The Trans-Planckian Problem in the Healthy Extension of Horava-Lifshitz Gravity

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(Dated: May 10, 2014)

Planck scale physics may influence the evolution of cosmological fluctuations in the early stages of cosmological evolution. Because of the quasi-exponential redshifting, which occurs during an inflationary period, the physical wavelengths of comoving scales that correspond to the present large-scale structure of the Universe were smaller than the Planck length in the early stages of the inflationary period. This trans-Planckian effect was studied before using toy models. The Horava-Lifshitz (HL) theory offers the chance to study this problem in a candidate UV complete theory of gravity. In this paper we study the evolution of cosmological perturbations according to HL gravity assuming that matter gives rise to an inflationary background. As is usually done in inflationary cosmology, we assume that the fluctuations originate in their minimum energy state. In the trans-Planckian region the fluctuations obey a non-linear dispersion relation of Corley-Jacobson type. In the "healthy extension" of HL gravity there is an extra degree of freedom which plays an important role in the UV region but decouples in the IR, and which influences the cosmological perturbations. We find that in spite of these important changes compared to the usual description, the overall scale-invariance of the power spectrum of cosmological perturbations is recovered. However, we obtain oscillations in the spectrum as a function of wavenumber with a relative amplitude of order unity and with an effective frequency which scales nonlinearly with wavenumber. Taking the usual inflationary parameters we find that the frequency of the oscillations is so large as to render the effect difficult to observe.

PACS numbers:

I. INTRODUCTION

The inflationary scenario [1] has become the current paradigm to describe the early evolution of the universe. The success of this theory comes from a number of predictions which have been confirmed by the observations, the main one being the causal mechanism for the generation of the primordial cosmological perturbation which inflation provides [2]. Inflation is generally modelled by General Relativity (GR) coupled to scalar field matter. The scalar field (called inflaton) is responsible for the accelerated expansion of space. The success of inflation as a theory for the origin of structure comes from the fact that the wavelengths of fixed comoving modes which are of cosmological interest today, namely wavelengths corresponding to the present large-scale structure and to the cosmic microwave background anisotropies, were sub-Hubble at the beginning of the period of inflation. They cross the Hubble horizon about 50 - 60 Hubble times, or e-folds, before the end of inflation. Thus, inflation must last at least that long to solve the problems of the standard cosmological model (SCM) [3, 4].

Most inflationary models have a period of accelerated expansion which lasts much more than the 50 - 60 e-folds required solve the SCM problems. If the period of inflation last for more than about 70 e-folds (this number depends very slightly on the energy scale at which inflation takes place - to get the above number we are assuming that it is the Grand Unification scale), then all of the wavelengths important today were smaller that the Planck length at the beginning of inflation. In this regime we cannot trust the calculations since General Relativity will no longer yield a good description of the physics. This is the known trans-Planckian problem for cosmological fluctuations [3]. A correct computation of both the generation and evolution of fluctuations in the trans-Planckian regime must be done in the correct UV completion of the theory.

This problem is named after the analogous trans-Planckian problem for Hawking radiation in black-hole physics
In the cosmological context this problem was discussed in the literature in many different toy model modifications of General Relativity coupled to a canonical scalar field. These models describe the evolution on sub-Planckian wavelengths by imposing ad-hoc modifications of the usual physics, e.g. modified dispersion relations or non-commutativity (see also [10]). It is possible to construct toy models in which the standard predictions of inflation are changed, in others they are maintained either fully or up to small corrections. For example, imposing the usual initial conditions on a time-like “new physics hypersurface” given by the physical wavelength being some fixed UV cutoff scale smaller or equal to the Planck scale yields a scale-invariant power spectrum of curvature fluctuations with small amplitude superimposed oscillations. If, on the other hand, we assume that a short period of inflation is preceded by a short non-singular bounce and (before that) the time reverse of the background evolution, then a deep red spectrum with index $n_s = -3$ results. However, to go beyond toy model studies, we must be able to perform the analysis of cosmological fluctuations in a full quantum gravity theory.

Horava-Lifshitz (HL) gravity is a candidate theory of quantum gravity which provides a well motivated framework for describing the evolution of the universe on trans-Planckian scales. It is an theory of gravity in 3+1 dimensions which uses the same metric degrees of freedom as General Relativity but which is power-counting renormalizable (with respect to the scaling symmetry to be introduced below). This is achieved by picking a preferred time direction, thus abandoning the full space-time diffeomorphism invariance (reducing the symmetry to simply spatial diffeomorphisms), and by introducing an anisotropic scaling of Lifshitz type. The loss of space-time isotropy has as consequence the absence of Lorentz symmetry, a symmetry which appears as an emergent symmetry in the infrared (IR), where GR is recovered.

As a consequence of the loss of the space-dependent time reparametrization symmetry of General relativity, one of the four gauge symmetries for cosmological perturbations (namely the space dependent time rescaling symmetry) is lost, and since the theory presents the same number of basic degrees of freedom as General Relativity, there is an extra scalar perturbation mode (scalar in terms of its transformation under spatial rotations) (11). Since it was proposed by Horava in its most simple version [15], many other versions of the theory were proposed in the literature. We have mainly two versions: the "projectable", where the lapse function depends only on time to mimic the reduced symmetry of the problem; or the "non-projectable" version, where the lapse function depends on space and time as in GR. In the "projectable" version the extra degree of freedom of the theory is dynamical and ghost-like or tachyonic. This version is also plagued by "strong coupling" problems [19, 20], which may spoil the validity of this setup.

In the original form of the "non-projectable" version, the extra degree of freedom is non-dynamical at linear order about a spatially homogeneous and isotropic background. "Strong coupling" problems may arise. The most studied type of a "non-projectable" theory (one which contains all of the terms which are allowed by power-counting renormalizability and symmetry) is the "healthy extension" [23] that appears to fix these "strong couplings" problems and pathologies associated with the extra degree of freedom that appear in the other versions of the theory [14]. Fluctuations in the healthy extension of HL gravity were first considered in [23]. As it was shown in [26], the extra degree of freedom now becomes dynamical. With appropriate choices of the parameters in the HL Lagrangian, the extra degree of freedom is neither ghost-like nor tachyonic. It is important only in the UV and decouples in the IR. This is interesting from the point of view of cosmology since we know, from observations, that we can only have one propagating scalar metric degree of freedom in the IR. On the other hand, the fact that in the UV there are two propagating degrees of freedom makes the theory a promising one to study in the context of the trans-Planckian problem for fluctuations [15].

In the presence of spatial curvature, HL gravity can lead to bouncing cosmological solutions [29]. Here, however, we will assume the existence of scalar field matter which leads to an inflationary phase during the evolution of the early universe. We will use the "healthy extension" of the HL gravity to study the trans-Planckian problem for fluctuations. We are interested in checking, based on a well motivated description of the UV and IR theory, if the predictions of inflation for cosmological perturbations are altered. We will have to take into account two effects: the change in the physics in the UV region that will be manifest in a modified dispersion relation of Corley-Jacobson type [30] (as the ones which were studied in [14]); and the presence of an extra degree of freedom that will be dynamical in the UV (for wavelengths smaller that Hubble radius in our case). This extra degree of freedom can be interpreted as an "entropy" perturbation, in analogy with what occurs in multi-field inflation models [31], but in this case the extra field is a scalar perturbation of the metric. The entropy mode is coupled to the "adiabatic perturbation" and will influence its dynamics while the wavelength is sub-Hubble. Thus, we need to analyse the coupled system of differential equations for both scalar perturbations modes in the different regimes of the theory in order to evaluate the final power spectrum of the "adiabatic" mode.

The article is organized as follows. In Section III we give a brief description of the "healthy extension" of HL, setting up the theory we are using and calculating linear perturbations about a homogeneous and isotropic background. We show how the reduced symmetry group gauges only one of the degrees of freedom, leaving one extra degree of freedom, by calculating the second order action. In Section IV we study the cosmological perturbations. First we find the
canonical variables of the problem and show that one of these corresponds to the usual Mukhanov-Sasaki variable and the other decouples in the IR. We take the UV limit of the theory to find the contributions of the trans-Planckian physics. The initial state is calculated by minimizing the vacuum energy and the solutions to the inhomogeneous equations of motion are found in each region of the problem. Finally, the power spectrum is calculated. We find that the overall scale-invariance of the spectrum is maintained, but that there are superimposed oscillations with an effective frequency depending on the wavenumber (very different from the oscillations which are seen in the approach of [11] to the trans-Planckian problem). The amplitude of these oscillations is not suppressed (again in contrast to what is seen in [11]). However, the frequency of oscillation is very large making the effect extremely difficult to measure.

II. HEALTH EXTENSION OF HORAVA-LIFSHITZ

A. Background

In HL theory it is natural to use the ADM (Arnowitt, Desner and Misner) formalism that separates space-time into time and spatial foliations:

$$ds^2 = -N^2 dt^2 + g_{ij} (dx^i - N^i dt) (dx^j - N^j dt),$$

where $g_{ij}(t, x)$ is the metric of the spatial section, $N$ is the lapse function and $N^i(t, x)$ is the shift vector. As mentioned, the lapse function can be restricted to depend only on time, yielding the "projectable" theory, or else it is taken to depend on both space and time, yielding the "non-projectable" version. The "healthy extension" is a non-projectable version of the theory with all of the terms allowed by the residual symmetries and by power-counting renormalizability included. Thus $N(t, x)$ and we describe the same lapse function as in GR. In this case, we must include in the action terms proportional to the quantity:

$$a_i = \frac{\partial_i N}{N}.$$  (2)

Because of the anisotropic scaling, the classical scaling dimensions of the coordinates of the theory (when the scaling dimension is $z = 3$ as it is in four space-time-dimensional HL gravity) are, in units of spatial momenta: $[t]_s = -3$, $[x_i]_s = -1$. With this we have that $[\nabla \equiv \partial_\mu \partial^\mu] = 2$. The ADM fields have scaling dimension: $[g_{ij}]_s = [N]_s = 0$ and $[N]_s = 2$.

The action of HL gravity is constructed by including all terms with scaling dimension smaller or equal to six, in order to obtain a theory which is power-counting renormalizable, and which are invariant under the spatial diffeomorphisms, i.e. under elements of the symmetry group $\text{Diff}_L$. The "healthy extension" action can be written as, following [26] (see also [32, 33]):

$$S = \chi^2 \int dt^3 x \sqrt{\mathcal{g}} \left( \mathcal{L}_{\text{kin}} - \mathcal{L}_V - \mathcal{L}_E + \chi^{-2} \mathcal{L}_M \right),$$

where $\chi^2 = 1/16\pi G$. The kinetic term is the one that contains time derivatives of the metric in a covariant way with respect to the new foliating preserving symmetry, $\text{Diff}_L$:

$$\mathcal{L}_{\text{kin}} = K_{ij} K^{ij} - \lambda K^2,$$

where $K_{ij} = (1/2N) (\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i)$ is the extrinsic curvature of the constant time hypersurfaces. The potential term is constructed by including all terms that preserve $\text{Diff}_L$ until the sixth spatial derivative:

$$\mathcal{L}_V = 2\Lambda - \mathcal{L}_V = 2\Lambda - \mathcal{L}_V = 2\Lambda - \frac{1}{\chi^2} \left( g_{2} R^2 + q_{3} R_{ij} R^{ij} \right) + \frac{1}{\chi^2} \left[ q_{4} R^3 + q_{5} R_{ij} R^{ij} + q_{6} R_{ij} R^{ij} \right] + g_{7} R^2 R + g_{8} (\nabla_i R_{jk}) (\nabla^i R^{jk}) + \cdots$$

where $\Lambda$ is the cosmological constant and has dimension of momentum squared and the coefficients are dimensionless. The new components proportional to $a_i$ that appear in the "healthy extension" are:

$$\mathcal{L}_E = -\eta_{a_i} a^i + \frac{1}{\chi^2} \left( \eta_{2} a_i \Delta a^i + \eta_{3} R \nabla_i a^i + \ldots \right) + \frac{1}{\chi^4} \left[ \eta_{4} a_i \Delta^2 a^i + \eta_{5} \Delta R \nabla_i a^i + \eta_{6} R^2 \nabla_i a^i + \ldots \right].$$
where the coefficients are dimensionless. As we are interested in obtaining an inflationary background cosmology, we include matter in the form of a scalar field, with matter Lagrangian [34]

\[ \mathcal{L}_M = \frac{1}{2N^2} \left( \dot{\phi} - N^i \partial_i \phi \right)^2 + V(\phi, \partial_i \phi, g_{ij}) , \]  

(7)

where the potential terms are given by:

\[ V(\phi, \partial_i \phi, g_{ij}) = V_0 + V_1 P_0 + V_2 P_1^2 + V_3 P_1^3 + V_4 \phi P_2 + V_5 P_5 P_2 + V_6 P_1 P_2 , \]  

(8)

with \( P_0 \equiv (\nabla \phi)^2 \), \( P_1 \equiv \Delta^i \phi \) and \( \Delta \equiv g^{ij} \nabla_i \nabla_j \). The scalar field has scaling dimension \( [\phi]_s = (d - z)/2 = 0 \). In the way it is written above, the coefficients are not dimensionless as the previous coefficients are. If we want to write them as dimensionless coefficients we have to make the following substitution: the order 2 coefficient \( V_1 \) is already dimensionless; the order four coefficients are changed to \( V_2 \rightarrow V_2/\chi^2 \) and \( V_4 \rightarrow V_4/\chi^2 \), with \( V_2 \) and \( V_4 \) dimensionless; and the dimension six coefficients are changed to \( V_3 \rightarrow V_3/\chi^4 \), \( V_5 \rightarrow V_5/\chi^4 \) and \( V_6 \rightarrow V_6/\chi^4 \), with \( V_3 \), \( V_5 \) and \( V_6 \) dimensionless.

The non-projectable version of HL can contain more than 60 terms. Here we only considered the ones that are important for the scalar part of the perturbations, the perturbations we interested in. In the IR the GR action is recovered when \( \lambda = 1 \).

We can now obtain the Hamiltonian and momentum constraints. Varying the action with respect to \( N^i (t, x^j) \), we have the momentum constraint:

\[ \nabla_i \pi^{ij} = \frac{k^2}{4} J^i , \]  

(9)

where \( \pi^{ij} \equiv \frac{\delta \mathcal{L}}{\delta \dot{g}_{ij}} = -K^{ij} + \lambda K g^{ij} \) and \( J_i \equiv -N \frac{\delta \mathcal{L}_E}{\delta N} = \frac{1}{N} \left( \phi - N^k \nabla_k \phi \right) \nabla_i \phi \). Note that (9) is a local constraint.

The Hamiltonian constraint can be obtained by varying the action with respect to the lapse function, \( N(t, x^i) \):

\[ \mathcal{L}_{cin} + \mathcal{L}_{pot} + \mathcal{L}_E + N \frac{\delta \mathcal{L}_E}{\delta N} = \frac{1}{4 \chi^2} J^i , \]  

(10)

with \( J^i \equiv 2 \left( N \frac{\delta \mathcal{L}_E}{\delta N} + \mathcal{L}_\phi \right) \) and the new terms from "healthy extension":

\[ N \frac{\delta \mathcal{L}_E}{\delta N} = 2\eta \nabla_i a^i - \frac{2\eta_2}{\chi^2} \Delta \nabla_i a^i + \frac{\eta_3}{\chi^2} \Delta R - \frac{2\eta_4}{\chi^4} \Delta^2 \nabla_i a^i + \frac{\eta_5}{\chi^4} \Delta^2 R + \frac{\eta_6}{\chi^6} \Delta R^2 + \ldots . \]  

(11)

This constraint is a local constraint, unlike what happens in the "projectable" version of the theory.

One might think that the presence of two local constraints should lead to a reduction of the number of degrees of freedom of the theory to have the same number as in General Relativity. However, this is not the case. The local Hamiltonian constraint is not able to decouple the extra degree of freedom in all the regimes of the theory, only in the IR. We can see this by studying cosmological perturbation theory.

**B. Perturbations**

Perturbing about an isotropic and homogeneous universe described by the spatially flat FRW metric,

\[ ds^2 = -dt^2 + a^2 \delta_{ij} dx^i dx^j , \]  

(12)

the background values and scalar perturbation of the ADM variables are given by (3):

\[ N + \delta N \left( t, x^k \right) = 1 + \nu \left( t, x^k \right) , \]  

(13)

\[ N_i + \delta N_i \left( t, x^k \right) = \partial_i \mathcal{B} \left( t, x^k \right) , \]  

(14)

\[ g_{ij} + \delta g_{ij} \left( t, x^k \right) = a^2 \delta_{ij} + a^2 \left( t \right) \left[ -2\psi \left( t, x^k \right) \delta_{ij} + a^2 \left( t \right) \right] . \]  

(15)

We must also consider the perturbation in the scalar field, given by:

\[ \phi \left( t, x^k \right) = \phi_0 \left( t \right) + \delta \phi \left( t, x^k \right) . \]  

(16)
The invariance under $Diff_F$, allow us to set $E = 0$. Because of the reduced symmetry we can no longer constrain another degree of freedom of the perturbations and use the standard gauges used in GR\(^3\). Thus we are left with more degrees of freedom than we would have in GR: $\nu, B, \psi$ and $\delta \phi$. We can reduce the number of them by solving the constraints of the theory.

Expanding the momentum and Hamiltonian constraints to first order, we can use them to solve only for two degrees of freedom (in a spatially flat background), namely $\nu$ and $B$. In the momentum space, we can rewrite the equation using the physical momentum $\bar{k} \equiv k/a$ (as done in \(^{22}\)):

$$d \left( \bar{k} \right) B_k(t) = -(3\lambda - 1) \left[ \frac{\ddot{\phi}_0}{\chi^2 k^2} + 2f_1 \left( \bar{k} \right) \right] \dot{\psi}_k(t) - (3\lambda - 1) \left( H \frac{\ddot{\phi}_0}{\chi^2 k^2} \delta \dot{\phi}_k(t) - 4(3\lambda - 1)f_2 \left( \bar{k} \right) H \psi_k(t) - \right. \left(3\lambda - 1 \right) \left[ V_{\alpha,\phi} \left( \phi_0 \right) + V_4 \left( \phi_0 \right) \bar{k}^4 H + 3(3\lambda - 1) \phi_0 H^2 - \frac{\ddot{\phi}_0}{\chi^2 k^2} - \phi_0 f_1 \left( \bar{k} \right) \right] \right),$$

$$d \left( \bar{k} \right) \nu_k(t) = (\lambda - 1) \frac{\ddot{\phi}_0}{\chi^2 k^2} \delta \dot{\phi}_k(t) - 4(3\lambda - 1) H \dot{\psi}_k(t) + 4(\lambda - 1)f_2 \left( \bar{k} \right) \bar{k}^2 \psi_k(t) + \right. \left\{ (3\lambda - 1) \phi_0 H + (\lambda - 1) \left[ V_{\alpha,\phi} \left( \phi_0 \right) + V_4 \left( \phi_0 \right) \bar{k}^4 \right] \right\} \frac{\delta \phi_k(t)}{\chi^2},$$

where we introduced the following functions to simplify these equations:

$$f_1 \left( \bar{k} \right) \equiv -\eta + \eta_2 \frac{\bar{k}^2}{\chi^2} + \eta_4 \frac{\bar{k}^4}{\chi^4}, \quad f_2 \left( \bar{k} \right) \equiv -1 + \eta_3 \frac{\bar{k}^2}{\chi^2} + \eta_4 \frac{\bar{k}^4}{\chi^4},$$

$$d \left( \bar{k} \right) \equiv 4(3\lambda - 1) H^2 + (\lambda - 1) \left[ \frac{\ddot{\phi}_0}{\chi^2} + 2f_1 \left( \bar{k} \right) \right],$$

$$= 4(3\lambda - 1) H^2 \left[ 1 + \frac{\lambda - 1}{2(3\lambda - 1) \chi^2 H^2} + \frac{\lambda - 1}{2(3\lambda - 1)} \left( -\eta + \eta_2 \frac{\bar{k}^2}{\chi^2} + \eta_4 \frac{\bar{k}^4}{\chi^4} \right) \frac{\bar{k}^2}{H^2} \right],$$

Since we have the function $d \left( \bar{k} \right)$ in both solutions, $B_k(t)$ and $\nu_k(t)$ are regular in the limit $\lambda \to 1$ as long as $H \neq 0$. The second expression in \(^{21}\) is valid only when $\lambda \neq 1/3$ and $H \neq 0$. From this, we can see that the value of the physical momentum that separates the regions of high and low energy is proportional to $H$, the square of the corresponding length being $\frac{\Lambda^{-1}}{2(3\lambda - 1) H^2}$. We are going to see later that in the de Sitter case this corresponds exactly to the Hubble horizon, the scale where the cosmological perturbations freeze out. Thus, the scale that separates the UV and IR limits in the "healthy extension" of HL gravity it is the same as the event horizon of de Sitter inflation. Beyond this length scale the evolution is similar to GR.

We can write the second order Lagrangian for the perturbations using the constraints \(^{18}\) in the following way:

$$\delta_2 S_{\text{scalar}} = \chi^2 \int dt \frac{d^3k}{(2\pi)^3} a^3 \left( c_{\phi} \delta \dot{\phi}_k^2 + c_{\psi} \psi_k^2 + c_{\phi,\psi} \delta \dot{\phi}_k \dot{\psi}_k + f_{\phi} \delta \phi_k \delta \phi_k + f_{\phi,\psi} \delta \phi_k \psi_k \right) + f_{\phi,\psi} \delta \phi_k \dot{\psi}_k + \bar{f}_{\phi,\psi} \dot{\psi}_k \delta \phi_k - m_{\phi}^2 \delta \phi_k^2 - m_{\psi}^2 \psi_k^2 - m_{\phi,\psi} \psi_k \delta \phi_k \right),$$

where the coefficients are given in Appendix 1. We can see that this action has two dynamical degrees of freedom with mixed kinetic terms. They cannot be combined into one degree of freedom, the Mukhanov-Sasaki variable, $-\zeta \equiv \psi + \frac{\ddot{H}}{\ddot{\phi}_0} \delta \phi$, as is done in the case of cosmological perturbations in GR\(^3\)\(^1\) or even in the "non-projectable" version of HL gravity with detailed balance.\(^{21}\)

We cannot diagonalize the entire action. Similar to what it is done in multi-field inflation, we are going to make a rotation in field space to find the true propagating and canonical variables of the problem.\(^{46}\). For this we will diagonalize the kinetic term of the action. We can write the action in vector and matrix form as:

$$\delta_2 S = \chi^2 \int dt \frac{d^3k}{(2\pi)^3} a^3 \left[ \left( \delta \phi_k \right) \dot{\psi}_k \right] \left( \begin{array}{c} c_{\phi} \frac{1}{2} c_{\phi,\psi} \frac{1}{2} c_{\phi,\psi} \end{array} \right) \left( \delta \dot{\phi}_k \right) \left( \begin{array}{c} \ddot{\phi}_k \psi_k \end{array} \right) + \left( \delta \phi_k \right) \dot{\psi}_k \left( \begin{array}{c} \ddot{\phi}_k \psi_k \end{array} \right) \left( \begin{array}{c} f_{\phi} \dot{\phi}_k \end{array} \right) \left( \begin{array}{c} f_{\phi,\psi} \dot{\phi}_k \end{array} \right) \left( \begin{array}{c} \ddot{\phi}_k \psi_k \end{array} \right)$$

$$+ \left( \delta \phi_k \right) \psi_k \left( \begin{array}{c} \ddot{\phi}_k \psi_k \end{array} \right) \left( \begin{array}{c} \ddot{\phi}_k \psi_k \end{array} \right) \left( \begin{array}{c} \ddot{\phi}_k \psi_k \end{array} \right) \left( \begin{array}{c} \ddot{\phi}_k \psi_k \end{array} \right),$$

(23)
Diagonalizing the kinetic term which we call $C$, we find that the eigenvalues are given by \[ 47:\]

\[
d (\tilde{k}) \lambda_{1,2} = \frac{(3\lambda - 1)}{\chi^2} \left\{ \left( H^2 + \dot{\phi}_0^2 \right) + f_1 (\tilde{k}) \tilde{k}^2 \left[ \frac{(\lambda - 1)}{2(3\lambda - 1)} + 2\chi^2 \right] \right\} \\
\pm \sqrt{\frac{(3\lambda - 1)^2}{\chi^4} \left\{ (H^2 + \dot{\phi}_0^2)^2 + 2 \left( H^2 - \dot{\phi}_0^2 \right) \left[ \frac{(\lambda - 1)}{2(3\lambda - 1)} - 2\chi^2 \right] + f_1^2 (\tilde{k}) \tilde{k}^4 \left[ \frac{(\lambda - 1)}{2(3\lambda - 1)} - 2\chi^2 \right]^2 \right\}}.
\]

Given these eigenvalues we have to determine the rotation matrix $S$ that diagonalizes $C$ and then apply it to the entire action. Imposing the normalization condition, the eigenvectors of $C$ can be determined and the rotation angle can be inferred from:

\[
\tan(2\theta) = \frac{c_{\phi\psi}}{c_{\psi} - c_{\phi}}.
\]

We need to apply the diagonalization matrix in the entire action to write it in terms of the new variables. As a result of this algebra we find that, up to total derivative terms, we have that the action in terms of the new basis with diagonalized kinetic term is given by:

\[
\delta_2 S = \int \frac{d^3k}{(2\pi)^3} a^3 \left\{ c_{3\phi_3} + c_{2\phi_4} + \tilde{F}_{\phi_3\phi_3} \tilde{\phi}_3 \tilde{\phi}_3 + F_{\phi_4\phi_4} \tilde{\phi}_4 \tilde{\phi}_4 + F_{\phi_3\phi_4} \tilde{\phi}_3 \tilde{\phi}_4 + \tilde{F}_{\phi_3\phi_4} \tilde{\phi}_3 \tilde{\phi}_4 - M^2_{\phi_3\phi_3} - M^2_{\phi_3\phi_4} - M^2_{\phi_4\phi_4} \right\}.
\]

where the new rotated variables are given by

\[
\begin{pmatrix} \phi_3 \\ \phi_4 \end{pmatrix} = S^t c_2
\]

and the new coefficients are given in Appendix 2.

### III. COSMOLOGICAL PERTURBATIONS AND THE POWER SPECTRUM

Because of the power-counting renormalizability, HL gravity can contain terms up to sixth order in spatial derivatives. The terms of highest order in momentum dominate the evolution in the UV. As pointed out in \[32, 36\] this will lead to a scale invariant initial power spectrum of vacuum fluctuations. The reason is that the form of the state which minimizes the HL Hamiltonian is different from the one which minimizes the usual Hamiltonian of a Lorentz-invariant theory. The difference in Hamiltonians also affects the evolution of the state. We have to investigate the evolution of the perturbations that are generated inside the trans-Planck region until the IR region where we recover the GR evolution and verify how the new physics in the UV modifies the power spectrum.

Another effect in the ”healthy extension” is the presence of an extra degree of freedom that will only be dynamical in the UV region of the theory, inside the Hubble radius. The coupling of this extra mode to the adiabatic mode can also be a source of modifications of the power spectrum of the adiabatic fluctuation mode. In the following we will investigate these effects.

#### A. Equations of Motion

It is convenient to perform a change of variables to absorb the multiplicative factors in front the kinetic term and absorb the scale factor. The rescaled variables are:

\[
u_k = a \sqrt{\lambda_1} \phi_3, \quad u_k = a \sqrt{\lambda_2} \phi_4.
\]

Up to terms which are total derivatives, the action becomes:

\[
\delta_2 S \propto \chi^2 \int d\eta \frac{d^3k}{(2\pi)^3} \left\{ u_k'^2 + v_k'^2 + \frac{\tilde{F}_{\phi_3\phi_4} a}{\sqrt{\lambda_1 \lambda_2}} u_k'v_k + \frac{\tilde{F}_{\phi_3\phi_4} a}{\sqrt{\lambda_1 \lambda_2}} u_k'^4 + \omega^2 u_k'^2 + \omega^2 v_k'^2 - M^2_{uu} u_k v_k \right\},
\]

...
The evolution of these variables is of the form of coupled oscillators with time dependent mass, where the right side the coefficients of Appendix 2 and (38,39).

In this way, we will determine the coefficients in this regime, using

We want to study the dynamics in the UV limit, as we are interested in investigating the trans-Planckian problem and the power spectrum at Hubble horizon exit. In this way, we will determine the coefficients in this regime, using the coefficients of Appendix 2 and (38,39).

After these calculations, some consistency checks must be done to see if the system describes the expected physics:

1. First consistency check: We need to check if the variable \( \phi_3 \) is the Mukhanov-Sasaki variable, the one that we want to be the right canonical variable described by the perturbations in the IR. In order to check this, let's evaluate \( \phi_3 \) and \( \phi_4 \):

\[
\phi_3^2 = \frac{1}{2}(1 + \cos 2\theta) \delta \phi_k^2 + \frac{1}{2}(1 - \cos 2\theta) \psi_k^2 + 2\theta \delta \phi_k \psi_k ,
\]

\[
\phi_4^2 = \frac{1}{2}(1 - \cos 2\theta) \delta \phi_k^2 + \frac{1}{2}(1 + \cos 2\theta) \psi_k^2 - 2\theta \delta \phi_k \psi_k ,
\]

and using \( (39) \) we can see that in the IR and when \( H^2 \gg \phi_3^2 / \lambda^2 \) we have:

\[
\phi_3 \sim \delta \phi_k + \frac{H}{\dot{\phi}_0} \psi_k , \quad \phi_4 \sim \frac{H}{\dot{\phi}_0} \delta \phi_k - \psi_k .
\]

We can see that the new variable \( \phi_3 \) has the form of the usual Mukhanov-Sasaki variable in the IR and it will hence describe the right observable.

2. Second consistency check: Now, we need to verify that the extra variable \( \phi_4 \) decouples in the IR, which means that its mass diverges in this limit, which then leads to the presence of a single propagating degree of freedom. Considering \( H^2 \gg \phi_0^2 \), we obtain that in the IR the eigenvalues are:

\[
\lambda_1 \to \frac{1}{2\lambda^2} , \quad \lambda_2 \to -\eta \left( \frac{k}{aH} \right)^2 ,
\]

and in the UV:

\[
\lambda_1 \to \frac{1}{2\lambda^2} , \quad \lambda_2 \to \frac{3\lambda - 1}{\lambda - 1} .
\]

From \( (38) \), \( (34) \) and \( (32) \) we can see that the mass terms of the equations are divided by the respective eigenvalue. So, from \( (35) \) we infer that \( \lambda_2 \) goes to zero in the IR when \( k \to 0 \) and its mass goes to infinity. The extra degree of freedom \( \phi_4 \) decouples in this limit, as required for the theory to have an IR behaviour compatible with observations.

Having verified the consistency relations we need to evaluate the coefficients appearing in the equations of motion.

The evolution of these variables is of the form of coupled oscillators with time dependent mass, where the right side the coefficients of Appendix 2 and (38,39).
B. UV Limit

In the UV, the most important contributions come from the highest orders in $k$. Therefore we will only consider the highest order contributions in each coefficient:

$$
\begin{align*}
F_{\phi_3} &\propto O(1/k^2); \\
F_{\phi_4} &\propto O(1/k^2); \\
\tilde{F}_{\phi_3\phi_4} &\propto O(\text{const.}); \\
M_{\phi_3}^2 &\propto O(k^6); \\
M_{\phi_4}^2 &\propto O(k^6); \\
M_{\phi_3\phi_4}^2 &\propto O(k^4);
\end{align*}
$$

(40)

The contributions of $F_{\phi_3}$ and $F_{\phi_4}$ are negligible in comparison with $M_{\phi_3}^2$ and $M_{\phi_4}^2$, which are also present in $\omega_3^2$ and in $\omega_4^2$, respectively. Hence we can simplify the equations of motion in the UV by considering $\omega_3^2 \sim -2M_{\phi_3}^2 + \mathcal{H}' + \mathcal{H}^2$ and $\omega_4^2 \sim -2M_{\phi_4}^2 + \mathcal{H}' + \mathcal{H}^2$:

$$
\begin{align*}
&u''_k + \left( \frac{\alpha^2 k^6}{a^4} + \frac{\beta^2 k^4}{a^2} + c^2_k - \mathcal{H}' - \mathcal{H}^2 \right) u_k = -v'_k a \Xi_2 + v_k \left( \frac{\mathcal{X}_3 k^4}{a^2} + \Xi_3 a^2 \right), \\
v''_k + \left( \frac{\delta^2 k^6}{a^4} + \frac{\epsilon^2 k^4}{a^2} + c'_k - \mathcal{H}' - \mathcal{H}^2 \right) v_k = -u'_k a \Xi_2 + u_k \left( \frac{\mathcal{X}_3 k^4}{a^2} + \Xi_3 a^2 \right),
\end{align*}
$$

(42)

(43)

where in the interaction terms we also considered the constant terms (which are the leading ones) of $F_{\phi_3\phi_4}$ and $\tilde{F}_{\phi_3\phi_4}$ since they are accompanied by a time derivative term. Without a careful analysis of the influence of these terms we cannot neglect them. The remaining coefficients are given in Appendix 3 in terms of the dimensionless coefficients. We can see that the terms have the expected scaling dimension since we know that $[\omega^2] = 2$.

The coefficient functions of the $u_k$ and $v_k$ terms on the left hand side of the above equations determine the dispersion relations of the mode functions in the UV. The dominant term is the $k^6$ term. These dispersion relations encode the change in the physics in the UV due to the presence of the higher space derivative terms in the HL action. These dispersion relations are of the Coley-Jacobson type and have been used in [7] to study the trans-Planckian problem for cosmological perturbations in the context of toy models.

As is obvious from the above equations, in the "healthy extension" of HL gravity we have to consider two effects in the evolution of fluctuations in the UV regime which are different from what is obtained using GR: firstly, the UV modification of the dispersion relation introduced by the UV complete action leads to a different form of the initial modification of the dispersion relation introduced by the UV complete action leads to a different form of the initial dispersion relation of the modes on trans-Planckian scales, and secondly there is an extra degree of freedom for scalar metric fluctuations which is active in the UV. Through the mode coupling, fluctuations in this second mode induce growing perturbations of the curvature fluctuations. If we imagine imposing initial conditions for all fluctuation modes at some initial time $t_i$, then short wavelengths are subject to the presence of the source term for a longer period than long wavelengths. Hence, one might expect that mode mixing will lead to important deviations from scale-invariance.

The effects of modified dispersion relations on the spectrum of fluctuations in inflationary cosmology was studied in detail in [7] (see also [33] for a review). It was found that if the dispersion relation is of the type for which the adiabatic evolution of the mode function is maintained, then no deviations from scale invariance of the perturbation spectrum results. The dispersion relation which we obtain is in this class. Hence, we expect that deviations from scale-invariance of the spectrum of curvature fluctuations in an inflationary universe in HL gravity can only come from the mode mixing effect.

After these preliminary discussions, we turn to the solution of the above system of coupled equations. In the linear approximation which we are using each Fourier mode evolves independently. The behaviour of the mode functions is different in the three regions of space-time indicated in Figure 1. Region I is the region where the $k^6$ term in the dispersion relation dominates, i.e. where $\lambda \ll l_c$. The wavelength $l_c$ where the $k^6$ term ceases to dominate is given by the mass parameters in the higher space derivative terms in the HL action. Based on cosmic ray constraints we know that $l_c$ must be smaller than the Planck length $l_{pl}$. In principle there could be a small region of wavelengths where the $k^4$ term in the dispersion relation dominates. However, this would require tuning of parameters. We will assume that we can neglect the term in the dispersion relation proportional to $k^4$, and this is justified if the conditions $\alpha^2 \geq \beta^2 \beta^2$ (for $u_k$) and $\delta^2 \geq \epsilon^2 \epsilon^2$ (for $v_k$) are satisfied. We will hence assume that for $\lambda \gg l_c$ the linear term in the dispersion relation dominates. Thus there is a region (Region II with $l_c \ll \lambda \ll l_H$, $l_H$ being the Hubble radius) where the dispersion relation is linear, as in standard GR theory. However, since the UV/IR scale is given by the Hubble radius, $l_H$, in Region II the extra degree of freedom is still dynamical and the interaction between the degrees of freedom must be considered. Region III is the IR domain where wavelengths are larger than the Hubble radius. In Region III where $\lambda \gg l_H$ the extra degree of freedom disappears. The perturbations will freeze out and $\omega^2_{k,III} = a/\alpha$. Hence, in this region the evolution of curvature fluctuations is as in GR. We wish to calculate the power spectrum at the time $\eta_2(k)$ where the perturbation mode $k$ becomes larger than $l_H$. 


To construct the general solution to this system, we need to match the different solutions from different regions at the transition times. Considering an expansion of the universe of power-law inflation type, \(a(\eta) = l_0 \eta^{(1-b)/2}\) (note that we recover de Sitter inflation when \(b = 3\)), the matching must be performed at the times:

\[
|\eta_1^u|^{(1-b)/2} = \left(\frac{\alpha}{c_s}\right)^{1/2} \frac{k}{l_0}, \quad |\eta_1^v|^{(1-b)/2} = \left(\frac{\delta}{c_s}\right)^{1/2} \frac{k}{l_0},
\]

which is the transition between Regions I and II for the two modes, with \(l_1^u = (2\pi)(\alpha/c_s)^{1/2}\) and \(l_1^v = (2\pi)(\delta/c_s)^{1/2}\). Considering the definition of the coefficients \((32)\), we can see that:

\[
l_c = \left(\frac{4 \tilde{V}_6}{\frac{3}{2}g_7 - 2g_5}\right)^{1/4} l_{pl} = Cl_{pl}, \quad \tilde{l}_c = \left(8 g_7 - 3 g_8 + 2 \frac{\eta_5^2}{\eta_4}\right)^{1/4} l_{pl} = \tilde{C}l_{pl}.
\]

From the constraints imposed by Gamma-Rays \((38)\), \(l_c\) and \(\tilde{l}_c\) cannot be larger than the Planck length. We do not want them to be much smaller, either, otherwise the effective field theory description of HL gravity becomes questionable. Hence, we assume that the two lengths are a bit smaller but close to the Planck length, and hence \(C\) and \(\tilde{C}\) are of order one.

The transition between Regions II and III takes place at the times

\[
|\eta_2^u| = \frac{\sqrt{b^2 - 1}}{2} \frac{1}{c_s k}, \quad |\eta_2^v| = \frac{\sqrt{b^2 - 1}}{2} \frac{1}{c_s k},
\]

with \(l_H^u = 4\pi c_s l_0 |\eta|^{(3-b)/2}/\sqrt{b^2 - 1}\) and \(l_H^v = 4\pi \tilde{c}_s l_0 |\eta|^{(3-b)/2}/\sqrt{b^2 - 1}\).

### 1. Initial Conditions

Before solving this system of equations, we need to determine the initial conditions. Since in inflationary cosmology initial classical fluctuations redshift, it is usually assumed that the initial conditions are given by quantum vacuum perturbations, and that the system that will evolve following the classical equations afterwards. In order to determine this initial state, we will make use of the Hamiltonian formalism and canonically quantize the system. Since the Hamiltonian in HL gravity is different from that in GR on the UV scales which are relevant to us, the vacuum state (defined as the state which minimizes the Hamiltonian \((3)\)) will differ.

From \((29)\), it follows that the conjugate momenta for the fluctuation modes are:

\[
p_{u_k} = \frac{\partial \delta \mathcal{L}_k}{\partial u_k'} = 2u_k' + \frac{F_{\phi_3 \phi_4 \phi}}{\sqrt{\lambda_1 \lambda_2}} v_k, \quad p_{v_k} = \frac{\partial \delta \mathcal{L}_k}{\partial v_k'} = 2v_k' + \frac{\tilde{F}_{\phi_3 \phi_4 \phi}}{\sqrt{\lambda_1 \lambda_2}} u_k.
\]
Hence we can write the Hamiltonian of the system for each mode:

$$\hat{H}_k = p_{u_k}u_k' + p_{v_k}v_k' - \delta \mathcal{L}_k = (u_k'^2 - \omega_u^2 u_k') + (v_k'^2 - \omega_v^2 v_k') + M^2 u_k v_k.$$

(48)

The canonical quantization can now be done. We promote the variables to operators that satisfy the canonical commutation relations:

$$[\hat{u}_k(\eta), \hat{u}_{k'}(\eta)] = [\hat{p}_{u_k}(\eta), \hat{p}_{u_{k'}}(\eta)] = 0,$$

(49)

$$[\hat{u}_k(\eta), \hat{p}_{u_{k'}}(\eta)] = \delta^{(3)}(k - k'),$$

(50)

that come from $[\phi(x, t), p_\phi(y, t)] = i\delta^{(3)}(x - y)$ (all the others are equal to zero). We have the same relations for $v_k$.

As the Region I is sub-Planck, the initial conditions $t_i$ are given in the deep UV. In this limit, the term that dominates is the one proportional to $k^0$ and the expansion of the universe can be neglected. The interaction term is also neglected initially and our system can be considered as a free system in a Minkowski space-time. With this $p_{u_k} = 2u_k$ and $p_{v_k} = 2v_k$.

The fields of the non-interacting system can be expanded in terms of creation and annihilation operators in the standard way:

$$u(x) = \sqrt{\hbar} \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_u}} \left( \hat{a}_ke^{ikx} + \hat{a}^\dagger_k e^{-ikx} \right),$$

(51)

$$v(x) = \sqrt{\hbar} \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_v}} \left( \hat{b}_ke^{ikx} + \hat{b}^\dagger_k e^{-ikx} \right),$$

(52)

where $\omega_u^l = k^6\alpha^2/\alpha^4$ and $\omega_v^l = k^6\delta^2/\alpha^4$. We can write the Fourier components of the fields in terms of the annihilation and creation operators:

$$u_k = \sqrt{\frac{\hbar}{2\omega_u}} \left( \hat{a}_k + \hat{a}^\dagger_{-k} \right), \quad p_{u_k} = -i \sqrt{\frac{\hbar\omega_u^l}{2}} \left( \hat{a}_k - \hat{a}^\dagger_{-k} \right),$$

(53)

$$v_k = \sqrt{\frac{\hbar}{2\omega_v^l}} \left( \hat{b}_k + \hat{b}^\dagger_{-k} \right), \quad p_{v_k} = -i \sqrt{\frac{\hbar\omega_v^l}{2}} \left( \hat{b}_k - \hat{b}^\dagger_{-k} \right).$$

(54)

The creation and annihilation operators obey the commutation relations $[\hat{a}_k, \hat{a}^\dagger_{k'}] = \delta^{(3)}(k - k')$ and $[\hat{a}_k, \hat{a}^\dagger_{k'}] = \delta^{(3)}(k - k')$, with all the others equal to zero. These commutation relations are valid if the following normalization condition holds:

$$u_k' u_k' - u_k u_k' = 2i, \quad v_k' v_k' - v_k v_k' = 2i.$$ 

(55)

which is the Wronskian of the classical solutions. This normalization allows us to fix the amplitude of $u_k(\eta)$ and $v_k(\eta)$ such that it is compatible with the Heisenberg uncertainty principle.

We can now determine the initial conditions for our modes. Because of the normalization conditions (55), this task is reduced to the determination of the effective frequency at the initial time. As mentioned before, we choose our initial state to be the one that minimizes the energy. This is a good definition for the initial vacuum state since it generalizes how the vacuum state is usually defined to the case of a modified dispersion relation. As these fluctuations are initially oscillating like they would in a Minkowski space-time, our initial conditions prescription can be viewed as an application of the Einstein Equivalence Principle to our problem. Making the ansatz

$$u_k = r_k(\eta)e^{i\gamma_k}, \quad v_k = \bar{r}_k(\eta)e^{i\gamma_k},$$

(56)

we see that in order for these variables in this form to obey the commutation relations (50) and the Wronskian condition (55), we have that $r_k^2 = 1$ and $\bar{r}^2 = 1$.

The energy of the system is:

$$E_k = \frac{1}{2} \left( \left| u_k' \right|^2 + \omega_u^l |u_k|^2 \right) + \frac{1}{2} \left( \left| v_k' \right|^2 + \omega_v^l |v_k|^2 \right)$$

(57)

$$= \frac{1}{2} \left( \left| r_k' \right|^2 + \frac{1}{r_k^2} + \omega_u^l |r_k|^2 \right) + \frac{1}{2} \left( \left| \bar{r}_k' \right|^2 + \frac{1}{\bar{r}_k'^2} + \omega_v^l |\bar{r}_k'|^2 \right).$$

(58)
Considering the minimal quantum fluctuations allowed by the uncertainty principle, the minimum energy is $E_{u,k}(\eta) = \omega^I_u$ and $E_{v,k}(\eta) = \omega^I_v$ which corresponds to:

$$\begin{align*}
r_k(\eta) &= \omega^I_u^{-1/2}, \\
\dot{r}_k(\eta) &= \omega^{-1/2}, \\
\ddot{r}_k(\eta) &= 0, \\
\dot{r}_k(\eta) &= 0.
\end{align*} \tag{59}$$

With this, the initial conditions are:

$$\begin{align*}
\nu_k(\eta) &= \frac{1}{\sqrt{\omega^I_0(\eta)}} = \frac{a(\eta)}{\alpha^{1/2}k^{3/2}}, \\
v_k(\eta) &= \frac{1}{\sqrt{\omega^I_v(\eta)}} = \frac{a(\eta)}{\beta^{1/2}k^{3/2}}, \\
\nu'_k &= i\sqrt{\omega^I_u} = i\frac{\alpha^{1/2}k^{3/2}}{a(\eta)}, \\
v'_k &= i\sqrt{\omega^I_v} = i\frac{\beta^{1/2}k^{3/2}}{a(\eta)}.
\end{align*} \tag{61}$$

2. Solving the Equation of Motion

Having determined the initial conditions we can now solve the equations of the system [44] in each region and then match them. The only degree of freedom that is propagating in the IR is $u_k$ and this is the observable one that we are interested in obtaining the power spectrum in order to compare with the results obtained in standard inflation. Because of this, we are only interested in calculating the solution of $u_k$. The perturbation $v_k$ acts as an "entropy" perturbation, analogous to what happens in multi-field inflation models [31], and will source the "adiabatic" perturbation $u_k$ in Regions I and II, decoupling in Region III.

The equations of motion that we want to solve are:

$$\begin{align*}
\nu'' + \left(\frac{\alpha^2k^6}{a^4} + c^2_s k^2 - \frac{a}{a}\right)u_k &= S_u, \\
v'' + \left(\frac{\beta^2k^6}{a^4} + c^2_s k^2 - \frac{a}{a}\right)v_k &= S_v.
\end{align*} \tag{63}$$

where

$$S_u = -v_k a\Sigma_2 + v_k \left(\frac{\gamma^2 k^4}{a^2} + \Xi_3 a^2\right), \quad S_v = -v_k a\Omega_2 + u_k \left(\frac{\zeta^2 k^4}{a^2} + \Omega_3 a^2\right)$$

are the interaction terms.

The homogeneous equations of motion have the same form of the equations of motion studied in [51] in the case of the Corley-Jacobson dispersion relation. In that work it was found that the evolution depends sensitively on the signs of the coefficients of the terms in the dispersion relation. In our case we know from the construction of the "healthy" HL action that the coefficients $\alpha$, $\delta$, $c_s$, and $\zeta_s$ must be positive to avoid exponential, ghost and tachyonic instabilities. If we evaluate the "adiabaticity coefficient" for our dispersion relation in the UV region and in the case of de Sitter inflation, we obtain

$$Q(k, \eta) = \left|\frac{\alpha^2 k^4 \eta^2 + c^2_s l_0^4 a^2}{\alpha^2 k^4 \eta^2 - c^2_s l_0^4 a^2}\right|.$$ \tag{66}

and it is hence clear that $Q(k, \eta) > 1$ which implies that the adiabaticity condition is satisfied. Hence, we know that the state tracks the instantaneous vacuum state, and thus the homogeneous solutions approach those of GR for $\lambda > l_{Pl}$. The homogeneous evolution alone does not lead to differences in the power spectrum of cosmological perturbations compared to what is obtained in standard inflation.

On the other hand, in both Regions I and II these equations are going to have both a homogeneous and a particular solution, because these are the UV regions where the extra degree of freedom is dynamical and mode mixing occurs.

Let us now turn to the effect of mode interactions in the two UV regions, and proceed to evaluate the analytic solutions of the equations of motion in the presence of the interaction in each region. For simplicity, we will choose $\alpha = \delta$ and $c_s = \zeta_s$, in order to only have a common value of $l_c$ and $l_H$ for both modes.
In Region I, the equation of motion reduces to:

\[
\dddot{u}_{I,k} + \frac{\alpha^2 k^6}{a^4} u_{I,k} = S_u, \tag{67}
\]

\[
\dddot{v}_{I,k} + \frac{\delta^2 k^6}{a^4} v_{I,k} = S_v. \tag{68}
\]

We will only show here the calculations for \(u_{I,k}\), since the solution for \(v_{I,k}\) is identical, only changing the coefficients.

We are going to solve this equation using the Green’s function method with Born approximation. This means that our solution is going to be the general solution of the homogeneous equation \(L u = 0\), where \(L = (d^2/d\eta^2) + \omega^2\), plus a particular solution of the inhomogeneous equation. The particular solution is constructed using the fundamental solution of \(LG(\eta, a) = \delta(\eta - a)\), where \(G(\eta, a)\) is the Green’s function. The solution is:

\[
u_k(\eta) = \tilde{u}_k(\eta) + \int_{-\infty}^{+\infty} f_u(a) G_u(\eta, a) \, da \tag{69}
\]

where \(\tilde{u}_k\) is the homogeneous one that obeys the initial conditions \(\tilde{u}\), and second term on the right hand side is the particular solution constructed from the Green’s function.

The homogeneous solution has two independent solutions:

\[
u_{I,k}^h(\eta) = A_1|\eta|^{1/2} J_{1/2 h}(z) + A_2|\eta|^{1/2} J_{-1/2 h}(z), \tag{70}
\]

where \(z = g|\eta|^2/b\) if \(\alpha = \omega = \) const. For the \(v_{I,k}\) equation we would have \(\tilde{z} = \tilde{g}|\eta|^2/b\) where \(\tilde{g} = \delta k^3/l_0^2\). To determine \(A_1\) and \(A_2\) we use the initial conditions discussed earlier. Using the Wronskian of the Bessel functions, \([J_{\nu-1} J_{\nu}](\eta) = 2\sin(\pi/2b)/\pi z\), we obtain

\[
A_1 = \frac{g\pi}{2b\sin(\pi/(2b))} |\eta|^{(b-1)/2} u_{I,k}(\eta) J_{-1/2-b}(z_i) \left[ 1 - \frac{J_{1/2-b}(z_i)}{J_{-1/2-b}(z_i)} \right], \tag{71}
\]

\[
A_2 = \frac{g\pi}{2b\sin(\pi/(2b))} |\eta|^{(b-1)/2} u_{I,k}(\eta) J_{1/2-b}(z_i) \left[ 1 + \frac{J_{1/2-b}(z_i)}{J_{-1/2-b}(z_i)} \right], \tag{72}
\]

unless \(1/2b\) an integer, which we assume is not the case.

In this regime, \(z_i\) is large and proportional to \(l_H/\lambda(\eta_i) \gg 1\). Thus, we can use the asymptotic expansion of the Bessel functions for large arguments, \(J_\nu(x) = (2/\pi x)^{1/2} \cos(x - \nu\pi/2 - \pi/4)\), and rewrite the coefficients as:

\[
A_1 \approx \mp i \left( \frac{\pi g}{2b} \right)^{1/2} |\eta|^{(b-1)/2} u_{I,k}(\eta) J_{1/2-b} z_i e^{\pm i z_i}, \quad A_2 \approx \pm i \left( \frac{\pi g}{2b} \right)^{1/2} |\eta|^{(b-1)/2} u_{I,k}(\eta) J_{-1/2-b} z_i e^{\pm i z_i}, \tag{73}
\]

where \(z_i(\eta_i) \equiv z_i(\eta_i) + (\pi/4b) - (\pi/4)\) and \(u_{I,k}(\eta_i) \equiv z_i(\eta_i) - (\pi/4b) - (\pi/4)\).

We can rewrite the particular solution (in the case where we have two independent solutions for the homogeneous equation \(\Box \eta = \delta\)) as:

\[
\delta u(\eta) = u_{1,k}(\eta) \int_{\eta_i}^{\eta(k)} d\eta' W_1^{-1} u_{2,k}(\eta') f_u(\eta') - u_{2,k}(\eta) \int_{\eta_i}^{\eta(k)} d\eta' W_1^{-1} u_{1,k}(\eta') f_u(\eta'), \tag{74}
\]

where \(W\) is the Wronskian of the solutions with index 1 and 2 which in Region I is \(W = -2A_1A_2b\sin(\pi/2b)/\pi\). The integration limits in Region I are from \(\eta_i\) until the cutoff \(\eta(k)\), which is the transition between Regions I and II. To write the interaction, \(S_u\) we use the Born approximation where only the homogeneous solution of \(v_k\) is considered. The solution for \(v_k^h\) is the same as the one for \(u_k^h\) given in \(\Box\).

We can now evaluate the particular solution. This is done in Appendix 4, where we considered the general case and the case of de Sitter inflation. Here we only discuss the second case, because in this limit we can evaluate the integrals exactly.

As we can see from \(\Box\) and \(\Box\) that the integrals only depend on the combination \(k\eta\) and that the integration limit \(k\eta_i = const\). This shows the integrands for different modes are related by time translation, as is the late time cutoff. The cutoff corresponding to the transition between Regions I and II also obeys this symmetry. The only source of different evolution between different modes is the initial time which is fixed for all modes \(\Box\). If the integrals are
dominated by earliest times, we would expect a mode mixing to lead to a large deviation from scale-invariance, whereas if the integrals are dominated by the contribution close to the transition point between Regions I and II then scale-invariance will be preserved. We now have to check if the dependency on \( k \) which arises from the initial condition is strong enough to affect the power spectrum in an important way.

The terms \( 1/(k\eta_i)^n \) in (112), where \( n = 3, 2 \) and 6, respectively, are always very small since they depend on \( (\lambda(\eta_i)/l_c)^n \) which is small since the initial length scale is much smaller than \( l_c \). Therefore these terms can be neglected and the solution is given by:

\[
\delta u_I(\eta_1) = \pm \frac{1}{c_s^{3/2} k^{1/2}} \left\{ -\frac{\Xi_2}{8\pi^2} l_c e^{-2i\eta_1} \left( \cos \left( z_1 - \frac{\pi}{3} \right) c_1^- + \cos \left( z_1 - \frac{\pi}{6} \right) c_2^- \right) + \left[ \frac{3\gamma^2}{c_s} \frac{1}{l_c^2} e^{-2i\eta_1} - \frac{3\sqrt{T}}{16\pi^2} \Xi_3 l_c^2 e^{-2i\eta_1} \right] \left( \cos \left( z_1 - \frac{\pi}{3} \right) c_1^+ + \cos \left( z_1 - \frac{\pi}{6} \right) c_2^+ \right) \right\},
\]

(75)

where \( c_1^\pm = e^{\pm i\xi_1} \pm (-1)^{5/6} e^{\pm i\eta_1 + 5\pi/3} \) and \( c_2^\pm = e^{\pm i\xi_1} \pm (-1)^{7/6} e^{\pm i\eta_1} \) are oscillating terms in \( k \). We can read off the important result that the solution maintains the overall scale invariance of the power spectrum since the overall amplitude is proportional to \( k^{-1/2} \). The phase \( z_1 \) is also independent of scale. However, we have an oscillatory dependence on \( e^{\pm i\xi_2} \) (since \( z_i = z_i(k) \)) coming from the initial integration limit (the initial time). Note that the oscillations have a frequency which is proportional to \( k^3 \).

Note that the overall amplitude of the inhomogeneous term is smaller than that of the homogeneous one, as one can check by inserting the expressions for the various constants.

We now move on to a study of the solutions in Region 2 in which the equation of motion has a linear dispersion relation:

\[
u'' + \frac{2}{c_s^2} k^2 u_{IIk} = S_u,
\]

(76)

with a homogeneous solution given as a linear combination of two basis solutions

\[
u_{IIk}(\eta) = B_1 e^{ic_s k \eta} + B_2 e^{-ic_s k \eta}.
\]

(77)

As our homogeneous solution we take the one which matches with the full solution at the end of Region I (the effect of the mode mixing in Region II will yield the particular solution in Region II). We need to match the solutions \( u_{IIk} \) and \( u_{IIk} \) when the transition between the regions occur at \( \eta_1 \), to determine the coefficients \( B_1 \) and \( B_2 \). This gives us:

\[
B_1 = \mp \frac{u_{IIk}(\eta_1) e^{-ic_s k \eta_1}}{2 \sin(\pi/2b)} \left[ \eta_1 (-1-b)^{-2} e^{\pm i\xi_1} \left( e^{-i\eta_1} - e^{\mp i(\pi/2b)} e^{-i\xi_1} \right) + \frac{e^{-ic_s k \eta_1} i}{c_s k} \left( \delta u_I(\eta_1) - \frac{i}{c_s k} \delta u_I'(\eta_1) \right) \right],
\]

(78)

\[
B_2 = \mp \frac{u_{IIk}(\eta_1) e^{ic_s k \eta_1}}{2 \sin(\pi/2b)} \left[ \eta_1 (-1-b)^{-2} e^{\pm i\xi_1} \left( e^{i\eta_1} - e^{\mp i(\pi/2b)} e^{i\xi_1} \right) + \frac{e^{ic_s k \eta_1} i}{c_s k} \left( \delta u_I(\eta_1) - \frac{i}{c_s k} \delta u_I'(\eta_1) \right) \right].
\]

(79)

To determine the particular solution in this region, we use the same procedure as in Region I. The particular solution is calculated in Appendix 4. Again, considering the case of de Sitter inflation, it follows from (114) that both of the integration limits are constant, \( k\eta_1 = (c_s/\alpha)^{1/2} l_0 = const. \) and \( k\eta_2 = \sqrt{2}/c_s = const. \). The integration range will hence be independent of \( k \). On the other hand, a dependence on \( k \) enters via the homogeneous solution in the source term. The k-dependence of the particular solution comes completely from the coefficients \( B_1 \) and \( B_2 \) (the coefficients analogous to the above \( B_1 \) and \( B_2 \) which appear in the homogeneous solution for \( v_2 \)) which are both proportional to \( k^{-1/2} \), indicating that the overall amplitude of the inhomogeneous term will respect scale invariance. However, like in the contribution to the inhomogeneous term in Region I, we still have oscillation.

Evaluating the 9 integrals present in the solution (114), and using the asymptotic expansion of the exponential integral function for large arguments, \( Ei(z) \sim e^{-z}/z + ... \) [33, pg. 231] we consider the dominant contribution from each term. Considering that the contributions of the particular solution in Region I and its derivative in the coefficients \( B_1 \) and \( B_2 \), given in (77), are smaller that of the homogeneous one, we can neglect them. With this approximation, the particular solution becomes:

\[
\delta u_{II}(\eta_2) = \mp \frac{i}{c_s^{3/2} k^{1/2}} e^{\pm i\xi_1} \left[ -\frac{\sqrt{2}}{2} \Xi_2 l_H \ln \left( \frac{1}{2\pi^2} \frac{l_H}{l_c} \right) \alpha + \frac{\pi^2}{2} \lambda + \frac{\sqrt{2}}{4\pi^2} \Xi_3 l_H \left( \tilde{a} \sin(2\sqrt{2}) - \alpha \right) \right],
\]

(80)
later in the evolution of the universe. The power spectrum is given by:
\[ P(k) = \frac{2}{l_H^3} \delta^2 \left( \frac{l}{\lambda(\eta)} \right)^2, \]

where \[ \delta = \cos \left( \frac{\sqrt{2}(l_H/l_c) + y_1}{x_1} \pm \frac{\pi}{6} \right), \]
\[ \lambda = \sin \left( \frac{\sqrt{2}(l_H/l_c) + y_1}{x_1} \right) - \sin \left( \frac{\sqrt{2}(l_H/l_c) + x_1 \pm \pi/6} \right), \]

are constants containing terms that depend on \( z_1 \) which is independent of \( k \).

Comparing term by term, this particular solution dominates over the one from Region I. The particular solution is oscillatory in \( k \) (with a frequency which is proportional to \( k^3 \)) via the overall coefficient \( e^{\pm ix_i} \). Comparing the amplitude of the particular solution with that of the homogeneous one, we see that the overall amplitudes (outside the square bracket of (80)) are the same. The relative amplitudes of the oscillatory and constant terms in the mode functions is given by the coefficients inside the square bracket of (80). Taking the new scale appearing in the HL Lagrangian to be comparable to the Planck scale we find that the amplitudes of the first two terms are of the order one, not suppressed by factors of \( H/m_\phi \) as the oscillations obtained in [11] are. The amplitude of the third term is suppressed by the inflationary slow-roll parameter.

Since
\[ z_i = \frac{\sqrt{2}}{3} \left( \frac{l_c}{\lambda(\eta_i)} \right)^2 \left( \frac{l_H}{\lambda(\eta_i)} \right), \]

we see that the frequency of oscillation is set by the initial time and is very large in units of \( \frac{H}{\lambda(\eta)} \) (since the second factor on the right hand side of (82) is much larger than unity). Hence, these oscillations will be very hard to detect observationally given a finite frequency resolution of an experiment.

To evaluate the power spectrum of curvature fluctuations we need to have the solution in Region III and match it with the full solution in Region II at \( l_H \) crossing.

In Region III, we have the equation:
\[ u''_{III} + \frac{a''}{a} u_{III} = 0 \]

with solution \( u_{III} = C a(\eta) \). Now, we need to match the solutions when the mode exits the Hubble radius at \( \eta_2 \) to determine \( C \). At \( \eta_2 \) the solution in Region II is given by:
\[ u_{II}(\eta) = \pm \frac{1}{c_s^{3/2} k^{1/2}} e^{i x_i} \left( e^{i c_s (\eta_2 - \eta_1)} h + e^{-i c_s (\eta_2 - \eta_1)} h^* \right) + \frac{1}{2} \left( e^{i c_s (\eta_2 - \eta_1)} + e^{-i c_s (\eta_2 - \eta_1)} \right) \left( \delta u_I(\eta_1) - \frac{i}{c_s k} \delta u'_I(\eta_1) \right) + \delta u_{II}(\eta_2), \]

where \( h = e^{-i y_1} - e^{-i(x_1 \pm \pi/2)} \) and \( h^* \) its complex conjugate. Thus, the coefficient is \( C = u_{II}(\eta_2)/a \).

3. Power Spectrum

Finally we can evaluate the power spectrum of the \( \dot{\phi}_2 \), the Mukhanov-Sasaki variable. This is the only propagating mode in the IR and will be the "adiabatic" perturbation produced during inflation. The dimensionless power spectrum will be calculated at Hubble horizon crossing. Afterwards the perturbations freeze until re-enter the Hubble horizon later in the evolution of the universe. The power spectrum is given by:
\[ k^3 P_\phi = k^3 |C|^2 = \frac{k^3 \eta_2^2}{l_H^2} |u_{II}(\eta_2)|^2 \]
\[ = \frac{4\pi^2 k}{l_H^3} \frac{1}{c_s^{3/2} k^{1/2}} e^{i x_i} \left( e^{i c_s (\eta_2 - \eta_1)} h + e^{-i c_s (\eta_2 - \eta_1)} h^* \right) + \frac{1}{2} \left( e^{i c_s (\eta_2 - \eta_1)} + e^{-i c_s (\eta_2 - \eta_1)} \right) \left( \delta u_I(\eta_1) - \frac{i}{c_s k} \delta u'_I(\eta_1) \right) + \delta u_{II}(\eta_2) \right|^2. \]

We can see from this solution that the amplitude of the power spectrum will be scale invariant since all terms are proportional to \( k^{1/2} \), as we can see in (80) and (83). As already pointed out, we will have an oscillation in this power spectrum coming from terms that depend on \( z_i \) and its mixing with the other terms in the sum. The relative amplitude of the oscillations is of order one, show a characteristic \( k \)-dependent frequency, but are too rapid to be detectable by an experiment with finite frequency resolution.
IV. CONCLUSION

We have studied the generation and early evolution of cosmological fluctuations in the “healthy extension” of Horava-Lifshitz gravity, assuming that there is matter which leads to an inflationary background cosmology. Since HL gravity can be viewed as a consistent UV completion of gravity, it being a power-counting renormalizable model for quantum gravity, inflation in the context of HL gravity provides an interesting background to study the trans-Planckian problem for inflationary cosmological perturbations.

There are two reasons which could lead to large deviations from the usual predictions of standard inflationary cosmology: firstly, the dispersion relation on trans-Planckian scales is non-standard, and secondly there is a second scalar metric fluctuation mode which acts as a source term in the evolution of the curvature fluctuations. The time interval during which mode mixing occurs depends on the wavenumber $k$ of the fluctuation mode.

The result of our analysis is that the overall scale-invariance of cosmological perturbations is maintained in HL inflation. On the other hand, there are oscillations in the amplitude of the spectrum as a function of $k$ whose relative amplitude is of order one, and whose frequency shows a characteristic scaling proportional to $k^3$, the power being determined by the power of the highest power of the spatial derivative term in the HL action. Physically, the oscillations are a consequence of the fact that setting initial conditions for all Fourier modes of the fluctuation spectrum come from the particular solution of the mode evolution equation (induced by mode mixing).

The reason why scale-invariance of the spectrum of cosmological perturbations is maintained in HL inflation is that the dispersion relation for the fluctuation modes satisfies the adiabaticity condition, and that the non-trivial initial vacuum state evolves into the usual one once the wavelength drops to sub-Planckian values. The oscillations in the amplitude are so high, however, that it is unlikely that an experiment with finite frequency resolution will be able to detect these oscillations.

The coefficients of the second order Lagrangian of the Healthy extension of Horava-Lifshitz gravity are

\[
d (\bar{k}) c_\phi = 2 (3 \lambda - 1) \frac{H^2}{\lambda^2} + (\lambda - 1) f_1 (\bar{k}) \frac{\bar{k}^2}{\lambda^2}, \quad d (\bar{k}) c_\psi = 2 (3 \lambda - 1) \left[ \frac{\phi_0^2}{\lambda^2} + 2 f_1 (\bar{k}) \bar{k}^2 \right], \quad (86)
\]

\[
d (\bar{k}) c_{\phi\phi} = 4 (3 \lambda - 1) H \frac{\phi_0^2}{\lambda^2}, \quad d (\bar{k}) f_\phi = - (3 \lambda - 1) H \frac{\phi_0^2}{\lambda^2} - (\lambda - 1) \left[ V_{0,\phi} + V_4 (\phi_0) \bar{k}^4 \right] \frac{\phi_0}{\lambda^2}, \quad (87)
\]

\[
d (\bar{k}) f_\psi = -24 (3 \lambda - 1) H A + 12 (3 \lambda - 1)^2 H^3 - 6 \lambda (3 \lambda - 1) \frac{\phi_0^2}{\lambda^2} H - 12 (3 \lambda - 1) \frac{V_0}{\lambda^2} (\phi_0) H, \quad (88)
\]

\[
d (\bar{k}) f_{\phi\psi} = 6 (\lambda - 1) \frac{\phi_0^2}{\lambda^2} \Lambda - 3 (3 \lambda - 1) (3 \lambda + 1) \frac{\phi_0^2}{\lambda^2} H^2 - 3 \frac{3}{2} (\lambda - 1) \frac{\phi_0^4}{\lambda^4} H^2 + 3 (\lambda - 1) \frac{\phi_0}{\lambda^2} H - 2 (\lambda - 1) [3 f_1 (\bar{k}) + 2 f_2 (\bar{k})] \frac{\phi_0^2 k^2}{\lambda^2}, \quad (89)
\]

\[
d (\bar{k}) f_{\phi\phi} = 4 (3 \lambda - 1) \frac{V_0,\phi}{\lambda^2} (\phi_0) H - (3 \lambda - 1) \frac{\phi_0^2}{\lambda^2} - 2 (3 \lambda - 1) f_1 (\bar{k}) \frac{\phi_0 k^2}{\lambda^2} + 4 (3 \lambda - 1) \frac{V_4 (\phi_0) H k^4}{\lambda^2}. \quad (91)
\]
Appendix 2: Rotated Coefficients

The coefficients of the rotated second order Lagrangian are

\[ F_{\phi_3} = \frac{1}{2} (1 + \cos 2\theta) f_{\phi} + \frac{1}{2} (1 - \cos 2\theta) f_{\psi} + \frac{\sin 2\theta}{2} (f_{\phi\psi} + \tilde{f}_{\phi\psi}) . \]  
(92)

\[ F_{\phi_4} = \frac{1}{2} (1 - \cos 2\theta) f_{\phi} + \frac{1}{2} (1 + \cos 2\theta) f_{\psi} - \frac{\sin 2\theta}{2} (f_{\phi\psi} + \tilde{f}_{\phi\psi}) . \]  
(93)

\[ F_{\phi_3\phi_4} = \frac{\sin 2\theta}{2} (f_{\psi} - f_{\phi}) + \frac{1}{2} (1 + \cos 2\theta) \tilde{f}_{\phi\psi} + \frac{1}{2} (1 - \cos 2\theta) f_{\phi\psi} . \]  
(94)

\[ \tilde{F}_{\phi_3\phi_4} = \frac{\sin 2\theta}{2} (f_{\psi} - f_{\phi}) + \frac{1}{2} (1 + \cos 2\theta) \tilde{f}_{\phi\psi} + \frac{1}{2} (1 - \cos 2\theta) f_{\phi\psi} . \]  
(95)

\[ M_{\phi_3}^2 = \frac{1}{2} (1 + \cos 2\theta) m_{\phi}^2 + \frac{1}{2} (1 - \cos 2\theta) m_{\psi}^2 + \frac{\sin 2\theta}{2} m_{\phi\psi}^2 . \]  
(96)

\[ M_{\phi_4}^2 = \frac{1}{2} (1 - \cos 2\theta) m_{\phi}^2 + \frac{1}{2} (1 + \cos 2\theta) m_{\psi}^2 - \frac{\sin 2\theta}{2} m_{\phi\psi}^2 . \]  
(97)

\[ M_{\phi_3\phi_4}^2 = \frac{\sin 2\theta}{2} (m_{\psi}^2 - m_{\phi}^2) + \cos 2\theta m_{\phi\psi}^2 . \]  
(98)

where:

\[ \sin 2\theta = \frac{c_{\phi\psi}}{\sqrt{(c_{\psi} - c_{\phi})^2 + c_{\phi\psi}^2}}, \quad \cos 2\theta = \frac{c_{\psi} - c_{\phi}}{\sqrt{(c_{\psi} - c_{\phi})^2 + c_{\phi\psi}^2}}. \]  
(99)

The general form of the \( c_x \) coefficients is quite involved and will not be written down here.

Appendix 3: Coefficients of the UV EOM

The coefficients of the UV equation of motion are given by:

\[ \alpha^2 = 4 \tilde{V}_6 \chi, \quad \beta^2 = \frac{2}{\chi^2} (\tilde{V}_2 + \tilde{V}_4, \phi), \quad \gamma^2 = \frac{\tilde{V}_4}{2\eta_4} - 2V_1 , \quad \gamma^2 = \sqrt{\frac{\lambda - 1}{\lambda - 1}} \frac{\eta_5}{\eta_4} \tilde{V}_4 \chi^2 . \]  
(100)

\[ \delta^2 = \frac{\lambda - 1}{3\lambda - 1} \frac{1}{\chi^2} \left[ (8g_7 - 3g_8) + \frac{\eta_5^2}{\eta_4} \right], \quad \epsilon^2 = \frac{\lambda - 1}{3\lambda - 1} \frac{(8g_7 - 3g_8)}{\chi^2} , \quad \zeta^2 = \frac{\lambda - 1}{3\lambda - 1}, \quad \zeta^2 = \gamma^2 . \]  
(101)

\[ \Xi_2 = -\Omega_2 = \sqrt{\frac{\lambda - 1}{3\lambda - 1}} \frac{\phi_0}{2\chi} \left( \frac{3\lambda - 1}{\lambda - 1} + 3 + 2 \frac{\eta_5}{\eta_4} \right) , \]  
(102)

\[ \Xi_3 = \sqrt{\frac{\lambda - 1}{3\lambda - 1}} \frac{\phi_0}{\chi} \left( 3 + 2 \frac{\eta_5}{\eta_4} \right) , \quad \Omega_3 = \sqrt{\frac{\lambda - 1}{3\lambda - 1}} \phi_0 . \]  
(103)

Appendix 4: Particular Solutions

Region I:
We are going to evaluate the particular solution, \( \delta u_I = \Gamma_1 + \Gamma_2 \). Making the change of variable \( \bar{\eta} = k\eta \), we have

\[
\Gamma_1 = \mp i \left( \frac{\pi}{2b} \right)^{3/2} \frac{1}{\sin(\pi/2b)} \frac{\bar{\eta}^{1/2}}{k^{1/2}} J_{1/2b} (z_1) \times 
\]

\[
\times \left\{ -\frac{\Xi_2 \alpha}{b \bar{b}_0} \left[ e^{\pm izi} \int_{\bar{\eta}_i(k)}^{\bar{\eta}_i} d\bar{\eta}^3 J_{1/2b} (z) J_{-1/2b} (z) - e^{\mp izi} \int_{\bar{\eta}_i(k)}^{\bar{\eta}_i} d\bar{\eta}^3 J_{1/2b} (z) J_{-1/2b} (z) \right] 
\right\}
\]

\[
\Gamma_2 = \mp i \left( \frac{\pi}{2b} \right)^{3/2} \frac{1}{\sin(\pi/2b)} \frac{\bar{\eta}^{1/2}}{k^{1/2}} J_{-1/2b} (z_1) \times 
\]

\[
\times \left\{ -\frac{\Xi_2 \alpha}{b \bar{b}_0} \left[ e^{\pm izi} \int_{\bar{\eta}_i(k)}^{\bar{\eta}_i} d\bar{\eta}^3 J_{1/2b} (z) J_{1/2b} (z) - e^{\mp izi} \int_{\bar{\eta}_i(k)}^{\bar{\eta}_i} d\bar{\eta}^3 J_{1/2b} (z) J_{1/2b} (z) \right] 
\right\}
\]

where \( z = \alpha k^{3-b} \bar{\eta}^b / b_0^2 \). In the case of de Sitter inflation, this particular solution reduces to:

\[
\Gamma_1 = \pm i \left( \frac{\pi}{6} \right)^{3/2} \frac{\bar{\eta}^{1/2}}{k^{1/2}} J_{1/2b} (z_1) \times 
\]

\[
\times \left\{ -\frac{\Xi_2 \alpha}{3b_0} \left[ e^{\pm izi} \int_{\bar{\eta}_i(k)}^{\bar{\eta}_i} d\bar{\eta}^3 J_{1/2b} (z) J_{-1/2b} (z) - e^{\mp izi} \int_{\bar{\eta}_i(k)}^{\bar{\eta}_i} d\bar{\eta}^3 J_{1/2b} (z) J_{-1/2b} (z) \right] 
\right\}
\]

\[
\Gamma_2 = \pm i \left( \frac{\pi}{6} \right)^{3/2} \frac{\bar{\eta}^{1/2}}{k^{1/2}} J_{-1/2b} (z_1) \times 
\]

\[
\times \left\{ -\frac{\Xi_2 \alpha}{3b_0} \left[ e^{\pm izi} \int_{\bar{\eta}_i(k)}^{\bar{\eta}_i} d\bar{\eta}^3 J_{1/2b} (z) J_{1/2b} (z) - e^{\mp izi} \int_{\bar{\eta}_i(k)}^{\bar{\eta}_i} d\bar{\eta}^3 J_{1/2b} (z) J_{1/2b} (z) \right] 
\right\}
\]

where \( z = \alpha \bar{\eta}^3 / 3b_0^2 \).

We can now evaluate the 12 integrals that we have in this solution. We are only computing them for de Sitter inflation. The solution of these integrals are polynomials multiplied by generalized hypergeometric functions. This functions can be written in terms of Meijer functions, a general function that contains most of the special functions.
as a special case and that it is defined as a path integral in the complex plane \([40, 41]\). For example, the integral:

\[
i_1 = i_8 = \int_{\tilde{\eta}(k)}^{\tilde{\eta}(l)} d\tilde{\eta}^2 J_{-1/2b}(z) J_{1/2b}(z)
= \frac{\tilde{\eta}^3}{3} F_3 \left( 1/2, 1/2; 5/6, 7/6, 9/6; -\tilde{\eta}^2 \right) = \frac{\tilde{\eta}^3}{6\sqrt{\pi}} G_{2,4}^{1,2} \left( \frac{\tilde{\eta}^2}{2} \right) 1/2, 1/2, 0, 1/6, -1/6, 1/2 \right)
\]

(109)

All integrals give the generalized hypergeometric function with \(p = 2\) and \(q = p + 1 = 3\) which is a special case of the Meijer function with \(m = 1, n = 2 = p\) and \(q = 4\), except for the last integral in (107) (and its counterpart in 108) that gives a hypergeometric function with \(p = 3\) and \(q = 4\) that it is a special case of a Meijer function with \(m = 1, n = 3 = p\) and \(q = 5\).

To be able to work with these results we make use of the asymptotic expansion of the Meijer function for large \(z\). For the Meijer functions present here, we use Theorem 1.8.5, the case (1.8.13) in [41], verifying that the functions obtained obey all the conditions imposed in the theorem:

\[
G_{p,q}^{m,n}(\tilde{\eta}^2 \left| \begin{array}{c}
a_1, \ldots, a_m, a_{m+1}, \ldots, a_p \\
b_1, \ldots, b_n, b_{n+1}, \ldots, b_q \end{array} \right. ) \sim D_{p,q}^{m,n}(\mu) H_{p,q}^\mu \left[ z^2 \exp[i\pi(\mu^* - 2\lambda)] \right],
\]

(110)

where, \(\mu^* = q - m - n\), \(\nu^* = -p + m + n\) and \(\arg(z^2) = 0\). In all of our cases \(\mu^* = 1 = \mu^*\) and \(D_{p,q}^{m,n}(\lambda) = A_{q}^{m,n} = (2\pi i)\mu^*\exp i\pi \sum_{j=1}^{n} a_j - \sum_{j=m+1}^{q} b_j\) as follows from (1.7.23) and (1.7.2) of [41]. The function \(H_{p,q}^\mu\) can be written as a series:

\[
H_{p,q}^\mu[x] = \exp \left( (p - q)x \frac{\tilde{\eta}^2}{\alpha} \right) x^{\rho^*} \left[ \frac{(2\pi)^\frac{\tilde{\eta}^2}{\alpha}}{(q - p)^{1/2}} + O(x^{-1/2}) \right],
\]

(111)

with \(\rho^* = \sum_{j=m+1}^{q} b_j - \sum_{j=m+1}^{q} a_j + (p - q + 1)/2 / (q - p)\).

This asymptotic expansion was applied in all of the results and, after some algebra, the particular solution simplifies to:

\[
\Gamma_1 + \Gamma_2 = \frac{1}{4} \left[ \frac{1}{\alpha} \right]^{1/2} \left( \frac{2\alpha}{1 + l_c} e^{-2iz \eta} - 1/\eta_1 e^{-2iz \eta} \right) \left( \cos \left( \frac{z}{3} \right) - \frac{1}{3} \right) + \left( \frac{1}{\eta_1} e^{-2iz \eta} - 1/\eta_1 e^{-2iz \eta} \right) \left( \cos \left( \frac{z}{3} \right) - \frac{1}{3} \right) + \left( \frac{1}{\eta_1} e^{-2iz \eta} - 1/\eta_1 e^{-2iz \eta} \right) \left( \cos \left( \frac{z}{3} \right) - \frac{1}{3} \right)
+ \left( \frac{1}{\eta_1} e^{-2iz \eta} - 1/\eta_1 e^{-2iz \eta} \right) \left( \cos \left( \frac{z}{3} \right) - \frac{1}{3} \right)
+ \left( \frac{1}{\eta_1} e^{-2iz \eta} - 1/\eta_1 e^{-2iz \eta} \right) \left( \cos \left( \frac{z}{3} \right) - \frac{1}{3} \right)
\]

(112)

Region II:

To derive the particular solution, \(\delta u_{II} = \tilde{\Gamma}_1 + \tilde{\Gamma}_2\), we make the change of variable \(\tilde{\eta} = k\eta\), and get:

\[
\delta u_{II} = -i \frac{1}{2} \left[ -i \frac{1}{\tilde{\eta}} \right]^{\tilde{\eta}_0} \left[ \tilde{B}_1 e^{ic \eta^2} - \tilde{B}_2 e^{-ic \eta^2} \right] \int_{\tilde{\eta}_1}^{\tilde{\eta}_2} d\tilde{\eta} e^{-2i \tilde{\eta} \eta} + \int_{\tilde{\eta}_1}^{\tilde{\eta}_2} d\tilde{\eta} e^{2i \tilde{\eta} \eta} \left( \tilde{B}_1 e^{ic \eta^2} \tilde{B}_2 e^{-ic \eta^2} \right)
+ \frac{\gamma^2}{l_6^2} \left[ \left( \tilde{B}_1 e^{ic \eta^2} + \tilde{B}_2 e^{-ic \eta^2} \right) \int_{\tilde{\eta}_1}^{\tilde{\eta}_2} d\tilde{\eta} k^{2-3} \eta^{1-2} + \int_{\tilde{\eta}_1}^{\tilde{\eta}_2} d\tilde{\eta} k^{2-3} \eta^{1-2} \right] \left( \tilde{B}_1 e^{ic \eta^2} \tilde{B}_2 e^{-ic \eta^2} \right)
+ \frac{\gamma^2}{l_6^2} \left[ \left( \tilde{B}_1 e^{ic \eta^2} + \tilde{B}_2 e^{-ic \eta^2} \right) \int_{\tilde{\eta}_1}^{\tilde{\eta}_2} d\tilde{\eta} k^{2-3} \eta^{1-2} + \int_{\tilde{\eta}_1}^{\tilde{\eta}_2} d\tilde{\eta} k^{2-3} \eta^{1-2} \right] \left( \tilde{B}_1 e^{ic \eta^2} \tilde{B}_2 e^{-ic \eta^2} \right)
\]

(113)

where \(\tilde{B}_1\) and \(\tilde{B}_2\) are the coefficients of the homogeneous equation in Region II for \(v_c\). They are equal to [79], since the coefficients of the equations of motions of \(u\) and \(v\) were set to be the same, for simplicity.

For de Sitter inflation, the solution is:

\[
\delta u_{II} = -i \frac{1}{2} \left[ -i \frac{1}{\tilde{\eta}} \right]^{\tilde{\eta}_0} \left[ \tilde{B}_1 e^{ic \eta^2} - \tilde{B}_2 e^{-ic \eta^2} \right] \int_{\tilde{\eta}_1}^{\tilde{\eta}_2} d\tilde{\eta} e^{-2i \tilde{\eta} \eta} + \int_{\tilde{\eta}_1}^{\tilde{\eta}_2} d\tilde{\eta} e^{2i \tilde{\eta} \eta} \left( \tilde{B}_1 e^{ic \eta^2} \tilde{B}_2 e^{-ic \eta^2} \right)
+ \frac{\gamma^2}{l_6^2} \left[ \left( \tilde{B}_1 e^{ic \eta^2} + \tilde{B}_2 e^{-ic \eta^2} \right) \int_{\tilde{\eta}_1}^{\tilde{\eta}_2} d\tilde{\eta} e^{-2i \tilde{\eta} \eta} + \int_{\tilde{\eta}_1}^{\tilde{\eta}_2} d\tilde{\eta} e^{2i \tilde{\eta} \eta} \right] \left( \tilde{B}_1 e^{ic \eta^2} \tilde{B}_2 e^{-ic \eta^2} \right)
+ \frac{\gamma^2}{l_6^2} \left[ \left( \tilde{B}_1 e^{ic \eta^2} + \tilde{B}_2 e^{-ic \eta^2} \right) \int_{\tilde{\eta}_1}^{\tilde{\eta}_2} d\tilde{\eta} e^{-2i \tilde{\eta} \eta} + \int_{\tilde{\eta}_1}^{\tilde{\eta}_2} d\tilde{\eta} e^{2i \tilde{\eta} \eta} \right] \left( \tilde{B}_1 e^{ic \eta^2} \tilde{B}_2 e^{-ic \eta^2} \right)
\]

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[43] There are also analyses using effective field theory which discuss limits on the possible magnitude of trans-Planckian effects.

[44] Trans-Planckian effects, however, almost by definition go beyond what can be discussed in an effective field theory approach based on General Relativity. Bounds on the magnitude of trans-Planckian coming from demanding that the enhanced fluctuations in the UV do not destroy the inflationary background are discussed in [15].

[45] For reviews, see [17].

[46] There is still a debate in the literature if the “healthy extension” is indeed free of “strong coupling” problems. More on this matter can be found in [24].

[47] Note that the extra fluctuating degree of freedom can be eliminated by introducing a local U(1) gauge symmetry [27]. Fluctuations in this context have been analyzed in [28].

[48] Since we know from the theory of cosmological perturbations in GR that the curvature fluctuations at late times are given by a combination of metric and matter perturbations, we expect that also here the canonical variables will be a combination of the matter perturbation $\delta\phi$ and the scalar perturbation $\psi$ of the metric.
We did not rescale the variable $\phi_k$ as was done in [26], since without the rescaling the observables have the right dimension, as will be seen in the following section.

We remind the reader that in the toy models discussed in [7] with dispersion relations leading to non-adiabatic mode evolution is was precisely the fact that short wavelength spend a longer period of time on sub-Planck wavelengths that led to the blue tilt of the spectrum of curvature perturbations obtained.

This is the usual feature in inflationary cosmology that the only source of violation of the time translation symmetry between the evolution of different modes is the initial condition at a fixed time.