that supersymmetry is only broken at the massless level. Using Jacobi’s “aequatio” the total partition function is

$$Z \sim -48 L^2 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} [j(\tau) + r_\Gamma(1) - 720]$$

(3)

where $L \to \infty$ regularizes the constant modes in the uncompactified dimensions and $\mathcal{F}$ is the fundamental region of the modular group

$$\mathcal{F} = \{ \tau = \tau_1 + i\tau_2 \in \mathcal{H} \mid -\frac{1}{2} \leq \tau_1 < \frac{1}{2}, |\tau| > 1 \}$$

(4)

When we consider these models in a target-space of the form $\mathbb{R} \times S^1$ the generic partition function can be written

$$Z(R) \sim -48 L^2 (2\pi R) \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{m,n} \exp \left\{ -\frac{2\pi R^2}{\tau_2} |m\tau + n| \right\} [j(\tau) + r_\Gamma(1) - 720]$$

(5)

We shall now study the analyticity properties of the partition function. It is easy to see from the Fourier series for $j(\tau)$

$$j(\tau) = q^{-1} + 744 + \sum_{n=1}^{\infty} c_n q^n, \quad c_n \in \mathbb{Z}$$

(6)

that any non-analytic behaviour in the partition function may come exclusively from the term $q^{-1}$. This term would give a divergent contribution for the zero mode of the solitonic sum if we do not take the prescription of performing first the $\tau_1$ integration in a vicinity of infinity (i.e. a region $U_\epsilon$ defined as the set of points in $\mathcal{F}$ such that $\tau_2 > \epsilon \geq 1$). For the $R$-dependent part we apply the same prescription. After performing the $\tau_1$ integration on $U_\epsilon$ of the term $q^{-1} \times \sum_{m,n}$ only two terms survive, namely those related to $m = 1$, $n = -1$ and $m = -1$, $n = 1$ in the sum. So the only possible non-analytic term in the sum will be proportional to

$$2 \int_{\epsilon}^{\infty} \frac{d\tau_2}{\tau_2^{3/2}} e^{-\pi \tau_2 \left(2 R^2 + \frac{1}{\pi R^2} - 2\right)}$$

(7)

Since the argument of the exponential has a double root at $R_0 = \frac{1}{\sqrt{2}}$ we can write the last expression as

$$2 \int_{\epsilon}^{\infty} \frac{d\tau_2}{\tau_2^{3/2}} e^{-\pi \tau_2 \left(\frac{\pi^2 - 2}{\pi R_0^2}\right)^2}$$

(8)
In spite of its naive appearance, the analyticity properties of this integral as a function of $R$ are nontrivial. One can easily see that it is finite and continuous at $R = R_0$, but its first derivative with respect to $R$ is discontinuous at that point. Denoting by $I_\epsilon(R)$ the last integral, we have

$$
\frac{dI_\epsilon}{dR}(R) = -\frac{4\pi}{R_0} \left( \frac{R^2 - R_0^2}{RR_0} \right) \left( \frac{R^2 + R_0^2}{R^2} \right) \int_\epsilon^\infty \frac{d\tau_2}{\tau_2^{1/2}} e^{-\pi\tau_2} \left( \frac{R^2 - R_0^2}{RR_0} \right)^2
$$

(9)

Now the integral can be calculated for any $R \neq R_0$ by performing a change of variables, the result being

$$
\frac{dI_\epsilon}{dR}(R) = -\frac{4\sqrt{\pi}}{R_0} \left( \frac{R^2 + R_0^2}{R^2} \right) \text{sig}(R - R_0) \Gamma \left( \frac{1}{2}, \pi\epsilon \left( \frac{R^2 - R_0^2}{RR_0} \right)^2 \right)
$$

(10)

where $\Gamma(a, x)$ is the incomplete gamma function and $\text{sig}(x)$ the sign function. The jump in the first derivative is then given by

$$
\frac{dI_\epsilon}{dR}(R_0^+) - \frac{dI_\epsilon}{dR}(R_0^-) = -\frac{16\pi}{R_0}
$$

(11)

Since any other term in (9) is analytic for any $R$, the partition function itself suffers from a discontinuity in the first derivative at the point $R_0$. This is the only point in which the partition function is not analytic.

Let us study now the $R$-duality properties of $Z(R)$. It is easy to see, applying Poisson resummation formula, that the solitonic sum is invariant under the replacement

$$
R \rightarrow \frac{1}{2R}
$$

(12)

so the partition function satisfies

$$
Z(R) = Z \left( \frac{1}{2R} \right)
$$

(13)

the critical point is thus the self-dual point, $R_0 = R_{self\text{-dual}}$.

3 Massless Fields in $R \times S^1$ and Strings

The vacuum energy per degree of freedom of a bosonic massless field in $R \times S^1$ is given by

$$
\Lambda(R) = -L(2\pi R)(2\pi)^{-2} \int_0^{+\infty} \frac{ds}{s^2} \sum_{r=-\infty}^{+\infty} \exp \left\{ -\frac{2\pi R^2}{s} r^2 \right\} + \Lambda_0 = -\frac{L}{12R} + \Lambda_0
$$

(14)

where $\Lambda_0$ is the ultraviolet divergent vacuum energy of the field in $R^2$ that would correspond to the $r = 0$ term in the sum. As was done in [2, 3] we introduce the unity in the form

$$
1 = \int_{-\frac{1}{2}}^{+\frac{1}{2}} d\theta
$$

(15)
and apply the techniques of [11, 12, 13] to get the following relationship

\[
\int_0^{+\infty} ds \int_{-\frac{1}{2}}^{+\frac{1}{2}} d\theta s^{-2} \sum_{r=-\infty}^{+\infty} \exp\left\{-\frac{2\pi R^2}{s} \right\}
= \int_F \frac{d^2\tau}{\tau_2} \sum_{m,n \in \mathbb{Z}} \exp\left\{-\frac{2\pi R^2}{\tau_2} \right\}
\]

where we have defined \(\tau = \tau_1 + i\tau_2 = \theta + is\). This means that the temperature dependent one-loop free energy for a single massless boson in two dimensions (with the identification \(1/T = 2\pi R\)) is equivalent to the temperature dependent Helmholtz free energy of the continuum Liouville theory coupled to conformal matter with \(c = 1\).

Let us now introduce the unity as

\[
1 = \frac{1}{744} \int_{-\frac{1}{2}}^{+\frac{1}{2}} j(\tau) d\tau_1
\]

Then we would have the following would-be equality

\[
\tilde{\Lambda}_1(R) = \int_F \frac{d^2\tau}{\tau_2} \sum_{m,n \in \mathbb{Z}} \exp\left\{-\frac{2\pi R^2}{\tau_2} \right\}
\]

\[
\frac{1}{744} \int_F \frac{d^2\tau}{\tau_2} j(\tau) \sum_{m,n \in \mathbb{Z}} \exp\left\{-\frac{2\pi R^2}{\tau_2} \right\} =: \tilde{\Lambda}_2(R)
\]

If we included the zero mode in both of the double sums we would see that (18) would not be correct because

\[
\int_F \frac{d^2\tau}{\tau_2} \neq \frac{1}{744} \int_F \frac{d^2\tau}{\tau_2} j(\tau) = \frac{720}{744} \int_F \frac{d^2\tau}{\tau_2}
\]

\(R\)-duality symmetry for each of the integrals in (18), which is actually a property of the solitonic sector in the integrand, reads [4]

\[
\tilde{\Lambda}_i(R) = \frac{1}{2R^2} \tilde{\Lambda}_i(\frac{1}{2R}) - (1 - \frac{1}{2R^2}) \times (zero\ mode\ contribution)_i
\]

where \(\tilde{\Lambda}_i\) represents any of the two integrals in (18) and \((zero\ mode\ contribution)_i\) is given respectively by the two sides of the inequality (19). Therefore \(R\)-duality symmetry implies that the integrals in (18) cannot be equal for any value of \(R\). To be more concrete, since \(\lim_{R \to 0^+} \tilde{\Lambda}_i(R)\) goes as \((1/R^2) \times (zero\ mode\ contribution)_i\), (18) is an inequality for “small” \(R\). This implies that we cannot naively use the theorem of [11, 12, 13]. The main subtlety in our case is the fact that on the left hand side of

\[
\frac{1}{744} \int_0^{+\infty} s^{-2} ds \int_{-\frac{1}{2}}^{+\frac{1}{2}} j(\theta + is) d\theta \sum_{r=-\infty}^{+\infty} \exp\left\{-\frac{2\pi R^2}{s} \right\}
\]

\[
\frac{1}{744} \int_F \frac{d^2\tau}{\tau_2} j(\tau) \sum_{m,n \in \mathbb{Z}} \exp\left\{-\frac{2\pi R^2}{\tau_2} \right\}
\]
a precise order of integration has been taken \cite{14}. The application of the theorem of \cite{11, 12, 13} to this case must be seen as a redefinition of the integral on the planar region $S = \{ \tau \mid \tau_2 > 0, -\frac{1}{2} \leq \tau_1 < \frac{1}{2} \}$ \cite{4} in a way consisting in mapping the region $\{ S - F \}$ into $F$ an infinite number of times plus a prescription about how to integrate over a neighborhood of infinity in each of these infinitely many copies of $F$. So we have an infinite number of changes of variables (labeled by a couple of coprime integer numbers) and a prescription that by inverting the mappings would correspond to a way of integrating in the vicinity of zero. The validity of these changes of variables depends on the behaviour of the integrand on the left hand side of the equation (22) as a function of the modular parameter. From the modular invariance of $j(\tau)$ one can conclude that when $R > 1$ the integrand is regular (bounded) at zero. Therefore this complicated way of obtaining the integral over $S$ will give the Riemann integral which is equal to the iterated integration at least in the region in which $R > 1$. Furthermore, the fact that we know that $\tilde{\Lambda}_2(R)$ has only one singular point means that $\tilde{\Lambda}_1(R) = \tilde{\Lambda}_2(R)$ for $R \geq R_0$. Taking this together with (20) we get that $\tilde{\Lambda}_1(R) > \tilde{\Lambda}_2(R)$ for $R < R_0$.

4 Conclusions

We have shown that the description of string theories as collections of fields breaks down when the typical length of the target space is smaller than the Planck length, $\sqrt{\alpha'}$. In the cases we have treated the physical explanation for this phenomenon can be enlightened by seeing that equation (18) is equivalent to knowing when

$$M(R) =: \int \frac{d^2 \tau}{\tau_2} J(\tau) \sum_{m,n \in \mathbb{Z}} \exp \left\{ -\frac{2\pi R^2}{\tau_2} |m\tau + n|^2 \right\} \neq 0 \quad (22)$$

where $J(\tau) = j(\tau) - 744$. We have proven that this integral vanishes when $R \geq R_0$ and, by duality symmetry that

$$M(R) = (1 - \frac{1}{2R^2}) \frac{24\pi}{3} \quad R \leq R_0 \quad (23)$$

Moreover, the jump of the first derivative at $R_0$ can be easily seen equal to $16\pi/R_0$ as computed in section 2. For all of our examples this stems from the fact that duality in addition with the description of the string as a collection of quantum fields gives the partition function as a function of $R$ for all $R > 0$.

Physically, $M(R)$ when $R \geq R_0$ would measure the quantum-field degrees of freedom of the model resulting from taking the left-moving part of the bosonic string after compactifying it using the Leech lattice \cite{6} and modding out the states at the massless level \cite{15, 6}. In other words, $M(R)$ measures nothing from a quantum-field theoretical point of view. However in \cite{10} it has been shown that $J(\tau)$ is the partition function corresponding to a string realization
of the symmetry encoded in the finite group $F_1$ usually called the Monster. What we see is that below the Planck length strings have physical degrees of freedom which correspond to non-propagating states. Therefore, it appears that Atkin-Lehner symmetry [6, 7] might be, as a stringy effect, more fundamental than previously thought.

One may guess that the one loop vacuum energy of the continuum Liouville theory coupled to conformal matter with $c = 1$ stems from the contribution of the so-called "co-dimension two" states [5] whose propagator cannot be defined. Neither we know how to prove this guess nor whether there is any relationship between the kind of physical information in $F_1$ and these states. What we know is that

$$\int_F \frac{d^2\tau}{\tau_i^2} = -\frac{1}{24} \int_F \frac{d^2\tau}{\tau_i^2} J(\tau) \quad (24)$$

namely, we have a relation between the vacuum energy corresponding to the $c = 1$ non-critical string and the Monster group.

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