GLOBAL WELL-POSEDNESS OF THE NAVIER-STOKES EQUATIONS WITH NAVIER-SLIP BOUNDARY CONDITIONS IN A STRIP DOMAIN

QUANRONG LI
College of Mathematics and Statistics, Shenzhen University
Shenzhen, 518060, Guangdong, China

SHIJIN DING*
South China Research Center for Applied Mathematics and Interdisciplinary Studies
South China Normal University, Guangzhou, 510631, Guangdong, China

(Communicated by Alain Miranville)

Abstract. This paper is concerned with the existence and uniqueness of the strong solution to the incompressible Navier-Stokes equations with Navier-slip boundary conditions in a two-dimensional strip domain where the slip coefficients may not have defined sign. In the meantime, we also establish a number of Gagliardo-Nirenberg inequalities in the corresponding Sobolev spaces which will be applicable to other similar situations.

1. Introduction. Navier-Stokes equations is one of the most classical mathematical models in fluid dynamics and is also the basic system in the study of most complex fluids. Since being derived by the famous physicists C. Navier and G. Stokes, it has attracted the attentions of considerable number of mathematicians and physicists during the past over 100 years. Precisely, the incompressible Navier-Stokes equation reads as follows

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho \mathbf{v}) &= 0, \\
\partial_t (\rho \mathbf{v}) + \text{div}(\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p &= \text{div} \mathbf{S} + \rho \mathbf{f}, \\
\partial_t (\rho E) + \text{div}(\rho \mathbf{v} E + \mathbf{v} p) &= \text{div} (\mathbf{S} + \kappa \nabla \theta) + \rho \mathbf{f} \cdot \mathbf{v}, \\
\text{div} \mathbf{v} &= 0,
\end{align*}
\]

in $\Omega \times (0,T)$. (1.1)

where $\rho$, $\mathbf{v}$, $\theta$, $E$ are density, velocity, absolute temperature and total energy of the fluid, respectively and $p$, $\mathbf{S}$, $\mathbf{f}$ stand for the pressure, stress tensor and external force, respectively. We point out that the first three equations are deduced by the conservation of mass, momentum and energy, respectively. For the derivation

2020 Mathematics Subject Classification. Primary: 35Q30, 35Q35; Secondary: 76N10.

Key words and phrases. Navier-Stokes equations, Navier-slip conditions, strip domain, global existence, uniqueness.

Li’s research is supported by the National Natural Science Foundation of China (No.11901399) and the Natural Science Foundation of Shenzhen University (2019084). Ding’s research is supported by the National Natural Science Foundation of China (No.11371152, No.11571117, No.11871005 and No.11771155), Natural Science Foundation of Guangdong Province (No.2017A030313003 and No.2021A1515010355) and Science and Technology Program of Guangzhou (No.2019060001).

* Corresponding author.
of the Navier-Stokes system, we refer the reader to books by G. P. Galdi[11], by R. Temam[24] and by P. L. Lions[18].

If the density and temperature are constants and the stress tensor is taken to be the simplest one
\[ \mathbb{S} = \mu (\nabla v + \nabla^T v), \]
then system (1.1) reduces to the following
\[
\begin{cases}
\partial_t v + v \cdot \nabla v + \nabla p = \mu \Delta v + f, \\
\text{div} v = 0,
\end{cases}
\]
in \( \Omega \times (0, T). \) (1.2)

To seek solutions for (1.2) and study the properties of the solutions, it is necessary to impose some conditions, such as \( \Omega \) is a bounded domain in \( \mathbb{R}^d, d \geq 2, \) and the velocity satisfies the Dirichlet boundary condition \( v|_{\partial \Omega} = \varphi \) and initial conditions \( v(0) = v_0; \) or \( \Omega = \mathbb{R}^d, d \geq 2, \) then give the data of \( v \) in the far field and the initial time, which is called Cauchy problem. In all these cases, to our knowledge, the uniqueness of the weak solution to the system (1.2) in 3D with general initial data \( v_0, \) or equivalently, the higher order regularity of the weak solution, is still an open problem.

It should be noted that most of the existing results mainly focus on the Dirichlet boundary conditions, i.e. no-slip boundary conditions. However, there are many other kinds of boundary conditions which also match with the reality. For example, hurricanes and tornadoes do slip along the ground and lose energy as they slip[4]. In 1827, the famous mathematician and physicist C. Navier[20] first considered the slip phenomena and proposed the following boundary conditions, called Navier-slip boundary conditions:
\[
\begin{cases}
v \cdot n = 0, \\
2\mu \mathbb{D}(v)n \cdot \tau = k(x)v \cdot \tau,
\end{cases}
on \partial \Omega, \quad (1.3)
\]
where \( \mathbb{D}(v) = \frac{1}{2}(\nabla v + \nabla^T v), \) \( n \) and \( \tau \) are unit outer normal vector and tangential vector of the boundary \( \partial \Omega. \) In (1.3), \( k(x) \) is a physical parameter, which can be a constant, function in \( L^\infty(\partial \Omega)[16] \) and even a smooth metrix[12]. Here we consider the case that \( k(x) \) is constant, called the slip coefficient.

We should also mention that the most known discussions on the Navier-slip boundary value problems are for the “classical” cases in which the slip coefficients are non-positive, that is, \( k(x) \leq 0 \) in the boundary conditions (1.3), which is according with the friction effect. The pioneers in analysing the Navier-Stokes equations with Navier-slip boundary conditions should be Solonnikov and Ščadilov [23], who considered the linearized equations in steady case. Afterwards, B. da Veiga [25] established the existence and the regularity of the weak solutions for the nonlinear problem in the upper half space, while C. Amrouche et al.[3, 2] gave the corresponding results in bounded domain and external domain.

What we are interested in this paper is for the “non-classical” cases in which the slip coefficients may be positive, and the domain is unbounded. As being pointed out by Serrin[22] in 1959, \( k(x) \) is unnecessary to be negative. Moreover, there do be some phenomena in the real world with \( k(x) > 0. \) For example, the effective slip length \( \alpha \) on the flat gas-liquid interface is always positive[13]. Navier-slip boundary conditions (1.3) with \( k(x) > 0 \) is also widely applied in the numerical modeling of fluid with rough boundary, such as in aeronautical dynamics or in the permeable boundary, where (1.3) are called Beavers-Joseph law[6, 3], in the weather forecast and Hemodynamics[6, 7]), or some case where the boundary accelerates the fluid[19, 4]. The readers could refer to Y. Xiao et al.[26, 27] and the reference therein for some results on the vanishing viscosity limit of the time-depending Navier-Stokes
equations. For more details in physical applications and numerical analysis, please refer to [1, 5, 6, 14, 15, 21, 22].

In 2017, H. Li and X. Zhang [17] established the global well-posedness for the 3D compressible Navier-Stokes equations in a strip domain with Dirichlet boundary condition on the upper plane and Navier-slip boundary condition on the bottom. However, the slip coefficient must be a negative constant. In 2018, Xin and the authors of this paper published a paper [9] on the stability analysis for the Navier-slip boundary value problems for this "non-classical" cases. We found in [9] that if some of the slip coefficients are positive, the kinetic energy generated on the boundary may cause instability if the viscosity is not large enough. So, we defined, in [9], a critical viscosity expressed only by the slip coefficients to distinguish the stability from the instability. However, in that paper we did not give the detailed proof for the global existence and uniqueness of strong solutions for the so called "non-classical" cases. We find that the proof of the global well-posedness is not a trivial problem mainly because of the unboundedness of the domain and the boundary conditions. Moreover, in this paper, we have also derived a number of Gagliardo-Nirenberg inequalities which may be applicable to other similar cases.

The rest of this paper will be arranged as follows: in Section 2, some notations will be given and the definition together with the main theorems will be stated; Section 3 is arranged for preliminary, that is, the proof of a series of Gagliardo-Nirenberg inequality; the global existence of the unique weak solution to system (2.1) will be established in Section 4 and the proof of higher order regularity to the weak solution, so that the weak solution is in fact a strong one, will be given in Section 5.

2. Notations and main results. Precisely, we consider the following initial boundary value problem

\[ \begin{align*}
\partial_t u + u \cdot \nabla u + \nabla p &= \mu \Delta u + f, \quad \text{in } \Omega \times (0, T), \\
\nabla \cdot u &= 0, \quad \text{in } \Omega \times (0, T), \\
u \cdot n &= 0, \quad \text{on } \partial \Omega \times (0, T) \\
2\mu D(u) \cdot \tau &= k(x, y) u \cdot \tau, \quad \text{on } \partial \Omega \times (0, T) \\
u(0) &= u_0, \quad \text{in } \Omega.
\end{align*} \]

(2.1)

where \( k(x, 1) = k_1, k(x, 0) = k_0 \) are constants and \( \Omega := \mathbb{R} \times (0, 1) \). For convenience, we denote

\[ \begin{align*}
H &:= \{ v \in L^2 | \nabla \cdot v = 0, \ v \cdot n = 0, \ \text{on } \partial \Omega \}, \\
V &:= \{ v \in H^1 | \nabla \cdot v = 0, \ v \cdot n = 0, \ \text{on } \partial \Omega \}, \\
W &:= \{ v \in V \cap H^2 | v \text{ satisfies (1.3)} \}. \end{align*} \]

In the meantime, we denote \( L^2(0, 1) \) and \( H^k(0, 1) \) by \( L^2 \) and \( H^k \), for simplicity. Without confusion, we will also write \( L^p(\Omega) \) and \( H^k(\Omega) \) by \( L^p \) and \( H^k \), respectively. The integral form \( \int_\Omega f dx dy \) will be simply denoted by \( \int f \). In addition, the scalar function and vector function will be denoted by \( f \) and \( \mathbf{f} \) for distinction, such as \( \mathbf{f} = (f^1, f^2) \), but the product functional space \( (X)^2 \) will also be denoted by \( X \). For example, the vector function \( \mathbf{u} \in (H^1)^2 \) will be still denoted by \( \mathbf{u} \in H^1 \). The usual notations will be used as in general unless extra statement. Since that \( \mu > 0 \) plays no role in the proof of the global well-posedness, we take \( \mu = 1 \), for simplification.
We will first prove the global existence of the unique weak solution to (2.1), and then improve the regularity to reach the global strong solution. Now we give the definition of weak solutions.

**Definition 2.1.** \( u \) is a weak solution to the initial boundary value problem (2.1) defined in \( \Omega \times (0, T) \), if it satisfies

(i) \( u \in L^\infty (0, T; H) \cap L^2 (0, T; V) \); (ii) \( u(x, y, 0) = u_0(x, y) \);

(iii) For any \( v \in V \), there holds that

\[
\frac{d}{dt} \int_\Omega u(t) \cdot v + \int_\Omega D(u(t)) : D(v) + \int_\Omega u(t) \cdot \nabla u(t) \cdot v = \int_{\partial \Omega} k(x, y) u(t) \cdot v dS + \int_\Omega f \cdot v.
\]

We do not mention the space of the initial data and the external force here, but these will be clear when the existence of the solutions is established. For the first step, we prove the following global well-posedness in weak sense

**Theorem 2.2.** For any \( u_0 \in H, T > 0, \) and \( f \in L^1 (0, T; L^2) \), there exists an unique weak solution \( u \in L^\infty (0, T; H) \cap L^2 (0, T; V) \) to the initial boundary value problem (2.1), which satisfies

\[
\sup_{0 \leq t \leq T} \| u(t) \|_{L^2}^2 + \int_0^T \| u(t) \|_{H^1}^2 \leq C \left( T, \| u_0 \|_{L^2}, \| f \|_{L^1 (0, T; L^2)} \right).
\]

Then, for higher regularity on the initial data and the external force, we prove the following global well-posedness

**Theorem 2.3.** For any \( u_0 \in W, T > 0 \) and \( f \in H^1 (0, T; L^2) \), initial boundary value problem (2.1) exists an unique strong solution \( u \in L^\infty (0, T; H^2) \) satisfying

\[
\sup_{0 \leq t \leq T} (\| u(t) \|_{H^2} + \| \nabla p(t) \|_{L^2}) \leq C \left( T, \| u_0 \|_{H^2}, \| f(t) \|_{H^1 (0, T; L^2)} \right).
\]

Before continuing, we would like to have some words on the main result of this paper.

**Remark 2.4.** (1) Theorem 2.3 is valid without any smallness constraint on the initial data, which is consistent with the existed results in dimension 2. (2) The global existence of the unique strong solution is valid without any constraint on the sign of the slip coefficients.

3. Preliminary. Note that the domain \( \Omega \) is unbounded and the boundary \( \partial \Omega \) is non-compact, which lead to the difficulty in finding the smooth orthonormal basis for the construction of Galerkin approximate solutions. Thus, we first find solutions in a subdomain \( \Omega_L := (-L, L) \times (0, 1) \) with the similar Navier-slip boundary conditions on \( (-L, L) \times [0, 1] \), where \( k(-L, y) = k(L, y) = 0 \). We infer that the definition of weak solution is similar to that on \( \Omega \) and denote the constraint of \( H, V, W \) in \( \Omega_L \) by \( H_L, V_L, W_L \), respectively. Without lose of any generality, we take \( L \geq 1 \).

To apply the Galerkin method in proving the existence of the unique solution, we need the following two lemmas, which are similar to Lemma 2.1 and Lemma 2.2 of [8], respectively.

**Lemma 3.1.** Assume that \( v \in H^2 (\Omega_L) \) satisfies \( v \cdot n = 0 \) on the boundary \( \partial \Omega_L \). Then it holds on the boundary that

\[
2D(v) n \cdot \tau = \text{curl} v,
\]
Proof. Refer to the proof of Lemma 2.1 in [8].

Lemma 3.2. There exists a basis \{w_1, w_2, \cdots, w_n, \cdots\} \subset H^2(\Omega_L) to V_L, such that

\[
2\mathcal{D}(w_m)u \cdot \tau = k(x, y)w_m \cdot \tau, \quad \text{on } \partial\Omega_L, \quad m = 1, 2, \cdots.
\]

Moreover, \{w_1, w_2, \cdots, w_n, \cdots\} is also an orthonormal basis of H_L.

Proof. The main idea of the proof, which consists of three steps, is quite different from that of Lemma 2.2 in [8], for the domain here is a rectangular region.

**STEP 1.** For some positive constant \(\beta\) large enough, and \(g \in H_L\), consider the auxiliary problem

\[
\begin{cases}
-\Delta u + \nabla p + \beta u = g, & \text{in } \Omega_L, \\
\text{div} u = 0, & \text{in } \Omega_L, \\
u \cdot n = 0, & \text{on } \partial\Omega_L, \\
\mathcal{D}(u)u \cdot \tau = k(x, y)u \cdot \tau, & \text{on } \partial\Omega_L,
\end{cases}
\tag{3.1}
\]

of which the variational form is to seek \(u \in V_L\) and \(\lambda \neq 0\), such that for any \(v \in V_L\), there holds

\[
\int_{\Omega_L} \nabla u \cdot \nabla v + \beta \int_{\partial\Omega_L} u \cdot v + \int_{\partial\Omega_L} k(x, y)(u \cdot \tau)(v \cdot \tau) = \int_{\Omega} g \cdot v.
\tag{3.2}
\]

Note that the bilinear form

\[
a(u, v) := \int_{\Omega_L} \nabla u \cdot \nabla v + \beta \int_{\partial\Omega_L} u \cdot v + \int_{\partial\Omega_L} k(x, y)(u \cdot \tau)(v \cdot \tau)
\]

is continuous and symmetric on \(V_L \times V_L\). In particular, when \(v = u\), one gets for any \(\varepsilon > 0\) that

\[
\int_{\partial\Omega_L} k(x, y)(u \cdot \tau)^2 = \int_{\Omega_L} \left[\left((k_0 + k_1)y - k_0\right)u^1\right]^2_y
\geq -\varepsilon\|\partial_y u^1\|^2_{L^2(\Omega_L)} + (k_0 + k_1 - \varepsilon^{-1}\max\{k_1^2, k_0^2\})\|u^1\|^2_{L^2(\Omega_L)}.
\]

Then, as long as \(\beta\) being so large that

\[
\beta > \beta_0 := \varepsilon^{-1}\max\{k_1^2, k_0^2\} - (k_0 + k_1), \quad \varepsilon \in (0, 1),
\]

the bilinear form \(a(\cdot, \cdot)\) satisfies

\[
a(u, u) \geq (1 - \varepsilon)\|\nabla u\|^2_{L^2(\Omega_L)} + (\beta - \beta_0)\|u\|^2_{L^2(\Omega_L)} \geq c_0\|u\|^2_{H^1(\Omega_L)}.
\]

This indicates that \(a(\cdot, \cdot)\) is coercive on \((V_L, V_L)\). By Riesz Representation Theorem, there exists an unique \(u \in V_L\), such that \(a(u, v) = (g, v)\), for any \(v \in V_L\). A linear invertible operator \(L\) then defined by this problem, which can be expressed by \(Lu = g\), and also by \(u = L^{-1}g \in V_L\).

**STEP 2.** Denote \(S = L^{-1}\). It is clear that the embedding map \(V_L \hookrightarrow H_L\) is compact. Then, operator \(S\) is a bounded, linear, compact operator from \(H_L\) to \(H_L\). The symmetry of \(S\) can also be deduced by the symmetry of bilinear form \(a(\cdot, \cdot)\).

Therefore, the spectral theory of operators implies that \(S\) possesses countable real positive eigenvalues and there are corresponding eigenfunctions which consist of an orthonormal basis of \(H_L\).

Let \(\eta > 0\) be the real positive eigenvalue of \(S\) and \(w \in H_L\) be the corresponding eigenfunction. Then one has \(Sw = \eta w\) if and only if \(Lw = \lambda w\), where \(\lambda = \frac{1}{\eta}\).
Then, we infer that there exist countable positive eigenvalues \( \{ \lambda_j \}_{j=1}^\infty \) of \( \mathcal{L} \) with the corresponding eigenfunctions \( \{ w_j \}_{j=1}^\infty \) constitute a basis of \( \mathbf{V}_L \), which, in the meantime, is also an orthonormal basis of \( \mathbf{H}_L \).

Denote \( \Lambda = -\beta + \lambda \). Then the eigenvalue problem

\[
\begin{cases}
-\Delta w + \nabla p = \Lambda w, & \text{in } \Omega_L, \\
\text{div}w = 0, & \text{in } \Omega_L, \\
w \cdot n = 0, & \text{on } \partial \Omega_L, \\
\mathbb{D}(w)n \cdot \tau = k(x,y)w \cdot \tau, & \text{on } \partial \Omega_L,
\end{cases}
\]  
(3.3)

possesses countable eigenvalues \( \{ \lambda_j \}_{j=1}^\infty \), satisfying \(-\beta < \Lambda_1 < \Lambda_2 < \cdots \) (Ref.[10])§6.2).

STEP 3. Now, we apply bootstrap method to improve regularity of the eigenfunctions \( \{ w_j \}_{j=1}^\infty \). As \( w \) satisfies (3.3), there exists steam function \( \psi \) such that \( w = (-\partial_y, \partial_x)\psi \). Further, denote \( \omega := \partial_x w^2 - \partial_y w^1 \). Then \( \psi \) satisfies the following Dirichlet problem

\[
\begin{cases}
-\Delta \psi = -\omega, & \text{in } \Omega_L, \\
\psi = 0, & \text{on } \partial \Omega_L.
\end{cases}
\]  
(3.4)

Let \( g(x,y) := [(k_0 + k_1)y - k_0]w^1(x,y) \). In virtue of (3.3) together with Lemma 3.1, we deduce that \( W = \omega - g \) satisfies the Dirichlet boundary value problem

\[
\begin{cases}
-\Delta W = \Lambda \omega + \Delta g, & \text{in } \Omega_L, \\
W = 0, & \text{on } \partial \Omega_L,
\end{cases}
\]  
(3.5)

Note that \( w \in H^1(\Omega_L) \), namely, the right-hand side of (3.5)_1 belongs to \( H^{-1}(\Omega_L) \). Then it follows from the elliptic estimate that \( W \in H^1_0(\Omega_L) \), which further implies that \( \omega \in H^1(\Omega_L) \). Consequently, applying the theory of elliptic equations to system (3.4) yields \( \psi \in H^3(\Omega_L) \), which indicates \( w \in H^2(\Omega_L) \). The proof of this lemma is completed.

In general, the uniform constants in Gagliardo-Nirelberg inequalities depend on the shape or the size of the domain. To obtain the independency of \( L \) in the energy estimates, we need the following Gagliardo-Nirelberg inequalities, of which the uniform constants are independent of the horizontal length \( L \). The authors believe that these inequalities will be applicable in other similar situations.

**Lemma 3.3.** ( \( L^2(\Omega_L) \) estimate) There exists a constant \( C > 0 \), being independent of \( L \), such that for any \( u \in \mathbf{V}_L \), there holds

\[
\|u\|_{L^2(\Omega_L)} \leq C \|\partial_y u\|_{L^2(\Omega_L)}.
\]  
(3.6)

**Proof.** First prove for \( u^2 \). Since \( u^2(x,0) = u^2(x,1) = 0 \), there holds \( u^2(x,y) = \int_0^y u^2(x,\theta)d\theta \). Then

\[
\int_{\Omega_L} |u^2(x,y)|^2dx\,dy = \int_{\Omega_L} \left( \int_0^y u^2(x,\theta)d\theta \right)^2dx\,dy \leq \int_{\Omega_L} |u^2_0(x,y)|^2dx\,dy.
\]

To \( u^1 \), it follows from the incompressible condition that

\[
\int_0^1 u^1_y(x,y)dy = -\int_0^1 u^2_y(x,y)dy = 0.
\]
i.e. $\int_0^1 u^1(x,y) dy$ is a constant. Besides, it is clear that

$$0 = \int_{\partial \Omega_L} xu \cdot n dS = \int_{\Omega_L} \text{div}(xu) = \int_{\Omega_L} u^1(x,y) dx dy,$$

which means $\int_0^1 u^1(x,y) dy \equiv 0$. The it follows from the Poincaré inequality on the vertical direction that

$$\int_0^1 |u^1(x,y)|^2 dy \leq C \int_0^1 |u_y^1(x,y)|^2 dy$$

holds for some constant $C > 0$. Integrating this inequality with respect to $x$ completes the proof of this lemma.

**Corollary 3.4. ($L^2(\Omega)$ estimate)** There exists a constant $C > 0$, such that for any $u \in V$, it is valid that

$$\|u\|_{L^2} \leq C\|\partial_y u\|_{L^2}. \quad (3.7)$$

**Lemma 3.5. ($L^4(\Omega_L)$ estimate)** There exists a constant $C > 0$, being independent of $L$, such that for any $u \in \mathcal{W}_L$, there holds

$$\|u\|_{L^4(\Omega_L)}^4 \leq C\|u\|_{L^2(\Omega_L)} \|\nabla u\|_{L^2(\Omega_L)}. \quad (3.8)$$

**Proof.** Note that the boundary conditions (2.1)$_3$ and (2.1)$_4$ can be rewritten as

$$u^1(\pm L, y) = 0; \quad u^2(x, 0) = u^2(x, 1) = 0;$$

$$\partial_y u^1(x, 0) = -k_0 u^1(x, 0), \quad \partial_y u^1(x, 1) = k_1 u^1(x, 1); \quad \partial_x u^2(\pm L, y) = 0.$$

We first claim that if $f \in H^1(\Omega_L)$ satisfies $f(-L, y) = f(x, 0) = 0$, then

$$\|f\|_{L^2}^2 \leq 2\|f\|_{L^2} \|\nabla f\|_{L^2}. \quad (3.9)$$

In fact, we have

$$|f(x, y)|^2 = 2 \int_{-L}^x f(s, y) f_x(s, y) ds \leq 2 \|f(y)\|_{L^2(-L, L)} \|f_x(y)\|_{L^2(-L, L)},$$

and

$$|f(x, y)|^2 = 2 \int_0^y f(x, \theta) f_y(x, \theta) d\theta \leq 2 \|f(x)\|_{L^2(0, 1)} \|f_y(x)\|_{L^2(0, 1)}.$$

Multiplying the above two equations, integrating over $\Omega_L$, and using Hölder inequality, we get

$$\int_{\Omega_L} |f(x, y)|^4 \leq 4 \int_0^1 \|f(y)\|_{L^2(-L, L)} \|f_x(y)\|_{L^2(-L, L)} \int_{-L}^L \|f(x)\|_{L^2(0, 1)} \|f_y(x)\|_{L^2(0, 1)}$$

$\leq 4\|f\|_{L^2(\Omega_L)}^2 \|f_x\|_{L^2(\Omega_L)} \|f_y\|_{L^2(\Omega_L)}.$

Then (3.9) follows.

Now, we prove (3.8) for $u^1$. Denote $\zeta(y)$ to be a smooth cut-off function on $\mathbb{R}$ satisfies (1) when $|y| \leq 1$, $\zeta(y) \equiv 1$: (2) when $|y| \geq 2$, $\zeta(y) \equiv 0$: (3) for any $y \in \mathbb{R}$ there holds $\zeta(y) \in [0, 1]$ and $|\zeta'(y)| \leq 2$: (4) $\zeta(y) = \zeta(-y)$. Further denote $v^1(x, y) := \zeta(2y) \hat{u}^1(x, y)$, where

$$\hat{u}^1(x, y) := \begin{cases} e^{ky} u^1(x, y), & y \in [0, 1], \\ e^{-ky} u^1(x, -y), & y \in [-1, 0]. \end{cases}$$
Then \(v^1 \in H^1([-L, L] \times [-1, 1])\) with \(v^1(x, -1) = v^1(x, 1) = v^1(\pm L, y) = 0\). It can be deduced by (3.9) together with the symmetry of \(v^1\) that
\[
\left(\int_{\Omega_L} |v^1(x, y)|^4 \right)^{1/2} \leq 2\sqrt{2} \left(\int_{\Omega_L} |v^1(x, y)|^2 \right)^{1/2} \left(\int_{\Omega_L} |\nabla v^1(x, y)|^2 \right)^{1/2}.
\]

In virtue of the definitions of \(v^1\) and \(\zeta\), one can rewrite the above inequality as
\[
\left(\int_{-L}^{L} \int_{-1}^{1} |v^1(x, y)|^4 \right)^{1/2} \leq C(k_0) \left(\|u^1\|_{L^2(\Omega_L)} \|\nabla u^1\|_{L^2(\Omega_L)} + \|u^1\|^2_{L^2(\Omega_L)}\right).
\]

Similarly, it is valid that
\[
\left(\int_{-L}^{L} \int_{-1}^{1} |v^1(x, y)|^4 \right)^{1/2} \leq C(k_1) \left(\|u^1\|_{L^2(\Omega_L)} \|\nabla u^1\|_{L^2(\Omega_L)} + \|u^1\|^2_{L^2(\Omega_L)}\right).
\]

Adding them up and using (3.6) yield inequality (3.8) for \(u^1\).

In what follows, we prove inequality (3.8) for \(u^2\). Write \(v^2 := \zeta \left(\frac{x}{L} + 1\right) \tilde{u}^2\) with
\[
\tilde{u}^2(x, y) := \begin{cases} 
 u^2(x, y), & x \in [-L, L] \\
 u^2(-x - 2L, y), & x \in [-3L, -L].
\end{cases}
\]

Then \(v^2(x, y)\) satisfies \(v^2(-3L, y) = v^2(L, y) = v^2(x, 0) = v^2(x, 1) = 0\).

By (3.9) and the symmetry of \(v^2\), we have
\[
\left(\int_{\Omega_L} |v^2(x, y)|^4 \right)^{1/2} \leq 2\sqrt{2} \|v^2(x, y)\|_{L^2(\Omega_L)} \|\nabla v^2(x, y)\|_{L^2(\Omega_L)}.
\]

Similarly, according to the definition of \(v^2, \zeta\), we rewrite the above inequality as
\[
\left(\int_{-L}^{0} \int_{0}^{1} |u^2(x, y)|^4 \right)^{1/2} \leq C \left(\|u^2\|_{L^2(\Omega_L)} \|\nabla u^2\|_{L^2(\Omega_L)} + \|u^2\|^2_{L^2(\Omega_L)}\right).
\]

We should point out here that the constant \(C\) depends on \(L^{-1}\) because of the derivation of \(\zeta \left(\frac{x}{L} + 1\right)\). However, since our final end is to take \(L \to +\infty\), so, without lose of any generality, we take \(L \geq 1\), and then \(C\) is independent of \(L\).

Besides, we also have
\[
\left(\int_{0}^{L} \int_{0}^{1} |u^1(x, y)|^4 \right)^{1/2} \leq C \left(\|u^2\|_{L^2(\Omega_L)} \|\nabla u^2\|_{L^2(\Omega_L)} + \|u^2\|^2_{L^2(\Omega_L)}\right).
\]

Adding them up and using (3.6) deduce inequality (3.8) for \(u^2\).

As the result of Lemma 3.5 is independent of \(L\), we take \(L \to \infty\) and yield the desired \(L^4\) estimate for functions in \(W\) as follows.

**Corollary 3.6. (\(L^4(\Omega)\) estimate)** There exists a constant \(C > 0\), such that for any \(u \in W\), it holds that
\[
\|u\|^2_{L^4} \leq C\|u\|_{L^2} \|\nabla u\|_{L^2}.
\]

**Lemma 3.7. (\(L^2(\Omega_L)\) estimate for gradient)** There exists constant \(C > 0\), being independent of \(L\), such that for any \(u \in W_L\), there holds
\[
\|\nabla u\|_{L^2(\Omega_L)}^2 \leq C\|u\|_{L^2(\Omega_L)} \|\nabla u\|_{H^1(\Omega_L)}.
\]
Proof. In fact, in virtue of integrating by parts and Hölder inequality, we have
\[ \int_{\Omega} |\nabla u|^2 = - \int_{\Omega} u \cdot \Delta u + k_1 \int_{-L}^{L} |u^1(x,1)|^2 + k_0 \int_{-L}^{L} |u^1(x,0)|^2 \]
\[ \leq \int_{\Omega} |u||\Delta u| + \int_{\Omega} [(k_1 + k_0)y - k_0)|u^1(x,y)|^2]dydxdy \]
\[ \leq \|u\|_{L^2(\Omega_L)} \|\Delta u\|_{L^2(\Omega_L)} + C\|u\|_{L^2(\Omega_L)}^2 + C\|u\|_{L^2(\Omega_L)} \|\nabla u\|_{L^2(\Omega_L)} \]
\[ \leq C\|u\|_{L^2(\Omega_L)} \|\nabla u\|_{H^1(\Omega_L)}, \]
in which (3.6) has been used in the last inequality.

**Corollary 3.8. (L^2(\Omega) estimate for gradient)** There exists a constant \( C > 0 \), such that for any \( u \in \mathcal{W} \), it is true that
\[ \|\nabla u\|^2_{L^2} \leq C\|u\|^2_{L^2} \|\nabla u\|^2_{H^1}. \] (3.12)

**Lemma 3.9. (Korn’s inequality on \( \Omega_L \))** There exists a constant \( C > 0 \), being independent of \( L \), such that for any \( u \in \mathcal{V}_L \), it holds that
\[ \|\mathcal{D}(u)\|_{L^2(\Omega_L)} \geq C\|u\|_{H^1(\Omega_L)}. \] (3.13)

**Proof.** Suppose that \( u \in \mathcal{V}_L \cap H^2(\Omega_L) \). Then one has
\[ \int_{\Omega_L} |\nabla u + \nabla^T u|^2 = 2\int_{\Omega_L} |\nabla u|^2 + 2\sum_{i,j=1}^2 \int_{\Omega_L} \partial_i u^i \partial_j u^j. \] (3.14)

In virtue of integration by parts, we get
\[ \sum_{i,j=1}^2 \int_{\Omega_L} \partial_i u^i \partial_j u^j = -2\sum_{i,j=1} \int_{\partial\Omega_L} \partial_i \partial_j u^i u^j + 2\sum_{i,j=1} \int_{\partial\Omega_L} \partial_i u^i \partial_j n^j dS. \]

Then, it follows from the incompressibility and boundary condition \( u \cdot n = 0 \) that the right-hand side of the above equality is 0. Consequence, (3.13) is true for \( u \in \mathcal{V}_L \cap H^2(\Omega_L) \) from (3.14) together with (3.6). As \( \mathcal{V}_L \cap H^2(\Omega_L) \) is dense in \( \mathcal{V}_L \), (3.13) is also valid for \( u \in \mathcal{V}_L \). □

In addition, we have

**Corollary 3.10. (Korn’s inequality on \( \Omega \))** There exists a constant \( C > 0 \), such that for any \( u \in \mathcal{V} \), there holds
\[ \|\mathcal{D}(u)\|_{L^2} \geq C\|u\|_{H^1}. \] (3.15)

**Lemma 3.11. (L^\infty(\Omega_L) estimate)** There exists a constant \( C > 0 \), being independent of \( L \), such that for any \( u \in \mathcal{W}_L \), there holds
\[ \|u\|^2_{L^\infty(\Omega_L)} \leq C\|u\|^2_{L^2(\Omega_L)} \|u\|^2_{H^2(\Omega_L)}. \] (3.16)

**Proof.** First prove (3.16) for \( u^1 \). Similar to the proof of Lemma 3.5, we denote \( v^1(x,y) = \zeta(2y)\tilde{u}^1(x,y) \), where
\[ \tilde{u}^1(x,y) := \begin{cases} e^{k_0y}u^1(x,y), & y \in [0,1] \\ e^{-k_0y}u^1(x,-y), & y \in [-1,0]. \end{cases} \]

Then, \( v^1(x,y) \) satisfies
\[ v^1(\pm L, y) = v^1(x, \pm 1) = 0, \]
and hence we have
\[ |u^1(x, y)|^2 = 2 \int_{-1}^{y} v_y^1(x, \theta)v^1(x, \theta)d\theta = 2 \int_{-L}^{L} \int_{-1}^{y} |v_y^1v^1 + v_y^1v_x^1|(s, \theta)d\theta ds \leq 2 \int_{-L}^{L} \int_{-1}^{1} |v_y^1v^1 + v_y^1v_x^1|(x, y)dxdy. \]

As \( v^1 \) is symmetric in the vertical direction and vanishes on the boundary, we use integration by parts together with Hölder inequality and yield
\[ \sup_{(x, y) \in \Omega_L} |u^1(x, y)|^2 \leq C\|v^1\|_{L^2(\Omega_L)}\|\nabla^2 v^1\|_{L^2(\Omega_L)}. \]

Further, in virtue of the definition of \( v^1 \), we infer that
\[ \sup_{(x, y) \in [-L, L] \times [0/2]} |u^1(x, y)|^2 \leq C\|u^1\|_{L^2(\Omega_L)}\|u^1\|_{H^2(\Omega_L)} \]  \( (3.17) \)

Similarly, we have
\[ \sup_{(x, y) \in [-L, L] \times [1/2]} |u^1(x, y)|^2 \leq C\|u^1\|_{L^2(\Omega_L)}\|u^1\|_{H^2(\Omega_L)} \]  \( (3.18) \)

This completes the proof of \( (3.16) \) for \( u^1 \).

For \( u^2 \), denote \( \tilde{v}^2 := \zeta(\frac{x + L}{4L})\tilde{v}^2 \), in which
\[ \tilde{v}^2(x, y) := \begin{cases} u^2(x, y), & x \in [-L, L] \\ u^2(-x - 2L, y), & x \in [-3L, -L]. \end{cases} \]

Obviously, \( \tilde{v}^2(x, y) \) satisfies
\[ v^2(-3L, y) = v^2(L, y) = v^2(x, 0) = v^2(x, 1) = 0, \]
and hence, we have
\[ |u^2(x, y)|^2 \leq 2 \int_{-3L}^{L} \int_{-1}^{1} |v_y^2v^2 + v_y^2v_x^2|(x, y)dxdy. \]

Since \( v^2 \) is symmetric in the horizontal direction and vanishes on the boundary, similarly, we obtain
\[ \sup_{(x, y) \in \Omega_L} |v^2(x, y)|^2 \leq C\|v^2\|_{L^2(\Omega_L)}\|\nabla^2 v^2\|_{L^2(\Omega_L)} \]  \( (3.19) \)

Further applying the definition of \( v^2 \) leads to
\[ \sup_{(x, y) \in [-L, 0] \times [0, 1]} |u^1(x, y)|^2 \leq C\|u^2\|_{L^2(\Omega_L)}\|u^2\|_{H^2(\Omega_L)} \]  \( (3.20) \)

Similarly, we also have
\[ \sup_{(x, y) \in [0, L] \times [0, 1]} |u^2(x, y)|^2 \leq C\|u^2\|_{L^2(\Omega_L)}\|u^2\|_{H^2(\Omega_L)} \]  \( (3.21) \)

The proof of this lemma is finished.

As constant \( C > 0 \) in inequality \( (3.16) \) is independent of \( L \), we take \( L \to \infty \) and yield

**Corollary 3.12. (\( L^\infty(\Omega) \) estimate)** There exists a constant \( C > 0 \), such that for any \( u \in W \), it holds
\[ \|u\|_{L^\infty} \leq C\|u\|_{L^2}\|u\|_{H^2}. \]  \( (3.22) \)
4. Proof of Theorem 2.2. In this section, we prove the existence and uniqueness of the weak solution to initial boundary value problem (2.1) in Ω.

In virtue of Lemma 3.2, the function space $V_L$ possess a basis $\{v_j\}_{j=1}^\infty \subset H^2(\Omega_L)$, which is also an orthonormal basis of $H_L$. For any fixed $m \in \mathbb{N}_+$, we seek approximate solutions in the form $u_m(t) = \sum_{j=1}^m g_j^m(t) v_j$, satisfying

$$\frac{d}{dt} \int_{\Omega_L} u_m(t) \cdot v_k + 2 \int_{\Omega_L} D(u_m(t)) : D(v_k) + \int_{\Omega_L} u_m(t) \cdot \nabla u_m(t) \cdot v_k = \int_{\partial\Omega_L} k(x, y) u_m(t) \cdot v_k dS + \int_{\Omega_L} f \cdot v_k$$

(4.1)

for any $k = 1, 2, \ldots, m$ and the initial data

$$u_m(0) = \sum_{j=1}^m (u_0, v_j) v_j.$$  

(4.2)

Combining (4.1) and (4.2) gives a Cauchy problem of ODEs for $(g_1^m(t), g_2^m(t), \ldots, g_m^m(t))$, in which the nonlinear terms is the zeroth-order ones. By the classical theory of the first order ODEs, it possesses a unique solution $(g_1^m(t), g_2^m(t), \ldots, g_m^m(t)) \in C^1[0, T_m)$, with $T_m$ being the maximum life time. Hence, there exists a unique solution $u_m(t) \in C^1([0, T_m); V_L)$ to problem (4.1)-(4.2). In order to take $m \to \infty$ and extend $T_m$ to $T$, we first get the uniform energy estimate.

Multiplying $g_k^m(t)$ to both sides of (4.1) and adding them up from $k = 1$ to $k = m$, integrating the results by parts yield

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega_L} |u_m(t)|^2 + 2 \int_{\Omega_L} |D(u_m(t))|^2 = \int_{\partial\Omega_L} k(x, y)|u_m(t)|^2 + \int_{\Omega_L} f \cdot u_m(t).$$

(4.3)

Note that

$$\int_{\partial\Omega_L} k(x, y)|u_m(t)|^2 = \int_{\Omega_L} (k_1|u_1(x, 1, t)|^2 + k_0|u_1(x, 0, t)|^2) dx$$

$$= \int_{\Omega_L} \left[ ((k_1 + k_0)y - k_0)|u_1(x, y, t)|^2 \right] y dx dy$$

$$\leq C \|u_m(t)\|_{L^2(\Omega_L)}^2 + \|\nabla u_m(t)\|_{L^2(\Omega_L)}^2,$$

(4.4)

$$\int_{\Omega_L} f(t) \cdot u_m(t) \leq \|u_m(t)\|_{L^2(\Omega_L)} \|f(t)\|_{L^2(\Omega_L)}.$$  

(4.5)

Substituting (4.4) and (4.5) into (4.3), together with using Korn’s inequality (3.13) and taking $\varepsilon$ small sufficiently, one has

$$\frac{d}{dt} \|u_m(t)\|_{L^2(\Omega_L)}^2 + \|u_m(t)\|_{H^1(\Omega_L)}^2$$

$$\leq C \|u_m(t)\|_{L^2(\Omega_L)}^2 + C \|u_m(t)\|_{L^2(\Omega_L)} \|f(t)\|_{L^2(\Omega_L)}.$$

(4.6)

Then, applying Gronwall’s inequality gives

$$\sup_{0 \leq t \leq T} \|u_m(t)\|_{L^2(\Omega_L)}^2 + \int_0^T \|u_m(t)\|_{H^1(\Omega_L)}^2$$

$$\leq C \left( \|u_0\|_{L^2(\Omega_L)}^2 + \|f(t)\|_{L^1(0, T; L^2(\Omega_L))}^2 \right).$$

(4.7)

Now, we are on the position taking the limit $m \to \infty$. (4.7) means that the sequence $\{u_m(t)\}_{m=1}^\infty$ is bounded in $L^\infty(0, T; H_L) \cap L^2(0, T; V_L)$, and hence the maximum life time $T_m$ can be extended to $T$. Consequently, there exists a function $u^L \in \cdots$
Since for any \( \mathbf{v} \in \mathbf{V}_L \), the subsequence \( \{ \mathbf{u}_m(t) \}_{m=1}^{\infty} \) satisfies

\[
\frac{d}{dt} \int_{\Omega_L} \mathbf{u}_m(t) \cdot \mathbf{v} + 2 \int_{\Omega_L} \mathbf{D}(\mathbf{u}_m(t)) : \mathbf{D}(\mathbf{v}) + \int_{\Omega_L} \mathbf{u}_m(t) \cdot \nabla \mathbf{u}_m(t) \cdot \mathbf{v} = \int_{\partial \Omega_L} k(x,y) \mathbf{u}_m(t) \cdot \mathbf{v} dS + \int_{\Omega_L} \mathbf{f} \cdot \mathbf{v},
\]

(4.8)

where the first term in the right-hand side can be rewritten as

\[
\int_{\Omega_L} [(k_1 + k_0)y - k_0] \mathbf{u}_m(t) \cdot \mathbf{v} = (k_1 + k_0) \int_{\Omega_L} \mathbf{u}_m(t) \cdot \mathbf{v} + \int_{\Omega_L} [(k_1 + k_0)y - k_0] \mathbf{v} \mathbf{u}_m \cdot \mathbf{v} + \int_{\Omega_L} [(k_1 + k_0)y - k_0] \mathbf{v} \mathbf{u}_m \cdot \mathbf{v}.
\]

(4.9)

Note that, \( G^m(t) := \int_{\Omega_L} \mathbf{u}_m(t) \cdot \mathbf{v} \) uniformly converges to \( \int_{\Omega_L} \mathbf{u}^L(t) \cdot \mathbf{v} \) for a.e. \( t \in (0,T) \), and \( G^m(t) \) is derivable with respect to \( t \in [0,T] \). Then, \( \int_{\Omega_L} \mathbf{u}^L(t) \cdot \mathbf{v} \) is derivable with respect to \( t \in [0,T] \) and

\[
\frac{d}{dt} \int_{\Omega_L} \mathbf{u}^L(t) \cdot \mathbf{v} = \lim_{m \to \infty} \frac{d}{dt} \int_{\Omega_L} \mathbf{u}_m(t) \cdot \mathbf{v}.
\]

In addition,

\[
G^m(t) = \int_0^t \int_{\Omega_L} \mathbf{D}(\mathbf{u}_m(t)) : \mathbf{D}(\mathbf{v}) + \int_0^t \int_{\Omega_L} \mathbf{D}(\mathbf{u}^L(t)) : \mathbf{D}(\mathbf{v}), \text{ uniformly for } t \in (0,T)
\]

as \( m \to \infty \), and \( G^m(t) \) is derivable with respect to \( t \in [0,T] \). Then, \( \int_0^t \int_{\Omega_L} \mathbf{D}(\mathbf{u}^L(t)) : \mathbf{D}(\mathbf{v}), \text{ uniformly for } t \in [0,T] \).

The convergence of the other terms in (4.8) can be analyzed similarly. Thus, taking \( m \to \infty \) deduces that \( \mathbf{u}^L \) satisfies

\[
\frac{d}{dt} \int_{\Omega_L} \mathbf{u}^L(t) \cdot \mathbf{v} + 2 \int_{\Omega_L} \mathbf{D}(\mathbf{u}^L(t)) : \mathbf{D}(\mathbf{v}) + \int_{\Omega_L} \mathbf{u}^L(t) \cdot \nabla \mathbf{u}^L(t) \cdot \mathbf{v} = \int_{\partial \Omega_L} k(x,y) \mathbf{u}^L(t) \cdot \mathbf{v} dS + \int_{\Omega_L} \mathbf{f} \cdot \mathbf{v}, \text{ for any } \mathbf{v} \in \mathbf{V}_L,
\]

(4.10)

and by weak lower continuity[24], there holds

\[
\sup_{0 \leq t \leq T} \| \mathbf{u}^L(t) \|^2_{L^2(\Omega_L)} + \int_0^T \| \mathbf{u}^L(t) \|^2_{H^1(\Omega_L)} \leq C(T, \| \mathbf{u}_0 \|_{L^2(\Omega_L)}, \| \mathbf{f} \|_{L^1(0,T;L^2(\Omega_L))} ),
\]

(4.11)

where \( C \) is a constant being independent of \( L \). Consequently, we point out that the existence of the weak solution and estimate (2.2) follows so long as taking \( L \to \infty \).
To see this, denote $u_L^t = \zeta(2x/L)u^t$, where $\zeta(\cdot)$ is defined in the proof of (3.8). Then, $u_L^t$ satisfies
\begin{align*}
\text{div}u_L^t &= \frac{2}{L}\zeta'u_L^{1,L}, \text{ in } L^2(0,T;L^2(\Omega)) \\
u_L^t \cdot n &= 0, \text{ on } \partial\Omega, \\
\sup_{0 \leq t \leq T} \|u_L^t(t)\|_{L^2(\Omega)}^2 + \int_0^T \|u_L^t(t)\|_{H^1(\Omega)}^2 &\leq C \left( T, \|u_0\|_{L^2(\Omega)}, \|f\|_{L^1(0,T;L^2(\Omega))} \right), \quad (4.14)
\end{align*}
Therefore, there exists a subsequence, still denoted by $\{u_L^t\}$, and a function $u_*$, such that as $L \to +\infty$,
\begin{align*}
u_L^t &\to u_* \text{ weakly* in } L^\infty(0,T;L^2(\Omega)), \\
u_L^t &\to u_* \text{ weakly in } L^2(0,T;H^1(\Omega)).
\end{align*}
Next, choosing test function in (4.10) by $v^* = \zeta(2x/L)v$, where $v \in V$, we find that $u_L^t$ satisfies weak form
\begin{align*}
\frac{d}{dt} \int_{\Omega} u_L^t(t) \cdot v + 2 \int_{\Omega} \text{div}(u_L^t(t)) \cdot \text{div}(v) + \int_{\Omega} u_L^t(t) \cdot \nabla u_L^t(t) \cdot v - \int_{\partial\Omega} k(x,y)u_L^t(t) \cdot v dS \\
&= 2 \int_{\Omega} \text{div}(u_L^t) : \text{div}(v) - \text{div}(u_L^t) : \text{div}(v) + \int_{\Omega} u_L^t \cdot (\nabla u_L^t - \nabla u^t) \cdot v + \int_{\partial\Omega} f \cdot v. \\
&\leq \frac{C}{L} \|u_L^t\|_{H^1(\Omega_L)} \|v\|_{H^1(\Omega)} \to 0, \text{ as } L \to +\infty
\end{align*}
In view of $|\nabla[\zeta(2x/L)]| \leq \frac{C}{L}$, and the similar analysis technique in $m \to +\infty$, we have
\begin{align*}
\int_{\Omega} \text{div}(u_L^t) : \text{div}(v) - \text{div}(u_L^t) : \text{div}(v) &= \int_{\Omega} \nabla[\zeta(2x/L)] \cdot [\nabla v \cdot u_L^t - \nabla u_L^t \cdot v] \\
&\leq \frac{C}{L} \|u_L^t\|_{H^1(\Omega_L)} \|v\|_{H^1(\Omega)} \to 0, \text{ as } L \to +\infty.
\end{align*}
Note also that $|\zeta(2x/L) - 1| \to 0$, as $L \to +\infty$. We infer
\begin{align*}
\int_{\Omega} u_L^t \cdot (\nabla u_L^t - \nabla u^t) \cdot v \\
&= \int_{\Omega} \zeta(2x/L - 1)u_L^t \cdot \nabla u_L^t \cdot v + \int_{\Omega} (u_L^t \cdot \nabla \zeta(2x/L))(u^t \cdot v) \\
&\leq \left( |\zeta(2x/L) - 1| + \frac{C}{L} \right) \|u_L^t\|_{H^1(\Omega)} \|u_L^t\|_{H^1(\Omega_L)} \|v\|_{H^1(\Omega)} \\
&\to 0, \text{ as } L \to +\infty.
\end{align*}
The convergence of the other terms in (4.14) is the same as the convergence from (4.8) to (4.10). Therefore, $u_* \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ satisfies
\begin{align*}
\frac{d}{dt} \int_{\Omega} u_*(t) \cdot v + 2 \int_{\Omega} \text{div}(u_*(t)) \cdot \text{div}(v) + \int_{\Omega} u_*(t) \cdot \nabla u_*(t) \cdot v \\
&= \int_{\partial\Omega} k(x,y)u_*(t) \cdot v dS + \int_{\Omega} f \cdot v, \\
&\leq C \left( T, \|u_0\|_{L^2(\Omega_L)} \|f\|_{L^1(0,T;L^2(\Omega_L))} \right), \quad (4.17)
\end{align*}
which can be given by (4.14).
Besides, it is easy to see that, the righthand-side of (4.12) converges to 0 in $L^2(0, T; L^2(\Omega))$ as $L \to +\infty$, that is, $\text{div}\, u = 0$.

In conclusion, we have proved that $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$ is a weak solution to problem (2.1), in the sense of Definition 2.1.

Finally, we prove the uniqueness of the weak solution. Assume that there are two weak solutions $u_1(t), u_2(t) \in L^\infty(0, T; H) \cap L^2(0, T; V)$ to problem (2.1), satisfying weak formulation (4.16), estimate (4.17) and the initial data $u_1(0) = u_2(0) = u_0$.

Then, the difference $u(t) := u_1(t) - u_2(t)$ satisfies weak formula

$$
\frac{d}{dt} \int_\Omega \bar{u}(t) \cdot v + 2 \int_\Omega \nabla \bar{u}(t) : \nabla v + \int_\Omega u_1(t) \cdot \nabla \bar{u}(t) \cdot v + \int_\Omega \bar{u} \cdot \nabla u_2(t) \cdot v
= \int_{\partial \Omega} k(x, y) \bar{u}(t) \cdot v dS.
$$

Specially take $v = \bar{u}$. Then, integrating by parts, we reach

$$
\frac{d}{dt} \|\bar{u}(t)\|_{L^2}^2 + \|\bar{u}(t)\|_{H^1}^2 \leq C \int_\Omega |\bar{u}(t)|^2 |\nabla u_2(t)| + C \int_{\partial \Omega} k(x, y) |\bar{u}(t)|^2 dS.
$$

Using the skill in (4.9), we find that

$$
\int_{\partial \Omega} k(x, y) |\bar{u}(t)|^2 dS \leq C \|\bar{u}(t)\|_{L^2}^2 + C \|\bar{u}(t)\|_{H^1} \|\bar{u}(t)\|_{H^1}^2
\leq \varepsilon \|\bar{u}(t)\|_{H^1}^2 + C \|\bar{u}(t)\|_{L^2}^2.
$$

In addition, it follows from (3.10) that

$$
\int_\Omega |\bar{u}|^2 |\nabla u_2(t)| \leq \|\bar{u}(t)\|_{L^2}^2 \|\nabla u_2(t)\|_{L^2}
\leq \varepsilon \|\bar{u}(t)\|_{H^1}^2 + C \|\bar{u}(t)\|_{L^2}^2 \|\nabla u_2(t)\|_{L^2}^2
$$

Then, substituting (4.20) and (4.21) into (4.19) with $\varepsilon$ being small sufficiently implies that

$$
\frac{d}{dt} \|\bar{u}(t)\|_{L^2}^2 \leq C (1 + \|u_2(t)\|_{H^1}^2) \|\bar{u}(t)\|_{L^2}^2
$$

Applying Gronwall’s inequality to (4.22) with estimate (4.11) and the fact that $\bar{u}(0) \equiv 0$, we yield $\bar{u} \equiv 0$, which completes the proof of uniqueness, and also the proof of Theorem 2.2.

5. Proof of Theorem 2.3. In order to show that the weak solution is in fact a strong solution, we should obtain higher order energy estimates. We start with the approximate solutions in Section 4 with $u_0 \in W$ and $f \in H^1(0, T; L^2(\Omega))$.

Firstly, applying operator $\frac{d}{dt}$ to (4.1) and multiplying both sides of (4.1) by $\frac{d}{dt} \partial_t u_m(t)$, similar to (4.20), one gets

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega_L} |\partial_t u_m(t)|^2 + \int_{\Omega_L} |D(\partial_t u_m)(t)|^2
$$

$$
\leq \int_{\Omega_L} |\partial_t u_m(t)|^2 |\nabla u_m(t)| + \int_{\Omega_L} |\partial_t f \cdot \partial_t u_m(t)| + \int_{\partial \Omega_L} k(x, y) |\partial_t u_m(t)|^2
$$

$$
\leq \varepsilon \|\nabla \partial_t u_m(t)\|_{L^2(\Omega_L)}^2 + ||\partial_t f(t)||_{L^2(\Omega_L)}^2 + (\|u_m(t)\|_{H^1(\Omega_L)}^2 + 1) \|\partial_t u_m(t)\|_{L^2(\Omega_L)}^2.
$$
With \( \varepsilon \) small sufficiently, integrating the result over \((0, t)\) and using Korn’s inequality (3.13), we reach

\[
\| \partial_t u_m(t) \|_{L^2(\Omega_L)}^2 - \| \partial_t u_m(0) \|_{L^2(\Omega_L)}^2 + \int_0^t \| \partial_t u_m(s) \|_{H^1(\Omega_L)}^2 ds \\
\leq \int_0^t \left( \| u_m(s) \|_{H^1(\Omega_L)}^2 + 1 \right) \| \partial_t u_m(s) \|_{L^2(\Omega_L)}^2 ds + \| f \|_{H^1(0,T; L^2(\Omega_L))}^2.
\]  

(5.2)

To use Gronwall’s inequality, we should estimate \( \| \partial_t u_m(0) \|_{L^2(\Omega_L)}^2 \).

In fact, multiplying (4.1) by \( \frac{d}{dt} g_k^n(t) \), adding them up from \( k = 1 \) to \( k = m \), and using integration by parts formula, we have

\[
\int_{\Omega_L} |\partial_t u_m(t)|^2 \\
= -\int_{\Omega_L} \Delta u_m(t) \cdot \partial_t u_m(t) + \int_{\Omega_L} f \cdot \partial_t u_m(t) + \int_{\Omega_L} u_m(t) \cdot \nabla u_m(t) \cdot \partial_t u_m(t) \\
\leq \frac{3}{4} \| \partial_t u_m(t) \|_{L^2(\Omega_L)}^2 + \| u_m(t) \cdot \nabla u_m(t) \|_{L^2(\Omega_L)}^2 + \| \Delta u(t) \|_{L^2(\Omega_L)}^2 + \| f \|_{L^2(\Omega_L)}^2.
\]

Taking \( t \to 0 \) and using (3.16), we get

\[
\| \partial_t u_m(0) \|_{L^2(\Omega_L)} \leq C \left( \| u_0 \|_{H^2(\Omega_L)}^2 + \| f(0) \|_{L^2(\Omega_L)}^2 \right).
\]

Now, substituting it into (5.2) and using Gronwall’s inequality together with (4.7) give

\[
\sup_{0 \leq t \leq T} \| \partial_t u_m(t) \|_{L^2(\Omega_L)}^2 + \int_0^T \| \partial_t u_m(t) \|_{H^1(\Omega_L)}^2 dt \\
\leq C \left( T, \| u_0 \|_{H^2(\Omega_L)}, \| f(0) \|_{L^2(\Omega_L)}, \| f \|_{H^1(0,T; L^2(\Omega_L))} \right).
\]  

(5.3)

Besides, there holds that

\[
\| f(0) \|_{L^2(\Omega_L)} = \int_0^T \partial_t \left( \frac{s-T}{T} \| f(s) \|_{L^2(\Omega_L)}^2 \right) ds \leq C \left( T, \| f(t) \|_{H^1(0,T; L^2(\Omega_L))} \right).
\]

Then, (5.3) can further be simplified as

\[
\sup_{0 \leq t \leq T} \| \partial_t u_m(t) \|_{L^2(\Omega_L)}^2 + \int_0^T \| \partial_t u_m(t) \|_{H^1(\Omega_L)}^2 dt \\
\leq C \left( T, \| u_0 \|_{H^2(\Omega_L)}, \| f \|_{H^1(0,T; L^2(\Omega_L))} \right).
\]

Secondly, multiplying both sides of (4.1) by \( \frac{d}{dt} g_k^n(t) \) and adding them up respect to \( k \) from 1 to \( m \), using integration by parts formula, one obtains

\[
\frac{d}{dt} \int_{\Omega_L} |\nabla(u_m)(t)|^2 + \int_{\Omega_L} |\partial_t u_m(t)|^2 \leq \int_{\Omega_L} |u_m(t)|^2 |\nabla \partial_t u_m(t)| + \int_{\Omega_L} |f||\partial_t u_m(t)| \\
+ \int_{\partial \Omega_L} k(x,y) u_m(t) \cdot \partial_t u_m(t).
\]  

(5.4)

Similar to (4.20), one gets

\[
\int_{\partial \Omega_L} k(x,y) u_m(t) \cdot \partial_t u_m(t) \leq C |\nabla \partial_t u_m(t)|_{L^2(\Omega_L)}^2 + C |u_m(t)|_{H^1(\Omega_L)}^2;
\]

\[
\int_{\Omega_L} |f(t)||\partial_t u_m(t)| \leq \varepsilon \| \partial_t u_m(t) \|_{L^2(\Omega_L)}^2 + C |f(t)|_{L^2(\Omega_L)}^2.
\]
In addition, using Hölder inequality and Lemma 3.5 yields
\[
\int_{\Omega_L} |u_m(t)|^2 \|\nabla \partial_t u_m(t)\|_{L^2(\Omega_L)}^2 \leq \|\nabla \partial_t u_m(t)\|_{L^2(\Omega_L)}^2 + \|u_m(t)\|_{L^2(\Omega_L)}^2 \|u_m(t)\|_{H^1(\Omega_L)}^2.
\]
Substituting the above three into (4.7) with small \(\varepsilon\) reaches
\[
\frac{d}{dt} \int_{\Omega_L} |D(u_m(t))|^2 + \int_{\Omega_L} |\partial_t u_m(t)|^2 \leq \|(\nabla \partial_t u_m, f)(t)\|_{L^2(\Omega_L)}^2 + \|u_m(t)\|_{H^1(\Omega_L)}^2.
\]
Integrating (5.5) with respect to \(t\) over \((0, T)\) gives
\[
\sup_{0 \leq t \leq T} \left( \|u_m(t)\|^2_{H^1(\Omega_L)} + \|\partial_t u_m(t)\|^2_{L^2(0, T; L^2(\Omega_L))} \right) \leq C \left( T, \|u_0\|_{H^2(\Omega_L)}, \|f\|_{H^1(0, T; L^2(\Omega_L))} \right).
\]
Taking \(m \to +\infty\) and \(L \to +\infty\) like that in Section 4, we deduce that the unique weak solution \(u \in L^\infty(0, T; H) \cap L^2(0, T; V)\) satisfies \(\partial_t u \in L^\infty(0, T; H) \cap L^2(0, T; V)\) and
\[
\|u\|_{L^\infty(0, T; V)} + \|\partial_t u\|_{L^\infty(0, T; H) \cap L^2(0, T; V)} \leq C(T, \|u_0\|_{H^2(\Omega)}, \|f\|_{H^1(0, T; L^2(\Omega))}).
\]
Thirdly, to establish the \(H^2\) regularity, we need the following Stokes estimate.

**Proposition 5.1.** Assume that \(u \in H^1\) is the weak solution to the following boundary value problem
\[
\begin{cases}
-\Delta u + \nabla p = F, & \text{in } \Omega, \\
\nabla \cdot u = 0, & \text{in } \Omega, \\
u \cdot n = 0, & \text{on } \partial \Omega, \\
2D(u) n \cdot \tau = k(x, y) u \cdot \tau, & \text{on } \partial \Omega,
\end{cases}
\]
where \(F \in L^2\) and \(k(x, 1) = k_1, k(x, 0) = k_0\) are constants. Then \(u \in H^2\) and satisfies
\[
\|u\|_{H^2} + \|\nabla p\|_{L^2} \leq C \left( \|F\|_{L^2} + \|u\|_{L^2} \right),
\]
where \(C > 0\) depends only on \(k_0, k_1\).

**Proof.** The proof of this proposition consists of 4 steps.

**Step 1.** For any positive constant \(\beta\) large enough and functions \(u, F, k(x, y)\) given in (5.8), we consider the auxiliary problem:
\[
\begin{cases}
-\Delta w + \beta w = \text{curl}F + \beta \text{curl}u := \text{curl}\Phi, & \text{in } \Omega, \\
w = k(x, y) u \cdot \tau := g, & \text{on } \partial \Omega.
\end{cases}
\]
Since \(\text{curl}\Phi \in H^{-1}\), we define the bilinear form
\[
B[w, \tilde{w}] = \int_{\Omega} \nabla w : \nabla \tilde{w} + \beta \int_{\Omega} w \tilde{w},
\]
for \(w, \tilde{w} \in H^1_0 := \{ w \in H^1 | w = 0 \text{ in } \partial \Omega \}\). As the inhomogeneous Dirichlet boundary value problem problem (5.10) can be rewritten as a homogeneous one via homogenization method, without lose of any generality, we assume that \(g = 0\). It is easy to check that the bilinear form \(B\) is continuous and coercive on \(H^1_0\), and hence it
follows from the Lax-Milgram theorem that there exists an unique \( w \in H^1_y \) being the weak solution to system (5.10), i.e.

\[
\int_{\Omega} \nabla w \cdot \nabla \bar{w} + \beta \int_{\Omega} w \bar{w} = - \int_{\Omega} \Phi \cdot \nabla \bar{w},
\]

(5.12)

holds for any \( \bar{w} \in H^1_y \). Here \( \nabla \bar{w} := (-\partial_y, \partial_x) \).

Take \( \bar{w} = w - [(k_0 + k_1) y - k_0] u^1 \). Then \( \bar{w} \in H^1_y \). Substituting it into (5.12) and using Cauchy inequality gives

\[
\int_{\Omega} |\nabla w|^2 + \beta \int_{\Omega} |w|^2 \leq \frac{1}{2} \int_{\Omega} |\nabla w|^2 + \frac{\beta}{2} \int_{\Omega} |w|^2 + C \int_{\Omega} (|\Phi|^2 + |u|^2 + |\nabla u|^2),
\]

which indicates that

\[
\|w\|_{H^1} \leq C (\|\Phi\|_{L^2} + \|u\|_{H^1}).
\]

(5.13)

**Step 2.** For \( w \) constructed in **Step 1**, consider the following boundary value problem

\[
\begin{cases}
-\Delta \Psi = -w, & \text{in } \Omega, \\
\Psi = 0, & \text{on } \partial \Omega.
\end{cases}
\]

(5.14)

By the classical elliptic equation theory, problem (5.14) possesses a unique solution \( \Psi \in H^3 \). In what follows, we deduce \( H^3 \)-estimate for \( \Psi \).

Multiplying (5.14) by \( \Psi \), integrating by parts over \( \Omega \) and using Poincaré inequality, we get

\[
\|\Psi\|_{H^1} \leq C \|w\|_{L^2}.
\]

Applying \( \partial_x \) to (5.14)1, similarly, we deduce

\[
\|\Psi_x\|_{H^1} \leq C \|w\|_{L^2}.
\]

Moreover, it follows from (5.14)1 that \( \Psi_{yy} = w - \Psi_{xx} \). Then we also have

\[
\|\Psi_{yy}\|_{L^2} \leq \|w\|_{L^2} + \|\Psi_{xx}\|_{L^2} \leq C \|w\|_{L^2}.
\]

In conclusion, we yield \( \|\Psi\|_{H^2} \leq C \|w\|_{L^2} \).

Note again that \( \Psi_x \) satisfies

\[
\begin{cases}
-\Delta \Psi_x = -w_x, & \text{in } \Omega, \\
\Psi_x = 0, & \text{on } \partial \Omega.
\end{cases}
\]

(5.15)

Then, by the analysis above, one has \( \|\Psi_x\|_{H^2} \leq C \|w_x\|_{L^2} \). To obtain estimates for \( \Psi_{yyy} \), we apply \( \partial_y \) to (5.14)1 and yield \( \Psi_{yyy} = \partial_y w - \partial_y \Psi_{xx} \), which leads to

\[
\|\Psi_{yyy}\|_{L^2} \leq \|w\|_{H^1} + \|\Psi_x\|_{H^2} \leq C \|w\|_{H^1}.
\]

Thus, there holds \( \|\Psi\|_{H^3} \leq C \|w\|_{H^1} \).

**Step 3.** Now, take \( v = \text{curl} \Psi \). Then \( v \in H^2 \) satisfies

\[
\|v\|_{H^2} \leq C \|w\|_{H^1}
\]

(5.16)

and the relationship \( w = \text{curl} v \). Furthermore, substituting this relationship into (5.10), we reach

\[
\begin{cases}
-\Delta v + \beta \text{curl} v = \text{curl}(F + \beta u), & \text{in } \Omega, \\
\text{curl} v = k(x, y) u \cdot \tau, & \text{on } \partial \Omega,
\end{cases}
\]

(5.17)

In virtue of Hodge decomposition, equation (5.17)1 is equivalent to

\[
-\Delta v + \beta v + \nabla q = F + \beta u, \text{ in } \Omega.
\]
In the meantime, by Lemma 3.1, boundary condition (5.17) is equivalent to
\[ 2\mathbb{D}(v)n \cdot \tau = k(x, y)u \cdot \tau, \quad \text{on } \partial \Omega. \]

Besides, it follows from the definition of \( v \) that \( \text{div} v = 0 \). In addition, since \( \Psi|_{\partial \Omega} = 0 \), we have \( \nabla \Psi \cdot \tau|_{\partial \Omega} = 0 \), which is equivalent to
\[ -\text{curl} \Psi \cdot n|_{\partial \Omega} = 0, \quad \text{i.e. } v \cdot n|_{\partial \Omega} = 0. \]

In conclusion, \( v \in H^2 \) is a solution of problem
\[
\begin{aligned}
-\Delta v + \beta v &= F + \beta u, & \text{in } \Omega, \\
\text{div} v &= 0, & \text{in } \Omega, \\
v \cdot n &= 0, & \text{on } \partial \Omega, \\
2\mathbb{D}(v)n \cdot \tau &= k(x, y)u \cdot \tau, & \text{on } \partial \Omega. \\
\end{aligned}
\tag{5.18}
\]

**Step 4.** (5.8) indicates that \( u \) is also a weak solution to (5.18). Thus, \( u = v \) is true if the solution of problem (5.18) is unique. In fact, one can see that \( u - v \) satisfies
\[
\begin{aligned}
-\Delta(u - v) + \beta(u - v) + \nabla(p - q) &= 0, & \text{in } \Omega, \\
\text{div}(u - v) &= 0, & \text{in } \Omega, \\
(u - v) \cdot n &= 0, & \text{on } \partial \Omega, \\
2\mathbb{D}(u - v)n \cdot \tau &= k(x, y)u \cdot \tau, & \text{on } \partial \Omega. \\
\end{aligned}
\]

It can be deduced from the standard energy method that \( u - v \equiv 0 \), i.e. \( u = v \in H^2 \). Consequently, in virtue of (5.16) and (5.13), we infer that
\[ \|u\|_{H^2} \leq C (\|F\|_{L^2} + \|u\|_{H^1}). \tag{5.19} \]

In particular, substituting (3.12) into (5.19) and using Cauchy inequality, we get
\[ \|u\|_{H^2} \leq C (\|F\|_{L^2} + \|u\|_{L^2}). \tag{5.20} \]

The final work is to deduce estimate for \( \nabla p \), which can be directly implied by (5.8)\(^1\) and (5.20).

The proof of this proposition is completed. \( \square \)

With this Stokes estimate in hand, we are able to state and prove the regularity of the solution.

**Proof of Theorem 2.3.** By Theorem 2.2, the initial boundary problem (2.1) has a unique weak solution \( u(t) \in L^\infty(0, T; \mathbf{H}) \cap L^2(0, T; \mathbf{V}) \) satisfies estimate (2.2). Thus, we still need to prove estimate (2.3) and that a weak solution of problem (2.1) with estimate (2.3) is in fact a strong solution to problem (2.1).

Rewrite problem (2.1) by
\[
\begin{aligned}
-\Delta u + \nabla p &= -\partial_t u - u \cdot \nabla u + f, & \text{in } \Omega, \\
\nabla \cdot u &= 0, & \text{in } \Omega, \\
u \cdot n &= 0, & \text{on } \partial \Omega, \\
2\mathbb{D}(u)n \cdot \tau &= k(x, y)u \cdot \tau, & \text{on } \partial \Omega. \\
\end{aligned}
\tag{5.21}
\]

In view of (5.7), \( u \in \mathbf{V} \) a.e. \( t \in (0, T) \) is the unique weak solution to (5.21). Then, it follows from proposition 5.1 that \( u(t) \in H^2 \) for a.e. \( t \in (0, T) \), and that
\[ \|u(t)\|_{H^2} + \|\nabla p(t)\|_{L^2} \lesssim \|f(t)\|_{L^2} + \|\partial_t u\|_{L^2} + \|u(t)\cdot \nabla u(t)\|_{L^2} + \|u(t)\|_{L^2}. \tag{5.22} \]

In addition, applying (3.22) gives
\[ \|u(t) \cdot \nabla u(t)\|_{L^2} \leq \|u(t)\|_{L^\infty} \|\nabla u(t)\|_{L^2} \leq \|u(t)\|_{L^2}^{1/2} \|u(t)\|_{H^2}^{1/2} \|\nabla u(t)\|_{L^2}. \]
(2.3) follows from substituting (5.23) into (5.22) with \( \varepsilon \) small enough and using estimate (5.7).

Finally, we recover the strong form of the weak solution in the sense of Definition 2.1. that is, \( u \in L^\infty(0, T; \mathcal{W}) \) satisfies

\[
\frac{d}{dt} \int_{\Omega} u(t) \cdot v + 2 \int_{\Omega} \mathfrak{D}(u(t)) : \mathfrak{D}(v) + \int_{\Omega} u(t) \cdot \nabla u(t) \cdot v
\]

\[
= \int_{\partial\Omega} k(x, y)u(t) \cdot v dS + \int_{\Omega} f(t) \cdot v,
\]

(5.24)

for any \( v \in \mathcal{V} \). For any \( \phi \in C^\infty_0(\Omega) \), take \( v = \nabla^T \phi \) in (5.24), where \( \nabla^T = (-\partial_y, \partial_x) \).

Then \( v \in \mathcal{V} \cap C^\infty_\partial \). Integrating by parts in (5.24) gives

\[
\int_{\Omega} \partial_t u(t) \cdot v + \int_{\Omega} -\Delta u(t) \cdot v + \int_{\Omega} u(t) \cdot \nabla u(t) \cdot v = \int_{\Omega} f(t) \cdot v,
\]

(5.25)

where the conditions \( \text{div} u(t) = 0 \) and \( v \in C^\infty_\partial \) have been used. Further integrating by parts, we get

\[
- \int_{\Omega} \text{curl} [\partial_t u(t) + u(t) \cdot \nabla u(t) - \Delta u(t) - f(t)] \cdot \phi = 0.
\]

(5.26)

As \( \phi \in C^\infty_0 \) is arbitrary, there holds

\[
\text{curl} [\partial_t u(t) + u(t) \cdot \nabla u(t) - \Delta u(t) - f(t)] = 0,
\]

which implies that there exists \( p \in H^1(\Omega) \), such that

\[
- [\partial_t u(t) + u(t) \cdot \nabla u(t) - \Delta u(t) - f(t)] = \nabla p,
\]

i.e.

\[
\partial_t u(t) + u(t) \cdot \nabla u(t) + \nabla p = \Delta u(t) + f(t).
\]

(5.27)

On the other hand, taking arbitrary \( v \in \mathcal{V} \) in (5.24) and integrating by parts, in view of that \( u(t) \) is a solution to (5.27), it turns to

\[
\int_{\partial\Omega} 2\mathfrak{D}(u(t))n \cdot v dS = \int_{\partial\Omega} k(x, y)u(t) \cdot v dS.
\]

(5.28)

Note also that \( v \cdot n = 0 \) on \( \partial\Omega \). Then \( v = (v \cdot \tau)\tau \) on \( \partial\Omega \). Thus, (5.28) can be rewritten as

\[
\int_{\partial\Omega} 2\mathfrak{D}(u(t))n \cdot \tau(v \cdot \tau) dS = \int_{\partial\Omega} k(x, y)u(t) \cdot \tau(v \cdot \tau) dS.
\]

The arbitrariness of \( v \) then implies boundary condition (2.1)4, which is suitable in the sense of trace for \( u(t) \in L^\infty(0, T; \mathcal{W}) \). In addition, (2.1)2 and (2.1)3 are valid automatically for \( u(t) \in \mathcal{V} \).

Acknowledgments. The authors would like to thank Prof. Yan Guo from Brown University for the invaluable comments on the present paper and Li also want to express sincere appreciation to Prof. Guo for his kindly assist and academic direction during the period of Li’s visitation to Brown University.
REFERENCES

[1] Y. Achdou, O. Pironneau and F. Valentin, Effective boundary conditions for laminar flow over periodic rough boundaries, *J. Comput. Phys.*, **147** (1998), 187–218.

[2] C. Amrouche and A. Rejeb, \( L^p \) theory for Stokes and Navier-Stokes equations with Navier boundary condition, *J. Differ. Equ.*, **256** (2014), 1515–1547.

[3] C. Amrouche and N. Seloula, On the Stokes equations with the Navier-type boundary conditions, *Differ. Equ. Appl.*, **3** (2011), 581–607.

[4] S. Antontsev and H. de Oliveira, Navier-Stokes equations with absorption under slip boundary conditions: existence, uniqueness and extinction in time, *RIMS Kōkyūroku Bessatsu, B1* (2007), 21–41.

[5] E. Bänsch, Finite element discretization of the Navier-Stokes equations with free capillary surface, *Numer. Math.*, **88** (2001), 203–235.

[6] G. Beavers and D. Joseph, Boundary conditions at a naturally permeable wall, *J. Fluid Mech.*, **30** (1967), 197–207.

[7] D. Chauhan and K. Shekhawat, Heat transfer in Couette flow of a compressible Newtonian fluid in the presence of a naturally permeable boundary, *J. Phys.D: Appl. Phys.*, **26** (1993), 933–936.

[8] T. Clopeau, A. Mikelić and R. Robert, On the vanishing viscosity limit for the 2D incompressible Navier-Stokes equations with the friction type boundary conditions, *Nonlinearity*, **11** (1998), 1625–1656.

[9] S. Ding, Q. Li and Z. Xin, Stability analysis for the incompressible Navier-Stokes equations with Navier boundary conditions, *J. Math. Fluid Mech.*, **20** (2018), 603–629.

[10] L. Evans, *Partial Differential Equations*, Amer. Math. Soc., Providence RI, 1998.

[11] G. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations*, Springer Monographs in Mathematics, 2nd Edition, 2011.

[12] G. Gie and J. Kelliher, Boundary layer analysis of the Navier-Stokes equations with generalized Navier boundary conditions, *J. Differ. Equ.*, **253** (2012), 1862–1892.

[13] A. Haase, J. Wood, R. Lammertink, J. Snoeijer, Why bumpy is better: the role of the dissipation distribution in slip flow over a bubble mattress, *Phys. Rev. Fluid.*, **1** (2016), 054101.

[14] W. Jäger and A. Mikelić, On the Roughness-induced effective boundary conditions for an incompressible viscous flow, *J. Differ. Equ.*, **170** (2001), 96–122.

[15] V. John, Slip with friction and penetration with resistance boundary conditions for the Navier-Stokes equation-numerical test and aspect of the implementation, *J. Comput. Appl. Math.*, **147** (2002), 287–300.

[16] J. Kelliher, Navier-Stokes equations with Navier boundary conditions for a bounded domain in plane, *SIAM J. Math. Anal.*, **38** (2006), 210–232.

[17] H. Li and X. Zhang, Stability of plane Couette flow for the compressible Navier-Stokes equations with Navier-slip boundary conditions, *J. Differ. Equ.*, **263** (2017), 1160–1187.

[18] P. Lions, *Mathematical Topics in Fluid Mechanics*, Volume I, Incompressible Models, Oxford Science Publications, 1998.

[19] J. Magnaudet, M. Riverot and J. Fabre, Accelerated flows past a rigid sphere or a spherical bubble. Part 1. Steady straining flow, *J. Fluid Mech.*, **284** (1995), 97–135.

[20] C. Navier, Sur les lois de l’équilibre et du mouvement des corps élastiques, *Mem. Acad. R. Sci. Inst. France*, **6** (1827), 369.

[21] T. Qian, X. Wang and P. Sheng, Molecular scale contact line hydrodynamics of immiscible flows, *Phys. Rev. E.*, **68** (2003), 016306.

[22] J. Serrin, Mathematical Principles of Classical Fluid Mechanics, Encyclopedia of Physics VIII/1, Springer-Verlag, Berlin, 1959.

[23] V. Solonnikov and V. Ščadilov, A certain boundary value problem for the stationary system of Navier-Stokes equations, *Trudy Mat. Inst. Steklov.*, **125** (1973), 196–210; translation in *Proc. Steklov Inst. Math.*, **125** (1973), 186–199.

[24] R. Temam, *Navier–Stokes Equations: Theory and Numerical Analysis*, AMS Chelsea edition, Providence RI, 2001.

[25] H.da Veiga, On the regularity of flows with Ladyzhenskaya shear-dependent viscosity and slip or nonslip boundary conditions, *Commun. Pure Appl. Math.*, **LVIII** (2005), 552–577.

[26] Y. Xiao and Z. Xin, On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition, *Commun. Pure Appl. Math.*, **60** (2007), 1027–1055.
[27] Y. Xiao and Z. Xin, On the inviscid limit of the 3D Navier-Stokes equations with generalized Navier-slip boundary conditions, Commun. Math. Stat., 1 (2013), 259–279.

Received April 2019; revised June 2021; early access July 2021.

E-mail address: quanrong.li@szu.edu.cn
E-mail address: dingsj@scnu.edu.cn