ON TOPOLOGICAL BRAUER CLASSES OVER 8-COMPLEXES
WITH PERIODS DIVISIBLE BY 4

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ABSTRACT. We determine the index of the topological Brauer class \( \beta_n \), the canonical generator of \( H^3(X; \mathbb{Z}) \), where \( X \) is the 8th skeleton of the Eilenberg-Mac Lane space \( K(\mathbb{Z}/n, 2) \), and \( 4|n \). This makes an important complement to a theorem on the topological period-index problem over 8-complexes, due to the author.

1. INTRODUCTION

We continue the study of topological period-index problem over 8-complexes started in [6], where we considered \( \text{sk}_8(K(\mathbb{Z}/n, 2)) \), the 8th skeleton of the Eilenberg-Mac Lane space \( K(\mathbb{Z}/n, 2) \). In this paper we study the topological period-index problem on this finite CW-complex, for \( n \) such that \( 4|n \).

A topological Brauer class over a topological space \( X \) is an element \( \alpha \in H^3(X; \mathbb{Z})_{\text{tor}} \), the torsion subgroup of \( H^3(X; \mathbb{Z}) \), or the topological Brauer group. This group is also defined as the group of Brauer equivalence classes of \( \mathbb{C} \)-Azumaya algebras over \( X \), with the group operation defined by tensor products.

The period of \( \alpha \), \( \text{per} (\alpha) \), is simply the order of \( \alpha \) in the group \( H^3(X; \mathbb{Z}) \), whereas its index \( \text{ind} (\alpha) \) is the greatest common divisor of degrees of the \( \mathbb{C} \)-Azumaya algebras contained in \( \alpha \).

The preceding definitions are motivated by their algebraic analogs. We refer to the introduction of [1] for the algebraic version of the definitions as well as the period-index conjecture, of which the topological counterpart was first considered by Antieau and Williams in [1]:

For a given Brauer class \( \alpha \) of a finite CW complexes \( X \), find the sharp lower bound of \( e \) such that

\[
\text{ind} (\alpha) | \text{per} (\alpha)^e
\]

holds for all finite CW complex \( X \) in \( C \) and all elements \( \alpha \in \text{Br}(X) \).

It is known, in both the algebraic and topological versions, that \( \text{per} (\alpha) | \text{ind} (\alpha) \), and the two integers have the same prime divisors. Therefore such an \( e \) always exists.

For further explanations of the preceding definitions and backgrounds on the topological period-index problem, see [1], [2] and [3]. All definitions and notations in this paper are consistent with those in [3]. In particular, let \( \beta_n \) be the canonical generator of \( H^3(K(\mathbb{Z}/n, \mathbb{Z})) \), i.e., the image of the identity class in \( H^2(K(\mathbb{Z}/n, \mathbb{Z})) \) under the Bockstein homomorphism. Without risk of ambiguity, we use \( \beta_n \) to denote the restriction of itself on \( H^3(\text{sk}_8(K(\mathbb{Z}/n, 2)), \mathbb{Z}) \). The bulk of this paper is devoted to proving the following.

2010 Mathematics Subject Classification. Primary 55S45; Secondary 55R20.

Key words and phrases. Brauer groups, twisted K-theory, period-index problems.
Proposition 1.1. When $4|n$, we have

$$\text{ind}(\beta_n) = \epsilon_3(n)n^3.$$ 

The expression $\epsilon_p(n)$ denotes the greatest common divisor of a prime $p$ and a positive integer $n$.

In [6], we obtained the following

Theorem 1.2 (Theorem 1.6, [6]). Let $X$ be a topological space of homotopy type of an 8-dimensional CW-complex, and let $\alpha \in H^3(X; \mathbb{Z})_{\text{tor}}$ be a topological Brauer class of period $n$. Then

$$\text{ind}(\alpha) = \epsilon_2(n)\epsilon_3(n)n^3.$$ 

In addition, if $X$ is the 8-th skeleton of $K(\mathbb{Z}/n, 2)$, and $\alpha$ is the restriction of the fundamental class $\beta_n \in H^3(K(\mathbb{Z}/n, 2), \mathbb{Z})$, then

$$\begin{cases} \text{ind}(\alpha) = \epsilon_2(n)\epsilon_3(n)n^3, & 4 \nmid n, \\ \epsilon_3(n)n^3|\text{ind}(\alpha), & 4|n. \end{cases}$$

In particular, the sharp lower bound of $\epsilon$ such that $\text{ind}(\alpha)|n^\epsilon$ for all $X$ and $\alpha$ is 4.

Proposition 1.1 extends Theorem 1.2 and gives the following

Theorem 1.3. Let $X$ be a topological space of homotopy type of an 8-dimensional CW-complex, and let $\alpha \in H^3(X; \mathbb{Z})_{\text{tor}}$ be a topological Brauer class of period $n$. Then

$$\text{ind}(\alpha) = \epsilon_2(n)\epsilon_3(n)n^3.$$ 

In addition, if $X$ is the 8-th skeleton of $K(\mathbb{Z}/n, 2)$ then

$$\begin{cases} \text{ind}(\beta_n) = \epsilon_3(n)n^3, & 4 \nmid n, \\ \text{ind}(\beta_n) = \epsilon_2(n)\epsilon_3(n)n^3, & 4|n. \end{cases}$$

In particular, the sharp lower bound of $\epsilon$ such that $\text{ind}(\alpha)|n^\epsilon$ for all $X$ and $\alpha$ is 4.

In [1, 2, 3] and [6] we considered the twisted Atiyah-Hirzebruch spectral sequence (1.1) of a CW-complex $X$ and a topological Brauer class $\alpha$, which we denote by $(\hat{E}_s^*, \hat{d}_s^*)$. The spectral sequence converges to the twisted K-theory $K_\alpha(X)$, and satisfies

$$\hat{E}_2^{s,t} \cong H^s(X; K^t(\text{pt})).$$

In other words, we have

$$\hat{E}_2^{s,t} \cong \begin{cases} H^s(X; \mathbb{Z}), & t \text{ even,} \\ 0, & t \text{ odd.} \end{cases}$$

The following result can be derived immediately from (1.1).

Theorem 1.4. Let $X$ be a finite CW-complex and let $\alpha \in \text{Br}(X)$. Consider $\hat{E}_s^{*,*}$, the twisted Atiyah-Hirzebruch spectral sequence with respect to $\alpha$ with differentials $\hat{d}_r^{*,t}$ with bi-degree $(r, -r + 1)$. In particular, $\hat{E}_2^{0,0} \cong H^0$, and any $\hat{E}_r^{0,0}$ with $r > 2$ is a subgroup of $\mathbb{Z}$ and therefore generated by a positive integer. The subgroup $\hat{E}_3^{0,0}$ (resp. $\hat{E}_\infty^{0,0}$) is generated by $\text{per}(\alpha)$ (resp. $\text{ind}(\alpha)$).
Following Theorem 1.4 it can be easily deduced from Theorem B of [1] and Theorem 1.2 that when $X = K(\mathbb{Z}/n, 2)$ and $\alpha$ is $\beta_n$, the differentials $\hat{d}_r^{0,0}$ is surjective for $r < 7$, and when $4 \nmid n$, also for $r = 7$. Theorem 1.3 however, provides the first known example of a $\hat{d}_r^{0,0}$ that is NOT surjective.

Proposition 1.5. Suppose $4|n$. In the twisted Atiyah-Hirzebruch spectral sequence $(\hat{E}_r^{*,*}, \hat{d}_r^{*,*})$ of the space $K(\mathbb{Z}/n, 2)$ and the class $\beta_n$, $\hat{d}_7^{0,0}$ has cokernal of order $2$.

This is easily deduced from the preceeding paragraph and Theorem 1.3 once we recall the cohomology of $K(\mathbb{Z}/n, 2)$, as in Section 2. See Figure 1.

We recall some more notations from [6]. Let $m, n$ be integers. Then $\mathbb{Z}/n$ is a closed normal subgroup of $SU_{mn}$ in the sense of the following monomorphism of Lie groups:

$$\mathbb{Z}/n \hookrightarrow SU_{mn} : t \mapsto e^{2\pi\sqrt{-1}t/n}\mathbf{I}_{mn},$$

where $\mathbf{I}_r$ is the identity matrix of degree $r$. We denote the quotient group by $P(n,mn)$, and consider its classifying space $BP(n,mn)$. It follows immediately from Bott’s periodicity theorem that we have

$$(1.4) \quad \pi_i(BP(n,mn)) \cong \begin{cases} \mathbb{Z}/n, & \text{if } i = 2, \\ \mathbb{Z}, & \text{if } 2 < i < 2mn + 1, \text{ and } i \text{ is even}, \\ 0, & \text{if } 0 < i < 2mn, \text{ and } i \text{ is odd}. \end{cases}$$

The space $BP(n,mn)$ plays an important role in the topological period-index problem. For a finite CW-complex $X$ and a topological Brauer class $\alpha$ of period $n$, there is an element $\alpha'$ in $H^2(X; \mathbb{Z}/n)$, which is sent to $\alpha$ by the Bockstein homomorphism.
Consider the following lifting diagram

\[
\begin{array}{ccc}
\mathcal{B} P(n, mn) & \rightarrow & K(\mathbb{Z}/2, n) \\
X & \xrightarrow{\alpha} & K(\mathbb{Z}/2, n)
\end{array}
\]

where the vertical arrow is the projection from \( \mathcal{B} P(n, mn) \) to the bottom stage of its Postnikov tower. We have the following

**Proposition 1.6.** [Proposition 4.3, [6]] Let \( X, \alpha \) be as above. Furthermore, suppose that \( H^2(X; \mathbb{Z}) = 0 \). Then \( \alpha \) is classified by an Azumaya algebra of degree \( mn \) if and only if the lifting in diagram (1.5) exists.

In Section 2 we recollect some facts on Eilenberg-Mac Lane spaces. Section 3 is a collection of technical lemmas on the cohomology of \( \mathcal{B} P(n, mn) \). In Section 4 we consider a \( k \)-invariant in the Postnikov decomposition of \( \mathcal{B} P(n, mn) \), for \( 4 \nmid n \) and suitable \( m \), and prove Proposition 1.1.

### 2. Recollection of Facts on Eilenberg-Mac Lane Spaces

All the facts recollected here are either well-known or easily deduced from [5].

The integral cohomology of \( K(\mathbb{Z}/n, 2) \) in degree \( \leq 8 \) is isomorphic to the following graded commutative ring:

\[
\mathbb{Z}[\beta_n, Q_n, R_n, \rho_n]/(n\beta_n, \epsilon_2(n)\beta_n^2, \epsilon_2(n)nQ_n, \epsilon_3(n)nR_n, \epsilon_3(n)\rho_n),
\]

where \( \deg(\beta_n) = 3, \deg(Q_n) = 5, \deg(R_n) = 7, \) and \( \deg(\rho_n) = 8 \). In other words, there is exactly one generator in each of the degrees 3, 5, 6, 7, which are, respectively, \( \beta_n, Q_n, \beta_n^2, R_n \), of order \( n, \epsilon_2(n)n, \epsilon_2(n), \epsilon_3(n)n \), and 2 generators in degree 8, \( \beta_n Q_n \) and \( \rho_n \), of order \( \epsilon_2(n) \) and \( \epsilon_3(n) \), respectively. (See (2.5) of [6].)

For \( n \geq 3 \), the ring \( H^*(K(\mathbb{Z}, n); \mathbb{Z}) \) in degree \( \leq n + 3 \) is isomorphic to the following graded rings:

\[
\begin{cases}
\mathbb{Z}[\iota_n, \Gamma_n]/(2\Gamma_n), n > 3, \\
\mathbb{Z}[\iota_3, \Gamma_3]/(2\Gamma_3, \Gamma_3 - \iota_3^2), n = 3,
\end{cases}
\]

where \( \iota_n \), of degree \( n \), is the so-called fundamental class, and \( \Gamma_n \), of degree \( n + 3 \), is a class of order 2. (See (2.1) of [6].)

**Proposition 2.1.** The \( \Gamma_n \in H^{n+3}(K(\mathbb{Z}, n); \mathbb{Z}) \) as above for all \( n \geq 3 \), stabilize to the same stable cohomology operation \( \text{Sq}_3^1 \in H^3(K(\mathbb{Z}); \mathbb{Z}) \) of order 2, where \( K(R) \) denotes the Eilenberg-Mac Lane spectrum associated to a unit ring \( R \). Moreover, the mod 2 reduction of \( \text{Sq}_3^1 \) is the well-understood Steenrod square \( \text{Sq}_3^1 \). This is to say that the following diagram in the homotopy category of spectra commutes:

\[
\begin{array}{ccc}
K(\mathbb{Z}) & \xrightarrow{\text{Sq}_3^1} & \Sigma^3 K(\mathbb{Z}) \\
\downarrow & & \downarrow \\
K(\mathbb{Z}/2) & \xrightarrow{\text{Sq}_3^1} & \Sigma^3 K(\mathbb{Z}/2)
\end{array}
\]

where the vertical arrows are the obvious ones.
For a proof see Lemma 2.1 of [6].

3. THE GROUP $H^7(\mathcal{B}P(n, mn); \mathbb{Z})$

In this section we collect some technical lemmas on the cohomology group $H^7(\mathcal{B}P(n, mn); \mathbb{Z})$.

Consider the Postnikov tower of $\mathcal{B}P(n, mn)$ where $\epsilon_2(n) | m$, $n > 1$:

\[
\begin{array}{ccc}
K(\mathbb{Z}, 4) & \longrightarrow & \mathcal{B}P(n, mn)[5] \cong K(\mathbb{Z}/n, 2) \times K(\mathbb{Z}, 4) \xrightarrow{\kappa_5} K(\mathbb{Z}, 7) \\
\downarrow & & \downarrow \\
K(\mathbb{Z}/n, 2) & \longrightarrow & K(\mathbb{Z}, 5)
\end{array}
\]

where $\kappa_3$ and $\kappa_5$ are the k-invariant of the space $\mathcal{B}P(n, m)$. The equation $\kappa_3 = 0$ follows from Proposition 4.11 of [6].

Consider the projection

\[
\mathcal{B}P(n, mn) \rightarrow \mathcal{B}P(n, mn)[6] \cong K(\mathbb{Z}/n, 2) \times K(\mathbb{Z}, 4),
\]

where $\epsilon_2(n) | m$. For future reference we take notes of the induced homomorphism between integral cohomology rings as follows:

\[
\beta_n \times 1 \mapsto x'_1, \quad R_n \times 1 \mapsto R_n(x'_1), \quad 1 \times \iota_4 \mapsto e'_2, \quad 1 \times \Gamma_4 \mapsto \text{Sq}^3_2(e'_2),
\]

where $x'_1$ and $e'_2$ are the additive generators of $H^3(\mathcal{B}P(n, mn); \mathbb{Z}) \cong \mathbb{Z}/n$ and $H^4(\mathcal{B}P(n, mn); \mathbb{Z}) \cong \mathbb{Z}$, respectively. Here $R_n$ is regarded as a cohomology operation. Recall Proposition 2.1 to make sense of the last expression.

The diagram (3.1) and (3.2) imply the following

**Lemma 3.1.** Suppose $\epsilon_2(n) | m$. Then we have

\[
H^7(\mathcal{B}P(n, mn); \mathbb{Z}) \cong H^7(K(\mathbb{Z}/n, 2) \times K(\mathbb{Z}, 4)) / (\kappa_5).
\]

In particular, the group $H^7(\mathcal{B}P(n, mn); \mathbb{Z})$ is generated by $R_n(x'_1)$, $x'_1 e'_2$, and $\text{Sq}^3_2(e'_2)$.

Consider the short exact sequence of Lie groups

\[
1 \rightarrow \mathbb{Z}/n \rightarrow SU_{mn} \rightarrow P(n, mn) \rightarrow 1,
\]

from which arises a fiber sequence

\[
\mathcal{B}SU_{mn} \rightarrow \mathcal{B}P(n, mn) \rightarrow K(\mathbb{Z}/n, 2)
\]

considered at the beginning of Section 6 of [6]. We denote the associated Serre spectral sequence by $(\hat{\mathcal{E}}^s_*, \hat{s} d_*)$.

For $k > 2$, it is well-known that

\[
H^*(\mathcal{B}U_2) = \mathbb{Z}[c_2, \cdots, c_k],
\]

where $c_i$ is the $i$th universal Chern class of degree $2i$.

**Lemma 3.2.** Suppose $\epsilon_2(n) | m$. The differential $\hat{s} d_5^{0, 4} = 0$. In particular, $c_2 \in sE_2^{0, 4}$ is a permanent cocycle.
Figure 2. Low dimensional differentials of the spectral sequence $S_{*}^{*,*}$, when $\epsilon_3(n)n|m$, $n > 1$. The dashed arrows represent trivial differentials.

Proof. Diagram 3.1 implies

$$H^5(BP(n, mn); \mathbb{Z}) \cong H^5(K(\mathbb{Z}/n, 2); \mathbb{Z}) \cong \mathbb{Z}/2\epsilon_2(n).$$

Hence we have $S_{3,4}^0 = 0$. There is no other non-trivial differential out of $S_{*}^{*,*}$ for obvious degree reasons, so $c_2$ is a permanent cocycle. □

Later we will need the following

Lemma 3.3 (Lemma 6.1, [6]). Suppose $\epsilon_2(n)n|m$. Recall that $H^3(K(\mathbb{Z}/n, 2); \mathbb{Z})$ is generated by an element $\beta_n$, and that $H^3(K(\mathbb{Z}/n, 2); \mathbb{Z}) \cong \mathbb{Z}/\epsilon_3(n)n$ is generated by $R_n$. In the spectral sequence $S_{*}^{*,*}$, we have $S_{3,6}^{0,6}(c_3) = 2\epsilon_2(\beta_n)$ with kernel generated by

$$\frac{n}{\epsilon_2(n)}c_3,$$

and

$$S_{d_7}^{0,6} \left( \frac{n}{\epsilon_2(n)}c_3 \right) = \frac{\epsilon_3(n)c_3(m/n)}{\epsilon_3(mn)}nR_n.$$

All the other differentials out of $S_{*}^{0,6}$ are trivial.

In particular $S_{d_3}^{0,6}$ is the only non-trivial differential out of $S_{*}^{0,6}$ when $\epsilon_3(n)n|m$.

See Figure 2 for the differentials mentioned in the lemmas above.

Lemma 3.4. Suppose $\epsilon_2(n)n|m$.

1. The element $Sq_3^2(e'_2)$ is a linear combination of $x_1'e_2'$ and $R_n(x_1')$.
2. The element $R_n(e'_2)$ is of order $\epsilon_3(n)n$.
3. The cardinality of the group $H^7(BP(n, mn); \mathbb{Z})$ is $\epsilon_2(n)\epsilon_3(n)n$.

Proof. As indicated in Figure 2 we have the exact sequence

$$0 \to S_{E^*}^{7,0} \to H^7(BP(n, mn); \mathbb{Z}) \to S_{E^*}^{3,4} \to 0,$$
where $S^7 E^\infty_{\infty}$ and $S^{3,4} E^\infty_{\infty}$ are generated by $R_n(x'_1)$ and $x'_1 e'_2$, respectively. Hence (1) and (2) follows. To verify the statement (3), it suffices to check the cardinality of $S^{3,4} E^\infty_{\infty}$ and $S^7 E^\infty_{\infty}$.

\[ \square \]

4. THE K-INARIANT

Consider the space $BP(n,mn)[6]$, the 6th level of the Postnikov tower of $BP(n,mn)$. We assume throughout this section that $\epsilon_2(n) | m$, $n > 1$, for when this holds, we have the homotopy equivalence

$$BP(n,mn)[6] = BP(n,mn)[5] \simeq K(\mathbb{Z}/n,2) \times K(\mathbb{Z},4).$$

(Proposition 4.11, [6].) We consider the following map

$$BP(n,mn) \rightarrow BP(n,mn)[6] \rightarrow K(\mathbb{Z},4)$$

such that both arrows above are the obvious projections, and we denote by $Y$ its homotopy fiber. Therefore we have a fiber sequence

\[ (4.1) \]

$$Y \rightarrow BP(n,mn) \rightarrow K(\mathbb{Z},4).$$

By construction the second arrow induces an isomorphism of homotopy groups

$$\pi_4(BP(n,mn)) \cong \pi_4(K(\mathbb{Z},4)).$$

This isomorphism lies in the long exact sequence of homotopy groups of the fiber sequence (4.1), from which, together with (1.4), we deduce

\[ (4.2) \]

$$\pi_i(Y) \cong \begin{cases} \mathbb{Z}/n, n = 2, \\
\mathbb{Z}, & \text{if } 6 \leq i < 2mn + 1, \text{ and } i \text{ is even,} \\
0, & \text{if } 0 < i < 2mn, \text{ and } i \text{ is odd, or } i = 4. \end{cases}$$

The fiber sequence (4.1) induces another one

\[ (4.3) \]

$$K(\mathbb{Z},3) \xrightarrow{g} Y \rightarrow BP(n,mn).$$

Consider the projection from $Y$ to the bottom level of its Postnikov tower

$$g : Y \rightarrow K(\mathbb{Z}/n,2).$$

Lemmas 4.1.

1. The induced homomorphisms

$$g^* : H^k(K(\mathbb{Z}/n,2); \mathbb{Z}) \rightarrow H^k(Y; \mathbb{Z})$$

are isomorphisms for $k \leq 5$.

2. The homomorphism $H^6(g; \mathbb{Z})$ is injective. Furthermore, we have

$$H^6(Y; \mathbb{Z}) = g^*(H^6(K(\mathbb{Z}/n,2); \mathbb{Z})) \oplus (\omega) \cong H^6(K(\mathbb{Z}/n,2); \mathbb{Z}) \oplus \mathbb{Z},$$

where $\omega$ generates the summand $\mathbb{Z}$.

3. The induced homomorphism

$$g^* : H^7(K(\mathbb{Z}/n,2); \mathbb{Z}) \rightarrow H^7(Y; \mathbb{Z})$$

is surjective.

4. $H^6(Y; \mathbb{Z}/2) = g^*(H^6(K(\mathbb{Z}/n,2); \mathbb{Z}/2)) + (\bar{\omega})$,

where an integral cohomology class with an overhead bar denotes its reduction in cohomology with coefficients in $\mathbb{Z}/2$. 
Proof. It follows from (4.2) that we have the following partial picture of its Postnikov tower

\[
\begin{align*}
Y[7] & \longrightarrow K(\mathbb{Z}, 9) \\
\downarrow & \\
K(\mathbb{Z}, 6) & \longrightarrow Y[6] \\
\downarrow & \\
Y[2] & = K(\mathbb{Z}/n, 2) \longrightarrow K(\mathbb{Z}, 7)
\end{align*}
\]

The statements (1), (2) and (3) follow from a simple observation on the fiber sequence

\[K(\mathbb{Z}, 6) \rightarrow Y[6] \rightarrow K(\mathbb{Z}/n, 2)\]

and the Serre spectral sequence associated to it. Finally, (4) follows from (2), (3), and Künneth formula.

The equations (2.1) and (2.2) together with the Künneth formula give us

\[H^7(K(\mathbb{Z}/n, 2) \times K(\mathbb{Z}, 4); \mathbb{Z}) = (R_n \times 1) \oplus (\beta_n \times \iota_4) \oplus (1 \times \Gamma_4) \cong \mathbb{Z}/\epsilon_3(n)n \oplus \mathbb{Z}/n \oplus \mathbb{Z}/2,\]

where \(R_n \times 1, \beta_n \times \iota_4\) and \(1 \times \Gamma_4\) generate the three summands, respectively.

When \(n\) is even, it follows from Theorem 6.8 of [6] that up to an invertible coefficient, we have

\[(4.5) \quad \kappa_5 \equiv \lambda R_n \times 1 + \lambda_2 \beta_n \times \iota_4 + 1 \times \Gamma_4 \mod 2 - \text{torsion},\]

an element in \(H^7(K(\mathbb{Z}/n, 2) \times K(\mathbb{Z}, 4); \mathbb{Z})\), where \(\lambda\) is invertible in \(\mathbb{Z}/\epsilon_3(n)n\).

To narrow down the choices of \(\kappa_5\), we have the following

**Lemma 4.2.** Suppose \(\epsilon_2(n)n|m, n > 1\). In \(H^7(K(\mathbb{Z}/n, 2) \times K(\mathbb{Z}, 4); \mathbb{Z})\) we have

\[\kappa_5 \equiv 1 \times \Gamma_4 \mod (R_n \times 1, \beta_n \times \iota_4).\]

**Proof.** Assume that the lemma is false. Since \(1 \times \Gamma_4\) is of order 2, we have

\[\kappa_5 \in (R_n \times 1, \beta_n \times \iota_4).\]

Therefore, we have

\[H^7(BP(n, mn); \mathbb{Z}) \cong H^7(K(\mathbb{Z}/n, 2) \times K(\mathbb{Z}, 4); \mathbb{Z})/(\kappa_5) \cong (R_n \times 1) \oplus (\beta_n \times \iota_4) \oplus (1 \times \Gamma_4)/(\kappa_5) \cong [(R_n \times 1) \oplus (\beta_n \times \iota_4)/(\kappa_5)] \oplus (1 \times \Gamma_4) \cong H^7(BP(n, mn); \mathbb{Z}) \oplus (1 \times \Gamma_4),\]

where the last step follows from (1) of Lemma 3.4 This is a contradiction since every group in sight is finite.

To prove Proposition 1.1 we are only interested in the case \(4|n\). Following Theorem 3 of [3] and Theorem 1.2 it suffices to consider the case \(n = 2^r, m = 2^{2r}\), where \(r > 1\). We assume this for the rest of this section.
Lemma 4.3. We have
\[ \kappa_5 \equiv \lambda_2 \beta_n \times \iota_4 \pmod{(R_n \times 1, 1 \times \Gamma_4)}, \]
where the coefficient \( \lambda_2 \) is invertible in \( \mathbb{Z}/n \).

Proof. We argue by contradiction. Suppose that \( \lambda_2 \) is not invertible in \( \mathbb{Z}/n \). Notice that, for our choice of \( m \) and \( n \), equation (4.5) implies
\[ \kappa_5 \equiv 2^{r-1} \lambda R_n \times 1 + \lambda_2 \beta_n \times \iota_4 + 1 \times \Gamma_4 \pmod{2 - \text{torsion}}. \]
Since \( \lambda_2 \) is not invertible, \( \kappa_5 \) has order less than \( 2^r \). On the other hand, it follows from (4.3) that the group \( H^7(K(\mathbb{Z}/n, 2) \times K(\mathbb{Z}, 4); \mathbb{Z}) \) has cardinality \( 2^{2r+1} \).

Therefore the group
\[ H^7(BP(n, mn); \mathbb{Z}) \cong H^7(K(\mathbb{Z}/n, 2) \times K(\mathbb{Z}, 4); \mathbb{Z})/\langle \kappa_5 \rangle \]
has cardinality greater than \( 2^{r+1} \), contradicting Lemma 3.4 \( \square \)

Notice that \( \kappa_5 \) is determined by the Postnikov tower merely up to multiplication by an invertible coefficient. By the choice we make of \( m \) and \( n \), this means that we are free to multiply \( \kappa_5 \) by any odd integer. Hence, we are enabled by Lemma 4.3 to normalize (4.5) by specializing \( \lambda_2 = 1 \):
\[ \kappa_5 = 2^{r-1} \lambda R_n \times 1 + \beta_n \times \iota_4 + 1 \times \Gamma_4 \pmod{2 - \text{torsion in } (R_n \times 1, 1 \times \Gamma_4)}, \]
where \( \lambda \) is odd. However, since \( R_n \) is of order \( 2^r \), and \( 2^{r-1} \equiv 2^{r-1} \lambda \pmod{2^r} \) for all odd integer \( \lambda \), the preceding equation becomes
\[ \kappa_5 = 2^{r-1} R_n \times 1 + \beta_n \times \iota_4 + 1 \times \Gamma_4 \pmod{2 - \text{torsion in } (R_n \times 1, 1 \times \Gamma_4)}. \]

Combining (4.6) \( \kappa_5 = 2^{r-1} R_n \times 1 + \beta_n \times \iota_4 + 1 \times \Gamma_4 \pmod{2 - \text{torsion in } (R_n \times 1)} \)

In other words, we have the following

Lemma 4.4. The abelian group \( H^7(BP(n, mn); \mathbb{Z}) \) is additively generated by \( R_n(x'_1) \), \( x'_1 e'_2 \) and \( Sq^3_{12}(e'_2) \), modulo the relation
\[ \mu R_n(x'_1) + x'_1 e'_2 + Sq^3_{12}(e'_2) = 0, \]
where \( \mu \) is either 0 or \( 2^{r-1} \). Moreover, only one of the two possible relations holds.

Proof. It remains to show the uniqueness. Indeed, if both relations hold, then we have
\[ 2^{r-1} R_n(x'_1) = 0, \]
and
\[ x'_1 e'_2 + Sq^3_{12}(e'_2) = 0. \]

Then the abelian group \( H^7(BP(n, mn); \mathbb{Z}) \) is generated by \( R_n(x'_1) \) and \( Sq^3_{12}(e'_2) \), whose orders divide \( 2^{r-1} \) and 2, respectively.

Therefore the cardinality of \( H^7(BP(n, mn); \mathbb{Z}) \) divides \( 2^r \), contradicting Lemma 3.4 \( \square \)

We turn to the hard work of determining \( \mu \). Consider the fiber sequence (4.3). Let \( (E^*_n(Z), d'_n^{*}) \) and \( (E^*_n(Z/2), d'_n^{*}) \) denote the associated cohomological Serre spectral sequences with coefficients in \( \mathbb{Z} \) and \( \mathbb{Z}/2 \), respectively.
We first consider the case with coefficients in \( \mathbb{Z}/2 \). This is easier since \( \mu \equiv 0 \) mod 2 whatever. We study the homomorphism

\[
h^* : H^6(Y; \mathbb{Z}/2) \to H^6(K(\mathbb{Z}, 3); \mathbb{Z}/2)
\]

with the spectral sequence \( E_\infty^*, \ast \) (\( \mathbb{Z}/2 \)), from which, with a little luck, we obtain enough information on the homomorphism

\[
h^* : H^6(Y; \mathbb{Z}) \to H^6(K(\mathbb{Z}, 3); \mathbb{Z})
\]

to determine a particular differential of the spectral sequence \( E_\infty^*(\mathbb{Z}) \), which in turn determines the coefficient \( \mu \).

As in Lemma 4.1 we use overhead bars to denote the mod 2 reductions of integral cohomology classes.

**Lemma 4.5.** The homomorphism

\[
h^* : H^6(Y; \mathbb{Z}/2) \to H^6(K(\mathbb{Z}, 3); \mathbb{Z}/2)
\]

is surjective. Furthermore, we have

\[
h^*(\bar{e}) = i_3^2.
\]

**Proof.** Consider the spectral sequence \( (E_\infty^*, \ast, d_\ast^*) \), refer to Figure 3 for the relevant differentials.

It follows from (1) of Lemma 4.1 that \( d_4^{0,3} \) is the first nontrivial differential out of the bidegree \((0, 3)\) and it is an isomorphism:

\[
d_4^{0,3} : E_4^{0,3}(\mathbb{Z}) \cong \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \cong E_4^{4,0}(\mathbb{Z})
\]

sending the generator \( i_3 \) of \( H^3(K(\mathbb{Z}, 3); \mathbb{Z}) \) to \( \pm e_2' \), the generator of \( H^4(BP(n, mn); \mathbb{Z}) \).

Passing to \( (E_\infty^*(\mathbb{Z}/2), d_\ast^*) \) as shown in Figure 3 it follows from (4.10) that

\[
i_3 \in H^3(K(\mathbb{Z}, 3); \mathbb{Z}/2) \cong E_2^{0,3}(\mathbb{Z}/2)
\]

is transgressive (cf. Section 6.2 of [7]) such that

\[
d_4^{0,3}(i_3) = e_2' \in H^4(BP(n, mn); \mathbb{Z}/2) \cong E_2^{4,0}(\mathbb{Z}/2) \cong E_4^{4,0}(\mathbb{Z}/2).
\]

Therefore, \( i_3^2 = \text{Sq}^3(i_3) \) is also transgressive and we have

\[
d_7^{0,6}(i_3^2) = d_7^{0,6}(\text{Sq}^3(i_3)) = \text{Sq}^3(d_4^{0,3}(i_3)) = \text{Sq}^3(e_2').
\]

It follows from (4.11) that we have

\[
d_4^{3,3}(x_1' i_3) = x_1' e_2' \in H^7(BP(n, mn); \mathbb{Z}/2) \cong E_2^{7,0}(\mathbb{Z}/2) \cong E_4^{7,0}(\mathbb{Z}/2).
\]

Taking the mod 2 reduction of \( \kappa_5 \) as in (4.7), we have \( \mu \equiv 0 \) mod 2 since \( r > 1 \). Therefore it follows from Lemma 4.4 that we have

\[
x_1' e_2' + \text{Sq}^3(e_2') = 0 \in H^7(BP(n, mn); \mathbb{Z}/2).
\]

This relation, together with (4.13), shows

\[
\text{Sq}^3(e_2') \equiv 0 \in E_5^{7,0}(\mathbb{Z}/2) \cong H^7(BP(n, mn); \mathbb{Z}/2)/(x_1' e_2').
\]

Then it follows from (4.12) that \( d_7^{0,6}(i_3^2) = 0 \). Hence, \( i_3^2 \) is a permanent cocycle, which proves that \( i_3^2 \) is in the image of \( h^* \), and in particular, that \( h^* \) is surjective in dimension 6 and with coefficients in \( \mathbb{Z}/2 \).

To verify (4.9), notice that \( g \) factors as

\[
Y \to BP(n, mn) \to K(\mathbb{Z}/n, 2)
\]
Figure 3. Differentials of the spectral sequence $E_2^{**}(\mathbb{Z}/2)$. The dashed arrow represents a trivial differential.

since the map $Y \to BP(n, mn)$ induces an isomorphism $\pi_2(Y) \cong \pi_2(BP(n, mn))$. In particular, it follows that

$$g \circ h : K(\mathbb{Z}, 3) \to Y \to K(\mathbb{Z}/n, 2)$$

is null homotopic. The equation (4.10) then follows from the surjectivity of $h^*$ and (4) of Lemma 4.1, which says

$$H^6(Y; \mathbb{Z}/2) = g^*(H^6(K(\mathbb{Z}/n, 2); \mathbb{Z}/2)) + (\bar{\omega}).$$

Passing to the integral case, we have the following

**Lemma 4.6.** The homomorphism

$$h^* : H^6(Y; \mathbb{Z}) \to H^6(K(\mathbb{Z}, 3); \mathbb{Z})$$

is surjective. Furthermore, $h^*(\omega) = i_3^2$.

**Proof.** Since $H^6(K(\mathbb{Z}, 3); \mathbb{Z}) \cong \mathbb{Z}/2$ has only two elements, 0 and $i_3^2$, we have either $h^*(\omega) = i_3^2$ or $h^*(\omega) = 0$. Lemma 4.1 shows that the latter is impossible.

We proceed to determine $\kappa_5$.

**Proposition 4.7.**

$$\kappa_5 = \beta_n \times i_4 + 1 \times \Gamma_4.$$

**Proof.** Consider the spectral sequence $(E_\infty^{**, *}(\mathbb{Z}), d_\infty^{**, *})$. (The picture of the differentials looks the same as Figure 3 but with all the overhead bars removed, and $\text{Sq}^3$ replaced by $\text{Sq}^3_2$.) For obvious degree reasons the only possibly nontrivial differentials hitting the bidegree $(7, 0)$ are $d_7^{0,6}$ and $d_4^{3,3}$. It follows from (4.10) and the Leibniz rule that

$$d_4^{3,3} (x_1^i t_3) = x_1^i e_2.$$
and
\[(4.15) \quad E^7_0(Z) = E^7_0(Z) \cong H^7(BP(n, mn); \mathbb{Z})/(x'_1 e'_2)\]
It follows from \((4.10)\) that \(\iota_3\) is transgressive. Therefore, so is \(Sq^3_3(e_3) = \iota_2^3\). (See Proposition 2.1.) Furthermore, since transgressions commute with stable cohomology operations, we have
\[(4.16) \quad d^7_7(e_3) = Sq^3_3(Z) = H^7(BP(n, mn); \mathbb{Z})/(x'_1 e'_2),\]
where the last step follows from \((4.10)\). On the other hand, it follows from Lemma 4.6 that \(\iota_2^3\) is a permanent cocycle. Therefore, \((4.15)\) and \((4.16)\) implies that \(Sq^3_3(Z) = \iota_2^3\). for some integer \(\nu\).

It follows from Lemma 4.4 that \(2x'_1 e'_2 = 0\). Therefore we only need to chose from \(\nu = 0\) and \(\nu = 1\). If \(\nu = 0\), then \(d^3_3(e'_2) = 0\). Applying Lemma 4.4 again, we see that
\[x'_1 e'_2 = 2^{-1} R_n(x'_1),\]
which implies that \(H^7(BP(n, mn); \mathbb{Z})\) is generated by \((R_n(x'_1))\), contradicting (2) and (3) of Lemma 3.4. It then follows that \(\nu = 1\) and we have
\[x'_1 e'_2 + d^3_3(e'_2) = 0 \in H^7(BP(n, mn); \mathbb{Z}).\]

By Lemma 4.4, the proof is complete.

\[\Box\]

\textbf{Proof of Proposition 1.1} Consider the following homotopy lifting problem
\[(4.18) \quad BP(n, mn) \quad \xrightarrow{sk_8(K(Z/n, 2))} K(Z/n, 2)\]
which is equivalent to
\[(4.19) \quad BP(n, mn)[7] \quad \xrightarrow{=} K(Z/n, 2)\]
It follows from \([6]\) that we have the following
\[(4.20) \quad BP(n, mn)[6] \quad \xrightarrow{f_5} BP(n, mn)[5] \simeq K(Z/n, 2) \times K(Z, 4) \xrightarrow{\kappa_5} K(Z, 7)\]
\[\xrightarrow{f_5=Id} K(Z/n, 2)\]
where the map $f_5$ is the obvious inclusion. Therefore $f_5^*$ annihilates all cohomology classes of $K(\mathbb{Z}, 4)$ in positive degrees, in particular, $\iota_4$ and $\Gamma_4$. Therefore, by Proposition 4.7 we have

$$f_5^*(\kappa_5) = f_5^*(\beta_n \times \iota_4 + 1 \times \Gamma_4) = 0,$$

and the dashed arrow exists. Proposition 1.1 then follows from Theorem 1.2 and Proposition 1.6.

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