Abstract—In this paper, we explored the algebraic structures of solution spaces for Gaussian latent factor analysis when the population covariance matrix $\Sigma_x$ is generated due to a latent Gaussian star graph. In particular, we found sufficient and necessary conditions under which the solutions to constrained minimum trace factor analysis (CMTF A) and constrained minimum determinant factor analysis (CMDFA) are still star. The later one (CMDFA) is also the problem of Wyner’s common information, which has been under extensive study in recent years. In addition, we further showed that the solution to CMTF A under the star constraint can only have two cases, i.e., the number of latent variable can be only one (star) or $n - 1$, where $n$ is the dimension of the observable vector.

Index Terms—Factor Analysis, MTFA, CMTFA, CMDFA

I. INTRODUCTION

Factor Analysis (FA) is a commonly used tool in multivariate statistics to represent the correlation structure of a set of observables in terms of significantly smaller number of variables called “latent factors”. With the growing use of data mining, high dimensional data and analytics, factor analysis has already become a prolific area of research [1] [2].

In traditional factor analysis of real $n$-dimensional random vector $X \in \mathbb{R}^n$ with Gaussian distribution $\mathcal{N}(0, \Sigma_x)$, where $\Sigma_x$ is the known population covariance matrix, the objective is to decompose $\Sigma_x$ as $\Sigma_x = (\Sigma_x - D) + I_n$, where $\Sigma_x - D$ is Gramian and $D$ is diagonal. Classical approaches to solve this problem are Minimum Rank Factor Analysis (MRFA) [3] and Minimum Trace Factor Analysis (MTFA) [4]. As the name suggests MRFA seeks to minimize the rank of $\Sigma_x - D$ and MTFA minimizes the trace of $\Sigma_x - D$. MTFA solution could lead to negative values for the diagonal entries of the matrix $D$. To solve this problem Constrained Minimum Trace Factor Analysis (CMTFA) was proposed [5], that imposes extra constraint of requiring $D$ to be Gramian. Computational aspects of CMTFA and uniqueness of its solution was discussed in [6].

Since $X$ is Gaussian random vector, factor analysis of $X$ can be modelled by the following equation,

$$X = AY + Z$$  \hspace{1cm} (1)

where $A_{n \times k}$ is a real matrix, $Y_{k \times 1}$, $k < n$ is the vector of independent latent variables and $Z_{n \times 1}$ is a Gaussian vector of zero mean and covariance matrix $\Sigma_x = D$. We have,

$$I(X; Y) = H(X) - H(X|Y) = H(X) - H(Z)$$  \hspace{1cm} (2)

where $I(X; Y)$ is the mutual information between $X$ and $Y$, $H(X)$, $H(Z)$ are entropies of $X$ and $Z$ and $H(X|Y)$ is the conditional entropy of $X$ given $Y$. Characterizing the common information between $X$ and $Y$ [7] [8] [9], $\min_{A, \Sigma_x} I(X; Y)$ is an equivalent problem to $\max_{Z} H(Z)$. Moharrer and Wei in [10] established relationship between the common information problem and MTFA, and named the problem Constrained Minimum Determinant Factor Analysis (CMDFA), because $\min_{A, \Sigma_x} I(X; Y)$ for the above model is equivalent to $\min_{Z} -log|\Sigma_x|$.

In [11] the same condition was found on the subspace of $\Sigma_x$ for MTFA solution to be a star as the one we give through this paper for CMTFA solution to be a star. For clarification, to make a star $\Sigma_x$ must be decomposable as a sum of a rank 1 matrix plus a diagonal matrix. The condition they found for MTFA was only a sufficient condition, we proved the same condition both sufficient and necessary for CMTFA. Another major difference between their work and ours is that we also characterized the solution of CMTFA and analysed the conditions on $\Sigma_x$ when the CMTFA solution of $\Sigma_x$ is not a star, which they did not address in their MTFA solution.

In this paper we generate a certain family by requiring $\Sigma_x$ has a latent star structure as shown in [59]. We analysed the solution space of both CMTFA and CMDFA for $\Sigma_x$. The novel contribution of this paper is we prove that there are only two solutions possible for CMTFA, one of which we prove is still a star, which means the optimal solution has the dimension $k = 1$. We also give the necessary and sufficient condition for the structure of the solution when it is not a star. For CMDFA we give the if and only if condition for the solution to be a star.

The rest of the paper is organized as follows: section II gives the necessary and sufficient conditions for two possible CMTFA solutions of $\Sigma_x$. Section III gives if and only if condition for CMDFA solution of $\Sigma_x$ to be a star. The last section has the conclusion.

II. SOLUTION TO CMTFA PROBLEMS UNDER A STAR CONSTRAINT

In line with the affine model in [1], we impose the following prior constraints on the generation of $X \in \mathbb{R}^n$.

$$\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} [Y] + \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$$  \hspace{1cm} (3)

where

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• \( Y \sim \mathcal{N}(0, 1) \)
• \( 0 < |\alpha_j| < 1, \ j = 1, 2, \ldots, n \).
• \( \{ Z_j \} \) are independent Gaussian random variables with \( Z_j \sim \mathcal{N}(0, 1 - \alpha_j^2) \).

We denote the real column vector \( \vec{\alpha} = [\alpha_1, \ldots, \alpha_n]' \in \mathbb{R}^n \).

Using \( || \vec{x} ||_I \) we have,

\[
\Sigma_x = \vec{\alpha} \vec{\alpha}' + \Sigma_z
\] (4)

If for any element of \( \vec{\alpha} \) the following condition holds, we call it a non-dominant element, otherwise it's a dominant element.

\[
|\alpha_i| \leq \sum_{j \neq i} |\alpha_j| \quad i = 1, 2, \ldots, n
\] (5)

It is easy to see that there can be only one dominant element in a vector and that has to be the element with the biggest absolute value among all. We call \( \vec{\alpha} \) dominant if its biggest element in terms of absolute value is dominant, otherwise \( \vec{\alpha} \) is non-dominant.

Let us, without the loss of generality, assume that \( \alpha_1 \) has the largest absolute value and thus all dominance is defined with respect to it.

The following necessary and sufficient condition for CMTFA solution was set in \([12]\).

The point \( d^* \) is a solution of the CMTFA problem if and only if \( \lambda(d^*) = 0, \ d^* \geq 0 \) and there exist \( t_i \in N(\Sigma_x - D^*) \), \( i = 1, \ldots, r \) such that the following holds,

\[
1 = \sum_{i=1}^{r} t_i^2 = \sum_{j \in I(d^*)} \mu_j \bar{x}_j
\] (6)

where \( r \leq n, 1 \) is \( n \) dimensional column vector with all the components equal to 1, \( \{ t_i \} \in N(\Sigma_x - D^*) \), \( i = 1, \ldots, r \) are \( n \) dimensional column vectors forming the rank \((n - k)\) matrix \( T \), \( t_i^2 \) is the Hadamard product of vector \( t_i \) with itself, \( \lambda(d^*) \) is the minimum eigenvalue of \( \Sigma_x - D^* \), \( d^* \) is the vector having all the diagonal entries of the diagonal matrix \( D^* \), \( I(d^*) = \{ i : d^*_i = 0, i \leq n \} \), \( \mu_j, j \in I(d^*) \) are non-negative numbers and \( \{ \bar{x}_j, j \in I(d^*) \} \) are column vectors in \( \mathbb{R}^n \) with the components equal to 0 except for the \( j \)th component which is equal to 1.

We have proved that if we apply CMTFA to \( \Sigma_x \) in (7) then the solution is either the rank 1 matrix \( \Sigma_{l,ND} \) given in (8) or the rank \( n - 1 \) matrix \( \Sigma_{l,DM} \) given in (9).

\[
\Sigma_x = \begin{bmatrix}
1 & \alpha_1 \alpha_2 & \ldots & \alpha_1 \alpha_n \\
\alpha_2 \alpha_1 & 1 & \ldots & \alpha_2 \alpha_n \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_n \alpha_1 & \alpha_n \alpha_2 & \ldots & 1
\end{bmatrix}
\] (7)

\[
\Sigma_{l,ND} = \begin{bmatrix}
\alpha_1^2 & \alpha_1 \alpha_2 & \ldots & \alpha_1 \alpha_n \\
\alpha_2 \alpha_1 & \alpha_2^2 & \ldots & \alpha_2 \alpha_n \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_n \alpha_1 & \alpha_n \alpha_2 & \ldots & \alpha_n^2
\end{bmatrix}
\] (8)

\[
\Sigma_{l,DM} = \begin{bmatrix}
(\Sigma_{l,DM})_{11} & \alpha_1 \alpha_2 & \ldots & \alpha_1 \alpha_n \\
\alpha_2 \alpha_1 & (\Sigma_{l,DM})_{22} & \ldots & \alpha_2 \alpha_n \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_n \alpha_1 & \alpha_n \alpha_2 & \ldots & (\Sigma_{l,DM})_{nn}
\end{bmatrix}
\] (9)

where

\[
(\Sigma_{l,DM})_{ii} = |\alpha_i| \left( \sum_{j \neq i} |\alpha_j| \right), \quad i = 2, \ldots, n
\]

In next two subsections we present the analytical details of those two solutions in terms of vector \( \vec{\alpha} \) being dominant or non-dominant.

A. Dominant Case

In this subsection we analyse the conditions under which the CMTFA solution of \( \Sigma_x \) is not a star. Following two Lemmas are essential to understand the Theorem to follow.

Lemma 1. \( \Sigma_{l,DM} \) is a rank \( n - 1 \) matrix.

We basically showed that, \( \sum_{i=1}^{n} s_i (\Sigma_{l,DM})_{i} = 0 \) where, \( (\Sigma_{l,DM})_{i} \) is the \( i \)th row of \( \Sigma_{l,DM} \) and \( s_i = \{ 1, -1 \} \). The detailed proof is given in Appendix D.

Lemma 2. There exists a column vector \( \Phi = [\Phi_1, \Phi_2, \ldots, \Phi_n]' \) such that \( \Sigma_{l,DM} \Phi = 0 \), where \( \Phi_i \in \{ -1, 1 \}, 1 \leq i \leq n \).

In the two chambered proof of Lemma 2 we first find a vector \( \Phi \), \( \Phi_i \in \{ -1, 1 \} \) such that \( (\Sigma_{l,DM})_{i} \Phi = 0 \). Then we show that the same vector is orthogonal to the other rows of \( \Sigma_{l,DM} \) as well. The detailed proof is given in Appendix E.

Theorem 1. \( \Sigma_{l,DM} \) is the CMTFA solution of \( \Sigma_x \) if and only if \( \vec{\alpha} \) is dominant.

Proof of Theorem 7. To prove the Theorem we refer to necessary and sufficient condition set in (6). Rank of \( \Sigma_{l,DM} \) is \( n - 1 \), so its minimum eigenvalue is 0. Since each \( 0 < |\alpha_j| < 1, 0 < (\Sigma_{l,DM})_{ii} < 1, i = 1, \ldots, n \). Hence all the diagonal entries \( d_i \) of \( D \) are positive. As a result, the set \( I(d^*) \) is empty and the second term in the right hand side of (6) vanishes.

The dimension of the null space of \( \Sigma_{l,DM} \) is 1. It will suffice for us to prove the existence of a column vector \( \Phi_i, \Phi_i \in \{ 1, 1 \}, 1 \leq i \leq n \) such that \( \Sigma_{l,DM} \Phi = 0 \). Lemma 2 gives that proof.

B. Non-Dominant Case

This subsection is dedicated to the analytical details of the conditions under which the CMTFA solution of star structured \( \Sigma_x \) is also star. The following Lemma is essential to prove the Theorem to follow.

Lemma 3. There exists rank \( n - 1 \) matrix \( T_{n \times n} \) such that the column vectors of \( T \) are in the null space of \( \Sigma_{l,ND} \) and the \( L_2 \)-norm of each row of \( T \) is 1.
Proof of Lemma 3
Its trivial to find the following basis vectors for the null space of \( \Sigma_{t,ND} \).

\[
v_1 = \begin{bmatrix}-\frac{\alpha_1}{\alpha_1} \\
1 \\
0 \\
\vdots \\
0
\end{bmatrix}, \quad v_2 = \begin{bmatrix}-\frac{\alpha_1}{\alpha_1} \\
0 \\
1 \\
\vdots \\
0
\end{bmatrix}, \quad \ldots \quad v_{n-1} = \begin{bmatrix}-\frac{\alpha_1}{\alpha_1} \\
0 \\
0 \\
\vdots \\
1
\end{bmatrix}
\]

We define matrix \( V \) as,

\[
V = \begin{bmatrix}
-\frac{\alpha_2}{\alpha_1} & -\frac{\alpha_3}{\alpha_1} & \cdots & -\frac{\alpha_n}{\alpha_1} \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
\end{bmatrix}
\]

The columns of \( V \) given in (11) span the null space of \( \Sigma_{t,ND} \). To prove the lemma, it will suffice for us to find a diagonal matrix \( B_{n \times n} \) such that the follows holds.

\[
T_{n \times n} = V_{n \times n}^T B_{n \times n}
\]

where, \( L_2 \)-norm of each row of \( T \) is 1. Using (12),

\[
TT' = VBB'V'
\]

We define the symmetric matrix \( \beta = BB' \), and we require the diagonal matrix \( \beta \) to have only non-negative entries.

Since we want each diagonal element of \( TT' \) to be 1, we have the following \( n \) equations,

\[
\frac{\alpha_2^2}{\alpha_1^2} \beta_{11} + \frac{\alpha_3^2}{\alpha_1^2} \beta_{22} + \cdots + \frac{\alpha_n^2}{\alpha_1^2} \beta_{n-1,n-1} + \\
\left( \frac{c_2 \alpha_2}{\alpha_1} + \frac{c_3 \alpha_3}{\alpha_1} + \cdots + \frac{c_n \alpha_n}{\alpha_1} \right)^2 \beta_{nn} = 1
\]

\[
\beta_{ii} + \frac{\alpha_1^2}{\alpha_i^2} \beta_{nn} = 1, \quad i = 1, \ldots, n - 1
\]

Solving, (14) we get,

\[
\beta_{nn} = \frac{\alpha_1^2 - \alpha_2^2 - \alpha_3^2 - \cdots - \alpha_n^2}{\sum_{i \neq j, i \neq 1,j \neq 1} c_i c_j \alpha_i \alpha_j}
\]

Since the diagonal entries of \( \beta \) can only be non-negative, we have the following three cases.

\[
\alpha_1^2 - \alpha_2^2 - \alpha_3^2 - \cdots - \alpha_n^2 = 0 \\
\alpha_1^2 - \alpha_2^2 - \alpha_3^2 - \cdots - \alpha_n^2 > 0 \\
\alpha_1^2 - \alpha_2^2 - \alpha_3^2 - \cdots - \alpha_n^2 < 0
\]

It will suffice for us to prove that for all of the above cases there exist \( \{ c_j, 2 \leq j \leq n \} \) that make \( \beta_{ii} \geq 0, i = 1, \ldots, n \).

Case 1: is straightforward. If \( \alpha_1^2 - \alpha_2^2 - \alpha_3^2 - \cdots - \alpha_n^2 = 0 \) then using (14) and (15) we get, \( \beta_{nn} = 0 \) and \( \beta_{11} = \beta_{22} = \cdots = \beta_{n-1,n-1} = 1 \).

Before we move on to the remaining two cases, without the loss of generality, we can re-arrange the elements of \( \tilde{\alpha} \) such that,

\[
|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_n|
\]

Normalizing each element by \( |\alpha_1| \) gives us the following,

\[
1 \geq |\tilde{\alpha}_2| \geq |\tilde{\alpha}_3| \geq \cdots \geq |\alpha_n|
\]

where \( \tilde{\alpha}_j = \frac{\alpha_j}{\alpha_1}, 1 \leq j \leq n \). We define,

\[
S_{min} = \min_A \left| \sum_{j \in A} |\tilde{\alpha}_j| - \sum_{j \in A^c} |\tilde{\alpha}_j| \right|
\]

where \( A \subset \{2,3,\ldots, n\} \) and \( A^c = \{2,3,\ldots, n\} - A \)

Let, of all the possibilities \( A = A^* \) be the subset of \( \{2,3,\ldots, n\} \) that gives us \( S_{min} \). Assuming that \( A^* \) has \( l \) elements, let the set \( A^* \) be \( A^* = \{a_1, a_2, \ldots, a_l\} \). We define the set \( F \) as,

\[
F = \{ F_{a_1}, \ldots, F_{a_l}, \quad F_{a_i} = |\tilde{\alpha}_{a_i}|, a_i \in A^* \}
\]

Now under this ordered and normalized settings, we have

Case 2: \( 1 - \tilde{\alpha}_2^2 - \tilde{\alpha}_3^2 - \cdots - \tilde{\alpha}_n^2 > 0 \)

We can select \( c_2, c_3, \ldots, c_n \) in a such way that \( c_i \tilde{\alpha}_i = |\tilde{\alpha}_i| \) to make \( \beta_{nn} > 0 \). Equation (15) dictates that to ensure the other diagonal entries of \( \beta \) are non-negative, the following must hold,

\[
1 - \tilde{\alpha}_2^2 - \tilde{\alpha}_3^2 - \cdots - \tilde{\alpha}_n^2 \leq 1 \\
\sum_{i \neq j, i \neq 1,j \neq 1} c_i c_j \tilde{\alpha}_i \tilde{\alpha}_j \\
\Rightarrow 1 \leq (|\tilde{\alpha}_2| + |\tilde{\alpha}_3| + \cdots + |\tilde{\alpha}_n|)^2 \\
\Rightarrow 1 \leq |\tilde{\alpha}_2| + |\tilde{\alpha}_3| + \cdots + |\tilde{\alpha}_n|
\]

which means such \( \beta_{nn} \) exists if and only if \( \tilde{\alpha}_1 \) is non-dominant.

Because of the ordered representation, that essentially means \( \alpha \) has to be non-dominant.

Case 3: \( 1 - \tilde{\alpha}_2^2 - \cdots - \tilde{\alpha}_n^2 < 0 \)

Using the lemma we proved in Appendix C of this paper, if we select \( c_i \in \{1, -1\} \) such that \( \sum_{j=2} c_j \tilde{\alpha}_j = S_{min} \) then,

\[
\sum_{i \neq j, i \neq 1,j \neq 1} c_i c_j \tilde{\alpha}_i \tilde{\alpha}_j < 0.
\]

And for such selection of \( c_i \) we have,

\[
\tilde{\alpha}_2^2 + \tilde{\alpha}_3^2 + \cdots + \tilde{\alpha}_n^2 + \sum_{i \neq j, i \neq 1,j \neq 1} c_i c_j \tilde{\alpha}_i \tilde{\alpha}_j = \left( \sum_{i=2} c_i \tilde{\alpha}_i \right)^2 = S_{min}^2 \leq 1
\]

The last inequality is due to the lemma we proved in Appendix A of this paper, that shows \( S_{min} \leq F_{a_i}, a_i \in A^* \). So, we have,

\[
\beta_{nn} = 0 \quad \text{and} \quad \beta_{11} = \beta_{22} = \cdots = \beta_{n-1,n-1} = 1
\]
Both the terms $1 - \alpha_1^2 - \alpha_2^2 - \cdots - \alpha_n^2$ and \[\sum_{i \neq j, i \neq 1, j \neq 1} c_i c_j \alpha_i \alpha_j\] are negative. Hence, \[\beta_{nn} = \frac{1 - \alpha_1^2 - \alpha_2^2 - \cdots - \alpha_n^2}{\sum_{i \neq j, i \neq 1, j \neq 1} c_i c_j \alpha_i \alpha_j} \leq 1\] (22)

Theorem 2. $\Sigma_t,ND$ is the CMTFA solution of $\Sigma_x$ if and only if $\vec{\alpha}$ is non-dominant.

The theorem states that the CMTFA solution to a star connected network is a star itself if and only if there is no dominant element in the vector $\vec{\alpha}$.

Proof of Theorem 2. We still use the same necessary and sufficient condition set in (6). $\Sigma_t,ND$ is rank 1, so its minimum eigenvalue is 0. Since each $0 < |\alpha_i| < 1, 1 - \alpha_i^2 > 0, 1 \leq i \leq n$. As a result, the set $I(d^*)$ is empty. So, the second term on the right side of (6) vanishes. The dimension of the null space of $\Sigma_t,ND$ is $n - 1$. Lemma 3 proves that there exists rank $n - 1$ matrix $T_{n \times n}$ such that the column vectors of $T$ are in the null space of $\Sigma_t,ND$ and the $L_2$-norm of each row of $T$ is 1. That essentially completes the proof.

III. CONDITIONS UNDER WHICH CMDFA SOLUTION OF $\Sigma_x$ PRODUCES A STAR

This section analyses the conditions under which the CMDFA solution of $\Sigma_x$ is a star. The definition of dominance and non-dominance slightly differs in CMDFA than it was in CMTFA. We define the real column vector, $\bar{\theta} = [\theta_1, \ldots, \theta_n]'$, $\theta_i = \frac{\alpha_i}{\sqrt{1 - \alpha_i^2}}, 1 \leq i \leq n$. We define the element $\theta_i$ to be non-dominant, if equation (23) holds, otherwise $\theta_i$ is dominant.

\[|\theta_i| \leq \sum_{j \neq i} |\theta_j| \quad i = 1, 2, \ldots, n\] (23)

In CMDFA the dominance was defined in terms of individual vector component, in CMDFA it is defined in terms of the square root of their signal to noise ratio ($\sqrt{\text{SNR}}$). Note that there can be only one dominant element in vector $\bar{\theta}$ and that to be the element with the biggest absolute value among all. We call $\bar{\theta}$ dominant if its biggest element in terms of absolute value is dominant, otherwise $\bar{\theta}$ non-dominant.

Let us, without the loss of generality, assume $\theta_1$ has the largest absolute value and thus all dominance is defined with respect to it. The following necessary and sufficient condition for CMDFA solution was set in (10).

The point $d^*$ is a solution of the CMDFA problem if and only if $\lambda(d^*) = 0$, and their exists matrix $T$ such that its column vectors are in the null space of $(\Sigma_x - D^*)$ and the $L_2$ norm of the $i$th row of $T$ is $\frac{1}{1 - \alpha_i^2}$, where $\lambda(d^*)$ is the minimum eigenvalue of $(\Sigma_x - D^*)$ and $d^*$ is the vector having all the diagonal entries of the diagonal matrix $D^*$.

The following Lemma is needed to prove the Theorem to follow.

Lemma 4. There exists rank $n - 1$ matrix $T_{n \times n}$ such that the column vectors of $T$ are in the null space of $\Sigma_t,ND$ and the $L_2$-norm of the $i$th row of $T$ is $\frac{1}{1 - \alpha_i^2}$, $1 \leq i \leq n$.

Proof of Lemma 4. Since, it is the same $\Sigma_t,ND$ as was the solution for CMTFA non-dominant case, the basis vectors for the null space remain the same $v_1, v_2, \ldots, v_n$. The matrix $V$ remain the same except for $c_i S$. Here we define them as,

\[c_i = \frac{\bar{c}_i}{\sqrt{1 - \alpha_i^2}}, \quad i = 2, \ldots, n\] (24)

where $\bar{c}_i \in \{1, -1\}$. Still the columns of $V$ with the newly defined $c_i S$, span the null space of $\Sigma_t,ND$. To prove this Lemma, it will suffice for us to find a diagonal matrix $B_{n \times n}$ such that the following holds.

\[T_{n \times n} = V_{n \times n} B_{n \times n}\] (25)

where the $L_2$-norm of the $i$th row of $T$ is $\frac{1}{1 - \alpha_i^2}$. Using (25),

\[TT' = VBB'V' = V\beta V'\] (26)

Like before, we require the diagonal matrix $\beta$ to have only non-negative entries. Based on the conditions imposed on the matrix $T$, we have the following $n$ equations,

\[\frac{\alpha_1^2}{\alpha_1^2} \beta_{11} + \frac{\alpha_2^2}{\alpha_1^2} \beta_{22} + \cdots + \frac{\alpha_n^2}{\alpha_1^2} \beta_{nn} = 1 - \frac{1}{1 - \alpha_1^2}\]

\[\left(\frac{\alpha_2 \alpha_2}{\alpha_1^2} + \frac{\alpha_3 \alpha_3}{\alpha_1^2} + \cdots + \frac{\alpha_n \alpha_n}{\alpha_1^2}\right) \beta_{nn} = 1 - \frac{1}{1 - \alpha_1^2}\] (27)

\[\beta_{ii} + c_{i+1}^2 \beta_{nn} = 1 - \frac{1}{1 - \alpha_{i+1}^2}, \quad i = 1, \ldots, n - 1\] (28)

Solving, (27) with the help of (28) we get,

\[\beta_{nn} = \frac{\alpha_1^2}{1 - \alpha_1^2} - \frac{\alpha_2^2}{1 - \alpha_1^2} - \cdots - \frac{\alpha_n^2}{1 - \alpha_1^2}\]

\[\sum_{i \neq j, i \neq 1, j \neq 1} c_i c_j \alpha_i \alpha_j = \frac{\theta_1^2 - \theta_2^2 - \cdots - \theta_n^2}{\sum_{i \neq j, i \neq 1, j \neq 1} c_i c_j \alpha_i \alpha_j}\] (29)

To ensure that the diagonal entries of $\beta$ are non-negative, we need to consider the following three cases.

\[\theta_2^2 - \theta_2^2 - \cdots - \theta_n^2 > 0\]

\[\theta_2^2 - \theta_2^2 - \cdots - \theta_n^2 > 0\]

\[\theta_2^2 - \theta_2^2 - \cdots - \theta_n^2 < 0\]

It will suffice for us to prove that for all of the above cases there exist $\{\bar{c}_i, 2 \leq i \leq n\}$ that make $\bar{\beta}_{ii} \geq 0, i = 1, \ldots, n$.

Case 1: is straightforward. If $\theta_1^2 - \theta_2^2 - \cdots - \theta_n^2 = 0$ Then using (27) and (28) we get, $\beta_{nn} = 0$ and $\beta_{ii} = \frac{1}{1 - \alpha_{i+1}^2}, \quad i = 1, \ldots, (n - 1)$. Before we move on to other two cases, without the loss of generality, we can arrange the elements of $\bar{\theta}$ such that,

\[|\theta_1| \geq |\theta_2| \geq \cdots \geq |\theta_n|\] (30)
Normalizing each element by $|\theta_i|$ gives us the following.

$$1 \geq |\bar{\theta}_2| \geq |\bar{\theta}_3| \geq \cdots \geq |\bar{\theta}_n|$$ \hspace{1cm} (31)

where, $\bar{\theta}_j = \frac{\theta_j}{\theta_i}, 1 \leq j \leq n$.

We define,

$$S_{\text{min}} = \min_A \left| \sum_{j \in A} |\bar{\theta}_j| - \sum_{j \notin A^*} |\bar{\theta}_j| \right|$$ \hspace{1cm} (32)

where $A \subset \{1, 2, \ldots, n\}$ and $A^* = \{1, 2, \ldots, n\} \setminus A$.

Let, of all the possibilities $A = A^*$ be the subset of $\{1, 2, \ldots, n\}$ that gives us $S_{\text{min}}$ from (32). Assuming that $A^*$ has $l$ elements, let the set $A^* = \{a_1, a_2, \ldots, a_l\}$. We define the set $F$ as,

$$F = \{F_{a_1}, \ldots, F_{a_l}, \ F_{a_i} = |\bar{\theta}_{a_i}|, a_i \in A^*\}$$

Now, under the above ordered and normalized settings, **Case 2**: $1 - \bar{\theta}_2 - \bar{\theta}_3 - \cdots - \bar{\theta}_n > 0$

We can select $c_2, c_3, \ldots, c_n$ in a way such that $c_i \bar{\theta}_i = |\bar{\theta}_i|$ to make $\beta_{mn} > 0$. Equation (28) dictates that to ensure the other diagonal entries of $\beta$ are non-negative, the following must hold,

$$1 - \frac{\bar{\theta}_2^2 - \bar{\theta}_3^2 - \cdots - \bar{\theta}_n^2}{\sum_{i \neq j, i \neq 1, j \neq 1} \bar{c}_i \bar{c}_j \bar{\theta}_i \bar{\theta}_j} \leq 1$$

$$\Rightarrow 1 \leq (|\bar{\theta}_2| + |\bar{\theta}_3| + \cdots + |\bar{\theta}_n|)^2$$

$$\Rightarrow 1 \leq |\bar{\theta}_2| + |\bar{\theta}_3| + \cdots + |\bar{\theta}_n|$$ \hspace{1cm} (33)

Which means such $\beta_{mn}$ exists if and only if $|\bar{\theta}_1|$ is non-dominant. Because of the ordered representation, that essentially means the vector $\bar{\theta}$ has to be non-dominant.

**Case 3**: $1 - \bar{\theta}_2 - \cdots - \bar{\theta}_n < 0$

Based on the proof in Appendix C of this paper, if we select $c_i \in \{1, -1\}$ such that $\sum_{j=2}^n c_i \bar{\theta}_j = S_{\text{min}}$ then,

$$\sum_{i \neq j, i \neq 1, j \neq 1} \bar{c}_i \bar{c}_j \bar{\theta}_i \bar{\theta}_j < 0$$

And for such selection of $c_i$ we have,

$$\bar{\theta}_2^2 + \bar{\theta}_3^2 + \cdots + \bar{\theta}_n^2 + \sum_{i \neq j, i \neq 1, j \neq 1} \bar{c}_i \bar{c}_j \bar{\theta}_i \bar{\theta}_j = \left( \sum_{i=2}^n c_i \bar{\theta}_i \right)^2 = S_{\text{min}}^2 \leq 1$$ \hspace{1cm} (34)

The last inequality is due to the lemma we proved in Appendix A of this paper. So, we have,

$$\bar{\theta}_2^2 + \bar{\theta}_3^2 + \cdots + \bar{\theta}_n^2 + \sum_{i \neq j, i \neq 1, j \neq 1} \bar{c}_i \bar{c}_j \bar{\theta}_i \bar{\theta}_j \leq 1$$

$$\Rightarrow 1 - \bar{\theta}_2^2 - \bar{\theta}_3^2 - \cdots - \bar{\theta}_n^2 \geq \sum_{i \neq j, i \neq 1, j \neq 1} \bar{c}_i \bar{c}_j \bar{\theta}_i \bar{\theta}_j$$

Both the terms $1 - \bar{\theta}_2^2 - \bar{\theta}_3^2 - \cdots - \bar{\theta}_n^2$ and $\sum_{i \neq j, i \neq 1, j \neq 1} \bar{c}_i \bar{c}_j \bar{\theta}_i \bar{\theta}_j$ are negative. Hence,

$$\beta_{mn} = \frac{1 - \bar{\theta}_2^2 - \bar{\theta}_3^2 - \cdots - \bar{\theta}_n^2}{\sum_{i \neq j, i \neq 1, j \neq 1} \bar{c}_i \bar{c}_j \bar{\theta}_i \bar{\theta}_j} \leq 1$$ \hspace{1cm} (35)

**Theorem 3.** CMDFA solution of $\Sigma_x$ is $\Sigma_{t,ND}$ if and only if $\bar{\theta}$ is non-dominant.

The theorem states that the CMDFA solution to a star connected network is a star itself if and only if there is no dominant element in the vector $\theta$.

**Proof of Theorem 3.** Now we refer back to the necessary and sufficient condition for CMDFA solution at the beginning of this section. Since, $\Sigma_{t,ND}$ in rank 1, it’s minimum eigenvalue is 0. And Lemma 3 proves the existence of rank $n-1$ matrix $T_{n \times n}$ such that the column vectors of $T$ are in the null space of $\Sigma_{t,ND}$ and the $L_2$-norm of the $i$th row of $T$ is $\frac{1}{1-\alpha_i^2}, 1 \leq i \leq n$. That completes the proof of Theorem 3. \hfill \square

**IV. Conclusion**

In this paper we characterized the solution space of both CMTFA and CMDFA. We showed that the CMTFA solution of a star structured population matrix can have either a rank 1 or a rank $n-1$ solution and nothing in between. We proved both the solutions with sufficient and necessary conditions. We established the necessary and sufficient conditions for CMDFA solution to a star structured population matrix to be a star.

**APPENDIX A**

Let $e_1, e_2, \ldots, e_n$ be a set of $n$ positive numbers. We define,

$$S_{\text{min}} = \min_A \left| \sum_{i \in A} e_i - \sum_{j \in A^*} e_j \right|$$ \hspace{1cm} (36)

where $A \subset \{1, 2, 3, \ldots, n\}$ and $A^c = \{1, 2, 3, \ldots, n\} \setminus A$.

Let of all the possibilities $A^*$ be the subset of $\{1, 2, 3, \ldots, n\}$ that gives us $S_{\text{min}}$. Assuming the set $A^*$ has $l$ elements, let the sets $A^*$ and $(A^*)^c$ be $A^* = \{a_1, a_2, \ldots, a_l\}$ and $(A^*)^c = \{a_1^c, a_2^c, \ldots, a_{n-l}^c\}$. Let $F$ and $G$ be following two sets,

$$F = \{F_{a_1}, \ldots, F_{a_l}, \ F_{a_i} = e_{a_i}, a_i \in A^*\}$$

$$G = \{G_{a_1^c}, \ldots, G_{a_{n-l}^c}, \ G_{a_i^c} = e_{a_i^c}, a_i^c \in (A^*)^c\}$$

We define,

$$M + S_{\text{min}} = \sum_{a_i \in A^*} F_{a_i}, \ M = \sum_{a_i^c \in (A^*)^c} G_{a_i^c}$$ \hspace{1cm} (37)

$$F_{\text{avg}} = \frac{1}{l}(M + S_{\text{min}}), \ G_{\text{avg}} = \frac{1}{n-l}M$$ \hspace{1cm} (38)

$$F_{\text{min}} = \min_{a_i \in A^*} F_{a_i}$$ \hspace{1cm} (39)

**Lemma 5.** $S_{\text{min}} \leq F_{\text{min}}$

**Proof of Lemma 5.** Let us assume $F_{\text{min}} < S_{\text{min}}$ and has the value $F_{\text{min}} = S_{\text{min}} - \epsilon$ where $0 < \epsilon < S_{\text{min}}$. \hfill \square
Now, if we deduct \( F_{\min} \) from set \( F \) and add it to the set \( G \), then we will have,
\[
(M + S_{\min} - F_{\min}) - (M + F_{\min}) = |S_{\min} - 2F_{\min}|
\]
\[
= |S_{\min} - 2|e|
< S_{\min}
\]
which is not possible. So, \( F_{\min} \geq S_{\min} \).

**APPENDIX B**

**Lemma 6.** For any set of positive numbers \( e_1, e_2, \ldots, e_n \),
\[
\sum_{i \neq j} e_i e_j \leq n(n - 1)e^2_{\text{avg}}
\]
(40)
where, \( e_{\text{avg}} = \frac{1}{n} \sum_{i=1}^{n} e_i \).

**Proof of Lemma 6.** Without the loss of generality, we can write the set of numbers in terms of their average in the following way: \( e_{\text{avg}} + k_1, e_{\text{avg}} + k_2, \ldots, e_{\text{avg}} + k_p, e_{\text{avg}} - j_1, e_{\text{avg}} - j_2, \ldots, e_{\text{avg}} - j_q \), where, \( p + q = n, k_i \geq 0, j_i \geq 0 \).

It is straightforward to see, \( \sum_{i=1}^{p} k_i = \sum_{i=1}^{q} j_i \). We define,
\[
\psi_1 = (e_{\text{avg}} + k_1) [(e_{\text{avg}} + k_2) + \cdots + (e_{\text{avg}} + k_p) + \cdots + (e_{\text{avg}} + j_q)]
\]
\[
= (e_{\text{avg}} + k_1) [(p + q - 1)e_{\text{avg}} + (k_2 + \cdots + k_p) - \cdots - (j_1 + \cdots + j_q)]
\]
\[
\psi_2 = (e_{\text{avg}} + k_2) [(p + q - 2)e_{\text{avg}} + (k_3 + \cdots + k_p) - \cdots - (j_1) + \cdots + j_q)]
\]
\[
\vdots
\]
\[
\psi_{p-1} = (e_{\text{avg}} + k_{p-1}) [(q + 1)e_{\text{avg}} + k_p - \cdots - (j_1 + \cdots + j_q)]
\]
\[
\psi_p = (e_{\text{avg}} + k_p) [qe_{\text{avg}} - (j_1 + \cdots + j_q)]
\]
\[
\psi_{p+1} = (e_{\text{avg}} - j_1) [(q - 1)e_{\text{avg}} - (j_2 + \cdots + j_q)]
\]
\[
\vdots
\]
\[
\psi_{p+q-1} = (e_{\text{avg}} - j_{q-1}) [e_{\text{avg}} - j_q]
\]
Using the above equations,
\[
\sum_{i \neq j} e_i e_j = 2[\psi_1 + \cdots + \psi_{p+q-1}]
\]
\[
= 2 \left[ (1 + \cdots + (p + q - 1) - e_{\text{avg}}(p + q - 1) \sum_{i=1}^{p} k_i + \cdots \sum_{i=1}^{p} k_i \sum_{i=1}^{q} j_i - \sum_{i=1}^{p} k_i \sum_{i=1}^{q} j_i \right]
\]

**APPENDIX C**

**Lemma 7.** If we select \( \{c_j\}_{j=1}^{n} \), \( c_j \in \{1, -1\} \) such that,
\[
\sum_{j=1}^{n} c_j e_j = S_{\min}
\]
then,
\[
\sum_{i \neq j} c_i c_j e_i e_j < 0
\]

(54)
(55)
We can write the left hand side of the equation (55) as,
\[
= \sum_{a_i, a_j \in A^*, a_i \neq a_j} F_{a_i} F_{a_j} + \sum_{a_i^*, a_j^* \in (A^*)^c, a_i^* \neq a_j^*} G_{a_i^*} G_{a_j^*}
- 2(F_{a_i} + \ldots + F_{a_j})(G_{a_i^*} + \ldots + G_{a_j^*})
\]
For \( l = 1 \) the term \( \sum_{a_i, a_j \in A^*, a_i \neq a_j} F_{a_i} F_{a_j} \) does not exist. Similarly for \( n - l = 1 \) the term \( \sum_{a_i^*, a_j^* \in (A^*)^c, a_i^* \neq a_j^*} G_{a_i^*} G_{a_j^*} \) does not exist. For \( l \geq 2 \) applying Lemma 4 in equation (56) we get,
\[
\sum_{i \neq j} c_i c_j e_i e_j \\
\leq l(l-1)F_{avg}^2 + (n-l)(n-l-1)F_{avg}^2 - 2M(M + S_{min})
= \frac{l(l-1)}{l^2}(M + S_{min})^2 + \frac{n-l-1}{n-l} |M^2 - 2M(M + S_{min})|
\]
\( F_{min} \) is the smallest element in the set \( F \), so we can write,
\[
F_{min} \leq \frac{M + S_{min} - F_{min}}{l-1}
\]
Combining the above two results,
\[
(\Sigma_{l,DM})_1 = \sum_{g=2}^{n} \gamma_1 \gamma_g (\Sigma_{l,DM})_{g|1}
\]
\Rightarrow \( (\Sigma_{l,DM})_1 - \sum_{g=2}^{n} \gamma_1 \gamma_g (\Sigma_{l,DM})_{g|1} = 0 \)
\Rightarrow \( (\Sigma_{l,DM})_1 - \sum_{g=2}^{n} (-1)^{S_{g}} (\Sigma_{l,DM})_{g|1} = 0 \)
where
\[
S_{g} = \begin{cases} 
1, & \gamma_1 \gamma_g = -1 \\
2, & \gamma_1 \gamma_g = 1 
\end{cases}
\]
\[\square\]
APPENDIX E

Proof of Lemma 2. It is obvious to see that the following selection of the elements of vector \( \Phi \) makes it orthogonal to \( (\Sigma_{l,DM})_1 \), i.e. \( (\Sigma_{l,DM})_1 \Phi = 0 \). Where \( (\Sigma_{l,DM})_1 \) is the 1st row of \( \Sigma_{l,DM} \).
\[
\Phi_i = \begin{cases} 
-1, & \alpha_1 \alpha_i > 0, i \neq 1 \\
1, & \text{otherwise}
\end{cases}
\]
Now it will be sufficient to prove that any vector \( \Phi \) orthogonal to \( (\Sigma_{l,DM})_1 \) is also orthogonal to all the other rows of \( \Sigma_{l,DM} \), i.e. \( (\Sigma_{l,DM})_i \Phi = 0 \) for \( 2 \leq i \leq n \). Let \( \Phi = [\Phi_1, \Phi_2, ..., \Phi_n]^T \) be a column vector such that \( \Phi_i \in \{-1, 1\}, 1 \leq i \leq n \) and \( (\Sigma_{l,DM})_1 \Phi = 0 \).
Let \( \gamma_i \in \{-1, 1\} \) be the sign of \( \alpha_i \), i.e. \( \alpha_i = \gamma_i |\alpha_i| \).
Now for any row \( g, g \neq 1 \),
\[
(\Sigma_{l,DM})_g \Phi = \Phi_g (\Sigma_{l,DM})_{g|1} + \sum_{h \neq g} \Phi_h (\Sigma_{l,DM})_{gh}
= \Phi_g |\alpha_g| (|\alpha_1| - \sum_{i \neq g, h} |\alpha_i|) + \sum_{h \neq g} \Phi_h |\alpha_g| |\alpha_h|
= (\Phi_g + \Phi_1 \gamma_1 |\alpha_g| |\alpha_1| + \sum_{h \neq g} (\gamma_1 \gamma_h \Phi_h - \Phi_g) |\alpha_g| |\alpha_h|)
\]
\( \Phi_g = \Phi_h \Rightarrow \gamma_1 \gamma_g = \gamma_1 \gamma_h \Rightarrow \gamma_g = \gamma_h \Rightarrow \gamma_g \gamma_h \Phi_h - \Phi_g = 0 \).

Else if \( \Phi_g \neq \Phi_h \Rightarrow \gamma_1 \gamma_g \neq \gamma_1 \gamma_h \Rightarrow \gamma_g \neq \gamma_h \Rightarrow \gamma_g \gamma_h \Phi_h - \Phi_g = 0 \).

Similarly, If \( \Phi_g = \Phi_i \Rightarrow \alpha_1 \alpha_g < 0 \Rightarrow \gamma_1 \neq \gamma_g \Rightarrow \Phi_g + \Phi_1 \gamma_1 \gamma_1 = 0 \)
Else if \( \Phi_g \neq \Phi_i \Rightarrow \alpha_1 \alpha_g > 0 \Rightarrow \gamma_1 = \gamma_g \Rightarrow \Phi_g + \Phi_1 \gamma_1 \gamma_1 = 0 \)
Plugging these results in equation (59), we get

\[(\Sigma_{t,DM})_g \Phi = 0\]

REFERENCES

[1] Y. Chen, X. Li, and S. Zhang, “Structured latent factor analysis for large-scale data: Identifiability, estimability, and their implications,” arXiv preprint arXiv:1712.08966, 2017.
[2] D. Bertsimas, M. S. Copenhaver, and R. Mazumder, “Certifiably optimal low rank factor analysis,” Journal of Machine Learning Research, vol. 18, no. 29, pp. 1–53, 2017.
[3] H. H. Harman, Modern factor analysis. University of Chicago Press, 1976.
[4] W. Ledermann, “On a problem concerning matrices with variable diagonal elements,” Proceedings of the Royal Society of Edinburgh, vol. 60, no. 1, pp. 1–17, 1940.
[5] P. Bentler and J. A. Woodward, “Inequalities among lower bounds to reliability: With applications to test construction and factor analysis,” Psychometrika, vol. 45, no. 2, pp. 249–267, 1980.
[6] J. M. Ten Berge, T. A. Snijders, and F. E. Zegers, “Computational aspects of the greatest lower bound to the reliability and constrained minimum trace factor analysis,” Psychometrika, vol. 46, no. 2, pp. 201–213, 1981.
[7] G. Xu, W. Liu, and B. Chen, “A lossy source coding interpretation of wyner’s common information,” IEEE Transactions on Information Theory, vol. 62, no. 2, pp. 754–768, 2016.
[8] A. Wyner, “The common information of two dependent random variables,” IEEE Transactions on Information Theory, vol. 21, no. 2, pp. 163–179, 1975.
[9] S. Satpathy and P. Cuff, “Gaussian secure source coding and wyner’s common information,” in Information Theory (ISIT), 2015 IEEE International Symposium on. IEEE, 2015, pp. 116–120.
[10] A. Moharrer and S. Wei, “Algebraic properties of solutions to common information of gaussian graphical models,” in Communication, Control, and Computing (Allerton), 2017 55th Annual Allerton Conference on. IEEE, 2017.
[11] J. Saunderson, V. Chandrasekaran, P. A. Parrilo, and A. S. Willsky, “Diagonal and low-rank matrix decompositions, correlation matrices, and ellipsoid fitting,” SIAM Journal on Matrix Analysis and Applications, vol. 33, no. 4, pp. 1395–1416, 2012.
[12] G. Della Riccia and A. Shapiro, “Minimum rank and minimum trace of covariance matrices,” Psychometrika, vol. 47, no. 4, pp. 443–448, 1982.