Discrete homotopies and the fundamental group

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Abstract
We generalize and strengthen the theorem of Gromov that every compact Riemannian manifold of diameter at most $D$ has a set of generators $g_1, \ldots, g_k$ of length at most $2D$ and relators of the form $g_i g_m = g_j$. In particular, we obtain an explicit bound for the number $k$ of generators in terms of the number “short loops” at every point and the number of balls required to cover a given semi-locally simply connected geodesic space. As a consequence we obtain a fundamental group finiteness theorem (new even for Riemannian manifolds) that implies the fundamental group finiteness theorems of Anderson and Shen-Wei. Our theorem requires no curvature bounds, nor lower bounds on volume or 1-systole. We use the method of discrete homotopies introduced by the first author and V. N. Berestovskii. Central to the proof is the notion of the “homotopy critical spectrum” that is closely related to the covering and length spectra. Discrete methods also allow us to strengthen and simplify the proofs of some results of Sormani-Wei about the covering spectrum.

Keywords: fundamental group, finiteness theorem, discrete homotopy, Gromov generators, length spectrum, covering spectrum, homotopy critical spectrum

1 Introduction

In ([14], [15]), Gromov proved the following: If $M$ is a compact Riemannian manifold of diameter $D$ then $\pi_1(M)$ has a set of generators $g_1, \ldots, g_k$ represented
by loops of length at most $2D$ and relations of the form $g_i g_m = g_j$. Among uses for this theorem are fundamental group finiteness theorems: If $\mathcal{X}$ is any collection of spaces with a global bound $N$ for the number of elements of $\pi_1(M)$ represented by loops of length at most $2D$ in any $X \in \mathcal{X}$ then $\pi_1(X)$ has at most $N$ generators and $N^3$ relators for any $X \in \mathcal{X}$. Therefore there are only finitely many possible fundamental groups of spaces in $\mathcal{X}$. This strategy was employed by Michael Anderson ([1]) to show that compact $n$-manifolds with global lower bounds on volume and Ricci curvature, and diameter $\leq D$, have finitely many possible fundamental groups. Shen-Wei ([24]) reached the same conclusion, replacing the lower volume bound by a positive uniform lower bound on the 1-systole (the infimum of lengths of non-null closed geodesics).

In this paper we generalize and strengthen Gromov’s theorem by giving an explicit bound for the number $k$ of generators in terms of the number of “short loops” and the number of balls required to cover a space, at a given scale. As a consequence we are able to prove a curvature-free finiteness theorem (Corollary 2) for fundamental groups of compact geodesic spaces that generalizes and strengthens both of the previously mentioned finiteness theorems. Our most general theorem (Theorem 3) applies to certain deck groups $\pi_\varepsilon(X)$ of covering maps that measure the fundamental group at a given scale. As an application of Theorem 3 we generalize a finiteness theorem stated by Sormani-Wei (Proposition 7.8, [21]). Note that their proof implicitly relies on some kind of generalization of Gromov’s theorem, although details are not given in their paper.

We will first state our main result for fundamental groups, saving the more general theorem from which it follows until we provide a little background concerning the method of discrete homotopies.

For any path $c$ in a metric space $X$, we define $||c||$ to be the infimum of the lengths of paths in the fixed endpoint homotopy class of $c$. For any $L > 0$, let $\Gamma(X, L)$ be the supremum, over all possible basepoints $\ast$, of the number of distinct elements $g \in \pi_1(X, \ast)$ such that $|g| \leq L$. For a compact geodesic space there may be no rectifiable curves in the homotopy class of a path (cf. [3]), and certainly $\Gamma(X, L)$ need not be finite (e.g. a geodesic Hawaiian Earring). If $X$ is semilocally simply connected then $|g|$ and $\Gamma(X, L)$ are always both finite (Theorem 25 Corollary 49), and the 1-systole of $X$ is positive if $X$ is not simply connected (Corollary 41). We denote by $C(X, r, s)$ (resp. $C(X, s)$) the minimum number of open $s$-balls required to cover a closed $r$-ball in $X$ (resp. $X$).

**Theorem 1** Suppose $X$ is a semilocally simply connected, compact geodesic space of diameter $D$, and let $\varepsilon > 0$. Then for any choice of basepoint, $\pi_1(X)$ has a set of generators $g_1, \ldots, g_k$ of length at most $2D$ and relations of the form $g_i g_m = g_j$ with

$$k \leq \frac{8(D + \varepsilon)}{\varepsilon} \cdot \Gamma(X, \varepsilon) \cdot C \left( X, \frac{\varepsilon}{4} \right)^{\frac{8(D + \varepsilon)}{\varepsilon}}.$$
In particular, if the 1-systole of $X$ is $\sigma > 0$ then we may take

$$k \leq \frac{8(D + \sigma)}{\sigma} C \left( X, D, \frac{\sigma}{4} \right)^{\frac{8(D + \sigma)}{\sigma}}.$$

The previously mentioned theorem of Shen-Wei is an immediate corollary of this theorem since for any $r, s > 0$, $C(X, r, s)$ is uniformly controlled in any precompact class of spaces by Gromov’s Precompactness Criterion, and the lower bound on Ricci curvature provides the precompactness ([15]). On the other hand, Anderson showed (Remark 2.2(2), [1]) that if $M$ has Ricci curvature $\geq - (n - 1)k^2$, diameter $\leq D$ and volume $\geq v$ then for any basepoint, the subgroup of $\pi_1(M)$ generated by loops of length less than $Dv$ has order bounded by above by $\frac{\nu_k(2D)}{v}$. Therefore Theorem 1 implies Anderson’s Finiteness Theorem. In fact, we have:

**Corollary 2** Let $X$ be any Gromov-Hausdorff precompact class of semilocally simply connected compact geodesic spaces. If there are numbers $\varepsilon > 0$ and $N$ such that for every $X \in X$, $\Gamma(X, \varepsilon) \leq N$, then there are finitely many possible fundamental groups for spaces in $X$.

We should point out a subtle but important difference between Anderson’s final step (i.e. from Remark 2.2(2) to the finiteness theorem) and our proof. Anderson’s final step depends on the fact that the universal covering space also has Ricci curvature $\geq - (n - 1)k^2$ and hence one may use Bishop’s volume comparison theorem in the universal covering space. One can “translate” his argument into one that relies instead on global control of the numbers $C(X, r, s)$ in the universal cover. However, this approach requires that one know that the collection of all universal covers of all spaces in the class is (pointed) Gromov-Hausdorff precompact. The Shen-Wei and Sormani-Wei theorems also rely on precompactness of the universal covering spaces. But without a lower bound on the 1-systole it is in general impossible to conclude from precompactness of a class of spaces that the collection of their universal covers is precompact (Example [14]), so we cannot use this strategy in our proof.

The basic construction of [4], [5] goes as follows in the special case of a metric space $X$. For $\varepsilon > 0$, an $\varepsilon$-chain is a finite sequence $\alpha := \{x_0, ..., x_n\}$ such that $d(x_i, x_{i+1}) < \varepsilon$ for all $i$. We define the length of $\alpha$ to be $L(\alpha) = \sum_{i=1}^{n} d(x_i, x_{i-1})$, and define the size of $\alpha$ to be $\nu(\alpha) := n$. The reversal of $\alpha$ is the chain $\overline{\alpha} := \{x_n, ..., x_0\}$. A basic move on an $\varepsilon$-chain $\alpha$ consists of either adding or removing a single point, as long as the resulting chain is still an $\varepsilon$-chain. An $\varepsilon$-homotopy between $\varepsilon$-chains $\alpha$ and $\beta$ with the same endpoints is a finite sequence of $\varepsilon$-chains $\langle \alpha = \eta_0, \eta_1, ..., \eta_k = \beta \rangle$ such that all $\eta_i$ have the same endpoints and for all $i$, $\eta_i$ and $\eta_{i+1}$ differ by a basic move. The resulting equivalence classes are denoted
\[\alpha\] for simplicity we will usually write \([x_0, ..., x_n]_\varepsilon\) rather than \([\{x_0, ..., x_n\}]_\varepsilon\). If \(\alpha = \{x_0, ..., x_n\}\) and \(\beta = \{y_0, ..., y_m\}\) are \(\varepsilon\)-chains then the concatenation \(\alpha \ast \beta\) is the \(\varepsilon\)-chain \(\{x_0, ..., x_n = y_0, ..., y_m\}\). It is easy to check that there is a well-defined operation induced by concatenation: \([\alpha]_\varepsilon \ast [\beta]_\varepsilon := [\alpha \ast \beta]_\varepsilon\). We define two \(\varepsilon\)-loops \(\lambda_1\) and \(\lambda_2\) to be freely \(\varepsilon\)-homotopic if there exist \(\varepsilon\)-chains \(\alpha\) and \(\beta\) starting at a common point \(x_0\), such that \(\alpha \ast \lambda_1 \ast \tau\) is \(\varepsilon\)-homotopic to \(\beta \ast \lambda_2 \ast \tau\).

Fix a basepoint \(\ast\) in \(X\) (change of basepoint is algebraically immaterial for connected metric spaces—see [3]). The set of all \(\varepsilon\)-homotopy classes \([\alpha]_\varepsilon\) of \(\varepsilon\)-loops starting at \(\ast\) forms a group \(\pi_\varepsilon(X)\) with operation induced by concatenation of \(\varepsilon\)-loops. The group \(\pi_\varepsilon(X)\) can be regarded as a kind of fundamental group that measures only “holes at the scale scale of \(\varepsilon\).” An \(\varepsilon\)-loop \(\alpha = \{x_0, ..., x_n = x_0\}\) that is \(\varepsilon\)-homotopic to the trivial loop \(\{x_0\}\) is called \(\varepsilon\)-null.

The set of all \(\varepsilon\)-homotopy classes \([\alpha]_\varepsilon\) of \(\varepsilon\)-chains \(\alpha\) in \(X\) starting at \(\ast\) will be denoted by \(X_\varepsilon\). The “endpoint mapping” will be denoted by \(\phi_\varepsilon : X_\varepsilon \rightarrow X\). That is, if \(\alpha = \{\ast = x_0, x_1, ..., x_n\}\) then \(\phi_\varepsilon([\alpha]_\varepsilon) = x_n\). Since \(\varepsilon\)-homotopic \(\varepsilon\)-chains always have the same endpoints, the function \(\phi_\varepsilon\) is well-defined. We choose \([\ast]_\varepsilon\) for the basepoint of \(X_\varepsilon\) so \(\phi_\varepsilon\) is basepoint preserving. For any \(\varepsilon\)-chain \(\alpha\) in \(X\), let

\[|[\alpha]_\varepsilon| := \inf\{L(\gamma) : \gamma \in [\alpha]_\varepsilon\}. \tag{1}\]

The above definition allows us to define a metric on \(X_\varepsilon\) so that \(\pi_\varepsilon(X)\) acts by isometries induced by concatenation (Definition [12]). When \(X\) is connected, \(\phi_\varepsilon\) is a regular covering map with deck group \(\pi_\varepsilon(X)\), and when \(X\) is geodesic the metric coincides with the usual lifted length metric (Proposition [22]). For any \(\delta \geq \varepsilon > 0\) there is a natural mapping \(\phi_{\delta\varepsilon} : X_\varepsilon \rightarrow X_\delta\) given by \(\phi_{\delta\varepsilon}([\alpha]_\varepsilon) = [\alpha]_\delta\). This map is well defined because every \(\varepsilon\)-chain (resp. \(\varepsilon\)-homotopy) is a \(\delta\)-chain (resp. \(\delta\)-homotopy). One additional very important feature of geodesic spaces is that any \(\varepsilon\)-chain has a “midpoint refinement” obtained by adding a midpoint between each point in the chain and its successor (which is clearly an \(\varepsilon\)-homotopy), producing an \(\varepsilon/2\)-chain in the same \(\varepsilon\)-homotopy class. Refinement is often essential for arguments involving limits, since being an \(\varepsilon\)-chain is not a closed condition. For this reason, many arguments in this paper do not carry over to general metric spaces.

The main relationship between \(\pi_\varepsilon(X)\) and the fundamental group \(\pi_1(X)\) involves a function \(\Lambda\) defined as follows (see also Proposition 78, [5]). Given any continuous path \(c : [0, 1] \rightarrow X\), choose \(0 = t_0 < \cdots < t_n = 1\) fine enough that every image \(c([t_i, t_{i+1}])\) is contained in the open ball \(B(c(t_i), \varepsilon)\). Then \(\Lambda([c]) := [c(t_0), ..., c(t_n)]_\varepsilon\) is well-defined by Corollary [20]. Note that \(\Lambda\) is “length non-increasing” in the sense that \(|\Lambda([c])| \leq |[c]|\). Restricting \(\Lambda\) to the fundamental group at any base point yields a homomorphism \(\pi_1(X) \rightarrow \pi_\varepsilon(X)\).

When \(X\) is geodesic, \(\Lambda\) is surjective since the successive points of an \(\varepsilon\)-loop \(\lambda\) may be joined by geodesics to obtain a path loop whose class goes to \([\lambda]_\varepsilon\). The kernel of \(\Lambda\) is precisely described in [19]: for the purposes of this paper we need only know that if \(X\) is a compact semilocally simply connected geodesic space then for small enough \(\varepsilon\), \(\Lambda\) is a length-preserving isomorphism (Theorem 25).
All of our theorems about the fundamental group are directly derived from the
next theorem via Λ.

**Theorem 3** Let $X$ be a compact geodesic space of diameter $D$, and $\varepsilon > 0$. Then

1. $\pi_\varepsilon(X)$ has a finite set of generators $G = \{[\gamma_1]_\varepsilon, \ldots, [\gamma_k]_\varepsilon\}$ such that $L(\gamma_i) \leq 2(D + \varepsilon)$ for all $i$, and relators of the form $[\gamma_i]_\varepsilon[\gamma_j]_\varepsilon = [\gamma_m]_\varepsilon$.

2. For any $L > 0$ there are at most $C(X, \varepsilon)\frac{4L}{\varepsilon}$ distinct elements $[\alpha]_\varepsilon$ of $\pi_\varepsilon(X)$ such that $|[\alpha]_\varepsilon| < L$, and in particular we may take

$$k \leq C(X, \varepsilon)\left(\frac{2(D+\varepsilon)}{\varepsilon}\right)^{\frac{8(D+\varepsilon)}{\varepsilon}}$$

in the first part.

3. Suppose, in addition, that for any basepoint $*$ and $0 < \delta < \varepsilon$ there are at most $M$ distinct non-trivial elements $[\alpha]_\delta \in \pi_\delta(X)$ such that $|[\alpha]_\delta| < \varepsilon$. Then the number of generators of $\pi_\delta(X)$ with relators as in the first part may be taken to be at most

$$M \left[\frac{8(D+\varepsilon)}{\varepsilon}\right] \left[C(X, \varepsilon)\frac{4L}{\varepsilon}\right]^{\frac{8(D+\varepsilon)}{\varepsilon}}$$

The proof of the second part of the theorem is a nice illustration of the utility of discrete methods. Fix any covering $B$ of $X$ by $N := C(X, \frac{\varepsilon}{4})\frac{4L}{\varepsilon}$-balls. Applying Lemma 16 and a midpoint refinement, we may represent any element of $\pi_\varepsilon(X)$ by an $\varepsilon$-loop $\alpha$ such that $\nu(\alpha) \leq \frac{4L}{\varepsilon} + 2$. We may choose one $B \in B$ containing each point in the loop. Since the first and last balls may always be chosen to be the same (containing the basepoint), each $\alpha$ corresponds to a sequential choice of at most $\frac{4L}{\varepsilon}$ balls in $B$. But Proposition 16 tells us that if any two loops share the same sequence of balls (so corresponding points are distance < $\frac{\varepsilon}{2}$ apart), they must be $\varepsilon$-homotopic. So there is at most one class $[\alpha]_\varepsilon$ for each such sequence of balls, and there are at most $N\frac{4L}{\varepsilon}$ different sequences of balls.

The proof of the first part of part of the theorem requires the construction of a metric simplicial 2-complex called an $(\varepsilon, \delta)$-chassis for a compact geodesic space $X$, which is described in the last section. For small enough $\delta > 0$, any $(\varepsilon, \delta)$-chassis has edge group isomorphic to $\pi_\varepsilon(X)$ (although the two spaces may not have the same homotopy type!). In this way our proof of Theorem 1 is quite different from Gromov’s proof of his theorem. However, it is interesting to note that he exploits the fact that the set of lengths of minimal loops representing fundamental group elements in a compact Riemannian manifold is discrete. Our proof depends on discreteness of what we call the “homotopy critical spectrum”, which is closely related to the length spectrum and covering spectrum of Sormani-Wei. We will now describe this spectrum and related concepts (and questions) that are of independent interest.
**Definition 4** An \( \varepsilon \)-loop \( \lambda \) in a metric space \( X \) is called \( \varepsilon \)-critical if \( \lambda \) is not \( \varepsilon \)-null, but is \( \delta \)-null for all \( \delta > \varepsilon \). When an \( \varepsilon \)-critical \( \varepsilon \)-loop exists, \( \varepsilon \) is called a homotopy critical value; the collection of these values is called the homotopy critical spectrum.

When \( X \) is a geodesic space the functions \( \phi_{\varepsilon \delta} : X_\delta \to X_\varepsilon \) are all covering maps, which are homeomorphisms precisely if there are no critical values \( \sigma \) with \( \varepsilon > \sigma \geq \delta \) (Lemma 23). In a compact geodesic space, the homotopy critical spectrum is discrete in \((0, \infty)\) (more about this below) and therefore indicates the exact values \( \varepsilon > 0 \) where the equivalence type of the \( \varepsilon \)-covering maps changes.

In ([20], [21], [22], [23]), Sormani-Wei independently studied covering maps that encode geometric information, and defined the “covering spectrum” to be those values at which the equivalence type of the covering maps changes. They utilized a classical construction of Spanier ([25]) for locally pathwise connected topological spaces that provides a covering map \( \pi^\delta : \tilde{X}^\delta \to X \) corresponding to the open cover of a geodesic space \( X \) by open \( \delta \)-balls, which they called the \( \delta \)-cover of \( X \). As it turns out, despite the very different construction methods, when \( \delta = \frac{3\varepsilon}{2} \) and \( X \) is a compact geodesic space, this covering map is isometrically equivalent to our covering map \( \phi_{\varepsilon} : X_\varepsilon \to X \) ([19]). In fact, \( \ker \Lambda \) is precisely the Spanier group for the open cover of \( X \) by \( \frac{3\varepsilon}{2} \)-balls. It follows that in the compact case, the covering spectrum and homotopy critical spectrum differ precisely by a factor of \( \frac{3}{2} \). However, we will not use any prior theorems about the covering spectrum, but rather will directly prove stronger results about the homotopy critical spectrum. Moreover, as should be clear from the present paper and ([19]), discrete methods have many advantages, including simplicity, amenability with the Gromov-Hausdorff metric, and applicability to non-geodesic spaces. For example, Sormani-Wei ([21], Theorem 4.7) show that the covering spectrum is contained in \( \frac{1}{3} \) times the length spectrum (set of lengths of closed geodesics) for geodesic spaces with a universal cover. We not only show that this statement is true (replacing \( \frac{1}{3} \) by \( \frac{1}{3} \) in our notation) without assuming a universal cover, we identify precisely the very special closed geodesics that contribute to the homotopy critical spectrum:

**Definition 5** An essential \( \varepsilon \)-circle in a geodesic space consists of the image of an arclength parameterized (path) loop of length \( 3\varepsilon \) that contains an \( \varepsilon \)-loop that is not \( \varepsilon \)-null.

Being an essential circle is stronger than it may seem at first: an essential circle is the image of a closed geodesic that is not null-homotopic, which is also a metrically embedded circle in the sense that its metric as a subset of \( X \) is the same as the intrinsic metric of the circle (Theorem 45). As Example 42 shows, even in flat tori this is not always true for the image of a closed geodesic, even when it is the shortest path in its homotopy class. We prove:

**Theorem 6** If \( X \) is a compact geodesic space then \( \varepsilon > 0 \) is a homotopy critical value of \( X \) if and only if \( X \) contains an essential \( \varepsilon \)-circle.
This theorem is connected to a problem with a long history in Riemannian geometry: to relate the spectrum of the Laplace-Beltrami operator and the length spectrum to one another and to topological and geometric properties of the underlying compact manifold. For example, an important open question is whether the “weak” length spectrum (i.e. without multiplicity) is completely determined by the Laplace spectrum. To this mix one may add the covering/homotopy critical spectrum (with or without multiplicity, see below), which up to multiplied constant is a subset of the length spectrum. The analog of the main question has already been answered: de Smit, Gornet, and Sutton recently showed that the covering spectrum is not a spectral invariant (13).

We are now in a position to propose yet another spectrum: the circle spectrum, consisting of the lengths of metrically embedded circles, which according to Theorem 6 and Example 42 is (up to multiplied constant) generally strictly intermediate between the homotopy critical spectrum and the length spectrum. For multiplicity one may consider either free homotopies or use Definition 40.

One may take this further, partitioning the length spectrum according to the degree to which a closed geodesic deviates from being metrically embedded, but we will not pursue these directions in the present paper. Also, in 19 we show that essential circles can be used to create a new set of generators for the fundamental group of a compact, semilocally simply connected space, which we conjecture has minimal cardinality.

Essential circles give a nice geometric picture, but their discrete analogs, which we will define now, are more useful for the type of problems we are presently considering.

**Definition 7** An $\varepsilon$-triad in a geodesic space $X$ is a triple $T := \{x_0, x_1, x_2\}$ such that $d(x_i, x_j) = \varepsilon$ for all $i \neq j$; when $\varepsilon$ is not specified we will simply refer to a triad. We denote by $\alpha_T$ the loop $\{x_0, x_1, x_2, x_0\}$. We say that $T$ is essential if some midpoint refinement of $\alpha_T$ is not $\varepsilon$-null. Essential $\varepsilon$-triads $T_1$ and $T_2$ are defined to be equivalent if a midpoint refinement of $\alpha_{T_1}$ is freely $\varepsilon$-homotopic to a midpoint refinement of either $\alpha_{T_1}$ or $\alpha_{T_2}$.

Of course $\alpha_T$ is not an $\varepsilon$-chain; that is why we use a midpoint refinement. We show that if one joins the corners of an essential $\varepsilon$-triad by geodesics then the resulting geodesic triangle is an essential $\varepsilon$-circle (Proposition 36). Conversely, given an essential $\varepsilon$-circle, every triad on it is an essential $\varepsilon$-triad (Corollary 39). We may now define essential $\varepsilon$-circles to be equivalent if their corresponding essential $\varepsilon$-triads are equivalent, and Theorem 6 allows us to define the multiplicity of a homotopy critical value $\varepsilon$ to be the number of non-equivalent essential $\varepsilon$-triads (or $\varepsilon$-circles).

We prove that “close” essential triads are equivalent:

**Proposition 8** Suppose $T = \{x_0, x_1, x_2\}$ is an essential $\varepsilon$-triad in a geodesic space $X$ and $T' = \{x'_0, x'_1, x'_2\}$ is any set of three points such that $d(x_i, x'_i) < \varepsilon$ for all $i$. If $T'$ is an essential triad then $T'$ is an $\varepsilon$-triad equivalent to $T$.

Now suppose we cover a compact geodesic space $X$ by $N$ open metric balls of radius $r$. If $T$ is an essential $\varepsilon$-triad with $\varepsilon \geq 3r$ then there are three distinct
balls $B_1, B_2, B_3$ in the cover, each containing one of the points of the triad. By Proposition 8 any triad having one point in each of $B_1, B_2, B_3$ is either not essential or is an $\varepsilon$-triad equivalent to $T$. We obtain:

**Corollary 9** Let $X$ be a compact metric space with diameter $D$ and $a > 0$. Then there are at most

$$\left( C \left( X, \frac{a}{3} \right) \right)^3$$

non-equivalent essential triads that are $\varepsilon$-triads for some $\varepsilon \geq a$.

Naturally one wonders how optimal this estimate is and whether it can be improved (see also Example 15). From Gromov’s Precompactness Theorem we immediately obtain:

**Theorem 10** Let $\mathcal{X}$ be a Gromov-Hausdorff precompact collection of compact geodesic spaces. For every $a > 0$, there is a number $N$ such that for any $X \in \mathcal{X}$ the number of homotopy critical values of $X$ greater than $a$, counted with multiplicity, is at most $N$.

One consequence is that the homotopy critical spectrum of any compact geodesic space is discrete in $(0, \infty)$, which is essential for the proof of our main theorem. In [29], a version of Theorem 10 is proved assuming that all spaces in question have a universal cover. The arguments there are indirect and without an explicit bound, since they first show that the set of corresponding covering spaces is itself Gromov-Hausdorff pointed precompact, then proceed by contradiction. Obtaining even better control over the distribution of critical values for specific classes of geodesic spaces is likely to be an interesting problem. For example, it was shown in [22] that limits of compact manifolds with non-negative Ricci curvature have finite covering spectra. The proof depends on deep results about the local structure of such spaces ([9], [10], [11]). That the limiting spaces have finite covering spectra implies that they have a universal cover in the categorial sense, but leaves open the interesting question of whether they are semilocally simply connected.

Gromov’s Betti Numbers Theorem inspires the following question: Is there a number $C(n)$ such that if $M$ is a Riemannian $n$-manifold with nonnegative sectional curvature then $M$ has at most $C(n)$ homotopy critical values, counted with multiplicity?

## 2 Basic Discrete Homotopy Tools

As is typical for metric spaces, the term “geodesic” in this paper refers to an arclength parameterized length minimizing curve (and a geodesic space is one in which every pair of points is joined by a geodesic). This is distinguished from the traditional term “geodesic” in Riemannian geometry, which is only a local isometry; we will refer to such a path in this paper as “locally minimizing”. The term “closed geodesic” will refer to a function from a standard circle into $X$ such
that the restriction to any sufficiently small arc is an isometry onto its image. 
We begin with a few results for metric spaces in general, including the definition 
of a natural metric on the space $X_\varepsilon$. While the lifting of a geodesic metric to 
a covering space is a well-known construction (see below), to our knowledge 
Definition 12 gives the first method to lift the metric of a general metric space 
to a covering space in such a way that the covering map is uniformly a local 
isometry and the deck group acts as isometries. In a metric space $X$ we denote 
by $B(x, r)$ the open metric ball $\{y : d(x, y) < r\}$.

**Proposition 11** Let $\alpha$, $\beta$ be $\varepsilon$-chains such that the endpoint of $\alpha$ is the beginning 
point of $\beta$. Then

1. (Positive Definite) $|[\alpha]_\varepsilon| \geq 0$ and $|[\alpha]_\varepsilon| = 0$ if and only if $\alpha$ is $\varepsilon$-null.

2. (Triangle Inequality) $|[\alpha * \beta]_\varepsilon| \leq |[\alpha]_\varepsilon| + |[\beta]_\varepsilon|$

**Proof.** That $|[\alpha]_\varepsilon| \geq 0$ and that $|[\alpha]_\varepsilon| = 0$ when $\alpha$ is $\varepsilon$-null are both immediate 
consequences of the definition. In general, if $|[\alpha]_\varepsilon| = 0$ then this means that 
for every $\delta > 0$ there is some $\varepsilon$-chain $\xi = \{y_0, ..., y_n\}$ such that $[\alpha]_\varepsilon = [\xi]_\varepsilon$ 
and $L(\xi) < \delta$. In particular we may take $\delta < \varepsilon$. Now for any $i < j$ we have $d(y_i, y_j) \leq \sum_{k=i+1}^{j} d(y_k, y_{k-1}) \leq L(\xi) < \delta < \varepsilon$ and $\alpha$ is $\varepsilon$-homotopic to the $\varepsilon$-chain $\{y_0, y_n\}$. By the same argument, $d(y_0, y_n) < \delta$ for all $\delta > 0$ and therefore 
d$(y_0, y_n) = 0$ and $y_0 = y_n$. That is, $\alpha$ is $\varepsilon$-homotopic to $\{y_0\}$.

For the Triangle Inequality, simply note that if $\alpha'$ is $\varepsilon$-homotopic to $\alpha$ and 
$\beta'$ is $\varepsilon$-homotopic to $\beta$, then $\alpha' * \beta'$ is $\varepsilon$-homotopic to $\alpha * \beta$. Therefore

$$|[\alpha * \beta]_\varepsilon| \leq L(\alpha' * \beta') = L(\alpha') + L(\beta').$$

Passing to the infimum $|[\alpha * \beta]_\varepsilon| - L(\beta') \leq |[\alpha]_\varepsilon| \Rightarrow |[\alpha * \beta]_\varepsilon| - |[\beta]_\varepsilon| \leq |[\alpha]_\varepsilon|$. Similarly $|[\alpha * \beta]_\varepsilon| - |[\alpha]_\varepsilon| \leq |[\beta]_\varepsilon|$. \]

**Definition 12** For $[\alpha]_\varepsilon$, $[\beta]_\varepsilon \in X_\varepsilon$ we define $d([\alpha]_\varepsilon, [\beta]_\varepsilon) = \inf \{L(\kappa) : \alpha * \kappa * \beta \text{ is } \varepsilon\text{-null} \} = |[\alpha * \beta]_\varepsilon|$.

The second equality above follows from the fact that $\alpha * (\alpha * \beta) = \alpha * \beta$ is $\varepsilon$-null. 
Proposition 11 implies that $d$ is a metric; we will always use this metric on $X_\varepsilon$.

**Proposition 13** Let $X$ be a metric space and $\varepsilon > 0$. Then

1. The function $\phi_\varepsilon : X_\varepsilon \rightarrow X$ preserves distances of length less than $\varepsilon$ and 
is injective when restricted to any open $\varepsilon$-ball. In particular, $\phi_\varepsilon$ is an 
isometry onto its image when restricted to any open $\frac{\varepsilon}{2}$-ball.

2. For any $\varepsilon$-loop $\lambda$ at $*$, the function $\tau_\lambda : X_\varepsilon \rightarrow X_\varepsilon$ defined by $\tau_\lambda([\alpha]_\varepsilon) = [\lambda * \alpha]_\varepsilon$ is an isometry such that $\tau_\lambda \circ \phi_\varepsilon = \phi_\varepsilon$. 

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Proof. As in the proof of the positive definite property, if \(d([\alpha]_\varepsilon, [\beta]_\varepsilon) < \varepsilon\) then \([\lambda \ast \beta]_\varepsilon\) must contain the chain \(\{y_0, y_1\}\) with \(d(y_0, y_1) = d([\alpha]_\varepsilon, [\beta]_\varepsilon) < \varepsilon\), where \(y_0\) and \(y_1\) are the endpoints of \(\alpha\) and \(\beta\). That \(\phi_\varepsilon\) is injective on any \(\varepsilon\)-ball was proved in greater generality in [5], but the argument is simple enough to repeat here. If \([\alpha]_\varepsilon, [\beta]_\varepsilon \in B(\gamma, \varepsilon)\) where \(\gamma = \{* = x_0, \ldots, x_n\}\), then we may take \(\alpha = \gamma \ast \{x_n, y_0\}\) and \(\beta = \gamma \ast \{x_n, y_1\}\). Then \(\phi_\varepsilon([\alpha]_\varepsilon) = \phi_\varepsilon([\beta]_\varepsilon)\) if and only if \(y_0 = y_1\), which is true if and only if \([\alpha]_\varepsilon = [\beta]_\varepsilon\).

To prove the second part, note that for any \([\alpha]_\varepsilon \in X_\varepsilon\),

\[
\tau_\lambda ([\lambda \ast \alpha]_\varepsilon) = [\lambda \ast \lambda \ast \alpha]_\varepsilon = [\alpha]_\varepsilon,
\]

showing that \(\tau_\lambda\) is onto. Next, for any \([\beta]_\varepsilon \in X_\varepsilon\) we have

\[
d(\tau_\lambda ([\alpha]_\varepsilon), \tau_\lambda ([\beta]_\varepsilon)) = d([\lambda \ast \alpha]_\varepsilon, [\lambda \ast \beta]_\varepsilon) = |\lambda \ast [\lambda \ast \alpha \ast \lambda \ast \beta]_\varepsilon|
\]

\[
= |\lambda \ast [\lambda \ast \alpha \ast \lambda \ast \beta]_\varepsilon| = |[\lambda \ast \beta]_\varepsilon| = d([\alpha]_\varepsilon, [\beta]_\varepsilon).
\]

Since \(\lambda \ast \alpha\) has the same endpoint as \(\alpha\), we also have that \(\tau_\lambda \circ \phi_\varepsilon = \phi_\varepsilon\). ■

The second part of the proposition shows that the group \(\pi_\varepsilon(X)\) acts by isometries on \(X_\varepsilon\). This action is \textit{discrete} in the sense of [18]; that is, if for any \([\alpha]_\varepsilon\) and \(\lambda\) we have that \(d(\tau_\lambda([\alpha]_\varepsilon), [\alpha]_\varepsilon) < \varepsilon\) then \(\tau_\lambda\) is the identity—i.e. \(\lambda\) is \(\varepsilon\)-null. Being discrete is stronger than being free and properly discontinuous, and hence when \(\phi_\varepsilon : X_\varepsilon \to X\) is surjective, \(\phi_\varepsilon\) is a regular covering map with covering group \(\pi_\varepsilon(X)\) (via the faithful action \([\lambda]_\varepsilon \to \tau_\lambda\)). Surjectivity of \(\phi_\varepsilon\) for all \(\varepsilon\) is clearly equivalent to \(X\) being “chain connected” in the sense that every pair of points in \(X_\varepsilon\) is joined by an \(\varepsilon\)-chain for all \(\varepsilon\). Chain connected is equivalent to what is sometimes called “uniformly connected” and is in general weaker than connected (see [5] for more details).

For consistency, we observe that our metric on \(X_\varepsilon\) is compatible with the uniform structure defined on \(X_\varepsilon\) in [5]. A basis for that uniform structure consists of all sets (called \textit{entourages}) \(E^*_\delta\), with \(0 < \delta \leq \varepsilon\), where \(E^*_\delta\) is defined as all ordered pairs \((([\alpha]_\varepsilon), [\beta]_\varepsilon)\) such that \([\alpha \ast [\alpha \ast \lambda \ast \beta]_\varepsilon]_\varepsilon = \{y, z\}_\varepsilon\) for some \(y, z\) with \(d(y, z) < \delta\). That is, \(E^*_\delta = \{([\alpha]_\varepsilon, [\beta]_\varepsilon) : d([\alpha]_\varepsilon, [\beta]_\varepsilon) < \delta\}\), which is a basis for the uniform structure of the metric defined in Definition 12. So the two uniform structures are identical.

We next consider a useful basic result showing that uniformly close \(\varepsilon\)-chains are \(\varepsilon\)-homotopic.

\textbf{Definition 14} Let \(X\) be a metric space. Given \(\alpha = \{x_0, \ldots, x_n\}\) and \(\beta = \{y_0, \ldots, y_n\}\) with \(x_i, y_i \in X\), define \(\Delta(\alpha, \beta) := \max_i d(x_i, y_i)\). For any \(\varepsilon > 0\), if \(\alpha\) is an \(\varepsilon\)-chain we define \(E_\varepsilon(\alpha) := \min_i \{\varepsilon - d(x_i, x_{i+1})\} > 0\). When no confusion will result we will eliminate the \(\varepsilon\) subscript.

\textbf{Proposition 15} Let \(X\) be a metric space and \(\varepsilon > 0\). If \(\alpha = \{x_0, \ldots, x_n\}\) is an \(\varepsilon\)-chain and \(\beta = \{y_0, \ldots, y_n = x_n\}\) is such that \(\Delta(\alpha, \beta) < \frac{E_\varepsilon(\alpha)}{2}\) then \(\beta\) is an \(\varepsilon\)-chain that is \(\varepsilon\)-homotopic to \(\alpha\).
**Proof.** We will construct an $\varepsilon$-homotopy $\eta$ from $\alpha$ to $\beta$. By definition of $E(\alpha)$ and the triangle inequality, each chain below is an $\varepsilon$-chain, and hence each step below is legal. Here and in the future we use the upper bracket to indicate that we are adding a point, and the lower bracket to indicate that we are removing a point in each basic step.

$$
\alpha = \{x_0, x_1, \ldots, x_n\} \rightarrow \{x_0, y_1, x_1, \ldots, x_n\} \rightarrow \{x_0, y_1, x_1, y_2, x_2, \ldots, x_n\} \\
\rightarrow \{x_0, y_1, x_1, y_2, x_2, \ldots, x_n\} \rightarrow \{x_0, y_1, y_2, x_2, y_2, x_2, \ldots, x_n\} \\
\rightarrow \{x_0, y_1, y_2, y_2, x_2, y_2, x_2, \ldots, x_n\} \rightarrow \{x_0, y_1, y_2, y_2, y_2, x_2, y_2, x_2, \ldots, x_n\} \rightarrow \cdots \rightarrow \beta
$$

In order to properly use Proposition 15, one needs chains of the same size, and the next lemma helps with this.

**Lemma 16** Let $L, \varepsilon > 0$ and $\alpha$ be an $\varepsilon$-chain in a metric space $X$ with $L(\alpha) \leq L$. Then there is some $\alpha' \in [\alpha]_\varepsilon$ such that $L(\alpha') \leq L(\alpha)$ and $\nu(\alpha') = \left\lceil \frac{2L}{\varepsilon} + 1 \right\rceil$.

**Proof.** If $\alpha$ has one or two points then we may simply repeat $x_0$, if necessary, (which doesn’t increase length) to obtain $\alpha'$ with $\nu(\alpha') = \left\lceil \frac{2L}{\varepsilon} + 1 \right\rceil$. Otherwise, let $\alpha := \{x_0, \ldots, x_n\}$ with $n \geq 2$. Suppose that for some $i$, $d(x_{i-1}, x_i) + d(x_i, x_{i+1}) < \varepsilon$. Then $d(x_{i-1}, x_{i+1}) < \varepsilon$ and the point $x_i$ may be removed to form a new $\varepsilon$-chain $\alpha_1$ that is $\varepsilon$-homotopic to $\alpha$ with $L(\alpha_1) \leq L(\alpha)$. After finitely many such steps, we have a chain $\alpha_0$ that is $\varepsilon$-homotopic to $\alpha$ and not longer, which either has two points (then proceed as above), or $\alpha_0$ has the property that for every $i$, $d(x_{i-1}, x_i) + d(x_i, x_{i+1}) \geq \varepsilon$. By pairing off terms we see that $L(\alpha_0) \geq \left\lceil \frac{\nu(\alpha_0)}{2} \right\rceil \varepsilon$ and hence

$$
\nu(\alpha_0) \leq \left\lceil \frac{2L(\alpha_0)}{\varepsilon} + 1 \right\rceil \leq \left\lceil \frac{2L}{\varepsilon} + 1 \right\rceil.
$$

As before, repeat $x_0$ enough times to make $\alpha'$ with $\nu(\alpha') = \left\lceil \frac{2L}{\varepsilon} + 1 \right\rceil$.

The above lemma can be used like a discrete version of Ascoli’s Theorem. That is, if one has a sequence of $\varepsilon$-chains of length at most $L$ (i.e., “equicontinuous”) then one can assume that all the chains have the same finite size $n$. In a compact space, one can then choose a subsequence so that the $i^{th}$ elements in each chain form a convergent sequence for all $0 \leq i \leq n$. For example, one may use this method in conjunction with Proposition 15 to obtain the following:

**Corollary 17** If $X$ is a compact metric space, $\varepsilon > 0$, and $\alpha$ is an $\varepsilon$-chain then there is some $\beta \in [\alpha]_\varepsilon$ such that $L(\beta) = ||[\alpha]_\varepsilon||$.

We next move onto the relationship between paths and chains.
Definition 18 Let $\alpha := \{x_0, ..., x_n\}$ be an $\varepsilon$-chain in a metric space $X$, where $\varepsilon > 0$. A stringing of $\alpha$ consists of a path $\hat{\alpha}$ formed by concatenating paths $\gamma_i$ from $x_i$ to $x_{i+1}$ where each path $\gamma_i$ lies entirely in $B(x_i, \varepsilon)$. If each $\gamma_i$ is a geodesic then we call $\hat{\alpha}$ a chording of $\alpha$.

Note that by uniform continuity, any path $c$ defined on a compact interval may be subdivided into an $\varepsilon$-chain $\alpha$ such that $c$ is a stringing of $\alpha$, and in any geodesic space every $\varepsilon$-chain has a chording.

Proposition 19 If $\alpha$ is an $\varepsilon$-chain in a chain connected metric space $X$ then the unique lift of any stringing $\hat{\alpha}$ starting at the basepoint $[\ast]$ in $X_{\varepsilon}$ has $[\alpha]_{\varepsilon}$ as its endpoint.

Proof. Let $\alpha_i := \{x_0, ..., x_i\}$, with $\alpha_n = \alpha$. We will prove by induction that the endpoint of the lift of a stringing $\hat{\alpha}_i$ is $[\alpha_i]_{\varepsilon}$. The case $i = 0$ is trivial; suppose the statement is true for some $i < n$ and consider some stringing $\alpha_{i+1}$. Then the restriction to a segment of $\alpha_{i+1}$ is a stringing $\hat{\alpha}_i$ and by the inductive step the lift of $\hat{\alpha}_i$ ends at $[\alpha_i]_{\varepsilon}$. By definition of stringing, $\alpha_{i+1}$ is obtained from $\hat{\alpha}_i$ by adding some path $c$ from $x_i$ to $x_{i+1}$ that lies entirely within $B(x_i, \varepsilon)$. By Proposition 13 $\phi_{\varepsilon}$ is bijective from the set $B([\alpha_i]_{\varepsilon}, \varepsilon)$ onto $B(x_i, \varepsilon)$. Therefore the lift $\tilde{c}$ of $c$ starting at $[\alpha_i]_{\varepsilon}$, must be contained entirely in $B([\alpha_i]_{\varepsilon}, \varepsilon)$. By uniqueness of lifts, the endpoint of the lift of $\alpha_{i+1}$ must be the endpoint $[\beta]_{\varepsilon}$ of $\tilde{c}$. Note that $\phi_{\varepsilon}([\beta]_{\varepsilon}) = x_{i+1}$; i.e. the endpoint of $\beta$ is $x_{i+1}$. Next, $[\beta]_{\varepsilon} \in B([\alpha_i]_{\varepsilon}, \varepsilon)$ means that there is some $\varepsilon$-chain $\sigma = \{y_0, ..., y_m\}$ such that $[\alpha_i]_{\varepsilon} = [y_0, ..., y_m, x_i]_{\varepsilon}$ and $[\beta]_{\varepsilon} = [y_0, ..., y_m, x_{i+1}]_{\varepsilon}$. Since $[\alpha_{i+1}]_{\varepsilon}$ is also clearly in $B([\alpha_i]_{\varepsilon}, \varepsilon)$ (just take $\sigma = \alpha_i$) and $\phi_{\varepsilon}([\alpha_{i+1}]_{\varepsilon}) = x_{i+1} = \phi_{\varepsilon}([\alpha_i]_{\varepsilon})$, the injectivity of $\phi_{\varepsilon}$ on $B([\alpha_i]_{\varepsilon}, \varepsilon)$ shows that $[\beta]_{\varepsilon} = [\alpha_{i+1}]_{\varepsilon}$. ■

Corollary 20 If $\alpha$ and $\beta$ are $\varepsilon$-chains in a chain connected metric space $X$ such that there exist stringings $\hat{\alpha}$ and $\hat{\beta}$ that are path homotopic then $\alpha$ and $\beta$ are $\varepsilon$-homotopic.

Proof. Choose the basepoint to be the common starting point of $\alpha$ and $\beta$. Since $\hat{\alpha}$ and $\hat{\beta}$ are path homotopic, the endpoints $[\alpha]_{\varepsilon}$ and $[\beta]_{\varepsilon}$ of their lifts must be equal. ■

A straightforward result in elementary homotopy theory of connected, locally path connected spaces is that two path loops $c_1$ and $c_2$ are freely homotopic if and only if for some paths $p_i$ from some particular point to the start/endpoint of $c_i$, $p_1 * c_1 * p_1$ is fixed-endpoint homotopic to $p_2 * c_2 * p_1$. Hence free $\varepsilon$-homotopy, as defined in the Introduction, is the correct discrete analog of continuous free homotopy. We have used this form because imitating the standard continuous version is notationally tricky for chains. The following lemma will be used later, and is the discrete analog of “rotation” of a path loop in itself.

Lemma 21 Let $\alpha := \{x_0, ..., x_n = x_0\}$ be an $\varepsilon$-loop in a metric space. Then $\alpha$ is freely $\varepsilon$-homotopic to $\alpha^P := \{x_{P(0)}, x_{P(1)}, ..., x_{P(n-1)}, x_{P(0)}\}$, where $P$ is any cyclic permutation of $\{0, 1, ..., n-1\}$. 

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Proof. It suffices to consider the cycle $P$ that adds one to each index, mod$(n)$. Let $\beta = \{ * = y_0, ..., y_m = x_0 \}$ be an $\varepsilon$-chain. Here, and in the future, we will denote $\varepsilon$-homotopies in the following form, where bracket on top denotes insertion and bracket on the bottom denotes deletion:

\[
\beta \ast \alpha \ast \overline{\beta} = \{ y_0, ..., y_m, x_1, ..., x_0, y_m-1, ..., y_0 \}
\]

\[
\rightarrow \{ y_0, ..., y_m, x_1, ..., x_0, \overbrace{x_0}^{\alpha}, y_m-1, ..., y_0 \}
\]

\[
\rightarrow \{ y_0, ..., y_m, x_1, ..., x_0, \overbrace{x_1}^{\alpha}, x_0, y_m-1, ..., y_0 \}
\]

\[
\rightarrow \{ y_0, ..., y_m, x_1, ..., x_0, \overbrace{x_1}^{\alpha}, x_1, x_0, y_m-1, ..., y_0 \}
\]

That is, $\beta \ast \alpha \ast \overline{\beta}$ is $\varepsilon$-homotopic to $\eta \ast \alpha^P \ast \eta$, where $\eta = \{ y_0, ..., y_m = x_0, x_1 \}$.

The rest of this section is devoted to basic results that are true for geodesic spaces. The situation for metric spaces in general is much more complicated—for example, the homotopy critical spectrum of a compact metric space may not be discrete (14).

The following statement is easy to check: Let $f : X \to Y$ be a bijection between geodesic spaces $X$ and $Y$. Then the following are equivalent: (1) $f$ is an isometry. (2) $f$ is a local isometry (i.e. for each $x \in X$ the restriction of $f$ to some $B(x, \varepsilon)$ is an isometry onto $B(f(x), \varepsilon)$). (3) $f$ is a length-preserving homeomorphism (i.e. if $c$ is a rectifiable path in $X$ then $f \circ c$ is rectifiable and $L(f \circ c) = L(c)$).

Recall that if $f : X \to Y$ is a covering map, $Y$ is a geodesic space, and $X$ is a connected topological space, then the lifted length metric on $X$ is defined by $d(x, y) = \inf \{ L(f \circ c) \}$, where $c$ is a path joining $x$ and $y$. When $Y$ is proper (i.e. its closed metric balls are compact) then $X$, being uniformly locally isometric to $Y$, is locally compact and complete. Hence by a classical result of Cohn-Vossen, $X$ with the lifted length metric is also a proper geodesic space. In this case it also follows from what was stated previously that the lifted metric is the unique geodesic metric on $X$ such that $f$ is a local isometry. In particular, if $g : Z \to Y$ is a covering map, where $Z$ is geodesic, and $h : X \to Z$ is a covering equivalence then $h$ is an isometry.

**Proposition 22** If $X$ is a geodesic space then the metric on $X_\varepsilon$ given in Definition 13 is the lifted length metric. In particular if $X$ is proper then $X_\varepsilon$ is a proper geodesic space.

**Proof.** Let $[\alpha]_\varepsilon, [\beta]_\varepsilon \in X_\varepsilon$. Then

\[
d([\alpha]_\varepsilon, [\beta]_\varepsilon) = \inf \{ L(\kappa) : [\alpha \ast \kappa \ast \overline{\beta}]_\varepsilon \text{ is } \varepsilon\text{-null} \}
\]

Let $\alpha = \{ x_0, ..., x_n \}$, $\beta = \{ y_0, ..., y_k \}$ and $\kappa = \{ x_n = z_0, ..., z_m = y_k \}$. Let $\hat{\alpha}$ and $\hat{\kappa}$ be chordings of $\alpha$ and $\kappa$, and note that the length of the chain $\kappa$ is the same as the length of the curve $\hat{\kappa}$. Moreover, when $[\alpha \ast \kappa \ast \overline{\beta}]_\varepsilon$ is $\varepsilon$-null, $[\alpha \ast \kappa]_\varepsilon = [\beta]_\varepsilon$.

We will now apply Proposition 19 a couple of times. First, the lift of $\hat{\alpha}$ starting
at \([*]\) ends at \([\alpha]\) and the lift of \(\hat{\alpha} \ast \hat{\kappa}\) (which is a chording of \(\alpha \ast \kappa\)) starting at \([*]\) ends at \([\alpha \ast \kappa]\) starting at \([\alpha]\). By uniqueness, the lift of \(\hat{\alpha} \ast \hat{\kappa}\) starting at \([*]\) must be the concatenation of the lift of \(\hat{\alpha}\) starting at \([*]\) with the lift \(\hat{\kappa}\) of \(\kappa\) starting at \([\alpha]\). That is, \(\hat{\kappa}\) is a path in \(X\) starting at \([\alpha]\) and ending at \([\beta]\), with \(L(\phi \circ \hat{\kappa}) = L(\hat{\kappa}) = L(\kappa)\). This shows that the metric of Definition 12 is a geodesic metric, and since we already know that \(\phi\) is a local isometry, by our previous comments on uniqueness, it must be the lifted length metric.

One consequence of Proposition 22 is that each \(X\) is path connected. Then it follows from the results of [5] that the maps \(\phi_{\epsilon \delta}: X_{\delta} \to X_{\epsilon}\) are also regular covering maps (in general surjectivity is the only question, and this requires \(X\) to be chain connected).

**Lemma 23** If \(X\) is a geodesic space then the covering map \(\phi_{\epsilon \delta}: X_{\delta} \to X_{\epsilon}\) is injective if and only if there are no homotopy critical values \(\sigma\) with \(\delta \leq \sigma < \epsilon\).

**Proof.** If there is such a critical value \(\sigma\) then there is a \(\sigma\)-loop \(\lambda\) that is not \(\sigma\)-null but is \(\epsilon\)-null. That is, \([\lambda]_{\sigma} \neq [*]_{\sigma}\) but \([\lambda]_{\epsilon} = [*]_{\epsilon}\), i.e. \(\phi_{\sigma}\) is not injective. But since \(\phi_{\sigma \delta}\) is surjective, \(\phi_{\epsilon \delta} = \phi_{\sigma \epsilon} \circ \phi_{\sigma \delta}\) is not injective. Conversely, if \(\phi_{\epsilon \delta}\) is not injective then there is some \(\delta\)-loop \(\lambda\) that is not \(\delta\)-null but is \(\epsilon\)-null. Let \(\sigma := \sup\{\tau: [\lambda]_{\tau} \neq [*]_{\tau}\}\); so \(\delta \leq \sigma < \epsilon\). If \(\lambda\) were \(\sigma\)-null then any \(\sigma\)-null homotopy would also be a \(\tau\)-homotopy for \(\tau < \sigma\) sufficiently close to \(\sigma\). So \(\lambda\) is not \(\sigma\)-null; hence \(\sigma\) is a homotopy critical value and \(\sigma < \epsilon\).

**Remark 24** Unfortunately the false statement that every free homotopy class in a compact geodesic space has a shortest path, and this path is a closed geodesic, is present in both editions of [14], [15], despite the intermediate publication of two kinds of counterexamples in ([3]). In one example, a homotopy class in an infinite dimensional “weakly flat” torus contains no rectifiable curves at all. One can also see this sort of thing in the Hawaiian Earring with its geodesic metric. In another example in [13] there is a loop that is rectifiable and shortest in its homotopy class, but is not a closed geodesic. It can easily be checked directly (and as is equally obvious from the fact that it is not a closed geodesic), this loop is not metrically embedded, as predicted by Theorem 38. The false statement in ([14], [13]) is evidently used in the proofs of Theorem 3.4 in [20] and Theorem 2.7 in [22], although it can be worked around using methods analogous to those found in [23] to solve a similar problem. Using \(\epsilon\)-chains avoids all such rectifiability issues. The next theorem clarifies the situation.

**Theorem 25** If \(X\) is a compact semilocally simply connected geodesic space then the homotopy critical spectrum has a positive lower bound. If \(\epsilon > 0\) is any such lower bound then

1. \(\phi_{\epsilon}: X_{\epsilon} \to X\) is the universal covering map of \(X\).

2. The function \(\Lambda\) is length preserving and hence the restriction to \(\pi_1(X) \to \pi_{\epsilon}(X)\) is an isomorphism.
3. Every path has a shortest path in its fixed-endpoint homotopy class, which is either constant or locally minimizing.

4. Every path loop has a shortest path in its free homotopy class, which is either constant or a closed geodesic.

Proof. Proposition 69 and Theorem 77 from [5] imply that for all sufficiently small \( \sigma > 0 \), \( \phi_\sigma \) is the simply connected covering map of \( X \). Lemma 23 shows that \( \phi_\varepsilon \) is equivalent to \( \phi_\sigma \) since there are no homotopy critical values between \( \varepsilon \) and \( \sigma \), proving the first part. Now let \( c : [0, a] \to X \) be a path: take \( * = c(0) \) to be the basepoint. By definition, \( \Lambda([c]) = [\alpha]_\varepsilon \), where \( \alpha := \{ c(t_0), \ldots, c(t_n) \} \) for a partition \( \{ 0 = t_0, \ldots, t_n = a \} \) that is sufficiently fine. By Corollary 17 there is some \( \beta \in [\alpha]_\varepsilon \) such that \( ||[\alpha]_\varepsilon|| = L(\beta) \). Let \( \sigma' \) be any chording of \( \beta \); so \( ||[\alpha]_\varepsilon|| = L(\beta) = L(\sigma') \). Now according to Proposition 19, the unique lifts of \( c \) and \( \sigma' \) at the basepoint of \( X_\varepsilon \) have the same endpoint, and therefore form a loop. But since \( X_\varepsilon \) is simply connected, this means that \( c \) and \( \sigma' \) are homotopic, and we have that \( ||c|| \leq L(\sigma') = ||[\alpha]_\varepsilon|| \). Since we already have the other inequality, the second part is finished. Moreover, we have shown that \( ||c|| \) is actually realized by any chording of a shortest \( \varepsilon \)-loop in \( \Lambda([c]) \). If any segment \( \sigma \) of such a shortest loop having length at most \( \frac{\varepsilon}{2} \) were not a geodesic then the endpoints of \( \sigma \) could be joined by a shorter geodesic \( \sigma' \). But then the loop formed by these two paths would lie in a ball of radius \( \frac{\varepsilon}{2} \), and hence would lift as a loop. That is, we could replace \( \sigma \) by \( \sigma' \) while staying in the same homotopy class, a contradiction. This proves the third part, and the proof of the fourth part is similar. }

Remark 26 In the previous theorem, if \( X \) is already simply connected then the proof shows that the homotopy critical spectrum is empty (the statement of the theorem is still correct in this case, since any real number is a lower bound for the empty set). Conversely, if \( X \) is compact and semilocally simply connected with empty covering spectrum, then \( X \) is simply connected. The latter implication is not true without the assumption that \( X \) is semilocally simply connected (see Example 14).

Definition 27 Let \( c : [0, L] \to X \) be an arclength parameterized path in a metric space. A subdivision \( \varepsilon \)-chain of \( c \) is an \( \varepsilon \)-chain \( \{ x_0, \ldots, x_n \} \) of the form \( x_i := c(t_i) \) for some subdivision \( t_0 = 0 < \cdots < t_n = L \) such that for all \( i \), \( t_{i+1} - t_i < \varepsilon \) (we will refer to this condition as \( \varepsilon \)-fine). If \( X \) is a geodesic space and \( \alpha \) is a chain in \( X \) then a refinement of \( \alpha \) consists of a chain \( \beta \) formed by inserting between each \( x_i \) and \( x_{i+1} \) some subdivision chain of a geodesic joining \( x_i \) and \( x_{i+1} \). If \( \beta \) is an \( \varepsilon \)-chain we will call \( \beta \) an \( \varepsilon \)-refinement of \( \alpha \).

Since \( c \) is 1-Lipschitz, any subdivision \( \varepsilon \)-chain is indeed an \( \varepsilon \)-chain. Obviously a refinement of an \( \varepsilon \)-chain \( \alpha \) is \( \varepsilon \)-homotopic to \( \alpha \) (just add the points one at a time) and hence any two refinements of \( \alpha \) are \( \varepsilon \)-homotopic. A special case is the midpoint refinement defined in the Introduction.

Definition 28 If \( X \) is a metric space and \( \varepsilon > 0 \), an \( \varepsilon \)-loop of the form \( \lambda = \alpha * \tau * \pi \), where \( \nu(\tau) = 3 \), will be called \( \varepsilon \)-small. Note that this notation includes the case when \( \alpha \) consists of a single point—i.e. \( \lambda = \tau \).
Note that any $\varepsilon$-small loop is $\varepsilon$-null, although it may or may not be $\delta$-null for smaller $\delta$.

**Proposition 29** Let $X$ be a geodesic space and $0 < \varepsilon < \delta$. Suppose $\alpha, \beta$ are $\varepsilon$-chains and $(\gamma_0, \ldots, \gamma_n)$ is a $\delta$-homotopy such that $\gamma_0 = \alpha$ and $\gamma_n = \beta$. Then $[\beta]_\varepsilon = [\lambda_1 \ast \ast \ast \lambda_r \ast \alpha \ast \lambda_{r+1} \ast \ast \ast \lambda_n]_\varepsilon$, where each $\lambda_i$ is an $\varepsilon$-refinement of a $\delta$-small loop.

**Proof.** We will prove by induction that for every $k \leq n$, an $\varepsilon$-refinement $\gamma'_k$ of $\gamma_k$ is $\varepsilon$-homotopic to $\lambda_1 \ast \ast \ast \alpha \ast \ast \ast \lambda_k$, where each $\lambda_i$ is an $\varepsilon$-refinement of a $\delta$-small loop. The case $k = 0$ is trivial. Suppose the statement is true for some $0 \leq k < n$. The points required to $\varepsilon$-refine $\gamma_k$ to $\gamma'_k$ will be denoted by $m_i$. Suppose that $\gamma_{k+1}$ is obtained from $\gamma_k$ by adding a point $x$ between $x_i$ and $x_{i+1}$. Let $\{x_i, a_1, \ldots, a_k, x\}$ be an $\varepsilon$-refinement of $\{x, x\}$ and $\{x, b_1, \ldots, b_m, x_{i+1}\}$ an $\varepsilon$-refinement of $\{x, x_{i+1}\}$, so

$$\gamma'_{k+1} = \{x_0, m_0, \ldots, x_i, a_1, \ldots, a_k, x, b_1, \ldots, b_m, x_{i+1}, m_{r+1}, \ldots, x_j\}$$

is an $\varepsilon$-refinement of $\gamma_{k+1}$. Defining $\mu_{k+1} := \{x_0, m_0, \ldots, x_i\}$ and

$$\kappa_{k+1} = \{x_i, a_1, \ldots, a_k, x, b_1, \ldots, b_m, x_{i+1}, m_r, \ldots, x_j\}$$

we have

$$[\gamma'_{k+1} ]_\varepsilon = [\mu_{k+1} \ast \kappa_{k+1} \ast \mu_{k+1} \ast \gamma_k]_\varepsilon$$

and since the homotopy is a $\delta$-homotopy, $\lambda_{k+1} := \mu_{k+1} \ast \kappa_{k+1} \ast \mu_{k+1}$ is a refinement of a $\delta$-small loop. The case when a point is removed from $\gamma_k$ is similar, except that the $\delta$-small loop is multiplied on the right. ■

**Example 30** Since circles play an important role in this paper, we’ll conclude this section with the simple example of the geodesic circle $C$ of circumference 1. If $\varepsilon > \frac{1}{2}$ then since all points in $C$ are of distance at most $\frac{1}{2}$, every $\varepsilon$-loop is $\varepsilon$-null: just remove the points (except the endpoints) one by one. The group $\pi_1(C)$ is trivial and $\phi_\varepsilon : C_\varepsilon \rightarrow C$ is an isometry. On the other hand, if $\varepsilon > 0$ is fairly small, it should be intuitively clear that it is impossible to “cross the hole” with an $\varepsilon$-homotopy, since any basic move “spans a triangle” with side lengths equal to $\varepsilon$; therefore $\pi_\varepsilon(C)$ should be the non-trivial (and in fact Theorem 25 tells us that it will be $\pi_1(C)$). One can check that in fact the homotopy critical spectrum of $C$ is $\{\frac{3}{2}\}$—see [7] for a nice argument involving “discrete winding numbers”; this also follows from results in the next section.

## 3 Essential Triads and Circles

**Definition 31** If $c$ is an arclength parameterized loop, we say that $c$ is $\varepsilon$-null if every (or equivalently, some) $\varepsilon$-subdivision chain of $c$ is $\varepsilon$-null.

**Lemma 32** Every arclength parameterized loop of length less than $3\varepsilon$ in a geodesic space $X$ is $\varepsilon$-null.
Proof. Let $c : [0, L] \to X$ be arclength parameterized with $c(0) = c(L) = p$ and $0 < L < 3\varepsilon$. Then there exists an $\varepsilon$-fine subdivision $\{0 = t_0, t_1, t_2, t_3 = L\}$. Since $d(c(t_1), c(t_3)) = d(c(t_1), c(t_0)) < \varepsilon$, we may simply remove $c(t_2)$ and then $c(t_1)$ to get an $\varepsilon$-null homotopy. ■

The next corollary is proved by simply joining the points in the loop by geodesics and concatenating them to obtain an arclength parameterized loop of length less than $3\varepsilon$.

**Corollary 33** If $\lambda$ is an $\varepsilon$-loop in a geodesic space $X$ of length less than $3\varepsilon$ then $\lambda$ is $\varepsilon$-null.

**Remark 34** If $C$ is the image of a rectifiable loop of length $L$ in a metric space $X$ then by the basic theory of curves in metric spaces, for every point $x$ on $C$ there are precisely two possible arclength parameterizations $c : [0, L] \to X$ of $C$ such that $c(0) = c(L) = x$.

**Proposition 35** The image $C$ of a rectifiable path loop of length $L = 3\varepsilon$ in a geodesic space $X$ is an essential $\varepsilon$-circle if and only if either arclength parameterization of it is not $\varepsilon$-null.

Proof. Let $c : [0, L] \to C$ be an arclength parameterization of $C$. If $C$ is not an essential $\varepsilon$-circle then by definition, every $\varepsilon$-chain in it is $\varepsilon$-null. But then any $\varepsilon$-subdivision of $c$, being an $\varepsilon$-chain, must be $\varepsilon$-null. Hence $c$ is by definition $\varepsilon$-null. Conversely, suppose that $C$ is essential, and so contains an $\varepsilon$-loop $\alpha = \{x_0, ..., x_n = x_0\}$ that is not $\varepsilon$-null, with $x_i := c(t_i)$. We will show that $\alpha$ is $\varepsilon$-homotopic to a concatenation of chains that are subdivision $\varepsilon$-chains of $c$ or reversals of $c$. Then at least one of those subdivision chains must be not $\varepsilon$-null, finishing the proof. Form a path as follows: choose a shortest segment $\sigma_i$ of $c$ between $x_{i-1}$ and $x_i$. By “segment” we mean the restriction of $c$ to a closed interval, or a path of the form $c |_{[t, L]} * c |_{[0, s]}$ (i.e. when it is shorter to go through $x_0$). Let $\tilde{c} := \sigma_1 * \cdots * \sigma_n$. Since each $\sigma_i$ has length at most $\frac{\varepsilon}{4}$, by adding points $b_j$ that bisect each segment $\sigma_i$ we see that $\alpha$ is $\varepsilon$-homotopic to a subdivision $\varepsilon$-chain $\tilde{\alpha} := \{x_0, b_1, x_1, ..., b_n, x_n\}$ of $\tilde{c}$. On the other hand, $\tilde{c}$ is path homotopic (in the image of $c$, in fact) to its “cancelled concatenation” $\sigma_1 * \cdots * \sigma_n$. Recall that the cancelled concatenation $c_1 * c_2$ is formed by starting with the concatenation $c_1 * c_2$ and removing the maximal final segment of $c_1$ that is equal to an initial segment of $c_2$ with reversed orientation (see [6], p. 1771, for more details). It is not hard to check by induction that $\sigma_1 * \cdots * \sigma_n$ is of the form $(k_1 * \cdots * k_m) * d$, where the following are true: $k_i = c$ or $k_i = \overline{c}$ for all $i$ (and it is possible that $m = 0$, meaning there are no $k_i$ factors), and for some $0 \leq s < L$, $d$ is of the form $c |_{[0, s]}$ or $c |_{[s, L]}$. Since $\alpha$ is a loop, $\sigma_1 * \cdots * \sigma_n$ has no nontrivial term $\overline{c}$, and hence consists of concatenations of $c$ or $\overline{c}$. Since $\tilde{c}$ is a stringing of $\tilde{\alpha}$, Corollary [20] implies that $\tilde{\alpha}$, hence $\alpha$, is $\varepsilon$-homotopic to any subdivision $\varepsilon$-chain of $\sigma_1 * \cdots * \sigma_n$. ■

A geodesic triangle consists of three geodesics $\gamma_1, \gamma_2, \gamma_3$ such that for some three points $v_1, v_2, v_3, \gamma_i$ goes from $v_i$ to $v_{i+1}$, with addition of vertices mod 3).
A geodesic triangle may be considered as a loop by taking the arclength parameterization of the concatenation of the geodesics; as far as being \( \varepsilon \)-null is concerned, the specific orientation clearly doesn’t matter. We say the triangle is \( \varepsilon \)-null if such a parameterization \( \varepsilon \)-null.

**Proposition 36** Let \( T \) be an \( \varepsilon \)-triad in a geodesic space. Then any two \( \varepsilon \)-refinements of \( \alpha_T \) are \( \varepsilon \)-homotopic. Moreover, the following are equivalent:

1. \( T \) is essential.
2. No \( \varepsilon \)-refinement of \( \alpha_T \) is \( \varepsilon \)-null.
3. Every geodesic triangle having \( T \) as a vertex set is an essential \( \varepsilon \)-circle.

**Proof.** Let \( T := \{x_0, x_1, x_2\} \) and \( \beta = \{x_0, m_0, x_1, m_1, x_2, m_2, x_0\} \) be a midpoint refinement of \( \alpha_T \). If \( m'_0 \) is another midpoint between \( x_0 \) and \( x_1 \) then the \( \varepsilon \)-chain \( \{x_0, m'_0, x_1, m_1, x_2, m_2, x_0\} \) has length at most \( 2\varepsilon < 3\varepsilon \) and is \( \varepsilon \)-null by Corollary 33. Therefore \( \beta \) is \( \varepsilon \)-homotopic to \( \{x_0, m'_0, x_1, m_1, x_2, m_2, x_0\} \). A similar argument shows that the other two midpoints may be replaced, up to \( \varepsilon \)-homotopy. In other words, any two midpoint refinements of \( \alpha_T \) are \( \varepsilon \)-homotopic. But any \( \varepsilon \)-refinement of \( \alpha_T \) has a common refinement with a midpoint refinement, so by the comments after Definition 27, any two \( \varepsilon \)-refinements of \( \alpha_T \) are \( \varepsilon \)-homotopic.

1 \( \Rightarrow \) 2. If \( T \) is essential then by definition some midpoint refinement of \( \alpha_T \) is not \( \varepsilon \)-null. By the very first statement of this proposition, any other \( \varepsilon \)-refinement of \( \alpha_T \) is not \( \varepsilon \)-null. 2 \( \Rightarrow \) 3. Suppose \( C := (\gamma_0, \gamma_1, \gamma_2) \) is any geodesic triangle having \( T \) as a vertex set. Then the subdivision chain of \( C \) consisting of the vertices and midpoints of the geodesics is an \( \varepsilon \)-refinement of \( \alpha_T \) and is not \( \varepsilon \)-null by assumption. Since \( C \) also has length \( 3\varepsilon \), by definition \( C \) is an essential \( \varepsilon \)-circle, and 3 is proved. 3 \( \Rightarrow \) 1. Form a geodesic triangle, hence an essential \( \varepsilon \)-circle \( C \), having the points of \( T \) as vertices. Then any midpoint refinement of \( T \) is an \( \varepsilon \)-subdivision of \( C \), which by Proposition 35 is not \( \varepsilon \)-null. By definition, \( T \) is essential. ■

An immediate consequence of Proposition 36 is the following:

**Corollary 37** The following statements are equivalent for two essential \( \varepsilon \)-triads \( T_1, T_2 \) in a geodesic space:

1. \( T_1 \) is equivalent to \( T_2 \).
2. Every \( \varepsilon \)-refinement of \( \alpha_{T_1} \) is freely \( \varepsilon \)-homotopic to every \( \varepsilon \)-refinement of either \( \alpha_{T_2} \) or \( \alpha_{T_2}^{-1} \).
3. Some \( \varepsilon \)-refinement of \( \alpha_{T_1} \) is freely \( \varepsilon \)-homotopic to some \( \varepsilon \)-refinement of either \( \alpha_{T_2} \) or \( \alpha_{T_2}^{-1} \).

**Proof of Proposition 36.** Note that by Corollary 34 we may use any \( \delta \)-refinement in the arguments that follow. Suppose that \( T' \) is a \( \delta \)-triad; by the triangle inequality, \( \delta < \frac{4\varepsilon}{3} \). Suppose first that \( \delta \geq \frac{4\varepsilon}{3} \). By the triangle inequality, \( L((x_0, x'_0, x'_1, x_1, x_0)) < \frac{10}{3}\varepsilon < 3\delta \), and therefore any \( \delta \)-refinement of this
chain is $\delta$-null by Corollary \ref{corollary:delta-null}. Since a similar statement applies to the loops $\{x_1, x_1', x_2, x_1\}$ and $\{x_0, x_2, x_0', x_0\}$, it follows that any $\delta$-refinement of $\alpha_T$ is freely $\delta$-homotopic to a $\delta$-refinement of $\alpha_T$. Since $T$ is an essential $\varepsilon$-triad and $\varepsilon < \delta$, any midpoint refinement of $\alpha_T$ and hence any midpoint refinement of $\alpha_T'$ is $\delta$-null. That is, $T'$ is not essential.

Now suppose that $\delta < \frac{3}{4}\varepsilon$. By the triangle inequality, $L(\{x_0, x_0', x_1, x_0\}) < 3\varepsilon$ and therefore any $\varepsilon$-refinement of this chain is $\varepsilon$-null by Corollary \ref{corollary:delta-null}. Since a similar statement applies to the loops $\{x_1, x_1', x_2, x_1\}$ and $\{x_0, x_2, x_0', x_0\}$, it follows that any $\varepsilon$-refinement of $\alpha_T$ is freely $\varepsilon$-homotopic to an $\varepsilon$-refinement of $\alpha_T$. Since no $\varepsilon$-refinement of $\alpha_T$ is $\varepsilon$-null, neither is any $\varepsilon$-refinement of $\alpha_T'$. On the other hand, if $\sigma > \varepsilon$, $\alpha_T$ is $\sigma$-null and hence $\alpha_T'$ is also $\sigma$-null. Therefore if $T'$ is an essential triad then $T'$ cannot be a $\sigma$-triad for any $\sigma > \varepsilon$. On the other hand, if $T'$ were an essential $\sigma$-triad for some $\sigma < \varepsilon$ then any midpoint refinement of $\alpha_T'$ would have to be $\varepsilon$-null, a contradiction. \hfill \blacksquare

**Theorem 38** Let $X$ be a geodesic space, $\varepsilon > 0$, $L = 3\varepsilon$ and $c : [0, L] \to X$ be arclength parameterized. If the image of $c$ is an essential $\varepsilon$-circle $C$ then $c$ is not null-homotopic and $C$ is metrically embedded.

**Proof.** That $c$ is not null-homotopic is immediate from Corollary \ref{corollary:delta-null}. For the second part we will start by showing that the restriction of $c$ to the interval $[\frac{L}{4}, \frac{3L}{4}]$ is a geodesic, hence a metric embedding. If not then $d(c\left(\frac{L}{4}\right), c\left(\frac{3L}{4}\right)) < \frac{L}{2}$. We will get a contradiction to Proposition \ref{proposition:triangle} by proving that the $\varepsilon$-loop $\alpha = \{x_0, x_1, x_2, x_3, x_0\}$ for the subdivision $\{0, \frac{L}{4}, \frac{L}{2}, \frac{3L}{4}, L\}$ is $\varepsilon$-null. Let $m$ be a midpoint between $x_1$ and $x_3$. By our assumption (and since $c$ is arclength parameterized), $\xi := \{x_1, x_2, x_3, m, x_1\}$ is an $\varepsilon$-chain and has length strictly less than $L$ and hence by Corollary \ref{corollary:delta-null} is $\varepsilon$-null. By adding points one at a time we have $\alpha$ is $\varepsilon$-homotopic to $\{x_0, x_1, x_2, x_3, m, x_3, x_0\}$, which is $\varepsilon$-homotopic to

$$\{x_0, x_1, x_2, x_3, m, x_1, m, x_3, x_0\} = \{x_0, x_1\} * \xi * \{x_1, m, x_3, x_0\}$$

which is $\varepsilon$-homotopic to $\beta = \{x_0, x_1, m, x_3, x_0\}$. But once again, since $d(x_1, x_3) < \frac{L}{2}$, $\beta$ is $\varepsilon$-null.

Now for any $s_0 \in [0, L]$ we may “shift” the parameterization of $c$ to a new curve $c_{s_0} : [0, L] \to X$ that is the unique arclength monotone reparameterization of the concatenation $c|_{[s_0, L]} * c|_{[0, s_0]}$. Applying the above argument for arbitrary $s_0$ we obtain the following. For every $x = c(s), y = c(t) \in C$, with $s < t$, $d(x, y)$ is the minimum of the lengths of the two curves $c|_{[s, t]}$ and $c|_{[t, L]} * c|_{[0, s]}$.

Define $r := \frac{L}{2\pi}$, and let $K$ be the standard Euclidean circle of radius $r$ (with the geodesic metric). Now we may define $f : C \to K$ by $f(c(t)) = (r \cos \frac{t}{r}, r \sin \frac{t}{r})$. Given that $c$ is arclength parameterized, and what we proved above, it is straightforward to check that $f$ is a well-defined isometry. \hfill \blacksquare

**Corollary 39** Every $\varepsilon$-triad on an essential $\varepsilon$-circle is essential. Moreover, if $C_1, C_2$ are essential $\varepsilon$-circles in a geodesic space then the following are equivalent:
1. $C_1$ and $C_2$ have arclength parameterizations with subdivision $\varepsilon$-chains that are freely $\varepsilon$-homotopic.

2. For some triads $T_i$ on $C_i$, $T_1$ is equivalent to $T_2$.

3. For any triads $T_i$ on $C_i$, $T_1$ is equivalent to $T_2$.

4. For any arclength parameterizations $c_i$ of $C_i$, any subdivision $\varepsilon$-chain of $c_1$ is freely $\varepsilon$-homotopic to any subdivision $\varepsilon$-chain of either $c_2$ or $\overline{c_2}$.

**Proof.** A triad $T$ on $C$ must be an $\varepsilon$-triad since by Theorem 38, $C$ is metrically embedded—in fact from the same theorem it follows that the segments of $C$ between the points of $T$ must be geodesics. Therefore the midpoints of these geodesics give a midpoint refinement of $\alpha_T$ that is also an $\varepsilon$-subdivision of a parameterization of $C$, and hence is not $\varepsilon$-null. That is, $T$ is essential.

We next show that any two triads $T = \{x_0, x_1, x_2\}$ and $T' = \{x'_0, x'_1, x'_2\}$ on $C$ are equivalent. First note that $T$ is equivalent to any reordering of its points. In fact, any reordering may be obtained by a cyclic permutation (which is covered by Lemma 21 applied to any midpoint refinement of $\alpha_T$ and/or a swap of $x_1$ and $x_2$ (which by definition doesn’t affect equivalence since it simply reverses $\alpha_T$). Now applying some reordering of $T$ we may suppose that the points are arranged around the circle in the following order: \{x_0, x'_0, x_1, x'_1, x_2, x'_2, x_0\}, which is an $\varepsilon$-refinement of $\alpha_T$. By Lemma 21 this $\varepsilon$-chain is freely $\varepsilon$-homotopic to $\{x'_0, x_1, x'_1, x_2, x'_2, x_0, x'_0\}$, which is an $\varepsilon$-refinement of $\alpha_T$. So the first part of the corollary is finished by Corollary 37.

1 $\Rightarrow$ 2. Choose arclength parameterizations $c_i$ of $C_i$ with subdivision $\varepsilon$-chains $\lambda_i$ starting at points $z_i$ that are freely $\varepsilon$-homotopic. Choose one of the two triads, call it $T_i$, in each $C_i$ starting at $z_i$, that is also a subdivision chain of $c_i$. By the comments after Definition 27 we see that $\lambda_i$ and the midpoint refinement of $\alpha_T_i$ on $C_i$ are $\varepsilon$-homotopic. Hence midpoint refinements of $\alpha_T_i$ and $\alpha_T_2$ are freely $\varepsilon$-homotopic, so $T_1$ is equivalent to $T_2$. 2 $\Rightarrow$ 3 is an immediate consequence of the first part of this corollary. 3 $\Rightarrow$ 4. Consider the triads $T_i = \{c_i(0), c_i(\varepsilon), c_i(2\varepsilon)\}$. By reversing one of the parameterizations, if necessary, we may suppose that $T_i$ is freely $\varepsilon$-homotopic to $T_2$. But then midpoint refinements of $T_i$ are subdivision $\varepsilon$-chains of $c_1$ that are freely $\varepsilon$-homotopic. 4 $\Rightarrow$ 1 simply follows from the definition.

**Definition 40** An essential $\varepsilon$-circle $C_1$ and an essential $\delta$-circle $C_2$ are said to be equivalent if $\varepsilon = \delta$ and the four equivalent conditions in the previous corollary hold. When $\varepsilon$ is not determined we will just refer to essential circles.

**Proof of Theorem 3** If there is an essential $\varepsilon$-circle $C$ then there is an arclength parameterization $c : [0, 3\varepsilon] \to C$. Since $c$ is not $\varepsilon$-null, by definition a subdivision of $[0, 3\varepsilon]$ into fourths results in an $\varepsilon$-loop $\alpha$ that is not $\varepsilon$-null. But for any $\delta > \varepsilon$, Lemma 32 (applied to $\delta$) shows that $\alpha$ must be $\delta$-null for all $\delta > \varepsilon$ and hence has $\varepsilon$ as its critical value.

For the converse, suppose that $\lambda$ is $\varepsilon$-critical. We will start by showing that for all $\varepsilon < \delta < 2\varepsilon$ there is a midpoint refinement of a $\delta$-small loop that is not
ε-null. In fact, since λ is ε-critical, it is δ-null and therefore by Proposition 29 can be written as a product of midpoint refinements of δ-small loops. If all of these loops were ε-null, then λ would also be ε-null, a contradiction. Now for every i we may find \((ε + \frac{1}{i})\)-small loops \(λ_i = μ_i \ast \{x_i, y_i, z_i, x_i\} \ast μ_i\) such that midpoint subdivisions \(θ_i = \{x_i, m_i, y_i, n_i, z_i, p_i, x_i\}\) are not ε-null. By choosing a subsequence if necessary, we may suppose that all six sequences converge to a limiting midpoint subdivision chain \(μ = \{x, m, y, n, z, p, x\}\) of length at most \(3ε\). But according to Proposition 15, for large enough \(i\), \(μ\) is ε-homotopic to \(μ_i\), which means that \(μ\) is not ε-null. This means that the chain \(\{x, y, z, x\}\) must have length equal to \(3ε\). Since \(d(x, y), d(y, z), d(x, z) ≤ ε\) it follows that \(\{x, y, z\}\) is a triad and hence is essential. By Proposition 30 any geodesic triangle having corners \(\{x, y, z\}\) is an essential ε-circle. ■

**Corollary 41** Suppose \(X\) is a compact geodesic space with 1-systole \(σ_1\). Then

1. \(\frac{σ_1}{3}\) is a lower bound for the homotopy critical spectrum of \(X\).

2. If \(X\) is semilocally simply connected and not simply connected then \(σ_1 > 0\) and \(ε := \frac{σ_1}{3}\) is the smallest homotopy critical value of \(X\).

**Proof.** Every parameterized essential circle is a closed geodesic that is not null-homotopic by Theorem 38, the first part is immediate. If \(X\) is semilocally simply connected and not simply connected, Theorem 25 implies that for some \(ε > 0\), \(φ_ε : X_ε \rightarrow X\) is the simply connected covering map of \(X\) and \(ε\) is the smallest homotopy critical value of \(X\). By Theorem 6, \(X\) contains an essential circle, which is the image of a closed geodesic \(γ\) of length \(3ε\). If \(γ\) were null-homotopic then \(γ\) would lift as a loop, contradicting Proposition 19 and the fact that any subdivision ε-chain of it is not ε-null. This implies that \(σ_1 ≤ 3ε\). Now \(X\) can be covered by open sets with the property that every loop in the set is null-homotopic in \(X\). Therefore any loop of diameter smaller then the Lebesgue number of this cover is by definition contained in a set in the cover, hence null-homotopic, which implies \(σ_1 > 0\). Now suppose that \(δ := \frac{σ_1}{4} < ε\). If \(γ\) were a non-null homotopic closed geodesic of length \(σ_1\), then \(γ\) could not lift as a loop to to the simply connected space \(X_ε\). Hence by Proposition 19 any ε-subdivision chain \(α\) of \(γ\) has the property that \([α]_ε = [σ]_ε\). This contradicts Corollary 33 ■

**Example 42** Let \(Y\) denote the flat torus obtained by identifying the sides of a rectangle of dimensions \(0 < 3a ≤ 3b\). When \(a < b\), \(a\) and \(b\) are distinct homotopy critical values: For \(ε > b\), \(Y_ε = Y\), for \(a < ε ≤ b\), \(Y_ε\) is a flat metric cylinder over a circle of length \(3a\), and for \(ε ≤ a\), \(Y_ε\) is the plane. There are infinitely many essential a-circles and b-circles, but all essential a-circles are equivalent and all essential b-circles are equivalent (Corollary 29). When \(a = b\), \(a\) is the only homotopy critical value; both circles “unroll” simultaneously and the covers go directly from trivial to universal. There are still two equivalence classes of essential circles, but since the circles have the same length, \(a\) is a homotopy critical value of multiplicity 2. Now fix \(a = b = \frac{1}{9}\) (i.e. \(Y\) comes
from a unit square). The closed geodesic determined by a straight path starting at the bottom left corner of the square having a slope of $\frac{1}{2}$ is a Riemannian isometric embedding of a circle of length $\sqrt{5}$, which is the shortest path in its homotopy class. However, the distance between the images of any two antipodal points is only $\frac{1}{\sqrt{2}}$, so this closed geodesic is not metrically embedded, hence not an essential circle. The diagonal of the square produces an $\varepsilon$-circle $C$ with $\varepsilon = \frac{\sqrt{2}}{3}$, which is the shortest path in its homotopy class, is metrically embedded and not null-homotopic, but is not essential. In fact, $C$ can be homotoped to the concatenation of the two circles of which the torus is a product. Hence any $\varepsilon$-loop $\lambda$ on $C$ can be $\varepsilon$-homotoped to a loop $\lambda'$ in those circles. But each of these circles is not $\varepsilon$-essential ($\varepsilon = \frac{\sqrt{2}}{3} > \frac{1}{3}$) so $\lambda'$, hence $\lambda$, is $\varepsilon$-null.

Note that if one adds a thin handle to the torus it will obstruct standard homotopies between some essential circles, but not $\varepsilon$-homotopies. This shows that using traditional homotopies rather than $\varepsilon$-homotopies in the definition of equivalence can “overcount” multiplicity. In [21], the multiplicity of a number $\delta$ in the covering spectrum is defined for compact spaces with a universal cover (in the categorial sense, not necessarily simply connected) as the minimum number of generators of a certain type in a certain subgroup of the “revised fundamental group” (Definition 6.1). We will not recall the definition of these groups here because they require a universal cover and this assumption is unnecessary for our work.

**Example 43** We will now recall the construction of a space $V$ that is known to contain a path loop $L$ that is homotopic to arbitrarily small loops but is not null-homotopic (see [8] or [26]), giving it a geodesic metric in the process. The Hawaiian Earring $H$ consists of all circles of radius $\frac{1}{i}$ in the plane centered at $(0, \frac{1}{i})$, $i \in \mathbb{N}$, with the subspace topology. The induced geodesic metric on $H$ measures the distance between any two points in $H$ as the length of the shortest path in $H$ joining them. It is easy to check that this metric is compatible with the subspace topology. Now take the cone on $H$, which also has a geodesic metric compatible with the topology of the cone (see, for example, the survey article [17] for details about geodesic metrics on glued spaces and cones). Glue two copies of this space together at the point $(0,0)$ in $H$. One can check that every $\varepsilon$-loop is $\varepsilon$-null for every $\varepsilon$, so the homotopy critical spectrum is empty even though the space is not simply connected. This example is related to Corollary [77] in the following way: one wonders if the requirement that $X$ be semilocally simply connected in the second part is required. If the path loop $L$ mentioned above had a closed geodesic in its homotopy class then we would have a counterexample to the second part of Corollary [77] with the weaker hypothesis. However, such a thing is not guaranteed—see Remark [77].

**Example 44** Let $X_n$ be the geodesic space consisting of circles of radii $\frac{1}{i}$ for $1 \leq i \leq n$ joined at a point. These spaces are Gromov-Hausdorff convergent to a geodesic Hawaiian Earring, but their universal covers consist of infinite trees with valencies tending to infinity, and hence are not Gromov-Hausdorff (pointed) precompact. One can “thicken” these examples into a family of Rie-
mannian 2-manifolds with same property. It seems like an interesting question to characterize when precompactness of a class of geodesic spaces (even a single space!) implies precompactness of the collection of all covering spaces.

The following example makes one wonder whether Corollary 9 is optimal.

Example 45 Let $S_n$ denote the space consisting of two points joined by $n$ edges of length $\frac{3}{2}$, with the geodesic metric. Each pair of edges determines a circle of length $3$, so there is a single critical value $1$ of multiplicity $\left( \frac{n}{2} \right) = \frac{1}{2}(n^2 + n)$.

On the other hand, we can cover the space using one open $\frac{1}{3}$-ball at each of the two vertices and $2$ additional $\frac{1}{3}$-balls on each edge for a total of $2(n + 1)$.

The estimate from Corollary 9 is $\frac{4}{3}n^3 + 2n^2 + \frac{2}{3}n$ and at any rate each edge requires at least one ball, so one cannot do better than a degree $3$ polynomial.

Another example that can be checked in a similar fashion is the $1$-skeleton of a regular $n$-simplex with every edge length equal to $1$, with the geodesic metric.

In this example each boundary of a $2$-face is isometric to a standard circle of circumference $3$. There is a single critical value $1$ of multiplicity $\left( \frac{n + 1}{3} \right) = \frac{1}{6}(n^3 - n)$. But any cover by open $\frac{1}{3}$-balls will again require at least one ball for each of the $\left( \frac{n + 1}{2} \right)$ edges and therefore the best that Corollary 9 can provide is a polynomial of order $6$ in $n$.

4 $(\varepsilon, \delta)$-Chassis

In this section, $X$ will be a compact geodesic space of diameter $D$, $\varepsilon > 0$ is fixed, and $0 < \delta < \sigma$ will be positive numbers with $\sigma \leq \varepsilon$, on which we will place additional requirements to reach stronger conclusions. An $(\varepsilon, \delta)$-chassis is defined to be a simplicial $2$-complex that has for its vertex set a $\delta$-dense set $V := \{v_0, ..., v_m\}$ (i.e. for every $x \in X$ there is some $v_i$ such that $d(x, v_i) < \delta$). We let $v_i$ and $v_j$ be joined by an edge if and only if $d(v_i, v_j) < \varepsilon$ and let $v_i, v_j, v_k$ span a $2$-simplex if and only if all three pairs of vertices are joined by an edge. Next, let $K$ be the $1$-skeleton of $C$ and denote the edge joining $v_i$ and $v_j$ by $e_{ij}$, $i < j$. Define the length of $e_{ij}$ to be $d(v_i, v_j)$ (distance in $X$), the length of an edge path to be the sum of the lengths of its edges, and the simplicial distance $d_S(v_i, v_j)$ between vertices $v_i \neq v_j$ to be the length of a shortest edge path joining them.

Every edge path in $C$ starting at $v_0$ (which we take for the basepoint) is equivalent to a chain of vertices $\{v_0 = v_{i_0}, ..., v_{i_k}\}$, which has a corresponding $\varepsilon$-chain $\{v_0 = v_{i_0}, ..., v_{i_k}\}$ in $X$. Now the basic moves in an edge homotopy in $C$ (replacing one side of a simplex by the concatenation of the other two, removal of an edge followed by its reversal, or vice versa) correspond precisely to the basic moves in an $\varepsilon$-homotopy. In other words, the function that takes the edge-homotopy class $[v_0 = v_{i_0}, ..., v_{i_k} = v_0]$ of a loop to the $\varepsilon$-homotopy
Lemma 46 If $\delta < \frac{\varepsilon}{4}$ then $C$ is connected and $E$ is surjective. In fact, if $\beta = \{v_0, y_1, \ldots, y_{n-1}, v_b\}$ is an $\varepsilon$-chain joining points in $V$ in $X$, then $[\beta]_\varepsilon$ contains a “simplicial” $\sigma$-chain $\alpha$ (i.e. a chain having all points in the vertex set $V$) such that

$$L(\alpha) \leq L(\beta) + 2 \left( \frac{8L(\beta)}{\sigma} \right) \delta$$

Proof. Given any $v_a, v_b \in V$, let $c$ be a geodesic joining them in $X$. We may subdivide $c$ into $n$ segments with endpoints $x_k, x_{k+1}, 0 \leq k \leq n$, of length at most $\frac{\varepsilon}{n}$. For each $m$ we may choose a point $v_m \in V$ such that $d(x_m, v_m) < \delta$. Since $\delta < \frac{\varepsilon}{4}$, the triangle inequality implies that $v_m$ and $v_{m+1}$ are joined by an edge in $C$, and hence $v_a, v_b$ are joined by an edge path in $C$. Surjectivity will follow from the last statement, since we may take $v_a = v_b = v_0$ and then resulting $\alpha$ is an $\varepsilon$-loop with $[\alpha]_\varepsilon$ in the image of $E$. By refinement we may suppose $\beta$ is a $\frac{\varepsilon}{4}$-chain, and applying Lemma 16 we may assume that $n = \left\lceil \frac{8L(\beta)}{\sigma} + 1 \right\rceil$. For each $i$ we may choose some $v_{j_i}$ such that $d(v_{j_i}, x_i) < \delta$ (letting $v_{j_0} = v_a$ and $v_{j_n} = v_b$). Since $\delta < \frac{\varepsilon}{4}$, Proposition 14 now implies that $\beta$ is $\sigma$-homotopic to the $\sigma$-chain $\alpha := \{v_{j_0}, \ldots, v_{j_n}\}$ and hence $[\beta]_\varepsilon = E([v_{j_0}, \ldots, v_{j_n}]_\varepsilon)$.

Moreover, the triangle inequality implies that $L(\alpha) \leq L(\beta) + 2n\delta$, completing the proof. 

Lemma 47 If $\delta < \min\{\frac{\varepsilon}{4}, \frac{\varepsilon}{32\beta}\}$ then for any $v_a, v_b \in V$, $d(v_a, v_b) \leq d_S(v_a, v_b) \leq d(v_a, v_b) + \frac{\varepsilon}{2}$. 

Proof. The left inequality is obvious. Subdivide a geodesic in $X$ joining $v_a, v_b$ to produce an $\varepsilon$-chain $\beta$ of length equal to $d(v_a, v_b)$. Taking $\sigma = \varepsilon$ in Lemma 16 produces a simplicial chain $\alpha$ of length at most $L(\beta) + \frac{\varepsilon}{2}$ joining $v_a$ and $v_b$. 

Lemma 48 If $\phi_{\varepsilon, \sigma}$ is a bijection and $\delta \lt \min\{\frac{\varepsilon-\sigma}{2}, \frac{\varepsilon}{4}\}$ then $E$ is injective. 

Proof. Suppose $[v_0 = v_{i_0}, \ldots, v_{i_k} = v_0] \in \ker E$. This means that the $\varepsilon$-chain $\alpha := \{v_0 = v_{i_0}, \ldots, v_{i_k} = v_0\}$ is $\varepsilon$-null in $X$. The problem, of course, is that the $\varepsilon$-null-homotopy may not involve only simplicial $\varepsilon$-chains and hence does not correspond to a simplicial null-homotopy in $C$. However, by Lemma 16 we may assume that $\alpha$ is in fact an $\varepsilon$-null simplicial $\sigma$-chain. By our choice of $\sigma$, $\alpha$ is in fact $\sigma$-null. Let $\{\alpha := \eta_0, \ldots, \eta_m = \{v_0\}\}$ be a $\sigma$-homotopy and $A$ be the set of all points $a$ such that $a$ is in some chain $\eta_i$. For each $a \in A$ let $\alpha' \in V$ be such that $d(a, \alpha') < \delta < \frac{\varepsilon-\sigma}{2}$, provided that if $a$ is already in $V$ then $\alpha' := a$. Finally, define $\eta_k := \{v_0 = x_{k_1}, \ldots, x_{k_r} = v_0\}$ whenever $\eta_k := \{v_0 = x_{k_1}, \ldots, x_{k_r} = v_0\}$; by definition, $\eta'_k$ is a simplicial chain and since $\alpha$ is already simplicial $\eta'_0 = \eta_0 = \alpha$. Moreover, $d(x_{k_1}, x'_{k_1}) = \sigma + 2(\frac{\varepsilon-\sigma}{2}) = \varepsilon$. That is, $\{\alpha := \eta_0, \ldots, \eta_m = \{v_0\}\}$ is an $\varepsilon$-homotopy via simplicial chains, and so is equivalent to a simplicial homotopy in $C$. 

We will now recall the well-known method of choosing generators and relations for \(\pi_E(C)\), while adding a geometric twist. First, we obtain a maximal subtree \(T\) of the 1-skeleton \(K\) as follows. Choose some \(v_k\) of maximal simplicial distance from \(v_0\) and connect \(v_k\) to \(v_0\) by a shortest simplicial path \(\Gamma_1\); \(\Gamma_1\) is the starting point in the construction of \(T\). Since \(\Gamma_1\) is minimal it must be simply connected, hence a tree; if it is maximal then we are done. Otherwise there is at least one vertex not in \(\Gamma_1\), and we choose one, \(v_j\), of maximal simplicial distance from \(v_0\). Let \(\Gamma_2\) be a minimal simplical path from \(v_j\) to \(v_0\). If at some point \(\Gamma_2\) meets (for the first time) any vertex \(w\) already in \(T\), then we replace the segment of \(\Gamma_2\) from \(w\) to \(v_0\) by the unique shortest segment of \(\Gamma_1\) from \(w\) to \(v_0\). In doing so we do not change the length of \(\Gamma_2\) and ensure that the union of \(\Gamma_1\) and \(\Gamma_2\) is still a tree. We iterate this process until all vertices are in the tree. The resulting maximal tree \(T\) has the property that every vertex \(v_j\) in \(K\) is connected to \(v_0\) by a unique simplicial path contained in \(T\) having length at most the simplicial diameter \(D_S\) of \(C\).

Now \(\pi_E(C)\) has generators and relations defined as follows (see, for example, [2], Section 6.4): The generators are concatenations of the form \([g_{ij}] = [p * e_{ij} * q]\), where \(e_{ij}\) is an edge that is in \(K\) but not in \(T\) and \(p\) (resp. \(q\)) is the unique shortest simplicial path in \(T\) from \(v_j\) to \(v_0\) (resp. \(v_0\) to \(v_i\)). The relations are of the form \([g_{ij}] [g_{jk}] = [g_{ik}]\), provided \(v_i, v_j, v_k\) span a 2-simplex in \(K\) with \(i < j < k\). Note that the simplicial length of \(g_{ij}\) is at most \(2D_S + \varepsilon\).

**Proof of Theorem** Since the homotopy critical values are discrete, we may always choose \(\sigma < \varepsilon\) so that \(\phi_{\sigma}\) is injective. We may then choose \(\delta\) so that all of the requirements of the above lemmas all hold. Then the resulting generators of \(\pi_E(C)\) correspond under the isomorphism \(E\) to classes \([\gamma_{ij}]_e\) in \(X\) such that the length of each \(\gamma_{ij}\) is at most \(2D_S + \varepsilon + \frac{\varepsilon}{2} < 2(D + \varepsilon)\). This proves the first part of the theorem, and the second part was proved in the Introduction.

For the third part, we begin by choosing an \(\frac{\varepsilon}{4}\)-dense set \(W = \{w_1, \ldots, w_s\}\) in \(X\) and an arbitrary \(\delta\)-chain \(\mu_{ij}\) from \(w_i\) to \(w_j\) with \(\mu_{ij} = \overline{\mu_{ij}}\). Given any \(\delta\)-loop \(\lambda = \{v_0 = x_0, \ldots, x_n = v_0\}\) of length at most \(2(D + \varepsilon)\), choose a subchain \(\mu = \{y_0 = v_0, \ldots, y_r = v_0\}\) (i.e. \(y_j = x_{i_j}\) for some increasing \(i_j\)) with the following property: If \(\lambda_j\) denotes the \(\delta\)-chain \(\{y_j = x_{i_j}, x_{i_j+1}, \ldots, x_{i_j+1} = y_j+1\}\) (i.e. the “segment” of \(\lambda\) from \(y_j\) to \(y_j+1\)) then for any \(j\), \(L(\lambda_j) < \frac{\varepsilon}{4}\) and \(L(\lambda_j) + L(\gamma_{j+1}) \geq \frac{\varepsilon}{4}\). This can be accomplished by iteratively removing points to form the subsequence, in a way similar to what was done in the proof of Lemma [4]. The same counting argument as in that proof gives us \(r \leq \frac{2L(\lambda)}{\varepsilon} \leq n(D + \varepsilon)\).

For each \(y_j\), choose some \(y'_j \in W\) such that \(d(y_j, y'_j) < \frac{\varepsilon}{4}\). There is now a corresponding \(\delta\)-chain \(\lambda'\) that is a concatenation of paths \(\mu_{i_{k}, j_{k}}\), where \(y_k = w_{j_k}\) and \(y_{k+1} = w'_{j_k}\). Next, let \(\gamma_j\) be a \(\delta\)-chain from \(y'_k\) to \(y_j\) of length at most \(\frac{\varepsilon}{4}\). It is not hard to check that \(\lambda\) is \(\delta\)-homotopic to \(\beta_{0} * \cdots * \beta_{0} * \lambda'\), where \(\beta_{0} := \overline{\lambda_0} * \gamma_1 * \mu_{i_0,j_0}\) and for \(k > 0\),

\[
\beta_{k} := \overline{\mu_{i_{k}j_{k}}} * \cdots * \overline{\mu_{i_{k}j_{k}}} * \gamma_{k} * \lambda_{k} * \gamma_{k+1} * \mu_{i_{k+1}j_{k+1}} * \cdots * \mu_{i_{k}j_{k}}.
\]

Let us count the ways to obtain \(\lambda\). First, \(\lambda'\) corresponds to a sequential choice of \(r\) elements of \(W\), so there are at most \(s^r\) possibilities. Next, \(\lambda\) is obtained
from $\lambda'$ by $r$ concatenations, each of which involves a choice of the element $[\gamma_k \ast \lambda_k \ast \gamma_{k+1} \ast \mu_{i_{k+1}j_{k+1}}]_{\delta} \in \pi_{\delta}(X, w)$ for some $w \in W$ with $L(\gamma_k \ast \lambda_k \ast \gamma_{k+1} \ast \mu_{i_{k+1}j_{k+1}}) < \varepsilon$. So there are at most $r \cdot M$ distinct choices to change from $\lambda'$ to $\lambda$.

From the second part of Theorem 3 and Theorem 25 we may immediately derive the following corollary:

**Corollary 49** Let $X$ be a compact, semilocally simply connected geodesic space. If $\varepsilon > 0$ is a lower bound for the homotopy critical spectrum of $X$ then for any $L > 0$, $\Gamma(X, L) \leq C(X, \varepsilon)^{4L\varepsilon}$.

**Proof of Theorem 1.** By Theorem 25 if $\delta < \varepsilon$ is sufficiently small, the function $\Lambda : \pi_1(X) \to \pi_{\delta}(X)$ is a length-preserving isomorphism. Then the desired generators are those corresponding to the generators of $\pi_{\delta}(X)$ given by the third part of Theorem 3 except that, a priori those generators have length $2(D + \delta)$. However, since $X$ is compact and semilocally simply connected, the proof is finished by a standard application of Ascoli’s Theorem. The statement about the 1-systole follows from Theorem 25.

**Example 50** The product of a circle with smaller and smaller spheres has 1-systole, but not volume, bounded below. On the other hand, Vitali Kapovitch has pointed out (see [16], Section 0.4) that examples from [1] can be modified to have a global lower bound on volume and Ricci curvature with 1-systole going to 0. These examples show that the finiteness theorems of Anderson and Shen-Wei are independent. At the same time, each of these examples satisfies the conditions of Theorem 1.

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**References**

[1] Anderson, Michael T. Short geodesics and gravitational instantons. J. Differential Geom. 31 (1990), no. 1, 265–275.

[2] Armstrong, M. A. *Topology*. Undergraduate Texts in Mathematics. Springer-Verlag, Berlin, 1983.

[3] Berestovskii, Valera; Plaut, Conrad; Stallman, Cornelius, Geometric groups. I. Trans. Amer. Math. Soc. 351 (1999), no. 4, 1403–1422. MR1458295

[4] Berestovskii, Valera; Plaut, Conrad, Covering group theory for topological groups. Topology Appl. 114 (2001), no. 2, 141–186. MR1834930

[5] Berestovskii, Valera; Plaut, Conrad, Uniform universal covers of uniform spaces. Topology Appl. 154 (2007), no. 8, 1748–1777. MR2317077
[6] Berestovski˘ı, Valera; Plaut, Conrad, Covering \( \mathbb{R} \)-trees, \( \mathbb{R} \)-free groups, and dendrites. Adv. Math. 224 (2010), no. 5, 1765–1783. MR2646109

[7] Byrd, Fred; Carlile, Chris; Happ, Alex, Homotopy critical values of the circle, REU project.

[8] Cannon, J. W.; Conner, G. R. The combinatorial structure of the Hawaiian earring group. Topology Appl. 106 (2000), no. 3, 225–271. MR1775709

[9] Cheeger, J.; Colding, T., On the structure of spaces with Ricci curvature bounded below I, J. Diff. Geom. 46 (1997) 406-480. MR98k:53044

[10] Cheeger, J.; Colding, T., On the structure of spaces with Ricci curvature bounded below II, J. Diff. Geom. 54 (2000), no. 1, 13-35. MR2003a:53043

[11] Cheeger, J.; Colding, T., On the structure of spaces with Ricci curvature bounded below III, J. Diff. Geom. 54 (2000), no. 1, 37-74. MR2003a:53044

[12] Conant, Jim; Curnutte, Victoria; Jones, Corey; Plaut, Conrad; Pueschel, Kristen; Walpole, Maria; Wilkins, Jay, Discrete homotopy theory and critical values of geodesic spaces, preprint.

[13] de Smit, B.; Gornet, R; Sutton, C., Sunada’s method and the covering spectrum, J. Diff. Geom. 86 (2010) 501-537.

[14] Gromov, M., Structures métriques pour les variétés riemanniennes. Edited by J. Lafontaine and P. Pansu. Textes Mathématiques 1. CEDIC, Paris, 1981. MR0682063

[15] Gromov, M., Metric structures for Riemannian and non-Riemannian spaces. With appendices by M. Katz, P. Pansu and S. Semmes. Translated from the French by Sean Michael Bates. Progress in Mathematics, 152. Birkhäuser Boston, Inc., Boston, MA, 1999. MR1699320

[16] Nabutovsky, A., Effective universal coverings and local minima of the length functional on loop spaces, GAFA Vol. 20 (2010) 545-570.

[17] Plaut, Conrad Metric spaces of curvature \( \geq k \). Handbook of geometric topology, 819–898, North-Holland, Amsterdam, 2002. MR1886682

[18] Plaut, Conrad, Quotients of uniform spaces. Topology Appl. 153 (2006), no. 14, 2430–2444. MR2243722

[19] Plaut, Conrad; Wilkins, Jay, Gromov-Hausdorff convergence of regular covering maps, preprint.

[20] Sormani, Christina; Wei, Guofang, Hausdorff convergence and universal covers. Trans. Amer. Math. Soc. 353 (2001), no. 9, 3585–3602. MR1837249

[21] Sormani, Christina; Wei, Guofang, The covering spectrum of a compact length space. J. Differential Geom. 67 (2004), no. 1, 35–77. MR2153481

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[22] Sormani, Christina; Wei, Guofang, Universal covers for Hausdorff limits of noncompact spaces. Trans. Amer. Math. Soc. 356 (2004), no. 3, 1233–1270. MR2021619

[23] Sormani, Christina; Wei, Guofang, The cut-off covering spectrum. Trans. Amer. Math. Soc. 362 (2010), no. 5, 2339–2391. MR2584603

[24] Shen, Zhong Min; Wei, Guofang, On Riemannian manifolds of almost nonnegative curvature. Indiana Univ. Math. J. 40 (1991), no. 2, 551–565. MR1119188

[25] Spanier, Edwin H., *Algebraic topology*. Springer-Verlag, New York-Berlin, 1981.

[26] A. Zastrow, The second van-Kampen theorem for topological spaces, Topology Appl. 59 (1994) 201–232.