On propagation of Love waves in an infinite transversely isotropic poroelastic layer

C Nageswara Nath¹, J Manoj Kumar² and S Ahmed Shah³

¹ Department of Mathematics, CMR Technical Campus, Hyderabad - 501 401, India
² Department of Mathematics, BVRIT College of Engineering for Women, Hyderabad 500 090, India
³ Department of Mathematics, Deccan College of Engineering and Technology, Hyderabad 500 001, India

E-mail: nagesh.nath@rediffmail.com

Abstract. The present paper is devoted for the study of propagation of Love waves in a compressible viscous liquid layer of thickness $h_2$ and co-efficient of viscosity $\eta_l$ bounded by an infinite transversely isotropic poroelastic layer and a transversely poroelastic half-space. The equations of motion in transversely isotropic poroelastic solid have been formulated in the frame work of Biot’s theory. The secular equation has been obtained and is discussed in the limiting case $h_2 \to 0, \eta_l \to 0$ such that the ratio $h_2/\eta_l$ is a constant. Further, secular equation is observed for a case of the poroelastic layer is in welded contact with half-space when the thickness of liquid layer is zero for finite co-efficient of viscosity. Several other special cases have been discussed.

1. Introduction

The study propagation of Love waves has been a topic of interest to crustal, earthquake, and engineering seismologists for several years. Recently, exploration geophysicists have become interested in Love waves as a result of current activity in shear-wave exploration. Deresiewicz [1, 2] investigated the problem of propagation of Love waves in liquid-filled porous space. Bhangar [5] studied propagation of Love waves in a system composed of a compressible viscous liquid layer sandwiched between homogeneous, elastic layer and homogeneous isotropic half-space. Wave propagation along the interaction of a poroelastic layer and half-space is examined by Tajuddin and Ahmed [3]. Using Biot’s theory, Malla Reddy and Tajuddin [4] presented an analysis of waves propagating along the edges of a thin poroelastic flat plate of infinite length which is in a state of plane stress. Propagation of waves in a viscous liquid layer bounded by two poroelastic half-spaces is studied by Nageswara Nath et. al. [8]. Possible bonding between the poroelastic half-spaces is discussed by considering three different limiting forms of the secular equation and it is shown that these three forms respectively represent welded, smooth and loosely bonded interface of poroelastic half-spaces. It is observed that, this secular equation is same for all the three types of bonding for each permeable and impermeable surface. In the present analysis, propagation of Love waves in viscous liquid layer sandwiched between transversely isotropic poroelastic layer and half-space is studied in the frame of work Biot’s theory. The governing equations in transversely isotropic poroelastic layer have been derived. In what follows, a theory of wave propagation in a compressible viscous liquid layer of thickness...
h₂ and of shear viscosity coefficient η bounded between a poroelastic layer of thickness h₁ and a poroelastic half-space which are homogeneous and transversely isotropic is discussed in the absence of dissipation. A secular equation for Love waves has been derived. Several limiting forms of secular equation are obtained when (i) h₂ → 0, η → 0 such that h₂/η = P, a constant (ii) η is finite and h₂ → 0 (iii) η → 0 for finite thickness of h₂. It is observed that the limiting case (ii) represents Loves waves for a case of poroelastic layer in welded contact with poroelastic half-space and limiting case (iii) represents Love waves in an inviscid liquid layer of finite thickness in contact with transversely isotropic poroelastic layer.

2. Basic equations, formulation and solution of the problem
Let (x, y, z) rectangular coordinates. Consider a compressible viscous liquid layer of thickness h₂ and of shear viscosity coefficient η separating poroelastic layer of thickness h₁ and poroelastic half-space. Both of them are assumed to be homogeneous and transversely isotropic. The physical parameters of poroelastic layer and half-space are denoted by a superscript j (1, 2) enclosed in parentheses. The parameters with superscript (1), (2) refer to poroelastic layer and half-space, respectively. The parameters of liquid layer will be without any superscript. The boundaries of the viscous layer are given by z = h₁ and z = h₁ + h₂. The equations of motion of a homogeneous, transversely isotropic poroelastic solid in the presence of dissipation b are:

\[
\begin{align*}
\left[(A + 2N) \frac{\partial^2 u_x}{\partial x^2} + N \frac{\partial^2 u_x}{\partial y^2} + L \frac{\partial^2 u_x}{\partial z^2}\right] u_x + (A + 2N) \frac{\partial^2 u_y}{\partial x \partial y} + (F + L) \frac{\partial^2 u_z}{\partial x \partial y} \\
M \frac{\partial^2 u_x}{\partial x^2} + M \frac{\partial^2 u_y}{\partial y^2} + Q \frac{\partial^2 u_z}{\partial y^2} + R \frac{\partial^2 \xi}{\partial y^2} = \frac{\partial^2}{\partial x^2} (\rho_1 u_x + \rho_12 U_x) - \frac{\partial^2}{\partial y^2} (u_x - U_x), \\
M \frac{\partial^2 u_y}{\partial x \partial y} + M \frac{\partial^2 u_z}{\partial y \partial z} + Q \frac{\partial^2 u_z}{\partial y^2} + R \frac{\partial^2 \xi}{\partial y^2} = \frac{\partial^2}{\partial y^2} (\rho_1 u_y + \rho_12 U_y) + \frac{\partial^2}{\partial t^2} (u_y - U_y), \\
(L + F) \frac{\partial^2 u_x}{\partial x \partial y} + (L + F) \frac{\partial^2 u_y}{\partial y \partial z} = \left[L \frac{\partial^2 u_x}{\partial z^2} + L \frac{\partial^2 u_y}{\partial y^2} + C \frac{\partial^2 \xi}{\partial z^2}\right] u_z + Q \frac{\partial^2 \xi}{\partial z^2} = \frac{\partial^2}{\partial z^2} (\rho_1 u_z + \rho_12 U_z) - \frac{\partial^2}{\partial t^2} (u_z - U_z) + \frac{\partial^2}{\partial x^2} (\rho_1 u_z + \rho_12 U_z),
\end{align*}
\]

where \((u_x, u_y, u_z)\) and \((U_x, U_y, U_z)\) are displacements of the solid and liquid media, respectively, while ε and c are dilatations of the solid and liquid respectively; A, N, Q, R, F, L, M and C are all poroelastic constants and are the mass coefficients [6]. The stresses \(\sigma_{ij}\) and the liquid pressure \(s\) of the transversely isotropic poroelastic solid[7] are

\[
\begin{align*}
\sigma_{xx} &= Pe_{xx} + Ae_{yy} + Fe_{zz} + M \varepsilon, \\
\sigma_{yy} &= Ae_{xx} + Pe_{yy} + Fe_{zz} + M \varepsilon, \\
\sigma_{zz} &= Fe_{xx} + Fe_{yy} + Ce_{zz} + Q \varepsilon, \\
\sigma_{xy} &= Ne_{xy}, \\
\sigma_{yz} &= Le_{yz}, \\
\sigma_{xz} &= Le_{xz}, \\
s &= Me_{xx} + Me_{yy} + Qe_{zz} + R \varepsilon.
\end{align*}
\]

For Love waves, the displacement is only along \(y\) direction thus the non-zero displacement component of the solid and liquid media are \((0, u_y, 0)\) and \((0, U_y, 0)\) respectively. These
displacements are functions of $x$, $y$ and time $t$. Then the equations of motion of transversely isotropic poroelastic solid [6], that is equation (1) reduces to

$$N \frac{\partial^2 u_y}{\partial x^2} + L \frac{\partial^2 u_y}{\partial z^2} = \frac{\partial^2}{\partial x^2} (\rho_{11} u_y + \rho_{12} U_y) + b \frac{\partial}{\partial t} (u_y - U_y),$$

$$0 = \frac{\partial^2}{\partial z^2} (\rho_{12} u_y + \rho_{22} U_y) - b \frac{\partial}{\partial t} (u_y - U_y).$$

(3)

The propagation mode shapes of solid and liquid and are taken as

$$u_y = f(z)e^{i(kx+\omega t)}, \quad U_y = g(z)e^{i(kx+\omega t)},$$

(4)

where $t$ is time, $\omega$ is circular frequency, $k$ is wave number and $i$ is the complex unity. Substitution of equation (4) into equation (3) yields

$$L f'' - k^2 N f = -\omega^2 (M_{11} f + M_{12} g),$$

$$0 = -\omega^2 (M_{12} f + M_{22} g),$$

(5)

where

$$M_{11} = \rho_{11} - \frac{ib}{\omega}, \quad M_{12} = \rho_{12} + \frac{ib}{\omega}, \quad M_{22} = \rho_{22} - \frac{ib}{\omega}.$$

(6)

From second equation of (5), we obtain

$$g = -\frac{M_{12}}{M_{22}} f.$$

(7)

Substitution of equation (7) into the first equation of (5), we obtain

$$f'' + \gamma^2 f = 0,$$

(8)

where

$$\gamma^2 = \left( -\frac{k^2 N}{L} + \frac{\omega^2 N}{LV_3^2} \right), \quad V_3^2 = \left( \frac{NM_{22}}{M_{11}M_{22} - M_{12}^2} \right).$$

(9)

In equation (9), $V_3$ is shear wave velocity [6]. On simplification, equation (8) gives

$$f(z) = C_1 e^{i\gamma z} + C_2 e^{-i\gamma z},$$

(10)

where $C_1$ and $C_2$ are constants. Hence from equation (7), $g(z)$ can be obtained as

$$g(z) = -\frac{M_{12}}{M_{22}} \left( C_1 e^{i\gamma z} + C_2 e^{-i\gamma z} \right).$$

Substituting $f(z)$ from equation (10) into first equation of (4), the displacement $u_y$ is

$$u_y = \left( C_1 e^{i\gamma z} + C_2 e^{-i\gamma z} \right) e^{i(kx+\omega t)}.$$

(11)

Following equations (2) and (11), the only non-zero stress can be obtained as

$$\sigma_{yz} = \left[ C_1 \left( \frac{i\gamma L}{2} \right) e^{i\gamma z} + C_1 \left( \frac{i\gamma L}{2} \right) e^{-i\gamma z} \right] e^{i(kx+\omega t)}.$$

(12)

From equation (4), it can be shown that the normal strains of solid and liquid are zero, hence the dilatations of solid and liquid media vanish. Since the dilatations of solid and liquid are zero,
the liquid pressure $s$ developed in the solid-liquid aggregate following equation (2) is zero. Thus, no distinction is made between a pervious and an impervious surface of the solid in case of Love waves. In the absence of body forces, the equations of motion for viscous compressible liquid are

$$\rho_l \left( \frac{\partial \mathbf{V}}{\partial t} \right) = -\nabla p + \eta_l \nabla^2 \mathbf{V},$$  

where $\mathbf{V} = (u_x, u_y, u_z)$ is the velocity vector, $\rho_l$ is density of liquid, $\eta_l$ is coefficient of viscosity, $p$ is over pressure.

For Love waves, $\mathbf{V} = (0, u_y, 0)$ and $\nabla \cdot \mathbf{V} = 0$. Hence, equation (13) reduces to

$$\left( \frac{\partial u_y}{\partial t} \right) = \nu_l \nabla^2 u_y,$$

where $\nu_l = \eta_l / \rho_l$ is the kinematic viscosity.

Solution of equation (14) can be obtained as

$$u_y = (D_1 i\omega \cos \alpha_l z + D_2 i\omega \sin \alpha_l z)e^{i(kx + \omega t)},$$

where

$$\alpha_l^2 = -k^2 - \frac{i\omega}{\nu_l}.$$

Following (15), stresses in viscous liquid layer can be shown as

$$\tau_{yz} = i\omega \eta_l \alpha_l (-D_1 \sin \alpha_l z + D_2 \cos \alpha_l z)e^{i(kx + \omega t)}.$$

3. Secular equation

For contact between the poroelastic half-spaces and the liquid layer, we assume that the stresses and displacement components are continuous at the interfaces $z = h_1$ and $z = h_1 + h_2$ whereas, stresses vanish at free surface $z = 0$. Thus the boundary conditions are

$$\sigma_{yz}^{(1)} = 0, \text{ at } z = 0,$$

$$\sigma_{yz}^{(1)} = \tau_{yz}, i\omega u_y^{(1)} = u_y, \text{ at } z = h_1,$$

$$\tau_{yz} = \sigma_{yz}^{(2)}, u_y = i\omega u_y^{(2)}, \text{ at } z = h_1 + h_2.$$  

Eqs. (17) results in a system of five homogeneous algebraic equations in five constants. By eliminating these constants the secular equation of wave propagation in a viscous liquid layer bounded by transversely isotropic poroelastic layer and half-space can be obtained as

$$\left[ \frac{i\gamma L^{(1)}}{2} \tan \left( \frac{\gamma h_1}{2} \right) - \frac{\nu_l}{4} \alpha^2 \omega^2 \right] \tan \left( \frac{\gamma h_1}{2} \right)$$

$$+ i\omega \eta_l \alpha_l \left[ \frac{\gamma L^{(1)}}{2} \tan \left( \frac{\gamma h_1}{2} \right) - \frac{\gamma L^{(2)}}{2} \right] = 0.$$

We can discuss the secular equation (18) in the following limiting cases:

(i) The thickness of liquid layer and coefficient of viscosity tends to zero such that their ratio is a constant i.e. $\eta_l \rightarrow 0$ and $h_2 \rightarrow 0$ such that $\eta_l h_2 = P$ a constant, then the secular equation (18) reduces to

$$\left[ 1 + \frac{1}{\nu_l} \frac{\gamma h_1}{2} P \right] \gamma L^{(1)} \tan \left( \frac{\gamma h_1}{2} \right) = i\gamma L^{(2)}.$$
(ii) The thickness of liquid layer is zero for a finite coefficient of viscosity, the secular equation (18) reduces to

\[ \gamma^{(1)} L^{(1)} \tan \left( \gamma^{(1)} h_1 \right) = i \gamma^{(2)} L^{(2)}. \] (20)

Here, equation (20) represents secular equation of Loves waves for a case of poroelastic layer in welded contact with poroelastic half-space, where each of them is homogeneous and transversely isotropic.

(iii) When co-efficient of viscosity tends to zero for finite thickness of liquid layer, the secular equation (19) reduces to

\[ \tan \left( \gamma^{(1)} h_1 \right) = 0. \] (21)

From equation (21), it is clear that Love wave energy is uncoupled with poroelastic half-space hence it represents secular equation of Love waves in an inviscid liquid layer of finite thickness in contact with transversely isotropic poroelastic layer.

4. Results and discussions

Secular equations (19)-(21) are investigated numerically by considering two distinct poroelastic materials with parameters N=6.4 Gpa, L= 8 Gpa (poroelastic plate) and N=3 Gpa, L= 5 Gpa (poroelastic half-space). Poroelastic medium is dissipative in nature and thus the wave number \( k \) is complex. The waves generated obey diffusion type process and therefore get attenuated. Let \( k = k_r + ik_i \), where \( k_r \) is real and \( k_i \) is the imaginary part of the wave number \( k \). The real and imaginary part of the wave number corresponds to propagation and attenuation of waves. Hence the phase velocity \( C_P \) and attenuation \( \delta \) are

\[ C_P = \frac{\omega}{\text{Re}(k)}, \quad \delta = \frac{\text{Im}(k)}{\text{Re}(k)}. \] (22)

**Figure 1.** Phase velocity as a function of non-dimensional wave number - Viscous fluid bounded between poroelastic layer and half-space when \( P \) is a constant.

**Figure 2.** Attenuation as a function of non-dimensional wave number - Viscous fluid bounded between poroelastic layer and half-space when \( P \) is a constant.

The variations in phase velocity against wave number in viscous liquid layer bounded between poroelastic layer and poroelastic half-space both are transversely isotropic is depicted in figure 1. First four modes are plotted for fixed constant \( P=1 \). The phase velocity decreases gradually as wave number increases over the range 0.2 to 1. Figure 2 presents attenuation for different values of for fixed ratio \( P \) i.e. for \( P = 0.1, 1, 5 \) and 10. There is gradual decrease in attenuation.
for $P=0.1$ and it is steady for $P=1$ and 5. When $P=10$, there is an increase in attenuation in the range of wave number 0.2 to 0.4 and then a steady decrease till 1. Phase velocity at the interface of poroelastic layer and half-space as a function of wave number is presented in figure 3. From figure 1 it is clear that phase velocity in this case is same as that of viscous liquid layer bounded between poroelastic layer and half-space. Hence, it can be observed that there is no effect of viscous liquid layer on propagation of Love waves. Attenuation at the interface of poroelastic layer and half-space is presented in figure 4. It is observed that attenuation is high when wave number is 0.2 and it decreases gradually as the wave number increases from 0.2 to 1. Phase velocity in an inviscid liquid layer of finite thickness in contact with transversely isotropic poroelastic layer is depicted in figure 5. A similar behavior in phase velocity is observed as in the earlier cases i.e. a gradual decrease in the range 0.2 to 1 of wave number.

5. Conclusion
Propagation of Love waves in viscous liquid layer bounded between transversely isotropic poroelastic layer and half-space is discussed. Several limiting cases of secular equation have been discussed. For the limiting case of co-efficient of viscosity tends to zero for finite thickness
of liquid layer, it is observed that Love wave energy is uncoupled with poroelastic half-space. It is found that there is no influence of viscous liquid layer on phase velocity of Love waves when the liquid layer is bounded between poroelastic layer and half-space.

6. References

[1] Deresiewicz H 1960 The effect of boundaries on wave propagation in a liquid-filled porous solid-I Bull. Seis. Soc. Am. 50 599-607
[2] Deresiewicz H and Rice J T 1962 The effect of boundaries on wave propagation in a liquid-filled porous solid-II Bull. Seis. Soc. Am. 52 595-626
[3] Tajuddin M and Ahmed S I 1991 Dynamic interaction of a poroelastic layer and a half-space J. Acoust. Soc. Am. 89 1169-75
[4] Malla Reddy P and Tajuddin M. 2003 Edge waves in poroelastic plate under plane-stress conditions J. Acoust. Soc. Am. 114 185-93
[5] Banghar A R 1978 On propagation and attenuation of Love waves Proc. Indian Acad. Sci. 88 133-46
[6] Biot M A 1956 The theory of propagation of elastic waves in fluid-saturated porous solid J. Acous. Soc. Am. 28 168-78
[7] Biot M A and Willis D G 1957 The elastic co-efficients of the theory of consolidation Journal of Applied Mechanics 24 594-601
[8] NageswaraNath C, Manoj Kumar J and Tajuddin M 2011 On the parametric model of loose bonding between two poroelastic half spaces Journal of Vibration and Control 18 1261-74