Interest Rates and Information Geometry

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The space of probability distributions on a given sample space possesses natural geometric properties. For example, in the case of a smooth parametric family of probability distributions on the real line, the parameter space has a Riemannian structure induced by the embedding of the family into the Hilbert space of square-integrable functions, and is characterised by the Fisher-Rao metric. In the nonparametric case the relevant geometry is determined by the spherical distance function of Bhattacharyya. In the context of term structure modelling, we show that minus the derivative of the discount function with respect to the maturity date gives rise to a probability density. This follows as a consequence of the positivity of interest rates. Therefore, by mapping the density functions associated with a given family of term structures to Hilbert space, the resulting metrical geometry can be used to analyse the relationship of yield curves to one another. We show that the general arbitrage-free yield curve dynamics can be represented as a process taking values in the convex space of smooth density functions on the positive real line. It follows that the theory of interest rate dynamics can be represented by a class of processes in Hilbert space. We also derive the dynamics for the central moments associated with the distribution determined by the yield curve. (26 June 2000)

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1. Introduction

The theory of interest rates has gone through two major developments in recent decades. Following initial investigations by Merton (1973) and others, the first decisive advance culminated in the work of Vasicek (1977) who was able to give a fairly general characterisation of the arbitrage-free dynamics of a family of discount bonds, indexed by their maturity. The well-known model that bears his name appears as an exact solution obtained with specialising assumptions. In the wake of Vasicek’s work were a number of other specific interest rate models, of varying degrees of usefulness and tractability, including, for example, the CIR model (Cox et al. 1985) and its generalisations. The next significant line of development, following the general martingale characterisation of arbitrage-free
asset pricing by Harrison & Kreps (1979) and Harrison & Pliska (1981), was in-
stigated with the recognition by Ho & Lee (1986) that the initial term structure
might be specified essentially arbitrarily, a feature that has important practical
implications. This insight was incorporated into the HJM framework (Heath et
al. 1992), which constituted a major advance in the subject, providing a general
model-independent basis for the analysis of interest rate dynamics and the pricing
of interest rate derivatives.

Since then there have been numerous further developments. These include,
for example, the infinite dimensional or ‘string-type’ models of Kennedy (1994),
Santa-Clara & Sornett (1997) and others, the positive interest rate models of Fle-
saker & Hughston (1996), the potential approach of Rogers (1997), the so-called
market models (Brace et al. 1996, 1997; Jamshidian 1997), and the geometric
analysis of the space of yield curves undertaken by Björk & Svensson (1999).

Nevertheless, no criterion has emerged, based on the extensive econometric
evidence available, that allows in a rational way for the identification of a clearly
preferred class of models. On these grounds it makes sense to try to cast the
general interest rate framework into a new form, with the idea that certain models
might thus become recognisable as more natural on mathematical and economic
grounds.

With this end in mind, the purpose of the present article is to propose a novel
application of information geometry to interest rate theory. The main results are
(i) the construction of a geometric measure for how ‘different’ two term structures
are from one another; (ii) a characterisation of the evolutionary trajectory of the
term structure as a measure-valued process; (iii) the derivation of dynamics for
the principal moments of the term structure; and (iv) a reformulation of arbitrage-
free interest rate dynamics in terms of a class of processes on Hilbert space.

The paper is organised as follows. In §2 we review the basic idea of informa-
tion geometry and its role in estimation theory. The geometry of the normal
distribution is considered in detail as an illustration. In §3 a remarkable char-
acterisation of the discount function in terms of an abstract probability density
function is introduced in Proposition 1. This allows us to apply information geo-
metric techniques to determine the deviation between different term structures
within a given model. In this connection, in §4 we consider a class of flat rate
models as examples.

The material of the first four sections of the paper is essentially static, i.e., set in
the present, whereas in §5 we investigate the dynamics of the density function that
generates the term structure. This is carried out in such a way that the resulting
dynamics is manifestly arbitrage-free. Our key result here is formula (5.15), in
which we establish that the dynamics of the term structure can be characterised
as a measure-valued process. This idea is developed further in Proposition 2.

In §6 we introduce an analogue of the classical principal components analysis
for yield curves, and in Propositions 3 and 4 we derive formulae for the evolution
of the first two moments of the term structure density process. Then, making
use of the information geometry developed earlier, in §7 we map the dynamics
developed in §5 to Hilbert space. Our main result here is Proposition 5, which
shows how this can be achieved.

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2. Information geometry

Because some of the mathematical techniques we employ here may not be familiar to those working in finance, it will be appropriate to begin with a few background remarks. It has long been known (see, e.g., Amari 1985; Kass 1989; Murray & Rice 1993) that a useful approach to statistical inference is to regard a parametric model as a differentiable manifold equipped with a metric. The recognition that a parametric family of probability distributions has a natural geometry associated with it arose in the work of Mahalanobis (1936), Bhattacharyya (1943) and Rao (1945) over half of a century ago.

Suppose, for example, that $X$ is a continuous random variable taking values on the real line $\mathbb{R}^1$, and that $\rho(x)$ is a density function for $X$. Because $\rho(x)$ is nonnegative and has integral unity, it follows that the square-root likelihood function

$$\xi(x) = \sqrt{\rho(x)} \quad (2.1)$$

exists for all $x$, and satisfies the normalisation condition

$$\int_{-\infty}^{\infty} (\xi(x))^2 \, dx = 1. \quad (2.2)$$

We see that $\xi(x)$ can be regarded as a unit vector in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^1)$. Now let $\rho_1(x), \rho_2(x)$ denote a pair of density functions on $\mathbb{R}^1$, and $\xi_1(x), \xi_2(x)$ the corresponding Hilbert space elements. Then the inner product

$$\cos \phi = \int_{-\infty}^{\infty} \xi_1(x)\xi_2(x) \, dx \quad (2.3)$$

defines an angle $\phi$ which can be interpreted as the distance between the two probability distributions. More precisely, if we write $\mathcal{S}$ for the unit sphere in $\mathcal{H}$, then $\phi$ is the spherical distance between the points on $\mathcal{S}$ determined by the vectors $\xi_1(x)$ and $\xi_2(x)$. The maximum possible distance, corresponding to nonoverlapping densities, is given by $\phi = \pi/2$. This follows from the fact that $\xi_1(x)$ and $\xi_2(x)$ are nonnegative functions, and thus define points on the positive orthant of $\mathcal{S}$. We remark that an alternative way of expressing (2.3) is

$$\cos \phi = 1 - \frac{1}{2} \int_{-\infty}^{\infty} ((\xi_1(x) - \xi_2(x))^2 \, dx, \quad (2.4)$$

which makes it apparent that the angle $\phi$ measures the extent to which the two distributions are distinct.

The spherical distance of Bhattacharyya introduced above is applicable in a nonparametric context. In the case of a parametric family of probability distributions we can develop matters further. Let us write $\rho(x, \theta)$ for the parameterised density function. Here $\theta$ stands for a set of parameters $\theta^i \ (i = 1, \cdots, r)$. By varying $\theta$ we obtain an $r$-dimensional submanifold $\mathcal{M}$ in $\mathcal{S}$ determined by the unit vectors $\xi(x, \theta) \in \mathcal{H}$. The parameters $\theta^i$ are local coordinates for $\mathcal{M}$.

The key point that we require in the following (cf. Dawid 1977) is that the spherical geometry of $\mathcal{S}$ induces a Riemannian geometry on $\mathcal{M}$, for which the metric tensor $g_{ij}(\theta)$ is given, in local coordinates, by

$$g_{ij}(\theta) = \int_{-\infty}^{\infty} \frac{\partial \xi(x, \theta)}{\partial \theta^i} \frac{\partial \xi(x, \theta)}{\partial \theta^j} \, dx. \quad (2.5)$$
By use of definition (2.1), we see that an alternative expression for $g_{ij}(\theta)$ is
\[ g_{ij}(\theta) = \frac{1}{4} \int_{-\infty}^{\infty} \rho(x, \theta) \frac{\partial \ln \rho(x, \theta)}{\partial \theta^i} \frac{\partial \ln \rho(x, \theta)}{\partial \theta^j} dx, \tag{2.6} \]
which shows (cf. Brody & Hughston 1998) that the metric $g_{ij}$ is, apart from the factor of $\frac{1}{4}$, the Fisher information matrix, i.e., the covariance matrix of the parametric gradient of the log-likelihood function (Fisher 1921). We refer to $g_{ij}(\theta)$ as the Fisher-Rao metric on the statistical model $\mathcal{M}$.

The significance of the Fisher-Rao metric in estimation theory is well known. Suppose that $\tau(\theta)$ is some given function of the parameters, and that the random variable $T$ represented by the function $T(x)$ on $\mathbb{R}^1$ is an unbiased estimator for $\tau(\theta)$ in the sense that
\[ \int_{-\infty}^{\infty} \rho(x, \theta) T(x) dx = \tau(\theta). \tag{2.7} \]
The variance of the estimator $T$ is defined, as usual, by
\[ \text{Var}[T] = \int_{-\infty}^{\infty} \rho(x, \theta) (T(x) - \tau(\theta))^2 dx. \tag{2.8} \]
Then a set of fundamental bounds on $\text{Var}[T]$, independent of the choice of the estimator $T(x)$, can be obtained by applying the operator $\sum \alpha^i \partial_i$ to (2.7), letting $\alpha^i$ be arbitrary. By use of (2.1) and the Schwartz inequality for $L^2(\mathbb{R}^1)$, we obtain
\[ g_{ij} \text{Var}[T] \geq \frac{1}{4} \frac{\partial \tau}{\partial \theta^i} \frac{\partial \tau}{\partial \theta^j}. \tag{2.9} \]
This matrix inequality is interpreted as saying that if we subtract the right side from the left, the result is nonnegative definite. It follows that if the random variables $\Theta^i$ ($i = 1, \ldots, r$) are unbiased estimators for the parameters $\theta^i$, satisfying
\[ \int_{-\infty}^{\infty} \rho(x, \theta) \Theta^i(x) dx = \theta^i, \tag{2.10} \]
then the covariance matrix of the estimators is bounded by the inverse Fisher information matrix:
\[ \text{Cov}[\Theta^i, \Theta^j] \geq \frac{1}{4} g^{ij}. \tag{2.11} \]

The Riemannian metric (2.5) introduced above can be used to define a distance measure between two distributions belonging to a given parametric family. This measure is invariant in the sense that it is unaffected by a reparameterisation of the distributions. The distance is calculated by integrating the infinitesimal line element $ds$ along the geodesic connecting the two points in the statistical manifold $\mathcal{M}$, where
\[ ds^2 = \sum_{i,j} g_{ij} d\theta^i d\theta^j. \tag{2.12} \]
The geodesics with respect to a given metric $g_{ij}$ are the solutions of the differential equation
\[ \frac{d^2 \theta^i}{du^2} + \Gamma^i_{jk} \frac{d\theta^j}{du} \frac{d\theta^k}{du} = 0 \tag{2.13} \]
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Figure 1. Geodesic curves for normal distributions. The statistical manifold \( M \) in this case is the upper half plane parameterised by \( \mu \) and \( \sigma \). We have \(-\infty < \mu < \infty\) and \(0 < \sigma < \infty\). The shortest path joining the two normal distributions \( N(\mu_1, \sigma_1) \) and \( N(\mu_2, \sigma_2) \) is given by the unique semi-circular arc through the given two points and centred on the boundary line \( \sigma = 0 \).

for the curve \( \theta^i(u) \) in \( M \), subject to the given boundary conditions at the two end points. Here, we have written

\[
\Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}),
\]

(2.14)

where \( \partial_i = \partial/\partial \theta^i \), and the inverse metric \( g^{ij} \), also appearing in (2.11), satisfies \( g^{ij} g_{jk} = \delta^i_k \), where \( \delta^i_k \) is the Kronecker delta. Note that in equations (2.13) and (2.14) above, and elsewhere henceforth in this article, we employ the standard Einstein summation convention on repeated indices.

Let us consider, as an explicit example, the manifold \( M \) corresponding to the normal distributions \( N(\mu, \sigma) \) on \( \mathbb{R}^1 \), with mean \( \mu \) and standard deviation \( \sigma \). For the parameterised density function we have

\[
\rho(x, \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right).
\]

(2.15)

A straightforward computation, making use of (2.6), gives

\[
ds^2 = \frac{1}{\sigma^2}(d\mu^2 + 2d\sigma^2)
\]

(2.16)

for the line element, which is defined on the upper half-plane \(-\infty < \mu < \infty\), \(0 < \sigma < \infty\). The resulting Riemannian geometry is that of hyperbolic space, which is a homogeneous manifold with constant negative curvature. The geometry of this space has been studied extensively, and has many intriguing properties. For the distance function in the case of a pair of normal distributions \( N(\mu_1, \sigma_1) \), \( N(\mu_2, \sigma_2) \) we obtain

\[
D(\rho_1, \rho_2) = \frac{1}{\sqrt{2}} \log \frac{1 + \delta_{1,2}}{1 - \delta_{1,2}},
\]

(2.17)

where the function \( \delta_{1,2} \), defined by

\[
\delta_{1,2} = \sqrt{\frac{(\mu_2 - \mu_1)^2 + 2(\sigma_2 - \sigma_1)^2}{(\mu_2 - \mu_1)^2 + 2(\sigma_2 + \sigma_1)^2}},
\]

(2.18)

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Figure 2. The system of admissible term structures. A smooth positive interest term structure can be regarded as a point in $D(\mathbb{R}_1^+)$, the convex space consisting of smooth density functions on $\mathbb{R}_1^+$. The points of $D(\mathbb{R}_1^+)$ are in one-to-one correspondence with rays lying in the positive orthant $S_+$ of the unit sphere $S$ in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_1^+)$. 

lies between 0 and 1. The geodesics, in particular, are given in general by semicircular arcs centred on the boundary line $\sigma = 0$ (this line itself is not part of the manifold $\mathcal{M}$). An exceptional situation arises when $\mu_1 = \mu_2$, for which the geodesic is a straight line given by constant $\mu$, and we have

$$D(\rho_1, \rho_2) = \frac{1}{\sqrt{2}} \left| \log \frac{\sigma_1}{\sigma_2} \right|. \quad (2.19)$$

We refer the reader to Burbea (1986), where metric and distance computations have been carried out explicitly for other families of distributions.

3. Discount bond densities

Our goal now is to make use of the analysis presented in the previous section to construct a natural metric on the space of yield curves. In doing so we shall take advantage of a remarkable ‘probabilistic’ characterisation of discount bonds, which we here proceed to describe.

Let $t = 0$ denote the present, and $P_{0T}$ a smooth family of discount bonds, where $T$ is the maturity date ($0 \leq T < \infty$). For positive interest we require

$$0 < P_{0T} \leq 1, \quad \frac{\partial}{\partial T} P_{0T} < 0, \quad (3.1)$$

and we assume that $P_{0T} \to 0$ as $T$ goes to infinity. A term structure that satisfies these conditions will be said to be ‘admissible’. These conditions can, in fact, be relaxed slightly: $P_{0T}$ need not be strictly smooth, nor strictly decreasing; but for most of the present discussion we shall stick with the assumptions indicated.

The interesting point that arises here, of which we shall make extensive use in the discussions that follow, is that the discount function $P_{0T}$ can be viewed as a complementary probability distribution. In other words, we think of the maturity date as an abstract random variable $X$, and for its distribution we write

$$\mathbb{P}[X < T] = 1 - P_{0T}. \quad (3.2)$$

It should be clear that this can be done if and only if the positive interest rate conditions given in (3.1) hold. As a consequence we are able to embody the positive interest property in a fundamental way in the structure of the theory.
Indeed, this basic economic property is essential if we wish to treat the yield curve consistently and naturally as a kind of mathematical object in its own right. Now let us introduce the function \( \rho(T) \) defined by

\[
\rho(T) = -\frac{\partial}{\partial T} P_0 T. \tag{3.3}
\]

Clearly, we have \( \rho(T) > 0 \) and

\[
\int_0^\infty \rho(T) dT = 1, \tag{3.4}
\]

from which we infer that \( \rho(T) \) can be consistently viewed as a probability density function. It follows from the defining equation (3.3) that the term structure density \( \rho(T) \) is the product of the instantaneous forward rate and the discount function itself. Now clearly if \( \rho_1(T) \) and \( \rho_2(T) \) are admissible term structure densities, and if \( A \) and \( B \) are nonnegative constants satisfying \( A + B = 1 \), then \( A\rho_1(T) + B\rho_2(T) \) is also an admissible term structure density. Putting these ingredients together, we see that the term structure of interest rates can be given the following general characterisation.

**Proposition 1.** The system of admissible term structures is isomorphic to the convex space \( \mathcal{D}(\mathbb{R}_1^+) \) of smooth density functions on the positive real line.

At first glance it may seem odd to think of the discount function in this manner. However, it gives us the advantage of being able to apply the tools of information geometry in an unexpected way, as we indicate in what follows.

In particular, there is a one-to-one map from the space \( \mathcal{D}(\mathbb{R}_1^+) \) of such term structure densities to the positive orthant \( \mathcal{S}_+ \) of the unit sphere \( \mathcal{S} \) in the Hilbert space \( \mathcal{H} \), as indicated in Figure 2. Therefore, given two yield curves we can calculate the distance between them. This can be carried out either in a nonparametric sense, by use of the Bhattacharyya spherical distance, or in a parametric sense, by use of the Fisher-Rao distance. In the former case first we calculate the corresponding term structure densities \( \rho_1(T) \) and \( \rho_2(T) \). These are then mapped to \( \mathcal{S}_+ \) by taking the square-roots, and their distance \( \phi(\rho_1, \rho_2) \) is given by

\[
\phi(\rho_1, \rho_2) = \cos^{-1} \int_0^\infty \sqrt{\rho_1(T)\rho_2(T)} dT. \tag{3.5}
\]

In the parametric case we regard the given parametric family of yield curves as defining a statistical model \( \mathcal{M} \subset \mathcal{S}_+ \), and the distance between the two yield curves within the given family is then defined by the Fisher-Rao metric.

### 4. Flat term structures

To provide some illustrations of the principles set forth in the previous section we consider here properties of yield curves for which the term structure is flat. Such yield curves, which are of various types, are on the whole too simple for use in practical modelling. Nevertheless, they are of interest as examples, because many of the relevant computations can be carried out explicitly.

In this connection we begin by introducing a representation of the discount
function as a Laplace transform

\[ P_{0T} = \int_0^\infty e^{-rT} \psi(r) \, dr \]  

(4.1)

for some function \( \psi(r) \). Thus we think of the discount function \( P_{0T} \) as being given by a weighted superposition of elementary discount functions, each of the form \( e^{-rT} \) for some value of \( r \). Taking the limit \( T \to 0 \), we find that \( \psi(r) \) must satisfy \( \int_0^\infty \psi(r) \, dr = 1 \). In general the inverse Laplace transform \( \psi(r) \) need not be positive. However, if we restrict our consideration to nonnegative functions, then \( \psi(r) \) can be interpreted as a density function, and by various choices of \( \psi(r) \) we are led to some interesting candidates for term structures.

First we consider the case where \( \psi(r) \) is a Dirac \( \delta \)-function concentrated at a point, that is, \( \psi(r) = \delta(r - R) \). A direct substitution gives \( P_{0T} = \exp(-RT) \), corresponding to a ‘flat’ term structure with a continuously compounded rate \( R \) for each value of the maturity date \( T \). If the density function \( \psi(r) \) is given by an exponential distribution \( \psi(r) = \tau \exp(-\tau r) \), with parameter \( \tau \), then one sees that \( \tau \) must have dimensions of time, and a short calculation gives \( P_{0T} = \tau/\left(\tau + T\right) \), which also corresponds to a flat term structure, in this case with a simple percentage yield of \( \tau^{-1} \) for all maturities. We see that the characteristic time-scale \( \tau \) allows us to define an interest rate \( R = \tau^{-1} \), which turns out to be the characteristic interest rate of the resulting structure, and we can write \( P_{0T} = 1/(1 + RT) \) for the discount function.

We note that flatness is not a completely unambiguous notion, because having a uniform continuously compounded yield for all maturities is not the same thing as having a uniform simple yield for all maturities. Both define plausible albeit quite distinct systems of discount bonds. This example illustrates how by superposing term structures of the elementary form \( \exp(-RT) \) for various maturities, we can obtain other reasonable looking and well behaved term structures. We mention one more example, which contains the previous two examples as special cases. Consider the standard gamma distribution, with parameters \( \kappa \) and \( \lambda \), defined for nonnegative values of \( r \) by the density function

\[ \psi(r) = \frac{1}{\Gamma(\kappa)} \frac{\lambda^\kappa r^{\kappa-1}}{\kappa} \exp(-\lambda r). \]  

(4.2)

In this case, we can verify that the resulting system of discount bonds is given by

\[ P_{0T} = \left(\frac{\lambda}{\lambda + T}\right)^\kappa, \]  

(4.3)

which assumes a more recognisable form if we set \( \lambda = \kappa \tau \), where \( \tau \) again defines a characteristic time scale, and \( \kappa \) is a dimensionless number. Then we have

\[ P_{0T} = \left(\frac{RT}{\kappa}\right)^{-\kappa}, \]  

(4.4)

where \( R = \tau^{-1} \). The system of discount bonds arising here can also be interpreted as a flat term structure, in this case with a constant annualised rate of interest \( R \) assuming compounding at the frequency \( \kappa \) over the life of each bond (\( \kappa \) need not be an integer). It is not difficult to check that for \( \kappa = 1 \) this reduces to the case of a flat rate on the basis of a simple yield, whereas in the limit \( \kappa \to \infty \) we recover the case of a flat rate on the basis of continuous compounding.
Now we shall apply the ideas of statistical geometry to make comparisons between various term structures of the form (4.4). For density function $\rho(T) = -\partial_T P_{0T}$ in this case we obtain

$$\rho(T, R) = R \left(1 + \frac{RT}{\kappa}\right)^{-(\kappa + 1)}.$$  \hfill (4.5)

Here we find it convenient to label the density function by the flat rate $R$. Note that in the limit $\kappa \to \infty$ we have $\rho(T, R) \to R e^{-RT}$. First consider the nonparametric separation between different term structures in this model via spherical distance of Bhattacharyya given in formula (3.5), where in the present example we write $\rho_i(T) = \rho(T, R_i)$ for $i = 1, 2$. A direct integration leads to the expression

$$\phi(\rho_1, \rho_2) = \cos^{-1} \left(\frac{\sqrt{R_1 R_2}}{R_1 - R_2} \log \frac{R_1}{R_2}\right)$$  \hfill (4.6)

for the distance when $\kappa = 1$, whereas in the limit $\kappa \to \infty$ (continuous compounding) we have

$$\phi(\rho_1, \rho_2) = \cos^{-1} \left(\frac{2\sqrt{R_1 R_2}}{R_1 + R_2}\right).$$  \hfill (4.7)

It is interesting to observe that the bracketed term in (4.7) is given by the ratio of the geometric and arithmetic means of the two rates.

Alternatively, we can view (4.5) as a parametric family of distributions, parameterised by the flat rate $R$. Then it is natural to consider the Fisher-Rao distance between the two term structures characterised by $R_1$ and $R_2$. A straightforward calculation then leads to a simple distance formula given by

$$D(R_1, R_2) = \sqrt{\frac{\kappa}{\kappa + 2} \log \frac{R_2}{R_1}},$$  \hfill (4.8)

where we have assumed $R_2 \geq R_1$.

5. Interest rate dynamics

The formalism we have developed so far is essentially a static one, set in the present. Now we turn to the problem of developing a dynamical theory of interest rates. The idea is that, at each instant of time, the yield curve is characterised by a term structure density according to the scheme described in the previous sections. Then, as time passes, the density function evolves randomly. As a consequence we obtain a measure-valued process. In particular, we obtain a process on $D(R^+_1)$. Our goal in this section is to determine a set of conditions on this process necessary and sufficient to ensure that the resulting interest rate dynamics will be arbitrage-free.

We shall assume the reader is familiar with the general theory of interest rate dynamics as laid out, for example, in Carverhill (1994), Rogers (1994), Hughston (1996), Baxter (1997), Musiela & Rutkowski (1997), Brody (2000) or Hunt & Kennedy (2000). For the general discount bond dynamics, let us write

$$dP_{IT} = \mu_{IT} dt + \Sigma_{IT} \cdot dW_t,$$  \hfill (5.1)
where $\mu_{tT}$ and $\Sigma_{tT}$ are the absolute drift and absolute volatility processes, respectively, for a bond with maturity $T$. Here, $W_t$ is a vector Brownian motion, and $\Sigma_{tT}$ is a vector process, and there is an inner product implied between $\Sigma_{tT}$ and $dW_t$, signified by a dot. We need not specify the dimensionality of the Brownian motion, which might be infinite, and indeed in some respects the infinite dimensional setting is the most natural one. In fact, it suffices for our purposes merely to assume that $P_{tT}$ is a one-parameter family of continuous semi-martingales on the given probability space, with respect to the given filtration. However, for simplicity of exposition we shall stick to the case where the relevant stochastic basis is generated by a multidimensional Brownian motion. Here, as in Flesaker & Hughston (1997a,b), we regard the discount bond dynamics as the natural starting position, rather than, say, the instantaneous forward rate dynamics (Heath et al. 1992), which we need not consider here directly. We shall assume nevertheless, as in the HJM framework, that the processes $\mu_{tT}$ and $\Sigma_{tT}$ are both smooth in the variable $T$, and that sufficiently strong technical conditions are in place to ensure that the instantaneous forward rate processes are semimartingales.

In order to extend the analysis of the previous section it is convenient to introduce what is sometimes conveniently referred to as the ‘Musiela parameterisation’, given by

$$B_{tx} = P_{t,t+x},$$

(5.2)

where $T = t + x$ represents the maturity date of the bond, and hence $x$ is the time left until maturity. Thus $B_{tx}$ is the value at time $t$ of a discount bond that has $x$ years left to mature. This choice of parameterisation has already been shown to be useful in the geometric analysis of interest rates (Björk & Svensson 1999, Björk & Christensen 1999, Björk & Gombani 1999, Björk 2000). We note that $B_{t0} = 1$ for all $t$, and that $B_{tx} \to 0$ as $x \to \infty$. It follows that

$$\rho_t(x) = -\frac{\partial}{\partial x} B_{tx},$$

(5.3)

is a measure-valued process in the sense that, for each value of $t$ the random function $\rho_t(x)$ satisfies $\rho_t(x) > 0$ and the normalisation condition

$$\int_0^\infty \rho_t(x) dx = 1.$$  

(5.4)

Here we have chosen the notation $\rho_t(x)$ that makes the $x$ dependence more prominent, to emphasise the fact that, for each value of $t$, and conditional on information given up to time $t$, $\rho_t(x)$ is a density function, though we might have written $\rho_{tx}$ instead. As a consequence $\rho_t(x)$ describes a process on $D(\mathbb{R}_+^\infty)$. By consideration of (5.1) and (5.2) we deduce for the dynamics of $B_{tx}$ that

$$d B_{tx} = dP_{tT|T=t+x} + \frac{\partial}{\partial x} B_{tx} dt,$$

(5.5)

and thus, by use of (5.1), that

$$d B_{tx} = (P_{t,t+x} + \partial_x B_{tx}) dt + \Sigma_{t,t+x} \cdot dW_t,$$

(5.6)

where $\partial_x = \partial / \partial x$. Differentiating this expression with respect to $x$ and introducing the measure-valued process $\rho_t(x)$ according to formula (5.3) we therefore obtain

$$d \rho_t(x) = (-\partial_x \mu_{t,t+x} + \partial_x \rho_t(x)) dt - \partial_x \Sigma_{t,t+x} \cdot dW_t.$$  

(5.7)
A further simplification is then achieved by introducing the notation
\[ \beta_{tx} = -\partial_x \mu_{t,t+x} \quad (5.8) \]
and
\[ \omega_{tx} = -\partial_x \Sigma_{t,t+x} \quad (5.9) \]
which gives us
\[ \text{d}\rho_t(x) = (\beta_{tx} + \partial_x \rho_t(x)) \, \text{d}t + \omega_{tx} \cdot \text{d}W_t. \quad (5.10) \]

In the foregoing discussion we have not yet imposed the arbitrage-free condition. This is given by the drift constraint
\[ \mu_{tT} = r_t P_{tT} + \Sigma_{tT} \cdot \lambda_t, \quad (5.11) \]
where \( \lambda_t \) is the process for the market price of risk. We note that \( \lambda_t \), like \( \Sigma_{tT} \), is a vector process. However, \( \lambda_t \) does not depend on the maturity \( T \). The absence of arbitrage ensures the existence of \( \lambda_t \). For our purposes we do not need to insist that the bond market is complete: all we require is the existence of a pricing kernel, or equivalently the existence of a self-financing ‘natural numéraire’ portfolio with value process \( N_t \) such that \( P_{tT}/N_t \) is a martingale for each value of \( T \) (cf. Flesaker & Hughston 1997c). The numéraire process satisfies
\[ \frac{\text{d}N_t}{N_t} = (r_t + \lambda_t^2) \text{d}t + \lambda_t \cdot \text{d}W_t, \quad (5.12) \]
and the corresponding pricing kernel is given by \( 1/N_t \). As a consequence of the constraint (5.11) we then have
\[ \mu_{t,t+x} = r_t B_{tx} + \Sigma_{t,t+x} \cdot \lambda_t, \quad (5.13) \]
and therefore, by differentiation of this expression with respect to \( x \), we obtain
\[ \beta_{tx} = r_t \rho_t(x) + \omega_{tx} \cdot \lambda_t. \quad (5.14) \]
Inserting (5.14) in (5.10) we are thus able to express the dynamics of the density function \( \rho_t(x) \) in the form
\[ \text{d}\rho_t(x) = (r_t \rho_t(x) + \partial_x \rho_t(x)) \, \text{d}t + \omega_{tx} \cdot (\text{d}W_t + \lambda_t \text{d}t). \quad (5.15) \]

Before proceeding further, let us verify, as a consistency check, that the dynamics given by (5.15) preserves the normalisation condition on \( \rho_t(x) \), given by (5.4). Integrating the right hand side of (5.15) with respect to \( x \) and equating the drift and volatility terms separately to zero leads to the relations
\[ r_t + \int_0^\infty \partial_x \rho_t(x) \, \text{d}x = 0 \quad (5.16) \]
and
\[ \int_0^\infty \omega_{tx} \, \text{d}x = 0, \quad (5.17) \]
which must hold for all \( t \). Condition (5.16) is satisfied because \( \rho_t(x) \to 0 \) as \( x \to \infty \) and
\[ \rho_t(0) = r_t. \quad (5.18) \]
Condition (5.17) is satisfied because, by definition, we have $\omega_{tx} = -\partial_x \Sigma_{t,t+\tau}$, and the absolute volatility $\Sigma_{t,t+\tau}$ vanishes both as $t \to 0$ (a maturing bond has a definite value and thus has no absolute volatility), and as $t \to \infty$ (a bond with infinite maturity has no value, and hence no absolute volatility).

Summing up matters so far, we see that in (5.15) we are able to cut the standard HJM arbitrage-free interest rate dynamics in the form of a measure-valued process $\rho_t(x)$ subject to the constraints (5.16) and (5.17). At first glance the role of the short rate $r_t$ in (5.15) seems anomalous, because it might appear that this has to be specified separately. However, by virtue of (5.18) we can incorporate $r_t$ directly into the dynamics of $\rho_t(x)$.

In fact, there is another way of expressing (5.15) which is very suggestive, and ties in naturally with the Hilbert space approach to dynamics introduced in §7. First we note that (5.16) can be rewritten in the form

$$r_t = -\int_0^\infty \rho_t(x) \partial_x \ln \rho_t(x) dx.$$  

(5.19)

In other words, $r_t$ is minus the expectation of the gradient of the log-likelihood function. Here the expectation is taken with respect to $\rho_t(x)$ itself. Writing $E_\rho$ for this abstract expectation, we have

$$d\rho_t(x) = \rho_t(x) \left( \partial_x \ln \rho_t(x) - E_\rho[\partial_x \ln \rho_t(x)] \right) dt + \omega_{tx} \cdot dW_t^\ast,$$  

(5.20)

where $dW_t^\ast = dW_t + \lambda_t dt$. We note that $W_t^\ast$ has the interpretation of being a Brownian motion with respect to the risk-neutral measure associated with the given pricing kernel. In the risk-neutral measure, for which the term involving $\lambda_t$ effectively disappears, the remaining drift for $\rho_t(x)$ is determined by the deviation of $\partial_x \ln \rho_t(x)$ from its abstract mean.

Let us now examine more closely the volatility term $\omega_{tx}$ appearing in (5.20), with a view to gaining a better understanding of the significance of the volatility constraint (5.17). Because $\rho_t(x)$ must remain positive for all values of $x$, the coefficient of $dW_t^\ast$ in (5.20) must be of the form

$$\omega_{tx} = \rho_t(x) \sigma_{tx},$$  

(5.21)

for some bounded process $\sigma_{tx}$, to ensure that $\omega_{tx}$ dies off appropriately for values of $x$ such that $\rho_t(x)$ approaches zero. As a consequence, we can write (5.15) in the quasi-lognormal form

$$d\rho_t(x) = \left( r_t + \partial_x \ln \rho_t(x) \right) dt + \sigma_{tx} \cdot dW_t^\ast,$$  

(5.22)

and for the constraint (5.17) we have

$$E_\rho[\sigma_{tx}] = 0,$$  

(5.23)

which can be satisfied by writing

$$\sigma_{tx} = \nu_{tx} - E_\rho[\nu_{tx}],$$  

(5.24)

where $\nu_{tx}$ is an exogenously specifiable unconstrained process. Here, for any process $A_{tx}$ we define $E_\rho[A_{tx}] = \int_0^\infty \rho_t(x) A_{tx} dx$. The results established above can then be summarised as follows.

**Proposition 2.** The general admissible term structure evolution based on the
information set generated by a multidimensional Brownian motion \( W_t \) is given by a measure-valued process \( \rho_t(x) \) in \( D(\mathbb{R}_+^1) \) satisfying

\[
\frac{d\rho_t(x)}{\rho_t(x)} = \left( \partial_x \ln \rho_t(x) - E[\partial_x \ln \rho_t(x)] \right) dt \\
+ (\nu_{tx} - E[\nu_{tx}]) \cdot (dW_t + \lambda_t dt), \tag{5.25}
\]

where the processes \( \lambda_t \) and \( \nu_{tx} \) are specified exogenously, along with the initial term structure density \( \rho_0(x) \).

An advantage of the particular expression (5.25) given for the dynamics above is that the preservation of the normalisation condition on \( \rho_t(x) \) is evident by inspection, because this is equivalent to the relation

\[
E[\rho_t(x)] = 0. \tag{5.26}
\]

An alternative expression for (5.25), which brings out more explicitly the nonlinearities in the dynamics, is given by

\[
d\rho_t(x) = \left( \partial_x \rho_t(x) + \rho_t(0) \rho_t(x) \right) dt \\
+ \rho_t(x) \left( \nu_{tx} - \int_0^\infty \rho_t(y) \nu_{ty} dy \right) \cdot dW^*_t, \tag{5.27}
\]

where \( dW^*_t = dW_t + \lambda_t dt \) as defined earlier.

6. Principal moment analysis

The characterisation of the yield curve as an abstract probability density enables us to develop a rigorous analogue of the classical ‘principal component’ analysis often used in the study of yield curve dynamics. To this end we let \( \rho_t(x) = -\partial_x P_{t,t+x} \) be the density process associated with an admissible family of discount bond prices, and define the moment processes

\[
\bar{x}_t = \int_0^\infty x \rho_t(x)dx \tag{6.1}
\]

and

\[
\bar{x}^{(n)}_t = \int_0^\infty x^n \rho_t(x)dx \tag{6.2}
\]

for \( n \geq 2 \), along with the central moment processes

\[
\tilde{x}^{(n)}_t = \int_0^\infty (x - \bar{x}_t)^n \rho_t(x)dx. \tag{6.3}
\]

It is important to note that in some cases the relevant moments may not exist. For example, in the case of a continuously compounded flat yield curve given at \( t = 0 \) by the density function \( \rho_0(x) = Re^{-Rx} \), we have \( \bar{x}_0 = R^{-1} \), \( \bar{x}^{(2)}_0 = R^{-2} \), \( \bar{x}^{(3)}_0 = 3R^{-3} \), and \( \bar{x}^{(4)}_0 = 9R^{-4} \) for the first four central moments. On the other hand, in the example of the simple flat term structure for which \( \rho_0(x) = R/(1 + Rx)^2 \) we find that none of the moments exist, on account of the fatness of the tail of the distribution. In fact, for the flat rate term structures with compounding frequency \( \kappa \) the moments exist only up to order \( \kappa - 1 \).
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The first four moments, if they exist, are the mean, variance, skewness and kurtosis of the distribution of the abstract random variable $X$ characterising the yield curve, and we refer to these (and other) moments as the ‘principal moments’ of the given term structure. At $t = 0$ the mean $\bar{x}_0$ determines a characteristic time-scale associated with the given term structure, and its inverse $1/\bar{x}_0$ can be thought of as an associated characteristic yield. The difference $\bar{x}_0^{(2)} - (\bar{x}_0)^2$ then measures the departure of the given term structure from flatness on a continuously compounded basis. This is on account of the fact that in the case of an exponential distribution the variance is given by the square of the mean.

It is legitimate to conjecture that for some purposes the specification of, e.g., the first three or four moments will be sufficient to provide an accurate representation of the term structure. One way of implementing this idea is to introduce the entropy $S_\rho$ of the given distribution, defined by

$$ S_\rho = -\int_0^\infty \rho(x) \ln \rho(x) dx. \quad (6.4) $$

Because $\rho(x)$ has dimensions of inverse time, $S_\rho$ is defined only up to an overall additive constant. Therefore, the difference of the entropies associated with two yield curves has an invariant significance.

For yield curve calibration we propose that $\rho(x)$ should be chosen such that $S_\rho$ is maximised subject to the constraints of the data available. For example, if we are given as data only the mean $\bar{x}_0$, then the maximum entropy term structure is $\rho_0(x) = R e^{-Rx}$, where $R = 1/\bar{x}_0$.

It is also of great interest to study the dynamics of the principal characteristics in the case of a general admissible arbitrage-free term structure. We examine here, in particular, the mean and the variance processes. For this purpose we introduce a simplified notation $v_t = \bar{x}_t^{(2)}$ for the variance process, i.e.,

$$ v_t = \int_0^\infty x^2 \rho_t(x) dx - (\bar{x}_t)^2, \quad (6.5) $$

where the mean process $\bar{x}_t$ is given as in (6.1). We assume that both $\rho_t(x)$ and the discount bond volatility $\Sigma_t$ fall off to zero sufficiently rapidly to ensure that $\lim_{x \to \infty} x^n \rho_t(x) = 0$ and $\lim_{x \to \infty} x^n \Sigma_t = 0$ for $n = 1, 2$, and that the integrals $\int_0^\infty x^n \rho_t(x) dx$ and $\int_0^\infty x^{n-1} \Sigma_t dx$ exist for $n = 1, 2$. A straightforward calculation then leads us to the following conclusion:

**Proposition 3.** The first principal moment $\bar{x}_t$ of an admissible, arbitrage-free term structure satisfies the dynamical law

$$ d\bar{x}_t = (r_t \bar{x}_t - 1) dt + \bar{\Sigma}_t \cdot dW_t^*, \quad (6.6) $$

where $\bar{\Sigma}_t = \int_0^\infty \Sigma_{t,t+x} dx$.

**Proof.** Starting with (5.22) and (6.1) we have

$$ d\bar{x}_t = \int_0^\infty x d\rho_t(x) dx 
= \left( r_t \bar{x}_t + \int_0^\infty x \partial_x \rho_t(x) dx \right) dt - \left( \int_0^\infty x \partial_x \Sigma_{t,t+x} dx \right) \cdot dW_t^* \quad (6.7) $$
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by use of (5.9). Then, integrating by parts and using the assumed asymptotic behaviours for $\rho_t(x)$ and $\Sigma_{t,t+x}$, we obtain the desired result.

We note that there is a critical level $\bar{x}_t^*$ for the first principal moment given by

$$\bar{x}_t^* = \frac{1}{r_t} (1 - \lambda_t \cdot \bar{\Sigma}_t).$$

(6.8)

When $\bar{x}_t > \bar{x}_t^*$ the drift of $\bar{x}_t$ is positive, and the drift increases further as $\bar{x}$ increases. On the other hand, when $\bar{x}_t < \bar{x}_t^*$, the drift of $\bar{x}_t$ is negative, and the drift decreases further as $\bar{x}_t$ decreases. For the variance process, we have:

**Proposition 4.** The second principal moment $v_t$ of an admissible, arbitrage-free term structure satisfies the dynamical law

$$dv_t = \left( r_t(v_t - \bar{x}_t^2) - \Sigma_t^2 \right) dt + 2 \left( \bar{\Sigma}_t^{(1)} - \bar{x}_t \bar{\Sigma}_t \right) \cdot dW_t^*,$$

(6.9)

where $\Sigma_t^{(1)} = \int_0^\infty x \Sigma_{t,t+x} dx$.

**Proof.** Starting with formula (6.5) for $v_t$ we have

$$dv_t = \int_0^\infty x^2 d\rho_t(x) dx - d(\bar{x}_t^2).$$

(6.10)

For the first term we obtain

$$\int_0^\infty x^2 \rho_t(x) dx = \left( r_t \int_0^\infty x^2 \rho_t(x) dx + \int_0^\infty x^2 \partial_x \rho_t(x) dx \right) dt$$

$$- \left( \int_0^\infty x^2 \partial_x \Sigma_{t,t+x} \right) \cdot dW_t^*,$$

(6.11)

where we have used (5.9) and (5.22). As a consequence of the assumed asymptotic behaviour of $\rho_t(x)$ and $\Sigma_{t,t+x}$, this becomes

$$\int_0^\infty x^2 \rho_t(x) dx = \left( r_t \bar{x}_t^{(2)} - 2 \bar{x}_t \right) dt + 2 \bar{\Sigma}_t^{(1)} \cdot dW_t^*,$$

(6.12)

after an integration by parts. For the second term in (6.10) we have

$$d(\bar{x}_t^2) = 2 \bar{x}_t d\bar{x}_t + (d\bar{x}_t)^2$$

(6.13)

by Ito’s lemma, and thus

$$d(\bar{x}_t^2) = \left( 2 r_t \bar{x}_t^2 - 2 \bar{x}_t + \Sigma_t^2 \right) dt + 2 \bar{x}_t \bar{\Sigma}_t \cdot dW_t^*$$

(6.14)

by use of Proposition 3. Combining (6.12) and (6.14), and using the definition (6.5) we obtain (6.9).

In this case we recall that the difference $v_t - \bar{x}_t^2$ acts as a simple measure of the extent to which the distribution deviates from the ‘flat’ term structure. As a consequence we see that the effect of the dynamics here is that the second principal moment of the term structure tends to increase, i.e., has a positive drift, providing $v_t - \bar{x}_t^2$ is already above the level given by

$$v_t - \bar{x}_t^2 = \frac{1}{r_t} \left( \Sigma_t^2 - 2 \lambda_t \cdot (\bar{\Sigma}_t^{(1)} - \bar{x}_t \bar{\Sigma}_t) \right).$$

(6.15)

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7. Hilbert space dynamics for term structures

Now that we have examined some of the advantages of expressing the arbitrage-free interest rate term structure dynamics as a randomly evolving density function, let us consider how we transform to the Hilbert space representation for density functions considered in §2. Denote by $\xi_{tx}$ the process for the square-root likelihood function, defined by

$$\rho_t(x) = \xi_{tx}^2. \quad (7.1)$$

It follows then, by Ito’s lemma, that

$$d\rho_t(x) = 2\xi_{tx}d\xi_{tx} + (d\xi_{tx})^2, \quad (7.2)$$

and hence $(d\rho_t(x))^2 = 4\xi_{tx}^2(d\xi_{tx})^2$. By rearranging $(7.2)$ we thus obtain

$$d\xi_{tx} = \frac{1}{2\xi_{tx}}d\rho_t(x) - \frac{1}{8\xi_{tx}^2}(d\rho_t(x))^2 \quad (7.3)$$

for the dynamics of the process $\xi_{tx}$, and hence

$$d\xi_{tx} = \left(\frac{\partial_x\xi_{tx}}{2\xi_{tx}} + \frac{1}{2}r_t\xi_{tx} - \frac{1}{8\xi_{tx}^2}\omega_{tx}^2\right)dt + \frac{1}{2\xi_{tx}}\omega_{tx} \cdot dW_t^*, \quad (7.4)$$

where $\omega_{tx}^2 = \omega_{tx} \cdot \omega_{tx}$. Now suppose we define $\sigma_{tx}$ by the quotient

$$\sigma_{tx} = \frac{\omega_{tx}}{\xi_{tx}}, \quad (7.5)$$

as before, and set $\sigma_{tx}^2 = \sigma_{tx} \cdot \sigma_{tx}$. Then the process for the square-root density $\xi_{tx}$ can be written in the form

$$d\xi_{tx} = \left(\frac{\partial_x\Sigma_{tx,t+x}}{2\xi_{tx}} + \frac{1}{2}r_t\xi_{tx} - \frac{1}{8\xi_{tx}^2}\sigma_{tx}^2\right)dt + \frac{1}{2\xi_{tx}}\sigma_{tx} \cdot dW_t^*. \quad (7.6)$$

We recall that the volatility process $\sigma_{tx}$ arising again in this connection, which is given more explicitly by the ratio

$$\sigma_{tx} = \frac{\partial_x\Sigma_{tx,t+x}}{\partial_x B_{tx}}, \quad (7.7)$$

can be specified exogenously, subject only to the condition that it has mean zero in the measure $\rho_t(x)$, which implies that $\sigma_{tx}$ can be written in the form $(5.24)$.

We would now like to interpret the Hilbert space dynamics in equation $(7.6)$ more directly in a geometrical fashion. For this purpose we find it expedient to introduce an index notation, using Greek letters to signify Hilbert space operations (cf. Brody & Hughston 1998).

Thus if the function $\psi(x)$ is an element of $H = L^2(\mathbb{R}^1_+)$, we denote it by $\psi^\alpha$, and if $\varphi(x)$ belongs to the dual Hilbert space $H^*$ we denote this by $\varphi_\alpha$. Furthermore, their inner product is written

$$\psi^\alpha \varphi_\alpha = \int_0^\infty \psi(x)\varphi(x)dx. \quad (7.8)$$

There is a preferred symmetric quadratic form $g_{\alpha\beta}$ on $H$, given by $g_{\alpha\beta} \psi^\alpha \psi^\beta = \int_0^\infty (\psi(x))^2dx$, which thus establishes an isomorphism between $H$ and $H^*$, given by $\psi^\alpha \rightarrow \psi_\alpha = g_{\alpha\beta} \psi^\beta$. Intuitively, one can think of $g_{\alpha\beta}$ as corresponding to the
Figure 3. Interest rate dynamics. At each instant of time the interest rate term structure can be represented as a point on the positive orthant of the unit sphere $S$ in the Hilbert space $H = L^2(R_+^1)$. The associated arbitrage-free interest rate dynamics gives rise to a stochastic trajectory on this space, which is foliated by hypersurfaces corresponding to level values of the short-term interest rate.

delta function $\delta(x, y)$, and then we have

$$g_{\alpha\beta}\psi^\alpha\varphi^\beta = \int_0^\infty \psi(x)\delta(x, y)\varphi(y)dx dy. \quad (7.9)$$

There are a number of Hilbert space technicalities that have to be considered for a complete exposition of the matter, but that is not our immediate concern.

If $\xi(x) > 0$ belongs to the positive orthant of $L^2(R_+^1)$ then the corresponding indexed quantity $\xi^\alpha$ has the interpretation of a ‘state vector’. In that case we can think of symmetric quadratic forms as representing certain classes of random variables. The expectation of the random variable $H_{\alpha\beta}$ in the state $\xi^\alpha$ is

$$E_\xi[H] = \frac{H_{\alpha\beta}\xi^\alpha\xi^\beta}{\xi^\gamma\xi^\gamma}. \quad (7.10)$$

Therefore, a state vector determines a mapping from random variables to real numbers, through (7.10). For a normalised state vector we have $\xi_\alpha\xi^\alpha = 1$, although for some purposes it is convenient to relax the normalisation condition. In particular, we notice that the expectation (7.10) only depends on the direction of $\xi^\alpha$.

Now suppose $\xi(x)$ is a positive function. In that case, the derivative $\partial_x$ can be thought of as a linear operator $D^\alpha_{\beta}$ on $H$, and we have an endomorphism given by $\xi^\alpha \rightarrow D^\alpha_{\beta}\xi^\beta$. By making use of this, we can now interpret, in the language of Hilbert space geometry, the first two terms appearing in the drift in the dynamical equation (7.6).

Let us begin by noting first that (5.16) can be rewritten in the form

$$\int_0^\infty \xi_{tx}\partial_x\xi_{tx}dx = -\frac{1}{2}r_t. \quad (7.11)$$

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This allows us to interpret the short term interest rate process \( r_t \) in terms of the mean of the symmetric part of the operator \( D^\alpha_\beta \) in the state \( \xi_t^\alpha \), i.e.,

\[
\frac{D^\alpha_\beta \xi_t^\alpha}{g^\alpha_\beta \xi_t^\alpha} = -\frac{1}{2} r_t, \tag{7.12}
\]

where \( D^\alpha_\beta = g^\alpha_\gamma D^\gamma_\beta \). Therefore, if we let \( D_{(\alpha\beta)} \) denote the symmetric part of the operator \( D^\alpha_\beta \), then the abstract random variable in \( \mathcal{H} \) corresponding to the short rate \( r_t \) is given by \( r_{\alpha\beta} = -2D_{(\alpha\beta)} \). Similarly we can represent the abstract random variable \( x \) for the time left until maturity in \( \mathcal{H} \) by a symmetric matrix \( X_{\alpha\beta} \). It is interesting to note that the random variables \( X_{\alpha\beta} \) for the maturity date and \( r_{\alpha\beta} \) for the short term interest rate are not ‘compatible’. Two random variables \( A \) and \( B \) are said to be compatible if the expression \( \{\{A, C\}, B\} - \{A, \{C, B\}\} \) vanishes for any random variable \( C \), where \( \{A, B\} = AB + BA \) denotes the anticommutator (Segal 1947). The lack of compatibility here indicates that the abstract probability system containing both \( r_{\alpha\beta} \) and \( X_{\alpha\beta} \) as random variables is not Kolmogorovian. However, the algebra of random variables generated by \( X_{\alpha\beta} \) is Kolmogorovian.

Now, let \( \eta(x) \) be an arbitrary element of \( L^2(\mathbb{R}_1^+) \), and let \( \eta^\alpha \) be the corresponding Hilbert space vector. Then clearly we have

\[
\int_0^\infty \eta(x) \left[ \partial_x \xi_{tx} + \frac{1}{2} r_t \xi_{tx} \right] dx = \eta^\alpha \left[ D^\alpha_\beta \xi_t^\beta - \left( \frac{D^\beta_\gamma \xi_t^\gamma \xi_t^\beta}{g^\beta_\delta \xi_t^\beta \xi_t^\delta} \right) \delta^\alpha_\beta \right]. \tag{7.13}
\]

In other words, the first two terms of the drift in (7.6) can be replaced by the expression \( \tilde{D}^\alpha_\beta \xi_t^\beta \), where

\[
\tilde{D}^\alpha_\beta = D^\alpha_\beta - \left( \frac{D^\beta_\gamma \xi_t^\gamma \xi_t^\beta}{g^\beta_\delta \xi_t^\beta \xi_t^\delta} \right) \delta^\alpha_\beta, \tag{7.14}
\]

where \( \delta^\alpha_\beta \) is the Kronecker delta. Clearly, we have \( \tilde{D}_{\alpha\beta} \xi^\alpha \xi^\beta = 0 \).

With this in mind, let us now proceed to the interpretation of the volatility process \( \sigma_{tx} \). Again, \( \sigma_{tx} \) has the character of a linear operator acting on \( \xi_{tx} \), subject to the constraint \( E_{\rho}[\sigma_{tx}] = 0 \). This can be consistently enforced if there exists a symmetric process \( \nu_{\alpha\beta} \) such that

\[
\int_0^\infty \eta(x) \xi_{tx} \sigma_{tx} dx = \eta^\alpha \left( \nu_{\alpha\beta} \xi^\beta_t - E[\xi^\beta_t] \right). \tag{7.15}
\]

The symmetric operator-valued random process \( \nu_{\alpha\beta} \), whose existence is thus implied, is ‘primitive’ in the sense that it is unconstrained and can be specified exogenously. If we write

\[
\sigma_{\alpha\beta} = \nu_{\alpha\beta} - E[\xi^\alpha_{tx}] g^\alpha_\beta,
\]

we obtain

\[
\int_0^\infty \eta(x) \xi_{tx} \sigma_{tx} dx = \eta^\alpha \sigma^\alpha_\beta \xi^\beta_t, \tag{7.17}
\]

and also

\[
\int_0^\infty \eta(x) \xi_{tx} \sigma^2_{tx} dx = \eta^\alpha \sigma^\alpha_\beta \sigma^\beta_\gamma \xi^\gamma_t. \tag{7.18}
\]
Therefore, putting the various ingredients together, we obtain:

**Proposition 5.** The dynamics of the Hilbert space vector \( \xi^\alpha_t \) that characterises the term structure in an admissible, arbitrage-free interest rate framework is governed by the stochastic differential equation

\[
d\xi^\alpha_t = \left( \tilde{D}^\alpha_{\beta} - \frac{1}{8} \sigma^\alpha_{t\gamma} \cdot \sigma^\gamma_{t\beta} \right) \xi^\beta_t \, dt + \frac{1}{2} \sigma^\alpha_{t\beta} \xi^\beta_t \cdot (dW_t + \lambda_t \, dt),
\]

where \( \tilde{D}^\alpha_{\beta} \) is given as in (7.14), and the adapted operator-valued process \( \sigma_{t\alpha\beta} \) is expressible in the form

\[
\sigma_{t\alpha\beta} = \nu_{t\alpha\beta} - \left( \frac{\nu_{t\gamma\delta} \xi^\gamma_t \xi^\delta_t}{g_{t\gamma\delta} \xi^\gamma_t \xi^\delta_t} \right) g_{t\alpha\beta},
\]

where \( \nu_{t\alpha\beta} \) is an arbitrary adapted operator-valued process.

This result shows that the evolution of the yield curve can be viewed consistently as a process on the positive orthant of the unit sphere in Hilbert space, and thus gives rise to an entirely new way of understanding the dynamics of the term structure. The purpose of the quadratic term in the drift of (7.19) is to keep the process on the sphere, and in the absence of the term involving the operator \( D^\alpha_{\beta} \) we would have a general local martingale on the sphere \( \mathcal{S} \) with respect to the risk-neutral measure, where the martingale property on \( \mathcal{S} \) is characterised in a standard way by use of the techniques of stochastic differential geometry (see, e.g., Emery 1989, Ikeda & Watanabe 1989, Hughston 1996). The term involving the operator \( D^\alpha_{\beta} \) splits into a symmetric and an antisymmetric part. The drift generated by the antisymmetric part of \( D^\alpha_{\beta} \) is generated by a symmetry of the sphere \( \mathcal{S} \). The drift generated by the symmetric part of \( D^\alpha_{\beta} \), on the other hand, is a negative gradient vector field orthogonal to surfaces in \( \mathcal{S} \) generated by level values of the short rate \( r_t \). This term therefore creates a tendency for the vector \( \xi^\alpha \) to drift towards a lower interest rate, a property of the negative gradient field which is then counterbalanced by the effects of the diffusive term.

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