Maxwell interpolation and close similarities between liquids and holographic models

M. Baggioli\textsuperscript{1} and K. Trachenko\textsuperscript{2}

\textsuperscript{1} Crete Center for Theoretical Physics, Institute for Theoretical and Computational Physics, Department of Physics, University of Crete, 71003 Heraklion, Greece and
\textsuperscript{2} School of Physics and Astronomy, Queen Mary University of London, Mile End Road, London, E1 4NS, UK

We show that liquids and certain holographic models are strikingly similar in terms of several detailed and specific properties related to their energy spectra. We consider two different holographic models and ascertain their similarity with liquids on the basis of emergence of the gap in transverse momentum space and the functional form of the dispersion relation. Furthermore, we find that the gap increases with temperature, the relaxation time governing the gap decreases with temperature and, finally, the gap is inversely proportional to the relaxation time as in liquids. On this basis, we propose that Maxwell-Frenkel approach to liquids can be used to understand holographic models and their strongly-coupled field theory counterparts in a non-perturbative way.

INTRODUCTION

Many interesting effects in quantum field theory (QFT) are related to strongly-coupled dynamics. These problems can not be solved by the perturbative approaches commonly used in QFT. However, the proposal of the correspondence between QFT and gravity models (AdS-CFT correspondence) has opened the way to approach strongly-coupled QFT problems by addressing the corresponding weakly-interacting gravitational duals \cite{1, 2}.

For the same reason of strong coupling, a theory of liquids was believed to be impossible to construct in a general form \cite{3}. Perturbation theories do not apply to liquids because the inter-atomic interactions are strong. Solid-based approaches seemingly do not apply to liquids either: its unclear how to apply the traditional harmonic expansion around the equilibrium positions because the equilibrium lattice does not exist due to particle rearrangements that enable liquids to flow. This combination of strong interactions and large particle displacements has proved to be the ultimate problem in understanding liquids theoretically, and is known as the “absence of a small parameter”.

The absence of traditional simplifying features in the liquid description does not mean that the problem can not be solved in some other way, including attempting the first-principles approach using the equations of motion. However, this involves solving a large number of non-linear equations, an exponentially complex problem not currently tractable \cite{4}.

Recent progress in understanding liquids followed from considering what kind of collective modes (phonons) can propagate in liquids and supercritical fluids \cite{4}. It has been ascertained that solid-like transverse modes can propagate in liquids but, interestingly, they develop a gap in $k$, or momentum space, with the gap growing with the inverse of liquids relaxation time \cite{4}. This enables us to discuss liquid thermodynamics on the basis of phonons, as is done in the solid-state theory.

Here, we find the same effect of the $k$-gap emerging in a very different setup: holographic gravitational models dual to strongly coupled condensed matter systems \cite{6}. We analyze two holographic models (HMs)\textsuperscript{1} and find that they are strikingly similar to liquids in terms of their energy spectra. Our analysis shows that transverse modes in the gravity models develop a gap in momentum space as in liquids. We define the corresponding relaxation time and show that, similarly to liquids, it decreases with temperature. Finally, we find that the gap in momentum space is proportional to the inverse of relaxation time as in liquids. This striking and detailed similarity suggests that the non-perturbative Maxwell-Frenkel approach to liquids can be applied to holographic models and strongly-coupled field theories.

We start with recalling how liquid transverse modes develop a gap in momentum space. This programme starts with Maxwell interpolation:

$$\frac{d\mathcal{S}}{dt} = \frac{\mathcal{P}}{\eta} + \frac{1}{G} \frac{d\mathcal{P}}{dt} \quad (1)$$

where $\mathcal{S}$ is shear strain, $\eta$ is viscosity, $G$ is shear modulus and $\mathcal{P}$ is shear stress.

Eq. (1) reflects the Maxwell’s proposal \cite{12} that the shear response in a liquid is the sum of viscous and elastic responses given by the first and second right-hand side terms. Notably, the dissipative term containing the viscosity is not introduced as a small perturbation: both elastic and viscous deformations are treated in (1) on equal footing.

Frenkel proposed \cite{13} to represent the Maxwell interpolation by introducing the operator $A$ as $A = 1 + \tau \frac{d}{dt}$ so that Eq. (1) can be written as $\frac{d\mathcal{S}}{dt} = \frac{1}{\eta} A \mathcal{P}$. Here, $\tau$ is the Maxwell relaxation time $\frac{\eta}{G}$. Frenkel’s idea was to generalize $\eta$ to account for liquid’s short-time elasticity

\textsuperscript{1} We consider two specific holographic models. We believe that the features discussed here are applicable to more general systems: they have been observed in different contexts in Refs. 3-11.
as $\frac{1}{\eta} \rightarrow \frac{1}{\eta} \left(1 + \tau \frac{d}{d\tau}\right)$ and use this $\eta$ in the Navier-Stokes equation as $\nabla^2 \mathbf{v} = \frac{1}{\eta} \rho \frac{d\mathbf{v}}{d\tau}$, where $\mathbf{v}$ is velocity, $\rho$ is density and the full derivative is $\frac{d}{d\tau} = \frac{d}{dt} + \mathbf{v} \nabla$. We have carried this idea forward [4] and, considering small $\mathbf{v}$, wrote

$$c^2 \frac{\partial^2 \mathbf{v}}{\partial x^2} = \frac{\partial^2 \mathbf{v}}{\partial t^2} + \frac{1}{\tau} \frac{\partial \mathbf{v}}{\partial t} \tag{2}$$

where $\mathbf{v}$ is the velocity component perpendicular to the $x$ direction. Eq. (2) can also be obtained by starting with the solid-like elastic equation for the non-decaying wave and, using Maxwell interpolation [1], generalizing the shear modulus to include the viscous response [15]. This shows that the hydrodynamic approach commonly applied to liquids is not a unique starting point and that the solid-like elastic approach is equally legitimate, implying an interesting symmetry of the liquid description.

In contrast to the Navier-Stokes equation, Eq. (2) contains the second time derivative of $\mathbf{v}$ and hence allows for propagating waves. We solved Eq. (2) [4] by seeking a plane-wave solution as $\mathbf{v} = v_0 \exp(i(kx - \omega t))$. This gives

$$\omega^2 + \frac{i\omega}{\tau} - c^2 k^2 = 0 \tag{3}$$

and the following dispersion relation

$$\omega = -\frac{i}{2\tau} \pm \sqrt{c^2 k^2 - \frac{1}{4\tau^2}} \tag{4}$$

An important property is the emergence of the gap in $k$-space: in order for $\omega$ in (4) to be real, $k > k_g$ should hold, where the $k$-gap

$$k_g = \frac{1}{2c\tau} \tag{5}$$

increases with the temperature because $\tau$ decreases.

Recently [5], detailed evidence for the $k$-gap was presented on the basis of molecular dynamics simulations. It has been ascertained that $k_g$ increases with the inverse of liquid relaxation time in a wide range of temperature and pressure for different liquids and supercritical fluids, as [5] predicts.

The gap in $k$-space, or momentum space is interesting. Indeed, the two commonly discussed types of dispersion relations are either gapless as for photons and phonons, $E = k$ ($c = 1$), or have the energy (frequency) gap for massive particles, $E = \sqrt{k^2 + m^2}$, where the gap is along the $Y$-axis. On the other hand, (4) implies that the gap is in momentum space and along the $X$-axis, similar to the hypothesized tachyon particles with imaginary mass [17]. The inset in Figure 1 illustrates this point.

We now focus on two specific models [19] and [20] which describe, using the AdS-CFT correspondence [1], the gravity duals of strongly-coupled relativistic viscoelastic systems. Their gravitational bulk actions read

$$S^I = \int d^4x \sqrt{-g} \left[ R - \frac{3}{2} L^I - \mathcal{L}^I \right] \tag{6}$$

where

$$\mathcal{L}^I = \frac{m^2}{2} \partial_\mu \phi^I \partial^\mu \phi^I, \quad \mathcal{L}^I = \frac{1}{6} H^I_{abc} H^I_{abc} \tag{7}$$

and:

$$\phi^I = x^I, \quad H^I_{1xx} = H^I_{1yr} = M \tag{8}$$

Both models admit a simple asymptotic AdS black brane solution whose event horizon is located at $r = r_h$ and controls the values of the field theory temperature $T$ (see the Supplementary material for details).

The first model [19] represents a simple relativistic holographic massive gravity theory whose dual field theory displays a finite relaxation time for the momentum operator $\tau_{rel}^{-1} = \Gamma$ which is inversely proportional to the graviton mass $\sim m^2$ (see [22] for details).

The second model [20] can be thought as the holographic dual of a finite number of dynamical elastic lines defects embedded in a strongly coupled fluid state. The density of defects is controlled by the parameter $M$ in

![Figure 1: The dispersion relation of the transverse collective modes obtained numerically in the first HM [19]. The temperature increases from blue ($T/m \sim 0.141$) to red ($T/m \sim 0.171$). The shaded region ($T/m > 0.156$) displays the presence of the k-gap. The inset illustrates the possible dispersion relations and dependencies of energy $E$ on the momentum $k$. Top curve shows the dispersion relation for a massive particle. Middle curve shows the gapless dispersion relation for a massless particle (photon) or a phonon in solids. Bottom curve shows the dispersion relation [4] with the gap in $k$-space, illustrating the results of Ref. [5].](image-url)
and a UV cutoff $C$ is present in the model for technical reasons due to the holographic renormalization procedure. This cutoff may be related to the height of the potential energy barrier $U$ present in liquids.

The first important result from the first model is the emergence of the $k$-gap. The numerical study of the transverse fluctuations results in the identification of the QNMs spectrum shown in Fig. 1. At high enough temperature, we observe the emergence of the gap in $k$-space. Increasing the temperature $T$ increases the $k$-gap, the same effect as in liquids (see (5)) derived from the Maxwell-Frenkel approach.

The temperature at which the $k$-gap opens up in Fig. 1 corresponds to $T/m > 0.156$. At smaller values of $T/m$, the spectrum has a mass gap as is the case for a massive particle. This can be attributed to the competition between the effective mass term $m$ and the dissipative $1/k$ term: adding the mass term to the Lagrangian describing the $k$-gap in [4] modifies the dispersion relation as [15]:

$$\omega = \sqrt{k^2 + m^2 - \frac{1}{4\tau^2}}$$

According to [9], the dispersion relation is linear and gapless for $m = \frac{1}{2\pi}$, whereas the gap in $k$-space opens up for $\frac{1}{2\pi} > m$. The nature of the effective mass $m$ within the model [19] will be discussed elsewhere.

In the second model [20] and in the limit of large cutoff $C \equiv C/r_h \gg 1$, we can extract the dispersion relation of the transverse modes analytically from the equation:

$$(1 - i \frac{\omega}{\omega_g}) \omega + i \frac{C}{r_h} k^2 = 0$$

which is identical to Eq. [3] describing the transverse modes in liquids on the basis of Maxwell-Frenkel approach result. The $k$-gap is given by:

$$k_g^2 = \frac{\omega_g r_h}{4(C - 1)}$$

where the meaning of the various quantities is explained in the Supplementary material in detail.

Similarly to the first model, the second HM model results in the emergence of the $k$-gap which increases with temperature. The gap reduces with the cutoff value as discussed below in more detail and disappears in the limit of infinite cutoff $C \to \infty$.

Before proceeding with the analysis of the $k$-gap further, we discuss the relaxation time $\tau$ and its physical meaning in liquids and HMs. There are two dynamical regimes in the liquid state where liquid properties are qualitatively different. In the low-temperature regime, particle dynamics combines solid-like oscillatory motion and diffusive jumps between quasi-equilibrium positions. Frenkel’s theory has identified the time between these jumps with the Maxwell relaxation time $\frac{\eta}{\rho}$, and this has become an accepted view since [21]. In the low-temperature regime, the diffusion constant $D$ is approximately $D_l = \frac{a^2}{\tau}$, where $a$ is of the order of interatomic separations, and decreases with $\tau$. In this regime, $D_l$ is inversely proportional to viscosity $\eta$, and $\eta$ itself decreases with temperature [13]. In the high-temperature gas-like regime, particles lose the oscillatory component of their motion and move as in a gas (low- and high-temperature liquid regimes are separated by the Frenkel line recently introduced and extending to the supercritical state [31]). In this regime, the relaxation time $\tau$ is related to the time between momentum-transferring particle collisions which sets the length of the mean free path. The diffusion constant $D_h = \frac{\eta}{\rho}$ is proportional to $\eta$, and $\eta$ itself increases with temperature [13].

A hydrodynamic description of relativistic fluids is characterised by (a) proportionality between $D$ and $\eta$ as $D = \eta/(\epsilon + p)$ and (b) viscosity increasing with temperature. As discussed above, this implies that these systems are in the high-temperature gas-like dynamical regime from the point of view of condensed matter physics. The relationship between $D$ and $\tau$ in this regime can be derived by expanding $\omega$ in (4) in the hydrodynamic limit of small $k \omega \ll 1$, resulting in a simple relation:

$$D = c^2 \tau$$

We note that relaxation time $\tau$ can also be obtained from $\text{Im}(\omega)$ at large $k$, where $\omega$ is given by the dispersion relation [4]. $\text{Im}(\omega)$ approaches a constant value in the limit of large $k$ (see [24] for more details).

The same relation (12) follows from considering $\tau$ in $D_l = \frac{a^2}{\tau}$ as the time between particle collisions in the high-temperature regime, in which case $a = c\tau$ becomes the distance travelled ballistically. For relativistic fluids, (12) follows from combining $D = \eta/(\epsilon + p)$ with the speed of transverse phonons $c^2 = G/(\epsilon + p)$ to yield $D = c^2 \frac{\eta}{G}$ and subsequently noting that $\frac{\eta}{G}$ is relaxation time from Maxwell interpolation [1].

The last point calls for two important observations related to Maxwell interpolation [1]. First, our earlier identification of $\frac{\eta}{G}$ with relaxation time for relativistic fluids is based on Maxwell interpolation [1]. Notably, Maxwell interpolation was discussed and later developed in the low-temperature liquid-like dynamical regime only, with $G$ being the solid-like high-frequency shear modulus governed by interatomic interactions. However, we propose that Maxwell interpolation also applies to the

$^2$ The same expression was already considered in [14] in the study of the causality of relativistic dissipative fluid dynamics.
high-temperature gas-like regime. Indeed, the idea of the system being able to support two types of response, viscous and elastic, is general enough and applicable to the high-temperature gas-like state as well, but with the proviso that in this state $G$ in [1] describes a purely kinetic term $\propto T$ due to particle inertia ($\eta$ remains to be defined in the usual way).

Second, the emergence of the $k$-gap in liquids due to Maxwell interpolation differs from the HMs in one important respect. In liquids as well as solids, there is an “ultraviolet” cutoff related to the shortest interatomic separation $a$ and the largest, Debye, frequency in the system $\omega_D$ or the shortest vibration period $\tau_D$. When $\tau$ in [5] reduces at high temperature and approaches $\tau_D$, the $k$-gap extends to the entire first zone or, equivalently, the wavelength of propagating shear modes becomes comparable to the interatomic separation. At this point, all transverse modes disappear from the liquid spectrum [3]. On the other hands, no equivalent cutoff exists in HMs. Therefore, the $k$-gap for propagating shear modes in HMs is not bounded from above.

We now return to our analysis of properties of the $k$-gap. We compare [12] to the direct numerical analysis of the first HM. The model has an additional contribution to $\tau$ due to momentum relaxation time $\tau_{rel} = \Gamma^{-1}$, resulting in $\omega = -i\Gamma - iDk^2$. Applying the inverse Matthiessen’s rule $\tau^{-1} = c^2/D + \Gamma$, we obtain:

$$\tau = \frac{D}{c^2 - D\Gamma}$$ (13)

In the first HM, we fit the calculated dispersion curves with $k$-gaps in Figure 1 to $\omega$ predicted on the basis of Maxwell interpolation in Eq. (4) fixing the speed to its relativistic limit $c = 1$. We find that the fits are of high quality. Using the fits, we extract the corresponding $\tau$ in [4] and plot $\tau$ in Figure 2. We subsequently compute $D$ from the imaginary part of $\omega$ at shortest momenta numerically and calculate $\tau$ using (12) and (13) as well as an approximate analytical equation for $D$ [23] (see Supplementary Material). The resulting curves are shown in Figure 2. We observe that $\tau$ from the fit of $\omega$ to [4] and $\tau$ calculated from Eq. (13) agree with high accuracy. We further observe that relaxation time $\Gamma$ obtained by fitting the numerical data and all $\tau$ calculated from the diffusion constant $D$ coincide at high temperature (shaded region) as expected because $\Gamma \to 0$, resulting in coinciding [12] and [13].

For a specific value of $T/m$ (see more details in the Supplementary material), the Green functions and, therefore, the dispersion relation $\omega(k)$, can be extracted analytically [24]. The corresponding $\tau$, calculated by matching the $k$-gap dispersion relation [4] with the analytic poles of the Green function (see expression [17]), is shown in red in Fig. 2 and agrees well with $\tau$ calculated from [13].

Several important implications follow from our analysis and from Figure 2. First, $\tau$ decreases with $T$ as is the case in liquids. Second, the dispersion relations in the HMs with the $k$-gap emerging in Fig. 1 agree with the liquid gapped dispersion relation resulting from Maxwell interpolation [4] as discussed above. Third, close agreement between $\tau$ from the fit using [4] and $\tau$ calculated in the HM implies that Maxwell relaxation time $\tau_M$ that governs the $k$-gap in [4] can be identified with the relaxation time governing fluid dynamics in the HM. This, in turn, suggests that the ideas involved in Maxwell interpolation and its extension by Frenkel can be used to analytically treat HMs and strongly-coupled field.

All of the above implications apply to the second HM where the results can be derived analytically. Indeed, we have earlier derived the dispersion relation [11] with the $k$-gap [11] in this model and observed that they are identical to those in liquids. Furthermore, as shown in the Supplemental Material, we (a) derive the temperature dependence of $\tau$ analytically as $\tau = \frac{1}{\omega_g} = \frac{1}{#} $ (# is a number depending on the dimensionless cutoff $C/T$) and observe that $\tau$ decreases with the temperature as in liquids; (b) analytically show that $\tau = \frac{1}{\omega_g}$ as in [12]; and (c) show that in the limit of large cutoff and zero defects density $M = 0$, this result coincides with the Maxwell-Frenkel result $\tau = \frac{2}{\omega_g}$.

We can now ascertain the validity of the main result for the $k$-gap [3] predicting $k_g \propto \frac{1}{T}$. We plot $k_g$ as a function of $\frac{1}{T}$ in Figure 3. In the first model, we observe a linear dependence with good accuracy. In the second model, the dependence is linear for large cutoff. For smaller cutoff, $k_g$ vs $\frac{1}{T}$ departs from linearity. The analogy with liquids can explain this departure as follows. As discussed earlier,
smaller cutoff in the HM corresponds to smaller activation barrier in liquids and hence smaller \( \tau \), resulting in faster-than-linear increase of \( k_g \) according to \([5]\).

![Graph showing k-gap as a function of \( \tau \) for different cutoffs.](image)

**FIG. 3**: The \( k \)-gap as a function of \( \frac{1}{\tau} \) in the first HM \([19]\) (top) and the second HM \([20]\) (bottom). Different colors are for different cutoffs \( C/T = 10, 50, 1000 \) (blue, orange, green). The straight dashed line is a guide for the eye.

We make a remark regarding relaxation time \( \tau \) that governs the \( k \)-gap in the Maxwell-Frenkel approach. From a practical perspective, the introduction of \( \tau \) in \([1-5]\) enables us to discuss collective modes in liquids and their role in liquid dynamical and thermodynamic properties \([4]\). From a general-theoretical perspective of treating strongly-interacting and dynamically disordered systems, the introduction of \( \tau \) simplifies and solves an exponentially complex problem of coupled non-linear oscillators describing the motion of liquid particles in the strongly anharmonic multi-well potential \([4]\). Although not derived from first-principles (recall that a first-principles description of liquids is exponentially complex and hence is non-tractable), \( \tau \) is an important liquid property directly linked to viscosity that enables to provide relationships between different liquid properties \([4]\). Therefore, the introduction of \( \tau \) is a non-perturbative way to treat strong interactions and in this sense is particularly suitable to address strong interactions in field theories.

In summary, we have shown that liquids and holographic models are strikingly similar in terms of several detailed and specific properties. Similarly to liquids, we find that (a) the HMs develop the \( k \)-gap with the same dispersion relation, (b) the \( k \)-gap in the HMs increases with temperature, (c) the relaxation time \( \tau \) governing the \( k \)-gap in the HMs coincides with the relaxation time calculated from the gapped dispersion relation following from Maxwell interpolation, (d) \( \tau \) in the HMs decreases with temperature as in liquids, and (e) the \( k \)-gap is inversely proportional to the relaxation time. These close similarities suggest that the ideas involved in Maxwell interpolation and its Frenkel development can serve as a constructive approach to treat holographic models and their strongly-coupled field theory counterparts.

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where $f$ event horizon position (AdS black brane solution: $\phi$ solution features [28–33]. The scalars admit the simple background [26, 27] which indeed display interesting viscoelastic features [28]. This theory is a particular case of the general holographic massive gravity models [26] which indeed display interesting viscoelastic features [28, 33]. The scalars admit the simple background solution $\phi^I = x^I$. The theory has a simple asymptotically AdS black brane solution:

$$\begin{align*}
\frac{ds^2}{f(r)} &= f(r) dt^2 + dx^2 + dy^2
\end{align*}$$

where $f(r) = 1 - \frac{r^2}{r_h} + \frac{m^2 r^3}{r_h} - m^2 r^2$ and $r = r_h$ is the event horizon position ($f(r_h) = 0$). The temperature and energy density of the system are given by:

$$\begin{align*}
T &= \frac{6 - 2 m^2 r^2}{8 \pi r_h}, \quad \epsilon = \frac{1}{r_h^3} \left(1 - m^2 r_h^2\right)
\end{align*}$$

and the entropy density reads $s = 2 \pi / r_h^2$.

The computations presented in the main text considers the QNMs spectrum in the transverse sector which can be obtained numerically solving the equations for the perturbations around the background just presented. Using standard holographic techniques, we are able to extract the retarded Green functions. Their poles $\omega_i(k)$ are the QNMs of the system. For more details see the companion paper [24]. At a specific value of $T/m$, at which the energy density vanishes $\epsilon = 0$ and $\Gamma = c^2 / D$, the system enjoys an enhanced symmetry which allows us to compute the Green function for the transverse modes analytically [23]. As a result we are able to find analytically the $k$-gap form as:

$$\omega = -\frac{3}{2} i \pm \sqrt{k^2 - \frac{1}{4}}$$

The corresponding relaxation time $\tau$ at this self-dual point is shown as a red bullet in fig.2. Finally, in the limit of large $T/m$ analytical formulas for the diffusion constant $D$ and the momentum dissipation rate $\Gamma$ have been obtained [22, 23]:

$$\begin{align*}
\Gamma &= \frac{m^2}{2 \pi T} + \ldots \\
D &= \frac{1}{4 \pi T} \left[1 + \frac{1}{24} \left(9 + \sqrt{3} \pi - 9 \log 3\right) \frac{m^2}{8 \pi^2 T^2}\right] + \ldots
\end{align*}$$

Both the previous formulas are in agreement with our numerics in the regime $T/m \gg 1$.

Holographic model II

Let us now analyze the features of the second holographic model proposed in Ref. [20]. The setup represents the gravity dual for a finite number of elastic defects immersed in a fluid background phase. Momentum is a perfectly conserved quantity and the theory necessitates a finite UV cutoff which will be denoted as $C$. The gravitational bulk action is defined as:

$$S = \frac{1}{2 \kappa_4^2} \int d^4 x \sqrt{-g} \left( R + \frac{6}{L^2} - \frac{1}{12} H_{abc}^I H^{abc}_I\right)$$

The two forms admit the simple solution in terms of their field strengths:

$$H_{txr}^1 = H_{tyr}^2 = M$$

where $M$ physically encodes the density of the aforementioned elastic defects and all associated elastic properties.
of the system. The system admits a similar black brane solution (notice the different definition of the \( r \) coordinates as compared to the previous model) whose temperature reads:

\[
T = \frac{r_h}{4\pi} \left( 3 - \frac{M^2}{2r_h^2} \right)
\]  
(22)

In the limit of \( M = 0 \) and large cutoff \( \bar{C} \), the fundamental equation for the transverse spectrum of excitations is presented in the main text in (10) where:

\[
\omega_g = \frac{r_h}{\bar{C} - 1 + \frac{1}{2}(\ln 3 - \frac{\pi}{3\sqrt{3}})}
\]  
(23)

Equation (10) can be solved explicitly, giving:

\[
\omega_{\pm} = -\frac{\omega_g}{2} i \pm \sqrt{\frac{(C - 1)\omega_g}{r_h} k^2 - \frac{\omega_g^2}{4}}
\]  
(24)

which is identical to the solution of the Maxwell-Frenkel equation for liquids

\[
\omega = -\frac{i}{2\tau} \pm \sqrt{c^2 k^2 - \frac{1}{4\tau^2}}
\]  
(25)

with a finite \( k \)-gap [11]. Moreover, we immediately obtain the expressions for the speed \( c \) and relaxation time \( \tau \):

\[
c^2 = \frac{(C - 1)\omega_g}{r_h} = \frac{9(4\pi - 3\bar{C})}{2\pi (18 + \sqrt{3\pi} - 9\log(3))} - 27\bar{C}
\]  
(26)

\[
\tau \equiv \frac{1}{\omega_g} = \frac{27 (C - 2 \pi (18 + \sqrt{3\pi} - 9\log(3)))}{48 \pi^2 T}
\]  
(27)

where we have used \( \bar{C} \equiv C/T \).

Let us highlight two important features of the model:

(a) at infinite cutoff \( \bar{C} \to \infty \), the speed becomes relativistic \( c = 1 \), the relaxation time \( \tau \) diverges and the \( k \)-gap closes as a result of (24) and (b) the \( k \)-gap increases with temperature and the relaxation time decreases with it as \( \# / T \) as explained in the main text.

The diffusion constant in the hydrodynamic limit is given by:

\[
D = \frac{3}{4\pi T} \left( \frac{3\bar{C}}{4\pi T} - 1 \right)
\]  
(28)

The last three equations imply

\[
\tau = \frac{D}{c^2}
\]  
(29)

as discussed in the main text.

In the limit of large cutoff \( \bar{C} \to \infty \) and zero defects density \( M = 0 \), we have the Maxwell-Frenkel result for liquids:

\[
\tau = \frac{\eta}{G}
\]  
(30)

where \( G \) is the elastic shear modulus in the presence of defects:

\[
G = (\bar{C} - 1) M^2 r_h
\]  
(31)

and \( \eta \) is the viscosity defined as \( \eta = D \chi_{PP} \). The latter is the momentum susceptibility that can be computed as:

\[
\chi_{PP} = \epsilon + p - 2K = M^2 \left( \bar{C} - \frac{3}{2} \right) r_h + 3 r_h^3
\]  
(32)

where \( \epsilon \) is the energy density, \( p \) the pressure and \( K \) the bulk modulus. For more details we refer to the companion paper [24].