Abstract
Using a model spectral function of the electron the Hall conductivity in the normal metallic state of the Pr$_{2-x}$Ce$_x$CuO$_4$ (PCCO) superconductor is calculated neglecting the current vertex-correction. The result is qualitatively consistent with the experiment. Consequently the reason becomes clear why the Fermi-liquid theory fails to explain the anomaly of the Hall conductivity. The inconsistency of the fluctuation-exchange approximation also becomes clear.

Keywords: Hall conductivity; Spectral function method; Fermi-liquid theory; Fluctuation-exchange approximation.

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1. Introduction
Various anomalous properties in the normal metallic state of cuprate superconductors cannot be explained by the Fermi-liquid theory which is the standard theory for interacting electrons. One of the most striking and puzzling anomalies is the strong temperature dependence of the Hall coefficient [1]. In the Fermi-liquid theory for a single-band case the Hall coefficient is a measure for the carrier density of the metal and almost temperature-independent [2]. Thus many attempts have been made to explain the Hall coefficient from non-Fermi-liquid viewpoints [1,3,4]. However, such attempts seem to be unsuccessful. On the other hand, it is claimed using the fluctuation-exchange
(FLEX) approximation \[5\] that the anomalous behavior of the coefficient (ABC) comes from the current vertex-correction (CVC). Since the Hall coefficient of the FLEX approximation is evaluated within the Fermi-liquid framework, such a claim contradicts the standard argument leading to almost temperature-independent Hall coefficient.

Recently the transport anomalies have been discussed on the basis of hidden Fermi-liquid \[6, 7\] or anisotropic marginal Fermi-liquid \[8, 9\] model. In these discussions the anisotropy in the temperature-dependence of the imaginary part of the electron self-energy in momentum space is crucial but the CVC is neglected. To explain the Hall coefficient the anisotropy of the real part is also important as discussed in the following. Anyway, the essential part of the transport coefficients is expressed in terms of the spectral function of the electron whose non-trivial part is characterized by the self-energy. For example, this is proved experimentally \[10\] for a CDW pseudo-gap material where an anomalous temperature dependence of the Hall coefficient is explained using the spectral function measured by the angle-resolved photoemission spectroscopy (ARPES). We support this spectral function method.

In this letter we demonstrate the effectiveness of the spectral function method for the Hall conductivity using a simple model of the spectral function. At the same time it becomes clear why the Fermi-liquid theory fails to explain the anomaly of the Hall conductivity. We also show the inconsistency of the FLEX approximation. Since the comparison of our result and that by the FLEX approximation \[11\] for the Pr$_{2-x}$Ce$_x$CuO$_4$ (PCCO) superconductor reveals the inconsistency, we choose this material as the target of our calculation.

2. Conductivity formula

We start from the full Green function for electrons

\[
G^R(p,\varepsilon) = \frac{1}{\varepsilon - E(p,\varepsilon) + i/2\tau(p,\varepsilon)},
\]

or

\[
G^A(p,\varepsilon) = \frac{1}{\varepsilon - E(p,\varepsilon) - i/2\tau(p,\varepsilon)},
\]

where the real-part of the self-energy is renormalized into the dispersion $E(p,\varepsilon)$ and the imaginary-part into the life-time $\tau(p,\varepsilon)$. Here we have made no assumption on the Green function. Namely, the following discussion is free from the nature of the quasi-particle, Fermi liquid or non-Fermi liquid.

The spectral function of the electron is given as

\[
\rho_p(\varepsilon) = \frac{1}{\pi} \frac{1/2\tau(p,\varepsilon)}{[\varepsilon - E(p,\varepsilon)]^2 + [1/2\tau(p,\varepsilon)]^2}.
\]
The DC conductivities are essentially determined by this function at \( \varepsilon = 0 \) as discussed in the following.

The integrand of the momentum-integration of the DC conductivity \( \sigma_{xx} \) in the absence of the magnetic field is proportional to \( v_x \cdot v_x \), where \( v_x \) is the \( x \)-component of the group velocity of the renormalized quasi-particle, and the effect of the CVC by the residual interaction among quasi-particles is expressed as the renormalization of the life-time \( \tau(p, \varepsilon) \) into the transport life-time. The proof of this is given in [12] for spherically symmetric case and in [13] for anisotropic case. The same is true for the DC Hall conductivity \( \sigma_{xy} \) linear in the magnetic field \( H \). The proof is given in [14] for spherically symmetric case and in [15] for anisotropic case. The FLEX approximation [5] cannot meet this exact result, since it violates the harmony [12, 13] between the imaginary part of the self-energy and the CVC. Such a harmony is a consequence of the local electron-number conservation [16]. However, the Baym-Kadanoff scheme including the FLEX approximation only assures the global conservation [17].

Although the renormalization in the transport life-time has a significant effect on the conductivities by removing forward scattering contributions ir-
relevant to the change in the velocity of electrons in the case where some long-wavelength scattering (see [18] for phonons and [19] for ferromagnetic spin-fluctuations) is dominant, we can neglect [20, 21] such a renormaliza-
tion for our target material, since the scattering across the Fermi surface is dominant in our case.

Our numerical calculation is done for 2D square lattice with the lattice constant \( a = 1 \). By the symmetry we only need the information for the quarter of the Brillouin zone: \( 0 \leq p_x \leq \pi \) and \( 0 \leq p_y \leq \pi \). In the following the summation over \( p \) is restricted within this quarter.

Neglecting the CVC the DC conductivity \( \sigma_{xx} \) in the absence of the mag-
netic field is given by [22]

\[
\sigma_{xx} = \frac{e^2}{\pi} \cdot \frac{4}{R(T)},
\]

(4)

with

\[
\frac{1}{R(T)} \equiv \sum_p v_x^2 G^R G^A = \sum_p v_x^2 \frac{E^2}{E^2 + (1/2 \tau)^2},
\]

(5)

where \( G^R = G^R(p, 0), \ G^A = G^A(p, 0), \ E = E(p, 0), \ \tau = \tau(p, 0) \) and \( v_x = \partial E/\partial p_x \). Here we have assumed the Fermi degeneracy for simplicity.

In the same manner the DC Hall conductivity \( \sigma_{xy} \) proportional to the weak magnetic field \( H \) is given by [22]

\[
\sigma_{xy}/H = \frac{|e|}{\pi} \cdot 4S(T),
\]

(6)
with
\[
S(T) = \sum_{p} v_x \left( v_y \frac{\partial v_y}{\partial p_y} - v_y \frac{\partial v_x}{\partial p_y} \right) \frac{G^R G^A G^R - G^A}{2t},
\]
where \( v_y = \partial E / \partial p_y \). Thus, if we have the data of the dispersion \( E \) and the life-time \( \tau \) at \( \epsilon = 0 \), we can get the DC conductivities from these formulae. The necessary data are obtained from the spectral function.

If the ARPES supplies such data, we can immediately calculate the conductivities. However, the purpose of this letter is to demonstrate the effectiveness of the spectral-function method so that we phenomenologically use a simple model for the spectral function for simplicity. To derive the model spectral function from a microscopic model is another task to do but not discussed in this letter. Although we introduce the model spectral function under the assumption that the dominant scattering is due to the anti-ferromagnetic spin-fluctuation, some qualitatively similar spectral function can be obtained from the other frameworks [6, 7, 8, 9].

3. Model spectral function

Here we use the dispersion for renormalized quasi-particles
\[
E = -2t(\cos p_x + \cos p_y) + 4t' \cos p_x \cdot \cos p_y - 2t''(\cos 2p_x + \cos 2p_y) - \mu, \tag{8}
\]
obtained by the band calculation [23]. For PCCO we adopt \( t = -0.438 \text{eV}, \ t' = 0.156 \text{eV} \) and \( t'' = 0.098 \text{eV} \) [23] in the numerical calculations in this letter.

The Fermi surface for this dispersion is shown in Fig. 1. The factor \( f(p) \equiv v_x(v_x \partial v_y / \partial p_y - v_y \partial v_x / \partial p_y) \) appearing in \( \sigma_{xy} \) is shown in Fig. 2.

For the life-time we adopt the model
\[
\frac{1}{2\tau} = w(p) \cdot g_1 \cdot T + \left[ 1 - w(p) \right] \cdot g_2 \cdot T^2, \tag{9}
\]
similar to the multi-patch model [24] where the Brillouin zone is divided into hot and cold patches. We expect that the life-time is determined by the coupling to the anti-ferromagnetic spin-fluctuation. In the cold patches the relevant spin-fluctuation spectrum is broad and its temperature-dependence is weak so that \( 1/\tau \sim T^2 \) [20, 21, 31]. On the other hand, in the hot patches it is peaked around the nesting-vector and its integrated weight depends on the temperature so that \( 1/\tau \sim T \) [20, 21, 31]. For simplicity we did not
implement the gradual change between hot and cold patches employed in [24]. Namely \( w(p) \) is a step function in our case where \( w(p) = 1 \) if \((p - p_1)^2 < r^2\) or \((p - p_2)^2 < r^2\) with \(p_1 \equiv (\pi, 0)\) and \(p_2 \equiv (0, \pi)\) and \(w(p) = 0\) otherwise. The parameter \(r^2\) is chosen to be 1.07.

We set \(g_2 = 100/eV\) in accordance with [9]. Here the temperature \(T\) is measured in eV. On the other hand, our choice, \(g_1 = 0.1\), is smaller than [9] and [24]. Such a smallness is explained by the nested spin fluctuation [31].

4. Numerical results

Before performing the numerical calculation we can make a rosy prediction that the non-monotonic temperature dependence of \(\sigma_{xy}\), which we want to derive, is obtained, if the temperature dependences of \(1/\tau\) are different between the regions with positive \(f(p)\) and the regions with negative \(f(p)\) in the Brillouin zone. After the numerical calculation we confirm the prediction as shown in Fig. 3. The result for \(\mu = 0\) actually shows the non-monotonic temperature dependence qualitatively similar to that observed in the experiment [11]. More direct comparison between the experiment [11] and our numerical calculation can be done for \(\sigma_{xy}/\sigma_{xx}\) shown in Fig. 4 which qualitatively explains the experiment.

5. Remarks

Our simple formula based on the spectral function of the electron qualitatively explains the non-monotonic temperature-dependence of the DC Hall conductivity in the normal state of PCCO superconductors.

As shown in [22] this formula leads to the constant Hall coefficient if the spectral function \(\rho_p(\varepsilon)\) is delta-function-like and isotropic in momentum space. The Fermi-liquid theory for the Hall coefficient [2] also assumes the delta-function-like spectral function so that it also leads to almost temperature-independent Hall coefficient. While the Fermi-liquid theory exploits only the information on the Fermi surface, our scheme exploits the entire Brillouin zone. Thus the Fermi-liquid theory fails to explain the ABC. Although some anisotropy leads to weak temperature dependence of the Hall coefficient [32], it is too weak to explain the anomalous temperature dependence observed in experiments.

The above-mentioned delta-function-like spectral function is justified only for weakly correlated systems in infinite momentum space at extremely low temperatures. Our calculation in the main text is performed without such a delta-function assumption. We have to use a broad spectral function in a finite momentum space (the 1st Brillouin zone) in the case of strongly correlated system at the room temperature. Thus the understanding of the
ABC is out of the scope of the Fermi-liquid theory. Our expecting spectral function is schematically shown in Fig. 5.

The normal state of cuprate superconductors is an incoherent metal [25, 26, 27, 28]. The spectral function method discussed in this letter is an attempt to describe the transport properties of incoherent metals where we have to consider the entire Brillouin zone not only the Fermi surface. The other approaches [29, 30] to incoherent metals are also intensively discussed recently.

Although our spectral function employed in this letter is a simple model, it captures the features of actual one observed by the ARPES experiments [31]. Thus our result for the Hall angle in Fig. 4 qualitatively explains the anomalous non-monotonic temperature dependence observed in experiments. Thus we can conclude that the ABC is explained without the CVC if we employ the correct spectral function.

On the other hand, the Hall angle calculated without the CVC in the FLEX approximation, the inset of Fig. 2 in [11], is totally different from the one observed by experiments. The failure of this calculation does not mean the necessity of the CVC. It only means that the spectral function obtained by the FLEX approximation is incorrect. The reason for the incorrectness is widely known [33]. We have also discussed [34] that the FLEX approximation is not applicable to the system with the Fermi degeneracy, since it violates the Pauli principle [33] and the local electron-number conservation [17].

Consequently, the spectral function is the key to explain the ABC. The FLEX approximation fails to explain the ABC, since the correct spectral function is not obtained by this approximation. The Fermi-liquid theory fails to explain the ABC, since the delta-function assumption is not justified for strongly correlated systems at the room temperature.

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References

[1] P. W. Anderson, Phys. Rev. Lett. 67, 2092 (1991).
[2] H. Kohno and K. Yamada, Prog. Theor. Phys. 80, 623 (1988).
[3] P. W. Anderson, The Theory of Superconductivity in the High-Tc Cuprate Superconductors (Princeton University Press, Princeton, 1997).
[4] P. A. Lee, N. Nagaosa and X.-G. Wen, Rev. Mod. Phys. 78, 17 (2006).
[5] H. Kontani, Rep. Prog. Phys. 71, 026501 (2008).
[6] P. W. Anderson and P. A. Casey, *Phys. Rev. B* **80**, 094508 (2009).

[7] P. A. Casey and P. W. Anderson, *Phys. Rev. Lett.* **106**, 097002 (2011).

[8] J. Kokalj and R. H. McKenzie, *Phys. Rev. Lett.* **107**, 147001 (2011).

[9] J. Kokalj, N. E. Hussey and R. H. McKenzie, *Phys. Rev. B* **86**, 045132 (2012).

[10] D. V. Evtushinsky, A. A. Kordyuk, V. B. Zabolotnyy, D. S. Inosov, B. Büchner, H. Berger, L. Patthey, R. Follath and S. V. Borisenko, *Phys. Rev. Lett.* **100**, 236402 (2008).

[11] G. S. Jenkins, D. C. Schmadel, P. L. Bach, R. L. Greene, X. Béchamp-Laganière, G. Roberge, P. Fournier, H. Kontani and H. D. Drew, *Phys. Rev. B* **81**, 024508 (2010).

[12] K. Yamada and K. Yosida, *Prog. Theor. Phys.* **76**, 621 (1986).

[13] H. Maebashi and H. Fukuyama, *J. Phys. Soc. Jpn.* **67**, 242 (1998).

[14] M. Khodas and A. M. Finkel’stein, *Phys. Rev. B* **68**, 155114 (2003).

[15] O. Narikiyo, [arXiv:1406.5831](http://arxiv.org/abs/1406.5831).

[16] H. Maebashi and H. Fukuyama, *J. Phys. Soc. Jpn.* **66**, 3577 (1997).

[17] H. Maebashi and Y. Takada, *Phys. Rev. B* **84**, 245134 (2011).

[18] J. M. Ziman, *Electrons and Phonons* (Oxford University Press, Oxford, 1960).

[19] T. Moriya, *Spin Fluctuations in Itinerant Electron Magnetism* (Springer-Verlag, Berlin, 1985).

[20] R. Hlubina and T. M. Rice, *Phys. Rev. B* **51**, 9253 (1995).

[21] B. P. Stojković and D. Pines, *Phys. Rev. B* **55**, 8576 (1997).

[22] H. Fukuyama, H. Ebisawa and Y. Wada, *Prog. Theor. Phys.* **42**, 494 (1969).

[23] I. A. Nekrasov, E. Z. Kuchinskii and M. V. Sadovskii, *J. Phys. Chem. Solids* **72**, 371 (2011).

[24] A. Perali, M. Sindel and G. Kotliar, *Eur. Phys. J. B* **24**, 487 (2001).
[25] C. M. Varma, P. B. Littlewood, S. Schmitt-Rink, E. Abrahams and A. E. Ruckenstein, Phys. Rev. Lett. 63, 1996 (1989).

[26] M. Imada, J. Phys. Soc. Jpn. 63, 851 (1994).

[27] O. Narikiyo and K. Miyake, J. Phys. Soc. Jpn. 63, 4169 (1994).

[28] O. Narikiyo, Physica C 267, 204 (1996).

[29] S. A. Hartnoll, Nature Physics 11, 54 (2015).

[30] A. Lucas and S. Sachdev, Phys. Rev. B 91, 195122 (2015).

[31] O. Narikiyo and K. Miyake, Physica C 307, 254 (1998).

[32] O. Narikiyo, arXiv:cond-mat/0006028.

[33] Y. M. Vilk and A. M. S. Tremblay, J. Phys. I (France) 7, 1309 (1997).

[34] O. Narikiyo, arXiv:1301.5996
Figure 1: Fermi surfaces for $\mu = -0.1, 0, 0.1$ with $t = -0.438, t' = 0.156, t'' = 0.098$ where all the energies are represented in eV. Only the quarter of the Brillouin zone ($0 \leq p_x \leq \pi$ and $0 \leq p_y \leq \pi$) is depicted.
Figure 2: The factor $f(p) = v_x(v_x \partial v_y/\partial p_y - v_y \partial v_x/\partial p_y)$ in the quarter of the Brillouin zone. In the white region $f(p) > 0$ and $f(p) < 0$ in the black region.
Figure 3: Temperature dependencies of $S(T)$ for $\mu = -0.1, 0, 0.1$ with $\gamma^2 = 1.07$ where $S(T) \propto \sigma_{xy}$. The temperature is represented in eV. The summation over $\mathbf{p}$ is carried out using $3000 \times 3000$ mesh in the quarter of the Brillouin zone.
Figure 4: The temperature dependence of $S(T)R(T)$ for $\mu = 0$ with $r^2 = 1.07$ where $S(T)R(T) \propto \sigma_{xy}/\sigma_{xx}$. 
Figure 5: Energy dependence of spectral function.