Even Power Weighted Generalized Exponential Distribution

Rana Hadi Mutlik 1*, Dr. Awatif Rezzoky Al-Dubaicy 2

1,2 Department of Mathematics, Education College, Mustansiriyah University, Baghdad, Iraq
emails: 1 rana09iq@gmail.com 2 draraldubaicy@yahoo.com

Abstract. In this search, derivation a new even power weighted generalized Exponential distributions with some statistical properties are discussed, such as [cumulative dist, probability generating function, moment generating function, reliability, and Entropy functions] and other properties The scale parameter for this distr. has been estimated, using two methods, [method of moment and maximum likelihood], and simulation study has been to compare by MSE criteria, for the performance of the two estimation.

1. Introduction
In some situations, it was noted that the classical distributions were not flexible for the data sets related to the field of biomedical, engineering, financial, environmental, computer science, Economy, and in other sciences [1,10], therefore continually needed to obtain a flexible model for applications in these areas.

there are many generalization of the exponential distribution in the literature, three parameters generalized exponential distribution was studied by Gupta (1999)[7], the theoretically properties of two parameter exponentiated exponential distribution and compare them with respect to the well properties of the gamma and Weibull distributions were discussed by Gupta (2007) [6].

K- generalized Exponential distribution which among other things includes generalized exponential and Weibull distributions was presented by Rather (2017)[11]. Fisher at (1934) [4] proposed a new generalization of classical distribution called weighted distribution for any random variable associated with probability function f (x, θ) as follows:

\[ f_w(x; \theta) = \frac{w(x) f(x; \theta)}{E(w(x))} \]  

(1)

Under the condition \(E(w(x)) = \int_{\infty}^{\infty} w(x) f(x; \theta) dx < \infty\), where \(w(x)\) and \(\theta\) are the positive weighted function and parameter vector respectively.

After that, there are many researchers studying the weight function and the weighted distributions. Gupta et.al. (2009) [10] introduced a classed of weighted exponential distributions. Hantoosh A, (2013)[2] Double Weighted distribution. Dey et.al. (2015)[3] investigated various properties and methods of estimation of the weighted exponential distribution. In (2015) Rezzoky and Nadhel [12] considered the generalized exponential distribution (GED) with one and double weighted functions. In 2017 Rather N. [11] introduced a New Generalized of Exponential Distribution with Applications

Here, a new generalization of the exponential distribution with even power-weighted function derived. The theoretically properties and estimation of this model are discussed, in addition, the expression of entropy function is also derived. Finally, simulation is performed to compare the performances of the two estimation methods and real data set have been analyzed. The remainder of this paper is organized as follows: In Section 2 drive an even power weighted generalized exponential distribution...
and its properties, in Section 3, estimation. The scale parameter by two methods [MLE ,MOM] Section 4. Includes simulation and conclusions.

2. Even Power Weighted distribution

The Even-Power weighted distribution is given by: \[ f_{w}(x) = \frac{(w(x))^{2r}f(x)}{W_D} \quad -\infty < x < +\infty, \quad r \in \mathbb{Z}^+ \] (2)

Where \( W_D = E[(w(x))^{2r}] = \int_{-\infty}^{\infty}(w(x))^{2r}f(x)\,dx \)

2.1. Even Power Weighted Generalized Exponential Distribution (EPWGED)

In this section the probability density function pdf, cumulative distribution function cdf and other properties for EPWGE distribution are obtained. Consider the pdf of the generalized exponential distribution as follows \[ f(x; \alpha, \lambda) = \alpha\lambda e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha-1}, \quad x, \alpha, \lambda > 0 \] (3)

Where \( \alpha \) and \( \lambda \) are shape and scale parameters respectively, and the cumulative distribution function as:

\[ F(x) = \lambda (1 - e^{-\lambda x})^\alpha \]

Here, assume that the shape parameter \( \alpha \) known and equal 2 then the pdf becomes as:

\[ f(x; \alpha, \lambda) = 2\lambda e^{-\lambda x} (1 - e^{-\lambda x}), \quad x, \alpha, \lambda > 0 \] (3)

and the cumulative distribution function as follows

\[ F(x) = (1 - e^{-\lambda x})^2 \] (4)

The weight function used is

\[ w(x) = x \] (5)

And according to (3)&(5)

\[ W = E[(w(x))^{2r}] = 2\lambda \int_{0}^{\infty} x^{2r} e^{-\lambda x}dx - 2\lambda \int_{0}^{\infty} x^{2r} e^{-2\lambda x}dx \]

This implies that

\[ W = \frac{\Gamma(2r + 1)(2^{2r+1} - 1)}{(2\lambda)^{2r}} \]

Then, by equation (2), the pdf of even power weighted generalized exponential distribution denoted by (EPWGE) distribution can be written as follows:

\[ f_{w}(x; \lambda) = \frac{(2\lambda)^{2r+1}}{\Gamma(2r+1)(2^{2r+1} - 1)} x^{2r} e^{-\lambda x} (1 - e^{-\lambda x}), \quad x, \lambda > 0 \]

Where, \( \lambda \) is a scale parameter. In this paper assume that \( r = 1, \) after simplification, the equation above becomes

\[ f_{w}(x; \lambda) = \frac{4\lambda^3}{7} x^2 e^{-\lambda x} (1 - e^{-\lambda x}), \quad x, \lambda > 0 \] (6)

Where the cdf of EPWGE distribution is given as follows
\[ F_w(x) = \int_0^x f_w(t; \lambda) \, dt \]

Therefore,
\[ F_w(x) = \frac{2}{7} \lambda^2 e^{-\lambda x} x^2 (e^{-\lambda x} - 2) + \frac{2}{7} \lambda e^{-\lambda x} x(e^{-\lambda x} - 4) + \frac{1}{7} e^{-\lambda x}(e^{-\lambda x} - 8) + 1 \]  

(7)

2.1.1. Some important properties.

2.1.1.1 Moments and moment generating function

Consider \( X \sim EPWGE(\lambda) \), then the moment generating function of \( X \) (denoted as \( M_X(t) \)) is given as follows
\[ M_X(t) = E(e^{tx}) = \int_0^\infty \frac{4 \lambda^3}{7} x^2 e^{-(\lambda-t)x} (1 - e^{-\lambda x}) \, dx \]

Re-writing last equation as
\[ M_X(t) = \frac{1}{7} \left\{ \int_0^\infty x^2 e^{-(\lambda-t)x} \, dx - \int_0^\infty x^2 e^{-(2\lambda-t)x} \, dx \right\} \]

Using change of variables \( y = (\lambda - t)x \) in part one and \( z = (2\lambda - t)x \) in part two of right side of equation above, respectively, we have
\[ M_X(t) = \frac{8 \lambda^3}{7} \left( \frac{1}{(\lambda-t)^3} - \frac{1}{(2\lambda-t)^3} \right) \]  

(8)

Now, the \( p \)-moments for any probability distribution \( f(x) \) is obtained by the following form
\[ \mu_p = E(x^p) = \int_{-\infty}^{\infty} x^p f(x) \, dx \]

So, by eq. (6) we get on the \( p \)-moments of EPWGE distribution as
\[ \mu_p = \int_0^\infty \frac{4 \lambda^3}{7} x^{p+2} e^{-\lambda x} (1 - e^{-\lambda x}) \, dx \]

Consequently,
\[ \mu_p = \frac{4 \Gamma(p+3)}{7 \lambda^p} \left( 1 - 2^{-(p+3)} \right) \]  

(9)

Then where
\[ p = 1 \text{ implies } \mu_1 = \frac{45}{14 \lambda} = E(x) \]  

(10)
\[ p = 2 \text{ implies } \mu_2 = \frac{93}{7 \lambda^2} \]  

(11)
\[ p = 3 \text{ implies } \mu_3 = \frac{135}{2 \lambda^3} \]  

(12)
\[ p = 4 \text{ implies } \mu_4 = \frac{5715}{14 \lambda^4} \]  

(13)

and the variance of \( X \) can be obtained as
\[ \text{Var}(x) = \mu_2 - \mu_1^2 \]

So,
\[ \text{Var}(x) = \frac{579}{196 \lambda^2} \]  

(14)

Moreover, the coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) of the EPWGE distribution are obtained respectively using equations (10-13), as follows
CV = \sqrt{\frac{\mu_2}{\mu_1^2} - 1} = 0.5347

CS = \frac{\mu_3 - 3\mu_2 \mu_1 + 2\mu_1^3}{\sqrt{(\mu_2 - \mu_1^2)^3}} = 1.1434

CK = \frac{\mu_4 - 4\mu_3 \mu_1 + 6\mu_2 \mu_1^2 - 3\mu_1^4}{(\mu_2 - \mu_1^2)^2} = 5.0086

2.1.1.2 Entropy, Reliability, Hazard rate, reversed hazard and probability generating functions

A measure to quantify the uncertainty of an event was proposed by Shannon [13]. For any continuous random variable \( x \) associated with pdf \( f(x) \) the Entropy function defined as

\[ H(X) = -E\left(\ln(f(x))\right), \]

Now, by equation (6), get

\[ H(X;\lambda) = -E\left\{ \ln\left(\frac{4\lambda^3}{7}\right) + 2\ln(x) - \lambda x + \ln(1 - e^{-\lambda x}) \right\}, \quad x, \lambda > 0 \]

Using equation (10), get

\[ H(X;\lambda) = -\ln\left(\frac{4\lambda^3}{7}\right) + \frac{45}{14} - 2E(\ln(x)) - E\left(\ln(1 - e^{-\lambda x})\right), \quad x, \lambda > 0 \]

In the right side of equation (15) using one of numerical integration to solve it.

Figure 1. The behavior of entropy function.

Figure 1. The relationship between the parameter \( \lambda \) and the values of entropy. The scale parameter \( \lambda \) plays an important rate in determining the values of entropy, mainly it decreases as the parameter value increases.
The reliability function \( R_w(x) \) which also known as survival function is the probability of an item not failing prior to time \( t \). The reliability function of a random variable \( x \) which associated with EPWGE distribution is obtained as \( R_w(x) = 1 - F_w(x) \) and by equation (7) it is given as follows

\[
R_w(x) = -\frac{2}{7} \lambda^2 e^{\lambda x} x^2 (e^{\lambda x} - 2) - \frac{2}{7} \lambda e^{\lambda x} x (e^{\lambda x} - 4) - \frac{1}{7} e^{\lambda x} (e^{\lambda x} - 8)
\]  
\[
(16)
\]

The hazard rate function which also known as force of mortality in actuarial statistics of a random variable \( X \) which associated with EPWGE distribution is defined as

\[
h_w(x) = \frac{f_w(x)}{R_w(x)}
\]

Then by equations (7) and (16), it is obtained as:

\[
h_w(x) = \frac{4\lambda^3 x^2 (1 - e^{-\lambda x})}{2\lambda^2 x^2 (2 - e^{-\lambda x}) + 2\lambda x (4 - e^{-\lambda x}) + (8 - e^{-\lambda x})}
\]

\[
(17)
\]

The reversed hazard function is given as

\[
l_w(x) = \frac{f_w(x)}{F_w(x)}
\]

and from the equations (6,7), the reversed hazard as in the following form

\[
l_w(x) = \frac{4\lambda^3 x^2 e^{-\lambda x} (1 - e^{-\lambda x})}{2\lambda^2 e^{-\lambda x} x^2 (e^{-\lambda x} - 2) + 2\lambda e^{-\lambda x} x (e^{-\lambda x} - 4) + e^{-\lambda x} (e^{-\lambda x} - 8) + 7}
\]

\[
(18)
\]

Similarity the probability generating function of \( X \) given as [5]:

\[
P_X(t) = E(t^X)
\]

\[
(19)
\]

\[
\text{Therefore,}
\]

\[
P_X(t) = \frac{8\lambda^3}{7} \left( \frac{1}{\lambda - \ln(t)} \right)^3 - \frac{1}{(2\lambda - \ln(t))^3}
\]

\[
2.1.1.3 \textit{Mode, Median and Limiting}
\]

The behavior of the density function in (6) is investigated when the variable \( x \) go to zero and infinite. Therefore, \( \lim_{x \to 0} f(x) \) and \( \lim_{x \to 0} f(x) \) are given in the following forms, respectively

\[
\lim_{x \to 0} f(x) = \frac{4\lambda^3}{7} \lim_{x \to 0} (x^2) \cdot \lim_{x \to 0} (e^{-\lambda x}) \cdot \lim_{x \to 0} (1 - e^{-\lambda x}) = 0
\]

\[
\lim_{x \to 0} f(x) = \frac{4\lambda^3}{7} \lim_{x \to 0} (x^2) \cdot \lim_{x \to 0} (e^{-\lambda x}) \cdot \lim_{x \to 0} (1 - e^{-\lambda x}) = 0
\]

Consequently, it is clear that the model has a unique mode. From equation (6), we have

\[
\ln(f(x)) = \ln \left( \frac{4\lambda^3}{7} \right) + 2\ln(x) - \lambda x + \ln \left( 1 - e^{-\lambda x} \right)
\]

\[
\frac{df}{dx} = \frac{2}{x} - \lambda + \frac{\lambda e^{-\lambda x}}{1 - e^{-\lambda x}}
\]

\[
\frac{d^2f}{dx^2} = -\left( \frac{2}{x^2} + \frac{\lambda^2 e^{-\lambda x}}{(1 - e^{-\lambda x})^2} \right) < 0
\]
Therefore, the value of \( x \) \((x \neq 0, x \neq \infty)\) which satisfies the following equation represents the mode of EPWGE distribution.

\[
2 - \lambda x + \frac{\lambda x e^{-\lambda x}}{1 - e^{-\lambda x}} = 0 \tag{20}
\]

Now, the value of \( x \) which it satisfies the following equation \( F_w(x) = \frac{1}{2} \), represents the median of EPWGE distribution, using equation (7), we get

\[
\frac{2}{7}\lambda^2 e^{-\lambda x} x^2 (e^{-\lambda x} - 2) + \frac{2}{7}\lambda e^{-\lambda x} x (e^{-\lambda x} - 4) + \frac{1}{7} e^{-\lambda x} (8 - e^{-\lambda x}) = \frac{1}{2}
\]

Consequently,

\[
e^{-\lambda x} \left( \lambda^2 x^2 (2 - e^{-\lambda x}) + \lambda x (4 - e^{-\lambda x}) + \frac{1}{2} (e^{-\lambda x} - 8) \right) = \frac{7}{4}
\]

From this, we have

\[
x = \frac{1}{\lambda} \left[ \ln \left( \frac{4}{7} \right) + \ln \left( \lambda^2 (2 - e^{-\lambda x}) x^2 + \lambda (4 - e^{-\lambda x}) x + \frac{1}{2} (8 - e^{-\lambda x}) \right) \right] \tag{21}
\]

Figure 2. Represent mode, median with some values of \( \lambda \).

Figure 2. Shows the relationship between the scale parameter \( \lambda \) and the values of Mode and median. We observe that the values of mode and median are increasing for increments of \( \lambda \).

3. Estimation

In this section, the estimation of a scale parameter \( \lambda \) of EPWGE is discussed for \( x_1, x_2, \ldots, x_n \) be the random sample from EPWGE distribution.

3.1. Method of Moments Estimator (MM)

The moments method was introduced in the first time by Chebyshev in 1887 in the proof of the central limit theorem. The idea is that take known fact about population parameter and extend this idea to the sample, then
\[ \mu_p = \frac{\sum_{i=1}^{n} x^p}{n} \] be the p-th sample moment, and by equation (9), we have

\[ \mu_p = \frac{4\Gamma(p + 3)}{7\lambda^p} (1 - 2^{-(p+3)}) \]

and by equations (10), we get

\[ \frac{45}{14\lambda} = \frac{\sum_{i=1}^{n} x_i}{n} \]

So, the moment estimator of \( \lambda \) as:

\[ \hat{\lambda}_{MM} = \frac{45n}{14 \sum_{i=1}^{n} x_i} \] (22)

3.2. Maximum Likelihood Estimator (MLE)

Maximum Likelihood is a relatively simple method of constructing an estimator for an unknown parameter it was introduced by R. A. Fisher in 1912. Estimation is a method that determines values for the parameters of a model. Then Likelihood function for equation (6) is given as

\[ L = L(x_1, x_2, ..., x_n; \lambda) = \left( \frac{4}{7} \right)^n \lambda^{3n} \prod_{i=1}^{n} x_i^2 e^{-\lambda \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} (1 - e^{-\lambda x_i}) \]

Taking natural logarithm of above equation, we have

\[ \ln L = n \ln \left( \frac{4}{7} \right) + 3n \ln(\lambda) + 2 \sum_{i=1}^{n} \ln(x_i) - \lambda \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \ln(1 - e^{-\lambda x_i}) \]

Derive both sides of above equation with respect to \( \lambda \), we get

\[ \frac{\partial \ln L}{\partial \lambda} = \frac{3n}{\lambda} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{x_i e^{-\lambda x_i}}{(1 - e^{-\lambda x_i})} \]

The Maximum Likelihood estimator for the scale parameter \( \lambda \) is obtained by equating the above equation equal to zero, then

\[ \frac{1}{3n} \sum_{i=1}^{n} x_i - \frac{1}{3n} \sum_{i=1}^{n} \frac{x_i e^{-\lambda x_i}}{(1 - e^{-\lambda x_i})} = \frac{1}{\lambda} \]

So, the ML estimator as:

\[ \hat{\lambda}_{MLE} = 3n \left\{ 2 \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{x_i}{(1 - e^{-\lambda x_i})} \right\}^{-1} \] (23)

4. Simulations

Simulation study using to generate a certain distribution of data in order to find the best estimate [12]. Here, the results of numerical four experiments, based on Monte Carlo in MATLAB version 2019a to compare the performance of the two estimators with sample size (n=10, 25, 75) and the parameter default values (\( \lambda = 0.5, 1, 2, 2.5 \)), using the solution of the equation (23) by Newton Raphson method and equation (22), and depend on equation (7) to generate the r.v. \( x \) as follow;

let \( F(x) = U \), where \( U \) is a random variable on interval (0,1)

\[ U = \frac{2}{7} \lambda^2 e^{-\lambda x} x^2 (e^{-\lambda x} - 2) + \frac{2}{7} \lambda e^{-\lambda x} x (e^{-\lambda x} - 4) + \frac{1}{7} e^{-\lambda x} (e^{-\lambda x} - 8) + 1 \]

After simplicity, we have

\[ U^* = \frac{2}{7} e^{-\lambda x} \left( \lambda^2 x^2 (e^{-\lambda x} - 2) + \lambda x (e^{-\lambda x} - 4) + \frac{1}{2} e^{-\lambda x} - 8 \right) \]

Where, \( U^* = 1 - U \). Taking ln for both sides of above equation, we have

\[ \ln(U^*) = \ln \left( \frac{2}{7} \right) - \lambda x + \ln \left( \lambda^2 x^2 (e^{-\lambda x} - 2) + \lambda x (e^{-\lambda x} - 4) + \frac{1}{2} e^{-\lambda x} - 8 \right) \]

Solving the following equation numerically to generate \( x \) which distributed EPWGE.
\[-\lambda x + \ln\left(\frac{2}{e}\right) - \ln(U^*) + \ln\left(\lambda^2 x^2 (e^{-\lambda x} - 2) + \lambda x (e^{-\lambda x} - 4) + \frac{1}{2} (e^{-\lambda x} - 8)\right)\]

Then MSE of parameter estimation replication (500) are given in table below.

**Table 1.** The MSE for estimate scale parameter \(\lambda\) for EPWGE distributions.

| \(\lambda\) = 0.5 |  |  |
|---|---|---|
| \(n\) | MLE | MM | Best |
| 10  | 0.0023 | 0.0017 | MM  |
| 25  | 0.0002 | 0.0004 | MLE |
| 75  | 0.0001 | 0.0002 | MLE |
| \(\lambda\) = 1 |  |  |
| \(n\) | MLE | MM | Best |
| 10  | 0.0198 | 0.0202 | MLE |
| 25  | 0.0022 | 0.0023 | MLE |
| 75  | 0.0006 | 0.0007 | MLE |
| \(\lambda\) = 2 |  |  |
| \(n\) | MLE | MM | Best |
| 10  | 0.2229 | 0.2240 | MLE |
| 25  | 0.0595 | 0.0548 | MLE |
| 75  | 0.0007 | 0.0009 | MLE |
| \(\lambda\) = 2.5 |  |  |
| \(n\) | MLE | MM | Best |
| 10  | 0.1256 | 0.1157 | MM |
| 25  | 0.0083 | 0.0097 | MLE |
| 75  | 0.0003 | 0.0010 | MLE |

5. **Conclusions**

We conclude that as sample size increases, the MSE decrease that is quite inevitable and also verifies the consistency properties of the estimators. The estimation performance of Maximum Likelihood method is superior, so it is better than method of moments to estimate the scale parameter of EPWGE distribution in this case.

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