EMBEDDING OF THE OPERATOR SPACE OH AND THE LOGARITHMIC ‘LITTLE GROTHENDIECK INEQUALITY’

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ABSTRACT. We use Voiculescu’s concept of free probability to construct a completely isomorphic embedding of the operator space OH in the predual of a von Neumann algebra. We analyze the properties of this embedding and determine the operator space projection constant of \( \text{OH}_n \):

\[
\frac{1}{108} \sqrt{\frac{n}{1 + \ln n}} \leq \inf_{P: B(\ell_2) \to \text{OH}_n, P^2 = P} \|P\|_{cb} \leq 288 \pi \sqrt{\frac{2n}{1 + \ln n}}.
\]

The lower estimate is a recent result of Pisier and Shlyakhtenko that improves an estimate of order \( 1/(1 + \ln n) \) of the author. The additional factor \( 1/\sqrt{1 + \ln n} \) indicates that the operator space \( \text{OH}_n \) behaves differently than its classical counterpart \( \ell_2^n \). We give an application of this formula to positive sesquilinear forms on \( B(H) \). This leads to logarithmic characterization of \( C^* \)-algebras with the weak expectation property introduced by Lance.

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0. Introduction and notation

Probabilistic techniques and concepts play an important role in the theory of Banach spaces and operator algebras. For example, the Khintchine inequality and the ‘little Grothendieck inequality’ are fundamental tools in Banach space theory. In the theory of operator algebras probabilistic concepts are important in Takesaki’s proof of Sakai’s theorem (see [T1]) and in Connes’ characterization of injective von Neumann algebras (see [C1]). Pisier/Haagerup’s non-commutative version of Grothendieck’s inequality (see e.g. [PH]) and Grothendieck’s inequality for exact operator spaces in [JP] use probabilistic techniques. This latter result is inspired by ‘Grothendieck’s program for operator algebras’ which motivates fundamental research in the theory of operator spaces. In Pisier/Shlyakhtenko’s Grothendieck theorem for operator spaces (see [PS]) and for the results in this paper the use of free probability is crucial.

We follow Grothendieck’s ideas and investigate positive sesquilinear forms on \( C^* \)-algebras. Let us first recall the so-called ‘little Grothendieck inequality’ on \( C(K) \)-spaces. Grothendieck showed that for every bounded linear map \( \nu : C(K) \to \ell_2 \) there is a probability measure \( \mu \) on \( C(K) \) such that (0.1) holds for all \( f \in C(K) \):

\[
(0.1) \quad \|\nu(f)\|^2 \leq \frac{2}{\sqrt{\pi}} \|v\|^2 \int_K |f|^2 d\mu.
\]

The constant \( \frac{2}{\sqrt{\pi}} \) is optimal for complex \( C(K) \) spaces. Bounded positive sesquilinear forms on \( C(K) \) (i.e. possibly degenerate scalar products) are in one-to-one correspondence with bounded linear maps \( \nu : C(K) \to \ell_2 \) via \( B(f, g) = (\nu(f), \nu(g)) \). For a probability measure \( \mu \) we may define the positive sesquilinear form \( B_\mu \) as follows

\[
B_\mu(f, g) = \int_K \bar{f} g d\mu.
\]

According to Grothendieck’s work, \( B_\mu \) is the prototype of an integral linear form. Integral forms are continuous functionals on the injective Banach space tensor product \( C(K) \otimes_\varepsilon C(K) \cong C(K \times K) \). Indeed, we have

\[
\left| \int \sum_{k=1}^n \bar{f}_k g_k d\mu \right| \leq \sup_{t \in K} \sum_{k=1}^n |\bar{f}_k(t) g_k(t)| \leq \sup_{t, s \in K} |\sum_{k=1}^n \bar{f}_k(t) g_k(s)|
\]

for all finite sequences \( (f_k) \) and \( (g_k) \). We say that a positive sesquilinear form \( B \) is majorized by a bilinear form \( \tilde{B} \), in short \( B \leq \tilde{B} \), if

\[
B(x, x) \leq |\tilde{B}(x, x)|
\]

holds for all \( x \). Therefore, the ‘little Grothendieck inequality’ implies that every bounded positive, sesquilinear form is majorized by an integral linear form (even a positive, integral, sesquilinear form).

We will now discuss the analogue of this result in the context of \( C^* \)-algebras. Let \( A_1 \subset B(H) \) and \( A_2 \subset B(K) \) be \( C^* \)-algebras. For \( C^* \)-algebras we shall replace the Banach space injective tensor norm by the smallest \( C^* \)-tensor norm \( A_1 \otimes_{\min} A_2 \) on \( A_1 \otimes A_2 \). This norm is given by the inclusion \( A_1 \otimes_{\min} A_2 \subset B(H \otimes K) \). In this context a bilinear form \( B : A_1 \times A_1 \to \mathbb{C} \) is called an integral form if there exists a constant \( C > 0 \) such that

\[
\left\| \sum_{k=1}^n B(x_k, y_k) \right\| \leq C \left\| \sum_{k=1}^n x_k \otimes y_k \right\|_{A_1 \otimes_{\min} A_2}
\]
holds for all finite sequences \((x_k) \subset A_1, (y_k) \subset A_2\). The smallest possible constant is the norm of \(B\) as a linear functional on \(A_1 \otimes_{\min} A_2\) and will be denoted by \(\|B\|_I\). From the theory of operator spaces it is clear that the ‘appropriate’ substitute for ‘bounded bilinear form’ is the notion of a ‘jointly completely bounded form’. A bilinear form \(B : A_1 \times A_2 \to \mathbb{C}\) is jcb (jointly completely bounded) if there exists a constant \(C\) such that

\[
(0.2) \quad \left\| \sum_{k,j} B(a_k, b_j) x_k \otimes y_j \right\|_{M_m \otimes_{\min} M_m} \leq C \left\| \sum_k a_k \otimes x_k \right\|_{M_m(A_1)} \left\| \sum_j b_j \otimes y_j \right\|_{M_m(A_2)}
\]

holds for all finite sequences \((a_k) \subset A_1, (b_j) \subset A_2\) and \((x_k), (y_j) \subset M_m\). The jcb-norm \(\|B\|_{jcb}\) is given by the infimum over all \(C\) satisfying (0.2). Following Grothendieck’s categorial approach it is natural to ask whether every positive, sesquilinear jcb form is majorized by an integral form. In contrast to the commutative case this fails in the noncommutative setting:

**Theorem 1.** There exists a positive, integral, sesquilinear form on \(\mathcal{B}(H)\) which cannot be majorized by an integral form. More precisely, for every \(n \in \mathbb{N}\) there exists a positive, integral, sesquilinear form \(B\) on \(\mathcal{B}(H)\) of rank \(n\) such that

\[
(0.3) \quad (1 + \ln n) \|B\|_{jcb} \leq C \|\tilde{B}\|_I
\]

holds for all \(\tilde{B}\) satisfying \(B \leq \tilde{B}\).

The factor \((1 + \ln n)\) in (0.3) is optimal. Indeed, Pisier/Shlyakhtenko [PS] showed that if \(B\) a positive, integral, sesquilinear form on \(\mathcal{B}(H)\) of rank \(n\), then there exists an integral, sesquilinear form \(\tilde{B}\) with \(B \leq \tilde{B}\) such that

\[
(0.4) \quad \|\tilde{B}\|_I \leq C (1 + \ln n) \|B\|_{jcb}.
\]

In fact, Pisier/Shlyakhtenko [PS] improved an estimate of the author (see [J3]) of the order \((1 + \ln n)^2\) in (0.4). Following Pisier’s work (see [Psl]), positive sesquilinear jcb forms are closely connected to completely bounded linear maps with values in the operator space \(\text{OH}\) (for definitions see below). Our approach to Theorem 1 is probabilistic in nature. We find an embedding of the operator space \(\text{OH}\) in a noncommutative \(L_1\) space, imitating the classical embedding of \(\ell_2\) via Gaussian variables. The properties of this embedding of \(\text{OH}\) then yield the logarithmic term.

We recall some operator space notation before giving more details. An operator space \(F\) comes either with a concrete isometric embedding \(\iota : F \to \mathcal{B}(H)\) or with a sequence \((\|\cdot\|_m)\) of matrix norms on \((M_m(F))\) such that

\[
\|[x_{ij}]\|_{M_m(F)} = \|[\iota(x_{ij})]\|_{\mathcal{B}(H^m)}.
\]

Ruan’s axioms (see e.g. [DR2]) describe axiomatically those sequences of matrix norms which can occur from an isometric embedding in \(\mathcal{B}(H)\). The morphisms in this category are completely bounded linear maps \(u : E \to F\), i.e. linear maps such that

\[
\|u\|_{cb} = \sup_m \|\text{id} \otimes u : M_m(E) \to M_m(F)\|
\]

is finite. We denote by \(\mathcal{CB}(E, F)\) the Banach (operator) space of completely bounded maps equipped with this norm. Hilbertian operator spaces are of particular interest. For example the column and row spaces of matrices

\[
K_c = \mathcal{B}(\mathbb{C}, K) \subset \mathcal{B}(K) \quad \text{and} \quad K^r = \mathcal{B}(K, \mathbb{C}) \subset \mathcal{B}(K)
\]
play a fundamental rôle. Pisier discovered that the sequence of norms on $M_m \otimes K$ obtained by the complex interpolation method

\[(0.5) \quad M_m(K^{oh}) = [M_m(K^c), M_m(K^\tau)]_\frac{1}{2}\]

defines a sequence of matrix norms on $M_m \otimes K$ satisfying Ruan’s axioms. Thus (0.5) defines an operator space structure on $K$ denoted by $K^{oh}$. In particular, for $K = \ell_2$ we find a sequence of operators $(T_k) \subset B(\ell_2)$ satisfying

$$\left\| \sum_{k=1}^\infty x_k \otimes T_k \right\|_{M_m(B(\ell_2))} = \sup_{\|a\|_2,\|b\|_2 \leq 1} \left( \sum_{k=1}^\infty \text{tr}(ax_k^* bx_k) \right)^{\frac{1}{2}}$$

for all sequences $(x_k) \in M_m$. The operator Hilbert space $OH = \ell_2^{oh}$ is the span of the $(T_k)$’s. It is still unclear how to construct ‘concrete’ operators $(T_k)$ satisfying this equality. The investigations in this paper may be considered as a starting point in this direction. We need the concept of the standard dual $E^*$ of an operator $E$. The operator space structure on $E^*$ is given by the isometric equality

$$M_m(E^*) = CB(E, M_m).$$

Here a matrix $[x_{ij}^*]$ of functionals corresponds to the linear map $x \mapsto [x_{ij}^*(x)]$. In particular, duals of $C^*$-algebras and preduals of von Neumann algebras carry a natural operator space structure.

With the help of the operator space dual it is easy to explain why the notion of jcb forms is the natural analogue of bounded bilinear forms on Banach spaces. Indeed, a bilinear form $B : F \times E \to \mathbb{C}$ is jcb if and only if the corresponding linear map $T_B : E \to F^*$, $T_B(x)(y) = B(y, x)$ is completely bounded. We will now explain the connection to the operator space OH. Given a positive sesquilinear form $B : \mathcal{E} \times \mathcal{E} \to \mathbb{C}$, we may consider $L = \{x \mid B(x, x) = 0\}$. We denote by $K$ the Hilbert space obtained by completion of $E/L$ with respect to induced scalar product $(x + L, x + L) = B(x, x)$. Then the natural map $v : E \to K^{oh}$ defined by $v(x) = x + L$ is completely bounded if and only if $B$ is jcb (see [PW]). This equivalence yields a one to one correspondence between, sesquilinear jcb forms and completely bounded maps with values in OH. It allows us to derive Theorem 1 from properties of the operator space OH.

A key new ingredient is a formula of Pusz/Woronowicz for the square root of two sesquilinear forms (see [PW] and [PS]). In section 3, we show how this formula (and its new dual version) provides a concrete realization of OH as a subspace of a quotient of the direct sum $R \oplus C$. Let us be more specific. Inspired by [PW] we consider the probability measure $d\mu(t) = dt/(\pi \sqrt{t(1-t)})$ on $[0, 1]$ and the two measures $dv_1(t) = t^{-1}d\mu(t)$, $dv_2(t) = (1-t)^{-1}d\mu(t)$. Then the direct sum

$$\mathcal{H} = L_2^c(\nu_1; \ell_2) \oplus L_2^c(\nu_2; \ell_2)$$

is an operator space. On $\mathcal{H}$, we define the map $Q : \mathcal{H} \to L_0(\mu; \ell_2)$ by

$$Q(x_1, x_2)(t) = x_1(t) + x_2(t) \in \ell_2.$$ 

It is easily checked that $M_m(\mathcal{H}/S) = M_m(\mathcal{H})/M_m(S)$ defines a sequence of matrix norms satisfying Ruan’s axioms and therefore

$$G = \mathcal{H}/\ker(Q)$$

is an operator space. Then, we may consider the subspace $F \subset G$ of equivalence classes $(x_1, x_2) = \ker(Q)$ such that $x_1 + x_2$ is a $\mu$-almost everywhere a constant element in $\ell_2$. The operator space structure of OH is encoded in $\mu$ and the two densities $1/t$ and $1/(1 - t)$:
Theorem 2. $F$ is 2-completely isomorphic to OH.

The embedding of $G$ into the predual of a von Neumann algebra uses free probability. We first extend Voiculescu’s inequality for the norm of the sums of free independent random variables to the operator-valued setting. This operator-valued version of Voiculescu’s inequality provides estimates for the cb-norm of linear maps. Voiculescu’s inequality naturally involves three terms. Using a central limit procedure, we may eliminate one of them. These methods can be used to show that arbitrary quotients of $R \oplus C$ embed in the predual of a von Neumann algebra. In a subsequent paper [J4] we will elaborate this fact and construct an embedding of OH in the predual of a hyperfinite factor. For our applications, it is important to know that the underlying von Neumann algebra is QWEP. Let us recall that a $C^*$-algebra has the weak expectation property (WEP) if there exists a (complete) contraction $P : \mathcal{B}(H) \to A^{**}$ such that $P|_A = id_A$. A $C^*$-algebra is QWEP if it is the quotient $A = A/I$ of some $C^*$-algebra $A$ with WEP by a two-sided ideal $I$. It is an open question whether every $C^*$-algebra is QWEP (see [Ki2]).

Theorem 3. $G$ is completely isomorphic to a completely complemented subspace of the predual $N_*$ of a von Neumann algebra $N$ with QWEP. In particular, OH embeds into $N_*$.

Recently Pisier [Ps7] showed that no embedding of OH is possible in the predual of a semifinite von Neumann algebra. Type III von Neumann algebras are indeed necessary for embedding OH in noncommutative $L_1$ spaces.

The factor $(1 + \ln n)$ is a result of norm calculations in the predual of the tensor product of two von Neumann algebras. This approach is again motivated by Grothendieck’s work on absolutely 1-summing maps. Let us denote by $\pi_1^B$ the absolutely summing norm for Banach spaces (see [Ps1]). Let $g_1, \ldots, g_n, g'_1, \ldots, g'_n$ be independent, normalized, complex Gaussian variables. Following Grothendieck’s work we know that

$$\pi_1^B(id_{\ell_2^n}) = \frac{\pi}{4} \left\| \sum_{i=1}^n g_i g'_i \right\|_{L_1([0,1]^2)} \sim \frac{\sqrt{\pi}}{2} \sqrt{n}.$$ 

The first ‘equality’ remains true in the operator space context if we replace independent Gaussian random variables by a suitable tensor product of noncommutative random variables. However, calculating this tensor norm in the noncommutative context is more involved. We show that it can be calculated as an element of a 4-term quotient of classical Banach spaces (see section 5). The outcome of these norm calculations provides the logarithmic factor:

Theorem 4. Let $u : G \to N_*$ be the embedding from Theorem 3 and $(f_k)$ be the unit vectors basis in $F$ and $n \in \mathbb{N}$. Then

$$\left\| \sum_{k=1}^n u(f_k) \otimes u(f_k) \right\|_{(N \otimes N)_*} \sim \sqrt{n(1 + \ln n)}.$$

In section 4, we show that Theorem 4 implies an estimate on the completely 1-summing norm of the identity map on $\text{OH}_n$ (see section 4 for a definition). Using the well-known concept of
trace duality, we obtain estimates for the operator space projection constant
\[ \lambda_{cb}(OH_n) = \inf_{P : B(\ell_2) \to OH_n, P|_{OH_n} = id} \|P\|_{cb}. \]

**Corollary 5.** Let \( n \in \mathbb{N} \). Then \( \lambda_{cb}(OH_n) \sim \sqrt{\frac{n}{1 + \ln n}}. \)

If \( P : B(\ell_2) \to OH_n \) is the optimal projection, then \( B(x, y) = (P(x), P(y)) \) provides an example for Theorem 1 where the logarithmic term is necessary (see again section 4). I learned from C. le Merdy that the analogue of the ‘little Grothendieck inequality’ fails for the reduced \( C^* \)-algebra of the free group in \( n \)-generators. Using Haagerup’s characterization of \( C^* \)-algebras with WEP in terms of selfpolar forms, we can show in section 2 that WEP is the crucial property.

**Theorem 6.** Let \( A \) be a \( C^* \)-algebra and \( \alpha > 0 \). \( A \) has WEP if and only if there exists a constant \( C_\alpha \) such that every positive sesquilinear form \( B \) of rank \( n \) on \( A \) is majorized by an integral sesquilinear form \( \tilde{B} \) satisfying
\[ \|\tilde{B}\|_I \leq C_\alpha(1 + \ln n)^\alpha \|B\|_{jcb}. \]

Moreover, if \( A \) is not a subalgebra of \( C(K, M_m) \) for some \( m \), then this condition is satisfied only for \( \alpha \geq 1 \).

In section 1 we provide some background and notation. Theorem 6 is proved in section 2. The Pusz/Woronowicz formula and its application to OH is contained in section 3. Recently, alternative pairs of measures have been found (see [JX2]) which lead to nicer representations of OH but are beyond the scope of this paper. In section 4 we prove Theorem 1 assuming the probabilistic result Theorem 3 (see section 7) and the norm calculation in section 5. In section 6, we investigate different notions of \( K \)-functionals, in particular 3-term \( K \)-functionals. These \( K \)-functionals arise naturally in the context of Voiculescu’s inequality in section 7. At the end of the paper we show that the free product of von Neumann algebras with QWEP is again QWEP (see Theorem 7.15). This result might be of independent interest.

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### 1. Preliminaries

We use standard notation in operator algebras as in [Ta, KR, Si]. Let \( A_1 \subset B(H) \) and \( A_2 \subset B(K) \) be \( C^* \)-algebras. The norm on \( A \otimes B \) induced by the inclusion \( A \otimes B \subset B(H \otimes_2 K) \) is called the minimal \( C^* \)-norm. We use \( A \otimes_{\text{min}} B \) for the completion of \( A \otimes B \) with respect to that norm. Here we have used \( H \otimes_2 K \) for the unique tensor product making \( H \otimes K \) a Hilbert space (often called the Hilbert-Schmidt norm). Let \( N \subset B(H) \), \( M \subset B(K) \) be von Neumann algebras. We denote by \( N \otimes M \) the closure of \( N \otimes_{\text{min}} M \) in the weak operator topology. For a \( C^* \)-algebra \( A \), we denote by \( A^{op} \) the \( C^* \)-algebra defined on the same underlying Banach space but with the reversed multiplication \( x \circ y = yx \). By \( \hat{A} \), we denote the \( C^* \)-algebra obtained from the complex multiplication \( \lambda.x = \bar{\lambda}x \) on \( A \). Thus \( A \) and \( \hat{A} \) coincide as real Banach algebras. We see that the map \( j : A^{op} \to \hat{A} \) given by \( j(x) = x^* \) is a \( C^* \)-isomorphism.
The notation $E \otimes \varepsilon F$ is used for the completion of $E \otimes F$ with respect to the largest tensor norm in the category of Banach spaces. Similarly, we will use $E \otimes \pi F$ for the completion of $E \otimes F$ with respect to the biggest tensor norm in the category of Banach spaces. For Hilbert spaces $H$ and $K$, the space of trace class operators is denoted by $H \otimes \pi K$ or $S_1(H,K)$. Similarly, we use the notation $H \otimes \varepsilon K = K(H,K)$ for the space of compact operators. We note the trivial inclusions

$$H \otimes \pi K \subset H \otimes_2 K \subset H \otimes \varepsilon K.$$  

We assume the reader to be familiar with standard operator space terminology which can be found in the monographs [L1R2] or [Ps6]. We will need some basic facts about the column Hilbert space $K^c = B(\mathbb{C},K)$ and the row Hilbert space $K^r = B(K,\mathbb{C})$ of a given Hilbert space $K$. Given an element $x = [x_{ij}] \in M_m(K^c)$, the norm is given by

$$(1.1) \quad \|x\|_{M_m(K^c)} = \|x\|_{\mathcal{B}(\ell_2^m,\ell_2^m(K))} = \|x^*x\|_{\mathcal{B}(\ell_2^m)}^{\frac{1}{2}} = \left\| \left[ \sum_k (x_{ki},x_{kj}) \right]_{ij} \right\|_{M_m}^{\frac{1}{2}}.$$  

Here $(x,y)$ denotes the scalar product of $x$ and $y$. We will assume that scalar products are antilinear in the first component. Similarly, we have

$$(1.3) \quad \|x\|_{M_m(K^r)} = \left\| \left[ \sum_k (x_{ik},x_{jk}) \right]_{ij} \right\|_{M_m}^{\frac{1}{2}}.$$  

We use the standard notation $C = \ell_2^c$, $C_n = (\ell_2^n)^c$ and $R = \ell_2^r$, $R_n = (\ell_2^n)^r$ for the column and row Hilbert spaces. We refer to [Ps4], [Ps6] for more details on the operator Hilbert space $K^{oh} = [K^c,K^r]_\frac{1}{2}$ and interpolation norms. For $K = \ell_2$ and $K = \ell_2^n$, we will simply write $OH = \ell_2^{oh}$ and $OH_n = (\ell_2^n)^{oh}$. We denote by $(e_k)$ the natural unit vector basis. Given a sequence $(x_k) \subset \mathcal{B}(H)$, we define the associated linear map $u : OH \to \mathcal{B}(H)$ defined by $u(e_k) = x_k$ and have

$$(1.4) \quad \|u : OH \to \text{Im}(u)\|_{cb} = \|u\|_{cb} = \left\| \sum_k e_k \otimes x_k \right\|_{\mathcal{B}(H) \otimes_{\min} \mathcal{B}(H)}^{\frac{1}{2}}.$$  

The Hilbertian operator spaces $K^c$, $K^r$ and $K^{oh}$ are homogeneous, i.e. for $s \in \{c, r, oh\}$ and every bounded linear map $u : K^s \to K^s$, we have

$$\|u\|_{cb} = \|u\|.$$  

In terms of general operator space notation, let us recall that a complete contraction $u : E \to F$ is given by a completely bounded map with $\|u\|_{cb} \leq 1$. We say that two operator spaces $E$ and $F$ are $\lambda$-cb isomorphic if there exists a linear isomorphism $u : E \to F$ such that $\|u\|_{cb},\|u^{-1}\|_{cb} \leq \lambda$. An operator space $E$ is $\lambda$-completely complemented in an operator space $F$ if there exist completely bounded maps $u : E \to F$ and $v : F \to E$ such that $vu = id_E$ and $\|u\|_{cb},\|v\|_{cb} \leq \lambda$.

Quotient operator spaces $G = V/E$ are important in this paper. We refer to the introduction for the definition of the operator space structure $M_m(G) = M_m(V)/M_m(E)$. If $T : V \to X$ is a completely bounded map which vanishes on $F$, then $T$ induces a unique map $\hat{T} : V/E \to X$ defined by $\hat{T}(x + F) = T(x)$. We have $\|T\|_{cb} = \|\hat{T}\|_{cb}$.

The injective tensor product $E \otimes_{\min} F$ of two operator spaces $E \subset \mathcal{B}(H)$, $F \subset \mathcal{B}(K)$ is the completion of $E \otimes F$ with respect to the norm induced by the inclusion map $E \otimes_{\min} F \subset \mathcal{B}(H \otimes_2 K)$. 

The norm does not depend on the underlying completely isometric embedding. The projective tensor product \( E \hat{\otimes} F \) is defined such that
\[
(E \hat{\otimes} F)^* = CB(E, F^*) \cong CB(F, E^*) .
\]
See [25, 26, 27] for details. Recall that a bilinear form \( B : E \times F \to \mathbb{C} \) is jcb if and only if its linear extension to \( B : E \hat{\otimes} F \to \mathbb{C} \) is continuous. Moreover, \( B \) induces the linear map \( T_B : E \to F^* \), \( T_B(e)(f) = B(e, f) \) which satisfies
\[
\|B\|_{jcb} = \|B : E \hat{\otimes} F \to \mathbb{C}\| = \|T_B : E \to F^*\|_{cb} .
\]
Note that the operator space projective tensor product is indeed projective, i.e. \( E/F \hat{\otimes} X = E \hat{\otimes} X/F \hat{\otimes} X \). Moreover, if \( N \) is an injective von Neumann algebra and \( E_1 \) is completely isometrically embedded in \( E_2 \), then we have an isometric inclusion
\[
N^* \hat{\otimes} E_1 \subset N^* \hat{\otimes} E_2 .
\]
The direct sum \( E \oplus_p F \) of given operator spaces \( E \) and \( F \) is defined for \( p = \infty \) and a matrix \([x_{kl}]\) with \( x_{kl} = (e_{kl}, f_{kl})\) by
\[
\|[x_{kl}]\|_{M_{\infty}(E \oplus_p F)} = \max\{\|[e_{kl}]\|_{M_{\infty}(E)}, \|[f_{kl}]\|_{M_{\infty}(E)}\} .
\]
The operator space \( E \oplus_1 F \) is defined by its canonical inclusion in \((E^* \otimes_\infty F^*)^*\). For \( 1 \leq p \leq \infty \), the operator space structure is given by complex interpolation \( E \oplus_p F = [E \oplus_\infty F, E \oplus_1 F]_p \).

We refer to [11, 12, 13] and [14, 15, 16, 17] for the general theory of noncommutative \( L_p \)-spaces. Let \( \phi \) be a normal, semifinite faithful weight with modular automorphism group \( \sigma^\phi_t \). For \( 0 < p < \infty \) the space \( L_p(N) = L_p(N, \phi) \) is defined as a subset of (unbounded) operators affiliated with \( N \rtimes_{\sigma^\phi_t} \mathbb{R} \). To be more precise, we denote by \( \theta_s \) the dual action and \( \tau \) the unique semifinite, normal, faithful trace on \( N \rtimes_{\sigma^\phi_t} \mathbb{R} \) such that \( \tau \circ \theta_s = e^{-s}\tau \). The Haagerup \( L_p \)-space is defined by
\[
L_p(N) = L_p(N, \phi) = \{d \mid d \tau \text{-measurable and } \theta_s(d) = \exp(-\frac{s}{p})d\} .
\]

We have an operator valued weight \( T(x) = \int_\mathbb{R} \theta_s(x)ds \) from \( N \rtimes_{\sigma^\phi_t} \mathbb{R} \) to \( N \). For a functional \( \phi \in N^* \), \( \phi \circ T \) defines a density \( d_\phi \in L_1(N) \) such that \( \phi \circ T(x) = \tau(d_\phi x) \). The tracial functional (different from \( \tau \)) on \( L_1(N) \) is defined by
\[
tr(d_\phi) = \phi(1) .
\]
For an element \( x \) in \( L_p(N) \) the norm is given by \( \|x\|_p = (tr(|x|^p))^{\frac{1}{p}} \). Let \( M \subset N \) be a von Neumann subalgebra with a faithful, normal conditional expectation \( E : N \to M \). Then we have natural inclusion mappings \( i_p : L_p(M) \to L_p(N) \). Indeed, let us assume that \( \phi \) is a normal, faithful state. According to [22] we have
\[
\sigma^\phi_t \circ E = E \circ \sigma^{\phi_{\circ E}}_t
\]
for all \( t \). Thus we have a natural inclusion \( M \rtimes_{\sigma^\phi_t} \mathbb{R} \subset N \rtimes_{\sigma^{E_{\circ E}}_t} \mathbb{R} \). The restriction of \( \theta_s \) is the corresponding dual automorphism group and similarly for \( \tau \). This yields an isometric inclusion \( L_p(M) \subset L_p(N) \) (see [21] for details).
$L_2(N)$ is a Hilbert space with scalar product $(x, y) = tr(x^*y)$. We will use the notation $L_2^s(N)$ instead of $L_2(N)^s$ for $s \in \{c, r, oh\}$. In the theory of operator spaces it is customary to use the ‘$(i,j)-(i,j)$-duality’

$$\langle [x^*_j], [x_{ij}] \rangle = \sum_{i,j=1}^n x^*_{ij}(x_{ij})$$

between matrices $[x^*_j] \in S_1^\wedge \otimes X^*$ and $[x_{ij}] \in M_n(X)$. Unfortunately, this is not consistent with the natural trace $tr$ on $n \times n$-matrices, which corresponds to

$$\langle [x^*_j], [x_{ij}] \rangle = \sum_{i,j=1}^n x^*_j(x_{ji}) .$$

This forces us to define the operator space structure on $L_1(N)$ by its action on $N^{op}$. Since $N$ and $N^{op}$ coincide as Banach spaces, we may consider $\iota : L_1(N) \to (N^{op})^*$ defined by

$$\iota(d)(y) = tr(dy) = \phi_d(y) .$$

Here $\phi_d$ is the linear functional associated with the density $d$ in $L_1(N)$. This implies that

$$\iota(L_1(N)) = N^{op} .$$

If $\phi$ is a semifinite, normal, faithful weight, then $\phi_n = tr_n \otimes \phi$ is a semifinite, normal, faithful weight on $M_n(N)$. Moreover, $tr_n \otimes \tau$ is the unique trace satisfying $(tr_n \otimes \tau) \circ \theta_s = e^{-s}(tr_n \otimes \tau)$ and $tr_n \otimes tr : L_1(M_n(N), tr \otimes \phi) \to \mathbb{C}$ still yields the evaluation at 1. Therefore, we get

$$\|\iota(x_{ij})\|_{S_1^\wedge \otimes N^{op}} = \sup_{\|y_{ij}\|_{M_n(N^{op})} \leq 1} \left| \sum_{ij=1}^n \iota(x_{ij})(y_{ij}) \right| = \sup_{\|y_{ij}\|_{M_n(N)}} \left| \sum_{ij=1}^n tr(y_{ji}x_{ij}) \right| = \sup_{\|y_{ij}\|_{M_n(N)} \leq 1} \left| tr_n \otimes tr([y_{ij}][x_{ij}]) \right| = \|x_{ij}\|_{L_1(M_n \otimes N, tr_n \otimes \phi)} .$$

The use of $N^{op}$ enables us to ‘untwist’ the duality bracket and we have

$$S_1^\wedge \otimes L_1(N, \phi) = L_1(M_n \otimes N, tr_n \otimes \phi) .$$

Here we distinguish between the predual $N_1^{op}$ and the concrete realization of $N_1^{op}$ as space of operators $L_1(N, \phi)$.

Column and row space interchange their roles when combined with the projective tensor. The parallel duality send columns to columns. Therefore the dualities $R_n = C_n^\wedge$, $(L_1(N) \otimes R_n)^* = N_1 \otimes C_n$ and $(L_1(N) \otimes S_1^\wedge)^* = N_1 \otimes M_n$ imply that

$$\left\| \sum_{k=1}^n x_k^* x_k \right\|_{L_1(N)} = \left\| \sum_{k=1}^n x_k \otimes e_{1,k} \right\|_{L_1(N) \otimes S_1^\wedge} = \left\| \sum_{k=1}^n x_k \otimes e_k \right\|_{L_1(N) \otimes R_n} .$$

We will frequently use this well-known (though surprising) switch between the $L_1$- and $L_\infty$-theory. Given an arbitrary Hilbert space $H$, we denote by

$$( , ) : (L_1(N) \otimes H) \otimes (L_1(N) \otimes H) \to L_{1,2}^2(N), (d_1 \otimes h_1, d_2 \otimes h_2) = d_1d_2(h_1, h_2)$$

the vector-valued extension of the scalar product. Then (1.9) implies that

$$\left\| (x^*, x) \right\|_{L_1(N)} = \left\| x \right\|_{L_1(N) \otimes H^c} \text{ and } \left\| (x, x^*) \right\|_{L_1(N)} = \left\| x \right\|_{L_1(N) \otimes H^c}$$
for all $x = \sum_k x_k \otimes h_k$ with $x^* = \sum_k x_k^* \otimes h_k \in L_1(N) \otimes H$. Instead of the opposite structure $N^{\text{op}}$, we will often work with an antilinear duality bracket. More precisely, the map $\bar{\iota} : N \to L_1(N, \phi)$ given by
\begin{equation}
\bar{\iota}(y)(x) = \text{tr}(x^*y) = \langle \langle x, y \rangle \rangle
\end{equation}
is a complete isometry. The symbol $\langle \langle x, y \rangle \rangle = \text{tr}(x^*y)$ is reserved for this antilinear duality bracket. Let us illustrate this duality in connection with (1.10). The dual space of the first column in $L_1(N \bar{\otimes} B(\ell_2))$ with respect to the antilinear duality bracket is the first column in $N \bar{\otimes} B(\ell_2)$. For such that a column with entries $(x_k)$, the sum $\sum_k x_k^* x_k$ is converging in the weak operator topology. We use the suggestive notation $B(\ell_2) \otimes C$ and $B(\ell_2) \otimes R$. Given an arbitrary Hilbert spaces $H$, we may still consider columns and rows after fixing a unit vector. We deduce from (1.10) that we have complete isometries
\begin{equation}
L_1(N) \hat{\otimes} H^r = N \bar{\otimes} H^c \quad \text{and} \quad L_1(N) \hat{\otimes} H^c = N \bar{\otimes} H^r.
\end{equation}
In the sequel, we will use
\begin{equation}
S^a_2[L_2^{\text{sh}}(N)] = L_2(M_n \otimes N).
\end{equation}
Equality (1.13) follows immediately from [Ps5] in the hyperfinite, semifinite case. In the general case, we may first assume that $N$ is $\sigma$-finite. Let $\phi$ be a faithful normal state with density $d \in L_1(N)$. Then the map $v : N \to L_2(N), v(x) = d^{1/2} x d^{1/2}$ is bounded. Note that $\bar{v}^* v$ is the map $M_{d^{1/2}, d^{1/2}} : N \to L_1(N)$ given by $M_{d^{1/2}, d^{1/2}}(x) = d^{1/2} x d^{1/2}$. Hence [Ps4] Corollary 2.4 implies that
\begin{equation}
[M_{d^{1/2}, d^{1/2}}(N), L_1(N)]_{1/2} = L_2^{\text{sh}}(N)
\end{equation}
completely isometrically. Therefore, we deduce from [K] and [Ps5] Corollary 1.4 that
\begin{equation}
L_2(M_n \otimes N) = [(\text{id} \otimes M_{d^{1/2}, d^{1/2}})(M_n \otimes N), L_1(M_n \otimes N)]_{1/2} = S_2^a[L_2^{\text{sh}}(N)].
\end{equation}
Since every von Neumann algebra admits a strictly semifinite normal weight, (i.e. a weight which is an orthogonal sum of states), (1.13) follows by approximation for arbitrary von Neumann algebras.

Let $N$ be a semifinite von Neumann algebra and $\tau$ a normal, faithful, semifinite trace. Then the classical $L_p$-space $L_p(N, \tau) = [N, L_1(N^{\text{op}}, \tau)]_{1/p}$ is (completely) isometrically isomorphic to the Haagerup $L_p$-space $L_p(N, \tau)$ (see [K] for the explicit isomorphism). Moreover, (1.8) and (1.13) below also hold in the category of semifinite $L_p$-spaces.

An important result of Effros and Ruan (see e.g. [ER2]) shows that the projective tensor product is compatible with von Neumann algebras. Indeed, given von Neumann algebras $N$ and $M$ then
\begin{equation}
(N_\ast \hat{\otimes} M_\ast)^\ast = N \bar{\otimes} M.
\end{equation}
Indeed, in the $\sigma$-finite with n.s.f. states $\phi$ and $\psi$, the modular group of $\phi \otimes \psi$ is given by $\sigma_t^{\phi \otimes \psi} = \sigma_t^\phi \otimes \sigma_t^\psi$. We may then use $L_1(N) \cong N^{\text{op}}_\ast$, $L_1(M) \cong M^{\text{op}}_\ast$. This yields a completely isometric isomorphism
\begin{equation}
L_1(N \bar{\otimes} M, \phi \otimes \psi) \cong (N \bar{\otimes} M)^{\text{op}}_\ast = N^{\text{op}}_\ast \hat{\otimes} M^{\text{op}}_\ast \cong L_1(N, \phi) \hat{\otimes} L_1(M, \psi).
\end{equation}
The general case follows by approximation from the $\sigma$-finite case.
2. A logarithmic characterization of $C^*$-algebras with WEP

The notion of $(2, oh)$-summing maps introduced in [Ps4] will be an important tool in this section. It allows us to find the smallest integral, sesquilinear form majorizing a given positive, sesquilinear form. A linear map $u : E \to OH$ is called $(2, oh)$-summing if there exists a constant $C > 0$ such that

$$\sum_k \|u(x_k)\|^2 \leq C^2 \sum_k \|\bar{x}_k \otimes x_k\|_{E \otimes_{\min} E}$$

holds for all finite sequences $(x_k) \subset E$. The $(2, oh)$-summing norm is defined by $\pi_{2, oh}^i(u) = \inf C$, where the infimum is taken over all $C$ satisfying (2.1).

**Lemma 2.1.** Let $E$ be an operator space and $B : \tilde{E} \times E \to \mathbb{C}$ be a positive sesquilinear form. Let $L = \{x \in E : B(x, x) = 0\}$ and $K$ the completion of $E/L$ with respect to the induced scalar product $(x + L, y + L) = B(x, y)$. Then the linear map $u : E \to K^{oh}$, $u(x) = x + L$ satisfies

i) $B(x, y) = (u(x), u(y))$ for all $x, y \in E$,
ii) $\|u\|_{2, oh}^i = \|B\|_{jcb}$,
iii) $\inf_{B \leq \tilde{B}} \|\tilde{B}\|_I = \pi_{2, oh}^i(u)^2$.

**Proof:** Equality i) is obvious from the definition of $u$. Assertion ii) is proved in [Ps4, Corollary 2.4]. For the estimate $\geq$ in iii), we assume that $\tilde{B}$ is an integral form such that $B \leq \tilde{B}$. Let $z_k$ be scalars such that $|z_k| = 1$ and $z_k \tilde{B}(x_k, x_k) = \tilde{B}(x_k, x_k)$. Then, we deduce from a version of the Cauchy-Schwarz inequality due to Haagerup (see [Ps7] (7.2)) that

$$\sum_k B(x_k, x_k) \leq \sum_k |\tilde{B}(x_k, x_k)| = \sum_k \tilde{B}(x_k, z_k x_k) \leq \|\tilde{B}\|_I \sum_k \|\bar{x}_k \otimes z_k x_k\|_{\min}$$

$$\leq \|\tilde{B}\|_I \left( \sum_k \|\bar{x}_k \otimes x_k\|_{\min} \right) \left( \sum_k \|\bar{z}_k \bar{x}_k \otimes z_k x_k\|_{\min} \right) = \|\tilde{B}\|_I \left( \sum_k \|\bar{x}_k \otimes x_k\|_{\min} \right) .$$

holds for all $(x_k) \subset E$. Taking the infimum over all $\tilde{B} \geq B$, we obtain

$$\sum_k \|u(x_k)\|^2 = \sum_k B(x_k, x_k) \leq \inf_{B \leq \tilde{B}} \|\tilde{B}\|_I \sum_k \|\bar{x}_k \otimes x_k\|_{\min} .$$

This implies that $\pi_{2, oh}^i(u)^2 \leq \inf_{B \leq \tilde{B}} \|\tilde{B}\|_I$. Conversely, we assume (2.1). We apply a variant of the Grothendieck-Pietsch separation argument in the context of $(2, oh)$-summing maps, see [Ps4] Theorem 5.7]. For this we shall assume that the operator space $E$ is given by a concrete representation $E \subset B(H)$. Then there exists an index set $I$, an ultrafilter $\mathcal{U}$ and nets $(a_i), (b_i)$ in the unit sphere of $S_4(H)$ such that

$$\|u(x)\| \leq \pi_{2, oh}^i(u) \lim_{i, \mathcal{U}} \|a_i x b_i\|_2$$

holds for all $x \in E$. For fixed $i$, we note that

$$\|a_i x b_i\|_2^2 = \text{tr}(a_i x b_i b_i^* x^* a_i^*) = \text{tr}(x^* a_i^* a_i x b_i b_i^*) = ((b_i b_i^*)^t, (x^* \otimes x)(a_i^* a_i)) .$$

We refer to [Ps6, ER2] for the fact that $B_i(x, y) = \text{tr}(x^* a_i^* a_i y b_i b_i^*) = ((b_i b_i^*)^t, (x^* \otimes y)(a_i^* a_i))$ is nuclear with norm $\leq 1$. Integral forms are closed under pointwise limits with respect to bounded
nets of nuclear forms. Therefore
\[ \tilde{B}(x, y) = \pi_2^{oh}(u)^2 \lim_{t \to 0} \left( x^* a_i y b_i^* \right) \]
is a positive integral bilinear form with \( \| \tilde{B} \|_I \leq \pi_2^{oh}(u)^2 \). Inequality \( \ref{2.2} \) shows that \( B(x, x) = (u(x), u(x)) \) is dominated by \( \tilde{B} \).

**Lemma 2.2.** Let \( A \) be a C*-algebra with WEP and \( u : A \to OH_n \). Then
\[ \pi_2^{oh}(u) \leq C(1 + \ln n)^{\frac{1}{2}} \| u \|_{cb} \]

**Proof:** We recall that \( A \) has WEP, if there exists a contraction \( P : \mathcal{B}(H) \to A^{**} \) such that \( A \subset A^{**} \subset \mathcal{B}(H) \) and that \( P|_A = id_A \). It is well-known that \( P \) is indeed completely contractive (see e.g. [J2, Lemma 2.1]). Then \( v = u^{**}P : \mathcal{B}(H) \to OH_n \) satisfies
\[ \| v \|_{cb} \leq \| P \|_{cb} \| u^{**} \|_{cb} \leq \| u \|_{cb} . \]
We can apply \( \ref{0.4} \) for \( B(x, y) = (v(x), v(y)) \) and deduce the assertion from Lemma 2.1.

**Lemma 2.3.** Let \( N \) be a von Neumann algebra. Let \( a, b \in L_4(N) \) and \( M_{ab} : N \to L_2^{oh}(N) \) defined by \( M_{ab}(x) = axb \). Then
\[ \left\| M_{ab} : N \to L_2^{oh}(N) \right\|_{cb} \leq \| a \|_4 \| b \|_4 . \]

**Proof:** Let \( x \in M_n(N) \) and \( a, b \in L_4(N) \). According to [PS5, Lemma 1.7], we have
\[ \|(id \otimes M_{ab})(x)\|_{M_n(L_2^{oh}(N))} = \sup_{\alpha, \beta} \|(\alpha \otimes a)x(\beta \otimes b)\|_{S_2^4[L_2^{oh}(N)]} . \]
Here the supremum is taken over all \( \alpha, \beta \) in the unit ball of \( S_2^4 \). The assertion follows from \( \ref{1.13} \) and Hölder’s inequality
\[ \|(\alpha \otimes a)x(\beta \otimes b)\|_{S_2^4[L_2^{oh}(N)]} = \|(\alpha \otimes a)x(\beta \otimes b)\|_{L_2(M_n \otimes N)} \leq \|\alpha \otimes a\|_4 \|x\|_{M_n(N)} \|\beta \otimes b\|_4 \]
\[ \leq \|a\|_4 \|b\|_4 \]

**Proof of Theorem 6:** We use the one-to-one correspondence between completely bounded linear maps \( u : A \to OH \) and positive sesquilinear jcb-forms on \( A \) (see Lemma 2.4). Theorem 6 means that \( A \) has WEP if and only if
\[ \pi_2^{oh}(u) \leq C(1 + \ln n)^{\frac{1}{2}} \| u \|_{cb} \]
holds for every linear map \( u : A \to OH_n \). If \( A \) has WEP, then Lemma 2.2 implies that \( \ref{2.3} \) holds for \( \beta = \frac{1}{2} \) and a universal constant \( C \). Conversely, we assume that \( \ref{2.3} \) holds for some \( \beta > 0 \) and some constant \( C \). Let us consider the von Neumann algebra \( N = A^{**} \). Recall from [10] that \( N \) is in standard form on \( L_2(N) \). This means in particular that \( N \) acts on \( L_2(N) \) by left multiplication \( \pi(x)h = xh \) and \( J(h) = h^* \) is an antilinear isometry \( J \) such that \( N' = JNJ \) (see [T3]). Let \( h \in L_2(N) \) be a unit vector. Then, we can find \( a, b \in L_4(N) \) of norm 1 such that \( h = ab \). According to Lemma 2.3, the maps \( M_{ab}^* : N \to L_2^{oh}(N) \) and \( M_{bb} : N \to L_2^{oh}(N) \) are complete contractions. Now, we consider elements \( x_1, \ldots, x_n \) in \( A \). Let \( P \) and \( Q \) be orthogonal
projections onto \( \text{span}\{M_{a^*a}(x_k)|1 \leq k \leq n\} \) and \( \text{span}\{M_{bb^*}(x_k)|1 \leq k \leq n\} \), respectively. Then \( PM_{a^*a} \) and \( QM_{bb^*} \) have rank as most \( n \). Therefore we may apply (2.3) and deduce

\[
(h, \sum_{k=1}^{n} x_{k} J x_{k} J h) = (h, \sum_{k=1}^{n} x_{k} h x_{k}^*) = \sum_{k=1}^{n} tr(h^* x_{k} h x_{k}^*) = \sum_{k=1}^{n} tr(a^* x_{k} ab x_{k}^*)
\]

(2.4)

\[
\leq \left( \sum_{k=1}^{n} \|P(a^* x_{k} a)\|^2 \right)^{\frac{1}{2}} \leq C^2 (1 + \ln n)^{2\beta} \left\| \sum_{k=1}^{n} \bar{x}_{k} \otimes x_{k} \right\|_{\text{min}}.
\]

In order to eliminate the log-term, we use Haagerup’s trick and consider the positive operator

\[
[(\sum_{k=1}^{n} x_{k} J x_{k} J)^* (\sum_{k=1}^{n} x_{k} J x_{k} J)]^m
\]

\[
= \sum_{k_1, \ldots, k_{2m}=1}^{n} x_{k_1}^* (J x_{k_1}^* J) x_{k_2} (J x_{k_2} J) \cdots x_{k_{2m-1}}^* (J x_{k_{2m-1}}^* J) x_{k_{2m}} (J x_{k_{2m}} J)
\]

\[
= \sum_{k_1, \ldots, k_{2m}=1}^{n} (J x_{k_1}^* x_{k_2} \cdots x_{k_{2m-1}}^* x_{k_{2m}} J) x_{k_1} x_{k_2} \cdots x_{k_{2m-1}} x_{k_{2m}}
\]

We apply (2.4) to the finite family \( x_{k_1, \ldots, k_{2m}} = x_{k_1}^* x_{k_2} \cdots x_{k_{2m-1}}^* x_{k_{2m}} \) and deduce that

\[
(h, [(\sum_{k=1}^{n} x_{k} J x_{k} J)^* (\sum_{k=1}^{n} x_{k} J x_{k} J)]^m h)
\]

\[
= \sum_{k_1, \ldots, k_{2m}=1}^{n} (h, x_{k_1}^* x_{k_2} \cdots x_{k_{2m-1}}^* x_{k_{2m}} J x_{k_1}^* x_{k_2} \cdots x_{k_{2m-1}}^* x_{k_{2m}} J h)
\]

\[
\leq C^2 (1 + \ln n)^{2\beta} \left\| \sum_{k_1, \ldots, k_{2m}=1}^{n} \bar{x}_{k_1, \ldots, k_{2m}} \otimes \bar{x}_{k_1, \ldots, k_{2m}} \right\|_{\text{min}}
\]

\[
= C^2 (1 + \ln n)^{2\beta} \left\| \left( \sum_{k=1}^{n} \bar{x}_{k} \otimes x_{k} \right)^m \left( \sum_{k=1}^{n} \bar{x}_{k} \otimes x_{k} \right) \right\|_{\text{min}} = C^2 (1 + 2m \ln n)^{2\beta} \left\| \sum_{k=1}^{n} \bar{x}_{k} \otimes x_{k} \right\|_{\text{min}}^{2m}.
\]

Taking the supremum over \( ||h|| \leq 1 \), we deduce from positivity that

\[
\left\| \sum_{k=1}^{n} x_{k} J x_{k} J \right\| = \left\| (\sum_{k=1}^{n} x_{k} J x_{k} J)^* (\sum_{k=1}^{n} x_{k} J x_{k} J) \right\|^{\frac{1}{2m}}
\]

\[
\leq C^2 \left( 1 + 2m \right)^{\frac{2\beta}{2m}} (1 + \ln n)^{\frac{2\beta}{2m}} \left\| \sum_{k=1}^{n} \bar{x}_{k} \otimes x_{k} \right\|_{\text{min}}^{\frac{1}{2m}}.
\]

Taking the limit for \( m \to \infty \), we obtain (in the language of [PS2])

\[
(2.5) \quad \left\| \sum_{k=1}^{n} L_{x_{k}} R_{x_{k}}^* \right\| = \left\| \sum_{k=1}^{n} x_{k} J x_{k} J \right\| \leq \left\| \sum_{k=1}^{n} \bar{x}_{k} \otimes x_{k} \right\|_{\mathcal{A}^\otimes_{\text{min}} A} = \left\| \sum_{k=1}^{n} \bar{x}_{k} \otimes x_{k} \right\|_{\mathcal{B}(H)^{\otimes_{\text{min}} \mathcal{B}(H)}}.
\]
Let us recall an equality proved by Pisier [PS2, Theorem 2.1] (and [HR, Theorem 2.9] in the non-semifinite case)

\[ \left\| \sum_{k=1}^{n} \bar{x}_k \otimes x_k \right\|_{A \otimes_{\max} A} = \left\| (x_1, \ldots, x_n) \right\|_{[R_n(A), C_n(A)]}^{2} = \left\| \sum_{k=1}^{n} L_{x_k} R_{x_k} \right\|_{L_2(A^{\ast \ast})}. \]

Combining this with (2.5), we deduce that

\[ \left\| \sum_{k=1}^{n} \bar{x}_k \otimes x_k \right\|_{A \otimes_{\max} A} \leq \left\| \sum_{k=1}^{n} \bar{x}_k \otimes x_k \right\|_{B(H) \otimes_{\min} B(H)} \leq \left\| \sum_{k=1}^{n} \bar{x}_k \otimes x_k \right\|_{B(H) \otimes_{\max} B(H)}. \]

According to [HR, Theorem 3.7] A has WEP. In section 4 we will show that \( \alpha = 2\beta = 1 \) is the best possible exponent for non sub-homogeneous \( C^* \)-algebras.

**Remark 2.4.** Let us consider the special case where \( A = N \) is a von Neumann algebra. We have shown above that if \( N \) satisfies the logarithmic Grothendieck inequality, then

\[ \left\| (x_1, \ldots, x_n) \right\|_{[R_n(N), C_n(N)]} \leq \left\| \sum_{k=1}^{n} \bar{x}_k \otimes x_k \right\|_{N \otimes_{\min} N}^{2}. \]

According to Pisier’s characterization [PS2] (see [PS4, Theorem 2.9] for a simple proof in the semifinite case), we deduce that \( N \) is injective. Thus for von Neumann algebras, we don’t have to use Haagerup’s deep (and unfortunately unpublished) results.

3. The Pusz/Woronowicz formula and the operator space \( OH \)

In this section we will show the connection between the Pusz/Woronowicz formula for square roots of sesquilinear forms and subspaces of quotients of Hilbert spaces. For our understanding of the problem this concrete formula (and its dual version) plays an important role. Following [PW] we consider two positive commuting operators \( A, B \) on a Hilbert space. According to [PW]

\[ (AB)^{\frac{1}{2}} x, x = \inf_{x=a(t)+b(t)} \int_{0}^{1} \left( \frac{a(t), a(t)}{t} + \frac{(Bb(t), b(t))}{1-t} \right) \frac{dt}{\pi \sqrt{t(1-t)}}. \]

Here the infimum is taken over piecewise constant functions in \( H \); see [PW, Appendix] for a proof. Let us denote by \( H_{\sqrt{AB}} \) the Hilbert space \( H \) equipped with the scalar product \( \sqrt{AB}(x, y) = (\sqrt{AB}x, y) \). Motivated by (3.1), we define the probability \( \mu \) and the measures \( \nu_1, \nu_2 \) on \([0, 1]\) as follows:

\[ d\mu(t) = \frac{dt}{\pi \sqrt{t(1-t)}}, \quad \nu_1(t) = t^{-1} d\mu(t) \quad \text{and} \quad d\nu_2(t) = (1-t)^{-1} d\mu(t). \]

We denote by \( H_A \) and \( H_B \) the space \( H \) equipped with the Hilbertian norm \( \| x \|_{H_A} = (Ax, x)^{\frac{1}{2}} \), respectively. If \( A \) and \( B \) are invertible, the canonical inclusion maps \( H_A \subset H \) and \( H_B \subset H \) are continuous. Then we may define the linear map \( Q : L_2(\nu_1, H_A) \oplus_2 L_2(\nu_2, H_B) \rightarrow L_0(\mu; H) \) by \( Q(f, g)(t) = f(t) + g(t) \in H \). We denote by

\[ K = L_2(\nu_1, H_A) \oplus_2 L_2(\nu_2, H_B)/\ker(Q) \]

the quotient space and

\[ E = \{(f, g) + \ker(Q) \mid Q(f, g) \text{ constant a.e.}\}. \]
Lemma 3.1. If $A$ and $B$ are invertible, then $H_{\sqrt{AB}}$ is isometrically isomorphic to the subspace $E \subset K$.

The dual version of Lemma 3.1 is based on a characterization of linear functionals on $E$.

Lemma 3.2. Let $A$ and $B$ be bounded and invertible. Linear functionals $\phi : E \to \mathbb{C}$ are in one-to-one correspondence with pairs $(f, g) \in L_2(\nu_A, H_A) \oplus L_2(\nu_B, H_B)$ such that

\[
Af(t) = \frac{Bg(t)}{1-t} \quad \mu \text{ a.e., } \quad \phi(x) = \int_0^1 (Af(t), x) \, d\mu(t).
\]

Moreover,

\[
\|\phi\| = \inf (\|f\|_{L_2(\nu_A, H_A)} + \|g\|_{L_2(\nu_B, H_B)})^\frac{1}{2}.
\]

Here the infimum is taken over all pairs satisfying (3.3).

Proof: By the Hahn-Banach theorem, the norm one functionals on $E$ are in one to one correspondence with the restrictions $\hat{\phi}_E$ of norm one functionals $\hat{\phi} : L_2(\nu_A, H_A) \oplus L_2(\nu_B, H_B) \to \mathbb{C}$. A norm one functional $\hat{\phi}$ is given by a couple $(f, g)$ such that $\|\hat{\phi}\|^2 = \|f\|^2_{L_2(\nu_A, H_A)} + \|g\|^2_{L_2(\nu_B, H_B)}$ and

\[
\hat{\phi}(h_1, h_2) = \int_0^1 (Af(t), h_1(t)) \frac{d\mu(t)}{t} + \int_0^1 (Bg(t), h_2(t)) \frac{d\mu(t)}{1-t}.
\]

Thus the functional $\hat{\phi}$ vanishes for all $(h, -h)$ if and only if $\frac{Af(t)}{t} = \frac{Bg(t)}{1-t}$ $\mu$-almost everywhere. Finally, given $x \in E$, we see that $\sqrt{t} x \in L_2(\nu_A, H_A)$ and $(1 - \sqrt{t}) x \in L_2(\nu_B, H_B)$. Since $\hat{\phi}$ is an extension of $\phi$, we deduce that

\[
\phi(x) = \hat{\phi} Q(\sqrt{t} x, \sqrt{1-t} x) = \int_0^1 \left( \frac{Af(t)}{t}, \sqrt{t} x \right) + \left( \frac{Bg(t)}{1-t}, (1 - \sqrt{t}) x \right) d\mu(t) = \int_0^1 \left( \frac{Af(t)}{t}, x \right) d\mu(t). \]

Lemma 3.3. Let $A,A^{-1}, B,B^{-1}$ be bounded and $y \in H$. Then

\[
((AB)^\frac{1}{2} y, y) = \inf \int_0^1 \left( \frac{Af(t)}{t}, f(t) \right) d\mu(t) + \int_0^1 \left( \frac{Bg(t)}{1-t}, g(t) \right) d\mu(t),
\]

where the infimum is taken over all tuples $(f, g)$ of $H$-valued measurable functions satisfying $\frac{Af(t)}{t} = \frac{Bg(t)}{1-t} \mu$ a.e. and

\[
B^\frac{1}{2} y = \int_0^1 \frac{A^\frac{1}{2} f(t)}{t} \, d\mu(t).
\]
Proof: Since $\sqrt{AB}$ defines a scalar product, we have
$$\|y\|_{H_{\sqrt{AB}}} = \| (AB)^{1/2} y \|_{\sqrt{AB}} = \sup_{\|x\|_{\sqrt{AB}} \leq 1} |((AB)^{1/2} y, x)|.$$  
Thus the norm of $y$ in $H_{\sqrt{AB}}$ coincides with the norm of the linear functional
$$\phi_y(x) = ((AB)^{1/2} y, x)$$
on $H_{\sqrt{AB}}$. According to Lemma 3.2 we can find $(f,g)$ such that (3.3) is satisfied and
$$((AB)^{1/2} y, y) = \|\phi_y\|^2 = \int_0^1 (Af(t), f(t)) \frac{d\mu(t)}{t} + \int_0^1 (Bg(t), g(t)) \frac{d\mu(t)}{1-t}.$$  
From (3.3) and the definition of $\phi_y$ we deduce that
$$B^{1/2} y = A^{-1/2} (AB)^{1/2} y = A^{-1/2} \left[ \frac{1}{t} \int_0^1 Af(t) \frac{d\mu(t)}{t} \right] = \left[ \frac{1}{t} \int_0^1 A^{1/2} f(t) \frac{d\mu(t)}{t} \right] .$$  
Conversely, any pair satisfying these conditions induces the same functional $\phi_y$ and thus provides an upper estimate for the norm of $y$. □

As pointed out in the introduction, the operator space $OH_n$ will be obtained by amplifications of these densities. To be more specific, we consider the map $Q : L_2(\nu_1; \ell_2^n) \oplus_1 L_2(\nu_2; \ell_2^n) \to L_0(\mu; \ell_2^n)$ defined by
$$Q(f,g)(t) = f(t) + g(t) \in \ell_2^n.$$ We denote by $G_n$ the quotient space
$$G_n = L_2^c(\nu_1; \ell_2^n) \oplus_1 L_2^c(\nu_2; \ell_2^n) / \ker(Q) .$$ In the limiting case $n = \infty$ we will simply write $G$. The interesting subspaces $F_n$, $F$ of $G_n$ are given by those equivalence classes $(f,g) + \ker Q$ such that $Q(f,g)$ is a constant function with values in $\ell_2^n$, $\ell_2$. We denote by $f_1, \ldots, f_n$ the canonical unit vector basis in $F_n$ given by $f_k = (\sqrt{\nu_1} e_k, (1 - \sqrt{\nu_2}) e_k) + \ker(Q)$. The following lemma is proved using the polar decomposition and the density of invertible matrices.

Lemma 3.4. Let $x_1, \ldots, x_n \in M_m$. Then
$$\left\| \sum_{k=1}^n \tilde{x}_k \otimes x_k \right\|_{\min M_m} \leq \sup_{\|a\|_4 \leq 1, \|b\|_4 \leq 1, a > 0, b > 0} \left( \sum_{k=1}^n \|bx_k a\|_2^2 \right)^{1/2} .$$
Here $a > 0$ means that $a \geq 0$ and $a$ is invertible.

Lemma 3.5. Let $(e_k)$ be the natural unit vector basis of $OH_n$. The identity map $id : F_n \to OH_n$ given by $(f_k) = e_k$ has cb-norm less than $\sqrt{2}$.

Proof: Let $x_1, \ldots, x_n \in M_m$ and assume $\| \sum_{k=1}^n x_k \otimes f_k \|_{M_m(F_n)} < 1$. By the definition of the matrix norms for quotient operator spaces, we find elements $f \in M_m(L_2^c(\nu_1; \ell_2^n))$ and $g \in M_m(L_2^c(\nu_2; \ell_2^n))$ such that
$$x_k = f_k(t) + g_k(t) \mu\text{-a.e.}$$
From Lemma 3.4, we deduce that
\[ \| z \|_{M_n} \leq 1. \]
Let \( a, b \) be positive, invertible, norm one elements in \( S_1^m \). On the Hilbert space \( H = \ell_2^n(S_2^m) \) with the scalar product
\[ ((x), (y)) = \sum_{k=1}^n \text{tr}(x_k y_k) \]
we define \( A(x_k) = (x_k a^4) \) and \( B(x_k) = (b^4 x_k) \). Clearly, these operators commute and we deduce from (3.1) that
\[
\sum_{k=1}^n \| b x_k a \|_2^2 = \sum_{k=1}^n \text{tr}(a^* x_k^* b^* b x_k a) = \sum_{k=1}^n \text{tr}(a^2 x_k^* b^2 x_k) = ((AB)^{1/2}(x), (x))
\]
\[
\leq \int_0^1 (A(f_k(t)), (f_k(t))) \frac{d\mu(t)}{t} + \int_0^1 (B(g_k(t)), (g_k(t))) \frac{d\mu(t)}{1-t}
\]
\[
= \int_0^1 \sum_{k=1}^n \text{tr}(a^4 f_k(t) f_k(t)) \frac{d\mu(t)}{t} + \int_0^1 \sum_{k=1}^n \text{tr}(g_k(t) b^4 g_k(t)) \frac{d\mu(t)}{1-t}
\]
\[
= \text{tr} \left( a^4 \int_0^1 \sum_{k=1}^n f_k(t) f_k(t) \frac{d\mu(t)}{t} \right) + \text{tr} \left( b^4 \int_0^1 \sum_{k=1}^n g_k(t) g_k(t) \frac{d\mu(t)}{1-t} \right)
\]
\[
\leq \| a^4 \|_1 \left\| \int_0^1 \sum_{k=1}^n f_k(t) f_k(t) \frac{d\mu(t)}{t} \right\|_{M_n} + \| b^4 \|_1 \left\| \int_0^1 \sum_{k=1}^n g_k(t) g_k(t) \frac{d\mu(t)}{1-t} \right\|_{M_n} \leq 2.
\]
From Lemma 3.4, we deduce that \( \| \text{id} : F_n \to \text{OH}_n \|_{cb} \leq \sqrt{2} \). □

**Lemma 3.6.** Let \( (f_k^*)_{k=1}^n \) be the dual basis of \( F_n^* \) satisfying \( f_j^* (f_j) = \delta_{kj} \). The identity map \( \text{id} : F_n^* \to \text{OH}_n \) given by \( \text{id}(f_k^*) = e_k \) has cb-norm less than \( \sqrt{2} \).

**Proof:** We have to consider a norm one element \( z \in M_n(F_n^*) \cong CB(F_n, M_m) \). Then, we may write \( z = \sum_{k=1}^n z_k \otimes f_k^* \). Let us denote the corresponding complete contraction which satisfies \( z_k = u_z(f_k) \in M_m \) as \( u_z : F_n \to M_m \). According to Wittstock’s theorem there exists a complete contraction \( v : G_n \to M_m \). Since \( G_n \) is a quotient space, we see that \( vQ : L^2_2(\nu_1; \ell_2^n) \oplus_1 L^2_2(\nu_2; \ell_2^n) \to M_m \) is a complete contraction. Thus there are \( x \in M_m(L^2_2(\nu_1; \ell_2^n)) \) and \( y \in M_m(L^2_2(\nu_2; \ell_2^n)) \) of norm less than 1 such that
\[
vQ(h^1, h^2) = \int_0^1 \sum_{k=1}^n x_k(t) h_k^1(t) \frac{d\mu(t)}{t} + \int_0^1 \sum_{k=1}^n y_k(t) h_k^2(t) \frac{d\mu(t)}{1-t}.
\]
Again, we can use the fact that \( v \) vanishes on \( \ker(Q) \) and get
\[
\frac{x_k(t)}{t} = \frac{y_k(t)}{1-t} \mu \text{-a.e.}
\]
for all $k = 1, \ldots, n$. In order to identify our original map $u_z$, we compute

$$u_z(f_k) = vQ(e_k \sqrt{t}, e_k(1 - \sqrt{t})) = \int_0^1 \sqrt{t} x_k(t) \frac{d\mu(t)}{t} + \int_0^1 (1 - \sqrt{t}) y_k(t) \frac{d\mu(t)}{1 - t} = \int_0^1 x_k(t) \frac{d\mu(t)}{t}.$$  

Now, let us consider invertible positive elements $a, b \in S_4^n$. As above, we define the operators $A(z_k)_{k=1}^n = (z_k a^4)_{k=1}^n$ and $B(z_k)_{k=1}^n = (b^4 z_k)_{k=1}^n$. We consider $\tilde{x}_k(t) = b^2 x_k(t) a^{-2}$ and get

$$B^\sharp((z_k)_{k=1}^n) = (b^2 z_k)_{k=1}^n = \int_0^1 (b^2 x_k(t))_{k=1}^n \frac{d\mu(t)}{t} = \int_0^1 A^\sharp((\tilde{x}_k(t))_{k=1}^n) \frac{d\mu(t)}{t}.$$

On the other hand for $\tilde{x} = (\tilde{x}_k)_{k=1}^n$, we deduce that

$$\int_0^1 (A\tilde{x}(t), \tilde{x}(t)) \frac{d\mu(t)}{t} = \int_0^1 (\tilde{x}(t), A\tilde{x}(t)) \frac{d\mu(t)}{t} = \int_0^1 \sum_{k=1}^n tr(\tilde{x}_k^*(t) \tilde{x}_k(t) a^4) \frac{d\mu(t)}{t}$$

$$= \int_0^1 \sum_{k=1}^n tr(a^{-2} x_k^*(t) b^2 b^2 x_k(t) a^{-2} a^4) \frac{d\mu(t)}{t}$$

$$= tr(b^4 \int_0^1 \sum_{k=1}^n x_k(t) x_k^*(t) \frac{d\mu(t)}{t})$$

$$\leq \|b^4\|_1 \left\| \int_0^1 \sum_{k=1}^n x_k(t) x_k^*(t) \frac{d\mu(t)}{t} \right\|_{M_m}$$

$$\leq \|x\|_{M_m(L_2(\nu_1, \ell_2^n))}^2 \leq 1.$$  

Similarly, we define $\tilde{y}_k(t) = b^{-2} y_k(t) a^2$, $\bar{y}(t) = (\tilde{y}_k(t))_{k=1}^n$ and get

$$\frac{A((\tilde{x}_k(t))_{k=1}^n)}{t} = \frac{(b^2 x_k(t) a^2)_{k=1}^n}{t} = \frac{(b^2 y_k(t) a^2)_{k=1}^n}{1 - t} = \frac{B((\tilde{y}_k(t))_{k=1}^n)}{1 - t} \mu\text{-a.e.} .$$

The same calculation as above yields

$$\int_0^1 (B\bar{y}(t), \bar{y}(t)) \frac{d\mu(t)}{1 - t} \leq \|a^4\|_1 \|y\|_{M_m(L_2(\nu_1, \ell_2^n))}^2 \leq 1 .$$  

Therefore Lemma 3.3 implies that

$$\sum_{k=1}^n \|b z_k a\|_2^2 = \left\| (\sqrt{A} B(x_k)_{k=1}^n) \right\|_2^2 \leq \int_0^1 (A\tilde{x}(t), \tilde{x}(t)) \frac{d\mu(t)}{t} + \int_0^1 (B\bar{y}(t), \bar{y}(t)) \frac{d\mu(t)}{1 - t} \leq 2 .$$

Since, $a, b$ are arbitrary, we deduce from Lemma 3.4 that

$$\left\| \sum_{k=1}^n \bar{z}_k \otimes z_k \right\|_{M_m(F_n^*)} \leq \sqrt{2} \left\| \sum_{k=1}^n z_k \otimes f_k^* \right\|_{M_m(F_n^*)} .$$  

**Corollary 3.7.** $F_n$ is 2-completely isomorphic to $OH_n$.  

Proof: Since \( \text{OH}_n \) is selfdual, see [Ps4], it suffices to apply Lemma 3.5 and Lemma 3.6

\[
\|\text{id} : F_n \rightarrow \text{OH}_n\|_{cb} \|\text{id} : \text{OH}_n \rightarrow F_n\|_{cb} = \|\text{id} : F_n \rightarrow \text{OH}_n\|_{cb} \|\text{id} : F_n^* \rightarrow \text{OH}_n\|_{cb} \leq 2.
\]

**Remark 3.8.** A similar result holds in the context of \( L_p \) spaces. We use the standard notation \( H^p = [H^c, H^r]^1_p \) and \( H^{p'} = [H^r, H^c]^1_p \). Let \( 2 \leq p \leq \infty \) and \( p' \leq q \leq p \). We consider \( 2 \leq r \leq \infty \) such that \( \frac{1}{r} + \frac{1}{p'} = \frac{1}{2} \). As above, we consider the subspace \( F_n(p) \) of ‘constant’ functions in the quotient space \( G_n = L^p_2(\nu_1; \ell^q_2) \oplus L^p_2(\nu_2; \ell^q_2)/\ker(Q) \). The same proofs as in Lemma 3.5 and in Lemma 3.6 applies with the help of the formula

\[
\left\| \sum_{k=1}^n x_k \otimes e_k \right\|_{S^p_m[\text{OH}_n]} = \sup_{\|a\|_{2_\alpha} \leq 1, \|b\|_{2_\alpha} \leq 1} \left( \sum_{k=1}^n \|bx_ka\|^2 \right)^{\frac{1}{2}}.
\]

Therefore, we have

\[
\|\text{id} : F_n(p) \rightarrow \text{OH}_n\|_{cb} \leq 2^{\frac{1}{2} - \frac{1}{p'}} \quad \text{and} \quad \|\text{id} : F_n(p)^* \rightarrow \text{OH}_n\|_{cb} \leq 2^{\frac{1}{2} - \frac{1}{p'}}.
\]

These inequalities imply that

\[
d_{cb}(F_n(p), \text{OH}_n) \leq 2^{\frac{1}{2} - \frac{1}{p'}}.
\]

We obtain a concrete embedding of \( \text{OH} \) as a subspace of a quotient of \( S_p = L_p(B(\ell_2), tr) \) (with \( q = p \)) and of \( S_{p'} \) (with \( q = p' \) and using \( H^{p'} = H^{p' \prime} \)). For an independent, alternative approach we refer to [Xu].

**Remark 3.9.** A slight modification of this approach yields the space \( C_p = [C, R]^1_p \). Indeed, let \( \alpha = \frac{1}{p} \). Using the substitution \( u = t^{-1} - 1 \), we find

\[
\int_0^1 \frac{1}{(1-t) + tB t^\alpha (1-t)^{1-\alpha}} \frac{dt}{u + B u^{1-\alpha}} = \int_0^\infty \frac{1}{u + B u^{1-\alpha}} = c(\alpha) B^{1-\alpha}.
\]

Following [PW], we deduce for arbitrary commuting operators \( A, B \) that

\[
(x, A^{1-\alpha} B^\alpha x) = \inf_{x=f(t)+g(t)} \int_0^1 \frac{(f(t), Af(t))}{t} \frac{dt}{c(\alpha) t^\alpha (1-t)^{1-\alpha}} + \int_0^1 \frac{(g(t), Bg(t))}{1-t} \frac{dt}{c(\alpha) t^\alpha (1-t)^{1-\alpha}}.
\]

Similar as in Lemma 3.3, it then easily follows that the space \( F_\alpha \) of ‘constants’ in the quotient \( L_2^p(t^{-1} \mu_\alpha) \oplus L_2^p((1-t)^{-1} \mu_\alpha)/\ker(Q) \) satisfies \( \|\text{id} : F_\alpha \rightarrow [C, R]^1_p\|_{cb} \leq \sqrt{2} \). The analogue of Lemma 3.6 is considerably more involved and again based on the dual Pusz/Woronowicz formula. In the forthcoming publication [JX2] we will develop better tools for finding the ‘right pair of densities’.

Assuming Theorem 3, we immediately obtain an embedding of \( \text{OH} \) in a noncommutative \( L_1 \) space.

**Theorem 3.10.** \( \text{OH} \) embeds into the predual of a von Neumann algebra with QWEP.
We obtain an isomorphism $id : F_n \to OH_n$ from Corollary 3.11. The density of $\bigcup F_n$ in 

$$F \subset L_2^\infty(\nu_1; \ell_2) \oplus_1 L_2^\infty(\nu_2; \ell_2)/\ker(Q) = G$$

we obtain an isomorphism $id : F \to OH$ with $\|id\|_cb \leq \sqrt{2}$. Similarly, the density of $\bigcup_n OH_n$ in $OH$ implies that $\|id^{-1}\|_cb \leq \sqrt{2}$. Hence, $OH$ is 2-cb-isomorphic to $F$ and Theorem 3 implies the assertion.

**Corollary 3.11.** There exists a constant $C > 0$ such that for all $n \in \mathbb{N}$, there is an integer $m$ and an injective linear map $u : OH_n \to S_{1}^{m}$ such that 

$$\|u\|_cb \|u^{-1} : u(OH_n) \to OH_n\|_cb \leq C.$$  

**Proof:** This is an immediate consequence of the strong principle of local reflexivity in [EJR] and the fact that $N$ is QWEP.

**4. The Projection Constant of the Operator Space OH$_n$**

In this section, we will provide the proof of Theorem 1, assuming the probabilistic result Theorem 3. The main tool is a characterization of the completely summing norms for linear maps between certain operator spaces, see Lemma 4.4 and Lemma 4.5. Some notation is required. For an operator space $E$ we denote by $\iota_E : E \to E^{**}$ the canonical completely isometric embedding of $E$ into its bidual $E^{**}$. Let $E$ and $F$ be operator spaces. The $\Gamma_\infty$-norm of a linear map $v : F \to E$ is defined by

$$\Gamma_\infty(v) = \inf \|\alpha\|_cb \|\beta\|_cb.$$ 

Here the infimum is taken over all $\alpha : F \to B(H)$, $\beta : B(H) \to E^{**}$ such that $\iota_E v = \alpha \beta$. For a finite rank map $v : F \to E$, we also define

$$\gamma_\infty(v) = \inf \|\alpha\|_cb \|\beta\|_cb.$$ 

Here the infimum is taken over all $m \in \mathbb{N}$, $\beta : F \to M_m$ and $\alpha : M_m \to E$ such that $v = \alpha \beta$. Following Effros/Ruan (see e.g. [ER2]) a linear map $v : E \to F$ is said to be (completely) 1-summing if

$$\pi_1^\alpha(v) = \left\|id_{S_1} \otimes v : S_1 \otimes_{\min} E \to S_1 \hat{\otimes} F\right\|$$

is finite. The $\gamma_\infty$-norm is related to the 1-summing norm via trace duality.

**Lemma 4.1.** Let $u : E \to F$ be a linear map. Then

$$\pi_1^u(u) = \sup \{|tr(vu)| \ | \gamma_\infty(v) \leq 1\}.$$  

**Proof:** Indeed, we have

$$\pi_1^u(u) = \sup \left\{|(id \otimes u)(x), y| \ | \|x\|_{S_1^{**} \otimes_{\min} E} \leq 1, \|y\|_{M_m(F^{*})} \leq 1\right\}$$

$= \sup \left\{|tr(T_yuT_x)| \ | \|T_x : M_m \to E\|_cb \leq 1, \|T_y : M_m^{*} \to F^{*}\|_cb \leq 1\right\}$$

$= \sup \left\{|tr(vu)| \ | \gamma_\infty(v) \leq 1\}.$

We will need the following result from [EJR].
Lemma 4.2 (EJR). Let $E$ and $F$ be finite dimensional operator spaces and $v : F \rightarrow E$. Then
\[ \gamma_\infty(v) = \Gamma_\infty(v). \]

The connection between the 1-summing norm and the operator space projective tensor norm works only for subspaces of noncommutative $L_1$ spaces. Therefore, we follow [P5] and define
\[ d_{SL_1}(E) = \inf_{w : E \rightarrow E_1 \subset S_1} \| w \|_{cb} \| w^{-1} \|_{cb}. \]

We are interested in estimates for finite dimensional spaces. This leads to the following definition for infinite dimensional operator spaces
\[ d_{SL_1}(Y) = \sup_{E \subseteq Y} d_{SL_1}(E). \]

Here the supremum is taken over all finite dimensional subspaces $E \subseteq Y$. The following fact follows immediately from the definition.

Lemma 4.3. Let $x \in X \otimes Y$ and $\varepsilon > 0$. Then there are finite dimensional subspaces $E \subseteq X$ and $F \subseteq Y$ such that
\[ \| x \|_{E \hat{\otimes} F} \leq (1 + \varepsilon) \| x \|_{X \hat{\otimes} Y}. \]

Lemma 4.4. Let $X$ and $Y$ be operator spaces, $E \subseteq X$ and $F \subseteq Y$ be finite dimensional subspaces. Let $x \in E \otimes F$ be a tensor with associated linear map $T_x : E^* \rightarrow Y$, $T_x(e^*) = (e^* \otimes \text{id})(x)$. Then
\[ \pi_1^0(T_x) \leq d_{SL_1}(Y) \| x \|_{X \hat{\otimes} Y}. \]

Proof: According to Lemma 4.3, we can find $\hat{F} \supset F$ such that
\[ \| x \|_{X \hat{\otimes} F} \leq (1 + \varepsilon) \| x \|_{X \hat{\otimes} Y}. \]

Let $w : \hat{F} \rightarrow F_1 \subset S_1$ be a linear isomorphism with completely contractive inverse $w^{-1} : F_1 \rightarrow \hat{F}$. In order to estimate the 1-summing norm, we have to consider complete contractions $\beta : Y \rightarrow M_m$ and $\alpha : M_m \rightarrow E^*$. By Wittstock’s extension theorem, there is a complete contraction $\hat{\beta} : S_1 \rightarrow M_m$ such that $\hat{\beta}|_{F_1} = \beta w^{-1}$. Then $\alpha \beta$ corresponds to an element $z \in E^* \otimes_{\min} M_n$ of norm less than one. The injectivity of the projective tensor product on $S_1$ (see [5]) yields
\[ \| \text{tr}(T_x \alpha \beta) \| = \| \text{tr}(\alpha \beta T_x) \| = \| \text{tr}(\alpha \beta w T_x) \| = \| (z, (id_E \otimes w)(x)) \| \]
\[ \leq \| z \|_{E^* \otimes_{\min} M_n} \| (id_E \otimes w)(x) \|_{E \hat{\otimes} S_1} \leq \| (id_E \otimes w)(x) \|_{X \hat{\otimes} F_1} \]
\[ \leq \| (id_X \otimes w)(x) \|_{X \hat{\otimes} F_1} \leq \| w \|_{cb} \| x \|_{X \hat{\otimes} F} \leq (1 + \varepsilon) \| w \|_{cb} \| x \|_{X \hat{\otimes} Y}. \]

Taking the infimum over all $w$, we may replace $\| w \|_{cb}$ by $d_{SL_1}(\hat{F})$ and then by $d_{SL_1}(Y)$. \[ \blacksquare \]

We have a partial converse to this inequality.

Lemma 4.5. Let $M$ and $N$ be von Neumann algebras such that $N$ is QWEP. Let $Y$ be an operator space and $w : Y \rightarrow M_*$ be a complete contraction. Let $X$ be an operator space, $E \subseteq X$ be a finite dimensional subspace and $u : E \rightarrow N_*$ be a complete contraction. If $x \in E \otimes Y$ and $T_x : E^* \rightarrow Y$ is the associated map, then
\[ \pi_1^0(T_x : E^* \rightarrow Y) \geq \| (u \otimes w)(x) \|_{N_* \hat{\otimes} M_*}. \]
**Proof:** Let us assume that $N = A/I$ where $A$ has WEP and let $\pi : A \to N$ be the quotient homomorphism. Then the map $\pi^* : N^* \to A^*$ has a completely contractive left inverse. Indeed, let $z$ be a central projection in $A^{**}$ such that $I^{**} = zA^{**}$. Then we have a canonical isomorphism $(A/I)^* \cong N^*$. Therefore the mapping $v : A^* \to (A/I)^* \cong N^*$ given by $v(x) = zx$ satisfies $id_{N^*} = v\pi^*$. Since $N_*$ is completely complemented in $N^*$, we also have a completely contractive left inverse for the restriction $\pi^*|_{N_*}$. This implies that

$$\pi^* \otimes id_{M_*} : N_* \hat{\otimes} M_* \to A^* \hat{\otimes} M_*$$

is completely isometric. We consider the tensor $(\pi^*u \otimes w)(x) \in A^* \hat{\otimes} M_*$. By the Hahn-Banach theorem we may find

$$y \in (A^* \hat{\otimes} M_*)^* = CB(M_*, A^{**}) \cong M^{**} \hat{\otimes} A^{**}$$

of norm less than one such that

$$\langle y, (\pi^*u \otimes w)(x) \rangle = \|(\pi^*u \otimes w)(x)\|_{A^* \hat{\otimes} M_*}.$$ Let $\varepsilon > 0$. According to Kaplansky's density theorem (see [EJR] [ER2] for details), we may find a finite rank element $y_\varepsilon \in M \otimes A$ of norm $\leq 1$ such that

$$\|(\pi^*u \otimes w)(x)\|_{N_* \hat{\otimes} M_*} \leq (1 + \varepsilon)\|\langle y_\varepsilon, (\pi^*u \otimes w)(x) \rangle\|.$$ The tensor $y_\varepsilon$ corresponds to a complete contraction $S_{y_\varepsilon} : M_* \to A$ (see [ER2] Theorem 7.3.2]). We need the finite dimensional subspace $F = S_{y_\varepsilon}wT_x(E^*)$. We consider $\tilde{T} = S_{y_\varepsilon}wT_x : E^* \to F$ as a map with values in its range $F$. Using rank one tensors it is easy to check that

$$\langle y_\varepsilon, (\pi^*u \otimes w)(x) \rangle = tr(u^*\pi S_{y_\varepsilon}wT_x) = tr(u^*\pi|_F \tilde{T}).$$

We apply Lemma 4.1, Lemma 4.2 and basic properties of the 1-summing norm:

$$|tr(u^*\pi S_{y_\varepsilon}wT_x)| \leq \gamma_\infty(u^*\pi|_F)\pi^0_1(\tilde{T}) \leq \Gamma_\infty(u^*\pi|_F)S_{y_\varepsilon}wT_x \leq \Gamma_\infty(u^*\pi|_F)\pi^0_1(T_x).$$

Since $A$ is WEP, there is a complete contraction $P : B(H) \to A^{**}$ such that $P|_A = id_A$ where $A \subset A^{**} \subset B(H)$. This implies that

$$\Gamma_\infty(u^*\pi|_F) = \Gamma_\infty(u^{***}\pi|_F) = \Gamma_\infty(u^{***}P\pi|_F) \leq \|u^{***}P\|_{cb} = \|u\|_{cb}.$$ Thus, we get $\|(u \otimes w)(x)\|_{N_* \hat{\otimes} M_*} \leq (1 + \varepsilon)\pi^0_1(T_x)$. Letting $\varepsilon \to 0$, the assertion follows.

In the following we use the notation $G$ for the spaces introduced in section 3. The following proposition will be proved in section 7.

**Proposition 4.6.** There exists a von Neumann algebra $N$ with QWEP and a completely contractive injective map $u : G \to N_*$ such that

$$\|(u \otimes w)(x)\|_{N_* \hat{\otimes} N_*} \geq \frac{1}{9} \|x\|_{G \hat{\otimes} G}$$

for all $x \in G \otimes G$. Moreover, $\|u^{-1} : u(G) \to G\|_{cb} \leq 3$ and $u(G)$ is completely complemented in $N_*$. 


Corollary 4.7. Let $F_n \subset G$ be a finite dimensional subspace and $x \in F_n \otimes G$. Then
\[
\frac{1}{9} \|x\|_{G \otimes G} \leq \pi_1^0(T_x : F_n^* \to G) \leq 3 \|x\|_{G \otimes G}
\]

Proof: Since $G$ is 3-cb isomorphic to a subspace of $N_*$ and $N$ is QWEP, the strong principle of local reflexivity in \[EJR\] implies that $d_{SL_1}(G) \leq 3$. Thus by Lemma 4.4
\[
\pi_1^0(T_x : F_n^* \to G) \leq 3 \|x\|_{G \otimes G}.
\]

Now, we prove the converse inequality. By Proposition 4.6, we may apply Lemma 4.5 to get
\[
\|x\|_{G \otimes G} \leq 9 \| (u \otimes u)(x) \|_{N_* \otimes N_*} \leq 9 \pi_1^0(T_x : F_n^* \to G).
\]

Corollary 4.8. Let $(f_i)$ be the canonical unit vector basis in the space $F_n$ constructed in section 3. Let $u : OH_n \to OH_n$ be a linear map represented by a matrix $[a_{ij}]$. Then
\[
\frac{1}{6} \pi_1^0(u) \leq \left\| \sum_{i,j=1}^n a_{ij} f_i \otimes f_j \right\|_{G_n \otimes G_n} \leq 18 \pi_1^0(u).
\]

Proof: Let $u : OH_n \to OH_n$ be a linear map represented by a matrix $(a_{ij})$. We deduce from Lemma 3.5, Lemma 3.6 and Corollary 4.7 that
\[
\pi_1^0(u : OH_n \to OH_n) \leq 2 \pi_1^0(u : F_n^* \to F_n) \leq 6 \left\| \sum_{i,j=1}^n a_{ij} f_i \otimes f_j \right\|_{G_n \otimes G_n}.
\]

Conversely, we have
\[
\left\| \sum_{i,j=1}^n a_{ij} f_i \otimes f_j \right\|_{G_n \otimes G_n} \leq 9 \pi_1^0(u : F_n^* \to F_n) \leq 18 \pi_1^0(u : OH_n \to OH_n).
\]

The following norm calculations in $G_n \otimes G_n$ will be postponed to the next section.

Proposition 4.9. Let $[a_{ij}]$ be an $n \times n$ matrix. Then
\[
\left\| \sum_{i,j=1}^n a_{ij} f_i \otimes f_j \right\|_{G_n \otimes G_n} \leq 18 \sqrt{1 + \ln n} \left( \sum_{i,j=1}^n |a_{ij}|^2 \right)^{\frac{1}{2}}.
\]

Moreover, for $n \geq 7$
\[
\left\| \sum_{i=1}^n f_i \otimes f_i \right\|_{G_n \otimes G_n} \geq (16 \sqrt{2} \pi)^{-1} \sqrt{n(1 + \ln n)}.
\]

As an application, we derive an independent proof of (0.4).
Corollary 4.10. Let $u : \mathcal{B}(H) \to OH$ be a completely bounded map. Then
\begin{equation}
\left( \sum_{k=1}^{n} \| u(x_k) \|_{OH}^2 \right)^{\frac{1}{2}} \leq 108 \sqrt{1 + \ln n} \| u \|_{cb} \left\| \sum_{k=1}^{n} \bar{x}_k \otimes x_k \right\|_{\mathcal{B}(H) \otimes \min \mathcal{B}(H)}^{\frac{1}{2}}
\end{equation}
for all $n \in \mathbb{N}$ and $x_1, ..., x_n \in \mathcal{B}(H)$. Moreover, let $B : \mathcal{B}(H) \times \mathcal{B}(H) \to C$ be a positive sesquilinear form of rank $n$. Then there exists a positive integral sesquilinear form $\tilde{B}$ such that $B \leq \tilde{B}$ and
\[ \| \tilde{B} \|_{1} \leq 2 \times 108^2 (1 + \ln n) \| B \|_{jcb} . \]

Proof: Let $u : \mathcal{B}(H) \to OH$ be a completely bounded map and $x_1, ..., x_n \in \mathcal{B}(H)$. We can find an orthogonal projection $P : OH \to \text{span}\{u(x_k)\}_{1 \leq k \leq n}$ of rank at most $n$. Since OH is homogeneous, we may assume that $P(OH) = OH_n$. A glance at (1.4) shows that
\[ (4.1) \sum_{k=1}^{n} \| u(x_k) \|_{OH}^2 \leq 2 \times 108^2 (1 + \ln n) \| B \|_{jcb} . \]
Now, we apply trace duality. Let $(a_{ij}) \in \ell_{2}^{n}(OH_n) = \ell_{2}^{n}$ be of norm one such that
\[ \left( \sum_{k=1}^{n} \| u(x_k) \|_{OH}^2 \right)^{\frac{1}{2}} = \sum_{k,l} a_{kl} = tr(aw) . \]
We deduce from Lemma 4.1, Corollary 4.8 and Proposition 4.9 that
\[ \left( \sum_{k=1}^{n} \| u(x_k) \|_{OH}^2 \right)^{\frac{1}{2}} = \left( \sum_{k=1}^{n} \| P u(x_k) \|_{OH}^2 \right)^{\frac{1}{2}} = \sum_{k,l} a_{kl} = tr(aw) . \]
This completes the proof of the first assertion. Following [Ps4 (9.3)] (and ultimately [To]) this implies that
\[ \left( \sum_{k=1}^{n} \| u(x_k) \|_{OH}^2 \right)^{\frac{1}{2}} \leq 2 \times 108^2 (1 + \ln n) \| u \|_{cb} \left\| \sum_{k=1}^{n} \bar{x}_k \otimes x_k \right\|_{\mathcal{B}(H) \otimes \min \mathcal{B}(H)}^{\frac{1}{2}} . \]
Now, we apply a typical trace duality argument.
Corollary 4.11. Let $n \in \mathbb{N}$. Then

$$\gamma_\infty(id_{OH_n}) \leq 288\sqrt{2}\pi \sqrt{\frac{n}{1 + \ln n}}.$$  

Proof: Using a well-known averaging trick, we have

$$\gamma_\infty(id_{OH_n}) \pi_1^0(id_{OH_n}) = n. \tag{4.2}$$

(Indeed, according to Lemma 4.1 there exists $v$ with $\gamma_\infty(v) = 1$ and $\pi_1^0(id) = |\text{tr}(v)|$. Let $\sigma$ be the normalized Haar measure on the unitary group $U_n$. Then $\tilde{v} = \int_{U_n} wvu^{-1}d\sigma(u)$ satisfies $\gamma_\infty(\tilde{v}) \leq 1$ and $n\tilde{v} = \text{tr}(v)id$. This implies that $\gamma_\infty(id)\pi_1^0(id) = \gamma_\infty(id)|\text{tr}(v)| \leq n$. The converse inequality is obvious.) For $n \geq 7$ we deduce from Corollary 4.8 and Proposition 4.9 that

$$\pi_1^0(id_{OH_n}) \geq \frac{1}{18} \sum_{i=1}^{n} f_i \otimes f_i \geq \frac{1}{288\sqrt{2}\pi \sqrt{n(1 + \ln n)}}. \tag{4.3}$$

For $n \leq 7$ we use the well-known Banach space estimate $\pi_1^0(id_{OH_n}) \geq \frac{2}{\sqrt{\pi}} \sqrt{n}$. Hence (4.3) is valid for all $n \in \mathbb{N}$ and the assertion follows from (4.2). \hfill \square

We are ready for the proof of Corollary 5.

Corollary 4.12. Let $OH_n \subset B(\ell_2)$, then there exists a projection $P : B(\ell_2) \to OH_n$ such that

$$\|P\|_{cb} \leq 288\sqrt{2}\pi \sqrt{\frac{n}{1 + \ln n}}.$$  

In particular,

$$\frac{1}{108\sqrt{\frac{n}{1 + \ln n}}} \leq \lambda_{cb}(OH_n) \leq 288\sqrt{2}\pi \sqrt{\frac{n}{1 + \ln n}}.$$  

Proof: We write $id_{OH_n} = vw$, with $w : OH_n \to B(H)$ and $v : B(H) \to OH_n$, and

$$\|v\|_{cb}\|w\|_{cb} \leq 288\sqrt{2}\pi \sqrt{\frac{n}{1 + \ln n}}.$$  

According to Wittstock’s extension theorem, we can find an extension $\hat{w} : B(\ell_2) \to B(H)$ with the same cb-norm as $w$. Then $P = vw$ is the corresponding projection. The lower estimate follows easily from Corollary 4.10 (see also [PS]). \hfill \square

Corollary 4.13. The order $(1 + \ln n)$ in (0.4) is best possible. Moreover, there is a sesquilinear jcb form which cannot be majorized by an integral form.

Proof: Let $\iota : OH_n \to B(\ell_2)$ be a completely isometric embedding and $x_k = \iota(e_k)$. According to Corollary 4.12 we can find a projection $P : B(\ell_2) \to OH_n$ of cb-norm

$$\|P\|_{cb} \leq 288\sqrt{2}\pi \sqrt{\frac{n}{1 + \ln n}}.$$
We define $B_n(x, y) = (P(x), P(y))$ and assume that $B_n \leq \bar{B}$. Then, we deduce from Lemma 2.4 that

$$n = \sum_{k=1}^{n} \|e_k\|^2 = \sum_{k=1}^{n} B_n(x_k, x_k) \leq \|\bar{B}\|_I \left\| \sum_{k=1}^{n} \bar{x}_k \otimes x_k \right\|_{B(H)^{\otimes n}B(H)} = \|\bar{B}\|_I.$$ 

This implies that

$$\|B_n\|_{jcb} = \|P\|^2_{cb} \leq (288\sqrt{2}\pi)^2 \frac{n}{1 + \ln n} \quad \text{and} \quad n \leq \|\bar{B}\|_I.$$

Combining these estimates, we deduce that

$$(1 + \ln n) \|B_n\|_{jcb} \leq (288\sqrt{2}\pi)^2 \|\bar{B}\|_I.$$ 

For the second assertion, we define $B = \sum_{k\in\mathbb{N}} k^{-2} \frac{k^4}{2^{k^4}} B_{2^k^4}$. The triangle inequality shows that $B$ is jcb. However every if $B \leq \bar{B}$, then $k^{-2} \frac{k^4}{2^{k^4}} B_{2^k^4} \leq \bar{B}$ implies that $\|\bar{B}\|_I \geq ck^{-2}k^4$ for all $k \in \mathbb{N}$. $\blacksquare$

**Remark 4.14.** This argument also shows that $\sqrt{1 + \ln n}$ in Lemma 2.2 is best possible. Moreover, we see that $\alpha = 1$ in Theorem 5 is best possible for non sub-homogeneous $C^*$-algebras.

Indeed, since $\gamma_\infty(id_{OH_n}) \leq 288\pi\sqrt{2n/(1 + \ln n)}$, we may find a complete contraction $w : OH_n \to M_m$ and a linear map $u : M_m \to OH_n$ such that $uw = id_{OH_n}$ and $\|u\|_{cb} \leq 288\pi\sqrt{2n/(1 + \ln n)}$. Moreover, it is well-known (see e.g. [JNRX]) that $M_m$ is $(1 + \varepsilon)$ completely complemented in a non subhomogeneous $C^*$-algebra $A$, i.e. there is a complete contraction $\alpha : M_m \to A$ and a map $\beta : A \to M_m$ of cb-norm $\leq (1 + \varepsilon)$ such that $\beta \alpha = id$. Then $B(x, y) = (u\beta(x), \beta(y))$ provides the ‘counterexample’ on $A$.

5. **Norm calculations in a quotient space**

Although the calculations in this section are of technical nature, the idea is very simple: The influence of the singularities at the corners $(0, 1)$ and $(1, 0)$ is minimized by rectangular decompositions. The next lemma justifies the use of these decompositions.

**Lemma 5.1.** $G_n \hat{\otimes} G_n$ is isometrically isomorphic to the quotient space of

$$L_2(\nu_1 \otimes \nu_1; \ell_2^n) \oplus_1 L_2(\nu_2; \ell_2^n) \oplus \pi L_2(\nu_2; \ell_2^n) \otimes_\pi L_2(\nu_1; \ell_2^n) \oplus_1 L_2(\nu_2 \otimes \nu_2; \ell_2^n)$$

with respect to

$$S = \{(f, g, h, k) \mid f(t, s) + g(t, s) + h(t, s) + k(t, s) = 0 \quad \mu \otimes \mu \text{ - a.e.}\}.$$ 

**Proof:** By the properties of the projective operator space tensor product, we have

$$G_n \hat{\otimes} G_n = (L_2(\nu_1; \ell_2^n) \oplus_1 L_2(\nu_2; \ell_2^n)) \hat{\otimes} (L_2(\nu_1; \ell_2^n) \oplus_1 L_2(\nu_2; \ell_2^n))/\ker(Q \otimes Q).$$

We note that $H^c \hat{\otimes} K^c = H \otimes_2 K = H^r \hat{\otimes} K^r$ and $H^c \otimes K^r = H \otimes_\pi K = H^r \otimes K^c$. Therefore the properties of $\oplus_1$ and $\hat{\otimes}$ imply that

$$(L_2(\nu_1; \ell_2^n) \oplus_1 L_2(\nu_2; \ell_2^n)) \hat{\otimes} (L_2(\nu_1; \ell_2^n) \oplus_1 L_2(\nu_2; \ell_2^n))$$

$$= (L_2(\nu_1; \ell_2^n) \hat{\otimes} L_2(\nu_1; \ell_2^n)) \oplus_1 (L_2(\nu_1; \ell_2^n) \hat{\otimes} L_2(\nu_2; \ell_2^n))$$
where the supremum is taken over all measurable functions \((f, g, h, k)\) such that
\[
\frac{f(t, s)}{ts} = \frac{g(t, s)}{(1-t)(1-s)} = \frac{h(t, s)}{t(1-s)} = \frac{k(t, s)}{(1-t)s} \mu \otimes \mu \text{ a.e.}
\]
and
\[
\begin{align*}
\max\{\|f\|_{L^2(\nu_1 \otimes \nu_1)}, &\|g\|_{L^2(\nu_2 \otimes \nu_2)}\} \leq 1, \\
\max\{\|h\|_{L^2(\nu_1) \otimes L^2(\nu_2)}, &\|k\|_{L^2(\nu_2) \otimes L^2(\nu_1)}\} \leq \sqrt{n}.
\end{align*}
\]

**Proof:** Let \((f, g, h, k)\) be given as above. Consider a decomposition of \(a\) in matrix valued functions \(a = a^1(t, s) + a^2(t, s) + a^3(t, s) + a^4(t, s)\) such that
\[
\begin{align*}
\|a^1\|_{L^2(\nu_1 \otimes \nu_1; \ell^2_2)} &+ \|a^2\|_{L^2(\nu_2 \otimes \nu_2; \ell^2_2)} \\
+ \|a^3\|_{L^2(\nu_1; \ell^2_2) \otimes L^2(\nu_2; \ell^2_2)} &+ \|a^4\|_{L^2(\nu_2; \ell^2_2) \otimes L^2(\nu_1; \ell^2_2)} \leq (1 + \varepsilon) \left\| \sum_{i,j=1}^n a_{ij} f_i \otimes f_j \right\|_{G_n \hat{\otimes} G_n}.
\end{align*}
\]

Then the Cauchy Schwarz inequality implies that
\[
\left| \int \sum_{i=1}^n a_{ii}^2(t, s) f(t, s) \frac{d\mu(t) d\mu(s)}{t} \right| \leq \left( \int \sum_{i=1}^n |a_{ii}^2(t, s)|^2 \frac{d\mu(t) d\mu(s)}{t} \right)^{\frac{1}{2}} \left( \int \sum_{i=1}^n |f(t, s)|^2 \frac{d\mu(t) d\mu(s)}{t} \right)^{\frac{1}{2}} \\
= \|a^1\|_{L^2(\nu_1 \otimes \nu_1; \ell^2_2)} \sqrt{n} \|f\|_{L^2(\nu_1 \otimes \nu_1)} \leq \sqrt{n} \|a^1\|_{L^2(\nu_1 \otimes \nu_1; \ell^2_2)}.
\]

Similarly,
\[
\left| \int \sum_{i=1}^n a_{ii}^2(t, s) g(t, s) \frac{d\mu(t) d\mu(s)}{1-t 1-s} \right| \leq \sqrt{n} \|a^2\|_{L^2(\nu_2 \otimes \nu_2; \ell^2_2)}.
\]

For every operator \(h : L_2 \to L_2\), we recall that \(\|h \otimes id_{\ell^2_2}\| = \|h\|.\) Hence, we deduce from trace duality and \((5.2)\) that
\[
\left| \int \sum_{i=1}^n a_{ii}^3(t, s) h(t, s) \frac{d\mu(t) d\mu(s)}{1-t 1-s} \right| \leq \|h \otimes id_{\ell^2_2}\|_{L^2(\nu_1; \ell^2_2) \otimes L^2(\nu_2; \ell^2_2)} \|a^3\|_{L^2(\nu_1; \ell^2_2) \otimes L^2(\nu_2; \ell^2_2)} \\
\leq \|h\|_{L^2(\nu_1) \otimes L^2(\nu_2)} \|a^3\|_{L^2(\nu_1; \ell^2_2) \otimes L^2(\nu_2; \ell^2_2)}.
\]
Similarly,
\[
\left| \int \sum_{i=1}^{n} a_{ii}^4(t,s) k(t,s) \frac{d\mu(t)}{1-t} \frac{d\mu(s)}{1-s} \right| \leq \|k\|_{L_2(\nu_1)} \|a^4\|_{L_2(\nu_1)} \leq \sqrt{n} \|a^4\|_{L_2(\nu_1)} \cdot
\]

Therefore, we get
\[
\int \sum_{i=1}^{n} a_{ii} f(t,s) \frac{d\mu(t)}{t} \frac{d\mu(s)}{s} \leq \int \sum_{i=1}^{n} a_{ii}^1 f(t,s) \frac{d\mu(t)}{t} \frac{d\mu(s)}{s} + \int \sum_{i=1}^{n} a_{ii}^2 g(t,s) \frac{d\mu(t)}{1-t} \frac{d\mu(s)}{1-s} + \int \sum_{i=1}^{n} a_{ii}^3 h(t,s) \frac{d\mu(t)}{1-t} \frac{d\mu(s)}{1-s} \leq \sqrt{n}(1+\varepsilon) \left\| \sum_{i,j=1}^{n} a_{ij} f_i \otimes f_j \right\|_{G_n \otimes G_n} .
\]

We will now prove the lower estimate in Proposition 4.9.

**Lemma 5.3.** Let \( n \geq 7 \). Then
\[
\left\| \sum_{i=1}^{n} f_i \otimes f_i \right\|_{G_n \otimes G_n} \geq \frac{1}{16\sqrt{2\pi}} \sqrt{n(1+\ln n)} .
\]

**Proof:** Let \( 0 < \delta < \frac{1}{2} \), to be determined later. We consider the rectangle \( I = [\delta, \frac{1}{2}] \times [\frac{1}{2}, 1-\delta] \) and the function
\[
v(t,s) = \frac{1}{ts + (1-t)(1-s)} 1_I .
\]

Following Corollary 5.2, we define \( f(t,s) = ts v(t,s) \) and \( g(t,s) = (1-t)(1-s) v(t,s) \) and observe that
\[
\int_I f(t,s)^2 \frac{d\mu(t)}{t} \frac{d\mu(s)}{s} + \int_I g(t,s)^2 \frac{d\mu(t)}{1-t} \frac{d\mu(s)}{1-s} = \int_I v(t,s)^2 [ts + (1-t)(1-s)] d\mu(t) d\mu(s)
\]
\[
= \int_I \frac{1}{ts + (1-t)(1-s)} d\mu(t) d\mu(s)
\]
\[
\leq 4\pi^{-2} \int_{\delta}^{1/2} \int_{\frac{1}{2}}^{1-\delta} \frac{1}{\sqrt{s}} \sqrt{1-s} \frac{ds}{\sqrt{t}} dt = 8\pi^{-2} \int_{\delta}^{1/2} \int_{\frac{1}{2}}^{1-\delta} \frac{ds}{\sqrt{s}} \frac{dt}{\sqrt{t}} \leq 16\pi^{-2} \int_{\delta}^{1/2} \sqrt{s} \frac{dt}{1-\delta} 
\]
\[
\leq 16\pi^{-2} \int_{\delta}^{1/2} \sqrt{\frac{dt}{t}} \leq 16\pi^{-2} (-\ln \delta) .
\]
In order to estimate the norm for \( h(t, s) = t(1 - s)v(t, s) \), we use (1.1). Hence it suffices to estimate the \( L_2 \)-norm:

\[
\|h\|_{L_2(v_1 \otimes v_2)}^2 \leq 4 \int_I \min(t^{-2}, (1 - s)^{-2}) t^2(1 - s)^2 \, dv_1(t)dv_2(s)
\]

\[
\leq 8\pi^{-2} \int_\delta^{1-\delta} \int_\delta^{1-\delta} \min(t^{-2}, s^{-2})t^2ds \sqrt{t/s} = 16\pi^{-2} \int_\delta^{1-\delta} \int \sqrt{ds} \frac{dt}{t\sqrt{t}}
\]

\[
\leq \frac{32}{3\pi^2} \int_\delta^{1-\delta} \frac{t^2 \, dt}{t\sqrt{t}} \leq \frac{16}{3\pi^2}.
\]

Finally, we need an \( L_2 \)-norm estimate of \( k(t, s) = (1 - t)sv(t, s) \):

\[
\|k\|_{L_2(v_2 \otimes v_1)}^2 \leq 4 \int_I \min(t^{-2}, (1 - s)^{-2}) (1 - t)^2s^2 \, d\mu(t) \, d\mu(s)
\]

\[
\leq 8\pi^{-2} \int_\delta^{1-\delta} \int_\delta^{1-\delta} \min(t^{-2}, s^{-2}) \, dt \, ds \leq 16\pi^{-2} \int_\delta^{1-\delta} \int \frac{s \, dt}{\sqrt{s} t^2 \sqrt{t}} \leq 32\pi^{-2} \int_\delta^{1-\delta} \frac{\sqrt{t} \, dt}{t^2 \sqrt{t}} \leq 32\pi^{-2}\delta^{-1}.
\]

We note that

\[
\int_I f(t, s) \frac{d\mu(t)}{t} \frac{d\mu(s)}{s} = \int_\delta^{1-\delta} \int_\delta^{1-\delta} \frac{1}{ts + (1 - t)(1 - s)} d\mu(t)d\mu(s)
\]

\[
\geq \frac{1}{2\pi^2} \int_\delta^{1-\delta} \int_\delta^{1-\delta} \min(t^{-1}, s^{-1}) \, dt \, ds \sqrt{t/s} = \frac{2}{\pi^2} \int_\delta^{1-\delta} (\sqrt{t} - \sqrt{s}) \, dt \sqrt{t/s}
\]

\[
\geq \frac{1}{\pi^2} \int_\delta^{1-\delta} \frac{dt}{t} \frac{d}{\delta} = \frac{(-\ln 8\delta)}{\pi^2}.
\]

We define \( \delta = \frac{1}{ne} \), \( C = \sqrt{\max\left\{ \frac{32}{\pi^2}, \frac{16}{\pi^2}, \frac{16}{\pi^2} \right\}} = \frac{4\pi}{\sqrt{\ln e}} \) and \( \tilde{f} = \frac{f}{C\sqrt{\ln e}} \). For \( n \geq 6 \) we have \( \ln ne \geq 6 \) and hence (5.1) and (5.2) are satisfied for the corresponding quadruple \((\tilde{f}, \tilde{g}, \tilde{k}, \tilde{h})\). Note that \(-\ln 8\delta = \ln ne \geq \frac{1}{4} \ln ne \) for \( n \geq 7 \). The assertion follows from

\[
\frac{-\ln 8\delta}{\pi^2 C\sqrt{\ln ne}} \geq \frac{\ln ne}{4\pi^2 C\sqrt{\ln ne}} = \frac{\sqrt{1 + \ln n}}{16\pi^2}.
\]

For very large \( n \) we can choose \( C = 4/\pi \) and asymptotically get \( \geq (1 - \epsilon_n)\sqrt{1 + \ln n} \).

The rest of this section is devoted to the upper estimate. For a measure \( \nu \) and positive measurable densities \( g, h \), we use the \( L_p \)-sum

\[
L_2(g\nu) +_p L_2(h\nu) = L_2(g\nu \oplus_p L_2(h\nu)/\ker Q
\]

where \( Q(f_1, f_2) = f_1 + f_2 \). Given a measurable function \( k \), we define the norm of \( k \) in \( L_2(g\nu) +_p L_2(h\nu) \) as the norm of the equivalence class \([k, 0]\). For \( p = 2 \), this is again a Hilbert space and we have an explicit formula (see [BLL, Theorem 5.2.2 and Theorem 5.4.4]).
Lemma 5.4. Let \( \nu \) be a measure and \( g, h \) strictly positive measurable functions. For a measurable function \( k \), the norm of \( k \) in \( L_2(g \nu) + L_2(h \nu) \) is given by

\[
\| k \|_{L_2(g \nu) + L_2(h \nu)} = \left( \int \frac{|k|^2}{(g^{-1} + h^{-1})} \, d\nu \right)^{\frac{1}{2}}.
\]

Using this formula, the following estimates are established in a very similar way to the estimates in Lemma 5.3. We leave them to the interested reader.

Corollary 5.5. Let \( 0 < \delta < \frac{1}{2} \). Then

\[
\begin{align*}
(5.3) & \quad \left\| 1_{[\frac{1}{2}, 1]} \otimes 1_{[\frac{1}{2}, 1-\delta]} \right\|_{L_2(\nu_1 \otimes \nu_1) + L_2(\nu_2 \otimes \nu_2)} \leq 4\sqrt{2\pi}^{-1} (-\ln \delta)^{\frac{1}{2}} \\
(5.4) & \quad \left\| 1_{[\frac{1}{2}, 1-\delta]} \otimes 1_{[\frac{1}{2}, 1]} \right\|_{L_2(\nu_1 \otimes \nu_1) + L_2(\nu_2 \otimes \nu_2)} \leq 4\sqrt{2\pi}^{-1} (-\ln \delta)^{\frac{1}{2}} \\
(5.5) & \quad \left\| 1_{[0, \frac{1}{2}]} \otimes 1_{[0, \frac{1}{2}]} \right\|_{L_2(\nu_1 \otimes \nu_1) + L_2(\nu_2 \otimes \nu_2)} \leq 2\sqrt{2} \\
(5.6) & \quad \left\| 1_{[\frac{1}{2}, 1]} \otimes 1_{[\frac{1}{2}, 1]} \right\|_{L_2(\nu_1 \otimes \nu_1) + L_2(\nu_2 \otimes \nu_2)} \leq 2\sqrt{2}.
\end{align*}
\]

The next inequality yields the upper estimate in the logarithmic ‘little Grothendieck inequality’.

Lemma 5.6. Let \( a \) be an \( n \times n \) matrix, then

\[
\left\| \sum_{i,j=1}^n a_{ij} f_i \otimes f_j \right\|_{G_n \otimes G_n} \leq 18 \sqrt{1 + \ln n} \| a \|_2.
\]

Proof: Given \( a \in L_2^0 \) and \( 0 < \delta < \frac{1}{2} \), we decompose \( a = a^1(t, s) + a^2(t, s) \) where

\[
a^1(t, s) = a \otimes \left( 1_{[0, \frac{1}{2}]}(t)1_{[0, \frac{1}{2}]}(s) + 1_{[\frac{1}{2}, 1]}(t)1_{[\frac{1}{2}, 1-\delta]}(s) + 1_{[\frac{1}{2}, 1-\delta]}(t)1_{[\frac{1}{2}, 1]}(s) + 1_{[\frac{1}{2}, 1]}(t)1_{[\frac{1}{2}, 1]}(s) \right)
\]

and

\[
a^2(t, s) = a \otimes 1 - a^1(t, s).
\]

According to Corollary 5.5, we get

\[
\| a^1 \|_{L_2(\nu_1 \otimes \nu_1 ; \ell_2^2)} \leq (4\sqrt{2} + 8\sqrt{2\pi}^{-1} (-\ln \delta)^{\frac{1}{2}}) \| a \|_2.
\]

In order to estimate \( a^2 \), we note that

\[
\left\| 1_{[0, \frac{1}{2}]} \otimes 1_{[\frac{1}{2}, 1]} \right\|_{L_2(\nu_1 \otimes \nu_1)} = \|1_{[0, \frac{1}{2}]}\|_{L_2(\nu_2)} \|1_{[\frac{1}{2}, 1]}\|_{L_2(\nu_1)} \leq \frac{2\pi^\frac{1}{2} \sqrt{2}}{\sqrt{\pi}} = \frac{2\pi}{\sqrt{\pi}} \delta^{\frac{1}{2}}.
\]

Similarly, we get

\[
\max \left\{ \|1_{[0, \frac{1}{2}]} \otimes 1_{[1-\delta, 1]}\|_{L_2(\nu_2) \otimes \nu_2}, \|1_{[\frac{1}{2}, 1]} \otimes 1_{[0, \delta]}\|_{L_2(\nu_1) \otimes \nu_2}, \|1_{[\frac{1}{2}, 1]} \otimes 1_{[0, \delta]}\|_{L_2(\nu_2) \otimes \nu_2} \right\} \leq \frac{2\pi}{\sqrt{\pi}} \delta^{\frac{1}{2}}.
\]

Using \( L_2(\ell_2^0) \otimes \nu_2 = (L_2^0 \otimes \nu_2) \otimes ((\ell_2^0)^c \otimes (\ell_2^0)^c) \), we deduce that

\[
\| a^2 \|_{L_2(\nu_1; \ell_2^0) \otimes \nu_2} \leq \frac{4\pi^{\frac{1}{2}}}{\sqrt{\pi}} \delta^{\frac{1}{2}} \| a \|_1 \leq \frac{4\pi^{\frac{1}{2}}}{\sqrt{\pi}} \delta^{\frac{1}{2}} \sqrt{n} \| a \|_2.
\]
Proof of Proposition 4.9: Combine Lemma 5.6 and Lemma 5.3.

6. K-functionals

In section 7 we will see that Voiculescu’s inequality leads to three terms. By duality, we have to consider three term $K$-functionals. However, the quotient structure discussed before only involves two terms. In this section we justify the abstract central limit procedure relating two $K$-functionals. In the following, $N$ will be a semifinite von Neumann algebra with a normal faithful trace $\tau$. We fix a positive $\tau$-measurable operator $d \in L_0(N, \tau)$ with full support. The corresponding strictly semifinite weight is given by

$$\varphi(x) = \tau(dx).$$

We will use the standard notation

$$n_\varphi = \{ x \in N \mid \varphi(x^*x) < \infty \}.$$

Similarly, we will use the notation $n_\psi = \{ x \in N \mid \psi(x^*x) < \infty \}$ for every operator-valued weight $\psi$ defined on $N$. The arguments in this section generalize to arbitrary strictly semifinite weights, but for our applications it suffices to consider $N = L_\infty(\bar{\mu}; M_2)$ (more precisely $L_\infty(N \times [0, 1], \bar{\mu}; M_2)$, where $\bar{\mu} = m \otimes \mu$ is given as the tensor product of the counting measure $m$ and the measure $\mu$ defined by (3.2) in section 3). The three term $K$-functional $K_t$ is defined on (a subspace of) $L_0(N, \tau)$ as follows:

$$\|x\|_{K_t} = \inf_{x = x_1 + x_2 d_{\frac{1}{2}} + d_{\frac{3}{2}} x_3} \frac{1}{2} \|x_1\|_1 + \|x_2\|_2 + \|x_3\|_2.$$

We will use the quotient map $q_t : L_1(N) \oplus L_2(N) \oplus L_2(N) \to L_0(N, \tau)$ given by

$$q_t(x_1, x_2, x_3) = t^{\frac{1}{2}} x_1 + x_2 d_{\frac{1}{2}} + d_{\frac{3}{2}} x_3.$$

The operator space structure of $K_t$ is then defined as the quotient space

$$K_t = K_t(N, d) = L_1(N) \oplus_1 L_2^*(N) \oplus_1 L_2^*(N)/\text{ker}(q_t).$$

Let us start with some elementary properties. (Note that the intersection in the following lemma depends on $d$.)

Lemma 6.1. i) The dual of $K_t$ with respect to the antilinear duality bracket is $n_\varphi \cap n_\varphi^*$ equipped with the operator space structure of $N \cap L_2^*(N) \cap L_2^*(N)$.

ii) Let $M$ be another von Neumann algebra and

$$\psi(m \otimes x) = m\varphi(x)$$

the induced operator valued weight on $M \otimes N$. The dual space of $L_1(M) \hat{\otimes} K_t$ is $n_\psi \cap n_\psi^*$ equipped with the operator space structure of $M \otimes N \cap M \otimes L_2^*(N) \cap M \otimes L_2^*(N)$.

iii) Let $e_n = 1_{[\frac{n-1}{n}, n]}(d)$ denote the spectral projections of $d$. The maps $P_n(x) = e_n x e_n$ extend to complete contractions on $K_t$ such that $\bigcup_n (id \otimes P_n)(L_1(M) \hat{\otimes} K_t)$ is norm dense in $L_1(M) \hat{\otimes} K_t$. 

We can choose $\delta = \frac{1}{e_n}$ and the assertion follows from Lemma 5.1.
Proof: We follow the well-known principle in interpolation theory that the dual (unit ball) of a sum is the intersection (of the dual unit balls). More precisely, let $l : \mathbb{K}_{t} \to \mathbb{C}$ be a linear functional. Since $L_{1}(N) = N$, we find an element $y_{1} \in N$ such that $l(q_{t}(x_{1},0,0)) = \tau(x_{1}^{*}y_{1})$ for all $x \in L_{1}(N)$. Similarly, we find $y_{2} \in L_{2}(N), y_{3}(N)$ such that

$$l(0,x_{2},0) = \tau(x_{2}^{*}y_{2}) \quad \text{and} \quad l(0,0,x_{3}) = \tau(x_{3}^{*}y_{3})$$

for all $x_{2}, x_{3} \in L_{2}(N)$. Since $q_{t}(xe_{n}d^{\frac{1}{2}}, t^{-\frac{1}{2}}xe_{n},0) = 0$, we deduce that

$$\tau(d^{\frac{1}{2}}e_{n}x^{*}y_{1}) = t^{\frac{1}{2}}\tau(e_{n}x^{*}y_{2})$$

holds for all $x$ and $n$. This implies that $y_{1}d^{\frac{1}{2}} = t^{\frac{1}{2}}y_{2}$ and hence $\tau(dy_{1}y_{1}) = t\|y_{2}\|^{2}$ is finite. Similarly, we find that $d^{\frac{1}{2}}y_{1} = t^{\frac{1}{2}}y_{3}$, and hence $y_{1}$ and $y_{3}$ are in $n_{\psi}$. This yields an isometric embedding

$$\bar{\mathbb{K}_{t}} = \{(y, t^{-\frac{1}{2}}yd^{\frac{1}{2}}, t^{-\frac{1}{2}}d^{\frac{1}{2}}y) \mid y \in n_{\psi} \cap n_{\psi}^{*} \} \subset N \oplus_{\infty} L_{2}(N) \oplus_{\infty} L_{2}(N).$$

Repeating the same argument for $L_{1}(M) \hat{\otimes} \mathbb{K}_{t}$, we apply (1.12) and obtain an isometric embedding

$$L_{1}(M) \hat{\otimes} \mathbb{K}_{t} \subset M \hat{\otimes} N \oplus_{\infty} M \bar{\otimes} L_{2}(N) \oplus_{\infty} M \bar{\otimes} L_{2}(N).$$

Moreover, for finite rank tensors $z = \sum_{i} m_{i} \otimes x_{i}$, we have

$$\|\psi(z^{*}z)\|^{2}_{M} = \left\| \sum_{i,k} \phi(x_{k}^{*}x_{i})m_{k}^{*}m_{i} \right\|_{M} = \left\| \sum_{i} m_{i} \otimes x_{i}d^{\frac{1}{2}} \right\|^{2}_{M \hat{\otimes} L_{2}(N)}.$$

By weak$^{*}$-density of the finite rank tensor in $n_{\psi}$, we obtain ii). Assertion i) follows immediately by applying ii) for the matrix algebras $M = M_{n}, n \in \mathbb{N}$. For the proof of iii), we observe that $P_{n}(x) = e_{n}, xe_{n}$ is a complete contraction on $L_{1}(N), L_{2}^{*}(N)$ and $L_{2}^{*}(N)$ for all $n \in \mathbb{N}$. Moreover, $(P_{n}, P_{n}, P_{n})(ker(q_{t})) \subset ker(q_{t})$. Since $(e_{n})$ converges strongly to 1, we have point-norm convergence of $(P_{n})$ to the identity in $L_{1}(N), L_{2}(N)$, respectively. This implies that

$$\bigcap_{n} id \otimes P_{n}(L_{1}(M) \hat{\otimes} \mathbb{K}_{t}) \text{ is norm dense.}$$

The two term $K$-functional is defined as follows:

$$K = K(N,d) = L_{2}^{*}(N) \oplus L_{2}^{*}(N)/ ker(q) \quad \text{,} \quad q(x_{2}, x_{3}) = x_{2}d^{\frac{1}{2}} + d^{\frac{1}{2}}x_{3}. $$

The identity map $I_{t} : K \to \mathbb{K}_{t}$ is completely contractive. (If $x = x_{2}d^{\frac{1}{2}} + d^{\frac{1}{2}}x_{3}$, then we may choose $x_{1} = 0$ in the definition of $\mathbb{K}_{t}$.)

Proposition 6.2. $K(N,d)$ is a direct limit of the $K(e_{n}N e_{n}, e_{n}d)$’s. Let $(t_{k})$ be a sequence with $\lim_{k} t_{k} = \infty$ and let $U$ be a free ultrafilter. Then $K$ is completely contractively complemented in $\prod_{k, U} \mathbb{K}_{t_{k}}$.

Proof: We will not repeat the argument for the first assertion, which is very similar to the proof of Lemma 6.1 iii). Similarly as in Lemma 6.1 we can show the dual $\bar{K}$ is the subspace of $L_{2}^{*}(N) \hat{\otimes} L_{2}^{*}(N)$ consisting of pairs $(y_{2}, y_{3})$ such that $d^{\frac{1}{2}}y_{2} = y_{3}d^{\frac{1}{2}}$. The linear mapping $I = (I_{t_{k}}) : K \to \prod_{k, U} \mathbb{K}_{t_{k}}$ is clearly a contraction. The only difficulty is to construct a right inverse. Using the $P_{n}$’s, we may assume that $d$ and $d^{-1}$ are bounded. For $y \in N$ we may define

$$l_{t_{k}}(y)(q_{t_{k}}(x_{1}, x_{2}, x_{3})) = \tau((t_{k}^{-\frac{1}{2}}x_{1}^{*} + d^{\frac{1}{2}}x_{2}^{*} + x_{3}d^{\frac{1}{2}})y).$$
Using the duality between sums and intersections, we deduce from Lemma 6.1 that
\[
\lim_k \| (id \otimes \ell_k)(y) \|_{M_m(\mathcal{K}_{tk})} = \lim_k \max \left\{ \frac{1}{2} \| y \|_{M_m(N)}, \| (1 \otimes d^\frac{1}{2})y \|_{M_m(L_2^p(N))}, \| (1 \otimes d^\frac{1}{2})y \|_{M_m(L_2^p(N))} \right\}
\]
holds for all \( y \in M_m(N) \). This shows that the map \( \ell = (\ell_k) : L \rightarrow \prod_k \mathcal{K}_{tk} \), defined on the subspace
\[
L = \{ (yd^\frac{1}{2}, d^\frac{1}{2}y) \mid y \in N \} \subset \mathcal{K}^* ,
\]
is a complete contraction. Since \( N d^\frac{1}{2} \) is dense in \( L_2(N) \) and the mapping \( T(x) = d^\frac{1}{2}xd^{-\frac{1}{2}} \) is bounded, we deduce that \( L \) is norm dense in \( \mathcal{K}^* \). (Here we use that \( d \) and \( d^{-1} \) are bounded; in general we have only weak*-density.) We will use the obvious inclusion \( \prod_{t_k, d} \mathcal{K}_{tk} \rightarrow \prod_{t_k, d} \mathcal{K}_{tk} \). By continuity we may extend \( \ell \) to a complete contraction \( \ell : \mathcal{K}^* \rightarrow \prod_{t_k, d} \mathcal{K}_{tk} \) such that
\[
(6.2) \quad \ell((yd^\frac{1}{2}, d^\frac{1}{2}y))(I((x_2, x_3) + \ker(q))) = \lim_{k, d} \ell_{t_k}(y)(q_{t_k}(x_1, x_2, x_3)) = \tau((d^\frac{1}{2}x_2^* + xd^\frac{1}{2})y)
\]
holds for all \( x_2, x_3 \in L_2(N) \) and \( y \in N \). For \( \eta \in \mathcal{K}^* \) we use the notation \( \tilde{\eta}(x) = \eta(x) \). Then, we may define the adjoint \( \ell' : \prod_{t_k, d} \mathcal{K}_{tk} \rightarrow \mathcal{K}^{**} \) by
\[
\eta(\ell'(\xi)) = \ell(\tilde{\eta}(\xi)) .
\]
It is easily checked that \( \ell' \) is also completely contractive. Using the reflexivity of \( K \) and (6.2), we deduce that \( id_K = \ell' I \).

In the preceding sections, we had to work with two densities. We refer to [6X2] for a more systematic explanation of why two densities are necessary for homogeneous operator spaces in \( QS(R \oplus C) \). Technically, two densities are easily obtained from the classical 2 \( \times \) 2 matrix trick.

Let \( (\Omega, \Sigma, \tilde{\mu}) \) be a measure space and \( N = L_\infty(\Omega, \Sigma, \tilde{\mu}; M_2) \). The natural trace is given by \( \tau(x) = \int tr_2(x(\omega))d\tilde{\mu}(\omega) \). Let \( d_1, d_2 \) be two non-singular densities in \( L_0(N, \tau) \) of \( \tau \)-measurable operators. Then the diagonal
\[
d = \begin{pmatrix}
  d_1 & 0 \\
  0 & d_2
\end{pmatrix}
\]
belongs to the space \( L_0(N, \tau) \) and \( \varphi(x) = \tau(dx) \) is faithful.

**Lemma 6.3.** Let \( N \) be as above. The (1, 2)-corner of \( K \) is completely complemented in \( K \) and completely isometrically isomorphic to
\[
K(d_1, d_2) = L_2^p(\tilde{\mu}) \oplus_1 L_2(\tilde{\mu})/(\{(f, g) \mid \int f d_1^\frac{1}{2} + g d_2^\frac{1}{2} = 0\}) .
\]

**Proof:** The orthogonal projection \( P(x) = \begin{pmatrix}
  0 & x_{12} \\
  0 & 0
\end{pmatrix} \) is completely contractive on \( L_2^p(N) \) and \( L_2(N) \). Moreover, we have \( (P, P) ker(q) \subset ker(q) \) and thus \( P \) extends to a complete contraction \( \tilde{P} \) on the quotient space \( L_2^p(N) \oplus L_2^p(N)/ker(q) \). The range of \( \tilde{P} \) is given by pairs \( (x, y) + ker(q) \)
such that \( x = \begin{pmatrix} 0 & x_{12} \\ 0 & 0 \end{pmatrix} \) and \( y = \begin{pmatrix} 0 & y_{12} \\ 0 & 0 \end{pmatrix} \). Then, we observe that

\[
xd_{2}^{\frac{1}{3}} + d_{4}^{\frac{1}{3}}y = \begin{pmatrix} 0 & x_{12}d_{2}^{\frac{1}{3}} + d_{1}^{\frac{1}{3}}y_{12} \\ 0 & 0 \end{pmatrix}.
\]

Thus \( \tilde{\mathcal{P}}(L_{2}^{c}(N) \oplus L_{2}^{c}(N)/\ker(q)) \) is completely isometrically isomorphic to \( K(d_{1}, d_{2}) \).

\[\text{Lemma 6.4.} \quad \text{Let} \ (\Omega, \Sigma, \mu) \text{ be a measure space and} \ \hat{\nu}_{1} = d_{1}^{-1} \mu, \ \hat{\nu}_{2} = d_{2}^{-1} \mu. \text{ Then} \]

\[
L_{2}^{c}(\hat{\nu}_{1}) \oplus L_{2}^{c}(\hat{\nu}_{2})/\{(f, g) \mid f + g = 0 \ a.e.\}
\]

is completely isometrically isomorphic to \( K(d_{1}, d_{2}) \).

\[\text{Proof:} \quad \text{Let} \ S = \{(f, g) \mid f d_{2}^{\frac{1}{3}} + d_{1}^{\frac{1}{3}}g = 0\}. \text{ The complete isometry is induced by the map} \]

\[
i : L_{2}^{c}(\hat{\nu}_{1}) \oplus L_{2}^{c}(\hat{\nu}_{2}) \rightarrow K(d_{1}, d_{2}), \ i(f, g) = (fd_{2}^{\frac{1}{3}}, d_{1}^{\frac{1}{3}}g) + S
\]

because \( \ker(i) = \{(f, g) \mid f + g = 0 \ a.e.\} \).

\[\text{7. Sums of free mean zero variables}\]

In this section we will consider free products in the sense of \[\text{VDN}\] to prove the probabilistic estimates we require. The estimates for cb-norms are obtained by considering free products with amalgamation (over matrix algebras). Duality will then provide the complementation for the \( 3 \)-term \( K \)-functional. Let us recall the notion of operator-valued free probability needed in this context. We assume that a von Neumann algebra \( B \) is given along with a family \( (A_{j}) \) of von Neumann algebras \( A_{j} \) which all contain \( B \). In addition, we assume that there are normal, faithful conditional expectations \( E_{i} : A_{i} \rightarrow B. \) Let \( M \) be a \( C^{*} \)-algebra containing \( B \) with a conditional expectation \( E : M \rightarrow B. \) We also assume that \( \pi_{i} : A_{i} \rightarrow M \) are \( * \)-homomorphisms such that \( E \circ \pi_{i} = E_{i} \) and \( \pi_{i}|_{B} = id. \) Now, the image algebras \( B_{i} = \pi_{i}(A_{i}) \) are free over \( B \) if

\[
E(b_{1} \cdots b_{n}) = 0
\]

holds for all \( n, b_{1} \in B_{i_{1}}, \ldots, b_{n} \in B_{i_{n}} \) with \( i_{1} \neq i_{2} \neq \cdots \neq i_{n} \) and \( E(b_{1}) = \cdots = E(b_{n}) = 0. \) The scalar case corresponds to \( B = \mathbb{C} \) and a state \( E : M \rightarrow \mathbb{C}. \) Then the \( C^{*} \)-algebra generated by the \( B_{i} \)'s is isomorphic to the free product \( *_{i \in I}(A_{i}, E_{i}) \) (see \[\text{BD}\] for details). We are more interested in the von Neumann algebra free product which will be described later. We refer the reader to \[\text{Vo1, Dk3, BD}\] for a detailed description of the free product with amalgamation and the Fock space construction, an essential tool for our estimates. Let us use the standard notation

\[
\hat{A}_{i} = (1 - E_{i})(A_{i})
\]

for the \( B \)-bimodule of mean 0 elements. Following \[\text{JJ}\] we use the notation \( L_{\infty}^{c}(A_{i}, E_{i}) \) for the completion of \( A_{i} \) with respect to the norm \( \|x\|_{L_{\infty}^{c}(A_{i}, E_{i})} = \|E_{i}(x^{*}x)\|^{\frac{1}{2}}. \) In \[\text{BD}\] the space \( L_{\infty}^{c}(A_{i}, E_{i}) \) is denoted by \( L_{2}(A_{i}, E_{i}) \). We consider the \( B \) bimodules

\[
\mathcal{H}_{i} = L_{\infty}^{c}(A_{i}, E_{i}) \otimes B.
\]

The Fock space is the \( B \)-bimodule

\[
\mathcal{H} = B \oplus \sum_{n \geq 1, i_{1} \neq \cdots \neq i_{n}} \mathcal{H}_{i_{1}} \otimes_{B} \cdots \otimes_{B} \mathcal{H}_{i_{n}}.
\]
Let us denote by $Q_{i_1,...,i_n}$ the orthogonal projection onto the submodule $\mathcal{H}_{i_1} \otimes_B \cdots \otimes_B \mathcal{H}_{i_n}$. We denote by $Q_0$ the projection onto $B$. As in [La] we use the notation $\mathcal{L}(\mathcal{H})$ for the $C^*$-algebra of adjointable right module maps. Indeed, a right module map $T : \mathcal{H} \to \mathcal{H}$ is called adjointable if there exists $S : \mathcal{H} \to \mathcal{H}$ such that

$$\langle S(x), y \rangle = \langle x, T(y) \rangle$$

for all $x, y \in \mathcal{H}$. (Here $(.,.)$ is the $B$-valued sesquilinear form.) Note that the $Q_{i_1,...,i_n}$ are adjointable $B$-bimodule maps, and that $E(T) = Q_0 T Q_0$ defines a conditional expectation from $\mathcal{L}(\mathcal{H})$ onto $B$. The free product with amalgamation may be constructed by defining the *-homomorphism $\pi_i : A_i \to \mathcal{L}(\mathcal{H})$ as follows: If $a \in B$, the $\pi_i(a)$ acts by left multiplication on $\mathcal{H}$. For $a$ with $E_i(a) = 0$ and $i_1 \neq i$ we have

$$\pi_i(a)(h_{i_1} \otimes \cdots \otimes h_{i_n}) = a \otimes h_{i_1} \otimes \cdots \otimes h_{i_n}.$$ 

For $i_1 = i$ we have

$$\pi_i(a)(h_{i_1} \otimes \cdots \otimes h_{i_n}) = (ah_{i_1} - E_i(ah_{i_1})) \otimes h_{i_2} \otimes \cdots \otimes h_{i_n} + E_i(ah_{i_1})h_{i_2} \otimes \cdots \otimes h_{i_n}.$$ 

Then $*_{i\in I}A_i$ is defined as the $C^*$-algebra generated by $\pi_i(A_i)$. It turns out that then the image algebras $B_i = \pi_i(A_i)$ are free over $E$. The conditional version of Voiculescu’s inequality ([Vo2]) reads as follows:

**Proposition 7.1.** Let $a_i \in \hat{A}_i$, such that only finitely many $a_i$’s are different from 0. Then

$$\left| \sum_i \pi_i(a_i) \right| \leq \sup_i \|a_i\| + \left| \sum_i E_i(a^*_i a_i) \right|^{\frac{1}{2}} + \left| \sum_i E_i(a_i a^*_i) \right|^{\frac{1}{2}}.$$ 

For our estimates, we follow Voiculescu [Vo2] and define the projections

$$P_i = \sum_{i_1 \neq \cdots \neq i_n} Q_{i_1 \cdots i_n}.$$ 

**Lemma 7.2.** Let $a \in \hat{A}_i$. Then $(1 - P_i)\pi_i(a)(1 - P_i) = 0$.

**Proof:** Given $h_{i_1} \otimes \cdots \otimes h_{i_n} \in \mathcal{H}_{i_1} \otimes_B \cdots \otimes_B \mathcal{H}_{i_n}$ and $i_1 \neq i$, we observe that

$$h = \pi_i(a)(h_{i_1} \otimes \cdots \otimes h_{i_n}) = a \otimes h_{i_1} \cdots h_{i_n}$$

is an element of $\mathcal{H}_{i} \otimes_B \mathcal{H}_{i_1} \otimes_B \cdots \otimes_B \mathcal{H}_{i_n}$. Thus $P_i(h) = h$ and $(1 - P_i)(h) = 0$. By linearity this yields the assertion.

**Corollary 7.3.** Let $a \in A_i$. Then $(1 - P_i)\pi_i(a)(1 - P_i) = E_i(a)(1 - P_i)$.

**Proof:** This is obvious from Lemma 7.2 by writing

$$\pi_i(a) = \pi_i(a - E_i(a)) + \pi_i(E_i(a)) = \pi_i(a - E_i(a)) + E_i(a).$$ 

We will now give the easy proof of Voiculescu’s inequality:

**Proof of 7.1** We deduce from Lemma 7.2 that

$$\sum_i \pi_i(a_i) = \sum_i P_i \pi_i(a_i) P_i + \sum_i (1 - P_i) \pi_i(a_i) P_i + \sum_i P_i \pi_i(a_i)(1 - P_i)$$

$$+ \sum_i (1 - P_i) \pi_i(a_i)(1 - P_i)$$
The calculation for the third term is the same. An algebraic free product. This definition follows a general scheme for

Let \( \pi \) be a faithful state on \( B \). Lemma 7.4. For simplicity we will assume in the following that

Now, we consider the second term. By orthogonality, positivity and the module property, we deduce from Corollary 7.3 that

Proof: The idea of the proof is very simple. For \( a \in \hat{A} \) we define the modified right action

\[
\pi_i^r(a)(h_{i_1} \otimes \cdots \otimes h_{i_m}) = \begin{cases} h_{i_1} \otimes \cdots \otimes h_{i_m} \otimes a & \text{if } i_m \neq i \\ 0 & \text{if } i_m = i \end{cases}
\]
It is elementary to check that $\pi_i^*(a)$ commutes with $\pi_k(a_k)$ for all $a \in \tilde{A}_i$ and $a_k \in A_k$. Therefore the right action commutes with the von Neumann algebra $\bar{\pi}_{i \in I}(A_i, E_i)$. Let $C$ be the algebra generated by the $\pi_i^*(A_i)$’s and the identity. Then $\text{CQ}_\emptyset$ is dense in $\mathcal{H}$. Let $x$ be a positive element in the von Neumann algebra generated $\bar{\pi}_{i \in I} A_i$ such that $E_\emptyset(x) = 0$. Then $x^{\frac{1}{2}}E_\emptyset = 0$. Since $x^{\frac{1}{2}}$ commutes with elements in $C$ and $\text{CQ}_\emptyset$ is dense in $\mathcal{H}$, we obtain $x^{\frac{1}{2}}h = 0$ for all $h \in \mathcal{H}$. Thus $x = 0$. The careful reader will have observed that the only shortcoming of this argument is that the right actions $\pi_i^*(a)$ are not necessarily continuous and therefore the passage from elements in $\bar{\pi}_{i \in I}(A_i, E_i)$ to elements in the von Neumann algebra is not justified. In order to avoid this difficulty, we consider the right action on $\mathcal{H} \otimes_B L_2(B)$. By assumption the states $\varphi_i = \varphi \circ E_i$ are faithful. Let $a \in A_i$ be an element in the domain of $\Delta_i^{-1/2}$, where $\Delta_i$ is the generator of the modular group $\sigma_i^\phi$. Then there exists a constant $C$ such that

$$\varphi(a^* y^* ya) \leq C \varphi(y^* y)$$

holds for all $y \in A_i$. We define the right action $R_a : K \otimes_B L_2(B) \to K \otimes_B A_i \otimes_B L_2(B)$ by $R(a)x = x \otimes a$. We observe that

$$\|R(a)x\|_{\frac{1}{2}}^2 = \varphi(a^* < x^*, x > a) \leq C \varphi(< x^* x >) = C \|x\|^2_{K \otimes_B L_2(B)}.$$  

In particular, $R(a)$ is continuous on $K \otimes_B L_2(B)$. Let $D_i^\frac{1}{2}$ be the cyclic and separating vector for $\varphi_i$ in $L_2(A_i)$. If $(a_s)$ is a bounded net converging strongly to $a$, then

$$\lim_s \|h \otimes (a - a_s)\|^2_{\frac{1}{2}} = \lim_s \|< h, h >_{\frac{1}{2}} (a - a_s)D_i^\frac{1}{2}\|_{\frac{1}{2}} = 0.$$  

It is well-known (see [KR]) that the algebra of analytic elements is strongly dense in $A_i$. Moreover, since $\sigma_i^\phi \circ E = E \circ \sigma_i^\phi$, we see that for analytic $x$ the expectation $E(x)$ is also analytic. Therefore elements in $\tilde{A}_i$ can be approximated by analytic elements in $\tilde{A}_i$. We replace the algebra $C$ from above by the algebra generated by right actions $\pi_i^*(a_i)$ where the $a_i$’s are analytic elements in $\tilde{A}_i, i \in I$. Then $\text{CQ}_\emptyset$ is dense in $\mathcal{H} \otimes_B L_2(B)$ and the argument at the beginning of this proof is justified.

In the following, we use $\mathcal{M}_\text{free} = \bar{\pi}_{i \in I}(A_i, E_i)$ and denote by $\phi = \varphi \circ E_\emptyset$ the normal faithful state on $\mathcal{M}_\text{free}$. We denote by $D_\phi$ ($D_\varphi, D_{\varphi_i}$) the density of $\phi$ in $L_1(\mathcal{M}_\text{free})$ (the density of $\varphi$ in $L_1(B)$, the density of $\varphi_i = \varphi \circ E_i$ in $L_1(A_i)$, respectively). These notation suggest that all these densities and states are compatible. Indeed, we deduce from (1.6) that

$$\sigma_i^\phi \circ E = E \circ \sigma_i^\phi.$$  

The same argument holds also for the inclusion $\pi_i(A_i) \subset \bar{\pi}_{i \in I}(A_i, E_i)$. Moreover, following [BD] Lemma 1.1], we have a conditional expectation $\mathcal{E}_i : \mathcal{M}_\text{free} \to \pi_i(A_i)$ given by

$$\mathcal{E}_i(x) = (Q_i + Q_\emptyset)x(Q_i + Q_\emptyset) \in \mathcal{L}(B \oplus \tilde{A}_i) = \mathcal{L}(L_\Delta^\infty(A_i, E_i))$$

such that

$$\mathcal{E}_i(\pi_i(a_1))a_2 = a_1a_2$$

holds for all $a_1, a_2 \in A_i$. Moreover, if $\pi_L : A_i \to \mathcal{L}(A_i)$ denotes the left action of $A_i$ on $\mathcal{L}_\Delta^\infty(A_i, E_i)$, then $\mathcal{E}_i(\mathcal{M}_\text{free}) \subset \pi_L(A_i)$. Obviously we have $E_i \circ \mathcal{E}_i = E_\emptyset$ and hence $\phi = \varphi_i \circ \mathcal{E}_i$. Applying (1.6) once more, we deduce that

$$\sigma_i^\phi \circ \mathcal{E}_i = \mathcal{E}_i \circ \sigma_i^\phi.$$
holds for all \( i \in I \). Now, the converse of Voiculescu’s inequality is easily verified:

**Lemma 7.5.** Let \( (a_i) \) be a sequence of elements \( a_i \in \hat{A}_i \) such that only finitely many elements are non-zero. Then

\[
(7.3) \quad \left\| \sum_i \pi_i(a_i)D_{\phi} \right\|_{L_1(A_i)} \leq \sum_i \left\| a_iD_{\phi} \right\|_{L_1(A_i)} ,
\]

\[
(7.4) \quad \left\| \sum_i \pi_i(a_i)D_{\phi} \right\|_{L_1(A_i)} \leq \left\| \left( \sum_i D_{\phi}E_i(a_i^*a_i)D_{\phi} \right)^{1/2} \right\|_{L_1(B)} ,
\]

\[
(7.5) \quad \left\| \sum D_{\phi}\pi_i(a_i) \right\|_{L_1(A_i)} \leq \left\| \left( \sum_i D_{\phi}E_i(a_i^*a_i)D_{\phi} \right)^{1/2} \right\|_{L_1(B)} .
\]

**Proof:** Using \( \varphi \circ E_0 = \phi \), we have a natural family of embeddings \( t_p : L_p(B) \to L_p(M_{f_{free}}) \) satisfying

\[
(7.6) \quad t_p(D_{\phi}^{1-\theta}bD_{\phi}^{\theta}) = D_{\phi}^{1-\theta}bD_{\phi}^{\theta}
\]

for all \( b \in B \) (see [JX1]). Since \( \varphi_i \circ \mathcal{E}_i = \phi \) we also have a family of isometric embeddings \( t_p : L_p(A_i) \to L_p(M_{f_{free}}) \) such that

\[
(7.7) \quad t_p(D_{\phi}^{1-\theta}aD_{\phi}^{\theta}) = D_{\phi}^{1-\theta}aD_{\phi}^{\theta}
\]

for all \( a \in A_i \). The triangle inequality implies (7.3). For the proof of (7.4), we consider \( z = \sum_i \pi_i(a_i) \) with \( a_i \in \hat{A}_i \). Note that by freeness

\[
E_0(z^*z) = \sum_{j,i} E_0(\pi_j(a_j)^*\pi_i(a_i)) = \sum_i E_i(a_i^*a_i) .
\]

Since \( E_0 \) is a conditional expectation we deduce from \( \| E_0(x^*x) \|_1 \leq \| x \|_\infty \) and by duality (see [Ji Corollary 2.12]) that

\[
(7.8) \quad \left\| D_{\phi}^{1/2}E_0(z^*z)D_{\phi}^{1/2} \right\|_{1/2} \leq \left\| D_{\phi}^{1/2}z^*zD_{\phi}^{1/2} \right\|_{1/2} .
\]

We deduce with (7.6) that

\[
\left\| \sum_i \pi_i(a_i)D_{\phi} \right\|_{1/2} \leq \left\| \sum_i D_{\phi}E_i(a_i^*a_i)D_{\phi} \right\|_{1/2} = \left\| \sum_i D_{\phi}E_i(a_i^*a_i)D_{\phi} \right\|_{1/2} .
\]

Assertion (7.5) follows by taking adjoints. \( \blacksquare \)

We now reformulate these inequalities in terms of a complementation result. Let \( \mathcal{N} \) be a von Neumann algebra and \( \mathcal{E} : \mathcal{N} \to B \) be a normal faithful conditional expectation onto a von Neumann subalgebra \( B \). Let \( \varphi \) be a normal faithful state on \( B \). We denote by \( D_{\varphi \circ \mathcal{E}} \) the density of \( \varphi \circ \mathcal{E} \). As for the third term K-functional, we define a new norm on \( L_1(\mathcal{N}) \) by

\[
\| x \|_{K_{\mathcal{N},\mathcal{E}}} = \inf_{x = x_1+x_2+x_3} \inf_n \| x_1 \|_1 + \sqrt{n} \| x_2 \|_{L^1_i(\mathcal{N},\mathcal{E})} + \sqrt{n} \| x_3 \|_{L^2_i(\mathcal{N},\mathcal{E})} .
\]

Following [Ji], the space \( L^1_i(\mathcal{N},\mathcal{E}) \) is defined as the closure of \( \mathcal{N}D_{\varphi \circ \mathcal{E}} \) with respect to the norm

\[
\| zD_{\varphi \circ \mathcal{E}} \|_{L^1_i(\mathcal{N},\mathcal{E})} = \| D_{\varphi \circ \mathcal{E}} \mathcal{E}(z^*z)D_{\varphi \circ \mathcal{E}} \|_{1/2} .
\]
The space \( L_1^*(N, E) \) is the space of adjoints of elements in \( L_1^*(N, E) \) defined by the norm
\[
\|x\|_{L_1^*(N, E)} = \|x^*\|_{L_1^*(N, E)}.
\]
For both spaces we have contractive injective inclusions \( L_1^*(N, E) \subset L_1(N) \) and \( L_1^*(N, E) \subset L_1(N) \) (see (7.3)). Therefore \( \mathbb{K}_n(N, E) \) is well-defined. The following observations are immediate consequences of [31, Corollary 2.12]. There we used the antilinear duality bracket. However, the adjoint map \( J(x) = x^* \) is an isometry on \( \mathbb{K}_n(N, E) \) and thus we may work with the usual trace duality.

**Lemma 7.6.** The dual of \( \mathbb{K}_n(N, E) \) with respect to the duality bracket
\[
(x, y)_n = \text{ntr}(xy)
\]
is \( N \) equipped with the norm
\[
\|y\|_{\mathbb{K}_n(N, E)} = \max\{\|y\|_N, n^{-\frac{1}{2}} \|\mathcal{E}(y^*y)\|^{\frac{1}{2}}, n^{-\frac{1}{2}} \|\mathcal{E}(yy^*)\|^{\frac{1}{2}}\}.
\]
In Voiculescu’s inequality it is very important to work with mean 0 elements. This will be achieved by a standard symmetrization process. We define \( A_i = \ell_2^\infty(N) = N \oplus N \). All the conditional expectations \( E_i \) coincide with \( E \) defined by
\[
E(x, y) = \frac{\mathcal{E}(x) + \mathcal{E}(y)}{2}.
\]

**Proposition 7.7.** The space \( \mathbb{K}_n(N, E) \) is 3-complemented in \( L_1(\mathbb{N}; \mathbb{N} \oplus N, E) \).

**Proof:** Let \( D_\phi, D_\varphi, D_{\varphi \circ E} \) and \( D_{\varphi \circ E}^\dagger \) be the density of \( \phi, \varphi, \varphi \circ E \) and \( \varphi \circ E \) on \( \mathcal{M}_{\text{free}}, B, N \) and \( A \), respectively. We may identify the space \( L_1(A) \) with \( L_1(N) \oplus \mathbb{Z} \oplus L_1(N) \). Then the state \( \varphi \circ E \) is given by
\[
\varphi \circ E(x, y) = \frac{\text{tr}(xD_{\varphi \circ E}) + \text{tr}(yD_{\varphi \circ E})}{2}.
\]
Therefore, we may assume that \( D_{\varphi \circ E} = (\frac{1}{2}D_{\varphi \circ E}, \frac{1}{2}D_{\varphi \circ E}) \in L_1(N) \oplus L_1(N) \). For \( x \in N \) we define \( \varepsilon x = (x, -x) \in A \). The embedding will be realized by the map \( T : L_1(N) \rightarrow L_1(\mathcal{M}_{\text{free}}) \) given by
\[
T(xD_{\varphi \circ E}) = \sum_{i=1}^n \pi_i(\varepsilon x)D_\phi.
\]
According to (7.3), we have
\[
\|T(xD_\varphi)\|_1 \leq n \|x\|D_{\varphi \circ E}\|_1(A) = \frac{n}{2}(\|xD_\varphi\|_{L_1(N)} + \|-xD_\varphi\|_{L_1(N)}) = n \|xD_\varphi\|_{L_1(N)}.
\]
Similarly, we deduce from the fact that \( \varepsilon x \) has mean 0 and (7.4) that
\[
\|T(xD_\varphi)\|_1 \leq \left\|D_\varphi \sum_{i=1}^n E_i(\pi_i(1 \otimes x^*x))D_\varphi\right\|_{L_1(B)}^{\frac{1}{2}} = \sqrt{n} \left\|(D_\varphi \mathcal{E}(x^*x)D_\varphi)^\frac{1}{2}\right\|_{L_1(B)}.
\]
For an analytic element \( x \in N_1 \), we deduce from (7.2) that
\[
(7.9) \quad T(D_{\varphi \circ E}x) = T(\sigma_{\varphi}^{-1}(x)D_\varphi^N) = \sum_{k=1}^n \sigma_{\varphi}^k(\pi_i(\varepsilon x))D_\phi = \sum_{k=1}^n D_\phi \pi_i(\varepsilon x).
\]
By continuity this equality extends to all elements \( x \in \mathcal{N} \). Hence, \((7.8)\) implies the missing inequality and we deduce that

\[
\|T : \mathbb{K}_n(\mathcal{N}, \mathcal{E}) \to L_1(\mathcal{M})\| \leq 1.
\]

We define the map \( S : \mathcal{N} \to \mathcal{M} \) by

\[
S(y) = \sum_{i=1}^{n} \pi_i(\varepsilon y).
\]

According to Proposition \((7.1)\) we have

\[
\|S : \mathbb{K}_n(\mathcal{N}, \mathcal{E})^* \to \mathcal{M}_{free}\| \leq 1.
\]

Moreover, using trace duality \((x, y)_{\mathcal{M}_{free}} = \text{tr}_{\mathcal{M}_{free}}(xy)\), we get

\[
(T(xD\varphi), S(y))_{\mathcal{M}_{free}} = \text{tr}_{\mathcal{M}_{free}}(T(xD\varphi)S(y)) = \sum_{i=1}^{n} \text{tr}_{\mathcal{M}_{free}}(\pi_i(\varepsilon x)D\varphi\pi_i(\varepsilon y)) = \sum_{i=1}^{n} \varphi(xy) = (xD\varphi, y)_n.
\]

We denote the restriction of \( S^* \) to \( L_1(\mathcal{M}_{free}) \) by \( S' = S^*|_{L_1(\mathcal{M}_{free})} \). Then we obtain a map \( S' : L_1(\mathcal{M}_{free}) \to \mathbb{K}_n(\mathcal{N}, \mathcal{E})^{**} \) such that \( S'T \) coincides with the natural inclusion map of \( \mathbb{K}_n(\mathcal{N}, \mathcal{E}) \) in its bidual. We want to show that \( S'(L_1(\mathcal{M}_{free})) \subset \mathbb{K}_n(\mathcal{N}, E) \). For the proof of this inclusion, we observe that \(*_{i\in I_n}(A, E_i)\) is strongly dense in \( \mathcal{M}_{free} \) and hence \(*_{i\in I_n}(A, E_i)D\phi\) is norm dense in \( L_1(\mathcal{M}_{free}) \). Thus it is enough to consider elements of the form \( x = zD\phi \), where \( z = \pi_{j_1}(a_1) \cdots \pi_{j_m}(a_m), a_k \in \tilde{A}_{j_k} \). Then, we have

\[
(S'(x), y) = \text{tr}(xS(y)) = \sum_{i=1}^{n} \text{tr}(zD\phi\pi_i((\varepsilon y))) = \sum_{i=1}^{n} \phi(\pi_i(\varepsilon y)\pi_{j_1}(a_1) \cdots \pi_{j_m}(a_m)D\phi).
\]

Thus for \( S'(x) \neq 0 \) to hold we must have \( m = 1 \) and we may assume that \( x = \pi_k((a_1, a_2))D\phi \) for some \( 1 \leq k \leq n \). In this case we obtain

\[
(S'(x), y) = \phi(\pi_k(\varepsilon y)\pi_k(a_1, a_2)) = \frac{1}{2}(\varphi(ya_1) - \varphi(ya_2)).
\]

This implies that \( S'(\pi_k((a_1, a_2))D\phi) = \frac{1}{2}(a_1 - a_2)D\varphi \). By density \( S'(L_1(\mathcal{M}_{free})) \subset L_1(\mathcal{N}) \).

A natural example of freeness with amalgamation is given by tensor products.

**Example 7.8.** Let \((C_i)_{i \in I}\) be von Neumann algebras and \((\phi_i)\) a family of normal faithful states. Let \( B \) be a von Neumann algebra. Let \( \pi_i : C_i \to \mathcal{N}(C_i, \phi_i) \) be the embedding and \( \phi = *_{i\in I}\phi_i \) be the free product state and \( E_{free} : B \bar{\otimes} \mathcal{N}(C_i, \phi_i) \to B \) be given by \( E_{free}(x \otimes y) = x\phi(y) \). Then \((id \otimes \pi_i)(B \bar{\otimes} C_i)\) are free over \( E_{free} \).

**Proof:** Let \( a = \sum_{k=1}^{m} x_k \otimes y_k \in B \otimes C_i \). We observe that

\[
\pi_i(a) - E(\pi_i(a)) = \sum_{k=1}^{m} x_k \otimes (\pi_i(y_k) - \phi_i(y_k)1).
\]

This shows that \((id - 1 \otimes \phi_i)(B \bar{\otimes} C) \cap B \otimes C = B \otimes \hat{C}_i \). Thus, given \( a_1, \ldots, a_n \) such that \( a_j \in (id - 1 \otimes \phi_i)(B \bar{\otimes} C) \cap B \otimes C \), we may write \( a_j = \sum_{k=1}^{m} x_{jk} \otimes \pi_{ij}(y_{jk}) \) with \( y_{jk} \in \hat{C}_{ij} \). (The
same \( m \) is achieved by adding 0’s.) If \( i_1 \neq \cdots \neq i_n \), the freeness of the \( \pi_i(C_i) \)'s implies that
\[
E(a_1 \cdots a_n) = \sum_{k_1, \ldots, k_n=1}^m x_{1k_1} \cdots x_{nk_n} \phi(y_{1k_1} \cdots y_{nk_n}) = 0.
\]

By Kaplansky’s density theorem, the unit ball of \( B \otimes_{\text{min}} C_i \) is strongly dense in \( B \otimes C_i \). Therefore mean 0 elements \( a_j \) in \( B \otimes C_i \) may be approximated in the strong operator topology by bounded nets \( a_j(\alpha) \) of elements in \( B \otimes_{\text{min}} C_i \). By continuity this implies that
\[
E(a_1 \cdots a_n) = \lim_{\alpha} E(a_1(\alpha) \cdots a_n(\alpha)) = 0 .
\]

According to [Vo1], it is well-known that free products of von Neumann algebras with states are in general not semifinite. Therefore, we have to work with Haagerup \( L_p \)-spaces in this context. We will now describe the natural isomorphism between the 3-term \( K \)-functional in the setting of Haagerup \( L_p \)-spaces and in the setting of semifinite \( L_p \)-spaces.

**Lemma 7.9.** Let \( N \) be a semifinite von Neumann algebra with trace \( \tau_N \). Let \( \psi \) be a faithful, normal state on \( N \) with density \( d_\psi \in L_1(N, \tau_N) \). Let \( B \) be \( \sigma \)-finite and semifinite. Then there is a natural isomorphism
\[
\mathbb{K}_N(B \otimes N, 1 \otimes \psi) \cong L_1(B) \otimes \mathbb{K}_N(N, d_\psi) .
\]

**Proof:** The trace on \( B \) is denoted by \( \tau_B \). Let \( d_\varphi \in L_1(B, \tau_B) \) be the density of a faithful normal state \( \varphi \) on \( B \). The density \( D_{\varphi \otimes \psi} \in L_0(B \otimes N) \) is supported because \( \varphi \otimes \psi \) is faithful. The complete isometry \( I : L_p(B \otimes N) \to L_p(B \otimes N, \tau_B \otimes \tau_N) \) between the Haagerup \( L_p \)-space and the semifinite \( L_p \)-space is given by
\[
I(zD_{\varphi \otimes \psi}) = z(d_\varphi^{\frac{1}{p}} \otimes d_\psi). 
\]

Since \( I \) commutes with the modular group (see also [IX]), we have
\[
I(D_{\varphi \otimes \psi}^\ast zD_{\varphi \otimes \psi}) = (d_\varphi^{\frac{1}{p}} \otimes d_\psi)^\ast(z(d_\varphi^{\frac{1}{p}} \otimes d_\psi)).
\]

We use \( \mathcal{E} = 1 \otimes \psi \) for the conditional expectation from \( B \otimes N \) onto \( B \otimes 1 \subset N \). In section 1 (see (1.10)) we discussed the \( L_2(B) \)-valued extension of the scalar product on the Hilbert space \( H = L_2(N) \). Given \( b_1, b_2 \in B \) and \( n_1, n_2 \in N \), we find
\[
(b_1 d_\varphi \otimes n_1 d_\psi^\ast, b_2 d_\varphi \otimes n_2 d_\psi^\ast) = d_\varphi(b_1')b_2d_\varphi(n_1d_\psi^\ast, n_2d_\psi^\ast) = d_\varphi b_1'b_2d_\varphi \psi(n_1^\ast n_2) = d_\varphi \mathcal{E}((b_1 \otimes n_1)^\ast(b_2 \otimes n_2))d_\varphi .
\]

Then \( \tau(d_\psi) = 1 \) and (1.10) imply that
\[
\|zD_{\varphi \otimes \psi}\|_{L_1(B \otimes N, \mathcal{E})} = \|D_{\varphi \otimes \psi} \mathcal{E}(z^\ast z)D_{\varphi \otimes \psi}\|_{\frac{1}{2}} = \|(d_\varphi \otimes d_\psi) \mathcal{E}(z^\ast z)(d_\varphi \otimes d_\psi)\|_{\frac{1}{2}} \leq \|d_\varphi \mathcal{E}(z^\ast z)d_\varphi\|_{\frac{1}{2}} = \left\|\mathcal{E}(z(d_\varphi \otimes d_\psi^\ast), z(d_\varphi \otimes d_\psi^\ast))\right\|_{L_2(B)}^{\frac{1}{2}} = \left\|z(d_\varphi \otimes d_\psi^\ast)\right\|_{L_1(B) \otimes L_2(N)} .
\]
This shows that \( \iota_r(x) = I(x(1 \otimes d_\psi^2)) \) is an isometry between \( L_1(B) \otimes \hat{\mathcal{L}}_2(N) \) and \( \hat{L}_1(\mathbb{B} \otimes N, \mathcal{E}) \).
Using adjoints, we see that \( \iota_c(x) = I((1 \otimes d_\psi^2)x) \) also is an isometry. Since these isometries are compatible with the map \( q_n \) defined before Proposition 6.1, we deduce that the map \( \iota_n(x) = \sqrt{n}I(x) \) yields an isometric isomorphism between \( \mathbb{K}_n(\mathbb{B} \otimes N) \) and \( L_1(B) \otimes \mathbb{K}_n(N, d_\psi) \).

**Corollary 7.10.** Let \( N \) be a semifinite von Neumann algebra and let \( \psi \) be a faithful normal state with density \( d_\psi \). Let \( \tilde{\psi}(x, y) = \frac{1}{2}(\psi(x) + \psi(y)) \) be the corresponding state on \( N \oplus N \). Then \( \mathbb{K}_n(N, d_\psi) \) is 3-completely complemented in \( L_1(\mathbb{K}_1 \otimes \cdots \otimes \mathbb{K}_n(N, \tilde{\psi})) \) for all \( n \in \mathbb{N} \).

**Proof:** According to Lemma 7.9 with \( B = \mathbb{C} \) it suffices to show the assertion for \( \mathbb{K}_n(N, \psi) \). Here \( \psi \) is considered as a conditional expectation onto \( C1 \) and the operator space structure is the one given by the isomorphism in Lemma 7.9. We define \( C = N \oplus N \) and denote by \( \pi_i : C \to \mathbb{K}_i \otimes \cdots \otimes \mathbb{K}_n(N, \tilde{\psi}) \) the natural embeddings. We shall write \( \phi = \pi_{1 \leq i \leq n} \tilde{\psi} \) for the free product state. The map \( T : \mathbb{K}_n(N, \psi) \to L_1(\mathbb{K}_1 \otimes \cdots \otimes \mathbb{K}_n(C_i, \psi)) \) is given by Proposition 7.7.

\[
T(xD_\psi) = \sum_{i=1}^n \pi_i(\varepsilon x)D_\psi.
\]

In order to show that the maps \( T \) and \( S' \) obtained in Proposition 7.7 are indeed completely bounded, we use operator-valued free products with respect to \( B = M_m \). Then we shall use \( N_m = \mathcal{L}_m(N) \) with the conditional expectation \( E_m(x \otimes y) = \psi(y)(x \otimes 1) \). Note that \( N_m \otimes N_m = M_m(C) \) and the conditional expectations \( E_i : M_m(C) \to M_m \) given by \( E_i(x \otimes y) = \tilde{\psi}(y)x \) are compatible with the definitions before Proposition 7.7. According to Example 7.8 we know that the algebras \( (id \otimes \pi_i)(M_m(C)) \) are free over \( 1 \otimes \phi \). \( M_m(\pi_{1 \leq i \leq n}(C, \tilde{\psi})) \) is generated as a von Neumann algebra by the algebras \( (id \otimes \pi_i)(M_m(C)) \). Hence \( M_m(\pi_{1 \leq i \leq n}(C, \tilde{\psi})) \) and \( \pi_{1 \leq i \leq n}(M_m(C), E_i) \) are isomorphic. Therefore we are in a position to apply Proposition 7.7. We use the normalized trace \( \varphi_m(x) = \frac{1}{m}tr(y) \) on \( B = M_m \). We observe that \( \varphi_m \circ E \) restricted to \( 1 \otimes N \) induces \( \psi \) and hence \( D_{\varphi_m \circ E} = D_{\varphi_m \circ \psi} \).
Moreover, in the Haagerup \( L_1 \)-spaces \( L_1(M_m(N), \varphi_m \circ \psi) \), \( L_1(M_m(\pi_{1 \leq i \leq n}(C_i, \tilde{\psi}))) \), \( \varphi_m \circ \phi \), we have \( D_{\varphi_m \circ \psi} = 1 \otimes D_\psi \), \( D_{\varphi_m \circ \phi} = 1 \otimes D_\phi \), respectively. We denote by \( T^{(m)} \), \( S^{(m)} \) the maps constructed in Proposition 7.7 for \( \mathbb{K}_m(N_m, E_m) \). We find that

\[
(id_{L_1(M_m)} \otimes T)(x(1 \otimes D_\psi)) = \sum_{i=1}^n (id \otimes \pi_i)(\varepsilon x)(1 \otimes D_\psi) = T^{(m)}(xD_{\varphi_m \circ \psi}).
\]

Hence \( \|id_{L_1(M_m)} \otimes T\| = \|T^{(m)}\| \leq 1 \) and \( T \) is a complete contraction. Using the concrete form of \( S^{(m)} \) constructed in Proposition 7.7 we find that \( S^{(m)} = id_{L_1(M_m)} \otimes S' \) and hence \( \|S'\|_{cb} \leq 3 \).

**Theorem 7.11.** Let \( N \) be a semifinite von Neumann algebra with trace \( \tau \) and \( d \) a positive density in \( L_0(N, \tau) \) with full support. Then \( K(N, d) \) is 3-completely complemented in the predual of a von Neumann algebra \( \mathcal{M} \). If \( N \) has QWEP, then there is such an \( \mathcal{M} \) with QWEP.

**Proof:** According to Proposition 6.2 we have \( K(N, d) = \lim_{m} K(N, e_m d) \). Here \( e_m = 1_{[1/m, m]}(d) \) are the spectral projections of \( d \). Using an ultraproduct, Lemma 7.14 and the reflexivity of \( K \), it therefore suffices to show the assertion for a density with \( \tau(d) < \infty \). By normalization we
may assume $\tau(d) = 1$. We define the state $\psi(x) = \tau(dx)$. According to Lemma 6.2, $K(N, d)$ is 1-completely contractively complemented in $\prod_{n, U} \mathbb{K}_n(N, d)$. By Corollary 7.10, $\prod_{n, U} \mathbb{K}_n(N, d)$ is 3-completely contractively complemented in $\mathcal{M}_n = \prod_{n, U} L_1(\hat{\psi}_1 \leq \hat{\psi} \leq n)$.

(7.11) $\mathcal{M}_n = \prod_{n, U} L_1(\hat{\psi}_1 \leq \hat{\psi} \leq n(N \oplus N, \tilde{\psi}))$ for every free ultrafilter $\mathcal{U}$ on the integers. This completes the proof in the general case. If we assume in addition that $N$ is QWEP, then $e_m N e_m$ is QWEP for every $m \in \mathbb{N}$. According to Theorem 7.15 (below), the von Neumann algebras $\hat{\psi}_1 \leq \hat{\psi} \leq n(N \oplus N, \tilde{\psi})$ are QWEP. Since $A$ QWEP implies that $A^{\text{op}}$ QWEP, this implies with (1.7) that $\mathcal{M}$ from (7.11) is QWEP. Using Lemma 7.14 (below) the QWEP property is stable under ultraproducts and the assertion follows.

By standard properties of the projective tensor product, we obtain the following application of Theorem 7.11.

**Corollary 7.12.** Let $N$ be a semifinite von Neumann algebra with trace $\tau$ such that $N$ is QWEP. Let $d$ be a density in $L_0(N, \tau)$ with full support. Then

$$K(N, d) \hat{\otimes} K(N, d)$$

is 9-completely complemented in

$$\mathcal{M}_n \hat{\otimes} \mathcal{M}_n$$

for some von Neumann algebra $\mathcal{M}$ with QWEP.

**Proof of Proposition 4.6 and Theorem 3:** We consider $\Omega = \mathbb{N} \times [0, 1]$ and $\tilde{\mu} = m \otimes \mu$, with $m$ the counting measure on $\mathbb{N}$ and $d\mu(t) = dt/(\pi \sqrt{t(1-t)})$. By Lemma 6.3 and Lemma 6.4 we see that $G$ is completely contractively complemented in $K(M_2(L_\infty(\mathbb{N} \times [0, 1], \tilde{\mu})), 1 \otimes d)$ where $d(t) = \begin{pmatrix} t & 0 \\ 0 & 1-t \end{pmatrix}$. Hence the assumptions of Theorem 7.11 and Corollary 7.12 are satisfied. This completes the proof of Theorem 3. The lower estimate in Proposition 4.6 follows immediately by complementation.

**Remark 7.13.** Using Speicher’s central limit theorem (see [Sp1]) it is not too difficult to identify the underlying von Neumann algebra of the embedding of OH (and indeed an arbitrary quotient of $R \oplus C$) as a free quasi-free state factor of Shlyakhtenko [S1]. In case of OH this turns out to be a free quasi-free factor of type III$_1$. After an early draft of this paper circulated, Pisier [Ps8] found a more direct approach to Theorem 7.11 without using the three term $K$-functional. This yields an easier way to identify OH in the predual of a type III factor. For related results in the $L_p$-setting and more information on the possible types of these factors, we refer to [Xu]. The approach via the three term $K$-functional is used in [J4] for a ‘concrete’ embedding of OH in the predual of the hyperfinite III$_1$ factor.

At the end of this section, we will provide the results on the QWEP property needed above.
Lemma 7.14. Let \((M^s)\) be a family of QWEP von Neumann algebra. Then the von Neumann algebra \(\mathcal{M} = (\prod_{s,t} M_s^t)^*\) also has QWEP.

Proof: According to Kirchberg [Ki2], we know that \(\prod M^s\) is QWEP and thus \((\prod M^s)^*\) is QWEP. Following Groh [Gr], we observe that the space of functionals \(\prod_{s,t} M_s^t\) on \(\prod M^s\) is left and right invariant under the action of \(\prod M^s\). Hence there is a central projection \(z_\mathcal{M}\) such that \(\mathcal{M} \cong z_\mathcal{M}(\prod M^s)^*\). Thus \(\mathcal{M}\) is QWEP.

Theorem 7.15. Let \(N\) and \(M\) be von Neumann algebras with QWEP and let \(\phi, \psi\) be normal faithful states on \(N, M\), respectively. Then the von Neumann algebra \((N, \phi)\#(M, \psi)\) is QWEP.

We need some preparation. The following result can alternatively be proved using Dykema’s deep analysis of free product of matrix algebras (see [Dk2]). We prefer a more direct approach using results from Shlyakhtenko [S1].

Lemma 7.16. Let \(A_1, A_2\) be matrix algebras with normal faithful states \(\phi_1\) and \(\phi_2\). Then the free product \((A_1, \phi_1)(A_2, \phi_2)\) is QWEP.

Proof: First we observe that we may assume \(A_1 = A_2\) and \(\phi_1 = \phi_2\). Indeed, we consider \((A, \phi) = (A_1 \otimes A_2, \phi_1 \otimes \phi_2)\). We denote by \(\pi_1\) (and \(\pi_2\)) the embedding of \(A\) in the first (respectively second) component of the free product. Then the von Neumann algebra generated by \(\pi_1(A_1 \otimes 1)\) and \(\pi_2(1 \otimes A_2)\) is isomorphic to the free product \((A_1, \phi_1)\#(A_2, \phi_2)\) and invariant under the action of the modular group of the free product state \(\phi \# \phi\). By Takesaki’s theorem (see e.g. [S1 Theorem 10.1]) we deduce the existence of a normal conditional expectation and hence it suffices to assume \(A_1 = A_2\), \(\phi_1 = \phi_2\).

Now, we assume \(A = M_n\) and that \(\phi_n(x) = \sum_{k=1}^{n} \lambda_k x_{kk}\) is the given state. We recall the notation \(l(h)\) for the creation operator on the full Fock space \(\mathcal{F}(H)\). On \(\mathcal{B}(\mathcal{F}(E_n^\mathbb{F})) \otimes M_n\) we consider the state \(\Phi = \phi_\Omega \otimes \phi_n\), where \(\phi_\Omega\) is given by the vacuum state. According to [S1 Theorem 5.2], we know that the \(C^*\)-algebra \(C^*(L)\) generated by the operator

\[
L = \sum_{k,l=1}^{n} \ell(h_{kl}) \otimes \sqrt{\lambda_k} e_{kl}
\]

is free from \(M_n\). We consider the semicircular operator

\[
s = L + L^* = \sum_{k,l=1}^{n} [\sqrt{\lambda_k} \ell(h_{kl}) + \sqrt{\lambda_l} \ell(h_{lk})^*] \otimes e_{kl}.
\]

For \(k \leq l\) we obtain the generalized semicircular elements

\[
y_{kl} = \sqrt{\lambda_k} \ell(h_{kl}) + \sqrt{\lambda_l} \ell(h_{lk})^* = \sqrt{\lambda_k} \ell(h_{kl}) + \sqrt{\lambda_l} \ell(h_{lk})^*.
\]

By orthogonality we deduce that the family \((y_{kl})_{k \leq l}\) is \(\ast\)-free. For fixed \(l \leq k\) the von Neumann algebra \(D_{kl}\) generated by \(y_{kl}\) is isomorphic to the von Neumann algebra \(T_{\lambda_l/\lambda_k}\) introduced in [S1 section4]. Moreover, the restriction \(\Phi_\Omega |_{D_{kl}}\) corresponds to the vacuum state \(\phi_{\lambda_k/\lambda_l}\) on \(T_{\lambda_l/\lambda_k}\). Hence the von Neumann algebra \(M\) generated by all the \(D_{kl}\)'s is isomorphic to \(\ast_{k \leq l}(T_{\lambda_k/\lambda_l}, \phi_{\lambda_k/\lambda_l})\). In particular, \(\Phi\) is faithful on the von Neumann algebra \(N = M \otimes M_n\) and the isomorphism between \(N\) and \(\ast_{k \leq l}(T_{\lambda_k/\lambda_l}, \phi_{\lambda_k/\lambda_l}) \otimes M_n\) sends \(\ast_{k \leq l}\phi_{\lambda_k/\lambda_l} \otimes \phi_n\) to \(\Phi\). In [S1]
Theorem 2.9] Shlyakhtenko investigated an automorphism group $\alpha_t$ of $T_{\lambda_k/\lambda_l}$ and showed that it satisfies the KMS condition at inverse temperature 1 with respect to $\phi_{\lambda_k/\lambda_l}$. In our normalization the modular group $\sigma_t^{\phi}$ of an arbitrary normal faithful state $\phi$ satisfies $\phi(xy) = \phi(y\sigma_t^{\phi}(x))$, i.e. the KMS condition at inverse temperature $-1$. This implies that $\sigma_t^{\phi_{\lambda_k/\lambda_l}} = \alpha_{-t}$. We refer to [SI section4] for the equation $\alpha_t(y_{kl}) = (\lambda_t/\lambda_k)^{-it} y_{kl}$. Thus we have

$$\sigma_t^{\phi_{\lambda_k/\lambda_l}}(y_{kl}) = \left(\frac{\lambda_l}{\lambda_k}\right)^{it} y_{kl}.$$  

(See [PS] for a direct argument.) We deduce from $\sigma_t^{\phi_u}(e_{kl}) = \lambda_k^i e_{kl} \lambda_l^{-it}$ that

$$\sigma_t^{\phi}(y_{kl} \otimes e_{kl}) = \left(\frac{\lambda_l}{\lambda_k}\right)^{it} \left(\frac{\lambda_k}{\lambda_l}\right)^{it} (y_{kl} \otimes e_{kl}) = y_{kl} \otimes e_{kl}.$$  

This implies that $s$ belongs to the centralizer $N^\Phi$ of the von Neumann $N$. The von Neumann algebra $W^*(s)$ generated by $s$ is isomorphic to $L_\infty[0,1]$. Let $u \in W^*(s)$ be a Haar unitary. Clearly, the algebras $uM_n u^*$ and $M_n$ are free with respect to $\Phi$. Moreover, we deduce from $\sigma_t^{\phi}(u) = u$ that

$$\phi(u x u^*) = \phi(x u^* u) = \phi(x).$$

Therefore the von Neumann algebra $M$ generated by $M_n$ and $uM_n u^*$ is isomorphic to the free product $(M_n, \phi_n) \bar{\otimes} (M_n, \phi_n)$. Since $M_n$ and $uM_n u^*$ are invariant under $\sigma_t^{\phi}$, we find a normal conditional expectation from $N = \bar{\otimes}_{k \leq 1}(T_{\lambda_k/\lambda_l}, \phi_{\lambda_k/\lambda_l}) \otimes M_n$ onto $M$. According to [PS] Lemma 2.5, the von Neumann algebra $\bar{\otimes}_{k \leq 1}(T_{\lambda_k/\lambda_l}, \phi_{\alpha_k/\lambda_l})$ is QWEP and the assertion follows. 

Remark 7.17. The preceding argument implies in particular that

$$M_n(\bar{\otimes}_{k \leq 1}(T_{\lambda_k/\lambda_l}, \phi_{\lambda_k/\lambda_l})) = (M_n, \phi_n) \bar{\otimes} L_\infty[0,1]$$

In the tracial situation we need $n + \frac{2(n-1)n}{2} = n^2$ semicircular random variables. This leads to the well-known isomorphism $M_n \bar{\otimes} L(\mathbb{Z}) = M_n(L(F_n^2))$ (see [VDN] Theorem 5.4.1]).

Our proof of Theorem 2.15 requires us to show that free products and ultraproducts are compatible. We will need some notation. Let $(A_j, \phi_j)_{j \in J}$ be a family of von Neumann algebras $A_j$ with normal faithful state $\phi_j$. For a free ultrafilter $U$ on $J$, we consider the von Neumann algebra $B = (\prod_{j \in U} A_j)^*$ and the ultraproduct state $\Phi(x_j) = \lim_{j \in U} \phi_j(x_j)$. Let us denote by $e$ the support of $\Phi$. Then $\Phi$ is a normal faithful state on $eBe$. We use the notation $\prod_{j \in U}[A_j, \phi_j] = e(\prod_{j \in U}(A_j)^*)e$ for this von Neumann algebra. (This is the non-tracial version of the usual von Neumann algebra ultraproduct of II$_1$-von Neumann algebras.)

Lemma 7.18. Let $A_1, A_2$ be von Neumann algebras with normal faithful states $\varphi_1$ and $\varphi_2$. Let $\pi_k : A_k \to \prod_{j}[A_{k,j}, \phi_{k,j}]$ be a faithful state preserving homomorphisms. Assume that the ultraproduct states $\Phi_k((x_j)) = \lim_{j \in U} \phi_{k,j}(x_j)$ satisfy $\sigma_t^{\Phi_k} \circ \pi_k = \pi_k \circ \sigma_t^{\Phi_k}$ for $k = 1, 2$. Then there is an injective $^*-$homomorphism

$$\alpha : (A_1, \varphi_1) \bar{\otimes} (A_2, \varphi_2) \to \prod_{j \in U}[(A_1,j, \phi_{1,j}) \bar{\otimes} (A_2,j, \phi_{2,j}), \phi_{1,j} * \phi_{2,j}]$$

together with a normal conditional expectation onto $\alpha((A_1, \varphi_1) \bar{\otimes} (A_2, \varphi_2))$. 
Expectation

Thus, for analytic elements, \((E_\star)^* = E\). Note that \(E_\star\) is a \(N\)-bimodule map and \(E_\star(\phi) = \phi \circ E = \psi\). Given an analytic element \(x \in eNe\), we see that

\[
(1 - f)x\psi = (1 - f)xE_\star(\phi) = (1 - f)E_\star(x\phi) = (1 - f)E_\star(\phi \sigma^\phi_{-i}(x)) = (1 - f)E_\star(\phi \sigma^\phi_{-i}(x)) = (1 - f)\psi \sigma^\phi_{-i}(x) = 0.
\]

Thus, for analytic elements, \((1 - f)x = 0\) and \((1 - f)x^*f = 0\). By the density of the analytic elements, this implies that \(xf = fx\) for all \(x \in eNe\).

Let \((A_1, \phi_1), (A_2, \phi_2)\) and \((A_{k,j}, \phi_{k,j})\) satisfy the assumptions. Let us denote by \(\psi_j = \phi_1,j \ast \phi_{2,j}\) the free product state and \(B_j = (A_{1,j}, \phi_{1,j}) \ast (A_{2,j}, \phi_{2,j})\). We denote by \(\pi_{k,j}\) the natural inclusion map of \(A_{k,j}\) in \(B_j\) using the first (second) component for \(k = 1\) (\(k = 2\), respectively). Then we find a mapping \(\hat{\rho}_k : \prod_{j \in \mathcal{I}} A_{k,j} \rightarrow \prod_{j \in \mathcal{I}} B_j\) given by \(\hat{\rho}_k(a_j) = (\pi_{k,j}(a_j))\). By density with respect to the strong operator topology, we may extend \(\hat{\rho}_k\) to a *-homomorphism from \((\prod_{j \in \mathcal{I}} (A_{k,j})_\star)^*\) to \((\prod_{j \in \mathcal{I}} (B_j)_\star)^*\), still denoted by \(\hat{\rho}_k\). For every \(j\) and \(k = 1, 2\) we have a normal conditional expectation \(E_{k,j} : B_j \rightarrow A_{k,j}\) such that \(\psi_j = \phi_{k,j} \circ E_{k,j}\). The predual maps provide contractions \((E_{k,j})_\star : (A_{k,j})_\star \rightarrow (B_j)_\star\) such that \(((E_{k,j})_\star)^* = E_{k,j}\). This induces (complete) contractions \((E_k)_\star : \prod_{j \in \mathcal{I}} (A_{k,j})_\star \rightarrow \prod_{j \in \mathcal{I}} (B_j)_\star\) such that \(((E_k)_\star)^* = E_k\). We follow Raynaud’s notation \((\cdot)^*\) (see [Ra]) for equivalence classes in ultraproducts and in \((\prod_{j \in \mathcal{I}} A_j)_\star\). In this notation we have

\[
(E_k)_\star(\phi_{k,j})^* = (\psi_j)^*.
\]

Let us denote by \(\Psi = (\psi_j)^*\) the ultraprodstate support. The support of \(\Psi\), \(\Phi_k\) is denoted by \(f, e_k\), respectively. By our preliminary observation we see that elements in \(\hat{\rho}_k(e_k(\prod_{j \in \mathcal{I}} (A_{k,j})_\star)^* e_k)\) commute with \(f\). Therefore, \(\hat{\rho}_k\) induces a *-homomorphism \(\rho_k(e_k x e_k) = f \hat{\rho}_k(e_k x e_k) f\) between the von Neumann algebras \(\prod_{j \in \mathcal{I}} A_{k,j}, \phi_{k,j}\) and \(\prod_{j \in \mathcal{I}} B_j, \psi_j\) for \(k = 1, 2\). For the modular group we use Raynaud’s [Ra] characterization

\[
\sigma^\Psi_t(f(y)^* f) = f(\sigma^\psi_{\psi_j}(y)^* f).
\]

A similar formula holds for the \(\Phi_k\)’s. We certainly have \(f \leq \rho_k(e_k)\). Thus we get

\[
\sigma^\Psi_t(f \hat{\rho}_k(e_k(x)^* e_k)^* f) = \sigma^\Psi_t(f \hat{\rho}_k((x)^* f) = \sigma^\Psi_t(f(\pi_{k,j}(x)^* f) = f(\sigma^\psi_{\pi_{k,j}}(x)^* f) = \rho_k(\sigma^{\Phi_k}_{\pi_{k,j}}(x)^* e_k).
\]

By density we find that \(\sigma^\Psi_t \circ \rho_k = \rho_k \circ \sigma^{\Phi_k}_t\) for \(k = 1, 2\). Thus we have found embeddings \(\alpha_k = \rho_k \pi_k\) of \(A_k\) in \(\prod_{j \in \mathcal{I}} B_j, \psi_j\) satisfying \(\Psi \circ \alpha_k = \varphi_k\). Using the assumption \(\sigma^{\Phi_k}_t \circ \pi_k = \pi_k \circ \sigma^{\varphi_k}_t\), we deduce that

\[
\sigma^\Psi_t \circ \alpha_k = \alpha_k \circ \sigma^{\varphi_k}_t.
\]

If we can show that \(\alpha_1(A_1)\) and \(\alpha_2(A_2)\) are free with respect to \(\Psi\), then the von Neumann algebra \(C\) generated by \(\alpha_1(A_1)\) and \(\alpha_2(A_2)\) is isomorphic to the free product \((A_1, \phi_1) \ast (A_2, \phi_2)\) and admits a normal conditional expectation as guaranteed by Takesaki’s theorem (see e.g. [St Theorem 10.1]). Let us show freeness. Consider \(a_r \in \hat{A}_r\) and assume \(i_1 \neq i_2 \cdots \neq i_m\). We may replace the \(a_r\)’s by their analytic approximations

\[
T^\psi_{i_r}(a_r) = \int_{\mathbb{R}} \sigma^\psi_{i_r} f_i(t) dt.
\]
where \( f_l(t) = \sqrt{\frac{t}{\pi}} e^{-lt^2} \). Note that \( T^\psi_l(a_r) \) still has mean 0. Since \( e_k(\prod_j A_{j,k})e_k \) is strongly dense in \( \prod_{j \in I} [A_{j,k}, \phi_k] \) we may apply Kaplansky’s density theorem to approximate \( \alpha_{ir}(a_r) \) by a bounded net of elements \( (a_{ir,j,s})_{s \in S} \) in the strong operator topology. We observe that \( \lim_s \phi_{is}((a_{ir,j,s})^*) = \varphi_i(a_r) = 0 \). Therefore \( ((a_{ir,j,s} - \phi_{ir,j}(a_{r,s})1)^*)_{s \in S} \) also provides a bounded net converging in the strong operator topology to \( \alpha_{ir}(a_r) \). We use the notation \( \widehat{a}_{ir,j,s} = a_{ir,j,s} - \phi_{ir,j}(a_r)1 \). Using the strong continuity of the modular group it is easy to show that \( T^\psi_l (f(\pi_{ir,j}(\widehat{a}_{ir,j,s}))^*) \) converges to \( T^\psi_l (\alpha_{ir}(a_r)) \) in the strong operator topology. We deduce from (7.12) that

\[
T^\psi_l (f(\pi_{ir,j}(\widehat{a}_{ir,j,s}))^*) = f(T^\psi_l (\pi_{ir,j}(\widehat{a}_{ir,j,s})))^* \, f.
\]

For an arbitrary family \( (y_j) \) we observe that

\[
(1 - f)(T^\psi_l (y_j))^* \Psi = (1 - f)(T^\psi_l (y_j)\psi_j)^* = (1 - f)(\psi_j\sigma_{-\frac{i}{l}}(T_l(y_j))^* = (1 - f)\Psi(\sigma_{-\frac{i}{l}}(T_l(y_j))^* = 0.
\]

In the last line we use the fact that the norm of \( \sigma_{-\frac{i}{l}}(T_l(y_j)) \) can be estimated as a function of \( ||y_j|| \) and \( l \). Applying this argument also for adjoints, we deduce that

\[
(7.14) \quad f(T^\psi_l (y_j))^* = (T^\psi_l (y_j))^* \, f.
\]

Using (7.13), (7.14) and the component-wise freeness we get

\[
\Psi \left( \alpha_{i_1} (T^\psi_{i_1} (a_{i_1})) \cdot \alpha_{i_2} (T^\psi_{i_2} (a_{i_2})) \cdots \cdot \alpha_{i_m} (T^\psi_{i_m} (a_{i_m})) \right)
\]

\[
= \Psi \left( \pi_{i_1,j}(\widehat{a}_{i_1,j,s_1})^* \cdot f(T_l^\psi \pi_{i_2,j}(\widehat{a}_{i_2,j,s_2}))^* \cdots \cdot \alpha_{i_m} (T^\psi_{i_m} (a_{i_m})) \right)
\]

\[
= \lim_{s_1} \Psi \left( \pi_{i_1,j}(\widehat{a}_{i_1,j,s_1})^* \cdot f(T_l^\psi \pi_{i_2,j}(\widehat{a}_{i_2,j,s_2}))^* \cdots \cdot \alpha_{i_m} (T^\psi_{i_m} (a_{i_m})) \right)
\]

\[
= \lim_{s_1} \lim_{s_2} \cdots \lim_{s_m} \Psi \left( \pi_{i_1,j}(\widehat{a}_{i_1,j,s_1})^* \cdot f(T_l^\psi \pi_{i_2,j}(\widehat{a}_{i_2,j,s_2}))^* \cdots \cdot \alpha_{i_m} (T^\psi_{i_m} (a_{i_m})) \right)
\]

\[
= \lim_{s_1} \lim_{s_2} \cdots \lim_{s_m} \Psi \left( \pi_{i_1,j}(\widehat{a}_{i_1,j,s_1})^* \cdot f(T_l^\psi \pi_{i_2,j}(\widehat{a}_{i_2,j,s_2}))^* \cdots \cdot \alpha_{i_m} (T^\psi_{i_m} (a_{i_m})) \right)
\]

\[
= \lim_{s_1} \lim_{s_2} \cdots \lim_{s_m} \pi_{i_1,j}(\widehat{a}_{i_1,j,s_1})^* \cdot f(T_l^\psi \pi_{i_2,j}(\widehat{a}_{i_2,j,s_2}))^* \cdots \cdot \pi_{i_m,j}(\widehat{a}_{i_m,j,s_m})^* = 0.
\]

Taking the limit for \( l_1, \ldots, l_m \to \infty \) yields the assertion. \( \blacksquare \)

**Proof of Theorem 7.15** Let \( A_1 \) and \( A_2 \) be von Neumann algebras with QWEP. According to (and a slight perturbation argument), we find state preserving embeddings \( \pi_1 : A_1 \to \prod_{j \in \mathcal{I}} [M_{m(j)}, \phi_j] \) and \( \pi_2 : A_2 \to \prod_{j \notin \mathcal{I}} [M_{m(j)}, \phi_j] \) together with state preserving conditional expectations \( E_1 : \prod_{j \in \mathcal{I}} [M_{m(j)}, \phi_j] \to A_1, E_2 : \prod_{j \notin \mathcal{I}} [M_{m(j)}, \phi_j] \to A_1 \). This implies that \( \pi_1(A_1), \pi_2(A_2) \) are invariant under the modular group of the ultraproduct states \( \Phi_1 = (\phi_j)^*, \Phi_2 = (\phi_j)^* \), respectively. We consider the index set \( I \times J \) with the ultrafilter \( \mathcal{U'} \) defined as follows: \( B \in \mathcal{U'} \) if and only \( \{i \mid \{j \mid (i, j) \in B \} \in \mathcal{U'} \} \in \mathcal{U'} \). We define \( (A_{1,1,1}, \phi_{1,1,1}) = (M_{m(i)}, \phi_i) \)
and \( (A_{2,ij}, \phi_{1,ij}) = (M_{m(j)}, \phi_j) \). According to Lemma 7.18, we deduce that \( (A_1, \varphi_1) \ast (A_2, \varphi_2) \) embeds in

\[
\prod_{(i,j) \in \mathcal{E}'} [(M_{m(i)}, \phi_i) \ast (M_{m(j)}, \phi_j), \phi_i \ast \phi_j]
\]

and admits a normal conditional expectation. Therefore Lemma 7.16 and Lemma 7.14 imply the assertion.

\[\square\]

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