An efficient and optimal higher-order scheme in general way for simple zeros

Ramandeep Behl\textsuperscript{a1}, Ali Saleh Alshomrani\textsuperscript{a}, Fouad Othman Mallawi and Mohammed Ali A. Mahnashi\textsuperscript{a}

\textsuperscript{a}Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia

E-mail: \textsuperscript{a}ramanbeh187@yahoo.in

Abstract. The inspiration behind of this paper is to suggest an iteration function of sixteenth-order in a general way which is applicable to every earlier optimal multi-point eighth-order formula provided whose first substep should be the classical Newton’s method. The presented technique meet the expectation of classical Kung-Traub conjecture regarding the optimality of multi-point methods without memory. Further, we fully investigated the theoretical and computational properties of the proposed scheme through the main theorem which demonstrates the convergence order and the term of asymptotic error. Computational consequences that the proposed methods are superior than the earlier studies of sixteenth-order in terms of approximated roots, terms of asymptotic error, residual errors in the considered functions, variation in two consecutive iterations, etc.

Keywords: Nonlinear equations, Simple roots, Newton’s method, Computational convergence order.

1. Introduction

Establishment of multi-point iterative schemes for obtaining the solutions of scalar equations of the following form

\[ f(x) = 0, \] (1)

(where \( f : \mathbb{D} \subset \mathbb{C} \rightarrow \mathbb{C} \) is a holomorphic/analytic function in the region including the required zero \( \alpha \)) is a foremost chore in the area of computational methods and numerical analysis. The main focus of the multi-point functions is to attain as high as possible convergence order by consuming a specific number of functional values. In addition, they use only first order derivative of the assumed function. Moreover, they conquer the theoretical restrictions of one-point iterative schemes about their computational efficiency and convergence order (form more explanations please see [11,17]).

With the progression of advanced computer arithmetic, digital computer and symbolic computation, researchers have been paid the special attention to the construction of optimal sixteenth-order schemes. The faster convergence towards the required root and attaining the required precision of accuracy with in a little number of iterations are the main reasons behind this importance.

\textsuperscript{1} Corresponding author: Ramandeep Behl
We have a few number of sixteenth-order schemes having optimal convergence, some of them can be found in the researcher articles presented by Neta [10], Kung and Traub [8], Sharma et al. [12], Guem and Kim [6,7] and Ullah et al. [18]. Most of them are the modification/extensions of Newton’s or Newton-like iterative formula at the cost of extra values of function/s and/or first order derivative/s or additional number of substep/s to the original iterative schemes.

In addition, no one has proposed a general scheme which is applicable on every optimal eighth-order formula to induce further optimal sixteenth-order convergence, according to our knowledge. Therefore, optimal techniques which are applicable to every iterative scheme of specific order to attain further high order formula instead of obtaining a higher-order extension of a known iterative scheme of particular order are more challenging, interesting, demanding and tough job in this area.

In this paper, we pursue to develop a technique that is applicable to every optimal scheme of eighth-order to attain further optimal sixteenth-order instead of applying that approach only on any particular iterative technique. The main advantages of the presented scheme is that it is applicable to every eighth-order optimal scheme whose first substep should be Newton’s formula. Then, it is quite obvious to choose any scheme from [1–5, 8, 9, 13–16, 19], etc. to further obtain applicable to every eighth-order optimal scheme whose first substep should be Newton’s formula.

2. Construction of the proposed optimal scheme

Here, we propose a new general technique in the following way:

\[
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
z_n &= \gamma(x_n, y_n), \\
t_n &= \eta(x_n, y_n, z_n), \\
x_{n+1} &= t_n - \frac{f(t_n)(x_n - y_n)^2(y_n - z_n)(x_n - z_n)^2}{(y_n - t_n)\beta_1(t_n - z_n)(z_n - y_n) + \beta_2} + f[y_n, t_n](t_n - x_n)^2(t_n - z_n)(x_n - z_n)^2,
\end{aligned}
\]

where \(\gamma, \eta, \beta_1, \beta_2, f, f'\) are two optimal iterative methods of four and eighth order, respectively. Then, the convergence of proposed technique will attain the optimal sixteenth-order without considering any additional functional evaluations. It is important to note that only the first coefficients \(\delta_0, \xi_0\) from \(\gamma(x_n, y_n)\) and \(\eta(x_n, y_n, z_n)\), respectively, contributes to their role in the establishment of the needed asymptotic error constant, which is mentioned in Theorem 2.1.

**Theorem 2.1** Let us assume that \(f : \mathbb{D} \subset \mathbb{C} \rightarrow \mathbb{C}\) is holomorphic function in the region \(\mathbb{D}\) containing the simple zero \(x = x_0\) is enough close to \(x\) for the assured convergence. In addition, we consider that \(\gamma(x_n, y_n)\) and \(\eta(x_n, y_n, z_n)\) are two optimal iterative methods of four and eighth order, respectively. Then, the convergence of proposed technique (2) reaches to sixteenth-order.

**Proof:** Let us consider \(e_n = x_n - \alpha\) be the error at nth step. By adopting the Taylor’s series expansion, we expand the functions \(f(x_n)\) and \(f'(x_n)\) in the neighborhood \(x = \alpha\) with the presumption \(f'\neq 0\) which yield:

\[
f(x_n) = f'\left(\alpha + \sum_{j=2}^{16} c_j e_n^j + O(e_n^{17})\right)
\]
and

\[ f'(x_n) = f'(\alpha) \left[ 1 + \sum_{j=2}^{16} j c_j e_n^{j-1} + O(e_n^{17}) \right], \] (4)

where \( c_j = \frac{f^{(j)}(\alpha)}{f^{(1)}(\alpha)} \) for \( j = 2, 3, \ldots, 16 \), respectively.

By using the expression (3) and (4) in the first step, we get

\[ y_n - \alpha = c_2 e_n^2 + 2(c_3 - c_2^2) e_n^3 + (4c_3^2 - 7c_3 c_2 + 3c_4) e_n^4 + \sum_{i=1}^{12} M_i e_n^{i+4} + O(e_n^{17}), \] (5)

where \( M_i = M_i(c_2, c_3, \ldots, c_{16}) \) are given in terms of \( c_2, c_3, \ldots, c_{16} \) with explicitly written three coefficients \( M_1 = 20c_3 e_n^2 - 8c_4^2 - 10c_4 c_2 - 6c_5^2 + 4c_6, \) \( M_2 = 16c_3^3 - 52c_3 c_2^2 + 28c_4 c_3^2 + (33c_3^3 - 13c_3) c_2 - 17c_3 c_2 + 5c_4, \) \( M_3 = 2(9c_3^3 - 2c_3) e_n^2 + 16c_3^3 + 36c_4 c_3^2 - 64c_5 c_3^2 + (8c_6 - 46c_3 c_4) c_2 + 6c_4^2 - 3c_7 + 11c_3 c_5 - 9c_3^3), \) etc.

Similarly, we have

\[ f(y_n) = f'(\alpha) [c_2 e_n^2 + 2(c_3 - c_2^2) e_n^3 + (5c_3^2 - 7c_3 c_2 + 3c_4) e_n^4 + 2(6c_4^2 - 12c_3 c_2 + 5c_4 c_2 + 3c_3^2 - 2c_3) e_n^5 + \sum_{i=1}^{11} M_i e_n^{i+4} + O(e_n^{17})]. \] (6)

As in the starting, we consider that \( \gamma(x_n, y_n) \) and \( \eta(x_n, y_n, z_n) \) are two optimal iterative methods of four and eighth order, respectively. Then, it is quite obvious that they will satisfy the error equations of the following forms

\[ z_n - \alpha = \sum_{l=0}^{12} \delta_l e_n^{l+4} + O(e_n^{17}), \] (7)

and

\[ t_n - \alpha = \sum_{l=0}^{8} \xi_l e_n^{l+8} + O(e_n^{17}), \] (8)

respectively, where \( \delta_0, \xi_0 \neq 0 \).

By adopting the Taylor series expansion, we further yield

\[ f(z_n) = f'(\alpha) [\delta_0 e_n^8 + \delta_1 e_n^9 + \delta_2 e_n^8 + \delta_3 e_n^7 + (\delta_4 c_2 + \delta_5) e_n^8 + (2\delta_0 \delta_1 c_2 + \delta_5) e_n^9 + (\delta_4^2 + 2\delta_0 \delta_2) c_2 + \delta_6) e_n^{10} + (2\delta_1 \delta_2 + \delta_0 \delta_3) c_2 + \delta_7) e_n^{11} + (\delta_0 \delta_3 + 2\delta_4 \delta_0 c_2 + \delta_8) e_n^{12} + (3\delta_1 \delta_2^2 c_3 + 2\delta_0 \delta_0 c_2 + 2\delta_2 \delta_3 c_2 + \delta_1 \delta_4 c_2 + \delta_9) e_n^{13} + H_1 e_n^{14} + H_2 e_n^{15} + H_3 e_n^{16} + O(e_n^{17})] \] (9)

and

\[ f(t_n) = f'(\alpha) [\xi_0 e_n^8 + \xi_1 e_n^9 + \xi_2 e_n^{10} + \xi_3 e_n^{11} + \xi_4 e_n^{12} + \xi_5 e_n^{13} + \xi_6 e_n^{14} + \xi_7 e_n^{15} + (A_2 e_n^{16} + \xi_8) e_n^{16} + O(e_n^{17})], \] (10)

where \( H_1 = 3\delta_2 \delta_3^2 c_3 + 2\delta_0 \delta_0 c_2 + 3\delta_3^2 c_3 c_2 + \delta_2^2 c_2 + 2\delta_2 \delta_4 c_2 + 2\delta_1 \delta_5 c_2 + \delta_1 10, \) \( H_2 = \delta_3^2 c_3 + 6\delta_0 \delta_2 \delta_1 c_3 + 2(\delta_0 \delta_4 + \delta_2 \delta_5 + \delta_1 \delta_6 + \delta_0 \delta_7) c_2 + 3\delta_0^2 \delta_3 c_3 + \delta_1 \) and \( H_3 = \delta_3^2 c_4 + 3\delta_1 \delta_5^2 c_3 + 3\delta_5 \delta_0 c_2 + \delta_8 \delta_0 \delta_2 c_2 + 3\delta_5 \delta_0 \delta_2 c_3 + 6\delta_1 \delta_5 \delta_0 c_3 + \delta_1^2 c_2 + 2\delta_3 \delta_5 c_2 + 2\delta_2 \delta_6 c_2 + \delta_0 \delta_7 c_2 + 3\delta_2^2 \delta_3 c_2 + \delta_1 12. \)

By using the expressions (3) – (10), we attain

\[ \frac{f(t_n)(x_n - y_n)^2(x_n - z_n)^2(y_n - z_n)}{f(y_n, t_n)[(t_n - z_n)^2(x_n - z_n)^2 - (t_n - y_n)[(t_n - z_n)(z_n - y_n)]]} = \xi_0 e_n^8 + \xi_1 e_n^9 + \xi_2 e_n^{10} + \xi_3 e_n^{11} + \xi_4 e_n^{12} + \xi_5 e_n^{13} + \xi_6 e_n^{14} + \xi_7 e_n^{15} - (c_2 \xi_0 (c_3 \delta_0 - \xi_0) + \xi_8) e_n^{16} + O(e_n^{17}). \] (11)
Finally, by using the expressions (8) and (11) in the final substep of the suggested formula (2), we obtain

$$e_{n+1} = \frac{-c_2(c_3 \delta_0 - \xi_0)}{\xi_0} e_n^{16} + O(e_n^{17}).$$  \hspace{1cm} (12)

The above expression (12) declares that the iterative technique (2) attains an optimal sixteenth-order in the sense of Kung-Traub conjecture because it uses only five functional values each step. Hence, it completes the proof. \hfill \Box

**Remark 2.2** It is quite natural that one may expect that the term of asymptotic error of the presented scheme (2) may be reliant on $\delta_0, \delta_1, \ldots, \delta_{12}$ and $\xi_0, \xi_1, \ldots, \xi_8$. But, it is clear from the expression (12) that the term asymptotic error dependent only on $\delta_0$ and $\xi_0$.

### 2.1. Particular cases of our scheme

Here, we described some particular cases of our technique (2). Therefore, we consider

(i) We assume an optimal eighth-order scheme given by Behl and Motsa in [1]. Then, we further have

$$\begin{align*}
    w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
    z_n &= w_n - \frac{f(w_n) - f(x_n)}{f(x_n) - 2f(w_n)f'(x_n)}, \\
    t_n &= z_n - \frac{f(w_n) - f(z_n)}{f'(x_n)} \left[ 1 + \frac{b_1 f(x_n)(4f(z_n) + f(w_n))}{2(2f(w_n) - f(x_n))} - \frac{f(w_n)}{2f(x_n)} \right] f(x_n) \\
    x_{n+1} &= t_n - \frac{f(t_n)(x_n - y_n)^2(y_n - z_n)(x_n - z_n)^2}{(y_n - t_n) \left[ \beta_1(t_n - z_n)(z_n - y_n) + \beta_2 \right] + f[y_n, t_n](t_n - x_n)^2(t_n - z_n)(x_n - z_n)^2},
\end{align*}$$  \hspace{1cm} (13)

where $b_1 \in \mathbb{R}$, $\beta_1$ and $\beta_2$ define earlier. Let us consider $b_1 = -\frac{1}{2}$ in the above scheme, denoted by $(PM1)$.

(ii) Again, we consider another optimal eighth-order scheme suggested by Džunić and Petković [5]. Then, we obtain

$$\begin{align*}
    w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
    z_n &= w_n - \frac{f(w_n) - f(x_n)}{f(x_n) - 2f(w_n)f'(x_n)}, \\
    t_n &= z_n + \frac{f(z_n) + f(w_n)}{f(w_n) - f(z_n)} \left( 2f(z_n) - f(x_n) \right) f(x_n) \\
    x_{n+1} &= t_n - \frac{f(t_n)(x_n - y_n)^2(y_n - z_n)(x_n - z_n)^2}{(y_n - t_n) \left[ \beta_1(t_n - z_n)(z_n - y_n) + \beta_2 \right] + f[y_n, t_n](t_n - x_n)^2(t_n - z_n)(x_n - z_n)^2},
\end{align*}$$  \hspace{1cm} (14)

where $\beta_1$ and $\beta_2$ define earlier. Let us call the above scheme by $(PM2)$.

In the similar fashion, we can develop many new and interesting optimal sixteenth-order schemes by considering any optimal eighth-order scheme from the literature whose first substep employs the classical Newton’s method.

### 3. Numerical experiments

Here, we demonstrate the efficiency and convergence behavior of our schemes (13) and (14). Therefore, we assume a good collection of standard examples that are mentioned in the table 1.
We cross verify the theoretical convergence order of them in the sense of computational convergence order and $|x_{n+1} - x_n|/|x_n - x_{n-1}|^{16}$. In addition, we depicted the iteration indexes ($n$), residual error ($|f(x_n)|$), approximated zeros ($x_n$), variation between the two consecutive iterations $|x_{n+1} - x_n|/|x_n - x_{n-1}|^{16}$, the computational convergence order ($\rho$) and the term of asymptotic error $\eta = \lim_{n \to \infty} |x_{n+1} - x_n|/(x_n - x_{n-1})^{16}$. For computing ($\rho$), we adopt the following formula

$$\rho = \left(\frac{x_{n+1} - x_n}{x_n - x_{n-1}}\right) \eta, \quad n = 1, 2, 3.$$ 

The values of all the above parameters are considered up to minimum 1000 significant digits in order to obtain least round off error. Because of restricted page space, we depicted the values of $x_n$, $|x_{n+1} - x_n|/|x_n - x_{n-1}|^{10}$, $\rho$ and $\eta$ up to 20, 10, 5 and 10 significant digits, respectively. Finally, the residual error $|f(x_n)|$ and variation between the two consecutive iterations $|x_{n+1} - x_n|$ are mentioned up to 2 significant digits with exponent power.

Now, we contrast our sixteenth-order schemes to optimal existing sixteenth-order iterative formulas which were suggested by Neta [10], Geum and Kim. [6, 7], Sharma et al. [12] and Ullah et al. [18], among them, we consider the schemes namely, expression (5), expression (Y1) (for more details please have a look [7]), expression (K2) (for more details please see [6]) and expression (K2) (for more details please see [6]).

We performed all the calculations and computations in Mathematica 11 (programming package) with multiple precision arithmetic. The $\alpha(\pm \beta)$ stands for $\alpha \times 10^{\mp \beta}$ in the tables 2–5.

**Table 1.** Test problems

| $f(x)$                           | $\alpha$     |
|----------------------------------|--------------|
| $f_1(x) = \exp(-x) - \cos(x)$; [18] | $\alpha = 0$ |
| $f_2(x) = \tan^{-1}(x) - x + 1$; [11] | $\alpha = 2.132677252728851316 \ldots$ |
| $f_3(x) = \cos \left( x^2 - 2x + \frac{16}{7} \right) - \log \left( x^2 - 2x + \frac{25}{7} \right) - 1$; [7] | $\alpha = 1 + \frac{\sqrt{2}}{7}i$ |
| $f_4(z) = x^5 + x^4 + 4x^2 - 15$; [9] | $\alpha = 1.347280989683049815 \ldots$ |

**Remark 3.1** From the numerical results obtained from Table 2–5, it is straightforward to say that our proposed methods not only converge faster towards the required zero but they have also minimum residual errors of the involved functions and smaller asymptotic error constants as compare to the existing optimal sixteenth-order schemes.

**4. Conclusions**

In the earlier research articles, many researchers presented modification/improvement of some special iterative schemes like Ostrowski’s or King’s iteration function, etc. having higher-order optimal convergence. Here, we suggested a new scheme which can be used to obtain optimal methods of sixteenth-order from any eighth-order iteration function employs first substep as the classical Newton’s method. Our scheme also fulfill the Kung-Traub conjecture for optimality of multi-point methods without memory. Further, we also investigated the theoretical properties of our scheme through the main theorem which demonstrates convergence of sixteenth-order. Computational consequence that the proposed methods are superior than the earlier studies of sixteenth-order in terms of approximated roots, terms of asymptotic error, residual errors in the considered functions, variation in two consecutive iterations, etc.
Table 2. (Convergence performance of distinct sixteenth-order optimal schemes on problem $f_1(x)$)

| Cases | $n$ | $x_n$ | $|f(x_n)|$ | $|x_{n+1} - x_n|$ | $\frac{x_{n+1} - x_n}{(x_n - x_{n-1})^2}$ | $\eta$ | $\rho$ |
|-------|-----|-------|------------|-----------------|---------------------------------|--------|--------|
| NM    | 0   | $\frac{1}{5}$ | 1.4(−1)   | 1.7(−1)         |                                  |        |        |
|      | 1   | $-5.1948307561794135490(−9)$ | 5.2(−9)   | 5.2(−9)         | 14655.18121                     | 353.7239212 | 13.922 |
|      | 2   | $-9.949628851358858861(−131)$ | 9.9(−131) | 9.9(−131)       | 353.7239212                     |        | 16.000 |
| GM1   | 0   | $\frac{1}{5}$ | 1.4(−1)   | 1.7(−1)         |                                  |        |        |
|      | 1   | $85387752909615785932(−5)$ | 8.6(−5)   | 8.6(−5)         | 2.457394120(+8)                 | 122.7287948 | 7.9043 |
|      | 2   | $-1.1808133172876590710(−63)$ | 1.2(−63)  | 1.2(−63)        | 122.7287948                     |        | 16.000 |
| GM2   | 0   | $\frac{1}{5}$ | 1.4(−1)   | 1.7(−1)         |                                  |        |        |
|      | 1   | $-16693144777480342681(−4)$ | 1.7(−4)   | 1.7(−4)         | 4.634489453(+8)                 | 2553.24552  | 10.524 |
|      | 2   | $-9.283239854707091605(−57)$ | 9.3(−57)  | 9.3(−57)        | 2553.24552                      |        | 16.000 |
| SM    | 0   | $\frac{1}{5}$ | 1.4(−1)   | 1.7(−1)         |                                  |        |        |
|      | 1   | $-2.2683738028220060104(−11)$ | 2.3(−11)  | 2.3(−11)        | 63.99331795                     | 2.779989711 | 14.250 |
|      | 2   | $-1.3660832205669711141(−170)$ | 1.4(−170) | 1.4(−170)       | 2.779989711                     |        | 16.000 |
| MM    | 0   | $\frac{1}{5}$ | 1.4(−1)   | 1.7(−1)         |                                  |        |        |
|      | 1   | $-2.7900513204475033618(−7)$ | 2.8(−7)   | 2.8(−7)         | 787083.604                      | 7.437597052(−5) | 3.1174 |
|      | 2   | 1.0028309168035609541(−109) | 1.0(−109) | (−)             | 7.437597052(−5)                 |        | 16.000 |
| PM1   | 0   | $\frac{1}{5}$ | 1.4(−1)   | 1.7(−1)         |                                  |        |        |
|      | 1   | $-2.0249703596403502482(−12)$ | 2.0(−12)  | 2.0(−12)        | 5.712663943                     | 2.05462963(−2) | 12.851 |
|      | 2   | $-1.61891123286257209589(189)$ | 1.6(−189) | 1.6(−189)       | 2.05462963(−2)                  |        | 16.000 |
| PM2   | 0   | $\frac{1}{5}$ | 1.4(−1)   | 1.7(−1)         |                                  |        |        |
|      | 1   | $-4.5319401539435624015(−152)$ | 4.5(−152) | 4.5(−152)       | 12.78510126                     | 2.843739283   | 15.161 |
|      | 2   | $-9.004196998022241907(−182)$ | 9.0(−182) | 9.0(−182)       | 2.843739283                     |        | 16.000 |

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Table 3. (Convergence performance of distinct sixteenth-order optimal schemes on problem $f_2(x)$)

| Cases | $n$ | $x_n$ | $|f(x_n)|$ | $|x_{n+1} - x_n|$ | $\frac{|x_{n+1} - x_n|}{|x_n|}$ | $\eta$ | $\rho$ |
|-------|----|-------|----------|----------------|-----------------|------|------|
| NM    | 0  | 2.2   | 5.6(-2)  | 6.8(-2)       | 3.96354324(-11) | 5.423178200(-11) | 16.196 |       |
|       | 1  | 2.1322677252728851316 | 5.1(-30) | 6.3(-30) | 3.196354324(-11) | 5.423178200(-11) | 16.196 |       |
|       | 2  | 2.1322677252728851316 | 2.5(-478) | 3.1(-478) | 5.423178200(-11) | 16.000 |       |
| GM1   | 0  | 2.2   | 5.6(-2)  | 6.8(-2)       | 3.861217878(-12) | 1.341563314(-11) | 16.334 |       |
|       | 1  | 2.1322677252728851316 | 8.1(-30) | 1.1(-30) | 5.461217878(-12) | 1.341563314(-11) | 16.334 |       |
|       | 2  | 2.1322677252728851316 | 3.1(-491) | 4.1(-491) | 1.341563314(-11) | 16.000 |       |
| GM2   | 0  | 2.2   | 5.6(-2)  | 6.8(-2)       | 3.857865144(-10) | 5.131435342(-10) | 16.254 |       |
|       | 1  | 2.1322677252728851316 | 4.2(-29) | 5.1(-29) | 5.857865144(-10) | 5.131435342(-10) | 16.254 |       |
|       | 2  | 2.1322677252728851316 | 8.2(-463) | 1.0(-462) | 5.131435342(-10) | 16.000 |       |
| SM    | 0  | 2.2   | 5.6(-2)  | 6.8(-2)       | 3.75289654(-12)  | 1.257691879(-11) | 16.180 |       |
|       | 1  | 2.1322677252728851316 | 1.2(-30) | 1.5(-30) | 7.75289654(-12)  | 1.257691879(-11) | 16.180 |       |
|       | 2  | 2.1322677252728851316 | 8.5(-489) | 1.0(-488) | 1.257691879(-11) | 16.000 |       |
| MM    | 0  | 2.2   | 5.6(-2)  | 6.8(-2)       | 3.64545545(-11)  | 1.083260223(-10) | 16.211 |       |
|       | 1  | 2.1322677252728851316 | 9.9(-29) | 1.2(-29) | 6.45455545(-11)  | 1.083260223(-10) | 16.211 |       |
|       | 2  | 2.1322677252728851316 | 1.8(-473) | 2.2(-473) | 1.083260223(-10) | 16.000 |       |
| PM1   | 0  | 2.2   | 5.6(-2)  | 6.8(-2)       | 3.22564869(-12)  | 3.296902569(-12) | 16.146 |       |
|       | 1  | 2.1322677252728851316 | 3.6(-31) | 4.4(-31) | 2.22564869(-12)  | 3.296902569(-12) | 16.146 |       |
|       | 2  | 2.1322677252728851316 | 4.7(-498) | 5.8(-498) | 3.296902569(-12) | 16.000 |       |
| PM2   | 0  | 2.2   | 5.6(-2)  | 6.8(-2)       | 1.03021841(-11)  | 1.577823986(-11) | 16.166 |       |
|       | 1  | 2.1322677252728851316 | 1.6(-30) | 2.0(-30) | 1.03021841(-11)  | 1.577823986(-11) | 16.166 |       |
|       | 2  | 2.1322677252728851316 | 7.3(-487) | 9.0(-487) | 1.577823986(-11) | 16.000 |       |

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Table 4. (Convergence performance of distinct sixteenth-order optimal schemes on problem $f_3(x)$)

| Cases | n   | $x_n$     | $|f(x_n)|$ | $|x_{n+1} - x_n|$ | $\frac{|x_{n+1} - x_n|}{|x_n|}$ | $\eta$  | $\rho$ |
|-------|-----|-----------|-----------|-----------------|-------------------------------|--------|--------|
| NM    | 0   | 1.1 + 0.7i| 3.4(-1)   | 2.1(-1)         |                               |        |        |
|       | 1   | 1.0000000 + 0.8819171...i | 4.6(-13)  | 2.6(-13)       | 0.02196382890 | 0.001863941353 | 14.431 |
|       | 2   | 1.0000000 + 0.8819171...i | 1.5(-204) | 8.8(-205)      | 0.001863941353 |        |        |
| GM1   | 0   | 1.1 + 0.7i| 3.4(-1)   | 2.1(-1)        |                               |        |        |
|       | 1   | 1.0000000 + 0.8819171...i | 1.0(-11)  | 5.7(-12)       | 0.4761416462 | 0.2594339569 | 15.614 |
|       | 2   | 1.0000000 + 0.8819171...i | 5.1(-181) | 2.9(-181)      | 0.2594339569 |        |        |
| GM2   | 0   | 1.1 + 0.7i| 3.4(-1)   | 2.1(-1)        |                               |        |        |
|       | 1   | 1.0000000 + 0.8819171...i | 1.1(-10)  | 6.2(-11)       | 5.215976282 | 1.374013647 | 15.152 |
|       | 2   | 1.0000000 + 0.8819171...i | 1.2(-163) | 6.6(-164)      | 1.374013647 |        |        |
| SM    | 0   | 1.1 + 0.7i| 3.4(-1)   | 2.1(-1)        |                               |        |        |
|       | 1   | 1.0000000 + 0.8819171...i | 7.8(-13)  | 4.4(-13)       | 0.03729549729 | 0.003334054674 | 14.464 |
|       | 2   | 1.0000000 + 0.8819171...i | 1.3(-200) | 7.5(-201)      | 0.003334054674 |        |        |
| MM    | 0   | 1.1 + 0.7i| 3.4(-1)   | 2.1(-1)        |                               |        |        |
|       | 1   | 1.0000000 + 0.8819171...i | 1.7(-11)  | 9.4(-12)       | 0.7907935289 | 68.70582276 | 18.840 |
|       | 2   | 1.0000000 + 0.8819171...i | 4.5(-175) | 2.6(-175)      | 68.70582276 |        |        |
| PM1   | 0   | 1.1 + 0.7i| 3.4(-1)   | 2.1(-1)        |                               |        |        |
|       | 1   | 1.0000000 + 0.8819171...i | 2.4(-13)  | 1.3(-13)       | 0.01128684515 | 0.7331984610 | 18.655 |
|       | 2   | 1.0000000 + 0.8819171...i | 1.4(-206) | 8.2(-207)      | 0.7331984610 |        |        |
| PM2   | 0   | 1.1 + 0.7i| 3.4(-1)   | 2.1(-1)        |                               |        |        |
|       | 1   | 1.0000000 + 0.8819171...i | 2.0(-13)  | 1.1(-13)       | 0.009554537761 | 0.6361170374 | 18.670 |
|       | 2   | 1.0000000 + 0.8819171...i | 8.7(-208) | 4.9(-208)      | 0.6361170374 |        |        |
Table 5. (Convergence performance of distinct sixteenth-order optimal schemes on problem $f_4(x)$)

| Cases | Cases | $x_n$ | $|f(x_n)|$ | $|x_{n+1} - x_n|$ | $\frac{|x_{n+1} - x_n|}{|x_n - x_{n-1}|}$ | $\eta$ |
|-------|-------|-------|-----------|----------------|---------------------------------|-------|
| $NM$  | 0     | 1.4   | 2.1       | 5.3(-2)       |                                  |       |
|       | 1     | 1.3474280989683049818 | 2.6(-19) | 76.98284914 | 144.0313037                  | 16.213 |
|       | 2     | 1.3474280989683049815 | 1.4(-294) | 7.1(-296) | 144.0313037                  | 16.000 |
| $GM1$ | 0     | 1.4   | 2.1       | 5.3(-2)       |                                  |       |
|       | 1     | 1.3474280989683049808 | 2.6(-17) | 7.0(-19) | 205.9741908                  | 113.0363485 | 15.796 |
|       | 2     | 1.3474280989683049815 | 1.4(-287) | 3.4(-289) | 113.0363485                  | 16.000 |
| $GM2$ | 0     | 1.4   | 2.1       | 5.3(-2)       |                                  |       |
|       | 1     | 1.3474280989683049811 | 1.4(-17) | 3.8(-19) | 110.9011373                  | 2816.150413 | 17.098 |
|       | 2     | 1.3474280989683049815 | 1.8(-290) | 4.8(-292) | 2816.150413                  | 16.000 |
| $SM$  | 0     | 1.4   | 2.1       | 5.3(-2)       |                                  |       |
|       | 1     | 1.3474280989683049815 | 2.7(-21) | 7.3(-23) | 0.02150906587                | 0.03180719351 | 16.133 |
|       | 2     | 1.3474280989683049815 | 8.1(-355) | 2.2(-356) | 0.03180719351                | 16.000 |
| $MM$  | 0     | 1.4   | 2.1       | 5.3(-2)       |                                  |       |
|       | 1     | 1.3474280989683049814 | 4.7(-18) | 1.3(-19) | 36.98448540                  | 261.4002160 | 16.664 |
|       | 2     | 1.3474280989683049815 | 3.9(-299) | 1.0(-300) | 261.4002160                  | 16.000 |
| $PM1$ | 0     | 1.4   | 2.1       | 5.3(-2)       |                                  |       |
|       | 1     | 1.3474280989683049815 | 4.2(-21) | 1.1(-22) | 0.0335471585                 | 0.07545269758 | 16.275 |
|       | 2     | 1.3474280989683049815 | 2.3(-351) | 6.3(-353) | 0.07545269758                | 16.000 |
| $PM2$ | 0     | 1.4   | 2.1       | 5.3(-2)       |                                  |       |
|       | 1     | 1.3474280989683049815 | 2.1(-21) | 5.8(-23) | 0.01702960087                | 0.00486056658 | 15.574 |
|       | 2     | 1.3474280989683049815 | 2.9(-357) | 7.9(-359) | 0.00486056658                | 16.000 |