A note on relationship between some bounding inequalities in stability analysis of time-delay systems

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Abstract

In this paper, an extension of the generalized free matrix based inequality is introduced in a unified form suitable for the estimation of integrals and sums of quadratic functions. The equivalences of several known variants are shown, including the free matrix based inequalities and its simplified form. Secondly, the relationship between the (simplified) free matrix based estimation and the combination of the Bessel-based inequality with different bounding inequalities affine in the length of the intervals are investigated.

Key words: Integral inequalities; Summation inequalities; Time-delay systems; Free-matrix-based inequality; Extended reciprocally convex approach.

1 Introduction

The stability of time-delay systems is often analyzed by means of appropriate Lyapunov-Krasovskii functionals, with the help of which tractable stability criteria can be derived (see e.g. Chen et al. (2016a) - Zhang et al. (2017) and the references therein). In these investigations, the lower estimation for integrals (in case of continuous time systems) or sums (in case of discrete time systems) of positive quadratic terms plays a crucial role. A possible tool is the Jensen’s inequality, but in the past years several results were published to reduce its conservatism. The so called free-matrix-based

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(FMB) and generalized free-matrix-based (GFMB) inequalities has been proposed recently for this purpose (see Zeng et al. (2015a), Chen et al. (2016a), Zhang et al. (2016a), Lee et al. (2017a), Lee et al. (2017b)). Another line of the improvement of Jensen’s inequality is represented by the Wirtinger inequality, the Bessel-Legendre and Bessel-Chebyshev inequalities (see e.g. Seuret et al. (2013), Seuret et al. (2014), Hien et al. (2015), Gyurkovics et al. (2016), Gyurkovics et al. (2017)). The relationship between the different approaches may be of interest. It has been proven in paper Gyurkovics (2015) that the Wirtinger inequality of Seuret et al. (2013) and the FMB inequality of Zeng et al. (2015a) are equivalent with superiority of the Wirtinger inequality. This result has been generalized lately by Chen et al. (2016b). However, the Jensen’s and Wirtinger’s inequalities and their generalizations depend in a non convex way on the length of the intervals, thus they can only be applied in combination with a convexifying inequality as e.g. reciprocally convex combination lemma of Park et al. (2011) when time-varying-delay systems are considered. Therefore, such combinations have to be taken into account to obtain a real comparison. The main purpose of the present paper is twofold. First, an extension of the generalized free matrix based inequality is to be formulated in a unified form, and the relationship between its variants is to be investigated. Secondly, the estimations obtained by FMB and by the combination of the Bessel-based inequality with different bounding inequalities affine in the length of the intervals are to be compared. Throughout this paper, $S_n^+$ is the set of positive definite symmetric matrices of size $n \times n$, and $\text{He}(A) = A + A^T$, where $A^T$ is the transpose of $A$.

2 Generalized free-matrix-based approach and its variants

In this section, a generalized free-matrix-based estimation will be formulated by a lemma that will serve as a base of comparisons of different related approaches that can be applied for time-delay systems both in continuous and discrete time cases. In this formulation, we shall keep in mind that, in the application for the stability analysis of systems with time-varying delays, one splits an interval to two subsequent subintervals. Let $E_i (i = 0, 1, 2)$ be the Euclidean space of functions $\varphi : D_i \subset \mathbb{R} \rightarrow \mathbb{R}$ with the scalar product $\langle \cdot, \cdot \rangle_i$ containing the elements $\Pi_{0i}(t) \equiv 1$, $t \in D_i$, respectively, and possessing with the following two properties:

(P1) If $\varphi, \psi \in E_i$, then $\varphi \psi \in E_i$ and $\langle \varphi, \psi \rangle_i = \langle \Pi_{0i}, \varphi \psi \rangle_i$;
(P2) If for $\varphi \in E_i \varphi(t) \geq 0$ for all $t \in D_i$, then $\langle \Pi_{0i}, \varphi \rangle_i \geq 0$.

Remark 1 Typically, $D_i \subset \mathbb{R}$ or $\mathbb{Z}$, $D_0 = [a, b]$, $D_1 = [a, c]$, and $D_2 = [c, b]$, and the scalar products are defined as $\langle \varphi, \psi \rangle_i = \int_{D_i} \varphi(t) \psi(t) dt$ or $\langle \varphi, \psi \rangle_i = \sum_{t \in D_i} \varphi(t) \psi(t)$, having properties (P1) and (P2).
Let $\Pi_{0i}, \Pi_{1i}, ... , \Pi_{\nu_i}$ be an orthogonal system in $E_i$ for some non-negative integer $\nu$, and consider $f \in E_0^0$. Suppose that $D_1 \cap D_2 = \emptyset$, $D_1 \cup D_2 = D_0$ and $\langle \varphi, \psi \rangle_0 = \langle \varphi_1, \psi_1 \rangle + \langle \varphi_2, \psi_2 \rangle_2$, where $\varphi_i$ and $\psi_i$ are the restrictions of $\varphi$ and $\psi$ respectively to $D_i$. Set $M_1 = (\nu + 1)n$, $M_2 = 2M_1$ and consider $\underline{w} \in R^{M_2}$, with

$$
\underline{w} = \begin{bmatrix} 
\underline{w}^1 \\
\underline{w}^2
\end{bmatrix} = \begin{bmatrix} 
\text{col} \{ \langle f_1, \Pi_{01} \rangle, ... , \langle f_1, \Pi_{\nu_1} \rangle \} \\
\text{col} \{ \langle f_2, \Pi_{02} \rangle, ... , \langle f_2, \Pi_{\nu_2} \rangle \}
\end{bmatrix},
$$

where $f_i$ is the restriction of $f$ to $D_i$. Set furthermore symmetric matrices $\Psi^i (i = 1, 2)$ as

$$
\Psi^i = \begin{bmatrix} 
Z^{i_0}_{00} & \cdots & Z^{i_0}_{0\nu} & N^i_0 \\
\vdots & \ddots & \vdots & \vdots \\
Z^{i_\nu_0}_{\nu_0} & \cdots & Z^{i_\nu_0}_{\nu_\nu} & N^i_\nu \\
N^i_0 & \cdots & N^i_\nu & W
\end{bmatrix},
$$

where $Z^{i}_{kl} \in R^{M_2 \times M_2}$, $N^i_k \in R^{M_2 \times n}$, $k, l = 0, ..., \nu$.

**Lemma 2** (GFMB inequality.) If $W \in S^n_+$, and

$$
\Psi^i \geq 0,
$$

then the following generalized free-matrix-based inequality holds true for constants $\rho^i_k \geq \| \Pi_{ki} \|^2_i$:

$$
\langle f_i, W f_i \rangle \geq -\chi^T_i \left( \sum_{k=0}^{\nu} \rho^i_k Z^{i}_{kk} \right) \chi_i - \text{He} \left( \chi^T_i N^i \underline{w} \right),
$$

where $\chi_i \in R^{M_2}$ ($i = 1, 2$) is arbitrary $N^1 = \left( \tilde{N}^1, 0 \right) = (N^1_0, ..., N^1_\nu, 0, ..., 0) \in R^{M_2 \times M_2}$ and $N^2 = \left( 0, \tilde{N}^2 \right) = (0, ..., 0, N^2_0, ..., N^2_\nu) \in R^{M_2 \times M_2}$.

**Proof.** For $i = 1, 2$, set $\xi_i = \text{col} \{ \Pi_{0i} \chi_i, ..., \Pi_{\nu_i} \chi_i, f_i \}$. Then $\xi_i(t) \in R^{(\nu+1)M_2+n}$ and $\xi_i(t)^T \Psi^i \xi_i(t) \geq 0$. We obtain that

$$
0 \leq \langle \Pi_{0i}, \xi_i^T \Psi^i \xi_i \rangle = \langle \xi_i, \Psi^i \xi_i \rangle = \sum_{k=0}^{\nu} \sum_{l=0}^{\nu} \langle \Pi_{ki}, \Pi_{li} \rangle \chi^T_i Z^{i}_{kl} \chi_i + 2 \sum_{k=0}^{\nu} \chi^T_i N^i_k \langle f_i, \Pi_{ki} \rangle + \langle f_i, W f_i \rangle.
$$

Since $\Psi^i \geq 0$, $Z^i_{kk} \geq 0$ is true. Taking into consideration the orthogonality of $\Pi_{ki}$'s and the definition $\langle f_i, \Pi_{ki} \rangle = w^i_k$ [2] can be obtained by rearranging (3).

**Remark 3** Frequently, Lemma 2 is applied for the derivative (in continuous-time case) or the forward difference (in discrete-time case) of a function under the choice of $\{ \Pi_{ki} \}^\nu_k=0$ as orthogonal polynomials. Using partial integra-
tion, i.e. the Abel-lemma, and expressing the derivatives in a chosen basis of polynomials, one can derive analogous estimations. (See the details how such derivations can be performed e.g. in Seuret et al. (2014), Gyurkovics et al. (2016), Lee et al. (2013), Chen et al. (2016c), etc. for continuous time, and Gyurkovics et al. (2017), Hien et al. (2015), Zhang et al. (2016b), etc. for discrete time.)

2.1 Independent functions based GFMB inequality

Recently, Lee et al. (2017a) and Lee et al. (2017b) derived a GFMB estimation in Lemma 3 and in Lemma 7 by considering certain linearly independent system of polynomials instead of orthogonal ones. We shall formulate a unified paraphrase of these lemmas as follows.

Lemma 4 (IFB-GFMB inequality.) Let \( \{p_{ki}\}_{k=0}^{\nu} \) be a system of linearly independent functions defined on \( E_i \), \( \tilde{w}_i^k = \langle f, p_{ki} \rangle \) and \( \gamma_{ki} = \langle p_{ki}, p_{li} \rangle \). If \( W \in S^+ \), and (1) holds true, then the following independent-functions-based generalized free-matrix-based inequality holds true:

\[
\langle f, W f \rangle \geq -\sum_{k=0}^{\nu} \text{He} \left( \chi_i^T N_i \tilde{w}_i^k \right) - \chi_i^T \left( \sum_{k=0}^{\nu} \gamma_{kk} Z_{ik}^2 + \sum_{k=0}^{\nu} \sum_{l=k+1}^{\nu} \text{He} \left( \gamma_{kl} Z_{kl} \right) \right) \chi_i.
\]

(4)

where \( \chi_i \in \mathbb{R}^{M^2} \) \((i = 1, 2)\) is arbitrary.

First we show that, although (4) contains more free parameters than (2), it does not give a better lower estimation.

Theorem 5 Suppose that \( \{p_{ki}\}_{k=0}^{\nu} \) and \( \{\Pi_{ki}\}_{k=0}^{\nu} \) span the same subspace of \( E_i \). Let \( \rho_i^k = ||\Pi_{ki}||^2 \). Then estimations GFMB and IFB-GFMB are equivalent.

Proof. Since the orthogonal system of functions is also linearly independent, it is enough to show that GFMB implies IFB-GFMB. For simplicity, we omit index \( i \) in the proof. Let \( \Psi \geq 0 \) be given. We will show that a \( \Psi' \geq 0 \) exists such that

\[
-\chi^T \left( \sum_{k=0}^{\nu} \Pi_k ||\Pi_k||^2 Z_{kk} \right) \chi - \text{He} \left( \chi^T \tilde{N}_k \langle f, \Pi_k \rangle \right) \\
= -\chi^T \left( \sum_{k=0}^{\nu} \gamma_{kk} Z_{kk} + \sum_{k=0}^{\nu} \sum_{l=k+1}^{\nu} \text{He} \left( \gamma_{kl} Z_{kl} \right) \right) \chi - \sum_{k=0}^{\nu} \text{He} \left( \chi^T N_k \langle f, p_k \rangle \right).
\]

(5)

In fact, there is an invertible matrix \( C \in \mathbb{R}^{(\nu+1)\times(\nu+1)} \) such that \( \text{col} \{p_0, ..., p_\nu\} \)
\[ \gamma_{kl} = \langle p_k, p_l \rangle = \sum_{j=0}^{\nu} c_{kj} c_{lj} \|\Pi_j\|^2. \]

Let \( \tilde{\Psi} \) be defined by \( \tilde{Z}_{jj} = \sum_{k=0}^{\nu} c_{kj} c_{kl} Z_{kl}, \tilde{N}_j = \sum_{k=0}^{\nu} c_{kj} N_k \) and \( \tilde{Z}_{kl} = \tilde{N}_k W^{-1} \tilde{N}_T \), if \( k \neq l \). Then we obtain that
\[
\chi^T \left( \sum_{k=0}^{\nu} \sum_{l=0}^{\nu} \gamma_{kl} Z_{kl} \right) \chi = \chi^T \sum_{j=0}^{\nu} \|\Pi_j\|^2 \tilde{Z}_{jj} \chi, \tag{6}
\]
and
\[
\sum_{j=0}^{\nu} \text{He} \left( \chi^T \tilde{N}_j w_j \right) = \sum_{k=0}^{\nu} \left( \chi^T N_k \sum_{j=0}^{\nu} c_{kj} \langle f, \Pi_j \rangle + \sum_{j=0}^{\nu} \langle f, \Pi_j \rangle^T c_{kj} N_k^T \chi \right). \tag{7}
\]

Equation (6) follows from (6) and (7). Now, it has to be shown only that \( \tilde{\Psi} \geq 0 \). By Schur complements, this is equivalent to
\[
\tilde{\Phi} = \begin{bmatrix}
\tilde{Z}_{00} - \tilde{N}_0 W^{-1} \tilde{N}_0^T & \ldots & \tilde{Z}_{0\nu} - \tilde{N}_0 W^{-1} \tilde{N}_\nu^T \\
\vdots & \ddots & \vdots \\
\tilde{Z}_{\nu 0} - \tilde{N}_\nu W^{-1} \tilde{N}_0^T & \ldots & \tilde{Z}_{\nu \nu} - \tilde{N}_\nu W^{-1} \tilde{N}_\nu^T
\end{bmatrix} \succeq 0.
\]

The non-diagonal blocks are zeros because of the definition of \( \tilde{Z}_{kl} \) for \( k \neq l \). The diagonal blocks can be written as
\[
\tilde{\phi}_{jj} = \tilde{Z}_{jj} - \tilde{N}_j W^{-1} \tilde{N}_j^T = \sum_{k=0}^{\nu} \sum_{l=0}^{\nu} c_{kj} c_{lj} \left( Z_{kl} - N_k W^{-1} N_l^T \right).
\]

Let \( \tilde{y} \in \mathbb{R}^{M_2} \) be arbitrary and let \( y_j = \text{col} \{ c_{0j} \tilde{y}, \ldots, c_{\nu j} \tilde{y} \} \), then \( \tilde{y}^T \tilde{\phi}_{jj} \tilde{y} = y_j^T \Phi y_j \) holds true with matrix \( \Phi \) defined by blocks \( \phi_{kl} = Z_{kl} - N_k W^{-1} N_l^T \). By Schur complements, \( \Psi \succeq 0 \) is equivalent to \( \Phi \succeq 0 \), therefore \( \tilde{\phi}_{jj} \succeq 0 \) for all \( j \), i.e. matrix \( \tilde{\Psi} \) is positive semi-definite indeed.

Having regard to the equivalence of GFMB and IFB-GFMB, we will consider only GFMB in the sequel.

### 2.2 Simplified GFMB and Bessel based inequalities

In what follows we shall need the following notations. For \( i = 1, 2 \), set \( \mathcal{W}_i = \text{diag} \left\{ \frac{1}{\rho_0^i}, \ldots, \frac{1}{\rho_{\nu}^i} \right\} \otimes W, \mathcal{W}_{i-} = \mathcal{W}_i^{-1}, \tilde{\mathcal{W}}_1 = \text{diag} (\mathcal{W}_1, 0), \tilde{\mathcal{W}}_2 = \text{diag} (0, \mathcal{W}_2) \), and \( \tilde{\mathcal{W}}_{i-} \) is defined analogously with \( \mathcal{W}_{i-} \).
Corollary 6 (S-GFMB) If \( W \in S_n^+ \),
\[
\langle f_i, W f_i \rangle \geq - \text{He} \left( \chi_i^T N_i w \right) - \chi_i^T N_i \hat{W}_i - N_i^T \chi_i,
\]
(8)
where \( \chi_i \) and \( N_i \) are the same as in Lemma 2. Moreover, the right hand side of (8) is always greater or equal than the right hand side of (2).

Proof. The right hand side of (2) can be written as
\[
- \text{He} \left( \chi_i^T N_i w \right) - \chi_i^T N_i \hat{W}_i - N_i^T \chi_i = - \left( \chi_i^T N_i w \right) - \chi_i^T N_i \hat{W}_i - N_i^T \chi_i.
\]
(9)
It follows from \( \Psi_i \geq 0 \) that the choice of matrices \( Z_{ki} = N_i^W W^{-1} N_k^T \) is admissible, thus (9) implies (8). The second part follows from \( Z_{ki} \geq N_i^W W^{-1} N_k^T \).

Remark 7 Corollary 6 is a straightforward extension of Lemma 5 of Zhang et al. (2017) (continuous time case), and Lemma 2/6 of Zhang et al. (2016b) (discrete time case).

Corollary 8 (BBI) If \( W \in S_n^+ \),
\[
\langle f_i, W f_i \rangle \geq \text{w}^i_T \hat{W}_i \text{w}^i.
\]
(10)
Moreover, the right hand side of (10) is always greater or equal than the right hand side of (2) and (8).

Proof. The right hand side of (8) can be written as
\[
- \chi_i^T \left( \sum_{k=0}^\nu \rho_k Z_{ik} \right) \chi_i - \text{He} \left( \chi_i^T N_i w \right) = - \chi_i^T \left( \sum_{k=0}^\nu \rho_k \left( Z_{ik} - N_i^i W^{-1} N_i^{(T)} \right) \right) \chi_i
\]
\[
- \text{He} \left( \chi_i^T N_i w \right) - \chi_i^T \left( \sum_{k=0}^\nu N_i^i \hat{W}_i - N_i^{(T)} \right) \chi_i.
\]
(11)
for \( i = 1, 2 \). Since the first term in the second line of (11) is non-positive, and it is equal to zero, if \( N_i^T \chi_i = - \hat{W}_i w \) the statements of the corollary follow.

Remark 9 The estimation of Corollary 8 is the same as that of Lemma 1 of Gyurkovics et al. (2016), which implies among others the Bessel-Legendre inequality of Seuret et al. (2014), Gyurkovics et al. (2016), and the Bessel-Chebyshev inequality of Gyurkovics et al. (2017).

The preceding results can be summarized as follows.

Theorem 10 The estimations (1)-(2) of Lemma 2, (8) of Corollary 6 and (10) of Corollary 8 are equivalent, and the tightest estimation is obtained under the choice of free parameters yielding (10).
2.3 Simplified FMB

Choosing $\chi_i = w$ we obtain the following estimation from Corollary 6.

**Corollary 11 (S-FMB)** If $W \in S_n^+$,

$$\langle f_i, W f_i \rangle \geq -w^T \left( H^i(N^i) - N^i \hat{W}_i - N_i^T \right) w,$$

where $N^i$ is the same as in Lemma 2.

**Remark 12** Estimation (12) of Corollary 11 is a general formulation of several FMB results. By appropriate choice of the matrices, of $\nu$ and of the polynomials, one obtains from S-FMB among others Lemma 3 of Chen et al. (2016a), Theorem 1 of Chen et al. (2016c), Lemma 4 of Zeng et al. (2015a), Lemma 1 of Zeng et al. (2015b), Lemma 2 of Liu et al. (2017), Lemma 3 of Seuret et al. (2014), Lemma 4 of Zhang et al. (2016a) (continuous time case) and Lemma 2 of Chen et al. (2016a), Corollaries 1-3 of Chen et al. (2016b), Lemmas 1-3 of Lee et al. (2017), Lemmas 2-3 of Wan et al. (2016), Lemma 2 of Chen et al. (2016a) (discrete time case).

**Theorem 13** The estimations (8) of Corollary 6 and (12) of Corollary 11 are equivalent.

**Proof.** It is enough to show that for any given $\chi_i$ and $N_i$ there is a $\tilde{N}_i$ such that the right hand side of (12) equals to the right hand side of (8) taken with $N_i = \tilde{N}_i$. In fact, there exist an orthogonal matrix $Q$ and a scalar $\eta$ such that $\chi_i = \eta Q w$. Substituting it into (8) and taking $\tilde{N}_i = \eta Q N_i$ results in (12).

**Remark 14** Choosing $\chi_i = w$, one can derive from GFMB the FMB inequality, and its equivalence can be proven in the same way as Theorem 13.

Having regard the previous equivalences, we will consider only S-FMB and BBI in the sequel.

3 Estimations for two connected intervals

Specify now the Euclidean spaces $E_i (i = 0, 1, 2)$ according to Remark 11. Denoting the lengths of the intervals as $h = b - a$, $h_1 = c - a$, $h_2 = b - c$, where $a < c < b$, one can verify that

$$\rho_j^0 = \frac{h}{2j+1}, \quad \rho_j^1 = \frac{h_1}{2j+1}, \quad \rho_j^2 = \frac{h_2}{2j+1},$$
satisfy condition $\rho^i_k \geq \|\Pi_k\|^2_i$. Introducing the notations $\alpha = \frac{h_1}{h}$, $\beta = \frac{h_2}{h}$, $\mathcal{W} = \text{diag} \{1, 3, ..., 2\nu + 1\} \otimes W$, we obtain from Corollary 11 that

\[
\langle f, Wf \rangle_0 = \langle f_1, Wf_1 \rangle_1 + \langle f_2, Wf_2 \rangle_2 \geq \frac{1}{h} w^T \Omega_F(\alpha, \hat{N}_1, \hat{N}_2) w. \tag{13}
\]

where

\[
\Omega_F(\alpha, \hat{N}_1, \hat{N}_2) = \text{He} \left( -h \begin{bmatrix} \hat{N}_1 & 0 \\ 0 & \hat{N}_2 \end{bmatrix} + (-h \hat{N}_1)(\alpha \mathcal{W}^{-1})(-h \hat{N}_1)^T + (-h \hat{N}_2)(\beta \mathcal{W}^{-1})(-h \hat{N}_2)^T \right).
\]

On the other hand, it follows from Corollary 8 that

\[
\langle f, Wf \rangle = \langle f_1, Wf_1 \rangle_1 + \langle f_2, Wf_2 \rangle_2 \geq \frac{1}{h} w^T \Omega_B(\alpha) w. \tag{14}
\]

where

\[
\Omega_B(\alpha) = \begin{bmatrix} \frac{1}{\alpha} \mathcal{W} & 0 \\ 0 & \frac{1}{\beta} \mathcal{W} \end{bmatrix}.
\]

From the results of the previous subsections, it follows that (14) yields a tighter lower bound than (13). However, if (14) is applied to the stability analysis of systems with time-varying delays, the obtained estimation is non-convex in the lengths of the intervals. To avoid the non-convexity, one has to apply a further lower estimation of the right hand side of (14). Therefore, a real comparison can be obtained by relating (13) to the combination of (14) and some convexifying lower bound.

In what follows, we shall investigate the application of the classical reciprocally convex combination (RCC) lemma of Park et al. (2011), the extended reciprocally convex combination (ERC) lemma of Seuret et al. (2016), and Lemma 3 of Liu et al. (2016), which will be referred to as (M-LSX). We shall apply the (ERC) lemma with two special choices of the matrices as well. These estimations are summarized in the lemma below.

**Lemma 15** If $W \in S_n^+$, then for $k = 1, \ldots, 5$,

\[
\Omega_B(\alpha) \geq \Omega_k(\alpha), \forall \alpha \in (0, 1),
\]

where $\Omega_k$ is given in different cases as follows:

- **(M-LSR):** with arbitrary $V_1, V_2 \in \mathbb{R}^{M_2 \times M_1}$,

\[
\Omega_1(\alpha) = \Omega_1(\alpha, V_1, V_2) = \text{He} \left( \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \right) - \alpha V_1 \mathcal{W}^{-1} V_1^T - \beta V_2 \mathcal{W}^{-1} V_2^T; \tag{15}
\]

8
• (ERC): with arbitrary \( Y_1, Y_2 \in \mathbb{R}^{M_2 \times M_1} \) and \( X_1, X_2 \in \mathbb{S}^+_{M_1} \), satisfying for \( \alpha = 0, 1 \),

\[
\begin{bmatrix}
\mathcal{W} & 0 \\
0 & \mathcal{W}
\end{bmatrix} - \alpha \begin{bmatrix}
X_1 & Y_1 \\
Y_1^T & 0
\end{bmatrix} - \beta \begin{bmatrix}
0 & Y_2 \\
Y_2^T & X_2
\end{bmatrix} \succeq 0,
\]

(16)

\[
\Omega_2(\alpha) = \Omega_2(\alpha, X_1, X_2, Y_1, Y_2) = \begin{bmatrix}
\mathcal{W} + \beta X_1 & \alpha Y_1 + \beta Y_2 \\
* & \mathcal{W} + \alpha X_2
\end{bmatrix};
\]

(17)

• (SERC): with arbitrary \( Y_1, Y_2 \in \mathbb{R}^{M_1 \times M_1} \) and \( \hat{X}_1 = \mathcal{W} - Y_1 \mathcal{W}^{-1} Y_1^T, \hat{X}_2 = \mathcal{W} - Y_2^T \mathcal{W}^{-1} Y_2 \),

\[
\Omega_3(\alpha) = \Omega_3(\alpha, Y_1, Y_2) = \begin{bmatrix}
\mathcal{W} + \beta \hat{X}_1 & \alpha \hat{Y}_1 + \beta \hat{Y}_2 \\
* & \mathcal{W} + \alpha \hat{X}_2
\end{bmatrix};
\]

(18)

• (MERC): with arbitrary \( Y \in \mathbb{R}^{M_1 \times M_1} \) and \( \overline{X}_1 = \mathcal{W} - Y \mathcal{W}^{-1} Y^T, \overline{X}_2 = \mathcal{W} - Y^T \mathcal{W}^{-1} Y \),

\[
\Omega_4(\alpha) = \Omega_4(\alpha, Y) = \begin{bmatrix}
\mathcal{W} + \beta \overline{X}_1 & Y \\
* & \mathcal{W} + \alpha \overline{X}_2
\end{bmatrix};
\]

(19)

• (RCC): with arbitrary \( Y \in \mathbb{R}^{M_1 \times M_1} \) satisfying (16) with \( X_1 = X_2 = 0 \) and \( Y_1 = Y_2 = Y \),

\[
\Omega_5(\alpha) = \Omega_5(\alpha, Y) = \begin{bmatrix}
\mathcal{W} & Y \\
Y & \mathcal{W}
\end{bmatrix}.
\]

(20)

**Theorem 16** The estimation (13) and the estimations of Lemma 15 are related as follows:

(A) (DS-FMB) is equivalent to (DBBI) & (M-LSR);
(B) (DBBI) & (M-LSR) implies (DBBI) & (SERC), but not conversely;
(C) (DBBI) & (ERC) is equivalent to (DBBI) & (SERC);
(D) (DBBI) & (SERC) implies (DBBI) & (MERC), but not conversely;
(E) (DBBI) & (MERC) implies (DBBI) & (RCC), and \( \Omega_5(\alpha, Y) \leq \Omega_4(\alpha, Y) \) if \( Y \) is chosen according to (RCC).

**Proof.** The proof of (A) consists of verifying that

\[
\Omega_B(\alpha, \hat{N}_1, \hat{N}_2) = \Omega_1(\alpha, -h \hat{N}_1, -h \hat{N}_2).
\]
In order to prove (B), we can verify first that the choice of

$$V_1 = \begin{bmatrix} W \\ Y_2^T \end{bmatrix}, \quad \text{and} \quad V_2 = \begin{bmatrix} Y_1 \\ W \end{bmatrix},$$

yields $\Omega_1(\alpha) = \Omega_3(\alpha)$, i.e. $\Omega_3(\alpha)$ can be obtained as a special case of $\Omega_1(\alpha)$. To show the second part of the statement, let us write $V_1 = \text{col} \{ W + \Xi_1, V_{12} \}$ and $V_2 = \text{col} \{ V_{21}, W + \Xi_2 \}$. Then

$$\Omega_1(\alpha, V_1, V_2) - \Omega_3(\alpha, Y_1, Y_2) = \begin{bmatrix} -\Xi_1 W^{-1} \Xi_1^T & \Xi_1 W^{-1} V_{12} + V_{12} - Y_1 \\ * & \text{He}(\Xi_2) - V_{12} W^{-1} V_{12}^T + Y_1^T W^{-1} Y_1 \end{bmatrix}.$$ 

If $\Xi \neq 0$, then there is a $y_1 \in \mathbb{R}^{M_1}$ such that $y_1^T (\Omega_1(\alpha)_{11} - \Omega_3(\alpha)_{11}) y_1 < 0$ for $\alpha = 1$, and for continuity reasons, for $\alpha \in (1 - \delta, 1], (0 < \delta < 1)$ independently of the choice of $Y_1, Y_2$. At the same time, if $\Xi_2$ is such that $\Xi_2 + \Xi_2^T - V_{12} W^{-1} V_{12}^T > 0$, then there is a $y_2 \in \mathbb{R}^{M_1}$ such that $y_2^T (\Omega_1(\alpha)_{22} - \Omega_3(\alpha)_{22}) y_2 > 0$ independently of the choice of $Y_1, Y_2$. Therefore, $\Omega_1(\alpha, V_1, V_2) - \Omega_3(\alpha, Y_1, Y_2)$ may be indefinite for some $V_1, V_2$ independently of the choice of $Y_1, Y_2$. This means that neither $\Omega_1(\alpha) \geq \Omega_3(\alpha)$, nor $\Omega_1(\alpha) \leq \Omega_3(\alpha)$ is true in general.

To show (C) we observe that it follows from (16) that $X_1 \leq W - Y_1^T W^{-1} Y_1^T$, and $X_2 = W - Y_2^T W^{-1} Y_2$. Thus $\Omega_3(\alpha)$ is a special case of $\Omega_2(\alpha)$, and $\Omega_2(\alpha) \leq \Omega_3(\alpha)$ for any $Y_1, Y_2$, which proves (C).

Since $\Omega_4(\alpha, Y) = \Omega_3(\alpha, Y, Y)$, the first assertion (D) is obvious. To show the second part of the statement, let us write $Y_i = Y + Y_i$ ($i = 1, 2$). Then a straightforward computation shows that $\Omega_3(1, Y + Y_1, Y + Y_2) - \Omega_4(1, Y)$ is indefinite, if $Y_1 \neq 0$, thus neither $\Omega_3(\alpha) \geq \Omega_4(\alpha)$, nor $\Omega_3(\alpha) \leq \Omega_4(\alpha)$ is true.

The proof of (E) is immediate, since $\Omega_5(\alpha, Y) = \Omega_4(\alpha, Y)$, if $Y \in \mathbb{R}^{M_1 \times M_1}$ is such that (16) is satisfied with $X_1 = X_2 = 0, Y_1 = Y_2 = Y$, and $\overline{X}_1 = \overline{X}_2 = 0$ is substituted in $\Omega_4$. Since $\overline{X}_1, \overline{X}_2 \geq 0$ for the above $Y$, inequality $\Omega_5(\alpha) \leq \Omega_4(\alpha)$ is true.

**Remark 17**

1.) Lemma 4 of [Zhang et al. (2016a)] is a special case of the estimation (DBBI) & (MERC).

2.) We have seen that neither of the estimations obtained by (M-LSR) and (SERC) is better than the other. Nevertheless, (M-LSR) may be advantageous in the analysis of systems with time-varying delays. Similar note is due with respect to the estimations (SERC) and (MERC) on favour of (SERC).
4 Conclusions

We introduced an extension of the generalized free matrix based inequality in a unified form suitable for the estimation of integrals and sums of quadratic functions. The equivalences of several known variants were proven, including the free-matrix-based inequalities and its simplified form. It was shown that the Bessel-based estimation is at least as good as any of the others, while the S-FMB estimation is at least as good as GFMB, IFB-GFMB. Secondly, the relationship between the S-FMB estimation and the combination of the Bessel-based inequality with different bounding inequalities being affine in the length of the intervals were intensively investigated.

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