Early thermalization of quark-gluon matter by elastic 3-to-3 scattering

Xiao-Ming Xu
Department of Physics, Shanghai University, Baoshan, Shanghai 200444, China
E-mail: xmxu@mail.shu.edu.cn

Abstract. The early thermalization is crucial to the quark-gluon plasma as a perfect liquid and results from many-body scattering. We calculate squared amplitudes for elastic parton-parton-parton scattering in perturbative QCD. Transport equations with the squared amplitudes are established and solved to obtain the thermalization time of initially produced quark-gluon matter and the initial temperature of quark-gluon plasma. We find that the thermalization times of quark matter and gluon matter are different.

1. Introduction

Quark-gluon matter initially produced in Au-Au collisions at RHIC energies is not in a thermal state and has not a temperature since gluon and quark distributions are anisotropic in momentum space. After a period of time $t_{\text{therm}}$, quark-gluon matter is in a thermal state and has a temperature while the distributions are isotropic, i.e. quark-gluon matter is a quark-gluon plasma. The transition from initially produced quark-gluon matter to the quark-gluon plasma is a thermalization process and $t_{\text{therm}}$ is the thermalization time. It had not been expected that the thermalization time is small before the measured elliptic flow coefficient of hadrons was first reported. In order to explain the measured data, hydrodynamic calculations assume early thermalization and ideal relativistic fluid flow [1, 2, 3]. Early thermalization is that a thermal state is achieved with a time less than 1 fm/$c$ from the moment when quark-gluon matter is initially created in heavy-ion collisions. Different thermalization times have been given in different hydrodynamic models [4]. Strongly interacting matter in thermal equilibrium can very easily flow in any direction without any breakup, i.e. this matter is a perfect liquid. Strong interaction of particles inside dense matter establishes a thermal state rapidly. This means that the quark-gluon plasma formed by the early thermalization is a perfect liquid. Therefore, the early thermalization is crucial to the perfect liquid the quark-gluon plasma.

The early thermalization is an assumption in hydrodynamic calculations. We need to understand the early thermalization. Then, two questions are raised. One is how initially produced quark-gluon matter establishes a thermal state, another is when initially produced quark-gluon matter establishes a thermal state. To answer the two questions, we examine the occurrence probabilities of two-parton scattering, three-parton scattering and so on in quark-gluon matter. This is motivated by the thought that thermalization is related to two-body scattering [5, 6, 7, 8]. One thousand and five hundred partons are generated from HIJING within $-0.3 < z < 0.3$ fm in the longitudinal direction and $r < 6.4$ fm in the transverse
direction in central Au-Au collisions at \( \sqrt{s_{NN}} = 200 \text{ GeV} \). This volume of partons corresponds to a parton number density of 19.4 fm\(^{-3}\). The maxima of the numbers of \( n \)-parton scattering with \( n = 2, \ldots, 7 \) are 750, 500, 375, 300, 250 and 214, respectively. The occurrence probability can be represented by the number of \( n \)-parton scattering divided by the sum of the maximum numbers of two-, three-, \ldots, and seven-parton scattering. The occurrence probability increases with increasing interaction range. At an interaction range of 0.62 fm the 2-parton scattering and 3-parton scattering have occurrence probabilities of 0.3 and 0.2, respectively [9]. Therefore, the three-parton scattering is important due to the high parton number density. One effect of elastic three-gluon scattering is the early thermalization of gluon matter initially produced in central Au-Au collisions [10]. The early thermalization is an effect of many-body interaction!

The occurrence probabilities for 4-parton, 5-parton, 6-parton, 7-parton, 8-parton, 9-parton, 10-parton and 11-parton scattering are 0.146, 0.11, 0.09, 0.075, 0.044, 0.023, 0.009 and 0.002, respectively. The occurrence probabilities for 12-parton scattering and 13-parton scattering are negligible. In a sphere with a radius the same as the interaction range the number of partons is at most 14. This means that at most 14-parton scattering is allowed. As a consequence, the occurrence probability of 15-parton scattering is zero and the one of the 14-parton scattering is very small. The sum of the occurrence probabilities from the 2-parton scattering equals 1 and the occurrence probabilities form a convergent series. From the occurrence probabilities the elastic 4-parton (5-parton) scattering is expected to give a smaller contribution to thermalization than the elastic 3-parton (4-parton) scattering and the contribution of the elastic \( n \)-parton scattering with \( n \geq 6 \) may be neglected.

Thermalization of quark-gluon matter at a high number density is governed by the two-parton scattering and the three-parton scattering in the present work. This is the answer to the first question. An answer to the second question is given in the next two sections.

2. Elastic three-parton scattering

Thermalization of water (air) is governed by elastic two-body scattering of water molecules (air molecules). Thermalization by elastic two-body scattering is conventional. Therefore, two-parton scattering is first taken into account to examine thermalization of quark-gluon matter. Elastic parton-parton scattering was studied by Cutler and Sivers in [11] and by Cambridge, Kripfganz and Raufi in [12] in perturbative QCD. Spin- and color-averaged squared amplitudes of order \( \alpha_s^4 \) were derived by hand and expressions for the squared amplitudes are presented in terms of the Mandelstam variables in [11, 12]. It is very convenient to use the expressions.

However, it is impractical to derive squared amplitudes for elastic parton-parton-parton scattering by hand and the squared amplitudes have to be derived from Fortran code. This is because many Feynman diagrams even at order \( \alpha_s^4 \) are involved in the elastic 3-to-3 scattering. For example, there are 72 diagrams for elastic scattering of one gluon and two identical quarks and 76 diagrams for elastic gluon-quark-antiquark scattering [13]. Only one Feynman diagram has two four-gluon vertices and it is shown in figure 1. The squared amplitude for the diagram takes the shortest expression among all diagrams involved in elastic parton-parton-parton scattering at the tree level. Including the average over the spin and color states of the three initial gluons, the spin- and color-summed squared amplitude is [14]

\[
\frac{118098 \pi^4 \alpha_s^4}{s^2}
\]

where \( s \) is the square of the sum of the four-momenta of the three initial gluons. The expression has only one term, but the squared amplitudes for other diagrams have more terms with the reasons as follows. The other diagrams must have at least one triple-gluon vertex or the trace of a product of Dirac matrices. According to the Feynman rules the triple-gluon vertex consists
of three terms. The trace of a product of even four Dirac matrices gives three terms. Then, the squared amplitude for each of the other Feynman diagrams or any interference term of different diagrams must have at least three terms.

According to the numbers of the triple-gluon vertex and the four-gluon vertex, Feynman diagrams are classified and their squared amplitudes are organized. For instance, the 72 diagrams for elastic gluon-quark-quark scattering are separated into 3 classes [13]. The first, second and third classes contain 40 diagrams with no triple-gluon vertex and no four-gluon vertex, 24 diagrams with one triple-gluon vertex and no four-gluon vertex and 8 diagrams with two triple-gluon vertices or one four-gluon vertex, respectively. The 76 diagrams for elastic gluon-quark-antiquark scattering are also divided into 3 classes even though quark-antiquark annihilation may occur [13]. The first class contains 40 diagrams with no triple-gluon vertex and no four-gluon vertex, 20 diagrams with no quark-antiquark annihilation and 20 diagrams with the annihilation. The second class contains 28 diagrams with one triple-gluon vertex and no four-gluon vertex, 12 with no quark-antiquark annihilation and 16 with the annihilation. The third class contains 8 diagrams with two triple-gluon vertices or one four-gluon vertex: 4 with no quark-antiquark annihilation and 4 with the annihilation.

\[
\frac{\partial f_{g1}}{\partial t} + \vec{v}_1 \cdot \vec{\nabla} f_{g1} = \frac{1}{2E_1} \int \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_3}{(2\pi)^3} \frac{d^3p_4}{(2\pi)^3} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)
\times \left\{ \frac{g_s}{2} \left| \mathcal{M}_{gg \rightarrow gg} \right|^2 [f_{g1}f_{g2}(1 + f_{g3})(1 + f_{g4}) - f_{g3}f_{g1}(1 + f_{g2})(1 + f_{g4})] + g_s \left| \mathcal{M}_{g1 \rightarrow g2} \right|^2 + \left| \mathcal{M}_{g1 \rightarrow g3} \right|^2 + \left| \mathcal{M}_{g1 \rightarrow g4} \right|^2 \right\}
\times \left\{ [f_{g1}f_{g2}(1 + f_{g3})(1 + f_{g4}) - f_{g3}f_{g1}(1 + f_{g2})(1 + f_{g4})] \right\} - \frac{1}{2E_1} \int \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_3}{(2\pi)^3} \frac{d^3p_4}{(2\pi)^3} \frac{d^3p_5}{(2\pi)^3} \frac{d^3p_6}{(2\pi)^3} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)
\]

Figure 1. Scattering of three gluons. The wiggly lines stand for gluons.

3. Thermalization

We establish transport equations for quark-gluon matter which consists of gluons, quarks and antiquarks with up and down flavors. We assume that the quark distribution is symmetric in flavor and is identical with the antiquark distribution. The quark and gluon distribution functions are denoted by \( f_{qi} \) and \( f_{gi} \), respectively, where \( i \) labels the \( i \)th quark or gluon. With the elastic parton-parton scattering and the elastic parton-parton-parton scattering, the transport equation for gluons is [13]
\[\times (2\pi)^4 \delta^4(p_1 + p_2 + p_3 - p_4 - p_5 - p_6) \left\{ \begin{array}{c}
\frac{g_{GQ}}{2} | M_{gusgus} |^2 \\
\frac{g_{GQ}}{2} | M_{gusgus} |^2 + | M_{gusgus} |^2 + | M_{gusgus} |^2 + | M_{gusgus} |^2 \\
+ | M_{gusgus} |^2 + | M_{gusgus} |^2 + | M_{gusgus} |^2 + | M_{gusgus} |^2 \\
+ | M_{gusgus} |^2 + | M_{gusgus} |^2 + | M_{gusgus} |^2 + | M_{gusgus} |^2 \\
\end{array} \right\} + \left[ \begin{array}{c}
\frac{1}{4} | M_{uusuuu} |^2 + \frac{1}{2} | M_{uusuuu} |^2 + | M_{uusuuu} |^2 + | M_{uusuuu} |^2 \\
\frac{1}{4} | M_{uusuuu} |^2 + \frac{1}{2} | M_{uusuuu} |^2 + | M_{uusuuu} |^2 + | M_{uusuuu} |^2 \\
\end{array} \right] \right\},
\]

and the transport equation for up-quarks is

\[
\frac{\partial f_{q_1}}{\partial t} + \vec{v}_i \cdot \vec{\nabla}_r f_{q_1} = \frac{1}{2E_1} \int \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_3}{(2\pi)^3} \frac{d^3p_4}{(2\pi)^3} \frac{d^3p_5}{(2\pi)^3} \frac{d^3p_6}{(2\pi)^3} \left( \begin{array}{c}
\frac{g_{GQ}}{4} | M_{ugugug} |^2 \\
\frac{g_{GQ}}{4} | M_{ugugug} |^2 + | M_{ugugug} |^2 + | M_{ugugug} |^2 + | M_{ugugug} |^2 \\
\frac{1}{4} | M_{uusuuu} |^2 + \frac{1}{2} | M_{uusuuu} |^2 + | M_{uusuuu} |^2 + | M_{uusuuu} |^2 \\
\frac{1}{4} | M_{uusuuu} |^2 + \frac{1}{2} | M_{uusuuu} |^2 + | M_{uusuuu} |^2 + | M_{uusuuu} |^2 \\
\end{array} \right) \right\},
\]

where \( \vec{v}_1 \) is the parton velocity; \( g_G \) and \( g_{Q} \) are the color-spin degeneracy factors; \( p_i(i = 1, \ldots, 6) \) denote the four-momenta of initial and final partons; \( E_i \) is the energy component of \( p_i \); \( | M_{ab\rightarrow c\gamma} |^2 \) and \( | M_{abcightarrow a\gamma} |^2 \) are the squared amplitudes for \( a + b \rightarrow c \) and \( a + b \rightarrow c' + b' + c' \), respectively. The squared amplitude \( | M_{gusgus} |^2 \) was obtained in [10]; \( | M_{gusgus} |^2 \) in [15]; \( | M_{gusgus} |^2 \) and \( | M_{gusgus} |^2 \) in [13]; \( | M_{gusgus} |^2 \) and \( | M_{gusgus} |^2 \) have not been obtained and are temporarily set as zero. Similar equations for down-quarks, up-antiquarks and down-antiquarks can be established.

Parton-parton scattering affects only two partons’ momenta while parton-parton-parton scattering changes three partons’ momenta. If the 3-to-3 scattering has the same occurrence
probability as the 2-to-2 scattering, the 3-to-3 scattering changes partons’ momenta faster than the 2-to-2 scattering. Frequent changes of momenta establish a thermal state. Indeed, the elastic 3-to-3 scattering drives anisotropic matter described by the transport equations towards global thermal equilibrium [14], which is similar to the $H$-theorem. Solutions of the transport equations will show that a thermal state is established from a moment.

In the region $-0.3 < z < 0.3$ fm in the longitudinal direction and $r < 6.4$ fm in the transverse direction, 1500 gluons and 1000 fermions including up quarks, down quarks, up antiquarks and down antiquarks are generated from HIJING for central Au-Au collisions at $\sqrt{s_{NN}} = 200$ GeV. These gluons, quarks and antiquarks at $t = 0.2$ fm/$c$ form initially produced quark-gluon matter and are anisotropically distributed in momentum space. Solving the transport equations, we found that at $t = 0.68$ fm/$c$ gluon matter has an isotropic momentum distribution function, a thermal state with the temperature $T = 0.5$ GeV and a thermalization time of the order of 0.48 fm/$c$ [13]. A thermal state of quark matter with $T = 0.3$ GeV is established at a later time of 1.56 fm/$c$. In other words, the thermalization time 1.36 fm/$c$ of quark matter is longer than gluon matter [13]. The elastic scattering of $gg$, $gq$, $gq$, $qqq$ and $qqq$ gives contributions to gluon matter and quark matter nearly identical in thermalization. The difference of the thermalization times of gluon matter and quark matter mainly comes from the difference that elastic $gg$ ($gg$) scattering has a larger squared amplitude than elastic $qqq$ or $qqq$ ($qq$ or $qg$) scattering and gluon matter is denser than quark matter.

4. Conclusions

Early thermalization is an intrinsic property of initially produced quark-gluon matter at the high number density and is an effect of many-body scattering. The elastic 3-to-3 scattering is important at such a number density. The elastic gluon-gluon scattering and the elastic gluon-gluon-gluon scattering lead to the early thermalization of gluon matter. The thermalization time of quark matter differs from that of gluon matter and we need to include the elastic scattering of both $ggq$ and $ggq$ in future work to explore whether quark matter thermalizes rapidly or not.

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