Bricks for the mixed high-order virtual element method: projectors and differential operators

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Abstract

We present the essential instruments to deal with Virtual Element Method (VEM) for the resolution of partial differential equations in mixed form. Functional spaces, degrees of freedom, projectors and differential operators are described emphasizing how to build them in a virtual element framework and for a general approximation order. To achieve this goal, it was necessary to make a deep analysis on polynomial spaces and decompositions. Finally, we exploit such “bricks” to construct virtual element approximations of Stokes, Darcy and Navier-Stokes problems and we provide a series of examples to numerically verify the theoretical behavior of high-order VEM.

Keywords: Virtual Element Method, Mixed Problems, Polygonal meshes, Projectors, High order.

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1. Introduction

The virtual element method (VEM) was introduced in [20, 11] as an extension of finite element method to general polygonal/polyhedral meshes.

The virtual element discrete spaces are similar to the usual polynomial spaces with the addition of suitable (and unknown) non-polynomial functions that are implicitly defined as the solutions of a proper PDE on each element of the decomposition. The main idea of the method is to define approximated discrete forms computable only via degree of freedom values. Moreover, it does not approximate non-polynomial test and trial functions at the integration points, but it exploits some polynomial projections which are exactly computed starting from the degrees of freedom. Using such projections, VEM can handle very general polygonal/polyhedral meshes without the need to integrate complex non-polynomial functions on the elements (as polygonal FEM do) and without loss of accuracy. We refer to [20, 2, 12, 9, 13, 15, 27] for an in-depth theoretical analysis of the virtual elements features.

The Virtual Element Method has been developed successfully for a large range of mathematical and engineering problems, we mention, as sample, the very brief list of papers [10, 28, 32, 18, 46, 6, 33, 7], while for the specific topic of implementation aspects related to the VEM we mention [43, 25, 14, 24, 4, 8, 42, 35, 45, 19]. Concerning the mixed PDEs we refer to [41, 30, 29, 37, 31] as a sample of VEM papers dealing with such kind of problem, and to [40, 34, 26, 36] as a representative list of papers treating the same topic with different polytopal technologies.

In the context of Stokes or Darcy flows and in many physical applications such as the models for precipitation and for flows in root-soil (see for instance [5, 1, 23, 22] and the references therein), the use of general polytopal meshes can be very useful so a virtual element approach is particularly appealing.

The potentiality of the VEM is not limited to the meshing aspect. Indeed, the flexibility of the virtual element framework has been exploited to construct a $H^1$-conforming virtual element...
space particularly suited for the mixed problems \[ [3, 16, 44, 17, 21]. By choosing a suitable pressure space, the virtual element approach leads to an exactly divergence-free discrete velocity kernel. Such property is really important to solve PDEs associated with incompressible fluid flows and we further underline that such property is not shared by most of the standard mixed finite element methods, where it is imposed only in a weak sense \[ [39, 38].

In the present contribution we show in detail the practical aspects of the high-order schemes developed in \[ [16, 44, 17]. We stress that such definition of the virtual elements and the associated degrees of freedom is more involved with respect to the plain-vanilla \( H^1 \)-conforming VEM space \[ [20]. Then, since a user manual can be very helpful for people with some experience in implementing the Virtual Element scheme, we focus on the explicit construction of “VEM bricks” (projectors and discrete forms) to deal with such kind of discretization. More specifically, we give the practical instructions for the computation of the \( L^2 \)-projection, the \( \nabla \)-projection and the \( \varepsilon \)-projection via the degrees of freedom in the spirit of the hitchhiker’s guide \[ [11]. Moreover, we make a wide variety of numerical tests to show the practical performance of the method for the mixed problems, underlining the robustness of the scheme with respect to high-order degree of accuracy.

The paper is organized as follows. In Section 2 we introduce same definitions and preliminaries and we fix the nations. In Section 3 we review the family of Virtual Elements presented in \[ [44] and we introduce the stiffness matrices associated with the mixed problems. In Section 4 we explicitly show how to compute the polynomial projections using (as unique information) the degree of freedom values. In Section 5 we analyse the algebraic form of the linear system arising from the virtual element discretization and we present several numerical tests which highlight the actual performance of our approach for the Stokes, the Darcy and the Navier–Stokes equation also for high-order polynomial degree. Finally, we draw some conclusions.

**Notation.** We will follow the usual notation for the differential operators. Hence the symbols \( \nabla \) and \( \Delta \) denote the gradient and Laplacian for scalar functions, while \( \Delta, \nabla, \) and \( \text{div} \) denote the vector Laplacian, the gradient operator and the divergence for vector fields, respectively.

Furthermore for a vector field \( v \) we denote with \( \varepsilon(v) \) the symmetric part of the gradient of \( v \), i.e.

\[
\varepsilon(v) := \frac{\nabla v + (\nabla v)^T}{2}.
\]

Finally \( \text{div} \) denotes the vector valued divergence for matrix fields.

**2. Definitions & preliminaries**

In this section we introduce the basic mathematical notation and tools to deal with the Virtual Element Method. From now on let \( E \subset \mathbb{R}^2 \) be a polygon, we will denote by \( x_E, h_E, |E| \) the centroid, the diameter and the measure of \( E \), respectively. Let \( n \in \mathbb{N} \) and let \( P_n(E), [P_n(E)]^2, [P_n(E)]^{2 \times 2} \) be the space of scalar, vectorial and matrix polynomials defined on \( E \) of degree less or equal to \( n \), respectively (with the extended notation \( P_{-1}(E) = \{ 0 \} \)). The dimension of such spaces are

\[
\dim(P_n(E)) = \pi_n := \frac{(n+1)(n+2)}{2}, \quad \dim([P_n(E)]^2) = 2 \pi_n, \quad \dim([P_n(E)]^{2 \times 2}) = 4 \pi_n.
\]

One of the main tool exploited in the VEM is the so-called scaled-monomials. Given a multi-index \( \alpha = (\alpha_1, \alpha_2) \) with \( |\alpha| = \alpha_1 + \alpha_2 \), a scaled monomial is defined as

\[
m_{\alpha} := \left( \frac{x - x_E}{h_E} \right)^{\alpha_1} \left( \frac{y - y_E}{h_E} \right)^{\alpha_2}.
\]

From now on we refer to the null monomial by \( m_\emptyset \), i.e. \( m_\emptyset = 0 \).

With a slight abuse of notation we may denote the scaled monomial \( m_{\alpha} \) with the notation \( m_i \), where the relation between the one dimensional index \( i \) and the multi-index \( \alpha \) is given by the natural correspondence

\[
1 \mapsto (0, 0), \quad 2 \mapsto (1, 0), \quad 3 \mapsto (0, 1), \quad 4 \mapsto (2, 0), \quad 5 \mapsto (1, 1), \quad \ldots
\]

(1)
It is easy to show that the set
\[ M_n(E) := \{ m_\alpha : 0 \leq |\alpha| \leq n \} := \{ m_i : 1 \leq i \leq \pi_n \}, \] (2)
is a basis for \( \mathbb{P}_n(E) \). Moreover for any \( m \leq n \) we denote with
\[ \hat{\mathbb{P}}_{n/m}(E) := \text{span}(m_\alpha : m + 1 \leq |\alpha| \leq n) \]
i.e. the set of polynomial of degree \( n \) which monomials have degree strictly greater than \( m \).

The definition of scaled monomial can be extended to the vectorial monomial. Let \( \alpha := (\alpha_1, \alpha_2) \) and \( \beta := (\beta_1, \beta_2) \) be two multi-indexes, then we define a vectorial scaled monomial as
\[ m_{\alpha, \beta} := \begin{pmatrix} m_\alpha \\ m_\beta \end{pmatrix}. \]
Also in this case, it is easy to show that the set
\[ [M_n(E)]^2 := \{ m_{\alpha, \emptyset} : 0 \leq |\alpha| \leq n \} \cup \{ m_{\emptyset, \beta} : 0 \leq |\beta| \leq n \} := \{ m_i : 1 \leq i \leq 2 \pi_n \}, \] (3)
is a basis for the vectorial polynomial space \([\mathbb{P}_n(E)]^2\), where we implicitly use the natural correspondence between on dimensional indexes and double multi-indexes (extending correspondence (1)).

**Remark 2.1.** Note that the following polynomials decomposition holds
\[ [\mathbb{P}_n(E)]^2 = \nabla \mathbb{P}_{n+1}(E) \oplus x^\perp \mathbb{P}_{n-1}(E), \]
where \( x^\perp := (y, -x)^t \).
In particular for each \( p_n \in [\mathbb{P}_n(E)]^2 \), there exist unique \( p_{n+1} \in \hat{\mathbb{P}}_{n+1}(E) \) and \( q_{n-1} \in \mathbb{P}_{n-1}(E) \) such that
\[ p_n = \nabla p_{n+1} + x^\perp q_{n-1}. \] (4)

The decomposition in Remark 2.1 is essential to define projector operators and consequently to proceed with a virtual element analysis for a large variety of PDEs. Unfortunately, finding such decomposition for a generic vectorial polynomial \( p_n \) is not an easy task, but, if we consider scaled monomials, we found a straightforward recipe to get it.

**Proposition 2.1.** Consider the vectorial monomials \( m_{\alpha, \emptyset} \) and \( m_{\emptyset, \beta} \in [M_n(E)]^2 \), with \( \alpha = (\alpha_1, \alpha_2) \) and \( \beta = (\beta_1, \beta_2) \). Then referring to (4), the following scaled decompositions hold
\[ m_{\alpha, \emptyset} := \frac{h_E}{|\alpha| + 1} \nabla m_{(\alpha_1 + 1, \alpha_2)} + \frac{\alpha_2}{|\alpha| + 1} m^\perp m_{(\alpha_1, \alpha_2 - 1)} \], (5)
\[ m_{\emptyset, \beta} := \frac{h_E}{|\beta| + 1} \nabla m_{(\beta_1, \beta_2 + 1)} - \frac{\beta_1}{|\beta| + 1} m^\perp m_{(\beta_1 - 1, \beta_2)}, \] (6)
where \( m^\perp := (m_{(0,1)}, -m_{(1,0)})^t \).

**Proof.** We show the decomposition in Equation (5), the one in (6) follows the same strategy. We compute the following quantities
\[ \nabla m_{(\alpha_1 + 1, \alpha_2)} = \frac{1}{h_E} \begin{pmatrix} (\alpha_1 + 1) m_{(\alpha_1, \alpha_2)} \\ \alpha_2 m_{(\alpha_1 + 1, \alpha_2 - 1)} \end{pmatrix} \] (7)
and
\[ m^\perp m_{(\alpha_1, \alpha_2 - 1)} = \begin{pmatrix} m_{(\alpha_1, \alpha_2)} \\ -m_{(\alpha_1 + 1, \alpha_2 - 1)} \end{pmatrix}. \] (8)
Note that the coefficient \( 1/h_E \) in (7) is due to the chain derivative rule. Therefore the choice of the multi-indexes on the right hand side of Equation (5) produces two vectorial polynomials with the same monomial at the same components. Notice that, according to (4), the gradient component of the decomposition has strictly positive degree. Decomposition (5) comes from a proper linear combination of (7) and (8). \( \square \)
Remark 2.2. Proposition 2.1 is an easy tool to compute the decomposition (4) for any general polynomial \( p_n \in [P_n(E)]^2 \) and represents a key ingredient in the implementation of the proposed schemes.

Finally we consider the matrix polynomial space \([P_n(E)]^{2 \times 2}\) and we define the matrix scaled monomials

\[ M_{\alpha, \beta, \gamma, \delta} := \begin{pmatrix} m_{\alpha} & m_{\beta} \\ m_{\gamma} & m_{\delta} \end{pmatrix}, \]

where \( \alpha, \beta, \gamma, \delta \) are multi-indexes. We build a basis of \([P_n(E)]^{2 \times 2}\) starting from matrix scaled monomials in the natural way (using again the usual correspondence between indexes and multi-indexes)

\[
[M_n(E)]^{2 \times 2} := \{ M_{\alpha, 0, 0, 0} : 0 \leq |\alpha| \leq n \} \cup \{ M_{0, \beta, 0, 0} : 0 \leq |\beta| \leq n \} \cup \\
\{ M_{0, 0, \gamma, 0} : 0 \leq |\gamma| \leq n \} \cup \{ M_{0, 0, 0, \delta} : 0 \leq |\delta| \leq n \}
\]

(9)

that clearly is a basis for \([P_n(E)]^{2 \times 2}\).

Remark 2.3. Note that in Definitions (2), (3) and (9) we consider both the index and the multi-index notations. Both notations will be employed indifferently when we are dealing with scaled monomials.

A key ingredient in the VEM construction is represented by the polynomial projections that will play a fundamental role in the construction of the approximated virtual elements form. For any \( n \in \mathbb{N} \) we define the following polynomial projections:

- the \( L^2 \)-projection \( \Pi_0^0 : [L^2(E)]^2 \to [P_n(E)]^2 \), defined for all \( v \in [L^2(E)]^2 \) by

\[
\int_E q_n \cdot (v - \Pi_0^0 v) \, dE = 0 \quad \text{for all } q_n \in [P_n(E)]^2,
\]

with obvious extension for matrix functions \( \Pi_0^0 : [L^2(E)]^{2 \times 2} \to [P_n(E)]^{2 \times 2} \),

- the \( \nabla \)-projection \( \Pi_\nabla^0 : [H^1(E)]^2 \to [P_n(E)]^2 \), defined for all \( v \in [H^1(E)]^2 \) by

\[
\left\{ \begin{aligned}
\int_E \nabla q_n : \nabla (v - \Pi_\nabla^0 v) \, dE &= 0 \quad \text{for all } q_n \in [P_n(E)]^2, \\
\int_{\partial E} q_0 : (v - \Pi_\nabla^0 v) \, d\mathbf{e} &= 0 \quad \text{for all } q_0 \in [P_0(E)]^2,
\end{aligned} \right.
\]

(11)

- the \( \varepsilon \)-projection \( \Pi_\varepsilon^0 : [H^1(E)]^2 \to [P_n(E)]^2 \), defined for all \( v \in [H^1(E)]^2 \) by

\[
\left\{ \begin{aligned}
\int_E \varepsilon(q_n) : \varepsilon (v - \Pi_\varepsilon^0 v) \, dE &= 0 \quad \text{for all } q_n \in [P_n(E)]^2, \\
\int_{\partial E} q : (v - \Pi_\varepsilon^0 v) \, d\mathbf{e} &= 0 \quad \text{for all } q \in ([P_0(E)]^2, \mathbf{x}^\perp).
\end{aligned} \right.
\]

(12)

3. The Virtual Element Approximation

In the present section we summarize a short overview of the \( H^1 \)-conforming Virtual Elements for the mixed problems. We will make use of various tools from the virtual element technology, that will be described briefly. We refer the reader to [16, 44, 17] for a deeper analysis.

Let \( \{\Omega_h\}_h \) be a sequence of decompositions of \( \Omega \) into general polygonal elements \( E \) with

\[ h := \sup_{E \in \Omega_h} h_E. \]

For all \( h \) we suppose that each element \( E \) in \( \Omega_h \) fulfills the following assumptions:
We here summarize the main properties of the virtual space $V$ analysis:

Referring to [44] we introduce the Virtual Element space $\mathcal{V}^E_h$.

### 3.1. Virtual elements spaces

Let $k \geq 2$ the polynomial degree of accuracy of the method. We consider on each element $E \in \Omega_h$ the (enlarged) finite dimensional local virtual space

$$
\mathcal{U}^E_h := \left\{ \mathbf{v} \in [H^1(E) \cap C^0(E)]^2 \mid \mathbf{v}_e \in [P_k(e)]^2 \text{ for all } e \in \partial E, \right. \\
\left. \begin{array}{l}
- \Delta \mathbf{v} - \nabla s \in \mathbf{x}^\perp \mathcal{P}_{k-1}(E), \\
\text{div } \mathbf{v} \in \mathcal{P}_{k-1}(E),
\end{array} \\
\text{for some } s \in L^2_0(E) \right\}.
$$

Referring to [44] we introduce the Virtual Element space $\mathcal{V}^E_h$ as the restriction of $\mathcal{U}^E_h$ defined by:

$$
\mathcal{V}^E_h := \left\{ \mathbf{v} \in \mathcal{U}^E_h \mid (\mathbf{v} - \Pi^E_h \mathbf{v}, \mathbf{x}^\perp \hat{p}_{k-1})_{E} = 0 \text{ for all } \hat{p}_{k-1} \in \mathcal{P}_{k-1 \setminus k-3}(E) \right\}.
$$

We here summarize the main properties of the virtual space $\mathcal{V}^E_h$ (we refer [44, 17] for a detailed analysis):

- **dimension**: the dimension of $\mathcal{V}^E_h$ is

$$\dim(\mathcal{V}^E_h) = 2n_E k + \frac{(k-1)(k-2)}{2} + \frac{(k+1)k}{2} - 1$$

where $n_E$ is the number of vertexes of $E$;

- **degrees of freedom**: Let $\text{NDoF} := \dim(\mathcal{V}^E_h)$, the linear operators

$$
\mathbf{D}_\mathbf{V} := \{ \mathbf{D}_{\mathbf{V},i} \}_{i=1}^{\text{NDoF}},
$$

split into four subsets (see Figure 1) constitute a set of DoFs for $\mathcal{V}^E_h$:

- $\mathbf{D}^{\text{vertex}}$: the values of $\mathbf{v}$ at the vertexes of the polygon $E$,
- $\mathbf{D}^{\text{edge}}$: the values of $\mathbf{v}$ at $k-1$ distinct internal points of the $(k+1)$-point Gauss–Lobatto rule on every edge $e \in \partial E$,
- $\mathbf{D}^{\text{m}^+}$: the moments of $\mathbf{v}$

$$
\mathbf{D}^{\text{m}^+}_i(\mathbf{v}) := \frac{1}{|E|} \int_E \mathbf{v} \cdot \mathbf{m}^+ m_i \, dE \quad \text{for } 1 \leq i \leq \pi_{k-3},
$$

- $\mathbf{D}^{\text{div}}$: the moments of $\text{div } \mathbf{v}$

$$
\mathbf{D}^{\text{div}}_i(\mathbf{v}) := \frac{h_E}{|E|} \int_E (\text{div } \mathbf{v}) m_i \, dE \quad \text{for } 2 \leq i \leq \pi_{k-1};
$$

- **projections**: referring to (10), (11) and (12), the DoFs $\mathbf{D}_\mathbf{V}$ allow us to compute exactly the polynomial projections

- $\Pi^E_k : \mathcal{V}^E_h \rightarrow [P_k(E)]^2$ (see Subsection 4.1),
- $\Pi_k : \mathcal{V}^E_h \rightarrow [P_k(E)]^2$ (see Subsection 4.2),
- $\Pi^1_k : \mathcal{V}^E_h \rightarrow [P_k(E)]^2$ (see Subsection 4.3),
- $\Pi^1_{k-1} : \nabla(\mathcal{V}^E_h) \rightarrow [P_{k-1}(E)]^{2 \times 22}$ (see Subsection 4.4),
- $\Pi^1_{k-1} : \varepsilon(\mathcal{V}^E_h) \rightarrow [P_{k-1}(E)]^{2 \times 2}$ (see Remark 4.3),
in the sense that, given any $v \in V^E_h$, we are able to compute the polynomials

$$\Pi_1^k v, \quad \Pi_2^k v, \quad \Pi_3^k v, \quad \Pi_{k-1}^k \nabla v, \quad \Pi_{k-1}^0 \varepsilon(v)$$

only using, as unique information, the degree of freedom values $D^V$ of $v$.

For future reference, we collect all the $2kn_E$ boundary DoFs (the first two items above) and denote them with $D^\partial := \{D^\partial_i\}_{i=1}^{2kn_E}$. Moreover we denote with $D^e := \{D^e_i\}_{i=1}^{2k}$ the DoFs of $D^\partial$ relative to the (closed) edge $e$. Note that in the case $k = 2$ the set of DoFs $D^m$ is empty.

The basis functions $\varphi_i \in V^E_h$ are defined as usual as the canonical basis functions $D^V,j(v_i) = \delta_{ji}$ for $i,j = 1,\ldots,N_{\text{DoF}}$.

in particular for any $v \in V^E_h$ we have the Lagrange identity

$$v = \sum_{i=1}^{N_{\text{DoF}}} D_{V,i}(v) \varphi_i.$$

The global virtual element space is obtained by combining the local spaces $V^E_h$ accordingly to the local degrees of freedom, as in standard finite elements and considering the homogeneous boundary conditions. Therefore we get

$$V_h := \{v \in [H_0^1(\Omega)]^2 \text{ s.t. } v|_E \in V^E_h \text{ for all } E \in \Omega_h\}. \quad (14)$$

The pressure space is simply given by the piecewise polynomial functions

$$Q_h := \{q \in L_0^2(\Omega) \text{ s.t. } q|_E \in \mathbb{P}_{k-1}(E) \text{ for all } E \in \Omega_h\}, \quad (15)$$

with local DoFs:

- $D_Q$: the coefficients $\{c_i\}_{i=1}^{\pi_{k-1}}$ of $q|_E$ with respect to the re-scaled basis $\mathbb{M}_{k-1}(E)$, i.e.

$$q = \sum_{i=1}^{\pi_{k-1}} c_i \frac{h_E}{|E|^{m_i}}.$$

**Remark 3.1.** As observed in [44, 17], by definitions (14) and (15), it holds

$$\text{div } V_h \subseteq Q_h.$$

This entails a series of important advantages: the numerical scheme leads to an exactly divergence-free discrete velocity solution for incompressible fluid. Moreover the proposed family of virtual elements is uniformly stable both for the Darcy and the Stokes problem.
3.2. Discrete forms and load term approximation

In this subsection we briefly describe the construction of a discrete version of the local stiffness matrices arising from the mixed problems such as Stokes, Darcy and the Navier–Stokes Equations (presented in Section 5). These physical problems share the same algebraic structure once a discretization is introduced. In particular we need to define the following stiffness matrices:

\[
\begin{align*}
\left(K_h^{0,E}\right)_{ij} &= a_h^{0,E}(\varphi_i, \varphi_j) \approx \int_E \varphi_i \cdot \varphi_j \, dE & \text{for } i, j = 1, \ldots, \text{NDoF}, \\
\left(K_h^{\nabla,E}\right)_{ij} &= a_h^{\nabla,E}(\varphi_i, \varphi_j) \approx \int_E \nabla \varphi_i \cdot \nabla \varphi_j \, dE & \text{for } i, j = 1, \ldots, \text{NDoF}, \\
\left(K_h^{\varepsilon,E}\right)_{ij} &= a_h^{\varepsilon,E}(\varphi_i, \varphi_j) \approx \int_E \varepsilon(\varphi_i) : \varepsilon(\varphi_j) \, dE & \text{for } i, j = 1, \ldots, \text{NDoF},
\end{align*}
\]

arising from the Darcy and the Stokes equation (in the “gradient form” or in the “epsilon form”) respectively. Following the standard procedure in VEM literature [20, 16, 44, 17], we introduce the computable discrete local bilinear forms:

\[
\begin{align*}
a_h^{0,E}(\varphi_i, \varphi_j) &:= \int_E \Pi_h^0 \varphi_i \cdot \Pi_h^0 \varphi_j \, dE + |E| S ((I - \Pi_h^0) \varphi_i, (I - \Pi_h^0) \varphi_j) \\
a_h^{\nabla,E}(\varphi_i, \varphi_j) &:= \int_E \nabla (\Pi_h^{\nabla}) \varphi_i : \nabla (\Pi_h^{\nabla}) \varphi_j \, dE + S ((I - \Pi_h^{\nabla}) \varphi_i, (I - \Pi_h^{\nabla}) \varphi_j) \\
a_h^{\varepsilon,E}(\varphi_i, \varphi_j) &:= \int_E \varepsilon (\Pi_h^{\varepsilon}) \varphi_i : \varepsilon (\Pi_h^{\varepsilon}) \varphi_j \, dE + S ((I - \Pi_h^{\varepsilon}) \varphi_i, (I - \Pi_h^{\varepsilon}) \varphi_j)
\end{align*}
\]

where

\[
S(u_h, v_h) := \sum_{i=1}^{\text{NDoF}} D_{V,i}(u_h) D_{V,i}(v_h)
\]

is the inner product of the vectors containing the DoF's values of \(u_h\) and \(v_h\) respectively.

Let \(a_h^{E}(\cdot, \cdot)\) be one of the discrete bilinear forms listed above. It is straightforward to check that the definitions (10), (11), (12) and (22) imply that the discrete forms \(a_h^{E}(\cdot, \cdot)\) satisfies the consistency and stability properties [20].

As usual we define the global approximated bilinear form by adding the local contributions:

\[
a_h(u_h, v_h) := \sum_{E \in \Omega_h} a_h^{E}(u_h, v_h), \quad \text{for all } u_h, v_h \in V_h.
\]

For the treatment of the Navier–Stokes equation we also need to define a computable approximation of the convective trilinear form

\[
c_h^{E}(w_h; u_h, v_h) \approx \int_E (\nabla w_h \cdot u_h) \cdot v_h \, dE.
\]

Referring to (10) we set for all \(w_h, u_h, v_h \in V_h^{E}:

\[
c_h^{E}(w_h; u_h, v_h) := \int_E \left((\Pi_h^{\nabla} - 1) \nabla u_h \right) (\Pi_h^0 w_h) \cdot \Pi_h^0 v_h \, dE
\]

and note that all quantities in the previous formula are computable. The global approximated trilinear form is defined by simply summing the local contributions:

\[
c_h(w_h; u_h, v_h) := \sum_{E \in \Omega_h} c_h^{E}(w_h; u_h, v_h), \quad \text{for all } w_h, u_h, v_h \in V_h.
\]

Finally, for any given function \(f \in [L^2(E)]^2\), we introduce the computable approximation of the right-hand side

\[
(f_h, v_h)_E \approx (f, v_h)_E \quad \text{for all } v_h \in V_h^{E}
\]
by taking
\[ f_{h|E} := \Pi^0_h \mathbf{f}. \] (25)
Therefore for any loads \( \mathbf{f} \in [L^2(E)]^2 \) and \( g \in L^2(E) \), the vectors of the local load terms are simply defined by [44, 17]
\[
(f^E)_{i} := \int_{E} \Pi^0_h \mathbf{f} \cdot \varphi_i \, dE \quad \text{for } i = 1, \ldots, \text{NDof},
\]
\[
(g^E)_{\ell} := \frac{h_E}{|E|} \int_{E} g m_{\ell} \, dE \quad \text{for } \ell = 1, \ldots, \pi_{k-1}.
\] (26)

3.3. Divergence of functions in \( V^E_h \)

In this subsection we show how to exactly determine the divergence of a function \( \mathbf{v}_h \in V^E_h \) starting from its degrees of freedom. First of all we recall from (13) that \( \text{div} \mathbf{v}_h \in P_{k-1}(E) \), therefore we can write this polynomial on the scaled monomial basis \( M_{k-1}(E) \), i.e.
\[
\text{div} \mathbf{v}_h = \sum_{i=1}^{\pi_{k-1}} d_i \mathbf{m}_i.
\]
To find the unknown coefficients \( d_i \), we test \( \text{div} \mathbf{v}_h \) against all the elements \( \mathbf{m}_j \) of \( M_{k-1}(E) \), obtaining
\[
\sum_{i=1}^{\pi_{k-1}} d_i \int_{E} \mathbf{m}_i \cdot \mathbf{m}_j \, dE = \int_{E} (\text{div} \mathbf{v}_h) \mathbf{m}_j \, dE \quad \text{for } j = 1, \ldots, \pi_{k-1}. \] (27)
All terms in Equation (27) are computable although the function \( \mathbf{v}_h \) is virtual. Indeed, the left-hand side is an integral of monomials, while the right-hand sides are:
- if \( j \geq 2 \) by definition of \( \mathbf{D}^{\text{div}} \):
  \[
  \int_{E} (\text{div} \mathbf{v}_h) \mathbf{m}_j \, dE = \frac{|E|}{h_E} \mathbf{D}^{\text{div}}_{j} (\mathbf{v}_h)
  \]
- if \( j = 1 \), by the divergence theorem follows that
  \[
  \int_{E} (\text{div} \mathbf{v}_h) \, dE = \int_{\partial E} \mathbf{v}_h \cdot \mathbf{n} \, de = \sum_{e \in \partial E} \mathbf{v}_h \cdot \mathbf{n}_e \, de
  \]
so it is computable via the DoFs values \( \mathbf{D}^r(\mathbf{v}_h) \) that are exactly the values of \( \mathbf{v}_h \) at the \((k+1)\) Gauss–Lobatto quadrature points on each edge \( e \in \partial E \).

Remark 3.2. We stress that the algebraic form of Equation (27) consists of a linear system \( \mathbf{A} \mathbf{d} = \mathbf{b} \), where
\[
\mathbf{A} := \int_{E} \mathbf{m}_i \mathbf{m}_j \, dE, \quad \mathbf{b} := \int_{E} (\text{div} \mathbf{v}_h) \mathbf{m}_j \, dE \quad \text{and } \quad \mathbf{d} := (d_i)_{i=1}^{\pi_{k-1}}.
\]
The same algebraic structure: matrix which elements are integrals of polynomials, right hand-side consisting of computable integrals involving the virtual functions, will be at the basis of the computations of all projections in Section 4.

The argument above give us a recipe to compute exactly the “divergence form” involved in the classic velocity-pressure problem, i.e.
\[
b(\mathbf{v}_h, q_h) := \int_{\Omega} (\text{div} \mathbf{v}_h) q_h \, dE \quad \text{for all } \mathbf{v}_h \in V_h \text{ and } q_h \in Q_h. \] (28)
Indeed we get
\[
b(\mathbf{v}_h, q_h) = \sum_{E \in \Omega_h} \int_{E} (\text{div} \mathbf{v}_h) q_h \, dE
\]
that is explicitly computable. Moreover, direct computation and the definition of the bases functions for the spaces $V^E_h$ and $Q^E_h$, show that the local matrix 

\[
(B^E)_{i\ell} = \frac{h_E}{|E|} \int_E (\text{div} \, \varphi_i) m_\ell \, dE \quad \text{for } i = 1, \ldots, \text{NDoF} \text{ and } \ell = 1, \ldots, \pi_{k-1}
\]

has the simple form

\[
B^E := \begin{bmatrix}
 b_1 & \cdots & b_{2kn_E} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
 0 & 0 & \cdots & \cdots & I \\
\end{bmatrix}
\]

(29)

where

\[
b_i = \frac{h_E}{|E|} \int_{\partial E} \varphi_i \cdot n \, de \quad \text{for } i = 1, \ldots, 2kn_E.
\]

### 4. How to make projectors

In this section we focus on the definition of the projection operators described in Section 2. In particular we will exploit how to compute such projections via the degrees of freedom even if we are dealing with virtual functions.

Let $\Pi_k : V^E_h \rightarrow P_k(E)$ denote one of the projections (10), (11) and (12). For a given virtual function $v_h \in V^E_h$, since by definition $\Pi_k v_h$ is a vector valued polynomial of degree $k$, it can be written in terms of the monomial basis $[M_k(E)]^2$, i.e.

\[
\Pi_k v_h = \sum_{i=1}^{2kn_E} \zeta_i m_i.
\]

(30)

Once we find the unknown coefficients $\zeta_i$, we uniquely determine such projection.

We further underline that a generic function $v_h \in V^E_h$ is a continuous vectorial polynomial of degree $k$ on each edge in $\partial E$, i.e. $v_h|_e \in [P_k(e)]^2$. Such polynomial is uniquely determined by $D^e$. Consequently each time we are considering a virtual function $v_h$ on $\partial E$, it has to be considered as a known function.

**Remark 4.1.** We stress that the choice of the Gauss–Lobatto quadrature points on each edge as DoFs is particularly convenient: we can compute the integral of a polynomial of degree $2k-1$ on each edge $e$ directly from its $k+1$ degrees of freedom $D^e$, and this feature will greatly simplify the implementation of the method.

### 4.1. $\nabla$-projection $\Pi_\nabla$

Let us start our analysis with the $\nabla$-projection $\Pi_\nabla$ (cf. Definition (11)). The target is to determine the coefficients $\zeta_i$ in (30) relative to $\Pi_\nabla$. To achieve this goal, we replace the generic vectorial polynomial $p_h \in V^E_h$ is a continuous vectorial polynomial of degree $k$ on each edge in $\partial E$, i.e. $p_h|_e \in [P_k(e)]^2$. Such polynomial is uniquely determined by $D^e$. Consequently each time we are considering a virtual function $p_h$ on $\partial E$, it has to be considered as a known function.

\[
\Pi_k v_h = \sum_{i=1}^{2kn_E} \zeta_i m_i.
\]

(30)

It is easy to show that Equation (31) is a set of linearly independent equations which uniquely determine the coefficients $\zeta_i$. Moreover, Equation (31) can be seen as a linear system in the unknowns $\zeta_i$ in a similar way we have done in Remark 3.2 to find the divergence of $v_h$. To find such polynomial coefficients, we have to understand if it is possible to compute the quantities where the virtual function $v_h$ appears. Since the second condition in Equation (31) involves integral over the boundary of the polygon, it is clear that is computable. Let us consider the
first one. Substituting the definition in Equation (30) and moving the virtual function on the right-hand side, we get
\[
\sum_{i=1}^{2\pi_h} \zeta_i \int_E \nabla m_i : \nabla m_j \, dE = \int_E \nabla v_h : \nabla m_j \, dE.
\]
The left-hand side is computable since it involves integral of vectorial monomials over the polygon \(E\). We have to show that the right-hand side is also computable via the degrees of freedom of \(v_h\). Integrating by parts and we obtain
\[
\int_E \nabla v_h : \nabla m_j \, dE = - \int_E v_h \cdot \Delta m_j \, dE + \int_{\partial E} v_h \cdot (\nabla m_j \cdot n) \, de.
\]  
(32)

As we already said in the introduction of this section, the virtual function \(v_h\) is a known vectorial polynomial on the boundary so integrals over edges are computable, in particular we integrate a polynomial of degree 2\(k - 1\) on \(e\) (see Remark 4.1). Let us focus on the term inside the polygon \(E\). We observe that \(\Delta m_j\) is a vectorial monomial then the following relation holds
\[
\Delta m_j = \frac{a_1}{h^2_E} m_{\alpha_1, \emptyset} + \frac{a_2}{h^2_E} m_{\alpha_2, \emptyset} + \frac{a_3}{h^2_E} m_{\emptyset, \alpha_3} + \frac{a_4}{h^2_E} m_{\emptyset, \alpha_4}.
\]
Finding the coefficients \(a_s\) and the multi-indexes \(\alpha_s\) is easy since we are dealing with scaled monomials. We substitute such expression in the first integral of Equation (32) and we get
\[
\int_E v_h \cdot \left( \frac{a_1}{h^2_E} m_{\alpha_1, \emptyset} + \frac{a_2}{h^2_E} m_{\alpha_2, \emptyset} + \frac{a_3}{h^2_E} m_{\emptyset, \alpha_3} + \frac{a_4}{h^2_E} m_{\emptyset, \alpha_4} \right) \, dE.
\]  
(33)

Let us consider one of these integrals, similar considerations can be done for the other terms. We exploit Proposition 2.1 and we integrate by parts
\[
\int_E v_h \cdot m_{\alpha_1, \emptyset} \, dE = \int_E v_h \cdot (b_1 \nabla m_{\beta_1} + b_2 m^+ \cdot m_{\beta_2}) \, dE
\]  
\[
\quad = b_1 \int_E v_h \cdot \nabla m_{\beta_1} \, dE + b_2 \int_E v_h \cdot m^+ m_{\beta_2} \, dE,
\]  
(34)

where we have defined the scalar coefficients \(b_s\) and the multi-indexes \(\beta_s\) from Proposition 2.1. According to such proposition \(-1 \leq |\beta_1| \leq k - 3\) so the last integral is the degree of freedom \(D^{m^+}\) and we can compute it. The first integral requires additional steps:
\[
\int_E v_h \cdot \nabla m_{\beta_1} \, dE = - \int_E (\nabla v_h) m_{\beta_1} \, dE + \int_{\partial E} (v_h \cdot n) m_{\beta_1} \, de
\]  
\[
\quad = - \int_E (\nabla v_h) m_{\beta_1} \, dE + \sum_{e \in \partial E} \int_e (v_h \cdot n_e) m_{\beta_1} \, de.
\]  
(35)

The integral inside the element is \(D^{\nabla}\) degree of freedom since \(1 \leq |\beta_1| \leq k - 1\). The integral over the boundary is computable since the virtual function \(v_h\) is a vectorial polynomial on each edge \(e\) (we are integrating a polynomial of degree 2\(k - 1\), that is computable by Remark 4.1).

The explicit computation of the local stiffness matrix \(K_h^{\nabla, E}\) (cf. (17)) now follows the guidelines given in [11].
from the relations in Definition (10) written in terms of vectorial scaled monomials

\[ \left\{ \begin{array}{l}
\int_E \varepsilon(v_h - \Pi_k^\nu v_h) \cdot \varepsilon(m_j) \, dE = 0 \quad \text{for all } m_j \in [M_k(E)]^2 \setminus \mathbb{K}_k^\nu(E), \\
\int_{\partial E} (v_h - \Pi_k^\nu v_h) \cdot m_j \, d\tau = 0 \quad \text{for all } m_j \in \mathbb{K}_k^\nu(E),
\end{array} \right. \tag{36}\]

It is easy to prove that Equation (36) defines a set of linearly independent conditions which uniquely determine the unknown coefficients \( \zeta_i \). Also in this case, Equation (36) is in the algebraic form detailed in Remark 3.2. Consequently, to find the projection we have to directly solve a linear system.

**Remark 4.2.** The set \([M_k(E)]^2 \setminus \mathbb{K}_k^\nu(E)\) contains the set of scaled monomials which does not belong to the kernel of the operator \( \varepsilon \) (the so-called rigid body motions) so that such conditions do not become trivial \((0=0)\).

We analyse only the first conditions in Equation (36), for the boundary ones we recall that the virtual functions are explicitly known on \( \partial E \) and the boundary integrals are exactly computable in the sense of Remark 4.1. Therefore we have the linear system

\[ \sum_{i=0}^{2\pi h} \zeta_i \int_E \varepsilon(m_i) \cdot \varepsilon(m_j) \, dE = \int_E \varepsilon(v_h) : \varepsilon(m_j) \, dE. \]

The left-hand side is computable since it involves only vectorial scaled monomials. We have to verify if the right-hand side is computable from the DoF’s values of the virtual function \( v_h \).

Simple integration by parts and yields

\[ \int_E \varepsilon(v_h) : \varepsilon(m_j) \, dE = - \int_E v_h \cdot \text{div}(\varepsilon(m_j)) \, dE + \int_{\partial E} v_h \cdot (\varepsilon(m_j) \cdot n_e) \, d\tau \\
= - \int_E v_h \cdot \text{div}(\varepsilon(m_j)) \, dE + \sum_{e \in \partial E} \int_{e} v_h \cdot (\varepsilon(m_j) \cdot n_e) \, d\tau, \]

As usual, the boundary term is computable (we integrate on each edge \( e \) a polynomial of degree \( 2k - 1 \)). Concerning the element integral, notice that \( \text{div}(\varepsilon(m_j)) \) can be written as

\[ \text{div}(\varepsilon(m_j)) = \frac{a_1}{h_E} m_{\alpha_1,0} + \frac{a_2}{h_E} m_{\alpha_2,0} + \frac{a_3}{h_E} m_{\theta,\alpha_3} + \frac{a_4}{h_E} m_{\theta,\alpha_4}. \]

Now we can proceed as before, see Equations (34) and (35). Again the local stiffness matrix \( K_h^{\varepsilon,E} \) (cf. (18)) is computed following in a rather slavish way the reference [11].

**4.3. \( L^2 \)–projection \( \Pi_k^\ell \) projection**

In this subsection we verify the computability of the \( L^2 \)–projection operator \( \Pi_k^\ell \) (cf. Definition (10)). In particular we will exploit the so-called enhanced property of the virtual space (13). As we have done for the previous polynomial projections, we look for the unknown coefficients from the relations in Definition (10) written in terms of vectorial scaled monomials

\[ \int_E (v_h - \Pi_k^\ell v_h) \cdot m_j \, dE = 0 \quad \text{for all } m_j \in [M_k(E)]^2. \tag{37} \]
It is easy to show that such conditions are sufficient to find the unknown polynomial coefficients \( \zeta_i \) (cf. (30) with respect to \( \Pi_k \)). We proceed as before and we put in Equation (31) the polynomial \( \Pi_k \v_h \) written in terms of vectorial scaled monomials, i.e.

\[
\sum_{i=1}^{2\pi_k} \zeta_i \int_E m_i \cdot m_j \, dE = \int_E \v_h \cdot m_j \, dE.
\]

The left-hand side is computable, while the right-hand side involves the virtual function \( \v_h \) so we have to verify if it is computable via the degrees of freedom of \( \v_h \). Fist we exploit Proposition 2.1 and we get

\[
\int_E \v_h \cdot m_j \, dE = \int_E \v_h \cdot (b_1 \nabla m_{\beta_1} + b_2 m^+ m_{\beta_2}) \, dE
\]

\[
= b_1 \int_E \v_h \cdot \nabla m_{\beta_1} \, dE + b_2 \int_E \v_h \cdot m^+ m_{\beta_2} \, dE,
\]

(a) In the first one we integrate by parts and we get

\[
\int_E \v_h \cdot \nabla m_{\beta_1} \, dE = - \int_E (\text{div } \v_h) m_{\beta_1} \, dE + \int_{\partial E} (\v_h \cdot n) m_{\beta_1} \, dE
\]

\[
= - \int_E (\text{div } \v_h) m_{\beta_1} \, dE + \sum_{e \in \partial E} \left( \int_e (\v_h \cdot n_e) m_{\beta_1} \, dE \right).
\]

Such integrals are computable. The way of computing the fist integral depends on the degree of \( m_{\beta_1} \). If \( |\beta_1| \leq k - 1 \), it is a degrees of freedom, \( \text{D}^{\text{div}} \). In all the other cases, \( k \leq |\beta_1| \leq k + 1 \), we already prove that we can find the exact expression of \( \text{div } \v_h \), see Subsection 3.3, so, since \( \text{div } \v_h \in \mathbb{P}_{k-1}(E) \), we compute such integral exactly. Regarding the boundary term, we recall that \( \v_h \) is a known vectorial polynomial on each edge \( e \). However, contrary to the previous cases, we are integrating a polynomial of degree \( 2k + 1 \), therefore the \((k+1)\) Gauss–Lobatto values (i.e. the DoFs \( \text{D}^v \)) are not sufficient to compute exactly this integral. In such case we need to reconstruct the polynomial \( \v_h \) on each edge \( e \) and then employ a quadrature rule of degree \( 2k + 1 \).

(b) Also this integral depends on the degree of the monomial \( m_{\beta_2} \). Indeed, if \( |\beta_2| \leq k - 3 \), it is a degrees of freedom \( \text{D}^{m^+} \). Otherwise, when \( k - 2 \leq |\beta_2| \leq k - 1 \), we have to exploit the enhancing condition and compute such integral via the projection operator \( \Pi_k^{\nabla} \), i.e.

\[
\int_E \v_h \cdot m^+ m_{\beta_2} \, dE = \int_E \Pi_k^{\nabla} \v_h \cdot m^+ m_{\beta_2} \, dE \quad k - 2 \leq |\beta_2| \leq k - 1.
\]

Now the the local matrix \( K_{h}^{0,E} \) (cf. (16)) is built using the guide [11].

4.4. \( \Pi_{k-1}^{0} \nabla \) projection

In this subsection we verify the computability of the \( L^2 \)-projection operator \( \Pi_{k-1}^{0} \nabla : \nabla(\mathbb{V}_h) \rightarrow [\mathbb{P}_{k-1}(E)]^{2 \times 2} \). We consider the basis \([\mathbb{M}_{k-1}(E)]^{2 \times 2}\) and we also write the projection \( \Pi_{k-1}^{0} \nabla \v_h \) in terms of such basis functions:

\[
\Pi_{k-1}^{0} \nabla \v_h = \sum_{i=1}^{4\pi_{k-1}} \zeta_i M_i.
\]

Then, starting from the matrix counterpart of conditions (10), we get these set of equations

\[
\sum_{i=1}^{4\pi_{k-1}} \zeta_i \int_E M_i : M_j \, dE = \int_E \nabla \v_h : M_j \, dE \quad \text{for } j = 1, \ldots, 4\pi_{k-1}.
\]
As for all the other projection operators, Equation (38) defines a linear system whose unknowns are the coefficients $\zeta$ of $\Pi_{k-1}^0 \nabla v_h$.

The left-hand side of Equation (38) is computable since it involves only matrix scaled monomials. Regarding the right-hand side we proceed as for the other cases. Integration by parts yields
\[
\int_E \nabla v_h : M_j \, dE = -\int_E v_h \cdot (\text{div} \, M_j) \, dE + \int_{\partial E} v_h \cdot (M_j \, n) \, de
\]
\[
= -\int_E v_h \cdot (\text{div} \, M_j) \, dE + \sum_{e \in \partial E} \int_e v_h \cdot (M_j \, n_e) \, de.
\]
The integral over the edges $e$ is computable since the virtual function $v_h$ is a vectorial polynomial on the edges (in particular we integrate along the edge $e$ a polynomial of degree $2k-1$, see Remark 4.1). To show the computability of the internal integral, we observe that
\[
\text{div} \, M_j = \frac{a_1}{h_E} m_{\alpha_1, \emptyset} + \frac{a_2}{h_E} m_{\alpha_2, \emptyset} + \frac{a_3}{h_E} m_{\emptyset, \alpha_3} + \frac{a_4}{h_E} m_{\emptyset, \alpha_4},
\]
where, as before, the coefficients $a_j$ and the multi-indexes $\alpha_s$ can be easily found since we are dealing with scaled monomials. Consequently, the computability of $\Pi_{k-1}^0$ follows from same arguments of $\Pi_k^\text{V}$, see Equation (33).

**Remark 4.3.** The computation of $\Pi_{k-1}^0(e(v))$ follows the same strategy of $\Pi_{k-1}^\text{V}$ so we omit the explicit construction of such projection operator.

5. Numerical Results

We present three numerical experiments to exploit the behaviour of the proposed virtual elements family for the mixed problems such as Stokes, Darcy, Brinkman and Navier–Stokes equations. More specifically we assess the actual performance of the virtual element method for high-order polynomial degrees (up to $k = 6$).

Given $f \in [L^2(\Omega)]^2$ and $g \in L^2(\Omega)$, and referring to (14), (15), (23), (28), (25), the virtual elements approximation of the general mixed problem has the form
\[
\begin{aligned}
&\text{find } (u_h, p_h) \in V_h \times Q_h \text{ s.t.} \\
&a_h(u_h, v_h) + b(v_h, p_h) = (f_h, v_h) \quad \text{for all } v_h \in V_h, \\
&b(u_h, q_h) = (g, q_h) \quad \text{for all } q_h \in Q_h.
\end{aligned}
\] (39)

We now explore the algebraic structure of the mixed problem above. Consider a generic polygonal mesh $\Omega_h := \{ E_i \}_{i=1}^{n_P}$ of $\Omega$. We denote with $K_h$ the global counterpart of one of the local stiffness matrices in (16), (17) and (18), and with $B$ (cf. (29)), $f_h$, $g$ (cf. (26)) the global version of the div-matrix and discrete right-hand side respectively. Denoting with $\text{GNDof}$ the total amount of global velocity DoFs, the velocity and pressure solutions can be expressed in terms of the global bases
\[
u_h = \sum_{i=1}^{\text{GNDof}} \chi_i \varphi_i \quad \text{and} \quad p_h = \sum_{j=1}^{n_P} \sum_{\ell=1}^{h_E \pi_{k-1}} q_{\ell j} m_{\ell}
\]
and the vectors
\[
\chi := (\chi_i)_i \quad \text{and} \quad \varrho := [\varrho^1 \ldots \varrho^{n_P}]
\]
are the unknowns of the associated discrete problem. The discretization of the mixed problem results in a linear algebraic system of the form
\[
\begin{bmatrix}
K_h & B^T \\
0 & \sigma^T
\end{bmatrix}
\begin{bmatrix}
\chi \\
\varrho
\end{bmatrix}
= 
\begin{bmatrix}
f_h \\
0
\end{bmatrix}
\] (40)
where \( \sigma := [\sigma_1 \mid \ldots \mid \sigma^n] \)

with
\[
(\sigma^j)_\ell = \frac{h_{E_j}}{|E_j|} \int_{E_j} m_\ell \, dE_j \quad \text{for } j = 1, \ldots, n_P \text{ and } \ell = 1, \ldots, \pi_{k-1}.
\]

Note that the second block-row in (40) represents the “zero averaged pressure” constraint and \( \lambda \) is the associated Lagrange multiplier.

Concerning the order of accuracy of the method, let \((u, p)\) and \((u_h, p_h)\) be the continuous and its corresponding discrete solution given by (39). Then, the expected rate of convergence for the errors are

- \(H^1\)-velocity error:
  \[
  \| \nabla u - \nabla u_h \|_{L^2(\Omega)} \lesssim h^k |u|_{k+1} + h^{k+2} |f|_{k+1},
  \]

- \(L^2\)-velocity error:
  \[
  \| u - u_h \|_{L^2(\Omega)} \lesssim h^{k+1} |u|_{k+1} + h^{k+3} |f|_{k+1},
  \]

- \(L^2\)-pressure error:
  \[
  \| p - p_h \|_{L^2(\Omega)} \lesssim h^k |u|_{k+1} + h^k |p|_k + h^{k+2} |f|_{k+1}.
  \]

Since the VEM velocity solution \( u_h \) is not explicitly known point-wise inside the elements, in order to check the actual performance of the method, we compute the errors comparing \( u \) with a suitable polynomial projection of the discrete solution \( u_h \). Note that the pressure variable \( p_h \) is a piecewise polynomial so we can use it directly to compute the pressure error. Therefore we consider the following computable error quantities:

- \( \text{error}(u, H^1) := \sqrt{\sum_{E \in \Omega_h} \| \nabla u - \nabla \Pi^V \nabla u_h \|_{L^2(E)}^2} \),
- \( \text{error}(u, L^2) := \sqrt{\sum_{E \in \Omega_h} \| u - \Pi^W u_h \|_{L^2(E)}^2} \),
- \( \text{error}(p, L^2) := \| p - p_h \|_{L^2(\Omega)} \).

In the experiments we consider three sequences of finer meshes of the unit square \([0, 1]^2\), see Figure 2:

- \text{quad}, a mesh composed by structured squares;
- \text{hexa}, a mesh composed by distorted hexagons;
- \text{voro}, a Voronoi tessellation composed by general polygons.

We associate with each discretization a mesh-size
\[
h := \frac{1}{n_P} \sum_{i=1}^{n_P} h_{E_i}.
\]
5.1. Stokes Equations

In this test we solve the virtual Stokes problem (in the “epsilon form”)

\[
\begin{align*}
\text{find } (u_h, p_h) &\in \mathbf{V}_h \times Q_h \text{ s.t.} \\
a_\epsilon^h(u_h, v_h) + b(v_h, p_h) &= (f_h, v_h) \quad \text{for all } v_h \in \mathbf{V}_h, \\
b(u_h, q_h) &= 0 \quad \text{for all } q_h \in Q_h.
\end{align*}
\]

(41)

where the bilinear operators \( a_\epsilon^h(\cdot, \cdot) \) and \( b(\cdot, \cdot) \) are defined in Equations (21) and (28), respectively, while the discrete load is defined in (25). The load term \( f \) and the boundary conditions of the continuous formulation of Problem (41) are chosen in such a way that the exact solution is

\[
\begin{align*}
\text{u}(x, y) &= \left( 
\begin{array}{c}
0.5 \sin(2\pi x) \sin(2\pi y) \\
-0.5 \sin(2\pi y) \sin(2\pi x)
\end{array}
\right), \\
p(x, y) &= \sin(2\pi x) \cos(2\pi y).
\end{align*}
\]

In Figures 3, 4 and 5 we show the convergence lines with different VEM approximation degrees (up to \( k = 6 \)) for the sequences of meshes listed above. The trend of the error is the expected one and it is astonishingly stable with high approximation degrees \( k \). Indeed, the error \( \text{error}(u, H^1) \) reaches small values, close to the machine precision, for \( k = 6 \) and the finest mesh. Here we do not show the results for \( \text{error}(u, L^2) \) since they have the expected trend too.
5.2. Darcy Equations

In this subsection we are solving the Darcy problem

\[
\begin{align*}
\text{find } (u_h, p_h) & \in V_h \times Q_h \text{ s.t.} \\
\mathcal{A}_h(u_h, v_h) + b(v_h, p_h) & = 0 \quad \text{for all } v_h \in V_h, \\
b(u_h, q_h) & = (g, q_h) \quad \text{for all } q_h \in Q_h.
\end{align*}
\]

(42)

where the bilinear operators \( \mathcal{A}_h(\cdot, \cdot) \) and \( b(\cdot, \cdot) \) are defined in Equations (19) and (28), respectively. We the continuous version of (42) whose exact solution is the pair

\[
\begin{align*}
u(x, y) & = \begin{pmatrix}
-\pi \sin(\pi x) \cos(\pi y) \\
-\pi \cos(\pi x) \sin(\pi y)
\end{pmatrix}, \\
p(x, y) & = \cos(\pi x) \cos(\pi y).
\end{align*}
\]

In Figures 6, 7 and 8 we provide the convergence lines. The trend of the error is the expected one for the cases \( k = 2, 3, 4 \) and 5. However, in the last step of the convergence lines for \( k = 6 \) associate with the error \( \| u_h \|_{L^2} \) the lines does not follow the theoretical trend. This fact is probably due to the matrix conditioning. Indeed, we are solving a large linear system of high degree and the error is close to the machine precision. We get the expected trend also for error \( \| u_h \|_{H^1} \), but, as for error \( \| u_h \|_{L^2} \), in the last steps of \( k = 6 \) it has a plateau around \( 10^{-12} \).
5.3. Navier-Stokes Equations

In this subsection we are solving the Navier-Stokes problem (in the “gradient form”)

\[
\begin{aligned}
\text{find } & (u_h, p_h) \in V_h \times Q_h \text{ s.t. } \\
& a_h(u_h, v_h) + c_h(u_h; u_h, v_h) + b(v_h, p_h) = (f_h, v_h) \\
& b(u_h, q_h) = 0
\end{aligned}
\]

for all \( v_h \in V_h \),

for all \( q_h \in Q_h \). (43)
where the bilinear operators $a_h^{\nabla}(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are defined in Equations (20) and (28), respectively, $c_h(\cdot, \cdot, \cdot)$ is the trilinear operator defined in Equation (24) and the discrete right-hand side is defined in (25). Also in this case we fix the load and the boundary condition in such a way that the exact solution related to (43) is

$u(x, y) = \left( \begin{array}{c}
-0.5 \cos(x) \cos(x) \cos(y) \sin(y) \\
0.5 \cos(y) \cos(y) \cos(x) \sin(x)
\end{array} \right), \quad p(x, y) = \sin(x) - \sin(y) \).$

In Figures 9, 10 and 11, we show the convergence lines for $k = 2, 3, 4$ and 5. For quad meshes the trend of such errors is the expected one, see Figure 9. In the case of the sets of hexa and voro meshes, we recover the expected trend for $k = 2, 3$ and 4. When we consider a degree $k = 5$, the last part of the convergence lines do not follow the theoretical trend. This fact is not so evident for the error $(u, H^1)$, but it becomes clearer in error $(p, L^2)$. Such bad behaviour is not addicted to the robustness of the virtual element method, but it is due to the machine precision. The error error $(u, L^2)$ has the expected trend, but it suffers for $k = 5$ at the last step in a similar way as error $(u, H^1)$.

6. Conclusion

In this paper we focus on the technical aspects of VEM when we are considering partial differential equations in mixed form. More specifically, we gave the essential “bricks” to make both projectors and differential operators starting from the proposed virtual element spaces.
This deep analysis allowed us to manage high VEM approximation order and solve a wide variety of problems (Stokes, Brinkman, Darcy and Navier–Stokes). Numerical results show that VEM are particularly robust with high-order, since we reach error values close to the machine precision when we are taking high degree and fine meshes.

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