Design of robust iterative learning control schemes for systems with polytopic uncertainties and sector-bounded nonlinearities

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Abstract. This paper deals with designing of iterative learning control schemes for uncertain systems with static nonlinearities. More specifically, the nonlinear part is supposed to be sector bounded and system matrices are assumed to range in the polytope of matrices. For systems with such nonlinearities and uncertainties the repetitive process setting is exploited to develop a linear matrix inequality based conditions for computing the feedback and feedforward (learning) controllers. These controllers guarantee acceptable dynamics along the trials and ensure convergence of the trial-to-trial error dynamics, respectively. Numerical examples illustrate the theoretical results and confirm effectiveness of the designed control scheme.

1. Introduction
Iterative learning control (ILC) is an effective method for providing a significant improvements of the tracking performance for systems that operate in a repetitive mode. The novel feature of this control scheme is to take a benefit of learning of the feedforward signals for subsequent task executions by iteratively updating them based on the control error accumulated in the previous task executions. As a result of successive repetitions or trials, a high performance feedforward signal is provided and hence a small tracking error is obtained by removing the repetitive part of the error. Specifically, the objective in ILC design is to construct the input such that the corresponding output precisely tracks a reference signal that is specified over a finite time interval. A particular feature is the availability to the control law of temporal information that would be non-causal outside the ILC setting, e.g., at sample instant

$$p + \lambda, \lambda > 0,$$

can be used provided it has been generated on a previous trial.

Since the original work of Arimoto [2], ILC has remained as a significant area of control systems research with many algorithms experimentally verified in the research laboratory and applied in many industrial applications - see [1, 5] for many practical systems and processes where ILC has been successfully applied. Particular application examples include industrial robotics, see, e.g., [9], where the pick and place operation common in many mass manufacturing processes is an immediate fit to ILC, and wafer stage motion systems, see, e.g., [8]. Also, the survey paper [18] focuses on run-to-run control, as often found in the chemical process industries.

A literature review, see, e.g., [1, 5], shows that a popular approach used for ILC design is to first apply a feedback control law to produce acceptable along the trial dynamics and then
apply ILC to force trial-to-trial error convergence of the resulting system. This means that the objective of stabilization of the system in the time domain and convergence of the ILC scheme in the trial domain can be written as convergence conditions on the tracking error and control input
\[
\lim_{k \to \infty} \| e_k \| = 0, \quad \lim_{k \to \infty} \| u_k - u_\infty \| = 0,
\]
where \( \| \cdot \| \) is an appropriate vector norm and \( u_\infty \) is termed the learned control. The reason for the condition on the control input sequence is to prevent large increases in the control vector from one trial to the next.

The main drawback of the classical synthesis procedure is that it does not lead to an optimal combination of the feedback and feedforward control actions. In particular, since the convergence condition includes terms related to the feedback controller, it should be treated simultaneously with the learning controller. Therefore, an obvious alternative to classical approach is to use a repetitive process setting for analysis [12–14]. This seems to be a natural setting for ILC analysis and design since the dynamics evolve in two independent directions and information in the temporal domain is limited to a finite duration. The main advantage of this approach is that gives a systematic way to simultaneously consider behaviour along the time axis and from trial-to-trial [15].

The analysis of linear repetitive processes has received considerable attention in the literature, see, e.g. [14]. Although this approach is relatively well known in the ILC area, there remain open problems to be solved. In this paper, the repetitive setting is applied and the results on the inclusion of nonlinear dynamics for uncertain systems are developed and hence the results of [11] are significantly extended.

As it is well known, an important class of nonlinear systems is a feedback system whose forward path contains a linear time invariant subsystem and the feedback path contains a memoryless nonlinearity, see [16] or [17] for some further details on this class of systems. In particular, systems where the nonlinearity satisfies certain sector condition (i.e. Lur’e systems) are treated. Such a description can be used to model many nonlinear dynamics arising in applications, such as systems with saturation and deadzone nonlinearities that can be treated as a linear system in connection with a nonlinearity satisfying some sector conditions. Hence, based on sector nonlinearity properties, the paper objective is to extend an approach of [11] to ILC design for uncertain systems with sector bounded nonlinearities in the repetitive process setting. Obviously, such results are required since practical control specifications require to involve the size of perturbation which affect system matrices.

This paper uses a repetitive setting to develop robust ILC design conditions for uncertain discrete linear systems with sector-bounded nonlinearities. It has to be emphasized that design is performed in discrete domain only, since practical implementation requires the control and error signals to be stored in a digital memory. Furthermore, a set of transformations is used to express the design conditions in terms of linear matrix inequalities (LMIs) for calculation purposes. A numerical example is also given to demonstrate the effectiveness of the design.

This paper is organized as follows. In Section 2, repetitive processes will be addressed in more detail as they are the basis for the main result in this paper and Section 3, an ILC scheme for uncertain systems with vector-valued nonlinear dynamics is embedded into the repetitive process setting. The main result of the paper on design in the presence of polytopic uncertainties and sector-valued nonlinearities is developed in Section 4. A numerical example is given in Section 5 to illustrate the effectiveness of the new design algorithm and is based on the model of an inverted pendulum with nonlinearities. The main results are summarized in Section 6 together with some possible areas for further research.

The notation used throughout the paper is fairly standard. The null and identity matrices with the required dimensions are denoted by 0 and I, respectively, and the notation \( X \succ Y \)
Lemma 1. [7] Given a symmetric matrix \( \Gamma \in \mathbb{R}^{p \times p} \) and two matrices \( \Lambda, \Sigma \) of column dimension \( p \), there exists a matrix \( W \) such that the following inequality holds

\[
\Gamma + \text{sym}\{\Lambda^T W \Sigma\} < 0,
\]

if, and only if the following two projection inequalities with respect to \( W \) are satisfied

\[
\Lambda^T \Gamma \Lambda^T < 0, \quad \Sigma^T \Gamma \Sigma^T < 0,
\]

where \( \Lambda^\perp \) and \( \Sigma^\perp \) denote the orthogonal complement of \( \Lambda \) and \( \Sigma \), respectively; i.e., \( \Lambda \Lambda^\perp = 0 \) and \( \Sigma \Sigma^\perp = 0 \).

Lemma 2. [4] Let \( \Omega_1(\omega) \) and \( \Omega_2(\omega) \) be two arbitrary quadratic forms of \( \omega \in \mathbb{R}^n \), then the following two conditions are equivalent

(i) \( \omega^T \Omega_1(\omega) \omega \leq 0, \forall \omega \neq 0 \) such that \( \omega^T \Omega_2(\omega) \omega \leq 0 \).

(ii) \( \exists \tau \geq 0 \) such that \( \Omega_1(\omega) - \tau \Omega_2(\omega) < 0 \).

Before formulating the design problem, we introduce some useful preliminaries on linear repetitive processes needed to develop the proposed main contributions.

2. Discrete Linear Repetitive Processes

Linear repetitive processes are one of the most important classes of two-dimensional (2D) linear systems of both industrial and algorithmic interest [14, 15]. The essential unique characteristic of such processes is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length and here denoted by \( \alpha \). On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the next pass profile. This characteristic shows that linear repetitive processes have clear 2D system structure and hence it is natural to exploit links between these processes and ILC schemes.

Following [14], the state-space model of a discrete linear repetitive process has the following form over \( 0 \leq p \leq \alpha - 1, \ k \geq 0, \ \alpha < +\infty \)

\[
\begin{align*}
x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + B_0y_k(p), \\
y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + D_0y_k(p),
\end{align*}
\]

where on pass \( k \), \( x_k(p) \in \mathbb{R}^n \) is the state vector, \( y_k(p) \in \mathbb{R}^m \) is the output (pass profile) vector and \( u_k(p) \in \mathbb{R}^l \) is the control input vector. In addition, the terms \( B_0y_k(p) \) and \( D_0y_k(p) \) represent the contribution of the previous pass profile to the current pass state and output vectors respectively.

Also, it is required to specify the boundary conditions, i.e., the state initial vector on each pass and the initial pass profile (i.e., on pass 0). Without loss of generality, it is assumed that the state initial vector at the start of each new pass is zero and that the entries in initial pass profile vector \( y_0(p) \) are known functions of \( p \) over the pass length.

It is important to note that several sets of necessary and sufficient conditions for stability of discrete linear repetitive processes of the form considered here are known [14]. From the

\[ \text{Lemma 1.} \quad \text{Lemma 2.} \]
practical point of view, the most important are those which guarantee asymptotic stability or stability along the pass. Asymptotic stability guarantees a bounded sequence of pass profiles (output signals) for a bounded initial pass profile over the finite and fixed pass length $\alpha$, whereas stability along the pass is stronger since it requires this property uniformly, that is, for all possible values of the pass length and hence, it is not surprising that asymptotic stability is a necessary condition for stability along the pass.

The following result characterizes stability along the pass of processes described by (1).

**Lemma 3.** [14] A discrete linear repetitive process represented by (1) is stable along the pass if and only if

1. $\rho(D_0) < 1$,
2. $\rho(A) < 1$,
3. all eigenvalues of $G(z) = C(zI - A)^{-1}B_0 + D_0$, $\forall|z| = 1$ have modulus strictly less than unity.

Obviously, the main problem to solve is to find the efficient condition for checking the third condition only. More precisely, the third condition requires to check if all eigenvalues of $\mathcal{G}(e^{j\omega})$, i.e., the characteristic loci, lie inside the unit circle in the complex plane. In addition, transformation of this condition into an LMI based condition is not trivial. However, as shown in [10, 13], by making extensive use of the Kalman-Yakubovich-Popov lemma the third condition in Lemma 3 can be converted into the LMIs, leading to the following result that provides the basis for the control law design results developed in this paper.

**Lemma 4.** [10] A discrete linear repetitive process described by (1) is stable along the pass if there exist $Q > 0$ and $P > 0$ such that the following LMI

$$
\begin{bmatrix}
-Q & QA^T & QC^T & 0 \\
AQ & -Q & 0 & B_0P \\
CQ & 0 & -P & D_0P \\
0 & PB_0^T & PD_0^T & -P
\end{bmatrix} < 0, \tag{2}
$$

is feasible. Clearly, if the above inequality is feasible, then

$$
\begin{bmatrix}
-Q & QA^T \\
AQ & -Q
\end{bmatrix} < 0, \text{ and } \begin{bmatrix}
-P & D_0P \\
PD_0^T & -P
\end{bmatrix} < 0,
$$

and it is straightforward to conclude that conditions i) and ii) of Lemma 3 hold. Furthermore, by the Schur’s complement formula, it follows that (2) can be converted to the form

$$
\begin{bmatrix}
A & I \\
C & 0
\end{bmatrix} \begin{bmatrix}
Q & 0 \\
0 & -Q
\end{bmatrix} \begin{bmatrix}
A & I \\
C & 0
\end{bmatrix}^T + \begin{bmatrix}
B_0 & 0 \\
D_0 & I
\end{bmatrix} \begin{bmatrix}
P & 0 \\
0 & -P
\end{bmatrix} \begin{bmatrix}
B_0 & 0 \\
D_0 & I
\end{bmatrix}^T < 0.
$$

Using the result of Kalman-Yakubovich-Popov lemma (see [10, 13] for details of required transformations) it follows that there exists $P > 0$ such that

$$
\begin{bmatrix}
C(e^{j\omega}I - A)^{-1} & I \\
D_0 & I
\end{bmatrix} \begin{bmatrix}
B_0 & 0 \\
D_0 & I
\end{bmatrix} \begin{bmatrix}
P & 0 \\
0 & -P
\end{bmatrix} \begin{bmatrix}
B_0^T & D_0^T \\
0 & I
\end{bmatrix} \begin{bmatrix}
C(e^{j\omega}I - A)^{-T}C^T \\
I
\end{bmatrix} < 0,
$$

is satisfied $\forall \omega \in [-\pi, \pi]$. This last inequality implies that condition iii) of Lemma 3 must hold. On the other side, the LMI of (2) can be derived by applying the Lyapunov theory. This approach, see [14] for more details, uses the following candidate Lyapunov function for processes described by (1)

$$
V(k,p) = V_1(k,p) + V_2(k,p) = x_{k+1}^T(p)Qx_{k+1}(p) + y_k^T(p)Py_k(p), \tag{3}
$$

for some $Q > 0$ and $P > 0$. This function is a combination of two independent functions due to the two-dimensional character of the systems considered. Also

$$\Delta V_1(k, p) = x_{k+1}^T(p + 1)Qx_k(p) + 1 - x_k^T(p)Qx_k(p),$$

$$\Delta V_2(k, p) = y_{k+1}^T(p)Py_k(p) - y_k^T(p)Py_k(p),$$

and the associated increment for (3) is calculated as

$$\Delta V(k, p) = \Delta V_1(k, p) + \Delta V_2(k, p). \tag{4}$$

Using this approach, a version of (2) is obtained by using the below result.

**Lemma 5.** [14] A discrete linear repetitive process described by (1) is stable along the pass if

$$\Delta V(k, p) < 0. \tag{5}$$

Clearly, based on the above inequality one can obtain the following condition for checking of stability along the pass that allows us to handle the considered control problem

$$\begin{bmatrix} x_{k+1}(p) \\ y_{k+1}(p) \end{bmatrix}^T \begin{bmatrix} A & B_0 \\ C & D_0 \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} x_{k+1}(p) \\ y_{k+1}(p) \end{bmatrix} < 0. \tag{6}$$

3. Problem formulation

Consider a class of discrete-time systems with sector-bounded nonlinearities and time-invariant polytopic uncertainties represented by the following state-space model in the ILC setting as

$$\begin{align*}
x_{k+1}(p + 1) &= A(v)x_{k+1}(p) + B(v)u_{k+1}(p) + f(x_{k+1}(p), p), \\
y_{k+1}(p) &= C(v)x_{k+1}(p), \tag{7}
\end{align*}$$

where $k$ is the trial number, $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}^m$ is the output vector and $u_k(p) \in \mathbb{R}^l$ is the control input vector. $A(v)$, $B(v)$ and $C(v)$ are constant matrices of appropriate dimensions and belong to the following uncertainty polytope of the form

$$\left\{ \left[ A(v) B(v) C(v) \right] = \sum_{i=1}^{N} v_i \left[ A_i B_i C_i \right], v_i \geq 0, \sum_{i=1}^{N} v_i = 1 \right\}. \tag{8}$$

In the above description, $A_i$, $B_i$, $C_i$ define $N^{th}$ vertices of a polytope in which it is assumed that the actual matrices $A$, $B$ and $C$ lie.

**Remark 1.** It has to be emphasized that known results on designing of ILC schemes for polytopic systems require to impose some constraints on the system matrices. In particular it is assumed that $C(v)$ is fixed, i.e., $C(v) = C$. The result developed in this paper avoids this drawback by application of the approach presented in [6].

Furthermore, it is assumed that $f(x_{k+1}(p), p) \in \mathbb{R}^n$ is a given nonlinear function that satisfy the following condition

$$\|f(x_{k+1}(p), p)\| \leq \zeta \|Mx_{k+1}(p)\|,$$

where $\zeta > 0$ is the bounding parameter on the uncertain disturbances function $f$, $M \in \mathbb{R}^{n \times n}$ is known constant matrix and $\| \cdot \|$ is Euclidean norm - see [16] or [17] for details on this condition. Maximize parameter $\zeta$ leads to an increased the level of robustness of the system for non-linear
disturbances, so the parameter $\zeta$ can be defined as a degree of robustness. A given function $f(x_{k+1}(p), p)$ satisfies the following constraints

$$f^T(x_{k+1}(p), p)f(x_{k+1}(p), p) \leq \zeta^2 f^T_{k+1}(p)M^TMx_{k+1}(p). \quad (9)$$

The above quadratic constraints can be written in the following matrix form

$$\begin{bmatrix} x_{k+1}(p) \\ f(x_{k+1}(p), p) \end{bmatrix}^T \begin{bmatrix} -\zeta^2 M^TM & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_{k+1}(p) \\ f(x_{k+1}(p), p) \end{bmatrix} \leq 0. \quad (10)$$

Linear repetitive process with the nonlinear disturbances is stable along the pass with $\zeta > 0$ if and only if all three conditions of Lemma 3 are satisfied for all $f(x_{k+1}(p), p)$ such that inequality (9) holds.

The primary problem here is to find a practical guideline for design of ILC schemes for systems represented by (7). To proceed assume that $y_d(p)$ denotes the reference trajectory and then the error on trial $k$ is

$$e_k(p) = y_d(p) - y_k(p).$$

The control signal can be written as the sum of current input signal $u_k(p)$ and $\Delta u_{k+1}(p)$

$$u_{k+1}(p) = u_k(p) + \Delta u_{k+1}(p),$$

where $\Delta u_{k+1}(p)$ denotes the modification term added to the previous pass input.

To formulate the design problem in a discrete linear repetitive process setting assume that the modification term in the ILC law takes the form

$$\Delta u_{k+1}(p) = K_1 \eta_{k+1}(p) + K_2 e_k(p). \quad (11)$$

In the above equation $K_1$ is the controller in the feedback loop and $K_2$ is the learning controller in the feedforward loop. Clearly, the main problem is to provide the procedure for computing both $K_1$ and $K_2$. To proceed, suppose that the vectors $\eta_{k+1}(p)$ and $\psi_{k+1}(p)$ are defined as

$$\eta_{k+1}(p) = x_{k+1}(p-1) - x_k(p-1),$$

$$\psi_{k+1}(p) = f(x_{k+1}(p-1), p-1) - f(x_k(p-1), p-1). \quad (12)$$

The above vectors $\eta_{k+1}(p)$ and $\psi_{k+1}(p)$ represent the the difference between the current state $x_{k+1}$ and the state in the previous iteration $x_k$, and the difference between the non-linear distortion in the current $f(x_{k+1})$ and previous iterations $f(x_k)$. Based on previous equations the controlled dynamics can be written as

$$\eta_{k+1}(p+1) = A(v)\eta_{k+1}(p) + B_0(v)e_k(p) + \psi_{k+1}(p),$$

$$e_{k+1}(p) = C(v)\eta_{k+1}(p) + D_0(v)e_k(p), \quad (13)$$

where

$$A(v) = A(v) + B(v)K_1, \quad B_0(v) = B(v)K_2,$$

$$C(v) = - C(v)A(v) - C(v)B(v)K_1, \quad D_0(v) = I - C(v)B(v)K_2. \quad (14)$$

The obtained model (13) in the state space form based on the discrete linear repetitive model given by equation (1). The state vector is given as $\eta_{k+1}(p)$, pass profile vector as $e_k(p)$ and $\psi_{k+1}(p)$ is a nonlinear term. Treated as a repetitive process, the stability along the pass property will force the trial-to-trial error to converge to zero, see the next section.
4. Main result

In this section, the stability theory for discrete linear repetitive processes is used to develop a new ILC design algorithm for systems with polytopic uncertainties for systems with sector-bounded nonlinearities. To proceed, observe that the result of (9) and equation (12) yields

$$\psi_{k+1}(p) \psi_{k+1}(p) \leq \zeta^2 \eta_{k+1}(p) M^T M \eta_{k+1}(p),$$

and then the above inequality which can be written in the matrix form as

$$\begin{bmatrix} \eta_{k+1}(p) \\ \epsilon_{k+1}(p) \\ \psi_{k+1}(p) \end{bmatrix}^T \begin{bmatrix} -\zeta^2 M^T M & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \eta_{k+1}(p) \\ \epsilon_{k+1}(p) \\ \psi_{k+1}(p) \end{bmatrix} \leq 0. \quad (16)$$

Hence, based on presented transformations, we have the following theorem that provides the sufficient condition for stability along the pass (with degree $\zeta > 0$) of ILC scheme represented by (13).

**Theorem 1.** Consider an ILC scheme described as a discrete linear repetitive process of the form (13). Then stability along the pass holds with degree $\zeta > 0$ and hence trial-to-trial error convergence occurs if there exist matrices $Q > 0$, $P > 0$ of appropriate dimensions and a positive scalar $\tau$ such that the following inequality is feasible

$$\Omega_1 - \tau \Omega_2 < 0, \quad (17)$$

where

$$\Omega_1 = \begin{bmatrix} \Delta^T(v) Q \Delta(v) - Q + C^T(v) PC(v) & \Delta^T(v) Q B_0(v) + D_0^T(v) PC(v) & \Delta^T(v) Q \\ B_0^T(v) Q \Delta(v) + C^T(v) PD_0(v) & B_0^T(v) Q B_0(v) + D_0^T(v) PD_0(v) - I & B_0^T(v) Q \\ Q A(v) & Q B_0(v) & Q \end{bmatrix}, \Omega_2 = \begin{bmatrix} -\zeta^2 M^T M & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}.$$

*Proof.* Assume that the inequality of (17) is feasible for some $Q > 0$, $P > 0$ and $\tau > 0$. Then, by the result of Lemma 2, the inequality (17) holds if

$$\omega^T \Omega_1 \omega \leq 0, \forall \omega \neq 0 \text{ such that } \omega^T \Omega_2 \omega \leq 0.$$

Next, set $\omega = [\eta_{k+1}(p), \epsilon_{k+1}(p), \zeta_{k+1}(p)]^T$ and then $\omega^T \Omega_2 \omega \leq 0$ is equivalent to (16). Furthermore, observe that $\omega^T \Omega_1 \omega \leq 0$ holds when

$$\begin{bmatrix} \eta_{k+1}(p + 1) \\ \epsilon_{k+1}(p) \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} \eta_{k+1}(p + 1) \\ \epsilon_{k+1}(p) \end{bmatrix} - \begin{bmatrix} \eta_{k+1}(p) \\ \epsilon_{k}(p) \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} \eta_{k+1}(p) \\ \epsilon_{k}(p) \end{bmatrix} < 0.$$

Obviously, by application of Lemma 5 to a system represented by (13) we can establish that feasibility of the above inequality guarantees stability along the pass and the proof is complete. \qed

Unfortunately, the condition of Theorem 1 cannot be implemented due to their infinite-dimensional nature in the parameter $v$. Moreover, the inequality (17) is not LMI as it is bilinear in $Q$, $P$ and the matrices defining the controller (11). This means no computationally effective method exists for checking (17). However, by using a particular set of transformations, the inequality (17) can be converted into an LMI form where the matrix variables $Q$ and $P$ are separated from the linear repetitive process matrices. Therefore, the affine parameter dependent matrix variables of the following form can be introduced

$$Q(v) = \sum_{i=1}^{N} v_i Q_i, \quad P(v) = \sum_{i=1}^{N} v_i P_i. \quad (18)$$

The above matrix variables can take $N$ different values corresponding to the vertices of the polytope (8). As the result, less conservatism can be attained.
Theorem 2. Consider an ILC scheme described as a discrete linear repetitive process of the form (11) and (13). Then stability along the pass holds for a given $\tau > 0$ with degree $\zeta > 0$ and hence trial-to-trial error convergence occurs if there exist matrices $Q_i > 0$, $P_i > 0$, $N_1$, $N_2$, $G_1$ and $G_2$ of appropriate dimensions such that the following LMIs are feasible

$$
\Pi_{ii} < 0, \quad (i = 1, 2, \ldots, N),
$$

$$
\Pi_{ij} + \Pi_{ji} < 0, \quad (i, j = 1, 2, \ldots, N),
$$

where

$$
\Pi_{ij} = 
\begin{bmatrix}
-\hat{Q}_{ij} & 0 & 0 & 0 & \hat{Q}_{ij}M^T & 0 & 0 & 0 \\
(*) & -\frac{1}{\tau}\hat{P}_{ij} & 0 & 0 & 0 & 0 & 0 & 0 \\
(*) & (*) & -I & I & 0 & 0 & 0 & 0 \\
(*) & (*) & (*) & -\hat{Q}_{ij} & 0 & 0 & A_iG_1 + B_iN_1 & B_iN_2 \\
(*) & (*) & (*) & (*) & -\beta I & 0 & 0 & 0 \\
(*) & (*) & (*) & (*) & (*) & (*) & (* & G_2 - C_iB_iN_1 \\
(*) & (*) & (*) & (*) & (*) & (*) & (*) & -G_2 - G_2^T \\
\end{bmatrix},
$$

and $\beta = 1/\sqrt{\zeta}$. Moreover, when the above LMIs hold then the required controller gains $K_1$ and $K_2$ of (11) are computed as

$$
K_1 = N_1G_1^{-1}, \quad K_2 = N_2G_2^{-1}.
$$

Proof. Suppose that the LMIs (19) and (20) are feasible. Then $G_1 + G_2^T > 0$, and $G_2 + G_2^T > 0$, which implies that the matrices $G_1$ and $G_2$ are non-singular and hence invertible. Furthermore, from (8), (19) and (20)

$$
\sum_{i=1}^{N} v_i^2 \Pi_{ii} + \sum_{i=1}^{N} \sum_{i<j} v_i v_j (\Pi_{ij} + \Pi_{ji}) = \sum_{i=1}^{N} \sum_{j=1}^{N} v_i v_j \Pi_{ij} = 
$$

$$
= 
\begin{bmatrix}
-\hat{Q}(v) & 0 & 0 & 0 & \hat{Q}(v)M^T & 0 & 0 & 0 \\
(*) & -\frac{1}{\tau}\hat{P}(v) & 0 & 0 & 0 & 0 & 0 & 0 \\
(*) & (*) & -I & I & 0 & 0 & 0 & 0 \\
(*) & (*) & (*) & -\hat{Q}(v) & 0 & 0 & \Upsilon_2(v) & B(v)N_2 \\
(*) & (*) & (*) & (*) & -\beta I & 0 & 0 & 0 \\
(*) & (*) & (*) & (*) & (*) & (*) & (-G_1 - G_1^T) & G_2 - C(v)B(v)N_2 \\
(*) & (*) & (*) & (*) & (*) & (*) & (*) & -G_2 - G_2^T \\
\end{bmatrix},
$$

where $\Upsilon_1(v) = -C(v)(A(v)G_1 + B(v)N_1)$, $\Upsilon_2(v) = A(v)G_1 + B(v)N_1$. Next, observe that the resulting matrix in (21) can be rewritten as

$$
\begin{bmatrix}
\Phi(v) \\
J^T(v) + G^T R^T(v) \\
\end{bmatrix} < 0
$$

where $G = \text{diag}(G_1, G_2)$ and

$$
\Phi(v) = 
\begin{bmatrix}
-\hat{Q}(v) & 0 & 0 & 0 & \hat{Q}(v)M^T & 0 \\
(*) & -\frac{1}{\tau}\hat{P}(v) & 0 & 0 & 0 & 0 \\
(*) & (*) & -I & I & 0 & 0 \\
(*) & (*) & (*) & -\hat{Q}(v) & 0 & 0 \\
(*) & (*) & (*) & (*) & -\beta I & 0 \\
(*) & (*) & (*) & (*) & (*) & -\hat{P}(v) \\
\end{bmatrix},
$$

$$
J^T(v) + G^T R^T(v) = 
\begin{bmatrix}
\hat{Q}(v)M^T & 0 \\
(*) & 0 \\
(*) & (*) & 0 \\
(*) & (*) & (*) & 0 \\
(*) & (*) & (*) & (*) & 0 \\
\end{bmatrix}.
$$
\[
R(v) = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
A(v) + B(v)K_1 & B(v)K_2 \\
-C(v)(A(v) + B(v)K_1) & I - C(v)B(v)K_2
\end{bmatrix}, \quad J(v) = \begin{bmatrix}
\hat{Q}(v) & 0 \\
0 & \hat{P}(v)
\end{bmatrix}.
\]

Since \( \Phi(v) < 0 \), a direct application of Lemma 1 yields
\[
\Phi(v) + R(v)J^T(v) + J(v)R^T(v) < 0.
\]

Also, the above inequality can be rewritten as
\[
\begin{bmatrix}
-\hat{Q}(v) & 0 & 0 & \hat{Q}(v)A^T & \hat{Q}(v)M^T & \hat{Q}(v)C^T \\
0 & -\frac{1}{\tau}\hat{P}(v) & 0 & P(v)B_0^T & 0 & \hat{P}(v)D_0^T \\
0 & 0 & -I & I & 0 & 0 \\
A\hat{Q}(v) & B_0P(v) & I & -\hat{Q}(v) & 0 & 0 \\
M\hat{Q}(v) & 0 & 0 & 0 & -\beta I & 0 \\
C\hat{Q}(v) & D_0\hat{P}(v) & 0 & 0 & 0 & -\hat{P}(v)
\end{bmatrix} \prec 0,
\]

where \( A, B_0, C \) and \( D_0 \) are defined in (14). Next, post- and pre-multiply the above inequality by \( \text{diag}\{\hat{Q}^{-1}(v), I, I, I, I, I\} \) and its transpose, respectively. Set \( \tilde{Q}(v) = \tau Q(v), \tilde{P}(v) = \tau P(v) \) and apply the Schur’s complement formula to obtain
\[
\frac{1}{\tau} \begin{bmatrix}
A^TQ(v)A - Q(v) + C^TP(v)C & A^TQ(v)B_0 + D_0^TP(v)C & A^TQ(v) \\
B_0^TQ(v)A + C^TP(v)D_0 & B_0^TQ(v)B_0 + D_0^TP(v)D_0 - P(v)B_0^TP(v)B_0 & B_0^TQ(v) \\
Q(v)A & Q(v)B_0 & Q(v)
\end{bmatrix} + \begin{bmatrix}
-\zeta^2M^TM & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & I
\end{bmatrix} \prec 0. \quad (22)
\]

Observe that the above inequality is just a version of (17) where the affine parameter dependent matrix variables of (18) are used. In view of Theorem 1 the resulting repetitive process is stable along the pass and hence ILC scheme is convergent and the proof is complete.

5. Simulation results

In this section, an example is given to illustrate the validity of our theoretical results on design of ILC schemes for uncertain systems with nonlinearities. The considered system, taken from [3], is an inverted pendulum system which consists of a pendulum rod with a fixed pivot point attached to a motor-driven shaft - see Figure 1. Based on motion equations of the pendulum, a simplified

![Figure 1. Inverted pendulum scheme.](image-url)
A model of an inverted pendulum system can be provided, see [3] for more details, and state space model is

\[
x_{k+1}(p + 1) = A(v)x_{k+1}(p) + B(v)(u_{k+1}(p) + \zeta \sin(y_{k+1}(p))), \\
y_{k+1}(p) = C(v)x_{k+1}(p),
\]

(23)

where \( x(p) = [x_1^T(p) \ x_2^T(p)]^T \), \( x_1(p) \) is the angular displacement, \( x_2(p) \) is the velocity and \( u(p) \) is the field current of the DC motor. The matrices \( A(v), B(v) \) and \( C(v) \) are assumed to be given as

\[
A(v) = v_1 \begin{bmatrix} 1 & 0.0953 \\ 0 & 0.9048 \end{bmatrix} + v_2 \begin{bmatrix} 1 & 0.0951 \\ 0 & 0.9048 \end{bmatrix}, \\
B(v) = v_1 \begin{bmatrix} 0.0481 \\ 0.952 \end{bmatrix} + v_2 \begin{bmatrix} 0.0479 \\ 0.952 \end{bmatrix}, \\
C(v) = v_1 \begin{bmatrix} 1.0001 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0.999 \\ 0 \end{bmatrix},
\]

where \( v_1, v_2 \) are uncertainty parameters, satisfying \( v_1 + v_2 = 1 \). Additionally, it is assumed that the nonlinear function \( f(x_{k+1}(p), p) \) is given by the following equation

\[
f(x_{k+1}(p), p) = \zeta B \sin(y_{k+1}(p)) = \zeta B \sin(x_1(p)),
\]

and satisfies the below constraints

\[
f^T(x_{k+1}(p), p)f(x_{k+1}(p), p) = \zeta^2 B^T \sin^2(x_1(p)) \leq \zeta^2 \begin{bmatrix} x_1(p) \\ x_2(p) \end{bmatrix}^T \begin{bmatrix} 0.9487 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(p) \\ x_2(p) \end{bmatrix}.
\]

Based on the above inequalities one can find that the matrix \( M \) of (9) is given as

\[
M = \begin{bmatrix} 0.9487 & 0 \\ 0 & 0 \end{bmatrix}.
\]

Executing the design procedure given in Theorem 2 for \( \tau = 1.5 \) gives the feedback \( (K_1) \) and learning \( (K_2) \) controllers as \( K_1 = [-20.8012 \ -1.9818] \), \( K_2 = 17.1929 \).

The designed ILC algorithm was simulated over 15 trials. As the reference signal we took a little change (approx. 1 degree) in the angular position of the pendulum during 2 seconds - see Figure 2. Since the sampling frequency is 100 Hz, the trial length is 200 points, i.e. \( \alpha = 200 \). The simulation results are depicted in Figure 2. The reference signal is marked with a solid line. The dashed lines marked the signal after the first and the second iteration. Obviously, fast convergence is observed. Furthermore, on completion of each trial the corresponding root mean square (RMS) value of the tracking error was computed using \( \text{RMS} = \sqrt{\frac{1}{N} \sum_{p=1}^{N} e(p)^2} \), where \( N = 200 \) is the number of sampled data in the trial time. The result of these computations are depicted in Figure 3. Clearly, Figure 3 shows the convergence of error signal in subsequent repetitions of the process. Also noise (RMS ranges between \( 10^{-4} \) and \( 10^{-3} \)) is included and hence the convergence curve stays at noise level after 5 trials.

6. Conclusions

In this paper a systematic procedure for design of ILC schemes with polytopic uncertainties for systems with sector-bounded nonlinearities using the repetitive process setting has been investigated. Developed results are based on the transformation to the problem of robust stability along the pass for linear repetitive processes. Then, the set of transformations is
Figure 2. The response of the control system for a given signal after 1st and 2nd trial.

Figure 3. The error convergence.

provided to allow the formulation of the problem in the form of linear matrix inequalities. Provided conditions allows to use the affine parameter dependent matrix variables which often introduce less conservatism. The theoretical findings have been illustrated by simulation results from an inverted pendulum system model and demonstrate the benefits of the developed design procedure. Future work will focus on imposing additional design specifications such as constraints imposed on control and output signals. Additionally, future work directions include design of ILC schemes for time-delay and non-minimum phase systems.
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