We discuss general aspects of the possibility of Bose condensation of diquark pairs in systems of dense matter. Lattice field theory simulations are presented for model four-fermion theories which are expected to manifest the phenomenon, and results from measurements of both diquark two-point and one-point functions presented. Whilst initial results are promising, there remain systematic effects needing to be understood.

There has been much recent interest in the possibility of Bose condensates formed from quark pairs close to the Fermi surface in dense matter; such condensates, which carry non-zero baryon number, can lead to exotic scenarios for dynamical breaking of the color gauge symmetry. So far most work in this direction has relied on effective field theory descriptions of the strong interaction. For any problem involving the QCD ground state it would be nice to have a calculation which in principle involves no uncontrolled approximations; this is therefore a natural area for the application of the methods of lattice gauge theory. In this talk we describe the first steps we have made in this direction; our initial results have already appeared.

When thinking about possible diquark condensates \( \langle qq \rangle \neq 0 \) which might form there are several issues to consider. Firstly, is the condensate wavefunction gauge invariant? In the QCD scenarios the condensate is formed from a \( 3 \otimes 3 \) of the color gauge group, which necessarily breaks the local symmetry, leading to a dynamical Higgs mechanism making some or all of the gluons massive. This is the phenomenon of color superconductivity. In other models, relevant to this discussion, another possibility is that the condensate wavefunction is gauge invariant; in this case condensate formation breaks only global symmetries, and yields a superfluid state. Next, is the condensate necessarily a spacetime scalar? We normally expect condensates to share the spacetime quantum numbers of the perturbative vacuum, but in systems of dense matter there is a preferred rest frame, making it possible to consider rotationally non-invariant condensates. Another possibility which should not be excluded \textit{a priori} is that the condensate spontaneously breaks parity invariance. Finally, the most crucial aspect of the condensate wavefunction is that it respects the Pauli Exclusion Principle, i.e. it must be antisymmetric under exchange of all
possible quantum numbers between the quarks. It is this requirement which makes the ground state so exquisitely sensitive to the number of light quark flavors present in QCD\textsuperscript{1}; more generally it may also result in a sensitivity to which representation of the gauge group is carried by the quarks.

The most generic symmetry we expect to be be broken by a diquark condensate is the global U(1)\textsubscript{V} of baryon number:

\begin{equation}
q \mapsto e^{i\alpha} q \quad ; \quad \bar{q} \mapsto \bar{q} e^{-i\alpha}.
\end{equation}

One issue we should address is that spontaneous breaking of a global vectorlike symmetry like this is usually forbidden by the Vafa-Witten theorem\textsuperscript{3}. It is convenient to classify the models in which diquark condensation might occur by the various escape clauses offered by the theorem. The theorem does not apply if the path integral measure is not positive definite. This is the case for QCD with chemical potential $\mu \neq 0$, since for this case the fermion determinant is complex. Unfortunately it is precisely this feature which makes Monte Carlo simulation extremely difficult. The theorem also fails if the theory includes a Yukawa coupling to a scalar degree of freedom. Two model field theories satisfying this criterion are the Gross-Neveu (GN) model in 2+1 dimensions, and the Nambu–Jona-Lasinio (NJL) model in 3+1 dimensions, and these will be the main focus of this talk. The final escape route is if baryon number is part of some larger non-vectorlike symmetry. An example is SU(2) lattice gauge theory, where the global symmetry group enlarges from U(1)\textsubscript{V} $\otimes$ U(1)\textsubscript{A} to U(2) due to the pseudoreal nature of the 2 representation. Simulations of SU(2) lattice gauge theory with $\mu \neq 0$ will be discussed separately\textsuperscript{4}.

The GN and NJL models, which are essentially identical apart from the number of spatial dimensions, are relativistic generalisations of the model originally considered in the BCS mechanism for superconductivity. In continuum notation the Lagrangian may be written:

\begin{equation}
\mathcal{L} = \bar{\psi} (\partial \psi + m) - g^2 \left[ (\bar{\psi} \psi)^2 - (\bar{\psi} \gamma_5 \vec{\tau} \psi)^2 \right].
\end{equation}

The four-fermi terms can be replaced by Yukawa couplings to scalar and pseudoscalar auxiliary fields. The lattice transcription of (2) is discussed in detail elsewhere\textsuperscript{2}. In addition to the U(1) baryon number symmetry (1), the model has a SU(2)\textsubscript{L} $\otimes$ SU(2)\textsubscript{R} axial symmetry in the chiral limit $m \to 0$:

\begin{equation}
\psi_L \mapsto U \psi_L \; , \; \bar{\psi}_L \mapsto \bar{\psi}_L U^\dagger \; ; \; \psi_R \mapsto V \psi_R \; , \; \bar{\psi}_R \mapsto \bar{\psi}_R V^\dagger,
\end{equation}

with $U, V$ independent SU(2)\textsubscript{V} matrices.

The most relevant features of the model for our purposes are that for strong coupling the SU(2)\textsubscript{L} $\otimes$ SU(2)\textsubscript{R} symmetry spontaneously breaks to SU(2)\textsubscript{V} by
formation of a chiral condensate $\langle \bar{\psi} \psi \rangle$. The spectrum in the broken phase contains both “baryons”, namely the elementary fermions which now have a dynamically generated mass, and “mesons”, namely $\psi \bar{\psi}$ composites, which include 3 Goldstone pions. It turns out that for the 2+1 dimensional GN case the model has an interacting continuum limit at the critical coupling required for symmetry breaking. Crucially, the model can be formulated on a lattice and simulated for $\mu \neq 0$; it is found that in the broken phase a first order chiral symmetry restoring transition occurs for some critical $\mu_c$.

We have considered the possible formation of a diquark condensate with wavefunction

$$qq = \psi^{tr} C \gamma_5 \otimes \tau_2 \otimes \tau_2 \psi;$$

the operators in the tensor product denote that the diquark is a spacetime scalar, and antisymmetric in both implicit (due to lattice fermion doubling) and explicit flavor indices (note that $C \gamma_5$ is also antisymmetric). Condensation of (4) spontaneously breaks baryon number (1) but not axial (3) symmetry. We have examined two possible signals for $\langle qq \rangle \neq 0$.

For large spatial separation the diquark pair propagator is proportional to the square of the condensate by the cluster property:

$$G(x) = \langle qq(0) \bar{q} \bar{q}(x) \rangle = \langle qq(0) \bar{q} \bar{q}(x) \rangle_c + \langle qq \rangle \langle \bar{q} \bar{q} \rangle \Rightarrow \lim_{x \to \infty} G(x) = |\langle qq \rangle|^2. \quad (5)$$

A non-zero condensate should therefore reveal itself as a plateau in the large-$t$ behaviour of the diquark timeslice propagator. In Fig. 1 we show the behaviour of $G(t)$ from simulations of the GN model on a $16^2 \times 40$ lattice, performed with a value $\mu = 0.8 > \mu_c$, so that chiral symmetry is restored. There is clear evidence for a stable plateau, especially when compared to $G(t)$ for non-interacting fermions, also shown with closed symbols. The square root of the plateau height is plotted, together with $\langle \bar{\psi} \psi \rangle$, as a function of $\mu$ in Fig. 2. We observe a large increase in the $\langle qq \rangle$ signal going from the chirally broken phase into the
symmetric phase, consistent with the notion that the condensates “compete.”

We also observe a small but non-zero \( \langle qq \rangle \) in the low density phase, and a parity violating pseudoscalar condensate in the high density phase, although probably both signals are finite volume artifacts.

Unfortunately the height of the plateau remains roughly constant as the spatial volume of the lattice is increased, whereas naively we expect it to be extensive. It is therefore not clear whether a true condensation is occurring, or whether there are unexplained finite volume effects. A similar behaviour is seen in 3+1 dimensional simulations, suggesting that this is not a low dimensional artifact as originally thought.

To attempt to clarify these ambiguities we have now performed direct measurements of \( \langle qq \rangle \) in the simulations. This involves including explicit diquark source terms in the action and using a Gor’kov representation, ie:

\[
S_{\text{ferm}} = \bar{\psi}M\psi + j\psi^{tr}\tau_2\psi + j\bar{\psi}\tau_2\bar{\psi}^{tr} = (\bar{\psi},\psi^{tr}) \begin{pmatrix} \frac{1}{2}M & \frac{j}{2}M^{tr} \\ \frac{j}{2}M & \frac{1}{2}M^{tr} \end{pmatrix} (\bar{\psi}^{tr},\psi)
\]

\[\equiv \Psi^{tr}A[j,\bar{j}]\Psi; \quad (6)\]

\[Z[j,\bar{j}] = \langle \text{Pf}(A[j,\bar{j}]) \rangle. \quad (7)\]

The diquark condensate is now defined by

\[
\langle qq \rangle = \frac{1}{V} \left. \frac{\partial \ln Z}{\partial j} \right|_{j,\bar{j}=0} = \lim_{j,\bar{j} \to 0} \frac{1}{V} \langle \frac{1}{2} \text{tr} \left\{ A^{-1} \begin{pmatrix} 0 & 0 \\ 0 & \tau_2 \end{pmatrix} \right\} \rangle, \quad (8)
\]

which is straightforward to implement. Our results are “quenched” in the sense that we have included \( j \neq 0 \) in the measurement routines but not as yet in the update algorithm. We show \( \langle qq(j) \rangle \) for various \( \mu \) for the GN model in Fig. 3 and for the NJL model in Fig. 4. In the low density chirally broken phase the signal extrapolates linearly to zero as \( j \to 0 \); in the high density phase the signal is larger and considerably less linear, although still vanishing in the zero source limit. Preliminary indications are that the curvature of \( \langle qq(j) \rangle \) is still more marked in the 3+1 dimensional simulation. We have found evidence for small finite volume effects, but the question of whether a diquark condensate forms, as revealed by a non-zero intercept in the thermodynamic limit, is still open.

Whilst we have still not found unambiguous evidence for diquark condensation, it is clear something interesting is happening in the dense phase, as revealed both by the long range timelike order in the behaviour of the two-point function, and the non-linear behaviour of the one-point function with \( j \). Systematic effects due to the influence of Goldstone modes, the relatively small number of states close to the Fermi surface on a finite lattice in a small
number of dimensions, and the quenched nature of the \langle gg \rangle measurement may all need to be understood. Interestingly, evidence for diquark condensation is much more compelling in simulations of SU(2) gauge theory. It is important, however, to understand how a BCS-like mechanism manifests itself in simple four-fermion models before QCD can be tackled (recall that in this case the condensate is not even gauge invariant). In future work we plan to explore the spectroscopy of the model using the Gor'kov representation, and hopefully measure the gap, which is after all the quantity closest to physics.

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