A refined and unified version of the inverse scattering method for the Ablowitz–Ladik lattice and derivative NLS lattices

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Abstract

We refine and develop the inverse scattering theory on a lattice in such a way that the Ablowitz–Ladik lattice and derivative NLS lattices as well as their matrix analogs can be solved in a unified way. The inverse scattering method for the (matrix analog of the) Ablowitz–Ladik lattice is simplified to the same level as that for the continuous NLS system. Using the linear eigenfunctions of the Lax pair for the Ablowitz–Ladik lattice, we can construct solutions of the derivative NLS lattices such as the discrete Gerdjikov–Ivanov (also known as Ablowitz–Ramani–Segur) system and the discrete Kaup–Newell system. Thus, explicit solutions such as the multisoliton solutions for these systems can be obtained by solving linear summation equations of the Gel’fand–Levitan–Marchenko type. The derivation of the discrete Kaup–Newell system from the Ablowitz–Ladik lattice is based on a new method that allows us to generate new integrable systems from known systems in a systematic manner. In an appendix, we describe the reduction of the matrix Ablowitz–Ladik lattice to a vector analog of the modified Volterra lattice from the point of view of the inverse scattering method.

Keywords: Lax pair, inverse scattering, integrable space discretization, N-soliton solution, nonlinear Schrödinger (NLS), Ablowitz–Ladik lattice (integrable space-discrete NLS), derivative NLS, Kaup–Newell system, vector modified Volterra lattice
1 Introduction

The cubic nonlinear Schrödinger (NLS) equation [1, 2] is probably the most prominent example of an integrable partial differential equation in $1 + 1$ space-time dimensions. The inverse scattering method for the NLS equation devised by Zakharov and Shabat [1] was reformulated by Ablowitz, Kaup, Newell and Segur [3, 4] in a user-friendly and broadly-applicable manner. Since then various extensions of the NLS equation have been obtained within the framework of the inverse scattering method. Among them, we mention two kinds of extensions: (i) space-discrete NLS systems\(^1\) wherein the spatial variable is discretized [6, 7] and (ii) derivative NLS systems wherein the nonlinear terms involve differentiation with respect to the spatial variable [8–12]. For these extensions, the inverse scattering method can be applied on a case-by-case basis, but its application requires more steps and is apparently more complicated than that for the original NLS system [3, 4].

The main objective of this paper is twofold. First, we develop the inverse scattering method on a lattice and correct the widespread impression that it is essentially more complicated than the inverse scattering method on the line. Second, we show that the derivative NLS systems can be solved using the inverse scattering method for the NLS system; however, this paper focuses on the space-discrete case.\(^2\) To be specific, we consider (a matrix generalization of) the Ablowitz–Ladik lattice [6] that is an integrable space discretization of the NLS system. The inverse scattering method for the Ablowitz–Ladik lattice reported in the existing literature involves some onerous processes peculiar to the discrete case; in fact, they are redundant and can be avoided. For this purpose, we only need to start with an eigenvalue problem that is trivially equivalent to the Ablowitz–Ladik eigenvalue problem up to a similarity transformation and inversion of the spatial coordinate. Then, all the key quantities such as the scattering data become even functions of the conventional spectral parameter; thus, we can use its square as a new (and more essential) parameter. This considerably simplifies the subsequent computations. Moreover, by the inversion of the spatial coordinate, we no longer need to normalize the “integral kernels” of the linear eigenfunctions (Jost solutions) to express the potentials in the Ablowitz–Ladik eigenvalue problem explicitly. This is in contrast to the conventional approach wherein one has to first introduce the “integral kernels” and then normalize them; in the literature (see, e.g., [15,16]), they are usually denoted as $K(n, m)$, $\bar{K}(n, m)$ and $\kappa(n, m)$, $\bar{\kappa}(n, m)$, respectively. Thus, we can successfully refine the inverse

\(^1\)The problem of how to discretize the continuous time variable in integrable space-discrete systems has a long history, see [5] and references therein.

\(^2\)Relevant results on continuous derivative NLS systems can be found in [13, 14].
scattering method associated with the (matrix) Ablowitz–Ladik eigenvalue problem; both the potentials and the linear eigenfunctions are determined from the scattering data through a set of linear summation equations in the most transparent manner.

In our previous paper [17], we proposed a systematic method of generating new integrable systems from known systems through inverse Miura maps. As a result, two derivative NLS systems, namely, the Gerdjikov–Ivanov (also known as Ablowitz–Ramani–Segur) system [10, 11] and the Chen–Lee–Liu system [9], were constructed from the Lax representation for the NLS system; the same prescription applies to the space-discrete case. Thus, the inverse scattering method for the Ablowitz–Ladik lattice can also provide the solutions of the space-discrete Gerdjikov–Ivanov system and the space-discrete Chen–Lee–Liu system in a unified way. In this paper, we propose yet another method of generating new integrable systems from known systems. In particular, by applying this new method to the Ablowitz–Ladik lattice, we obtain a lattice system that is essentially equivalent to the space-discrete Kaup–Newell system studied in [20, 21]; its solutions can be expressed in terms of the linear eigenfunctions associated with the Ablowitz–Ladik lattice. Thus, the space-discrete Kaup–Newell system can also be solved using the inverse scattering method for the Ablowitz–Ladik lattice; note that among the derivative NLS systems, the Kaup–Newell system [8] is the most important for physical applications.

The main body of this paper is organized as follows. In section 2, we start with the Lax representation for the Ablowitz–Ladik lattice; using its linear eigenfunctions, we derive two derivative NLS lattices. First, we apply the method proposed in [17] and derive the space-discrete Gerdjikov–Ivanov system. Second, we propose a new systematic method and obtain the space-discrete Kaup–Newell system. We can also derive and solve the space-discrete Chen–Lee–Liu system, but we do not present it in this paper; the interested reader is referred to appendix B of [17] and section 3 of [20]. In section 3, we present a streamlined version of the inverse scattering method for the Ablowitz–Ladik lattice. In section 4, we combine the results of sections 2 and 3 and show that the derivative NLS lattices can be solved by the inverse scattering method associated with the Ablowitz–Ladik eigenvalue problem. Their multisoliton solutions can be derived from the linear summation equations in a straightforward manner. The last section, section 5, is devoted to concluding remarks.

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3 The original Miura map transforms the modified KdV equation (or, more generally, its one-parameter generalization called the Gardner equation) to the KdV equation [18].

4 The term was coined after Lax’s work on the KdV hierarchy [19].
In this paper, we consider the general case where the dependent variables take their values in matrices \([7, 22]\) in such a way that the operations such as addition and multiplication in the equations of motion make sense. In appendix A, we consider the reduction of the matrix Ablowitz–Ladik lattice to a vector analog of the modified Volterra lattice and discuss the effect of the reduction on the scattering data; this considerably refines our previous results given in \([23, 24]\).

2 Ablowitz–Ladik lattice and derivative NLS lattices

2.1 Lax representation for the Ablowitz–Ladik lattice

We start with the matrix Ablowitz–Ladik eigenvalue problem written in the nonstandard form:

\[
\begin{bmatrix}
\Psi_{1,n} \\
\Psi_{2,n}
\end{bmatrix}
= 
\begin{bmatrix}
zI & zQ_n \\
z^{-1}R_n & z^{-1}I
\end{bmatrix}
\begin{bmatrix}
\Psi_{1,n+1} \\
\Psi_{2,n+1}
\end{bmatrix},
\]

(2.1)

where \(z\) is a constant spectral parameter and \(I\) is the identity matrix of arbitrary size. For simplicity, we assume that all the entries in (2.1), such as the potentials \(Q_n\) and \(R_n\), are \(l \times l\) square matrices. It is also possible to consider the more general case where \(Q_n\) is an \(l_1 \times l_2\) matrix and \(R_n\) is an \(l_2 \times l_1\) matrix; however, the results for that case can be easily obtained by setting some rows and columns in \(Q_n\) and \(R_n\) as identically zero.

The time evolution of the linear eigenfunction can be introduced in such a way that it is compatible with the eigenvalue problem (2.1). The most illustrative example is given by

\[
\begin{bmatrix}
\Psi_{1,n} \\
\Psi_{2,n}
\end{bmatrix}
t = 
\begin{bmatrix}
(-z^2 + 1)bI + bQ_{n-1}R_n & -z^2bQ_{n-1} - aQ_n \\
-bR_n - z^{-2}aR_{n-1} & (1 - z^{-2})aI + aR_{n-1}Q_n
\end{bmatrix}
\begin{bmatrix}
\Psi_{1,n} \\
\Psi_{2,n}
\end{bmatrix}.
\]

(2.2)

Here, \(a\) and \(b\) are arbitrary scalar constants; in fact, they may depend on the time variable \(t\) in an arbitrary manner (see, e.g., \([26, 27]\)), but we do not discuss it in this paper. The compatibility condition for the overdetermined linear system, (2.1) and (2.2), with the isospectral condition \(z_t = 0\) implies the time evolution equations for \(Q_n\) and \(R_n\):

\[
\begin{aligned}
Q_{n,t} - aQ_{n+1} + bQ_{n-1} + (a - b)Q_n + aQ_nR_nQ_{n+1} - bQ_{n-1}R_nQ_n = 0, \\
R_{n,t} - bR_{n+1} + aR_{n-1} + (b - a)R_n + bR_nQ_nR_{n+1} - aR_{n-1}Q_nR_n = 0.
\end{aligned}
\]

(2.3a, 2.3b)

5 Actually, in the simplest \(2 \times 2\) matrix case, (2.1) can be identified with a special case of the eigenvalue problem studied in \([25]\).
We call (2.3) the (matrix) Ablowitz–Ladik lattice/system [6, 7]; (2.1) and (2.2) comprise its Lax representation. The symbol $O$ on the right-hand side of the equations implies that the dependent variables can take their values in matrices. In fact, there exist infinitely many ways to define such an isospectral time evolution (cf. [28, 29]), depending on the choice of the temporal Lax matrix in (2.2) as a Laurent polynomial in $z^2$; they provide the positive flows of the Ablowitz–Ladik hierarchy (cf. appendix A of [20] for the negative flows) and each of them is uniquely determined by its linear part or, equivalently, the dispersion relation.

2.2 Space-discrete Gerdjikov–Ivanov system

In this subsection, using the method proposed in [17], we derive the space-discrete Gerdjikov–Ivanov system from the Lax representation for the Ablowitz–Ladik lattice. The result is essentially the same as that given in [17], but we restate it here for the self-containedness and readability of the paper.

We consider a $2l \times l$ matrix-valued solution to the pair of linear equations (2.1) and (2.2) such that $\Psi_{1,n}$ is an $l \times l$ invertible matrix. Then, in terms of the $l \times l$ matrix $P_n := \Psi_{2,n}^{-1} \Psi_{1,n}$, (2.1) and (2.2) can be rewritten as a pair of discrete and continuous matrix Riccati equations for $P_n$,

$$R_n = \mu P_n - P_{n+1} + \mu P_n Q_n P_{n+1}, \quad (2.4a)$$

$$P_{n,t} = -b R_n - \mu^{-1} a R_{n-1} + (1 - \mu^{-1}) a P_n + (\mu - 1) b P_n + a R_{n-1} Q_n P_n - b P_n Q_{n-1} R_n + \mu b P_n Q_{n-1} P_n + a P_n Q_n P_n, \quad (2.4b)$$

where $\mu := z^2$. The first relation (2.4a) defines the Miura map $(Q_n, P_n) \mapsto (Q_n, R_n)$. Using (2.4a), we can eliminate $R_n$ and $R_{n-1}$ in (2.3a) and (2.4b) to obtain a closed system for $(Q_n, P_n)$, i.e., the space-discrete Gerdjikov–Ivanov system [20]:

$$Q_{n,t} - a Q_{n+1} + b Q_{n-1} + (a - b) Q_n + a Q_n (\mu P_n - P_{n+1}) Q_{n+1} - b Q_{n-1} (\mu P_n - P_{n+1}) Q_{n+1} + a Q_{n-1} Q_n P_{n+1} Q_{n+1} Q_n = O, \quad (2.5a)$$

$$P_{n,t} - b P_{n+1} + a P_{n-1} + (b - a) P_n - b P_n (Q_{n-1} - \mu Q_n) P_{n+1} + a P_{n-1} (Q_{n-1} - \mu Q_n) P_{n+1} + b P_n Q_{n-1} Q_n P_{n+1} - a P_{n-1} Q_{n-1} P_n Q_n P_n = O. \quad (2.5b)$$

2.3 Space-discrete Kaup–Newell system

In this subsection, we propose yet another method of generating new integrable systems from known systems using a fundamental set of linear eigenfunctions. The method is applicable to a matrix Lax representation of arbitrary size, but for brevity we describe it in the simplest case of a $2 \times 2$
Lax representation as well as its block-matrix generalization involving two potentials. In this paper, we consider the space-discrete case (see [14] for the continuous case).

Suppose that a lattice system admits the Lax representation

\[
\begin{bmatrix}
\Psi^{(j)}_{1,n} \\
\Psi^{(j)}_{2,n}
\end{bmatrix}
= \begin{bmatrix}
L_{11,n} & L_{12,n} \\
L_{21,n} & L_{22,n}
\end{bmatrix}
\begin{bmatrix}
\Psi^{(j)}_{1,n+1} \\
\Psi^{(j)}_{2,n+1}
\end{bmatrix}, \quad (2.6a)
\]

\[
\begin{bmatrix}
\Psi^{(j)}_{1,n} \\
\Psi^{(j)}_{2,n}
\end{bmatrix}_t
= \begin{bmatrix}
M_{11,n} & M_{12,n} \\
M_{21,n} & M_{22,n}
\end{bmatrix}
\begin{bmatrix}
\Psi^{(j)}_{1,n} \\
\Psi^{(j)}_{2,n}
\end{bmatrix}, \quad (2.6b)
\]

where all the entries are assumed to be square matrices of the same size. Because the method requires a full set of linearly independent eigenfunctions, the superscript \((j)\) with \(j = 1\) or \(2\) is used to designate the two eigenfunctions. We apply a gauge transformation defined using one eigenfunction to the other eigenfunction as

\[
\begin{bmatrix}
\Psi^{(2)}_{1,n} \\
\Psi^{(2)}_{2,n}
\end{bmatrix}
\mapsto \begin{bmatrix}
\Psi^{(1)}_{1,n}^{-1} & O \\
O & \Psi^{(1)}_{2,n}^{-1}
\end{bmatrix}
\begin{bmatrix}
\Psi^{(2)}_{1,n} \\
\Psi^{(2)}_{2,n}
\end{bmatrix} = \begin{bmatrix}
\Psi^{(1)}_{1,n}^{-1}\Psi^{(2)}_{1,n} \\
\Psi^{(1)}_{2,n}^{-1}\Psi^{(2)}_{2,n}
\end{bmatrix}.
\]

Then, the Lax representation (2.6) is transformed to the degenerate form:

\[
\begin{bmatrix}
\Psi^{(1)}_{1,n}^{-1}\Psi^{(2)}_{1,n} \\
\Psi^{(1)}_{2,n}^{-1}\Psi^{(2)}_{2,n}
\end{bmatrix} = \begin{bmatrix}
I - \Psi^{(1)}_{1,n}^{-1}L_{12,n}\Psi^{(1)}_{2,n+1} & \Psi^{(1)}_{1,n}^{-1}L_{12,n}\Psi^{(1)}_{2,n} \\
\Psi^{(1)}_{2,n}^{-1}L_{21,n}\Psi^{(1)}_{1,n+1} & I - \Psi^{(1)}_{2,n}^{-1}L_{21,n}\Psi^{(1)}_{1,n+1}
\end{bmatrix}
\begin{bmatrix}
\Psi^{(1)}_{1,n+1}^{-1}\Psi^{(2)}_{1,n} \\
\Psi^{(1)}_{2,n+1}^{-1}\Psi^{(2)}_{2,n+1}
\end{bmatrix},
\]

\[
\begin{bmatrix}
\Psi^{(1)}_{1,n}^{-1}\Psi^{(2)}_{1,n} \\
\Psi^{(1)}_{2,n}^{-1}\Psi^{(2)}_{2,n}
\end{bmatrix}_t = \begin{bmatrix}
-\Psi^{(1)}_{1,n}^{-1}M_{12,n}\Psi^{(1)}_{2,n} & \Psi^{(1)}_{1,n}^{-1}M_{12,n}\Psi^{(1)}_{2,n} \\
\Psi^{(1)}_{2,n}^{-1}M_{21,n}\Psi^{(1)}_{1,n} & -\Psi^{(1)}_{2,n}^{-1}M_{21,n}\Psi^{(1)}_{1,n}
\end{bmatrix}
\begin{bmatrix}
\Psi^{(1)}_{1,n+1}^{-1}\Psi^{(2)}_{1,n} \\
\Psi^{(1)}_{2,n+1}^{-1}\Psi^{(2)}_{2,n+1}
\end{bmatrix}.
\]

Indeed, there are only \(2 + 2\) independent quantities in these Lax matrices. Thus, we can express them in terms of the components of the linear eigenfunction as

\[
\Psi^{(1)}_{1,n}^{-1}L_{12,n}\Psi^{(1)}_{2,n+1} = \left(\Psi^{(1)}_{1,n}^{-1}\Psi^{(2)}_{1,n} - \Psi^{(1)}_{1,n+1}^{-1}\Psi^{(2)}_{1,n+1}\right) \left(\Psi^{(1)}_{2,n+1}^{-1}\Psi^{(2)}_{2,n+1} - \Psi^{(1)}_{2,n+1}^{-1}\Psi^{(2)}_{2,n+1}\right)^{-1},
\]

\[
\Psi^{(1)}_{2,n}^{-1}L_{21,n}\Psi^{(1)}_{1,n+1} = \left(\Psi^{(1)}_{2,n}^{-1}\Psi^{(2)}_{2,n} - \Psi^{(1)}_{2,n+1}^{-1}\Psi^{(2)}_{2,n+1}\right) \left(\Psi^{(1)}_{1,n+1}^{-1}\Psi^{(2)}_{1,n+1} - \Psi^{(1)}_{1,n+1}^{-1}\Psi^{(2)}_{1,n+1}\right)^{-1},
\]

and

\[
\Psi^{(1)}_{1,n}^{-1}M_{12,n}\Psi^{(1)}_{2,n} = \left(\Psi^{(1)}_{1,n}^{-1}\Psi^{(2)}_{1,n} - \Psi^{(1)}_{1,n}^{-1}\Psi^{(2)}_{1,n}\right) \left(\Psi^{(1)}_{2,n}^{-1}\Psi^{(2)}_{2,n} - \Psi^{(1)}_{2,n}^{-1}\Psi^{(2)}_{2,n}\right)^{-1},
\]

\[
\Psi^{(1)}_{2,n}^{-1}M_{21,n}\Psi^{(1)}_{1,n} = \left(\Psi^{(1)}_{2,n}^{-1}\Psi^{(2)}_{2,n} - \Psi^{(1)}_{2,n}^{-1}\Psi^{(2)}_{2,n}\right) \left(\Psi^{(1)}_{1,n}^{-1}\Psi^{(2)}_{1,n} - \Psi^{(1)}_{1,n}^{-1}\Psi^{(2)}_{1,n}\right)^{-1}.
\]
Note that (2.6a) also implies the two important relations:

\[ \Psi_{1,n}^{(1)} - L_{11,n}^{(1)} = I - \Psi_{1,n}^{(1)} - L_{12,n}^{(1)} \Psi_{2,n+1}^{(1)}, \]  
(2.11)

\[ \Psi_{2,n}^{(1)} - L_{22,n}^{(1)} = I - \Psi_{2,n}^{(1)} - L_{21,n}^{(1)} \Psi_{1,n+1}^{(1)}. \]  
(2.12)

Typically, \( L_{11,n} \) and \( L_{22,n} \) are ultralocal functions of \( L_{12,n} \) and \( L_{21,n} \), such as

\[ L_{11,n} = \alpha I + \beta L_{12,n}L_{21,n}, \quad L_{22,n} = \gamma I + \delta L_{21,n}L_{12,n}, \]

where \( \alpha, \beta, \gamma \) and \( \delta \) are scalar functions of the spectral parameter. Thus, we can try to solve (2.11) and (2.12) to express \( \Psi_{1,n}^{(1)} - L_{12,n}^{(1)} \Psi_{2,n+1}^{(1)} \) and \( \Psi_{2,n}^{(1)} - L_{21,n}^{(1)} \Psi_{1,n+1}^{(1)} \) in terms of \( \Psi_{1,n}^{(1)} - L_{12,n}^{(1)} \Psi_{2,n+1}^{(1)} \) and \( \Psi_{2,n}^{(1)} - L_{21,n}^{(1)} \Psi_{1,n+1}^{(1)} \). This is relatively easy if either \( L_{11,n} \) or \( L_{22,n} \) is a constant scalar matrix, e.g., \( \beta \) or \( \delta \) vanishes in the above example. Then, using (2.7) and (2.8), we can also express

\[ \Psi_{1,n}^{(1)} - L_{12,n}^{(1)} \Psi_{2,n+1}^{(1)} = \left( \Psi_{1,n}^{(1)} - L_{12,n}^{(1)} \Psi_{2,n+1}^{(1)} \right)^{-1}, \]

\[ \Psi_{1,n+1}^{(1)} - L_{12,n}^{(1)} \Psi_{2,n+1}^{(1)} = \left( \Psi_{1,n}^{(1)} - L_{12,n}^{(1)} \Psi_{2,n+1}^{(1)} \right)^{-1} \Psi_{2,n}^{(1)} - L_{21,n}^{(1)} \Psi_{1,n+1}^{(1)}; \]

\[ \Psi_{2,n}^{(1)} - L_{21,n}^{(1)} \Psi_{1,n+1}^{(1)} = \left( \Psi_{2,n}^{(1)} - L_{21,n}^{(1)} \Psi_{1,n+1}^{(1)} \right)^{-1}, \]

\[ \Psi_{2,n+1}^{(1)} - L_{21,n}^{(1)} \Psi_{1,n+1}^{(1)} = \left( \Psi_{2,n}^{(1)} - L_{21,n}^{(1)} \Psi_{1,n+1}^{(1)} \right)^{-1} \Psi_{2,n}^{(1)} - L_{21,n}^{(1)} \Psi_{1,n+1}^{(1)}; \] etc.

recursively in terms of

\[ u_n := \Psi_{1,n}^{(1)} - \Psi_{1,n}^{(2)}, \quad v_n := \Psi_{2,n}^{(1)} - \Psi_{2,n}^{(2)}. \]  
(2.13)

In general, \( M_{12,n} \) and \( M_{21,n} \) are local (but not ultralocal) functions of \( L_{12,n} \) and \( L_{21,n} \). Therefore, with the aid of the above relations, (2.9) and (2.10) can be rewritten as a closed lattice system for \( u_n \) and \( v_n \).

Let us illustrate the method using the matrix Ablowitz–Ladik lattice (2.3) as an example. By setting

\[ L_{11,n} = zI, \quad L_{22,n} = z^{-1}I, \]

(2.11) and (2.12) provide the useful relations

\[ \Psi_{1,n+1}^{(1)} - L_{12,n}^{(1)} \Psi_{2,n+1}^{(1)} = z \left( I - \Psi_{1,n}^{(1)} - L_{12,n}^{(1)} \Psi_{2,n+1}^{(1)} \right)^{-1}, \]  
(2.14)

\[ \Psi_{2,n+1}^{(1)} - L_{21,n}^{(1)} \Psi_{1,n+1}^{(1)} = z^{-1} \left( I - \Psi_{2,n}^{(1)} - L_{21,n}^{(1)} \Psi_{1,n+1}^{(1)} \right)^{-1}. \]  
(2.15)
The Lax representation, (2.1) and (2.2), implies simple relations between off-diagonal elements of the temporal and spatial Lax matrices, i.e.

\[ M_{12,n} + z^{-1} a L_{12,n} + zbL_{12,n-1} = O, \]
\[ M_{21,n} + zbL_{21,n} + z^{-1} a L_{21,n-1} = O, \]

which can be rewritten as

\[
\begin{align*}
\Psi_{1,n}^{(1)} - \Psi_{1,n}^{(2)} + z^{-1} a \Psi_{1,n}^{(1)} L_{12,n} \Psi_{2,n}^{(1)} & = O, \\
\Psi_{2,n}^{(1)} - \Psi_{2,n}^{(2)} + zb \Psi_{2,n}^{(1)} L_{21,n} \Psi_{1,n}^{(1)} & = O.
\end{align*}
\]

Substituting (2.14) and (2.15) and subsequently (2.7)–(2.10) into the above relations, we arrive at a closed system for \( \Psi_{1,n}^{(1)} \Psi_{1,n}^{(2)} \) and \( \Psi_{2,n}^{(1)} \Psi_{2,n}^{(2)} \).

**Proposition 2.1.** Consider two linearly independent solutions of (2.1) and (2.2) and write them as in (2.6) with \( j = 1 \) or 2. Then, \( u_n \) and \( v_n \) defined in (2.13) satisfy the following system:

\[
\begin{align*}
\begin{cases}
  u_{n,t} + \frac{a}{\mu} (u_n - u_{n+1}) \left[ I + (v_n - u_n)^{-1}(u_n - u_{n+1}) \right]^{-1} \\
  + \mu b \left[ I - (u_{n-1} - u_n)(v_n - u_n)^{-1} \right]^{-1} (u_{n-1} - u_n) = O, \\
  v_{n,t} + \mu b (v_n - v_{n+1}) \left[ I + (u_n - v_n)^{-1}(v_n - v_{n+1}) \right]^{-1} \\
  + \frac{a}{\mu} \left[ I - (v_{n-1} - v_n)(u_n - v_n)^{-1} \right]^{-1} (v_{n-1} - v_n) = O,
\end{cases}
\end{align*}
\]

(2.16a, 2.16b)

where \( \mu = z^2 \).

**Remarks:**

(i) The system (2.16) with \( \mu b = (a/\mu)^\ast \) allows both the complex conjugation reduction \( v_n = \sigma u_n^\ast \) and the Hermitian conjugation reduction \( v_n = \sigma u_n^\dagger \), where \( \sigma = \pm 1 \). In addition, by setting \( v_n = -u_n \), (2.16) with \( \mu b = a/\mu \) reduces to a single matrix equation,

\[ u_{n,t} = u_n \left[ (u_{n+1} + u_n)^{-1} - (u_n + u_{n-1})^{-1} \right] u_n, \]

up to a rescaling of \( t \). In the scalar case, this belongs to Yamilov’s list of Volterra-type lattices in [30]; in the matrix case, it allows further reductions
to various multicomponent systems (cf. [31]).

(ii) The system (2.16) provides a space-discrete analog of the system studied by Svinolupov and Sokolov [32]; their system gives a matrix generalization of the Heisenberg ferromagnet model written in a two-component form [33,34]. Indeed, (2.16) in the scalar case is closely related to the lattice Heisenberg ferromagnet model [35] and its simplest higher symmetry [36,37].

(iii) The system (2.16) in the scalar case appeared in the recent paper [38] (also see [39]). In this context, it is natural to rewrite (2.16) as a two-component system for the pair of variables $u_n$ and $v_n^{-1}$ (cf. (4.8) in [33]).

In (2.16), the two variables $u_n$ and $v_n$ interact with each other through the quantity $(v_n - u_n)^{-1}$, which can be used as a new dependent variable. Indeed, a direct calculation shows the following two propositions.

**Proposition 2.2.** Let $u_n$ and $v_n$ satisfy the system (2.16). Then, the new pair of variables $q_n$ and $r_n$,

$$q_n := \Delta_+^n u_n (= u_{n+1} - u_n), \quad r_n := (v_n - u_n)^{-1},$$

satisfies the space-discrete Kaup–Newell system [20]:

$$\begin{align*}
q_{n,t} - \Delta_+^n \left[ \frac{a}{\mu} (I - q_n r_n)^{-1} q_n + \mu b (I + q_{n-1} r_n)^{-1} q_{n-1} \right] &= O, \quad (2.17a) \\
(r_{n,t} - \Delta_+^n \left[ \mu b (I + r_n q_{n-1})^{-1} r_n + \frac{a}{\mu} (I - r_{n-1} q_{n-1})^{-1} r_{n-1} \right] &= O, \quad (2.17b)
\end{align*}$$

where $\Delta_+^n$ denotes the forward difference operator.

**Proposition 2.3.** Let $u_n$ and $v_n$ satisfy the system (2.16). Then, the new pair of variables $\tilde{q}_n$ and $\tilde{r}_n$,

$$\tilde{q}_n := (v_n - u_n)^{-1}, \quad \tilde{r}_n := \Delta_+^n v_{n-1} (= v_n - v_{n-1}),$$

also satisfies the space-discrete Kaup–Newell system (2.17) for $\tilde{q}_n$ and $\tilde{r}_n$. 

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3 Inverse scattering method for the Ablowitz–Ladik lattice

3.1 Revisiting the Ablowitz–Ladik eigenvalue problem

In this section, we describe the inverse scattering method associated with the matrix Ablowitz–Ladik eigenvalue problem (2.1),

\[
\begin{bmatrix}
\Psi_{1,n} \\
\Psi_{2,n}
\end{bmatrix} =
\begin{bmatrix}
zI & zQ_n \\
z^{-1}R_n & z^{-1}I
\end{bmatrix}
\begin{bmatrix}
\Psi_{1,n+1} \\
\Psi_{2,n+1}
\end{bmatrix},
\] (3.1)

Here, the potentials \(Q_n\) and \(R_n\) are assumed to decay sufficiently rapidly at spatial infinity:

\[
\lim_{n \to \pm \infty} Q_n = \lim_{n \to \pm \infty} R_n = O. 
\] (3.2)

The matrix generalization of the Ablowitz–Ladik lattice [6] first considered in the early 1980s [7] is still a topic of interest in discrete integrable systems (see [20, 40, 41] and references therein). In our previous papers [23, 24], we presented the inverse scattering method for the matrix Ablowitz–Ladik lattice while assuming some symmetry conditions on the potentials \(Q_n\) and \(R_n\). Here, we remove such assumptions and consider the general case of \(l \times l\) square matrices \(Q_n\) and \(R_n\); recall that the results on rectangular matrix potentials can be obtained by setting some rows/columns in \(Q_n\) and \(R_n\) as zero.

The inverse scattering method reported here bypasses some redundant computation and consideration contained in the existing literature on the same subject, so we believe that this is the most streamlined version. The results in the previous work [23, 24] can be reproduced by imposing some reduction conditions on the scattering data; this is briefly sketched in appendix A. In addition, we fix some minor inconsistencies in [23, 24], though they do not affect the main results of these papers.

All the flows of the matrix Ablowitz–Ladik hierarchy are associated with the same eigenvalue problem (3.1), so they can be solved together by the inverse scattering method. However, in the following, we concentrate on the matrix Ablowitz–Ladik lattice (2.3) to illustrate the method in an easy-to-read manner. We stress that in contrast to other methods of obtaining special solutions, the inverse scattering method can provide the general solution formulas. Moreover, the method can determine not only the potentials \(Q_n\) and \(R_n\) but also a fundamental set of linear eigenfunctions, which will be used in section 4.
3.2 Jost solutions and relevant quantities

To analyze the general case of the matrix potentials $Q_n$ and $R_n$, we consider the adjoint equation:

$$
\begin{bmatrix}
\Phi_{1,n+1} & \Phi_{2,n+1}
\end{bmatrix} =
\begin{bmatrix}
\Phi_{1,n} & \Phi_{2,n}
\end{bmatrix}
\begin{bmatrix}
zI & zQ_n \\
z^{-1}R_n & z^{-1}I
\end{bmatrix}.
$$

(3.3)

Indeed, a discrete analog of Lagrange’s identity,

$$
\begin{align*}
&\begin{bmatrix}
\Phi_{1,n} & \Phi_{2,n}
\end{bmatrix}
\begin{bmatrix}
\Psi_{1,n} & \Psi_{2,n}
\end{bmatrix}
-
\begin{bmatrix}
\Phi_{1,n+1} & \Phi_{2,n+1}
\end{bmatrix}
\begin{bmatrix}
\Psi_{1,n+1} & \Psi_{2,n+1}
\end{bmatrix}
\end{align*}
$$

implies that (3.1) and (3.3) can be said to be adjoint to each other. Thus, we can introduce an $l \times l$ matrix function $W[\cdot, \cdot]$ for a pair of solutions to (3.1) and (3.3) as

$$
W[\Phi_n, \Psi_n] :=
\begin{bmatrix}
\Phi_{1,n} & \Phi_{2,n}
\end{bmatrix}
\begin{bmatrix}
\Psi_{1,n} \\
\Psi_{2,n}
\end{bmatrix} = \Phi_{1,n} \Psi_{1,n} + \Phi_{2,n} \Psi_{2,n},
$$

which is $n$-independent:

$$
W[\Phi_n, \Psi_n] = W[\Phi_{n+1}, \Psi_{n+1}].
$$

In addition to the rapidly decaying boundary conditions (3.2), we assume that the spatial Lax matrix defining the eigenvalue problem is invertible:

$$
\det (I - Q_n R_n) \neq 0, \ \forall n \in \mathbb{Z}.
$$

(3.4)

Because $\log [\det (I - Q_n R_n)]$ is a conserved density for the matrix Ablowitz–Ladik hierarchy [7], this assumption is preserved under the time evolution. Thus, a set of column-vector (or row-vector) solutions to the eigenvalue problem (3.1) (or (3.3)) that are linearly independent at some lattice site, say $n = n_0$, remain independent for all $n \in \mathbb{Z}$.

---

Note that if we consider a square matrix solution $\Psi_n$ to the eigenvalue problem $\Psi_n = L_n(z) \Psi_{n+1}$, then its inverse $\Phi_n := \Psi_n^{-1}$ satisfies the eigenvalue problem $\Phi_{n+1} = \Phi_n L_n(z)$. 

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We introduce Jost solutions $\phi_n(z)$, $\phi^*_n(z)$ and $\psi_n(z)$, $\psi^*_n(z)$ at a fixed time that satisfy (3.1) and the boundary conditions,

\[
\begin{align*}
    z^n \phi_n & \to \begin{bmatrix} I \\ O \end{bmatrix} \quad \text{as } n \to -\infty \quad (3.5a) \\
    z^{-n} \phi^*_n & \to \begin{bmatrix} O \\ -I \end{bmatrix}
\end{align*}
\]

and

\[
\begin{align*}
    z^{-n} \psi_n & \to \begin{bmatrix} O \\ I \end{bmatrix} \quad \text{as } n \to +\infty. \quad (3.5b) \\
    z^n \psi^*_n & \to \begin{bmatrix} I \\ O \end{bmatrix}
\end{align*}
\]

The time evolution of the Jost solutions will be considered in subsection 3.4. Note that the overbar does not mean complex conjugation in this paper. Similarly, we introduce adjoint Jost solutions $\phi^\text{ad}_n(z)$, $\phi^*\text{ad}_n(z)$ and $\psi^\text{ad}_n(z)$, $\psi^*\text{ad}_n(z)$ that satisfy (3.3) and the boundary conditions,

\[
\begin{align*}
    z^n \phi^\text{ad}_n & \to \begin{bmatrix} O & -I \end{bmatrix} \quad \text{as } n \to -\infty \quad (3.6a) \\
    z^{-n} \phi^*\text{ad}_n & \to \begin{bmatrix} I & O \end{bmatrix}
\end{align*}
\]

and

\[
\begin{align*}
    z^{-n} \psi^\text{ad}_n & \to \begin{bmatrix} I \\ O \end{bmatrix} \quad \text{as } n \to +\infty. \quad (3.6b) \\
    z^n \psi^*\text{ad}_n & \to \begin{bmatrix} O \\ I \end{bmatrix}
\end{align*}
\]

Because the $l + l (= 2l)$ columns of the Jost solutions $\psi_n$ and $\psi^*_n$ form a fundamental set of solutions to the eigenvalue problem (3.1), we can set on the unit circle $|z| = 1$ as

\[
\begin{align*}
    \phi_n(z) & = \overline{\psi}_n(z)A + \psi_n(z)B, \quad (3.7a) \\
    \phi^*_n(z) & = \overline{\psi}_n(z)\overline{B} - \psi_n(z)\overline{A}. \quad (3.7b)
\end{align*}
\]

Here, $A$, $B$, $\overline{B}$ and $\overline{A}$ are $n$-independent $l \times l$ matrices, which depend on the spectral parameter $z$ and are called scattering data. According to the asymptotic behaviors of the Jost solutions (3.5)–(3.6), we can express them on $|z| = 1$ as

\[
\begin{align*}
    A & = W[\psi^\text{ad}_n, \phi_n], \quad (3.8a) \\
    B & = W[\psi^\text{ad}_n, \phi^*_n], \quad (3.8b) \\
    \overline{B} & = W[\psi^\text{ad}_n, \phi^*_n], \quad (3.8c) \\
    \overline{A} & = -W[\psi^\text{ad}_n, \phi^*_n]. \quad (3.8d)
\end{align*}
\]
We can rewrite the eigenvalue problem (3.1) in the following equivalent forms:

\[
\begin{bmatrix}
    z^{-n}\Psi_{1,n} \\
    z^{-n}\Psi_{2,n}
\end{bmatrix} = \begin{bmatrix}
    z^2 I & z^2 Q_n \\
    R_n & I
\end{bmatrix} \begin{bmatrix}
    z^{-(n+1)}\Psi_{1,n+1} \\
    z^{-(n+1)}\Psi_{2,n+1}
\end{bmatrix}, \quad (3.9a)
\]

\[
\begin{bmatrix}
    z^n\Psi_{1,n} \\
    z^n\Psi_{2,n}
\end{bmatrix} = \begin{bmatrix}
    I & Q_n \\
    z^{-2} R_n & z^{-2} I
\end{bmatrix} \begin{bmatrix}
    z^{n+1}\Psi_{1,n+1} \\
    z^{n+1}\Psi_{2,n+1}
\end{bmatrix}. \quad (3.9b)
\]

Thus, in view of the boundary conditions (3.5), \(z^n\phi_n, z^{-n}\phi_n, z^n\psi_n\) and \(z^n\overline{\psi}_n\) depend on \(z\) only through \(z^2\). Similarly, (3.3) can be rewritten as

\[
\begin{bmatrix}
    z^{n+1}\Phi_{1,n+1} \\
    z^{n+1}\Phi_{2,n+1}
\end{bmatrix} = \begin{bmatrix}
    z^2 I & z^2 Q_n \\
    R_n & I
\end{bmatrix} \begin{bmatrix}
    z^{-n}\Phi_{1,n+1} \\
    z^{-n}\Phi_{2,n+1}
\end{bmatrix}, \quad (3.10a)
\]

\[
\begin{bmatrix}
    z^{-(n+1)}\Phi_{1,n+1} \\
    z^{-(n+1)}\Phi_{2,n+1}
\end{bmatrix} = \begin{bmatrix}
    I & Q_n \\
    z^{-2} R_n & z^{-2} I
\end{bmatrix} \begin{bmatrix}
    z^{-n}\Phi_{1,n+1} \\
    z^{-n}\Phi_{2,n+1}
\end{bmatrix}. \quad (3.10b)
\]

so \(z^{-n}\psi_n^{ad}\) and \(z^n\overline{\psi}_n^{ad}\) depend on \(z\) only through \(z^2\). Therefore, relations (3.8) imply that the scattering data \(A, B, \overline{B}\) and \(\overline{A}\) are even functions of \(z\) (cf. [37]); they can be denoted as \(A(\mu), B(\mu), \overline{B}(\mu)\) and \(\overline{A}(\mu)\), where

\[\mu = z^2.\]

We introduce the following representations of the Jost solutions \(\psi_n\) and \(\overline{\psi}_n\):

\[
z^{-n}\psi_n = \prod_{i=n}^{\infty} \begin{bmatrix}
    \mu I & \mu Q_i \\
    R_i & I
\end{bmatrix} \begin{bmatrix}
    O \\
    I
\end{bmatrix} =: \begin{bmatrix}
    O \\
    I
\end{bmatrix} + \sum_{k=0}^{\infty} \mu^{k+1} K(n, n+k), \quad (3.11a)
\]

\[
z^n\overline{\psi}_n = \prod_{i=n}^{\infty} \begin{bmatrix}
    I & Q_i \\
    \mu^{-1} R_i & \mu^{-1} I
\end{bmatrix} \begin{bmatrix}
    I \\
    O
\end{bmatrix} =: \begin{bmatrix}
    I \\
    O
\end{bmatrix} + \sum_{k=0}^{\infty} \mu^{-k-1} \overline{K}(n, n+k), \quad (3.11b)
\]

which are assumed to be uniformly convergent in \(|\mu| \leq 1\) and \(|\mu| \geq 1\), respectively (cf. (3.2)). Here and hereafter, the order of the matrix product is defined as

\[
\prod_{i=n}^{m} X_i := X_n X_{n+1} \cdots X_m, \quad \prod_{i=n}^{m} X_i := X_m X_{m-1} \cdots X_n, \quad m \geq n,
\]

\[\text{In the area of orthogonal polynomials, the Ablowitz–Ladik eigenvalue problem in a similar rewritten form was studied by G. Baxter in the early 1960s after the seminal work of G. Szegő, see [42,43].} \]
and the “integral kernels” $K(n, m)$ and $\tilde{K}(n, m)$ are $\mu$-independent $2l \times l$ matrices denoted in terms of $l \times l$ matrices as

$$K(n, m) = \begin{bmatrix} K_1(n, m) \\ K_2(n, m) \end{bmatrix}, \quad \tilde{K}(n, m) = \begin{bmatrix} \tilde{K}_1(n, m) \\ \tilde{K}_2(n, m) \end{bmatrix}, \quad m \geq n.$$ 

We substitute (3.11a) and (3.11b) into (3.9a) and (3.9b), respectively. Noting that they are identities in $\mu$, we can express the “integral kernels” recursively in terms of the potentials $Q_n$ and $R_n$; the most important relations are

$$K_1(n, n) = Q_n, \quad (3.12)$$

$$K_2(n, n) = \sum_{j=n}^{\infty} R_j Q_{j+1}, \quad (3.13)$$

and

$$\tilde{K}_1(n, n) = R_n, \quad \tilde{K}_2(n, n) = \sum_{j=n}^{\infty} Q_j R_{j+1}.$$ 

In a way similar to (3.11), we can also express the other (adjoint) Jost solutions as power series in either $\mu$ or $\mu^{-1}$. Indeed, noting the identity,

$$\begin{bmatrix} \mu I & \mu Q_n \\ R_n & I \end{bmatrix} \begin{bmatrix} \mu^{-1} (I - Q_n R_n)^{-1} & -Q_n (I - R_n Q_n)^{-1} \\ -\mu^{-1} R_n (I - Q_n R_n)^{-1} & (I - R_n Q_n)^{-1} \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix},$$

we obtain

$$z^n \phi_n = \prod_{i=-\infty}^{\infty} \begin{bmatrix} (I - Q_i R_i)^{-1} & -\mu Q_i (I - R_i Q_i)^{-1} \\ -R_i (I - Q_i R_i)^{-1} & \mu (I - R_i Q_i)^{-1} \end{bmatrix} \begin{bmatrix} I \\ O \end{bmatrix}, \quad (3.14a)$$

$$z^{-n} \bar{\phi}_n = \prod_{i=-\infty}^{n-1} \begin{bmatrix} \mu^{-1} (I - Q_i R_i)^{-1} & -Q_i (I - R_i Q_i)^{-1} \\ -\mu^{-1} R_i (I - Q_i R_i)^{-1} & (I - R_i Q_i)^{-1} \end{bmatrix} \begin{bmatrix} O \\ -I \end{bmatrix}, \quad (3.14b)$$

$$z^{-n} \psi_n^{ad} = \prod_{i=n}^{\infty} \begin{bmatrix} (I - Q_i R_i)^{-1} & -\mu Q_i (I - R_i Q_i)^{-1} \\ -R_i (I - Q_i R_i)^{-1} & \mu (I - R_i Q_i)^{-1} \end{bmatrix},$$

$$z^n \bar{\psi}_n^{ad} = \prod_{i=n}^{\infty} \begin{bmatrix} \mu^{-1} (I - Q_i R_i)^{-1} & -Q_i (I - R_i Q_i)^{-1} \\ -\mu^{-1} R_i (I - Q_i R_i)^{-1} & (I - R_i Q_i)^{-1} \end{bmatrix}, \quad \text{etc.}$$
Thus, (3.8a) and (3.8d) imply that $A(\mu)$ and $\overline{A}(\mu)$ can be written explicitly as

$$A(\mu) = \begin{bmatrix} I & O \end{bmatrix} \prod_{i=-\infty}^{\infty} \begin{bmatrix} (I - Q_i R_i)^{-1} & -\mu Q_i (I - R_i Q_i)^{-1} \\ -R_i (I - Q_i R_i)^{-1} & \mu (I - R_i Q_i)^{-1} \end{bmatrix} \begin{bmatrix} I \\ O \end{bmatrix},$$

(3.15a)

$$\overline{A}(\mu) = \begin{bmatrix} O & I \end{bmatrix} \prod_{i=-\infty}^{\infty} \begin{bmatrix} \mu^{-1} (I - Q_i R_i)^{-1} & -Q_i (I - R_i Q_i)^{-1} \\ -\mu^{-1} R_i (I - Q_i R_i)^{-1} & (I - R_i Q_i)^{-1} \end{bmatrix} \begin{bmatrix} O \\ I \end{bmatrix}.$$

(3.15b)

Therefore, as long as $Q_n$ and $R_n$ decay sufficiently rapidly as $n \to \pm \infty$, $z^n \phi_n$ and $z^{-n} \psi^\text{ad}_n$ are analytic on and inside the unit circle ($|\mu| \leq 1$), and $z^{-n} \overline{\phi}_n$ and $z^n \overline{\psi}^\text{ad}_n$ are analytic on and outside the unit circle ($|\mu| \geq 1$). Consequently, $A(\mu)$ and $\overline{A}(\mu)$ can be analytically continued for $|\mu| \leq 1$ and $|\mu| \geq 1$, respectively. A more precise discussion on the analytical properties of the Jost solutions can be made using a discrete analog of the approach in [4]; that is, we can rewrite the eigenvalue problem in the form of linear summation equations and discuss the convergence of their Liouville–Neumann-type series solutions. However, we omit such a discussion in this paper.

### 3.3 Gel’fand–Levitan–Marchenko equations

We multiply (3.7a) and (3.7b) from the right by $z^n A(\mu)^{-1}$ and $z^{-n} \overline{A}(\mu)^{-1}$, respectively, to obtain

$$[z^n \phi_n](\mu) A(\mu)^{-1} = [z^n \overline{\psi}_n](\mu) + [z^{-n} \overline{\psi}_n](\mu) B(\mu) A(\mu)^{-1} \mu^n,$$

(3.16a)

$$[z^{-n} \overline{\phi}_n](\mu) \overline{A}(\mu)^{-1} = -[z^{-n} \psi_n](\mu) + [z^n \overline{\psi}_n](\mu) \overline{B}(\mu) \overline{A}(\mu)^{-1} \mu^{-n}.$$

(3.16b)

Here, “$(\mu)$” emphasizes that the argument of the functions is $\mu (= z^2)$ rather than $z$.

Then, we substitute the summation representations (3.11) into the right-hand side of (3.16a) and operate with

$$\frac{1}{2\pi i} \oint_C d\mu \mu^{m-n} \ (m \geq n)$$

(3.17)

---

8Here, we use the term “analytic continuation” loosely. See appendix B of [44] for a rigorous treatment of “analytic continuation” in the delicate case, e.g., when $Q_n$ and $R_n$ do not decay exponentially fast as $n \to \pm \infty$.

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on both sides. Here, $C$ denotes the counterclockwise contour along the unit circle $|\mu| = 1$. Thus, we obtain

$$J(n, m) = K(n, m) + \sum_{k=0}^{\infty} K(n, n + k) F_C(m + k + 1),$$

where

$$J(n, m) := \frac{1}{2\pi i} \oint_C [z^n \phi_n](\mu) A(\mu)^{-1} \mu^{m-n} d\mu,$$

$$F_C(m) := \frac{1}{2\pi i} \oint_C B(\mu) A(\mu)^{-1} \mu^m d\mu.$$

Because of the analyticity of $[z^n \phi_n](\mu)$ and $A(\mu)$ in $|\mu| \leq 1$, we can evaluate $J(n, m)$ using the residue theorem. Recall that the inverse of the matrix $A(\mu)$ is given by

$$A(\mu)^{-1} = \frac{1}{\det A(\mu)} \tilde{A}(\mu),$$

where the tilde denotes the adjugate (i.e., transposed cofactor) matrix. Thus, the singularities of the integrand in (3.19) are determined by the zeros of $\det A(\mu)$. For simplicity, we assume that the matrix function $A(\mu)^{-1}$ only has isolated simple poles in $|\mu| < 1$, denoted as $\{\mu_1, \mu_2, \ldots, \mu_N\}$, and is regular on $|\mu| = 1$. In fact, the more general case where $A(\mu)^{-1}$ also has higher order poles can be recovered by taking a suitable coalescence limit of two or more simple poles afterward.

In the neighborhood of $\mu = \mu_j$, we can expand $A(\mu)$ and $A(\mu)^{-1}$ as (cf. [45, 46])

$$A(\mu) = A(\mu_j) + (\mu - \mu_j) A'(\mu_j) + O((\mu - \mu_j)^2), \quad \det A(\mu_j) = 0,$$

$$A(\mu)^{-1} = \frac{1}{\mu - \mu_j} A_j^{(-1)} + A_j^{(0)} + O(\mu - \mu_j), \quad A_j^{(-1)} \neq O,$$

where

$$A(\mu_j) A_j^{(-1)} = O, \quad A(\mu_j) A_j^{(0)} + A'(\mu_j) A_j^{(-1)} = I.$$

Thus, using (3.5b), (3.6b) and (3.8a), we obtain

$$z^{-n} \psi_n \left[ z^n \phi_n A_j^{(-1)} z^n \psi_n \right] = \left[ A(\mu) A_j^{(-1)} O \right]$$

$$= \left[ O \ O \right] \text{ at } \mu = \mu_j.$$

We should not assume such a strong condition as $\det A(\mu)$ has only simple zeros, which was assumed in our previous papers [23, 24]. Indeed, a zero of multiplicity $k$ of $\det A(\mu)$ may be cancelled by a zero of multiplicity $k - 1$ of $\tilde{A}(\mu)$ to give a simple pole of $A(\mu)^{-1}$. However, this correction does not affect the validity of the formulas in [23, 24].
Because \( z^{-n} \psi_n^{\text{ad}} \) consisting of \( l \) rows satisfies the boundary condition in (3.6b) and the eigenvalue problem (3.10b), the rank of \( [z^{-n} \psi_n^{\text{ad}}](\mu_j) \) is equal to \( l \) for all \( n \in \mathbb{Z} \). Similarly, the rank of \( [z^n \phi_n](\mu_j) \) is \( l \) for all \( n \in \mathbb{Z} \). Therefore, there exists an \( l \times l \) matrix \( C_j \) such that

\[
[z^n \phi_n](\mu_j) A_j^{-1} = [z^{-n} \psi_n](\mu_j) C_j^m. \tag{3.21}
\]

Here, \( C_j \) must be \( n \)-independent, because both \( z^n \phi_n \) and \( z^n \psi_n \) satisfy the same eigenvalue problem (3.9b). The matrix \( C_j \) can be intuitively considered as \( B(\mu_j) \lim_{\mu \to \mu_j} (\mu - \mu_j) A(\mu)^{-1} \), but it is, in general, different from the naive residue \( \lim_{\mu \to \mu_j} (\mu - \mu_j) B(\mu) A(\mu)^{-1} \). Indeed, \( B(\mu) \) can have a discontinuity at \( \mu = \mu_j \).

Because \( \phi_n A_j^{(-1)} \) and \( \psi_n C_j \) vanish exponentially for \( n \to -\infty \) and \( n \to +\infty \) respectively, each nonzero column vector of the \( 2l \times l \) matrix \( \phi_n A_j^{(-1)} = \psi_n C_j \) at \( z^2 = \mu_j \) gives a bound state in the potentials \( Q_n \) and \( R_n \).

Therefore, using the residue theorem with the aid of (3.20) and (3.21), we can compute the right-hand side of (3.19) as

\[
J(n, m) = \sum_{j=1}^{N} [z^{-n} \psi_n](\mu_j) C_j^m \mu_j^m
\]

\[
= \sum_{j=1}^{N} \left\{ \begin{bmatrix} O \\ I \end{bmatrix} + \sum_{k=0}^{\infty} \mu_j^{k+1} K(n, n + k) \right\} C_j^m \mu_j^m
\]

\[
= - \begin{bmatrix} O \\ F_D(m) \end{bmatrix} - \sum_{k=0}^{\infty} K(n, n + k) F_D(m + k + 1),
\]

where

\[
F_D(m) := - \sum_{j=1}^{N} C_j^m \mu_j^m.
\]

Substituting this expression for \( J(n, m) \) into (3.18), we obtain a linear summation equation of the Gel’fand–Levitan–Marchenko type,

\[
\tilde{R}(n, m) + \begin{bmatrix} O \\ F(m) \end{bmatrix} + \sum_{k=0}^{\infty} K(n, n + k) F(m + k + 1) = \begin{bmatrix} O \\ O \end{bmatrix}, \quad m \geq n. \tag{3.22}
\]

Here, \( F(m) \) is defined as

\[
F(m) := F_C(m) + F_D(m)
\]

\[
= \frac{1}{2\pi i} \oint_C B(\mu) A(\mu)^{-1} \mu^m d\mu - \sum_{j=1}^{N} C_j^m \mu_j^m. \tag{3.23}
\]
Note that $F_C$ and $F_D$ correspond to the contributions of the continuous and discrete spectra, respectively.\footnote{In this section, we often follow the notation of Ablowitz et al. [6, 15, 16].}

**Remark.** Using the expressions (3.14a) and (3.15a), we can evaluate $[z^n \phi_n](\mu)$ and $A(\mu)$ in the limit $\mu \to 0$ as (cf. [15, 25, 40])

\[
\lim_{\mu \to 0} [z^n \phi_n](\mu) = \left[ \begin{array}{c} I \\ -R_{n-1} \end{array} \right] \prod_{i=-\infty}^{\infty} (I - Q_i R_i)^{-1},
\]

\[
\lim_{\mu \to 0} A(\mu) = \prod_{i=-\infty}^{\infty} (I - Q_i R_i)^{-1}.
\]

Because $\lim_{\mu \to 0} A(\mu)$ is invertible (cf. (3.4)), we have $\mu_j \neq 0$, $j = 1, 2, \ldots, N$. Note that instead of (3.17), we can operate with $\frac{1}{2\pi i} \oint_C d\mu \mu^{-1}$ on (3.16a) with (3.11). Thus, we can express $\lim_{\mu \to 0} [z^n \phi_n](\mu) A(\mu)^{-1}$ as

\[
\lim_{\mu \to 0} [z^n \phi_n](\mu) A(\mu)^{-1} = \left[ \begin{array}{c} I \\ F(n-1) \end{array} \right] + \sum_{k=0}^{\infty} \left[ \begin{array}{c} K_1(n,n+k) \\ K_2(n,n+k) \end{array} \right] F(n+k).
\]

From the above three relations, we obtain

\[
\left[ \begin{array}{c} I \\ -R_{n-1} \end{array} \right] \prod_{i=n}^{\infty} (I - Q_i R_i) = \left[ \begin{array}{c} I \\ F(n-1) \end{array} \right] + \sum_{k=0}^{\infty} \left[ \begin{array}{c} K_1(n,n+k) \\ K_2(n,n+k) \end{array} \right] F(n+k).
\]

The nonlocal quantities on the left-hand side can be used to transform the matrix Ablowitz–Ladik lattice (2.3) to other systems [7].

Next, we substitute the summation representations (3.11) into the right-hand side of (3.16b) and operate with

\[
\frac{1}{2\pi i} \oint_C d\mu \mu^{n-m-2} \quad (m \geq n)
\]

on both sides. Thus, we obtain

\[
\mathcal{J}(n,m) = -K(n,m) + \left[ \begin{array}{c} \mathcal{F}_C(m) \\ O \end{array} \right] + \sum_{k=0}^{\infty} \mathcal{K}(n,n+k) \mathcal{F}_C(m+k+1),
\]

(3.25)
where
\[ J(n, m) := \frac{1}{2\pi i} \oint_C [z^{-n} \Phi_n](\mu) \overline{A}(\mu)^{-1} \mu^{n-m-2} d\mu, \quad (3.26) \]
\[ F_C(m) := \frac{1}{2\pi i} \oint_C B(\mu) \overline{A}(\mu)^{-1} \mu^{-m-2} d\mu. \]

Because of the analyticity of \([z^{-n} \Phi_n](\mu)\) and \(\overline{A}(\mu)\) in \(|\mu| \geq 1\), we can evaluate \(J(n, m)\) using the residue theorem. We assume that \(\overline{A}(\mu)^{-1}\) only has isolated simple poles in \(|\mu| > 1\), denoted as \(\{\mu_1, \mu_2, \ldots, \mu_N\}\), and is regular on \(|\mu| = 1\).

In the neighborhood of \(\mu = \mu_j\), we expand \(\overline{A}(\mu)\) and \(\overline{A}(\mu)^{-1}\) as
\[ \overline{A}(\mu) = \overline{A}(\mu_j) + (\mu - \mu_j) \overline{A}'(\mu_j) + O((\mu - \mu_j)^2), \quad \text{det} \overline{A}(\mu_j) = 0, \]
\[ \overline{A}(\mu)^{-1} = \frac{1}{\mu - \mu_j} \overline{A}^{-1}(\mu_j) + \overline{A}^{(0)}(\mu_j) + O(\mu - \mu_j), \quad \overline{A}^{-1}(\mu_j) \neq O, \quad (3.27) \]
where
\[ \overline{A}(\mu_j) \overline{A}^{-1}(\mu_j) = O, \quad \overline{A}(\mu_j) \overline{A}^{(0)}(\mu_j) + \overline{A}'(\mu_j) \overline{A}^{-1}(\mu_j) = I. \]

Thus, using (3.5b), (3.6b) and (3.8d), we obtain
\[ z^n \overline{\psi}_n \left[ z^{-n} \overline{\Phi}_n \overline{A}^{-1}(\mu_j) \right] = \left[ -\overline{A}(\mu) \overline{A}^{-1}(\mu_j) O \right] \]
\[ = \left[ O O \right] \quad \text{at} \quad \mu = \mu_j. \]

In the same way as the derivation of (3.21), there exists an \(n\)-independent \(l \times l\) matrix \(\overline{C}_j\) such that
\[ [z^{-n} \overline{\Phi}_n](\mu_j) \overline{A}^{-1}(\mu_j) = [z^n \overline{\psi}_n](\mu_j) \overline{C}_j \mu_j^{-n}. \quad (3.28) \]

Therefore, using the residue theorem with the aid of (3.27) and (3.28), we can compute the right-hand side of (3.26) as
\[ J(n, m) = -\sum_{j=1}^N [z^n \overline{\psi}_n](\mu_j) \overline{C}_j \mu_j^{-m-2} \]
\[ = -\sum_{j=1}^N \left\{ \begin{bmatrix} I \\ O \end{bmatrix} + \sum_{k=0}^{\infty} \mu_j^{-k-1} \overline{K}(n, n + k) \right\} \overline{C}_j \mu_j^{-m-2} \]
\[ = -\left[ \overline{F}_D(m) \right] - \sum_{k=0}^{\infty} \overline{K}(n, n + k) \overline{F}_D(m + k + 1), \quad (3.29) \]

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where

\[ F_D(m) := \sum_{j=1}^{\infty} C_j \mu_j^{-m-2}. \]

Substituting this expression for \( \mathcal{J}(n,m) \) into (3.25), we obtain another linear summation equation of the Gel’fand–Levitan–Marchenko type,

\[-K(n,m) + \left[ \frac{F(m)}{O} \right] + \sum_{k=0}^{\infty} \mathcal{R}(n,n+k)F(m+k+1) = \left[ \frac{O}{O} \right], \quad m \geq n.\]

(3.29)

Here, \( F(m) \) is defined as

\[ F(m) := F_C(m) + F_D(m) \]

\[ = \frac{1}{2\pi i} \oint_C B(\mu) \mathcal{A}(\mu)^{-1} \mu^{-m-2} d\mu + \sum_{j=1}^{\infty} C_j \mu_j^{-m-2}. \]  

(3.30)

Remark. Using the expressions (3.14b) and (3.15b), we can evaluate \( [z^{-n} \Phi_n](\mu) \) and \( \mathcal{A}(\mu) \) in the limit \( |\mu| \to \infty \) as (cf. [40])

\[ \lim_{|\mu| \to \infty} [z^{-n} \Phi_n](\mu) = \left[ \frac{Q_{n-1}}{-I} \right] \prod_{i=\infty}^{n-1} (I - R_i Q_i)^{-1}, \]

\[ \lim_{|\mu| \to \infty} \mathcal{A}(\mu) = \prod_{i=\infty}^{\infty} (I - R_i Q_i)^{-1}. \]

Because \( \lim_{|\mu| \to \infty} \mathcal{A}(\mu) \) is invertible (cf. (3.4)), \( \mathcal{A}(\mu)^{-1} \) does not have poles at infinity, which is indeed consistent with the computation for \( \mathcal{J}(n,m) \). Note that instead of (3.24), we can operate with \( \frac{1}{2\pi i} \oint_C d\mu \mu^{-1} \) on (3.16b) with (3.11). Thus, we can express \( \lim_{|\mu| \to \infty} [z^{-n} \Phi_n](\mu) \mathcal{A}(\mu)^{-1} \) as

\[ \lim_{|\mu| \to \infty} [z^{-n} \Phi_n](\mu) \mathcal{A}(\mu)^{-1} = -\left[ \frac{-F(n-1)}{I} \right] + \sum_{k=0}^{\infty} \left[ \frac{K_1(n,n+k)}{K_2(n,n+k)} \right] F(n+k). \]

From the above three relations, we obtain

\[ \left[ \frac{-Q_{n-1}}{I} \right] \prod_{i=n}^{\infty} (I - R_i Q_i) = \left[ \frac{-F(n-1)}{I} \right] - \sum_{k=0}^{\infty} \left[ \frac{K_1(n,n+k)}{K_2(n,n+k)} \right] F(n+k). \]
The nonlocal quantities on the left-hand side can be used to transform the matrix Ablowitz–Ladik lattice (2.3) to other systems [7].

The Gel’fand–Levitan–Marchenko equations (3.22) and (3.29) relate the scattering data to the potentials \( Q_n \) and \( R_n \) through (3.12) and (3.13). The required set of the scattering data is given by \( B(\mu)A(\mu)^{-1} - 1 \) for \( |\mu| = 1 \), \( \{\mu_j, C_j\}_{j=1,2,...,N} \) and \( \{\pi_j, \overline{C_j}\}_{j=1,2,...,N} \), which define \( F(m) \) and \( \overline{F}(m) \) as in (3.23) and (3.30). Because \( F(m) \) and \( \overline{F}(m) \) are “linear” in the scattering data, we can consider a linear superposition of different sets of scattering data at this level. In fact, we will show in the next subsection that \( F(m) \) and \( \overline{F}(m) \) satisfy linear evolution equations.

### 3.4 Time evolution

Under the rapidly decaying boundary conditions (3.2), the temporal part of the Lax representation (2.2) for the Ablowitz–Ladik lattice (2.3) has the asymptotic behavior:

\[
\begin{bmatrix}
\Psi_{1,n} \\
\Psi_{2,n}
\end{bmatrix}
t \sim \begin{bmatrix}
(-\mu + 1)bI & O \\
O & (1 - \mu^{-1})aI
\end{bmatrix}
\begin{bmatrix}
\Psi_{1,n} \\
\Psi_{2,n}
\end{bmatrix}
as \ n \to \pm \infty.
\]

This can be used to fix the time dependence of the leading order terms in \( \Psi_{1,n} \) and \( \Psi_{2,n} \) (cf. (3.5)). Thus, we can introduce the explicitly time-dependent Jost solutions \( \phi_n^{(t)} \) and \( \overline{\phi}_n^{(t)} \) as

\[
z^n\phi_n^{(t)} := e^{(-\mu+1)bt}z^n\phi_n \rightarrow e^{(-\mu+1)bt} \begin{bmatrix} I \\ O \end{bmatrix}
as \ n \to -\infty \quad (3.31a)
\]

and \( \psi_n^{(t)} \) and \( \overline{\psi}_n^{(t)} \) as

\[
z^{-n}\psi_n^{(t)} := e^{(1-\mu^{-1})at}z^{-n}\psi_n \rightarrow e^{(1-\mu^{-1})at} \begin{bmatrix} O \\ I \end{bmatrix}
as \ n \to +\infty, \quad (3.31b)
\]

respectively; they satisfy both the eigenvalue problem (2.1) and the time-evolution equation (2.2).
To determine the time dependence of the scattering data, we rewrite the defining relations (3.7) as

\[ \phi_n(t) = \psi_n(t) A + \psi_n(t) B e^{-[(\mu-1)b+(1-\mu^{-1})a]t}, \]
\[ \bar{\phi}_n(t) = \psi_n(t) \bar{A} e^{[(\mu-1)b+(1-\mu^{-1})a]t} - \psi_n(t) \bar{A}. \]

Note that \( \phi_n(t), \bar{\phi}_n(t), \psi_n(t), \) and \( \bar{\psi}_n(t) \) satisfy the same equation (2.2) and the columns of \( \psi_n(t) \) and \( \bar{\psi}_n(t) \) are linearly independent. Thus, the time dependences of \( A, B \) and \( \bar{A}, \bar{B} \) for \( |\mu| = 1 \) are given by

\[ A(\mu, t) = A(\mu, 0), \quad B(\mu, t) = B(\mu, 0)e^{[(\mu-1)b+(1-\mu^{-1})a]t} \quad (3.32) \]

and

\[ \bar{A}(\mu, t) = \bar{A}(\mu, 0), \quad \bar{B}(\mu, t) = \bar{B}(\mu, 0)e^{-[(\mu-1)b+(1-\mu^{-1})a]t} \quad (3.33) \]

respectively. This implies that \( A(\mu, t), \bar{A}(\mu, t), B(\mu, t)e^{-[(\mu-1)b+(1-\mu^{-1})a]t} \) and \( \bar{B}(\mu, t)e^{[(\mu-1)b+(1-\mu^{-1})a]t} \) are generating functions of the integrals of motion.

Because the “analytic continuation” of \( A(\mu) \) and \( \bar{A}(\mu) \) into the regions \( |\mu| \leq 1 \) and \( |\mu| \geq 1 \), respectively, is unique and remains time-independent (cf. (3.15)), the positions of the simple poles of \( A(\mu)^{-1} \) and \( \bar{A}(\mu)^{-1} \) and the corresponding residues, \( \{\mu_j, A_j^{-1}\} \) and \( \{\bar{\mu}_j, \bar{A}_j^{-1}\} \) are also time-independent. For (3.21) and (3.28), we can apply a similar discussion as used to obtain (3.32) and (3.33), so the time dependences of \( C_j \) and \( \bar{C}_j \) are given by

\[ C_j(t) = C_j(0)e^{[(\mu_j-1)b+(1-\mu_j^{-1})a]t} \quad (3.34) \]

and

\[ \bar{C}_j(t) = \bar{C}_j(0)e^{-[(\bar{\mu}_j-1)b+(1-\bar{\mu}_j^{-1})a]t}, \quad (3.35) \]

respectively.

Substituting (3.32)–(3.35) into (3.23) and (3.30), we obtain the explicitly
time-dependent forms of $F(n)$ and $\bar{F}(n)$ as

$$F(n, t) = \frac{1}{2\pi i} \oint_C B(\mu, 0) A(\mu, 0)^{-1} \mu^n e^{[(\mu^{-1})^{(b+1-\mu^{-1})}a]t} d\mu$$

$$- \sum_{j=1}^{N} C_j(0) \mu_j^n e^{[(\mu_j^{-1})^{(b+1-\mu_j^{-1})}a]t},$$

(3.36)

$$\bar{F}(n, t) = \frac{1}{2\pi i} \oint_C \bar{B}(\mu, 0) \bar{A}(\mu, 0)^{-1} \mu^{-n-2} e^{-(\mu^{-1})^{(b+1-\mu^{-1})}a]t} d\mu$$

$$+ \sum_{j=1}^{\tilde{N}} \bar{C}_j(0) \mu_j^{-n-2} e^{-(\bar{\mu}_j^{-1})^{(b+1-\bar{\mu}_j^{-1})}a]t}.$$

(3.37)

Thus, it is easy to see that $F(n, t)$ and $\bar{F}(n, t)$ satisfy the pair of uncoupled linear evolution equations:

$$\begin{cases}
\frac{\partial F(n, t)}{\partial t} - bF(n + 1, t) + aF(n - 1, t) + (b - a)F(n, t) = O, \\ 
\frac{\partial \bar{F}(n, t)}{\partial t} - a\bar{F}(n + 1, t) + b\bar{F}(n - 1, t) + (a - b)\bar{F}(n, t) = O.
\end{cases}$$

(3.38a, 3.38b)

Note that these equations coincide with the linear part of the equations for $R_n$ and $Q_n$ (see (2.3)). In addition, $F(n, t)$ and $\bar{F}(n, t)$ are required to decay rapidly as $n \to +\infty$ so that the Gel'fand–Levitan–Marchenko equations (3.22) and (3.29) are well-posed.

Because of the linear nature of the sum terms in (3.36) and (3.37), we can take a coalescence limit of two or more simple poles of $A(\mu)^{-1}$ and $\bar{A}(\mu)^{-1}$ directly. Thus, we obtain the following generalized expressions for $F(n, t)$ and $\bar{F}(n, t)$:

$$F(n, t) = \frac{1}{2\pi i} \oint_C B(\mu, 0) A(\mu, 0)^{-1} \mu^n e^{[(\mu^{-1})^{(b+1-\mu^{-1})}a]t} d\mu$$

$$- \sum_{j=1}^{N} \sum_{k=0}^{M_j} C_j^{(k)}(0) \left( \frac{\partial}{\partial \mu_j} \right)^k \mu_j^n e^{[(\mu_j^{-1})^{(b+1-\mu_j^{-1})}a]t},$$

$$\bar{F}(n, t) = \frac{1}{2\pi i} \oint_C \bar{B}(\mu, 0) \bar{A}(\mu, 0)^{-1} \mu^{-n-2} e^{-(\mu^{-1})^{(b+1-\mu^{-1})}a]t} d\mu$$

$$+ \sum_{j=1}^{\tilde{N}} \sum_{k=0}^{L_j} \bar{C}_j^{(k)}(0) \left( \frac{\partial}{\partial \bar{\mu}_j} \right)^k \mu_j^{-n-2} e^{-(\bar{\mu}_j^{-1})^{(b+1-\bar{\mu}_j^{-1})}a]t}.$$

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These expressions encompass the most general case where $A(\mu)^{-1}$ and $\overline{A}(\mu)^{-1}$ have arbitrarily higher order poles. Moreover, they satisfy the same linear equations (3.38) as the original expressions (3.36) and (3.37).

Instead of using the partial differentiation with respect to $\mu_j$ and $\overline{\mu}_j$, one can consider the matrix functions $X^ne^{bt(X-I)+at(I-X^{-1})}$ and $Y^{-n-2}e^{-bt(Y-I)-at(I-Y^{-1})}$ with constant invertible matrices $X$ and $Y$ in Jordan normal form. Indeed, they satisfy the linear equations of the form (3.38), so linear combinations of the independent elements of each matrix function can be used to replace the sum terms in $F(n,t)$ and $\bar{F}(n,t)$. Readers interested in such an approach are referred, e.g., to [41,47–52].

### 3.5 Exact linearization

To reconstruct the potentials $Q_n$ and $R_n$ from the scattering data through (3.12) and (3.13), we rewrite the Gel’fand–Levitan–Marchenko equations (3.22) and (3.29) as “closed” linear summation equations for $K_1(n,m)$ and $\overline{K}_2(n,m)$. Thus, the general solution formulas for the matrix Ablowitz–Ladik lattice (2.3) can be presented in the form:

\[
Q_n = K_1(n,n), \tag{3.39a}
\]

\[
R_n = \overline{K}_2(n,n), \tag{3.39b}
\]

\[
K_1(n,m) = \bar{F}(m) - \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} K_1(n,n+i)F(n+i+k+1)\bar{F}(m+k+1), \quad m \geq n, \tag{3.39c}
\]

\[
\overline{K}_2(n,m) = -F(m) - \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \overline{K}_2(n,n+i)\bar{F}(n+i+k+1)\bar{F}(m+k+1), \quad m \geq n. \tag{3.39d}
\]

Here, the time dependence of the functions is suppressed; $F(n)$ and $\bar{F}(n)$ are solutions of the linear uncoupled system (3.38) and decay rapidly as $n \to +\infty$. More generally, the set of formulas (3.39) can provide the solutions for any flow of the matrix Ablowitz–Ladik hierarchy if $F(n)$ and $\bar{F}(n)$ satisfy the linear part of the equations for $R_n$ and $Q_n$, instead of (3.38). Hence, (3.39) realizes an exact linearization of the matrix Ablowitz–Ladik hierarchy in the sense of [10] (also see [53,54] and Proposition A.2). As long as $F(n)$ and $\bar{F}(n)$ decay rapidly as $n \to +\infty$, so do $Q_n$ and $R_n$ determined by (3.39). However, the requirement that $Q_n$ and $R_n$ should also decay as $n \to -\infty$ imposes nontrivial conditions on $F(n)$ and $\bar{F}(n)$, which will be touched upon in the next subsection.
3.6 Multisoliton solutions

To construct exact solutions in explicit form, we consider the special case of $B(\mu) = \overline{B}(\mu) = 0$ on $|\mu| = 1$; this is preserved under the time evolution (cf. (3.32) and (3.33)) and corresponds to the reflectionless potentials (cf. (3.7)). Moreover, we assume that $A(\mu)^{-1}$ and $\overline{A}(\mu)^{-1}$ only have simple poles (see [41] for the more general case). Thus, we can set

$$F(n, t) = -\sum_{j=1}^{N} C_j(t) \mu_j^n, \quad \overline{F}(n, t) = \sum_{j=1}^{N} \overline{C}_j(t) \overline{\mu}_j^{-n-2}, \quad (3.40)$$

where the time dependences of $C_j$ and $\overline{C}_j$ are given by (3.34) and (3.35). We also set

$$K_1(n, m; t) = \sum_{j=1}^{N} G_j(n, t) \overline{\mu}_j^{-m-2}, \quad \overline{K}_2(n, m; t) = \sum_{j=1}^{N} H_j(n, t) \mu_j^n, \quad (3.41)$$

and substitute all these expressions into (3.39c) and (3.39d); recalling that $|\mu_j| < 1$ ($j = 1, 2, \ldots, N$) and $|\overline{\mu}_j| > 1$ ($j = 1, 2, \ldots, \overline{N}$), we can evaluate the infinite sum. Thus, we obtain a linear algebraic system for determining $G_j$ and that for $H_j$ as

$$\begin{bmatrix} G_1 \overline{\mu}_1^{-n-2} & G_2 \overline{\mu}_2^{-n-2} & \cdots & G_N \overline{\mu}_N^{-n-2} \end{bmatrix} \begin{bmatrix} U_{11} & \cdots & U_{1\overline{N}} \\ \vdots & \ddots & \vdots \\ U_{N1} & \cdots & U_{N\overline{N}} \end{bmatrix} = \begin{bmatrix} C_1 \mu_1^n \\ \vdots \\ C_N \mu_N^n \end{bmatrix}, \quad (3.42a)$$

and

$$\begin{bmatrix} H_1 \mu_1^n & H_2 \mu_2^n & \cdots & H_N \mu_N^n \end{bmatrix} \begin{bmatrix} V_{11} & \cdots & V_{1N} \\ \vdots & \ddots & \vdots \\ V_{N1} & \cdots & V_{NN} \end{bmatrix} = \begin{bmatrix} C_1 \mu_1^n & C_2 \mu_2^n & \cdots & C_N \mu_N^n \end{bmatrix}. \quad (3.42b)$$

Here, all the entries in (3.42) are $l \times l$ matrices; the block matrices $U = (U_{jk})_{1 \leq j, k \leq \overline{N}}$ and $V = (V_{jk})_{1 \leq j, k \leq N}$ are defined as

$$U_{jk} := \delta_{jk} I - \sum_{i=1}^{N} \frac{\mu_i^{n+1} \overline{\mu}_i^{-n-3}}{1 - \frac{\mu_i}{\overline{\mu}_j}} \left( 1 - \frac{\mu_i}{\overline{\mu}_k} \right) C_i(t) \overline{C}_k(t),$$
and

\[ V_{jk} := \delta_{jk}I - \sum_{i=1}^{\bar{N}} \frac{\bar{p}_i^{-n-3} \mu_i^{n+1}}{(1 - \mu_j / \bar{p}_i)(1 - \mu_k / \bar{p}_i)} \bar{C}_i(t)C_k(t), \]

respectively. Here, \( \delta_{jk} \) denotes the Kronecker delta. Thus, using (3.39a), (3.39b) and (3.41), we obtain

\[ Q_n(t) = K_1(n, n; t) \]

\[ = \begin{bmatrix} G_1 \bar{p}_1^{-n-2} & \cdots & G_{\bar{N}} \bar{p}_{\bar{N}}^{-n-2} \end{bmatrix} \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} \]

\[ = \begin{bmatrix} \bar{C}_1(t) \bar{p}_1^{-n-2} & \cdots & \bar{C}_{\bar{N}}(t) \bar{p}_{\bar{N}}^{-n-2} \end{bmatrix} \begin{bmatrix} U_{11} & \cdots & U_{1\bar{N}} \\ \vdots & \ddots & \vdots \\ U_{\bar{N}1} & \cdots & U_{\bar{N}\bar{N}} \end{bmatrix}^{-1} \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix}, \]

(3.43a)

\[ R_n(t) = K_2(n, n; t) \]

\[ = \begin{bmatrix} H_1 \mu_1^n & \cdots & H_\bar{N} \mu_\bar{N}^n \end{bmatrix} \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} \]

\[ = \begin{bmatrix} C_1(t) \mu_1^n & \cdots & C_{\bar{N}}(t) \mu_{\bar{N}}^n \end{bmatrix} \begin{bmatrix} V_{11} & \cdots & V_{1\bar{N}} \\ \vdots & \ddots & \vdots \\ V_{\bar{N}1} & \cdots & V_{\bar{N}\bar{N}} \end{bmatrix}^{-1} \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix}. \]

(3.43b)

This provides the multisoliton solutions of the nonreduced matrix Ablowitz–Ladik lattice (2.3); some additional conditions on \( \{\mu_j, C_j\}_{j=1,2,\ldots,\bar{N}} \) and \( \{\bar{p}_j, \bar{C}_j\}_{j=1,2,\ldots,\bar{N}} \) need to be satisfied for (3.43) to exhibit solitonic behavior.

In the simplest nontrivial case of \( N = \bar{N} = 1 \), we obtain the one-soliton solution of (2.3) in the form:

\[ Q_n(t) = \bar{D}(n, t) \begin{bmatrix} I - \frac{\mu_1}{\bar{p}_1} \left( 1 - \frac{\mu_1}{\bar{p}_1} \right) \bar{D}(n, t) \bar{D}(n, t) \end{bmatrix}^{-1}, \]

(3.44a)

\[ R_n(t) = D(n, t) \begin{bmatrix} I - \frac{\mu_1}{\bar{p}_1} \left( 1 - \frac{\mu_1}{\bar{p}_1} \right) D(n, t) D(n, t) \end{bmatrix}^{-1}, \]

(3.44b)
where $\bar{D}(n, t) := \bar{C}_1(0)\bar{\mu}_1^{-n-2}e^{-(\bar{\mu}_1^{-1})b+(1-\bar{\mu}_1^{-1})a)t}$ and $D(n, t) := C_1(0)\mu_j^n e^{[(\mu_1-1)b+(1-\mu_1^{-1})a]t}$.

For this solution to decay also as $n \to -\infty$, we require that $\lim_{n \to -\infty} Q_n(t)l = 0$ for any $n$-independent column vector $l$ of dimension $l$ and similar for $R_n(t)$.

Thus, considering the Maclaurin series for $(I - X)^{-1}$ where $X$ is the corresponding matrix in (3.44a) or (3.44b), we obtain the following conditions on the kernels of the $l \times l$ matrices $\bar{C}_1$ and $C_1$:

$$\text{Ker} \left( \bar{C}_1 C_1 \right) = \text{Ker} \left( \bar{C}_1 \right)$$

and

$$\text{Ker} \left( C_1 \bar{C}_1 C_1 \right) = \text{Ker} \left( C_1 \right).$$

Note that these conditions remain invariant under the time evolution (cf. (3.34) and (3.35)). They can also be written in a more easy-to-understand form:

$$\text{Ker} \left( \bar{C}_1 \right) \cap \text{Im} \left( \bar{C}_1 \right) = \text{Ker} \left( \bar{C}_1 \right) \cap \text{Im} \left( C_1 \right) = \{0\}.$$  

Consequently, we have $\text{rank} \left( \bar{C}_1 \right) = \text{rank} \left( C_1 \right)$.

For general values of $N$ and $\bar{N}$, it is rather difficult to grasp the condition that $Q_n$ and $R_n$ given by (3.43) should also decay as $n \to -\infty$. Thus, we take a different route. In view of the first component of (3.29) and the second component of (3.22), relations (3.40) and (3.41) together with (3.11) imply that

$$[z^n \bar{\psi}_n] (\bar{\mu}_j) C_j \bar{\mu}_j^{-n-2} = \begin{bmatrix} G_j(n, t) \bar{\mu}_j^{-n-2} \\ \ast \end{bmatrix}, \quad j = 1, 2, \ldots, \bar{N},$$

$$[z^{-n} \bar{\psi}_n] (\mu_j) C_j \mu_j^n = \begin{bmatrix} \ast \\ H_j(n, t) \mu_j^n \end{bmatrix}, \quad j = 1, 2, \ldots, N.$$  

Thus, $G_j$ and $H_j$ are closely related to the bound-state eigenfunctions. Owing to the connection formulas (3.28) and (3.21) as well as the boundary conditions (3.5a), $G_j \bar{\mu}_j^{-n-2}$ and $H_j \mu_j^n$ must decay as $n \to -\infty$. If this is satisfied, then $Q_n$ and $R_n$ in (3.43) naturally vanish as $n \to -\infty$. The relations
(3.42) for determining $G_j \bar{\mu}_j^{n-2}$ and $H_j \mu_j^n$ can be rewritten as

$$
\begin{bmatrix}
G_1 & G_2 & \cdots & G_N
\end{bmatrix}
- \begin{bmatrix}
G_1 & G_2 & \cdots & G_N
\end{bmatrix}

\times
\begin{bmatrix}
\frac{1}{\beta_1 - \mu_1} C_1 \bar{C}_1 & \cdots & \frac{1}{\beta_{N} - \mu_1} C_1 \bar{C}_N
\vdots & \ddots & \vdots
\frac{1}{\beta_1 - \mu_N} C_N \bar{C}_1 & \cdots & \frac{1}{\beta_{N} - \mu_N} C_N \bar{C}_N
\end{bmatrix}
= \begin{bmatrix}
\bar{C}_1 & \bar{C}_2 & \cdots & \bar{C}_N
\end{bmatrix}
$$

(3.45a)

and

$$
\begin{bmatrix}
H_1 & H_2 & \cdots & H_N
\end{bmatrix}
- \begin{bmatrix}
H_1 & H_2 & \cdots & H_N
\end{bmatrix}

\times
\begin{bmatrix}
\frac{\mu_1^{n+1} \beta_1^{n-2}}{\beta_1} I & \frac{\mu_{N}^{n+1} \beta_{1}^{n-2}}{\beta_{1}} I
\vdots & \ddots & \vdots
\frac{\mu_1^{n+1} \beta_{N}^{n-2}}{\beta_{N}} I & \frac{\mu_{N}^{n+1} \beta_{N}^{n-2}}{\beta_{N}} I
\end{bmatrix}
= \begin{bmatrix}
C_1 & C_2 & \cdots & C_N
\end{bmatrix}
$$

(3.45b)

Then, we multiply both sides of (3.45a) from the right by an $n$-independent column vector of dimension $l \times N$ and consider the limit $n \to -\infty$. Thus, we obtain the condition

$$\text{Ker} \begin{bmatrix}
\frac{1}{\beta_1 - \mu_1} C_1 \bar{C}_1 & \cdots & \frac{1}{\beta_{N} - \mu_1} C_1 \bar{C}_N
\vdots & \ddots & \vdots
\frac{1}{\beta_1 - \mu_N} C_N \bar{C}_1 & \cdots & \frac{1}{\beta_{N} - \mu_N} C_N \bar{C}_N
\end{bmatrix}
\subseteq \text{Ker} \begin{bmatrix}
\bar{C}_1 & \bar{C}_2 & \cdots & \bar{C}_N
\end{bmatrix}.$$  

(3.46a)

Similarly, from (3.45b), we obtain

$$\text{Ker} \begin{bmatrix}
\frac{1}{\beta_1 - \mu_1} \bar{C}_1 C_1 & \cdots & \frac{1}{\beta_{N} - \mu_1} \bar{C}_1 C_N
\vdots & \ddots & \vdots
\frac{1}{\beta_1 - \mu_N} \bar{C}_N C_1 & \cdots & \frac{1}{\beta_{N} - \mu_N} \bar{C}_N C_N
\end{bmatrix}
\subseteq \text{Ker} \begin{bmatrix}
C_1 & C_2 & \cdots & C_N
\end{bmatrix}.$$  

(3.46b)
The two conditions (3.46a) and (3.46b) can be combined and simplified to provide more easy-to-understand conditions on the scattering data in the reflectionless case. For this purpose, we need the following lemma.

**Lemma 3.1.** Define the \((M - 1) \times M\) matrix elements \(d_{ij} \in \mathbb{C}\) as

\[
d_{ij} := \frac{\langle a_i, b_j \rangle}{\lambda_i - \nu_j}, \quad i \in \{i_1, i_2, \ldots, i_{M-1}\}, \quad j \in \{j_1, j_2, \ldots, j_M\}.
\]

Here, \(\lambda_i\) and \(\nu_j\) are parameters, \(a_i\) and \(b_j\) are nonzero vectors of dimension \(l\) and \(\langle \cdot, \cdot \rangle\) stands for the scalar product. For \(d_{ij}\) to be well-defined, we assume \(\lambda_i \neq \nu_j\) for all \(i\) and \(j\), but we do not require \(\lambda_{i\alpha} \neq \lambda_{i\beta}\) or \(\nu_{j\alpha} \neq \nu_{j\beta}\) for \(\alpha \neq \beta\). Instead, we assume the following condition: for any subset \(\{k_1, k_2, \ldots, k_\gamma\} \subseteq \{j_1, j_2, \ldots, j_M\}\) such that \(\nu_{k_1} = \nu_{k_2} = \cdots = \nu_{k_\gamma}\), the vectors

\[
b_{k_1}, b_{k_2}, \ldots, b_{k_\gamma}
\]

are linearly independent. Then, if the equality

\[
\sum_{\alpha=1}^{M} (-1)^{\alpha-1} \begin{vmatrix} d_{i_1 j_1} & \cdots & d_{i_1 j_{\alpha-1}} & d_{i_1 j_{\alpha+1}} & \cdots & d_{i_1 j_M} \\ \vdots & \ddots & \vdots & \vdots \\ d_{i_{M-1} j_1} & \cdots & d_{i_{M-1} j_{\alpha-1}} & d_{i_{M-1} j_{\alpha+1}} & \cdots & d_{i_{M-1} j_M} \end{vmatrix} b_{j_\alpha} = 0
\]

is valid, all the scalar coefficients must be zero, where \(\cdot\) stands for the determinant. In other words, the above vector equation holds true only in the trivial case; note that this equation can be written compactly as

\[
\begin{vmatrix} b_{j_1} & \cdots & b_{j_M} \\ d_{i_1 j_1} & \cdots & d_{i_1 j_M} \\ \vdots & \ddots & \vdots \\ d_{i_{M-1} j_1} & \cdots & d_{i_{M-1} j_M} \end{vmatrix} = 0,
\]

using the Laplace expansion formally.

We omit the proof of this lemma. To obtain useful information from the conditions (3.46), we first remove the trivial subspace of the kernels commonly contained on both sides. From a given \(l \times l\) matrix \(W\), we extract the maximum number of linearly independent column vectors to form an \(l \times \text{rank}(W)\) matrix \(W^{(c)}\). Similarly, we extract the maximum number of linearly independent row vectors from \(W\) to form a \(\text{rank}(W) \times l\) matrix \(W^{(r)}\). 

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With this notation, (3.46a) and (3.46b) can be rewritten in more compact forms as

\[
\text{Ker} \begin{bmatrix}
\frac{1}{\mu_1 - \mu_1} C_1^{(r)} & \cdots & \frac{1}{\mu_N - \mu_1} C_N^{(r)} \\
& \ddots & \vdots \\
\frac{1}{\mu_1 - \mu_N} C_1^{(c)} & \cdots & \frac{1}{\mu_N - \mu_N} C_N^{(c)}
\end{bmatrix} \subseteq \text{Ker} \begin{bmatrix}
\overline{C}_1^{(c)} & \overline{C}_2^{(c)} & \cdots & \overline{C}_N^{(c)}
\end{bmatrix}
\]

(3.47a)

and

\[
\text{Ker} \begin{bmatrix}
\frac{1}{\mu_1 - \mu_1} \overline{C}_1^{(r)} C_1^{(c)} & \cdots & \frac{1}{\mu_N - \mu_1} \overline{C}_1^{(r)} C_N^{(c)} \\
& \ddots & \vdots \\
\frac{1}{\mu_1 - \mu_N} \overline{C}_1^{(r)} C_1^{(c)} & \cdots & \frac{1}{\mu_N - \mu_N} \overline{C}_1^{(r)} C_N^{(c)}
\end{bmatrix} \subseteq \text{Ker} \begin{bmatrix}
C_1^{(c)} & C_2^{(c)} & \cdots & C_N^{(c)}
\end{bmatrix}.
\]

(3.47b)

With the aid of Lemma 3.1, we can prove the following two propositions.

**Proposition 3.2.** Assume that

\[
\sum_{j=1}^{N} \text{rank} \left( C_j \right) \leq \sum_{j=1}^{N} \text{rank} \left( \overline{C}_j \right),
\]

and the condition (3.47a) is satisfied. Then, the above inequality becomes an equality,

\[
\sum_{j=1}^{N} \text{rank} \left( C_j \right) = \sum_{j=1}^{N} \text{rank} \left( \overline{C}_j \right),
\]

(3.48)

and the matrix on the left-hand side of (3.47a) must be invertible, i.e.

\[
\begin{vmatrix}
\frac{1}{\mu_1 - \mu_1} C_1^{(r)} \overline{C}_1^{(c)} & \cdots & \frac{1}{\mu_N - \mu_1} C_1^{(r)} \overline{C}_N^{(c)} \\
& \ddots & \vdots \\
\frac{1}{\mu_1 - \mu_N} C_1^{(r)} \overline{C}_1^{(c)} & \cdots & \frac{1}{\mu_N - \mu_N} C_1^{(r)} \overline{C}_N^{(c)}
\end{vmatrix} \neq 0.
\]

(3.49)

**Proposition 3.3.** Assume that

\[
\sum_{j=1}^{N} \text{rank} \left( C_j \right) \geq \sum_{j=1}^{N} \text{rank} \left( \overline{C}_j \right),
\]

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and the condition (3.47b) is satisfied. Then, the above inequality becomes an equality,
\[
\sum_{j=1}^{N} \text{rank}(C_j) = \sum_{j=1}^{N} \text{rank}(\overline{C}_j),
\]
and the matrix on the left-hand side of (3.47b) must be invertible, i.e.
\[
\begin{vmatrix}
\frac{1}{\mu_1} \overline{C}_1^{(t)} C_1^{(c)} & \cdots & \frac{1}{\mu_N} \overline{C}_1^{(t)} C_N^{(c)} \\
\vdots & \ddots & \vdots \\
\frac{1}{\mu_1} \overline{C}_N^{(t)} C_1^{(c)} & \cdots & \frac{1}{\mu_N} \overline{C}_N^{(t)} C_N^{(c)}
\end{vmatrix} \neq 0. 
\tag{3.50}
\]

By combining Propositions 3.2 and 3.3, we arrive at the following theorem.

**Theorem 3.4.** Assume that (3.43) provides the multisoliton solutions of the matrix Ablowitz–Ladik lattice (2.3), which decay as \( n \to \pm \infty \) and produce the bound states of the associated eigenvalue problem (3.1). Then, the scattering data must satisfy the three conditions (3.48)–(3.50).

Note that the time evolution does not change these conditions (cf. (3.34) and (3.35)).

**Remark.** Here, we only considered the case where the time variable \( t \) is fixed at some finite value. In the limits \( t \to \pm \infty \), the solitons generally separate from one another and restore their original shapes. Thus, for (3.43) to describe proper multisoliton collisions through the passage of time, we have to impose additional conditions, i.e., the above conditions for subsets of \( \{\mu_j, C_j\}_{j=1,2,\ldots,N} \) and \( \{\mu_j, \overline{C}_j\}_{j=1,2,\ldots,N} \). Some relevant results were obtained independently in [41].

### 3.7 Complex conjugation reduction

When \( b = a^* \), the matrix Ablowitz–Ladik lattice (2.3) allows the complex conjugation reduction \( R_n = \sigma Q_n^* \) with a real constant \( \sigma \) (cf. [7]). In particular, the simplest reduction \( R_n = -Q_n^* \) can be realized in formulas (3.39) by identifying \( \overline{F}(n) \) with the complex conjugate of \( F(n) \), i.e.
\[
\overline{F}(n) = \{F(n)\}^*,
\tag{3.51}
\]
which is naturally preserved under the time evolution (3.38) with \( b = a^* \). Indeed, this relation can be derived by exploiting the symmetry of the eigenvalue problem (3.1) with \( R_n = -Q_n^* \); that is, if
\[
\begin{bmatrix}
\Psi_{1,n}(z) \\
\Psi_{2,n}(z)
\end{bmatrix}
\]
is an eigenfunction, then
\[
\pm \begin{bmatrix}
-\Psi_{2,n}(1/z^*) \\
\Psi_{1,n}(1/z^*)
\end{bmatrix}^*
\]
gives another eigenfunction of the same problem. Thus, we can reflect this symmetry in the Jost solutions and the scattering data to confirm (3.51).

In particular, the \( N \)-soliton solution of the matrix Ablowitz–Ladik equation,
\[
Q_{n,t} - aQ_{n+1} + a^*Q_{n-1} + (a - a^*)Q_n - aQ_nQ_n^*Q_{n+1} + a^*Q_{n-1}Q_n^*Q_n = 0,
\]
is obtained by setting \( b = a^* \), \( N = N \) and
\[
\overline{\mu}_j = \frac{1}{\mu_j^*}, \quad \overline{C}_j(t)\overline{\mu}_j^{-2} = -\{C_j(t)\}^*, \quad j = 1, 2, \ldots, N
\]
in formula (3.43a) with (3.34). The reduced set of scattering data is required to satisfy the conditions (3.49) and (3.50); in fact, they are equivalent under this reduction. In addition, (3.49) (or (3.50)) for subsets of \( \{\mu_j, C_j\}_{j=1,2,\ldots,N} \) should also be satisfied. Throughout this paper, we do not discuss the issue of regularity of solutions and the term “soliton solution” is used in a broad sense. That is, it may have singularities at some values of the independent variables \( n \) and \( t \).

4 Solution formulas for the derivative NLS lattices

4.1 Solutions of the space-discrete Gerdjikov–Ivanov system

In this subsection, we solve the space-discrete Gerdjikov–Ivanov system derived in subsection 2.2 by using the results in section 3. Note that the nonzero parameter \( \mu \) in the space-discrete Gerdjikov–Ivanov system (2.5) is nonessential; it can be fixed at any nonzero value, say 1, using a simple point
transformation and rescalings of the parameters $a$ and $b$ (cf. [17, 20]). In
addition, as is clear from the defining relation of the Miura map (2.4a), the
limit $\mu \to 0$ is trivial and need not be considered separately. Thus, in the
following, we consider the space-discrete Gerdjikov–Ivanov system (2.5) with
$\mu = 1$:

\[
\begin{align*}
Q_{n,t} & - aQ_{n+1} + bQ_{n-1} + (a - b)Q_n + aQ_n (P_n - P_{n+1}) Q_{n+1} \\
- bQ_{n-1} (P_n - P_{n+1}) Q_n + aQ_n P_n Q_n P_{n+1} Q_{n+1} - bQ_{n-1} P_n Q_n P_{n+1} Q_n = O, \\
P_{n,t} & - bP_{n+1} + aP_{n-1} + (b - a)P_n - bP_n (Q_{n-1} - Q_n) P_{n+1} \\
+ aP_{n-1} (Q_{n-1} - Q_n) P_n + bP_n Q_{n-1} P_n Q_n P_{n+1} - aP_{n-1} Q_{n-1} P_n Q_n P_n = O.
\end{align*}
\]

In section 3, we developed the inverse scattering method associated with
the matrix Ablowitz–Ladik eigenvalue problem (3.1) under the vanishing
boundary conditions on the potentials $Q_n$ and $R_n$ (cf. (3.2)). The remaining
unknown $P_n$ in (4.1) can be determined from a linear eigenfunction through
the simple formula

\[
P_n = \Psi_{2,n} \Psi_{1,n}^{-1} |_{\mu(z^2) = 1}.
\]

Because there is some arbitrariness in choosing the linear eigenfunction, we
need to specify boundary conditions to determine $P_n$ uniquely. Thus, we
assume that not only $Q_n$ but also $P_n$ decays rapidly as $n \to \pm \infty$:

\[
\lim_{n \to \pm \infty} Q_n = \lim_{n \to \pm \infty} P_n = O.
\]

This is consistent if we choose the linear eigenfunction appearing in the right-
hand side of (4.2) as

\[
\begin{bmatrix}
\Psi_{1,n} \\
\Psi_{2,n}
\end{bmatrix} := \psi_n,
\]

and assume that the scattering data $B(\mu)$ vanishes at $\mu = 1$, i.e., $B(1) = O$
(see (3.5) and (3.7a)). In fact, the Jost solution $\psi_n$ as well as $\phi_n$ does not
satisfy the time part of the Lax representation (2.2), so it is more appropriate
to use the explicitly time-dependent Jost solutions introduced in subsection 3.4. However, the overall multiplicative factor $e^{(-\mu+1)bt}$ as introduced in
(3.31) plays no role in formula (4.2), so in view of (3.11b), we can express $P_n$ as

\[
P_n = \left\{ \sum_{k=0}^{\infty} K_2(n, n+k) \right\} \left\{ I + \sum_{k=0}^{\infty} K_1(n, n+k) \right\}^{-1}.
\]

This expression enables us to determine $P_n$ from the set of scattering data
with the aid of the Gel’fand–Levitan–Marchenko equations (3.22) and (3.29);
however, for later convenience, we take an alternative approach.
Because the space-discrete Gerdjikov–Ivanov system (4.1) is “symmetric” with respect to $Q_n$ and $P_n$, there must be a formula for expressing $Q_n$ in a manner similar to (4.2). Such a formula can be established by identifying an appropriate Ablowitz–Ladik eigenvalue problem that is gauge equivalent to the original problem (3.1); the corresponding gauge transformation is often referred to as a Bäcklund–Darboux transformation. Then, in the new gauge, the roles of $Q_n$ and $P_n$ are swapped and $P_n$ appears directly as a potential in the Ablowitz–Ladik eigenvalue problem. In other words, there exists another Miura map from the space-discrete Gerdjikov–Ivanov system (4.1) to the Ablowitz–Ladik lattice in the form $(Q_n, P_n) \mapsto (\bar{Q}_n, \bar{P}_n)$.

Let us consider the original Ablowitz–Ladik eigenvalue problem (3.1) with $R_n = P_n - P_{n+1} + P_n Q_n P_{n+1}$ (cf. (2.4a)):

$$(AL1) : \begin{bmatrix} \Psi_{1,n} \\ \Psi_{2,n} \end{bmatrix} = \begin{bmatrix} zI & zQ_n \\ z^{-1}(P_n - P_{n+1} + P_n Q_n P_{n+1}) & z^{-1} I \end{bmatrix} \begin{bmatrix} \Psi_{1,n+1} \\ \Psi_{2,n+1} \end{bmatrix}. \quad (4.4)$$

Indeed, this can be rewritten as another Ablowitz–Ladik eigenvalue problem:

$$(AL2) : \begin{bmatrix} \Phi_{1,n} \\ \Phi_{2,n} \end{bmatrix} = \begin{bmatrix} zI & z(-Q_n + Q_{n+1} + Q_n P_{n+1} Q_{n+1}) \\ z^{-1} P_{n+1} & z^{-1} I \end{bmatrix} \begin{bmatrix} \Phi_{1,n+1} \\ \Phi_{2,n+1} \end{bmatrix}, \quad (4.5)$$

using the gauge transformation defined as

$$\begin{bmatrix} \Phi_{1,n} \\ \Phi_{2,n} \end{bmatrix} := \begin{bmatrix} (z^{-2} - 1) \Psi_{1,n} - Q_n (I - P_n Q_n)^{-1} (\Psi_{2,n} - z^{-2} P_n \Psi_{1,n}) \\ (I - P_n Q_n)^{-1} (\Psi_{2,n} - z^{-2} P_n \Psi_{1,n}) \end{bmatrix} =: g_n \begin{bmatrix} \Psi_{1,n} \\ \Psi_{2,n} \end{bmatrix}. \quad (4.6)$$

The explicit form of the transformation matrix $g_n$ is unimportant; only its asymptotic behavior as $n \to \pm \infty$ is needed:

$$\lim_{n \to \pm \infty} g_n = \begin{bmatrix} (z^{-2} - 1) I & O \\ O & I \end{bmatrix}.$$ 

For the original Jost solutions for (AL1) defined as (3.5), the Jost solutions for (AL2) are given as

$$\phi_n^{(AL2)}(z) = \frac{1}{z^{-2} - 1} g_n \phi_n^{(AL1)}(z), \quad \overline{\phi}_n^{(AL2)}(z) = g_n \overline{\phi}_n^{(AL1)}(z), \quad (4.7a)$$

$$\psi_n^{(AL2)}(z) = g_n \psi_n^{(AL1)}(z), \quad \overline{\psi}_n^{(AL2)}(z) = \frac{1}{z^{-2} - 1} g_n \overline{\psi}_n^{(AL1)}(z). \quad (4.7b)$$

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Indeed, they satisfy both the eigenvalue problem (4.5) and the boundary conditions (3.5). The case of \( z^2 = 1 \) can be understood in the corresponding limit.

In view of the aforementioned condition \( B(1) = 0 \), it is natural to modify the defining relations (3.7) of the scattering data for (AL1) on \( |\mu| = 1 \) as

\[
\begin{align*}
\phi_n^{(AL1)} &= \psi_n^{(AL1)} A(\mu) + \psi_n^{(AL1)} (\mu^{-1} - 1) B(\mu), \quad (4.8a) \\
\overline{\phi}_n^{(AL1)} &= \psi_n^{(AL1)} \overline{B}(\mu) - \psi_n^{(AL1)} \overline{A}(\mu). \quad (4.8b)
\end{align*}
\]

Then, relations (4.7) between the Jost solutions for (AL1) and those for (AL2) imply that the defining relations of the scattering data for (AL2) on \( |\mu| = 1 \) become

\[
\begin{align*}
\phi_n^{(AL2)} &= \psi_n^{(AL2)} A(\mu) + \psi_n^{(AL2)} B(\mu), \quad (4.9a) \\
\overline{\phi}_n^{(AL2)} &= \psi_n^{(AL2)} (\mu^{-1} - 1) \overline{B}(\mu) - \psi_n^{(AL2)} \overline{A}(\mu). \quad (4.9b)
\end{align*}
\]

The bound-state eigenvalues are determined by the positions of the simple poles of \( A(\mu)^{-1} \) in \( |\mu| < 1 \) and \( \overline{A}(\mu)^{-1} \) in \( |\mu| > 1 \); the more general case of higher order poles can be recovered by taking a suitable coalescence limit. Because of the uniqueness of the “analytic continuation”, the bound-state eigenvalues are common to (AL1) and (AL2):

\[
\begin{align*}
\mu_j^{(AL1)} &= \mu_j^{(AL2)} = \mu_j, \quad j = 1, 2, \ldots, N, \\
\mu_j^{(AL1)} &= \mu_j^{(AL2)} = \overline{\mu}_j, \quad j = 1, 2, \ldots, \overline{N}.
\end{align*}
\]

Owing to (4.7), the corresponding matrices \( C_j \) and \( \overline{C}_j \) (cf. (3.21) and (3.28)) for (AL1) and (AL2) can be expressed as

\[
\begin{align*}
C_j^{(AL1)} &= (\mu_j^{-1} - 1) C_j, \quad C_j^{(AL2)} = C_j, \quad j = 1, 2, \ldots, N, \quad (4.10a) \\
\overline{C}_j^{(AL1)} &= \overline{C}_j, \quad \overline{C}_j^{(AL2)} = (\overline{\mu}_j^{-1} - 1) \overline{C}_j, \quad j = 1, 2, \ldots, \overline{N}. \quad (4.10b)
\end{align*}
\]

By combining the above relations, the functions \( F(m) \) and \( \overline{F}(m) \) for (AL1) and (AL2) (cf. (3.23) and (3.30)) can be written as

\[
\begin{align*}
F^{(AL1)}(m) &= F(m - 1) - F(m), \quad F^{(AL2)}(m) = F(m), \quad (4.11a) \\
\overline{F}^{(AL1)}(m) &= \overline{F}(m), \quad \overline{F}^{(AL2)}(m) = \overline{F}(m + 1) - \overline{F}(m), \quad (4.11b)
\end{align*}
\]

in terms of the original \( F(m) \) and \( \overline{F}(m) \) defined as (3.23) and (3.30). Clearly, the modification of the scattering data for (AL1) and (AL2) as described
above does not change their time dependences given by (3.32)–(3.35). Thus, each of the pairs \((F, F)\), \((F^{(AL1)}, F^{(AL1)})\) and \((F^{(AL2)}, F^{(AL2)})\) satisfies the same linear evolutionary system (3.38).

We can construct the Gel’fand–Levitan–Marchenko equations for the space-discrete Gerdjikov–Ivanov system (4.1) by combining (3.39a) and (3.39c) for (AL1) and (3.39b) and (3.39d) for (AL2), i.e.

\[
Q_n = K_1^{(AL1)}(n, n),
\]

\[
P_{n+1} = \tilde{K}_2^{(AL2)}(n, n),
\]

\[
K_1^{(AL1)}(n, m) = F^{(AL1)}(m) - \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} K_1^{(AL1)}(n, n + i) F^{(AL1)}(n + i + k + 1) \times F^{(AL1)}(m + k + 1), \quad m \geq n,
\]

\[
\tilde{K}_2^{(AL2)}(n, m) = -F^{(AL2)}(m) - \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \tilde{K}_2^{(AL2)}(n, n + i) F^{(AL2)}(n + i + k + 1) \times F^{(AL2)}(m + k + 1), \quad m \geq n.
\]

Substituting (4.11) and changing the notation slightly, we obtain

\[
Q_n = \mathcal{K}(n, n), \quad (4.12a)
\]

\[
P_n = \overline{\mathcal{K}}(n, n), \quad (4.12b)
\]

\[
\mathcal{K}(n, m) = F(m) + \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \mathcal{K}(n, n + i) \{F(n + i + k + 1) - F(n + i + k)\} \times F(m + k + 1), \quad m \geq n, \quad (4.12c)
\]

\[
\overline{\mathcal{K}}(n, m) = -F(m - 1) - \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \overline{\mathcal{K}}(n, n + i) \{F(n + i + k + 1) - F(n + i + k)\} \times F(m + k), \quad m \geq n. \quad (4.12d)
\]

Note that \(F\) and \(\overline{F}\) satisfy the linear part of the equations for \(P_n\) and \(Q_n\), which is (3.38) for the space-discrete Gerdjikov–Ivanov system (4.1).

In the case of \(B(\mu) = \tilde{B}(\mu) = O\) on \(|\mu| = 1\), which corresponds to the reflectionless potentials for both (AL1) and (AL2), we can solve the Gel’fand–Levitan–Marchenko equations (4.12) to obtain the soliton solutions in closed form. The derivation is essentially the same as in the Ablowitz–Ladik case described in subsection 3.6; naturally, the multisoliton solutions of the space-discrete Gerdjikov–Ivanov system (4.1) can be obtained directly by applying
the correspondence relations (4.10) to (3.43), i.e.

\[ Q_n(t) = \begin{bmatrix} \overline{C_1(t)\overline{\mu}_1^{-n-2}} & \cdots & \overline{C_N(t)\overline{\mu}_N^{-n-2}} \end{bmatrix} \begin{bmatrix} \mathcal{U}_{11} & \cdots & \mathcal{U}_{1N} \\ \vdots & \ddots & \vdots \\ \mathcal{U}_{N1} & \cdots & \mathcal{U}_{NN} \end{bmatrix}^{-1} \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix}, \]  

(4.13a)

\[ P_n(t) = \begin{bmatrix} C_1(t)\mu_1^{n-1} & \cdots & C_N(t)\mu_N^{n-1} \end{bmatrix} \begin{bmatrix} \mathcal{V}_{11} & \cdots & \mathcal{V}_{1N} \\ \vdots & \ddots & \vdots \\ \mathcal{V}_{N1} & \cdots & \mathcal{V}_{NN} \end{bmatrix}^{-1} \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix}. \]  

(4.13b)

Here, all the entries in (4.13) are \( l \times l \) matrices; the block matrices \( \mathcal{U} = (\mathcal{U}_{jk})_{1 \leq j,k \leq N} \) and \( \mathcal{V} = (\mathcal{V}_{jk})_{1 \leq j,k \leq N} \) are defined as

\[ \mathcal{U}_{jk} := \delta_{jk}I - \sum_{i=1}^{N} \frac{(1 - \mu_i)\mu_i^n\overline{\mu}_k^{-n-3}}{(1 - \overline{\mu}_i\overline{\mu}_j)(1 - \mu_i\overline{\mu}_k)} C_i(t)\overline{C_k(t)}, \]

\[ \mathcal{V}_{jk} := \delta_{jk}I + \sum_{i=1}^{N} \frac{(1 - \overline{\mu}_i^{-1})\mu_i^{-n-2}\mu_k^n}{(1 - \mu_j\overline{\mu}_i)(1 - \mu_k\overline{\mu}_i)} \overline{C_i(t)}C_k(t), \]

and the time dependences of \( C_j \) and \( \overline{C}_j \) are given by (3.34) and (3.35). Note that the three conditions (3.48)–(3.50) must be satisfied for (4.13) to describe proper multisoliton solutions decaying as \( n \to \pm\infty \) (cf. Theorem 3.4). In addition, we need to impose similar conditions for subsets of the soliton parameters so that the solitons interact with each other properly throughout the time evolution.

When \( b = a^* \), the space-discrete Gerdjikov–Ivanov system (4.1) allows the complex conjugation reduction \( P_n = i\sigma Q_{n-1/2}^* \) with a real constant \( \sigma \) [20]. That is, two originally uncoupled systems, (4.1) with \( n \in \mathbb{Z} \) and (4.1) with \( n \in \mathbb{Z} + 1/2 \), can be related by this reduction to give a single equation with \( n \in \mathbb{Z}/2 \). Clearly, the value of \( \sigma \) is nonessential, so we set \( \sigma = 1 \) and consider the reduction \( P_n = iQ_{n-1/2}^* \). This reduction can be realized at the level of formulas (4.12) by setting

\[ \overline{F}(n) = -i \left\{ F \left( n - \frac{1}{2} \right) \right\}^*, \]

which is consistent with the time evolution (3.38) with \( b = a^* \). In particular,
the $N$-soliton solution of the space-discrete Gerdjikov–Ivanov equation,

$$Q_{n,t} - aQ_{n+1} + a^*Q_{n-1} + (a - a^*)Q_n + iaQ_n \left( Q_{n-\frac{1}{2}}^* - Q_{n+\frac{1}{2}}^* \right) Q_{n+1}$$

$$- ia^*Q_{n-1} \left( Q_{n-\frac{1}{2}}^* - Q_{n+\frac{1}{2}}^* \right) Q_n - aQ_nQ_{n-\frac{1}{2}}Q_{n+\frac{1}{2}}Q_{n+1} + a^*Q_{n-1}Q_{n-\frac{1}{2}}Q_{n+\frac{1}{2}}Q_n = 0,$$

is obtained by setting $b = a^*$, $\overline{N} = N$ and

$$\overline{m}_j = \frac{1}{\mu_j}, \quad \overline{C}_j(t)\overline{m}_j^{-2} = i \left\{ C_j(t)\mu_j^{-\frac{1}{2}} \right\}^*, \quad j = 1, 2, \ldots, N$$

in formula (4.13a) with (3.34). The imaginary unit (roman $i$) should not be confused with the index of summation (italic $i$) in the definition of $\Psi_{jk}$. The reduced set of scattering data is required to satisfy the condition (3.49) (or (3.50)) and its smaller versions corresponding to subsets of the solitons.

### 4.2 Solutions of the space-discrete Kaup–Newell system

In this subsection, we solve the space-discrete Kaup–Newell system (2.17) derived in subsection 2.3 by applying the results in section 3. Because the parameter $\mu$ is nonessential in (2.17), we set $\mu = 1$ and consider the space-discrete Kaup–Newell system in the form:

$$\begin{align*}
q_{n,t} - \Delta_n^+ \left[ a(I - q_n r_n)^{-1} q_n + b(I + q_{n-1} r_n)^{-1} q_{n-1} \right] &= O, \quad (4.14a) \\
r_{n,t} - \Delta_n^+ \left[ b(I + r_n q_{n-1})^{-1} r_n + a(I - r_{n-1} q_{n-1})^{-1} r_{n-1} \right] &= O. \quad (4.14b)
\end{align*}$$

Recall that $\Delta_n^+$ denotes the forward difference operator: $\Delta_n^+ f_n := f_{n+1} - f_n$. We assume vanishing boundary conditions at spatial infinity:

$$\lim_{n \to \pm \infty} q_n = \lim_{n \to \pm \infty} r_n = O. \quad (4.15)$$

Propositions 2.1 and 2.2 imply that the solution of (4.14) can be obtained as

$$\begin{align*}
q_n &= \Delta_n^+ \left( \Psi_{1,n}^{(1)} \Psi_{1,n}^{(2)} \right)_{\mu = 1}, \\
r_n &= \left( \Psi_{2,n}^{(1)} \Psi_{2,n}^{(2)} - \Psi_{1,n}^{(1)} \Psi_{1,n}^{(2)} \right)^{-1}_{\mu = 1}.
\end{align*} \quad (4.16)$$

using two linearly independent eigenfunctions of the linear problem, (2.1) and (2.2), associated with the Ablowitz–Ladik lattice (cf. (2.6)); this is in contrast to the space-discrete Gerdjikov–Ivanov system, which can be derived using only one linear eigenfunction as described in subsection 2.2. Proposition 2.3
implies that \( r_n \) in (4.16) can be rewritten in the difference form as \( q_n \); to see this explicitly, we need to identify an appropriate linear problem for the Ablowitz–Ladik lattice, which is gauge equivalent to the original problem. It turns out that the gauge transformation connecting (AL1) and (AL2) considered in subsection 4.1 plays the desired role.

Suppose that the two eigenfunctions appearing in (4.16) satisfy (AL1). Moreover, we choose the linear eigenfunction appearing in (4.2) as

\[
P_n = \Psi_{2,n}^{(1)} - \Psi_{1,n}^{(1)}.
\]

Recall that (AL1) and (AL2) involve this \( P_n \) and are connected through the gauge transformation (4.6). Thus, the two linear eigenfunctions for (AL2) can be introduced as

\[
\begin{bmatrix}
\Phi_{1,n}^{(1)} \\
\Phi_{2,n}^{(1)}
\end{bmatrix} := \frac{1}{z^{-2} - 1} g_n \begin{bmatrix}
\Psi_{1,n}^{(1)} \\
\Psi_{2,n}^{(1)}
\end{bmatrix}, \quad \begin{bmatrix}
\Phi_{1,n}^{(2)} \\
\Phi_{2,n}^{(2)}
\end{bmatrix} := g_n \begin{bmatrix}
\Psi_{1,n}^{(2)} \\
\Psi_{2,n}^{(2)}
\end{bmatrix},
\]

so that both of them become nontrivial in the limit \( \mu (= z^2) \to 1 \). Note that (4.5) and (4.6) imply that

\[
\Phi_{2,n}^{(2)} = z^{-1} P_{n+1} \Psi_{1,n+1}^{(1)} + z^{-1} (I - P_{n+1} Q_{n+1}) \Phi_{2,n+1}^{(1)},
\]

\[
\Phi_{2,n}^{(2)} = z^{-1} (z^{-2} - 1) P_{n+1} \Psi_{1,n+1}^{(2)} + z^{-1} (I - P_{n+1} Q_{n+1}) \Phi_{2,n+1}^{(2)}.
\]

Thus, the ratio between these quantities in the limit \( \mu (= z^2) \to 1 \) satisfies

\[
\Phi_{2,n}^{(2)} \Psi_{2,n}^{(1)} - \Phi_{2,n}^{(2)} \Psi_{2,n}^{(1)} \to 1 = \frac{\Psi_{2,n}^{(2)} - \Psi_{2,n}^{(1)}}{\Psi_{2,n}^{(2)} + \Psi_{2,n}^{(1)}},
\]

\[
\Phi_{2,n}^{(2)} \Psi_{2,n}^{(1)} - \Phi_{2,n}^{(2)} \Psi_{2,n}^{(1)} \to 1 = \frac{\Psi_{2,n}^{(2)} - \Psi_{2,n}^{(1)}}{\Psi_{2,n}^{(2)} + \Psi_{2,n}^{(1)}}.
\]

Therefore, the formula for determining \( r_n \) in (4.16) can be replaced with

\[
r_n = -\Delta_n^+ \left( \Phi_{2,n}^{(2)} \Phi_{2,n}^{(1)} \right) \bigg|_{\mu \to 1},
\]

which uses the two linear eigenfunctions for (AL2).
We set
\[
\begin{bmatrix}
\Psi_{1,n}^{(1)} \\
\Psi_{2,n}^{(1)}
\end{bmatrix} := \psi_n^{(AL1)},
\begin{bmatrix}
\Psi_{1,n}^{(2)} \\
\Psi_{2,n}^{(2)}
\end{bmatrix} := \psi_n^{(AL1)},
\] (4.19)
so that (cf. (4.17) and (4.7b))
\[
\begin{bmatrix}
\Phi_{1,n}^{(1)} \\
\Phi_{2,n}^{(1)}
\end{bmatrix} := \psi_n^{(AL2)},
\begin{bmatrix}
\Phi_{1,n}^{(2)} \\
\Phi_{2,n}^{(2)}
\end{bmatrix} := \psi_n^{(AL2)}.
\] (4.20)

In view of (3.5), (4.8) and (4.9), this choice is indeed consistent with the boundary conditions (4.15). To be precise, we should use the explicitly time-dependent Jost solutions introduced in subsection 3.4, which satisfy not only the Ablowitz–Ladik eigenvalue problem (3.1) but also the time-evolution equation (2.2); however, this makes no difference in the limit \( \mu \to 1 \) (cf. (3.31b)), so the above choice is valid in formulas (4.16) and (4.18). Thus, with the aid of (3.11), the solution of the space-discrete Kaup–Newell system (4.14) can be expressed as
\[
g_n = \Delta_n^+ \left\{ \left[ I + \sum_{k=0}^{\infty} \bar{K}_1^{(AL1)}(n, n + k) \right]^{-1} \sum_{k=0}^{\infty} K_1^{(AL1)}(n, n + k) \right\},
\] (4.21a)
\[
r_n = -\Delta_n^+ \left\{ \left[ I + \sum_{k=0}^{\infty} K_2^{(AL2)}(n - 1, n - 1 + k) \right]^{-1} \sum_{k=0}^{\infty} K_2^{(AL2)}(n - 1, n - 1 + k) \right\}.
\] (4.21b)

Recall that the infinite sums in (4.21) are assumed to be convergent; this is satisfied if the potentials in (AL1) and (AL2) decay sufficiently rapidly as \( n \to \pm \infty \) (cf. (3.11)). To realize an exact linearization of the space-discrete Kaup–Newell system (4.14), we introduce new quantities \( \mathcal{K}(n, m) \) and \( \bar{\mathcal{K}}(n, m) \) for \( m \geq n \) as
\[
\mathcal{K}(n, m) := \left[ I + \sum_{k=0}^{\infty} \bar{K}_1^{(AL1)}(n, n + k) \right]^{-1} \sum_{s=m}^{\infty} K_1^{(AL1)}(n, s),
\]
\[
\bar{\mathcal{K}}(n, m) := -\left[ I + \sum_{k=0}^{\infty} K_2^{(AL2)}(n - 1, n - 1 + k) \right]^{-1} \sum_{s=m-1}^{\infty} \bar{K}_2^{(AL2)}(n - 1, s),
\]
so that \( q_n = \Delta_n^+ \mathcal{K}(n, n) \) and \( r_n = \Delta_n^+ \mathcal{\tilde{K}}(n, n) \). Let us derive the linear summation equation for \( \mathcal{K}(n, m) \) from the Gel’fand–Levitan–Marchenko equations (3.22) and (3.29) for (AL1). From (3.22), we have

\[
\mathcal{\tilde{K}}_{1}^{(AL1)}(n, p) + \sum_{j=0}^{\infty} \mathcal{\tilde{K}}_{1}^{(AL1)}(n, n+j) F^{(AL1)}(p + j + 1) = O, \quad p \geq n.
\]

Thus, taking the sum with respect to \( p \), we obtain the relation

\[
\sum_{p=n+k}^{\infty} \mathcal{\tilde{K}}_{1}^{(AL1)}(n, p) = -\sum_{j=0}^{\infty} \left[ \sum_{s=n+j}^{\infty} \mathcal{\tilde{K}}_{1}^{(AL1)}(n, s) - \sum_{s=n+j+1}^{\infty} \mathcal{\tilde{K}}_{1}^{(AL1)}(n, s) \right] \times \sum_{p=n+k}^{\infty} F^{(AL1)}(p + j + 1). \quad (4.22)
\]

From (3.29), we have

\[
\mathcal{K}_{1}^{(AL1)}(n, s) = F^{(AL1)}(s) + \sum_{k=0}^{\infty} \mathcal{K}_{1}^{(AL1)}(n, n+k) F^{(AL1)}(s) - \sum_{k=0}^{\infty} \left[ \sum_{p=n+k}^{\infty} \mathcal{K}_{1}^{(AL1)}(n, p) \right] \left[ F^{(AL1)}(s+k) - F^{(AL1)}(s+k+1) \right], \quad s \geq n,
\]

where (a variant of) the summation by parts formula is used. Thus, taking the sum with respect to \( s \) and using the fact that \( F^{(AL1)}(n) \) decays rapidly as \( n \to +\infty \), we obtain

\[
\sum_{s=m}^{\infty} \mathcal{K}_{1}^{(AL1)}(n, s) = \left[ I + \sum_{k=0}^{\infty} \mathcal{K}_{1}^{(AL1)}(n, n+k) \right] \sum_{s=m}^{\infty} F^{(AL1)}(s) - \sum_{k=0}^{\infty} \left[ \sum_{p=n+k}^{\infty} \mathcal{K}_{1}^{(AL1)}(n, p) \right] F^{(AL1)}(m+k), \quad m \geq n.
\]

Substituting (4.22) into the last term and multiplying both sides from the left by \( \left[ I + \sum_{k=0}^{\infty} \mathcal{K}_{1}^{(AL1)}(n, n+k) \right]^{-1} \), we arrive at the linear summation equation for \( \mathcal{K}(n, m) \):

\[
\mathcal{K}(n, m) = \sum_{s=m}^{\infty} F^{(AL1)}(s) + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} [\mathcal{K}(n, n+j) - \mathcal{K}(n, n+j+1)] \times \sum_{p=n+k}^{\infty} F^{(AL1)}(p + j + 1) F^{(AL1)}(m+k), \quad m \geq n,
\]

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which can be rewritten using (4.11) as

\[ K(n, m) = \sum_{s=m}^{\infty} F(s) + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} [K(n, n+j) - K(n, n+j+1)] F(n+j+k) F(m+k), \quad m \geq n. \]

In a similar way, we can derive the linear summation equation for \( K(n, m) \) from the Gel’fand–Levitan–Marchenko equations (3.22) and (3.29) for (AL2) as

\[ \bar{K}(n, m) = \sum_{s=m-1}^{\infty} F^{(AL2)}(s) + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} [\bar{K}(n, n+j) - \bar{K}(n, n+j+1)] \]

\[ \times \sum_{p=n+k-1}^{\infty} F^{(AL2)}(p+j+1) F^{(AL2)}(m+k-1) \]

\[ = \sum_{s=m-1}^{\infty} F(s) - \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} [\bar{K}(n, n+j) - \bar{K}(n, n+j+1)] \bar{F}(n+j+k) F(m+k-1), \quad m \geq n, \]

where (4.11) is used. Combining the above results, we obtain a set of formulas for the solutions of the space-discrete Kaup–Newell system (4.14), which tend to zero as \( n \to +\infty \), in the form [21]:

\[ q_n = \Delta^+ K(n, n), \quad (4.23a) \]

\[ r_n = \Delta^+ \bar{K}(n, n), \quad (4.23b) \]

\[ K(n, m) = \bar{F}(m) + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} [K(n, n+j) - K(n, n+j+1)] \]

\[ \times [\bar{F}(n+j+k+1) - \bar{F}(n+j+k+2)] \bar{F}(m+k) - \bar{F}(m+k+1), \quad m \geq n, \]

\[ \bar{K}(n, m) = F(m) - \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} [\bar{K}(n, n+j) - \bar{K}(n, n+j+1)] \]

\[ \times [\bar{F}(n+j+k) - \bar{F}(n+j+k+1)] [F(m+k) - F(m+k+1)], \quad m \geq n. \]

\[ (4.23c) \]

\[ (4.23d) \]

Here, the functions \( F(n) \) and \( \bar{F}(n) \) are defined as \( F(n) := \sum_{s=n-1}^{\infty} F(s) \) and \( \bar{F}(n) := \sum_{s=n}^{\infty} \bar{F}(s) \), respectively; they satisfy the same linear evolutionary
system as $F(n)$ and $F(n)$ (cf. (3.38)) and decay rapidly as $n \to +\infty$. Note that the set of formulas (4.23) can provide the solutions for any flow of the space-discrete Kaup–Newell hierarchy if $F(n)$ and $\overline{F}(n)$ satisfy the corresponding linear system.

In the same way as for the other lattice systems, we can derive the multisoliton solutions of the space-discrete Kaup–Newell system (4.14) from the set of formulas (4.23); this corresponds to the special case of reflectionless potentials for both (AL1) and (AL2). For simplicity, we assume that $A(\mu)^{-1}$ and $\overline{A}(\mu)^{-1}$ only have simple poles and set

$$ F(n, t) = \sum_{j=1}^{N} C_j(t) \mu_j^n, \quad \overline{F}(n, t) = \sum_{j=1}^{N} \overline{C}_j(t) \overline{\mu}_j^{-n}, \quad (4.24) $$

where the time dependences of $C_j$ and $\overline{C}_j$ are given as (3.34) and (3.35), i.e.

$$ C_j(t) = C_j(0) e^{[(\mu_j-1) + (1-\mu_j^{-1})]t}, \quad \overline{C}_j(t) = \overline{C}_j(0) e^{[-(\overline{\mu}_j-1) + (1-\overline{\mu}_j^{-1})]t}. $$

We also set

$$ K(n, m; t) = \sum_{j=1}^{N} G_j(n, t) \mu_j^{-m}, \quad \overline{K}(n, m; t) = \sum_{j=1}^{N} H_j(n, t) \mu_j^m, \quad (4.25) $$

and substitute the expressions (4.24) and (4.25) into (4.23c) and (4.23d). Because $|\mu_j| < 1$ ($j = 1, 2, \ldots, N$) and $|\overline{\mu}_j| > 1$ ($j = 1, 2, \ldots, N$), we can evaluate the infinite sum to obtain linear algebraic systems for determining $G_j$ and $H_j$, respectively. They can be written as

$$ \begin{bmatrix} G_1 \overline{\mu}_1^{-n} & G_2 \overline{\mu}_2^{-n} & \cdots & G_N \overline{\mu}_N^{-n} \end{bmatrix} \begin{bmatrix} U_{11} & \cdots & U_{1N} \\ \vdots & \ddots & \vdots \\ U_{N1} & \cdots & U_{NN} \end{bmatrix} = \begin{bmatrix} \overline{C}_1 \overline{\mu}_1^{-n} & \overline{C}_2 \overline{\mu}_2^{-n} & \cdots & \overline{C}_N \overline{\mu}_N^{-n} \end{bmatrix} \quad (4.26a) $$

and

$$ \begin{bmatrix} H_1 \mu_1^n & H_2 \mu_2^n & \cdots & H_N \mu_N^n \end{bmatrix} \begin{bmatrix} V_{11} & \cdots & V_{1N} \\ \vdots & \ddots & \vdots \\ V_{N1} & \cdots & V_{NN} \end{bmatrix} = \begin{bmatrix} C_1 \mu_1^n & C_2 \mu_2^n & \cdots & C_N \mu_N^n \end{bmatrix}. \quad (4.26b) $$
Here, all the entries in (4.26) are \( l \times l \) matrices; the block matrices \( \mathbf{U} = (\mathbf{U}_{jk})_{1\leq j,k \leq N} \) and \( \mathbf{V} = (\mathbf{V}_{jk})_{1\leq j,k \leq N} \) are defined as

\[
\mathbf{U}_{jk} := \delta_{jk} \mathbf{I} - \sum_{i=1}^{N} \frac{(1 - \mathbf{U}_{ij}^{-1})(1 - \mathbf{U}_{ik}^{-1})}{(1 - \mathbf{U}_{ii})(1 - \mathbf{U}_{kk})} \mu_i^{n+1} \mathbf{C}_i(t) \mathbf{C}_k(t),
\]

and

\[
\mathbf{V}_{jk} := \delta_{jk} \mathbf{I} + \sum_{i=1}^{N} \frac{(1 - \mathbf{U}_{ij})(1 - \mathbf{U}_{ik})(1 - \mathbf{U}_{ii})}{(1 - \mathbf{U}_{ii})(1 - \mathbf{U}_{kk})} \mu_i^{-n} \mathbf{C}_i(t) \mathbf{C}_k(t),
\]

respectively. Thus, with the aid of (4.23a), (4.23b) and (4.25), we obtain the multisoliton solutions of the space-discrete Kaup–Newell system (4.14) in the difference form:

\[
q_n(t) = \Delta_n^+ \left\{ \mathbf{C}_1(t) \mathbf{C}_2^{-n} \cdots \mathbf{C}_N(t) \mathbf{C}_N^{-n} \right\}^{-1} \left[ \begin{array}{c} \mathbf{U}_{11} \cdots \mathbf{U}_{1N} \\ \vdots \quad \ddots \\ \mathbf{U}_{N1} \cdots \mathbf{U}_{NN} \end{array} \right] \left[ \begin{array}{c} I \\ \vdots \\ I \end{array} \right],
\]

(4.27a)

\[
r_n(t) = \Delta_n^+ \left\{ \mathbf{C}_1(t) \mu_1^n \cdots \mathbf{C}_N(t) \mu_N^n \right\}^{-1} \left[ \begin{array}{c} \mathbf{V}_{11} \cdots \mathbf{V}_{1N} \\ \vdots \quad \ddots \\ \mathbf{V}_{N1} \cdots \mathbf{V}_{NN} \end{array} \right] \left[ \begin{array}{c} I \\ \vdots \\ I \end{array} \right].
\]

(4.27b)

When \( b = a^* \), the space-discrete Kaup–Newell system (4.14) allows not only the complex conjugation reduction \( r_n = \text{i} \sigma q_n^{*} \) but also the Hermitian conjugation reduction \( r_n = \text{i} \sigma q_n^{\dagger} \), where \( \sigma \) is a real constant [20]. Each reduction relates two originally uncoupled systems, (4.14) with \( n \in \mathbb{Z} \) and (4.14) with \( n \in \mathbb{Z} + 1/2 \), to provide a single equation with \( n \in \mathbb{Z}/2 \). Clearly, the value of \( \sigma \) is nonessential, so we set \( \sigma = 1 \) and consider the Hermitian conjugation reduction \( r_n = \text{i} q_n^{\dagger} \). This reduction can be realized at the level of formulas (4.23) by setting

\[
\mathcal{F}(n) = \text{i} \left\{ \mathcal{F} \left( n + \frac{1}{2} \right) \right\}^{\dagger},
\]

which is consistent with the time evolution (cf. (3.38) with \( b = a^* \)). In particular, the \( N \)-soliton solution of the space-discrete Kaup–Newell equation,

\[
q_n, t = \Delta_n^+ \left[ a \left( I - \text{i} q_n^{\dagger} \right)^{-1} q_n + a^* \left( I + \text{i} q_{n-1}^{\dagger} \right)^{-1} q_{n-1} \right] = O, \quad (4.28)
\]
is obtained by setting $b = a^*$, $\bar{N} = N$ and
\[
\bar{\mu}_j = \frac{1}{\mu_j}, \quad \bar{C}_j(t) = i \left\{ C_j(t) \mu_j^{\frac{1}{2}} \right\}^\dagger, \quad j = 1, 2, \ldots, N
\]
in formula (4.27a), where $C_j(t) = C_j(0)e^{[(\mu_j-1)a^*+(1-\mu_j^*)a]t}$. The imaginary unit (roman $i$) should not be confused with the index of summation (italic $i$) in the definitions of $U_{jk}$ and $V_{jk}$. The $N$-soliton solution thus obtained is a space-discrete analog of the $N$-soliton solution of the continuous matrix Kaup–Newell equation reported in [14]. Note that the square matrix equation (4.28) can be further reduced to a vector equation by setting all but one of the columns (or rows) of $q_n$ to zero; its $N$-soliton solution can be obtained in the same way as described in [55].

5 Concluding remarks

In this paper, we have developed the inverse scattering method associated with the matrix Ablowitz–Ladik eigenvalue problem and its applications to space-discrete analogs of derivative NLS systems. In particular, the most streamlined version of the inverse scattering method on a lattice is formulated, which can avoid redundant processes present in the existing literature. Thus, we are now able to understand the inverse scattering method for the Ablowitz–Ladik lattice as a direct discrete analog of the inverse scattering method for the continuous NLS system; in essence, the discrete case is no longer more complicated than the continuous case. Moreover, we can characterize the space-discrete derivative NLS systems using the potentials and linear eigenfunctions appearing in the Lax representation for the Ablowitz–Ladik lattice. On the basis of this characterization, we can solve the space-discrete derivative NLS systems by preparing two relevant copies of the inverse scattering formulas for the Ablowitz–Ladik lattice and considering a Bäcklund–Darboux transformation between them. This provides a unification of the inverse scattering method for the NLS system and that for the derivative NLS systems in the discrete setting; such a unification is also possible in the continuous case (see [13, 14]). The multisoliton solutions of the space-discrete derivative NLS systems can be obtained in a straightforward manner within this unified framework; they reduce to the multisoliton solutions of the derivative NLS systems in the continuous space limit. Note that the space-discrete Kaup–Newell system allows the introduction of the potential variables with respect to the discrete spatial coordinate and can be rewritten locally in terms of these variables; our solution formulas reflect
this property accurately, that is, by construction any solution is written in the difference form using the forward difference operator.

We assumed vanishing boundary conditions at spatial infinity, namely, as $n \to +\infty$ and $n \to -\infty$. In the case of matrix-valued dependent variables, this assumption imposes highly nontrivial conditions on the scattering data, which become almost trivial in the scalar case (cf. (3.48)). For simplicity, we derived such conditions in the reflectionless case of the potentials; however, they are expected to be valid in the general case, because in the limits $t \to \pm \infty$ the contribution of the continuous spectrum would become negligible (cf. (3.36) and (3.37) with $b = a^\ast$). Note also that our approach of solving the derivative NLS systems using the NLS eigenvalue problem is applicable under other boundary conditions that are amenable to the inverse scattering method or its generalizations. In addition, although we mainly considered the first nontrivial flows of the integrable hierarchies, our approach can be applied, with minor amendments, to the higher flows as well as the negative flows of the hierarchies.

\section{A Reduction to the vector modified Volterra lattice}

In this appendix, we consider the reduction of the matrix Ablowitz–Ladik lattice (2.3) to the vector modified Volterra lattice:

\begin{equation}
q_{n,t} = (1 + \langle q_n, q_n \rangle)(q_{n+1} - q_{n-1}).
\end{equation}

(A.1)

Here, $q_n$ is a row vector of arbitrary dimension and $\langle \cdot , \cdot \rangle$ stands for the scalar product. The scalar (i.e., one-component) modified Volterra lattice was introduced by Hirota [56] and the two-component case was studied by Ablowitz and Ladik [6]. The vector generalization (A.1) was recognized as an integrable system in the late 1990s (see, e.g., references in [57]); it can also be considered as the simplest space-discrete analog of the vector modified KdV equation [32,58]:

\begin{equation}
q_t = q_{xxx} + 6\langle q, q \rangle q_x.
\end{equation}

(A.2)

As was shown in our previous paper [23] (also see [24]), the inverse scattering method formulated for the matrix Ablowitz–Ladik lattice (2.3) can be applied to the vector modified Volterra lattice (A.1) by imposing suitable reduction conditions on the scattering data. Here, we revisit this problem and derive the reduction conditions in a more convincing manner.
We introduce a set of $2^{M-1} \times 2^{M-1}$ matrices \( \{e_j, e_k\} \) that are linearly independent and satisfy the anticommutation relations:

\[
\{e_j, e_k\} := e_j e_k + e_k e_j = -2\delta_{jk}I. \tag{A.3}
\]

Note that they are the generators of the Clifford algebra. We set the matrix-valued dependent variables \( Q_n \) and \( R_n \) as

\[
Q_n = q_n^{(1)} I + \sum_{j=1}^{2M-1} q_n^{(j+1)} e_j, \tag{A.4a}
\]

\[
R_n = -q_n^{(1)} I + \sum_{j=1}^{2M-1} q_n^{(j+1)} e_j. \tag{A.4b}
\]

Then, because of (A.3), they satisfy the important relation:

\[
Q_n R_n = R_n Q_n = -\langle q_n, q_n \rangle I.
\]

Here, \( q_n = (q_n^{(1)}, \ldots, q_n^{(2M)}) \). Using this relation, it is easy to see that the choice (A.4) reduces the matrix Ablowitz–Ladik lattice (2.3) with \( a = b = 1 \) to the vector modified Volterra lattice (A.1). Lax representations involving the generators of the Clifford algebra were introduced in the pioneering papers [59, 60], but our choice (A.4) is more efficient and useful because it contains the unit matrix \( I \). As a natural analog of the complex conjugate, we define the Clifford conjugate for the linear span of \( \{I, e_1, e_2, \ldots, e_{2M-1}\} \) as

\[
\hat{Q}_n = q_n^{(1)} I - \sum_{j=1}^{2M-1} q_n^{(j+1)} e_j, \quad \hat{R}_n = -q_n^{(1)} I - \sum_{j=1}^{2M-1} q_n^{(j+1)} e_j.
\]

Note that \( \hat{Q}_n = Q_n, R_n = -\hat{Q}_n, Q_n \hat{Q}_n = \hat{Q}_n Q_n = \langle q_n, q_n \rangle I, \) etc. In short, the Clifford conjugate denoted by \( \hat{\cdot} \) changes the sign of the coefficients of \( \{e_1, e_2, \ldots, e_{2M-1}\} \). This definition of the Clifford conjugate is very useful in the following discussion.

Let us discuss how the reduction (A.4) constrains the scattering data for the matrix Ablowitz–Ladik lattice. The main role is played by the quantity \( P_n = \Psi_{2n,1} \Psi_{1,n}^{-1} \) defined from the Lax representation, (2.1) and (2.2), which satisfies the pair of discrete and continuous matrix Riccati equations (2.4). We can show in an inductive manner that under the reduction (A.4), \( P_n \) also takes its values in the linear span of \( \{I, e_1, e_2, \ldots, e_{2M-1}\} \). We assume that the expression,

\[
P_n = -p_n^{(1)} I + \sum_{j=1}^{2M-1} p_n^{(j+1)} e_j, \tag{A.5}
\]

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is valid at some value of \( n \) and write its coefficients as \( \mathbf{p}_n := (p_n^{(1)}, \ldots, p_n^{(2M)}) \).

Then, noting the relations derivable from the anticommutation relations (A.3),

\[
\begin{align*}
\hat{P}_n \hat{P}_n &= \hat{P}_n P_n = \langle \mathbf{p}_n, \mathbf{p}_n \rangle I, \\
\hat{Q}_n \hat{P}_n + P_n \hat{Q}_n &= -2 \langle \mathbf{p}_n, \mathbf{q}_n \rangle I, \\
\hat{Q}_n \hat{P}_n \hat{Q}_n &= \left( \hat{Q}_n \hat{P}_n + P_n \hat{Q}_n \right) \hat{Q}_n - P_n \hat{Q}_n \hat{Q}_n = -2 \langle \mathbf{p}_n, \mathbf{q}_n \rangle \hat{Q}_n - \langle \mathbf{q}_n, \mathbf{q}_n \rangle P_n, \\
\left( I - \mu \hat{Q}_n \hat{P}_n \right) (I - \mu P_n Q_n) &= (1 + 2\mu \langle \mathbf{p}_n, \mathbf{q}_n \rangle + \mu^2 \langle \mathbf{p}_n, \mathbf{p}_n \rangle \langle \mathbf{q}_n, \mathbf{q}_n \rangle) I,
\end{align*}
\]

we can rewrite (2.4a) as

\[
P_{n+1} = (I - \mu P_n Q_n)^{-1} \left( \mu P_n + \hat{Q}_n \right)
= \frac{1}{1 + 2\mu \langle \mathbf{p}_n, \mathbf{q}_n \rangle + \mu^2 \langle \mathbf{p}_n, \mathbf{p}_n \rangle \langle \mathbf{q}_n, \mathbf{q}_n \rangle} \left( \mu P_n + \hat{Q}_n - \mu^2 \hat{Q}_n \hat{P}_n P_n - \mu \hat{Q}_n \hat{P}_n \hat{Q}_n \right)
= \frac{1}{1 + 2\mu \langle \mathbf{p}_n, \mathbf{q}_n \rangle + \mu^2 \langle \mathbf{p}_n, \mathbf{p}_n \rangle \langle \mathbf{q}_n, \mathbf{q}_n \rangle} \left[ \mu \left( 1 + \langle \mathbf{q}_n, \mathbf{q}_n \rangle \right) P_n + (1 + 2\mu \langle \mathbf{p}_n, \mathbf{q}_n \rangle - \mu^2 \langle \mathbf{p}_n, \mathbf{p}_n \rangle) \hat{Q}_n \right].
\]

Thus, the expression (A.5) is also valid for \( P_{n+1} \), and the coefficients satisfy the recursion relation written in the vector form:

\[
\mathbf{p}_{n+1} = \frac{\mu \left( 1 + \langle \mathbf{q}_n, \mathbf{q}_n \rangle \right) \mathbf{p}_n - (1 + 2\mu \langle \mathbf{p}_n, \mathbf{q}_n \rangle - \mu^2 \langle \mathbf{p}_n, \mathbf{p}_n \rangle) \mathbf{q}_n}{1 + 2\mu \langle \mathbf{p}_n, \mathbf{q}_n \rangle + \mu^2 \langle \mathbf{p}_n, \mathbf{p}_n \rangle \langle \mathbf{q}_n, \mathbf{q}_n \rangle}.
\]

In a similar way, we can show using (2.4a) that if the expression (A.5) is valid for \( P_{n+1} \), then it is also valid for \( P_n \). Therefore, under a suitable boundary condition on \( P_n \), such as \( \lim_{n \to -\infty} P_n = O \) or \( \lim_{n \to +\infty} P_n = O \), so that the expression (A.5) is valid at the boundary, \( P_n \) indeed takes its values in the linear span of \( \{ I, e_1, e_2, \ldots, e_{2M-1} \} \) for all \( n \in \mathbb{Z} \). Moreover, using (2.4b), we can easily check that this property is preserved under the time evolution.

Substituting the expressions

\[
R_n = a_n P_n + b_n P_{n+1}, \\
Q_n = -\tilde{R}_n = -a_n \hat{P}_n - b_n \hat{P}_{n+1}
\]

into (2.4a), we can express the scalar coefficients \( a_n \) and \( b_n \) in terms of \( \mathbf{p}_n \) and \( \mathbf{p}_{n+1} \). Thus, \( \mathbf{q}_n \) can be written explicitly using \( \mathbf{p}_n \) and \( \mathbf{p}_{n+1} \), which defines a Miura map to the vector modified Volterra lattice (A.1). The corresponding modified system, i.e., the closed differential-difference equation for \( \mathbf{p}_n \) can be obtained from (2.4b), but we do not present it here (cf. [57,61]).
Thus, if we take the limit of \( n \) the boundary condition \( \lim_{n \to \pm \infty} \mu^{-n} P_n = 0 \) defines the other boundary value \( \lim_{n \to \pm \infty} \mu^{-n} P_n \), which belongs to the linear span of \( \{ I, e_1, e_2, \ldots, e_{2M-1} \} \). The defining relation (3.7a) of the scattering data together with (3.5) implies that on the unit circle \( |\mu| = 1 \),

\[
\begin{bmatrix}
    z^n I & O \\
    O & z^{-n} I
\end{bmatrix}
\begin{bmatrix}
    \phi_n \\
    \psi_n
\end{bmatrix} =
\begin{bmatrix}
    I & O \\
    O & \mu^{-n} I
\end{bmatrix}
\begin{bmatrix}
    z^n \psi_n A + \left[ \mu^n I \ O \ I \right] z^{-n} \psi_n B
\end{bmatrix}
\to
\begin{cases}
    \begin{bmatrix} I \\ O \end{bmatrix} & \text{as } n \to -\infty, \\
    \begin{bmatrix} A(\mu) \\ B(\mu) \end{bmatrix} & \text{as } n \to +\infty.
\end{cases}
\]

Thus, if we set \( \mu^{-n} P_n = (z^n \phi_{2,n})(z^n \phi_{1,n})^{-1} \) using the above linear eigenfunction, we find that \( \lim_{n \to \pm \infty} \mu^{-n} P_n = B(\mu)A(\mu)^{-1} \) takes its values in the linear span of \( \{ I, e_1, e_2, \ldots, e_{2M-1} \} \). More precisely, we should use the explicitly time-dependent Jost solution \( \phi_n(t) \) as introduced in (3.31a), but this makes no difference in the above discussion. In addition, the established property of \( B(\mu)A(\mu)^{-1} \) is preserved under the time evolution.

For \( |\mu| < 1 \), we can still consider the difference equation (2.4a) for \( P_n \) with the boundary condition \( \lim_{n \to \pm \infty} P_n = 0 \) by setting \( P_n = (z^n \phi_{2,n})(z^n \phi_{1,n})^{-1} \). However, in this case the other boundary value \( \lim_{n \to \pm \infty} \mu^{-n} P_n \) does not exist in general. Thus, we need to take a more delicate limit for \( n \) and \( \mu \) to extract meaningful information from \( P_n \). Let us evaluate the \( n \to +\infty \) behavior of the Jost solution \( [z^n \phi_n](\mu) \) in the neighborhood of \( \mu = \mu_j \) \( (|\mu_j| < 1) \) at which \( A(\mu)^{-1} \) has a simple pole. Recalling the results in subsections 3.2 and 3.3, we obtain that

\[
[z^n \phi_n](\mu) \times (\mu - \mu_j)A(\mu)^{-1} = \left[ \begin{array}{c}
(\mu - \mu_j) \{ I + o(1) \} + o(\mu_j^n) \\
(\mu - \mu_j) o(1) + C_j \mu_j^n + o(\mu_j^n)
\end{array} \right].
\]

Thus, if we take the limit of \( n \to +\infty \) and \( \mu \to \mu_j \) while maintaining the balance condition \( |\mu - \mu_j| \sim |\mu_j^n| \), we arrive at the desired formula:

\[
P_n = \frac{1}{\mu - \mu_j} C_j \mu_j^n + o(1).
\]
Hence, $C_j$ also takes its values in the linear span of $\{I, e_1, e_2, \ldots, e_{2M-1}\}$; naturally, this property is preserved under the time evolution.

Combining the above results, we conclude that $F(n, t)$ as given in (3.36) with $a = b = 1$ (or its generalization corresponding to the case of higher order poles of $A(\mu)^{-1}$) can be expressed in the form:

$$F(n, t) = f^{(1)}(n, t)I - \sum_{j=1}^{2M-1} f^{(j+1)}(n, t)e_j. \quad (A.7)$$

We remark that the quantity $P_n := \Psi_{1,n}^{1,2} \Psi_{1,n}^{2,1}$ defined from another linear eigenfunction of the Ablowitz–Ladik eigenvalue problem (2.1) satisfies

$$Q_n = \mu^{-1}P_n - P_{n+1} + \mu^{-1}P_nR_nP_{n+1},$$

or equivalently,

$$P_{n+1} = (I - \mu^{-1}P_nR_n)^{-1}(\mu^{-1}P_n - Q_n).$$

Because $Q_n$ and $R_n$ in (A.4) are related as $R_n = -\hat{Q}_n$, we can identify $P_n$ with the Clifford conjugate of $P_n$ as

$$P_n(\mu) = -\hat{P}_n(\mu^{-1}).$$

Here, we assume that the boundary conditions for $P_n$ and $P_n$ are compatible in the above identification, e.g., $\lim_{n \to -\infty} P_n = \lim_{n \to -\infty} P_n = 0$. Thus, if we set $P_n = (z^{-n}\hat{\phi}_{1,n})(z^{-n}\hat{\phi}_{2,n})^{-1}$ using the Jost solution $\hat{\phi}_n$ as introduced in (3.5a) (or the explicitly time-dependent one $\hat{\phi}_{n}^{(t)}$ in (3.31a)) and compare it with $P_n = (z^n\phi_{1,n})(z^n\phi_{2,n})^{-1}$, we obtain the following relations:

- $\bar{B}(\mu)\bar{A}(\mu)^{-1}$ on $|\mu| = 1$ is the Clifford conjugate of $B(\mu^{-1})A(\mu^{-1})^{-1}$,
- $\bar{\mu}_j = \frac{1}{\mu_j}, \quad \bar{C}_j = -\frac{1}{\mu_j^2}\hat{C}_j, \quad j = 1, 2, \ldots, N(= N)$, up to renumbering.

Therefore, $F(n, t)$ as given in (3.37) with $a = b = 1$ is equal to the Clifford conjugate of $F(n, t)$:

$$\bar{F}(n, t) = \bar{F}(n, t) = f^{(1)}(n, t)I + \sum_{j=1}^{2M-1} f^{(j+1)}(n, t)e_j. \quad (A.8)$$
Here, the vector $f_n(t) := (f^{(1)}(n, t), \ldots, f^{(2M)}(n, t))$ satisfies the linear evolution equation $f_{n,t} = f_{n+1} - f_{n-1}$ (cf. (3.38)) and decays rapidly as $n \to +\infty$.

In fact, (A.7) and (A.8) provide not only necessary but also sufficient conditions for the corresponding potentials $Q_n$ and $R_n$ to be expressed as (A.4) without using $e_je_k, e_ie_je_k$, etc.

**Proposition A.1.** If $F(n, t)$ and $\bar{F}(n, t)$ are given as (A.7) and (A.8), then the potentials $Q_n$ and $R_n$ reconstructed from the set of exact linearization formulas (3.39) can be expressed in the form (A.4).

Before proving Proposition A.1, we need to state one proposition and two lemmas. We first show that the set of formulas (3.39) realizes an exact linearization of the matrix Ablowitz–Ladik lattice.

**Proposition A.2.** Suppose that $F(n, t)$ and $\bar{F}(n, t)$ satisfy the pair of linear evolution equations (3.38) and decay sufficiently rapidly as $n \to +\infty$. Then, the Liouville–Neumann-type series for the potentials $Q_n$ and $R_n$ defined by formulas (3.39) solve the matrix Ablowitz–Ladik lattice (2.3) exactly.

**Remark.** Only the case $a = b = 1$ is relevant to the vector modified Volterra lattice (A.1), but we find it more convenient to prove Proposition A.2 for general values of $a$ and $b$.

**Proof of Proposition A.2.** Using (3.39), we obtain the Liouville–Neumann-
type series for \( Q_n \) and \( R_n \) in the form:

\[
Q_n = \bar{F}(n) - \sum_{i_1, i_2 = 0}^{\infty} \bar{F}(n + i_1)F(n + i_1 + i_2 + 1)\bar{F}(n + i_2 + 1)
\]

\[
+ \sum_{i_1, i_2, i_3, i_4 = 0}^{\infty} \bar{F}(n + i_1)F(n + i_1 + i_2 + 1)\bar{F}(n + i_2 + i_3 + 1)F(n + i_3 + i_4 + 1)\bar{F}(n + i_4 + 1)
\]

\[- \ldots\]

\[
= \sum_{k=0}^{\infty} (-1)^k Q_n^{(k)},
\]

(A.9a)

\[
R_n = -F(n) + \sum_{i_1, i_2 = 0}^{\infty} F(n + i_1)\bar{F}(n + i_1 + i_2 + 1)F(n + i_2 + 1)
\]

\[- \sum_{i_1, i_2, i_3, i_4 = 0}^{\infty} F(n + i_1)\bar{F}(n + i_1 + i_2 + 1)F(n + i_2 + i_3 + 1)\bar{F}(n + i_3 + i_4 + 1)F(n + i_4 + 1)
\]

\[+ \ldots\]

\[
= \sum_{k=0}^{\infty} (-1)^{k+1} R_n^{(k)}.
\]

(A.9b)

Here,

\[
Q_n^{(0)} := \bar{F}(n), \quad R_n^{(0)} := F(n),
\]

(A.10a)

and \( Q_n^{(k)} \) and \( R_n^{(k)} \) for \( k \geq 1 \) are defined as

\[
Q_n^{(k)} := \sum_{i_1, i_2, \ldots, i_{2k} = 0}^{\infty} \bar{F}(n + i_1)F(n + i_1 + i_2 + 1)\bar{F}(n + i_2 + i_3 + 1)F(n + i_3 + i_4 + 1)
\]

\[
\times \ldots \bar{F}(n + i_{2k-2} + i_{2k-1} + 1)F(n + i_{2k-1} + i_{2k} + 1)\bar{F}(n + i_{2k} + 1),
\]

(A.10b)

\[
R_n^{(k)} := \sum_{i_1, i_2, \ldots, i_{2k} = 0}^{\infty} F(n + i_1)\bar{F}(n + i_1 + i_2 + 1)F(n + i_2 + i_3 + 1)\bar{F}(n + i_3 + i_4 + 1)
\]

\[
\times \ldots F(n + i_{2k-2} + i_{2k-1} + 1)\bar{F}(n + i_{2k-1} + i_{2k} + 1)F(n + i_{2k} + 1).
\]

(A.10c)

Note that \( Q_n^{(k)} \) and \( R_n^{(k)} \) are “polynomials” of degree \( 2k + 1 \) in \( F \) and \( \bar{F} \) with their arguments bounded below but unbounded above. We only consider the region of \((n, t)\) where the series in (A.9) are absolutely convergent and admit termwise differentiation by \( t \).
In the following, we use the shift operator $E_n$ as well as its inverse $E_n^{-1}$ defined as

$$E_nf(n) := f(n + 1), \quad E_n^{-1}f(n) := f(n - 1),$$

and the forward difference operator $\Delta^+_a$ in each index of summation $i_a$,

$$\Delta^+_af(\ldots, i_a, \ldots) := f(\ldots, i_a + 1, \ldots) - f(\ldots, i_a, \ldots).$$

Then, using (3.38), we can compute the time derivative of $Q_n^{(k)}$ for $k \geq 1$ to obtain

$$\left[\partial_t - aE_n + bE_n^{-1} + (a - b)I\right]Q_n^{(k)}$$

$$= a \sum_{i_1, i_2, \ldots, i_{2k} = 0}^{\infty} \left\{ \tilde{F}(n + i_1 + 1)F(n + i_1 + i_2 + 1)\tilde{F}(n + i_2 + i_3 + 1) \cdots \tilde{F}(n + i_{2k} + 1)
- \tilde{F}(n + i_1)F(n + i_1 + i_2 + 1)\tilde{F}(n + i_2 + i_3 + 1) \cdots \tilde{F}(n + i_{2k} + 1)
+ \cdots
- \tilde{F}(n + i_1 + 1)F(n + i_1 + i_2 + 2) \cdots F(n + i_{2k-1} + i_{2k} + 2)\tilde{F}(n + i_{2k} + 2) \right\}$$

$$+ b \sum_{i_1, i_2, \ldots, i_{2k} = 0}^{\infty} \left\{ -\tilde{F}(n + i_1 - 1)F(n + i_1 + i_2 + 1)\tilde{F}(n + i_2 + i_3 + 1) \cdots \tilde{F}(n + i_{2k} + 1)
+ \tilde{F}(n + i_1)F(n + i_1 + i_2 + 2)\tilde{F}(n + i_2 + i_3 + 1) \cdots \tilde{F}(n + i_{2k} + 1)
- \tilde{F}(n + i_1 + 1)F(n + i_1 + i_2 + 2) \cdots F(n + i_{2k-1} + i_{2k} + 2)\tilde{F}(n + i_{2k} + 2) \right\},$$

where the arguments shifted from the original ones in $Q_n^{(k)}$ are underscored.
with a wavy line. We can further rewrite it as
\[
\begin{align*}
& a \sum_{i_1, i_2, \ldots, i_{2k} = 0}^{\infty} \left\{ \Delta^+_{i_1} \left[ \bar{F}(n + i_1)F(n + i_1 + i_2)\bar{F}(n + i_2 + i_3 + 1) \cdots \bar{F}(n + i_{2k} + 1) \right] \\
& \quad + \Delta^+_{i_1} \left[ \bar{F}(n + i_1)F(n + i_1 + i_2 + 1)\bar{F}(n + i_2 + i_3 + 1)F(n + i_3 + i_4) \cdots \bar{F}(n + i_{2k} + 1) \right] \\
& \quad + \cdots \\
& \quad + \Delta^+_{i_{2k-1}} \left[ \bar{F}(n + i_1) \cdots \bar{F}(n + i_{2k-2} + i_{2k-1} + 1)F(n + i_{2k-1} + i_{2k})\bar{F}(n + i_{2k} + 1) \right] \\
& \quad - \sum_{\alpha = 1}^{k} \Delta^+_{i_{2\alpha - 1}} \left[ \bar{F}(n + i_1)F(n + i_1 + i_2 + 1) \cdots \bar{F}(n + i_{2\alpha - 1} + i_{2\alpha - 1} + 1) \\
& \quad \times F(n + i_{2\alpha - 1} + i_{2\alpha - 1}) \cdots \bar{F}(n + i_{2\alpha - 1} + 1) \right] \right\} \\
& + b \sum_{i_1, i_2, \ldots, i_{2k} = 0}^{\infty} \left\{ \Delta^+_{i_2} \left[ \bar{F}(n + i_1)F(n + i_1 + i_2 + 1)\bar{F}(n + i_2 + i_3) \cdots \bar{F}(n + i_{2k} + 1) \right] \\
& \quad + \Delta^+_{i_1} \left[ \bar{F}(n + i_1) \cdots \bar{F}(n + i_3 + i_4 + 1)\bar{F}(n + i_4 + i_5) \cdots \bar{F}(n + i_{2k} + 1) \right] \\
& \quad + \cdots \\
& \quad + \Delta^+_{i_{2k}} \left[ \bar{F}(n + i_1)F(n + i_1 + i_2 + 1) \cdots \bar{F}(n + i_{2k} + 1) \right] \\
& \quad - \sum_{\beta = 1}^{k} \Delta^+_{i_{2\beta + 1}} \left[ \bar{F}(n + i_1 - 1)\bar{F}(n + i_1 + i_2) \cdots \bar{F}(n + i_{2\beta} + i_{2\beta} + 1) \right] \left[ \bar{F}(n + i_{2\beta + 1} + i_{2\beta + 2} + 1) \cdots \bar{F}(n + i_{2\beta + 1} + 1) \right] \right\} \\
& = a \sum_{i_1, i_2, \ldots, i_{2k} = 0}^{\infty} \sum_{1 \leq \alpha \leq \beta \leq k} \Delta^+_{i_{2\alpha - 1}} \Delta^+_{i_{2\beta}} \left[ \bar{F}(n + i_1)F(n + i_1 + i_2 + 1) \cdots \bar{F}(n + i_{2\alpha - 2} + i_{2\alpha - 1} + 1) \\
& \quad \times F(n + i_{2\alpha - 1} + i_{2\alpha})\bar{F}(n + i_{2\alpha} + i_{2\alpha + 1} + 1) \cdots \bar{F}(n + i_{2\beta} + 1) \\
& \quad \times \bar{F}(n + i_{2\beta} + i_{2\beta + 1} + 1) \cdots \bar{F}(n + i_{2\beta + 1} + 1) \right] \\
& \quad + b \sum_{i_1, i_2, \ldots, i_{2k} = 0}^{\infty} \sum_{1 \leq \alpha \leq \beta \leq k} \Delta^+_{i_{2\alpha - 1}} \Delta^+_{i_{2\beta}} \left[ \bar{F}(n + i_1 - 1)F(n + i_1 + i_2) \cdots \bar{F}(n + i_{2\alpha - 2} + i_{2\alpha - 1}) \\
& \quad \times F(n + i_{2\alpha - 1} + i_{2\alpha})\bar{F}(n + i_{2\alpha} + i_{2\alpha + 1} + 1) \cdots \bar{F}(n + i_{2\beta} + 1) \\
& \quad \times \bar{F}(n + i_{2\beta} + i_{2\beta + 1} + 1) \cdots \bar{F}(n + i_{2\beta + 1} + 1) \right].
\end{align*}
\]
which is equal to

\[
\begin{aligned}
a \sum_{1 \leq \alpha \leq \beta \leq k} & \left[ - \sum_{i_1, \ldots, i_{2\alpha-2}=0}^{\infty} F(n+i_1)F(n+i_1+i_2+1) \cdots F(n+i_{2\alpha-2}+1) \\
& \times \sum_{i_{2\alpha}, \ldots, i_{2\beta-1}=0}^{\infty} F(n+i_{2\alpha})F(n+i_{2\alpha}+i_{2\alpha+1}+1) \cdots F(n+i_{2\beta-1}+1) \\
& \times \sum_{i_{2\beta+1}, \ldots, i_{2k}=0}^{\infty} F(n+i_{2\beta+1}+1)F(n+i_{2\beta+1}+i_{2\beta+2}+2) \cdots F(n+i_{2k}) \right] \\
+ b \sum_{1 \leq \alpha \leq \beta \leq k} & \left[ \sum_{i_1, \ldots, i_{2\alpha-2}=0}^{\infty} F(n+i_1-1)F(n+i_1+i_2) \cdots F(n+i_{2\alpha-2}) \\
& \times \sum_{i_{2\alpha}, \ldots, i_{2\beta-1}=0}^{\infty} F(n+i_{2\alpha})F(n+i_{2\alpha}+i_{2\alpha+1}+1) \cdots F(n+i_{2\beta-1}+1) \\
& \times \sum_{i_{2\beta+1}, \ldots, i_{2k}=0}^{\infty} F(n+i_{2\beta+1})F(n+i_{2\beta+1}+i_{2\beta+2}+1) \cdots F(n+i_{2k}+1) \right] \\
= -a \sum_{1 \leq \alpha \leq \beta \leq k} Q_n^{(\alpha-1)} \mathcal{R}_n^{(\beta-\alpha)} Q_{n+1}^{(k-\beta)} + b \sum_{1 \leq \alpha \leq \beta \leq k} Q_n^{(\alpha-1)} \mathcal{R}_n^{(\beta-\alpha)} Q_n^{(k-\beta)}. 
\end{aligned}
\]

Thus, we obtain

\[
[\partial_t - a E_n + b E_n^{-1} + (a-b)I] Q_n^{(k)} \\
= -a \sum_{1 \leq \alpha \leq \beta \leq k} Q_n^{(\alpha-1)} \mathcal{R}_n^{(\beta-\alpha)} Q_{n+1}^{(k-\beta)} + b \sum_{1 \leq \alpha \leq \beta \leq k} Q_n^{(\alpha-1)} \mathcal{R}_n^{(\beta-\alpha)} Q_n^{(k-\beta)}, \quad k \geq 1. 
\] (A.11a)

Similarly, we also obtain

\[
[\partial_t - b E_n + a E_n^{-1} + (b-a)I] \mathcal{R}_n^{(k)} \\
= -b \sum_{1 \leq \alpha \leq \beta \leq k} \mathcal{R}_n^{(\alpha-1)} Q_n^{(\beta-\alpha)} \mathcal{R}_{n+1}^{(k-\beta)} + a \sum_{1 \leq \alpha \leq \beta \leq k} \mathcal{R}_n^{(\alpha-1)} Q_n^{(\beta-\alpha)} \mathcal{R}_n^{(k-\beta)}, \quad k \geq 1. 
\] (A.11b)

Because of (A.9), (A.10a) and (3.38), relations (A.11) imply that

\[
\begin{align*}
Q_{n,t} - a Q_{n+1} + b Q_{n-1} + (a-b)Q_n &= -a Q_n R_n Q_{n+1} + b Q_{n-1} R_n Q_n, \\
R_{n,t} - b R_{n+1} + a R_{n-1} + (b-a)R_n &= -b R_n Q_n R_{n+1} + a R_{n-1} Q_n R_n.
\end{align*}
\]
These are exactly the equations of motion for the matrix Ablowitz–Ladik lattice (2.3).

In the following, we set $a = b = 1$, which simplifies (3.38) to

\[
\begin{align*}
\partial_t F(n, t) &= F(n + 1, t) - F(n - 1, t), & (A.12a) \\
\partial_t \bar{F}(n, t) &= \bar{F}(n + 1, t) - \bar{F}(n - 1, t). & (A.12b)
\end{align*}
\]

Here, $F(n, t)$ and $\bar{F}(n, t)$ are required to decay rapidly as $n \to +\infty$. For the moment, we do not impose the conditions (A.7) and (A.8), and consider the general case where $F(n, t)$ and $\bar{F}(n, t)$ are independent matrix functions.

Then, we can rewrite (A.11) as recurrence relations for $Q_n^{(k)}$ and $R_n^{(k)}$ that were originally defined as (A.10), i.e.

\[
Q_n^{(k)} = (I + E_n \partial_t - E_n^2)^{-1} \left[ - \sum_{1 \leq \alpha \leq \beta \leq k} Q_n^{(\alpha - 1)} R_n^{(\beta - \alpha)} Q_n^{(k - \beta)} + \sum_{1 \leq \alpha \leq \beta \leq k} Q_n^{(\alpha - 1)} R_n^{(\beta - \alpha)} Q_n^{(k - \beta)} \right],
\]

\[
R_n^{(k)} = (I + E_n \partial_t - E_n^2)^{-1} \left[ - \sum_{1 \leq \alpha \leq \beta \leq k} R_n^{(\alpha - 1)} Q_n^{(\beta - \alpha)} R_n^{(k - \beta)} + \sum_{1 \leq \alpha \leq \beta \leq k} R_n^{(\alpha - 1)} Q_n^{(\beta - \alpha)} R_n^{(k - \beta)} \right]
\]

for $k \geq 1$, and $Q_n^{(0)} = \bar{F}(n, t)$ and $R_n^{(0)} = F(n, t)$. Here, the inverse operator $(I + E_n \partial_t - E_n^2)^{-1}$ is defined using the Maclaurin series as

\[
(I + E_n \partial_t - E_n^2)^{-1} := \sum_{j=0}^{\infty} E_n^j (E_n - \partial_t)^j,
\]

where the action of $\partial_t$ on $F(n, t)$ and $\bar{F}(n, t)$ is given by (A.12).

In fact, the linear operator $I + E_n \partial_t - E_n^2$ has a nontrivial kernel that contains $F(n, t)$ and $\bar{F}(n, t)$ as well as their spatial/temporal shifts, so it is not invertible in general; however, this does not cause any problem in obtaining (A.13). To confirm this, it is sufficient to note that no “polynomials” of degree $2k + 1$ ($k \geq 1$) in $F$ and $\bar{F}$ can vanish by the action of $I + E_n \partial_t - E_n^2$, as long as their spatial arguments are bounded below.

**Lemma A.3.** For general $F(n)$ and $\bar{F}(n)$ satisfying the linear evolution equations (A.12), assume that the following equality is valid:

\[
(I + E_n \partial_t - E_n^2) \sum_{i_1, i_2, \ldots, i_{2k+1} = 0}^{\infty} \gamma_{i_1 i_2 \ldots i_{2k+1}} F(n + i_1 + \alpha_1) F(n + i_2 + \alpha_2) F(n + i_3 + \alpha_3) \times \cdots F(n + i_{2k} + \alpha_{2k}) \bar{F}(n + i_{2k+1} + \alpha_{2k+1}) = O.
\]

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Here, $\gamma_{i_1i_2...i_{2k+1}} \in \mathbb{C}$ and $\alpha_j \in \mathbb{Z}_{\geq 0}$ are constants, and $k \geq 1$. Then, the equality must be trivial, i.e.

$$\gamma_{i_1i_2...i_{2k+1}} = 0, \quad \forall i_1, i_2, \ldots, i_{2k+1} \in \mathbb{Z}_{\geq 0}. $$

This lemma can be easily verified by ordering the nonzero terms involved in the summation, e.g., using sums of the arguments such as

$$(n + i_1 + \alpha_1) + (n + i_2 + \alpha_2) + (n + i_3 + \alpha_3) + \cdots + (n + i_{2k+1} + \alpha_{2k+1}),$$

$$(n + i_2 + \alpha_2) + (n + i_3 + \alpha_3) + \cdots + (n + i_{2k+1} + \alpha_{2k+1}),$$

etc.

Note that both $E_n \partial_t$ and $E_n^2$ increase the values of the arguments in each term, while $I$ leaves them invariant. Thus, the equality implies that the coefficient of the first term in the ordering, which has the minimum values of the arguments, must be zero. Therefore, there is no first term and all $\gamma_{i_1i_2...i_{2k+1}}$ must vanish.

□

Lemma A.3 guarantees that the difference between $Q_n^{(k)}$ defined as (A.10b) and the right-hand side of (A.13a) does not belong to the kernel of $I + E_n \partial_t - E_n^2$. Thus, (A.13a) is indeed an exact equality; the same applies to (A.13b).

On the basis of the recurrence relations (A.13), we can prove Proposition A.1 by induction. To this end, we need to use one lemma, which is a direct consequence of the anticommutation relations (A.3) for the generators of the Clifford algebra (cf. (A.6)).

**Lemma A.4.** If the square matrices $X$ and $Y$ take their values in the linear span of $\{I, e_1, e_2, \ldots, e_{2M-1}\}$, then

$$X \hat{Y} + \hat{Y} X$$

is a scalar matrix and coincides with $\hat{X} Y + \hat{Y} X$. Thus, up to an overall factor, it can be considered as the definition of an inner product. Here, $\hat{\cdot}$ denotes the Clifford conjugate, which changes the sign of the coefficients of $\{e_1, e_2, \ldots, e_{2M-1}\}$.

**Proof of Proposition A.1.** Because of (A.9), we need only show that, for all $k \in \mathbb{Z}_{\geq 0}$,

$$Q_n^{(k)} \text{ and } R_n^{(k)}$$

take their values in the linear span of $\{I, e_1, e_2, \ldots, e_{2M-1}\}$ and satisfy the Clifford conjugation relation $R_n^{(k)} = \hat{Q}_n^{(k)}$. 

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Note that this is true for \( k = 0 \) (cf. (A.7), (A.8) and (A.10a)). We assume that \((\clubsuit)\) is true for \( k \leq L \) and then proceed to show \((\clubsuit)\) for \( k = L + 1 \). Lemma A.4 implies that the quantity appearing in (A.13a) with \( k = L + 1 \),

\[
\sum_{\alpha=1}^{\beta} Q_{n+1}^{(\alpha-1)} R_{n+1}^{(\beta-\alpha)} = \frac{1}{2} \sum_{\alpha=1}^{\beta} \left[ Q_{n+1}^{(\alpha-1)} R_{n+1}^{(\beta-\alpha)} + Q_{n+1}^{(\beta-\alpha)} R_{n+1}^{(\alpha-1)} \right],
\]

is a scalar matrix and coincides with the quantity appearing in (A.13b) with \( k = L + 1 \),

\[
\sum_{\alpha=1}^{\beta} R_{n+1}^{(\alpha-1)} Q_{n+1}^{(\beta-\alpha)} = \frac{1}{2} \sum_{\alpha=1}^{\beta} \left[ R_{n+1}^{(\alpha-1)} Q_{n+1}^{(\beta-\alpha)} + R_{n+1}^{(\beta-\alpha)} Q_{n+1}^{(\alpha-1)} \right],
\]

\[
= \frac{1}{2} \sum_{\alpha=1}^{\beta} \left[ Q_{n+1}^{(\alpha-1)} Q_{n+1}^{(\beta-\alpha)} + Q_{n+1}^{(\beta-\alpha)} Q_{n+1}^{(\alpha-1)} \right],
\]

\[
1 \leq \beta \leq L + 1,
\]

is scalar matrix and coincides with the quantity appearing in (A.13b) with \( L + 1 \),

\[
\sum_{\beta=\alpha}^{L+1} R_{n+1}^{(\beta-\alpha)} Q_{n+1}^{(L+1-\beta)} = \sum_{\beta=\alpha}^{L+1} Q_{n+1}^{(\beta-\alpha)} R_{n+1}^{(L+1-\beta)} = \text{scalar}, \quad 1 \leq \alpha \leq L + 1.
\]

Thus, the recurrence relations (A.13) imply that \((\clubsuit)\) holds true for \( k = L + 1 \). Therefore, by complete induction, \((\clubsuit)\) is true for all \( k \in \mathbb{Z}_{\geq 0} \). \(\square\)

**Remark.** By considering the continuous space limit of Proposition A.1, we can obtain similar results on the corresponding reductions of the continuous matrix NLS hierarchy. In particular, we can identify the reduction conditions on the scattering data to solve the vector modified KdV equation (A.2) and the vector sine-Gordon equation \([59]\) by the inverse scattering method (see \([62]\)). In addition, it is an easy task to consider a continuous analog of Proposition A.2 that realizes an exact linearization of the matrix NLS system.

Proposition A.1 demonstrates that under the restrictions (A.7) and (A.8) on \( F(n,t) \) and \( \overline{F(n,t)} \), we can solve the vector modified Volterra lattice (A.1) by the inverse scattering method. Moreover, if the vector dependent variable \( q_n \) is real-valued, we need to restrict the components of the vector.
\( f_n(t) = (f^{(1)}(n, t), \ldots, f^{(2M)}(n, t)) \) appearing in (A.7) and (A.8) to be real-valued. Recall that in the reflectionless case of the potentials, if \( A(\mu)^{-1} \) only has isolated simple poles, \( F(n, t) \) is given as (cf. (3.36) with \( a = b = 1 \))

\[
F(n, t) = -\sum_{j=1}^{N} C_j(0) \mu_j^n e^{(\mu_j - \mu_j^{-1})t},
\]

where \( 0 < |\mu_j| < 1 \). Thus, the configuration of \( \{\mu_1, \mu_2, \ldots, \mu_N\} \) must be symmetric with respect to the real \( \mu \)-axis, that is, they either take real values or occur in complex conjugate pairs. Up to a reordering, they can be classified into the following three types:

(I) \( \mu_{2k-1} = \mu_{2k}^* = a_k e^{i\theta_k}, \; 0 < a_k < 1, \; 0 < \theta_k < \pi, \; k = 1, 2, \ldots, N_1 \),

(II) \( 0 < \mu_j < 1, \; j = 2N_1 + 1, \ldots, 2N_1 + N_2 \),

(III) \( -1 < \mu_j < 0, \; j = 2N_1 + N_2 + 1, \ldots, 2N_1 + N_2 + N_3 (= N) \).

Each of the corresponding matrices \( \{C_1(0), C_2(0), \ldots, C_N(0)\} \) takes its values in the linear span of \( \{I, e_1, e_2, \ldots, e_{2M-1}\} \), wherein the coefficients are all real for type (II) and type (III). For type (I), the coefficients associated with \( \mu_{2k-1} \) and those associated with \( \mu_{2k} \) form a complex conjugate pair so that the coefficients of \( \{I, e_1, e_2, \ldots, e_{2M-1}\} \) in (A.14) become real-valued [23, 24].

The nature of the three types (I)–(III) can be unveiled by constructing the corresponding one-soliton solutions. Type (II) provides the trivial vector analog of the one-soliton solution of the scalar modified Volterra lattice,

\[
q_n(t) = \frac{\sinh \alpha}{\cosh [\alpha n + 2(\sinh \alpha)t + \delta]} u, \quad \langle u, u \rangle = 1.
\]

Type (III) provides a very similar solution,

\[
q_n(t) = \frac{(-1)^n \sinh \alpha}{\cosh [\alpha n - 2(\sinh \alpha)t + \delta]} u, \quad \langle u, u \rangle = 1,
\]

which reflects the form-invariance of the vector modified Volterra lattice (A.1) under the transformation \( q_n \to (-1)^n q_n, \; t \to -t \). For type (I), we need to impose an additional condition to exclude a breather solution so that a pure soliton solution with a time-independent profile can be obtained [23, 24]. Thus, the one-soliton solution of type (I) is given by

\[
q_n(t) = \frac{ce^{i\beta n + 2i(\cosh \alpha \sin \beta)t} + c^*e^{-i\beta n - 2i(\cosh \alpha \sin \beta)t}}{\cosh [\alpha n + 2(\sinh \alpha \cos \beta)t + \delta]}
\]

with \( \langle c, c \rangle = 0 \) and \( 2\langle c, c^* \rangle = (\sinh \alpha)^2 \). This is indeed the most general one-soliton solution, because it reduces to type (II) and type (III) in the limit \( \beta \to 0 \) and \( \beta \to \pi \), respectively.
Proposition A.1 enables us to formulate the inverse scattering method for the vector modified Volterra lattice (A.1), as well as the relevant continuous systems, with satisfactory rigor. As by-products, it can also generate numerous nontrivial identities for some rational functions of \( \{\mu_1, \mu_2, \ldots, \mu_N\} \). Indeed, for \( F(n,t) \) and \( \tilde{F}(n,t) \) given as (A.7) and (A.8), the coefficients of \( e_j e_k \) (\( j \neq k \)), \( e_i e_j e_k \) (\( i \neq j \neq k \neq i \)), etc. in \( Q_n^{(k)} \) for \( k \geq 1 \) defined as (A.10b) must vanish (cf. (♣)). For the reflectionless case of the potentials, we can express these coefficients explicitly in terms of the scattering data, which indeed provide nontrivial identities. If \( A(\mu)^{-1} \) only has isolated simple poles, \( F(n) \) and \( \tilde{F}(n) \) are given as

\[
F(n) = - \sum_{j=1}^{N} C_j \mu_j^n, \quad \tilde{F}(n) = - \sum_{j=1}^{N} \hat{C}_j \mu_j^n.
\]

(A.15)

Here, \( \{\mu_1, \mu_2, \ldots, \mu_N\} \) are pairwise distinct and satisfy \( 0 < |\mu_j| < 1 \), and \( \hat{\cdot} \) denotes the Clifford conjugate. The time dependence of \( C_j \) and \( \hat{\cdot} \) is irrelevant in this context and thus is omitted. Substituting (A.15) into (A.10b) and computing the multiple sum with respect to \( i_1, i_2, \ldots, i_{2k} \), we obtain

\[
Q_n^{(k)} = - \sum_{j_1, j_2, \ldots, j_{2k+1} = 1}^{N} \frac{\mu_{j_1}^n \mu_{j_2}^{n+1} \mu_{j_3}^{n+1} \cdots \mu_{j_{2k}}^{n+1} \mu_{j_{2k+1}}^{n+1} \hat{C}_{j_1} C_{j_2} \hat{C}_{j_3} C_{j_4} \cdots \hat{C}_{j_{2k}} C_{j_{2k+1}}}{(1 - \mu_{j_1} \mu_{j_2}) (1 - \mu_{j_2} \mu_{j_3}) \cdots (1 - \mu_{j_{2k}} \mu_{j_{2k+1}})}
\]

for \( k \geq 1 \). Thus, we can state the following proposition.

**Proposition A.5.** For \( \{e_1, e_2, \ldots, e_{2M-1}\} \) satisfying the anticommutation relations \( e_j e_k + e_k e_j = -2 \delta_{jk} I \), all the coefficients of quadratic or higher terms such as \( e_j e_k \) (\( j \neq k \)) and \( e_i e_j e_k \) (\( i \neq j \neq k \neq i \)) in the quantity,

\[
\sum_{\{j_1, j_2, \ldots, j_{2k+1}\} = \{i_1, i_2, \ldots, i_{2k+1}\}} \frac{\hat{C}_{j_1} C_{j_2} \hat{C}_{j_3} C_{j_4} \cdots \hat{C}_{j_{2k}} C_{j_{2k+1}}}{\mu_{j_1} (1 - \mu_{j_1} \mu_{j_2}) (1 - \mu_{j_2} \mu_{j_3}) \cdots (1 - \mu_{j_{2k}} \mu_{j_{2k+1}})}
\]

vanish identically. Here, \( \{i_1, i_2, \ldots, i_{2k+1}\} \) is any (repeated) combination of positive integers \( 1 \leq i_1 \leq i_2 \leq \cdots \leq i_{2k+1} \) and

\[
C_j = c_j^{(1)} I + \sum_{a=1}^{2M-1} c_j^{(a+1)} e_a, \quad \hat{C}_j = c_j^{(1)} I - \sum_{a=1}^{2M-1} c_j^{(a+1)} e_a.
\]

Because \( \{c_j^{(1)}, c_j^{(2)}, \ldots, c_j^{(2M)}\} \) for \( j \in \{i_1, i_2, \ldots, i_{2k+1}\} \) at a fixed time can be chosen arbitrarily, each identity obtained this way splits into as many identities as the number of different products of \( \{c_j^{(i)}\} \) involved in the identity.
For example, when \( \{i_1, i_2, \ldots, i_{2k+1}\} \) are pairwise distinct, we can simply set them as \( \{1, 2, \ldots, 2k + 1\} \). Thus, the coefficients of the highest products of \( \{e_j\} \) such as \( e_1 e_2 \cdots e_{2k+1} \) provide the identity:

\[
\sum_{\{j_1, j_2, \ldots, j_{2k+1}\} = \{1, 2, \ldots, 2k+1\}} \sgn \left( \begin{array}{c} 1 \\ j_1 \\ 2 \\ j_2 \\ \vdots \\ 2k+1 \\ j_{2k+1} \end{array} \right) \mu_{j_1} (1 - \mu_{j_1} \mu_{j_2}) (1 - \mu_{j_2} \mu_{j_3}) \cdots (1 - \mu_{j_{2k}} \mu_{j_{2k+1}}) = 0.
\]

Considering the lower products such as \( e_1 e_2 \cdots e_{2k} \), \( e_1 e_2 \cdots e_{2k-1} \), \ldots, \( e_1 e_2 \), we obtain various extensions of the above identity, wherein the sign of the permutation is replaced with more elaborate functions that still take values in \( \{+1, -1\} \).

Proposition A.5 can be generalized in many different directions. First, we can consider the case where \( A(\mu)^{-1} \) also has second or higher order poles. Thus, \( \mu_n^j \) in (A.15) can be replaced with more general functions decaying as \( n \to +\infty \). Second, instead of using Proposition A.1 designed for solving the vector modified Volterra lattice (A.1), we can consider corresponding results for other integrable systems having a Lax representation of the same type. That is, any integrable system with vector dependent variables can provide a similar result if it can be obtained as a reduction of a matrix-valued integrable system using the generators of the Clifford algebra. In particular, starting with the solution formulas for the matrix generalizations of the NLS, derivative NLS, modified KdV as well as their discrete analogs [21], we can obtain various interesting variants of Proposition A.5.

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