Uniqueness of $p(f)$ and $P[f]$

Kuldeep Singh Charak$^1$, Banarsi Lal$^2$

Department of Mathematics, University of Jammu, Jammu-180 006, INDIA.
1 E-mail: kscharak7@rediffmail.com
2 E-mail: banarsiverma644@gmail.com

Abstract
Let $f$ be a non-constant meromorphic function, $a(\neq 0, \infty)$ be a meromorphic function satisfying $T(r,a) = o(T(r,f))$ as $r \to \infty$, and $p(z)$ be a polynomial of degree $n \geq 1$ with $p(0) = 0$. Let $P[f]$ be a non-constant differential polynomial of $f$. Under certain essential conditions, we prove the uniqueness of $p(f)$ and $P[f]$ when $p(f)$ and $P[f]$ share $a$ with weight $l \geq 0$. Our result generalizes the results due to Zang and Lu, Banerjee and Majumdar, Bhoosnurmath and Kabbur and answers a question of Zang and Lu.

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1 Introduction

Let $f$ and $g$ be two non-constant meromorphic functions and $k$ be a non-negative integer. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all $a$-points of $f$, where an $a$-point of multiplicity $m$ is counted $m$ times if $m < k$ and $k + 1$ times if $m > k$. If $E_k(a,f) = E_k(a,g)$, we say that $f$ and $g$ share the value $a$ with weight $m$ of order $k$.

We write “$f$ and $g$ share $(a,k)$” to mean that “$f$ and $g$ share the value $a$ with weight $k$”. Since $E_k(a,f) = E_k(a,g)$ implies $E_p(a,f) = E_p(a,g)$ for any integer $p(0 \leq p < k)$, clearly if $f$ and $g$ share $(a,k)$, then $f$ and $g$ share $(a,p)$, $0 \leq p < k$. Also we note that $f$ and $g$ share the value $a$ IM(ignoring multiplicity) or CM(counting multiplicity) if and only if $f$ and $g$ share $(a,0)$ or $(a,\infty)$, respectively.

A differential polynomial $P[f]$ of a non-constant meromorphic function $f$ is defined as

$$P[f] := \sum_{i=1}^{m} M_i[f],$$

where $M_i[f] = a_i \prod_{j=0}^{k}(f^{(i)})^{n_{ij}}$ with $n_{ij}$ as non-negative integers and $a_i(\neq 0)$ are meromorphic functions satisfying $T(r,a_i) = o(T(r, f))$ as $r \to \infty$. The numbers $\overline{d}(P) = \max_{1 \leq i \leq m} \sum_{j=0}^{k} n_{ij}$ and $\underline{d}(P) = \min_{1 \leq i \leq m} \sum_{j=0}^{k} n_{ij}$ are respectively called the degree and lower degree of $P[f]$. If $\overline{d}(P) = \underline{d}(P) = d$ (say), then we say that $P[f]$ is a homogeneous differential polynomial of degree $d$.

For notational purpose, let $f$ and $g$ share 1 IM, and let $z_0$ be a zero of $f - 1$ with multiplicity $q$ and a zero of $g - 1$ with multiplicity $p$. We denote by $N_E^1(r,1/(f-1))$, the counting function of the zeros of $f-1$ when $p = q = 1$. By $N_E^2(r,1/(f-1))$, we denote the counting function of the zeros of $f-1$ when $p = q \geq 2$ and by $N_L(r,1/(f-1))$, we denote the counting function of the zeros of $f-1$ when $p > q \geq 1$, each point in these counting functions is counted only once; similarly, the terms $N_E^1(r,1/(g-1))$, $N_E^2(r,1/(g-1))$ and $N_L(r,1/(g-1))$. Also, we denote by $N_L^{>k}(r,1/(g-1))$, the reduced counting function of those zeros of $g-1$ such that $p > q = k$, and similarly the term $N_L^{g>k}(r,1/(f-1))$.

Inspired by a uniqueness result due to Mues and Steinmetz [10]: “If $f$ is a non-constant entire function sharing two distinct values ignoring multiplicity with $f'$, then $f \equiv f'$”, the study of the uniqueness of $f$ and $f^{(k)}$, $f^n$ and $(f^{(m)})^{(k)}$, $f$ and $P[f]$ is carried out by numerous authors. For example, Zang and Lu [12] proved:

**Theorem A.** Let $k, n$ be the positive integers, $f$ be a non-constant meromorphic function, and $a(\neq 0, \infty)$ be a meromorphic function satisfying $T(r,a) = \frac{\overline{d}}{\underline{d}}$.
$o(T(r, f))$ as $r \to \infty$. If $f^n$ and $f^{(k)}$ share a IM and
\[(2k + 6)\Theta(\infty, f) + 4\Theta(0, f) + 2\delta_{2+k}(0, f) > 2k + 12 - n,
\]
or $f^n$ and $f^{(k)}$ share a CM and
\[(k + 3)\Theta(\infty, f) + 2\Theta(0, f) + \delta_{2+k}(0, f) > k + 6 - n,
\]
then $f^n \equiv f^{(k)}$.

In the same paper, T. Zhang and W. Lu asked the following question:

**Question 1:** What will happen if $f^n$ and $(f^{(k)})^m$ share a meromorphic function $a(\neq 0, \infty)$ satisfying $T(r, a) = o(T(r, f))$ as $r \to \infty$?

S.S. Bhoosnurmath and Kabbur [3] proved:

**Theorem B.** Let $f$ be a non-constant meromorphic function and $a(\neq 0, \infty)$ be a meromorphic function satisfying $T(r, a) = o(T(r, f))$ as $r \to \infty$. Let $P[f]$ be a non-constant differential polynomial of $f$. If $f$ and $P[f]$ share a IM and
\[(2Q + 6)\Theta(\infty, f) + (2 + 3\delta(P))\delta(0, f) > 2Q + 2\delta(P) + 7,
\]
or if $f$ and $P[f]$ share a CM and
\[3\Theta(\infty, f) + (\delta(P) + 1)\delta(0, f) > 4,
\]
then $f \equiv P[f]$.

Banerjee and Majumder [2] considered the weighted sharing of $f^n$ and $(f^m)^{(k)}$ and proved the following result:

**Theorem C.** Let $f$ be a non-constant meromorphic function, $k, n, m \in \mathbb{N}$ and $l$ be a non-negative integer. Suppose $a(\neq 0, \infty)$ be a meromorphic function satisfying $T(r, a) = o(T(r, f))$ as $r \to \infty$ such that $f^n$ and $(f^m)^{(k)}$ share $(a, l)$. If $l \geq 2$ and
\[(k + 3)\Theta(\infty, f) + (k + 4)\Theta(0, f) > 2k + 7 - n,
\]
or $l = 1$ and
\[\left( k + \frac{7}{2} \right)\Theta(\infty, f) + \left( k + \frac{9}{2} \right)\Theta(0, f) > 2k + 8 - n,
\]
or $l = 0$ and
\[(2k + 6)\Theta(\infty, f) + (2k + 7)\Theta(0, f) > 4k + 13 - n,
\]
then $f^n \equiv (f^m)^{(k)}$.

Motivated by such uniqueness investigations, it is rational to think about the problem in more general setting: Let $f$ be a non-constant meromorphic function,
Let $P[f]$ be a non-constant differential polynomial of $f$, $p(z)$ be a polynomial of degree $n \geq 1$ and $a(\neq 0, \infty)$ be a meromorphic function satisfying $T(r, a) = o(T(r, f))$ as $r \to \infty$. If $p(f)$ and $P[f]$ share $(a, l)$, $l \geq 0$, then is it true that $p(f) \equiv P[f]$?

Generally this is not true, but under certain essential conditions, we prove the following result:

**Theorem 1.1.** Let $f$ be a non-constant meromorphic function, $a(\neq 0, \infty)$ be a meromorphic function satisfying $T(r, a) = o(T(r, f))$ as $r \to \infty$, and $p(z)$ be a polynomial of degree $n \geq 1$ with $p(0) = 0$. Let $P[f]$ be a non-constant differential polynomial of $f$. Suppose $p(f)$ and $P[f]$ share $(a, l)$ with one of the following conditions:

(i) $l \geq 2$ and

\[(Q+3)\Theta(\infty, f) + 2n\Theta(0, p(f)) + \overline{d}(P)\delta(0, f) > Q + 3 + 2\overline{d}(P) - d(P) + n, \quad (1.1)\]

(ii) $l = 1$ and

\[\left( Q + \frac{7}{2} \right) \Theta(\infty, f) + \frac{5n}{2} \Theta(0, p(f)) + \overline{d}(P)\delta(0, f) > Q + \frac{7}{2} + 2\overline{d}(P) - d(P) + \frac{3n}{2}, \quad (1.2)\]

(iii) $l = 0$ and

\[(2Q + 6)\Theta(\infty, f) + 4n\Theta(0, p(f)) + 2\overline{d}(P)\delta(0, f) > 2Q + 6 + 4\overline{d}(P) - 2d(P) + 3n. \quad (1.3)\]

Then $p(f) \equiv P[f]$.

**Example 1.2.** Consider the function $f(z) = \cos \alpha z + 1 - 1/\alpha^4$, where $\alpha \neq 0, \pm 1, \pm i$ and $p(z) = z$. Then $p(f)$ and $P[f] \equiv f^{(iv)}$ share $(1, l)$, $l \geq 0$ and none of the inequalities (1.1), (1.2) and (1.3) is satisfied, and $p(f) \neq P[f]$. Thus conditions in Theorem 1.1 can not be removed.

**Remark 1.3.** Theorem 1.1 generalizes Theorem A, Theorem B, Theorem C (and also generalizes Theorem 1.1 and Theorem 1.2 of [2]) and provides an answer to a question of Zhang and Lu [12].

The main tool of our investigations in this paper is Nevanlinna value distribution theory [5].

## 2 Proof of the Main Result

We shall use the following results in the proof of our main result:

**Lemma 2.1.** [3] Let $f$ be a non-constant meromorphic function and $P[f]$ be a differential polynomial of $f$. Then

\[ m \left( r, \frac{P[f]}{f^{d(P)}} \right) \leq (\overline{d}(P) - d(P))m \left( r, \frac{1}{f} \right) + S(r, f), \quad (2.1) \]
$$N \left( r, \frac{P[f]}{f^{d(P)}} \right) \leq (d(P) - d(P)) N \left( r, \frac{1}{f} \right) + Q \left[ N(r, f) + N \left( r, \frac{1}{f} \right) \right] + S(r, f),$$

(2.2)

$$N \left( r, \frac{1}{P[f]} \right) \leq Q N(r, f) + (d(P) - d(P)) m \left( r, \frac{1}{f^{d(P)}} \right) + N \left( r, \frac{1}{f^{d(P)}} \right) + S(r, f),$$

(2.3)

where $Q = \max_{1 \leq i \leq m} \{ n_{i0} + n_{i1} + 2n_{i2} + \ldots + kn_{ik} \}$.

**Lemma 2.2.** Let $f$ and $g$ be two non-constant meromorphic functions.

(i) If $f$ and $g$ share $(1,0)$, then

$$N_L \left( r, \frac{1}{f - 1} \right) \leq N \left( r, \frac{1}{f} \right) + N(r, f) + S(r),$$

(2.4)

where $S(r) = o(T(r))$ as $r \to \infty$ with $T(r) = \max\{T(r, f); T(r, g)\}$.

(ii) If $f$ and $g$ share $(1,1)$, then

$$2N_L \left( r, \frac{1}{f - 1} \right) + 2N_L \left( r, \frac{1}{g - 1} \right) + N_E^2 \left( r, \frac{1}{f - 1} \right) - N_{f>2} \left( r, \frac{1}{g - 1} \right)$$

$$\leq N \left( r, \frac{1}{g - 1} \right) - N \left( r, \frac{1}{f - 1} \right).$$

(2.5)

**Proof of Theorem 1.1:** Let $F = p(f)/a$ and $G = P[f]/a$. Then

$$F - 1 = \frac{p(f) - a}{a} \text{ and } G - 1 = \frac{P[f] - a}{a}.$$  

(2.6)

Since $p(f)$ and $P[f]$ share $(a, l)$, it follows that $F$ and $G$ share $(1, l)$ except at the zeros and poles of $a$. Also note that

$$N(r, F) = N(r, f) + S(r, f) \text{ and } N(r, G) = N(r, f) + S(r, f).$$

Define

$$\psi = \left( \frac{F''}{F'} - \frac{2F'}{F - 1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G - 1} \right).$$

(2.7)

Claim: $\psi \equiv 0$.

Suppose on the contrary that $\psi \not\equiv 0$. Then from (2.7), we have

$$m(r, \psi) = S(r, f).$$

By the Second fundamental theorem of Nevanlinna, we have

$$T(r, F) + T(r, G) \leq 2N(r, f) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{F - 1} \right) + N \left( r, \frac{1}{G} \right)$$

$$+ N \left( r, \frac{1}{G - 1} \right) - N_0 \left( r, \frac{1}{F'} \right) - N_0 \left( r, \frac{1}{G'} \right) + S(r, f),$$

(2.8)

$$5$$
where \( N_0(r, 1/F') \) denotes the counting function of the zeros of \( F' \) which are not the zeros of \( F(F - 1) \) and \( N_0(r, 1/G') \) denotes the counting function of the zeros of \( G' \) which are not the zeros of \( G(G - 1) \).

**Case 1.** When \( l \geq 1 \).

Then from (2.7), we have,

\[
N_{E_1}^1 \left( r, \frac{1}{F - 1} \right) \leq N \left( r, \frac{1}{\psi} \right) + S(r, f) \\
\leq T(r, \psi) + S(r, f) \\
= N(r, \psi) + S(r, f) \\
\leq \overline{N}(r, F) + \overline{N}_2 \left( r, \frac{1}{F} \right) + \overline{N}_2 \left( r, \frac{1}{G} \right) + \overline{N}_L \left( r, \frac{1}{F - 1} \right) \\
+ \overline{N}_L \left( r, \frac{1}{G - 1} \right) + N_0 \left( r, \frac{1}{F'} \right) + N_0 \left( r, \frac{1}{G'} \right) + S(r, f).
\]

and so

\[
\overline{N} \left( r, \frac{1}{F - 1} \right) + \overline{N} \left( r, \frac{1}{G - 1} \right) = N_{E_1}^1 \left( r, \frac{1}{F - 1} \right) + N_{E_2}^2 \left( r, \frac{1}{F - 1} \right) + N_L \left( r, \frac{1}{F - 1} \right) \\
+ \overline{N}_L \left( r, \frac{1}{G - 1} \right) + \overline{N} \left( r, \frac{1}{G - 1} \right) + S(r, f) \\
\leq \overline{N}(r, F) + \overline{N}_2 \left( r, \frac{1}{F} \right) + \overline{N}_2 \left( r, \frac{1}{G} \right) + 2N_L \left( r, \frac{1}{F - 1} \right) \\
+ 2\overline{N}_L \left( r, \frac{1}{G - 1} \right) + \overline{N}_E \left( r, \frac{1}{F - 1} \right) + \overline{N} \left( r, \frac{1}{G - 1} \right) \\
+ N_0 \left( r, \frac{1}{F'} \right) + N_0 \left( r, \frac{1}{G'} \right) + S(r, f). \quad (2.9)
\]

**Subcase 1.1:** When \( l = 1 \).

In this case, we have

\[
\overline{N}_L \left( r, \frac{1}{F - 1} \right) \leq \frac{1}{2} N \left( r, \frac{1}{F'} | F \neq 0 \right) \leq \frac{1}{2} \overline{N}(r, F) + \frac{1}{2} \overline{N} \left( r, \frac{1}{F} \right), \quad (2.10)
\]

where \( N \left( r, \frac{1}{F'} | F \neq 0 \right) \) denotes the zeros of \( F' \), that are not the zeros of \( F \).
From (2.5) and (2.10), we have

\[
2N_L \left( r, \frac{1}{F-1} \right) + 2N_L \left( r, \frac{1}{G-1} \right) + N_F^2 \left( r, \frac{1}{F-1} \right) + N \left( r, \frac{1}{G-1} \right) \\
\leq N \left( r, \frac{1}{G-1} \right) + N_L \left( r, \frac{1}{F-1} \right) + S(r, f) \\
\leq N \left( r, \frac{1}{G-1} \right) + \frac{1}{2}N(r, F) + \frac{1}{2}N \left( r, \frac{1}{F} \right) + S(r, f) \\
\leq N \left( r, \frac{1}{G-1} \right) + \frac{1}{2}N(r, f) + \frac{1}{2}N \left( r, \frac{1}{p(f)} \right) + S(r, f).
\]

(2.11)

Thus, from (2.9) and (2.11), we have

\[
N \left( r, \frac{1}{F-1} \right) + N \left( r, \frac{1}{G-1} \right) \leq N(r, f) + N_2 \left( r, \frac{1}{F} \right) + N_2 \left( r, \frac{1}{G} \right) \\
+ \frac{1}{2}N(r, f) + \frac{1}{2}N \left( r, \frac{1}{p(f)} \right) + N \left( r, \frac{1}{G-1} \right) \\
+ N_0 \left( r, \frac{1}{F} \right) + N_0 \left( r, \frac{1}{G} \right) + S(r, f) \\
\leq N(r, f) + N_2 \left( r, \frac{1}{F} \right) + N_2 \left( r, \frac{1}{G} \right) \\
+ \frac{1}{2}N(r, f) + \frac{1}{2}N \left( r, \frac{1}{p(f)} \right) + T(r, G) \\
+ N_0 \left( r, \frac{1}{F} \right) + N_0 \left( r, \frac{1}{G} \right) + S(r, f).
\]

(2.12)

From (2.3), (2.8) and (2.12), we obtain

\[
T(r, F) \leq 3N(r, f) + \frac{1}{2}N \left( r, \frac{1}{F-1} \right) + \frac{1}{2}N \left( r, \frac{1}{G-1} \right) \\
+ \frac{1}{2}N \left( r, \frac{1}{p(f)} \right) + S(r, f) \\
\leq \frac{7}{2}N(r, f) + 2N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{G} \right) + \frac{1}{2}N \left( r, \frac{1}{p(f)} \right) + S(r, f) \\
\leq \frac{7}{2}N(r, f) + \frac{5}{2}N \left( r, \frac{1}{p(f)} \right) + N \left( r, \frac{1}{p(f)} \right) + S(r, f) \\
\leq \left( Q + \frac{7}{2} \right) N(r, f) + \frac{5}{2}N \left( r, \frac{1}{p(f)} \right) + (\bar{d}(P) - d(P))T(r, f) + \bar{d}(P)N \left( r, \frac{1}{F} \right) + S(r, f) \\
\leq \left[ \left( Q + \frac{7}{2} \right) \{ 1 - \Theta(\infty, f) \} + \frac{5n}{2} \{ 1 - \Theta(0, p(f)) \} + \bar{d}(P) \{ 1 - \delta(0, f) \} \right] T(r, f) \\
+ (\bar{d}(P) - d(P))T(r, f) + S(r, f).
\]
Thus from (2.9), we obtain
\[
nT(r, f) = T(r, F) + S(r, f)
\]
\[
\leq \left[ \left( Q + \frac{7}{2} \right) \{1 - \Theta(\infty, f)} + \frac{5n}{2} \{1 - \Theta(0, p(f))\} + \mathcal{A}(P)\{1 - \delta(0, f)\} \right] T(r, f)
\]
\[
+ (\mathcal{A}(P) - \mathcal{D}(P))T(r, f) + S(r, f).
\]

Thus
\[
\left[ \left( Q + \frac{7}{2} \right) \Theta(\infty, f) + \frac{5n}{2} \Theta(0, p(f)) + \mathcal{A}(P)\delta(0, f) \right] - \left( Q + \frac{7}{2} + 2\mathcal{A}(P) - \mathcal{D}(P) + \frac{3n}{2} \right) T(r, f) \leq S(r, f).
\]

That is,
\[
\left( Q + \frac{7}{2} \right) \Theta(\infty, f) + \frac{5n}{2} \Theta(0, p(f)) + \mathcal{A}(P)\delta(0, f) \leq Q + \frac{7}{2} + 2\mathcal{A}(P) - \mathcal{D}(P) + \frac{3n}{2},
\]
which violates \textbf{(1.2)}.

\textbf{Subcase 1.2:} When \( l \geq 2 \).
In this case, we have
\[
2\mathcal{N}_L \left( r, \frac{1}{F - 1} \right) + 2\mathcal{N}_E \left( r, \frac{1}{G - 1} \right) + \mathcal{N}_E^2 \left( r, \frac{1}{F - 1} \right) + \mathcal{N} \left( r, \frac{1}{G - 1} \right) \leq \mathcal{N} \left( r, \frac{1}{G - 1} \right) + S(r, f).
\]

Thus from (2.9), we obtain
\[
\mathcal{N} \left( r, \frac{1}{F - 1} \right) + \mathcal{N} \left( r, \frac{1}{G - 1} \right) \leq \mathcal{N}(r, f) + \mathcal{N}_E \left( r, \frac{1}{F} \right) + \mathcal{N}_E \left( r, \frac{1}{G} \right) + \mathcal{N} \left( r, \frac{1}{G - 1} \right)
\]
\[
+ N_0 \left( r, \frac{1}{F^2} \right) + N_0 \left( r, \frac{1}{G^2} \right) + S(r, f)
\]
\[
\leq \mathcal{N}(r, f) + \mathcal{N}_E \left( r, \frac{1}{F} \right) + \mathcal{N}_E \left( r, \frac{1}{G} \right) + T(r, G)
\]
\[
+ N_0 \left( r, \frac{1}{F^2} \right) + N_0 \left( r, \frac{1}{G^2} \right) + S(r, f). \quad (2.13)
\]

Now from (2.3), (2.8) and (2.13), we obtain
\[
T(r, F) \leq 3\mathcal{N}(r, f) + N \left( r, \frac{1}{F} \right) + \mathcal{N}_E \left( r, \frac{1}{F} \right) + \mathcal{N} \left( r, \frac{1}{G} \right) + \mathcal{N}_E \left( r, \frac{1}{G} \right) + S(r, f)
\]
\[
\leq 3\mathcal{N}(r, f) + 2\mathcal{N} \left( r, \frac{1}{F} \right) + \mathcal{N} \left( r, \frac{1}{G} \right) + S(r, f)
\]
\[
\leq 3\mathcal{N}(r, f) + 2\mathcal{N} \left( r, \frac{1}{p(f)} \right) + \mathcal{N} \left( r, \frac{1}{P(f)} \right) + S(r, f)
\]
\[
\leq (Q + 3)\mathcal{N}(r, f) + 2\mathcal{N} \left( r, \frac{1}{p(f)} \right) + (\mathcal{A}(P) - \mathcal{D}(P))T(r, f) + \mathcal{A}(P)N \left( r, \frac{1}{f} \right) + S(r, f)
\]
\[
\leq [(Q + 3)\{1 - \Theta(\infty, f)} + 2n\{1 - \Theta(0, p(f))\} + \mathcal{A}(P)\{1 - \delta(0, f)\}]T(r, f)
\]
\[
+ (\mathcal{A}(P) - \mathcal{D}(P))T(r, f) + S(r, f).
\]
\[ nT(r, f) = T(r, F) + S(r, f) \]
\[ \leq [(Q + 3)\{1 - \Theta(\infty, f)\} + 2n\{1 - \Theta(0, p(f))\} + \overline{d}(P)\{1 - \delta(0, f)\}]T(r, f) \]
\[ + (\overline{d}(P) - \underline{d}(P))T(r, f) + S(r, f). \]

Thus
\[ \{(Q + 3)\Theta(\infty, f) + 2n\Theta(0, p(f)) + \overline{d}(P)\delta(0, f)\} - \{(Q + 3 + 2\overline{d}(P) - \underline{d}(P) + n)\}T(r, f) \leq S(r, f). \]

That is,
\[ (Q + 3)\Theta(\infty, f) + 2n\Theta(0, p(f)) + \overline{d}(P)\delta(0, f) \leq Q + 3 + 2\overline{d}(P) - \underline{d}(P) + n, \]
which violates (1.1).

**Case 2.** When \( l = 0 \).

Then, we have
\[ N^1_E \left( r, \frac{1}{F - 1} \right) = N^1_E \left( r, \frac{1}{G - 1} \right) + S(r, f), \]
\[ N^2_E \left( r, \frac{1}{F - 1} \right) = N^2_E \left( r, \frac{1}{G - 1} \right) + S(r, f), \]
and also from (2.7), we have
\[ \overline{N} \left( r, \frac{1}{F - 1} \right) + \overline{N} \left( r, \frac{1}{G - 1} \right) \leq N^1_E \left( r, \frac{1}{F - 1} \right) + \overline{N} \left( r, \frac{1}{F - 1} \right) + S(r, f) \]
\[ + N \left( r, \frac{1}{G - 1} \right) + S(r, f) \]
\[ \leq N^1_E \left( r, \frac{1}{F - 1} \right) + \overline{N} \left( r, \frac{1}{F - 1} \right) + N \left( r, \frac{1}{G - 1} \right) + S(r, f) \]
\[ \leq \overline{N}(r, F) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{G} \right) + 2\overline{N} \left( r, \frac{1}{F - 1} \right) \]
\[ + \overline{N} \left( r, \frac{1}{G - 1} \right) + N \left( r, \frac{1}{G - 1} \right) + N_0 \left( r, \frac{1}{F'} \right) \]
\[ + N_0 \left( r, \frac{1}{G'} \right) + S(r, f). \]  
(2.14)
From (2.3), (2.4), (2.8) and (2.14), we obtain
\[ T(r, F) \leq 3\bar{N}(r, f) + \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{G}\right) \]
\[ + 2\bar{N}_L\left(r, \frac{1}{F-1}\right) + \bar{N}_L\left(r, \frac{1}{G-1}\right) + S(r, f) \]
\[ \leq 3\bar{N}(r, f) + 2\bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{G}\right) + 2\bar{N}\left(r, \frac{1}{F}\right) \]
\[ + 2\bar{N}(r, F) + \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}(r, G) + S(r, f) \]
\[ \leq 6\bar{N}(r, f) + 4\bar{N}\left(r, \frac{1}{F}\right) + 2\bar{N}\left(r, \frac{1}{G}\right) + S(r, f) \]
\[ \leq 6\bar{N}(r, f) + 4\bar{N}\left(r, \frac{1}{p(f)}\right) + 2\bar{N}\left(r, \frac{1}{P(f)}\right) + S(r, f) \]
\[ \leq (2Q + 6)\bar{N}(r, f) + 4\bar{N}\left(r, \frac{1}{p(f)}\right) + 2(\bar{d}(P) - \bar{d}(P))T(r, f) + 2\bar{d}(P)\bar{N}\left(r, \frac{1}{f}\right) + S(r, f) \]
\[ \leq [(2Q + 6)\{1 - \Theta(\infty, f)\} + 4n\{1 - \Theta(0, p(f))\} + 2\bar{d}(P)\{1 - \delta(0, f)\}]T(r, f) \]
\[ + 2(\bar{d}(P) - \bar{d}(P))T(r, f) + S(r, f). \]

That is,
\[ nT(r, f) = T(r, F) + S(r, f) \]
\[ \leq [(2Q + 6)\{1 - \Theta(\infty, f)\} + 4n\{1 - \Theta(0, p(f))\} + 2\bar{d}(P)\{1 - \delta(0, f)\}]T(r, f) \]
\[ + 2(\bar{d}(P) - \bar{d}(P))T(r, f) + S(r, f). \]

Thus
\[ [(2Q + 6)\Theta(\infty, f) + 4n\Theta(0, p(f)) + 2\bar{d}(P)\delta(0, f)] - \{2Q + 6 + 4\bar{d}(P) - 2\bar{d}(P) + 3n\}]T(r, f) \leq S(r, f). \]

That is,
\[ (2Q + 6)\Theta(\infty, f) + 4n\Theta(0, p(f)) + 2\bar{d}(P)\delta(0, f) \leq 2Q + 6 + 4\bar{d}(P) - 2\bar{d}(P) + 3n, \]

which violates (1.3).

This proves the claim and thus \( \psi \equiv 0 \). So (2.7) implies that
\[ \frac{F''}{F'} - \frac{2F'}{F - 1} = \frac{G''}{G'} - \frac{2G'}{G - 1}, \]
and so we obtain
\[ \frac{1}{F - 1} = \frac{C}{G - 1} + D, \]

(2.15)

where \( C \neq 0 \) and \( D \) are constants.
Here, the following three cases can arise:

**Case (i)**: When $D \neq 0$, $-1$. Rewriting \((2.15)\) as

$$
\frac{G - 1}{C} = \frac{F - 1}{D + 1 - DF},
$$

we have

$$
N(r, G) = N\left(r, \frac{1}{F - (D + 1)/D}\right).
$$

In this subcase, the Second fundamental theorem of Nevanlinna yields

\[
nT(r, f) = T(r, F) + S(r, f)
\]

\[
\leq N(r, F) + N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{F - (D + 1)/D}\right) + S(r, f)
\]

\[
\leq N(r, F) + N\left(r, \frac{1}{F}\right) + N(r, G) + S(r, f)
\]

\[
\leq 2N(r, f) + N\left(r, \frac{1}{p(f)}\right) + S(r, f)
\]

\[
= \left[2\{1 - \Theta(\infty, f)\} + n\{1 - \Theta(0, p(f))\}\right]T(r, f) + S(r, f).
\]

Thus

\[
\left[\left\{2\Theta(\infty, f) + n\Theta(0, p(f))\right\} - 2\right]T(r, f) \leq S(r, f).
\]

That is,

$$
2\Theta(\infty, f) + n\Theta(0, p(f)) \leq 2,
$$

which contradicts \(1.1, 1.2\) and \(1.3\).

**Case (ii)**: When $D = 0$. Then from \((2.15)\), we have

\[
G = CF - (C - 1).
\]

\[(2.16)\]

So if $C \neq 1$, then

\[
N\left(r, \frac{1}{G}\right) = N\left(r, \frac{1}{F - (C - 1)/C}\right).
\]
Now the Second fundamental theorem of Nevanlinna and (2.3) gives
\[ nT(r, f) = T(r, F) + S(r, f) \]
\[ \leq N(r, F) + \mathcal{N} \left( r, \frac{1}{F} \right) + \mathcal{N} \left( r, \frac{1}{F - (C - 1)/C} \right) + S(r, f) \]
\[ \leq N(r, F) + \mathcal{N} \left( r, \frac{1}{F} \right) + \mathcal{N} \left( r, \frac{1}{G} \right) + S(r, f) \]
\[ \leq N(r, F) + \mathcal{N} \left( r, \frac{1}{p(f)} \right) + \mathcal{N} \left( r, \frac{1}{P[f]} \right) + S(r, f) \]
\[ \leq N(r, F) + \mathcal{N} \left( r, \frac{1}{p(f)} \right) + (\mathcal{d}(P) - \mathcal{d}(P))m \left( r, \frac{1}{f} \right) \]
\[ + N \left( r, \frac{1}{\mathcal{d}(P)} \right) + S(r, f) \]
\[ \leq (Q + 1)N(r, f) + \mathcal{N} \left( r, \frac{1}{p(f)} \right) + (\mathcal{d}(P) - \mathcal{d}(P))T(r, f) \]
\[ + \mathcal{d}(P)N \left( r, \frac{1}{f} \right) + S(r, f) \]
\[ \leq [(Q + 1)\{1 - \Theta(\infty, f)\} + n\{1 - \Theta(0, p(f))\} + \mathcal{d}(P)\{1 - \delta(0, f)\}]T(r, f) \]
\[ + (\mathcal{d}(P) - \mathcal{d}(P))T(r, f) + S(r, f). \]

Thus
\[ [(Q + 1)\Theta(\infty, f) + n\Theta(0, p(f)) + \mathcal{d}(P)\delta(0, f)] - \{Q + 1 + 2\mathcal{d}(P) - \mathcal{d}(P)\}]T(r, f) \leq S(r, f). \]

That is,
\[ (Q + 1)\Theta(\infty, f) + n\Theta(0, p(f)) + \mathcal{d}(P)\delta(0, f) \leq Q + 1 + 2\mathcal{d}(P) - \mathcal{d}(P), \]

which contradicts (1.1), (1.2) and (1.3).

Thus, \( C = 1 \) and so in this case from (2.16), we obtain \( F \equiv G \) and so
\[ p(f) \equiv P[f]. \]

**Case (iii) :** When \( D = -1 \). Then from (2.15) we have
\[ \frac{1}{F - 1} = \frac{C}{G - 1} - 1. \]

So if \( C \neq -1 \), then
\[ \mathcal{N} \left( r, \frac{1}{G} \right) = \mathcal{N} \left( r, \frac{1}{F - C/(C + 1)} \right), \]

and as in the Subcase (ii), we find that
\[ nT(r, f) \leq (Q + 1)N(r, f) + \mathcal{N} \left( r, \frac{1}{p(f)} \right) + (\mathcal{d}(P) - \mathcal{d}(P))T(r, f) \]
\[ + \mathcal{d}(P)N \left( r, \frac{1}{f} \right) + S(r, f). \]
Thus
\[
[(Q+1)\Theta(\infty, f)+n\Theta(0, p(f))+\overline{d}(P)\delta(0, f)]-\{Q+1+2\overline{d}(P)-\overline{d}(P)\}]T(r, f) \leq S(r, f).
\]
That is,
\[
(Q + 1)\Theta(\infty, f) + n\Theta(0, p(f)) + \overline{d}(P)\delta(0, f) \leq Q + 1 + 2\overline{d}(P) - \overline{d}(P),
\]
which contradicts (1.1), (1.2) and (1.3).

Thus, \( C = -1 \) and so in this case from (2.1) we obtain \( FG \equiv 1 \) and so \( p(f)P[f] = a^2. \) Thus, in this case \( N(r, f) + N(r, 1/f) = S(r, f). \)

Now, by using (2.1) and (2.2), we have
\[
(n + \overline{d}(P))T(r, f) \leq T\left( r, \frac{a^2}{f^n + \overline{d}(P)} \right) + S(r, f)
\]
\[
\leq T\left( r, \left[ 1 + \frac{a_{n-1}}{f} + \ldots + \frac{a_1}{f^{n-1}} \right] \frac{P[f]}{\overline{d}(P)} \right) + S(r, f)
\]
\[
\leq (n - 1)T(r, f) + T\left( r, \frac{P[f]}{\overline{d}(P)} \right) + S(r, f)
\]
\[
= (n - 1)T(r, f) + m\left( r, \frac{P[f]}{\overline{d}(P)} \right) + N\left( r, \frac{P[f]}{\overline{d}(P)} \right) + S(r, f)
\]
\[
\leq (n - 1)T(r, f) + (\overline{d}(P) - \overline{d}(P))m\left( r, \frac{1}{f} \right) + (\overline{d}(P) - \overline{d}(P))N\left( r, \frac{1}{f} \right)
\]
\[
+ Q \left[ N(r, f) + N\left( r, \frac{1}{f} \right) \right] + S(r, f)
\]
\[
\leq (n - 1)T(r, f) + (\overline{d}(P) - \overline{d}(P))T(r, f) + S(r, f).
\]

Thus
\[
(1 + \overline{d}(P))T(r, f) \leq S(r, f),
\]
which is a contradiction.

\[\Box\]

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