RELATIVE SUBCOPURE-INJECTIVE MODULES

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Abstract. In this paper, copure-injective modules are examined from an alternative perspective. For two modules $A$ and $B$, $A$ is called $B$-subcopure-injective if for every copure monomorphism $f : B \rightarrow C$ and homomorphism $g : B \rightarrow A$, there exists a homomorphism $h : C \rightarrow A$ such that $hf = g$. The class $\mathbb{CP}^{-1}(A) = \{B : A$ is $B$-subcopure-injective$\}$ is called the subcopure-injectivity domain of $A$. We obtain characterizations of copure-injective modules, right CDS rings and right V-rings with the help of subcopure-injectivity domains. Since subcopure-injectivity domains clearly contains all copure-injective modules, studying the notion of modules which are subcopure-injective only with respect to the class of copure-injective modules is reasonable. We refer to these modules as sc-indigent. We studied the properties of subcopure-injectivity domains and of sc-indigent modules and investigated these modules over some certain rings.

1. Introduction and preliminaries

Throughout this paper, $R$ will denote an associative ring with identity, and modules will be unital right $R$-modules, unless otherwise stated. As usual, the category of right $R$-modules is denoted by $\text{Mod} - R$.

Some new studies in module theory have focused on to approach to the injectivity from the point of relative notions. The injectivity domain $\mathbb{In}^{-1}(A)$ for a module $A$, is the class of all modules $B$ such that $A$ is $B$-injective [1]. Given $A$ and $B$ modules, $A$ is called $B$-subinjective if for every monomorphism $f : B \rightarrow C$ and homomorphism $g : B \rightarrow A$, there exists a homomorphism $h : C \rightarrow A$ such that $hf = g$. Instead of using the injectivity domain, in latest articles, authors have proposed to consider an alternative sight so-called subinjectivity domain $\mathbb{In}^{-1}(A)$, contains of modules $B$ such that $A$ is $B$-subinjective (2). It is clear that injectivity of $A$ is equivalent to that $\mathbb{In}^{-1}(A) = \text{Mod} - R$. If $B$ is injective, then $A$ is exactly $B$-subinjective. So by [2] Proposition 2.3), the class of injective modules is the smallest...
possible subinjectivity domain. The recent studies of non-injective modules have been made to figure out the notion of modules that are subinjective only with respect to the class of injective modules. This kind of non-injective modules are called indigent in [2]. So far, it is not known whether the existence of indigent modules for an arbitrary ring, but a positive answer is known for some rings, such as Noetherian rings ([3, Proposition 3.4]).

A submodule $A$ of a right $R$-module $B$ is said to be pure if for every left $R$-module $K$ the natural induced map $i \otimes 1_K : A \otimes K \to B \otimes K$ is a monomorphism. Recall that a module $A$ is said to be $B$-pure-injective if for every pure monomorphism $f : C \to B$ and every homomorphism $g : C \to A$, there exists a homomorphism $h : B \to A$ such that $hf = g$. A module $A$ is said to be pure-injective if it is $B$-pure-injective for every module $B$. As an analogue to the injectivity profile of [12], the pure-injectivity profile of a ring is introduced in [5]. The pure-injectivity domain $\mathcal{PI}^{-1}(A)$ of a module $A$, consists of those modules $B$ such that $A$ is $B$-pure-injective. Inspired by the notion of subinjectivity, the notion of pure-subinjectivity introduced in [11]. A module $A$ is called $B$-pure-subinjective if for every pure monomorphism $f : B \to C$ and homomorphism $g : B \to A$, there exists a homomorphism $h : C \to A$ such that $hf = g$. The pure-subinjectivity domain of a module $A$ is the class $\mathcal{PSI}^{-1}(A) = \{ B : A$ is $B$-pure-subinjective $\}$. If $B$ is pure-injective, then $A$ is exactly $B$-pure-subinjective. So by [11, Theorem 2.4], for a module $A$, the class $\mathcal{PSI}^{-1}(A)$ must contain the class of pure-injective modules at least. In [11], modules whose pure-subinjectivity domain consists of only pure-injective modules is called pure-subinjectively poor (ps-poor for short).

An $R$-module $A$ is said to be finitely embedded (or cofinitely generated) if $E(A) = E(S_1) \oplus E(S_2) \oplus \ldots \oplus E(S_n)$, where $S_1, S_2, \ldots, S_n$ are simple $R$-modules (see [16]). If an $R$-module $A$ is isomorphic to $\prod_{i \in I} E(S_\alpha_i)$, where $I$ is some index set, then $A$ is called a cofree module (see [6]). A right $R$-module $A$ is said to be cofinitely related if there is an exact sequence $0 \to A \to B \to C \to 0$ of $R$-modules with $B$ finitely embedded, cofree and $C$ finitely embedded (see [6]). As a dual notion of purity, by using cofinitely related modules, the notion of copurity is introduced in [7]. An exact sequence of $R$-modules $0 \to A \to B \to C \to 0$ is called a copure exact sequence if every cofinitely related right $R$-module is injective relative to this sequence.

Following idea on pure-injectivity profile of [6], in [15], the copure-injectivity profile of a ring is introduced. For two modules $A$ and $B$, $A$ is called $B$-copure-injective if for every copure monomorphism $f : C \to B$ and a homomorphism $g : C \to A$, there exists a homomorphism $h : B \to A$ such that $hf = g$. $A$ is copure-injective if it is injective with respect to every copure exact sequences (see [8]). The copure-injectivity domain $\mathcal{CPI}^{-1}(A)$ of $A$ is the class of modules $B$ such that $A$ is $B$-copure-injective. In [15], copure-injectively-poor (shortly copi-poor) modules introduced as modules with minimal copure-injectivity domain and studied properties of copi-poor modules. The existence of copi-poor modules are
studied and investigated over some certain rings, but we do not know whether copi-poor modules exist over arbitrary rings (see [15]).

Inspired by the notion of pure-subinjectivity from [11], in this paper we initiate the study of an alternative perspective on the analysis of the copure-injectivity of a module, as we introduce the notions of relative subcopure-injectivity and assign to every module its subcopure-injectivity domain. The aim of this paper is to investigate the viability of obtaining valuable information about a ring $R$ from the perspective of subcopure-injectivity domain.

In Section 2, relative subcopure-injectivity and subcopure-injectivity domains of modules introduced. We investigate the properties of the notion of subcopure-injectivity and we compare subcopure-injectivity domains with (copure-)injectivity domains. We obtain characterizations of copure-injective modules, right CDS rings and right V-rings with the help of subcopure-injectivity domains.

In section 3, we introduced and studied the concept of cc-injective modules in terms of relative subcopure-injective modules. We give examples of cc-injective modules and compare cc-injective modules with cotorsion modules in Example [19]. We prove that $R$ is a right V-ring if and only if every cc-injective right $R$-module is injective. We investigate when the class of $B$-subcopure-injective modules is closed under extensions.

An $R$-module is copure-injective if and only if its subcopure-injectivity domain consists of $Mod - R$. Since subcopure-injectivity domains clearly contain all copure-injective modules, it is reasonable to investigate modules which are subcopure-injective only with respect to the class of copure-injective modules. It is thus to keep in line with [11], we refer to these modules as sc-indigent. In Section 4 of this paper, we studied and investigated sc-indigent modules over some certain rings. We compared sc-indigent modules with indigent modules and ps-poor modules.

2. Relative subcopure-injective modules

In this section, we study the $B$-subcopure-injective modules for a module $B$ and examine its fundamental properties.

**Definition 1.** For two modules $A$ and $B$, $A$ is called $B$-subcopure-injective if for every copure monomorphism $f : B \to C$ and homomorphism $g : B \to A$, there exists a homomorphism $h : C \to A$ such that $hf = g$. The class $\mathbb{CPI}^{-1}(A) = \{B : A$ is $B$-subcopure-injective$\}$ is called the subcopure-injectivity domain of $A$.

Hiremath proved in [8, Theorem 7] that every module can be embedded as a copure submodule in a direct product of cofinitely related modules. By [8, Proposition 3], every cofinitely related module is copure-injective and every direct product of copure-injective modules is copure-injective. This gives the below result that we use frequently in the sequel.

**Lemma 2.** For every module $A$, there exists a copure monomorphism $\alpha : A \to C$ with $C$ is copure-injective.
Our next Lemma gives a characterization of the B-subcopure-injective modules for a module B.

**Lemma 3.** Let A and B be two modules. The following conditions are equivalent:

1. A is B-subcopure-injective.
2. For every homomorphism \( g : B \rightarrow A \) and every copure monomorphism \( \alpha : B \rightarrow C \) with C copure-injective, there exists \( h : C \rightarrow A \) such that \( h\alpha = g \).
3. For every homomorphism \( g : B \rightarrow A \) and every copure monomorphism \( \alpha : B \rightarrow C \) with \( C \) direct product of cofinitely related modules, there exists \( h : C \rightarrow A \) such that \( h\alpha = g \).
4. For every \( g : B \rightarrow A \) there exist a copure monomorphism \( \alpha : B \rightarrow C \) with C copure-injective and \( h : C \rightarrow A \) such that \( h\alpha = g \).

**Proof.** (1) \( \Rightarrow \) (2) Obvious. (2) \( \Rightarrow \) (3) It follows from [8, Proposition 3].

(3) \( \Rightarrow \) (4) Let \( g : B \rightarrow A \) be a homomorphism. By Lemma 2, there exists a copure monomorphism \( \alpha : B \rightarrow C \) with C copure-injective, whence \( C \) is a direct summand of \( F \) where \( F = \prod_{i \in I} F_i \) with each \( F_i \) cofinitely related by [8, Theorem 8]. So \( i\alpha : B \rightarrow F \) is copure monomorphism where \( i : C \rightarrow F \). By (3), there exists \( h : F \rightarrow A \) such that \( (hi)\alpha = h(i\alpha) = g \), where \( i\alpha : B \rightarrow F \).

(4) \( \Rightarrow \) (1) Let \( g : B \rightarrow A \) be a homomorphism and \( \tilde{\alpha} : B \rightarrow D \) a copure monomorphism. By (4), there exists a monic copure map \( \alpha : B \rightarrow C \) with \( C \) copure-injective and a homomorphism \( h : C \rightarrow A \) such that \( h\alpha = g \). So by the copure-injectivity of \( C \), there exists a homomorphism \( h : D \rightarrow C \) such that \( \alpha = \tilde{\alpha} h \). Then \( hh : D \rightarrow A \) and \( hh\tilde{\alpha} = h\alpha = g \). Hence, \( A \) is B-subcopure-injective.

**Proposition 4.** Let \( A \) be an R-module. The following conditions are equivalent:

1. \( A \) is copure-injective.
2. \( \text{CPI}^{-1}(A) = \text{Mod} - R \).
3. \( A \) is A-subcopure-injective.

**Proof.** (1) \( \Rightarrow \) (2) For any R-module \( B \) and any copure-injective module \( A \), every copure monomorphism \( \alpha : B \rightarrow D \) and a homomorphism \( g : B \rightarrow A \), there exists a homomorphism \( h : D \rightarrow A \) such that \( h\alpha = g \). Hence, \( A \) is B-subcopure-injective and so \( B \in \text{CPI}^{-1}(A) \). Consequently, \( \text{CPI}^{-1}(A) = \text{Mod} - R \).

(2) \( \Rightarrow \) (3) Obvious.

(3) \( \Rightarrow \) (1) Assume that \( A \) is A-subcopure-injective. For any copure monomorphism \( \alpha : A \rightarrow B \) with \( B \) copure-injective and \( 1_A : A \rightarrow A \), there exists a homomorphism \( g : B \rightarrow A \) such that \( g\alpha = 1_A \). Thus \( \alpha \) splits. This means that \( A \) is copure-injective.

The next result asserts that subcopure-injectivity domain \( \text{CPI}^{-1}(A) \) of \( A \) how small can be. It should contain the copure-injective modules at least.

**Proposition 5.** \( \bigcap_{A \in \text{Mod} - R} \text{CPI}^{-1}(A) = \{ C \in \text{Mod} - R \mid C \text{ is copure-injective} \} \).
Proof. Suppose that each $R$-module is $B$-subcopure-injective for an $R$-module $B$. Then, by Proposition 4, $B$ is copure-injective. Conversely, let $A$ be any $R$-module and $B$ a copure-injective module. Let $g : B \rightarrow A$ be a homomorphism and $\alpha : B \rightarrow C$ a copure monomorphism. Since $B$ is copure-injective, the splitting map $\beta : C \rightarrow B$ gives the homomorphism $\beta \alpha = 1_B$. So $\beta (\alpha g) = (\beta \alpha) g = g$. Hence $B \in \mathcal{CPI}^{-1}(A)$ for any $R$-module $A$. □

Clearly, $\mathcal{CPI}^{-1}(A)$ contains $\mathcal{In}^{-1}(A)$ for any module $A$. The following example shows that equality need not hold.

Example 6. Let $G = \mathbb{Z}(n)$ be a cyclic group of order $n$. Since $G$ is finite it is cofinitely related and so it is copure-injective $\mathbb{Z}$-module [8, Proposition 3]. So $G \in \mathcal{CPI}^{-1}(G)$ by Proposition 4. But $G \notin 2\mathcal{In}^{-1}(G)$, otherwise $G$ would be an injective $\mathbb{Z}$-module.

It is natural to investigate conditions to get the coincidence of the injectivity, and subcopure-injectivity domains, either for a certain class of modules or all the modules in $\text{Mod} - R$. We start by proving that, for all modules, subcopure-injectivity domains are the same as their subinjectivity domains over a right $V$-ring. Recall that a ring $R$ is a right $V$-ring if and only if all exact sequences in $\text{Mod} - R$ are copure if and only if all copure-injective modules are injective (see [8, Proposition 5]).

Corollary 7. Let $R$ be a ring. The following conditions are equivalent:

1. $R$ is a right $V$-ring.
2. $\mathcal{CPI}^{-1}(A) = \mathcal{In}^{-1}(A)$ for each $R$-module $A$.
3. $\mathcal{CPI}^{-1}(A) \subseteq \mathcal{In}^{-1}(A)$ for each $R$-module $A$.

Proof. (1) $\Rightarrow$ (2) It is easy since for any module $A$, over a right $V$-ring its extension is copure.
(2) $\Rightarrow$ (3) It is obvious.
(3) $\Rightarrow$ (1) For a copure injective right $R$-module $A$, by Proposition 4, $A \in \mathcal{CPI}^{-1}(A)$. By (3), $A \in 2\mathcal{In}^{-1}(A)$. This says that $A$ is injective, and so $R$ is a right $V$-ring by [8, Proposition 5]. □

Proposition 8. Let $A$ be a module. The following conditions are equivalent:

1. $A$ is copure-injective.
2. $\mathcal{CPI}^{-1}(A)$ is closed under copure submodules.
3. $\mathcal{CPI}^{-1}(A) = \mathcal{CPI}^{-1}(A)$.
4. $\mathcal{CPI}^{-1}(A) \subseteq \mathcal{CPI}^{-1}(A)$.

Proof. The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are clear since $\mathcal{CPI}^{-1}(A) = \mathcal{CPI}^{-1}(A) = \text{Mod} - R$. 

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For a copure-injective extension $C$ of $A$, $C \in \mathfrak{CPI}^{-1}(A)$, so $A$ is also in $\mathfrak{CPI}^{-1}(A)$ by (2). Then by Proposition 4, $A$ is copure-injective.

(3) $\Rightarrow$ (4) It is clear.

(4) $\Rightarrow$ (1) For a copure-injective extension $C$ of $A$, $C \in \mathfrak{CPI}^{-1}(A)$. This implies that $A$ is $C$-copure-injective i.e. $C = A \oplus B$ for some submodule $B$ of $A$, whence $A$ is copure-injective.

The rings for which every right $R$-module is copure-injective are called right CDS, [8, Corollary 18]. As a result of Proposition 8, we get the following Corollary.

**Corollary 9.** Let $R$ be a ring. The following conditions are equivalent:

1. $R$ is right CDS.
2. $\mathfrak{CPI}^{-1}(A) = \mathfrak{In}^{-1}(A)$ for each $R$-module $A$.
3. $\mathfrak{CPI}^{-1}(A) \subseteq \mathfrak{In}^{-1}(A)$ for each $R$-module $A$.

**Proof.** (2) $\Rightarrow$ (3) It is clear.

(1) $\Rightarrow$ (2) Let $A$ be an $R$-module. Since $R$ is a right CDS ring, $A$ is copure-injective. The rest follows from Proposition 8.

(3) $\Rightarrow$ (1) For any right $R$-module $A$, $\mathfrak{CPI}^{-1}(A) \subseteq \mathfrak{In}^{-1}(A)$ by the hypothesis. Thus every right $R$-module $A$ is copure-injective by Proposition 8 whence $R$ is right CDS.

**Remark 10.** If $A$ is $R$-subcopure-injective, for a ring $R$ and a module $A$, then $\mathfrak{CPI}^{-1}(A)$ and $\text{Mod}_R$ need not be equal. For example if $R$ is copure-injective ring that is not CDS, then for every module $A$, $A$ is $R$-subcopure-injective by Proposition 8. But by the definition of right CDS ring, we can find a module $A$ that is not copure-injective.

**Proposition 11.** Let $A$ be a module. The following conditions are equivalent:

1. $A$ is injective.
2. $\mathfrak{CPI}^{-1}(A) = \mathfrak{In}^{-1}(A)$.
3. $\mathfrak{CPI}^{-1}(A) \subseteq \mathfrak{In}^{-1}(A)$.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) By the copure-injectivity of $E(A)$, $E(A) \in \mathfrak{CPI}^{-1}(A)$. By (3), $E(A) \in \mathfrak{In}^{-1}(A)$, and hence $A$ is injective.

**Corollary 12.** Let $R$ be a ring. The following conditions are equivalent:

1. $R$ is semisimple.
2. $\mathfrak{CPI}^{-1}(A) = \mathfrak{In}^{-1}(A)$ for each $R$-module $A$.
3. $\mathfrak{CPI}^{-1}(A) \subseteq \mathfrak{In}^{-1}(A)$ for each $R$-module $A$.

**Proof.** (2) $\Rightarrow$ (3) It is clear.

(1) $\Rightarrow$ (2) Let $A$ be an $R$-module. Since $R$ is semisimple, $A$ is injective. The rest follows from Proposition 11.
(3) \(\Rightarrow\) (1) For any right \(R\)-module \(A\), \(\text{CPI}^{-1}(A) \subseteq \mathfrak{m}^{-1}(A)\) by the hypothesis. Thus every right \(R\)-module \(A\) is injective by Proposition 11, whence \(R\) is semisimple.

In general, factors of copure-injective modules need not be copure-injective (see, [S, Remark 24]). But if \(R\) is a Dedekind domain, every copure factor of copure-injective module is copure-injective by [S, Corollary 28]. Hence, by the following Proposition, \(\text{CPI}^{-1}(A)\) is closed under copure homomorphic images over Dedekind domains for a module \(A\).

**Proposition 13.** \(\text{CPI}^{-1}(A)\) is closed under copure quotients for any module \(A\) if and only if every copure homomorphic image of a copure-injective module is copure-injective.

**Proof.** Let \(B\) be a copure submodule of copure-injective module \(A\). Since \(A \in \text{CPI}^{-1}(\frac{A}{B})\), by the hypothesis \(\frac{A}{B} \in \text{CPI}^{-1}(\frac{A}{B})\), and so \(\frac{A}{B}\) is copure-injective. Conversely, let \(A\) be a module and \(C\) a copure submodule of \(B\) with \(B \in \text{CPI}^{-1}(A)\). By Lemma 3 there exists a copure monomorphism \(\alpha : B \to D\) with \(D\) copure-injective. Let \(f : \frac{B}{C} \to A\) be any homomorphism. Consider the following pushout diagram:

\[
\begin{array}{ccc}
0 & \to & B & \xrightarrow{\alpha} & D & \xrightarrow{\pi} & \frac{D}{D} & \to & 0 \\
\pi & \downarrow & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \frac{B}{C} & \xrightarrow{\alpha'} & \frac{D}{C} & \xrightarrow{\pi'} & \frac{
\ast}{\ast} & \to & 0 \\
\end{array}
\]

where \(\pi : B \to \frac{B}{C}\) is the natural epimorphism. By commutativity of the following diagram:

\[
\begin{array}{ccc}
B & \xrightarrow{\alpha} & D \\
\downarrow & & \downarrow \pi'' \\
\frac{B}{C} & \xrightarrow{\alpha''} & \frac{D}{C}
\end{array}
\]

and the pushout diagram property, there exists a map \(\phi : E \to \frac{D}{C}\) such that \(\phi\pi' = \pi''\) and \(\phi\alpha' = \alpha''\). Since \(A\) is \(B\)-subcopure-injective, there exists a homomorphism \(\varphi : D \to A\) such that \(\varphi\alpha = f\pi\). Then, \(\varphi(C) = \varphi\alpha(C) = f\pi(C) = f(0) = 0\). Hence, \(\text{Ker}(\varphi\pi') \subseteq \text{Ker}\varphi\), and so there exists \(\psi : \frac{D}{C} \to A\) such that \(\psi\pi'' = \varphi\). For every \(x \in B\), \(\psi(x + C) = \psi\pi''(x) = \varphi(x) = f\pi(x) = f(x + C)\). Thus \(\psi\) extends \(f\). Then by the hypothesis, \(\frac{B}{C}\) is copure-injective, so by Lemma 3, \(\frac{B}{C} \in \text{CPI}^{-1}(A)\). \qed
Proposition 14. $\mathcal{P}\mathcal{P}^{-1}(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} \mathcal{P}\mathcal{P}^{-1}(A_i)$ for any set of modules $\{A_i\}_{i \in I}$.

Proof. Let $B \in \mathcal{P}\mathcal{P}^{-1}(\bigcap_{i \in I} A_i)$, $i \in I$ and $f : B \to A_i$ be a homomorphism. Then there exists a homomorphism $g : C \to \bigcap_{i \in I} A_i$ such that $g\alpha = i_{A_i}f$, where $\alpha : C \to A$ is the monic map with $C$ copure-injective and $i_{A_i} : A_i \to \bigcap_{i \in I} A_i$ is the inclusion map. Let $\pi_{A_i} : \bigcap_{i \in I} A_i \to A_i$ denote the natural projection. Since $\pi_{A_i}g\alpha = \pi_{A_i}i_{A_i}f = f$, $f$ is extended to $\pi_{A_i}g$. Therefore $B \in \mathcal{P}\mathcal{P}^{-1}(A_i)$ for any $i \in I$. Conversely, let $B \in \mathcal{P}\mathcal{P}^{-1}(A_i)$ for all $i \in I$ and $f : B \to \bigcap_{i \in I} A_i$. Hence for each $i \in I$, there exists $g_i : C \to A_i$ with $g_i\alpha = \pi_{A_i}f$. Now define $g : C \to \bigcap_{i \in I} A_i$ by $x \mapsto g_i(x)$. Since $g\alpha = f$, $g$ extends $f$. Thus, $B \in \mathcal{P}\mathcal{P}^{-1}(\bigcap_{i \in I} A_i)$. \hfill \Box

Corollary 15. Let $B$ be a module. Then $B$-subcopure-injective modules are closed under direct summands and finite direct sums.

Proof. Let $A$ be a module with decomposition $A = \oplus_{i=1}^n A_i$. By Proposition 14, $B \in \mathcal{P}\mathcal{P}^{-1}(A)$ if and only if $B \in \bigcap_{i=1}^n \mathcal{P}\mathcal{P}^{-1}(A_i)$. Now the result follows. \hfill \Box

The following shows that Proposition 14 do not hold for infinite direct sums.

Example 16. Let $K_i = \mathbb{Z}_{p_i}$ and $G = \bigoplus_{i \in \mathbb{N}} \mathbb{Z}_{p_i}$, where $p_i$ is a prime integer for all $i \in \mathbb{N}$. Since every $\mathbb{Z}_{p_i}$ is pure-injective, every $\mathbb{Z}_{p_i}$ is copure-injective by Proposition 9. So $G \in \mathcal{P}\mathcal{P}^{-1}(\mathbb{Z}_{p_i})$ for all $i \in \mathbb{N}$. But $G \notin \mathcal{P}\mathcal{P}^{-1}(G)$ since $G$ is not copure-injective by Examples-(ii)]).

Proposition 17. If $B \in \mathcal{P}\mathcal{P}^{-1}(A)$, then every direct summand of $B$ is in $\mathcal{P}\mathcal{P}^{-1}(A)$.

Proof. Suppose $C$ is a direct summand of $B$, and let $f : C \to A$ be a homomorphism. By Lemma 2, there exist copure monomorphisms $i : B \to D$ and $j : C \to E$ with $D$ and $E$ copure-injective. Consider the following diagram:

\[
\begin{array}{ccc}
0 & \longrightarrow & C \\
& & \downarrow j \\
& & E \\
& & \downarrow i \\
& & D
\end{array}
\]

where $i_C : C \to B$ the inclusion map. Since $D$ is copure-injective, there exists $h : E \to D$ such that $hj = ii_C$. Let $\pi_C : B \to C$ be the projection map. Since $A$ is $B$-subcopure-injective, there exists a homomorphism $g : D \to A$ such that $gi = f\pi_C$. Then, $(gh)j = g(hj) = gii_C = f\pi_Ci_C = f$, and so by Lemma 3, $A$ is $C$-subcopure-injective. \hfill \Box
3. CC-INJECTIVE MODULES

In this section, we introduced and studied the concept of cc-injective modules in terms of relative subcopure-injective modules.

A module \( C \) is said to be co-absolutely co-pure (c.c. in short) if every exact sequence of modules ending with \( C \) is copure, equivalently \( \text{Ext}^1_R(C, A) = 0 \) for every co-finitely related module \( A \). Clearly every projective module is c.c. But the converse need not be true, for instance, the additive group \( \mathbb{Q} \) is a c.c. \( \mathbb{Z} \)-module but \( \mathbb{Q} \) is not projective as a \( \mathbb{Z} \)-module (see, [9, Example on page 290]).

**Definition 18.** A right module \( A \) is called cc-injective if \( \text{Ext}^1_R(B, A) = 0 \) for any c.c. module \( B \).

Recall that a module \( A \) is called cotorsion if \( \text{Ext}^1_R(B, A) = 0 \) for every flat module \( B \). A module \( A \) is called linearly compact if any family of cosets having the finite intersection property has a nonempty intersection. A commutative ring is called classical if the injective hull \( E(S) \) of all simple modules \( S \) are linearly compact (see [17, §3]).

**Example 19.** (1) By definition, any cofinitely related module is cc-injective.
(2) By [9, Remark 15], c.c. modules need not be flat in general. By [9, Corollary 14] c.c. modules are flat over a commutative ring. So, in this case every cotorsion module is cc-injective.
(3) By [9, Remark 12], flat modules need not be c.c. Over a commutative classical ring flat modules are c.c. by [9, Proposition 11]. So, in this case every cc-injective module is cotorsion.

**Remark 20.** Over a commutative ring \( \mathbb{R} \) every simple \( \mathbb{R} \)-module is cotorsion by [13, Lemma 2.14]. So by Example 19(2), every simple \( \mathbb{R} \)-module is cc-injective.

**Lemma 21.** Every copure-injective module is cc-injective.

**Proof.** Let \( A \) be a copure-injective module and \( B \) a c.c. module. By [9, Proposition 5], there exists a copure exact sequence \( 0 \rightarrow D \rightarrow P \rightarrow B \rightarrow 0 \) with \( P \) projective. If we apply \( \text{Hom}(\_ , A) \) to this sequence, we have \( \text{Hom}(P, A) \rightarrow \text{Hom}(D, A) \rightarrow \text{Ext}^1_R(B, A) \rightarrow \text{Ext}^1_R(P, A) = 0 \). Since \( A \) is copure-injective, \( \text{Hom}(P, A) \rightarrow \text{Hom}(D, A) \) is epic, and so \( \text{Ext}^1_R(B, A) = 0 \) for any c.c. module \( B \). Hence \( A \) is cc-injective. \( \square \)

**Proposition 22.** For a ring \( \mathbb{R} \), the following conditions are equivalent:

(1) \( \mathbb{R} \) is a right \( V \)-ring.
(2) Every copure-injective right \( \mathbb{R} \)-module is injective.
(3) Every cc-injective right \( \mathbb{R} \)-module is injective.

**Proof.** (1) \( \Leftrightarrow \) (2) It follows by [8, Proposition 5].
(3) \( \Rightarrow \) (2) It immediately from Lemma 21.
(1) \(\Rightarrow\) (3) Let \(A\) be a cc-injective \(R\)-module and \(B\) any \(R\)-module. Since \(R\) is right \(V\), \(B\) is a c.c. module by [9, Proposition 4]. Thus \(\text{Ext}^1_R(B, A) = 0\) for any \(R\)-module \(B\), and so \(A\) is injective.

**Proposition 23.** Let \(B\) be an \(R\)-module and \(\alpha : B \rightarrow C\) a copure monomorphism with \(C\) copure-injective. If \(C/\text{im}(\alpha)\) is c.c., then every cc-injective module is \(B\)-subcopure-injective.

**Proof.** Let \(A\) be a cc-injective module and \(C/\text{im}(\alpha)\) a c.c. module. Applying functor \(\text{Hom}(-, A)\) to the exact sequence \(0 \rightarrow B \rightarrow C \rightarrow C/\text{im}(\alpha) \rightarrow 0\), we have \(\text{Hom}(C, A) \rightarrow \text{Hom}(B, A) \rightarrow \text{Ext}^1_R(C/\text{im}(\alpha), A)\). Since \(C/\text{im}(\alpha)\) is c.c., \(\text{Ext}^1_R(C/\text{im}(\alpha), A) = 0\) and so \(\text{Hom}(C, A) \rightarrow \text{Hom}(B, A)\) is epic. Hence \(A\) is \(B\)-subcopure-injective by Lemma [3].

**Theorem 24.** Let \(A\) and \(B\) be two modules. Consider the following conditions:

1. \(A\) is \(B\)-subcopure-injective.
2. For every homomorphism \(g : B \rightarrow A\), there exist a monomorphism \(\alpha : B \rightarrow C\) with \(C\) copure-injective and a homomorphism \(h : C \rightarrow A\) such that \(h\alpha = g\).
3. For every homomorphism \(g : B \rightarrow A\), there exist a monomorphism \(\alpha : B \rightarrow C\) with \(C\) cc-injective and a homomorphism \(h : C \rightarrow A\) such that \(h\alpha = g\).
4. For every homomorphism \(g : B \rightarrow A\) and for any extension \(\alpha : B \hookrightarrow C\) with \(C/B\) is c.c., there exists \(h : C \rightarrow A\) such that \(h\alpha = g\).

Then (1) \(\Leftrightarrow\) (2) \(\Rightarrow\) (3) \(\Rightarrow\) (4). Also, if \(D/\text{im}(\alpha)\) is c.c. for a copure monomorphism \(\alpha : B \rightarrow D\) with \(D\) copure-injective, then (4) \(\Rightarrow\) (1).

**Proof.** (1) \(\Rightarrow\) (2) Obvious by Lemma [3].

(2) \(\Rightarrow\) (3) It follows from Lemma [21] since every copure-injective module is cc-injective.

(2) \(\Rightarrow\) (1) Let \(\alpha : B \rightarrow C\) be a copure-monomorphism and \(g : B \rightarrow A\) a homomorphism. By (2), exists a monomorphism \(\beta : B \rightarrow D\) with \(D\) copure-injective and a homomorphism \(h : D \rightarrow A\) such that \(h\beta = g\). Since \(D\) is copure-injective, there exists a homomorphism \(f : C \rightarrow D\) such that \(f\alpha = \beta\). Hence, \((hf)\alpha = h\beta = g\), and so (1) follows.

(3) \(\Rightarrow\) (4) Let \(C\) be an extension of \(B\) with \(C/B\) is c.c. and \(g : B \rightarrow A\) a homomorphism. So, \(0 \rightarrow B \xrightarrow{g} C \rightarrow C/B \rightarrow 0\) is copure exact. Then consider the exact sequence with \(E\) cc-injective:

\[
0 \rightarrow \text{Hom}_R(C/B, E) \rightarrow \text{Hom}_R(C, E) \xrightarrow{\alpha^*} \text{Hom}_R(B, E) \rightarrow \text{Ext}^1_R(C/B, E) = 0
\]

Since, \(\alpha^*\) is surjective, by (3), there exists a monomorphism \(f : B \rightarrow E\) and a homomorphism \(h : E \rightarrow A\) such that \(hf = g\). Since \(\alpha^*\) is surjective, there exists a homomorphism \(\beta : C \rightarrow E\) such that \(\beta\alpha = f\). Hence, \(h(\beta\alpha) = hf = g\), and so (4) follows.
Let $\alpha : B \to D$ be a copure monomorphism with $D$ copure-injective and $D/\text{im}(\alpha)$ is c.c. So, by (4), for any homomorphism $g : B \to A$ there exists $h : D \to A$ such that $h\alpha = g$. Thus $A$ is $B$-subcopure-injective by Lemma 3.

Now we investigate when the class of $B$-subcopure-injective modules is closed under extensions.

**Proposition 25.** Let $B$ be an $R$-module and $\alpha : B \to C$ a copure monomorphism with $C$ copure-injective. The class of $B$-subcopure-injective modules is closed under extensions if and only if for every exact sequence $0 \to A' \to A \to C \to 0$ with $A'$ $B$-subcopure-injective, $A$ is $B$-subcopure-injective.

**Proof.** Let $0 \to A' \to A \to C \to 0$ be an exact sequence with $A'$ $B$-subcopure-injective. Since $C$ is copure-injective, it is $B$-subcopure-injective. By the hypothesis, $A$ is $B$-subcopure-injective. Conversely, let $0 \to A' \to A \overset{\pi}{\to} A'' \to 0$ be an exact sequence with $A'$ and $A''$ $B$-subcopure-injective. Then by Lemma 3 for every map $g : B \to A$, there exists a map $h : C \to A''$ such that $\pi g = h\alpha$ where $\alpha : B \to C$ is the copure monomorphism with $C$ copure-injective. If we consider the pullback diagram:

$$
\begin{array}{ccc}
0 & \longrightarrow & A' \\
\downarrow & & \downarrow \\
0 & \longrightarrow & A'
\end{array}
\begin{array}{ccc}
A' & \overset{\pi}{\longrightarrow} & A'' \\
& & h \downarrow \\
D & \overset{f}{\longrightarrow} & C & \overset{h}{\longrightarrow} & 0
\end{array}
$$

there exists a homomorphism $\gamma : B \to D$ such that $f\gamma = g$ and $\beta\gamma = \alpha$. By hypothesis, $D$ is $B$-subcopure-injective, so by Lemma 3 there exists a homomorphism $h' : C \to D$ such that $h'\alpha = \gamma$. Thus, $fh'\alpha = f\gamma = g$ and so, $A$ is $B$-subcopure-injective by Lemma 3. 

A ring $R$ is said to be right co-noetherian if every homomorphic image of a finitely embedded $R$-module is finitely embedded, equivalently for each simple right $R$-module $S$ the injective hull $E(S)$ is Artinian (see [10, Theorem]). Over a commutative noetherian ring, the injective hull of each simple right $R$-module is Artinian by [14, Exercise 4.17]. Thus every commutative Noetherian ring is co-noetherian. In the following, for an ideal $I$, we deal with an $R$-module structure of an $R/I$-module.

**Proposition 26.** Let $R$ be a right co-noetherian ring and $f : R \to S$ a ring epimorphism. If $A$ is $cc$-injective $S$-module, then $A$ is $cc$-injective $R$-module.

**Proof.** Let $A$ be a $cc$-injective $S$-module. Since $f : R \to S$ is a ring epimorphism, $S \cong R/I$ for some ideal $I$ of $R$ and so $A$ can be considered as $R/I$-module. Let $C$ be an extension of $A$ by a c.c. module $F$ as $R$-modules. Since $F$ is c.c., the exact sequence $0 \to A \to C \to F \to 0$ is copure. Then $A \cap CI = AI$ for each right ideal $I$ by [7, proposition 16]. Since $A$ is an $R/I$-module, $A \cap CI = AI = 0$, and so $\frac{A+CI}{CI} \cong A$. Thus we have the following commutative diagram.
Since $C \otimes_A \frac{R}{I} \cong \frac{C \otimes_A \frac{R}{I}}{C \otimes_\frac{R}{I} I}$ is c.c. as an $R/I$-module, so the second exact sequence splits and so does the first. Hence $\text{Ext}^1_R(F, A) = 0$, and $A$ is cc-injective $R$-module.

4. SC-INDIGENT MODULES

Indigent (resp. ps-poor) modules were introduced and some results about them were obtained in [2] (resp. [11]). Proposition 5 says that subcopure-injectivity domain of any module $A$ contains all copure-injective modules, so studying the notion of modules which are subcopure-injective only with respect to the class of copure-injective modules is reasonable. It is thus to keep in line with [2], we refer to these modules as subcopure-injectively indigent (sc-indigent for short). In this section, sc-indigent modules investigated over certain rings and compared these modules with indigent modules and ps-poor modules.

Definition 27. A module $A$ is said to be subcopure-injectively indigent (sc-indigent for short), if $\text{CPI}(A)$ consists of only copure-injective modules.

Remark 28. Let $A$ be a module with decomposition $A = B \oplus C$. If $B$ is sc-indigent, then so is $A$, by Proposition [4]

Proposition 29. For a ring $R$, the following conditions are equivalent:

1. $R$ is right CDS.
2. Every $R$-module is sc-indigent.
3. There exists a copure-injective sc-indigent $R$-module.
4. $0$ is an sc-indigent $R$-module.
5. $R$ has an sc-indigent module and every sc-indigent $R$-module is copure-injective.
6. $R$ has an sc-indigent module and every factor of an sc-indigent $R$-module is sc-indigent.
7. $R$ has an sc-indigent module and every summand of an sc-indigent $R$-module is sc-indigent.

Proof. The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (5) are clear since every $R$-module is copure-injective.
The implications (2) $\Rightarrow$ (4) and (2) $\Rightarrow$ (6) $\Rightarrow$ (7) are clear.
(4) $\Rightarrow$ (2) It immediately from Remark [28].
(2) $\Rightarrow$ (3) The copure-injective extension $C$ of any module $A$ is sc-indigent.
(3) $\Rightarrow$ (1) Let $C$ be a copure-injective sc-indigent module and $A$ a module. Since $C$ is $A$-subcopure-injective, $A$ is copure-injective. Then $R$ is a right CDS ring.
(5) ⇒ (1) By (5), there exist an sc-indigent module $B$. Then $A \oplus B$ is also sc-indigent for any module $A$ by Remark 28. So $A$ is copure-injective by (5). Also $A$ is copure-injective. Thus $R$ is a right CDS ring.

(7) ⇒ (2) Let $A$ be an $R$-module. Then $A \oplus B$ is an sc-indigent module for some sc-indigent module $B$. Hence, $A$ is sc-indigent by the hypothesis.

Remark 30. Over a commutative uniserial ring $R$, every $R$-module is sc-indigent since such rings are CDS by [4, Theorem 10.4].

Remark 31. An sc-indigent module need not be indigent. Consider the ring $R = \mathbb{Z}/p^2\mathbb{Z}$, for some prime integer $p$. $R$ is an artinian principal ideal ring. Hence it is a CDS-ring by [4, Theorem 10.4]. So every $R$-module is sc-indigent. Since $\mathbb{Z}/p^2\mathbb{Z}$ is injective $\mathbb{Z}/p^2\mathbb{Z}$-module, $\mathbb{Z}/p^2\mathbb{Z} = \text{Mod} \ R$. But since $R$ is not a semisimple ring, $\mathbb{Z}/p^2\mathbb{Z}$ is not an indigent $R$-module.

Remark 32. An indigent module need not be sc-indigent. Let $R$ be a commutative Noetherian ring which is not CDS and $\Gamma$ a complete set of representatives of finitely presented right $R$-modules. Set $F := \bigoplus_{S_i \in \Gamma} S_i$. Thus the character module $F^+$ of $F$ is a pure-injective indigent $R$-module by [3, Proposition 3.4]. Since $R$ is commutative, $F^+$ is copure-injective by [8, Proposition 9], and so $\text{CPI}^{-1}(F^+) = \text{Mod} \ R$. But since $R$ is not a CDS-ring, $F^+$ is not an sc-indigent $R$-module.

Proposition 33. Indigent modules and sc-indigent modules coincide over a right V-ring $R$.

Proof. Let $R$ be a right V-ring. Then by Corollary 7, $\text{CPI}^{-1}(A) = \mathfrak{n}^{-1}(A)$ for any $R$-module $A$. Hence $A$ is indigent if and only if $A$ is sc-indigent by [8, Proposition 5].

Proposition 34. A module $A$ is sc-indigent if and only if $\prod_{i \in I} A_i$ is sc-indigent where $A_i = A$ for all $i \in I$.

Proof. Clear by Proposition 14.

By Remark 28 and Proposition 34, sc-indigent rings are characterized as follows:

Corollary 35. For a ring $R$, the following are equivalent:

1. $R_R$ is sc-indigent.
2. Any direct product of copies of $R$ is sc-indigent.
3. Every free $R$-module is sc-indigent.
4. There exists a cyclic projective sc-indigent $R$-module.

Theorem 36. Let $R$ be a ring, $B$ an $R$-module and $A$ an $R/I$-module for any ideal $I$ of $R$. If $B/BI \in \text{CPI}^{-1}(A_{R/I})$, then $B \in \text{CPI}^{-1}(A_R)$.

Proof. Let $B/BI \in \text{CPI}^{-1}(A_{R/I})$, and $C$ be a copure extension of $B$ and $g : B \to A$ an $R$-homomorphism. Since copure short exact sequences of $R$-modules form a proper class by [7, Proposition 8], $B/BI$ can be embedded in $C/C I$ as
a copure submodule via \( f : B/BI \to C/CI \) defined by \( f(b + BI) = b + CI \) for any \( b \in B \). Since \( BI \subseteq \text{Ker}(g) \), there exists a homomorphism \( h : B/BI \to A \) such that \( h\pi_B = g \) where \( \pi_B : B \to B/BI \). By assumption, there exists an \( R/I \)-homomorphism \( \tilde{h} : C/CI \to A \) such that \( \tilde{h}f = g \). Since \( h \) is also an \( R \)-homomorphism and \( \tilde{h}\pi_Ci_B = g \) where \( \pi_C : C \to C/CI \) and \( i_B : B \to C \) is the inclusion. Thus \( B \in \mathcal{CPI}^{-1}(A_R) \). \( \square \)

**Corollary 37.** Let \( I \) be an ideal of a ring \( R \) and \( A \) and \( B \) be \( R/I \)-modules. Then the following statements hold:

1. \( B \in \mathcal{CPI}^{-1}(A_R) \) if and only if \( B \in \mathcal{CPI}^{-1}(A_{R/I}) \).
2. \( A \) is a copure-injective \( R \)-module if and only if \( A \) is a copure-injective \( R/I \)-module.
3. \( A \) is an sc-indigent \( R \)-module if and only if \( A \) is an sc-indigent \( R/I \)-module.

**Proof.** (1) If \( A_R \) is \( B \)-subcopure-injective, then clearly it is a \( B \)-subcopure-injective \( R/I \)-module. The converse follows by Theorem 36.

(2) By using Proposition 4, (2) follows from (1).

(3) Clear by (1) and (2). \( \square \)

Recall [11] that a module \( A \) is called ps-poor if pure-subinjectivity domain of \( A \) consists of only pure-injective modules. Over a commutative classical ring \( R \), by [8, Corollary 17], pure-injective modules and copure-injective modules coincide. Hence, the following result is immediate.

**Proposition 38.** Let \( R \) be a commutative classical ring. Then an \( R \)-module \( A \) is sc-indigent if and only if \( A \) is ps-poor.

Since by [16, Theorem 2] and [17, Proposition 4.1], every commutative (co-)noetherian ring is classical, we have the following result.

**Corollary 39.** Let \( R \) be a commutative (co-)noetherian ring. Then an \( R \)-module \( A \) is sc-indigent if and only if \( A \) is ps-poor.

**Remark 40.** ps-poor abelian groups and sc-indigent abelian groups coincide by Corollary 39.

**Corollary 41.** Every finitely embedded \( \mathbb{Z} \)-module is copure-injective but not sc-indigent.

**Proof.** Let \( A \) be a finitely embedded \( \mathbb{Z} \)-module. Then \( A \) is cofinitely related by [6, Proposition 17]. So \( A \) is copure-injective by [3, Proposition 3]. Since \( \mathbb{Z} \) is not a CDS ring, by Proposition 29, \( A \) is not an sc-indigent module. \( \square \)

**Proposition 42.** If a ring \( R \) has an sc-indigent cc-injective module \( B \), then every module with its copure injective extension has c.c cokernel is copure-injective.
Proof. Let $A$ be an $R$-module with the exact sequence $0 \to A \to C \to C/A \to 0$, where $A \to C$ is a copure extension of $A$ with $C$ is copure-injective. Consider the sequence $0 \to \text{Hom}(C/A, B) \to \text{Hom}(C, B) \to \text{Hom}(A, B) \to \text{Ext}^1(C/A, B)$. Since $C/A$ is c.c., $\text{Ext}^1(C/A, B) = 0$. So by Lemma 3, $A \in \mathfrak{CPI}^{-1}(B)$, that is $A$ is copure-injective.

Acknowledgement. The author is very grateful to the anonymous referees for carefully reading the original version of this paper and for providing several very helpful comments and suggestions.

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