A NOTE ON THE CLASSIFICATION OF GAMMA FACTORS

ROMÁN SASYK

Abstract. One of the earliest invariants introduced in the study of finite von Neumann algebras is the property \(\Gamma\) of Murray and von Neumann. In this note we prove that it is not possible to classify separable II\(_1\) factors satisfying the property \(\Gamma\) up to isomorphism by a Borel measurable assignment of countable structures as invariants. We also show that the same holds true for the full II\(_1\) factors.

1. Introduction

In this note we continue with the line of research initiated by the author in collaboration with A. Törnquist in \cite{19}, \cite{20} and \cite{21}, where we applied the notion of Borel reducibility from descriptive set theory to study the complexity of the classification problem of several different classes of separable von Neumann algebras.

Recall that if \(E\) and \(F\) are equivalence relations on standard Borel spaces \(X\) and \(Y\), respectively, we say that \(E\) is Borel reducible to \(F\) if there is a Borel function \(f: X \to Y\) such that

\[(\forall x, x' \in X) x Ex' \iff f(x)Ff(x'),\]

and if this is the case we write \(E \leq_B F\). Thus if \(E \leq_B F\) then the points of \(X\) can be classified up to \(E\) equivalence by a Borel assignment of invariants that we may think of as \(F\)-equivalence classes. \(E\) is smooth if it is Borel reducible to the equality relation on \(\mathbb{R}\). While smoothness is desirable, it is most often too much to ask for. A more generous class of invariants which seems natural to consider are countable groups, graphs, fields, or other countable structures, considered up to isomorphism. Thus, following \cite{14}, we will say that an equivalence relation \(E\) is classifiable by countable structures if there is a countable language \(\mathcal{L}\) such that \(E \leq_B \simeq_{\text{Mod}(\mathcal{L})}\), where \(\simeq_{\text{Mod}(\mathcal{L})}\) denotes isomorphism in \(\text{Mod}(\mathcal{L})\), the Polish space of countable models of \(\mathcal{L}\) with universe \(\mathbb{N}\).

In \cite{20} it was proved that the isomorphism relation in the set of finite von Neumann algebras is not classifiable by countable structures. Nonetheless, it can certainly be the case that some subclasses of finite factors are possible to classify by countable structures. For instance, Connes’ celebrated Theorem \cite{3}, says that the set of infinite dimensional injective finite factors has only one element on its isomorphism class, namely the hyperfinite II\(_1\) factor \(R\). In contrast with the injective case, in this note we show that it is not possible to obtain a reasonable classification up to isomorphisms for a well studied family of finite factors that includes \(R\). In order to state our results we observe first that the set of finite factors can be split in

\[2000\] Mathematics Subject Classification. 46L36; 03E15; 37A15.

Key words and phrases. von Neumann algebras; descriptive set theory; Gamma factors.

The author acknowledges support from the following grants: PICT 2012-1292 (ANPCyT), and UBACyT 2011-2014 (UBA).
two disjoint subsets: those who satisfy the property $\Gamma$ of Murray and von Neumann and those who are full. The first set contains the hyperfinite $II_1$ factor $R$ and more generally, the class of McDuff factors, i.e. those factors of the form $M \otimes R$ for $M$ a $II_1$ factor. On the other hand the set of full factors contains the free group factors $L(\mathbb{F}_n)$. In this article we show that the $II_1$ factors constructed in [20] are full. As a consequence, Theorem 7 in [20] strengthens to prove:

**Theorem 1.1.** The isomorphism relation for full type $II_1$ factors is not classifiable by countable structures.

It remained then to analyze the complexity of the classification of $II_1$ factors with the property $\Gamma$. In this note we address this problem by showing that:

**Theorem 1.2.** The isomorphism relation for McDuff factors is not classifiable by countable structures.

An immediate consequence is:

**Corollary 1.3.** The isomorphism relation for type $II_1$ factors satisfying the property $\Gamma$ of Murray and von Neumann is not classifiable by countable structures.

We end this introduction by mentioning that the study of the connections between logic and operator algebras has recently attracted many researchers from both fields. As a consequence, in the past five years there has been a burst of activity in proving results along the lines of the ones presented in this note and first unveiled in [19], [20] and [21]. We refer the reader who wants to learn more on these exciting new developments to the recent survey of I. Farah [9].

2. **Gamma factors**

We start by recalling the definitions of the objects we study in this article. Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space and denote by $\mathcal{B}(\mathcal{H})$ the space of bounded operators on $\mathcal{H}$, which we give the weak topology. A separable von Neumann algebra is a weakly closed self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. The set of von Neumann algebras acting on $\mathcal{H}$ is denoted $vN(\mathcal{H})$. A von Neumann algebra $M$ is said to be finite if it admits a finite faithful normal tracial state, i.e. a linear functional $\tau : M \to \mathbb{C}$ such that: $\tau(x^*x) \geq 0$, $\tau(x^*x) = 0$ iff $x = 0$, $\tau(1) = 1$, $\tau(xy) = \tau(yx)$ and the unit ball of $M$ is complete with respect to the norm given by the trace $\|x\|_2 = \tau(x^*x)$. If a finite von Neumann algebra is also a factor, i.e. its center is trivial, then it has a unique such a trace. A finite von Neumann factor that is not a matrix algebra is called a type $II_1$ factor. This terminology is due to the general classification of von Neumann algebras according to types (see [4, Chapter 5.1] for an historical account of the theory of types).

In this note we will be interested in $II_1$ factors arising from the so called group-measure space construction, that we proceed to describe. For that, let $G$ be a countably infinite discrete group which acts in a measure preserving way on a Borel probability space $(X, \mu)$. For each $g \in G$ and $\zeta \in L^2(X, \mu)$ the formula

$$\sigma_g(\zeta)(x) = \zeta(g^{-1} \cdot x)$$

defines a unitary operator on $L^2(X, \mu)$.

We identify the Hilbert space $\mathcal{H} = L^2(G, L^2(X, \mu))$ with the Hilbert space of formal sums $\sum_{g \in G} \zeta_g \xi_g$, where the coefficients $\zeta_g$ are in $L^2(X, \mu)$ and satisfy
\[\sum_{g} \|\zeta_g\|_{L^2(X,\mu)}^2 < \infty,\] and \(\zeta_g\) are indeterminates indexed by the elements of \(G\). The inner product on \(\mathcal{H}\) is given by
\[
\langle \sum_{g \in G} \zeta_g(x)\xi_g, \sum_{g \in G} \zeta'_g(x)\xi'_g \rangle = \sum_{g \in G} \langle \zeta_g, \zeta'_g \rangle_{L^2(X,\mu)}.
\]
Both \(L^\infty(X,\mu)\) and \(G\) act by left multiplication on \(\mathcal{H}\) by the formulas
\[
f(\zeta_g(x)\xi_g) = ((f(x)\zeta_g(x))\xi_g,
\]
\[
u_h(\zeta_g(x)\xi_g) = \sigma_h(\zeta_g(x))\xi_{hg},
\]
where \(f \in L^\infty(X,\mu), \zeta_g(x) \in L^2(X,\mu)\) and \(g, h \in G\). Thus if we denote by \(\mathcal{FS}\) the set of finite sums,
\[
\mathcal{FS} = \{ \sum_{g \in G} f_g u_g : f_g \in L^\infty(X,\mu), f_g = 0, \text{except for finitely many } g \},
\]
then each element in \(\mathcal{FS}\) defines a bounded operator on \(\mathcal{H}\). Moreover, multiplication and involution in \(\mathcal{FS}\) satisfy the formulas
\[
(f_g u_g)(f_h u_h) = f_g \sigma_g(f_h) u_{gh}
\]
and
\[
(f u_g)^* = \sigma_{g^{-1}}(f^*) u_{g^{-1}}
\]
and so \(\mathcal{FS}\) is a \(*\)-algebra. By definition, the \textit{group-measure space von Neumann algebra} is the weak operator closure of \(\mathcal{FS}\) on \(\mathcal{B}(\mathcal{H})\) and it is denoted by \(L^\infty(X,\mu)\rtimes_\sigma G\). The trace on \(\mathcal{FS}\), defined by
\[
\tau(\sum_{g \in G} f_g u_g) = \int_X f \, d\mu,
\]
extends to a faithful normal tracial state in \(L^\infty(X)\rtimes_\sigma G\) by the formula \(\tau(T) = \langle T(\xi_e), \xi_e \rangle\), where \(e\) represents the identity of \(G\).

**Definition 2.1** (Murray-von Neumann [15]). A finite von Neumann algebra \(M\) has the property \(\Gamma\) if given \(x_1, \ldots, x_n \in M\), and \(\varepsilon > 0\) there exists \(u \in \mathcal{U}(M), \tau(u) = 0\) such that
\[
\|x_i u - u x_i\|_2 < \varepsilon, \text{ for all } 1 \leq i \leq n.
\]

It follows immediately from its definition that the hyperfinite \(II_1\) factor \(R\) is a \(\Gamma\)-factor. Moreover, it is clear that any finite factor of the form \(M \otimes N\) with \(N\) a \(\Gamma\)-factor is also a \(\Gamma\)-factor. In particular, \textit{McDuff factors}, i.e., factors of the form \(M \otimes R\), are \(\Gamma\)-factors. The paradoxical decomposition of the free groups \(F_n, n \geq 2\) is the key ingredient [15 Lemmas 6.2.1, 6.2.2] to show that the corresponding group von Neumann factors \(L(F_n), n \geq 2\) do not have the property \(\Gamma\). More generally, Effros showed in [8] that if \(G\) is a discrete ICC group and \(L(G)\) has the property \(\Gamma\), then \(G\) is inner amenable, (so in particular, free groups are not inner amenable). That the converse of Effros’ theorem is false is a recent result of Vaes [21].

If \(M\) is finite von Neumann algebra with trace \(\tau\), \(\text{Aut}(M, \tau)\), the set of \(\tau\)-preserving automorphisms of \(M\) is a Polish group. A basis for that topology is given by the sets \(\mathcal{V}_{T,u_1,\ldots,u_n,\varepsilon} = \{ S \in \text{Aut}(M, \tau) : \|S(a_i) - T(a_i)\|_2 \leq \varepsilon, \forall 1 \leq i \leq n \} \).

\(^1\text{ICC stands for infinite conjugacy classes. } G \text{ is ICC if and only if } L(G) \text{ is a factor.}\)
\( u, \ a_i \in M \). \( \text{Inn}(M) \) denotes the set of inner automorphisms of \( M \), i.e. those of the form \( \text{Ad}(u), \ u \in \mathcal{U}(M) \).

**Definition 2.2** (Connes [2]). A finite von Neumann algebra \( M \) is full if \( \text{Inn}(M) \) is closed in \( \text{Aut}(M) \).

In [2, Corollary 3.8], Connes showed that a II\(_1\) factor \( M \) is full if and only if \( M \) does not have the property \( \Gamma \). It follows that for each \( n \geq 2 \), \( L(F_n) \) is a full factor.

In order to discern when group measure space von Neumann algebras are full we need the following:

**Definition 2.3** (Schmidt [22]). Let \( G \) be a discrete group and let \( \sigma \) be an ergodic measure preserving action of \( G \) on a probability space \( (X, \mu) \). A sequence \( (B_n)_{n \in \mathbb{N}} \) of measurable subsets of \( X \) is asymptotically invariant if \( \mu(B_n \triangle \sigma_g(B_n)) \to 0 \), for all \( g \in G \).

The sequence is trivial if \( \mu(B_n)(1 - \mu(B_n)) \to 0 \).

The action \( \sigma \) is strongly ergodic if every asymptotically invariant sequence is trivial.

The relation between strong ergodicity and fullness has been studied by several authors. For the purpose of this note, it is enough to mention the following Theorem of Choda [1]:

**Theorem 2.4.** Let \( G \) be a discrete group that is not inner amenable, and let \( \sigma \) be a strongly ergodic measure preserving action of \( G \) on a probability space \( (X, \mu) \). Then \( L^\infty(X, \mu) \rtimes_\sigma G \) is a full factor.

**Remark 2.5.** The condition that is really used in the proof of Theorem 2.4 is that \( L(G) \) is full.

It is known that a group is amenable if and only if it does not admit strongly ergodic actions [22], while a group has the property (T) of Kazhdan if and only if every m.p. ergodic action of it is strongly ergodic [6]. We describe now a concrete example of a strongly ergodic action of \( F_2 \) that we will use in this work. Since \( F_2 \) can be identified with the finite index subgroup of \( SL(2, \mathbb{Z}) \) generated by the matrices \( \left\{ \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\} \) (see [7, II.B.25]), it follows that \( F_2 \) naturally acts on \( T^2 \). Let denote such an action by \( \sigma \) and by \( T_a, T_b \) the automorphisms corresponding to the generators \( a, b \) of \( F_2 \). This action is clearly measure preserving, and one of the main results in [22] is that \( \sigma \) is strongly ergodic. Inspired by earlier work of Gaboriau and Popa and \( \text{Törnquist} \) ([12], [23]), in [20] we used this action as the starting point for showing that II\(_1\) factors are not classifiable by countable structures. More precisely, the set

\[
\text{Ext}(\sigma) = \{ S \in \text{Aut}(T^2, \mu) : T_a, T_b, S \text{ generates a free action of } F_3 \}
\]

was shown in [23, §3] to be a dense \( G_\delta \) subset of \( \text{Aut}(T^2, \mu) \). Thus \( \text{Ext}(\sigma) \) is a standard Borel space. For each \( S \in \text{Ext}(\sigma) \) denote by \( \sigma_S \) the corresponding \( F_3 \) action and \( M_S \in \text{vN}(L^2(F_3, L^2(T^2, \mu))) \) the corresponding group-measure space von Neumann algebra

\[
M_S = L^\infty(T^2) \rtimes_{\sigma_S} F_3.
\]
Theorem 2.6. The equivalence relation on \( \text{Ext}(\sigma) \) given by \( S \simeq_{\mathcal{F}_1} S' \) if \( M_S \) is isomorphic to \( M_{S'} \) is not classifiable by countable structures.

In [20] it was shown that \( S \to M_S \) is a Borel map from \( \text{Ext}(\sigma) \) to \( \nu N(L^2(\mathbb{F}_3, L^2(\mathbb{T}^2, \mu))) \).

Thus Theorem 1.1 is an immediate consequence of the previous theorem and of the next:

Lemma 2.7. For each \( S \in \text{Ext}(\sigma) \), \( M_S \) is a full factor.

Proof. Let \( (B_n)_{n\in\mathbb{N}} \) be an asymptotically invariant sequence for the \( \mathbb{F}_2 \)-action \( \sigma_S \). Then \( (B_n)_{n\in\mathbb{N}} \) is an asymptotically invariant sequence for the action restricted to the subgroup generated by \( \{T_a, T_b\} \). By construction, this is the \( \mathbb{F}_2 \)-action \( \sigma \) described above, thus it is strongly ergodic by [22, §4]. It follows that \( (B_n)_{n\in\mathbb{N}} \) is trivial and then \( \sigma_S \) is strongly ergodic. Since \( \mathbb{F}_3 \) is not inner amenable, the result now follows from Theorem 2.4. \( \square \)

In order to prove Theorem 1.2 we require the following Theorem of Popa ([18, Theorem 5.1]):

Theorem 2.8. If \( M_1 \) and \( M_2 \) are full type \( \Pi_1 \) factors such that \( M_1 \otimes R \) is isomorphic to \( M_2 \otimes R \) then there exists \( t \in \mathbb{R}_{>0} \) such that \( M_1 \) is isomorphic to \( M_1^t \).

Remark 2.9. By interchanging the roles of \( M_1 \) and \( M_2 \) one can assume that \( t \in (0, 1] \). In which case \( M_1^t \) is by definition the type \( \Pi_1 \) factor \( pM_2p \) where \( p \in \mathcal{P}(M_2) \) is any projection of trace equal to \( t \) in \( M_2 \).

Theorem 2.10. The assignment \( M_S \to M_S \otimes R \) is a Borel reduction of \( \simeq_{\mathcal{F}_1} \) to isomorphism of McDuff factors.

Proof. It is fairly straightforward to prove that the map \( M_S \to M_S \otimes R \) is a Borel assignment (see for instance [13, Corollary 3.8]). We are left to show that if \( M_S \otimes R \) is isomorphic to \( M_S' \otimes R \), then \( M_S \) is isomorphic to \( M_S' \).

Let us fix \( S, S' \in \text{Ext}(\sigma) \). Lemma 2.7 shows that \( M_S \) and \( M_S' \) are full factors. By Theorem 2.8 \( M_S \otimes R \) is isomorphic to \( M_S' \otimes R \) if and only if there exists \( t > 0 \) such that \( M_{S'} \) is isomorphic to \( (M_S)^t \). The proof is over once we show that \( t = 1 \).

For this we make use of the celebrated theorem of Popa on \( \Pi_1 \) factors with trivial fundamental group [17, 16]. (see also Connes’s account in the Bourbaki Séminaire [5]). Indeed by [16 Proposition], there exists a projection \( p \in \mathcal{P}(L^\infty(\mathbb{T}^2)), \tau(p) = t \), such that the inclusion of von Neumann algebras \( (L^\infty(\mathbb{T}^2) \subset L^\infty(\mathbb{T}^2) \rtimes_{\sigma_S} \mathbb{F}_3) \) is isomorphic to the inclusion of von Neumann algebras \( pL^\infty(\mathbb{T}^2) \subset p (L^\infty(\mathbb{T}^2) \rtimes_{\sigma_S} \mathbb{F}_3) \).

Feldman-Moore’s Theorem [10] applies to conclude that the action \( \sigma_S \) is stable orbit equivalent to the action \( \sigma_{S'} \), with compression constant \( c = t \). Since \( \mathbb{F}_3 \) has non trivial Atiyah’s \( \ell^2 \)-betti numbers, Gaboriau’s Theorem on \( \ell^2 \)-betti numbers for orbit equivalence relations [11, Theorem 3.12] then implies that \( t = 1 \). \( \square \)

Proof of Theorem 1.3. Since \( M_S \to M_S \otimes R \) is a Borel reduction of \( \simeq_{\mathcal{F}_1} \) to isomorphism of McDuff factors and the equivalence relation \( \simeq_{\mathcal{F}_1} \) is not classifiable by countable structures, it follows that the equivalence relation of isomorphism of McDuff factors is not classifiable by countable structures. \( \square \)
References

1. M. Choda, *Inner amenability and fullness*, Proceedings of the American Mathematical Society 86 (1982), no. 4, 663–666.
2. A. Connes, *almost periodic states and factors of type III₁*, Journal of Functional Analysis 16 (1974), 415–445.
3. ———, *Classification of injective factors, cases III₁, II∞, III₁, λ ≠ 1*, Annals of Mathematics 104 (1976), 73–115.
4. ———, *Noncommutative geometry*, Academic Press, 1994.
5. A. Connes, *Nombres de Betti L² et facteurs de type II₁* (d’après D. Gaboriau et S. Popa), Astérisque (2004), no. 294, ix, 321–333.
6. A. Connes and B. Weiss, *Property T and asymptotically invariant sequences*, Israel Journal of Mathematics 37 (1980), 209–210.
7. P. de la Harpe, *Topics in geometric group theory*, Chicago Lectures in Mathematics, University of Chicago Press, 2000.
8. E. Effros, *Property T and inner amenability*, Proceedings of the American Mathematical Society 47 (1975), 483–486.
9. I. Farah, *Logic and operator algebras*, Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II, Kyung Moon Sa, Seoul, 2014, pp. 15–39.
10. J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras I and II*, Trans. Amer. Math. Soc. 234 (1977), no. 2, 289–359.
11. D. Gaboriau, *Invariants L² de relations d équivalence et de groupes*, Publ. math., Inst. Hautes Études Sci 95 (2002), 93–150.
12. D. Gaboriau and S. Popa, *An uncountable family of nonorbit equivalent actions of Fₙ*, Journal of the American Mathematical Society 18 (2005), 547–559.
13. U. Haagerup and C. Winsløw, *The Effros-Maréchal topology in the space of von Neumann algebras I*, American Journal of Mathematics 120 (1998), no. 3, 567–617.
14. G. Hjorth, *Classification and orbit equivalence relations*, Mathematical Surveys and Monographs, vol. 75, American Mathematical Society, 2000.
15. F. Murray and J. von Neumann, *On rings of operators, IV*, Annals of Mathematics (2) 44 (1943), 716–808.
16. S. Popa, *On the fundamental group of type II₁ factors*, Proceedings of the National Academy of Sciences 101 (2004), no. 3, 723–726.
17. ———, *On a class of type II₁ factors with Betti numbers invariants*, Annals of Mathematics 163 (2006), 809–899.
18. ———, *On Ozawa’s property for free group factors*, International Mathematics Research Notices. IMRN 11 (2007), Art. ID rnm036, 10.
19. R. Sasyk and A. Törnquist, *Borel reducibility and classification of von Neumann algebras*, Bulletin of Symbolic Logic 15 (2009), no. 2, 169–183.
20. ———, *The classification problem for von Neumann factors*, Journal of Functional Analysis 256 (2009), 2710–2724.
21. ———, *Turbulence and Araki-Woods factors*, Journal of Functional Analysis 259 (2010), 2238–2252.
22. K. Schmidt, *Asymptotically invariant sequences and an action of SL(2, Z) on the 2-sphere*, Israel Journal of Mathematics 37 (1980), 193–208.
23. A. Törnquist, *Orbit equivalence and actions of Fₙ*, Journal of Symbolic Logic 71 (2006), 265–282.
24. S. Vaes, *An inner amenable group whose von Neumann algebra does not have property Gamma*, Acta Mathematica 208 (2012), 389–394.

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Argentina

and

Instituto Argentino de Matemáticas-CONICET, Saavedra 15, Piso 3 (1083), Buenos Aires, Argentina

E-mail address: rasayk@dm.uba.ar