Polarization modes of gravitational wave for viable \( f(R) \) models

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Received: 16 June 2017 / Accepted: 8 November 2017 / Published online: 16 November 2017
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Abstract In this paper, we study the gravitational wave polarization modes for some particular \( f(R) \) models using Newman-Penrose formalism. We find two extra scalar modes of gravitational wave (longitudinal and transversal modes) in addition to two tensor modes of general relativity. We conclude that gravitational waves correspond to class \( II_6 \) under the Lorentz-invariant \( E(2) \) classification of plane null waves for these \( f(R) \) models.

Keywords \( f(R) \) gravity · Gravitational wave polarizations

1 Introduction

Gravitational waves (GWs) are fluctuations in the fabric of spacetime produced by the motion of massive celestial objects. The scientific curiosity and struggles to detect these waves by the Earth based detectors lead to the invention of laser interferometer detectors such as LIGO, VIRGO, GEO and LISA (Bassan 2014). The most promising source for these detectors is merging the compact binaries composed of neutron star-neutron star, neutron star-black hole and black hole-black hole. These orbiting systems loose their energy in the form of GWs which speed up their orbital motion and this process ends up at the merging of orbiting objects. Recently, LIGO scientific and Virgo collaborations (Abbott et al. 2016) detected these waves and provided two observational evidences (with signals known as GW150914 and GW151226) for GWs each of which is the result of a pair of colliding black holes.

Polarization of a wave gives information about the geometrical orientation of oscillations. A common method to discuss polarization modes (PMs) of GWs is the linearized theory consisting of metric perturbations around Minkowski background. Newman and Penrose (1962) introduced tetrad and spinor formalism in general relativity (GR) to deal with radiation theory. Eardley et al. (1973) used this formalism for linearized gravity and showed that six Newman-Penrose (NP) parameters for plane null waves represent six polarization modes (amplitudes) of these GWs. They also introduced Lorentz-invariant \( E(2) \) classification of plane null waves.

Hawking (1971) found an upper bound for the energy of gravitational radiation emitted by the collision of two black holes. Wagoner (1984) investigated gravitational radiation emitted by accreting neutron stars. Cutler and Flanagan (1994) explored the extent of accuracy of the distance to source and masses as well as spin of two bodies measured by the detectors LIGO and VIRGO from the gravitational wave signal. Turner (1997) worked on GWs produced by inflation and discussed the potential of cosmic microwave background anisotropy as well as laser interferometers (LIGO, VIRGO, GEO and LISA) for the detection of GWs. Langlois et al. (2000) studied the evolution of GWs for a brane embedded in five-dimensional anti-de Sitter universe and showed that a discrete normalizable massless graviton mode exists during slow roll inflation.

Recent indications of accelerated expansion of the universe caused by dark energy introduced much interest in cosmology. The mysteries of dark energy and dark matter (invisible matter) leads to modified theories of gravity obtained by either modifying matter part or geometric part of the Einstein-Hilbert action. A direct generalization of GR is the \( f(R) \) theory in which the Ricci scalar \( R \) in
the Einstein-Hilbert action is replaced by its generic function $f(R)$. De Felice and Tsujikawa (2010) presented a comprehensive study on various applications of $f(R)$ theory to cosmology and astrophysics. Starobinsky (1980) proposed the first inflationary model in $f(R)$ gravity compatible with anisotropies of cosmic microwave background radiation. Hu and Sawicki (2007) proposed a class of $f(R)$ models without cosmological constant that satisfy cosmological and solar system tests for small field limit of the parameter space. Tsujikawa (2008) explored observational consequences of $f(R)$ models that satisfy the local gravity constraints. Bamba et al. (2010) introduced $f(R)$ model which explains inflation and late cosmic expansion at the same time.

A lot of work has been done for PMs of GWs in $f(R)$ as well as in other modified theories. Capozziello et al. (2008) investigated PMs of GWs in $f(R)$ gravity and concluded that for every $f(R)$ model there is an extra mode than GR called massive longitudinal mode. They also worked out the response function of GWs with LISA. Alves et al. (2009) discussed PMs of GWs for particular $f(R)$ model concluding the same results. They showed that five non-zero PMs exist for a specific form of quadratic gravity. The topic of gravitational radiation for linearized $f(R)$ theory has also been discussed in literature Berry and Gair (2011), Näf and Jetzer (2011). Capozziello and Stabile (2015) studied GWs in the context of general fourth order gravity and discussed the states of polarization and helicity. Kausar et al. (2016) found that for any $f(R)$ model there are two extra modes as compared to GR. Alves et al. (2016) explored these modes in $f(R, T)$ as well as $f(R, T^\phi)$ theories concluding that the earlier one reduces to $f(R)$ in vacuum while PMs for the later depend on the expression of $T^\phi$.

Herrera et al. (2015a, 2015b) studied the presence of gravitational radiation in GR for perfect as well as dissipative dust fluid with axial symmetry using super-Poynting vector and showed that both fluids do not produce gravitational radiation. We have investigated that the axial dissipative dust acts as a source of gravitational radiation in $f(R)$ theory (Sharif and Siddiqa 2017). This paper is devoted to find PMs for some viable dark energy models of this gravity. The paper is organized as follows. In the next section, we write down field equations and dark energy models of $f(R)$ gravity. We then find PMs of GWs for three models in its subsections. Finally, we conclude our results.

## 2 Dark energy models in $f(R)$ gravity

The $f(R)$ gravity action is defined as

$$S = \frac{1}{16\pi} \int \sqrt{-g} f(R) \, d^4x + S_M,$$

where $S_M = \int \sqrt{-g} L_M \, d^4x$ denotes the matter action and $L_M$ represents the matter Lagrangian. To discuss PMs of GWs, one needs to investigate the linearized far field vacuum field equations. The vacuum field equations for the action (1) are given by

$$F(R) R_{\beta\gamma} - \frac{1}{2} f(R) g_{\beta\gamma} - \nabla_\beta \nabla_\gamma F(R) + g_{\beta\gamma} \Box F(R) = 0,$$

where $F = \frac{df}{dR}$ and $\Box = \nabla^2$ is the D’Alembertian operator. The trace of this equation is

$$RF(R) - 2f(R) + 3\Box F(R) = 0.$$

We assume that waves are traveling in $z$-direction, i.e., each quantity can be a function of $z$ and $t$.

Various models of $f(R)$ gravity have been proposed in literature Starobinsky (1980), Hu and Sawicki (2007), Tsujikawa (2008), Bamba et al. (2010) describing the phenomena of early inflation and late cosmic expansion. The model proposed by Hu and Sawicki (2007) is reduced to the model considered by Alves et al. (2009) in the weak field regime (i.e., when $R \ll m^2$, $m$ stands for mass). Thus Hu and Sawicki model which satisfies the cosmological and solar system tests has been indeed examined for PMs of GWs in the low curvature case or Minkowski background. Similarly, the Starobinsky model having consistency with the temperature anisotropies measured by CMBR (De Felice and Tsujikawa 2010) has also been analyzed for PMs of GWs by Kausar et al. (2016).

Amendola et al. (2007) derived the conditions for cosmological viability of some dark energy models in $f(R)$ gravity. They divided $f(R)$ models into four classes according to the existence of a matter dominated era and the final accelerated expansion phase or geometrical properties of the $m(r)$ curves where $m(r) = \frac{Rf}{R^2 \frac{df}{dR}}$. They concluded that models of class I are not physical, class II models asymptotically approach to de Sitter universe, class III contains models showing strongly phantom era and models of class IV represent non-phantom acceleration ($\omega > -1$). They argued that only models belonging to class II are observationally acceptable with the final outcome of $\Lambda$CDM model. Here we consider these observationally acceptable models having the similar geometry of $m(r)$ curves to discuss the PMs of GW. There are four models among the considered models that fall in class II while the model $R + \alpha R^{-n}$ has already been discussed by Alves et al. (2009) so we discuss the remaining three models in this paper.

### 2.1 Polarization modes for $f(R) = R + \xi R^2 - \Lambda$

We consider the model $f(R) = R + \xi R^2 - \Lambda$, it is assumed that $\xi$ (an arbitrary constant) and $\Lambda$ (cosmological constant)
have positive values (Amendola et al. 2007). This model corresponds to $\Lambda$CDM model in the limit $\xi \to 0$ and Starobinsky inflationary model for $\Lambda \to 0$. In this case, Eq. (3) yields
\[ R(1 + 2\xi R) - 2(R + \xi R^2 - \Lambda) + 3\Box(1 + 2\xi R) = 0, \] (4)
which on simplification gives
\[ \Box R - \frac{1}{6\xi} R = -\frac{\Lambda}{3\xi}. \] (5)

For the sake of simplicity, we consider gravitational waves moving in one direction, i.e., $z$-direction. Thus Eq. (5) can be interpreted as a non-homogeneous two-dimensional wave equation or Klein-Gordon equation and its solution can be found using different methods like Fourier transform and Green’s function etc. Here we obtain its solution following the technique used to solve Klein-Gordon and Sine-Gordon equations given in Rajaraman (1982) which is simple as compared to other methods. According to this method, any static solution is a wave with zero velocity and for the systems with Lorentz invariance, once a static solution is known, moving solutions are trivially obtained by boosting, i.e., transforming to a moving coordinate frame. Since we are considering the vacuum field equations and the background metric is Minkowski, so we can apply Lorentz transformations to the Ricci scalar $R$ (a Lorentz invariant quantity). Hence static solution of Eq. (5) is obtained by solving
\[ \frac{d^2 R}{dz^2} - \frac{1}{6\xi} R = -\frac{\Lambda}{3\xi}, \] (6)
whose solution is
\[ R(z) = c_1 e^{\sqrt{\frac{m}{v^2} + c_2}} e^{-\sqrt{\frac{m}{v^2}}} + 2\Lambda, \] (7)
where $c_1$, $c_2$ are constants of integration and $m = \frac{1}{6\xi}$.

Since our system is Lorentz invariant, the time dependent solution is obtained from the static solution through Lorentz transformation as
\[ R(z, t) = c_1 e^{\sqrt{\frac{m}{v^2}}} e^{-\sqrt{\frac{m}{v^2}} + 2\Lambda}, \] (8)
where $\sqrt{1 - \xi^2}$ is the Lorentz factor and $v$ represents the velocity of wave propagation. Also, Eq. (2) can be rewritten as
\[ R_{\beta\gamma} = \frac{1}{F(R)} \left[ \frac{1}{2} f(R) g_{\beta\gamma} + \nabla_{\beta} \nabla_{\gamma} F(R) - g_{\beta\gamma} \Box F(R) \right]. \] (9)
Replacing the values of $f(R)$ and $F(R)$, we obtain its linearized form as
\[ R_{\beta\gamma} = \frac{1}{2}(R - \Lambda + 2\xi \Lambda R) g_{\beta\gamma} + 2\xi \nabla_{\beta} \nabla_{\gamma} R - 2\xi \Delta g_{\beta\gamma} \Box R. \] (10)

The non-zero components of Ricci tensor are
\[ R_{tt} = \frac{1}{6(1 - v^2)} \left[ 3v^2(R - \Lambda) - (R + 3\Lambda) \right] - \frac{\xi}{6} R, \] (11)
\[ R_{xx} = \frac{1}{6} (R + \Lambda) + \xi \Lambda R = R_{yy}, \] (12)
\[ R_{zz} = \frac{1}{6(1 - v^2)} \left[ 3(R - \Lambda) - v^2(R + \Lambda) \right] + \xi \Lambda R, \] (13)
\[ R_{iz} = -\frac{vR}{3(1 - v^2)}. \] (14)

With the help of Eqs. (47) and (48), we have
\[ \Psi_2 = \frac{1}{12} R, \quad \Psi_3 = \frac{1}{2} R_{\mu\nu}, \quad \Phi_{22} = -\frac{1}{2} R_{tt}. \] (15)

Now, we find the expressions of $\Psi_3$ and $\Phi_{22}$ using Eq. (46). For $\Psi_3$, it yields
\[ \Psi_3 = \frac{1}{2} R_{\mu\nu} = \frac{1}{2} R_{\mu\nu} l^\mu m^\nu, \] (16)
which can also be written as
\[ \Psi_3 = \frac{1}{2} \left( R_{tt} l^t m^t + R_{xx} l^x m^x + R_{yy} l^y m^y + R_{zz} l^z m^z \right. \]
\[ \left. + R_{iz} l^i m^z \right) \] (17)

From Eqs. (43) and (44), the component form of vectors $l$, $m$ and $\bar{m}$ can be written as
\[ k^\mu = \frac{1}{\sqrt{2}} (1, 0, 0, 1), \quad l^\mu = \frac{1}{\sqrt{2}} (1, 0, 0, -1), \] (18)
\[ m^\mu = \frac{1}{\sqrt{2}} (0, 1, i, 0), \quad \bar{m}^\mu = \frac{1}{\sqrt{2}} (0, 1, -i, 0). \] (19)

Substituting all the required values in Eq. (17), we obtain
\[ \Psi_3 = 0. \] (20)

Similarly, Eq. (46) for $\Phi_{22}$ yields
\[ \Phi_{22} = -\frac{1}{2} R_{tt} = -\frac{1}{2} R_{\mu\nu} l^\mu l^\nu \]
\[ = -\frac{1}{2} \left( R_{tt} l^t l^t + 2R_{xz} l^x l^z + R_{zz} l^z l^z \right). \]
Replacing the Ricci tensor components and components of $l^\mu$, the above equation leads to
\[ \Phi_{22} = -\frac{R}{12} \left( \frac{1 + v}{1 - v} \right) + \frac{\Lambda (2v^2 + 3)}{12(1 - v^2)}. \] (21)

Notice that $\Psi_4 \neq 0$ represents the tensor modes of GWs. Since there is no expression of $\Psi_4$ in terms of Ricci tensor, so it cannot be evaluated with the help of available values of Ricci tensor and Ricci scalar (Alves et al. 2016). It can be
observed that for $Λ$CDM model (when $ξ → 0$) $Ψ_2$ and $Φ_{22}$ remain non-zero.

The model, $f(R) = R + \xi R^2 - \Lambda$, is always viable and reduces to GR when both $ξ$ as well as $\Lambda$ approach to zero. In GR, there are only two tensor modes of polarization associated with $\Re Ψ_4$ and $\Im Ψ_4$, i.e., we have only $Ψ_4$ non-zero among six NP parameters. From Eq. (4), we have $R = 0$ for $ξ → 0$, $Λ → 0$, hence GR results are retrieved.

### 2.2 Polarization modes for $f(R) = R^p (\ln α R)^q$

This model is observationally acceptable for $p = 1$ and $q > 0$ (Amendola et al. 2007). Here we assume that $q = 1$ such that the model becomes $f(R) = R \ln α R$. Substituting the values of $f(R)$ and $F(R)$ in Eq. (3), it gives

\[ 3\Box \ln α R - R \ln α R + R = 0. \]  

(22)

Assuming $\ln α R = φ$, this equation transforms to

\[ \Box φ = \frac{e^φ}{3α} (φ - 1), \]  

(23)

which can also be written as

\[ \Box φ = \frac{∂U}{∂φ}; \quad U(φ) = \frac{e^φ}{3α} (φ - 2). \]  

(24)

First we seek for a static solution, i.e., consider $φ = φ(z)$ such that integration of Eq. (24) gives

\[ \frac{1}{2} \left( \frac{dφ}{dz} \right)^2 = U(φ). \]  

(25)

Substituting the value of $U(φ)$ and then integrating, it follows that

\[ φ(z) = 2 \left[ 1 + \text{InverseErf} \left( \frac{ez}{\sqrt{3απ}} + \frac{ec^3}{\sqrt{2απ}} \right) \right]. \]  

(26)

where $c_3$ is a constant of integration, $e = 2.71828$ and Erf is defined by

\[ \text{Erf}(z) = \frac{2}{\sqrt{π}} \int_0^z e^{-s^2} ds. \]  

(27)

Using Lorentz transformation, we obtain time dependent solution given by

\[ φ(z, t) = 2 \left[ 1 + \text{InverseErf} \left( \frac{e(z - vt)}{\sqrt{1 - v^2} \sqrt{3απ}} + \frac{ec^3}{\sqrt{2απ}} \right) \right]. \]  

(28)

The expression for Ricci scalar is obtained as

\[ R(z, t) = \frac{1}{α} \exp \left( 2 \left[ 1 + \text{InverseErf} \left( \frac{e(z - vt)}{\sqrt{1 - v^2} \sqrt{3απ}} + \frac{ec^3}{\sqrt{2απ}} \right) \right] \right). \]  

(29)

The non-zero components of the Ricci tensor have the form

\[ R_{tt} = -\frac{R}{6(v^2 - 1)} \left[ (3v^2 - 1) \ln α R - 2 \right]. \]  

\[ R_{sx} = \frac{R}{6} \left( \ln α R + 2 \right) = R_{yy}, \]  

\[ R_{iz} = \frac{Rv(\ln α R - 1)}{3(v^2 - 1)(\ln α R + 1)}, \]  

\[ R_{zz} = \frac{R}{6(v^2 - 1)} \left[ (v^2 - 3) \ln α R + 2v^2 \right]. \]

Finally, the NP parameters for this case are

\[ Ψ_2 = \frac{1}{12} R, \quad Ψ_3 = 0, \]  

(30)

\[ Ψ_{22} = -\frac{R}{12} \left( 1 + \frac{v}{1 - v} \right) \ln α R + 1. \]

Here Ψ₄ is also a non-vanishing NP parameter as discussed in the previous case.

### 2.3 Polarization modes for $f(R) = R^p e^{qR}$

This model is observationally acceptable for $p = 1$, so we take $f(R) = Re^{qR}$ (Amendola et al. 2007). This model reduces to GR when $q = 0$ and consequently gives no additional PMs. Thus to find extra PMs, we consider $q ≠ 0$ in further calculations. For this model, the trace equation (3) becomes

\[ \text{Re}^q \left( 1 - \frac{q}{R} \right) - 2 \text{Re}^q + 3\Box \left( e^{qR} \left( 1 - \frac{q}{R} \right) \right) = 0. \]  

(31)

In low curvature regime, we have $R ≪ q$ which reduces the above equation to the following

\[ \Box \left( \frac{1}{R} e^{qR} \right) + \frac{1}{3} e^{qR} = 0. \]  

(32)

Replacing $\frac{1}{R} = u$ and $u = u(z)$ for static solution, we obtain

\[ \frac{d^2}{dz^2} (ue^{qu}) + \frac{1}{3} e^{qu} = 0. \]  

(33)

Solving the double derivative of the above equation, it becomes

\[ (1 + qu) \frac{d^2 u}{dz^2} + q(qu + 2) \left( \frac{du}{dz} \right)^2 + \frac{1}{3} = 0. \]

The is a non-homogeneous non-linear second order differential equation and does not provide an exact analytic solution unless we make some assumptions to simplify it. Since we are working in the weak-field regime, so $R$ is very small. Assuming $q$ to be very large, we have $qu = \frac{q}{R}$ (as $u = \frac{1}{R}$) to be
very large such that $(qu + 1) \approx qu$ as well as $(qu + 2) \approx qu$ and the above equation reduces to

$$u \frac{d^2 u}{dz^2} + qu \left( \frac{du}{dz} \right)^2 + \frac{1}{3q} = 0.$$ 

Here $\frac{1}{3q} \rightarrow 0$ as $q$ is very large, hence it reduces to

$$u \frac{d^2 u}{dz^2} + qu \left( \frac{du}{dz} \right)^2 = 0$$

whose solution yields $(u = 1/R)$

$$R(z) = \left( \frac{1}{q} \ln \left[ q \left( c_4 z + c_5 \right) \right] \right)^{-1},$$

where $c_4$ and $c_5$ are integration constants. The non-static solution becomes

$$R(z, t) = \left( \frac{1}{q} \ln \left[ q \left( c_4 \left( \frac{z - vt}{1 - v^2} \right) + c_5 \right) \right] \right)^{-1}.$$ 

The non-vanishing components of the Ricci tensor are

$$R_{tt} = \frac{q[q(z - vt)^2 + 2\ln[q(c_4 \left( \frac{z - vt}{1 - v^2} \right) + c_5)]c_4^2}{2[\ln[q(c_4 \left( \frac{z - vt}{1 - v^2} \right) + c_5)]^2((tv - z)c_4 - \sqrt{1 - v^2}c_5)^2}

+ \frac{q[2q\sqrt{1 - v^2}(z - vt)c_4c_5 - qc_4^2(v^2 - 1)]}{2[\ln[q(c_4 \left( \frac{z - vt}{1 - v^2} \right) + c_5)]^2((tv - z)c_4 - \sqrt{1 - v^2}c_5)^2},$$

$$R_{xx} = R_{yy} = \frac{-q[q(z - vt)^2 - 2(v^2 - 1)\ln[q(c_4 \left( \frac{z - vt}{1 - v^2} \right) + c_5)]c_4^2}{2[\ln[q(c_4 \left( \frac{z - vt}{1 - v^2} \right) + c_5)]^2((tv - z)c_4 - \sqrt{1 - v^2}c_5)^2}

- \frac{q[2q\sqrt{1 - v^2}(z - vt)c_4c_5 - qc_4^2(v^2 - 1)]}{2[\ln[q(c_4 \left( \frac{z - vt}{1 - v^2} \right) + c_5)]^2((tv - z)c_4 - \sqrt{1 - v^2}c_5)^2}. $$

$$R_{zz} = \frac{-q[q(z - vt)^2 - 2v^2\ln[q(c_4 \left( \frac{z - vt}{1 - v^2} \right) + c_5)]c_4^2}{2[\ln[q(c_4 \left( \frac{z - vt}{1 - v^2} \right) + c_5)]^2((tv - z)c_4 - \sqrt{1 - v^2}c_5)^2}

- \frac{q[2q\sqrt{1 - v^2}(z - vt)c_4c_5 - q^2(v^2 - 1)]}{2[\ln[q(c_4 \left( \frac{z - vt}{1 - v^2} \right) + c_5)]^2((tv - z)c_4 - \sqrt{1 - v^2}c_5)^2},$$

$$R_{zz} = \frac{-q\frac{vc^2}{1 - v^2}}{(1 - v^2)(c_4 \left( \frac{z - vt}{1 - v^2} \right) + c_5).$$

The corresponding NP parameters are

$$\Psi_2 = \frac{1}{12} R,$$ 

$$\Psi_3 = 0,$$ 

$$\Psi_{22} = -\frac{Rc_4^2(v - 1)^2}{4((tv - z)c_4 - \sqrt{1 - v^2}c_5)^2},$$ 

$$\Psi_4$$ is also non-zero.
pipelines and analysis techniques for the detection of future GW events are continuously made for accurate measurements. Recently, two more events of GWs, GW170104 (Abbott et al. 2017a) and GW170817 (Abbott et al. 2017b) have been detected by the advanced interferometers. The event GW170104 is consistent with merging black holes of masses 31 $M_\odot$ and 19 $M_\odot$ in GR while the second one GW170817 is consistent with the binary neutron star inspiral having masses in the range 1.17 $M_\odot$–1.60 $M_\odot$. The signal GW170817, has the association with GRB170817A detected by Fermi-GBM and provides the first direct evidence of a link between these mergers and short γ-ray bursts. It is expected that future GW observations made by a network of the Earth based interferometers could actually measure the polarization of GWs and thus constrain $f(R)$ deviations from GR.

Acknowledgements We would like to thank the Higher Education Commission, Islamabad, Pakistan for its financial support through the Indigenous Ph.D. 5000 Fellowship Program Phase-II, Batch-III. We are also grateful to the anonymous referee for his constructive comments.

Appendix

In this appendix, we first briefly describe the Newman-Penrose formalism (Newman and Penrose 1962) to discuss gravitational waves and then PMs as well as classification of null waves is developed (Eardley et al. 1973).

Newman and Penrose developed a new technique in GR with the help of tetrad formalism and applied this to resolve the issue of outgoing gravitational radiation. They defined the following relations between the Cartesian ($\hat{t}, \hat{x}, \hat{y}, \hat{z}$) and null-tetrads ($k, l, m, \tilde{m}$)

\[ k = \frac{1}{\sqrt{2}}(\hat{t} + \hat{z}), \quad l = \frac{1}{\sqrt{2}}(\hat{t} - \hat{z}), \]

\[ m = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{y}), \quad \tilde{m} = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{y}), \]

which satisfy the relations

\[ -k.l = m.\tilde{m} = 1, \quad k.m = k.\tilde{m} = l.m = l.\tilde{m} = 0. \]

Any tensor can be transformed from Cartesian to null basis by the formula (Alves et al. 2009)

\[ S_{abce} = S_{a\beta\gamma\alpha} a^\alpha b^\beta c^\gamma, \]

where ($a, b, c, \ldots$) vary over the set {$k, l, m, \tilde{m}$} and ($\alpha, \beta, \gamma, \ldots$) vary over the set {$t, x, y, z$}. In Newman and Penrose (1962), the irreducible parts of the Riemann tensor, also called the NP parameters, are defined by ten $\Psi$'s, nine $\Phi$'s and a term $\Lambda$ (these are all algebraically independent). Eardley et al. (1973) showed that for plane null waves (due to differential and symmetry properties of the Riemann tensor) these NP quantities are reduced to the set \{ $\Psi_2$, $\Psi_3$, $\Psi_4$, $\Phi_{22}$ \}. This set consists of six NP parameters or PMs because $\Psi_3$ and $\Psi_4$ are complex and thus represent two independent modes. They also give formulas of these NP quantities in terms of null-tetrad components of the Riemann tensor as

\[ \Psi_2 = -\frac{1}{6} R_{lkkl}, \quad \Psi_3 = -\frac{1}{2} R_{lkln}, \quad \Phi_{22} = -R_{lm\tilde{m}n}. \]  

(47)

Following are some helpful relations of null-tetrad components of the Riemann and Ricci tensors

\[ R_{lk} = R_{lkkl}, \quad R_{ll} = 2R_{lm\tilde{m}n}, \]

\[ R_{lm} = R_{lklm}, \quad R_{ln} = R_{kl\tilde{m}n}, \quad R = -2R_{lk}. \]  

(48)

The classification of weak plane null waves (Eardley et al. 1973) obtained for standard observer (i.e., each observer sees the waves traveling in $z$-direction and each observer measures the same frequency) is given in Table 1.

| Classes | Condition for NP parameters |
|---------|-----------------------------|
| $H_6$   | $\Psi_2 \neq 0$             |
| $H_5$   | $\Psi_2 = 0$ and $\Psi_3 \neq 0$ |
| $N_3$   | $\Psi_2 = 0 = \Psi_3$, $\Psi_4 \neq 0$ and $\Phi_{22} \neq 0$ |
| $N_2$   | $\Psi_2 = 0 = \Psi_3$, $\Phi_{22} \neq 0$ and $\Psi_4 \neq 0$ |
| $O_1$   | $\Psi_2 = 0 = \Psi_3$, $\Psi_4 \neq 0$ and $\Phi_{22} \neq 0$ |
| $O_0$   | $\Psi_2 = 0 = \Psi_3 = \Phi_{22} = \Psi_4$ |

Table 1 The E(2) classes of weak plane null waves

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