Quantum Loop Modules and Quantum Spin Chains

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Abstract. We construct level-0 modules of the quantum affine algebra $U_q(\hat{sl}_2)$, as the $q$-deformed version of the Lie algebra loop module construction. We give necessary and sufficient conditions for the modules to be irreducible. We construct the crystal base for some of these modules and find significant differences from the case of highest weight modules. We also consider the role of loop modules in the recent scheme for diagonalising certain quantum spin chains using their $U_q(\hat{sl}_2)$ symmetry.

In memory of Rolf Adams.
1. Introduction

Since the advent of quantum mechanics, representation theory and mathematical physics have enjoyed close ties — indeed, in some areas this is so self-evident as to go almost un-noticed. In the subject of exactly solved models of two-dimensional statistical mechanics, it has not always been so. But developments in the last decade have been rapid. The quantum inverse scattering method — an algebraic implementation of the Bethe Ansatz and associated Yang-Baxter equations — culminated in the theory of quantum groups [1–5]. Representation theory of infinite dimensional Lie algebras is paramount in Conformal Field Theory [6,7]. In the massive case, Corner transfer matrices (CTMs), invented by Baxter [8,9], pointed to the involvement once more of representation theory, through the appearance of character formulae [10,11].

In [12] was given a new scheme for diagonalising the XXZ (six-vertex) Hamiltonian spin chain, in the anti-ferromagnetic regime, using the Quantum affine algebra $U_q(\widehat{sl}_2)$. The approach of that paper has been extended to higher spin chains [13], to the higher rank case [14] and to the SOS models of Andrews, Baxter and Forrester [15,16]. In the six-vertex case, the scheme has been used to give general expressions for the $n$-point correlation functions [17]: it also provides a means to derive $q$-difference equations for correlation functions for all the models [18,19]. The six-vertex model, and its higher-spin and higher-rank generalisations, are related to the quantum affine algebra $U'_q(\widehat{sl}_2)$ of Drinfeld and Jimbo [2], since their Boltzmann weights form the $R$-matrices which intertwine tensor products of finite-dimensional representations. For the SOS models the relation is more technical: the Boltzmann weights are the connection matrices which intertwine vertex operators — themselves intertwiners — discovered by Frenkel and Reshetikhin [20]. An important ingredient in the use of the quantum affine algebra for the solution of these models is the fact that the eigenvectors of their CTMs may be identified with the weight vectors of level-$k$ infinite-dimensional modules, and their CTMs with the derivation operator $d$ [21,22]. In this way, physical calculations are reduced to problems of representation theory.

CTMs act on a semi-infinite spin chain rather than the full infinite chain of the row transfer matrix (RTM) and its associated Hamiltonian. The scheme presented in [12] is that one selects an arbitrary point in the infinite chain, and then employs the eigenvectors of a level-1 module and its dual (which is level $-1$) to represent the states of the left and right hand semi-infinite parts, respectively. The state space of the entire chain is then a tensor product and is a level-0 module. This tensor product is highly reducible; the most obvious reduction being the decompositions into $n$-particle states. One may shift the selected point at which translational symmetry is broken, one site at a time, using the theory of quantum vertex operators [20]. This gives a viable representation-theoretic realisation of the translation operator $T$. Since the derivation $d$ was already identified with a Hamiltonian spin chain (the CTM) whose coefficients are linear in the position along the chain, the usual Hamiltonian spin chain becomes identified with a multiple of the operator $(TdT^{-1} - d)$. We shall give some further relevant details, for $H_{XXZ}$, in a later
These developments highlight the importance of further study of level-0 $U_q(\hat{sl}_2)$ modules. In the Lie algebra case it is shown in [23,24] that the only integrable level-0 modules, with finite dimensional weight spaces, are loop modules (up to isomorphism). Since our interest is in the generic case of $U_q(\hat{sl}_2)$, we confine our attention here to loop modules. They are, of course, constructed by defining an appropriate module action on the affinization of a finite dimensional $U_q(\hat{sl}_2)$ module — see (3.1) below. For the most part, this paper considers the fundamental properties of the loop modules, such as their generation from “highest weight” components, reducibility, tensor products and the construction of a crystal base. We do, however, return to the problem which motivated this study in section 6 of the paper. There we apply our results to a better understanding of the diagonalisation scheme for $H_{XXZ}$, referred to above.

The plan of the paper is as follows. In section 2 we give the necessary definitions of the quantum algebras and the presentations employed herein. Section 3 defines loop modules and their character function: we also derive an important determinant formula to be used later. Section 4 is devoted to questions about the fundamental structure of the loop modules and their tensor products. In section 5 we construct a crystal base for a subset of the loop modules. We show that the loop modules do not enjoy the nice existence and uniqueness properties (modulo the crystal lattice) of a crystal base, which hold for the highest weight modules. Section 6 returns to the problem which motivated this work: the diagonalisation of $H_{XXZ}$ using the quantum affine symmetry. We give some new results, particularly about the preservation of the crystal lattice by the particle creation operators. Some concluding comments are made in section 7.

2. Notations

2.1 Definition of $U_q(\hat{sl}_2)$. We follow Jimbo [5]. $U_q(\hat{sl}_2)$ is an associative algebra generated by $e$, $f$, $t$, which satisfy the defining relations

$$tet^{-1} = q^2 e, \quad tft^{-1} = q^{-2} f, \quad [e,f] = t - t^{-1}.$$

The representations we use are built on the basic $(l+1)$-dimensional irreducible $U_q(sl_2)$ modules $V_l$, with weight vectors $v^l_k$ ($k = 0, \ldots, l$), defined by the module action

$$e \cdot v^l_k = [l-k+1]v^l_{k-1},$$
$$f \cdot v^l_k = [k+1]v^l_{k+1},$$
$$t \cdot v^l_k = q^{-2k}v^l_k,$$

(2.2)

where $[n] = (q^n - q^{-n})/(q - q^{-1})$, $v^l_0 = 0$ is the highest weight vector, and $v^l_k = 0$ if $k < 0$ or $k > l$.

For $U'_q(\hat{sl}_2)$ the generators are $e_i$, $f_i$, $t_i$, ($i = 0, 1$), and they satisfy

$$t_ie_j = q^{A_{ij}}e_jt_i, \quad t_if_j = q^{-A_{ij}}f_jt_i,$$
\[t_i t_j = t_j t_i, \quad [e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q - q^{-1}}, \quad (2.3)\]

\[e_i^3 e_j - (q^2 + 1 + q^{-2}) e_i^2 e_j e_i + (q^2 + 1 + q^{-2}) e_i e_j e_i^2 - e_j e_i^3 = 0, \quad (i \neq j), \]

\[f_i^3 f_j - (q^2 + 1 + q^{-2}) f_i^2 f_j f_i + (q^2 + 1 + q^{-2}) f_i f_j f_i^2 - f_j f_i^3 = 0, \quad (i \neq j).\]

where \(A_{ij}\) is the generalised Cartan matrix for the affine Lie algebra \(\widehat{sl}_2\)

\[A_{ij} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \quad (2.4)\]

Finally, the full quantum affine algebra \(U_q(\widehat{sl}_2)\) is obtained by adding the generator \(q^d\). \(d\) is the derivation, for which

\[d, e_i = \delta_{i,0} e_i, \quad d, f_i = -\delta_{i,0} f_i, \quad d, t_i = 0. \quad (2.5)\]

Formulae for the crystal base depend on the normalisation of the roots via the parameter \(q_i = q^{(\alpha_i, \alpha_i)}\). Since we are only interested in the case of \(U_q(\widehat{sl}_2)\), we follow [12] and choose \((\alpha_i, \alpha_i) = 1\). Consequently, our conventions for \(\widehat{sl}_2\) are as follows. The Cartan subalgebra \(H\) is spanned by \(\{h_0, h_1, d\}\) and \(\alpha_0, \alpha_1\) are the roots. They are related to the fundamental weights by \(\alpha_0 = 2\Lambda_0 - 2\Lambda_1 + \delta, \omega_1 = 2\Lambda_1 - 2\Lambda_0\); we also write \(\rho = \Lambda_0 + \Lambda_1\). The invariant form on \(H^*\) is given by \((\Lambda_i, \Lambda_j) = \delta_{ij} \delta_{j1}/4, (\Lambda_i, \delta) = 1/2, (\delta, \delta) = 0\). The weight lattice and its dual are \(P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\delta\) and \(P^* = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \mathbb{Z}d\), with \((\Lambda_i, h_j) = \delta_{ij}, (\Lambda_i, d) = 0, (\delta, h_i) = 0, (\delta, d) = 1\). We identify \(P^*\) with a subset of \(P\) via \((, )\), so that \(2\alpha_i = h_i\) and \(4\rho = 4d + h_1\).

2.2 Drinfeld’s realisation. We shall require the realisation of \(U_q'(\widehat{sl}_2)\) due to Drinfeld [3]. \(U_q'(\widehat{sl}_2)\) is the associative algebra over \(\mathbb{C}\) with generators \(K^\pm, \{x^\pm_k \mid k \in \mathbb{Z}\}, \{h_k \mid k \in \mathbb{Z}_{\neq 0}\}\), and central elements \(\gamma^\pm\), satisfying the following defining relations:

\[[h_k, h_l] = \delta_{k,-l} \frac{[2k](\gamma^k - \gamma^{-k})}{k(q - q^{-1})},\]

\[[K, h_k] = 0,\]

\[Kx^\pm_k K^{-1} = q^{\pm2}x^\pm_k,\]

\[[h_k, x^\pm_l] = \pm k^{-1}[2k]\gamma^\mp(k+|k|)/2x^\pm_{k+l},\]

\[x^\pm_{k+1}x^\pm_l - q^{\pm2}x^\pm_lx^\pm_{k+1} = q^{\pm2}x^\pm_{l+1}x^\pm_k - x^\pm_{l+1}x^\pm_k,\]

\[[x^+_k, x^-_l] = \frac{\gamma^{k-l}\psi_{k+l} - \phi_{k+l}}{q - q^{-1}},\]

where only \(\{\psi_k \mid k \in \mathbb{Z}_{\geq 0}\}\) and \(\{\phi_k \mid k \in \mathbb{Z}_{\leq 0}\}\) are non-zero; they are given by the formal expansions

\[\psi(u) = \sum_{k=0}^{\infty} \psi_k u^k = K \exp \left( (q - q^{-1}) \sum_{k=1}^{\infty} h_k u^k \right),\]

\[\phi(u) = \sum_{k=0}^{\infty} \phi_{-k} u^{-k} = K^{-1} \exp \left( -(q - q^{-1}) \sum_{k=1}^{\infty} h_{-k} u^{-k} \right).\]
The isomorphism between the two presentations (2.3, 2.6) is given by
\[\begin{align*}
e_0 & \mapsto x_1^-K^{-1}, & f_0 & \mapsto \gamma Kx_1^+, & t_0 & \mapsto \gamma K^{-1}, \\
e_1 & \mapsto x_0^+, & f_1 & \mapsto x_0^-, & t_1 & \mapsto K. \end{align*}\] (2.8)

For \(U_q(\widehat{sl}_2)\) we must add the derivation \(d\). The isomorphism (2.8) gives the commutators of \(d\) with \(x_0^+, x_1^+, x_1^-\) and \(K\). It is readily checked that these imply
\[\begin{align*}
[d, x_k^+] &= kx_k^+, & (k & \in \mathbb{Z}), \\
[d, h_k] &= kh_k, & (k & \in \mathbb{Z}_{\neq 0}), \\
[d, \psi_k] &= k\psi_k, & (k & \in \mathbb{Z}_{\geq 0}), \\
[d, \phi_k] &= k\phi_k, & (k & \in \mathbb{Z}_{\leq 0}). \end{align*}\] (2.9)

### 2.3 Co-algebra structure

We define a coproduct \(\Delta\) and an antipode \(S\) for the presentation (2.3) by
\[\begin{align*}
\Delta(e_i) &= e_i \otimes 1 + t_i \otimes e_i, & S(e_i) &= -t_i^{-1}e_i, \\
\Delta(f_i) &= f_i \otimes t_i^{-1} + 1 \otimes f_i, & S(f_i) &= -f_it_i, \\
\Delta(t_i) &= t_i \otimes t_i & S(t_i) &= t_i^{-1}. \end{align*}\] (2.10)

\(\Delta\) is related to the co-product used in Chari and Pressley [25] by transposition; in the terminology of Kashiwara [26] it is the upper co-product. No general formula for the co-product is known in the Drinfeld presentation. However, some partial results are given in [25]. We recall them here for convenience. Denote by \(H\) the subalgebra of \(U = U_q(\widehat{sl}_2)\) generated by \(d, \gamma, K, h_k, (k \in \mathbb{Z}_{\neq 0})\), let \(N_{\pm}\) be the subalgebras generated by the \(x_k^\pm, (k \in \mathbb{Z})\), and \(X_{\pm}\) the linear span of the \(x_k^\pm\). Then we have that \(U = N_-HN_+\). Furthermore, it is shown in [25], proposition 4.4, that the co-product \(\Delta\) satisfies the following:

(i) \(\text{mod}(UX_+ \otimes UX_+^2),\)
\[\begin{align*}
\Delta(x_k^+) &= x_k^+ \otimes 1 + \sum_{i=0}^{k} \psi_i \otimes \psi_{k-i}, & (k & \geq 0), \\
\Delta(x_{-k}^+) &= x_{-k}^+ \otimes 1 + \sum_{i=0}^{k-1} \phi_{-i} \otimes \phi_{i-k}, & (k & > 0). \end{align*}\] (2.11)

(ii) \(\text{mod}(UX_-^2 \otimes UX_+)\),
\[\begin{align*}
\Delta(x_k^-) &= 1 \otimes x_k^- + \sum_{i=0}^{k-1} x_{-i}^- \otimes \psi_i, & (k & > 0), \\
\Delta(x_{-k}^-) &= 1 \otimes x_{-k}^- + \sum_{i=0}^{k} x_{i-k}^- \otimes \phi_{-i}, & (k & \geq 0). \end{align*}\] (2.12)
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(iii) $\text{mod}(UX_- \otimes UX_+ + UX_+ \otimes UX_-),$

\[
\Delta(\psi_k) = \sum_{i=0}^{k} \psi_i \otimes \psi_{k-i}, \quad (k \geq 0),
\]

\[
\Delta(\phi_{-k}) = \sum_{i=0}^{k} \phi_{-i} \otimes \phi_{-k-i}, \quad (k \geq 0),
\]  

(2.13)

3. Quantum Loop Modules

3.1 Definitions. Finite dimensional representations of $U'_q(\hat{\mathfrak{sl}}_2)$ are constructed by first defining an evaluation map depending on a parameter $a \in \mathbb{C}^\times$ (i.e. $a \neq 0$) [25,27]. Let $L = \mathbb{C}[t, t^{-1}]$ be the ring of Laurent polynomials in the formal variable $t$. If $V$ is a vector space over $\mathbb{C}$, we define the loop space $L(V)$ as

\[
L(V) = L \otimes V
\]  

(3.1)

as in [24]. The evaluation map which we employ herein for the construction of loop modules is defined, for the presentation (2.3), as follows:

**Definition 1.** For any $a \in \mathbb{C}^\times$ let $\varphi_a : U'_q(\hat{\mathfrak{sl}}_2) \rightarrow L(U_q(\mathfrak{sl}_2))$ be given by

\[
\begin{align*}
\varphi_a(e_0) &= t \otimes af, \\
\varphi_a(f_0) &= t^{-1} \otimes a^{-1}e, \\
\varphi_a(e_1) &= 1 \otimes e, \\
\varphi_a(f_1) &= 1 \otimes f, \\
\varphi_a(t_0) &= 1 \otimes K^{-1}, \\
\varphi_a(t_1) &= 1 \otimes K.
\end{align*}
\]  

(3.2)

It is easy to check that:

**Lemma 3.1.** $\varphi_a$ is an algebra homomorphism and the image of the Drinfeld generators $\gamma, x_k^{\pm}$, is given by

\[
\begin{align*}
\varphi_a(\gamma) &= 1, \\
\varphi_a(x_k^{+}) &= t^k \otimes a^k K^k e, \\
\varphi_a(x_k^{-}) &= t^k \otimes a^k fK^k.
\end{align*}
\]  

(3.3)

We use $\varphi_a$ to construct loop representations of $U_q(\mathfrak{sl}_2)$. Since $\varphi_a(\gamma) = 1$, these modules will be of level 0.

**Definition 2.** Let $(\pi, V)$ be a representation of $U_q(\mathfrak{sl}_2)$. Then we construct the loop representation $\hat{\pi}_a$ of $U'_q(\hat{\mathfrak{sl}}_2)$, $a \in \mathbb{C}^\times$, on the space $L(V)$ as follows:

\[
\hat{\pi}_a = (\text{id} \otimes \pi) \circ \varphi_a,
\]  

(3.4)

i.e., $\hat{\pi}_a$ is a composition of maps

\[
U'_q(\hat{\mathfrak{sl}}_2) \xrightarrow{\varphi_a} L(U_q(\mathfrak{sl}_2)) \xrightarrow{\text{id} \otimes \pi} L(\text{End}(V)) = L \otimes \text{End}(V).
\]  

(3.5)

This representation $\hat{\pi}_a$ extends to a representation of $U_q(\mathfrak{sl}_2)$, by setting:

\[
\hat{\pi}_a(d) = \frac{t d}{dt} \otimes 1.
\]  

(3.6)

We define more general quantum loop modules by a simple generalisation of the loop group construction given in [24].
**Definition 3.** Let \((\pi^i, V_i), (i = 1, \ldots, k)\), be irreducible \((l_i + 1)\)-dimensional highest weight representations of \(U_q(\mathfrak{sl}_2)\) and let \(a_i \in \mathbb{C}^*\). Then the module \(V(l; a) = L(V_{l_1} \otimes \cdots \otimes V_{l_k})\) is defined via the \(U_q(\mathfrak{sl}_2)\) action
\[
\pi_{l,a} = P_L \circ (\hat{\pi}_{a_1} \otimes \cdots \otimes \hat{\pi}_{a_k}) \circ \Delta_k
\]
where \(\Delta_k\) is the \(k\)-fold iterated co-product and \(P_L\) is the linear map
\[
P_L : L(\text{End}(V)) \otimes \cdots \otimes L(\text{End}(V)) \longrightarrow L(\text{End}(V) \otimes \cdots \otimes \text{End}(V))
\]
which identifies the powers of the formal variable \(t\) in each factor.

We remark that the action of \(d\) on loop modules is given by \(d \cdot (t^n \otimes v) = n(t^n \otimes v)\). One could define twisted loop modules by modifying this action to \(d \cdot (t^n \otimes v) = (n + c)(t^n \otimes v), c \in \mathbb{C}^*\), but we shall not choose to make this generalisation. Also, one can use arbitrary highest weight \(U_q(\mathfrak{sl}_2)\) modules, but we shall not consider this complication either. Obviously one recovers finite-dimensional representations of \(U'_q(\mathfrak{sl}_2)\) by dropping the derivation and setting \(t = 1\).

**3.2 Dual modules.** We shall need dual loop modules, which we now define. First, given a representation \((\pi, V)\) of \(U_q(\mathfrak{sl}_2)\), the dual representation \((\pi^*, V^*)\) is defined by the action
\[
\langle x \cdot u^*, v \rangle = \langle u^*, S(x) \cdot v \rangle, \quad \forall u^* \in V^*, u \in V, x \in U_q(\mathfrak{sl}_2).
\]
Recall also that the loop space \(L(V) = L \otimes V\) is turned into a representation \((\hat{\pi}_a, L(V))\) of \(U_q(\mathfrak{sl}_2)\) by pulling back a representation \((\pi, V)\) of \(U_q(\mathfrak{sl}_2)\) using the evaluation map \(\varphi_a\). We can use the dual space \(V^*\) in the same way. It is easy to check that
\[
(id \otimes S) \circ \varphi_{a^2} = \varphi_a \circ S,
\]
where on the left hand side the antipode is acting in \(U_q(\mathfrak{sl}_2)\) and on the right hand side it acts in \(U'_q(\mathfrak{sl}_2)\). The definition of a dual loop module may now be given:

**Definition 4.** Let \((\pi, V)\) be a representation of \(U_q(\mathfrak{sl}_2)\) and \((\pi^*, V^*)\) its dual. Then we construct the loop representation \(\hat{\pi}_a^*, a \in \mathbb{C}^*\), on \(L(V^*)\) as follows:
\[
\hat{\pi}_a^* = (id \otimes \pi^* \circ S^{-1}) \circ \varphi_a \circ S
\]
In constructing the dual to Definition 3, one should note that taking the dual of a tensor product reverses the order of the factors and we want the dual of \(L(V_{l_1} \otimes \cdots \otimes V_{l_k})\) to act on \(L((V_{l_1} \otimes \cdots \otimes V_{l_k})^*)\).

**Definition 5.** Let \((\pi^i, V_i), (i = 1, \ldots, k)\), be \((l_i + 1)\)-dimensional highest weight representations of \(U_q(\mathfrak{sl}_2)\) and let \(a_i \in \mathbb{C}^*\). Then the module \(V^*(l; a) = L(V_{l_k}^* \otimes \cdots \otimes V_{l_1}^*)\) is defined via the \(U_q(\mathfrak{sl}_2)\) action
\[
\pi_{l,a}^* = P_L \circ (\hat{\pi}_{a_k}^* \otimes \cdots \otimes \hat{\pi}_{a_1}^*) \circ \Delta_k
\]
with $\Delta_k$ and $P_L$ as before.

Observe that $V^*(l; a) \subset (V(l; a))^*$. (3.10) is an equality between maps $U'_q(sl_2) \rightarrow L \otimes U_q(sl_2)$. It follows that if $(l, a)$ and $(l', a')$ are related to each other by reversal of order, then there is an isomorphism of loop modules

$$V^*(l, a) \sim \rightarrow V(l', q^2a')$$ (3.13)

3.3 Character formula. Using the formulae for $\Delta(\psi_m)$, $\Delta(\phi_m)$ and $\Delta(x_k^+)$ given in section 2.3, it is easy to see that the “highest weight components” $\Omega_{t,n} = t^n \otimes u_0^l \otimes \cdots u_0^k$ are eigenvectors of $\psi_m$ and $\phi_m$, $(m \in \mathbb{Z}_{\geq 0})$, and are annihilated by the subalgebra $N_+$. Write

$$\psi_m \cdot \Omega_{t,n} = \chi_{t,a}(\psi_m)\Omega_{t,n},$$
$$\phi_m \cdot \Omega_{t,n} = \chi_{t,a}(\phi_m)\Omega_{t,n},$$
$$N_+ \cdot \Omega_{t,n} = 0,$$ (3.14)

where the eigenvalues are in the ring $L$. (That is, we may regard $V(l; a)$ as a free module of finite rank over the ring $L$.) Let $H$ be the subalgebra of $U'_q(sl_2)$ generated by $\psi_k, \phi_k, k \in \mathbb{Z}_{\geq 0}$, and let $H_0$ be the quotient of $H$ by its center. We can extend the function $\chi_{t,a}$ to a homomorphism of commutative algebras, $\chi_{t,a} : H_0 \rightarrow L$, which preserves the grading. (Strictly speaking we should write $\chi_{t,a}(p(\psi_k))$, etc., $p$ being the canonical projection to $H_0$, but the above notation is clear.) This leads naturally to:

**Definition 6.** The character $\chi_{t,a}$ of the loop module $V(l; a)$ is defined by the eigenvalues $\chi_{t,a}(\psi_m), \chi_{t,a}(\phi_m)$ extended to a homomorphism of commutative algebras

$$\chi_{t,a} : H_0 \rightarrow L.$$ (3.15)

For $V(l; a)$, the eigenvalue formulae (3.14) have the explicit form

$$\psi_0 \cdot (t^n \otimes v_0^l) = q^l(t^n \otimes v_0^l),$$
$$\phi_0 \cdot (t^n \otimes v_0^l) = q^{-l}(t^n \otimes v_0^l),$$
$$\psi_m \cdot (t^n \otimes v_0^l) = (q^l - q^{-l})(aq^l)^m(t^{m+n} \otimes v_0^l),$$
$$\phi_m \cdot (t^n \otimes v_0^l) = (q^{-l} - q^l)(aq^l)^{-m}(t^{-m+n} \otimes v_0^l).$$ (3.16)

Further formulae which we shall need are

$$\psi(u) \cdot (t^n \otimes v_0^l) = \left(\frac{q^l - aut}{1 - aq^lut}\right)(t^n \otimes v_0^l).$$
(3.17)

$$\phi(u) \cdot (t^n \otimes v_0^l) = \left(\frac{q^{-l} - (aut)^{-1}}{1 - (aq^lut)^{-1}}\right)(t^n \otimes v_0^l).$$

These are, of course, formal expansions in the spirit of (2.7). It follows that

$$h_m \cdot (t^n \otimes v_0^l) = m^{-1}[m]a^m(t^{m+n} \otimes v_0^l)$$ (3.18)
From (2.13) we know that \(\Delta(\psi_k) \cdot \Omega_{l,n} = \sum_{j=0}^{k} (\psi_j \cdot \upsilon_l^j) \otimes (\psi_{k-j} \cdot \Omega_{l,n})\), where \(l' = l_2, \ldots, l_k\) and \(a' = a_2, \ldots, a_k\). Moreover, this may be extended to the \(k\)-fold iteration of the co-product. This convolution property enables general formulæ for the eigenvalues to be constructed quite readily, either as explicit formulæ, or by using the multiplicative property which is implied, namely

\[
\psi(u) \cdot \Omega_{l,n} = \left( \prod_{j=1}^{k} \frac{q^{l_j} - a_j u t}{1 - a_j q^{l_j} u t} \right) \Omega_{l,n}, \tag{3.19}
\]

\[
\phi(u) \cdot \Omega_{l,n} = \left( \prod_{j=1}^{k} \frac{q^{-l_j} - (a_j u t)^{-1}}{1 - (a_j q^{l_j} u t)^{-1}} \right) \Omega_{l,n}.
\]

**Lemma 3.2.** Let \(V(l; a)\) be a quantum loop module. Then the character function has the additive property \(\chi_{l;a}(h_m) = \sum_{i=1}^{k} \chi_{l_i; h_m}, (m \in \mathbb{Z})\).

**Proof.** This follows immediately from equations (2.7) and (3.19). \(\square\)

We proceed to an important theorem concerning the character function.

**Theorem 1.** The image of \(\chi_{l;a}\) is a Laurent subring of \(L\) i.e., \(\chi_{l;a}(H_0) = C[t^r, t^{-r}],\) for some integer \(r > 0\).

**Proof.** From (3.18) we find

\[
\chi_{l;a}(h_m) = m^{-1}(q - q^{-1})^{-1} t^m \left( \sum_{j=1}^{k} (a_j q^{l_j})^m - \sum_{j=1}^{k} (a_j q^{-l_j})^m \right). \tag{3.20}
\]

Let \(y_i, i = 1, \ldots, p\) be the distinct elements of the set \(\{a_j q^{l_j}, -a_j q^{-l_j} \mid 1 \leq j \leq k\}\), and let \(\mu_i\) be the multiplicity of \(y_i\), then

\[
\chi_{l;a}(h_{2m-1}) = (2m - 1)^{-1}(q - q^{-1})^{-1} t^{2m-1} \sum_{i=1}^{p} \mu_i y_i^{2m-1}. \tag{3.21}
\]

The determinant of the matrix \((y_i^{2m-1})_{1 \leq i, m \leq p}\) is proportional to a Vandermonde determinant, so that the characters \(\chi_{l;a}(h_{2m-1})\) cannot all vanish simultaneously in the range \(1 \leq m \leq p\). Similarly, \(\chi_{l;a}(h_{2m+1})\) do not all vanish simultaneously in the same range. Therefore there exists integers \(r, s \in \{1, \ldots, 2p - 1\}\) such that \(\chi_{l;a}(h_r) \neq 0\) and \(\chi_{l;a}(h_s) \neq 0\). The rest of the proof proceeds exactly as in Lemma 4.1, p. 328 of [23]. \(\square\)

In the sequel we shall assume that that \(r = 1\), i.e. that the map \(\chi_{l;a}\) is surjective, unless otherwise stated.

### 3.4 A determinant formula

Our strategy is as follows. First we shall prove, in Lemma 4.1, that the module \(V(l; a)\), with the dimensions of the factors in the order \(l_1 \leq \ldots \leq l_k\), is generated by \(\Omega_{l,n}\) provided that

\[
a_j/a_i \neq q^{(l_i + l_j - 2p + 2)}, \quad 0 < p \leq l_i, \quad i < j. \tag{3.22}
\]
The dual module $V^*(l; a)$ is isomorphic to $V(l'; a')$, with $(l'; a')$ obtained from $(l; a)$ by reversal of order: the proof used in Lemmas 3.4 & 4.1 may be modified to show that $V(l'; a')$ is is generated by $\Omega_{l', n}$, but now under the conditions

$$a_j/a_i \neq q^{-(l_i + l_j - 2p + 2)}, \quad 0 < p \leq l_j, \quad i < j. \quad (3.23)$$

Duality and isomorphism of the two modules so generated leads to their irreducibility, and then further to the fact that all $V(l''; a'')$, where $(l''; a'')$ is obtained from $(l; a)$ by arbitrary permutation, are irreducible and generated by $\Omega_{l'', n}$ (Theorem 2). Finally we show that, when the conditions of Theorem 2 are not satisfied, the modules are reducible (Theorem 3).

We recall some definitions and results from [25] which generalize the conditions (3.22, 3.23) to arbitrary $(l; a)$.

**Definition 7.** Given an $(m+1)$-dimensional irreducible $U_q(sl_2)$ module $V_m$, and a parameter $a \in \mathbb{C}^\times$, the associated $q$-string $S_m(a)$ is the set $S_m(a) = \{a^{-1}q^m, \ldots, a^{-1}q^{1-m}\}$.

**Definition 8.** The $q$-strings $S_1$ and $S_2$ are said to be in general position if either $S_1 \cup S_2$ is not a $q$-string, or $S_1 \subset S_2$ or $S_2 \subset S_1$.

**Lemma 3.3.** The $q$-strings $S_m(a)$ and $S_n(b)$ are in general position if and only if 

$$\{b/a \neq q^{\pm(m+n-2p+2)} \mid 0 < p \leq \min(m, n)\}$$

We shall show that the necessary and sufficient condition for a loop module to be generated by $\Omega_{l,n}$ and irreducible is that all the $q$-strings are in general position.

As a preliminary to Lemma 4.1, we prove a result concerning the coefficient matrix $A$ which appears in equation (4.8) below. This matrix has the definition

$$A_{r,j} = \sum_{s=1}^{r} b^s_j d_{r-s,j+1}, \quad 1 \leq r \leq k, \quad 1 \leq j \leq k, \quad (3.24)$$

where $b_1 = q^{l_1-2} a_1$, (0 $\leq i < l_1$), $b_j = q^{l_j} a_j$, (j $> 1$) and the $d_{s,j}$ are numerical factors in the character formula $\psi_s \cdot (v_0^{l_1} \otimes \cdots \otimes v_0^{l_k}) = d_{s,j} (t^s \otimes v_0^{l_1} \otimes \cdots \otimes v_0^{l_k})$, together with $d_{s,k+1} = \delta_{s,0}$. The columns of $A$ are discrete convolutions, as are the $d_{s,j}$. So it is natural to introduce generating functions for the columns: $A_j(u) = \sum_{r=1}^{\infty} A_{r,j} u^{r-1}$, since they factorise:

$$A_j(u) = \left( \sum_{r=0}^{\infty} b^r_j u^r \right) \left( \sum_{r=0}^{\infty} d_{r,j+1} u^r \right)$$

$$= \frac{b_j}{1 - ub_j} \prod_{i=j+1}^{k} \frac{q^{l_i} - u q^{-l_i} b_i}{1 - ub_i}. \quad (3.25)$$

In this calculation we have used (3.17). We use this to evaluate the determinant of $A$ in an elementary way.
Lemma 3.4.

\[ \det A = \prod_r b_r \prod_{r<s} (q^{l_r} b_r - q^{-l_s} b_s) \]  \hspace{1cm} (3.26)

Proof. The determinant is a multinomial in the variables \( b_j \); for \( q = 1 \), it is a Vandermonde determinant with the stated factorisation. From (3.25) we can see that the highest and lowest powers of \( q \) which can occur are correctly given by (3.26). Therefore (3.26) is proved if we can show that the determinant has the stated linear factors \((q^{l_r} b_r - q^{-l_s} b_s), (r < s)\), since no further factors involving \( q \) are then possible. To obtain these factors, we show that if \( b_r = q^{-2l_s} b_s \), there exist constants \( c_r, \ldots, c_s \), not all zero, such that \( \sum_{i=r}^s c_i A_i(u) \equiv 0 \), which implies that the columns of the matrix are linearly dependent for any \( k \). After substituting and simplifying, these equations read

\[ \sum_{i=r}^{s-1} c_i b_i \prod_{j=i+1}^s \frac{q^{l_j} - u q^{-l_s} b_j}{1 - u b_j} + c_s b_s \prod_{j=r+1}^s \frac{1}{1 - u b_j} \equiv 0. \]  \hspace{1cm} (3.27)

If \( b_r = q^{-2l_s} b_s \), there is a cancellation in the first term, which becomes

\[ \frac{c_r b_r}{1 - u b_s} \prod_{j=r+1}^{s-1} \frac{q^{l_j} - u q^{-l_s} b_j}{1 - u b_j} \equiv 0. \]  \hspace{1cm} (3.28)

There is a common factor \((1 - u b_s)\) in the denominators in (3.27), which we discard; the other factors are \((1 - u b_i), (r < i < s)\). Rationalising, we obtain \( \sum_{i=r}^s c_i P_i(u) \equiv 0 \). But the polynomials \( P_i(u) \) are all of degree \( s-r-1 \), and \( s-r+1 \) such polynomials are necessarily linearly dependent. \( \square \)

4. Reducibility and Irreducibility

4.1 Cyclic property. The following result is of central importance.

Lemma 4.1. Suppose that \((l, a)\) is such that the \( q \)-strings \( S_{l_i} (a_i) \) are all in general position relative to each other, and that the dimensions of the terms in the tensor product are in one of the orders \( l_1 \leq \cdots \leq l_k \) or \( l_1 \geq \cdots \geq l_k \). Then the loop module \( V(l; a) \) is generated by any one of the following vectors:

\[ \Omega_{l,n} = t^n \otimes v_{l_1}^0 \otimes \cdots \otimes v_{l_k}^0, \]  \hspace{1cm} (4.1)

where the \( v_{l_i}^0 \) are highest weight vectors of the irreducible \( U_q(sl_2) \) modules \( V_{l_i} \).

Proof. We shall give the proof for the first ordering only. It is enough to show that all the vectors

\[ t^n \otimes y_1 v_{l_1}^0 \otimes \cdots \otimes y_k v_{l_k}^0, \quad (y_i \in N_-), \]  \hspace{1cm} (4.2)
are in the submodule generated by $\Omega_{t,n}$. By Theorem 1 and our assumption that $r = 1$, we know that for any $m, n \in \mathbb{Z}$ there exists a $Q_{m-n} \in H_0$ such that $\chi_{t,a}(Q_{m-n}) = t^{m-n}$, giving

$$Q_{m-n} \cdot \Omega_{t,n} = \Omega_{t,m}, \quad (m \in \mathbb{Z}). \tag{4.3}$$

Therefore we have only to show that any one of the vectors (4.2) is in the submodule generated by $\Omega_{t,n}$, $(n \in \mathbb{Z})$. We prove this by induction on $k$. The case $k = 1$ is obvious from the definitions. Let $l' = (l_2, \ldots, l_k), \ a' = (a_2, \ldots, a_k)$ and define

$$\Omega_{l',n} = t^n \otimes v_0^{l_2} \otimes \cdots \otimes v_0^{l_k}, \tag{4.4}$$

First we prove that $V(l, a)$ is generated by $v_{i_1}^{l_1} \otimes \Omega_{l',n}$ where $v_{i_1}^{l_1}$ is the lowest weight vector in $V_{i_1}$. By the induction hypothesis, $V(l', a')$ is generated by $\Omega_{l',n}$. Recall that $U_q(sl_2) = N_- H N_+$. For any $w \in V(l', a')$ there exists $x \in N_-$ such that $w = x \cdot \Omega_{l',n}$. Using (2.12):

$$\Delta(x) \cdot (v_{i_1}^{l_1} \otimes \Omega_{l',n}) = v_{i_1}^{l_1} \otimes w. \tag{4.5}$$

This means that $v_{i_1}^{l_1} \otimes V(l', a')$ is generated by $v_{i_1}^{l_1} \otimes \Omega_{l',n}$. Now

$$x_0^+ \cdot (v_{i_1}^{l_1} \otimes w) = (x_0^+ \cdot v_{i_1}^{l_1}) \otimes w + (K \cdot v_{i_1}^{l_1}) \otimes (x_0^+ w). \tag{4.6}$$

So $v_{i_1}^{l_1-1} \otimes V(l', a')$ is also generated from $v_{i_1}^{l_1} \otimes \Omega_{l',n}$. A finite induction completes the first step of the proof.

The second step consists in showing that for each $0 \leq i < l_1$, the vector $v_{i+1}^{l_1} \otimes \Omega_{l',n}$ is generated by the vectors $v_{i}^{l_1+1} \otimes \Omega_{l',n+r}$, for $1 \leq r \leq k$. Repeating this process, we will show that $v_{i_1}^{l_1} \otimes \Omega_{l',n}$ is in the submodule generated by $v_{0}^{l_1} \otimes \Omega_{l',n}$. From the first step and (4.3), this will complete the proof. Now

$$x^{-}_r \cdot (t^m \otimes v_{i_1}^{l_1} \otimes \Omega_{l'} = t^m \otimes v_{i}^{l_1} \otimes (x^{-}_r \cdot v_{i_1}^{l_1}) + \sum_{s=1}^{r} t^m \otimes (x^{-}_s \cdot v_{i}^{l_1}) \otimes (x^{-}_{r-s} \cdot \Omega_{l'}). \tag{4.7}$$

Iterating this process, we get

$$x^{-}_r \cdot (t^m \otimes v_{i_1}^{l_1} \otimes \Omega_{l'}) = \sum_{j=1}^{k} A_{rj} t^{m+s} \otimes v_{i}^{l_1} \otimes \cdots \otimes (f \cdot v_{i}^{l_j}) \otimes \cdots \otimes v_{0}^{l_k}, \tag{4.8}$$

where the $A_{rj}$ were defined in (3.24). From Lemma 3.4 we conclude that the equations do indeed have a solution under the stated conditions.

Finally, we should note that there is a double induction, starting with the case that $\Omega_{l',n} = t^n \otimes v_{0}^{l_2}$. In the course of the process, every one of the conditions (3.22) are required - precisely half of the general position conditions.
The remaining conditions (3.23) are required to repeat the proof with the other ordering \( l_1 \geq \ldots \geq l_k \). The main difference is that one notes, from the decomposition (4.9), that generation from the highest component \( \Omega_{l,n} \) is equivalent to generation from the lowest: \( \Omega_{l,n} = t^n \otimes v_{l_1}^1 \otimes \cdots \otimes v_{l_k}^1 \). The first step becomes a proof that \( V(l, a) \) is generated by \( v_{l_1}^1 \otimes \Omega_{l',n} \), whilst in the second step one replaces \( x_r \) in equations (4.7) and (4.8) by \( x^+ \) to show that for each \( 0 < i \leq l_1 \), the vector \( v_{l_1-1}^i \otimes \Omega_{l',n} \) is a linear combination of vectors \( v_{l_1}^i \otimes \Omega_{l',n+r} \), for \( 1 \leq r \leq k \).

**4.2 Irreducibility.** We know from \( U_q(sl_2) \) theory that

\[
V_m \otimes V_n = \bigoplus_{p=0}^{\min(m,n)} V_{m+n-2p} \tag{4.9}
\]

Since \( U_q(sl_2) \) is a subalgebra of \( U_q(\widehat{sl_2}) \), the loop modules have such a \( U_q(sl_2) \) weight decomposition, with highest component \( V_N, N = \sum_{j=1}^k l_j \). The simplest case, \( V(1,1; a, b) \), is instructive. We have the following:

(i) If \( a/b = q^2 \) and \( q^4 \neq 1 \), then \( \Omega = v_0^1 \otimes v_0^1 \) generates the whole module, but the vector \( \Omega_1 = v_0^1 \otimes v_1^1 - qv_1^1 \otimes v_0^1 \) generates a one-dimensional submodule.

(ii) If \( b/a = q^2 \) and \( q^4 \neq 1 \), then \( \Omega = v_0^1 \otimes v_0^1 \) generates a three-dimensional submodule, whilst the vector \( \Omega_1 = v_0^1 \otimes v_1^1 - qv_1^1 \otimes v_0^1 \) generates the whole module.

(iii) \( V^*(1,1; a, b) \) has a complementary structure in each case.

(iv) The above illustrates why it is necessary to prove that the modules \( V(l; a) \) and \( V(l'; a') \) are both generated from their highest component, even if \( l' = l \).

Obviously, modules which satisfy the conditions of Lemma 4.1 cannot have proper submodules which contain the highest component \( V_N \) of (4.9). This leads immediately to the following result.

**Lemma 4.2.** Under the conditions of Lemma 4.1, the modules \( V(l, a) \) and \( V^*(l, a) \) are irreducible.

**Proof.** Suppose that \( W \) is a proper submodule of \( V(l, a) \), then its annihilator \( W^0 \) is a proper submodule of \( V^*(l, a) \) which does contain the highest component. This is not possible. \( \square \)

**Theorem 2.** Suppose that \( V(l; a) \) satisfies the conditions of Lemma 4.1, and that \( l' = \sigma(l), a' = \sigma(a) \), where \( \sigma \) is a permutation of \( k \) objects. Then \( V(l'; a') \) is irreducible and is generated by any vector of the form (4.1).

**Proof.** We have that \( V(l; a) \) is generated by \( \Omega_{l,n} \), and is irreducible. Moreover, \( \Omega_{l',n} \) generates a submodule of \( V(l'; a') \). Define a map \( V(l; a) \rightarrow V(l'; a') \) by \( x \cdot \Omega_{l,n} \rightarrow x \cdot \Omega_{l',n} \), for all \( x \in U_q(sl_2) \). This is obviously a homomorphism of \( U_q(sl_2) \) modules which is injective because \( V(l; a) \) is irreducible. We must show that it is surjective. This follows from the fact that it preserves the grading, and the image of the finite-dimensional weight space at each grading level already has dimension \( \prod_{j=1}^k l_j \). \( \square \)
Corollary 4.1. Suppose that \( V(l; a) \) is irreducible and that \( l' = \sigma(l), a' = \sigma(a) \). for some permutation \( \sigma \). Then \( V(l'; a') \) is irreducible.

Corollary 4.2. Suppose that \( V(l; a) \) and \( V(l'; a') \) are irreducible. Then the following are equivalent:
(i) the characters are proportional: \( \chi_{l,a} = c\chi_{l',a'} \).
(ii) \( V(l; a) \) is isomorphic to \( V(l'; a') \).
(ii) there is a permutation \( \sigma \) and a \( s \in C^\times \) such that \( l' = \sigma(l) \) and \( a' = s\sigma(a) \).

4.3 Reducibility. Let us now show that the conditions of general position are necessary as well as sufficient for irreducibility. The crucial step is to consider the module \( V(m, n; a, b) \).

Lemma 4.3. Let \( \Omega_p^s = t^s \otimes \Omega_p \), where \( \Omega_p \) is the \( U_q(sl_2) \) highest weight vector of the component \( V_{m+n-2p} \) in (4.9) and \( \Omega_p^s \) is a weight vector in \( V(m, n; a, b) \). Then we have \( N_+ \cdot \Omega_p^s = 0 \), and \( h_k \cdot \Omega_p^s = c\Omega_p^{s+k} \) \((\forall k \in Z)\), if and only if \( b/a = q^{m+n-2p+2} \), \((0 < p \leq \min(m, n))\).

Proof. The proof involves a long computation. We give details only in the case that \( m \leq n \). Then, the \( U_q(sl_2) \) highest weight vector \( \Omega_p \) is given by

\[
\Omega_p = \sum_{i=0}^{p} (-1)^i q^{i(m-i+1)}[m-i]!(n-p+i)!v_i^n \otimes v_{p-i}^n. \tag{4.10}
\]

By construction, we have \( x_0^+ \cdot \Omega_p^s = 0 \). By explicit computation we will show that

\[
x_{\pm1}^+ \cdot \Omega_p^s = 0, \quad h_{\pm1} \cdot \Omega_p^s = c_{\pm} \Omega_p^{s \pm 1}, \tag{4.11}
\]

for some constants \( c_{\pm} \), if and only if \( b/a = q^{m+n-2p+2} \). This done, it is an easy induction to extend the condition to all the generators \( x_k^+ \), \((k \in Z)\) of \( N_+ \) using the commutators

\[
[h_{-1}, x_k^+] = (q + q^{-1})x_{k-1}^+, \quad [h_1, x_k^+] = (q + q^{-1})x_{k+1}^+. \tag{4.12}
\]

By direct computation, one first shows that \( x_{-1}^+ \cdot \Omega_p^s = 0 \) if and only if \( b/a = q^{m+n-2p+2} \). This uses the isomorphism (2.8) to find \( \Delta(x_{-1}^+) = \Delta(K^{-1}K^{-1}f_0) = \Delta(K^{-1}K^{-1})(f_0 \otimes t_0^{-1} + 1 \otimes f_0) \). Therefore,

\[
(\varphi_a \otimes \varphi_b)\Delta(x_{-1}^+) = (K^{-1} \otimes K^{-1})(t^{-1}a^{-1}e \otimes K + t^{-1} \otimes b^{-1}e) \tag{4.13}
\]

which allows the computation to be completed. Now consider the vector \( \tilde{\Omega}_p^s = h_{-1} \cdot \Omega_p^s \). From \( [h_{-1}, x_0^+] = (q + q^{-1})x_{-1}^+ \) we find that \( e_1 \cdot \tilde{\Omega}_p^s = 0 \), i.e., \( \tilde{\Omega}_p^s \) is
also a $U_q(sl_2)$ highest weight vector of the same $U_q(sl_2)$ weight as $\Omega_p^s$. Moreover, $d \cdot \hat{\Omega}_p^s = (s-1)\hat{\Omega}_p^s$, which shows that $\hat{\Omega}_p^s$ is proportional to $\Omega_p^{s-1}$.

We must also prove the analogous result for $k = 1$. First we need that $h_1 \cdot \Omega_p^s = c\Omega_p^{s+1}$ if and only if $b/a = q^{m+n-2p+2}$. This time we use $h_1 = [x_0^+, x_1^-]K^{-1}\gamma$ together with (2.8) to find

$$(\varphi_a \otimes \varphi_b)\Delta(h_1) = (1 - q^2)atfe \otimes 1 + (1 - q^2)bt \otimes fe +$$

$$+ (q^2 - q^2)atfK \otimes e + at\left(\frac{K - K^{-1}}{q - q^{-1}}\right) \otimes 1 + bt \otimes \left(\frac{K - K^{-1}}{q - q^{-1}}\right). \quad (4.14)$$

A long computation gives the desired result. Since $[h_1, x_0^+] = (q + q^{-1})x_1^+$, this gives $x_1^+ \cdot \Omega_p^s = 0$. The second statement, $h_k \cdot \Omega_p^s = c\Omega_p^{s+k}$ ($\forall k \in \mathbb{Z}$), is now easily proved. \hfill \square

**Lemma 4.4.** $V(m, n; a, b)$ is irreducible if and only if $S_m(a)$ and $S_n(b)$ are in general position.

The “if” part is the subject of Theorem 2. From Lemma 4.3 $V(m, n; a, b)$ has a submodule generated by $\Omega_p^s$ if $b/a = q^{m+n-2p+2}$, for $0 < p \leq \text{min}(m, n)$. Recall that $V^*(m, n; a, b)$ is isomorphic to $V(n, m; q^2b, q^2a)$, so it has a submodule generated in the same way when $a/b = q^{m+n-2p+2}$. Together these possibilities exhaust the conditions of the lemma. Moreover, $V(m, n; a, b)$ and $V^*(m, n; a, b)$ must be both irreducible or both reducible. \hfill \square

**Theorem 3.** $V(l; a)$ is irreducible if and only if all the $q$-strings $S_{l_i}(a_i)$ are in general position relative to each other.

**Proof.** Assume first that some pair $S_{l_i}(a_i), S_{l_j}(a_j)$, ($i \neq j$), are not in general position, but that $V(l; a)$ is irreducible. By Theorem 2 we can assume that $j = i+1$. But then Lemma 4.4 shows that there is a proper submodule. The sufficiency of the condition was already proved. \hfill \square

**4.4 Integrability.** For completeness we also note:

**Theorem 4.** Every irreducible loop module $V(l; a)$ is integrable.

**Proof.** The argument in proposition 2.2 of [24] can be very easily generalized to the quantum case. \hfill \square

**4.5 Tensor Products.** In the previous section we saw that $U_q(sl_2)$ loop modules share many of the properties with finite-dimensional $U_q(sl_2)$ modules, particularly in the way the parameters determine their irreducibility. Tensor products of irreducible finite-dimensional $U_q(sl_2)$ modules are again irreducible in the generic case. However, this is not true for loop modules. Let $l = (l_1, \ldots, l_k)$, $a = (a_1, \ldots, a_k)$, and $l' = (l'_1, \ldots, l'_{k'})$, $a' = (a'_1, \ldots, a'_{k'})$. We put $(l, l') = (l_1, \ldots, l_k, l'_1, \ldots, l'_{k'})$ and $(a, a') = (a_1, \ldots, a_k, a'_1, \ldots, a'_{k'})$. Define the linear map, for $s, s' \in \mathbb{C}^\times$,

$$p_{s, s'} : V(l; a) \otimes V(l'; a') \to V(l, l'; sa, s'a') \quad (4.15)$$
Quantum Loop Modules and Quantum Spin Chains

5. Crystal Base

5.1 Basic notions. We recall from [26] some basic notions concerning the upper and lower crystal lattices and bases, here restricted to integrable $U_q(\widehat{sl}_2)$ modules. They are related by an appropriate choice of coalgebra structure. The coproduct and antipode defined in (2.10) will be called upper coproduct $\Delta_+$ and antipode $S_+$. The lower coproduct $\Delta = \Delta_-$ and the corresponding antipode $S_-$ are defined by

$$
\Delta_-(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i, \quad S_-(e_i) = -e_i t_i,
$$

$$
\Delta_-(f_i) = f_i \otimes 1 + t_i \otimes f_i, \quad S_-(f_i) = -t_i^{-1} f_i,
$$

$$
\Delta_-(t_i) = t_i \otimes t_i, \quad S_-(t_i) = t_i^{-1}.
$$

To define crystal lattices and bases one considers $U_q(\widehat{sl}_2)$ as an algebra over the rational function field $\mathbb{Q}(q)$, where $q$ is now a formal variable, or a transcendental number over $\mathbb{Q}$ [26].

One checks readily that this is a $U_q(\widehat{sl}_2)$-linear map, using the coassociativity of the co-product. In the generic case we may assume that $V(l; a)$, $V(l'; a')$ and $V(l, l'; sa, s' a')$ are all irreducible. Thus we see that $V(l; a) \otimes V(l'; a')$ has uncountably many quotients. This is already noted, for the Lie algebra case, in [24].

In view of corollary 4.2, we may restrict ourselves to the case that $s' = 1$. We also define the algebraic direct integral,

$$
\int \oplus V(l, l'; sa, a') \, ds,
$$

as in [24], to be the space of algebraic maps

$$
\omega : \mathbb{C}^\times \longrightarrow L\left( \bigotimes_{i=1}^{k} V_{l_i} \otimes \bigotimes_{i=1}^{k'} V_{l'_i} \right),
$$

equipped with the $U_q(\widehat{sl}_2)$ action given by

$$
(x \cdot w)(s) = P_L \circ (\pi_{L, sa} \otimes \pi_{L', a'}) \circ \Delta(x) w(s).
$$

Theorem 5. $\int \oplus V(l, l'; sa, a') \, ds$ is isomorphic to $V(l; a) \otimes V(l'; a')$.

Proof. Assign to each $(t^n \otimes w) \otimes (t'^n \otimes w')$ the map given by $\omega(s) = s^n t^{n+n'} \otimes w \otimes w'$. This map is $U_q(\widehat{sl}_2)$-linear, and its kernel is obviously trivial. Clearly the direct integral is generated by the map $\omega_{\Omega, \Omega'}(s) = s^n \Omega \otimes \Omega'$, where $\Omega$ and $\Omega'$, with degree $n$, $n'$, are the generators of $V(l; a)$, $V(l'; a')$ respectively. Therefore the $U_q(\widehat{sl}_2)$-linear map is also surjective.

5. Crystal Base

5.1 Basic notions. We recall from [26] some basic notions concerning the upper and lower crystal lattices and bases, here restricted to integrable $U_q(\widehat{sl}_2)$ modules. They are related by an appropriate choice of coalgebra structure. The coproduct and antipode defined in (2.10) will be called upper coproduct $\Delta_+$ and antipode $S_+$. The lower coproduct $\Delta = \Delta_-$ and the corresponding antipode $S_-$ are defined by
Let $M$ be an integrable $U_q(sl_2)$ module with finite-dimensional weight spaces $M_\nu$, and let $A(q)$ be the subring of $\mathbb{Q}(q)$ consisting of rational functions regular at $q = 0$. We describe the construction of the lower crystal lattice/base. For $i = 0, 1$, any weight vector $u \in M$ belongs to a $U_q(sl_2)$ subalgebra generated by $e_i$, $f_i$, and $M$ can therefore be uniquely decomposed as

$$M = \bigoplus_{0 \leq n \leq (h_i, \nu)} f_i^{(n)} \cdot (\ker e_i \cap M_\nu),$$

where $e_i^{(k)} = e_i^k / [k]!$, $f_i^{(k)} = f_i^k / [k]!$. One defines linear maps $\bar{e}_i$, $\bar{f}_i$, in $\text{End}(M)$ as follows [26].

$$\bar{f}_i \cdot (f_i^{(n)} \cdot u) = f_i^{(n+1)} \cdot u, \quad \bar{e}_i \cdot (f_i^{(n)} \cdot u) = f_i^{(n-1)} \cdot u,$$ (5.3)

for $u \in \ker e_i \cap M_\nu$ with $0 \leq n \leq (h_i, \nu)$. There is a similar definition in [26] of operators $\bar{e}_i^{up}$ and $\bar{f}_i^{up}$ suitable for the upper crystal base. The crystal lattice and crystal base at $q = 0$ are defined using these modified Chevalley generators $\bar{e}_i$, $\bar{f}_i$.

The pair $(L, B)$ is a lower crystal base of the integrable module $M$ if and only if the following properties hold.

(i) $L$ is a free $A(q)$-module such that $M = L \otimes_{A(q)} \mathbb{Q}(q)$.

(ii) $B$ is a base of the $\mathbb{Q}$-vector space $L/qL$.

(iii) $\bar{e}_i L \subset L$, $\bar{f}_i L \subset L$ and $\bar{e}_i B \subset B \cup \{0\}$, $\bar{f}_i B \subset B \cup \{0\}$.

(iv) Corresponding to the weight space decomposition $M = \bigoplus M_\nu$, we have $L = \bigoplus L_\nu$ and $B = \bigcup B_\nu$ with $L_\nu = L \cap M_\nu$ and $B_\nu = B \cap M_\nu$.

(v) If $b, b' \in B$ then $b' = \bar{f}_i \cdot b$ if and only if $b = \bar{e}_i \cdot b'$.

$L$ is called the crystal lattice, and $B$ is the crystal base. The properties of a crystal base $(L, B)$ of an integrable $U_q(sl_2)$ module $M$, when it exists, are captured in the definition of the crystal graph $\mathcal{G}$. The vertices of $\mathcal{G}$ are the elements of $B$. To each pair $b, b' \in B$ such that $b' = \bar{f}_i \cdot b$, there corresponds an arrow from $b$ to $b'$, labeled by $i$.

An integrable highest weight module $V(\lambda)$ has the standard crystal base at $q = 0$ described as follows.

$$L(\lambda) = \sum A(q) \bar{f}_{i_1} \cdots \bar{f}_{i_k} \cdot u_\lambda, \quad B(\lambda) = L(\lambda) \mod qL(\lambda) \setminus \{0\},$$ (5.4a)

$$L^{up}(\lambda)_\nu = q^{(\lambda, \lambda) - (\nu, \nu)} L(\lambda)_\nu, \quad B^{up}(\lambda)_\nu = q^{(\lambda, \lambda) - (\nu, \nu)} B(\lambda)_\nu.$$ (5.4b)

It is easy to see that the crystal base of the $U_q(sl_2)$ module $V_l$ defined in (2.2) is given by $v^l_k$, $k = 0, \ldots, l$.

5.2 Construction for loop modules. We shall now present some results on the crystal bases of loop modules. We shall see that not all the loop modules $V(l; a)$ admit a crystal base. Furthermore, when a crystal base exists, it may not be unique, and the crystal graph is not always connected (recall that uniqueness of $B$ is always mod $qL$). We shall consider only the lower crystal base, the upper one is related to the lower one by the relations such as those in (5.4b) above [26, 28].

We start with two basic observations.
Lemma 5.1.
(i) Suppose the loop module \( V(l; \alpha) \) has a crystal base \((L, B)\). Then every \( b \in B \) is proportional \((\mod qL)\) to some vector of the form \( t^p \otimes v_{j_1}^{l_1} \otimes \cdots \otimes v_{j_k}^{l_k} \), where \( p \in \mathbb{Z} \) and \( 0 \leq j_i \leq l_i, \ i = 1, \ldots, k \).

(ii) Assume \((L, B)\) and \((L', B')\) are two crystal bases of a loop module \( V(l; \alpha)\), with \( L = L'\). Let \( S \) be a connected subgraph of the crystal graph \( G \), and let \( S \) and \( S' \) be the subsets of \( B \) and \( B' \) corresponding to \( S \). Then \( S' = \alpha S \mod qL)\), \( \alpha \in \mathbb{Q} \).

Proof. Let \( U_1 \) be the \( U_q(sl_2) \) subalgebra of \( U_q(sl_2) \) generated by \( e_1 \) and \( f_1 \). The structure of the module \( V(l; \alpha) \) restricted to \( U_1 \oplus \mathbb{C} d \) is as follows:

\[
V(l; \alpha) = \bigoplus_{p \in \mathbb{Z}} M_p,
\]

where \( M_p \) is the eigenspace of \( d \) with eigenvalue \( p \) and

\[
M_p \cong M = V_{l_1} \otimes \cdots \otimes V_{l_k},
\]

as a \( U_q(sl_2) \) module. We know by proposition 6 of \([26]\) that the crystal base of \( M \) may be taken as \( B^{(1)} = \{ v_{j_1}^{l_1} \otimes \cdots \otimes v_{j_k}^{l_k} \} \). By the above-mentioned point (iv) of the definition of a crystal base, \( B \) must have a decomposition:

\[
B = \bigcup_{p \in \mathbb{Z}} B_p,
\]

with \( B_p \) the subset of eigenvectors of \( d \) with eigenvalue \( p \). Now \( B \) has to be a crystal base with respect to \( U_1 \), and by the uniqueness theorem for crystal bases of finite-dimensional representations of \( U_q(sl_2) \) \([26]\), every \( b \in B_p \) is proportional to some vector of \( t^p \otimes B^{(1)} \mod qL) \). This proves (i).

Suppose \((L, B)\) and \((L', B')\) are two crystal bases of \( V(l; \alpha)\). Then by (i) every \( b \in B \) is proportional to some \( b' \in B' \). Take \( b_0 \in S \), \( b'_0 \in S' \) such that \( b'_0 = \alpha b_0, \alpha \in \mathbb{Q} \). By the assumption of connectedness, every \( b \in S \) can be written as \( b = P \cdot b_0 \), where \( P \) is a monomial in the variables \( \tilde{e}_i, \tilde{f}_i, i = 0, 1 \). Therefore, \( \alpha b = P \cdot b'_0 \in S' \), and (ii) is proved.

Next we note the following obvious, but important \([29]\)

Lemma 5.2. Let \( a \in \mathbb{Q} \), and \( L \) be the free \( A(q) \)-module generated by \( t^p \otimes v_k^m \), \( p \in \mathbb{Z}, 0 \leq k \leq m \). The loop module \( V(m; a) \) has a crystal base \((L, B)\) given by \( B = \{ t^p \otimes a^p v_k^m, p \in \mathbb{Z}, 0 \leq k \leq m \} \). Its crystal graph is connected and the crystal base \((L, B)\) is unique \(\mod qL) \) up to an overall multiplicative constant.

The crystal graph of \( V(m; a) \) is given on figure 1. The solid arrows correspond to the action of \( \tilde{f}_1 \), the dashed ones to \( \tilde{f}_0 \).

The remainder of this section is devoted to a study of \( V(m, n; a, b) \). We assume that \( a, b \in \mathbb{Q} \). Because \( q \) is transcendent over \( \mathbb{Q} \), this automatically implies that
$S_m(a)$ and $S_n(b)$ are in general position, and thus that $V(m, n; a, b)$ is irreducible. We begin with a simple remark. Suppose $f(q) \in \mathbb{Q}(q)$, and suppose the term in the Laurent expansion of $f(q)$ about $q = 0$ with the lowest exponent of $q$ is $aq^N$, where $a \in \mathbb{C}^\times$, $N \in \mathbb{Z}$. We abbreviate this by $f(q) \approx aq^N$. Then we have, for $k \in \mathbb{Z}_{\geq 0}$

$$[k] \approx q^{-k+1}, \quad [k]^{-1} \approx q^{k-1}. \quad (5.8)$$

![Figure 1. The crystal graph of $V(1; a)$](image)

By proposition 6 of [26], we know that the crystal base of $V_m \otimes V_n$ may be taken as $B = \{v^m_j \otimes v^n_k \mid 0 \leq j \leq m, 0 \leq k \leq n\}$. Hence the crystal base of $V(m, n; a, b)$ considered as a $U_q(sl_2)$ module is given by

$$B_1 = \{t^p \otimes v^m_j \otimes v^n_k \mid 0 \leq j \leq m, 0 \leq k \leq n, p \in \mathbb{Z}\}, \quad (5.9)$$

and the action of $\tilde{f}_1$ on $B_1$ is:

$$\tilde{f}_1 \cdot (t^p \otimes v^m_j \otimes v^n_k) = t^p \otimes v^m_{j+1} \otimes v^n_k \mod qL_1, \quad m - j > k, \quad (5.10a)$$

$$\tilde{f}_1 \cdot (t^p \otimes v^m_j \otimes v^n_k) = t^p \otimes v^m_j \otimes v^n_{k+1} \mod qL_1, \quad m - j \leq k. \quad (5.10b)$$

Denote by $L_1$ the vector space over $\mathbb{Q}(q)$ spanned by $B_1$. The generators $e_0$ and $f_0$ act on $V(m, n; a, b)$ as the operators

$$e_0 = t \otimes (a^{-1}e_1 \otimes 1 + b^{-1}t^{-1}_1 \otimes e_1), \quad f_0 = t^{-1} \otimes (a^{-1}e_1 \otimes 1 + b^{-1}t^{-1}_1 \otimes e_1). \quad (5.11)$$

**Lemma 5.3.** Let $p \in \mathbb{Z}$ and $k \in \{0, 1, \ldots, \min(m, n)\}$, and define $\omega^p_k = t^p \otimes v^m_m \otimes v^n_{n-k}$. Then

$$V(m, n; a, b) \cap \text{Ker } e_0 \ni \omega^p_k \mod qL_1. \quad (5.12)$$
Proof. Consider the linear combination

\[ \Omega^p_k = t^p \otimes \sum_{j=0}^{k} \alpha_j v_{m-j}^n \otimes v_{n+j-k}^n. \]  

We obtain after some simple manipulations

\[ e_0 \cdot \Omega^p_k = t^{p+1} \otimes \sum_{j=1}^{k} (\alpha_j a[m - j + 1]q^{-n+2k-2j}) v_{m-j+1}^m \otimes v_{n+j-k}^n. \]

Thus \( \Omega^p_k \in \text{Ker} \ e_0 \) if and only if

\[ \alpha_j a[m - j + 1]q^{-n+2k-2j} = 0, \quad \forall j, \]  

Therefore,

\[ \alpha_j = -\frac{b \ [n + j - k]}{a \ [m - j + 1]} q^{2j} \alpha_{j-1}. \]  

If we set \( \alpha_0 = 1 \) = coefficient of \( \omega^p_k \) in \( \Omega^p_k \), we get

\[ \alpha_j = (-1)^j \left( \frac{b}{a} q^{n-2k} \right)^j \prod_{l=1}^{j} \frac{[n + l - k]}{[m - l + 1]} q^{2l}, \]

and we find \( \alpha_j \approx \text{const} \ q^{j(m-k+1)}. \) \( \square \)

Lemma 5.4. Let \( p \in \mathbb{Z}, \ 0 \leq j \leq m \) and \( 0 \leq k \leq n \). Then

\[ \tilde{f}_0 \cdot (t^p v_j^m \otimes v_{n-k}^n) = t^{p-1} a^{-1} v_{j-1}^m \otimes v_{n-k}^n \mod qL_1, \quad j > k, \]  

\[ \tilde{f}_0 \cdot (t^p v_j^m \otimes v_{n-k}^n) = t^{p-1} b^{-1} v_j^m \otimes v_{n-k-1}^n \mod qL_1, \quad j \leq k. \]  

Proof. Let us start by showing (5.18a). We have

\[ f_0 \cdot (t^p \otimes v_j^m \otimes v_{n-k}^n) \]  

\[ = t^{p-1} \otimes (a^{-1} [m - j + 1] v_{j-1}^m \otimes v_{n-k}^n + b^{-1} q^{-m+2j} [k + 1] v_j^m \otimes v_{n-k-1}^n). \]

The proof goes by induction on \( j \). The case \( j = m \) follows from the previous lemma. Assume (5.18) has been proved for \( j + 1, j + 2, \ldots, m \). Then

\[ a^{j-m} t^p \otimes v_j^m \otimes v_{n-k}^n = \tilde{f}_0^{m-j} \cdot \omega_k^{p+m-j} = f_0^{(m-j)} \cdot \omega_k^{p+m-j} \mod qL_1. \]  

But (5.3) tells us that

\[ \tilde{f}_0 \cdot (t^p \otimes v_j^m \otimes v_{n-k}^n) = [m - j + 1]^{-1} f_0 \cdot (t^p \otimes v_j^m \otimes v_{n-k}^n), \]  

(5.21)
therefore the properly normalized second term of (5.19) belongs to $qL_1$, since
\[ [m - j + 1]^{-1} q^{-m+2j} [k + 1] \approx q^{j-k}. \] (5.22)

Next we prove the case $j = k$ of (5.18b). Put
\[ \Omega_j = t^p \otimes \sum_{k=0}^{j} (-1)^k \left( \frac{b}{a} \right)^k q^k v^m_{j-k} \otimes v^n_{n-j+k}. \] (5.23)

Note that $\Omega_j = t^p \otimes v^m_j \otimes v^n_{n-j} \mod qL_1$. It is easy to see that
\[ f_0 \cdot \Omega_j = t^{p-1} \otimes b^{-1} q^{-m+2j} [j + 1] v^m_j \otimes v^n_{n-j-1} \]
\[ - t^{p-1} \otimes \sum_{k=1}^{j} (-1)^k \left( \frac{b}{a} \right)^k q^k b^{-1} \beta_{jk} v^m_{j-k} \otimes v^n_{n-j+k-1}, \] (5.24)

where
\[ \beta_{jk} = q^{-1} [m - j + k] - q^{-m+2j-2k} [j - k + 1] \approx q^{-m+j-k} - q^{-m+j-k} = 0. \] (5.25)

For the coefficient of the first term of (5.24) we find $q^{-m+2j} [j + 1] \approx q^{-m+j}$. As above, (5.24) has to be normalized by multiplying it with the factor $[m - j + 1]^{-1} \approx q^{m-j}$. This proves that $f_0 \cdot \Omega_j = t^{p-1} b^{-1} v^m_j \otimes v^n_{n-j-1} \mod qL_1$.

It remains to prove (5.18b) when $j < k$. Assume it has been proved already for $k - 1, k - 2, \ldots, j$. The case $k = j$ has just been dealt with. By the induction hypothesis,
\[ b^{-k+j-m+j} t^p \otimes v^m_j \otimes v^n_{n-k} = f_0^{(k-j+m-j)} \cdot \omega_k^{p+m+k-2j} \mod qL_1. \] (5.26)

Hence the normalizing factor is $[m + k - 2j + 1]^{-1}$. Consider the two terms on the r.h.s. of (5.19). The coefficient of the first one becomes $[m + k - 2j + 1]^{-1} [m - j + 1] \approx q^{k-j}$, and the second one $[m + k - 2j + 1]^{-1} q^{-m+2j} [k + 1] \approx q^0$. \]

We can summarize the previous two lemmas as follows. Put $v^p_{jk} = t^p \otimes v^m_j \otimes v^n_k$. Then the action of the modified Chevalley generators on this base can be read from (5.10, 5.18):
\[ \tilde{f}_0 \cdot v^p_{jk} = a^{-1} v^p_{j-1,k} \quad j > n - k; \] (5.27a)
\[ \tilde{f}_0 \cdot v^p_{jk} = b^{-1} v^p_{j,k-1} \quad j \leq n - k. \] (5.27b)
\[ \tilde{f}_1 \cdot v^p_{jk} = v^p_{j+1,k} \quad m - j > k; \] (5.28a)
\[ \tilde{f}_1 \cdot v^p_{jk} = v^p_{j,k+1} \quad m - j \leq k. \] (5.28b)

Because there are structure constants different from 1, we call it the pseudo-crystal base. Its importance stems from the fact that each vector of any crystal base must be proportional to some $v^p_{jk}$ by Lemma 5.1(i). We also define in the obvious way the notion of pseudo-crystal graph. To illustrate this, we give in figure 2 the pseudo-crystal graph of $V(3, 2; a, b)$. The graph itself can be thought of as being 3-dimensional, the two horizontal dimensions corresponding to the indices $j, k$, and the vertical dimension to $p$. The figure is the projection of the graph on the $p = 0$ plane. The solid arrows correspond to the action of $\tilde{f}_1$, the dashed ones to $\tilde{f}_0$, which has of course also a vertical component not shown here.
Definition 9. Let \( j, k \in \mathbb{Z} \) be such that \( 0 \leq j \leq m, \ 0 \leq k \leq n \). We say that \( V(m, n; a, b) \) has an escalator at \((j, k)\), if for every \( p \in \mathbb{Z} \), \( \tilde{f}_0 \tilde{f}_1 \cdot v_{jk}^p = c v_{jk}^{p-1} \), where \( c \) is either \( a^{-1} \) or \( b^{-1} \). If there exists \((j, k)\) such that \( V(m, n; a, b) \) has an escalator at \((j, k)\), but we do not want to specify the value(s) of \((j, k)\), then we just say that \( V(m, n; a, b) \) has an escalator.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure2.png}
\caption{\( V(3, 2; a, b) \)}
\end{figure}

Lemma 5.5. \( V(m, n; a, b) \) has an escalator if and only if \( m \neq n \). More precisely: if \( m > n \), for every \( k \in \{0, \ldots, n\} \), there exists \( j \) such that \( V(m, n; a, b) \) has an escalator at \((j, k)\), and if \( m < n \), for every \( j \in \{0, \ldots, m\} \), there exists \( k \) such that \( V(m, n; a, b) \) has an escalator at \((j, k)\).

Proof. There are two possibilities for an escalator: either

\begin{align}
\tilde{f}_1 \cdot v_{jk}^p &= v_{j+1,k}^p & m - j > k \tag{5.29a} \\
\tilde{f}_0 \cdot v_{j+1,k}^p &= a^{-1} v_{jk}^{p-1} & j + 1 > n - k \tag{5.29b}
\end{align}

or

\begin{align}
\tilde{f}_1 \cdot v_{jk}^p &= v_{j,k+1}^p & m - j \leq k \tag{5.30a} \\
\tilde{f}_0 \cdot v_{j,k+1}^p &= b^{-1} v_{jk}^{p-1} & j \leq n - k - 1 \tag{5.30b}
\end{align}

In order for the first one, (5.29), to be realized, we must have \( n - k - 1 < j < m - k \), so that \( m > n \). For the second one, (5.30), we must have \( m - j \leq k \leq n - j - 1 \), hence \( m < n \). Conversely, if \( m \neq n \), it is clear that we can find \( j, k \) satisfying the conditions in (5.29) or (5.30).
The escalators are easy to recognize on the projection of the pseudo-crystal graph. They correspond to the pairs of vertices in the projection which are connected by two arrows of opposite direction, one coming from $\tilde{f}_1$ and the other from $\tilde{f}_0$. In figure 3, we give the projections of the pseudo-crystal graphs of $V(2, 2; a, b)$, $V(2, 1; a, b)$ and $V(4, 2; a, b)$.

For $0 \leq j \leq m$, $0 \leq k \leq n$, we define the quantity $d_{jk} = \min(k, m - j)$. Then
for each \( p \in \mathbb{Z} \), \( N \in \{0, \ldots, \min(m,n)\} \), the vectors \( v_{jk}^p \) such that \( d_{jk} = N \) form the crystal base of an irreducible submodule of the algebra generated by \( e_1 \) and \( f_1 \).  

**Theorem 6.** Let \( L \) be the free \( A(q) \)-module generated by \( v_{jk}^p \), with \( p \in \mathbb{Z} \), 0 \( \leq \) \( j \leq m \), 0 \( \leq k \leq n \).

(i) If \( m = n \) and \( \sqrt{b/a} \in \mathbb{Q} \), then the vectors 

\[
\begin{align*}
    b_{jk}^{p,\epsilon} &= e^{p+d_{jk}}(ab)^{p/2}(b/a)^{d_{jk}/2}v_{jk}^p,
\end{align*}
\]

where \( \epsilon = \pm 1 \), define two crystal bases \((L, B^{(\pm 1)})\) and \((L, B^{(-1)})\) of \( V(m,n;a,b) \), which are not proportional.

(ii) If \( m \neq n \), then there exists no crystal base of \( V(m,n;a,b) \) unless \( a = b \), in which case the vectors 

\[
    b_{jk}^p = a^p v_{jk}^p,
\]

define a crystal base \((L, B)\), and every other crystal base is proportional to \((L, B)\).

**Proof.** The remark about \( d_{jk} \) preceding the theorem proves that \( B^{(\pm 1)} \) are stable under \( \tilde{e}_1, \tilde{f}_1 \). Assume \( j > m - k \). Then

\[
\begin{align*}
    \tilde{f}_0 \cdot b_{jk}^{p,\epsilon} &= e^{p+d_{jk}} a^{-1}(ab)^{p/2}(b/a)^{d_{jk}/2}v_{j-1,k}^{p-1},
\end{align*}
\]

and \( d_{j-1,k} = d_{jk} + 1 \). Indeed, \( d_{j-1,k} = \min(k, m-j+1) \), but \( k < m-j+1 \) together with \( j > m-k \) implies \( m-j < k < m-j+1 \), which is impossible for \( k \in \mathbb{Z} \). Thus \( k \geq m - j + 1 > m - j \), and \( d_{j-1,k} = m - j + 1 = \min(k, m-j) + 1 = d_{jk} + 1 \). From this it follows that the r.h.s. of (5.33) is equal to \( b_{j-1,k}^{p-1,\epsilon} \). The case \( j \leq m - k \) is similar. This proves (i).

Now we show (ii). By Lemma 5.5, \( V(m,n;a,b) \) has \( \min(m,n) \) escalators. Consider the subgraph \( S_{jk} \) of the pseudo-crystal graph corresponding to the escalator at \((j,k)\). This subgraph coincides with the crystal graph of \( V(1;a')\). Thus if \( V(m,n;a,b) \) has a crystal base \((L,B)\), with the decomposition (5.7) into eigenvectors of \( d \), and if \( b \in B_p \) is a vertex of \( S_{jk} \), then Lemma 5.1 and Lemma 5.2 imply

\[
    b = \alpha_{jk} x^p v_{jk}^p,
\]

where \( x = a \) or \( b \) according to whether \( m > n \) or \( m < n \), and \( \alpha_{jk} \in \mathbb{Q} \) is independent of \( p \). Next, observe that (5.34) holds for every \( b \in B_p \) such that \( b \) is proportional to \( v_{jk'}^p \) with \( d_{jk'} = d_{jk} \), as \( B \) must be crystalline for \( \tilde{e}_1, \tilde{f}_1 \). Hence there are constants \( \alpha_i \in \mathbb{Q}, i = 1, \ldots, \min(m,n) \), such that for every \( b \in B_p \), \( b = \alpha_i x^p v_{jk}^p \) with \( i = d_{jk} \).

Suppose now \( m > n \), and let us concentrate our attention on the last two rows \( k = n - 1, n \) of the projection of the pseudo-crystal graph. The reader may well look at the examples on the figures above, which illustrate our arguments. The left side of these two rows contains the two solid lines corresponding to the vertices with \( d_{jk} = n-1 \) and \( n \). They are connected on the last row by a horizontal dashed arrow corresponding to the action of \( \tilde{f}_0 \). The effect of \( \tilde{f}_0 \) in this case is given by
exists a vertex of to the SE corner of degree Thus, we have shown that because the escalator now prove that it is also a necessary condition, by showing that, if there exists a monomial \( P \) be an integrable \( U_q(\widehat{sl}_2) \) module, \( v \in M \), then
\[
\hat{e}_i \cdot v = C_i(v, q) e_i \cdot v, \quad \hat{f}_i \cdot v = D_i(v, q) f_i \cdot v, \quad i = 0, 1,
\]
where \( C_i(v, q), D_i(v, q) \in \mathbb{Q}(q) \). This follows from comparing (5.3) with (2.2). Thus if \( M \) has a crystal base \( (L, B) \), \( u \in B \), then \( \hat{e}_i \hat{f}_j \cdot u = \epsilon \hat{f}_j \hat{e}_i \cdot u \), where \( i \neq j \) and \( \epsilon \in \{0, 1\} \), in view of (2.3). Now if \( \epsilon = 0 \), then (5.35) implies \( f_j \cdot u \in \text{Ker} e_i \), and \( \hat{f}_j \hat{e}_i \cdot u = C(u, q) f_j e_i \cdot u = C(u, q) e_i f_j \cdot u = 0 \), with \( C(u, q) \in \mathbb{Q}(q) \). Hence we have shown:
\[
\hat{e}_i \hat{f}_j \cdot u = \hat{f}_j \hat{e}_i \cdot u, \quad u \in B, \quad i \neq j. \tag{5.36}
\]

**Theorem 7.**

(i) The crystal graph of \( V(m, n; a, b) \) is connected if and only if \( m \neq n \).

(ii) If \( m = n \), it has two components.

**Proof.** Suppose \( m \neq n \). Then by Lemma 5.5, \( V(m, n; a, b) \) has an escalator, and looking at (5.29, 5.30) we see that its crystal graph has an infinite subgraph \( \mathcal{E} \) of the form
\[
\cdots \rightarrow b^p_{j,k} \rightarrow b^p_{j+1,k} \rightarrow b^{p-1}_{j,k} \rightarrow b^{p-1}_{j+1,k} \rightarrow \cdots \tag{5.37}
\]
or
\[
\cdots \rightarrow b^p_{j,k} \rightarrow b^p_{j,k+1} \rightarrow b^{p-1}_{j,k} \rightarrow b^{p-1}_{j,k+1} \rightarrow \cdots \tag{5.38}
\]
To prove the connectedness of the crystal graph, it is enough to show that there exists a path from every vertex \( b^p_{j,k} \) to \( \mathcal{E} \). This last fact is proved as follows. If \( j > 0 \), \( b^p_{j,k'} = (\hat{f}_1)^{2j+k-m} b^p_{0,m-j} \), since \( b^p_{0,m-j} \in \text{Ker} e_1 \mod qL \), and similarly, \( b^p_{0,m-j} = (\hat{f}_0)^{2n-m+j} b^{p+2n-m+j}_{mn} \). Therefore, \( \forall p \in \mathbb{Z} \), the vertex \( b^p_{j,k'} \) is connected to the SE corner of degree \( p + 2n - m + j \), i.e. \( b^{p+2n-m+j}_{mn} \). On the other hand, because the escalator \( \mathcal{E} \) is infinite, and contains vertices of all degrees \( p \in \mathbb{Z} \), there exists a vertex of \( \mathcal{E} \) connected to the same SE corner of degree \( p + 2n - m + j \). Thus, we have shown that \( m \neq n \) is a sufficient condition for connectedness. We now prove that it is also a necessary condition, by showing that, if \( m = n \), \( \forall p \in \mathbb{Z} \), \( \forall j, k \), the vertices \( b^p_{j,k} \) and \( b^{p-1}_{j,k} \) are not connected. Assume for a contradiction that there exists a monomial \( P(\hat{e}_i, \hat{f}_i), i = 0, 1 \), such that
\[
b^{p-1}_{j,k} = P(\hat{e}_i, \hat{f}_i) b^p_{j,k}. \tag{5.39}
\]
Consider the length of the word \( P \), i.e. the number of its letters \( \tilde{e}_i, \tilde{f}_i \). Let \( P_0 \) be the monomial of minimal length satisfying (5.39). Then \( P_0 \) is a function of the letters \( \tilde{f}_i \) only, not the \( \tilde{e}_i \). Indeed, suppose that \( P_0 \) involves both \( \tilde{e}_i \) and \( \tilde{f}_i \). Then by virtue of (5.36) above, one can permute the factors \( \tilde{f}_i \) and \( \tilde{e}_j \) for \( i \neq j \), without changing the length, until one of the sequences \( \tilde{e}_i \tilde{f}_i, \tilde{f}_i \tilde{e}_i, i = 0, 1 \) appears. (Such a sequence is bound to appear, for the only possibility to avoid this is when \( P_0 = \tilde{e}_n \tilde{f}_n \) or \( \tilde{e}_m \tilde{f}_m \). (5.40))

But this is forbidden by (5.39): the eigenvalue of \( t_1 \) acting on \( b_{jk}^p \) and \( b_{jk}^{p-1} \) being the same, we must have \( t_1 P_0 t_1^{-1} = P_0 \), while this does not hold in (5.40).)

Now \( \tilde{e}_i \tilde{f}_i = \tilde{f}_i \tilde{e}_i = 1 \), and hence we get a monomial satisfying (5.39) shorter than \( P_0 \), contradicting minimality. Taking into account the fact that there is a difference of 1 in the degrees \( p \) and \( p - 1 \) of the vectors in (5.39), so that we must also have \([d, P_0] = -P_0\), which means that \( P_0 \) contains \( \tilde{f}_0 \) with multiplicity 1, we are left with the two possibilities \( P_0 = \tilde{f}_0 \tilde{f}_1, \tilde{f}_1 \tilde{f}_0 \). But these are also ruled out by Lemma 5.5.

It remains to prove (ii). We do this by studying \( G_m \), the crystal graph of \( V(m, m; a, b) \). More precisely we will show that \( \forall p \in \mathbb{Z}, \forall j, k \), the vertices \( b_{jk}^p \) and \( b_{jk}^{p-2} \) are connected. This implies the statement in the theorem, as the projection of the crystal graph on the \( p = 0 \) plane is connected.

The case \( m = 1 \) is obvious by looking at \( G_1 \). Assume it has been shown for \( G_m \). To prove it for \( G_{m+1} \), it is enough to show that every one of the vertices of \( G_{m+1} \setminus G_m \) has the required property. But this is clear from an inspection of the graph.  

6. Eigenstates of the XXZ Hamiltonian

6.1 Basic constructions. The XXZ Hamiltonian is formally defined as

\[
H_{XXZ} = -\frac{1}{2} \sum_{k=-\infty}^{\infty} \left( \sigma_{k+1}^x \sigma_k^x + \sigma_{k+1}^y \sigma_k^y + \Delta \sigma_{k+1}^z \sigma_k^z \right),
\]

where \( \sigma_k^x, \sigma_k^y, \sigma_k^z \), etc., are Pauli matrices acting at site \( k \), and the four-fermion coupling constant \( \Delta \) is related to the quantum algebra parameter \( q \) by \( \Delta = (q + q^{-1})/2 \). Similarly, the generator of the (two-sided) corner transfer matrix has the formal definition

\[
H_{CTM} = -\frac{q}{1 - q^2} \sum_{k=-\infty}^{\infty} k \left( \sigma_{k+1}^x \sigma_k^x + \sigma_{k+1}^y \sigma_k^y + \Delta \sigma_{k+1}^z \sigma_k^z \right).
\]

It is shown in [21] that the left-hand half \( (k > 0) \) of the CTM may be identified with the derivation \( d \) of \( U_q(\widehat{sl}_2) \), acting on a standard level-1 highest weight module. There are two such modules, \( V(\Lambda_0) \) and \( V(\Lambda_1) \), which correspond to the
two possible choices of boundary condition in the anti-ferromagnetic regime. The right-hand half \((k < 0)\) of the CTM acts similarly, but in a dual module, since the eigenvalues of this half of \(H_{\text{CTM}}\) are negated. Let \(T\) be a shift by one lattice unit. Then the naive definitions of \(H_{\text{XXZ}}\) and \(H_{\text{CTM}}\) make natural the identification

\[
\frac{q}{1 - q^2} H_{\text{XXZ}} = T \cdot H_{\text{CTM}} \cdot T^{-1} - H_{\text{CTM}} = T \cdot d \cdot T^{-1} - d.
\]  

Given this, the space of states involves the level-0 modules \(V(\Lambda_i) \otimes V^*(\Lambda_j), (i, j = 0, 1)\), i.e., loop modules.

We turn to the necessary definitions for the vertex operators (VOs). They are intertwiners of \(U_q(\widehat{sl}_2)\) modules. The appropriate VOs for \(H_{\text{XXZ}}\) intertwine the level-1 standard modules and the loop module \(V(1; a)\). In the sequel we use the notations of [12], thus we denote the variable \(t\) by \(z\) and the module \(V(1; a)\) by \(V_{za}\). There are two basic types.

(i) Type I:

\[
\tilde{\Phi}^V_\lambda (z) : V(\lambda) \rightarrow V(\mu) \otimes V_z, \\
\tilde{\Phi}^\lambda_\mu (z) : V(\mu) \otimes V_z \rightarrow V(\lambda).
\]  

(ii) Type II:

\[
\tilde{\Phi}^V_\lambda (z) : V(\lambda) \rightarrow V_z \otimes V(\mu), \\
\tilde{\Phi}^\lambda_\mu (z) : V_z \otimes V(\mu) \rightarrow V(\lambda).
\]  

The existence and the uniqueness of such vertex operators, once the overall normalisation is fixed, is proved in [28] in a general setting. Strictly speaking the image is a completion of the appropriate space, but we shall not worry about this. The grading is preserved, with the derivation acting on the tensor products as \(d \otimes 1 + 1 \otimes d\).

The translation operator \(T\) uses the type-I VOs, but as \(U'_q(\widehat{sl}_2)\) intertwiners (formally, set \(z = 1\)). Type-I VOs preserve the crystal base, a property which is important in making the physical connection between translation and the action of these VOs [22]. Explicitly, a shift to the right by one lattice unit is the composition of maps:

\[
T : V(\lambda) \otimes V^*(\lambda') \xrightarrow{\Phi^V_\lambda \otimes \text{id}} V(\mu) \otimes V \otimes V^*(\lambda') \xrightarrow{\text{id} \otimes \Phi^\lambda_{\mu'}} V(\mu) \otimes V^*(\mu')
\]  

Note that translation by one lattice site changes the boundary condition.

6.2 Particle states. In [12], the vacuum state is identified as the canonical element of \(V(\Lambda_0) \otimes V^*(\Lambda_0)\) corresponding to the identity map of \(\text{Hom}(V(\Lambda_0), V(\Lambda_0))\):

\[
W_0 = \sum_k u_k \otimes u^*_k.
\]  

Here \(\{u_k\}\) is a basis of \(V(\Lambda_0)\), and \(\{u^*_k\}\) the dual basis of \(V^*(\Lambda_0)\). \(W_0\) is a trivial one-dimensional \(U_q(\widehat{sl}_2)\) module, whose action is given by the co-unit \(\varepsilon\).
The creation operator $\varphi^*_{\pm}(z)$ is defined by the action

$$
\varphi^*_{\pm}(z) \cdot (u \otimes u^*) = \tilde{\Phi}^\mu_{V\lambda}(z)(v_\pm \otimes u) \otimes u^*.
$$

Plainly, $\tilde{\Phi}^\mu_{V\lambda}(z) \otimes id$ is an intertwining operator

$$
\tilde{\Phi}^\mu_{V\lambda}(z) \otimes id : V_z \otimes V(\Lambda_1) \otimes V^*(\Lambda_0) \longrightarrow V(\Lambda_{1-i}) \otimes V^*(\Lambda_0).
$$

We already noted that the trivial module $W_0$ is embedded canonically in $V(\Lambda_0) \otimes V^*(\Lambda_0)$. Now remember that the intertwiner $\tilde{\Phi}^\mu_{V\lambda}(z)$ is really a formal Laurent expansion: $\tilde{\Phi}^\mu_{V\lambda}(z)(v_+ \otimes u) = \sum_n z^n \tilde{\Phi}^\mu_{V\lambda,n}(z^n v_\pm \otimes u)$. Therefore, from (6.9) and the properties of the co-unit, we see that the one-particle states $\varphi^*_{\pm}(z) \cdot W_0$ are the image under $(\tilde{\Phi}^\lambda_{V\Lambda_0}(z) \otimes id)$ of $V_z \otimes W_0$, since $\{z^n \otimes v_\pm\}$ is the basis of $V_z$. $V_z$ is irreducible and the map $\tilde{\Phi}^\lambda_{V\Lambda_0}(z) \otimes id$ is a module homomorphism, so the one-particle states form a module which we identify with $V_z$.

Now we iterate this process: the two-particle states are the image under $\tilde{\Phi}^\lambda_{V\Lambda_1}(z) \otimes id$ of $V_z \otimes V_z \otimes W_0$ and may be identified with the tensor product $V_z \otimes V_z$. We know that such tensor products are infinitely reducible, with quotients $V(1, 1; s, 1)$, $s \in \mathbb{C}^\times$. Therefore the 2-particle states are also infinitely reducible, and similarly for general $N$. The complex number $s$ corresponds to the relative momentum of the particles via $s = \exp(i(p_1 - p_2))$. In the Lie algebra case, it is shown in [24] that the variables $a_i$ must be unimodular in order that the modules be unitarizable, corresponding to real momenta.

### 6.3 Crystal base

It was conjectured in [12] that the n-particle states, which are a set of linear maps in $\text{Hom}(V(\Lambda_i), V(\Lambda_j))$, preserve the crystal lattice, even though the type-II VOs which are employed in their definition do not enjoy this property. In fact, the space of states $V_{\lambda,\lambda'}$ is defined in section 7.1 of [12] using this assumption and it is observed that $W_0 \in V_{\lambda,\lambda}$. Apart from the technical consideration, one expects that a reasonable physical theory should have this property since the conventional approach to the calculation of physical quantities such as correlation functions is that they are the infinite limit of a low-temperature expansion — here an expansion in the variable $q$ about $q = 0$. The theory of the crystal base is precisely about how one might make such expansions even though the underlying field, $Q(q)$, contains elements with arbitrarily large negative power of $q$.

Here we shall obtain some further partial results, extending the proof to 1-particle states and presenting further evidence to support the conjecture in general. We recall some results from [12].

**Lemma 6.1.** Let $T_\lambda \in \text{End}(V(\lambda))$ denote the linear map

$$
T_\lambda u_\nu = q^{2\rho,\lambda-\nu} u_\nu, \quad u_\nu \in V(\lambda)_\nu.
$$

Then the lower crystal lattice $L^*(\lambda)$ of $V^*(\lambda)$ is characterised as

$$
L^*(\lambda) = \{ u \in V^*(\lambda) | \langle u, T_\lambda L(\lambda) \rangle \} \in A(q).
$$
Proof. This result is proved in [12], proposition 6.2, for the upper crystal lattice. For the lower crystal lattice one simply uses (5.4b). □

This has an implication for the vacuum state. If one chooses the basis \{u_k\} in (6.7) to be also a crystal base of \(V(\Lambda_0)\), then the elements of the dual basis provide a crystal base when multiplied by appropriate powers of \(q\): \(b_k^* = q^{-(2\rho,\lambda - \nu_k)}u_k^*\). So

\[
W_0 = \sum_k q^{(2\rho,\lambda - \nu_k)}b_k \otimes b_k^*.
\] (6.12)

One sees that the crystal base of \(W_0\) is simply \(b_{\Lambda_0} \otimes b_{\Lambda_0}^*\). To treat the one-particle states \(\varphi_{\pm}(z) \cdot W_0\), we need a preliminary idea.

**Lemma 6.2.** Let \(L(\lambda)\) be the lower crystal lattice of \(V(\lambda)\), and let \(\tilde{\Phi}_{V_{\lambda}}^\mu(z) : V_z \otimes V(\lambda) \rightarrow V(\mu)\) be a type-II vertex operator, with respect to the lower coproduct. Then for any \(u \in T_{\lambda}L(\lambda)\), we have \(\tilde{\Phi}_{V_\lambda}^\mu(z) \cdot (v_{\pm} \otimes u) \in L(\mu)\).

**Proof.** Without loss of generality we assume that \(u\) is a weight vector of weight \(\nu\). We use induction on the height \((2\rho, \lambda - \nu)\), noting that \((2\rho, \lambda - \nu) = m_0 + m_1\) when \(\lambda - \nu = m_0\alpha_0 + m_1\alpha_1\). We need only consider the case of \(\tilde{\Phi}_{V_{\lambda}}^{\Lambda_1}(z)\); the intertwiner \(\tilde{\Phi}_{V_{\lambda}}^{\Lambda_1}(z)\) may be obtained from it by a Dynkin diagram automorphism. The case that the height is zero involves only the highest weight vector \(u_{\Lambda_0}\). Consider the images \(e_i \cdot \tilde{\Phi}_{V_{\lambda}}^{\Lambda_1}(z)(v_{\pm} \otimes u_{\Lambda_0}) = \tilde{\Phi}_{V_{\lambda}}^{\Lambda_1}(z)(e_i \cdot v_{\pm} \otimes u_{\Lambda_0})\) by the intertwining property. Substituting for \(e_i \cdot v_{\pm}\) in each case, we obtain the equations

\[
e_0 \cdot \tilde{\Phi}_{V_{\lambda}}^{\Lambda_1}(z)(v_+ \otimes u_{\Lambda_0}) = z\tilde{\Phi}_{V_{\lambda}}^{\Lambda_1}(z)(v_- \otimes u_{\Lambda_0}),
\]

\[
e_0 \cdot \tilde{\Phi}_{V_{\lambda}}^{\Lambda_1}(z)(v_- \otimes u_{\Lambda_0}) = 0,
\]

\[
e_1 \cdot \tilde{\Phi}_{V_{\lambda}}^{\Lambda_1}(z)(v_+ \otimes u_{\Lambda_0}) = 0,
\]

\[
e_1 \cdot \tilde{\Phi}_{V_{\lambda}}^{\Lambda_1}(z)(v_- \otimes u_{\Lambda_0}) = \tilde{\Phi}_{V_{\lambda}}^{\Lambda_1}(z)(v_+ \otimes u_{\Lambda_0}),
\]

which implies that each weight component of the image is in a spin one-half \(U_q(sl_2)\) sub-module with respect to both \(i = 0\) and \(i = 1\). Therefore the modified Chevalley generators \(\tilde{f}_i\) have the same action as the \(f_i\) on the image, which proves the initial step of the induction by (5.4a).

For the inductive step, consider an arbitrary weight vector \(u \in q^{(2\rho,\lambda - \nu)}L(\Lambda_0)_{\nu}\), whose height is one more than the current level of the induction. Then \(u = \tilde{f}_i u'\) for some \(i\) and some \(u'\), and the latter has the decomposition \(u' = \sum_{j \geq 0} f_i^{(j)} u_j, u_j \in \text{Ker}(e_i) \forall j\). So \(u = \sum_{j \geq 0} f_i^{(j+1)} u_j\), and it is sufficient to consider the case that there is only one term in the sum, i.e., \(u' = f_i^{(j)} u_j\), so that \(f_i u' = [j+1] u\). The crucial point is that

\[
q^{-1} u' \in q^{(2\rho,\lambda - \nu')}L(\Lambda_0)_{\nu'}.
\] (6.14)

From the intertwining property \(f_i \cdot \tilde{\Phi}_{V_{\lambda}}^{\Lambda_1}(z)(v_{\pm} \otimes u') = \tilde{\Phi}_{V_{\lambda}}^{\Lambda_1}(z)(f_i \cdot v_{\pm} \otimes u') + \tilde{\Phi}_{V_{\lambda}}^{\Lambda_1}(z)(t_i \cdot v_{\pm} \otimes f_i \cdot u')\), which implies

\[
\tilde{\Phi}_{V_{\lambda}}^{\Lambda_1}(z)(v_{\pm} \otimes u) = \frac{f_i \cdot \tilde{\Phi}_{V_{\lambda}}^{\Lambda_1}(z)(v_{\pm} \otimes u') - \tilde{\Phi}_{V_{\lambda}}^{\Lambda_1}(z)(f_i \cdot v_{\pm} \otimes u')}{q^{1[j+1]}},
\] (6.15)
The induction hypothesis applies to the right hand side. Moreover, \( e_i^{j+1} u' = 0 \) and either \( e_i v_\pm = 0 \) or \( e_i^2 v_\pm = 0 \). By the intertwining property,

\[
e_i^M \cdot \Phi_{\Lambda_0}^A(z)(v_\pm \otimes u') = 0,
\]

\[
\begin{cases}
M = j + 1 & \text{if } e_i v_\pm = 0, \\
M = j + 2 & \text{if } e_i^2 v_\pm = 0.
\end{cases}
\] (6.16)

Write \( \Phi_{\Lambda_0}^A(z)(v_\pm \otimes u') = \sum_n w_n \). It is easy to see that \( (f_i/[j + 1]) \cdot w_n \approx q^{j-M+1} f_i \cdot w_n \), giving

\[
(f_i/[j + 1]) \cdot \Phi_{\Lambda_0}^A(z)(v_\pm \otimes u') \approx q^{j+1-M} f_i \cdot \Phi_{\Lambda_0}^A(z)(v_\pm \otimes u').
\] (6.17)

Similar considerations apply to the other term on the right hand side of (6.14). The result follows immediately.

From (6.12) and the preceding lemma, it follows trivially that

**Theorem 8.** The one-particle states \( \varphi_{\pm}(z) \cdot W_0 \in \text{Hom}(V(\Lambda_0), V(\Lambda_1)) \) preserve the crystal lattice.

One sees from the foregoing the difficulty in proceeding further. Formally any state is a sum of the form

\[
\Psi(z_1, \cdots, z_n) = \sum_{k,l} c_{k,l}(z_1, \cdots, z_n) b_k^* \otimes b_l^*
\] (6.18)

and we want to show that \( c_{k,l}(z_1, \cdots, z_n) \in A(q) \). These coefficients are obtained from the definition (6.8) as infinite sums, with individual terms involving arbitrary negative powers of \( q \). We expect that they should be analytic functions in some annulus which contains the physical region \( |z_1| = \cdots = |z_n| = 1 \). But, as discussed in [12], the formal definition of the necessary matrix elements is by meromorphic continuation from the region \( |z_1| \gg \cdots \gg |z_n| \), around poles at \( z_i/z_{i+1} = q^{1-2} \).

Some method to perform the continuation is required for further progress to be made. Probably the necessary information is contained in the fact that the creation-annihilation operators provide a lattice realization of the Zamolodchikov algebra. For example, the creation operators are known to satisfy (eqn.(7.10a) of [12])

\[
\varphi_{\epsilon_1}^*(z_1) \varphi_{\epsilon_2}^*(z_2) = \sum_{\epsilon_1^1 \epsilon_2^2} (R_{\epsilon_1^1 \epsilon_2^2} \cdot V^*(z_1/z_2)) \epsilon_1^2 \epsilon_2 \varphi_{\epsilon_1^1}^*(z_2) \varphi_{\epsilon_2^2}^*(z_1)
\] (6.19)

where the \( R \) matrix (eqn.(6.18) of [12]) is meromorphic in the annulus \( q^2 < |z| < q^{-2} \). Moreover, the information about analyticity comes from general considerations and is needed to compute the normalisation of the \( R \) matrix — which contains most of the analytic information — by the method of Frenkel and Reshetikhin [20]. The commutation relations in turn place severe constraints on the coefficients appearing in (6.18), possibly sufficiently strong to allow a proof of the conjecture about preservation of the crystal base for arbitrary \( n \). But at present we are unable to find a way to construct such a proof.
7. Conclusions

This paper has been concerned mainly with the study of loop modules of the quantum algebra $U_q(\widehat{sl}_2)$. The motivation for this study is the importance of representation theory in the solution of two-dimensional lattice models of statistical mechanics. The simplest loop modules — the affinization of an irreducible $U_q(sl_2)$ module — are of central importance in the papers cited in the introduction and are extensively studied in [29]. Nevertheless, some important and interesting questions are raised even in the solution of the six-vertex model. In [12] one sees that new features emerge in the $n$-particle sector with $n \geq 2$. The $n$-particle sectors are built on a certain symmetrisation, under the $R$-matrix symmetry, of the tensor product of $n$ copies of a simple $U_q(\widehat{sl}_2)$ module $V$. Unlike the 1-particle sector, an understanding of the structure of these sectors at $q = 0$ left some unexplained puzzles.

In the present paper we have defined rather general quantum loop modules, although certainly not the most general possible. In this regard our choice is dictated by the physically interesting questions raised by the cited works. Since the $U_q(\widehat{sl}_2)$ modules constructed herein are also $U'_q(\widehat{sl}_2)$ modules, it is natural that our treatment should parallel the works of Chari and Pressley to some extent. In fact, the conditions we find for the modules to be cyclic are exactly those of the corresponding Lie algebra case; similarly the conditions for irreducibility are the same as for $U'_q(\widehat{sl}_2)$ modules. One sees however that there is a sharp distinction between the irreducibility properties of loop modules constructed by affinizing a tensor product of irreducible $U_q(sl_2)$ modules $V_1, \cdots, V_n$, and the tensor product of the affinizations of the individual $V_j$. The latter is highly reducible.

The crystal base theory of $U_q(\widehat{sl}_2)$ modules has great significance in the physical interpretation of the theory. It is the connecting link between the use of representation theory and the traditional method of low-temperature expansions. For this reason we have studied the next simplest case after $V(m; a)$ in some detail. In particular, the modules $V(m, n; a, b)$ do not have a crystal base in general unless $m = n$. In the latter case, the crystal graph is also not connected, although the module is irreducible. Presumably these properties extend to general loop modules $V(l; a)$. This would be entirely consistent with the findings of Reshetikhin [30] (using the Bethe Ansatz) and Idzumi et.al. [13] (using the quantum affine symmetry) for the higher spin generalization of the six-vertex model. The important point is that even though the basic representation are all spin $k/2$, $k \geq 2$, the elementary excitations of the system are all spin $1/2$ particles.

Section 6 also contains incomplete results. Nevertheless considerable new light is thrown onto the solution of the six-vertex model using the quantum affine symmetry, presented in [12]. Here we just recall our results for the two-particle sector. We saw in section 6 that it is isomorphic to a tensor product $V_z \otimes V_z$, which is highly reducible into copies of $V(1, 1; s, 1), s \in \mathbb{C}^\times$. Physically, this means that states of given total momentum $s = s_1s_2$ — recall that $s_j = \exp(\text{i}u_j), u \in \mathbb{R}$ — still are infinitely degenerate, with internal momentum given by the ratio $s = s_1/s_2$. This fact is exactly mirrored in the possibilities for constructing the crystal base.
It is well-known that the tensor product of irreducible finite-dimensional $U'_q(\hat{sl}_2)$ modules is irreducible. But the crystal base for the $n$-particle sectors which was proposed in [12] has an infinite number of disconnected components, which correspond to only the fraction $1/n!$ of the number of such components which might be expected from taking the tensor product of $n$ copies of the crystal base for $V_z$. There is no contradiction here: it simply illustrates the completely different reducibility properties of the tensor product of loop modules.

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