An algorithmic approach to construct crystallizations of 3-manifolds from presentations of fundamental groups

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Abstract

For a given presentation \( \langle S | R \rangle \) with number of generators is equal to the number of relations, we have defined weight of the pair \((\langle S | R \rangle, R)\). For such a pair with two generators and two relations, we present an algorithm which gives all the \(n\)-vertex crystallizations such that the crystallizations yield the pair, where \(n\) is the weight of the pair. In particular, the algorithm determines whether such a crystallization exists or not. Using the algorithm, we present crystallizations of some 3-manifolds and also discuss some examples when such crystallizations do not exist. Consider the closed connected orientable 3-manifold \(M\langle m, n, k \rangle\) having fundamental group \(\langle x_1, x_2, x_3 | x_1^m = x_2^n = x_3^k = x_1x_2x_3 \rangle\) for \(m \geq n \geq k \geq 2\). We have generalized our algorithm for presentations with three generators and certain class of relations. This gives us a crystallization of \(M\langle m, n, k \rangle\) with \(4(m + n + k - 3) + 2\delta_2 + 2\delta_3\) vertices for \(m \geq 3\), where \(\delta_j\) is the Kronecker delta. Those crystallizations are minimal and unique with respect to the given presentations. If \(n = 2\) or \(k \geq 3\) and \(m \geq 4\) then our crystallization of \(M\langle m, n, k \rangle\) is vertex-minimal for all the known cases.

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1 Introduction

For \(d \geq 1\), a \((d + 1)\)-colored graph \((\Gamma, \gamma)\) represents a pure \(d\)-dimensional simplicial cell complex \(\mathcal{K}(\Gamma)\) which has \(\Gamma\) as its dual graph. For a certain class of such graphs, the underlying space \(|\mathcal{K}(\Gamma)|\) is a closed connected \(d\)-manifold. In that case, the \((d + 1)\)-colored graph \((\Gamma, \gamma)\) is called a crystallization of the \(d\)-manifold \(|\mathcal{K}(\Gamma)|\). In [Pezzana 74], Pezzana showed existence of crystallization for every closed connected PL-manifold. In [Gagliardi 79], Gagliardi introduced an algorithm to finding a presentation of the fundamental group of a closed connected \(d\)-manifold \(M\) from a crystallization of \(M\). The components of the graph restricted over two colors give the relations and the components of the graph restricted over remaining colors give generators of the presentation. In [Epstein 61], Epstein proved that the fundamental group of a 3-manifold has a presentation with the number of relations is less than or equal to the number of generators. For a pair \((\langle S | R \rangle, R)\) with \(#S = #R\), we have defined its weight \(\lambda(\langle S | R \rangle, R)\) (cf. Definition 2.2). From the construction of the fundamental group (cf. Subsection 2.5), it is clear that, if \((\Gamma, \gamma)\) is a crystallization
of a closed connected orientable 3-manifold and yields a presentation \((S \mid R), R\) then \(#V(\Gamma) \geq \lambda((S \mid R), R)\). In this article, we are interested in the converse of this. More explicitly, given a presentation \((S \mid R)\) with two generators and two relations, we want to construct all the crystallizations \((\Gamma, \gamma)\) with \(\lambda((S \mid R), R)\) vertices which yield \((S \mid R), R\). For such a presentation \((S \mid R)\) of a group, we present an algorithm (cf. Section 3) which gives all the crystallizations \((\Gamma, \gamma)\) such that \((\Gamma, \gamma)\) yield \((S \mid R), R\) and are minimal (cf. Definition 2.3) with respect to \((S \mid R), R\). In particular, the algorithm determines whether such a crystallization \((\Gamma, \gamma)\) exists or not. Let \(M \not\cong L(p, q)\) be a closed connected orientable prime 3-manifold and fundamental group has a presentation \((S \mid R)\) with two generators and two relations. Then, using this algorithm we can construct all possible crystallizations of \(M\) which yield \((S \mid R), R\) and are minimal with respect to \((S \mid R), R\) (cf. Theorem 3.1). If \(M \cong L(p, q)\) then it gives all such crystallizations of \(L(p, q')\) for some \(q' \in \{1, \ldots, p - 1\}\).

We have defined generalized quaternion spaces \(S^3/Q_{4m}\) for \(m \geq 2\) and a closed connected orientable 3-manifold \(M\langle m, n, k \rangle\) having fundamental group \(\langle m, n, k \rangle\) for \(m, n, k \geq 2\) (cf. Subsection 2.4). As an application of the algorithm, we construct such crystallizations of some closed connected orientable 3-manifolds, namely, \(M\langle m, n, 2 \rangle\) for \(m, n \geq 2\), lens spaces and a hyperbolic 3-manifold. We have generalized the algorithm for presentations with three generators and certain class of relations (cf. Section 5.1). As an application, we construct a \(4(m + 2)\)-vertex crystallization of the generalized quaternion space \(S^3/Q_{4m}\) and a \(4(m + n + k - 3)\)-vertex crystallization of the 3-manifold \(M\langle m, n, k \rangle\) for \(m, n, k \geq 3\). For \((m, n, k) \neq (3, 3, 3)\), these crystallizations are vertex-minimal when the numbers of vertices are at most 30, in fact, there are no known crystallizations of these manifolds which have less number of vertices than our constructed ones (cf. Remark 5.4). We construct also a \((4m + 4n - 2)\)-vertex crystallization of the 3-manifold \(M\langle m, n, 2 \rangle\) for \(m, n \geq 3\). The crystallizations of the 3-manifolds \(S^3/Q_{4m}\), \(M\langle m, n, 2 \rangle\) and \(M\langle m, n, k \rangle\) for \(m, n, k \geq 3\) are minimal and unique with respect to the given presentations (cf. Theorem 5.1, Theorem 5.2 and Theorem 5.3).

## 2 Preliminaries

### 2.1 Colored graphs

A multigraph \(\Gamma = (V(\Gamma), E(\Gamma))\) is a finite connected graph which can have multiple edges but no loops where \(V(\Gamma)\) and \(E(\Gamma)\) denote the set of vertices and edges of \(\Gamma\) respectively. For \(n \geq 1\), an \(n\)-path is a tree with \((n + 1)\) distinct vertices and \(n\) edges. If \(a_i\) and \(a_{i+1}\) are adjacent in an \(n\)-path for \(1 \leq i \leq n\) then the \(n\)-path is denoted by \(P_n(a_1, a_2, \ldots, a_{n+1})\). For \(n \geq 2\), an \(n\)-cycle is a closed path with \(n\) distinct vertices and \(n\) edges. If vertices \(a_i\) and \(a_{i+1}\) are adjacent in an \(n\)-cycle for \(1 \leq i \leq n\) (addition is modulo \(n\)) then the \(n\)-cycle is denoted by \(C_n(a_1, a_2, \ldots, a_n)\). A graph \(\Gamma\) is called \((d + 1)\)-regular if the number of edges adjacent to each vertex is \((d + 1)\).

First we call \(\Delta_d = \{0, 1, \ldots, d\}\) the color set. An edge coloring with \((d + 1)\) colors on the graph \(\Gamma = (V(\Gamma), E(\Gamma))\) is a map \(\gamma : E(\Gamma) \to \Delta_d\) such that \(\gamma(e) \neq \gamma(f)\) whenever \(e\) and \(f\) are adjacent (i.e., \(e\) and \(f\) are adjacent to a common vertex). An \((d + 1)\)-colored graph is a pair \((\Gamma, \gamma)\) where \(\Gamma\) is a multigraph and \(\gamma\) is an edge coloring on the graph \(\Gamma\) with \((d + 1)\) colors.

Let \((\Gamma, \gamma)\) be a \((d + 1)\)-colored connected graph with color set \(\Delta_d\). If \(B \subseteq \Delta_d\) with \(k\) elements then the graph \((V(\Gamma), \gamma^{-1}(B))\) is a \(k\)-colored graph with coloring \(\gamma|_{\gamma^{-1}(B)}\). This colored graph is denoted by \(\Gamma_B\). If \(\Gamma_{\Delta_d \setminus \{c\}}\) is connected for all \(c \in \Delta_d\) then \((\Gamma, \gamma)\) is called
2.2 Spherical and hyperbolic 3-manifolds

A 3-manifold $M$ is called a spherical 3-manifold if $M \cong S^3/\Gamma$ where $\Gamma$ is a finite subgroup of $SO(4)$ acting freely by rotations on the 3-sphere $S^3$. Therefore, spherical 3-manifolds are prime, orientable and closed. Spherical 3-manifolds are sometimes called elliptic 3-manifolds or Clifford-Klein 3-manifolds. In [Thurston 80, Chapter 3], Thurston conjectured that a closed 3-manifold with finite fundamental group is spherical, which is also known as elliptization conjecture. In [Perelman 03], Perelman proved the elliptization conjecture.

A 3-manifold is called a hyperbolic 3-manifold if it equipped with a complete Riemannian metric of constant sectional curvature $-1$. In other words, it is the quotient of three-dimensional hyperbolic space by a subgroup of hyperbolic isometries acting freely and properly discontinuously. In [Aschenbrenner et al. 13, Theorem 2.2], the authors proved the following.

**Proposition 2.1.** Let $M$ and $N$ be two orientable, closed, prime 3-manifolds and let $\varphi : \pi_1(M) \to \pi_1(N)$ be an isomorphism.

(i) If $M$ and $N$ are not lens spaces then $M$ and $N$ are homeomorphic.

(ii) If $M$ and $N$ are not spherical then there exists a homeomorphism which induces $\varphi$.

2.3 Weights of presentations of groups

Given a set $S$, let $F(S)$ denote the free group generated by $S$. So, any element $w$ of $F(S)$ is of the form $w = x_1^{\varepsilon_1} \cdots x_m^{\varepsilon_m}$, where $x_1, \ldots, x_m \in S$ and $\varepsilon_i = \pm 1$ for $1 \leq i \leq m$ and $(x_{j+1}, \varepsilon_{j+1}) \neq (x_j, -\varepsilon_j)$ for $1 \leq j \leq m - 1$. For $R \subseteq F(S)$, let $N(R)$ be the smallest normal subgroup of $F(S)$ containing $R$. Then, the quotient group $F(S)/N(R)$ is denoted by $\langle S \mid R \rangle$.

For a presentation $P = \langle S \mid T \rangle$ with $N(T) = N(R)$, the pair $(P, R)$ denotes the presentation $P$ with the relation set $R$. So, if $T \neq R$ and $N(T) = N(R)$ then $\langle S \mid T \rangle = \langle S \mid R \rangle$ but as a pair $((S \mid T), T) \neq ((S \mid R), R)$.

For a finite subset $R$ of $F(S)$, let

$$ \overline{R} := \{ w \in N(R) : N((R \setminus \{ r \}) \cup \{ w \}) = N(R) \text{ for each } r \in R \}. $$

(2.1)

Observe that $\overline{\emptyset} = \emptyset$ and if $R \neq \emptyset$ is a finite set then $w := \prod_{r \in R} r \in \overline{R}$ and hence $\overline{R} \neq \emptyset$.

For $w = x_1^{\varepsilon_1} \cdots x_m^{\varepsilon_m} \in F(S)$, $m \geq 1$, let

$$ \varepsilon(w) := \begin{cases} 0 & \text{if } m = 1, \\ |\varepsilon_1 - \varepsilon_2| + \cdots + |\varepsilon_{m-1} - \varepsilon_m| + |\varepsilon_m - \varepsilon_1| & \text{if } m \geq 2. \end{cases} $$

Consider the map $\lambda : F(S) \to \mathbb{Z}^+$ define inductively as follows.

$$ \lambda(w) := \begin{cases} 2 & \text{if } w = \emptyset, \\ 2m - \varepsilon(w) & \text{if } w = x_1^{\varepsilon_1} \cdots x_m^{\varepsilon_m}, (x_m, \varepsilon_m) \neq (x_1, -\varepsilon_1), \\ \lambda(w') & \text{if } w = x_1^{\varepsilon_1} w' x_1^{-\varepsilon_1}. \end{cases} $$

(2.2)

Since $|\varepsilon_i - \varepsilon_j| \geq 0$ or 2, $\varepsilon(w)$ is an even integer and hence $\lambda(w)$ is also even. For $w \in F(S)$, $\lambda(w)$ is said to be the weight of $w$. Observe that $\lambda(w_1 w_2) = \lambda(w_2 w_1)$ for $w_1, w_2 \in F(S)$. In
A group is called a binary polyhedral group

2.4 Binary polyhedral groups and generalized quaternion spaces

Let \( S = \{x_1, \ldots, x_s\} \) and \( R = \{r_1, \ldots, r_s\} \subseteq F(S) \). Let \( r_{s+1} \) be an element in \( \overline{R} \) of minimum weight. Then, the number

\[ \lambda(S | R, R) := \lambda(r_1) + \cdots + \lambda(r_s) + \lambda(r_{s+1}). \]

is called the weight of the pair \((S | R, R)\).

Let \( w = \alpha_{1}^{\varepsilon_1} \alpha_{2}^{\varepsilon_2} \cdots \alpha_{m}^{\varepsilon_m} \in F(S := \{x_1, \ldots, x_s\}) \) where \( \varepsilon_i \in \{+1, -1\} \) for \( 1 \leq i \leq m \). Then, we define

(i) \( w_{ij}^{(2)} := \) total number of appearances of \( x_i^{-1} x_j \) and \( x_j^{-1} x_i \) in \( \alpha_{m}^{\varepsilon_m} \alpha_{1}^{\varepsilon_1} \alpha_{2}^{\varepsilon_2} \cdots \alpha_{m}^{\varepsilon_m} \), for \( x_i, x_j \in S \) and \( 1 \leq i \neq j \leq 3 \),

(ii) \( w_{ij}^{(3)} := \) total number of appearances of \( x_i x_j^{-1} \) and \( x_j x_i^{-1} \) in \( \alpha_{m}^{\varepsilon_m} \alpha_{1}^{\varepsilon_1} \alpha_{2}^{\varepsilon_2} \cdots \alpha_{m}^{\varepsilon_m} \), for \( x_i, x_j \in S \) and \( 1 \leq i \neq j \leq 3 \),

(iii) \( w_{i(j+1)}^{(2)} := \) total number of appearances of \( x_i^{-1} x_{j+1} \) and \( x_{j+1} x_i^{-1} \) in \( \alpha_{m}^{\varepsilon_m} \alpha_{1}^{\varepsilon_1} \alpha_{2}^{\varepsilon_2} \cdots \alpha_{m}^{\varepsilon_m} \), for \( x_i, x_{j+1} \in S \),

(iv) \( w_{i(j+1)}^{(3)} := \) total number of appearances of \( x_j^{-1} x_i \) and \( x_i x_j^{-1} \) in \( \alpha_{m}^{\varepsilon_m} \alpha_{1}^{\varepsilon_1} \alpha_{2}^{\varepsilon_2} \cdots \alpha_{m}^{\varepsilon_m} \), for \( x_i, x_j \in S \).

Observe that \( \lambda(w) = \sum w_{ij}^{(c)} \) the sum over \( 1 \leq i \neq j \leq s + 1 \) and \( 2 \leq c \leq 3 \).

2.4 Binary polyhedral groups and generalized quaternion spaces

A group is called a binary polyhedral group if it has a presentation of the form \( \langle x_1, x_2, x_3 \mid x_i^n = x_i^m = x_i^k = 1 \rangle \) for some integer \( m, n, k \geq 2 \). This group is denoted by \( \langle m, n, k \rangle \).

This group is known to be fundamental group group of the 3-manifold \( M \). In this article, we will denote such a 3-manifold by \( M \).

\[ \lambda(m, n, k) \approx M(m, n, k) \]

for every permutation \( \overline{mn}k \) of \( mnk \). Thus, we can assume that \( m \geq n \geq k \).

Clearly, the group \( \langle x_1, x_2 \mid x_1^{-n} x_2 = x_2^{-n} x_1 - x_1 x_2 x_1^{-1} x_2^{-1} \rangle \) is isomorphic to the abelianized of \( \langle m, n, k \rangle \). Therefore, \( \langle m, n, k \rangle = (5, 3, 2) \) or \( (7, 3, 2) \) implies abelianization of \( \langle m, n, k \rangle \) is trivial. Thus, \( M(5, 3, 2) \) and \( M(7, 3, 2) \) are homology spheres, in fact, \( M(5, 3, 2) \) is the Poincaré homology sphere. Since \( \langle m, 2, 2 \rangle \approx Q_{4m} \), \( P_{24} := (3, 3, 2) \), \( P_{32} := (4, 3, 2) \) and \( P_{210} := (5, 3, 2) \) are finite group, by the proof of elliptization conjecture, |K(\Gamma)| is spherical for these groups. It is not difficult to prove that, if \( (m, n, k) = (6, 3, 2) \), or \( (3, 3, 3) \), then abelianization of \( \langle m, n, k \rangle \approx Z \oplus H \), for some group \( H \). Therefore, in these three cases, the 3-manifold \( M(m, n, k) \) has a handle and in all the other cases, \( M(m, n, k) \) is handle-free.
A group is called a generalized quaternion group or dicyclic group if it has a presentation of the form \( \langle x_1, x_2 \mid x_1^{2m} = x_2^2 = 1, x_1^m = x_2^2x_1x_2 = x_1^{-1} \rangle \) for some integer \( m \geq 2 \). This group has order \( 4m \) and is denoted by \( Q_{4m} \).

**Claim:** For \( m \geq 2 \), \( Q_{4m} \) has a presentation \( \langle S \mid R \rangle \), where \( S = \{ x_1, x_2, x_3 \} \) and \( R = \{ x_1^{-m}x_3^{-1}x_2^{-1}, x_2x_1^{-1}x_3^{-1}, x_3x_2^{-1}x_1^{-1} \} \).

Observe that \( x_2x_1^{-1}x_3^{-1} = 1 = x_3x_2^{-1}x_1^{-1} \) implies \( x_2x_1^{-1} = x_3 = x_1x_2 \), i.e., \( x_1^{-1} = x_2x_1x_2 \). Again, \( x_1^{-m}x_3^{-1}x_2^{-1} = 1 \) and \( x_2x_1^{-1}x_3^{-1} = 1 \) implies \( x_1^{m} = x_2^2 \). Now, \( x_2 = x_1x_2x_1 = x_1(x_1x_2x_1) = x_1^2x_2^2 \).

Let \( K \) be a pure simplicial cell complex whose geometric carrier is \( M \langle m, 2, 2 \rangle \). This proves the claim.

The 3-manifold \( M \langle m, 2, 2 \rangle \) is called generalized quaternion space. Then, by the proof of elliptization conjecture, \( M \langle m, 2, 2 \rangle \) is a spherical and homeomorphic to \( S^3/Q_{4m} \).

### 2.5 Crystallizations

A CW-complex \( X \) is said to be regular if the attaching maps which define the incidence structure of \( X \) are homeomorphisms. Given a regular CW-complex \( X \), let \( \mathcal{X} \) be the set of all closed cells of \( X \) together with the empty set. Then, \( \mathcal{X} \) is a poset, where the partial ordering is the set inclusion. This poset \( \mathcal{X} \) is said to be the face poset of \( X \). Clearly, if \( X \) and \( Y \) are two finite regular CW-complexes with isomorphic face posets then \( X \) and \( Y \) are homeomorphic. A regular CW-complex \( X \) is said to be simplicial if the boundary of each cell in \( X \) is isomorphic (as a poset) to the boundary of a simplex of same dimension. A simplicial cell complex \( K \) of dimension \( d \) is a poset isomorphic to the face poset \( \mathcal{X} \) of a \( d \)-dimensional simplicial CW-complex \( X \). The topological space \( X \) is called the geometric carrier of \( K \) and is also denoted by \( |K| \). If a topological space \( M \) is homeomorphic to \( |K| \), then \( K \) is said to be a pseudotriangulation of \( M \).

Let \( K \) be a simplicial cell complex with partial ordering \( \preceq \). If \( \beta \leq \alpha \in K \) then we say \( \beta \) is a face of \( \alpha \). If all the maximal cells of a \( d \)-dimensional simplicial cell complex \( K \) are \( d \)-cells then it is called pure. Maximal cells in a pure simplicial cell complex \( K \) are called the facets of \( K \). The 0-cells in a simplicial cell complex \( K \) are said to be the vertices of \( K \). If \( u \) is a face of \( \alpha \) and \( u \) is a vertex then we say \( u \) is a vertex of \( \alpha \). Clearly, a \( d \)-dimensional simplicial cell complex \( K \) has at least \( d + 1 \) vertices. If a \( d \)-dimensional simplicial cell complex \( K \) has exactly \( d + 1 \) vertices then \( K \) is called contracted.

Let \( K \) be a pure \( d \)-dimensional simplicial cell complex. Consider the graph \( \Lambda(K) \) whose vertices are the facets of \( K \) and edges are the ordered pairs \( (\{ \sigma_1, \sigma_2 \}, \gamma) \), where \( \sigma_1, \sigma_2 \) are facets, \( \gamma \) is a \((d - 1)\)-cell and is a common face of \( \sigma_1, \sigma_2 \). The graph \( \Lambda(K) \) is said to be the dual graph of \( K \). Observe that \( \Lambda(K) \) is in general a multigraph without loops. On the other hand, for \( d \geq 1 \), if \( (\Gamma, \gamma) \) is a \((d + 1)\)-colored graph with color set \( \Delta_d = \{0, \ldots, d\} \) then we define a \( d \)-dimensional simplicial cell complex \( \mathcal{K}(\Gamma) \) as follows. For each \( v \in V(\Gamma) \), we take a \( d \)-simplex \( \sigma_v \) and label its vertices by \( 0, \ldots, d \). If \( u, v \in V(\Gamma) \) are joined by an edge \( e \) and \( \gamma(e) = i \), then we identify the \((d - 1)\)-faces of \( \sigma_u \) and \( \sigma_v \) opposite to the vertices labelled by \( i \), so that equally labelled vertices are identified together. Since there is no identification within a \( d \)-simplex, this gives a simplicial CW-complex \( W \) of dimension \( d \). So, the face poset \( \mathcal{K}(\Gamma) \) of \( W \) is a pure \( d \)-dimensional simplicial cell complex. We say that \( (\Gamma, \gamma) \) represents the simplicial cell complex \( \mathcal{K}(\Gamma) \). Clearly, the number of \( i \)-labelled vertices of \( \mathcal{K}(\Gamma) \) is equal to the number of components of \( \Gamma_{\Delta_d \setminus \{i\}} \) for each \( i \in \Delta_d \). Thus, the simplicial cell complex \( \mathcal{K}(\Gamma) \) is contracted if and only if \( \Gamma \) is contracted (cf. [Ferri et al. 86]).

A crystallization of a connected closed \( d \)-manifold \( M \) is a \((d + 1)\)-colored contracted graph.
(\(\Gamma, \gamma\)) such that the simplicial cell complex \(K(\Gamma)\) is a pseudotriangulation of \(M\). Thus, if \((\Gamma, \gamma)\) is a crystallization of a \(d\)-manifold \(M\) then the number of vertices in \(K(\Gamma)\) is \(d+1\). On the other hand, if \(K\) is a contracted pseudotriangulation of \(M\) then the dual graph \(\Lambda(K)\) gives a crystallization of \(M\). Clearly, if \((\Gamma, \gamma)\) is a crystallization of a closed \(d\)-manifold \(M\) then, either \(\Gamma\) has two vertices (in which case \(M\) is \(S^d\)) or the number of edges between two vertices is at most \(d - 1\). In [Pezzana 74], Pezzana showed the following.

**Proposition 2.3** (Pezzana). Every connected closed PL-manifold admits a crystallization.

Thus, every connected closed pl \(d\)-manifold has a contracted pseudotriangulation, i.e., a pseudotriangulation with \(d+1\) vertices. From [Cavicchioli et al. 80], we know the following.

**Proposition 2.4** (Cavicchioli-Grasselli-Pezzana). Let \((\Gamma, \gamma)\) be a crystallization of a \(d\)-manifold \(M\). Then \(M\) is orientable if and only if \(\Gamma\) is bipartite.

Let \(\Delta_d = \{0, \ldots, d\}\) be the color set of a \((d+1)\)-colored graph \((\Gamma, \gamma)\). For \(0 \leq i \neq j \leq d\), \(g_{ij}\) denote the number of connected components of the graph \(\Gamma_{\{i,j\}}\). In [Gagliardi 79a], Gagliardi proved the following.

**Proposition 2.5** (Gagliardi). Let \((\Gamma, \gamma)\) be a contracted \(4\)-colored graph with the color set \(\Delta_3\). Then, \((\Gamma, \gamma)\) is a crystallization of a connected closed \(3\)-manifold if and only if

(i) \(g_{ij} = g_{kl}\) for \(\{i, j, k, l\} = \Delta_3\), and

(ii) \(g_{01} + g_{02} + g_{03} = 2 + \frac{\#V(\Gamma)}{2}\).

Let \((\Gamma, \gamma)\) be a crystallization (with the color set \(\Delta_d\)) of a connected closed \(d\)-manifold \(M\). So, \(\Gamma\) is an \((d+1)\)-regular graph. Choose two colors, say, \(i\) and \(j\) from \(\Delta_d\). Let \(\{G_1, \ldots, G_{s+1}\}\) be the set of all connected components of \(\Gamma_{\Delta_d\setminus\{i,j\}}\) and \(\{H_1, \ldots, H_{t+1}\}\) be the set of all connected components of \(\Gamma_{\{i,j\}}\). Since \(\Gamma\) is regular, each \(H_p\) is an even cycle. Note that, if \(d = 2\) then \(\Gamma_{\{i,j\}}\) is connected and hence \(H_1 = \Gamma_{\{i,j\}}\). Consider a set \(\tilde{S} = \{x_1, \ldots, x_s, x_{s+1}\}\) of \(s+1\) elements. For \(1 \leq k \leq t+1\), consider the word \(\tilde{r}_k\) in \(F(\tilde{S})\) as follows. Choose a vertex \(v_1\) in \(H_k\). Let \(H_k = v_1e_1^i e_2^j v_3 e_3^i e_4 v_4 \cdots e_{2l}^j v_{2l} e_{2l}^j v_1\), where \(e_i^j\) and \(e_q^j\) are edges with colors \(i\) and \(j\) respectively. Define

\[
\tilde{r}_k := x_{k_2}^{\pm 1} x_{k_3}^{\pm 1} \cdots x_{k_{2l}}^{\pm 1} x_{k_1}^{\pm 1},
\]  

(2.3)

where \(G_{k_h}\) is the component of \(\Gamma_{\Delta_d\setminus\{i,j\}}\) containing \(v_h\). For \(1 \leq k \leq t+1\), let \(r_k\) be the word obtained from \(\tilde{r}_k\) by deleting \(x_{s+1}^{\pm 1}\)’s in \(\tilde{r}_k\). So, \(r_k\) is a word in \(F(S)\), where \(S = \tilde{S} \setminus \{x_{s+1}\}\). In [Gagliardi 79b], Gagliardi proved the following.

**Proposition 2.6** (Gagliardi). For \(d \geq 2\), let \((\Gamma, \gamma)\) be a crystallization of a connected closed \(d\)-manifold \(M\). For two colors \(i, j\), let \(s, t, x_p, r_q\) be as above. If \(\pi_1(M, x)\) is the fundamental group of \(M\) at a point \(x\), then

\[
\pi_1(M, x) \cong \begin{cases} 
\langle x_1, x_2, \ldots, x_s \mid r_1 \rangle & \text{if } d = 2, \\
\langle x_1, x_2, \ldots, x_s \mid r_1, \ldots, r_t \rangle & \text{if } d \geq 3.
\end{cases}
\]

In this case, we will say \((\Gamma, \gamma)\) yields \((\langle S \mid R \rangle, R)\), where \(S = \{x_1, \ldots, x_s\}\) and \(R = \{r_1, \ldots, r_t\}\). From Proposition 2.5, it is clear that, if \((\Gamma, \gamma)\) is a crystallization of a \(3\)-manifold then \(s = t\). Note that, there may have a relation \(r \in R\) such that \(r \in R \setminus \{r\}\) and in that case \(\langle S \mid R \rangle \cong \langle S \mid R \setminus \{r\} \rangle\).
Lemma 2.7. If \((\Gamma, \gamma)\) is a crystallization of a 3-manifold such that \((\Gamma, \gamma)\) yields \((\langle S \mid R \rangle, R)\) then \(\#V(\Gamma) \geq \lambda(\langle S \mid R \rangle, R)\).

Proof. Since the crystallization yields \((\langle S \mid R \rangle, R)\), from the above discussion, we know the crystallization yields the relations in \(R \cup \{w\}\) where \(w \in \overline{R}\). Thus, the lemma follows from the construction of \(\tilde{\gamma}\) as in Eq. (2.3) and Definition 2.2.

Definition 2.8. A crystallization \((\Gamma, \gamma)\) of a 3-manifold is called minimal with respect to the pair \((\langle S \mid R \rangle, R)\) if \(\#V(\Gamma) = \lambda(\langle S \mid R \rangle, R)\).

3 Constructions of crystallizations and an algorithm

In this section, we first discuss that, for a pair \((\langle S \mid R \rangle, R)\), how to construct \(\lambda(\langle S \mid R \rangle, R)\)-vertex crystallizations which yield \((\langle S \mid R \rangle, R)\) and then we present an algorithm which gives such crystallizations. In particular, the algorithm determines whether such a crystallization exists or not. Since we are interested in orientable 3-manifolds, by Proposition 2.4, \(\Gamma\) is bipartite. Thus, we use black dots ‘•’ for vertices in one part and white dots ‘◦’ for vertices in the other part in \(\Gamma\).

3.1 Constructions

Let \(\langle S \mid R \rangle\) be a presentation of a group with \(\#S = \#R = 2\). We will construct all possible crystallizations from the pair \((\langle S \mid R \rangle, R)\) by the following steps.

Step 1: If a crystallization yields \((\langle S \mid R \rangle, R)\) then the crystallization yields the relations in \(R \cup \{w\}\) where \(w \in \overline{R}\). Since we are interested in \(\lambda(\langle S \mid R \rangle, R)\)-vertex crystallizations, \(w \in \overline{R}\) is of minimum weight. Let \(w_i \in \overline{R}, 1 \leq i \leq k\) be all the elements with minimum weight (as there are only finite number of words in \(\overline{R}\) with minimum weight). Let \(R_i = R \cup \{w_i\}\) for \(1 \leq i \leq k\). For each \(R \in \{R_1, \ldots, R_k\}\), we will construct all possible crystallizations which yield the relations in \(R\). Choose a \(R \in \{R_1, \ldots, R_k\}\).

Step 2: If possible, let \((\Gamma, \gamma)\) be a crystallization of a 3-manifold which yields the relations in \(R\). Without loss of generality, let \(G_1, G_2, G_3\) be the components of \(\Gamma_{\{0,1\}}\) such that \(G_i\) represents the generator \(x_i\) for \(1 \leq i \leq 2\) and \(G_3\) represents \(x_3\) (cf. Eq. (2.3) for construction of \(\tilde{\gamma}\)). Let \(n_i\) be the total number of appearance of \(x_i\) in the three relations in \(R\) for \(1 \leq i \leq 2\) and \(n_3 = \lambda(\langle S \mid R \rangle, R) - (n_1 + n_2)\). Then, the total number of vertices in \(G_i\) should be \(n_i\). Let \(G_i = C_{n_i}(x^{(1)}_i, \ldots, x^{(n_i)}_i)\) for \(1 \leq i \leq 3\). Clearly, each \(n_i\) is even and \(n_1 + n_2 + n_3 = \#V(\Gamma)\). Without loss of generality, we can assume that \(x_i^{(2j-1)}x_i^{(2j)} \in \gamma^{-1}(1)\) and \(x_i^{(2j)}x_i^{(2j+1)} \in \gamma^{-1}(0)\) with \(x_i^{(n_i+1)} = x_i^{(1)}\) for \(1 \leq j \leq n_i/2\) and \(1 \leq i \leq 3\). Here and after, the additions and subtractions at the point ‘*’ in \(x_i^{(s)}\) are modulo \(n_i\) for \(1 \leq i \leq 3\).

Step 3: We know that \(\#V(\Gamma)\) is always even and there is no 2-cycle in \(\Gamma_{\{i,j\}}\) for \(0 \leq i \leq 1\) and \(2 \leq j \leq 3\) as \(\#V(\Gamma) = \lambda(\langle S \mid R \rangle, R)\). If \(\#V(\Gamma)\) is of the form \(4n\) for some \(n \in \mathbb{N}\), then \(g_{01} + g_{02} + g_{03} = 2n + 2\). This implies, \(g_{02} + g_{03} = 2n - 1\). Without loss of generality, consider \(g_{02} = g_{13} = n - 1\) and \(g_{03} = g_{12} = n\). Therefore, all the components of \(\Gamma_{\{0,3\}}\) (resp., \(\Gamma_{\{1,2\}}\)) are 4-cycles. But, there are two choices for \(\Gamma_{\{0,2\}}\) (resp., \(\Gamma_{\{1,3\}}\)). Either \(\Gamma_{\{0,2\}}\) has one 8-cycle and remaining 4-cycles or \(\Gamma_{\{0,2\}}\) has two 6-cycles and remaining 4-cycles. Similar arguments hold for \(\Gamma_{\{1,3\}}\). On the other hand, If \(\#V(\Gamma)\) is of the form \(4n + 2\) for some \(n \in \mathbb{N}\), then \(g_{01} + g_{02} + g_{03} = 2n + 3\). This implies, \(g_{02} + g_{03} = 2n\) and hence \(g_{02} = g_{13} = g_{03} = g_{12} = n\). Therefore, \(\Gamma_{\{i,j\}}\) has one 6-cycle and remaining all 4-cycles for \(0 \leq i \leq 1\) and \(2 \leq j \leq 3\).

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Step 4: Since components of $\Gamma_{(2,3)}$ yield the relations in $\mathcal{R}$, without loss of generality, let the colors 2 and 3 be the colors ‘$i$’ and ‘$j$’ as in construction of $\bar{r}$ for $r \in \mathcal{R}$ (cf. Eq. (2.3). Let $m_{ij}^{(c)} := \sum_{w \in \mathcal{R}} w_{ij}^{(c)}$ for $1 \leq i \neq j \leq 3$ and $2 \leq c \leq 3$, where $w_{ij}^{(c)}$ as in Subsection 2.3. Then, the number of edges of color $c$ between $G_i$ and $G_j$ is $m_{ij}^{(c)}$ for $2 \leq c \leq 3$ and $1 \leq i \neq j \leq 3$. Therefore, $2(m_{12}^{(c)} + m_{13}^{(c)} + m_{23}^{(c)}) = \#V(\Gamma)$ for $2 \leq c \leq 3$. Thus, $g_{01} + g_{02} + g_{03} = \#V(\Gamma)/2 + 2 = m_{12}^{(c)} + m_{13}^{(c)} + m_{23}^{(c)} + 2$. Since $g_{01} = 3$, $g_{02} = g_{13}$ and $g_{03} = g_{12}$, we have $g_{0c} + g_{1c} = m_{12}^{(c)} + m_{13}^{(c)} + m_{23}^{(c)} - 1$ for $2 \leq c \leq 3$.

Step 5: Since $\Gamma_{(0,1,c)}$ is connected for $2 \leq c \leq 3$, $\#\{m_{ij}^{(c)} \geq 1, 1 \leq i \neq j \leq 3\} \geq 2$.

Case 1: Let $\#\{m_{ij}^{(c)} \geq 1, 1 \leq i \neq j \leq 3\} = 3$, i.e., $m_{ij}^{(c)} \geq 1$, where $1 \leq i \neq j \leq 3$ for some $c \in \{2,3\}$. Then, the maximum number of bi-colored 4-cycles in $\Gamma_{(0,1,c)}$ with two edges of color $c$ is $(m_{12}^{(c)} - 1) + (m_{13}^{(c)} - 1) + (m_{23}^{(c)} - 1) = m_{12}^{(c)} + m_{13}^{(c)} + m_{23}^{(c)} - 3$. Since $g_{0c} + g_{1c} = m_{12}^{(c)} + m_{13}^{(c)} + m_{23}^{(c)} - 1$, from the arguments in Step 3, $\Gamma_{(0,1,c)}$ must have $m_{12}^{(c)} + m_{13}^{(c)} + m_{23}^{(c)} - 3$ number of bi-colored 4-cycles and two 6-cycles with some edges of color $c$. Therefore, from the arguments in Step 3, if $\#V(\Gamma) = 4n$ for some $n \in \mathbb{N}$ then $\Gamma_{(0,3)}$ (resp., $\Gamma_{(1,2)}$) is union of 4-cycles and $\Gamma_{(c-2,c)}$ is of the form $C_6 \sqcup (n - 3)C_4$. But, if $\#V(\Gamma) = 4n + 2$ for some $n \in \mathbb{N}$ then $\Gamma_{(i,c)}$ is of the form $C_6 \sqcup (n - 1)C_4$ for $0 \leq i \leq 1$.

Case 2: Let $\#\{m_{ij}^{(c)} \geq 1, 1 \leq i \neq j \leq 3\} = 2$ for some $c \in \{2,3\}$. Then, assume $\{i,j,l\} = \{1,2,3\}$ such that $m_{jl}^{(c)} = 0$. Then, the maximum number of bi-colored 4-cycles in $\Gamma_{(0,1,c)}$ with two edges of color $c$ is $(m_{12}^{(c)} - 1) + (m_{13}^{(c)} - 1) = m_{12}^{(c)} + m_{13}^{(c)} - 2$. Since $\#V(\Gamma) = \lambda((S \mid R), R)$ and $m_{ij}^{(c)} = 0$, $\Gamma_{(0,1,c)}$ does not have a bi-colored 6-cycle with some edges of color $c$. Therefore, $g_{0c} + g_{1c} = m_{12}^{(c)} + m_{13}^{(c)} - 1$ and the arguments in Step 3 implies $\Gamma_{(0,1,c)}$ must have $m_{12}^{(c)} + m_{13}^{(c)} - 2$ number of bi-colored 4-cycles and one 8-cycle with some edges of color $c$. Thus, $\Gamma_{(0,3)}$ (resp., $\Gamma_{(1,2)}$) is union of 4-cycles and $\Gamma_{(c-2,c)}$ is of the form $C_8 \sqcup (n - 2)C_4$. In this case, $\#V(\Gamma) = 4n$ for some $n \in \mathbb{N}$. Therefore, it is clear that, $m_{ij}^{(c)}$ edges of color $c$ between $G_i$ and $G_j$ yield $m_{ij}^{(c)} - 1$ number of bi-colored 4-cycles for $1 \leq i \neq j \leq 3$ and $2 \leq c \leq 3$.

Step 6: Now, we will construct $\Gamma_{(0,1,2)}$ and will show that $\Gamma_{(0,1,2)}$ is unique up to an isomorphism. Choose $\{n_i, n_j, n_l\} = \{n_1, n_2, n_3\}$ such that $n_i \geq n_j$ and $n_i \geq n_l$. Clearly, there are edges of color 3 between the pairs $(G_i, G_j)$ and $(G_i, G_l)$. Without loss of generality, let $x_i^{(1)} x_l^{(1)} x_i^{(n_i)} x_j^{(1)} \in \gamma^{-1}(2)$.

Case 1: If $m_{jl}^{(2)} = 0$ then the path $P_5(x_j^{(n_j)} x_j^{(1)} x_i^{(n_i)} x_i^{(1)} x_l^{(1)} x_j^{(n_j)})$ must be a part of the 8-cycle of $\Gamma_{(0,2)}$. Since the $m_{ij}^{(2)}$ (resp., $m_{il}^{(2)}$) edges of color $c$ between the pair $(G_i, G_j)$ (resp., $(G_i, G_l)$) yield $m_{ij}^{(2)} - 1$ (resp., $m_{il}^{(2)} - 1$) number of bi-colored 4-cycles, we have $x_i^{(1)} x_l^{(1)}, \ldots, x_l^{(m_{il}^{(2)})} x_l^{(m_{il}^{(2)})}$ and $x_i^{(n_i)} x_j^{(1)}, \ldots, x_i^{(n_i + 1 - m_{ij}^{(2)})} x_j^{(m_{ij}^{(2)})} \in \gamma^{-1}(2)$. In this case $m_{ij}^{(2)} = n_l, m_{ij}^{(2)} = n_i$ and $n_i = n_l + n_j$.

Case 2: If $m_{jl}^{(2)} \geq 1$ then $x_j^{(n_j)} x_l^{(n_l)} \in \gamma^{-1}(2)$ to complete the 6-cycle in $\Gamma_{(0,2)}$. Therefore, by similar reasoning as above, $\{x_i^{(1)} x_j^{(1)}, \ldots, x_l^{(m_{il}^{(2)})} x_l^{(m_{il}^{(2)})}\}$, $\{x_i^{(n_j)} x_j^{(1)}, \ldots, x_i^{(n_i + 1 - m_{ij}^{(2)})} x_j^{(m_{ij}^{(2)})}\}$ and $\{x_j^{(n_l)} x_l^{(1)}, \ldots, x_j^{(n_l + 1 - m_{jl}^{(2)})} x_l^{(m_{jl}^{(2)})}\}$ are the sets of edges of color 2. In this case, $m_{ij}^{(2)} + m_{il}^{(2)} = n_i, m_{ij}^{(2)} + m_{jl}^{(2)} = n_j$ and $m_{il}^{(2)} + m_{jl}^{(2)} = n_l$. 

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Thus, $\Gamma_{\{0,1,2\}}$ is unique up to an isomorphism. We use white dot ‘o’ for vertex $x_i^{(1)}$ and black dot ‘•’ for vertex $x_i^{(ii)}$. Since $\Gamma_{\{0,1,2\}}$ is bipartite graph, $x_i^{(2p_i-1)}$, $x_j^{(2p_j-1)}$, $x_l^{(2p_l)}$ are denoted by white dots ‘o’ and $x_j^{(2p_j)}$, $x_j^{(2p_j)}$, $x_l^{(2p_l-1)}$ are denoted by black dots ‘•’ for $1 \leq p_i \leq n_i/2$, $1 \leq p_j \leq n_j/2$ and $1 \leq p_l \leq n_l/2$.

**Step 7:** Now, we are ready to construct a crystallization $(\Gamma, \gamma)$ which yields the relations in $\mathcal{R}$. For a given set of relation $\mathcal{R}$, we construct $\Gamma_{\{0,1,2\}}$ uniquely. For the same arguments as above, $m_{ij}^{(3)}, m_{il}^{(3)} \geq 1$. Choose an edge $x_i^{(2q_i-1)}x_j^{(2q_j)} \in \gamma^{-1}(1)$. There are $n_i/2$ choices for such edge as $1 \leq q_i \leq n_i/2$. Now, choose two edges $x_j^{(2q_j-1)}x_j^{(2q_j)}$ and $x_i^{(2q_i-1)}x_i^{(2q_i)} \in \gamma^{-1}(1)$ in $G_j$ and $G_l$ respectively. Then, either $x_i^{(2q_i-1)}x_j^{(2q_j)}x_j^{(2q_j)}x_i^{(2q_i)} \in \gamma^{-1}(3)$ or $x_i^{(2q_i-1)}x_j^{(2q_j-1)}x_j^{(2q_j-1)}x_i^{(2q_i)} \in \gamma^{-1}(3)$. Thus, either $P_3(x_j^{(2q_j-1)}, x_j^{(2q_j)}, x_i^{(2q_i-1)}, x_i^{(2q_i)}, x_i^{(2q_i-1)})$ or $P_5(x_j^{(2q_j)}, x_j^{(2q_j-1)}, x_i^{(2q_i)}, x_i^{(2q_i-1)}, x_i^{(2q_i)}, x_i^{(2q_i-1)}, x_i^{(2q_i)}$) is a path in $\Gamma_{\{1,3\}}$. Therefore, by similar arguments as in Step 6, there are unique choices for the remaining edges of color 3. Since $1 \leq q_i \leq n_i/2$, $1 \leq q_j \leq n_j/2$ and $1 \leq q_l \leq n_l/2$, we have $2(n_i/2)(n_j/2)(n_l/2) = (n_1n_2n_3)/4$ choices for the 4-colored graph.

**Step 8:** For a set $\mathcal{R}$ of relations there are $(n_1n_2n_3)/4$ choices for the 4-colored graph. If there is a choice, for which $(\Gamma, \gamma)$ yields the relations in $\mathcal{R}$ then $(\Gamma, \gamma)$ is a regular bipartite 4-colored graph which satisfies all the properties of Proposition 2.5. Therefore, $(\Gamma, \gamma)$ is a crystallization of a closed connected orientable 3-manifold $M$ whose fundamental group is $(\langle S \mid R \rangle, R)$. Now, we choose a different $\mathcal{R} \in \{R_1, \ldots, R_k\}$ and repeat the process from Step 2 to find all possible $\Lambda(\langle S \mid R \rangle, R)$-vertex crystallizations which yield $(\langle S \mid R \rangle, R)$. If there is no such a 4-colored graph for each $\mathcal{R} \in \{R_1, \ldots, R_k\}$ then there is no crystallization of a closed, connected orientable 3-manifold which yields $(\langle S \mid R \rangle, R)$ and is minimal with respect to $(\langle S \mid R \rangle, R)$.

**Theorem 3.1.** Let $M$ be a closed connected orientable prime 3-manifold with fundamental group $(\langle S \mid R \rangle, R)$, where $\#S = \#R = 2$. Let $(\Gamma, \gamma)$ be a crystallization constructed from the pair $(\langle S \mid R \rangle, R)$ by using the above construction. Then, we have the following.

(i) If $\langle S \mid R \rangle \not\cong \mathbb{Z}_p$ then $|\mathcal{K}(\Gamma)| \cong M$.

(ii) If $\langle S \mid R \rangle \cong \mathbb{Z}_p$ then $|\mathcal{K}(\Gamma)| \cong L(p, q')$ for some $q' \in \{1, \ldots, p-1\}$.

(iii) $(\Gamma, \gamma)$ is minimal with respect to the pair $(\langle S \mid R \rangle, R)$.

**Proof.** Since $(\Gamma, \gamma)$ yields the presentation $(\langle S \mid R \rangle, R)$, by Proposition 2.4, $\pi_1(|\mathcal{K}(\Gamma)|, *) \cong \langle S \mid R \rangle$. Since $\Gamma$ is regular and bipartite, $|\mathcal{K}(\Gamma)|$ is a closed connected orientable 3-manifold. Since $M$ is prime, fundamental group of $M$ is not a free product of two group. Therefore, $|\mathcal{K}(\Gamma)|$ is also prime as $M$ and $|\mathcal{K}(\Gamma)|$ have same fundamental group. Part (i) now follows from Proposition 2.1.

If $\pi_1(|\mathcal{K}(\Gamma)|, *) \cong \pi_1(M, *) \cong \mathbb{Z}_p$ then, by the proof of elliptization conjecture, $|\mathcal{K}(\Gamma)|$ is spherical and hence $|\mathcal{K}(\Gamma)| \cong S^3/\mathbb{Z}_p \cong L(p, q')$ for some $q' \in \{1, \ldots, p-1\}$. This proves part (ii).

Part (iii) follows from the construction.

#### 3.2 Algorithm 1

Now, we state an algorithm to find crystallizations of 3-manifolds for a presentation $(\langle S \mid R \rangle, R)$ with two generators and two relations, such that the crystallizations yield the pair and have number of vertices equal to the weight of the pair $(\langle S \mid R \rangle, R)$. 9
(i) Find the set \( \{w_i \in \overline{R}, 1 \leq i \leq k \} \) such that \( \lambda(w_i) \) is minimum. Let \( \mathcal{R} = R \cup \{w_1\} \) and consider a class of graphs \( \mathcal{C} \) which is empty.

(ii) For \( \mathcal{R} \), (a) find \( m^{(c)}_{ij} \) for \( 2 \leq c \leq 3 \) and \( 1 \leq i \neq j \leq 3 \), (b) find \( n_1, n_2, n_3 \) and (c) choose \( i, j, l \) such that \( n_i \geq n_j \) and \( n_i \geq n_l \).

(iii) Consider three bi-colored cycles \( G_i = C_{n_i}(x_i^{(1)}, \ldots, x_i^{(m_i)}) \) for \( 1 \leq i \leq 3 \) such that \( x_i^{(2j-1)}x_i^{(2j)} \) has color 1 and \( x_i^{(2j)}x_i^{(2j+1)} \) has color 0 with the consideration \( x_i^{(n_i+1)} = x_i^{(1)} \) for \( 1 \leq j \leq n_i/2 \) and \( 1 \leq i \leq 3 \).

(iv) The edges \( x_i^{(1)}x_i^{(1)}, \ldots, x_i^{(m_i)}x_i^{(m_i)} \) and \( x_i^{(n_i)}x_i^{(1)}, \ldots, x_i^{(n_i+m_i)}x_i^{(m_i)} \) have color 2. If \( n_j + n_l \neq n_i \) then the edges \( x_j^{(n_j)}x_i^{(n_i)}, \ldots, x_j^{(n_j+m_j)}x_i^{(m_j)} \) have also color 2.

(v) For \( 1 \leq q_i \leq n_i/2 \), \( 1 \leq q_j \leq n_j/2 \) and \( 1 \leq q_l \leq n_l/2 \), choose a bi-colored path of colors 1 and 3 from the two paths \( P_i(x_i^{(2q_i-1)}, x_i^{(2q_i)}, x_i^{(2q_i-1)}, x_i^{(2q_i)}, x_i^{(2q_i-1)}, x_i^{(2q_i)}) \) and \( P_l(x_l^{(2q_l)}, x_l^{(2q_l-1)}, x_l^{(2q_l)}, x_l^{(2q_l-1)}, x_l^{(2q_l)}) \) and complete the 4-colored graph as there is a unique choice for the 4-colored graph with this path. There are \( (n_1n_2n_3)/4 \) choices for 4-colored graphs. If some graphs yield \( (S \mid R), R \), put them in the class \( \mathcal{C} \).

(vi) If \( \mathcal{R} = R \cup \{w_i\} \), for some \( i \in \{1, \ldots, k-1\} \), choose \( \mathcal{R} = R \cup \{w_{i+1}\} \) and go to step (ii). If \( \mathcal{R} = R \cup \{w_k\} \) then \( \mathcal{C} \) is the collection of all crystallizations which yield \( (S \mid R), R \) and are minimal with respect to \( (S \mid R), R \). If \( \mathcal{C} \) is empty, there does not exist such a crystallization.

### 4 Applications of Algorithm 1

From a given presentation \( \langle S \mid R \rangle \) with two generators and two relations, we will find all the possible \( \lambda(\langle S \mid R \rangle), R \)-vertex crystallizations which yield \( (S \mid R), R \). We will also discuss the case, where no such a crystallization exists.

#### 4.1 Existence of crystallizations

In this subsection, we present some applications of Algorithm 1 when the algorithm determines the existence of crystallizations for a pair \( (S \mid R), R \).

**Example 4.1 (Crystallizations of \( M(m, n, 2) \) for \( m, n \geq 2 \)).** Recall that binary polyhedral group \( (m, n, k) \) has a presentation \( \langle x_1, x_2, x_3 \mid x_1^m = x_2^m = x_3^m = x_1x_2x_3 \rangle \) for some integer \( m, n, k \geq 2 \). If \( k = 2 \) then \( x_3 = x_1x_2 \) and hence \( x_1^m = x_1x_2x_1x_2 \) and \( x_2^m = x_1x_2x_1x_2 \). Therefore, \( (m, n, 2) \) has a presentation \( \langle S \mid R \rangle \), where \( S = \{x_1, x_2\} \) and \( R = \{x_1^{-1}x_1^{-1}, x_1^{-1}x_2^{-1}, x_2^{-1}x_1^{-1}, x_2^{-1}x_2^{-1}\} \). Clearly, \( x_1^mx_2^n \) is the only element in \( \overline{R} \) of minimum weight.

Therefore, \( \mathcal{R} = R \cup \{x_1^mx_2^n\} \). Thus, \( m^{(c)}_{12} = 3, m^{(c)}_{13} = 2m-1, m^{(c)}_{23} = 2n-1 \) for \( 2 \leq c \leq 3 \), \( (n_1, n_2, n_3) = (2m+2, 2n+2, 2m+2n-2) \) and \( G_i = C_{n_i}(x_i^{(1)}, \ldots, x_i^{(n_i)}) \) for \( 1 \leq i \leq 3 \) as in Figure 1. Choose \( (n_1, n_2, n_3) = (n_3, n_1, n_2) \). Thus, \( \{x_1^{(1)}, x_2^{(1)}, \ldots, x_1^{(2n-1)}, x_2^{(2n-1)}\} , \{x_3^{(2m+2)}, x_3^{(2m+3)}, x_3^{(2m+4)}, x_3^{(2m+5)}\} \) and \( \{x_2^{(2m+2)}, x_2^{(2m+3)}, x_2^{(2m+4)}\} \) give a regular bipartite 4-colored graph \( (\Gamma, \gamma) \) which yields \( (S \mid R), R \) (cf. Lemma 1.3). Therefore, by Theorem 3.1 \( (\Gamma, \gamma) \) (cf. Figure 1) is a crystallization of the closed connected orientable 3-manifold \( M(m, n, 2) \) (cf. Subsection 2.4) and minimal with respect to \( (S \mid R), R \).
Remark 4.2. Observe that in Example 4.1, (i) \( \#V(\Gamma) = 4m + 4n + 2 = (2m + 2) + (2n + 2) + (2m + 2n - 2) = \lambda((S \mid R), R) \), (ii) \( \Gamma_{i,j} = (m + n - 2)C_4 \sqcup C_6 \) for \( 0 \leq i \leq 1 \) and \( 2 \leq j \leq 3 \) as \( \#V(\Gamma) = 4(m + n - 1) + 2 \) and (iii) the \( m_{ij}^{(c)} \) edges of color \( c \) between \( G_i \) and \( G_j \) yield \( m_{ij}^{(c)} - 1 \) number of bi-colored 4-cycles for \( 1 \leq i \neq j \leq 3 \) and \( 2 \leq c \leq 3 \).

Lemma 4.3. Let the presentation \((\langle S \mid R \rangle, R)\) and \( q_1, q_2, q_3, P_5 \) be as in Example 4.1. Then, the choices of the triplet \( q_1, q_2, q_3 \) and the path \( P_5 \) are unique, for which the 4-colored graph \((\Gamma, \gamma)\) yields \((\langle S \mid R \rangle, R)\).

Proof. Since \( x_1^{m-1}x_2^{-1} x_1^{-1}x_2^{-1} \) (resp., \( x_1^{m}x_2^{-n} \)) is a relation which contains \( x_1^{m-1} \) (resp., \( x_1^{m} \)), \( m - 2 \) (resp., \( m - 1 \)) edges of color 3 between \( G_1 \) and \( G_3 \) are involved to yield \( x_1^{m-1} \) (resp., \( x_1^{m} \)). Since \( \Gamma \) is bipartite, \( m_{13}^{(3)} = 2m - 1 \) and \( G_1 \) has \( 2m + 2 \) vertices, either white dots ‘\( \circ \)’ vertices (resp., black dots ‘\( \bullet \)’ vertices) or black dots ‘\( \cdot \)’ vertices (resp., white dots ‘\( \circ \)’ vertices) in \( G_1 \) are involved to yield \( x_1^{m-1} \) (resp., \( x_1^{m} \)). Therefore, exactly one of the vertices \( x_2^{(2n+2)}, x_2^{(2n)} \) is the starting vertex for one of the two relations above as \( m_{12}^{(3)} = 3 \) and \( \{ x_2^{(2n+2)}, x_2^{(2n)} \} \) is the set of white dots ‘\( \circ \)’ vertices. Thus, the black dot ‘\( \bullet \)’ vertex \( x_2^{(2n+1)} \) is the starting vertex for the other relation. Up to an automorphism, we can assume \( x_2^{(2n)} \) and \( x_2^{(2n+1)} \) are the starting vertices for the two relations above. Therefore, \( x_1^{(2n)} \) and \( x_1^{(2n+1)} \) are joined with vertices of \( G_3 \) by edges of color 3. Since the relation with starting vertex \( x_2^{(2n+1)} \) contains \( x_1^{m-1} \) or \( x_1^{m} \), either \( \{ x_1^{(2m+1)} x_2^{(2n+1)} x_3^{(2n+1)}, x_3^{(2n)}, x_3^{(2n+1)}, x_3^{(2n+2)}, \ldots, x_3^{(2m+2n+6)} \} \) or \( \{ x_1^{(2m+1)} x_2^{(2m+2n+2)}, x_3^{(2m+2n+4)}, x_3^{(2m+2n+6)}, \ldots, x_3^{(2m+7)} x_3^{(2n+6)} \} \) is the set of edges of color 3. But, in the later case, \( x_1^{(2m)} x_3^{(1)} \in \gamma^{-1}(3) \) as \( x_1^{(2m)} \) and \( x_1^{(2m+1)} \) are joined with vertices of \( G_3 \) by edges of color 3. Then, the relation with starting vertex \( x_1^{(2n)} \) contains \( x_1 x_2 w x_2^{-1} \) for some \( w \in F(S) \), which is a contradiction. Therefore, \( x_1^{(2m+1)} x_3^{(1)}, x_1^{(2m-1)} x_3^{(2n+2)}, x_3^{(2n+4)}, x_3^{(2n+6)}, \ldots, x_3^{(2m+7)} x_3^{(2n+6)} \in \gamma^{-1}(3) \) and hence \( x_1^{(2m)} x_3^{(1)}, x_1^{(2m-2)} x_3^{(2n+3)}, \ldots, x_1^{(2m+2n-5)} x_3^{(2n+6)} \in \gamma^{-1}(3) \) since one relation
contains \( x_1^m \) and other contains \( x_1^{m-1}, x_1^2, x_1^3, \ldots, x_1^{2m-2} \) \( \in \gamma^{-1}(3) \) and hence \( x_1^m, x_1^3, x_1^2 \) are joined with \( G_2 \) with edges of color 3. Thus, \( x_1^{(2m+2)} \) \( x_3^{(2m+1)}, x_3^{(2m-1)}, x_3^{(2m-2)} \) \( \in \gamma^{-1}(3) \) as \( m_{13}^{(3)} = 2m - 1 \). Therefore, \( x_2^{(2m+1)} \) is the starting vertex for the relation \( x_1^m x_2^{-m} \). In the other word, we can say \( x_2^{(2m+1)} \) is the starting vertex for the relation \( x_1^m x_2^{-m} \). Therefore, \( x_1^{(2m+2)} \) is the starting vertex for the relation \( x_2^{(2m+1)} x_1^{-m} x_1^{-1} x_2^{-1} \). Thus, by the similar arguments as above, \( x_2^{(2m+1)} x_3^{(2m+2)}, x_2^{(2m+2)} x_3^{(2m+1)}, x_2^{(2m+1)} x_3^{(2m+3)}, \ldots, x_2^{(2m+2)} x_3^{(2m+1)} \) \( \in \gamma^{-1}(3) \). Therefore, \( x_3^{(2m+1)} x_2^{(2m-1)} x_2^{(2m-2)} x_1^{(2m-3)}, x_3^{(4)} x_2^{(2m-2)} x_1^{(2m-3)}, x_3^{(4)} x_2^{(2m-2)} x_1^{(2m-3)} \) \( \in \gamma^{-1}(3) \). Thus, the lemma follows. 

**Example 4.4 (Crystallization of \( L(kq - 1, q) \), for \( k, q \geq 2 \)).** We know \( \mathbb{Z}_{kq-1} \) has a presentation \( \langle S \mid R \rangle \), where \( S = \lbrace x_1, x_2 \rbrace \) and \( R = \lbrace x_1^q x_2^{-1}, x_2^q x_1^{-1} \rbrace \). Clearly, \( x_1^q x_2^{-1} \) is the only element in \( R \) of minimum weight.

![Figure 2: Crystallization of \( L(kq - 1, q) \) for \( k, q \geq 2 \)](image)

Therefore, \( R = R \cup \lbrace x_1^{q-1} x_2^{k-1} \rbrace \). Thus, \( m_{12}^{(c)} = 2, m_{13}^{(c)} = 2q - 2, m_{23}^{(c)} = 2k - 2 \) for \( 2 \leq c \leq 3 \). \( (n_1, n_2, n_3) = (2q, 2k, 2q + 2k - 4) \) and \( G_i = C_{n_i}(x_i^{(1)}, \ldots, x_i^{(n_i)}) \) for \( 1 \leq i \leq 3 \) as in Figure 2. Choose \( (n_1, n_2, n_3) = (n_3, n_1, n_2) \). Thus, \( \lbrace x_2^{(1)} x_3^{(1)}, \ldots, x_3^{(2k-2)} x_2^{(2k-2)} \rbrace \), \( \lbrace x_3^{(2q+2k-4)} x_1^{(1)}, \ldots, x_3^{(2k-4)} x_1^{(1)} \} \) and \( \lbrace x_1^{(2q)} x_2^{(2q-1)} x_3^{(2q)}, x_2^{(2k-1)} x_1^{(2q)} x_2^{(2k)} \} \) are the sets of edges of color 2. Here the only choices for the triplet \( (q_1, q_2, q_3) = (q - 1, k, k - 1) \) and the path \( P_3(x_1^{(2q-2)}, x_2^{(2q-3)}, x_3^{(2q-2)}, x_3^{(2k-3)}, x_2^{(2k-2)}, x_1^{(2k)}) \) give a regular bipartite 4-colored graph \( (\Gamma, \gamma) \) which yields \( \langle (S \mid R), R \rangle \). Therefore, by Theorem 3.1, \( (\Gamma, \gamma) \) (cf. Figure 3) is a crystallization of a closed connected orientable 3-manifold and minimal with respect to \( \langle (S \mid R), R \rangle \). Here \( (\Gamma, \gamma) \) is a crystallization of \( L(kq - 1, q) \) as \( (\Gamma, \gamma) \) is isomorphic to the graph \( M_{k,q} \) (cf. [Basak and Datta 14 Subsection 5.1]) and \( M_{k,q} \) is a crystallization of \( L(kq - 1, q) \).

**Remark 4.5.** Observe that in Example 4.3 \( (i) \# V(\Gamma) = 4(q + k - 1) = 2q + 2k + 2(q + k - 2) = \lambda((S \mid R), R) \) and \( (ii) \Gamma_{(0,3)} \) (resp., \( \Gamma_{(1,3)} \)) is union of \( (q + k - 1) \) 4-cycles and \( \Gamma_{(0,2)} \) (resp., \( \Gamma_{(1,3)} \)) is of type \( (q + k - 4)C_4 \cup 2C_6 \) as \( \# V(\Gamma) = 4(q + k - 1) \) and \( m_{ij}^{(c)} \geq 1 \) for \( 2 \leq c \leq 3 \) and \( 1 \leq i \neq j \leq 3 \).
Lemma 4.6. Let the presentation \( \langle S \mid R \rangle \) and \( q_1, q_2, q_3, P_5 \) be as in Example 4.4. Then, the choices of the triplet \((q_1, q_2, q_3)\) and the path \(P_5\) are unique, for which the 4-colored graph \((\Gamma, \gamma)\) yields \(\langle S \mid R \rangle\).

Proof. Clearly, either \(x_2^{(2k)}\) or \(x_2^{(2k-1)}\) is the starting vertex for the relation \(x_1^q x_2^{-1}\). Up to an automorphism, we can assume \(x_2^{(2k)}\) is the starting vertices for the relation. Therefore, all the black dots '•' vertices in \(G_1\) are involved to yield the relation as \(\Gamma\) is bipartite. Thus, either \(\{x_1^{(2q)} x_3^{(2k-1)}, x_1^{(2q-2)} x_3^{(2k-3)}, \ldots, x_1^{(4)} x_3^{(2q+2k-5)}\}\) or \(\{x_1^{(2q)} x_3^{(2q+2k-5)}, x_1^{(2q-2)} x_3^{(2q+2k-7)}, \ldots, x_1^{(2q-4)} x_3^{(2k-1)}\}\) is the set of edges of color 3. Since the \(2q-2\) edges of color 3 between \(G_1\) and \(G_3\) form 2q-3 bi-colored 4-cycles in \(\Gamma_{(0,1,3)}\), in the first case, \(x_1^{(2q-1)} x_3^{(2k)}, x_1^{(2q-3)} x_3^{(2k-2)}, \ldots, x_1^{(5)} x_3^{(2q+2k-6)} \in \gamma^{-1}(3)\). This gives a relation \(x_1^{q-2} wx_2^{-1}\) other than \(x_1^q x_2^{-1}\) for some \(w \in F(S)\). But, there does not exist such a relation. So, \(x_1^{(2q)} x_3^{(2q+2k-5)} x_1^{(2q-2)} x_3^{(2q+2k-7)}, \ldots, x_1^{(2q-4)} x_3^{(2k-1)} \in \gamma^{-1}(3)\) and hence \(x_1^{(2q+2k-6)} x_3^{(2q+2k-8)}, x_1^{(2q+2k-8)} x_3^{(2q)} x_1^{(2q+2k-10)}, \ldots, x_1^{(2q-2)} x_3^{(2q)} \in \gamma^{-1}(3)\). Since \(x_2^{(2k)}\) is the starting vertex for the relation \(x_1^q x_2^{-1}\), we have \(x_1^{(2q-2)} x_2^{(2k)} \in \gamma^{-1}(3)\) and hence \(x_1^{(2q-1)} x_2^{(1)} x_1^{(2q-3)} x_3^{(2k-2)} \in \gamma^{-1}(3)\) as all components of \(\Gamma_{(0,3)}\) are 4-cycle.

By similar arguments as above, \(x_2^{(2k-1)} x_3^{(2k-3)}, x_2^{(2k-3)} x_3^{(2k-4)}, \ldots, x_2^{(2)} x_3^{(2q+2k-4)} \in \gamma^{-1}(3)\) as \(x_1^{(2q-1)}\) is the starting vertex for the relation \(x_1^q x_2^{-1}\). Thus, the lemma follows. 

Example 4.7 (Crystalization of \(L((k-1)q + 1, q)\) for \(k, q \geq 2\)). We know \(\mathbb{Z}_{(k-1)q+1}\) has a presentation \(\langle S \mid R \rangle\), where \(S = \{x_1, x_2\}\) and \(R = \{x_1^q x_2^{-1}, x_1^{q-1} x_2^{-k}\}\).

![Figure 3: Crystallization of \(L((k-1)q + 1, q)\) for \(k, q \geq 2\)](image)

Clearly, \(x_1 x_2^k\) is the only element in \(\overline{R}\) of minimum weight. Therefore, \(\overline{R} = R \cup \{x_1 x_2^k\}\). Thus, \(m_{(c)} = 2, m_{(0)} = 2q-2, m_{(c)} = 2k-2\) for \(2 \leq c \leq 3\), \((n_1, n_2, n_3) = (2q, 2k, 2q+2k-4)\).

Choose \((n_1, n_2, n_3) = (n_1, n_1, n_2)\). Therefore, the 3-colored graph with colors 0, 1, 2 as in previous example. Here, the only choices for the triplet \((q_1, q_2, q_3) = (q-1, k, 1)\) and the path \(P_5\) give a regular bipartite 4-colored graph \((\Gamma, \gamma)\) which yields \(\langle (S \mid R) \rangle\) (cf. Lemma 4.8). Therefore, by Theorem 3.1 \((\Gamma, \gamma)\) (cf. 

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Figure 3 is a crystallization of a closed connected orientable 3-manifold and minimal with respect to $(\langle S\mid R\rangle, R)$. Here $(\Gamma, \gamma)$ is a $4(k+q-1)$-vertex crystallization of $L((k-1)q-1,q)$ as $(\Gamma, \gamma)$ is isomorphic to the graph $\mathcal{N}_{k-1,q}$ (cf. Basak and Datta [14] Subsection 5.2) and $\mathcal{N}_{k-1,q}$ is a crystallization of $L((k-1)q-1,q)$.

Lemma 4.8. Let the presentation $(\langle S\mid R\rangle, R)$ and $q_1, q_2, q_3, P_3$ be as in Example 4.7. Then, the choices of the triplet $(q_1, q_2, q_3)$ and the path $P_3$ are unique, for which the 4-colored graph $(\Gamma, \gamma)$ yields $(\langle S\mid R\rangle, R)$.

Proof. Without loss of generality, we choose $x_2^{(2k-1)}$ as the starting vertices for the relation $x_1^2x_2^{-1}$. Thus, either $\{x_1^{(2q-1)}x_3^3, x_1^{(2q-3)}x_3^{(2k+2)}, \ldots, x_1^{(3)}x_3^{(2q+2k-4)}\}$ or $\{x_1^{(2q-1)}x_3^{(2q+2k-4)}, x_1^{(1)}x_3^{(2q+2k-6)}, \ldots, x_1^{(2q-5)}x_3^{(26)}\}$ is the set of edges of color 3. By the similar arguments as in above example, we get a relation $x_3^2x_1^{-1}$ in the first case. Therefore, we have to choose the second case. Again, by similar arguments as above examples, the lemma follows.

Example 4.9 (Crystallization of a hyperbolic 3-manifold). Let $(\langle S\mid R\rangle)$ be a presentation of a group, where $S = \{x_1, x_2\}$ and $R = \{x_1^4x_2x_1x_2^{-1}x_4x_2^{-1}x_1x_3x_1x_2^{-1}, x_1^3x_2x_1x_2^{-1}x_1x_2x_1x_2^{-1}x_1x_2^{-1}\}$. Clearly, $x_2^3x_2^3x_1^{-1}x_2x_2^{-1} \in R$ is only element of minimum weight.

Therefore, $\mathcal{R} = R \cup \{x_2^3x_2^3x_1^{-1}x_2x_2^{-1}\}$. Thus, $m_{12}^{(c)} = 11$, $m_{13}^{(c)} = 17$, $m_{13}^{(c)} = 7$ for $c \leq 3$, $(n_1, n_2, n_3) = (28, 18, 24)$ and $G_1 = C_{n_1}(x_1^{(1)}, \ldots, x_1^{(n_1)})$ for (1) $i \leq 3$ as in Figure 4. Choose $(n_1, n_2, n_3) = (n_1, n_2, n_3)$. Thus, $\{x_1^{(1)}x_1^{(1)}, \ldots, x_1^{(1)}x_2^{(1)}\}$, $\{x_1^{(28)}x_3^{(1)}, \ldots, x_1^{(12)}x_3^{(17)}\}$ and $\{x_1^{(24)}x_2^{(1)}, \ldots, x_1^{(32)}x_2^{(1)}\}$ are the sets of edges of color 2. Now, only choices for the triplet $(q_1, q_2, q_3) = (8, 5, 10)$ and the path $P_3$ gives a bipartite oriantable prime hyperbolic 3-manifold (see the list in http://www.dms.umontreal.ca/~math/Logicels/Magma/text414.htm). Therefore, by Proposition 2.1, $|\mathcal{K}(\Gamma)|$ is homeomorphic to the hyperbolic 3-manifold as in the list. Thus, using the algorithm we get a crystallization of the hyperbolic 3-manifold from a given presentation.

Remark 4.10. Observe that in Example 4.9 (i) $\#V(\Gamma) = 70 = 28 + 26 + 16 = \lambda(\langle S\mid R\rangle, R)$ and (ii) $\Gamma_{(i,j)} = 16C_4 \cup C_6$ for $0 \leq i \leq 1, 2 \leq j \leq 3$ as $\#V(\Gamma) = 70 = 4 \times 17 + 2$.

Lemma 4.11. Let the presentation $(\langle S\mid R\rangle, R)$ and $q_1, q_2, q_3, P_5$ be as in Example 4.9. Then, the choices of the triplet $(q_1, q_2, q_3)$ and the path $P_5$ are unique, for which the 4-colored graph $(\Gamma, \gamma)$ yields $(\langle S\mid R\rangle, R)$.

Proof. From the discussions in the previous lemmas it is clear that, if $x_j^{e_1}x_i^{e_2}x_j^{e_2}$ is a part of a relation for some $e_1, e_2 \in \{1, -1\}$, $m \geq 2$ and $1 \leq i \neq j \leq 2$ then $x_j^{e_1}x_i^{e_2}$, there are exactly $m - 1$ vertices in $G_1$ which are joined to $G_i$ with both edges of colors 2 and 3. There are three words of type $x_3^2x_2^{-1}x_1^{-1}$ and three words of type $x_3^2x_2^{-1}x_1^{-1}$ in the relations in $\mathcal{R}$. Therefore, exactly $3(4 - 1) + 3(3 - 1) = 15$ vertices of $G_3$ are joined to $G_1$ with both edges of colors 2 and 3. By similar arguments, $3(3 - 1) + 3(2 - 1) = 5$ vertices of $G_3$ are joined to $G_2$ with both edges of colors 2 and 3. We can re-write the relations in $\mathcal{R}$ as $x_4^2((x_1x_2^{-1})^2x_4^2x_2^2x_2^{-1}x_1x_2^{-1}x_1), x_4^2x_2^2x_1^2((x_1x_2^{-1})^2x_1^2x_2^{-1}x_1), x_4^2x_2^2x_1^2x_2^{-1}$. Therefore, $3(3 - 1) + 3(2 - 1) = 9$ vertices of $G_2$ are joined to $G_1$ with both edges of colors 2 and 3. Since $m_{13}^{(2)} = m_{13}^{(3)} = 17$ and 15 vertices of $G_3$ are joined to $G_1$
with both edges of colors 2 and 3, \(x_3^{(1+m)}, \ldots, x_3^{(15+m)}\), \(0 \leq m \leq 2\) are the choices of these vertices. If \(m = 1\) then, either \(x_3^{(1)}\) or \(x_3^{(17)}\) is joined to \(G_1\) with edge of color 3 as \(m_{13}^{(3)} = 17\). Since exactly 15 vertices of \(G_3\) are joined to \(G_1\) with both edges of colors 2 and 3, \(m \neq 1\). Up to an automorphism, we can assume \(x_3^{(1)}\) is not such a vertex, i.e., \(m = 2\). Therefore, \(x_3^{(1)}\) and \(x_3^{(2)}\) are joined to \(G_2\) with edges of color 3. Let \(x_3^{(3)} x_1^{(10+n)}, \ldots, x_3^{(19)} x_1^{(10+n)} \in \gamma^{-1}(3)\) for some integer \(n\). If \(-4 \leq n \leq 4\) then there is a relation containing \(x_1\) and if \(n \geq 8\) or \(n \leq -8\) then there is no relation containing \(x_1\) and both are not possible. Therefore, \(n = \pm 6\). If \(m = 2\) and \(n = -6\) then \(x_3^{(3)} x_1^{(20)}, \ldots, x_3^{(19)} x_1^{(4)} \in \gamma^{-1}(3)\). This contradicts that 9 vertices of \(G_2\) are joined to \(G_1\) with both edges of colors 2 and 3. Therefore, \(m = 2\) and \(n = 6\) and hence \(x_3^{(3)} x_1^{(4)}, x_3^{(4)} x_1^{(4)}, \ldots, x_3^{(19)} x_1^{(16)} \in \gamma^{-1}(3)\). Since \(x_3^{(18)}\) and \(x_3^{(19)}\) are already joined to \(G_1\) with edges of color 3, the remaining vertices \(x_3^{(20)}, \ldots, x_3^{(24)}\) are joined to \(G_2\) with both edges of colors 2 and 3. Since the relations with starting vertices \(x_3^{(18)}\) and \(x_3^{(19)}\) yield \(x_2 w\) for some \(w \in F(S), x_2^{(20)} x_2^{(14+k)}, \ldots, x_2^{(24)} x_2^{(28+k)} \in \gamma^{-1}(3)\) for \(k = -4\). So,
Remark 4.12. If \((\Gamma, \gamma)\) is a crystallization of a 3-manifold then the regular genus of \(\Gamma\) is the integer \(\rho(\Gamma) = \min\{g_0, g_2, g_3\} - 1\). The crystallizations constructed in Example 4.1 for \(n = k = 2\) and in Example 4.4 and Example 4.7 are of handle-free manifolds. Thus, by [Casali 99 Proposition 4] and from the catalogue in [Bandieri et al. 13], these crystallizations are vertex-minimal regular genus two crystallizations when number of vertices of the crystallizations are at most 42. The crystallizations constructed in Example 4.4 and Example 4.7 are vertex-minimal for all known cases, in fact, the crystallizations of \(L((k-1)q + 1, q)\) are vertex-minimal when \((k-1)q + 1\) are even (cf. [Basak and Datta 14], [Casali and Cristofori 14] and [Swartz 13]).

4.2 Non existence of crystallizations

Here, we present some applications of Algorithm 1 when the algorithm determines the non existence of crystallizations for a pair \((\langle S \mid R \rangle, R)\).

Example 4.13 (For a presentation of \(\mathbb{Z}_6\)). Let \(\langle S \mid R \rangle\) be a presentation of the cyclic group \(\mathbb{Z}_6\), where \(S = \{x_1, x_2\}\) and \(R = \{x_1^2x_2^{-1}, x_1^3x_2\}\). Clearly, \(x_2^3\) is the only element in \(R\) of minimum weight. Therefore, \(R = R \cup \{x_2^3\}\) and let \((\Gamma, \gamma)\) be such a crystallization. Thus, \(m_{11} = 1, m_{13} = 5, m_{23} = 3\) for \(2 \leq c \leq 3\) and \((n_1, n_2, n_3) = (6, 4, 8)\). Therefore, by choosing \((n_i, n_j, n_l) = (n_3, n_1, n_2)\), we have \(\Gamma_{\{0,1,2\}}\) as in Figure 5.

![Figure 5: The graph \(\Gamma_{\{0,1,2\}}\)](image)

If \(x_3^{(2q_1-1)}x_3^{(2q_2)} = x_3^{(7)}x_3^{(8)}, x_3^{(1)}x_3^{(2)}\) or \(x_3^{(5)}x_3^{(6)}\) then, for each of two choices of the path \(P_5\), either \(x_3^{(1)}\) or \(x_3^{(3)}\) is joined with \(G_1\) with edges of color 3. Since the graph is bipartite \(x_2^{(2)}x_3^{(1)}\) or \(x_2^{(2)}x_3^{(3)}\) can not be edge in \(\Gamma\). Therefore, no components of \(\Gamma_{\{2,3\}}\) yield the relation \(x_2^{2}\). For the same reason, we can not choose \(x_3^{(2q_1-1)}x_3^{(2q_2)} = x_3^{(3)}x_3^{(4)}\) and the path \(P_5(x_1^{(2q_1-1)}, x_1^{(2q_2)}, x_3^{(3)}, x_3^{(4)}x_2^{(2q_2)}, x_2^{(2q_2-1)})\). Therefore, the only choice remain is \(x_3^{(2q_1-1)}x_3^{(2q_2)} = x_3^{(3)}x_3^{(4)}\) and the path \(P_5(x_1^{(2q_1)}, x_1^{(2q_1-1)}, x_1^{(3)}x_3^{(4)}x_3^{(3)}, x_3^{(2q_2-1)}x_2^{(2q_2)}\). Thus, \(x_3^{(2)}\) is joined with \(G_2\) and \(x_2^{(4)}x_3^{(2)} \in \gamma^{-1}(3)\) (since \(\Gamma\) is bipartite and has no double edge). Then, \(P_3(x_1^{(6)}, x_1^{(4)}x_3^{(2)}, x_2^{(2)}x_2^{(2)})\) is a part of a component of \(\Gamma_{\{2,3\}}\), which yields the word \(x_3^{(1)}x_3^{(2)}\). Since \(x_3^{(1)}x_3^{(2)}\) is not a part of relations in \(R\), this choice is also not possible. Thus, there is no crystallization of a closed connected 3-manifold which yields \((\langle S \mid R \rangle, R)\) and is minimal with respect to \((\langle S \mid R \rangle, R)\).
Example 4.14 (For a presentation of $\mathbb{Z}_{m+n+1}$ for $m, n \geq 1$). Let $(S \mid R)$ be the presentation of $\mathbb{Z}_{m+n+1}$, where $S = \{x_1, x_2\}$ and $R = \{x_1^{m+1}x_2^{-1}, x_1x_2^{n}\}$ and $m, n \geq 1$. Clearly, $\{x_1^{m_1}x_2^{n_1} : m_1 + m_2 = m, m_1, m_2 \geq 1\}$ is the set of all elements in $R$ of minimum weight. So, $R = R \cup \{x_1^{m_1}x_2^{-1}, x_1x_2^{n}\}$ and let $(\Gamma, \gamma)$ be such a crystallization. Thus, $m_1 = 3, m_2 = 2m - 1$ and $m_3 = 2n - 1$ for $2 \leq c \leq 3$ and $(n_1, n_2, n_3) = (2m + 2, 2n + 2, 2m + 2n - 2)$. By choosing $(n_1, n_2, n_3) = (n_3, n_1, n_2)$, we have $\Gamma_{\{0,1,2\}}$ as in Figure 1. From the discussions in the proof of Lemma 4.11, it is clear that, since there is a minimum weight. So, $(S \mid R)$, $R$ is isomorphic to $\gamma$. In this section, we generalize Algorithm 1 for a presentation $(S \mid R)$ of a 3-manifold which yields $(\langle S \mid R \rangle, R)$ and is minimal with respect to $(\langle S \mid R \rangle, R)$.

5 Generalization of Algorithm 1

In Section 3, we have computed crystallizations of 3-manifolds from a given presentation $(S \mid R)$ with two generators and two relations. For such presentation, $\Gamma_{\{0,1,2\}}$ and $\Gamma_{\{0,1,3\}}$ were unique up to an isomorphism. But, if the given presentation $(S \mid R)$ has no more than two relations, then $\Gamma_{\{0,1,2\}}$ and $\Gamma_{\{0,1,3\}}$ may have many choices. But, there are some classes of presentations, for which $\Gamma_{\{0,1,2\}}$ and $\Gamma_{\{0,1,3\}}$ are unique up to an isomorphism. In this section, we generalize Algorithm 1 for a presentation $(S \mid R)$ with three generators and a certain class of relations. Let $r \in R$ be an element of minimum weight and $R' = R \cup \{r\}$. Let $m_{ij}^{(c)} := \sum_{w \in R'} w_{ij}^{(c)}$ for $1 \leq i \neq j \leq 4$ and $2 \leq c \leq 3$, where $w_{ij}^{(c)}$ as in Subsection 2.3. Let $C_R := \{w_1, \ldots, w_k\}$ be the set of all elements such that (i) weight of $w \in R$ is minimum and (ii) for each $R \cup \{w\}$, we have $m_{ij}^{(c)} = \sum_{w \in R \cup \{w\}} w_{ij}^{(c)} \geq 1$, where $1 \leq i \neq j \leq 4$ and $2 \leq c \leq 3$. Let $(\Gamma, \gamma)$ be a crystallization of a 3-manifold which yields $\gamma = R \cup \{w\}$, where $w \in C_R$ and minimal with respect to $(\langle S \mid R \rangle)$. Without loss of generality, let $\Gamma_{\{0,1\}} = \bigcup_{i=1}^{k} G_i$ such that $G_i$ represents the generator $x_i$ for $1 \leq i \leq 3$ and $G_4$ represents $x_4$ (cf. Eq. 2.24 for construction of $\tilde{r}$). Let $n_i$ be the total number of appearance of $x_i$ in the four relations in $R$ for $1 \leq i \leq 3$ and $n_4 = \lambda(\langle S \mid R \rangle, R) - (n_1 + n_2 + n_3)$. Then, the total number of vertices in $G_i$ should be $n_i$. Assume $G_i = C_{n_i} (x_i^{(1)}, \ldots, x_i^{(n_i)})$ for $1 \leq i \leq 4$. Clearly, each $n_i$ is even and $n_1 + n_2 + n_3 + n_4 = \#V(\Gamma)$. Without loss of generality, we can assume that $x_i^{(j-1)}x_i^{(j)} \in \gamma^{-1}(1)$ and $x_i^{(j)}x_i^{(j+1)} \in \gamma^{-1}(0)$ with $x_i^{(n_i+1)} = x_i^{(1)}$ for $1 \leq j \leq n_i/2$ and $1 \leq i \leq 4$. Here and after, the additions and subtractions at the point ‘$\ast$’ in $x_i^{(\ast)}$ are modulo $n_i$ for $1 \leq i \leq 4$. Let the colors 2 and 3 be the colors ‘$i$’ and ‘$j$’ as in construction of $\tilde{r}$ for $r \in R$ (cf. Eq. 2.23). Then, the number of edges of color $c$ between $G_i$ and $G_j$ is $m_{ij}^{(c)}$ for $2 \leq c \leq 3$ and $1 \leq i \neq j \leq 4$. Therefore, $m := \sum_{1 \leq i \neq j \leq 4} m_{ij}^{(c)} = \#V(\Gamma)/2$. Now, the maximum number of bi-colored 4-cycles in $\Gamma_{\{0,1,3\}}$ with two edges of color $c$ is $\sum_{1 \leq i \neq j \leq 4} (m_{ij}^{(c)} - 1) = m - 6$. Again, by Proposition 2.5, $4g_c + 9_{1c} = \#V(\Gamma)/2 + 2$, i.e.,

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$g_0 + g_1 = m - 2$. Since $G_i$ and $G_j$ are connected by an edge of color $c$ for $1 \leq i \neq j \leq 4$ and $2 \leq c \leq 3$, we have at least four distinct bi-colored paths $P_5$ with some edges of color $c$ in $\Gamma_{0,1,c}$ which touch $G_i, G_j; G_l$ for all distinct $i, j, l \in \{1, 2, 3, 4\}$. Therefore, we must have $m = 6$ bi-colored 4-cycles and four bi-colored 6-cycles with some edges of color $c$ in $\Gamma_{0,1,c}$. Thus, $m_{ij}^{(c)}$ edges of color $c$ between $G_i$ and $G_j$ yield $m_{ij}^{(c)} - 1$ number of bi-colored 4-cycles in $\Gamma_{0,1,c}$ for $1 \leq i \neq j \leq 4$ and $2 \leq c \leq 3$. Therefore, two bi-colored 6-cycles with some edges of color $c$ in $\Gamma_{0,1,c}$ give unique choices for the remaining edges of color $c$. Without loss, we can assume $\Gamma_{0,2}$ has a 6-cycle $C_6(x_1^{(1)}, x_1^{(n_1)}, x_4^{(n_1)}, x_4^{(1)}, x_2^{(n_2)}, x_2^{(1)})$. Then, join $x_1^{(1)} x_2^{(1)} \ldots, x_1^{(m_1^{(2)})} x_2^{(m_1^{(2)})}$ by edges of color 2. Without loss of generality, choose $x_3^{(p)} \in G_3$ such that $C_6(x_1^{(m_1^{(2)})}, x_1^{(m_1^{(2)} + 1)}, x_3^{(p)}, x_3^{(p + 1)}, x_2^{(m_1^{(2)} + 1)}, x_2^{(m_1^{(2)})})$ is a bi-colored cycle with three edges of color 2. Therefore, we have a unique choice for $\Gamma_{0,2}$ up to an isomorphism. The choices of two 6-cycles with three edges of color 3 in $\Gamma_{0,1,3}$ give all possible 4-colored graphs. If some graphs yield $((S | R), R)$ then these satisfy all the properties of Proposition 2.2 and hence crystallizations of some 3-manifolds. By similar arguments as in the proof of Theorem 3.1 if $M$ is a closed connected prime manifold with fundamental group $((S | R), R)$ and $(\Gamma, \gamma)$ is a crystallization, constructed from the pair $((S | R), R)$ then $(\Gamma, \gamma)$ is a crystallization of $M$.

5.1 Algorithm 2

We now present an algorithm for a presentation $\langle S | R \rangle$ with $\#S = \#R = 3$ and $C_R \neq \phi$. This algorithm gives all crystallizations which yield the relation set $R \cup \{w\}$, where $w \in C_R$ and are minimum with respect to $(S | R)$.

(i) Find the set $\{w_i \in \overline{R}, 1 \leq i \leq k\}$ such that $\lambda(w_1)$ is minimum and for each $R \cup \{w_i\}$, we have $m_{ij}^{(c)} = \sum_{w \in \overline{R} \cup \{w_i\}} w_{ij}^{(c)} \geq 1$, where $1 \leq i \neq j \leq 4$ and $2 \leq c \leq 3$. Let $R = R \cup \{w_1\}$ and consider a class of graphs $C$ which is empty.

(ii) For $R$, (a) find $m_{ij}^{(c)}$ for $2 \leq c \leq 3$ and $1 \leq i \neq j \leq 4$, (b) find $n_1, n_2, n_3, n_4$.

(iii) Consider four bi-colored cycles $G_i = C_{n_i}((x_i^{(1)}), \ldots, x_i^{(n_i)})$ for $1 \leq i \leq 4$ such that $x_i^{(2j - 1)} x_i^{(2j)}$ has color 1 and $x_i^{(2j)} x_i^{(2j + 1)}$ has color 0 with the consideration $x_i^{(n_i + 1)} = x_i^{(1)}$ for $1 \leq j \leq n_i/2$ and $1 \leq i \leq 4$.

(iv) The sets $\{x_1^{(1)} x_2^{(1)} \ldots, x_1^{(m_1^{(2)})}, x_2^{(m_1^{(2)})}, x_3^{(n_1 + 1)} x_4^{(n_1 + 1 + m_1^{(2)})}, x_4^{(n_1 + 1 + m_1^{(2)})}\}$ and $\{x_2^{(n_2)} x_3^{(n_1 + 1)} x_3^{(n_1 + 1 + m_1^{(2)})}, x_3^{(n_1 + 1 + m_1^{(2)})}\}$ contain edges of color 2. Without loss of generality, choose $x_3^{(p)} \in G_3$ such that $C_6(x_1^{(m_1^{(2)})}, x_1^{(m_1^{(2)} + 1)}, x_3^{(p)}, x_3^{(p + 1)}, x_2^{(m_1^{(2)} + 1)}, x_2^{(m_1^{(2)})})$ is a bi-colored cycle with three edges of color 2. So, the edges of the sets $\{x_2^{(m_1^{(2)} + 1)}, x_3^{(p)}, \ldots, x_2^{(m_1^{(2)} + m_2^{(2)})}, x_3^{(p + m_2^{(2)})}, x_4^{(m_2^{(2)})}, \ldots, x_3^{(p + m_2^{(2)} + m_3^{(2)}), x_3^{(m_2^{(2)} + m_3^{(2)})}}, x_1^{(m_2^{(2)} + 1)}, x_3^{(p)}, \ldots, x_1^{(m_2^{(2)} + m_3^{(2)})}\}$ have also color 2.

(v) For each $1 \leq q_1 \leq n_1$, choose $q_2 \in G_2$ such that the set of color 3 edges is $\{x_1^{(q_1)}, x_2^{(q_2)}, \ldots, x_1^{(q_1 - 1 + m_1^{(3)})}, x_2^{(q_2 - 1 + m_1^{(3)})}\}$. Then, choose $q_3 \in G_3$ and $q_4 \in G_4$ such that, either $\{x_1^{(q_1 - 1)}, x_3^{(q_3)}, x_1^{(q_1 + m_1^{(3)})}, x_4^{(q_4)}\}$ or $\{x_1^{(q_1 - 1)}, x_4^{(q_4)}, x_1^{(q_1 + m_1^{(3)})}, x_3^{(q_3)}\}$ contains edges of color 3. There are $n_1 n_2/2 (n_3/2) (n_4/2) = (n_1 n_2 n_3 n_4)/4$ choices for choosing those vertices and edges. Then, for each case, complete the 4-colored graph as there is a unique choice for the 4-colored graph with those $m_1^{(3)} + 2$ edges. If some graphs yield $((S | R), R)$, put them in the class $C$. 

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(vi) If \( \mathcal{R} = R \cup \{ w_i \} \), for some \( i \in \{ 1, \ldots, k-1 \} \), choose \( \mathcal{R} = R \cup \{ w_{i+1} \} \) and go to step (ii). If \( \mathcal{R} = R \cup \{ w_k \} \) then \( C \) is the collection of all crystallizations which yield \((\langle S \mid R \rangle, R)\) and are minimal with respect to \((\langle S \mid R \rangle, R)\).

### 5.2 Crystallizations of \( M(m, n, k) \)

Recall that \( M(m, n, k) \) is a closed orientable 3-manifold with fundamental group \( \langle m, n, k \rangle \) which has a presentation \( \langle S \mid R \rangle \), where \( S = \{ x_1, x_2, x_3 \} \) and \( R_{mnk} = \{ x_1^{m-1}x_3^{1-x_2^{1}}, x_2^{n-1}x_1^{1-x_3^{1}}, x_3^{k-2}x_2^{1} \} \). Clearly, \( x_1^{m-2}x_3^{n-2}x_2^{k-2}x_1^{1}x_3^{2}x_2^{1} \) is the only element in \( R_{mnk} \) of minimum weight. Observe that

\[
\lambda(x_1^{m-2}x_3^{n-2}x_2^{k-2}x_1^{1}x_3^{2}x_2^{1}) = \begin{cases} 
2m & \text{if } k = n = 2, m \geq 3 \\
2m + 2n - 6 & \text{if } k = 2, m, n \geq 3 \\
2m + 2n + 2k - 12 & \text{if } m, n, k \geq 3.
\end{cases}
\]

Therefore

\[
\lambda(\langle S \mid R_{mnk} \rangle, R_{mnk}) = \begin{cases} 
4(m+2) & \text{if } k = n = 2, m \geq 3 \\
4(m+n) - 2 & \text{if } k = 2, m, n \geq 3 \\
4(m+n-3) & \text{if } m, n, k \geq 3.
\end{cases}
\]

Since for the set \( R_{mnk} \cup \{ x_1^{m-2}x_3^{n-2}x_2^{k-2}x_1^{1}x_3^{2}x_2^{1} \}, m_{ij}^{(c)} \geq 1 \), where \( 1 \leq i \neq j \leq 4 \) and \( 2 \leq c \leq 3 \), we have \( x_1^{m-2}x_3^{n-2}x_2^{k-2}x_1^{1}x_3^{2}x_2^{1} \in C_{R_{mnk}} \). Thus, we can apply Algorithm 2. Recall that \( M(2, 2) \cong S^3/Q_{4m} \).

**Theorem 5.1.** For \( m \geq 3 \), \( S^3/Q_{4m} \) has a crystallization with 4\((m+2)\) vertices which is unique and minimal with respect to \((\langle S \mid R_{mn} \rangle, R_{mn2})\).

**Proof.** Since \( C_{R_{mn}} \neq \emptyset \), we can apply the Algorithm 2. Let \( \mathcal{R} = R_{mn2} \cup \{ x_1^{m-2}x_3^{1-x_2^{1}}x_1^{1-x_2^{1}} \} \).

Thus, \( m_{12}^{(2)} = m_{23}^{(2)} = m_{24}^{(2)} = m_{12}^{(3)} = m_{34}^{(3)} = m_{13}^{(3)} = 1 \), \( m_{12} = m_{12}^{(2)} = m_{34}^{(2)} = m_{13} = m_{24} = 2 \) and \( m_{14}^{(2)} = m_{14}^{(3)} = 2m - 3 \). Observe that, \( (n_1, n_2, n_3, n_4) = (2m, 4, 4, 2m) \) and \( G_{i} = C_{m}(x_{i}^{1}, \ldots, x_{i}^{m}) \) for \( 1 \leq i \leq 4 \) as in Figure 8. Choose \( x_{3}^{(p)} = x_{2}^{(2)} \) as in Algorithm 2, then the 3-colored graph with the color set \{ 0, 1, 2 \} as in Figure 8 which is unique up to an isomorphism. For the choices \( q_1, q_2, q_3, q_4 = (5, 3, 1, 2) \) and \( \{ x_{4}^{(1)}, x_{4}^{(2)}, x_{1}^{1}, x_{3}^{1} \} \) we get a 6-colored graph which yields \((\langle S \mid R_{mn2} \rangle, R_{mn2})\). Therefore, for each \( m \geq 3 \), we get a crystallization \((\Gamma_{1}, \gamma_{1})\) of the 3-manifold \( S^3/Q_{4m} \).

Now, we show that the crystallization \((\Gamma_{1}, \gamma_{1})\) is unique. Here we choose the colors 2 and 3 are the colors ‘i’ and ‘j’ as in construction of \( \bar{\tau} \) for \( r \in \mathcal{R} \) (cf. Eq. (2.3)). From the construction of \( \bar{\tau} \) for \( r \in \mathcal{R} \), it is clear that, either \( x_{2}^{(1)} \) or \( x_{2}^{(2)} \) is the starting vertex \( x_{1} \) of the component of \( \Gamma_{(2,3)} \), which yields the relation \( x_{1}^{m-1}x_{3}^{1-x_2^{1}}x_1^{1-x_2^{1}} \) (resp., \( x_{1}^{m-2}x_{3}^{1-x_2^{1}}x_1^{1-x_2^{1}} \). If possible let \( x_{2}^{(1)} \) be the starting vertex to yield the relation \( x_{1}^{m-1}x_{3}^{1-x_2^{1}}x_1^{1-x_2^{1}} \) then \( x_{2}^{(2)} \) is the starting vertex to yield the relation \( x_{1}^{m-2}x_3^{n-2}x_2^{k-2}x_1^{1}x_3^{2}x_2^{1} \) (since \( \Gamma_{1} \) is bipartite). Since \( m_{14}^{(3)} \) edges of color 3 yield \( m_{14}^{(3)} - 1 \) bi-colored 4-cycle in \( \Gamma_{(0,1,3)} \), we have \( x_{1}^{(2m)} x_{4}^{(2m-2)}, x_{4}^{(4)} x_{4}^{(4m-4)}, \ldots, x_{1}^{(8)} x_{4}^{(6)} \in \gamma^{-1}(3) \). Since \( x_{1}^{(2m)} x_{4}^{(2m)} \), the component of \( \Gamma_{(2,3)} \) with starting vertex \( x_{2}^{(1)} \) yields a relation \( x_{1}^{m-1}w \) for some \( w \in F(S) \), which is not possible. Thus, \( x_{2}^{(2)} \) is the starting vertex to yield the relation \( x_{1}^{m-2}x_3^{n-2}x_2^{k-2}x_1^{1}x_3^{2}x_2^{1} \) and \( x_{2}^{(1)} x_{4}^{(1)}, x_{4}^{(1)} x_{4}^{(2m-1)}, \ldots, x_{1}^{(8)} x_{4}^{(6)} \in \gamma^{-1}(3) \). Therefore, \( x_{1}^{(3)} x_{4}^{(1)}, x_{1}^{(4)} x_{4}^{(2)} \in \gamma^{-1}(3) \) as \( m_{14} = 2m - 3 \). To yield the relation \( x_{1}^{m-1}x_{3}^{1-x_2^{1}}x_1^{1-x_2^{1}} \), we
have \(x_1^{(6)}x_3^{(1)}, x_2^{(2)}x_4^{(3)}\) \(\gamma^{-1}(3)\). Since \(x_1^{(3)}x_4^{(1)}\) \(\gamma^{-1}(3)\), we have \(x_2^{(4)}x_3^{(2)}\) \(\gamma^{-1}(3)\) and \(C_4(x_1^{(3)}, x_2^{(2)}, x_3^{(1)}, x_4^{(1)})\) is the component of \(\Gamma_{(2,3)}\) which yields the relation \(x_3x_2^{-1}x_1^{-1}\) with starting vertex \(v_1 = x_3^{(3)}\). Thus, we have unique choices for the remaining edges of color 3 as in Figure 6. Since \(x_1^{m-2}x_3^{-1}x_1^{-1}x_2^{-1}\) is the only element in \(R_{m22}\) of minimum weight, the theorem follows.

Note: For \(m = 2\), there is no crystallization of \(S^3/Q_{4m}\) with \(4(m + 2) = 16\) vertices (cf. Basak and Datta [14]).

**Theorem 5.2.** For \(m, n \geq 3\), \(M(m,n,2)\) has a crystallization with \(4(m + n) - 2\) vertices which is unique and minimal with respect to \((S \mid R_{mn2}, R_{mn2})\).

**Proof.** Since \(C_{R_{mn2}} \neq \emptyset\), we can apply the Algorithm 2. Let \(R = R_{mn2} \cup \{x_1^{m-2}x_3^{-1}x_2^{-1}x_1^{-1}x_2^{-1}\}\). Thus, \(m_{13}^{(2)} = m_{23}^{(3)} = m_{34}^{(3)} = 1, m_{24}^{(2)} = 2n - 4, m_{24}^{(3)} = 2n - 3, m_{12}^{(2)} = m_{23}^{(2)} = m_{13}^{(3)} = 2\) and \(m_{14}^{(2)} = 2m - 3, m_{14}^{(3)} = 2m - 4\). Observe that, \((n_1, n_2, n_3, n_4) = (2m, 2n, 4, 2(m + n - 3))\) and \(G_i = C_n(x_i^{(1)}, \ldots, x_i^{(n)})\) for \(1 \leq i \leq 4\) as in Figure 7. Choose \(x_3^{(p)} = x_3^{(q)}\) as in Algorithm 2, then the 3-colored graph with the color set \(\{0, 1, 2\}\) as in Figure 7 which is unique up to an isomorphism. For the choices \((q_1, q_2, q_3, q_4) = (4, 2n - 2, 1, 1)\) and \(\{x_1^{(3)}x_4^{(1)}, x_1^{(6)}x_3^{(1)}\}\), we get a 4-colored graph which yields \((S \mid R_{mn2}, R_{mn2})\). Therefore, for each \(m, n \geq 3\), we get a crystallization \((\Gamma, \gamma)\) of the 3-manifold \(M(m,n,2)\).

Now, we show that the crystallization \((\Gamma, \gamma)\) is unique. Here we choose the colors 2 and 3 are the colors ‘i’ and ‘j’ as in construction of \(\tilde{r}\) for \(r \in R\) (cf. Eq. (2.3)). By similar arguments as in previous theorem, \(x_2^{(2)}\) is the starting vertex to yield the relation \(x_1^{m-1}x_3^{-1}x_2^{-1}\).
and hence $x_1(2) x_4(2m + 2n - 6), x_1(1) x_4(2m + 2n - 5), \ldots, x_1(8) x_4(2n) \in \gamma^{-1}(3)$ and $x_1(7) x_4(2n - 1) \notin \gamma^{-1}(3)$. Therefore, $x_1(3) x_4(1) \in \gamma^{-1}(3)$ as $m(3) = 2m - 4$. Since $m(2) = 1$, to yield the relation $x_1^{m-1} x_3^{1} x_2^{1} x_2^{1}$, we have $x_1(6) x_3(1) x_2(2n - 3) \in \gamma^{-1}(3)$. Similarly, $x_3(3)$ is the starting vertex to yield the relation $x_2^{3} x_2^{1} x_3^{1}$ and hence $x_2(2) x_4(2n - 4), x_2(4) x_4(2n - 5), \ldots, x_2(2n - 3) x_4(2) \in \gamma^{-1}(3)$ and $x_2(2n - 2) x_4(1) \notin \gamma^{-1}(3)$. Therefore, $x_2(2) x_4(2n - 3), x_2(1) x_4(2n - 4) \in \gamma^{-1}(3)$ as $m(3) = 2n - 3$ and $x_1(7) x_3(4), x_1(4) x_2(2n - 2), x_2(1) x_4(2n - 2) \in \gamma^{-1}(3)$ to yield the relation $x_1^{m-2} x_3^{1} x_2^{n-2} x_1^{1} x_2^{1}$. Now, we have unique choices for the remaining edges of color 3 as in Figure 7. Since $x_1^{m-2} x_3^{1} x_2^{n-2} x_1^{1} x_2^{1}$ is the only element in $R_{mn2}$ of minimum weight, the theorem follows.

**Theorem 5.3.** For $m, n, k \geq 3$, $M\langle m, n, k \rangle$ has a crystallization with $4(m + n + k - 3)$ vertices which is unique and minimal with respect to $(\langle S \mid R_{mnk} \rangle, R_{mnk})$.

**Proof.** Since $C_{R_{mn2}} \neq \emptyset$, we can apply the Algorithm 2. Let $R = R_{mnk} \cup \{x_1^{m-2} x_3^{1} x_2^{n-2} x_1^{1} x_3^{k-2} x_2^{1}\}$. Thus, $m(2) = m(2) = m(2) = m(2) = m(2) = m(2) = m(2) = 2$, $m(2) = m(2) = 2m - 4, m(2) = m(2) = m(2) = 2k - 4$. Again, we have $m(2) = (2m, 2n, 2k, 2(m + n - 6))$ and $G_i = C_{n_i}(x_1^{(1)}, \ldots, x_1^{(m_i)})$ for $1 \leq i \leq 4$ as in Figure 8. Choose $x_3^{(p)} = x_3^{(2)}$ as in Algorithm 2, then the 3-colored graph with the color set $\{0, 1, 2\}$ as in Figure 8, which is unique up to an isomorphism. For the choices $(q_1, q_2, q_3, q_4) = (4, 2n - 2, 5, 1)$ and (1)
the remaining edges of color 3 as in Figure 8. Since $x^{(1)} \times^{(6)}$, we get a 4-colored graph which yields $(S \mid R_{m,n,k}, R_{m,n,k})$. Therefore, for each $m, n, k \geq 3$, we get a crystallization $(\Gamma, \gamma)$ of the 3-manifold $M(m, n, k)$.

Figure 8: Crystallization of $M(m, n, k)$ for $m, n, k \geq 3$

Now, we show that the crystallization $(\Gamma, \gamma)$ is unique. Here we choose the colors 2 and 3 are the colors ‘i’ and ‘j’ as in construction of $\tilde{r}$ for $r \in \mathcal{R}$ (cf. Eq. (2.3)). By similar arguments as in previous theorem, $x^{(2)}, x^{(3)}$, $x^{(3)}$, $x^{(3)}$, $x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}$ are the starting vertices to yield the relations $x^{m-1}, x^{m-1}, x^{m-1}, x^{m-1}, x^{m-1}, x^{m-1}, x^{m-1}, x^{m-1}, x^{m-1}$ respectively. Therefore, as in previous theorem, $\{x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}\}$, $\{x^{(2)}, x^{(2)}, x^{(2)}, x^{(2)}, x^{(2)}, x^{(2)}, x^{(2)}, x^{(2)}\}$, $\{x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}\}$, $\{x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}\}$, $\{x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}, x^{(3)}\}$

Remark 5.4. By [Casali 99] Proposition 4, the vertex-minimal crystallizations of prime and handle-free 3-manifolds with at most 30 vertices are known (see [Bandieri et al. 11], [Casali and Cristofori 08] and [Lins 95]). Thus, our crystallizations of the 3-manifolds $M(m, n, k)$ for $m \geq 3$ and $M(m, n, k, k)$ for $m \geq 4$ and $n, k \geq 3$ are minimal crystalliza-
tions when the numbers of vertices of the crystallizations are at most 30. Till now, there are no known crystallizations of $M(m, 2, 2)$ for $m \geq 3$ and $M(m, n, k)$ for $m \geq 4$ and $n, k \geq 3$ which have less number of vertices than our constructed ones.

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References

[Aschenbrenner et al. 13] M. Aschenbrenner, S. Friedl and H. Wilton, 3-manifold groups, arXiv: math/1205.0202v3, 2013, 149 pages.

[Bandieri et al. 13] Paola Bandieri, Paola Cristofori and Carlo Gagliardi, A census of genus two 3-manifolds up to 42 coloured tetrahedra, Discrete Math. 310 (2010), 2469–2481.

[Bandieri et al. 11] P. Bandieri, M. R. Casali, P. Cristofori, L. Grasselli and M. Mulazzani, Computational aspects of crystallization theory: complexity, catalogues and classification of 3-manifolds, Atti Sem. Mat. Fis. Univ. Modena 58 (2011), 11–45.

[Basak and Datta 14] B. Basak and B. Datta, Crystallizations of 3-manifolds, Electron. J Combin. 21 (1) (2014), #P1.61, 1–25.

[Bondy and Murty 08] J. A. Bondy and U. S. R. Murty, Graph Theory, Springer, New York, 2008.

[Casali 99] M. R. Casali, classification of non-orientable 3-manifolds admitting decompositions into $\leq 26$ coloured tetrahedra, Acta Appl. Math. 54 (1999), 75–97.

[Casali and Cristofori 08] M. R. Casali and P. Cristofori, A catalogue of orientable 3-manifolds triangulated by 30 coloured tetrahedra, J. Knot Theory Ramification 17 (2008), 1–23.

[Casali and Cristofori 14] M. R. Casali and P. Cristofori, A note about complexity of lens spaces, Forum Math. (2014), DOI:10.1515/forum-2013-0185, published online February 19, 2014, 14 pages.

[Cavicchioli et al. 80] A. Cavicchioli, L. Grasselli and M. Pezzana, Su di una decomposizione normale per le $n$-varietà chiuse, Boll. Un. Mat. Ital. 16-A (1980), 1146-1165.

[Epstein 61] D. B. A. Epstein, Finite presentations of groups and 3-manifolds, Quart. J. Math. Oxford 12 (1961), 205–212.

[Perelman 03] G. Perelman, Finite extinction time for the solutions to the Ricci flow on certain three-manifolds, arXiv:math/0307245v1, 2003, 7 pages.

[Swartz 13] E. Swartz, The average dual surface of a cohomology class and minimal simplicial decompositions of infinitely many lens spaces, arXiv:1310.1991v2, 2013, 6 pages.