Universality of classical and quantum SAT-UNSAT transitions of convex continuous satisfaction problems

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Here we investigate the single-layer linearized perceptron near the SAT-UNSAT transition point as a prototypical model of the convex continuous satisfaction problems. The simplicity of the model allows us to take into account the effects of the quantum fluctuation, which have not been fully investigated before. We found that the classical and quantum models have different critical exponents and thus have different universality classes. We also briefly discuss the effects of the random field.

I. INTRODUCTION

The purpose of the constraint satisfaction problem is to find out solutions satisfying given constraints. There are many such solutions if the number of constraints is sufficiently smaller than the number of degrees of freedom. On increasing the number of constraints, the number of solutions decreases, and eventually, the solution ceases to exist at a certain point. This is the so-called satisfaction (SAT) unsatisfaction (UNSAT) transition [1–3]. The SAT-UNSAT transition becomes a genuine phase transition in the thermodynamic limit where both the number of degrees of freedom and the number of constraints go to infinity [4].

Recently, the SAT-UNSAT transition of continuous variables has attracted much attention in connection with the sphere packing problem [3, 5–16]. The sphere packing problem is thought as a constraint satisfaction problem to find a configuration with the constraint that spheres do not overlap [17, 18]. The packing fraction at which such solutions cease to exist is called the jamming transition point $\varphi_J$ [10, 17, 19]. Several physical quantities, such as the contact number, shear modulus, correlation length, and relaxation time, exhibit the critical behavior near $\varphi_J$ [19, 22].

A promising way to study phase transitions is first to consider a solvable mean-field model [23]. In the case of the SAT-UNSAT transition of the continuous constraint satisfaction problem, a prototypical mean-field model is the single-layer perceptron [2, 24]. For the classical model of the perceptron, the static critical properties of the model have already been well investigated by using the replica method [3, 11, 16]. In particular, a non-convex version of the model is shown to have the same critical exponents of those of particle systems near the jamming transition point $\varphi_J$ [11]. However, due to the complexity of the model, there are several unsolved problems, even for the convex case, where the cost function has a unique minimum. For instance, out-of-equilibrium dynamics of the perceptron has been actively studied recently because of its relevance to machine learning and the jamming transition [25, 28]. However, the current formalism based on the dynamical density functional theory is numerically highly demanding, which makes it difficult to estimate the dynamical critical exponent [28]. Another theoretically interesting question is how the quantum fluctuation affects the nature of the SAT-UNSAT transition of the continuous satisfaction problem. Unfortunately, a quantum version of the perceptron is difficult to solve analytically and previous studies relied on the Scher–Giamarchi–Le Doussal Expansion [15] or Monte Carlo sampling [16]. In particular, the scaling behavior of the quantum SAT-UNSAT transition in the UNSAT side has not been fully investigated yet [15, 19]. Considering those difficulties of the perceptron, it is desirable to first consider a more analytically tractable model.

In this work, we revisit a simplified problem: a linearized version of the perceptron [29, 30]. The model can be solved analytically without relying on the replica method [30]. Furthermore, its dynamical properties have been already well investigated [29, 31, 32]. We first investigate the classical version of the model near SAT-UNSAT transition point. The model shows the same scaling as that of the original perceptron of the convex case [3]. Next, to demonstrate the usefulness of the model, we consider the quantum version of the model. We show that the susceptibility against quantum fluctuation affects the nature of the SAT-UNSAT transition of the convex continuous satisfaction problems. Unfortunately, a quantum version of the perceptron is difficult to solve analytically and previous studies relied on the Scher–Giamarchi–Le Doussal Expansion [15] or Monte Carlo sampling [16]. In particular, the scaling behavior of the quantum SAT-UNSAT transition in the UNSAT side has not been fully investigated yet [15, 19]. Considering those difficulties of the perceptron, it is desirable to first consider a more analytically tractable model.

This paper is organized as follows. In Sec. II, we investigate the classical model. We characterize the criticality in terms of the condensation transition as previously done for $p = 2$-spin spherical model [33]. In Sec. II, we investigate the quantum model. In Sec. IV, we investigate the model with a random field. Finally, in Sec. V, we summarize the work and discuss possible future works.

II. CLASSICAL MODEL

We consider the following continuous constraint satisfaction problem. Let $x = (x_1, \ldots, x_N)$ be the state vector of norm $x \cdot x = N$. The problem is if there exists the jamming transition [25, 28]. However, the current formalism based on the dynamical density functional theory is numerically highly demanding, which makes it difficult to estimate the dynamical critical exponent [28]. Another theoretically interesting question is how the quantum fluctuation affects the nature of the SAT-UNSAT transition of the continuous satisfaction problem. Unfortunately, a quantum version of the perceptron is difficult to solve analytically and previous studies relied on the Scher–Giamarchi–Le Doussal Expansion [15] or Monte Carlo sampling [16]. In particular, the scaling behavior of the quantum SAT-UNSAT transition in the UNSAT side has not been fully investigated yet [15, 19]. Considering those difficulties of the perceptron, it is desirable to first consider a more analytically tractable model.

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\( \mathbf{x} \) such that

\[
\mathbf{x} \cdot \xi^\nu = 0 \quad \text{for } \nu = 1, \ldots, M, \tag{1}
\]

where \( \xi^\nu = \{\xi_1^\nu, \ldots, \xi_N^\nu\} \) denotes a \( N \) dimensional random vector. \( \xi^\nu \) is an i.i.d Gaussian random number of zero mean and unit variance. The inequality version of the problem is referred to as the perceptron and has already been well investigated \[2, 3, 11, 34\]. It is known that the perceptron exhibits a sharp phase transition from the satisfiable (SAT) phase, where all constraints are satisfied, to the unsatisfiable (UNSAT) phase, where some constraints are violated \[2\]. Later, we show that our model also exhibits a similar SAT-UNSAT transition.

To solve the problem, we consider the quadratic cost function:

\[
V_N = \frac{1}{2N} \sum_{\nu=1}^{M} (\mathbf{x} \cdot \xi^\nu)^2 + \frac{\mu}{2} (N - \mathbf{x} \cdot \mathbf{x}), \tag{2}
\]

where \( \mu \) denotes the Lagrange multiplier to impose the spherical constraint \( \mathbf{x} \cdot \mathbf{x} = N \). The cost function Eq. (2) can be considered as a special case of the linealized perceptron \[29, 32\] with the spherical constraint. Eq. (2) is also very similar to that of the perceptron of \( \sigma = 0 \), where the cost function of the model is convex \[3\]. Later, we show that the current model indeed exhibits the same scaling as that of the perceptron of \( \sigma = 0 \). When the conditions Eqs. (1) are satisfied, one obtains \( V_N = 0 \) and vice versa. After some manipulations, Eq. (2) is rewritten as

\[
V_N = \frac{1}{2} \mathbf{x} \cdot W \cdot \mathbf{x} + \frac{\mu}{2} (N - \mathbf{x} \cdot \mathbf{x}), \tag{3}
\]

where \( W \) is a \( N \times N \) symmetric matrix whose \( ij \) component is given by

\[
W_{ij} = \frac{1}{N} \sum_{\nu=1}^{M} \xi_i^\nu \xi_j^\nu. \tag{4}
\]

To investigate the model, we diagonalize the matrix \( W \) and expand the potential by the normal modes:

\[
V_N = \sum_{i=1}^{N} \frac{\lambda_i - \mu}{2} u_i^2 + \frac{\mu}{2} N, \tag{5}
\]

where \( \lambda_i \) denotes the \( i \)-th eigenvalue of \( W \). We will order \( \lambda_i \) such that

\[
\lambda_1 < \lambda_2 < \cdots < \lambda_N. \tag{6}
\]

Since \( W \) is a Wishart matrix, in the thermodynamic limit \( N \to \infty \), its distribution is given by the Marcenko-Pastur law \[34, 35\]:

\[
\rho(\lambda) = \theta(1 - \alpha)(1 - \alpha) \delta(\lambda) + g(\lambda)
\]

\[
g(\lambda) = \begin{cases} \frac{1}{\pi} \sqrt{\frac{(\lambda - \lambda_-)(\lambda - \lambda_+)}{\lambda}} & \lambda \in [\lambda_- , \lambda_+] , \\ 0 & \text{otherwise} \end{cases} \tag{7}
\]

where \( \theta(x) \) denotes the Heaviside step function, \( \alpha = M/N \) denotes the number of the constraints per degree of freedom, and

\[
\lambda_{\pm} = (\sqrt{\alpha} \pm 1)^2. \tag{8}
\]

It is easy to show that the ground state energy of Eq. (5) is given by

\[
\frac{V_{GS}}{N} = \frac{\lambda_{\min}}{2} = \begin{cases} 0 & \alpha < 1 \\ (\frac{1}{\alpha} - 1)^{\frac{1}{2}} & \alpha > 1 \end{cases}, \tag{9}
\]

where \( \lambda_{\min} \) denotes the minimal eigenvalue of \( W \). When \( \alpha < 1 \), \( V_{GS} = 0 \), implying that all constraints Eq. (1) are satisfied. On the contrary, when \( \alpha > 1 \), \( V_{GS} > 0 \), implying that some constraints are unsatisfied. Therefore, the model exhibits the SAT-UNSAT transition at \( \alpha_c = 1 \) \[30\]. At the transition point, the system is isostatic: the number of degrees of freedom is the same as that of the constraints \( N = M \) \[3\]. The isostaticity has been previously reported for the perceptron for \( \sigma < 0 \) \[3, 11\].

Now we characterize the criticality around \( \alpha_c \). For this purpose, we first consider the model in equilibrium at temperature \( T \) and take the limit \( T \to 0 \) at the end of the calculation. From the equipartition theorem \[30\], we get

\[
\langle u_i^2 \rangle = \frac{k_B T}{\lambda_i - \mu}, \tag{10}
\]

where \( k_B \) denotes the Boltzmann constant. Hereafter, we set \( k_B = 1 \) to simplify the notation. Since \( \langle u_i^2 \rangle \geq 0 \), \( \mu \) should satisfy

\[
\mu \leq \lambda_{\min}. \tag{11}
\]

Since an orthogonal transformation preserves the inner product, the spherical constraint \( \mathbf{x} \cdot \mathbf{x} = N \) is written as

\[
1 = \frac{\mathbf{x} \cdot \mathbf{x}}{N} = \frac{\mathbf{u} \cdot \mathbf{u}}{N} = \int_{-\infty}^{\infty} d\lambda \rho_N(\lambda) \frac{T}{\lambda - \mu}, \tag{12}
\]

where we have defined the distribution of \( \lambda_i \):

\[
\rho_N(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i). \tag{13}
\]

In the limit \( N \to \infty \), \( \rho_N(\lambda) \) converges to Eq. (7). Therefore, we get

\[
1 = -(1 - \alpha) \theta(1 - \alpha) \frac{T}{\mu} + \int_{\lambda_-}^{\lambda_+} d\lambda g(\lambda) \frac{T}{\lambda - \mu}. \tag{14}
\]

We first investigate the behavior of the model in the SAT phase \( \alpha < 1 \). For \( T \ll 1 \), the dominant contribution comes from the first term, thus we get

\[
1 = -(1 - \alpha) T \frac{\mu}{\mu} \Rightarrow \mu = -T(1 - \alpha), \tag{15}
\]
which has the smallest curvature \( \tau \). As we approach the transition point

Therefore, the relaxation time diverges as

\[ \tau \sim (1 - \alpha)^{-1} = T^{-1}(1 - \alpha)^{-1}. \]

Therefore, the relaxation time diverges as \( \tau \sim (1 - \alpha)^{-\beta} \) with the critical exponent \( \beta = 1 \). The result seems to be consistent with the previous research for the linealized perceptron with the Langevin dynamics [29]. For the perceptron, to the best of our knowledge, the relaxation time in the SAT phase has not been calculated yet, due to the complexity of the dynamical equation [26 28].

For \( \alpha > 1 \), \( \mu \) is to be determined by

\[ 1 = F(\mu), \]

where

\[ F(\mu) = T \int_{\lambda_-}^{\lambda_+} d\lambda \frac{g(\lambda)}{\lambda - \mu}. \]

One can show that \( F(\mu) \) takes its maximum value at \( \mu = \lambda_- \):

\[ F(\lambda_-) = \frac{T}{T_c}, \]

where

\[ T_c = \left[ \int_{\lambda_-}^{\lambda_+} d\lambda \frac{g(\lambda)}{\lambda - \lambda_-} \right]^{-1} = \sqrt{\alpha} - 1, \]

see Fig. 1 (a). Below \( T_c \), however, Eq. (18) has no solution, which is the signature of the condensation to the lowest eigenmode \( \lambda_- [33 37 38] \). In the case of the \( p = 2 \)-spin spherical mode, the condensation transition occurs as a consequence of the underlying spin-glass transition [33]. Here we assume that the same is true for our model and identify \( T_c \) with the spin-glass transition point. For \( T < T_c \), the first and the other terms in Eq. (12) should be treated separately, as in the case of the Bose-Einstein condensation [35]:

\[ 1 = \frac{(u_1)^2}{N} + \frac{1}{N} \sum_{i=2}^{N} \langle u_i^2 \rangle = \frac{(u_1)^2}{N} + F(\lambda_-). \]  

From the above equation and Eq. (20), we get

\[ \frac{\langle u_1^2 \rangle}{N} = 1 - \frac{T}{T_c}. \]

This can be identified with the Edward–Anderson order parameter [33 39]. To characterize the criticality in the limit \( T \to 0 \), following [3], we define the susceptibility against the thermal fluctuation:

\[ \chi = \lim_{T \to 0} \frac{1}{T} \left[ 1 - \frac{\langle u_1^2 \rangle}{N} \right] = \frac{1}{\sqrt{\alpha} - 1}. \]

The susceptibility diverges as \( \chi \propto (\alpha - 1)^{-1} \) on approaching the SAT-UNSAT transition point. The same critical exponent has been previously reported for the perceptron for \( \sigma = 0 [3] \).

Finally, we comment on the marginal stability. As in the SAT phase, the \( i \)-th eigenvalue \( \lambda_i \) of the Hessian of the interaction potential Eq. (5) is calculated as

\[ \tilde{\lambda}_i = \left( \frac{\partial V_N}{\partial u_i^2} \right) = \lambda_i - \mu. \]

Using \( \langle u_1^2 \rangle = 1/(\lambda_- - \mu) \) and Eq. (23), we get in the spin-glass phase

\[ \mu = \lambda_- - \frac{1}{N} \frac{T_c}{T} \to \lambda_- . \]

Therefore, the system is marginally stable \( \lambda_{\min} = \lambda_- - \lambda_- = 0 \) in the thermodynamic limit \( N \to \infty [40] \). In the terminology of the effective medium theory (EMT), the term \( \mu \) that shifts all eigenvalues, as in Eq. (23), is referred to as the pre-stress [41]. Using Eq. (8) and the marginal stability \( \lambda_{\min} = \lambda_- - \mu = 0 \), one obtain a well-known square root scaling \( \alpha - 1 \propto \sqrt{T} [41] \). Note that in the case of the EMT, the marginal stability is an assumption, but in the case of the current model, the value of \( \mu \) is fine-tuned automatically to achieve the marginal stability as a consequence of the spin-glass transition for \( T < T_c \). The distribution of \( \tilde{\lambda}_i \) is calculated as

\[ \rho(\tilde{\lambda}) = \rho(\lambda = \tilde{\lambda} + \mu) \frac{d\lambda}{d\tilde{\lambda}} = \rho(\lambda + \lambda_-), \]
where $\rho(\lambda)$ is given by Eq. (7). To compare with previous researches \cite{19, 42}, we define the vibrational density of states: the distribution of the eigenfrequency $\omega_i = \sqrt{\lambda_i}$,

\[
D(\omega) = \rho(\lambda) = \omega^2 \frac{d\lambda}{d\omega} = \frac{\omega^2 \sqrt{\lambda_+ - \lambda_- - \omega^2}}{\pi(\omega^2 + \lambda_-)}.
\]

(28)

One can deduce the scaling behavior of $D(\omega)$ at $T = 0$ near the transition point $\alpha \approx 1$ as follows:

\[
D(\omega) \sim \begin{cases} 
\text{const} & \omega \gg \omega_* \\
(\omega/\omega_*)^2 & \omega \ll \omega_*
\end{cases},
\]

(29)

where $\omega_* = (\alpha - 1)/2$. The same scaling has been previously derived for the perceptron for $\sigma \leq 0$ \cite{34}, the EMT for a disordered lattice \cite{41}, and numerical simulation of sphere packing near the jamming transition point \cite{19, 42, 43}.

### III. QUANTUM MODEL

Now, we consider the quantum version of the model:

\[
H = \sum_{i=1}^{N} \frac{\mu_i^2}{2} + \sum_{i=1}^{N} \left[ \lambda_i - \mu \frac{u_i^2}{2} \right],
\]

(30)

where we omit the constant term $\mu N/2$ to simplify the notation. We require the standard canonical commutation relation \cite{44, 45}:

\[
[u_i, u_j] = 0, \quad [p_i, p_j] = 0, \quad [u_i, p_j] = \hbar \delta_{ij},
\]

(31)

where $\hbar$ is the plank constant. Here we use $\hbar$ as a control parameter to control the strength of the quantum fluctuation. Following the standard operation of quantum statistical mechanics \cite{30}, one can calculate the partition function for the $i$-th harmonic oscillator as

\[
Z_i = \text{Tr} e^{-\beta H} = \sum_{n=0}^{\infty} e^{-\hbar \sqrt{\lambda_i - \mu} n(n+1/2)} = \left[ 2 \sinh \left( \frac{\beta \hbar \sqrt{\lambda_i - \mu}}{2} \right) \right]^{-1},
\]

(32)

where $\beta = 1/T$ denotes the inverse temperature. Then, the second moment is

\[
\langle u_i^2 \rangle = \frac{2}{\beta} \frac{d \log Z_i}{d \mu} = \left[ 2 \sqrt{\lambda_i - \mu} \frac{\hbar \sqrt{\lambda_i - \mu}}{h} \tanh \left( \frac{\hbar \sqrt{\lambda_i - \mu}}{2T} \right) \right]^{-1}.
\]

(33)

In the high temperature limit, we get

\[
\frac{2 \sqrt{\lambda_i - \mu}}{h} \tanh \left( \frac{\hbar \sqrt{\lambda_i - \mu}}{2T} \right) \sim \frac{\lambda_i - \mu}{T}.
\]

(34)

Substituting it back into Eq. (33), we recover the classical result Eq. (10). Instead, here we first take the $T \to 0$ limit and then observe the asymptotic behavior for $\hbar \ll 1$. At $T = 0$, Eq. (33) reduces to

\[
\langle u_i^2 \rangle = \frac{\hbar}{2\sqrt{\lambda_i - \mu}}.
\]

(35)

As before, $\mu$ is determined by the spherical constraint:

\[
1 = \frac{1}{N} \sum_{i=1}^{N} \langle u_i^2 \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{\hbar}{2\sqrt{\lambda_i - \mu}}.
\]

(36)

Repeating the same analysis of that of the classical model, one can see that for $\alpha < 1$ and $\hbar \ll 1$, Eq. (36) reduces to

\[
1 = \frac{1}{N} \sum_{i=1}^{(1-\alpha)N} \langle u_i^2 \rangle = (1 - \alpha) \frac{\hbar}{2\sqrt{-\mu}},
\]

(37)

which leads to

\[
\mu = -\frac{\hbar^2 (1 - \alpha)^2}{4},
\]

\[
\langle u_i^2 \rangle = \begin{cases} 
(1 - \alpha)^{-1} & i = 1, \ldots, (\alpha - 1)N \\
0 & \text{otherwise}
\end{cases}.
\]

(38)

We find the same exponent for $\langle u_i^2 \rangle$ and the different exponent for $\mu$ from those of the classical model, see Eqs. (15) and (16). The similar result has been previously obtained for the perceptron for $\sigma = 0$ \cite{18}, where the authors mentioned the differences in the critical exponent between the classical and quantum models. For $\alpha > 1$, the condensation (spin-glass) transition occurs at a finite $\hbar = h_c$. As before, this transition point is calculated as

\[
1 = \frac{\hbar}{2} \int_{\lambda_-}^{\lambda_+} d\lambda \left\{ \frac{g(\lambda)}{\sqrt{\lambda - \lambda_-}} \right\} = \frac{2}{\pi} \int d\lambda g(\lambda)(\lambda - \lambda_-)^{-1/2},
\]

(39)

see Fig.1(b). In the limit $\alpha \to 1$, $h_c$ vanishes as

\[
h_c \propto -\frac{1}{\log(\alpha - 1)}.
\]

(40)

For $\hbar < h_c$, we define the order parameter

\[
\langle u_i^2 \rangle = 1 - \frac{\hbar}{2} \sum_{i=2}^{N} \frac{1}{\sqrt{\lambda_i - \lambda_-}},
\]

(41)

and susceptibility w.r.t the quantum fluctuation:

\[
\chi = \lim_{\hbar \to 0} \frac{1}{\hbar} \left( 1 - \frac{\langle u_i^2 \rangle}{N} \right) = \frac{1}{2} \int_{\lambda_-}^{\lambda_+} d\lambda \frac{g(\lambda)}{\sqrt{\lambda - \lambda_-}}.
\]

(42)
In the limit $\alpha \to 1$, $\chi$ diverges logarithmically
\begin{equation}
\chi \sim -\log(\alpha - 1),
\end{equation}
instead of the power-law found in the classical model Eq. (24). To the best of our knowledge, this is the first result that reveals qualitative differences between the thermal and quantum fluctuations near the continuous SAT-UNSAT transition point.

Finally, we would briefly comment on the marginal stability [40]. As in the case of the classical model, the minimal eigenvalue of the Hessian is calculated as
\begin{equation}
\tilde{\lambda}_1 = \left< \frac{\partial V_N}{\partial u_1^2} \right> = \lambda_- - \mu.
\end{equation}
Repeating a similar analysis as in the previous section, one can show that $\tilde{\lambda}_1 > 0$ for $h > h_c$, and $\tilde{\lambda}_1 = 0$ for $h \leq h_c$, implying that the density of states is gapped for $h > h_c$ even at $T = 0$. In particular, at $\alpha = \alpha_c = 1$, the gap vanishes only in the limit $h \to 0$, see Fig. 1 (b).

This gap was not reported in a previous calculation for the perceptron based on the Scherr–Giamarchi–Le Doussal Expansion, which is the expansion by $h$ with fixed $h/T$ [15]. Further studies of the quantum version of the perceptron with more accurate approximations would be beneficial to clarify the origin of this discrepancy.

IV. EFFECTS OF RANDOM FIELD

Finally, we consider the model with the random field:
\begin{equation}
V_N = \sum_{i=1}^{N} \frac{\lambda_i - \mu}{2} u_i^2 + \sum_{i=1}^{N} h_i u_i,
\end{equation}
where $h_i$ is an i.i.d random variable of zero mean and variance $\Delta$. In equilibrium at temperature $T$, we get
\begin{equation}
\langle u_i^2 \rangle = \frac{T}{\lambda_i - \mu} + \frac{\Delta}{(\lambda_i - \mu)^2},
\end{equation}
where the overline denotes the average for $h_i$, and $\Delta = \bar{h}_i^2$. Hereafter we consider the model at $T = 0$, and observe the asymptotic behavior for $\Delta \ll 1$. As before, $\mu$ is determined by the spherical constraint:
\begin{equation}
1 = \frac{1}{N} \sum_{i=1}^{N} \frac{\Delta}{(\lambda_i - \mu)^2}.
\end{equation}
Repeating the same analysis of that of the classical model, in the limit $\Delta \ll 1$, we get for $\alpha > 1$
\begin{equation}
\mu = -(1 - \alpha)^{1/2} \Delta^{1/2}, \quad \langle u_i^2 \rangle = \begin{cases} (1 - \alpha)^{-1} & i = 1, \ldots, (\alpha - 1)N \\ 0 & \text{otherwise} \end{cases}.
\end{equation}
We find the same exponent for $\langle u_i^2 \rangle$ and the different exponent for $\mu$ from both the classical and quantum models.

The condensation (spin-glass) transition point for $\alpha > 1$ is estimated as
\begin{equation}
\Delta_c = \frac{1}{\int d\lambda g(\lambda)(\lambda - \lambda_-)^2} = 0,
\end{equation}
meaning that the transition does not occur at finite $\Delta$, see Fig. 1 (c). In a previous work, we investigated a similar equation as Eq. (47) and found that the transition at finite $\Delta$ occurs only for $\rho(\lambda) \sim (\lambda - \lambda_-)^n$ with $n > 1$ [46], which is not satisfied by the eigenvalue distribution of the current model Eq. (7).

V. SUMMARY AND DISCUSSIONS

In this work, we investigated the convex continuous SAT-UNSAT transition of a special case of the linearized perceptron. Since the interaction potential has a quadratic form, the model can be easily analyzed by expanding the potential by the normal modes. We successfully characterized the criticality near the SAT-UNSAT transition point. The simplicity of the model allows us to investigate the quantum effects, which have not been fully investigated before. We found different critical behaviors from those of the classical model. In particular, the susceptibility of the order parameter diverges logarithmically when approaching the transition point from the UNSAT side. This is qualitatively different behavior from that of the classical model, where the power-law divergence is observed. Finally, we investigated the model with the random field. We found different critical exponents from both the classical and quantum models.

There are still several important points that deserve further investigation. Here we give a tentative list:

- The dynamics of the linearized perceptron has been already well investigated [29, 31, 32]. Furthermore, the simplicity of the model may allow us to derive a full dynamical solution as done for the $p$-spin spherical models [17, 38]. It would be interesting to revisit these results and compare them with recent numerical simulations of particle systems near the jamming (SAT-UNSAT) transition point [22, 49, 50].

- We found that $D(\omega)$ has a finite gap for $h > 0$, which has not been reported before [15]. It is interesting to investigate how this gap affects the low temperature behavior, in particular, the temperature dependence of the specific heat.

- In this work, we considered the quadratic cost function Eq. (2). A natural generalization is to consider the $p$-body interaction as in the case of the $p$-spin spherical model [51, 52]:
\begin{equation}
V_N = \frac{1}{N^p} \sum_{\mu=1}^{M} (x \cdot \xi^{\mu})^p + \frac{\mu}{2} (N - x \cdot x).
\end{equation}
By analogy from the $p$-spin spherical mode, we expect that the model exhibits the one-step (or higher) replica symmetric breaking (1RSB) for $p > 2$. Thus, the phase diagram of the model is more similar to that of the structural glasses. It would be interesting to investigate how the 1RSB transition affects the SAT-UNSAT transition.

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