Distributed Averaging Problems on Signed Networks with Directed Topologies

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Abstract—This paper aims at addressing distributed averaging problems for signed networks in the presence of general directed topologies that are represented by signed digraphs. A new class of improved Laplacian potential functions is proposed by presenting two notions of any signed digraph: induced unsigned digraph and mirror (undirected) signed graph, based on which two distributed averaging protocols are designed using the nearest neighbor rules. It is shown that with any of the designed protocols, signed-average consensus (respectively, state stability) can be achieved if and only if the associated signed digraph of signed network is structurally balanced (respectively, unbalanced), regardless of whether weight balance is satisfied or not. Additionally, the convergence analysis of signed networks can be implemented via the Lyapunov stability approach, which benefits from bridging the relationship between convergence behaviors of directed signed networks and properties of improved Laplacian potential functions. Illustrative examples are presented to demonstrate the validity of our theoretical results for distributed averaging of directed signed networks.

Index Terms—Directed topology, distributed averaging, signed-average consensus, signed network, structural balance.

I. INTRODUCTION

Networked systems generally consist of multiple interacting agents to deal with problems that are difficult to address for a single agent. One of the most considered classes of networked systems is called unsigned networks (also called conventional or traditional networks), where cooperative interactions among agents are only contained. To characterize unsigned networks, unsigned graphs can be conveniently employed such that their nodes and edges with positive weights are adopted to represent agents and cooperative interactions among agents, respectively. By leveraging this property, distributed control has attracted considerable attention in unsigned networks (see, e.g., [1]–[6]). In particular, consensus is one of the fundamental distributed control problems in unsigned networks such that all the agents cooperate with each other to accomplish a common objective.

Of specific interest in consensus is the average consensus of unsigned networks such that all nodes converge to the average value of their initial states. In the consensus literature, average consensus generally can provide a solution for the distributed averaging problems on unsigned networks, which has been extensively investigated (see, e.g., [7]–[12]). In [7], a distributed control protocol has been proposed to guarantee the average consensus of unsigned networks under unsigned digraphs that are both strongly connected and weight balanced.

The average consensus problems of unsigned networks under arbitrary strongly connected topologies have been discussed in [8], with which a novel protocol is designed by introducing a “surplus” variable in comparison with the protocol of [7]. Toward this problem, another new protocol has been designed by leveraging the left eigenvector of the Laplacian matrix associated with the zero eigenvalue [9]. Regardless of unsigned networks subject to time-varying communication topologies, the average consensus problems have been explored in [10], [11]. Besides, a distributed protocol has been proposed to ensure the average consensus of unsigned networks in the presence of time-varying delays [12].

Recently, signed networks emerging from social networks have drawn great interests because of its potential applications (see [13] for more details). Different from unsigned networks, signed networks contain not only cooperative interactions but also antagonistic interactions, which should be described under signed digraphs such that the edges with positive (respectively, negative) weights can be leveraged to represent the cooperative (respectively, antagonistic) interactions among nodes. Owing to the antagonistic interactions among nodes, signed networks may naturally generate more plentiful collective behaviors than unsigned networks. In [14], a basic framework of addressing distributed control issues on signed networks is established. It has been shown that, under strongly connected communication topologies, signed networks achieve bipartite consensus when they are associated with structurally balanced signed digraphs; and the state stability emerges, otherwise. Bipartite consensus represents that all nodes converge to two different values with the same modulus but opposite signs, and it includes consensus as a trivial case. In addition, not only has bipartite consensus been extended to signed networks with general linear dynamics [15]–[17], but also many other convergence behaviors of signed networks have been explored, such as bipartite flocking [18], interval bipartite consensus [19]–[21], modulus consensus [22], [23], bipartite containment tracking [24] and finite-time bipartite consensus [25], [26].

However, there have been presented quite limited results for solving distributed averaging problems on signed networks. A main reason is that the average consensus may no longer work due to the presence of antagonistic interactions. To overcome this problem, distributed averaging of signed networks mainly concentrate on designing distributed protocols to accomplish signed-average consensus, instead of average consensus. In [14], it has been revealed that signed networks can be ensured to reach signed-average consensus when the associated signed digraphs are strongly connected and weight balanced. Nevertheless, it is worth pointing out that the protocol of [14] is
ineffective for the signed-average consensus once the weight balance is broken. To the best of our knowledge, there have been no studies on signed-average consensus issues of signed networks under any directed topologies.

In this paper, we contribute to solving distributed averaging problems for arbitrary directed signed networks in the presence of strongly connected communication topologies. We provide two notions of any signed digraph: induced unsigned digraph and mirror signed graph, with which we simultaneously introduce a new class of improved Laplacian potential functions to measure the total disagreements among nodes. By benefiting from improved Laplacian potential functions, we can induce two distributed control protocols based on the nearest neighbor rules for signed networks. Our two proposed protocols can ensure the signed-average consensus (respectively, state stability) of the signed network if and only if its associated signed digraph is structurally balanced (respectively, unbalanced), no matter whether the weight balance is satisfied or not. Besides, we can establish a relationship between convergence behaviors of signed networks and improved Laplacian potential functions. This brings benefits in deriving the convergence analysis for our two proposed protocols by using the Lyapunov stability approach. In particular, since unsigned networks are considered as a special case of signed networks, our developed results can be applied to unsigned networks under any directed topologies. Two illustrative examples are exhibited to demonstrate the effectiveness of our theoretical results.

The rest of the paper is organized as follows. In Section II, the problem statement of distributed averaging control is provided for signed networks. In Section III, we propose some notions of signed digraphs: including induced unsigned digraph, mirror signed graph and improved Laplacian potential functions. In Section IV, two control protocols are designed based on the nearest neighbor rules to ensure the signed-average consensus of signed networks. Simulation examples and conclusions are given in Sections V and VI, respectively.

Notations: For a positive integer $n$, $\mathcal{F}_n = \{1, 2, \ldots, n\}$, $1_n = [1, 1, \ldots, 1]^T \in \mathbb{R}^n$, $0_n = [0, 0, \ldots, 0]^T \in \mathbb{R}^n$, and $\text{diag}(d_1, d_2, \ldots, d_n)$ is diagonal matrix whose diagonal elements are $d_1, d_2, \ldots, d_n$ and non-diagonal elements are zero. For a matrix $M \in \mathbb{R}^{n \times n}$, $\det(M)$ and $\mathcal{N}(M)$ represent the determinant and null space of $M$, respectively. We denote $M > 0$ (or $M \geq 0$) and $M < 0$ (or $M \leq 0$) as the positive (or semi-positive) definite matrix and negative (or semi-negative) definite matrix, respectively. For a real number $a \in \mathbb{R}$, let $|a|$ and sgn$(a)$ represent the absolute value and the sign function of $a$, respectively. The set of all $n$-by-$n$ gauge transformations is given by

$$D = \{D_n = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\} : \sigma_i \in \{-1, 1\}, i \in \mathcal{F}_n\}.$$

II. PROBLEM STATEMENT

Consider signed networks with a collection of $n$ nodes given by $\mathcal{V} = \{v_i : i \in \mathcal{F}_n\}$. Let every node $v_i$ have single-integrator dynamics described by

$$\dot{x}_i(t) = u_i(t), \quad i \in \mathcal{F}_n \quad (1)$$

where $x_i(t) \in \mathbb{R}$ and $u_i(t) \in \mathbb{R}$ are the information state and control protocol of $v_i$, respectively. The problem of our interest is to design protocols to achieve the distributed averaging of all nodes in the presence of cooperative-antagonistic interactions among nodes. For the convenience of our following analyses, we denote $x_i(0) = x_{i0}, \forall i \in \mathcal{F}_n$ and

$$x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T,$$

$$x_0 = [x_{10}, x_{20}, \ldots, x_{n0}]^T.$$

We say that the signed network (1) achieves signed-average consensus if for any initial states $x_{i0} \in \mathbb{R}$, $\forall i \in \mathcal{F}_n$, there exist some scalars $\sigma_i \in \{\pm 1\}$ such that

$$\lim_{t \to \infty} x_i(t) = \frac{\sigma_i}{n} \sum_{j=1}^{n} \sigma_j x_{j0}, \quad i \in \mathcal{F}_n \quad (2)$$

where $\sigma_i$ assigns the sign to each node $v_i$. It is worth pointing out that the signed-average consensus may reflect the effects of both cooperative and antagonistic interactions among nodes by the selections of scalars $\sigma_i, \forall i \in \mathcal{F}_n$. This gives an alternative solution to the distributed averaging problem of nodes involved in signed networks. In particular, if $\sigma_i = 1, \forall i \in \mathcal{F}_n$, then the signed-average consensus collapses into the traditional average consensus of (unsigned) networks.

Another fact we need to highlight is that the signed-average consensus (2) represents a specific type of bipartite consensus. In general, we say that the signed network (1) reaches bipartite consensus if for any $x_{i0} \in \mathbb{R}$, $\forall i \in \mathcal{F}_n$, there exist some scalars $\sigma_i \in \{\pm 1\}$ and $c \neq 0$ such that (see also [14] Definition 1)

$$\lim_{t \to \infty} x_i(t) = \sigma_i c, \quad i \in \mathcal{F}_n$$

where $c$ depends closely on $x_{i0}, \forall i \in \mathcal{F}_n$. For signed networks, the (state) stability is usually considered a counterpart problem of bipartite consensus. Namely, for any $x_{i0} \in \mathbb{R}, \forall i \in \mathcal{F}_n$, the signed network (1) is said to achieve stability if

$$\lim_{t \to \infty} x_i(t) = 0, \quad i \in \mathcal{F}_n.$$

Though many notable results have been derived for bipartite consensus of signed networks, none of them can be adopted to address the signed-average consensus problems in the presence of directed topologies. This issue will be solved in the current paper to achieve the distributed averaging of nodes in directed signed networks. A new protocol design method for distributed averaging will be introduced simultaneously, for which an idea of a new class of improved Laplacian potential functions will be explored for a signed directed graph (digraph for short).

Remark 1. By resorting to the state vector $x(t)$, we know that signed-average consensus (respectively, bipartite consensus) of (1) refers to $\lim_{t \to \infty} x(t) = (1_n^T D_n x_{0n}/n) D_n 1_n$ (respectively, $\lim_{t \to \infty} x(t) = c D_n 1_n$, where $D_n \in D$ denotes some gauge transformation, and that stability of (1) means $\lim_{t \to \infty} x(t) = 0$. It has been revealed in the literature (see, e.g., [14]) that $D_n$ is generally related to the sign patterns of signed networks. In addition, the behaviors of signed networks depend heavily on their sign patterns. These will be also developed for distributed averaging of signed networks with general directed topologies.
III. Graph-Theoretic Analysis of Signed Networks

We first introduce notions and properties related with signed digraphs, together with their induced graphs. Then we explore the Laplacian matrices of signed digraphs and their motivated Laplacian potential functions of the signed network \( (\mathcal{G}, E, A) \). In the following discussions, the time variable \( t \) will be omitted for simplicity when no confusions may be caused.

A. Notions and Properties of Signed Digraphs

A signed digraph is represented by \( \mathcal{G} = (\mathcal{V}, E, A) \) consisting of a node set \( \mathcal{V} = \{v_1, v_2, \ldots, v_n\} \), an edge set \( E \subseteq \{(v_i, v_j) : v_i, v_j \in \mathcal{V}\} \) and an adjacency weight matrix \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) such that \( a_{ij} \neq 0 \Leftrightarrow (v_j, v_i) \in E \) and \( a_{ii} = 0 \), otherwise. Let \( \mathcal{G} \) have no self-loops, i.e., \( a_{ii} = 0, \forall i \in \mathcal{F}_n \), and \( \mathcal{G} \) be digon sign-symmetric, i.e., \( a_{ij}a_{ji} \geq 0, \forall i, j \in \mathcal{F}_n \). For each \( v_i \), let us define its in-degree and out-degree as \( \text{deg}_\text{in}(v_i) = \sum_{j=1}^{n} |a_{ij}| \) and \( \text{deg}_\text{out}(v_i) = \sum_{j=1}^{n} |a_{ji}| \), respectively. If \( \text{deg}_\text{out}(v_i) = \text{deg}_\text{in}(v_i), \forall i \in \mathcal{F}_n \), then \( \mathcal{G} \) is called a weight balanced signed digraph. In particular, when \( A = A_T \) holds, \( \mathcal{G} \) collapses into an undirected signed graph. If we denote \( \Delta = \text{diag}(\text{deg}_\text{in}(v_1), \text{deg}_\text{in}(v_2), \ldots, \text{deg}_\text{in}(v_n)) \) as the in-degree matrix of \( \mathcal{G} \), then the Laplacian matrix of \( \mathcal{G} \), denoted by \( L = \Delta - A \).

An edge \( e_h = (v_j, v_i) \in E \) represents the accessibility of the information of \( v_j \) by \( v_i \), by which \( v_j \) is called a neighbor of \( v_i \). All neighbors of \( v_i \) are collected in \( N(i) = \{j : (v_j, v_i) \in E\} \).

A directed path \( \mathcal{P} \) of length \( k \) is formed by a finite sequence of edges satisfying \( e_i = (v_{m_{i-1}}, v_{m_i}), \forall i \in \mathcal{F}_k \), where \( v_{m_0}, v_{m_1}, \ldots, v_{m_k} \) are distinct nodes. By contrast, an undirected path \( \mathcal{P} \) of length \( k \) allows \( e_i = (v_{m_{i-1}}, v_{m_i}) \) or \( e_i = (v_{m_i}, v_{m_{i+1}}), \forall i \in \mathcal{F}_k \).

We proceed to present a useful lemma to reveal the relations between the connectivity and sign pattern properties of \( \mathcal{G} \) and of \( \hat{\mathcal{G}} \).

**Lemma 1.** For any signed digraph \( \mathcal{G} \), the Laplacian matrix \( \hat{L} \) of its mirror signed graph \( \hat{\mathcal{G}} \) can be given by

\[
\hat{L} = WL + L^T W \frac{2}{2}.
\]

**Proof:** See Appendix A.

**Lemma 2.** For a strongly connected signed digraph \( \mathcal{G} \) and its mirror signed graph \( \hat{\mathcal{G}} \), the following results hold:

1. \( \hat{\mathcal{G}} \) is connected with \( \hat{E} \) given by \( \hat{E} = E \cup \hat{E} \), where \( \hat{E} = \{(v_j, v_i) : (v_i, v_j) \in E\} \);
2. \( \hat{\mathcal{G}} \) has the same sign pattern as \( \mathcal{G} \) such that \( \hat{\mathcal{G}} \) is structurally balanced (respectively, unbalanced) if and only if \( \mathcal{G} \) is structurally balanced (respectively, unbalanced).

**Proof:** See Appendix B.

As a consequence of Lemma 2 the following lemma shows the relationship between the Laplacian matrices of \( \mathcal{G} \) and of \( \hat{\mathcal{G}} \) from the viewpoint of their null spaces.

**Lemma 3.** For a strongly connected signed digraph \( \mathcal{G} \) and its mirror signed graph \( \hat{\mathcal{G}} \), \( \mathcal{N}(\hat{L}) = \mathcal{N}(L) \) holds, and moreover,

1. \( \mathcal{N}(\hat{L}) = \text{span}\{D_n1_n\} \) if and only if \( \mathcal{G} \) is structurally balanced, where \( D_n \in \mathbb{D} \) is such that \( \hat{L} = D_nLD_n \);
2. \( \mathcal{N}(\hat{L}) = 0_n \) if and only if \( \mathcal{G} \) is structurally unbalanced.

**Proof:** See Appendix C.

B. Improved Laplacian Potential Function and Its Properties

The Laplacian potential function can be exploited to measure the total disagreements of all nodes, which plays a central role in investigating the distributed control problems of signed networks. On the specific, the distributed control protocol is induced from the gradient-based feedback of Laplacian potential functions (see e.g., \([1], [7], [14]\)). Besides, the Laplacian potential function is employed to derive the convergence analysis for dynamic behaviors of signed networks (see e.g., \([7], [14], [25], [27], [28] \)), in which the weight balance assumption on signed digraphs is necessary. We can easily see that the requirement of weight balance may generate limitations on
using Laplacian potential functions to deal with the distributed control problems of signed networks. To avoid this drawback, we propose a new class of Laplacian potential functions for signed networks that is named as improved Laplacian potential function and is provided in the following definition.

**Definition 3 (Improved Laplacian Potential Function):** For any signed digraph \( G \), an improved Laplacian potential function \( \Phi_e(x) \) of the signed network (1) is defined as

\[
\Phi_e(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \det(\mathcal{T}_{ij}) |a_{ij}|(x_i - \text{sgn}(a_{ij})x_j)^2. \tag{4}
\]

We provide an expression of \( \Phi_e(x) \) with matrices \( L \) and \( W \), and give an useful property of \( \Phi_e(x) \) in the following lemma.

**Lemma 4.** For the signed network (1) under a signed digraph \( G \), its improved Laplacian potential function \( \Phi_e(x) \) satisfies

\[
\Phi_e(x) = x^T(WL + L^T W)x. \tag{5}
\]

Furthermore, when the signed digraph \( G \) is strongly connected, the following results hold.

1) If \( G \) is structurally balanced, then \( \Phi_e(x) \) is semi-positive definite, and \( \Phi_e(x) = 0 \) implies \( x = cD_n1_n \) for some \( c \in \mathbb{R} \).

2) If \( G \) is structurally unbalanced, then \( \Phi_e(x) \) is positive definite, and \( \Phi_e(x) = 0 \) denotes \( x = 0_n \).

**Proof:** See Appendix D. 

**Remark 2.** From [14], the Laplacian potential function \( \Phi(x) \) is given by

\[
\Phi(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|(x_i - \text{sgn}(a_{ij})x_j)^2.
\]

When the signed digraph is weight balanced, \( \Phi(x) \) satisfies

\[
\Phi(x) = x^T(L + L^T)x. \tag{6}
\]

It is worth noticing that the equation (6) fails once the weight balance condition is broken, which may lead to constraints on the application of \( \Phi(x) \) when the associated signed digraph is weight unbalanced (see e.g., [14, 22, 27, 28]). In contrast to \( \Phi(x) \), the improved Laplacian potential function \( \Phi_e(x) \) has a series of coefficients \( \det(\mathcal{T}_{ii}), \forall i \in F_n \). From Lemma 4 we can develop that the equation (5) holds regardless of whether the associated signed digraph is weight balanced or not, which makes it possible to employ \( \Phi_e(x) \) to solve distributed control problems of signed networks under both weight balanced and unbalanced signed digraphs. This greatly extends the range of application for Laplacian potential functions in the studies of distributed control problems of signed networks.

As a consequence of Lemma 4, we can induce the following corollary for improved Laplacian potential functions.

**Corollary 1.** Consider a strongly connected, signed digraph \( G \). If \( G \) is weight balanced, then \( \Phi_e(x) \) satisfies

\[
\Phi_e(x) = \alpha \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|(x_i - \text{sgn}(a_{ij})x_j)^2 = \alpha x^T(L + L^T)x \tag{7}
\]

for some \( \alpha > 0 \).

**Proof:** Since the signed digraph \( G \) is strongly connected and weight balanced, it follows from [29] that

\[
\det(\mathcal{T}_{i1}) = \det(\mathcal{T}_{22}) = \cdots = \det(\mathcal{T}_{nn}) = \alpha
\]

where \( \alpha > 0 \). We thus can derive \( W = \alpha I_n \). It is immediate to obtain (7) from (5). This proof is complete.

From (6) and (7), it is obvious that \( \Phi(x) \) is only a particular case of \( \Phi_e(x) \) when the signed digraph \( G \) is strongly connected and weight balanced.

### IV. DISTRIBUTED AVERAGING PROTOCOLS AND RESULTS

#### A. Protocols and Consensus Analyses

In this subsection, we aim to propose two distributed control protocols such that signed networks achieve the signed-average consensus (respectively, stability) if and only if their associated signed digraphs are structurally balanced (respectively, unbalanced), regardless of whether the weight balance condition is satisfied or not. Benefiting from improved Laplacian potential functions, we can develop the convergence analysis for the two protocols by employing the Lyapunov stability approach.

For every \( v_i \), we propose the first control protocol based on the nearest neighbor rule as

\[
u_i = - \sum_{j \in N(i)} |\hat{a}_{ij}|(x_i - \text{sgn}(\hat{a}_{ij})x_j), \; \forall i \in F_n. \tag{8}
\]

We should point out that the protocol (8) can also be obtained by the gradient-based feedback of \( \Phi_e(x) \). By using \( L \), we can write (1) and (8) as a compact form

\[
x = -\hat{L}x. \tag{9}
\]

If the protocol (8) is applied, then signed-average consensus results can be provided in the following theorem.

**Theorem 1.** Consider the system (9) under a signed digraph \( G \) that is strongly connected. Then, the system (9) can achieve
1) the signed-average consensus if and only if \( G \) is structurally balanced;
2) the state stability if and only if \( G \) is structurally unbalanced.

**Proof:** We choose a Lyapunov function candidate for the system (9) as

\[
V(x) = x^T x.
\]

Taking the derivation of \( V \) along (9) yields

\[
\dot{V}(x) = \dot{x}^T x + x^T \dot{x} = -x^T (\dot{L}^T + \dot{L})x = -x^T (WL + L^T W)x = -\Phi_e(x).
\]

\( \blacksquare \)

#### 1. When the signed digraph \( G \) is structurally balanced, one can know \( V \leq 0 \) from Lemma 4. With LaSalle’s Theorem [30 Theorem 4.4], we can obtain that the trajectories converge to the largest invariant set \( S = \{ x \in \mathbb{R}^n | V(x) = 0 \} \) as \( t \to \infty \). This, together with Lemma 4, can guarantee \( S = \{ x \in \mathbb{R}^n | x = cD_n1_n, c \in \mathbb{R} \} \), which implies that the system (9) can achieve bipartite consensus when \( G \) is structurally balanced.

In the following, we investigate the converged value of the system (9). Since \( G \) is strongly connected and structurally balanced, its mirror signed graph \( \hat{G} \) is connected and structurally balanced;
balanced from Lemma 2. We can deduce that \( \nu_l = D_n 1_n \) and \( \nu_r = D_n 1_n \) are the left and right eigenvectors of \( \hat{L} \) associated with zero eigenvalue, respectively. The terminal state of the system (9) is given by

\[
x(\infty) = \lim_{t \to \infty} e^{-Lt} x(0) = \frac{\nu_r \nu_l^T}{\nu_l^T \nu_r} x(0) = \frac{\nu_r \nu_l^T}{\nu_l^T \nu_r} x(0) = \frac{1}{n} D_n 1_n.
\]

With (10), it is immediate to develop that the system (9) can achieve the signed-average consensus.

2) When the signed digraph \( G \) is structurally unbalanced, it follows from Lemma 2 that \( \hat{G} \) is structurally unbalanced and \( \hat{L} \) is a positive definite matrix, which leads to \( \dot{V}(x) < 0 \). This, together with [30, Theorem 4.1], can guarantee that the system (9) is asymptotically stable.

“\( \Rightarrow \)”:
1) We adopt the proof by contradiction to develop this result and assume that \( \hat{G} \) is structurally unbalanced. It follows from Lemma 2 that \( \hat{G} \) is also structurally unbalanced and \( \hat{L} \) is Hurwitz stable. Hence, the system (9) is asymptotically stable, which causes a contradiction on the system (9) achieving the signed-average consensus. Conversely, the signed digraph \( \hat{G} \) is structurally balanced.

2) Assume that \( \hat{G} \) is structurally balanced. Then, the system (9) can achieve the bipartite consensus which causes a contradiction that the system (9) can achieve state stability. Thus, \( \hat{G} \) is structurally unbalanced. We complete this proof.

Remark 3. According to theorem 1, we know that the signed-average consensus problems of signed networks under strongly connected signed digraphs can be equivalently converted into the bipartite consensus problems of signed networks under the associated mirror signed graphs, which gives a new viewpoint to study distributed averaging problems of signed networks.

Different from the protocol (8), we give the second protocol based on the nearest neighbor rule as

\[
u_i = -\sum_{j \in N(i)} \det(T_{ij}) |a_{ij}| (x_i - \text{sgn}(a_{ij}) x_j), \quad i \in F_n. \tag{11}
\]

Remark 4. In comparison with the existing protocol (14)

\[
u_i = -\sum_{j \in N(i)} |a_{ij}| (x_i - \text{sgn}(a_{ij}) x_j), \quad i \in F_n, \tag{12}
\]

the protocol (11) includes a gain \( \det(T_{ij}) \), \( i \in F_n \). When \( \hat{G} \) is strongly connected and weight balanced, it follows from [29] that there exists a positive real number \( \alpha \) such that \( \det(T_{11}) = \det(T_{22}) = \cdots = \det(T_{nn}) = \alpha \) holds. Therefore, the protocol (11) becomes

\[
u_i = -\alpha \sum_{j \in N(i)} |a_{ij}| (x_i - \text{sgn}(a_{ij}) x_j), \quad i \in F_n. \tag{13}
\]

Compared with the protocols (12) and (13), we can easily find that the protocol (12) is a particular case of the protocol (13), in which the gain \( \alpha \) just has an effect on the convergence rate.

Applying the protocol (11) to the system (1) yields

\[
\dot{x} = -WLx. \tag{14}
\]

The following lemma discloses the distribution of eigenvalues for the matrix \( WL \).

Lemma 5. Consider a strongly connected signed digraph \( G \). The following results hold.

1) \( WL \) has a zero eigenvalue and all other eigenvalues with positive real parts if and only if \( \hat{G} \) is structurally balanced.
2) All eigenvalues of \( WL \) have positive real parts if and only if \( \hat{G} \) is structurally unbalanced.

Proof: Because the matrix \( W \) is a diagonal matrix and all diagonal elements are positive real numbers. We immediately obtain these results from [21, Theorems 4.1 and 4.2].

With Lemma 5, we are in position to present signed-average consensus results of signed networks in the following theorem.

Theorem 2. Consider the system (14) under a signed digraph \( G \) that is strongly connected. Then, the system (14) can reach

1) the signed-average consensus if and only if \( \hat{G} \) is structurally balanced;
2) the state stability if and only if \( \hat{G} \) is structurally unbalanced.

Proof: We construct a Lyapunov function candidate

\[
\dot{V}(x) = x^T x.
\]

Taking the derivation of \( V \) along (14) yields

\[
\dot{V}(x) = \dot{x}^T x + x^T \dot{x} = -x^T (LT W + WL)x = -\Phi_e(x).
\]

1) When \( \hat{G} \) is structurally balanced, there exists a matrix \( D_n \in \mathbb{D} \) such that \( \hat{L} = D_n LD_n \) holds. From [29], we can realize \( \det(T_{11}), \det(T_{22}), \cdots, \det(T_{nn}) = \alpha \), which implies \( \det(T_{ij}) \sum_{j=1}^{n} |a_{ij}| = \sum_{j=1}^{n} \det(T_{jj}) |a_{jj}|, \forall i \in F_n \). We can further induce \( \alpha \) to \( \frac{1}{n} T^T W L = \frac{1}{n} D_n LD_n \), leads to \( \frac{1}{n} T^T W L = \frac{1}{n} D_n WD_n LD_n = \frac{1}{n} D_n WLD_n = \frac{1}{n} \).

We can thus develop that \( \nu_l = D_n 1_n \) and \( \nu_r = D_n 1_n \) are the left and right eigenvectors of \( WL \) associated with zero eigenvalue, respectively. The rest of proof is similar to the proof of Theorem 1.

2) When \( \hat{G} \) is structurally unbalanced, the proof is same as the proof of Theorem 1 and we omit it for simplicity.

Remark 5. From Theorems 1 and 2, we can solve the signed-average consensus issues for any directed signed networks, in which two distributed control algorithms are proposed to ensure the signed-average consensus no matter whether the associated signed digraphs are weight balanced or not. It greatly extends the existing bipartite consensus results of signed networks (see e.g., [14]). Because unsigned networks are a particular case of signed networks, our proposed protocols (8) and (11) can also solve average consensus issues of unsigned networks although their associated unsigned digraphs are weight unbalanced.

B. Application of Improved Laplacian Potential Functions

In this subsection, we target at introducing an application of improved Laplacian potential functions. From [14] Remark 2, we realize that the Lyapunov stability method can be exploited to develop the convergence analysis of the protocol (12) when the associated signed digraph is strongly connected and weight balanced. However, this analysis method is ineffective once the
weight balance condition is broken. Motivated by the proofs of Theorems 1 and 2, we find that this drawback can be removed with the help of improved Laplacian potential functions. Let us elaborate on this statement.

Employing the protocol (12) to the system (1) yields

\[ \dot{x} = -Lx. \]  

(15)

From [14], it is immediate to obtain the following convergence results for the system (15).

**Corollary 2.** Consider the system (15) whose communication topology is described by a strongly connected signed digraph \( G \). Then, the following results hold.

1) The system (15) achieves the bipartite consensus if and only if \( G \) is structurally balanced.

2) The system (15) achieves the state stability if and only if \( G \) is structurally unbalanced.

Next, we can provide the proof of Corollary 2 by exploiting the Lyapunov stability method, no matter whether the weight balance is satisfied or not.

**Proof:** We choose a Lyapunov function candidate as

\[ V_1(x) = x^T W x. \]

We can validate that \( V_1(x) \) is positive definite via the positive definiteness of \( W \). Taking the derivation of \( V_1(x) \) along the system (15) causes

\[ \dot{V}_1(x) = -x^T (WL + L^T W)x = -\Phi_v(x). \]

Following by the proof of Theorems 1 and 2, we realize that the system (15) can achieve the bipartite consensus if and only if \( G \) is structurally balanced and otherwise, the system (15) can reach the state stability. We complete this proof. \( \blacksquare \)

Through the above discussions, we can extend the Lyapunov stability method to deal with distributed control problems of directed signed networks regardless of whether their associated signed digraphs are weight balanced or not. In addition, it can also provide an alternative approach to construct the Lyapunov function for convergence analysis of signed networks, in which the requirement of weight balance on signed digraphs (see e.g., 25, 27, 28) can be removed.

V. SIMULATIONS

In this section, we introduce two examples to illustrate the developed theoretical results. We employ two signed digraphs in Fig. 1 to describe the interactions among nodes and adopt the initial states of nodes as

\[ x(0) = [1, 2, 3, 4, 5, 6]^T. \]

It is obvious from Fig. 1 that \( G_a \) and \( G_b \) are both strongly connected and weight balanced.

**Example 1.** Consider the communication topology for the system (1) described by \( G_a \) in Fig. 1. Because \( G_a \) is structurally balanced, there exists a gauge transformation \( D_b = \text{diag} \{1, 1, 1, -1, -1, -1\} \). The system (1) can achieve signed-average consensus if the terminal states of nodes satisfy

\[ \lim_{t \to \infty} x_i(t) \in \{1.5, -1.5\}, \quad i \in \mathcal{F}_6. \]

By employing the protocol (12), we can plot the state evolution of the system (1) in Fig. 2. This figure depicts that the states \( x_i, i \in \mathcal{F}_6 \) can reach bipartite consensus with the modulus value 0.4348. Clearly, the signed-average consensus is not achieved with (12). By applying the protocols (8) and (11) to the system (1), the state evolutions of all nodes can be described in Figs. 2b) and 2c), respectively. From Figs. 2b) and 2c), the states of nodes polarize with the polarized values 1.5 and -1.5, which coincides with the developed signed-average consensus results of Theorems 1 and 2.

**Example 2.** We employ the signed digraph \( G_b \) to represent the communication topology of the system (1). Different from \( G_a \), \( G_b \) is structurally unbalanced. When the protocols (12), (8) and (11) are applied to the system (1), the state evolutions of all nodes are plotted in Figs. 3a), 3b) and 3c), respectively. We can easily see from Figs. 3a)-c) that all nodes converge to zero. The simulation tests are consistent with the developed state stability results of Theorems 1 and 2.

VI. CONCLUSIONS

In this paper, we have investigated the distributed averaging problems of signed networks under arbitrary signed digraphs. We have introduced the improved Laplacian potential function for signed networks, with which two distributed protocols are designed to make sure the signed-average consensus of signed networks no matter whether their associated signed digraphs...
are weight balanced or not. Benefitting from improved Laplacian potential functions, we have developed the convergence analysis for our two proposed protocols by using the Lyapunov stability approach, which proposes a novel viewpoint to solve distributed control problems of signed networks. In addition, we have introduced two simulation examples to illustrate the effectiveness of our developed results.

APPENDIX

A. Proof of Lemma 7
Proof: Let \( \hat{L} \) and \( \hat{\Delta} = \text{diag}\{\hat{\Delta}_{11}, \hat{\Delta}_{22}, \ldots, \hat{\Delta}_{nn}\} \) denote the Laplacian matrix and in-degree matrix of \( \hat{G} \), respectively. The diagonal element of \( 1/2(WL + L^TW) \) is given by
\[
\hat{\Delta}_{ii} = \frac{\det(T_{ii}) \sum_{j=1}^{n} |a_{ij}| + \det(T_{jj}) \sum_{j=1}^{n} |a_{ji}|}{2}, \quad \forall i \in \mathcal{F}_n.
\]
Define a vector \( w = [\det(T_{11}), \det(T_{22}), \ldots, \det(T_{nn})]^T \in \mathbb{R}^n \) and \( w \) satisfies \( w^T L = 0^T \) from (29). We thus can induce
\[
\det(T_{ii}) \sum_{j=1}^{n} |a_{ij}| = \sum_{j=1}^{n} \det(T_{jj}) |a_{ji}|, \quad \forall i \in \mathcal{F}_n.
\]
With (17), the equation (16) can be rewritten as
\[
\hat{\Delta}_{ii} = \frac{\det(T_{ii}) \sum_{j=1}^{n} |a_{ij}| + \sum_{j=1}^{n} \det(T_{jj}) |a_{ji}|}{2} = \hat{\Delta}_{ii}
\]
Let \( \hat{\Delta} = \text{diag}\{\hat{\Delta}_{11}, \hat{\Delta}_{22}, \ldots, \hat{\Delta}_{nn}\} \). It can further derive
\[
\frac{1}{2}(WL + L^TW) = \hat{\Delta} - \frac{WA + A^T}{2} = \hat{\Delta} - \hat{\Delta} = \hat{\Delta}
\]
which satisfies the definition of the Laplacian matrix. Hence, \( 1/2(WL + L^TW) \) is a valid Laplacian matrix of \( \hat{G} \).

B. Proof of Lemma 2
Proof: 1) Due to the strong connectivity of \( G \), all elements \( \det(T_{ii}), \det(T_{ii}), \ldots, \det(T_{nn}) \) are positive real numbers from (29). This, together with (3), ensures
\[
a_{ij} \neq 0 \Rightarrow \hat{a}_{ij} \neq 0 \quad \text{and} \quad \hat{a}_{ij} \neq 0, \quad \forall i, j \in \mathcal{F}_n
\]
which leads to that the mirror signed graph \( \hat{G} \) is connected.

2) Because \( \hat{G} \) is strongly connected, the mirror signed graph \( \hat{G} \) is connected and undirected. It follows immediately from (3) that
\[
\begin{align*}
\{ a_{ij} > 0 \text{ or } a_{ji} > 0 \} & \iff \hat{a}_{ij} > 0 \text{ and } \hat{a}_{ji} > 0 \\
\{ a_{ij} < 0 \text{ or } a_{ji} < 0 \} & \iff \hat{a}_{ij} < 0 \text{ and } \hat{a}_{ji} < 0, \quad \forall i, j \in \mathcal{F}_n.
\end{align*}
\]
With (31), we realize that \( G \) is structurally balanced if and only if all semi-cycles of \( G \) are positive. By employing (19), we can derive that all semi-cycles of \( G \) are positive if and only if all semi-cycles of \( G \) are positive. Therefore, \( \hat{G} \) is structurally balanced if and only if \( \hat{G} \) is structurally balanced.

Since structural balance and unbalance are mutually exclusive properties, we can directly develop that \( \hat{G} \) is structurally unbalanced if and only if \( \hat{G} \) is structurally unbalanced.

C. Proof of Lemma 3
Proof: 1) “⇒”: According to (14), we can know \( \mathcal{N}(L) = \text{span}\{D_{n1} \} \) when \( G \) is structurally balanced. On one hand, for \( \forall x \in \mathcal{N}(L) \), it follows \( x^T L x = x^T (WL + L^TW)x = 0_n \) which leads to \( x \in \mathcal{N}(\hat{L}) \). We thus can obtain \( \mathcal{N}(L) \subseteq \mathcal{N}(\hat{L}) \). On the other hand, since \( G \) is structurally balanced and strongly connected, a direct consequence of Lemma 2 is that \( \hat{G} \) is also structurally balanced and connected. It follows from Lemma 1 and [14, Lemma 1] that rank \( \hat{L} = n - 1 \) holds. Hence, we can deduce \( \mathcal{N}(\hat{L}) = \mathcal{N}(L) = \text{span}\{D_{n1} \} \).

“⇒”: Because of \( \mathcal{N}(\hat{L}) = \mathcal{N}(L) = \text{span}\{D_{n1} \} \), it can induce \( LD_{n1} = 0_n \) and rank(L) = n - 1. We can further derive \( LD_{n1} L_{n1} = \mathcal{T}_{n1} = 0_n \) which implies \( LD_{n1} L_{n1} = \mathcal{T} \). It is immediate to know that \( G \) is structurally balanced from [21, Theorem 4.1].

2) “⇐”: Since \( G \) is structurally unbalanced, all eigenvalues of \( L \) and \( \hat{L} \) have positive real parts. Hence, \( \mathcal{N}(\hat{L}) = \mathcal{N}(L) = 0_n \) holds.

“⇒”: Owing to \( \mathcal{N}(\hat{L}) = \mathcal{N}(L) = 0_n \), we can realize that the equation \( Lx = 0_n \) have no non-zero solution and thus obtain rank(L) = n. It follows from [14, Corollary 3] that \( G \) is structurally unbalanced.

D. Proof of Lemma 4
Proof: For the first statement, we can calculate
\[
x^T WL x = \sum_{i=1}^{n} \det(T_{ii}) x_i \sum_{j=1}^{n} |a_{ij}| (x_i - \text{sgn}(a_{ij}) x_j)
\]
\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} \det(T_{ii}) |a_{ij}| x_i (x_i - \text{sgn}(a_{ij}) x_j). 
\]
By employing (17), we can further deduce
\[ x^T W L x = \sum_{i=1}^{n} \det(L_{ii}) x_i \sum_{j=1}^{n} |a_{ij}| (x_j - \text{sgn}(a_{ij}) x_j) \]
\[ = \sum_{i=1}^{n} \det(L_{ii}) x_i^2 \sum_{j=1}^{n} |a_{ij}| - \sum_{i=1}^{n} \sum_{j=1}^{n} \det(L_{ij}) |a_{ij}| \text{sgn}(a_{ij}) x_i x_j \]
\[ = \sum_{i=1}^{n} x_i^2 \sum_{j=1}^{n} \det(L_{jj}) |a_{ij}| - \sum_{i=1}^{n} \sum_{j=1}^{n} \det(L_{ij}) |a_{ij}| \text{sgn}(a_{ij}) x_i x_j \]
\[ = \sum_{j=1}^{n} \sum_{i=1}^{n} \det(L_{ij}) |a_{ij}| x_j (x_j - \text{sgn}(a_{ij}) x_i). \] (21)

With (20) and (21), we have
\[ x^T (WL + L^T W)x = 2x^T W L x \]
\[ = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \det(T_{ij}) |a_{ij}| x_i (x_i - \text{sgn}(a_{ij}) x_j) \right. \]
\[ + \left. \sum_{i=1}^{n} \sum_{j=1}^{n} \det(L_{ij}) |a_{ij}| x_j (x_j - \text{sgn}(a_{ij}) x_i) \right\} \]
\[ = \sum_{i=1}^{n} \sum_{j=1}^{n} \det(L_{ii}) |a_{ij}| (x_i - \text{sgn}(a_{ij}) x_j)^2 = \Phi_e(x). \]

Hence, the equation (5) holds. Based on (5), the remaining of this proof can be immediately derived from Lemmas 1-3.

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