THE RECTIFIABILITY OF THE ENTROPY DEFECT MEASURE FOR BURGERS EQUATION

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Abstract. We consider bounded weak solutions to the Burgers equation for which every entropy dissipation is representable by a measure and we prove that all these measures are concentrated on the graphs of countably many Lipschitz curves. The main tool is the Lagrangian representation, which is an extension of the method of characteristics to the non-smooth setting.

1. Introduction

We consider the Cauchy problem for the Burgers equation:

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0, & (t, x) \in [0, T] \times \mathbb{R} \\ u(0, \cdot) = u_0, \end{cases}$$

with $T > 0$ and $u_0 \in L^\infty(\mathbb{R})$. It is known since the early stages [Kru70] of the theory of scalar conservation laws that the Cauchy problem (1) is well-posed in the setting of bounded entropy solutions, namely weak solutions with the following additional constraint: for every smooth convex entropy $\eta : \mathbb{R} \to \mathbb{R}$ and relative flux $q : \mathbb{R} \to \mathbb{R}$ defined up to a constant by $q'(v) = \eta'(v)v$, it holds

$$\mu_\eta := \partial_t \eta(u) + \partial_x q(u) \leq 0$$

in the sense of distributions. The celebrated Oleinik one-sided Lipschitz estimate [Ole63] implies that bounded entropy solutions to (1) belong to $BV_{\text{loc}}((0, T] \times \mathbb{R})$ and the structure of the solution is well described by means of the theory of BV functions.

In this paper we investigate the structure of more general weak solutions introduced in the following definition:

Definition 1.1. We say that $u$ is a bounded weak solutions to (1) with finite entropy production if $u \in C^0([0, T]; L^1(\mathbb{R})) \cap L^\infty([0, T] \times \mathbb{R})$ and for every convex entropy $\eta$ and corresponding flux $q$

$$\mu_\eta := \partial_t \eta(u) + \partial_x q(u) \in \mathcal{M}([0, T] \times \mathbb{R}),$$

where $\mathcal{M}$ denotes the set of finite Radon measures.

The interest toward these solutions is clearly motivated in [LO18]. We report the main motivations here for completeness: weak solutions with finite entropy production arise as domain of the $\Gamma$-limit of a sequence of functionals introduced in [BBMN10] to study large deviations principles for stochastic approximations of entropy solutions (see also [Mar10]). Moreover there is a strong analogy between the solutions introduced above and the weak solutions of the eikonal equation

$$|\nabla \phi| = 1$$

in the plane arising as domain of the $\Gamma$-limit as $\varepsilon \to 0$ of the Aviles-Giga functionals:

$$AG_\varepsilon(\phi) := \int \left(\frac{\varepsilon}{2} |\nabla^2 \phi|^2 + \frac{1}{2\varepsilon} (|\nabla \phi|^2 - 1)^2\right).$$

The same analogy holds for a very related model about thin ferro-magnetic films studied in [RS01, RS03], see also [Riv02]. At a formal level the link between conservation laws and the eikonal equation is the following (see for example [DLO03]): setting $m = \nabla \perp \phi$ it holds $\text{div} \ m = 0$. Moreover we can

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impose the constraint $|m| = 1$ introducing a phase $\theta \in \mathbb{S}^1$ so that $m = (\cos \theta, \sin \theta)$ and the eikonal equation reduces to the $\mathbb{S}^1$-valued scalar conservation law

$$\partial_{x_1} \cos \theta + \partial_{x_2} \sin \theta = 0.$$ 

This analogy has been pushed much further when the notion of entropy has been transferred from the theory of conservation laws to the eikonal equation [JK00, ADLM99, DMKO01] and this allowed to provide a kinetic formulation for (2), see for example [JP01]. The kinetic formulation for scalar conservation laws has been introduced in [LPT94] in the context of entropy solutions: in our setting it reads as follows (see for example [DLOW03]): given a bounded weak solution to (1) with finite entropy production $u$ there exists a Radon measure $\mu \in \mathcal{M}([0,T] \times \mathbb{R} \times \mathbb{R})$ such that

$$\partial_t \chi + v \partial_x \chi = \partial_v \mu,$$

where $\chi(t, x, v) := \begin{cases} 1 & \text{if } 0 < v \leq u(t, x), \\ -1 & \text{if } u(t, x) \leq v < 0, \\ 0 & \text{otherwise}. \end{cases}$

The kinetic measure $\mu$ encodes all the entropy production measures $\mu_\eta$ by means of the following formula:

$$\langle \mu_\eta, \phi \rangle = \int_{[0,T] \times \mathbb{R} \times \mathbb{R}} \eta''(v) \phi(t, x) d\mu.$$

In particular we will consider the measure

$$\nu := (p_{t,x})^2 |\mu| = \sup_{\|\eta''\|_{L^\infty} \leq 1} \mu_\eta,$$

where $p_{t,x} : [0,T] \times \mathbb{R}^2 \to [0,T] \times \mathbb{R}$ is the standard projection on the first two components and the wedge denotes the supremum operator in the set of measures. The equality in (4) follows from (3) (see for example [LO18]).

In contrast with the case of entropy solutions, bounded weak solutions with finite entropy production are not locally BV: they belong to $B^{1/3,3}_{\infty,\text{loc}}$ [GP13] and this regularity is optimal [DLW03]. The same result has been established recently in the case of the eikonal equation [GL20]. Nevertheless these solutions share several fine properties with functions of bounded variation: in [Lec04, DLOW03] it has been proved that the set

$$J := \left\{ (t, x) \in [0,T] \times \mathbb{R} : \limsup_{r \to 0} \frac{\nu(B_r(x))}{r} > 0 \right\}$$

is $\mathcal{H}^1$-rectifiable and it admits strong traces on both sides. In particular for every entropy $\eta$ it holds

$$\mu_\eta \ll J = \left( (\eta(u^+) - \eta(u^-)) n_t + (q(u^+) - q(u^-)) n_x \right) \mathcal{H}^1 \ll J,$$

where $n = (n_t, n_x)$ is the normal to $J$ and $u^\pm$ are the traces. Moreover every point in $J^c$ is a point of vanishing mean oscillation of $u$. Recently, it has been proved in [LO18] that the set $S$ of non-Lebesgue points of $u$ has Hausdorff dimension at most 1, almost filling the gap with the case of BV solutions, where we would have $\mathcal{H}^1(J \setminus S) = 0$ (see also [Mar19] for the same result for more general conservation laws and [Sil19] for the case of entropy solutions, where every point in $J^c$ is actually a continuity point of $u$). Another relevant property of BV solutions is that for every smooth entropy $\eta$ the measure $\mu_\eta$ is concentrated on $J$. This result has been proved also for entropy solutions for more general conservation laws in one space dimensions [DLR03, BM17], for continuous entropy solutions in several space dimensions [Sil19, BBM17] and for continuous solutions to general conservation laws in one space dimension [Daf06]. In both settings of conservation laws and of the eikonal equation, the proof of this concentration property is considered as a main step to prove the $\Gamma$-convergence of the families of functionals discussed above, see [BBMN10, Lec05].

The main result of this work establishes that this property holds for bounded weak solutions to the Burgers equation with finite entropy production:
Theorem 1. Let $u$ be a bounded weak solution to (1) with finite entropy production. Then there exist countably many Lipschitz curves $\gamma_i : [0, T] \to \mathbb{R}$ such that $\nu$ (and therefore every entropy dissipation measure $\mu_\eta$) is concentrated on

$$J' := \bigcup_{i \in \mathbb{N}} \{(t, x) \in [0, T] \times \mathbb{R} : x = \gamma_i(t)\}.$$

We observe that by definition of $J$ it follows that $\nu_J^c$ does not charge any set with finite $\mathcal{H}^1$-measure, and therefore any $1$-rectifiable set; in particular, in the statement of Theorem 1, we can replace the set $J'$ with the set $J$ defined in (5) and recover the representation (6) for $\mu_\eta = \mu_\eta|_J$.

The fine properties of solutions to (1) and (2) that we discussed and the regularizing effect for conservation laws can be attributed to the characteristics structure that these equations own. The main tool to get Theorem 1 is the Lagrangian representation, which is an extension of the method of characteristics to deal with nonsmooth solutions. It has been introduced first for entropy solutions to general scalar conservation laws in [BBM17], relying on the transport-collapse scheme by Brenier [Bre84]. Then this notion has been extended to cover the case of bounded weak solutions with finite entropy production in [Mar19], building on the kinetic formulation introduced in [DLOW03]: in the particular case of the Burgers equation it takes the form provided in Definition 2.1. We would like to point out that the notion of Lagrangian representation is inspired by the superposition principle for nonnegative measure-valued solutions to the continuity equation (see [AC14]). In particular it shares with it the main feature that the evolution of the solution $u$ is described as the result of the evolutions of the single particles along the characteristics. Finally we mention that the notion of Lagrangian representation is available for solutions with finite entropy production to the eikonal equation too [Mar20].

As a consequence of Theorem 1 and (6) we obtain the following result:

Theorem 2. Let $u$ be a bounded weak solution to the Burgers equation with finite entropy production. Then it holds

$$|(p_{t,x})_{\sharp}\mu| = (p_{t,x})_{\sharp}|\mu|.$$

In particular, denoting by $\bar{\eta}(u) = u^2/2$, if $\mu_\bar{\eta} \leq 0$ then $\mu \leq 0$, namely if $u$ dissipates the quadratic entropy $\bar{\eta}$, then $u$ is the entropy solution to (1).

The second part of this statement is known even under milder assumptions: in [Pan94] the result has been proved for all bounded weak solutions to (1) and in [DLOW04] it has been extended to weak solutions in $L^4_{loc}$. Both proofs rely on the link between entropy solutions to (1) and viscosity solutions to the Hamilton-Jacobi equation

$$v_t + \frac{v^2}{2} = 0.$$

The interest in looking for alternative proofs is motivated by the problem of extending this result to systems of conservation laws, where the link with the Hamilton-Jacobi equation is not available: see [KV19] for a recent result in this direction.

We finally observe that both Theorem 1 and Theorem 2 can be proved with the same strategy and minor modifications in the case of bounded weak solutions with finite entropy production to conservation laws with uniformly convex fluxes.

2. Lagrangian representation

In this section we present the notion of Lagrangian representation introduced in [Mar19] and we discuss some of its properties. Since we consider bounded solutions we can assume without loss of generality that $u$ takes values in $[0, 1]$.

For every function $f : \mathbb{R} \to [0, 1]$ we denote its hypograph and its epigraph by

$$E_f := \{(x, v) \in \mathbb{R} \times [0, 1] : v \leq f(x)\} \quad \text{and} \quad E_f^c := \{(x, v) \in \mathbb{R} \times [0, 1] : v \geq f(x)\}.$$
respectively. Moreover we denote by
\[ 
\Gamma := \{ \gamma = (\gamma_x, \gamma_y) \in BV([0, T]; \mathbb{R} \times [0, 1]) : \gamma_x \text{ is Lipschitz} \}. 
\]
It will be useful to consider the standard decomposition of the measure \( Df \in \mathcal{M}(\mathbb{R}) \), where \( f \in BV(\mathbb{R}, \mathbb{R}) \) (see for example [AFP00]). We will adopt the following notation:
\begin{equation}
(7) \quad Df = D^{ac}f + D^c f + D^i f = \tilde{D}f + D^i f,
\end{equation}
where \( D^{ac}f \), \( D^c f \) and \( D^i f \) denote the absolutely continuous part, the Cantor part and the atomic part of \( Df \) respectively; we refer to \( \tilde{D}f \) as the diffuse part of \( Df \).

**Definition 2.1.** Let \( u \) be a weak solution to (1) with finite entropy production. We say that the Radon measure \( \omega_h \in \mathcal{M}(\Gamma) \) is a Lagrangian representation of the hypograph of \( u \) if the following conditions hold:

1. for every \( t \in [0, T) \) it holds
\begin{equation}
(8) \quad (e_t)_\# \omega_h = \mathcal{L}^2 \mathcal{L}_{E_u(t)},
\end{equation}
where \( e_t \) denotes the evaluation map defined by
\[ 
e_t : \Gamma \to \mathbb{R}, \quad \gamma \mapsto \gamma(t) \]
2. the measure \( \omega_h \) is concentrated on the set of curves \( \gamma \in \Gamma \) such that
\begin{equation}
(10) \quad \dot{\gamma}_x(t) = \gamma_v(t) \quad \text{for a.e. } t \in [0, T);
\end{equation}
3. it holds the integral bound
\begin{equation}
(11) \quad \int_{\Gamma} \text{TotVar}_{(0,T)} \gamma_v d\omega_h(\gamma) < \infty.
\end{equation}
Similarly we say that \( \omega_e \in \mathcal{M}(\Gamma) \) is a Lagrangian representation of the epigraph of \( u \) if Conditions (2) and (3) hold and (1) is replaced by
\begin{equation}
(12) \quad (e_t)_\# \omega_e = \mathcal{L}^2 \mathcal{L}_{E_u(t)} \quad \text{for every } t \in [0, T].
\end{equation}

A useful property of these representations is that the kinetic measure (and therefore any entropy dissipation measure \( \mu_\gamma \)) can be decomposed along the characteristics.

Given \( \gamma \in \Gamma \) we consider
\[ 
\mu_\gamma = (I, \gamma)_\# \tilde{D} \mu_{\gamma} + \mathcal{H}^1 \mathcal{L} E^+_{\gamma} - \mathcal{H}^1 \mathcal{L} E^-_{\gamma} \in \mathcal{M}([0, T] \times \mathbb{R} \times [0, 1]),
\]
where
\[ 
E^+_{\gamma} := \{(t, x, v) : \gamma_x(t) = x, \gamma_v(t-) < \gamma_v(t+), v \in (\gamma_v(t-), \gamma_v(t+))\},
\]
\[ 
E^-_{\gamma} := \{(t, x, v) : \gamma_x(t) = x, \gamma_v(t+) < \gamma_v(t-), v \in (\gamma_v(t+), \gamma_v(t-))\},
\]
\( I : [0, T] \to [0, T] \) denotes the identity and \( \tilde{D} \mu_{\gamma} \) denotes the diffuse part of the measure \( D \mu_{\gamma} \).

**Proposition 3.** Let \( u \) be a weak solution to (1) with finite entropy production. Then there exist \( \omega_{h}, \omega_{e} \) Lagrangian representations of the hypograph and of the epigraph of \( u \) respectively enjoying the additional properties:

1. \begin{equation}
(13) \quad \int_{\Gamma} \mu_\gamma d\omega_{h}(\gamma) = \mu = - \int_{\Gamma} \mu_\gamma d\omega_{e}(\gamma),
\end{equation}
2. \begin{equation}
(14) \quad \int_{\Gamma} |\mu_\gamma| d\omega_{h}(\gamma) = |\mu| = \int_{\Gamma} |\mu_\gamma| d\omega_{e}(\gamma).
\end{equation}
Proof. Let us prove the properties about \( h \) and we denote the set of these curves by \( \Gamma \).

Moreover it follows from (13) and (14) that we can separately represent the negative and the positive parts of \( \mu \):

\[
\int_{\Gamma} \text{TotVar}_{[0,T]} \gamma^2 d\omega_h(\gamma) = \int_{\Gamma} \mu^- d\omega_h(\gamma) \quad \text{and} \quad \int_{\Gamma} \text{TotVar}_{[0,T]} \gamma^2 d\omega_e(\gamma).
\]

The assertion of Proposition 3 regarding \( \omega_h \) is proved in [Mar19] for more general conservation laws and it is straightforward to adapt the same argument to get the existence of an \( \omega_e \) as in the statement. Alternatively, in order to prove the part concerning \( \omega_e \), we can consider a Lagrangian representation \( \hat{\omega}_h \) of the hypograph of \( \hat{u} = 1 - u \), which is a weak solution with finite entropy production to

\[
\hat{u}_t + g(\hat{u})_x = 0, \quad \text{with} \quad g(z) = -\frac{(z - 1)^2}{2}.
\]

Let \( T : \Gamma \to \Gamma \) be defined by

\[
T(\gamma)(t) = (\gamma_x(t), 1 - \gamma_v(t)).
\]

Then it is straightforward to check that the measure \( \omega_e := T_\sharp \hat{\omega}_h \) satisfies the requirements in Proposition 3.

The following lemma is an application of Tonelli theorem:

**Lemma 4.** For \( \omega_h \)-a.e. \( \gamma \in \Gamma \) it holds that for \( \mathcal{L}^1 \)-a.e. \( t \in [0,T] \)

1. \((t, \gamma_x(t))\) is a Lebesgue point of \( u \);
2. \( \gamma_v(t) < u(t, \gamma_x(t)) \).

We denote by \( \Gamma_h \) the set of curves \( \gamma \in \Gamma \) such that the two properties above hold. Similarly for \( \omega_e \)-a.e. \( \gamma \in \Gamma \) it holds that for \( \mathcal{L}^1 \)-a.e. \( t \in [0,T] \)

1. \((t, \gamma_x(t))\) is a Lebesgue point of \( u \);
2. \( \gamma_v(t) > u(t, \gamma_x(t)) \)

and we denote the set of these curves by \( \Gamma_e \).

**Proof.** Let us prove the properties about \( \Gamma_h \). We denote by \( S \subset [0,T] \times \mathbb{R} \) the set of non-Lebesgue points of \( u \) and for every \( t \in [0,T] \) we set

\[
e_t^x : \Gamma \to \mathbb{R}, \quad \gamma \mapsto \gamma_x(t)
\]

By (8) it follows that for every \( t \in [0,T] \) it holds \((e_t^x)_* \omega_h \leq \mathcal{L}^1\). Since \( \mathcal{L}^2(S) = 0 \) it holds

\[
0 = \left( \mathcal{L}^1 \otimes (e_t^x)_* \omega_h \right)(S)
\]

\[
= \int_0^T \omega_h(\{ \gamma \in \Gamma : (t, \gamma_x(t)) \in S \}) dt
\]

\[
= \int_{\Gamma} \mathcal{L}^1(\{ t \in [0,T] : (t, \gamma_x(t)) \in S \}) d\omega_h(\gamma),
\]

where the last equality follows by Tonelli theorem. Similarly by (8) it follows that for every \( t \in [0,T] \)

\[
\omega_h(\{ \gamma \in \Gamma : \gamma_v(t) \geq u(t, \gamma_x(t)) \}) = (e_t)_* \omega_h(\{(x, v) \in \mathbb{R} \times [0,1] : v \geq u(t, x) \}) = 0,
\]

where \( e_t \) is defined in (9). Therefore by Tonelli theorem it holds

\[
0 = \int_0^T \omega_h(\{ \gamma \in \Gamma : \gamma_v(t) \geq u(t, \gamma_x(t)) \}) dt
\]

\[
= \int_{\Gamma} \mathcal{L}^1(\{ t \in [0,T] : \gamma_v(t) \geq u(t, \gamma_x(t)) \}) d\omega_h(\gamma).
\]

By (16) and (18) it follows that \( \omega_h \) is concentrated on \( \Gamma_h \). The case of \( \Gamma_e \) is analogous. \qed
The following result is a well-known property of traces on Lipschitz curves:

**Lemma 5.** Assume that $\tilde{\gamma}_x : (t^-, t^+) \subset [0, T] \to \mathbb{R}$ is a Lipschitz curve and that for $\mathcal{L}^1$-a.e. $t \in (t^-, t^+)$ the point $(t, \tilde{\gamma}_x(t))$ is a Lebesgue point of $u \in L^\infty([0, T] \times \mathbb{R})$. Then

$$\lim_{\delta \to 0} \int_{t^-}^{t^+} \frac{1}{\delta} \int_{\gamma_x(t)}^{\gamma_x(t) + \delta} |u(t, x) - u(t, \tilde{\gamma}_x(t))|dxdt = 0. \quad (19)$$

**Proof.** For $t \in (t^-, t^+)$ we set

$$v_\delta(t) := \frac{1}{\delta} \int_0^\delta |u(t, \tilde{\gamma}_x(t) + y) - u(t, \tilde{\gamma}_x(t))|dy,$$

so that (19) is equivalent to $v_\delta$ to converge to 0 in $L^1(t^-, t^+)$. For every $t \in [t^+ + \delta, t^- - \delta]$ we have (20)

$$|u(t, \tilde{\gamma}_x(t) + y) - u(t, \tilde{\gamma}_x(t))| \leq \frac{1}{2\delta} \int_{t^- - \delta}^{t^- + \delta} (|u(t, \tilde{\gamma}_x(t) + y) - u(t', \tilde{\gamma}_x(t'))| + |u(t', \tilde{\gamma}_x(t')) - u(t, \tilde{\gamma}_x(t))|) dt'.$$

Since $|v_\delta| \leq 1$, integrating (20) with respect to $t$ we get

$$\int_{t^-}^{t^+} v_\delta(t)dt \leq 2\delta + \int_{t^- - \delta}^{t^- + \delta} \frac{1}{2\delta^2} \int_0^{t^- + \delta} |u(t, \tilde{\gamma}_x(t) + y) - u(t', \tilde{\gamma}_x(t'))|dt'dydt$$

$$+ \int_{t^- + \delta}^{t^+ - \delta} \frac{1}{2\delta} \int_{t^- - \delta}^{t^- + \delta} |u(t', \tilde{\gamma}_x(t')) - u(t, \tilde{\gamma}_x(t))|dt'dt.$$

By Lebesgue differentiation theorem, $\frac{1}{2\delta} \int_{t^- - \delta}^{t^- + \delta} |u(t', \tilde{\gamma}_x(t')) - u(t, \tilde{\gamma}_x(t))|dt' \to 0$ as $\delta \to 0$ for $\mathcal{L}^1$-a.e. $t \in (t^-, t^+)$ therefore the last term in (21) converges to zero by dominated convergence theorem. By applying Tonelli theorem to the second term in the right-hand side of (21) we get

$$\int_{t^- + \delta}^{t^+ - \delta} \frac{1}{2\delta^2} \int_0^{t^- + \delta} |u(t, \tilde{\gamma}_x(t) + y) - u(t', \tilde{\gamma}_x(t'))|dt'dydt' \leq$$

$$\int_{t^- + \delta}^{t^+ - \delta} \frac{1}{2\delta^2} \int_{B(2+L)\delta(t', \tilde{\gamma}_x(t'))} |u(z) - u(t', \tilde{\gamma}_x(t'))|dzdt',$$

where $L$ is the Lipschitz constant of $\tilde{\gamma}_x$ and $B(2+L)\delta(t', \tilde{\gamma}_x(t'))$ denotes the ball in $\mathbb{R}^2$ of radius $(2 + L)\delta$ and center $(t', \tilde{\gamma}_x(t'))$. Since by assumption $\mathcal{L}^1$-a.e. $t \in (t^-, t^+)$ is a Lebesgue point of $u$, the right-hand side of (22) converges to 0 by the dominated convergence theorem and this concludes the proof. \[ \square \]

The following proposition formalizes the intuition that a curve lying in the hypograph of a function cannot cross from above a curve lying in the epigraph of the same function.

**Proposition 6.** Let $\tilde{\gamma} \in \Gamma_h, G \subset \Gamma_e$ and $t > 0$ be such that for every $\gamma \in G$ it holds $\tilde{\gamma}_x(t) < \gamma_x(t)$. Then for every $t \in [\tilde{t}, T]$ it holds

$$\omega_e(\{\gamma \in G : \gamma_x(t) < \tilde{\gamma}_x(t)\}) = 0.$$

**Proof.** Given $\delta > 0$ we define $\phi_\delta : \mathbb{R} \to \mathbb{R}$ by

$$\phi_\delta(x) = \begin{cases} 1 & \text{if } x \leq 0, \\ \frac{1}{2}(\delta - x) & \text{if } x \in (0, \delta), \\ 0 & \text{if } x \geq \delta, \end{cases}$$

and for every $t \in [\tilde{t}, T]$ we consider

$$\Psi_\delta(t) = \int_G \phi_\delta(\gamma_x(t) - \tilde{\gamma}_x(t))d\omega_e(\gamma).$$

Clearly for every $\delta > 0$ and every $t \in [\tilde{t}, T]$ it holds

$$\omega_e(\{\gamma^2 \in G : \gamma^2_x(t) < \gamma^1_x(t)\}) \leq \Psi_\delta(t) \quad (23)$$
and by assumption
\[ \Psi_\delta(t) \leq \omega_c(\gamma \in G : \gamma_x(t) \in [\gamma_x(\tilde{t}), \tilde{\gamma}_x(t) + \delta]) \leq \delta. \]
Since \( \omega_c \) is concentrated on curves satisfying (10), for every \( t \in [\tilde{t}, T] \)
\[ \Psi_\delta(t) = \Psi_\delta(\tilde{t}) + \int_{\tilde{t}}^{t} \frac{1}{\delta} \int_{G(\delta,t')} (\bar{\gamma}_u(t') - \gamma_u(t'))d\omega_c(\gamma)dt' \leq \delta + \int_{\tilde{t}}^{t} \frac{1}{\delta} \int_{G(\delta,t')} (\bar{\gamma}_u(t') - \gamma_u(t'))^+d\omega_c(\gamma)dt', \]
where \( G(\delta,t') := \{ \gamma \in G : \gamma_x(t') \in [\gamma_x(t'), \tilde{\gamma}_x(t') + \delta] \} \). Since \( \tilde{\gamma} \in \Gamma_\delta \) for \( L^1 \)-a.e. \( t' \in (t^-, t^+) \) it holds \( \bar{\gamma}_u(t') < u(t', \tilde{\gamma}_x(t')) \). Moreover for \( \omega_c \)-a.e. \( \gamma \in \Gamma \) and every \( t \in [0, T] \) it holds \( \gamma_u(t) \in [0, 1] \), therefore
\[ \Psi_\delta(t) \leq \delta + \int_{\tilde{t}}^{t} \frac{1}{\delta} \int_{G(\delta,t')} (u(t', \tilde{\gamma}_x(t')) - \gamma_u(t'))^+d\omega_c(\gamma)dt' \]
\[ \leq \delta + \int_{\tilde{t}}^{t} \frac{1}{\delta} \int_{0}^{\delta} |u(t', \gamma_x(t')) - u(t', \gamma_x(t') + y)|dydt', \]
where the last inequality follows by (12). The claim follows by Lemma 5 and (23) by letting \( \delta \to 0 \). \( \square \)

For every \((\tilde{t}, \bar{x}) \in [0, T] \times \mathbb{R} \) we denote by
\[ G^l_{\tilde{t}, \bar{x}} := \{ \gamma \in \Gamma_\delta : \gamma_x(\tilde{t}) < \bar{x} \}, \quad G^r_{\tilde{t}, \bar{x}} := \{ \gamma \in \Gamma_\delta : \gamma_x(\tilde{t}) > \bar{x} \}. \]

**Corollary 7.** Let \((\tilde{t}, \bar{x}) \in [0, T] \times \mathbb{R} \) and consider \( G^l_{\tilde{t}, \bar{x}}, G^r_{\tilde{t}, \bar{x}} \) as above. Then there exists a Lipschitz function \( f_{\tilde{t}, \bar{x}} : [\tilde{t}, T] \to \mathbb{R} \) such that such that for every \( t \in [\tilde{t}, T] \) it holds
\[ \omega_h(\{ \gamma \in G^l_{\tilde{t}, \bar{x}} : \gamma_x(t) > f_{\tilde{t}, \bar{x}}(t) \}) = 0 \quad \text{and} \quad \omega_c(\{ \gamma \in G^r_{\tilde{t}, \bar{x}} : \gamma_x(t) < f_{\tilde{t}, \bar{x}}(t) \}) = 0. \]

**Proof.** If \( G^l_{\tilde{t}, \bar{x}} = \emptyset \) we set \( f_{\tilde{t}, \bar{x}}(t) = \bar{x} \) and (25) follows by (10) since for \( \omega_c \)-a.e. \( \gamma \in \Gamma, \) it holds \( \gamma_x(t) \geq 0 \) for \( L^1 \)-a.e. \( t \in (0, T) \). If \( G^l_{\tilde{t}, \bar{x}} \neq \emptyset \), then we set
\[ f_{\tilde{t}, \bar{x}} = \sup_{\gamma \in G^l_{\tilde{t}, \bar{x}}} \gamma_x. \]
The first condition in (25) is trivially satisfied and by Proposition 6 for every \( \bar{\gamma} \in G^l_{\tilde{t}, \bar{x}} \) it holds
\[ \omega_c(\{ \gamma \in G^r_{\tilde{t}, \bar{x}} : \gamma_x(t) < \bar{\gamma}_x(t) \}) = 0. \]

Let \( \{\bar{t}_i\}_{i \in \mathbb{N}} \) be an enumeration of \( \mathbb{Q} \cap [\tilde{t}, T) \) and for every \( i, n \in \mathbb{N} \) let \( \gamma^{i,n} \in G^l_{\tilde{t}, \bar{x}} \) be such that
\[ \gamma_x^{i,n}(t_j) \geq f_{\tilde{t}, \bar{x}}(t_i) - \frac{1}{n} \]
For every \( \gamma \in G^l_{\tilde{t}, \bar{x}} \) the \( x \)-component \( \gamma_x \) is 1-Lipschitz, hence \( f_{\tilde{t}, \bar{x}} = \sup_{i,n \in \mathbb{N}} \gamma_x^{i,n} \). Therefore it follows by (27) that for every \( t \in [\tilde{t}, T] \) it holds
\[ \omega_c(\{ \gamma \in G^r_{\tilde{t}, \bar{x}} : \gamma_x(t) < f_{\tilde{t}, \bar{x}}(t) \}) = 0. \]

Thanks to Corollary 7 we can identify the candidate rectifiable set \( J^- \) on which \( (p_{\tilde{t}, \bar{x}})^- \mu^- \) is concentrated: let \( \{\bar{t}_i\}_{i \in \mathbb{N}} \) be an enumeration of \([0, T] \cap \mathbb{Q} \) and \( \{\bar{x}_j\}_{j \in \mathbb{N}} \) be an enumeration of \( \mathbb{Q} \). Then we set
\[ J^- := \bigcup_{i,j \in \mathbb{N}} C_{f_{\tilde{t}, \bar{x}}, j}, \quad \text{where} \quad C_{f_{\tilde{t}, \bar{x}}} := \{ (t, x) \in [\tilde{t}, T] \times \mathbb{R} : x = f_{\tilde{t}, \bar{x}}(t) \}. \]
3. Concentration of the entropy dissipation

In this section we prove Theorem 1 and Theorem 2. In the next lemma we couple the two representations \( \omega_h, \omega_e \) taking into account that they represent the same measure \( \mu^- \) as in (15). We denote by \( X = [0, T] \times \mathbb{R} \times [0, 1] \) and consider the measures \( \omega_h \otimes \mu^-_\gamma \) and \( \omega_e \otimes \mu^+_\gamma \) defined on the set \( \Gamma \times X \) by

\[
\omega_h \otimes \mu^-_\gamma(G \times E) = \int_G \mu^-_\gamma(E) d\omega_h(\gamma) \quad \text{and} \quad \omega_e \otimes \mu^+_\gamma(G \times E) = \int_G \mu^+_\gamma(E) d\omega_e(\gamma),
\]

for every measurable \( E \subset X, G \subset \Gamma \).

**Lemma 8.** Denote by \( p_1, p_2 : (\Gamma \times X)^2 \to \Gamma \times X \) the standard projections. Then there exists a plan \( \pi^- \in \mathcal{M}((\Gamma \times X)^2) \) with marginals

\[
(p_1)_\sharp \pi^- = \omega_h \otimes \mu^-_\gamma, \\
(p_2)_\sharp \pi^- = \omega_e \otimes \mu^+_\gamma,
\]

concentrated on the set

\[
\mathcal{G} := \left\{ ((\gamma, t, x, v), (\gamma', t', x', v')) \in (\Gamma \times X)^2 : t = t', \gamma_x(t) = x = x' = \gamma'_x(t'), v = v' \right. \\
\left. \quad v \in [\gamma_0(t+), \gamma_0(t-)] \cap [\gamma'_0(t-), \gamma'_0(t+)] \right\}.
\]

**Proof.** First we observe that by definition, \( \omega_h \otimes \mu^-_\gamma \) is concentrated on the set

\[
\mathcal{G}^-_h := \left\{ (\gamma, t, x, v) \in \Gamma \times X : \gamma_x(t) = x, v \in [\gamma_0(t+), \gamma_0(t-)] \right\}
\]

and \( \omega_e \otimes \mu^+_\gamma \) is concentrated on the set

\[
\mathcal{G}^+_e := \left\{ (\gamma, t, x, v) \in \Gamma \times X : \gamma_x(t) = x, v \in [\gamma_0(t-), \gamma_0(t+)] \right\}.
\]

Denoting by \( p_X : \Gamma \times X \to X \) the standard projection it follows from (15) that

\[
(p_X)_\sharp(\omega_h \otimes \mu^-_\gamma) = \mu^- = (p_X)_\sharp(\omega_e \otimes \mu^+_\gamma).
\]

By the disintegration theorem (see for example [AFP00]) there exist two measurable families of probability measures \( (\mu^-_{t,x,v})_{(t,x,v) \in X}, (\mu^+_{t,x,v})_{(t,x,v) \in X} \in \mathcal{P}(\Gamma \times X) \) such that

\[
(31) \quad \omega_h \otimes \mu^-_\gamma = \int_X \mu^-_{t,x,v} d\mu^- \quad \text{and} \quad \omega_e \otimes \mu^+_\gamma = \int_X \mu^+_{t,x,v} d\mu^-,
\]

and for \( \mu^- \)-a.e. \( (t, x, v) \) the measures \( \mu^-_{t,x,v} \) and \( \mu^+_{t,x,v} \) are concentrated on the set

\[
\mathcal{P}_X^{-1}(\{t, x, v\}) = \{ (\gamma, t', x', v') \in \Gamma \times X : t' = t, x' = x, v' = v \}.
\]

Moreover, since \( \omega_h \otimes \mu^-_\gamma \) is concentrated on the set \( \mathcal{G}^-_h \) and \( \omega_e \otimes \mu^+_\gamma \) is concentrated on the set \( \mathcal{G}^+_e \), we have that for \( \mu^- \)-a.e. \( (t, x, v) \) the measure \( \mu^-_{t,x,v} \) is concentrated on \( \mathcal{P}_X^{-1}(\{t, x, v\}) \cap \mathcal{G}^-_h \) and \( \mu^+_{t,x,v} \) is concentrated on \( \mathcal{P}_X^{-1}(\{t, x, v\}) \cap \mathcal{G}^+_e \). We eventually set

\[
\pi^- := \int_X \left( \mu^-_{t,x,v} \otimes \mu^+_{t,x,v} \right) d\mu^-.
\]

From (31) it directly follows (30) and by the above discussion for \( \mu^- \)-a.e. \( (t, x, v) \in X \) the measure \( \mu^-_{t,x,v} \otimes \mu^+_{t,x,v} \) is concentrated on \( (\mathcal{P}_X^{-1}(\{t, x, v\}) \cap \mathcal{G}^-_h) \times (\mathcal{P}_X^{-1}(\{t, x, v\}) \cap \mathcal{G}^+_e) \), therefore \( \pi^- \) is concentrated on

\[
\bigcup_{(t,x,v)\in X} (\mathcal{P}_X^{-1}(\{t, x, v\}) \cap \mathcal{G}^-_h) \times (\mathcal{P}_X^{-1}(\{t, x, v\}) \cap \mathcal{G}^+_e) = \mathcal{G}
\]

and this concludes the proof. \( \square \)

The following elementary lemma is about functions of bounded variation of one variable: we refer to [AFP00] for the theory of BV functions.
Lemma 9. Let $v : (a, b) \to \mathbb{R}$ be a BV function and denote by $D^-v$ be the negative part of the measure $Dv$. Then for $D^-v$ a.e. $\bar{x} \in (a, b)$ there exists $\delta > 0$ such that
\[\bar{v}(x) > \bar{v}(\bar{x}) \quad \forall x \in (\bar{x} - \delta, \bar{x}) \quad \text{and} \quad \bar{v}(x) < \bar{v}(\bar{x}) \quad \forall x \in (\bar{x}, \bar{x} + \delta).\]

The proof of Theorem 1 is obtained considering separately the positive and the negative parts of $\mu$; in the following theorem we deal with $\mu^-$.  

Theorem 10. The measure $(p_{t,x})_t \mu^-$ is concentrated on the set $J^-$, defined in (29).

Proof. Step 1. For every $(\bar{t}, \bar{x}) \in [0, T) \times \mathbb{R}$ we consider the measure
\[\pi^\gamma_{t,x} := \pi^-((G^t_{\bar{t},x} \times [\bar{t}, T] \times \mathbb{R} \times [0, 1]) \times (G^t_{\bar{t},x} \times [\bar{t}, T] \times \mathbb{R} \times [0, 1])),\]
where $G^t_{\bar{t},x}, G^t_{\bar{t},x}$ are defined in (24) and we set
\[p^1_{t,x} : (\Gamma \times [0, T] \times \mathbb{R} \times [0, 1])^2 \to [0, T] \times \mathbb{R},\]
\[(\gamma, t, x, v, \gamma', t', x', v') \mapsto (t, x)\]
We prove that the measure $(p^1_{t,x})_t \pi^\gamma_{t,x}$ is concentrated on the graph of $f_{t,x}$, namely $C_{f_{t,x}}$.

First we observe that
\[\begin{align*}
(p^1_{t,x})_t \pi^\gamma_{t,x} &\leq (p^1_{t,x})_t [\pi^-((G^t_{\bar{t},x} \times [\bar{t}, T] \times \mathbb{R} \times [0, 1]) \times (\Gamma \times X))] \\
&= \omega_h G^t_{\bar{t},x} \otimes \mu^-((\bar{t}, T) \times \mathbb{R} \times [0, 1]).
\end{align*}\]
Since for $\omega_h$ a.e. $\gamma \in G^t_{\bar{t},x}$ it holds $(t, x, \gamma(t)) \notin \Omega^+_{t,x}$ for every $t \in [\bar{t}, T]$, then for $\omega_h$ a.e. $\gamma \in G^t_{\bar{t},x}$ it holds $\mu^-((\Omega^+_{t,x} \times [0, 1]) = 0$ and therefore it follows by (32) that
\[(p^1_{t,x})_t \pi^\gamma_{t,x}(\Omega^+_{t,x}) = 0.\]
In the same way we get
\[(p^2_{t,x})_t \pi^\gamma_{t,x}(\Omega^-_{t,x}) = 0,\]
where
\[p^2_{t,x} : (\Gamma \times [0, T] \times \mathbb{R} \times [0, 1])^2 \to [0, T] \times \mathbb{R},\]
\[(\gamma, t, x, v, \gamma', t', x', v') \mapsto (t', x').\]
Finally, since $\pi^-$ is concentrated on $G$, then
\[(p^1_{t,x} \otimes p^2_{t,x})_t \pi^- \in \mathcal{M}([0, T] \times \mathbb{R})^2)\]
is concentrated on the graph of the identity on $[0, T] \times \mathbb{R}$ and in particular $(p^1_{t,x})_t \pi^\gamma_{t,x} = (p^2_{t,x})_t \pi^\gamma_{t,x}$.
Therefore it follows from (33) and (34) that $(p^1_{t,x})_t \pi^\gamma_{t,x}$ is concentrated on
\[[[0, T] \times \mathbb{R}) \setminus (\Omega^+_{t,x} \cup \Omega^-_{t,x}) = C_{f_{t,x}}.\]

Step 2. We prove that for $\pi^-$ a.e. pair $(\gamma, t, x, v, \gamma', t', x', v') \in (\Gamma \times X)^2$ there exists $\delta > 0$ such that for every $s \in (t - \delta, t)$ it holds $\gamma(s) < \gamma'_s(s)$.

By applying Lemma 9 to the BV function $\gamma_v : (\bar{t}, T) \to [0, 1]$ we get that for $(\bar{t}, T)$ a.e. $t \in (\bar{t}, T)$ there exists $\delta > 0$ such that for every $s \in (t - \delta, t)$ it holds $\gamma_v(s) < \gamma_v(t)$. Moreover for every $t$ such that $\gamma_v$ has a negative jump and for every $v \in [\gamma_v(t+), \gamma_v(t-)]$ there exists $\delta > 0$ such that for every $s \in (t - \delta, t)$ it holds $\gamma_v(s) < v$. Since for every $\gamma \in \Gamma$ the measure $\mu^-\gamma$ has no atoms and the set
\[E_\gamma := \{(t, x, v) \in X : \gamma_v(t-) = v > \gamma_v(t+)\}\]
is countable, then $\mu^-\gamma(E_\gamma) = 0$ and in particular
\[\omega_h \otimes \mu^-\gamma (\{(\gamma, t, x, v) \in \Gamma \times X : \gamma_v(t-) = v > \gamma_v(t+)\}) = 0.\]
Therefore it follows by the argument above that for \( \omega_h \otimes \mu^{-}_\gamma \)-a.e. \( (\gamma, t, x, v) \) there exists \( \delta_h > 0 \) such that for every \( s \in (t - \delta_h, t) \) it holds

\[
\gamma_v(s) > v \quad \text{and hence} \quad \gamma_x(s) < \gamma_x(t) - v(t - s)
\]

by (10). In the same way we obtain that for \( \omega_e \otimes \mu^{+}_\gamma \)-a.e. \( (\gamma', t', x', v') \) there exists \( \delta_e > 0 \) such that for every \( s \in (t' - \delta_e, t') \) it holds

\[
\gamma_x(s) > \gamma_x(t') - v'(t' - s).
\]

Finally, since \( \pi^- \) is concentrated on \( G \), then for \( \pi^{-}_e \)-a.e. \( (\gamma, t, x, v, \gamma', t, x', v') \) we set \( \delta = \min(\delta_h, \delta_e) > 0 \), so that for every \( s \in (t - \delta, t) \) it holds

\[
\gamma_x(s) < \gamma_x(t) - v(t - s) = x - v(t - s) = x' - v'(t' - s) \gamma_x'(t') - v'(t' - s) < \gamma_x'(s)
\]

and this concludes Step 2.

By Step 2 it follows that for \( \pi^{-}_e \)-a.e. \( (\gamma, t, x, v, \gamma', t, x', v') \) there exists \( (\tilde{i}, \tilde{x}, j) \in ([0, T] \cap \mathbb{Q}) \times \mathbb{Q} \) such that

\[
(\gamma, t, x, v, \gamma', t, x', v') \in (G^e_{\tilde{i}, \tilde{x}, j} \times [\tilde{i}, T] \times \mathbb{R} \times [0, 1]) \times (G^e_{\tilde{i}, \tilde{x}, j} \times [\tilde{i}, T] \times \mathbb{R} \times [0, 1])
\]

so that \( \pi^- \) is concentrated on

\[
\bigcup_{i,j \in \mathbb{N}} (G^e_{\tilde{i}, \tilde{x}, j} \times [\tilde{i}, T] \times \mathbb{R} \times [0, 1]) \times (G^e_{\tilde{i}, \tilde{x}, j} \times [\tilde{i}, T] \times \mathbb{R} \times [0, 1]).
\]

Therefore by Step 1 it follows that \((p^1_{t,x})_t \pi^- \) is concentrated on

\[
\bigcup_{i,j \in \mathbb{N}} C_{\tilde{i}, \tilde{x}, j} = J^-.
\]

We finally observe that from (30) and (15) it follows that

\[
(p^1_{t,x})_t \pi^- = (p_{t,x})_t ((p^1_{t,x})_t \pi^-) = (p_{t,x})_t (\omega_h \otimes \mu^{-}_\gamma) = (p_{t,x})_t \mu^{-}
\]

and therefore the proof is completed. \( \square \)

The analogous statement for the positive part of \( \mu \) follows from the observation that reversing the direction of time turns entropy solutions into anti-entropic solutions.

**Proposition 11.** The measure \((p_{t,x})_t \mu^+ \) is concentrated on a 1-rectifiable set \( J^+ \).

**Proof.** We consider the two reflections \( R : [0, T] \times \mathbb{R} \to [0, T] \times \mathbb{R} \) and \( \bar{R} : X \to X \) defined by

\[
R(t, x) = (T - t, -x) \quad \text{and} \quad \bar{R}(t, x, v) = (T - t, -x, v).
\]

Observe that \( R^{-1} = R \) and \( \bar{R}^{-1} = \bar{R} \). From the kinetic formulation for \( u \):

\[
\chi(t, x, v) := \begin{cases} 
1 & \text{if } u(t, x) > v \\
0 & \text{otherwise}
\end{cases}, \quad \partial_t \chi + v \partial_x \chi = \partial_v \mu
\]

we deduce that \( \tilde{\chi} := \chi \circ \bar{R} \) satisfies

\[
(35) \quad \partial_t \tilde{\chi} + v \partial_x \tilde{\chi} = \partial_v \tilde{\mu}, \quad \text{with} \quad \tilde{\mu} = -\bar{R}_2 \mu
\]

and it is the kinetic formulation of

\[
\tilde{u}(t, x) = \int_0^1 \tilde{\chi}(t, x, v) dv = (u \circ R)(t, x).
\]

In particular \( \tilde{u} \) is a weak solution with finite entropy production to the Burgers equation (1) and by Theorem 10 the measure \((\pi_{t,x})_t \tilde{\mu}^- \) is concentrated on a 1-rectifiable set \( J^- \). From (35) it follows that \( \mu^+ = \bar{R}_2 \mu^- \) and therefore that \((p_{t,x})_t \mu^+ \) is concentrated on \( J^+ := R(J^-) \), which is obviously \( \sqrt{1} \)-rectifiable. \( \square \)
Theorem 1 immediately follows from Theorem 10 and Proposition 11 setting 
\[ J' = J^- \cup J^+ . \]

We conclude by proving Theorem 2, which essentially follows from the observation that every negative shock of \( u \) is entropic and every positive shock is anti-entropic.

\textbf{Proof.} By (3) it follows that 
\[ \left| (p_t, x) \right| \mu \mu_\eta \lor \mu_- \eta , \]
where \( \bar{\eta}(v) = \nu^2/2 \). Therefore in order to prove the theorem, it is sufficient to check that for \( \nu \text{-a.e.} \) \((t, x) \in [0, T] \times \mathbb{R} \) the supremum in (4) is attained at \( \bar{\eta} \) or at \( -\bar{\eta} \). Let us denote by \( \bar{J}^+ \subset J \) the set of points for which \( u^- < u^+ \) and by \( \bar{J}^- \subset J \) the set of points for which \( u^+ < u^- \). In view of Theorem 1 and (6) it holds \( \nu(([0, T] \times \mathbb{R}) \setminus (\bar{J}^+ \cup \bar{J}^-)) = 0 \). We claim that 
\[ \eta_n \bar{J}^+ = \mu_\eta \quad \text{and} \quad \eta_n \bar{J}^- = \mu_- \eta. \]

We consider the second equality: by the Rankine-Hugoniot conditions the normal to \( J \) in (6) is determined for \( H^1 \text{-a.e.} \) \((t, x) \in J \) by 
\[ \mathbf{n} = (\mathbf{n}_t, \mathbf{n}_x) = \frac{1}{\sqrt{1 + \lambda^2}}(\lambda, -1), \quad \text{where} \quad \lambda = \frac{u^- + u^+}{2} . \]

In particular the density in (6) takes the form 
\[ \frac{1}{\sqrt{1 + \lambda^2}} \left( q(u^-) - q(u^+) \right) \lambda (\eta(u^-) - \eta(u^+)) = \frac{1}{\sqrt{1 + \lambda^2}} \int_{u^-}^{u^+} (q'(v) - \lambda \eta'(v)) dv \]
\[ = \frac{1}{\sqrt{1 + \lambda^2}} \int_{u^-}^{u^+} (v - \lambda) \eta'(v) dv \]
\[ = \frac{(v - \lambda)^2}{2} \left( u^- \right)_{u^+} - \frac{1}{\sqrt{1 + \lambda^2}} \int_{u^-}^{u^+} \frac{(v - \lambda)^2}{2} \eta''(v) dv , \]
which is maximized by \( \eta = -\bar{\eta} \) in the set \( \{ \eta \in C^2(\mathbb{R}) : \| \eta'' \|_{C^0} \leq 1 \} \). Being the first inequality of (36) completely analogous, this concludes the proof. \( \square \)

\textbf{References}

[AC14] L. Ambrosio and G. Crippa. Continuity equations and ODE flows with non-smooth velocity. \textit{Proc. Roy. Soc. Edinburgh Sect. A}, 144(6):1191–1244, 2014.
[ADLM99] L. Ambrosio, C. De Lellis, and C. Mantegazza. Line energies for gradient vector fields in the plane. \textit{Calc. Var. Partial Differential Equations}, 9(4):327–255, 1999.
[AFF00] L. Ambrosio, N. Fusco, and D. Pallara. \textit{Functions of Bounded Variation and Free Discontinuity Problems}. Oxford Science Publications. Clarendon Press, 2000.
[BBM17] S. Bianchini, P. Bonicatto, and E. Marconi. A lagrangian approach to multidimensional conservation laws. \textit{preprint SISSA 36/MATE}, 2017.
[BBMN10] G. Bellettini, L. Bertini, M. Mariani, and M. Novaga. \( \Gamma \)-Entropy Cost for Scalar Conservation Laws. \textit{Archive for Rational Mechanics and Analysis}, 195:261–309, 2010.
[BM17] S. Bianchini and E. Marconi. On the structure of \( L^\infty \) entropy solutions to scalar conservation laws in one-space dimension-entropy solutions to scalar conservation laws in one-space dimension. \textit{Archive for Rational Mechanics and Analysis}, Jun 2017.
[Bre84] Y. Brenier. Averaged multivalued solutions for scalar conservation laws. \textit{SIAM J. Numer. Anal.}, 21(6):1013–1037, 1984.
[Da06] C. M. Dafermos. Continuous solutions for balance laws. \textit{Ric. Mat.}, 55(1):79–91, 2006.
[DLO03] C. De Lellis and F. Otto. Structure of entropy solutions to the eikonal equation. \textit{J. Eur. Math. Soc. (JEMS)}, 5(2):107–145, 2003.
[DLOW03] C. De Lellis, F. Otto, and M. Westdickenberg. Structure of entropy solutions for multi-dimensional scalar conservation laws. \textit{Arch. Ration. Mech. Anal.}, 170(2):137–184, 2003.
[DLOW04] C. De Lellis, F. Otto, and M. Westdickenberg. Minimal entropy conditions for burgers equation. \textit{Quarterly of Applied Mathematics}, 62(4):687–700, 2004.
[DLR03] C. De Lellis and T. Rivièrè. The rectifiability of entropy measures in one space dimension. \textit{J. Math. Pures Appl. (9)}, 82(10):1343–1367, 2003.
[DLW03] C. De Lellis and M. Westdickenberg. On the optimality of velocity averaging lemmas. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 20(6):1075–1085, 2003.

[DMKO01] A. DeSimone, S. Müller, R.V. Kohn, and F. Otto. A compactness result in the gradient theory of phase transitions. *Proc. Roy. Soc. Edinburgh Sect. A*, 131(4):833–844, 2001.

[GL20] F. Ghiraldin and X. Lamy. Optimal Besov differentiability for entropy solutions of the eikonal equation. *Comm. Pure Appl. Math.*, 73(2):317–349, 2020.

[GP13] F. Golse and B. Perthame. Optimal regularizing effect for scalar conservation laws. *Rev. Mat. Iberoam.*, 29(4):1477–1504, 2013.

[JK00] W. Jin and R. V. Kohn. Singular perturbation and the energy of folds. *J. Nonlinear Sci.*, 10(3):355–390, 2000.

[JP01] P. E. Jabin and B. Perthame. Compactness in Ginzburg-Landau energy by kinetic averaging. *Comm. Pure Appl. Math.*, 54(9):1096–1109, 2001.

[Kru70] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81(123):228–255, 1970.

[KV19] S. G. Krupa and A. F. Vasseur. On uniqueness of solutions to conservation laws verifying a single entropy condition. *J. Hyperbolic Differ. Equ.*, 16(1):157–191, 2019.

[Lec04] M. Lecumberry. *Geometric structure of micromagnetic walls and shock waves in scalar conservation laws*. PhD thesis, Université de Nantes, 2004.

[Lec05] M. Lecumberry. Geometric structure of magnetic walls. In *Journées “Équations aux Dérivées Partielles”*, pages Exp. No. I, 11. École Polytech., Palaiseau, 2005.

[LO18] X. Lamy and F. Otto. On the regularity of weak solutions to Burgers’ equation with finite entropy production. *Calc. Var. Partial Differential Equations*, 57(4):Art. 94, 19, 2018.

[LPT94] P.-L. Lions, B. Perthame, and E. Tadmor. A kinetic formulation of multidimensional scalar conservation laws and related equations. *J. Amer. Math. Soc.*, 7(1):169–191, 1994.

[Mar10] M. Mariani. Large deviations principles for stochastic scalar conservation laws. *Probab. Theory Related Fields*, 147(3-4):607–648, 2010.

[Mar19] E. Marconi. On the structure of weak solutions to scalar conservation laws with finite entropy production. *ArXiv:1909.07257*, 2019.

[Mar20] E. Marconi. On the structure of weak solutions to the eikonal equation with finite entropy production. In preparation.

[Ole63] O. A. Oleinik. Discontinuous solutions of non-linear differential equations. *Amer. Math. Soc. Transl. (2)*, 26:95–172, 1963.

[Pan94] E. Y. Panov. Uniqueness of the solution of the Cauchy problem for a first-order quasilinear equation with an admissible strictly convex entropy. *Mat. Zametki*, 55(5):116–129, 159, 1994.

[Riv02] T. Rivièrem Parois et vortex en micromagnétisme. In *Journées “Équations aux Dérivées Partielles” (Forges-les-Eaux, 2002)*, pages Exp. No. XIV. 15. Univ. Nantes, Nantes, 2002.

[RS01] T. Rivièrem and S. Serfaty. Limiting domain wall energy for a problem related to micromagnetics. *Comm. Pure Appl. Math.*, 54(3):294–338, 2001.

[RS03] T. Rivièrem and S. Serfaty. Compactness, kinetic formulation, and entropies for a problem related to micromagnetics. *Comm. Partial Differential Equations*, 28(1-2):249–269, 2003.

[Sil19] L. Silvestre. Oscillation properties of scalar conservation laws. *Comm. Pure Appl. Math.*, 72(6):1321–1348, 2019.

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