THE NUMBER OF CLOSED IDEALS OF $\mathcal{L}(\ell_p \oplus \ell_q)$

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ABSTRACT. We prove that for $1 < p < q < \infty$, the algebra $\mathcal{L}(\ell_p \oplus \ell_q)$ of all bounded linear operators on $\ell_p \oplus \ell_q$ has $2^{2^\omega}$ many closed ideals.

1. Introduction

Given Banach spaces $X$ and $Y$, we call a subspace $\mathcal{J}$ of the space of bounded operators $\mathcal{L}(X,Y)$ an ideal if $BTA \in \mathcal{J}$ for all $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $T \in \mathcal{J}$. In the case that $X = Y$, this coincides with the standard algebraic definition of $\mathcal{J}$ being an ideal in the algebra of bounded operators $\mathcal{L}(X)$. In this paper we will only be considering closed ideals. For example, if $X$ and $Y$ are any Banach spaces, then the space of compact operators from $X$ to $Y$ and the space of strictly singular operators from $X$ to $Y$ are both closed ideals in $\mathcal{L}(X,Y)$. If $X$ contains a complemented subspace $Z$ such that $Z$ is isomorphic to $Z \oplus Z$, then the closure of the set of operators in $\mathcal{L}(X)$ which factor through $Z$ is a closed ideal. In the case $1 < p < \infty$ with $p \neq 2$, there are infinitely many (even uncountably many) distinct complemented subspaces of $L_p$ which are isomorphic to their square, and thus there are infinitely many distinct closed ideals in $\mathcal{L}(L_p)$.

Obviously, constructing infinitely many closed ideals for $\mathcal{L}(\ell_p \oplus \ell_q)$ or $\mathcal{L}(\ell_p \oplus c_0)$ with $1 \leq p < q < \infty$ requires different techniques than just considering complemented subspaces, and it was a long outstanding question from Pietsch’s book [4] whether these spaces have infinitely many distinct closed ideals. For the cases $1 \leq p < q < \infty$, the operators which factor through $\ell_p$ and the operators which factor through $\ell_q$ are distinct closed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$, and all other proper closed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$ correspond to closed ideals in $\mathcal{L}(\ell_p, \ell_q)$. Progress on constructing new ideals in $\mathcal{L}(\ell_p, \ell_q)$ proceeded through building finitely many ideals at a time (see [6] and [7]) until it was shown using finite-dimensional versions of Rosenthal’s $X_{p,w}$ spaces that there is a chain of $2^\omega$ distinct closed ideals in $\mathcal{L}(\ell_p, \ell_q)$ for all $1 < p < q < \infty$ [9]. For $1 < p < \infty$, $p \neq 2$, $\ell_p \oplus \ell_2$ is isomorphic to a complemented subspace of $L_p$, and thus there are at least $2^\omega$ distinct closed ideals in $\mathcal{L}(L_p)$. Other new constructions for building infinitely many closed ideals soon followed. Wallis observed [10] that the techniques of [9] extend to prove the existence of a continuum chain of closed ideals for $\mathcal{L}(\ell_p, c_0)$ in the range $1 < p < 2$, and for $\mathcal{L}(\ell_1, \ell_q)$ in the range $2 < q < \infty$. Then, using ordinal indices, Sirotnik and Wallis proved that there is an $\omega_1$-chain of closed ideals in $\mathcal{L}(\ell_1, \ell_q)$ for $1 < q \leq \infty$ as well as in $\mathcal{L}(\ell_1, c_0)$ and in $\mathcal{L}(\ell_p, \ell_\infty)$ for $1 \leq p < \infty$ [8]. Using RIP matrices, both chains and

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anti-chains of $2^\omega$ distinct closed ideals were constructed in $\mathcal{L}(\ell_p, c_0)$, $\mathcal{L}(\ell_p, \ell_\infty)$, and $\mathcal{L}(\ell_1, \ell_p)$ for all $1 < p < \infty$ [2].

Recently, using the infinite-dimensional $X_{n,w}$ spaces of Rosenthal and almost disjoint sequences of integers, Johnson and Schechtman proved that there is an anti-chain of $2^\omega$ distinct closed ideals in $\mathcal{L}(L_p)$ for $1 < p < \infty$ with $p \neq 2$ [3]. In particular, the cardinality of the set of closed ideals in $\mathcal{L}(L_p)$ is exactly $2^{2\omega}$. The technique of using almost disjoint sequences can be naturally combined with the construction in [2] to prove that there are anti-chains of $2^\omega$ distinct closed ideals in $\mathcal{L}(\ell_p, c_0)$, $\mathcal{L}(\ell_p, \ell_\infty)$, and $\mathcal{L}(\ell_1, \ell_p)$ for all $1 < p < \infty$ [1]. The goal of this paper is to prove that there are anti-chains of $2^\omega$ distinct closed ideals in $\mathcal{L}(\ell_p, \ell_q)$ for all $1 < p < q < \infty$. Hence, the cardinality of the set of closed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$ is exactly $2^{2\omega}$ for all $1 < p < q < \infty$. This is achieved by first constructing an anti-chain of closed ideals of size $2^\omega$, generated by a single operator, from which a bigger anti-chain of size $2^{2\omega}$ is built. Since in the previous paper [9], the continuum many distinct closed ideals in $\mathcal{L}(\ell_p, \ell_q)$ form a chain, this extension requires additional work and the techniques in [3] cannot be immediately applied.

2. A general approach

Let $X$ and $Y$ be Banach spaces and let $T \subset \mathcal{L}(X, Y)$. The ideal generated by $T$ is the smallest closed ideal in $\mathcal{L}(X, Y)$ containing $T$ and is denoted by $\mathcal{J}^T(X, Y)$. That is, $\mathcal{J}^T(X, Y)$ is the closure in $\mathcal{L}(X, Y)$ of the set

$$\left\{ \sum_{j=1}^{n} A_j T_j B_j : n \in \mathbb{N}, (A_j)_{j=1}^{n} \subset \mathcal{L}(Y), (T_j)_{j=1}^{n} \subset T, (B_j)_{j=1}^{n} \subset \mathcal{L}(X) \right\}$$

consisting of finite sums of operators factoring through members of $T$. When $T = \{\{T\}\}$ consists of a single operator $T \in \mathcal{L}(X, Y)$, then we write $\mathcal{J}^T(X, Y)$ instead of $\mathcal{J}^T(X, Y)$.

In [2], for each $1 < p < \infty$, a collection of operators $(T_N)_{N \in \mathbb{N}} \subset \mathcal{L}(\ell_p, c_0)$ is constructed such that $\mathcal{J}^{T_M}(\ell_p, c_0) \neq \mathcal{J}^{T_N}(\ell_p, c_0)$ whenever $M \triangle N$ is infinite. For a non-empty family $\mathcal{A}$ of subsets of $\mathbb{N}$, let $\mathcal{J}_{\mathcal{A}}$ be the closed ideal of $\mathcal{L}(\ell_p, c_0)$ generated by $\{T_N : N \in \mathcal{A}\}$. There are at most $2^\omega$ closed ideals in $\mathcal{L}(\ell_p, c_0)$ which are generated by a single operator. However, it was observed in [1] that if $\mathcal{C}$ is an almost disjoint family of infinite subsets of $\mathbb{N}$, then $\{\mathcal{J}_{\mathcal{A}} : \mathcal{A} \subset \mathcal{C}, \mathcal{A} \neq \emptyset\}$ is an anti-chain of $2^\omega$ distinct closed ideals in $\mathcal{L}(\ell_p, c_0)$.

Let $X$ and $Y$ be Banach spaces. In this section we formulate a general criterion for the condition that $\mathcal{L}(X, Y)$ has at least $2^\omega$ distinct closed ideals. Let $U$ be the 1-unconditional sum $U = \left( \bigoplus_{n=1}^{\infty} E_n \right)_W$ of a sequence $(E_n)_{n=1}^{\infty}$ of finite-dimensional spaces. This means that $W$ has a 1-unconditional basis $(e_n)$ and

$$U = \left\{ (x_n) : x_n \in E_n \text{ for all } n \in \mathbb{N} \text{ and } \left\| (x_n) \right\|_U = \left\| \sum_{n=1}^{\infty} \left\| x_n \right\|_e e_n \right\|_W < \infty \right\}.$$ 

For $n \in \mathbb{N}$ we call $E_n$ the $n^{th}$ component of $U$. We shall assume that $U$ is complemented in $X$ and let $P : X \to U$ be a bounded projection onto $U$.

Secondly, we assume that there is an 1-unconditional sum $V = \left( \bigoplus_{n=1}^{\infty} F_n \right)_W$, of some sequence $(F_n)_{n=1}^{\infty}$ of Banach spaces which embeds into $Y$.

We are then given a sequence $(T_n)$ of operators $T_n : E_n \to F_n$ with $\sup \|T_n\| < \infty$ and we put $T : X \to Y$, $T(x) = \left( T_n(P(x)) \right)_{n=1}^{\infty} \in V \subset Y$. For $N \subset \mathbb{N}$ we define
Rosenthal \[5\] that there exists a constant $K$ such that $\|\cdot\|_{\ell_p}$ is 3-valued, disjoint sets. Thus, we can and will think of

\[
L_n = T \circ Q_N \circ P.
\]

For each non-empty $A \subset \mathbb{N}$, we let $\mathcal{J}_A$ be the closed ideal of $\mathcal{L}(X,Y)$ generated by $\{T_M : M \in A\}$. The following Proposition was observed in [1].

**Proposition 1.** Assume that for some $c > 0$ the sequence $(T_n)_{n=1}^\infty$ satisfies the following condition for each $M, N \subset \mathbb{N}$.

1. If $M \setminus N$ is infinite then $T_M \not\in \mathcal{J}_S^T$, and moreover $\text{dist}(T_M, \mathcal{J}^T_N) \geq c$.
2. If $N \setminus M$ is finite then $\mathcal{J}_N \subset \mathcal{J}_M^T$.

Let $\mathcal{C}$ be an almost disjoint family of infinite subsets of $\mathbb{N}$. This means that for $A, B \in \mathcal{C}$, $A \neq B$, it follows that $A \cap B$ is finite.

Then $\mathcal{J}_A \neq \mathcal{J}_B$, for any two distinct non-empty subsets $A$ and $B$ of $\mathcal{C}$. In particular, since we can choose $\mathcal{C}$ to have cardinality $2^{2^\infty}$, $\mathcal{L}(X,Y)$ has at least $2^{2^\infty}$ many distinct closed ideals.

### 3. Preliminaries

We recall some notation and results from [9].

#### 3.1. Finite-dimensional versions of Rosenthal’s $X_{p,w}$ space.

In this paper we shall use finite-dimensional versions of Rosenthal’s $X_{p,w}$ spaces, and we will only need the result about the existence of well-isomorphic and well-complemented copies in $\ell_p$. We begin with some definitions.

Given $2 < p < \infty$, $0 < w \leq 1$ and $n \in \mathbb{N}$, we denote by $E_{p,w}^{(n)}$ the Banach space $(\mathbb{R}^n, \|\cdot\|_{p,w})$, where

\[
\| (a_j)_{j=1}^n \|_{p,w} = \left( \sum_{j=1}^n |a_j|^p \right)^{\frac{1}{p}} \lor w \left( \sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}}.
\]

We write $\{e_j^{(n)} : 1 \leq j \leq n\}$ for the unit vector basis of $E_{p,w}^{(n)}$, and we denote by $\{e_j^{(n)*} : 1 \leq j \leq n\}$ the unit vector basis of the dual space $(E_{p,w}^{(n)})^*$, which is biorthogonal to the unit vector basis of $E_{p,w}^{(n)}$.

Given $1 < p < 2$, $0 < w \leq 1$ and $n \in \mathbb{N}$, we fix once and for all a sequence $f_j^{(n)} = f_{p,w,j}^{(n)}$, $1 \leq j \leq n$, of independent symmetric, 3-valued random variables with $\|f_j^{(n)}\|_{L_p} = 1$ and $\|f_j^{(n)}\|_{L_2} = \frac{1}{w}$ for $1 \leq j \leq n$ (these two equalities determines the distribution of a 3-valued symmetric random variable). We then define $E_{p,w}^{(n)}$ to be the subspace span$\{f_j^{(n)} : 1 \leq j \leq n\}$ of $\ell_p$. It follows from the work of Rosenthal [5] that there exists a constant $K_p > 0$ which only depends on $p$, so that for all scalars $(a_j)_{j=1}^n$ we have

\[
\frac{1}{K_p} \left\| \sum_{j=1}^n a_j e_j^{(n)*} \right\| \leq \left\| \sum_{j=1}^n a_j f_j^{(n)} \right\|_{L_p} \leq \left\| \sum_{j=1}^n a_j e_j^{(n)*} \right\|,
\]

where $\{e_j^{(n)*} : 1 \leq j \leq n\}$ is the unit vector basis of the dual space $(E_{p,w}^{(n)})^*$ as defined above and $p'$ is the conjugate index of $p$. Since the random variables $f_j^{(n)}$ are 3-valued, $E_{p,w}^{(n)}$ is a subspace of the span of indicator functions of $3^n$ pairwise disjoint sets. Thus, we can and will think of $E_{p,w}^{(n)}$ as a subspace of $\ell_3^n$. The following result follows directly from Rosenthal’s work [5].
Proposition 2. [9, Proposition 1] Let $1 < p < 2$, $0 < w \leq 1$ and $n \in \mathbb{N}$. Then

(i) $\{f_j^{(n)} : 1 \leq j \leq n\}$ is a normalized, 1-unconditional basis of $F_{p,w}^{(n)}$.

(ii) There exists a projection $\mathcal{P}_{p,w}^{(n)} : \ell_p^w \to \ell_p^w$ onto $\mathcal{F}_{p,w}^{(n)}$ with $\|\mathcal{P}_{p,w}^{(n)}\| \leq K_p$.

(iii) For each $1 \leq k \leq n$ and for every $A \subset \{1, \ldots, n\}$ with $|A| = k$ we have

$$\frac{1}{K_p} \left( k^\frac{1}{p} \wedge \frac{1}{w} k^\frac{1}{2} \right) \leq \left\| \sum_{j \in A} f_j^{(n)} \right\| \leq k^\frac{1}{p} \wedge \frac{1}{w} k^\frac{1}{2}.$$  

4. The spaces $Y_{p,\bar{k},\bar{w}}$

For $1 < p < 2$, $\bar{w} = (w_n) \subset (0,1]$ and $\bar{k} = (k_n) \subset \mathbb{N}$ we let

$$Y_{p,\bar{k},\bar{w}} = \left( \bigoplus_{n=1}^\infty F_{p,w_n}^{(k_n)} \right) \ell_p.$$  

Note that $Y_{p,\bar{k},\bar{w}}$ is a $K_p$-complemented subspace of $\ell_p$. Indeed, the diagonal operator

$$P_{p,\bar{k},\bar{w}} = \text{diag} \left( P_{p,w_n}^{(k_n)} : \ell_p \cong \left( \bigoplus_{n=1}^\infty \ell_p^{w_n} \right) \ell_p \right),$$

where $P_{p,w_n}^{(k_n)}$ was defined in Proposition 2, is a projection onto $Y_{p,\bar{k},\bar{w}}$. Furthermore, $\{f_j^{(n)} : n \in \mathbb{N}, 1 \leq j \leq n\}$ is a normalized, 1-unconditional basis of $Y_{p,\bar{k},\bar{w}}$.

Note that as $Y_{p,\bar{k},\bar{w}}$ is a complemented subspace of $\ell_p$, we have by Pelczyński’s Decomposition Theorem that $Y_{p,\bar{k},\bar{w}}$ is isomorphic to $\ell_p$. However, we shall never make this identification, and instead consider $Y_{p,\bar{k},\bar{w}}$ as a complemented subspace of $\ell_p$ with corresponding projection $P_{p,\bar{k},\bar{w}}$ fixed as above.

Before continuing, we recall the following definitions. Let $X$ be a Banach space with a fixed normalized, 1-unconditional basis $(x_i)$ (finite or infinite). Let $N = \mathbb{N}$ if $\dim(X) = \infty$, and $N = \{1,2,\ldots,\dim(X)\}$ otherwise. We define the fundamental function $\varphi_X : N \to \mathbb{R}$ of $X$ by setting

$$\varphi_X(k) = \sup \left\{ \left\| \sum_{i \in A} x_i \right\| : A \subset N, |A| \leq k \right\} , \quad k \in N.$$  

We then extend the definition of $\varphi_X$ to the real interval $I = [1, \dim(X)]$, respectively $I = [1, \infty)$ by linear interpolation. We next introduce the lower fundamental function $\lambda_X : N \to \mathbb{R}$ of $X$ defined by

$$\lambda_X(k) = \inf \left\{ \left\| \sum_{i \in A} x_i \right\| : A \subset N, |A| \geq k \right\} , \quad k \in N,$$  

and extend the definition to $I$ by linear interpolation. The lower estimate in Lemma 3 of the lower fundamental function of $Y_{p,\bar{k},\bar{w}}$ follows easily from [9, Lemma 3] and its proof.

Lemma 3. Let $\bar{k} = (k_n)$ be strictly increasing in $\mathbb{N}$ and $\bar{w} = (w_n)$ be decreasing in $(0,1]$. Then for all $m \in \mathbb{N}$ we have

$$\lambda_{p,\bar{k},\bar{w}}(m) \geq \frac{1}{K_p} \left( \left( \frac{m}{2} \right)^{1/p} \wedge \left( \sum_{j=1}^{s-1} k_j + \frac{t}{w_2} \right)^{1/2} \right),$$

where $s = s(m) \in \mathbb{N}$ is maximal so that $\sum_{j=1}^{s-1} k_j \leq m/2$ and $t = m/2 - \sum_{j=1}^{s-1} k_j$.  

In particular, if $m \leq k_1$ then
\[
\lambda_{Y_p, \mathfrak{m}}(m) \geq \frac{1}{2K_p} \left( m^{1/p} \wedge \frac{m^{1/2}}{w_1} \right).
\]

Remark. It is important to note that for $m \in \mathbb{N}$, there is a lower estimate for $\lambda_{Y, \mathfrak{m}}(m)$ which only depends on $w_1, w_2, \ldots, w_s$, where $s = s(m)$ is defined as in the lemma.

4.1. Corollary of the Key Lemma (Lemma 4) in [9]. Recall the following result from [9].

**Lemma 4.** ([9, Lemma 4]) Let $Y$ be an infinite-dimensional Banach space with a normalized, 1-unconditional basis $(f_j)$. For each $m \in \mathbb{N}$ let $G_m$ be an $m$-dimensional Banach space with a normalized, 1-unconditional basis $\{g^{(m)}_i : 1 \leq i \leq m \}$. Assume that

\begin{align*}
(2) & \quad \lim_{k \to \infty} \sup_{m \geq k} \frac{\varphi_{G_m}(k)}{k} = 0, \quad \text{and} \\
(3) & \quad \lim_{m \to \infty} \frac{\varphi_{G_m}(m)}{\lambda_Y(cm)} = 0 \quad \text{for all } c > 0.
\end{align*}

If $(B_m : G_m \to Y)_{m=1}^{\infty}$ is a sequence of operators with $\sup_m \|B_m\| \leq 1$, then
\[
\frac{1}{m} \sum_{i=1}^{m} \left\| B_m(g^{(m)}_i) \right\|_{\infty} \to 0 \quad \text{as } m \to \infty.
\]

Here $\|y\|_{\infty} = \sup |y_j|$ for $y = \sum_j y_j f_j \in Y$.

For $1 < p < 2$, $k_0 \in \mathbb{N}$ and $w_0 \in (0, 1]$ we put
\[
\mathcal{Y}_{p, k_0, w_0} = \{ (y_{p, k_0, w_0} : w = (w_n) \subset (0, w_0) \text{ is decreasing, } \mathcal{K} = (k_n) \in \mathcal{N}, k_0 \leq k_1 \}.
\]

If $Y \in \mathcal{Y}_{p, k_0, w_0}$, say $Y = (\bigoplus_{n=1}^{\infty} F_{p, w_n}(k_n))_{\ell_p}$, we call $F_{p, w_n}(k_n)$ the $n^{th}$ component of $Y$.

The following result can be deduced from [9, Lemma 4]:

**Corollary 5.** For all $\varepsilon \in (0, 1)$ there exist $k_0 = k_0(\varepsilon) \in \mathbb{N}$, $w_0 = w_0(\varepsilon) \in (0, 1)$, $c = c(\varepsilon) > 0$ and $\delta = \delta(\varepsilon) > 0$ so that:

For all $m \in \mathbb{N}$ with $cm \geq 1$, for all $m$-dimensional spaces $G$ with a normalized and 1-unconditional basis $(g_j)_{j=1}^{m}$ dominated by the unit vector basis of $\ell_p^m$, for all $Y \in \mathcal{Y}_{p, k_0, w_0}$ with the property that $\varphi_Y(G)/\lambda_Y(cm) < \delta$ and for all $B \in \mathcal{L}(G, Y)$ with $\|B\| \leq 1$ it follows that
\[
\frac{1}{m} \sum_{j=1}^{m} \|B(g_j)\|_{\infty} < \varepsilon.
\]

**Proof.** Assume our claim is not true. Then there is an $\varepsilon_0 \in (0, 1)$ so that:

(*\ ) For all $k_0 \in \mathbb{N}$, $w_0 \in (0, 1)$, $c > 0$ and $\delta > 0$, there exist $m \in \mathbb{N}$ with $cm \geq 1$, an $m$-dimensional space $G$ with a normalized and 1-unconditional basis $(g_j)_{j=1}^{m}$
dominated by the unit vector basis of \( \ell_p^m \), a space \( Y \in \mathcal{Y}_{p,k_0,w_0} \) and an operator \( B \in \mathcal{L}(G,Y) \) with \( \|B\| \leq 1 \) so that
\[
\frac{\varphi_G(m)}{\lambda_Y(cm)} < \delta \quad \text{and} \quad \frac{1}{m} \left\| \sum_{j=1}^{m} B(g_j) \right\|_\infty > \varepsilon_0.
\]

Since the space \( G \) in \((*)\) is finite-dimensional, we can assume after an arbitrary small perturbation of the operator \( B \in \mathcal{L}(G,Y) \) that the image of \( G \) is a subset of the sum of only finitely many components of \( Y \). Condition \((*)\) can therefore be reformulated into:

\[(**) \quad \text{For all } k_0 \in \mathbb{N}, w_0 \in (0,1), c > 0 \text{ and } \delta > 0, \text{ there exist } m \in \mathbb{N} \text{ with } cm \geq 1, \text{ an } m\text{-dimensional space } G \text{ with a normalized and 1-unconditional basis } (g_j)_{j=1}^{m} \#
\]
dominated by the unit vector basis of \( \ell_p^m \), an \( n \in \mathbb{N} \), a strictly increasing sequence \( (k_j)_{j=1}^{n} \subset \mathbb{N} \) with \( k_0 \leq k_1 \), a decreasing sequence \( (w_j)_{j=1}^{n} \subset (0,w_0] \>
\]and an operator \( B \in \mathcal{L}(G,Y) \) where \( Y = \left( \bigoplus_{j=1}^{n} F_{p,w_j}^{(k_j)} \right)_{\ell_p} \), so that
\[
\frac{\varphi_G(m)}{\lambda_Y(cm)} < \delta \quad \text{and} \quad \frac{1}{m} \left\| \sum_{j=1}^{m} B(g_j) \right\|_\infty > \varepsilon_0.
\]

Using now \((**)\) we can by induction find decreasing sequences \((\delta_l)\) and \((c_l)\) in \((0,1)\) both converging to zero, and strictly increasing sequences \((k_j)\) and \((m_l)\) in \( \mathbb{N} \) so that for each \( l \in \mathbb{N} \), we have \( c_l m_l \geq 1 \), there is an \( m_l\)-dimensional space \( G_l \) with a normalized, 1-unconditional basis \((g_j^{(l)})_{j=1}^{m_l}) \), dominated by the unit vector basis of \( \ell_p^{m_l} \), there are positive integers \( n_l \) and \( k_{n_l-1} < k_{n_l-1} < k_{n_l-1} < \dots < k_{n_l} \) so that \( k_{n_l+1} < k_{n_l+2} < \dots < k_{n_l} \) and finally there is an operator \( B_l \in \mathcal{L}(G_l,Y_l) \) with \( \|B_l\| \leq 1 \), where \( Y_l = \left( \bigoplus_{j=1}^{m_l} F_{p,w_j}^{(k_j)} \right)_{\ell_p} \), so that
\[
\frac{\varphi_{G_l}(m_l)}{\lambda_Y(c_l m_l)} < \delta \quad \text{and} \quad \frac{1}{m_l} \left\| \sum_{j=1}^{m_l} B_l(g_j^{(m_l)}) \right\|_\infty > \varepsilon_0.
\]

Now we let \( Y = \left( \bigoplus_{j=1}^{\infty} F_{p,w_j}^{(k_j)} \right)_{\ell_p} \). We let \( \widetilde{G}_m = G_l \), \( \widetilde{B}_m = B_l \) and \( \widetilde{g}_j^{(m)} = g_j^{(l)} \) for \( 1 \leq j \leq m_l \) if \( m = m_l \) for some \( l \in \mathbb{N} \), and we let \( \widetilde{G}_m = \ell_p\), \( \widetilde{B}_m = 0 \) and \( \widetilde{g}_j^{(m)} = g_j^{(l)} \) be the unit vector basis of \( \ell_p^m \) if \( m \notin \{m_l : l \in \mathbb{N} \} \). Then it follows that
\[
\lim_{k \to \infty} \sup_{m \geq k} \frac{\varphi_{\widetilde{G}_m}(k)}{k} = 0 \quad \text{and} \quad \lim_{m \to \infty} \frac{\varphi_{\widetilde{G}_m}(m)}{\lambda_Y(cm)} = 0 \quad \text{for all } c > 0
\]
but
\[
\lim_{m \to \infty} \frac{1}{m} \left\| \sum_{j=1}^{m} B(\widetilde{g}_j^{(m)}) \right\|_\infty \geq \varepsilon_0.
\]

This contradicts [9, Lemma 4].

The goal is to use Corollary 5 to prove the following.

**Proposition 6.** Let \((\varepsilon_n)\) be a null sequence in \((0,1)\). Then there exist a strictly increasing sequence \( \tilde{k} = (k_n) \in \mathbb{N} \), and a decreasing sequence \( \overline{m} = (w_n) \in (0,1] \) so
that the following holds: For all \( m \in \mathbb{N} \), for all \( B \in \mathcal{L}(F_{p,w_n}^{(k_m)}, Y_m) \) with \( \|B\| \leq 1 \), where \( Y_m = \left( \bigoplus_{n=1}^{\infty} F_{p,w_n}^{(k_m)} \right)_{\ell_p} \), it follows that

\[
\frac{1}{k_m} \sum_{j=1}^{k_m} \|B(f_j^{(k_m)})\|_\infty < \varepsilon_m .
\]

We will need the following lemma.

**Lemma 7.** Let \( n \in \mathbb{N} \), \( 1 < r \leq \infty \) and \( \varepsilon > 0 \), and let

\[
N \geq \left[ \frac{2n}{(\varepsilon/2)^{1+1/r}} \right] .
\]

Let \( H \) be any \( N \)-dimensional space with a normalized and 1-unconditional basis \( (h_j) \) which is dominated by the unit vector basis of \( \ell_1^n \), and let \( B \in \mathcal{L}(H, \ell_\infty^n) \) with \( \|B\| \leq 1 \). Then it follows that

\[
\frac{1}{N} \left\| \sum_{j=1}^{N} B(h_j) \right\|_\infty < \varepsilon .
\]

**Proof.** For \( x \in H \), we will write \( B(x) \) as \( (B_i(x))_{i=1}^{n} \) \( \in \mathbb{R}^n \). For each \( i = 1, 2, \ldots, n \) we put

\[
A_i^+ = \{ j \in \{1, 2, \ldots, n\} : B_i(h_j) \geq \varepsilon/2 \}
\]

and

\[
A_i^- = \{ j \in \{1, 2, \ldots, n\} : B_i(h_j) \leq -\varepsilon/2 \} .
\]

Then on the one hand we have

\[
\left\| B \left( \sum_{j \in A_i^+} h_j \right) \right\|_\infty \leq \left\| \sum_{j \in A_i^+} h_j \right\| \leq |A_i^+|^{1/r} ,
\]

and on the other hand

\[
\left\| B \left( \sum_{j \in A_i^+} h_j \right) \right\|_\infty \geq \left| \left( \sum_{j \in A_i^+} B_i(h_j) \right) \right| \geq \varepsilon |A_i^+|/2 .
\]

It follows therefore that

\[
|A_i^+| \leq (\varepsilon/2)^{-1/r} .
\]

Since the same inequality also holds for \( |A_i^-| \), we deduce that

\[
\left| \{ j : \|B(h_j)\|_\infty \geq \varepsilon/2 \} \right| \leq 2n(\varepsilon/2)^{-1/r} ,
\]

and thus

\[
\frac{1}{N} \sum_{j=1}^{N} \|B(h_j)\|_\infty \leq \frac{1}{N} \left( \frac{N}{2} + 2n \left( \frac{\varepsilon}{2} \right)^{-1/r} \right) \leq \frac{\varepsilon}{2} + \frac{1}{2} \frac{2n}{(\varepsilon/2)^{1+1/r}} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \frac{1}{N} (\varepsilon/2)^{1+1/r} \leq \varepsilon .
\]

\[ \square \]

**Proof of Proposition 6.** We put \( k_{0,n} = k_0(\varepsilon_n/2) \), \( w_{0,n} = w_0(\varepsilon_n/2) \), \( c_n = c(\varepsilon_n/2) \) and \( \delta_n = \delta(\varepsilon_n/2) \) satisfying the conclusion of Corollary 5.

Inductively for every \( n \in \mathbb{N} \) we choose numbers \( k_n \in \mathbb{N} \) and \( w_n \in (0,1) \) so that
(a) \( c_n k_n \geq 1 \) and \( \frac{1}{2K_p} (c_n k_n)^{1/p} > \frac{1}{\delta_n} w_n \),
(b) \( k_n \geq k_{0,n} \) and \( w_n \leq w_{0,n} \),
(c) if \( n > 1 \) then \( k_n > k_{n-1} \) and \( w_n \leq w_{n-1} \),
(d) if \( n > 1 \) then for any operator \( B \in \mathcal{L}(F_{p,w_n}, (\bigoplus_{j=1}^{n-1} F_{p,\tilde{w}_j})\ell_p) \) with \( \|B\| \leq 1 \) it follows that
\[
\frac{1}{k_n} \sum_{j=1}^{k_n} \|B(f_{(k_n)}^j)\|_\infty \leq \frac{\varepsilon_n}{2},
\]
e (e) if \( n > 1 \) then for every \( Y \in \mathcal{Y}_{p,k_n-1,w_{n-1}} \) (i.e., \( Y = (\bigoplus_{j=1}^{k_n-1} F_{p,\tilde{w}_j})\ell_p \)) with \( k_{n-1} \leq \tilde{k}_1 < \tilde{k}_2 < \ldots \) and \( w_{n-1} = \tilde{w}_1 \geq \tilde{w}_2 \geq \ldots \), and for any \( B \in \mathcal{L}(F_{p,w_n-1}, Y) \) with \( \|B\| \leq 1 \), it follows that
\[
\frac{1}{k_{n-1}} \sum_{j=1}^{k_{n-1}} \|B(f_{(k_{n-1})}^j)\|_\infty \leq \frac{\varepsilon_{n-1}}{2}.
\]
If this is done, then we observe that for \( m \in \mathbb{N} \) and \( B \in \mathcal{L}(F_{p,w_m}, (\bigoplus_{j=m+1}^{n} F_{p,\tilde{w}_j})\ell_p) \), \( \|B\| \leq 1 \), we can write \( B = B_1 + B_2 \) with \( B_1 \in \mathcal{L}(F_{p,w_m}, (\bigoplus_{j=m+1}^{n-1} F_{p,\tilde{w}_j})\ell_p) \) and \( B_2 \in \mathcal{L}(F_{p,w_n}, (\bigoplus_{j=m+1}^{\infty} F_{p,\tilde{w}_j})\ell_p) \), and deduce from (d) (applied to \( n = m \)) that
\[
\frac{1}{k_m} \sum_{j=1}^{k_m} \|B_1(f_{(k_m)}^j)\|_\infty \leq \frac{\varepsilon_n}{2},
\]
and from (e) (applied to \( n = m + 1 \)) that
\[
\frac{1}{k_m} \sum_{j=1}^{k_m} \|B_2(f_{(k_m)}^j)\|_\infty \leq \frac{\varepsilon_n}{2}
\]
which implies our claim.

We are now choosing recursively \( w_n \) and \( k_n \) so that (a)–(e) are satisfied.

We first choose \( w_1 = w_{0,1} \) and then \( k_1 \geq k_{0,1} \) large enough to satisfy (a). Then (b) is satisfied and the other conditions (c)–(e) are vacuous for \( n = 1 \). So assume we have chosen for some \( n \in \mathbb{N} \), numbers \( k_1 < k_2 < \ldots < k_n \) in \( \mathbb{N} \) and \( 1 \geq w_1 \geq \ldots \geq w_n \) in \((0,1)\). We then first choose \( w_{n+1} \) so that \( 0 < w_{n+1} \leq w_{0,n+1} \), \( w_{n+1} \leq w_n \) and \( w_{n+1} \) is small enough so that
\[
(c_n k_n)^{1/p} \leq \frac{(c_n k_n)^{1/2}}{w_{n+1}}.
\]
Then by (a) we have
\[
\frac{1}{2K_p} (c_n k_n)^{1/p} > \frac{1}{\delta_n} w_n \frac{1}{w_{n+1}} k_n^{1/2}.
\]
Then we choose \( k_{n+1} \) so that the following three inequalities hold:
\[
k_{n+1} \geq k_{0,n+1},
\]
\[
k_{n+1} \geq \left[ \frac{2 \sum_{j=1}^{k_n} j}{(c_{n+1}/2)^{1/p-n-2}} \right],
\]
and I where the projection $e$ of $p$ of $L$ of $p_{n+1}$. We define for $n$ from (7) and Lemma 7. In order to verify (e) (for $n+1$), let $Y \in Y_{p,k_n,w_n}$. Then by Lemma 3, (5) and Proposition 2 (iii) we have

$$\lambda_Y(c_n k_n) \geq \frac{1}{2K_p} \left( (c_n k_n) \wedge \frac{(c_n k_n)^{1/2}}{w_{n+1}} \right) \frac{1}{\delta_n} \frac{k_n^{1/2}}{w_n} \geq \frac{1}{\delta_n} \varphi_{F_*}^w(k_n),$$

and thus

$$\frac{\varphi_{F_*}^w(k_n)}{\lambda_Y(c_n k_n)} \leq \delta_n,$$

which implies (d) by Corollary 5 and the choice of $k_n, w_n, c_n$ and $\delta_n$. 

5. The Main Result

Let $(\varepsilon_n)$ be a null sequence in $(0,1)$, and let $(k_n) \subset \mathbb{N}$ and $(w_n) \subset (0,1)$ satisfy the conclusion of Proposition 6. We define for $n \in \mathbb{N}$

$$T_n : \ell_p^{k_n} \to \ell_2^{k_n} \text{ by } T_n(x) = f_{p,2}^{(k_n)} \circ f_{p,k_n,w_n}(x),$$

where the projection $f_{p,k_n,w_n}$ onto $f_{p,w_n}$ was introduced in Proposition 2, and $\ell_2^{(k_n)} : F_{p,w_n} \to \ell_2^{k_n}$ is the formal identity that maps $f_{p,2}^{(k_n)}$ to the unit basis vector $e_2^{(k_n)}$ of $\ell_2^{k_n}$. Note that $f_{p,2}^{(k_n)}$ is bounded in norm by the cotype-2 constant of $L_p$. We next set

$$T = \text{diag}(T_n) : \ell_p = (\bigoplus_{n=1}^{\infty} \ell_p^{k_n}) \ell_p \to Z_q = (\bigoplus_{n=1}^{\infty} \ell_2^{k_n}) \ell_q, \quad (x_n) \mapsto (T_n(x_n)).$$

For $N \subset \mathbb{N}$ we put $T_N = T \circ Q_N$, where $Q_N$ is the canonical projection of elements $\ell_p = (\bigoplus_{n=1}^{\infty} \ell_p^{k_n}) \ell_p$ onto the components corresponding to elements of $N$. We are now ready to state our main result.

**Theorem 8.** For nonempty subsets $M, N$ of $\mathbb{N}$, it follows that

1. if $M \setminus N$ is infinite then $T_M \notin J^{TN}$, and moreover $\text{dist}(T_M, J^{TN}) \geq 1$, and
2. if $N \setminus M$ is finite then $J^{TN} \subset J^{TM}$.

We will need the following lemma from [9]. Recall that $p'$ is the conjugate index of $p$.

**Lemma 9.** [9, Lemma 5] Given $1 < p < 2$ and $p < q < \infty$, let $n \in \mathbb{N}$, $w \in (0,1]$ and $F = F_{p,w}$. Let $y = \sum_{j=1}^{n} y_j f_j$ in $F$ with $\|y\|_p \leq 1$, and let $\tilde{y} = \sum_{j=1}^{n} y_j f_{2,j} \in \ell_2^n$. If $\|y\|_\infty = \max_j |y_j| \leq \sigma \leq 1$ and $w \leq \sigma^{\frac{1}{p'}} - \frac{\sigma}{p} = \sigma^{\frac{1}{p'} - \frac{1}{2}}$, then

$$\|\tilde{y}\|_{\ell_2^n}^2 \leq D \sigma r \cdot \|y\|_F^p,$$

where $D$ only depends on $p$ and $q$, and $r = \min \left\{ \frac{1}{2} - \frac{\sigma}{q}, \frac{1}{2} - \frac{\sigma}{p} \right\}$.

**Proof of Theorem 8.** We first verify the second claim and assume that $N = N' \cup A$, where $N' \subset M$ and $A \subset \mathbb{N} \setminus M$ is finite. Then it is clear that $T_{N'} \in J^{TM}$ and since every nontrivial ideal contains all finite rank operators, we also have $T_A \in J^{TM}$, and thus $T_N = T_{N'} + T_A \in J^{TM}$.
In order to show (1), assume that $M \setminus N$ is infinite and let $(m(l))_{l=1}^{\infty}$ be a strictly increasing subsequence of $M \setminus N$. For $l \in \mathbb{N}$ we define the following element of the dual ball of $\mathcal{L}(\ell_p, Z_q)$. For $S \in \mathcal{L}(\ell_p, Z_q)$, we put

$$\Psi_l(S) = \frac{1}{k_m(l)} \sum_{j=1}^{k_m(l)} \langle S(f_j^{(m(l))}), e_{2,j}^{(m(l))} \rangle .$$

Let $\Psi$ be a $w^*$-accumulation point of the sequence $(\Psi_l)$ in $\mathcal{L}(\ell_p, Z_q)^*$.

Since $T_M(f_j^{(k_m(l))}) = e_{2,j}^{(k_m(l))}$ for all $l \in \mathbb{N}$ and all $j = 1, 2, \ldots, k_m(l)$, it follows that $\Psi_l(T_M) = 1$ for all $l \in \mathbb{N}$, and thus $\Psi(T_M) = 1$.

We will now show that $\lim_{l \to \infty} \Psi_l(A \circ T_N \circ B) = 0$ for all $A \in \mathcal{L}(Z_q)$ and $B \in \mathcal{L}(\ell_p)$.

It is clearly enough to show this when $\|A\| \leq 1$ and $\|B\| \leq 1$. Note that for $l \in \mathbb{N}$

$$|\Psi_l(A \circ T_N \circ B)| \leq \frac{1}{k_m(l)} \sum_{j=1}^{k_m(l)} \left| \langle AT_N B(f_j^{(m(l))}), e_{2,j}^{(m(l))} \rangle \right|$$

$$= \frac{1}{k_m(l)} \sum_{j=1}^{k_m(l)} \left| \langle IPQ_N B(f_j^{(m(l))}), A^* e_{2,j}^{(m(l))} \rangle \right|$$

$$\leq \frac{1}{k_m(l)} \sum_{j=1}^{k_m(l)} \|IB_l(f_j^{(m(l))})\|,$$

where $P = P_{p, w}$ was defined in Section 4, $I = \text{diag} \left( \left( f_j^{(k_n)} \right) : (\bigoplus F_{p, w}^{(k_n)})_{\ell_p} \to Z_q \right)$ is the formal identity, and where for $l \in \mathbb{N}$ we put $B_l = P \circ Q_N \circ B |_{F_{p, w}^{(k_m(l))}}$. Since $m(l) \notin N$, we can and will think of $B_l$ being an operator defined on $F_{p, w}^{(k_m(l))}$ into the space $Y^{(l)} := (\bigoplus_{n=1, n \neq m(l)}^{\infty} F_{p, w}^{(k_n)})_{\ell_p}$.

Fix $\rho \in (0, 1)$ and choose $n_0 \in \mathbb{N}$ so that $w_n \leq \sigma^{1/2} - \frac{1}{2}$ for all $n \geq n_0$, and put

$$C = \left( \sum_{n=1}^{n_0} k_n^2 \right)^{1/2} .$$

For $l \in \mathbb{N}$ with $m(l) > n_0$, and for $1 \leq j \leq k_m(l)$ write $x = B_l(f_j^{(k_m(l))})$ as

$$x = \sum_{n=1, n \neq m(l)}^{\infty} x_{n,j} f_j^{(k_m(l))}$$

and put

$$\sigma_j^{(l)} := \rho \lor \|B_l(f_j^{(k_m(l))})\|_{\ell_q} = \rho \lor \sup_{n \in \mathbb{N}, n \neq m(l)} \max_{1 \leq j \leq k_n} |x_{n,j}| .$$

We compute, abbreviating $\sigma = \sigma_j^{(l)}$,

$$\|IB_l(f_j^{(k_m(l))})\|_{Z_q} = \left( \sum_{n=1, n \neq m(l)}^{\infty} \left( \sum_{j=1}^{k_n} x_{n,j}^2 \right)^{q/2} \right)^{1/q} .$$
The number of closed ideals of \( \mathcal{L}(\ell_p \oplus \ell_q) \)

\[
\left( \sum_{n=1}^{n_0} \left( \sum_{j=1}^{k_n} x_{n,j}^2 \right)^{q/2} \right)^{1/q} + \left( \sum_{n=n_0+1, n \neq m(l)}^{\infty} \left( \sum_{j=1}^{k_n} x_{n,j}^2 \right)^{q/2} \right)^{1/q}
\]

\[
\leq C \max_{n \leq n_0, j \leq k_n} |x_{n,j}| + D^\frac{1}{q} \sigma^\frac{q}{r} \left( \sum_{n>n_0, n \neq m(l)} \left( \sum_{j=1}^{k_n} x_{n,j} f_j(k_n) \right)^p \right)^{\frac{1}{p}}
\]

(By definition of \( C \) and by Lemma 9 with \( r = \min \{ \frac{p}{2}, \frac{q}{2}, \frac{q}{p'} \} \).

\[
\leq C \| B_l(f_j^{k_n(l)}) \|_\infty + D^\frac{1}{q} \sigma^\frac{q}{r} \| B_l(f_j^{k_n(l)}) \|_{Y(l)}
\]

\[
\leq C \| B_l(f_j^{k_n(l)}) \|_\infty + D^\frac{1}{q} \sigma^\frac{q}{r} .
\]

Thus

\[
\frac{1}{k_{m(l)}} \sum_{j=1}^{k_{m(l)}} \| I \circ B_l(f_j^{k_{m(l)}}) \|
\]

\[
\leq \frac{C}{k_{m(l)}} \sum_{j=1}^{k_{m(l)}} \| B_l(f_j^{k_{m(l)}}) \|_\infty + \frac{D^\frac{1}{q}}{k_{m(l)}} \sum_{j=1}^{k_{m(l)}} (\sigma_j^{(l)})^\frac{1}{r}
\]

\[
\leq \frac{C}{k_{m(l)}} \sum_{j=1}^{k_{m(l)}} \| B_l(f_j^{k_{m(l)}}) \|_\infty + D^\frac{1}{q} \left( \frac{1}{k_{m(l)}} \sum_{j=1}^{k_{m(l)}} (\sigma_j^{(l)})^\frac{1}{r} \right)^\frac{1}{q},
\]

Note that \( r \leq q \), and so the map \( t \rightarrow t^\frac{1}{r} \) is concave, which justifies the last line. It follows from the above that

\[
\limsup_{t \to \infty} \Psi(t) (A \circ T_N \circ B) \leq D^\frac{1}{q} p^\frac{1}{r},
\]

which proves our claim since \( \rho \in (0, 1) \) was arbitrary. \( \square \)

Combining now Proposition 1 and Theorem 8, and its dualization, with the main results in [1] we obtain the following Corollary.

**Corollary 10.** Let \( 1 \leq p < q \leq \infty \), but \( \{p, q\} \neq \{1, \infty\} \). Then the space \( \mathcal{L}(\ell_p, \ell_q) \) has \( 2^{2^r} \) many closed ideals, and if \( 1 < p < \infty \), the space \( \mathcal{L}(\ell_p, c_0) \) has \( 2^{2^r} \) many closed ideals.

**Proof.** If \( 1 < p < 2 \) and \( p < q < \infty \), the follows from Proposition 1 and Theorem 8. If \( 2 \leq p < q < \infty \) we note that \( 1 < q' < 2 \) and \( q' < p' < \infty \), and the claim follows from the fact that \( (\cdot)^* : \mathcal{L}(\ell_p, \ell_q) \to \mathcal{L}(\ell_{q'}, \ell_{p'}) \), \( T \mapsto T^* \), is a norm isometry and extends to a bijections between the closed ideals of \( \mathcal{L}(\ell_p, \ell_q) \) and the closed ideals of \( \mathcal{L}(\ell_{q'}, \ell_{p'}) \). If \( 1 < p < \infty \), then Corollary 1 in [1] states that \( \mathcal{L}(\ell_p, c_0) \) as well as \( \mathcal{L}(\ell_{q'}, \ell_{p'}) \) have \( 2^{2^r} \) closed ideals, and if \( 1 < q < \infty \), then it follows from Remark 1 in [1], that \( \mathcal{L}(\ell_1, \ell_q) \) has \( 2^{2^r} \) closed ideals. \( \square \)

**Remark.** For the algebras \( \mathcal{L}(\ell_1, c_0) \) and \( \mathcal{L}(\ell_1, \ell_\infty) \) the best result known to us is still the result obtained by Sirotkin and Wallis [8] which states that \( \mathcal{L}(\ell_1, c_0) \) and \( \mathcal{L}(\ell_1, \ell_\infty) \) both have an uncountable chain of closed ideals.
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