NEW RECURRENCE RELATIONS AND MATRIX EQUATIONS FOR ARITHMETIC FUNCTIONS GENERATED BY LAMBERT SERIES

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ABSTRACT. We consider relations between the pairs of sequences, \((f, g_f)\), generated by the Lambert series expansions, \(L_f(q) = \sum_{n \geq 1} f(n)q^n/(1-q^n)\), in \(q\). In particular, we prove new forms of recurrence relations and matrix equations defining these sequences for all \(n \in \mathbb{Z}^+\). The key ingredient to the proof of these results is given by the statement of Euler’s pentagonal number theorem expanding the series for the infinite \(q\)-Pochhammer product, \((q; q)_\infty\), and for the first \(n\) terms of the partial products, \((q; q)_n\), forming the denominators of the rational \(n\)th partial sums of \(L_f(q)\). Examples of the new results given in the article include new exact formulas for and applications to the Euler phi function, \(\phi(n)\), the Möbius function, \(\mu(n)\), the sum of divisors functions, \(\sigma_1(n)\) and \(\sigma_\alpha(n)\), for \(\alpha \geq 0\), and to Liouville’s lambda function, \(\lambda(n)\).

1. Introduction

1.1. Overview and motivation. Our new results provide exact matrix-based formulas for a wide range of classical special arithmetic functions expanded in well-known Lambert series expansions of the form defined in the next subsection. The first form of the exact formulas for these special case arithmetic functions, \(f(n)\) are stated through a matrix factorization result of the following form for all natural numbers \(n \geq 0\):

\[
(f(k))_{1 \leq k \leq n} = A_n^{-1} \cdot (B_m(f))_{0 \leq m < n}.
\]

The \(n \times n\) matrices \(A_n\) and \(A_n^{-1}\) implicit to the last equation are always independent of the choice of the function \(f(n)\). Moreover, the right-hand-side vector whose entries are given by \(B_m(f)\) is independent of the \(A_n\) and depends only on a finite sum determined by \(f\) for all \(m, n \geq 0\).

The matrix equation in (1) reflects a more general class of so-termed Lambert series factorization results of the form

\[
\sum_{n \geq 1} \frac{f(n)q^n}{1-q^n} = \frac{1}{C(q)} \sum_{n \geq 0} \sum_{k=1}^{n} s_{n,k} f(k) \cdot q^n,
\]

where the invertible, lower triangular matrix \(A_n\) in the form of (1) corresponds to the entries \(s_{n,k}\), which are also independent of the function \(f\), for fixed \(n \geq 1\). In the cases of these more general factorizations presented in this article, which we derive and prove by separate means here, we always have that the series expansion of the factorization parameter \(C(q)\) is defined through the \(q\)-Pochhammer symbol as \(C(q) \equiv (q; q)_\infty\).

Examples of the new formulas for the special arithmetic functions, \(\mu(n)\), \(\phi(n)\), and \(\lambda(n)\), that we are able to obtain through the forms of the new matrix factorization
There are many well-known Lambert series for special arithmetic functions of the form \( \phi_f \) for prescribed functions \( f \).

The results we prove within the article also include new recurrence relations for the computations of the average order of special arithmetic functions, denoted by \( \lfloor B_{m,f} \rfloor \), for fixed functions \( f(n) \) and its corresponding \( g_f(n) \) when \( x \geq 1 \). The next subsections make the expansions of the Lambert series expansions we consider and the main theorems proved within the article precise.

1.2. Lambert series generating functions. In this article, we consider new recurrence relations and matrix equations related to Lambert series expansions of the form [3, §27.7] [1, §17.10]

\[
L_f(q) := \sum_{n \geq 1} \frac{f(n)q^n}{1 - q^n} = \sum_{m \geq 1} g_f(m)q^m, \quad |q| < 1,
\]

for prescribed functions \( f : \mathbb{Z}^+ \to \mathbb{C} \), and some \( g_f : \mathbb{Z}^+ \to \mathbb{C} \) where \( g_f(m) = \sum_{d|m} f(d) \). There are many well-known Lambert series for special arithmetic functions of the form in (2). Examples include the following series where \( \mu(n) \) denotes the Möbius function, \( \phi(n) \) denotes Euler’s totient function, \( \sigma_{\alpha}(n) \) denotes the generalized sum of divisors function, and \( \lambda(n) \) denotes Liouville’s function [3, §27.7]:

\[
\sum_{n \geq 1} \frac{\mu(n)q^n}{1 - q^n} = q, \quad (f, g_f) := (\mu(n), [n = 1]_\delta)
\]

\footnote{Notation: Iverson’s convention compactly specifies boolean-valued conditions and is equivalent to the Kronecker delta function, \( \delta_{i,j} \), as \( [n = k]_\delta \equiv \delta_{n,k} \). Similarly, \( [\text{cond} = \text{True}]_\delta \equiv \delta_{\text{cond}, \text{True}} \) in the remainder of the article.}
\[
\sum_{n \geq 1} \frac{\phi(n)q^n}{1 - q^n} = \frac{q}{(1 - q)^2}, \quad (f, g_f) := (\phi(n), n)
\]
\[
\sum_{n \geq 1} \frac{n^{\alpha}q^n}{1 - q^n} = \sum_{m \geq 1} \sigma_{\alpha}(n)q^n, \quad (f, g_f) := (n^{\alpha}, \sigma_{\alpha}(n))
\]
\[
\sum_{n \geq 1} \frac{\lambda(n)q^n}{1 - q^n} = \sum_{m \geq 1} q^{\sigma_{\alpha}(m)n^2}, \quad (f, g_f) := (\lambda(n), [n \text{ is a positive square}]_g).
\]

1.3. New Results. We have two interesting cases of (2) to consider:

1. The case where \(f(n)\) is our arithmetic function of interest that we wish to study, i.e., in the first, second, and fourth equations in (3); and
2. The case where \(g_f(n)\) is the interesting arithmetic function we wish to study, i.e., in the third equation from (3).

1.3.1. Case 1. In the first case, for each \(n \geq 1\) and \(i \leq n\) we are able to form the matrix solutions for \(f(n)\) given in Theorem 1.3, which are expanded in terms of the sequences in the next definition.

**Definition 1.1.** For integers \(n \geq 0\), the sequences, \(a_f(n)\) and \(a_{n,i}\), are defined to be (cf. Remark 1.2 on page 4)\(^2\)

\[
a_f(n) = \sum_{i=1}^{n} f(i) \left( \sum_{(k,s):(s+1)i+k(3k\pm 1)/2=n} (-1)^k \cdot [n = 0]_k \right),
\]
\[
a_{n,i} = \sum_{b=\pm 1} \sum_{s=0}^{[\sqrt{24(n-(s+1)i)/6}]} (-1)^s \cdot \left[ \frac{24n - (s + 1)i + 1 - b}{6} \in \mathbb{Z} \right].
\]

For \(n \geq 1\), we define the \(n \times n\) matrices, \(A_n\) and \(A_n^{-1}\), in terms of these sequences as follows:

\[
A_n := (a_{i,j})_{1 \leq i,j \leq n}, \quad A_n^{-1} := \left( a_{i,j}^{(-1)} \right)_{1 \leq i,j \leq n}.
\]

The matrices, \(A_n\) and \(A_n^{-1}\), are independent of the choice of the function \(f\) for all \(n\) and are each invertible, lower triangular square matrices with ones on their diagonals. The independence of these matrices on the choice of \(f\) implicit to the expansions in (2) leads to the Lambert series matrix factorization result phrased by Theorem 1.3 below.

The motivation for defining the sequences, \(a_f(n)\) and \(a_{n,i}\), in Definition 1.1 is to provide a compact notation for expressing the left-hand-side terms, \(A_n[f(1) \cdots f(n)]^T\), of the matrix equation corresponding to the non-homogeneous recurrence relations for \(g_f(n)\) given in Theorem 1.3 and in Theorem 1.4 (see also Lemma 2.1 on page 7).

\(^2\) The floored terms, \([kr + 1 - b]/6\), for \(b = \pm 1\) in this and in subsequent formulas in the article correspond to solving for the upper bounds on \(k\) in the following inequalities (cf. footnote 4 on page 9):

\[0 \leq \frac{k(3k + b)}{2} \leq n \iff 0 \leq k \leq \frac{\sqrt{24n + 1} - b}{6}.
\]
The bracket notation for coefficient extraction of a (formal) power series is defined to be \([a_n]F(q) := f_n\), where \(F(q) := \sum_{n \geq 0} f_n q^n\) represents the ordinary generating function of the sequence, \([f_n]_{n \geq 0}\). This notation is also employed in Section 2 below.

The first few cases of the matrices, \(A_n \in \mathbb{Z}_{n \times n}\), and their corresponding inverses, \(A_n^{-1} \in \mathbb{N}_{n \times n}\), are shown in Table 1. In general, we see that for all \(n \geq 2\), we have that

\[
A_{n+1}^{-1} = \begin{bmatrix}
A_n^{-1} & 0 \\
\frac{A_n^{-1}}{r_{n+1,n}} & \cdots & \frac{A_n^{-1}}{r_{n+1,1}} & 1
\end{bmatrix},
\]

where the first several special cases of the sequences, \(\{r_{n+1,n}, r_{n+1,n-1}, \ldots, r_{n+1,1}\}\), are given in Table 2. The statement of the next theorem employs these sequences and matrix forms. The proof of Theorem 1.3 is given in Section 2 below.

**Remark 1.2** (Short Author’s Note). Since the first submission of the manuscript the entries, \(a_{n,i}\) and \(a_{n,i}^{-1}\), in (4) have been determined in closed-form through joint work by Merca and Schmidt on Lambert series factorizations (2017). In particular, we have that \(a_{n,i}\) corresponds to\(^3\)

\[
s_o(n, i) - s_e(n, i) = [q^n] \frac{q^i}{1 - q^i} (q; q)_\infty,
\]

where \(s_o(n, k)\) and \(s_e(n, k)\) are respectively the number of \(k\)'s in all partitions of \(n\) into an odd (even) number of distinct parts. Similarly, we can derive an exact formula for the inverse matrix entries as

\[
a_{n,i}^{-1} = \sum_{d | n} p(d - i)\mu(n/d),
\]

where \(p(n)\) denotes Euler’s partition function which is generated by the reciprocal of the \(q\)-Pochhammer symbol as \(p(n) = [q^n] (q; q)_\infty^{-1}\) for all \(n \geq 0\).

**Theorem 1.3** (Matrix Factorization Equations for \(f(n)\)). For all \(n \geq 1\), we have the following matrix factorization equations exactly generating the arithmetic functions, \(f(n)\), in the definition of (2):

\[
\begin{bmatrix}
f(1) \\
f(2) \\
\vdots \\
f(n)
\end{bmatrix} = A_n^{-1} \left[ g_f(m+1) - \sum_{b=\pm 1} \sum_{k=1}^\left\lfloor \frac{4m+1+b}{6} \right\rfloor (-1)^{k+1} g_f(m+1-k(3k+b)/2) \right] \quad 0 \leq m < n.
\]

\(^3\)Notation: The bracket notation for coefficient extraction of a (formal) power series is defined to be \([q^n]F(q) := f_n\) when \(F(q) := \sum_{n \geq 0} f_n q^n\) represents the ordinary generating function of the sequence, \([f_n]_{n \geq 0}\). This notation is also employed in Section 2 below.
Table 1. The first few special cases of the matrices, $A_n$ and $A_n^{-1}$

| $n$ | $A_n$ | $A_n^{-1}$ |
|-----|-------|------------|
| 1   | $[1]$ | $[1]$      |
| 2   | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ |
| 3   | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ |
| 4   | $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \end{bmatrix}$ |
| 5   | $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 4 & 3 & 2 & 1 & 1 \end{bmatrix}$ |

Table 2. The bottom row sequences in the matrices, $A_n^{-1}$

| $n$ | $\{r_{n,n-1}, r_{n,n-2}, \ldots, r_{n,1}\}$ |
|-----|---------------------------------|
| 2   | $\{1\}$                        |
| 3   | $\{1, 1\}$                     |
| 4   | $\{2, 1, 1\}$                  |
| 5   | $\{4, 3, 2, 1\}$               |
| 6   | $\{5, 3, 2, 2, 1\}$            |
| 7   | $\{10, 7, 5, 3, 2, 1\}$        |
| 8   | $\{12, 9, 6, 4, 3, 2, 1\}$     |
| 9   | $\{20, 14, 10, 7, 5, 3, 2, 1\}$|
| 10  | $\{25, 18, 13, 10, 6, 5, 3, 2, 1\}$|
| 11  | $\{41, 30, 22, 15, 11, 7, 5, 3, 2, 1\}$|
| 12  | $\{47, 36, 26, 19, 14, 10, 7, 5, 3, 2, 1\}$|

1.3.2. Case 2. In the second case, we have recurrence relations for $g_f(n)$ in (2), of the form stated in Theorem 1.4. Moreover, if we define $\Sigma_{g_f,x} := \sum_{n \leq x} g_f(n)$ to be the average order of $g_f(n)$, then we can also prove easily by induction that $\Sigma_{g_f,x}$ itself also satisfies the related form of the recurrence relation given in Corollary 1.5.
Theorem 1.4 (Recurrence Relations for $g_f(n)$). For all $n \geq 1$, we have the following recurrence relation for $g_f(n)$ expanded in terms of the sequences from Definition 1.1:

$$g_f(n + 1) = \sum_{b=\pm 1} \sum_{k=1}^{\left\lfloor \frac{\sqrt{24n+1}-b}{6} \right\rfloor} (-1)^{k+1} g_f(n + 1 - k(3k + b/2) + a_f(n + 1)).$$

Corollary 1.5 (Recurrence Relations for $\Sigma_{g_f,x}$). Let the $x^{th}$ partial sums of the function, $g_f(n)$, i.e., its average order, be defined by $\Sigma_{g_f,x} := \sum_{n\leq x} g_f(n)$. Then for all $n \geq 1$, we have that

$$\Sigma_{g_f,n+1} = \sum_{b=\pm 1} \left( \sum_{k=1}^{\left\lfloor \frac{\sqrt{24n+1}-b}{6} \right\rfloor+1} (-1)^{k+1} \Sigma_{g_f,n+1-k(3k+b)/2} + \sum_{k=1}^{n} a_f(k+1) \right).$$ (7)

Each of Theorem 1.4 and Corollary 1.5 are proved in Section 2.

1.3.3. Algorithms for computing the functions, $f(n)$ and $g_f(n)$, in polynomial time. Since the determinant of a $(n - 1) \times (n - 1)$ matrix can be computed in $O(n^3)$ time, if $g_f(n)$ can be computed in constant time, then by the theorem, we have a $O(n^4)$ polynomial time algorithm to compute any function $f(n)$ in (2). If we instead use Gaussian elimination with back substitution, we can compute the functions, $f(n)$, in $O(n^5)$ time. Similarly, if $g_f(n)$ can be computed in $O(h_{g_f}(n))$ time, then we can compute any function $f(n)$ in (2) in $O(h_{g_f}(n)\sqrt{n} + n^4) / O(h_{g_f}(n)\sqrt{n} + n^3)$ time.

However, we note that each of the special arithmetic functions on the left-hand-side of (3) can be computed more efficiently using sieves and other prime factorization algorithms. Despite more efficient known methods for computing the classical arithmetic functions involved in the expansions of (3), this observation is still useful since it implies that there are now known polynomial-time algorithms for computing the pairs, $(f(n), g_f(n))$, in any Lambert series expansion provided a polynomial-time method of computation for either one of the functions in the corresponding pair.

1.4. Organization of the article. The proofs of Theorem 1.3, Theorem 1.4, and Corollary 1.5 stated in the last subsection are given first in Section 2. In Section 3, we provide several concrete examples of the applications of these new results to the classical arithmetic functions of the sum of divisors function, $\sigma_1(n)$, the Euler phi function, $\phi(n)$, the Môbius function, $\mu(n)$, and to the Liouville lambda function, $\lambda(n)$. In the concluding remarks we give in Section 4, we suggest an approach to analogs of the new results proved within the article for a generalized class of Lambert series expansions, $L_f(\alpha, \beta; a, b, c, d; q)$, which are suggested as a future avenue of research on this topic.

2. Proofs of the theorems

In order to obtain recurrence relations between the sequences implicit to the definition of (2), we first observe that for all $m \geq 0$ we have the next series expansions of the partial sums of the Lambert series, $L_f(q)$, where $(q; q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n)$ denotes the $q$-Pochhammer symbol [3, \S 17.2], and where the functions $\mathrm{poly}_{i,m}(f; q)$ denote polynomials in $q$ with coefficients depending on $f$ for $i = 1, 2$ and whose degree is linear in the fixed index $m$. 
Lemma 2.1 (Partial Sums of the Lambert Series, \( L_f(q) \)). For a fixed pair of functions \( (f(n), g_f(n)) \) in the expansions of (2) and for all integers \( m \geq 0 \) we have that

\[
g_f(m + 1) = [q^m] \left( \frac{1}{q} \times \sum_{n=1}^{m+1} \frac{f(n)q^n}{1 - q^n} \right) = [q^m] \left( \frac{1}{q} \cdot (q; q)_{m+1} \frac{\left[ f(1)q + f(2)q^2 + \cdots + f(n)q^{m+1} \right]}{(1 - q)(1 - q^2) \cdots (1 - q^{m+1})} \right)
\]

\[
= [q^m] \left( \frac{\sum_{1 \leq i \leq m+1} a_f(i)q^i + q^{m+2} \cdot \text{poly}_{1,m}(f; q)}{1 + \sum_{b=\pm 1} \sum_{k=1}^{\frac{\sqrt{1+4m+8m^2+1}}{6}} (-1)^k q^{k(3k+b)/2} + q^{m+2} \cdot \text{poly}_{2,m}(f; q)} \right).
\]

Proof. To justify (8a), we observe that for all integers \( m \geq 1 \) and \( 1 \leq i \leq m \), we have that

\[
[q^i] \left( L_f(q) - \sum_{n>0} \frac{f(n)q^n}{1 - q^n} \right) = 0,
\]

i.e., that the \( m^{th} \) partial sums of \( L_f(q) \) accurately generate \( f(k) \) for \( 1 \leq k \leq m \), which is easy enough to see by considering the numerator multiples, \( q^n \), of the geometric series, \( (1 - q^n)^{-1} \), in the individual Lambert series terms from (2). The result in (8b) follows immediately from (8a) by combining the terms in the first partial sum, and implies the third result in (8c) in two key ways.

First, the respective form of the denominator terms in (8c) follows from the statement of Euler’s pentagonal number theorem, which states that [1, §19.9, Thm. 353]

\[
(q; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = 1 + \sum_{n \geq 1} (-1)^n \left( q^{k(3k-1)/2} + q^{k(3k+1)/2} \right)
\]

\[
= 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - \cdots .
\]

In particular, we see that the pentagonal number theorem shows that

\[
[q^i](1 - q)(1 - q^2) \cdots (1 - q^n) = \begin{cases} (-1)^k, & \text{if } i = \frac{k(3k+1)}{2}, \\ 0, & \text{otherwise}. \end{cases}
\]

for all \( i \leq n \) by a contradiction argument. Since \( (1 - q^i) \) is a factor of \( (q; q)_n \) for all \( 1 \leq i \leq n \), we see that both of the numerator and denominator of (8b) are polynomials in \( q \), each with degree greater than \( m + 1 \). This implies the correctness of the denominator polynomial form stated in (8c).

Secondly, since the geometric series in \( q^i \) is expanded by

\[
\frac{1}{1 - q^i} = \sum_{s \geq 0} q^{si},
\]

for each finite \( i \geq 1 \), we have by the definition of \( a_f(n) \) in Definition 1.1 that the first \( m + 1 \) terms of the numerator expansion in (8a) are correct. Thus since the numerator in (8c) is polynomial in \( q \), it is also correct in form. \( \Box \)
Proof of Theorem 1.4. We use (8c) in Lemma 2.1 to prove our result. If we let $\text{Num}_m(q)$ and $\text{Denom}_m(q)$ denote the numerator and denominator polynomials in (8c), respectively, we see that by definition, $\deg_q \{\text{Num}_m(q)\} < \deg_q \{\text{Denom}_m(q)\}$. For any sequence, $(f_n)_{n \geq 0}$, generated by a rational generating function of the form
\[
\sum_{n \geq 0} f_n q^n = \frac{a_0 + a_1 q + a_2 q^2 + \cdots + a_{k-1} q^{k-1}}{1 - b_1 q - b_2 q^2 - \cdots - b_k q^k},
\]
for some fixed finite integer $k \geq 1$, we can prove that $f_n$ satisfies at most a $k$-order finite difference equation with constant coefficients of the form [2, §2.3]
\[
f_n = \sum_{i=1}^{\min(k,n)} b_i f_{n-i} + a_n [0 \leq n < k]_\delta.
\]
Then since we define $f(n) = 0$ for all $n < 1$ in (2), and since the $m^{\text{th}}$ partial sums of $L_f(q)$ generate $g_f(i)$ for all $1 \leq i \leq m$ by the lemma, and since $g_f(i) = 0$ for all $i < 1$, we see that (8c) implies our result. □

Proof of Theorem 1.3. The theorem is a consequence of Definition 1.1 applied to Theorem 1.4. Specifically, by rearranging terms in the result from the previous theorem, we see that
\[
A_n \begin{bmatrix} f(1) \\ f(2) \\ \vdots \\ f(n) \end{bmatrix} = \begin{bmatrix} B_{g_f,0} \\ B_{g_f,1} \\ \vdots \\ B_{g_f,n-1} \end{bmatrix},
\]
where (6) defines the sequence of $B_{g_f,m}$. Then by the definition of $a_n,i$ given in Definition 1.1 (cf. Remark 1.2), it is easy to see that $A_n$ is lower triangular with ones on its diagonals, and so is invertible for all $n \geq 1$. Thus by applying $A_n^{-1}$ to both sides of (i), we have proved (6) in the statement of the theorem. □

Proof of Corollary 1.5. We can show directly by computation that the statement is true for $n = 1$. For some $j \geq 1$, suppose that the hypothesis in (7) is true for $n = j$. Then we see that
\[
\tilde{\Sigma}_{g_f,j+1} = \sum_{b=\pm 1} \left\lfloor \frac{\sqrt{24n+25-b}}{6} \right\rfloor \sum_{k=1}^{\frac{j+1}{2}} (-1)^k \left[ \Sigma_{g_f,j+1-k(3k+b)/2} + g_f(j + 2 - k(3k + b)/2) \right] + \sum_{k=1}^{j+1} a_f(k + 1)
\]
\[
= \Sigma_{g_f,j+1} + \sum_{b=\pm 1} \sum_{k=1}^{\left\lfloor \frac{\sqrt{24n+25-b}}{6} \right\rfloor} g_f(j + 2 - k(3k + b)/2) + a_f(j + 2), \text{ by hypothesis}
\]
\[
= \Sigma_{g_f,j+1} + g_f(j + 2)
\]
The second to last of the previous equations follows from Theorem 1.4, the fact that $\left\lfloor (\sqrt{24n+25-b})/6 \right\rfloor \geq \left\lfloor (\sqrt{24n+1-b})/6 \right\rfloor$, and since $g_f(i) = 0$ for all $i < 1$. □
3. Examples of the new results

3.1. The generalized sum-of-divisors functions. For any \( n, x \geq 0 \), we have the following recurrence relations following from the results proved in Theorem 1.4 and Corollary 1.5\(^4\):

\[
\sigma_1(n + 1) = \sum_{b=\pm 1}^{\lfloor \sqrt[6]{24n+1} - b \rfloor} \sum_{k=1}^{\lfloor \sqrt[6]{24n+1} - b \rfloor} (-1)^{k+1} \sigma_1(n + 1 - k(3k + b)/2) \\
+ (-1)^k (n + 1) [n + 1 = k(3k \pm 1)/2] \delta
\]

\[
\sum_{n,x+1} = \sum_{b=\pm 1}^{\lfloor \sqrt[6]{24n+1} - b \rfloor} \sum_{k=1}^{\lfloor \sqrt[6]{24n+1} - b \rfloor} (-1)^{k+1} \sum_{n,x+1-k(3k+b)/2} + \sum_{k=1}^{\lfloor \sqrt[6]{24n+1} - b \rfloor} (-1)^{k+1} k(3k + b)/2
\]

Notice that the previous two equations imply exact closed-form formulas for \( \sigma_1(m) \) and \( \sum_{n,m} \) at each \( m \geq 1 \), and similarly for fixed \( m \) and all \( 1 \leq n \leq m \). Moreover, since we conjecture that the zeros of the polynomials, \( \tilde{Q}_n(q) := q^n \cdot Q_n(1/q) \), or alternately the reciprocal zeros of the polynomials

\[
Q_n(q) := 1 - \sum_{b=\pm 1}^{\lfloor \sqrt[6]{24n+1} - b \rfloor} \sum_{k=1}^{\lfloor \sqrt[6]{24n+1} - b \rfloor} (-1)^k q^{k(3k+b)/2},
\]

have maximum magnitude of a little over 1 (depending on \( n \)), we remark as another potential application, which is suggested for further work based on the results in this article, that it may be possible to use these results to obtain better error bounds on the known average order sums [3, §27.11]

\[
\sum_{n \leq x} \sigma_1(n) = \frac{\pi^2}{12} x^2 + O(x \log x) \\
\sum_{n \leq x} \sigma_{\alpha}(n) = \frac{\zeta(\alpha + 1)}{\alpha + 1} x^{\alpha+1} + O(x^\beta), \; \alpha > 0, \alpha \neq 1, \beta = \max(1, \alpha),
\]

using the explicit sequence formulas in (5).

\(^4\)Here, we make use of a natural conjecture from (5) that \( a_n(m) = (-1)^k (m+1) [m = k(3k \pm 1)/2] \delta \), where the pentagonal numbers, \( \omega_{k,b} := \frac{k(3k+b)}{2} \), between 1 and \( n + 1 \) are given by the following sets for each respective \( b = \pm 1 \) (see footnote 2 on page 3):

\[
\left\{ \omega_{1,b}, \omega_{2,b}, \ldots, \omega_{\lfloor \sqrt[6]{24n+1}-b \rfloor/6, b} \right\}.
\]
3.2. Euler’s totient function. We provide a computation of (6) to demonstrate the utility to our method:

\[
\begin{bmatrix}
\phi(1) \\
\phi(2) \\
\phi(3) \\
\phi(4) \\
\phi(5) \\
\phi(6) \\
\phi(7)
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 1 & 0 & 0 & 0 \\
4 & 3 & 2 & 1 & 1 & 0 & 0 \\
5 & 3 & 2 & 2 & 1 & 1 & 0 \\
10 & 7 & 5 & 3 & 2 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
0 \\
-1 \\
4 \\
-2 \\
2
\end{bmatrix} =
\begin{bmatrix}
1 \\
1 \\
2 \\
2 \\
4 \\
2 \\
6
\end{bmatrix}.
\]

In this case, we can solve for the right-hand-side vector, \((B_{g_f,n})\), explicitly. More precisely, when \((f, g_f) := (\phi(n), n)\) from (3), we see by straightforward summation that

\[
B_{n,m} = m + 1 - \frac{1}{8} \left( 8 - 5 \cdot (-1)^{u_1} - 4(-2 + (-1)^{u_1} + (-1)^{u_2}) m + 2(-1)^{u_1} u_1(3u_1 + 2) + (-1)^{u_2}(6u_2^2 + 8u_2 - 3) \right),
\]

where \(u_1 \equiv u_1(m) := \lfloor \sqrt{24m + 1} \rfloor / 6\) and \(u_2 \equiv u_2(m) := \lfloor (\sqrt{24m + 1} - 1) / 6 \rfloor\).

The first terms of the sequence, \(\{B_{n,m}\}_{m \geq 0}\), corresponding to the Lambert series over Euler’s phi function are given by

\[
\{B_{n,m}\}_{m \geq 0} = \{1, 1, 0, -1, -2, -2, -2, -1, 0, 1, 2, 3, 3, 3, 3, 3, 2, 1, 0, -1, -2, -3, \ldots \}\.
\]

3.3. The Möbius function. We similarly provide a computation of (6) to demonstrate the utility to our method in this special case:

\[
\begin{bmatrix}
\mu(1) \\
\mu(2) \\
\mu(3) \\
\mu(4) \\
\mu(5) \\
\mu(6) \\
\mu(7)
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 1 & 0 & 0 & 0 \\
4 & 3 & 2 & 1 & 1 & 0 & 0 \\
5 & 3 & 2 & 2 & 1 & 1 & 0 \\
10 & 7 & 5 & 3 & 2 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
-1 \\
-1 \\
0 \\
0 \\
1 \\
0
\end{bmatrix} =
\begin{bmatrix}
1 \\
-1 \\
-1 \\
0 \\
0 \\
1 \\
-1
\end{bmatrix}.
\]

In this case the vector, \((B_{g_f,m})\), is given by the formula

\[
B_{[n=1]_\delta,m} = [m = 0]_\delta + \sum_{b=\pm 1}^{\left\lfloor \sqrt{\frac{24m + 1}{6}} \right\rfloor} \sum_{k=1}^{\left\lfloor (\sqrt{24m + 1} - 1) / 6 \right\rfloor} (-1)^k [m + 1 - k(3k + b)/2 = 1]_\delta.
\]

The first terms of the sequence, \(\{B_{g_f,m}\}_{m \geq 0}\), for the Lambert series over the Möbius function are given by

\[
\{B_{[n=1]_\delta,m}\}_{m \geq 0} = \{1, -1, -1, 0, 0, 1, 0, 0, 1, 0, 0, 0, -1, 0, 0, -1, 0, 0, 0, 0, \ldots \}\.
\]
3.4. **Liouville’s lambda function.** Finally, we provide a computation of (6) to demonstrate the utility to our method in the case of the Liouville lambda function:

\[
\begin{bmatrix}
\lambda(1) \\
\lambda(2) \\
\lambda(3) \\
\lambda(4) \\
\lambda(5) \\
\lambda(6) \\
\lambda(7)
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 1 & 0 & 0 & 0 \\
4 & 3 & 2 & 1 & 1 & 0 & 0 \\
5 & 3 & 2 & 2 & 1 & 1 & 0 \\
10 & 7 & 5 & 3 & 2 & 1 & 1
\end{bmatrix} \begin{bmatrix}
1 \\
-1 \\
-1 \\
1 \\
-1 \\
0 \\
0
\end{bmatrix}.
\]

In this case the vector, \((B_{g_f, m})\), is given by the formula

\[
B_{[n \text{ is a positive square], } m} = \left[ \sqrt{m + 1} \in \mathbb{Z} \right] \delta - \sum_{b=\pm 1} \sum_{k=1}^{\lfloor 2(n+1)^2/6 \rfloor} (-1)^k \left[ \sqrt{m + 1 - k(3k + b)/2} \in \mathbb{Z} \right] \delta.
\]

The first few terms of the sequence, \(\{B_{g_f, m}\}_{m \geq 0}\), for the Lambert series over Liouville’s function are given by

\[
\{B_{[n = k^2], m} \}_{m \geq 0} = \{1, -1, -1, 1, -1, 0, 0, 1, 2, -1, 0, 0, -1, 1, 0, 0, -1, -1, -1, 0, \ldots \}.
\]

4. **Conclusions**

4.1. **Summary.** We have given proofs of several new recurrence relations and matrix equations for the sequences, \(f(n)\) and \(g_f(n)\), implicit to the Lambert series expansions defined in (2) where one of \(f(n)\) or \(g_f(n)\) is typically an interesting arithmetic function we wish to study. The key ingredients to the proofs of these results are the definitions of the matrices, \(A_n\), in Definition 1.1, and Euler’s pentagonal theorem applied to the partial sums of the left-hand-side series for \(L_f(q)\) defined by (2).

The special case examples from (3) for \(\sigma_1(n)\), \(\phi(n)\), \(\mu(n)\), and \(\lambda(n)\) given in Section 3 are easily extended to form related results for Lambert series of other special functions, such as those given for the logarithmic derivatives of the Jacobi theta functions, \(\vartheta_3(z, q)\), cited in the reference [3, §20.5(ii)]. There are also well-known Lambert series expansions involving von Mangoldt’s function, \(\Lambda(n)\), \(|\mu(n)|\), the number of distinct primes dividing \(n\), \(\omega(n)\), and Jordan’s totient function, \(J_i(n)\), which provide still other applications of our new results. To the best of our knowledge these results, and certainly the interpretations of their proofs, are new in the literature.

4.2. **Generalizations.** We can generalize these results to form analogous matrix equations and new recurrence relations for the sequences, \(\tilde{f}(n)\) and \(\tilde{g}(n)\), implicit to the expansions of generalized Lambert series of the form

\[
L_f(\alpha, \beta; a, b, c, d; q) := \sum_{n \geq 1} \tilde{f}(n) \alpha^n q^{(an+b)} / (1 - \beta q^{(an+b)}) = \sum_{m \geq 1} \tilde{g}(m) q^m,
\]

for some \(\alpha, \beta, a, b, c, d \in \mathbb{C}\) with \(\alpha, \beta, a, c, d \neq 0\) such that \(\max(|\alpha q^a|, |\beta q^c|) < 1\). In particular, if we know the forms of the coefficients of the power series expansions of the infinite \(q\)-Pochhammer products, \((\beta q^b; q^a)_\infty\), with respect to \(q\) (which may or may not
be obvious depending on the application), then we can extend the proof of Lemma 2.1 to form new proofs of analogous results for these generalized Lambert series expansions.

One immediate application of these generalized results is a Lambert series generating function for the sums of squares function, $r_2(n)$, given by [1, §17.10, Thm. 311] [3, §27.13(iv)]

$$\sum_{n \geq 1} \frac{4 \cdot (-1)^{n+1} q^{2n+1}}{1 - q^{2n+1}} = \sum_{m \geq 1} r_2(m) q^m, \quad (\tilde{f}, \tilde{g}) := ((-1)^{n+1}, r_2(n)).$$

However, we point out that the coefficients of the power series expansion of the $q$-Pochhammer symbol, $(q; q^2)_\infty$, in $q$ does not appear to have a known closed-form formula, only related $q$-series expansions proved in the references. Nonetheless, the study of the analogous results to those proved within this article corresponding to these generalized cases is an interesting new direction for more careful future study.

**References**

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[3] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark. *NIST Handbook of Mathematical Functions*. Cambridge University Press, 2010.