A semi-classical inverse problem II: reconstruction of the potential

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February 13, 2008

1 Introduction

This paper is the continuation of [4], where Victor Guillemin and I proved the following result: the Taylor expansion of the potential $V(x)$ ($x \in \mathbb{R}$) at a non degenerate critical point $x_0$ of $V$, satisfying $V'''(x_0) \neq 0$, is determined by the semi-classical spectrum of the associated Schrödinger operator near the corresponding critical value $V(x_0)$. Here, I prove results which are stronger in some aspects: the potential itself, without any analyticity assumption, but with some genericity conditions, is determined from the semi-classical spectrum. Moreover, our method gives an explicit way to reconstruct the potential.

Inverse spectral results for Sturm-Liouville operators are due to Borg, Gelfand, Levitan, Marchenko and others (see for example [8]). They need the spectra of the differential operator with two different boundary conditions in order to recover the potential. Our results are different in several aspects:

- They are local using only the part of the spectrum included in some interval $]-\infty, E[\] in order to get $V$ in the inverse image $\{x|V(x) < E\}$ of this interval.
- They need only approximate spectra.
- They still apply if the operator is essentially self-adjoint.

After having completed the present work, I founded that similar methods were already used by David Gurarie [7] in order to recover a surface of revolution from the joint spectrum of the Laplace operator and the momentum operator $L_z$. Our genericity assumptions are weaker and more explicit:

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• David Gurarie assumes that the potential is a Morse function with pairwise different critical values, while we assume only a weak non degeneracy condition (see Section 10.1.1).

• His argument for the separation of spectra associated to the different wells is less explicit than ours which uses the semi-classical trace formula (see Section 11.3).

• He does not say a word about the problem of a non generic symmetry defect and explicit non isomorphic potentials with the same semi-classical spectra (Section 7 and Assumption 3 in Theorem 5.1).

For a recent review on the use of semi-classics in inverse spectral problems, the reader could look at [9].

2 Motivation I: surfaces of revolution

Let us consider a surface of revolution with a metric

\[ ds^2 = dx^2 + a^4(x)dy^2 \]

with \( x \in [0, L] \) and \( y \in \mathbb{R}/2\pi\mathbb{Z} \). We assume that \( a(0) = a(L) = 0, a(x) > 0 \) for \( 0 < x < L \) and \( a \) is smooth. The volume element is given by \( dv = a^2(x)|dx|dy| \). The Laplace operator is:

\[ \Delta = -\frac{\partial^2}{\partial x^2} - \frac{2a'}{a} \frac{\partial}{\partial x} - \frac{1}{a^4} \frac{\partial^2}{\partial y^2}. \]

Using the change of function \( f = Fa \), we get the operator \( P = a\Delta a^{-1} \) which is formally symmetric w.r. to \( |dx|dy| \):

\[ P = -\frac{\partial^2}{\partial x^2} + \frac{a''}{a} - \frac{1}{a^4} \frac{\partial^2}{\partial y^2}. \]

If \( F(x, y) = \varphi(x)\exp(il\gamma) \) with \( l \in \mathbb{Z} \), we define \( Q_l \) as follows

\[ PF = l^2(Q_l\varphi)e^{il\gamma}, \]

and putting \( h = l^{-1} \), we get

\[ Q_h\varphi = -\hbar^2 \varphi'' + \left(a^{-4} + \hbar^2 W\right) \varphi \]

with \( W = \frac{a''}{a} \). It implies that the knowledge of the joint spectrum of \( \Delta \) and \( \partial_y \) is closely related to the spectra of \( Q_h \) for \( h = 1/l \) with \( l \in \mathbb{Z} \setminus 0 \). This relates our paper to Gurarie’s result [7].
3 Motivation II: effective surface waves Hamiltonian

In our paper [2], we started with the following acoustic wave equation

\[
\begin{cases}
    u_{tt} - \text{div}(n \, \text{grad} u) = 0 \\
    u(x, 0, t) = 0
\end{cases}
\]

in the half space \( X = \mathbb{R}^{d-1} \times (-\infty, 0]_z \) where \( n(z) : \mathbb{R}_- \to \mathbb{R}_+ \) is a non negative function which satisfies

\[
0 < n_0 := \inf n(z) < n_\infty := \lim_{z \to -\infty} n(z).
\]

This equation describes the propagation of acoustic waves in a medium which is stratified: the variations of the density are on much smaller scales vertically than horizontally\(^2\). This equation admits solutions of the form \( \exp(i(\omega t - x \xi))v(z) \) provided that \( v \) is an eigenfunction of the operator \( L_\xi \) on the half line \( z \leq 0 \) defined as follows:

\[
L_\xi v := -\frac{d}{dz} \left(n(z) \frac{dv}{dz}\right) + n(z)|\xi|^2 v
\]

with Dirichlet boundary conditions and eigenvalue \( \omega^2 \). These solutions are exponentially localized near the boundary provided that \( \omega^2 \) is in the discrete spectrum of \( L_\xi \) contained in \( J := [n_0|\xi|^2, n_\infty|\xi|^2] \).

Let us denote by \( \lambda_1(\xi) < \lambda_2(\xi) < \cdots < \lambda_j(\xi) < \cdots \) the spectrum of \( L_\xi \) in the interval \( J \) and \( v_j(\xi, z) \) the associated normalized eigenfunctions. The unitary map from \( L^2(\partial X) \) into \( L^2(X) \) defined by

\[
T_j(a) := (2\pi)^{-(d-1)} \int_{\mathbb{R}^{d-1}} \hat{a}(\xi)v_j(\xi, z)e^{ix\xi} d\xi,
\]

with \( \hat{a}(\xi) := \int_{\mathbb{R}^{d-1}} a(x)e^{-ix\xi} dx \), satisfies:

\[
PT_j = T_j \text{Op}(\lambda_j),
\]

where \( P = -\text{div}(n \, \text{grad} u) \) with Dirichlet boundary conditions and \( \text{Op}(\lambda_j) \) is an elliptic pseudo-differential operator of degree 2 and of symbol \( \lambda_j \). So that, for each \( j = 1, \ldots \), we get an effective surface wave Hamiltonian with the Hamiltonian \( \lambda_j \). The map \( T : \oplus_{j=1}^\infty L^2(\partial X) \to L^2(X) \) given by \( T = \oplus_{j=1}^\infty T_j \) is an injective isometry.

\(^1\) \( u = u(x, z, t) \) is the pressure, \( n = K/\rho \) with \( \rho \) the density and \( K > 0 \) the incompressibility assumed to be a constant. The acoustic wave equation is a simplification of the elastic wave equation which holds if the medium is fluid.

\(^2\) In [2], we took a more complicated function \( n(x, z) = N(x, z/\varepsilon, z) \) with \( N \) smooth and \( \varepsilon \) small.
We see that the high frequency surface waves are associated to the semi-classical spectrum of a Schrödinger type operator

\[ \mathcal{L}_\hbar = -\hbar^2 \frac{d}{dz} \left( n(z) \frac{d}{dz} \right) + n(z), \]

with \( \hbar = \| \xi \|^{-1} \). One can try to recover \( n(z) \) from the propagation of surface waves: this is equivalent to get the operator \( \mathcal{L}_\hbar \) from its semi-classical spectrum.

4 Some notations

The following notations will be used everywhere in this paper. The interval \( I \) is defined by \( I = [a, b] \) with \(-\infty \leq a < b \leq +\infty \). The potential \( V : I = [a, b] \to \mathbb{R} \) is a smooth function with \(-\infty < E_0 := \inf V < E_\infty = \liminf_{x \to \partial I} V(x) \). We will denote by \( \hat{H} \) any self-adjoint extension of the operator \(-\hbar^2 \frac{d^2}{dx^2} + V(x)\) defined on \( C^\infty_0(I) \). The discrete spectrum of \( \hat{H} \) will be denoted by

\[ (E_0 <) \lambda_1(\hbar) < \lambda_2(\hbar) < \cdots < \lambda_l(\hbar) < \cdots. \]

The semi-classical limit is associated to the classical Hamiltonian \( H = \xi^2 + V(x) \) and the dynamics \( dx/dt = \xi, \; d\xi/dt = -V'(x) \).

Definition 4.1 We say that \( \mu_l(\hbar) \) is a semi-classical spectrum of \( \hat{H} \mod o(\hbar^N) \) in \([E_0, E]\) if, for any \( F < E \),

\[ \left( \sum_{\lambda_i(\hbar) \leq F} |\lambda_i(\hbar) - \mu_l(\hbar)|^2 \right)^{\frac{1}{2}} = o(\hbar^{N-\frac{1}{2}}). \]

If we have a uniform approximation of the eigenvalues up to \( o(\hbar^N) \), it is also a semi-classical spectrum of \( \hat{H} \mod o(\hbar^N) \) in the previous \( L^2 \) sense because the number of eigenvalues in \([-\infty, F]\) is \( O(\hbar^{-1}) \).

5 A Theorem for one well potentials

Theorem 5.1 Let us assume that the potential \( V : I \to \mathbb{R} \) satisfies:

1. A single well below \( E \): there exists \( E \leq E_\infty \) so that, for any \( y \leq E \), the sets \( I_y := \{ x \mid V(x) \leq y \} \) are connected. The intervals \( I_y \) are compact for \( y < E \). There exists a unique \( x_0 \) so that \( V(x_0) = E_0 \) (= \( \inf_{x \in I} V(x) \)). For any \( y \) with \( E_0 < y \leq E \), if the interval \( I_y \) is defined by \( I_y = [f_-(y), f_+(y)] \), we have \( V'(x_0) = 0, \; V'(x) < 0 \) for \( f_-(E) < x < x_0 \) and \( V'(x) > 0 \) for \( x_0 < x < f_+(E) \).
2. A genericity hypothesis at the minimum: there exists $N \geq 2$ so that the $N$-th derivative $V^{(N)}(x_0)$ does not vanish.

3. A generic symmetry defect: if there exists $x_\pm$, satisfying $f_-(E) < x_- < x_+ < f_+(E)$ and $\forall n \in \mathbb{N}$, $V^{(n)}(x_-) = (-1)^n V^{(n)}(x_+)$, then $V$ is globally even w.r. to $x_0 = (x_- + x_+)/2$ on the interval $I_E$. This is true for example if $V$ is real analytic.

Then the spectra modulo $o(h^2)$ in the interval $]-\infty, E[$ of the Schrödinger operators $\hat{H}_h$, for a sequence $h_j \to 0^+$, determine $V$ in the interval $I_E$ up to a symmetry-translation $V(x) \to V(c \pm x)$.

![Figure 1: The potential $V$ and the functions $f_+$ and $f_-$](image)

6 One well potentials: Bohr-Sommerfeld rules and a $\Psi DO$ trace formula

From [3], we know that the semi-classical spectrum (i.e. the spectrum up to $O(h^\infty)$) of $\hat{H}_h$ in the interval $]E_0, E[$ is given by

$$\Sigma(h) = \{y \mid E_0 < y < E \text{ and } S(y) \in 2\pi h \mathbb{Z}\}$$

where, for $E_0 < y < E$, the function $S$ admits the formal series expansion $S(y) \equiv S_0(y) + h\pi + h^2 S_2(y) + h^4 S_4(y) + \cdots$ (the formal series $S$ will be called the semi-classical action and the remainder term in the expansion is uniform in every compact sub-interval of $]E_0, E[$) with

- $S_0(y) = \int_{\gamma_y} \xi dx$ with $\gamma_y = \{(x, \xi)|H(x, \xi) = y\}$ oriented according to the classical dynamics and

$$\frac{dS_0}{dy}(y) = \int_{f_-(y)}^{f_+(y)} \frac{dx}{\sqrt{y - V(x)}}$$
is the period $T(y)$ of the trajectory of energy $y$ for the classical Hamiltonian $H$.

- If $t$ is the time parametrization of $\gamma_y$,

$$S_2(y) = -\frac{1}{12} \frac{d}{dy} \int_{\gamma_y} V''(x) dt ,$$

which can be rewritten as:

$$S_2(y) = -\frac{1}{12} \frac{d}{dy} \left( \int_{f^+(y)}^{f^-(y)} \frac{V''(x) dx}{\sqrt{y-V(x)}} \right).$$

- For $j \geq 1$, $S_{2j}(y)$ is a linear combination of expressions of the form

$$\left( \frac{d}{dy} \right)^n \int_{\gamma_y} P(V', V'', \cdots) dt ,$$

where $dt$ is the differential of the time on $\gamma_y$; outside the caustic set $dt = dx/2\xi$.

In what follows, we will use only $S_0$ and $S_2$. It will be convenient to relate the semi-classical action to the spectra by using the following trace formula:

**Theorem 6.1 (ΨDO trace formula)** Let $f \in C_\infty([-E_0, E])$ and $F(y) := -\int_y^\infty f(u) du$, we have, with $Z = T^*I$:

$$\text{Trace} F(\hat{H}) = \frac{1}{2\pi\hbar} \left( \int_Z F(H) dxd\xi + \hbar^2 \int_{E_0}^E f(y)(S_2(y) + \hbar^2 S_4(y) + \cdots) dy \right) + O(\hbar^\infty) .$$

This formula implies that $S_0$ and $S_2$ are determined by the semi-classical spectrum mod $o(\hbar^2)$ in $]-\infty, E[.$

This Theorem is closely related to (but a bit stronger) than what is proved in my paper [3]. The trace formula contains implicitly the Maslov index.

### 7 Two potentials with the same semi-classical spectra

We introduced a genericity Assumption 3 on symmetry defects in Theorem 5.1. The Figure 2 shows two one well potentials with the same semi-classical spectra mod $O(\hbar^\infty)$. The fact that they have the same semi-classical spectra comes from the description of Bohr-Sommerfeld rules in Section 6.

It would be nice to prove that they do NOT have the same spectra!
Figure 2: The (graphs of the) two potentials are the same in the sets II and III, they are mirror image of each other in I (green curve and dotted green curve), the potential is even in the set II.

8 One well potentials: the proof of Theorem 5.1

8.1 Some useful Lemmas

Lemma 8.1 The semi-classical spectra modulo $o(h^2)$ in $]E_0, E[$ determine the actions $S_0(y)$ and $S_2(y)$ for $y \in ]E_0, E[.$

It is a consequence of Theorem 6.1.

Lemma 8.2 If $V$ satisfies Assumption 2 in Theorem 5.1, we have:

$$\lim_{y \to E_0} \int_{\gamma_y} V''(x) dt = \pi \sqrt{2V''(x_0)}.$$  

This holds even if the minimum is degenerate.

The Lemma is clear if $V''(x_0) > 0$: the limit is then $V''(x_0)$ times the period of small oscillations of a pendulum which is $\pi / \sqrt{2V''(x_0)}$.

Let us consider the case of an isolated degenerate minimum with $V(x) = E_0 + a(x - x_0)^N(1 + o(1))$ ($a > 0, \ N > 2$), we can check that the integral to be evaluated is $O \left( (y - E_0)^{\frac{3}{2} - \frac{1}{N}} \right) = o(1)$.

\footnote{I do not know if this is still true without the genericity Assumption 2 in Theorem 5.1 it is the only place where I use it}
Lemma 8.3 We have
\[ \lim_{y \to 0} \left( \frac{1}{f'_+(y)} - \frac{1}{f'_-(y)} \right) = 0. \]

Lemma 8.4 If \( x_0 \) is the unique point where \( V(x_0) = \inf V = E_0 \), the first eigenvalue of \( \hat{H}_\hbar \) satisfies \( \lambda_1(\hbar) = E_0 + \hbar \sqrt{V''(x_0)/2} + o(\hbar) \).

This is well known if \( V''(x_0) > 0 \) and is still true otherwise by comparison: if \( E_0 \leq V(x) \leq A(x - x_0)^2 \) with \( A > 0 \), near \( x_0 \) then \( E_0 < \lambda_1(\hbar) \leq 2\pi \hbar \sqrt{A} \).

8.2 Rewriting \( V \) using \( F \) and \( G \)

We will denote by \( F = \frac{1}{2}(f_+ + f_-) \) and \( G = \frac{1}{2}(f_+ - f_-) \).

- The function \( F \) is smooth on \( ]E_0, E[ \), continuous on \( [E_0, E[ \) (smooth in the non degenerate case \( V''(x_0) > 0 \) as a consequence of the Morse Lemma), with \( F(E_0) = x_0 \), and is constant if and only if \( V \) is even w.r. to \( x_0 \). More generally, if \( F \) is constant on some interval, \( V \) is even on the inverse image of that interval. We call \( F \) the parity defect.

Lemma 8.5 Under the Assumption 3 in Theorem 5.1, the function \( F' \) is determined up to \( \pm \) by its square.

- The function \( G \) is smooth on \( ]E_0, E[ \), continuous at \( y = E_0 \). We have \( G(E_0) = 0 \). It is clear that, from \( F \) and \( G \), we can recover the restriction of \( V \) to \( I_E \).

8.3 How to get \( V \) from \( S_0 \) and \( S_2 \)

Let us consider, for \( E_0 < y < E \),
\[ I(y) := \int_{f_-(y)}^{f_+(y)} dx \frac{dx}{\sqrt{y - V(x)}} \]
and
\[ J(y) = \int_{f^-}^{f^+(y)} \frac{V''(x)dx}{\sqrt{y-V(x)}}. \]

We have \( I(y) = dS_0(y)/dy \) and \( S_2(y) = -(1/12)dJ(y)/dy \). This implies that \( S_0, S_2 \) and the limit \( J(E_0) \) determine \( I \) and \( J \). The limit \( J(E_0) \) is determined by \( V''(x_0) \) (Lemma 8.2) which is determined by the first semi-classical eigenvalue (Lemma 8.4). We can express \( I \) and \( J \) using \( F \) and \( G \). Using the change of variables \( x = f_+(u) \) for \( x > x_0 \) and \( x = f_-(u) \) for \( x < x_0 \), we get:

\[ I(y) = 4 \int_{E_0}^{y} \frac{G'(u)du}{\sqrt{y-u}} \]
\[ J(y) = \int_{E_0}^{y} \frac{d}{du} \left( \frac{1}{f'_+(u)} - \frac{1}{f'_-(u)} \right) \frac{du}{\sqrt{y-u}}. \]

Using Abel’s result [1] (and Appendix A), we can recover \( G' \) and

\[ \frac{d}{dy} \left( \frac{1}{f'_+(y)} - \frac{1}{f'_-(y)} \right) = \frac{d}{dy} \left( \frac{2G'}{G'^2 - F'^2} \right). \]

Using Lemma 8.3 we recover \( F'^2 \). The Assumption 3 implies that there exists an unique square root to \( F'^2 \) up to signs. From that we recover \( G' \) and \( \pm F' \) and hence \( \pm F \) and \( G \) modulo constants. This gives \( V \) up to change of \( x \) into \( c \pm x \).

9 Taylor expansions

From the previous section, we see that the semi-classical spectra determine \( F'^2 \) and \( G \) even without assuming the hypothesis 3 of Theorem 5.1 on symmetry defect. It is not difficult to see that, if \( V \) satisfies the hypothesis 2 of Theorem 5.1 the parity defect \( F \) is a smooth function of \( y^{2/N} \). We have the following:

**Lemma 9.1** Let us give two formal powers series \( a = \sum_{j=0}^{\infty} a_j t^j \) and \( b = \sum_{j=0}^{\infty} b_j t^j \) which satisfy \( a^2 = b \). The equation \( f^2 = b \) has exactly two solutions as formal powers series: \( f = \pm a \).

From this Lemma, we deduce the:

**Theorem 9.1** Under the Assumptions 1 and 2 of Theorem 5.1 but without Assumption 3, the Taylor expansion of \( V \) at a local minimum \( x_0 \) is determined (up to mirror symmetry) by the semi-classical spectrum modulo \( o(h^2) \) in a fixed neighbourhood of \( E_0 \).

In some aspects, this result is stronger than the one obtained in [4], but it requires the knowledge of the semi-classical spectrum in a fixed neighbourhood of \( E_0 \), while, in [4], we need only \( N \) semi-classical eigenvalues in order to get \( 2N \) terms in the Taylor expansion.
A Theorem for a potential with several wells

Figure 4: a 2 wells potential $V$

We will extend our main result to cases including that of Figure 4, a two wells potential with three critical values, $E_0 = 0$, $E_1$ and $E_2$. We can take any boundary condition at $x = 0$.

10.1 The genericity Assumptions

In what follows, we choose $E$ so that $E_0 < E \leq E_\infty$ and define $I_E = \{x | V(x) < E\}$. The goal is to determine the restriction of $V$ to $I_E$ from the semi-classical spectrum in $]-\infty, E]$.

We need the following Assumptions which are generically satisfied. We introduce a:

Definition 10.1 Two smooth functions $f, g : J \to \mathbb{R}$ are weakly transverse if, for every $x_0$ so that $f(x_0) = g(x_0)$, there exists an integer $N$ such that the $N$th derivative $(f - g)^{(N)}(x_0)$ does not vanish.

10.1.1 Assumption on critical points

- for any point $x_0$ so that $V'(x_0) = 0$ and $V(x_0) < E$, there exists $N \geq 2$ so that, the $N$-th derivative $V^{(N)}(x_0)$ does not vanish.

- The critical values associated to different critical points are distinct.

The wells: Let us label the critical values of $V$ below $E_\infty$ as $E_0 < E_1 < \cdots < E_k < \cdots < E_\infty$ and the corresponding critical points by $x_0$, $x_1$, \ldots. The critical values can only accumulate at $E_\infty$ because the critical points are isolated.
and hence only a finite number of them lies in \( \{ x | V(x) < E_\infty - c \} \) for any \( c > 0 \).

Let us denote, for \( k = 1, 2, \cdots \) by \( J_k = ]E_{k-1}, E_k[ \).

**Definition 10.2** A well of order \( k \) is a connected component of \( \{ x | V(x) < E_k \} \).

Let us denote by \( N_k \) the number of wells of order \( k \).

For any \( k \), \( H^{-1}(J_k) \) is an union of \( N_k \) topological annuli \( A_j^k \) and the map \( H : A_j^k \to J_k \) is a submersion whose fibers \( H^{-1}(y) \cap A_j^k \) are topological circles \( \gamma_j^k(y) \) which are periodic trajectories of the classical dynamics: if \( y \in J_k \), \( H^{-1}(y) = \bigcup_{j=1}^{N_k} \gamma_j^k(y) \). We will denote by \( T_j^k(y) = \int_{\gamma_j^k} dt \), the corresponding classical periods.

We will often remove the index \( k \) in what follows.

The semi-classical spectrum in \( J_k \) is the union of \( N_k \) spectra which are given by Bohr-Sommerfeld rules associated to actions \( S_j^k(y) \) given as in Section 6.

10.1.2 A generic symmetry defect

If there exists \( x_- < x_+ \), satisfying \( V(x_-) = V(x_+) < E \) and, \( \forall n \in \mathbb{N}, V^{(n)}(x_-) = (-1)^n V^{(n)}(x_+) \), then \( V \) is globally even on \( I_E \).

10.1.3 Separation of the wells

For any \( k = 1, 2, \cdots \) and any \( j \) with \( 1 \leq j < l \leq N_k \), the classical periods \( T_j(y) \) and \( T_l(y) \) are weakly transverse in \( J_k \). This is assumed to hold also at \( E_{k-1} \) if \( x_k \) is a local non degenerate minimum of \( V \) (in this case, the period of the new periodic orbit is smooth at \( (E_{k-1})_+ \)).

10.2 Quartic potentials

If \( V \) is a polynomial of degree four with two wells like \( V(x) = x^4 + ax^3 + bx^2 \) with \( b < 0 \), the periods of the two wells (between \( E_1 \) and \( E_2 \) (= 0)) are identical. This is because, on the complex projective compactification \( X_E \) (with \( E < 0 \)) of \( \xi^2 + V(x) = E \), the differential \( dx/\xi \) is holomorphic and the real part of \( X \) consists of 2 homotopic curves in \( X_E \). One can check directly that all other actions \( S_{2j} \), \( j \geq 1 \) coincide; this is also proved for example in [5] p. 191.

10.3 The statement of the result

Our result is:

**Theorem 10.1** Under the three Assumptions in Sections 10.1.1, 10.1.2 and 10.1.3, \( V \) is determined in the domain \( I_E : = \{ x | V(x) < E \} \) by the semi-classical spectrum in \( ]-\infty, E[ \) modulo \( o(h^4) \) up to the following moves: \( I_E \) is an union of open intervals \( I_{E,m} \), each interval \( I_{E,m} \) is defined up to translation and the restriction of \( V \) to each \( I_{E,m} \) is defined up to \( V(x) \to V(c-x) \).
Remark 10.1 We need $o(h^4)$ in the previous Theorem while we needed only $o(h^2)$ in the one well case. This is due to the way we are able to separate the spectra associated to the different wells.

11 The case of several wells: the proof of Theorem 10.1

11.1 What can be read from the Weyl's asymptotics?

Lemma 11.1 Under the Assumption 10.1.1, the singular (non smooth) points of the function $y \rightarrow A(y) = \int_{H(x,\xi) \leq y} dxd\xi$ are exactly the critical values $E_0, E_1, \cdots$ of $V$. Moreover,

- the function $A(y)$ is smooth on $]E_k - c, E_k]$, with $c > 0$, if and only if $x_k$ is a local minimum of $V$,
- From the singularity of $A(y)$ at $E_k$, one can read the value of $V''(x_k)$.

The function $A(y)$ is determined by the semi-classical spectrum, this is a consequence of the Weyl asymptotics:

$$\#\{\lambda_i(h) \leq y\} \sim \frac{A(y)}{2\pi h}.$$ 

This implies that the critical values $E_k$ of $V$ are determined by the semi-classical spectrum.

11.2 The scheme of the reconstruction

The proof is by “induction” on $E$.

We start by constructing the piece of $V$ where $V(x) \leq E_1$ using Theorem 5.1.

We want then to construct $V$ where $E_1 \leq V(x) \leq E_2$.

There are two cases:

1. $x_1$ is not an extremum: then we are able to extend the proof of Theorem 5.1 using the fact that we know, using Section 11.4, the limits of $\int_{y} V''(x)dt$ and $f_{\pm}'(y)$ as $y \rightarrow E_1^+$. We can reduce to an Abel transform starting from $E_1$ using

$$\int_{V(x) \leq y} = \int_{V(x) \leq E_1} + \int_{E_1 \leq V(x) \leq y}$$

where the first part is known from the knowledge of $V(x)$ in $\{x|V(x) \leq E_1\}$. 
2. \( x_1 \) is a local minimum: using the separation of spectra (Section 11.3) and Theorem 5.1 we can construct the 2 wells of order 2 if we know \( V''(x_1) \). But the estimate

\[
A(y) = A(E_1) + \pi \sqrt{2/V''(x_1)}(y - E_1) + a(y - E_1) + o(y - E_1)
\]

shows that the singularity of \( A(y) \) at \( y = E_1 \) determines \( V''(x_1) \).

We then proceed to the interval \([E_2, E_3]\). A new case arises when \( x_2 \) is a local maximum. Then we need to glue together the wells of order 2. This case works then as before.

### 11.3 Separation of spectra

![Figure 5: The primitive periods as functions of \( y \) for the Example of Figure 4](image)

Let us start with a:

**Lemma 11.2** Let us give some open interval \( J \) and assume that we have a function

\[
F(x) = \sum_{j=1}^{N} a_j(x)e^{iS_j(x)/\hbar}
\]

with the functions \( S_j \) and \( S_k \) weakly transverse for any \( j \neq k \). If for any compact interval \( K \subset J \), we have

\[
\int_{K} |F|^2(x)dx = o(1)
\]

then all \( a_j \)'s vanish identically.

If \( P = \text{Op}(p) \) with \( p \in C^\infty_o(T^*J) \), using the \( L^2 \) \( \hbar \)-uniform continuity of \( P \), we have

\[
PF(x) = \sum_{j=1}^{N} p(x, S'_j(x))a_j(x)e^{iS_j(x)/\hbar} = o(1).
\]
One sees that the $a_j$'s vanish by choosing $p$ in an appropriate way, i.e. supported near a point $(x_0, S'_{j_0}(x_0))$.

**Lemma 11.3** Let us consider the distributions $D_k(h)$ on $J_k$ defined by $D_k(h) = \sum_{\lambda(h) \in I_k} \delta(\lambda(h))$, then $D_k$ is microlocally in $T^*J_k$ a locally finite sum of WKB functions $D_{j,l}$ associated to the Lagrangian manifolds $t = lS'_j(y)$ with $j = 1, \ldots, N_k$ and $l \in \mathbb{Z}$. We have

$$D_{j,l} = \frac{1}{2\pi \hbar} e^{is_{j,h}(y)/\hbar} S'_{j,h}(y),$$

and

$$D_{j,l} = \frac{(-1)^l}{2\pi \hbar} e^{is_{j,0}(y)/\hbar} T_j(y) \left(1 + i\hbar S_{j,2}(y) + O(\hbar^2)\right),$$

with $S_{j,h} \equiv \sum_{k=0}^{\infty} \hbar^k S_{j,k}$ the semi-classical actions associated to the $j$-th well.

This is a formulation of the semi-classical trace formula (see Appendix C).

**Lemma 11.4** If $\mu_t(h)$ is a semi-classical spectrum modulo $o(h^4)$ and $\tilde{D}_k(h) = \sum_{\mu_t(h) \in I_k} \delta(\mu_t(h))$, then, for any pseudo-differential operator $P = \text{Op}_h(p)$, with $p \in C^\infty_o(T^*J_k)$, we have

$$\|P(D_k - \tilde{D}_k)\|_{L^2(J_k)} = o(h).$$

It is enough to prove it for $p = \chi(E)\rho(t)$ and then it is elementary because $P\delta(\lambda) = \hbar^{-1}\chi(\lambda)\rho((y - \lambda)/\hbar)$.

From the three previous Lemmas, it follows that, with Assumption [10.1.3] the spectrum in $J_k$ modulo $o(h^4)$ determine the periods $T_j(y)$ and the actions $S_{j,2}(y)$.

### 11.4 Limit values of some integrals

Using the trick of Section [8.3] we can use Abel’s result (Section [12.3]) once we know the following limits (or asymptotic behaviours) as $y \to E^+_j$ ($j = 0, 1, \cdots$):

- $f^j_{\pm}(y)$
- $\int_{H^{-1}(y)} V''(x)dt$ where $H = \xi^2 + V(x)$ is the classical Hamiltonian. Here $H^{-1}(y)$ is oriented so that $dt > 0$.
- $f^j_{\pm}(y)$

All of them are determined by the knowledge of $V$ in the set $\{x|V(x) \leq E_j\}$.

It is clear, except for the second one; we have:

**Lemma 11.5** Let us assume that $V$ satisfies Assumption 1 of Section [10.1]. If $E_j$ is a critical value of $V$ which is not a local minimum and $\tau(z) := \int_{H^{-1}(E_j+z)} V''(x)dt - \int_{H^{-1}(E_j-z)} V''(x)dt$, then $\lim_{z \to 0^+} \tau(z) = 0$. 

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Proof.–

We cut the integrals into pieces. One piece near each critical point and another piece far from them. Far from the critical points, the convergence is clear.

- **Local maximum:** let us take a critical point where \( V(x) = E_j - A(x - x_0)^{2N}(1 + o(1)) \) with \( N \geq 1 \) and \( A > 0 \). We use a smooth change of variable \( x = \psi(y) \) with \( \psi(0) = x_0 \) so that \( V(\psi(y)) = E_j - y^{2N} \). We are reduced to check that

\[
\lim_{\varepsilon \to 0^+} \left( \int_0^1 \frac{W(y)dy}{\sqrt{\varepsilon + y^{2N}}} - \int_{\varepsilon^{1/2N}}^1 \frac{W(y)dy}{\sqrt{y^{2N} - \varepsilon}} \right) = 0 ,
\]

assuming that \( W(y) = O(y^{2N-2}) \).

- **Other critical points:** let us take a critical point where \( V(x) = E_j + A(x - x_0)^{2N+1}(1 + o(1)) \) with \( N \geq 1 \) and \( A > 0 \). We use the same method.

\[
\square
\]

12 Extensions to other operators

12.1 The statement

Let us indicate in this Section how to extend the previous results to the operator

\[
L_h = -h^2 \frac{d}{dx} \left( n(x) \frac{d}{dx} \right) + n(x)
\]

which was found in Section 3. We want to recover the function \( n(x) \). Let us sketch the one well case for which we will get:

**Theorem 12.1** Assuming that

- the function \( n(x) \) admits a non degenerate minimum \( n(x_0) = E_0 > 0 \),
- the function \( n(x) \) has no critical values in \( ]E_0, E_1[ \) with \( E_1 \leq \lim \inf_{x \to \partial I} n(x) \),
- the function \( n(x) \) has a generic symmetry defect as in Theorem 5.1,

then the function \( n \) is determined in \( \{ x | n(x) \leq E_1 \} \) by the semi-classical spectrum of \( L_h \) modulo \( o(h^2) \).

The proof works along the same lines as that of Theorem 5.1 except that we get an integral transform which is not exactly Abel’s transform.
12.2 The Weyl symbol and the actions

The Weyl symbol $l$ of $L$ can be computed, using the Moyal product, as $l = \xi \ast n \ast \xi + n$. We get:

$$l(x, \xi) = n(x)(1 + \xi^2) + \frac{\hbar^2}{4} n''(x) .$$

The action $S_0$ satisfies:

$$\frac{dS_0}{dy}(y) = T(y) = \int_{n(x) \leq y} \frac{dx}{\sqrt{n(x)(y - n(x))}} .$$

The action $S_2$ is given from [3] by

$$S_2(y) = -\frac{1}{12} \frac{d}{dy} \int_{\gamma_y} \left( y n'' - 2 \left( \frac{y}{n} - 1 \right) n'^2 \right) dt - \frac{1}{4} \int_{\gamma_y} n'' dt ,$$

which we rewrite:

$$S_2(y) = -\frac{1}{12} \frac{d}{dy} J(y) - \frac{1}{4} K(y) .$$

- The integral $J$:

$$J(y) = \int_{x_+(y)}^{x_-(y)} \left( y n'' - 2 \left( \frac{y}{n} - 1 \right) n'^2 \right) \frac{dx}{n(y - n)}$$

Using $x = f_{\pm}(y)$ as in Section [3] and

$$\Phi(y) = \frac{1}{f_{\pm}'(y)} - \frac{1}{f_{\pm}(y)} ,$$

we get $J(y) = (\mathcal{J}\Phi)(y)$, with

$$(\mathcal{J}\Phi)(y) = \int_{E_0}^{y} \left( y \Phi'(u) - 2 \left( \frac{y}{u} - 1 \right) \Phi(u) \right) \frac{du}{\sqrt{u(y - u)}} .$$

- The integral $K$:

$$K(y) = \int_{E_0}^{y} \Phi'(u) \frac{du}{\sqrt{u(y - u)}}$$

and

$$K(y) = 2 \frac{d}{dy} \int_{E_0}^{y} \Phi'(u) \frac{\sqrt{y - u}}{\sqrt{u}} du$$

which is rewritten as:

$$K(y) = 2 \frac{d}{dy} (\mathcal{K}\Phi)(y) .$$
12.3 An integral transform

Lemma 12.1 If $0 < E_0 < E_1$, the kernel of $A := J + 6K$ on the space of continuous function on $[E_0, E_1]$ at most two dimensional and all functions in this kernel are smooth.

Proof.– we have

$$A\Phi(y) = \int_{E_0}^{y} \left( (7y - 6u)\Phi(u) - 2 \left( \frac{y}{u} - 1 \right) \Phi(u) \right) \frac{du}{\sqrt{u(y - u)}}. \quad (3)$$

We compute $T \circ A$ with the operator $T$ defined by $T\psi(y) = \int_{E_0}^{y} \frac{\psi(u)du}{\sqrt{y - u}}$. We will need the easy:

Lemma 12.2 We have:

$$\int_{E_0}^{y} \frac{udu}{\sqrt{y - u}} \int_{E_0}^{u} f(t) \frac{dt}{\sqrt{u - t}} = \frac{\pi}{2} \int_{E_0}^{y} (t + y) f(t) dt,$$

and

$$\int_{E_0}^{y} \frac{du}{\sqrt{y - u}} \int_{E_0}^{u} f(t) \frac{dt}{\sqrt{u - t}} = \pi \int_{E_0}^{y} f(t) dt.$$

Applying the previous formulae, we get:

$$T \circ A \Phi(y) = \frac{\pi}{2} \int_{E_0}^{y} [(t + y)(7\Phi'(t) - 2\frac{\Phi(t)}{t}) + 2(-6t\Phi'(t) + 2\Phi(t))] \frac{dt}{\sqrt{t}}.$$

Taking two derivatives:

$$\frac{\pi}{y^{3/2}} \frac{d^2}{dy^2} ((T \circ A) \Phi)(y) = y^2 \Phi''(y) + 4y \Phi'(y) - \Phi(y).$$

From $S_2$ and $A\Phi(E_0)$, we get $A\Phi$, then we get $P(\Phi)$ where $P\phi = y^2 \phi'' + 4y \phi' - \phi$ is a non singular linear differential equation (remind that $E_0 > 0$). So, if we know also $\Phi(E_0)$ and the asymptotic behaviour of $\Phi'(E_0)$, we can get $\Phi$. Let us assume $n''(x_0) = a > 0$. Then we have:

- $A\Phi(E_0) = 2\pi \sqrt{aE_0}$
- $\Phi(E_0) = 0$
- $\Phi'(y) \sim 4\sqrt{a}/\sqrt{y - E_0}$.

\[\square\]
Appendix A: Abel’s result

Let us consider the linear operator $T$ which acts on continuous functions on $[E_0, E]$ defined by:

$$Tf(x) = \int_{E_0}^{x} \frac{f(y)dy}{\sqrt{x-y}}.$$  

Then $T^2f(x) = \pi \int_{E_0}^{x} f(y)dy$. This implies that $T$ is injective! This is the content of [1].

Appendix B: a proof of the $\Psi DO$ trace formula of Section 6

For this Section, one can read [6]. This can be seen as a complement and a partial rewriting of my paper [3] with a better trace formula. The formula we will prove is more general than that in Section 6. It is valid even for several wells. Let us state it:

**Theorem 12.2** Let $f \in C_0^\infty(J_k)$ and $F(y) := -\int_y^{\infty} f(u)du$, we have, with $Z = T^*I$, modulo $O(h^\infty)$:

$$\text{Trace} F(\hat{H}) \equiv \frac{1}{2\pi \hbar} \left( \int_Z F(H) dx d\xi + h^2 \int_{J_k} f(y) \left( \sum_{j=1}^{N_k} (S_{2,j}^k(y) + h^2 S_{4,j}^k(y) + \cdots) \right) dy \right).$$

**Proof.**

1. **Reduction to $N_k = 1$:** we can decompose both the lefthandside and the righthandside according to the $N_k$ wells: for the lhs, it uses the fact that the classical spectrum splits into $N_k$ parts; for the rhs, it is enough to decompose the first integral terms according to the connected component of $H < E_k$.

2. **Reduction from $N_k = 1$ to one well:** the whole Moyal symbol of $F(\hat{H})$ is $\equiv F(E_0)$ in $\{ H \leq E_{k-1} \}$.

3. **The harmonic oscillator case ($\hat{H} = \Omega$):**

$$\text{Trace} F(\Omega) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \tilde{F} \left( (n + \frac{1}{2}) \hbar \right)$$

with $\tilde{F}$ even and coinciding with $F$ on the positive axis. We get with Poisson summation formula:

$$\text{Trace} F(\Omega) = \frac{1}{2\pi \hbar} \int \int F \left( \frac{x^2 + \xi^2}{2} \right) dx d\xi + O(h^\infty).$$
4. The case where $F$ is compactly supported: using Poisson summation formula as in [3], we get

$$\text{Trace} F(\hat{H}) = \frac{1}{2\pi\hbar} \int F(y)S'(y)dy$$

and we get this case by integration by part.

5. The final step: we can assume that $H = \frac{(x-x_0)^2 + \xi^2}{2} + E_0$ near $(x_0,0)$ and we split $F = F_0 + F_1$ where

$$F_0(H) \equiv F_1 \left( \frac{(x-x_0)^2 + \xi^2}{2} + E_0 \right).$$

The formula then follows from the two particular cases computed before.

For the convenience of the reader, we regive also the way to get $S_2$ from the Moyal formula.

Defining $F^*(H)$ by $F(\hat{H}) = \text{Op}_\text{Weyl}(F^*(H))$ we know that, with $z_0 = (x_0, \xi_0)$ and $H_0 = H(z_0),

$$F^*(H)(z_0) = F(H_0) + \frac{1}{2}F''(H_0)(H-H_0)^2(z_0) + \frac{1}{6}F'''(H_0)(H-H_0)^3(z_0) + O(h^4).$$

Computing the Moyal powers of $H - H_0$ at the point $z_0 \mod O(h^4)$, gives

$$F^*(H) = F(H) - \hbar^2 \left( \frac{1}{8}f'(H)\det(H'') + \frac{1}{24}f''(H)H''(X_H, X_H) \right) + O(h^4).$$

If $\alpha = \iota(X_H)H''$, we have $d\alpha = 2\det(H'')d\xi \wedge dx$, and we get, by Stokes and with $\gamma_y$ oriented according to the dynamics:

$$\int_{\gamma_y} \alpha = 2 \int_{H \leq y} \det(H'')dx d\xi$$

and the final result for $S_2(y)$ using an integration by part and the formula $dtdy = dxd\xi$:

$$S_2(y) = -1/24 \int_{\gamma_y} \det(H'')dt.$$ 

Appendix C: the semi-classical trace formula

In this Section, we want to give a proof of Lemma 11.3.
We want to evaluate mod $O(h^\infty)$ the sums:

$$D(y) := \frac{1}{h} \sum_{l \in \mathbb{Z}} \rho \left( \frac{y - S^{-1}(2\pi l)}{h} \right),$$

where $S : \mathbb{R} \to \mathbb{R}$ is an extension to $\mathbb{R}$ of the given function $S_j$ on $\Delta$ which is $\equiv \text{Id}$ near infinity. This is the equal to $D_{\Delta, \rho}(y)$ up to $O(h^\infty)$. Using the Poisson summation formula and defining

$$F_y(x) = \int_{\mathbb{R}} \rho \left( \frac{y - S^{-1}(hy)}{h} \right) e^{-ixy} dy,$$

we get

$$D(y) = \frac{1}{2\pi h} \sum_{m \in \mathbb{Z}} F_y(m). \quad (4)$$

Using the change of variable, $y - S^{-1}(hy) = hz$ or $y = S(y - hz)/h$, we get:

$$F_y(x) = \int \rho(z) e^{-ixS(y-hz)/h} S'(y-hz) dz.$$

Using the fact that all moments of $\rho$ vanish and Taylor expanding $S(y - hz)$ w.r. to $h$, we get

$$F_y(x) = e^{-ixS(y)/h} S'(y) \hat{\rho}(-xS'(y)) + O(h^\infty).$$

If the support of $\hat{\rho}$ is close enough to $S'(y)$, we get the final answer taking the contribution of $m = -1$ to Equation (4). This way, we get the formula of Lemma 11.3.

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