A lower bound for Torelli-$K$-quasiconformal homogeneity

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Abstract A closed hyperbolic Riemann surface $M$ is said to be $K$-quasiconformally homogeneous if there exists a transitive family $\mathcal{F}$ of $K$-quasiconformal homeomorphisms. Further, if all $[f] \subset \mathcal{F}$ act trivially on $H_1(M; \mathbb{Z})$, we say $M$ is Torelli-$K$-quasiconformally homogeneous. We prove the existence of a uniform lower bound on $K$ for Torelli-$K$-quasiconformally homogeneous Riemann surfaces. This is a special case of the open problem of the existence of a lower bound on $K$ for (in general non-Torelli) $K$-quasiconformally homogeneous Riemann surfaces.

Keywords Quasiconformal homogeneity · Riemann surface · Mapping class group · Torelli group

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1 Introduction

$K$-Quasiconformal homeomorphisms of a Riemann surface $M$ generalize the notion of conformal maps by bounding the dilatation at any point of $M$ by $K < \infty$. Let $\mathcal{F}$ be the family of all $K$-quasiconformal homeomorphisms of $M$. If for any points $p, q \in M$, there is a map $f \in \mathcal{F}$ such that $f(p) = q$, that is, the family $\mathcal{F}$ is transitive, then $M$ is said to be $K$-quasiconformally homogeneous. Quasiconformal homogeneity was first studied by Gehring and Palka in [1] in 1976 for genus zero surfaces and analogous higher dimensional manifolds. Gehring and Palka also showed that the only 1-quasiconformally homogeneous (i.e. $\mathcal{F}$ is transitive with all maps conformal) genus zero surfaces are non-hyperbolic. It was also found that there do exist genus zero surfaces which are $K$-quasiconformal for $1 < K < \infty$.

A more recent question is whether there exists a uniform lower bound on $K$ for $K$-quasiconformally homogeneous hyperbolic manifolds. Using Sullivan’s Rigidity Theorem,
Bonfert-Taylor, Canary, Martin, and Taylor showed in [2] that for dimension $n \geq 3$, there exists such a universal constant $\mathcal{K}_n > 1$ such that for any $K$-quasiconformally homogeneous hyperbolic $n$-manifold other than $\mathbb{D}^n$, we have $K \geq \mathcal{K}_n$. In [2] it is also shown that $K$-quasiconformally homogeneous hyperbolic $n$-manifolds for $n \geq 3$ are precisely the regular covers of closed hyperbolic orbifolds. For two-dimensional surfaces, such a classification is shown to be false with the construction of $K$-quasiconformally homogeneous surfaces which are not quasiconformal deformations of regular covers of closed orbifolds in [3].

In dimension two, Bonfert-Taylor, Bridgeman, Canary, and Taylor showed the existence of such a bound for a specific class of closed hyperbolic surfaces which satisfy a fixed-point condition in [4]. Bonfert-Taylor, Martin, Reid, and Taylor showed in [5] the existence of a similar bound $\mathcal{K}_c > 1$ such that if $M \neq \mathbb{H}^2$ is a $K$-strongly quasiconformally homogeneous hyperbolic surface, that is, each member of the transitive family of $K$-quasiconformally homogeneous maps is homotopic to a conformal automorphism of $M$, then $K \geq \mathcal{K}_c$. Kwakkel and Markovic proved the conjecture of Gehring and Palka for genus zero surfaces by showing the existence of a lower bound on $K$ for hyperbolic genus zero surfaces other than $\mathbb{D}^2$ in [6]. Additionally, it was shown by Kwakkel and Markovic that for surfaces of positive genus, only maximal surfaces can be $K$-quasiconformally homogeneous (Proposition 2.6 of [6]).

Here, we consider a special case of the problem for closed hyperbolic Riemann surfaces of arbitrary genus pertaining to the homological action of the family of quasiconformal maps. Recall that the mapping class group of a surface $M$, denoted $\text{MCG}(M)$, consists of homotopy classes of orientation-preserving homeomorphisms of $M$. In general, $K$-quasiconformal homeomorphisms are representatives of any mapping class in $\text{MCG}(M)$. The Torelli subgroup $\mathcal{I}(M) \leq \text{MCG}(M)$ contains those elements of $\text{MCG}(M)$ which act trivially on $H_1(M; \mathbb{Z})$, the first homology group of $M$ (see e.g. §2.1 and §7.3 of [7]). This means that the image of any closed curve $c \subset M$ under a Torelli map must be some curve homologous to $c$. We define a closed Riemann surface $M$ to be Torelli-$K$-quasiconformally homogeneous if it is $K$-quasiconformally homogeneous and there exists transitive family $\mathcal{F}$ of $K$-quasiconformal homeomorphisms which consists of maps whose homotopy classes are in the Torelli group of $M$. That is, all $f \in \mathcal{F}$ are homologically trivial. Farb, Leninger, and Margalit found bounds on the dilatation of related pseudo-Anosov maps on a Riemann surface in [8]. In particular, they give the following result as Proposition 2.6:

**Proposition 1.1** If $g \geq 2$, then $L(\mathcal{I}(M)) > .197$, where $L$ is the logarithm of the minimal dilatation of pseudo-Anosov maps in $\mathcal{F}(M)$.

This result relies on several lemmas for pseudo-Anosov maps that can also be applied to Torelli-$K$-quasiconformal maps once we show that appropriate $K$-quasiconformal maps must exist. After proving a proposition that allows us to avoid the assumption of pseudo-Anosov maps, we will use an argument similar to that in the proof of Proposition 1.1 to give the following result on Torelli-$K$-quasiconformally homogeneous surfaces:

**Theorem 1.1** There exists a universal constant $K_T > 1$ such that if $M$ is a Torelli-$K$-quasiconformally homogeneous closed hyperbolic Riemann surface, then $K \geq K_T$.

A related proof of the result has very recently been given by Vlamis, along with similar results for other subgroups of $\text{MCG}(M)$. The result by Nicholas Vlamis is obtained in “Quasiconformal homogeneity and subgroups of the mapping class group”, available at arXiv:1309.7026.
In addition to bettering our estimate for $K_T$, proving the existence of a bound on $K$ for the general case of non-Torelli maps is of course still an important open question. Should it be found that such a bound must exist, another interesting result may be a comparison between the value of this bound and our bound $K_T$. We may also wish to seek more specific details on which types of surfaces can be Torelli-$K$-quasiconformally homogeneous with small values of $K$, and if we can obtain stricter bounds for different types of surfaces.

## 2 Preliminary notions

First, we will introduce some relevant concepts, definitions, and lemmas which will be used throughout the paper.

### 2.1 Definitions

Let $M$ be a closed hyperbolic Riemann surface of genus $g \geq 0$. Let $c$ be a shortest geodesic on $M$. The injectivity radius $\iota(M)$ is the infimum over all $p \in M$ of the largest radius for which the exponential map at $p$ is injective (see §2.1 of [6]). In particular, $|c| \geq 2\iota(M)$, where $|c|$ denotes the length of $c$.

Let $\gamma \subset M$ be a closed curve. We denote by $[\gamma]$ its homotopy class. Recall that in any homotopy class, there exists a unique geodesic whose length bounds the length of all elements of $[\gamma]$ from below (see e.g. [9]). We define the geometric intersection number of two closed curves $a, b \subset M$ by

$$i(a, b) = \min \{\#(\gamma \cap \gamma')\}, \quad (2.1)$$

where the minimum is taken over all closed curves $\gamma, \gamma' \subset M$ with $[\gamma] = a$ and $[\gamma'] = b$ (as defined in [6]). From the discussion on page 804 of [8], the intersection number of a pair of homologous curves must be even.

Next, let $X$ and $Y$ be complete metric spaces. A map $f : X \to Y$ is said to be a $K$-quasi-isometry if for some $R > 0$ we have

$$R + Kd(x, x') \geq d(f(x), f(x')) \geq \frac{d(x, x')}K - R$$

for all $x, x' \in X$ (see [10]). A quasi-geodesic in a metric space $X$ is a quasi-isometric map

$$\gamma : [a, b] \to X.$$ 

It is known that the image of a geodesic under a quasiconformal homeomorphism is a quasi-geodesic (Theorem 5.1 of [11]), and that a quasi-geodesic is within a bounded distance of a unique hyperbolic geodesic.

### 2.2 Previous results

Let $M$ be as above, and recall that the Torelli group of $M$ is denoted $\mathcal{I}(M)$. First, we have Lemmas 2.2, 2.3, and 2.4, respectively from [8]. These will be instrumental in our final proof in Sect. 4.

**Lemma 2.1** Suppose that $[f] \in \mathcal{I}(M)$, that $c$ is a separating curve, and that $[f(c)] \neq [c]$. Then $i(f(c), c) \geq 4$. 
Fig. 1 Illustration of Lemma 2.3, with \( c \) and \( c' \) homologous with intersection number 2. The labels \( a, b, a', \) and \( b' \) are used in the proof of Theorem 1.1. (Adapted from [8])

**Lemma 2.2** Suppose that \([f] \in \mathcal{I}(M)\), that \( c \) is a nonseparating curve, and that \([f(c)] \neq [c]\). Then at least one of \( i(f(c), c) \) and \( i(f^2(c), c) \) is at least 2.

**Lemma 2.3** Suppose \( c \) and \( c' \) are homologous nonseparating curves with \( i(c, c') = 2 \). Suppose that \( d, d', e, \) and \( e' \) are the boundary components of a 4-holed sphere as shown in Fig. 1. Then \( d \) and \( d' \) are separating in \( M \), and \([e] = -[e] = [c] = [c']\) in \( H_1(M; \mathbb{Z})\).

Next, we have Lemmas 2.2 and 2.3 from [6], respectively:

**Lemma 2.4** Let \( M \) be a K-quasiconformally homogeneous hyperbolic surface and \( \iota(M) \) its injectivity radius. Then \( \iota(M) \) is uniformly bounded from below (for \( K \) bounded from above) and \( \iota(M) \to \infty \) for \( K \to 1 \).

In particular, if \( c \) is the shortest curve on \( M \), then \( |c| \to \infty \) as \( K \to 1 \).

**Lemma 2.5** Let \( \gamma \) a simple closed geodesic in \( M \). Let \( f : M \to M \) be a K-quasiconformal homeomorphism and \( \gamma' \) the simple closed geodesic homotopic to \( f(\gamma) \). Then \( \frac{1}{K} |\gamma| \leq |\gamma'| \leq K |\gamma| \).

This will allow us to bound the lengths of geodesics under \( K \)-quasiconformal maps in terms of their pre-images. We also recall the following classical result:

**Proposition 2.1** There exists a function \( \delta(K) > 0 \) such that \( \delta(K) \to 0 \) as \( K \to 1 \), which satisfies the following. Let \( f : S_1 \to S_2 \) be a \( K \)-quasiconformal map between two hyperbolic Riemann surfaces and suppose that \( \gamma \) is a geodesic on \( S_1 \). Then \( f(\gamma) \) is contained in a \( \delta(K) \)-neighborhood of the unique geodesic on \( S_2 \) homotopic to \( f(\gamma) \).

That is, the image of a geodesic under a \( K \)-quasiconformal map is contained within a collar of a geodesic, the width of which tends to 0 as \( K \) tends to 1. Finally, Proposition 1.16 from [12] gives us the following:

**Proposition 2.2** Let \( S \) be a closed Riemann surface of genus \( g \geq 2 \) equipped with its hyperbolic metric. Then the shortest curve \( c \) on \( S \) has length \( |c| \leq 2 \log(4g - 2) \).

These two facts will allow us to prove the main proposition in the next section.
3 Existence of a suitable $f \in \mathcal{F}$

Let $S$ be a closed hyperbolic Riemann surface with shortest geodesic $c$. Suppose that $S$ is $K$-quasiconformally homogeneous with transitive family of $K$-quasiconformal maps $\mathcal{F}$.

In [8], a lower bound was obtained for the dilatation of pseudo-Anosov maps. Recall the condition in Lemmas 2.1 and 2.2, that we have some map $f$ such that $[c] \neq [f(c)]$. Pseudo-Anosov maps are always homotopically nontrivial, so the existence of such an $f$ is known a priori. For general $K$-quasiconformal homeomorphisms, this is not necessarily the case, but our proof of Theorem 1.1 makes use of the aforementioned lemmas with the curve $c$, as well as a nearby geodesic. Thus, we must show that there exists a map $f \in \mathcal{F}$ that sends both $c$ and a neighboring geodesic to curves not homotopic to their preimages for surfaces with sufficiently small $K$. We phrase this as follows:

**Proposition 3.1** There exists a universal constant $K_0 > 1$ such that if $S$ is a $K_0$-quasiconformally homogeneous closed Riemann surface of genus $g \geq 2$, and $\mathcal{F}$ is a transitive family of $K_0$-quasiconformal homeomorphisms of $S$, then if $c$ is the shortest geodesic on $S$ and $d$ another geodesic on $S$ whose length is at most $2|c|$ and such that the distance between $c$ and $d$ on $S$ is at most $\frac{1}{|c|}$, there exists $f \in \mathcal{F}$ such that $[c] \neq [f(c)]$ and $[d] \neq [f(d)]$.

We will prove this in three parts. First, we show that surfaces lacking a $f \in \mathcal{F}$ such that $[f(c)] \neq [c]$ and $[f(d)] \neq [d]$ must be contained in the union of small neighborhoods of $c$ and $d$. Then, we will exhibit a bound on the area of such a surface. Finally, we show that for sufficiently small $K$ this bound cannot hold.

**Claim** Let $S$ be as in Proposition 3.1. If for all $f \in \mathcal{F}$, we have that $f$ fixes at least one of $[c]$ or $[d]$, then $S$ is contained in the union of a $\delta$-neighborhood of $c$ and a $\delta$-neighborhood of $d$, where $\delta = \delta(K)$ depends on $K$ and $\delta \to 0$ as $K \to 1$.

**Proof** By hypothesis, we can choose some $x \in S$ such that $x$ is in a $\frac{1}{|c|}$-neighborhood of both $c$ and $d$. Since $\mathcal{F}$ is transitive, for any point $y \in S$ we can find a map $f \in \mathcal{F}$ such that $f(x) = y$. By Proposition 2.1, we know that each $f \in \mathcal{F}$ sends any point on $c$ or $d$ to a point contained in a $\delta^*$-neighborhood of a geodesic, where $\delta^* \to 0$ as $K \to 1$. By continuity, we know that each image of $x$ will be in a slightly larger neighborhood of a geodesic, say a $\delta(K)$-neighborhood. Notice that as $|c|$ increases (as $K \to 1$ by Lemma 2.4), we have $\delta \to \delta^*$ because $c$ and $d$ are separated by a distance $\frac{1}{|c|}$.

Now, if all maps $f \in \mathcal{F}$ fix at least one of $[c]$ or $[d]$, the image of $x$ must remain in a $\delta$-neighborhood of at least one of these curves. It follows that $S$ is contained in the union of a $\delta$-neighborhood of $c$ and a $\delta$-neighborhood of $d$, where $\delta$ depends only on $K$. By Proposition 2.1, together with Lemma 2.4, we can make $\delta$ arbitrarily small by sending $K \to 1$. These neighborhoods are collars around the curves $c$ and $d$ of total width $2\delta$, and as remarked above, sending $K \to 1$ will send $\delta \to 0$. This completes the proof of the claim. \(\square\)

In the rest of the proof, we show that when $\delta$ is small, $S$ cannot be contained in these two $\delta(K)$-neighborhoods of $c$ and $d$.

**Claim** Let $S$ be as above. Then:

$$\text{Area}(S) < 2\pi \left( \frac{3\log(4g - 2)}{\delta} + 2 \right) (\cosh(2\delta) - 1).$$

**Proof** Since $S$ is contained in small neighborhoods of the two curves $c$ and $d$, we will bound the area of the neighborhoods of each curve from above as follows. Each collar can be covered...
Covering the $\delta$-neighborhood of the curve $\gamma$ with disks of radius $2\delta$. The hyperbolic right triangle has base $\delta$ (half the separation between the disks) and height $h \geq \delta$. The hypotenuse is the radius of a disk. We do this for curves $c$ and $d$ by hyperbolic disks (2-balls) of radius $2\delta$ whose centers lie on the main curve (See Fig. 2). We arrange them such that each disk is separated by a distance of $2\delta$ from the adjacent disks. That gives a total of less than $|c| + 1$ disks (taking the smallest integer greater than $\frac{|c|}{2\delta}$) for curve $c$, and $|d| + 1$ disks for $d$. One disk for each of $c$ and $d$ will be less than $2\delta$ away from one of its neighbors if $|c|$ or $|d|$ is not an integer multiple of $2\delta$.

In order to show that the disks cover the entire collar, we must show that the height $h$ of the hyperbolic right triangle (i.e. half of the width of the area covered by the disks) in Fig. 2 is at least $\delta$. In that case, the disks will cover a collar around their respective curve of total width at least $2\delta$ (since the disks are convex), thus covering the $\delta$-neighborhood of the curve. This follows from the hyperbolic Pythagorean theorem, which gives

$$\cosh(2\delta) = \cosh(\delta) \cosh(h).$$

Indeed, supposing otherwise and applying the appropriate identities, we have

$$h < \delta \Rightarrow \cosh(h) < \cosh(\delta) \Rightarrow \frac{\cosh(2\delta)}{\cosh(\delta)} < \cosh(\delta) \Rightarrow 2 \cosh^2(\delta) - 1 < \cosh^2(\delta) \Rightarrow \cosh^2(\delta) < 1,$$

contradicting the fact that $\cosh(x) \geq 1$ for all $x \in \mathbb{R}$. Thus, the disks cover the $\delta$-neighborhoods of $c$ and $d$, whence they cover $S$.

Recall that the area of a hyperbolic disk of radius $r$ is $2\pi\left(\cosh r - 1\right)$. Since our collection of disks bounds the area of $S$ from above, we have:

$$\text{Area}(S) < \left(\frac{|c|}{2\delta} + 1\right) 2\pi(\cosh(2\delta) - 1) + \left(\frac{|d|}{2\delta} + 1\right) 2\pi(\cosh(2\delta) - 1) \quad (3.1)$$

where $\left(\frac{|c|}{2\delta} + 1\right)$ and $\left(\frac{|d|}{2\delta} + 1\right)$ bound the number of disks from above, and $2\pi(\cosh(2\delta) - 1)$ is the area of each disk. Since by hypothesis we have that $|d| \leq 2|c|$, we can rewrite this as:

$$\text{Area}(S) < \left(\frac{3|c|}{2\delta} + 2\right) 2\pi(\cosh 2\delta - 1) \quad (3.2)$$

From Proposition 2.2 we also have that $|c| \leq 2 \log(4g - 2)$. Together with (3.2), this gives:
as desired.

Armed with the inequality (3.3), we can proceed to the final proof of Proposition 3.1.

\textbf{Proof (Proposition 3.1)} Let $S$ be as above. We need to show that there exists a universal constant $K_0 > 1$ such that if $K \leq K_0$, then for some $f \in \mathcal{F}$, we have both $[f(c)] \neq [c]$ and $[f(d)] \neq [d]$. We show that $K$ is bounded from below by some $K_0 > 1$ for surfaces with families in which no such $f$ exists. Recall that the area of a hyperbolic surface $S$ of genus $g \geq 2$ is given by:

$$\text{Area}(S) = 4\pi(g - 1).$$

(3.4)

Now, from the previous claim we have an upper bound for the area of our surface $S$ in terms of the genus $g$ and $\delta = \delta(K)$. Combining (3.3) and (3.4), we have:

$$4\pi(g - 1) < 2\pi \left( \frac{3\log(4g - 2)}{\delta} + 2 \right) (\cosh(2\delta) - 1).$$

(3.5)

This inequality follows from the upper bound on the area from Claim 3, the area of a hyperbolic surface from (3.4), and the upper bound on the lengths of $c$ from Proposition (2.2). Simplifying, we obtain:

$$\frac{2(g - 1)}{\log(4g - 2)} < \left( \frac{3}{\delta} + \frac{2}{\log(4g - 2)} \right) (\cosh(2\delta) - 1) < \left( \frac{3}{\delta} + 2 \right) (\cosh(2\delta) - 1).$$

(3.6)

We have $g \geq 2$, and so (3.6) gives:

$$\frac{2}{\log(6)} < \left( \frac{3}{\delta} + 2 \right) (\cosh(2\delta) - 1).$$

(3.7)

Notice that for the upper bound in (3.7), we have

$$\lim_{\delta \to 0} \left( \frac{3}{\delta} + 2 \right) (\cosh(2\delta) - 1) = 0.$$

(3.8)

Now, (3.7) and (3.8) show that there exists a uniform lower bound on $\delta$. By Lemma 2.4 and Proposition 2.1, we can choose $K > 1$ such that $\delta$ becomes arbitrarily small, which sends the right-hand side of (3.7) to 0. Thus, there must be a universal lower bound $K_0 > 1$ on $K$. If the transitive family $\mathcal{F}$ does not include maps that send $c$ to non-homotopic curves, then $K > K_0$.

Thus for $1 < K \leq K_0$, the transitive family $\mathcal{F}$ must include a map that sends both $c$ and $d$ to a non-homotopic curve.

Using Proposition 3.1, will now prove Theorem 1.1.

\section{Proof of Theorem 1.1}

\textbf{Proof} Let $S$ be a closed hyperbolic Riemann surface, and suppose $S$ is Torelli-K-quasiconformally homogeneous with family of homeomorphisms $\mathcal{F}$. Suppose $1 < K \leq K_0$ from Proposition 3.1. Let $c$ be a simple closed geodesic on $S$ of minimal length. Using
Proposition 3.1, choose some \( f \in \mathcal{F} \) such that \( [c] \neq [f(c)] \) and \( f \) is also homotopically non-trivial on any geodesic in a \( \frac{1}{|c|} \)-neighborhood of \( c \).

We have the following two cases: (1) \( i(c, f(c)) \geq 4 \) or \( i(c, f^2(c)) \geq 4 \), or (2) \( c \) is nonseparating and \( i(c, f(c)) \) and \( i(c, f^2(c)) \) are both less than 4. Notice that these cover all possibilities: if \( c \) is separating, then Lemma 2.1 gives us that \( i(c, f(c)) \geq 4 \). If \( c \) is nonseparating, then either one of \( i(c, f(c)) \) or \( i(c, f^2(c)) \) is greater than 4, which is case 1, and otherwise we have case 2.

**Case 1** Let \( h \) be either \( f \) or \( f^2 \), where \( i(c, h(c)) \geq 4 \). Since \( i(c, h(c)) \geq 4 \), any member of the homotopy class \( [h(c)] \) will have at least 4 intersections with \( c \). In particular, let \( c' \) be the geodesic homotopic to \( h(c) \). The intersection points \( c \cap c' \) cut \( c \) and \( c' \) into arcs. Since there are at least 4 such points, there is an arc \( a \) of \( c' \) which satisfies

\[
|a| \leq \frac{|c'|}{4} \leq \frac{K^2|c|}{4}.
\]

(4.1)

The second inequality follows from Lemma 2.5, with \( K^2 \) since \( h \) is possibly \( f^2 \), and \( f^2 \) is a \( K^2 \)-quasiconformal homeomorphism. The endpoints of \( a \) cut \( c \) into two arcs, one of which, say \( b \), has length \( |b| \leq |c|/2 \). The union \( a \cup b \) is a simple closed curve. It must be homotopically nontrivial since otherwise we could, by homotopy, reduce the number of intersections of \( c \) and \( c' \) below \( i(c, c') \). Now, recall \( c \) is the shortest closed geodesic, so \( |c| \leq |a \cup b| \). Then we have:

\[
|c| \leq |a| + |b| \leq \frac{K^2|c|}{4} + \frac{|c|}{2} \Rightarrow 1 \leq \frac{K^2}{4} + \frac{1}{2} \quad (4.2)
\]

Thus, \( 2 \leq K^2 \Rightarrow K \geq \sqrt{2} \).

**Case 2** By Lemma 2.2, either \( i(c, f(c)) = 2 \) or \( i(c, f^2(c)) = 2 \). Let \( h \) be either \( f \) or \( f^2 \), where \( i(c, h(c)) = 2 \). Now, let \( c' \) be the geodesic homotopic to \( h(c) \); we still have \( i(c, c') = 2 \). Let \( d \) and \( d' \) be the separating curves from Lemma 2.3 with \( c \) and \( c' \) as the homologous pair (since \( f \) is Torelli). Alternatively, the intersection points \( c \cap c' \) define two arcs of \( c \), say \( a \) and \( a' \), and two arcs of \( c' \), say \( b \) and \( b' \) as in Fig. 1. The curves \( d \) and \( d' \) are then \( d \sim a \cup b \) and \( d' \sim a' \cup b' \). Now,

\[
|d| + |d'| = |a| + |b| + |a'| + |b'| = |c| + |c'| \leq |c| + K^2|c| \quad (4.3)
\]

by Lemma 2.5 and since \( h \) may be \( f^2 \), which is a \( K^2 \)-quasiconformal homeomorphism. It follows that at least one of \( d \) and \( d' \), say \( d \), has length bounded above by half of \( |c| + |c'| \):

\[
|d| \leq \frac{|c| + K^2|c|}{2} \quad (4.4)
\]

We now consider \( d_1 \), the geodesic homotopic to \( d \), which is a separating curve, and the geodesic \( e_1 \) homotopic to \( e \). We have \( |d_1| \leq |d| \) and \( |e_1| \leq |e| \). To continue, we require the following lemma, which will be proven at the end:

**Lemma 4.1** For sufficiently small \( K \), the curves \( c \) and \( d_1 \) are within a distance of \( \frac{1}{|c|} \) from each other.

Now, suppose \( K > 1 \) is small enough so Lemma 4.1 can be used. Applying the lemma and Proposition 3.1, we see that \( [f(d_1)] \neq [d_1] \). We can now apply Lemma 2.1, so that
$i(d_1, f(d_1))$ is at least 4. As in Case 1, if $\tilde{a}$ is the shortest arc of the geodesic homotopic to $f(d_1)$ cut off by $d_1$ and $\tilde{b}$ is the shortest arc of $d_1$ cut off by $\tilde{a}$, then

$$|\tilde{a} \cup \tilde{b}| \leq |d_1| \left( \frac{K}{4} + \frac{1}{2} \right). \quad (4.5)$$

Note that we can use $K$ instead of possibly $K^2$ since $d$ is separating, so Lemma 2.1 applies with $f$. Recall that we also have

$$|c| \leq |\tilde{a} \cup \tilde{b}|. \quad (4.6)$$

Combining (4.4), (4.5), (4.6), and the fact that $|d_1| \leq |d|$, we see that

$$|c| \leq \left( \frac{K}{4} + \frac{1}{2} \right) \left( \frac{|c| + K^2 |c|}{2} \right) \quad (4.7)$$

Which then gives

$$1 \leq \left( \frac{K}{4} + \frac{1}{2} \right) \left( \frac{1 + K^2}{2} \right) \Rightarrow K^3 + 2K^2 + K - 6 \geq 0. \quad (4.8)$$

The cubic polynomial in $K$ has one real root, and so

$$K \geq -\frac{2}{3} + \frac{1}{3} \sqrt{82 - 9\sqrt{83}} + \frac{1}{3} \sqrt{82 + 9\sqrt{83}} \approx 1.218 \quad (4.9)$$

approximated from below.

Now, since these bounds are independent of genus, we have that $K > 1.218$. We can then see that for some $K_T > 1$, any such surface $S$ cannot be Torelli-$K$-quasiconformally homogeneous with $K \leq K_T$. Thus we have a universal constant $K_T > 1$ such that any Torelli-$K$-quasiconformally homogeneous closed hyperbolic Riemann surface of genus $g \geq 2$ must have $K \geq K_T > 1$. This completes the proof.

$\square$

**Proof (Lemma 4.1)** First, consider the pair of pants defined by curves $c$, $d_1$, and $e_1$, as in Fig. 1. Notice that as $K$ approaches 1, we eventually have:

$$|c| \leq |d_1| \leq \frac{3|c|}{2}, \quad |c| \leq |e_1| \leq \frac{3|c|}{2}. \quad (4.10)$$

This follows from (4.4), which can also apply with the same bounds to the curve $e_1$, and the fact that $c$ is the shortest geodesic on $S$.

Consider now the right-angled hyperbolic hexagon formed by inserting perpendiculars connecting each of $c$, $d_1$, and $e_1$, and cutting off a fundamental hexagon from this pair of pants. Then there are three alternating sides of length $\frac{|c|}{2}$, $\frac{|d_1|}{2}$, and $\frac{|e_1|}{2}$. By Lemma 2.4 we also have that as $K \rightarrow 1$, $|c| \rightarrow \infty$. Recall from [10] the Law of Cosines for right-angled hyperbolic hexagons:

$$\cosh(z') = \coth(x) \coth(y) + \frac{\cosh(z)}{\sinh(x) \sinh(y)}, \quad (4.11)$$

where $x$, $y$, and $z$ are the lengths of alternate sides of the hexagon, and $z'$ is the length of the side opposite the side of length $z$. Let $x$, $y$, and $z$ correspond to the sides of length $\frac{|c|}{2}$, $\frac{|d_1|}{2}$, and $\frac{|e_1|}{2}$ respectively, and so $z'$ is the distance between curves $c$ and $d_1$. We wish to show that $z' \rightarrow 0$ faster than $\frac{1}{|c|} \rightarrow 0$ as $K \rightarrow 1$. Notice from (4.10) that, as $|c| \rightarrow \infty$, $x$, $y$, and $z$ all increase within a factor of $\frac{3}{2}$ from each other. Thus, the $\coth(x) \coth(y)$ term proceeds exponentially to 1, and the $\frac{\cosh(z)}{\sinh(x) \sinh(y)}$ term exponentially approaches 0.
Now, the right-hand side of (4.11) proceeds to unity, and so \( \cosh(z') \to 1 \). We can then see that \( z' \), the distance between curves \( c \) and \( d_1 \), approaches zero exponentially in \( |c| \) as \( |c| \) increases. Therefore as \( K \to 1 \), the curves \( c \) and \( d_1 \) will approach each other exponentially in \( |c| \), and so for sufficiently small \( K \), they will be closer than \( \frac{1}{|c|} \), as desired. \( \Box \)

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