Dynamical Equations of Spinning Particles: Feynman’s Proof

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ABSTRACT

In this letter, we discuss the extension of Feynman’s derivation of the equation of motion to the case of spinning particles. We show that a spinning particle interacts only with the electromagnetic and gravitational fields. In the absence of the electromagnetic interactions, we rederive Papapetrou’s equations for spinning particles in the background of the conformal gravity. We also find that the effect of spin coupled to non-constant electromagnetic fields leads to further corrections to the Lorentz force equations. Some discussions of these results are given at the end.

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The derivation of the dynamical equations of the spinning particles in external fields has attracted the interests of physicists for over fifty years.\textsuperscript{1–6} The goal in attacking this problem is to study the dynamical effects of the spin precession, the spin-spin interactions (the Stern-Gerlach effects) and the spin-orbits couplings for particles in external fields. The most notable results of this search are the Bargmann-Michel-Telegdi (BMT) equations\textsuperscript{2} and the Papapetrou equations.\textsuperscript{3} Based on what Frenkel suggested,\textsuperscript{1} Bargmann, Michel and Telegdi discussed the precession of spinning particles in external electromagnetic fields,\textsuperscript{2} while Papapetrou conjectured the dynamical equations of spinning particles in general relativity by considering a rotational mass-energy distribution in the limit of vanishing volume but with the angular momentum remaining finite.\textsuperscript{3} Still various discussions in looking for the dynamics of spinning particles remain active from different points of view\textsuperscript{5,6,12,4} and all of these are less direct or attractive when compared to Feynman’s rederivation of Lorentz force equations.\textsuperscript{7}

Since Dyson\textsuperscript{7} presented the Feynman’s proof of the homogeneous Maxwell equations and Lorentz force equation for a Newtonian particle, the generalizations to the case of the spinless particle (with and without the internal structure) in both special and general relativity have been studied.\textsuperscript{8–13} The conventional theories have not been completely rederived. In particular, Tanimura\textsuperscript{9} showed that the particle worldlines are not parametrized by the proper time of motion and the particle does not follow a geodesic. In this paper, we will take a further generalization to the case of spinning particles and reexamine closely the above mentioned features. We rederive both the Lorentz force equations and the Papapetrou equations in a simple and direct way. Specially, our approach offers a systematic study of both spin-charge and spin-gravitational coupling to all orders.
By postulating the Poisson brackets of the Newtonian variables of particle motion, Feynman, according to Dyson, obtained the homogeneous Maxwell equations and the Lorentz force equations. The key points in his proof are to use the associative conditions, namely the Jacobi identity of the brackets, and the so-called second Leibniz rule:

\[
\frac{d}{d\tau} [A, B] = \left[ \frac{dA}{d\tau}, B \right] + [A, \frac{dB}{d\tau}],
\]

(1)

where the \( \tau \) is a parameter of the particle trajectory. Note that using the Jacobi identity one may derive Eq. (1) if one assumes the existence of Hamiltonian evolution. Therefore, this procedure may be re-formulated in terms of the symplectic language, in which the symplectic two-form, by definition yields the particle’s Poisson brackets and their associated Jacobi identity. Moreover, the homogeneous Maxwell equations follow trivially from the closure of the symplectic two-form and the Lorentz force equations may be derived simply from one or two line computations. We shall use this new formulation of the Feynman’s approach to study spinning particles.

Since the spin degree of freedom arises from symmetry transformations of space-time, it is generally not possible to formulate the spin variables purely in terms of Newtonian coordinates of the particle motion. This difficulty prevents one from directly employing Feynman’s proof. However, in two spatial dimensions, the canonical structure of a spinning particle is explicitly known and the spin degree of freedom indeed can be purely represented by the Newtonian variables of the particle motion.

We shall begin with a discussion of a spinning charged particle in 2+1 dimensional flat space-time. In this paper, we assume the particle without internal
structure (extension to the case of internal degrees of freedom is straightforward).
For a spinning particle or an anyon with spin $-s$ in a 2+1 dimensional flat spacetime, with the metric $\eta_{ab} = \text{diag}(+-\cdots)$, the symplectic structure is given by

$$\omega = dx^a \wedge dp_a + \frac{1}{2} s f_{ab} dp^a \wedge dp^b + \frac{1}{2} \epsilon F_{ab} dx^a \wedge dx^b. \tag{2}$$

Where $p^a = m \dot{x}^a$, $p^2 = \eta_{ab} p^a p^b$ ($\epsilon_{012} = 1$), $f_{ab} = \epsilon_{abc} p^c / (p^2)^{3/2}$, and $x^a (a = 0, 1, 2)$ are the position variables of a particle (the overdot denotes the $\tau$-derivative and $m$ is the particle’s mass). The spin vector $S^a$ for the particles, as shown in Ref. [14, 16], is given by

$$S^a = -s \frac{p^a}{\sqrt{p^2}}. \tag{3}$$

The closure condition $d\omega = 0$ tells us that the antisymmetric tensor $F_{ab}$ is a function of $x^a$ only and satisfies the homogeneous Maxwell equations:

$$\partial_c F_{ab} + \partial_a F_{bc} + \partial_b F_{ca} = 0. \tag{4}$$

By definition, $\omega$ gives the Poisson brackets or the commutation rules as

$$[x^a, x^b] = is(\tilde{M}^{-1} f)^{ab},$$

$$[p^a, x^b] = i(M^{-1})^{ab},$$

$$[p^a, p^b] = i\epsilon (M^{-1} F)^{ab}, \tag{5}$$

where $M_{ab} = \eta_{ab} + es(Ff)_{ab} \equiv \eta_{ab} + esF_{ac}f^c_b$. $\tilde{M}$ denotes the transpose of $M$.

Using the second Leibniz rule (1) and Eqs. (5), we have,

$$\frac{d}{d\tau} (M^{-1})^{ab} = i^{-1} \left( \frac{1}{m} [p^a, x^b] + [\dot{p}^a, x^b] \right), \tag{6}$$

$$= \frac{1}{m} (M^{-1} F)^{ab} + i^{-1} [\dot{p}^a, x^b].$$

In principle, one may deduce from Eq. (6) the equation of motion of the spinning
particle for arbitrary value of spins and external field $F$. It is, of course, very
difficult to simplify these equations in general to obtain the desired equations. We
shall instead derive the equations of motion in terms of a series in powers of spin.
Since the $\tau$ derivative of $p^a$ is also hidden in the left hand side of Eq. (6), it is easy
to deduce the equations of motion from the first equation of (5) by applying the
Leibniz rule (1). Thus

$$ms\frac{d}{d\tau}(\widetilde{M}^{-1}f)^{ab} = i^{-1}\left([p^a, x^b] + [x^a, p^b]\right),$$  \hspace{1cm} (7)

$$= (M^{-1})^{ab} - (M^{-1})^{ba}.$$

We expand Eq. (7) into a power series of $s$ and keep only the lowest nontrivial
order:

$$\frac{df^{ab}}{d\tau} = \frac{e}{m} \left[ (Ff)^{ba} - (Ff)^{ab} \right] + es\frac{d}{d\tau}(fFf)^{ab}$$
$$+ \frac{e^2 s}{m} \left[ (Fff)^{ab} - (Fff)^{ba} \right] + O(s^2),$$  \hspace{1cm} (8)

which may be further simplified into

$$\frac{dY^a}{d\tau} = -\frac{e}{m} F^{ab} Y_b - \frac{es}{m} Y^a Y \cdot \dot{F} + O(s^2),$$  \hspace{1cm} (9)

where $Y^a = \frac{1}{2} \epsilon^{abc} f_{bc}$ and $F_a = \frac{1}{2} \epsilon_{abc} F^{bc}$. Thus, we have the equations of motion
for both the position and spin variables (3):

$$\frac{dp^a}{d\tau} + \frac{e}{m} F^{ab} p_b = 3\frac{p^a}{2m} \ln(p^2) + \frac{e}{m} p^a \frac{d}{d\tau} S^b(\partial^c F_b)p_c + O(s^2)$$
$$= -\frac{e}{2m} p^a \frac{d}{d\tau} S^b(\partial^c F_b)p_c + O(s^2),$$  \hspace{1cm} (10)

$$\frac{dS^a}{d\tau} = -\frac{e}{m} F^{ab} S_b + O(s^2).$$  \hspace{1cm} (11)

Clearly, Eq. (11) is the Bargmann-Michel-Telegdi equations in 2+1 dimensions.\textsuperscript{2,16}

Since $\partial F$ is always coupled to the spin, $\partial F$ term will not give any contributions
to order \(s^2\) to the equations of motion for the spin. We will see this again below. However, the equations (10) are physically wrong although it follows from our starting point. First of all, they are not the Hamiltonian equations, because the \(\tau\)-evolution is not generated by the \(\tau\)-Hamiltonian, which is proportional to the first integral of motion of Eqs. (10), i.e. \(A = p^2 + eS \cdot F\). Secondly, Eqs. (10) are not the typical Hamiltonian equations of motion because of explicit \(\tau\)-derivative appearing in the right hand side of the equations. Alternatively speaking, using the rules in Eq. (5) we cannot derive Eqs. (10) and their counterpart \(m\dot{x}^a = p^a\) as the Heisenberg equations. For example, \(\dot{x}^a \neq i^{-1}[x^a, \zeta A]\) for any constant \(\zeta\). This apparent inconsistency arises as the consequence of the spin coupled to the gradient of the external electromagnetic field, although it is perfectly consistent for a slowly varying field \(F\), as we will demonstrated below.

For a slowly varying field \(F\), Eqs. (11) remain unchanged and Eqs. (10) yield:

\[
\frac{d}{d\tau} p^2 + \mathcal{O}(\partial F, s^2) = 0,
\frac{dp^a}{d\tau} = -\frac{e}{m} F^{ab} p_b + \mathcal{O}(\partial F, s^2),
\]

(12)
i.e. we have the (correct) worldline-equation and the Lorentz force equations (upto possible terms which depend on gradients of \(F\)). As shown in Ref. [16], the \(\tau\)-Hamiltonian \(G\) which generates \(\tau\)-evolution for this system can be obtained from (2) and (12):

\[
G = -\frac{1}{2m} \left( p^2 - 2eS \cdot F - m^2 \right) + \mathcal{O}(s^2).
\]

(13)

Notice that Eq. (13) is not the first integral of motion of (12), but rather is that of a similar equation:\[6,17\]

\[
\frac{dp^a}{d\tau} + \frac{e}{m} F^{ab} p_b = \frac{e}{m} S^{b} \partial^a F_b + \mathcal{O}(s^2).
\]

(14)
These equations in Eqs. (12) or (14) are indeed realized quantum mechanically as the Heisenberg equations:\textsuperscript{16}

\[
\frac{d\eta^a}{d\tau} = \frac{1}{i} [\eta^a, G], \tag{15}
\]

where \(\eta^a = (x^a, p^a)\).

Notice that one needs only the rules in Eq. (5) to order \(s\) to verify the equivalence between Eq. (15) and (14). However, when we used the first equation in Eq. (5) to derive Eq. (10), we actually use the rules in Eq. (5) to order \(s^2\). This suggests that the postulated symplectic structure or the Poisson brackets in Eq. (5) are not compatible with the definition of \(m\dot{x}^a \equiv p^a\) to order \(s^2\). In other word, it is necessary to either find the proper higher order corrections to \(\omega\) or modify the relation \(m\dot{x}^a \equiv p^a\) to resolve the physically unacceptable situation mentioned above. In fact, one may find that if one assumes \(m\dot{x}^a = p^a + emsk^a \equiv p^a + emse^{abc}\partial_b(S \cdot F)Y_c\), in stead of having the equation (7) or (10), one has a new equation by applying the rule (1)

\[
ms \frac{d}{d\tau} (\widetilde{M}^{-1}f)^{ab} = i^{-1} \left( [p^a, x^b] + [x^a, p^b] + ems \{[k^a, x^b] + [x^a, k^b]\} \right),
\]

\[
= (M^{-1})^{ab} - (M^{-1})^{ba} + ems \left( \frac{\partial k^b}{\partial p_b} - \frac{\partial k^a}{\partial p_a} \right) + \mathcal{O}(s^3). \tag{16}\]

Similarly, we have

\[
\frac{dY^a}{d\tau} = -\frac{e}{m} F^{ab} Y_b + \frac{es}{m} \left[ 3Y^a Y \cdot \dot{F} - p \cdot Y \partial^a (Y \cdot F) \right] + \mathcal{O}(s^2). \tag{17}\]

One may easily show that Eq. (17) leads to both the Hamiltonian in Eq. (13) and Eq. (14) or (12).
Equivalently, one can keep the relation \( m\dot{x}^a = p^a \) and change the symplectic structure to

\[
\tilde{\omega} = dx^a \wedge dp_a + \frac{1}{2} s f_{ab} dp^a \wedge dp^b + \frac{1}{2} e F_{ab} dx^a \wedge dx^b - em s dx^a \wedge dk_a,
\]

which is compatible with the relation \( m\dot{x}^a = p^a \) to order \( s^2 \). In general, it is very difficult to find the (exact) symplectic structure which is compatible with \( m\dot{x}^a \equiv p^a \) to arbitrary order of \( s \). However, what we have provided is a systematic way of constructing such model. Namely, we can repeatedly do the above procedure.

Now we consider the case of the space-time with metric \( g_{ab}(x) \). To avoid lengthy computations, we first consider the case of pure gravitational interactions. The (simplest) symplectic structure will take the following form (the numeral tensor density \( \epsilon_{abc} \) is assumed):\(^{17}\)

\[
\omega_g = dx^a \wedge dp_a + \frac{1}{2} \sqrt{g} f_{ab} dp^a \wedge dp^b + \frac{1}{2} dx^a \wedge d(S^b_{ab} \partial_b \ln p^2),
\]

where \( g = \det g_{ab}, S_{ab} = -s \sqrt{g} p^2 f_{ab}, \) and the index is raised or lowered by \( g_{ab} \). The closure condition of \( \omega_g \) yields

\[
(p^2 g^{ab} - 3 p^a p^b)dg_{ab} = 0,
\]

which is satisfied only by the conformal flat metric, where \( g^{ab} \) is the inverse of \( g_{ab} \).

For a conformal flat metric \( g_{ab} = \eta_{ab} e^{\phi(x)} \), we may rewrite \( \omega \) in term of \( x^a \), \( u^a = m\dot{x}^a \):

\[
\omega_g = e^\phi \left( \tilde{\eta}_{ab} dx^a \wedge du^b - \frac{1}{2} u^2 S_{ab}(u) du^a \wedge du^b + \frac{1}{2} E_{ab} dx^a \wedge dx^b \right),
\]

\(^{17}\)
where
\[ \tilde{\eta}_{ab} = \eta_{ab} + \frac{1}{2} \partial S_{ac} \phi^c \]
\[ E_{ab} = u_a \phi_b - u_b \phi_a + \frac{1}{2} (S_{ac} \phi^c_b - S_{bc} \phi^c_a) \] (22)
\[ S^{ab} = \epsilon^{abc} S_c \exp(-\phi), \quad \phi_a = \partial_a \phi, \quad S_a = -s \frac{u_a}{\sqrt{u^2}}. \]

Notice \( S^{ab} \) is the tensor with respect to the metric \( g_{ab} \) and the index is lowered or raised by the flat metric \( \eta_{ab} \) and we will use this convention from now on except for the explicit indication.

The two-form in Eq. (21) gives the commutation rules:
\[ [x^a, x^b] = ise^{-2\phi}(\tilde{\eta}^{-1})^{ca} f_{cd}(N^{-1})^{bd}, \]
\[ [u^a, x^b] = ie^{-\phi}(N^{-1})^{ab}, \] (23)
\[ [u^a, u^b] = ie^{-2\phi}(\tilde{\eta}^{-1})^{ac} E_{cd}(N^{-1})^{bd}, \]

where \( N_{ab} = \tilde{\eta}_{ab} + s E_{ac}(\tilde{\eta}^{-1})^{dc} f_{db} \exp(-\phi) \) and \( \tilde{N} \) denotes the transpose of \( N \). Again, we seek an expansion in term of a series of spin. The first two of these rules (all one needs) become:
\[ [x^a, x^b] = ise^{-2\phi} f_{ab} + \mathcal{O}(s^3) \]
\[ [u^a, x^b] = ie^{-\phi}(\eta^{ab} + K^{ab} + D^{ab}) + \mathcal{O}(s^3), \] (24)

where
\[ K^{ab} = \frac{1}{2} \frac{\partial S^{ac}}{\partial u_b} \phi_c + \frac{1}{u^2} u^a \phi_c S^{cb} \]
\[ D^{ab} = (KK)^{ab} + \frac{1}{2u^2} \left( (SS)^{bc}(\phi^a \phi_c + \phi^a_c) + S^{ad} \phi_{dc} S^{cb} \right). \] (25)

Applying the rule (1) on the first equation in Eq. (24) and using the second
one in the same equations, we have

\[ \dot{Y}^a - 2Y^a \dot{\phi} = \frac{1}{s} \epsilon^{abc}(K_{bc} + D_{bc}) + \mathcal{O}(s^2), \]  
(26)

which may be simplified into

\[ \dot{Y}^a - \frac{1}{2}(u \cdot Y \phi^a + Y^a \dot{\phi}) = \frac{1}{2(u^2)^{\frac{3}{2}}} S^{ab} \left( \frac{1}{2} \phi_b \phi_b - \phi_b \right) + \mathcal{O}(s^2). \]  
(27)

Immediately, we may deduce from the equations (27)

\[ \frac{d}{d\tau} \left( u^2 e^\phi \right) + \mathcal{O}(s^2) = 0 \]
\[ \frac{du^a}{d\tau} - \frac{1}{2} (u^2 \phi^a + u^a \dot{\phi}) = \frac{1}{2} S^{ab} \left( \frac{1}{2} \phi_b \phi_b - \phi_b \right) + \mathcal{O}(s^2). \]  
(28)

The \( \tau \)-Hamiltonian \( G_g \), which is proportional to the first integral of motion, may be determined by the method shown in Ref. [16] and a similar calculation yields:

\[ G_g = -\frac{1}{2m} \left( u^2 e^\phi - m^2 \right) + \mathcal{O}(s^2). \]  
(29)

One may check that Eqs. (28) are indeed the \( \tau \)-Hamiltonian equations of motion by using the commutation rules (23) and the Heisenberg equations:

\[ \frac{dx^a}{d\tau} = \frac{1}{i} [x^a, G_g] \]
\[ \frac{du^a}{d\tau} = \frac{1}{i} [u^a, G_g]. \]  
(30)

Recalling the forms of the Christoffel symbol and Riemann tensor in the case
of the conformal metric:

\[
\Gamma^a_{bc} = \frac{1}{2}(\delta^a_c \phi_b + \delta^a_b \phi_c - \eta_{bc} \phi^a) \\
R^a_{bcd} = \frac{1}{4} \delta^a_d (\phi_b \phi_c - \eta_{bc} \phi^m \phi_m - 2 \phi_{bc}) + \frac{1}{4} \eta_{bc} (\phi^a \phi_d - 2 \phi_d^a) - (d \leftrightarrow c),
\]

we can rewrite the equations of motion in Eqs. (28) in a more familiar form,

\[
\frac{d u^a}{d \tau} + \frac{1}{m} \Gamma^a_{bc} u^b u^c + \frac{1}{2} R^a_{bcd} S^{cd} = O(s^2).
\]  \hspace{1cm} (32)

The equations of motion for the spin may also be obtained,

\[
\frac{D S^{ab}}{D \tau} \equiv \frac{d S^{ab}}{d \tau} + \frac{1}{m} \Gamma^{a}_{cd} S^{cb} u^d + \frac{1}{m} \Gamma^{b}_{cd} S^{ac} u^d = O(s^2).
\]  \hspace{1cm} (33)

One recognizes that Eqs. (32) and (33) are in fact the Papapetrou equations at 2+1 dimensions.\textsuperscript{3,17} In particular, Eqs. (32) is formally similar to the Lorentz force law (12), in which the field strength \( F_{ab} \), the scalar charge \( e \) are replaced by the space-time curvature, the tensorial coupling \( S^{ab} \), respectively, while \( S^{ab} \) is covariantly constant of motion.

For the case of including the external electromagnetic field, the computation is straightforward but lengthy, and we here only state the results. The symplectic structure for this case is

\[
\omega_{\text{new}} = \omega_g + \frac{1}{2} F_{ab} dx^a \wedge dx^b - e m s d x^a \wedge d k_a,
\]  \hspace{1cm} (34)

where \( k_a \) is given by

\[
k_a = \frac{1}{2} \epsilon_{abc} S^{mn} \frac{D F_{mn}}{D x_b} \gamma^c e^{-\phi},
\]  \hspace{1cm} (35)
and
\[
\frac{DF_{mn}}{Dx^l} = \frac{\partial F_{mn}}{\partial x^l} - \Gamma^l_{im} F_{kn} - \Gamma^l_{in} F_{mk}
\]
\[
\frac{1}{2} \epsilon_{mn} \frac{DF_{mn}}{Dx^a} = \partial^a F^b - F^b \phi^a - \frac{1}{2} F^a \phi^b + \frac{1}{2} F^c \phi^c \eta^{ab}.
\] (36)

The equations of motion for both the particle’s positions and spin variables are:
\[
\frac{du^a}{d\tau} + \frac{1}{m} \Gamma^a_{bc} u^b u^c + \frac{e}{m} F_{ab} u_c e^{-\phi} + \frac{1}{2} R^a_{bcd} u^b S^{cd} = \frac{e}{2m} e^{-\phi} S^{cd} \frac{DF_{cd}}{Dx^a} + \mathcal{O}(s^2)
\] (37)
\[
\frac{dS^{ab}}{d\tau} + \frac{1}{m} \Gamma^a_{cd} S^{cb} u^d + \frac{1}{m} \Gamma^b_{ac} S^{ac} u^d = -\frac{e}{m} \epsilon^{abc} F_{cd} S^{d} e^{-2\phi} + \mathcal{O}(s^2).
\]

Note that these equations in Eq. (37) are obviously generally covariant and the first one has been obtained previously.\(^6,17\) Again, the equations (37) are consistent with the (appropriate) Heisenberg equations (Hamiltonian evolution) as mentioned above, where the \(\tau\)-Hamiltonian is
\[
G_{\text{new}} = -\frac{1}{2m} \left( u^2 e^\phi - 2e S \cdot F e^{-\phi} - m^2 \right) + \mathcal{O}(s^2).\] (38)

Some remarks are in order. If one postulates the existence of Hamiltonian equations of motion, one may easily obtain the desired equations by finding the Hamiltonian \(G\) from the relation \(m \dot{x}^a = i^{-1}[x^a, G] = p^a.\)\(^17\) Namely, if one proves that the second Leibniz rule is equivalent to the existence of Hamiltonian equations of motion, Feynman’s proof will not be necessary. But with use of second Leibniz rule(Feynman’s ideas), it is practically easier to determine the dynamics of spinning particles if one is interested in the higher order spin-spin interactions. We are able to carry out a general proof that the second Leibniz rule does indeed lead to the existence of the Hamiltonian evolution. This proof and its application for
constructing the dynamics of spinning particle in 3+1 dimensional space-time will be published elsewhere.\textsuperscript{18}

As demonstrated above, the inconsistency arises only in the case of that the spin is coupled to the fast varying electromagnetic field, i.e. the gradients of the external field strength $F$. This is due to the fact that the symplectic structure for spinning particles is not exactly known and what we had is not compatible with the postulated relation $m\dot{x}^a \equiv p^a$. We believe that with proper modification of the symplectic structure of spinning particles, we can completely resolve this matter. We have shown that starting with modifying the relation between the phase space variables $p^a$ and the velocities of particles, the second Leibniz rule leads to desired equations, which are also Heisenberg equations. This procedure demonstrates a systematic method to study the dynamical effects of spin coupled to the external fields to any order $s$.

In the absence of the electromagnetic interactions, the geodesic equation for particle’s position (with the standard spin-curvature term of Papapetrou\textsuperscript{3}) is obviously valid. This feature was not rediscovered in Ref. [9] as the consequence of allowing for interactions with external scalar field. This is because if a relativistic particle interacts with the scalar field, its position will not follow a geodesic. The argument offered by Tanimura\textsuperscript{9} is irrelevant, since the geodesic condition $g_{ab}\dot{x}^a \dot{x}^b = 1$ appears as the constraint equation and it will require Dirac’s procedure to further test its consistency with the postulated Poisson brackets. Our parameter $\tau$ may be interpreted as the proper time of particle’s motion.

We have demonstrated that the only possible fields that can consistently act on a quantum mechanical spinning particle in 2+1 dimensions are gauge and grav-
itational fields. Therefore, there will be a lack of continuity when we take the limit of spin going to zero, as we have learned from Ref. [9], where the scalar interactions may be allowed. In other words, a spinless particle may not be viewed as a spinning particle at the limit of $s \to 0$, because of possible scalar interactions. Finally, since our derivation is limited to the case of the conformal gravitational backgrounds, it will be interesting to see if one can carry out the same analysis for a general gravitational field; this will also shed light on how an anyon interacts with the gravitational fields. Generalization to 3+1 dimensions will be discussed elsewhere.\textsuperscript{18} Although the spin is no longer relevant to the 1+1 dimensional space-time, a special parameter arises in a similar position to the spin at 2+1. When we apply our approach to this case, we find even more peculiar features.\textsuperscript{19}

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