On a new formulation of nonlocal image filters involving the relative rearrangement

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Abstract

Nonlocal filters are simple and powerful techniques for image denoising. In this paper we study the reformulation of a broad class of nonlocal filters in terms of two functional rearrangements: the decreasing and the relative rearrangements.

Independently of the dimension of the image, we reformulate these filters as integral operators defined in a one-dimensional space corresponding to the level sets measures.

We prove the equivalency between the original and the rearranged versions of the filters and propose a discretization in terms of constant-wise interpolators, which we prove to be convergent to the solution of the continuous setting.

For some particular cases, this new formulation allows us to perform a detailed analysis of the filtering properties. Among others, we prove that the filtered image is a contrast change of the original image, and that the filtering procedure behaves asymptotically as a shock filter combined with a border diffusive term, responsible for the staircasing effect and the loss of contrast.

Keywords: Nonlocal image filters, Neighborhood filter, Bilateral filter, decreasing rearrangement, relative rearrangement, denoising.

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ be an open and bounded set, $u \in L^\infty(\Omega)$, and consider the nonlocal family of filters defined by

$$F_h u(x) = \frac{1}{C(x)} \int_{\Omega} K_h(u(x) - u(y)) w(x,y) u(y) dy,$$

(1)

where $h$ is a positive constant, and $C(x) = \int_{\Omega} K_h(u(x) - u(y)) w(x,y) dy$ is a normalization factor.

Functions $K_h(\xi) = K(\xi/h)$ and $w$ are the kernels of the filter. A usual choice for $K$ is the Gaussian $K(\xi) = \exp(-\xi^2)$, while different choices of $w$ give rise to several well known nonlocal filters, e.g.,

- The Neighborhood filter, see [8], for $w(x,y) \equiv 1$.

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• The Yaroslavsky filter \[40, 41\], for \(w(x, y) \equiv \chi_{B_\rho(x)}(y)\), the characteristic function of a ball centered at \(x\) of radius \(\rho > 0\).

• The SUSAN [36] and Bilateral filters [38], for

\[
w(x, y) = e^{-\frac{|x-y|^2}{\rho^2}}, \quad \rho > 0.
\]

• The weighted Bilateral filter [24], with, for some function \(\bar{w}\) usually related to the depth map of \(u\),

\[
w(x, y) = e^{-\frac{|x-y|^2}{\rho^2}} \bar{w}(y), \quad \rho > 0,
\]

and its corresponding weighted Neighborhood filter, for \(w(x, y) = \bar{w}(y)\).

These filters have been introduced in the last decades as efficient alternatives to local methods such as those expressed in terms of nonlinear diffusion partial differential equations (PDE's), among which the pioneering approaches of Perona and Malik [28], álvarrez, Lions and Morel [2] and Rudin, Osher and Fatemi [34] are fundamental. We refer the reader to [10] for a review and comparison of these methods.

Nonlocal filters have been analyzed from different points of view. For instance, Barash [6], Elad [16], Barash et al. [7], and Buades et al. [9] investigate the asymptotic relationship between the Yaroslavsky filter and the Perona-Malik PDE. Gilboa et al. [21] study certain applications of nonlocal operators to image processing. In [29], Peyré establishes a relationship between the non-iterative nonlocal filtering schemes and thresholding in adapted orthogonal basis. In a more recent paper, Singer et al. [35] interpret the Neighborhood filter as a stochastic diffusion process, explaining in this way the attenuation of high frequencies in the processed images.

In [19] we heuristically introduced a denoising algorithm based in the Neighborhood filter but computed only on the level sets of the image, implying a large gain of computational effort. Later, in [20], we reformulated this nonlocal filter in terms of the decreasing rearrangement of the initial image, denoted by \(u_*\), which is defined as the inverse of the distribution function \(m_u(q) = |\{x \in \Omega : u(x) > q\}|\), see Section 2 for the precise definition and some of its properties.

Realizing that the structure of level sets of \(u\) is invariant through the Neighborhood filter operation as well as through the decreasing rearrangement of \(u\) allowed us to rewrite (1), for \(w \equiv 1\), in terms of the one-dimensional integral expression

\[
NF^*_h u(x) = \frac{\int_0^{[\bar]{\Omega}} K_h(u(x) - u_*(s))u_*(s)ds}{\int_0^{[\bar]{\Omega}} K_h(u(x) - u_*(s))ds},
\]

which is computed jointly for all the pixels in each level set \(\{x : u(x) = q\}\).

Perhaps, the most important consequence of using the rearrangement was, apart from the large dimensional reduction, the reinterpretation of the Neighborhood filter as a local algorithm. Thanks to this we proved, among others, the following properties for the most usual nonlinear iterative variant of the Neighborhood filter, see [18]:

• The asymptotic behavior of the NF as a shock filter of the type introduced by álvarrez et al. [1], combined with a contrast loss effect.

• The contrast change character of the NF, i.e. the existence of a continuous and increasing function \(g : \mathbb{R} \to \mathbb{R}\) such that \(NF^h(u(x)) = g(u(x))\).
In this article we extend the use of rearranging techniques to other nonlocal filters. Indeed, as noticed in [20], even if the kernel $w$ is non-constant we may still use our approach by introducing the relative rearrangement of the kernel with respect to the image, see Section 2 for definitions.

In this way, we may express the general nonlocal filters embedded in formula (1) in terms of one-dimensional integral expressions of the form

$$F^*_h u(x) = \frac{\int_0^{\|\Omega\|} K_h(u(x) - u_*(s)) w(x, \cdot) u_*(s) ds}{\int_0^{\|\Omega\|} K_h(u(x) - u_*(s)) w(x, \cdot) u_*(s) ds},$$

where $v_\cdot u$ denotes the relative rearrangement of $v$ with respect to $u$.

This sophisticated reformulation of the nonlocal filter in terms of one-dimensional integration may be of limited computational use since, in general, the filtering transformation requires computing for each pixel, $x$, the expensive term involving the relative rearrangement.

However, there are some particular filters, like the weighted Neighborhood filter, for which this reformulation offers a large gain of computational effort and gives a notable analytical insight into the filter functioning. In addition, in general, since the computation of (2) is based on the number of level lines (quantized levels) of $u$, when this number is small formula (2) may be more efficient than the direct implementation of (1).

The plan of the article is the following. In Section 2 we introduce the notion of decreasing rearrangement and relative rearrangement and establish the equivalence between the usual pixel-based expression of the filter (1) and its rearranged formulation (2).

In Section 3, we provide a fully discrete algorithm to approximate by constant-wise functions the filter $F^*_h u(x)$ given by (2), and thus the original equivalent filter $F_h u(x)$ given by (1). We also prove the convergence of this discretization to the solution of the continuous setting.

In Section 4, we analyze the particular cases in which the kernel $w(x, y)$ only depend on the integration variable $y$, and thus may be considered as a weight function. In this situation, we are able to extend most of the results proved for the Neighborhood filter in [20], that is, for $w \equiv 1$. In particular, we show the asymptotic behavior of these filters as shock filters when $h \to 0$.

2 Nonlocal filters in terms of functional rearrangements

2.1 The decreasing rearrangement

Let us denote by $|E|$ the Lebesgue measure of any measurable set $E$. For a Lebesgue measurable function $u : \Omega \to \mathbb{R}$, the function $q \in \mathbb{R} \to m_u(q) = |\{x \in \Omega : u(x) > q\}|$ is called the distribution function corresponding to $u$.

Function $m_u$ is non-increasing and therefore admits a unique generalized inverse, called the decreasing rearrangement. This inverse takes the usual pointwise meaning when the function $u$ has not flat regions, i.e. when $|\{x \in \Omega : u(x) = q\}| = 0$ for any $q \in \mathbb{R}$. In general, the decreasing rearrangement $u_* : [0, |\Omega|] \to \mathbb{R}$ is given by:

$$u_*(s) = \begin{cases} 
\text{ess sup}\{u(x) : x \in \Omega\} & \text{if } s = 0, \\
\inf\{q \in \mathbb{R} : m_u(q) \leq s\} & \text{if } s \in (0, |\Omega|), \\
\text{ess inf}\{u(x) : x \in \Omega\} & \text{if } s = |\Omega|.
\end{cases}$$
We shall also use the notation \( \Omega_* = (0, |\Omega|) \). Notice that since \( u_* \) is non-increasing in \( \Omega_* \), it is continuous but at most a countable subset of \( \Omega_* \). In particular, it is right-continuous for all \( t \in (0, |\Omega|) \).

The notion of rearrangement of a function is classical and was introduced by Hardy, Littlewood and Polya \cite{22}. Applications include the study of isoperimetric and variational inequalities \cite{30, 5, 26}, comparison of solutions of partial differential equations \cite{37, 3, 39, 11, 12, 4}, and others. We refer the reader to the textbook \cite{23} for the basic definitions.

Two of the most remarkable properties of the decreasing rearrangement are the equi-measurability property,

\[
\int_{\Omega} F(u(y))\,dy = \int_0^{[\Omega]} F(u_*(s))\,ds.
\]

for any Borel function \( F : \mathbb{R} \to \mathbb{R}_+ \), and the contractivity

\[
\|u_* - v_*\|_{L^p(\Omega_*)} \leq \|u - v\|_{L^p(\tilde{\Omega})},
\]

for \( u, v \in L^p(\tilde{\Omega}), p \in [1, \infty] \).

### 2.2 Motivation

Apart from the pure mathematical interest, the reformulation of nonlocal filters in terms of functional rearrangements is useful for computational purposes, specially when the level lines of \( u \) are left invariant through the filter, i.e. when \( u(x) = u(y) \) implies \( F_h(u)(x) = F_h(u)(y) \). Thus, in these cases, the filter is computed only for each (quantized) level line, instead of for each pixel, meaning a large gaining of computational effort.

In the following lines, we provide a heuristic derivation of the nonlocal filter rearranged version, as first noticed in \cite{20}. Under suitable assumptions, the coarea formula states

\[
\int_{\Omega} g(y)|\nabla u(y)|\,dy = \int_{-\infty}^{\infty} \int_{u=t} g(y)\,d\Gamma(y)\,dt.
\]

Taking \( g(y) = K_h(u(x) - u(y))w(x,y)u(y)/|\nabla u(y)| \), and using \( u(x) \in [0, Q] \) for all \( x \in \Omega \) we get

\[
I(x) := \int_{\Omega} K_h(u(x) - u(y))w(x,y)u(y)\,dy = \int_{0}^{Q} K_h(u(x) - t) t \int_{u=t}^{\tilde{\Omega}} \frac{w(x,y)}{|\nabla u(y)|} d\Gamma(y)\,dt.
\]

Introducing the change of variable \( t = u_*(s) \) we find

\[
I(x) = -\int_{0}^{[\Omega]} K_h(u(x) - u_*(s))u_*(s)\frac{du_*(s)}{ds} \int_{u=u_*(s)}^{\tilde{\Omega}} \frac{w(x,y)}{|\nabla u(y)|} d\Gamma(y)\,ds
\]

\[= \int_{0}^{[\Omega]} K_h(u(x) - u_*(s))u_*(s)w(x, \cdot)_*u_*(s)\,ds. \tag{4}\]

Here, the notation \( v_{*u} \) stands for the relative rearrangement of \( v \) with respect to \( u \) which, under regularity conditions, may be expressed as

\[
v_{*u}(s) = \frac{\int_{u=u_*(s)}^{\tilde{\Omega}} \frac{v(y)}{|\nabla u(y)|} d\Gamma(y)}{\int_{u=u_*(s)}^{\tilde{\Omega}} \frac{1}{|\nabla u(y)|} d\Gamma(y)}, \tag{5}\]
see the next section for details. Transforming \( C(x) \) in a similar way allows us to deduce

\[
F_h u(x) = \frac{\int_0^{|\Omega|} K_h(u(x) - u_*(s)) u_*(s) w(x, \cdot)_{*u}(s) ds}{\int_0^{|\Omega|} K_h(u(x) - u_*(s)) w(x, \cdot)_{*u}(s) ds}.
\]  

(6)

2.3 The relative rearrangement

The relative rearrangement was introduced by Mossino and Temam \[25\] as the directional derivative of the decreasing rearrangement. Thus, if for \( u \in L^1(\Omega) \) and \( v \in L^p(\Omega) \), with \( p \in [1, \infty] \), we consider the function \( w : \Omega_* \rightarrow \mathbb{R} \) given by

\[
w(s) = \int_{u > u_*(s)} v(x) dx + \int_0^{s - |u > u_*(s)|} (v|_{u = u_*(s)})_*(\sigma) d\sigma,
\]

then the relative rearrangement of \( v \) with respect to \( u \), \( v_{*u} \), is defined as

\[v_{*u} := \frac{d}{ds} w \in L^p(\Omega_*).\]

This identity may be also understood as the weak \( L^p(\Omega_*) \) directional derivative (weak* \( L^\infty(\Omega_*) \)), if \( p = \infty \)

\[v_{*u} = \lim_{t \to 0} \frac{(u + tv)_{*} - u_{*}}{t}.
\]

(7)

Under the additional assumptions \( u \in W^{1,1}(\Omega) \) and \(|\{ y \in \Omega : \nabla u(y) = 0\}| = 0\), i.e. the non-existence of flat regions of \( u \), the identity (7) is well defined. In this case the relative rearrangement represents an averaging procedure of the values of \( v \) on the level lines of \( u \) labeled by the superlevel sets measure, \( s \).

When formula (5) does not apply, that is in flat regions of \( u \), we may resort to identity (7) to interpret the relative rearrangement in flat regions as the decreasing rearrangement of \( v \) restricted to such sets.

After the seminal work of Mossino and Temam \[25\], the relative rearrangement was further studied by Mossino and Rakotoson \[27\] and applied to several types of problems, among which those related to variable exponent spaces and functional properties, see Rakotoson et al. \[17, 18, 32, 33\], or nonlocal formulations of plasma physics problems related to nuclear fusion devices, see Díaz et al. \[13, 14, 15\].

In the rest of the article, we shall make an extensive use of results appearing in the monograph on the relative rearrangement by Rakotoson \[31\].

2.4 A general result

The main assumption we implicitly made for the heuristic deduction of formula (6) is the condition \(|\{ y \in \Omega : \nabla u(y) = 0\}| = 0\), i.e. the non-existence of flat regions of \( u \), which gives sense on one hand to formula (5), and on the other hand, allow us to obtain the strictly decreasing behaviour of \( u_{*} \) which justifies the change of variable in (4).

Our first result is that formulas (1) and (6) are equivalent under weaker hypothesis. Due to the nature of our application, we keep the assumption on the boundedness of \( u \) and \( w \) in \( L^\infty \), although these can be also weakened to less regular \( L^p \) spaces.
Theorem 1 Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set, $d \geq 1$, $K \in L^\infty(\mathbb{R}, \mathbb{R}_+)$ and $w \in L^\infty(\Omega \times \Omega, \mathbb{R}_+)$. Assume that $u \in L^\infty(\Omega)$ is, without loss of generality, non-negative. Consider $F_hu(x)$ given by (7) and

$$F_h^*u(x) = \frac{\int_0^{[\Omega]} K_h(u(x) - u_*(s))u_*(s)w(x, \cdot)\nu(s)ds}{\int_0^{[\Omega]} K_h(u(x) - u_*(s))w(x, \cdot)\nu(s)ds}. \quad (8)$$

Then $F_h^*u(x) = F_hu(x)$ for a.e. $x \in \Omega$.

Proof. Let $f \in L^\infty(\mathbb{R})$ and $b \in L^\infty(\Omega)$. We start showing

$$\int_{\Omega} f(u(y))b(y)dy = \int_0^{[\Omega]} f(u_*(s))b_*(s)ds. \quad (9)$$

Consider the sets of flat regions of $u$ and $u_*$,

$$P = \bigcup_{i \in D} P_i, \quad P_i = \{y \in \Omega : u(y) = q_i\}, \quad (10)$$

and $P_\ast = \bigcup_{i \in D} P_i^\ast$, with $P_i^\ast = \{s \in \Omega_\ast : u(s) = q_i\}$, where the subindices set $D$ is, at most, countable. According to [31 Lemma 2.5.2], we have

$$\int_0^{[\Omega]} f(u_*(s))b_*(s)ds = \int_{\Omega \setminus P} f(u_*(m_0(u(y))))b(y)dy + \sum_{i \in D} \int_{P_i} M_{b_i}(h_i)(y)b(y)dy, \quad (11)$$

where $b_i = b|_{P_i}$, $h_i(s) = f(u_*(s^i + s))$ for $s \in [s^i, s^\prime_i] := P_i^\ast$, and

$$M_{b_i}(h_i)(y) = \begin{cases} h_i(m_0(b_i(y))) & \text{if } y \in P_i \setminus Q_i, \\ \frac{1}{|Q^j|} \int_{s^j_i} h_i(s)ds & \text{if } y \in Q^j_i, \end{cases}$$

where $Q^j = \bigcup_{j \in D'} Q^j_j$, with $Q^j_j$ the flat regions of $b_i$ and $[s^j_i, s^\prime_j] := Q^j_j$. In (11), since the functions $u_*$ and $m_0$ are strictly decreasing and inverse of each other in the set $\Omega \setminus P$, we obtain

$$\int_{\Omega \setminus P} f(u_*(m_0(u(y))))b(y)dy = \int_{\Omega \setminus P} f(u(y))b(y)dy. \quad (12)$$

In the flat regions of $u$ and $u_*$ we have, on one hand,

$$\int_P f(u(y))b(y)dy = \sum_{i \in D} \int_{P_i} f(u(y))b(y)dy = \sum_{i \in D} f(q_i) \int_{P_i} b(y)dy. \quad (13)$$

And, on the other hand, since $h_i(s) = f(u_*(s^i + s)) = f(q_i)$ for $s \in P_i^\ast$,

$$\sum_{i \in D} \int_{P_i} M_{b_i}(h_i)(y)b(y)dy = \sum_{i \in D} \left( \int_{P_i \setminus Q_i} h_i(m_0(b_i(y)))b(y)dy + \sum_{j \in D'} \frac{1}{|Q^j|} \int_{s^j_i} h_i(s)ds \int_{Q^j_i} b(y)dy \right)$$

$$= \sum_{i \in D} \left( f(q_i) \int_{P_i \setminus Q_i} b(y)dy + f(q_i) \sum_{j \in D'} \int_{Q^j_i} b(y)dy \right)$$

$$= \sum_{i \in D} f(q_i) \int_{P_i} b(y)dy.$$
Therefore, both sides of (12) are equal.

Finally, for fixed \( x \in \Omega \) set \( b(y) = w(x, y) \) and first, \( f(t) = K_h(u(x) - t)t \), for \( t \geq 0 \) to obtain, using the identity (13), the equality between the numerators of (11) and (3), and second, \( f(t) = K_h(u(x) - t) \) to obtain the equality between the denominators of those expressions. \( \square \)

**Remark 1** As deduced in the proof, identity (8) follows from [27, Lemma 2.5.2]. In fact, a little more than (8) may be obtained. Let \( f, b \in L^\infty(\Omega) \). Then, if \( f \) is constant in the flat regions of \( u \), that is \( f|_{P_i} = f_i = \text{const} \), then

\[
\int_0^{\Omega} f(s)B_n(s)ds = \int_\Omega f(m_u(u(y)))b(y)dy + \sum_{i \in D} f(q_i) \int_{P_i} b(y)dy. \tag{12}
\]

### 3 Constant-wise discretization and convergence

In this section we provide a fully discrete algorithm to approximate the filter \( F_h^* \) given by (8), and thus the original equivalent filter \( F_h \) given by (11), as proved in Theorem 1.

In Theorem 2 we prove that if the initial image \( u \) has a finite number of flat regions, that is if the set \( D \) given in (10) is finite, then we may approximate \( u \) and \( w \) by constant-wise functions \( u_n, w_m \) which have a finite number of levels and such that

\[
F_{h,m}^* u_n(x) \to F_h^* u(x)
\]

where \( F_{h,m}^* \) is the discrete version of \( F_h^* \), see (14).

This result gives sense to Theorem 3 in which we produce a finite discrete formula for the approximation of \( F_h^* u(x) \), and thus of \( F_h u(x) \). This formula is what can actually be used for the numerical experimentation.

**Theorem 2** Let \( u, w \in L^\infty(\Omega) \) be nonnegative and assume that \( u \) has a finite number of flat regions. Then, there exist sequences of constant-wise functions \( u_n, w_m \), with a finite number of flat regions, such that \( u_n \to u \) strongly in \( L^\infty(\Omega) \), \( w_m \to w \) strongly in \( L^\infty(\Omega \times \Omega) \), and

\[
F_{h,m}^* u_n(x) \to F_h^* u \quad \text{a.e. in } \Omega \quad \text{and strongly in } L^\infty(\Omega), \tag{13}
\]

where

\[
F^*_h u(x) = \frac{\int_0^{\Omega|} K_h(u(x) - u_*(s))u_*(s)w_m(x, \cdot)_{su}(s)ds}{\int_0^{\Omega|} K_h(u(x) - u_*(s))w_m(x, \cdot)_{su}(s)ds}. \tag{14}
\]

**Proof.** We split the proof in several steps.

**Step 1.** We use the construction of the sequence of constant-wise functions \( u_n \) given in [31, Th. 7.2.1]. In our case, the construction is simpler because \( u \) has a finite number of flat regions, implying that \( u_n \) has a finite number of levels.

In any case, this construction is such that \( u_n \to u \) a.e. in \( \Omega \) and strongly in \( L^\infty(\Omega) \), and \( w_{su_n} \to w_{su} \) weakly* in \( L^\infty(\Omega_s) \). Besides, due to the strong continuity of the decreasing rearrangement (3), we also have \( (u_n)_s \to u_s \) strongly in \( L^\infty(\Omega_s) \). Therefore, we readily see first that

\[
F^*_h u_n(x) \to F_h^* u(x) \quad \text{for a.e. } x \in \Omega,
\]
and then, due to the dominated convergence theorem,

$$F_h^* u_n \to F_h^* u \quad \text{strongly in } L^\infty(\Omega). \quad (15)$$

**Step 2.** We consider a sequence of constant-wise functions with a finite number of levels, $w_m$, such that $w_m \to w$ strongly in $L^\infty(\Omega \times \Omega)$. Due to the contractivity property of the relative rearrangement, see [25], we also have, for a.e. $x \in \Omega$, $w_m(x,\cdot) \to w(x,\cdot)$ strongly in $L^\infty(\Omega)$, for any $v \in L^\infty(\Omega)$. Thus, as $m \to \infty$,

$$F_{h,m}^* u_n(x) = F_h^* u_n(x) \quad \text{for a.e. } x \in \Omega,$$

and, again, the dominated convergence theorem implies

$$F_{h,m}^* u_n \to F_h^* u_n \quad \text{strongly in } L^\infty(\Omega). \quad (16)$$

**Step 3.** In view of (15) and (16), we have

$$|F_{h,m}^* u_n - F_h^* u| \leq |F_{h,m}^* u_n - F_h^* u_n| + |F_h^* u_n - F_h^* u| \to 0$$

as $m \to \infty$ and $n \to \infty$, so (13) follows. □

**Remark 2** Theorem 2 may be extended to the case in which $u$ has a countable number of flat regions. However, the construction in [31] then implies that each element of the sequence of constant-wise functions $u_n$ has also a countable number of levels. Since our aim is providing a finite discretization for numerical implementation, such a sequence is not appropriate.

In the following theorem we produce a discrete numerical formula for computing the nonlocal filter for each pair $(u_n, w_m)$ of the sequences given in Theorem 2.

The main difficulty of computing formula (14) the determination of the relative rearrangement. However, in the case of constant-wise functions with a finite number of levels this computation is simplified thanks to identity (7), which may be easily applied to this situation, as shown in [31, Th.7.3.4].

In few words, for the case of constant-wise functions $u$ and $v$, the relative rearrangement $v^* u$ may be computed as the decreasing rearrangement of $v$ restricted to the level sets of $u$.

**Theorem 3** Let $u \in L^\infty(\Omega)$ be a constant-wise function quantized in $n$ levels labeled by $q_i$, with max$(u) = q_1 > \ldots > q_n = 0$. That is $u(x) = \sum_{i=1}^n q_i \chi_{E_i}(x)$, where $E_i$ are the level sets of $u$,

$$E_i = \{x \in \Omega : u(x) = q_i\}, \quad i = 1, \ldots, n.$$

Similarly, let $w \in L^\infty(\Omega \times \Omega)$ be constant-wise and quantized in $m$ levels $r_j$, with max$(w) = r_1 > \ldots > r_m = \min(w) \geq 0$. For each $x \in \Omega$, consider the partition of $E_i$ given by $F_j^i(x) = \{y \in E_i : w(x,y) = r_j\}$. Then, for each $x \in E_k$, $k = 1, \ldots, n$

$$F_h^* u(x) = \frac{\sum_{i=1}^n \sum_{j=1}^m K_h(q_k - q_i) r_j |F_j^i(x)|}{\sum_{i=1}^n \sum_{j=1}^m K_h(q_k - q_i) r_j |F_j^i(x)|}. \quad (17)$$
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Proof. Since \( u \) is constant-wise, the decreasing rearrangement of \( u \) is constant-wise too, and given by

\[
    u_*(s) = \sum_{i=1}^{n} q_i \chi_{I_i}(s),
\]

with \( I_i = [a_{i-1}, a_i) \) for \( i = 1, \ldots, n \), and \( a_0 = 0, a_1 = |E_1|, a_2 = |E_1| + |E_2|, \ldots, a_n = \sum_{i=1}^{n} |E_i| = \Omega \). It is convenient to introduce here the cumulative sum of sets measures

\[
    \text{cum}(E_0, 0) = 0, \quad \text{and} \quad \text{cum}(E_0, i) = \sum_{k=1}^{i} |E_k|, \quad i = 1, \ldots, n,
\]

where the symbol \( \circ \) denotes the summation variable. Thus, \( a_i = \text{cum}(E_0, i) \).

We consider, for fixed \( x \in \Omega \) and \( t > 0 \), the function \( H(y) := u(y) + tw(x, y) \). Since both \( u \) and \( w \) are constant-wise with a finite number of levels, we have, for \( t \) small enough

\[
    q_{i+1} < q_i + tr_j < q_i - 1,
\]

implying that each level set of \( H \) is included in one and only one level set of \( u \). Thus, we have the orderings

- For each \( j \), if \( i_1 > i_2 \) and \( y \in F^{i_1}_j(x) \), \( y \in F^{i_2}_j(x) \) then \( H(y) < H(\bar{y}) \).
- For each \( i \), if \( j_1 > j_2 \) and \( y \in F^i_{j_1}(x) \), \( y \in F^i_{j_2}(x) \) then \( H(y) < H(\bar{y}) \).

With these observations we may compute the decreasing rearrangement of \( H \) as follows. For instance, for \( s \in I_1 = [0, |E_1|) \) we have (omitting \( x \) from the notation \( F^i_j(x) \))

\[
    H_*(s) = \begin{cases} 
    q_1 + tr_1 & \text{if } s \in [0, F^1_1]) = [\text{cum}(F^1_0, 0), \text{cum}(F^1_0, 1)), \\
    q_1 + tr_2 & \text{if } s \in [|F^1_1|, |F^1_1| + |F^1_2|]) = [\text{cum}(F^1_0, 1), \text{cum}(F^1_0, 2)), \\
    \vdots & \vdots \\
    q_1 + tr_m & \text{if } s \in [\text{cum}(F^1_0, m - 1), \text{cum}(F^1_0, m)) = [\text{cum}(F^1_0, m - 1), |E_1|]. 
\end{cases}
\]

In general, we may write for \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \),

\[
    H_*(s) = q_i + tr_j \quad \text{if } s \in J^i_j(x),
\]

where \( J^i_j(x) := \left[ b^i_{j-1}(x), b^i_j(x) \right] \), with \( b^i_j(x) := a_{i-1} + \text{cum}(F^i_0(x), j) \). Observe that \( b^i_m(x) = a_i \).

Finally, since \( J^i_j(x) \subset E_i \) we have for \( s \in J^i_j(x) \)

\[
    \frac{H_*(s) - u_*(s)}{t} = r_j, \quad \text{implying} \quad w(x, \cdot)_{u_*(s)}(s) = r_j.
\]

We are now in disposition to compute formula (3). For \( x \in E_k \),

\[
    \int_{0}^{\Omega} K_h(u(x) - u_*(s))u_*(s)w(x, \cdot)_{u_*(s)}(s)ds = \sum_{i=1}^{n} \sum_{j=1}^{m} K_h(q_k - q_i)q_i r_j |J^i_j(x)|
\]

In a similar way we obtain

\[
    C(x) = \int_{0}^{\Omega} K_h(u(x) - u_*(s))w(x, \cdot)_{u_*(s)}(s)ds = \sum_{i=1}^{n} \sum_{j=1}^{m} K_h(q_k - q_i) r_j |J^i_j(x)|
\]

and therefore, using the definition of the sets \( J^i_j(x) \) we obtain, (17). □
3.1 Examples

As it clear from formula (17), the main difficulty for its computation is the determination of the measures of the sets $F^i_j(x)$, which must be computed for each $x \in \Omega$.

The formula also provides the complexity of the algorithm. If $N$ is the number of pixels of the image, then the complexity is of the order $O(Nnm)$, where $n$ is the number of levels of the image and $m$ is the number of levels of the kernel. Let us examine some examples.

The Neighborhood filter. In this case, $w(x, y) \equiv 1$, and therefore $j = 1$ and $F^i_j(x) = E_i$ is independent of $x$ for all $i = 1, \ldots, n$. Thus, formula (17) is computed only on the level sets of $u$, that is, for all $x \in E_k$

$$F^*_h u(x) = \frac{\sum_{i=1}^{n} K_h(q_k - q_i)q_i |E_i|}{\sum_{i=1}^{n} K_h(q_k - q_i)|E_i|}$$

In this case, the complexity is of order $O(n^2)$.

The weighted Neighborhood filter. Here, $w(x, y) \equiv \bar{w}(y)$, and therefore $F^i_j(x)$ is independent of $x$ for all $i = 1, \ldots, n$, $j = 1, \ldots, m$. Thus, formula (17) is computed again only on the level sets of $u$, that is, for all $x \in E_k$

$$F^*_h u(x) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{m} K_h(q_k - q_i)q_i r_j |F^i_j|}{\sum_{i=1}^{n} \sum_{j=1}^{m} K_h(q_k - q_i)r_j |F^i_j|}$$

The complexity is of order $O(n^2m)$.

The Yaroslavsky filter. In this case, $w(x, y) = \chi_{B_{\rho}(x)}(y)$, and therefore there are only two levels $r_1 = 1$, $r_2 = 0$ of $w$ corresponding to the sets

$$F^1_i(x) = \{y \in E_i : |x - y| < \rho\}, \quad F^2_i(x) = \{y \in E_i : |x - y| \geq \rho\}.$$ 

Thus, formula (17) reduces to: for each $x \in E_k$, $k = 1, \ldots, n$

$$F^*_h u(x) = \frac{\sum_{i=1}^{n} K_h(q_k - q_i)q_i |F^1_i(x)|}{\sum_{i=1}^{n} K_h(q_k - q_i)|F^1_i(x)|}$$

The complexity is of order $O(Nn)$.

The Bilateral filter. In this case, $w(x, y) = \exp(|x - y|^2/\rho^2)$ and therefore there is a continuous range of levels for $w$. However, for computational porpouses the range of $w$ is quantized to some finite number of levels, determined by the size of $\rho$. Thus, the full formula (17) must be used in this case. The resulting complexity is of order $O(Nnm)$.

4 The Neighborhood and weighted Neighborhood filters

The next results are particularized to the cases in which the introduction of the rearranged formulation (6) implies an important gain in the algorithmic complexity. This happens when the function $w$ is a weight function instead of window functions, i.e. if $w(x, y) \equiv w(y)$. 


This simplification is the case of, for instance, the Neighborhood filter \((w \equiv 1)\) or the weighted Neighborhood filter (non-negative \(w \in L^\infty(\Omega)\)). In these cases, since one application of the filter is usually not enough, the following iterative scheme is considered. Set \(u_0 = u\), the initial image. For \(n \in \mathbb{N}\),

\[
 u_{n+1}(x) = \frac{1}{C_n(x)} \int_\Omega K_h(u_n(x) - u_n(y)) u_n(y) w(y) dy,
\]

(18)

with \(C_n(x) = \int_\Omega K_h(u_n(x) - u_n(y)) w(y) dy\).

Straightforward adaptations of the proofs of Theorems 1 and 2 and Corollaries 1 and 2 of [20], proved for \(w \equiv 1\), imply similar results for the general case of a weight functions \(w \geq 0\). We list here the most salient properties stated in these results, to which we shall refer as to Properties (P). We use the following notation for the level sets of \(u\) given in terms of the levels of \(u_*\):

\[
 L_t(u) = \{ y \in \Omega : u(y) = u_*(t) \}, \quad t \in \bar{\Omega}.*
\]

1. The iterative scheme (18) may be computed only on the level sets of \(u\) as follows: if \(x \in L_t(u)\) for some \(t \in [0, |\Omega|]\), we set \(u_{n+1}(x) = v_{n+1}(t)\), with

\[
 v_{n+1}(t) = \frac{1}{c_n(t)} \int_0^{[\Omega]} K_h(v_n(t) - v_n(s)) v_n(s) w_{*u}(s) ds,
\]

(19)

\[
 c_n(t) = \int_0^{[\Omega]} K_h(v_n(t) - v_n(s)) w_{*u}(s) ds,
\]

and \(v_0 = u_*\).

2. Under suitable assumptions on the kernel \(K\), in which the Gaussian kernel is included, if \(v_0 \in W^{1,p}(0,|\Omega|)\) for some \(p \geq 1\) then

(i) \(v_{n+1} \in W^{1,p}(0,|\Omega|)\), and if \(v'_0(t) = 0\) then \(v'_{n+1}(t) = 0\).

(ii) \(v''_{n+1} \leq 0\) a.e. in \((0,|\Omega|)\), and if \(v'_0(t) < 0\) then \(v'_{n+1}(t) < 0\).

(iii) If \(v_0 \in C^m([0,|\Omega|])\) and \(K \in C^m(\mathbb{R})\) then \(v_{n+1} \in C^m([0,|\Omega|])\) for all \(n\).

(iv) For each \(n \in \mathbb{N}\), there exists a strictly increasing function \(g : \mathbb{R} \to \mathbb{R}\), a contrast change, such that \(u_{n+1}(x) = g(u(x))\), where \(u_{n+1}\) is given by (18).

In the following theorem we establish a correspondence between the nonlocal diffusion scheme (19) and local diffusion, establishing an asymptotic behavior of the filter, when \(h \to 0\) as a shock filter.

Although more general assumptions on \(K\) may be prescribed, see Remark 3 we estate the following result for the Gaussian kernel, for clarity. We also ask for further regularity on \(u_*\) and \(w_{*u}\).

**Theorem 4** Let \(v_0 = u_* \in C^3(\bar{\Omega}_*)\) and \(w_{*u} \in C^2(\bar{\Omega}_*)\) be such that \(v'_0 < 0\) and \(w_{*u} > 0\) in \([0,|\Omega|]\). Set \(K(\xi) = e^{-\xi^2}\). Then, for all \(t \in \bar{\Omega}_*\), there exist positive constants \(\alpha_1\), and \(\alpha_2\), independent of \(h\) such that

\[
 v_{n+1}(t) = v_n(t) + \alpha_1 \frac{k_h(t) v'_n(t)}{w_{*u}(t)} (h + O(h^{3/2})) - \alpha_2 \frac{v''_n(t)}{(v'_n(t))^2} h^2
\]

\[
 + \alpha_2 \frac{w'_{*u}(t)}{w_{*u}(t) v'_n(t)} h^2 + O(h^{5/2}),
\]

(20)
with
\[ \tilde{k}_h(t) = \frac{w_{su}(|\Omega|)K_h(v_n(t) - v_n(|\Omega|))}{v_n'(0)} - \frac{w_{su}(0)K_h(v_n(t) - v_n(0))}{v_n'(0)}, \]  
(21)
and with \( \alpha_1 \approx 1/\sqrt{\pi} \), and \( \alpha_2 \approx 1 \).

For \( w \equiv 1 \), and thus \( w_{su} \equiv 1 \), this result was proven in [20]. The second and third terms at the right hand side of (20) were interpreted, respectively, as a (border) loss of contrast, and an anti-diffusive shock filter term similar to that introduced by Alvarez and Mazorra [1].

However, for a general weights \( w \), the fourth term is more difficult to interpret due specially to the unclear meaning of the derivative \( w'_{su} \). Just to gain some insight, let us assume that \( w(x) = f(u(x)) \) for some contrast change function, \( f \). Then, \( w_{su} = f(u_*) \). By point 2(iii) of Properties (P), for each step \( n \) there exists another contrast change, \( g \), such that \( v_n = g(u_*) \). Then we have

\[ \frac{w'_{su}(t)}{w_{su}(t)v_n'(t)} = \frac{f'(u_*(t))}{f(u_*(t))g'(u_*(t))}, \]

which corresponds to a nonnegative source term.

**Proof of Theorem 4.** We may rewrite the iterative scheme (19) as
\[ v_{n+1}(t) - v_n(t) = \frac{1}{c_n(t)} \int_0^{|\Omega|} K_h(v_n(t) - v_n(s))(v_n(s) - v_n(t))w_{su}(s)ds. \]
(22)
Due to (P) we have \( v_n' < 0 \) in \((0, |\Omega|)\), and \( v_n(0, |\Omega|) \subset v_0(0, |\Omega|) \). Let us denote the inverse of \( v_n \) by \( v_n^{-1} \). Using the change of variable \( s = v_n^{-1}(q) \) and writing \( t = v_n^{-1}(z) \), we obtain from (22)
\[ v_{n+1}(t) - v_n(t) = \frac{I_1(z)}{I_2(z)}, \]
(23)
with
\[ I_1(z) = \int_{v_n(|\Omega|)}^{v_n(0)} K_h(z - q)(q - z)w_{su}(v_n^{-1}(q))v_n'(v_n^{-1}(q))dq, \]
\[ I_2(z) = \int_{v_n(|\Omega|)}^{v_n(0)} K_h(z - q)v_n'(v_n^{-1}(q))w_{su}(v_n^{-1}(q))dq. \]
Using the Gaussian explicit form of \( K \) and integrating by parts, we obtain
\[ I_1(z) = \frac{h^2}{2} (\tilde{k}_h(v_n^{-1}(z)) + \int_{v_n(|\Omega|)}^{v_n(0)} K_h(z - q)f(q)dq), \]
(24)
with \( \tilde{k}_h \) given by (21), and
\[ f(q) = \frac{w_{su}(v_n^{-1}(q))}{(v_n'(v_n^{-1}(q)))^2} - \frac{w_{su}(v_n^{-1}(q))v_n''(v_n^{-1}(q))}{(v_n'(v_n^{-1}(q)))^3}. \]

Let us also introduce
\[ g(q) = \frac{w_{su}(v_n^{-1}(q))}{v_n'(v_n^{-1}(q))}. \]


On a new formulation of nonlocal image filters involving the relative rearrangement

By assumption, \( f \) and \( g \) are bounded in \([v_n(|\Omega|), v_n(0)]\) and by (P) they are also continuously differentiable in \((v_n(|\Omega|), v_n(0))\).

Consider the interval \( J_h = \{ q : |z - q| < \sqrt{h} \} \). By well known properties of the Gaussian kernel, we have

\[
\kappa(h) := \int_{J_h} K_h(z - q) dq < \int_{\mathbb{R}} K_h(q) dq = h\sqrt{\pi}, \tag{25}
\]

and

\[
K_h(z - q) \leq e^{-1/h} \quad \text{if} \quad q \in J_h^C = \{ q : |z - q| \geq \sqrt{h} \}. \tag{26}
\]

In particular, from (26) we get

\[
\left| \int_{J_h^C} K_h(z - q) f(q) dq \right| < O(h^\alpha) \quad \text{for any} \quad \alpha > 0. \tag{27}
\]

Taylor’s formula implies

\[
\int_{v_n(0)}^{v_n(|\Omega|)} K_h(z - q) f(q) dq = \int_{J_h} K_h(z - q)(f(z) + O(\sqrt{h})) dq + \int_{J_h^C} K_h(z - q) f(q) dq.
\]

Therefore, from (24), (25) and (27) we deduce

\[
I_1(z) = \frac{h^2}{2} \left( k(v_n^{-1}(z)) + f(v_n^{-1}(z))\kappa(h) + O(h^{3/2}) \right).
\]

Similarly,

\[
I_2(z) = \int_{v_n(0)}^{v_n(|\Omega|)} K_h(z - q) g(q) dq = \int_{J_h} K_h(z - q)(g(z) + O(\sqrt{h})) dq + \int_{J_h^C} K_h(z - q) g(q) dq
\]

\[
= g(v_n^{-1}(z))\kappa(h) + O(h^{3/2}).
\]

Then, the result follows from (23) substituting \( z \) by \( v_n(t) \). \( \square \)

**Remark 3** Theorem 4 may be extended to Lipschitz continuous decaying kernels satisfying the growth condition

\[
\mathcal{K}(s) \leq \frac{k_0}{1 + |s|^p}, \quad \text{for some} \quad p > 1.
\]

See [20] for details.

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