APPLICATIONS OF SUPERSYMMETRIC MATRIX MODELS

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Matrix models have wide applications in nuclear theory, condensed matter theory and quantum field theory. I discuss supersymmetric extensions of matrix models and their applications to branched polymers, the meander problem, and superstrings in lower dimensions.

1 Introduction

Matrix models have wide applications in nuclear theory, condensed matter theory, high energy theory and quantum gravity since the seminal paper by Wigner. An extension to supermatrices can be found in Refs. 2, 3, 4.

I report in this talk some results on another supersymmetric extension of matrix models, which is based on one complex bosonic matrix $B$ and one fermionic matrix $F$ and is along the line proposed by Marinari and Parisi. I discuss applications of the supersymmetric matrix models to branched polymers, the meander problem, and superstrings in lower dimensions.

2 Supersymmetric matrix models

The supersymmetric matrix models in the $D = 0$ dimensional target space are built out of the “superfields”

$$ W_a = (B, F), \quad \bar{W}_a = (\bar{B}^\dagger, \bar{F}) , $$

where $a = 1, 2$ while $B$ and $F$ are general complex bosonic and fermionic (i.e. Grassmann valued) $N \times N$ matrices, respectively. In other words, the hermitean conjugated $B^\dagger \neq B$ and the Grassmann involuted $\bar{F} \neq F$.

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There is no need of introducing a superspace coordinate \( \theta \) which would be dimensionless in \( D = 0 \), since the propagators for both bosonic and fermionic matrices coincide:

\[
\left\langle B_{ij}B_{kl}^\dagger \right\rangle_{\text{Gauss}} = \frac{1}{N} \delta_{il} \delta_{kj}, \quad \left\langle F_{ij}\bar{F}_{kl} \right\rangle_{\text{Gauss}} = \frac{1}{N} \delta_{il} \delta_{kj}.
\]  

Hence, the supersymmetry reduces in \( D = 0 \) simply to rotations between the \( B \)- and \( F \)-components. The proper transformation reads

\[
\delta \epsilon B^\dagger = \bar{F} \epsilon, \quad \delta \epsilon F = -\epsilon B, \quad \delta \bar{\epsilon} B = \bar{\epsilon} F, \quad \delta \bar{\epsilon} \bar{F} = -B^\dagger \bar{\epsilon},
\]

where \( \epsilon \) and \( \bar{\epsilon} \) are Grassmann valued. Note that it is a huge symmetry since the parameters \( \epsilon \) and \( \bar{\epsilon} \) are \( N \times N \) matrices.

The simplest Gaussian supersymmetric potential reads

\[
V_{\text{Gauss}} = N \text{tr} \bar{W}W,
\]

where

\[
\bar{W}W = \sum_{a=1}^{2} \bar{W}_a W_a = B^\dagger B + \bar{F}F,
\]

which reproduces the propagators (2). It is obviously invariant under the rotation (3)–(4). It is also clear from Eq. (3) why one needs complex matrices in \( D = 0 \): the trace of the square of a fermionic matrix vanishes.

Any potential, which is symmetrically constructed from the “superfields” (3), is supersymmetric so that contributions from the loops of the bosonic and fermionic matrix fields are mutually cancelled which is the key property of the supersymmetry.

A general interaction potential, which is invariant under the matrix supersymmetry transformation (3)–(4), reads

\[
V_{\text{gen}} (\bar{W}W) = N \sum_{k \geq 1} \frac{g_k}{k} \text{tr} (\bar{W}W)^k,
\]

where \( g_k \) are the coupling constants. This invariance can be seen from

\[
\delta \epsilon \bar{W}W = \delta \epsilon (B^\dagger B + \bar{F}F) = \bar{F} \epsilon B - \bar{F} \epsilon B = 0.
\]

The supersymmetric matrix model with the potential (8) describes branched polymers as is shown in the next Section.
The supersymmetry transformations (3), (4) can be formalized by introducing the (matrix) supercharges

\[ Q_{ij} = \sum_{k=1}^{N} \left( F_{ik} \frac{\partial}{\partial B_{jk}} - \frac{\partial}{\partial F_{kj}} B_{k}^{\dagger} \right), \quad \bar{Q}_{ij} = \sum_{k=1}^{N} \left( \frac{\partial}{\partial B_{ki}} \bar{F}_{kj} - B_{ik} \frac{\partial}{\partial F_{kj}} \right), \]

so that

\[ \delta \ldots = \left[ \text{tr} \bar{Q} \epsilon, \ldots \right], \quad \delta \ldots = \left[ \text{tr} \epsilon Q, \ldots \right]. \] (10)

Their commutators read

\[ \{ Q_{ij}, Q_{mn} \} = \{ \bar{Q}_{ij}, \bar{Q}_{mn} \} = 0, \] (11)

\[ \{ Q_{ij}, \bar{Q}_{mn} \} = -\delta_{in} \sum_{k=1}^{N} \left( B_{mk} \frac{\partial}{\partial B_{jk}} + \frac{\partial}{\partial B_{km}} B_{k}^{\dagger} \right) - \sum_{k=1}^{N} \left( F_{ik} \frac{\partial}{\partial F_{nk}} - \frac{\partial}{\partial F_{nk}} \bar{F}_{kn} \right) \delta_{mj}. \] (12)

3 Application to branched polymers

The partition function of the supersymmetric matrix model with the potential (7) reads

\[ Z[g] \equiv \int dW d\bar{W} e^{-V_{\text{gen}}(\bar{W} W)} = 1. \] (13)

It is equal to 1 because of the cancellation of bosonic and fermionic loops due to the supersymmetry.

Likewise, all the supersymmetric correlators vanish, for example

\[ \left\langle \frac{1}{N} \text{tr} (\bar{W} W)^{n} \right\rangle \equiv \int dW d\bar{W} e^{-V_{\text{gen}}(\bar{W} W)} \frac{1}{N} \text{tr} (\bar{W} W)^{n} = 0. \] (14)

Physical quantities of the model are described by the correlators of the pure bosonic matrices, which are nontrivial. Their generating function is

\[ G(\lambda) = \left\langle \frac{1}{N} \text{tr} \frac{1}{\lambda - B^{\dagger} B} \right\rangle = \frac{1}{\lambda} + \sum_{n=1}^{\infty} \frac{1}{\lambda^{n+1}} G_{n} \] (15)

with

\[ G_{n} \equiv \left\langle \frac{1}{N} \text{tr} (B^{\dagger} B)^{n} \right\rangle. \] (16)

\(^6\)Here and below the order reflects matrix multiplication.
The imaginary part of \( G(\lambda) \) determines the distribution of eigenvalues of the matrix \( B^\dagger B \) and, therefore, the spectrum of the proper statistical model.

The correlators \( (16) \) can be calculated using the Schwinger–Dyson equations and the supersymmetry Ward identities. As a result, \( G(\lambda) \) obeys at large \( N \) the closed equation

\[
(\lambda G(\lambda) - 1) V'(\lambda - 1/G(\lambda)) = \lambda G^2(\lambda),
\]

whose limit of \( \lambda \to \infty \) yields the equation

\[
G_1 V'(G_1) = 1
\]

for \( G_1 \). This equation is quadratic for a quartic potential.

The fact that a closed equation is obtained for the propagator \( G_1 \) is a consequence of the cancellations between bosonic and fermionic loops. Some of the diagrams which survive the cancellation are shown in Fig. \( \text{1} \). For obvious reasons, they are called the “cactus diagrams”. Note that the diagrams have orientation: the cactus loops can only proliferate on the exterior of already
existing loops. This is in contradistinction to related bubble diagrams one encounters in the large-\(N\) limit of pure bosonic or fermionic vector models.

The critical behavior of the model arises when Eq. (18) holds simultaneously with

\[ [G_c V'(G_c)]' = \ldots = [G_c V'(G_c)]^{(m-1)} = 0, \]

which can always be achieved by tuning \(m\) couplings \(g_k\) of the potential \(\mathcal{F}K\).

Near the critical point, \(G_1\) behaves as

\[ G_1 \simeq G_c - \text{const} \cdot (\alpha_c - \alpha)^{1/m}, \]

where \(\alpha\) stands for an overall scale of \(g_k\)'s. The susceptibility \(\chi \equiv \partial G_1 / \partial \alpha\) at the critical point scales as

\[ \chi \sim (\alpha_c - \alpha)^{-\gamma_{\text{str}}} \]

with \(\gamma_{\text{str}} = 1 - 1/m\), which coincides with the (multi-)critical index of the branched polymers.\(^7\)

This relation to branched polymers becomes explicit by noting that the cactus graphs which survive the supersymmetric cancellation have an interpretation as branched polymers, with the couplings \(-g_k\) associated with the branching weights. Cutting each loop in a succession, such that only the standing line is attached to the vertex, produces a branched polymer graph of a "chiral" type since branching only occurs at one side of the open line, corresponding to the fact that the cactus loops can only be attached to the exterior of already existing loops. This is illustrated in Fig. 1. Equation (18) can then be rederived purely combinatorially.

### 4 Application to the meander problem

The meander problem is to calculate combinatorial numbers associated with the crossings of an infinite river (Meander) and a closed road by \(2n\) bridges.\(^5\) Neither the river nor the road intersects with itself. These meander numbers, \(M_n\), obviously describe the number of different foldings of a closed strip of \(2n\) stamps or of a closed polymer chain.

The generating function of the meander numbers can be represented via the following correlator in the supersymmetric matrix model:

\[ M(c) \equiv \sum_{n=1}^{\infty} c^{2n} M_n = \lim_{N \to \infty} \left\langle \frac{1}{N} \text{tr} BB^\dagger \ln \left( \int d\phi_1 d\phi_2 e^{-S} \right) \right\rangle_{\text{Gauss}}, \]

\(^6\)See Ref. 9 for an introduction to the subject.
where $\phi_1$ and $\phi_2$ are $N \times N$ hermitean matrices, the action $S$ is given by

$$S = \frac{N}{2} \text{tr} \phi_1^2 + \frac{N}{2} \text{tr} \phi_2^2 - cN \text{tr} (\phi_1 B^\dagger \phi_2 B) - cN \text{tr} (\phi_1 \bar{F} \phi_2 F)$$  \hspace{1cm} (23)$$

and the Gaussian averaging is with respect to the action (5). The presence of the log in Eq. (22) leaves only one loop of the field $\phi$ associated with the road while the supersymmetry kills the loops of the field $W$ associated with the river. The limit of large $N$ is needed to keep only the planar graphs as in the original meander problem.

Expanding in $c$, the coupling constant of the quartic interaction, the meander numbers can be represented as the sum over words built out of two letters:

$$M_n = \sum_{a_2, \cdots, a_{2n-1}, a_{2n}=1} \left\langle \frac{1}{N} \text{tr} BW_{a_2} \cdots W_{a_{2n-1}} \bar{W}_{a_{2n}} \right\rangle_{\text{Gauss}}$$

$$\times \left\langle \frac{1}{N} \text{tr} \bar{W}_{a_{2n}} W_{a_{2n-1}} \cdots \bar{W}_{a_2} B \right\rangle_{\text{Gauss}},$$  \hspace{1cm} (24)

where the order of matrices is essential for the fermionic components. Equation (24) is a nice representation of the meander numbers which looks more natural than the one based on the replica trick in a pure bosonic model.

Equation (24) can be represented in an alternative form by introducing noncommutative variables $u, v$ and $u^\dagger, v^\dagger$ which are annihilation and creation operators in a Hilbert space with the vacuum $|\Omega\rangle$ and obey the Cuntz algebra

$$uu^\dagger = 1, \quad vv^\dagger = 1, \quad uu^\dagger = 0, \quad vu^\dagger = 0,$$  \hspace{1cm} (25)

as well as the completeness condition

$$u^\dagger u + v^\dagger v = 1 - |\Omega\rangle \langle \Omega|.$$  \hspace{1cm} (26)

There are no more relations between the noncommutative variables.

Denoting

$$u_a = (u, v), \quad \bar{u}_a = (u, -v),$$  \hspace{1cm} (27)

the generating function (22) can alternatively be represented as the vacuum expectation value

$$M(c) = \langle \Omega | \bar{G} u^\dagger G^\dagger | \Omega \rangle = -\langle \Omega | \bar{G} v^\dagger G^\dagger | \Omega \rangle,$$  \hspace{1cm} (28)

6
where $G$ is given by the continued fraction

$$G(u) = \frac{1}{1 - \sqrt{c}u_{a_1}} \frac{1}{1 - \sqrt{c}u_{a_2}} \frac{1}{1 - \sqrt{c}u_{a_3}} \frac{1}{1 - \sqrt{c}u_{a_4}} \ldots}.$$ \hspace{1cm} (29)

Here $u$ and $\bar{u}$ interchange in the consequent lines. The following notations are used in Eq. (28):

$$G \equiv G(u), \quad \bar{G} \equiv G(\bar{u}), \quad G^\dagger \equiv G(u^\dagger), \quad \bar{G}^\dagger \equiv G(\bar{u}^\dagger).$$ \hspace{1cm} (30)

The two expressions on the right hand side of Eq. (28) are equal due to the supersymmetry.

Though Eq. (28) reproduces the meander numbers when expanded in $c$, it seems to look like a reformulation rather than a solution to the problem since it is not clear how to deal with functions of noncommutative variables.

5 Application to superstrings?

The critical index of the string susceptibility $\gamma_{\text{str}}$ for a superstring embedded in a $D$-dimensional space had been calculated from the super-Liouville theory and reads

$$\gamma_{\text{str}} = \frac{D - 1 - \sqrt{(1 - D)(9 - D)}}{4}.$$ \hspace{1cm} (31)

Many (not yet successful) attempts of discretizing superstring are performed starting from \[4\]. A progress has been achieved \[12\] only for the simplest case of pure 2-dimensional supergravity which can be associated with a super-eigenvalue model. It reveals the super-Virasoro algebra associated with the Neveu–Schwarz sector of the superstring.

The idea to verify whether or not a super-Virasoro algebra can be realized in the supermatrix models is to construct the matrix generators (cf. (9), (12))

$$L_{ij} = \sum_{k=1}^{N} \left( \frac{\partial}{\partial B_{k_i}} B_{kj} - \frac{\partial}{\partial F_{k_i}} F_{kj} \right), \quad G_{ij} = \sum_{k=1}^{N} \left( \frac{\partial}{\partial B_{k_i}} F_{kj} + \frac{\partial}{\partial F_{k_i}} B_{kj} \right).$$ \hspace{1cm} (32)

Here $L_{ij}$ is Grassmann even and $G_{ij}$ is Grassmann odd.
The operators $L_{ij}$ and $G_{ij}$ obey the commutation relations

\[
[L_{ij}, L_{mn}] = \delta_{in} L_{mj} - L_{in} \delta_{mj},
\]

(33)

\[
[G_{ij}, L_{mn}] = \delta_{in} G_{mj} - G_{in} \delta_{mj},
\]

(34)

\[
\{G_{ij}, G_{mn}\} = \delta_{in} L_{mj} + L_{in} \delta_{mj}.
\]

(35)

These can be derived by explicitly commuting the operators (32).

The commutator (33) itself implies the Virasoro algebra

\[
[L_s, L_t] = (t - s) L_{s+t}, \quad L_s = \text{tr} (L (\bar{W} W)^s)
\]

(36)

for $s, t \geq 0$ as $N \to \infty$. Given (36), the potential (7) can then be recovered from the Virasoro constraints

\[
0 = \int dW d\bar{W} L_s e^{-V_{\text{gen}}(\bar{W} W)} = L_s [g] Z [g].
\]

(37)

The conjecture is that the whole matrix algebra (33)–(35) implies, as $N \to \infty$, the super-Virasoro algebra associated with the Ramond sector of the superstring in the $D = 0$ dimensional target space:

\[
[L_s, L_t] = (t - s) L_{s+t}, \quad [G_r, L_s] = \left(\frac{s}{2} - r\right) G_{r+s}, \quad \{G_r, G_s\} = 2 L_{r+s}.
\]

(38)

An explicit form of the operators $L_s$ and $G_r$ can be constructed starting from

\[
L_0 = \text{tr} L, \quad G_0 = \text{tr} G, \quad L_1 = a \text{ tr } L \bar{W} W + b \text{ tr } G \bar{W} \wedge W
\]

(39)

where

\[
\bar{W} \wedge W \equiv \sum_{a,b=1}^2 e^{ab} \bar{W}_a W_b = B\dagger F - \bar{F} B
\]

(40)

and using (38). For example, one gets

\[
G_1 = c \text{ tr } G \bar{W} W + d \text{ tr } L \bar{W} \wedge W,
\]

\[
L_2 = c(c + d) \text{ tr } L (\bar{W} W)^2 + c(c - d) \text{ tr } G \bar{W} \wedge W \bar{W} W
\]

(41)

with

\[
c = -2b, \quad d = 2a + 4b
\]

(42)

and so on. The constant $b \neq 0$ in $L_1$ since $L_2$ vanishes otherwise. In order for this procedure to be successful, all the operators $L_s$ and $G_r$ are to be nonvanishing.

The action of the proper supersymmetric matrix model should then be determined to reproduce these super-Virasoro operators. It should involve both $\bar{W} W$ and $\bar{W} \wedge W$ given respectively by Eqs. (40) and (38), and, therefore, Grassmann odd coupling constants in addition to those in (7).
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