Superconformal index of $\mathcal{N} = 3$ orientifold theories

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Abstract

We analyze the superconformal index of the $\mathcal{N} = 3$ supersymmetric generalized orientifold theories recently proposed. In the large $N$ limit we derive the index from the Kaluza–Klein modes in $\text{AdS}_5 \times S^5/\mathbb{Z}_k$, which are obtained from ones in $\text{AdS}_5 \times S^5$ by a simple $\mathbb{Z}_k$ projection. For the ordinary $\mathbb{Z}_2$ orientifold case the agreement with the gauge theory calculation is explicitly confirmed, and for $\mathbb{Z}_{k \geq 3}$ we perform a few consistency checks with known results for $\mathcal{N} = 3$ theories. We also study finite $N$ corrections by analyzing wrapped D3-branes and discrete torsions in the dual geometry.

Keywords: SUSY gauge theory, superconformal index, AdS/CFT correspondence, superconformal field theory

1. Introduction

Various four-dimensional superconformal field theories have been extensively studied for decades, with the exception of one with just twelve supercharges. This is because the $\mathcal{N} = 3$ vector multiplet is the only free multiplet of $\mathcal{N} = 3$ supersymmetry. It has the same field contents as those of the $\mathcal{N} = 4$ vector multiplet, and thus an $\mathcal{N} = 3$ theory is necessarily enhanced to the $\mathcal{N} = 4$ one as long as it has perturbative description. From the viewpoint of AdS/CFT correspondence [1], non-trivial $\mathcal{N} = 3$ superconformal field theories have been expected to exist from the analysis of type IIB supergravity solution [2]. They have shown that an $\text{AdS}_5$ solution with such superconformal symmetry can be constructed by appropriate truncation from a maximally supersymmetric one, and also implied that some non-perturbative symmetry is mandatory to realize a genuine $\mathcal{N} = 3$ theory. (See [3] for a recent study of such a supergravity solution.)

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A detailed realization of such genuine $\mathcal{N} = 3$ theories has recently been completed in [4]. Starting with the D3-brane realization of the $\mathcal{N} = 4$ $U(N)$ supersymmetric Yang–Mills theory (SYM), the authors reduced its supersymmetry by a simple orbifold projection at special points of the marginal coupling $\tau$. For a generic value of $\tau$ a $\mathbb{Z}_k$ subgroup of the $SL(2, \mathbb{Z})$ Montonen–Olive duality of the parent $\mathcal{N} = 4$ theory is a perturbative symmetry, and the orbifold by it leads to the usual orientifold theories. However, the symmetry is enhanced to $\mathbb{Z}_k$ with $k = 3, 4, 6$ at some special values of $\tau$, and we can define new orbifolds, which we refer to as $\mathbb{Z}_k$ orientifolds. In the F-theoretic description, the $\mathbb{Z}_k$ symmetry can be interpreted as the rotation of the internal torus.

A general analysis of genuine $\mathcal{N} = 3$ theories that does not rely on explicit construction of the theory was performed in [5] by using $\mathcal{N} = 3$ superconformal symmetry and its representations. They proved that a genuine $\mathcal{N} = 3$ theory does not have exactly marginal deformations. This is consistent with the fact that they can be defined only at special values of $\tau$. This was also confirmed in [6], in which a comprehensive analysis of supersymmetric deformations in superconformal theories was given. Another important property is that the dimension of Coulomb branch operators must be integer greater than 2. This was confirmed in rank-1 $\mathcal{N} = 3$ theories in [7, 8].

A purpose of this paper is to analyze the $\mathbb{Z}_k$ orientifold theories defined in [4] from their holographic duals. Namely, type IIB theory in $AdS_5 \times S^5/\mathbb{Z}_k$ with $k = 3, 4, 6$. We first derive the superconformal index of such generalized orientifold theories in the large $N$ limit by means of Kaluza–Klein analysis. Fortunately, the $SL(2, \mathbb{Z})$ symmetry is realized as a classical symmetry in type IIB supergravity, which facilitates specification of the orientifold. Since the Kaluza-Kline modes in $AdS_5 \times S^5$ were already examined in [9], it is straightforward to calculate the index of those on the orientifold by a simple projection. We check that the index determined by the truncation precisely reproduces that of the corresponding gauge theory when $k = 2$, in which case the gauge theory is $\mathcal{N} = 4$ SYM with orthogonal or symplectic gauge group, and its large $N$ index is computable.

Then we move on to analysis on finite $N$ corrections of the index. In the perturbative cases with $k = 1$ and 2 the leading correction to the superconformal index in the finite $N$ theory comes from wrapped D3-branes. If they wrap topologically trivial cycles they are called giant or dual-giant gravitons, and give negative contribution to the index. If the cycle is topologically non-trivial they contribute positively to the index. We assume this relation persists in general cases and determine which type of wrapped branes gives the leading correction to the index by analyzing the discrete torsion and the three-cycle homology.

2. $\mathbb{Z}_k$ orientifolds

The orientifold construction of $\mathcal{N} = 3$ theories in [4] starts from the type IIB realization of the $\mathcal{N} = 4$ supersymmetric $U(N)$ Yang–Mills theory. We realize it as the theory on a stack of $N$ D3-branes placed in the flat ten-dimensional space–time. It possesses the superconformal symmetry $PSU(2, 2|4)$, whose bosonic subgroup is

$$SU(2, 2) \times SU(4)_{R} \subset PSU(2, 2|4).$$

The $R$-symmetry, $SU(4)_{R}$, is the rotation of $\mathbb{R}^6$ transverse to the D3-branes. The four supercharges $Q_{I}$ $(I = 1, 2, 3, 4)$ with positive chirality belong to the fundamental representation of $SU(4)_{R}$. We represent $SU(4)_{R}$ irreducible representations by using integral Dynkin labels $(r_1, r_2, r_3)$. The integers $r_{\alpha}$ are eigenvalues of Cartan generators $R_{\alpha}$, whose action on $Q_{I}$ are
\[ R_1 = (1, -1, 0, 0), \quad R_2 = (0, 1, -1, 0), \quad R_3 = (0, 0, 1, -1). \]  
(2.2)

(We mean by \((\lambda_1, \lambda_2, \lambda_3, \lambda_4)\) that the eigenvalues for \(Q_I\) are \(\lambda_I\).) In our convention the Dynkin labels of the fundamental and the anti-fundamental representations are \((1, 0, 0)\) and \((0, 0, 1)\), respectively. Unlike the \(\mathcal{N} \leq 3\) case the \(R\)-symmetry group is not \(U(\mathcal{N}_R)\) but \(SU(4)_R\).

For the conformal group, we use the Cartan generators of the compact subgroup \(U(1) \times SU(2)_1 \times SU(2)_2 \subset SU(2, 2)\): the conformal energy \(E\), the \(SU(2)_1\) Cartan \(J_1\), and the \(SU(2)_2\) Cartan \(J_2\). We normalize these generators so that the supercharges \(Q_I\) carry \(E = \frac{1}{2}\) and \(J_{1/2} = 0, \pm 1\).

Let us consider an orientifold group generated by \(g\) that acts on the supercharges as

\[
g : (Q_1, Q_2, Q_3, Q_4) \rightarrow (e^{i\theta_1} Q_1, e^{i\theta_2} Q_2, e^{i\theta_3} Q_3, e^{i\theta_4} Q_4). \tag{2.3}
\]

To preserve three supercharges \(Q_I (I = 1, 2, 3)\), we need to set \(\theta_1 = \theta_2 = \theta_3 = 0\). When \(g\) is an element of \(SU(4)_R\) it is impossible to obtain a genuine \(\mathcal{N} = 3\) theory because \(\theta_{1,2,3} = 0\) requires the fourth angle \(\theta_4\) to also be zero, and the whole \(\mathcal{N} = 4\) supersymmetry is preserved.

To realize genuine \(\mathcal{N} = 3\) supersymmetry, we need to combine \(SU(4)_R\) with the duality symmetry, which rotates all \(Q_I\) by the same angle.

The \(SL(2, \mathbb{R})\) symmetry of type IIB supergravity can be linearly realized by introducing auxiliary scalar fields \(V^a_\alpha \) [12, 13]. In such a formalism the theory has the \(SL(2, \mathbb{R})\) global and \(U(1)\) local symmetries, and the scalar fields \(V^a_\alpha (a = 1, 2)\) are transformed as \(2, 1\) under the symmetry. The moduli space \(SL(2, \mathbb{R})/U(1)\) is parameterized by the axio-dilaton \(\tau = \chi + i e^{-\psi} = V^2_1/V^1_1\), which is \(U(1)\) invariant and transformed under \(SL(2, \mathbb{R})\) as

\[
\tau \rightarrow \tau' = \frac{a \tau + b}{c \tau + d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1. \tag{2.4}
\]

The expectation values of \(V^a_\alpha\) break \(SL(2, \mathbb{R}) \times U(1)\) down to a \(U(1)\) global symmetry, which is denoted in [4] by \(U(1)_Y\). This plays an essential role in the orientifold construction of \(\mathcal{N} = 3\) theories. We denote its generator by \(Y\).

The three-form fields \(H^R_{3,2}\) and \(H^S_{3,2}\), and the corresponding \((p, q)\)-string charges

\[
q^1 = \frac{1}{2\pi} \int H^R_{3,2}, \quad q^2 = \frac{1}{2\pi} \int H^S_{3,2}, \tag{2.5}
\]

form \(SL(2, \mathbb{R})\) doublets. Two charges are often combined into the complex charge

\[ q^C = V^a_\alpha q^a. \tag{2.6} \]

The quantized values of \(q^C\) form the two-dimensional charge lattice with the modulus \(\tau\) on the complex plane. The associated torus is the internal torus of the F-theory description of type IIB theory. Because the \(U(1)_Y\) symmetry rotates the lattice the symmetry is broken to a discrete subgroup \(\Gamma' \subset U(1)\). For a generic value of \(\tau\) this is \(\Gamma' = \mathbb{Z}_2\), while at special values of \(\tau\) larger symmetry is realized; \(\Gamma' = \mathbb{Z}_k\) with \(k = 3, 4, 6\) for \(\tau = e^{2\pi i/k}\) up to \(SL(2, \mathbb{Z})\). The orientifold construction of genuine \(\mathcal{N} = 3\) theories works only for these special values of \(\tau\).

To describe the \(U(1)_Y\) transformation, it is convenient to combine two three-form fields into a complex field \(H^C_3 = V^1_3 H^R_3 + V^2_3 H^S_3\). We also express the spin 3/2 and spin 1/2 fermion fields as complex fields \(\psi_\mu^C\) and \(\overline{\chi}\), respectively. If we normalize the \(U(1)_Y\) generator \(Y\) so that \(Y(V^a_\alpha) = +1\) the complex fields carry the following charges:

\[
Y(\psi_\mu^C) = +\frac{1}{2}, \quad Y(H^C_3) = +1, \quad Y(\overline{\chi}) = +\frac{3}{2}, \quad Y(\overline{\psi}_\mu^C) = +2. \tag{2.7}
\]
where $\delta \tau$ denotes the fluctuation around the expectation value of $\tau$. The non-trivial $U(1)_Y$ charge of the gravitino implies that the supercharge is also rotated by $U(1)_Y$. In the context of the boundary $\mathcal{N} = 4$ theory all the four supercharges $Q_I$ ($I = 1, 2, 3, 4$) carries $Y = +1/2$.

Let $z_i (i = 1, 2, 3)$ be complex coordinates of the transverse space $\mathbb{R}^6 = \mathbb{C}^3$. We consider the abelian orbifold group that is generated by the combination of the spacial rotation $\Theta = \begin{pmatrix} e^{i\phi_1} & e^{i\phi_2} & e^{i\phi_3} & e^{i\phi_4} \end{pmatrix}$ and the $U(1)_Y$ rotation $e^{i\varphi Y}$. This acts on the supercharges as the rotation (2.3) with the angles

$$
\begin{align*}
\theta_1 &= \frac{1}{2} (\phi_1 - \phi_2 - \phi_3 + \phi_4), \\
\theta_2 &= \frac{1}{2} (\phi_1 + \phi_2 - \phi_3 + \phi_4), \\
\theta_3 &= \frac{1}{2} (-\phi_1 - \phi_2 + \phi_3 + \phi_4), \\
\theta_4 &= \frac{1}{2} (\phi_1 + \phi_2 + \phi_3 + \phi_4).
\end{align*}
$$

We set the same value to all $\psi_i$

$$
\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = \varphi \quad \text{(2.10)}
$$

and denote the corresponding rotation by $g = e^{i\varphi A}$, where the generator $A$ acts on $Q_I$ as

$$
A = (0, 0, 0, +2). \quad \text{(2.11)}
$$

As we mentioned the $U(1)_Y$ symmetry is broken to its discrete subgroup, and only restricted values of $\varphi$ is allowed. To realize $\mathbb{Z}_k$ orientifold, we take

$$
\varphi = \frac{2\pi}{k}, \quad k = 2, 3, 4, 6. \quad \text{(2.12)}
$$

The fourth supercharge $Q_4$ is rotated by $g$ as

$$
Q_4 \rightarrow e^{i\varphi A}Q_4, \quad \text{(2.13)}
$$

which is trivial for $k = 2$ and non-trivial for $k = 3, 4, 6$. Accordingly while the orbifold projection for the former value of $k$ gives another $\mathcal{N} = 4$ theory, the projections for the latter values give genuine $\mathcal{N} = 3$ theories.

It is natural to ask what kind of gauge theory is realized on the D3-branes sitting at the $\mathbb{Z}_k$ orientifold plane. For $k = 2$ the projection is nothing but the ordinary perturbative orientifold associated with the reversal of string orientation. Leaving string modes invariant under the orientifold action leads to the orthogonal or symplectic gauge group depending on the intrinsic parity of open strings. When the number of D3-branes before the projection is $2N$ the gauge group is $SO(2N)$ or $Sp(N)$, while $SO(2N + 1)$ gauge theory is realized for $2N + 1$ D3-branes. For $\mathcal{N} = 3$ theories this question is rather difficult to answer due to the fact that the theory is defined in a non-perturbative fashion (see [5] for recent analysis).

Before ending this section, let us explicitly define the superconformal index, which will be analyzed in the following sections. To define the index, we should choose one of the supercharges. We use the one with the following quantum numbers.

\begin{center}
\textbf{J. Phys. A: Math. Theor.} \textbf{49} (2016) 435401 Y Imamura and S Yokoyama
\end{center}
The choice of the supercharge breaks $SU(4)_R$ symmetry into $SU(3) \times U(1)$. Furthermore, the orientifold projection with $k \geq 3$ further breaks the $SU(3)$ into $SU(2) \times U(1)$. Thus it is convenient to define the following additional $SU(4)_R$ Cartan generators.

$$X = 3R_1 + 2R_2 + R_3 = (+3, -1, -1, -1),$$

$$T_8 = R_2 + 2R_3 = (0, +1, +1, -2).$$  

(2.15)

$X$ appears in the algebra

$$\Delta = \{Q, Q^\dagger\} = E - J_1 - \frac{1}{2}X,$$  

(2.16)

and the $SU(3) \times U(1)$ subgroup respected by the choice of $Q$ is specified as the maximal subalgebra of $SU(4)_R$ commuting with $X$. We use $T_8$ together with $R_2$ as Cartan generators of $SU(3)$, and we define the $\mathcal{N} = 4$ superconformal index by

$$I(t, y, p, q) = \text{tr}[-t^2 e^{-t^{\Delta}_{2E+K} p^R q^T}] .$$

(2.17)

3. Large $N$ limit

A $\mathbb{Z}_k$ orientifold theory is characterized by $k$, the order of the orientifold group, and $N$, the size of the gauge group. Precisely speaking, the latter does not make sense for $k = 3, 4, 6$ because of the absence of the perturbative description. Even so we can define $N$ as the dimension of the Coulomb branch divided by 6. This is often referred to as the rank of the theory. From the viewpoint of the brane system the rank is the number of mobile D3-branes. In this section we analyze the superconformal index in the large $N$ limit by using the AdS/CFT correspondence.

In general, $k$ and $N$ do not uniquely specify the theory because there may be more than one theory distinguished by discrete torsions. The difference among these theories does not appear in the large $N$ limit. We discuss finite $N$ corrections in the next section, and here we do not pay attention to the discrete torsions.

The Kaluza–Klein spectrum of type IIB supergravity in $AdS_5 \times S^5$ was determined in [9]. They belong to the reducible representation

$$\bigoplus_{n=1}^{\infty} \mathcal{S}[0; 0]^{(0,n,0)}$$

(3.1)

of the superconformal algebra. $\mathcal{S}[j_1, j_2]^{(i_1, i_2, r_1)}$ denotes the superconformal irreducible representation with highest weight state that carries the specified quantum numbers. Each irreducible representation in (3.1) is decomposed into 36 irreducible representations of the bosonic subgroup $SU(2, 2) \times SU(4)$ for generic $n$, which are listed in the table in [9].

$$\mathcal{S}[0; 0]^{(0,n,0)} = \mathcal{C}[0; 0]^{(0, n, 0)} + \mathcal{C}[1; 1]^{(1, n-1, 0)} + \mathcal{C}[1; 0]^{(2, n-2, 0)} + \mathcal{C}[1; 0]^{(1, n-1, 1)} + \mathcal{C}[1; 0]^{(1, n-2, 1)} + \mathcal{C}[2; 0]^{(1, n-3, 1)} + \mathcal{C}[2; 0]^{(2, n-3, 0)} + \cdots .$$

(3.2)

We denote the $SU(2, 2) \times SU(4)_Y \times U(1)_Y$ irreducible representations by $\mathcal{C}[j_1; j_2]^{(i_1, i_2, r_1)}$. Only nine of them in the decomposition saturating the BPS bound $\Delta \geq 0$ are shown.

5
explicitly in (3.2), and \( \cdots \) denotes the other terms that do not contribute to the index. For small \( n \), some of representations on the right hand side are absent. By using this decomposition it is easy to calculate the superconformal index. Let \( S_n \) be the contribution of an \( \mathcal{N} = 4 \) superconformal multiplet \( S(0; 0^{(0,(0,0))}) \). It is given by

\[
S_n(t, y, p, q) = \sum (-1)^F \frac{2E+h}{Y} \xi_{\text{th}}(y) \chi_{(r,r',y)}(p, q, t), \quad (3.3)
\]

where the sum is taken over the nine components explicitly shown in (3.2). The \( SU(2) \) character \( \chi_{\text{th}}(y) \) and the \( SU(3) \) character \( \chi_{(u,v,w)}(u, v, w) \) are defined so that

\[
\chi_{(u,v,w)}(u, v, w) = y^u + y^{v-2} + \cdots + y^{-w},
\]

\[
\chi_{(0,0,0)}(p, q) = q \left( p + \frac{1}{p} \right) + \frac{1}{q}, \quad \chi_{(0,0,1)}(p, q) = \frac{1}{q} \left( p + \frac{1}{p} \right) + q. \quad (3.4)
\]

By summing up \( S_n \) for \( n = 1, 2, \ldots \), we obtain the total single particle index \([14]\).

\[
I^{\text{KK}}(t, y, p, q) = \sum_{n=1}^{\infty} S_n(t, y, p, q). \quad (3.5)
\]

An explicit expression is obtained by setting \( z = 1 \) and \( q' = q \) in (3.10) below.

In the large \( N \) limit, the index of the \( \mathcal{N} = 3 \) theory is obtained by the \( \mathcal{Z}_k \) projection that eliminate \( \mathcal{Z}_k \) non-invariant modes. For this purpose it is convenient to extend the index by introducing the fugacity for \( U(1)_y \). We combine \( Y \) and \( X \) to preserve \( Q \) used for the definition of the index (2.17), and insert the factor \( z^{-1/2} \). In addition, we rewrite the fugacity \( q \) by \( q = q^{'z^{-2/3}} \). Then, by using the relation

\[
Y - \frac{1}{6} X = A + \frac{2}{3} T_8 \quad (3.6)
\]

we can rewrite the factor \( q^{'z^{-1/2}} \) in the index as \( q^{'T_8} z^A \). As the result we obtain the index.

\[
I^{\text{KK}}(t, y, p, q', z) = \text{tr} \left[ (-1)^F t^{2E+h} y^F p^{R_E} q^{'T_8} z^A \right]. \quad (3.7)
\]

The \( U(1)_y \) charge of each Kaluza–Klein mode is given in [9], and are shown in the decomposition (3.2). It is straightforward to repeat the calculation with the extra fugacity. The contribution of each superconformal multiplet is

\[
S_n(t, y, p, q', z) = \sum (-1)^F \frac{2E+h}{Y} \xi_{\text{th}}(y) \chi_{(r,r',y)}(p, q', z^{-2/3}) z^{-1/6}, \quad (3.8)
\]

where the summation is again taken over the nine components explicitly shown in (3.2). Although fractional powers of \( z \) appear in the expression (3.8) we can easily check that powers of \( z \) in the \( z \)-expansion of \( S_n \) are actually always \( n \) mod 2. Namely, \( S_n \) satisfies

\[
S_n(t, y, p, q', -z) = (-1)^n S_n(t, y, p, q', z). \quad (3.9)
\]
By summing up all the contributions for \( n = 1, 2, \ldots \) we obtain

\[
I_{\text{KK}}(t, y, p, q', z) = \sum_{n=1}^{\infty} S_n(t, y, p, q', z)
\]

\[
= \frac{1}{(1 - t^3)(1 - t^{31})} \left( 1 - t^4 \left( \frac{z}{p^2} + \frac{z}{q^2} + \frac{z^3}{q^3} \right) + (1 + z) t^6 \right)
\]

\[
= \frac{1}{(1 - t^3)(1 - t^{31})} \left( 1 - t^6 \frac{z}{z} \right) - \frac{1 - t^9 \frac{z}{z}}{(1 - t^3)(1 - t^{31})}.
\]

(3.10)

Once we obtain this expression, it is straightforward to perform the projection. We define \( I_{m}^{\text{KK}} \) by expanding \( I_{\text{KK}} \) with respect to \( z \) as

\[
I_{m}^{\text{KK}}(t, y, p, q', z) = \sum_{m \in \mathbb{Z}} I_{m}^{\text{KK}}(t, y, p, q') z^m.
\]

(3.11)

Then the index of \( \mathbb{Z}_k \) orientifold theory is obtained by

\[
I_{m}^{\text{KK}}(t, y, p, q') = \sum_{m \in k \mathbb{Z}} I_{m}^{\text{KK}}(t, y, p, q').
\]

(3.12)

For reference we show the explicit form of the index for \( p = q' = 1 \):

\[
I_{m}^{\text{KK}}(t, y, 1, 1) = \frac{a_k(t) + b_k(t) \chi_1(y)}{(1 - t^3 y)(1 - t^3 y^{-1})}.
\]

(3.13)

The coefficients \( a_k \) and \( b_k \) are given in Table 1, including \( a_1 \) and \( b_1 \) for the unprojected index \( I_{m}^{\text{KK}} \equiv I_{m}^{\text{KK}} \).

Some comments are in order. The complete agreement of the index \( I_{\text{KK}} \) before the projection and the index of \( \mathcal{N} = 4 \) \( U(N) \) theory is confirmed in [14]. The gauge theory index that should be compared to the one-particle index of the Kaluza–Klein modes is not the full index defined as the trace over all states in \( S^3 \) Hilbert space, but the plethystic logarithm of the full index. In the following, we always mean by the index the plethystic logarithm.

Note that the gauge group for \( k = 1 \) is not \( SU(N) \) but \( U(N) \). The diagonal \( U(1) \) corresponds to \( S_1 \).
\[
S_4 = \frac{1}{z} \left( t^2 q + \frac{1}{q} t^2 \right) - \frac{1}{z} \left( t^2 q + \frac{1}{q} t^2 \right) + \frac{1}{z} \left( t^2 q + \frac{1}{q} t^2 \right).
\]

This consists of only \( z^{\pm 1} \) terms and is always projected out for \( k \geq 2 \).

When \( k = 2 \), \( S_{2n+1} \) are projected out due to the relation (3.9), and the index is given by

\[
I_{S_{2n}}^{KK} = \sum_{n=1}^{\infty} S_{2n}.
\]

The projection does not divide \( S_n \) into two or more parts. This fact is consistent with the fact that \( \mathbb{Z}_2 \) projection is the usual orientifold projection and it does not break the \( \mathcal{N} = 4 \) supersymmetry. This index precisely agrees with the large \( N \) limit of the index of \( \mathcal{N} = 4 \) \( SO(N) \) and \( Sp(N) \) gauge theories, which is calculated in appendix B.

The difference of the supersymmetry between \( k = 1, 2 \) and \( k = 3, 4, 6 \) is also seen by looking at \( S_2 \), which corresponds to the energy--momentum multiplet of \( \mathcal{N} = 4 \) theory. In [5] the branching of the \( \mathcal{N} = 4 \) energy--momentum multiplet into \( \mathcal{N} = 3 \) multiplets are examined. The bottom components belonging to \( 20 \) of \( SU(4)_{kr} \) is decomposed into \( 8_{0} + 6_{4} + 6_{4} \) of \( SU(3) \times U(1) \). This can be seen in the \( z \)-expansion of \( S_2 \)

\[
S_2 = \frac{1}{z^2} \frac{3 t^4 - 4 t^5 - t^6 + 2 t^7}{(1 - t^3)^2} + \frac{2 t^4 - 2 t^5 - 4 t^6 + 4 t^7 + 2 t^8 - 2 t^9}{(1 - t^3)^2} + \frac{z^2 t^4 - 2 t^6 + t^8}{(1 - t^3)^2}.
\]

(We set \( y = p = q' = 1 \).) The \( z^0 \) term, which is never projected out, corresponds to the \( \mathcal{N} = 3 \) energy--momentum multiplet, while the terms containing \( z^\pm 2 \) correspond to the multiplets that include the exactly marginal deformation and the fourth supersymmetry current. The latter is projected out when \( k \geq 3 \), and genuine \( \mathcal{N} = 3 \) theories are realized.

For rank 1 case a \( \mathbb{Z}_2 \) orientifold theory has a Coulomb branch operator with dimension \( k \) [8]. This is expected to be the case for large \( N \) limit, too, because the \( \mathbb{Z}_2 \) invariant coordinates of \( \mathbb{C}^3 / \mathbb{Z}_2 \) are given as order \( k \) monomials of the coordinates \( z_i \) that have dimension 1. The corresponding contribution is found in the index. For \( \mathbb{Z}_k \) orientifold \( I_{kk}^{KK} \) survive the projection and when \( k \geq 2 \) the leading terms in \( I_{kk}^{KK} \) are given by

\[
I_{kk}^{KK} = t^{2k} q^k + O(t^{2k+1}),
\]

\[
I_{kk}^{KK} = t^{2k} q^{-2k} + O(t^{2k+1}).
\]

These are consistent with the existence of the \( \mathcal{N} = 3 \) short representation

\[
\mathcal{S}[0; 0; 0^{(0, k; 2k)}] \quad \text{(for } I_{kk}^{KK}), \quad \mathcal{S}[0; 0; 0^{(k, k; 2k)}] \quad \text{(for } I_{kk}^{KK}),
\]

where \( \mathcal{S}[j_1; j_2; \ell; r; c; r'] \) is the \( \mathcal{N} = 3 \) representation with highest weights specified. The quantum numbers \( (r_1, r_2; r) \) are the \( SU(3)_{kr} \) integral Dynkin weights \( (r_1, r_2) \) and \( U(1)_{kr} \) charge \( r \). These are \( \mathcal{N} = 3 \) chiral representations, and their bottom components are identified with the Coulomb branch operators with dimension \( k \).

4. Finite \( N \) corrections

Now let us turn to finite \( N \) corrections. It is instructive to start from the \( k = 1 \) and \( k = 2 \) cases where we can calculate the index on the gauge theory side.
For $k = 1$ the theory is the $\mathcal{N} = 4$ $U(N)$ SYM. The index of the finite $N$ theory is related to $I^{KK} = I_{U(\infty)}$ by

$$I_{U(N)} = I^{KK} - \chi_{(N+1,0)} r^{2(N+1)} + \mathcal{O}(r^{2N+3}). \quad (4.1)$$

(See (A.5).) The leading correction is of order $r^{2N+2}$ and its coefficient is negative. On the gauge theory side this comes from the fact that the gauge invariant operators of the form

$$\text{tr}\Phi^m \quad (4.2)$$

are independent only for $m = 1, 2, \ldots, N$, and ones with $m \geq N + 1$ can be decomposed into the independent ones.

On the gravity side, they correspond to objects with angular momentum $\ell \sim m$. When $\ell$ is of order $N$ they should be treated not as point-like gravitons but as extenden objects. A giant graviton [15] (a spherical D3-brane expanding in the $S^3$) and a dual giant graviton [16, 17] (a spherical D3-brane expanding in the $AdS_5$) are two extremes of such objects. The configuration relevant to the leading finite $N$ correction to the index is a giant graviton [18–20], whose radius depending on $\ell$ is bounded by the radius of $S^3$, which is the same as the AdS radius

$$L = (4\pi N g_{\text{str}})^{1/4}.$$

(4.3)

The mass $M_{D3}$ of a D3-brane wrapped around a large $S^3 \subset S^5$ in the unit $1/L$ is

$$M_{D3}L = 2\pi^2 L^4 T_{D3} = N. \quad (4.4)$$

At the second equality we used $T_{D3} = 1/((2\pi)^3 l_s^3 g_{\text{str}})$. There are no giant gravitons whose energy exceeds this bound. This is easily generalized to the $Z_k$ orientifold. Because a giant graviton wraps a topologically trivial cycle we can use covering space $S^5$ in the calculation of the bound, and the only difference is that the number of the flux $N$ is replaced by $kN$.

The $k = 2$ case is investigated in [10] in detail. An important feature of $k \geq 2$ case is that we have different choices of discrete torsion classified by

$$\Gamma_{\text{tor}}^{(k)} = H^3(S^3/\mathbb{Z}_k, \mathbb{Z}^3 \oplus \mathbb{Z}_k). \quad (4.5)$$

$\mathbb{Z} \oplus \mathbb{Z}$ represents the sheaf of a pair of integers corresponding to two three-form field strength $H_{k}^{RR}$ and $H_{k}^{NS}$, which has the $\mathbb{Z}_k$ monodromy around the non-trivial cycle in $S^3/\mathbb{Z}_k$. When $k = 2$ the internal space is $\mathbb{RP}^5 = S^5/\mathbb{Z}_2$ and the monodromy around the non-trivial cycle is $(p, q) \rightarrow (-p, -q)$. Because this monodromy does not mix the R-R and NS-NS fluxes the torsion in the $k = 2$ case is factorized into the direct sum

$$\Gamma_{\text{tor}}^{(k=2)} = H^3(\mathbb{RP}^5, \mathbb{Z}_2) \oplus H^3(\mathbb{RP}^5, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2. \quad (4.6)$$

There are four cases, which we denote by $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$ where the first and the second components represent the discrete torsion of $H_{k}^{RR}$ and $H_{k}^{NS}$, respectively. They correspond to the following gauge groups [10]:

$$(0, 0) : SO(2N), \quad (1, 0) : SO(2N + 1), \quad (0, 1) : Sp(N), \quad (1, 1) : Sp(N). \quad (4.7)$$

The latter three are transformed among them by the $SL(2, \mathbb{Z})$ Montonen–Olive duality, while the $SO(2N)$ theory corresponding to the trivial discrete torsion is self-dual.

Let us first consider the $SO(2N)$ theory corresponding to the trivial discrete torsion. The index is

4 The gauge group of the theory is actually $O(2N)$ rather than $SO(2N)$. We treat $\mathbb{Z}_2 = O(2N)/SO(2N)$ as a global symmetry and we assume the trivial holonomy in the index calculation.
The leading correction of order $t^{2N}$ is given by
\[ I_{SO(2N)} = I_{KK} + \chi_{(N,0)} t^{2N} + O(t^{2N+1}). \] (4.8)

(See (A.7).) We have the leading correction of order $t^{2N}$ with the positive coefficient. On the gauge theory side this is identified with the contribution of baryonic operators of the form
\[ B = \text{Puff} \Phi \] (4.9)
with dimension $E = N$. On the gravity side, these correspond to D3-branes wrapped on the topologically non-trivial cycle. Let $H^{(k)}$ be the 3-cycle homology of the internal space $S^3/Z_k$. When $k = 2$ it is
\[ H^{(k=2)} = H_3(\mathbb{RP}^3, \mathbb{Z}) = \mathbb{Z}_2. \] (4.10)
The non-trivial element of this homology group is $\mathbb{RP}^3 \subset \mathbb{RP}^5$. As we have shown above the mass of a D3-brane wrapped on a large $S^3$ is the same as the number of flux passing through $S^3$ in the unit of $1/L$, and it is $2N$ in the $k = 2$ case. By taking account of $\text{Vol}(\mathbb{RP}^3) = \text{Vol}(S^3)/2$ we obtain the dimension $E \sim N$, which is consistent with the correction in (4.8).

When the discrete torsion is non-trivial, the corresponding $SO(2N + 1)$ theory (or $Sp(N)$ theory) has the index
\[ I_{SO(2N+1)} = I_{Sp(N)} = I_{KK} - (\chi_{(2N+2,0)} + \cdots) t^{4N+4} + O(t^{4N+5}). \] (4.11)
(See (A.8).) This time we have the leading correction of order $t^{4N+4}$ with the negative coefficient, which corresponds to operators with dimension $E \sim 2N$. (We are interested in the $O(N)$ behavior of the dimension, and do not pay attention to $O(1)$ shift.) This is explained in the same way as the $k = 1$ case by the bound for the giant graviton. What is important here is that the positive contribution of baryonic operator of dimension $E \sim N$, which gives the leading correction in the $SO(2N)$ case, is absent. In the context of the gauge theory this is simply because we cannot define operators in the form (4.9). On the gravity side this is explained by the topological obstruction to wrapped D3-branes due to the non-vanishing discrete torsion [10].

Let $K^{(k)}$ be the charge lattice of the electric and magnetic charges on a D3-brane wrapped on the cycle corresponding to the generating element of $H^{(k)}$, which has the topology $S^3/Z_k$.
\[ K^{(k)} \equiv H_0(S^3/Z_k, \bar{\mathbb{Z}} \oplus \bar{\mathbb{Z}}). \] (4.12)
When $k = 2$ it is given by
\[ K^{(k=2)} = H_0(\mathbb{RP}^3, \bar{\mathbb{Z}} \oplus \bar{\mathbb{Z}}) = H_0(\mathbb{RP}^3, \bar{\mathbb{Z}} \oplus H_0(\mathbb{RP}^3, \bar{\mathbb{Z}})) = \mathbb{Z}_2 \oplus \mathbb{Z}_2. \] (4.13)
The three form fluxes, which are classified by $\Gamma^{(k)}_{\text{tot}}$, induce electric and magnetic charges on a wrapped D3-brane. This is described by the natural pairing
\[ f : H^{(k)} \times \Gamma^{(k)}_{\text{tot}} \rightarrow K^{(k)}. \] (4.14)
The flux conservation and the compactness of the wrapped D3-brane require the pairing to vanish. For the discrete torsion (1, 0), (0, 1), and (1, 1), wrapped D3-branes give non-trivial element of $K^{(k=2)}$ and such wrappings are prohibited.

In the following we generalize the argument above to $k \geq 3$ cases. For a general $k$ the 3-cycle homology classifying wrapped D3-branes is determined by using the Gysin exact sequence as
\[ H^{(k)} \equiv H_1(S^3/Z_k, \mathbb{Z}) = \mathbb{Z}_k. \] (4.15)
We can obtain $K^{(k)}$ from the lattice $\mathbb{Z} \oplus \mathbb{Z}$ of electric and magnetic charges by taking account of the non-trivial monodromy around the non-trivial cycle in $S^3/Z_k$. We identify elements in...
To analyze the discrete torsion \((4.5)\), it may be convenient to visualize the elements of \(\Gamma_{\text{tor}}^{(k)}\) as their Poincaré duals, which are classified by the two-cycle homology group
\[
\Gamma_{\text{tor}}^{(k)} \simeq H_2(S^3/\mathbb{Z}_4, \mathbb{Z} \oplus \mathbb{Z}^2).
\]

For \(k \geq 3\) the orientifold group mixes the NS-NS and R-R three-form fluxes, and the torsion does not factorize. An element of this group is regarded as a string worldsheet \(\Sigma\) carrying the NS-NS and R-R charges. To visualize \(\Sigma\) it is convenient to describe \(S^3/\mathbb{Z}_4\) as a Hopf fibration over \(\mathbb{C}P^2\). The Chern class of the fiber is \(k \in H^2(\mathbb{C}P^2, \mathbb{Z}) = \mathbb{Z}\). Let \(B \sim \mathbb{C}P^4\) be a non-trivial two-cycle in \(\mathbb{C}P^2\). The restriction of the Hopf fibration also has the Chern class \(k\), and has the topology \(S^3/\mathbb{Z}_4\). By using the Mayer–Vietoris exact sequence we can show the isomorphism
\[
\Gamma_{\text{tor}}^{(k)} = H_2(S^3/\mathbb{Z}_4, \mathbb{Z} \oplus \mathbb{Z}^2) \simeq H_2(S^3/\mathbb{Z}_4, \mathbb{Z} \oplus \mathbb{Z}^2),
\]
and we can focus on the subspace \(S^3/\mathbb{Z}_4 \subset S^3/\mathbb{Z}_4\).

Roughly speaking, a non-trivial element of \(\Gamma_{\text{tor}}^{(k)}\) can be represented as a worldsheet \(\Sigma\) wrapped on a section over \(B\). Of course, due to the non-vanishing Chern class, there is no global section over \(B\). Let \(B'\) be the punctured sphere obtained by removing a point \(p\) from \(B\), and \(C_p\) the fiber over \(p\). We can define a section \(\Sigma'\) over \(B'\), which has the topology of a punctured sphere as \(B'\). The reason why we cannot fill in the puncture of \(\Sigma'\) to obtain a global section over \(B\) is that the boundary of \(\Sigma'\) is not just a point-like puncture but \(S^1\) winding around \(C_p\).

In fact, we can remove this puncture to obtain \(\Sigma\) for \(k \geq 2\). Let us first consider \(k = 2\) case. The boundary of \(\Sigma'\) winds \(C_p\) twice. This means that two parts of the boundary of \(\Sigma'\) meet along \(C_p\). Due to the non-trivial monodromy of the string charges around \(C_p\), the two worldsheets have opposite string charge to each other, and the boundary can be consistently covered by a Möbius strip wrapped around \(C_p\). As the result we obtain \(\Sigma = \mathbb{R}P^2\), which wraps the non-trivial cycle in \(\mathbb{R}P^5\).

A generalization to \(k \geq 3\) is straightforward. The Chern class of the fibration over \(B\) is now \(k\), and the boundary of the worldsheet \(\Sigma'\) winds \(k\) times around \(C_p\). Namely, \(k\) worldsheets, which are different parts of \(\Sigma'\), meet along \(C_p\). We can stitch them consistently into a \(k\)-valent junction because their \((p, q)\) string charges are related by the \(\mathbb{Z}_4\) monodromy and sum up to zero.

Once we fix \(\Sigma\) for \(k \geq 2\), an element of the discrete torsion is specified by the \((p, q)\) string charge of the worldsheet. If we move \(\Sigma\) around the fibers its charges get transformed by the monodromy and thus charges transformed to one another by \(\mathbb{Z}_4\) must be identified. After this identification, we obtain the same groups as \(K^{(3)}\) in \((4.16)\). This is also directly shown from the isomorphism \(H_i(\text{tor}) \cong H_{n-i-1}(\text{tor})\) for an \(n\)-dimensional manifold.

By using the worldsheet representation of the discrete torsion, we can easily determine the pairing \(f : H \times \Gamma_{\text{tor}} \to K\) as the intersection of the worldsheet and the D3-brane worldvolume. Let us consider case by case.

- \(k = 1\)
  
  \(k = 1\) gives \(N = 4\) \(U(N)\) SYM. The 3-cycle homology is trivial and there exist no baryonic operators. The leading term of the finite \(N\) correction is due to the bound of giant gravitons at \(E \sim N\).
The perturbative orientifold theories. The 3-cycle homology is $H^2 = H^{(k=2)} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the discrete torsion group and the charge lattice is $\Gamma_{\text{tor}}^{(k=2)} = K^{(k=2)} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The pairing is

$$f : \mathbb{Z}_2 \times (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \to (\mathbb{Z}_2 \oplus \mathbb{Z}_2).$$

This vanishes for a wrapped D3-brane only when the discrete torsion is trivial. Then we have positive correction at $E \sim N$. Otherwise, the leading correction is negative one at $E \sim 2N$.

* $k = 3$

$H^3 = H^{(k=3)} = \mathbb{Z}_3$ and the discrete torsion group and the charge lattice is $\Gamma_{\text{tor}}^{(k=3)} = K^{(k=3)} = \mathbb{Z}_3$. The pairing is

$$f : \mathbb{Z}_3 \times \mathbb{Z}_3 \to \mathbb{Z}_3,$$

and this vanishes for non-trivial 3-cycle only when the discrete torsion vanishes. Then we have the positive correction at $E \sim N$. Otherwise, we have the negative contribution at $E \sim 3N$.

* $k = 4$

$H^4 = H^{(k=4)} = \mathbb{Z}_4$ and the discrete torsion group and the charge lattice is $\Gamma_{\text{tor}}^{(k=4)} = K^{(k=4)} = \mathbb{Z}_2$. The pairing is

$$f : \mathbb{Z}_4 \times \mathbb{Z}_2 \to \mathbb{Z}_2.$$

When the discrete torsion is trivial any winding number of a D3-brane is allowed, and the leading correction is positive one at $E \sim N$. Even when the discrete torsion is non-trivial, we still have wrapped D3-branes with even wrapping number, and we have positive correction at $E \sim 2N$.

* $k = 6$

$H^6 = H^{(k=6)} = \mathbb{Z}_6$ and the discrete torsion group and the charge lattice is $\Gamma_{\text{tor}}^{(k=6)} = K^{(k=6)} = 0$. Because of the absence of the non-trivial discrete torsion, any winding of D3-branes is allowed, and we have positive correction at $E \sim N$.

We summarize these results in Table 2.

---

Table 2. Finite $N$ corrections to the index. The negative and positive signs show that the leading correction comes from the bound on the giant gravitons and D3-branes wrapped on non-trivial cycles, respectively. The right-most column shows the $O(N)$ part of the dimension of operators that give the leading corrections.

| $k$ | $H^{(k)}$ | $K^{(k)} = \Gamma_{\text{tor}}^{(k)}$ | torsion | sign | dim |
|-----|-------------|----------------------------------|---------|------|-----|
| 1   | 0           | 0                                | 0       | $\sim N$ |
| 2   | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | (0, 0) | $\sim N$ |
|     |             | (1, 0), (0, 1), (1, 1) | $\sim 2N$ |
| 3   | $\mathbb{Z}_3$ | $\mathbb{Z}_3$ | 0 | $\sim N$ |
|     |             | 1, 2 | $\sim 3N$ |
| 4   | $\mathbb{Z}_4$ | $\mathbb{Z}_2$ | 0 | $\sim N$ |
|     |             | 1 | $\sim 2N$ |
| 6   | $\mathbb{Z}_6$ | 0 | 0 | $\sim N$ |

---

The analysis here is based on the assumption that when the pairing vanish we can wrap a D3-brane consistently. However, there are some arguments that this naive expectation may not be correct. In [11] it is pointed out that in the $k = 4$ case non-trivial wrapping seems to be allowed only when the discrete torsion is trivial. The authors would like to thank Tachikawa for notifying of this point.
5. Conclusions and discussion

In this paper we have investigated the superconformal index of $\mathcal{N}=3$ $\mathbb{Z}_k$ orientifold superconformal theories by using AdS/CFT correspondence. The result in the large $N$ limit has been obtained by a simple $\mathbb{Z}_k$ projection of the Kaluza–Klein modes. We have checked that the result is consistent with some known properties of genuine $\mathcal{N}=3$ theories. We have also discussed finite $N$ corrections based on the assumption that corrections with positive and negative coefficients originate from D3-branes wrapped on non-trivial and trivial cycles, respectively. We have determined the signature and large $N$ behavior of the exponent of the leading term in the $t$-expansion by examining the discrete torsion of the three-form fluxes.

It would be interesting to investigate more on BPS operators of $\mathcal{N}=3$ theories predicted from the index determined in this paper. When $k = 2$, the microscopic origin of such BPS operators lies in BPS strings ending on D3-branes at the orbifold singularity. Thus it may be natural to expect that BPS operators in $\mathcal{N}=3$ theory originate junctions connecting D3-branes.

In this paper we treated the wrapped branes classically, and obtained only the coefficients of $O(N^0)$ term in the exponent. To determine the exponent including $O(1)$ part and $SU(3)_R$ representation we need to perform quantum mechanical analysis of the collective motion of the wrapped branes. It would be interesting to determine the configuration of such wrapped branes.

We represent the discrete torsion as two-cycles that are Poincaré dual to the three-form flux, and study how the wrapped branes are affected by the discrete torsion. The same two-cycles can be used to realize domain walls. We can wrap $(p, q)$ five branes on the cycles. They divide the four-dimensional spacetime into two parts. On the two sides of a domain wall the discrete torsions are different by the amount depending on the homology class of the wrapped fivebrane. It may be interesting to study properties of such domain walls by using the brane realization mentioned above.

We studied the discrete torsion in type IIB picture. In [4] discrete torsion in the M-theory dual was investigated, and more variety than those of type IIB picture were found. It may be interesting to study the relation between discrete torsions in type IIB theory and M-theory. It may play important role when we discuss $S^1$ compactification of the $\mathcal{N}=3$ theory to three-dimensional theories.

As another example of wrapped branes on torsion-cycles in the dual internal geometry, one can consider a wrapped D1-brane on $H_1(S^5/\mathbb{Z}_k)$. This object will correspond to a monopole with fractional charge in the field theory side. It would be interesting to investigate the properties of monopole with fractional charge from a purely fieldtheoretical viewpoint, as done for BPS states [21, 22].

We hope to come back to these issues in near future.

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Note added. After completion of this work, we have noticed a paper by Aharony and Tachikawa [11], which also analyzes $\mathcal{N} = 3$ theories from a holographic point of view and has some overlap with this work.

Appendix A. The indices for small $N$

In this appendix we summarize the indices of $\mathcal{N} = 4$ theories with small $N$, which are used in the main text.

A general formula for the total index is [14]

$$I^{\mathcal{N}=4} = \int [dU] \exp \left[ \sum_{m=1}^{\infty} \frac{1}{m} f(y^m, p^m, q^m) \chi_{\text{adj}}(U^m) \right]$$  \hspace{1cm} (A.1)

where $f$ is the index for the letters of the $\mathcal{N} = 4$ $U(1)$ gauge theory, which are the elementary BPS operators of the theory, given by

$$f(t, y, p, q) = \frac{t^2 \chi_{(1,0)} - t^3 \chi_{1} - t^4 \chi_{(0,1)} + 2t^6}{(1 - t^3)(\frac{1}{1 - t^y})},$$  \hspace{1cm} (A.2)

where $\chi_{(m,n)} \equiv \chi_{(m,n)}(p, q)$ and $\chi_{1} \equiv \chi_{1}(y)$ are the $SU(3)$ and the $SU(2)$ characters, respectively. $[dU]$ is the Haar measure of the gauge group and $\chi_{\text{adj}}(U^m)$ is the character for the adjoint representation thereof, which describe the contributions of holonomy. The letter index (A.2) contains both contributions from a $\mathcal{N} = 1$ vector multiplet and three $\mathcal{N} = 1$ chiral ones, whose zero point contributions to the index cancel. Note also that the holonomy contribution is already factored out from the letter index (A.2) since all the fields belong to the adjoint representation. See also [23, 24].

For $U(N)$ gauge group, $\chi_{\text{adj}}$ and $\int [dU]$ are given by

$$\chi_{\text{adj}}(U) = N + \sum_{m=n} e^{i(\alpha_m - \alpha_n)},$$  \hspace{1cm} (A.3)

$$\int [dU] = \frac{1}{N!} \prod_{n=1}^{N} \int_0^{2\pi} \frac{d\alpha_m}{2\pi} \prod_{n=m} (1 - e^{i(\alpha_m - \alpha_n)}),$$  \hspace{1cm} (A.4)

where $\alpha_m$ ($m = 1, \ldots, N$) is the $m$th holonomy. It is straightforward to obtain the index in the form of $t$-expansion by direct calculation. For $U(N)$ gauge groups with $N = 1, 2, 3, 4$ we obtain

$$I_{U(1)} = \int^{\mathcal{K}} - \chi_{2} t^4 + \chi_{1} \chi_{1,0} t^5 + (-\chi_{3,0} + \chi_{1,1}) t^6 - \chi_{1} \chi_{0,1} t^7 + O(t^8),$$

$$I_{U(2)} = \int^{\mathcal{K}} - \chi_{3,0} t^6 + \chi_{1} \chi_{2,0} t^7 + (-\chi_{4,0} + \chi_{2,1} - \chi_{0,2} - \chi_{1,2}) t^8 + O(t^9),$$

$$I_{U(3)} = \int^{\mathcal{K}} - \chi_{4,0} t^8 + O(t^9),$$

$$I_{U(4)} = \int^{\mathcal{K}} - \chi_{5,0} t^{10} + O(t^{11}).$$  \hspace{1cm} (A.5)

These results show the pattern in (4.1).
For $SO(2N)$ gauge group the adjoint character and the Haar measure are given by
\[
\chi_{\text{adj}}(U) = N + \sum_{m < n} \left( e^{i(\alpha_m + \alpha_n)} + e^{-i(\alpha_m + \alpha_n)} \right) + \sum_{m \neq n} e^{i(\alpha_m - \alpha_n)},
\]
\[
\int [dU] = \frac{1}{2^{N-1}N!} \prod_{m=1}^{N} \int_{0}^{2\pi} \frac{d\alpha_m}{2\pi} \prod_{m < n} \left( 1 - e^{i(\alpha_m + \alpha_n)} \right) \prod_{m \neq n} \left( 1 - e^{i(\alpha_m - \alpha_n)} \right),
\]
and the indices for small $N$ are
\[
I_{SO(2i)} = I_{KK}^{SO(2i)} + \chi_{(1,0)} t^2 - \chi_{(1,0)} t^3 - \chi_{(2,0)} t^4 + 2 \chi_{(1,0)} \chi_{(2,0)} t^5 + O(t^6),
\]
\[
I_{SO(4i)} = I_{KK}^{SO(4i)} + \chi_{(2,0)} t^4 - \chi_{(1,0)} \chi_{(2,0)} t^5 + (\chi_{(1,0)} + 1) t^6 + \chi_{(2,0)} + \chi_{(0,1)} t^7 + O(t^8),
\]
\[
I_{SO(6i)} = I_{KK}^{SO(6i)} + \chi_{(3,0)} t^6 - \chi_{(2,0)} t^7 + (\chi_{(1,0)} + \chi_{(2,0)}) t^8 + O(t^9),
\]
(A.6)

We see the pattern of (4.8).

For $SO(2N + 1)$ gauge group
\[
\chi_{\text{adj}}(U) = N + \sum_{m < n} \left( e^{i(\alpha_m + \alpha_n)} + e^{-i(\alpha_m + \alpha_n)} \right) + \sum_{m \neq n} e^{i(\alpha_m - \alpha_n)} + \sum_{m=1}^{N} \left( e^{i\alpha_m} + e^{-i\alpha_m} \right),
\]
\[
\int [dU] = \frac{1}{2^{N}N!} \prod_{m=1}^{N} \int_{0}^{2\pi} \frac{d\alpha_m}{2\pi} \prod_{m < n} \left( 1 - e^{i(\alpha_m + \alpha_n)} \right) \prod_{m \neq n} \left( 1 - e^{i(\alpha_m - \alpha_n)} \right) \prod_{m=1}^{N} \left( 1 - e^{i\alpha_m} \right),
\]
and the indices for small $N$ are
\[
I_{SO(1i)} = I_{KK}^{SO(1i)} - \chi_{(2,0)} t^4 + \chi_{(1,0)} \chi_{(2,0)} t^5 + O(t^6),
\]
\[
I_{SO(3i)} = I_{KK}^{SO(3i)} - (\chi_{(1,0)} + \chi_{(0,1)}) t^6 + \chi_{(2,0)} + \chi_{(1,0)} t^7 + O(t^8),
\]
\[
I_{SO(5i)} = I_{KK}^{SO(5i)} - (\chi_{(0,1)} + \chi_{(2,0)} + 1) t^8 + O(t^{10}),
\]
(A.7)

We find the pattern in (4.11).

Appendix B. Large $N$ limit of $SO(N)$ index

In this appendix we give a brief derivation of large $N$ limit of an index for $N = 4$ SYM with orthogonal and symplectic gauge groups. For this purpose it is sufficient to consider the case when the gauge group is $SO(2N)$, since the large $N$ index of $SO(2N)$ matches those of $SO(2N + 1)$ and $Sp(N)$. In this appendix we use the same convention as used in [14].

For this end we first restrict the range of holonomy from 0 to $\pi$ by using the symmetry of the index under flip of sign of each holonomy for convenience. Then under the large $N$ limit the holonomy distributes densely between 0 and $\pi$, whose density function we denote by $\rho$: 
\[
\rho(\alpha) = \frac{1}{N} \sum_{m=1}^{N} \delta(\alpha - \alpha_m),
\]

(B.1)
whose normalization is
\[ \int_0^\pi \rho(\alpha) \, d\alpha = 1. \]  
(B.2)

Since the range of \( \alpha \) is restricted from 0 to \( \pi \), the density function can be expanded by cosine functions so that
\[ \rho(\alpha) = \frac{1}{2\pi} \rho_0 + \frac{1}{\pi} \sum_{k=1}^\infty \rho_k \cos(k\alpha) \]  
(B.3)

where \( \rho_k \) is given by
\[ \rho_k = 2 \int_0^\pi \rho(\alpha) \cos(-k\alpha). \]  
(B.4)

By using the Fourier components of the density function, the large \( N \) form of the Haar measure can be written as
\[ [dU] \to \prod_{k=1}^\infty d\rho_k \exp^{-N^2\sum_{k=1}^\infty \frac{1}{2k^2} + N\sum_{k=1}^\infty \frac{1}{2k^2} t_k} \]  
(B.5)

up to an overall constant, and the character is of a form such that
\[ \chi_{\text{adj}}(U^m) \to N^2 \sum_{k=1}^\infty \frac{1}{2k} \rho_k^2 - N \sum_{k=1}^\infty \frac{1}{2k^2} \rho_{2k}. \]  
(B.6)

Plugging these into the gauge index (A.1) and performing the integrations by Gaussian ones we obtain the large \( N \) index as
\[ I_{\mathcal{N}=4}^{SO(\infty)} \sim \prod_{m \geq 1} \frac{e^{(1-f)(y^m, y^m, v^m, w^m)^2}}{\sqrt{1 - f(t^m, y^m, v^m, w^m)}} \]  
(B.7)

up to an overall constant factor. Employing the method given in [14] the index of single gauge-invariant operators is determined as
\[ I_{\mathcal{N}=4}^{SO(\infty)} = -\frac{1}{4} + \frac{(1-f(t, y, v, w))^2}{4(1-f(t^2, y^2, v^2, w^2))} + \frac{1}{2} I_{U(\infty)}^{N=4} \]  
(B.8)

where \( I_{U(\infty)}^{N=4} \) is the large \( N \) limit of single gauge-invariant operator index of \( \mathcal{N} = 4 \) SYM with \( U(N) \) gauge group given by
\[ I_{U(\infty)}^{N=4} = \frac{vt^2}{1-vt^2} + \frac{t^2/w}{1-t^2/w} + \frac{w^2/v}{1-w^2/v} - \frac{yt^3}{1-yt^3} - \frac{t^3/y}{1-t^3/y}. \]  
(B.9)

Note that the constant term in (B.8) is determined so that vacuum contribution vanishes. This large \( N \) index of the orthogonal gauge group (B.8) exactly agrees with the single particle index describing the supergravity multiplet in \( AdS_5 \times \mathbb{R}P^5 \) given by \( t^{KK}_{E_7} \) by setting
\[ p = \frac{1}{\sqrt{w}}, \quad q = \sqrt{w} \, v, \quad z = \frac{1}{v}. \]  
(B.10)

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