STABILITY OF THREE- AND FOUR-BODY COULOMB SYSTEMS

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ABSTRACT

We discuss the stability of three- and four-particle system interacting by pure Coulomb interactions, as a function of the masses and charges of the particles. We present a certain number of general properties which allow to answer a certain number of questions without or with less numerical calculations.

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1 Introduction

In this talk, I would like to speak of the problem of the stability of three- and four-body non-relativistic purely Coulombic systems. A system will be said to be stable if its energy is lower than the energy of any subdivision in subsystems. This is a restrictive definition of stability, because besides that there are other useful notions: “metastability” and “quasi-stability” on which we shall say only a few words later.

The works I will present are due, in what concerns the three-body case, to J.-M. Richard, T.T. Wu and myself. The four-body work is due to J.-M. Richard in collaboration with various other persons (including J. Fröhlich!)

The reasons why I decided to choose this subject for this workshop dedicated to Walter Thirring are that I know that Walter is interested in that topic and that the tools which are used are found precisely in Walter’s celebrated quantum mechanics course: concavity, scaling, and the Feynman-Hellmann theorem. One tries to avoid, as much as possible, numerical calculations, or to use already existing numerical calculations in particular cases. I will speak of:

i) three-body systems with equal absolute value of the charge, i.e., $-e + e + e$, or $+e - e - e$, since it is clear that $+e + e + e$ is unbound. Then binding or no binding will depend on the masses;

ii) three-body systems with unequal charges;

iii) four-body systems with charges with equal absolute value. It will be mostly $+e + e - e - e$. However, I shall say a word on $+e - e - e - e$.

2 Three-body case: equal $|\text{charges}|$

The problem we discuss now is whether a system of three charged particles (1,2,3), 1 having charge $+e$ and 2 and 3 charges $-e$, is stable or will dissociate into a two-body system and an isolated particle, (1,2)+3 or (1,3)+2. The system will be stable if the algebraic binding energy of the (123) system is strictly less than the binding energy of both (1,2) and (1,3). If, on the other hand, the infinimum of the spectrum of the (123) system coincides with the lowest of the (12) and (13) binding energies the system will be unstable.

This is an old problem which has been treated in many particular cases. For instance, long ago, Bethe has shown that the hydrogen negative ion ($pe^{-}e^{-}$) has one bound state \cite{1}, and Hill has shown that there is only one such bound state with natural parity \cite{2}, and Drake has also shown that there exists an unnatural parity state \cite{3} and finally Grosse and Pittner \cite{4} have shown also that this unnatural parity state is unique. In what follows we shall treat only the natural parity states, i.e., states such that $P = (-1)^L$, where $L$ is the total orbital angular
momentum (we neglect spin interactions!). For three particles there is no problem with the Pauli principle even if two of them are identical fermions, since we can adjust the spin.

Wheeler [5] has also shown that the system $e^+ e^- e^-$ is bound, and, more generally, Hill [3] has shown that any three-body system in which the two particles with the same sign of the charge have the same mass is stable. This covers the two previous cases.

As an example of an unstable system (there are many others!), we can give the proton-electron-negative muon system, for which a heuristic proof was given by Wightman in his thesis [6] and a rigorous proof was given by a collaboration including Walter Thirring himself [7].

Richard, Wu and myself [8] have tried to organise the results on stability, and, by using simple properties, save numerical calculations. From the reactions we had from experts on numerical calculations we believe that this was not totally useless. The three-body Schrödinger equation reads

$$\frac{-1}{2m_1} \Delta_1 \psi - \frac{1}{2m_2} \Delta_2 \psi - \frac{1}{2m_3} \Delta_3 \psi + \left[ -\frac{e^2}{r_{12}} - \frac{e^2}{r_{13}} + \frac{e^2}{r_{23}} \right] \psi = E\psi \tag{1}$$

and the corresponding two-body equations can be obtained by omitting some terms. It is obvious that we have scaling properties:

i) the charges can be multiplied by some arbitrary number without changing the stability problem;

ii) the masses can also be multiplied by an arbitrary number, so that the stability problem depends only on the ratio of the masses, i.e., of 2 parameters.

It will be convenient to introduce some variables:

- the inverse of the masses

$$x_1 = \frac{1}{m_1} \quad x_2 = \frac{1}{m_2} \quad x_3 = \frac{1}{m_3} \tag{2}$$

then the ground state energy of the system will be concave in $x_1, x_2, x_3$, and, in particular, concave in $x_1$ when $x_2$ and $x_3$ are fixed (and circular permutations!);

- the constrained inverse of the masses

$$\alpha_1 = \frac{x_1}{x_1 + x_2 + x_3} \text{ etc.} \tag{3}$$

such that

$$\alpha_1 + \alpha_2 + \alpha_3 = 1 \tag{4}$$
With these new variables, any system of three particles can be represented by a point in a triangle, \( \alpha_1, \alpha_2, \alpha_3 \) being the distances to the sides of the triangle. Figure 1 represents such a triangle with a few points representing some three-body systems.

\[
\begin{align*}
\alpha_1 &= 0 \\
\alpha_2 &= \alpha_3 = 0 \\
\end{align*}
\]

Figure 1:

It is of course sufficient, for the time being (i.e., for equal charges of 2 and 3) to consider the left half of the triangle, i.e., to assume \( m_2 \geq m_3 \). Let us remember that since we have, according to Hill’s theorem, strict stability for \( m_2 = m_3 \), i.e., \( \alpha_2 = \alpha_3 \), there will be some neighbourhood of the line \( \alpha_2 = \alpha_3 \) where we shall have stability. However, not all systems will be stable. We have already mentioned the \( pe^-\mu^- \) system as unstable. Another point where instability is obvious is the left summit marked 3, where we have two infinitely heavy particles with opposite charge producing zero attraction on the third particle. There is, therefore, an instability region in the left half triangle.

We have proved three theorems on the instability region in the left half triangle.

**Theorem I**

The instability region in the left half triangle is star-shaped with respect to summit 3.

The proof is based on the Feynman-Hellmann theorem combined with scaling. take a point \( P \) (Fig. 2) where the system is unstable or at the limit of stability. First we use the variables \( x_1, x_2, x_3 \). From the Feynman-Hellmann theorem, \( \frac{dE_{123}}{dx_3} > 0 \), if \( x_1 \) and \( x_2 \) are fixed. The binding energy of the subsystem 12 is fixed. Hence the residual binding can only increase (algebraically). \( x_3 \) moves from \( x_3(P) \) to infinity. The image of this in the rescaled \( \alpha \) variable is the segment, \( P3 \), where \( \alpha_1/\alpha_2 = \text{constant} \). If there is no binding at \( P \) there is no binding on the whole segment.

**Theorem II**

In the left-half triangle, the instability region is convex.

Take two points \( P' \) and \( P'' \) on the border of the stability domain inside the triangle with the \( \alpha \) variables. At \( P' \) and \( P'' \) we have \( E_{P'}(12) = E_{P'}(123), E_{P''}(12) = E_{P''}(123) \). It is possible to find a linear rescaling \( P \to M \) such that \( E_{M'}(12) = E_{M'}(123) = E_{M''}(123) = E_{M''}(123) \). Then
one can interpolate linearly between $M'$ and $M''$:

$$M_\lambda = \lambda M' + (1 - \lambda) M'' , \quad 0 < \lambda < 1 .$$

For any $M_\lambda$, $E_{M_\lambda}(12) = \text{const.}$ and $E_{M_\lambda}(123)$ is concave in $\lambda$, and therefore $E_{M_\lambda}(123) \geq E_{M'}(123) = E_{M''}(123)$. Returning to the original variables $\alpha_1 \alpha_2 \alpha_3$ and noticing that the scaling is linear we see that, on $P'' P'''$ we have $E(123) \geq E(12)$. Hence we have instability (Fig. 2).

There is, in fact, a more refined theorem, which we found, following a question by the late V.N. Gribov during a seminar in Budapest in 1996.

**Theorem III**

The domain (in the left-half triangle) where

$$\frac{E(123)}{E(12)} \leq 1 + \epsilon , \quad \epsilon > 0$$

is convex.

The meaning of this theorem is that the lines along which the relative binding is constant have a definite convexity. Note the sign of the inequality because $E(123)$ and $E(12)$ are both negative. We believe that the proof is essentially obvious, since, in the previous theorem, one goes through a rescaling, replacing $P$ and $P'$ by $M$ and $M'$ where the two-body energies are equal. Theorem II is of course becoming a special case of Theorem III, with $\epsilon \rightarrow 0$. The dotted line on Fig. 2 corresponds to some positive value of $\epsilon$.

Let us give a very simple application of Theorem II. We know that, according to Glaser et al., the system $p_\infty A^- e^-$ is unstable if $m_{A^-} > 1.57 m_{e^-}$ ($p_\infty$ means a proton with infinite mass). Similarly, we know that, from the work of Armour and Schrader [1] the system $p_\infty A^+ B^-$ is unstable if $m_{A^+}/m_{B^-} < 1.51$.

This means that $p_\infty e^- e^+$ is unstable (not because of annihilation that we neglect, but of dissociation into $p_\infty e^-$ and $e^+$). In Fig. 3, $p_\infty A^+ B^-$ and $p_\infty A^- e^-$ with the limit masses correspond respectively to $X$ and $Y$. Any point to the left of the segment $XY$ corresponds,
according to Theorem II to an unstable system. Therefore, $p e^+ e^-, p \mu^+ \mu^-$, with the actual mass of the proton, are unstable, and one can go up to $p z^- z^+$ which will be unstable if $m_p/m_z > 2.2$.

Figure 3:

We obtain too that the system $p \mu^- e^-$, with the actual proton mass is unstable, and also (disregarding again annihilation) $p \bar{p} e^-, p \bar{p} \mu^-$. One can also use convexity to get results in the opposite direction, i.e., prove that certain three-body systems are stable. We know that the systems represented by a point on the vertical bissector of the triangle are stable. In practice we know more than that, namely we have an estimate or more exactly a lower bound of the absolute value of the binding energy of many systems by using variational calculations and, in fact, by playing with convexity again it is possible to have a lower bound of the absolute value of the binding energy at any point on the bissector which corresponds to $\alpha_2 = \alpha_3$, $0 \leq \alpha_2 \leq 1$.

Now we use convexity along a horizontal line $\alpha_1 = \text{const}$. The systems $\alpha_1, \alpha_2, \alpha_3$, $\alpha_1, \alpha_3, \alpha_2$ represented in Fig. 2 by $Q$ and $Q'$, are of course completely equivalent. Hence

$$E_{123}(\alpha_1, \alpha_2, \alpha_3) = E_{123}(\alpha_1, \alpha_3, \alpha_2) < E_{123}\left(\alpha_1, \frac{\alpha_2 + \alpha_3}{2}, \frac{\alpha_2 + \alpha_3}{2}\right),$$

by convexity and

$$E_{123}\left(\alpha_1, \frac{\alpha_2 + \alpha_3}{2}, \frac{\alpha_2 + \alpha_3}{2}\right) = \left(1 + g(\alpha_1)\right) E_{12}\left(\alpha_1, \frac{\alpha_2 + \alpha_3}{2}\right) = \left(1 + g(\alpha_1)\right) E_{12}\left(\alpha_1, \frac{1 - \alpha_1}{2}\right),$$

where $g$ represents the relative excess in binding energy. We are assured of stability if

$$E_{12}(\alpha_1, \alpha_2) > \left(1 + g(\alpha_1)\right) E_{12}\left(\alpha_1, \frac{1 - \alpha_1}{2}\right)$$

i.e., if

$$\frac{e^2}{2} \frac{1}{\alpha_1 + \alpha_2} < \left(1 + g(\alpha_1)\right) \frac{e^2}{2} \frac{2}{1 + \alpha_1}$$
In this way, it is possible to prove that the system \( pd\mu^- \), important for fusion processes, is stable, through it is off the diagonal. One would like to show also in this way that \( \pi^+\mu^-\mu^+ \) and \( \mu^+\pi^+\pi^- \) are stable, but these considerations are not sufficient. There is a hint that they are stable because, from explicit calculations at \( \alpha_1 = 0 \) one sees that this method tends to give a band of stability which is two times narrower than the real one and this is just what one needs.

3 Three-body case. Unequal charges

On this topic, Richard, Wu and myself have published one paper [10] and one in preparation, of which I shall give some of the really new results. We have the right to take \( q_1 = 1 \), the charge of the particle which is opposite to the other two, of the same sign, \( q_2 \) and \( q_3 \).

A) Unequal charges, but \( q_2 = q_3 \)

This is the simplest case, very similar to the case of all equal charges. For fixed \( q_2 = q_3 \) we can again represent a system with the variables \( \alpha_1\alpha_2\alpha_3 \), and, on the bissector of summit 1, the energies of the subsystems (12) and (13) are equal. The fact that the instability regions are star-shaped with respect to 3 for the left-half of the triangle and to 2 for the right-half persists, and as well the convexity of the instability regions. There are two major differences which are:

i) that if \( q_2 = q_3 < 1 \), all three-body systems are stable, because near summit 3, for instance, the subsystem (12) is very compact and exerts a Coulomb attraction at long distances on particle 3; it may seem strange that as \( q_2 \to 1 \) part of the triangle becomes unstable, but this is just due to the fact that the binding energy, in that region, tends to zero as \( q_2 \to 1 \);

ii) that if \( q_2 = q_3 \) is large enough, stability disappears completely.

Figure 4 summarizes the situation. For \( q_2 > 1 \) but very close to 1, there is no qualitative difference but for a certain critical value \( 1 < q_{2c} < 1.1 \), the stability band breaks into two pieces, and from calculations by Hill and collaborators [11] stability near \( \alpha_2 = \alpha_3 = 1/2 \), \( \alpha_1 = 0 \) disappears completely for \( q_2 \geq 1.1 \), and from the calculations of Hogrève it disappears near \( \alpha_2 = \alpha_3 = 0 \), \( \alpha_1 = 1 \) for \( q_2 > 1.24 \). From convexity, it hence disappears completely along the segment joining \( \alpha_2 = \alpha_3 = 0 \) and \( \alpha_2 = \alpha_3 = 1/2 \), and from the star-shaped property, there is no stability at any point in the triangle for \( q_2 > 1.24 \).
B) Unequal charges, but $q_2 \neq q_3$ fixed

First we continue to use the $\alpha_1, \alpha_2, \alpha_3$ variables to describe the three-body system for fixed charges. A fundamental difference is that the bissector of summit 1 of the triangle no longer plays a special role. It is, instead, the line along which $E_{12} = E_{13}$, i.e.,

$$\frac{q_2^2}{\alpha_1 + \alpha_2} = \frac{q_3^2}{\alpha_1 + \alpha_3} \quad \text{or} \quad q_2^2 \left(1 - \alpha_2\right) = q_3^2 \left(1 - \alpha_3\right),$$

which becomes important. This line goes through the point $\alpha_2 = \alpha_3 = 1$, symmetric of summit 1 with respect to the line $\alpha_1 = 0$ (Fig. 5). The line divides the triangle into two subregions. If we decide to take $q_2 \geq q_3$, $E_{12} < E_{13}$ in the left region which contains the summit 1.

If $q_2$ and $q_3$ are both less than 1, we have again stability everywhere. If $q_2 \geq 1$ with $q_3 < 1$, part of the triangle becomes unstable. Various scenarios are shown in Fig. 6. Notice that summit 3, on the left is unstable, together with some neighbourhood. This is because
\[ q_3 < q_2 < 1 \]
\[ q_3 < q_2 < 1 \quad q_3 \text{ not too small} \]
\[ q_3 < q_2 = 1 \quad q_3 \text{ small} \]
\[ q_3 < q_2 < q_2 \quad q_3/q_2 \text{ not too small} \]
\[ q_3 < 1 < q_2 \quad q_3/q_2 \text{ small} \]

Figure 6:

\[ \alpha_1 \cong \alpha_2 \cong 0 \] corresponds to a very compact system with either very weak rapidly decreasing attraction on particle 3 (if \( q_2 = 1 \)) or repulsion (if \( q_2 > 1 \)).

For \( q_2 \) and \( q_3 \) both \( > 1 \), we have only a partial understanding of the situation. If they are both not too large, a neighbourhood of the dividing line will survive as stability region plus a neighbourhood of summit 1. Summits 2 and 3 will definitely be unstable.

We believe that if \( q_2 \) and \( q_3 \) are sufficiently large there is no stability at all, for any mass, but this is very difficult to implement quantitatively except in two places:

1. at summit 1, which corresponds to \( \alpha_2 = \alpha_3 = 0 \), and is the Born-Oppenheimer limit, for which Hogrèvé [12] has shown that one has instability if either \( q_2 \) or \( q_3 > 1.24 \);

2. at the point \( \alpha_1 = 0, \alpha_2 = \alpha_3 = 1/2 \) where Lieb’s theorem [13] applies: if \( 1/q_2 + 1/q_3 \leq 1 \) there is no stability.

This implies by convexity that if \( q_2 \) and \( q_3 > 2 \) one has instability for \( \alpha_2 = \alpha_3 \), and by the star property for \( \alpha_2 < \alpha_3 \) (if \( q_2 > q_3 \)). For a considerable improvement, see D.

C) \( q_2 \) and \( q_3 \) variable, fixed masses

Instead of holding charges fixed one can fix the masses and study stability in the \( q_2, q_3 \) plane. One particular case is \( m_2 = m_3 = \infty \) where one has the Born-Oppenheimer limit and one has the diagram calculated by Hogrèvé [12] (Fig. 7).
In the $q_2, q_3$ plane there is again for the general mass case a dividing line where the binding energies of the two subsystems (12) and (13) are equal:

$$q_2^2 \frac{m_2}{m_1 + m_2} = q_3^2 \frac{m_3}{m_1 + m_3}.$$ 

In the two sectors thus defined there are two instability regions for which we have been able to derive a new concavity property:

**Theorem IV**

Define $z_2 = 1/q_2$, $z_3 = 1/q_3$, the image of $q_2 > 0$ $q_3 > 0$ is $z_2 > 0$ $z_3 > 0$. Then, in the $z$ variables the two instability regions are **convex** (Fig. 8). The proof is based on a rescaling such that the binding energy of the relevant subsystem remains constant on a segment in the $z_2z_3$ plane.

**D) An illustration: the instability of the systems $\alpha p e^-$ or $\alpha p \mu^-$**

In the Born-Oppenheimer limit it is known that such systems are unstable [14]. Spruch and collaborators [15] have given arguments which seem to indicate that this might remain true for the actual masses of the protons and of the $\alpha$ particle, but, to our knowledge, a completely rigorous proof does not exist.
STEP 1  Take \( m_2 = m_3 \). Then if \( q_2 = q_3 \) we have instability if \( q_2 = q_3 \geq 1.24 \) and \( m_2 = m_3 = 0 \) and \( m_2 = m_3 = \infty \), and by concavity for any \( m_2 = m_3 \). Now consider the segment \( q_2 = 1.24, 0 < q_3 < 1.24 \). Along this segment the subsystem with the most negative binding energy is (12) and this energy is constant. There is no stability for \( q_2 = q_3 = 1.24 \), and no stability for \( q_2 = 1.24, q_3 \) very close to zero, because then particle 3 is submitted to a very weak force and is therefore very far away most of the time while 12 is overall repulsive for 3. By concavity there is no stability for \( q_2 = 1.24, 0 < q_3 < 1.24 \). The same kind of argument applies to \( 1.24 < q_3 < \infty \), because one has instability for \( q_3 \to \infty, q_2 > 1 \), and one can use concavity in the inverse charge. The conclusion is that if \( m_2 = m_3 \), one has no stability if either \( q_2 \geq 1.24 \) or \( q_3 \geq 1.24 \).

STEP 2  Assume \( q_2 \geq q_3 \). Then, in the whole sector, \( m_2 > m_3 \), the lowest two-body threshold is given by the (12) system, therefore in the \( \alpha \) triangle the region \( \alpha_2 < \alpha_3 \), is completely unstable for \( q_2 > 1.24 \) by use of the star shaped property.

The systems
\[
\begin{align*}
\alpha e^- & \quad \alpha \mu^- \\
n e^- & \quad \alpha d \mu^- \\
\alpha e^- & \quad \alpha t \mu^-
\end{align*}
\]
satisfy precisely the conditions: \( q_2 > 1.24, \quad q_3 = 1, \quad m_2 > m_3 \), and are therefore unstable, in the sense we have given to “instability”. Notice that the proof would fail if the particle with charge 2 was lighter than the particle with charge 1.

However, as pointed out by for instance Gerstein [16], some of the levels of these systems are “quasi stable” in the Born-Oppenheimer approximation in the sense that the minimum of the Born-Oppenheimer potential is below the value it takes for infinite separation between the two nuclei where one of the limit atomic states is excited and degenerate with the other one. Gerstein [17] went as far as estimating the lifetimes of these quasi-stable states and showed that the lifetime increases drastically when the proton is replaced by a triton. One should also mention the metastability, where the Born-Oppenheimer curve has a minimum above zero [12].

4  Four-body case: equal charges

Most of what I will say concerns systems of two positive and two negative charges of absolute value \( e \).

However, let me start with the case
\[
pe^- e^- e^- ,
\]
i.e., a doubly negative hydrogen ion.

Such a state according to a review by Hogrèве [18] does not seem to exist. In the limit of an infinitely heavy proton, the Lieb bound on \( n \), the number of electrons around a charge \( Z \) [13], \( n < 2Z + 1 \), which is a strict inequality, gives \( n < 3 \). In fact no doubly negative atomic
ions seem to exist in nature, while singly negative ions may (like $H^-$) or may not exist (like the case of the rare gases).

We return now to systems with charges $-e-e+e+e$, and first of all $m^-m^-M^+M^+$, i.e., two negatively charged particles with equal mass and two positively charged particles with equal masses. A familiar example is the hydrogen molecule $e^-e^-p^+p^+$. A more exotic example is the positronium molecule $e^-e^-e^+e^+$.

It has been realized by Jurg Fröhlich that up to very recently there did not exist any rigorous proof of the stability of the hydrogen molecule. It was believed to be stable because of experiment of course, and of Born-Oppenheimer calculations.

Two groups (Fröhlich et al., Richard) investigated this problem and finally joined their efforts to produce a completely rigorous proof [19].

The simplest approach, whose idea comes from J.-M. Richard [20] consists of starting from the work of Øre, which is valid by scaling for a system $A^-A^-A^+A^+$. Øre used a very simple variational trial function, of the form

$$
\psi = \exp\left(-\frac{1}{2}\left(r_{13} + r_{14} + r_{23} + r_{24}\right)\right) \cosh\left[\frac{\beta}{2}\left(r_{13} - r_{14} - r_{23} + r_{24}\right)\right]
$$

Notice that the distances between particles with same charge sign do not appear. All integrals can be carried analytically and it is found that the energy is less than

$$
2.0168 \ E_0 (A^+A^-)
$$

The system is therefore stable because it cannot dissociate into $A^+A^- + A^+A^+$. It cannot dissociate either in $A^+A^-A^+ + A^-$, because between these two systems there is a long-distance Coulomb force, producing unavoidably infinitely many bound states.

If we take now

$$
x_e + x_p = \frac{1}{m_e} + \frac{1}{m_p} = \frac{2}{m_A},
$$

we see that the binding energy of $e^-p$ is the same as that of $A^+A^-$. However,

$$
E(x_e, x_e, x_p, x_p) = E(x_p, x_p, x_e, x_e) < E\left(\frac{x_p + x_e}{2}, \frac{x_p + x_e}{2}, \frac{x_p + x_e}{2}, \frac{x_p + x_e}{2}\right),
$$

by concavity in the inverse masses.

So,

$$
E(p^+, p^+, e^-, e^-) < E(A^+, A^+, A^-, A^-) < 2E(A^+A^-)
$$

Hence, $ppee$ is stable.
One can wonder if stability remains if the masses of two particles of the same charge are different, i.e.,

\[ A^+ B^+ C^- C^- \]

Then there is still a unique possible dissociation threshold:

\[ A^+ C^- + B^+ C^- \]

Øre has predicted explicitly that the system \( p e^+ e^- \) is stable [22], and this has been observed experimentally by Schräder and collaborators [23].

It is also easy, from the upper bound of the energy of \( e^+ e^- e^- \) to show that the systems \( p d e^- e^- \), \( p t e^- e^- \), \( d t e^- e^- \) are stable. This is implicit in the work of Richard [20], established in the thesis of Seifert [24] and I present here my own version. By concavity we have

\[
E(x_A, x_B, x_C, x_C) < E\left(\frac{x_A + x_B}{4}, \frac{x_A + x_B}{4}, x_C\right)
\]

\[
< E\left(\frac{x_A + x_B}{4} + \frac{x_C}{2}, \frac{x_A + x_B}{4} + \frac{x_C}{2}, \frac{x_A + x_B}{4} + \frac{x_C}{2}\right)
\]

\[
< -2.0168 \left(\frac{1}{4} x_A + \frac{1}{4} x_B, x_C\right)
\]

If the inequality

\[
-2.0168 \left(\frac{1}{4} x_A + \frac{1}{4} x_B, x_C\right) < - \left(\frac{1}{2} x_A + \frac{1}{2} x_B + \frac{1}{2} x_C\right)
\]

is satisfied, the system is stable. If \( m_A > m_B > m_C \) one finds that this condition is certainly satisfied if \( m_B > 5m_C \).

Using the more refined bound [19]

\[
E_{A^+ A^- A^- A^-} < -2.06392E(A^+ A^-)
\]

which uses more sophisticated trial function and must be “cleaned” from numerical roundup errors, one gets

\[ m_B > 2.45m_C \]

However, Varga and collaborators [25], using trial functions leading to integrals which can be expressed analytically, and adjusting parameters, have found that one has stability for any \( m_A \) and \( m_B \), including the case where one or two of them are less than \( m_C \). By a tedious but feasible exercize, one could, using concavity, transform this calculation, which is unavoidably done for discrete values of the masses, into a very inelegant proof. Let us hope that someone, in the future, will find a still more clever trial function and avoid this.

Let us mention, finally, a conjecture by Jean-Marc Richard: “A four-body system with two positive-charge and two negative-charge particles (equal in absolute value) is stable if one three-body subsystem is stable”.

12
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