Some Numerical Results on the Rank of Generic Three-Way Arrays over $\mathbb{R}$

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Abstract. The aim of this paper is the introduction of a new method for the numerical computation of the rank of a three-way array, $X \in \mathbb{R}^{I \times J \times K}$, over $\mathbb{R}$. We show that the rank of a three-way array over $\mathbb{R}$ is intimately related to the real solution set of a system of polynomial equations. Using this, we present some numerical results based on the computation of Gröbner bases.

Key words: Tensors; three-way arrays; Candecomp/Parafac; Indscal; generic rank; typical rank; Veronese variety; Segre variety; Gröbner bases.

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1. Introduction

Let $X \in \mathbb{R}^{I \times J \times K}$ be a tensor of order 3, sometimes named a three-way array or a three-mode data set. A rank 1 or a decomposed tensor is

$$D = a \otimes b \otimes c,$$

where $a \in \mathbb{R}^I$, $b \in \mathbb{R}^J$ and $c \in \mathbb{R}^K$, and $\otimes$ is the tensor product, sometimes named also outer product. $X$ can be expressed as a sum of decomposed tensors given in (1),

$$X = \sum_{\alpha=1}^{r} D_\alpha.$$

The rank of $X$ is defined to be the minimal integer $r$, see for instance Kruskal (1977, 1989). In data analysis, this implies that the rank of a three-way array is the smallest number of components that provide a perfect fit in Candecomp/Parafac (CP), see for instance, (Carroll and Chang, 1970, and Harshman, 1970). In statistics CP is considered a natural extension of singular value decomposition or principal components analysis to three-way data.

There is quite a literature concerning the value of maximal rank, generic rank or typical rank of three-way arrays in the area of statistics, algebraic complexity theory and algebraic geometry. Some references, among others, are: Ja’Ja’ (1979),
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Kruskal (1977, 1983, 1989), Strassen (1983), Ten Berge (1991, 2000, 2004a, 2004b), Ten Berge and Kiers (1999), Ten Berge and Stegeman (2006), Comon and Ten Berge (2008), Bürgisser et al (1997), Catalisano et al (2002), Friedland (2008) and Abo et al. (2006). Friedland (2008) provides an up to date survey with some new results on the generic rank of three-way arrays.

First, we give the following

**Definition 1**: A dataset is called *generic* if its elements are randomly generated from a continuous distribution.

The generic and typical ranks are defined in the following way by Comon and Ten Berge (2008): Given that the rank of $I \times J \times K$ arrays is bounded, one can partition the arrays according to the rank values. Generic rank is defined to be true almost everywhere; while typical ranks are associated with the rank values that occur with positive probability. So, if there is a single typical rank, then it may be called generic rank; that is, a generic rank is typical, but the converse is not true.

Ten Berge (2000) classified three-way arrays into three classes: very tall, tall and compact. Let $X \in \mathbb{R}^{I \times J \times K}$ be a tensor of order 3 with $I \geq J \geq K$. The array $X$ is called very tall when $I \geq KJ$; $X$ is tall when $KJ - J < I \leq KJ - 1$; $X$ is compact when $I \leq KJ - J$. The generic rank of the very tall arrays is very well known and easiest to prove: it is $KJ$. Ten Berge (2000) showed that all tall three-way arrays have generic rank $I$; and the tallest among the compact arrays, that is when $I = KJ - J$, have typical rank $\{I, I + 1\}$. Ten Berge and Stegeman (2006) provided some further results on the compact case. Friedland (2008) showed that:

- typical rank$(12 \times 4 \times 4) \geq 12$,
- typical rank$(11 \times 4 \times 4) \geq 11$,
- typical rank$(I \times J \times K) \geq I$ for $(I, J, K) = ((J-1)^2+1, J, J)$ when $J \geq 2$.

These results are all based on mathematical proofs. However, the rank computation problem has also been approached from a numerical point of view: Comon and ten Berge (2008) and Friedland (2008) applied Terracini’s lemma, based on the numerical calculation of the maximal rank of the Jacobian matrix of (2), to obtain numerically the generic rank of some three-way arrays. The numerical method based on Terracini’s lemma, when used to evaluate rank over $\mathbb{R}$, gives the generic rank when the typical rank is single-valued, and the smallest typical rank value otherwise.

Two well known facts are:

- There is no known method to calculate the rank of a given three-way dataset, Martin (2004, AIM tensor workshop);
- A three-way array over $\mathbb{R}$ may have a different rank than the same array considered over $\mathbb{C}$, (Kruskal, 1989).

We shall be concerned by the numerical computation of the rank of a three-way array over $\mathbb{R}$ only.

Computationally, the most primitive approach to the numerical evaluation of the typical rank of three-way arrays is based on the alternating least square (ALS) min-
imization algorithm: It is to run ALS many times to convergence on many generic three-way arrays of a given format, and to check whether or not the fit is perfect for a given number of components. But as a referee remarked, this approach has 2 problems: First, we do not know how many three-way arrays of a given format to examine before a valid inference can be drawn. For instance, when 100000 arrays have been examined and all seem to have the same rank \( \alpha \), it does not follow that \( \alpha \) is indeed the generic rank for that array format. After all, a different rank may occur with an extremely small yet positive probability. Second, the decision of when to terminate ALS is hazardous, because even if the residual sum of squares is, say, \( e^{\exp(-32)} \), this does not prove that it is zero; in fact, it may have zero as infimum. The present paper relieves us from both above mentioned problems: It offers a straightforward method of determining the rank of any given array over \( \mathbb{R} \), based on inspection of the number of real roots of a system of certain polynomial equations.

The real solution set of a system of polynomial equations is called semi-algebraic set in real algebraic geometry, see Basu, Pollack and Roy (2006) or Friedland (2008). Semi-algebraic sets are open sets and are composed of a finite union of connected components, where each component is called a basic semi-algebraic set. The main problem can be reformulated as: For a given tensor \( \mathbf{X} \) over \( \mathbb{R} \) calculate the number of connected components where each component is characterized by a unique real rank value. Our numerical results will shed some light on this. The numerical results on simulated datasets will be obtained by computing the Gröbner bases using Maple 12 of the system of polynomial equations characterizing the dataset. We note that generic datasets and random numbers are generated from integers between \(-99\) and \(99\).

The paper is organized as follows. In section 2 we present the main lemma which provides a necessary and sufficient condition that a three-way array can be expressed as a sum of a fixed number of decomposed tensors. All results in this paper will be based on this lemma. In section 3, we show how the lemma can be applied to compute the rank of a generic tensor over \( \mathbb{R} \) numerically for some cases. In section 4, we show another application of the lemma for the computation of rank for nongeneric particular datasets. In section 5, we show how the lemma can be applied to compute the rank of generic \( I \) symmetric \( J \times J \) arrays, named INDSCAL arrays, over \( \mathbb{R} \). And finally in section 6 we conclude.

2. MAIN LEmMA
Let \( \mathbf{X} \in \mathbb{R}^{I \times J \times K} \) be a three-way dataset and \( 2 \leq K \leq J \leq I \). The lemma provides a necessary and sufficient condition that the tensor \( \mathbf{X} \) can be expressed as a sum of \( I \)
decomposed tensors; that is

\[ \mathbf{X} = \sum_{\alpha=1}^{I} D_{\alpha}, \]

\[ = \sum_{\alpha=1}^{I} \mathbf{a}_{\alpha} \otimes \mathbf{b}_{\alpha} \otimes \mathbf{c}_{\alpha}, \quad (3) \]

where \( \{ \mathbf{a}_{\alpha} | \alpha = 1, \ldots, I \} \) is a basis for \( \mathbb{R}^{I} \), \( \mathbf{c}_{\alpha} \in \mathbb{R}^{K} \) and \( \mathbf{b}_{\alpha} \in \mathbb{R}^{J} \). Note that if (3) is true, then \( \text{rank}(\mathbf{X}) \leq I \). We denote by \( \mathbf{X}_{k} \in \mathbb{R}^{I \times J} \) the \( k \)th slice in \( \mathbf{X} \) for \( k = 1, \ldots, K \).

We note that (3) can be written as

\[ \mathbf{X}_{k} = \sum_{\alpha=1}^{I} c_{k\alpha} \mathbf{a}_{\alpha} \otimes \mathbf{b}_{\alpha} \quad \text{for} \quad k = 1, \ldots, K, \]

\[ = \mathbf{A} \mathbf{D}(\mathbf{c}_{k}) \mathbf{B}' \quad \text{for} \quad k = 1, \ldots, K, \quad (4) \]

where \( \mathbf{A} = (\mathbf{a}_{1} \mathbf{a}_{2} \ldots \mathbf{a}_{I}) \in \mathbb{R}^{I \times I} \), \( \mathbf{B} = (\mathbf{b}_{1} \mathbf{b}_{2} \ldots \mathbf{b}_{I}) \in \mathbb{R}^{J \times I} \), \( \mathbf{C} = (c_{\alpha}) \in \mathbb{R}^{K \times I} \) and \( \mathbf{D}(\mathbf{c}_{k}) = \text{Diag}(\mathbf{c}_{k}) \in \mathbb{R}^{I \times I} \) is a diagonal matrix with diagonal elements \( c_{k\alpha} \). Note that the vector \( \mathbf{c}_{k} \in \mathbb{R}^{I} \) represents the \( k \)th row of \( \mathbf{C} \).

We consider the system of polynomial equations

\[ s'_{\alpha} \mathbf{X}_{k} = c_{k\alpha} b'_{\alpha} \quad \text{for} \quad k = 1, \ldots, K \text{ and } \alpha = 1, \ldots, I, \quad (5) \]

where \( \{ s_{\alpha} | \alpha = 1, \ldots, I \} \) is a basis for \( \mathbb{R}^{I} \) and \( \mathbf{c}_{\alpha} \in \mathbb{R}^{K} \), and \( \mathbf{b}_{\alpha} \in \mathbb{R}^{J} \). We note that (5) can be written as

\[ \mathbf{S}' \mathbf{X}_{k} = \mathbf{D}(\mathbf{c}_{k}) \mathbf{B}' \quad \text{for} \quad k = 1, \ldots, K, \quad (6) \]

where \( \mathbf{S} \) has columns \( s_{\alpha} \).

**Lemma 1:** (6) is a necessary and sufficient condition for (4).

**Proof:** Let \( \mathbf{I} = \mathbf{A} \mathbf{S}' \), then (4) is true if and only if (6) is true.

**Remark 1:** a) To see if (5) is true, we solve the system of polynomial equations

\[ s'_{\alpha} \mathbf{X}_{k} = c_{k\alpha} b'_{\alpha} \quad \text{for} \quad k = 1, \ldots, K \]

for \( s \in \mathbb{R}^{I} \), \( \mathbf{b} \in \mathbb{R}^{J} \) and \( \mathbf{c} \in \mathbb{R}^{K} \).

b) We note that (7) has two indeterminacies: It can be rewritten as \( s'_{\alpha} \mathbf{X}_{k} = c_{k\alpha} b_{s} \) for \( k = 1, \ldots, K \), where for instance, \( s_{\alpha} = \lambda s \) for any scalar \( \lambda \neq 0 \), \( c_{k\alpha} = \mu c_{k} \) for any scalar \( \mu \neq 0 \), and \( b_{s} = \lambda \mathbf{b}/\mu \). To eliminate these indeterminacies, hereafter, we fix

\[ c_{1} = 1 \quad \text{and} \quad s_{I} = 1. \quad (8) \]
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c) Theorem 2.4 of Friedland (2008) provides another characterization for (3) or (4): It states that each slice $X_k \in \text{span}(a_1 \otimes b_1, \ldots, a_I \otimes b_I)$ and the rank($X$) equals the minimal dimension of the $\text{span}(a_1 \otimes b_1, \ldots, a_I \otimes b_I)$.

d) The necessary condition, when rank($X$) = $I$, which was shown to be also sufficient afterwards, was used many times by Ten Berge and his coworkers, Ten Berge (2000), Ten Berge (2004a), and Ten Berge, Sidiropoulos and Rocci (2004).

3. Rank computation

We shall suppose in the sequel that $X \in \mathbb{R}^{I \times J \times K}$ is a generic three-way array and $2 \leq K \leq J \leq I \leq KJ$. Then we have the following well known inequality: rank($X$) \geq $I$. We will check if $X$ has rank $I$. By the Main Lemma, the tensor $X$ has rank $I$, if for parameter vectors $s \in \mathbb{R}^I$, $b \in \mathbb{R}^J$ and $c \in \mathbb{R}^K$ the system of polynomial equations (7) subject to (8) have $I$ real solutions $(c_{\alpha}, b_{\alpha}, s_{\alpha})$ for $\alpha = 1, \ldots, I$, such that the elements of the set $\{s_{\alpha} | \alpha = 1, \ldots, I\}$ is a basis for $\mathbb{R}^I$; that is, (7) with (8) has $I$ real isolated solutions. Let us see how can we know if this is true. The system of polynomial equations (7) with (8) is equivalent to

$$s'(X_k - c_k X_1) = 0' \text{ for } k = 2, \ldots, K. \quad (9)$$

So the number of equations, $neq$, in (9) is

$$neq = (K - 1)J, \quad (10)$$

and the number of degrees of freedom or the number of free variables, $df$, is

$$df = (I - 1) + (K - 1), \quad (11)$$

because of (8) there are $(K - 1)$ free $c_k$’s and $(I - 1)$ free $s_i$’s.

We are interested in the study of the number of solutions of (9) over $\mathbb{R}$ for generic data. We distinguish three cases named, minimal when $neq = df$, overdetermined when $df < neq$, and, underdetermined when $df > neq$. We note that Abo et al. (2006) also distinguished three cases that they named subabundant, superabundant and equiabundant: these were used for induction purposes.

3.1. Case 1: Minimal System($neq = df$). When $I = (K - 1)(J - 1) + 1$, $neq = df$, and the system (9) is called minimal. The number of real solutions is bounded; an upper bound is provided by Khovanskii’s theorem, see Sturmfels (2002),

**Theorem 1 (Khovanskii):** Consider $n$ polynomials in $n$ variables involving $m$ distinct monomials in total. The number of isolated roots in the positive orthant $(\mathbb{R}_+)^n$ of any such system is at most $2^{\binom{m}{2}}(n + 1)^m$. 
In our case \( n = \text{neq} = df = (K - 1)J \) and \( m = I - 1 + (K - 1)(I - 1) = K(I - 1) \). The number of isolated roots in the positive orthant \((\mathbb{R}_+)^d\) of any such system is at most \(2^{2^{df}}(df + 1)^m\) and \( m \) is the number of distinct monomials in the system (9). So (9) may or may not have \( I \) real isolated solutions. In case (9) has \( I \) real isolated solutions, then \( \text{rank}(X) = I \); otherwise we embed it, which is discussed later on.

**Example 1:** \( I \times I \times 2 \) arrays: \( \text{neq} = df = I \)

This class of arrays is discussed in detail by Ten Berge (1991), who showed that the typical rank of such arrays is \( \{I, I + 1\} \). To check if the rank of a generic \( I \times I \times 2 \) array is \( I \), it suffices to solve (9), which reduces to finding the real roots of the determinantal equation \( \det(X_2 - c_2X_1) = 0 \). If \( \det(X_2 - c_2X_1) = 0 \) has \( I \) real roots, then \( \text{rank}(X) = I \), otherwise \( \text{rank}(X) = I + 1 \). Simulation results for 5000 generic \( 3 \times 3 \times 2 \) arrays produced one real root 51.76% and 3 real roots 48.24% of the time. So we can deduce that \( \text{Pr}(\text{rank (3} \times 3 \times 2 \text{ array}) = 3) \approx 48.24\% \) and \( \text{Pr}(\text{rank (3} \times 3 \times 2 \text{ array}) = 4) \approx 51.76\% \).

**Example 2:** \( I \times J \times 3 \) arrays with \( I = 2J - 1 \): \( \text{neq} = df = 2J \)

a) \( 5 \times 3 \times 3 \) arrays: \( \text{neq} = df = 6 \). This class of arrays is also discussed in Ten Berge (2004a), where Ten Berge showed that generic \( 5 \times 3 \times 3 \) arrays have either rank 5 or rank 6 with positive probability. Further, he showed that a closed form solution for the case when the array has rank 5 corresponds to finding the number of real roots of a sixth degree polynomial equation: if there are 6 real roots, then the array has rank 5, otherwise its rank is 6. Table 1 displays the number of real roots obtained by solving the system (9) for 1000 simulated generic arrays. First, we note that the solution set of (9) always admitted 6 roots, as expected according to Ten Berge (2004a); further, the number of real solutions is an even number or zero. Second, \( \text{Pr}(\text{rank (5} \times 3 \times 3 \text{ array}) = 5) \approx 6.8\% \) and \( \text{Pr}(\text{rank (5} \times 3 \times 3 \text{ array}) = 6) \approx 93.2\% \).

| Table 1: Simulation results for 1000 generic \( 5 \times 3 \times 3 \) arrays. |
|-------------------------|---|---|---|---|
| real roots | 0 | 2 | 4 | 6 |
| counts | 47 | 501 | 384 | 68 |

b) \( 7 \times 4 \times 3 \) arrays: \( \text{neq} = df = 8 \). Table 2 displays the number of real roots obtained by solving the system (9) for 1000 simulated generic arrays. First, we note that the solution set of (9) always admitted 10 roots and the number of real solutions is an even number or zero. Second, \( \text{Pr}(\text{rank (7} \times 4 \times 3 \text{ array}) = 7) \approx 4.2\% \).

| Table 2: Simulation results for 1000 generic \( 7 \times 4 \times 3 \) arrays. |
|-------------------------|---|---|---|---|---|---|
| real roots | 0 | 2 | 4 | 6 | 8 | 10 |
| counts | 16 | 268 | 456 | 218 | 40 | 2 |

c) \( 9 \times 5 \times 3 \) arrays: \( \text{neq} = df = 10 \). Table 3 displays the number of real roots obtained by solving the system (9) for 1000 simulated generic arrays. First, we note
that the solution set of (9) always admitted 15 roots and the number of real solutions
is an odd number. Second, \( \Pr(\text{rank } (9 \times 5 \times 3 \text{ array}) = 9) \approx 6\% \) and
\( \Pr(\text{rank } (9 \times 5 \times 3 \text{ array}) = 10) \approx 94\% \). This latter result follows from
Ten Berge (2000, Result 5) or see Example 5 describing tallest compact arrays, by embedding
9 \times 5 \times 3 \text{ arrays into } 10 \times 5 \times 3 \text{ arrays.}

### Table 3: Simulation results for 1000 generic \( 9 \times 5 \times 3 \) arrays.

| real roots | 1   | 3   | 5   | 7   | 9   | 11  | 13  |
|------------|-----|-----|-----|-----|-----|-----|-----|
| counts     | 34  | 290 | 404 | 212 | 51  | 8   | 1   |

#### Example 3: \( I \times J \times 4 \) arrays with \( I = 3J - 2 \): \( \text{neq} = df = 3J \)

Numerical computations showed that \( \neq (\text{roots of } 10 \times 4 \times 4 \text{ arrays}) = 20; \neq (\text{roots of } 13 \times 5 \times 4 \text{ arrays}) = 35 \) and \( \neq (\text{roots of } 16 \times 6 \times 4 \text{ arrays}) = 56 \). Table 4 shows that \( \Pr(\text{rank } (10 \times 4 \times 4 \text{ array}) = 10) \approx 7.8\% \).

### Table 4: Simulation results for 1000 generic \( 10 \times 4 \times 4 \) arrays.

| real roots | 0   | 2   | 4   | 6   | 8   | 10  | 12  | 14  |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|
| counts     | 2   | 78  | 284 | 342 | 216 | 58  | 14  | 6   |

#### Remark 2: a) To calculate a Gröbner basis for (9) in Example 2 for \( I \times J \times 3 \) arrays
with \( I = 2J - 1 \), we used pure lexicographic order given by the following sequence
\( (s_1, \ldots, s_{I-1}, c_3, c_2) \) of the free variables. In all cases the Gröbner basis, denoted by \( G_\beta \),
consisted of \( (K-1)J \) polynomials having the following form: \( G_1(c_2) = 0 \), \( G_2(c_2, c_3) = \text{poly}_2(c_2) + c_3 = 0 \), \( G_3,\alpha(c_2, s_\alpha) = \text{poly}_\alpha(c_2) + s_\alpha = 0 \) for \( \alpha = 1, \ldots, I - 1 \). It is
important to note that this particular form of the Gröbner basis polynomials, \( G_\beta \),
shows that the degree of the polynomial \( G_1(c_2) = 0 \), denoted by \( \deg G_1(c_2) \), represents
the number of roots of the system (9). An introduction to Gröbner basis can be found in,
among others, Cox et al. (2007). Example 6 show quite in detail the Gröbner
basis application to a generic array.

b) The Maple 12 commands to do the computations in Example 2 are shown in
Appendix 1.

c) For \( I \times I \times 2 \) arrays and \( I \geq 2 \), \( \text{det}(X_2 - c_2 X_1) = G_1(c_2) = 0 \), where \( G_1(c_2) = 0 \) is the first element of the Gröbner basis. This phenomenon will be also seen for tallest
compact arrays, see Examples 4, 5 and 6.

A reviewer noted that the right hand side of (7) is a Segre variety, which is
the image of the Segre map, \( \Sigma_{(K-1),(J-1)} \). The Segre map sends an element of the projective space \( P^{(K-1)} \times P^{(J-1)} \) into \( P^{KJ-1} \). While the left hand side of (7) is a
linear space of projective dimension \( I - 1 = (K - 1)(J - 1) \). So, (7), will represent the
intersection of the linear space with the Segre map, and the number of intersections
is the degree of the Segre variety given in (12), see for instance Harris (1992, p. 233).

This result is summarized in the following
**Theorem 2:** Let \( I = (K - 1)(J - 1) + 1 \) and \( 2 \leq K \leq J \leq I \), then for generic data the number of roots (real or complex) of the polynomial system (9) is

\[
deg G_1(c_2) = \binom{K - 1 + J - 1}{K - 1}.
\]  

(12)

**Corollary 1:** For minimal systems and \( 3 \leq K \leq J \leq I \), \( I < \deg G_1(c_2) \).

*Proof:* Let \( n = J - 1 \) and \( m = K - 1 \). We have to show that

\[
mn + 1 \leq \frac{(m + n)!}{n!m!} \quad \text{for} \quad 2 \leq m \leq n.
\]

It is true for \( m = 2 \). For \( m \geq 3 \), we have

\[
\frac{(m + n)!}{n!m!} = \frac{(n + 1)(n + 2)}{m} \frac{(n + m - 2)}{3} \cdots \frac{(n + m - 1)}{2} \frac{(n + m)}{1} \geq \frac{(n + m - 1)}{2} \frac{(n + m)}{1}.
\]

So, it is sufficient to show that \((n + m)(n + m - 1) \geq 2(mn + 1)\), which is easily seen to be true.

**Corollary 2:** The typical rank of arrays with a minimal system have more than one rank value and the minimum attained value is \( I \).

*Proof:* The rank of a generic array with a minimal system is \( I \), if the number of real roots of \( G_1(c_2) \) is greater than or equal to \( I \); otherwise its rank is greater than \( I \).

We note that Corollary 2 generalizes Friedland (2008), who showed that: typical rank \( (I \times J \times K) \geq I \) for \((I, J, K) = ((J - 1)^2 + 1, J, J)\) when \( J \geq 2 \).

**3.2. Case 2: Underdetermined System** \((df > neq)\). When \((K - 1)(J - 1) + 2 \leq I \leq IJ\), \( df > neq \), and the system (9) is called underdetermined. The upper bound for the number of isolated roots of (9) is infinity; so (9) may or may not have \( I \) real isolated solutions: So the attained minimum bound for the rank of a generic three-way array is, \( b_{min} = I \). Before discussing two general classes studied in detail by Ten Berge (2000), we introduce some notation.

The system (9) can be written as

\[
s' \Gamma = s' [(X_2 - c_2X_1), (X_3 - c_3X_1), ..., X_K - c_KX_1] = 0',
\]

(13)
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where the number of columns of the matrix $\Gamma$ is

$$n_{\text{col}} \Gamma = (K-1)J,$$

$$= \text{neq}$$

and the number of rows of $\Gamma$ is

$$n_{\text{row}} \Gamma = I,$$

and $\Gamma$ is a matrix function of the parameters $c_2, \ldots, c_K$. We also define

$$\text{nbil} = K - 1,$$

which represents the minimal number of $c_k$ parameters that can be specialized to make the system of polynomial equations (13) linear. In algebraic geometry, the replacement of variables by specific values is called specialization.

**Example 4: Tall arrays:** $df - \text{neq} \geq \text{nbil}$

These are arrays when $(K-1)J < I \leq KJ$ and $I \geq J \geq K$, as shown by Ten Berge (2000, Result 2). This implies that (15) > (14), that is $I > (K-1)J$, or, $df - \text{neq} \geq \text{nbil} = K - 1$, where $\text{nbil}$ is given in (16). By assigning random values to the $(K-1)c_k$’s in (13), we reduce (13) to a system of linear equations, which will have a solution for any generic data; so (13) will admit $I$ real and isolated solutions; from which we deduce that the generic rank of tall arrays is $I$.

**Example 5: Tallest compact arrays:** $n_{\text{col}} \Gamma = n_{\text{row}} \Gamma$ and $K \geq 3$

These are arrays when $I = J(K-1)$, $I \geq J \geq K$ and $K \geq 3$. Note that we exclude $I \times I \times 2$ arrays for $I \geq 2$ discussed in Example 1. Ten Berge (2000, Results 3, 4 and 5) discussed this case.

When $I = (K-1)J$ and $K \geq 3$, it implies that (14) = (15), that is, $\Gamma$ is a square matrix. Solving (13) for $c_k$’s for $k = 2, \ldots, K$ is equivalent to solving $\det(\Gamma) = 0$. The leading monomial in $\det(\Gamma) = 0$ is $\prod_{k=2}^{K} c_k^I$. If $J$ is an odd integer, then (13) will have infinite number of real solutions: Assign random continuous numbers to $c_k$’s for $k = 3, \ldots, K$, and solve for $c_2$. This corresponds to Result 5 in Ten Berge (2000), which states: When $I = J(K-1)$ and $I \geq J \geq K$ and $K \geq 3$ and $J$ is odd, then the typical rank is $I$. If $J$ is an even integer, then (13) may have infinite number of real solutions or finite number of real solutions or 0 real solution. For instance for $J = 4$ and $K = 3$, the polynomial $f(c_2, c_3) = 3c_2^4c_3^4 + 1$ has 0 real solution, the polynomial $f(c_2, c_3) = 3c_2^3c_3^3 - 1$ has infinite number of real and distinct solutions, and the polynomial $f(c_2, c_3) = 3c_2^4(c_3^4 - 1)$ has a finite number of real solutions. Ten Berge (2000) specifically discussed the case of $8 \times 4 \times 3$ arrays, where he stated that
typical rank of such arrays is \( \{8,9\} \) and for randomly sampled data the rank of 9 is extremely rare. Similarly, Friedland (2008, Th.7.2) showed that typical rank of \( 12 \times 4 \times 4 \) arrays has more than one value. We conducted a limited simulation study on generic \( 8 \times 4 \times 3 \) and \( 12 \times 4 \times 4 \) arrays; and each time we got \( I \) real isolated solutions. The simulation study was done in the following way: For a generic dataset let \( f(c_2, c_3, \ldots, c_K) = \det(\Gamma) = 0 \); assign random values to the parameters \( c_3, \ldots, c_K \), then solve for \( c_2 \). This shows that for generic data, when \( I = J(K - 1) \) and \( K \geq 3 \) the rank is \( I \) with very high probability. Also, see example 6.

### 3.3. Example 6

We consider a simulated generic dataset of size \( 7 \times 4 \times 3 \) having the following three slices

\[
X_1 := \begin{pmatrix}
[-50, -38, -98, -93, -32, 8, 44] \\
[-22, -18, -77, -76, -74, 69, 92] \\
[45, 87, 57, -72, -4, 99, -31] \\
[-81, 33, 27, -2, 27, 29, 67]
\end{pmatrix}
\]

\[
X_2 := \begin{pmatrix}
[99, -25, 24, -61, 31, 25, 50] \\
[60, 51, 65, -48, -50, 94, 10] \\
[-95, 76, 86, 77, -80, 12, -16] \\
[-20, -44, 20, 9, 43, -2, -9]
\end{pmatrix}
\]

\[
X_3 := \begin{pmatrix}
[80, -70, 70, -13, 18, -33, 14] \\
[19, 41, -32, 82, -59, -68, 16] \\
[88, 91, -1, 72, 12, -67, 9]
\end{pmatrix}
\]

Our aim is to find the rank of \( X \), by representing it as in (6). This dataset has a minimal system of polynomial equations. We solve equation (9) via Gröbner basis using the lexicographic order \( (s_1, s_2, s_3, s_4, s_5, s_6, c_3, c_2) \). The first two polynomials of the Gröbner basis are

\[
G_1(c_2) = 0 =
-25879797508399058663603818114838724165583573114256294
-1987946767932180125365724555441379125561037244553323732\times c_2
-7447583055793225423658520296147635567495579387052082486\times c_2^2
+1847729223934054741969006645285810935999768448664319668\times c_3^2
+16286857667624818445824550468648649537661407605447407344\times c_4^2
+2232467132520956119892266581321656237924922929459424662\times c_5^2
-930445947744549163542468526018116402981920664188422515202\times c_6^2
+1034990365268175640254342731156079689724071145674294956746\times c_7^2
-1399215109838269848671482913176200716552825195700591390075\times c_8^2
+15534670079465049050115016130585172314320583287574900147\times c_9^2
\]
Some Numerical Results on the Rank of Generic Three-Way Arrays over \( \mathbb{R} \)

\[
+645072630378953757678717001000719217821315452777261680268 * c_2^{10} \\
G_2(c_2, c_3) = 0 = \\
235473386557912292024617338928506186026357940595072461145844743562575 \\
1722433261178738081490708008202015469486345955342202731803730371551646 \\
21434092461426076416955806798240434553933253975374522448571968951154 \\
464613254918229577275827687109510431558799243 577326297824844206156762654692158360323213910443263049581048689547 \\
-3097250590883846738228688978288835179830460928245185696410303546337 \\
36319437905844914114465485452317942363486873159383866283249006746894 \\
4016303197698866195617484294254067684300497233927857246816557510045452 \\
2954903024479564397921820954609128071823958346296586509337880753 \\
4275787231229464827038338375800527668710333131072705903255691990320 * c_2 \\
-1903803609627374035621320987836160402010627448257906499213001947264 \\
956720338867341748658480731859038594412417938078985776898706276221196 \\
3512792430363763258398867657772402447319430435982073861960197197620136 \\
52879058280068724250615748388816805478213287696041214145895488077794 \\
70397142616700793931054210823332398086941172270258543479004885596 * c_2 \\
+8450512390839763624967624579807928666582130379782638299 \\
901903452364927022770162589839438882861673620370340791082680428040404 \\
313313263033835885717828701688967159875905276468573274982627052888023 \\
53553012079933331611443254584580454967073437680409194666521803651767 \\
11931233839650588912264378749384868202790932902763246688455431 * c_3 \\
+43366804297153934702523725213515455841035812345961936162313074515 \\
66891380993068458789021032092183097107068630380980654723975384806834 \\
39954726574692859506766667419402862021536450184791968545054733408451 \\
83860627646148107457210809071860477611130187777513815112951258325196 \\
900057165616628550982377743546465518896751125305750805486831937 * c_2 \\
-296671224810910263649016390891844003965888668669822500893093424968 \\
602525252447247366817542743454974240434783232091550470123701293339 \\
71081285674562111731271369111860472809751620103390836422947014561918 \\
59674217080070318784112863111550989738603355921960217645649903785096 \\
98954852342306523795902988595527306271824462024682645883868369 * c_2^{10} \\
+8713668020931759975202209402993411331481527445489159653748216588295 \\
782666926472483631627325229251382286919828084602035659664769625951 \\
618445548964513923257211749460347876004398825197326658391216038859 \\
547938768254023880112832091100880202921766725381340769795219852772659 \\
31253203195178455569943590163548845586486897095823801767254199 * c_2^{10} \\
-78305675171958244526488263512192141094477411091651712476760732721351 \\
782888387170740108769516955504305407699176179301277874671231305494467
Some Numerical Results on the Rank of Generic Three-Way Arrays over $\mathbb{R}$

The polynomials $G_3(s_6, c_2) = 0, G_3(s_5, c_2) = 0, \ldots, G_3(s_1, c_2) = 0$ have the same form as $G_2(c_2, c_3) = 0$ given above.

The polynomial $G_1(c_2) = 0$ has only four real roots, which are: $-1.871987136, -0.3332612900, -0.2556946431, 0.2733107997$; so the rank of the dataset is greater than 7. We embed it by joining the following vectors to the three slices: $v_1' = (1\ 0\ 0\ 0), v_2 = v_3 = 0$. The embedded dataset is $X_e = (X_1' v_1')', X_e^2 = (X_2' v_2')'$ and $X_e^3 = (X_3' v_3')'$. The rank of the embedded dataset will be calculated by two distinct methods.

First, for the embedded dataset we see that $n \text{col}\Gamma = n \text{row}\Gamma =8$, so we can calculate the determinant of $\Gamma$ as in Example 5, which is:

$$
\det(\Gamma) = 0 = 111296195967997* c_2^4 - 163212875913821* c_2^3 - 288078435761246* c_2^2* c_3 + 188384423078426* c_2^3 + 139757151961919* c_2^2* c_3 - 123855533958927* c_2^2* c_3^2 + 3188520736473* c_2 + 1745777654358* c_2* c_3 + 145702375007129* c_2* c_3^2 + 15415625186696* c_2* c_3^3 - 30068441704134* c_3 - 78231890782721* c_3^2 - 9292669314727* c_3^3 + 24148992371016* c_3^4.
$$

Following the argument in Example 5, we note that there is a slight possibility that there will not be eight distinct values of $(\tilde{c}_2, \tilde{c}_3)$ such that the $\det(\Gamma) = 0$, because it is of degree 4. So, in general, following this approach of computing we can not assert that typical rank of generic 7 × 4 × 3 arrays is $\{7, 7 + 1\}$. However, let us continue our computation as in Example 5. We obtain the $C$ matrix of Lemma 1, rounded to 2 decimal digits,
Consider nongeneric dataset of size $4 \times 4 \times 3$
Some Numerical Results on the Rank of Generic Three-Way Arrays over $\mathbb{R}$

To see if the rank of $X$ is 4, we solve the system (9) composed of 8 polynomial equations in 5 variables via Gröbner basis using the pure lexicographic order $(s_1, s_2, s_3, c_2)$. The elements of the Gröbner basis are

$$
G_1(c_2) = 0 = -266104 + 1131869c_2 + 1855673c_2^2 - 10091484c_2^3 + 3934656c_2^4;
$$

$$
G_2(c_2, c_3) = 0 = 701501546752102136 - 61657878275323159c_2 - 700780737688415568c_2^2
$$
$$
+ 308891236767911424c_2^3 + 628616789525725c_3;
$$

$$
G_{3,3}(c_2, s_3) = 0 = -7901195868326608845932181098557
$$
$$
+ 377170837437434006703954200886c_2 + 901077269210427745705210304730192c_2^2
$$
$$
- 390331538460948950190867958544016c_2^3 + 1099565644871457602013709282455s_3;
$$

$$
G_{3,2}(c_2, s_2) = 0 = 393531030464214280234416428219949
$$
$$
- 190977009897456095062042069799807c_2 + 29064381129999539517569775784236c_2^2
$$
$$
- 57940861139941085694575004742848c_2^3 + 5497828224357288010068514912275s_2;
$$

$$
G_{3,1}(c_2, s_1) = 0 = -29281575292540618957256320186316
$$
$$
- 39599675315017556116317567116147c_2 + 390527303469098244882389504161956c_2^2
$$
$$
- 165798278803217428934162052760128c_2^3 + 1099565644871457602013709282455s_1.
$$

The polynomial $G_1(c_2) = 0$ is of degree 4 and it has four real roots, which are:

$[-3369565217, .2929292929, .2962962963, 2.312500000]$. So the rank of the dataset is 4 by the main Lemma. Such datasets have been characterized by their defining equations in Landsberg and Manivel (2006).

5. INDSCAL ARRAYS

Let $X \in \mathbb{R}^{I \times J \times J}$ be a tensor of order 3, where the $i$th slice $X_i \in \mathbb{R}^{J \times J}$ for $i = 1, ..., I$ is symmetric. INDSCAL, proposed by Carroll and Chang (1970), is a statistical method used in psychometrics to analyse such arrays. For this reason, we shall name such an array an INDSCAL array to distinguish it from a general three-way array $Y \in \mathbb{R}^{I \times J \times K}$ discussed above, where such a decomposition is usually named
Some Numerical Results on the Rank of Generic Three-Way Arrays over $\mathbb{R}$

PARAFAC, see Harshman (1970). A rank 1 INDSCAL array or a decomposed tensor is

$$D = a \otimes b \otimes b,$$  \hfill (18)

where $a \in \mathbb{R}^I$ and $b \in \mathbb{R}^J$.

The following theoretical results are known for generic INDSCAL data $X \in \mathbb{R}^{I \times J \times J}$: a) By Zellini (1979), see also Rocci and Ten Berge (1994), if $I \geq J(J + 1)/2$, then $\text{rank}(X) = J(J + 1)/2$. b) $I \times 2 \times 2$ and $I \times 3 \times 3$ arrays are studied by Ten Berge, Sidiropoulos and Rocci (2004). The rank computation problem has also been approached from a numerical point of view by Comon and ten Berge (2008), who applied applied Terracini’s lemma, based on the numerical calculation of the maximal rank of the Jacobian matrix of (2), to obtain numerically the generic rank of some INDSCAL three-way arrays. The numerical method based on Terracini’s lemma, when used to evaluate rank over $\mathbb{R}$, gives the generic rank when the typical rank is single-valued, and the smallest typical rank value otherwise.

For INDSCAL data (7) becomes

$$s'X_k = b_kb' \text{ for } k = 1, ..., J,$$  \hfill (19)

for $X_k \in \mathbb{R}^{I \times J}$, $s \in \mathbb{R}^I$ and $b \in \mathbb{R}^J$.

We note that (19) has two indeterminacies, scale and sign: It can be rewritten as

$$\tilde{s}'X_k = \tilde{b}_k\tilde{b}' \text{ for } k = 1, ..., J, \text{ where for instance, } \tilde{s} = \lambda s \text{ for any scalar } \lambda > 0 \text{ and } \tilde{b} = \lambda^{1/2}b. \text{ The second indeterminacy is the sign indeterminacy of } b: \text{ replacing } b \text{ by } -b \text{ in (19) does not change the equality in (19). To eliminate both indeterminacies, hereafter, we fix }$$

$$b_1 = 1.$$  \hfill (20)

We will represent the set of solutions of (19) subject to (20) by $V$ (Veronese variety).

We are interested in the study of the number of solutions of (19) subject to (20) over $\mathbb{R}$ for generic INDSCAL data for $2 \leq J, I \leq J(J + 1)/2$. We distinguish three cases named, minimal when $I = 1 + J(J - 1)/2$, overdetermined when $I > 1 + J(J - 1)/2$, and, underdetermined when $I < 1 + J(J - 1)/2$. The overdetermined systems is similar to the one discussed above.

**Theorem 3 (minimal system=Veronese variety):** Let $I = 1 + J(J - 1)/2$ and $2 \leq J \leq I$, then for generic INDSCAL data the number of roots (real or complex) of the polynomial system (19) is

$$\text{deg}V = 2^{J-1}.$$  \hfill (21)
**Proof:** Let \([b_1, ..., b_J]\) be an element of the projective space \(P^{(J-1)}\). We note that the right hand side of (19) is a Veronese variety of degree \(d = 2\), which is the image of the Veronese map, \(\nu_2\), defined by

\[
\nu_2 : P^{(J-1)} \rightarrow P^N,
\]

by sending

\[
[b_1, ..., b_J] \rightarrow [b_1^2, b_1 b_2, ..., b_J b_{J-1}, b_J^2],
\]

where the image has \(N + 1 = (J-1+2)/2\) elements composed of binomials in \(b_1, ..., b_J\). While the left hand side of (19) is a general linear space of projective dimension \(I - 1\). The number of intersections of the general linear space with the Veronese variety is finite, when \(I - 1 = N - (J - 1)\); that is

\[
I = 1 + J(J - 1)/2.
\]  

(22)

When (22) is true, the finite number of intersections is the degree of the Veronese variety given in (21), see for instance Harris (1992, p. 231).

**Corollary 1:** The typical rank of INDSCAL arrays with a minimal system have more than one rank value and the minimum attained value is \(I\).

**Proof:** For minimal systems and \(2 \leq J \leq I, I \leq degV\). The rank of a generic INDSCAL array with a minimal system is \(I\), if the number of real roots of \(V\) is greater than or equal to \(I\); otherwise its rank is greater than \(I\).

### 5.1. Example 7.

We consider a simulated generic dataset of size \(4 \times 3 \times 3\) having the following 4 slices

\[
X_1 := \begin{pmatrix}
54, 107, 161 \\
107, 58, 13 \\
161, 13, 134
\end{pmatrix}
\]

\[
X_2 := \begin{pmatrix}
114, -49, -125 \\
-49, -144, -76 \\
-125, -76, -8
\end{pmatrix}
\]

\[
X_3 := \begin{pmatrix}
-44, 7, -48 \\
7, -36, -11 \\
-48, -11, -154
\end{pmatrix}
\]

\[
X_4 := \begin{pmatrix}
50, 92, -4 \\
92, 100, 1 \\
-4, 1, -100
\end{pmatrix}
\]

INDSCAL \(4 \times 3 \times 3\) arrays have been studied in detail by Ten Berge, Sidiropoulos, and Rocci (2004), where it is shown that if a certain polynomial of degree 4 has 4 real roots, then \(rank(X) = 4\), otherwise the rank is 5.

The Gröbner basis with pure lexicographic order given by the following sequence \((b_1, b_2, s_1, s_2, s_3, s_4)\) of the free variables is formed of 6 polynomials listed below. The first polynomial \(G_4(s_4) = 0\) is of degree 4, as shown by ten Berge, Sidiropoulos and Rocci (2004) and Theorem 3, and it has 2 real roots -0.1881015674e-2, 0.7632125093e-1, so the rank of the dataset is greater than 4.
5.2. Example 8. We consider a simulated generic INDSCAL dataset of size $7 \times 4 \times 4$ having the following 7 slices

$$X_1 := \begin{bmatrix}
140, 86, -110, -4 \\
86, -182, 70, 36 \\
-110, 70, 104, 183 \\
-4, 36, 183, 148 \\
\end{bmatrix} \quad X_2 := \begin{bmatrix}
-20, 100, 173, -56 \\
100, 128, 101, 75 \\
173, 101, 124, 65 \\
-56, 75, 65, -158 \\
\end{bmatrix}$$

$$X_3 := \begin{bmatrix}
178, 15, -186, 52 \\
15, 196, 119, -148 \\
-186, 119, -138, 43 \\
52, -148, 43, -110 \\
\end{bmatrix} \quad X_4 := \begin{bmatrix}
-8, -137, 21, 20 \\
-137, -60, 64, 5 \\
21, 64, -24, -14 \\
20, 5, -14, -128 \\
\end{bmatrix}$$

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$$G_4(s_4) = 73376693603411773654444527 - 42937278219369270858661345768 s_4$$

$$- 211863296775796994233864209576 s_3^2 + 1486920579131214046506874714272 s_3^4 + 65728692033647334980166676348496 s_3$$

$$G_3(s_3, s_4) = 17560973913802573904674715803175900627366683113948054931$$

$$- 161223257377160925514526689016693788919657357280054060638 s_4$$

$$- 583769607238054101491021518154115595431615363709926796 s_4^2$$

$$+ 13183979237017459316244423593146597525756973974101183796472 s_4^3$$

$$+ 523880678525191567165441234720094289579153419952152373008 s_3$$

$$G_2(s_2, s_4) = -9628239825303370993360207993471191965769478430007385299$$

$$+ 110682528234335582777544894686418609856834063010831993176 s_4$$

$$+ 70595008022931051200292293272762726240114885506041880318901 s_4^2$$

$$+ 91182012996299227366092221532631872879721728393799255462476 s_4^3$$

$$+ 2095522431410076628661764938880377158316613679808609492032 s_2$$

$$G_1(s_1, s_4) = 93849239408544344920105218300023585460128289680648177$$

$$- 196482270901995622589398714750515451903512627004147123640 s_1$$

$$+ 19141224660419798871046145092036847545567187777703796830 s_2^2$$

$$+ 689540537276625267152783462787012635706612460573819932520176 s_3$$

$$+ 26194030392625957835827206173600471447895767099767616654 s_4$$

$$G_2(b_2, s_4) = -1550288230487019146543847206172324321617571775035413463431$$

$$- 6621188602916478638439782073183667186014336250277499222104 s_4$$

$$- 3324980827880602990137773057985895339331874506372137213739628 s_2^2$$

$$+ 44167907819442655873419676087304190072683931413772512430408456 s_3$$

$$+ 13097015196312978917913603086800235723947883549888038903252 s_4$$

$$G_1(b_1, s_4) = -1514819905854866108143372567736179608809277026174154396973$$

$$- 34540316274377016240902795956549557726567542396897273802580 s_4$$

$$- 769664897486097483396981151215439423133970629224509594873506300 s_3$$

$$+ 161342893355178852699731325084429928324620333696539365418650824 s_4$$

$$+ 1905020392190978751696524085352761559846964879982600682912 b_1$$
Some Numerical Results on the Rank of Generic Three-Way Arrays over $\mathbb{R}$

$$X_5 := \begin{pmatrix} [194, -164, -36, -6] \\ [-164, 2, -114, -110] \\ [-36, -114, 64, -127] \\ [-6, -110, -127, -18] \end{pmatrix}$$

$$X_6 := \begin{pmatrix} [-22, 74, 85, -40] \\ [74, -198, 23, -53] \\ [85, 23, -152, 18] \\ [-40, -53, 18, -48] \end{pmatrix}$$

$$X_7 := \begin{pmatrix} [-94, 109, -16, 90] \\ [109, 124, 164, -93] \\ [-16, 164, 98, -134] \\ [90, -93, -114, 184] \end{pmatrix}$$

The Gröbner basis with pure lexicographic order given by the following sequence $(s_1, ..., s_7, b_1, b_2, b_3)$ of the free variables is formed of 10 polynomials, but only the first one is shown below. The first polynomial $G_3(b_3) = 0$ is of degree 8, as shown in Theorem 3, and it has 2 real roots -4.615952848, 1.035693119, so the rank of the dataset is greater than 7.

$$G_3(b_3) = -267319790697212354162205439965563724346086890209668628287611296 -418573483979735109514695930195818332286961955303805928337210144*b_3 -53224562968122644847846329140305933491773608156555442814188832*b_3^2 +19026026023031128397470525128614232688395842607775283676963072480*b_3^3 +172806709139583658797792038234309205181892588602072553465939944*b_3^4 +164461253658569745584828839860332360925297521683057843681458288*b_3^5 +95090716874891491062104108972298040778579595301835996945880112*b_3^6 +2457546154202810650701573516399968058219521923088768430008456228*b_3^7 +2240887382441309183839416634048576470976843441637962999441259*b_3^8$$

6. Conclusion

We introduced a new method to compute ranks of three-way arrays, by showing that it is intimately related to the solution set of a system of polynomial equations, which is a well developed and active area of mathematics known as algebraic geometry. The new method was used to compute numerically the ranks of some sizes of three-way arrays over $\mathbb{R}$ via Gröbner basis.

The problem of computing the rank of overdetermined systems by solving embedded polynomial systems is a work in progress.

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Appendix 1

Below the matrix $Y_k = X_k'$.

$>$ K := 3;
Some Numerical Results on the Rank of Generic Three-Way Arrays over $\mathbb{R}$

> J := 8;
> I := 15;
> with(LinearAlgebra);
> Y1 := RandomMatrix(J, I);
> Y2 := RandomMatrix(J, I);
> Y3 := RandomMatrix(J, I);
> S := Vector(1 .. I, 1);
> for h from 1 to I-1 do
    S[h] := s_h end do;
> M1 := Y2-c_2*Y1;
> P1 := M1.S;
> M2 := Y3-c_3*Y1;
> P2 := M2.S;
> poly := [seq(P1[l], l = 1 .. J), seq(P2[n], n = 1 .. J)];
> with(Groebner);
> liste := seq(s_h, h = 1 .. I-1);
> polyG := Basis(poly, plex(liste, c_3, c_2));
> polyG[1];
> S := [solve(polyG[1])];
> nops(S);
> fsolve(polyG[1]);
> nops([fsolve(polyG[1])]);
> fsolve(polyG[1], a, complex);
> nops([fsolve(polyG[1], c_3, complex)]);

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