ON BILINEAR HILBERT TRANSFORM ALONG TWO POLYNOMIALS

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Abstract. We prove that the bilinear Hilbert transform along two polynomials $B_{P,Q}(f,g)(x) = \int_\mathbb{R} f(x - P(t))g(x - Q(t))\frac{dt}{t}$ is bounded from $L^p \times L^q$ to $L^r$ for a large range of $(p,q,r)$, as long as the polynomials $P$ and $Q$ have distinct leading and trailing degrees. The same boundedness property holds for the corresponding bilinear maximal function $M_{P,Q}(f,g)(x) = \sup_{\epsilon > 0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |f(x - P(t))g(x - Q(t))| dt$.

1. Introduction

The Hilbert transform along a curve $\gamma : \mathbb{R} \rightarrow \mathbb{R}^n$ is defined by

$$(1.1) \quad H_{\gamma}(f)(x) := \int_\mathbb{R} f(x - \gamma(t))\frac{dt}{t}, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

Here $\mathcal{S}(\mathbb{R}^n)$, $n \in \mathbb{N}$, denotes the space of Schwartz functions on $\mathbb{R}^n$. Stein ([13]) raised the question that under what condition on $\gamma$ is $H_{\gamma}$ bounded from $L^p(\mathbb{R}^n)$ to itself for some $p$. Among many curves, a simple but important two dimensional example is the curve $\gamma_{a,b}(t) = (ta, tb)$, where $a,b$ are distinct natural numbers. For this particular type of curve, (1.1) becomes

$$(1.2) \quad H_{\gamma_{a,b}}(f)(x_1, x_2) = \int_\mathbb{R} f(x_1 - ta, x_2 - tb)\frac{dt}{t}, \quad f \in \mathcal{S}(\mathbb{R}^2).$$

The $L^2$-boundedness of $H_{\gamma_{a,b}}$ was first proved by Fabes [1] and Stein and Wainger [15], using different methods. Nagel et.al. [10, 11] obtained the $L^p$-boundedness for $p \in (1, \infty)$. It turns out $\gamma_{a,b}$ is the model curve for the very general "well-curved" curves ([17]).

The purpose of this article is to investigate a bilinear analogue of $H_{\gamma_{a,b}}$. Given two polynomials $P$ and $Q$ on $\mathbb{R}$, define the bilinear Hilbert transform along $P, Q$ by

$$(1.3) \quad B_{P,Q}(f,g)(x) := \int_\mathbb{R} f(x - P(t))g(x - Q(t))\frac{dt}{t}, \quad f, g \in \mathcal{S}(\mathbb{R}).$$

In the above definition, instead of just $ta$ and $tb$, two arbitrary polynomials are involved, which provides a more general framework. A natural question is that under what condition on $P$ and $Q$ does $B_{P,Q}$ satisfy any $L^p$ estimates. For this problem, we can assume without loss of generality that both $P$ and $Q$ contain no constant term. There are already some positive results in the literature. For example, when $P$ and $Q$ are distinct linear polynomials, $B_{P,Q}$ is in fact the famous bilinear Hilbert transform, whose boundedness was proved by Lacey and Thiele in a pair of breakthrough papers ([4, 5]). Xiaochun Li first studied the case $P(t) = t, Q(t) = td$, $d \in \mathbb{N}$, and showed that $B_{P,Q}$ is bounded from $L^2 \times L^2$ to $L^1$ ([8]).

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Together with Lechao Xiao, Li later [9] obtained the $L^p$ estimates in full range when $P(t) = t$ and $Q$ is any polynomial without linear term. Following the approach in [8,9], we obtain the theorems below which can be viewed as an extension of Li-Xiao’s result to a larger range of pairs of polynomials.

**Definition.** The correlation degree of any two polynomials $P$ and $Q$ is defined as the smallest natural number $d$ such that any non-zero real root of $P'(x) - Q'(x)$ has multiplicity at most $d$.

**Theorem 1.1.** Given two polynomials $P$ and $Q$ without constant terms, we can always write them as

\[
P(t) = a_d t^{d_1} + a_{d_1 - 1} t^{d_1 - 1} + \cdots + a_{e_1} t^{e_1}, 1 \leq e_1 \leq d_1, a_d a_{e_1} \neq 0 \tag{1.4}
\]

\[
Q(t) = b_d t^{d_2} + b_{d_2 - 1} t^{d_2 - 1} + \cdots + b_{e_2} t^{e_2}, 1 \leq e_2 \leq d_2, b_d b_{e_2} \neq 0. \tag{1.5}
\]

Assume $d_1 \neq d_2$ and $e_1 \neq e_2$. Then there is a constant $C_{P,Q}$ depending on $P$ and $Q$ (and of course $p,q,r$) such that $B_{P,Q}$ defined in [13] satisfies $\|B_{P,Q}(f,g)\|_r \leq C_{P,Q} \|f\|_p \|g\|_q$ for any $f, g \in S(\mathbb{R})$, whenever $p,q \in (1,\infty)$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Here $d$ is the correlation degree of $P$ and $Q$.

**Remarks.** 1. In the expressions (1.4) and (1.5), we can call $d_1$ and $d_2$ the leading degrees, as they are the degrees of the leading terms. Similarly, $e_1$ and $e_2$ may be called trailing degrees if we name $a_{e_1} t^{e_1}$ and $b_{e_2} t^{e_2}$ as trailing terms. So the condition imposed on $P$ and $Q$ in the theorem can be phrased in words as “$P$ and $Q$ have distinct leading and trailing degrees”.

2. We conjecture that the constant $C_{P,Q}$ in the theorem may be chosen to be independent of the coefficients of the polynomials. This seems to be a hard and technical problem, whose solution may involve the ideas in the proof of uniform estimate for the bilinear Hilbert transform ([2,6,18]).

3. For any fixed natural number $d$, there exist polynomials $P$ and $Q$ with correlation degree $d$ such that $B_{P,Q}$ is unbounded whenever $r < \frac{d}{d + 1}$ (see Section 3.2 in [9] for an example). In this sense the lower bound for $r$ given in Theorem 1.1 is sharp up to the endpoint. However, if we fix the polynomials $P$ and $Q$, the lower bound of $r$ in Theorem 1.1 may not be the best. For instance, let $P(t) = t^6$ and $Q(t) = 3t^4 - 3t^2$. Then $B_{P,Q}$ is the zero operator, which is trivially bounded for $r > \frac{1}{2}$. But the correlation degree of $P$ and $Q$ is 2. It is interesting to find a way to determine the lowest $r$ for any given $P$ and $Q$. This task requires improvement on Lemma 2.4 (see Section 4).

As a byproduct of the proof of Theorem 1.1 we obtain the same estimate for the bilinear maximal function $M_{P,Q}$ defined by

\[
M_{P,Q}(f,g)(x) := \sup_{\epsilon > 0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |f(x - P(t))g(x - Q(t))| \mathrm{d}t. \tag{1.6}
\]

**Theorem 1.2.** Let $P, Q$ and $p, q, r$ satisfy the conditions stated in Theorem 1.1. Then $M_{P,Q}$ is bounded from $L^p \times L^q$ to $L^r$.

Just like the relationship between $B_{P,Q}$ and $H_{\gamma_{a,b}}$, $M_{P,Q}$ can be viewed as a bilinear analogue of the the maximal function associated with $H_{\gamma_{a,b}}$.

\[
M_{\gamma_{a,b}}(f)(x_1, x_2) := \sup_{h > 0} \frac{1}{2h} \int_{-h}^{h} |f(x_1 - t^a, x_2 - t^b)| \mathrm{d}t, \quad f \in S(\mathbb{R}^2).
\]
The $L^p$-boundedness of $M_{a,b}$ was proved in [12] (see [14, 16, 17] for further developments on more general curves), and Theorem 1.2 is the parallel result in the bilinear setting.

The rest of the paper is organized as follows. In section 2 we make careful decompositions on our operator, and after throwing away the paraproduct part, reduce Theorem 1.1 to two estimates (Proposition 2.4 and Proposition 2.5): a scale-type decay estimate when $p = q = 2$, and a moderate blow-up estimate for general $p$ and $q$. The decay estimate will be proved in section 3 and 4 using TT*-method and $\sigma$-uniformity method. In the last section, we show how to obtain the moderate blow-up estimate by adapting methods from [9], and prove Theorem 1.2.

Throughout the paper we use $C$ to denote a positive constant (which may depend on $P$ and $Q$) whose value is allowed to change from line to line. $A \leq B$ means $A$ is bounded from $L^p \times L^q$ to $L^r$, whenever $p, q \in (1, \infty)$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $r > \frac{d}{d+1}$, where $d$ is the correlation degree of $A$ after using Taylor expansion or stationary phase method. $\chi_E$ will be used to denote the indicator function of a set $E$.

2. Decomposition and Reduction

Pick an odd function $\rho \in S(\mathbb{R})$ supported in the set $\{ x : |x| \in (\frac{1}{2}, 2) \}$ with the property that $t^{-1} = \sum_{j \in \mathbb{Z}} 2^j \rho(2^j t)$ for any $t \neq 0$. Then we can write $B_{P,Q}(f,g)(x) = \sum_{j \in \mathbb{Z}} T_j(f,g)(x)$, where

$$T_j(f,g)(x) := \int f(x - P(t))g(x - Q(t))2^j \rho(2^j t) \, dt$$

$$= \iint \hat{f}(\xi)\hat{g}(\eta)e^{2\pi i (\xi + \eta)x}m_j(\xi, \eta) \, d\xi d\eta,$$

and

$$m_j(\xi, \eta) := \int 2^j \rho(2^j t)e^{-2\pi i (\xi P(t) + \eta Q(t))} \, dt.$$

We first prove that each $T_j$ is bounded.

**Lemma 2.1.** Let $P$ and $Q$ be two arbitrary polynomials. Then each $T_j$ is bounded from $L^p \times L^q$ to $L^r$, whenever $p, q \in (1, \infty)$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $r > \frac{d}{d+1}$, where $d$ is the correlation degree of $P$ and $Q$.

**Proof.** We only consider the operator $T_0$, as the other cases are similar. The idea of the proof is based on Lemma 9.1 in [3]. Note that when $r \geq 1$ the boundedness of $T_0$ follows from Minkowski inequality. So we assume now $r < 1$. Since $|t| \approx 1$, we can restrict $x$ and the support of $f$ and $g$ to a bounded interval $I_{P,Q}$. When the Jacobian $Q'(t) - P'(t) \neq 0$ for all $t$ in the support of $\rho$, $T_0$ is bounded from $L^1 \times L^1$ to $L^1$ by changing variables $u = x - P(t)$ and $v = x - Q(t)$. Thus $T_0$ is bounded from $L^1 \times L^1$ to $L^r$ by Cauchy-Schwarz inequality.

Now we focus on the case that there is a root of $Q'(t) - P'(t)$ lying in the support of $\rho$. Let $t_0$ be such a root and $I(t_0)$ be a small neighborhood of $t_0$. It suffices to prove that

$$\int_{I_{P,Q}} \left| \int_{I(t_0)} f(x - P(t))g(x - Q(t))\rho(t) \, dt \right|^r \, dx \lesssim \|f\|_p^r \|g\|_q^r.$$
for \( p, q \in (1, \infty), \frac{1}{p} + \frac{1}{q} = \frac{1}{r}, r > \frac{d}{d+1} \). Because of the restriction on \( I(t_0) \), the function \( \rho \) in (2.3) can be dropped. Let \( \rho_0 \) be a bump function supported in \( \{ t : |t| \in (\frac{1}{2}, 2) \} \) and satisfies \( \sum_{j \in \mathbb{Z}} \rho_0(2^j t) = 1 \) for all \( t \in \mathbb{R} \). Then (2.3) will be proved once we can show that there is some \( \epsilon > 0 \) such that

\[
(2.4) \quad \int_{I_{P,Q}} \left| \int f(x - P(t))g(x - Q(t))\rho_0(2^j(t - t_0)) \, dt \right|^r \, dx \lesssim 2^{-\epsilon j} \| f \|^r_p \| g \|^r_q
\]

holds for all large positive \( j \). Changing variable \( t - t_0 \to t \) and translating \( f \) and \( g \) by \( P(t_0) \) and \( Q(t_0) \) respectively, (2.4) becomes

\[
(2.5) \quad \int_{I_{P,Q}} \left| \int f(x - P_1(t))g(x - Q_1(t))\rho_0(2^j t) \, dt \right|^r \, dx \lesssim 2^{-\epsilon j} \| f \|^r_p \| g \|^r_q,
\]

where \( P_1(t) := P(t + t_0) - P(t_0) \) and \( Q_1(t) := Q(t + t_0) - Q(t_0) \). By the support of \( \rho_0 \), \( |t| \simeq 2^{-j} \). This implies that \( P_1(t) \lesssim 2^{-j} \) and \( Q_1(t) \lesssim 2^{-j} \) by mean value theorem. So we can for free restrict \( x \) to an interval of length \( \simeq 2^{-j} \). Let \( I_N \) be such an interval and define

\[
T_N(f, g)(x) = \chi_{I_N}(x) \int f(x - P_1(t))g(x - Q_1(t))\rho_0(2^j t) \, dt.
\]

It remains to show

\[
(2.6) \quad \| T_N(f, g) \|_r \lesssim 2^{-\epsilon j} \| f \|^r_p \| g \|^r_q.
\]

By Fubini theorem, \( T_N \) is bounded with norm \( \lesssim 2^{-j} \) when \( r = 1 \). Next we aim to get a slow increasing \( L^1 \times L^1 \to L^\frac{4}{3} \) norm. By Cauchy-Schwarz inequality,

\[
(2.7) \quad \left\| T_N(f, g)(x) \right\| \frac{1}{2} \, dx \lesssim 2^{-j/2} \| T_N(f, g) \|_1^{\frac{1}{2}}.
\]

\( \| T_N(f, g) \|_1 \) can be calculated by changing variables \( u = x - P_1(t) \) and \( v = x - Q_1(t) \). Using Taylor expansion and the fact that \( t_0 \) has multiplicity at most \( d \), the Jacobian \( Q'_1(t) - P'_1(t) \) is bounded below by \( 2^{-dj} \). Therefore

\[
(2.8) \quad \| T_N(f, g) \|_1 \lesssim 2^{dj} \| f \|_1 \| g \|_1.
\]

Combining (2.7) and (2.8), we get

\[
(2.9) \quad \| T_N(f, g) \|_\frac{4}{3} \lesssim 2^{(d-1)j} \| f \|_1 \| g \|_1.
\]

Interpolating (2.9) with the \( L^1 \)-norm, we obtain (2.6).

By lemma 2.1, to prove Theorem 1.1 it suffices to prove the following theorem.

**Theorem 2.2.** Let \( P \) and \( Q \) be two polynomials with distinct leading and trailing degrees. Then there is a large \( N \) depending on \( P \) and \( Q \) such that \( \sum_{|j| > N} T_j(f, g)(x) \) is bounded from \( L^p \times L^q \) to \( L^r \) for all \( p, q \in (1, \infty), \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \).

From the definition (2.1), we see that \( j > N \) corresponds to small \( |t| \), in which case the trailing term dominates each polynomial; \( j < -N \) corresponds to large \( |t| \), in which case \( P \) and \( Q \) behave almost the same as their leading term. We will only deal with \( \sum_{j>N} T_j(f, g)(x) \) since the other case is similar.

Let \( P, Q \) be polynomials written as (1.4) and (1.5). In When \( j \) is large (i.e. \( |t| \) is close to 0), the trailing terms \( a_{e_1} t^{e_1} \) and \( b_{e_2} t^{e_2} \) dominate \( P(t) \) and \( Q(t) \), respectively. Since all the constants in our proof are allowed to depend on the coefficients of \( P \) and \( Q \), we may assume without loss of generality that \( a_{e_1} = b_{e_2} = 1 \). For notation
simplicity, from now on we denote \( a := e_1 \) and \( b := e_2 \). Recall that \( e_1 \neq e_2 \) and thus we may assume \( a < b \). With these new notations, we can write \( P(t) = t^a + P_a(t) \) and \( Q(t) = t^b + Q_a(t) \), where \( P_a(t) \) (resp. \( Q_a(t) \)) consists of terms whose degree is higher than \( a \) (resp. \( b \)). As \( P_a(t) \) and \( Q_a(t) \) are small when \( j > N \) and can be viewed as error terms. We urge the reader to ignore them in the first reading of this paper.

The overall idea of the proof is to look at the size of the symbol \( m_j(\xi, \eta) \) defined in (2.2), which can be estimated by stationary phase method after proper cut-off and rescaling. By a change of variable,

\[
m_j(\xi, \eta) = \int \rho(t)e^{-2\pi i\left(\frac{\xi}{2^m}t^a + \frac{\eta}{2^n}t^b + e_p(t) + e_q(t)\right)} dt,
\]

where

\[
e_p(t) := 2^a P_a(2^{-a}t);
\]

\[
e_q(t) := 2^b Q_a(2^{-b}t).
\]

Clearly \( |e_p(t)| \leq 2^{-N}\left|t^a\right| \) and \( |e_q(t)| \leq 2^{-N}\left|t^b\right| \) as \( j > N \). The expression (2.10) suggests that we need to consider the sizes of \( \overline{\Phi} \) and \( P(t) \) in (2.2), which can be estimated by stationary phase method after proper cut-off and rescaling. By a change of variable,

\[
M_j(\xi, \eta) := m_j(\xi, \eta)\overline{\Phi}\left(\frac{\xi}{2^{aj+m}}\right)\overline{\Phi}\left(\frac{\eta}{2^{bj+n}}\right)
\]

is the bilinear operator with symbol

\[
m_j(\xi, \eta) := m_j(\xi, \eta)\overline{\Phi}\left(\frac{\xi}{2^{aj+m}}\right)\overline{\Phi}\left(\frac{\eta}{2^{bj+n}}\right)
\]

In estimating its size, the symbol \( M_{j,m,n} \) can be viewed roughly as

\[
\int \rho(t)e^{-2\pi i\left(\frac{\xi}{2^m}t^a + \frac{\eta}{2^n}t^b + e_p(t) + e_q(t)\right)} dt\overline{\Phi}\left(\frac{\xi}{2^{aj+m}}\right)\overline{\Phi}\left(\frac{\eta}{2^{bj+n}}\right)
\]

which decays rapidly if \( |m-n| \) is large. In fact, \( \sum_{j>N} \sum_{|m-n|\geq 1} T_{j,m,n}(f, g)(x) \) can be reduced to the paraproduct studied in [7] (see Section 7.2 in [9] for details). To deal with the remaining \( |m-n| \leq 1 \) case, we can assume without loss of generality that \( m = n \). For notation simplicity, denote \( M_{j,m} := M_{j,m,n} \) and \( T_{j,m} := T_{j,m,n} \). Using oddness of \( \rho \) and Taylor expansion, \( \sum_{j>N} \sum_{m\leq 0} T_{j,m} \) can also be reduced to the paraproduct in [7]. Thus we will only focus on the most difficult case in proving Theorem 2.2, handing the operator \( \sum_{j>N} \sum_{m>0} T_{j,m} \). Our goal is to prove
There exists a positive
Proposition 3.4.
\[ \| (3.2) \approx \int \frac{\text{interval of length}}{x} \text{partitions in time spaces, we see that} \]
Proposition 2.5. 
For \((3.1)\) whenever 
Proposition 3.3.
\[ \| \sum_{j>N} T_{j,m}(f,g) \| \lesssim \| f \|_p \| g \|_q. \]

By interpolation, the above theorem follows from two propositions below.

**Proposition 2.4.**
\[ \left\| \sum_{j>N} T_{j,m}(f,g) \right\|_1 \lesssim 2^{-\epsilon m} \| f \|_2 \| g \|_2 \text{ for some } \epsilon > 0. \]

**Proposition 2.5.** For \(p, q \in (1, \infty)\), \(\frac{1}{r} = \frac{1}{p} + \frac{1}{q}\),
\[ \left\| \sum_{j>N} T_{j,m}(f,g) \right\|_{r,\infty} \lesssim m \| f \|_p \| g \|_q. \]

### 3. TT* Method

We prove Proposition 2.4 in this section and the next.

Since we can for free insert cut-offs on \(\hat{f}\) and \(\hat{g}\) according to the support of \(M_{j,m}\), in proving Proposition 2.4 we only need to consider the estimate for a single scale, i.e.

**Proposition 3.1.** \(\| T_{j,m}(f,g) \|_1 \lesssim 2^{-\epsilon m} \| f \|_2 \| g \|_2 \) for any \(j > N\) and \(m > 0\).

By rescaling, Proposition 3.1 is a consequence of

**Proposition 3.2.** For any \(j > N\) and \(m > 0\), \(\| B_{j,m}(f,g) \|_1 \lesssim 2^{-\epsilon m} \| f \|_2 \| g \|_2\), where
\[ B_{j,m}(f,g)(x) := 2^{-\frac{(b-a)}{2}} \int \rho(t) f * \Phi \left( \frac{x}{2^{(b-a)j}} - 2^m (t^a + \epsilon_p(t)) \right) g * \Phi(x - 2^m (t^b + \epsilon_Q(t))) \ dt \]

This proposition follows from the two estimates below.

**Proposition 3.3.** \(\| B_{j,m}(f,g) \|_1 \lesssim 2^{-\frac{(b-a)j-m}{2}} \| f \|_2 \| g \|_2 \) for any \(j > N\) and \(m > 0\).

**Proposition 3.4.** There exists a positive \(\delta\) such that \(\| B_{j,m}(f,g) \|_1 \lesssim 2^{-\epsilon m} \| f \|_2 \| g \|_2\) whenever \((b-a)j > (1-\delta)m\).

Proposition 3.3 is efficient when \(m\) is large and Proposition 3.4 is useful for small \(m\). The proofs for the above two propositions require different methods.

We prove Proposition 3.3 in this section, using a TT* method. More precisely, we aim to obtain a \(L^2 \times L^2 \rightarrow L^2\) bound with good decay. By making suitable partitions in time spaces, we see that \(x\) can be assumed to be supported in an interval of length \(\approx 2^{(b-a)j+m}\). This observation indicates that it suffices to prove
\[ \| B_{j,m}(f,g) \|_2 \lesssim 2^{-\frac{(b-a)j-m}{2}} \| f \|_2 \| g \|_2. \]

Rewrite \(B_{j,m}\) as
\[ B_{j,m}(f,g)(x) = 2^{-(b-a)j} \int \int \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i \left( \frac{\xi}{2^{(b-a)j}} + \eta \right)} \hat{I}_{p,m} \hat{\Phi}(\xi) \hat{\Phi}(\eta) \ d\xi \ d\eta, \]
where
\[ I_{\rho, m} := \int \rho(t) e^{-2\pi i 2^m (\xi(t^n + \epsilon_P(t)) + \eta(t^b + \epsilon_Q(t)))} dt \]

Let \( \phi(t) := \xi(t^n + \epsilon_P(t)) + \eta(t^b + \epsilon_Q(t)) \) and \( t_0 \) be a solution of \( \phi'(t) = 0 \). Let \( \phi(\xi, \eta) := \phi(t_0) \). By stationary phase method,
\[ I_{\rho, m}(\xi, \eta) \tilde{\phi}(\xi, \eta) \sim 2^{-m} e^{i2^m \phi(\xi, \eta)}. \]

Thus we can regard \( B_{j, m} \) as
\[ B_{j, m}(f, g)(x) \sim 2^{-\frac{(b-a)j}{2}} e^{i2^m \phi(x, \eta)} \]

Then
\[ \| B_{j, m} \|_2^2 = \int B_{j, m}(x) \overline{B_{j, m}(x)} \, dx \]

where
\[ F_\tau(\xi) := \hat{f}(\xi) \hat{\phi}(\xi - \tau) \hat{\Phi}(\xi) \]
\[ G_\tau(\eta) := \hat{g}(\eta) \hat{\phi}(\eta + \frac{\pi}{2(b-a) \tau}) \hat{\Phi}(\eta + \frac{\pi}{2(b-a) \tau}) \]
\[ Q_\tau(\xi, \eta) := \phi(\xi, \eta) - \phi(\xi - \tau, \eta + \frac{\pi}{2(b-a) \tau}) \]

We claim that whenever \( \xi, \eta, x, \tau, y + \frac{\pi}{2(b-a) \tau} \in \text{supp} \tilde{\Phi} \), we have
\[ |\partial_\xi \partial_\eta Q_\tau(\xi, \eta)| \gtrsim |\tau| \]

Let's briefly justify (3.7). By the definition of \( Q_\tau \) and mean value theorem, we need to show that \( |\partial_\xi^2 \partial_\eta \phi(\xi, \eta)| \) and \( |\partial_\xi \partial_\eta^2 \phi(\xi, \eta)| \) are bounded below by some positive \( C \). Let \( t_0 \) be a root of \( F_0(t) := \phi'(t) = D_1(\xi(t^n + \epsilon_P(t)) + \eta(t^b + \epsilon_Q(t))) \), and \( t_1 \) be a root of \( F_1(t) := D_1(\xi(t^n + \eta t^b) \eta). \) Let \( \phi^*(\xi, \eta) := \xi t_0^n + \eta t_1^b \). Then
\[ \phi(\xi, \eta) = \phi(t_0) = \phi^*(\xi, \eta) + \text{Err}(\xi, \eta), \]
where
\[ \text{Err}(\xi, \eta) := \xi(t_0^n - t_1^n) + \eta(t_0^n - t_1^n) + \xi \epsilon_P(t_0) + \eta \epsilon_Q(t_0). \]
Clearly the mixed derivatives of \( \phi^*(\xi, \eta) \) are bounded below by some positive \( C \). It remains to show that \( |\text{Err}(\xi, \eta)| \leq C^{-1} \) for some large \( C \). Since \( F_0 \) and \( F_1 \) are “close”, the difference of their inverses \( t_0 - t_1 \) (and its derivatives) is also very small (see Definition A.1 and Lemma A.2 in [3] for details). By this observation and the facts that \( |\epsilon_P(t_0)| \) and \( |\epsilon_Q(t_0)| \) are tiny when \( N \) is large enough, we conclude that \( |\text{Err}(\xi, \eta)| \) is very small compared with 1. This finishes the justification of (3.7).

By (3.7) and Hömander principle (Theorem 1.1 in [3]),
\[ \int \int F_\tau(\xi) G_\tau(\eta) e^{i2^m Q_\tau} d\xi d\eta \lesssim \min \{ \|f\|_2^2, \|g\|_2^2, 2^{-\frac{m}{2}} |\tau|^{-\frac{1}{2}} \|F_\tau\|_2 \|G_\tau\|_2 \}. \]
Therefore,
\[
\|B_{j,m}\|^2 \lesssim 2^{-(b-a)j-m} \int \min\{\|f\|_{L^2}^2, 2^{-\frac{j}{3}}c, 2^{-\frac{j}{4}}c \} \, dt \\
\lesssim 2^{-(b-a)j-m} \left( \int_{|\tau|<\tau_0} \|f\|_{L^2}^2 \, d\tau + \int_{\tau_0 \leq |\tau| \leq 1} \|f\|_{L^2}^2 \, d\tau \right) \\
\lesssim 2^{-(b-a)j-m} \left( \tau_0 \|f\|_{L^2}^2 + 2^{-\frac{j}{3}}c \|f\|_{L^2}^2 \right) \\
\lesssim 2^{-(b-a)j-m} 2^{\frac{(b-a)j-m}{3}} \|f\|_{L^2}^2,
\]
from which (3.2) follows.

4. \sigma\text{-}UNIFORMITY METHOD

We prove Proposition 3.4 and hence finish the proof of Proposition 2.3 in this section. Let \( I \subseteq \mathbb{R} \) be a fixed interval. Let \( \sigma \in (0,1) \) and \( \mathcal{Q} \) be a collection of real-valued functions.

**Definition.** A function \( f \in L^2(I) \) is called \( \sigma \)-uniform in \( \mathcal{Q} \) if
\[
\left| \int_I f(\xi)e^{-iq(\xi)} \, d\xi \right| \leq \sigma \|f\|_{L^2(I)}
\]
for all \( q \in \mathcal{Q} \).

The main tool of proving Proposition 3.4 is the following theorem, whose proof can be found in Theorem 6.2 in [8].

**Theorem 4.1.** Let \( L \) be a bounded sub-linear functional from \( L^2(I) \) to \( \mathbb{C} \). Let \( \mathcal{S}_\sigma \) be the set of all \( L^2 \) functions that are \( \sigma \)-uniform in \( \mathcal{Q} \) and \( U_\sigma := \sup_{f \in \mathcal{S}_\sigma} \frac{|L(f)|}{\|f\|_{L^2(I)}} \). Then for all functions \( f \in L^2(I) \),
\[
|L(f)| \lesssim \max \left\{ U_\sigma, \frac{Q_0}{\sigma} \right\} \|f\|_{L^2(I)},
\]
where \( Q_0 := \sup_{q \in \mathcal{Q}} L(e^{i\theta q}) \).

Now we start to estimate \( U_\sigma \). Recall that we can assume \( x \) is restricted in an interval of length \( \sim \frac{1}{2^{(b-a)j}+m} \). We fix such an interval and partition it into \( 2^m \) intervals of length \( \sim \frac{1}{2^{(b-a)j}} \), which are denoted by
\[
I_k = [\alpha_k - 2^{(b-a)j}, \alpha_k + 2^{(b-a)j}], k = 1, 2, \ldots, 2^m.
\]
To each \( I_k \) we assign an enlarged interval
\[
I'_k = [\alpha_k - C(2^{(b-a)j} + 2^m), \alpha_k + C(2^{(b-a)j} + 2^m)]
\]
such that \( x - 2^m(t^b + \epsilon_Q(t)) \in I'_k \) whenever \( x \in I_k \) and \( t \in \text{supp}\rho \). So \( B_{j,m} \) can be partitioned accordingly as
\[
B_{j,m}(f,g)(x) = 2^{-\frac{(b-a)j}{2}} \sum_{k=1}^{2^m} \chi_{I_k}(x) \int f * \Phi \left( \frac{x}{2^{(b-a)j}} \right) - 2^m(t^a + \epsilon_P(t)) \chi_{I'_k} \Phi(x - 2^m(t^b + \epsilon_Q(t))) \rho(t) \, dt \\
= 2^{-\frac{(b-a)j}{2}} \sum_{k=1}^{2^m} \chi_{I_k}(x) \int \hat{f}(\xi) \hat{\Phi}(\xi) e^{2\pi i \left( \frac{\xi}{2^{(b-a)j}} + \eta \right) x} \hat{g}_k(\eta) I_{\rho,m}(\xi, \eta) \, d\xi d\eta,
\]
where
where
\[ g_k(x) := \chi_{I_k'} g \ast \Phi(x), \]
and \( I_{p,m} \) is defined as (3.4). Since \( |\xi| \simeq 1 \), \( I_{p,m} \) has rapid decay unless \( |\eta| \simeq 1 \). So we may insert a cut-off function \( \Phi(\eta) \) for free in the above integrand.

Pair \( B_{j,m} \) with an \( h \in L^\infty \),
\[
\langle B_{j,m}(f,g), h \rangle = 2^{-\frac{(b-a)}{2}} \sum_{k=1}^{2^m} \int \chi_{I_k}(x)h(x)e^{2\pi i q x}
\int \left( \hat{f}(\xi)\hat{\Phi}(\xi)e^{2\pi i \frac{\xi}{2(\eta + \alpha_k)}} \hat{\Phi}(\eta)g_k(\eta)I_{p,m}(\xi, \eta) \right) d\xi d\eta.
\]

Thanks to the cut-off \( \chi_{I_k}(x) \), and we can replace \( e^{2\pi i \frac{\xi}{2(\eta + \alpha_k)}} \) with \( e^{2\pi i \frac{\xi}{2(\eta + \alpha_k)} a_k} \) using Taylor expansion. Thus essentially,
\[
\langle B_{j,m}, h \rangle \sim 2^{-\frac{(b-a)}{2}} \sum_{k=1}^{2^m} \int h_k(\eta)\Gamma_k(\eta)g_k(\eta) d\eta,
\]
where
\[
h_k(x) := \chi_{I_k} h(x), \quad \text{and} \quad \Gamma_k(\eta) := \hat{\Phi}(\eta) \int \hat{f}(\xi)\hat{\Phi}(\xi)e^{2\pi i \frac{\xi}{2(\eta + \alpha_k)}} I_{p,m}(\xi, \eta) d\xi.
\]

As before, we can replace \( I_{p,m} \) with \( 2^{-\frac{\alpha}{2}}e^{2\beta \phi(\xi, \eta)} \) and thus
\[
\Gamma_k(\eta) \sim 2^{-\frac{\alpha}{2}}\hat{\Phi}(\eta) \int \hat{f}(\xi)\hat{\Phi}(\xi)e^{i(2\beta \phi(\xi, \eta) + \frac{\xi}{2(\eta + \alpha_k \alpha_k)}} d\xi.
\]

Let \( Q := \{ A(\xi, \eta) + B\xi \} \), where \( A, B \in \mathbb{R}, \ |A| \simeq a^m \), and \( \epsilon(\xi) \) and its derivatives are \( \lesssim 2^{-CN} \). Then \( 2^m \phi(\xi, \eta) + \frac{\xi}{2(\eta + \alpha_k \alpha_k) a_k} \in \mathcal{Q} \) for large \( N \). Let \( f \) be \( \sigma \)-uniform in \( Q \). Then
\[
\|\Gamma_k\|_\infty \lesssim 2^{-\frac{\alpha}{2}} \sigma \|f\|_2,
\]
and thus
\[
\langle B_{j,m}(f,g), h \rangle \lesssim 2^{-\frac{(b-a)}{2}} \sum_{k=1}^{2^m} \|\Gamma_k\|_\infty \|h\|_2 \|g_k\|_2
\]
\[
\lesssim \sigma \|f\|_2 \|h\|_\infty \left( \sum_k \|g_k\|_2 \right)^2
\]
\[
\lesssim \begin{cases} \sigma \|f\|_2 \|g\|_2 \|h\|_\infty & \text{when } (b-a)j \geq m \\ \sigma 2^{m-(b-a)} \|f\|_2 \|g\|_2 \|h\|_\infty & \text{when } (b-a)j \leq m. \end{cases}
\]

This finishes the computation of \( U_{\sigma} \).

Now we turn to \( U_{\sigma} \). Let \( f(\xi) = e^{i \xi(\xi)} \) for some \( q \in \mathcal{Q} \). Let \( h \in L^\infty \) be a function supported on an interval of length \( \simeq 2^{(b-a)x+m} \) as before. Define
\[
\Lambda_q(g,h) := \langle B_{j,m}, h \rangle
\]
\[
= 2^{-\frac{(b-a)}{2}} \int \hat{f}(\xi) e^{i(\xi(\xi) + B\xi)} e^{i\xi \left( \frac{\xi}{2(\eta + \alpha_k a_k)} - 2^m(\eta + \epsilon(\eta)) \right)} d\xi
\]
\[
ge^* \left( x - 2^m (\epsilon + \epsilon Q(t)) \rho(t) \right) dt dx.
\]
Our goal is to show

\[
|\Lambda_q(g, h)| \lesssim 2^{-cm}\|g\|_2\|h\|_\infty.
\]

This means that \(Q_0 \lesssim 2^{-cm}\). Combining this with (4.2), Proposition 3.4 will be proved by Theorem 4.1.

To prove (4.3), we will use the strategy similar to the previous cases: rescaling, stationary phase, and (local) TT*. Let

\[
|\tilde{\Lambda}_q(g, h)| := \int \mathcal{F}(y, t) g \ast \hat{\Phi} \left( y - \frac{t^b + \epsilon Q(t)}{2(b-a)\lambda} \right) \rho(t) dh(y)dy,
\]

where \(\hat{\Phi}(\xi) := \hat{\Phi} \left( \frac{\xi}{2(b-a)\lambda + m} \right)\) and

\[
\mathcal{F}(y, t) := \frac{2\pi}{Q} \int \hat{\Phi}(\xi) e^{iA\left( \xi \frac{y}{\rho} + \epsilon(\xi) + C'(y - (t^a + \epsilon P(t)) + B')\xi \right)} d\xi.
\]

By rescaling, (4.3) follows from the estimate

\[
|\tilde{\Lambda}_q(g, h)| \lesssim 2^{-cm}\|g\|_2\|h\|_\infty
\]

for any \(h \in L^\infty\) supported in an interval of length \(\simeq 1\).

Write

\[
\mathcal{F}(y, t) = \frac{2\pi}{Q} \int \hat{\Phi}(\xi) e^{iA\left( \xi \frac{y}{\rho} + \epsilon(\xi) + C'(y - (t^a + \epsilon P(t)) + B')\xi \right)} d\xi,
\]

where \(C' := \frac{m}{A} \simeq 1\) and \(B' := \frac{b}{B}\). For simplicity we drop \(C'\) from now on. Let \(\zeta(z)\) be the solution of \(\left( \frac{\zeta'}{\zeta} + \epsilon(\zeta) + z\zeta' \right) = 0\) and \(\beta(z) := \zeta(z) \frac{2\beta}{A} + \epsilon(\zeta(z)) + z\zeta(z)\). Then stationary phase methods give that

\[
\mathcal{F}(y, t) \sim e^{iA\beta(y - (t^a + \epsilon P(t)) + B')} \hat{\Phi}(\zeta(y - (t^a + \epsilon P(t)) + B')).
\]

Since the term \(\hat{\Phi}(\zeta(y - (t^a + \epsilon P(t)) + B'))\) can be dropped by Fourier expansion, we have

\[
\tilde{\Lambda}_q(g, h) \sim \int e^{iA\beta(y - (t^a + \epsilon P(t)) + B')} g \ast \hat{\Phi} \left( y - \frac{t^b + \epsilon Q(t)}{2(b-a)\lambda} \right) \rho(t) dh(y)dy.
\]

This finishes the use of the stationary phase method. The last step is to use TT* method to obtain the decay. Change variable \(s = t^b + \epsilon Q(t)\). Define three new functions \(\kappa, l\) and \(\tilde{\rho}\) by \(t = \kappa(s), l(s) = \kappa(s)^a + \epsilon P(\kappa(s))\) and \(\tilde{\rho}(s)ds = \rho(t)dt\). Then

\[
\tilde{\Lambda}_q(g, h) = \int e^{iA\beta(y - l(s) + B')} g \ast \hat{\Phi} \left( y - \frac{s}{2(b-a)\lambda} \right) \tilde{\rho}(s)ds h(y)dy
\]

\[
\lesssim \|\Delta(h)\|_2\|g\|_2,
\]

where

\[
\Delta(h)(y) := \int e^{iA\beta(y + \frac{s}{2(b-a)\lambda} - l(s) + B')} h \left( y + \frac{s}{2(b-a)\lambda} \right) \tilde{\rho}(s)ds.
\]

It remains to show

\[
\|\Delta(h)\|_2^2 \lesssim 2^{-cm}\|h\|_\infty^2.
\]

A straightforward calculation gives

\[
\|\Delta(h)\|_2^2 = \iint e^{iAO_x(u, v)} H_+(u)\Theta_+(v)dudv d\tau,
\]

where \(\frac{(s-a)}{\lambda} \simeq \frac{(y-a)}{\lambda} + \epsilon\).
where
\[ H_\tau(u) := h(u)h\left(u + \frac{\tau}{2(\theta - a_j)}\right), \]
\[ \Theta_\tau(v) := \tilde{\rho}(v)\tilde{\rho}(v + \tau), \]
and
\[ O_\tau(u, v) := \beta(u - l(v) + B') - \beta \left(u + \frac{\tau}{2(\theta - a_j)} - l(v + \tau) + B'\right). \]

By the same idea in the proof of (3.7), we see that the mixed partial derivatives of \( O_\tau(u, v) \) is bounded below by \( C|\tau| \). By the operator version of van der Corput lemma (see for example Lemma 5.8 in [9]), we have
\[ (4.7) \quad \int e^{iA_O(u, v)} H_\tau(u)\Theta_\tau(v) dudv \lesssim \min\{1, |2^m\tau|^{-\epsilon}\} \|H_\tau\|_2\|\Theta_\tau\|_2. \]

By definitions, it is easy to see that \( \|H_\tau\|_2 \lesssim \|h\|_{C_0}^2 \) and \( \|\Theta_\tau\|_2 \lesssim 1 \). So we can break the integral against \( \tau \) in (4.1) into two parts as before: \( |\tau| \leq \tau_0 \) and \( \tau_0 < |\tau| \lesssim 1 \), and use the estimate (4.7) to obtain the desired result (4.5).

5. \( L^r \) estimates and the maximal function

We start to prove Proposition 2.5 and then finish the proof of Theorem 1.1. Rewrite \( T_{j,m} \) as
\[ T_{j,m}(f, g)(x) = \int f * \Phi_{a+j+m} \left(x - \frac{t^n + \epsilon p(t)}{2a_j}\right) g * \Phi_{b+j+m} \left(x - \frac{t^b + \epsilon q(t)}{2^b b_j}\right) \rho(t) dt, \]
where \( \Phi_k(x) := 2^k \Phi(2^k x) \). Let
\[ T^m(f, g)(x) := \sum_{j>N} \int \left| f * \Phi_{a+j+m} \left(x - \frac{t^n + \epsilon p(t)}{2a_j}\right) g * \Phi_{b+j+m} \left(x - \frac{t^b + \epsilon q(t)}{2^b b_j}\right) \rho(t) \right| dt. \]

It suffices to prove the boundedness of \( T^m \) with norm \( \lesssim m \).

Given any measurable sets \( F_1, F_2, F_3 \) of finite measure, define
\[ \Omega := \bigcup_{i=1}^2 \left\{ x : \#(\mathcal{M} \cap F_i) > C \frac{|F_i|}{|F_3|} \right\}, \]
where \( \mathcal{M} \) denotes the Hardy-Littlewood maximal operator. Let \( F'_3 := F_3 \setminus \Omega \), which has measure no less than \( \frac{|F_3|}{|F_3|} \) when \( C \) is chosen large enough. By standard interpolation, we need to show that
\[ (5.3) \quad |(T^m(f, g), h)| \lesssim m |F_1|^{\frac{1}{p}} |F_2|^{\frac{1}{q}} |F_3|^{1 - \frac{1}{r}}, \]
for all \( |f| \leq \chi_{F_1}, |g| \leq \chi_{F_2} \), \( p, q \in (1, \infty), \frac{1}{r} = \frac{1}{p} + \frac{1}{q} + \frac{1}{2} \).}

We first remove some error terms related with \( \Omega \). Define \( \Omega_k := \{ x : \text{dist}(x, \Omega^c) \geq 2^{-k} \} \) and let \( \psi_k(x) = \chi_{\Omega_k^c} \hat{\psi}(x) \), where \( \hat{\psi} \in \mathcal{S}(\mathbb{R}) \) is Fourier supported in \( [-2^k, 2^k] \). It turns out that in proving (5.3) we can replace \( T^m(f, g) \) with
\[ (T')^m(f, g)(x) := \sum_{j>N} \int \left| \psi_{a+j+m} f \phi_{a+j+m} \left(x - \frac{t^n + \epsilon p(t)}{2a_j}\right) \right| \]
\[ \left| \psi_{b+j+m} g \phi_{b+j+m} \left(x - \frac{t^b + \epsilon q(t)}{2^b b_j}\right) \rho(t) \right| dt. \]
This is because the difference of these two operators has good control. See Lemma 6.3 in [9], whose proof is based on a discussion about whether $x - t$ (or $x - \frac{t^a + \epsilon_p(t)}{2^n}$) belongs to $\Omega$ or not. That proof can be easily modified to include the $x - \frac{t^a + \epsilon_p(t)}{2^n}$ case. So we focus on proving the following variant of (6.3), with $T^m$ being replaced by $(T')^m$:

$$|(T')^m(f, g), h)| \lesssim m|F_1|^\frac{2}{3} |F_2|^\frac{1}{3} |F_3|^{1 - \frac{1}{3}}.$$  

Time-frequency analysis must be employed to prove (5.5). For any integers $n, j$, define $I_{n,j} := [2^{-j} n, 2^{-j}(n + 1))$. Let $1^*_n(x) := \chi_{I_{n,j}} \ast \theta_{j+m}(x)$, where $\theta_k \in \mathcal{S}(\mathbb{R})$ is Fourier supported on $[-2^{-10k}, 2^{-10k}]$. Then

$$(T')^m(f, g)(x) = \sum_{j > N} \int \left| \sum_{n \in \mathbb{Z}} f_{n, m, j} \left( x - \frac{t^a + \epsilon_p(t)}{2^n} \right) \right| \left| \sum_{n \in \mathbb{Z}} g_{n, m, j} \left( x - \frac{t^b + \epsilon_Q(t)}{2^n} \right) \right| dt,$$

where

$$f_{n, m, j}(x) := 1_{n, a_j}^* \psi_{a_j + m} f \ast \Phi_{a_j + m}(x);$$

$$g_{n, m, j}(x) := 1_{n, b_j}^* \psi_{b_j + m} g \ast \Phi_{b_j + m}(x).$$

Let $S_0 := \{(j, n) \in \mathbb{Z}^2 : j > N\}$. For any $S \subseteq S_0$, define $S_j := \{n \in \mathbb{Z} : (j, n) \in S\}$ and

$$\Lambda_S(f, g) := \sum_{j > N} \int \left| \sum_{n \in S_j} f_{n, m, j} \left( x - \frac{t^a + \epsilon_p(t)}{2^n} \right) \right| \left| \sum_{n \in S_j} g_{n, m, j} \left( x - \frac{t^b + \epsilon_Q(t)}{2^n} \right) \right| |\rho(t)| dt dx,$$

We aim to prove that for any finite $S \subseteq S_0$,

$$\Lambda_S(f, g) \lesssim m|F_1|^\frac{2}{3} |F_2|^\frac{1}{3} |F_3|^{1 - \frac{1}{3}},$$

from which (6.3) follows. The strategy is to organize elements in $S$ into union of subsets called maximal trees. On each tree $T \in S$, $\Lambda_T(f, g)$ can be controlled. Let’s perform some reductions on $\Lambda_T(f, g)$ as in [9]. By a change of variable $u = x - \frac{t^a + \epsilon_p(t)}{2^n}$,

$$\Lambda_T(f, g) = \sum_{j > N} \int \left| \sum_{n \in T_j} f_{n, m, j} (u - tr(t)) \rho(t) \right| dt \left| \sum_{n \in T_j} g_{n, m, j} (u) \right| du,$$

where $tr(t) := \frac{t^a + \epsilon_p(t)}{2^n} - \frac{t^b + \epsilon_Q(t)}{2^n}$. Since $tr(t) \simeq \frac{QA}{2^n}$, we have

$$\int \left| \sum_{n \in T_j} f_{n, m, j} (u - tr(t)) \rho(t) \right| dt \lesssim \mathfrak{M} \left( \sum_{n \in T_j} f_{n, m, j} (u) \right).$$

From here the translation determined by $t$ disappears and thus we can use the same calculations as in [9]. We omit the details. This finishes the proof of Proposition 2.5 and Theorem 1.1.
Now we show how to use Lemma 2.1 and Theorem 2.3 to obtain the boundedness of the bilinear maximal function $M_{P,Q}$, proving Theorem 1.2. By triangle inequality, it suffices to consider the following operator
\begin{equation}
T^*(f,g)(x) := \sup_{j \in \mathbb{Z}} T_j(f,g)(x),
\end{equation}
where $T_j$ is defined as in (2.1) and $f,g$ are non-negative. By Lemma 2.1 and symmetry, we can further assume that the supremum is taken over $j > N$ for some large $N$.

As before, decompose $T_j = \sum_{(m,n) \in \mathbb{Z}^2} T_{j,m,n}$ (see (2.13)). Let $E := \{|m-n| \gtrsim 1\} \cup \{\max\{m,n\} \leq 0\}$. Using Fourier expansion and integration by parts (or Taylor expansion), it is easy to see that
\[ \sup_{j > N} \left| \sum_{(m,n) \in E} T_{j,m,n}(f,g)(x) \right| \lesssim M f(x) M g(x). \]
By Hölder inequality and the boundedness of $M$, $\sup_{j > N} | \sum_{(m,n) \in E} T_{j,m,n}(f,g)(x) |$ is bounded from $L^p \times L^q$ into $L^r$.

For $(m,n) \in \mathbb{Z}^2 \setminus E$, we can assume without loss of generality that $m = n$. In this case we bound $\sup_{j > N} | T_{j,m,m}(f,g)(x) |$ crudely by $\sum_{j > N} | T_{j,m,m}(f,g)(x) |$. Since each $\sum_{j > N} | T_{j,m,m}(f,g)(x) |$ is bounded with $2^{-c \kappa m}$ decay in norm by Theorem 2.3, we conclude that $\sup_{j > N} \left| \sum_{(m,n) \in \mathbb{Z}^2 \setminus E} T_{j,m,n} \right|$ is bounded. This finishes the proof of Theorem 1.2.

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