Learning Tractable Probabilistic Models in Open Worlds

Amélie Levray · Vaishak Belle

Abstract Large-scale probabilistic representations, including statistical knowledge bases and graphical models, are increasingly in demand. They are built by mining massive sources of structured and unstructured data, the latter often derived from natural language processing techniques. The very nature of the enterprise makes the extracted representations probabilistic. In particular, inducing relations and facts from noisy and incomplete sources via statistical machine learning models means that the labels are either already probabilistic, or that probabilities approximate confidence. While the progress is impressive, extracted representations essentially enforce the closed-world assumption, which means that all facts in the database are accorded the corresponding probability, but all other facts have probability zero. The CWA is deeply problematic in most machine learning contexts. A principled solution is needed for representing incomplete and indeterminate knowledge in such models, imprecise probability models such as credal networks being an example.

In this work, we are interested in the foundational problem of learning such open-world probabilistic models. However, since exact inference in probabilistic graphical models is intractable, the paradigm of tractable learning has emerged to learn data structures (such as arithmetic circuits) that support efficient probabilistic querying. We show here how the computational machinery underlying tractable learning has to be generalized for imprecise probabilities. Our empirical evaluations demonstrate that our regime is also effective.

This work was supported by the EPSRC grant Towards Explainable and Robust Statistical AI: A Symbolic Approach.

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Keywords Tractable Learning · Credal Networks · Sum-product Networks · Open-World Assumption · Missing Values

1 Introduction

Large-scale probabilistic representations, including statistical knowledge bases and graphical models, are increasingly in demand. They are built by mining massive sources of structured and unstructured data, the latter often derived from natural language processing techniques. Commercial and scientific projects such as NELL, ProbBase and DeepDive [Mitchell et al., 2015; Sa et al., 2017; Wu et al., 2012] are populated with hundreds of thousands of facts, thereby informing search and retrieval capabilities in high-impact applications from fields such as biology, physics, social sciences and AI.

The very nature of the enterprise makes the extracted representations probabilistic. In particular, inducing relations and facts from noisy and incomplete sources via statistical machine learning models [Bordes et al., 2011; Socher et al., 2013; Wang et al., 2013; Raedt et al., 2015] means that the labels are either already probabilistic, or that probabilities approximate confidence. While the progress is impressive, as argued in [Ceylan et al., 2016], extracted representations essentially enforce the closed-world assumption (CWA) [Reiter, 1977], which means that all tuples in the database are accorded the corresponding probability, but all other facts have probability zero. The CWA is deeply problematic in most machine learning contexts. For example, sources from which such representations are obtained are frequently updated, and it is clearly problematic to attach a prior of zero to some fact that we later want to consider plausible. Dropping the CWA would mean embracing the open world property, where missing values and missing tuples are treated in a systematic manner. Thus, in general, a principled solution is needed for representing incomplete and indeterminate knowledge in such models: representative examples include imprecise probability models such as credal networks (as considered in [Ceylan et al., 2016]) where a well-defined semantic theory justifies the computation of conditional probabilities in the absence of information.

In this work, we are interested in the foundational problem of learning such open-world probabilistic models. Unfortunately, exact inference in probabilistic graphical models is intractable [Valiant, 1979; Bacchus et al., 2009]. Naturally, then, owing to the intractability of inference, learning also becomes challenging, since learning typically uses inference as a sub-routine [Koller and Friedman, 2009]. Moreover, even if such a representation is learned, prediction will suffer because inference has to be approximated.

Tractable learning is a powerful new paradigm that attempts to learn representations that support efficient probabilistic querying. Much of the initial work focused on low tree-width models [Bach and Jordan, 2002], but later, building on properties such as local structure [Chavira and Darwiche, 2008], data structures such as arithmetic circuits (ACs) emerged. These circuit learners can also represent high tree-width models and enable exact inference for a range of queries in time polynomial in the circuit size. Sum-product networks (SPNs) [Poon and Domingos, 2012] are
stances of ACs with an elegant recursive structure – essentially, an SPN is a weighted sum of products of SPNs, and the base case is a leaf node denoting a tractable probability distribution (e.g., a univariate Bernoulli distribution). In so much as deep learning models can be understood as graphical models with multiple hidden variables, SPNs can be seen as a tractable deep architecture. Of course, learning the architecture of standard deep models is very challenging (Bengio, 2009), and in contrast, SPNs, by their very design, offer a reliable structure learning paradigm. While it is possible to specify SPNs by hand, weight learning is additionally required to obtain a probability distribution, but also the specification of SPNs has to obey conditions of completeness and decomposability, all of which makes structure learning an obvious choice. Since SPNs were introduced, a number of structure learning frameworks have been developed for those and related data structures, e.g., (Gens and Domingos, 2013; Hsu et al., 2017; Liang et al., 2017). Recently, (Mauá et al., 2017) proposed a unification of the credal network semantics and SPNs, resulting in so-called credal SPNs, and study algorithms and complexity results for inference tasks.

In this work, we investigate the learning of such open-world SPNs. We show how the computational machinery underlying the learning of SPNs has to be generalized for imprecise probabilities. In our empirical evaluations, we show that our learning regime is effective, providing perhaps for the first time a scalable and tractable learner for open-worlds grounded in imprecise probability theory.

2 Preliminaries

This section reviews some definitions on sum-product networks (SPNs) and its extension to the imprecise case: credal sum-product networks (CSPNs).

2.1 Sum-Product networks (SPNs)

SPNs are rooted acyclic graphs whose internal nodes are sums and products, and leaf nodes are tractable distributions, such as Bernoulli and Gaussian distributions (Poon and Domingos, 2012; Gens and Domingos, 2013). More formally,

**Definition 1 (Syntactic definition)** An SPN over the set of variables $X_1, \ldots, X_n$ is a rooted directed acyclic graph whose leaves are the indicators $x_1, \ldots, x_n$ and $\overline{x}_1, \ldots, \overline{x}_n$, and whose internal nodes are sums and products nodes. Each edge $(i, j)$ from a sum node $i$ has a non-negative weight $\omega_{ij}$.

**Definition 2 (Semantic definition)** The value of a product node is the product of the values of its children. The value of a sum node is $\sum_{j \in \text{Ch}(i)} \omega_{ij} \nu_{ij}$, where $\text{Ch}(i)$ are the children of $i$ and $\nu_{ij}$ is the value of node $j$. The value of an SPN is the value of its root.

SPNs allow for time linear computation of conditional probabilities, among other inference computations, by means of a bottom-up pass from the indicators to the root. An SPN is therefore a function of the indicator variables $S(x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n)$. Evaluating a SPN for a given configuration of the indicator $\lambda = (x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n)$
is done by propagating from the leaves to the root. When all indicators are set to 1 then \( S(\lambda) \) is the partition function of the unnormalised probability distribution. (If \( S(\lambda = 1) \) then the distribution is normalised.)

The scope of an SPN is defined as the set of variables appearing in it. An essential property of SPNs as deep architecture with tractable inference is the one of validity. An SPN is said to be valid \cite{poon2012} if it satisfies the two following conditions:

1. an SPN is complete: if and only if all children of the same sum node have the same scope.
2. an SPN is consistent: if and only if no variable appears negated in one child of a product node and non-negated in another.

SPNs are essentially based on the notion of network polynomials \cite{darwiche2013}; an SPN can be expanded to a network polynomial as seen in Example 1. Intuitively, if an SPN is valid, its expansion includes all monomials present in its network polynomial.

**Example 1** Figure 1 illustrates an example of a Sum-Product network over two Boolean variables.

![Figure 1](image)

The function \( S \) of the indicators is written as

\[
S(x_1, x_2, \overline{x_1}, \overline{x_2}) = \cdot7 \cdot (.4 x_1 + .6 \overline{x_1}) \cdot (.9 x_2 + .1 \overline{x_2}) + \cdot3 \cdot (.2 x_1 + .8 \overline{x_1}) \cdot (.9 x_2 + .1 \overline{x_2})
\]

Its expansion into a network polynomial is given by

\[
\begin{align*}
\cdot7 \cdot (.4 + .9 + .3 + .2 + .9) x_1 x_2 + \cdot7 \cdot (.4 + .1 + .3 + .2 + .1) x_1 \overline{x_2} + \\
\cdot7 \cdot (.6 + .9 + .3 + .8 + .9) \overline{x_1} x_2 + \cdot7 \cdot (.6 + .1 + .3 + .8 + .1) \overline{x_1} \overline{x_2}
\end{align*}
\]

\[
= .306 x_1 x_2 + .034 x_1 \overline{x_2} + .594 \overline{x_1} x_2 + .066 \overline{x_1} \overline{x_2}
\]

It's worth noting also that the SPN in Figure 1 is valid.

### 2.2 Credal sum-product networks (CSPNs)

In recent work \cite{maua2017}, so-called credal sum-product networks (CSPNs) were proposed, which are a class of imprecise probability models that unify the credal
network semantics with SPNs. That work was particularly aimed at analysing the robustness of classification in SPNs learned from data: first they learn a standard SPN (Gens and Domingos, 2013), and from that they obtain the CSPN by an \( \varepsilon \)-contamination of the SPN (cf. (Mauá et al., 2017)).

A CSPN is defined in the same way as a SPN except that it allows weights on sum nodes to vary in a closed and convex set. More formally,

**Definition 3 (Credal Sum-Product network)** A CSPN over variables \( X_1, \ldots, X_n \) is a rooted directed acyclic graph whose leaves are the indicators \((x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n})\) and whose internal nodes are sums and products. Each edge \((i, j)\) emanating from a sum node \(i\) is associated to a credal set \(K_i\).

The key notion here, is the one of **credal set**.

**Definition 4 (Credal set)** A credal set is a convex set of probability distributions.

A credal set can be interpreted as a set of imprecise beliefs meaning that the true probability measure is in that set but due to lack of information, that cannot be determined. In order to characterise a credal set, one can use a (finite) set of extreme points (edges of the polytope representing the credal set), probability intervals or linear constraints.

**Example 2** We illustrate an example of a CSPN in Figure 2. In this example, along the lines of the discussions in (Mauá et al., 2017), the weights are given as a convex set of different types. That is, weights \(w_1, w_2\) are probability intervals, as are weights \(w_5, w_6\). In contrast, \(w_3, w_4\) and \(w_7, w_8\) are sets of extreme points (i.e. described as the convex hull by the set of extreme points). We have to ensure that the constraints associated with probability intervals realise a normalised distribution.

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Fig. 2 Credal Sum-Product network example
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Although (Mauá et al., 2017) do not consider the notion of validity for their structures, we can lift the existing formulation and define a CSPN to be **valid** if it satisfies the same conditions as would valid SPNs, namely **consistency** and **completeness**.

Inference in CSPN boils down to the computation of minimum and maximum values for \(\lambda\), that is, a given configuration of the indicators. This amounts to computing lower and upper likelihood of evidence, which can be performed in almost

1 It is important to note that the number of extreme points can reach \(N!\) where \(N\) is the number of interpretations (Wallner, 2007).
the same fashion as for SPNs. More precisely, we evaluate a CSPN from leaves to
ward the root. The value of a product node is the product of the values of its children
(as in SPNs). The value of a sum node is given by optimising (either maximising or
minimising) the following equation:
\[
\sum_{j \in \text{Ch}(i)} w_{ij} \cdot v_j
\]
where the weights \(w_{ij}\) are chosen from the credal set \(w_i\) (more precisely, weights are
chosen subject to constraints that define a probability distribution within the credal
set). The value of a CSPN, denoted \(S(x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_m)\), is the value of its root
node.\(^2\)

Since the inference computation is crucial, we study reasoning in CSPNs from
(Mauá et al., 2017) in more detail in the following section.

3 Inference in credal SPN

As just discussed, (Mauá et al., 2017) provide ways to compute the minimum and
maximum values for a given configuration \(\lambda\) of the indicator variables, as well as ways
to compute conditional expectations, which we describe in the sequel. In particular,
they propose to use a linear program solver to solve (maximise or minimise) and find
the best weight to propagate. We explain later that it is easier to learn intervals of
probability degrees when clustering.

3.1 Inference using a linear program solver

Interval-based probability distributions (IPD for short) are a very natural and common
way to specify imprecise and incomplete information. In an IPD \(\text{IP}\), every interpre-
tation \(\omega_i \in \Omega\) is associated with a probability interval \(\text{IP}(\omega_i) = [l_i, u_i]\) where \(l_i\) (resp.
\(u_i\)) denotes the lower (resp. upper) bound of the probability of \(\omega_i\).

**Definition 5 (Interval-based probability distribution)** Let \(\Omega\) be the set of possible
worlds. An interval-based probability distribution \(\text{IP}\) is a function that maps every
interpretation \(\omega_i \in \Omega\) to a closed interval \([l_i, u_i]\) \subseteq [0, 1].

Given a credal set, if the weights are depicted as extreme points, we can obtain an
IPD by taking into account for each value the minimum degree of the extreme points
(for the lower bound) and the maximum degree of the extreme points (for the upper
bound). The result, as any other interval-based probability distribution, should satisfy
the following constraints in order to ensure that the underlying credal set is not empty
and every lower/upper probability bound is reachable.

\[
\sum_{\omega_i \in \Omega} l_i \leq 1 \leq \sum_{\omega_i \in \Omega} u_i
\]

\(^2\) We remark that in this paper for simplicity, we restrict ourselves to Boolean variables, but this is easily
generalised to many-valued variables.
\[ \forall \omega_i \in \Omega, \ l_i + \sum_{\omega_j \neq i} u_j \geq 1 \quad \text{and} \quad u_i + \sum_{\omega_j \neq i} l_j \leq 1 \tag{3} \]

Let us illustrate an interval-based probability distribution in the context of a credal SPN.

**Example 3** Let \( S \) be a sum node with three children \( D_S = \{cl_1, cl_2, cl_3\} \). Table 1 provides an example of interval-based probability distribution.

| \( S \) | \( P(S) \) |
|-------|--------|
| \( cl_1 \) | \([0, .4]\) |
| \( cl_2 \) | \([.1, .55]\) |
| \( cl_3 \) | \([.1, .65]\) |

This imprecise probability distribution satisfies the two conditions of Equations 2 and 3.

From intervals, as depicted in Table 1, it is easy to compute, using a linear program solver, \( \min_w S_w(\lambda) \) and \( \max_w S_w(\lambda) \) as seen in the following example.

**Example 4 (Continued)** Given Table 1 let us consider that the weights associated to the children of the clusters have been optimised to .4 for cluster 1, .5 for cluster 2 and .1 for cluster 3. Thus, we encode the linear program as follow:

Maximize  
\[ \text{value} = .4 \cdot c_{l1} + .5 \cdot c_{l2} + .1 \cdot c_{l3} \]

Subject To  
\[ c_{0} : \quad c_{l1} + c_{l2} + c_{l3} = 1 \]

Bounds  
\[ 0 \leq c_{l1} \leq 0.4 \]
\[ 0.1 \leq c_{l2} \leq 0.55 \]
\[ 0.1 \leq c_{l3} \leq 0.65 \]

End

4 Learning a credal sum-product network

Algorithms to learn sum-product networks are maturing; we can broadly categorise them in terms of discriminative training and generative learning. Discriminative training (Gens and Domingos, 2012) learns conditional probability distributions while generative learning (Gens and Domingos, 2013) learn a joint probability distribution. In the latter paradigm, (Gens and Domingos, 2013) propose the algorithm Learn-SPN that starts with a single node representing the entire dataset, and recursively
adds product and sum nodes that divide the dataset into smaller datasets until a stopping criterion is met. Product nodes are created using group-wise independence tests, while sum nodes are created performing clustering on the row instances. The weights associated with sum nodes are learned as the proportion of instances assigned to a cluster.

In this paper, starting from LearnSPN, we propose a generative learning method that deals with missing values in the dataset. The degenerate case is when an entire tuple is missing. The outcome of the learning approach is a credal SPN. The idea, like with (Gens and Domingos, 2013), is to first search independent set of variables and then clusters the instances. That is, it builds product nodes when a dependent set of variables is found and sum nodes when clustering the instances. In this section, we detail the mathematical justifications for learning a credal SPN. In the next section, we evaluate the learned CSPN on several datasets.

4.1 Learning the structure

The structure of a CSPN is built in the same way as the structure of a SPN. It is done by recursively creating product nodes using independence tests and creating sum nodes by clustering instances. The idea is captured in Figure 3.

The difference to the standard approach arises in the way that we treat missing values. In the sequel, in order to illustrate the different parts of the algorithm, we will refer to the following table containing a dataset with missing values. More precisely,
let us consider a dataset that contains 600 instances on 7 (Boolean) variables. We consider the particular case of MAR data (i.e. missing at random) on two variables $A$ and $C$ with 1% of missing data depicted in Table 2.

**Table 2** MAR dataset example

| instance number | A | B | C | D | E | F | G |
|-----------------|---|---|---|---|---|---|---|
| 1               | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2               | ? | 1 | ? | 0 | 0 | 1 | 1 |
| 3               | ? | 1 | ? | 0 | 0 | 0 | 1 |
| 4               | ? | 0 | 1 | 0 | 0 | 0 | 0 |
| 5               | ? | 0 | ? | 1 | 0 | 0 | 0 |
| 6               | ? | 1 | ? | 0 | 1 | 1 | 1 |
| 7               | ? | 1 | ? | 0 | 0 | 0 | 1 |
| 8               | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| ...             |   |   |   |   |   |   |   |

### 4.1.1 Group-wise independence test

We first recall that the independence test used in LearnSPN [Gens and Domingos, 2013] is given by the following equation

$$G(X_1, X_2) = 2 \sum_{x_1} \sum_{x_2} C(x_1, x_2) \ast \log \frac{C(x_1, x_2) \ast |T|}{C(x_1)C(x_2)}, \quad (4)$$

where we sum over the domain of each variable, the function $C(x_1, x_2)$ counts the occurrences (number of instances) matching the values $x_1, x_2$ (analogously, $C(x_1)$ counts the occurrences of $x_1$) and $|T|$ is the number of instances in the current partition.

This G-value is compared to a threshold. The lower the G-value is the most likely chance there is that the two variables are independent.

The count function in LearnSPN algorithm takes into account a complete dataset. Given an incomplete dataset (i.e., with missing values), the count function has to be reconsidered. Let us first take an example using the MAR dataset of Table 2. Consider variable $A$, and then we build a set of variables that are dependent to $A$. More precisely, it involves the computation of $G$ (Equation (4)) for $A$ and $B$, $A$ and $C$, and so on. To illustrate the calculation, suppose we consider only complete instances (594 instances) where the counts of $(A, B)$ and $(A, C)$ are depicted in Table 3 in the gray columns. Given a $p$-value of 0.0015 for the threshold, and a $G$-value $G(A, B) = 0.00457$, we identify that $A$ and $B$ are independent in the current partition. This then raises the question: how are we to deal with the missing values?

In our case, we consider two count functions, that we denote $\overline{C}$ and $\overline{C}$. The first function takes into account only instances where the values of the variables involved in the independence test are complete (much like the count function $C$ in Equation (4)). The second function $\overline{C}$ takes into account all instances. More precisely, $\overline{C}(x_1)$
counts the number of instances matching \( x_1 \) and adding to that every instance where \( X = \text{?} \) is present. Formally, \( C \) is as follow:

\[
C(x_1, x_2) = |\text{insts} \models (x_1, x_2)| + |\text{insts} \models (? , x_2)| + |\text{insts} \models (x_1 , ?)| \quad (5)
\]

Consider, for example, the counts for (A,B) and (A,C) in Table 3. Every value of B is available. In contrast, for A and C we have some missing values. We can use Equation (4) and compute the \( G \)-value for the two functions \( C \) and \( \overline{C} \).

**Table 3** Counts for the tuples (A,B) and (A,C)

|       | \( C(a, b) \) | \( C(a, \text{?}) \) | \( \overline{C}(a, b) \) | \( \overline{C}(a, \text{?}) \) | \( C(\text{?}, b) \) | \( \overline{C}(\text{?}, b) \) | \( C(\text{?}, \text{?}) \) | \( \overline{C}(\text{?}, \text{?}) \) |
|-------|---------------|-------------------|--------------------------|--------------------------|----------------|--------------------------|----------------|--------------------------|
| (a, b) | 210           | 214               | 242                      | 248                      |               |                           |               |                           |
| (a, ?) | 32            | 34                | 352                      | 358                      |               |                           |               |                           |
| (\text{?}, b) | 304          | 308               | 514                      | 522                      |               |                           |               |                           |
| (\text{?}, ?) | 48          | 50                | 80                       | 84                       |               |                           |               |                           |
| (a, c) | 150           | 156               | 200                      | 212                      |               |                           |               |                           |
| (\text{?}, c) | 92          | 97                | 394                      | 404                      |               |                           |               |                           |
| (\text{?}, ?) | 302         | 307               |                           |                           |               |                           |               |                           |

Here, \( G(A, B) = 0.00457 \) for \( C \) and \( G(A, B) = -2.61699 \) for \( \overline{C} \). As for (A,C) we have \( G(A, C) = 33.3912 \) for \( C \) and \( G(A, B) = 28.6183 \) for \( \overline{C} \) which implies that in both cases A and B are independent and A and C are dependent.

Putting it together, to determine if two variables are dependent, similar to SPN learning, we use the group-wise independence test, except that the count function is now a mean between the two count functions \( C \) and \( \overline{C} \). Formally,

\[
G(X_1, X_2) = 2 \sum_{x_1} \sum_{x_2} m(C, \overline{C})(x_1, x_2) \cdot \log \frac{m(C, \overline{C})(x_1, x_2) \cdot |T|}{m(C, \overline{C})(x_1) \cdot m(C, \overline{C})(x_2)}, \quad (6)
\]

where \( m \) is the mean function.

Intuitively, \( \overline{C} \) acts as a gatekeeper that takes into account missing information and readjusts the \( G \)-value accordingly. A more elaborate measure other than the mean could be applied too, of course; for example, we could take into account the proportion of missing values and define a measure that is based on that proportion.

**4.1.2 Clustering instances**

When no independent set of variables is found, we perform hard EM to cluster the instances.

Intuitively, a cluster is a set of instances where variables mostly have the same values. However, when dealing with missing values, it is less obvious how one is to evaluate an instance to a cluster. To that end, we will be using two measures of likelihood. Let us motivate that using an example: consider the following cluster \( cl \) which contains 40 instances including 36 complete instances and the cluster \( cl/2 \) which contains 70 instances including 51 complete instances. Statistics of \( cl/1 \) and \( cl/2 \) are shown in Table 4.
Let us consider the following complete instance: 0 0 1. In this case, we only use one measure which is the lower measure of log-likelihood, denoted $\log_{\text{cl}}(\text{inst})$. And then only take into account the statistics of the known values as shown in Table 4. More precisely, we compute the sum for each variable $\text{var}$ of $\text{inst}$, with value $\text{val}$ being the log of the proportion of instances in the cluster having the same value. This is expressed as:

$$\log_{\text{cl}}(\text{inst}) = \sum_{\text{var}} \log((w_{\text{val}} + \text{smoo})/(\text{nb}_{\text{inst}} + 2 \times \text{smoo}))$$  (7)

where $w_{\text{val}}$ is the number of instances where the value for the variable $\text{var}$ is also the value of the instance, $\text{val}$. $\text{nb}_{\text{inst}}$ is the number of instances where the value for variable $\text{var}$ is known. Also, $2 \times \text{smoo}$ is the smoothing value multiplied with the number of possible values of $\text{var}$; since we consider a Boolean dataset, it is 2.

Given the complete instance $\text{inst}_1 0 0 1$, the computation of the lower log-likelihood of $\text{inst}_1$ for the cluster $\text{cl}$, which we denote $\log_{\text{cl}}(\text{inst}_1)$, is given by $\log_{\text{cl}}(001) = -0.1434$ and $\log_{\text{cl}}(001) = -1.8787$.

Now let us consider the incomplete instance $\text{inst}_2 0 0 \?$, in which case the computation of the lower log-likelihood is given by $\log_{\text{cl}}(00?) = -0.0812$ and $\log_{\text{cl}}(00?) = -0.9914$.

For this example it is clear that the best cluster for $\text{inst}_2$ is $\text{cl}$. But note that statistics on incomplete instances for B are not used, and that is also true for the statistics for C. This then motivates a second measure of log-likelihood, called upper log-likelihood and denoted $\log_{\text{u}}(\text{inst})$. First, we compute for each variable $\text{var}$ (whose value is unknown), the log value of the worst case scenario:

$$\tilde{\log}_{\text{cl}}(\text{inst}) = \sum_{\text{var}} \log((w + \text{smoo})/(\text{nb}_{\text{inst}} + 2 \times \text{smoo})) - \log((\text{inc} + \text{smoo})/(\text{all}_{\text{inc}} + 2 \times \text{smoo}))$$ (8)

where $w$ is the number of instances where the value for the variable $\text{var}$ is that value that is poorest fit (i.e., the pessimistic scenario). The term $\text{inc}$ here specifically corresponds to the number of instances in the cluster where the value is unknown but has been assigned a number that corresponds to the worst case scenario when adding to the cluster. Then, $\text{all}_{\text{inc}}$ is the number of instances where the value is unknown. This value is then added to the lower log-likelihood. Formally,

$$\log_{\text{u}}(\text{inst}) = \tilde{\log}_{\text{cl}}(\text{inst}) + \log_{\text{cl}}(\text{inst})$$ (9)

Intuitively, we take into account all instances for the computation of the upper log-likelihood, and consider incomplete instances in a manner that does not exaggerate the uncertainty relative to the complete instances.

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**Table 4** Statistics for clusters $\text{cl}_1$ and $\text{cl}_2$

|        | $\text{cl}_1$ | $\text{cl}_2$ |
|--------|---------------|---------------|
| $\text{A}$ | 40 | 60 |
| $\text{B}$ | 30 | 6 |
| $\text{C}$ | 5 | 61 |
| $\text{D}$ | 10 | 45 |
| $\text{E}$ | 9 | 6 |

---
In our example, with \textit{inst} \_2 0 0 \_?, consider that the two incomplete instances (missing values for the variable \textit{C}) in \textit{cl1} have been added by considering the worst case scenario as \textit{C} = 0. The computation of the log-likelihood for the unknown values of \textit{inst} \_2 is \( \log_{\text{cl1}}(00?) = -0.8543 \). For the second cluster, there is no incomplete instance to consider, and so \( \log_{\text{cl2}}(00?) = -0.8873 \).

The upper log-likelihood is then given by

\[
\begin{align*}
- \log_{\text{cl1}}(00?) &= -0.9355 \\
- \log_{\text{cl2}}(00?) &= -1.8787
\end{align*}
\]

A new cluster is created only if for each cluster the lower log-likelihood for the instance does not exceed the penalty for that cluster. In LearnSPN, the cluster penalty is chosen between \{0.2, 0.4, 0.6, 0.8\}, we use the same penalty in LearnCSPN.

Intuitively, the lower bound can be seen as a certainty bound whereas the upper bound can be seen as a possibility bound. Specifically, the lower bound refers to the SPN that would be learned if we only consider the complete instances. And the upper bound denotes the learning regime if every incomplete instance has been made complete in a way that fits the cluster.

The overall algorithm that clusters the set of instances is given in Algorithm 1.

\begin{algorithm}
\caption{Find best clusters for instance inst}
\begin{algorithmic}
\Require \textit{inst}, set of clusters \textit{nbcs}
\Ensure A set of clusters \textit{clusters for inst}
\State Let \texttt{bestC LL} be the best current log likelihood of \textit{inst} with a cluster
\State \texttt{bestC LL} = \text{newClusterPenalizedLL}
\State Let \texttt{bestC LL worstcasescenario} be the best current log likelihood of the worst case scenario of \textit{inst} with a cluster
\State \texttt{bestC LL worstcasescenario} = \text{newClusterPenalizedLL}
\State Let \texttt{clusters for inst} initialized to an empty set
\State \texttt{Have found cluster} = false
\For {each \textit{cluster} in \textit{nbcs}}
\State \texttt{cll} ← log likelihood of \textit{inst} in \textit{cluster}
\State \texttt{worstcase} ← log likelihood of the worst case scenario of \textit{inst} in \textit{cluster}
\If {\texttt{cll} > \texttt{bestC LL}}
\State \texttt{Have found cluster} = true
\State \texttt{bestC LL} = \text{cll}
\State \texttt{bestC LL worstcasescenario} = \text{worstcase}
\For {each \texttt{clus} in \texttt{clusters for inst}}
\If {\texttt{clus.LL} < \texttt{bestC LL worstcasescenario}}
\Remove \texttt{clus} from \texttt{clusters for inst}
\EndIf
\EndFor
\ElseIf {\texttt{cll} > \texttt{bestC LL worstcasescenario}}
\EndIf
\EndFor
\If {not \texttt{Have found cluster}}
\State \texttt{c} = \text{newCluster}
\State \texttt{nbcs.add(c)}
\State \texttt{clusters for inst.add(c)}
\EndIf
\EndIf
\end{algorithmic}
\end{algorithm}
4.2 Weights accorded to sum nodes

In LearnSPN, the weights associated to edges from a sum node to its children are given by the number of instances in the cluster normalised by the total number of instances. This way, they obtain a normalised probability distribution. As we noted in Section 2.2, the syntactic difference between SPN and credal SPN is the weight associated to edges from sum nodes. In particular, we are trying to learn credal sets of probability degrees, in the form of extreme points in particular. As explained in the clustering process, we allow an instance to belong to multiple clusters if this instance is incomplete. This implies that the intersection between clusters might not be empty and that taking the proportion of instances to determine the weight might not be coherent.

Our first attempt to address this problem was to obtain one extreme point at each EM run; however, the issue here is that the number of clusters may vary. Imposing an upper bound on this number explicitly, as a prior, is of course problematic as the clustering process is data dependent. From this observation, we chose to learn intervals instead of extreme points and then extract the extreme points from the learned intervals. Let us consider an example for parameter learning in a credal network.

Example 5 Let us consider Figure 4 which depicts a small credal network over 3 variables and a dataset where each row is an instance that represents a record at time \( t \) of the variables (or perhaps the record from an expert about the variables). Some of these instances are incomplete, denoted with a ‘?’. 

![Figure 4: Credal network structure and a corresponding dataset](image)

Since the structure is given, we can easily learn the imprecise probability distributions associated to nodes \( A, B \) and \( C \). The lower bound of an interpretation is given by the proportion of **known instances** compatible with the interpretation. For example, \( \mathcal{IP}(A = 1) = 1/2 \). As for the upper bound, the degree is given by the proportion of instances that *are or could* potentially (owing to the missing information) be compatible with the interpretation. For example, \( \mathcal{UP}(A = 1) = 3/4 \).

Likewise, the lower (respectively, upper bound) for a cluster is given by the number of complete instances (respectively, the number of incomplete instances). Unfor-
Unfortunately, this method does not ensure that all interval bounds are reachable. Let us illustrate such an example.

**Example 6** Let us consider a partition of 80 instances, which contains 30 incomplete instances (and so, 50 complete). Let us say that the partition has been clustered into 5 clusters. One extreme case could be that the first cluster contains 10 complete instances and 30 incomplete. The other clusters split the remaining complete instances. This means that the lower and upper bound of the interval associated with the second, third and forth cluster are the same. Only the first cluster has a different bounds, indeed $IP(c1) = [.125, .5]$. Yet, the probability of the three other clusters add up to .5 which implies that .125 cannot be reached.

A simple and effective way to deal with this problem is to readjust the bound to be able to satisfy the conditions of a well-defined interval-based probability distribution (as given by Equations (2) and (3)). More precisely, from the learned interval, we extract a set of extreme points that defines a convex set included in the actual convex set determined by the intervals (cf. Example 7). Note that the extraction process does not extract all the extreme points. This is purposeful and is motivated by two reasons:

- the number of extreme points can be extremely high (at most $n^a$) (Wallner, 2007);
- between two extreme points, the change in the probability degrees might be small which would not gravely affect the overall log-likelihood or conditional probability.

**Example 7** In this example, we illustrate the convex set covered in the process of extraction of extreme points. Note that in case of a small amount of clusters, the two reasons discussed above do not manifest so strongly. Let us consider a sum node with 3 clusters with the following interval weights:

- $c_1 [.2,.4]$
- $c_2 [.3,.55]$
- $c_3 [.2,.48]$

There are 6 extreme points as shown below:

- $P_1 [0.4, 0.4, 0.2]$
- $P_2 [0.4, 0.3, 0.3]$
- $P_3 [0.25, 0.55, 0.2]$
- $P_4 [0.2, 0.55, 0.25]$
- $P_5 [0.22, 0.3, 0.48]$
- $P_6 [0.2, 0.32, 0.48]$

![Fig. 5 Convex set of extreme points extracted](image-url)
In this example, it is easy to extract all extreme points since it only involves 3 clusters. However, on a huge dataset we might obtain (say) a hundred clusters. In this case, the number of extreme points can reach 100!. To avoid the computation of all the extreme points and also avoid having to consider all of them in the computation of the log-likelihood, we restrict ourselves to a subset of the extreme points which defines a sub convex set as illustrated in Figure 5. In our example, this results in considering 3 extreme points over the 6 possible ones, and the grey areas are the convex sets of degrees that are ignored.

Nonetheless, the data structure used to store the extreme points can be easily optimised for also storing the maximum weight which can augment the efficiency of inference. Analogously, the set of extreme points we select can be improved to minimise the convex regions that are ignored.

The overall algorithm to learn a credal SPN is given in Algorithm 2.

Algorithm 2 LearnCSPN(T, V)
- Require: set of instances T and set of variables V
- Ensure: a CSPN representing an imprecise probability distribution over V learned from T

1: if |V| = 1 then
2: return univariate imprecise distribution estimated from the variable's values in T
3: else
4: partition V into approximately independent subsets V_j > (See Subsubsection 4.1.1)
5: if success then
6: return ∏ j LearnCSPN(T, V_j)
7: else
8: partition T into subsets of similar instances T_i > (See Subsubsection 4.1.2)
9: return ∑ i K_i * LearnCSPN(T_i, V) * (where K_i is computed as described in Subsection 4.2)
10: end if
11: end if

4.3 Properties of the learning algorithm LearnCSPN

LearnCSPN of Algorithm 2 holds some interesting properties as stated in the following lemmas.

Lemma 1 LearnCSPN learns a complete CSPN.

Proof Consider the partition P on n variables {V_1, ..., V_n} where P is clustered as c clusters; each resulting cluster is then a new partition P_i (i ∈ {1, ..., c}) on the same n variables. If P_i is clustered at the next step, then in the same way, each cluster is a new partition P_{ij} on the same variables. Therefore, the scope of the sum node is the set of variables {V_1, ..., V_n}. If at the next step, a set of independent variables is found, new partitions are created on subsets of variables {V_1, ..., V_k} and {V_{k+1}, ..., V_n}. The scope of the product node created is still the set of variables {V_1, ..., V_n}. Hence, the CSPN is complete.
Analogously, the learned CSPN can be shown to be consistent.

**Lemma 2** LearnCSPN learns a consistent CSPN.

Using the same reasoning as for Lemma 1, we can proved that the CSPN is consistent.

**Proof** Consider the partition $P$ on the variables $\{V_1, \ldots, V_n\}$ where $P$ is divided into two new partitions $P_1$ on variables $\{V_1, \ldots, V_k\}$ and $P_2$ on variables $\{V_{k+1}, \ldots, V_n\}$. If $P_1$ (respectively $P_2$), is clustered in the next step, then each cluster is a new partition on the variables $\{V_1, \ldots, V_k\}$ (respectively, on variables $\{V_{k+1}, \ldots, V_n\}$). Thus, the scope of $P_1$ (respectively $P_2$) is $\{V_1, \ldots, V_k\}$ (respectively, $\{V_{k+1}, \ldots, V_n\}$) and does not share any variable with $P_2$ (respectively $P_1$). If at the next step, a set of independent variables are identified in $P_1$, then new partitions are created: $\{V_1, \ldots, V_i\}$ and $\{V_{i+1}, \ldots, V_k\}$. The scope of $P_1$ remains the set of variables $\{V_1, \ldots, V_k\}$. Hence, the CSPN is consistent.

Thus, the following theorem holds.

**Theorem 1** LearnCSPN learns a valid CSPN.

The proof is direct and follows from Lemmas 1 and 2.

LearnCSPN holds a second and fundamental property wrt the downward compatibility of CSPNs with SPNs. Indeed, when learning a CSPN from a complete dataset, the weights are no longer interval but single degrees. Therefore, the heuristic developed is reduced to the heuristic of LearnSPN. This is formally stated by the following theorem.

**Theorem 2** LearnCSPN on a complete dataset is equivalent to LearnSPN.

Indeed, when there is no missing value in the dataset, the count function for the independence test matches the count function of LearnSPN. Analogously, the computation of the log-likelihood of an instance in a cluster corresponds to the computation of the lower log-likelihood. In that case, we only add an instance to one cluster and the lower and upper bound of intervals are the same. This results in a single extreme point that matches the weights that LearnSPN would have obtained for the same partition.

4.4 Complexity remarks

The previous result stated that LearnCSPN returns a valid CSPN. We can now be precise in our analysis of the learned CSPN, and its structure. In fact, if we remove leaves then the CSPN is a tree-shaped network. This is formally stated in the next proposition.

**Proposition 1** LearnCSPN learns a CSPN where internal nodes have at most one parent.
Proof (Sketch) Following the scheme of the algorithm given in Figure 3, it is easy to see that each partition has been created either by clustering the instances of a (unique) partition (represented by the parent node) or by creating (distinct) sets of variables via the independence test on a (unique) partition (also given by the parent node).

This proposition allows us to leverage a strong result on the polynomial time computational complexity for conditional expectations, as proved in Mauá et al. (Mauá et al., 2017):

**Theorem 3** (Mauá et al., 2017) Computing lower/upper conditional expectations of a variable in CSPNs takes at most polynomial time when each internal node has at most one parent.

From Proposition 1 and Theorem 3, we can infer the next corollary.

**Corollary 1** Computing lower and upper conditional expectations of a variable in CPSNs learned using LearnCSPN takes at most polynomial time.

A second result from Mauá et al. (2017) is also worth noting:

**Theorem 4** (Mauá et al., 2017) Consider a CSPN \( \{S_w : w \in C\} \), where \( C \) is the Cartesian product of finitely-generated polytopes \( C_i \), one for each sum node \( i \). Computing min\( _w S_w(\lambda) \) and max\( _w S_w(\lambda) \) takes \( O(sL) \) time, where \( s \) is the number of nodes and arcs in the shared graphical structure and \( L \) is an upper bound on the cost of solving a linear program \( \min_{w_i} \sum_j c_{ij} w_{ij} \) subject to \( w_i \in C_i \).

Intuitively, this result says that the computational complexity associated with upper-log-likelihoods in a tree-shaped CSPN remains linear. In the case of that work, it was due to calling a linear solver on each of the sum nodes, to optimise and propagate values to the root node. And this is all the more true when dealing with extreme points, and so we get that the computational complexity for log-likelihoods is linear in the number of extreme points multiplied by the number of nodes.

**Corollary 2** Computing upper/lower log-likelihood of evidence takes \( O(n * N) \) time where \( n \) is the number of nodes and \( N \) is the maximum number of extreme points in the learned CSPN.

In the next section, we evaluate LearnCSPN on various datasets.

5 Experiments

In our experimental evaluations, we have applied LearnCSPN on various datasets discussed below. Where applicable, we compare our results to the learning of SPNs, obtained via the LearnSPN algorithm (Gens and Domingos, 2013).
5.1 Datasets

In our results, we measure the accuracy of the learned CSPNs using three different log-likelihood measures:

- lower log-likelihood is given by considering only lower bounds of the weights associated with sum-nodes;
- upper log-likelihood is given by considering only upper bounds of the weights;
- **max log-likelihood** is given by considering the best extreme points.

We performed LearnCSPN using the same sets of datasets considered in [Gens and Domingos, 2013]. In Table 5, we recall the characteristics of the datasets in terms of number of variables (|V|), number of instances in the training set (Train), number of instances in the validation set (Valid) and number of instances in the test set (Test).

| Dataset    | |V| | Train | Valid | Test |
|------------|------|-----|-------|-------|------|
| NLTCS      | 16   | 16181| 2137  | 3236  |
| Plants     | 69   | 17412| 2321  | 3482  |
| Audio      | 100  | 15000| 2000  | 3000  |
| Jester     | 100  | 9000 | 1000  | 4116  |
| Netflix    | 100  | 15000| 2000  | 3000  |
| Retail     | 135  | 22041| 2938  | 4408  |
| Pumsb-star | 163  | 12262| 1635  | 2452  |
| DNA        | 180  | 1600 | 400   | 1186  |
| MSWeb      | 294  | 29441| 3270  | 5000  |
| Book       | 500  | 8700 | 1159  | 1739  |
| EachMovie  | 500  | 4524 | 1002  | 591   |
| WebKB      | 839  | 2803 | 558   | 838   |
| Reuters-52 | 889  | 6532 | 1028  | 1540  |
| 20 Newsgr. | 910  | 11293| 3764  | 3764  |
| BBC        | 1058 | 1670 | 225   | 330   |
| Ad         | 1556 | 2461 | 327   | 491   |

For the purpose of this paper, to be able to precisely study the effect of missing values, we do not consider incomplete datasets but simply modify the existing datasets by removing values from instances on 1% and 5% of the set of instances. The number of values removed on each instance can vary and is randomly chosen. (A degenerate case with all missing values on an instance may happen, but we do not explicitly test for it.)

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4 We note that for both the lower and upper log-likelihoods, the sum of the weights does not equal 1, and it is renormalised during the computation of probabilities.

5 Out of the 20 datasets present in [Gens and Domingos, 2013], we considered 16 datasets, which span various types of domains, and where variable numbers were as many as 1600, and the number of instances were as many as 28k. We did not consider 4 of them, one of which has about 300k instances, as our evaluations frequently ran out of memory on our machine as we increased the percentage of missing values. The issue is thus solely related to computational power: reasonable subsets of these datasets are not problematic.
5.2 Learning credal SPNs

In Table 6, we present the comparison of learnSPN and LearnCSPN on our datasets. We also show the time of the learning computation, and the size (in terms of nodes) of the learnt SPNs and CSPNs. The heuristics used in learnSPN are hard incremental EM over a naive Bayes mixture model to cluster the instances and a G-test of pairwise independence for the variable splits. The heuristics for LearnCSPN are the one given in Section 4. In both algorithms, we ran ten restarts through the subset of instances four times in random order.

We present in bold the results on the size of the learnt CSPN when it is a significantly improvement (i.e., less nodes in the CSPN) in terms of size when compared to the SPN. And we also present the best log-likelihood values in bold when comparing learnSPN and LearnCSPN.

Table 6 Log-likelihood given missing values

| Dataset     | 0% missing values | 1% of inst | 5% of inst | Log Likelihood | LearnSPN lower | upper | max | LearnCSPN lower | upper | max |
|-------------|-------------------|------------|------------|----------------|----------------|-------|-----|-----------------|-------|-----|------|
| NLTC        |                   |            |            |                | 6.114          | 6.150 | 6.134 | 6.111           | 6.111 | 6.111 | 6.111 |
| Plants      |                   | -32.988    |            | -32.988        | 47.881 / 3851  | 891.506 | 548.58 | 891.506 / 5561 | 891.506 | 548.58 |
| Audio       |                   | -40.524    |            | -40.524        | 770.685 / 354911 | 1281.92 / 202349 | 1281.92 / 202349 | 1281.92 / 202349 | 1281.92 / 202349 |
| Jester      |                   | 0.544      |            | -0.544         | 1022.399 / 31769 | 13309 / 101089 | 13309 / 101089 | 13309 / 101089 | 13309 / 101089 |
| Netflix     |                   | -37.408    |            | -37.408        | 1130.976 / 39447 | 689.406 / 5381 | 591.95 / 4985 | 591.95 / 4985 | 591.95 / 4985 |
| Plants      |                   | 0.544      |            | 0.544          | 1022.399 / 31769 | 13309 / 101089 | 13309 / 101089 | 13309 / 101089 | 13309 / 101089 |
| Audio       |                   | -40.524    |            | -40.524        | 770.685 / 354911 | 1281.92 / 202349 | 1281.92 / 202349 | 1281.92 / 202349 | 1281.92 / 202349 |
| Jester      |                   | 0.544      |            | 0.544          | 1022.399 / 31769 | 13309 / 101089 | 13309 / 101089 | 13309 / 101089 | 13309 / 101089 |
| Netflix     |                   | -37.408    |            | -37.408        | 1130.976 / 39447 | 689.406 / 5381 | 591.95 / 4985 | 591.95 / 4985 | 591.95 / 4985 |
| Plants      |                   | 0.544      |            | 0.544          | 1022.399 / 31769 | 13309 / 101089 | 13309 / 101089 | 13309 / 101089 | 13309 / 101089 |
| Audio       |                   | -40.524    |            | -40.524        | 770.685 / 354911 | 1281.92 / 202349 | 1281.92 / 202349 | 1281.92 / 202349 | 1281.92 / 202349 |
| Jester      |                   | 0.544      |            | 0.544          | 1022.399 / 31769 | 13309 / 101089 | 13309 / 101089 | 13309 / 101089 | 13309 / 101089 |
| Netflix     |                   | -37.408    |            | -37.408        | 1130.976 / 39447 | 689.406 / 5381 | 591.95 / 4985 | 591.95 / 4985 | 591.95 / 4985 |
| Plants      |                   | 0.544      |            | 0.544          | 1022.399 / 31769 | 13309 / 101089 | 13309 / 101089 | 13309 / 101089 | 13309 / 101089 |
| Audio       |                   | -40.524    |            | -40.524        | 770.685 / 354911 | 1281.92 / 202349 | 1281.92 / 202349 | 1281.92 / 202349 | 1281.92 / 202349 |
| Jester      |                   | 0.544      |            | 0.544          | 1022.399 / 31769 | 13309 / 101089 | 13309 / 101089 | 13309 / 101089 | 13309 / 101089 |
| Netflix     |                   | -37.408    |            | -37.408        | 1130.976 / 39447 | 689.406 / 5381 | 591.95 / 4985 | 591.95 / 4985 | 591.95 / 4985 |

From Table 6 it is clear that in most datasets (13/16), the best log-likelihood is given by the maximum log-likelihood computed on the learnt CPSN. And this is observed regardless of whether we consider a small amount of missing data (1%) or a slightly bigger one (5%). For datasets where learnSPN gives the best log-likelihood,
e.g., Plants and Ad, we notice that the number of nodes in the learnt CSPN (way) is much lesser than the learnt SPN. (Also note that the difference between log-likelihood values in that case is not much.)

One drawback of LearnCSPN is the computational time during learning. As explained in Section 4, it is due to the consideration of an instance in multiple clusters as well as the computation of more values during the clustering process. This can be improved by considering more efficient data structures to store the clustering information, which would make for interesting future work.

6 Related works

Mauá et al. (Mauá et al., 2017) developed credal sum-product networks, an imprecise extension of sum-product networks, as a motivation to avoid unreliability and overconfidence when performing classification (e.g., in handwritten digit recognition tasks). In their work, they consider $\epsilon$-contamination to obtain the credal sets associated to edges of sum nodes in order to evaluate the accuracy of CSPNs in distinguishing between robust and non-robust classifications. More precisely, they look for the value of $\epsilon$ such that the CSPN obtained by local contamination on each sum nodes results in single classification under maximality. It differs from our work as we learn both the structure and weights of a CSPN based on missing values in the dataset. Intuitively, to compute tasks like classification with a CSPN learned using Algorithm 2 it searches for the optimal CSPN given the evidence and return a set of classes that are maximal.

As a credal representation (Levi, 1980), open-world probabilistic databases (Ceylan et al., 2016) (OpenPBDs) are also closely related to credal networks. They extend probabilistic databases using the notion of credal sets. Their idea is to assume that facts that do not belong to the probabilistic database should not be considered as false, and in this sense, the semantics is open-world. Consequently, these facts are associated with a probability interval $[0, \lambda]$ where $\lambda$ is the threshold probability. This relates to our work in the sense that the probability for missing facts (that would be illustrated by all instances having a missing value for the fact) varies in a closed set $[0, \lambda']$ where the threshold $\lambda'$ is defined by a combination of the weights associated to sum nodes.

SPNs have received a lot of attention recently. For example, in (Hsu et al., 2017) the authors propose the first online structure learning technique for SPNs with Gaussian leaves. It consists in assuming that all variables are independent at first. Then the network structure is updated as a stream of data points is processed. They introduce a correlation in the network in the form of a multivariate Gaussian or a mixture distribution. In (Bueff et al., 2018), Bueff et al. tackle the problem of tractable learning and querying in mixed discrete-continuous domains. By combining the power and flexibility of WMI with SPNs, they learn both parametric and non-parametric models and their mixtures. Molina et al. (Molina et al., 2018) developed mixed sum-product networks as a novel combination of non-parametric probability distributions and deep probabilistic models. They provide effective learning using the Rényi Maximum Correlation Coefficient, tractable inference and enhanced interpretability. In
their work, they show that MSPNs are competitive to parameterized distributions as well as mixed graphical models.

There are of course other approaches to representing imprecise probabilistic information, such as the well known belief functions (Shafer, 1976; Denoeux, 2013). Possibilistic frameworks (Zadeh, 1999) have also been introduced to deal with imprecise and more specifically incomplete information. And in that way, many works have been devoted to study properties and build bridges between the two frameworks (Haddad et al., 2017; Benferhat et al., 2017).

Finally, it is worth remarking that open-world semantics and missing values is a major focus in database theory (Libkin, 2016; Console et al., 2017), and it would be interesting to see if our work can be related to and impact those studies.

7 Conclusions

In this work, we were motivated by the need for a principled solution for learning probabilistic models in the presence of missing information. Rather than making a closed-world assumption that might ignore or discard the corresponding facts, which is deeply problematic, we argued for an open-world interpretation, which resulted in the foundational problem of learning open-world probabilistic models. In particular, given the intractability of inference, we were interested in leveraging the exciting paradigm of tractable learning. Consequently, by lifting the credal network semantics, we proposed and developed a learning regime for credalSPNs. Our empirical results show that the regime performs very strongly on our datasets.

Directions for future work include pushing the applicability of LearnCSPNs on a large-scale text data application, while possibly considering relational extensions (Nath and Domingos, 2013) to leverage extracted and mined relations that may be obtained by natural language processing techniques. For the immediate future, we are interested in optimising the algorithm for minimising the discarded convex regions when extracting extreme points and evaluate the inference accuracy.

References

Bacchus F, Dalmao S, Pitassi T (2009) Solving #SAT and Bayesian inference with backtracking search. J Artif Intell Res (JAIR) 34:391–442
Bach FR, Jordan MI (2002) Thin junction trees. In: Advances in Neural Information Processing Systems, pp 569–576
Benferhat S, Levray A, Tabia K (2017) Approximating MAP inference in credal networks using probability-possibility transformations. In: 29th IEEE International Conference on Tools with Artificial Intelligence, ICTAI 2017, Boston, MA, USA, November 6-8, 2017, IEEE Computer Society, pp 1057–1064, DOI 10.1109/ICTAI.2017.00162, URL https://doi.org/10.1109/ICTAI.2017.00162
Bengio Y (2009) Learning deep architectures for AI. Foundations and trends in Machine Learning 2(1):1–127

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6 The inference part has been dealt with and will be posted in an extended report
Bordes A, Weston J, Collobert R, Bengio Y (2011) Learning structured embeddings of knowledge bases. In: Burgard W, Roth D (eds) Proceedings of the Twenty-Fifth AAAI Conference on Artificial Intelligence, AAAI 2011, San Francisco, California, USA, August 7-11, 2011, AAAI Press, URL http://www.aaai.org/ocs/index.php/AAAI/AAAI11/paper/view/3659
Bueff A, Speichert S, Belle V (2018) Tractable querying and learning in hybrid domains via sum-product networks. 1807.05464
Ceylan II, Darwiche A, den Broeck GV (2016) Open-world probabilistic databases. In: Baral C, Delgrande JP, Wolter F (eds) Principles of Knowledge Representation and Reasoning: Proceedings of the Fifteenth International Conference, KR 2016, Cape Town, South Africa, April 25-29, 2016., AAAI Press, pp 339–348, URL http://www.aaai.org/ocs/index.php/KR/KR16/paper/view/12908
Chavira M, Darwiche A (2008) On probabilistic inference by weighted model counting. Artificial Intelligence 172(6-7):772–799
Console M, Guagliardo P, Libkin L (2017) On querying incomplete information in databases under bag semantics. In: Sierra C (ed) Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI 2017, Melbourne, Australia, August 19-25, 2017, ijcai.org, pp 993–999, DOI 10.24963/ijcai.2017/138, URL https://doi.org/10.24963/ijcai.2017/138
Darwiche A (2013) A differential approach to inference in bayesian networks. CoRR abs/1301.3847, URL http://arxiv.org/abs/1301.3847
Denoeux T (2013) Maximum likelihood estimation from uncertain data in the belief function framework
Gens R, Domingos P (2013) Learning the structure of sum-product networks. In: International Conference on Machine Learning, pp 873–880
Gens R, Domingos PM (2012) Discriminative learning of sum-product networks. In: Bartlett PL, Pereira FCN, Burges CJC, Bottou L, Weinberger KQ (eds) Advances in Neural Information Processing Systems 25: 26th Annual Conference on Neural Information Processing Systems 2012. Proceedings of a meeting held December 3-6, 2012, Lake Tahoe, Nevada, United States., pp 3248–3256, URL http://papers.nips.cc/paper/4516-discriminative-learning-of-sum-product-networks
Haddad M, Leray F, Levray A, Tabia K (2017) Learning the parameters of possibilistic networks from data: Empirical comparison. In: FLAIRS Conference, AAAI Press, pp 736–741
Hsu W, Kalra A, Poupart P (2017) Online structure learning for sum-product networks with gaussian leaves. arXiv preprint arXiv:170105265
Koller D, Friedman N (2009) Probabilistic graphical models: principles and techniques. MIT press
Levi I (1980) The Enterprise of Knowledge: An Essay on Knowledge, Credal Probability, and Chance. MIT Press
Liang Y, Bekker J, Van den Broeck G (2017) Learning the structure of probabilistic sentential decision diagrams. In: Proceedings of the 33rd Conference on Uncertainty in Artificial Intelligence (UAI)
Libkin L (2016) Sql’s three-valued logic and certain answers. ACM Trans Database Syst 41(1):1:1–1:28, DOI 10.1145/2877206, URL http://doi.acm.org/10.1145/2877206
Mauá DD, Cozman FG, Conaty D, de Campos CP (2017) Credal sum-product networks. In: Antonucci A, Corani G, Couso I, Destercke S (eds) Proceedings of the Tenth International Symposium on Imprecise Probability: Theories and Applications, Lugano, Switzerland, 10-14 July 2017., Proceedings of Machine Learning Research, vol 62, pp 205–216, URL http://proceedings.mlr.press/v62/mau%C3%A117a.html

Mitchell TM, Cohen WW, Jr ERH, Talukdar PP, Betteridge J, Carlson A, Mishra BD, Gardner M, Kisiel B, Krishnamurthy J, Lao N, Mazaitis K, Mohamed T, Nakashole N, Plataniotis EA, Ritter A, Samadi M, Settles B, Wang RC, Wijaya D, Gupta A, Chen X, Saparov A, Greaves M, Welling J (2015) Never-ending learning. In: Bonet B, Koenig S (eds) Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence, January 25-30, 2015, Austin, Texas, USA., AAAI Press, pp 2302–2310, URL http://www.aaai.org/ocs/index.php/AAAI/AAAI15/paper/view/10049

Molina A, Vergari A, Di Mauro N, Natarajan S, Esposito F, Kersting K (2018) Mixed sum-product networks: A deep architecture for hybrid domains. In: AAAI

Nath A, Domingos PM (2015) Learning relational sum-product networks. In: Bonet B, Koenig S (eds) Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence, January 25-30, 2015, Austin, Texas, USA., AAAI Press, pp 2878–2886, URL http://www.aaai.org/ocs/index.php/AAAI/AAAI15/paper/view/10027

Poon H, Domingos PM (2012) Sum-product networks: A new deep architecture. CoRR abs/1202.3732, URL http://arxiv.org/abs/1202.3732 1202.3732

Raedt LD, Dries A, Thon I, den Broeck GV, Verbeke M (2015) Inducing probabilistic relational rules from probabilistic examples. In: Yang Q, Wooldridge M (eds) Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence, IJCAI 2015, Buenos Aires, Argentina, July 25-31, 2015, AAAI Press, pp 1835–1843, URL http://ijcai.org/Abstract/15/261

Reiter R (1977) On closed world data bases. In: Gallaire H, Minker J (eds) Logic and Data Bases, Symposium on Logic and Data Bases, Centre d'études et de recherches de Toulouse, France, 1977., Plenum Press, New York, Advances in Data Base Theory, pp 55–76

Sa CD, Ratner A, Ré C, Shin J, Wang F, Wu S, Zhang C (2017) Incremental knowledge base construction using deepdive. VLDB J 26(1):81–105, DOI 10.1007/s00778-016-0437-2, URL https://doi.org/10.1007/s00778-016-0437-2

Shafer G (1976) A Mathematical Theory of Evidence. Princeton University Press, Princeton

Socher R, Chen D, Manning CD, Ng AY (2013) Reasoning with neural tensor networks for knowledge base completion. In: Burges CJC, Bottou L, Ghahramani Z, Weinberger KQ (eds) Advances in Neural Information Processing Systems 26: 27th Annual Conference on Neural Information Processing Systems 2013. Proceedings of a meeting held December 5-8, 2013, Lake Tahoe, Nevada, United States., pp 926–934

Valiant LG (1979) The complexity of enumeration and reliability problems. SIAM Journal on Computing 8(3):410–421
Wallner A (2007) Extreme points of coherent probabilities in finite spaces. Int J Approx Reasoning 44(3):339–357, DOI 10.1016/j.ijar.2006.07.017, URL https://doi.org/10.1016/j.ijar.2006.07.017

Wang WY, Mazaitis K, Cohen WW (2013) Programming with personalized pagerank: a locally groundable first-order probabilistic logic. In: He Q, Iyengar A, Nejdl W, Pei J, Rastogi R (eds) 22nd ACM International Conference on Information and Knowledge Management, CIKM’13, San Francisco, CA, USA, October 27 - November 1, 2013, ACM, pp 2129–2138, DOI 10.1145/2505515.2505573, URL https://doi.org/10.1145/2505515.2505573

Wu W, Li H, Wang H, Zhu KQ (2012) Probase: a probabilistic taxonomy for text understanding. In: Candan KS, Chen Y, Snodgrass RT, Gravano L, Fuxman A (eds) Proceedings of the ACM SIGMOD International Conference on Management of Data, SIGMOD 2012, Scottsdale, AZ, USA, May 20-24, 2012, ACM, pp 481–492, DOI 10.1145/2213836.2213891, URL https://doi.org/10.1145/2213836.2213891

Zadeh LA (1999) Fuzzy sets as a basis for a theory of possibility. Fuzzy sets and systems 100:9–34