Quantization of Galois theory,
Examples and observations

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Abstract

Heiderich [5] discovered that we can apply the Hopf Galois theory also to non-linear equations. We showed in [15] that so far as we consider linear difference-differential equations, Galois group is a linear algebraic group. We give three examples of non-linear difference-differential equations in which quantum groups naturally arise as Galois groups.

1 Introduction

The pursuit of $q$-analogue of hypergeometric functions goes back to the 19th century. Galois group of a $q$-hypergeometric function is not a quantum group but it is a linear algebraic group. This shows that we consider a $q$-deformations of the hypergeometric equation, Galois theory is not quantised. In fact, generally we know that Galois group of a linear difference equation is a linear algebraic group. Y. André [2] was the first who studied linear difference-differential equations in the framework of non-commutative geometry. He encountered only linear algebraic groups treating linear difference-differential equations. See also Hardouin [3]. We clarified the situation in [15]. So far as we study linear difference-differential equations, how twisted or non-commutative the ring of difference and differential operators are, Galois group according to general Hopf Galois theory is a linear algebraic group.

So it is natural to wonder how about considering non-linear difference-differential equations. We proposed to study the $q$-Painlevé equations in [15]. We answer this question in the following way. We see quantization of Galois group for much simpler equations than the $q$-Painlevé equations (Sections 4, 5 and 6).

Except for Lie algebras, all the rings or algebras are associative $Q$-algebras and contain the unit element. Morphisms between them are unitary. For a commutative algebra $A$, we denote by $(Alg/A)$ the category of $A$-algebras, which we sometimes denote by
(\text{CAlg}/A) to emphasize that we are dealing with commutative $A$-algebras. In fact, to study quantum groups, we have to also consider non-commutative $A$-algebras. We denote by (\text{NCAlg}/A) the category of not necessarily commutative $A$-algebras $B$ such that $A$ (or to be more logic, the image of $A$ in $B$) is contained in the center of $B$.

2 Foundation of a general Galois theory [11], [13], [14]

2.1 Notation

Let us recall basic notation. Let $(\mathbb{R}, \delta)$ be a difference ring so that $\delta : \mathbb{R} \to \mathbb{R}$ is a derivation of a commutative ring $\mathbb{R}$ of characteristic 0. When there is no danger of confusion of the derivation $\delta$, we simply say the differential ring $\mathbb{R}$ without referring to the derivation $\delta$. We often have to talk, however, about the abstract ring $\mathbb{R}$ that we denote by $\mathbb{R}^\natural$. For a commutative ring $S$ of characteristic 0, the power series ring $S[[X]]$ with derivation $d/dX$ gives us an example of differential ring.

2.2 General Galois theory of differential field extensions

Let us start by recalling our general Galois theory of differential field extensions.

2.2.1 Universal Taylor morphism

Let $(\mathbb{R}, \delta)$ be a differential ring and $S$ a commutative ring. A Taylor morphism is a differential morphism

$$(\mathbb{R}, \delta) \to (S[[X]], d/dX).$$

Given a differential ring $(\mathbb{R}, \delta)$, among the Taylor morphisms (1), there exists the universal one. In fact, for an element $a \in \mathbb{R}$, we define the power series

$$\iota(a) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n(a)X^n \in \mathbb{R}^\natural[[X]].$$

Then the map

$$\iota : (\mathbb{R}, \delta) \to (\mathbb{R}^\natural[[X]], d/dX)$$

is the universal Taylor morphism.

2.2.2 Galois hull $\mathcal{L}/\mathcal{K}$ for a differential field extension $L/k$

Let $(L, \delta)/(k, \delta)$ be a differential field extension such that the abstract field $L^\natural$ is finitely generated over the abstract base field $k^\natural$. We constructed the Galois hull $\mathcal{L}/\mathcal{K}$ in the following manner.

We take a mutually commutative basis

$$\{D_1, D_2, \ldots, D_d\}$$
of the $L^\sharp$-vector space $\text{Der}(L^\sharp/k^\sharp)$ of $k^\sharp$-derivations of the abstract field $L^\sharp$. So we have

$$[D_i, D_j] = D_i D_j - D_j D_i = 0 \quad \text{for } 1 \leq i, j \leq d.$$ 

Now we introduce a partial differential structure on the abstract field $L^\sharp$ using the derivations $\{D_1, D_2, \cdots, D_d\}$. Namely we set

$$L^\sharp := (L^\sharp, \{D_1, D_2, \cdots, D_d\})$$

that is a partial differential field. Similarly we define a differential structure on the power series ring $L^\sharp[[X]]$ with coefficients in $L^\sharp$ by considering the derivations $\{D_1, D_2, \cdots, D_d\}$ that operate on the coefficients of the power series. In other words, we work with the differential ring $L^\sharp[[X]]$. So the power series ring $L^\sharp[[X]]$ has differential structure defined by the differentiation $d/dX$ with respect to the variable $X$ and the set $\{D_1, D_2, \cdots, D_d\}$ of derivations. Since there is no danger of confusion of the choice of the differential operator $d/dX$, we denote this differential ring by $L^\sharp[[X]]$.

We have the universal Taylor morphism

$$\iota: L \to L^\sharp[[X]] \quad (3)$$

that is a differential morphism. We added further the $\{D_1, D_2, \cdots, D_d\}$-differential structure on $L^\sharp[[X]]$ or we replace the target space $L^\sharp[[X]]$ of the universal Taylor morphism (3) by $L^\sharp[[X]]$ so that we have

$$\iota: L \to L^\sharp[[X]].$$

In Definition 2.1 below, we work in the differential ring $L^\sharp[[X]]$ with differential operators $d/dX$ and $\{D_1, D_2, \cdots, D_d\}$. We identify the differential field $L^\sharp$ with the set of power series consisting only of constant terms. Namely,

$$L^\sharp = \left\{ \sum_{n=0}^{\infty} a_n X^n \in L^\sharp[[X]] \mid \text{The coefficients } a_n = 0 \text{ for every } n \geq 1 \right\}.$$ 

Therefore $L^\sharp$ is a differential sub-field of the differential ring $L^\sharp[[X]]$. The differential operator $d/dX$ kills $L^\sharp$. Similarly, we set

$$k^\sharp := \left\{ \sum_{n=0}^{\infty} a_n X^n \in L^\sharp[[X]] \mid \text{The coefficients } a_0 \in k \text{ and } a_n = 0 \text{ for every } n \geq 1 \right\}.$$ 

So all the differential operators $d/dX, D_1, D_2, \cdots, D_d$ act trivially on $k^\sharp$ and so $k^\sharp$ is a differential sub-field of $L^\sharp$ and hence of the differential algebra $L^\sharp[[X]]$.

**Definition 2.1.** The Galois hull $\mathcal{L}/\mathcal{K}$ is the differential sub-algebra of $L^\sharp[[X]]$, where $\mathcal{L}$ is the differential sub-algebra generated by the image $\iota(L)$ and $L^\sharp$ and $\mathcal{K}$ is the sub-algebra generated by the image $\iota(k)$ and $L^\sharp$. So $\mathcal{L}/\mathcal{K}$ is a differential algebra extension with differential operators $d/dX$ and $\{D_1, D_2, \cdots, D_d\}$. 

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2.2.3 Universal Taylor morphism for a partial differential ring

The universal Taylor morphism has a generalization for partial differential ring. Let

$$(R, \{\partial_1, \partial_2, \cdots, \partial_d\})$$

be a partial differential ring. So $R$ is a commutative ring of characteristic 0 and $\partial_i : R \to R$ are mutually commutative derivations. For a ring $S$, the power series ring

$$(S[[X_1, X_2, \cdots, X_d]], \{\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \cdots, \frac{\partial}{\partial X_d}\})$$

gives us an example of partial differential ring.

A Taylor morphism is a differential morphism

$$(R, \{\partial_1, \partial_2, \cdots, \partial_d\}) \to (S[[X_1, X_2, \cdots, X_d]], \{\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \cdots, \frac{\partial}{\partial X_d}\}).$$

(4)

For a differential algebra $(R, \{\partial_1, \partial_2, \cdots, \partial_d\})$, among Taylor morphisms (4), there exists the universal one $\iota_R$ given below.

**Definition 2.2.** The universal Taylor morphism is a differential morphism

$$\iota_R : (R, \{\partial_1, \partial_2, \cdots, \partial_d\}) \to (R^*[X_1, X_2, \cdots, X_d]], \{\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \cdots, \frac{\partial}{\partial X_d}\})$$

(5)

defined by the formal power series expansion

$$\iota_R(a) = \sum_{n \in \mathbb{N}^d} \frac{1}{n!} \partial^n a X^n$$

for an element $a \in R$, where we use the standard notation for multi-index.

Namely, for $n = (n_1, n_2, \cdots, n_d) \in \mathbb{N}^d$,

$$|n| = \sum_{i=1}^d n_i,$$

$$\partial^n = \partial_1^{n_1} \partial_2^{n_2} \cdots \partial_d^{n_d}$$

$$n! = n_1! n_2! \cdots n_d!$$

and

$$X^n = X_1^{n_1} X_2^{n_2} \cdots X_d^{n_d}.$$
2.2.4 The functor $\mathcal{F}_{L/k}$ of infinitesimal deformations for a differential field extension

For the partial differential field $L^♯$, we have the universal Taylor morphism

$$\iota_L^♯ : L^♯ \to L^♯[[W_1, W_2, \ldots, W_d]] = L^♯[[W]],$$

(6)

where we replaced the variables $X$’s in (5) by the variables $W$’s for a notational reason. The universal Taylor morphism (6) gives a differential morphism

$$L^♯[[X]] \to L^♯[[W_1, W_2, \ldots, W_d]][[X]].$$

(7)

Restricting the morphism (7) to the differential sub-algebra $\mathcal{L}$ of $L^♯[[X]]$, we get a differential morphism $\mathcal{L} \to L^♯[[W_1, W_2, \ldots, W_d]][[X]]$ that we denote by $\iota$. So we have the differential morphism

$$\iota : \mathcal{L} \to L^♯[[W_1, W_2, \ldots, W_d]][[X]].$$

(8)

Similarly for every commutative $L^♯$-algebra $A$, thanks to the differential morphism

$$L^♯[[W]] \to A[[W]]$$

we have the canonical differential morphism

$$\iota : \mathcal{L} \to A[[W_1, W_2, \ldots, W_d]][[X]].$$

(9)

We define the functor

$$\mathcal{F}_{L/k} : (Alg/L^♯) \to (Set)$$

from the category $(Alg/L^♯)$ of commutative $L^♯$-algebras to the category $(Set)$ of sets, by associating to an $L^♯$-algebra $A$, the set of infinitesimal deformations of the canonical morphism $\iota$. So

$$\mathcal{F}_{L/k}(A) = \{ f : \mathcal{L} \to A[[W_1, W_2, \ldots, W_d]][[X]] | f \text{ is a differential morphism congruent to the canonical morphism } \iota \text{ modulo nilpotent elements such that } f = \iota \text{ when restricted on the sub-algebra } \mathcal{K} \}. $$

2.2.5 Group functor $\text{Inf-gal}(L/k)$ of infinitesimal automorphisms for a differential field extension

The Galois group in our Galois theory is the group functor

$$\text{Inf-gal}(L/k) : (Alg/L^♯) \to (Grp)$$

defined by

$$\text{Inf-gal}(L/k)(A) = \{ f : \mathcal{L} \otimes_{L^♯} A[[W]] \to \mathcal{L} \otimes_{L^♯} A[[W]] | f \text{ is a differential } \mathcal{K} \otimes_{L^♯} A[[W]]\text{-automorphism continuous with respect to the } W\text{-adic topology and congruent to the identity modulo nilpotent elements } \}$$

for a commutative $L^♯$-algebra $A$. Here the completion is taken with respect too the $W$-adic topology. See Definition 2.19 in [S].

Then the group functor $\text{Inf-gal}(L/k)$ operates on the functor $\mathcal{F}_{L/k}$ in such a way that the operation $(\text{Inf-gal}(L/k), \mathcal{F}_{L/k})$ is a principal homogeneous space (Theorem 2.20, [S]).
2.2.6 Origin of the group structure

We explained the origin of the group functor Inf-gal. We illustrate it by an example.

Example 2.3. Let us consider a differential field extension

\[ L/k := \mathbb{C}(y), \delta)/\mathbb{C} \]

such that \( y \) is transcendental over the field \( \mathbb{C} \) and

\[ \delta(y) = y \quad \text{and} \quad \delta(\mathbb{C}) = 0 \tag{10} \]

so that \( k = \mathbb{C} \) is the field of constants of \( L \).

The universal Taylor morphism

\[ \iota : L \to L^\sharp[[X]] \]

maps \( y \in L \) to

\[ Y := y \exp X \in L^\sharp[[X]]. \]

Since the field extension \( L^\sharp/k^\sharp = \mathbb{C}(y)/\mathbb{C} \), taking \( d/dy \in \text{Der}(L^\sharp/k^\sharp) \) as a basis of 1-dimensional \( L^\sharp \)-vector space \( \text{Der}(L^\sharp/k^\sharp) \), we get \( L^\sharp := (L^\sharp, d/dy) \). As we have a relation

\[ y \frac{\partial Y}{\partial y} = Y \tag{11} \]

that is an equality in the power series ring \( L^\sharp[[X]] \) so that the Galois hull \( L/K \) is

\[ L = K[\exp X], \quad K = L^\sharp \subset L^\sharp[[X]] \tag{12} \]

by definition of the Galois hull.

Now let us see the infinitesimal deformation functor \( \mathcal{F}_{L/k} \). To this end, we Taylor expand the coefficients of the power series in \( L^\sharp[[X]] \) to get

\[ \iota : L \to L^\sharp[[X]] \to L^\sharp[[W]][[X]] = L^\sharp[[W, X]] \]

so that \( \iota(y) = (y + W) \exp X \in L^\sharp[[W, X]]. \)

It follows from (11) and (12), for a commutative \( L^\sharp \)-algebra \( A \) an infinitesimal deformation \( \varphi \in \mathcal{F}_{L/k} \) is determined by the image

\[ \varphi(Y) = cY \in A[[W, X]], \]

where \( c \in A \). Moreover any invertible element \( c \in A \) infinitesimally close to 1 defines an infinitesimal deformation so that we conclude

\[ \mathcal{F}_{L/k}(A) = \{ c \in A \mid c - 1 \text{ is nilpotent} \}. \tag{13} \]

Where does the group structure come from?
To see this, we have to look at the dynamical system defined by the differential equation (10). Geometrically the differential equation (10) gives us a dynamical system on the line $\mathbb{C}$.

$$y \mapsto Y = y \exp X$$

describes the dynamical system. Observe the dynamical system through algebraic differential equations. is equivalent to considering the deformations of the Galois hull. So the (infinitesimal) deformation functor measures the ambiguity of the observation. In other words, the result due to our method is (13). In terms of the initial condition, it looks as

$$y \mapsto cY |_{X=0} = cy \exp X |_{X=0} = cy.$$ 

Namely,

$$y \mapsto cy. 
(14)$$

If we have two transformations (14)

$$y \mapsto cy, \quad y \mapsto c'y$$

the composite transformation corresponds to the product

$$y \mapsto cc'y.$$ 

### 2.3 Difference Galois theory

If we replace the universal Taylor morphism by the universal Euler morphism, we can construct a general Galois theory of difference equations ([8], [9]).

#### 2.3.1 Universal Euler morphism

Let $(R, \sigma)$ be a difference ring so that $\sigma : R \to R$ is an endomorphism of a commutative ring $R$. When there is no danger of confusion of the endomorphism $\sigma$, we simply say the difference ring $R$ without referring to the endomorphism $\sigma$. We often have to talk however about the abstract ring $R$ that we denote by $R^S$. For a commutative ring $S$, we denote by $F(\mathbb{N}, S)$ the ring of functions on the set

$$\mathbb{N} = \{0, 1, 2, \cdots \}$$

taking values in the ring $R$. For a function $f \in F(\mathbb{N}, S)$, we define the shifted function $\Sigma f \in F(\mathbb{N}, S)$ by

$$(\Sigma f)(n) = f(n + 1) \quad \text{for every } n \in \mathbb{N}.$$ 

Hence the shift operator

$$\Sigma : F(\mathbb{N}, S) \to F(\mathbb{N}, S)$$

is an endomorphism of the ring $F(\mathbb{N}, S)$ so that $(F(\mathbb{N}, S), \Sigma)$ is a difference ring.

**Remark 2.4.** In this paragraph 2.3 and the next 2.3.1, in particular for the existence of the universal Euler morphism, we do not need the commutativity assumption of the underlying ring.
Let \((R, \sigma)\) be a difference ring and \(S\) a ring. An Euler morphism is a difference morphism
\[
(R, \sigma) \rightarrow (F(\mathbb{N}, S), \Sigma).
\] (15)
Given a difference ring \((R, \sigma)\), among the Euler morphisms (15), there exists the universal one. In fact, for an element \(a \in R\), we define the function \(u[a] \in F(\mathbb{N}, R^\natural)\) by
\[
u[a](n) = \sigma^n(a) \quad \text{for } n \in \mathbb{N}.
\]
Then the map
\[
\nu: (R, \sigma) \rightarrow (F(\mathbb{N}, R^\natural), \Sigma) \quad a \mapsto u[a]
\] (16)
is the universal Euler morphism (Proposition 2.5, [8]).

2.3.2 Galois hull \(L/K\) for a difference field extension \(L/k\)
Let \((L, \sigma)/k\sigma\) be a difference field extension such that the abstract field \(L^\natural\) is finitely generated over the abstract base field \(k^\natural\). We constructed the Galois hull \(L/K\) as in the differential case. Namely, we take a mutually commutative basis
\[
\{D_1, D_2, \ldots, D_d\}
\]
of the \(L^\natural\)-vector space \(\text{Der}(L^\natural/k^\natural)\) of \(k^\natural\)-derivations of the abstract field \(L^\natural\). We introduce the partial differential field
\[
L^\natural := (L^\natural, \{D_1, D_2, \ldots, D_d\}).
\]
Similarly we define a differential structure on the ring \(F(\mathbb{N}, L^\natural)\) of functions taking values in \(L^\natural\) by considering the derivations
\[
\{D_1, D_2, \ldots, D_d\}.
\]
In other words, we work with the differential ring \(F(\mathbb{N}, L^\natural)\). So the ring \(F(\mathbb{N}, L^\natural)\) has a difference-differential structure defined by the shift operator \(\Sigma\) and the set
\[
\{D_1, D_2, \ldots, D_d\}.
\]
of derivations. Since there is no danger of confusion of the choice of the difference operator \(\Sigma\), we denote this difference-differential ring by
\[
F(\mathbb{N}, L^\natural).
\]
We have the universal Euler morphism
\[
\nu: L \rightarrow F(\mathbb{N}, L^\natural)
\] (17)
that is a difference morphism. We added further the \(\{D_1, D_2, \ldots, D_d\}\)-differential structure on \(F(\mathbb{N}, L^\natural)\) or we replace the target space \(F(\mathbb{N}, L^\natural)\) of the universal Euler morphism (17) by \(F(\mathbb{N}, L^\natural)\) so that we have
\[
\nu: L \rightarrow F(\mathbb{N}, L^\natural).
\]
In Definition 2.5 below, we work in the difference-differential ring $F(N, L^2)$ with difference operator $\Sigma$ and differential operators $\{D_1, D_2, \cdots, D_d\}$. We identify with $L^2$ the set of constant functions on $N$. Namely,

$$L^2 = \{ f \in F(N, L^2) \mid f(0) = f(1) = f(2) = \cdots \in L^2 \}.$$  

Therefore $L^2$ is a difference-differential sub-field of the difference-differential ring $F(N, L^2)$. The action of the shift operator on $L^2$ being trivial, the notation is adequate. Similarly, we set

$$k^2 := \{ f \in F(N, L^2) \mid f(0) = f(1) = f(2) = \cdots \in k \subset L^2 \}.$$  

So both the shift operator and the derivations act trivially on $k^2$ and so $k^2$ is a difference-differential sub-field of $L^2$ and hence of the difference-differential algebra $F(N, L^2)$.

**Definition 2.5.** The Galois hull $L/K$ is a difference-differential sub-algebra extension of $F(N, L^2)$, where $L$ is the difference-differential sub-algebra generated by the image $\iota(L)$ and $L^2$ and $K$ is the sub-algebra generated by the image $\iota(k)$ and $L^2$. So $L/K$ is a difference-differential algebra extension with difference operator $\Sigma$ and derivations $\{D_1, D_2, \cdots, D_d\}$.

### 2.3.3 The functor $\mathcal{F}_{L/k}$ of infinitesimal deformations for a difference field extension

For the partial differential field $L^2$, we have the universal Taylor morphism

$$\iota_{L^2}: L^2 \to L^2[[W_1, W_2, \cdots, W_d]] = L^2[[W]].$$  

(18)

The universal Taylor morphism (18) gives a difference-differential morphism

$$F(N, L^2) \to F(N, L^2[[W_1, W_2, \cdots, W_d]]).$$  

(19)

Restricting the morphism (19) to the difference-differential sub-algebra $\mathcal{L}$ of $F(N, L^2)$, we get a difference-differential morphism $\mathcal{L} \to F(N, L^2[[W_1, W_2, \cdots, W_d]])$ that we denote by $\iota$. So we have the difference-differential morphism

$$\iota: \mathcal{L} \to F(N, L^2[[W_1, W_2, \cdots, W_d]]).$$  

(20)

Similarly for every commutative $L^2$-algebra $A$, thanks to the differential morphism

$$L^2[[W]] \to A[[W]],$$

we have the canonical difference-differential morphism

$$\iota: \mathcal{L} \to F(N, A[[W_1, W_2, \cdots, W_d]]).$$  

(21)

We define the functor

$$\mathcal{F}_{L/k}: (Alg/L^2) \to (Set)$$

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from the category \((\text{Alg}/L^2)\) of commutative \(L^2\)-algebras to the category \((\text{Set})\) of sets, by associating to a commutative \(L^2\)-algebra \(A\), the set of infinitesimal deformations of the canonical morphism \((20)\). So

\[
\mathcal{F}_{L/k}(A) = \{ f : \mathcal{L} \to F(N, A[[W_1,W_2,\ldots,W_d]]) \mid f \text{ is a differential morphism congruent to the canonical morphism } \iota \text{ modulo nilpotent elements such that } f = \iota \text{ when restricted on the sub-algebra } K \}\.
\]

See Definition 2.13 in [8], for a rigorous definition.

### 2.3.4 Group functor \(\text{Inf-gal}(L/k)\) of infinitesimal automorphisms for a difference field extension

The Galois group in our Galois theory is the group functor

\[
\text{Inf-gal}(L/k) : (\text{Alg}/L^2) \to (\text{Grp})
\]

defined by

\[
\text{Inf-gal}(L/k)(A) = \{ f : \mathcal{L} \hat{\otimes}_{L^2} A[[W]] \to \mathcal{L} \hat{\otimes}_{L^2} A[[W]] \mid f \text{ is a difference-differential } K \hat{\otimes}_{L^2} A[[W]] \text{-automorphism continuous with respect to the } W\text{-adic topology and congruent to the identity modulo nilpotent elements } \}
\]

for a commutative \(L^2\)-algebra \(A\). Here the completion is taken with respect to the \(W\)-adic topology. See Definition 2.19 in [8].

Then the group functor \(\text{Inf-gal}(L/k)\) operates on the functor \(\mathcal{F}_{L/k}\) in such a way that the operation \((\text{Inf-gal}(L/k), \mathcal{F}_{L/k})\) is a principal homogeneous space (Theorem 2.20, [8]).

### 2.4 Introduction of more precise notations

So far, we explained general differential Galois theory and general difference Galois theory. To go further we have to make our notations more precise.

For example, we defined the Galois hull for a differential field extension in Definition 2.1 and the Galois hull for a difference field extension in Definition 2.5. Since they are defined by the same principle, we denoted both of them by \(\mathcal{L}/K\). We have to, however, distinguish them.

**Definition 2.6.** We denote the Galois hull for a differential field extension by \(\mathcal{L}_\delta/K_\delta\) and we use the symbol \(\mathcal{L}_\sigma/K_\sigma\) for the Galois hull of a difference field extension.

We also have to distinguish the functors \(\mathcal{F}_{L/k}\) and \(\text{Inf-gal}(L/k)\) in the differential case and in the difference case: we add the suffix \(\delta\) for the differential case and the suffix \(\sigma\) for the difference case so that

1. We use \(\mathcal{F}_{\delta L/k}\) and \(\text{Inf-gal}_\delta(L/k)\) when we deal with differential algebras.
2. We use \(\mathcal{F}_{\sigma L/k}\) and \(\text{Inf-gal}_\sigma(L/k)\) for difference algebras.
We denoted, according to our convention, for a commutative algebra $A$ the category of commutative $A$-algebras by $(\text{Alg}/A)$. As we are going to consider the category of not necessarily commutative $A$-algebras. This notation is confusing. So we clarify the notation.

**Definition 2.7.** We often denote the category of commutative $A$-algebras by $(\text{CAlg}/A)$.

### 3 Hopf Galois theory

Picard-Vessiot theory is a Galois theory of linear differential or difference equations. The idea of introducing Hopf algebra in Picard-Vessiot theory is traced back to Sweedler [10]. Specialists in Hopf algebra succeeded in unifying Picard-Vessiot theories for differential equations and difference equations [1]. They further succeeded in generalizing the Picard-Vessiot theory for difference-differential equations, where the operators are not necessarily commutative. Heiderich [5] combined the idea of Picard-Vessiot theory via Hopf algebra with our general Galois theory for non-linear equations [11], [8]. His general theory includes a wide class of difference and differential algebras.

There are two major advantages in his theory.

1. Unified study of non-linear differential equations and difference equations.
2. Generalization of universal Euler morphism and Taylor morphism.

Let $C$ be a field. For $C$-vector spaces $M, N$, we denote by $cM(M, N)$ the set of $C$-linear maps from $M$ to $N$.

**Example 3.1.** Let $\mathcal{H} := C[\mathbb{G}_a] = C[t]$ be the $C$-Hopf algebra of the coordinate ring of the additive group scheme $\mathbb{G}_{a,\mathcal{C}}$ over the field $C$. Let $A$ be a commutative $C$-algebra and

$$\Psi \in cM(A \otimes_C \mathcal{H}, A) = cM(A, cM(\mathcal{H}, A))$$

so that $\Psi$ defines two $C$-linear maps

1. $\Psi_1: A \otimes_C \mathcal{H} \to A$,
2. $\Psi_2: A \to cM(\mathcal{H}, A)$.

**Definition 3.2.** We say that $(A, \Psi)$ is an $\mathcal{H}$-module algebra if the following equivalent conditions are satisfied.

1. The $C$-linear map $\Psi_1: A \otimes_C \mathcal{H} \to A$ defines an operation of the $C$-algebra $\mathcal{H}$ on the $C$-algebra $A$,

2. The $C$-linear map $\Psi_2: A \to cM(\mathcal{H}, A)$ is a $C$-algebra morphism, the dual $cM(\mathcal{H}, A)$ of co-algebra $\mathcal{H}$ being a $C$-algebra.
Concretely the dual algebra $C^\mathcal{M}(\mathcal{H},A)$ is the formal power series ring $A[[X]]$.

It is a comfortable exercise to examine that $(A, \Psi)$ is an $\mathcal{H}$-module algebra if and only if $A$ is a differential algebra with derivation $\delta$ such that $\delta(C) = 0$. When the equivalent conditions are satisfied, for every element $a$ in the algebra $A$, $\Psi(a \otimes t) = \delta(a)$ and the $C$-algebra morphism

$$\Psi_2 : A \to C^\mathcal{M}(\mathcal{H},A) = A[[X]]$$

is the universal Taylor morphism. So

$$\Psi_2(a) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n(a) X^n \in A[[X]]$$

for every $a \in A$. See Heiderich [5], 2.3.4.

In Example 3.1, we explained the differential case. If we take an appropriate bialgebra for $\mathcal{H}$, we get difference structure and the universal Euler morphism. See [5], 2.3.1. More generally we can take any bialgebra $\mathcal{H}$ to get an algebra $A$ with operation of the algebra $\mathcal{H}$ and a morphism

$$\Psi_2 : A \to C^\mathcal{M}(\mathcal{H},A)$$

generalizing the universal Taylor morphism and Euler morphism. So we can define the Galois hull $\mathcal{L}/\mathcal{K}$ and develop a general Galois theory for a field extension $L/k$ with operation of the algebra $\mathcal{H}$. In the differential case as well as in the difference case, the corresponding bialgebra $\mathcal{H}$ is co-commutative so that the dual algebra $C^\mathcal{M}(\mathcal{H},A)$ is a commutative algebra. Consequently the Galois hull $\mathcal{L}/\mathcal{K}$ that are sub-algebras in the commutative algebra $C^\mathcal{M}(\mathcal{H},A)$. In these case the Galois hull is an algebraic counter part of the geometric object, algebraic Lie groupoid. See Malgrange [6]. Therefore the most fascinating quion is

**Question 3.3.** Let us consider a non-co-commutative bialgebra $\mathcal{H}$ and assume that the Galois hull $\mathcal{L}/\mathcal{K}$ that is a sub-algebra of the dual algebra $C^\mathcal{M}(\mathcal{H},A)$, is not a commutative algebra. Does the Galois hull $\mathcal{L}/\mathcal{K}$ quantize the algebraic Lie groupoid?

We answer affirmatively the question by analyzing examples in $q$-SI $\sigma$-differential field extensions.

**Remark 3.4.** How non-co-commutative the bialgebra $\mathcal{H}$ may be, so far as one considers linear equations, the Galois hull $\mathcal{L}/\mathcal{K}$ is a commutative sub-algebra of the non-commutative algebra $C^\mathcal{M}(\mathcal{H},A)$. Hence one does not encounter quantum groups, except for linear algebraic groups, studying generalized Picard-Vessiot theories. See Hardouin [3] and Umemura [15].

Let $C$ be a field, $q$ an element of $C$. We use a standard notation of $q$-binomial coefficients. To this end, let $Q$ be a variable over the field $C$.

We set $[n]_Q = \sum_{i=0}^{n-1} Q^i \in C[Q]$ for positive integer $n$. We need also $q$-factorial

$$[n]_Q! := \prod_{i=1}^{n} [i]_Q$$

for a positive integer $n$ and $[0]_Q! := 1$.  

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So \([n]_Q \in C[Q]\). The \(Q\)-binomial coefficient is defined for \(m, n \in \mathbb{N}\) by

\[
\binom{m}{n}_Q = \begin{cases} 
\frac{[m]_Q!}{[m-n]_Q! [n]_Q!} & \text{if } m \geq n, \\
0 & \text{if } m < n.
\end{cases}
\]

Then we can show that the rational function

\[
\binom{m}{n}_Q \in C(Q)
\]

is in fact a polynomial or

\[
\binom{m}{n}_Q \in C[Q].
\]

We have a ring morphism

\[C[Q] \to C[q], \quad Q \mapsto q \quad (22)\]

over \(C\) and we denote the image of the polynomial

\[
\binom{m}{n}_Q
\]

under morphism \((22)\) by

\[
\binom{m}{n}_q.
\]

### 3.1 \(q\)-skew iterative \(\sigma\)-differential algebra [3], [4]

First non-trivial example of a Hopf Galois theory dependent on non-co-commutative Hopf algebra is Galois theory of \(q\)-skew iterative \(\sigma\)-differential field extensions, abbreviated as \(q\)-SI \(\sigma\)-differential field extensions.

#### 3.1.1 Definition of \(q\)-SI \(\sigma\)-differential algebra

**Definition 3.5.** Let \(C\) be a field of characteristic 0 and \(q \neq 0\) an element of the field \(C\). A \(q\)-skew iterative \(\sigma\)-differential algebra \((A, \sigma, \theta^i) = (A, \sigma, \{\theta^i\}_{i \in \mathbb{N}})\), a \(q\)-SI \(\sigma\)-differential algebra for short, consists of a \(C\)-algebra \(A\) that is eventually non-commutative, a \(C\)-endomorphism \(\sigma : A \to A\) of the \(C\)-algebra \(A\) and a family

\[
\theta^i : A \to A \quad \text{for } i \in \mathbb{N}
\]

of \(C\)-linear maps satisfying the following conditions.

1. \(\theta^{(0)} = \text{Id}_A\),
2. \(\theta^{(i)} \sigma = q^i \sigma \theta^{(i)}\) for every \(i \in \mathbb{N}\),
3. \(\theta^{(i)}(ab) = \sum_{l+m=i} \sigma^m(\theta^{(l)}(a))\theta^{(m)}(b)\),
\[ \theta^{(i)} \circ \theta^{(j)} = \binom{i+j}{i}_q \theta^{(i+j)}. \]

We say that an element \( a \) of the \( q \)-SI \( \sigma \)-differential algebra \( A \) is a constant if \( \sigma(a) = a \) and \( \theta^{(i)}(a) = 0 \) for every \( i \geq 1 \).

A morphism of \( q \)-SI \( \sigma \)-differential \( C \)-algebra morphism compatible with the endomorphisms \( \sigma \) and the derivations \( \theta^* \).

Both differential algebras and difference algebras are \( q \)-SI \( \sigma \)-differential algebras.

### 3.1.2 Difference algebra and a \( q \)-SI \( \sigma \)-differential algebra

Let \( A \) be a commutative \( C \)-algebra and \( \sigma : A \to A \) be a \( C \)-endomorphism of the ring \( A \). So \( (A, \sigma) \) is a difference algebra. If we set \( \theta^{(0)} = \text{Id}_A \) and

\[ \theta^{(i)}(a) = 0 \text{ for every element } a \in A \text{ and for } i = 1, 2, 3, \ldots. \]

Then \( (A, \sigma, \theta^*) \) is a \( q \)-SI \( \sigma \)-differential algebra.

Namely we have a functor of the category \( \text{(Diff}^\prime \text{ceAlg/C)} \) of \( C \)-difference algebras to the category \( \text{(q-SI}\sigma\text{-diff}^\prime \text{ialAlg/C)} \) of \( q \)-SI \( \sigma \)-differential algebras over \( C \):

\[ \text{(Diff}^\prime \text{ceAlg/C)} \to \text{(q-SI}\sigma\text{-diff}^\prime \text{ialAlg/C)}. \]

Let \( t \) be a variable over the field \( C \) and let us now assume

\[ q \neq 1 \quad \text{for every integer } n \in \mathbb{N}. \] (23)

We denote by \( \sigma : C(t) \to C(t) \) the \( C \)-automorphism of the rational function field \( C(t) \) sending the variable \( t \) to \( qt \). We consider a difference algebra extension \( (A, \sigma)/(C(t), \sigma) \).

If we set

\[ \theta^{(1)}(a) = \frac{\sigma(a) - a}{(q-1)t} \quad \text{for every element } a \in A \]

and

\[ \theta^{(i)} = \frac{1}{[i]_q!} \theta^{(1)} \quad \text{for } i = 2, 3, \ldots. \]

Then \( (A, \sigma, \theta^*) \) is a \( q \)-SI \( \sigma \)-differential algebra. Therefore if \( q \in C \) satisfies (23), then we have a functor

\[ \text{(Diff}^\prime \text{ceAlg}((C(t), \sigma)) \to \text{(q-SI}\sigma\text{-diff}^\prime \text{ialAlg)}. \] (24)

### 3.1.3 Differential algebra and \( q \)-SI \( \sigma \)-differential algebra

Let \( (A, \theta) \) be a differential algebra such that the field \( C \) is a subfield of the ring \( C_A \) of constants of the differential algebra \( A \). We set

\[ \theta^{(0)} = \text{Id}_A, \]

\[ \theta^{(i)} = \frac{1}{i!} \theta^i \quad \text{for } i = 1, 2, 3, \ldots. \]
Then \((A, \text{Id}_A, \theta^*)\) is a \(q\)-SI \(\sigma\)-differential algebra for \(q = 1\). In other words, we have a functor

\[
(\text{Diff}^\prime\text{ialAlg}/C) \to (q\text{-SI}\sigma\text{-diff}^\prime\text{ialAlg}/C)
\]

of the category of (commutative) differential \(C\)-algebras to the category of \(q\)-SI \(\sigma\)-differential algebras over \(C\). We have shown that both difference algebras and differential algebras are particular instances of \(q\)-SI \(\sigma\)-differential algebra.

### 3.1.4 Example of \(q\)-SI \(\sigma\)-differential algebra

We are going to see \(q\)-SI \(\sigma\)-differential algebras on the border between commutative algebras and non-commutative algebras. The example below seems to suggest that it looks natural to seek \(q\)-SI \(\sigma\)-differential algebras in the category of non-commutative algebras.

An example of \(q\)-SI \(\sigma\)-differential algebra arises from a commutative \(C\)-difference algebra \((S, \sigma)\). We need, however, a non-commutative ring, the twisted power series ring \((S, \sigma)[[X]]\) over the difference ring \((S, \sigma)\) that has a natural \(q\)-SI \(\sigma\)-differential algebra structure.

Namely, let \((S, \sigma)\) be the \(C\)-difference ring so that \(\sigma : S \to S\) is a \(C\)-algebra endomorphism of the commutative ring \(S\). We introduce the following twisted formal power series ring \((S, \sigma)[[X]]\) with coefficients in \(S\) that is the formal power series ring \(S[[X]]\) as an additive group with the following commutation relation

\[
aX = X\sigma(a) \quad \text{for every } a \in S.
\]

So more generally

\[
aX^n = X^n\sigma^n(a)
\]

for every \(n \in \mathbb{N}\). The multiplication of two formal power series is defined by extending \(^{(25)}\) by linearity. Therefore the twisted formal power series ring \((S, \sigma)[[X]]\) is non-commutative in general. By commutation relation \(^{(25)}\), we can identify

\[
(S, \sigma)[[X]] = \{ \sum_{i=0}^{\infty} X^i a_i \mid a_i \in S \text{ for every } i \in \mathbb{N} \}
\]

as additive groups.

We are going to see that the twisted formal power series ring has a natural \(q\)-SI \(\sigma\)-differential structure. We define first a ring endomorphism

\[
\hat{\Sigma} : (S, \sigma)[[X]] \to (S, \Sigma)[[X]]
\]

by setting

\[
\hat{\Sigma}(\sum_{i=0}^{\infty} X^i a_i) = \sum_{i=0}^{\infty} X^i q^i \sigma(a_i) \quad \text{for every } i \in \mathbb{N},
\]

for every element

\[
\sum_{i=0}^{\infty} X^i a_i \in (S, \sigma)[[X]]
\]
The operators $\Theta^* = \{\theta^{(l)}\}_{l \in \mathbb{N}}$ are defined by

$$
\Theta^{(l)}(\sum_{i=0}^{\infty} X^i a_i) = \sum_{i=0}^{\infty} X^i \left( \frac{i + l}{q} \right) a_{i+l} \quad \text{for every } i \in \mathbb{N}.
$$

(27)

Hence the twisted formal power series ring $(S, \sigma)[[X]], \hat{\Sigma}, \Theta^*$ is a non-commutative $q$-SI $\sigma$-differential ring. We denote this $q$-SI $\sigma$-differential ring simply by $(S, \sigma)[[X]]$. See [5], 2.3. In particular, if we take as the coefficient difference ring $S$ the difference ring $(\mathbb{F}(\mathbb{N}, A), \Sigma)$ in 2.3, where $\Sigma : \mathbb{F}(\mathbb{N}, A) \to \mathbb{F}(\mathbb{N}, A)$ is the shift operator, we obtain the $q$-SI $\sigma$-differential ring

$$(\mathbb{F}(\mathbb{N}, A), \Sigma)[[X]], \hat{\Sigma}, \Theta^*).$$

(28)

**Remark 3.6.** We assumed that the coefficient difference ring $(S, \sigma)$ is commutative. The commutativity assumption on the ring $S$ is not necessary. Consequently we can use non-commutative ring $A$ in (28).

### 3.1.5 Universal Hopf morphism for a $q$-SI $\sigma$-differential algebra

We introduced in [2.3] the difference ring of functions $(F(\mathbb{N}, A), \Sigma)$ on the set $\mathbb{N}$ taking values in a ring $A$. It is useful to denote the function $f$ by a matrix

$$
\begin{bmatrix}
0 & 1 & 2 & \cdots \\
0 & 1 & 2 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
$$

For an element $b$ of a difference algebra $(R, \sigma)$ or a $q$-SI $\sigma$-differential algebra $(R, \sigma, \theta^*)$, we denote by $u[b]$ a function on $\mathbb{N}$ taking values in the abstract ring $R^\#$ such that

$$u[b](n) = \sigma^n(b) \quad \text{for every } n \in \mathbb{N}$$

so that

$$u[b] = \begin{bmatrix} 0 & 1 & 2 & \cdots \\ b & \sigma^1(b) & \sigma^2(b) & \cdots \end{bmatrix}.$$

So $u[b] \in F(\mathbb{N}, R^\#)$.

**Proposition 3.7** (Proposition 2.3.17, Heiderich [5]). For a $q$-SI $\sigma$-differential algebra $(R, \sigma, \theta^*)$, hence in particular for an iterative $q$-difference ring $R$, there exists a canonical morphism, which we call the universal Hopf morphism

$$
u: (R, \sigma, \theta^*) \to \left( (F(\mathbb{N}, R^\#), \Sigma)[[X]], \hat{\Sigma}, \hat{\Theta}^* \right), \quad a \mapsto \sum_{i=0}^{\infty} X^i u[\theta^{(i)}(a)]$$

(29)

of $q$-SI $\sigma$-differential algebras.
We can also characterize the universal Hopf morphism as the solution of a universal mapping property.

When \( q = 1 \) and \( \sigma = \text{Id}_R \) and \( R \) is commutative so that the \( q \)-SI \( \sigma \)-differential ring \((R, \text{Id}_R, \theta^*)\) is simply a differential algebra as we have seen in \(3.1.3\) the universal Hopf morphism \((29)\) is the universal Taylor morphism in \(2\). Similarly a commutative difference ring is a \( q \)-SI \( \sigma \)-differential ring with trivial derivations as we noticed in \(3.1.2\). In this case the universal Hopf morphism \((29)\) is nothing but the universal Euler morphism \((16)\). Therefore the universal Hopf morphism unifies the universal Taylor morphism and the Universal Euler morphism.

Let us recall the following fact.

**Lemma 3.8.** Let \((R, \sigma, \theta^*)\) be a \( q \)-SI \( \sigma \)-differential domain. If the endomorphism \( \sigma: R \to R \) is an automorphism, then the field \( Q(R) \) of fractions of \( R \) has the unique structure of \( q \)-SI \( \sigma \)-differential field.

If moreover \( R \) is an iterative \( q \)-difference algebra, then the field \( Q(R) \) of fractions of \( R \) is also an iterative \( q \)-difference field.

**Proof.** See for example, Proposition 2.5 of [4].

We can interpret the Example in \(3.1.4\) from another viewpoint. We constructed there from a difference ring \((S, \sigma)\) a \( q \)-SI \( \sigma \)-differential algebra \(((S, \sigma)[[X]], \hat{\Sigma}, \hat{\Theta}^*)\). We notice that this procedure is a particular case of Proposition \(3.7\). In fact, given a difference ring \((S, \sigma)\). So as in \(3.1.2\) by adding the trivial derivations, we get the \( q \)-SI \( \sigma \)-differential algebra \((S, \sigma, \theta^*)\), where

\[
\theta^{(0)} = \text{Id}_S, \\
\theta^{(i)} = 0 \quad \text{for } i \geq 1.
\]

Therefore we have the universal Hopf morphism

\[
(S, \sigma, \theta^*) \to (F(\mathbb{N}, S^\mathbb{Z})[[X]], \hat{\Sigma}, \hat{\Theta}^*)
\]

by Proposition \(3.7\). So we obtained the \( q \)-SI \( \sigma \)-differential algebra \((F(\mathbb{N}, S^\mathbb{Z})[[X]], \hat{\Sigma}, \hat{\Theta}^*)\) as a result of composite of two functors. Namely,

(1) The functor: (Category of Difference algebras) \to (Category of \( q \)-SI \( \sigma \)-differential algebras) of adding trivial derivations

(2) The functor: (Category of \( q \)-SI \( \sigma \)-differential algebras) \to (Category of \( q \)-SI \( \sigma \)-differential algebras), \( A \mapsto B \) if there exists the universal Hopf morphism \( A \to B \).

**3.1.6 Galois hull \( L/K \) for a \( q \)-SI \( \sigma \)-differential field extension**

We can develop a general Galois theory for \( q \)-SI \( \sigma \)-differential field extensions analogous to our theories in \([12],[13]\) and \([14]\) thanks to the universal Hopf morpism. Let \( L/k \) be an extension of \( q \)-SI \( \sigma \)-differential fields such that the abstract field \( L^\Sigma \) is finitely generated over the abstract field \( k^\Sigma \). Let us assume that we are in characteristic 0. General theory in
works, however, also in characteristic $p \geq 0$. We have by Proposition 3.7 the universal Hopf morphism

$$\iota : (L, \sigma, \theta^*) \to \left( (F(N, L^\natural), \Sigma)[[X]], \hat{\Sigma}, \hat{\Theta}^* \right)$$

(30)

so that the image $\iota(L)$ is a copy of the iterative $q$-difference field $L$. We have another copy of $L^\natural$. The set

$$\left\{ f = \sum_{i=0}^{\infty} X^i a_i \in F(N, L^\natural)[[X]] \mid a_i = 0 \text{ for every } i \geq 1 \text{ and } \Sigma(a_0) = a_0 \right\}$$

(31)

forms the sub-ring of constants in the $q$-SI $\sigma$-differential algebra of the twisted power series

$$\left( (F(N, L^\natural), \Sigma)[[X]], \hat{\Sigma}, \hat{\Theta}^* \right).$$

We identify $L^\natural$ with the ring of constants through the following morphism. For an element $a \in L^\natural$, we denote the constant function $f_a$ on $N$ taking the value $a \in L^\natural$ so that

$$L^\natural \to \left( (F(N, L^\natural), \Sigma)[[X]], \hat{\Sigma}, \hat{\Theta}^* \right), \quad a \mapsto f_a$$

(32)

is an injective ring morphism. We may denote the sub-ring in (31) by $L^\natural$. In fact, as an abstract ring it is isomorphic to the abstract field $L^\natural$ and the endomorphism $\hat{\Sigma}$ and the derivations $\hat{\Theta}^*(i)$, $(i \geq 1)$ operate trivially on the sub-ring.

We are now exactly in the same situation as in 2.2.2 of the differential case and in 2.3.2 of the difference case. We choose a mutually commutative basis $\{D_1, D_2, \ldots, D_d\}$ of the $L^\natural$-vector space $\text{Der}(L^\natural/k\natural)$ of $k$-derivations. So $L^\natural := (L^\natural, \{D_1, D_2, \ldots, D_d\})$ is a differential field.

So we introduce derivations $D_1, D_2, \ldots, D_d$ operating on the coefficient ring $F(N, L^\natural)$. In other words, we replace the target space $F(N, L^\natural)[[X]]$ by $F(N, L^\natural)[[X]]$. Hence the universal Hopf morphism in Proposition 3.7 becomes

$$\iota : L \to F(N, L^\natural)[[X]].$$

In the twisted formal power series ring $(F(N, L^\natural)[[X]], \hat{\Sigma}, \hat{\Theta}^*)$, we add differential operators $D_1, D_2, \ldots, D_d$.

So we have a set $\mathcal{D}$ of the following operators on the ring $(F(N, L^\natural), \Sigma)[[X]]$.

1. The endomorphism $\hat{\Sigma}$.

$$\hat{\Sigma}(\sum_{i=0}^{\infty} X^i a_i) = \sum_{i=0}^{\infty} X^i q^i(\Sigma(a_i)),$$

$$\Sigma : F(N, L^\natural) \to F(N, L^\natural)$$

being the shift operator of the ring of functions on $N$. 18
(2) The $q$-skew $\hat{\Sigma}$-derivations $\hat{\Theta}^{(i)}$'s in (27).

\[
\hat{\Theta}^{(i)} \left( \sum_{i=0}^{\infty} X^i a_i \right) = \sum_{i=0}^{\infty} X^i \binom{l+i}{l} q^i a_{i+l} \quad \text{for every } l \in \mathbb{N}.
\]

(3) The derivations $D_1, D_2, \cdots, D_d$ operating through the coefficient ring $F(N, L^\sharp)$ as in (29).

Hence we may write $F(N, L^\sharp)$, where

\[ D = \{ \hat{\Sigma}, D_1, D_2, \cdots, D_d, \hat{\Theta}^* \} \text{ and } \hat{\Theta}^* = \{ \hat{\Theta}^{(i)} \}_{i \in \mathbb{N}}. \]

We identify using inclusion (32)

\[ L^\sharp \to F(N, L^\sharp)[[X]]. \]

We sometimes denote the image $a_f$ of an element $a \in L^\sharp$ by $a^\sharp$.

We are ready to define Galois hull as in Definition 2.1.

**Definition 3.9.** The Galois hull $L/K$ is a $D$-invariant sub-algebra of $F(N, L^\sharp)[[X]]$, where $L$ is the $D$-invariant sub-algebra generated by the image of $L$ and $L^\sharp$ and $K$ is the $D$-invariant sub-algebra generated by the image of $K$ and $L^\sharp$. So $L/K$ is a $D$-algebra extension.

As in 2.4, if we have to emphasize that we deal with $q$-SI $\sigma$-differential -algebras, we denote the Galois hull by $L_{\sigma \theta}/K_{\sigma \theta}$.

We notice that we are now in a totally new situation. In the differential case, the universal Taylor morphism maps the given fields to the commutative algebra of the formal power series ring so that the Galois hull is an extension of commutative algebras. Similarly for the universal Euler morphism of a difference rings. The commutativity of the Galois hull comes from the fact in the differential and the difference case, the theory depends on the co-commutative Hopf algebras. When we treat the $q$-SI $\sigma$-differential algebras, the Hopf algebra $H$ is not co-comutatives so that the Galois hull $L/K$ that is an algebra extension in the non-commutative algebra of twisted formal power series algebra, the dual algebra of $H$. So even if we start from a ( commutative ) field, extension $L/k$, the Galois hull can be non-commutative. See the Examples in sections 4, 5 and 6. We also notice that when $L/k$ is a Picard-Vessiot extension fields in $q$-SI $\sigma$-differential algebra, the Galois hull is commutative [15].

As the Galois hull is a non-commutative, if we limit ourselves to the category of commutative $L^\sharp$-algebras ($\text{Alg}/L^\sharp$), we can not detect non-commutative nature of the $q$-SI $\sigma$-differential field extension. So it is quite natural to extend the functors over the category of non-commutative algebras.

**3.1.7 Infinitesimal deformation functor $F_{L/k}$ for a $q$-SI $\sigma$-differential field extension.**

We pass to the task of defining the infinitesimal deformation functor $F_{L/k}$ and the infinitesimal automorphism functor Inf-gal ($L/k$).
We have the universal Taylor morphism

\[ \iota_L^*: L^* \to (L^*[W_1, W_2, \cdots, W_d], \{ \frac{\partial}{\partial W_1}, \frac{\partial}{\partial W_2}, \cdots, \frac{\partial}{\partial W_d} \}) \]  

(33)

as in (3). So by (33), we have the canonical morphism

\[ (F(\mathbb{N}, L^*)[[X]], \mathcal{D}) \to (F(\mathbb{N}, L^*[W])[[X]], \mathcal{D}), \]  

(34)

where in the target space

\[ \mathcal{D} = \{ \Sigma, \frac{\partial}{\partial W_1}, \frac{\partial}{\partial W_2}, \cdots, \frac{\partial}{\partial W_d}, \Theta^* \} \]

by abuse of notation.

For an \( L^* \)-algebra \( A \), the structure morphism \( L^* \to A \) induces the canonical morphism

\[ (F(\mathbb{N}, L^*[[X]], \mathcal{D}) \to (F(\mathbb{N}, A[[W]])[[X]], \mathcal{D}). \]  

(35)

The composite of the \( \mathcal{D} \)-morphisms (34) and (35) gives us the canonical morphism

\[ (F(\mathbb{N}, L^*)[[X]], \mathcal{D}) \to (F(\mathbb{N}, A[[W]])[[X]], \mathcal{D}). \]  

(36)

The restriction of the morphism (36) to the \( \mathcal{D} \)-invariant sub-algebra \( L \) gives us the canonical morphism

\[ \iota: (L, \mathcal{D}) \to (F(\mathbb{N}, A[[W]]), \mathcal{D}). \]  

(37)

We can define the functors exactly as in paragraphs 2.2.4 for the differential case and 2.3.3 for the difference case.

**Definition 3.10.** [Introductory definition] We define the functor

\[ \mathcal{F}_{L/k}: (\text{Alg}/L^*) \to (\text{Set}) \]

from the category \((\text{Alg}/L^*)\) of \( L^* \)-algebras to the category \((\text{Set})\) of sets, by associating to an \( L^* \)-algebra \( A \), the set of infinitesimal deformations of the canonical morphism (36).

Hence

\[ \mathcal{F}_{L/k}(A) = \{ f: (L, \mathcal{D}) \to (F(\mathbb{N}, A[[W_1, W_2, \cdots, W_d]])[[X]], \mathcal{D}) \mid f \text{ is a \( \mathcal{D} \)-morphism congruent to the canonical morphism } \iota \text{ modulo nilpotent elements such that } f = \iota \text{ when restricted to the sub-algebra } K \}. \]

The introductory definition 3.10 is exact, analogous to Definitions in 2.2.4 and 2.3.3 and easy to understand. Since as we explained in 3.1.6 we, however, have to consider also deformations over non-commutative algebras, the notation is confusing.

We have to treat both the category of commutative algebras and that of non-commutative algebras.
Definition 3.11. All the associative algebras that we consider are unitary and the morphisms between them are assumed to be unitary. For a commutative algebra $R$, we denote by $(C\text{Alg}/R)$ the category of associative commutative $R$-algebras. We consider also the category of not necessarily commutative $R$-algebras. To be more precise we denote by $(N\text{Calg}/R)$ the category of associative $R$-algebras $A$ such that (the image of) $R$ is in the center of $A$. When there is no danger of confusion the category of commutative algebras is denoted simply by $(\text{Alg}/R)$.

Let us come back to the $q$-SI $\sigma$-differential field extension $L/k$. We can now give the infinitesimal deformation functors in an appropriate language.

Definition 3.12. The functor $F_{L/k}$ defined in 3.10 will be denoted by $\mathcal{CF}_{L/k}$. So we have $\mathcal{CF}_{L/k} : (C\text{Alg}/L) \to (\text{Set})$.

We extend formally the functor $\mathcal{CF}_{L/k}$ in 3.10 from the category $(C\text{Alg}/L)$ to the category $(N\text{Calg}/L)$. Namely, we define the functor $\mathcal{NCF}_{L/k} : (N\text{Calg}/L) \to (\text{Set})$ by setting

$$\mathcal{NCF}_{L/k}(A) = \{ \varphi : L \to F(N, A[[W]])[[X]] \mid \varphi \text{ is an injective } q\text{-SI } \sigma\text{-differential morphism compatible with the derivations } \partial/\partial W_1, \partial/\partial W_2, \ldots, \partial/\partial W_d. \}$$

for $A \in \text{Ob}(N\text{Calg}/L)$. So it follows from the definition that the restriction of the functor $\mathcal{NCF}_{L/k}$ to the sub-category $(C\text{Alg}/L)$ is $\mathcal{CF}_{L/k}$ so that

$$\mathcal{NCF}_{L/k} \mid (C\text{Alg}/L) = \mathcal{CF}_{L/k}.$$ In the examples, we consider $q$-SI $\sigma$-differential structure, differential structure and difference structure of a given field extension $L/k$ and we study Galois groups with respect to the structures. So we have to clarify which structure is in question. For this reason, when we treat $q$-SI $\sigma$-differential structure, we sometimes add suffix $\sigma\theta$ to indicate that we treat the $q$-SI $\sigma$-differential structure as in 2.4. For example $\mathcal{NCF}_{\sigma\theta,L/k}$.

3.1.8 Definition of quantum Galois group

The definition of the group functor $\text{Inf-gal}(L/k)$ is similar.

Definition 3.13. The Galois group in our Galois theory is the group functor $\text{Inf-gal}(L/k) : (\text{Alg}/L) \to (\text{Grp})$ defined by

$$\text{Inf-gal}(L/k)(A) = \{ f : L \otimes L^2 A[[W]] \to L \otimes L^2 A[[W]] \mid f \text{ is a } K \otimes L^2 A[[W]]\text{-automorphism compatible with } \mathcal{D},$$

continuous with respect to the $W$-adic topology

and congruent to the identity modulo nilpotent elements \}$

for an $L^2$-algebra $A$. See Definition 2.19 in [8].

Then the group functor $\text{Inf-gal}(L/k)$ operates on the functor $\mathcal{F}_{L/k}$ in such a way that the operation $(\text{Inf-gal}(L/k), \mathcal{F}_{L/k})$ is a principal homogeneous space.
4 The first example, the field extension \( \mathbb{C}(t)/\mathbb{C} \)

From now on, we assume \( C = \mathbb{C} \). The arguments below work for an algebraic closed field \( C \) of characteristic 0. So \( q \) is a non-zero complex number.

4.1 Analysis of the example

Let \( t \) be a variable over \( \mathbb{C} \). The field \( \mathbb{C}(t) \) of rational functions has various structures: the differential field structure, the \( q \)-difference field structure and the \( q \)-SI \( \sigma \)-differential structure that we are going to define. We are interested in the Galois group of the field extension \( \mathbb{C}(t)/\mathbb{C} \) with respect to these structures. Let \( \sigma: \mathbb{C}(t) \to \mathbb{C}(t) \) be the \( \mathbb{C} \)-automorphism of the rational function field \( \mathbb{C}(t) \) sending \( t \) to \( qt \). So \( (\mathbb{C}(t), \sigma) \) is a difference field. We assume \( q^n \neq 1 \) for every positive integer \( n \). We define a \( \mathbb{C} \)-linear map \( \theta^{(1)}: \mathbb{C}(t) \to \mathbb{C}(t) \) by setting

\[
\theta^{(1)}(f(t)) := \frac{\sigma(f) - f}{\sigma(t) - t} = \frac{f(qt) - f(t)}{(q - 1)t}
\]

for \( f(t) \in \mathbb{C}(t) \).

For an integer \( n \geq 2 \), we set

\[
\theta^{(n)} := \frac{1}{[n]_q} (\theta^{(1)})^n.
\]

It is convenient to define

\[
\theta^{(0)} = \text{Id}_{\mathbb{C}(t)}.
\]

It is well-known and easy to check that \( (\mathbb{C}(t), \sigma, \theta^{*}) = (\mathbb{C}(t), \sigma, \{\theta^{(i)}\}_{i \in \mathbb{N}}) \) is a \( q \)-SI \( \sigma \)-differential algebra.

**Lemma 4.1.** Galois group of the difference Picard-Vessiot extension \( (\mathbb{C}(t), \sigma)/(\mathbb{C}, \text{Id}_\mathbb{C}) \) is the multiplicative group \( \mathbb{G}_m \).

**Proof.** Since \( t \) satisfies the linear difference equation \( \sigma(t) = qt \) over \( \mathbb{C} \) and the field \( C_{\mathbb{C}(t)} \) of constant of \( \mathbb{C}(t) \) is \( \mathbb{C} \), the extension \( (\mathbb{C}(t), \sigma)/(\mathbb{C}, \text{Id}_\mathbb{C}) \) is a difference Picard-Vessiot extension. The result follows from the definition of the Galois group. \( \square \)

When \( q \to 1 \), the limit of the \( q \)-SI \( \sigma \)-differential ring \( (\mathbb{C}(t), \sigma, \theta^{*}) \) is the differential algebra \( (\mathbb{C}(t), d/dt) \). We denote by \( AF_{1k} \), the algebraic group of affine transformations of the affine line so that

\[
AF_{1 \mathbb{C}} = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \left| a, b \in \mathbb{C}, a \neq 0 \right. \right\}.
\]

Then

\[
AF_{1 \mathbb{C}} \simeq \mathbb{G}_m \times \mathbb{G}_a,
\]

where

\[
\mathbb{G}_m \simeq \left\{ \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \in AF_{1 \mathbb{C}} \left| a \in \mathbb{C}^* \right. \right\},
\]

\[
\mathbb{G}_a \simeq \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \in AF_{1 \mathbb{C}} \left| b \in \mathbb{C} \right. \right\}.
\]
Lemma 4.2. The Galois group of differential Picard-Vessiot extension \((\mathbb{C}(t), d/dt)/\mathbb{C}\) is \(\mathbb{G}_a\).  

Proof. We consider the linear differential equation

\[ Y' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Y, \]  \hspace{1cm} (38)

where \(Y\) is a 2 \times 2-matrix with entries in a differential extension field of \(\mathbb{C}\). Then \(\mathbb{C}(t)/\mathbb{C}\) is the Picard-vessiot extension for (38),

\[ Y = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \]

being a fundamental solution of (38). The result is well-known and follows from the definition of Galois group.

The extension \((\mathbb{C}(t), \sigma, \theta^*)/\mathbb{C}\) is not a Picard-Vessiot extension so that we cannot treat it in the framework of Picard-Vessiot theory. We can apply, however, Hopf Galois theory of Heiderich [5].

Proposition 4.3. The Galois group of the extension \((\mathbb{C}(t), \sigma, \theta^*)/\mathbb{C}\) is isomorphic to the formal completion \(\hat{\mathbb{G}}_{m,\mathbb{C}}\) of the multiplicative group \(\mathbb{G}_{m,\mathbb{C}}\).

Theory of Umemura [11] and Heiderich [2] single out only the Lie algebra. Proposition 4.3 should be understood in the following manner. We have two specializations of the \(q\)-SI \(\sigma\)-differential field extension \((\mathbb{C}(t), \sigma, \theta^*)/\mathbb{C}\).

(i) \(q \to 1\) giving the differential field extension \((\mathbb{C}(t), d/dt)/\mathbb{C}\). See 2.2.2

(ii) Forgetting \(\theta^*\), or equivalently specializing

\[ \theta^{(i)} \to 0 \hspace{1cm} \text{for} \ i \geq 1, \]

we get the difference field extension \((\mathbb{C}(t), \sigma)/\mathbb{C}\). See 3.1.2

We can summarise the behavior of the Galois group under the specializations.

(1) Proposition 4.3 says that the Galois group of \((\mathbb{C}(t), \sigma, \theta^*)/\mathbb{C}\) is the multiplicative group \(\mathbb{G}_{m,\mathbb{C}}\). This describes the Galois group at the generic point.

(2) By Lemma 4.1, Galois group of the specialization (i) is the multiplicative group \(\mathbb{G}_{m,\mathbb{C}}\).

(3) Galois group of the specialization (ii) is the additive group \(\mathbb{G}_{a,\mathbb{C}}\) by Lemma 4.2

Proof. Let us set \(L = (\mathbb{C}(t), \sigma, \theta^*)\) and \(k = (\mathbb{C}, \sigma, \theta^*)\). By definition of the universal Hopf morphism (29),

\[ \iota: (L, \sigma, \theta^*) \to \left( F(\mathbb{N}, L^3)[[X]], \hat{\Sigma}, \hat{\Theta}^* \right), \hspace{1cm} \iota(t) = tQ + X \in F(\mathbb{N}, L^3)[[X]], \]

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where

\[ Q \in F(\mathbb{N}, L^\natural) \]

is a function on \( \mathbb{N} \) taking values in \( C \subset L^\natural \) such that

\[ Q(n) = q^n \quad \text{for } n \in \mathbb{N}. \]

We denote the function \( Q \) by

\[ Q = \begin{bmatrix} 0 & 1 & 2 & \cdots \\ 1 & q & q^2 & \cdots \end{bmatrix} \]

according to the convention in [8]. We take the derivation \( \partial/\partial t \in \text{Der}(L^\natural/k^\natural) \) as a basis of the 1-dimensional \( L^\natural \)-vector space \( \text{Der}(L^\natural/k^\natural) \) of \( k^\natural \)-derivations of \( L^\natural \). So \( (\partial/\partial t)(\iota(t)) = Q \)

is an element of the Galois hull \( \mathcal{L} \). Therefore

\[ \mathcal{L} = L^\natural < X, Q >_{\text{alg}}, \]

which is the \( L^\natural \)-sub-algebra of \( F(\mathbb{N}, L^\natural)[[X]] \) generated by \( X \) and \( Q \). Since \( QX = qXQ \), the Galois hull \( \mathcal{L} \) is a non-commutative \( L^\natural \)-algebra. Now we consider the universal Taylor expansion

\[ (L^\natural, \partial/\partial t) \to L^\natural[[W]] \]

and consequently we have

\[ \iota: \mathcal{L} \to F(\mathbb{N}, L^\natural)[[X]] \to F(\mathbb{N}, L^\natural[[W]])[[X]]. \]  \hspace{1cm} (39)

We study infinitesimal deformation of \( \iota \) in (39). Let \( A \) be a commutative \( L^\natural \)-algebra

\[ \varphi: \mathcal{L} \to F(\mathbb{N}, A[[W]])[[X]] \]

be an infinitesimal deformation of

\[ \iota: \mathcal{L} \to F(\mathbb{N}, A[[W]])[[X]], \]

for a commutative \( L^\natural \)-algebra \( A \) so that

\[ \varphi \in \mathcal{F}_{L/k}(A). \]

**Sublemma 4.4.**  (1) There exists a nilpotent element \( n \in A \) such that \( \varphi(Q) = (1 + n)Q \).

(2) \( \varphi(X) = X \).

Sublemma proves Proposition 4.3 \( \Box \)

**Proof of Sublemma.** The elements \( X, Q \in \mathcal{L} \) satisfy the following equation.

\[ \frac{\partial X}{\partial W} = \frac{\partial Q}{\partial W} = 0, \]

\[ \hat{\Sigma}(X) = qX, \quad \hat{\Sigma}(Q) = qQ, \]

\[ \hat{\Theta}^{(1)}(X) = 1, \quad \hat{\Theta}^{(i)}(X) = 0 \quad \text{for } i \geq 2, \]

\[ \hat{\Theta}^{(i)}(Q) = 0 \quad \text{for } i \geq 1. \]
So $\varphi(X), \varphi(Q)$ satisfy the same equations as above, which shows

\[
\varphi(X) = X + fQ \in F(\mathbb{N}, A[[W]])[[X]],
\]
\[
\varphi(Q) = eQ \in F(\mathbb{N}, A[[W]])[[X]],
\]

where $f, e \in A$. Since $\varphi$ is an infinitesimal deformation of $\iota$, $f$ and $1 - e$ are nilpotent elements in $A$. It remains to show $f = 0$. In fact, it follows from the equation

\[
QX = qXQ
\]

that

\[
\varphi(Q)\varphi(X) = q\varphi(X)\varphi(Q)
\]
or

\[
eQ(X + fQ) = q(X + fQ)eQ.
\]

So we have

\[
eQfQ = qfQeQ
\]

and so

\[
efQ^2 = qfeQ^2.
\]

Therefore

\[
ef = qfe.
\]

Since $e$ is a unit, $e - 1$ being nilpotent in $A$,

\[
f - qf = 0,
\]

so that

\[
(1 - q)f = 0.
\]

As $1 - q$ is a non-zero complex number, $f = 0$ as we wanted.

We have shown that the functor

\[
\mathcal{F}_{L/k} : (\text{Alg}/L^2) \to (\text{Set})
\]
is a principal homogeneous space of the group functor $\hat{\mathbb{G}}_{m, \mathbb{C}}$. See paragraph 2.2.6 Origin of the group structure as well as paragraph 4.2 below.

During the proof of Proposition 4.3, we have proved the following

**Proposition 4.5.** The Galois hull $\mathcal{L}$ coincides with the sub-algebra

\[
L^2 < X, Q >_{\text{alg}}
\]
of $F(\mathbb{N}, L^2)[[X]]$ generated by $L^2$, $X$ and $Q$. The commutation relation of $X$ and $Q$ is

\[
QX = qXQ.
\]

In particular, if $q \neq 1$, then the Galois hull is non-commutative.
We are ready to describe the non-commutative deformations.

**Lemma 4.6.** If $q \neq 1$, we have

$$\mathcal{NC}_L/k(A) = \{ e \in A, f \in A, \ | fe = qef \text{ and } 1 - e, f \text{ are nilpotent} \}$$

for every $A \in \text{Ob}(NC\text{Alg}/L^\natural)$.

**Proof.** Since $q \neq 1$, it follows from the argument of the proof of Sublemma 4.4 that if we take

$$\varphi \in \mathcal{NC}_L/k_\sigma$$

for $A \in \text{Ob}(NC\text{Alg}/L^\natural)$,

then $\varphi(X) = X + f$ and $\varphi(Q) = eQ$, $f, e \in A$.

Since $\varphi$ is an infinitesimal deformation of $\iota$, $f$ and $1 - e$ are nilpotent.

It follows from $QX = qXQ$ that

$$eQ(X + f) = q(X + f)eQ$$

so $ef = qfe$. \hfill \Box

We are going to see in 4.2 that theoretically, we can identify

$$\mathcal{NC}_L/k(A) = \left\{ \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix} \ | \ e \in A, f \in A, fe = qef \text{ and } 1 - e, f \text{ are nilpotent} \right\}. \quad (40)$$

**Corollary 4.7** (Corollary to the proof of Lemma 4.6). When $q = 1$ that is the case excluded in our general study, we consider the $q$-SI $\sigma$-differential differential field

$$(\mathbb{C}(t), \text{Id}, \sigma^*)$$

as in 3.1.3. So $\theta^*$ is the iterative derivation;

$$\theta^{(0)} = \text{Id},$$

$$\theta^{(i)} = \frac{1}{i!} \frac{d^i}{dt^i} \text{ for } i \geq 1.$$ 

Then we have

$$\mathcal{L}_{\sigma^*} \simeq \mathcal{L}_{d/dt}, \quad (41)$$

$$\mathcal{NC}_F(\mathbb{C}(t), \text{Id}, \sigma^*/\mathbb{C})(A) = \{ f \in A \ | \ f \text{ is a nilpotent element} \} \quad (42)$$

for $A \in \text{Ob}(NC\text{Alg}/L^\natural)$.

**Proof.** In fact if $q = 1$, then

$$\begin{bmatrix} 0 & 1 & 2 & 3 & \cdots \\ 1 & 1 & 1 & 1 & \cdots \end{bmatrix} = 1 \in \mathbb{C}.$$  

So $\mathcal{L}_{q, \sigma}$ is generated by $X$ over $\mathcal{K}$. Therefore $\mathcal{L}_{\sigma^*} \simeq \mathcal{L}_{d/dt}$. Since $Q = 1 \in \mathcal{K}, \varphi(Q) = Q$ for an infinitesimal deformation

$$\varphi \in \mathcal{NC}_F_{\sigma, \theta^*}(A)$$

and we get (42). \hfill \Box
4.1.1 Quantum group enters

To understand Lemma 4.6, it is convenient to introduce a quantum group.

**Definition 4.8.** We work in the category $({\text{NCAlg}}/\mathbb{C})$. Let $A$ be a not necessarily commutative $\mathbb{C}$-algebra. We say that two subsets $S, T$ of $A$ are mutually commutative if for every $s \in T$, $t \in T$, we have $[s, t] = st - ts = 0$.

For $A \in \text{Ob}({\text{NCAlg}}/L^o)$, we set

$$H_q(A) = \{ \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix} \mid e, f \in A, \ e \text{ is invertible in } A, \ qef = fe \}.$$ 

**Lemma 4.9.** For two matrices

$$Z_1 = \begin{bmatrix} e_1 & f_1 \\ 0 & 1 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} e_2 & f_2 \\ 0 & 1 \end{bmatrix} \in H_q(A),$$

if $\{e_1, f_1\}$ and $\{e_2, f_2\}$ are mutually commutative, then the product matrix

$$Z_1Z_2 \in H_q(A).$$

**Proof.** Since

$$Z_1Z_2 = \begin{bmatrix} e_1e_2 & e_1f_2 + f_1 \\ 0 & 1 \end{bmatrix},$$

we have to prove

$$qe_1e_2(e_1f_2 + f_1) = (e_1f_2 + f_1)e_1e_2.$$ 

This follows from the mutual commutativity of $\{e_1, f_1\}$ and $\{e_2, f_2\}$, and the conditions $qe_1f_1 = f_1e_1$, $qe_2f_2 = f_2e_2$. \hfill \Box

**Lemma 4.10.** For a matrix

$$Z = \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix} \in H_q(A),$$

if we set

$$\tilde{Z} = \begin{bmatrix} e^{-1} & -e^{-1}f \\ 0 & 1 \end{bmatrix} \in M_2,$$

then

$$\tilde{Z} \in H_{q^{-1}}(A) \text{ and } \tilde{Z}Z = Z\tilde{Z} = I_2.$$

**Proof.** We can check it by a simple calculation. \hfill \Box

**Remark 4.11.** If $q^2 \neq 1$, for $f \neq 0 \tilde{Z} \notin H_q(A)$. In fact let us set

$$\tilde{Z} = \begin{bmatrix} \tilde{e} & \tilde{f} \\ 0 & 1 \end{bmatrix}$$

so that $\tilde{e} = e^{-1}, \tilde{f} = -fe^{-1}$. Then $\tilde{e}\tilde{f} = e^{-1}(-fe^{-1}) = -qfe^{-2}$ and $\tilde{f}\tilde{e} = -fe^{-1}e^{-1} = -fe^{-2}$. So $q\tilde{e}\tilde{f} = -q^2fe^{-2} \neq -fe^{-2} = \tilde{f}\tilde{e}$, if $q^2 \neq 1$, then and $\tilde{Z} \in H_{q^{-1}}(A)$. 

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For an algebra $R$, a sub-algebra $S$ of $R$ and a sub-set $T \subset R$, we denote by $S < T >_{alg}$ the sub-algebra of $R$ generated over $S$ by $T$.

**Lemma 4.12.** Let $u$ and $v$ be symbols over $\mathbb{C}$. We have shown that we find a $\mathbb{C}$-Hopf algebra

$$\mathcal{H}_q = \mathbb{C} < u, u^{-1}, v >_{alg} / (uv - q^{-1}vu)$$

(43)

as an algebra so that

$$uu^{-1} = u^{-1}u = 1, \quad u^{-1}v = qvu^{-1}.$$ 

Definition of the algebra $\mathcal{H}_q$ determines the multiplication

$$m: \mathcal{H}_q \otimes_\mathbb{C} \mathcal{H}_q \to \mathcal{H}_q,$$

the unit

$$\eta: \mathbb{C} \to \mathcal{H}_q,$$

that is the composition of natural morphisms

$$\mathbb{C} \to \mathbb{C} < u, u^{-1}, v >_{alg}$$

and

$$\mathbb{C} < u, u^{-1}, v >_{alg} \to \mathbb{C} < u, u^{-1}, v >_{alg} / (uv - q^{-1}vu) = \mathcal{H}_q.$$ 

The product of matrices gives the co-multiplication

$$\Delta: \mathcal{H}_q \to \mathcal{H}_q \otimes_\mathbb{C} \mathcal{H}_q,$$

that is a $\mathbb{C}$-algebra morphism defined by

$$\Delta(u) = u \otimes u, \quad \Delta(u^{-1}) = u^{-1} \otimes u^{-1}, \quad \Delta(v) = u \otimes v + v \otimes 1,$$

for the generators $u, u^{-1}, v$ of the algebra $\mathcal{H}_q$, the co-unit is a $\mathbb{C}$-algebra morphism

$$\epsilon: \mathcal{H}_q \to \mathbb{C}, \quad \epsilon(u) = \epsilon(u^{-1}) = 1, \quad \epsilon(v) = 0$$

for the generators $u, u^{-1}, v$ of the algebra $\mathcal{H}_q$. The antipode

$$i: \mathcal{H}_q \to \mathcal{H}_q$$

is a $\mathbb{C}$-anti-algebra morphism given by

$$i(u) = u^{-1}, \quad i(u^{-1}) = u, \quad (v) = -u^{-1}v.$$ 

Let us set

$$\mathcal{H}_q L^2 := \mathcal{H}_q \otimes_\mathbb{C} L^2,$$

so that $\mathcal{H}_q L^2$ is an $L^2$-Hopf algebra. We notice that for an $L^2$-algebra $A$.

$$\mathcal{H}_q L^2(A) := \text{Hom}_{L^2\text{-alg}}(\mathcal{H}_q L^2, A) = \left\{ \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix} \mid e, f \in A, ef = q^{-1}fe, \ e \text{ is invertible} \right\}.$$
Remark 4.13. We know by general theory that the antipode $i: H \rightarrow H$ that is a linear
map making a few diagrams commutative, is necessarily an anti-endo-morphism of the
algebra $H$ so that

$$i(ab) = i(b)i(a) \text{ for all elements } a, b \in H \text{ and } i(1) = 1.$$

See Manin [7], section 1, 2.

The Hopf algebra $\hat{H}_q$ is a $q$-deformation of the affine algebraic group $AF_1^\mathbb{C}$ of affine
transformations of the affine line.

Anyhow, we notice that the quantum group appears in this very simple example
showing that quantum groups are indispensable for a Galois theory of $q$-SI $\sigma$-differential
field extensions.

4.2 Observations on the Galois structures of the field extension
\( \mathbb{C}(t)/\mathbb{C} \)

Let us now examine that the group structure in 2.2.6 arising from the variation of initial
conditions coincides with the quantum group structure defined in 4.1.1.

To see this, we have to clearly understand the initial condition of a formal series

$$f(W, X) = \sum_{i=0}^{\infty} X^i a_i(W) \in F(\mathbb{N}, A[[W]])[[X]]$$

so that the coefficients $a_i$’s, which are elements of $F(\mathbb{N}, A[[W]])$, are functions on $\mathbb{N}$ taking
values in the formal power series ring $A[[W]]$. The initial condition of $f(W, X)$ is the value
of the function $f(0, 0) = a_0(0) \in F(\mathbb{N}, A)$ at $n = 0$ which we may denote by

$$f(0, 0)|_{n=0} \in A.$$

For $A \in Ob(NCAlg/L^2)$, we take an infinitesimal deformation $\varphi \in NC\mathcal{F}_{L/k}(A)$ so that
the morphism $\varphi: \mathcal{L} \rightarrow F(\mathbb{N}, A[[W]])[[X]]$ is determined by the image $\varphi(y)$ of $y \in L \subset \mathcal{L}$, the $q$-SI $\sigma$-differential field $L$ being a sub-algebra of $\mathcal{L}$ by the universal Hopf morphism.
It follows from Lemma 4.6 that there exist $e, f \in A$ such that $qef = fe$, the elements $e - 1, f$ are nilpotent and such that

$$\varphi(t) = (e(t + W) + f) Q + X. \quad (44)$$

The above equality (44) says that in the level of the initial condition, the dynamical
system $y \mapsto \varphi(y)$ looks as

$$t \mapsto \text{the initial condition of } \varphi(y) = et + f. \quad (45)$$

The composition of two mutually commutative transformations of the form (45) is nothing
but the multiplication of $2 \times 2$ matrices. Therefore the quantum group structure is the
same as in the group structure in 2.2.6.

The Hopf algebra $H$ in ?? defines a functor

$$\hat{H}_q: (NCAlg/L^2) \rightarrow (Set)$$
such that
\[ \hat{\mathcal{H}}_{q, L^2}(A) = \{ \psi : \mathcal{H}_q \otimes_{C} L^2 \to A \mid \psi \text{ is a } L^2\text{-algebra morphism such that } \psi(u) - 1, \psi(v) \text{ are nilpotent} \} \]

for \( A \in (\text{NCAlg}/L^2) \). In other words \( \hat{\mathcal{H}}_{q, L^2} \) is the formal completion of the quantum group \( \mathcal{H}_q \otimes_{C} L^2 = \mathcal{H}_q L^2 \). We can summarize our results in the following form.

**Theorem 4.14.** The formal quantum group \( \hat{\mathcal{H}}_{q, L^2} \) operates on the functor \( \mathcal{NCF}_{L/k} \) in such a way that there exists a functorial isomorphisms

\[ \mathcal{NCF}_{L/k} \simeq \hat{\mathcal{H}}_{q, L^2}. \]

The restriction of the functor \( \mathcal{NCF}_{L/k} \) on the subcategory \( (\text{CAlg}/L^2) \) gives the functorial isomorphism

\[ \mathcal{NCF}_{L/k}|_{(\text{CAlg}/L^2)} \simeq \hat{\mathbb{G}}_m L^2. \]

Or equivalently,

1. The infinitesimal Galois group of the q-SI \( \sigma \)-differential extension \( (\mathbb{C}(t), \sigma, \theta^*)/\mathbb{C} \) on the category \( (\text{NCAlg}/L^2) \) of not necessarily commutative \( L^2 \)-algebra is isomorphic to the formal quantum group \( \hat{\mathcal{H}}_{q, L^2} \).
2. The infinitesimal Galois group of the q-SI \( \sigma \)-differential extension \( (\mathbb{C}(t), \sigma, \theta^*)/\mathbb{C} \) on the category \( (\text{Alg}/L^2) \) of commutative \( L^2 \)-algebras is isomorphic to the formal group \( \hat{\mathbb{G}}_m \).

The operation of formal quantum group requires a precision.

**Remark 4.15.** We should be careful about the operation of quantum formal group. To be more precise, for \( \varphi \in \mathcal{F}_{L/k}(A) \) and \( \psi \in \hat{\mathcal{H}}_{q, L^2}(A) \) so that we have

\[ \varphi(t) = (e(t + W) + f)Q + X \in F(\mathbb{N}, A[[W]])[[X]] \]

with \( e, f \in A \) and we imagine the matrix

\[ \begin{bmatrix} \psi(u) & \psi(v) \\ 0 & 1 \end{bmatrix} \in M_2(A) \]

corresponding to \( \psi \). If the sub-sets of the algebra \( A \), \( \{ \psi(u), \psi(v) \} \) and \( \{ e, f \} \) are commutative, the product

\[ \psi \cdot \varphi = \varpi \in \mathcal{F}_{L/k}(A) \]

is defined to be

\[ \varpi(t) = (\psi(u)e(t + W) + \psi(u)f + \psi(v))Q + X \in F(\mathbb{N}, A[[W]])[[X]]. \]
5 The second example, the $q$-SI $\sigma$-differential field extension $(\mathbb{C}(t, t^\alpha), \sigma, \theta^*)/\mathbb{C}$

5.1 Commutative deformations

As in the previous section, let $t$ be a variable over $\mathbb{C}$ and we assume that the complex number $q$ is not a root of unity if we do not mention other assumptions on $q$. Sometimes we write the condition that $q$ is not a root of unity, simply to recall it. We work under the condition that $\alpha$ is an irrational complex number so that $t$ and $t^\alpha$ are algebraically independent over $\mathbb{C}$. Therefore the field $\mathbb{C}(t, t^\alpha)$ is isomorphic to the rational function field of two variables over $\mathbb{C}$. We denote by $\sigma$ the $\mathbb{C}$-automorphism of the field $\mathbb{C}(t, t^\alpha)$ such that

$$\sigma(t) = qt \quad \text{and} \quad \sigma(t^\alpha) = q^\alpha t^\alpha.$$ 

Let us set $\theta(0) := \text{Id}_{\mathbb{C}(t, t^\alpha)}$, the map $\theta(1) := \sigma - \text{Id}_{\mathbb{C}(t, t^\alpha)} : \mathbb{C}(t, t^\alpha) \to \mathbb{C}(t, t^\alpha)$

and

$$\theta(n) = \frac{1}{[n]_q!}(\theta(1))^n \quad \text{for} \quad n = 2, 3, \cdots.$$ 

So the the $\theta(i)$'s are $\mathbb{C}$-linear operators on $\mathbb{C}(t, t^\alpha)$ and $L := (\mathbb{C}(t, t^\alpha), \sigma, \theta^*)$

is a $q$-SI $\sigma$-differential field. The restriction of $\sigma$ and $\theta^*$ to the subfield $\mathbb{C}$ are trivial. We denote the $q$-SI $\sigma$-differential field extension $L/\mathbb{C}$ by $L/k$. We denote $t^\alpha$ by $y$ so that the abstract field $\mathbb{C}(t, t^\alpha) = \mathbb{C}(t, y)$ is isomorphic to the rational function field of 2 variables over $\mathbb{C}$. We take the derivations $\partial/\partial t$ and $\partial/\partial y$ as a basis of the $k^2$-vector space $\text{Der}(L^2/k^2)$ of $k^2$-derivations of $L^2$. Hence $L^2 = (L^2, \{\partial/\partial t, \partial/\partial y\})$ as in [15].

Let us list the fundamental equations.

$$\sigma(t) = qt, \quad \sigma(y) = q^\alpha y, \quad \theta(1)(t) = 1, \quad \theta(1)(y) = [\alpha]_q \frac{y}{t}. \quad (46, 47)$$

We are going to determine the Galois group $\text{NCInf-gal}(L/k)$. Before we start, we notice that since by Proposition [15] the Galois hull of the extension $\mathbb{C}(t), \sigma, \theta^*)/\mathbb{C}$ is not a commutative algebra and since $(\mathbb{C}(t)$ is a sub-field of $\mathbb{C}(t, t^\alpha)$, the Galois hull of the $q$-SI $\sigma$-differential field extension $(\mathbb{C}(t, t^\alpha), \sigma, \theta^*)/\mathbb{C}$ is not a commutative algebra either. Consequently the $q$-SI $\sigma$-differential field extension $\mathbb{C}(t, t^\alpha)/\mathbb{C}$ is not a Picard-Vessiot extension (See [15]). So we have to go beyond the general theory of Heiderich [5], Umemura [15] for the definition of the Galois group $\text{NCInf-gal}(L/k)$.

It follows from general definition that the universal Hopf morphism

$$\iota : L \to F(N, L^2)[[X]]$$
is given by

\[ \iota(a) = \sum_{n=0}^{\infty} X^n u[\theta^n(a)] \in F(\mathbb{N}, L^2)[[X]] \]

for \( a \in L \). Here for \( b \in L \), we denote by \( u[b] \) the element

\[ u[b] = \begin{bmatrix} 0 & 1 & 2 & \cdots \\ b & \sigma(b) & \sigma^2(b) & \cdots \end{bmatrix} \in F(\mathbb{N}, L^2). \]

It follows from the definition above of the universal Hopf morphism \( \iota \),

\[ \iota(y) = \sum_{n=0}^{\infty} X^n \binom{\alpha}{n}_q t^{-n} Q^{\alpha-n} y, \]

where we use the following notations. For a complex number \( \beta \in \alpha + \mathbb{Z} \),

\[ [\beta]_q = \frac{q^\beta - 1}{q - 1} \]

and

\[ \binom{\alpha}{n}_q = \frac{[\alpha]_q [\alpha - 1]_q \cdots [\alpha - n + 1]_q}{[n]_q!}. \]

\[ Q = \begin{bmatrix} 0 & 1 & 2 & \cdots \\ 1 & q & q^2 & \cdots \end{bmatrix} \quad \text{and} \quad Q^\alpha = \begin{bmatrix} 0 & 1 & 2 & \cdots \\ 1 & q^\alpha & q^{2\alpha} & \cdots \end{bmatrix}. \]

We set

\[ Y_0 := \sum_{n=0}^{\infty} X^n \binom{\alpha}{n}_q t^{-n} Q^{\alpha-n} \]

so that

\[ \iota(y) = Y_0 y \quad \text{in} \ F(\mathbb{N}, L^2)[[X]]. \quad (48) \]

Considering \( k^2 \)-derivations \( \partial/\partial t, \partial/\partial y \) in \( L^2 \) and therefore in \( F(\mathbb{N}, L^2) \) or in \( F(\mathbb{N}, L^2)[[X]] \), we generate the Galois hull \( \mathcal{L} \) by \( \iota(L) \) and \( L^2 \) so that \( \mathcal{L} \subset F(\mathbb{N}, L^2)[[X]] \) is invariant under \( \Sigma \), the \( \Theta^i \) and \( \{ \partial/\partial t, \partial/\partial y \} \). We may thus consider

\[ \mathcal{L} \hookrightarrow F(\mathbb{N}, L^2)[[X]]. \]

By the universal Taylor morphism

\[ L^2 = (L^2, \{ \partial/\partial t, \partial/\partial y \}) \to L^2[[W_1, W_2]]. \]

We identify \( \mathcal{L} \) by the canonical morphism

\[ \iota: \mathcal{L} \to F(\mathbb{N}, L^2)[[X]] \to F(\mathbb{N}, L^2[[W_1, W_2]])[[X]]. \]

We study first the deformations of \( \iota \) on the category \((CAlg/L^2)\) of commutative \( L^2 \)-algebras and then generalize the argument to the category \((NCAlg/L^2)\) of not necessarily commutative \( L^2 \)-algebras.

For a commutative \( L^2 \)-algebra \( A \), let \( \varphi: \mathcal{L} \to F(\mathbb{N}, A[[W_1, W_2]]][[X]] \) be an infinitesimal deformation of the canonical morphism \( \iota: \mathcal{L} \to F(\mathbb{N}, L^2[[W_1, W_2]])[[X]] \) so that both \( \iota \) and \( \varphi \) are compatible with operators \( \{ \Sigma, \Theta^*, \partial/\partial W_1, \partial/\partial W_2 \} \).
Lemma 5.1. The infinitesimal deformation $\varphi$ is determined by the images $\varphi(Y_0)$, $\varphi(Q)$ and $\varphi(X)$.

Proof. The Galois hull $\mathcal{L}/\mathcal{K}$ is generated over $\mathcal{K} = L^t$ by $\iota(t) = tQ + X$ and $\iota(y) = Y_0y$ with operators $\Theta^*$, $\Sigma$ and $\partial/\partial t$, $\partial/\partial y$ along with localizations. This proves the Lemma. □

Let us set $Z_0 := \varphi(Y_0) \in F(N, A[[W_1, W_2]])([X])$ and expand it into a formal power series in $X$:

$$Z_0 = \sum_{n=0}^{\infty} X^n a_n, \quad \text{with } a_n \in F(N, A[[W_1, W_2]]) \quad \text{for every } n \in \mathbb{N}.$$  

It follows from (46) and (48)

$$\Sigma(Z_0) = q^\alpha Z_0$$

so that

$$\sum_{n=0}^{\infty} X^n q^n \Sigma(a_n) = q^\alpha \sum_{n=0}^{\infty} X^n a_n. \quad (49)$$

Comparing the coefficient of the $X^n$'s in (49) we get

$$\Sigma(a_n) = q^{\alpha - n} a_n \quad \text{for } n \in \mathbb{N}.$$  

So $a_n = b_n Q^{\alpha - n}$ with $b_n \in A[[W_1, W_2]]$ for $n \in \mathbb{N}$. Namely we have

$$Z_0 = \sum_{n=0}^{\infty} X^n b_n Q^{\alpha - n} \quad \text{with } b_n \in F(N, A[[W_1, W_2]]). \quad (50)$$

It follows from (47)

$$\sigma(y) - y = \theta^{(1)}(y)(q - 1)t$$

and so by (46)

$$(q^\alpha - 1)y = \theta^{(1)}(y)(q - 1)t.$$  

Applying the canonical morphism $\iota$ and the deformation $\varphi$, we get

$$(q^\alpha - 1)Y_0 = \Theta^{(1)}(Y_0)(q - 1)(tQ + X) \quad (51)$$

as well as

$$(q^\alpha - 1)Z_0 = \Theta(Z_0)(q - 1)(teQ + X). \quad (52)$$

Substituting (50) into (52), we get a recurrence relation among the $b_m$'s;

$$b_{m+1} = \frac{[\alpha - m]_q}{[m + 1]_q (e(t + W_1))} b_m.$$  

Hence

$$b_m = \left(\frac{\alpha}{m}\right)_q (e(t + W_1))^{-m} b_0 \quad \text{for every } m \in \mathbb{N}, \quad (53)$$
where \( b_0 \in A[[W_1, W_2]] \) and every coefficient of the power series \( b_0 - 1 \) are nilpotent. Since
\[
\frac{\partial Y_0}{\partial y} = \frac{\partial}{\partial W_2} \left( \sum_{n=0}^{\infty} X^n \binom{\alpha}{n} (t + W_1)^{-n} Q^{\alpha-n} \right) = 0,
\]
we must have
\[
0 = \varphi(\frac{\partial Y_0}{\partial y}) = \frac{\partial \varphi(Y_0)}{\partial W_2} = \frac{\partial Z_0}{\partial W_2}
\]
and consequently
\[
\frac{\partial b_0}{\partial W_2} = 0
\]
so that
\( b_0 \in A[[W_1]]. \)
by (50). Therefore, we have determined the image
\[
Z_0 = \varphi(Y_0) = \sum_{n=0}^{\infty} X^n \binom{\alpha}{n} (e(t + W_1))^{-n} Q^{\alpha-n} b_0
\]
by (53), where all the coefficients of the power series \( b_0 - 1 \) are nilpotent.

5.2 The functor \( \mathcal{NCF}_{L/k} \) of infinitesimal deformations restricted on the category \( (CAlg/L^2) \) of commutative \( L^2 \)-algebras

We can summarize what we have proved as follows.

**Proposition 5.2.** There exists a functorial inclusion on the category \( (CAlg/L^2) \) of commutative \( L^2 \)-algebras
\[
\mathcal{NCF}_{L/k}(A) \hookrightarrow \hat{G}_{II}(A) := \{(e, b(W_1)) \in A \times A[[W_1]] \mid \text{all the coefficients of } b(W_1) \text{ and } e - 1 \text{ are nilpotent} \} \quad (54)
\]
for every commutative \( L^2 \)-algebra \( A \).

**Proof.** In fact, we send a deformation \( \varphi \in \mathcal{NCF}_{L/k}(A) \) to \((e, b_0(W)) \in A \times A[[W_1]]\) that is an element of the sub-set \( \hat{G}_{II}(A) \) of \( A \times A[[W_1]] \).

**Conjecture 5.3.** If \( q \) is not a root of unity, the inclusion in Proposition 5.2 is the equality.

The set
\[
\hat{G}_{II}(A) = \left\{ (e, b(W_1)) \in A \times A[[W_1]] \mid e - 1 \in A \text{ and all the coefficients of the power series } b(W_1) - 1 \text{ are nilpotent} \right\}
\]
has a natural group structure functorial in \( A \in Ob(Alg/L^2) \).
Namely for two elements \((e_1, b_1(W_1)), (e_2, b_2(W_1)) \in \hat{G}_{II}(A)\), the product is given by
\[
(e_1, b_1(W_1)) \times (e_2, b_2(W_2)) = (e_1 e_2, 1(e_2 W_1 + e_2 - 1)b_2(W_1)).
\]
the unit being \((1, 1) \in \hat{G}_{II}(A)\) and the inverse
\[
(e, b(W_1))^{-1} = (e^{-1}, b(e^{-1} W_1 + e^{-1} - 1)^{-1}).
\]
is an element of \( \hat{G}_{II}(A) \).
Proposition 5.4. If Conjecture 5.3 is true, we have isomorphism
\[ \text{NCInf-gal} \left( \frac{L}{k} \right) \mid_{(\text{Alg}/2)} \simeq \hat{G}_{II}. \]
of group functors.

Remark 5.5. We explain a background of Conjecture 5.3.

Lemma 5.6. The Galois hull \( L \) is a localization of the following ring
\[ L^2 \left[ Q, X, \frac{1}{tQ + X} \left[ \frac{\partial^n}{\partial t^n} Y_0 \right] \right]_{t \in \mathbb{N}}. \]

Proof. Since \( \iota(t) = tQ + X \), as we have seen in the first exampe,
\[ L^2[Q, X] \left[ \frac{\partial^n}{\partial t^n} Y_0 \right]_{t \in \mathbb{N}} \subset \mathcal{L}. \]
We show that the ring
\[ L^2[Q, X] \left[ \frac{\partial^n}{\partial t^n} Y_0 \right]_{t \in \mathbb{N}} \]
is closed under the operations \( \Sigma, \Theta^{(i)}, \partial/\partial t \) and \( \partial/\partial y \) of \( F(\mathbb{N}, L^2)[[X]] \). Evidently the ring is closed under the last two operators. Since the operators \( \Sigma \) and \( \partial^n/\partial t^n \) operating on \( F(\mathbb{N}, L^2)[[X]] \) mutually commute, it follows from (46)
\[ \Sigma \left( \frac{\partial^n}{\partial t^n} Y_0 \right) = \frac{\partial^n}{\partial t^n} \Sigma(Y_0) = \frac{\partial^n}{\partial t^n} (q^\alpha Y_0) = q^\alpha \frac{\partial^n}{\partial t^n} Y_0. \]
So the ring is closed under \( \Sigma \). Similarly since the operators \( \Theta^{(1)} \) and \( \partial^n/\partial t^n \) mutually commute on \( F(\mathbb{N}, L^2)[[X]] \),
\[ \Theta^{(1)} \left( \frac{\partial^n}{\partial t^n} Y_0 \right) = \frac{\partial^n}{\partial t^n} \Theta^{(1)}(Y_0) \]
\[ = \frac{1}{y} \frac{\partial^n}{\partial t^n} \Theta^{(1)}(y) \]
\[ = \frac{1}{y} \frac{\partial^n}{\partial t^n} \Theta^{(1)}(\iota(y)) \]
\[ = \frac{1}{y} \frac{\partial^n}{\partial t^n} \iota(\sigma^{(1)}(y)) \]
\[ = \frac{1}{y} \frac{\partial^n}{\partial t^n} \iota \left( \frac{\sigma(y) - y}{(q - 1)t} \right) \]
\[ = \frac{1}{y} \frac{\partial^n}{\partial t^n} \left( \frac{q^\alpha Y_0 y - Y_0 y}{(q - 1)(tQ + X)} \right) \]
\[ = \frac{1}{y} \frac{\partial^n}{\partial t^n} \left( \frac{q^\alpha Y_0 - Y_0}{(q - 1)(tQ + X)} \right), \]
which is an element of the ring. \( \square \)
Conjecture 5.3 arises from experience that if \(q\) is not a root of unity, it is very hard to find a non-trivial algebraic relations among the partial derivatives \(\frac{\partial^n Y_0}{\partial t^n}\) for \(n \in \mathbb{N}\) over \(L\) so that we could guess that there would be none.

In fact, assume that we could prove our guess. Let \(\varphi: \mathcal{L} \to F(\mathbb{N}, A[[W_1, W_2]][[X]])\) be an infinitesimal deformation of \(\iota\). So as we have seen

\[
Z_0 = \varphi(Y_0) = \sum_{n=0}^{\infty} X^n \binom{\alpha}{n}_q (et)^{-n} Q^{\alpha-n} b(W_1)
\]

with \(b(W_1) \in A[[W_1]]\). There would be no constraints among the partial derivatives \(\partial^n b(W_1)/\partial W_1^n\), \(n \in \mathbb{N}\) and hence we could choose any power series \(b(W_1) \in A[[W_1]]\).

### 5.3 The functor \(\mathcal{NCF}_{L/k}\) of non-commutative deformations

We study the functor \(\mathcal{NCF}_{L/k}(A)\) of non-commutative deformations

\[
\mathcal{NCF}_{L/k}: ((\text{NCAlg}/L^\sharp)) \to (\text{Set}).
\]

For a not necessarily commutative \(L^\sharp\)-algebra \(A \in \text{Ob}((\text{NCAlg}/L^\sharp))\), let

\[
\varphi: \mathcal{L} \to F(\mathbb{N}, A[[W_1, W_2]][[X]])
\]

be an infinitesimal deformation of the canonical morphism

\[
\iota: \mathcal{L} \to F(\mathbb{N}, A[[W_1, W_2]][[X]]).
\]

Both \(t\) and \(y\) are elements of the field \(\mathbb{C}(t, t^\alpha) = \mathbb{C}(t, y)\) so that \([t, y] = ty - yt = 0\). So for the deformation \(\varphi \in \mathcal{NCF}_{L/k}(A)\) we must have

\[
[\varphi(t), \varphi(y)] = \varphi(t)\varphi(y) - \varphi(y)\varphi(t) = 0.
\]

When we consider the non-commutative deformations, the commutativity \((56)\) gives a constraint for the deformation. To see this we need a Lemma.

**Lemma 5.7.** For every \(l \in \mathbb{N}\), we have

\[
q^l \binom{\alpha}{l}_q + \binom{\alpha}{l - 1}_q = \binom{\alpha}{l}_q + q^{\alpha-l+1} \binom{\alpha}{l-1}_q.
\]

**Proof.** This follows from the definition of \(q\)-binomial coefficient. \(\square\)

**Lemma 5.8.** Let \(A\) be a not necessarily commutative \(L^\sharp\)-algebra in \(\text{Ob}(\text{NCAlg}/L^\sharp)\). Let \(e, f \in A\) such that \(e - 1\) and \(f\) are nilpotent. We set \(A := (e(t + W_1) + f)Q + X\) and for a power series \(b(W_1) \in A[[W_1]]\), we also set

\[
\mathcal{Z} := \sum_{n=0}^{\infty} X^n \binom{\alpha}{n}_q (e(t + W_1) + f)^{-n} Q^{\alpha-n} b(W_1)
\]

so that \(A\) and \(Z\) are elements of \(F(\mathbb{N}, A[[W_1]][[X]])\). The following conditions are equivalent.
(1) \([A, Z] := AZ - ZA = 0.\)

(2) \([e(t + W_1) + f, b(W_1)] = 0.\)

**Proof.** We formulate condition (1) in terms of coefficients of the power series in \(X\). Assume condition (1) holds so that we have

\[
\left((e(t + W_1 + f))Q + X\right) \left(\sum_{n=0}^{\infty} X^n \left(\frac{\alpha}{n}\right)_q (e(f + W_1) + f)^{-n} Q^{\alpha-n} b(W_1)\right)
= \left(\sum_{n=0}^{\infty} X^n \left(\frac{\alpha}{n}\right)_q (e(f + W_1) + f)^{-n} Q^{\alpha-n} b(W_1)\right) ((e(t + W_1 + f))Q + X).
\]

Comparing degree \(l\) terms in \(X\) of (57), we find that condition (1) is equivalent to

\[
q^l \left(\frac{\alpha}{l}\right)_q (e(t + W_1) + f)^{-l+1} Q^{\alpha-l+1} b(W_1)
+ \left(\frac{\alpha}{l-1}\right)_q (e(t + W_1) + f)^{-l+1} Q^{\alpha-l+1} b(W_1)
= \left(\frac{\alpha}{l}\right)_q (e(t + W_1) + f)^{-l} b(W_1)(e(t + W_1) + f) Q^{\alpha-l+1}
+ \left(\frac{\alpha}{l-1}\right)_q q^{\alpha-l+1} (e(t + W_1) + f)^{-l+1} Q^{\alpha-l+1} b(W_1).
\]

That is equivalent to

\[
q^l \left(\frac{\alpha}{l}\right)_q (e(t + W_1) + f) b(W_1)
+ \left(\frac{\alpha}{l-1}\right)_q (e(t + W_1) + f) b(W_1)
= \left(\frac{\alpha}{l}\right)_q b(W_1)(e(t + W_1) + f)
+ \left(\frac{\alpha}{l-1}\right)_q q^{\alpha-l+1} (e(t + W_1) + f) b(W_1)
\]

for every \(l \in \mathbb{N}\). Condition (59) for \(l = 0\) is condition (2). Hence condition (1) implies condition (2). Conversely, condition (1) follows from (2) in view of (59) and Lemma 5.7. \(\square\)

Now let us come back to the infinitesimal deformation (55) of the canonical morphism \(\iota\). The argument in Section 4 allows us to determine the restriction \(\varphi\) on the subalgebra generated by \(\iota(t) = tQ + X\) over \(L\) invariant under the \(\Theta^{(i)}\)'s, \(\Sigma\) and \(\{\partial/\partial t, \partial/\partial y\}\) in \(F(\mathbb{N}, L)[[X]]\). So there exist \(e, f \in A\) such that \(ef = q^{-1} fe, e - 1, f\) are nilpotent and such that

\[
\varphi(Q) = eQ \quad \text{and} \quad \varphi(X) = fX + Q,
\]
that are equations in $F(N, A[[W_1, W_2]])[[X]]$. In particular
\[
\varphi(t) = \varphi(tQ + X) = (et + f)Q + X = (e(t + W_1) + f)Q + X,
\]
where we naturally identify rings
\[
F(N, L^2)[[X]] \to F(N, A[[W_1, W_2]])[[X]] \to F(N, A[[W_1, W_2]])[[X]]
\]
through the canonical maps.

Then the argument in the commutative case allows us to show that there exists a power series $b_0(W_1) \in A[[W_1]]$ such that
\[
\varphi(Y_0) = \sum_{n=0}^{\infty} X^n \binom{\alpha}{n} q (e(t + W_1) + f)^{-n} Q^{\alpha-n} b_0(W_1).
\]
such that all the coefficients of the power series $b_0(W_1)$ are nilpotent. As we deal with the not necessarily commutative algebra $A$, the commutation relation in $L$ gives a constraint. Namely since $\iota(y) = yY_0$ and $ty = yt$ in $L$ so that $\iota(t)\iota(y) = \iota(y)\iota(t)$, we get $\iota(t)(yY_0) = (yY_0)t(t)$ in $L$ and $\varphi(tQ + X)\varphi(Y_0) = \varphi(Y_0)\varphi(tQ + X$. So we consequently have
\[
AZ_0 = Z_0A \quad \text{in} \quad F(N, A[[W_1, W_2]])[[X]],
\]
(60)

setting
\[
A := e((t + W_1) + f)Q + X, \quad Z_0 := \sum_{n=0}^{\infty} X^n \binom{\alpha}{n} q (e(t + W_1) + f)^{\alpha-n} b_0(W_1).
\]

**Lemma 5.9.** We have $[e(t + W_1) + f, b_0(W_1)] = 0$.

**Proof.** This follows from (60) and Lemma 5.8.

**Definition 5.10.** We define a functor
\[
QG_{II_q} : ((NCAlg/L^2)) \to (Set)
\]
by putting
\[
QG_{II_q}(A) = \{ \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix}, b(W_1) \in M_2(A) \times A[[W_1]] \mid e, f \in A, ef = q^{-1}fe, \text{ e is invertible in } A, b(W_1) \in A[[W_1]], [e(t + W_1) + f, b(W_1)] = 0 \}
\]
for $A \in Ob(NCAlg/L^2)$.

The functor $QG_{II_q}$ is almost a quantum group. We also need the formal completion $\widehat{QG_{II_q}}$ of the quantum group functor $QG_{II_q}$ so that
\[
\widehat{QG_{II_q}} : (NCAlg/L^2) \to (Set)
\]
is given by
\[ \widehat{G}_{II q}(A) = \{ \left[ \begin{array}{cc} e & f \\ 0 & 1 \end{array} \right], b(W_1) \in G_{II q}(A) \mid e^{-1}, f \text{ and all the coefficients of } b(W_1) \text{ are nilpotent} \} \]
for \( A \in \text{Ob}(NCAlg/L^{\#}) \).

Using Definition 5.10, we have shown the following

**Proposition 5.11.** There exists a functorial inclusion
\[ \mathcal{NCF}_{L/k}(A) \hookrightarrow \widehat{G}_{II q}(A) \]
sending \( \phi \mathcal{NCF}_{L/k}(A) \) to \( \left( \left[ \begin{array}{cc} e & f \\ 0 & 1 \end{array} \right], b_0(W_1) \right) \in \widehat{G}_{II q}(A) \).

We show that \( \widehat{G}_{II q} \) is a quantum formal group over \( L^{\#} \). In fact, we take two elements
\[(G, \xi(W_1)) = \left( \left[ \begin{array}{cc} e & f \\ 0 & 1 \end{array} \right], \xi(W_1) \right), \quad (H, \eta(W_1)) = \left( \left[ \begin{array}{cc} g & h \\ 0 & 1 \end{array} \right], \eta(W_1) \right)\]
of \( \widehat{G}_{II q}(A) \) so that \( e, f, g, h \in A \) so that
\[ ef = q^{-1}fe, \quad gh = q^{-1}hg, \]
the elements \( e \) and \( g \) are invertible and such that
\[ [e(t + W_1) + f, \xi(W_1)] = 0, \quad [g(t + W_1) + h, \eta(W_1)] = 0. \quad (61) \]

When the following two subsets of the ring \( A \)

1. \( \{e, f\} \cup (\text{the subset of all the coefficients of the power series } \xi(W_1)) \),
2. \( \{g, h\} \cup (\text{the subset of all the coefficients of the power series } \eta(W_1)) \),

are mutually commutative, we define the product of \( (G, \xi(W_1)) \) and \( (H, \eta(W_1)) \) by
\[ (G, \xi(W_1)) \star (H, \eta(W_1)) = (GH, \xi(gW_1 + (g - 1)t + h)\eta(W_1)). \]

**Lemma 5.12.** The product \( (GH, \xi(gW_1 + (g - 1)t + h)\eta(W_1)) \) is indeed an element of \( \widehat{G}_{II q}(A) \).

**Proof.** First of all, we notice that the constant term \((g - 1)t\) of the linear polynomial in \( W_1 \)
\[ gW_1 + (g - 1)t + h \quad (62) \]
is nilpotent so that we can substitute (62) into the power series $\xi(W_1)$. Therefore

$$\xi(gW_1 + (g - 1)t + h)\eta(W_1)$$

is a well-determined element of the power series ring $A[[W_1]]$. We have seen in Section 4 that if $\{e, f\}$ and $\{g, h\}$ are mutually commutative, then the product $GH$ of matrices $G, H \in \mathcal{H}_qL^\flat(A)$ is in $\mathcal{H}_qL^\flat(A)$. Since $GH = \begin{bmatrix} eg & eh + f \\ 0 & 1 \end{bmatrix}$, it remains to show

$$[eg(t + W_1) + eh + f, \xi(gW_1 + (g - 1)t + h)\eta(W_1)] = 0.$$  

(63)

The proof of (63) is done in several steps.

First we show

$$[\xi(gW_1 + (g - 1)t + h), \eta(W_1)] = 0.$$  

(64)

This follows, in fact, from the mutual commutativity of the subsets (1) and (2) above, and the second equation of (61).

Second we show

$$[eg(t + W_1) + eh + f, \xi(gW_1 + (g - 1)t + h)] = 0.$$  

(65)

To this end, we notice

$$eg(t + W_1) + eh + f = e(gW_1 + (g - 1)t + h) + et + f.$$  

(66)

So we have to show

$$[e(gW_1 + (g - 1)t + h) + et + f, \xi(gW_1 + (g - 1)t + h)] = 0.$$  

(67)

This follows from the first equation of (61) and the mutual commutativity of the subsets (1) and (2).

We prove third

$$[eg(t + W_1) + eh + f, \eta(W_1)] = 0.$$  

(68)

In fact, this a consequence of the second equation of (61) and the mutual commutativity of the subsets (1) and (2). Equality (63) is a consequence of (64), (65) and (68).

One can check associativity for the multiplication by a direct calculation. The unit element is given by

$$(I_2, 1) \in \widehat{QG}_{11}(L^\sharp).$$

The antipode is given by the formula below. For an element

$$(G, b(W_1)) = \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix}, b(W_1) \in \widehat{QG}_{11}(A),$$

we set

$$(G, b(W_1))^{-1} := \begin{bmatrix} e^{-1} & -e^{-1}f \\ 0 & 1 \end{bmatrix}, b(e^{-1}W_1) + (e^{-1}t - e^{-1}f)^{-1}b(W_1) \in \widehat{QG}_{11}(A),$$

then we have

$$(G, b(W_1))^{-1} \ast (G, b(W_1)) = (G, b(W_1)) \ast (G, b(W_1))^{-1} = (I_2, 1).$$
Conjecture 5.13. If $q$ is not a root of unity, the injection in Proposition 5.11 is an equality for every $A \in \text{Ob}(\text{CAlg}/L^2)$.

Proposition 5.14. Conjecture 5.13 implies Conjecture 5.3

Proof. Let us assume Conjecture 5.13. Take an element $(e, \xi(W_1)) \in \hat{G}_{II}(A)$ for $A \in \text{Ob}(\text{CAlg}/L^2)$. Since $A$ is commutative, the commutation relation in Lemma 5.9 imposes no condition on $\xi(W_1)$, $(e, \xi(W_1)) \in \hat{QG}_{II,q}(A)$. Conjecture 5.13 says that if $q$ is not a root of unity, $(e, \xi(W_1))$ arise from an infinitesimal deformation $\iota: L \to F(\mathbb{N}, A[[W_1,W_2]][[X]])$.

Conjecture 5.13 says that we can identify the functor $NCF_{L/k}$ with the quantum formal group $\hat{QG}_{II,q}$. To be more precise, the argument in the first Example studied in 4 allows us to define a formal $\mathbb{C}$-Hopf algebra $\hat{I}_q$ and hence

$$\hat{I}_q L^2 := \hat{I}_q \otimes \mathbb{C} L^2,$$

which is a functor on the category $(NCAlg/L^2)$ so that we have a functors isomorphism

$$\hat{I}_q L^2(A) \cong \hat{QG}_{II,q}(A) \quad \text{for every } L^2\text{-algebra } A \in \text{Ob}(NCAlg/L^2).$$

5.4 Summary on the Galois structures of the field extension $\mathbb{C}(t, t^\alpha)/\mathbb{C}$

Let us summarize our results on the $(\mathbb{C}(t, t^\alpha)/\mathbb{C})$.

(1) Difference field extension $(\mathbb{C}(t, t^\alpha), \sigma)/\mathbb{C}$. This is a Picard-Vessiot extension with Galois group $\mathbb{G}_m \times \mathbb{G}_m$.

(2) Differential field extension $(\mathbb{C}(t, t^\alpha), d/dt)/\mathbb{C}$. This is not a Picard-Vessiot extension. The Galois group

$$\text{Inf-gal}(L/k): (CAlg/L^2) \to (\text{Grp})$$

is isomorphic to $\hat{G}_{mL^2} \times \hat{G}_{aL^2}$, where $\hat{G}_{mL^2}$ and $\hat{G}_{aL^2}$ are formal completion of the multiplicative group and the additive group. So as group functors on the category $(CAlg/L^2)$, we have

$$\hat{G}_{mL^2}(A) = \{b \in A \mid b - 1 \text{ is nilpotent}\}$$

and

$$\hat{G}_{aL^2}(A) = \{b \in A \mid b \text{ is nilpotent}\}$$

for a commutative $L^2$-algebra $A$. 

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(3) Commutative deformation of $q$-SI $\sigma$-differential extension $(\mathbb{C}(t, t^\alpha), \sigma, \theta^*)/\mathbb{C}$. If $q$ is not a root of unity, $\text{Inf-gal}(L/k)$ is an infinite dimensional formal group such that we have
\[ 0 \to A[[W_1]]^* \to \text{Inf-gal}(L/k)(A) \to \hat{\mathbb{G}}_m(A) \to 0, \]
where $A[[W_1]]^*$ denotes the multiplicative group
\[ \{a \in A[[W_1]] \mid \text{all the coefficients of power series } a - 1 \text{ are nilpotent}\}. \]
modulo Conjecture 5.13.

(4) Non-commutative Galois group. If $q$ is not a root of unity, the infinitesimal deformation functor is isomorphic to a quantum formal group:
\[ \mathcal{NCF}_{L/k} \simeq \hat{QG}_{11q}, \]
modulo Conjecture 5.13.

We should be careful about the group structure. Quantum formal group structure in $\hat{QG}_{qL}$ coincides with the group structure defined from the initial conditions as in Remark 4.13. So we might say that non-commutative Galois group is the quantum formal group $\hat{QG}_{11q}$.

(5) Let us assume $q$ is not a root of unity. If we have a $q$-difference field extension $(L, \sigma)/(k, \sigma)$ such that $t \in L$ with $\sigma(t) = qt$, then we can define the operator $\theta^{(1)} : L \to L$ by setting
\[ \theta^{(1)}(a) := \frac{\sigma(a) - a}{qt - t}. \]
We also assume the field $k$ is $\theta^{(1)}$ invariant. Defining the operator $\theta^{(n)} : L \to L$ by
\begin{align*}
\theta^{(0)} &= \text{Id} \
\theta^{(n)} &= \frac{1}{[n]_q!}(\theta^{(1)})^n
\end{align*}
for every positive integer $n$ so that we have a $q$-SI $\sigma$-differential field extension $(L, \sigma, \theta^*)/(k, \sigma, \theta^*)$.

Here arises a natural question of comparing the Galois groups of the difference field extension $(L, \sigma)/(k, \sigma)$ and $q$-SI $\sigma$-differential field extension $(L, \sigma, \theta^*)/(k, \sigma, \theta^*)$. As the $q$-SI $\sigma$-differential field extension is constructed from the difference field extension in a more or less trivial way, one might imagine that they coincide or they are not much different.

This contradicts Conjecture 5.13. Let us take our example $\mathbb{C}(t, t^\alpha)/\mathbb{C}$. Assume Conjecture 5.13 is true. Then the Galois group for the $q$-SI $\sigma$-differential extension is $\hat{QG}_{11qL}$ that is infinite dimensional, whereas the the Galois group is of the difference field extension is of dimension 2.
6 The third example, the field extension $\mathbb{C}(t, \log t)/\mathbb{C}$

We assume $q$ is a complex number not equal to 0. Let us study the field extension $L/k := \mathbb{C}(t, \log t)/\mathbb{C}$ from various viewpoints as in Sections 4 and 5.

6.1 $q$-difference field extension $\mathbb{C}(t, \log t)/\mathbb{C}$.

We consider $q$-difference operator $\sigma: L \to L$ such that $\sigma$ is the $\mathbb{C}$-automorphism of the field $L$ satisfying

$$\sigma(t) = qt \quad \text{and} \quad \sigma(\log t) = \log t + \log q. \quad (71)$$

It follows from (71) that if $q$ is not a root of unity, then the field of constants of the difference field $(\mathbb{C}(t, \log t), \sigma)$ is $\mathbb{C}$ and hence $(\mathbb{C}(t, \log t), \sigma)/\mathbb{C}$ is a Picard-Vessiot extension with Galois group $\mathbb{G}_m \times \mathbb{G}_a$.

6.2 Differential field extension $(\mathbb{C}(t, \log t), d/dt)/\mathbb{C}$.

As we have

$$\frac{dt}{dt} = 1 \quad \text{and} \quad \frac{d \log t}{dt} = \frac{1}{t}.$$ 

So both differential field extensions $\mathbb{C}(t, \log t)/\mathbb{C}(t)$ and $\mathbb{C}(t)/\mathbb{C}$ are Picard-Vessiot extensions with Galois group $\mathbb{G}_a$. The differential extension $\mathbb{C}(t, \log t)/\mathbb{C}$ is not, however, a Picard-Vessiot extension. Therefore we need general differential Galois theory [11] to speak of the Galois group of the differential field extension $\mathbb{C}(t, \log t)/\mathbb{C}$.

The universal Taylor morphism

$$\iota: L \to L^\#[[X]]$$

sends

$$\iota(t) = t + X, \quad (72)$$

$$\iota(\log t) = \log t + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{n} \left( \frac{X}{t} \right)^n \in L^\#[[X]]. \quad (73)$$

Writing $\log t$ by $y$, we take $\partial/\partial t$, $\partial/\partial y$ as a basis of $L^\# = \mathbb{C}(t, y)^2$-vector space $\text{Der}(L^\#/k^\#)$ of $k^\#$-derivations of $L^\#$. It follows from (72), (73) that

$$\mathcal{L} = \text{a localization of the algebra } L^\#[t + X, \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \left( \frac{X}{t} \right)^n ] \subset L^\#[[X]].$$

We argue as in [4,2] and Section 5. For a commutative algebra $A \in \text{Ob}(\mathcal{C}\text{Alg}/L^\#)$ and $\varphi \in \mathcal{F}_{L/k}(A)$, there exist nilpotent elements $a, b \in A$ such that $a, b$ such that

$$\varphi(t + X) = t + W_1 + X + a,$$

$$\varphi(\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \left( \frac{X}{t + W_1} \right)^n ) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \left( \frac{X}{t + W_1 + a} \right)^n + b.$$
Therefore we arrived at the dynamical system

$$
\begin{align*}
\varphi(t) &= t + X + W_1 + a, \\
\varphi(y) &= y + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \left( \frac{X}{t + W_1 + a} \right)^n + b.
\end{align*}
$$

(74)

In terms of initial conditions, dynamical system (74) reads

$$
\begin{align*}
\{ t, y, \mapsto \{ t + a, \\
y, \mapsto \{ y + b, 
\end{align*}
$$

where $a, b$ are nilpotent elements of $A$. So we conclude

$$\text{Inf-gal} \left( L/k \right)(A) = \hat{G}_a(A) \times \hat{G}_a(A)$$

for every commutative $L^\natural$-algebra $A$. Consequently we get

$$\text{Inf-gal} \left( L/k \right) \simeq (\hat{G}_a, C) \times (\hat{G}_a, C) \otimes_C L^\natural.$$

### 6.3 $q$-SI $\sigma$-differential field extension $(\mathbb{C}(t, \log t), \sigma, \theta^*)/\mathbb{C}$

$\sigma: \mathbb{C}(t, \log t) \to \mathbb{C}(t, \log t)$ is the automorphism in Subsection 6.1. We set $\theta^{(0)} = \text{Id}_{\mathbb{C}(t, \log t)}$ and

$$\theta^{(1)} = \frac{\sigma - \text{Id}_{\mathbb{C}(t, \log t)}}{(q - 1)t}$$

so that $\theta^{(1)}: \mathbb{C}(t, \log t) \to \mathbb{C}(t, \log t)$ is a $\mathbb{C}$-linear map. We farther introduce

$$\theta^{(i)} := \frac{1}{[i]_q!} (\theta^{(1)})^i: \mathbb{C}(t, \log t) \to \mathbb{C}(t, \log t)$$

that is a $\mathbb{C}$-linear map for $i = 1, 2, 3 \cdots$. Hence if we denote the set $\{\theta^{(i)}\}_{i \in \mathbb{N}}$ by $\theta^*$, then $(\mathbb{C}(t, \log t), \sigma, \theta^*)$ is a $q$-SI $\sigma$-differential ring.

The universal Hopf morphism

$$\iota: \mathbb{C}(t, \log t) \to F(\mathbb{N}, L^\natural)[[X]]$$

sends, by Proposition 3.7, $t$ and $y$ respectively to

$$\begin{align*}
\iota(t) &= tQ + X \\
\iota(y) &= y + (\log q)N + \frac{\log q}{q - 1} \sum_{n=1}^{\infty} X^n (-1)^{n+1} \frac{1}{[n]_q} (tQ)^{-n} 
\end{align*}$$

that we identify with

$$y + W_2 + (\log q)N + \frac{\log q}{q - 1} \sum_{n=1}^{\infty} X^n (-1)^{n+1} \frac{1}{[n]_q} (t + W_1)^{-n} Q^{-n}, \in F(\mathbb{N}, A[[W_1, W_2]])[[X]],$$

where $a$, $b$ are nilpotent elements of $A$. So we conclude

$$\text{Inf-gal} \left( L/k \right)(A) = \hat{G}_a(A) \times \hat{G}_a(A)$$

for every commutative $L^\natural$-algebra $A$. Consequently we get

$$\text{Inf-gal} \left( L/k \right) \simeq (\hat{G}_a, C) \times (\hat{G}_a, C) \otimes_C L^\natural.$$
We identify further $t + X$ with $t + W_1 + X \in L^2[[W_1, W_2]][[X]]$ and

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{[n]_q} \left( \frac{X}{t} \right)^n$$

with

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{[n]_q} \left( \frac{X}{t + W_1} \right)^n \in L^2[[W_1, W_2]][[X]].$$

### 6.3.1 Commutative deformations $F_{L/k}$ for $(\mathbb{C}(t, \log t), \sigma, \theta^*)/\mathbb{C}$

Now the argument of Section 5 allows us to describe infinitesimal deformations on the category of commutative $L^2$-algebras $(CAlg/L^2)$. Let $\varphi: \mathcal{L} \to F(N, A[[W_1, W_2]][[X]]$ be an infinitesimal deformation of the canonical morphism $\iota: \mathcal{L} \to F(N, A[[W_1, W_2]][[X]]$ for $A \in \text{Ob}(CAlg/L^2)$. Then there exist $e \in A$ and $b(W_1) \in A[[W_1]]$ such that $e - 1$ and all the coefficients of the power series $b(W_1)$ are nilpotent and such that

$$\varphi(t + W_1 + X) = e(t + W_1) + X,$$

where

$$\varphi\left( \sum_{n=1}^{\infty} X^n(-1)^{n+1} \frac{1}{[n]_q} (t + W_1)^{-n} Q^{-n} \right) = \sum_{n=1}^{\infty} X^n(-1)^{n+1} \frac{1}{[n]_q} (e(t + W_1))^{-n} Q^{-n} + b(W_1).$$

**Proposition 6.1.** We have an injection

$$F_{L/k}(A) \to \hat{G}_{III} := \{ (e, b(W_1)) \in A \times A[[W_1]] \mid e - 1 \text{ is nilpotent, all the coefficients of } b(W_1) \text{ are nilpotent} \}.$$

**Conjecture 6.2.** If $q$ is not a root of unity, then the injection in Proposition 6.1 is an equality.

$\hat{G}_{III}$ is a group functor on $(CAlg/L^2)$. In fact, for $A \in \text{Ob}(Alg/L^2)$, we define the product of two elements

$$(e, b(W_1)), (g, c(W_1)) \in \hat{G}_{III}(A)$$

by

$$(e, b(W_1)) \ast (g, c(W_1)) := (eg, b(gW_1 + (g - 1)t) + c(W_1)).$$

Then the product is a well-determined element of $\hat{G}_{III}(A)$, the product is associative, the unit element is $(1, 0) \in \hat{G}_{III}(A)$ and the inverse $(e, b(W_1))^{-1} = (e^{-1}, -b(e^{-1}W_1 + (e^{-1} - 1)t))$.

So if Conjecture 6.2 is true, we have a non-splitting exact sequence

$$0 \to A[[W_1]]_+ \to \text{Inf-gal}(L/k)(A) \to \hat{G}_{mL^2}(A) \to 1,$$

where $A[[W_1]]_+$ denote the additive group of the power series in $A[[W_1]]$ whose coefficients are nilpotent element.
6.3.2 Non-commutative deformations $\mathcal{NCF}_{L/k}$ for $(\mathbb{C}(t, \log t), \sigma, \theta^r)/\mathbb{C}$

The arguments in Section 5 allows us to prove analogous results for $q$-SI $\sigma$-differential field extension $(\mathbb{C}(t, \log t), \sigma, \theta^r)/\mathbb{C}$. We write assertions without giving detailed proofs. For, the proofs are same.

**Definition 6.3.** We introduce a functor

$$\hat{\mathcal{Q}}_{III}^{q}: ((NCAlg/L)) \rightarrow (Set)$$

by setting

$$\hat{\mathcal{Q}}_{III}^{q}(A) := \{(H, \varphi(W_1)) \in \mathcal{S}_q L^2(A) \times A[[W_1]] \mid (1) H = \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix} \in \mathcal{S}_q(A) \text{ so that} \ e f = q f e, \ e - 1, f \in A \text{ are nilpotent.} \ (2) \text{ All the coefficients of } \varphi(W_1) \text{ are nilpotent.} \ (3) [e(t + W_1) + f, \varphi(W_1)] = 0.\}$$

$\hat{\mathcal{Q}}_{III}^{q}$ is a quantum formal group. Namely, for

$$(G, \varphi(W_1)), (H, \psi(W_1)) \in \hat{\mathcal{Q}}_{III}^{q}(A)$$

such that the two subsets

- all the entries of matrix $H$, all the coefficients of the power series $\varphi(W_1)$;
- all the entries of matrix $H$, all the coefficients of the power series $\psi(W_1)$

of $A$ are mutually commutative, we define their product by

$$(G, \varphi(W_1)) \star (H, \psi(W_1)) := (G H, \varphi(g W_1 + (g - 1)t + h) + \psi(W_2)),$$

where

$$H = \begin{bmatrix} g & h \\ 0 & 1 \end{bmatrix}.$$

Then the product of two elements is a well-determined element in the set $\hat{\mathcal{Q}}_{III}^{q}(A)$ and the product is associative. The unit element is $(\text{Id}_2, 0) \in \hat{\mathcal{Q}}_{III}^{q}(A)$. The inverse

$$(G, \varphi(W_1))^{-1} = (G^{-1}, -\varphi(e^{-1}W_1 + (e^{-1} - 1)t - e^{-1}f) \in \hat{\mathcal{Q}}_{III}^{q-1}(A),$$

where

$$G = \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix}.$$

**Proposition 6.4.** We have a functorial injection

$$\mathcal{NCF}_{L/k}(A) \rightarrow \hat{\mathcal{Q}}_{III}^{q}(A)$$

that send $\varphi \in \mathcal{NCF}_{L/k}(A)$ to $\begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix}, b(W_1))$. Here

$$\varphi(t + W_1 + X) = e(t + W_1) + f + X,$$

$$\varphi(\sum_{n=1}^{\infty} X^n(-1)^{n+1} \frac{1}{n!}(e(t + W_1) + f)^{-n}Q^{-n} + b(W_1)).$$
We also have a Conjecture.

**Conjecture 6.5.** If $q$ is not a root of unity, then the injection in Proposition 6.4 is an equality. So

$$\mathcal{NCF}_{L/k} \simeq \widehat{G}_{III}.$$  

Therefore quantum Galois group of the $q$-SI $\sigma$-differential extension is the wuantum formal group $\widehat{G}_{III,q}$.

**Remark 6.6.** The argument in 5.3 allows us to prove that Conjecture 6.5 implies Conjecture 6.13.

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