Cosmological perturbations in a healthy extension of Hořava gravity

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In Hořava’s theory of gravity, Lorentz symmetry is broken in exchange for renormalizability, but
the original theory has been argued to be plagued with problems associated with a new scalar mode
stemming from the very breaking of Lorentz symmetry. Recently, Blas, Pujolàs, and Sibiryakov have
proposed a healthy extension of Hořava gravity, in which the behavior of the scalar mode is improved.
In this paper, we study scalar modes of cosmological perturbations in extended Hořava gravity.
The evolution of metric and density perturbations is addressed analytically and numerically. It is
shown that for vanishing non-adiabatic pressure of matter the large scale evolution of cosmological
perturbations converges to that described by a single constant, ζ, which is an analog of a curvature
perturbation on the uniform-density slicing commonly used in usual gravitational theories. The
subsequent evolution is thus determined completely by the value of ζ.

I. INTRODUCTION

Recently, Hořava has proposed a new theory of quantum gravity [1]. The basic idea of the theory is abandoning
the Lorentz invariance, with which the theory is made power-counting renormalizable. The broken Lorentz invariance
results in a preferred foliation of spacetime by three-dimensional spacelike hypersurfaces, and thus allows
to add higher spatial curvature terms to the gravitational Lagrangian as well as to modify the kinetic term
of the graviton. Obviously, some notable features of Hořava gravity become manifest at high energies, which
motivates studying the early universe based on Hořava’s theory [2, 3]. The effect of the broken Lorentz invariance,
however, persists down to low energies. This not only brings interesting consequences regarding in particular
dark matter in the universe [3], but also causes potential problems stemming from an additional scalar degree
of freedom of the graviton that inevitably appears as a result of the reduced symmetry of the theory [6–10].
Some of the troubles are cured in the projectable version of Hořava gravity [3, 11] and some are still controversial.
Several problems have also been reported at a phenomenological level, including a large isocurvature perturbation [12]
and the absence of static stars [13] (see also [14]).

It is argued by Blas, Pujolàs, and Sibiryakov that original Hořava gravity can be extended in such a way that the
lapse function N may depend on the spatial coordinate (i.e., the theory is not projectable) and terms constructed
from a 3-vector ∂i ln N are included in the action [15], which makes N dynamical [16]. With the appropriate
choice of the new terms, the resultant theory will be free from the problems and pathologies reported in the literature
[6–11]. The extended theory could still suffer from strong coupling at low energies [17], but it can be evaded
by taking into account higher spatial derivative terms in the action [18]. See also the recent paper [19] for the
strong-coupling issue in the extended version of Hořava gravity.

So far cosmological perturbations have been explored in projectable and non-projectable versions of Hořava
gravity in the vast literature [12, 20–24]. The aim of this paper is to go on in this direction and to study cosmological
perturbations in healthfully extended Hořava gravity of [13]. We begin with presenting the most general equations
for scalar perturbations on a flat cosmological background in extended Hořava gravity, and then solve them
both analytically and numerically. As for the other aspects of extended Hořava gravity, spherically symmetric
solutions have been worked out in [25].

The paper is organized as follows. In the next section we review how non-projectable Hořava gravity can be
healthfully extended, following [13]. Then, in Sec. III, the background evolution and perturbation equations are
given within the framework of extended Hořava gravity. In Sec. IV the perturbation equations are solved analytically
and numerically. We draw our conclusions in Sec. V.

II. A HEALTHY EXTENSION OF HOŘAVA GRAVITY

The constituent variables in Hořava gravity are N, N_i, and g_{ij}, which correspond to the lapse function, the shift
vector, and the spatial metric, respectively, in the (3 + 1) decomposition of a spacetime. The non-projectable ver-

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sion of Hořava gravity is allowed to include terms constructed from the 3-vector,
\[ a_i := \partial_t \ln N(t, \vec{x}), \] (1)
in the Lagrangian. Along with the line of Refs. 13, 26, we consider the extended Hořava gravity described by
\[ S = \frac{M_{Pl}^2}{2} \int dt d^3x \sqrt{g} N (\mathcal{L}_K - \mathcal{V}[g_{ij}, a_i]) + \int dt d^3x \sqrt{g} N \mathcal{L}_m. \]
The “kinetic term” is given by
\[ \mathcal{L}_K = K_{ij} K^{ij} - \lambda K^2, \] (2)
with the extrinsic curvature defined by \( K_{ij} = (g_{ij} - \nabla_i N_j - \nabla_j N_i)/2N, \) while the “potential” is given by \( \mathcal{V}[g_{ij}, a_i] = \sum_{z=1}^3 \mathcal{V}_z, \) where
\[ \mathcal{V}_1 = -R - \eta_0 a^i a^i, \] (3)
\[ \mathcal{V}_2 = M_{Pl}^{-2} (g_{ij} R_{ij} + g_3 R_{ij} R^{ij}) + 2 \eta_2 a^i \Delta a^i + 2 \eta_3 \nabla_i a^i + ..., \] (4)
\[ \mathcal{V}_3 = M_{Pl}^{-4} (g_{ij} \Delta R + g_5 \nabla_i \nabla_j R^{ij}) + 2 \eta_4 a^i \Delta a^i + 2 \eta_5 \nabla_i \Delta R a^i + ... . \] (5)
Here, the Ricci tensor \( R_{ij} \) and Ricci scalar \( R \) are constructed from \( g_{ij}, \) \( \nabla_i \) is the covariant derivative with respect to \( g_{ij}, \) \( \Delta := \nabla_i \nabla^i, \) and \( \partial_i := \partial_i. \) In \( \mathcal{V}_2 \) and \( \mathcal{V}_3 \) we only write explicitly the terms that will be relevant to scalar-type, linear perturbations on a flat cosmological background. For instance, terms such as \( a_i a_j R^{ij} \) and \( (a_i a^i)^2 \) are not relevant to linear perturbations on a flat cosmological background. A parity-violating term \( \epsilon^{ijk} R_{il} \nabla_j R^j_k, \) which appears in the original version of Hořava gravity with the detailed valance condition 1, is not relevant to scalar-type perturbations. The potential presented above is therefore the most general one in the situation we are considering. \( \mathcal{L}_m \) is the Lagrangian of matter. In this paper we simply assume that matter is minimally coupled to gravity.

The symmetry of the theory is invariance under the foliation-preserving diffeomorphism transformations: \( t \rightarrow t(t), \) \( x^i \rightarrow x^i(t, \vec{x}). \) Under the infinitesimal transformation, \( t \rightarrow t + f(t), x^i \rightarrow x^i + \xi^i(t, \vec{x}), \) the variables transform as
\[ N \rightarrow N - f N - \dot{f} N - \xi^i \partial_i N, \]
\[ N_i \rightarrow N_i - \nabla_i \xi^j N_j - \xi^j \nabla_j N_i - \dot{\xi}^j g_{ij} - f N_i - \dot{f} N_i, \]
\[ g_{ij} \rightarrow g_{ij} - g_{ij} f - g_{ik} \nabla_j \xi^k - g_{jk} \nabla_i \xi^k. \] (6)

As a result of broken general covariance, a new scalar degree of freedom appears in addition to the usual helicity-2 polarizations of the graviton, and this scalar graviton could be a ghost depending on the value of \( \lambda. \) Since the structure of the kinetic term 2 is the same as in the original version of Hořava gravity, the time kinetic term of the scalar graviton remains the same. Therefore, the condition for avoiding ghosts is \[ \mathcal{A} := \frac{3 \lambda - 1}{\lambda - 1} > 0. \] (7)
However, the sound speed squared is negative for \( \mathcal{A} > 0 \) in original Hořava gravity, indicating that the scalar graviton is unstable 3, 22. This fact itself does not necessarily mean that the theory is “unhealthy,” because whether or not an instability really causes a trouble depends upon its time scale 1, 14. In the above “healthy” extension of Hořava gravity, such an instability can be made absent from the beginning for \( \mathcal{A} > 0 \) by the appropriate choice of the potential terms of \( a_i \) 13.

The Hamiltonian constraint is derived by varying the action with respect to \( N: \)
\[ \mathcal{L}_K + \mathcal{V} + \delta \mathcal{V} + \frac{2}{M_{Pl}^2} \rho = 0, \] (8)
where
\[ \delta \mathcal{V} := 2 \eta_1 \nabla_i a^i - \frac{2 \eta_2}{M_{Pl}^2} \Delta \nabla_i a^i + \frac{\eta_3}{M_{Pl}} \Delta R - \frac{2 \eta_4}{M_{Pl}^2} \Delta^2 \nabla_i a^i + \frac{\eta_5}{M_{Pl}^2} \Delta^2 R + ... . \] (9)
We have written here explicitly the terms that are relevant at zeroth and linear order in our cosmological setting. The matter energy density is defined as \( \rho := -\mathcal{L}_m - N \delta \mathcal{L}_m / \delta N. \) Variation with respect to the shift vector gives the momentum constraint equations:
\[ \nabla_j \pi^{ij} = \frac{1}{M_{Pl}^2} J^i, \] (10)
where \( \pi^{ij} := K^{ij} - \lambda K g^{ij} \) and \( J^i := -N \delta \mathcal{L}_m / \delta N_i. \) Finally, the evolution equations are derived from variation with respect to \( g_{ij}: \)
\[ 2 \left( K_{ik} K^k_j - \lambda K K_{ij} \right) - \frac{g_{ij}}{N \sqrt{g}} \frac{\partial}{\partial t} \left( \sqrt{g} a^{kl} \right) g_{ik} g_{jl} - \frac{1}{N} \nabla^k (\pi_{ij} N_k) + \frac{1}{N} \nabla^k (\pi_{ik} N_j) + \frac{1}{N} \nabla^k (\pi_{kj} N_i). \]
where $T_{ij} := L_m g_{ij} - 2\delta L_m / \delta g^{ij}$. Here also we have written explicitly only the terms linear in $R_{ij}$ and $a_0$; the other possible terms are not relevant in the present paper.

The matter action is invariant under the infinitesimal transformation (6), which results in

$$\int d^3x \left[ \sqrt{-N} g_{ij} \mathcal{T}^{ij} + N \partial_i (\sqrt{g} \rho) + N_i \partial_t (\sqrt{g} a^k) \right] = 0, \quad (11)$$

and

$$\nabla^i T_{ij} + a_i \rho + a_i T_{i}^j - \frac{1}{N \sqrt{g}} \partial_i (\sqrt{g} J_i) - \frac{N}{N} \nabla_j J^j = \frac{f^j}{N} (\nabla_j N_i - \nabla_i N_j) = 0. \quad (12)$$

Note that the energy conservation equation (11) is given by the integration over the whole space, as the gauge parameter $f(t)$ is space-independent.

III. COSMOLOGY OF EXTENDED HÖRAV GRAVITY

A. Background equations

To describe the background evolution of the universe we write $N = 1$, $N_i = 0$, and $g_{ij} = a^2(t) \delta_{ij}$, assuming that the three-dimensional spatial section is flat. The Hamiltonian constraint reduces to

$$\frac{3(3\lambda - 1)}{2} H^2 = \frac{\rho}{M_{Pl}^2}, \quad (13)$$

where $H := \dot{a} / a$ and $\rho$ is the background value of the energy density of matter, while the evolution equation reads

$$-\frac{3\lambda - 1}{2} (3H^2 + 2\dot{H}) = \frac{p}{M_{Pl}^2}. \quad (14)$$

where $p$ is the background value of the isotropic pressure of matter, $T^i_{i,j} = p \delta^i_{j}$. From the above equations we may naturally identify the gravitational constant in Friedmann-Robertson-Walker cosmology as $8\pi G_c := 2M_{Pl}^2(3\lambda - 1)^{-1}$.

It follows from Eqs. (13) and (14) that

$$\dot{\rho} + 3H (\rho + p) = 0. \quad (15)$$

Obviously, the energy conservation equation derived from (11),

$$\int d^3x \ a^3 [\dot{\rho} + 3H (\rho + p)] = 0, \quad (16)$$

is satisfied. Note that in contrast to the case of projectable Hörava gravity, we now have the local Hamiltonian constraint and hence “dark matter as an integration constant” [3] does not appear.

We see that nothing special happens at the background level, except that the gravitational constant $G_c$ differs from the locally measured value of Newton's constant, $G_N = M_{Pl}^2(1 - \eta/2)^{-1}$ [13].

B. Cosmological perturbations

General scalar perturbations of the lapse function, shift vector, and the spatial metric can be written as

$$N = 1 + \phi(t, \vec{x}), \quad N_i = a^2 \partial_i \beta(t, \vec{x}), \quad g_{ij} = a^2 [1 - 2\psi(t, \vec{x})] \delta_{ij} + 2a^2 \partial_i \partial_j E(t, \vec{x}). \quad (17)$$

Note that in the cosmological background of Hörava gravity we are considering, linear perturbations can be decomposed into scalar, vector, and tensor perturbations because all the background quantities are homogeneous and isotropic. Under a scalar gauge transformation, $t \rightarrow t + f(t), \quad x^i \rightarrow x^i + \partial^i \xi(t, \vec{x})$, these perturbations transform as

$$\phi \rightarrow \phi - \dot{f}, \quad \beta \rightarrow \beta - \dot{\xi}, \quad \psi \rightarrow \psi + H f, \quad E \rightarrow E - \xi. \quad (18)$$

By using the spatial gauge transformation we may set $E = 0$. Note, however, that the temporal gauge degree of freedom does not help to remove $\phi$ or $\psi$ because $f$ is a function of $t$ only.

The linearized Hamiltonian constraint is given by

$$3(3\lambda - 1) H \left[ \dot{\psi} + H \phi + \frac{1}{3} \nabla^2 \beta \right] - 2 \nabla^2 \psi + \eta \nabla^2 \phi + \frac{\nabla^2 \delta H}{a^2} + \frac{\delta \rho}{M_{Pl}^2} = 0, \quad (19)$$

where

$$\delta H(t, \vec{x}) := -\frac{\eta_2}{M_{Pl}^2} \frac{\nabla^2}{a^2} \phi + \frac{2\eta_3}{M_{Pl}^2} \frac{\nabla^2}{a^2} \psi - \eta_1 \frac{\nabla^4}{M_{Pl}^2} \frac{\nabla^4}{a^2} \phi + \frac{2\eta_3}{M_{Pl}^2} \frac{\nabla^4}{a^2} \psi. \quad (20)$$
The momentum constraints at linear order read
\[
(3\lambda - 1) \left( \dot{\psi} + H\phi \right) + (\lambda - 1) \nabla^2 \beta = \frac{\delta J}{M_{Pl}^2},
\] (21)
where we write \( J_i = \partial_i \delta J \). The evolution equations can be written as
\[
\mathcal{G}_T \delta_{i}^{,j} - \left( \frac{\partial_i \partial_j}{\nabla^2} - \frac{1}{3} \delta_i^{,j} \right) \mathcal{G}_{TL} = \frac{\delta T^{,j}_{,i}}{M_{Pl}^2}.
\] (22)
where
\[
\mathcal{G}_T = (3\lambda - 1) \left[ \dot{\psi} + 3H \dot{\psi} + H \dot{\phi} + \left( 3H^2 + 2H \right) \phi \right] + (\lambda - 1) \nabla^2 \left( \dot{\beta} + 3H\beta \right) + \frac{2}{3} \mathcal{G}_{TL}
\] (23)
and
\[
\mathcal{G}_{TL} = \nabla^2 \left( \dot{\beta} + 3H\beta \right) + \frac{\nabla^2}{a^2} (\phi - \psi) + \frac{\nabla^2}{a^2} \mathcal{E},
\] (24)
with
\[
\delta \mathcal{E}(t, \vec{x}) = - \frac{8g_2 + 3g_3}{M_{Pl}^2} \frac{\nabla^2}{a^2} \psi - \frac{8g_4 - 3g_5}{M_{Pl}^2} \frac{\nabla^4}{a^4} \psi - \frac{\eta_1}{M_{Pl}^2} \frac{\nabla^2}{a^2} \phi - \frac{\eta_2}{M_{Pl}^2} \frac{\nabla^4}{a^4} \phi.
\] (25)
Neglecting for simplicity the anisotropic stress perturbation, we have
\[
\mathcal{G}_T = \frac{\delta \rho}{M_{Pl}^2},
\] (26)
\[
\mathcal{G}_{TL} = 0.
\] (27)

The perturbed energy conservation equation reduces to
\[
\int d^3x \ a^3 \left[ \delta \rho + 3H (\delta \rho + \delta p) - 3 \dot{\psi}(\rho + p) \right] + \int d^3x \ a^3 \left[ \dot{\rho} + 3H (\rho + p) \right] \ (\phi - 3 \dot{\psi}) = 0.
\] (28)

The second line vanishes thanks to the background equation. Combining the Hamiltonian constraint (19) and the evolution equations (26) and (27), and using the background equations, one finds
\[
M_{Pl}^{-2} \left[ \frac{\delta \rho + 3H (\delta \rho + \delta p) - 3 \dot{\psi}(\rho + p)}{a^2} \right] = \nabla^2 \frac{\left[ 2 \left( \dot{\psi} + H \dot{\phi} \right) - \eta \left( \dot{\phi} + H \phi \right) - \left( \dot{\mathcal{H}} + H \delta \mathcal{H} \right) \right]}{\eta} + \nabla^2 \left[ (1 - 3\lambda) H \beta - 2H \left( \dot{\beta} + 3H \beta \right) \right].
\] (29)

Since the right hand side is a total derivative, the first line of Eq. (28) also vanishes, and hence Eq. (28) is automatically satisfied. Note, however, that the local equation (29) is stronger than the integrated equation (28). The momentum conservation equation gives
\[
\delta J + 3H \delta J - (\rho + p) \phi - \delta p = 0.
\] (30)
This equation can also be derived using the momentum constraint (21) and the evolution equations (26) and (27) together with the background equations. Therefore, the energy and momentum conservation equations do not give rise to any independent equations.

In order to obtain a closed set of equations, we need a matter equation of motion. Suppose that the matter equation of motion takes of the form
\[
\delta \rho + 3H (\delta \rho + \delta p) - 3 \dot{\psi}(\rho + p) - \nabla^2 \frac{\left[ \delta J + \eta a^2 (\rho + p) \beta \right]}{a^2} = \nabla^2 \delta \mathcal{F}(t, \vec{x}),
\] (31)
where the concrete form of \( \delta \mathcal{F} \) depends on the matter field one is considering. A matter field that respects four-dimensional general covariance is conserved locally, and consequently, \( \delta \mathcal{F} = 0 \). However, in Hořava gravity it is natural to consider that matter fields respect only the foliation-preserving diffeomorphism, and hence are not conserved locally in general. For example, a scalar field Lagrangian presented in [22, 23] leads to \( \delta \mathcal{F} \neq 0 \). In this paper, we assume that \( \delta \mathcal{F} \) arises only from the ultraviolet (UV) effect and is suppressed by \( \Delta / M_{Pl}^2 \), where \( M_{m} \) is a typical mass scale. \( M_{m} \) is not necessarily of the same order of \( M_{Pl} \) and may be different for different matter fields, but here we simply assume that the scale \( M_{m} \) is sufficiently high. Using the momentum constraint (21), one obtains
\[
3(\lambda - 1) \left( \dot{\psi} + H \dot{\phi} + \frac{1}{3} \nabla^2 \beta \right) + \eta \left( \dot{\phi} + H \phi \right) + \left( \dot{\mathcal{H}} + H \delta \mathcal{H} \right) - 2H \delta \mathcal{E} + \frac{\delta \mathcal{F}}{M_{Pl}^2} = 0.
\] (32)

### C. Minkowski limit

Let us check that the perturbation equations derived in the previous subsection can reproduce the known result in the Minkowski background by taking the limit \( H \to 0 \). Since we are particularly interested in the stability of a scalar graviton in the infrared (IR) regime, we neglect the UV terms \( \delta \mathcal{H} \) and \( \delta \mathcal{E} \). Then, the Hamiltonian constraint simply gives \( \dot{\phi} = (2/\eta) \psi \). The trace part of the evolution equation (26) reduces to \( \mathcal{A} \dot{\psi} + \nabla^2 \beta = 0 \), while the traceless part (27) implies \( \beta = (1 - 2/\eta) \psi \). We thus obtain
\[
\mathcal{A} \dot{\psi} - \frac{2 - 2/\eta}{\eta} \nabla^2 \psi = 0.
\] (33)

This reproduces the result obtained in [13]. It can be seen that the scalar graviton is stable in the IR regime provided that
\[
0 < \eta < 2.
\] (34)
The propagation speed of the scalar graviton is given by
\[
c^2_s = \frac{2 - \eta}{\eta} \mathcal{A}.
\] (35)
In addition to the theoretical bound \([7] \) and \([31] \), low energy phenomenology can put constraints on the value of \(\lambda \) and \(\eta \). A cosmological bound comes from the fact that \(G_c \) does not coincide with \(G_N \). The bound on the difference is derived from the primordial abundance of He\(^4\): \(|G_c/G_N - 1| \lesssim 0.13 \) \([15, 28] \), which in turn gives a mild constraint on the parameters. This constraint can be further weakened by considering additional relativistic degree of freedom and/or large lepton asymmetry \([29] \) to compensate a change in the expansion rate of the universe. More stringent constraints will come from the parameterized post-Newtonian study. See Ref. \([18] \) for a preliminary discussion on this point.

IV. EVOLUTION OF COSMOLOGICAL PERTURBATIONS IN THE IR

In what follows, we are interested in perturbation modes whose spatial momenta are much smaller than \(M_P \) and \(M_m \), so that we ignore any UV terms in gravity and matter sectors.

A. Superhorizon evolution

We begin with examining the large scale evolution of perturbations analytically by neglecting gradient terms. It can be shown from the matter equation of motion with \(\delta F = 0\) that

\[
\zeta := -\psi - H \frac{\delta \rho}{\rho} \quad (36)
\]

is conserved on large scales, \(\zeta \approx \zeta_* \) constant, in the case of a negligible non-adiabatic pressure perturbation. This is the well-known fact when matter is conserved locally \([27] \). Using the Hamiltonian constraint, we obtain

\[
-\psi + \frac{H}{H} (\psi + H \phi) \approx \zeta_* . \quad (37)
\]

Equation (32) on large scales reads

\[
\mathcal{B} \left( \psi + H \phi \right) + \dot{\phi} + H \phi \approx 0, \quad (38)
\]

where \(\mathcal{B} := 3(\lambda - 1)/\eta (> 0)\). The above two equations are combined to give

\[
\frac{d^2 \tilde{\psi}}{d \ln a^2} - \left[ \frac{(1 - \epsilon)^2}{4} - \epsilon \mathcal{B} - \frac{d \epsilon}{d \ln a} \right] \tilde{\psi} \approx 0 , \quad (39)
\]

where \(\tilde{\psi} := (\psi + \zeta_*) \exp \left[ \frac{1}{2} \int (1 + \epsilon) d \ln a \right] \) and \(\epsilon := -d \ln H/d \ln a\). We may assume \(\epsilon > 0\) and \(de/d \ln a \approx 0\). (These assumptions are indeed true in a matter-dominated universe and a radiation-dominated universe.) Then, if \((1 - \epsilon)^2 - 4\epsilon \mathcal{B} < 0\), \(\tilde{\psi}\) oscillates and therefore \(\psi + \zeta_* \) decays as \(\sim a^{-(1 + \epsilon)/2}\). If \((1 - \epsilon)^2 - 4\epsilon \mathcal{B} > 0\), \(\tilde{\psi}\) may grow like \(\psi \sim a^p\) with \(p = \sqrt{(1 - \epsilon)^2 - 4\epsilon \mathcal{B}}/2\), but \(\psi + \zeta_* \) decays also in this case because \((1 + \epsilon)/2 > p\).

We therefore conclude that the large scale evolution of the non-decaying perturbation can be characterized by a single constant \(\zeta_*\), and

\[
\psi \approx -\zeta_* , \quad \phi \approx 0 , \quad (40)
\]

provided that the mode stays in the superhorizon regime for a sufficiently long time. This indicates that the subsequent evolution of cosmological perturbations is essentially determined from the value of \(\zeta_*\). This fact will be confirmed by a numerical calculation.

A remark is in order. In usual gravitational theories having general covariance, \(\zeta\) is conveniently computed from the matter sector by taking the gauge \(\psi = 0\). In Hořava gravity, however, \(\psi\) cannot be gauged away because the symmetry is reduced and the relevant gauge parameter is independent of \(\hat{x}\). In this sense, two scalar-type contributions, a scalar graviton \(\psi\) and an adiabatic matter fluctuation \(\delta \rho\), translate to \(\zeta\). Therefore, we need to work out the two physically different contributions in order to compute \(\zeta\) in Hořava gravity.

B. Subhorizon evolution

Let us define

\[
\delta := \frac{\delta \rho + 3H \delta J}{\rho} . \quad (41)
\]

Using the equation of motion of matter, the momentum conservation equation, the evolution equations, and the Hamiltonian constraint, we find, for a universe dominated by a fluid with \(w = p/\rho = \text{const.}\),

\[
\ddot{\delta} + (2 - 3w)H \dot{\delta} = \frac{\nabla^2 \delta}{a^2} + \frac{3}{2} (1 - w)(1 + 3w)H^2 \delta
+ (1 + w)\frac{3(\lambda - 1) + \eta \nabla^2}{3\lambda - 1} \frac{\delta}{a^2} \phi , \quad (42)
\]

which coincides with the corresponding equation in general relativity except the last term. The Hamiltonian and momentum constraints give

\[
2 \frac{\nabla^2 \phi}{a^2} - \eta \frac{\nabla^2}{a^2} \phi - 2H \nabla^2 \beta = \frac{\rho \delta}{M_P^2} , \quad (43)
\]

which can be regarded as a generalized Poisson equation. So far no approximation has been made other than neglecting the UV terms.

We now focus on the subhorizon evolution of \(\delta\) in a matter-dominated universe (\(w = 0\)). It is natural to assume that in the matter-dominated universe the metric perturbations are slowly-varying functions of time in the sense that their time derivatives give rise to the Hubble scale: \(\dot{\psi}, \dot{\phi} \sim H \psi, H \phi\). Under this “quasi-static”
approximation, the evolution equations imply that
\[ \psi \approx \phi, \quad \beta \approx 0, \quad (44) \]
on subhorizon scales. Equations (42)–(44) are then arranged to give the following equations:
\[ \ddot{\delta} + 2H\dot{\delta} = \frac{\nabla^2}{a^2} \phi, \quad (45) \]
\[ \frac{\nabla^2}{a^2} \phi = 4\pi G_N \rho \delta, \quad (46) \]
where \(8\pi G_N = M_{\text{Pl}}^{-2}(1 - \eta/2)^{-1}\). This gravitational constant coincides with effective Newton’s constant defined in \( \xi \) (so that it differs from the gravitational constant in the Friedmann equation, \( G_c \)). This is a natural extension of the result of \( \xi \), where the gravitational field of a static point source has been derived. Again, we have the relation \( \psi = \phi \), in contrast to the case of Lorentz-invariant scalar-tensor theories of gravity.

Equations (45) and (46) admit the analytic solution
\[ \delta = C_1 t^{(1-\sqrt{1+2\xi})/6} + C_2 t^{(1-\sqrt{1-2\xi})/6}, \quad (47) \]
where \( \xi := \frac{G_N}{G_c} = (3\lambda - 1)/(2 - \eta) \). The first term grows in time, and, for \( \xi \neq 1 \), the growth rate of the matter density perturbation is slightly different from the standard one (\( \delta \sim t^{2/3} \)).

Next, let us consider a radiation-dominated universe \( (w = 1/3) \) and study the evolution of \( \delta \) inside the sound horizon. From Eq. (15) we may estimate \( (\nabla^2/a^2) \phi \sim O(H^2 \delta) \), so that we have \( \delta + H \dot{\delta} \simeq (\nabla^2/3a^2) \delta \) inside the sound horizon. This coincides with the corresponding equation in general relativity, and the last term hinders the growth of the density perturbation in the radiation-dominated stage. Moving to the Fourier space and using the conformal time defined by \( \tau = dt/a \), this equation can easily be solved to give \( \delta = C_0 \cos(k\tau/\sqrt{3} + \theta_0) \), with \( C_0 \) and \( \theta_0 \) being integration constants.

Although the behavior of the density perturbation on subhorizon scales in the radiation-dominated universe is simple as seen above, the metric perturbations evolve in a non-trivial manner. Using the generalized Poisson equation (19), the traceless part of the evolution equations (27), and Eq. (32), with some manipulation and the approximation \( k^2 \gg O(1/\tau^2) \), we arrive at
\[ \mathcal{A} \left( \phi'' + \frac{2}{\tau} \phi' + c_g^2 k^2 \phi \right) \simeq \frac{6}{\eta} \frac{a^2 \rho}{M_{\text{Pl}}^2} \left( \delta' \frac{\delta'}{k^2 \tau} + \frac{\delta}{3} \right), \quad (48) \]
where \( \mathcal{A} \) := \( \partial_\tau \). The detailed derivation is presented in Appendix A. A solution to this equation is given by
\[ \phi = \Psi(t; k) + \frac{6k^2}{\eta \mathcal{A} - 3(2 - \eta)} \frac{a^2 \rho}{M_{\text{Pl}}^2} \left[ \delta + O(\mathcal{A}/k^2 \tau) \right], \quad (49) \]
where \( \Psi(t; k) \) is the solution to the source-free wave equation \( \Psi'' + (2/\tau) \Psi' + c_g^2 k^2 \Psi = 0 \), and hence
\[ \Psi = C_\Psi \cos(c_g k \tau + \theta_\Psi), \quad (50) \]
where \( C_\Psi \) and \( \theta_\Psi \) are integration constants. The mode \( \Psi \) can be naturally identified as the scalar graviton, and it decays as \( \sim a^{-3} \) inside the horizon, analogously to the usual helicity-2 gravitational wave mode in a cosmological setting. Note that the propagation speed of a scalar gravitational wave in a cosmological background is identical to \( c_g \) in the Minkowski background, and its stability is ensured for \( c_g^2 > 0 \). The \( \delta \) induced part of \( \phi \) decays as \( \sim a^{-1} \). \( \psi \) can be obtained from the generalized Poisson equation (19) as
\[ \psi \approx \frac{\eta}{2} \frac{\eta \mathcal{A} - 3(2 + \eta)}{2k^2 \eta \mathcal{A} - 3(2 - \eta)} \frac{a^2 \rho}{M_{\text{Pl}}^2} \delta. \quad (51) \]
The subhorizon evolution in a radiation-dominated universe is apparently characterized by four integration constants, but they are completely determined by a single constant, \( \xi \), and thus are in fact related to each other.

### C. Numerical solutions

We have solved numerically the perturbation equations in the IR without any other approximations. The procedure for doing so is described in some detail in Appendix B. The results are summarized in Figs. 1 and 2 all of which reproduce the analytic results obtained above.

In Fig. 1 we show the evolution of \( \psi \) and \( \phi \) in a matter-dominated universe. The initial condition is set by specifying four numbers \( \psi_0 = \psi(t_0), \chi_0 = \chi(t_0), \phi_0 = \phi(t_0) \), and \( \beta_0 = \beta(t_0) \) at some initial time \( t = t_0 \). It can be seen that the perturbation evolution starting with the two different initial conditions first converges to the solution (49) outside the horizon, and then to (44) inside.
the horizon. The evolution of \( \delta \) is shown in Fig. 2 for the same set of the initial conditions as in Fig. 1. One can confirm that the growth of \( \delta \) is in agreement with the analytic solution [17].

The same large scale behavior, \( \psi \approx -\zeta \) and \( \phi \approx 0 \), is found also in a radiation-dominated universe, as can be seen in Fig. 3. The oscillation on small scales in the radiation-dominated universe, shown in Fig. 4, is well approximated by the analytic solutions presented in the previous subsection. The \( \delta \) induced part and \( \Psi \) typically have the same amplitude in the plotted region, and \( \Psi \) dominates for larger \( \tau \). We have confirmed that different initial conditions with the same value of \( \zeta * \) and the same set of the other parameters result in the identical profile.

FIG. 2: Subhorizon evolution of the density perturbation (divided by \( \zeta * \)) in a matter-dominated universe, compared to the analytic solution [47].

FIG. 3: Evolution of \( \psi \) (solid line) and \( \phi \) (dashed line) with \( k = 0.01 \times a(t_0)H(t_0) \) in a radiation-dominated universe. The initial condition is given by \( \psi_0 = 1 = \phi_0, \beta_0 = 0, \) and \( \chi_0 = 0 \). The parameters are \( \lambda = 1.05 \) and \( \eta = 0.1 \), and \( \alpha_c \) is defined similarly to the previous figures: \( k = a_c H(a_c) \).

FIG. 4: Subhorizon evolution of \( \psi \) and \( \phi \) in a radiation dominated universe, compared to the analytic solutions [49-51]. The initial condition and parameters are the same as in Fig. 3.

V. CONCLUSIONS

In this paper, we have studied cosmological perturbations in a healthfully extended version of Hořava gravity. The most general perturbation equations have been derived without specifying the matter content. We then solved the resultant perturbation equations in the IR regime analytically and numerically, for a universe dominated by a single fluid with vanishing non-adiabatic fluctuations. It was found that the large scale evolution converges to that described by a constant \( \zeta * \), which is an analog of the curvature perturbation on uniform density hypersurfaces commonly used in the context of usual gravitational theories having general covariance. This implies that, although the system has two scalar degrees of freedom corresponding to a scalar graviton and an adiabatic matter fluctuation, it is sufficient to specify the value of \( \zeta * \) for predicting the late-time evolution of perturbations. Our analytic results were confirmed by numerical calculations.

It would be important to revisit the analysis of cosmological perturbations in extended Hořava gravity by employing the Hamiltonian formulation in order to get a more transparent understanding of the properties of the scalar graviton [3, 24]. This issue is left for a further study.

It was pointed out very recently that the IR limit of extended Hořava gravity is identical to Einstein-aether theory [20] if the aether vector is restricted to be hypersurface orthogonal [19, 31]. Hypersurface orthogonal solutions of Einstein-aether theory are also solutions to the IR limit of extended Hořava gravity, though the converse is not true. This observation is insightful for understanding the background cosmological dynamics, as a cosmological background aether field is hypersurface orthogonal. For example, the discrepancy between \( G_N \) and \( G_c \) is a generic feature of Einstein-aether theory. Note, however, that the aether field is no longer hypersurface
orthogonal at perturbative order. It would be interesting to compare the evolution of cosmological perturbations in extended Hořava gravity with that in Einstein-aether theory \[32, 33\].

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Appendix A: Master equations on small scales

We present a detailed derivation of the master equations governing the subhorizon evolution of \( \delta \) and \( \phi \) for general \( w (=\text{const.}) \).

Defining convenient variables as \( u := a^2 \dot{\psi} \), \( v := a^2 \phi \), and \( \beta = a^3 \dot{\beta} \), Eq. (13) can be written as

\[
2u - \eta v - 2a' \frac{a}{a} \beta + \frac{a^4 \rho}{M_{Pl}^2} \delta = 0, \tag{A1}
\]

while the traceless part of the evolution equations reads

\[
\dot{\beta} + v - u = 0, \tag{A2}
\]

where the conformal time \( \tau \) is used: \( \dot{t} := \partial_{\tau} \). Equation (32) implies

\[
\mathcal{B} \left( u'' - 2a' \frac{a}{a} u + a' \frac{a}{a} v - \frac{k^2}{3} \beta \right) + v' - \frac{a'}{a} v = 0. \tag{A3}
\]

Differentiating Eq. (A3) and using Eq. (A2) to remove \( \dot{\beta} \), we obtain

\[
\mathcal{B} \left[ u'' - 2a' \frac{a}{a} u + a' \frac{a}{a} v + \frac{k^2}{3} (v - u) \right] + v'' - \frac{a'}{a} v' \simeq 0, \tag{A4}
\]

where we used \( a''/a, (a'/a)^2 \ll k^2 \). The expression in the square brackets can be written, using Eq. (A1) and then Eq. (A2), as

\[
\frac{\eta}{2} \left( \frac{\eta}{2} - 2a' \frac{a}{a} v' \right) + \frac{2 - \eta}{2} \frac{k^2}{3} \delta - \frac{a^4 \rho}{2M_{Pl}^2} M + \frac{a'}{a} \left( u' - \frac{k^2}{3} \beta \right) = 0, \tag{A5}
\]

where

\[
M := \frac{\delta''}{k^2} - 6a' \frac{a}{a} \frac{\delta'}{k^2} - \frac{\delta}{3}, \tag{A6}
\]

and we used \( k^2 \gg O(\tau^{-2}) \) again. Using Eq. (A3), the expression in the second line can be evaluated as \( \simeq -\mathcal{B}^{-1}(a'/a) v' \). Thus, we arrive at

\[
\mathcal{A} \left( v'' - 2a' \frac{a}{a} v' + c_g k^2 v \right) = \frac{3 a^4 \rho}{\eta M_{Pl}^2} M, \tag{A7}
\]

or equivalently,

\[
\mathcal{A} \left( \phi'' + 2a' \frac{a}{a} \phi' + c_g^2 k^2 \phi \right) = \frac{3 a^4 \rho}{\eta M_{Pl}^2} M. \tag{A8}
\]

The evolution equation of \( \delta \), Eq. (12), can be written in terms of the conformal time as

\[
\delta'' + (1 - 3w) a' \frac{a}{a} \delta' + wk^2 \delta = \frac{3}{2} (1 - w)(1 + 3w) \left( \frac{a'}{a} \right)^2 \delta - \frac{3}{2} (1 + w) \frac{3(\lambda - 1) + \eta}{3\lambda - 1} k^2 \phi. \tag{A9}
\]

We should note that here we do not drop the \( O((a'/a)^2 \delta) \) term because \( O(k^2 \delta) \) term vanishes for \( w = 0 \), in contrast to the approximation made in deriving Eqs. (A1), (A3), and (A5), where we implicitly assumed that the coefficients of \( O(k^2) \) terms are not too far from the order of unity value. In particular, we are not considering the case with \( c_g^2 \ll 1 \). Using Eq. (A9), we have

\[
\mathcal{M} = -(1 + 3w) \left( \frac{a'}{a} \frac{\delta'}{k^2} + \frac{\delta}{3} \right) + O(\phi). \tag{A10}
\]

Note that \( \delta \sim (k/aH)^2 \phi \gg \phi \) on small scales.

In a matter-dominated universe, \( w = 0 \), the following solves Eqs. (A8) and (A9):

\[
\delta = C_1 a^{(-1+\sqrt{1+24 \xi})/4} + C_2 a^{(-1-\sqrt{1+24 \xi})/4}, \tag{A11}
\]

\[
\phi = -\frac{a^2}{k^2} \cdot 4\pi G_N \rho \delta + \Psi, \tag{A12}
\]

where \( \xi \) and \( G_N \) are defined in the main text, and \( \Psi \) is a solution to the wave equation \( \Psi'' + 2(a'/a) \Psi' + c_g^2 k^2 \Psi = 0 \), and hence can be identified as a scalar gravitational wave. \( \Psi \) decays as \( \sim a^{-1} \) on small scales. We assumed that the amplitude of \( \Psi \) is small enough not to source \( \delta \) via Eq. (A9). Neglecting \( \Psi \), Eqs. (A11) and (A12) reproduce the result obtained in the main text.

Appendix B: Solving the perturbation equations numerically

In this appendix we describe the procedure for solving the perturbation equations numerically. The UV terms \( \delta H, \delta \mathcal{E} \), and \( \delta \mathcal{F} \) are all neglected. We use the following variables: \( \chi := \psi, \ D := \delta \rho/(\rho + p), \) and \( \mathcal{J} := \delta J/(\rho + p) \). Initial data at \( t = t_0 \) are specified as follows:

\[
\psi(t_0) = \psi_0, \quad \chi(t_0) = \chi_0, \quad \phi(t_0) = \phi_0, \quad \beta(t_0) = \beta_0. \tag{B1}
\]

Using the constraint equations, we then obtain the initial conditions for \( D \) and \( \mathcal{J} \) at \( t = t_0 \). Choosing \( t_0 \) in the superhorizon regime, we can evaluate \( \zeta \) from Eq. (57).
The constraint equations can be solved for $\phi$ and $\beta$ (in the Fourier space) as follows:

\[
\mathcal{L} \phi = -\mathcal{A} \left[ \frac{2\chi}{H} + (\lambda - 1) \frac{\dot{H}}{H^2} (D + \mathcal{A} H J) \right] + \frac{k^2}{a^2 H^2} \psi, \tag{B2}
\]

\[
\mathcal{A}^{-1} \mathcal{L} k^2 \beta = \frac{k^2}{a^2 H^2} \left( \eta \chi + 2 H \psi + \eta \dot{H} J \right) - (3\lambda - 1) \frac{\ddot{H}}{H} (D + 3H J), \tag{B3}
\]

where $\mathcal{L} := \eta(k/a H)^2 + 2\mathcal{A}$. These two equations can be used to remove $\phi$ and $\beta$ in the other equations, so that a system of four first-order equations for $\psi$, $\chi$, $D$, and $J$ can be derived. Assuming $\delta \rho - c_s^2 \delta \rho = 0$ with $c_s^2 := \bar{p}/\bar{\rho}$, we obtain

\[
\dot{D} = 3\chi - \frac{k^2}{a^2} J - \frac{k^2}{a^2} \beta, \tag{B4}
\]

\[
\dot{J} = 3Hc_s^2 J + \phi + c_s^2 D, \tag{B5}
\]

from the matter equations, and

\[
\mathcal{A} \left[ \chi + 3H \chi + H \dot{\phi} + \left( 3H^2 + 2\dot{H} \right) \phi \right] - \frac{k^2}{a^2} (\psi - \phi) = -\mathcal{A} \dot{H} c_s^2 D, \tag{B6}
\]

from the evolution equations, together with $\dot{\psi} = \chi$. In the main text the above set of the equations is solved for $c_s^2 = w = \text{constant}$.

[1] P. Horava, Phys. Rev. D 79, 084008 (2009) [arXiv:0901.3775 [hep-th]].
[2] G. Calcagni, JHEP 0909, 112 (2009) [arXiv:0904.0829 [hep-th]].
[3] E. Kiritsis and G. Kofinas, Nucl. Phys. B 821, 467 (2009) [arXiv:0904.1334 [hep-th]].
[4] See, e.g., T. Takahashi and J. Soda, Phys. Rev. Lett. 102, 231301 (2009) [arXiv:0904.0554 [hep-th]]; R. Brandenberger, Phys. Rev. D 80, 043516 (2009) [arXiv:0904.2835 [hep-th]]; S. Mukohyama, JCAP 0906, 001 (2009) [arXiv:0904.2190 [hep-th]]; S. Mukohyama, K. Nakayama, F. Takahashi and S. Yokoyama, Phys. Lett. B 679, 6 (2009) [arXiv:0905.0055 [hep-th]].
[5] S. Mukohyama, Phys. Rev. D 80, 064005 (2009) [arXiv:0905.3563 [hep-th]].
[6] C. Charmousis, G. Niz, A. Padilla and P. M. Saffin, JHEP 0908, 070 (2009) [arXiv:0905.2579 [hep-th]].
[7] M. Li and Y. Pang, JHEP 0908, 015 (2009) [arXiv:0905.2751 [hep-th]].
[8] D. Blas, O. Pujolas and S. Sibiryakov, JHEP 0910, 029 (2009) [arXiv:0906.3046 [hep-th]].
[9] K. Koyama and F. Arroja, arXiv:0910.1998 [hep-th].
[10] M. Hennemann, A. Kleinhschmidt and G. L. Gomez, arXiv:0912.0399 [hep-th].
[11] S. Mukohyama, JCAP 0909, 005 (2009) [arXiv:0906.5069 [hep-th]].
[12] T. Kobayashi, Y. Urakawa and M. Yamaguchi, JCAP 0911, 105 (2009) [arXiv:0908.1005 [astro-ph.CO]].
[13] K. Izumi and S. Mukohyama, arXiv:0911.1814 [hep-th].
[14] J. Greenwald, A. Papazoglou and A. Wang, arXiv:1012.0011 [hep-th].
[15] D. Blas, O. Pujolas and S. Sibiryakov, arXiv:0909.3525 [hep-th].
[16] For a similar attempt see M. Lemoine, M. Lubo, J. Martin and J. P. Uzan, Phys. Rev. D 65, 023510 (2002) [arXiv:hep-th/0109128]; S. Shankaranarayanan and M. Lubo, Phys. Rev. D 72, 123513 (2005) [arXiv:hep-th/0507086].
[17] A. Papazoglou and T. P. Sotiriou, arXiv:0911.1209 [hep-th].
[18] D. Blas, O. Pujolas and S. Sibiryakov, arXiv:0912.0550 [hep-th].
[19] I. Kimpton and A. Padilla, arXiv:1003.5658 [hep-th].
[20] X. Gao, Y. Wang, R. Brandenberger and A. Riotto, arXiv:0905.3821 [hep-th]; X. Gao, Y. Wang, W. Xue and R. Brandenberger, arXiv:0911.3196 [hep-th].
[21] Y. S. Piao, Phys. Lett. B 681, 1 (2009) [arXiv:0904.4117 [hep-th]]; X. Gao, arXiv:0904.4187 [hep-th]; B. Chen, S. Pi and J. Z. Tang, JCAP 0908, 007 (2009) [arXiv:0905.2300 [hep-th]]; S. Koh, arXiv:0907.0850 [hep-th]; K. Yamamoto, T. Kobayashi and G. Nakamura, Phys. Rev. D 80, 063514 (2009) [arXiv:0907.1549 [astro-ph.CO]]; S. Maeda, S. Mukohyama and T. Shiromizu, Phys. Rev. D 80, 123538 (2009) [arXiv:0909.2149 [astro-ph.CO]]; B. Chen, S. Pi and J. Z. Tang, arXiv:1010.0338 [hep-th].
[22] A. Wang and R. Maartens, Phys. Rev. D 81, 024009 (2010) [arXiv:0907.1748 [hep-th]].
[23] A. Wang, D. Wands and R. Maartens, arXiv:0909.5167 [hep-th].
[24] J. O. Gong, S. Koh and M. Sasaki, arXiv:1002.1429 [hep-th].
[25] E. Kiritsis, arXiv:0911.3164 [hep-th].
[26] T. P. Sotiriou, M. Visser and S. Weinfurther, Phys. Rev. Lett. 102, 251601 (2009) [arXiv:0904.4464 [hep-th]]; T. P. Sotiriou, M. Visser and S. Weinfurther, JHEP 010, 033 (2009) [arXiv:0905.2798 [hep-th]].
[27] D. Wands, K. A. Malik, D. H. Lyth and A. R. Liddle, Phys. Rev. D 62, 043527 (2005) [arXiv:astro-ph/0005278].
[28] S. M. Carroll and E. A. Lim, Phys. Rev. D 70, 123525 (2004) [arXiv:hep-th/0407149].
[29] M. Kawasaki, F. Takahashi and M. Yamaguchi, Phys. Rev. D 66, 043516 (2002) [arXiv:hep-ph/0205101]; K. Yamamoto, Phys. Rev. D 68, 063507 (2003) [arXiv:hep-ph/0211163]; T. Chiba, F. Takahashi and M. Yamaguchi, Phys. Rev. Lett. 92, 011301 (2004) [arXiv:hep-ph/0304102]; F. Takahashi and M. Yamaguchi, Phys. Rev. D 69, 083506 (2004) [arXiv:hep-ph/0308173].
[30] C. Eling and T. Jacobson, Phys. Rev. D 69, 064005 (2004) [arXiv:gr-qc/0310044]; T. Jacobson, PoS QG-
PH, 020 (2007) [arXiv:0801.1547 [gr-qc]].

[31] T. Jacobson, arXiv:1001.4823 [hep-th].

[32] E. A. Lim, Phys. Rev. D 71, 063504 (2005) [arXiv:astro-ph/0407437]. B. Li, D. Fonseca Mota and J. D. Barrow, Phys. Rev. D 77, 024032 (2008)

[33] C. Armendariz-Picon, N. F. Sierra and J. Garriga, arXiv:0709.4581 [astro-ph].