The Electron in Three-Dimensional Momentum Space

Abstract We study the electron as a system composed of an electron and a photon and derive the leading-twist transverse-momentum-dependent distribution functions for both the electron and photon in the dressed electron, thereby offering a three-dimensional description of the dressed electron in momentum space. To obtain the distribution functions, we apply both the formalism of light-front wave function overlap representation and the diagrammatic approach; we discuss the comparison of our results between light-cone gauge and Feynman gauge, discussing the role of the Wilson lines to obtain gauge-independent results. We provide examples of plots of the computed distributions.

1 Introduction

The electron is a structureless particle; however, thanks to its quantum fluctuations, we can think of it as an object surrounded by a composite virtual cloud of photons, electrons and positrons. The latter can be interpreted as partons contained in a “dressed” electron, in analogy to the partonic structure of hadrons, and therefore described through the formalism of parton correlation functions, which can be parametrized in terms of different types of distribution functions [1, 2]. These functions not only allow us to view the dressed electron from a new perspective [3–5]; they can also help to shed light on several formal aspects of their QCD analogues, taking advantage of the perturbative nature of QED.

In this work we review the results obtained in Ref. [6] for the transverse-momentum dependent distribution functions (TMDs), which describe the distribution of QED partons in the dressed electron as a function of their longitudinal and transverse momentum, and therefore give access to the three-dimensional picture of the dressed electron in momentum space. In Sect. 2 we review the derivation of the leading-twist TMDs for the electron and the photon inside the electron and discuss the equivalence of the results in two different gauges, namely the Feynman and the light-cone gauge. We apply both the diagrammatic approach of Feynman diagrams and the representation in terms of overlap of light-front wave functions (LFWFs) [7]. In Sect. 3 we show some numerical results and plots of the TMDs, and we summarize our conclusions.

2 Dressed Electron TMDs

Let us consider a dressed electron of mass $m$, spin $S$ and 4-momentum $P$, composed of a bare electron of momentum $k$ and a photon of momentum $P - k$. We first focus on the distribution for the internal electron...
and we set its momentum, in light-cone coordinates, as \( k \equiv (x P^+, k^-, \mathbf{k}_\perp) \), with the longitudinal momentum fraction \( x \) defined as \( x \equiv k^+/P^+ \). If we probe the internal structure of the electron via the scattering of a virtual photon (see Fig. 1), the transverse-momentum dependent correlator function between initial and final electron states is given by [1, 8]

\[
\Phi(x, \mathbf{k}_\perp; P, S) = \int dk^- \Phi(k^-, P, S) = \int \frac{d\xi^- d^2 \xi_\perp}{(2\pi)^3} e^{ik^-\xi^-} \langle P, S|\bar{\psi}(0)\mathcal{U}(0,\xi)\psi(\xi)|P, S\rangle \Big|_{\xi^+ = 0}.
\]

From a diagrammatic point of view, the correlator \( \Phi \) corresponds to the lower part of the handbag diagrams in Fig. 1, i.e. the part obtained removing the external photons and the propagator of momentum \( k + q \). The gauge link operator (or Wilson line) \( \mathcal{U}(0,\xi) \) in Eq. (1) is needed to make the correlator a gauge-invariant object; it is defined as [9]

\[
\mathcal{U}(0,\xi) := \exp \left[ -ie \int_0^\xi d\eta^- A_\mu(\eta) \right], \quad (e > 0).
\]

The integration path lies in the \( \eta_\perp - \eta^- \) plane, since the plus component of the fields is fixed at 0 in Eq. (1); the path used in the TMD definition is shown in Fig. 2. It should be noticed that the path itself is gauge-dependent: in the Feynman gauge we have the boundary condition for this reason, the simple picture of the handbag diagram (a) in Fig. 1 is valid only if we take the (infinitely many, in principle) photons that can be absorbed or emitted by the incoming or outgoing particles. For this reason, the simple picture of the handbag diagram (a) in Fig. 1 is valid only if we take the \( O(1) \) expansion of the Wilson line; if we go to higher order, we need to consider also different diagrams, such as (b) and (c) in Fig. 1. We remark that in the present work we neglect all the contributions that come from vertex corrections and self-energy diagrams (which show up only at the end-point \( x = 1, \mathbf{k}_\perp = 0_{\perp} \), since they would require a regularization procedure, which is beyond the scope of the present work.

From now on, we consider a reference frame where \( P = (P^+, P^-, 0_{\perp}) \), with \( P^- = m^2/2P^+ \) from the on-shell condition; moreover, we can set \( S \equiv (S^+, S^-, S_{\perp}) \). Introducing also the light-like vector \( n_+ = (1, 0, 0_{\perp}) \) and the antisymmetric tensor \( \epsilon_\perp = -\epsilon_{\perp T} = 1 \), the eight leading-twist electron TMDs are implicitly defined through the following decomposition of the correlator (1) in terms of the elements of the Dirac space basis \( D := \{1, \gamma^\mu, \gamma^5, \gamma^5\gamma^\mu, i\sigma_{\mu\nu}\} \):

\[
\Phi(x, \mathbf{k}_\perp; S) \approx \frac{1}{2} \left[ f_1 T \gamma^5 h_{1L}^e + \frac{\epsilon_{\perp T}}{2m} \left\{ \gamma^5 - \frac{\mathbf{k}_\perp \cdot \mathbf{S}_{\perp}}{m} \right\} h_{1T}^e + \frac{\mathbf{k}_\perp \cdot \mathbf{S}_{\perp}}{m} \gamma^5 h_{1T}^e - \frac{\mathbf{k}_\perp \cdot \mathbf{S}_{\perp}}{m} \gamma^5 h_{1L}^e \right],
\]

where \( i, j = 1, 2 \) and \( a_{\perp}^\mu = (0, 0, a_{\perp}) \). From the above decomposition, one can recover the analytic expression of the TMDs by selecting proper projections of the correlator in Dirac space. Out of the eight TMDs in Eq. (3), the so-called Boer–Mulders function \( h_{1T}^e \) and the Sivers function \( f_{1T}^e \) are T-odd, i.e. they change sign under naive time reversal (which is defined as usual time-reversal but without interchange of initial and final states), while the other six are T-even. If one excludes the contribution at the end-point \( x = 1, \mathbf{k}_\perp = 0_{\perp} \), the T-odd TMDs \( h_{1T}^e \) and \( f_{1T}^e \) are vanishing [6]. This is due to the fact that only diagrams containing a loop at one side of the cut can potentially give a non-zero contributions to T-odd TMDs [10], but the only possibility to have such a loop is either to consider diagrams of higher perturbative order, or take into account also the end-point.

The evaluation of the correlator, and consequently of the TMDs, can be performed in two different ways, which yield equivalent results: the LFWF overlap representation and the diagrammatic approach with Feynman diagrams. Let us describe the former first. In the light-cone gauge \( A^+ = 0 \), we introduce the state \( |P, A\rangle \) for

![Fig. 1](image-url) Handbag diagrams contributing to the electron TMDs at order \( \alpha \).
the dressed electron with four-momentum $P$ and light-front helicity $\lambda$ [11], and expand it in the Fock space of free quanta [12], i.e.

$$ |P, \lambda \rangle = |e \rangle + |e\gamma \rangle, $$

where

$$|e\rangle = b^+_\lambda (P)|0\rangle, \quad |e\gamma \rangle = \int \frac{d^2k_\perp}{(2\pi)^2 \sqrt{x(1-x)}} \sum_{\lambda, \lambda_\gamma} \Psi^\lambda_{\lambda, \lambda_\gamma} (x, k_\perp | e \gamma; \ p_e, \ \lambda; \ p_\gamma, \ \lambda_\gamma).$$

The coefficients in the above expansion are the LFWFs $\Psi^{\lambda}_{\lambda, \lambda_\gamma}$, which depend on $\lambda$ and the light-front helicities $\lambda$ and $\lambda_\gamma$ of the internal electron and photon, respectively; they can be calculated exactly in QED using time-ordered perturbation theory in the infinite momentum frame, as outlined e.g. in [6]. Inserting Eqs. (4) and (5) in the correlator function (1), one obtains

$$\phi^{\Gamma}(x, k_\perp; \ A) \equiv \frac{1}{2} \text{Tr} [\Phi \Gamma] = \frac{1}{4(2\pi)^3 x P^+} \sum_{\lambda, \lambda_\gamma} \Psi^{\lambda \ast}_{\lambda', \lambda_\gamma'} (x, k_\perp) \bar{u}_{\lambda'} (p_\gamma) \Gamma u_{\lambda}(p_e) \Psi^\lambda_{\lambda, \lambda_\gamma} (x, k_\perp).$$

where $\Gamma \in D$; consequently, the TMDs themselves can be expressed as overlap of LFWFs. Since the LFWFs are eigenstates of the total spin of the partons in the $z$ direction, $J_z = \lambda/2 + \lambda_\gamma$, and of the total OAM $L_z = \lambda/2 - J_z$, this representation allows one to disclose the different contributions to the TMDs from the spin and OAM configurations of the partons.

In the Feynman diagram approach, instead, one needs to evaluate the handbag diagram in Fig. 1a, by applying the usual QED Feynman rules, and then factorize out the lower part, corresponding to the correlator. In the Feynman gauge (FG), as one considers also the longitudinal gauge link, we need to take into account also diagrams (b) and (c) shown in Fig. 1, which however are vanishing in the light-cone gauge (LCG). Nonetheless, we find perfect agreement between the results in the two gauges: the mismatch coming from the different number of diagrams to be considered is compensated by the fact that the photon propagator is proportional to the sum $d^{\mu\nu}(p) := \sum_{\lambda} \epsilon^\mu_{\lambda} (p) \epsilon^\nu_{\lambda} (p)$, which in the two gauges reads

$$d^{\mu\nu}(p) = -g^{\mu\nu} \text{ (FG),} \quad d^{\mu\nu}(p) = -g^{\mu\nu} + \frac{1}{p^+} (p^\nu n^\mu + p^\mu n^\nu) \text{ (LCG).}$$

We stress that in order to obtain the equivalence between the results for the two choices of the gauge, one needs to include the longitudinal gauge link when working in the Feynman gauge, while it is not necessary to take into account the transverse gauge link in the light-cone-gauge. There is however a contribution to the TMDs that comes from the transverse gauge link [13] and that it is worth to analyze, even though it shows up only at the end-point $x = 1$. It comes from the consideration that one must regularize the denominator of the photon propagator in light-cone gauge, i.e. Eq. (7), since we integrate over the photon momentum. One still has, however, the freedom of choosing a certain prescription for the regularization, among the following possibilities: the retarded and advanced prescriptions, where $p^+ \rightarrow p^+ \pm i \epsilon$, respectively, and the principal value (PV) prescription, where $1/p^+ \rightarrow 1/2[1/(p^+ + i \epsilon) + 1/(p^+ - i \epsilon)]$. While the PV prescription yields

1 One should distinguish between the $|e\gamma\rangle$ state, which describes the dressed electron as a 2-body system in Eq. (4), and the state of two free quanta $|e\gamma; \ p_e, \ \lambda; \ p_\gamma, \ \lambda_\gamma\rangle$. 

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**Fig. 2** Wilson line connecting two points at $0 = (0^-, 0)$ and $\xi = (\xi^-, \xi^\perp)$. The vertical line parallel to the transverse axis is taken at $\eta^- = \infty$. 

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\[2 \text{ Tr } \Psi^{\lambda}_{\lambda, \lambda_\gamma} (x, k_\perp | e \gamma; \ p_e, \ \lambda; \ p_\gamma, \ \lambda_\gamma).\]
a vanishing contribution to the TMDs, this is not the case for the other two prescriptions, where extra terms (proportional to $\delta(1-x)$) appear. The final result, however, should be prescription-independent, and coincide with the one found in the Feynman gauge, where the propagator (Eq. 7) does not need to be regularized; this is ensured by the contributions from the transverse gauge link since they exactly cancel the terms coming from the regularization of the propagator, as shown in Ref. [6].

We finally mention that an analysis similar to the one performed so far for the TMDs of the electron inside the electron can be extended also to the case of the internal photon. In this case, the active parton is the photon, and the relevant correlator reads

$$\Phi^{ij}(k; P, S) := \frac{1}{x P^+} \int \frac{d\xi^- d\xi^\perp}{(2\pi)^3} e^{ik\cdot\xi} \left\langle P, S| F^{+j}(0) F^{+i}(\xi)| P, S \right\rangle \mid_{\xi^+ = 0}. \tag{8}$$

The photon-field tensor $F^{\mu\nu}$ is by itself gauge-invariant; therefore, in this case, we do not need to insert a gauge link operator in the definition (8). This consideration allows us to state straightforwardly that the T-odd photon TMDs (namely $f_{1T}^\gamma$, $h_{1L}^\gamma$, $h_{1T}^\gamma$ and $h_{1T}^{\gamma T}$) are vanishing, at least at order $\alpha$ and still neglecting the end-point $x = 1, k_\perp = 0$. For the T-even TMDs, instead, one can use the same procedure applied to the case of the electron TMDs.

![Fig. 3](image-url) Density plots in the transverse momentum plane $k_x - k_y$ and at fixed value of $x = 0.5$ for $f_1^\gamma$ (upper-left panel), and the combinations $f_1^{\gamma} + (k_y/m) g_1^T$, $f_1^{\gamma} + (k_y/m) h_1^T$, and $(f_1^{\gamma} + (k_y^2 - k_x^2)/(2m^2) h_1^{\gamma T})/2$ (upper-right, lower-left and lower-right panel, respectively), all rescaled by a factor $2\pi^2/\alpha$. The legend in the bottom-right corner of each panel indicates the corresponding spin configurations: the grey and white discs refer to the dressed and bare electron, respectively; the empty discs refer to the unpolarized case; the circle inside the discs stands for polarization along the longitudinal axes; the arrows indicate polarization along the $y$ direction.
3 Results and Conclusions

In this section we briefly review some results of the calculations of the electron and photon TMDs for \((x, k_\perp) \neq (1, 0)\) presented in [6]; we recall that these are perturbative-QED exact results at \(\mathcal{O}(\alpha)\), hence they are model independent. We can form the following densities for electrons of definite longitudinal or transverse polarization:

\[
\rho(x, k_x, k_y, (\lambda, s_\perp), (A, S_\perp)) = \frac{1}{2} \left[ f_1^e + \lambda A g_{1L}^e + \lambda S_\perp k_{1\perp} \frac{1}{m} g_{1T}^e + \Lambda s_\perp k_{1\perp} \frac{1}{m} h_{1L}^e + s_\perp S_\perp h_{1T}^e \right],
\]

where \(s_\perp\) is the transverse spin of the internal electron. As an example, we show in Fig. 3 the densities in the transverse momentum plane, at fixed \(x = 0.5\), for the electron TMD \(f_1^e\) and for the combinations \(f_1^e + (k_y/m)g_{1T}^e\) and \(f_1^e + (k_y/m)h_{1L}^e\); they can all be interpreted as probability densities of finding the internal electron in a certain point in momentum space, in correspondence of a particular spin configuration. The TMD \(f_1^e\) describes the situation where both the internal and the dressed electron are unpolarized: it has a peak when \(|k_\perp|\) is around 0.2–0.3 keV and looks like a ring-shaped image with a radius of about 200 keV. The distributions \(f_1^e + (k_y/m)g_{1T}^e\) and \(f_1^e + (k_y/m)h_{1L}^e\), instead, describe the situation for longitudinally polarized electrons in an electron target transversely polarized along the \(y\)-direction and transversely polarized electrons in a longitudinally polarized target, respectively. We observe a dipole distortion along the direction of the transverse spin, which deforms the symmetric situation of the unpolarized case. Finally, we show the distributions within a 3Q light-cone picture of the nucleon. JHEP 05, 041 (2011)

This work can be extended further by including the contributions from the endpoints and the treatment of divergences; another possible direction of study would be to identify which observables are sensitive to the TMDs of a dressed electron.

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