Countable Strongly Annihilated Ideals in Commutative Rings

R. Mohamadian

Received: 11 October 2022 / Revised: 14 February 2023 / Accepted: 17 February 2023 / Published online: 17 April 2023 © The Author(s) under exclusive licence to Iranian Mathematical Society 2023

Abstract
In this paper, we introduce and study the concept of countable strongly annihilated ideal in commutative rings, in particular in rings of continuous functions. We show that a maximal ideal in \( C(X) \) is countable strongly annihilated if and only if it is a real maximal \( z^\circ \)-ideal. It turns out that \( X \) is an almost \( P \)-space if and only if countable strongly annihilated ideals and strongly divisible \( z \)-ideals coincide. We observe that an almost \( P \)-space \( X \) is Lindelöf if and only if every countable strongly annihilated ideal is fixed. We give a negative answer to a question raised by Gilmer and McAdam.

Keywords Countable strongly annihilated ideal · Real maximal ideal · Strongly divisible ideal · Almost \( P \)-space

Mathematics Subject Classification 13A18 · 54C40

1 Introduction
Throughout this paper, all rings are commutative with unity. Let \( R \) be a ring and \( a \in R \). The principal ideal generated by \( a \) denotes by \((a)\). By \( \text{rad}(R) \) we mean the prime radical of \( R \). Clearly, \( \text{rad}(R) \) is the set of all nilpotent elements of \( R \). If \( \text{rad}(R) = (0) \) then \( R \) is called a reduced ring. Elements of a ring \( R \) that are not zero divisors are called regular, and the set of all regular elements of \( R \) is denote by \( r(R) \). An ideal of \( R \) is called regular if it intersects \( r(R) \); otherwise, it is called nonregular. The classical ring of quotients of \( R \) denoted by \( q(R) \). For each \( S \subseteq R \) let \( P_S \) (resp. \( M_S \)) be the intersection of all minimal prime (resp. maximal) ideals of \( R \) containing \( S \). We use \( P_{\sigma} \) (resp. \( M_{\sigma} \)) instead of \( P_{\{a\}} \) (resp. \( M_{\{a\}} \)). A proper ideal \( I \) of a ring \( R \) is called a \( z^\circ \)-ideal

Communicated by Mohammad Reza Koushesh.

R. Mohamadian
mohamadian_r@scu.ac.ir

1 Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran
(resp. $z$-ideal) if for each $a \in I$, we have $P_a \subseteq I$ (resp. $M_a \subseteq I$). A proper ideal $I$ of a ring $R$ is called a $sz^0$-ideal (resp. $sz$-ideal) if for each $S \subseteq I$, we have $P_S \subseteq I$ (resp. $M_S \subseteq I$). If $I$ is an ideal of $R$, then $\operatorname{Min}(I)$ denotes the set of all prime ideals minimal over $I$ and if $a \in R$, then $I(a)$ denotes $a + I$ in ring of fraction $\frac{R}{I}$.

All topological spaces are completely regular Hausdorff. If $X$ is a space and $A \subseteq X$ then $A^\circ$ and $\overline{A}$ denote $\operatorname{int}(A)$ and $\operatorname{cl}(A)$ respectively. $C(X)$ is the ring of all continuous real valued functions on the space $X$ and $C^*_X(X)$ is the subring of $C(X)$ consisting of all bounded functions in $C(X)$. The space $X$ is called pseudocompact whenever $C^*_X(X) = C(X)$. For $f \in C(X)$, the zero-set of $f$ is the set $Z(f) = \{x \in X : f(x) = 0\}$. The set-theoretic complement of $Z(f)$ is denoted by $\operatorname{coz}(f)$. The support of a function $f \in C(X)$ is the $\operatorname{coz}(f)$. It is well known that $C(X)$ is a reduced ring. A topological space $X$ is said to be a $P$-space if $Z(f) = Z^0(f)$ for every $f \in C(X)$. The space $X$ is called an almost $P$-space if $Z(f) \neq \emptyset$ implies that $Z^0(f) \neq \emptyset$, for every $f \in C(X)$. It is well known that an ideal $I$ of $C(X)$ is $z^0$-ideal (resp. $z$-ideal) if $Z^0(f) = Z^0(g)$ (resp. $Z(f) = Z(g)$) $f \in I$ and $g \in C(X)$ implies that $g \in I$. For an ideal $I$ of $C(X)$, we write $Z[I]$ to designate the family of zero-sets $\{Z(f) : f \in I\}$. $\nu X$ is the Hewitt real compactification of $X$, $\beta X$ is the Stone–Čech compactification of $X$ and for any $p \in \beta X$, the maximal ideal $M_p$ (resp. the ideal $O_p$) is the set of all $f \in C(X)$ for which $p \in \operatorname{cl}_{\beta X}Z(f)$ (resp. $p \in \operatorname{int}_{\beta X}\operatorname{cl}_{\beta X}Z(f)$). For $p \in X$, we set $M_p = \{f \in C(X) : p \in Z(f)\}$ and $O_p = \{f \in C(X) : p \in Z^0(f)\}$. Clearly $M_p$ is a maximal ideal and hence it is a $z$-ideal and also $O_p$ is a $z^0$-ideal. A maximal ideal $M$ of $C(X)$ is called real if $\frac{C(X)}{M} \cong \mathbb{R}$ and it is well-known that $\nu X = \{p \in \beta X : M_p$ is a real maximal ideal$\}$. A space $X$ is called realcompact if $\nu X = X$. For more information about commutative rings, see [5], about general topology, see [12], about rings of continuous functions, see [15] and about $z$-ideals and $z^0$-ideals in $C(X)$ and commutative rings, see [1–3, 9–11, 22].

The paper is organized as follows. In Sect. 2, we introduce the concept of the countable strongly annihilated ideal in commutative rings and we investigate their relation to the strongly annihilated and weakly annihilated ideals. In Sect. 3, we will answer to a question raised by Gilmer and McAdam in [16]. In Sect. 4, we investigate the behavior of the countable strongly annihilated ideals in rings of continuous functions. It turns out that a maximal ideal $M$ of $C(X)$ is countable strongly annihilated if and only if it is a real maximal $z^0$-ideal.

## 2 Countable Strongly Annihilated Ideals

We start by recalling definition from [16]. An ideal $I$ of a ring $R$ is called weakly annihilated (resp. strongly annihilated) if for each $a \in I$ (resp. finite subset $\{a_1, \ldots, a_n\}$ of $I$) and $b \in R \setminus I$, there exists $c \in R$ such that $ca = 0$ (resp. $ca_i = 0$ for $i = 1, \ldots, n$) but $cb \neq 0$.

In Lemma 2.15 of [14], it is shown that in a reduced ring weakly annihilated ideals and $z^0$-ideals coincide and in Lemma 2.16 of the same reference it is also shown that strongly annihilated ideals and $sz^0$-ideals coincide. For details about the aforementioned concepts, see [14, 16, 17].
Definition 2.1 An ideal $I$ of a ring $R$ is called countable strongly annihilated if for each countable subset $\{a_1, a_2, \ldots\}$ of $I$ and $b \in R \setminus I$, there exists $c \in R$ such that $ca_n = 0$, for $n = 1, 2, \ldots$, but $cb \neq 0$.

Every countable strongly annihilated is strongly annihilated but not conversely, see Example 4.3. Every strongly annihilated is weakly annihilated, but not conversely, see Example 4.2 in [1]. For any ideal $I$ of $R$, $\text{Ann}(I)$ is countable strongly annihilated. Every minimal ideal of a ring is weakly annihilated if and only if countable strongly annihilated. Every element of a weakly annihilated ideal is zero divisor.

Remark 2.2 Every reduced principal ideal ring $R$ is a von Neumann regular ring if and only if every its ideal is countable strongly annihilated. To see this, first suppose that $R$ is a von Neumann regular ring and $I$ be an ideal of $R$. There is $a \in R$ such that $I = (a)$. In fact, there exists an idempotent $e \in R$ such that $I = \text{Ann}(e)$, that shows $I$ is countable strongly annihilated. The converse is clear by Corollary 1.14 of [10].

Remark 2.3 It is clear that the only weakly annihilated ideal of an integral domain is zero ideal, but the converse is false. For example, we consider the ring $\mathbb{Z}_4$.

In Question 5 of [17], Heinzer and Lantz raised the question: Is a maximal weakly annihilated ideal prime? It seems that this question is settled. However, here we give an affirmative answer to this question. Let us state the following easy, yet crucial, lemma.

Lemma 2.4 Let $I$ be a weakly annihilated ideal of a ring $R$. Then $(I : r)$ is a weakly annihilated ideal for each $r \notin I$.

Proof Assume that $a \in (I : r)$ and $b \notin (I : r)$. Hence $ar \in I$ and $br \notin I$. By hypothesis, there exists $c \in R$ such that $car = 0$ but $cbr \neq 0$. Take $d = cr$. This yields $da = 0$ and $db 
eq 0$. Thus $(I : r)$ is a weakly annihilated ideal. □

Proposition 2.5 Every maximal weakly annihilated ideal is prime.

Proof Let $I$ be a maximal weakly annihilated ideal of a ring $R$. We must show that $I$ is prime. Suppose that $ab \in I$ and $a \notin I$. Clearly, $I \subseteq (I : a) \neq R$ and $b \in (I : a)$. Now by maximality of $I$ and by Lemma 2.4, we conclude that $I = (I : a)$ and we are done. □

Similar to the proof of the above lemma and proposition, we can show that if $I$ is a countable strongly annihilated ideal then $(I : r)$ is a countable strongly annihilated ideal for each $r \notin I$ and every maximal countable strongly annihilated ideal is prime.

Let $M$ be a maximal ideal of a reduced ring $R$ and $R(x)$ means the localization of $R[x]$ at the prime ideal $P = M[x]$. If $I$ is a $z^\circ$-ideal of $R$ which contained in $M$, then $(I[x])_P$ may not be a $z^\circ$-ideal in $R(x)$. See Example 2.7. First we need the following lemma.

Lemma 2.6 Let $Q$, $P$ be two prime ideals in $R$, $Q \subseteq P$ and $R_P$ be the localization $R$ at $P$. If $Q_P$ is a weakly annihilated ideal in $R_P$, then $Q$ is a weakly annihilated ideal in $R$. 
**Proof** Suppose that \( a \in Q \) and \( b \in R \setminus Q \). Then \( \frac{a}{1} \in Q_P \) and \( \frac{b}{1} \in R_P \setminus Q_P \). By hypothesis, there is \( \frac{c}{d} \in R_P \) such that \( \frac{ac}{d} = \frac{0}{1} \) and \( \frac{db}{d} \neq \frac{0}{1} \). Hence, there is \( t_0 \notin P \) such that \( r at_0 = 0 \) and also \( rbt \neq 0 \), for every \( t \notin P \). Take \( d = rt_0 \), then \( da = 0 \) but \( db \neq 0 \) and we are done. \( \square \)

It is well known that if \( R \) is a reduced ring then \( R[x] \) is reduced and if \( P \) is a prime ideal of \( R \), then \( R_P \) is also a reduced ring.

**Example 2.7** Suppose that \( R \) is the same as the ring in Example 4.2 in [1]. It is shown that the ideal \( I \) is contained in a \( z^\circ \)-ideal, while there is no \( sz^\circ \)-ideal containing \( I \). Suppose \( A \) be the set of all \( z^\circ \)-ideals of \( R \) containing \( I \), which is not \( sz^\circ \)-ideal. By Zorn’s Lemma \( A \) has a maximal element \( J \). Now it can be easily seen that \( J \) is a prime \( z^\circ \)-ideal which is not a \( sz^\circ \)-ideal. Since \( J \) is not a \( sz^\circ \)-ideal, then by Proposition 3.9 of [1], we infer that \( J[x] \) is not a \( z^\circ \)-ideal in \( R[x] \). Now by the above lemma \( (J[x])_{M[x]} \) is not a \( z^\circ \)-ideal in \( R(x) \), where \( M \) is the maximal ideal of \( R \) which containing \( J \).

For \( f \in R[x] \), let \( C(f) \) denotes the set of all coefficients of \( f \).

**Proposition 2.8** The following statements are equivalent.

(a) \( I \) is a countable strongly annihilated ideal in \( R \).

(b) \( I[x] \) is a countable strongly annihilated ideal in \( R[x] \).

(c) \( I[[x]] \) is a countable strongly annihilated ideal in \( R[[x]] \).

**Proof** (\( a \Leftrightarrow b \)) Let \( S = \{f_n : n = 1, \ldots \} \) be a countable subset of \( I[x] \) and \( g \notin I[x] \). Hence \( C = \bigcup_{n=1}^{\infty} C(f_n) \) is a countable subset of \( I \) and there is also an element \( i_0 \) such that \( g_{i_0} \notin I \). By hypothesis, there exists \( r \in R \) such that \( rc = 0 \) and \( rg_{i_0} \neq 0 \). This means that \( rf_{i_0} = 0 \), for every \( n \), and \( rg \neq 0 \). The converse is straightforward. \( (a \Leftrightarrow c) \) Similar to implication \( (a \Leftrightarrow b) \). \( \square \)

**Proposition 2.9** The following statements are hold.

(a) If \( I \) is a countable strongly annihilated ideal in \( R \), then \( I_S \) is a countable strongly annihilated ideal in \( q(R) \).

(b) Let \( P \) be a nonregular ideal in \( R \). If \( P_S \) is a countable strongly annihilated ideal in \( q(R) \), then \( P \) is a countable strongly annihilated ideal in \( R \).

**Proof** It is straightforward. \( \square \)

An ideal \( I \) of a ring \( R \) is called \( \lambda - z^\circ \)-ideal whenever \( P_S \subseteq I \), for every \( S \subseteq I \) with \( |S| \leq \lambda \), where \( \lambda \) is a cardinal number, see [2]. It is obvious that every minimal prime ideal is a \( \lambda - z^\circ \)-ideal, where \( \lambda \leq |P| \). Refer to Example 4.20 to see a maximal ideal that is a \( c - z^\circ \)-ideal.

**Proposition 2.10** Every countable strongly annihilated ideal \( I \) of a reduced ring \( R \) is an \( \aleph_0 - z^\circ \)-ideal.

**Proof** Let \( S = \{a_n : n = 1, \ldots \} \) be a countable subset of \( I \) and \( b \in P_S \). If \( b \notin I \), then there is \( c \in R \) such that \( ca_n = 0 \), for any \( n \), and \( cb \neq 0 \). Since \( cb \notin \text{rad}(R) = (0) \), there is a minimal prime ideal \( Q \) such that \( cb \notin Q \). Now \( cS = 0 \in Q \) and \( c \notin Q \) conclude that \( S \subseteq Q \). This consequence that \( b \in Q \) which is a contradiction. \( \square \)
The converse of the above proposition is false. See Example 4.3.
Recall that a ring \( R \) satisfies property \( A \) if each finitely generated nonregular ideal has a nonzero annihilator. A ring \( R \) is said to have the annihilator condition (briefly a.c.) if for each finitely generated ideal \( I \) of \( R \) there exists an element \( b \in R \) with \( \text{Ann}(I) = \text{Ann}(b) \). If this element \( b \in R \) can be chosen in \( I \), then we say \( R \) satisfies the strong annihilator condition (briefly s.a.c.). For details, see [1, 18, 19, 21].

**Definition 2.11** We say that an ideal \( I \) of a ring \( R \) satisfies the countable strongly annihilator condition (briefly c.s.a.c.) if for any countable subset \( S \) of \( I \), there exists an element \( a \in I \) such that \( \text{Ann}(S) = \text{Ann}(a) \).

**Proposition 2.12** Let \( I \) be a weakly annihilated ideal satisfies the c.s.a.c., then \( I \) is countable strongly annihilated.

**Proof** Let \( S = \{a_n : n = 1, \ldots \} \) be a countable subset of \( I \) and \( b \notin I \). There is \( c \in I \) such that \( \text{Ann}(S) = \text{Ann}(c) \). By hypothesis, \( cd = 0 \) and \( db \neq 0 \), for some \( d \in R \). This implies that \( dS = 0 \) and we are through. \( \square \)

The converse is not true, in general. See the next example.

**Example 2.13** Let \( R = \mathbb{Z}[x, y] / (x^2, y^2) \) and suppose that \( J = (x, y)/(x^2, y^2) \). In Example 3.13 of [21] it is shown that \( J \) does not have c.s.a.c.. It is not hard to see that \( I(xy)I(f) = 0 \), for every \( I(f) \in J \) and \( I(xy)I(g) \neq 0 \), for every \( I(g) \notin J \). This shows that \( J \) is countable strongly annihilated.

**Proposition 2.14** Let \( I \) be an ideal of a Noetherian ring \( R \). Then
(a) \( I \) satisfies the c.s.a.c. if and only if satisfies the s.a.c.
(b) \( I \) is a strongly annihilated ideal if and only if countable strongly annihilated.

**Proof** (\( a \Rightarrow \)) It is obvious.
(\( a \Leftarrow \)) Let \( S = \{a_n : n = 1, \ldots \} \) be a countable subset of \( I \). There exists \( n \in \mathbb{N} \) such that \( (a_1, \ldots, a_n) = (a_1, \ldots, a_{n+1}) = \cdots \). Now by hypothesis there is \( c \in I \) such that \( \text{Ann}((a_1, \ldots, a_n)) = \text{Ann}(c) \). Since \( a_{n+1} = r_1a_1 + \cdots + r_n a_n \), where \( r_1, \ldots, r_n \in R \), we infer that \( \text{Ann}(S) = \text{Ann}(c) \) and this complete the proof. The proof of part (b) is similar to part (a). \( \square \)

Recall that a ring in which ideals are totally ordered by inclusion is called a chained ring. By Proposition 2.14 and Proposition 3.3 of [17] the proof of the next result is clear.

**Corollary 2.15** Let \( R \) be a Noetherian chained ring and \( M \) be the maximal ideal of \( R \). Then the following statements are equivalent.
(a) Every ideal in \( R \) is weakly annihilated.
(b) Every ideal in \( R \) is strongly annihilated.
(c) Every ideal in \( R \) is countable strongly annihilated.
(d) \( M \) consists of zero divisors.

A countable strongly annihilated ideal need not be an annihilator ideal. See Example 4.5.
Proposition 2.16 A countable generated ideal $I$ of a ring $R$ is countable strongly annihilated ideal if and only if it is a annihilator ideal.

Proof ($\Leftarrow$) It is clear.

($\Rightarrow$) Let $I = \langle a_1, a_2, \ldots \rangle$. If $b \notin I$, then there is $c_b \in R$ such that $c_b a_n = 0$, for any $n$, and $c_b \neq 0$. Suppose that $J$ is the ideal generated by $A = \{c_b : b \notin I\}$. It is obvious that $I \subseteq \text{Ann}(J)$. Now let $t \in \text{Ann}(J)$. If $t \notin I$, then there is $c_t \in R$ such that $c_t a_n = 0$ and $c_t t \neq 0$. Consequently, $c_t \in J$ and this implies that $c_t t = 0$, which is not true. Hence, $\text{Ann}(J) \subseteq I$ and thus $\text{Ann}(J) = I$. □

From [16] recall that an ideal $I$ of a ring $R$ is called universally contracted if for any extension ring $T$ of $R$, there is an ideal $J \subseteq T$ with $I = J \cap R$, this is equivalent to the condition that $IT \cap R = I$ for each unitary extension ring $T$ of $R$. The proof of the next result is similar to the proof of Proposition 3.3 of [16].

Proposition 2.17 Let $I$ be an ideal of the ring $R$ and consider the following conditions.

(a) $I$ is a countable strongly annihilated ideal in $R$.

(b) $I[x_1, x_2, \ldots]$ is universally contracted for each countable set \{x_1, x_2, \ldots\} of indeterminates over $R$.

(c) $I$ is universally contracted.

Then ($a \Rightarrow b \Rightarrow c$).

3 On a Question of Gilmer and McAdam

From [16] recall that an ideal $I$ of a ring $R$ is called universally proper if each finite subset of $I$ is annihilated by a nonzero element of $R$. Let $I$ be an ideal of $R$. If $\text{Ann}(I) \neq (0)$, then $I$ is universally proper, see Proposition 3.2 of [16]. By the same theorem, if a maximal ideal $M$ of $R$ is universally proper then $M$ is contracted from each simple ring extension of $R$.

In Question 4 of [16], Gilmer and McAdam raised the question: If $I$ is an ideal of $R$ such that $I$ is contracted from each simple ring extension of $R$, does it follow that $I$ is universally contracted? We found a negative to this question.

Example 3.1 Let $T = \mathbb{Z}_2[x, y, u, v] / (x^2, y^2, ux + vy - 1)$ and $I = (x^2, y^2, ux + vy - 1)$. Suppose that $\alpha = I(x), \beta = I(y), \gamma = I(u)$ and $\lambda = I(v)$. Then $T = \mathbb{Z}_2[\alpha, \beta, \gamma, \lambda]$, where $\alpha^2 = \beta^2 = 0$ and $\alpha \gamma + \beta \lambda = 1$. Now assume that $R = \mathbb{Z}_2[\alpha, \beta]$. Clearly, $T$ is a unitary extension ring of $R$. Note that $M = (\alpha, \beta)$ is a maximal ideal of $R$ and in Example 2.13 we show that $\text{Ann}(M) \neq (0)$. Hence, $M$ is a universally proper ideal of $R$ and so it is contracted from each simple ring extension of $R$. However, one can easily see that $MT = T$. This conclude that $M$ is not a contracted ideal from $T$ and so we are through.

4 Applications to $C(X)$

In this section we see that the countable strongly annihilated ideals in $C(X)$ are closely related to the real maximal ideals. As a matter of fact, every $z^\circ$-ideal is a countable
strongly annihilated ideal if and only if it is a real maximal ideal. An ideal \( I \) in a ring \( R \) is said to be strongly divisible if for every \( a_1, a_2, \ldots \) in \( I \) there exists \( c \in I \) and \( b_1, b_2, \ldots \) in \( R \) such that \( a_i = cb_i \), for \( i = 1, 2, \ldots \). We begin by the following lemma.

**Lemma 4.1** Every countable strongly annihilated ideal in \( C(X) \) is strongly divisible.

**Proof** Suppose that \( I \) is a countable strongly annihilated ideal. Since \( I \) is a \( z \)-ideal, it is enough to show that \( Z[I] \) is closed under countable intersection, by Lemma 4.1 of [6]. Hence, suppose that \( \{Z(f_n) : n \in \mathbb{N}\} \) be a subfamily of \( Z[I] \). Since \( I \) is a \( z \)-ideal, then \( f_n \in I \), for any \( n \in \mathbb{N} \) and we can also assume that \( |f_n| \leq 1 \). Define \( f = \sum_{n=1}^{\infty} \frac{|f_n(x)|}{2^n} \). Then \( f \in C(X) \) and \( \text{Ann}(f) = \bigcap_{n=1}^{\infty} \text{Ann}(f_n) \). If \( f \notin I \), then by hypothesis there is \( g \in C(X) \) such that \( gf_n = 0 \), for any \( n \in \mathbb{N} \) and \( gf \neq 0 \) which is a contradiction. \( \Box \)

The converse of the above lemma is not true. For example, we consider the maximal ideal \( M_0 \) in \( C(\mathbb{R}) \). Then \( M_0 \) is a real maximal and hence by Corollary 4.2 of [6] it is a strongly divisible ideal which is not a \( z^\circ \)-ideal. Let \( I \) be a countable generated \( z \)-ideal and \( Z[I] \) is closed under countable intersection, then it is well known that \( I = (e) \) for an idempotent \( e \), hence \( I \) is a countable strongly annihilated ideal.

**Proposition 4.2** A space \( X \) is an almost \( P \)-space if and only if countable strongly annihilated ideals and strongly divisible \( z \)-ideals coincide.

**Proof** \((\Rightarrow)\) Let \( I \) be a strongly divisible \( z \)-ideal and suppose that \( f_1, f_2, \ldots \) in \( I \) and \( g \notin I \). By hypothesis, there exists \( h \in I \) and \( g_1, g_2, \ldots \) in \( C(X) \) such that \( f_i = h g_i \), for any \( i = 1, 2, \ldots \). Since \( I \) is a \( z^\circ \)-ideal, we infer that \( \text{Ann}(h) \not\subset \text{Ann}(g) \) and hence there is \( k \in \text{Ann}(h) \) but \( k \notin \text{Ann}(g) \). Therefore \( k g \neq 0 \) and \( k f_i = 0 \), for any \( i = 1, 2, \ldots \). This shows that \( I \) is a countable strongly annihilated ideal. Now using the above lemma we are done.

\((\Leftarrow)\) For any \( x \in X \), the ideal \( M_x \) is a real maximal \( z \)-ideal and by hypothesis is a countable strongly annihilated ideal. Hence, \( M_x \) is a \( z^\circ \)-ideal and therefore \( X \) is an almost \( P \)-space. \( \Box \)

Recall that every minimal prime ideal is a \( \lambda - z^\circ \)-ideal, where \( \lambda \leq |P| \), but need not be countable strongly annihilated. See the next example.

**Example 4.3** We consider the ideal \( O_{\sigma} \) in \( C(\Sigma) \), where \( \Sigma \) is the space of 4M in [15]. By 4M.4, \( O_{\sigma} \) is a minimal prime \( z \)-ideal. We claim that \( O_{\sigma} \) is not countable strongly annihilated. By 4M.1, there is \( f \in C(\Sigma) \) such that \( Z(f) = \{\sigma\} \). If \( n \in \mathbb{N} \), then \( n \notin Z(f) \) and hence there are \( g_n, h_n \in C(\Sigma) \) such that \( n \in Z^\circ(g_n) \) and \( Z(f) \subseteq Z^\circ(h_n) \) with \( Z(g_n) \cap Z(h_n) = \emptyset \), for every \( n \in \mathbb{N} \). Note that \( h_n \in O_{\sigma} \), for every \( n \). Suppose that \( Z(k) = \bigcap_{n=1}^{\infty} Z(h_n) \), where \( k \in C(\Sigma) \). It is sufficient to show that \( k \notin O_{\sigma} \). Otherwise, \( \sigma \in Z^\circ(k) \) implies that there is a subset \( \emptyset \neq U \subseteq \mathbb{N} \), such that \( U \cup \{\sigma\} \subseteq \bigcap_{n=1}^{\infty} Z(h_n) \). Hence, there is a \( i_0 \in \mathbb{N} \) such that \( i_0 \in Z(h_n) \), for any \( n \). This conclude that \( Z(g_{i_0}) \cap Z(h_{i_0}) \neq \emptyset \) which is not true. Therefore, by Lemma 4.1, \( O_{\sigma} \) is not a countable strongly annihilated ideal.

**Proposition 4.4** A maximal ideal \( M \) of \( C(X) \) is countable strongly annihilated if and only if it is a \( z^\circ \)-ideal and real maximal ideal.
Proof ($\Rightarrow$) Since $M$ is a $z$-ideal, then similar to the proof of Lemma 4.1, $Z[M]$ is closed under countable intersection. Now by Theorem 5.14 of [15], $M$ is real.

($\Leftarrow$) Let $f_n \in M$, for any $n \in \mathbb{N}$ and $g \notin M$. Define $f(x) = \sum_{n=1}^{\infty} \frac{|f(x_n)|}{2^n}$. Then $f \in C(X)$ and $\text{Ann}(f) = \bigcap_{n=1}^{\infty} \text{Ann}(f_n)$. Since $M$ is real, by Theorem 5.14 of [15], we infer that $f \in M$. Now by hypothesis, there is $h \in C(X)$ such that $hf = 0$ and $hg \neq 0$. Hence, $hf_n = 0$, for any $n \in \mathbb{N}$ and we are done. \qed

Example 4.5 We consider the ideal $M_s$ of the space $S$ of $4N$ in [15]. By Proposition 4.4, $M_s$ is countable strongly annihilated. Now let there is a nonzero ideal $I$ in $C(S)$ such that $M_s = \text{Ann}(I)$. Assume that $0 \neq f \in I$. Let $s \neq x_\alpha \in S$ an arbitrary element of $S$. We define $f_\alpha(x) = 0$ if $x \neq x_\alpha$ and $f_\alpha(x_\alpha) = 1$. Then $f_\alpha \in C(S)$ and $s \in Z(f_\alpha)$ implies that $f_\alpha \in M_s$, for any $\alpha$. Hence $ff_\alpha = 0$, for any $\alpha$. But $f_\alpha(x_\alpha) = 1$ conclude that $f(x_\alpha) = 0$, for any $\alpha$. This shows that $S - \{s\} \subseteq Z(f)$ and, therefore, we have $S - \{s\} = Z(f)$. Hence, $coz(f) = \{s\}$, that is, $coz(f)$ is an open set which contains $s$. Therefore, by definition of topology on $S$, the zero set $Z(f)$ must be a countable set, which implies that $S$ is countable and this is a contradiction.

Corollary 4.6 Let $X$ be a pseudocompact space. Every maximal ideal of $C(X)$ is countable strongly annihilated if and only if it is a $z^\circ$-ideal.

Proof It is trivial, by Theorem 5.8 of [15] and Proposition 4.4. \qed

The condition of maximality in the above result is necessary. See the next example.

Example 4.7 Let $X^* = X \cup \{\alpha\}$ be the one-point compactification of $X$, which $X$ is a uncountable discrete space. The ideal $O_\alpha$ is a $s z^\circ$-ideal but not maximal. We define $f_n \in C(X^*)$ such that $Z(f_n) = X^* - \{x_n\}$, for any $x_n \in X$. Note that $x_n \neq x_m$, for $n \neq m$. Clearly, $f_n \in O_\alpha$, for any $n$. Suppose that $Z(f) = \bigcap_{n=1}^{\infty} Z(f_n)$, then $Z(f) = X^* - \{x_1, x_2, \ldots\}$. If $\alpha \in Z^\circ(f)$, then there is a finite subset $F$ of $X$ such that $\{\alpha\} \cup (X - F) \subseteq Z(f)$. This shows that $\{x_1, x_2, \ldots\} \subseteq\text{F}$ which is not true. Therefore, $f \notin O_\alpha$ and so $O_\alpha$ is not countable strongly annihilated.

Proposition 4.8 Let $I$ be a $z$-ideal in $C(X)$ and $M$ be a maximal ideal containing $I$. Then the following statements are equivalent.

(a) $\frac{M}{T}$ is a $z^\circ$-ideal in $\frac{C(X)}{T}$.

(b) $\frac{M}{T}$ is a nonregular ideal in $\frac{C(X)}{T}$.

(c) $\frac{M}{T}$ is a universally proper ideal in $\frac{C(X)}{T}$.

(d) $M = \bigcup_{P \in \text{Min}(I)} P$.

Proof (a $\Rightarrow$ b) It is clear.

(b $\Rightarrow$ c) We use part (3) of Theorem 3.2 in [16]. Let $I(f_1), \ldots, I(f_n)$ be a finite subset of $\frac{M}{T}$. Then by hypothesis $I(f_1^2 + \cdots + f_n^2)$ is a zero divisor of $\frac{C(X)}{T}$, thus there exists $I(g) \neq I$ such that $I((gf_1)^2 + \cdots + (gf_n)^2) = I$. Therefore, $(gf_1)^2 + \cdots + (gf_n)^2 \in I$ and since $I$ is a $z$-ideal then $gf_i \in I$, and hence $I(gf_i) = I$, for every $1 \leq i \leq n$. This complete the proof.

(b $\Rightarrow$ c) It is clear by part (3) of Theorem 3.2 in [16].
(b ⇒ a) Since $I$ is semiprime, then $\frac{C(X)}{I}$ is a reduced ring and it also satisfies the property $A$, by Lemma 4.1 of [4]. Now Lemma 1.22 of [10] shows that $\frac{M}{I}$ is a $z^\circ$-ideal in $\frac{C(X)}{I}$.

(a ⇔ d) It is clear by Theorem 4.2 of [4].

In Lemma 4.1 of [4], it is shown that if $I$ is a semiprime ideal of $C(X)$ then the factor ring $\frac{C(X)}{I}$ satisfies the property $A$. However, in the next example we show that the converse is not true.

**Example 4.9** Recall that every ideal in $\frac{C(\Sigma)}{I}$ is absolutely convex. Now let $f \in C(\Sigma)$ and $I = (f)$ such that $Z(f) = \{\sigma\}$. Since $Z(f)$ is not an open set, then $I$ is not a semiprime ideal. We claim that $\frac{C(\Sigma)}{I}$ is a $z^\circ$-ideal. Assume that $J$ be an ideal of $\frac{C(\Sigma)}{I}$ consisting of zero divisors generated by finitely many elements $I(f_1), \ldots, I(f_n)$, where $f_1, \ldots, f_n \in I$. Hence, $I(|f_i|) = I$ for every $1 \leq i \leq n$ and so there exists $f(|g|) \neq I$ such that $I(|g|f_1 + \cdots + |g|f_n) = I$. This implies that $|g|f_1 + \cdots + |g|f_n \in I$. Since $|g|f_i \leq |g|f_1 + \cdots + |g|f_n$, we infer that $|g|f_i \in I$, for every $1 \leq i \leq n$ and so we are through.

In the next result, $q(X)$ denotes the classical ring of quotients $C(X)$.

**Proposition 4.10** ([4], Corollary 5.5) The following statements are equivalent.

(a) Every maximal ideal of $q(X)$ is real.
(b) $X$ is a pseudocompact almost $P$-space.
(c) The set of all real maximal ideals of $C(X)$ coincides with the set of all maximal $z^\circ$-ideals of $C(X)$.

In the following proposition, we see another equivalent condition for pseudocompact almost $P$-spaces in terms of countable strongly annihilated ideals.

**Proposition 4.11** A space $X$ is a pseudocompact almost $P$-space if and only if every maximal ideal of $C(X)$ is countable strongly annihilated.

**Proof** ($\Rightarrow$) Let $M$ be a maximal ideal of $C(X)$. Since $X$ is an almost $P$-space, then $M$ is $z^\circ$-ideal, by Theorem 2.14 of [9]. Also because $X$ is pseudocompact, then by Corollary 4.6, we conclude that $M$ is countable strongly annihilated.

($\Leftarrow$) By Proposition 4.4, every maximal ideal is a $z^\circ$-ideal and real maximal, hence by Theorem 2.14 of [9], $X$ is an almost $P$-space and by Theorem 5.8 of [15], $X$ is a pseudocompact space. 

**Remark 4.12** (a) Let $X$ be a real compact almost $P$-space. Then every maximal ideal of $C(X)$ is fixed if and only if it is a countable strongly annihilated ideal.
(b) Let $X$ be a compact almost $P$-space. Then every maximal ideal of $C(X)$ is countable strongly annihilated.

**Remark 4.13** If every ideal of $C(X)$ is countable strongly annihilated then $X$ is a $P$-space. The converse of this fact is not true. For example, the space $\mathbb{N}$ is a $P$-space and since it is not pseudocompact, then by Proposition 4.11, there is a maximal ideal which is not countable strongly annihilated.
Proposition 4.14  The following statements are equivalent for an ideal $I$ of a $P$-space $X$.

(a) $I$ is a countable strongly annihilated ideal.
(b) $Z[I]$ closed under countable intersection.
(c) $I$ satisfies in c.s.a.c..

Proof  ($a \Rightarrow b$) It is clear by Lemma 4.1.

($b \Rightarrow c$) Let $S = \{f_n : n = 1, \ldots\}$ be a countable subset of $I$. There is $f \in C(X)$ such that $\text{Ann}(f) = \bigcap_{n=1}^{\infty} \text{Ann}(f_n)$ and $\bigcap_{n=1}^{\infty} Z(f_n) = Z(f) \in Z[I]$. Since $I$ is a $z$-ideal, then $f \in I$ and we are through.

($c \Rightarrow a$) Let $S = \{f_n : n = 1, \ldots\}$ be a countable subset of $I$ and $g \not\in I$. Suppose that $f \in I$ in which $\text{Ann}(f) = \bigcap_{n=1}^{\infty} \text{Ann}(f_n)$ and $\bigcap_{n=1}^{\infty} Z(f_n) = Z(f) \in Z[I]$. We define $h(x) = 1$ if $x \in Z(f)$ and $h(x) = 0$ if $x \in \text{coz}(f)$. Then $h$ is a continuous function on $X$. Clearly, $hf = 0$ and hence $hf_n = 0$, for every $n \in \mathbb{N}$. It is sufficient to show that $hg \neq 0$. Otherwise, $Z(f) \subseteq \text{coz}(h) \subseteq Z(g)$ implies that $Z(f) \subseteq Z^\circ(g)$. By 1D.1 of [15] there is $k \in C(X)$ such that $g = fk$ which conclude that $g \in I$ and a contradiction. \hfill $\square$

Corollary 4.15  The following statements are equivalent for a $z^\circ$-ideal $I$ of $C(X)$.

(a) $I$ is a countable strongly annihilated ideal.
(b) $Z[I]$ closed under countable intersection.
(c) $I$ satisfies in c.s.a.c.

Proof  It is similar to the proof of Proposition 4.14. \hfill $\square$

In Theorem 4.4 of [6] it is shown that $X$ is Lindelöf if and only if every strongly divisible ideal is fixed. If $X$ is an almost $P$-space, then the same result holds for countable strongly annihilated ideals, see the next proposition.

Proposition 4.16  An almost $P$-space $X$ is Lindelöf if and only if every countable strongly annihilated ideal is fixed.

Proof  ($\Rightarrow$) It is hold without assuming that the space is an almost $P$-space. Suppose that $X$ is Lindelöf and $I$ is a countable strongly annihilated ideal. In proving Lemma 4.1, we show that $Z[I]$ has the countable intersection property and hence $Z[I]$ is fixed, i.e., $I$ is fixed.

($\Leftarrow$) It is similar to the proof of Theorem 4.4 of [6]. Just to point out that since $X$ is an almost $P$-space, every $z$-ideal is a $z^\circ$-ideal and we use the above corollary. \hfill $\square$

It is clear that every real maximal ideal of $C(X)$ satisfy in c.s.a.c.. The proof of the next result is clear by Propositions 4.4 and 4.14.

Proposition 4.17  Let $R$ be a reduced ring with s.a.c.. For a maximal ideal $M$ of $R$, the following conditions are equivalent.

(a) $M$ is a universally contracted ideal.
(b) $M$ is a $sz^\circ$-ideal.
(c) $M$ is a $z^\circ$-ideal.
(d) $M$ is a nonregular ideal.
(e) Each finite subset of $M$ is annihilated by a nonzero element of $R$.
(f) $M[\alpha] \neq R[\alpha]$ for each simple ring extension $R[\alpha]$ of $R$.
(g) $M$ is a contracted proper ideal.
(h) $M$ is contracted from each simple ring extension of $R$.
(i) $M[\alpha]$ is a universally contracted ideal in $R[\alpha]$.
(j) $M[\alpha]$ is a $sz^\circ$-ideal in $R[\alpha]$.
(k) $M[\alpha]$ is a $z^\circ$-ideal in $R[\alpha]$.

In case $R = C(X)$ and $M$ is a maximal ideal of $C(X)$, the following statement is equivalent to the above statements.

(l) $M$ is a countable strongly annihilated ideal.

In case $R = C(X)$ and $X$ is a finite space, the following statements are equivalent to the above statements.

(m) $M[\alpha]$ is a countable strongly annihilated in $R[\alpha]$.
(n) $M[\alpha]$ is a $z^\circ$-ideal in $R[\alpha]$.

Proof

(a $\Leftrightarrow$ b) It is clear by Theorem 3.5 in [16].
(b $\Rightarrow$ c) It is clear by Proposition 1.2 in [16].
(c $\Rightarrow$ d) It is clear.
(d $\Rightarrow$ e) It is clear for $R$ satisfy the s.a.c..
(e $\Leftrightarrow$ f) and (f $\Leftrightarrow$ g) They are clear by Proposition 3.2 in [16].
(f $\Rightarrow$ h) Let $R[\alpha]$ be a simple ring extension of $R$. Clearly $M[\alpha]$ is an ideal of $R[\alpha]$. Hence $M[\alpha] \cap R$ is an ideal of $R$ which contains $M$. If $M[\alpha] \cap R = R$, then $R[\alpha] = M[\alpha]$ which is a contradiction. Therefore, $M[\alpha] \cap R = M$ by maximality of $M$ and we are done.
(h $\Rightarrow$ f) On the contrary, assume that there exists $\alpha$ such that $M[\alpha] = R[\alpha]$. We claim that $M$ is not contracted from $R[\alpha]$. Suppose that there is an ideal $I$ of $R[\alpha]$ such that $M = I \cap R$. Clearly, $M[\alpha] = R[\alpha] \subseteq I$. Hence, $I = R[\alpha]$ which implies that $M = R$ and it is a contradiction.
(e $\Rightarrow$ d) It is clear.
(d $\Rightarrow$ c) It is clear by Corollary 1.22 in [10].
(c $\Rightarrow$ b) It is obvious by Proposition 2.8 in [1].
(i $\Leftrightarrow$ j), (j $\Leftrightarrow$ k) and (k $\Leftrightarrow$ a) See Theorem 3.5 in [16].
(l $\Rightarrow$ c) It is obvious.
(c $\Rightarrow$ l) It is clear by Proposition 4.4.
(m $\Leftrightarrow$ l) In this case $C(X)$ is a Noetherian ring. Now by Corollary 3.16 and Proposition 3.19 in [2] we are done.
(n $\Leftrightarrow$ l) It is similar to the above item and by Corollary 3.16 and Proposition 3.20 in [2] we are done.

Corollary 4.18 Let $X$ be a compact space. A maximal ideal $M$ of $C(X)$ is $z^\circ$-ideal if and only if it is countable strongly annihilated.

Proof In this case by Theorem 4.11 of [15] every maximal ideal in $C(X)$ is of the form $M_p$, for a $p \in X$. Now by implication ($l \Leftrightarrow c$) of Proposition 4.17 we are done.

Proposition 4.19 The following statements are equivalent.
(a) $X$ is an almost $P$-space.
(b) Every $M_x$ is an $\aleph_0 - \varepsilon^0$-ideal.
(c) Every $M_x$ is a countable strongly annihilated ideal.

**Proof** $(a \Rightarrow b)$ Let $S = \{f_n : n = 1, \ldots\}$ be a countable subset of $M_x$ and suppose that $f \in M_x$ in which $\text{Ann}(f) = \bigcap_{n=1}^{\infty} \text{Ann}(f_n)$. We claim that $P_S \subseteq P_f$. Let $g \in P_S$ and $f \in Q$, where $Q$ is a minimal prime ideal of $C(X)$. Then there is $h \notin Q$ such that $hf = 0$ and hence $h f_n = 0$, for every $n \in \mathbb{N}$. This shows that $f_n \in Q$, for every $n$, and consequence that $g \in Q$. Therefore, $g \in P_f$. Since $M_x$ is a $\varepsilon^0$-ideal, we infer that $P_S \subseteq P_f \subseteq M_x$ which complete the proof.

$(b \Rightarrow c)$ Let $S = \{f_n : n = 1, \ldots\}$ be a countable subset of $M_x$ and $g \notin M_x$. Suppose that $f \in M_x$ in which $\text{Ann}(f) = \bigcap_{n=1}^{\infty} \text{Ann}(f_n)$ then $P_f \subseteq M_x$ and hence $g \notin P_f$. Therefore, there is a minimal prime ideal $Q$ such that $f \in Q$ but $g \notin Q$. Hence, there is $h \notin Q$ such that $hf = 0$ and $hg \neq 0$ and we are done.

$(c \Rightarrow a)$ It is clear by Theorem 2.14 of $[9]$. \hfill $\square$

**Example 4.20** We consider the space $\Sigma$ of $4$ in $[15]$. Let $f \in C(\Sigma)$ with $Z(f) = \{\sigma\}$. Since $f$ is a regular element, by Corollary 2.6 in $[8]$, then every element of $C(\Sigma)/I$ is either a unit or a zero divisor, where $I = (f)$. Hence, the maximal ideal $M_{\Sigma}$ is a nonregular ideal. One can easily see that every prime ideal containing $f$ is contained in $M_{\Sigma}$. This implies that, for every subset $S$ of $M_{\Sigma}$, the intersection of all minimal prime ideals containing $S$ is a subset of $M_{\Sigma}$. Since $\vert M_{\Sigma} \vert = c$, we conclude that $M_{\Sigma}$ is a $c - \varepsilon^0$-ideal.

**Remark 4.21** (a) Recall that $C_F(X)$ is the socle of $C(X)$ and it is well-known that $C_F(X) = \{f \in C(X) : \text{coz}(f) \text{ is finite}\}$, see $[20]$. $C_F(X)$ is always a $\varepsilon^0$-ideal but it is not necessarily a countable strongly annihilated ideal, even if $X$ is a $P$-space. For example, let $f_n \in C(\mathbb{N})$ such that $\text{coz}(f_n) = \{n\}$, for any $n \in \mathbb{N}$. Then $f_n \in C_F(X)$, but $\mathbb{N} = \text{coz}(f) = \bigcup_{n=1}^{\infty} \text{coz}(f_n) \notin C_F(X)$.

(b) $SC_F(X) = \{f \in C(X) : \text{coz}(f) \text{ is countable}\}$ is called the super socle of $C(X)$, the reader refer to $[13]$ for details. $SC_F(X)$ is always a $\varepsilon$-ideal and it is clear that it closed under countable intersection. If $X$ is a weak $P$-space (a space which every countable set is closed) then $SC_F(X)$ is a $\varepsilon^0$-ideal and so it is a countable strongly annihilated ideal.

(c) If $SC_F(X)$ is a $\varepsilon^0$-ideal, then it is not necessarily $X$ is a weak $P$-space. For example, let $X^* = X \cup \{\alpha\}$ be the one-point compactification of $X$, which $X$ is a uncountable discrete space. Obviously, $X^*$ is not a weak $P$-space. On the other hand if $A$ is an infinite countable subset of $X^*$, then $\text{cl}(A) = A \cup \{\alpha\}$ whence we conclude that $SC_F(X)$ is a $\varepsilon^0$-ideal.

(d) One can easily see that $SC_F(X)$ is weakly annihilated if and only if it is a countable strongly annihilated ideal.

(e) It is easy to see that a discrete space $X$ is finite if and only if $C_F(X)$ is a countable strongly annihilated ideal.

(f) Let $X$ be a countably compact space and every $\sigma$-compact subset of $X$ is closed. Then $C_K(X) = \{f \in C(X) : \overline{\text{coz}(f)} \text{ is compact}\}$ is a countable strongly annihilated ideal. To see this first note that $C_K(X)$ is always a $\varepsilon^0$-ideal. Now suppose that $S = \{f_n : n = 1, \ldots\}$ be a countable subset of $C_K(X)$ and let
\[ \text{coz}(f) = \bigcup_{n=1}^{\infty} \text{coz}(f_n). \] 
One can easily see that \( \text{coz}(f) = \bigcup_{n=1}^{\infty} (\text{coz}(f_n)). \) 
Hence, \( \text{coz}(f) \) is \( \sigma \)-compact and so it is Lindelöf. Furthermore, \( \text{coz}(f) \) is also a countably compact. Therefore, it is a compact subset of \( X \) and consequence that \( f \in C_K(X) \) and we are trough.

Acknowledgements The author is grateful to Dr. A. Azarang for his advice during the preparation of this paper. Also the author is grateful to the Research Council of Shahid Chamran University of Ahvaz financial support (GN:SCU.MM401.648).

References

1. Aliabad, A.R., Mohamadian, R.: On \( sz^\circ \)-ideals in polynomial rings. Commun. Algebra 39, 701–717 (2011)
2. Aliabad, A.R., Mohamadian, R.: On \( z \)-ideals and \( z^\circ \)-ideals of power series rings. J. Math. Ext. 7, 93–108 (2013)
3. Aliabad, A.R., Mohamadian, R.: Prime \( z \)-ideal rings (\( p_z \)-rings). Bull. Iran. Math. Soc. 48, 1177–1192 (2022)
4. Arjmandnezhad, Z., Azarpanah, F., Hesari, A.A., Salehi, A.R.: Characterizations of maximal \( z^\circ \)-ideals of \( C(X) \) and real maximal ideals of \( q(X) \). Quaest. Math. 45, 1–13 (2021)
5. Atiyah, M.F., Macdonald, I.G.: Introduction to Commutative Algebra. Addison-Wesley, Reading (1969)
6. Azarpanah, F.: Algebraic properties of some compact spaces. Real Anal. Exch. 25, 317–328 (2000)
7. Azarpanah F.: On almost \( P \)-spaces. Far East J. Math. Soc. Spec. Vol. 2000, 121–132 (2000)
8. Azarpanah, F., Esmaeilvandi, D., Salehi, A.R.: Depth of ideals of \( C(X) \). J. Algebra 528, 474–496 (2019)
9. Azarpanah, F., Karamzadeh, O.A.S., Rezaei, A.: On \( z^\circ \)-ideals in \( C(X) \). Fund. Math. 160, 15–25 (1999)
10. Azarpanah, F., Karamzadeh, O.A.S., Aliabad, A.R.: On ideals consisting entirely of zero divisors. Commun. Algebra 28, 1061–1073 (2000)
11. Azarpanah, F., Mohamadian, R.: \( \sqrt{z} \)-ideals and \( \sqrt{z^\circ} \)-ideals in \( C(X) \). Acta Math. Sin. (Engl. Ser.) 23, 989–996 (2007)
12. Engelking, R.: General Topology. PWN-Polish Science Publishers, Warsaw (1977)
13. Ghasemzadeh, S., Karamzadeh, O.A.S., Namdari, M.: The super socle of the rings of continuous functions. Math. Slov. 67(4), 1001–1010 (2017)
14. Ghashgai, E.: Variations of essentiality of ideals in commutative rings. J. Algebra Appl. 21(03), 2250056 (2022)
15. Gillman, L., Jerison, M.: Rings of Continuous Functions. The University Series in Higher Math., Van Nostrand, Princeton (1960)
16. Gilmer, R., McAdam, S.: Ideals contracted from each extension ring. Commun. Algebra 7, 287–311 (1979)
17. Heinzer, W., Lantz, D.: Universally contracted ideals in commutative rings. Commun. Algebra 12(10), 1265–1289 (1984)
18. Henriksen, M., Jerison, M.: The space of minimal prime ideals of a commutative ring. Trans. Am. Math. Soc. 115, 110–130 (1965)
19. Huckabo, J.A.: Commutative Ring with Zero Divisors. Marcel Dekker Inc (1988)
20. Karamzadeh, O.A.S., Rostami, M.: On the intrinsic topology and some related ideals of \( C(X) \). Proc. Am. Math. Soc. 93, 179–184 (1985)
21. Lucas, T.: Two annihilator conditions: property (\( A \)) and (\( AC \)). Commun. Algebra 14(3), 557–580 (1986)
22. Mason, G.: \( z \)-ideals and prime ideals. J. Algebra 26, 280–297 (1973)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.
