Thermodynamic efficiency of information and heat flow

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A basic task of information processing is information transfer (flow). Here we study a pair of Brownian particles each coupled to a thermal bath at temperature $T_1$ and $T_2$, respectively. The information flow in such a system is defined via the time-shifted mutual information. The information flow nullifies at equilibrium, and its efficiency is defined as the ratio of flow over the total entropy production in the system. For a stationary state the information flows from higher to lower temperatures, and its the efficiency is bound from above by $\frac{\max(T_1, T_2)}{T_1 T_2}$. This upper bound is imposed by the second law and it quantifies the thermodynamic cost for information flow in the present class of systems. It can be reached in the adiabatic situation, where the particles have widely different characteristic times. The efficiency of heat flow—defined as the heat flow over the total amount of dissipated heat—is limited from above by the same factor. There is a complementarity between heat- and information-flow: the setup which is most efficient for the former is the least efficient for the latter and vice versa. The above bound for the efficiency can be [transiently] overcome in certain non-stationary situations, but the efficiency is still limited from above. We study yet another measure of information-processing [transfer entropy] proposed in literature. Though this measure does not require any thermodynamic cost, the information flow and transfer entropy are shown to be intimately related for stationary states.

PACS numbers: 89.70.Cf, 02.50.-r, 05.10.Gg

I. INTRODUCTION

Relations between statistical thermodynamics and information science have long since been recognized; they have been the source of mutual fertilization, but occasionally also of confusion. Pertinent examples are the nature of the maximum entropy principle [1] and the analysis of the Maxwell demon concept [2]. Then there is the huge field of information processing devices. Their continuing miniaturization [3] is approaching already the nanometer scale, which makes it obligatory to study the information carriers as physical entities subject to the laws of statistical physics. Note that information processing constitutes a certain class of tasks (functionality) to be implemented on a physical system. Basic tasks of such kind are, inter alia, information erasure and information transfer.

The thermodynamic cost of information erasure received much attention [4, 5, 6, 7, 8, 9, 10, 11]; see 7 for a review. The information erasure is governed by the Landauer principle [4, 5, 6, 7], which—together with its limitations [8, 9, 10, 11]—has been investigated from different perspectives.

The task of information transfer is not uniquely defined, but has to be specified before one can start any detailed physical investigation (see below). Its thermodynamical cost in the presence of a thermal bath has been studied in [12, 13, 14, 15, 16, 17, 18, 19]. There is an essential difference between the problem of thermodynamic costs during information erasure versus those during information transfer. In the former case the information carrier has to be an open, or even dissipative system, while for the latter case the external bath frequently plays a role of a hindrance [16, 17, 19]. For a finite, conservative system the problem of thermodynamic costs is not well-defined, since for such systems the proper measures of irreversibility and dissipation are lacking; see, however, [24] in this context. Thus, the thermodynamic cost for information transfer is to be studied in specific macroscopic settings.

The current status of the problem is controversial: in the literature there are statements claiming both the existence [15, 18, 19] and the absence [12, 13, 14, 15, 16] of inevitable thermodynamic costs during information transfer. The arguments against the fundamental bounds on thermodynamic costs in information transfer [12, 13, 14, 15, 16] rely in essence on the known statistical physics fact that the entropy generated during a process can be nullified by nullifying the rate of this process [20]. However, this and related arguments leave unanswered the question on whether there are thermodynamic costs in a more realistic case of finite-rate information transfer. In practice, the information transfer should normally proceed at a finite rate.

The presently known arguments in favour of the existence of thermodynamic cost during information transfer are either heuristic [17, 18], or concentrate on those aspects of computation and communication, which require some energy for carrying out the task, but this energy is not necessarily dissipated; see [18, 21, 22, 23]. The present work aims at clarifying how processes of information and heat flow relate to dissipative characteristics such

1 The authors of [24] point out that although in principle the energy required for carrying out these tasks need not be dissipated, in practice it is normally dissipated, at least partially.
as entropy production or heat dissipation. In particular, we clarify which tasks of information processing do (not) require thermodynamic cost.

We shall work with the—supposedly—simplest set-up that allows to study the above problem: two (classical) Brownian particles each interacting with a thermal bath. The model combines two basic ingredients needed for studying the information transfer: randomness—which is necessary for the very notion of information—and directedness, i.e., the possibility of inducing a current of physical quantities via externally imposed gradients of temperature and/or various potentials.

For the present bi-partite problem, whose dynamics is formulated as a Markov process for random coordinates $X_1(t)$ and $X_2(t)$ of Brownian particle, the mutual information is given by the known Shannon expression $I[X_1(t) : X_2(t)] \approx \frac{1}{2} \ln \left( \frac{1}{|\Delta X_2(t) - \Delta X_1(t)|} \right)$. This is an ensemble property, which is naturally symmetric $I[X_1(t) : X_2(t)] = I[X_2(t) : X_1(t)]$, and has the same status as other macroscopic observables in statistical physics, e.g., the average energy. The information flow $I_{1 \rightarrow 2}$ is defined via the time-shifted mutual information via $I_{1 \rightarrow 2} = \partial_t I[X_1(t + \tau) : X_2(t)] \big|_{\tau \rightarrow 0}$. The full rate $\frac{d}{dt} I[X_1(t) : X_2(t)]$ of mutual information is now separated into the information that has flown from the first to the second particle $I_{1 \rightarrow 2}$ and vice versa: $\frac{d}{dt} I[X_1(t) : X_2(t)] = I_{1 \rightarrow 2} + I_{2 \rightarrow 1}$. We discuss this definition in section VI, explain its relation to prediction and clarify the meaning of a negative information flow. The usage of the time-shifted mutual information for quantifying the information flow was advocated in [29, 30, 31, 32].

The information flow is an asymmetric quantity, $I_{1 \rightarrow 2} \neq I_{2 \rightarrow 1}$, and it allows to distinguish between the source of information versus its recipient (such a distinction cannot be done via the mutual information, which is a symmetric quantity). Another important feature of information flow is that—for the considered bi-partite Markovian system—it nullifies in equilibrium: $I_{1 \rightarrow 2} = I_{2 \rightarrow 1} = 0$, because the information flow appears to be related to the entropy flow; see section VIII.

Once we have shown that the information flow is absent in equilibrium, we concentrate on the case when this flow is induced by a temperature gradient. Despite some formal similarities, a temperature gradient and a potential gradient—two main sources of non-equilibrium—are of different physical origin and enjoy different features; see [44] for a recent discussion. Hence we expect that the specific features of the information flow will differ depending on the type of non-equilibrium situation. In this context, the temperature-gradient situation is perhaps the first one to study, since the function of information transfer comes close to the function of a thermodynamic machine. This far reaching analogy reflects itself in the features of the efficiency of information transfer, which is defined as the useful product, i.e., the information flow, divided over the total waste, as quantified by the entropy production in the overall system. In the stationary two-temperature state, where both the information flow and entropy production are time-independent, the efficiency of information flow is limited from above by $\frac{\max(T_1, T_2)}{|T_1 - T_2|}$. This expression depends only on temperatures and does not depend on various details of the system (such as damping constants, inter-particle potentials, etc). The existence of such an upper bound implies a definite thermodynamic cost for information flow. Interestingly, the upper bound for the efficiency can be reached in the adiabatic situation, when the source of information is much slower than its recipient. This fact to some extent resembles the reachability of the Carnot bound for thermodynamic efficiency of heat-engines and refrigerators [20]. The upper bound $\frac{\max(T_1, T_2)}{|T_1 - T_2|}$ for the efficiency of information flow can be well surpassed in certain non-stationary, transient situations. Even then the efficiency of information flow is limited from above via the physical parameters of the system.

We shall argue that there is a clear parallel in the definition of information flow and heat flow. This is additionally underlined by the fact that in the stationary state both heat and information flow from higher temperatures to lower ones. Moreover, the efficiency of heat flow—defined accordingly as the ration of the heat flow and heat dissipation rate—appears to be limited from above by the same factor $\frac{\max(T_1, T_2)}{|T_1 - T_2|}$. There is however an important complementarity here: the upper bound for the efficiency of heat flow is is reached exactly for that setup which is the worst one for the efficiency of information flow; see section VIII.

The information flow $I_{1 \rightarrow 2}$ characterizes the predictability of (future of) 1 (first particle) from the viewpoint of 2 (second particle). Another aspect of information processing in stochastic systems concerns predicting by 1 its own future, and the help provided by 2 in accomplishing this task. This is essentially the notion of Granger-causality first proposed in econometrics for quantifying causal relation between coupled stochastic processes [33, 34], and formalized in information-theoretic terms via the concept of transfer entropy [33, 35]. It appears that this type of information processing does not require any thermodynamic cost, i.e., the transfer entropy—in contrast to information flow—does not necessarily nullify at equilibrium. Despite of this, there are interesting relations between the transfer entropy and information flow, which are partially uncovered in section VII.

The paper is organized as follows. Section II reminds the definition of entropy and mutual information; this reminder is continued in Appendix A. Section III defines the class of models to be studied. In section IV we discuss

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2 We stress that the employed notion of information concerns its syntactic aspects and does not concern its semantic [meaning] and pragmatic [purpose] aspects. Indeed, the problems we intend to study—physics of information carriers, thermodynamic cost of information flow, etc—refer primarily to syntactic aspects.
in detail the information-theoretic definition of information flow. This discussion is continued in section \[\text{V}\] with relating the information flow to the entropy flow. The same section recalls the concepts of entropy production and heat dissipation. The efficiency of information flow is defined and studied in section \[\text{VI}\]. Many of the obtained results will be illustrated via an exactly solvable example of coupled harmonic oscillators; see section \[\text{VIII}\]. Section \[\text{VIII}\] studies the efficiency of heat flow, while in section \[\text{IX}\] we compare the concept of information flow with the notion of transfer entropy. Our results are shortly summarized in section \[\text{X}\]. Some technical questions are relegated to Appendices.

II. BASIC CONCEPTS: ENTROPY AND MUTUAL INFORMATION

The purpose of this section is to recall the definition of entropy and mutual information.

A. Entropy

How can one quantify the information content of a random variable \(X\) with realizations \(x_1, \ldots, x_n\) and probabilities \(p(x_1), \ldots, p(x_n)\)? This is routinely done by means of the entropy

\[
S[X] = -\sum_{k=1}^{n} p(x_k) \ln p(x_k). \tag{1}
\]

\(S[X]\) is well known in physics. Its information-theoretic meaning, which is based on the law of large numbers, is reminded in Appendix \[\text{A}\]. One can arrive at the same form \[\text{I}\] by imposing certain axioms, which are intuitively expected from the notion of uncertainty or the information content \[\text{II}\].

For a continuous random variable \(X\) the entropy converges to

\[
S[X] = -\int dx \, P(x) \ln P(x) + \text{additive constant}, \tag{2}
\]

where \(P(x)\) is the probability density of \(X\), and where the additive constant is normally irrelevant, since it cancels out when calculating any entropy differences. The quantity \(e^{-\int dx \, P(x) \ln P(x)}\) characterizes the [effective] volume of the support of \(P(x)\) \[\text{III}\].

B. Mutual Information

We shall consider the task to predict some present state \(Y(t)\) of one subsystem from the present state \(X(t)\) of another subsystem and vice versa. For this purpose we introduce two dependent random variables \(X\) and \(Y\) with realizations \(x_1, \ldots, x_n\) and \(y_1, \ldots, y_n\) and joint probabilities \(p(x_k, y_l)\) \[\text{I}_k,l=1\].

Assume that we learned a realization \(x_l\) of \(X\). This allows to redefine the probabilities of various realizations of \(Y\): \(p(y_k) \rightarrow p(y_k|x_l)\), where \(p(y_k|x_l)\) is the conditional probability. Due to this redefinition also the entropy \(Y\) changes:

\[
S[Y] \rightarrow S[Y|x_l] = -\sum_{k=1}^{n} p(y_k|x_l) \ln p(y_k|x_l).
\]

Averaging \(S[Y|x_l]\) over \(p(x_l)\) we get

\[
S[Y|X] = -\sum_{k,l=1}^{n} p(y_k, x_l) \ln p(y_k|x_l).
\]

This conditional entropy characterizes the average residual entropy of \(Y\): \(S[Y|X] < S[Y]\). If there is a bijective function \(f()\) such that \(Y = f(X)\) \((X = f^{-1}(Y))\), then \(S[Y|X] = 0\). We have \(S[Y|X] = S[Y]\) for independent random variables \(X\) and \(Y\).

The mutual information \(I[Y : X]\) between \(X\) and \(Y\) is that part of entropy of \(Y\), which is due to the missing knowledge about \(X\). To define \(I[Y : X]\) we subtract the residual entropy \(S[Y|X]\) from the unconditional entropy \(S[Y]\):

\[
I[Y : X] = S[Y] - S[Y|X] \tag{3}
\]

\[
= \sum_{k,l=1}^{n} p(x_k, y_l) \ln \frac{p(x_k, y_l)}{p(x_k)p(y_l)} \tag{4}
\]

where \(p_X\) and \(p_Y\) are the marginal probabilities \(p_X(x_k) = \sum_{l=1}^{n} p(x_k, y_l)\) and \(p_Y(y_l) = \sum_{k=1}^{n} p(x_k, y_l)\).

The mutual information is non-negative, \(I[Y : X] \geq 0\), symmetric, \(I[Y : X] = I[X : Y]\), and characterizes the entropic response of one variable to fluctuations of another. For two bijectively related random variables \(I[X : Y] = S[X]\), and we return to the entropy. For independent random quantities \(X\) and \(Y\), \(I[X : Y] = 0\). Conversely, \(I[X : Y] = 0\) implies that \(X\) and \(Y\) are independent. Thus \(I[X : Y]\) is a non-linear correlation function between \(X\) and \(Y\).

The information-theoretic meaning of the mutual information \(I[X : Y]\) is recalled in Appendix \[\text{A}\]. \(I[X : Y]\) is related to the information shared via a noisy channel with input \(X\) and output \(Y\), or, alternatively, with input \(Y\) and output \(X\).

For continuous random variable \(X\) and \(Y\) with the joint probability density \(P(x, y)\), respectively, the mutual information reads

\[
I[X : Y] = \int dx \int dy \, P(x,y) \ln \frac{P(x,y)}{P_X(x)P_Y(y)}, \tag{5}
\]

where additive constants have cancelled after taking the difference in \[\text{II}\].

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3 Note that generally \(S[Y|x_l] < S[Y]\) does not hold, i.e., it is not true that any single observation reduces the entropy. Such a reduction occurs only in average.
III. MODEL CLASS: TWO COUPLED BROWNIAN PARTICLES

Consider two brownian particles with coordinates \( x = (x_1, x_2) \) interacting with two independent thermal baths at temperatures \( T_1 \) and \( T_2 \), respectively, and subjected to a potential [Hamiltonian] \( H(x) \). The corresponding time-dependent random variables will be denoted via \( (X_1(t), X_2(t)) \); their realizations are \( (x_1, x_2) \).

The overdamped limit of the Brownian dynamics is defined by the following two conditions [46]: i) The characteristic relaxation time of the (real) momenta \( \dot{m}\dot{x}_i \) is much smaller than the one of the coordinates. This condition is satisfied due to strong friction and/or small mass [46]. ii) One is interested in times which are much larger than the relaxation time of the momenta, but which can be much smaller than [or comparable to] the relaxation time of the coordinates. Under these conditions the dynamics of the system is described by Langevin equations [46]:

\[
0 = -\partial_t H - \Gamma_i \dot{x}_i + \eta_i(t),
\]

\[
\langle \eta_i(t) \eta_j(t') \rangle = 2 \Gamma_i T_i \delta(t - t') \quad i, j = 1, 2,
\]

where \( \Gamma_i \) are the damping constants, which characterize the coupling of the particles to the respective baths, \( \delta_{ij} \) is the Kronecker symbol, and where \( \dot{\partial}_i \equiv \partial / \partial x_i \). It is assumed that the relaxation time toward the total equilibrium (where \( T_2 = T_1 \)) is much larger than all considered times; thus for our purposes \( T_2 \) and \( T_1 \) are constant parameters.

Eq. (6) comes from the Newton equation (mass \( \times \) acceleration = conservative force + friction force + random force) upon neglecting the mass \( \times \) acceleration due to strong friction and/or small mass [46]. Among many other realizations, Eq. (6) may physically be realized via two coupled RLC circuits. Then \( \Gamma_i = R_i \) corresponds to the resistance of each circuit, \( x_i \) is the charge, while the noise \( \eta_i \) refers to the random electromotive force. The overdamped regime would refer here to small inductance \( L_i \) and/or large resistance \( R_i \), while the Hamiltonian part \( H(x_1, x_2) \) collects separate effects of capacitances \( C_1 \) and \( C_2 \), as well as capacitance-capacitance coupling, e.g., \( H = \frac{x_1^2}{2C_1} + \frac{x_2^2}{2C_2} + \kappa x_1 x_2 \) in the harmonic regime. This example will be studied in section VII.

Below we use the following shorthands:

\[
\partial_t \equiv \partial / \partial t, \quad \partial_i \equiv \partial / \partial x_i, \quad x = (x_1, x_2), \quad dx = dx_1 dx_2, \quad y = (y_1, y_2), \quad dy = dy_1 dy_2.
\]

The joint probability distribution \( P(x_1, x_2; t) \) satisfies the Fokker-Planck equation [46]

\[
\partial_t P(x; t) + \sum_{i=1}^{2} \partial_i J_i(x; t) = 0,
\]

\[
\Gamma_i J_i(x; t) = -P(x; t) \partial_i H(x) - T_i \partial_i P(x; t),
\]

where \( J_1, J_2 \) are the currents of probability. Eq. (8) is supplemented by the standard boundary conditions

\[
P(x_1, x_2; t) \to 0 \quad \text{when} \quad x_1 \to \pm \infty \quad \text{or} \quad x_2 \to \pm \infty. \tag{10}
\]

A. Chapman-Kolmogorov equation.

To put our discussion in a more general context, let us recall that the process described by [8, 9] is Markovian and satisfies the Chapman-Kolmogorov equation [46]:

\[
P(x; t + \tau) = \int dy P(x; t + \tau | y; t) P(y; t), \tag{11}
\]

where for \( \tau \to 0 \) the conditional probability density \( P(x; t + \tau | y; t) \) is written as [46]

\[
P(x; t + \tau | y; t) = \delta(x_1 - y_1) \delta(x_2 - y_2) + \tau \delta(x_2 - y_2) G_1(x_1 | y_1; x_2) + \tau \delta(x_1 - y_1) G_2(x_2 | y_2; x_1) + O(\tau^2), \tag{12}
\]

where for \( i = 1, 2 \) we have defined

\[
G_i(x_1 | y_1; x_2) = \frac{1}{\Gamma_i} \partial_i \left[ \delta(y_1 - x_1) \partial_i H(x) + T_i \partial_i \delta(y_1 - x_1) \right].
\]

Note from (12–14) that

\[
P(x_1; t + \tau | y; t) P(x_2; t + \tau | y; t)
\]

\[
= P(x_1; x_2; t + \tau | y; t) + O(\tau^2),
\]

which means that the conditional dependence of \( X_1(t + \tau) \) and \( X_2(t + \tau) \), given \( X_1(t) \) and \( X_2(t) \) vanishes with second order in \( \tau \).

Eqs. [8, 9] are recovered after substituting (12–14) into (11) and noting \( P(x; t + \tau) = P(x; t + \tau) + \tau \partial_t P(x; t) + O(\tau^2) \):

\[
\partial_t P(x; t) = \int dy_1 G_1(x_1 | y_1; x_2) P(y_1, x_2; t)
\]

\[
+ \int dy_2 G_2(x_2 | y_2; x_1) P(x_1, y_2; t). \tag{16}
\]

IV. INFORMATION-THEORETICAL DEFINITION OF INFORMATION FLOW

A. Task

We consider the task of predicting gain (or loss) of the future of subsystem 1 from the present state of subsystem 2. This task can by quantified by the information flow \( I_{2 \rightarrow 1} \), which is defined via the time-shifted mutual information

\[
I_{2 \rightarrow 1}(t) = \partial_{\tau} I[X_1(t + \tau) : X_2(t)] \big|_{\tau \to 0} \tag{17}
\]

\[
= \lim_{\tau \to 0} \frac{1}{\tau} \left( I[X_1(t + \tau) : X_2(t)] - I[X_1(t) : X_2(t)] \right).
\]

\( i_{2 \rightarrow 1}(t) \) is defined analogously with interchanging 1 and 2.
Recall that the mutual information $I[X_1(t + \tau) : X_2(t)]$ quantifies (non-linear) statistical dependencies between $X_1(t + \tau)$ and $X_2(t)$, i.e., it quantifies the extent of which the present (at time $t$) of $X_2$ can predict the future of $X_1$. Thus a positive $i_{2\rightarrow1}(t)$ means that the future of $X_1$ is more predictable for $X_2$ than the present of $X_1$. Thus, for $i_{2\rightarrow1}(t) > 0$, 2 is gaining control over 1, or that 1 gains autonomy with respect to 2. This is the meaning of negative information flow.

Noting from (5) that

$$I[X_1(t + \tau) : X_2(t)] = \int dx_1 dy_2 P_2(y_2; t) \times$$

$$P_{1|2}(x_1; t + \tau|y_2; t) \ln \frac{P_{1|2}(x_1; t + \tau|y_2; t)}{P_1(x_1; t + \tau)},$$

we work out $i_{2\rightarrow1}(t)$ with help of (12)[14]:

$$i_{2\rightarrow1} = \int dy \int dx_1 G_1(x_1|y_1:y_2) P(y; t) \ln \frac{P_{1|2}(x_1|y_2; t)}{P_1(x_1; t)}.$$  

Employing now (26) and the boundary conditions (10) we get from (18)

$$i_{2\rightarrow1}(t) = \int dx \left( \frac{\partial_1 P_1(x_1; t)}{P(x; t)} \right) \partial_1 J_1(x; t).$$  

Parametrizing the Hamiltonian as

$$H(x_1, x_2) = H_1(x_1) + H_2(x_2) + H_{12}(x_1, x_2),$$

where $H_{12}(x_1, x_2)$ is the interaction Hamiltonian, we obtain from (25)

$$i_{2\rightarrow1}(t) = \frac{1}{I_1} \int dx \left[ \partial_1 H_{12}(x) + T_{1} \partial_1 \ln P(x; t) \right] \partial_1 \ln \frac{P_1(x_1; t)}{P(x; t)}.$$  

The time-shifted mutual information was employed for quantifying information flow in reaction-diffusion systems [29], neuronal ensembles [30], coupled map lattices [31], and ecological dynamics [32].

### B. Basic features of information flow

1. As deduced from (19), the information flow is generally not symmetric

$$i_{2\rightarrow1}(t) \neq i_{1\rightarrow2}(t),$$

but the symmetrized information flow [17] is equal to the rate of mutual information

$$i_{2\rightarrow1}(t) + i_{1\rightarrow2}(t) = \frac{d}{dt} I[X_1(t) : X_2(t)].$$  

While the mutual information is symmetric with respect to inter-changings its arguments $I[X_1(t) : X_2(t)] = I[X_2(t) : X_1(t)]$ and it quantifies correlations between two random variables, the information flow is capable of distinguishing the source versus the recipient of information: $i_{2\rightarrow1}(t) > 0$ means that 2 is the source of information, and 1 is its recipient.

For $i_{2\rightarrow1}(t) > 0$ and $i_{1\rightarrow2}(t) > 0$ we have a feedback regime, where 1 and 2 are both sources and recipients of information. Now the interaction between 1 and 2 builds up the mutual information $I[X_1(t) : X_2(t)]$; see (22). In contrast, for $i_{2\rightarrow1}(t) < 0$ and $i_{1\rightarrow2}(t) < 0$ both particles are detached from each other, and the mutual information naturally decays.

For $i_{2\rightarrow1}(t) > 0$ and $i_{1\rightarrow2}(t) < 0$ we have one-way flow of information: 2 is source and 1 is recipient; likewise, for $i_{2\rightarrow1}(t) < 0$ and $i_{1\rightarrow2}(t) > 0$, 1 is source and 2 is recipient. A particular, but important case of this situation is when the mutual information is conserved: $\frac{d}{dt} I[X_1(t) : X_2(t)] = 0$. This is realized in a stationary case; see below. Now due to $i_{2\rightarrow1}(t) + i_{1\rightarrow2}(t) = 0$ the information behaves as a conserved resource (e.g., as energy): the amount of information lost by 2 is received by 1, and vice versa. Another example of one-way flow of information is $i_{2\rightarrow1}(t) > 0$ and $i_{1\rightarrow2}(t) \approx 0$.

2. The information flow $i_{2\rightarrow1}$ can be represented as [see (31)][17]

$$i_{2\rightarrow1} = \frac{dS[X_1(t)]}{dt} - \partial_x S[X_1(t + \tau) | X_2(t)]_{\tau \rightarrow 0}.$$  

The first term in the RHS of (23) is the change of the marginal entropy of $X_1$, while the second term is the change of the conditional entropy of $X_1$ with $X_2$ being frozen to the value $X_2(t)$. In other words, $i_{2\rightarrow1}(t)$ is that part of the entropy change of $X_1$ (between $t$ and $t + \tau$), which exists due to fluctuations of $X_2(t)$; see section [119]. This way of looking at the information flow is close to that suggested in [108]; see also [41][42] for related works. Appendix B studies in more detail the operational meaning of the freezing operation.

3. Eq. (27) implies the information flow $i_{2\rightarrow1}(t)$ can be divided in two components: a force-driven part $i_{2\rightarrow1}^F(t)$ and bath-driven (or fluctuation-driven) part $i_{2\rightarrow1}^B(t)$

$$i_{2\rightarrow1}(t) = i_{2\rightarrow1}^F(t) + i_{2\rightarrow1}^B(t),$$

$$i_{2\rightarrow1}^F(t) = \frac{1}{I_1} \int dx P[\partial_1 H_{12}] \partial_1 \ln \frac{P_1}{P},$$

$$i_{2\rightarrow1}^B(t) = \frac{T_1}{I_1} \int dx P_1 P_2 \partial_1 \ln P_2.$$  

where for simplicity we omitted all integration variables. The force-driven part $i_{2\rightarrow1}^F(t)$ nullifies together with the force $-\partial_1 H_{12}$ acting from the second particle on the first
particle. The fluctuation-driven part $i_{2-1}^B(t)$ nullifies together with the bath temperature $T_1$ (which means that the random force acting from the first bath is zero).

Note that although $i_{2-1}^B(t)$ is defined via the interaction Hamiltonian $H_{12}$, it does not suffer from the known ambiguity related to the definition of $H_{12}$. That is redefining $H_{12}(x)$ via $H_{12}(x) \rightarrow H_{12}(x) + f_1(x_1) + f_2(x_2)$, where $f_1(x_1)$ and $f_2(x_2)$ are arbitrary functions will not alter $i_{2-1}^B(t)$ (and will not alter $i_{2-1}(t)$, of course). Thus, changes in separate Hamiltonians $H_1(x_1)$ and $H_2(x_2)$ that do not alter the probabilities (e.g., sudden changes) do not influence $i_{2-1}^B(t)$. In contrast, sudden changes of the interaction Hamiltonian will, in general, contribute to $i_{2-1}^B(t)$.

The bath-driven contribution $i_{2-1}^B(t)$ into the information flow is negative, which means that for $2$ to be a source of information, i.e., for $i_{2-1}(t) > 0$, the force-driven part $i_{2-1}^P(t)$ should be sufficiently positive. In short, there is no transfer of information without force.

4. It is seen from (19) [or from (21)] that if the variables $X_1$ and $X_2$ are independent, i.e., $P(x; t) = P_1(x_1; t)P_2(x_2; t)$ the information flow $i_{2-1}$ nullifies:

$$i_{2-1}(t) = i_{1-2}(t) = 0 \quad \text{for} \quad P = P_1P_2.$$ (26)

In fact, both $i_{1-2}^P(t)$ and $i_{2-1}^P(t)$ nullify for $P = P_1P_2$.

If the two particles were interacting at times $t < t_{\text{switch}}$, but the interaction Hamiltonian $H_{12}$ [see (20)] is switched off at $t = t_{\text{switch}}$, the information flow $i_{2-1} = i_{2-1}^B(t)$ will be in general different from zero for times $t_{\text{relax}} + t_{\text{switch}} > t > t_{\text{switch}}$—where $t_{\text{relax}}$ is the relaxation time—since at these times the common probability will be still non-factorized, $P \neq P_1P_2$. However, since now $i_{2-1}(t) = i_{2-1}^B(t) < 0$ the particles can only decorrelate from each other: neither of them can be a source of information for another.

We note that $i_{2-1}^B(t) = -\frac{T_1}{T_2} \int dx_1 P_1(x_1)f_1(x_1)$ in (26) is proportional to the average Fisher information $F_1(x_1)$ [27]:

$$F_1(x_1) \equiv \int dx_2 P_{2|1}(x_2|x_1) [\partial_1 \ln P_{2|1}(x_2|x_1)]^2.$$ 1/$F_1(x_1)$ is the minimal variance that can be reached during any unbiased estimation of $x_1$ from observing $x_2$ [26]. In the sense of estimation theory $F_1(x_1)$ quantifies the information about $x_1$ contained in $x_2$, since $F_1(x_1)$ is larger for those distributions $P_{2|1}(x_2|x_1)$ whose support is concentrated at $x_2 \approx x_1$. 5. We conclude that once the interaction between 1 and 2 is switched off, 1 gets detached (diffuses away) from 2 by the rate equal to the diffusion constant $D_1$ times the average Fisher information about $x_1$ contained in $x_2$.

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5. For various applications of the Fisher information in the physics of Brownian motion see [45].

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FIG. 1: Time-discretized version of the joint dynamical evolution.

5. To visualize information flow, we time-discretize the two-particle system and consider the two time series $X_1(t), X_2(t)$ with $t = 1, 2, \ldots$. We model the system dynamics by a first order Markov process, as being the discrete analogue of a first order differential equation. It is described by the graphical model depicted in Fig. 1.

The graph is indeed meant in the sense of a causal structure [28], where an arrow indicates direct causal influence. In particular, the arrows on Fig. 1 mean that $X_1(t + 1)$ and $X_2(t + 1)$ become independent after conditioning over $(X_1(t), X_2(t))$; see (10) in this context. Assume, for the moment, that the arrow between $X_2(t)$ and $X_1(t + 1)$ was missing, e.g., $X_2(t)$ is simply a passive observation of $X_1(t)$. Then $X_2(t)$ and $X_1(t + 1)$ are conditionally independent given $X_1(t)$. Now the data processing inequality [27] imposes negativity of the (time-discretized) information flow

$$I[X_1(t + 1) : X_2(t)] - I[X_1(t) : X_2(t)] \leq 0,$$ (27)

because $X_1(t + 1)$ is obtained from $X_1(t)$ by a stochastic map (which can never increase the information about $X_2(t)$ without having access to this quantity). Expression (27) can, however, be properly below zero if the bath dissipates some of the information. If (27) is posi-
tive there must be an arrow from $X_2(t)$ to $X_1(t + 1)$, i.e., there is information flowing from $X_2$ to $X_1$. However, the presence of such an arrow does not guarantee positivity of $\mathcal{I}$ because the amount of information dissipated by the bath can exceed the one provided by the arrow. We can nevertheless interpret $I[X_1(t + 1) : X_2(t)] - I[X_1(t) : X_2(t)]$ as the information in the sense of a net effect.

V. THERMODYNAMIC ASPECTS OF INFORMATION FLOW

A. Information flow nullifies in equilibrium

So far we have discussed the formal definition of information flow $\mathcal{I}_{2\to1}$. This definition relates $\mathcal{I}_{2\to1}$ to the concept of mutual information. It is however expected that there should be a more physical way of understanding $\mathcal{I}_{2\to1}$, since the concepts of entropy and entropy flow are defined and discussed in thermodynamics of non-equilibrium systems [18, 49, 50, 53]. We now turn to this aspect.

Recall that the Fokker-Planck equation (8, 9) describes a system interacting with two thermal baths at different temperatures $T_1$ and $T_2$. If $T_1 = T_2 = T$ then the two-particle system relaxes with time to the Gibbs distribution with the common temperature $T$ [46]:

$$P_{eq}(x) = \frac{1}{Z} e^{-\beta H(x)}, \quad Z = \int dx e^{-\beta H(x)}. \quad (28)$$

An important feature of the equilibrium probability distribution [28] is that the currents of probability [9] do not depend on time and explicitly nullify in that state

$$J_i(x) = J_2(x) = 0. \quad (29)$$

This detailed balance feature—which is both necessary and sufficient for equilibrium—indicates that in the equilibrium state there is no transfer of any physical quantity, e.g., there is no transfer of energy (heat).

Eq. (14) implies that the same holds for $\mathcal{I}_{2\to1}$: in the equilibrium state there is no information flow,

$$\mathcal{I}_{2\to1} = \mathcal{I}_{1\to2} = 0. \quad (30)$$

In other words, the transfer of information should always be connected with a certain non-equilibrium situation. One may distinguish several types of such situations:

i) Non-stationary (transient) states, where the joint distribution $P(x_1, x_2; t)$ is time-dependent.

ii) Stationary state, but non-equilibrium states realized for $T_1 \neq T_2$. Now the probability currents $J_1$ and $J_2$ are not zero (only $\partial_1 J_1 + \partial_2 J_2 = 0$ holds), and so are the information transfer rates $\mathcal{I}_{2\to1}$ and $\mathcal{I}_{1\to2}$. However, since the state is stationary, their sum nullifies due to (22)

$$\mathcal{I}_{2\to1} + \mathcal{I}_{1\to2} = 0. \quad (30)$$

This on indicates on one-way flow of information. Below we clarify how this flow relates to the temperature difference.

iii) Non-equilibrium states can be maintained externally by time-varying conservative forces, or, alternatively, by time-independent non-conservative forces accompanied by cyclic boundary conditions. Here we do not consider this type of non-equilibrium.

B. Entropy production and heat dissipation

A non-equilibrium state will show a tendency towards equilibrium, or, as expressed by the second law, by the entropy production of the overall system (in our case the brownian particles plus their thermal baths). Since for the model [48, 50, 53] the baths are in equilibrium and the cause of non-equilibrium is related to the brownian particles, the overall entropy production can be expressed via the variables pertaining to the brownian particles [48, 49, 50]. Recall (16) and define for $i = 1, 2$

$$\frac{d_iQ}{dt} = -\int dx H(x) \partial_i J_i(x; t), \quad (31)$$
$$\frac{d_iS}{dt} = \int dx \ln P(x; t) \partial_i J_i(x; t), \quad (32)$$

and note that these quantities satisfy

$$\ell_i = \frac{d_iS}{dt} - \beta_i \frac{d_iQ}{dt} = \beta_i \Gamma_i \int dx J_i^2(x; t) P(x; t) \geq 0, \quad (33)$$

where $\ell_1 = \ell_2 = 0$ holds in the equilibrium only, where $J_1 = J_2 = 0$; see (20).

Our discussion in section 4.1A implies that $\frac{d_iQ}{dt}$ is the change of energy of the two-particle system due to the dynamics of $x_i$. Since in the overdamped regime this dynamics is driven by the bath at temperature $T_i$, we see that $\frac{d_iQ}{dt}$ is the heat received by the two-particle system from the bath at temperature $T_i$ [50, 53]. Thus the energy (heat) received by the bath is $-\frac{d_iQ}{dt}$. Likewise, $\frac{d_iS}{dt}$ is the change of entropy of the two-particle system due to the dynamics of $x_i$. Then (33) is the local version of the Clausius inequality [50, 53], which implies that $\ell_i$ is the local entropy produced per unit time due to the interaction with the bath at temperature $T_i$, while

$$\ell = \ell_1 + \ell_2 = \frac{dS}{dt} - \beta_1 \frac{d_1Q}{dt} - \beta_2 \frac{d_2Q}{dt} \quad (35)$$

is the total entropy produced per unit of time in the overall system: two equilibrium baths plus two brownian particles [48, 49, 50, 53].

Noting our discussion in Appendix B one can see that $\frac{\ell}{\beta_i \Gamma_i}$ is equal to the space-average $\int dx P(x; t) v_i^2(x; t)$, where $v_i(x; t)$ is the coarse-grained velocity; see [44, C4].
the non-equilibrium stationary state realized for $T \neq T_2$, since then $J_1(\mathbf{x}) \neq 0$. This is consistent with the interpretation of these states as “metastable” non-equilibrium states, where some heat flows between the two baths, but the temperatures $T_i$ do not change in time due to the macroscopic size of the baths (these temperatures would change for very large times, which are, however, beyond the time-scales considered here.)

Likewise, $T_i \ell_i$ is the heat dissipated per time due to the interaction with the bath at temperature $T_i$, while $T_1 \ell_1 + T_2 \ell_2$ is the total dissipated energy (heat) per time.

C. Equivalence between flow of mutual information and flow of entropy

The picture that emerges from the above consideration is as follows. The entropy is produced with the rate $\ell_1$ somewhere at the interface between the first brownian particle and its bath. Eq. (34) expresses the fact that $\ell_1$ is the rate by which a part of the produced entropy flows into the two-particle system. The rest of the produced entropy goes to the bath with the rate $-\beta_1 \frac{d_1 Q}{dt}$. This is consistent with the conservation of energy and the fact that the bath itself is in equilibrium: then $-\beta_1 \frac{d_1 Q}{dt}$ is the rate by which heat goes to the bath, and after dividing by $T_1$—since the bath is in equilibrium—this becomes the rate with which the bath receives entropy.

The corresponding argumentation can be repeated for the second particle and its thermal bath.

The thermodynamic definition of entropy flow eventually reads

$$i_{2\rightarrow 1} = \frac{d_1 S_1}{dt} - \frac{d_1 S_2}{dt}. \quad (36)$$

Once $\frac{d_1 S}{dt}$ is the entropy entering into the two-particle system via the first particle, then subtracting $\frac{d_1 S}{dt}$ from the entropy rate $\frac{dS}{dt}$ of the first particle itself, we get the entropy flow from the second particle to the first one.

It should be now clear that this definition of entropy flow is just equivalent to the definition of information flow (17). Eq. (36) can also be written as

$$i_{2\rightarrow 1} = \frac{d_1 S_1}{dt} + \frac{d_1 S_2}{dt} = \frac{d_1 S_1}{dt} - \frac{d_1 S_2}{dt}, \quad (37)$$

making clear again that $i_{2\rightarrow 1}$ is the change of mutual information due to dynamics of the first particle.

VI. EFFICIENCY OF INFORMATION FLOW

Once it is realized that non-zero information transfer is possible only out of equilibrium, the existence of such a transfer is related to entropy production. This is an entropic cost of the information transfer. One can define a dimensionless ratio:

$$\eta_{2\rightarrow 1} = \frac{i_{2\rightarrow 1}}{\ell}, \quad (38)$$

which is the desired output (= information transfer) over the irreversibility cost (= total entropy production). This quantity characterizes the efficiency of information transfer. A large $\eta$ is desirable, since it gives larger information transfer rate at lesser cost.

Note that $\eta_{2\rightarrow 1}$ is more similar to the coefficient of performance of thermal refrigerators—which is also defined as the useful output (heat extracted from a colder body) to the cost (work)—than to the efficiency of heat engines. The latter is defined as the useful output over the resource entered into the engine. We shall still call $\eta_{2\rightarrow 1}$ efficiency, but this distinction is to be kept in mind.

Below in several different situations we shall establish upper bounds on $\eta$. These determine the irreversibility (entropy) cost of information transfer. The usage of the efficiency for informational processes was advocated in [51].

A. Stationary case

In the stationary two-temperature scenario the joint probability $P(x_1, x_2)$ and the probability currents $J_1(x_1, x_2)$ and $J_2(x_1, x_2)$ do not depend on time. Thus many observables—e.g., the average energy of the two Brownian particles, their entropy, entropies of separate particles—do not depend on time either.

Employing (35), $i_{2\rightarrow 1} = -\frac{d_1 S_1}{d_1 S_2} = -\frac{d_1 S_2}{d_1 S_1}$ and (33) we get

$$\ell = (\beta_2 - \beta_1) \frac{d_1 Q}{dt}, \quad T_2 i_{2\rightarrow 1} + \frac{\ell}{\beta_2 - \beta_1} = T_2 \ell_2 \geq 0,$$

$$i_{2\rightarrow 1} = \frac{T_1 \ell}{T_2 - T_1} + \ell_2. \quad (39)$$

After interchanging the indices 1 and 2 we get the analogous equation for $i_{1\rightarrow 2}$. Recalling that (30) holds in the
stationary state, we get
\[ t_{2\rightarrow 1} = \frac{T_2\ell}{T_2 - T_1} - \ell_1. \] (40)

First of all, \( t_{2\rightarrow 1} \geq 0 \) for \( T_2 \geq T_1 \), which means that in the stationary two-temperature scenario the information (together with heat) flows from higher to lower temperature.

For the efficiency \( \eta_{2\rightarrow 1} \) we get from \( t_{2\rightarrow 1} \):
\[ \eta_{2\rightarrow 1} = \frac{T_1}{T_2 - T_1} + \frac{\ell_2}{\ell_1 + \ell_2} \] (41)
\[ = \frac{T_1}{T_2 - T_1} - \frac{\ell_1}{\ell_1 + \ell_2} \] (42)

This then implies (since \( \ell_1 \geq 0, \ell_2 \geq 0 \))
\[ \frac{T_1}{T_2 - T_1} \leq \eta_{2\rightarrow 1} \leq \frac{T_2}{T_2 - T_1}. \] (43)

For \( T_2 > T_1, \eta_{2\rightarrow 1} \) is positive and is bounded from above by \( \frac{T_2}{T_2 - T_1} \), which means that in the stationary state the ratio of the information transfer rate (over the entropy production rate) is bounded.

It is important to note from \( \frac{T_1}{T_2 - T_1} \) that for \( T_2 \) approaching \( T_1 \) from above, \( T_2 \rightarrow T_1 \), the efficiency of information transfer tends to plus infinity, since in this reversible limit the entropy production, which should be always positive, naturally scales as \( (T_1 - T_2)^2 \), while the information flow scales as \( T_2 - T_1 \). Thus very slow flow of information can be accompanied by very little entropy production such that the efficiency becomes very large. A similar argument led some authors to conclude that there is no fundamental cost for the information transfer at all. Our analysis makes clear that this interpretation would be misleading. It is more appropriate to say that in the reversible limit the thermodynamic cost, while becoming less restrictive (since the efficiency can be very large), is certainly still there, because the difference \( T_2 - T_1 \) is, after all, always finite; otherwise the very information flow would vanish.

It remains to stress that the relations \( \eta_{2\rightarrow 1} \) are general and do not depend on the details of the considered Brownian system. They will hold for any bi-partite Markovian system which satisfies to master equation (33) and local formulations of the second law (33, 34).

B. Reachability of the upper efficiency bound

In the remaining part of this section we show that the upper bound (13) for the efficiency of information flow is reached in a certain class of Brownian systems satisfying time-scale separation.

1. The stationary probability in the adiabatic case.

For \( T_1 = T_2 \), the stationary probability distribution \( P(x) \) of the two-particle system (8) is Gibbsian:
\[ P(x) \propto e^{-\beta_1 H(x)}. \]

For non-equal temperatures \( T_1 \neq T_2 \) a general expression for this stationary probability can be derived in the adiabatic situation, where \( x_2 \) changes in time much slower than \( x_1 \). This is ensured by
\[ \epsilon = \frac{2}{\Gamma_1}. \] (44)

Below we present a heuristic derivation of the stationary probability distribution \( P(x) \) in the order of magnitude estimate of \( J_1(x) \) nullifies. This means that \( J_1(x) \) has to be searched for
to first order in $\varepsilon = \Gamma_1 / \Gamma_2$. Assuming for the corrected probability $\tilde{P}(x) = P(x)[1 - \varepsilon A(x)] + O(\varepsilon^2)$ we get

$$J_1(x) = \varepsilon \frac{T_1}{T_2} P(x) \partial_x A(x),$$

(49)

where $A(x)$ does not depend on $\varepsilon$ and is to be found from the stationarity equation $\partial_t J_1 + \partial_x J_2 = 0$. Concrete expressions for $A(x)$ are presented in [53].

2. Entropy production and heat dissipation in the adiabatic stationary case.

Using (54) together with (47) and (49) we get for the partial entropy productions:

$$\ell_1 = \varepsilon \frac{T_1}{T_2} \int dx [\partial_x A(x)]^2 P(x),$$

(50)

$$\ell_2 = \frac{(T_1 - T_2)^2}{T_2 T_1 \Gamma_2} \int dx \phi_{1-2}^2(x) P(x).$$

(51)

Note that $\ell_1$ contains an additional small factor $\varepsilon$ as compared to $\ell_2$. This is natural, since the fast system $x_1$ is in a local thermal equilibrium. Thus in the considered order $\varepsilon^0$ the overall entropy production $\ell$ is dominated by the entropy production of the slow sub-system:

$$\ell_1 = 0, \quad \ell = \ell_2.$$

(52)

Thus the heat dissipation is simply $T_2 \ell = T_2 \ell_2$. The physical meaning of $\ell_1 = 0$ is that the fast system does not produce entropy [and does not dissipate heat], since in its fast characteristic times it sees the slow variable as a frozen [not fluctuating] field. Thus the fast system remains in local equilibrium.

3. Efficiency of information transfer in the adiabatic stationary case.

Employing (52) we can immediately see from (42) that the efficiency in the considered order of $\varepsilon$ reads:

$$\eta_{2 \rightarrow 1} = \frac{T_2}{T_2 - T_1}. $$

(53)

Thus, the efficiency reaches its maximal value when the hotter system is the slowest one.

Recall that in the adiabatic limit the information flow is of order $O(1/\Gamma_2)$, i.e., it is small on the characteristic time scale of the fast particle, but sizable on the characteristic time of the slow particle. In more detail, using (40) (47) we get for the information flow $\ell_{2 \rightarrow 1}$ from the slow to the fast particle:

$$\ell_{2 \rightarrow 1} = \frac{T_2 - T_1}{T_2 T_1 \Gamma_2} \int dx \phi_{1-2}^2(x) P(x).$$

(54)

The reachability of the upper bound for the efficiency $\eta_{2 \rightarrow 1}$ appears to resemble the reachability of the optimal Carnot efficiency for heat engines. However, for heat engines the Carnot efficiency is normally reached for processes that are much slower than any internal characteristic time of the engine working medium. In contrast, the information flow in (53) is sizable on the time-scale $\Gamma_2$ (which is one of the internal time-scales).

VII. EXACTLY SOLVABLE MODEL: TWO COUPLED HARMONIC OSCILLATORS

We exemplify the obtained results by the exactly solvable model of two coupled harmonic oscillators. We shall also employ this model to check whether the upper bound (43) may hold in the non-stationary situation.

The Hamiltonian [or potential energy] is given as

$$H(x) = \frac{a_1}{2} x_1^2 + \frac{a_2}{2} x_2^2 + b x_1 x_2,$$

(55)

where the constants $a_1$, $a_2$ and $b$ have to satisfy

$$a_1 > 0, \quad a_2 > 0, \quad a_1 a_2 > b^2,$$

(56)

for the Hamiltonian to be positively defined.

Let us assume that the probability of the two-particle system is Gaussian [this holds for the stationary probability as seen below]

$$- \ln P(x) = \frac{A_1}{2} x_1^2 + \frac{A_2}{2} x_2^2 + B x_1 x_2 + \text{constant},$$

(57)

Here we did not specify the irrelevant normalization constant, and the constants $A_1$, $A_2$ and $B$ read

$$\left( \begin{array}{cc} A_1 & B \\ B & A_2 \end{array} \right)^{-1} = \left( \begin{array}{cc} \langle x_1^2 \rangle & \langle x_1 x_2 \rangle \\ \langle x_1 x_2 \rangle & \langle x_2^2 \rangle \end{array} \right) = \left( \begin{array}{cc} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{array} \right),$$

(58)

where $\langle \ldots \rangle$ is the average over the probability distribution (57). All the involved parameters ($A_1$, $A_2$ and $B$) can be in general time-dependent. Note that in (57) we assumed $\langle x_1 \rangle = \langle x_2 \rangle = 0$.

For the Hamiltonian (55) and the probability (57) the information transfer reads from (21):

$$\ell_{1 \rightarrow 2} = \frac{\sigma_{12} (T_1 B - b)}{\Gamma_1 \sigma_{11}}.$$  

(59)

Likewise, we get for the local entropy production (53):

$$\ell_1 = \frac{T_1}{\Gamma_1} \left[ \sigma_{11} (\beta_1 a_1 - A_1)^2 + \sigma_{22} (\beta_1 b - B)^2 + 2 \sigma_{12} (\beta_1 a_1 - A_1)(\beta_1 b - B) \right],$$  

(60)

where the information transfer $\ell_{1 \rightarrow 2}$ and the entropy production $\ell_2$ are obtained from (59) and (60), respectively, by interchanging the indices 1 and 2.
A. Stationary case.

Provided that the stability conditions (55) hold, any initial probability of the two-particle system relaxes to the time-independent Gaussian stationary probability (57) [46]. The averages $\langle x^2_1 \rangle$, $\langle x_1 x_2 \rangle$, and $\langle x_1^2 \rangle$ are deduced from the stationarity equation $\partial_t J_1(x) + \partial_x J_2(x) = 0$:

$$\langle x^2_1 \rangle = \frac{a_2 T_1 (a_1 \gamma_1 + a_2 \gamma_2) + b^2 \gamma_1 (T_2 - T_1)}{(a_1 a_2 - b^2)(a_1 \gamma_1 + a_2 \gamma_2)},$$

(61)

$$\langle x^2_2 \rangle = \frac{a_1 T_2 (a_1 \gamma_1 + a_2 \gamma_2) + b^2 \gamma_2 (T_1 - T_2)}{(a_1 a_2 - b^2)(a_1 \gamma_1 + a_2 \gamma_2)},$$

(62)

$$\langle x_1 x_2 \rangle = \frac{-b(a_2 \gamma_2 T_1 + a_1 \gamma_1 T_2)}{(a_1 a_2 - b^2)(a_1 \gamma_1 + a_2 \gamma_2)},$$

(63)

$$\langle x_1^2 \rangle - \langle x_1 \rangle \langle x_2 \rangle = \frac{b^2 \gamma_1 \gamma_2 (T_1 - T_2)^2 + (a_1 \gamma_1 + a_2 \gamma_2)^2 T_1 T_2}{(a_1 a_2 - b^2)(a_1 \gamma_1 + a_2 \gamma_2)^2},$$

(64)

where

$$\gamma_1 = \frac{1}{\Gamma_1}, \quad \gamma_2 = \frac{1}{\Gamma_2},$$

and where the conditions $\langle x_1 \rangle = \langle x_2 \rangle = 0$ hold automatically. Let us introduce the following dimensionless parameters:

$$\varphi = \frac{\gamma_1 a_1}{\gamma_2 a_2} = \frac{\Gamma a_1}{a_2 \gamma}, \quad \xi = \frac{T_1}{T_2}, \quad \kappa = \frac{b^2}{a_1 a_2},$$

(65)

where $\varphi$ is the ratio of time-scales, while $\kappa$ characterizes the interaction strength; note that $0 < \kappa < 1$ due to (56).

Using (61)–(65) together with (59) and (60), we get for the information flow, the total entropy production, and the efficiency

$$\ell_{2-1} = \frac{a_2}{\Gamma_2} \frac{\kappa \varphi (1 - \xi) \left( \xi + \varphi \right)}{1 + \varphi},$$

(66)

$$\ell = \frac{a_2}{\Gamma_2} \frac{1 - \xi^2}{\xi} \frac{\kappa \varphi}{1 + \varphi},$$

(67)

$$\eta_{2-1} = \frac{\xi}{1 - \xi} \frac{(\xi + \varphi)(1 + \varphi)}{\kappa \varphi (1 - \xi)^2 + (1 + \varphi)^2}.$$

(68)

Let us for simplicity assume that $\xi < 1$, i.e., $T_2 > T_1$. Now $\ell_{2-1} > 0$. It is seen that both $\ell_{2-1}$ and $\ell$ are monotonically increasing function of $\varphi$. Thus both the information transfer and the overall entropy production maximize when the system attached to the hotter bath at the temperature $T_2$ is slower [the same holds for the overall heat dissipation]. The efficiency $\eta_{2-1} = \ell_{2-1}/\ell$ is also equal to its maximal value (63) in the same limit $\varphi \to \infty$; see (69).

The behaviour with respect to the dimensionless coupling $\kappa$ is different. Now $\ell_{2-1}$ and $\ell$ are again monotonically increasing functions of $\kappa$. They both maximize for $\kappa \to 1$, which—as seen from (61)–(63)—means at the instability threshold, where the fluctuations are very large. However, the efficiency $\eta_{2-1}$ is a monotonically decreasing function of $\kappa$. It maximizes (as a function of $\kappa$) in the almost uncoupled limit $\kappa \to 0$. Thus, as far as the behaviour with respect of the coupling constant is concerned, there is a complementarity between maximizing the information flow and maximizing its efficiency.

B. Non-stationary situation.

Once the upper bound (43) on the efficiency of information transfer is established in the stationary situation, we ask whether it might survive also in the non-stationary case. Below we reconsider the coupled harmonic oscillators and show that, although the upper limit (43) can be exceeded by some non-stationary states, the efficiencies $\eta_{2-1}$ and $\eta_{1-2}$ are still limited from above.

We assume that a non-equilibrium probability of the two harmonic oscillators is Gaussian, as given by (57) [68]. The information flow and the entropy production are given by (59) and (60), respectively. Any Gaussian probability can play the role of some non-stationary state, which is chosen, e.g., as an initial condition.

To search for an upper bound of the efficiency in the non-stationary situation, we assume a Gaussian probability (57) and maximize the efficiency (55) over the Hamiltonian, i.e., over the parameters $a_1, a_2$ and $b$ in (54). The stability conditions (56), which are necessary for the existence of the stationary state, are not necessary for studying non-stationary situations (imagine a non-stationary down-hill moving oscillator). Thus conditions (55) will not be imposed during the maximization over $a_1, a_2$ and $b$.

Note that $\ell_{2-1}$ in (59) does not depend on $a_1$ and $a_2$. Since we are interested in maximizing the efficiency (55), the first step in this maximization is to minimize the local entropy production $\ell_1$ [given by (60)] over $a_1$. This produces

$$a_1 = \frac{T_1 - b \sigma_{12}}{\sigma_{11}}, \quad \ell_1 = \frac{T_1 (\beta_1 b - B)^2}{\Gamma_1 A_2}.$$

(69)

The second entropy production $\ell_2$ is analogously minimized over $a_2$. The resulting expression for the efficiency reads:

$$\eta_{2-1} = \frac{\sigma_{12}}{\Gamma_1 \sigma_{11}} \frac{b_2 (T_1 b - B)^2}{\Gamma_1 A_2} + \frac{b_2}{\ell_1} \frac{(T_1 b - B)^2}{\ell_1}.$$

(70)

Next, we maximize this expression over $b$. Let us for simplicity assume $\sigma_{12} > 0$, since the conclusions do not change for $\sigma_{12} < 0$. The maximal value of $\eta_{2-1}$ reads

$$\eta_{2-1} = \frac{T_1}{2|T_2 - T_1|} \frac{[\text{sign}(T_2 - T_1) + \sqrt{1 + \chi}]}{\sqrt{1 + \chi}},$$

(71)

where

$$\chi = \frac{T_2 \Gamma_2 A_1}{T_1 \Gamma_1 A_2} = \frac{T_2 \Gamma_2 \sigma_{22}}{T_1 \Gamma_1 \sigma_{11}}.$$

(72)
and where the RHS of (71) is reached for
\[ b = -\frac{\sigma_{12}}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \left( \frac{T_1 + |T_2 - T_1|}{\sqrt{1 + \chi}} \right). \] (73)

The information flow \( \nu_{2 \rightarrow 1} \) at the optimal value of \( b \) reads
\[ \nu_{2 \rightarrow 1} = \frac{\sigma_{12}^2 |T_1 - T_2|}{\sigma_{11} \sqrt{1 + \chi}}. \] (74)

Note that the maximal value of \( \nu_{2 \rightarrow 1} \) does not require a strong inter-particle coupling. This coupling, as quantified by (73) can be small due to \( \sigma_{12} \to 0 \).

Now we assume that the temperatures \( T_1 \) and \( T_2 \) are fixed. It is seen from (71) that taking \( \chi \) sufficiently large—either due to a large \( \frac{T_2}{T_1} \), which means that the second particle is slow, or due to a large \( \frac{\sigma_{22}}{\sigma_{11}} \), which means that its dispersion is larger—we can achieve efficiencies as large as desired. In all these cases, for a sufficiently large \( \chi \), \( \eta_{2 \rightarrow 1} \) scales as \( \sqrt{\chi} \). Thus one can overcome the stationary bound (113) via some special non-stationary states and the corresponding Hamiltonians. We, however, see from (74) that increasing the efficiency due to \( \chi \to \infty \) leads to decreasing the information flow. This is the same trend as in the stationary case; see (113).

The large values of \( \eta_{2 \rightarrow 1} \) do not imply anything special for the efficiency of the inverse information transfer efficiency \( \eta_{1 \rightarrow 2} \). Indeed, the partially optimized \( \eta_{1 \rightarrow 2} \) reads analogously to (70)
\[ \eta_{1 \rightarrow 2} = \frac{\sigma_{12} \sqrt{T_2(B - b)}}{G_2 + \frac{\sigma_{11}(A_1B - b)^2}{A_2} + \sigma_{12}^2 (T_2B - b)^2}. \] (75)

At the values (69), (63), where \( \eta_{2 \rightarrow 1} \) extremizes, \( \eta_{1 \rightarrow 2} \) assumes a simple form
\[ \eta_{1 \rightarrow 2} = \frac{T_2}{2(T_1 - T_2)}, \]

i.e., \( \eta_{1 \rightarrow 2} \) depends only on the temperatures and can have either sign, depending on the sign of \( T_1 - T_2 \). Note that \( \eta_{1 \rightarrow 2} \) follows the same logic as in the stationary state: it is positive for \( T_1 > T_2 \).

**VIII. COMPLEMENTARITY BETWEEN HEAT- AND INFORMATION-FLOW**

Contrary to the notion of information flow, the heat flow from one brownian particle to another calls for an additional discussion. The main reason for this is that the brownian particles could, in general, be non-weakly coupled to each other, and then the local energy of a single particle is not well defined; for various opinions on this point see (59). In contrast, the entropy of a single particle is always well-defined; recall in this context feature 3 in section III. Nevertheless, the notion of a separate energy can be applied once there are physical reasons for selecting a particular form of the interaction Hamiltonian \( H_{12}(x_1, x_2) \) in (20), and the average value of \( H_{12}(x_1, x_2) \) is conserved in time, at least approximately. (A particular case of this is when \( H_{12}(x_1, x_2) \) is small.) Now \( H_1(x_1) \) can be defined as the local energy of the first particle, while the average energy change is
\[ \frac{d}{dt} \left[ \int dx_1 P_1(x_1)H_1(x_1) \right] = \frac{dE_1}{dt}. \]

The energy flow \( \epsilon_{2 \rightarrow 1} \) from the second particle to the first one is defined in full analogy with (37)
\[ \epsilon_{2 \rightarrow 1} = -\frac{dE_1}{dt} - \frac{d\bar{Q}}{dt}, \] (76)

where \( \frac{d\bar{Q}}{dt} \) is defined in (31). In words (76) means: the energy change \( \frac{dE_1}{dt} \) of the first particle is equal to the energy \( \epsilon_{2 \rightarrow 1} \) received from the second particle plus the energy \( d\bar{Q} \) put into the system via coupling of the first particle to its thermal bath.

Working out (76) we obtain
\[ \epsilon_{2 \rightarrow 1} = -\int dxJ_1(x, t)\partial_t H_{12}(x). \] (77)

The interpretation of (77) is straightforward via concepts introduced in Appendix C where we argue that \( \eta_1(x, t) = J_1(x, t) \) can be regarded as a coarse-grained velocity of the particle 1. The (77) becomes the average work done on the particle 1 by the force \( \partial_t H_{12}(x) \) generated the particle 2.

The symmetrized heat flow is equal to the minus change of the interaction energy
\[ \epsilon_{2 \rightarrow 1} + \epsilon_{1 \rightarrow 2} = -\frac{d}{dt} \int dxH_{12}(x)P(x, t). \] (78)

Thus the there is a mismatch between the heat flowing from 1 to 2, as compared to the energy flowing from 2 to 1. This mismatch is driven by the change of interaction energy, and it is small provided that the average interaction energy is [approximately] conserved in time.

It is important to stress that the ambiguity in the definition of heat flow, which was related to the choice of the local energy \( E_1 \), is absent in the stationary state, since the average interaction energy is conserved by construction. Now the choice of the single-particle energy is irrelevant, since any such definition will lead to time-independent quantity, which would then disappear from (76), \( \frac{dE_1}{dt} = 0 \), and from the RHS of (78).

As clear from its physical meaning, the efficiency \( \zeta_{2 \rightarrow 1} \) of heat flow is to defined as the ratio of the heat flow over the total heat dissipation
\[ \zeta_{2 \rightarrow 1} = \frac{\epsilon_{2 \rightarrow 1}}{T_1\ell_1 + T_2\ell_2}. \] (79)

Let us work out \( \zeta_{2 \rightarrow 1} \) for the two-temperature stationary case. Analogously to (111) we obtain
\[ \zeta_{2 \rightarrow 1} = \frac{T_1}{T_2 - T_1} + \frac{T_1\ell_1}{T_1\ell_1 + T_2\ell_2} \]
\[ = \frac{T_2}{T_2 - T_1} - \frac{T_2\ell_2}{T_1\ell_1 + T_2\ell_2} \] (80) (81)
Eq. (80) shows that, as expected, the heat flows from higher to lower temperatures. In addition, (80) (81) imply the same bounds as in (83):

\[
\frac{T_1}{T_2 - T_1} \leq \zeta_{2 \rightarrow 1} \leq \frac{T_2}{T_2 - T_1}.
\]  

(82)

Moreover, we note that the stationary efficiencies of information flow and heat flow are related as

\[
\zeta_{2 \rightarrow 1} \eta_{2 \rightarrow 1} = \frac{T_1 T_2}{(T_2 - T_1)^2}.
\]  

(83)

Note that the efficiencies \( e_{2 \rightarrow 1} \) and \( \eta_{2 \rightarrow 1} \) in (83) depend in general on the inter-particle and intra-particle potentials, temperatures, damping constants, etc. However, their product in the stationary state is universal, i.e., it depends only on the temperatures.

Eq. (83) implies the following complementarity: for fixed temperatures the set-up most efficient for the information transfer is the least efficient for the heat transfer and \textit{vice versa}. Recalling our discussion in section VII B we see that the upper bound \( \frac{T_2}{2} \) for the efficiency \( \zeta_{2 \rightarrow 1} \) is achieved for the adiabatic stationary state, where—on the contrary to the information transfer—the hotter system is faster, i.e., \( \ell_2 \to 0 \), but \( \ell_1 \) is finite.

Note from (77) (85) the following relation between between the heat flow (77) and the information flow (54):

\[
T_{12 \rightarrow 1} = e_{2 \rightarrow 1},
\]

which is valid in the adiabatic stationary state. Recall that \( T_1 \) here is the temperature of the bath interacting with the fast particle.

Let us illustrate the obtained results with the exactly solvable situation of two coupled harmonic oscillators; see section VII. Recalling (65) (68) and (77) we obtain

\[
e_{2 \rightarrow 1} = \frac{a_2 T_2}{T_2} \kappa \varphi \frac{1 - \xi}{1 + \varphi},
\]  

(84)

where we employed the dimensionless variables (65). Likewise, we get for the heat dissipation and the efficiency of the heat transfer [see (66) (68) for similar formulas]

\[
T_1 \ell_1 + T_2 \ell_2 = \frac{a_2 T_2}{T_2} \kappa \varphi (1 - \xi)^2 (\xi + \varphi),
\]  

(85)

\[
\zeta_{2 \rightarrow 1} = \frac{\xi (1 + \varphi)^2 + \kappa \varphi (1 - \xi)^2}{(1 + \varphi) (\xi + \varphi)(1 - \xi)}.
\]  

(86)

It is seen that the efficiency \( \zeta_{2 \rightarrow 1} \) maximizes at the instability threshold \( \kappa \to 1 \), in contrast to the efficiency \( \eta_{2 \rightarrow 1} \) of information transfer that maximizes at the weakest interaction; see section VII. As we already discussed above, as a function of the time-scale \( \varphi \), \( \zeta_{2 \rightarrow 1} \) maximizes for \( \varphi \to 0 \), again in contrast to the behaviour of \( \eta_{2 \rightarrow 1} \).

\section{Transfer Entropy}

\subsection{A. Definition}

We consider the task of predicting the future of \( X_1 \) from its own past (with or without the help of the present \( X_2 \)). Quantifying this task leads to a concept, which has been termed directed transinformation \cite{33} or transfer entropy \cite{36}. In the following we will use the latter name, as this seems to be more broadly accepted. Closely related ideas were expressed in \cite{37}. The notion of transfer entropy became recently popular among researchers working in various inter-disciplinary fields; see \cite{38} for a short review.

Our discussion of transfer entropy aims at two purposes. First, the information flow and transfer entropy are two different notions, and their specific differences should be clearly understood, so as to avoid any confusion \cite{39}. Nevertheless, in one particular, but important case, we found interesting relations between these two notions.

To introduce the idea of transfer entropy let us for the moment assume that the random quantities \( X_1(t) \) and \( X_2(t) \) (whose realizations are respectively the coordinates \( x_1 \) and \( x_2 \) of the brownian particles) assume discrete values and change at discrete instances of time:\( t, t + \tau, t + 2\tau, \ldots \). Recalling our discussion in section II we see that the conditional entropy \( S[X_1(t+\tau)|X_1(t)] \) is the entropy reduction (residual uncertainty) of \( X_1(t+\tau) \) due to knowing \( X_1(t) \). Likewise, \( S[X_1(t+\tau)|X_1(t), X_2(t)] \) characterizes the uncertainty of \( X_1(t+\tau) \) given both \( X_1(t) \) and \( X_2(t) \). The difference \( m_{2 \rightarrow 1} \) is the transfer entropy:

\[
m_{2 \rightarrow 1} \equiv \frac{1}{\tau} \left( S[X_1(t+\tau)|X_1(t)] - S[X_1(t+\tau)|X_1(t), X_2(t)] \right)
\]

\[
= \frac{1}{\tau} \left( I[X_1(t+\tau) : X_1(t), X_2(t)] - I[X_1(t+\tau) : X_1(t)] \right)
\]

\[
= \frac{1}{\tau} \sum_{x_1,y_1,y_2} p(y_1,y_2;t)p(x_1;t+\tau|y_1,y_2;t) \times \ln \frac{p(x_1;t+\tau|y_1,y_2;t)}{p_1(x_1;t+\tau|y_1;t)}.
\]  

(87)

\( m_{2 \rightarrow 1} \) measures the difference between predicting the future of \( X_2 \) from the present for both \( X_1 \) and \( X_2 \) (quantified by \( I[X_1(t+\tau)|X_1(t), X_2(t)] \)) and predicting the future of \( X_2 \) from its own present only (quantified by \( I[X_1(t+\tau)|X_1(t)] \)). Note that \( m_{2 \rightarrow 1} \) is always positive, since additional conditioning decreases the entropy. \( m_{2 \rightarrow 1} \) is also equal to the mutual information \( I[X_1(t+\tau) : X_2(t)|X_1(t)] \) shared between the present state of \( X_2 \) and the future state of \( X_1 \) conditioned upon the present state of \( X_1 \).

To explain the transfer entropy \textit{versus} information flow, we consider again the discretized version in Fig. II (where for simplicity we take \( \tau = 1 \)). By standard prop-
erities of mutual information, we have
\[ I[X_1(t + 1) : X_2(t)] \leq I[X_1(t + 1), X_1(t) : X_2(t)] \quad (88) \]
\[ = I[X_1(t) : X_2(t)] + I[X_1(t + 1) : X_2(t)|X_1(t)], \quad (89) \]
where the inequality in (88) is related to the strong subadditivity feature, and where the equality in (89) is the chain rule for the mutual information. Hence,
\[ m_{2-1} = I[X_1(t + 1) : X_2(t)] - I[X_1(t) : X_2(t)] \leq I[X_1(t + 1) : X_2(t)|X_1(t)] = m_{2-1}. \]

The LHS (left hand side) is the discrete version of information flow, the RHS the transfer entropy. If the arrow from \( X_2(t) \) to \( X_1(t + 1) \) was absent, the modified graph would impose that \( X_1(t + 1) \) and \( X_2(t) \) are conditionally independent, given \( X_1(t) \), i.e.,
\[ m_{2-1} = I[X_1(t + 1) : X_2(t)|X_1(t)] = 0. \]

This shows that the transfer entropy vanishes in this case, as it should be because there is no arrow transmitting information. Thus, \( m_{2-1} \) characterizes the strength of that arrow.

For continuous time [but still discrete variables \( X_1 \) and \( X_2 \)] we take \( \tau \to 0 \) in (54) producing
\[ m_{2-1} = \sum_{y_2, x_1 \neq y_1} p(y_2, y_1; t) g_1(x_1|y_1, y_2) \times \ln \frac{g_1(x_1|y_1, y_2)}{\sum_{x_2} g_1(x_1|y_1, z_2) P_2(z_2|x_1, y_1, t)}, \quad (90) \]
where \( g_1(x_1|y_1, y_2) \) is defined analogously to \( G_1 \) in (12), but with discrete random variables.

Eq. (90) cannot be translated to the continuous variable situation simply by interchanging the probabilities \( p \) with the probability densities \( P \), since attempting such a translation leads to singularities. The proper extension of (87) to continuous variables and continuous time reads
\[ m_{2-1} = \lim_{\tau \to 0} \frac{1}{\tau} \int dy P(y; t) \int P[dx_1^{t+\tau}|y; t] \times \ln \frac{P[dx_1^{t+\tau}|y; t]}{P[dx_1^{t+\tau}|y_1; t]}, \quad (91) \]
where \( P[dx_1^{t+\tau}|y; t] \) is the measure of all paths \( x_1(t) \) starting from \( y = (y_1, y_2) \) (from \( y_1 \)) at time \( t \) and ending somewhere at time \( t + \tau \), i.e., not the final point of the path is fixed, but rather the initial time and final times.\[ ^8 \]

This extension naturally follows general ideas of information theory in continuous spaces. For our situation the measures \( P[dx_1^{t+\tau}|y; t] \) and \( P[dx_1^{t+\tau}|y_1; t] \) refer to the stochastic process described by (61) [26]; see [43] for an introduction to such measures.

Eq. (61) is worked out in Appendix D producing
\[ m_{2-1} = \frac{1}{2T_1 T_2} \int dx P(x; t) \phi^{2-1}_2(x; t), \quad (92) \]
\[ \phi^{2-1}_2 \equiv \partial_t H_{12}(x) - \int dy_2 \partial_t H_{12}(x, y_2) P_2_{11}(y_2|x_1, t), \]
where \( \phi^{2-1}_2 \) is the force acting from 2 to 1 minus its conditional average; compare with (48).

B. Information flow versus entropy transfer: General differences.

Let us compare features of the entropy transfer \( m_{2-1} \) to those of information flow \( i_{2-1} \). We remind that difference between the information flow \( i_{2-1} \) and transfer entropy \( m_{2-1} \) stem from the fact that \( m_{2-1} \) refers to the prediction of the future of \( X_2 \) from its own past (with or without the help of the present of \( X_1 \), while \( i_{2-1} \) refers to the prediction gain (or loss) of the future of \( X_2 \) from the present of \( X_1 \). Thus for \( m_{2-1} \) the active agent is 1 predicting its own future, while for \( i_{2-1} \) the active agent is 2 predicting the future of 1.

1) Both \( m_{2-1} \) and \( i_{2-1} \) are invariant with respect to redefining the interaction Hamiltonian; see [25] and compare with information flow \( i_{2-1} \).

2) In contrast to the information flow \( i_{2-1} \), the entropy transfer \( m_{2-1} \) is always non-negative.

3) In contrast to \( i_{2-1} \), \( m_{2-1} \) does not nullify for factorized probabilities \( P(x) = P_1(x_1) P_2(x_2) \), provided that there is a non-trivial interaction \( H_{12} \). This because \( m_{2-1} \) is defined with respect to the transition probabilities; see [37].

4) \( m_{2-1} \) nullifies whenever there is no force acting from one particle to another. Recall that the force-driven part \( i^{2-1}_2 \) of the information flow also nullifies together with the force, albeit \( i^{2-1}_2 \) nullifies also for factorized probabilities; see [25].

5) In contrast to \( i_{2-1} \), \( m_{2-1} \) does not nullify at equilibrium. Thus, 2 can help 1 in predicting its future at absolutely no thermodynamic cost. However, as we have shown, there is a definite thermodynamic cost for 2 wanting to predict the future of 1 better than it predicts the present of 1.

6) \( m_{2-1} \) is not a flow, since it does not add up additively to time-derivative of any global quantity. However, obviously \( m_{2-1} \) does refer to some type of information processing. In fact, \( m_{2-1} \) underlies the notion of Granger-causality, which was first proposed in the context of econometrics [33, 34] (see [43] for a review); the ratio \( \frac{m_{2-1}}{i_{2-1}} \) quantifies the strength of causal influences.
from 2 to 1 relative to those from 1 to 2. The notion of Granger-causality is useful as witnessed by its successful empirical applications [33, 34, 38, 39, 43]. For \( \frac{m_{2 \rightarrow 1}}{m_{1 \rightarrow 2}} \gg 1 \) we shall tell that 2 is Granger-driving 1.

C. Information flow versus transfer entropy in the adiabatic stationary limit

Given the differences between the information flow and the entropy transfer it is curious to note that in the adiabatic stationary situation (see section VII B 1) there exist a direct relation between them. We recall that the adiabatic situation is special, since the efficiency of information flow reaches its maximal value there. Recall that this situation is defined (besides the long-time limit) by condition (44), which means that 2 is slow, while 1 is fast; at equilibrium, when \( T_1 = T_2 \), this slow versus fast separation becomes irrelevant. Reminding also the definition (29) of the force-driven part \( i_{2 \rightarrow 1}^F \) of the information flow, we see that

\[
\frac{m_{2 \rightarrow 1}}{m_{1 \rightarrow 2}} = \mathcal{O} \left( \frac{\Gamma_2}{\Gamma_1} \right) \gg 1, \quad \frac{i_{2 \rightarrow 1}^F}{i_{1 \rightarrow 2}^F} = \mathcal{O} \left( \frac{\Gamma_2}{\Gamma_1} \right) \gg 1. \tag{93}
\]

The first relation in (93) indicates that the slow system is Granger-driving the fast one, while the second relation implies that the same qualitative conclusion is got from looking at \( i_{2 \rightarrow 1}^F \). This point is strengthened by noting from (16, 92) that in the adiabatic, stationary, two-temperature situation we have

\[
m_{2 \rightarrow 1} = \frac{1}{2} i_{2 \rightarrow 1}^F. \tag{94}
\]

A less straightforward relation holds for the action of the fast system on the slow one

\[
m_{1 \rightarrow 2} = \frac{T_1}{T_2} \frac{1}{2} i_{1 \rightarrow 2}^F. \tag{95}
\]

Recall that for the considered adiabatic stationary state, it is the action of the fast on the slow that determines the magnitude of the information flow \( i_{1 \rightarrow 2} \) (the sign of \( i_{1 \rightarrow 2} \) is fixed by the temperature difference):

\[
i_{1 \rightarrow 2} = -i_{2 \rightarrow 1} = \frac{T_1 - T_2}{T_1} i_{1 \rightarrow 2}^F. \tag{96}
\]

It is seen that provided \( T_2 > T_1 \) (i.e., the slow system is attached to the hotter bath, a situation realized for the optimal information transfer) the Granger-driving qualitatively coincides with the causality intuition implied by the sign of information flow: both \( i_{2 \rightarrow 1} > 0 \) and \( \frac{m_{2 \rightarrow 1}}{m_{1 \rightarrow 2}} \gg 1 \) hold, which means that 2 predicts better the future of 1 (\( i_{2 \rightarrow 1} > 0 \)) and that 2 is more relevant for helping 1 to predict its own future (\( \frac{m_{2 \rightarrow 1}}{m_{1 \rightarrow 2}} \gg 1 \)).

It is tempting to suggest that only when the causality intuition deduced from \( i_{2 \rightarrow 1} \) agrees with that deduced from \( m_{2 \rightarrow 1} \), we are closer to gain a real understanding of causality (still without doing actual interventions). Interestingly, the present slow-fast two-temperature adiabatic system was considered recently from the viewpoint of other non-interventional causality detection methods reaching a similar conclusion: unambiguous causality can be detected in this system, if the slow variable is attached to the hot thermal bath [62].

For the harmonic-oscillator example treated in section VII we obtain

\[
m_{2 \rightarrow 1} = \frac{b^2}{2 A_2 T_1 \Gamma_1}, \quad \frac{m_{2 \rightarrow 1}}{m_{1 \rightarrow 2}} = \frac{bB}{A_2 \Gamma_1}. \tag{97}
\]

Employing formulas (57) [58] and (61) [64] for the stationary state we note the following relation

\[
\frac{i_{2 \rightarrow 1}^F}{i_{1 \rightarrow 2}^F} = \frac{T_1}{T_2} \frac{m_{2 \rightarrow 1}}{m_{1 \rightarrow 2}} \tag{98}
\]

\[
= \frac{1}{\kappa} \frac{(\xi - 1) + \varphi + 1}{\xi (\varphi + 1) + \frac{1}{\varphi} + 1}, \tag{99}
\]

where the dimensionless parameters \( \kappa \), \( \xi \), and \( \varphi \) are defined in (65). In the adiabatic situation (98) is naturally consistent with (95) [96], but for the considered harmonic oscillators it is valid more generally, i.e., for an arbitrary stationary state. Eq. (99) explicitly demonstrates the conflict in Granger-driving between making the oscillator 2 slow (i.e., \( \varphi \to \infty \)) and making it cold (i.e., \( \xi \to \infty \)): for \( \varphi \to \infty \), \( \frac{m_{2 \rightarrow 1}}{m_{1 \rightarrow 2}} \) tends to infinity, while for \( \xi \to \infty \), \( \frac{m_{2 \rightarrow 1}}{m_{1 \rightarrow 2}} \) tends to zero.

Note finally that

\[
\frac{i_{2 \rightarrow 1}^F}{m_{2 \rightarrow 1}} = \frac{\xi(1 + \varphi)(\xi + \varphi)}{\kappa \varphi + \xi (1 + \varphi)^2},
\]

which means that apart from the adiabatic limit (\( \varphi \to 0 \) or \( \varphi \to \infty \)) there is no straightforward relation between \( i_{2 \rightarrow 1}^F \) and \( m_{2 \rightarrow 1} \).

X. SUMMARY

We have investigated the task of information transfer implemented on a special bi-partite physical system (pair of Brownian particles, each coupled to a bath). Our main conclusions are as follows:

0. The information flow \( i_{2 \rightarrow 1} \) from one Brownian particle to another is defined via the time-shifted mutual information. For the considered class of systems this definition coincides with the entropy flow, as defined in statistical thermodynamics.

9 The usage of ratio \( \frac{m_{2 \rightarrow 1}}{m_{1 \rightarrow 2}} \) is obligatory, since \( m_{2 \rightarrow 1} \) does not characterize the absolute strength of the influence of 2 on 1. Note, e.g., that \( m_{2 \rightarrow 1} \) nullifies not only for independent process \( X_1(t) \) and \( X_2(t) \), but also for identical (strongly coupled) processes \( X_1(t) = X_2(t) \); see [87]. This point is made in [53].
1. The information flow $i_{2\rightarrow 1}$ is a sum of two terms: $i_{2\rightarrow 1} = i^B_{2\rightarrow 1} + i^E_{2\rightarrow 1}$, where the bath driven contribution $i^B_{2\rightarrow 1} \leq 0$ is the minus Fisher information, and where the force-driven contribution $i^E_{2\rightarrow 1}$ has to be positive and large enough for the particle 2 to be an information source for the particle 1.

2. No information flow from one particle to another is possible in equilibrium. This fact is recognized in literature [17], though by itself it does not yet point out to a definite thermodynamic cost for information transfer.

3. For a stationary non-equilibrium state created by a finite difference between two temperatures $T_1 < T_2$, the ratio of the information flow to the total entropy production—i.e., the efficiency of information flow—is limited from above by $\frac{T_2 - T_1}{T_2}$. This bound for the efficiency defines the minimal thermodynamic cost of information flow for the studied setup. Note that not the total amount of transferred information, but rather its rate is limited. Thus the thermodynamic cost accounts also for the time during which the information is transferred.

4. The upper bound $\frac{T_2 - T_1}{T_2}$ is reachable in the adiabatic limit, where the sub-systems have widely different characteristic times. The information flow is then small on the time-scale of the fast motion, but sizable on the time-scale of the slow motion.

5. The information transfer between two sub-systems (Brownian particles) naturally nullifies, if these system are not interacting, and were not interacting in the past. It is thus relevant to study how the efficiency and information flow depend on the inter-particle coupling strength. As functions of the inter-particle coupling strength, the efficiency and information flow demonstrate the following complementarity. The information flow is maximized at the instability threshold of the system (which is reached at the strongest coupling compatible with stability). On the contrary, the efficiency is maximized for the weakest coupling.

6. There are special two-temperature, but non-stationary scenarios, where the efficiency of information flow is much larger than $\frac{T_2 - T_1}{T_2}$, but it is still limited by the basic parameters of the system (the ratio of the time-scales and the ratio of temperatures).

7. Analogous consideration can be applied to the energy (heat) flow from one sub-system to another. The efficiency of the heat flow—which is defined as the heat flow over the total amount of the heat dissipated in the overall system—is limited from above by the same factor $\frac{T_2 - T_1}{T_2}$ (assuming that $T_1 < T_2$). However, in the stationary state there is a complementarity between heat flow and information flow: the setup which is most efficient for the information transfer is the least efficient for the heat transfer and vice versa.

8. There are definite relations between the information flow and the transfer entropy introduced in [35, 36]. The transfer entropy is not a flow of information, though it quantifies some type of information processing in the system, a processing that occurs without any thermodynamic cost.

**Acknowledgements**

The Volkswagenstiftung is acknowledged for financial support (grant "Quantum Thermodynamics: Energy and Information flow at nano-scale"). A. E. A. was supported by ANSEF and SCS of Armenia (grant 08-0166).

**APPENDIX A: INFORMATION-THEORETIC MEANING OF ENTROPY AND MUTUAL INFORMATION**

1. **Entropy**

Let us recall the information-theoretic meaning of entropy (1), i.e., in which specific operational sense $S[X]$ quantifies the amount of information contained in the random variable $X$. To this end imagine that $X$ is composed of $N$ random variables $\{X(1), \ldots, X(N)\}$, i.e., $X$ is a random process. We assume that this process is ergodic [25]. The simplest example of such a process is the case when $X(1), \ldots, X(N)$ are all independent and identical,

$$p(x(1), \ldots, x(N)) = \prod_{k=1}^{N} p(x(k)), \quad (A1)$$

where $x(k) = 1, \ldots, n$ parametrize realizations of $X(k)$. Note that for an ergodic process the entropy in the limit $N \gg 1$ scales as $\propto N$, e.g., $S[X] = -N\sum_{k=1}^{n} p(k) \ln p(k)$ for the above example (A1).

For $N \gg 1$, the set of $n^N$ realizations of the ergodic process $X$ can be divided into two subsets [25, 26, 27]. The first subset $\Omega(X)$ is called typical, since this is the minimal subset with the probability converging to 1 for $N \gg 1$ [25, 26, 27]; the convergence is normally exponential over $N$. The number of elements in $\Omega(X)$ grows asymptotically as $e^{S[X]}$ for $N \gg 1$. These elements have (nearly) equal probabilities $e^{-S[X]}$. Thus the number of elements in $\Omega(X)$ is generally much smaller than the overall number of realizations $e^{N \ln n}$.

The overall probability of those realizations which do not fall into $\Omega(X)$, scales as $e^{-\text{const} \cdot N}$, and is negligible in the thermodynamical limit $N \gg 1$. All these features are direct consequences of the law of large numbers, which holds for ergodic processes at least in its weak form [25, 26, 27]. Since in the limit $N \gg 1$ the realizations of the original random variable $X$ can be in a sense substituted by the typical set $\Omega(X)$, the number of elements in $\Omega(X)$ characterizes the information content of $X$ [25, 26, 27].

2. **Mutual information**

While the entropy $S[X]$ reflects the information content of the (noiseless) probabilistic information source, the mutual information $I[Y : X]$ characterizes the maximal information, which can be shared through a noisy channel, where $X$ and $Y$ correspond to the input and
output of the channel \(25, 26, 27\), respectively. To understand the qualitative content of this relation consider an ergodic process \(XY = \{X(1)Y(1), \ldots, X(N)Y(N)\}\), where \(x(l) = 1, \ldots, n\) and \(y(l) = 1, \ldots, n\) are the realizations of \(X(l)\) and \(Y(l)\), respectively.

One now looks at \(X(Y)\) as the input (output) of a noisy channel \(25, 24, 27\). In the limit \(N \to \infty\) we can study the typical sets \(\Omega(.)\) instead of the full set of realizations for the random variables. It appears for \(N \to \infty\) that the typical sets \(\Omega(X)\) and \(\Omega(Y)\) can be represented as union of \(M = e^{f(X)}\) non-overlapping subsets \(26:\)

\[
\Omega(X) = \bigcup_{\alpha=1}^{M} \omega_{\alpha}(X), \quad \Omega(Y) = \bigcup_{\alpha=1}^{M} \omega_{\alpha}(Y), \quad (A2)
\]

such that for \(N \gg 1\)

\[
p[y \in \omega_{\alpha}(Y) \vert x \in \omega_{\beta}(X)] = \delta_{\alpha\beta} e^{-S(Y|X)}, \quad (A3)
\]

\[
p[x \in \omega_{\alpha}(X) \vert y \in \omega_{\beta}(Y)] = \delta_{\alpha\beta} e^{-S(Y|X)}, \quad (A4)
\]

Note that the number of elements in \(\omega_{\alpha}(X)\) \(\omega_{\alpha}(Y)\) is asymptotically \(e^{S(Y|X)}\) \(e^{S(Y|X)}\). Eqs. \(A3, A4\) mean that the realizations from \(\omega_{\alpha}(X)\) correlate only with those from \(\omega_{\alpha}(X)\), and that all realizations within \(\omega_{\alpha}(X)\) and \(\omega_{\alpha}(Y)\) are equivalent in the sense of \(A3, A4\). It should be clear that once the elements of \(\omega_{\alpha}(X)\) are completely mixed during the mapping to \(\omega_{\alpha}(Y)\), the only reliable way of sending information through this noisy channel is to relate the reliably shared words to the sets \(\omega_{\alpha}\). Since there are \(e^{f(X)}\) such sets, the number of reliably shared words is limited by \(e^{f(X)}\). Note that the number of elements in the typical set \(\Omega_N(X)\) is equal to \(e^{S(X)}\), which in general is much larger than \(e^{f(X)}\).

**APPENDIX B: OPERATIONAL DEFINITION OF INFORMATION FLOW**

Here we demonstrate that the definition \(17\) of the information flow is recovered from an operational approach proposed in \(40\); see \(41, 42\) for related works.

Let us imagine that at some time \(t\) we suddenly increase the damping constant \(\Gamma_2\) to some very large value. As seen from \(4, 8, 9\), this will freeze the dynamics of the second particle, so that for \(t > \bar{t}\) the joint probability distribution \(\tilde{P}(x; t)\) satisfies the following modified Fokker-Planck equation

\[
\partial_t \tilde{P}(x; t) + \partial_j \tilde{J}_1(x; t) = 0, \quad (B1)
\]

\[
-\Gamma_1 \tilde{J}_1(x; t) = \tilde{P}(x; t) \partial_t H(x) + T_1 \partial_t \tilde{P}(x; t), \quad (B2)
\]

together with the boundary condition

\[
\tilde{P}(x; \bar{t}) = P(x; \bar{t}), \quad (B3)
\]

where \(P(x; \bar{t})\) satisfies the Fokker-Planck equation \(8, 9\). Other methods of freezing the dynamics of \(X_2(t)\) (e.g., switching on a strong confining potential \(H_2(x)\) acting on \(x_2\)) would work for the present purposes equally well.

The fact of freezing should be apparent from \(11\) whose solution can be represented [using also \(13\)] as

\[
\tilde{P}(x; t) = \tilde{P}_{1|2}(x_1|x_2; t) P_2(x_2; \bar{t}), \quad t \geq \bar{t}. \quad (B4)
\]

Note that once \(X_2\) is frozen, it becomes a random external field from the viewpoint of the dynamics of \(X_1(t)\). The entropy of a system in a random field is standardly calculated via averaging (over the field distribution) the entropy calculated at a fixed field:

\[
\tilde{S}_{1|2} = - \int dx P(x_2; t) \tilde{P}_{1|2}(x_1|x_2; t) \ln \tilde{P}_{1|2}(x_1|x_2; t).
\]

We select \(t \to \bar{t} + 0\) and note that the freezing directly marginal entropy rate of the first particle

\[
\frac{dS_1}{dt} = \frac{d\tilde{S}_{1|2}}{dt} \bigg|_{\bar{t}=t},
\]

a fact that follows from \(11, 13, 14\).

Now we subtract from the entropy rate of the first particle the rate of the conditional entropy \(\tilde{S}_{1|2}\):

\[
i_{2 \rightarrow 1}(t) \equiv \frac{dS_1}{dt} - \frac{d\tilde{S}_{1|2}}{dt} \bigg|_{\bar{t}=t}, \quad (B5)
\]

where the conditioning \(\bar{t} = t\) is done after taking \(\frac{d}{dt}\). Thus, \(i_{2 \rightarrow 1}(t)\) is that part of the entropy change of \(X_1\) (between \(t\) and \(t + \tau\)), which exists due to fluctuations of \(X_2(t)\); see section \(11\).

Employing \(11\) we note for the last part in \(13\)

\[
\left. \frac{d\tilde{S}_{1|2}}{dt} \right|_{\bar{t}=t} = \frac{d\tilde{S}}{dt} \bigg|_{\bar{t}=t},
\]

which means that \(i_{2 \rightarrow 1}(t)\) can be defined equivalently via the total entropy rate \(\frac{dS}{dt}\) of the overall system, with the second particle being frozen. Now using

\[
\frac{dS_1}{dt} = \int dx [\ln P_1(x_1; t)] \partial_t J_1(x; t),
\]

\[
\frac{dS}{dt} \bigg|_{\bar{t}=t} = \int dx [\ln P(x; t)] \partial_t J_1(x; t),
\]

we get back from \(13\) to \(19\) confirming that both definitions are equivalent.

**APPENDIX C: COARSE-GRAINED VELOCITIES FOR BROWNIAN PARTICLES**

Consider an ensemble of all realizations of the two-particle Brownian system which at time \(t\) have a coordinate vector \(x\). For this ensemble the average coarse-grained velocity for the particle with index \(j\) might
naively be defined as:

\[ v_j(x, t) = \lim_{\epsilon \to 0} \int dy \frac{y_j - x_j}{\epsilon} P(y, t + \epsilon|x, t). \]  

(C1)

However, it was pointed out by Nelson [57] that the absence of regular trajectories enforces one to define different velocities for different directions of time:

\[ v_{+j}(x, t) = \lim_{\epsilon \to +0} \int dy_j \frac{y_j - x_j}{\epsilon} P(y_j, t + \epsilon|x, t), \]  

(C2)

\[ v_{-j}(x, t) = \lim_{\epsilon \to +0} \int dy_j \frac{x_j - y_j}{\epsilon} P(y_j, t - \epsilon|x, t). \]  

(C3)

The physical meaning of these expressions is as follows: \( v_{+j}(x, t) \) is the average velocity to move anywhere starting from \( (x, t) \), whereas \( v_{-j}(x, t) \) is the average velocity to come from somewhere and to arrive at \( x \) at the moment \( t \). Since these velocities are defined already in the overdamped limit, \( \epsilon \) is assumed to be much smaller than the characteristic relaxation time of the (real) momentum which is small in the overdamped limit. Therefore, we call \( (C2, C3) \) coarse-grained velocities. It is known that for the overdamped brownian motion almost all trajectories are not smooth. This is connected to the chaotic influences of the bath(s) which randomize the real momenta on much smaller times, and this is also the reason for \( v_{+j}(x, t) \neq v_{-j}(x, t) \). The difference \( v_{+j}(x, t) - v_{-j}(x, t) \) thus characterizes the degree of the above non-smoothness. One now can show that [57, 58]:

\[ v_{+j}(x, t) = -\frac{1}{\Gamma_j} \partial_j H(x), \]

\[ v_{-j}(x, t) = -\frac{1}{\Gamma_j} [\partial_j H(x) + 2T_j \partial_j \ln P(x, t)]. \]

We now see that the probability times the average coarse-grained velocity \( \frac{1}{\Gamma_j} [v_{+j}(x, t) + v_{-j}(x, t)] \equiv v_j(x, t) \) amounts to the probability current of the Fokker-Planck equation [3, 4]:

\[ v_j(x, t) P(x; t) = J_j(x; t). \]

If one would take \( \epsilon \) in \( (C2, C3) \) much smaller than the characteristic relaxation time of the momentum—which would amount to applying definitions \( (C2) \) and \( (C3) \) to a smoother trajectory—then \( v_{+j}(x, t) \) and \( v_{-j}(x, t) \) would be equal to each other and equal to the average momentum; see [58] for more details.

**APPENDIX D: CALCULATION OF TRANSFER ENTROPY**

Here we calculate the entropy transfer as defined in [91]. The measures entering this equation read for a small \( \tau \)

\[ \mathcal{P}[dx_1 t^{+\tau}|y_1, y_2; t] = \mathcal{K} e^{\frac{1}{\tau[1]} f_1^{+\tau} d\sigma [\Gamma_1 x_1 + \partial_1 H(x_1(\sigma), y_1)]^2}, \]

(D1)

\[ \mathcal{P}[dx_1 t^{+\tau}|y_1; t] = \mathcal{K} e^{\frac{1}{\tau[1]} f_1^{+\tau} d\sigma [\Gamma_1 x_1 + h_1(x_1(\sigma))]^2}, \]

(D2)

where \( \mathcal{K} \) is the normalization constant, and where

\[ h_1(x_1) \equiv \int dy_2 \partial_1 H(x_1, y_2) P_{21}(y_2|x_1). \]

(D3)

Both time integrals \( \int_0^{+\tau} \) in \( (D1) \) \( (D2) \) are to be interpreted in the Ito sense [47]. Due to this the normalization constants for both path-integrals are identical.

To understand the origin of \( (D1), (D2) \) recall from [11] that in the small-\( \tau \) limit:

\[ P(x_1; t + \tau|y_1, y_2; t) = \delta(x_1 - y_1) \]

\[ + \frac{\tau}{\Gamma_1} \partial_1 [\delta(y_1 - x_1) \partial_1 H(x_1, y_2) + T_1 \partial_1 \delta(y_1 - x_1)], \]

(D4)

\[ P(x_1; t + \tau|y_1; t) = \delta(x_1 - y_1) \]

\[ + \frac{\tau}{\Gamma_1} \partial_1 [\delta(y_1 - x_1) h_1(x_1) + T_1 \partial_1 \delta(y_1 - x_1)]. \]

(D5)

For a small \( \tau \) we get

\[ 1 \mathcal{P}[dx_1 t^{+\tau}|y; t] \mathcal{P}[dx_1 t^{+\tau}|y; t] = \frac{h_1^2(y_1) - [\partial_1 H(y_1, y_2)]^2}{2T_1 \Gamma_1} \]

\[ + \frac{h_1(y_1) - \partial_1 H(y_1, y_2)}{T_1} \frac{x_1(t + \tau) - x_1(t)}{\tau}. \]

(D6)

The fact that the time-integrals were taken in the Ito sense is visible in the last term of \( (D6) \). Putting \( (D6) \) into \( (D1) \) and noting [for a small \( \tau \)]

\[ \int \mathcal{P}[dx_1 t^{+\tau}|y; t] \frac{x_1(t + \tau) - x_1(t)}{\tau} = -\frac{1}{\Gamma_1} \partial_1 H(y), \]

we end up at \( (D2) \).
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