CUTTING THE SAME FRACTION OF SEVERAL MEASURES

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Abstract. We study some measure partition problems: Cut the same positive fraction of $d + 1$ measures in $\mathbb{R}^d$ with a hyperplane or find a convex subset of $\mathbb{R}^d$ on which $d + 1$ given measures have the same prescribed value. For both problems positive answers are given under some additional assumptions.

1. Introduction

The famous “ham sandwich” theorem of Stone, Tukey, and Steinhaus [14, 13] asserts that every $d$ absolutely continuous probability measures in $\mathbb{R}^d$ can be simultaneously partitioned into equal parts by a single hyperplane.

In [3] M. Kano and S. Bereg raised the following question (in the planar case): If we are given $d + 1$ measures in $\mathbb{R}^d$ and want to cut the same (but unknown) fraction of every measure by a hyperplane then what assumptions on the measures allow us to do so? Certainly, additional assumptions are required because if the measures are concentrated near vertices of a $d$-simplex then such a fraction cut is impossible. A sufficient assumption is described below:

Definition 1.1. Let $\mu_0, \mu_1, \ldots, \mu_d$ be absolutely continuous probability measures on $\mathbb{R}^d$ and let $\varepsilon \in (0, 1/2)$. Call the set of measures $\varepsilon$-not-permuted if for any halfspace $H$ the inequalities $\mu_i(H) < \varepsilon$ for all $i = 0, 1, \ldots, d$ imply

$$\mu_i(H) \geq \mu_j(H),$$

for some $i < j$.

Remark 1.2. For $\varepsilon > 0$ consider all halfspaces $H$ in $\mathbb{R}^d$ such that $\mu_i(H) < \varepsilon$ for all $i$ and the values $\mu_i(H)$ are pairwise distinct. If we arrange the values $\mu_i(H)$ in the ascending order then we get some permutation of $\{0, 1, \ldots, d\}$. So the measures $\mu_i$ are $\varepsilon$-not-permuted if and only if in such a way we cannot get all possible permutations of the $d + 1$ element set.

Remark 1.3. A natural example of $\varepsilon$-not-permuted measures appears when the support of one measure lies in the interior of the convex hull of the union of supports of the other $d$ measures. In this case the measures are $\varepsilon$-not-permuted for sufficiently small $\varepsilon$.

Now we state the main result:

Theorem 1.4. Suppose $\mu_0, \mu_1, \ldots, \mu_d$ are absolutely continuous probability $\varepsilon$-not-permuted measures in $\mathbb{R}^d$ for some $\varepsilon \in (0, 1/2)$. Then there exists a halfspace $H$ such that

$$\mu_0(H) = \mu_1(H) = \cdots = \mu_d(H) \in [\varepsilon, 1/2].$$
Note that in [2, 6, 8] a similar problem was considered: Cut by a hyperplane a \textit{prescribed} fraction of each of \(d\) measures in \(\mathbb{R}^d\). Again, this cannot be done in general and some additional assumptions were needed.

A straightforward consequence of Theorem 1.4 follows by considering one measure concentrated near a point:

\textbf{Corollary 1.5.} Suppose \(\mu_1, \ldots, \mu_d\) are absolutely continuous probability measures in \(\mathbb{R}^d\) and \(p\) is a point in the convex hull of their supports. Then there exists a halfspace \(H\) such that

\[\mu_1(H) = \mu_2(H) = \cdots = \mu_d(H)\]

and \(p \in \partial H\).

In Section 4 we consider a discrete version of Theorem 1.4 replacing measures by finite point sets. This is in accordance with the initial statement of the problem in [3]. Finally, in Section 5 we consider a problem of cutting the same \textit{prescribed} fraction of every measure, this time allowing cutting with a convex subset of \(\mathbb{R}^d\).

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\section{2. Ham sandwich theorem for charges}

In order to prove Theorem 1.4 we need a version of the “ham sandwich” theorem [13] [14] for charges:

\textbf{Definition 2.1.} A difference \(\rho = \mu' - \mu''\) of two absolutely continuous finite measures on \(\mathbb{R}^d\) is called a \textit{charge}. In other words, the charge is represented by its density from \(L_1(\mathbb{R}^d)\).

\textbf{Theorem 2.2} (Ham sandwich for charges). Suppose we are given \(d\) charges \(\rho_1, \ldots, \rho_d\) in \(\mathbb{R}^d\), then there exists a (possibly degenerate) halfspace \(H\) such that for any \(i\)

\[\rho_i(H) = 1/2 \rho_i(\mathbb{R}^d).\]

\textbf{Remark 2.3.} A \textit{degenerate halfspace} is either \(\emptyset\) of the whole \(\mathbb{R}^d\). We cannot exclude degenerate halfspaces in this theorem when \(\rho_i(\mathbb{R}^d)\)'s are all zero. The reason is the same for which we cannot exclude the “\(\varepsilon\)-not-permuted” assumption from Theorem 1.4; see also Remark 3.1 below.

\textbf{Proof.} The classical proof from the book of Matoušek [11] passes for charges (the authors learned this fact long ago from Vladimir Dol’nikov). Identify \(\mathbb{R}^d\) with \(\mathbb{R}^d \times \{1\} \subset \mathbb{R}^{d+1}\). Parameterize halfspaces \(H \subset \mathbb{R}^{d+1}\) with boundary passing through the origin by their inner normals. So all halfspaces \(H = \tilde{H} \cap \mathbb{R}^d \times \{1\}\) (including degenerate) are parameterized by the unit sphere \(S^d\).

If we map every halfspace to the vector

\[(\rho_1(H) - \rho_1(\mathbb{R}^d \setminus H)), \ldots, \rho_d(H) - \rho_d(\mathbb{R}^d \setminus H))\]

then we obtain a continuous odd map \(P : S^d \to \mathbb{R}^d\); by the Borsuk–Ulam theorem [6] (see also [11]) one halfspace must be mapped to zero.

\[\square\]
3. Proof of Theorem 1.4

As the first attempt we try to apply the ham sandwich theorem for charges to
\[ \rho_1 = \mu_1 - \mu_0, \rho_2 = \mu_2 - \mu_1, \ldots, \rho_d = \mu_d - \mu_{d-1}. \]
This way we easily obtain a halfspace \( H \) such that \( \mu_0(H) = \mu_1(H) = \cdots = \mu_d(H) \). But the halfspace \( H \) may be degenerate or \( \mu_i(H) \) may be all zero. This is not what we want.

Remark 3.1. By the way, we see that starting from three measures \( \mu_i \) on \( \mathbb{R}^2 \) distributed along three rays emanating from vertices of a regular triangle and going outside it, we cannot cut the same fraction (possibly zero) of these measures by a non-degenerate halfplane. The same example generalizes to higher dimensions and shows that in the ham sandwich theorem for charges we cannot avoid using degenerate halfspaces when all charges satisfy \( \rho_i(\mathbb{R}^d) = 0 \).

So let us perturb the charges with a small positive parameter \( s \):
\[ \rho_i^s = (1 + s)\mu_i - \mu_{i-1}, \text{ for } i = 1, \ldots, d. \]

Now Theorem 2.2 gives a halfspace \( H \) with:
\[ \rho_1^s(H) = \cdots = \rho_d^s(H) = s/2. \]
This is equivalent to the following equalities:
\[ \mu_{i-1}(H) = (1 + s)\mu_i(H) - s/2 = \mu_i(H) + s(\mu_i(H) - 1/2). \]
Suppose \( \mu_i(H) < \varepsilon \) for all \( i = 0, \ldots, d \). Then we have \( \mu_i(H) < 1/2 \) and \( s(\mu_i(H) - 1/2) < 0 \). Hence
\[ \mu_{i-1}(H) < \mu_i(H), \text{ for } i = 1, \ldots, d. \]
This contradicts the assumption that the measures are \( \varepsilon \)-not-permuted.

Hence the inequality \( \mu_i(H) \geq \varepsilon \) for some \( i \) is guaranteed while we decrease \( s \) to 0, and in turn it guarantees (remember that we can interchange \( H \) and \( \mathbb{R}^d \setminus H \)) that \( H \) cannot approach degenerate halfspaces \( \emptyset \) and \( \mathbb{R}^d \). So we assume by compactness that \( H \) tends to a certain halfspace as \( s \to 0 \) and going to the limit in (3.2) together with (3.1) we obtain the conclusion.

4. Discrete version

In the paper [3] Bereg and Kano consider a discrete version of this theorem in the plane. They call a line \( \ell \) balanced if each half-plane bounded by \( \ell \) contains precisely the same number of points of each color.

**Theorem 4.1** (S. Bereg and M. Kano, [3]). Let \( S \) be a set of \( 3n \geq 6 \) points in the plane in general position colored in red/blue/green such that
(i) the number of points of each color is \( n \);
(ii) the vertices of the convex hull of \( S \) have the same color.
Then there exists a balanced line of \( S \).

Here we generalize the result of Bereg and Kano as follows:

**Theorem 4.2.** Let \( S \) be a set of \( (d+1)n \) points in \( \mathbb{R}^d \) in general position colored in colors 0, 1, \ldots, \( d \) so that:
(i) the number of points of each color is \( n \);
(ii) for any directed line \( \ell \) there exist two colors \( i \) and \( j \), \( i < j \) such that the farthest in the direction of \( \ell \) point of color \( i \) is not closer (in the direction of \( \ell \)) than any point of color \( j \).
Then there exists a balanced hyperplane \( h \), in other words, the hyperplane such that each half-space bounded by \( h \) contains precisely the same number of points of each color.

**Proof.** The proof follows almost directly from the continuous version. Replace each point by a ball solid ball centered in it and of radius \( r > 0 \) sufficiently small so that there is no hyperplane that intersects any \( d + 1 \) balls at the same time and for any partition of points of \( s \) by a hyperplane there exist a hyperplane that separates the corresponding balls accordingly. Since the points are in general position such \( r \) does exist.

Balls of each color generate an absolutely continuous measure. So we have \( d + 1 \) absolutely continuous measures; multiplying them by the same constant we make these measures probabilistic, denote them by \( \mu_0, \mu_1, \ldots, \mu_d \).

Now as in the proof of Theorem 4.3 we consider charges \( \rho_i^s = (1 + s)\mu_i - \mu_{i-1} \) and find a halfspace \( H \) such that \( \rho_i^s(H) = \cdots = \rho_d^s(H) = s/2 \).

We will show that \( \mu_i(H) \) could not be less than \( 1/n \) for all \( i \). Suppose it is so. Then again using equation \( 3.2 \) we get:

\[
(4.1) \quad \mu_{i-1}(H) < \mu_i(H), \text{ for } i = 1, \ldots, d.
\]

Consider a line \( \ell \) inner normal to \( H \) and colors \( i \) and \( j \) from condition (ii) for this line. Suppose \( \mu_i(H) < \mu_j(H) < 1/n \). The hyperplane \( \partial H \) intersects at least two balls of color \( j \), otherwise the inequality would be \( \mu_i(H) \geq \mu_j(H) \) by condition (ii). Under the above assumptions, every hyperplane can pass through at most \( d \) balls therefore there exist two colors \( k \) and \( m \) that do not intersect with \( \partial H \) and therefore with \( H \) (in the opposite case \( \mu_k(H) \) or \( \mu_m(H) \) would be at least \( 1/n \)). Hence we have \( \mu_k(H) = \mu_m(H) = 0 \) in contradiction with \( 4.1 \). So we for at least one \( i \) the measure \( \mu_i(H) \) is at least \( 1/n \).

As \( s \) goes to \( 0 \), the halfspace \( H \) tends to a certain halfspace \( H_0 \). Its border hyperplane \( h \) cuts equal positive fraction of every measure \( \mu_i \).

If \( h \) touches no ball where the measures are concentrated then we are done. But a problem can occur if \( h \) intersects some balls. Since the points in \( S \) are in general position, the hyperplane \( h \) touches at most \( d \) of them (denote the corresponding subset of \( X \) by \( I \)) and we can perturb \( h \) so that arbitrary subset \( J \subseteq I \) will be on one side of \( h \) while \( i \setminus J \) will be on the other side of \( h \). Thus we may “round” the fractions of the measure in any way we want and equalize the numbers of points in \( H \) for all the colors (compare with the proof of Corollary 3.1.3 in [11]).

**Remark 4.3.** As in remark 1.2 we can describe point sets satisfying condition (ii) in terms of permutations: For any directed line \( \ell \) the order of points with least (among the point of the same color) projection to \( \ell \) gives a permutation of colors \( \{0, 1, \ldots, d\} \); a colored point set satisfies condition (ii) if and only if we cannot get all permutations this way.

Let us give a simpler but still powerful sufficient assumption on \( S \):

**Corollary 4.4.** Let \( S \) be a set of \((d + 1)n\) points in \( \mathbb{R}^d \) in general position colored in colors \( 0, 1, \ldots, d \) so that:

(i) the number of points of each color is \( n \);
(ii) points of one color lie in the convex hull of the union of points of other \( d \) colors.

Then there exists a balanced hyperplane \( h \).

### 5. Cutting a Prescribed Fraction by a Convex Set

Let us state another problem about cutting the same fraction of several measures:

**Problem 5.1.** The dimension \( d \) and the number of measures \( k > 1 \) are given. For which \( \alpha > 0 \) for any absolutely continuous probability measures \( \mu_1, \ldots, \mu_k \) on \( \mathbb{R}^d \) it is always
intersection with \((0, \lambda)\) where \(\lambda\) defines either a halfspace or a ball and therefore their intersection is possible to find a convex subset \(C \subset \mathbb{R}^d\) such that
\[
\mu_1(C) = \cdots = \mu_k(C) = \alpha.
\]

For \(\alpha = 1/2\) and \(k = d\) a positive solution to this problem follows from the ham sandwich theorem. If \(\alpha > 1/2\) then considering two measures, one concentrated near the origin and the other concentrated uniformly near a unit sphere, we see that there is no solution.

If \(k > d + 1\) one can consider \(d + 2\) measures concentrated near the vertices of a simplex \(S\) and the mass center \(w\) of \(S\); in this case any such \(C\) must contain a neighborhood of \(w\) and therefore cannot cut \(\alpha\) of the corresponding measure.

In [15] Stromquist and Woodall solved a similar problem for \(k\) measures on a circle and cutting it by a union of \(k\) arcs.

Using the results on measure equipartitions from [12, 1, 9] we are able to solve Problem 5.1 for \(d + 1\) measures:

**Theorem 5.2.** Suppose \(\mu_0, \mu_1, \ldots, \mu_d\) are absolutely continuous probability measures on \(\mathbb{R}^d\) and \(\alpha \in (0, 1)\). It is always possible to find a convex subset \(C \subset \mathbb{R}^d\) such that
\[
\mu_0(C) = \mu_1(C) = \cdots = \mu_d(C) = \alpha,
\]
if and only if \(\alpha = 1/m\) for a positive integer \(m\).

**Proof.** First, we prove that it is possible if \(\alpha = 1/m\). Following [12] it is sufficient to consider the case when \(m\) is a prime and use induction. In this case we are going to apply [9, Theorem 1.3] to these measures and the space of functions
\[
L = \left\{a_0 + \sum_{i=1}^{d} a_i x_i + b \sum_{i=1}^{d} x_i^2 \right\}.
\]
This space has dimension \(d + 2\), which is sufficient to partition \(d + 1\) measures into equal parts by a generalized Voronoi partition. Recall that a generalized Voronoi partition corresponding to an \(m\)-tuple of pairwise distinct functions \(\{f_1, \ldots, f_m\} \subset L\) is defined by
\[
C_i = \{x \in \mathbb{R}^d : f_i(x) \leq f_j(x) \text{ for all } j \neq i\}.
\]
Assume without loss of generality that \(f_1(x)\) has the largest coefficient at \(\sum_{i=1}^{d} x_i^2\) among all \(f_j(x)\). Then the defining equations for \(C_1\) will look like
\[
(b_1 - b_j) \sum_{i=1}^{d} x_i^2 + \lambda(x) \leq 0,
\]
where \(\lambda(x)\) is a linear function and \(b_1 - b_j\) is nonnegative. Note that each of these equations defines either a halfspace or a ball and therefore their intersection \(C_1\) is convex.

Now we give a counterexample for \(d = 1\) and \(\alpha\) not of the form \(1/m\). Assume \(1/n > \alpha > \frac{1}{n+1}\). Let \(\mu_0\) to be the uniform measure on \((0, 1)\).

Let \(a_i, i = 1, \ldots, n\) be the points with coordinates \(\frac{i}{n+1}\) and \(\Delta_i\) be the intervals with centers at \(a_i\) an length \(\varepsilon < \alpha - \frac{1}{n+1}\). The support of the measure \(\mu_1\) is the union of intervals \(\Delta_i\) and \(\int_{\Delta_i} d\mu_1 = \frac{1}{n}\) for each \(i\) (Fig. 1).

It easy to see that each convex set \(C\) with \(\mu_0(C) = \alpha\) is an interval of length \(\alpha\) in intersection with \((0, 1)\) and it must contain at least one interval \(\Delta_i\). Therefore \(\mu_1(C) \geq \frac{1}{n} > \alpha\).

For \(d > 1\) we can extend the one-dimensional example. Consider a \(d\)-dimensional regular simplex with vertices \(v_0, v_1, \ldots, v_d\). Let the measures \(\mu_2, \ldots, \mu_d\) concentrate near the vertices \(v_2, \ldots, v_d\) respectively. Like we did it for \(d = 1\) let \(\mu_0\) be the uniform
measure on the tiny cylinder around the edge $v_0v_1$. The measure $\mu_1$ will look like in the one-dimensional case, but its support will be intervals on the segment that connects centers of faces $v_0v_2\ldots v_d$ and $v_1v_2\ldots v_d$. See Fig. 2 for the two-dimensional case.

Fig. 1. Fig. 2.

To conclude the proof we note the following. Assume that the measures $\mu_i$ are concentrated in $\delta$-neighborhoods of their respective vertices (for $i > 1$) or segments (for $i = 0, 1$). Then going to the limit $\delta \to +0$ and using the Blaschke selection theorem we assume that the corresponding $C_5$ tend in the Hausdorff metric to some $C_0$. This $C_0$ must intersect the segment $[v_0, v_1]$ by an interval of length at least $\alpha|v_0 - v_1|$. It also has to contain every vertex $v_i$ for $i = 2, \ldots, d$. Hence, similar to the one-dimensional case, $C_0$ contains in its interior one of the support segments of $\mu_1$. For small enough $\delta$, the convex set $C_0$ will also contain at least $1/n$ of the measure $\mu_1$, which is a contradiction.

Remark 5.3. For $d = 1$ this result follows from the theorem of Levy [10] about a segment on a curve. Note that Hopf [7] showed that the set of $\alpha$ for which the required segment does not exist is additive.

Remark 5.4. In [4, Theorem 3.2] it was proved that any two absolutely continuous probability measures on $S^2$ can be cut into pieces of measures $\alpha, \alpha, 1 - 2\alpha$ by a 3-fan. Using the central projection we see that any two absolutely continuous probability measures on $\mathbb{R}^2$ may be cut into pieces of measures $\alpha, \alpha, 1 - 2\alpha$ with a (possibly degenerate) 3-fan. It is clear that at least one of the $\alpha$ parts of the fan is a convex angle and therefore Problem 5.1 has a positive solution for $k = d = 2$ and any $\alpha \in (0, 1/2]$.

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