ASYMPTOTIC EXPANSIONS
FOR INFINITE WEIGHTED CONVOLUTIONS
OF LIGHT SUBEXPONENTIAL DISTRIBUTIONS

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Abstract. We establish some asymptotic expansions for infinite weighted convolutions of distributions having light subexponential tails. Examples are presented, some showing that in order to obtain an expansion with two significant terms, one needs to have a general way to calculate higher order expansions, due to possible cancellations of terms. An algebraic methodology is employed to obtain the results.

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1. Introduction. Subexponential distributions (Chistyakov, 1964), as the name suggests, have tails which decay at a subexponential rate, and hence, provide good models for heavy-tailed data. These distributions are defined by a property on their \( n \)-fold convolutions which make them useful in various problems involving sums of independent random variables. To be more precise, we need to introduce some notation. For a distribution \( F \) we write \( F = 1 - F \) its tail function and \( F \star n \) its \( n \)-fold convolution. Next if \( f \) and \( g \) are two functions, we write \( f \sim g \) to signify that the ratio \( f/g \) tends to 1 at infinity. A distribution function \( F \) is subexponential if and only if \( F \star 2 \sim 2F \), or, equivalently (Chistyakov, 1964), if \( F \star n \sim nF \) for all positive integers \( n \). Thus, these distributions have particularly nice analytic features with respect to convolution, and, as such, form an excellent class that allows both generality and tractability in asymptotic analyses of several stochastic models.

When dealing with problems involving extreme values, subexponential distributions have two possible limiting extremal behavior; for instance the Pareto type belong to the domain of max-attraction of the Fréchet distribution, while the Weibull type belong to the domain of max-attraction of the Gumbel distribution. Hence, as far as extremal behavior is concerned, some subexponential distributions are classified as light tail, in the sense that their extremes have a
limiting behavior which is of the same type as that for the extremes of a normal distribution say.

In this paper, we derive tail area expansions for infinite order weighted sums of light tailed, yet subexponential, random variables. For Pareto type distribution, such expansions were obtained in Barbe and McCormick (2005) but their form and the proofs differ sharply from those in the current paper.

Examples of applications of subexponentiality include transient renewal theory (Teugels, 1975; Embrechts and Goldie, 1982), random walks (Grübel, 1985; Veraverbeke, 1977), branching processes (Chistyakov, 1964; Athreya and Ney, 1972; Chover, Ney, and Waigner, 1972), queueing theory (Pakes, 1975), shot noise (Lebedev, 2002), infinite divisibility (Embrechts, Goldie, and Veraverbeke, 1979), ruin theory (Asmussen, 1997; Goldie and Klüppelberg, 1998; Tang and Tsitsiashvili, 2003), compound sums (Cline, 1987; Embrechts, 1985; Grübel, 1987), insurance risk theory (Embrechts, Klüppelberg and Mikosch, 1997), and heavy-tailed linear processes (Rootzén, 1986; Davis and Resnick, 1988; Geluk and De Vries, 2004; Chen, Ng, Tang, 2005). In these papers, inputs to the processes follow a subexponential distribution and an asymptotic analysis of an output is obtained, e.g., claim size distribution and ruin probability, age distribution and expected number of particles alive, service time distribution and stationary waiting time distribution, or innovation distribution and tail area for weighted averages. Mainly, these analyses are first order asymptotic results. Some second order results have been obtained. For compound or subordinated distributions, that is distributions of the form $\sum_{n \geq 0} p_n F^{*n}$ with $F$ subexponential, second order results have been obtained in Omey and Willekens (1987). For finite order convolutions, Baltrunas and Omey (1998) prove a second order result.

Throughout this paper, we consider an infinite weighted sum $\sum_{i \in \mathbb{Z}} c_i X_i$, where the $X_i$'s are independent and identically distributed random variables. We write $F$ the distribution function of the $X_i$'s and $G$ that of the weighted sum.

Davis and Resnick (1988) consider the tail of $G$ when $F$ is in the domain of max-attraction of the Gumbel distribution, and belongs to a class slightly larger than the subexponential one. To state in a compact form the part of their result which we will use, and because we will be mostly interested in distributions and not in random variables, we introduce the following notation. If $X$ has
distribution \( F \), we write \( M_c F \) the distribution of \( cX \). Thus, whenever \( c \) is positive, \( M_c F = F(\cdot/c) \), while \( M_0 F \) is the distribution function of the point mass at 0. Since it acts on functions, we call \( M_c \) a multiplication operator. The distribution of \( \sum_{i \in \mathbb{Z}} c_i X_i \) is the infinite weighted convolution \( \ast_{i \in \mathbb{Z}} M_{c_i} F \).

In this introduction, we also write \( c_{(1)} \) the largest weight, and \( \nu_+ \) its multiplicity in the sequence of weights \((c_i)\).

**Theorem.** (Davis and Resnick, 1988, Proposition 1.3). Let \( F \) be a subexponential distribution supported on the nonnegative half-line and in the domain of max-attraction of the Gumbel distribution. Assume furthermore that the \( c_i \)'s are nonnegative and that

\[ \sum_{i \in \mathbb{Z}} c_i^\delta \text{ is finite for some positive } \delta \text{ less than 1.} \tag{1.1} \]

Then

\[ G \sim \nu_+ M_{c(1)} F \] \( \tag{1.2} \)

We remark that Geluk and De Vries (2004) show a more general version of the above result; see also Chen, Ng, Tang (2005).

As the above result shows, in this case the largest weight controls the first order asymptotic equivalence. This stands in contrast to what happens in the regular variation heavy tail case, where all weights are on an equal footing and appear in the first order asymptotic equivalence; see Resnick (1987, p.227). One might propose that the next term in an expansion would be, say, \( k_2 M_{c(2)} F \) where \( c_{(2)} \) is the second largest coefficient and \( k_2 \) equals the multiplicity of \( c_{(2)} \) in the sequence \( (c_i) \). This intuition is correct for a narrow range of distributions. In section 2, we consider three regimes within the light subexponential tail one. In the heaviest of these regimes, the above intuition provides the right answer for the expansion; see Theorem 2.3.2. In the other two cases, the missing element in our discussion makes its appearance, namely, derivatives of the underlying distribution. In fact, our formalism will not only provide the second order term, but, more generally, arbitrary many terms provided the distribution \( F \) is smooth enough.

We conclude this introduction by noting sufficient conditions for a distribution to be subexponential have been obtained by several authors. Simple sufficient conditions in terms of the hazard
function \(- \log F\) are given in Pitman (1980), Teugels (1975) and Klüppelberg (1988). Another sufficient condition on \(F\) is given by Goldie (1978). Finally, Goldie and Klüppelberg (1998) and Goldie and Resnick (1988) give sufficient conditions for a distribution to be both subexponential and in the domain of max-attraction of the Gumbel distribution.

The paper is organized as follows. The next section contains our main results. A brief illustration of these results is given in section 3, mostly to show that the main discussion in Barbe and McCormick (2005) carries over to the setting of the current paper. The proofs of the main results are in the last section.

2. Main results. In order to state our asymptotic expansions, we need first to discuss more precisely the class of distribution functions which we will use, and some algebraic tools which will allow us to both state and compute these expansions. This will be done in the next two sub-sections. Our main results are presented in the third subsection.

Results for some specific random weights require extra developments and will be reported elsewhere.

2.1. Hazard rate and smoothly varying functions. Consider for the time being a distribution function \(F\) supported on the nonnegative half-line. Assuming it exists ultimately, the hazard rate \(h = F' / F\) will be of primary importance. Our analysis will show that how the hazard rate function compares with \(t^{-1} \log t\) asymptotically determines the form of the asymptotic expansion for tail areas.

Writing the tail function as 

\[
F(t) = F(t_0) \exp \left( - \int_{t_0}^t h(u) \, du \right),
\]

we see that if \(h(t) \sim \alpha / t\) then \(F\) is regularly varying with index \(\alpha\). On the other hand, if \(\lim_{t \to \infty} h(t) = \alpha\), then \(F(t) = e^{-\alpha(1+o(1))}\) has a behavior close to that of an exponential distribution. Thus, in order to study tail area behavior for weighted convolutions in the lighter than regularly varying case but heavier than exponential case, we are led to consider hazard rates which satisfy

\[
h \text{ is regularly varying,} \\
\lim_{t \to \infty} h(t) = +\infty \quad \text{and} \quad \lim_{t \to \infty} h(t) = 0. \tag{2.1.1}
\]
All the distribution functions satisfying assumption (2.1.1) are rapidly varying; indeed, writing Id for the identity function on the real line, for any $M$, the hazard rate is ultimately more than $M/Id$. Therefore, for $t$ large enough and positive $a$ less than 1, we have

$$\frac{\overline{F}(t)}{\overline{F}(at)} = \exp\left(-\int_{at}^{t} h(u) \, du\right) \leq \exp\left(-\int_{at}^{t} M/u \, du\right) = a^{M},$$

showing that $\lim_{t \to \infty} \overline{F}(t)/\overline{F}(at) = 0$. It follows that if $0 < a < b$ then $\overline{M}_a F = o(\overline{M}_b F)$.

The representation of distribution functions in terms of their hazard rate is closely related to the representation for the class $\Gamma$ of De Haan (1970); see Bingham, Goldie and Teugels (1989, §3.10). In particular, if (2.1.1) holds, then $\lim_{t \to \infty} \overline{F}(t+x/h(t))/\overline{F}(t) = e^{-x}$ and $F$ belongs to the domain of max-attraction of the Gumbel distribution (see Bingham, Goldie, Teugels, 1989, Theorem 8.13.4). Moreover, if the index of regular variation of the hazard rate is negative, then Pitman’s (1980) criterion implies that $F$ is subexponential. Consequently, if the hazard rate satisfies (2.1.1) and has a negative index of regular variation, then the corresponding distribution function satisfies the assumptions of Proposition 1.3 in Davis and Resnick (1988), and (1.2) holds provided (1.1) holds as well.

For most linear time series models considered in practice, the sequence has an infinite order moving average representation (see Brockwell and Davis, 1987, Theorem 3.1.3). In that setting, $F$ is the innovation distribution, and it is often unrealistic to assume that it is supported on the nonnegative half-line. A more common assumption then, is that of tail balance, asserting that $\overline{F}/\overline{M}_{-1} F$ has a positive finite limit at infinity. This assumption is not necessary for our results to hold, and some possible variations should be clear; but it makes results easier to state, and so we will use it. We then write $h_-$ the hazard rate of $\overline{M}_{-1} F$ when it exists. We then have the representation for the lower tail of $F$ given by

$$F(-t) = F(-t_0) \exp\left(-\int_{t_0}^{t} h_-(u) \, du\right),$$

for $t$ at least $t_0$. Note that if the distribution is tail balanced, then $\int_{t_0}^{t} (h - h_-)(u) \, du$ has a limit at infinity. This implies that if both $h$ and $h_-$ are regularly varying with index greater than $-1$, then they are asymptotically equivalent.
Throughout this paper we adopt, without further mention, the following convention.

**Convention.** If a distribution function $F$ is tail balanced and its hazard rate $h$ satisfies a property, then $h$ satisfies the same property. To emphasize this convention, we say that $F$ is strongly tail balanced.

For instance, if the property is 'h has a continuous $m$-th derivative', we mean that $h$ has also a continuous $m$-th derivative.

Assuming that the hazard rate is regularly varying is not always sufficient to obtain higher order results, in particular because higher order derivatives of the hazard rate will be involved. This motivates the following definition, introduced in Barbe and McCormick (2005). It somewhat extends that of a smoothly varying function as presented in Bingham, Goldie and Teugels (1989).

**Definition.** A function $h$ is smoothly varying of index $\alpha$ and order $m$ if it is ultimately $m$-times continuously differentiable and the $m$-th derivative $h^{(m)}$ is regularly varying of index $\alpha - m$. We write $\text{SR}_{\alpha,m}$ the set of all such functions.

Recall that $\text{Id}$ is the identity function on the real line. For any real number $\alpha$ and any nonnegative integer $m$, we write $(\alpha)_m$ for $\alpha(\alpha - 1) \cdots (\alpha - m + 1)$. Note that if $h$ is smoothly varying of index $\alpha$ and order $m$, then for any nonnegative integer $k$ at most $m$, the asymptotic equivalence $h^{(k)} \sim (\alpha)_k h/\text{Id}^k$ holds.

### 2.2. Laplace characters.

To effectively calculate asymptotic expansions, we need to recall the algebraic method introduced in Barbe and McCormick (2005) in their analysis of tail areas for linear and related processes based on innovations with regularly varying tails. We recall the terminology and concepts from that paper for which we shall have use. We write $D$ the derivation operator, that is, if $g$ is a differentiable function, $Dg$ is its derivative. We write $D^0$ the identity operator, and $D^i = DD^{i-1}$ for any positive integer $i$. For a distribution function $F$, we write $\mu_{F,i}$ its $i$-th moment.

Because they will appear in our asymptotic expansions, the basic algebraic objects that we need are the Laplace characters.
**Definition.** (Barbe and McCormick, 2005). Let $F$ be a distribution function having at least $m$ moments. Its Laplace character of order $m$ is the differential operator

$$L_{F,m} = \sum_{0 \leq i \leq m} \frac{(-1)^i}{i!} \mu_{F,i} D^i.$$

The algebraic role of these operators derives from their property as morphisms from the convolution semi-group of distribution functions with at least $m$ moments to the space of differential operators modulo the ideal generated by $D^{m+1}$. More specifically define a bilinear product on (equivalence classes of) differential operators modulo the ideal generated by $D^{m+1}$, denoted $\circ$, by

$$D^i \circ D^j = \begin{cases} D^{i+j} & \text{if } i + j \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

This is just an explicit expression for composition of differential operators modulo the ideal generated by $D^{m+1}$. Then, it is a simple matter, involving only the definition of convolution and the binomial formula, to show the useful relation

$$L_{K \ast H,m} = L_{K,m} \circ L_{H,m}.$$

We refer to Barbe and McCormick (2005) for a more extensive discussion of Laplace characters and their practical use.

**2.3. Asymptotic expansions.** To write concise formulas, we need to introduce a bit more specialized notation, $\nabla$. This symbol is used to delete a factor from a convolution product. More precisely, if $H = K_1 \ast K_2$ is a convolution product of distributions $K_1$ and $K_2$, then $H \nabla K_2 = K_1$. In particular $G \nabla M \ast F$ is the distribution of $\sum_{j \in \mathbb{Z} \setminus \{i\}} c_j X_j$.

To allow weights and random variables to be negative and still write concise statements, we introduce a partial ordering on real numbers, reflecting some form of tail behavior. If $a$ and $b$ are two real numbers, we write $a \prec b$ if $M_a F = o(M_b F)$. If there exists a positive $\epsilon$ such that $\epsilon M_a F \leq M_b F \leq \epsilon^{-1} M_a F$ ultimately, then we say that $a$ is equivalent to $b$ and write $a \equiv b$; provided the denominator does not vanish ultimately, this means that the ratio $M_a F / M_b F$ is ultimately bounded away from 0 and infinity. We also write $a \preceq b$ if
and only if \( a < b \) or \( a \equiv b \). Naturally, we define \( a > b \) as \( b < a \) and \( a \succeq b \) as \( b \preceq a \).

Since \( F \) is rapidly varying, if both \( a \) and \( b \) are positive, then \( a < b \) if \( a < b \), while \( a \equiv b \) if and only if \( a = b \). When \( a \) and \( b \) have different signs, the ordering introduced depends on how both tails of \( F \) compare. In particular, if \( F \) vanishes in a neighborhood of \( -\infty \), then \( a \equiv 0 \) whenever \( a \) is negative, and \( \preceq \) defines a total ordering on the real numbers. Note that the ordering is also total when \( F \) is tail-balanced or vanishes in a neighborhood of \( -\infty \).

We use the notation \((c_i)_{i \in \mathbb{N}^*}\) to represent the sequence \((c_i)_{i \in \mathbb{Z}}\) in nonincreasing order and without multiplicity; by the latter, we mean that \( c_i \) is strictly decreasing for the order, except in the case that it eventually reaches the minimal element 0 when it stays identically constant 0 from that point onward. For instance, if \( c_1 = c_2 = 1 \) while \( c_3 = 1/2 \) and \( c_4 = c_5 = 1/3 \) and all the other \( c_i \)'s are nonnegative less than 1/3, then \( c_{(1)} = 1 \), while \( c_{(2)} = 1/2 \) and \( c_{(3)} = 1/3 \) and so on. Note that any sequence \((c_i)_{i \in \mathbb{Z}}\) with limit 0 at infinity can be put in nonincreasing order indexed by the positive integers. We say that an element \( c_i \) of the sequence is maximal if it is equivalent to \( c_{(1)} \).

Since we will not restrict the sequence \((c_i)_{i \in \mathbb{Z}}\) to be nonnegative, we need to replace condition (1.1) by

\[
\sum_{i \in \mathbb{Z}} |c_i|^\delta \text{ is finite for some positive } \delta \text{ less than } 1.
\] (2.3.1)

The following convention will be convenient.

**Convention.** We say that the 'standard conditions' hold if (2.3.1) holds and either \( F \) vanishes on a neighborhood of \(-\infty\) or is strongly tail balanced.

Minor changes in our proof show that the results of this paper remain true if instead of requiring \( F \) to vanish in a neighborhood of \(-\infty\) we impose that \( \int_{-\infty}^0 x^m dF(x) \) is finite whenever we use the Laplace character of order \( m \) of \( F \) in these results.

Our first result covers most light subexponential distributions used in applications, that is those for which \(- \log F\) is regularly varying of index positive and less than 1. In particular, it includes Weibull distributions; however, the log-normal distribution is excluded.
Theorem 2.3.1. Assume that the standard conditions hold and
(i) \( h(t) \gg t^{-1} \log t \),
(ii) \( h \) is smoothly varying of negative index and order \( m \),
Then
\[
G = \sum_{i: c_i \equiv c(1)} L_{GZ_{c(i)}F, m M_{c_i} F} + o(h^m M_{c(1)} F).
\]

To convey the meaning of this formula, let us consider the case when the sequence \( c_i \) is nonnegative, indexed by the positive integers — that is \( c_i \) vanishes if \( i \) is negative — decreasing, with \( c_1 = 1 \).
Writing \( H = gZ_{g}F \), that is the distribution of \( \sum_{i \geq 2} c_i X_i \), the formula in Theorem 2.3.1 asserts that for any \( k \) at most \( m \),
\[
G = F - H_1 F' + \frac{H_2}{2} F'' + \cdots + (-1)^k \frac{H_k}{k!} F^{(k)} + o(h^k F).
\]

More generally, for a sequence \( c_i \) of arbitrary sign with maximal element \( c(1) \equiv 1 \), and assuming that \( F \) is tail-balanced, Theorem 2.3.1 yields the following. Let \( \nu_+ \) (respectively \( \nu_- \)) be the number of maximal positive (respectively negative) \( c_i \)'s which are maximal for the \( \prec \) ordering. Since \( F \) and \( M_{-1} F \) are rapidly varying and asymptotically of the same order, the maximal positive \( c_i \)'s, if they exist, are all equal, and equal to minus the maximal negative \( c_i \)'s if they exist. Consequently, Theorem 2.3.1 asserts that for \( k \) at most \( m \),
\[
G = \nu_+ \sum_{0 \leq i \leq k} \frac{(-1)^i}{i!} \mu_{GZ_{c(i)}F, M_{c(i)} F} F' + \cdots + \frac{(-1)^k}{k!} \mu_{GZ_{c(i)}F, M_{c(i)} F} F^{(k)} + o(h^k F).
\]

Qualitatively, this result is very much in agreement with Davis and Resnick’s result, showing that the largest weight drives the asymptotic behavior of \( G \).

For practical purposes, Theorem 2.3.1 is applicable over a broad range of Weibull-like subexponential distributions. However, its scope does not include the useful subexponential distribution given by the log-normal distribution. Moreover, Theorem 2.3.1 stands in contrast with the situation where \( F \) is regularly varying, where each weight contributes to a term of the asymptotic expansion (Barbe and
McCormick, 2005). The purpose of the remainder of this section is to show the delicate transition between regularly varying tails and those covered by Theorem 2.3.1.

Theorem 2.3.1 contains the assumption that the hazard rate is asymptotically much larger than $\text{Id}^{-1}\log t$, excluding distributions with tail $e^{-(\log t)^a}$ when $a$ is greater than 1 and at most 2. The next result covers this class of distributions with $a$ less than 2 (note 2 is not allowed yet!). It shows that this narrow class of distributions, as far as the asymptotic behavior of weighted convolution goes, retains some features of regularly varying tails, yet has rapidly varying tails.

**Theorem 2.3.2.** Assume that the standard conditions and (2.1.1) hold, and

(i) $h(t) = o(t^{-1}\log t)$ as $t$ tends to infinity

(ii) $\limsup_{t \to \infty} t h(t)^2/h(1/h(t)) < \infty$, \hspace{1cm} (2.3.2)

Then, for any positive integer $m$,

$$\overline{G} = \sum_{c_i \geq c(m)} \overline{M}_{c_i} F + o(\overline{M}_{c(m)} F).$$

Setting $\eta(t) = th(t)$, condition (2.3.2) asserts that the supremum limit of $\eta(t)/\eta(1/h(t))$ is finite. This is the case if $\eta$ is of order $\log^a t$.

Note that under the assumptions of Theorem 2.3.2, the index of regular variation of the hazard rate is $-1$. In the conclusion of this theorem, the largest term of the expansion is driven by the largest coefficient $c_i$, the second order term is driven by the second largest coefficient, and so on. Although formally the expansion has a similar form as a first order equivalence in the case of distributions with regularly varying tails, namely $\sum_{i \in \mathbb{Z}} \overline{M}_{c_i} F$, here, because the tails are rapidly varying, nonequivalent weights produce asymptotically distinct contributions to the expansion. When $F$ is rapidly varying, the $\overline{M}_{c_i} F$ are ordered by tail dominance, and this yields formally the formula presented in Theorem 2.3.2.

Our last result yields some very unexpected behavior and fills the gap between Theorems 2.3.1 and 2.3.2. It covers the intermediate range where the hazard rate is exactly of order $t^{-1}\log t$ and therefore $\log F$ is exactly of order $-\log^2 t$. In particular, it covers a distribution often used in practice: the log-normal. A close look at the conclusion of that theorem (particularly examining limiting cases as $\lambda$ tends to 0 or infinity) reveals a natural progression in the terms involved in the
expansions in Theorems 2.3.1 and 2.3.2 with only terms equivalent to \( c_{(1)} \) in the former result and as many as desired in the latter result (viz. \( \lambda = \infty \) and \( \lambda = 0 \)).

In this result \( \lfloor \cdot \rfloor \) denotes the integer part, that is the largest integer at most equal to the argument.

**Theorem 2.3.3.** Assume that the standard conditions hold and
(i) \( h(t) \sim \lambda t^{-1} \log t \) for some positive \( \lambda \),
(ii) \( h \) is smoothly varying (necessarily of index \( -1 \)) and order \( m \).

Then, for any positive integer \( k \) at most \( m \),

\[
\bar{G} = \sum_{i \in \mathbb{Z}} \mathbb{1}\{ c_i \geq c_{(1)} e^{-k/\lambda} \} \mathcal{L}_{G \mathbf{M}_{c_i} F, k + \lfloor \lambda \log(|c_i/c_{(1)}) \rfloor} \mathbf{M}_{c_i} F
+ o(h^k \overline{\mathbf{M}_{c_{(1)}} F}).
\]

With regard to this theorem, we note that the presence of the integer part serves to indicate that for small \( c_i \), the order of the Laplace character may be reduced. The indicator function ensures that no negative order appears as an order to a Laplace character. Since differentiation and multiplication operators change the asymptotic order of their argument in different ways, it may happen, for example, that \( \overline{\mathbf{M}_{c_{(2)}} F} \) is of higher order than \( D \overline{\mathbf{M}_{c_{(1)}} F} \) yielding a case where the second largest coefficient has dominance over the largest coefficient in determining the second order term. This will be illustrated in the next section.

### 3. Examples.

The purpose of this section is to illustrate briefly our three results, and show that, as was done more lengthily in Barbe and McCormick (2005) for regularly varying tails, the asymptotic scale in which an expansion is written influences which terms of our results should be kept.

The following four examples illustrate successively Theorems 2.3.2, 2.3.3 and 2.3.1. Throughout these examples, we consider a sequence of weights such that \( c_i \) is 0 if \( i \) is nonpositive and the sequence \( c_i \) is nonnegative and strictly decreasing for \( i \) positive. We also take \( c_1 \) to be 1. Finally, the summability condition (2.3.1) is assumed to hold. In the fifth example, we derive a three terms expansion for the marginal distribution of some linear processes.

**Example 1.** Define the functions

\[
e_1(t) = \exp(-\log^{3/2} t) \quad \text{and} \quad e_2(t) = e_1(t) \exp(-\log^{1/4} t),
\]
so that \( e_2 = o(e_1) \) at infinity. Consider a distribution function \( F \) such that \( F = e_1 + e_2 \) ultimately. Since \( F \sim e_1 \), Davis and Resnick (1988) shows that \( \bar{G} \sim F \sim e_1 \), because our sequence \( c_i \) is decreasing with \( c_1 = 1 \). What is the second order?

Note that for any positive number \( a \) less than 1,

\[
\log e_1(t/a) - \log e_2(t) \sim \frac{3}{2}(\log t)^{1/2} \log a.
\]

Consequently, \( M_a e_1 = o(e_2) \). Hence, for any \( i \) at least 2, we have \( M_{c_i} F = o(e_2) \). It follows from Theorem 2.3.2 that the relation \( F = e_1 + e_2 \), holding ultimately, shows that \( F \) provides not just a first order equivalence to \( F \) but actually a second order result in that \( \bar{G} = e_1 + e_2 + o(e_2) \).

**Example 2.** Let \( \theta \) be a positive real number. Take \( h(t) = 2\theta t^{-1} \log t \), so that \( F = \exp(-\theta \log^2 t) \) on \([1, \infty)\). Define \( \gamma = \sum_{i \neq 1} c_i \). Note \( \gamma \) is nonnegative and finite by our assumptions. We see that \( \mu_{G^\# C_i F, i} = \gamma \mu_{F, i} \). Since \( c_1 = 1 \), Theorem 2.3.3 shows that a two terms expansion is either

- **case 1**: \( F - \gamma \mu_{F, 1} F' \) if \( M_{c_2} F = o(F') \),
- **case 2**: \( F + M_{c_2} F ' \) if \( M_{c_2} F \gg F' \),
- **case 3**: \( F + M_{c_2} F - \gamma \mu_{F, 1} F' \) if \( M_{c_2} F \asymp F' \)

We are in the first case if \( c_2 \leq e^{-1/2\theta} \) and in case 2 otherwise. Case 3 cannot occur for this specific distribution. But case 3 may occur if, for instance, \( F(t) = \exp(-\log^2 t + \log \log t) \) ultimately! Note also that for the log-normal, \( h(t) = t^{-1} \log t + t^{-1} \), corresponding to \( \theta = 1/2 \).

**Example 3.** Assume again that \( c_1 = 1 \) and moreover that \( k_1 = 1 \). We write \( \mu_i \) for \( \mu_{G^\# C_i F, i} \). We suppose that the assumptions of Theorem 2.3.1 are satisfied for \( m = 3 \). The conclusion of Theorem 2.3.1 yields a seemingly four terms expansion,

\[
\bar{G} = F - \mu_1 F' + \frac{\mu_2}{2} F'' - \frac{\mu_3}{6} F''' + o(h^3 F) \quad \text{(3.0.1)}
\]

However, this formula hides the true number of significant terms in the expansion. To clarify this remark, we further investigate the meaning of (3.0.1) and calculate

\[
\begin{align*}
F' &= -h F, \\
F'' &= (-h' + h^2) F, \\
F''' &= (-h'' + 3h'h - h^3) F.
\end{align*}
\]
It is then natural to seek an expansion in the scale containing all the functions involved in these derivatives, that is

\[ \bar{F}, \ h\bar{F}, \ h^2\bar{F}, \ h^3\bar{F}, \ h'\bar{F}, \ hh'\bar{F}, \ h''\bar{F} \]

which are not \( o(h^3\bar{F}) \). Then, for example, a three terms expansion in that scale can be obtained from (3.0.1) and is given by

\[ \bar{G} = F + \mu_1 hF + \frac{\mu_2}{2} h^2F + o(h^3F). \]

In similar fashion, from (3.0.1), using the expression for the derivatives of \( F \), we calculate that a four terms expansion, where now all terms in the formulas below are significant, is either

\[ \bar{G} = F + \mu_1 hF + \frac{\mu_2}{2} h^2F - \frac{\mu_2}{2} h'F + o(h^3F) \]

or

\[ \bar{G} = F + \mu_1 hF + \frac{\mu_2}{2} h^2F - \frac{\mu_3}{6} h^3F + o(h^3F), \]

or

\[ \bar{G} = F + \mu_1 hF + \frac{\mu_2}{2} h^2F - \frac{\mu_2}{2} h'F - \frac{\mu_3}{6} h^3F + o(h^3F), \]

the first one occurring if \( h^3 = o(h') \), the second one occurring if \( h' = o(h^3) \) and the third one occurring if both \( h^3 \) and \( h' \) are of the same asymptotic order. Note that if the index of regular variation of \(- \log F\) is less than 1/2 then we are in the first case, while if it is more than 1/2 we are in the second case. When this index is equal to 1/2 we can be in any of the three cases a priori.

**Example 4.** The purpose of this example is to show that some cancellation may occur, analogous to that observed for regularly varying tails in Barbe and McCormick (2005).

Consider a distribution for which ultimately

\[ \bar{F}(t) = \exp(-\log^{3/2} t) - \exp\left(-\log^{3/2}(2t)\right). \]

Since

\[ \log^{3/2}(2t) = \log^{3/2} t + \frac{3}{2} \log^{1/2} t \log 2 \left(1 + o(1)\right), \]

we see that

\[ \bar{F}(t) \sim \exp(-\log^{3/2} t). \]
Assume that $c_1 = 1$ and $c_2 = 1/2$. Theorem 2.3.2 with $m = 2$ yields 
$$
\overline{G}(t) = \overline{F}(t) + \overline{F}(2t) + o(\overline{F}(2t)),
$$
apparently giving a two terms expansion. However, note the cancellation between the second order term of $\overline{F}(t)$ and the leading term of $\overline{F}(2t)$, so that the above expansion gives only the one term expansion
$$
\overline{G}(t) = \exp(-\log^{3/2} t) + o(\overline{F}(2t)).
$$

It is clear from this example that a construction similar to that done at the end of section 3.2 in Barbe and McCormick (2005) can be made: regardless whether we apply any of the three theorems of the previous section, for any fixed $m$, we can find a distribution and a sequence of weights such that the asymptotic expansion of $\overline{G}$ in the theorem gives in fact a one term expansion, because of cancellations. Hence, we see once more that obtaining a second order term for an infinite convolution is not that of obtaining a second order expansion, but that of obtaining arbitrarily accurate expansions.

**Example 5.** Consider a stationary linear process with innovations having a symmetric distribution $F$. The stationary distribution, $G$, is that of $\sum_{i \in \mathbb{Z}} c_i X_i$ for some proper constants $c_i$. Since $F$ is symmetric, the odd moments of $G$ vanish.

We assume that the sequence $(c_i)$ has a unique maximal element in the $\prec$ ordering, equal to 1 say.

To calculate the first two even moments, for any positive integer $k$, write $C_k = \sum_{i: c_i \prec 1} c_i^k$. Then
$$
\mu_{G \ast M_{c(1)} F, 2} = \mu_{G \ast F, 2} = C_2 \mu_{F, 2}
$$
$$
\mu_{G \ast M_{c(1)} F, 4} = \mu_{G \ast F, 4} = 3(C_2^2 - C_4^2) \mu_{F, 2}^2 + C_4 \mu_{F, 4}.
$$

If $F$ obeys the assumptions of Theorem 2.3.1 with $m = 6$ say, we have
$$
\overline{G} = \overline{F} + \frac{C_2^*}{2} \mu_{F, 2} \overline{F}(2) + \frac{3(C_2^* - C_4^*) \mu_{F, 2}^2 + C_4 \mu_{F, 4}}{12} \overline{F}(4) + O(\overline{F}(6)).
$$

This formula applies for instance to symmetric Weibull type densities $\alpha e^{-|x|^\alpha} / 2\Gamma(1/\alpha)$. It applies also to densities which are ultimately equal to $x^\beta e^{-x^\gamma}$; compare with Rootzen (1986) who obtained first order results.
4. Proofs. The proofs are based on the writing of the convolutions in terms of operators modelled after those introduced in Barbe and McCormick (2005). For every distribution function $K$, define the operator

$$T_K f(t) = \int_{-\infty}^{t/2} f(t - x) \, dK(x).$$

Then,

$$\mathcal{H} \ast K = T_K \mathcal{H} + T_H K + M_2(\mathcal{H} K). \quad (4.0.1)$$

Throughout the proofs, we write $F_i$ for $M_{c_i} F$. Without loss of generality, we assume that the index set of the sequence $(c_i)$ is $\mathbb{N}$ and not $\mathbb{Z}$.

Define $G_n = F_1 \ast \cdots \ast F_n$. By induction, formula (4.0.1) yields

$$G_n = T_{G_{n-1}} F_n + \sum_{2 \leq k \leq n} T_{F_n} \cdots T_{F_{n-k+2}} T_{G_{n-k}} F_{n-k+1} + M_2(G_{n-1} F_n) + \sum_{2 \leq k \leq n} T_{F_n} \cdots T_{F_{n-k+2}} M_2(G_{n-k} F_{n-k+1}). \quad (4.0.2)$$

Note. Throughout this section, we assume that (2.1.1) holds, possibly without mentioning it.

4.1. Some lemmas. Preliminary lemmas are presented under not necessarily sharpest conditions. Use of these lemmas appears mainly in the proof of Theorem 2.3.1. Our first lemma gives the exact order of derivatives of $\mathcal{F}$.

**Lemma 4.1.1.** Assume that $h$ is in $SR_{\alpha-1,m}$ for some $\alpha$ nonnegative and less than 1. If $\alpha$ vanishes, assume further that $th(t)$ tends to infinity at infinity. For any nonnegative integer $k$ at most $m$,

$$\mathcal{F}^{(k)} \sim (-1)^k h^k \mathcal{F}. \quad (4.1.1)$$

**Proof.** Using the Faà di Bruno formula (see Roman, 1980, p.809 for the formulation that we are using) $\mathcal{F}^{(k)}$ equals

$$\sum_{1 \leq i \leq k} \frac{1}{i!} \sum_{n_1 + \cdots + n_i = k \atop n_1, \ldots, n_i \geq 1} \frac{k!}{n_1! \cdots n_i!} h^{(n_1-1)} \cdots h^{(n_i-1)} (-1)^i \mathcal{F}. \quad (4.1.2)$$
Since $h$ is smoothly varying of order $m$ and $k$ is at most $m$, for any index $n_j$ involved in the sum, $h^{(n_j-1)} \sim (\alpha - 1)n_j^{-1}h/\text{Id}^{n_j-1}$. As usual in the theory of regular variation, when $(\alpha - 1)n_j-1$ vanishes, such asymptotic equivalence must be read as $h^{(n_j-1)} = o(h/\text{Id}^{n_j-1})$. Thus,

$$h^{(n_1-1)} \cdots h^{(n_i-1)} \sim (\alpha - 1)n_1^{-1} \cdots (\alpha - 1)n_i^{-1}h^i/\text{Id}^{k-i},$$

where we use $n_1 + \cdots + n_i = k$. The function $h^i/\text{Id}^{k-i}$, that is $(\text{Id}h)^i/\text{Id}^k$, is regularly varying of index $i\alpha - k$. Among such terms, the one with the largest order is obtained when $i = k$, even if $\alpha$ vanishes, for in this case $\text{Id}h$ tends to infinity at infinity. This forces all the $n_i$’s to be 1 and yields the result.

The next lemma is a Potter type bound which we have already stated and proved in section 2.

**Lemma 4.1.2.** Let $M$ be an arbitrary positive number. There exists $t_1$ such that for any $t$ at least $t_1$ and any $\lambda$ at least 1,

$$\frac{F(\lambda t)}{F(t)} \leq \lambda^{-M}.$$

Our next lemma shows that for $t$ large enough, the function $F(t-x)F(x)$ is nonincreasing on some interval $[M, t/2]$, where $M$ does not depend on $t$.

**Lemma 4.1.3.** Assume that $h$ is in $\text{SR}_{\alpha-1,m}$ for some $m$ at least 1. There exists some positive $t_1$ and $M$ such that for any $t$ at least $t_1$, the function $x \mapsto F(t-x)F(x)$ is nonincreasing in $[M, t/2]$.

**Proof.** Let $\delta$ be a positive real number such that $\alpha - 1 + \delta$ is negative. Lemma 2.2.4 in Barbe and McCormick (2005) shows that there exist $t_1$ and $M$ such that for any $t$ at least $t_1$ and $x$ in $[M, t/2]$,

$$\frac{1}{h(x)} \frac{d}{dx} \log(F(t-x)F(x)) = \left(\frac{h(t-x)}{h(x)} - 1\right) \leq \left(\left(\frac{t}{x} - 1\right)^{\alpha-1+\delta} - 1\right).$$

This upper bound is nonpositive in the given range of $x$, implying the result.
The following result will be instrumental.

**Lemma 4.1.4.** Let $K$ be a cumulative distribution function on the nonnegative half-line, such that for some $M$ positive, $\overline{K} \leq M\overline{F}$. For any interval $[a, b]$ and any nonnegative function $f$ continuous and nondecreasing on $[a, b]$,

$$\int_{[a,b]} f \, dK \leq M\left(\int_{[a,b]} f \, dF + f\overline{F}(b)\right).$$

We will sometimes use this lemma in its limiting form, as $b$ tends to infinity. In particular, if both $F$ and $K$ are supported by the nonnegative half-line, it implies that $\mu_{K,i} \leq M\mu_{F,i}$ for any positive integer $i$.

**Proof.** Assume that $f$ is differentiable. An integration by parts yields

$$\int_{[a,b]} f \, dK = f\overline{K}(a) - f\overline{K}(b) + \int_{[a,b]} f'\overline{K}(x) \, dx. \quad (4.1.1)$$

Hence, the left hand side of (4.1.1) is at most $M$ times

$$f\overline{F}(a) + \int_{[a,b]} f'\overline{F}(x) \, dx.$$ 

But taking $K$ to be $F$ in (4.1.1) yields that this sum is at most

$$\int_{[a,b]} f \, dF + f\overline{F}(b).$$

This prove the lemma when $f$ is differentiable. When it is only continuous, use that the differentiable functions are dense in the continuous ones in $(L^1[a, b], \, dK)$. \hfill \blacksquare

Combining the two previous lemmas yields the following asymptotic rate of decay on the tail of moment integrals for a class of distributions.

**Lemma 4.1.5.** Let $K$ be a distribution function such that $\overline{K} = O(\overline{F})$. Assume that the hazard rate function $h$ of $F$ satisfies (2.1.1). For any nonnegative integers $j$ and $n$,

$$\int_{1/2}^{\infty} x^j \, dK(x) = o(t^{-n}).$$
Proof. There exists some positive $\epsilon$ such that $\epsilon K \leq F$ ultimately. Using Lemma 4.1.4, we have for $t$ large enough

$$\epsilon \int_t^\infty x^j \, dK(x) \leq \int_t^\infty x^j \, dF(x)$$

$$= t^j F(t) + j t^j \int_1^\infty s^{j-1} F(ts) \, ds$$

$$\leq t^j F(t) + j t^j F(t) \int_1^\infty s^{-2} \, ds .$$

This bound is rapidly varying, hence it is $o(t^{-n})$ for any $n$.  

4.2. Proof of Theorems 2.3.1 and 2.3.3. In this subsection, we first prove Theorems 2.3.1 and 2.3.3 with the extra assumption that the weights $c_i$’s are nonnegative. We will remove this assumption afterwards. So, from now on, until stated otherwise, we assume that the weights are nonnegative.

We start by proving a lemma which allows us to neglect terms arising from the multiplication operator in formula (4.0.2).

Lemma 4.2.1. Assume that the hazard rate is regularly varying of negative index and that $\lim \inf_{t \to \infty} th(t) / \log t$ is positive (possibly infinite). Then, for any $k$ nonnegative, $M_2 F^2 = o(\log h)$.

Proof. Using the representation for $F$, for any $t$ at least $t_0$,

$$M_2 F^2 (t) = F(t_0) \exp \left( - \int_{t_0}^{t/2} h(u) \, du + \int_{t/2}^t h(u) \, du - k \log h(t) \right) .$$

Let $\epsilon$ be a positive real number. Using that $h$ is regularly varying of index $\alpha - 1$ for some $\alpha$ less than 1, we have ultimately

$$- \int_{t/2}^t h(u) \, du$$

$$= -th(t) \left( \int_{1/2}^{1/2} v^{\alpha-1} \, dv - \int_{1/2}^1 v^{\alpha-1} \, dv \right) (1 + o(1))$$

The right hand side above is less than some $-ath(t)$ for some positive $a$ by taking $\epsilon$ small enough.

If $\alpha$ is positive, and less than 1, then the result is clear since log $h(t) = o(th(t))$.  

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If $\alpha$ vanishes, $a$ can be taken as large as desired by taking $\epsilon$ small enough. By assumption $th(t)$ is ultimately at least like some positive multiple of $\log t$, implying that $\log h(t)$ grows at most as a multiple of $\log t$; moreover, $th(t)$ is is ultimately at least some positive multiple of $\log t$. This proves the lemma.

For any positive real number $M$, we write $B(F, M)$ the set of all distribution functions $K$ such that both $K$ and $M - 1$ are less than $M_1$ on the nonnegative half-line.

Our next lemma shows that remainder terms stay negligible under the action of some $T_K$ operators.

**Lemma 4.2.2.** Assume that $h$ is regularly varying with negative index of regular variation and $F$ is subexponential. Let $K$ be a distribution function in some $B(F, M)$. If $g$ is any function which is $o(h^m_F)$ then,$$T_K g = o(h^m_F).$$

**Proof.** Let $\epsilon$ be a positive real number. Let $t$ be large enough so that $|g| \leq \epsilon h^m F$ on $[t/2, \infty)$. Theorem 1.5.3 in Bingham, Goldie and Teugels (1987) implies that $h(t)$ is asymptotically equivalent to $\sup_{s \geq t} h(s)$. For $t$ large enough, we obtain for some positive $\rho$

$$|T_K g(t)| \leq \epsilon \int_{-\infty}^{t/2} (h^m F)(t - x) \, dK(x) \leq \epsilon 2^\rho h^m(t) \int_{-\infty}^{t/2} F(t - x) \, dK(x).$$

Lemma 4.1.4 implies that $\int_{-\infty}^{t/2} F(t - x) \, dK(x)$ is less than a constant times $T_F F(t) + M_2 F^2(t)$, which is at most a constant times $F^2(t)$, while $\int_{-\infty}^{0} F(t - x) \, dK(x)$ is at most $F(t)$. The conclusion follows using subexponentiality of $F$.

The next lemma is at the heart of both Theorems 2.3.1 and 2.3.3. We will use it to approximate of some $T$ operators by Laplace characters, when acting on derivatives of $F$. In the proof, we will use the absolute moments$$|\mu|_{K,i} = \int |x|^i \, dK(x).$$
Lemma 4.2.3. For any fixed integer \( p \) at most \( m \) and any positive \( M \),

\[
\lim_{\delta \to 0} \lim_{t \to \infty} \sup_{\delta/h(t)} \left| \frac{\int_{-\delta/h(t)}^{\delta/h(t)} F^{(p)}(t-x) \, dK(x) - L_{K,m-p} F^{(p)}(t)}{h^m F(t)} \right| = 0,
\]

where the supremum is taken over all distribution functions \( K \) belonging to \( B(F,M) \).

**Proof.** Taylor’s formula with remainder term asserts that

\[
F^{(p)}(t-x) = \sum_{0 \leq j \leq m-p-1} \frac{(-1)^j}{j!} x^j F^{(p+j)}(t) + (-1)^{m-p} \int_0^x \frac{y^{m-p-1}}{(m-p-1)!} F^{(m)}(t-x+y) \, dy.
\]

Integrating this equality between \(-\delta/h(t)\) and \(\delta/h(t)\) with respect to \(dK(x)\) shows that

\[
\left| \int_{-\delta/h(t)}^{\delta/h(t)} F^{(p)}(t-x) \, dK(x) - L_{K,m-p} F^{(p)}(t) \right|
\]

is at most

\[
\sum_{0 \leq j \leq m-p} \frac{|F^{(p+j)}(t)|}{j!} \int_{[-\delta/h(t),\delta/h(t)]} |x|^j \, dK(x) + \int_{-\delta/h(t)}^{\delta/h(t)} \left| \int_0^x \frac{y^{m-p-1}}{(m-p-1)!} (F^{(m)}(t-x+y) - F^{(m)}(t)) \, dy \right| \, dK(x).
\]

Combining Lemma 4.1.1 and the proof of 4.1.5, we see that for any positive \( \delta \),

\[
\lim_{t \to \infty} \sup_{K \in B(F,M)} \frac{|F^{(p+j)}(t)| \int_{[-\delta/h(t),\delta/h(t)]} |x|^j \, dK(x)}{h^m F(t)} = 0.
\]

Let \( \epsilon \) be a positive number. For any fixed \( \delta \), Lemma 4.1.1 and regular variation of \( h \) show that uniformly in \( z \) in \([-1,1]\) say,

\[
\frac{F^{(m)}(t-\delta z/h(t))}{F^{(m)}(t)} \text{ tends to } e^{\delta z}.\]

Therefore, we can find \( \delta \) such that, ultimately, for any \( z \) in \([-1,1]\),

\[
1 - \epsilon \leq \frac{F^{(m)}(t-\delta z/h(t))}{F^{(m)}(t)} \leq 1 + \epsilon.
\]
Consequently, ultimately and uniformly over \( B(F, M) \),

\[
\begin{align*}
\int_{-\delta/h(t)}^{\delta/h(t)} \int_0^x \frac{y^{m-p-1}}{(m-p-1)!} \left( \frac{F^{(m)}(t-x+y)}{F^{(m)}(t)} - 1 \right) dy \, dK(x) \\
&\leq \epsilon \int_{-\delta/h(t)}^{\delta/h(t)} \int_0^x \frac{y^{m-p-1}}{(m-p-1)!} dy \, dK(x) \\
&\leq \epsilon |\mu|_{K,m-p}.
\end{align*}
\]

Since \( \epsilon \) is arbitrary, Lemma 4.2.3 follows from Lemmas 4.1.1 and 4.1.4.

As announced, the preceding lemma yields an approximation of some \( T \) operator by Laplace characters, which we now state and prove.

**Lemma 4.2.4.** For any \( p \) at most \( m \),

\[
\lim_{t \to 0} \sup_{K \in B(F,M)} \left( \frac{T_K - L_{K,m}}{h^m F} \right)(t) = 0.
\]

**Proof.** Given Lemma 4.2.3, it suffices to show that for any positive \( \epsilon \) and any \( \delta \) small enough,

\[
\left| \int_{-\delta/h(t)}^{t/2} F^{(p)}(t-x) \, dK(x) \right| \leq \epsilon h^m F(t) \tag{4.2.1}
\]

and

\[
\left| \int_{-\infty}^{-\delta/h(t)} F^{(p)}(t-x) \, dK(x) \right| \leq \epsilon h^m F(t) \tag{4.2.2}
\]

ultimately and uniformly over \( B(F, M) \). Using Lemma 4.1.1 and convergence of the hazard rate to 0 at infinity, the left hand side of (4.2.1) is ultimately bounded by \( \int_{-\delta/h(t)}^{t/2} F(t-x) \, dK(x) \). By Lemma 4.1.4, this is less than

\[
M \int_{-\delta/h(t)}^{t/2} F(t-x) \, dF(x) + M F^2(t/2).
\]

For \( t \) large enough, we substitute \( dF(x) \) by \( F'(x) \, dx \) in this bound. Thus, Applying Lemmas 4.1.1, 4.1.3 and 4.2.1, this upper bound is ultimately at most

\[
MtF(t-\delta/h(t))F(\delta/h(t)) + o(h^m F(t)).
\]
This implies (4.2.1) because on one hand, $\mathcal{F}(t - \delta/h) \sim \mathcal{F}(t)e^{-\delta}$, and, on the other hand, $\mathcal{F}$ being rapidly varying, $\mathcal{F}(\delta/h) = o(h^p)$ for any positive $p$, and, in particular, $\text{Id}\mathcal{F}(\delta/h) = o(h^m)$.

To prove (4.2.2), we replace $\mathcal{F}(p)$ by $h^p\mathcal{F}$ using Lemma 4.1.1. Since $x$ is negative in the range of integration, we bound $h^m(t - x)$ by $\sup_{s \geq t} h^m(s)$, which in turn is equivalent to $h^m(t)$, and bound $\mathcal{F}(t - x)$ by $\mathcal{F}(t)$. Then (4.2.2) follows, and this completes the proof of Lemma 4.2.4.

Note that in Lemma 4.2.4, we could replace $\mathcal{F}$ by $M_a\mathcal{F}$ for any positive $a$, since the hazard rate of $M_a\mathcal{F}$ is $a^{-1}h(a\cdot)$ and has the same properties as $h$ as far as the proof of the lemma is concerned.

We now complete the proofs of both Theorems 2.3.3 and 2.3.1, assuming positivity of the weights. Without any loss of generality, we assume that $c(1)$ is 1. Note that in Theorem 2.3.3 the hazard rate is regularly varying of index $-1$. While, in Theorem 2.3.1, the hazard rate is regularly varying of index $\alpha - 1$ with $0 \leq \alpha < 1$.

Recall formula (4.0.2). Again, using Davis and Resnick's (1988) Proposition 1.3, $G_{n-k}\mathcal{F}_{n-k+1} = O(\mathcal{F})$. Thus, Lemma 4.2.1 implies

$$M_2(G_{n-k}\mathcal{F}_{n-k+1}) = o(h^m\mathcal{F}),$$

and by induction, using Lemma 4.2.2,

$$M_2(G_{n-k}\mathcal{F}_{n-k+1}) = o(h^m\mathcal{F}).$$

It follows that under either the assumptions of Theorem 2.3.3 or Theorem 2.3.1,

$$G_n = T_{G_{n-1}\mathcal{F}_{n}} + \sum_{2 \leq k \leq n} T_{F_n} \cdots T_{F_{n-k+2}} T_{G_{n-k}\mathcal{F}_{n-k+1}} + o(h^m\mathcal{F}).$$

Again, combining Davis and Resnick's (1988) Proposition 1.3 with Lemma 4.2.4, we see that

$$T_{G_{n-k}\mathcal{F}_{n-k+1}} = L_{G_{n-k},n}\mathcal{F}_{n-k+1} + o(h^m\mathcal{F}).$$

Any Laplace character applied to some $\mathcal{F}_i$ yields a linear combination of derivatives of $\mathcal{F}_i$. Therefore, using Lemmas 4.2.2 and 4.2.4
inductively, we see that
\[
G_n = \sum_{1 \leq k \leq n} L_{F_n,m} \circ \cdots \circ L_{F_{n-k+2},m} \circ L_{G_{n-k},m} F_{n-k+1} + o(h^m F)
\]
\[
= \sum_{1 \leq k \leq n} L_{G_{n-k+1},m} F_{n-k+1} + o(h^m F). \tag{4.2.3}
\]

Note in the first sum above the term indexed by \(k = 1\) is to be read as \(L_{G_{n-1},m} F_n\). Write \(H_n\) for the distribution function of \(\star_{i>n} M_{c_i} F\).

We claim that there exists \(n\) large enough so that \(H_n = o(h^m F)\).

Since our sequence \((c_i)\) is nonincreasing, Davis and Resnick’s (1988) Proposition 1.3 implies that \(H_n \asymp F_n + 1\) and
\[
\frac{M_{c_{n+1}} F}{h^m F}(t) = \exp \left( - \int_t^{t/c_{n+1}} h(u) du - m \log h(t) \right)
\]
\[
= \exp \left( - th(t) \frac{c_{n+1} - 1}{\alpha} (1 + o(1)) - m \log h(t) \right). \tag{4.2.4}
\]

Thus, our claim is now evident. Further, (4.2.4) shows that under the assumptions of Theorem 2.3.1, in order that \(H_n = o(h^m F)\), it suffices that \(n\) be chosen large enough so that \(c_{n+1} < 1\).

Next, note that \(G = G_n \star H_n\), so that
\[
\overline{G} = T_{H_n} \overline{G}_n + T_{G_n} \overline{F}_n + M_2(\overline{F}_n \overline{G}_n).
\]

Since \(\overline{F}_n = o(h^m F)\) and Davis and Resnick’s (1988) Proposition 1.3 asserts that \(\overline{G}_n \asymp F\), Lemmas 4.2.2 and 4.2.1 show that
\[
\overline{G} = T_{H_n} \overline{G}_n + o(h^m F).
\]

Thus, using Lemmas 4.2.4, 4.2.2 and formula (4.2.3), we obtain
\[
\overline{G} = \sum_{1 \leq k \leq n} L_{G_{n-k+1},m} F_k + o(h^m F).
\]

This proves Theorem 2.3.1 since we can take \(n\) to be the first integer for which \(c_{n+1} < 1\), that is to be \(k_1\).

This proves Theorem 2.3.3 as well for the following reason. First, recall that under the assumptions of that theorem, \(\alpha\) vanishes and in (4.2.4) we need to read \((c_{n+1}^{-\alpha} - 1)/\alpha\) as equal to \(- \log c_{n+1}\). Then, since \(h(t) \sim \lambda t^{-1} \log t\), equality (4.2.4) shows that in order to have
\[ F_{n+1} = o(h^m \overline{F}), \] we must have \((-\lambda \log 1/c_{n+1} + m)\) negative, that is \(c_{n+1} < e^{-m/\lambda}\). Hence we have proved that

\[
\overline{G} = \sum_{n \geq 1} \mathbb{1} \{ c_n \geq e^{-m/\lambda} \} L_{G \ast F_n} F_n^m + o(h^m \overline{F}).
\]

Finally, since \(F^{(i)}_n\) is of order \(h^i \overline{F}_n\), it is of order smaller than \(h^m \overline{F}\) as soon as \(c_n < e^{-(m-i)/\lambda}\), that is \(i > m + \lambda \log c_n\). Thus, the differential operators in the Laplace characters only make a nonnegligible contribution for \(0 \leq i \leq m + \lfloor \lambda \log c_n \rfloor\). Theorem 2.3.3 follows.

By an obvious symmetry, Theorems 2.3.1 and 2.3.3 follow if all the \(c_i\)'s are negative and \(F\) is strongly tail balanced.

To remove the sign restriction on the constants, we write \(H\) (respectively \(K\)) for the distribution of

\[
\sum_{i \in \mathbb{Z}, c_i < 0} c_i X_i \quad \text{(respectively } \sum_{i \in \mathbb{Z}, c_i > 0} c_i X_i \text{)}.
\]

Then \(G\) is \(H \ast K\) and

\[
\overline{G} = T_H \overline{K} + T_K \overline{H} + M_2(\overline{H} \overline{K}).
\]

Now that we have obtained the asymptotic expansions for \(\overline{H}\) and \(\overline{K}\), we conclude with nearly identical arguments.

4.3. Proof of Theorem 2.3.2. We begin by proving a lemma which allows us to neglect terms arising from the multiplication operator in formula (4.0.1). Recall that under the assumptions of Theorem 2.3.2, the hazard rate is regularly varying of index \(-1\).

Lemma 4.3.1. Assume that \(h\) is regularly varying of index \(-1\) and (2.1.1) holds. For any positive numbers \(a, b,\) and \(c,\)

\[
\overline{M}_a F \overline{M}_b F = o(M_c F).
\]

Proof. Since \(x \mapsto \overline{M}_a F\) is nondecreasing on \((0, \infty)\), it suffices to prove the lemma when \(0 < c \leq a \leq b\). Using the representation of
\( F \) in terms of its hazard rate, we have for any positive \( \delta \) less than 1 and for any \( t \) at least \( t_0 \),

\[
\log\left( \frac{M_a F}{M_c \bar{F}} \right)(t) \leq \int_{t/a}^{t/c} h(u) \, du - \int_{\delta t/b}^{t/b} h(u) \, du + \log \bar{F}(t_0) = th(t) (\log(a/c) + \log \delta) (1 + o(1)).
\]

Taking \( \delta \) small enough ensures that \( \log(\delta a/c) \) is negative. The result follows since \( th(t) \) tends to infinity with \( t \).

Our next lemma is the essential step in the proof. It shows that some \( T \) operators are close to the identity.

**Lemma 4.3.2.** Let \( M \) be a positive number, and let \( K \) be a distribution function in \( B(F, M) \). Then, under the assumptions of Theorem 2.3.2, for any positive real numbers \( a \) and \( b \),

\[
(T_K - \text{Id}) M_a F = o(M_b \bar{F})
\]

at infinity.

**Proof.** Let \( \delta \) be a positive number. Consider first the integral

\[
\int_{-\delta/h(t)}^{\delta/h(t)} \frac{F((t-x)/a) - F(t/a)}{F(t/b)} \, dK(x) = \frac{F(t/a)}{F(t/b)} \int_{-\delta/h(t)}^{\delta/h(t)} \left( \frac{F((t-x)/a)}{F(t/a)} - 1 \right) \, dK(x). \tag{4.3.1}
\]

If the absolute value of \( x \) is at most \( \delta/h(t) \), then

\[
\frac{F((t-x)/a)}{F(t/a)} = \exp\left( - \int_{t/a}^{(t-x)/a} h(u) \, du \right) = \exp\left( - \frac{t}{a} h(t/a) \int_0^{-x/t} \frac{h(t(1+v)/a)}{h(t/a)} \, dv \right),
\]

is ultimately between \( \exp(-2h(t/a)|x|/a) \) and \( \exp(2h(t/a)|x|/a) \).

Therefore, if \( \delta \) is small enough,

\[
\left| \int_{-\delta/h(t)}^{\delta/h(t)} \left( \frac{F((t-x)/a)}{F(t/a)} - 1 \right) \, dK(x) \right| \leq \frac{4}{a} h(t/a) \int_{-\delta/h(t)}^{\delta/h(t)} |x| \, dK(x) \leq \frac{4}{a} h(t/a) |\mu|_{K,1}.
\]

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It follows that the absolute value of the right hand side in (4.3.1) is ultimately at most
\[
\frac{4F(t/a)}{aF(t/b)} h(t/a) |\mu|_{K,1} .
\] (4.3.2)

Recall that \( h \) is regularly varying. So, up to the \( 4|\mu|_{K,1}/a \) factor, this upper bound is
\[
\exp\left(-\int_{t/b}^{t/a} h(u) \, du + \log h(t) + O(1)\right)
\]
\[
= \exp\left(-th(t) \log(b/a)(1 + o(1)) + \log h(t) + O(1)\right).
\]

Since \( h(t) = o(t^{-1} \log t) \) and \( th(t) \) tends to infinity, \( th(t) = o(\log h(t)) \). So (4.3.2) tends to 0 and so does the left hand side in (4.3.1).

It remains to show that
\[
\int_{-\delta/h(t)}^{\delta/h(t)} F\left(\frac{t-x}{a}\right) dK(x) = o\left(F(t/b)\right),
\]
\[
\int_{-\infty}^{-\delta/h(t)} F\left(\frac{t-x}{a}\right) dK(x) = o\left(F(t/b)\right),
\]
as well as
\[
F(t/a)K(\delta/h(t)) = o\left(F(t/b)\right),
\]
and
\[
F(t/a)K(-\delta/h(t)) = o\left(F(t/b)\right).
\]

Because of our assumption that both \( K \) and \( M_{-1}K \) are \( O(F) \), and the function \( x \mapsto F((t-x)/a) \) is nondecreasing, these assertions are implied by
\[
\frac{F(t/2a)F(\delta/h(t))}{F(t/b)} = \frac{F(t/2a)}{F(t/b)} h(t) \frac{F(1/h(t))}{h(t)} = o(1).
\]

This last estimate holds because (4.3.2) tends to 0 at infinity, \( F \) is rapidly varying and \( h \) is regularly varying.

From Proposition 1.3 in Davis and Resnick (1988) and using Lemma 4.3.1, we see that for any positive \( \delta \),
\[
M_2(G_{n-k}F_{n-k+1}) \sim M_2(M_{c_1}F \cdot M_{c_{n-k+1}}F) = o(M_{c}F).
\]
But then, using the definition of the $T$ operators,
\[
T_{F_{n-k+2}} M_2(G_{n-k} F_{n-k+1}) \leq M_2 M_2(G_{n-k} F_{n-k+1}) = o(M_2 F),
\]
and by induction, taking $\delta$ small enough, we see that
\[
M_2(G_{n-1} F_n) + \sum_{2 \leq k \leq n} T_{F_n} \cdots T_{F_{n-k+2}} M_2(G_{n-k} F_{n-k+1}) = o(F_n).
\]
Therefore, we have
\[
G_n = T_{G_{n-1}} F_n + \sum_{2 \leq k \leq n} T_{F_n} \cdots T_{F_{n-k+2}} T_{G_{n-k}} F_{n-k+1} + o(F_n).
\]
We apply Lemma 4.3.2 to obtain
\[
T_{G_{n-k}} F_{n-k+1} = F_{n-k+1} + o(F_n).
\]
Then, another use of Lemma 4.3.2 shows that
\[
T_{F_{n-k+2}} o(F_n) = o(F_n),
\]
and proceeding by induction, we obtain the expansion of $G_n$,
\[
G_n = \sum_{1 \leq k \leq n} F_{n-k+1} + o(F_n).
\]
To finish the proof, let $H_n$ denote the distribution function $\star_{i>n} F_i$, so that $G = G_n \star H_n$. Then, representation (4.0.1) gives
\[
\overline{G} = T_{H_n} G_n + T_{G_n} \overline{H}_n + M_2 G_n \overline{M}_2 H_n.
\]
Up to increasing $n$ we can assume that $c_{n+1} < c_n$. Note that by Proposition 1.3 in Davis and Resnick (1988), $\overline{H}_n$ is of order $\overline{F}_{n+1}$, which is $o(F_n)$. Then, by Lemma 4.3.1, $M_2 G_n \overline{M}_2 H_n = o(F_n)$. Since $\overline{H}_n = o(\overline{F}_n)$, Lemma 4.3.2 shows that $T_{G_n} \overline{H}_n = o(\overline{F}_n)$. Yet another application of Lemma 4.3.2 shows that
\[
T_{H_n} G_n = \sum_{1 \leq k \leq n} T_{H_n} F_{n-k+1} + o(F_n)
\]
\[
= \sum_{1 \leq k \leq n} F_{n-k+1} + o(F_n).
\]
This proves Theorem 2.3.2 when the weights are nonnegative.

Removal of the sign restriction is accomplished via the decomposition of a weighted average into two weighted averages in which each sum has all weights with the same sign as was done at the end of the proof of Theorems 2.3.1 and 2.3.3.

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