NONEXPANDING ATTRACTORS: CONJUGACY TO
ALGEBRAIC MODELS AND CLASSIFICATION IN
3-MANIFOLDS

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ABSTRACT. We prove a result motivated by Williams’s classification of expanding attractors and the Franks-Newhouse Theorem on codimension-1 Anosov diffeomorphisms: If Λ is a topologically mixing hyperbolic attractor with dim $E^u|_{\Lambda} = 1$ then either Λ is expanding or is homeomorphic to a compact abelian group (a toral solenoid); in the latter case $f|_{\Lambda}$ is conjugate to a group automorphism. As a corollary we obtain a classification of all 2-dimensional basic sets in 3-manifolds. Furthermore we classify all topologically mixing hyperbolic attractors in 3-manifolds in terms of the classically studied examples, answering a question of Bonatti in [1].

1. INTRODUCTION

In the study of hyperbolic dynamics, a major theme is that strong dynamical hypotheses impose a conjugacy between an abstract dynamical system and an algebraic, or at least highly structured, model. For instance, results of Franks and Manning established that every Anosov diffeomorphism of an infranil-manifold is conjugate to a hyperbolic infranil-automorphism [14, Theorem C]. Among of the oldest conjectures in modern dynamics is the hypothesis that every Anosov diffeomorphism is conjugate to a hyperbolic infranil-automorphism. A partial result towards this conjecture was obtained by Franks and Newhouse for codimension-1 Anosov systems. Recall an Anosov diffeomorphism is called codimension-1 if dim($E^\sigma$) = 1 for some $\sigma \in \{s, u\}$.

Theorem I (Franks-Newhouse [6], [15]; see also [9]). Let $f : M \to M$ be a codimension-1 Anosov diffeomorphism. Then $M$ is homeomorphic to a torus, and $f$ is conjugate to a hyperbolic toral automorphism.

Outside the realm of global hyperbolicity, that is, when dealing with proper hyperbolic subsets $\Lambda \subset M$, one often sees dynamics which is not conjugate to any algebraic system. However, in the case of expanding attractors, Williams showed in [23] that the restricted dynamics $f|_{\Lambda}$ is conjugate to the shift map on a generalized solenoid. Recall that by an expanding attractor we mean a hyperbolic attractor Λ such that dim(Λ) = dim($E^u|_{\Lambda}$). Also by a generalized solenoid (or n-solenoid) we mean a topological space $N$ (which Williams takes to be a branched n-manifold), and a surjective map $g : N \to N$, and define the generalized solenoid to be the inverse limit

$$\lim(N, g) := \lim\{N \xrightarrow{g} N \xrightarrow{g} N \xrightarrow{g} \ldots \}$$

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Theorem II ([23, Theorem A]). Assume $\Lambda$ is an $n$-dimensional expanding attractor for $f \in \text{Diff}(M)$. Then $f|_{\Lambda}$ is conjugate to the shift map of an $n$-solenoid.

Note that Theorem II, as originally stated in [23], required the additional hypothesis that the foliation $\{W^s_r(x) \mid x \in \Lambda\}$ was $C^1$ on some neighborhood of $\Lambda$. This was latter seen to be unnecessary (see for example [2]). While not algebraic, the conjugacy in Theorem II provides a significant insight into the topology of $\Lambda$ and the dynamics of $f|_{\Lambda}$.

In this article we present a result inspired in part by the Franks-Newhouse Theorem on codimension-1 Anosov diffeomorphisms, and somewhat dual to the conjugacy between the dynamics of 1-dimensional expanding attractors and shift maps on generalized solenoids established in [21] and [22]. In particular, we study non-expanding hyperbolic attractors $\Lambda$ for an embedding $f$, under the assumption that $\dim E^u|_{\Lambda} = 1$, and show that the dynamics $f|_{\Lambda}$ is conjugate to an automorphism of a compact abelian group. We take our dynamics to be generated by $C^r$ embeddings for $r \geq 1$.

**Theorem 1.1.** Let $\Lambda \subset U \subset M$ be a compact topologically mixing hyperbolic attractor for a $C^r$ embedding $f : U \to M$ such that $\dim E^u|_{\Lambda} = 1$. Then either $\Lambda$ is expanding, or is an embedded toral solenoid (see Section 4). In the latter case, $f|_{\Lambda}$ is conjugate to a leaf-wise hyperbolic solenoidal automorphism. In particular, if $\Lambda$ is locally connected then $\Lambda$ is homeomorphic to a torus and $f|_{\Lambda}$ is conjugate to a hyperbolic toral automorphism.

Using the primary result in [11] we conclude that the only 2-dimensional toral solenoids that may be embedded in a 3-manifold are homeomorphic to $\mathbb{T}^2$. In particular, we obtain the following.

**Corollary 1.2.** Let $M$ be a 3-manifold, and let $\Lambda \subset M$ be a basic set with $\dim(\Lambda) = 2$. Then either $\Lambda$ is a codimension-1 expanding attractor (or contracting repeller), or $\Lambda$ decomposes as a disjoint union

$$\Lambda = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k$$

where each $\Omega_j$ is homeomorphic to $\mathbb{T}^2$ and $f^k|_{\Omega_j}$ is conjugate to a hyperbolic automorphism of $\mathbb{T}^2$.

We note that the above corollary is a significantly stronger version of the main result in [7]. Indeed, in [7] the result corresponding to the second case in Corollary 1.2 requires the additional hypothesis that $\Lambda$ is embedded as a subset of a closed surface in $M$. Our result, on the other hand, rules out the possibility that $\dim(E^u|_{\Lambda}) = 1$ and $W^s(x) \cap \Lambda$ is a connected 1-dimensional set that is not a manifold, for example, a Sierpinski carpet.

It should also be noted that in the conclusion of Corollary 1.2, the $\mathbb{T}^2$ need not be smoothly embedded. Indeed in [12] a hyperbolic attractor is constructed as a nowhere differentiable torus embedded in a 3-manifold.

The motivation for this work was initially to answer a question by Bonatti [1] which can be paraphrased as follows: Do there exist examples of hyperbolic attractors in 3-manifolds besides the classical examples? We answer this question in the negative.
Theorem 1.3. Let $M$ be a 3-manifold, and let $\Lambda \subset U \subset M$ be a topologically mixing, hyperbolic attractor for a $C^r$ embedding $f: U \to M$. If

$$\dim \Lambda = 0: \text{ then } \Lambda \text{ is an attracting fixed point for } f;$$

$$\dim \Lambda = 1: \text{ then } \dim E^u|_\Lambda = 1 \text{ and } \Lambda \text{ is conjugate to the shift map on a } 1\text{-dimensional attractor as classified by Williams ([22]);}$$

$$\dim \Lambda = 2: \text{ then } \dim E^u|_\Lambda = 2 \text{ and if}$$

$$\dim E^u|_\Lambda = 1: \text{ then } \Lambda \text{ is homeomorphic to } \mathbb{T}^2 \text{ and } f|_\Lambda \text{ is conjugate to a hyperbolic toral automorphism;}$$

$$\dim E^u|_\Lambda = 2: \text{ then } \Lambda \text{ is a codimension-1 expanding attractor studied by Plykin ([17], [18]);}$$

$$\dim \Lambda = 3: \text{ then } \Lambda = M \cong \mathbb{T}^3 \text{ and } f \text{ is conjugate to a hyperbolic toral automorphism.}$$

We remark that in the case of 1-dimensional topologically mixing attractors (which are necessarily expanding), the proof of Theorem 1.1 provides a mechanism to determine if the attractor is algebraic, that is, if $f|_\Lambda$ is conjugate to a solenoidal automorphism. In particular the presence or absence of a global product structure as described in Section 5.3.2 determines whether or not a 1-dimensional attractor is algebraic. See Proposition 5.30.

2. Hyperbolic dynamics

We begin with background material in hyperbolic dynamics and attractors. Let $M$ be a smooth manifold endowed with a Riemannian metric. Given $U \subset M$ and a $C^r$ embedding $f: U \to M$, $r \geq 1$, we say a subset $\Lambda \subset U$ is invariant if $f(\Lambda) = \Lambda$. A compact invariant set $\Lambda$ is said to be hyperbolic if there exist a Riemannian metric on $M$ (called the adapted metric), a constant $\kappa < 1$, and a continuous $Df$-invariant splitting of the tangent bundle $T_xM = E^s(x) \oplus E^u(x)$ over $\Lambda$ so that for every $x \in \Lambda$ and $n \in \mathbb{N}$

$$\|Df_x^nv\| \leq \kappa^n\|v\|, \text{ for } v \in E^s(x)$$

$$\|Df_x^{-n}v\| \leq \kappa^n\|v\|, \text{ for } v \in E^u(x).$$

We set

$$V^\pm = \bigcap_{n \in \mathbb{N}} f^\pm n(U).$$

When $\Lambda$ is hyperbolic, there exists an $\epsilon > 0$ such that the sets

$$W^s_{\epsilon}(x) := \{y \in V^- \mid d(f^n(x), f^n(y)) < \epsilon, \text{ for all } n \geq 0\}$$

$$W^u_{\epsilon}(x) := \{y \in V^+ \mid d(f^{-n}(x), f^{-n}(y)) < \epsilon, \text{ for all } n \geq 0\}$$

are $C^r$ embedded open disks, called the local stable and unstable manifolds. Furthermore, if $d$ is the distance on $M$ induced by the adapted metric, there are $\lambda < 1 < \mu$ so that for $x \in \Lambda, y \in W^s_{\epsilon}(x), z \in W^u_{\epsilon}(x)$ and $n \geq 0$ we have

$$d(f^n(x), f^n(y)) \leq \lambda^nd(x, y)$$

$$d(f^{-n}(x), f^{-n}(z)) \leq \mu^{-n}d(x, z).$$

Note that (1) and (2) imply $f(W^s_{\epsilon}(f^{-1}(x))) \subset W^s_{\epsilon}(x)$ and $W^u_{\epsilon}(x) \subset f(W^u_{\epsilon}(f^{-1}(x)))$.

For $x \in \Lambda$ we also have the sets

$$W^s(x) := \{y \in V^- \mid d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty\}$$
and
\[ W^u(x) := \{ y \in V^+ \mid d(f^{-n}(x), f^{-n}(y)) \to 0 \text{ as } n \to \infty \} \]
called the global stable and unstable manifolds. Both \( W^u(x) \) and \( W^s(x) \) are \( C^r \) injectively immersed submanifolds. Note that in the case that \( f \) is invertible (that is, when \( f(U) = U \)), we have \( W^u(x) \cong \mathbb{R}^{\dim E^u(x)} \) and \( W^s(x) \cong \mathbb{R}^{\dim E^s(x)} \).

An invariant set \( \Lambda \) is said to be topologically transitive under \( f \) if it contains a dense orbit. Alternatively, a compact invariant subset \( \Lambda \subseteq M \) is topologically transitive if for all pairs of nonempty open sets \( U, V \subseteq \Lambda \), there is some \( n \) such that \( f^n(U) \cap V \neq \emptyset \). An invariant set \( \Lambda \) is called topologically mixing if for all pairs of nonempty open sets \( U, V \subseteq \Lambda \), there is some \( N \) such that \( f^n(U) \cap V \neq \emptyset \) for all \( n \geq N \).

A hyperbolic set \( \Lambda \) is a hyperbolic attractor if there exists an open neighborhood \( \Lambda \subseteq V \) such that \( \bigcap_{n \in \mathbb{N}} f^n(V) = \Lambda \). Alternatively, if \( \Lambda \) is a hyperbolic set, then it is an attractor if and only if \( W^u(x) \subseteq \Lambda \) for all \( x \in \Lambda \). When \( \Lambda \) is a hyperbolic attractor, the set \( \bigcup_{y \in \Lambda} W^s(y) \) is called the basin of \( \Lambda \). Note that if \( \Lambda \) is a topologically mixing hyperbolic attractor, then for each \( x \in \Lambda \), \( W^u(x) \) is dense in \( \Lambda \).

We recall from the introduction that a hyperbolic attractor \( \Lambda \) is called expanding if the topological dimension of \( \Lambda \) equals the dimension of the unstable manifolds. (For an introduction to topological dimension, see [10].) Alternatively, \( \Lambda \) is expanding if for every \( x \in \Lambda \) the set \( W^s(x) \cap \Lambda \) is totally disconnected.

2.1. Local product structure and Markov partitions. Recall that given a compact hyperbolic set, we may find \( 0 < \delta < \eta \) so that \( d(x, y) < \delta \) implies the intersection \( W^u_\eta(x) \cap W^s_\delta(y) \) is a singleton. We say that a hyperbolic set \( \Lambda \) has local product structure if for \( \eta, \delta \) above, \( d(x, y) < \delta \) implies \( W^u_\eta(x) \cap W^s_\eta(y) \subseteq \Lambda \). A compact hyperbolic set \( \Lambda \) is called locally maximal if there exists an open set \( \Lambda \subseteq V \) such that \( \Lambda = \bigcap_{n \in \mathbb{Z}} f^n(V) \). For compact hyperbolic sets, local maximality is equivalent to the existence of a local product structure [13]; in particular, hyperbolic attractors have local product structure.

**Definition 2.1.** Given a set \( \Lambda \) with local product structure and \( \delta \) and \( \eta \) as above, we say a closed set \( R \subseteq \Lambda \) is a rectangle or a local product chart if

1. \( \sup\{d(x, y) \mid x, y \in R\} < \delta \);
2. \( R \) is proper, that is, \( R \) is equal to the closure of its interior (in \( \Lambda \));
3. \( x, y \in R \) implies \( W^u_\eta(x) \cap W^s_\delta(y) \subseteq R \).

If \( R \) is a rectangle, we write \( W^u_R(x) := W^u_\eta(x) \cap R \).

If \( \Lambda \) is an attractor, we say an ambiently open set \( V \subseteq M \) is a local \( s \)-product neighborhood if the closure of \( V \cap \Lambda \) is a rectangle and for each \( x \in V \cap \Lambda \)
\[ V \subseteq \bigcup_{y \in W^s_R(x)} W^s_R(y). \]

For \( x \in V \cap \Lambda \) we notate
\[ W^s_R(x) := W^s_R(x) \cap V. \]

**Definition 2.2.** Given a hyperbolic set \( \Lambda \) with local product structure we say a collection of rectangles \( \mathcal{R} = \{ R_j \} \) is a Markov partition if

1. \( \Lambda = \bigcup_j R_j \);
2. for \( i \neq j \), \( R_j \cap R_i \subseteq \partial R_j \) where \( \partial \) denotes the topological boundary;
The following is adapted from entropy as a product of measures supported on the stable and unstable manifolds.

Basic sets

in particular, hyperbolic attractors admit Markov partitions. Also note that if $R$ is a rectangle and $\mathcal{R}$ is a Markov partition, then $f(R)$ is a rectangle and $f(\mathcal{R}) := \{f(R_j)\}$ is a Markov partition. In particular, we have the following.

Claim 2.3. If $\Lambda$ is locally maximal, then given any set $K \subset W^\sigma(x) \cap \Lambda$, compact in the internal topology of $W^\sigma(x)$, there is a rectangle containing $K$.

2.2. Disintegration of the measure of maximal entropy. For a hyperbolic set with local product structure we define a canonical isomorphism between subsets of the stable and unstable manifolds.

Definition 2.4 (Canonical Isomorphism). Let $\Lambda$ be a locally maximal hyperbolic set, $R$ a rectangle, and $x \in R$. Let $x' \in W^s_R(x)$, and let $D \subset W^s_R(x)$, $D' \subset W^s_R(x')$. Then $D$ and $D'$ are said to be canonically isomorphic if $y \in D \cap \Lambda$ implies $D' \cap W^u_R(y) \neq \emptyset$ and $y' \in D' \cap \Lambda$ implies $D \cap W^u_R(y') \neq \emptyset$.

Similarly, we may define a canonical isomorphism between subsets of local unstable manifolds.

Recall that a point $x \in M$ is said to be nonwandering if for every neighborhood $U$ of $x$, there is an $n$ so that $f^n(U) \cap U \neq \emptyset$. Let $NW(f)$ denote the nonwandering points of $f$. Recall that given an Axiom-A diffeomorphism, (respectively a locally maximal hyperbolic set $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(V)$) we have a partition, called the spectral decomposition, of the nonwandering points $NW(f) = \Omega_1 \cup \cdots \cup \Omega_k$ (respectively $NW(f|\Lambda) = \Omega_1 \cup \cdots \cup \Omega_k$) where each $\Omega_j$ is a transitive hyperbolic set for $f$ (see [13], [20]). Given a spectral decomposition, we call the partition elements $\Omega_j$ above basic sets. That is, a compact hyperbolic set $\Omega \subset NW(f)$ is a basic set, if $\Omega$ is open in $NW(f)$ and $f$ is topologically transitive on $\Omega$. Clearly topologically mixing hyperbolic attractors are basic sets.

Given a basic set, there is a canonical disintegration of the measure of maximal entropy as a product of measures supported on the stable and unstable manifolds. The following is adapted from [19].

Theorem 2.5 (Ruelle, Sullivan [19]). Let $\Omega$ be a basic set for $f$. Let $h$ be the topological entropy of $f|\Omega$. Then there is an $\epsilon > 0$ so that for each $x \in \Omega$ there is a measure $\mu^u_x$ on $W^u(x)$ and a measure $\mu^s_x$ on $W^s(x)$ such that:

a) $\text{supp}(\mu^u_x) = W^u(x) \cap \Omega$ and $\text{supp}(\mu^s_x) = W^s(x) \cap \Omega$,

b) $\mu^u_x$ and $\mu^s_x$ are invariant under canonical isomorphism (see Definition 2.4); that is, if $x' \in W^u(x)$ and $D \subset W^u(x)$, $D' \subset W^u(x')$ are canonically isomorphic then $\mu^u_x(D) = \mu^u_{x'}(D')$, and if $x' \in W^s(x)$ and $D \subset W^s(x)$, $D' \subset W^s(x')$ are canonically isomorphic then $\mu^s_x(D) = \mu^s_{x'}(D')$,

c) $f_* \mu^u_x = e^{-h} \mu^u_{f(x)}$ and $f^{-1}_* \mu^s_x = e^{-h} \mu^s_{f^{-1}(x)}$,

d) the product measure $\mu^u_x \times \mu^s_x$ is locally equal to Bowen’s measure of maximal entropy.
By 2.5(b) we drop the subscript and simply write μσ. By additivity, we may extend the definition of μσ to any set K ⊂ Wσ(x) for σ ∈ {s, u}. The following properties of μσ are corollaries to the proof of Theorem 2.5 in [19].

**Corollary 2.6.** Let Ω be a basic set with an infinite number of points. Then for σ ∈ {s, u}

— μσ is non-atomic and positive on non-empty open sets in Wσ(x) ∩ Λ;
— μσ(K) is finite for sets K ⊂ Wσ(x) compact in the internal topology of Wσ(x).

Furthermore, in the case dim E_u↾Λ = 1, we have the following.

**Corollary 2.7.** Let Λ be a hyperbolic attractor such that dim E_u↾Λ = 1. Then for any connected set K ⊂ Wu(x) (that is, an interval) we have μu(K) < ∞ if and only if its closure K in Wu(x) is compact in the internal topology of Wu(x).

**Proof.** If K is not compact in Wu(x), then K passes through some rectangle a countable number of times which implies that μu(K) = ∞. □

3. Limits of directed and inverse systems

We review basic constructions and properties of the direct and inverse limit objects in algebra and topology.

3.1. **Direct limits.** Given a topological space X and an injective continuous map f: X → X we construct the direct limit

\[ \text{lim}^+ (X, f) := \text{lim}^+ \{ X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \ldots \} \]

as follows. Endow N with the discrete topology and introduce the equivalence relation on X × N generated by the relation (x, k) ∼ (f(x), k + 1). Then we define

\[ \text{lim}^+ (X, f) := (X \times N)/\sim. \]

The map f: X → X naturally induces a homeomorphism τ_f: lim^+(X, f) → lim^+(X, f) by

\[ \tau_f: [(x, m)] \mapsto [(f(x), m)]. \]

Note, that for m ≥ 1 we have τ_f([(x, m)]) = [(x, m − 1)], whence it is natural to refer to τ_f as the left shift on lim^+(X, f).

We present an alternate, more explicit, construction of the set lim^+(X, f). For every j ∈ N define a homeomorphism

\[ h_j: X_j \to X \]

and consider the inclusion i_j: X_j ↪ X_{j+1} given by i_j = h_{j+1}^{-1} ∘ f ∘ h_j. We then have X_0 ⊂ X_1 ⊂ X_2 ⊂ ⋯ whence we define

\[ \lim^+ \{ X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \ldots \} = \bigcup_{n \in \mathbb{N}} X_n. \]

When the map f: X → X is open, the inclusions i_j induce a nested inclusion of topologies, the union of which correctly reconstructs the topology of the direct limit. Given ξ ∈ lim^+(X, f), we have that ξ ∈ X_j for some j whence we may define

\[ \tau_f(ξ) = h_j^{-1} ∘ f ∘ h_j(ξ). \]

One verifies this definition of τ_f coincides with that above.
By the second construction, we see that if $X$ is a $C^r$ manifold and $f$ a $C^r$ embedding, then $\lim(X, f)$ can be endowed with a $C^r$ differential structure under which $\tau_f$ is a $C^r$ diffeomorphism.

Given a group $G$ and a homomorphism $h: G \to G$ we define
\[
\lim(G, h) := \lim\{G \to G \xrightarrow{h} G \to \ldots\}
\]
as follows. Let $i_j: G_j \to G$ be a group isomorphism and define $h_j: G_j \to G_{j+1}$ by $h_j: g \mapsto i_j^{-1}(h(i_j(g)))$. Let $N$ be the normal subgroup of $\bigoplus_{k \in \mathbb{N}} G_k$ generated by the elements $\{g_j^{-1} h_j(g_j)\}$ for $g_j \in G_j$. Then
\[
\lim(G, h) = \left(\bigoplus_{k \in \mathbb{N}} G_k\right)/N
\]
with canonical left shift automorphism $\tau_h$ given by $\tau_h: [g_j] \mapsto [i_j^{-1} \circ h \circ i_j(g_j)]$. We denote $[(g, m)] := g_m + N$ for $g_m \in G_m$.

The following proposition is straightforward from the Van Kampen theorem.

**Proposition 3.1.** Let $X$ be a connected manifold, $f: X \to X$ an embedding, and $G = \pi_1(X)$. Then
\[
\begin{align*}
\pi_1(\lim(X, f)) &= \lim(G, f_*), \\
(\tau_f)_* &= \text{the map given by } (\tau_f)_*([(g, m)]) = [(f_*(g), m)].
\end{align*}
\]

The construction above allows us to embed every hyperbolic attractor as an attractor for an ambient diffeomorphism.

**Claim 3.2.** Let $\Lambda \subset U \subset M$ be a hyperbolic attractor for a $C^r$ embedding $f: U \to M$. Then there is a $C^r$ diffeomorphism $f': M' \to M'$, a hyperbolic attractor $\Lambda' \subset M'$ for $f'$, neighborhoods $N$ and $N'$ of $\Lambda$ and $\Lambda'$ respectively, and a $C^r$ diffeomorphism $h: N \to N'$ so that $h(\Lambda) = \Lambda'$ and $h \circ f|_N = f'|_{N'} \circ h$.

**Proof.** Let $N = \bigcup_{y \in \Lambda} W^s_\varepsilon(y)$. Then $f: N \to N$ is a $C^r$ embedding. Take $X = N$ and $M' := \lim(X, f)$. Then we have a canonical inclusion $\Lambda \subset X_0 \cong N$, where $X_0$ is as in (3). But then $\Lambda \subset X_0 \subset M'$ is a hyperbolic attractor for the $C^r$ diffeomorphism $\tau_f: M' \to M'$.

Note that in constructing the direct limit, we assumed the map $f: X \to X$ was injective to avoid pathological topological properties in the limiting object.

**3.2. Inverse limits.** Let $f: X \to X$ be a continuous map (which we typically take to be surjective). We then define the inverse limit
\[
\lim(X, f) := \lim\{X \xleftarrow{f} X \xleftarrow{f} \ldots\}
\]
to be the subset of $X^\mathbb{N} := \prod_{i \in \mathbb{N}} X$ satisfying
\[
(x_0, x_1, x_2, \ldots) \in \lim(X, f) \text{ if } x_j = f(x_{j+1})
\]
for all $j \in \mathbb{N}$. We then have an induced homeomorphism $\sigma_f: \lim(X, f) \to \lim(X, f)$ given by
\[
\sigma_f: (x_0, x_1, x_2, \ldots) \mapsto (f(x_0), f(x_1), f(x_2), \ldots) = (f(x_0), x_0, x_1, x_2, \ldots)
\]
hence it is natural to call $\sigma_f$ a right shift map. We will call the topological object $\lim(X, f)$ a generalized solenoid.
Note that even in the case that $X$ is a manifold and $f$ is a smooth map, we do not expect $\lim(X, f)$ to have a manifold structure. Indeed in the case that $f$ is a $C^\infty$ covering with degree greater than 1, the limit $\lim(X, f)$ will locally be the product of a Cantor set and a manifold.

If $G$ is a group, and $h$ a homomorphism we define

$$\lim(G, h) := \lim\{G \xrightarrow{h} G \xrightarrow{h} \ldots\}$$

to be the subgroup of $\prod_{n \in \mathbb{N}} G$ satisfying $(g_0, g_1, g_2, \ldots) \in \lim(G, h)$ if $g_j = h(g_{j+1})$, with the induced right shift automorphism

$$\sigma_h: (g_0, g_1, g_2, \ldots) \mapsto (h(g_0), g_0, g_1, g_2, \ldots).$$

4. Toral solenoids

We give a brief introduction to toral solenoids, the compact abelian groups obtained as the algebraic models in the conclusion of Theorem 1.1. For more detailed exposition, see, for example, [3]. For an explicit construction of toral solenoids embedded as hyperbolic attractors for differentiable dynamics, see [8].

Let $A \in \text{Mat}(k, \mathbb{Z})$ have non-zero determinant. Then considering the standard torus $\mathbb{T}^k := \mathbb{R}^k / \mathbb{Z}^k$ as a compact abelian group, the map $A: \mathbb{R}^k \to \mathbb{R}^k$ induces an endomorphism $A: \mathbb{T}^k \to \mathbb{T}^k$. We define a toral solenoid $S_A$ to be the topological group obtained via the inverse limit

$$S_A := \lim(\mathbb{T}^k, A) = \lim\{\mathbb{T}^k \xleftarrow{A} \mathbb{T}^k \xleftarrow{A} \ldots\}.$$ 

Note the above inverse limit is taken both as a limit of topological and algebraic objects, and that $S_A$ inherits the right shift automorphism $\sigma_A: S_A \to S_A$

$$\sigma_A: (x_0, x_1, x_2, \ldots) \mapsto (Ax_0, x_0, x_1, \ldots).$$

$S_A$ will fail to be a manifold in the case when $|\det(A)| > 1$, in which case we will call $S_A$ proper. Let $C_\xi$ denote the path component of $\xi$ in $S_A$. Then, even in the case $|\det(A)| > 1$, we can endow the path components $\{C_\xi\}$ with the smooth Euclidean structure pulled back from the projection to the zeroth coordinate $S_A \to \mathbb{T}^k$. With respect to this Euclidean structure the map $\sigma_A: C_\xi \to C_{\sigma_A(\xi)}$ is smooth. Furthermore, in the case when $A$ has no eigenvalues of modulus 1, the map $\sigma_A: C_\xi \to C_{\sigma_A(\xi)}$ is hyperbolic with respect to the pull-back metric, whence we say $\sigma_A$ is leaf-wise hyperbolic.

4.1. $\mathbb{R}^k$ and $\mathbb{Z}^k$ actions. We now define an $\mathbb{R}^k$ action and an induced $\mathbb{Z}^k$ action on $S_A$. (Compare to the immersions of $\mathbb{R}^k$ and $\mathbb{Z}^k$ into $S_A$ constructed in [3]).

**Definition 4.1.** We define the $\mathbb{R}^k$ action $\theta: \mathbb{R}^k \times S_A \to S_A$ by the rule

$$\theta_v: ([y_0], [y_1], [y_2], \ldots) \mapsto ([y_0 + v], [y_1 + A^{-1}v], [y_2 + A^{-2}v], \ldots)$$

for $v \in \mathbb{R}^k$ and $([y_0], [y_1], [y_2], \ldots) \in S_A$ where $[y]$ denotes the class of $y \in \mathbb{R}^k$ in the quotient $\mathbb{T}^k = \mathbb{R}^k / \mathbb{Z}^k$.

We then define the $\mathbb{Z}^k$ action $\vartheta: \mathbb{Z}^k \times S_A \to S_A$ to be the restriction of $\theta$ to the subgroup $\mathbb{Z}^k \subset \mathbb{R}^k$.

Let $p_0: S_A \to \mathbb{T}^k$ denote the projection in the zeroth coordinate.

**Claim 4.2.** The action $\theta$ has the following properties.

a) For each $\xi \in S_A$, the $\theta$-orbit of $\xi$ is dense.
b) The homomorphism \( \sigma_A \) is \( \theta \)-equivariant; that is,
\[
\sigma_A(\theta_v(\xi)) = \theta_{A\cdot}(\sigma_A(\xi))
\]
and
\[
\sigma_A^{-1}(\theta_v(\xi)) = \theta_{A^{-1}\cdot}(\sigma_A^{-1}(\xi)).
\]

c) For all \( \xi \in S_A \) and \( v \in \mathbb{R}^k \) we have
\[
p_0(\theta_v(\xi)) = p_0(\xi) + v.
\]
Here \([x] + v := [x + v]\) is the standard \( \mathbb{R}^k \) action on \( T^k \).

d) \( \theta \) commutes with the group operation; that is,
\[
\theta_v(\xi + \eta) = \theta_v(\xi) + \eta = \xi + \theta_v(\eta)
\]

Proof. 4.2(a) is essentially [3, Proposition 2.4]. 4.2(b), 4.2(c), 4.2(d) follow from (4). \( \square \)

Define \( \Sigma \) to be the 0-dimensional compact group \( \Sigma := p_0^{-1}(\{0\}) \). Note \( \Sigma \) is either the trivial group \{1\} in the case that \( \det(A) = \pm 1 \), or homeomorphic to a Cantor set in the case \( \det(A) > 1 \). By [3, Corollary 2.3] the map \( p_0: S_A \to T^k \) defines a principle \( \Sigma \)-bundle.

Let \( p: \mathbb{R}^k \to T^k \) denote the canonical projection. Given an \( m \in \mathbb{Z}^k \), we may find some curve \( \gamma: [0,1] \to \mathbb{R}^k \) with \( \gamma(0) = 0 \) and \( \gamma(1) = m \). Then \( p(\gamma) \) corresponds to a closed curve at \([0]\), hence determines an element of \( \pi_1(T^k, [0]) \). For \( \eta \in \Sigma \) we have that \( \gamma'(t) := \theta_{\gamma(t)}(\eta) \) is the unique lift of \( p(\gamma(t)) \) to \( S_A \) starting at \( \eta \). Then we have that \( \gamma'(1) = \theta_m(\eta) \). This motivates the construction in the next section.

4.2. A covering space for \((S_A, \sigma_A)\). Define the topological group \( \mathcal{S} \) to be the product \( \Sigma \times \mathbb{R}^k \). The group action \( \vartheta \) of \( \mathbb{Z}^k \) on \( S_A \) induces a group action \( \vartheta \) of \( \mathbb{Z}^k \) on \( \Sigma \). We define an embedding \( \alpha \) of \( \mathbb{Z}^k \) as a subgroup of \( \mathcal{S} \) by
\[
\alpha(n) := (\vartheta_{-n}(e), n)
\]
where \( e \) is the identity element of \( \Sigma \). Then \( \alpha \) naturally defines a \( \mathbb{Z}^k \) action on \( \mathcal{S} \) by
\[
n \cdot (\xi, v) := (\xi, v) + \alpha(n) = (\vartheta_{-n}(\xi), v + n)
\]
for \( n \in \mathbb{Z}^k, \xi \in \Sigma, \) and \( v \in \mathbb{R}^k \).

We also define maps \( \overline{\sigma}: \mathcal{S} \to \mathcal{S} \) given by
\[
\overline{\sigma}: (\xi, v) \mapsto (\sigma_A(\xi), A(v))
\]
and \( \overline{\theta}: \mathcal{S} \to S_A \) given by
\[
\overline{\theta}: (\xi, v) \mapsto \theta_v(\xi).
\]

We check that \( \overline{\sigma} \) is an injective endomorphism. Furthermore, \( \overline{\theta} \) is seen to be a homomorphism by Claim 4.2(d).

We have the following properties of the above construction.

Claim 4.3. \( \overline{\sigma}, \overline{\theta} \), and \( \alpha \) satisfy

a) \( N := \alpha(\mathbb{Z}^k) \) is a discrete subgroup isomorphic to \( \mathbb{Z}^k \);

b) \( \ker(\overline{\theta}) = N \), whence we have the canonical identification \( \mathcal{S}/N \cong S_A \) as topological groups;

c) \( \overline{\theta} \circ \overline{\sigma} = \sigma_A \circ \overline{\theta} \);

d) \( \overline{\sigma}(x + \alpha(n)) = \overline{\sigma}(x) + \alpha(A(n)) \).
Proof. 4.3(a) is clear. For 4.3(b), let \( \overline{\eta}(\xi, v) = e \) where \( e \) is the identity in \( \Sigma \subset S_A \). Then we have
\[
\theta_v(\xi) = e.
\]
In particular, \( v \in \mathbb{Z}^k \). Furthermore
\[
\alpha(v) = (\vartheta_{-v}(e), v) = (\vartheta_{v}^{-1}(e), v) = (\xi, v)
\]
hence \( \overline{\eta}(\xi, v) = e \) implies \( (\xi, v) \in N \). Similarly for any \( n \in \mathbb{Z}^k \) we have \( \overline{\eta}(\alpha(n)) = \vartheta_n(\vartheta_{-n}(e)) = e \) hence 4.3(b) holds.

We have
\[
\overline{\eta} \circ \overline{\sigma}(\xi, v) = \theta_{A^v}(\sigma_A \xi) = \sigma_A(\theta_v(\xi)) = \sigma_A \circ \overline{\eta}(\xi, v)
\]
whence 4.3(c) follows. Finally we have
\[
\overline{\sigma}(\alpha(n)) = (\sigma_A(\vartheta_{-n}(e)), A(n)) = (\vartheta_{-A(n)}(\sigma_A(e)), A(n)) = \alpha(A(n))
\]
from which 4.3(d) follows. \( \square \)

Now \( \overline{\sigma} : \overline{S} \rightarrow \overline{S} \) is an injective homomorphism, but will fail to be surjective whenever \( |\det(A)| > 1 \). We define the topological group \( \overline{S} \) as the direct limit
\[
\overline{S} := \lim_{\longrightarrow} \{ \overline{S} \overset{\overline{\sigma}}{\rightarrow} \overline{S} \overset{\overline{\sigma}}{\rightarrow} \overline{S} \overset{\overline{\sigma}}{\rightarrow} \ldots \}
\]
and define \( \overline{\sigma} : \overline{S} \rightarrow \overline{S} \) to be the left shift automorphism \( (s^\tau \in \Sigma \) in the notation of Section 3.1) induced by \( \overline{\sigma} : \overline{S} \rightarrow \overline{S} \); that is,
\[
\overline{\sigma}([s, l]) = ([\overline{\sigma}(s), l]) = [(s, l - 1)]
\]
where the second equality holds for \( l \geq 1 \). Furthermore, we define the torsion-free abelian group
\[
\mathbb{Z}^k[A^{-1}] := \lim_{\longrightarrow} \{ \mathbb{Z}^k \overset{A}{\rightarrow} \mathbb{Z}^k \overset{A}{\rightarrow} \mathbb{Z}^k \overset{A}{\rightarrow} \ldots \}
\]
and the (left shift) group homomorphism \( \tau_A : \mathbb{Z}^k[A^{-1}] \rightarrow \mathbb{Z}^k[A^{-1}] \). Note that by Claim 4.3(d) the diagram
\[
\begin{array}{cccccccc}
\mathbb{Z}^k & \overset{A}{\rightarrow} & \mathbb{Z}^k & \overset{A}{\rightarrow} & \mathbb{Z}^k & \overset{A}{\rightarrow} & \ldots \\
\downarrow{\overline{\sigma}} & & \downarrow{\overline{\sigma}} & & \downarrow{\overline{\sigma}} & & \ldots \\
\overline{S} & \overset{\overline{\sigma}}{\rightarrow} & \overline{S} & \overset{\overline{\sigma}}{\rightarrow} & \overline{S} & \overset{\overline{\sigma}}{\rightarrow} & \ldots
\end{array}
\]
commutes whence we may extend the embedding \( \alpha : \mathbb{Z}^k \rightarrow \overline{S} \) to an embedding \( \overline{\alpha} \) of \( \mathbb{Z}^k[A^{-1}] \) as a subgroup of \( \overline{S} \). Since \( \alpha(\mathbb{Z}^k) \) is a discrete subgroup of \( \overline{S} \), the homomorphism \( \overline{\alpha} \) embeds \( \mathbb{Z}^k[A^{-1}] \) as a discrete subgroup of \( \overline{S} \). More explicitly we have
\[
\overline{\alpha}([(n, m)]) := [(\alpha(n), m)]
\]
which is seen to be well defined by Claim 4.2(b). As above, the embedding \( \overline{\alpha} \) of \( \mathbb{Z}^k[A^{-1}] \) as a subgroup of \( \overline{S} \) defines a natural \( \mathbb{Z}^k[A^{-1}] \) action on \( \overline{S} \). We also define a group homomorphism \( \tilde{\eta} : \overline{S} \rightarrow S_A \) by
\[
\tilde{\eta} : [((\xi, v), l)] \mapsto \sigma_{\overline{\alpha}(\xi, v)}^{-1} \overline{\eta}(\xi, v).
\]
We enumerate properties of the above constructions.

**Proposition 4.4.** For \( \overline{S}, \overline{\sigma}, \overline{\alpha}, \) and \( \tilde{\eta} \) we have

a) \( \overline{N} := \alpha(\mathbb{Z}^k[A^{-1}]) \) is a discrete subgroup isomorphic to \( \mathbb{Z}^k[A^{-1}] \).
b) ker($\tilde{q}$) = $\tilde{N}$, whence we have the canonical identification $\tilde{S}/\tilde{N} \cong S_A$ as topological groups;

c) $\tilde{q} \circ \tilde{\sigma} = \sigma_A \circ \tilde{q}$;

d) $\tilde{\sigma}(x + \tilde{\alpha}(g)) = \tilde{\sigma}(x) + \tilde{\alpha}(\sigma_A(g))$ for $g \in \mathbb{Z}^k[A^{-1}]$.

**Proof.** 4.4(a) is clear. To see 4.4(b), let $\tilde{q}\left([((\xi, v), l)]\right) = e$. Then we have

$$\sigma_A^{-1}(\theta_v(\xi)) = e.$$ 

Applying $\sigma_A^l$ of both sides we have $\theta_v(\xi) = \sigma_A^l(e) = e$ whence $v \in \mathbb{Z}^k$. Taking $g = [(v, l)]$ we have

$$\tilde{\alpha}(g) = [([\alpha(v), l]) = [[[\theta_{-v}(e), v], l]] = [([\xi, v), l]]$$

hence $[([\xi, v), l]) = \tilde{\alpha}(g)$. Similarly one verifies that $\tilde{q}(\tilde{\alpha}(g)) = e$ for any $g \in \mathbb{Z}^k[A^{-1}]$. Hence 4.4(b) holds.

To see 4.4(c), note that for any $\tilde{\sigma} \in \tilde{S}$ and $l \in \mathbb{N}$ we have

$$\tilde{q} \circ \tilde{\sigma}([([\tilde{\sigma}, l)]]) = \sigma_A^{-l}(\tilde{\sigma}(\tilde{\sigma}(\tilde{\sigma})) = \sigma_A^{-l}(\tilde{\sigma}(\tilde{\sigma})) = \sigma_A(\tilde{\sigma}(\tilde{\sigma})) = \sigma_A(\tilde{q}(([\tilde{\sigma}, l])]).$$

Finally for $g = [(n, m)]$ we see

$$\tilde{\sigma}(\tilde{\alpha}(g)) = [([\sigma(n), m]) = [([\alpha(A(n)), m]) = \tilde{\alpha}(\sigma_A(g))$$

establishing 4.4(d). \[\square\]

We thus have that $\tilde{q}: \tilde{S} \to S_A$ is a covering map and that $\tilde{\sigma}$ lifts $\sigma_A$.

### 4.3. Metrization of $S_A$.

We conclude this section with the construction of a canonical metric on $S_A$ with respect to which $S_A$ behaves (metrically) like a hyperbolic set for $\sigma_A$ when $A$ is hyperbolic.

Firstly, let $\rho$ denote the standard metric on $\mathbb{R}^k$. Given a curve $\gamma: [0, 1] \to S_A$, there is a unique curve $\gamma^\prime: [0, 1] \to \mathbb{R}^n$ with $\gamma^\prime(0) = 0$ such that $\gamma(t) = \theta_{\gamma^\prime(t)}(\gamma(0))$. If $x, y \in S_A$ lie in the same path component, define $\Gamma(x, y)$ to be the set of all curves $\gamma: [0, 1] \to S_A$ with $\gamma(0) = x$ and $\gamma(1) = y$. Then define

$$\rho(x, y) := \inf_{\gamma \in \Gamma(x, y)} \rho(\gamma^\prime(0), \gamma^\prime(1)).$$

Secondly, for any $x, y \in S_A$ we define

$$\mathcal{J}(x, y) := \{ j \in \mathbb{Z} \mid p_0(\sigma_A^j(x)) \neq p_0(\sigma_A^j(y))\}$$

and

$$d_\Sigma(x, y) := \sum_{j \in \mathcal{J}(x, y)} 2^j.$$ 

Note that for any $x, y \in S_A$ and $v \in \mathbb{R}^k$ we have $d_\Sigma(\theta_v(x), \theta_v(y)) = d_\Sigma(x, y)$ and $d_\Sigma(\sigma_A(x), \sigma_A(y)) = \frac{1}{l} d_\Sigma(x, y)$.

Given $x, y \in S_A$, let $\Xi(x, y)$ be the set of all sequences $\xi = (x_0, y_0, x_1, y_1, \ldots, x_k, y_k)$ such that there is a curve $\gamma_j \subset S_A$ with endpoints $x_j$ and $y_j$ for $0 \leq j \leq l$. Then define

$$l(\xi) := \sum_{0 \leq j \leq l} \rho(x_j, y_j) + \sum_{0 \leq j \leq l-1} d_\Sigma(y_j, x_{j+1})$$

and

$$d(x, y) := \inf_{\xi \in \Xi(x, y)} \{l(\xi)\}.$$
Now, denoting
\[
\tilde{\Sigma} := \lim_{\rightarrow} \{ \Sigma \overset{\sigma}{\rightarrow} \Sigma \overset{\sigma}{\rightarrow} \Sigma \overset{\sigma}{\rightarrow} \ldots \} = \{(x_0, x_1, \ldots) \in S_A \mid A^j(x_0) = [0] \text{ for some } j \in \mathbb{N}\}
\]
we have \(\tilde{S} \cong \tilde{\Sigma} \times \mathbb{R}^k\). Hence if \(A \in \text{Mat}(k, \mathbb{Z})\) is a hyperbolic matrix, \(E^+\) and \(E^-\) denote the expanding and contracting subspaces, then \(\tilde{S} \cong \tilde{\Sigma} \times E^+ \times E^-\). Furthermore, if \(\tilde{d}\) is the lift of the metric \(d\) to \(\tilde{S}\) then \(\tilde{d}\) is equivalent to the product metric \(d_{\tilde{S}}|_{\tilde{\Sigma}} \times \rho|_{E^+} \times \rho|_{E^-}\), hence with respect to \(\tilde{d}\)
\[
\tilde{\sigma} : E^+ \to E^+
\]
is expanding and
\[
\tilde{\sigma} : \tilde{\Sigma} \times E^- \to \tilde{\Sigma} \times E^-
\]
is contracting.

5. Proof of Theorem 1.1

Since we are only concerned with the topology of \(\Lambda\) and the dynamics \(f|_\Lambda\) in Theorem 1.1, by Claim 3.2 we assume without loss of generality that \(f : M \to M\) is a diffeomorphism. To prove Theorem 1.1 we first present some preliminary observations and constructions that will enable us to build the essential dichotomy.

5.1. Preliminaries. Let \(\Lambda\) be a compact, topologically mixing, hyperbolic attractor for a diffeomorphism \(f : M \to M\) such that \(\dim E^u|_\Lambda = 1\). Let \(B\) denote the basin of \(\Lambda\), \(\tilde{B}\) the universal cover of \(B\), \(\pi : \tilde{B} \to B\) the covering projection, and \(\tilde{\Lambda} := \pi^{-1}(\Lambda)\). Let \(G := \pi_1(B)\) denote the fundamental group of \(B\), which we identify with the group of deck transformations for the covers \(\pi : \tilde{B} \to B\) and \(\pi|_{\tilde{\Lambda}} : \tilde{\Lambda} \to \Lambda\). Given subsets \(H \subset G\) and \(X \subset B\) denote by \(O_H(X) := \bigcup_{g \in H} g(X)\) the orbit of \(X\) under \(H\).

Let \(g\) be a Riemannian metric on \(M\). Note that for any lift \(\tilde{f}\) of \(f\), \(\tilde{\Lambda}\) is a hyperbolic set under the pull-back metric \(\pi^\ast(g)\). For \(x \in \Lambda\) and \(\sigma \in \{s, u\}\) denote by \(W^\sigma(x)\) the \(\sigma\)-manifold of \(x\) under the dynamics \(\tilde{f}\) in the metric \(\pi^\ast(g)\). Note also that \(W^\sigma(x)\) is the connected component of \(\pi^{-1}(W^\sigma(\pi(x)))\) containing \(x\). In addition note that for \(g \in G\) different from the identity, \(W^\sigma(g(x)) \cap W^\sigma(x) = \emptyset\).

Given a subset \(X \subset \tilde{\Lambda}\) we will write \(W^\sigma(X) := \bigcup_{x \in X} W^\sigma(x)\).

**Definition 5.1.** Let \(x \in \tilde{B}\). Define \(d^u_x\) to be the distance function on \(W^u(x)\) induced by restricting the metric \(\pi^\ast(g)\) to \(W^u(x)\). For simplicity we shall suppress the dependence on \(x\) and simply write \(d^u(x, y) := d^u_x(x, y)\) whenever \(y \in W^u(x)\).

Note that \(\tilde{B}\) admits a codimension-1 foliation \(W^s\) by the stable manifolds of \(\tilde{\Lambda}\). Since \(\pi_1(\tilde{B}) = \{1\}\), the foliation \(W^s\) is transversely orientable. (Indeed, one can always make \(W^s\) and \(\tilde{B}\) orientable by passing to double covers.) Fix a transverse orientation for \(W^s\). Note that neither \(G\) nor any lift of \(f\) is assumed to preserve this transverse orientation.

Given a compact, oriented, \(C^1\) curve \(\gamma \subset \tilde{B}\) that is everywhere transverse to the foliation \(W^s\), we define a signed length \(I^n(\gamma)\). Let \(\{\tilde{V}_i\}\) be a cover of \(\Lambda\) by local \(s\)-product neighborhoods (see Definition 2.1) and let \(\{\tilde{V}_{ij}\} = \pi^{-1}(\{V_i\})\) where for each \(j\), the set \(\tilde{V}_{ij}\) is homeomorphic to \(V_i\). Let \(\tilde{f}\) be a lift of \(f\). Then we may find some \(n > 0\) so that \(\gamma \subset \tilde{f}^{-n}(\bigcup_{ij} \tilde{V}_{ij})\).
Definition 5.2. Given $\gamma$ and $n$ as above, we define the signed unstable length $l^u(\gamma)$ as follows. We first define $l^u$ on a connected component $\gamma'$ of $\left(\gamma \cap \tilde{f}^{-n}(V_0)\right)$. Let

$$C = \pi \left(\tilde{f}^n(\gamma')\right)$$

and define

$$l^u(\gamma') = \text{sgn}(\gamma) e^{-nh} \mu^u \left(\{y \in W^u_{V_i}(z) \mid W^s_{V_i}(y) \cap C \neq \emptyset\}\right)$$

where $z$ is any point in $V_i \cap \Lambda$, $\mu^u$ is as in Theorem 2.5,

$$\text{sgn}(\gamma) = \begin{cases} 1, & \text{if } \gamma \text{ is positively orientated with respect to } W^s, \\ -1, & \text{if } \gamma \text{ is negatively orientated with respect to } W^s, \end{cases}$$

and $h$ is the topological entropy of $f|\Lambda$. We may then extend $l^u$ to all of $\gamma$ additively.

Theorem 2.5 shows that $l^u$ is well defined, independent of all choices above. Furthermore, given any piecewise-smooth oriented curve $\gamma \subset \tilde{B}$ we may partition $\gamma$ into a family of curves $\{\gamma_i\}$, each of which is everywhere tangent to $W^s$ or everywhere transverse to $W^s$; in the former case we define $l^u(\gamma_i) = 0$ while in the latter we use Definition 5.2. Thus we may extend the definition of $l^u$ to all piecewise-smooth oriented curve in $\tilde{B}$. Note that by Corollary 2.6, $l^u(\gamma)$ is non-zero on any curve $\gamma$ transverse to $W^s$ and $l^u(\{x\}) = 0$.

Now piecewise-smooth curves generate the group of piecewise-smooth simplicial 1-chains. Thus we may extend the function $l^u$ to a piecewise-smooth simplicial 1-cochain denoted by $\alpha^u$.

Claim 5.3. The cochain $\alpha^u$ is closed, hence exact.

Proof. A 1-cochain is closed if it is locally independent of path. This is clear by Theorem 2.5(b). \qed

Claim 5.3 has the following two corollaries.

Corollary 5.4. Given $x, y \in \tilde{B}$ and two oriented piecewise-smooth curves $\gamma_1, \gamma_2$ with end points $x$ and $y$ then $|l^u(\gamma_1)| = |l^u(\gamma_2)|$.

Proof. Changing orientation if necessary we may assume that the concatenation $\gamma_1 \cdot \gamma_2$ is a closed 1-chain. But then we have

$$0 = \alpha^u(\gamma_1 \cdot \gamma_2) = l^u(\gamma_1) + l^u(\gamma_2)$$

hence $l^u(\gamma_2) = -l^u(\gamma_1)$. \qed

Corollary 5.5. For each pair $x, y \in \tilde{\Lambda}$, the intersection $W^s(x) \cap W^u(y)$ contains at most one point.

Proof. If not we could find a piecewise smooth 1-cycle $\gamma$ with $|\alpha^u(\gamma)| > 0$, a contradiction since $\alpha^u$ is exact. \qed

The above corollaries motivate the following definitions.

Definition 5.6. We say a subset $V \subset \tilde{\Lambda}$ is a product chart if $x, y \in V$ implies $W^u(x) \cap W^s(y)$ is non-empty and $W^u(x) \cap W^s(y) \subset V$.

Definition 5.7. For $x \in \tilde{\Lambda}$ and $x' \in W^u(x)$ let $l^u(x, x') := l^u(\gamma_{xx'})$ where $\gamma_{xx'}$ is the unique oriented curve in $W^u(x)$ from $x$ to $x'$. For $x \in \tilde{\Lambda}$ and $L \in \mathbb{R}$, let $x +_u L$ denote the unique point $x' \in W^u(x)$ with $l^u(x, x') = L$. 
Definition 5.8. Given \( x, y \in \tilde{B} \) we define the pseudometric \( d^u(x, y) := |l^u(\gamma)| \) for any piecewise smooth curve \( \gamma \) with endpoints \( x \) and \( y \). Furthermore we define a metric on leaves of the foliation \( W^s \) by \( d^s(W^s(x), W^s(y)) := d^u(x, y) \).

Note that Corollary 5.4 guarantees \( d^u \) is well defined, and Corollaries 2.6 and 2.7 guarantees that the restriction of \( d^u \) to \( W^u(x) \) defines a complete metric consistent with the topology on \( W^u(x) \).

Definition 5.9. For \( x, y \in \tilde{\Lambda} \) we define \( \Xi(x, y) \) to be the set of sequences \( \xi = (x = x_0, y_0, \ldots, x_k, y_k = y) \) where \( y_j \in W^u(x_j) \) for \( 0 \leq j \leq k \) and \( x_{j+1} \in W^s(y_j) \) for \( 0 \leq j \leq k - 1 \). Then define \( d(x, y) := \inf_{\xi \in \Xi(x, y)} \left\{ \sum_{j=0}^{k} d^u(x_j, y_j) + \sum_{j=0}^{k-1} d^s(x_{j+1}, y_j) \right\} \) where \( d^s \) is the distance induced by the Riemannian metric in Definition 5.1 and \( d^u \) is the pseudometric constructed from the measure \( \mu^u \) in Definition 5.8. Clearly \( d \) defines a metric on \( \tilde{\Lambda} \) consistent with the ambient topology.

5.1.1. Global product relation. We now define a binary relation on points in \( \tilde{\Lambda} \).

Definition 5.10 (Global Product Relation). For \( x, y \in \tilde{\Lambda} \) we say \( x \sim y \) if \( y \in W^s(x) \) and

\[
W^u(y) \cap W^s(x') \neq \emptyset
\]

for all \( x' \in W^u(x) \).

Claim 5.11. \( \sim \) is an equivalence relation.

Proof. Clearly \( \sim \) is reflexive. To see that \( \sim \) is symmetric suppose \( x \sim y \), and that there exists some \( y' \in W^u(y) \) such that \( W^s(y') \cap W^u(x) = \emptyset \). Set \( L = l^u(y, y') \) and \( x' = x + u L \). Then

\[
l^u(y, W^u(y) \cap W^s(x')) = L
\]

hence \( y' = W^u(y) \cap W^s(x') \) contradicting the assumptions on \( y' \). Thus \( \sim \) is symmetric. A similar argument shows that \( \sim \) is transitive. \( \Box \)

We let \([x]\) denote the equivalence class of \( x \) under the relation \( \sim \).

Remark. The equivalence class \([x]\) represents the maximal subset of \( \tilde{\Lambda} \cap W^s(x) \) with global product structure, that is, admitting the canonical homeomorphism

\[
[x] \times W^u(x) \cong W^u([x])
\]

given by

\[
(y, x') \mapsto W^u(y) \cap W^s(x').
\]

Furthermore we have that \( W^u \) saturation of \([x]\) is \( \sim \)-saturated whence we have the equality

\[
W^u([x]) = [W^u(x)]
\]

and \( W^u([x]) \), with the quotient topology, is homeomorphic to \( \mathbb{R} \).

We enumerate a number of properties of the equivalence classes of \( \sim \).

Claim 5.12.
a) The equivalence classes are preserved under u-holonomy; in particular, the \( \mathbb{R} \)-action \( x \mapsto x + u_L \) on \( \tilde{\Lambda} \) descends to a well defined \( \mathbb{R} \) action \( [x] \mapsto [x] + u_L \).

b) Equivalence classes are preserved by the covering action of \( G \) and by any lift \( \tilde{f} \) of \( f \).

c) The equivalence classes of \( \sim \) are closed, both as subsets of the stable manifolds and hence as subsets of \( \tilde{\Lambda} \).

d) Let \( C^s(x) \) denote the connected component of \( \tilde{\Lambda} \cap W^s(x) \) containing \( x \). Then \( C^s(x) \subseteq [x] \).

e) Let \( y \in W^s(x) \) be such that \( C^s(y) \) contains points arbitrarily close to \( C^s(x) \). Then \( C^s(y) \subseteq [x] \).

**Proof.** 5.12(a) and 5.12(b) are trivial.

To see 5.12(c) let \( x_j \to x \) in \( W^s(x) \) where \( x_j \sim x_k \) for all \( j, k \in \mathbb{N} \). Suppose there is some \( x' \in W^s(x) \) so that \( W^s(x') \cap W^u(x_j) = \emptyset \) for some (hence all) \( j \). Let \( C \subseteq W^u(x) \) be a compact connected set containing \( x \) and \( x' \). Then there is some rectangle \( V \subset \Lambda \) containing \( \pi(C) \) by Claim 2.3. But then there is a product chart \( \tilde{V} \subset \pi^{-1}(V) \) containing \( C \) and \( x_j \) for a sufficiently large \( j \) contradicting the assumption that \( W^s(x') \cap W^u(x_j) = \emptyset \) for all \( j \). Hence 5.12(c) holds.

Fix an \( L > 0 \). Clearly the set

\[
V = \{ y \in W^s(x) \mid W^s(y + u_r) \cap W^u(x) \neq \emptyset \quad \text{for all } |r| \leq L \}
\]

is open in \( W^s(x) \). By a similar argument as above we see that \( V \) is closed hence \( C^s(x) \subseteq V \). Since \( L \) was arbitrary 5.12(d) follows.

For 5.12(e), let \( y \) satisfy the hypotheses and suppose \( y' \in W^u(y) \) is such that \( W^s(y') \cap W^u(x) = \emptyset \) and let \( L = l^u(y, y') \). Since \( \Lambda \) is compact we may find some \( \delta > 0 \) so that for every \( z \in \Lambda \) there is some rectangle \( V(z, L) \) containing both the sets \( W^s_{\tilde{z}}(z) \cap \Lambda \) and \( \pi(\{ \tilde{z} + u_r \mid r \leq |L| \}) \) where \( \tilde{z} \) is some lift of \( z \) to \( \tilde{\Lambda} \). By assumption we may find a \( w \in C^s(y) \) and \( x' \in C^s(x) \) so that \( d^w(w, x') < \delta \); setting \( w' = w + u_L \) we may find a product chart containing \( w, w' \), and \( x', \) hence \( W^s(w') \cap W^u(x') \neq \emptyset \). By 5.12(a) \( w' \in [y'] \) and by 5.12(d) \( x' \in [x] \) whence

\[
W^s(y') \cap W^u(x) = W^s(w') \cap W^u(x) = W^s(x' + u_L) \cap W^u(x) \neq \emptyset
\]
a contradiction. \( \square \)

We now define a metric on the quotient \( \tilde{\Lambda}/\sim \) and study its induced topology.

5.1.2. **Metrization of \( \tilde{\Lambda}/\sim \).** Denote by \( \tilde{\Omega} \) the set of equivalence classes of \( \tilde{\Lambda} \) under the equivalence relation \( \sim \) and by \( \Omega^s([x]) \) the set of equivalence classes of \( \Lambda \cap W^s(x) \) under \( \sim \). We introduce metric topologies on \( \tilde{\Omega} \) and \( \Omega^s \). Note that the pseudometric \( d^w \) on \( \Lambda \) descends to a pseudometric on \( \tilde{\Omega} \); that is, given two points \( [x], [y] \in \tilde{\Omega} \)

\[
d^w([x], [y]) := d^w(x, y)
\]
is well defined. We define a metric on each \( \Omega^s([x]) \) as follows.

**Definition 5.13.** Given \( [x] \in \tilde{\Omega} \) and \( [y] \in \Omega^s([x]) \) let

\[
r^s([x], [y]) := \sup \{ r > 0 \mid W^u(y) \cap W^s(x \pm_r u_r') \neq \emptyset \quad \forall 0 < r' < r \}
\]
and
\[
d_\Omega^*([x], [y]) = \begin{cases} 
\frac{1}{r^*([x], [y])}, & r^*([x], [y]) \neq \infty, \\
0, & r^*([x], [y]) = \infty.
\end{cases}
\]

Note that \(r^*([x], [y]) = r^*([y], [x]), r^*([x], [y]) > 0\), and that \(r^*([x], [y]) \neq \infty\) unless \([x] = [y]\). Furthermore,

**Lemma 5.14.** \(d_\Omega^*([x], [y])\) is a metric on \(\Omega^*([x])\).

**Proof.** We clearly have \(d_\Omega^*([x], [y]) = 0\) if and only if \([x] = [y]\) and \(d_\Omega^*([x], [y]) = d_\Omega^*([y], [x])\). Thus we need only prove the triangle inequality
\[
d_\Omega^*([x], [y]) \leq d_\Omega^*([x], [z]) + d_\Omega^*([z], [y]).
\]
By definition of \(r^*\) we have for \([y], [z] \in \Omega^*([x])\)
\[
r^*([x], [y]) \geq \min\{r^*([x], [z]), r^*([z], [y])\}.
\]
Thus
\[
d_\Omega^*([x], [y]) \leq \max\{d_\Omega^*([x], [z]), d_\Omega^*([z], [y])\}
\]
and (5) holds.

**Definition 5.15.** Given two points \([x], [y] \in \Omega\) let \(\Xi([x], [y])\) be the space of all sequences \(([x_0], [y_0], [x_1], [y_1], \ldots, [x_k], [y_k])\) in \(\Omega\) with
\[
1) \quad [x_0] = [x], \ \text{and} \ \ [y_k] = [y] \\
2) \quad y_j \in W^u(x_j) \text{ for } 0 \leq j \leq k \\
3) \quad x_j \in W^s(y_{j-1}) \text{ for } 1 \leq j \leq k.
\]
Given a \(\xi = ([x_0], [y_0], \ldots, [y_k]) \in \Xi([x], [y])\) define
\[
\ell(\xi) := \sum_{j=0}^{k} d_\Omega^u(x_j, y_j) + \sum_{j=1}^{k} d_\Omega^s([x_j], [y_{j-1}])
\]
and define \(d_\Omega([x], [y])\) by
\[
d_\Omega([x], [y]) := \inf \{\ell(\xi) \mid \xi \in \Xi([x], [y])\}.
\]
Clearly \(d_\Omega\) defines a metric on \(\Omega\).

**Corollary 5.16.** The group \(G = \pi_1(B)\) acts via isometries on \((\Omega, d_\Omega)\). Furthermore the dynamics on \(\Omega\) induced by the dynamics \(\hat{f}: \Lambda \to \Lambda\), which we also denote by \(\hat{f}: \Omega \to \Omega\), acts conformally: \(d_\Omega(f([x]), [y]) = e^{\hat{h}}d_\Omega([x], [y])\) for \([y] \in W^u([x])\) and \(d_\Omega(f([x]), [y]) = e^{-\hat{h}}d_\Omega([x], [y])\) for \([y] \in \Omega^s([x])\).

**Proof.** The pseudometric \(d^u\) is preserved under \(G\). Since \(d_\Omega^u\) is defined via \(d^u\), it is also preserved. Furthermore, we have that \(d^u\) transforms according to Theorem 2.5.

Note that since \(G\) acts via invertible isometries on \((\Omega, d_\Omega)\), it acts via homeomorphisms on \((\Omega, d_\Omega)\) despite the fact that the metric topology may not coincide with the quotient topology.

**Lemma 5.17.** For \(\Omega\) and \(\Omega^*([x])\) we have
\[
\text{a) the topology on } \Omega^*([x]) \text{ induced by the metric } d_\Omega^* \text{ is weaker than the quotient topology on } \Omega^*([x]) \text{ inherited as the quotient } \Omega^*([x]) = (\Lambda \cap W^s([x]))/\sim;
\]
b) the topology on $\tilde{\Omega}$ induced by the metric $d_{\tilde{\Omega}}$ is weaker than the quotient topology on $\tilde{\Omega}$ inherited as the quotient $\tilde{\Omega} = \tilde{\Lambda}/\sim$.

Furthermore for $\tilde{\Omega}$ and $\tilde{\Omega}^*([x])$ endowed with their metric topologies

- c) the quotient map $\tilde{\Lambda} \to \tilde{\Omega}$ is continuous;
- d) $\tilde{\Omega}$ and $\tilde{\Omega}^*([x])$ are Hausdorff;
- e) either $\tilde{\Omega}^*([x])$ is perfect for all $[x] \in \tilde{\Omega}$ or is a singleton for all $[x] \in \tilde{\Omega}$.

Proof. To see 5.17(a), fix a $t > 0$ and let $U := B_{d_{\tilde{\Omega}}}(\{x\}, t)$. Then

$$U = \left\{ y \in W^s(x) \cap \tilde{\Lambda} \mid W^u(y) \cap W^s(x \pm_u r') \neq \emptyset \quad \forall \ 0 < r' < \frac{1}{t} \right\}.$$

Clearly $U$ is open as a subset of $W^s(x) \cap \tilde{\Lambda}$ since for any $y \in U$ we may find an open product chart containing $y$ and $y \pm \frac{1}{t}$. 5.17(b) then follows from 5.17(a), and 5.17(c) follows from 5.17(b). 5.17(d) follows since the topologies are metric.

To see 5.17(e) assume $\tilde{\Omega}^*([x])$ is not perfect, hence contains an isolated point $[z]$. Then for all $z' \in W^u(z)$ we have $[z']$ is isolated in $\tilde{\Omega}^*([z'])$. Periodic points are dense in $\Lambda$, hence we may find some periodic $q \in \Lambda$ so that $[\tilde{q}] \in W^u([z])$ for some lift $\tilde{q}$ of $q$ in $\Lambda$. Furthermore, since $W^s(q) \cap \Lambda$ is dense in $\Lambda$ we may assume that if there is any $[z'] \in W^u([z])$ so that $\tilde{\Omega}^*([z']) \neq [z']$ then $\tilde{\Omega}^*([\tilde{q}]) \neq [\tilde{q}]$. Passing to an iterate of $f$ and choosing an appropriate lift $f: \tilde{\Lambda} \to \tilde{\Lambda}$ of $f$ we may assume that $f([\tilde{q}]) = [q]$. Since $f$ is conformally contracting on $\tilde{\Omega}^*([\tilde{q}])$, the assumption $\tilde{\Omega}^*([\tilde{q}]) \neq [\tilde{q}]$ contradicts that $[\tilde{q}]$ is isolated in $\tilde{\Omega}^*([\tilde{q}])$. Thus we conclude for every $[z'] \in W^u([z])$ that $\tilde{\Omega}^*([z']) = [z']$. Furthermore, we must have $\tilde{\Omega} = \bigcup \bigcup \tilde{\Omega}^*([x])$, hence we have $\tilde{\Omega}^*([x])$ is a singleton for every $[x] \in \tilde{\Omega}$. □

The proof of Theorem 1.1 will follow from considering two cases. In the first case we will assume that $G$ acts properly discontinuously on $\tilde{\Omega}$ and deduce that $\tilde{\Lambda}$ is expanding. We will then show that if $G$ fails to act properly discontinuously on $\tilde{\Omega}$ then $\tilde{\Lambda}$ has a product structure which will be used to obtain a conjugacy between $f|\Lambda$ and an automorphism of a toral solenoid.

5.2. Case 1: $G$ acts properly discontinuously on $\tilde{\Omega}$. The goal of this section is to prove the following proposition.

**Proposition 5.18.** Suppose $G$ acts properly discontinuously on $\tilde{\Omega}$. Then $\Lambda$ is expanding.

For $\tilde{x} \in \tilde{\Lambda}$ denote by $C^*(\tilde{x})$ the connected component of $\tilde{\Lambda} \cap W^s(\tilde{x})$ containing $\tilde{x}$. We define $r: \Lambda \to \mathbb{R} \cup \{\infty\}$ by

$$r: x \mapsto \sup_{\tilde{y} \in C^*(\tilde{x})}\{d^s(\tilde{x}, \tilde{y})\}$$

where $\tilde{x}$ is any lift of $x$ to $\tilde{\Lambda}$.

Given a metric space $(X, \rho)$ and a subset $Y \subset X$ we call

$$\text{diam}(Y) := \sup\{\rho(x, y) \mid x, y \in Y\}$$

the diameter of $Y$. For any $x \in X$ and $Y \subset X$ we write

$$\rho(x, Y) := \inf\{\rho(x, y) \mid y \in Y\}.$$
Definition 5.19. Let \( \{A_n\} \) be a countable sequence of subsets in a metric space \((X, \rho)\). We define the Kuratowski limit supremum by

\[
\lim_{n \to \infty} A_n := \{x \in X \mid \liminf_{n \to \infty} \rho(x, A_n) = 0\}.
\]

Clearly the Kuratowski limit supremum is a closed set for any collection \( \{A_n\} \).

Lemma 5.20. Let \((X, \rho)\) be a proper metric space; that is, one for which any closed ball \( \{y \in X \mid d(x, y) \leq R\} \) is compact. Let \( \{A_n\} \subset X \) be a countable sequence of subsets such that

1. \( \lim_{n \to \infty} \operatorname{diam}(A_n) \) is finite;
2. each \( A_n \) is connected;
3. there exists a Cauchy sequence \( \{x_n\} \) with \( x_n \in A_n \).

Then \( \lim_{n \to \infty} A_n \) is connected.

Note that the result need not hold if assumption 3 is omitted.

Proof. By assumption 3, we may fix \( x := \lim_{n \to \infty} x_n \in \lim_{n \to \infty} A_n \). By assumption 1 we may find an \( L \) and \( N \) so that for all \( n \geq N \), the inclusion \( A_n \subset B(x, L) \) holds. Suppose \( \lim_{n \to \infty} A_n \) is disconnected. Let \( N_1, N_2 \subset B(x, L) \) be two disjoint open sets such that \( x \in N_1 \), \( (\lim_{n \to \infty} A_n) \subset N_1 \cup N_2 \) and \( (\lim_{n \to \infty} A_n) \cap N_2 \neq \emptyset \).

By assumption 3 we may find an \( M \) such that for all \( n \geq M \), we have \( A_n \cap N_1 \neq \emptyset \). Furthermore, we may find an infinite subsequence \( \{n_j\} \) such that \( A_{n_j} \cap N_2 \neq \emptyset \). Since \( A_{n_j} \) is connected we have \( A_{n_j} \cap \partial(N_1) \neq \emptyset \). For each \( j \in \mathbb{N} \) pick some \( a_j \in A_{n_j} \cap \partial(N_1) \). Then since \( \partial(N_1) \) is compact we may find some \( y \in \partial(N_1) \) that is an accumulation point of \( \{a_j\} \). But this implies that \( y \in \lim_{n \to \infty} A_n \) whence \( \lim_{n \to \infty} A_n \cap \partial(N_1) \neq \emptyset \), a contradiction. \( \square \)

Lemma 5.21. The function \( r: \Lambda \to \mathbb{R}^+ \cup \{\infty\} \) is upper semicontinuous.

Proof. We prove the lemma for the pull-back of the function \( r: \bar{\Lambda} \to \mathbb{R}^+ \cup \{\infty\} \). Clearly the lemma holds at \( x \in \bar{\Lambda} \) if \( r(x) = \infty \). We assume otherwise.

The function \( r \) is clearly continuous along unstable leaves. Consequently, we need only show that for \( \bar{x}_i \in W^s_\ast(x) \), if \( \bar{x}_i \to x \) then \( r(x) \geq \lim_{i \to \infty} r(\bar{x}_i) \). Passing to a subsequence \( \{x_n\} \subset \{\bar{x}_i\} \) we may assume that

\[
\lim_{n \to \infty} r(x_n) = \lim_{i \to \infty} r(\bar{x}_i).
\]

If \( r(x) < \infty \) but the lemma failed at \( x \), we could find \( \epsilon > 0 \) and \( K \) so that for all \( n > K \) we have \( r(x_n) > r(x) + \epsilon \) and \( d^\ast(x, x_n) < \epsilon/3 \). Let \( \bar{C}^\ast(x_n) \) denote the connected component of \( C^\ast(x_n) \cap B_{d^\ast}(x, r(x) + \epsilon/3) \) containing \( x_n \). (Here \( B_{d^\ast}(x, R) \) denotes the \( d^\ast \)-ball in \( W^\ast(x) \) of radius \( R \).)

Let \( \Xi = \lim_{n \to \infty} \bar{C}^\ast(x_n) \). By Lemma 5.20, \( \Xi \) is connected and hence we have \( \Xi \subset C^\ast(x) \). On the other hand, the assumption on \( r(x_n) \) ensures that \( \bar{C}^\ast(x_n) \cap \partial(B_{d^\ast}(x, r(x) + \epsilon/3)) \neq \emptyset \) for all \( n \geq K \). Hence \( \Xi \cap \partial(B_{d^\ast}(x, r(x) + \epsilon/3)) \neq \emptyset \), contradicting the definition of \( r(x) \). \( \square \)

Corollary 5.22. Either \( r = 0 \) or \( r \equiv \infty \).

Proof. Suppose first that the range of \( r \) does not contain \( \infty \). Then by upper semicontinuity, \( r \) is globally bounded. Let \( M = \max\{r(x) \mid x \in \Lambda\} \). By hyperbolicity of \( f \) on \( \Lambda \) and boundedness of \( r \) we find an \( m \in \mathbb{N} \) so that

\[
f^m(\pi(C^\ast(\bar{x}))) \subset W^s_\ast(f^m(x))
\]
(where \(\tilde{x}\) is a lift of \(x\), hence \(r(f^{m+1}(x)) \leq \lambda r(f^m(x))\) for all \(x \in \Lambda\). On the other hand, since \(f\) is a homeomorphism, we should have
\[
\max\{r(f^m(x)) \mid x \in \Lambda\} = \max\{r(f^{m+1}(x)) \mid x \in \Lambda\}.
\]
But then \(M = \lambda M\) which implies \(M = 0\).

Now if \(r(x) \neq \infty\) then \(r(y) \neq \infty\) for all \(y \in W^u(x)\). Indeed, let \(\tilde{x}\) be a lift of \(x\), \(\tilde{y}\) the lift of \(y\) contained in \(W^u(\tilde{x})\), and \(L = \lambda^u(\tilde{x}, \tilde{y})\). Let \(U\) be a cover of \(C^u(\tilde{y})\). Then for every \(z \in C^u(\tilde{y})\) there is an \(\epsilon(z) > 0\) so that \(W^u_{\epsilon(z)}(z) \subset U\) for some \(U \in U\) and the set
\[
\{\tilde{z} + u \mid \tilde{z} \in W^u_{\epsilon(z)}(z), |z| \leq |L|\}
\]
is a product chart. But then \(\{V(z)\}\) covers \(C^u(\tilde{x})\), whence we conclude that \(U\) admits a finite subcover.

Thus if \(r(x) = \infty\) for some \(x \in \Lambda\), then \(r(y) = \infty\) for all \(y \in W^u(x)\). Since \(W^u(x)\) is dense in \(\Lambda\), the upper semicontinuity of \(r\) implies \(r \equiv \infty\).

We thus establish that \(\Lambda\) is expanding under the assumption that \(G\) acts properly discontinuously on \(\tilde{\Omega}\).

**Proof of Proposition 5.18.** Let \(\Omega = \tilde{\Omega}/G\) be the orbit space. Note that since \(G\) acts properly discontinuously, \(\Omega\) is Hausdorff. Denote the canonical projections by \(\pi: \Lambda \to \Lambda, q: \tilde{\Lambda} \to \tilde{\Omega}, \pi': \Omega \to \tilde{\Omega}\). Consider the diagram
\[
\begin{array}{ccc}
\tilde{\Lambda} & \xrightarrow{q} & \tilde{\Omega} \\
\pi \downarrow & & \downarrow \pi' \\
\Lambda & \to & \Omega
\end{array}
\]
Since the equivalence classes of \(\sim\) are \(G\)-invariant, the \(G\)-orbit of \(q(y)\) is equivalent to the \(G\)-orbit of \(q(g(y))\) for any \(g \in G\) and \(y \in \tilde{\Lambda}\). Thus we may find a map \(q'\) so that the diagram
\[
\begin{array}{ccc}
\tilde{\Lambda} & \xrightarrow{q} & \tilde{\Omega} \\
\pi \downarrow & & \downarrow \pi' \\
\Lambda & \xrightarrow{q'} & \Omega
\end{array}
\]
commutes.

Since \(\Lambda\) is compact and \(\Omega\) is Hausdorff, \(q'\) is proper, whence \(q\) is proper. Hence the equivalence classes of \(\sim\) must be compact subsets of \(\tilde{\Lambda}\). By Claim 5.12(d) and Corollary 5.22 this implies \(r \equiv 0\); hence the connected components of \(\Lambda \cap W^x(x)\) are singletons and \(\Lambda\) is expanding.

5.3. **Case 2:** \(G\) fails to act properly discontinuously on \(\tilde{\Omega}\). In the case that \(G\) fails to act properly discontinuously at some point in \(\tilde{\Omega}\), we show that \(\Lambda\) is homeomorphic to a toral solenoid and \(f|\Lambda\) is conjugate to a solenoidal automorphism.

5.3.1. **Metric properties of \(\tilde{\Omega}\).** We first enumerate some additional properties of the metric \(d_{\tilde{\Omega}}\) and the action of \(G\) on \(\tilde{\Omega}\).

**Claim 5.23.** The following hold in the metric space \((\tilde{\Omega}, d_{\tilde{\Omega}})\).

a) Let \(d_{\tilde{\Omega}}([x], [y]) < 1\). Then \(W^u(y) \cap W^s(x) \neq \emptyset\).
b) We have \([z_j] \to [x]\) in \((\Omega, d_\Omega)\) if and only if \(d^u([x],[z_j]) \to 0\) and
\[
d^u_\Omega([x],[W^u(z_j) \cap W^s(x)]) \to 0.
\]

Note we have \(W^u(z_j) \cap W^s(x) \neq \emptyset\) for sufficiently large \(j\) by 5.23(a).

c) Fix \(L \in \mathbb{R}\). If \(g_i([x]) \to [x]\) then \(g_i([x] +_u L) \to [x] +_u L\).

**Proof.** Fix \(R > 1\) so that \(d_{\Omega}([x],[y]) < \frac{1}{R}\). Let \(\xi = ([x_0],[y_0],\ldots,[y_k]) \in \Xi([x],[y])\) be as in Definition 5.15 with \(l(\xi) < \frac{1}{R}\). Then we clearly have \(d^u(x,x_j) < \frac{1}{R} < R\) for all \(0 \leq j \leq k\). Since we must also have \(d^u_\Omega([x_j],[y_{j-1}]) < \frac{1}{R}\) for \(1 \leq j \leq k\), we inductively see that \(W^u(y_j) \cap W^s(x) \neq \emptyset\), for each \(0 \leq j \leq k\) hence 5.23(a) holds.

For \([x_i]\) and \([y_i]\) as above, denote by \(H([y_i]) = H([x_i]) := [W^u(y_i) \cap W^s(x)] = \left[ W^u(x_i) \cap W^s(x) \right] \). We check that for each \(y_i\)
\[
(7) \quad d^u_\Omega([x],H([y_i])) \leq \frac{R}{R^2 - 1}.
\]
Indeed since \(d^u_\Omega([x_j],[y_{j-1}]) < \frac{1}{R}\) then
\[
R - \frac{1}{R} \leq r^s(H([x_j]),H([y_{j-1}]))
\]
from which we obtain
\[
d^u_\Omega(H([x_j]),H([y_{j-1}])) \leq \frac{R}{R^2 - 1}
\]
for all \(1 \leq j \leq k\). Furthermore, \(d^u_\Omega(H([x_j]),H([y_j])) = 0\) for all \(0 \leq j \leq k - 1\), hence applying (6) recursively one obtains (7). In particular
\[
(8) \quad d^u_\Omega([x],H([y])) \leq \frac{R}{R^2 - 1}.
\]
Hence, by setting \(y = z_j\) and letting \(R \to \infty\) in (8), we see that \(d_\Omega([z_j],[x]) \to 0\) implies \(d^u([z_j],[x]) \to 0\) and
\[
d^u_\Omega([x],H([z_j])) \to 0.
\]
Furthermore, we clearly have \(d^u([x],[z_j]) \to 0\) and \(d^u_\Omega([x],H([z_j])) \to 0\) implies that \(d_\Omega([x],[z_j]) \to 0\) hence both implications in 5.23(b) follow.

Note that \(g_i(x +_u L) = g_i(x) \pm_u L\) depending of whether \(g_i\) preserves the transverse orientation on \(W^s\). However for \(g\) such that \(l^u([x],g([x])) = t\) and
\[
d^u_\Omega([x],[W^u(g(x)) \cap W^s(x)]) < \frac{1}{|t|}
\]
g can not reverse the orientation since otherwise we would have
\[
W^s(x +_u t/2) \cap W^s(g(x +_u t/2)) = W^s(x +_u t/2) \cap W^s(g(x) -_u t/2) \neq \emptyset,
\]
a contradiction unless \(g\) is the identity. Thus we may assume that for \(g_i\) in 5.23(c), \(g_i(x +_u L) = g_i(x) +_u L\).

Now let \(L \in \mathbb{R}\) be given. By forgetting initial terms and invoking 5.23(a) and 5.23(b), we may assume that for all \(i\)
\[
H([g_i(x)]) := [W^u(g_i(x)) \cap W^s(x)]
\]
is defined, and \( d_{p_{q_1}}([x], H([g_i(x)])) < \frac{1}{|U|+1} \). Then by definition of \( r^s \) we have 
\[ W^u(g_i(x)) \cap W^s(x + u L) \neq \emptyset, \text{ hence } H([g_i(x)]) + u L = [W^u(g_i(x)) \cap W^s(x + u L)]. \]
As above we have 
\[ r^s([x], H([g_i(x)])) - L \leq r^s([x + u L], H([g_i(x)]) + u L) \]
hence 
\[ d_{p_{q_1}}([x + u L], H([g_i(x)]) + u L) \leq \frac{d_{p_{q_1}}([x], H([g_i(x)]))}{1 - L \cdot d_{p_{q_1}}([x], H([g_i(x)]))} \]
which by 5.23(b), establishes 5.23(c). \( \square \)

Given a subset \( S \subset G \) we say \( S \) acts properly discontinuously at \([x]\) if there is some open set \( U \ni [x] \) so that for each \( s \in S \), we have \( s(U) \cap U \neq \emptyset \) implies \( s = 1 \) for any \( s \in S \). Since \( G \) acts freely on \( \tilde{\Omega} \), every finite subset \( S \subset G \) acts properly discontinuously at every point of \( \tilde{\Omega} \).

**Lemma 5.24.** Suppose a set \( S \subset G \) acts properly discontinuously at one point \([x] \in \Omega\). Then \( S \) acts properly discontinuously at every point \([y] \in \tilde{\Omega}\).

**Proof.** Let \( U \subset \tilde{\Omega} \) be an open neighborhood of \([x]\) so that for each \( s \in S \), we have \( s(U) \cap U \neq \emptyset \) implies \( s = 1 \). Then \( S \) acts properly discontinuously at every point \([y] \in U\). By Claim 5.23(c), if \( S \) acts properly discontinuously at \([y]\) then it acts properly discontinuously at every point in \( W^u([y]) \), hence we have that \( S \) acts properly discontinuously at every point of \( W^u(U) \). But then \( S \) acts properly discontinuously at every point of \( \mathcal{O}_G(W^u(U)) = \tilde{\Omega} \). \( \square \)

Setting \( S = G \), we have the contrapositive.

**Corollary 5.25.** If \( G \) fails to act properly discontinuously at one point \([x] \in \Omega\) then it fails to act properly discontinuously at every point \([y] \in \tilde{\Omega}\).

Furthermore we have

**Corollary 5.26.** Let \( g_i([x]) \rightarrow [x] \) in \((\tilde{\Omega}, d_{\Omega})\) for some sequence \( \{g_i\} \subset G \). Then \( g_i([y]) \rightarrow [y] \) in \((\tilde{\Omega}, d_{\Omega})\) for any \([y] \in \tilde{\Omega}\).

**Proof.** Set \( S = \{g_i\} \). If \( g_i([y]) \) fails to converge to \([y]\) then there is a neighborhood \( U \) of \([y]\) and an infinite subset \( S' \subset S \) so that \( s(U) \cap U = \emptyset \) for all \( s \in S' \). But then \( S' \) acts properly discontinuously at \([y]\) which by Lemma 5.24 implies that \( S' \) acts properly discontinuously at \([x]\). But \( S' \) corresponds to an infinite subsequence of \( \{g_i\} \), contradicting that \( g_i([x]) \rightarrow [x] \). \( \square \)

We now show that the above convergence happens uniformly in \([y]\). Define a map \( \zeta : G \times \tilde{\Omega} \rightarrow [0, \infty) \) by 
\[ \zeta : (g, [x]) \rightarrow d_{\Omega}([x], g([x])). \]
Endowing \( G \) with the discrete topology, we have that \( \zeta \) is continuous. Since the metric topology on \( \tilde{\Omega} \) is weaker than the quotient topology, the quotient map \( q : \Lambda \rightarrow \tilde{\Omega} \) induces a continuous map \( q^*\zeta : G \times \Lambda \rightarrow [0, \infty) \). Now 
\[ q^*\zeta(g, x) = q^*\zeta(g, g'(x)) \]
for all \( g, g' \in G \) hence \( q^*\zeta \) induces a continuous map \( \overline{\zeta} : G \times \Lambda \rightarrow [0, \infty) \).

As a result we have,
**Lemma 5.27.** Assume \( g_i([x]) \to [x] \) for some \([x] \in \tilde{\Omega} \). Then given an \( \epsilon > 0 \), we may find an \( N \) so that for all \( i \geq N \) and \([y] \in \Omega \) we have \( d_\Omega([y], g_i([y])) < \epsilon \).

**Proof.** Assume the conclusion fails for some fixed \( \epsilon \). Passing to an infinite subsequence, we may assume the conclusion fails for all \( i \in \mathbb{N} \). Let \( \{y_i\} \subset A \) be such that \( \delta(g_i, y_i) > \epsilon \) and, again passing to a subsequence, let \( z \in A \) be a limit point of \( \{y_i\} \). Let \( \hat{z} \) be a lift of \( z \) and \( \{\hat{y}_i\} \) a lift of \( \{y_i\} \) such that \( \hat{z} \) a limit point of \( \{\hat{y}_i\} \).

Because the metric topology on \( \Omega \) is weaker than the quotient topology, we have that \( \hat{z} \) is a limit point of the sequence \( \{\hat{y}_i\} \) with respect to the metric topology. Passing to a subsequence we may assume \( d_\Omega([\hat{y}_i], [\hat{z}]) < \epsilon/4 \) for all \( i \) from which we obtain

\[
d_\Omega([\hat{z}], g_i([\hat{z}])) \geq d_\Omega([\hat{y}_i], g_i([\hat{y}_i])) - d_\Omega([\hat{z}], [\hat{y}_i]) - d_\Omega(g_i([\hat{y}_i]), g_i([\hat{z}]))
\]

hence \( d_\Omega([\hat{z}], g_i([\hat{z}])) > \epsilon/2 \) for all \( i \), contradicting Corollary 5.26. \( \square \)

### 5.3.2. Global product structure.

We now establish that when \( G \) fails to act properly discontinuously, the set \( \tilde{A} \) has a global product structure; that is, for all \( x, y \in \tilde{A} \) we have \( W^u(x) \cap W^s(y) \neq \emptyset \).

**Lemma 5.28.** Let \( G \) fail to act properly discontinuously on \( \tilde{\Omega} \). Then \( \Omega^s([x]) \) is a singleton for all \( x \) and \( \Lambda = W^u(x) \) for any \( x \in \tilde{A} \).

**Proof.** Fix some \( R > 1 \), and choose an \( R' > R \) with the property that \( \frac{R'}{(R^2 - 1)} \leq \frac{1}{R} \).

Suppose \( G \) fails to act properly discontinuously at \([x]\). Then we may find a subset \( \{g_i\} \subset G \) so that \( g_i([x]) \to [x] \) and \( d_\Omega([x], g_i([x])) < 1/R \) for all \( i \). Let \( N = B_{d_\Omega([x], 1/R)} \). As guaranteed by Lemma 5.27, we may remove initial terms of \( \{g_i\} \) so that \( d_\Omega([y], g_i([y])) \leq \frac{1}{R'} \) for all \( i \) and \([y] \in \tilde{\Omega} \).

For \([y] \in N\) define

\[
H_{g_i}([y]) := [W^u(g_i([y])) \cap W^s([x])].
\]

Then, as in (8) we have

\[
\frac{d_\Omega([y], H_{g_i}([y]))}{d_\Omega([x], g_i([x]))} \leq \frac{R'}{R} \leq \frac{1}{R}.
\]

But then by (6) we have

\[
d^*([x], H_{g_i}([y])) \leq \max\{d^*([x], [y]), d^*_\Omega([y], H_{g_i}([y]))\} \leq \frac{1}{R}
\]

hence \( H_{g_i}(N) \subset N \). Furthermore \( H_{g_i}^{-1} \) is defined on \( N \) and by the same argument as above \( H_{g_i}^{-1}(N) \subset N \). Hence \( H_{g_i}(N) = N \). In particular, setting \( L = l^u(x, g_i([x])) \) we have

\[
g_i(N) = N +_u L.
\]

Set

\[
D = \{y +_u l \mid |l| < R, [y] \in N\}.
\]

Since \( d^u([x], g_i([x])) < \frac{1}{R} < R \), we have \( g_i(N) \subset D \) and \( g_i^{-1}(N) \subset D \). Inductively, we see that for any \( k \) and \([x], [y] \in \bigcup_{|l| \leq k} g_i^l(D)\)
that $W^u(x) \cap W^s(y) \neq \emptyset$, hence $\bigcup_{|j| \leq k} g_j^t(D)$ is a product chart. In particular, we have equality between product charts $\bigcup_{j \in \mathbb{Z}} g_j^t(D) = W^u([x])$, thus showing that $[x]$ is isolated in $\Omega^u([x])$. By Lemma 5.17(e) we see that $\Omega^u([x]) = [x]$ for all $[x] \in \tilde{\Omega}$.

Considering $[x]$ as a subset of $\Lambda$ we have $\Lambda = \mathcal{O}_G(W^u([x]))$ and $\tilde{B} = \mathcal{O}_G(W^u(W^u([x])))$.

If $g \in G$ is such that $g(W^u([x])) \neq W^u([x])$ then
$$g(W^u(W^u([x]))) \cap (W^s(W^u([x]))) = \emptyset.$$ Thus we must have $\tilde{\Lambda} = W^u([x])$ since otherwise $\tilde{B}$ would not be connected. □

The following is immediate from Lemma 5.28.

**Corollary 5.29.** When $G$ fails to act properly discontinuously on $\tilde{\Omega}$ then $\tilde{\Lambda}$ admits a global product structure.

We now shift out attention back to $\tilde{\Lambda}$, under the assumption that $\tilde{\Lambda}$ admits a global product structure. Our objective is to prove the following.

**Proposition 5.30.** Assume $\tilde{\Lambda}$ has a global product structure. Then $f : \Lambda \to \Lambda$ is conjugate to a leaf-wise hyperbolic automorphism of a toral solenoid (see Section 4).

To prove Proposition 5.30 we need the following technical result. We note that the proof technique for Proposition 5.30, including Lemma 5.31, are adapted from [9].

5.3.3. **Global shadowing lemma.** Let $(\Upsilon, \tau)$ be a metrizable topological space. For a fixed $k$ let $\rho$ be the standard metric on $\mathbb{R}^k$. Furthermore let $\{d_x\}_{x \in \mathbb{R}^k}$ be a family of complete metrics on $\Upsilon$ (each inducing the topology $\tau$) such that

1. $d_x$-balls in $\Upsilon$ are precompact for all $x \in \mathbb{R}^k$.
2. the induced map $\mathbb{R}^k \times \Upsilon \times \Upsilon \to \mathbb{R}$ given by
   $$(x, \xi, \eta) \mapsto d_x(\xi, \eta)$$
   is continuous.

Let $\Omega = \mathbb{R}^k \times \Upsilon$ with projections $\pi_1 : \Omega \to \mathbb{R}^k$ and $\pi_2 : \Omega \to \Upsilon$. Given $x, y \in \Omega$ let $\Xi(x, y)$ be the set of sequences $\{x_0, y_0, \ldots, x_k, y_k\}$ such that

1. $x = x_0$ and $y = y_k$;
2. $\pi_1(x_j) = \pi_1(y_j)$ for all $0 \leq j \leq k$;
3. $\pi_2(x_j) = \pi_2(y_{j-1})$ for all $1 \leq j \leq k$.

Given an $\xi \in \Xi(x, y)$ define
$$l(\xi) := \sum_{j=0}^{k} d_{\pi_1(x_j)}(\pi_2(x_j), \pi_2(y_j)) + \sum_{j=1}^{k} \rho(\pi_1(x_j), \pi_1(y_{j-1}))$$
and define
$$(9) \quad d(x, y) := \inf_{\xi \in \Xi(x, y)} \{l(\xi)\}.$$ Clearly $d$ defines a metric on $\tilde{\Omega}$; furthermore, the continuity of the function $(x, \xi, \eta) \mapsto d_x(\xi, \eta)$ guarantees that the metric topology is consistent with the product topology of $\mathbb{R}^k \times \Upsilon$. 

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Given a metric space \((X, d)\), a homeomorphism \(f : X \to X\) is called **expanding** if there is some \(\mu > 1\) so that for all \(x, y \in X\), \(d(f(x), f(y)) \geq \mu d(x, y)\). A sequence \(\{x_j\}_{j \in \mathbb{Z}} \subset (X, d)\) is called an **\(L\)-pseudo orbit** for \(f\) if \(d(f(x_j), x_{j+1}) \leq L\) for all \(j\).

Given an \(L\)-pseudo orbit \(\{x_j\}\) we say a point \(x \in X\) **shadows** \(\{x_j\}\) if there is some \(\delta\) so that \(d(f^j(x), x_j) \leq \delta\) for all \(j\).

**Lemma 5.31** (Global Shadowing). Let \(h : \Omega \to \Omega\) be a product homeomorphism \(h : (x, \xi) \mapsto (h_1(x), h_2(\xi))\). Assume that \(h_1 : \mathbb{R}^k \to \mathbb{R}^k\) is expanding with respect to the metric \(\rho\). Furthermore assume that \(h_2 : \Upsilon \to \Upsilon\) is asymptotically exponentially contracting on bounded sets with respect to each metric \(d_x\); that is, given an \(R > 0\), \(\xi \in \Upsilon\), and \(x \in \mathbb{R}^k\) there are \(c > 0\) and \(\lambda < 1\), depending continuously on \((x, \xi) \in \Omega\), so that if \(d_x(\xi, \zeta) \leq R\) then

\[
d_{h_1^j(x)}(h_2^j(\xi), h_2^j(\zeta)) \leq c\lambda^j d_x(\xi, \zeta)
\]

for all \(j \geq 0\). Additionally assume that \(\Omega\) admits a properly discontinuous action by a subgroup \(G\) of the group of isometries of \((\Omega, d)\) such that \(h\) preserves \(G\)-orbits, and the quotient \(\Omega/G\) is compact. We have the following.

a) **Given a** \(C > 0\) **there is a** \(K > 0\) **so that if** \(d(x, y) \leq C\) **for any** \(x, y \in \Omega\), **then**

\[
d_{\pi_1(y)}(\pi_2(y), \pi_2(x)) \leq K.
\]

b) **Given an** \(R > 0\) **there are** \(c > 0\) **and** \(\lambda < 1\) (**depending only on** \(R\)) **so that**

\[
d_{h_1^j(x)}(h_2^j(\xi), h_2^j(\zeta)) \leq c\lambda^j d_x(\xi, \zeta).
\]

c) **Given any** \(L\)-pseudo orbit \(\{x_j\} \subset \Omega\) **in the metric** \(d\), **there is a point** \(x \in \Omega\)

**that shadows** \(\{x_j\}\).

d) **Fix** \(C > 0\). **Then there is a sequence** \(\{\epsilon_N\}\) **with** \(\epsilon_N \to 0\) **as** \(N \to \infty\) **such that**

\[
d(h^j(x), h^j(y)) \leq C\quad \text{for}\quad |j| \leq N
\]

**implies** \(d(x, y) \leq \epsilon_N\).

**Remark 5.32.** Note that Lemma 5.31 applies to \(\tilde{\Lambda}\) in the case that \(\tilde{\Lambda}\) has global product structure by choosing an \(x \in \tilde{\Lambda}\) and taking \(\Upsilon = \tilde{\Lambda} \cap W^s(x), \ k = 1, \rho = d^u\)

**in Definition 5.8,** and \(d_x\) the metrics \(d^u_x\) in Definition 5.1. Then the metric in Definition 5.9 corresponds to (9).

Furthermore, assuming the linear map \(A : \mathbb{R}^n \to \mathbb{R}^n\) is hyperbolic, we have that the cover \(\tilde{\mathcal{S}}\) constructed in Section 4 and endowed with the metric \(\tilde{d}\) constructed in Section 4.3 satisfies the hypotheses of Lemma 5.31 with \(\Upsilon = E^- \times \Sigma\) and \(k = \dim E^u\).

**Proof of Lemma 5.31(a) and 5.31(b).** The hypotheses of Lemma 5.31 guarantee that the numbers \(K, c,\) and \(\lambda\) can be chosen pointwise on \(\Omega\). Since the group \(G\) acts via isometries, we may assume they are constant on \(G\)-orbits. Since the quotient \(\Omega/G\) is compact, we may choose uniform \(K, c,\) and \(\lambda\). □

**Proof of Lemma 5.31(c).** Because \(h_1\) is an expanding homeomorphism, there is a unique fixed point \(p \in \mathbb{R}^k\). If \(\{x_j\}\) is an \(L\)-pseudo orbit for \(h\) in the metric \(\rho\), then \(\pi_1(x_j)\) is an \(L\)-pseudo orbit for \(h_1\) in the metric \(\rho\); that is, \(\rho(\pi_1(h(x_j)), \pi_1(x_{j+1})) \leq L\). But then

\[
\rho(\pi_1(h^{-j}(x_j)), \pi_1(h^{-j-1}(x_{j+1}))) \leq \mu^{-j-1}L
\]
from which we see that the sequence
\[
\{ \pi_1(x_0), \pi_1(h^{-1}(x_1)), \ldots, \pi_1(h^{-j}(x_j)), \ldots \}
\]
is Cauchy. Set
\[
y = \lim_{j \to \infty} \{ \pi_1(x_0), \pi_1(h^{-1}(x_1)), \ldots, \pi_1(h^{-j}(x_j)), \ldots \}.
\]

Let \( g: \mathbb{R} \to \mathbb{R} \) be the map \( g: x \mapsto \mu^{-1}(x + L) \). Then \( g \) is contracting, hence has a unique fixed point. In particular the sequence \( \{ x, g(x), g^2(x), \ldots \} \) is bounded for any \( x \). Set
\[
R := \sup\{ \rho(\pi_1(x_0), p), g(\rho(\pi_1(x_0), p)), g^2(\rho(\pi_1(x_0), p)), \ldots \} < \infty.
\]
Then for every \( j \leq 0 \) we have
\[
\rho(\pi_1(x_j), p) \leq R < \infty.
\]

Taking \( C = R + L + \mu R \) we have that \( d\left( (p, \pi_2(x_j)), (p, \pi_2(h(x_{j-1}))) \right) \leq C \) for all \( j \leq 0 \), whence from 5.31(a) we may find a \( K < \infty \) so that \( d_p(\pi_2(x_j), \pi_2(h(x_{j-1}))) \leq K < \infty \) for all \( j \leq 0 \). Thus from 5.31(b) the sequence
\[
\{ \pi_2(x_0), \pi_2(h(x_{-1})), \pi_2(h^2(x_{-2})), \ldots, \pi_2(h^j(x_{-j})), \ldots \}
\]
is Cauchy in the metric \( d_p \). Let
\[
z = \lim_{j \to \infty} \{ \pi_2(x_0), \pi_2(h(x_{-1})), \pi_2(h^2(x_{-2})), \ldots, \pi_2(h^j(x_{-j})), \ldots \}.
\]
Then one easily verifies that \( x := (y, z) \) shadows the \( L \)-pseudo orbit \( \{ x_j \} \). \qed

**Proof of Lemma 5.31(d)**. For \( C \), let \( K \) be as in 5.31(a) and let \( c, \lambda \) be as in 5.31(b) with \( R = K \). Fix \( x, y \in \Omega \) so that \( d(h^j(x), h^j(y)) \leq C \) for \( |j| \leq N \). Then we have
\[
\rho(\pi_1(h^j(x)), \pi_1(h^j(y))) \leq C
\]
for \( j \leq N \), hence
\[
\rho(\pi_1(x), \pi_1(y)) \leq \mu^{-N} C.
\]
Furthermore,
\[
d_{h_1^{-N}}(\pi_1(y)) \left( h_2^{-N} (\pi_2(y)), h_2^{-N} (\pi_2(x)) \right) \leq K
\]
hence
\[
d_y(\pi_2(y), \pi_2(x)) \leq cK\lambda^N
\]
from which we conclude that
\[
d(x, y) \leq cK\lambda^N + \mu^{-N} C
\]
and the conclusion follows with \( \epsilon_N := cK\lambda^N + \mu^{-N} C \). \qed
5.3.4. Proof of Proposition 5.30. We now return to the proof of Proposition 5.30. We have the following observation.

**Lemma 5.33.** When $\tilde{\Lambda}$ has global product structure the covering group $G := \pi_1(B)$ is torsion-free abelian.

**Proof.** Fix any $x \in \tilde{\Lambda}$. Since $\tilde{\Lambda}$ has global product structure, we may canonically identify $\tilde{\Lambda}$ with $W^u(x) \times (\tilde{\Lambda} \cap W^s(x))$. Let $\sim_s$ be the equivalence relation $z \sim_s y$ if $z \in W^s(y)$. Then we have a canonical identification of $W^u(x)$ with $\Lambda/\sim_s$ which induces a $G$-action on $W^u(x)$. By Theorem 2.5 and the construction of the pseudo-Markov partition where each $\tilde{\Lambda}$, we have that $G$ acts on $(W^u(x), d^n)$ via isometries. Furthermore the isometries are orientation-preserving since otherwise there would be a non-identity $g \in G$ and $y \in W^u(x)$ with $W^s(y) = W^s(g(y))$. Hence we naturally identify $G$ with a subgroup of the orientation-preserving isometries of $\mathbb{R}$ and the result follows. □

Note that $G$ need not be finitely generated. However, we can represent $G$ as the limit of a directed system of finitely generated, torsion-free abelian groups as follows. Let $\{\mathcal{R}_j\}$ be a Markov partition of $\Lambda$ and let $\{\overline{\mathcal{R}}_{j,\alpha}\}_{\alpha \in G}$ be a lift of the Markov partition where each $\overline{\mathcal{R}}_{j,\alpha}$ is homeomorphic to $\mathcal{R}_j$. Fix an $x \in \tilde{\Lambda}$. Recall that $W^u(\pi(x))$ is dense in $\Lambda$. For each $j$, distinguish a $\overline{\mathcal{R}}_j \in \{\overline{\mathcal{R}}_{j,\alpha}\}$ so that

$$W^u(x) \cap \text{int}(\overline{\mathcal{R}}_j) \neq \emptyset.$$ (10)

Set $D = \bigcup_j \overline{\mathcal{R}}_j$. Then $D$ is a fundamental domain for the covering $\tilde{\Lambda} \rightarrow \Lambda$.

Let $H \subset G$ be the subgroup generated by $\{\alpha \in G \mid \alpha(D) \cap D \neq \emptyset\}$. Since $D$ is compact and $G$ acts discontinuously, $H$ is finitely generated. Let $N := \mathcal{O}_H(D)$. Note $N$ is clopen in $\Lambda$ and $H = \{\alpha \in G \mid \alpha(N) = N\}$.

**Claim 5.34.** There is a lift $\tilde{f}$ of $f$ so that $\tilde{f}(N) \subset N$. Indeed $f_* H \subset H$, where $f_* : G \rightarrow G$ is the automorphism induced by the diffeomorphism $f: B \rightarrow B$.

**Proof.** Since $D$ is a lift of a Markov partition, by the definition of $H$ if $y \in N$ then $W^u(y) \subset N$. Choose a lift $\tilde{f}$ of $f : \Lambda \rightarrow \Lambda$ so that

$$\tilde{f}(x) \cap N \neq \emptyset$$ (11)

where $x$ is as chosen above.

Now $\tilde{f}(N) = \mathcal{O}_{f_* H}(\tilde{f}(D))$. We note that $f_* H$ is the subgroup of $G$ generated by the set $\mathcal{A} := \{\alpha \in G \mid \alpha(\tilde{f}(D)) \cap \tilde{f}(D) \neq \emptyset\}$. By (10) and (11) we have that $\text{int}(\tilde{f}(\overline{\mathcal{R}}_j)) \cap N \neq \emptyset$, hence by the Markov property (Definition 2.2(3)) we have $\tilde{f}(\overline{\mathcal{R}}_j) \subset N$ for each $j$. In particular $\tilde{f}(D) \subset N$. Hence we conclude that $\mathcal{A} \subset H$, $f_* H \subset H$, and $\tilde{f}(N) \subset N$. □

Note that $N$ is a covering of $\Lambda$ with covering group $H$. Also, for any $y \in \tilde{\Lambda}$ and $\tilde{f}$ as in Claim 5.34 there is some $m$ so that $\tilde{f}^m(y) \in N$. Consequently, we may reconstruct $\tilde{\Lambda}$ and the covering group $G$ as limits of the directed systems

$$\Lambda \cong \lim \left\{ N \overset{\tilde{f}}{\rightarrow} N \overset{\tilde{f}}{\rightarrow} N \overset{\tilde{f}}{\rightarrow} \ldots \right\}$$

and

$$G \cong \lim \left\{ H \overset{f_*}{\rightarrow} H \overset{f_*}{\rightarrow} H \overset{f_*}{\rightarrow} \ldots \right\}.$$
Fix an isomorphism $\Phi: H \to \mathbb{Z}^k$ and let $A: \mathbb{Z}^k \to \mathbb{Z}^k$ be the endomorphism $\Phi \circ f_* \circ \Phi^{-1}$. Considering $\mathbb{Z}^k$ as embedded in $\mathbb{R}^k$, $A: \mathbb{Z}^k \to \mathbb{Z}^k$ induces a linear automorphism on $\mathbb{R}^k$ and a surjective endomorphism on the quotient $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$, also denoted by $A$. Let $S_A$ and $\mathcal{S}$ be the solenoid and its cover constructed in Section 4, and let $\sigma_A$ and $\tilde{\sigma}$ be the respective shift automorphisms.

Fix an identification $G = \lim(H, f_*)$ whence $f_*: G \to G$ is identified with the shift map $\tau_{f_* \mid H}$. We have that the diagram

$$
\begin{array}{c}
\xymatrix{ H \ar[r]^{f_*} & H \\
\mathbb{Z}^k \ar[r]_{A} \ar[u]^{\Phi} & \mathbb{Z}^k \ar[u]_{\Phi} }
\end{array}
$$

commutes, hence

$$
\begin{array}{c}
\xymatrix{ H \ar[r]^{f_*} & H \ar[r]^{f_*} & H \ar[r]^{f_*} & \cdots \\
\mathbb{Z}^k \ar[r]_{A} \ar[u]^{\Phi} & \mathbb{Z}^k \ar[r]_{A} \ar[u]_{\Phi} & \mathbb{Z}^k \ar[r]_{A} \ar[u]_{\Phi} & \cdots }
\end{array}
$$

induces an isomorphism $\Phi: G \to \mathbb{Z}^k[A^{-1}]$. Furthermore, we have $\Phi \circ f_* \circ \Phi^{-1} = \tau_A$ where $\tau_A$ is as constructed in Section 4.2.

**Proof of Proposition 5.30.** Fix a lift $\tilde{f}$ of $f$. Let $S_A, \sigma_A, \mathcal{S}$, and $\tilde{\sigma}$ be as above. Let $D$ be a fundamental domain for the cover $\mathcal{S} \to S_A$; note that $D$ will be compact. Let $\Phi$ be the isomorphism between $G$ and $\mathbb{Z}^k[A^{-1}]$ above. We let $\mathbb{Z}^k[A^{-1}]$ act by addition on $\mathcal{S}$ via the action $\tilde{\alpha}$ in Section 4.2.

Given a $\xi \in \mathcal{S}$ we may find a sequence $\{\alpha_j\} \subset \mathbb{Z}^k[A^{-1}]$ so that

$$
(\ref{eq:claim})
\tilde{\sigma}^j(\xi) \in D + \alpha_j.
$$

**Claim 5.35.** There is an $L$ so that for any $x \in \tilde{\Lambda}$, $\xi \in \mathcal{S}$, and a sequence $\{\alpha_j\}$ satisfying (12), the sequence $\{(\Phi^{-1}(\alpha_j))(\tilde{f}^j(x))\}$ is an $L$-pseudo orbit.

**Proof.** Let

$$
\mathcal{A} = \{a \in \mathbb{Z}^k[A^{-1}] \mid (D + a) \cap \tilde{\sigma}(D) \neq \emptyset\}.
$$

By Proposition 4.4(d) if $\xi \in D + \alpha$ then $\tilde{\sigma}(\xi) \in (D + \alpha) + \tau_A(\alpha)$ for some $a \in \mathcal{A}$. In particular, $\alpha_{j+1} = \tau_A \alpha_j + a$ for some $a \in \mathcal{A}$. We set

$$
L = \sup\{d(\tilde{y}, (\Phi^{-1}(a))(\tilde{y})) \mid y \in \Lambda, a \in \mathcal{A}\}
$$

where $\tilde{y}$ is an arbitrary lift of $y$ to $\tilde{\Lambda}$. Finiteness of $\mathcal{A}$ guarantees $L < \infty$. Hence

$$
\begin{align*}
d\left(\tilde{f}^j((\Phi^{-1}(\alpha_j))(\tilde{f}^j(x))), (\Phi^{-1}(\alpha_{j+1}))(\tilde{f}^{j+1}(x))\right) \\
= d\left((f_*)(\Phi^{-1}(\alpha_j))(\tilde{f}^{j+1}(x)), (\Phi^{-1}(\alpha_{j+1}))(\tilde{f}^{j+1}(x))\right) \\
= d\left((\Phi^{-1}(\tau_A(\alpha_j)))(\tilde{f}^{j+1}(x)), (\Phi^{-1}(\alpha_{j+1}))(\tilde{f}^{j+1}(x))\right) \\
\leq \max_{a \in \mathcal{A}} \left\{d\left((\Phi^{-1}(\tau_A(\alpha_j)))(\tilde{f}^{j+1}(x)), (\Phi^{-1}(\tau_A(\alpha_j) + a))(\tilde{f}^{j+1}(x))\right)\right\} \\
\leq L.
\end{align*}
$$

Hence the claim holds. \qed
We define a map $\Psi: \tilde{S} \to \tilde{\Lambda}$ as follows. Fix a $p \in \tilde{\Lambda}$. Given $\xi \in \tilde{S}$ choose a sequence $\{\alpha_j\} \subset \mathbb{Z}^k[A^{-1}]$ satisfying (12). Then define $\Psi(\xi)$ to be the unique point $x$ in $\tilde{\Lambda}$ that shadows the $L$-pseudo orbit $\{(\tilde{\Phi}^{-1}(\alpha_j))(\tilde{f}^j(p))\}$. Note that Lemma 5.31(c) guarantees the point $x$ exists, whereas Lemma 5.31(d) guarantees that the point $x$ is unique. Furthermore, Lemma 5.31(d) guarantees that $\Psi: \tilde{S} \to \tilde{\Lambda}$ is continuous.

Claim 5.36. $\Psi$ is proper.

Proof. For $\xi \in \tilde{S}$ and $\alpha \in \mathbb{Z}^k[A^{-1}]$ we clearly have $\Psi(\xi + \alpha) = (\tilde{\Phi}^{-1}(\alpha))(\Psi(\xi))$, hence the map $\Psi: \tilde{S} \to \tilde{\Lambda}$ descends to a continuous map $h: S_A \to \Lambda$. □

Since $\Psi$ is proper, we have that $A$ is hyperbolic. Thus as in Remark 5.32, Lemma 5.31 applies to $\tilde{S}$. Hence, given a fundamental domain $D \subset \tilde{\Lambda}$, and $x \in \tilde{\Lambda}$, we choose $\{g_i\}$ so that $\tilde{f}^i(x) \in g_i(D)$. Then as above we define $\Psi'(x)$ to be the unique point $\xi \in \tilde{S}$ so that $\xi$ shadows the pseudo orbit $\{\tilde{\Phi}(g_i)(e)\}$ where $e$ is the identity in $\tilde{S}$. We thus obtain a map $\Psi': \tilde{\Lambda} \to \tilde{S}$.

One easily verifies

(1) $\Psi$ and $\Psi'$ are inverses;

(2) the diagram

\[
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\Psi} & \tilde{S} \\
\downarrow & & \downarrow \\
\tilde{\Lambda} & \xrightarrow{\tilde{f}} & \tilde{\Lambda}
\end{array}
\]

commutes;

(3) $\Psi$ and $\Psi'$ intertwine the covering actions of $G$ and $\mathbb{Z}^k[A^{-1}]$, that is,

- $\Psi(\alpha(\xi)) = (\tilde{\Phi}^{-1}(\alpha))(\Psi(\xi))$ for all $\alpha \in \mathbb{Z}^k[A^{-1}], \xi \in \tilde{S}$;
- $\Psi'(g(x)) = \Psi'(x) + \tilde{\Phi}(g)$ for all $g \in G, x \in \Lambda$.

Thus the homeomorphism $\Psi: \tilde{S} \to \tilde{\Lambda}$ induces a homeomorphism $h: S_A \to \Lambda$ such that the diagram

\[
\begin{array}{ccc}
S_A & \xrightarrow{\sigma_A} & S_A \\
\downarrow & & \downarrow
\Lambda & \xrightarrow{f} & \Lambda
\end{array}
\]

commutes. □

5.4. Proof of Theorem 1.1.

Proof of Theorem 1.1. By Proposition 5.18, Corollary 5.29, and Proposition 5.30 if $\Lambda$ is not an expanding attractor, then $\Lambda$ is homeomorphic to a toral solenoid, and $f|_\Lambda$ is conjugate to a solenoidal automorphism.

Furthermore we have $W^s(x) \cap \Lambda$ is perfect for every $x \in \Lambda$. Thus if $W^s(x) \cap \Lambda$ is locally connected, $\Lambda$ cannot be expanding, which by the above implies $\Lambda$ is homeomorphic to a solenoid. However, the only locally connected toral solenoids are in fact tori, that is, $S_A$ for $\det A = \pm 1$. □
6. Proof of Corollary 1.2 and Theorem 1.3

The following observation is straightforward (see, for example, [4, Lemma 2.4]).

**Lemma 6.1.** Let $\Lambda \subset M$ be a compact hyperbolic set for a diffeomorphism $g: M \to M$. The points $y \in \Lambda$ with the property that $E^\sigma(y) = \{0\}$ for some $\sigma \in \{s,u\}$ are periodic and isolated in $\Lambda$. In particular if $\Lambda$ is transitive and contains such a point then $\Lambda$ is finite.

We now prove the remaining results from the introduction.

**Proof of Corollary 1.2.** By [16, Theorem 3] and by passing to the inverse if necessary we may assume that $\Lambda$ is an attractor for $f$. We thus have $\dim E^u|_\Lambda \leq 2$. Furthermore, by Lemma 6.1, $\dim E^u|_\Lambda = 0$ would imply that $\dim(\Lambda) = 0$, whence we have $\dim E^u|_\Lambda \geq 1$. By the spectral decomposition and by passing to an appropriate iterate $f^n$ we may assume that $\Lambda$ is topologically mixing for $f^n$.

If $\dim E^u|_\Lambda = 2$ then $\Lambda$ is a codimension-1 expanding attractor. If $\dim E^u|_\Lambda = 1$ then by Theorem 1.1 we have that $\Lambda$ is an embedded toral solenoid. By [11, Theorem 1], no proper 2-dimensional solenoid may be embedded in a closed orientable 3-manifold.

If needed, we first argue on a double cover in the case $M$ is non-orientable. Also if needed we may pass to a compact manifold with boundary $N$ containing $\Lambda$, and glue a second copy of $N$ along the boundary to obtain a closed manifold containing $\Lambda$. We may then apply [11, Theorem 1] and thus obtain that $\Lambda$ is locally connected, hence $\Lambda \cong \mathbb{T}^2$ and $f^n$ is conjugate to a toral automorphism. □

**Proof of Theorem 1.3.** We note that for a hyperbolic attractor we always have $\dim E^u|_\Lambda \leq \dim \Lambda$.

If $\dim \Lambda = 0$ we have that $\dim E^u|_\Lambda = 0$ hence Lemma 6.1 implies that every $x \in \Lambda$ is periodic and isolated; hence we must have $\Lambda = \{x\}$ in order for $\Lambda$ to be topologically mixing. If $\dim \Lambda = 1$ then Lemma 6.1 implies $\dim E^u|_\Lambda = 1$; hence $\Lambda$ is an expanding 1-dimensional attractor and $\Lambda$ is conjugate to the shift map on a generalized 1-solenoid by Theorem II.

If $\dim \Lambda = 2$, Lemma 6.1 implies $1 \leq \dim E^u|_\Lambda \leq 2$. When $\dim E^u|_\Lambda = 1$ the fact that $\Lambda$ is topologically mixing implies $\Lambda$ is connected, whence $\Lambda$ is homeomorphic to $\mathbb{T}^2$ and $f|_\Lambda$ is conjugate to a hyperbolic toral automorphism by Corollary 1.2. When $\dim E^u|_\Lambda = 2$ then $\Lambda$ is a codimension-1 expanding attractor by definition.

Finally, when $\dim \Lambda = 3$ then Lemma 6.1 implies $1 \leq \dim E^u|_\Lambda \leq 2$. Furthermore, [10, Theorem 4.3] implies $\Lambda$ has non-empty interior, which by [5, Theorem 1] implies $\Lambda = M$. But then the result follows from Theorem I. □

**References**

[1] C. Bonatti, Problem in dynamical systems, [http://www.math.sunysb.edu/dynamics/bonatti_prob.txt](http://www.math.sunysb.edu/dynamics/bonatti_prob.txt), November 1999.

[2] H. G. Bothe, *Expanding attractors with stable foliations of class $C^0$*, in “Ergodic theory and related topics, III,” Lecture Notes in Math., 1514, Springer, Berlin, 1992, pp. 36–61.

[3] B. Brenken, *The local product structure of expansive automorphisms of solenoids and their associated $C^*$-algebras*, Canad. J. Math., 48 (1996), 692–709.

[4] A. Brown, *Constraints On Dynamics Preserving Certain Hyperbolic Sets*, Ergodic Theory Dynam. Systems, to appear.

[5] T. Fisher, *Hyperbolic sets with nonempty interior*, Discrete Contin. Dyn. Syst., 15 (2006), 433–446.
[6] J. Franks, *Anosov diffeomorphisms*, in “Global Analysis,” Amer. Math. Soc., Providence, R.I., 1970, pp. 61–93.
[7] V. Z. Grines, V. S. Medvedev, and E. V. Zhuzhoma, *On surface attractors and repellers in 3-manifolds*, Mat. Zametki, 78 (2005), 813–826.
[8] B. Günther, *Attractors which are homeomorphic to compact abelian groups*, Manuscripta Math., 82 (1994), 31–40.
[9] K. Hiraide, *A simple proof of the Franks-Newhouse theorem on codimension-one Anosov diffeomorphisms*, Ergodic Theory Dynam. Systems, 21 (2001), 801–806.
[10] W. Hurewicz and H. Wallman, “Dimension Theory,” Princeton University Press, Princeton, N. J., 1941.
[11] B. Jiang, S. Wang, and H. Zheng, *No embeddings of solenoids into surfaces*, Proc. Amer. Math. Soc., 136 (2008), 3697–3700.
[12] J. L. Kaplan, J. Mallet-Paret, and J. A. Yorke, *The Lyapunov dimension of a nowhere differentiable attracting torus*, Ergodic Theory Dynam. Systems, 4 (1984), 261–281.
[13] A. Katok and B. Hasselblatt, “Introduction to the modern theory of dynamical systems,” Cambridge University Press, Cambridge, 1995.
[14] A. Manning, *There are no new Anosov diffeomorphisms on tori*, Amer. J. Math., 96 (1974), 422–429.
[15] S. E. Newhouse, *On codimension one Anosov diffeomorphisms*, Amer. J. Math., 92 (1970), 761–770.
[16] R. V. Plykin, *The topology of basic sets of Smale diffeomorphisms*, Math. USSR-Sb., 13(2) (1971), 297–307.
[17] R. V. Plykin, *Hyperbolic attractors of diffeomorphisms*, Russian Math. Surveys, 35(3) (1980), 109–121.
[18] R. V. Plykin, *Hyperbolic attractors of diffeomorphisms (the nonorientable case)*, Russian Math. Surveys, 35(4) (1980), 186–187.
[19] D. Ruelle and D. Sullivan, *Currents, flows and diffeomorphisms*, Topology, 14 (1975), 319–327.
[20] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc., 73 (1967), 747–817.
[21] R. F. Williams, *One-dimensional non-wandering sets*, Topology, 6 (1967), 473–487.
[22] R. F. Williams, *Classification of one dimensional attractors*, in “Global Analysis,” Amer. Math. Soc., Providence, R.I., 1970, pp. 341–361.
[23] R. F. Williams, *Expanding attractors*, Inst. Hautes Études Sci. Publ. Math., (1974), 169–203.