Diffusion Approximation for Transport Equations with Dissipative Drifts

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Abstract

We study stochastic differential equations (SDEs) with a small perturbation parameter. Under the dissipative condition on the drift coefficient and the local Lipschitz condition on the drift and diffusion coefficients we prove the existence and uniqueness result for the perturbed SDE, also the convergence result for the solution of the perturbed system to the solution of the unperturbed system when the perturbation parameter approaches zero. We consider the application of the above-mentioned results to the Cauchy problem and the transport equations.

Keywords: diffusion process, dissipative drift, local Lipschitz condition, perturbation parameter, Cauchy problem, transport equation

1 Introduction

We consider Markov processes $X_t^\varepsilon$ which arise from small random perturbations of dynamical systems, imposing specific conditions on the coefficients of the diffusion process, i.e., the dissipativity and dissipativity for differences for the drift and the local Lipschitz condition for all coefficients. These kind of processes arise in different areas of natural sciences. The concept of dissipativity comes, in particular, from physics. Dissipative systems are systems which absorb more energy from the external world than they supply and such systems are contrasted with energy conserving systems like Hamiltonian systems. The dissipativity of dynamical systems as it is known in modern system and control community was introduced by Willems in [2].
Freidlin and Wentzell in their book [1] have developed the theory for random perturbations assuming that the coefficients satisfy Lipschitz condition and have a linear growth bound. They study the random perturbations by direct probabilistic methods and then deduce consequences concerning the corresponding problems for partial differential equations. They consider mainly schemes of random perturbations of the form

\[ \dot{X}_t^\epsilon = b(X_t^\epsilon, \epsilon \xi_t), \quad X_0^\epsilon = x \]

where \( \xi_t(\omega), t \geq 0 \) is a random process on a probability space with values in \( \mathbb{R}^l \), its trajectories are right continuous, bounded and have at most a finite number of points of discontinuity on every interval \([0, T], T < \infty\). At the points of discontinuity of \( \xi_t \), where as a rule, (1) can not be satisfied, is imposed the requirement of continuity of \( X_t^\epsilon \). Additionally \( \epsilon \) is a small numerical number and \( b(x, y) = (b^1(x, y), ..., b^r(x, y)) \), \( x \in \mathbb{R}^r, y \in \mathbb{R}^l \) is a vector field assumed to be jointly continuous in its variables. Let \( b(x, 0) = b(x) \), the random process \( X_t^\epsilon \) is considered as a result of small perturbations of the system

\[ \dot{x}_t = b(x_t), \quad x_0 = x \]

The following equation

\[ X_t^\epsilon = b(X_t^\epsilon) + \epsilon \sigma(X_t^\epsilon) \dot{w}_t, \quad X_0^\epsilon = x \]

can be considered as a special case of (1) with \( b(x, y) = b(x) + \sigma(x)y \). Here \( y \) is substituted by white noise process.

The precise meaning of (3) can be formulated in the language of stochastic integrals in the following way:

\[ X_t^\epsilon = x + \int_0^t b(X_s^\epsilon)ds + \epsilon \int_0^t \sigma(X_s^\epsilon)dw_s \]

Every solution of (3) is a Markov process (a diffusion process with drift vector \( b(x) \) and diffusion matrix \( \epsilon^2 \sigma(x) \sigma^*(x) \)).

Freidlin and Wentzell in their book [1] show that \( X_t^\epsilon \) converges to the solution \( x_t \) of the unperturbed system as \( \epsilon \to 0 \), moreover they discuss the application of this result to related partial differential equations. Particularly, they obtain results concerning the behaviour of solutions of boundary value problems as \( \epsilon \to 0 \) from the behaviour of \( X_t^\epsilon(w) \) as \( \epsilon \to 0 \).

In the theory of differential equations of parabolic type, much attention is devoted to the study of the behaviour, as \( \epsilon \to 0 \), for solutions of boundary value problems for equations of the form

\[ \frac{\partial v^\epsilon}{\partial t} = L^\epsilon v^\epsilon + c(x)v^\epsilon + g(x). \]

Here \( L^\epsilon \) is a differential operator with a small parameter at the derivatives of highest order:

\[ L^\epsilon = \frac{\epsilon^2}{2} \sum_{i,j=1}^{r} a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{r} b^i(x) \frac{\partial}{\partial x^i}. \]
Every operator $L^\epsilon$ (whose coefficients are assumed to be sufficiently regular) has an associated diffusion process $X^{\epsilon,x}_t$. This diffusion process can be given by means of the stochastic equation
\begin{equation}
\dot{X}^{\epsilon,x}_t = b(X^{\epsilon,x}_t) + \epsilon\sigma(X^{\epsilon,x}_t) \dot{\omega}_t, \quad X^{\epsilon,x}_0 = x, \quad (5)
\end{equation}
where $\sigma(x)\sigma^T(x) = (\sigma_{ij}(x))$, $b(x) = (b^1(x), \ldots, b^r(x))$.

In particular, they consider the Cauchy problem:
\begin{equation}
\frac{\partial v^\epsilon(t,x)}{\partial t} = L^\epsilon v^\epsilon(t,x) + c(x)v^\epsilon(t,x) + g(x), \quad v^\epsilon(0,x) = f(x), \quad (6)
\end{equation}
for $t > 0$, $x \in \mathbb{R}^r$ for $\epsilon > 0$ and together with it the problem for the first-order operator which is obtained for $\epsilon = 0$:
\begin{equation}
\frac{\partial v^0(t,x)}{\partial t} = L^0 v^0(t,x) + c(x)v^0(t,x) + g(x), \quad v^0(0,x) = f(x) \quad (7)
\end{equation}
A special case of Cauchy equation is the so called transport equation:
\begin{equation}
\frac{\partial v^\epsilon(t,x)}{\partial t} = L^\epsilon v^\epsilon(t,x), \quad v^\epsilon(0,x) = f(x),
\end{equation}
which equals (6) in the case that $g \equiv 0$ and $c \equiv 0$.

2 The model

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$ be the reference filtered probability space and $w$ is a given l-dimensional standard Brownian motion adapted to the defined filtration $(\mathcal{F}_t)_{t \in [0,T]}, 0 < T < +\infty$ being a finite horizon time. Here $\Omega$ is a nonempty set, which is interpreted as the space of elementary events. The second object, $\mathcal{F}$, is a $\sigma$-algebra of subsets of $\Omega$. Finally, $\mathbb{P}$ is a probability measure on the $\sigma$-algebra $\mathcal{F}$.

We consider
\begin{equation}
\dot{x}_t = b(x_t), \quad x_0 = x \quad (8)
\end{equation}
and the perturbed stochastic differential equation
\begin{equation}
\dot{X}^\epsilon_t = b(X^\epsilon_t) + \epsilon\sigma(X^\epsilon_t) \dot{\omega}_t, \quad X^\epsilon_0 = x \quad (9)
\end{equation}
in $\mathbb{R}^r$. Where $\epsilon$ is a small numerical number, $b(x) = (b^1(x), \ldots, b^r(x))$ is a vector field in $\mathbb{R}^r$ and $\sigma(x) = (\sigma^T_i(x))$ is a matrix having $l$ columns and $r$ rows. By a solution of this equation we understand a random process $X_t = X^\epsilon_t(w)$ which satisfies the relation
\begin{equation}
X^\epsilon_t = x + \int_0^t b(X^\epsilon_s)ds + \epsilon \int_0^t \sigma(X^\epsilon_s)dw_s,
\end{equation}
with probability 1 for every $t \in [0,T]$. We usually assume that the coefficients of our diffusion fulfill a Lipschitz condition and have a linear growth bound. Under those conditions
is proved that the solution to the stochastic differential equation exists and is unique. We modify the conditions on coefficients and prove that the existence and uniqueness result for the solution still holds (the proof of this result is based on the book by Gihman and Skorohod [3]) in one-dimensional case. For multi-dimensional situation it was analysed in classical book by Stroock and Varadhan [6]. We will assume that \( \sigma \) increases no faster than linearly and \( b \) satisfies dissipativity, the coefficients of (9) satisfy a local Lipschitz condition: for some \( K \)

\[
< y, b(y) > + \sum_{i,j} [\sigma_i^j(y)]^2 \leq K^2 (1 + |y|^2);
\]

for each \( N \) there exists an \( L_N \) for which

\[
\sum_i |b^i(y) - b^i(z)| + \sum_{i,j} |\sigma^i_j(y) - \sigma^i_j(z)| \leq L_N |y - z|
\]

with \( |y| \leq N, |z| \leq N. \)

After proving the existence and uniqueness result, we will show that the zeroth approximation for the process (9) with dissipative drift and locally Lipschitz coefficients holds, i.e the solution of (9) \( X^\epsilon_t \) converges to the solution of (8) \( x_t \) as \( \epsilon \to 0 \). The last approximation will be used to show that the solution to the Cauchy problem for \( \epsilon > 0 \) converges to the solution for \( \epsilon = 0 \) with weaker conditions, this convergence holds also for Transport equation.

Before stating and proving the main results, we would like to state a Gronwall-Lemma which is often used in the proofs.

**Lemma 2.1. [Gronwall]** Let \( m(t), t \in [0, T], \) be a nonnegative function satisfying the relation

\[
m(t) \leq C + \alpha \int_0^t m(s) ds, \ t \in [0, T]
\]

with \( C, \alpha \geq 0. \)

Then \( m(t) \leq Ce^{\alpha t}, \) for \( t \in [0, T]. \)

### 3 Main results

#### 3.1 Existence and Uniqueness of a Solution

We aim to show that under the weaker conditions on the coefficients, i.e. dissipativity for the drift and the local Lipschitz condition for all the coefficients, (9) has a solution and the solution is unique. This fact may be find in [3] but we will include the proof because certain steps in this proof we will use later. To prove the existence and uniqueness result we will need the following theorem:
Theorem 3.1. Assume that the coefficients $b_{i}(x)$, $b_{2}(x)$, $\sigma_{1}(x)$, $\sigma_{2}(x)$ of the equations

$$\dot{X}_{i,j}^{\ast} = b_{i}(X_{i,j}^{\ast}) + e\sigma_{i}(X_{i,j}^{\ast})\dot{\omega}_{t}, \ i = 1, 2$$

(11)
satisfy Lipschitz condition and linear growth condition, i.e., there exists a constant $K$ such that for $t \in [0, T]$, $x, y \in \mathbb{R}^{r}$, $i = 1, 2$

$$|b_{i}(x) - b_{i}(y)| + |\sigma_{i}(x) - \sigma_{i}(y)| \leq K|x - y|,$$

and that for some $N > 0$ with $|x| \leq N$ for all $j \in \mathbb{N}_{0} : 0 \leq j \leq r$, $b_{1}(x) = b_{2}(x)$ and $\sigma_{1}(x) = \sigma_{2}(x)$.

If $X_{i,1}^{\ast}$ and $X_{i,2}^{\ast}$ are solutions of (11) with the same initial condition $X_{0,1}^{\ast} = X_{0,2}^{\ast} = x$,

$M[x^{2}] < \infty$, and $\tau_{i}$ is the largest $t$ for which $\sup_{0 \leq s \leq t, 0 \leq j \leq r}|X_{i,j}^{\ast}| \leq N$, then $P\{\tau_{1} = \tau_{2}\} = 1$ and

$$P\{\sup_{0 \leq s \leq \tau_{1}}|X_{i,1}^{\ast} - X_{i,2}^{\ast}| = 0\} = 1.$$

Proof. Define $\gamma_{1}(t) := 1$, if $\sup_{0 \leq s \leq t, 0 \leq j \leq r}|X_{i,j}^{\ast}| \leq N$, and $\gamma_{1}(t) := 0$, if $\sup_{0 \leq s \leq t, 0 \leq j \leq r}|X_{i,j}^{\ast}| > N$. Then we get

$$\gamma_{1}(t) \sum_{j}^{t}|X_{i,1}^{\ast} - X_{i,2}^{\ast}| = \gamma_{1}(t) \int_{0}^{t} \sum_{j}^{t} [b_{i}^{1}(X_{s,j}^{\ast}) - b_{i}^{2}(X_{s,j}^{\ast})]ds$$

$$+ \gamma_{1}(t) \int_{0}^{t} \sum_{i,j}^{t} [\sigma_{i,1}^{1}(X_{s,j}^{\ast}) - \sigma_{i,2}^{1}(X_{s,j}^{\ast})]d\omega_{s}$$

$$= \gamma_{1}(t) \int_{0}^{t} \sum_{j}^{t} [b_{i}^{1}(X_{s,j}^{\ast}) - b_{i}^{2}(X_{s,j}^{\ast})]ds$$

$$+ \gamma_{1}(t) \int_{0}^{t} \sum_{i,j}^{t} [\sigma_{i,1}^{1}(X_{s,j}^{\ast}) - \sigma_{i,2}^{1}(X_{s,j}^{\ast})]d\omega_{s}$$

$$+ \gamma_{1}(t) \int_{0}^{t} \sum_{i,j}^{t} [b_{i}^{1}(X_{s,j}^{\ast}) - b_{i}^{2}(X_{s,j}^{\ast})]ds$$

$$+ \gamma_{1}(t) \int_{0}^{t} \sum_{i,j}^{t} [\sigma_{i,2}^{1}(X_{s,j}^{\ast}) - \sigma_{i,2}^{2}(X_{s,j}^{\ast})]d\omega_{s}.$$
satisfy the conditions (9). Then

\[ \text{Theorem 3.2.} \]

Let the coefficients of \( \sigma \) \( \gamma \) \( P \) inside the brackets. Taking the expectation and then using the Lipschitz condition and Cauchy-Schwarz-inequality, we can show that for \( \tau \), we can establish

\[ \gamma(t)[\sum_{j} X_{t,1}^{i,j} - X_{t,2}^{i,j}]^2 \leq 2\gamma(t) \left[ \int_{0}^{t} \sum_{j} [b_j'(X_{s,1}) - b_j'(X_{s,2})]ds \right]^2 \]

\[ + 2\gamma(t)\epsilon^2 \left[ \int_{0}^{t} \sum_{j} [\sigma_{i,j}'(X_{s,1}) - \sigma_{i,j}'(X_{s,2})]dw_j \right]^2. \]

Taking into account that \( \gamma(t) = 1 \) implies \( \gamma(s) = 1 \) for \( s \leq t \) we can write the \( \gamma_1(s)'s \) inside the brackets. Taking the expectation and then using the Lipschitz condition and Cauchy-Schwarz-inequality, we can show that for \( \epsilon \leq 1 \) there exists a constant \( L \) for which

\[ M \left[ \gamma(t)[\sum_{j} X_{t,1}^{i,j} - X_{t,2}^{i,j}]^2 \right] \leq M \left[ 4\int_{0}^{t} \gamma(s)K[|X_{s,1} - X_{s,2}|^2]ds \right] \]

\[ \leq 4K^2t \int_{0}^{t} M[\gamma(s)\sum_{j} X_{s,1}^{i,j} - X_{s,2}^{i,j}]^2]ds \]

\[ = L \int_{0}^{t} M[\gamma(s)\sum_{j} X_{s,1}^{i,j} - X_{s,2}^{i,j}]^2]ds. \]

Now we can use Lemma 2.1 for which \( C=0 \). It follows that

\[ M \left[ \gamma(t)[\sum_{j} X_{t,1}^{i,j} - X_{t,2}^{i,j}]^2 \right] = 0. \]

Considering the continuity of \( X_{t,1} \) and \( X_{t,2} \) we can establish

\[ P\{ \sup_{0 \leq s \leq T} \gamma(t)[\sum_{j} X_{t,1}^{i,j} - X_{t,2}^{i,j}]^2 = 0 \} = P\{ \sup_{0 \leq s \leq T} \gamma(t)[X_{t,1} - X_{t,2}]^2 = 0 \} = 1 \]

On the interval \([0, \tau]\) the processes \( X_{t,1} \) and \( X_{t,2} \) coincide with probability 1. Hence \( P\{\tau_2 \geq \tau_1\} = 1 \). Interchanging the indices 1 and 2 in the proof of the theorem, we can show analogously that \( P\{\tau_1 \geq \tau_2\} = 1 \). □

**Theorem 3.2.** Let the coefficients of (9) be defined and measurable for \( t \in [0, 1] \), and satisfy the conditions

1. For some \( K \)

\[ <y, b(y)> + \sum_{i,j} [\sigma_{i,j}'(y)]^2 \leq K^2(1 + |y|^2); \quad (12) \]

2. for each \( N \) there exists an \( L_N \) for which

\[ \sum_{i} |b_i'(y) - b_i'(z)| + \sum_{i,j} [\sigma_{i,j}'(y) - \sigma_{i,j}'(z)] \leq L_N|y - z| \quad (13) \]

with \( |y| \leq N, |z| \leq N \).

Then (9) has a unique solution in the sense that for two solutions \( X_{t,1} \) and \( X_{t,2} \)

\[ P\{ \sup_{0 \leq s \leq T} |X_{t,1} - X_{t,2}| = 0 \} = 1. \]
Proof. We will first start by showing the existence and afterwards we move to the uniqueness of the solution. Define $x_N^i$ (the i-th component of the vector $x_N$) as $x_N^i = x^i$ for $|x^i| \leq N$ and $x_N^i = N\text{sign}(x^i)$ for $|x^i| > N$, $b_N(y) = b^i(y)$ for $|b^i(y)| \leq N$ and $b_N(y) = N\text{sign}(b^i(y))$ for $|b^i(y)| > N$, $\sigma_{j,N}^i(y) = \sigma_j^i(y)$ for $|\sigma_j^i| \leq N$ and $\sigma_{j,N}^i(y) = N\text{sign}(\sigma_j^i(y))$ for $|\sigma_j^i| > N$.

By $X_{t,N}^i$ we denote the solution of

$$\dot{X}_{t,N}^i = b_N(X_{t,N}^i) + \epsilon\sigma_N(X_{t,N}^i)\dot{w}_t, \quad X_{t,N}^i = x_N.$$  \hspace{1cm} (14)

For this equation all conditions for existence are given, because we have a growth bound depending on $N$ and for the coefficients we also have a global Lipschitz condition.

Let $\tau_N$ be the largest value of $t$ for which $\sup_{0 \leq s \leq t} |X_{t,N}^i| \leq N$. Let $N' > N$. Since $b_N(y) = b_{N'}(y)$ and $\sigma_N(y) = \sigma_{N'}(y)$ for all $|b(y)| \leq N$, $|\sigma_{j,N}| \leq N$, we can now apply Theorem 3.1 to obtain $X_{t,N}^i = X_{t,N'}^i$, with probability 1 for $t \in [0, \tau_N]$.

Hence for $N' > N$:

$$P\{ \sup_{0 \leq t \leq T} |X_{t,N}^i - X_{t,N'}^i| > 0 \} \leq P\{ \tau_N > T \} = P\{ \sup_{0 \leq t \leq T} |X_{t,N}^i| > N \}.$$  

If we can show that the probability on the right hand side converges to zero for $N \to \infty$, then it will clearly follow that $X_{t,N}^i$ converges uniformly with probability 1 to some limit $X_t^i$ as $N \to \infty$.

Going to the limit in

$$X_{t,N}^i = x_N + \int_0^t b_N(X_{s,N}^i)ds + \epsilon \int_0^t \sigma_N(X_{s,N}^i)dw_s,$$

we convince ourselves that $X_t^i$ is equal with probability 1 to the continuous solution of (9).

So to finish the proof of the existence of a solution it remains to show that

$$\lim_{N \to \infty} P\{ \sup_{0 \leq t \leq T} |X_{t,N}^i| > N \} = 0.$$  \hspace{1cm} (15)

To do this we first define the function $\psi(y) = \frac{1}{1+|y|^2}$ and then we use the Ito formula. We obtain

$$M[|X_{t,N}^i|^2\psi(x_N)] - M[|x_N|^2\psi(x_N)]
= M\left[ \int_0^t 2\psi(x_N) < X_{s,N}^i, b(X_{s,N}^i) > + \epsilon^2\psi(x_N) \sum_{k=1}^r \sum_{i=1}^l |\sigma_{k,N}^i(X_{s,N}^i)|^2 dt \right]
\leq M\left[ \psi(x_N) \int_0^t 2K^2(1 + |X_{s,N}^i|^2) + \epsilon^2\psi(x_N)K^2(1 + |X_{s,N}^i|^2) dt \right]
\leq \psi(x_N)(2K^2 t + \epsilon^2 K^2 t) + (2K^2 + \epsilon^2 K^2) \int_0^t M[\psi(x_N)|X_{s,N}^i|^2] dt$$

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By having that we can use Lemma 2.1 to get

\[ M[\psi(x_N)|X^i_{t,N}|^2] \leq [\psi(x_N)(2K^2 t + c^2K^2 + |x_N|^2 \psi(x_N))e^{(2K^2 + c^2K^2)t)]. \]

Which means we have

\[ M[\psi(x_N) \sup_{0 \leq t \leq T} |X^i_{t,N}|^2] \leq C_1 \]

where \( C_1 \) is independent of \( N \).

We can moreover write

\[ P\{ \sup_{0 \leq t \leq T} |X^i_{t,N}| > N \} = P\{ \sup_{0 \leq t \leq T} |X^i_{t,N}|^2 > N^2 \psi(x_N) \} \]

\[ \leq P\{ \sup_{0 \leq t \leq T} |X^i_{t,N}|^2 > \delta N^2 \} + P\{ \psi(x_N) \leq \delta \} \]

\[ \leq \frac{C_1}{\delta N^2} + P\{ \psi(x_N) \leq \delta \}, \]

where the last inequality follows from the Chebychev inequality.

Consequently

\[ \lim_{N \to \infty} P\{ \sup_{0 \leq t \leq T} |X^i_{t,N}| > N \} \leq P\{ \psi(x_N) \leq \delta \}. \]

Since \( \delta \) is an arbitrary positive number and \( P\{ \psi(x_N) = 0 \} = 0 \), results from the preceding relation. This completes the proof of the existence of a solution to (9).

Next we want to prove the uniqueness of the solution. Let \( X^i_{t,1} \) and \( X^i_{t,2} \) be two solutions of (9). Denoting by \( \phi(t) \) the variable equal to 1 if \( \sup_{0 \leq s \leq t} |X^i_{s,1}| \leq N \) and \( \sup_{0 \leq s \leq t} |X^i_{s,2}| \leq N \) and equal to 0 otherwise. Using our second condition we can write

\[ M[|X^i_{t,1} - X^i_{t,2}|^2 \phi(t)] \leq 2M[\phi(t)(\int_0^t \sum_i |b'(X^i_{s,1} - b'(X^i_{s,2})|ds)^2] \]

\[ + 2M[\phi(t)(\int_0^t \sum_i |\sigma'j(X^i_{s,1}) - \sigma'j(X^i_{s,2})|dw_s)^2] \]

\[ \leq 2tM[\int_0^t \phi(s) \sum_i |b'(X^i_{s,1}) - b'(X^i_{s,2})|^2ds] \]

\[ + 2M[\int_0^t \phi(s) \sum_i |\sigma'j(X^i_{s,1}) - \sigma'j(X^i_{s,2})|^2ds] \]

\[ \leq (2T + 2)L_p^2 \int_0^t M(\phi(s)|X^i_{s,1} - X^i_{s,2}|^2ds). \]

Where we first used that \( a^2 + b^2 \geq 2ab \), then the Cauchy-Schwarz inequality and the properties of an Ito integral and afterwards the local Lipschitz continuity. Then we need to use Lemma 2.1 with \( C = 0 \) to get \( M[|X^i_{t,1} - X^i_{t,2}|^2 \phi(t)] = 0 \), which means

\[ P\{X^i_{t,1} \neq X^i_{t,2} \} \leq P\{ \sup_{0 \leq s \leq T} |X^i_{s,1}| > N \} + P\{ \sup_{0 \leq s \leq T} |X^i_{s,2}| > N \}. \]

This is because \( \phi(t) \) was zero for \( \sup_{0 \leq s \leq t} |X^i_{s,1}| > N \) or \( \sup_{0 \leq s \leq t} |X^i_{s,2}| > N \). From the
continuity of \( X_{t,1}^\epsilon \) and \( X_{t,2}^\epsilon \) follows their boundedness. Hence the probability on the right side of this inequality tend to zero as \( N \to \infty \), i.e., for all \( t \in [0,T] \): \( P\{X_{t,1}^\epsilon = X_{t,2}^\epsilon\} = 1 \) from which the uniqueness follows in the sense that \( P\{\sup_{0 \leq s \leq T} |X_{s,1}^\epsilon - X_{s,2}^\epsilon| = 0\} = 1 \).

3.2 Zeroth Order Approximation for Dissipative Case

After proving the existence and uniqueness of a solution to (9) with our conditions on coefficients we want to prove convergence of the solution of (9) \( X_t^\epsilon \) to the solution of (8) \( x_t \) as \( \epsilon \to 0 \) under dissipativity and dissipativity for differences for the drift vector and the local Lipschitz condition for all coefficients.

**Theorem 3.3.** Assume that the coefficients of (9) satisfy a local Lipschitz condition, \( \sigma \) increases no faster than linearly and \( b \) satisfies dissipativity and dissipativity for the differences:

1. For some \( K < y, b(y) < \sum_{i,j} [\sigma_i^j(y)]^2 \leq K^2(1 + |y|^2) \); (16)
2. \( < y - z, b(y) - b(z) > \leq K^2(1 + |y - z|^2) \); (17)

2. for each \( N \) there exists an \( L_N \) for which

\[
\sum_{i} |b_i^j(y) - b_i^j(z)| + \sum_{i,j} |\sigma_i^j(y) - \sigma_i^j(z)| \leq L_N |y - z| \tag{18}
\]

with \( |y| \leq N, |z| \leq N \).

Then for all \( t > 0 \) and \( \delta > 0 \) we have:

1. \( M|X_t^\epsilon - x_t| \leq \epsilon^2 a(t) \), and
2. \( \lim_{\epsilon \to 0} P\{\max_{0 \leq s \leq t} |X_s^\epsilon - x_s| > \delta\} = 0 \)

where \( a(t) \) is a monotone increasing function, which is expressed in terms of \( |x| \) and \( K \).

**Proof.** We start by showing that \( M|X_t^\epsilon|^2 \) is bounded uniformly in \( \epsilon \in [0,1] \). To show that we first apply Itô’s formula to get

\[
(1 + |X_t^\epsilon|^2) - (1 + |x|^2) = \sum_{k=1}^r \sum_{l=1}^I 2 \int_0^t |(X_s^\epsilon)^i| |\sigma_k^i(X_s^\epsilon)| dw_s^k + \int_0^t \left[ 2 < X_s^\epsilon, b(X_s^\epsilon) > \\
+ \epsilon^2 \sum_{k=1}^r \sum_{i,j=1, i \neq j}^I \sigma_k^i(X_s^\epsilon) \sigma_k^j(X_s^\epsilon) \\
+ \epsilon^2 \sum_{k=1}^r \sum_{i,j=1, i \neq j} \sigma_k^i(X_s^\epsilon) \sigma_k^j(X_s^\epsilon) \right] ds.
\]

Applying the mathematical expectation and adding \( (1 + |x|^2) \) on both sides we obtain:
\[ 1 + M|X^*_t|^2 = 1 + |x|^2 + 2 \int_0^t M <X^*_s, b(X^*_s)> ds + \epsilon^2 \int_0^t M \sum_{i,j} [\sigma^*_s(X^*_s)]^2 ds. \]

Using the Cauchy-Schwarz inequality, that \( \sigma^* \) in (9) increases no faster than linearly and the dissipativity for \( b \), the last relation implies the estimate

\[ \left| 1 + M|X^*_t|^2 \right| \leq 1 + |x|^2 + 2 \int_0^t M K^2 (1 + |X^*_s|^2) ds + \epsilon^2 \int_0^t M K^2 (1 + |X^*_s|^2) ds \]

\[ \leq 1 + |x|^2 + (2K^2 + \epsilon^2 K^2) \int_0^t (1 + M|X^*_s|^2) ds. \]

Next we use Lemma 2.1 we choose \( m(t) = 1 + M|X^*_t|^2, \ C = 1 + |x|^2 \) and \( \alpha = (2K^2 + \epsilon^2 K^2) \). By doing this we obtain

\[ 1 + M|X^*_t|^2 \leq (1 + |x|^2) \exp[(2K^2 + \epsilon^2 K^2)t]. \] (19)

By the inequality we proved that \( M|X^*_t|^2 \) is bounded uniformly in \( \epsilon \in [0,1] \). In the next step we want to use our result to prove that \( M|X^*_t - x_0|^2 \leq \epsilon^2 a(t) \). To do this we work through it very similarly to before.

We apply the Ito formula to the function \( |X^*_t - x_0|^2 \), which works the same way as it did with \( 1 + |X^*_t| \), just that the starting term vanishes, because \( X^*_0 = x = x_0 \). Next we apply the mathematical expectation on both sides of the equality to get

\[ M|X^*_t - x_0|^2 = 2 \int_0^t M <X^*_s - x_0, b(X^*_s) - b(x)> ds + \epsilon^2 \int_0^t M \sum_{i,j} [\sigma_s^*(X^*_s)]^2 ds. \]

In the proof of the existence we proved (15). Since \( X^*_{t,N} \) converges to \( X^*_t \) as \( N \to \infty \) we also know

\[ \lim_{N \to \infty} P \{ \sup_{0 \leq s \leq T} |X^*_{s,N}| > N \} = 0 \] (20)

and

\[ \lim_{N \to \infty} P \{ \sup_{0 \leq s \leq T} |x_s| > N \} = 0. \] (21)

From this follows that there exists an \( N \), such that

\[ P \{ \sup_{0 \leq s \leq T} |X^*_s| > N \} \leq \frac{\epsilon^2}{2} \] (22)

and

\[ P \{ \sup_{0 \leq s \leq T} |x_s| > N \} \leq \frac{\epsilon^2}{2}. \] (23)

In the following calculations we first split up our mathematical expectation in two different cases, then we use the Cauchy-Schwarz inequality and (16). Afterwards we apply
the local Lipschitz condition (18) and the dissipativity for the differences for $b$ (17). Then we estimate the probabilities we used in the inequality:

$$M|X_t^x - x_t|^2 = 2\int_0^t M < X_t^x - x_s, b(X_s^x) - b(x_s) > ds + \epsilon^2 \int_0^t M \sum_{i,j} |\sigma_{ij}(X_s^x)|^2 ds$$

$$\leq P\{ \max_{0 \leq s \leq T} |X_s^x|, \sup_{0 \leq s \leq T} |x_s| \leq N \}$$

$$2\int_0^t M \left[ |X_t^x - x_s|^2 \sum_i [b(X_t^x) - b(x_t^x)]^2 ds \right] \sup_{0 \leq s \leq T} |X_s^x|, \sup_{0 \leq s \leq T} |x_s| \leq N$$

$$+ P\{ \max_{0 \leq s \leq T} |X_s^x|, \sup_{0 \leq s \leq T} |x_s| \} > N \}$$

$$2\int_0^t M |X_t^x - x_s, b(X_s^x) - b(x_s) > | \sup_{0 \leq s \leq T} |X_s^x|, \sup_{0 \leq s \leq T} |x_s| > N] ds$$

$$+ \epsilon^2 K^2 \int_0^t (1 + M|X_t^x|^2) ds$$

$$\leq P\{ \max_{0 \leq s \leq T} |X_s^x|, \sup_{0 \leq s \leq T} |x_s| \} \leq N \}$$

$$2\int_0^t M \sqrt{|X_t^x - x_s|^2 L_{K^2}^2} |X_t^x - x_s|^2 ds$$

$$+ P\{ \max_{0 \leq s \leq T} |X_s^x|, \sup_{0 \leq s \leq T} |x_s| \} > N \}$$

$$2\int_0^t K^2 (1 + M|X_t^x - x_s|^2) ds$$

$$+ \epsilon^2 K^2 \int_0^t (1 + M|X_t^x|^2) ds$$

$$\leq 2L_N \int_0^t M|X_t^x - x_s|^2 ds$$

$$+ \{ P\{ \sup_{0 \leq s \leq T} |X_s^x| > N \} + P\{ \sup_{0 \leq s \leq T} |x_s| > N \} \} [2K^2 t + 2K^2 \int_0^t M|X_t^x - x_s|^2 ds]$$

$$+ \epsilon^2 K^2 \int_0^t (1 + M|X_t^x|^2) ds$$

$$\leq 2L_N \int_0^t M|X_t^x - x_s|^2 ds + \epsilon^2 2K^2 t + \epsilon^2 2K^2 \int_0^t M|X_t^x - x_s|^2 ds$$

$$+ \epsilon^2 K^2 \int_0^t (1 + M|X_t^x|^2) ds$$

$$\leq (2L_N + \epsilon^2 2K^2) \int_0^t M|X_t^x - x_s|^2 ds + \epsilon^2 2K^2 t + \epsilon^2 2K^2 \int_0^t (1 + M|X_t^x|^2) ds.$$
shev inequality that says that $P\{\xi(\omega) \geq a\} \leq \frac{M f(t)}{f(a)}$. By setting $\xi(\omega) = \max_{0 \leq s \leq t} |X_s^\varepsilon - x_s|$, $a = \delta$, $f(x) = x^2$ and applying the first assertion of the theorem we obtain

$$P\{\max_{0 \leq s \leq t} |X_s^\varepsilon - x_s| > \delta\} \leq \frac{M\max_{0 \leq s \leq t} |X_s^\varepsilon - x_s|^2}{\delta^2} \leq \frac{\varepsilon^2 a(t)}{\delta^2}.$$  

(24)

Taking limits on both sides in (24), we get

$$\lim_{\varepsilon \to 0} P\{\max_{0 \leq s \leq t} |X_s^\varepsilon - x_s| > \delta\} \leq \lim_{\varepsilon \to 0} \frac{\varepsilon^2 a(t)}{\delta^2} = 0$$

$\square$

3.3 Parabolic Differential equations with a Small Parameter: Cauchy Problem, Transport Equation

We aim to obtain results concerning the behavior of solution $s$ of Cauchy problem as $\varepsilon \to 0$ from the behavior of $X_t^\varepsilon(w)$ as $\varepsilon \to 0$. In the preceding section we have obtained a result concerning the behavior of solutions $X_t^\varepsilon(w)$ as $\varepsilon \to 0$, which will be used in the present section. We consider the Cauchy problem:

$$\frac{\partial v^\varepsilon(t, x)}{\partial t} = L^\varepsilon v^\varepsilon(t, x) + c(x)v^\varepsilon(t, x) + g(x), \quad v^\varepsilon(0, x) = f(x),$$

(25)

$t > 0$, $x \in \mathbb{R}^r$ for $\varepsilon > 0$ and together with it the problem for the first-order operator which is obtained for $\varepsilon = 0$:

$$\frac{\partial v^0(t, x)}{\partial t} = L^0 v^0(t, x) + c(x)v^0(t, x) + g(x), \quad v^0(0, x) = f(x).$$

(26)

Where $L^\varepsilon$ is a differential operator with a small parameter at the derivatives of highest order:

$$L^\varepsilon = \varepsilon^2 \sum_{i,j=1}^r a^{ij}(x) \frac{\partial^2}{\partial x^i x^j} + \sum_{i=1}^r b^i(x) \frac{\partial}{\partial x^i}.$$  

Every operator $L^\varepsilon$ (whose coefficients are assumed to be sufficiently regular) has an associated diffusion process $X_t^\varepsilon$. This diffusion process can be given by means of the stochastic equation

$$\dot{X}_t^\varepsilon = b(X_t^\varepsilon, x) + \varepsilon \sigma(X_t^\varepsilon, x) \dot{\omega}_t, \quad X_0^\varepsilon = x,$$

(27)

where $\sigma(x)\sigma^*(x) = (a^{ij}(x))$, $b(x) = (b^1(x), ..., b^r(x))$.

We assume the following conditions are satisfied:

1. the function $c(x)$ is uniformly continuous and bounded for $x \in \mathbb{R}^r$;

2. the coefficients of $L^\varepsilon$ satisfy a local Lipschitz condition, $b$ satisfies dissipativity and
dissipativity for the differences;
3. \( k^{-2} \sum \lambda_i^2 \leq \sum_{i,j=1}^r a^{ij}(x) \lambda_i \lambda_j \leq k^2 \sum \lambda_i^2 \) for any real \( \lambda_1, \lambda_2, \ldots, \lambda_r \)
and \( x \in \mathbb{R}^r \), where \( k^2 \) is a positive constant.

Under these conditions, the solutions of problems \((25)\) and \((26)\) exist and are unique.

Having these conditions we obtain the following result:

**Theorem 3.4.** If conditions (1)-(3) are satisfied, then the limit \( \lim_{\epsilon \to 0} v^\epsilon(t, x) = v^0(t, x) \)
exists for every bounded continuous initial function \( f(x), x \in \mathbb{R}^r \).
The function \( v^0(t, x) \) is a solution of problem \((26)\).

**Proof.** If condition (3) is satisfied, then there exists a matrix \( \sigma(x) \) with entries satisfying a local Lipschitz condition for which \( \sigma(x) \sigma^*(x) = (a^{ij}(x)) \). The solution of \((25)\) can be represented in the following way ([1] Chap.1, Sec.5):

\[
v^\epsilon(t, x) = M[f(X_t^\epsilon, x)] + M[g(X_0^\epsilon, x)] = f(X_0^\epsilon, x) + \int_0^t g(X_s^\epsilon, x) \exp\left\{\int_s^t c(X_u^\epsilon, x) \, du\right\} ds. \tag{28}
\]

This stays true for the changed conditions, because of the uniqueness of the solution.

From Theorem 3.3 follows the convergence of \( X_t^\epsilon, x \) to \( X_0^\epsilon, x \) in probability on the interval \([0, t]\) as \( \epsilon \to 0 \). Taking into account that there is a bounded continuous functional of \( X_t^\epsilon, x(\omega) \) under the sign of mathematical expectation in \((29)\), by the Lebesgue dominated convergence theorem, which we can use because the functional is bounded, we obtain

\[
\lim_{\epsilon \to 0} v^\epsilon(t, x) = \lim_{\epsilon \to 0} M[f(X_t^\epsilon, x)] + M[g(X_0^\epsilon, x)] = f(X_0^0, x) + \int_0^t g(X_s^0, x) \exp\left\{\int_s^t c(X_u^0, x) \, du\right\} ds.
\]

The function on the right side of the equality is a solution of \((26)\), this finishes the proof.

The special case is when \( c(x) \equiv g(x) \equiv 0 \), which gives us the Transport equation

\[
\frac{\partial v^\epsilon(t, x)}{\partial t} = L^*v^\epsilon(t, x), \quad v^\epsilon(0, x) = f(x),
\]

the solution of the transport equation can be written in the following form

\[
v^\epsilon(t, x) = M[f(X_t^\epsilon, x)]. \tag{29}
\]
As in the case of Cauchy problem passing to the limit when \( \epsilon \to 0 \) we get

\[
\lim_{\epsilon \to 0} v^\epsilon(t,x) = v^0(t,x)
\]

where \( v^0(t,x) \) is the solution of

\[
\frac{\partial v^0(t,x)}{\partial t} = L^0 v^0(t,x), \quad v^0(0,x) = f(x).
\]

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