INSTABILITY OF RATIONAL AND POLYNOMIAL CONVEXITY

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Abstract. It is shown that rational and polynomial convexity of totally real submanifolds is in general unstable under perturbations that are $C^\alpha$-small for any Hölder exponent $\alpha < 1$. This complements the result of Løw and Wold that these properties are $C^1$-stable.

1. Introduction and results

A smooth submanifold in $\mathbb{C}^n$ is called totally real if it is not tangent to any complex line. A compact totally real submanifold is polynomially or, respectively, rationally convex if and only if every continuous function on it can be uniformly approximated by polynomials or, respectively, rational functions (see e.g. [12, Theorem 1.2.10 and Corollary 6.3.3]).

Løw and Wold [9, Corollary 1] proved that a $C^1$-small perturbation of a polynomially convex totally real submanifold $M \subset \mathbb{C}^n$ remains polynomially convex. Their argument applies mutatis mutandis to prove that the same holds for rationally convex totally real submanifolds. The purpose of this note is to show that these conclusions are no longer true if the perturbations are only $C^\alpha$-small for any Hölder exponent $\alpha < 1$ (and the dimension of $M$ is $\geq 2$, see Remark 1.6).

Definition 1.1. A smooth submanifold $M \subset \mathbb{C}^n$ admits a $C^\alpha$-small perturbation with a certain property (e.g. being rationally convex or not) if there exist

(i) a submersion $\pi : U_M \rightarrow M$ from a tubular neighbourhood of $M$ onto $M$;

(ii) a sequence of smooth submanifolds $M_j \subset U_M$, $j \in \mathbb{N}$, having that property such that $M_j$ converge to $M$ in $C^\alpha$-topology as $j \rightarrow \infty$ and $\pi : M_j \rightarrow M$ is a diffeomorphism for all $j$.

In other words, an identification of a neighbourhood of $M$ in $\mathbb{C}^n$ with a neighbourhood of the zero section in its normal bundle can be chosen so that all $M_j$’s correspond to graphs of $C^\alpha$-small sections of that bundle. For $\alpha \geq 1$ (e.g. for deformation considered by Løw and Wold), this holds automatically for any such identification once $j$ is large enough, but in the properly Hölder range this is an additional assumption, cf. Remark 1.3(i).

Theorem 1.2. Let $T$ be a rationally convex totally real embedded 2-torus in $\mathbb{C}^2$. For every $\alpha < 1$, there exists a $C^\alpha$-small perturbation of $T$ that is totally real and not rationally convex.

The tori constructed in the proof of the theorem in §2 will be shown not to be rationally convex by an application of a result from symplectic geometry on the Maslov class rigidity of Lagrangian tori [11]. In particular, the perturbed tori are not isotopic (or even regularly homotopic) to $T$ through totally real tori because their Maslov class is different.

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Remark 1.3. (i) Non rationally convex tori \( C^\alpha \)-close to a rationally convex one can be obtained using spin tori [1 §6.4]. The spin tori associated to the embedded curves in \( \mathbb{C} \times (0, +\infty) \) shown in Fig. 1 define a totally real \( C^\alpha \)-small isotopy of the standard product torus in \( \mathbb{C}^2 \) to a spin torus that is manifestly not rationally convex. (Each self-intersection of the rightmost curve in Fig. 1 corresponds to a holomorphic annulus which has the same oriented boundary as an annulus on the spin torus and hence lies in its rationally convex hull by the argument principle.) On the other hand, this construction does not seem to yield \( C^\alpha \)-small perturbations in the sense of Definition 1.1 because any fixed normal projection onto the standard torus will no longer be diffeomorphic on the ‘kinks’ once they are small enough.

![Figure 1. Spin tori](image)

(ii) The deformation described in (i) can be applied to any other totally real surface \( S \subset \mathbb{C}^2 \) in the following way. First, smoothly approximate \( S \) by a real analytic surface \( S' \). For any embedded real analytic circle \( \gamma \subset S' \), there exists a biholomorphic map defined on a neighbourhood \( U \supset \gamma \) mapping \( S' \cap U \) into the product torus in \( \mathbb{C}^2 \) and taking \( \gamma \) to the fibre of the product torus. Hence, the part of the isotopy from (i) near the ‘kinks’ may be grafted onto \( S' \) making that surface non rationally convex. The perturbations in the proof of Theorem 1.2 will also be localised near an embedded circle on \( T \). However, there is no reason to expect that grafting them onto other surfaces will have the same effect because the relevant obstruction to rational convexity is global.

Corollary 1.4. The 2-torus \( \tilde{T} = \{ (z_1, z_2, z_3) \in \mathbb{C}^3 \mid |z_1| = |z_2| = 1, z_1z_2z_3 = 1 \} \) is totally real and polynomially convex in \( \mathbb{C}^3 \). For every \( \alpha < 1 \), there exists a \( C^\alpha \)-small perturbation of \( \tilde{T} \) that is totally real and neither polynomially nor rationally convex.

Proof. Every continuous function on \( \tilde{T} \) can be uniformly approximated by a Laurent polynomial in \( z_1 \) and \( z_2 \) which can be written as a polynomial in \( z_1, z_2, \) and \( z_3 = \frac{1}{z_1z_2} \). So \( \tilde{T} \) is polynomially convex in \( \mathbb{C}^3 \).

Let \( T^\text{st}_j \subset \mathbb{C}^2, j \in \mathbb{N} \), be a \( C^\alpha \)-small perturbation of the standard torus

\[
T^\text{st} = \{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1| = |z_2| = 1 \}
\]

that is not rationally convex. Such a perturbation exists by Theorem 1.2. Then

\[
\tilde{T}_j = \{ (z_1, z_2, z_3) \in \mathbb{C}^3 \mid (z_1, z_2) \in T^\text{st}_j, z_1z_2z_3 = 1 \}
\]

is a \( C^\alpha \)-small totally real perturbation of \( \tilde{T} \). If \( \tilde{T}_j \) were rationally convex, rational functions of \( (z_1, z_2) \) would be dense in the space of continuous functions on \( T^\text{st}_j \), which is impossible since those tori are not rationally convex. \( \square \)
Remark 1.5. The tori in the corollary are isotopic to \( \tilde{T} \) through totally real 2-tori in \( \mathbb{C}^3 \). This follows by a general position argument. Indeed, the space of complex lines has (real) codimension \( \geq 4 \) in the space of all real two-planes in \( \mathbb{C}^n \) for \( n \geq 3 \) and therefore a generic isotopy of embedded surfaces in \( \mathbb{C}^3 \) is through totally real surfaces.

Similar examples in higher dimensions are obtained by considering product tori of the form

\[ T_j = \{ z \in \mathbb{C}^n \mid (z_1, z_2) \in T_{j}^{\text{st}} \text{ and } |z_k| = 1 \text{ for } k \geq 3 \} \]

and setting

\[ \tilde{T}_j = \{ z \in \mathbb{C}^{n+1} \mid (z_1, \ldots, z_n) \in T_j, z_1 \cdots z_{n+1} = 1 \}. \]

Then \( T_j \) is a non rationally convex perturbation of the standard \( n \)-torus in \( \mathbb{C}^n \) and \( \tilde{T}_j \) is a non rationally convex perturbation of its polynomially convex lift to \( \mathbb{C}^{n+1} \).

Remark 1.6. Polynomial convexity is well-known to be \( C^0 \)-stable for 1-dimensional totally real tori, that is, for \((C^1\text{-})\)smooth closed curves \( \Gamma \in \mathbb{C}^n \). Indeed, if \( \Gamma \) is polynomially convex, then every continuous function on \( \Gamma \) can be approximated by polynomials and, in particular, there exist polynomials \( P_+ \) and \( P_- \) such that \( \Gamma \cap \{ P_+P_- = 0 \} = \emptyset \) and \( \Delta_{\Gamma} \arg P_+ = \pm 2\pi \) for some choice of orientation on \( \Gamma \). A curve \( \Gamma' \) sufficiently \( C^0 \)-close to \( \Gamma \) is homotopic to it in \( \mathbb{C}^n \setminus \{ P_+P_- = 0 \} \) and hence \( \Delta_{\Gamma'} \arg P_+ = \pm 2\pi \) for the induced orientation on \( \Gamma' \). However, if \( \Gamma' \) is not polynomially convex, it is the boundary of a 1-dimensional complex analytic set by the classical result of Wermer [13]. This implies that for all entire functions the variation of the argument along \( \Gamma' \) must have the same sign by the argument principle, a contradiction.

Remark 1.7. The result of Corollary 1.4 may be interpreted in the following way. Consider the 2-torus as the product \( S^1 \times S^1 \) with the angular coordinates \((\theta_1, \theta_2)\). Laurent polynomials of \( e^{i\theta_1} \) and \( e^{i\theta_2} \) are dense in the space of continuous functions on the torus. The theorem of Løw and Wold asserts that this property holds for any pair of functions (depending on both variables) that are sufficiently \( C^1 \)-close to the exponentials. On the other hand, this may no longer be so for functions that are \( C^\alpha \)-close to them even if they separate points on the torus and have \( \mathbb{C} \)-linearly independent differentials.

2. Proof of Theorem 1.2

By a result of Duval [3] (see also [4]), a totally real surface \( \Sigma \subset \mathbb{C}^n \) is rationally convex if and only if it is Lagrangian with respect to a Kähler form \( \omega \) on \( \mathbb{C}^n \) which can be assumed standard (i.e. equal to \( \omega_{\text{std}} = dd^c \|z\|^2 \)) outside of a sufficiently large ball. (We are only going to use the easier ‘only if’ part of this statement.) Let \( \omega_t = (1-t)\omega + tw_{\text{std}}, t \in [0, 1], \) be the linear homotopy of Kähler forms connecting \( \omega \) to \( \omega_{\text{std}} \). By Moser’s stability theorem, there exists a family of compactly supported diffeomorphisms \( f_t, t \in [0, 1], \) of \( \mathbb{C}^n \) with \( f_0 = \text{id} \) such that \( f_t^* \omega_t = \omega \). Therefore \( f_t(\Sigma) \) is a family of \( \omega_t \)-Lagrangian (and hence totally real) surfaces connecting \( \Sigma = f_0(\Sigma) \) to \( f_1(\Sigma) \), which is Lagrangian with respect to \( \omega_{\text{std}} \). It follows, in particular, that if \( \Sigma \) is a rationally convex totally real 2-torus in \( \mathbb{C}^2 \), then its Maslov class \( \mu \in H^1(\Sigma; \mathbb{Z}) \) must be non-zero by [4, Example 2.2].

Thus, to prove Theorem 1.2 it is enough to produce totally real \( C^\alpha \)-small perturbations of \( T \) with trivial Maslov class. For \( \alpha = 0 \), the existence of such tori may be deduced from Gromov’s \( h \)-principle for totally real embeddings (see e.g. [6 §19.3]).
It is quite probably possible to get Hölder estimates from the convex integration proof of the $h$-principle in [6]. In the case at hand, however, we will use a different approach going back to [7] and based on the cancellation/creation theorem for pairs of complex points due to Eliashberg and Kharlamov [5].

According to [11, Example 2.2], the minimal positive value of the Maslov class of a Lagrangian torus in $\mathbb{C}^2$ is always 2. Hence, we can find two oriented simple closed curves $\gamma_1$ and $\gamma_2$ on $T$ intersecting at a single point and such that $\mu([\gamma_1]) = 0$ and $\mu([\gamma_2]) = 2$. Let $I \subseteq \gamma_1$ be an arc on $\gamma_1$ containing its intersection point with $\gamma_2$.

**Example 2.1.** If $T = T^{st}$, one can take $\gamma_1 = T^{st} \cap \{z_2 = \bar{z}_1\}$ and $\gamma_2 = T^{st} \cap \{z_2 = 1\}$.

![Figure 2. First step in the modification of $T$.](image-url)

First, we can deform $T$ in a neighbourhood of $I$ so that the new surface has two positive (with respect to some orientation on $T$) complex points, one elliptic and one hyperbolic, at the ends of $I$, see Fig. 2. If the elliptic point is on the negative side of $\gamma_2$ with respect to the orientation on the torus, then $\mu([\gamma_2]) = 0$ on the modified surface. (To see this, note that the Maslov class takes value 2 on a small positive loop about the positive elliptic complex point and that this loop gives the homological difference in the complement of that complex point between the modified $\gamma_2$ and a representative $\gamma'_2$ of $[\gamma_2]$ on $T$ avoiding the neighbourhood affected by the isotopy.) Next, we can cancel these two complex points out again by an isotopic modification in a neighbourhood of the complementary arc $\gamma_1 \setminus I$. The Maslov class of the resulting totally real torus is trivial, as it takes value zero on the curves $\gamma'_1$ and $\gamma_2$ generating the first homology group.

To ensure that the modification in the preceding paragraph can be realised by a $C^\alpha$-small perturbation, we use the argument from the proof of the Eliashberg–Kharlamov theorem given in [10, §2.3] and [8, §10.4]. A neighbourhood $U$ of $\gamma_1$ can be biholomorphically mapped onto a domain in $\mathbb{C}^2$ so that $T \cap U$ is taken to the graph of a function $f : A \to \mathbb{C}$ on an annulus $A \subset \mathbb{C}$. (This follows by taking a diffeomorphism of $T \cap U$ onto a model totally real annulus and approximating it by a holomorphic map.) The required modifications are realised by adding to $f$ a suitably cut off solution of the equation $\frac{\partial u}{\partial z} = \varphi$ with the right hand side having arbitrarily small $L^p$-norm for any fixed $p > 0$. (More precisely, $\varphi$ may be chosen uniformly bounded by a constant depending on the geometry and having arbitrarily thin support.) By a classical result (see e.g. [1, Theorem 4.3.13]), the Cauchy transform of $\varphi$ is a solution of this $\overline{\partial}$-equation and its $C^{1 - \frac{2}{p}}$-norm is bounded by...
a constant times the $L^p$-norm of $\varphi$. (In [10, p. 735] and [8, Lemma 10.4.3], the $C^0$-version of the same estimate was used.) Thus, $u$ can be made $C^\alpha$-small for any fixed $\alpha < 1$ by taking $p$ large enough.

It should now be clear from the construction that all our deformations are graphs of sections with respect to a common bundle structure on a neighbourhood of $T$. □

**Remark 2.2.** It was shown in [2] that all Lagrangian 2-tori in $\mathbb{C}^2$ with the standard Kähler form are Lagrangian isotopic. So the space of rationally convex totally real 2-tori in $\mathbb{C}^2$ is in fact connected.

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