Reinhardt cardinals in inner models

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1 Introduction

A cardinal \( \kappa \) is Reinhardt if it is the critical point of an elementary embedding from the universe of sets to itself. Kunen [1] famously refuted the existence of Reinhardt cardinals using the Axiom of Choice (AC). It is a longstanding open problem whether Reinhardt cardinals are consistent if AC is dropped.

Noah Schweber [2] introduced the notion of a uniformly supercompact cardinal, a cardinal \( \kappa \) that is the critical point of an elementary embedding \( j : V \rightarrow M \) such that \( M^\alpha \subseteq M \) for all ordinals \( \alpha \). He posed the question of whether such a cardinal must be Reinhardt, and he also asked about the consistency strength of uniformly supercompact cardinals. Both questions remain open, but this note makes some progress on the matter.

Say a cardinal \( \kappa \) is weakly Reinhardt if it is the critical point of an elementary embedding \( j : V \rightarrow M \) such that \( j \upharpoonright P(\alpha) \in M \) for all ordinals \( \alpha \). This condition is equivalent to requiring that \( P(P(\alpha)) \subseteq M \) for all ordinals \( \alpha \). It seems to be weaker than demanding that \( M^{P(\alpha)} \subseteq M \) for all ordinals \( \alpha \).

**Theorem 2.2.** If there is a proper class of weakly Reinhardt cardinals, then there is an inner model with a proper class of Reinhardt cardinals.

As a corollary, we obtain a consistency strength lower bound for a large cardinal that looks a bit more like Schweber’s: for lack of a better term, say \( \kappa \) is ultrafilter Reinhardt if it is the critical point of an elementary embedding \( j : V \rightarrow M \) such that for all ordinals \( \alpha \), \( M^\alpha \subseteq M \) and \( \beta(\alpha) \subseteq M \). Here \( \beta(X) \) denotes the set of ultrafilters on \( X \).

**Proposition 2.3.** If \( \kappa \) is ultrafilter Reinhardt, then \( \kappa \) is weakly Reinhardt.

1.1 Preliminaries

Our background theory is von Neumann-Bernays-Gödel (NBG) set theory without AC. Even though we work without AC, for us a cardinal is an ordinal number that is not in bijection with any smaller ordinal. Of course, if AC fails, there are sets whose cardinality cannot be identified with a cardinal in this sense. Still, for any set \( Y \), one can define the Hartogs number of \( Y \), denoted by \( \aleph(Y) \), as the least cardinal \( \kappa \) such that there is no injection from \( \kappa \) to \( Y \).

2 The inner model \( N_\nu \)

If \( \nu \) is a cardinal and \( X \) is a set, \( \beta_\nu(X) \) denotes the set of \( \nu \)-complete ultrafilters on \( X \). In the context of choiceless large cardinal axioms, sufficiently complete ultrafilters on ordinals
can often be treated as “idealized ordinals.” The following lemma is a simple example of this phenomenon, although the pattern runs quite a bit deeper than this.

**Lemma 2.1.** If there is a weakly Reinhardt cardinal, then for all sufficiently large cardinals \( \nu \), for any ordinal \( \alpha \), \( \beta_\nu(\alpha) \) can be wellordered.

**Proof.** Assume not. Let \( j : V \to M \) witness that \( \kappa \) is weakly Reinhardt. By transfinite recursion, define a sequence of ordinals \( \delta_\xi \) for \( \xi \in \text{Ord} \), taking supreme at limit ordinals and, at successor stages, setting \( \delta_{\xi + 1} \) equal to the least ordinal \( \alpha > \text{ran}(\beta(\delta_\xi)) \) such that \( \beta_\delta(\alpha) \) cannot be wellordered. Let \( \epsilon_\xi = (\delta_\xi)^M \). Then \( j(\delta_\kappa) = \epsilon_{j(\kappa)} > \epsilon_{\kappa + 1} \). For each \( \gamma < \epsilon_{j(\kappa) + 1} \), let \( D_\gamma \) be the ultrafilter on \( \delta_{\kappa + 1} \) derived from \( j \) using \( \gamma \), so \( D_\gamma = \{ A \subseteq \delta_{\kappa + 1} : \gamma \in j(A) \} \).

Note that the function \( D(\gamma) = D_\gamma \) is simply definable from \( j \upharpoonright P(\delta_{\kappa + 1}) \), and so \( d \in M \). For any \( W \in \beta_{\epsilon_\kappa}(\epsilon_{j(\kappa) + 1}) \), \( D \) is constant on a set in \( W \) because \( W \) is \( \epsilon_{j(\kappa)} \)-complete and \( \text{ran}(D) \) has cardinality less than \( \epsilon_{j(\kappa)} \). Indeed, \( |\text{ran}(D)| < \aleph^M(\beta(\delta_\kappa)) \), the Hartogs numbers of \( \beta(\delta_\kappa) \) as computed in \( M \), since \( \text{ran}(D) \) is a wellorderable subset of \( \beta(\delta_\kappa) \) in \( M \). Moreover, \( \aleph^M(\beta(\delta_\kappa)) \leq \aleph^M(\beta(\epsilon_\kappa)) \) since \( \epsilon_\kappa \) is sup \( j(\delta_\kappa) \geq \delta_\kappa \), and \( \aleph^M(\beta(\epsilon_\kappa)) < \aleph_{\kappa + 1} < \epsilon_{j(\kappa)} \) by the definition of the ordinals \( \delta_\xi \) and the elementarity of \( j \).

Suppose \( U \in \beta_{\epsilon_\kappa}(\delta_{\kappa + 1}) \), and we will show that for \( j(U) \)-almost all \( \gamma \), \( D_\gamma = U \). Let \( D \) be the unique ultrafilter on \( \delta_{\kappa + 1} \) such that \( D_\gamma = D \) for \( j(U) \)-almost all \( \gamma < \epsilon_{j(\kappa) + 1} \). If \( A \subseteq U \), then for all \( \gamma \in j(A), A \in D_\gamma \), and hence for \( j(U) \)-almost all \( \gamma \), \( A \in D_\gamma \). It follows that \( A \in D \). This proves \( U \subseteq D \), and so \( U = D \). Therefore \( D \) is a surjection from the ordinal \( \epsilon_{j(\kappa) + 1} \) to \( \beta_{\epsilon_\kappa}(\delta_{\kappa + 1}) \), which contradicts that \( \beta_{\epsilon_\kappa}(\delta_{\kappa + 1}) \) cannot be wellordered. \( \square \)

Let us now define the inner model in which weakly Reinhardt cardinals become Reinhardt. Suppose \( \nu \) is a cardinal. Let \( \beta_\nu(\text{Ord}) = \bigcup_{\alpha \in \text{Ord}} \beta_\nu(\alpha) \) denote the class of \( \nu \)-complete ultrafilters on ordinals. For any class \( C \), we denote the class of all subsets of \( C \) by \( P(C) \). Finally, let

\[ N_\nu = L(P(\beta_\nu(\text{Ord}))) \]

Granting that sufficiently complete ultrafilters on ordinals are idealized ordinals, the models \( N_\nu \) are the corresponding idealizations of the inner model \( L(P(\text{Ord})) \).

**Theorem 2.2.** If there is a proper class of weakly Reinhardt cardinals, then for all sufficiently large cardinals \( \nu \), \( N_\nu \) contains a proper class of Reinhardt cardinals.

**Proof.** Let \( \nu \) be a cardinal large enough that for all ordinals \( \alpha \), \( \beta_\nu(\alpha) \) can be wellordered. Let \( N = N_\nu \). We claim that if \( \kappa > \nu \) is weakly Reinhardt, then \( \kappa \) is Reinhardt in \( N \). To see this, let \( j : V \to M \) witness that \( \kappa \) is weakly Reinhardt. We will show that \( j(N) = N \) and \( j \upharpoonright X \in N \) for all \( X \in N \). Hence \( j \upharpoonright N \) is an amenable class of \( N \) and in \( N \), \( j \upharpoonright N \) is an elementary embedding from the universe to itself. Letting \( C \) denote the collection of classes amenable to \( N \), it follows that \( (N, C) \) is a model of NBG with a proper class of Reinhardt cardinals.

We first show that \( j(N) = N \), or in other words, that \( N \) is correctly computed by \( M \). (Here we use that \( j(\nu) = \nu \) since \( \nu < \kappa \).) The closure properties of \( M \) guarantee that all ultrafilters on ordinals are in \( M \), and the elementarity of \( j \) implies that for all \( \alpha \), \( \beta_\nu(\alpha) \) is wellorderable in \( M \). Finally, since \( M \) is closed under wellordered sequences, \( P(\beta_\nu(\alpha)) \) is contained in \( M \). This implies that \( N \) is correctly computed by \( M \).

Finally, we show that for any \( X \in N \), \( j \upharpoonright X \in N \). For this, it suffices to show that for any ordinal \( \alpha \), \( j \upharpoonright P(\beta_\nu(\alpha)) \) is in \( N \). Since \( \beta_\nu(\alpha) \) is wellorderable in \( N \), it suffices to show
that \( j \upharpoonright P(\delta) \) belongs to \( N \) where \( \delta = |\beta(\alpha)|^N \). Then letting \( f : P(\beta(\alpha)) \to P(\delta) \) be a bijection in \( N \),
\[
j \upharpoonright P(\beta(\alpha)) = j(f)^{-1} \circ (j \upharpoonright P(\delta)) \circ f
\]
and \( j(f) \in N \) since \( N = j(N) \) by the previous paragraph. But \( j \upharpoonright P(\delta) \in N \) because it is encoded by the extender \( E = \{ D_\gamma : \gamma < j(\delta) \} \) where \( D_\gamma \) is the ultrafilter on \( \delta \) derived from \( j \) using \( \gamma \): indeed, if \( A \subseteq \delta \), then \( j(A) = \{ \gamma < j(\delta) : A \in D_\gamma \} \). Since \( E \) is a wellordered sequence of \( \nu \)-complete ultrafilters, \( E \in N \). \( \square \)

We now show that ultrafilter Reinhardt cardinals are weakly Reinhardt, so the same consistency strength lower bound applies to them.

**Proposition 2.3.** If \( \kappa \) is ultrafilter Reinhardt, then \( \kappa \) is weakly Reinhardt.

**Proof.** Suppose \( j : V \to M \) is elementary and for all ordinals \( \alpha \), \( M^\alpha \subseteq M \) and \( \beta(\alpha) \subseteq M \). We claim that for all ordinals \( \delta, j \upharpoonright P(\delta) \in M \). Consider the extender \( E = \{ D_\gamma : \gamma < j(\delta) \} \) given by letting \( D_\gamma = \{ A \subseteq \delta : \gamma \in j(A) \} \) be the ultrafilter derived from \( j \) using \( \gamma \). Then \( E \in M \), and hence \( j \upharpoonright P(\delta) \in M \), since for \( A \in P(\delta) \), \( j(A) = \{ \gamma < j(\delta) : A \in D_\gamma \} \). \( \square \)

Finally, observe that the inner models considered here yield models with Reinhardt cardinals that are a bit tamer than one might expect:

**Proposition 2.4.** If there is a proper class of Reinhardt cardinals, then there is an inner model with a proper class of Reinhardt cardinals in which every set is constructible from a wellordered sequence of ultrafilters on ordinals. \( \square \)

**Corollary 2.5.** If the existence of a proper class of Reinhardt cardinals is consistent, then it is consistent with \( V = L(P(\text{Ord})) \).

**Proof.** For any cardinal \( \lambda \), a \( \lambda \)-sequence \( \langle S_\alpha : \alpha < \lambda \rangle \) of subsets of \( P(\lambda) \) can be coded by a single subset of \( P(\lambda \times \lambda) \); namely, \( \{ \{ \alpha \} \times A : A \in S_\alpha \} \). So if every set is constructible from a wellordered sequence of ultrafilters on ordinals, then \( V = L(P(\text{Ord})) \). \( \square \)

More advanced techniques yield the following theorem, whose proof is omitted:

**Theorem 2.6.** If there is a Reinhardt cardinal, then for a closed unbounded class of cardinals \( \nu \), there is a Reinhardt cardinal in \( N_\nu(V_{\nu+1}) \). \( \square \)

Here \( N_\nu(V_{\nu+1}) \) is the smallest inner model \( N \) such that \( P(\beta(\text{Ord})) \cup V_{\nu+1} \subseteq N \). We also note that the proofs here easily generalize to show that if there is a proper class of Berkeley cardinals, then for sufficiently large \( \nu \), there is a proper class of Berkeley cardinals in \( N_\nu \).

## 3 Questions

Let us list some variants of Schweber’s original questions that seem natural given the results of this note.

**Question 3.1.** Are Reinhardt cardinals and weakly Reinhardt cardinals equiconsistent?

**Question 3.2.** Is the existence of a Reinhardt cardinal compatible with \( V = L(P(\text{Ord})) \)?
In the context of NBG, a cardinal $\kappa$ is Ord-\textit{supercompact} if for all ordinals $\alpha$, there is an elementary embedding $j : V \rightarrow M$ such that $j(\kappa) > \alpha$ and $M^\alpha \subseteq M$.

\textbf{Question 3.3.} Is NBG plus the existence of a proper class of Ord-supercompact cardinals equiconsistent with ZFC plus a proper class of supercompact cardinals?

\textbf{References}

[1] Kenneth Kunen. Elementary embeddings and infinitary combinatorics. \textit{J. Symbolic Logic}, 36:407–413, 1971.

[2] Noah Schweber. Supercompact and Reinhardt cardinals without choice. MathOverflow.