A short note on a conjecture of Okounkov about a \( q \)-analogue of multiple zeta values

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July 28, 2014

Abstract

In [Ok] Okounkov studies a specific \( q \)-analogue of multiple zeta values and makes some conjectures on their algebraic structure. In this note we want to compare Okounkov’s \( q \)-analogues to the generating function for multiple divisor sums defined in [BK].

1 Introduction

Multiple zeta values are natural generalizations of the Riemann zeta values that are defined for integers \( s_1 > 1 \) and \( s_i \geq 1 \) for \( i > 1 \) by

\[
\zeta(s_1, \ldots, s_l) := \sum_{n_1 > n_2 > \cdots > n_l > 0} \frac{1}{n_1^{s_1} \cdots n_l^{s_l}}.
\]

Because of its occurrence in various fields of mathematics and physics these real numbers are of particular interest. In [Ok] Okounkov discusses a conjectural connection from enumerative geometry of some Hilbert schemes to a specific \( q \)-analogue \( Z(s_1, \ldots, s_l) \) of the multiple zeta-values. He denotes by \( \text{qMZV} \) the \( \mathbb{Q} \)-algebra generated by these. In this short note we want to discuss the connection of these \( q \)-multiple zeta values to the algebra \( \mathcal{MD} \) of generating functions for multiple divisor sums \([s_1, \ldots, s_l]\) defined by the authors in [BK]. More precisely we have

**Theorem 1.1.** Let \( \mathcal{MD}^\sharp = \{[s_1, \ldots, s_l] \in \mathcal{MD} \mid s_i > 1 \forall i \text{ or } s_1 = \emptyset \}_Q \).

i) The sub vector space \( \mathcal{MD}^\sharp \) is in fact a sub algebra of \( \mathcal{MD} \).

ii) We have \( \text{qMZV} = \mathcal{MD}^\sharp \), in particular the \( \mathbb{Q} \)-vector space generated by the \( Z(s_1, \ldots, s_l) \) is closed under multiplication.

iii) We have \( \frac{d}{dq} Z(k) \in \text{qMZV} \) for all \( k \geq 2 \).

The first two statements are merely a reformulation results implicitely contained in [BK]. The third is direct consequence of some explicit formula given in [BK]. It gives some evidence to the conjecture of Okunkov, that the operator \( d \) is a derivation on \( \text{qMZV} \).
2 q-analogues of multiple zeta values

In the following we fix a subset $S \subset \mathbb{N}$, which we consider as the support for index entries, i.e. we assume $s_1, \ldots, s_l \in S$. For each $s \in S$ we let $Q_s(t) \in \mathbb{Q}[t]$ be a polynomial with $Q_s(0) = 0$ and $Q_s(1) \not\equiv 0$. We set $Q = \{Q_s(t)\}_{s \in S}$. A sum of the form

$$Z_Q(s_1, \ldots, s_l) := \sum_{n_1 > \cdots > n_l > 0} \prod_{j=1}^l \frac{Q_{s_j}(q^{n_j})}{(1 - q^{n_j})^s}$$

with polynomials $Q_s$ as before, defines a q-analogue of a multiple zeta-value of weight $k = s_1 + \cdots + s_l$ and length $l$. Observe only because of $Q_{s_1}(0) = 0$ this defines an element of $\mathbb{Q}[[q]]$. This notion is due to the identity

$$\lim_{q \to 1} (1 - q)^k Z_Q(s_1, \ldots, s_l) = \sum_{n_1 > \cdots > n_l > 0} \prod_{j=1}^l \lim_{q \to 1} \left( Q_{s_j}(q^{n_j}) \frac{(1 - q)^{s_j}}{(1 - q^{n_j})^s_j} \right) = Q_{s_1}(1) \cdots Q_{s_l}(1) \cdot \zeta(s_1, \ldots, s_l).$$

Here we used that $\lim_{q \to 1} (1 - q)^s/(1 - q^n)^s = 1/n^s$ and with the same arguments as in [BK] Proposition 6.4, the above identity can be justified for all $(s_1, \ldots, s_l)$ with $s_1 > 1$. Related definition for q-analogues of multiple zeta values are given in [LT], [Ta], [Zu] and [OOZ]. It is convenient to define $Z_Q(\emptyset) = 1$ and then we denote the vector space spanned by all these elements by

$$Z(Q, S) := \langle Z_Q(s_1, \ldots, s_l) \mid l \geq 0 \text{ and } s_1, \ldots, s_l \in S \rangle_Q.$$ (2.2)

Note by the above convention we have that $Q$ is contained in this space.

Lemma 2.1. If for each $r, s \in S$ there exists numbers $\lambda_j(r, s) \in \mathbb{Q}$ such that

$$Q_r(t) \cdot Q_s(t) = \sum_{1 \leq j \leq r+s} \lambda_j(r, s)(1 - t)^{r+j}Q_j(t),$$ (2.3)

then the vector space $Z(Q, S)$ is a Q-algebra.

Proof. We have to show that $Z_Q(s_1, \ldots, s_l) \cdot Z_Q(r_1, \ldots, r_m) \in Z(Q, S)$ and illustrate this in the $l = m = 1$ case because the higher length case will be clear after this. Suppose there is a representation of the form (2.3), then it is

$$Z_Q(r) \cdot Z_Q(s) = \sum_{n_1 > 0} \frac{Q_r(q^{n_1})}{(1 - q^{n_1})^r} \cdot \sum_{n_2 > 0} \frac{Q_s(q^{n_2})}{(1 - q^{n_2})^s}$$

$$= \sum_{n_1 > n_2 > 0} \cdots + \sum_{n_2 > n_1 > 0} \cdots + \sum_{n_1 = n_2 > n > 0} \frac{Q_r(q^n)Q_s(q^n)}{(1 - q^n)^{r+s}}$$

$$= Z_Q(r, s) + Z_Q(s, r) + \sum_{j \in S'} \lambda_j Z_Q(j) \in Z(S, Q).$$
We give three examples of $q$-analogues of multiple zeta values, which are currently considered by different authors where just the second and the third will be of interest in the rest of this note.

0) The polynomials $Q^T_s(t) = t^{s-1}$ are considered in [Ta] and sums of the form (2.1) with $s_1 > 1$ and $s_2, \ldots, s_l \geq 1$ are studied there.

i) In [BK] the authors choose $Q^E_s(t) = \frac{1}{(s-1)!}tP_{s-1}(t)$, where the $P_s(t)$ are the eulerian polynomials defined by

$$tP_{s-1}(t) = \sum_{d=1}^\infty d^{s-1}t^d$$

for $s \geq 0$. With this define for all $s_1, \ldots, s_l \in \mathbb{N}$

$$[s_1, \ldots, s_l] := \sum_{n_1 > \ldots > n_l > 0} \prod_{j=1}^l \frac{Q^E_{s_j}(q^{n_j})}{(1-q^{n_j})^{s_j}} .$$

and set

$$\mathcal{MD} = Z(\{Q^E_s(t)\}_{s \in \mathbb{N}}) .$$

These brackets are generating functions for multiple divisor sums and they occur in the Fourier expansion of multiple Eisenstein series.

ii) Okounkov chooses the following polynomials in [Ok]

$$Q^O_s(t) = \begin{cases} t^\frac{s}{2} & s = 2, 4, 6, \ldots \\ t^\frac{s-1}{2} (1 + t) & s = 3, 5, 7, \ldots . \end{cases}$$

and defines for $s_1, \ldots, s_l \in S = \mathbb{N}_{>1}$

$$Z(s) = \sum_{n_1 > \ldots > n_l > 0} \prod_{j=0}^l \frac{Q^O_{s_j}(q^{n_j})}{(1-q^{n_j})^{s_j}} .$$

We write for the space of the Okounkov $q$-multiple zetas

$$q\text{MZV} = Z(\{Q^O_s(t)\}_{s \in \mathbb{N}_{>1}}) .$$

**Proposition 2.2.** For the polynomials above we have

i) for $r, s \in \mathbb{N}$ and $Q^E_j(t) = \frac{1}{(s-1)!}tP_{s-1}(t)$

$$Q^E_r(t) \cdot Q^E_s(t) = \sum_{j=1}^r \lambda_{r, s}(1-t)^{r+s-j}Q^E_j(t) + \sum_{j=1}^s \lambda_{s, r}(1-t)^{r+s-j}Q^E_j(t) + Q^E_{r+s}(t) ,$$

where the coefficient $\lambda_{a, b} \in \mathbb{Q}$ for $1 \leq j \leq a$ is given by

$$\lambda_{a, b}^j = (-1)^{b-1} \binom{a+b-j-1}{a-j} \frac{B_{a+b-j}}{(a+b-j)!} .$$
ii) for \( r, s \in \mathbb{N}_{>1} \) it is
\[
Q^O_r(t) \cdot Q^O_s(t) = \begin{cases} 
Q^O_{r+s}(X) & , r + s \text{ even} \\
2Q^O_{r+s}(t) + (1-t)^2Q^O_{r+s-2}(t) & , r + s \text{ odd}.
\end{cases}
\]

In particular, because of Lemma 2.1, the vector spaces \( \mathcal{MD} \) and \( qMZV \) are \( \mathbb{Q} \)-algebras.

**Proof.** In [BK] the claim i) is proven. The cases in ii) are checked easily. \( \square \)

**Corollary 2.3.** \( \mathcal{MD}^\sharp = Z(\{Q^E_s\}_{s \in \mathbb{N}_{>1}}) \) is a sub algebra of \( \mathcal{MD} \).

**Proof.** Using Proposition 2.2 it is easy to see that it suffices to show that
\[
\lambda^1_{a,b} + \lambda^1_{b,a} = ((-1)^a-1 + (-1)^b-1) \left( \frac{a+b-2}{a-1} \right) \frac{B_{a+b-1}}{(a+b-1)!}
\]
vanishes for \( a, b > 1 \). This term clearly vanishes when \( a \) and \( b \) have different parity. In the other case \( a + b - 1 \) is odd and greater than 1, as \( a, b > 1 \). It is well known that in this case \( B_{a+b-1} = 0 \), from which we deduce that \( \lambda^1_{a,b} + \lambda^1_{b,a} = 0 \). \( \square \)

**Theorem 2.4.** Let \( Z(Q, \mathbb{N}_{>1}) \) be any family of \( q \)-analogues of multiple zeta values as in (2.2), where each \( Q_s(t) \in \mathbb{Q} \) is a polynomial with degree at most \( s-1 \), then
\[
Z(Q, \mathbb{N}_{>1}) = \mathcal{MD}^\sharp.
\]
and therefore all such families of \( q \)-analogues of multiple zeta values are \( \mathbb{Q} \)-subalgebras of \( \mathcal{MD} \). In particular \( qMZV = \mathcal{MD}^\sharp \).

**Proof.** To proof the first equality it is sufficient to show that for each \( s > 1 \) there are numbers \( \lambda_j \in \mathbb{Q} \) with \( 2 \leq j \leq s \) such that
\[
\frac{Q_s(t)}{(1-t)^s} = \sum_{j=2}^{s} \lambda_j \frac{Q^E_j(t)}{(1-t)^j}.
\]
The space of polynomials with at most degree \( s-1 \) and no constant term has dimension \( s-1 \). For \( 2 \leq j \leq s \) the polynomials \( (1-t)^{s-j}Q^E_j(t) \) are all linear independent since \( Q'(1) \neq 1 \) and therefore such \( \lambda_j \) exist. The second statement follows directly from the definition of \( qMZV \). \( \square \)

The following proposition allows one to write an arbitrary element in \( Z(Q, \mathbb{N}_{>1}) \) as an linear combination of \( [s_1, \ldots, s_l] \in \mathcal{MD}^\sharp \).

**Proposition 2.5.** Assume \( k \geq 2 \). For \( 1 \leq i, j \leq k-1 \) define the numbers \( b_{i,j}^k \in \mathbb{Q} \) by
\[
\sum_{j=1}^{k-1} \frac{b_{i,j}^k}{j!} t^j := \binom{t+k-1-i}{k-1}.
\]
With this it is for \( 1 \leq i \leq k-1 \) and \( Q^E_j(t) = \frac{1}{(j-1)!} tP_j(t) \)
\[
t^i = \sum_{j=2}^{k} b_{i,j-1}^k (1-t)^{k-j} Q^E_j(t).
\]
Proof. We want to show that
\[
\frac{t^i}{(1-t)^{k}} = \sum_{j=1}^{k-1} b_{k,j}^i \frac{t P_j(t)}{j! (1-t)^{j+1}}
\]
By the definition of the Eulerian Polynomials it is
\[
\sum_{j=1}^{k-1} b_{k,j}^i \frac{t P_j(t)}{j! (1-t)^{j+1}} = \sum_{j=1}^{k-1} b_{k,j}^i \sum_{d>0} d^j t^d
\]
\[
= \sum_{d>0} \left( \sum_{j=1}^{k-1} b_{k,j}^i d^j \right) t^d
\]
\[
= \sum_{d>0} \left( d - i + k - 1 \right) t^d
\]
The claim now follows directly from the easy to prove formula
\[
\frac{1}{(1-t)^{k}} = \sum_{n \geq 0} \binom{n + k - 1}{k - 1} t^n.
\]
\[
\square
\]
We give some examples how to write elements in $qMZV$ as linear combinations of elements in $\mathcal{MD}$. From the proposition we deduce for the length one case for all $k > 0$
\[
Z(2k) = \sum_{j=2}^{2k} b_{k,j-1}^{2k} [j] \quad \text{and} \quad Z(2k + 1) = \sum_{j=2}^{2k+1} \left( b_{k,j-1}^{2k+1} + b_{k+1,j-1}^{2k+1} \right) [j].
\]
Clearly this also suffices to give linear combinations in higher length.

Example 2.6. We give some examples
\[
Z(2) = [2], \quad Z(3) = 2[3],
\]
\[
Z(4) = [4] - \frac{1}{6} [2], \quad Z(5) = 2[5] - \frac{1}{6} [3],
\]
\[
Z(6) = [6] - \frac{1}{4} [4] + \frac{1}{30} [2], \quad Z(7) = 2[7] - \frac{1}{3} [5] + \frac{1}{45} [3],
\]
\[
Z(2, 2) = [2, 2], \quad Z(2, 4) = [2, 4] - \frac{1}{6} [2, 2].
\]
The $q$-expansion of modular forms are well known to give rise to $q$-analogues of Riemann zeta values. Let us denote by $M_Q = Q[G_4, G_6]$ and $\tilde{M}_Q = Q[G_2, G_4, G_6]$ the ring of modular and quasi-modular forms, where the Eisenstein series $G_2$, $G_4$ and $G_6$ are given by
\[
G_2 = -\frac{1}{24} + [2], \quad G_4 = \frac{1}{1440} + [4], \quad G_6 = -\frac{1}{60480} + [6].
\]
We clearly have the following inclusions of $Q$-algebras

$$M_Q \subset \tilde{M}_Q \subset qMZV \subset MD.$$  

where the second inclusion follows from

$$G_2 = -\frac{1}{24} + Z(2),$$
$$G_4 = \frac{1}{1440} + Z(2) + \frac{1}{6}Z(4),$$
$$G_6 = -\frac{1}{60480} + Z(6) + \frac{1}{4}Z(4) + \frac{1}{120}Z(2).$$

In the theory of modular forms the operator $d := q\frac{dq}{d}$ plays an important role and it is a well known fact that $\tilde{M}_Q$ is closed under $d$. In [BK] the authors showed the following

**Theorem 2.7.** The operator $d$ is a derivation on $MD$ that is compatible with the filtrations on $MD$ given by the weight and the length.

In [Ok] the following conjecture is stated by Okounkov

**Conjecture 2.8.** The operator $d$ is a derivation on $qMZV$.

For the derivative of a length one generating series of multiple divisor sums we have several identities. These will be used to make the following result which gives some evidence for the conjecture above.

**Proposition 2.9.** It is $dZ(k) \in qMZV$ for all $k \geq 2$.

**Proof.** In [BK] Theorem 3.5 the authors prove the following representation of the derivative $d[k-2]$:

$$\binom{k-2}{s_1-1} \frac{d[k-2]}{k-2} = [s_1] \cdot [s_2] - [s_1, s_2] - [s_2, s_1]$$
$$+ \binom{k-2}{s_1-1} [k-1] - \sum_{a+b=k+2, a>s_1} \left( \binom{a-1}{s_1-1} + \binom{a-1}{s_2-1} - \delta_{a, s_2} \right) [a, b].$$

where $s_1, s_2 > 0$ can be chosen arbitrary such that $k = s_1 + s_2$. First divide both sides by $(s_1-1)(k-2)^{-1}$. Whenever $k \geq 4$ all elements on the right of the resulting equation belong to $MD^2$ except for the term with $[k-1, 1]$. By direct calculation one obtains that for $s_1 = 1$ and $s_2 = k - 1$ the coefficient of $[k-1, 1]$ is $-(k-2)$ and for $s_2 = 2$ and $s_2 = k - 2$ it is $-2(k-2)$ and therefore $d[k-2]$ can be expressed as an element in $MD^2$.

Since $d$ is a derivation it satisfies the Leibniz rule. Therefore the above proposition allows us to derive further identities, e.g.

$$dZ(k, ..., k), d(Z(k_1, k_2) + Z(k_2, k_1)) \in qMZV.$$
Example 2.10. Some examples of representations of \( dZ(s) \) in qMZV.

\[
\begin{align*}
\text{d} Z(2) &= 3Z(4) + Z(2) - Z(2, 2), \\
\text{d} Z(3) &= 5Z(5) + Z(3) - 4Z(3, 2) - 6Z(2, 3), \\
\text{d} Z(4) &= 10Z(6) + 2Z(4) + 4Z(4, 2) - 8Z(2, 4) - 6Z(3, 3), \\
\text{d} Z(2, 2) &= -6Z(6) - 12Z(2, 2, 2) - 15Z(4, 2) + 3Z(2, 4) + 9Z(3, 3), \\
\text{d} Z(3, 3) &= 4Z(8) - 12Z(2, 3, 3) - 10Z(3, 2, 3) - 8Z(3, 3, 2) \\
&\quad + Z(3, 5) - Z(5, 3) + 8Z(6, 2) + 3Z(3, 3), \\
\text{d} Z(2, 2, 2) &= -24Z(2, 2, 2) + 9Z(3, 2, 3) + 6Z(3, 3, 2) \\
&\quad - 15Z(4, 2, 2) - 15Z(2, 4, 2) + 3Z(2, 2, 4) - 6Z(2, 6) + Z(5, 3) - 6Z(6, 2).
\end{align*}
\]

At the end we give some conjectured representations of \( dZ(s) \) in qMZV coming from numerical experiments and which were checked for the first 200 coefficients but which should be also provable by using the results in [BK].

\[
\begin{align*}
\text{d} Z(2, 3) &= 2Z(7) - 16Z(2, 2, 3) - 4Z(2, 3, 2) - 8Z(3, 2, 2) \\
&\quad - 15Z(4, 3) - 4Z(3, 4) + 4Z(5, 2) + 5Z(2, 5) + Z(3, 2) - Z(2, 3),
\end{align*}
\]

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