Analysis of stationary points and their bifurcations in the ABC flow

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Abstract

Analytical expressions for coordinates of stationary points and conditions for their existence in the ABC flow are received. The type of the stationary points is shown analytically to be saddle-node. Exact expressions for eigenvalues and eigenvectors of the stability matrix are given. Behavior of the stationary points along the bifurcation lines is described.

Keywords: ABC flow, stationary points, stability matrix, bifurcations

1. Introduction

The well-known Arnold–Beltrami–Childress (ABC) flow is a steady solution of the Euler equations for a steady incompressible flow of Newtonian fluids. Furthermore, the ABC flow can be considered as a solution of the Navier–Stokes equations if an external body force $f = -\nu \Delta \vec{V} = \nu \vec{V}$, just compensating viscous losses, is applied. Arnold [1] firstly suggested chaos in the field lines and, therefore, in trajectories in that three-dimensional steady flow. The chaotic aspect then has been developed and advanced for a wide class of two-dimensional unsteady flows under the name “chaotic advection” [2–9].

The ABC flow has been studied by many authors. Dombre et al. [10] investigated the ABC flow both analytically and numerically for different values of the real control parameters $A$, $B$, $C$. Henon [11] provided a numerical evidence for chaos in the special case at $A = \sqrt{3}$, $B = \sqrt{2}$ and $C = 1$. Independently

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Childress [12] considered the special case as a model for the kinematic dynamo effect where $A = B = C = 1$ (this version is useful in dynamo theory). In 1993, Zhao et al. [13] obtained an analytical criteria for existence of chaotic and resonant streamlines in the ABC flow by the Melnikov’ method [14]. In 1998, Huang et al. [15] obtained an explicit analytical criterion for existence of chaotic streamlines in the ABC flow. Ziglin et al. [16] proved that the ABC flow at $A = B = C$ has no real-analytic first integral. Later Maciejewski [17] has investigated non-integrability of the ABC flow at the condition $A^2 = B^2$ for $ABC \neq 0$, and it was proved that the set does not possess the first real meromorphic integral for $A^2/B^2 < 2$. The integrable case (when one of the parameters $A$, $B$ or $C$ vanishes) has been considered in Ref. [10]. Another case of the absence and existence of the first integrals has been considered in Refs. [18–20]. Brummell [21] has investigated some properties of the time-dependent ABC flow. In Ref. [22] obtained an explicit analytical criterion and numerical evidences for the existence of the $n:m$ resonances in the ABC flow.

Arnold and Korkina [23], Galloway and Frisch [24], and Moffatt and Proctor [25] performed numerical and analytical studies of the dynamo action at finite values of the conductivity. They have shown that the ABC flow can excite a magnetic field. However, it was shown that the generated magnetic field looks as a combination of cigar-like structures and can hardly be considered as a large-scale one. The Arnold-Beltrami-Childress flow is a prototype for the fast dynamo action, essential to the origin of magnetic field for large astrophysical objects, and dynamo properties can be investigated by varying the magnetic Reynolds number $R_m$. Recently, some properties the ABC flow have been investigated by Galloway [26]. Bouya [27] has investigated properties of the ABC flow for the extended region from $R_m < 1600$ to $R_m < 25000$. In 2013, Jones and Gilbert [28] have studied in detail symmetries of the various dynamo branches up to $R_m = 10^4$. In 2015, Bouya [29] has investigated the kinematic dynamo action up to $R_m = 5 \cdot 10^5$. Solution for stationary points in the special case has been obtained in Ref. [30]. It has been proved the existence of 1 point for two partial cases of parameters $A$, $B$, $C$: 1) $A = B = 1; 2) C = 1$ ($A^2 + B^2 = 1$).

In this paper, we find stationary points in the general case and criteria for their existence. The behavior of streamlines in vicinity of stationary points will be discussed as well. Exact analytic expression for eigenvalues and eigenvectors of the stability matrix will be given for the first time.

2. Analytical solution of the ABC flow

Let us write the autonomous set of differential equations for the ABC flow:

$$\frac{dx}{dt} = V_x = A \sin(z) + C \cos(y),$$

$$\frac{dy}{dt} = V_y = B \sin(x) + A \cos(z),$$

$$\frac{dz}{dt} = V_z = C \sin(y) + B \cos(x),$$

(1)
where $A$, $B$ and $C$ are real parameters. This flow is periodic on all three variables with period $2\pi$, so, we consider only one cubic box $x$, $y$, $z \in [-\pi, \pi]$. Let $A$, $B$ and $C$ are greater than or equal to zero. If any of $A$, $B$ and $C$ parameter is negative, we can translate the corresponding variable by $\pi$. By using some results from [10] we can assume

$$1 = A \geq B \geq C \geq 0,$$

then (1) can be rewritten as

$$\frac{dx}{dt} = V_x = \sin(z) + C \cos(y),$$
$$\frac{dy}{dt} = V_y = B \sin(x) + \cos(z),$$
$$\frac{dz}{dt} = V_z = C \sin(y) + B \cos(x).$$

Let us find the stationary points of the ABC flow system. Assuming $dx/dt = dy/dt = dz/dt = 0$, we get the set of algebraic equations

$$\sin(z_0) = -C \cos(y_0),$$
$$B \sin(x_0) = -\cos(z_0),$$
$$C \sin(y_0) = -B \cos(x_0),$$

where $M(x_0, y_0, z_0)$ is a stationary point. Squaring (4) and using identity $\cos^2(\alpha) + \sin^2(\alpha) = 1$, we get

$$\sin^2(z_0) = C^2(1 - \sin^2(y_0)),$$
$$B^2 \sin^2(x_0) = 1 - \sin^2(z_0),$$
$$C^2 \sin^2(y_0) = B^2(1 - \sin^2(x_0)).$$

After replacement

$$X = \sin^2(x_0), \quad Y = \sin^2(y_0), \quad Z = \sin^2(z_0),$$

we obtain the solution of (5) is the following form:

$$X = \frac{B^2 - C^2 + 1}{2B^2}, \quad Y = \frac{C^2 + B^2 - 1}{2C^2}, \quad Z = \frac{C^2 - B^2 + 1}{2}.$$ 

Substitute $X$, $Y$ and $Z$ from (7) to (6), we get

$$x_0 = \delta_x \arcsin(P_x), \quad x_0 = \delta_x \{\pi - \arcsin(P_x)\},$$
$$y_0 = \delta_y \arcsin(P_y), \quad y_0 = \delta_y \{\pi - \arcsin(P_y)\},$$
$$z_0 = \delta_z \arcsin(P_z), \quad z_0 = \delta_z \{\pi - \arcsin(P_z)\},$$

(8)
where

\[ P_x = \sqrt{\frac{B^2 - C^2 + 1}{2B^2}}, \quad \delta_x = \pm 1, \]
\[ P_y = \sqrt{\frac{B^2 + C^2 - 1}{2C^2}}, \quad \delta_y = \pm 1, \]
\[ P_z = \sqrt{\frac{-B^2 + C^2 + 1}{2}}, \quad \delta_z = \pm 1. \] (9)

To allow existence of solution of the set (4) the parameters \( P_x, P_y \) and \( P_z \) must be between zero and one. This is true if

\[ C^2 \geq 1 - B^2 \] (10)

in additional to inequality (2). Region of existence of solution in the parametric space \( B, C \) by the hatched area on Fig. 1.
Let us write (8) in the general form

\[
\begin{align*}
x_0 &= \delta_x \left[ \frac{1}{2} \left(1 - \gamma_x\right) \pi + \gamma_x \arcsin(P_x) \right], \\
y_0 &= \delta_y \left[ \frac{1}{2} \left(1 - \gamma_y\right) \pi + \gamma_y \arcsin(P_y) \right], \\
z_0 &= \delta_z \left[ \frac{1}{2} \left(1 - \gamma_z\right) \pi + \gamma_z \arcsin(P_z) \right].
\end{align*}
\]

(11)

The coefficients \(\delta_i\) and \(\gamma_i\) \((i = x, y, z)\) are determined by the following rule:

\[
\begin{align*}
\delta_i &= \begin{cases} 
+1, & i \in [0, \pi), \\
-1, & i \in [-\pi, 0),
\end{cases} \\
\gamma_i &= \begin{cases} 
+1, & i \in [-\pi/2, \pi/2), \\
-1, & i \in [-\pi, -\pi/2) \cup [\pi/2, \pi).
\end{cases}
\end{align*}
\]

(12)

There are 64 combinations of the coefficients \(\delta_i\) and \(\gamma_i\), but only 8 satisfy the initial equations (4), and they are presented in Table 1. Each point has its own unique combination of the coefficients \(\delta_i\), so each octant of the phase space box contains exactly one stationary point.

Table 1: The sign of \(\delta_i\) and \(\gamma_i\) for different stationary points.

| Solution | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|----------|---|---|---|---|---|---|---|---|
| Parameter | \(\delta_x\) | \(-\) | \(-\) | + | + | + | + | - |
|          | \(\gamma_x\) | + | + | + | - | + | - | - |
|          | \(\delta_y\) | + | - | - | + | - | - | + |
|          | \(\gamma_y\) | - | - | - | - | + | + | + |
|          | \(\delta_z\) | + | + | + | - | - | - | - |
|          | \(\gamma_z\) | + | + | - | - | - | - | + |

Now we briefly consider behavior of the stationary points (11) on the bifurcation line (10) separating region of existence of stationary points in the parametric space. It follows from (11) the following stationary points will merge: (1, 2), (3, 4), (5, 6) and (7, 8). So, (3) have 4 stationary points on the bifurcation line (10).

The set (3) is integrable at the point \((B = 1, C = 0)\). In this case the solutions (1, 2, 3, 4) and (5, 6, 7, 8) are merge and form stationary lines \((x = \pi/2, \text{any } y, z = -\pi)\) and \((x = -\pi/2, \text{any } y, z = 0)\).

3. Stability analysis of solutions of the ABC flow

Now we consider behavior of phase trajectories near the stationary points (11). Linearizing the set (3) in vicinity of the stationary points (11), we obtain equa-


\[
\Delta x = \cos \left( \delta_z \left[ \frac{1}{2} (1 - \gamma_z) \pi + \gamma_z \arcsin P_z \right] \right) \Delta z - \\
\delta_y C \sin \left( \frac{1}{2} (1 - \gamma_y) \pi + \gamma_y \arcsin P_y \right) \Delta y, \\
\Delta y = B \cos \left( \delta_x \left[ \frac{1}{2} (1 - \gamma_x) \pi + \gamma_x \arcsin P_x \right] \right) \Delta x - \\
\delta_z \sin \left( \frac{1}{2} (1 - \gamma_z) \pi + \gamma_z \arcsin P_z \right) \Delta z, \\
\Delta z = C \cos \left( \delta_y \left[ \frac{1}{2} (1 - \gamma_y) \pi + \gamma_y \arcsin P_y \right] \right) \Delta y - \\
\delta_x B \sin \left( \frac{1}{2} (1 - \gamma_x) \pi + \gamma_x \arcsin P_x \right) \Delta x, 
\]

where \( \Delta x = x - x_0 \), \( \Delta y = y - y_0 \) and \( \Delta z = z - z_0 \). We can omit in (13) the coefficient \( \delta_i \) in the arguments of cosines. Using the trigonometric identities \( \sin(\pi - \alpha) = \sin(\alpha) \), \( \cos(\pi - \alpha) = -\cos(\alpha) \) and expressing cosines by sines, we get

\[
\Delta x = \gamma_z \sqrt{1 - P_z^2} \Delta z - \delta_y C P_y \Delta y, \\
\Delta y = \gamma_x B \sqrt{1 - P_x^2} \Delta x - \delta_z P_z \Delta z, \\
\Delta z = \gamma_y C \sqrt{1 - P_y^2} \Delta y - \delta_x B P_z \Delta x. 
\]

The solution of equations (14) is determined by the roots of characteristic equation

\[
\begin{vmatrix}
-\lambda & -\delta_y C P_y & \gamma_z \sqrt{1 - P_z^2} \\
-\delta_z B P_z & -\lambda & -\delta_x P_z \\
\gamma_y C \sqrt{1 - P_y^2} & -\delta_x B P_z & -\lambda
\end{vmatrix} = 0.
\]

\[
\lambda^3 + \lambda \left( \delta_x \gamma_x B P_x \sqrt{1 - P_x^2} + \delta_z \gamma_z C P_z \sqrt{1 - P_z^2} + \delta_y \gamma_y B C P_y \sqrt{1 - P_y^2} \right) - \\
- BC \left( \gamma_x \gamma_y \gamma_z \sqrt{1 - P_x^2} \sqrt{1 - P_y^2} \sqrt{1 - P_z^2} - \delta_x \delta_y \delta_z P_x P_y P_z \right) = 0. 
\]

Numerical values of the coefficients \( \gamma_i \) and \( \delta_i \) given in Table 1 and their multiplication are presented in Table 2. Let

\[
Q = B P_x \sqrt{1 - P_x^2} + C P_z \sqrt{1 - P_z^2}, \\
W = B C \left( \sqrt{1 - P_x^2} \sqrt{1 - P_y^2} + P_x P_y P_z \right) = \\
= \frac{1}{\sqrt{2}} \left( B^2 + C^2 - 1 \right) \left( 1 - (B^2 - C^2)^2 \right). 
\]

Finally, the characteristic equation (15) has the following form:

\[
\lambda^3 - \lambda Q - \xi W = 0, 
\]

(17)
where \( \xi = \gamma_x\gamma_y\gamma_z = -\delta_x\delta_y\delta_z \) depends on choice of solution of (4)

\[
\xi = \begin{cases} 
+1, & \text{for solutions 1, 3, 5, 7}, \\
-1, & \text{for solutions 2, 4, 6, 8}.
\end{cases}
\] (18)

Table 2: The sign of multiplication of coefficients \( \gamma_i \) and \( \delta_i \).

| Solution | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|----------|---|---|---|---|---|---|---|---|
| Parameter | \( \delta_x\gamma_x \) | - | - | - | - | - | - | - |
|           | \( \delta_x\gamma_y \) | - | - | - | - | - | - | - |
|           | \( \delta_y\gamma_x \) | - | - | - | - | - | - | - |
|           | \( \delta_x\delta_y\delta_z \) | - | + | - | + | - | + | + |
|           | \( \gamma_x\gamma_y\gamma_z \) | + | - | + | - | + | - | - |

The equation (17) has two variants (for cases \( \xi = -1 \) and \( \xi = 1 \)). We consider each case individually. Let \( \xi = -1 \), then we get polynomial (17) in the form

\[ Y_1(\lambda) = \lambda^3 - \lambda Q + W = 0. \] (19)

The polynomial (19) has extreme points \( \lambda_1 = -\sqrt{\frac{Q}{3}} \) and \( \lambda_2 = \sqrt{\frac{Q}{3}} \). Since \( Y_1(0) > 0 \), then the polynomial (19) has one real negative root. The two other roots can be either complex conjugate pair with positive real part \( Y_1(\lambda_2) > 0 \) or real positive ones \( Y_1(\lambda_2) \leq 0 \). We will prove that \( Y_1(\lambda_2) \leq 0 \) for all values of parameters \( B \) and \( C \).

\[ Y_1(\lambda_2) = \frac{Q\sqrt{Q}}{3\sqrt[3]{3}} - Q\sqrt{\frac{Q}{3}} + W \leq 0. \] (20)

After simplification and substitution (16) to (20), we get

\[ \frac{1}{3\sqrt[3]{3}} (1 + B^2 + C^2) \sqrt{(1 + B^2 + C^2) \geq \sqrt{(B^2 + C^2 - 1)(1 - (B^2 - C^2)^2)}}. \] (21)

By squaring inequality (21) we obtain

\[ \frac{1}{27} (1 + B^2 + C^2)^3 \geq (B^2 + C^2 - 1) \left(1 - (B^2 - C^2)^2\right). \] (22)

Let

\[ q = B^2 + C^2, \quad p = B^2 - C^2. \] (23)

In the region of existence of the solution parameters we have \( q \in [1, 2], p \in [0, 1] \). Inequality (22) can be rewritten as follows:

\[ \frac{1}{27} (q + 1)^3 \geq 1 - p^2, \] (24)
Since $1 - p^2 \leq 1$, we can strengthen the inequality (24)

$$T(q) = (1 + q)^3 - 27(q - 1) \geq 0. \quad (25)$$

The inequality (25) is true, because $T(1) > 0$ and $T(q_{\text{min}}) = 0$, where $q_{\text{min}} = 2$ is a minimum point of $T(q)$. So equation (19) has 3 real roots (two are positive and one is negative). In the case $q = 2$ and $p = 0$ ($B = C = 1$) equation (19) has a multiple positive root.

Let us consider the case $\xi = +1$. The polynomial (17) can be written as

$$Y_2(\lambda) = \lambda^3 - \lambda Q - W = Y_1(-\lambda) = 0. \quad (26)$$

Because of $Y_2(\lambda) = Y_1(-\lambda)$, the polynomial (26) has two negative real roots and one positive real root. The streamlines around the stationary points are shown in Figs. 2 and 3.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Streamlines in a vicinity of the stationary point with two negative eigenvalues and a positive one.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Streamlines in a vicinity of the stationary point with one negative eigenvalue and two positive ones.}
\end{figure}

The equation for eigenvalues (27) on bifurcation lines reduces to $\lambda^3 + \alpha \lambda = 0$, so, one of the eigenvalues is zero. All these stationary points are plane saddles.

4. Eigenvalues

In this section we obtain an exact solution for eigenvalues of the stability matrix. We will use the Cardano method for the cubic equation expressed as

$$\lambda^3 + \alpha \lambda + \beta = 0, \quad (27)$$

where $\alpha = -Q$, $\beta = -\xi W$. Solution of (27) is defined by the discriminant

$$S = \frac{\beta^2}{4} + \frac{\alpha^3}{27} = \frac{216(q - 1)(1 - p^2) - 8(1 + q)^3}{1728}. \quad (28)$$
The discriminant $S$ is real and the type of roots of Eq. (27) can be defined if we
known the sign of $S$. Now we show that $S$ is always less or equal to zero. Let’s
consider the polynomial

$$I(q) = 216 \left( q - 1 \right) (1 - p^2) - 8 (1 + q)^3.$$ (29)

On the edges of the interval $q \in [1, 2]$ we have $I(q) \leq 0$. Consequently $I(q)$ is
less or equal to zero for all $q \in [1, 2]$ if $I(q_1) \leq 0$, where $q_1 = -1 + 3 \sqrt{1 - p^2}$ is
a maximum point of $I(q)$. Let us consider

$$I(q_1) = 216 \left( 3 \sqrt{1 - p^2} - 2 \right) (1 - p^2) - 8 \left( 3 \sqrt{1 - p^2} \right)^3 =$$

$$= \left( \sqrt{1 - p^2} - 1 \right) (1 - p^2).$$ (30)

Because of $\sqrt{1 - p^2} \leq 1$, we obtain that $I(q) \leq 0$ for all $q \in [1, 2]$.

In the case $S \leq 0$ the equation (27) has three real roots which can be found
by the following way:

$$\lambda_1 = 2 \sqrt{-\alpha} \cos \left( \frac{F}{3} \right),$$

$$\lambda_2 = 2 \sqrt{-\alpha} \cos \left( \frac{F}{3} + \frac{2\pi}{3} \right),$$

$$\lambda_3 = 2 \sqrt{-\alpha} \cos \left( \frac{F}{3} + \frac{4\pi}{3} \right),$$ (31)

where $F$ are defined by

$$F = \begin{cases} 
\arctan \left( \frac{2 \sqrt{|S|}}{-\beta} \right), & \beta < 0, \\
\arctan \left( \frac{2 \sqrt{|S|}}{-\beta} \right) + \pi, & \beta \geq 0.
\end{cases}$$ (32)

All roots and $F$ are defined via the parameters $B$ and $C$ as

$$\lambda_1 = 2 \sqrt{ \frac{1 + C^2 + B^2}{6} } \cos \left( \frac{F}{3} \right),$$

$$\lambda_2 = 2 \sqrt{ \frac{1 + C^2 + B^2}{6} } \cos \left( \frac{F}{3} + \frac{2\pi}{3} \right),$$

$$\lambda_3 = 2 \sqrt{ \frac{1 + C^2 + B^2}{6} } \cos \left( \frac{F}{3} + \frac{4\pi}{3} \right),$$ (33)

$$F = \begin{cases} 
\arctan \left( 2 \sqrt{ \frac{2 \sqrt{2} \sqrt{1728(B^2 + C^2 - 1)(1 - (B^2 + C^2)^2) - 8(1 + B^2 + C^2)^3)\sqrt{216(B^2 + C^2 - 1)(1 - (B^2 + C^2)^2) - 8(1 + B^2 + C^2)^3)}}{1728(B^2 + C^2 - 1)(1 - (B^2 + C^2)^2)}} - 1 \right), & \xi = 1, \\
\pi - \arctan \left( 2 \sqrt{ \frac{2 \sqrt{2} \sqrt{1728(B^2 + C^2 - 1)(1 - (B^2 + C^2)^2) - 8(1 + B^2 + C^2)^3)\sqrt{216(B^2 + C^2 - 1)(1 - (B^2 + C^2)^2)}}{1728(B^2 + C^2 - 1)(1 - (B^2 + C^2)^2)}} + 1 \right), & \xi = -1.
\end{cases}$$ (34)

where sign of $\xi$ is defined by the chosen solution (see (18) and Table 2) of eqs. (3).
5. Eigenvectors

We previously used the Cardano method for calculating eigenvalues of the polynomial appearing in (15). We now face the problem of finding the eigenvectors. Let us consider the eigenvector of this system

$$A \Psi = \lambda \Psi \Rightarrow (A - \lambda E) \Psi = 0,$$

where the matrix $A$ can be constructed from the system (14) as

$$A = \begin{pmatrix} 0 & -\delta_y CP_y & \gamma_z \sqrt{1 - P_z^2} \\ \gamma_z B \sqrt{1 - P_x^2} & 0 & -\delta_z P_z \\ -\delta_x BP_x & \gamma_y C \sqrt{1 - P_y^2} & 0 \end{pmatrix}.$$  (36)

Vector $\Psi$ has the form

$$\Psi = \begin{pmatrix} U \\ V \\ W \end{pmatrix}.$$  (37)

We use the Gauss’s method for the solution of (35) to simplify the factor $(A - \lambda E)$

$$(A - \lambda E) = \begin{pmatrix} -\lambda & -\delta_y CP_y & \gamma_z \sqrt{1 - P_z^2} \\ \gamma_z B \sqrt{1 - P_x^2} & -\lambda & -\delta_z P_z \\ -\delta_x BP_x & \gamma_y C \sqrt{1 - P_y^2} & -\lambda \end{pmatrix} \sim \begin{pmatrix} -\lambda & -\delta_y CP_y & \gamma_z \sqrt{1 - P_z^2} \\ 0 & -\delta_x CP_y \sqrt{1 - P_x^2} & -\delta_z P_z \end{pmatrix} \begin{pmatrix} \gamma_z B \sqrt{1 - P_x^2} \\ \gamma_y C \sqrt{1 - P_y^2} \\ 0 \end{pmatrix}.$$  (38)

Matrix (38) has one zero line and, therefore, $U$ and $V$ can be defined via $W$, and $W$ is free.

The system of linear equations is

$$-\lambda U - \delta_y CP_y V + \gamma_z \sqrt{1 - P_z^2} W = 0,$$

$$\left( -\delta_y \gamma_y CP_y \sqrt{1 - P_z^2} \right) V + \left( \gamma_z \gamma_x B \sqrt{1 - P_z^2} \right) W = 0.$$  (39)

and its solution is

$$V = \frac{\gamma_z \gamma_x B \sqrt{1 - P_z^2} \sqrt{1 - P_x^2} - \delta_z \lambda P_z}{\delta_y \gamma_y CP_y \sqrt{1 - P_z^2} + \lambda^2} W,$$

$$U = \frac{1}{\lambda} \left( -\delta_y CP_y \frac{\gamma_z \gamma_x B \sqrt{1 - P_z^2} \sqrt{1 - P_x^2} - \delta_z \lambda P_z}{\delta_y \gamma_y CP_y \sqrt{1 - P_z^2} + \lambda^2} + \gamma_z \sqrt{1 - P_z^2} \right) W.$$  (40)
If \( \lambda \neq 0 \) we can let
\[
W = \delta_y \gamma_x B C P_y \sqrt{1 - P_x^2} + \lambda^2.
\]
(41)

Components \( U, V \) and \( W \) of the eigenvector can be computed as follows:
\[
U = \frac{1}{\lambda} \left[ -\delta_y C P_y \left( \gamma_z \gamma_x B \sqrt{1 - P_x^2} \sqrt{1 - P_x^2} - \delta_z \lambda P_z \right) + \gamma_z \sqrt{1 - P_x^2} \left( \delta_y \gamma_x B C P_y \sqrt{1 - P_x^2} + \lambda^2 \right) \right],
\]
\[
V = \gamma_z \gamma_x B \sqrt{1 - P_x^2} \sqrt{1 - P_x^2} - \delta_z \lambda P_z,
\]
\[
W = \delta_y \gamma_x B C P_y \sqrt{1 - P_x^2} + \lambda^2.
\]
(42)

6. Conclusion

Analytical expressions for coordinates of the stationary points and conditions for their existence in the ABC flow are obtained. The coordinates are found from Eqs. (8). The points exist in the region above the bifurcation line: \( B^2 + C^2 = 1 \) (see Fig. 1). It has been proved that only 2 stationary points exist at the line \( B^2 + C^2 = 1 \). It is analytically shown that the stationary points are of a saddle-node type (see Figs. 2 and 3) at all values of the control parameters in the region of their existence. Exact expressions for the eigenvalues (33) and eigenvectors (42) of the stability matrix are given. The behavior of the stationary points (11) along the bifurcation lines (10) is considered.

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