Model-Free Sure Screening via Maximum Correlation

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Abstract

We consider the problem of screening features in an ultrahigh-dimensional setting. Using maximum correlation, we develop a novel procedure called MC-SIS for feature screening, and show that MC-SIS possesses the sure screen property without imposing model or distributional assumptions on the response and predictor variables. Therefore, MC-SIS is a model-free sure independence screening method as in contrast with some other existing model-based sure independence screening methods in the literature. Simulation examples and a real data application are used to demonstrate the performance of MC-SIS as well as to compare MC-SIS with other existing sure screening methods. The results show that MC-SIS outperforms those methods when their model assumptions are violated, and it remains competitive when the model assumptions hold.

Key Words: B-spline; Distance correlation; Optimal transformation; Sure screening property; Variable selection.

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1 Introduction

With rapid development of modern technology, various types of high-dimensional data are collected in a variety of areas such as next-generation sequencing and biomedical imaging data in bioinformatics, high-frequency time series data in quantitative finance, and spatial-temporal data in environmental studies. In those types of high-dimensional data, the number of variables $p$ can be much larger than the sample size $n$, which is referred to the ‘large $p$ small $n$’ scenario. To deal with this scenario, a commonly adopted approach is to impose the sparsity assumption that the number of important variables is small relative to $p$. Based on the sparsity assumption, a variety of regularization procedures have been proposed for high-dimensional regression analysis such as the lasso (Tibshirani, 1996), the smoothly clipped absolute deviation method (Fan and Li, 2001), and elastic net (Zou and Hastie, 2005). All these methods work when $p$ is moderate. However, when applied to analyze ultrahigh-dimensional data, their performances will deteriorate in terms of computational expediency, statistical accuracy and algorithmic stability (Fan et al., 2009). To address the challenges of ultrahigh dimensionality, a number of marginal screening procedures have been proposed under different model assumptions. They all share the same goal that is to reduce dimensionality from ultrahigh to high while retaining all truly important variables. When a screening procedure achieves this goal, it is said to have the sure screening property in the literature.

Fan and Lv (2008) proposed to use Pearson correlation for feature screening and showed that the resulting procedure possesses the sure screening property under the linear model assumption. They refer to the procedure as the Sure Independence Screening (SIS) procedure. Fan and Song (2010) extended SIS from linear models to generalized linear models by using maximum marginal likelihood values. Fan et al. (2011) developed a Nonparametric Independence Screening (NIS) procedure and proved that NIS has the sure screening property under the additive model. Li et al. (2012) proposed to use distance correlation to
rank the predictor variables, and showed that the resulting procedure, denoted as DC-SIS, has the sure screening property without imposing any specific model assumption. Compared with the other screening procedures discussed previously, DC-SIS is thus model-free. Distance correlation was introduced in Szekely et al. (2007), which uses joint and marginal characteristic functions to measure the dependence between two random variables.

From the review above, it is clear that the standard approach to developing a valid screening procedure consists of two steps. First, a proper dependence measure between the response and predictor variables needs to be defined and further used to rank-order all the predictor variables; and second, the sure screening property needs to be established for the screening procedure based on the dependence measure. The screening methods discussed in the previous paragraph differ from each other in these two steps. For example, SIS uses Pearson correlation as the dependence measure and possesses the sure screening property under linear models, whereas NIS uses the goodness of fit measure of the nonparametric regression between the response and predictor variable as the dependence measure and possesses the sure screening property under additive models.

For the purpose of screening in an ultrahigh dimensional setting, we argue that an effective screening procedure should employ a sensitive dependence measure and satisfy the sure screening requirement without model specifications. The goal of screening is not to precisely select the true predictors, instead, it is to reduce the number of predictor variables from ultrahigh to high while retaining the true predictor variables. Therefore, false positives or selections can be tolerated to a large degree, and sensitive dependence measures are more preferred than insensitive measures. In ultrahigh dimensional data, there usually does not exist information about the relationship between the response and predictor variables, and it is extremely difficult to explore the possible relationship due to the presence of a large number of predictors. Therefore, model assumptions should be avoided as much as possible in ultrahigh dimensional screening, and we should prefer screening
procedures that possess the sure screening property without model specifications. In other words, model-free sure screening procedures are more preferable. Among the existing screening procedures discussed previously, only DC-SIS does not require restrictive model assumption and therefore is model-free. However, the distance correlation measure used by DC-SIS may not be sensitive especially when sample size is small, because empirical characteristic functions are employed to estimate distance correlations.

A more sensitive dependence measure between the response and a predictor variable is maximum correlation, which was originally proposed by Gebelein (1941) and studied by Rényi (1959) as a general dependence measure between two random variables. Rényi (1959) gave a list of seven fundamental properties a reasonable dependence measure must have, and maximum correlation is one of a few measures that satisfy this requirement. The definition and estimation of maximum correlation involve maximizations over functions (see Section 2.1), and thus it is fairly sensitive even when sample size is small. Recently, there have been some revived interests in using maximum correlation as a proper dependence measure in high-dimensional data analysis (Bickel and Xu, 2009; Hall and Miller, 2011; Reshef et al., 2011; Speed, 2011).

In this paper, we propose to use maximum correlation as a dependence measure for ultrahigh dimensional screening, and prove that the resulting procedure has the sure screening property without imposing model specifications (see Theorem 1 in Section 2.4). We adopt the B-spline functions-based estimation method (Burman, 1991) to estimate maximum correlation. We refer to our proposed procedure as the maximum correlation-based sure independence screening procedure, or in short, the MC-SIS procedure. Numerical results show that MC-SIS is competitive to other existing model-based screening procedures, and is more sensitive and robust than DC-SIS when sample size is small or the distributions of the predictor variables have heavy tails.

The rest of the paper is organized as follows. In Section 2, we introduce maximum
correlation and the B-spline functions-based method for estimating maximum correlation, propose the MC-SIS procedure, and establish the sure screening property for MC-SIS. Section 3 presents results from simulation study and a real life screening application. Section 4 concludes the paper with additional remarks and future research. The proof of the main theorem is given in the Appendix section.

2 Independence Screening via Maximum Correlation

2.1 Maximum correlation and optimal transformation

Let \( Y \) denote the response variable and \( X = (X_1, \ldots, X_p) \) be the vector of predictor variables. We assume the supports of \( Y \) and \( X_j (j = 1, \ldots, p) \) are compact, and they are further assumed to be \([0,1]\) without loss of generality. For any given \( j \), consider a pair of random variables \((X_j, Y)\). The maximum correlation coefficient between \( X_j \) and \( Y \), denoted as \( \rho^*_j \), is defined as follows.

\[
\rho^*_j(X_j, Y) = \sup \{ \rho(\theta(Y), \phi(X_j)) : 0 < E\{\theta^2(Y)\} < \infty, 0 < E\{\phi^2(X_j)\} < \infty \},
\]  

(1)

where \( \rho \) is the Pearson correlation, and \( \theta \) and \( \phi \) are Borel-measurable functions of \( Y \) and \( X_j \). We further denote \( \theta^*_j \) and \( \phi^*_j \) as optimal transformations that attain the maximum correlation.

Maximum correlation coefficient enjoys the following properties (Rényi, 1959): (a) \( 0 \leq \rho^*_j(X_j, Y) \leq 1 \); (b) \( \rho^*_j(X_j, Y) = 0 \) if and only if \( X_j \) and \( Y \) are independent; (c) \( \rho^*_j(X_j, Y) = 1 \) if there exist Borel-measurable functions \( \theta^* \) and \( \phi^* \) such that \( \theta^*(Y) = \phi^*(X_j) \); and (d) if \( X_j \) and \( Y \) are jointly Gaussian, then \( \rho^*_j(X_j, Y) = |\rho(X_j, Y)| \). Some other properties of maximum correlation coefficient are discussed in Szekely and Mori.
Dembo et al. (2001), Bryc and Dembo (2005), and Yu (2008). Due to Property (d), it is clear that maximum correlation is a natural extension of Pearson correlation. Note that Pearson correlation does not possess Properties (b) and (c). Therefore, maximum correlation is a more proper measure of the dependence between two random variables than Pearson correlation.

Rényi (1959) established the existence of maximum correlation under certain sufficient conditions, and a different set of sufficient conditions are given in Breiman and Friedman (1985). Breiman and Friedman (1985) also showed that optimal transformations $\theta^*_j$ and $\phi^*_j$ can be obtained via the following minimization problem.

$$
\min_{\theta_j, \phi_j \in L_2(P)} e^2_j = E\{[\theta_j(Y) - \phi_j(X_j)]^2\},
$$

subject to

$$
E\{\theta_j(Y)\} = E\{\phi_j(X_j)\} = 0;
$$

$$
E\{\theta^2_j(Y)\} = 1.
$$

Here, $P$ denotes the joint distribution of $(X_j, Y)$ and $L_2(P)$ is the class of square integrable functions under the measure $P$. Let $e^*_j$ be the minimum of $e^2_j$. Breiman and Friedman (1985) derived two critical connections between $e^*_j$, squared maximum correlation $\rho^*_j$, and optimal transformation $\phi^*_j$, which we state as Fact 0 below.

**Fact 0.**

$$
e^*_j = 1 - \rho^*_j; \quad (3a)
$$

$$
E(\phi^*_j) = \rho^*_j. \quad (3b)
$$

**Fact 0** suggests that the minimization problem (2) is equivalent to the optimization problem (1). Furthermore, the squared maximum correlation coefficient is equal to the expectation of the squared optimal transformation $\phi^*_j$.

Various algorithms have been proposed in the literature to compute maximum corre-
lation, including Alternating Conditional Expectations (ACE) in Breiman and Friedman (1985), B-spline approximation in Burman (1991), and polynomial approximation in Bickel and Xu (2009) and Hall and Miller (2011). Equation (3b) indicates that maximum correlation coefficient $\rho_{j}^*$ can be calculated through the optimal transformation $\phi_{j}^*$. In this paper, we apply Burman’s approach to first estimate $\phi_{j}^*$, and then estimate $\rho_{j}^*$, which will be further used in screening.

2.2 B-spline estimation of optimal transformations

Let $S_n$ be the space of polynomial splines of degree $\ell \geq 1$ and $\{B_{jm}, m = 1, \ldots, d_n\}$ denote a normalized B-spline basis with $||B_{jm}||_{\text{sup}} \leq 1$, where $||\cdot||_{\text{sup}}$ is the sup-norm. We have $\theta_{nj}(Y) = \alpha_j^T B_j(Y), \phi_{nj}(X_j) = \beta_j^T B_j(X_j)$ for any $\theta_{nj}(Y), \phi_{nj}(X_j) \in S_n$, where $B_j(\cdot) = (B_{j1}(\cdot), \ldots, B_{jd_n}(\cdot))^T$ denotes the vector of $d_n$ basis functions. Additionally, we let $k$ be the number of knots where $k = d_n - \ell$. The population version of B-spline approximation to the minimization problem (2) can be written as follows.

\[
\min_{\theta_{nj}, \phi_{nj} \in S_n} \mathbb{E}\{[\theta_{nj}(Y) - \phi_{nj}(X_j)]^2\},
\]

subject to $\mathbb{E}\{\theta_{nj}(Y)\} = \mathbb{E}\{\phi_{nj}(X_j)\} = 0$; $\mathbb{E}\{\theta_{nj}^2(Y)\} = 1$. (4)

Burman (1991) applied a technique to remove the first constraint $\mathbb{E}\{\theta_{nj}(Y)\} = \mathbb{E}\{\phi_{nj}(X_j)\} = 0$ in the optimization problem above as follows. First, let $z_1, \ldots, z_{d_n-1}$ ($z_i = (z_{i1}, \ldots, z_{id_n})^T$ for $i = 1, \ldots, d_n - 1$) be $d_n$-dimensional vectors which are orthogonal to each other, orthogonal to the vector of 1s and $z_i^T z_i = 1$ for $i = 1, \ldots, d_n - 1$. Second, obtain matrix $D_j$ with the $(s, m)$ entry $D_{j,sm} = z_{sm}/(kb_{jm})$ where $b_{jm} = \mathbb{E}\{B_{jm}(X_j)\}$, for $s = 1, \ldots, d_n - 1$ and $m = 1, \ldots, d_n$. Third, let $\phi_{nj}(X_j) = \eta_j^T \psi_j(X_j)$ where $\psi_j(X_j) = D_j B_j(X_j)$. With this construction, it is easy to verify that $\mathbb{E}\{\phi_{nj}(X_j)\} = 0$, ...
and the minimization of $E[\{\theta_{nj}(Y) - \phi_{nj}(X_j)\}]^2$ subject to $E[\theta_{nj}^2(Y)] = 1$ ensures that $E[\theta_{nj}(Y)] = 0$. Burman (1991) showed the equivalence between the optimization problem (4) and the one stated below.

$$\min_{\theta_{nj}, \phi_{nj} \in \mathcal{S}_n} E[\{\theta_{nj}(Y) - \phi_{nj}(X_j)\}]^2,$$

subject to $E[\theta_{nj}^2(Y)] = 1$. (5)

For fixed $\theta_{nj}(Y)$ (i.e. fixed $\alpha_j$), the minimizer of (5) with respect to $\phi_{nj}(X_j)$ is

$$\phi_{nj}(X_j) = \psi_j^T(X_j)[E\{\psi_j(X_j)\psi_j^T(X_j)\}]^{-1}E\{\psi_j(X_j)B_j^T(Y)\}\alpha_j$$

By plugging $\phi_{nj}(X_j)$ back in (5), we obtain the following maximization problem,

$$\max_{\alpha_j \in \mathbb{R}^{d_n}} \alpha_j^T E\{B_j(Y)\psi_j^T(X_j)\}[E\{\psi_j(X_j)\psi_j^T(X_j)\}]^{-1}E\{\psi_j(X_j)B_j^T(Y)\}\alpha_j,$$

subject to $\alpha_j^T E\{B_j(Y)B_j^T(Y)\}\alpha_j = 1$. (6)

Following the notation in Burman (1991), we denote

$$A_{j00} = E\{B_j(Y)B_j^T(Y)\}, \quad A_{jXX} = E\{\psi_j(X_j)\psi_j^T(X_j)\},$$

$$A_{jX0} = E\{\psi_j(X_j)B_j^T(Y)\}, \quad \text{and} \quad A_{j0X} = A_{jX0}^T.$$

It is clear that (6) is a generalized eigenvalue problem, which can be solved by the largest eigenvalue and its corresponding eigenvector of $A_{j00}^{-1/2}A_{j0X}A_{jXX}^{-1}A_{jX0}A_{j00}^{-1/2}$. We denote the largest eigenvalue by $\lambda_{j1}^*$, which is equal to $||A_{j00}^{-1/2}A_{j0X}A_{jXX}^{-1}A_{jX0}A_{j00}^{-1/2}||$, where $|| \cdot ||$ is the operator norm, and further denote the corresponding eigenvector by $\alpha_j^*$. Let $\phi_{nj}^*(X_j) = \psi_j^T(X_j)[E\{\psi_j(X_j)\psi_j^T(X_j)\}]^{-1}E\{\psi_j(X_j)B_j^T(Y)\}\alpha_j^*$. $\phi_{nj}^*$ can be considered the spline approximation to the optimal transformation $\phi_j^*$ defined previously. Note that the target function in (6) is $E(\phi_{nj}^*)$, and we also have $E(\phi_{nj}^*) = \lambda_{j1}^*$. 

8
Given the data \( \{ Y_u \}_{u=1}^n \) and \( \{ X_{uj} \}_{u=1}^n \), we estimate \( A_{j00} \), \( A_{jXX} \), \( A_{jX0} \), and \( A_{j0X} \) as follows.

\[
\hat{A}_{j00} = n^{-1} \sum_{u=1}^n B_j(Y_u)B^T_j(Y_u), \quad \hat{A}_{jXX} = n^{-1} \sum_{u=1}^n \hat{\psi}_j(X_{uj})\hat{\psi}^T_j(X_{uj}),
\]

\[
\hat{A}_{jX0} = n^{-1} \sum_{u=1}^n \hat{\psi}_j(X_{uj})B_j(Y_u), \quad \text{and} \quad \hat{A}_{j0X} = \hat{A}_{jX0}^T,
\]

where \( \hat{\psi}_j(X_{uj}) = \hat{D}_jB_j(X_{uj}) \), the \((s, m)\) entry of \( \hat{D}_j \) is \( \hat{D}_{j,sm} = z_{sm}/(k\hat{b}_{jm}) \), and \( \hat{b}_{jm} = n^{-1} \sum_{u=1}^n B_{jm}(X_{uj}) \), for \( s = 1, \ldots, d_n - 1 \) and \( m = 1, \ldots, d_n \). Then, \( \lambda_{j1}^* \) is estimated by

\[
\hat{\lambda}_{j1}^* = \left\| \hat{A}_{j00}^{-1/2} \hat{A}_{j0X}^{-1} \hat{A}_{jXX}^{-1} \hat{A}_{j0X}^T \hat{A}_{j00}^{-1/2} \right\|.
\]

Based on the two relationships \((i)\) \( E(\phi_{nj}^*)^2 = (\rho_{nj}^*)^2 \) and \((ii)\) \( E(\phi_{nj}^*) = \lambda_{j1}^* \), and the fact that \( \phi_{nj}^* \) is the optimal spline approximation to \( \phi_{nj}^* \), we propose to screen important variables using the magnitudes of \( \hat{\lambda}_{j1}^* \) for \( 1 \leq j \leq p \).

### 2.3 MC-SIS procedure

Let \( \nu_n \) be a pre-specified threshold, and \( \hat{D}_{\nu_n} \) the collection of selected important variables. Then our proposed screening procedure can be defined as

\[
\hat{D}_{\nu_n} = \{ 1 \leq j \leq p: \hat{\lambda}_{j1}^* \geq \nu_n \}.
\]

Empirically, the threshold value \( \nu_n \) is often set to be \( n \) or \( [n / \log n] \), where \([a]\) denotes the integer part of \( a \). Since \( \hat{\lambda}_{j1}^* \) is the estimate of \( \lambda_{j1}^* \), which is an approximation to the squared maximum correlation coefficient \( \rho_{nj}^2 \), we refer to the procedure as the MC-SIS procedure. The sure screening property of the procedure will be discussed in next section.
2.4 Sure Screening Property

Adopting notations from Li et al. (2012), we use \( F(Y|X) \) to denote the conditional distribution of \( Y \) given \( X \) and \( \Psi_Y \), the support for \( Y \), and we define \( D = \{ j : F(y|X) \text{ functionally depends on } X_j \} \), \( I = \{ j : F(y|X) \text{ does not functionally depends on } X_j \} \). Let \( X_D = \{ X_j : j \in D \} \) and \( X_I = \{ X_j : j \in I \} \), which are referred to active set and inactive set, respectively. As discussed in the Introduction, a screening procedure is said to possess the sure screening property if it can retain the active set after screening.

We have established the sure screening property of the MC-SIS procedure under certain conditions. Before stating the theorem, we first list the conditions below.

(C1) If the transformations \( \theta_j \) and \( \phi_j \) with zero means and finite variances satisfy

\[
\theta_j(Y) + \phi_j(X_j) = 0 \ a.s., \ 	ext{then each of them is zero a.s.}
\]

(C2) The conditional expectation operators \( E\{\phi_j(X_j) \mid Y\} : H_2(X_j) \to H_2(Y) \) and \( E\{\theta_j(Y) \mid X_j\} : H_2(Y) \to H_2(X_j) \) are all compact operators. \( H_2(Y) \) and \( H_2(X_j) \) are Hilbert spaces of all measurable functions with zero mean, finite variance and usual inner product.

(C3) The optimal transformations \( \{\theta_j^*, \phi_j^*\}_{j=1}^p \) belong to a class of functions \( \mathcal{F} \), whose \( r \)th derivative \( f^{(r)} \) exists and is Lipschitz of order \( \alpha_1 \), that is, \( \mathcal{F} = \{ f : |f^{(r)}(s) - f^{(r)}(t)| \leq K|s - t|^{\alpha_1} \text{ for all } s, t \} \) for some positive constant \( K \), where \( r \) is a nonnegative integer and \( \alpha_1 \in (0, 1] \) such that \( d = r + \alpha_1 > 0.5 \).

(C4) The joint density of \( Y \) and \( X_j (j = 1, \ldots, p) \) is bounded and the marginal densities of \( Y \), and \( X_j \) are bounded away from zero.

10
(C5) The smallest maximum correlation of active predictors satisfies
\[
\min_{j \in D}(\rho_j^2) \geq 2c_1d_n^{-2\kappa}, \text{ for some constant } c_1 > 0 \text{ and } 0 \leq \kappa < d/(2d+1).
\]

(C6) There exist positive constant \(c_1\) and \(\xi \in (0, 1)\) such that \(d_n^{-d-1} \leq c_1(1-\xi)n^{-2\kappa}/C_1\) for some positive constant \(C_1\).

Conditions (C1) and (C2) are adopted from Breiman and Friedman (1985), which ensure that the optimal transformations exist. Conditions (C3) and (C4) are from Burman (1991), but modified for our two-variable scenario. Condition (C5) requires that the maximum correlation between the response and the active predictors cannot be too small. The following lemma shows that the correlations achieved by spline-based transformations are at the same level as the signal of maximum correlation \(\rho_j^*\).

**Lemma 1.** Under conditions (C3) – (C6), we have \(\min_{j \in D} \lambda_{j1}^* \geq c_1\xi d_n^{-2\kappa}\).

Based on condition (C1) – (C6), we establish the sure screening properties for MC-SIS:

**Theorem 1.** Define \(\zeta(d_n, n) = d_n^2\exp(-c_3n^{-1-4\kappa}d_n^{-4}) + d_n\exp(-c_4nd_n^{-7})\).

(a) Under conditions (C1) – (C4), for any \(c_2 > 0\), there exist positive constants \(c_3\) and \(c_4\) such that
\[
P \left( \max_{1 \leq j \leq p} |\hat{\lambda}_{j1}^* - \lambda_{j1}^*| \geq c_2d_n^{-2\kappa} \right) \leq O \left( p\zeta(d_n, n) \right). \tag{8}
\]

(b) Additionally, if conditions (C5) and (C6) hold, by taking \(\nu_n = c_5d_n^{-\kappa}\) with \(c_5 \leq c_1\xi/2\), we have that
\[
P(D \subseteq \hat{D}_{\nu_n}) \geq 1 - O \left( s_n\zeta(d_n, n) \right), \tag{9}
\]
where \(s_n\) is the cardinality of \(D\).

From Theorem 1, MC-SIS can handle dimensionality as high as \(\exp\{o(n^{-1-4\kappa}d_n^{-4}) + \ldots\\}\).
And we remark that the number of basis functions $d_n$ affects the final performance of MC-SIS. To obtain the sure screening property, an upper bound of $d_n$ is $o(n^{1/7})$. Since $d_n$ is determined by the choices of the degree of B-spline basis functions and the number of knots, different combinations of degree and the number of knots can lead to different screening results. Additionally, knots placement can further affect the behavior of B-spline functions. In practice, cubic splines with $d_n = \lceil \frac{n^{1/5}}{2} \rceil + 2$ (Fan et al., 2011) or $d_n = \lceil \frac{2n^{1/5}}{3} \rceil$ (Fan et al., 2013) are commonly used, and knots are usually equally spaced or placed at sample quantiles. Optimal choice of $d_n$ and knots placement are beyond the scope of this paper and can be an interesting topic for future research.

3 Numerical Results

We illustrate the MC-SIS procedure by studying its performance under different model settings and distributional assumptions of the predictor variables. For all examples, we compare MC-SIS with SIS, NIS, and DC-SIS. As mentioned at the end of Section 2.1, the ACE algorithm in Breiman and Friedman (1985) can also be used to calculate maximum correlation coefficient. Therefore, the ACE algorithm can also be used to perform maximum correlation-based screening, and we refer to the resulting procedure as the ACE-based MC-SIS procedure. We also include the ACE-based MC-SIS procedure in our simulation study. To avoid confusion, we refer to our proposed procedure as the spline-based MC-SIS procedure in this section. For each example, we set $p = 1000$ and choose $n \in \{200, 300, 400\}$. We use the Minimum Model Size (MMS) required to retain the entire active set and its robust estimate of standard deviation (RSD = IQR/1.34; IQR is interquartile range) as measures of the effectiveness of a screening method. When constructing B-spline basis functions, we consider different combinations of degree (ranging from 1 to 3) and number of knots (ranging from 2 to 5), and place knots at sample quantiles. We only report the best
results from the combinations of degree and knows considered in the simulation study.

**Example 1.** (1.a): \( Y = \beta^T X + \varepsilon \), with the first \( s \) components of \( \beta^* \) taking values \( \pm 1 \) alternatively and the remaining being 0, where \( s = 3, 6 \) or 12; \( X_k \) are independent and identically distributed as \( N(0, 1) \) for \( 1 \leq k \leq 950 \); \( X_k = \sum_{j=1}^{s} X_j (-1)^{j+1} / 5 + (1 - s \varepsilon_k / 25)^{1/2} \) where \( \varepsilon_k \) are independent and identically distributed as \( N(0, 1) \) for \( k = 951, \ldots, 1000 \); and \( \varepsilon \sim N(0, 3) \).

(1.b): \( Y = X_1 + X_2 + X_3 + \varepsilon \), where \( X_k \) are independent and identically distributed as \( N(0, 1) \) for \( k = 1 \), and \( 3 \leq k \leq 1000 \); \( X_2 = \frac{1}{3} X_1^3 + \tilde{\varepsilon} \), and \( \tilde{\varepsilon} \sim N(0, 1) \); and \( \varepsilon \sim N(0, 3) \).

The first example is from Fan et al. (2011) and the simulation results are presented in Table 1. Under model (1.a), SIS demonstrates the best performance across all cases, which is expected since SIS is specifically developed for linear models. Under the models (1.a) with \( s = 3 \) or 6, when \( n = 200 \), MC-SIS under-performs all other methods. However, when sample size increases to 300 or 400, MC-SIS becomes comparable to others. For the case with \( s = 12 \), MC-SIS under-performs other methods for all choices of \( n \). The cause for the relatively poor performance of MC-SIS is due to the weak signal. With \( s = 12 \), it requires more samples for MC-SIS to estimate maximum correlation coefficient, without taking advantages of linearity assumptions. In model (1.b), SIS fails as there exists a nonlinear relationship between \( X_1 \) and \( X_2 \). NIS demonstrates the best performance as NIS is designed for dealing with nonparametric additive models. The ACE-based MC-SIS procedure demonstrates the second best performance. The spline-based MC-SIS procedure and DC-SIS perform similarly.

**Example 2.** (2.a): \( Y = X_1 X_2 + X_3 X_4 + \varepsilon \); (2.b): \( Y = X_1^2 + X_2^3 + X_3^2 X_4 + \varepsilon \); (2.c): \( Y = X_1 \sin(X_2) + X_2 \sin(X_1) + \varepsilon \); (2.d): \( Y = X_1 \exp(X_2) + \varepsilon \); (2.e): \( Y = X_1 \log(|c_0 + X_2|) + \varepsilon \); (2.f): \( Y = X_1 / (c_0 + X_2) + \varepsilon \). Here \( X_1, \ldots, X_{1000} \) and \( \varepsilon \) are generated independently from \( N(0, 1) \), and \( c_0 = 10^{-4} \).
Table 1: MMS and RSD (in parenthesis) for Example 1

| Model | n   | SIS   | NIS   | DC-SIS | MC-SIS (ACE) | MC-SIS (B-spline) |
|-------|-----|-------|-------|--------|--------------|-------------------|
| 1.a   | 200 | 5.8(3.0) | 6.4(3.0) | 6.8(3.2) | 11.9(7.7) | 15.4(9.3) |
| (s = 3)| 300 | 4.6(0.9) | 4.9(1.5) | 5.1(1.5) | 5.9(3.0) | 7.5(3.7) |
|       | 400 | 3.3(0.0) | 3.4(0.0) | 3.6(0.8) | 3.6(0.8) | 5.1(2.2) |
| 1.a   | 200 | 57.4(2.4) | 68.7(9.7) | 60.2(3.7) | 140.5(60.8) | 115.0(34.3) |
| (s = 6)| 300 | 56.0(0.0) | 58.2(0.2) | 57.1(0.0) | 67.4(5.2) | 65.0(7.5) |
|       | 400 | 55.8(0.0) | 55.9(0.0) | 55.9(0.0) | 56.8(0.8) | 58.3(1.5) |
| 1.a   | 200 | 119.4(42.9) | 250.6(133.2) | 195.2(55.8) | 484.6(181.9) | 337.0(204.3) |
| (s = 12)| 300 | 73.4(7.5) | 120.6(35.3) | 80.3(10.6) | 211.2(108.4) | 189.3(86.2) |
|       | 400 | 64.5(0.8) | 82.216(6.7) | 69.7(1.5) | 118.2(90.8) | 118.0(42.7) |
| 1.b   | 200 | 443.6(455.2) | 26.5(6.7) | 136.1(113.4) | 56.8(32.8) | 159.9(98.3) |
|       | 300 | 394.5(379.7) | 7.3(0.0) | 59.9(48.5) | 21.9(5.4) | 88.1(45.0) |
|       | 400 | 410.0(361.2) | 3.2(0.0) | 41.1(36.8) | 5.64(0.8) | 31.7(9.9) |

The eight models considered in this example are non-additive, and the simulation results are presented in Table 2. Due to the presence of non-additive structures, we notice that SIS and NIS fail in all models, and increasing sample size does not help improve the performances of SIS and NIS for most models. Both MC-SIS and DC-SIS work well in this example, but MC-SIS outperforms DC-SIS for almost all the models in terms of MMS. Even when the sample size is as small as 200, MC-SIS can effectively retain the active set under models (2.c), (2.e) and (2.f). This example demonstrates the advantages of MC-SIS and DC-SIS over SIS and NIS for non-additive models as well as the effectiveness of MC-SIS over DC-SIS.

Example 3. The models considered in this example are modifications of the models considered in Example 2. First, the error term $\epsilon$ in each original model is removed; and second, the predictor variables $X_1, X_2, \ldots, X_p$ are drawn independently from $\text{Cauchy}(0, 1)$ instead of $\text{N}(0, 1)$. The resulting models are denoted as (3.a)-(3.f), correspondingly. Simulation results based on these models are presented in Table 2.

Intuitively, the absence of the error terms in the models is expected to help the screening methods, but the use of heavy-tailed distributions for the predictor variables is expected
| Model | n   | SIS       | NIS       | DC-SIS    | MC-SIS (ACE) | MC-SIS (B-spline) |
|-------|-----|-----------|-----------|-----------|--------------|--------------------|
| 2.a   | 200 | 709.3(239.0) | 651.5(285.5) | 440.6(231.2) | 248.7(242.5) | 331.1(249.3)      |
|       | 300 | 724.1(194.6) | 631.2(251.7) | 350.5(186.0) | 117.8(88.3)  | 187.1(164.6)      |
|       | 400 | 795.3(194.8) | 636.5(256.3) | 280.0(148.9) | 59.25(26.1)  | 98.5(78.5)        |
| 2.b   | 200 | 617.5(308.2) | 300.5(298.7) | 186.5(132.5) | 104.2(103.0) | 127.3(88.6)       |
|       | 300 | 608.5(305.0) | 277.8(250.0) | 163.6(150.2) | 78.4(44.6)   | 81.6(42.5)        |
|       | 400 | 597.4(291.6) | 262.0(228.9) | 114.7(103.7) | 54.9(13.9)   | 41.9(16.0)        |
| 2.c   | 200 | 574.5(352.2) | 511.7(389.0) | 113.6(80.2)  | 18.1(2.24)   | 12.8(5.2)         |
|       | 300 | 616.4(342.2) | 521.8(321.6) | 51.0(30.0)   | 8.4(0.8)     | 5.9(2.2)          |
|       | 400 | 622.4(306.3) | 547.8(337.9) | 21.4(14.0)   | 13.0(0.0)    | 2.9(0.8)          |
| 2.d   | 200 | 536.5(285.1) | 181.8(168.5) | 2.0(0.0)     | 2.3(0.8)     | 2.3(0.0)          |
|       | 300 | 268.6(307.1) | 172.8(190.9) | 2.0(0.0)     | 2.0(0.0)     | 2.0(0.0)          |
|       | 400 | 272.1(331.0) | 176.3(178.7) | 2.0(0.0)     | 2.0(0.0)     | 2.0(0.0)          |
| 2.e   | 200 | 580.2(152.8) | 512.2(405.6) | 191.0(152.8) | 55.1(20.3)   | 28.3(10.8)        |
|       | 300 | 588.7(299.4) | 641.0(295.3) | 107.1(70.3)  | 40.7(1.5)    | 7.5(3.7)          |
|       | 400 | 602.1(258.4) | 568.0(311.9) | 66.2(44.6)   | 19.8(0.0)    | 5.1(3.0)          |
| 2.f   | 200 | 928.8(59.3)  | 654.5(417.9) | 140.5(123.5) | 30.0(9.9)    | 16.8(6.2)         |
|       | 300 | 936.7(37.7)  | 768.8(292.0) | 61.6(46.6)   | 23.4(2.2)    | 3.8(1.5)          |
|       | 400 | 942.0(39.9)  | 821.7(175.2) | 60.9(22.8)   | 17.8(0.8)    | 2.7(0.8)          |

To hinder the methods. The exact performance of a screening method in this example depends on the trade-off between those two changes. Comparing Table 3 with Table 2, we can see that the performances of SIS and NIS have improved, though they are still far from being satisfactory. The performance of DC-SIS has improved in models (3.a) and (3.c), but has much deteriorated in the other models, which indicates that DC-SIS is susceptible to heavy-tailed distributions. In the presence of heavy tails, Condition (C1) in Li et al. (2012) is violated, and DC-SIS may not have the sure screening property. The performances of ACE-based and spline-based MC-SIS have improved or stayed about the same in all models except model (3.d), which indicates the robustness of MC-SIS towards heavy-tailed distributions.

**Example 4.** In this example, we consider a real data set that contains the expression levels of 6319 genes and the expression levels of a G protein-coupled receptor (Ro1) in 30 mice. The same data set has been analyzed in Hall and Miller (2009) and in Li et al. (2012) using...
Table 3: MMS and RSD (in parenthesis) for Example

| Model | n   | SIS           | NIS           | DC-SIS        | MC-SIS (ACE) | MC-SIS (B-spline) |
|-------|-----|---------------|---------------|---------------|--------------|------------------|
| 3.a   | 200 | 338.8(284.3)  | 296.6(175.4)  | 90.3(54.3)    | 124.1(39.2)  | 74.9(50.4)       |
|       | 300 | 310.2(241.6)  | 310.8(253.7)  | 64.6(32.5)    | 72.2(14.7)   | 20.7(6.0)        |
|       | 400 | 273.3(242.4)  | 303.1(260.6)  | 48.33(29.9)   | 41.5(7.1)    | 9.3(3.0)         |
| 3.b   | 200 | 617.5(305.2)  | 617.5(256.7)  | 478.9(286.6)  | 117.8(36.6)  | 50.8(14.4)       |
|       | 300 | 665.8(348.3)  | 689.2(256.2)  | 511.2(258.8)  | 72.0(8.6)    | 11.9(3.7)        |
|       | 400 | 619.8(297.0)  | 696.8(250.0)  | 507.8(265.1)  | 32.7(6.7)    | 9.1(2.2)         |
| 3.c   | 200 | 136.5(80.2)   | 106.6(70.7)   | 23.7(12.7)    | 11.9(5.2)    | 5.6(2.2)         |
|       | 300 | 116.1(82.1)   | 90.1(56.2)    | 13.4(6.3)     | 8.7(4.5)     | 4.0(2.4)         |
|       | 400 | 90.4(36.0)    | 67.9(39.2)    | 9.9(4.7)      | 7.3(3.2)     | 3.2(1.5)         |
| 3.d   | 200 | 409.5(367.0)  | 434.8(409.0)  | 412.3(401.1)  | 15.4(3.7)    | 5.1(1.5)         |
|       | 300 | 485.1(320.0)  | 486.7(411.0)  | 493.8(397.0)  | 7.8(2.4)     | 4.3(1.5)         |
|       | 400 | 460.8(342.0)  | 493.4(360.1)  | 480.7(407.3)  | 12.5(0.0)    | 3.3(1.5)         |
| 3.e   | 200 | 252.2(193.8)  | 250.2(228.5)  | 124.0(99.1)   | 55.8(11.4)   | 6.7(2.2)         |
|       | 300 | 332.9(332.7)  | 340.0(289.0)  | 188.7(120.9)  | 42.9(4.5)    | 4.9(1.5)         |
|       | 400 | 314.3(315.5)  | 334.6(308.6)  | 121.1(98.0)   | 37.8(4.1)    | 4.1(1.5)         |
| 3.f   | 200 | 779.8(172.0)  | 737.0(244.2)  | 507.7(249.6)  | 37.5(6.9)    | 5.5(2.2)         |
|       | 300 | 808.4(149.8)  | 855.7(120.9)  | 498.6(336.0)  | 28.7(4.5)    | 4.3(2.2)         |
|       | 400 | 806.7(149.1)  | 837.6(143.5)  | 432.6(281.9)  | 34.3(3.7)    | 3.7(1.5)         |

DC-SIS. The goal is to identify the most influential genes for Ro1.

We apply SIS, NIS, DC-SIS and ACE-based MC-SIS to select the top two most important genes, separately, and the results are reported in Table 4. Because the results of spline-based MC-SIS differ under different choices of degrees and knots, they are not included in the table. We note that when linear splines with 2 interior knots placed at sample quantiles are used, the top two genes selected by spline-based MC-SIS are Msa.2437.0 and Msa.2877.0, the latter of which is also selected by the other procedures as one of the top two genes. Additionally, we note that almost all of the used procedures, including spline-based MC-SIS, consistently ranked Msa.741.0, Msa.2134.0 and Msa.2877.0 among the top ranked genes. These three genes are worthy of further investigation.

To further compare the performances of the screening procedures, we fit regression models for the response, which is the expression level of Ro1, using the top two genes
Table 4: Top ranked genes for Example 4

|        | SIS          | NIS          | DC-SIS       | MC-SIS       |
|--------|--------------|--------------|--------------|--------------|
| Rank 1 | Msa.2877.0   | Msa.2877.0   | Msa.2134.0   | Msa.8081.0   |
| Rank 2 | Msa.964.0    | Msa.1160.0   | Msa.2877.0   | Msa.2437.0   |

Table 5: Adjusted $R^2$ of fitting 3 different models for Example 4

| Model      | SIS top 1 | SIS top 2 | NIS top 1 | NIS top 2 | DC-SIS top 1 | DC-SIS top 2 | MC-SIS top 1 | MC-SIS top 2 |
|------------|-----------|-----------|-----------|-----------|--------------|--------------|--------------|--------------|
| Linear     | 74.5%     | 82.3%     | 74.5%     | 75.8%     | 58.4%        | 77.6%        | 13.8%        | 16.9%        |
| Additive   | 80.0%     | 84.2%     | 80.0%     | 84.5%     | 65.7%        | 96.8%        | 58.9%        | 68.7%        |
| Transformation | 84.5% | 88.1% | 84.5% | 88.0% | 90.0% | 94.7% | 94.1% | 96.9% |

selected by the procedures. Three different models are considered, which are the linear regression model $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \epsilon$, the additive model $Y = \ell_1(X_1) + \ell_2(X_2) + \epsilon$, and the optimal transformation model $\theta^*(Y) = \phi^*_{1}(X_1) + \phi^*_{2}(X_2) + \epsilon$, where $\theta^*$, $\phi^*_{1}$, and $\phi^*_{2}$ are the optimal transformations (Breiman and Friedman, 1985). For each procedure, all three models are fitted using the top ranked gene as well as using the top ranked two genes, and the resulting adjusted $R^2$ values are reported in Table 5.

Under the linear model, as expected, SIS achieves the largest adjusted $R^2$ values, whereas the adjusted $R^2$ values of ACE-based MC-SIS are rather poor. The major cause for the difference between SIS and ACE-based MC-SIS is that the former is specifically developed for screening under the linear model, whereas the latter is for screening under the optimal transformation model. Under the additive model, when the top one gene is used, NIS achieves the largest adjusted $R^2$ value; and when the top two genes are used, DC-SIS achieves the largest adjusted $R^2$ value. Under the optimal transformation model, as expected, ACE-based MC-SIS achieves the largest adjusted $R^2$ values under both the top one gene and the top two gene scenarios. When plotting the expression levels of Ro1 against the expression levels of various selected genes, different patterns including linear and nonlinear patterns emerge for different screening methods. In practice, we believe that the top ranked genes by different methods are all worthy of further investigation.
4 Discussion

The performance and results of spline-based MC-SIS depend on the choice of degree and the number of knots for B-splines. Several methods have been proposed for knots selection when B-splines are used for nonparametric regression (Ruppert and Carroll, 2000; Miyata and Shen, 2003) in the literature, and Burman (1991) has investigated the selection of knots when B-splines are used for estimating maximum correlation under the optimal transformation model. How to extend these existing methods and develop novel methods for selecting degree and knots for spline-based MC-SIS will be an important research topic in the near future. Similar to other marginal screening procedures, MC-SIS may fail to identify important predictors when they are marginally independent with the response. There are two possible approaches to alleviating this drawback. The first approach is to use iterative strategy similar to what has been developed by Fan and Lv (2008) for iterative SIS, and the second is to screen variables in groups rather than individually. We will also investigate these two approaches for MC-SIS in our future research.

5 Proofs

5.1 Notation

\( n \) : sample size

\( p \) : dimension size

\( \ell \) : degree of polynomial spline

\( k \) : number of knots

\( d_n \) : dimension of B-spline basis
$\mathcal{D}$ : active set

$\mathcal{I}$ : inactive set

$\theta_j$ : transformation of response $Y$ for pair $(X_j, Y)$, $j = 1, 2, \ldots, p$

$\phi_j$ : transformation of $X_j$ for pair $(X_j, Y)$

$\rho_j$ : Pearson correlation of pair $(X_j, Y)$

$\epsilon_j^2$ : squared error by regressing $\phi_j$ on $\rho_j$

$\theta^*_j$ : optimal transformation of response $Y$ for pair $(X_j, Y)$

$\phi^*_j$ : transformation of $X_j$ for pair $(X_j, Y)$

$\rho^*_j$ : maximum correlation of pair $(X_j, Y)$

$\epsilon^*_j^2$ : squared error by regressing $\phi^*_j$ on $\theta^*_j$

$\theta^*_{nj}$ : spline approximation to optimal transformation $\theta^*_j$

$\phi^*_{nj}$ : spline approximation to optimal transformation $\phi^*_j$

$s_n$ : cardinality of active set $\mathcal{D}$

$\|\cdot\|$: operator norm

$\|\cdot\|_{sup}$: sup norm
5.2 Bernstein’s inequality and four facts

Lemma 2. (Bernstein’s inequality, Lemma 2.2.9, Van der Vaart and Wellner (1996)) For independent random variables $Y_1, \ldots, Y_n$ with bounded ranges $[-M, M]$ and 0 means,

$$P(|Y_1 + \ldots + Y_n| > x) \leq 2 \exp[-x^2/(2(v + Mx/3))]$$

for $v \geq \text{var}(Y_1 + \ldots + Y_n)$.

Under conditions (C3) and (C4), the following four facts hold when $\ell \geq d$.

Fact 1. There exists a positive constant $C_1$ such that (Burman, 1991)

$$E\{(\phi_j^* - \phi_{nj}^*)^2\} \leq C_1k^{-d}$$

(10)

Fact 2. There exists a positive constant $C_2$ such that (Huang et al., 2010)

$$E\{B^2_{jm}(\cdot)\} \leq C_2d_n^{-1}$$

(11)

Fact 3. There exist positive constants $c_{11}$, $c_{12}$ such that (Burman, 1991; Zhou et al., 1998)

$$c_{11}d_n^{-1} \leq \lambda_{\text{min}}(E\{B_j(\cdot)B_j^T(\cdot)\}) \leq \lambda_{\text{max}}(E\{B_j(\cdot)B_j^T(\cdot)\}) \leq c_{12}d_n^{-1}$$

$$c_{11}k^{-1} \leq \lambda_{\text{min}}(E\{\psi_j(X_j)\psi_j^T(X_j)\}) \leq \lambda_{\text{max}}(E\{\psi_j(X_j)\psi_j^T(X_j)\}) \leq c_{12}k^{-1}$$

(12)

Fact 4. There exists a positive constant $C_3$ such that (Burman, 1991; Faouzi et al., 1999)

$$C_3k^{-1} \leq b_{jm} \leq 1, \quad 0 \leq \hat{b}_{jm} \leq 1$$

(13)
**Remark 1.** The choice of knots plays a role in establishing the sure screening property. When the knots of the B-splines are placed at the sample quantiles, $\hat{b}_{jm}$ is positive. When knots are uniform placed, $\hat{b}_{jm}$ can be zero with a small probability. According to Burman (1991, section 6a), when the marginal density $f_{X_j}(x) > \gamma_0 > 0$ by condition (C4) for each $X_j$, we have $P(\hat{b}_{jm} = 0$ for some $m = 1, \ldots, d_n) \leq k \exp(-\gamma_0 n/k)$. The results in Burman (1991) are based on equally spaced knots, and our proof for MC-SIS use the same choice of knots, as the probability of $\hat{b}_{jm}$ being zero is a small probability, we just acknowledge $\hat{b}_{jm} > 0$ in the proof. In fact, sure screening property still hold when the event $\hat{b}_{jm} = 0$ is included.

**Remark 2.** With $\ell$ fixed, $k$ and $d_n$ are of the same order, we replace $k$ with $d_n$ in the following proof for convenience.

### 5.3 Proof of Lemma 1

**Proof.** By Cauchy-Schwarz inequality, we have

$$E(\phi_j^{*2}) \leq 2E\{(\phi_j^* - \phi_{nj}^*)^2\} + 2E(\phi_{nj}^{*2})$$

Therefore,

$$E(\phi_{nj}^{*2}) \geq \frac{1}{2} E(\phi_j^{*2}) - E\{(\phi_j^* - \phi_{nj}^*)^2\}$$

Lemma 1 follows from condition (C5) together with $E(\phi_{nj}^{*2}) = \lambda_{j1}^*$. 

### 5.4 Proof of eight basic results

We list and prove eight results (R1) – (R8) which together form the major parts in proving sure screening property of MC-SIS. For the rest of the paper, we use $P_n$ to denote the
sample average.

R1. With \(c_{11}\) in Fact 3, we have that,

\[
||A_{j00}^{-1/2}|| \leq c_{11}^{-1/2}d_n^{1/2}
\]  

(14)

**Proof.** \(||A_{j00}^{-1/2}|| = \lambda_{\text{min}}^{-1/2}(A_{j00})\), result follows by Fact 3.

R2. There exist positive constant \(c_{13}\) such that

\[
||A_{j0X}|| \leq c_{13}d_n^{-1/2}
\]  

(15)

**Proof.** Let \(u = (u_1, \ldots, u_{d_n})^T \in R^{d_n}\) with \(\sum_{m=1}^{d_n} u_m^2 = 1\).

\[
u^T \mathbb{E}\{B_j(X_j)B_j^T(Y)\} \mathbb{E}\{B_j(Y)B_j^T(X_j)\}u = \sum_{i=1}^{d_n} \left[ \int \{ \sum_{m=1}^{d_n} u_mB_{jm}(X_j)\}B_{ji}(Y)dF \right]^2 \\
\leq \int \{ \sum_{m=1}^{d_n} u_mB_{jm}(X_j)\}^2dF \times \sum_{i=1}^{d_n} \{ \int B_{ji}^2(Y)dF \} \\
\leq \lambda_{\text{max}}[\mathbb{E}\{B_j(X_j)B_j^T(X_j)\}] \times d_n \max E\{B_j^2(Y)\}
\]

Then, \(||E\{B_j(Y)B_j^T(X_j)\}|| \leq (c_{12}C_2/d_n)^{1/2}\) by Fact 2 and Fact 3.

It can be easily shown that, for \(u \in R^{d_n-1}\) with \(\sum_{i=1}^{d_n-1} u_i^2 = 1\),

\[
u^T D_j^T D_j^T u = \sum_{m=1}^{d_n} \frac{1}{k^2b_{jm}^2} \left( \sum_{i=1}^{d_n-1} u_i z_{im} \right)^2 \leq C_3^{-2} \sum_{m=1}^{d_n} \left( \sum_{i=1}^{d_n-1} u_i z_{im} \right)^2 \leq C_3^{-2}
\]

which indicates \(||D_j^T|| \leq C_3^{-1}\).

Then, \(||A_{j0X}|| \leq ||E\{B_j(Y)B_j^T(X_j)\}|| ||D_j^T|| \leq c_{13}d_n^{-1/2}\) with \(c_{13} = (c_{12}C_2)^{1/2}C_3^{-1}\).
R3. For any given constant $c_4$, there exists a positive constant $c_8$ such that

\[ P\{|\|\hat{A}_{j00}^{1/2}\| - 1/2 \| \geq (\frac{c_8 + 1}{c_{11}}) d_n\} \leq 2d_n^2 \exp(-c_4 n d_n^{-3}) \]  

(16)

**Proof.** Since $|\|\hat{A}_{j00}^{1/2}\|| = \sqrt{|\|P_n\{B_j(Y)B_j^T(Y)\}\|^{-1}}|$. R3 can be obtained via equation (26) in Fan et al. (2011), which is $P\{|\|P_n\{B_j(Y)B_j^T(Y)\}\|^{-1} \| \geq (c_8 + 1)c_{11} d_n\} \leq 2d_n^2 \exp(-c_4 n d_n^{-3})$. \hfill \Box

R4. There exist some positive constants $c_6$, $c_7$ such that,

\[ P\{|\|\hat{A}_{j0X}^{1/2}\| \geq c_6 d_n^{-1/2}\} \leq 4d_n^2 \exp(-c_7 n d_n^{-2}) \]  

(17)

**Proof.** As $|\|\hat{A}_{j0X}^{1/2}\|| = |\|P_n\{B_j(Y)B_j^T(X_j)\}\hat{D}_j^T\| \|\|P_n\{B_j(Y)B_j^T(X_j)\}\| |\|\hat{D}_j^T\| | \leq |\|P_n\{B_j(Y)B_j^T(X_j)\}\| |\|\hat{D}_j^T\| |$, we firstly deal with $|\|P_n\{B_j(Y)B_j^T(X_j)\}\||$.

For any square matrix $A$ and $B$, $|\|A + B\| | \leq |\|A\| | + |\|B\| |$. We have

\[ |\|A\| | - |\|B\| | \leq |\|A - B\| | \quad \text{and} \quad |\|B\| | - |\|A\| | \leq |\|B - A\| | \]

Then,

\[ |\|A\| | - |\|B\| | \leq |\|A - B\| | \]

Let $V_j = P_n\{B_j(Y)B_j^T(X_j)\} - E\{B_j(Y)B_j^T(X_j)\}$. It follows that,

\[ |\|P_n\{B_j(Y)B_j^T(X_j)\}\| | - |\|E\{B_j(Y)B_j^T(X_j)\}\| | \leq |\|V_j\| | \]

It is easy to verify that,

\[ |\|P_n\{B_j(Y)B_j^T(X_j)\}\| | - |\|E\{B_j(Y)B_j^T(X_j)\}\| | \leq d_n |\|V_j\| |_{sup} \]
Since \( \|B_{jm}(\cdot)\|_{\text{sup}} \leq 1 \) and using Fact 2, we have

\[
\text{var}(B_{jm_1}(Y)B_{jm_2}(X_j)) \leq E\{B_{jm_1}^2(Y)B_{jm_2}^2(X_j)\} \leq E\{B_{jm_1}^2(Y)\} \leq C_2d_n^{-1}
\]

By Bernstein’s inequality, for any \( \delta > 0 \),

\[
P\{|(P_n - E)\{B_{jm_1}(Y)B_{jm_2}(X_j)\}| \geq \delta/n\} \leq 2 \exp\{-\frac{\delta^2}{2(C_2nd_n^{-1} + 2\delta/3)}\} \quad (18)
\]

Therefore,

\[
P\{|\|P_n\{B_j(Y)B_j^T(X_j)\}\| - \|E\{B_j(Y)B_j^T(X_j)\}\|\| \geq d_n\delta/n\} \leq 2d_n^2 \exp\{-\frac{\delta^2}{2(C_2nd_n^{-1} + 2\delta/3)}\}
\]

Recalling R2, we have,

\[
P\{|\|P_n\{B_j(Y)B_j^T(X_j)\}\| \geq d_n\delta/n + (c_{12}C_2/d_n)^{1/2}\} \leq 2d_n^2 \exp\{-\frac{\delta^2}{2(C_2nd_n^{-1} + 2\delta/3)}\}
\]

By taking \( \delta = c_8(c_{12}C_2)^{1/2}nd_n^{-3/2} \), we obtain that for some positive constant \( c_4 \),

\[
P\{|\|P_n\{B_j(Y)B_j^T(X_j)\}\| \geq (c_8 + 1)(c_{12}C_2/d_n)^{1/2}\} \leq 2d_n^2 \exp(-c_4nd_n^{-2}) \quad (19)
\]

Next we deal with \( |\|D_j^T\|\|. \) Using Bernstein’s inequality, we obtain that,

\[
P\{|b_{jm} - b_{jm}| \geq \delta/n\} \leq 2 \exp\{-\frac{\delta^2}{2(C_2nd_n^{-1} + 2\delta/3)}\} \quad (20)
\]

Since \( b_{jm} \geq C_3k^{-1} \), by taking \( \delta = C_3w_1nd_n^{-1} \) for \( w_1 \in (0, 1) \), we have that there exists
some positive constant $c_5$ such that

$$P\{b_jm \leq C_3(1 - w_1)d_n^{-1}\} \leq 2 \exp(-c_5 n d_n^{-1}) \quad (21)$$

For $u = (u_1, \ldots, u_{d_n-1})^T \in \mathbb{R}^{d_n-1}$ with $\sum_{i=1}^{d_n-1} u_i^2 = 1$,

$$u^T \hat{D}_j \hat{D}_j^T u = \sum_{m=1}^{d_n} \frac{1}{k^2 b_jm} \left( \sum_{i=1}^{d_n-1} u_i z_{im} \right)^2 \leq \max_m \frac{1}{k^2 b_jm} \quad (22)$$

Combining (20), (21) and (22), we have that

$$P\{|\hat{D}_j^T| \geq C_3^{-1}(1 - w_1)^{-1}\} \leq P\{\max_m \frac{1}{k b_jm} \geq C_3^{-1}(1 - w_1)^{-1}\}$$

$$\leq P\{\min_m \hat{b}_j \leq C_3(1 - w_1)k^{-1}\}$$

$$\leq 2d_n \exp(-c_5 n d_n^{-1}) \quad (23)$$

Combining (19), (23), and $||\hat{A}_{j0X}|| \leq ||P_n\{B_j(Y)B_j^T(X_j)\}|| \cdot ||\hat{D}_j^T||$, we have

$$P\{|\hat{A}_{j0X}| \geq (c_8 + 1)(c_{12} C_2)^{1/2} d_n^{-1/2} C_3^{-1}(1 - w_1)^{-1}\}$$

$$\leq P\{|P_n\{B_j(Y)B_j^T(X_j)\}| \geq (c_8 + 1)(c_{12} C_2)^{1/2} d_n^{-1/2} + P\{|\hat{D}_j^T| \geq C_3^{-1}(1 - w_1)^{-1}\}$$

$$\leq 2d_n^2 \exp(-c_4 n d_n^{-2}) + 2d_n \exp(-c_5 n d_n^{-1}) \quad (24)$$

Result in R4 follows by choosing $c_6, c_7$ accordingly.

**R5.** There exist some positive constants $c_9, c_{10}$ such that, for any $\delta > 0$,

$$P\{|\hat{A}_{j0X} - A_{j0X}| \geq c_9 d_n^2 \delta^2 / n^2 + c_{10} d_n \delta / n\}$$

$$\leq 8d_n^2 \exp(-\frac{\delta^2}{2(C_2 n d_n^{-1} + 2\delta/3)}) + 4d_n \exp(-c_5 n d_n^{-1}) \quad (25)$$
Proof. It is easy to derive

\[
\| \hat{A}_{j0X} - A_{j0X} \| = \| P_n \{ B_j(Y)B_j^T(X_j) \} \hat{D}_j^T - E \{ B_j(Y)B_j^T(X_j) \} D_j^T \| \\
\leq \| (P_n - E) \{ B_j(Y)B_j^T(X_j) \} \| \| \hat{D}_j^T - D_j^T \| + \| E \{ B_j(Y)B_j^T(X_j) \} \| \| \hat{D}_j^T - D_j^T \|
\]

(26)

It is proved in R2 that \( \| E \{ B_j(Y)B_j^T(X_j) \} \| \leq (c_{12}C_2/d_n)^{1/2} \) and that \( \| D_j^T \| \leq C_3^{-1} \).

Combining (18) and the fact that

\[
\| (P_n - E) \{ B_j(Y)B_j^T(X_j) \} \| \leq d_n \| (P_n - E) \{ B_j(Y)B_j^T(X_j) \} \|_{sup},
\]

we have that,

\[
P \{ \| (P_n - E) \{ B_j(Y)B_j^T(X_j) \} \| \geq d_n \delta/n \} \leq 2d_n^2 \exp \left\{ -\frac{\delta^2}{2(C_2nd_n^{-1} + 2\delta/3)} \right\}. \quad (27)
\]

For \( u \in R^{d_n-1} \) with \( \sum_{i=1}^{d_n-1} u_i^2 = 1 \),

\[
u^T (\hat{D}_j - D_j)(\hat{D}_j - D_j)^T u = \sum_{m=1}^{d_n} \left( \frac{1}{kb_{jm}} - \frac{1}{kb_{jm}} \right)^2 \left( \sum_{i=1}^{d_n-1} u_i z_{im} \right)^2 \\
\leq C_3^{-2} \max_m (\hat{b}_{jm} - b_{jm})^2 \frac{\delta/n}{b_{jm}^2}
\]

(28)

From (20), (21) and (28), we have that,

\[
P \{ \| \hat{D}_j^T - D_j^T \| \geq C_3^{2-2} (1 - w_1)^{-1} d_n \delta/n \} \\
\leq P \{ C_3^{-1} \max_m (\hat{b}_{jm} - b_{jm}) \geq C_3^{-1} \frac{\delta/n}{C_3(1 - w_1)d_n^{-1}} \} \\
\leq P \{ \max_m (\hat{b}_{jm} - b_{jm}) \geq \delta/n \} + P \{ \min_m \hat{b}_{jm} \leq C_3(1 - w_1)d_n^{-1} \} \\
\leq 2d_n \exp \left\{ -\frac{\delta^2}{2(C_2nd_n^{-1} + 2\delta/3)} \right\} + 2d_n \exp(-c_3nd_n^{-1})
\]

(29)
Therefore, together with (26), (27), (29) and union bound of probability, we have

\[
P\{||\hat{A}_{j0x} - A_{j0x}|| \geq \frac{d_2^2 \delta^2 / n^2}{C_3^2 (1 - w_1)} + \frac{c_{12} C_2^{1/2} d_1^{1/2} \delta / n}{C_3^3 (1 - w_1)} + C_3^{-1} d_n \delta / n\}
\leq 4d_n^2 \exp\{-\frac{\delta^2}{2(C_2^2 d_n^{-1} + 2\delta / 3)}\} + 4d_n \exp\{-\frac{\delta^2}{2(C_2^2 d_n^{-1} + 2\delta / 3)}\} + 4d_n \exp(-c_5 nd_n^{-1})
\]

Result in R5 can be obtained by adjusting the values of \(c_9\) and \(c_{10}\).

R6. For given \(c_4\) and \(c_5\), there exist positive constants \(c_{15}\) and \(c_{16}\) such that,

\[
P\{||\hat{A}_{jXX}^{-1}|| \geq c_{16} d_n\}
\leq 2d_n^2 \exp(-c_4 nd_n^{-2}) + 2d_n^3 \exp(-c_{15} nd_n^{-7}) + 2d_n^3 \exp(-c_5 nd_n^{-1})
\]

**Proof.** Follow the proof in Lemma 5 of Fan et al. (2011), we have that,

\[|\lambda_{\min}(\hat{D}_j \hat{D}^T_j) - \lambda_{\min}(D_j D^T_j)| \leq d_n ||V_j||_{\sup}, \text{ where } V_j = \hat{D}_j \hat{D}^T_j - D_j D^T_j\]

The \((s, m)\) entry of \(V_j\) is

\[
(V_j)^{(s, m)} = \left| \sum_{i=1}^{d_n} \frac{z_{si} z_{mi}}{k^2} \left( \frac{1}{b_{ji}} - \frac{1}{\hat{b}_{ji}} \right) \right| = \left| \sum_{i=1}^{d_n} \frac{z_{si} z_{mi}}{k^2 b_{ji}^2} \left( \frac{b_{ji}^2 - \hat{b}_{ji}^2}{\hat{b}_{ji}^2} \right) \right|
\]

\[
\leq C_3^{-2} d_n \max_i \left| \frac{b_{ji}^2 - \hat{b}_{ji}^2}{\hat{b}_{ji}^2} \right| \leq 2C_3^{-2} d_n \max_i \left| \frac{b_{ji} - \hat{b}_{ji}}{\hat{b}_{ji}} \right|^2
\]

It is clear that \(||V_j||_{\sup} \leq 2C_3^{-2} d_n \max_i \left| (b_{ji} - \hat{b}_{ji}) / \hat{b}_{ji}^2 \right|^2\). Together with (20) and (21),
we have

\[
P\{|\lambda_{\min}(\hat{D}_j\hat{D}_j^T) - \lambda_{\min}(D_jD_j^T)| \geq 2C_3^{-4}(1 - w_1)^{-2}d_n^4\delta/n\}
\leq P\{2C_3^{-2}d_n^2 \max_i |\hat{b}_{ji} - \bar{b}_{ji}| \geq 2C_3^{-2}d_n^2\delta/n \times C_3^{-2}(1 - w_1)^{-2}d_n^2\}
\leq P\{\max_m |\hat{b}_{jm} - b_{jm}| \geq \delta/n\} + P\{\min_m \hat{b}_{jm} \leq C_3(1 - w_1)d_n^{-1}\}
\leq 2d_n \exp\left\{-\frac{\delta^2}{2(C_3^2d_n^{-1} + 2\delta/3)}\right\}
+ 2d_n \exp(-c_5nd_n^{-1})
\]

which indicates that there exists a positive constant \(c_{14}\),

\[
P\{|\lambda_{\min}(\hat{D}_j\hat{D}_j^T) - \lambda_{\min}(D_jD_j^T)| \geq c_{14}d_n^4\delta/n\}
\leq 2d_n \exp\left\{-\frac{\delta^2}{2(C_3^2d_n^{-1} + 2\delta/3)}\right\}
+ 2d_n \exp(-c_5nd_n^{-1})
\tag{31}
\]

Due to the facts that

\[
c_{11}k^{-1} \leq \lambda_{\min}(D_jE\{B_j(X_j)B_j^T(X_j)\}D_j^T) \leq \lambda_{\max}(E\{B_j(X_j)B_j^T(X_j)\})\lambda_{\min}(D_jD_j^T) \leq c_{12}k^{-1}\lambda_{\min}(D_jD_j^T)
\]

and that

\[
c_{11}k^{-1}\lambda_{\max}(D_jD_j^T) \leq \lambda_{\min}(E\{B_j(X_j)B_j^T(X_j)\})\lambda_{\max}(D_jD_j^T) \leq \lambda_{\max}(D_jE\{B_j(X_j)B_j^T(X_j)\}D_j^T) \leq c_{12}k^{-1}
\]

we have

\[
\frac{c_{11}}{c_{12}} \leq \lambda_{\min}(D_jD_j^T) \leq \lambda_{\max}(D_jD_j^T) \leq \frac{c_{12}}{c_{11}}
\]

By taking \(\delta = w_2/c_{14}nd_n^{-4} \times c_{11}/c_{12}\) in (31) for any \(w_2 \in (0, 1)\), there exists a positive
constant $c_{15}$ such that,

$$P\{|\lambda_{\min}(\hat{D}_j\hat{D}_j^T) - \lambda_{\min}(D_jD_j^T)| \geq w_2\lambda_{\min}(D_jD_j^T)\} \leq 2d_n \exp(-c_{15}nd_n^{-2}) + 2d_n \exp(-c_5nd_n^{-1})$$

By following a similar argument in proving inequality (26) in NIS (Fan et al., 2011), we have,

$$P\{\lambda_{\min}^{-1}(\hat{D}_j\hat{D}_j^T) \geq (c_8 + 1)c_{12}/c_{11}\} \leq 2d_n \exp(-c_{15}nd_n^{-2}) + 2d_n \exp(-c_5nd_n^{-1}) \quad (32)$$

Similarly, it is easy to obtain

$$P\{\lambda_{\min}^{-1}(P_n\{B_j(X_j)B_j^T(X_j)\}) \geq (c_8 + 1)c_{11}^{-1}d_n\} \leq 2d_n^2 \exp(-c_4nd_n^{-3}) \quad (33)$$

Due to the fact that $\lambda_{\max}(H^{-1}) = \lambda_{\min}^{-1}(H)$, we have

$$||\hat{A}_{jXX}^{-1}|| = \lambda_{\min}^{-1}(\hat{A}_{jXX}) \leq \lambda_{\min}^{-1}(P_n\{B_j(X_j)B_j^T(X_j)\}) \lambda_{\min}^{-1}(\hat{D}_j\hat{D}_j^T)$$

Together with (32) and (33), we can obtain that

$$P\{||\hat{A}_{jXX}^{-1}|| \geq (c_8 + 1)^2c_{12}c_{11}^{-2}d_n\} \leq P\{\lambda_{\min}^{-1}(P_n\{B_j(X_j)B_j^T(X_j)\}) \lambda_{\min}^{-1}(\hat{D}_j\hat{D}_j^T) \geq (c_8 + 1)^2c_{12}c_{11}^{-2}d_n\} \leq P\{\lambda_{\min}^{-1}(P_n\{B_j(X_j)B_j^T(X_j)\}) \geq (c_8 + 1)c_{12}/c_{11}\} + P\{\lambda_{\min}^{-1}(\hat{D}_j\hat{D}_j^T) \geq (c_8 + 1)c_{11}^{-1}d_n\} \leq 2d_n^2 \exp(-c_4nd_n^{-3}) + 2d_n \exp(-c_{15}nd_n^{-2}) + 2d_n \exp(-c_5nd_n^{-1})$$

Therefore, R6 follows by choosing $c_{16} = (c_8 + 1)^2c_{12}c_{11}^{-2}$. \qed
R7. For any \( \delta > 0 \), given positive constant \( c_4 \), there exists a positive constant \( c_{17} \) such that,

\[
P\{ \| \hat{A}_{j00}^{-1/2} - A_{j00}^{-1/2} \| \geq c_{17}d^{\delta/2}/n \} \leq 2d_n^2 \exp(-c_4nd_n^{-3}) + 2d_n^2 \exp\left\{-\frac{\delta^2}{2(C_2nd_n^{-1} + 2\delta/3)}\right\}
\]

(34)

**Proof.** Using perturbation theory (Kato, 1995), it is proved (Burman, 1991, Lemma 6.3) that for some \( c_{18} > 0 \),

\[
\| \hat{A}_{j00}^{-1/2} - A_{j00}^{-1/2} \| \leq c_{18}\tilde{\gamma}^{-3/2}\| \hat{A}_{j00} - A_{j00} \|
\]

(35)

where \( \tilde{\gamma} \) is the minimum of the smallest eigenvalues of \( \hat{A}_{j00} \) and \( A_{j00} \). \( \tilde{\gamma} \) is positive by definition. Therefore,

\[
\tilde{\gamma}^{-1} = \max\{\lambda_{\text{min}}^{-1}(\hat{A}_{j00}), \lambda_{\text{min}}^{-1}(A_{j00})\} = \max\{\| \hat{A}_{j00}^{-1} \|, \| A_{j00}^{-1} \|\}
\]

From Fact 3 and R3, we have,

\[
c_{12}d_n \leq \| A_{j00}^{-1} \| \leq c_{11}d_n
\]

(36a)

\[
P\{\| P_n\{ B_j(Y)B_j^T(Y)\} \|^{-1} \| \geq (c_8 + 1)c_{11}^{-1}d_n \} \leq 2d_n^2 \exp(-c_4nd_n^{-3})
\]

(36b)

Combining (36a) and (36b) yields

\[
P\{ \tilde{\gamma}^{-1} \geq \max\{(c_8 + 1)c_{11}^{-1}d_n, c_{11}^{-1}d_n\} \} \leq 2d_n^2 \exp(-c_4nd_n^{-3})
\]

which is,

\[
P\{ \tilde{\gamma}^{-1} \geq (c_8 + 1)c_{11}^{-1}d_n \} \leq 2d_n^2 \exp(-c_4nd_n^{-3})
\]

(37)
Additionally, as proved in equation (33) in Fan et al. (2011), we have large deviation bound for \( \| (P_n - E) \{ B_j(Y) B_j^T(Y) \} \| \). 

\[
P\{ \| (P_n - E) \{ B_j(Y) B_j^T(Y) \} \| \geq d_n \delta / n \} \leq 2d_n^2 \exp\left\{ -\frac{\delta^2}{2(C_2nd_n^{1} + 2\delta/3)} \right\}
\]

By (35), (37), (38) and under the union bound of probability, we have that,

\[
P\{ \| \hat{A}_{j00} - A_{j00} \| \geq c_18(c_8 + 1)c_1^{-1}d_n \} + P\{ \| \hat{A}_{j00} - A_{j00} \| \geq d_n \delta / n \}
\leq 2d_n^2 \exp(-c_4nd_n^{-3}) + 2d_n^2 \exp\left\{ -\frac{\delta^2}{2(C_2nd_n^{1} + 2\delta/3)} \right\}
\]

Therefore, R7 follows by choosing \( c_{17} = c_{18}(c_8 + 1)c_1^{-3/2} \).

**R8.** For any \( \delta > 0 \), given positive constant \( c_4 \), there exist a positive constant \( c_{19} \) such that,

\[
P\{ \| \hat{A}_{jXX} - A_{jXX} \| \geq c_{19}(d_n^{5}\delta^3/n^3 + d_n^{3}\delta/n) \} \leq 8d_n^2 \exp\left\{ -\frac{\delta^2}{2(C_2nd_n^{1} + 2\delta/3)} \right\} +
4d_n^2 \exp(-c_4nd_n^{-3}) + 2d_n \exp(-c_{15}nd_n^{-7}) + 6d_n \exp(-c_5nd_n^{-1})
\]

**Proof.** It’s obvious that

\[
\| \hat{A}_{jXX}^{-1} - A_{jXX}^{-1} \| \leq \| A_{jXX}^{-1} \| \| A_{jXX} - \hat{A}_{jXX} \| \| \hat{A}_{jXX} \|^{-1}
\]
and that
\[
\|A_{jXX} - A_{jXX}\| = \|D_j P_n (B_j (X_j) B_j^T (X_j)) D_j^T - D_j E\{B_j(X_j)B_j^T(X_j)\} D_j^T\|
\leq \|D_j^T - D_j\| \|(P_n - E)\{B_j(X_j)B_j^T(X_j)\}\| \|D_j^T - D_j\| + 2\|P_n\{B_j(X_j)B_j^T(X_j)\}\| \|D_j\|
\]
(42)

From the similar reasoning in proving (19) and (27), it is easy to obtain that
\[
P\{\|P_n\{B_j(X_j)B_j^T(X_j)\}\| \geq (c_8 + 1)c_{13} d_n^{-1}\} \leq 2d_n^2 \exp(-c_4 n d_n^{-3})
\]
(43)
\[
P\left(\|(P_n - E)\{B_j(X_j)B_j^T(X_j)\}\| \geq d_n \delta/n\right) \leq 2d_n^2 \exp\left\{-\frac{\delta^2}{2(C_2 n d_n^{-1} + 2\delta/3)}\right\}
\]
(44)

With \(c_{19}\) chosen properly, results in R8 follows by combining Fact 3, (29), (30), (41), (42), (43), (44) and the fact \|D_j^T\| < C_{\delta}^{-1}.

5.5 Proof of Theorem 1

Proof of Theorem 1. Recall that
\[
\lambda_{j1}^\star = \|A_{j00}^{-1/2} A_{j0X} A_{jXX}^{-1} A_{jX0} A_{j00}^{-1/2}\|
\]
and that
\[
\tilde{\lambda}_{j1}^\star = \|\widehat{A}_{j00}^{-1/2} \widehat{A}_{j0X} \widehat{A}_{jXX}^{-1} \widehat{A}_{jX0}^T \widehat{A}_{j00}^{-1/2}\|
\]

Let \(a = A_{j00}^{-1/2}, b = A_{j0X}, H = A_{jXX}^{-1}, a_n = A_{j00}^{-1/2}, b_n = A_{j0X}, H_n = A_{jXX}^{-1}\).
\[
\overline{\lambda}_{j_1} = \lambda_{j_1}^* = ||a_n^T b_n^T H_n b_n a_n|| - ||a^T b^T H b|| 
\leq ||(a_n - a)^T b_n^T H_n b_n (a_n - a)|| + 2||(a_n - a)^T b_n^T H_n b_n a|| + ||a^T (b_n^T H_n b_n - b^T H b)a||
\]

\[
\triangleq S_1 + S_2 + S_3
\]

(45)

We denote the terms in r.h.s as \( S_1, S_2 \) and \( S_3 \) respectively. Furthermore, we let the r.h.s of inequalities (17), (25), (30), (34), (40) as \( Q_4, Q_5, Q_6, Q_7, Q_8 \).

Note that

\[
S_1 \leq ||a_n - a||^2 ||b_n||^2 ||H_n||
\]

(46)

By (17), (30), (34), we have that there exist a positive constant \( c_{20} \) such that,

\[
P\{S_1 \geq c_{20} d_n^3 \delta^2 / n^2 \} \leq Q_4 + Q_6 + Q_7
\]

(47)

As to \( S_2 \),

\[
S_2 \leq ||a_n - a|| ||b_n||^2 ||H_n|| ||a||
\]

(48)

By (14), (17), (30), (34), we have that there exist a positive constant \( c_{21} \) such that,

\[
P\{S_2 \geq c_{21} d_n^3 \delta / n \} \leq Q_4 + Q_6 + Q_7
\]

(49)
As to $S_3$,

\[
S_3 \leq ||a||^2 ||b_n^T H_n B_n - b^T H b|| \\
\leq ||a||^2 (||b_n - b|| H_n (b_n - b)|| + 2||b_n - b|| H_n b|| + ||b^T (H_n - H)b||) \\
\triangleq ||a||^2 (S_{31} + 2S_{32} + S_{33})
\]

(50)

Note that

\[
S_{31} \leq ||b_n - b||^2 ||H_n||
\]

(51)

By (25), (30), we have that there exist a positive constant $c_{22}$ such that,

\[
P\{S_{31} \geq c_{22} d_n^3 (\delta^2 / n^2 + \delta / n)^2\} \leq Q_5 + Q_6
\]

(52)

As to $S_{32}$,

\[
S_{32} \leq ||b_n - b|| ||H_n|| ||b||
\]

(53)

By (15), (25), (30), (34), we have that there exist a positive constant $c_{23}$ such that,

\[
P\{S_{32} \geq c_{23} d_n^{5/2} (\delta^2 / n^2 + \delta / n)\} \leq Q_5 + Q_6
\]

(54)

As to $S_{33}$,

\[
S_{33} \leq ||b||^2 ||H_n - H||
\]

(55)

By (15), (40), we have that there exist a positive constant $c_{24}$ such that,

\[
P\{S_{33} \geq c_{24} (d_n^4 \delta^3 / n^3 + d_n^2 \delta / n)\} \leq Q_8
\]

(56)
Combining (14),(50),(51),(53),(55), we have
\[
P\{S_3 \geq c_{22} d_n^6 (\delta^2/n^2 + \delta/n)^2 + c_{23} d_n^{7/2} (\delta^2/n^2 + \delta/n) + c_{24} (d_n^5 \delta^3/n^3 + d_n^3 \delta/n)\}
\leq 2Q_5 + 2Q_6 + Q_8
\]

(57)

Define \(\varsigma(d_n, \delta) = c_{20} d_n^5 \delta^2/n^2 + c_{21} d_n^3 \delta/n + c_{22} d_n^6 (\delta^2/n^2 + \delta/n)^2 + c_{23} d_n^{7/2} (\delta^2/n^2 + \delta/n) + c_{24} (d_n^5 \delta^3/n^3 + d_n^3 \delta/n)\). Then from (45),(47),(49),(57), we have that due to symmetry,
\[
P\{|\hat{\lambda}_{j_1} - \lambda_{j_1}^*| \geq \varsigma(d_n, \delta)\} \leq 4Q_4 + 4Q_5 + 8Q_6 + 4Q_7 + 2Q_8
\]

(58)

By properly choosing the value of \(\delta\) (i.e., taking \(\delta = c_2 (c_{22} + c_{23})^{-1} d_n^{-5/2} n^{1-2\kappa}\)), we can make \(\varsigma(d_n, \delta) = c_2 d_n n^{-2\kappa}\), for any \(c_2 > 0\). Then, we have
\[
P\{|\hat{\lambda}_{j_1} - \lambda_{j_1}^*| \geq c_2 d_n n^{-2\kappa}\} \leq O\left(d_n^2 \exp\{-c_3 n^{1-4\kappa} d_n^{-4}\} + d_n \exp\{-c_4 n d_n^{-7}\}\right)
\]

(59)

The first part of Theorem 1 follows via the union bound of probability.

To prove the second part, we define a event
\[
\mathcal{A}_n \equiv \{\max_{j \in \mathcal{D}} |\hat{\lambda}_{j_1} - \lambda_{j_1}^*| \leq c_1 \xi d_n n^{-2\kappa}/2\}
\]

By Lemma we have
\[
\hat{\lambda}_{j_1}^* \geq c_1 \xi d_n n^{-2\kappa}/2, \forall j \in \mathcal{D}
\]

(60)

Thus, by choosing \(\nu_n = c_5 d_n n^{-2\kappa}\) with \(c_5 \leq c_1 \xi/2\). We have that \(\mathcal{D} \subseteq \hat{\mathcal{D}}_{\nu_n}\). Therefore,
\[
P(\mathcal{A}_n^c) \leq O\left(s_n \{d_n^2 \exp\{-c_3 n^{1-4\kappa} d_n^{-4}\} + d_n \exp\{-c_4 n d_n^{-7}\}\}\right)
\]

35
Then the probability bound for the second part of Theorem 1 is attained.

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