Two invariant submodels of rank 1 of the hydrodynamic type equations and exact solutions

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Abstract. In this paper, the hydrodynamic type equations with the state equation in the form of pressure separated into the sum of density and entropy functions are considered. The admissible group of transformation generate 12-dimensional Lie algebra. For two three-dimensional subalgebras the invariant submodels of rank 1 are calculated. The exact solutions for the hydrodynamic type equations are obtained.

1. Introduction
Solving non-linear high-order partial differential equations is a non-trivial problem. One of the methods for finding exact solutions to such equations is group analysis. Academician L.V. Ovsyannikov announced the Submodel Program [1], one of the problems of which is the search for exact solutions of gas dynamics equations. It is known that the gas dynamics equations with the equation of state in a general form admit an 11-parameter group of transformations. The Lie algebra contains 11 basic first-order derivation operators. An optimal system of dissimilar subalgebras of an 11-dimensional Lie algebra is known [1, 2]. In this paper, we consider the hydrodynamic type equations with pressure in the form of a sum of density and entropy functions. The admissible Lie Algebra is 12-dimensional. An optimal system of dissimilar subalgebras of a 12-dimensional Lie algebra was constructed in [3]. All two-dimensional subalgebras were considered — all invariant submodels of rank 2 were constructed [4, 5] and the reduction of partially invariant submodels of rank 3 of defect 1 to invariant submodels was proved [6]. We consider two three-dimensional subalgebras 3.42 \( a \neq 0, b = 0 \) and 3.47 from [3].

2. Theory
We consider the hydrodynamic type equations

\[
\begin{align*}
D \vec{u} + \rho^{-1} \nabla p &= 0, \\
D \rho + \rho \text{div} \vec{u} &= 0, \\
D p + \rho f_\rho \text{div} \vec{u} &= 0,
\end{align*}
\]

where the total differentiation operator has the form

\[
D = \partial_t + (\vec{u} \cdot \nabla);
\]
\( t \) is the time, \( \nabla = \partial x \) is the gradient with respect to the spatial independent variables \( \vec{x}, \vec{u} \) is the velocity vector, \( \rho \) is the density, and \( p \) is the pressure.

In the Cartesian coordinate system we have \([7]\)

\[
\vec{x} = x\vec{i} + y\vec{j} + z\vec{k},
\]
\[
\nabla = \vec{i}\partial_x + \vec{j}\partial_y + \vec{k}\partial_z,
\]
\[
\vec{u} = u\vec{i} + v\vec{j} + w\vec{k},
\]

where \( \vec{i}, \vec{j}, \) and \( \vec{k} \) is an orthonormal basis.

In the cylindrical coordinate system we have \([7]\)

\[
x = x, \quad y = r \cos \theta, \quad z = r \sin \theta,
\]
\[
u = U, \quad v = V \cos \theta - W \sin \theta, \quad w = V \sin \theta + W \cos \theta,
\]
\[
\vec{x} = x\vec{i} + r(\cos \theta\vec{j} + \sin \theta\vec{k}) = xe_x + re_r,
\]
\[
\vec{u} = Ue_x + Ve_r + We_\theta,
\]

where \( e_x, e_r, \) and \( e_\theta \) is an orthonormal basis.

We consider the hydrodynamic type equations (1) with the state equation which is the pressure represented as the sum of density and entropy functions

\[
p = f(\rho) + h(S), \quad f = \rho^2 F'(\rho).
\] (2)

The thermodynamic parameters of an ideal medium, the specific internal energy and temperature, have the form

\[
T = g'(S) - \rho^{-1}h'(S),
\]
\[
\varepsilon = F(\rho) - \rho^{-1}h(S) + g(S).
\] (3)

For the measurable parameters \( T, \varepsilon, \) and \( \rho \), we obtain the state equations of the form

\[
p = G(T, \rho), \quad \varepsilon = E(T, \rho),
\]

given in parametric form by formulas (2) and (3).

The entropy \( S \) is determined from (2). The last equation of system (1) can be replaced by the equation for entropy: \( DS = 0 \).

The hydrodynamic type equations (1) with the equation of state (2) are invariant under the action of the Galilean group extended by the uniform dilatation and pressure translation:

\[
1^o. \ \vec{x}' = \vec{x} + \vec{a} \text{ (space translations)};
\]
\[
2^o. \ \vec{x}' = \vec{x} + \vec{a}_0 \text{ (time translation)};
\]
\[
3^o. \ \vec{x}' = O\vec{x}, \vec{u}' = O\vec{u}, OO^T = E, \det O = 1 \text{ (rotations)};
\]
\[
4^o. \ \vec{x}' = \vec{x} + \vec{b}, \vec{u}' = \vec{u} + \vec{b} \text{ (Galilean translations)};
\]
\[
5^o. \ \vec{x}' = ct, \vec{u}' = c\vec{u} \text{ (uniform dilatation)};
\]
\[
6^o. \ p' = p + p_0 \text{ (pressure translation)}.
\] (4)
or in the cylindrical coordinate system:

\begin{align*}
X_1 &= \partial_x, \quad X_2 = \cos \theta \partial_r - \frac{1}{r} \sin \theta (\partial_\theta + W \partial_\varphi - V \partial_W), \\
X_3 &= \sin \theta \partial_r + \frac{1}{r} \cos \theta (\partial_\theta + W \partial_\varphi - V \partial_W), \quad X_4 = t \partial_x + \partial_U, \\
X_5 &= \cos \theta (t \partial_r + \partial_\varphi) - \frac{t}{r} \sin \theta \left( \partial_\theta + W \partial_\varphi - \frac{V}{t} \partial_W \right), \\
X_6 &= \sin \theta (t \partial_r + \partial_\varphi) + \frac{t}{r} \cos \theta \left( \partial_\theta + W \partial_\varphi - \frac{V}{t} \partial_W \right), \\
X_7 &= \partial_\theta, \\
X_8 &= \sin \theta (r \partial_x - x \partial_r + V \partial_U - U \partial_V) + \\
&\quad \cos \theta \left( W \partial_U - U \partial_W - \frac{x}{r} (\partial_\theta + W \partial_\varphi - V \partial_W) \right), \\
X_9 &= -\cos \theta (r \partial_x - x \partial_r + V \partial_U - U \partial_V) + \\
&\quad \sin \theta \left( W \partial_U - U \partial_W - \frac{x}{r} (\partial_\theta + W \partial_\varphi - V \partial_W) \right), \\
X_{10} &= \partial_t, \quad X_{11} = t \partial_t + x \partial_x + r \partial_r, \quad Y_1 = \partial_p.
\end{align*}

3. Results

The 3-dimensional subalgebra 3.42 for \( a \neq 0, b = 0 \) \cite{3} is considered. Its basic operators in the cylindrical coordinate system are as follows:

\begin{align*}
\begin{cases}
X_2 = \cos \theta \partial_r - \frac{1}{r} \sin \theta (\partial_\theta + W \partial_\varphi - V \partial_W), \\
X_3 = \sin \theta \partial_r + \frac{1}{r} \cos \theta (\partial_\theta + W \partial_\varphi - V \partial_W), \\
Y_1 + a X_4 + X_7 = \partial_p + a t \partial_x + a \partial_U + \partial_\theta. 
\end{cases}
\end{align*} (5)

We calculate the invariants of the subalgebra (5)

\begin{align*}
t, \quad U - \frac{x}{t}, \quad Q, \quad \vartheta_C + \theta - \frac{x}{at}, \quad \rho, \quad p - \frac{x}{at},
\end{align*}

where \( V = Q \cos \vartheta_C, \ W = Q \sin \vartheta_C \) \cite{8}. The representation of invariant solution can be written as

\begin{align*}
\begin{cases}
U = U_1(t) + \frac{x}{t}, \\
V = Q(t) \cos \left( \vartheta(t) - \theta + \frac{x}{at} \right), \\
W = Q(t) \sin \left( \vartheta(t) - \theta + \frac{x}{at} \right), \\
\rho = \rho(t), \quad p = \rho_1(t) + \gamma \frac{x}{at}, \quad S = S_1(t) + \gamma \frac{x}{at},
\end{cases}
\end{align*} (6)
We add the coefficient $\gamma$ in the representation of solution (6) for clear distinction between the 3-dimensional submodels of 11-dimensional and 12-dimensional Lie algebras, i.e. $\gamma = 0$ in the case of $L_{11}$, while $\gamma = 1$ in the case of $L_{12}$.

Substituting (6) into (1), with account taken of (2), we get invariant submodel of rank 1:

$$
\begin{align*}
U_1' &= -\frac{\gamma}{at}\rho^{-1} - \frac{U_1}{t} , \\
Q_1' \cos \vartheta_C - Q \sin \vartheta_C \left[ \frac{\vartheta'}{at} + \frac{U_1}{at} \right] &= 0 , \\
Q_1' \sin \vartheta_C + Q \cos \vartheta_C \left[ \frac{\vartheta'}{at} + \frac{U_1}{at} \right] &= 0 , \\
\rho_1 &= -\frac{\rho}{t} , \quad S_{1t} = -\frac{\gamma}{at}U_1 , \quad p_1 = f(\rho) + S_1 .
\end{align*}
$$

Multiplying the second equation (7) by $\sin \vartheta_C$ and adding the resultant expression to the third equation of (7) multiplied by $-\cos \vartheta_C$ we obtain the follow equation

$$
Q \left[ \frac{\vartheta'}{at} + \frac{U_1}{at} \right] = 0 .
$$

Next, multiplying the second equation (7) by $\cos \vartheta_C$ and adding the resultant expression to the third equation of (7) multiplied by $\sin \vartheta_C$ we obtain the follow equation

$$
Q_1' = 0 .
$$

Thus, $Q = Q_0$ is constant. The exact solution of system (1) with account taken of (2) is

$$
\begin{align*}
U &= \frac{x + x_0}{t} - \frac{\gamma t}{2a \rho_0} , \\
V &= Q_0 \cos \left( \frac{x + x_0}{at} - \theta + \frac{\gamma t}{2a^2 \rho_0} + \vartheta_0 \right) , \\
W &= Q_0 \sin \left( \frac{x + x_0}{at} - \theta + \frac{\gamma t}{2a^2 \rho_0} + \vartheta_0 \right) , \\
\rho &= \frac{\rho_0}{t} , \\
p &= \gamma \frac{x + x_0}{at} + f \left( \frac{\rho_0}{t} \right) + \frac{\gamma^2}{2a^2} t + p_0 ,
\end{align*}
$$

where $Q_0$ is arbitrary. In this solution we can remove non-essential constants using transformations $1^o, 3^o, 5^o, 6^o$ from (4). Thus, we have $\rho_0 = 1$, $\vartheta_0 = x_0 = \rho_0 = 0$, and we can rewrite the solution (8) in the form

$$
\begin{align*}
U &= \frac{x}{t} - \frac{\gamma t}{2a} , \\
V &= Q_0 \cos \left( \frac{x}{at} - \theta + \frac{\gamma t}{2a^2} \right) , \\
W &= Q_0 \sin \left( \frac{x}{at} - \theta + \frac{\gamma t}{2a^2} \right) , \\
\rho &= \frac{1}{t} , \quad p = \gamma \frac{x}{at} + \frac{\gamma^2}{2a^2} t + f \left( \frac{1}{t} \right) .
\end{align*}
$$

In the case $Q_0 \neq 0$ we can substitute $\tilde{V} = \frac{V}{Q_0}$, $\tilde{W} = \frac{W}{Q_0}$ in (9).
The basic operators of subalgebra 3.47 [3] in the Cartesian coordinate system are as follows

\[\begin{align*}
X_2 &= \partial_y, \\
X_3 &= \partial_z, \\
X_1 + bX_5 + Y_1 &= \partial_x + bt\partial_y + b\partial_v + \partial_p.
\end{align*}\] (10)

We calculate the invariants of the subalgebra (10)

\[t, \quad u, \quad v - bx, \quad w, \quad \rho, \quad p - x.\]

The representation of invariant solution can be written as

\[\begin{align*}
u &= u(t), \\
v &= v_1(t) + bx, \\
w &= w(t), \\
\rho &= \rho(t), \\
p &= p_1(t) + \gamma x, \\
S &= S_1(t) + \gamma x.
\end{align*}\] (11)

Substituting (11) into (1), with account taken of (2), we get invariant submodel of rank 1:

\[\begin{align*}
u_t &= -\gamma \rho^{-1}, \\
v_{1t} &= -bu, \\
w_t &= 0, \quad \rho_t = 0, \\
p_{1t} &= -\gamma u, \quad p_1 = f(\rho) + S_1.
\end{align*}\]

The exact solution of system equations (1), with account taken of (2), up to transformations (4) is

\[\begin{align*}
u &= -\gamma \rho^{-1} t, \\
v &= b \left( x + \frac{\gamma}{2\rho_0} t^2 \right), \\
w &= 0, \quad \rho = \rho_0, \quad p = \gamma \left( x + \frac{\gamma}{2\rho_0} t^2 \right).
\end{align*}\]

4. Conclusion

Thus, for the hydrodynamic type equations with the state equation in the form of pressure separated into the sum of density and entropy functions the invariant submodels of rank 1 3.42 \(a \neq 0, b = 0\) and 3.47 are constructed. The exact solutions for the hydrodynamic type equations are obtained.

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