An approximation formula for $n!$

Nedret Batir
Nevşehir University, Nevşehir, Turkey
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Abstract
We prove the following very accurate approximation formula for the factorial function:

$$n! \approx n^n e^{-n} \sqrt{2\pi \left( n + \frac{1}{6} + \frac{1}{72n} - \frac{1}{720n^2} + \frac{1}{2520n^3} - \frac{1}{65520n^4} \right)}.$$ 

This gives better results than the following approximation formula

$$n! \approx \sqrt{2\pi n^n e^{-n}} \sqrt{n + \frac{1}{6} + \frac{1}{72n} - \frac{31}{6480n^2} - \frac{139}{155520n^3} + \frac{9871}{6531840n^4}},$$

which is established by the author [5] and C. Mortici [16] independently, and gives similar results with

$$n! \approx \sqrt{\pi \left( \frac{n}{e} \right)^n \sqrt{16n^4 + \frac{32}{3} n^3 + \frac{32}{9} n^2 + \frac{176}{405} n - \frac{128}{1215}}}.$$ 

which is established by C. Mortici in his very new paper [8].

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1. Introduction

For a positive real number $z$ the gamma function $\Gamma$ and its logarithmic derivative $\psi$, so-called psi function or digamma function, are defined by

$$\Gamma(z) = \int_0^\infty u^{z-1}e^{-u}du, \quad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad (z > 0).$$

These two functions are considered to be the most important special functions next to the Riemann zeta function. As it is well known, the gamma function and factorials are related with the identity $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$. The problem of approximating the gamma function, in particular, the factorial function has been attracted the attention of many mathematicians recently, and a lot of paper concerning this problem have been published; see for example [1-20]. The most well known approximation formula for $n!$ is the classical Stirling formula

$$n! \approx n^n e^{-n} \sqrt{2\pi n}.$$  

Bauer [7] defined the sequence $(\delta_n)$ by the relation

$$n! = n^n e^{-n} \sqrt{2\pi(n + \delta_n)}$$

and numerical computations led him to infer that

$$\lim_{n \to \infty} \delta_n = 0.166666... = 1/6.$$  

Consequently, he conjectured the approximation formula

$$n! \approx \sqrt{2\pi n^n e^{-n} \sqrt{n + 1/6}}.$$  

In [2] the author shoved the superiority of (1.2) over the Stirling formula. In [6] the author obtained the best possible scaler pairs $(a, b)$ of real numbers such as the approximation $n! \approx n^n e^{-n-a} \sqrt{2\pi(n + b)}$ gives the best accurate values for $n!$. We note that (1.2) corresponds to $(a, b) = (0, 1/6)$. The author [3] proved (1.1) and established the following inequalities, for $n \in \mathbb{N}$

$$n^n e^{-n} \sqrt{2\pi \left(n + \frac{1}{6}\right)} < n! \leq n^n e^{-n} \sqrt{2\pi \left(n + \frac{e^2}{2\pi} - 1\right)}.$$  

Very recently and independently, the author [5] and C. Mortici [16] obtained the following approximation formula for $n!$ to improve (1.2).

$$n! \approx \sqrt{2\pi n^n e^{-n} \sqrt{n + \frac{1}{6} + \frac{1}{72} - \frac{31}{648n^2} - \frac{139}{155520n^3} + \frac{9871}{8331840n^4}}}.$$
An approximation formula for \( n! \)

As a first aim of this work we determine the best scalers \( \alpha, \beta, a, b \) in such a way that the following formula offers the best approximation for \( n! \).

\[
(1.3) \quad n! \approx n^n e^{-n} \sqrt{2\pi} \left( n + \frac{1}{6} + \frac{\alpha}{n + a} + \frac{\beta}{(n + b)^3} \right).
\]

In particular, we show that the best approximation of this form is obtained for

\[
\alpha = \frac{1}{72}, \beta = \frac{5929}{2332800}, a = \frac{31}{90}, b = \frac{3055123}{11205810}.
\]

Our second aim is to find the best scalers \( \alpha, \beta \) such that the following inequalities hold true:

\[
\alpha \cdot \sqrt{x + \frac{1}{6} + \frac{1}{72(x + \frac{31}{90})}} \leq \frac{\Gamma(x+1)}{x! e^{-x}} < \beta \cdot \sqrt{x + \frac{1}{6} + \frac{1}{72(x + \frac{31}{90})}}.
\]

In order to prove our main results we need the following elementary but very useful lemmas. The first lemma is proved in [19]. The algebraic and numerical computations have been carried out with the computer software Mathematica 8.

**Lemma 1.1.** If \( (\omega_n)_{n \geq 1} \) is convergent to zero and there exists the limit

\[
\lim_{n \to \infty} n^k (\omega_n - \omega_{n+1}) = c \in \mathbb{R},
\]

with \( k > 1 \), then there exists the limit

\[
\lim_{n \to \infty} n^{k-1} \omega_n = \frac{c}{k-1}.
\]

It is clear from this lemma that the speed of convergence of the sequence \( (\omega_n) \) is as higher as the value of \( k \) is greater.

**Lemma 1.2.** Let \( f \) be a function defined on an interval \( I \) and \( \lim_{x \to \infty} f(x) = 0 \). If \( f(x+1) - f(x) > 0 \) for all \( x \in I \), then \( f(x) < 0 \). If \( f(x+1) - f(x) < 0 \), then \( f(x) > 0 \).

**Proof.** Let \( f(x+1) - f(x) > 0 \) for all \( x \in I \). By mathematical induction we have \( f(x) < f(x+n) \) for all \( n \in \mathbb{N} \). Letting \( n \to \infty \), we have \( f(x) < \lim_{n \to \infty} f(x+n) = 0 \). The proof of second part of the lemma follows from the same argument. \( \square \)
2. Main results

The following two theorems are our main results.

**Theorem 2.1.** For all \( n \in \mathbb{N} \), the best approximation of the form (1.3) is obtained for
\[
\alpha = \frac{1}{72}, \beta = \frac{5929}{2332800}, a = \frac{31}{90}, b = \frac{3055123}{11205810},
\]
namely
\[
n! \approx n^{\gamma - n} \sqrt{2\pi \left( n + \frac{1}{6} + \frac{1}{72(n+\alpha)} - \frac{5929}{2332800(n+\frac{3055123}{11205810})} \right)}.
\]

**Proof.** We define for \( n \in \mathbb{N} \)
\[
T_n = \log n! - n \log n + n - \frac{1}{2} \log(2\pi)
\]
(2.1)
\[-\frac{1}{2} \log \left( n + \frac{1}{6} + \frac{\alpha}{n+a} + \frac{\beta}{(n+b)^2} \right).
\]

Successive differences of the sequence \((T_n)\) are given by
\[
T_n - T_{n+1} = n \log \left( 1 + \frac{1}{n} \right) - 1
\]
(2.2)
\[-\frac{1}{2} \log \left( \frac{n + \frac{1}{6} + \frac{\alpha}{n+a} + \frac{\beta}{(n+b)^2}}{n + \frac{7}{6} + \frac{\alpha}{n+1+a} + \frac{\beta}{(n+1+b)^2}} \right).
\]

If we expand the right hand side of (2.2) as a power series of \( n^{-1} \) we get
\[
T_n - T_{n+1} = \frac{1}{72} (1 - 72\alpha) \frac{1}{n^3} + \frac{1}{540} \left( -17 + 945\alpha + 810\alpha\beta \right) \frac{1}{n^4}
\]
\[+ \frac{1}{12960} \left( 641 - 33120\alpha - 43200\alpha a - 25920\alpha a^2 + 12960\alpha^2 \right)
\]
\[+ \frac{1}{27216} \left( -1831 + 94815\alpha + 160650\alpha a + 147420\alpha a^2 + 147420\beta \right)
\]
\[+ \frac{1}{11205810} \left( 55609 - 2981664\alpha - 6132672\alpha a + 79380\alpha a^2 + 68040\alpha a^3 + 147420\beta \right)
\]
\[+ \frac{1}{1831452} \left( -7402752\alpha a^2 - 5225472\alpha^3 - 1959552\alpha^4 + 4164048\alpha^2 \right)
\]
\[+ \frac{1}{5552064\alpha a^2} + 2939328\alpha a^2 - 653184\alpha a^3 - 7402752\beta
\]
An approximation formula for $n!$

\[
(2.3) \quad -15676416b\beta - 11757312b^2\beta + 1959552\alpha\beta \left( \frac{1}{n^7} + O \left( \frac{1}{n^8} \right) \right).
\]

Faster convergences are obtained by imposing the first six coefficients vanish. This results in

\[
(2.4) \quad \alpha = \frac{1}{72}, \quad \beta = \frac{5929}{2332800}, \quad a = \frac{31}{90}, \quad b = \frac{3055123}{11205810},
\]

that is, the best approximation of the form (1.3) is

\[
n! \approx n^n e^{-n} \sqrt{2\pi \left( n + \frac{1}{6} + \frac{1}{72(n + \frac{31}{90})} - \frac{5929}{2332800(n + \frac{3055123}{11205810})^3} \right)}.
\]

If we put the values of $\alpha, \beta, a$ and $b$ obtained in (2.4) in (2.3) we find

\[
(2.5) \quad T_n - T_{n+1} = \frac{39977573013907}{1097918369856000} \frac{1}{n^7} + O \left( \frac{1}{n^8} \right).
\]

By Lemma 1 this proves that

\[
\lim_{n \to \infty} n^6 T_n = \frac{39977573013907}{65875102191360000},
\]

namely, the sequence $(T_n)$ converges to zero like $n^{-6}$. Our second theorem provides new and elegant bounds for the gamma function.

**Theorem 2.2.** Let $x$ be a non-negative real number. Then we have

\[
(2.6) \alpha \cdot \sqrt{x + \frac{1}{6} + \frac{1}{72(x + \frac{31}{90})}} \leq \frac{\Gamma(x + 1)}{x^x e^{-x}} < \beta \cdot \sqrt{x + \frac{1}{6} + \frac{1}{72(x + \frac{31}{90})}}
\]

where $\alpha = \frac{1452}{1709} = 2.30951...$ and $\beta = \frac{\sqrt{2\pi}}{2} = 2.50663...$ are the best possible constants.

**Proof.** Let $g$ be as following

\[
g(x) = \log \Gamma(x + 1) - x \log x + x - \frac{1}{2} \log(2\pi) - \frac{1}{2} \log \left( x + \frac{1}{6} + \frac{1}{72(x + \frac{31}{90})} \right).
\]

Differentiation gives
\[ g'(x) = \psi(x + 1) - \log x \]
\[ - \frac{6(9833 + 129735x + 376650x^2 + 364500x^3)}{0541 + 438216x + 2310120x^2 + 5248800x^3 + 4374000x^4} \]

and

\[ g''(x) = \psi'(x + 1) - \frac{1}{x} - \frac{a(x)}{b(x)}, \]

where

\[ a(x) = -18(31 + 90x)^2(-2579729 + 7079760x + 46704600x^2 + 90396000x^3 + 65610000x^4) \]

and

\[ b(x) = (90541 + 438216x + 2310120x^2 + 5248800x^3 + 4374000x^4)^2. \]

Using the functional relation \( \psi'(x + 1) - \psi'(x) = -1/x^2 \), we obtain

\[ g''(x + 1) - g''(x) = \frac{P(x)}{Q(x)}, \]

where

\[ -P(x) = 243646234638147649 + 4312614519882734208x + 30540379834894285224x^2 + 113179004500484499840x^3 + 240030776577049466400x^4 + 299615216845238784000x^5 + 217070653175475840000x^6 + 84383307377280000000x^7 + 13611947136480000000x^8 \]

and

\[ Q(x) = x(1 + x)^2(31 + 90x)^2(121 + 90x)^2(77 + 552x + 1080x^2)^2 \cdot \left(1709 + 2712x + 1080x^2\right)^2. \]

Since both \( P \) and \( Q \) are positive in \((0, \infty)\), we have \( g''(x + 1) - g''(x) > 0 \) for \( x \geq 0 \). Utilizing the relation \( \lim_{x \to \infty} \psi'(x) = 0 \), we see that \( \lim_{x \to \infty} g''(x) = 0 \).

Consequently, by the help of Lemma 2 we prove that \( g' \) is strictly decreasing in \((0, \infty)\). It is well known that \( \lim_{x \to \infty} (\psi(x) - \log x) = 0 \); see, for example, [1], so that \( \lim_{x \to \infty} g'(x) = 0 \). This shows that \( g \) is strictly increasing in \((0, \infty)\). Also, by Stirling formula we get \( \lim_{x \to \infty} g(x) = 0 \). Consequently, we have

\[ g(0) = \frac{1}{2} \log(372/77) - \frac{1}{2} \log(2\pi) < g(x) < \lim_{x \to \infty} g(x) = 0, \]

which is equivalent to (2.6). \( \Box \)
3. Numerical computations

In this section we want to compare our estimation to some recent estimations due to the author and C. Mortici. First, we set

$$a_n = \left( \frac{n}{e} \right)^n \sqrt{2\pi \left( n + \frac{1}{6} + \frac{1}{72(n + \frac{31}{90})} - \frac{5929}{2332800(n + \frac{3055123}{11205810})^3} \right)}.$$  

(3.1)

and

$$b_n = \sqrt{2\pi \left( \frac{n}{e} \right)^n \sqrt{n + \frac{1}{6} + \frac{1}{72(n + \frac{31}{90})}}}.$$  

(3.2)

As shown in Table 1, the formula $n! \approx a_n$ gives better results than the following estimate

$$n! \approx \sqrt{2\pi \left( \frac{n}{e} \right)^n \sqrt{n + \frac{1}{6} + \frac{1}{72n} - \frac{31}{6480n^2} - \frac{139}{155520n^3} + \frac{9871}{6531840n^4}}} = \alpha_n,$$

(3.3)

which is established by the author [5] and C. Mortici [16] independently, and can be compared with the formula [5, 16]

$$n! \approx \sqrt{\pi \left( \frac{n}{e} \right)^n \sqrt{16n^4 + \frac{32}{3}n^3 + \frac{32}{9}n^2 + \frac{176}{405}n - \frac{128}{1215}}} = \beta_n$$

(3.4)

which is established by C. Mortici in his very new paper [10].

| $n$ | $|a_n - n!|$ | $|\alpha_n - n!|$ | $|\beta_n - n!|$ |
|-----|-------------|-----------------|-----------------|
| 1   | 0.0001023   | 0.0001633       | 0.0001633       |
| 2   | 0.0000078   | 0.000189        | 0.000189        |
| 10  | 0.00188     | 0.011978        | 0.000283        |
| 50  | $1.1399 \times 10^{51}$ | $3.0925 \times 10^{52}$ | $2.3968 \times 10^{50}$ |
| 100 | $6.0971 \times 10^{147}$ | $2.9144 \times 10^{144}$ | $1.2194 \times 10^{142}$ |

Table 3.1: A comparison between some terms of $a_n$, $\alpha_n$, and $\beta_n$. 

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References

[1] H. Alzer, On some inequalities for the gamma and psi function, Math. Comput., 66, 217, pp. 373-389, (1997).

[2] H. Alzer, On Ramanujan’s double inequality for the gamma function, Bull. London Math. Soc. 5, 35, pp. 601-607, (2003).

[3] N. Batir, Inequalities for the gamma function, Arch. Math(Basel), 91, pp. 554-563, (2008).

[4] N. Batir, Sharp inequalities for factorial n, Proyecciones, 1, 27, pp. 97-102, (2008).

[5] N. Batir, Very accurate approximations for the factorial function, J. Math. Inequal., 3, 4, pp. 335-344, (2010).

[6] N. Batir, Improving Stirling formula, Math. Commun., 16, No. 1, pp. 105-114, (2011).

[7] L. Bauer, Remark on Stirling’s formulas and on approximations for the double factorial, Mathematical Intelligences, 28, 2, pp. 10-21, (2006).

[8] E. A. Karatsuba, On the asymptotic representation of the Euler gamma function by Ramanujan, J. Comput. Appl. Math., 135, 2001, 225-240.

[9] C. Mortici, Ramanujan’s estimate for the gamma function via monotonicity arguments, Ramanujan J, DOI.10.1007/s11139-010-9265-y, (2011).

[10] C. Mortici, A new Stirling series as continued fraction, Numerical Algorithms, 56, no. 1, pp. 17-26, (2011).

[11] C. Mortici, Accurate estimates of the gamma function involving the psi function, Numer. Funct. Anal. Optim., 32, no. 4, pp. 469-476, (2011).

[12] C. Mortici, New sharp inequalities for approximating the factorial function and the digamma function, Miskolc Math., 11, no. 1, pp. 79-86, (2010).

[13] C. Mortici, Asymptotic expansions of the generalized Stirling approximation, Mathe. Comput. Model., 52, no. 9-10, pp. 1867-1868, (2010).
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[14] C. Mortici, Estimating gamma function in terms of digamma function, Math. Comput. Model., 52, no. 5-6, pp. 942-946, (2010).

[15] C. Mortici, On the Stirling expansion into negative powers of a triangular numbers, Math. Commun., 15, no. 2, pp. 359-364, (2010).

[16] C. Mortici, Sharp inequalities related to Gosper’s Formula, Comptes Rendus Mathematique, 348, no. 3-4, pp. 137-140, (2010).

[17] C. Mortici, A class of integral approximations for the factorial function, Computers and Mathematics with Applications, 59, no. 6, pp. 2053-2058, (2010).

[18] C. Mortici, New improvements of the Stirling formula, Appl. Math. Comput., 217, no. 2, pp. 699-704, (2010).

[19] C. Mortici, New approximations of the gamma function in terms of the digamma function, Applied Mathematics Letters, 23, no. 1, pp. 97-100, (2010).

[20] C. Mortici, An ultimate extremely accurate formula for approximation of the factorial function, Arch. Math. (Basel), 93, no. 1, pp. 37-45, (2009).

Necdet Batir
Department of Mathematics,
Faculty of Science and Arts,
 Nevşehir University, Nevşehir,
Turkey
e-mail : nbatir@hotmail.com