The Nonperturbative Color Meissner Effect in a Two-Flavor Color Superconductor

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Abstract

Color superconductivity in QCD breaks the $SU(3)$ color gauge group down to $SU(2)$, inducing masses in five of the eight gluons. This is a dynamical Higgs effect, in which the diquark condensate acts as the vacuum expectation value of a composite scalar field. In order to analyze this effect at low quark density, when gaps are large and generated nonperturbatively, we use instanton-induced quark interactions augmented with gauge-invariant interactions between quarks and perturbative gluons. The five gluon masses are found from the static limit of the relevant polarization operators, in which transversality is maintained via the Nambu-Goldstone modes of broken color symmetry. Working in the microscopic theory we calculate these masses to one-loop order and estimate their density dependence. Finally, we speculate that the Meissner effect may postpone the onset of color superconductivity to higher matter density than estimated previously.

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1 Introduction

It has recently been demonstrated that the color symmetry of Quantum Chromodynamics (QCD) might be spontaneously broken through the formation of a diquark condensate. This phase is metastable at zero and small matter density [1] but becomes stable and replaces the usual chiral-broken phase at some critical density [2, 3].

Condensation of diquarks is analogous to Cooper pairing of electrons in BCS theory, in which the massive photon is the microscopic embodiment of the Meissner effect. In QCD the gauge group is $SU(3)$, but the consequences are similar: gauge bosons (gluons) become massive due to their interaction with the correlated fermions (quarks). Not every gluon species becomes massive, since the symmetry breaking pattern is $SU(3) \rightarrow SU(2)$ and the three gluons of the residual $SU(2)$ remain massless. Among the five massive modes, four are degenerate and the mass of the fifth is $\sqrt{4/3}$ times that of the others. This is comparable to the symmetry-broken phase of the electroweak theory. Thus the direct analogy is to the Higgs mechanism, with the diquark field forming a composite scalar and its nonzero VEV generating a dynamical Higgs effect.

In this paper this mechanism is analyzed microscopically. As the source of diquark condensation we will use the effective quark action obtained by averaging over instanton configurations [4]. In this approach to the QCD vacuum, all information from the large background gauge fields (instantons) is encoded in the non-local form of the ’t Hooft interaction and the two instanton parameters, average size ($\rho$) and number density ($N/V$). This procedure replaces dynamical gluons with classical, nonperturbative field solutions, ignoring quantum gluonic fluctuations. The reintroduction of gluonic fluctuations will require a gauge-invariant perturbative modification of the ’t Hooft vertex. A direct consequence of gauge invariance is the transverse polarization operator, which we will compute explicitly. This is the main result of the paper as from it we obtain the gluon masses.

Since the dynamical Higgs effect is interesting regardless of density, we will first consider the vacuum alone (chemical potential $\mu = 0$), although in this case the superconductor is only metastable [1]. The gluon masses are found to be proportional to $g\Delta$ where $\Delta$ is the superconducting gap, the scale of which is a nonperturbative one set by the strength of the instanton background. Extending the formalism to finite chemical potential $\mu$ is
straightforward, given that the density dependence of the propagators and effective action are known [5]. However, in this paper we avoid exact calculations at $\mu \neq 0$. Instead, for simplicity we assume that the in-medium gluon masses are determined by the behavior of $g\Delta(\mu)$ up to and somewhat above the critical value of $\mu \approx 300$ MeV. At this density, the point at which the color superconductor becomes the stable phase, we find that the gluon masses are about 120 MeV.

For gluons the color-breaking Meissner masses are a primary ramification of the superconducting state, however this is not the only effect of a quark medium. There will be additional contributions, among them a Debye screening mass of the order of $g\mu$, which do not break color symmetry. When the density becomes large, the Debye mass increases and instantons are screened out of the picture. Yet diquark condensation persists, now due to perturbative gluon exchange [6], and at asymptotically large density the Meissner masses are also proportional to $g\mu$ [7,27]. So while one still has quark pairing, any matching between low and high density mechanisms remains unclear.

In this paper we consider the case of two massless flavors, which is expected to be relevant at finite-density chiral restoration given a relatively large strange quark mass [8,9]. Instantons are Euclidean pseudoparticles, and thus all calculations will be in Euclidean space.

In Section 2 we recall those features of the ordinary Higgs mechanism (based on elementary scalar fields) which will also be relevant for composite scalars. In Section 3 the instanton-induced action including perturbative gluons will be formulated. In Section 4 the diquark gap equation is reviewed, and in Section 5 the color current is determined. In Section 6 we sum a set of quark-quark correlation functions to recover the Nambu-Goldstone modes of the theory, a necessary exercise as these will mix with the longitudinal gluons. These ingredients are assembled into the gluon polarization operator in Section 7, which taken in the static limit corresponds to a mass. Section 8 compares the result to chiral symmetry breaking, and in Section 9 this result is discussed and conclusions are drawn.


2 Higgs Mechanism

The dynamical Higgs mechanism, though technically more involved, does not differ much from the ordinary one. Let us denote the quark bilinear combination

\[ \phi^\alpha = \epsilon_{\alpha\beta\gamma} \epsilon_{f\phi} \epsilon_{ij} \left( \psi_{L}^{\beta f_i} \psi_{L}^{\gamma g_j} + (L \to R) \right), \]  

(1)

where Greek letters denote color, \( f, g = 1, 2 \) flavor, and \( i, j = 1, 2 \) are spinor indices of the left (\( L \)) and right (\( R \)) components of the quark field \( \psi \). The resulting complex field \( \phi^\alpha \) is a Lorentz scalar isoscalar field belonging to the fundamental representation of the \( SU(3) \) group. To support gauge invariance it must couple to the gauge potential via the covariant derivative, \( (\nabla_\mu)_{\beta}^\alpha = \partial_\mu \delta^\alpha_\beta - igA_\mu^a(\lambda^a/2)_\beta^\alpha \), in which the \( \lambda^a \) denote the eight Gell-Mann matrices.

The kinetic energy term for the composite diquark field \( \phi^\alpha \) is the usual

\[ \mathcal{L} = Z_\phi^{-1} |\nabla_\mu \phi|^2 \]

\[ = Z_\phi^{-1} \left[ \partial_\mu \phi^\dagger_\alpha \partial_\mu \phi^\alpha + i g 2 A_\mu^a \left( \phi^\dagger_\alpha (\lambda^a)_{\beta}^\gamma \partial_\mu \phi^\beta - \partial_\mu \phi^\dagger_\alpha (\lambda^a)_{\beta}^\gamma \phi^\beta \right) + g^2 4 A_\mu^a A_\mu^b \phi^\dagger_\alpha (\lambda^a \lambda^b)_{\beta}^\gamma \phi^\beta \right]. \]  

(2)

The only difference with the standard case of the elementary field is that there is, in principle, a common ‘wave function renormalization’ factor, \( Z_\phi \). For elementary fields \( Z_\phi = 1 \) at the tree level, however, it deviates from unity even for the elementary field when one takes into account the perturbative virtual emission of particles. For the fields that are composite from the start there is no reason for \( Z_\phi \) to be unity. This quantity is \textit{a priori} unknown and should be determined from a dynamical calculation, as will follow below. We stress, however, that the relative weights of the three terms in Eq. (2) are fixed by gauge invariance.

If the scalar field \( \phi^\alpha \) develops a nonzero VEV signaling the diquark condensation,

\[ \langle \phi^\alpha \rangle = 2\Delta_0 \delta^{\alpha3}, \]  

(3)

(it can be always arranged along the third color axis and made real), then gluons obtain a mass matrix

\[ M^2_{ab} = 2g^2Z_\phi^{-1} \Delta_\phi^{\alpha3} (\lambda^a \lambda^b)_{\beta}^3 = \begin{cases} 0 & a = b = 1, 2, 3 \\ 2g^2Z_\phi^{-1} \Delta_\phi^3 & a = b = 4, 5, 6, 7 \\ \frac{8}{3}g^2Z_\phi^{-1} \Delta_\phi^3 & a = b = 8. \end{cases} \]  

(4)
The symmetry breaking pattern is, thus, $SU(3) \rightarrow SU(2)$; three gluons corresponding to the unbroken $SU(2)$ subgroup remain massless, four gluons obtain masses proportional to the Higgs VEV, and the fifth gluon is $\sqrt{4/3}$ times as heavy. This relation will be, of course, reproduced for composite Higgs fields as well, as it follows from symmetry considerations alone.

This simple elementary-Higgs model also provides an alternative way to find the gluon masses, which we will generalize to the case of the composite Higgs field. One can compute the gluon polarization operator $\Pi^{(a)}_{\mu\nu}(q)$, as seen in Fig. 1. In Fig. 1a we take the linear coupling of the gauge potential $A^a_\mu$, as given by the second term in Eq. (2), and iterate it twice. The intermediate state in this diagram is the would-be Nambu-Goldstone boson, which contributes to the polarization operator

$$\Pi^{(NG)}_{\mu\nu}^{ab}(q) = 2 \left[ Z^{-1}_\phi g(q)^\dagger_{\alpha}(\lambda^a)^\dagger_{\alpha} q_\mu \right] \left[ Z^{-1}_\phi g(q)^\dagger_{\beta}(\lambda^b)^\dagger_{\beta} q_\nu \right] = 2 g^2 Z^{-1}_\phi \Delta^2_0 q_\mu q_\nu (\lambda^a \lambda^b)^3. \tag{5}$$

This contribution is purely longitudinal.

Fig. 1b is the contact term; it gives a Kronecker delta contribution:

$$\Pi^{(contact)}_{\mu\nu}^{ab}(q) = -2 g^2 Z^{-1}_\phi \Delta^2_0 \delta_{\mu\nu} (\lambda^a \lambda^b)^3. \tag{6}$$

Combining the two we get a transverse polarization operator,

$$\Pi^{(full)}_{\mu\nu}^{ab}(q) = -2 g^2 Z^{-1}_\phi \Delta^2_0 \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) (\lambda^a \lambda^b)^3. \tag{7}$$

The fact that it is transverse reflects the color current conservation, even in case of a broken symmetry.
In the case of a composite Higgs field the tree diagrams of Fig. 1 will be replaced by loop diagrams, formed by quarks, however, the general setting will be rather similar to this simple case: there will be a pole contribution from the would-be Goldstone bosons in the intermediate state, and there will be a ‘contact’ term. The resulting polarization operator will be transverse, provided one takes a conserved color current.

3 Gauge Invariant Effective Action

Following a long tradition of instanton phenomenology, we assume the following:

- Instantons are the dominant nonperturbative contribution to low-energy QCD, and
- Low and high momentum scales are safely separable for quark and gluon fields.

These assumptions are supported a posteriori by instanton-based phenomenology of the vacuum, as reviewed in Ref. [10]. Instantons break chiral symmetry spontaneously and axial $U(1)$ anomalously, both at a satisfactory magnitude. Instanton-induced interactions also provide the necessary $qq$ attraction for a diquark condensate, nonperturbatively.

Here we use a formulation which not only relies on the separation of quark zero and free modes, but also explicitly includes both instantons and perturbative gluons. Gluons are separated into classical instantons and quantum corrections, which we write as

$$A_\mu(x) = \sum A^{(I)}_\mu(x) + \sum A^{(I)}_\mu(x) + a_\mu(x), \quad (8)$$

where the sums are over all instantons and anti-instantons, each given by the ’t Hooft solution in the singular gauge [11]. Although there is a certain distribution of instanton sizes (see, for example, [12]) we simply use the average value, $\rho \simeq 1/3$ fm, in all calculations.

The main idea of this paper is to examine the effects of color superconductivity on the gluonic excitations above the instanton vacuum, which means we must retain and analyze the $a^{a}_\mu$. Hereafter, the term ‘gluon’ will refer to the perturbative gauge fluctuation above the instanton background, $a^{a}_\mu$. 

6
The origin of the 't Hooft interaction is explained in the literature [11] and will not be repeated here. To summarize, the low momentum quarks are approximated by zero mode solutions in the presence of one instanton. Averaging over the instanton ensemble generates a vertex for dynamical quarks, mediated by the zero modes, which can be treated perturbatively when the instanton liquid is reasonably dilute. The relevant small parameter is the ratio of average instanton size ($\rho$) to the average inter-instanton spacing ($R$). From phenomenological [13], variational [14], and lattice calculations [12] one finds

$$\frac{\rho}{R} \simeq \frac{1}{3}.$$  

(9)

More details follow in Section 4.

We now consider the effective action itself. In the quark sector it is a non-local one of four-fermion operators, since quarks are connected to instantons via the quark zero modes whose spatial extent is of the order of the instanton size, $\rho$. Specifically, one has [4]:

$$S_{\text{INT}} = \lambda \int dU \, d^4z \, N_f \prod_f \left[ \int d^4x_f \, d^4y_f \, \bar{\psi}_f^\dagger(x_f) \, \partial \Phi(x_f - z, U) \bar{\Phi}(y_f - z, U) \, \partial \psi_f(y_f) \right],$$  

(10)

where $U$ is the instanton’s $2 \times N_c$ color/spin orientation matrix and $z$ is its position. The $\Phi(x)$ is the zero mode solution for fermions in the field of one instanton; in general it depends on the chemical potential. At $\mu = 0$ its Fourier transform is the form factor,

$$f(p) = 2x \left[ I_1(x)K_0(x) - I_0(x)K_1(x) + \frac{1}{x}I_1(x)K_1(x) \right]_{x = \rho p / 2},$$  

(11)

such that $f(0) = 1$. At $\mu \neq 0$ the form factor is also known explicitly [3].

Because of non-locality Eq. (10) is not gauge invariant. When calculating bulk vacuum properties one needn’t worry about quantum corrections as observables are seldom sensitive to them, however in this case the dependence is crucial. A limited literature does exist in which non-local interactions are modified to be gauge invariant [4, 10, 17, 18] and we follow the same procedure here. In particular, we will minimally modify the effective action (10) to suit the present needs.

We will take the non-local four-fermion interaction as the starting point. It then becomes a matter of multiplying each quark operator by a path-
ordered exponential in the background of the perturbative gluon field $a_\mu$, replacing as:

$$\psi(x) \rightarrow \psi(x)W(x, z),$$

$$W(x, y) = \mathcal{P}\exp \left( i\frac{g}{2} \int_y^x ds_\mu a_\mu^a \lambda^a \right), \quad (12)$$

where $a_\mu^a$ is the perturbative field. As has been pointed out in the cited works, the choice of path integrated over is not unique. Yet as long as these factors transform as

$$W(x, y) \rightarrow U(x)W(x, y)U^\dagger(y) \quad (13)$$

by virtue of path ordering, the action will be gauge invariant. This remains true when, as in this case, the interaction involves an explicit color average since overall color is conserved. We are also concerned only with the static limit, $q^2 \rightarrow 0$, in which results are independent of any particular choice of path.

Thus we write the modified interaction as a product over flavors,

$$S_{INT} = \lambda \int dU \ d^4 z \prod_f \left[ d^4 x_f \ d^4 y_f \right.$$ 

$$\times \psi_f^\dagger(x_f)W(x_f, z)\Phi(x_f - z)\Phi^\dagger(y_f - z)\Phi W(y_f, z)\psi_f(y_f) \left. \right], \quad (14)$$

noting that this includes terms of all orders in $ga_\mu$. Calculating the gluon polarization operator will require the linear and quadratic contributions.

To this interaction term one must add the usual quark kinetic term, minimally modified to preserve gauge invariance:

$$S_{KIN} = \int d^4 x \ \psi^\dagger \gamma_\mu (i\partial_\mu + \frac{g}{2} \lambda^a a^a_\mu) \psi. \quad (15)$$

Consequently, the color current obtained from the variation of the action in respect to $a_\mu^a$ will have two contributions: one is the ordinary one arising from the minimal coupling (15) and the other arising from the non-local interaction term (14).
4 Diquark Condensation

In this section some details of the color superconductor phase are reviewed. All expressions are Euclidean and the notation generally follows that of Ref. [5]; the reader is referred to this reference for details specific to this particular approach. The spontaneous breaking of chiral symmetry was also a central concern in that work, whereas here we are considering restored chiral symmetry. The main result of Refs. [19, 5, 20, 21, 22] is a competition between chiral and diquark condensates at zero temperature and nonzero quark chemical potential. In all cases, there is a phase transition from a low-density phase of spontaneously broken chiral symmetry to a high-density one of color superconductivity.

While technically involved, our previous results arise from an effective four-quark interaction which allows for pairing of quarks as in the BCS theory. The instanton model retains more of QCD’s features than ad hoc models, notably the anomalous breaking of axial $U(1)$ and transmutation of dimensions, and is constructed in a more systematic way from that underlying theory.

A prominent feature of the instanton approach used here is the determination of the effective four-fermion coupling constant $\lambda$. This constant is determined by a saddle-point evaluation and proves to be nonlinearly dependent on the background instanton density, $N/V$. A gap equation obtained from a set of Schwinger-Dyson-Gorkov equations are solved to first order in $\lambda$, done self-consistently to determine the quark pairing gap, $\Delta_0$. Standard quark propagators are necessarily split by a nonzero $\Delta_0$, since this introduces a color bias and quarks are no longer color degenerate. This detail will be crucial when quark loops are computed in Section 7.

Since this paper deals only with a phase of chiral-symmetric, diquark condensation the Schwinger-Dyson-Gorkov equations and corresponding diagrams are the simple ones of Fig. 4. We define the quark propagators as

$$\langle \psi^{f \alpha i}(p) \psi^{g \beta j \dagger}(p) \rangle = \frac{\delta^f_i \delta^g_j \delta^\alpha_\beta S_1(p)}{\delta^f_i \delta^g_j \delta^\alpha_\beta S_2(p)} \delta_\beta^j \delta_\alpha^\beta, \quad \alpha, \beta = 1, 2$$

and the anomalous Gorkov propagator as

$$\langle \psi^{f \alpha i}_L(p) \psi^{g \beta j}_L(-p) \rangle = \epsilon^f g \epsilon^{\alpha \beta} \epsilon^{ij} F(p).$$

(16)
Figure 2: Schwinger-Dyson-Gorkov diagrams for normal and anomalous quark propagators. The labels refer to the functions in the text.

In these expressions indices $f$ and $g$ refer to flavor, $i$ and $j$ to spin, the Greek letters to color, and $\psi_{L,R}$ are chiral spinors. Written in the chiral $L, R$ basis, the $4 \times 4$ propagator $S_1(p)$ is of the form:

$$S_1(p) = \begin{pmatrix} 0 & Z(p)S_0(p)^+ \\ Z(p)S_0(p)^- & 0 \end{pmatrix},$$

while $S_2(p)$ is the free propagator for color 3 and hence identical to the above with the function $Z(p)$ absent. The notation is $x^\pm = x_\mu \sigma_\mu^\pm$, where the $2 \times 2$ matrices $\sigma_\mu^\pm = (\pm i\sigma, 1)$ decompose the Dirac matrices into chiral components. The off-diagonal, bare propagator is therefore written $S_0(p)^\pm = [p^\pm]^{-1}$.

With these definitions, we have the scalar Schwinger-Dyson-Gorkov equations

$$Z(p) = 1 - \Delta(p)F(p)$$
$$F(p) = \frac{\Delta(p)Z(p)}{p^2}. $$

The momentum dependence of the gap,

$$\Delta(p) = \Delta_0 f(p)^2,$$

is given by the instanton-induced form factor, Eq. (11), which suppresses the interaction beyond $p > 1/\rho \simeq 600$ MeV. This pair of equations leads to a gap equation for $\Delta_0$:

$$\Delta_0^2 = \frac{2\lambda}{N_c(N_c-1)} \int \frac{d^4 p}{(2\pi)^4} \Delta(p)F(p).$$
which must be self-consistently solved with $\lambda$. This coupling constant is in turn determined by a saddle-point integration (exactly in the thermodynamic limit $N, V \to \infty$),

$$
\lambda = \frac{4N_c(N_c - 1)}{N/V} \Delta_0^2.
$$

Combining these two equations $\lambda$ may be eliminated and we obtain

$$
1 = 8 \left( \frac{N}{V} \right)^{-1} \int \frac{d^4 p}{(2\pi)^4} \frac{\Delta(p)^2}{p^2 + \Delta(p)^2}.
$$

The scale of the gap, therefore, is set by the instanton density $N/V$. To this order $N/V$ remains at its vacuum value, and although it will be affected by the finite density of quarks this is an $O(\lambda)$ correction to the instanton weight and not considered here.

Solved numerically, the gap is $\Delta_0 \simeq 400$ MeV in vacuum and drops to $\Delta_0 \simeq 200$ MeV at the phase transition from chiral broken to color superconducting matter, as detailed in Ref. [5].

5 Color Current

A color-conserving Noether current naturally follows from the modified interaction, including contributions from both $S_{KIN}$ (15) and $S_{INT}$ (14). Along with the standard quark-gluon coupling piece,

$$
\tilde{J}_a^\mu(q) = -\delta S_{INT} \left. \frac{\delta a^\mu_a(q)}{\delta a^\nu_{\mu}(q)} \right|_{a^\nu_{\mu}=0} = \int \frac{d^4 p}{(2\pi)^4} \left[ \psi_R^\dagger(p) \frac{\lambda^a}{2} \sigma^\mu \psi_L(p + q) + \psi_L^\dagger(p) \frac{\lambda^a}{2} \sigma^\mu \psi_R(p + q) \right],
$$

the current now includes a four-quark coupling to the gluon as shown in Fig. [3] and written in terms of four-momentum:

$$
\bar{J}_a^\mu(q) = \delta S_{INT} \left. \frac{\delta a^\mu_a(q)}{\delta a^\nu_{\mu}(q)} \right|_{a^\nu_{\mu}=0}
$$

\[1\] In Ref. [3] $\Delta_0$ was defined as half this paper’s (and the more standard) definition.
\begin{equation}
\lambda e^{f_1 f_2} e^{g_1 g_2} \int D i e^{-i x_f(p_f-p_f')} + i y_f(k_f+k_f') - i z(p_f'+k_f')
\times \left[ \int \frac{d^4 z}{D} e^{-i q \cdot z} \psi^\dagger L(p_1) \lambda^a(\psi^\dagger L(p_2)) \right] + (L \rightarrow R). \tag{25}
\end{equation}

Indices on quarks denote chirality and flavor. The measure is

\begin{equation}
D = dU d^4 z \prod_f d^4 x_f d^4 y_f d^4 p_f d^4 p'_f d^4 k_f d^4 k'_f \tag{26}
\end{equation}

and the form factors lie in the color/spin matrices

\begin{equation}
M(p, k)^{\alpha \beta} = U^{\alpha \epsilon} \epsilon^{\epsilon \beta} U^{\dagger \beta}. \tag{27}
\end{equation}

The $U$ are $2 \times N_c$ color orientation matrices, averaged in each vertex. Through use of the Dirac equation for $\psi$, one can explicitly verify that $q_\mu J^\mu = 0$ for $J_\mu = j_\mu + \tilde{j}_\mu$. While this condition would remain satisfied with a transverse addition to the color current, no such addition is motivated here.

In practice we are interested in pairing off two of the four quark legs of the second vertex. Since we are considering a phase where chiral symmetry is unbroken, a chirality-violating $\langle \psi^\dagger \psi \rangle$ loop cannot contribute. Thus we pair either $\langle \psi \psi \rangle$ or $\langle \psi^\dagger \psi^\dagger \rangle$ and obtain the effective current,

\begin{equation}
\tilde{j}^a \mu(q) = \frac{\Delta_u}{2} \int d^4 z d^4 x \frac{d^4 p_f d^4 p'_f d^4 p'}{(2\pi)^{12}} \times \left[ e^{-i x \cdot (p_1-p_2') - i z \cdot (p'_f+p_2)} \int \frac{d^4 z}{D} e^{-i q \cdot z} \psi^\dagger L(p_1) \lambda^a(\psi^\dagger L(p_2)) \right] \tag{28}
\end{equation}
\[ + e^{-i\varepsilon(p_1+p')-i\varepsilon(p'-p_2)} \int_z^x ds_\mu e^{-iq \cdot s} \psi_L(p_1) \mathcal{N}(p_2, p')^{\dagger} \lambda^a \psi_L(p_2) \]
\[ + (L \to R). \]

(28)

where here the flavor/color/spin structure is
\[ \mathcal{N}(p_1, p')^{ff',aa',ii'} = i \epsilon^{ff'} \epsilon^{aa'} \epsilon^{ii'} f(p) \psi_L(p_2). \]

(29)

The two terms in Eq. (28) correspond to pairing either two incoming or two outgoing quark legs of Fig. 3.

Gluon mass terms are found in the \( q^2 \to 0 \) limit. When the gluon couples to nonsingular composite quark modes one can set \( q^2 \) strictly to zero and the path integrals simplify to
\[ e^{-i(p' - p_2) \cdot \varepsilon} \int_x^z ds_\mu e^{-iq \cdot s} |_{q=0} = e^{-i(p' - p_2) \cdot \varepsilon} (z - x_i)_\mu \]
\[ = -i e^{-i(p' - p_2) \cdot \varepsilon} \frac{\partial}{\partial p'_\mu}. \]

(30)

This substitution, the result of which clearly depends only on the endpoints of the path \( s_\mu \) rather than any particular choice of path, leads to differentiation of the form factor. On the other hand, the vertex can also couple gluons to Nambu-Goldstone modes which are singular as \( q^2 \to 0 \). Their \( 1/q^2 \) behavior must be countered by expanding \( \bar{\psi} a \psi \) to order \( q_\mu \), but before this complication arises the Nambu-Goldstone modes must be specified.

6 Nambu-Goldstone Modes

A symmetry has been spontaneously broken and Nambu-Goldstone modes are certain to follow. With \( SU(3) \) being broken to \( SU(2) \) there are five massless modes and, directly analogous to the Higgs mechanism, they do not become additional degrees of freedom. Instead they mix with and are incorporated into the five massive gluons, thereby relevant in the color Meissner effect.

Since the associated condensate is \( \langle qq \rangle \), these massless modes will be quark bilinears. To determine their quantum numbers one need only perform a gauge rotation on the diquark condensate and catalog the five orthogonal
correlators which appear. Recalling the condensate direction as chosen in Eq. (17),
\[ i \epsilon_{ij} \epsilon_{fg} \langle \psi_L^{f} T_{\lambda} \lambda^2 \psi_L^g \rangle \sim \Delta_0, \]  
we can list the five diquark operators which couple to the Nambu-Goldstone excitations:
\[ \epsilon_{fg} \epsilon_{kl} \psi_L^{fT} \lambda^7 \psi_L^g, \quad i \epsilon_{fg} \epsilon_{kl} \psi_L^{fT} \lambda^7 \psi_L^g, \quad \epsilon_{fg} \epsilon_{kl} \psi_L^{fT} \lambda^5 \psi_L^g, \]
\[ i \epsilon_{fg} \epsilon_{kl} \psi_L^{fT} \lambda^5 \psi_L^g, \quad \epsilon_{fg} \epsilon_{kl} \psi_L^{fT} \lambda^2 \psi_L^g. \]  
Propagators for these quark-quark modes will exhibit a simple pole at \( q^2 = 0 \), behavior recovered by computing the corresponding correlation functions. In order to obtain this pole we must sum \( s \) and \( u \) channel contributions to all orders in the four-quark coupling \( \lambda \). Since connecting these modes to the gluon propagator will be itself of order \( g^2 \), we do not include any gluonic corrections to the instanton vertex. With both standard and anomalous quark propagators at our disposal we obtain the set of coupled Bethe-Salpeter-Gorkov equations diagramed in Fig. 4. The diagram shows not only standard two-body propagators, denoted \( \Gamma \), but also its anomalous analog, \( \Omega \). It is easy to see that \( \Omega \) will vanish when any of the external lines are quarks of color 3, since the vertices conserve color and the internal (Gorkov) propagators involve only colors 1 and 2.

Knowing the quantum numbers of the Nambu-Goldstone modes, we can immediately write down ans"atze for the four-point functions which will be

\[ \text{Figure 4: Bethe-Salpeter-Gorkov diagrams for summation of the quark-quark correlation functions.} \]
The integrals which result from these loops are required when we later compute the gluon polarization operator:

\[
\langle \bar{\psi}_{\chi f} (p) \psi_{\chi f'} (p') | \psi_{\chi}^{\beta j} (k) | \psi_{\chi}^{\beta' j'} (k') \rangle = - \frac{\epsilon_{f f'} \epsilon_{g g'} \epsilon_{i i'} \epsilon_{j j'}}{N_f^2 (N_c - 1)} [\epsilon_3 \lambda^a]_{\alpha \alpha'} [\epsilon_3 \lambda^b]_{\beta \beta'}
\times (2\pi)^4 \delta^4 (p + p' - k - k') f(p) f(p') f(k) f(k') \Gamma_{\chi \chi'}^{ab} (p + p') \tag{33}
\]

\[
\langle \bar{\psi}_{\chi} (p) | \psi_{\chi}^{\alpha i} (p') | \psi_{\chi}^{\alpha' j} (k) | \psi_{\chi}^{\alpha' j'} (k') \rangle = - \frac{\epsilon_{f f'} \epsilon_{g g'} \epsilon_{i i'} \epsilon_{j j'}}{N_f^2 (N_c - 1)} [\epsilon_3 \lambda^a]_{\alpha \alpha'} [\epsilon_3 \lambda^b]_{\beta \beta'}
\times (2\pi)^4 \delta^4 (p + p' + k + k') f(p) f(p') f(k) f(k') \Omega_{\chi \chi'}^{ab} (p + p') . \tag{34}
\]

Here, \([\epsilon_3]_{\alpha \beta} = \epsilon_{3\alpha \beta}\), and the \(\chi\) refer to \(L\) or \(R\) chirality with the chiral substructures defined as

\[
\Gamma = \begin{bmatrix} \Gamma_{LL} & \Gamma_{LR} \\ \Gamma_{RL} & \Gamma_{RR} \end{bmatrix}, \quad \Omega = \begin{bmatrix} \Omega_{LL} & \Omega_{LR} \\ \Omega_{RL} & \Omega_{RR} \end{bmatrix}, \tag{35}
\]

in which color indices have been suppressed. It is clear from their definitions that \(\Gamma_{LR} = \Gamma_{RL}, \Gamma_{LL} = \Gamma_{RR}, \Omega_{LR} = \Omega_{RL}, \text{ and } \Omega_{LL} = \Omega_{RR}\).

Written in terms of these chiral elements, the two diagrams of Fig. 4 correspond to the following set of equations:

\[
\begin{align*}
\Gamma_{ab}^{LL} (q) & = \hat{\lambda} \left[ 1 + \tilde{c}_1^{ab} \Gamma_{LR}^{ab} (q) \mathcal{I}_1 (q) + \tilde{c}_2^{ab} \Gamma_{RL}^{ab} (q) \mathcal{I}_2 (q) + \tilde{c}_3^{ab} \Omega_{LL}^{ab} (q) \mathcal{I}_3 (q) \right] \\
\Gamma_{ab}^{LR} (q) & = \hat{\lambda} \left[ \tilde{c}_1^{ab} \Gamma_{LL}^{ab} (q) \mathcal{I}_1 (q) + \tilde{c}_2^{ab} \Gamma_{RL}^{ab} (q) \mathcal{I}_2 (q) + \tilde{c}_3^{ab} \Omega_{LR}^{ab} (q) \mathcal{I}_3 (q) \right] \\
\Omega_{ab}^{LL} (q) & = \hat{\lambda} \left[ \tilde{c}_3^{ab} \Gamma_{LL}^{ab} (q) \mathcal{I}_3 (q) + \tilde{c}_2^{ab} \Omega_{LR}^{ab} (q) \mathcal{I}_2 (q) \right] \\
\Omega_{ab}^{LR} (q) & = \hat{\lambda} \left[ \tilde{c}_3^{ab} \Gamma_{LR}^{ab} (q) \mathcal{I}_3 (q) + \tilde{c}_2^{ab} \Omega_{LL}^{ab} (q) \mathcal{I}_2 (q) \right] , \tag{36}
\end{align*}
\]

where no sums are implied on color indices \(a, b\). The adjoint color vectors \(\{\tilde{c}_i\}\) determine the internal propagators and are found to be

\[
\begin{align*}
\tilde{c}_1 & = (0, 0, 0, 1, 1, 1, 1, 0) \\
\tilde{c}_2 & = (1, 1, 1, 0, 0, 0, 0, 1) \\
\tilde{c}_3 & = (-1, -1, -1, 0, 0, 0, 0, 1) \tag{37}
\end{align*}
\]

The integrals which result from these loops are:

\[
\mathcal{I}_1 (q^2) \equiv \frac{1}{\Delta_0^2} \int \frac{d^4 p}{(2\pi)^4} \frac{p_+ \cdot p_-}{p_+^2 p_-^2} [\Delta (p_+) \Delta (p_-)] [Z (p_+) + Z (p_-)]
\]
\[ \mathcal{I}_2(q^2) = \frac{2}{\Delta_0^2} \int \frac{d^4 p}{(2\pi)^4} [\Delta(p_+)\Delta(p_-)] F(p_+)F(p_-) \]
\[ \mathcal{I}_3(q^2) = \frac{2}{\Delta_0^2} \int \frac{d^4 p}{(2\pi)^4} \frac{p_+ \cdot p_-}{p_+^2 p_-^2} [\Delta(p_+)\Delta(p_-)] Z(p_+)Z(p_-), \]
(38)

where we have defined \((p_\pm)_\mu = p_\mu \pm \frac{1}{2} q_\mu\) and
\[ \bar{\lambda} \equiv \frac{\lambda}{N_c(N_c - 1)}. \]
(39)

The set of Bethe-Salpeter-Gorkov equations (36) can easily be solved for correlations functions of all adjoint colors. For colors \(a = b = 4 \ldots 8\), in direct analogy to similar calculations of pions in the instanton vacuum \([2,3]\), the \(q^2 = 0\) pole arises due to a cancellation in the denominator which is a direct consequence of the gap equation. The calculation follows identically for \(a = 4, 5, 6,\) and 7, where we find
\[ \Gamma_{44}^{44}(q^2) = \Gamma_{LL}^{44}(q^2) = \frac{\bar{\lambda}}{1 - \bar{\lambda}^2 \mathcal{I}_1(q^2)^2}. \]
(40)

After writing the gap equation \([23]\) in these terms,
\[ 1 = \bar{\lambda} \mathcal{I}_1(0), \]
(41)

we have
\[ \Gamma_{LR}^{44}(q^2) = \bar{\lambda} \left( 1 - [\bar{\lambda} \mathcal{I}_1(0)]^2 - q^2 \bar{\lambda}^2 \frac{\partial}{\partial q^2} \mathcal{I}_1(q^2)^2 \right)^{-1} \equiv \frac{Z_\phi}{q^2}, \]
(42)

where
\[ Z_\phi^{-1} = -2 \left. \frac{\partial \mathcal{I}_1(q^2)}{\partial q^2} \right|_{q^2 = 0} \]
\[ = \frac{1}{\Delta_0^2} \int \frac{d^4 p}{(2\pi)^4} \left\{ \frac{(\Delta(p))^2 - \frac{1}{2} \Delta(p) p \Delta'(p) + \frac{1}{2} p^2 \Delta'(p)^2}{p^2[p^2 + \Delta(p)^2]} \right. \]
\[ \left. - \frac{1}{2} \Delta(p)^2 \frac{[\Delta(p) - p \Delta'(p)]^2}{p^2[p^2 + \Delta(p)^2]^2} \right\}. \]
(43)

The primes denote differentiation with respect to \(p\). The value of \(Z_\phi\) will change with density, since not only does it depend on \(\Delta_0\) but the integrand
will exhibit functional dependence on \( \mu \). In vacuum numerical evaluation gives \( Z^\phi = 8.07 \times 10^{-3} \).

For the \( \Gamma^{88} \) and \( \Omega^{88} \) solving the coupled equations becomes more involved, since all diagrams in Fig. 4 are present. One finds

\[
\Gamma^{88}_{LR}(q^2) = \Omega^{88}_{LR}(q^2) = \frac{\lambda}{1 - \lambda^2 [I_2(q^2) + I_3(q^2)]^2} = \frac{Z^\phi}{q^2},
\]

where the second line is obtained by making use of the Schwinger-Dyson-Gorkov equations, Eqs. (19), in the limit of small \( q^2 \). This completes the set of Nambu-Goldstone modes. One can naturally compute the correlation function for the additional diquark correlators with \( a = b = 1, 2, 3 \), and find a crucial sign difference in the combination of integrals:

\[
\Gamma^{11}_{LR}(q^2) = \Omega^{11}_{LR}(q^2) = \frac{\lambda}{1 - \lambda^2 [I_2(q^2) - I_3(q^2)]^2} = \frac{1}{q^2 + m(q)^2}.
\]

The mass can be determined from an \( \mathcal{O}(q^2) \) expansion of the denominator, however this will not be done here as for our purposes it is enough to verify the \( a = 1, 2, \) and 3 modes are indeed massive.

### 7 Gluon Polarization Operator

With the gapped quark propagators, conserved current interactions, and composite modes defined in the previous sections, we now compute the leading modification to the gluon polarization operator. All diagrams prove to be color diagonal, and so we write

\[
\Pi^{ab}_{\mu\nu}(q^2) = \delta^{ab}\Pi_{\mu\nu}(q^2).
\]

Our interest is the static limit, \( q^2 \to 0 \), which may be considered an effective mass.

In the presence of a color-3, scalar diquark the gluons are divided into three classes. Gluons of adjoint colors 1, 2, and 3 belong to the residual \( SU(2) \)
Figure 5: Quark loop diagrams contributing to the gluon polarization operator.

gauge group and as such remain massless. Gluons 4, 5, 6, and 7 couple one gapped quark (of fundamental color 1 or 2) with the ungapped species and share a degenerate mass. Gluon 8, diagonal in fundamental color, obtains a mass proportional to the previous four. One polarization operator from each class will be explained here to avoid unnecessary repetition. All possible contributions to order $g^2$ are diagramed in Figs. 5 and 6 and, depending on the gluonic species, some of these diagrams vanish and others combine to cancel in the static limit.

We begin with the case of gluons 4–7, considering corrections to $\langle a_{\mu}^4 a_{\nu}^4 \rangle$. Diagrams (5b), (5f), and those involving (6c) require pairs of Gorkov propagators and thus vanish since the quarks of color 3 (to which these gluons couple) cannot propagate anomalously. There are four graphs which constitute the ‘contact’ term proportional to $\delta_{\mu\nu}$.

Diagram (5a) is a standard loop, where one of the quarks is gapped (color 1 in the case of gluon 4) and the other not (color 3). After subtracting off the vacuum part of this diagram, which remains a concern of gluon renormalization and is not relevant to the Meissner mass, we find

$$\Pi_{\mu\nu}^{(5a)}(q^2 \to 0) = -g^2 N_f \delta_{\mu\nu} \int \frac{d^4 p}{(2\pi)^4} \frac{\Delta(p)^2}{p^2[p^2 + \Delta(p)^2]}.$$  

(47)

The remaining integral is finite due to an power-law cut-off in the the function $\Delta(p)$ arising from the finite size of instantons.

Additional diagrams arise from the modified $S_{INT}$. Not only does this generate additional current interactions (29), but the interaction (14) itself
contains a contribution to the gluon two-point function. This, diagram (5c), is the second variation of the action (14) with respect to the fourth gluon field,

\[ \Pi_{\mu\nu}^{(5c)}(q^2 \to 0) = \frac{\delta^2 S_{INT}}{\delta a_\mu^4(q) \delta a_\nu^4(q)} \bigg|_{q^2=0}. \] (48)

Evaluated to order \( g^2 \) it is

\[ \Pi_{\mu\nu}^{(5c)}(q^2 \to 0) = \frac{1}{4} g^2 N_f \delta_{\mu\nu} \int \frac{d^4 p}{(2\pi)^4} \frac{\Delta(p) p \Delta'(p) - \frac{1}{2} \Delta(p) p^2 \Delta''(p)}{p^2 [p^2 + \Delta(p)^2]}. \] (49)

Diagrams (5d) and (5e), constructed with the additional current piece, are

\[ \Pi_{\mu\nu}^{(5d)}(q^2 \to 0) = \frac{1}{4} g^2 N_f \delta_{\mu\nu} \int \frac{d^4 p}{(2\pi)^4} \frac{\Delta(p) p \Delta'(p)}{p^2 [p^2 + \Delta(p)^2]} \]

\[ \Pi_{\mu\nu}^{(5e)}(q^2 \to 0) = \frac{1}{8} g^2 N_f \delta_{\mu\nu} \int \frac{d^4 p}{(2\pi)^4} \frac{p^2 \Delta'(p)}{p^2 [p^2 + \Delta(p)^2]} \] (50)

Eqs. (47), (49), and (50) comprise the microscopic equivalent of the Higgs contact term (6), and their sum is

\[ \Pi_{\mu\nu}^{(5)}(q^2 \to 0) = \]

\[ -g^2 N_f \delta_{\mu\nu} \int \frac{d^4 p}{(2\pi)^4} \Delta(p)^2 \left( -\frac{1}{4} \Delta(p) p \Delta'(p) - \frac{1}{3} \Delta(p) p^2 \Delta''(p) \right) \frac{p^2 [p^2 + \Delta(p)^2]}{p^2}. \] (51)

The integral can be trivially rewritten and then manipulated to yield

\[ \int \frac{d^4 p}{(2\pi)^4} \left\{ \frac{\Delta(p)^2}{p^2 [p^2 + \Delta(p)^2]} \right\} = Z^{-1} \phi_0. \] (52)

The final equality is achieved by integrating the second and third terms by parts.

The Nambu-Goldstone modes couple to the gluons as in Fig. 6. The construction \([\bar{a}(a) + \bar{b}(b)] \Gamma^{44}(q^2) \) \([\bar{a}(a) + \bar{b}(b)]\) supplies the \( q_\mu q_\nu / q^2 \) piece to ensure
Figure 6: Diagrams coupling the Nambu-Goldstone modes to gluons. Their sum squared contributes to the polarization operator.

\[
\Pi^{(6)}_{\mu\nu}(q^2 \to 0) = g^2 N_f q_\mu q_\nu Z_\phi \frac{Z_0}{q^2} \left[ \int \frac{d^4 p}{(2\pi)^4} \left\{ \frac{\frac{1}{2} \Delta(p)^2 [\Delta(p) - p \Delta'(p)]^2}{p^2 [p^2 + \Delta(p)^2]} \right. \right.
\]
\[
\left. \left. - \frac{\Delta(p)^2 - \frac{1}{2} \Delta(p) p \Delta'(p) + \frac{1}{4} p^2 \Delta'(p)^2}{p^2 [p^2 + \Delta(p)^2]} \right\} \right]^2 \delta_{\mu\nu} \frac{1}{q^2} Z_\phi^{-1} q_\mu q_\nu.
\]

(53)

We have now accounted for all contributions to gluons 4, 5, 6, and 7.

Analysis of the eighth gluon follows in a similar fashion, although with terms from every diagram. A superficial difference lies in the factors arising from the elements \( \lambda^8 \) which lead to a polarization \( 4/3 \) times the previous result. More subtle is the combination of diagrams (5a) and (5b). For \( a^8_\mu \) this sums to \( 4/3 \) times Eqs. (47), whereas for \( a^{1,2,3}_\mu \) they cancel one another. This cancellation is only manifest to \( \mathcal{O}(\Delta_0^4) \) here, due to the fact that the diagrams are constructed with a pair of one-loop, resummed quark propagators as determined in Section 3. This generates a \( \Delta(p)^2 \) term in each integrand denominator, in essence including higher-order terms in the perturbative expansion (in \( \lambda \)) which violate gauge invariance. After the vacuum pieces are subtracted, these diagrams contribute the following to \( a^8_\mu \):

\[
\Pi^{(5a-b)}_{\mu\nu}(q^2 \to 0) = \frac{1}{3} g^2 N_f \Delta_0^2 \delta_{\mu\nu} \int \frac{d^4 p}{(2\pi)^4} \left\{ \frac{p^2}{[p^2 + \Delta(p)^2]^2} \right. \right.
\]
\[
\left. \left. - \frac{2\Delta(p)^2}{[p^2 + \Delta(p)^2]^2} - \frac{1}{p^2} \right\} \right\}.
\]
\begin{equation}
\Pi^{ab}_{\mu\nu}(q^2 \to 0) = \begin{cases} 
0 & a, b = 1, 2, 3 \\
-g^2 N_f \Delta_0^2 Z_\phi^{-1} \delta^{ab} \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) & a, b = 4, 5, 6, 7 \\
-\frac{4}{3} g^2 N_f \Delta_0^2 Z_\phi^{-1} \delta^{ab} \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) & a, b = 8.
\end{cases}
\end{equation}

Transversality requires that the contact term be proportional to the wave function renormalization of the Nambu-Goldstone modes and this result, a Ward Identity for color superconductivity, is recovered here.

Finally, to determine a numerical value for the masses we must fix the coupling constant $g$. Evaluating in the instanton vacuum, one finds the large finite action \[ S_0 = \frac{8\pi^2}{g^2} \simeq 12, \]

or $g \simeq 2.6$ and the perturbative expansion parameter $\alpha_s = g^2/4\pi = 0.54$.

The gluon masses squared are thus

\begin{equation}
M_a^2 = \begin{cases} 
0 & a = 1, 2, 3 \\
2g^2 \Delta_0^2 Z_\phi^{-1} \simeq (150 \text{ MeV})^2 & a = 4, 5, 6, 7 \\
\frac{4}{3} g^2 \Delta_0^2 Z_\phi^{-1} \simeq (175 \text{ MeV})^2 & a = 8.
\end{cases}
\end{equation}

These masses apply to the vacuum, $\mu = 0$.

In order to estimate the finite-density behavior of the Meissner mass, we can simply take the values of $\Delta_0$ for a given $\mu$ from the results of Ref. \[14\]. As detailed in that paper, the instanton form factor \[1\] becomes density dependent and thus $\Delta(p)$ should be replaced by a complicated $\Delta(p, |\vec{p}|, \mu)$. However, the changes in $Z_\phi$ arising from the finite-$\mu$ modifications of the form factors are minor compared compared to the changes in the gap magnitude,
Figure 7: Scaling of $M$, $Z_{\phi}^{-1/2}$, and $\Delta_0$ with quark chemical potential $\mu$. Each quantity is shown relative to its vacuum value. For reference, diquark condensation is only stable in quark matter at and above $\mu \approx 300$ MeV.

$\Delta_0$; we numerically estimate this correction to be about 3%. For simplicity we therefore considered only the scaling from $\Delta_0(\mu)$ (taken from previous work $[5]$). Each non-zero gluon mass is proportional to $\Delta_0 Z_{\phi}^{-1/2}$ (see Eq. (57)) and therefore all will scale identically with density. The resulting gluon mass $M(\mu)$, as well as that of the renormalization constant $Z_{\phi}(\mu)^{-1/2}$, is shown in units of its vacuum value in Fig. 7. Note that although $Z_{\phi}$ rises with increasing quark density, this effect is not sufficient to overcome the falling gap and the masses continuously decrease. The first point of physical relevance would occur around $\mu \approx 300$ MeV, the common prediction for chiral restoration to a color superconductor. Results for matter at lower densities correspond to an unstable solution and are only of academic interest $[3, 19, 20, 21, 22]$. 

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8 Similarity with Chiral Symmetry Breaking

Apart from the gauge coupling $g$ the gluon masses (57) are determined by the combination $F_{qq}^2 \equiv 2\Delta_0^2 Z^{-1}_\phi$. This quantity is the analog of the $F_\pi^2$ constant in the chiral-broken phase, both in its physical meaning and algebraically.

In the chiral-broken phase the Nambu-Goldstone bosons are pions. The correlation function of the axial current, $j_{\mu5}^A = \bar{\psi} \gamma_\mu \gamma_5 \tau^A \psi$, in the massless quark limit has the transverse form

$$\langle j_{\mu5}^A(q) j_{\nu5}^B(-q) \rangle = \Pi_{\mu\nu}^{AB}(q) = -F_\pi^2 \delta^{AB} \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right).$$

(58)

The transversality of $\Pi_{\mu\nu}$ is the consequence of the conservation of the axial current; the $1/q^2$ pole is due to the pion in the intermediate state. Were there gauge bosons coupled to the quark axial current their mass would be equal to $F_\pi$ multiplied by the corresponding gauge coupling. In case of diquark (vs. quark-antiquark) condensation the relevant currents are color ones, and we obtain a similar form for the correlation function of two color currents, Eq. (55). One needs only to multiply $F_{qq}$ by the gauge coupling to deduce the gluon mass.

If instantons are dilute, the leading contribution to the Kronecker part of the polarization operator arises from the $\Pi_{\mu\nu}^{(4a)}$ piece, Eq. (47). It is the only contribution which diverges logarithmically if one neglects the momentum dependence of the gap $\Delta(p)$. For the same reason the leading contribution to the $Z_\phi^{-1}$ and $F_{qq}^2$ in the dilute limit comes from the pieces not containing the derivatives $\Delta'(p)$. One has therefore in the dilute limit:

$$F_{qq}^2 \approx 2 \int \frac{d^4p}{(2\pi)^4} \frac{\Delta(p)^2}{p^2 + \Delta(p)^2} \approx \frac{2\Delta_0^2}{8\pi^2} \log \frac{R^2}{\rho^2},$$

(59)

where $\Delta_0$ is the superconducting gap at zero momentum. It follows from the gap equation (23) that $\Delta_0 \sim \pi \rho \sqrt{N/V} = \pi \rho / R^2$. Similarly, the axial correlation function (58) computed in Ref. [23] gives (in the same approximation)

$$F_\pi^2 \approx 4N_c \int \frac{d^4p}{(2\pi)^4} \frac{M(p)^2}{[p^2 + M(p)^2]^2} \approx \frac{4N_c M_0^2}{8\pi^2} \log \frac{R^2}{\rho^2},$$

(60)

where $M(p)$ is the dynamical quark mass (the chiral gap) whose value at zero momentum is determined from a corresponding gap equation to be $M_0 \sim \pi \rho \sqrt{N/V N_c}$. 

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We see, thus, that not only have the Meissner mass and the $F_\pi$ constant analogous meaning, but their algebraic structure is quite similar. One expects, therefore, that the numerical value for the Meissner mass is of the order of $F_\pi$, and this expectation is confirmed by an exact numerical calculation of the previous section.

9 Discussion and Conclusions

We have analyzed the problem of spontaneous gauge symmetry breaking brought about by a diquark condensate. Since the broken symmetry is continuous and gauged the resulting Nambu-Goldstone modes do not remain in the spectra, instead mixing with the longitudinal components of the gauge fields to produce massive gauge bosons. This, a dynamical Higgs mechanism, can be called the color Meissner effect in the context of color superconductivity.

To reveal the gauge boson masses mathematically one has to compute the polarization operator, which ought to be transverse,

$$\Pi_{\mu\nu}^{ab}(q) = -M^2 \delta^{ab} \left( \delta_{\mu\nu} - \frac{q\mu q\nu}{q^2} \right),$$  

where the massless pole $1/q^2$ is the manifestation of the Nambu–Goldstone intermediate state. The coefficient, $M^2$ gives the mass of the gauge boson. In this paper, we explicitly solved this problem for the case of diquark condensation as induced by the instanton background, the effective action from which we found necessary to modify in order to maintain a conserved color current.

Through computing the gluon polarization operators to order $g^2$ we find the effective gluon masses to be on the order of the diquark gap. The three gluons comprising the residual $SU(2)$ group remain massless and hence a quark-gluon medium would become color-biased in such a phase. The analysis here was done in the limit of zero temperature and, initially, vanishing chemical potential $\mu$, though strictly speaking at zero $\mu$ the color superconductor is only metastable with the ground state being the usual chiral-broken phase. We then estimated finite-density dependence of the calculated quantities. At the critical density, where one expects the phase transition to the color superconducting phase, we deduce that the Meissner masses of the 4th,
5th, 6th, and 7th gluon are about 120 MeV and the 8th gluon has a mass about 140 MeV. These quantities would be of physical relevance should a low temperature, high density region become experimentally accessible. At a chemical potential low enough to leave the instanton background approximately unchanged ($\mu < 0.6$ GeV), the instanton effects analyzed here would still be present and likely dominant.

In computing the coupling of the Nambu-Goldstone modes to gluons we have established that their mixing is described by an effective theory in which the composite diquark is replaced by a complex scalar in the fundamental representation of the gauge group, $\phi^a$. The effective Lagrangian coincides with that of elementary scalar field covariantly coupled to gluons, aside from an overall factor $Z_{\phi}$ which is the ‘wave function renormalization’ of the composite scalar field. There are no a priori reasons for this factor to be close to unity (as it is in the case of a weakly-coupled elementary Higgs field), and indeed we find a substantial deviation from unity. Meanwhile, it is a crucial factor for the estimate of the Meissner mass.

If electromagnetic interactions were taken into account, the gluon $a^8_{\mu}$ will invariably mix with the (massless) photon. This mixing, estimated from general arguments to be rather small [24], reorganizes the fields into a massive ‘new gluon’ and massless ‘new photon’. Given that the gluonic sector alone can be recast as a Yang-Mills-Higgs theory, this additional result would complete the analogy to the electroweak sector of the Standard Model.

Finally, we would like to comment on the phase transition between ordinary chiral-broken phase and the color superconductivity. The point is that all estimates existing in the literature (see Ref. [5] and references therein) indicate that this phase transition happens alarmingly ‘early’: taken literally, the claim is that the interior region of a heavy nucleus is actually in a ‘boiling’ state. However, those estimates generally neglect the influence of the dense medium on the gluonic background fluctuations which induce the color superconductivity itself. The Meissner mass of about 150 MeV found here is a large quantity and, together with the Debye mass and other effects, will suppress instantons. Therefore, by taking into account back-influence effects one has a chance to ‘save’ ordinary nuclear matter from a premature phase transition by moving that transition to higher densities.

Although the instanton density is not expected to change significantly for any chemical potential below the inverse instanton size (600 MeV), at and beyond this point perturbative Meissner and screening masses will become
increasingly important. At asymptotically large densities perturbative gluon exchange is the source of diquark condensation \cite{6, 7, 25, 26, 27}, whereas in this work we have been concerned with the low density, nonperturbative regime. Determining the behavior at a moderate density scale – which would be an interpolation between the two – is necessary if one wishes to confidently consider signals of color superconductivity in any potentially realizable situation, be it in some future heavy-ion collider or a neutron star.

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