EMERGENT BEHAVIOR OF CUCKER-SMALE FLOCKING PARTICLES WITH TIME DELAYS

YOUNG-PIL CHOI AND ZHUCHUN LI

Abstract. We analyze Cucker-Smale flocking particles with delayed coupling, where different constant delays are considered between particles. By constructing a system of dissipative differential inequalities together with a continuity argument, we provide a sufficient condition for the flocking behavior when the maximum value of time delays is sufficiently small.

1. Introduction

Let \((x_i(t), v_i(t)) \in \mathbb{R}^d \times \mathbb{R}^d, i = 1, \ldots, N\) be position and velocity at time \(t\) of the \(i\)-th agent. Then the delayed Cucker-Smale particle system can be described by

\[
\frac{dx_i(t)}{dt} = v_i(t), \quad i = 1, \ldots, N, \quad t > 0,
\]

\[
\frac{dv_i(t)}{dt} = \frac{1}{N} \sum_{j \neq i} \psi(|x_j(t - \tau_{ji}) - x_i(t)|) (v_j(t - \tau_{ji}) - v_i(t)),
\]

subject to the initial data:

\[
(x_i(s), v_i(s)) = (x_0^i(s), v_0^i(s)), \quad i = 1, \ldots, N, \quad s \in [-\tau, 0].
\]

Here, \(\psi : \mathbb{R}_+ \to \mathbb{R}_+\) is a communication weight function, \(\tau_{ji} > 0\) denotes the interaction delay between \(i\)-th and \(j\)-th agents.

The main purpose of this paper is to study the effect of time delays in Cucker-Smale flocking particle system. For the proof, inspired by [1, 7], we construct a system of dissipative differential inequalities by using diameters of position and velocity. Using that together with a continuity argument, we provide a sufficient condition for the flocking behavior estimate under a smallness assumption on the time delays.

It is worth mentioning that there are a few literature on the flocking of Cucker-Smale type models with time delays. For example, a sufficient flocking condition for the Motsch-Tadmor variant of the model with processing delay is obtained in [8], see also [10] for that model without time delays. In [6], sufficient flocking condition for the Cucker-Smale model with noise and delay is derived in terms of noise intensity and delay length. In [4] the first author and his collaborator analyzed a Cucker-Smale model with delay and normalized communication weights where the communication weights received by any agent sum to 1. In another recent paper [9], the authors considered the Cucker-Smale model with processing time-varying delays only in velocities, in which the velocity is governed by

\[
\frac{dv_i(t)}{dt} = \frac{1}{N} \sum_{j \neq i} \psi(|x_j(t - \tau(t)) - x_i(t - \tau(t))|) (v_j(t - \tau(t)) - v_i(t)).
\]

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In comparison, in this paper we will consider the Cucker-Smale model with processing delays in velocities and positions. We also emphasize that the strategy used in requires the strictly positive lower bound assumption for the weight function $\psi$, which is not needed in our framework. We refer to and references therein for recent surveys on Cucker-Smale type flocking models.

We now introduce the main assumptions and state the main result.

**Assumption on $\psi$** - The communication weight $\psi$ is bounded, positive, non increasing and Lipschitz continuous on $\mathbb{R}_+$, with $\psi(0) = 1$.

**Assumption on $\tau_{ji}$** - The interaction delays are strictly positive and symmetric, i.e., $\tau_{ji} = \tau_{ij} > 0$ for all $i, j \in \{1, \cdots, N\}$ and $\tau := \max_{1 \leq i,j \leq N} \tau_{ji} < \infty$.

**Theorem 1.1.** Let $\{(x_i, v_i)\}_{i=1}^N$ be a global solution to the system (1.1). Suppose that there exist some constants $\tau_0 > 0$ and $\alpha > 0$ such that

$$\frac{d\nu(0)}{\alpha \psi(dX(0) + R_v\tau_0 + \alpha)} < 1.$$

Then there exists $\bar{\tau} \in (0, \tau_0)$ such that for all $\tau \in (0, \bar{\tau}]$ we have

$$\sup_{-\tau \leq t \leq +\infty} dX(t) < +\infty \quad \text{and} \quad d\nu(t) \leq C_0 e^{-c_1 \psi(dX(0) + R_v\tau_0 + \alpha)t}, \quad \forall t \geq 0,$$

where $C_0$ and $c_1$ are some positive constants.

**Remark 1.1.** The communication weight $\psi$ introduced in the seminal paper by Cucker and Smale is of the form

$$\psi(r) = \frac{1}{(1 + r^2)^{\beta/2}} \quad \text{with} \quad \beta > 0.$$

Thus for the long-range communication weight, i.e., $\beta < 1$, we can always find positive constants $\tau_0$ and $\alpha$ satisfying the assumption (1.3).

In the next section of this note, we will present the details for the proof of main result.

**2. Emergent behavior: Proof of Theorem 1.1**

2.1. Global existence and uniqueness of solutions to the system (1.1). In this part, we first prove the global-in-time existence and uniqueness of solutions for the system (1.1) so that all computations for the emergent behavior are justified. We notice from the above assumption on $\psi$ that the right-hand-side of (1.1) is locally Lipschitz continuous as a function of $(x_i(t), v_i(t))$. Thus, by the Cauchy-Lipschitz theorem, the particle system (1.1) admits a unique local-in-time $C^1$-solution. On the other hand, that local-in-time solution can be a global-in-time solution once we can show the uniform-in-time boundedness of the velocity since $\psi$ is bounded and Lipschitz.

In the lemma below, we show the uniform-in-time boundedness of the velocity which guarantees the global-in-time existence of the unique solution to the system (1.1). We are also going to use this estimate for the large-time behavior.

**Lemma 2.1.** Let $\{(x_i, v_i)\}_{i=1}^N$ be a solution to the system (1.1)-(1.2). Suppose that the initial velocity $v_{i0}, i = 1, 2, \cdots, N$ are continuous on the compact time interval $[-\tau, 0]$ and denote

$$R_v^\tau := \max_{s \in [-\tau, 0]} \max_{1 \leq i \leq N} |v_i(s)| > 0.$$

Then we have

$$R_v(t) := \max_{1 \leq i \leq N} |v_i(t)| \leq R_v^\tau \quad \text{for} \quad t \geq -\tau.$$

**Proof.** Although the proof is very similar to [3, Lemma 2.2], we provide the details here for the completeness of this paper. For any $\varepsilon > 0$, we set $R_v^{\tau, \varepsilon} := R_v^\tau + \varepsilon$ and $S^\varepsilon := \{t > 0 : R_v(t) <$
$R_v^{\tau,\varepsilon}$ for $s \in [0, t)$. By the continuity of $R_v(t)$ together with $R_v^\tau < R_v^{\tau,\varepsilon}$, we get $S^\varepsilon \neq 0$, and $T_*^\varepsilon := \sup S^\varepsilon > 0$ exists. We now claim $T_*^\varepsilon = \infty$. If not, it holds
\[
\lim_{t \to T_*^\varepsilon^-} R_v(t) = R_v^\varepsilon, \quad \text{and} \quad R_v(t) < R_v^{\tau,\varepsilon} \quad \text{for} \quad t < T_*^\varepsilon. \tag{2.1}
\]
On the other hand, it follows from (1.1) that
\[
\frac{1}{2} \frac{d |v_i(t)|^2}{dt} = \frac{1}{N} \sum_{j \neq i} \psi(|x_j(t) - x_i(t)|)(|v_j(t) - v_i(t)|) \leq \frac{1}{N} \sum_{j \neq i} \psi(|x_j(t) - x_i(t)|)(|v_j(t)|) |v_i(t)|
\]
This further yields
\[
\frac{1}{2} \frac{d |v_i(t)|^2}{dt} \leq \frac{1}{N} \sum_{j \neq i} \psi(|x_j(t) - x_i(t)|)(R_v^\varepsilon - |v_i(t)|) \leq R_v^\varepsilon - |v_i(t)|, \quad \text{a.e. on} \quad (0, T_*^\varepsilon),
\]
due to $R_v^\varepsilon \geq R_v(t)$, i.e., $R_v^\varepsilon \geq |v_i(t)|$ for all $1 \leq i \leq N$ and $t < T_*^\varepsilon$, and $0 \leq \psi \leq 1$. Applying Gronwall’s inequality to the above, we have
\[
\lim_{t \to T_*^\varepsilon^-} R_v(t) \leq (R_v(0) - R_v^{\tau,\varepsilon}) e^{-T_*^\varepsilon} + R_v^{\tau,\varepsilon} < R_v^{\tau,\varepsilon},
\]
since $R_v(0) < R_v^{\tau,\varepsilon}$. This contradicts (2.1), and thus $T_*^\varepsilon = \infty$. We finally pass to the limit $\varepsilon \to 0$ to conclude our desired result.

2.2. Construction of a Lyapunov functional. In this part, we construct the system of dissipative differential inequalities. For this, we introduce position and velocity diameters as
\[
d_X(t) := \max_{1 \leq i,j \leq N} |x_i(t) - x_j(t)| \quad \text{and} \quad d_V(t) := \max_{1 \leq i,j \leq N} |v_i(t) - v_j(t)|.
\]

Lemma 2.2. Let $\{(x_i, v_i)\}_{i=1}^N$ be a global solution to the system (1.1)-(1.2). Then the diameters functions $d_X(t)$ and $d_V(t)$ satisfy
\[
\frac{d}{dt} d_X(t) \leq d_V(t), \tag{2.2}
\]
for almost all $t > 0$, where $\Delta_N^\tau(t)$ is given by
\[
\Delta_N^\tau(t) := \frac{1}{N} \max_{1 \leq i,k \leq N} \sum_{k \neq i} |v_k(t) - v_i(t)|
\]
and satisfies
\[
\Delta_N^\tau(t) \leq C_{N,1} \int_{t-\tau}^{t} d_V(s) \, ds + \int_{t-\tau}^{t} \Delta_N^\tau(s) \, ds, \tag{2.3}
\]
for $t \geq \tau$, where $C_{N,1} := (N - 1)/N$.

Proof. We first easily find from (1.1) that
\[
\frac{d}{dt} d_X(t) \leq d_V(t).
\]
Next we derive the differential inequality for $d_V(t)$. Note that there exist at most countable number of increasing times $t_k$ such that we can choose indices $i$ and $j$ such that $d_V(t) = |v_i(t) - v_j(t)|$
on any time interval \((t_k, t_{k+1})\) since the number of particles is finite and continuity of the velocity trajectories. This allows us to estimate the time evolution of \(dv(t)\) as
\[
\frac{1}{2} \frac{d}{dt} dv(t)^2 = \frac{1}{2} \frac{d}{dt} |v_i(t) - v_j(t)|^2
\]
\[
= \left< v_i(t) - v_j(t), \frac{dv_i(t)}{dt} - \frac{dv_j(t)}{dt} \right>
\]
\[
= \frac{1}{N} \left< v_i(t) - v_j(t), \sum_{k \neq i} \psi(|x_k(t - \tau_k) - x_i(t)|)(v_k(t - \tau_k) - v_i(t)) \right>
\]
\[
- \frac{1}{N} \left< v_i(t) - v_j(t), \sum_{k \neq j} \psi(|x_k(t - \tau_k) - x_j(t)|)(v_k(t - \tau_k) - v_j(t)) \right>
\]
\[
=: I_1 + I_2.
\]

Before estimating the terms \(I_i, i = 1, 2\), we notice that
\[
|x_k(t - \tau_k) - x_i(t)| = |x_k(t) - x_i(t) + \int_t^{t-\tau_k} v_k(s) ds| \leq d_X(t) + R_v \tau.
\]

Using the above inequality, we estimate \(I_1\) as
\[
I_1 = \frac{1}{N} \sum_{k \neq i} \psi(|x_k(t - \tau_k) - x_i(t)|)(v_i(t) - v_j(t), v_k(t) - v_i(t))
\]
\[
+ \frac{1}{N} \sum_{k \neq i} \psi(|x_k(t - \tau_k) - x_i(t)|)(v_i(t) - v_j(t), v_k(t - \tau_k) - v_k(t))
\]
\[
\leq \psi(d_X(t) + R_v \tau) \sum_{k \neq i} \langle v_i(t) - v_j(t), v_k(t) - v_i(t) \rangle + \frac{d_V(t)}{N} \sum_{k \neq i} |v_k(t - \tau_k) - v_k(t)|,
\]
where we used \(\psi \leq 1\) and \(\langle v_i(t) - v_j(t), v_k(t) - v_i(t) \rangle \leq 0\) for \((i, j)\) with \(d_V = |v_i - v_j|\). Similarly, we can obtain
\[
I_2 \leq -\psi(d_X(t) + R_v \tau) \sum_{k \neq j} \langle v_i(t) - v_j(t), v_k(t) - v_j(t) \rangle + \frac{d_V(t)}{N} \sum_{k \neq j} |v_k(t - \tau_k) - v_k(t)|.
\]

This yields
\[
\frac{1}{2} \frac{d}{dt} dv(t)^2 \leq -\psi(d_X(t) + R_v \tau) dv(t)^2 + \frac{2d_V(t)}{N} \max_{1 \leq i \leq N} \sum_{k \neq i} |v_k(t - \tau_k) - v_k(t)|,
\]
for almost all \(t \geq 0\). Thus we have
\[
\frac{d}{dt} dv(t) \leq -\psi(d_X(t) + R_v \tau) dv(t) + 2\Delta_N(t),
\]
for almost all \(t \geq 0\). We next estimate the term \(\Delta_N(t)\). Note that
\[
|v_k(t - \tau_k) - v_k(t)| = \left| \int_{t-\tau_k}^{t} \frac{dv_k(s)}{ds} ds \right| \leq \int_{t-\tau}^{t} \left| \frac{dv_k(s)}{ds} \right| ds.
\]
This gives
\[
\Delta_N(t) \leq \frac{1}{N} \sum_{k=1}^{N} \int_{t-\tau}^{t} \left| \frac{dv_k(s)}{ds} \right| ds \text{ for } t \geq 0.
\]
(2.4)
On the other hand, it follows from (1.1) that
\[
\left| \frac{dv_k(s)}{ds} \right| = \left| \frac{1}{N} \sum_{\ell \neq k} \psi(|x_\ell(s) - x_k(s)|)(v_\ell(s) - v_k(s)) \right|
\leq \frac{dV(s)(N-1)}{N} + \frac{1}{N} \sum_{\ell \neq k} |v_\ell(s) - v_k(s)|
\leq C_{N,1}dV(s) + \Delta_N(s),
\] for \( t \geq 0 \). Combining the above estimates (2.4) and (2.5) concludes the desired result. \( \square \)

**Remark 2.1.** If there is no time delay, i.e., \( \tau = 0 \), then the differential inequality in Lemma 2.2 becomes the standard system of dissipative differential inequalities in \[.\]

**Remark 2.2.** It follows from Lemma 2.1 that
\[
\left| \frac{dv_k(s)}{ds} \right| = \left| \frac{1}{N} \sum_{\ell \neq k} \psi(|x_\ell(s) - x_k(s)|)(v_\ell(s) - v_k(s)) \right| \leq 2R^*_N.
\]
This gives the following estimate:
\[
\sup_{0 \leq t \leq \tau} \Delta_N(t) \leq \frac{1}{N} \sum_{k=1}^{N} \int_{t-\tau}^{t} \left| \frac{dv_k(s)}{ds} \right| ds \leq 2R^*_N \rightarrow 0 \quad \text{as} \quad \tau \rightarrow 0.
\]

2.3. **Proof of Theorem 1.1.** We are going to use the differential inequalities (2.2) together with a continuity argument to complete the proof of Theorem 1.1.

Set \( \mathcal{T} := \{ t \in [0, \infty) : d_X(s) < d_X(0) + \alpha \quad \text{for} \quad s \in [0, t) \} \).

It is clear from the continuity of the function \( d_X(t) \) that \( \mathcal{T} \neq \emptyset \), thus we can set \( T^\infty := \sup \mathcal{T} \).

- **Step A.** (Time-decay estimate of \( d_V(t) \) and \( \Delta_N(t) \)): According to (1.3), we first choose some positive constants \( \beta > 0 \) and \( 0 < c < 1 \) such that
\[
\frac{dV(0)}{\psi(d_X(0) + R_e \tau_0 + \alpha)} + \frac{2\beta}{1-c} < \alpha.
\]
Then we set \( T_* := \{ t \in [0, T^\infty) : d_V(s) < \left( d_V(0) + \frac{2\beta \psi^\infty}{1-c} \right) e^{-c\psi^\infty s} \quad \text{and} \quad \Delta_N(t) < \beta(\psi^\infty)^2 e^{-c\psi^\infty s} \quad \text{for} \quad s \in [0, t) \} \)
where we denoted \( \psi^\infty := \psi(d_X(0) + R_e \tau_0 + \alpha) \) for notational simplicity. Note that \( T_* \neq \emptyset \) for \( \tau \) small enough. Indeed, we find
\[
d_V(0) < d_V(0) + \frac{2\beta \psi^\infty}{1-c} \quad \text{and} \quad \sup_{0 \leq t \leq \tau} \Delta_N(t) < \beta(\psi^\infty)^2 e^{-c\psi^\infty \tau}
\]
for \( \tau > 0 \) small enough such that \( 2R^*_N \tau e^{-c\psi^\infty \tau} < \beta(\psi^\infty)^2 \) due to Remark 2.2. Thus by continuity of functions \( d_V(t) \) and \( \Delta_N(t) \) there exists \( \tau_1 > 0 \) such that \( 0 < T_*^\infty := \sup T_* \) for all \( \tau \in (0, \tau_1) \). Then in the rest of this step we are going to show \( T_*^\infty = T^\infty \). Suppose that \( 0 < T_*^\infty < T^\infty \). Then we have either
\[
\lim_{t \rightarrow T_*^\infty} d_V(t) = \left( d_V(0) + \frac{2\beta \psi^\infty}{1-c} \right) e^{-c\psi^\infty T_*^\infty} \quad \text{or} \quad \lim_{t \rightarrow T_*^\infty} \Delta_N(t) = \beta(\psi^\infty)^2 e^{-c\psi^\infty T_*^\infty}.
\]
On the one hand, it follows from Lemma 2.2 that
\[
\frac{d}{dt} d_V(t) \leq -\psi^\infty d_V(t) + 2\beta(\psi^\infty)^2 e^{-c\psi^\infty t}, \quad \text{a.e.} \quad t \in [0, T_*^\infty).
Applying Gronwall’s inequality yields
\[ d_V(t) \leq d_V(0)e^{-\psi t} + \frac{2\beta \psi}{1-c} \left( e^{-\psi t} - e^{-\psi t} \right), \quad t \in [0, T_\infty). \]

Taking the limit \( t \to T_\infty \) to the above inequality gives
\[ \lim_{t \to T_\infty} d_V(t) \leq d_V(0)e^{-\psi T_\infty} + \frac{2\beta \psi}{1-c} \left( e^{-\psi T_\infty} - e^{-\psi T_\infty} \right) < \left( d_V(0) + \frac{2\beta \psi}{1-c} \right) e^{-c\psi T_\infty}. \]

On the other hand, we find from (2.3) together with (2.6) that
\[ T \quad \text{Taking the limit} \quad \text{Applying Gronwall’s inequality yields} \]

\[
\Delta_N^\tau(t) \leq \left( C_{N,1} \left( d_V(0) + \frac{2\beta \psi}{1-c} + \beta(\psi)^2 \right) \right) \int_{t-\tau}^{t} e^{-c\psi s} ds
\]

\[
= \left( C_{N,1} \left( d_V(0) + \frac{2\beta \psi}{1-c} + \beta(\psi)^2 \right) \right) \left( \frac{e^{c\psi \tau} - 1}{c\psi} \right) e^{-c\psi t}
\]

for all \( t \in [0, T_\infty) \). We then now choose \( 0 < \tau_2 < \tau_1 \) such that
\[
\left( C_{N,1} \left( d_V(0) + \frac{2\beta \psi}{1-c} + \beta(\psi)^2 \right) \right) \left( \frac{e^{c\psi \tau} - 1}{c\psi} \right) < \beta(\psi)^2
\]

for \( \tau \in (0, \tau_2) \). This together with taking the limit \( t \to T_\infty \) yields
\[
\lim_{t \to T_\infty} \Delta_N^\tau(t) < \beta(\psi)^2 e^{-c\psi T_\infty}. \]

Hence both equalities (2.7) do not hold, and this concludes \( T_\infty = T_\infty \).

**Step B.** (Uniform-in-time bound estimate of \( d_X(t) \)): We are now ready to show that \( T_\infty = \infty \) when \( \tau > 0 \) is small enough. Note that for \( t \in [0, T_\infty) \) and \( \tau \in (0, \tau_2) \) it holds
\[
d_X(t) < d_X(0) + \alpha,
\]
\[
d_V(t) < \left( d_V(0) + \frac{2\beta \psi}{1-c} \right) e^{-c\psi t},
\]
\[
\Delta_N^\tau(t) < \beta(\psi)^2 e^{-c\psi T_\infty}. \]

Suppose not, i.e., \( T_\infty < \infty \), then we get
\[
\lim_{t \to T_\infty} d_X(t) = d_X(0) + \alpha.
\]

On the other hand, it follows from Lemma 2.2 together with the above estimate that
\[
d_X(t) \leq d_X(0) + \int_{0}^{t} d_V(s) ds
\]
\[
\leq d_X(0) + \left( d_V(0) + \frac{2\beta \psi}{1-c} \right) \int_{0}^{t} e^{-c\psi s} ds
\]
\[
= d_X(0) + \left( d_V(0) + \frac{2\beta \psi}{1-c} \right) \frac{1}{c\psi} \left( 1 - e^{c\psi t} \right)
\]

for \( t \in [0, T_\infty) \). This gives
\[
\lim_{t \to T_\infty} d_X(t) \leq d_X(0) + \left( d_V(0) + \frac{2\beta \psi}{1-c} \right) \frac{1}{c\psi} \left( 1 - e^{c\psi T_\infty} \right) < d_X(0) + \alpha.
\]

This is a contradiction and yields \( T_\infty = \infty \).
• **Step C.** (Exponential decay estimate of $d_V(t)$): By the discussion in Step A and Step B, we find $T^*_s = T^\infty = \infty$, that is, the following inequalities hold for $t \geq 0$:

$$d_X(t) \leq d_X(0) + \alpha \quad \text{and} \quad d_V(t) \leq \left( d_V(0) + \frac{2\beta \psi^\infty}{1-c} \right) e^{-c\psi^\infty t},$$

where $\psi^\infty$, $\beta$, and $c$ are appeared in Step A. This completes the proof.

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(Young-Pil Choi)

DEPARTMENT OF MATHEMATICS AND INSTITUTE OF APPLIED MATHEMATICS

INHA UNIVERSITY, 402–751, INCHEON, REPUBLIC OF KOREA

E-mail address: ypchoi@inha.ac.kr

(Zhuchun Li)

DEPARTMENT OF MATHEMATICS

HARBIN INSTITUTE OF TECHNOLOGY

E-mail address: lizhuchun@hit.edu.cn