On the stability of redundancy models

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Load-balancing strategies:

\[ \mu_1 \quad \mu_2 \quad \ldots \quad \mu_K \]

Exploit variability in the workload in different queues!
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Redundancy-d: A job is dispatched into several servers.

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Exploit variability in the workload in different queues!
Positive aspect: Exploits variability in the workload.

Negative aspect: There is additional workload added to the system.
Previous work

Theorem

Assume FCFS service policy and all the copies of a job are i.i.d. The system is stable $\iff \lambda < \mu K$.

[Gardener et al.]

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1Kristen Gardner, Samuel Zbarsky, Sherwin Doroudi, Mor Harchol-Balter, Esa Hyytiä, and Alan Scheller-Wolf. 2016. Queueing with redundant requests: exact analysis. Queueing Systems 83, 3-4 (2016), 227–259
Main objectives

Determine how the stability condition is impacted by:

- The scheduling policy implemented in the $K$ servers.
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Determine how the stability condition is impacted by:

- The scheduling policy implemented in the $K$ servers.
- The possible correlation between the $d$ copies of the same job.
1 Model description

2 I.i.d copies
   - PS service policy
   - ROS service policy
   - Priority policy

3 Identical copies
   - PS service policy
   - FCFS service policy
   - ROS service policy

4 Numerical results

5 Conclusions
Model description

- $K$ servers with capacity 1.
- Poisson arrivals with rate $\lambda$.
- Exponential service times with parameter $\mu$. 
Each arrival chooses \( d \) servers at random, \( s_1, \ldots, s_d \).

This job is said to be of type \( c = \{ s_1, \ldots, s_d \} \).

The set of types:

\[
\mathcal{C} := \{ c = \{ s_1, \ldots, s_d \} \subset S : s_i \neq s_j \ \forall i \neq j \} \text{ and } |\mathcal{C}| = \binom{K}{d}.
\]

Arrivals of type \( c \) at rate \( \lambda_c = \frac{\lambda}{\binom{K}{d}} \).
Arrival rate to a server $s$ is $\frac{d}{K}\lambda$.

Departure in server $s$ due to:

- Local copy has completed service.
- A copy of a job in the local queue has completed service in an other server.
The number of type-$c$ jobs at time $t$ is given by $N_c(t)$ and

$$\vec{N}(t) = (N_1(t), \ldots, N_{|C|}(t)) \in \mathbb{Z}_+^K$$
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\]

The number of copies in server $s$ at time $t$ is given by $M_s(t) = \sum_{c \in C(s)} N_c(t)$ and
\[
\vec{M}(t) = (M_1(t), \ldots, M_K(t)) \in \mathbb{Z}^+_K
\]
Service policies we consider:

- **PS (Processor Sharing):** service is equally shared among the copies in a server.
- **FCFS:** copies are served in order of arrival.
- **ROS (Random Order of Service):** An empty server picks a copy to serve at random.
- **Priority policy:** In each server, a priority law is fixed among the types it can serve.
We consider copies of a job to be:

1. **i.i.d copies.**

2. **Identical copies:** All $d$ copies of a job are identical replicas and have the same service time.
### Main results

#### Table: Summary of stability conditions

| Priority policy | PS     | FCFS   | ROS     | Priority policy |
|-----------------|--------|--------|---------|-----------------|
| i.i.d           | $\lambda < \mu K$ | $\lambda < \mu K$ | $\lambda < \mu K$ | $\lambda << \mu K$ |
| i.c.            | $\lambda < \mu \frac{K}{d}$ | $\lambda < \bar{\mu}$ | $\lambda < \mu K$ | $-$ |
|                 |        |        |         | $(\bar{\mu} < \mu(K - (d - 1)))$ |
Example: $K = 3$ and $d = 2$ copies, $C = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$

- I.i.d copies $\implies$ the departure rate of a type–c job is

$$\sum_{s \in C} \frac{\mu}{M_s(t)}$$
Theorem

Assume PS service policy and copies of a job are i.i.d.
The system is stable $\iff \lambda < \mu K$.

Proof:

- Show that fluid limit satisfies

$$
\frac{dm_{\text{max}}(t)}{dt} = \frac{\lambda}{K} - \mu \left( \sum_{c \in C(s)} \sum_{l \in S(c)} \frac{n_c}{m_l} \right) \leq \frac{\lambda}{K} - \mu d
$$
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Assume ROS service policy and copies of a job are i.i.d.

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Priority policy with $K=3$ servers and $d=2$ copies

$C = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. 

Server 1: FCFS, Server 2: $\{1, 2\} \preceq \{2, 3\}$, Server 3: $\{1, 3\} \preceq \{2, 3\}$.

\[
\frac{d|\bar{n}(t)|}{dt} = \lambda - (3\mu - \mu P(\text{server 1 is empty})).
\]
Priority policy with $K=3$ servers and $d=2$ copies

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Server 1: FCFS, Server 2: $\{1, 2\} \preceq \{2, 3\}$, Server 3: $\{1, 3\} \preceq \{2, 3\}$.

The system can be unstable when $\lambda < \mu K$. 

\[ \lambda = 2.9 < 3 = \mu K \]
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Identical copies assumption

- IID copies: \( \lambda < \mu K \).
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  - $d = 1 \implies K$ homogeneous servers with rate $\mu$.
  - $d = K \implies$ single server with rate $\mu K$.}

The performance decreases in $d$: no longer maximum stable.
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The performance decreases in \( d \): no longer maximum stable.
Example: \( K = 3 \) and \( d = 2 \) copies, \( C = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\} \)

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PS service policy with Identical copies

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- $\frac{da_{cis}(t)}{dt} = \frac{1}{M_s(t)}$.
- A job leaves the system due to a departure in server $s_{ci}^*(t) = \arg\max_{s \in c}\{a_{cis}(t)\}$.
- Departure rate of the $i$-th type-$c$ job: $\frac{\mu}{M_{s_{ci}^*}(t)(t)}$. 
Example: $K = 3$ and $d = 2$ copies, $C = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$

![Diagram showing states S1, S2, S3 with transitions]

- The drift of server $s$: $\frac{d m_s}{dt} = \lambda \frac{d}{K} - \sum_{c \in C(s)} \sum_{i=1}^{N_c(t)} \frac{\mu}{M_{s*}^{ci}(t)}.$
PS service policy with Identical copies

Example: \( K = 3 \) and \( d = 2 \) copies, \( C = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\} \)

- The drift of server \( S \): \( \frac{dm_s}{dt} = \lambda \frac{d}{K} - \sum_{c \in C(s)} \sum_{i=1}^{N_c(t)} \frac{\mu}{M_{s^*}(t)} \).
- When symmetric state \( (M_1 = M_2 = M_3) \): \( \frac{dm_s}{dt} = \lambda \frac{d}{K} - \mu \) which can be strictly positive when \( \lambda < \mu K \).
Theorem

Assume PS service policy and copies of a job to be identical copies. The system is stable $\iff \lambda < \mu \frac{K}{d}$.

Proof:

$\iff$

- Upper Bound $\tilde{N}^{UP}(t)$: the system where all copies need to be served.
- $\tilde{N}^{PS}(t) \leq_{st.} \tilde{N}^{UP}(t)$
- $\tilde{N}^{UP}(t)$ is stable iff $\lambda d < \mu K$
Theorem

Assume PS service policy and copies of a job to be identical copies. The system is stable $\iff \lambda < \mu \frac{K}{d}$.

Proof:

$\implies$)

- Lower Bound $\tilde{N}^{LB}(t)$: the departure rate of a job is determined by the capacity it gets at the server with the least number of copies: $\frac{\mu}{M_{s^*_c}(t)}$ where $s^*_c = \arg \min_{s \in S(c)} \{M_s(t)\}$.

- $\tilde{N}^{PS}(t) \geq_{st.} \tilde{N}^{LB}(t)$, since $\frac{\mu}{M_{s^*_c(t)}(t)} \leq \frac{\mu}{M_{s^*_c}(t)}$.

- The fluid limit of $\tilde{N}^{LB}(t)$ satisfies $\frac{d}{dt} m_{\min}(t) = \lambda \frac{d}{K} - \mu > 0$
Stability condition reduces at least to $\lambda < \mu(K - d + 1)$. 
Theorem

Under FCFS service policy and identical copies the system is stable if and only if

\[ \lambda < \bar{\mu} = \sum_{i \in \tilde{S}} \tilde{\Pi}_i i \mu \]

where \( \tilde{\Pi}_i \) is the fraction of time one sees departure rate \( i \mu \) when the system is congested.
FCFS system with Identical copies

The solution of the congested system:

- $K$ and $d = K - 1$, **Stability condition:** $\lambda < 2\mu$. 
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Example: $K = 4$ and $d = 2$. 

```
\begin{array}{cccc}
(1, 3) & (2, 4) & (1, 3) & (2, 4) \\
(1, 2) & (2, 3) & (2, 3) & (3, 4) \\
\end{array}
```
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The solution of the congested system:

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- For general $K$ and $d$ is hard to characterize.

Example: $K = 4$ and $d = 2$. The steady-state equations are:

\[
2\mu\pi(O_2, n, O_1) = \mu\pi(O_2, n + 1, O_1) + \mu \sum_{j=0}^{n} (\frac{1}{6})^{j+1}\pi(O_2, n - j, O_1) + \mu \sum_{s=1}^{4} (\frac{1}{3})^n\pi(O_2, n, O_1, 0, O_s) + \mu (\frac{1}{6})^{n+1}\pi(O_1, 0, O_2) + \mu \sum_{s=1}^{4} \sum_{j=0}^{n} \mu (\frac{1}{3})^j\pi(O_2, j, O_s, n - j, O_1)
\]

\[
3\mu\pi(O_3, m, O_2, n, O_1) = \mu\pi(O_3, m, O_2, n + 1, O_1) + \mu \sum_{s=1,2} \sum_{j=0}^{n} (\frac{1}{3})^{j+1}\pi(O_3, m + j + 1, O_s, n - j, O_1) + \mu \sum_{s=1}^{3} \sum_{j=0}^{m} (\frac{3}{6})^j\pi(O_s, m - j, O_2, n, O_1) + \mu (\frac{1}{3})^n (\frac{3}{6})^m \frac{1}{6} \sum_{s=1,2} \pi(O_2, n, O_1, 0, O_s) + \mu (\frac{1}{3})^{n+1} \sum_{s=1,2} \pi(O_3, m + n + 1, O_1, 0, O_s) + \mu \sum_{s=1,2} \sum_{j=0}^{n} (\frac{1}{3})^j (\frac{3}{6})^m \frac{1}{6} \pi(O_2, j, O_s, n - j, O_1) + \mu \sum_{j=0}^{n} (\frac{1}{6})^{j+2}(\frac{3}{6})^m \pi(O, n - j, O_1) + \mu (\frac{1}{6})^{n+2}(\frac{3}{6})^m \pi(O_1, 0, \tilde{O}),
\]
At a fluid scale, 
\[ P( \text{a job is served simultaneously in more than one server}) \to 0. \]

**Theorem**

Under ROS service policy and identical copies assumption, the system is stable \( \iff \lambda < \mu K \)

**Proof:**

- Show that fluid limit satisfies 
  \[ \frac{dm_{max}(t)}{dt} \leq \lambda \frac{d}{K} - \mu d \]
Table: Summary of stability conditions

| Priority policy | PS                        | FCFS                      | ROS                        | ROS                        |
|-----------------|---------------------------|---------------------------|----------------------------|---------------------------|
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| i.c.            | $\lambda < \mu \frac{K}{d}$ | $\lambda < \tilde{\mu}$ | $\lambda < \mu K$         | $-$                       |

($\tilde{\mu} < \mu(K - (d - 1))$)
Simulations for the mean number of jobs

Mean number of jobs with identical copies and $K = 5$. 

![Graph](image-url)
Simulations for the mean number of jobs

Mean number of jobs with identical copies and $K = 5$. 

The graph shows the number of jobs as a function of $\lambda/(\mu K)$ for $d=3$. The lines represent PS, FCFS, and ROS policies.
Simulations for the mean number of jobs

Mean number of jobs with identical copies and $K = 5$. 

![Graph showing the mean number of jobs with identical copies for different scheduling algorithms.](image-url)
LT approximation for FCFS with identical copies

Relative mean response time under low load of $\lambda$.

\[
\mathbb{E}(D^{LT, FCFS}) = \frac{1}{\mu} + \frac{3\lambda}{2\mu^2} \left( \frac{1}{K_d} \right),
\]

\[
\min \mathbb{E}(D^{LT, FCFS}) \text{ when } d^* = \arg \max_d \left\{ \binom{K}{d} \right\} = 2.
\]
Homogeneous servers for FCFS with identical copies

Mean number of jobs for $K=5$ servers with different $d$. The graph shows the relationship between $\frac{\lambda}{\mu(5)}$ and the mean number of jobs, with $d$ representing the number of identical copies.
$K = 3$ and $\mu = (1, 4, 8)$
Conclusions

- Redundancy systems under iid assumption:
  - FCFS, PS and ROS are maximum stable.
  - Priority queues lose stability.

- Redundancy system under identical copies assumption:
  Stability condition strongly depends on the scheduling policy.

- Heterogeneous servers can improve stability.
Redundancy systems under iid assumption:
Analyse sufficient conditions for which the system is maximum stable.
Future work

- Redundancy systems under iid assumption:
  Analyse sufficient conditions for which the system is maximum stable.

- Redundancy system under identical copies assumption:
  Characterize the stability condition when variable servers: heterogeneous speed servers, S&X model,...
Thank you!