On Approximating (Sparse) Covering Integer Programs

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Abstract

We consider approximation algorithms for covering integer programs of the form
\[ \min \langle c, x \rangle \text{ s.t. } Ax \geq b \text{ and } x \leq d; \]
where \( A \in \mathbb{R}_{\geq 0}^{m \times n} \), \( b \in \mathbb{R}_{\geq 0}^m \), and \( c, d \in \mathbb{R}_{\geq 0}^n \) all have nonnegative entries. We refer to this problem as CIP, and the special case without the multiplicity constraints \( x \leq d \) as CIP\(_\infty\). These problems generalize the well-studied Set Cover problem. We make two algorithmic contributions.

First, we show that a simple algorithm based on randomized rounding with alteration improves or matches the best known approximation algorithms for CIP and CIP\(_\infty\) in a wide range of parameter settings, and these bounds are essentially optimal. As a byproduct of the simplicity of the alteration algorithm and analysis, we can derandomize the algorithm without any loss in the approximation guarantee or efficiency. Previous work by Chen, Harris and Srinivasan [12] which obtained near-tight bounds is based on a resampling-based randomized algorithm whose analysis is complex.

Non-trivial approximation algorithms for CIP are based on solving the natural LP relaxation strengthened with knapsack cover (KC) inequalities [3, 27, 12]. Our second contribution is a fast (essentially near-linear time) approximation scheme for solving the strengthened LP with a factor of \( n \) speed up over the previous best running time [3]. To achieve this fast algorithm we combine recent work on accelerating the multiplicative weight update framework with a partially dynamic approach to the knapsack covering problem.

Together, our contributions lead to near-optimal (deterministic) approximation bounds with near-linear running times for CIP and CIP\(_\infty\).
1 Introduction

Set Cover is a fundamental problem in discrete optimization with many applications and connections to other problems. A number of variants and generalization of Set Cover have been studied over the years. In this paper we consider a general problem that captures many of these as special cases. This is the minimum cost covering integer program problem (CIP for short). It is a class of integer programs of the form

\[
\text{minimize } \langle c, x \rangle \text{ over } x \in \mathbb{Z}^n_\geq \text{ s.t. } Ax \geq b \text{ and } x \leq d,
\]

where \( A \in \mathbb{R}^{m \times n}_\geq, b \in \mathbb{R}^m_\geq, \) and \( c, d \in \mathbb{R}^n_\geq \) all have nonnegative coefficients. We let \( N = \|A\|_0 + \|b\|_0 + \|c\|_0 + \|d\|_0 > m + n \) denote the total number of nonzeros in the input. Let \( C \geq 1 \) be such that \( A_{i,j} \in \{0\} \cup [1/C, C] \) for all \( i \) and \( j \); \( \log C \) reflects the number of bits required to write down a coefficient of \( A_{i,j} \). Since we are interested in integer solutions we can assume, without loss of generality, that \( d \in \mathbb{Z}^n \) and \( A_{i,j} \leq b_i \) for all \( i, j \). An important special case of CIP is when there are no multiplicity constraints, in other words, \( d_j = \infty \) for all \( j \). We refer to this problem as CIP\(_\infty\).

CIP and CIP\(_\infty\) can be understood combinatorially as multiset multicover problems, particularly if we assume for convenience that \( A \) and \( b \) have integer entries. The elements that need to be covered correspond to the rows of \( A \), (say) the element \( e_i \) for row \( i \), for a total of \( m \) elements. For each element \( e_i, b_i \) is the requirement on the number of times \( e_i \) needs to be covered. Each column \( j \) of \( A \) correspond to a multiset \( S_j \). For each element \( e_i \) and multiset \( S_j \), \( A_{i,j} \) is the number of times \( S_j \) covers \( e_i \). Multiplicity constraints are specified by \( d \); \( d_j \) is the maximum number of copies of \( S_j \) that can be chosen. The cost of one copy of \( S_j \) is \( c_j \). The goal is to pick a minimum cost collection of multisets (with copies allowed) that together cover all the requirements of the elements, while respecting the multiplicity bounds on the sets.

Set Cover is a special case of CIP\(_\infty\) where \( A \) is a \( \{0,1\} \)-matrix and \( b = 1 \); \( A_{i,j} = 1 \) implies that element \( e_i \) is in set \( S_j \). In this setting the multiplicity bounds are irrelevant since at most one copy of a set is ever needed. Set Cover is NP-Hard and its approximability has been extensively studied. A simple greedy algorithm achieves an approximation of \( H_k \leq (1 + \ln k) \leq (1 + \ln m) \) where \( k = \max_i |S_i| \) is the maximum set size \([3, 4]\); here \( H_k = 1 + 1/2 + \ldots + 1/k \) is the \( k \)th harmonic number.\(^1\) Unles \( P = NP \) there is no \((1-\delta)\ln m\) approximation for any fixed \( \delta > 0 \) \([30, 18]\) where \( m \) is the number of elements.\(^2\) Another approximation bound for Set Cover is \( f \) where \( f \) is the maximum frequency \([22]\); the frequency of an element is the number of sets that contain it. This bound is achieved via the natural LP relaxation. For any fixed \( f \), Set Cover instances with maximum frequency \( f \) are hard to approximate to within a \( f - \epsilon \) factor under UGC \([2]\), and to within \( f - 1 - \epsilon \) under \( P \neq NP \) \([16] \).

The greedy algorithm for Set Cover can be easily generalized to CIP. Dobson \([17]\) analyzed this extension and showed that, when all entries of \( A, b \) are integers, it has an approximation bound of \( H_d \), where \( d = \max_{1 \leq j \leq n} \sum_{i} A_{i,j} \geq C \) is the maximum column sum.\(^3\) The analysis for the greedy bound is tight, and the dependency on the maximum coordinate \( C \) is undesirable for several reasons. In particular, the entries in \( A \) and \( b \) can be rational, and the greedy algorithm’s approximation

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\(^1\)We will be concerned with the setting of arbitrary costs for the sets. Unit-cost Set Cover admits some improved bounds and those will not be the focus here.

\(^2\)It is common to use \( n \) for the number of elements and \( m \) for the number of sets. However, in the setting of covering integer programs, it is natural to use our notation in accordance with the usual notation for optimization where \( n \) is the number of decision variables and \( m \) is the number of constraints. See \([22]\) among others.

\(^3\)Wolsey’s analysis \([42]\) for the Submodular Set Cover problem further generalizes Dobson’s result.
The question of obtaining an improved approximation ratio that did not depend on $C$ was raised in [17]. For CIP, when there are no multiplicity constraints, Raghavan and Thompson, in their influential work on randomized rounding, used the LP relaxation of [CIP] (which we refer to as Basic-LP) to obtain an $O(\log m)$ approximation [32]. Subsequent work has refined and improved this bound, and later we will describe recent approximation bounds by Chen et al. [12] that are much tighter w.r.t the sparsity of a given instance.

**Stronger LP Relaxation:** In the presence of multiplicity constraints, Basic-LP has an unbounded integrality gap even when $m = 1$, which corresponds to the Knapsack Cover problem. The input to this problem consists of $n$ items with item $i$ having cost $c_i$ and size $a_i$, and the goal is to find a minimum cost subset of the items whose total size is at least a given quantity $b$. To illustrate the integrality gap of the LP relaxation consider the following simple example from [5].

$$\min x_1 \text{ over } x_1, x_2 \geq 0 \text{ s.t. } Bx_1 + (B - 1)x_2 \geq B \text{ and } x_1, x_2 \leq 1.$$  

It is easy to see that the optimum integer solution has value 1 while the LP relaxation has value $1/B$, leading to an integrality gap of $B$. The example shows that the integrality gap is large even when $d = 1$, a natural and important setting.

To overcome this gap, Carr et al. [5] suggested the use of knapsack cover (KC) inequalities to strengthen the LP. We describe the idea. For each $S \subseteq [n]$, one can consider the residual covering constraints if we force $x_i = d_i$ for all $i \in S$. The residual system is called the knapsack covering constraint for $S$ and written as $A_S x \geq b_S$, where $b_S \in \mathbb{R}^n$ is defined by $b_{S,i} = \max \{0, b_i - \sum_{j \in S} A_{i,j}d_j\}$, and for $j \in [m]$, $A_{S,i,j}$ is defined by

$$A_{S,i,j} = \begin{cases} 0 & \text{if } j \in S, \\ \min \{A_{i,j}, b_{S,i}\} & \text{otherwise}. \end{cases}$$

That is, for $S \subseteq [n]$, we compute the residual demand $b_S$, zero the coefficients in $A$ of any contracted coordinate $j \in S$, and reduce each remaining coefficient in $A$ to be at most the residual covering demand.

A feasible integral solution $x$ to (CIP) satisfies $A_S x \geq b_S$ for all $S \subseteq [n]$. The following LP, then, is a valid linear relaxation of the integer program (CIP).

$$\text{minimize } \langle c, x \rangle \text{ over } x \in \mathbb{R}^n_{\geq 0} \text{ s.t. } x \leq d \text{ and } A_S x \geq b_S \text{ for all } S \subseteq [n].$$ (KC-LP)

Note that the knapsack cover constraints made the packing constraints $x \leq d$ redundant and expendable; (KC-LP) is a pure covering problem. Given a feasible solution $x$ to (KC-LP), one can (randomly) round $x$ to a feasible integer solution $y$ to (CIP). Kolliopoulos and Young [25] obtain an $O(\log \Delta_0)$ approximation via (KC-LP) where $\Delta_0$ is the maximum number of non-zeroes in any column of $A$; note that $\Delta_0 \leq m$. Recent tighter bounds [12] will be discussed shortly.

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4 Dobson also described a variant of greedy for rational data whose approximation ratio is $1 + \max_j (H_d + \ln \sum_i A_{i,j})$; $d_j$ is the number of non-zeroes in column $j$ and the entries of $A$ are assumed to be scaled such that the minimum non-zero entry of each row is at least 1.
Sparsity bounds and motivation: We are motivated by the following high-level question. Can one obtain near-tight approximation bounds for CIP and CIP\(_\infty\) that are efficient, simple and deterministic? Set Cover is the model here where a simple greedy algorithm or a simple primal-dual algorithm gives provably optimal worst-case approximation ratios with near-linear running time. We briefly discuss some existing results before stating our results.

The first issue is regarding the approximation ratios for CIP and CIP\(_\infty\). Currently the best bounds in terms of column sparsity are from the recent work of Chen, Harris, and Srinivasan [12]. To describe the bounds, we assume, without loss of generality, that the problem is normalized such that entries of \(A\) are in \([0, 1]\) and \(b \geq 1\). Following [12], we let \(\Delta_1\) denote the maximum number of non-zeroes in any column of \(A\), and let \(\Delta_0 = \max_j \sum_{i=1}^m A_{i,j}\) denote the maximum column sum. \(\Delta_0\) and \(\Delta_1\) are the \(\ell_0\) and \(\ell_1\) measures of column sparsity of \(A\). Similarly we let \(\Gamma_0\) and \(\Gamma_1\) denote the corresponding measures for row sparsity of \(A\). In the context of Set Cover, \(\Delta_0\) is the maximum set size and \(\Gamma_0\) is the maximum frequency. Under the normalization that \(A_{i,j} \in [0, 1]^{m \times n}\), we have \(\Delta_1 \leq \Delta_0 \leq m\), and in some cases \(\Delta_1 \ll \Delta_0\). We let \(b_{\min}\) denote \(\min_{i \in [m]} b_i\), and this measures the so-called “width” of the system. As \(b_{\min}\) increases, the problem gets easier in the case of CIP\(_\infty\). We summarize the relevant approximation ratios from [12], all of which are randomized:

- A \((1 + \ln \Delta_0 + O(\sqrt{\ln \Delta_0}))\)-approximation for CIP via (KC-LP).
- A \((1 + \frac{\ln(1 + \Delta_1)}{b_{\min}} + 4 \sqrt{\frac{\ln(1 + \Delta_1)}{b_{\min}}})\) approximation for CIP\(_\infty\) via Basic-LP.
- A bicriteria algorithm for CIP: given \(\epsilon > 0\), the algorithm outputs an integer solution \(z\) with cost at most \((1 + 4 \frac{\ln(1 + \Delta_1)}{b_{\min} \cdot \epsilon} + 5 \sqrt{\frac{\ln(1 + \Delta_1)}{b_{\min} \cdot \epsilon}})\) times the LP value, and satisfies the multiplicity constraints to within a \((1 + \epsilon)\)-factor, that is \(z \leq \lfloor (1 + \epsilon)d \rfloor\).

The algorithmic framework of [12] is based on resampling, which is motivated by the developments on the constructive version of the Lovász Local Lemma starting with the work of Moser and Tardos and continuing through several subsequent developments. Although the high-level algorithmic idea is not that complicated, the analysis is technically involved and randomization seems inherently necessary. One of the significant and novel contributions of [12] is to show, for the first time, that approximation bounds based on \(\Delta_1\) are feasible. Not only can \(\Delta_1\) be much smaller than \(\Delta_0\), it is also more robust to noise and perturbation. Noise is typically not an issue for combinatorial instances such as those arising in Set Cover but can be relevant in instances of CIP that arise from data with real numbers. We note that there are two regimes of interest for \(\Delta_1\). One regime is when \(\Delta_1\) is large (at least some fixed constant) in which case the approximation bounds tend towards \(\ln \Delta_1\) plus lower order terms. The other regime is when \(\Delta_1\) is small and tends to 0; in this regime the approximation ratio guarantees from preceding bounds for CIP\(_\infty\) tend to \(1 + O(\sqrt{\Delta_1})\). Approximation bounds in terms of row sparsity are also known for CIP. Pritchard and Chakrabarty [31] describe a \(\Gamma_0\) approximation for CIP (previous results obtained a similar bound in more restricted settings). For CIP\(_\infty\) a bound of \((1 + \Gamma_1)\) is implicit in [31] (see Proposition 7).

\(^5\)Note that the result of Dobson is based on a very different normalization.

\(^6\)These bounds were based on the latest version of [12] at the time of our work. Chen, Harris, and Srinivasan have improved their bounds in a recently updated preprint [13], essentially replacing the second order \(\sqrt{\cdots}\) terms with \(O(\ln \ln(\cdots))\) terms. We plan to do a careful comparison with the results in [13] in a future version of this work.
The second issue is with regards to efficiency. A \((1 - \epsilon)\)-approximation for the Basic-LP can be obtained in near-linear \(\tilde{O}(N/e^2)\) time \([43]\) or \(\tilde{O}(N/e + m/e^2 + n/e^3)\) randomized time \([4]\) (and more efficiently if there are no multiplicity constraints \([23, 40]\)). On the other hand, \((\text{KC-LP})\) is not as simple to solve because of the exponential number of implicit constraints. Carr et al. \([5]\) describe two methods to solve \((\text{KC-LP})\). The first is to use the Ellipsoid method via an approximate separation oracle\(^7\). The other is to use a Lagrangean relaxation based approximation scheme which yields a running time of \(O\left( nN \text{poly}\left( \frac{1}{\epsilon} \log C \right) \right)\) for a \((1 + \epsilon)\)-approximate solution. Kolliopoulos and Young \([25]\) explicitly raise the question of a fast approximation algorithm for CIP. The recent work of Chen et al. \([12]\) discussed above shows that a randomized rounding technique via a resampling framework yields near-optimal approximations, and they run in expected near-linear time. This is in contrast to some previous rounding algorithms \([35, 37]\) that were rather complex and slow. Hence the bottleneck for CIP is solving the LP relaxation \((\text{KC-LP})\).

### 1.1 Our Results

In this paper we address both the approximability and efficiency of CIP and CIP\(_\infty\).

Our first set of results is on rounding the fractional solution to the Basic-LP and \((\text{KC-LP})\). Our main contribution is to show that a very simple combination of randomized rounding followed by alteration, that has been previously considered for covering integer programs \([36, 34, 20]\), yields clean and (in some cases) improved bounds when compared to those of \([12]\). Our focus in this paper is in the regime where \(\Delta_0\) and \(\Delta_1\) are larger than some fixed (modest) constant. Under this assumption, we obtain the following improved approximation bounds:

- A \((\ln \Delta_0 + \ln \ln \Delta_0 + O(1))\)-approximation for CIP via \((\text{KC-LP})\)

- A \((\ln \Delta_1 + \ln \ln \Delta_1 + O(1))\)-approximation for CIP\(_\infty\) via Basic-LP. When \(b_{\min}\) is large the ratio improves to \(\left( \frac{\ln \Delta_1}{b_{\min}} + \ln \frac{\ln \Delta_1}{b_{\min}} + O(1) \right)\) under the assumption that \(\frac{\ln \Delta_1}{b_{\min}}\) is sufficiently large.

- A bicriteria algorithm for CIP via \((\text{KC-LP})\) given an error parameter \(\epsilon > 0\), the algorithm outputs a solution with cost at most \(\left( \frac{\ln \Delta_1}{b_{\min}} + \ln \frac{\ln \Delta_1}{b_{\min}} + O\left( \frac{1}{\epsilon} \right) \right)\) times the LP value, and satisfies the multiplicity constraints to within a \((1 + \epsilon)\)-factor\(^8\).

When \(\Delta_0\) is large, \([12]\) established a hardness lower bound of the form \(\ln \Delta_0 - c \ln \ln \Delta_0\) for some constant \(c\) by extending a result of Trevisan \([32]\) for Set Cover. Thus our improved bounds in this regime get closer to the lower bound in the second order term. We obtain a more substantial improvement for the bicriteria approximation as a function of \(\epsilon\), and it is important to observe that it is based on \((\text{KC-LP})\) while \([12]\) uses Basic-LP. Perhaps of greater interest than the precise improvements is the fact that the alteration algorithm is simple and easy to analyze. All we really need is a careful use of the lower tail of the standard Chernoff bound. A significant consequence of the simple analysis is that we are able to easily and efficiently derandomize the algorithm via the standard method of conditional expectations. This leads to simple deterministic algorithms without loss in the approximation bounds. We also believe that our analysis is insightful for the bound based on \(\Delta_1\); it is not easy to see why such a bound should be feasible in the first place.

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\(^7\)The LP is not solved exactly but the fractional solution output by the algorithm provides a lower bound and suffices for the current randomized rounding algorithms.

\(^8\)It is not hard to show that bounds based on \(\Delta_1\) are not feasible for CIP if multiplicity constraints are not violated.
Finally we note that our bound for CIP based on $\Delta_0$ is quite elementary and direct, and does not rely on the more involved analysis for the bound based on $\Delta_1$; this is not the case in [12].

Remark 1.1. For CIP$_\infty$ when $\Delta_1$ is sufficiently small we can show that the alteration approach gives an approximation ratio of $O\left(1 + O\left(\sqrt{\Delta_1 \log \frac{1}{\Delta_1}}\right)\right)$, which tends to 1 as $\Delta_1 \to 0$. This is slightly weaker than the bound from [12], which gives an approximation ratio of $O\left(1 + O(\sqrt{\Delta_1})\right)$. We believe that our analysis is of the alteration algorithm is not tight.

Remark 1.2. The alteration based algorithms have a single parameter $\alpha$ that controls the scaling of the variables in the randomized rounding step. Depending on the regime of interest we choose an appropriate $\alpha$ to obtain the best theoretical guarantee. One can try essentially all possible values of $\alpha$ (by appropriate discretization) and take the best solution. This makes the algorithm oblivious to the input parameters. The fixing step in the alteration algorithm depends on the objective function $c$ and is deterministic. One can make the fixing step oblivious to $c$ via randomization and known results on the Knapsack Cover problem.

Our second result is a fast approximation scheme for solving $(KC-LP)$, improving upon the previous bound in [5] by a factor of $n$. For polynomially-bounded $C$ (which is a reasonable assumption in many settings) the running time is near-linear for any fixed $\epsilon$. The precise result is stated in the theorem below. To achieve the result we develop an incremental dynamic data structure for the Knapsack Cover problem and combine it with other data structures following our recent line of work on speeding up MWU based approximation schemes for implicit positive linear programs.

Theorem 1.3. Let $\epsilon > 0$ be fixed, and consider an instance of $(KC-LP)$ and let OPT be the value of an optimum solution. There is a deterministic algorithm that in $\tilde{O}\left(\frac{N \log C}{\epsilon^3} + \frac{(m + n) \log C}{\epsilon^3}\right)$ time outputs $x \in \mathbb{R}^n$ such that $\langle c, x \rangle \leq OPT$, $x \leq d$, and for all $S \subseteq [n]$, $A_S x \geq (1 - \epsilon)b_S$.

Together, our results yield deterministic and fast approximation algorithms for CIP and CIP$_\infty$ that are near-optimal in a wide range of parameter settings. Our analysis demonstrates that random rounding plus alteration provides near-tight bounds for Set Cover, CIP and CIP$_\infty$. The rounding algorithm and analysis can be considered textbook material.

1.2 Techniques and other related work

Approximation algorithms and hardness results for Set Cover, its generalizations, and important special cases have been extensively studied in the literature. We refer the reader to approximation books for standard and well-known results [39, 41]. CIP and CIP$_\infty$ have been primarily addressed in previous work via LP relaxations and randomized rounding starting with the well-known work of Raghavan and Thompson [32]. Srinivasan has used sophisticated probabilistic techniques based on the Lovász Local Lemma (LLL) and its derandomization via pessimistic estimators to obtain bounds that depend on the sparsity of $A$ [35, 37]. These were also used, in a black box fashion, for CIP by Kolliopoulos and Young [25]. The more recent work by Chen et al. [12] is inspired by the ideas surrounding the Moser-Tardos resampling framework [29, 21] that led to constructive versions of LLL. They were the first to consider $\ell_1$-sparsity based bounds. Srinivasan [36] used randomized rounding with alteration for covering and packing problems. For packing problems the

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9In retrospect, the algorithms in [12] appear to be randomized and oblivious fixing schemes. However, the analysis is involved for various technical reasons.
alteration approach has led to a broad framework on contention resolution with several applications [3, 10]. For covering we are also inspired by the paper of Saket and Sviridenko [34] who described an alteration based algorithm for Set Cover that achieves an approximation ratio of $\Gamma_0(1 - e^{\frac{\ln \Delta_0}{\Gamma_0 - 1}})$ that addresses both column sparsity and row sparsity in a clean and unified fashion. The known hardness of approximation result of Set Cover that we already mentioned carries over to CIP when $\Delta_0$ is the parameter of interest. Chen et al. [12], building upon the results for Set Cover, showed near-optimal integrality gaps and hardness results for CIP and CIP$_\infty$ in various sparsity regimes as a function of the parameter $\Delta_1$. Together these results show that the current upper bounds for CIP and CIP$_\infty$ in terms of column and row sparsity are essentially optimal up to lower order terms.

KC inequalities are well-known in integer programming, and since the work of Carr et al. [5], there have been several uses in approximation algorithms. MWU based Lagrangean relaxation methods have been extensively studied to derive FPTAS’s for solving packing, covering and mixed packing and covering linear programs. We refer the reader to some recent papers [43, 28, 8, 9] for pointers. Building upon some earlier work of Young [43], we have recently demonstrated, via several applications [8, 7, 9], some techniques to speed up MWU based approximation schemes for a variety of explicit and implicit problems. The key idea is the use of appropriate data structures that mesh with the analysis and flexibility of the high-level MWU based algorithm. There are certain general principles in this approach and there are problem specific parts. In this paper our technical contribution is to adapt the FPTAS for the Knapsack Cover problem and make it dynamic in a manner that is suitable to the needs of the MWU based updates. This allows us to speed up the basic approach outlined in Carr et al. [5] and obtain a fast running time. Note that the rounding step for CIP loses a large factor in the approximation, and hence solving the LP to high-precision is not the main focus. Moreover, the large dependence on $\epsilon$ is mainly due to the FPTAS for knapsack cover. A different line of work based on Nesterov’s accelerated gradient descent achieve running time with a better dependence on $O(1/\epsilon)$ for solving positive LPs [4, 14]. Until recently, the running time of these algorithms did not have a good dependence on the combinatorial parameters. The work of Allen-Zhu and Orecchia has remedied this for explicit pure packing and covering LPs [1, 40]. However, it is not clear how well these techniques can be applied to implicit problems like the ones we consider here.

**Organization:** The paper has several technical results, and broadly consists of two parts. **Section 2** describes approximation results obtained via randomized rounding plus alteration and its derandomization. **Section 3** describes our fast approximation scheme to solve (KC-LP) and proves **Theorem 1.3**. The two parts can be read independently. We made the paper modular to keep it both readable and detailed, and suggest that the reader skip to section of interest rather than read the paper sequentially.

### 2 Randomized Rounding with Alteration

In this section we formally describe two versions of randomized rounding with alteration and analyze it for CIP$_\infty$ and CIP. The algorithms are simple and have been proposed and analyzed previously. We analyze them in a tighter fashion, especially in terms of the $\ell_1$ sparsity of $A$. Recall that we have normalized the problem such that all entries of $A$ are in $[0, 1]$ and $b \geq 1$. It is convenient to simplify the problem further and assume that $b = 1$. This can be done by scaling each row $i$ by $b_i$. This setting captures the essence of the problem and simplifies notation and the analysis.

As a preliminary step we state some known facts about the Knapsack Cover problem (which is the special case of CIP with $m = 1$) in the lemma below.
Lemma 2.1 (Carr et al. [3]). Consider an instance of Knapsack Cover of the form \( \min \langle c, x \rangle \) subject to \( \sum_{j=1}^{n} a_j x_j \geq 1, \ x \leq d, \) and \( x \in \mathbb{Z}_{\geq 0}^{n} \).

- Suppose \( y \) is a feasible solution to the Basic-LP relaxation. Then there is an integer solution \( z \) in the support of \( y \) such that \( \langle c, z \rangle \leq 2 \langle c, y \rangle \) and \( y_j \leq \lfloor 2x_j \rfloor \).

- Suppose \( y \) is a feasible solution to (KC-LP) (satisfying knapsack cover inequalities). Then there is an integer solution \( z \) in the support of \( y \) such that \( \langle c, z \rangle \leq 2 \langle c, y \rangle \) and \( z \leq d \).

Given \( y \), an integer vector \( z \) satisfying the stated conditions can be found in near-linear time.

The first algorithm we present is called round-and-fix, and pseudocode is given in Figure 1. The input consists of a nonnegative matrix \( A \in \mathbb{R}_{\geq 0}^{m \times n} \), a cost vector \( c \in \mathbb{R}_{\geq 0}^{n} \), a nonnegative vector \( x \in \mathbb{R}_{\geq 0}^{n} \) such that \( Ax \geq 1 \), and a positive parameter \( \alpha > 0 \). The goal is to output an integer vector \( z \in \mathbb{Z}_{\geq 0}^{n} \) such that \( Ax \geq 1 \) with cost \( \langle c, z \rangle \) comparable to \( \langle c, x \rangle \). The algorithm consists of a randomized rounding step, step 2 in Figure 1, and an alteration step, step 3.

In the randomized rounding step, round-and-fix scales up the fractional solution \( x \) by \( \alpha \) and independently rounds each coordinate \( i \) to \( \lfloor \alpha x_i \rfloor \) or \( \lceil \alpha x_i \rceil \) such that the expectation is \( \alpha x_i \). Let \( z \) be the random vector picked in the first step. \( z \) may leave several covering constraints \( i \in [m] \) unsatisfied (that is \( (Az)_i < 1 \)). These constraints are fixed in the subsequent alteration step. In the alteration step, each unsatisfied covering constraint \( i \) is addressed separately. Let \( y^{(i)} \) be an approximately optimum solution to cover the \( i \)th constraint by itself; we can find a constant factor approximation since this is the knapsack cover problem. Letting \( U \subseteq [m] \) denote the subset of unsatisfied constraints, we output the solution \( z' \) where \( z'_j = \max_{i \in U} \{ z_j, y^{(i)}_j \} \) for each \( j \in [n] \).

Remark 2.2. The fixing step for an unsatisfied constraint \( i \) can be done more carefully. The residual requirement after the randomized step is \( 1 - (Az)_i \), and we can solve a Knapsack Cover problem to satisfy this requirement. This is crucial when \( \Delta_1 \) is small but makes the analysis complex. For the bounds we seek in the regime when \( \Delta_0 \) and \( \Delta_1 \) are sufficiently large constants, it suffices to fix each unsatisfied constraint ignoring the contribution from the first step.

The alterations guarantee that all of the covering constraints are met. The cost of the solution is a random variable. Large values of \( \alpha \) decrease the expected cost for alterations but increase the initial cost of randomized rounding; small values of \( \alpha \) decrease the initial cost of randomized rounding but increase the expected cost of alterations. A careful choice of \( \alpha \) leads to an approximation guarantee independent of \( m \) and relative to either the \( \ell_0 \)-column sparsity of \( A \), \( \Delta_0 \overset{\text{def}}{=} \max_{i \in [n]} |\{ j \in [m] : A_{i,j} \neq 0 \}| \), or the \( \ell_1 \)-column sparsity of \( A \), \( \Delta_1 \overset{\text{def}}{=} \max_{j \in [n]} \sum_{i=1}^{m} A_{i,j} \).

In the sequel we will assume that \( \Delta_0 \) and \( \Delta_1 \) are greater than some fixed constant. We believe that this is the main regime of interest. Moreover, the assumption avoids notational and technical complexity. In Appendix A we consider the case when \( \Delta_1 \) is small.

We now state the specific theorems for CIP and \( \text{CIP}_{\infty} \) and prove them in subsequent sections. The first theorem analyzes the performance of round-and-fix in terms of \( \ell_1 \) sparsity of \( A \). Note that the theorem is directly relevant for \( \text{CIP}_{\infty} \). The bounds on the multiplicities in Theorem 2.3 are insufficient for CIP, but will prove useful for CIP when discussing the second algorithm later.

Theorem 2.3. Let \( A \in [0, 1]^{m \times n}, \ c \in \mathbb{R}_{\geq 0}^{n} \), and \( x \in \mathbb{R}_{\geq 0}^{n} \) such that \( Ax \geq 1 \). Let \( \alpha = \ln \Delta_1 + \ln \Delta_1 + O(1) \). In \( O(\|A\|_0) \) time, round-and-fix(\( A, c, x, \alpha \)) returns a randomized integral vector
### round-and-fix

\( A \in [0, 1]^{m \times n}, \ c \in \mathbb{R}^n_{\geq 0}, \ x \in \mathbb{R}^n_{\geq 0}, \ \alpha \geq 1 \)

// goal: given a fractional point \( x \in \mathbb{R}^n_{\geq 0} \) with \( Ax \geq 1 \), output an integral point \( z \in \mathbb{Z}^n_{\geq 0} \) with \( Az \geq 1 \) and \( \langle c, z \rangle \) comparable to \( \langle c, x \rangle \).

1. \( z \leftarrow \lfloor \alpha x \rfloor \), \( x' \leftarrow \alpha x - z \)
2. for \( j = 1 \) to \( n \)
   A. with probability \( x'_j \)
   i. \( z_j \leftarrow z_j + 1 \)
3. for all \( i \) such that \( (Az)_i < 1 \):
   // fix \( z \) s.t. \( (Az)_i \geq 1 \)
   A. Find (approximate) solution \( y^{(i)} \) for knapsack cover problem induced by constraint \( i \)
   B. \( z \leftarrow z \lor y^{(i)} \)
4. return \( z \)

Figure 1: An alteration based rounding algorithm for covering programs.

### contract-round-fix

\( A \in [0, 1]^{m \times n}, \ c \in \mathbb{R}^n_{\geq 0}, \ d \in \mathbb{N}^n, \ x \in \mathbb{R}^n_{\geq 0}, \ \alpha \geq 1, \ \epsilon \in [0, 1] \)

// goal: given a fractional point \( x \in [0, d] \) satisfying all knapsack covering constraints w/r/t the program \( \{Ay \leq 1, 0 \leq y \leq d\} \), output an integral point \( z \in \mathbb{Z}^n_{\geq 0} \) with \( Az \geq (1 - \epsilon)1 \), \( z \leq \lceil (1 + \epsilon)d \rceil \), and \( \langle c, z \rangle \) comparable to \( \langle c, x \rangle \).

1. \( S \leftarrow \{j \in [n] : \alpha x_j \geq d_j\} \), \( z'' \leftarrow d \land S \) // where \( (d \land S)_j = d_j \) if \( j \in S \) and 0 otherwise
2. \( b \leftarrow 1 - Az'' \)
3. \( M \leftarrow \{i \in [m] : b_i > \epsilon\} \), \( N \leftarrow [n] \setminus S \)
4. define \( A' \in [0, 1]^{M \times N} \) by \( A'_{i,j} = \min\{A_{i,j} / b_i, 1\} \)
5. \( z' \leftarrow \text{round-and-fix}(A', c \land N, x \land N, d - z'') \)
6. return \( z' + z'' \).

Figure 2: An alteration based rounding algorithm for covering programs with knapsack covering constraints.
\[ z \in \mathbb{Z}_{\geq 0}^n \text{ with coverage } Az \geq 1, \text{ expected cost } \mathbb{E}[(c, z)] \leq (\alpha + O(1))(c, x), \text{ and multiplicities } z < [\alpha x]. \]

The following theorem is an easy corollary of the preceding theorem since \( \Delta_1 \leq \Delta_0 \) when entries of \( A \) are from \([0,1]\). Nevertheless we state it separately since its proof is simpler and more direct and there is a tighter bound on the additive constant. We encourage a reader new to the alteration analysis to read the proof of this theorem before that of Theorem 2.3.

**Theorem 2.4.** Let \( A \in [0,1]^{m \times n}, c \in \mathbb{R}_{\geq 0}^n, \text{ and } x \in \mathbb{R}_{\geq 0}^n \) such that \( Ax \geq 1 \). Let \( \alpha = \ln \Delta_0 + \ln \ln \Delta_0 + O(1) \). In \( O(\|A\|_0) \) time, round-and-fix(\( A, c, x \)) returns a randomized integral vector \( z \in \mathbb{Z}_{\geq 0}^n \) with coverage \( Az \geq 1 \), expected cost \( \mathbb{E}[(c, z)] \leq (\alpha + O(1))(c, x) \), and multiplicities \( z \leq [\alpha x] \).

We prove Theorem 2.4 in Section 2.1 and we prove Theorem 2.3 in Section 2.2.

The upper bounds on the multiplicities, \( z < \alpha x + 1 \), are useful for handling multiplicity constraints via knapsack covering constraints. The second algorithm, called contract-round-fix and given in Figure 2, adds a preprocessing step to round-and-fix. The basic idea is by now standard and first proposed by Carr et al. \([3]\) to take advantage of the KC inequalities. At a high level, we want to simulate the standard randomized rounding without violating the multiplicity constraints. Note that if \( \alpha x \leq d \), then round-and-fix already returns an integral solution meeting the multiplicity constraints. If \( [\alpha x]_j \geq d_j \) for some coordinate \( j \), then the algorithm deterministically sets \( z_j = d_j \). After contracting the large coordinates, the knapsack covering constraints ensure that the remaining small coordinates still satisfy the residual covering problem. We apply round-and-fix to the residual instance. We prove the following theorems on the performance of contract-round-fix in terms of the \( \ell_0 \) and \( \ell_1 \) sparsity of \( A \).

**Theorem 2.5.** Let \( A \in [0,1]^{m \times n}, c \in \mathbb{R}_{\geq 0}^n, d \in \mathbb{N}^n \) and \( x \in \mathbb{R}_{\geq 0}^n \) such that \( x \) covers all the knapsack covering constraints \( w/r/t \) \( A \) and \( d \). Let \( \alpha = \ln \Delta_0 + \ln \ln \Delta_0 + O(1) \).

In \( \|A\|_0 \) time, contract-round-fix(\( A, c, d, x, \alpha, 0 \)) returns a randomized integral vector \( z \in \mathbb{Z}_{\geq 0}^n \) with coverage \( Az \geq 1 \), expected cost \( \mathbb{E}[(c, z)] \leq (\ln \Delta_0 + \ln \ln \Delta_0 + O(1))(c, x) \) and multiplicities \( z \leq d \).

**Theorem 2.6.** Let \( A \in [0,1]^{m \times n}, c \in \mathbb{R}_{\geq 0}^n, \epsilon \in (0,1], d \in \mathbb{N}^n \), and \( x \in \mathbb{R}_{\geq 0}^n \) such that \( x \) covers all the knapsack covering constraints \( w/r/t \) \( A \) and \( x \leq d \). Let \( \alpha = \ln \Delta_1 + \ln \ln \Delta_1 + O\left(\ln\left(\frac{1}{\epsilon}\right)\right)\).

In \( \|A\|_0 \) time, contract-round-fix(\( A, c, d, x, \alpha, \epsilon \)) returns a randomized integral vector \( z \in \mathbb{Z}_{\geq 0}^n \) with coverage \( (1 + \epsilon)Az \geq 1 \), expected cost \( \mathbb{E}[(c, z)] \leq (\ln \Delta_1 + \ln \ln \Delta_1 + O(\ln(1/\epsilon)))(c, x) \), and multiplicities \( z \leq d \).

Scaling up the output to Theorem 2.6 by a \((1 + \epsilon)\)-multiplicative factor and then rounding up to an integral vectors shifts the approximation error from the coverage constraints to the multiplicity constraints, as follows.

**Corollary 2.7.** Let \( A \in [0,1]^{m \times n}, c \in \mathbb{R}_{\geq 0}^n, \epsilon \in (0,1], d \in \mathbb{Z}^n \), and \( x \in \mathbb{R}_{\geq 0}^n \) such that \( x \) covers all the knapsack covering constraints \( w/r/t \) \( A \) and \( d \).

In \( \|A\|_0 \) time, one can compute a randomized integral vector \( z \in \mathbb{Z}_{\geq 0}^n \) with coverage \( Az \geq 1 \), expected cost \( \mathbb{E}[(c, z)] \leq (1 + \epsilon)(\ln \Delta_1 + O(\ln(1/\epsilon)))(c, x) \), and multiplicities \( z \leq [(1 + \epsilon)d] \).

In Section 2.3 we analyze contract-round-fix and prove Theorem 2.6.

Proofs of Theorem 2.3 and Theorem 2.6 can be adapted to obtain improved approximations when one considers \( Ax \geq b \) with \( b_{\text{min}} > 1 \). We prove the stronger bound in Section 2.5. We derandomize the algorithm from Theorem 2.3 in Section 2.4. The other results can be made deterministic in the same fashion.
2.1 $\ell_0$-column sparse covering problems

In this section, we prove Theorem 2.4. The main point of interest is the expected cost, and the high level approach is as follows. The expected cost of the rounded solution comes from either the randomized rounding step or the subsequent alterations, where the expected cost of randomized rounding is immediate. To analyze the expected cost of alterations, we first bound the fixing cost of any constraint $i$, with cost proportional to the restriction of the fractional solution $x$ to coordinates $j$ with nonzero coefficients ($A_{i,j} > 0$). Then, we analyze the probability of a constraint $i$ being unmet, and obtain a probability inversely proportional to $\Delta_0$. The expected cost of alteration is the sum, over each constraint $i$, of the product of probability of failing to meet the $i$th constraint and the cost of repairing $z$ to fix it. This sum cancels out nicely and shows that the expected cost of alteration is at most $\langle c, x \rangle$.

**Lemma 2.8.** Let $i \in [m]$, and let $a = \max_{i,j} A_{i,j}$. After randomized rounding in steps \((2.\ast)\),

$$P[(Az)_i < 1] \leq \exp\left(\frac{1 + \ln(a) - \alpha}{a}\right).$$

For $\alpha \geq \ln \Delta_0 + \ln \ln \Delta_0 + O(1)$, we have $P[(Az)_i < 1] \leq \frac{1}{2\Delta_0}$.

**Proof.** We apply the Chernoff inequality Lemma B.3 with $\beta = 1$ and $\mu = (Ax)_i \geq \alpha$. One can easily verify (numerically for instance) that if we choose $\alpha = \ln \Delta_0 + \ln \ln \Delta_0 + 4$, then the inequality is satisfied for $\Delta_0 \geq 2$. $\square$

**Proof of Theorem 2.4.** Let $z \in \mathbb{Z}_{\geq 0}^n$ be the randomized integral vector output by round-and-fix. By the alteration step, we have $Az \geq 1$. We need to bound the expected cost by $E[\langle c, z \rangle] \leq (\alpha + O(1))\langle c, x \rangle$, and the multiplicities by $z \leq \lceil \alpha x \rceil$.

The cost of $z$ comes from the randomized rounding in steps \((2.\ast)\) and from alterations for unmet constraints in steps \((3.\ast)\). The expected cost of the rounding step of $\alpha \langle c, x \rangle$. For each $i \in [m]$, the $i$th constraint is unmet with probability $\leq \frac{1}{2\Delta_0}$ by Lemma 2.8. When a constraint $i$ is unmet, by Lemma 2.1 we can fix $z$ to that $(Az)_i \geq 1$ with additional cost at most $2 \sum_{j : A_{i,j} \neq 0} c_j x_j$. Summed over all $i$, by (a) interchanging sums and (b) definition of $\Delta_0$, the expected cost from alteration is

$$\frac{1}{2\Delta_0} \sum_{i=1}^m 2 \sum_{j : A_{i,j} \neq 0} c_j x_j = \sum_{j=1}^n c_j \sum_{i : A_{i,j} \neq 0} \frac{x_j}{\Delta_0} \leq \langle c, x \rangle,$$

as desired. Between randomized rounding and alterations, the expected cost of $z$ is $(\alpha + 1)\langle c, x \rangle$, as desired.

It remains to bound the multiplicities of $z$. For a fixed coordinate $j \in [n]$, $z_j$ is set by either the randomized rounding in steps \((2.\ast)\) or by an alteration in \((3.\ast)\). The randomized rounding step sets $z_j$ to at most $\lceil \alpha x_j \rceil$. By Lemma 2.1 an alteration sets $z_j$ to at most $\lceil 2x_j \rceil$. Thus $z_j \leq \lceil \alpha x_j \rceil$ for $\alpha \geq 2$. $\square$

2.2 $\ell_1$-column sparse covering programs

In this section, we analyze the alteration-based algorithm round-and-fix for covering programs (without multiplicity constraints) and prove Theorem 2.3. To build some intuition for our analysis,
suppose all of the nonzero coordinates in \(A\) are big (and close to 1). Then we essentially have the \(\Delta_0\) setting analyzed more simply in Section 2.1. On the other hand, suppose \(A_{i,j} = a\) for all nonzero \(A_{i,j}\), for some small \(a < 1\). Here the gap between \(\Delta_0\) and \(\Delta_1\) is a multiplicative factor of \(\frac{1}{a}\), and the fixing costs from the \(\ell_0\)-setting sum to \(O\left(\frac{\Delta_1}{a} \langle c, x \rangle\right)\). To offset the increased costs, observe that because the coordinates are uniformly bounded by \(a\), the Chernoff bound tightens exponentially by a \(\frac{1}{a}\)-factor, giving a much better bound the probability of needing to fix each constraint \(i\). In general, the coefficients in a row \(i\) can be non-uniform, with some close to 1 and some much less than 1. The key to the analysis is identifying a certain threshold \(\rho_i\) for each covering constraint \(i\) that divides the coordinates in the \(i\)th row between “big” and “small”. We call \(\rho_i\) the \(i\)th (weighted) median coefficient.

Lemma 2.9. Suppose \(Ax \geq 1\). For each \(i\), there exists \(\rho_i \in (0, 1]\) such that

\[
\sum_{j: A_{i,j} \geq \rho_i} A_{i,j} x_j \geq \frac{1}{2} \quad \text{and} \quad \sum_{j: A_{i,j} \leq \rho_i} A_{i,j} x_j \geq \frac{1}{2}.
\]

Proof. Let \(\rho_i = \inf \left\{ \rho : \sum_{A_{i,j} > \rho} A_{i,j} x_j < \frac{1}{2} \right\}\). By choice of \(\rho_i\), we have \(\sum_{A_{i,j} \geq \rho_i} A_{i,j} x_j \geq \frac{1}{2}\) and

\[
\sum_{A_{i,j} \leq \rho_i} A_{i,j} x_j = (Ax)_i - \sum_{A_{i,j} > \rho_i} A_{i,j} x_j > 1 - \frac{1}{2} = \frac{1}{2},
\]

as desired. \(
\)

The next two lemma’s consider the fixing cost for constraint \(i\), obtaining a value inversely proportional to \(\rho_i\).

Lemma 2.10. Let \(i \in [m]\). There exists a vector \(y \in \mathbb{R}^n_{\geq 0}\) with coverage \(\sum_j A_{i,j} y_j \geq 1\), cost \(\langle c, y \rangle \leq \frac{2}{\rho_i} \sum_j c_j A_{i,j} x_j\), and coordinates bounded above by \(y \leq 2x\).

Proof. Let \(y \in \mathbb{R}^n_{\geq 0}\) be defined by \(y_j = 2x_j\) if \(A_{i,j} \geq \rho_i\), and \(y_j = 0\) otherwise. We have \(0 \leq y \leq 2x\), \((Ay)_i = 2 \sum_{j: A_{i,j} \geq \rho_i} A_{i,j} x_j \geq 1\) by choice of \(\rho_i\), and

\[
\langle c, y \rangle = 2 \sum_{j: A_{i,j} \geq \rho_i} c_j x_j \leq \frac{2}{\rho_i} \sum_{j: A_{i,j} \geq \rho_i} c_j A_{i,j} x_j \leq \frac{2}{\rho_i} \sum_j c_j A_{i,j} x_j,
\]

as desired. \(
\)

Lemma 2.11. Let \(i \in [m]\). In time near-linear in the number of nonzero coefficients in row \(i\), one can find an integral vector \(z \subseteq [n]\) with coverage \(\sum_{j=1}^n A_{i,j} z_j \geq 1\), cost \(\sum_{j=1}^n c_j z_j \leq \frac{4}{\rho_i} \sum_{j=1}^n c_j A_{i,j} x_j\), and multiplicities \(z \leq [4x]\).
Proof. Applying Lemma 2.1 to the vector $y$ of Lemma 2.10 gives the desired result.

The expected cost incurred from repairing $z$ for the sake of constraint $i$ is the probability of failing to meet constraint $i$ times the cost given in Lemma 2.11. When $\rho_i$ goes to zero the expected cost of fixing constraint $i$ is dominated by the multiplicative factor of $\frac{1}{\rho_i}$. For small $\rho_i$, we require a stronger concentration bound than Lemma 2.8 that decays exponentially in $\alpha$ and proportionally with $\rho_i$, to offset the increasing fixing costs.

Lemma 2.12. Let $i \in [m]$. After randomized rounding in steps (2.*),

$$P[(Az)_i < 1] \leq \exp\left(\frac{1 + \ln(\alpha/2) - \alpha/2}{\rho_i}\right).$$

Proof. From Lemma 2.9 $\sum_{j:A_{i,j} \leq \rho_i} A_{i,j}x_j \geq 1/2$. Applying the Chernoff inequality to this sum (Lemma B.4 with $\gamma = \rho_i$ and $\mu = \frac{\alpha}{2}$), we have

$$P[(Az)_i < 1] \leq P\left[\sum_{A_{i,j} \leq \rho_i} A_{i,j}z_j < 1\right] \leq \exp\left(\frac{1}{\rho_i}\left(1 + \ln \frac{\alpha}{2} - \frac{\alpha}{2}\right)\right),$$

as desired.

Lemma 2.13. Let $i \in [m]$ and $\alpha = \ln \Delta_1 + \ln \ln \Delta_1 + O(1)$. After randomized rounding in steps (2.*), $P[(Az)_i < 1] \leq C \frac{\rho_i}{\Delta_1}$ for some constant $C > 1$.

Proof. Let $C > 0$ be a constant to be determined later. We divide the analysis into two cases, depending on if $\rho_i \geq \frac{1}{C}$ or $\rho_i \leq \frac{1}{C}$.

Suppose $\rho_i \geq \frac{1}{C}$. In this case we simply use Lemma 2.8 as if for $a = 1$. We have

$$P[(Az)_i < 1] \leq \frac{1}{\Delta_1} \leq C \frac{\rho_i}{\Delta_1},$$

as desired.

Suppose now that $\rho_i \leq \frac{1}{C}$. By Lemma 2.12 we have $P[(Az)_i < 1] \leq \frac{\rho_i}{\Delta_1}$ if

$$\alpha \geq 2\rho_i \ln(\Delta_1) + 2\rho_i \ln\left(\frac{1}{\rho_i}\right) + 2 \ln(\alpha/2) + 2.$$

For $\alpha = \ln(\Delta_1) + \gamma$, as $\gamma \to \infty$ and $\rho_i \to 0$, the left side dominates the right side. In particular, for $C$ sufficiently large, and $\gamma$ a sufficiently large constant, we have $P[(Az)_i < 1] \leq \frac{\rho_i}{\Delta_1}$. ■

We conclude the section by completing the proof of Theorem 2.3.

Proof of Theorem 2.3. The algorithm ensures that $Az \geq 1$. The upper bound on the multiplicities of $z$ follows by the same argument as in proving Theorem 2.4. It remains to bound the expected cost $E[(c, z)]$.

The expected cost is the sum of the expected cost from the randomized rounding in steps (2.*), and the expected cost from alterations in steps (2.*). The expected cost of the rounding step is $\alpha(c, x)$. For each $i \in [m]$, the expected cost incurred by alterations for the $i$th covering constraint is
the product of the fixing cost for the $i$th constraint, bounded by \textbf{Lemma 2.11} and the probability of failing to meet the $i$th constraint, bounded by \textbf{Lemma 2.13}. Summing over all $i$, (a) interchanging sums, and (b) definition of $\Delta_1$, the expected cost of alteration is at most
\[
 C \sum_{i=1}^{m} \frac{\rho_i}{\Delta_1} \cdot \sum_{j=1}^{n} c_{j} A_{i,j} x_{j} \leq C \sum_{j=1}^{n} \frac{c_{j} x_{j}}{\Delta_1} \sum_{i=1}^{m} A_{i,j} \leq C \langle c, x \rangle
\]
for some constant $C \geq 1$. Summing the expected costs from randomized rounding and alterations, the expected cost of the output $z$ is $(\alpha + C) \langle c, x \rangle$, as desired.

\textbf{2.3 Handling multiplicity constraints}

In this section, we analyze the second algorithm, \textit{contract-round-fix}. \textit{contract-round-fix} takes as input a fractional solution that satisfies all the knapsack covering constraints w.r.t a system of covering constraints $Ax \geq 1$ and multiplicity constraints $x \leq d$ and adds a preprocessing step to \textit{round-and-fix} to handle multiplicity constraints.

We obtain different approximation factors depending on whether one desires a pure approximation, meeting the multiplicity constraints exactly, or a bicriteria factor that approximates the multiplicity constraints by a $(1 + \epsilon)$-multiplicative factor. We obtain an $\ell_0$-sparse bound in the former setting and an $\ell_1$-sparse bound in the latter.

The algorithms for either setting are similar, with a slight difference in the choice of parameters $\alpha$ and $\epsilon$. We adopt the following common notation when proving either bound:

- Let $S = \{ j \in [n] : \alpha x_j \geq d_j \}$ be the set of coordinates that are deterministically set to their multiplicity, and let $z'' = d \wedge S$ denote the corresponding integral vector.
- Let $\mathcal{N} = [n] \setminus S = \{ j : x_j < \alpha \}$ be the set of coordinates that are not deterministically rounded up to $d$.
- Let $\mathcal{M} = \{ i : (Az'')_i < 1 - \epsilon \}$ be the constraints that are not (sufficiently) covered by $z''$.
- Let $x' = x \wedge \mathcal{N}$ restrict $x$ to the remaining coordinates in $\mathcal{N}$, and let $x'' = x \setminus x' = x \wedge S$ denote the part of $x$ deterministically rounded up.
- Let $A'$ be the residual covering matrix w.r.t $z''$.
- Let $z'$ be the random vector output by \textit{round-and-fix} w.r.t $A'$ and $x'$.
- Let $z = z' + z''$ be the combined output.

\textbf{Proof of Theorem 2.5} Consider \textit{contract-round-fix} with $\epsilon = 0$ and $\alpha = \ln \Delta_0 + \ln \ln \Delta_0 + O(1)$.

The cost of $z''$ is bounded above by
\[
\langle c, z'' \rangle = \sum_{j \in S} c_j d_j \leq \sum_{j \in S} c_j \alpha x_j = \alpha \langle c, x'' \rangle,
\]
where $x''$ restricts $x$ to the coordinates in $S$.

The second vector, $z'$, is the output of \textit{round-and-fix} for the residual covering system w.r.t $z''$. Since $x$ satisfies the knapsack covering constraints, and $A' \in [0,1]^{M \times \mathcal{N}}$ is (rescaled) the residual system after deterministically rounding up the coordinates in $S$, we have $A' x' \geq 1$. All the coordinates in $A'$ lie between 0 and 1, and the $L_0$ column sparsity of $A'$ is bounded above by
\[
\max_{j \in \mathcal{N}} \{ i \in \mathcal{M} : A'_{ij} > 0 \} = \max_{j \in \mathcal{N}} \{ i \in \mathcal{M} : A_{ij} > 0 \} \leq \max_{j \in [n]} \{ i \in [m] : A_{ij} > 0 \} = \Delta_0.
\]
By Theorem 2.4, round-and-fix returns a vector $z''$ with coverage $A'z'' \geq 1$, multiplicities $z'' \leq [\alpha x''] \leq d \land \mathcal{N}$, and expected cost $\mathbb{E}[(c,z')] \leq (\alpha + O(1))(c,x')$.

It remains to combine the bounds for $z'$ and $z''$ and analyze $z$. The expected cost of $z$ is

$$\mathbb{E}[(c,z)] = \mathbb{E}[(c,z')] + \mathbb{E}[(c,z'')] \leq (\alpha + O(1))(c,x') + \alpha(c,x'') = (\alpha + O(1))(c,x).$$

For the multiplicity constraints, we have

$$z = z' + z'' \leq d \land S + d \land \mathcal{N} = d.$$

For the coverage, for each $i \in [m]$, we consider two cases depending on whether $i \in \mathcal{M}$ or not. If $i \notin \mathcal{M}$, then

$$(Az)_i \geq (Az'')_i \geq 1.$$

If $i \in \mathcal{M}$, then for $b_i = 1 - (Az'')_i$, we have

$$(Az)_i = (Az')_i + (Az'')_i = 1 - b_i + b_i(A'z'')_i \geq 1 - b_i + b_i = 1,$$

as desired. \hfill \blacksquare

Now we consider the bicriteria approximation based on $\ell_1$ sparsity.

**Proof of Theorem 2.6.** Consider contract-round-fix for a given $\epsilon > 0$.

The max column sum of $A'$ is

$$\Delta'_1 = \max_{j \in \mathcal{N}} \sum_{i \in \mathcal{M}} A'_{i,j} \leq \max_{j \in \mathcal{N}} \sum_{i \in \mathcal{M}} \frac{A_{i,j}}{b_i} \leq \max_{j \in \mathcal{N}} \frac{1}{\epsilon} \sum_{i \in \mathcal{M}} A_{i,j} \leq \frac{1}{\epsilon} \max_{j \in [m]} \sum_{i=1}^m A_{i,j} = \frac{\Delta_1}{\epsilon}.$$

By Theorem 2.3, $z'$ has coverage $A'z' \geq 1$, expected cost $\mathbb{E}[(c,z')] \leq \alpha'(c,x')$, and multiplicities $z' < \alpha x' + 1$ for $\alpha' = \ln \Delta'_1 + \ln \ln \Delta'_1 + O(1)$. Observe that, plugging in $\Delta_1/\epsilon$ for $\Delta'_1$, we have $\alpha' = \ln \Delta_1 + \ln \ln \Delta_1 + O(\ln(1/\epsilon)) = \alpha$. Thus, $\mathbb{E}[(c,z')] \leq \alpha(c,x')$ and $z' \leq \lfloor \alpha x' \rfloor \leq d$.

Then $z''$ has cost $\sum_{j \in S} c_j d_j \leq \alpha(c,x'')$, so the expected total cost is

$$\mathbb{E}[(c,z' + z'')] = \mathbb{E}[(c,z')] + \mathbb{E}[(c,z'')] \leq \alpha((c,x') + (c,x'')) = \alpha(c,x).$$

Since $z'$ and $z''$ have disjoint support, and both $z' \leq d$ and $z'' \leq d$ individually, we have $z' + z'' \leq d$.

Finally, for each constraint $i$, we have either $i \notin \mathcal{M}$, in which case $A(z' + z'')_i \geq A''_i \geq 1 - \epsilon$ by definition of $\mathcal{M}$; or $i \in \mathcal{M}$, in which case

$$A(z' + z'')_i \geq (Az'')_i + (1 - (Az'')_i)(A'z')_i \geq 1 - b_i + b_i = 1,$$

as desired. \hfill \blacksquare

### 2.4 Derandomization

In this section, we apply the method of conditional expectations to derandomize the alteration-based rounding schemes for column sparse covering problems. Recall that the round-and-fix algorithm consists of a randomized rounding step followed by an alteration step. In particular, randomization only enters when deciding whether to round a coordinate up or down initially. We thus apply the method of conditional expectations to the sequence of coin tosses that round each coordinate up or down.
For ease of exposition, we focus on derandomizing w.r.t \( \Delta_1 \). Fix a scalar \( \alpha > 0 \) and fractional solution \( x \in \mathbb{R}^n_{\geq 0} \) with \( Ax \geq 1 \). For each \( i \in [m] \), we define \( \psi_i(y) : \{0,1\}^n \rightarrow \mathbb{R}_{\geq 0} \) as the minimum of two functions, \( \psi_i(y) = \min\{\psi_i'(y), \psi_i''(y)\} \), where \( \psi_i' : \{0,1\}^n \rightarrow \mathbb{R}_{\geq 0} \) is defined by

\[
\psi_i'(y) = \alpha(Ax)_i \prod_{j=1}^n (\alpha Ax)_i^{-(A_{i,j}[\alpha x_j]+A_{i,j} y_j)},
\]

and \( \psi_i'' : \{0,1\}^n \rightarrow \mathbb{R}_{\geq 0} \) is defined by

\[
\psi_i''(y) = \left( \frac{\alpha(Ax)_i}{2} \right)^{1/\rho_i} \prod_{j: A_{i,j} \leq \rho_i} \left( \frac{\alpha(Ax)_i}{2} \right)^{-(A_{i,j}[\alpha x_j]+A_{i,j} y_j)}/\rho_i.
\]

We define \( \Phi : \{0,1\}^n \rightarrow \mathbb{R}_{\geq 0} \) by

\[
\Phi(y) = \langle c, [\alpha x] \rangle + \langle c, y \rangle + C \sum_{i=1}^m \psi_i(y) \sum_{j=1}^n c_j A_{i,j} x_j.
\]

where \( C \in \mathbb{R}_{>0} \) is the constant specified by Lemma 2.11. The formula for \( \Phi(y) \) appears involved, but has a simple interpretation as a pessimistic estimator for the expected cost as a function of the randomized rounding in steps (2).

**Lemma 2.14.** Let \( \alpha = \ln(\Delta_1) + \ln \ln(\Delta_1) + O(1) \), and let \( z \in \mathbb{Z}^n_{\geq 0} \) be the randomized integral vector produced by round-and-fix. Let \( y \in \{0,1\}^n \) be the random vector with independent coordinates where \( y_i = 1 \) if \( z_i \) is rounded up in step (2), and \( y_i = 0 \) otherwise. Then

\[
E[\Phi(y)] \leq (\alpha + O(1))\langle c, x \rangle.
\]

**Proof sketch.** The claim is implicit in the proof of Theorem 2.3 as \( E[\Phi(y)] \) is an intermediate upper bound on \( E[\langle c, z \rangle] \) in the full chain of inequalities in the proof. The expectation of the first two terms, \( \langle c, [\alpha x] \rangle + E[\langle c, y \rangle] \), gives the cost from the randomized rounding. For each \( i \), \( \psi_i \) is a pessimistic estimator for the probability that \( (Az)_i < 1 \), occurring in the proof of the Chernoff inequality and implicitly bounded from above when we invoked Lemma 2.3 and Lemma 2.12. The full relationship between \( E[\Phi(y)] \) and Theorem 2.3 is: \( E[\langle c, z \rangle] \leq E[\Phi(y)] \leq (\alpha + O(1))\langle c, x \rangle \). \(\square\)

**Lemma 2.15.** \( \langle c, [x] \rangle + \langle c, y \rangle + C \sum_{i: (A([\alpha x]+y))_i < 1} \sum_{j=1}^n c_j A_{i,j} x_j \leq \Phi(y) \).

**Proof.** Subtracting out common terms, the claim is equivalent to showing that

\[
C \sum_{i: (A([\alpha x]+y))_i < 1} \sum_{j=1}^n c_j A_{i,j} x_j \leq C \sum_{i=1}^m \sum_{j=1}^n c_j A_{i,j} x_j.
\]

Since \( \psi_i(y) \geq 0 \) for all \( i \), it suffices to show that \( \psi_i(y) \geq 1 \) whenever \( (A([\alpha x]+y))_i < 1 \). Indeed, if this is the case, then

\[
\psi_i'(y) = (\alpha Ax)_i^{1-(A([\alpha x]+y))_i} > (\alpha Ax)_i^0 = 1,
\]

and

\[
\psi_i''(y) = \left( \frac{\alpha Ax_i}{\rho_i} \right)^{1-(A([\alpha x]+y))_i/\rho_i} > \left( \frac{\alpha Ax_i}{\rho_i} \right)^0 = 1,
\]

as desired. \(\square\)
Theorem 2.16. Let \( x \in \mathbb{R}_{\geq 0}^n \) with \( Ax \geq \mathbb{1} \). In nearly linear deterministic time, one can compute a vector \( z \) with coverage \( Az \geq \mathbb{1} \), cost \( \langle c, z \rangle \leq (\ln \Delta_1 + \ln \ln \Delta_1 + O(1)) \), and multiplicities \( z < \alpha x + \mathbb{1} \).

Proof. We apply the method of conditional expectations to \( \Phi(y) \), where initially \( y \in \{0, 1\}^n \) is a randomized vector with independent coordinates and \( E[y] = \alpha x - |\alpha x| \). For \( j = 1, \ldots, m \), we fix \( y_j \) to either 0 or 1 as to not increase the conditional expectation \( E[\Phi(y) \mid y_1, \ldots, y_j] \).

Note that \( E[\Phi(y) \mid y_1, \ldots, y_j] \) is easily computable. Moreover, one can arrange a simple data structure such that upon advancing \( i \) to the next index, the next conditional expectation \( E[\Phi(y) \mid y_1, \ldots, y_j] \) can be recomputed in time that is linear in the number of nonzeros in the \( j \)th column of \( A \).

At the end, by Lemma 2.14 we have a fixed point \( \hat{y} \in \{0, 1\}^n \) such that \( \Phi(\hat{y}) \leq E[\Phi(y)] \leq (\alpha + O(1))\langle c, x \rangle \). We take the integral vector \( |\alpha x| + \hat{y} \) and fix the unmet constraints with Lemma 2.11. By Lemma 2.15 this solution has cost at most \( \Phi(\hat{y}) \leq (\alpha + O(1))\langle c, x \rangle \), as desired. ■

2.5 Improved bound for CIP\(_\infty\) when \( b_{\min} \) is large

We now consider the case when \( Ax \geq b \) where \( b_i \geq b_{\min} > 1 \) for \( i \in [m] \). Recall that \( \Delta_1 \) is the maximum column sum of \( A \). We can scale each row \( i \) by \( b_i \) to obtain a system \( A'x \geq \mathbb{1} \). If we let \( \Delta'_1 \) denote the maximum column sum of \( A' \), we see that \( \Delta'_1 \leq \Delta_1 / b_{\min} \). The analysis that we have already seen would yield an approximation ratio of \( \ln \Delta'_1 + \ln \ln \Delta'_1 + O(1) \) assuming that \( \Delta'_1 \) is sufficiently large. In fact one can obtain a better bound of the form \( \frac{\ln \Delta_1}{b_{\min}} + \ln \frac{\ln \Delta_1}{b_{\min}} + O(1) \).

Here we assume that \( \frac{\ln \Delta_1}{b_{\min}} \) is sufficiently large constant. The analysis closely mimics the one in Section 2.2 and we only highlight the main changes.

The algorithm round-and-fix generalizes in the obvious fashion to the setting when \( b \geq \mathbb{1} \). After the randomized rounding step each constraint \( i \) that is uncovered, that is, \( (Az)_i < b_i \) is greedily fixed by solving a Knapsack Cover problem. We analyze as follows. Lemma 2.9 easily generalizes to show that for each row \( i \), there is a median coefficient \( \rho_i (w/r/t x) \) such that \( \sum_{j:A_{i,j} \leq \rho_i} A_{i,j}x_j \geq b_i/2 \) and \( \sum_{j:A_{i,j} \geq \rho_i} A_{i,j}x_j \geq b_i/2 \). Lemma 2.11 also generalizes to show that the fixing cost for \( i \) is at most \( 4 \sum_{j:A_{i,j} \geq \rho_i} c_j A_{i,j}x_j \). We address the changes to Lemma 2.12 and Lemma 2.13. The probability that constraint \( i \) is not covered is:

\[
P[(Az)_i < b_i] \leq \exp\left(\frac{b_i(1 + \ln(\alpha/2) - \alpha/2)}{\rho_i}\right),
\]

where we used the Chernoff inequality given by Lemma 3.5 with \( \gamma = \rho_i, \mu = \alpha b_i/2 \) and \( \beta = b_i \). We also have the following bound.

\[
P[(Az)_i < b_i] \leq \exp(b_i(1 + \ln(\alpha) - \alpha)).
\]

Our goal is to show that if \( \alpha = \frac{\ln \Delta_1}{b_{\min}} + \ln \frac{\ln \Delta_1}{b_{\min}} + O(1) \) then \( P[(Az)_i < b_i] \leq C \frac{\rho_i}{\Delta_1} \) for sufficiently large but fixed constant \( C \). We consider two cases as before. If \( \rho_i \geq 1/C \) then

\[
P[(Az)_i < b_i] \leq \exp(b_i(1 + \ln(\alpha) - \alpha)) \leq \frac{1}{\Delta_1} \leq C \frac{\rho_i}{\Delta_1}.
\]
Now suppose $\rho_i < 1/C$. We see that $P[(Az)_i < b_i] \leq \rho_i/\Delta_1$ if

$$\alpha \geq 2\frac{\rho_i}{b_i} \ln(\Delta_1) + 2\frac{\rho_i}{b_i} \ln \left(\frac{1}{\rho_i}\right) + 2 \ln(\alpha/2) + 2.$$ 

One can argue as before that with $C$ and the $O(1)$ term in $\alpha$ chosen sufficiently large the inequality holds true. With these facts in place, the expected cost of the solution is $(\alpha + C)\langle c, x \rangle$.

3 Fast Algorithm for Solving the Knapsack-Cover LP

In this section we develop a fast approximation scheme for $[\text{KC-LP}]$. The algorithm is based on speeding up an MWU based scheme by a combination of technical ingredients.

3.1 Reviewing the MWU framework and its bottlenecks

In this section, we give an overview of a width-independent version of the multiplicative weight update (MWU) framework, as applied to the dual of $[\text{KC-LP}]$. Along the way, we review the techniques of Carr et al. [5] and recover their running time. With some standard techniques, we improve the running time to nearly quadratic, and we identify two bottlenecks that we need to overcome to remove the quadratic factor.

For the remainder of this paper, we assume that $\epsilon \geq \frac{1}{\text{poly}(n)}$, since beyond this point one can use the ellipsoid algorithm instead.

Following Carr et al. [5], we apply the MWU framework to the dual of the LP $[\text{KC-LP}]$, which is the following pure packing problem.

$$\max \sum_{S,i} y_{S,i} \cdot b_{S,i} \quad \text{over } y : \mathcal{P}([n]) \times m \to \mathbb{R}$$

s.t. $\sum_{i=1}^{m} \sum_{S \subseteq [n]} A_{S,i,j} y_{S,i} \leq c_j$ for all $j \in [n]$  \hspace{1cm} (D)

Here $\mathcal{P}([n]) = \{S : S \subseteq [n]\}$ denotes the power set of $[n]$. The preceding LP has one variable for every constraint $i$ and every set $S \subseteq [n]$, and corresponds to a single knapsack covering constraint in the primal LP $[\text{KC-LP}]$. The LP (D) can be interpreted as packing knapsack covering constraints into the variables.

The MWU framework is a monotonic and width-independent algorithm that starts with an empty solution $y = 0$ to the LP (D) and increases $y$ along a sequence of Lagrangian relaxations to (D). Each Lagrangian relaxation is designed to steer $y$ away from items $j$ for which the packing constraint is tight. For each item $j$, the framework maintains a weight $w_j$ that (approximately) exponentiates the load of the $j$th constraint with the current solution $y$; i.e.,

$$\ln(c_j w_j) \approx \frac{\log n}{\epsilon} \cdot \frac{\sum_{i=1}^{m} \sum_{S \subseteq [n]} A_{S,i,j} y_{S,i}}{c_j}$$

for each $j \in [n]$. Initially, we have $w_j = \frac{1}{c_j}$ for each $j$. Each iteration, the framework solves the following Lagrangian relaxation of (D)

$$\max \sum_{S \subseteq [n], i \in [m]} y_{S,i} \cdot b_{S,i} \quad \text{over } z : \mathcal{P}([n]) \times [m] \to \mathbb{R}_{\geq 0}$$
\[
\text{s.t. } \sum_{j=1}^{n} w_j \sum_{i=1}^{m} \sum_{S \subseteq [n]} A_{S,i,j} z_{S,i} \leq \sum_{j=1}^{n} w_j c_j. \quad (R)
\]

Observe that the above relaxation biases the solution \( z \) away from items \( j \) with large weight \( w_j \), which are the items \( j \) for which the packing constraint w/r/t \( y \) is tight. Given an approximate solution \( z \) to the above, we add \( \delta z \) to \( y \) for a carefully chosen value \( \delta > 0 \) (discussed in greater detail below). The next iteration encounters a different relaxation, where the weights are increased to account for the loads increased by \( z \). Note that the weights \( w_j \) are monotonically increasing over the course of the algorithm.

At the end of the algorithm, standard analysis shows that the vector \( y \) satisfies \( \langle b, y \rangle \geq (1 - O(\epsilon)) \text{OPT} \) and that \( (1 - O(\epsilon))y \) satisfies all the packing constraints (see for example [43, 11]). The error can be made one-sided by scaling \( y \) up or down. Moreover, it can be shown that at some point in the algorithm, an easily computable rescaling of \( w \) is an \( (1 \pm \epsilon) \)-relative approximation for the original LP [KC-LP] (see for example [5, 7]). Thus, although we may appear more interested in solving the dual packing LP \( (D) \) we are approximating the desired LP [KC-LP] as well.

The choice of \( \delta \) differentiates this “width-independent” MWU framework from other MWU-type algorithms in the literature. The step size \( \delta \) is chosen small enough that no weight increases by more than an \( \exp(\epsilon) \)-multiplicative factor, and large enough that some weight increases by (about) an \( \exp(\epsilon) \)-multiplicative factor. The analysis of the MWU framework reveals that at all times. In particular, each weight can increase by an \( \exp(\epsilon) \)-multiplicative factor at most \( O \left( \frac{\ln n}{\epsilon^2} \right) \) times, so there are most \( O \left( \frac{n \ln n}{\epsilon^2} \right) \) iterations total.

3.1.1 Reduction to knapsack cover

An important aspect of the Lagrangian approach is that the 1-constraint packing problem \( (R) \) is much simpler to solve than the many-constraint packing problem \( (D) \). It suffices to approximately identify the best bang-for-buck coordinate indexed by \( S \subseteq [n] \) and \( i \in [m] \); i.e., approximately maximizing the ratio

\[
\frac{b_{S,i}}{\sum_{S,i,j} w_j A_{S,i,j}}
\]

and setting \( z = \gamma e_{S,i} \) for \( \gamma \) as large as possible within the single packing constraint. Carr et al. \([3]\) calls this choice of \( S \) and \( i \) the “most violated inequality”.

Carr et al. \([3]\) reduces the above search problem to a family of knapsack cover problems as follows. Fix \( i \in [m] \). Expanding out the definitions of \( b_{S,i} \) and \( A_{S,i,j} \), finding the set \( S \) maximizing the above is shown to be equivalent to

\[
\text{minimize } \frac{1}{\alpha} \sum_{j \notin S} w_j \min \{ A_{i,j}, \alpha \} \text{ over } \alpha > 0 \text{ and } S \subseteq [n] \text{ s.t. } \sum_{j \in S} A_{i,j} d_j \leq b_i - \alpha.
\]

If we let \( \hat{b}_i = \sum_{j=1}^{n} A_{i,j} d_j - b_i \) denote the total “excess” for the \( i \)th covering constraint, then we can rewrite the above as follows:

\[
\text{minimize } \frac{1}{\alpha} \sum_{j \in S} w_j \min \{ A_{i,j}, \alpha \} \text{ over } \alpha > 0 \text{ and } S \subseteq [n] \text{ s.t. } \sum_{j \in S} A_{i,j} d_j \geq \hat{b}_i + \alpha. \quad (KC)
\]
For fixed $i$ and $\alpha > 0$, $[\text{KC}]$ is a knapsack covering problem. For the sake of a $(1 + O(\epsilon))$-multiplicative approximation to $[\text{R}]$, we can approximate the objective by a $(1 + O(\epsilon))$-multiplicative factor, but we must satisfy the covering constraint exactly.

A few basic observations by Carr et al. \[5\] allow us to guess $\alpha$ by exhaustive search. To obtain a $(1 + \epsilon)$-multiplicative approximation to $[\text{R}]$ we can afford to round $\alpha$ up to the next integer power of $(1 + \epsilon)$. Since the nonzero coefficients $A_{i,j}$ all lie in the range $[1/C, C]$, it suffices to check $\alpha$ for just $\log_{(1+\epsilon)} C = O(\log(C)/\epsilon)$ powers of $(1 + \epsilon)$. We let $A$ denote the set of $O\left(\frac{\log C}{\epsilon}\right)$ values of $\alpha$ of interest.

A constant factor approximation to $[\text{KC}]$ can be obtained in $O(n \log n_i)$ time \[15\], and there are approximation schemes (discussed in greater detail in Section 3.3) with running times on the order of $O(n_i + \text{poly}(1/\epsilon))$, where $n_i$ is the number of nonzeros in the $i$th row of $A$. Carr et al. \[5\] solves the relaxation $[\text{R}]$ by applying a FPTAS for knapsack cover to each choice of $i$ and $\alpha$. For larger values of $m$, by multiplying the running time of the FPTAS with the number of choices of $\alpha$ per $i$, and summing over $i \in [m]$, each instance of $[\text{R}]$ takes $\tilde{O}\left(\frac{N \log C}{\epsilon} + \frac{m \log C}{\text{poly}(\epsilon)}\right)$ time to approximate.

In a straightforward implementation of the MWU framework, each iteration also requires $O(n)$ time to adjust the weight of each item. With $O(n \log(n)/\epsilon^2)$ iterations total, we achieve a running time of $O\left(\frac{nN \log C}{\epsilon^3} + \frac{mn \log C}{\text{poly}(\epsilon)}\right)$. This gives the running time described in Carr et al. \[5\].

3.1.2 Thresholding

A standard technique called “lazy greedy”, “thresholded greedy”, or “lazy bucketing” in the literature immediately reduces the running time to $\tilde{O}\left( (N \log C + n^2) \text{poly}(1/\epsilon) \right)$. Observe that the optimum ratio is monotonically decreasing as the weights $w_j$ are monotonically increasing; equivalently, the optimum value of $[\text{KC}]$ is monotonically increasing for each $\alpha$ and $i$ as the weights $w_j$ are increasing. This allows us to employ the following thresholding technique, also used within MWU frameworks in \[19, 8, 7, 9\].

We maintain a threshold $\lambda > 0$ such that $\lambda$ is less than the optimal value of $[\text{KC}]$ for all $i \in [m]$ and $\alpha \in A$. The first value of $\lambda$ is obtained by applying a constant factor approximation algorithm to $[\text{KC}]$ for each $i \in [m]$ and $\alpha \in A$ and setting $\lambda$ to be a constant factor less than the minimum cost over all $i \in [m]$ and $\alpha \in A$. If $\lambda$ is a lower bound for $[\text{KC}]$, then any solution $S$ and $i$ with ratio $\leq (1 + O(\epsilon))\lambda$ leads to a $(1 + O(\epsilon))$-multiplicative approximation to $[\text{R}]$. Thus, for a fixed value of $\lambda$, we solve each $i \in [m]$ and $\alpha \in A$ in round-robin fashion, taking any $(1 + \epsilon)$-approximation $S$ with value $\leq (1 + O(\epsilon))\lambda$, or continuing to the next choice of $i$ and $\alpha$ if the returned approximation has value $\geq (1 + O(\epsilon))\lambda$. If all $i \in [m]$ and $\alpha \in A$ generate approximations of value $\geq (1 + O(\epsilon))\lambda$, then we can safely increase $\lambda$ to $(1 + \epsilon)\lambda$. Observe that since each weight $w_j$ increases by at most a $n^{O(1/\epsilon)}$-multiplicative factor over the entire algorithm, and then initial choice of $\lambda$ is within a constant factor of the optimal value for the initial values of $w_j$, $\lambda$ stays within a $n^{O(1/\epsilon)}$-multiplicative value of its initial value. In particular, $\lambda$ is bumped up at most $\log_{(1+\epsilon)}\left(n^{O(1/\epsilon)}\right) = O\left(\frac{\log n}{\epsilon^2}\right)$ times.

Each time we approximate an instance of $[\text{KC-LP}]$ for $i \in [m]$ and $\alpha \in A$, we either (a) find a good approximation to the relaxation $[\text{R}]$, or (b) declare that no solution has value $\leq (1 + \epsilon)\lambda$ for this choice of $i$ and $\alpha$ and put the choice of $i$ and $\alpha$ aside until the next value of $\lambda$. That is, each approximated knapsack cover problem can be charged to either an iteration of the framework, of which there are $O\left(\frac{n \log n}{\epsilon^2}\right)$, or a new threshold for this choice of $i$ and $\alpha$. This leads to a running
time on the order of $\tilde{O}\left(\frac{n^2}{\epsilon^2} + \frac{N \log C}{\epsilon^3} + (m + n) \text{poly}\left(\frac{1}{\epsilon}\right)\right)$.

3.2 Two bottlenecks

Our goal, as stated in Theorem 1.3, is a fast running time on the order of

$$\tilde{O}\left(\frac{N \log C}{\epsilon^3} + (m + n) \text{poly}(1/\epsilon)\right).$$

The bottleneck of $\tilde{O}\left(\frac{n^2}{\epsilon^2}\right)$ appears necessary for at least two basic reasons. First, there are $\Omega\left(\frac{n \log n}{\epsilon^2}\right)$ iterations, and each iteration requires a solution $z$ to the relaxation (R). Here $z$ is a vector indexed by $[m]$ and the power set of $[n]$ — an $m2^n$-dimensional space. Even the index of a nonempty coordinate $(S, i) \in \text{support } z$ requires $\Omega(\log(m2^n)) \geq \Omega(n)$ bits to write down. Thus writing out explicitly a solution to (R) in each iteration — let alone computing it — generates a $\Omega\left(\frac{n^2 \log n}{\epsilon^2}\right)$ lower bound. Stepping out of the MWU framework, LP duality tells us that (D) can be minimized by a vector $y$ with support $|\text{support } y| \leq n$. If $y$ has $n$ nonzeros, then even in a sparse explicit representation of $y$, we need $\Omega(n^2)$ to list the supporting indices. Thus the descriptive complexity of optimal solutions to (D) is seemingly at least $\Omega\left(n^2\right)$. A second bottleneck arises from the weight updates. By the formula (1), the logs of the weights should track the loads of each packing constraint. However, each iteration may increase the load of every packing constraint, so updating each weight explicitly can require $\Omega(n)$ time per iteration. Thus even the innocuous weight updates makes a faster running time discouraging.

3.3 Approximation schemes for knapsack cover

In this section, we review a classical FPTAS for knapsack cover and set the stage for a more sophisticated integration with the MWU framework. In the knapsack cover problem, we are given positive costs and sizes for $n$ items and a positive real-valued size $b$ of a knapsack; we want to find the minimum (sum) cost subset of items whose sizes sum to at least the size of the knapsack. In our setting, we have a sequence of such problems, and the costs and sizes are dictated via the MWU framework per equation (KC) for fixed $i \in [m]$ and $\alpha > 0$. The cost of an item $j$ is $\text{cost}(j) \equiv \frac{w_j}{\alpha} \min\{A_{i,j}, \alpha\}$, and the size of an item $j$ is $\text{size}(j) = A_{i,j}d_j$. For each $j$, $\text{cost}(j)$ depends linearly on the weight $w_j$ and $\text{size}(j)$ is held constant throughout the algorithm. We want to update our solution quickly when a weight $w_j$ is increased by the framework. We are allowed to output solutions that are within a $(1 + O(\epsilon))$-multiplicative factor greater than the optimal objective, but insist on filling the knapsack completely.

Fix $i \in [m]$ and $\alpha > 0$ and consider equation (KC). We let $n_i$ denote the number of nonzeros in the row $A_{i,j}$. Ignoring $j \in [n]$ such that $A_{i,j} = 0$, $n_i$ is the effective number of items in the current knapsack cover problem.

There are several known approximation schemes for knapsack cover that run in time $\tilde{O}(n_i + \text{poly}(1/\epsilon))$. Here we focus on the similar approaches of Ibarra and Kim [23] and Lawler [27]. A sketch of the algorithm (following [27] in particular) is given in Figure 3. The algorithm combines two basic ideas. First, a greedy heuristic can work fairly well. If every item has small
DP+greedy(cost(1),...,cost(n); size(1),...,size(n); b, β)

// we assume without loss of generality that each item has size size(j) ≤ b, and that
// β is a constant factor approximation for OPT.

1. Compute a collection of $O\left(\frac{1}{\epsilon^2}\right)$ pareto optimal sets $S$ over the expensive items
   (with cost $\text{cost}(j) ≥ \epsilon \beta$) w/r/t the truncated costs
   $\tilde{\text{cost}}(j) = \left\lfloor \frac{\text{cost}(j)}{\epsilon^2 \beta} \right\rfloor \epsilon^2 \beta$.

2. For each pareto-optimal set $S \in S$, greedily add cheap items (with cost $\text{cost}(j) < \epsilon \beta$) to $S$ in increasing order of cost-to-size ratio until $S$ fills the
   knapsack.

3. Return the best solution $S \in S$

Figure 3: High-level sketch of the algorithm by Lawler [27] for approximating knapsack cover problems.

cost relative to the optimum value, then the greedy algorithm repeatedly taking the item with
minimum cost-to-size ratio until the knapsack is filled is a good approximation. When costs are
large, the greedy heuristic can be modified to provide a constant factor approximation within the
same $O(n \log n)$ running time [15]. For expensive items (relative to the optimum value), one can
take advantage of the fact that (a) only a few expensive items can fit in any optimum solution, and
that (b) expensive items can have their costs discretized while changing their costs by only a small
relative factor. After discretization, the expensive items in the optimum solution can be efficiently
guessed by dynamic programming, and the discretization only introduces a small relative error.

We require the following facts about the algorithm DP+greedy. We state the running time w/r/t
the total number of items in a generic problem, denoted by $n$, which would be replaced by $n_i$ in our
particular setting.

Lemma 3.1 ([23, 27]).

1. If $\beta$ is a constant factor approximation of the optimum value, then
   DP+greedy returns a $(1 + O(\epsilon))$-multiplicative approximation to the minimum cost knapsack
   cover.

2. If the items are sorted in increasing order of cost-to-size ratio, and the prefix sums over the
   sorted list w/r/t size are precomputed, the greedy algorithm can be simulated in $O(\log n)$ time.
   After preprocessing all the cheap items (with cost $\text{cost}(j) < \epsilon \beta$) in this way, the greedy aug-
   mentation in line 3 can be implemented in $O(\log n)$ time for each set $S \in S$.

3. If, for each $k \in \mathbb{N}$, the expensive items of truncated cost $\tilde{\text{cost}}(j) = k \epsilon^2 \beta$ are sorted in decreasing
   order of size, then line 2 can be computed in $O\left(\frac{1}{\epsilon^2}\right)$ time.

Marrying DP+greedy with the MWU framework in $\tilde{O}\left(\text{poly}\left(\frac{1}{\epsilon}\right)\right)$ time per iteration has two
components. First, we want to make DP+greedy partially dynamic as the costs $\text{cost}(j)$ are increased
via increments to weights $w_j$. Second, after computing a good solution, we want to be able to

[15] to obtain a constant factor approximation. Faster or incomparable running times for maximum knapsack have
been achieved since by more sophisticated techniques [24, 33, 5]; we follow [23, 27] for the sake of simplicity.

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simulate a weight update along the solution in $\tilde{O}\left(\text{poly}\left(\frac{1}{\epsilon}\right)\right)$ amortized time. There are basic reasons (discussed earlier in Section 3.2) why neither component should be feasible.

### 3.4 Dynamically updating the minimum cost knapsack

The first goal is to be able to respond to weight updates and generate the solution to the next knapsack cover problem quickly. By Lemma 3.1, this boils down to two basic data structures. First, we need to be able to maintain items in increasing order of cost-to-size ratio along with the prefix sums w/r/t size in order to reduce the greedy algorithm to a binary search. This allows us to greedily augment each candidate set $S \in \mathcal{S}$ with inexpensive items in line 3 in $O(\log n)$ time per set. Second, we need to maintain, for each expensive item, all the expensive items with the same truncated cost in descending order of size. Note that a constant factor approximation $\beta$ is provided by the threshold $\lambda$ from the lazy greedy thresholding scheme of Section 3.1.2. Since only a constant factor is required, we actually set and maintain $\beta = 2^{\lfloor \log \lambda \rfloor}$.

We address the second point first, because it is much simpler. We need to maintain, for each $k \in \mathbb{N}$, the set of items with truncated cost $\tilde{\text{cost}}(j) = k\epsilon^2\beta$ sorted in descending order of size. This is very easy — when an item’s cost increases, we reinsert it into the appropriate sorted list in $O(n_i)$ time; when $\beta$ increases, we rebuild all the lists, from scratch, in $O(n_i \log n)$ time.

**Lemma 3.2.** In $O(\log n)$ time per weight update, and $O(n_i \log n)$ time per update to $\beta$, one can maintain, for each $k \in \mathbb{N}$, the expensive items of truncated cost $\tilde{\text{cost}}(j) = k\epsilon^2\beta$ sorted in descending order of size.

The next step is a data structure to facilitate the greedy algorithm in lines 1 and 3. If the items are sorted in ascending order or cost-to-size ratio, and the prefix sums w/r/t size are precomputed, then the greedy algorithm can be implemented by a binary search. In the face of dynamically changing costs, we can maintain both these values with dynamic tree data structures. Our setting is simpler because the range of possible costs of a particular item is known in advance, as follows.

**Observation 3.3.** For fixed $j \in [n]$ with $A_{i,j} \neq 0$, let

$$C_j = \left\{ \left(1 + \frac{\epsilon}{\alpha}\right)^k \min\{A_{i,j}, \alpha\} : k \in \{0, 1, \ldots, O\left(\frac{\log m}{\epsilon^2}\right)\} \right\}.$$ 

Then $\text{cost}(j) \in C_j$.

For $j \in [n]$ with $A_{i,j} \neq 0$, let $\mathcal{R}_j = \{\alpha/\text{size}(j) : \alpha \in C_j\}$ be the set of possible cost-to-size ratios for $j$. For any $j$ with $A_{i,j} \neq 0$, we have $|\mathcal{R}_j| = |C_j| \leq O\left(\frac{\log n}{\epsilon^2}\right)$.

Knowing all the possible ratios in advance allows for a simpler data structure. We will benefit from the simplicity later when we need to incorporate efficient weight updates. Consider the set $\mathcal{L} = \{(j, \alpha) : A_{i,j} \neq 0 \text{ and } \alpha \in \mathcal{R}_j\}$. $\mathcal{L}$ consists of all possible assignments of ratios to items. We have $|\mathcal{L}| \leq O\left(\frac{n_i \log n}{\epsilon^2}\right)$ and consider $\mathcal{L}$ as a sorted set based on the ratio, breaking ties arbitrarily.

We build a balanced binary tree over $\mathcal{L}$. We mark a leaf $(j, \alpha)$ as occupied iff $\text{cost}(j) \leq \epsilon\beta$ and $\text{cost}(j)/\text{size}(j) = \alpha$. For each internal node, we track the sum size of all items in leaves marked as occupied. The tree has depth $O(\log n)$, and in particular we can update the tree in $O(\log n)$ time when an item’s weight increases. The tree can be rebuilt $O(n_i \log n)$ time per update to $\beta$.

As the leaves are in ascending order of cost-to-size ratio, any set of items considered by the greedy algorithm corresponds to the occupied leaves of an interval in the range tree. The total size
of a greedy set is the total size of occupied leaves in the corresponding interval. To compute the sum size of a set of greedily selected items, we first decompose the corresponding interval in the range tree to the disjoint union of $O(\log n)$ subtrees. For each subtree, the root is labeled with the sum size of all occupied leaves in the subtree. Summing together these $O(\log n)$ values gives us the total size of the greedy sequence.

With this data structure, we can simulate the greedy algorithm in $O(\log n)$ time by running a binary search for the shortest prefix that fills the knapsack.

**Lemma 3.4.** In $O\left(\frac{n_i \log n}{\epsilon^2}\right)$ time initially, $O(\log n)$ time per weight update, and $O(n_i \log n)$ time per update to $\beta$, one can maintain a data structure that simulates the greedy algorithm over the cheap items (with cost $\text{cost}(j) < \epsilon \beta$) in $O(\log n)$ time.

### 3.5 Updating weights along a knapsack cover solution

In Section 3.4, we showed how to modify known FPTAS’s for knapsack cover so that it can respond to increases in costs quickly without redoing everything from scratch. This removes one bottleneck from the overall MWU framework, in that the Lagrangian relaxations can now be solved in $\tilde{O}(\text{poly}(1/\epsilon))$ amortized time per weight update.

There is still another bottleneck just as important as computing an approximation $S$ to the knapsack cover problem; namely, simulating a weight update $w/r/t$ the solution $S$. Let $S$ be a solution to (KC) for fixed $i$ and $\alpha$. The MWU framework dictates that the weights $w$ should update to a new set of weights $w'$ per the following formula:

$$w'_j = \begin{cases} 
w_j & \text{if } j \notin S \\
\exp\left(\frac{\epsilon A_{S,i,j}/c_j}{\max_{\hat{j} \in S} A_{S,i,\hat{j}}/c_{\hat{j}}}\right)w_j & \text{if } j \in S.
\end{cases}$$

The basic problem is as follows. By the above formula, every coordinate $j \in S$ has its weight adjusted. Updating these weights directly requires $O(|S|)$ time to visit each item. Since $|S|$ may be as large as $n$, and there are $O\left(\frac{n \log n}{\epsilon^2}\right)$ iterations, this already leads to a running time of $\tilde{O}(n^2 \log n/\epsilon^2)$. With no assumptions on $S$, this is seemingly the best that one can expect.

We circumvent this lower bound by taking advantage of the structure of our solution $S$. A solution $S$ to the knapsack cover problem (KC) for fixed $i$ and $\alpha$ consists of $O\left(\frac{1}{\epsilon}\right)$ expensive items and a greedily selected sequence of cheap items. Since there are only $O\left(\frac{1}{\epsilon}\right)$ expensive items, we can spend $O\left(\frac{1}{\epsilon}\right)$ time to update each of them individually. By contrast, there is no upper bound on the number of cheap items in our solution.

Recall the data structure by which we simulate the greedy algorithm in the previous section. The items in greedy order in a balanced binary tree so that any subsequence of this order decomposes into the leaf sets of $O(\log n)$ disjoint subtrees. The subtrees implicitly define canonical intervals such that

1. There are a total of $O\left(\frac{n_i \log n}{\epsilon^2}\right)$ canonical intervals.
2. Any item appears in at most $O(\log n)$ canonical intervals.
3. Any greedy sequence decomposes into $O(\log n)$ canonical intervals.

This is essentially the same setup as in [8], where intervals are fractionally packed into capacitated points on the real line. Similar techniques are also employed in [7]. We briefly discuss the high-level ideas and refer to previous work [8, 7] for complete details. There is a small technical adjustment required that is discussed at the end.

Decomposing the solution into a small number of known static sets is important because weight updates can be simulated over a fixed set efficiently. More precisely, the data structure lazy-inc, defined in [8] and inspired by techniques by Young [43], simulates a weight update over a fixed set of weights in such a way that the time can be amortized against the logarithm of the increase in each of the weights. The total increase of any weight is bounded above by invariants revealed in the analysis of the MWU framework. It is easy to make lazy-inc dynamic, allowing insertion and deletion into the underlying set, in $O(\log n)$ time per insertion or deletion [7].

We define an instance of lazy-inc at each node in the balanced binary tree over cheap items as defined above. Whenever a leaf is marked as occupied, it is inserted into each of $O(\log n)$ instances of lazy-inc at the ancestors of the leaf; when a leaf is marked as unoccupied, it is removed from each of these instances as well. Each instance of lazy-inc can then simulate a weight update over the marked leaves at its nodes in $O(\log n)$ amortized time.

Given a greedy sequence of cheap items, we divide the sequence into the disjoint union of the marked leaves of $O(\log n)$ subtrees as discussed above. For each subtree, we simulate a weight update over the leaves via the instance of lazy-inc at the root of the subtree.

One final technical modification is required to make the algorithm sound. Each instance of lazy-inc accrues a small amount of error. Within a fixed choice of $i$ and $\alpha$, the sum of errors for a single weight is small because an item is tracked by only $O(\log n)$ instances of lazy-inc. Across all the choices of $i$ and $\alpha$, however, a single weight may be managed by $O(m \log(C) \log(n)/\epsilon)$ instances of lazy-inc. A similar accrual of error across instances of lazy-inc also arises in [7].

A final feature of the lazy-inc data structure, made explicit in [7], is that one can “flush” the error of an instance in $O(1)$ time per tracked item. We use this feature within the context of the overall lazy greedy algorithm. Recall that by thresholding the optimum value and trying choices of $i$ and $\alpha$ in round robin fashion, we move on from a fixed choice of $i$ and $\alpha$ iff the optimum value for these parameters have gone up by a full $O(1 + \epsilon)$-multiplicative factor, which occurs for any particular $i$ and $\alpha$ at most $O(\log(n)/\epsilon^2)$ times. Thus, whenever we move on from a fixed choice of $i$ and $\alpha$, we “flush” all the lazy-inc data structures for this choice of $i$ and $\alpha$ in $O(n, \log n)$ time, so that no error is carried across different values of $i$ and $\alpha$.

**Lemma 3.5.** One can extend the data structures of Lemma 3.4 such that, given a solution $S$ generated by Lemma 3.3 and Lemma 3.4 to an instance of (KC) for $i \in [m]$ and $\alpha > 0$, one can simulate a weight update over $(i, S)$ in $O(\log n)$ amortized time per iteration. The extension induces an overhead of $O(\log^2 n)$ per weight update, $O(n, \log n)$ per $(1 + \epsilon)$-factor increase to $\lambda$, and $O(n, \log^2 n)$ per constant factor increase to $\lambda$.

### 3.6 Putting things together

In this section, we summarize the main points of the algorithm and account for the running time claimed in Theorem 1.3.

**Proof of Theorem 1.3** By known analyses (e.g., [8, Theorem 2.1]), the MWU framework returns a $(1 - O(\epsilon))$-multiplicative approximation to the packing LP (D) as long as we can approximate the relaxation (R) in each of $O(n \log n/\epsilon^2)$ iterations. Moreover, each weight $w_j$ increases by at most a $m^{O(\frac{1}{\epsilon})}$-multiplicative factor, and we can afford to approximate $w_j$ by a multiplicative factor...
(1 ± O(1))-multiplicative factor and to obtain (1 ± O(1))-multiplicative approximation to each relaxation. Thus, if we only propagate a change to \(w_j\) when it increases by a \((1 + \epsilon)\)-multiplicative factor, correctness still holds, and we can assume that each \(w_j\) changes at most \(O(\log(n)/\epsilon^2)\) times.

In Section 3.1.1, solving [R] is reduced to \(O(m \log(C)/\epsilon)\) instances of minimum knapsack cover. The instances of minimum knapsack cover are visited in a round robin fashion, where an instance is given a value \(\lambda > 0\) and only needs to output a set \(S\) with ratio \(\leq (1 + O(\epsilon))\lambda\) if there exists a set with ratio \(\leq (1 + \epsilon)\lambda\). For each instance of knapsack cover, we maintain an \((1 + \epsilon)\)-approximate solution that outputs an approximate solution (in the thresholded sense) in \(O(\frac{1}{\epsilon^4})\) time. By Lemma 3.1, Lemma 3.4, and Lemma 3.5, the data structure takes \(O(\log(n))\) time to update per weight update from the framework, \(O(n_i \log^2 n)\) time to update every time \(\lambda\) increases by a power of 2, and \(O(n_i \log n)\) time every time \(\lambda\) increases by a \((1 + \epsilon)\)-multiplicative factor. Moreover, if an instance returns a solution within a \((1 + O(\epsilon))\)-multiplicative factor of the threshold \(S\), then it can simulate a weight update along the corresponding knapsack cover constraint \((S, i)\) in \(O(\log)\) amortized time (per iteration).

The threshold \(\lambda\) increases by a constant factor \(O(\log(n)/\epsilon)\) times and by a \((1 + \epsilon)\)-multiplicative factor \(O(\log(n)/\epsilon^2)\) times. The total time spent updating the data structure for a single instance of knapsack cover for the \(i\)th constraint is \(O(n_i \log^3 n/e^2 + n_i \log^2 n/e^2)\). The total time over all instances is thus \(O(N \log^3(n) \log(C)/\epsilon^2 + N \log^2(n) \log(C)/\epsilon^3)\). Querying an instance takes \(O(1/\epsilon^4)\) time, can be charged to either an iteration of the MWU framework or bumping the instance up to the next threshold. Thus \(O(\frac{m \log C}{\epsilon^3} + \frac{n \log n}{\epsilon^6})\) time is spent querying instances of knapsack cover. The total running time is at most \(O\left(\frac{N \log^3(n) \log(C)}{\epsilon^3} + \frac{m \log C}{\epsilon^3} + \frac{n \log n}{\epsilon^6}\right)\), as desired.

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A The regime of small $\Delta_1$

In this section, we consider the regime where $\Delta_1$ is asymptotically small. For $\Delta_1$ sufficiently small, we show that round-and-fix obtains an approximation ratio of $1 + O\left(\sqrt{\Delta_1 \ln(1/\Delta_1)}\right)$. Note that this bound is slightly weaker than the $1 + O\left(\sqrt{\Delta_1}\right)$ bound obtained by Chen et al. [12].

Let $\delta > 0$ be a small parameter to be determined later. (We will eventually set $\delta = c\sqrt{\Delta_1 \ln(1/\Delta_1)}$ for some constant $c > 0$.) We set the scaling factor $\alpha$ to

$$\alpha = \frac{1 + \delta + \delta^2/2}{1 - \delta}.$$ 

Note that $\alpha \leq (1 + (1 + \delta)\delta)\left(1 + \delta + \frac{\delta^2}{2}\right) \leq 1 + 2\delta + O(\delta^2)$ for sufficiently small $\delta > 0$.

The analysis again depends on identifying a “threshold” coefficient $\rho_i$ for each row $i$, but is no longer the weighted median coordinate (weighted by $x_j$). One basic reason that the definition of $\rho_i$ must be modified is as follows. Recall that considering only the coverage induced by $\rho_i$-small coefficients allows us to amplify the Chernoff inequality exponentially by a factor of $1/\rho_i$. (This in turn offsets the $1/\rho_i$ multiplicative factor in the fixing cost.) To apply the Chernoff bound, we required that the expected coverage of $\rho_i$-small (when scaling by up $\alpha$ and rounding) is greater than 1. In our case, $\alpha = 1 + O(\delta)$ is very close to 1. To apply the Chernoff inequality to $\rho_i$-small coordinates, we require that the fractional coverage of $\rho_i$-small coordinates is at least $1/\alpha$; that is, very close to 1, rather than just 1/2. Thus, for $i$ in $[m]$, we let $\rho_i$ be the weighted rank-$\delta$ largest coefficient as follows.

**Lemma A.1.** For each $i \in [m]$, there exists a value $\rho_i \in [0, \Delta_1]$ such that

$$\sum_{A_{ij} \leq \rho_i} A_{ij} x_j \geq 1 - \delta \quad \text{and} \quad \sum_{A_{ij} \geq \rho_i} A_{ij} x_j \geq \delta.$$ 

Choosing $\rho_i$ to be the rank-$\delta$ coefficient boosts the randomized rounding, since the expected coverage from rounding $\rho_i$-small coordinates is at least $(1 - \delta)\alpha$. On the other hand, rounding the $\rho_i$-big coordinates (which have only $\delta$ fractional coverage) leads to an integral vector with coverage $\delta$. To obtain an integral vector with coverage 1, one can take $1/\delta$ copies of the integral vector with coverage $\delta$, increasing the cost by a $1/\delta$-multiplicative factor.
Lemma A.2. For each $i \in [m]$, one can compute in near-linear time an integral vector $z \in \mathbb{Z}^n_{\geq 0}$ with cost
\[ \langle c, z \rangle \leq O \left( \frac{1}{\delta \rho_i} \sum_{j=1}^{n} c_j A_{ij} x_j \right) \]
and coverage $(Az)_i \geq 1$.

Proof. Consider the problem
\[ \min \langle c, z \rangle \text{ over } z \in \mathbb{Z}^n_{\geq 0} \text{ s.t. } (Az)_i \geq 1. \]
Observe that each coefficient $A_{ij}$ is $\leq \Delta_1$, so this is just an instance of Knapsack Cover for which there is a PTAS and a simple constant factor greedy algorithm, and the integrality gap is 2. Thus it suffices to find a fractional vector $y \in \mathbb{R}^n_{\geq 0}$ with (say) $\langle c, y \rangle \leq \frac{1}{\delta \rho_i} \sum_{j=1}^{n} c_j A_{ij} x_j$ and coverage $(Ay)_i \geq 1$.

Define $y \in \mathbb{R}^n_{\geq 0}$ by
\[ y_j = \begin{cases} \frac{1}{\delta} x_j & \text{if } A_{ij} \geq \rho_i, \\ 0 & \text{otherwise}. \end{cases} \]

First, $y$ has coverage
\[ \sum_j A_{ij} y_j = \frac{1}{\delta} \sum_{A_{ij} \geq \rho_i} A_{ij} x_j \geq 1, \]
by (a) choice of $\rho_i$. Second, $y$ has cost
\[ \langle c, y \rangle = \frac{1}{\delta} \sum_{A_{ij} \geq \rho_i} c_j x_j \leq \frac{1}{\delta \rho_i} \sum_{A_{ij} \geq \rho_i} c_j A_{ij} x_j \leq \frac{1}{\delta \rho_i} \sum_j c_j A_{ij} x_j, \]
as desired.

Lemma A.2 establishes an upper bound on the fixing cost. It remains to obtain concentration bounds for the probability of the $i$th constraint failing to be covered. Note that we need a probability of failure proportional to $\rho_i$ for each $i$ to cancel the $\frac{1}{\rho_i}$ factor in Lemma A.2. In fact, if we can show that $P[(Az)_i < 1] \leq \rho_i$ for all $i$, then the total expected fixing cost over all $i$ would sum to
\[ \left( \frac{\Delta_1}{\delta} \right) \langle c, x \rangle, \]
as desired. We first obtain the following.

Lemma A.3. For each $i \in [m]$, $P[(Az)_i < 1] \leq e^{-\frac{\rho_i^2}{2}}$.

Proof. We apply the Chernoff inequality to the $\rho_i$-small coordinates. From the choice of $\alpha$ and $\rho_i$ we have $E \left[ \sum_{j : A_{ij} \leq \rho_i} A_{ij} z_j \right] \geq \frac{\alpha}{1 - \frac{1}{\delta}} \geq 1 + \delta + \frac{\delta^2}{2}$. Therefore,
\[ P[(Az)_i < 1] \leq P \left[ \sum_{A_{ij} \leq \rho_i} A_{ij} z_j < 1 - \epsilon \right] \]
\[
\leq \exp\left(\frac{1}{\rho_i} \left(1 + \ln\left(1 + \delta^2 + \frac{\delta^2}{2}\right) - 1 - \delta - \frac{\delta^2}{2}\right)\right)
\]
\[
\leq \exp\left(\frac{1}{\rho_i} \left(1 + \ln\left(e^\delta\right) - 1 - \delta - \frac{\delta^2}{2}\right)\right)
\]
\[
= \exp\left(-\frac{\delta^2}{2\rho_i}\right).
\]

by (a) Taylor expansion of \(e^\delta\) for \(\delta > 0\).

It remains to choose \(\delta\) such that the upper bound in Lemma A.3 is at most \(O(\rho_i)\). For sufficiently small \(\Delta_1\), it suffices to take \(\delta \geq 2\sqrt{\Delta_1 \ln(1/\Delta_1)}\), as follows.

**Lemma A.4.** Let \(\delta \geq 2\sqrt{\Delta_1 \ln(1/\Delta_1)}\). For sufficiently small \(\Delta_1\), we have

\[\Pr[\langle Az\rangle_i < 1] \leq \rho_i.\]

**Proof.** By Lemma A.3 it suffices to show that \(\exp\left(-\frac{\Delta_1 \ln(1/\Delta_1)}{\rho_i}\right) \leq \rho_i\). If \(\rho_i \leq \Delta_1^2\), then we have

\[
\exp\left(-\frac{\Delta_1 \ln(1/\Delta_1)}{\rho_i}\right) \leq \frac{\rho_i^2}{2\Delta_1^2 \ln^2(1/\Delta_1)} \leq \rho_i,
\]

as desired. If \(\rho_i \geq \Delta_1^2\), then since \(\rho_i \leq \Delta_1\), we have

\[
\exp\left(-\frac{2\Delta_1 \ln(1/\Delta_1)}{\rho_i}\right) \leq \exp(-2\ln(1/\Delta_1)) = \Delta_1^2 \leq \rho_i,
\]

as desired.

We now add up the various costs and obtain the desired approximation factor.

**Theorem A.5.** **round-and-fix** returns a randomized vector \(z\) with \(Az \geq 1\) and

\[
\mathbb{E}[\langle c, z \rangle] \leq \left(1 + (4 + o(1))\sqrt{\Delta_1 \ln(1/\Delta_1)}\right)\langle c, x \rangle.
\]

**Proof.** Let \(\delta = 2\sqrt{\Delta_1 \ln(1/\Delta_1)}\). The total expected cost is sum of the expected cost of randomized rounding,

\[
\alpha\langle c, x \rangle = \left(1 + 2\delta + O(\delta^2)\right)\langle c, x \rangle = \left(1 + (4 + o(1))\sqrt{\Delta_1 \ln(1/\Delta_1)}\right)\langle c, x \rangle,
\]

and the total expected fixing cost. The total expected fixing cost is at most

\[
O\left(\sum_i \Pr[(Az)_i < 1] \frac{1}{\delta \rho_i} \sum_j c_j A_{ij} x_j\right) = O\left(\frac{1}{\delta} \sum_j c_j x_j \sum_i A_{ij} \Pr[(Az)_i < 1] \frac{1}{\rho_i}\right)
\]
\[
\leq O\left(\frac{1}{\delta} \sum_j c_j x_j \sum_i A_{ij}\right) \leq O\left(\frac{\sqrt{\Delta_1} \langle c, x \rangle}{\ln(1/\Delta_1)}\right)
\]

by (a) Lemma A.4. Thus the expected costs of randomized rounding and subsequently fixing add up to \(\left(1 + (4 + o(1))\sqrt{\Delta_1 \ln(1/\Delta_1)}\right)\langle c, x \rangle\), as desired.
Remark A.6. Our preceding analysis does not appear to be tight. The expected fixing cost is \( O\left(\frac{\sqrt{\Delta_1(c,x)}}{\ln(1/\Delta_1)}\right) \) for the choice of \( \alpha = 1 + \Omega\left(\sqrt{\Delta_1 \log(1/\Delta_1)}\right) \). The two terms in the cost are not balanced. We note that our analysis did not take into account a more careful fixing process that considers the expected value of the residual requirement after the initial random step. Our efforts in doing such a careful analysis have not so far succeeded in obtaining the bound \( 1 + O\left(\sqrt{\Delta_1}\right) \) which we believe is the right bound for the algorithm. In fact it is quite conceivable that the resampling framework from [12] implies such a bound for the alteration algorithm.

B Chernoff inequalities

The standard lower tail bound in multiplicative Chernoff inequalities is the following.

**Lemma B.1.** Let \( X_1, \ldots, X_n \in [0,1] \) be independent with \( \mu = \mathbb{E}\left[\sum_{i=1}^n X_i\right] \), and \( \delta \in (0,1) \). Then
\[
P\left[\sum_{i=1}^n X_i < (1-\delta)\mu\right] \leq \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^\mu. \quad \text{If } X_1, \ldots, X_n \in [0,\gamma] \text{ for some } \gamma \leq 1 \text{ then the bound, by scaling, improves to } P\left[\sum_{i=1}^n X_i < (1-\delta)\mu\right] \leq \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu/\gamma}.
\]

Typically a simplified version of the upper bound, namely, \( \exp(-\delta^2/2) \) is used. However, in the interest of leading constants, we need to employ Lemma B.1 carefully. For this reason, and also for use as pessimistic estimators for derandomization in Section 2.4, we isolate explicit (intermediate) forms that we need. Several of the following inequalities are standard and we include proofs for the sake of completeness.

**Lemma B.2.** Let \( X \in [0,1] \) be a random variable and \( t \in \mathbb{R} \). Then \( \mathbb{E}[\exp(tZ)] \leq \exp(\mathbb{E}[Z](e^t - 1)) \).

**Proof.** Consider the function \( f(x) = \exp(tx) \). \( f \) is convex. By (a) convexity of \( f \), for any fixed value of \( Z \), we have
\[
\exp(tZ) = f((1 - Z) \cdot 0 + Z \cdot 1) \leq (1 - Z)f(0) + Zf(1) = 1 - Z + Ze^t = 1 + Z(e^t - 1).
\]
By (b) taking expectations of both sides and (c) applying \( 1 + x \leq e^x \), we have
\[
\mathbb{E}[tZ] \leq 1 + \mathbb{E}[Z](e^t - 1) \leq \exp(\mathbb{E}[Z](e^t - 1)),
\]
as desired. \( \blacksquare \)

**Lemma B.3.** Let \( X_1, \ldots, X_n \in [0,1]^n \) be independent random variables, let \( \mu = \mathbb{E}\left[\sum_{i=1}^n X_i\right] \), and \( \beta \in \mathbb{R}_{>0} \). If \( \beta < \mu \), then
\[
P\left[\sum_{i=1}^n X_i < \beta\right] \leq \left(\frac{\mu}{\beta}\right)^\beta \prod_{i=1}^n \mathbb{E}\left[\left(\frac{\beta}{\mu}\right)^{X_i}\right] \leq \exp\left(\beta + \beta \ln\frac{\mu}{\beta} - \mu\right).
\]

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Proof. This lemma can be proven by appropriate substitutions into the standard Chernoff inequality, but we instead prove it directly. Let \( t > 0 \) be determined later. By (a) Markov’s inequality, (b) independence of the \( Y_i \)’s, (c) Lemma B.2 w.r.t \( Z = X_i \), we have

\[
P \left[ \sum_{i=1}^{n} X_i < \beta \right] = P \left[ \exp \left( -t \sum_{i=1}^{n} X_i \right) > \exp(-t\beta) \right] \overset{(a)}{\leq} E \left[ \exp \left( -t \sum_{i=1}^{n} X_i \right) \right] \exp(t\beta)
\]

\[
\overset{(b)}{=} \left( \prod_{i=1}^{n} E[\exp(-tX_i)] \right) \exp(t\beta) \overset{(c)}{\leq} \left( \prod_{i=1}^{n} \exp(E[X_i]e^{-t} - 1) \right) \exp(t\beta) = \exp(\mu(e^{-t} - 1) + t\beta).
\]

Consider the exponent as a function of \( t \), \( f(t) = \mu(\exp(-t) - 1) + t\beta \). We have

\[f'(t) = 0 \iff \mu \exp(-t) = \beta \iff t = \ln \frac{\mu}{\beta}.
\]

Consider \( t = \ln \left( \frac{\mu}{\beta} \right) \). Note that \( \mu > \beta \iff t = \ln \frac{\mu}{\beta} > 0 \), as required when \( t \) was declared above.

For this choice of \( t \), we have

\[
P \left[ \sum_{i=1}^{n} X_i < \beta \right] \leq \cdots \leq \exp \left( \beta - \mu + \beta \ln \frac{\mu}{\beta} \right).
\]

\[\blacksquare\]

Lemma B.4. Let \( X_1, \ldots, X_n \in [0, \gamma] \) be independent random variables for \( \gamma > 0 \), let \( \mu = E \left[ \sum_{i=1}^{n} X_i \right] \). Then

\[
P \left[ \sum_{i=1}^{n} X_i < 1 \right] \leq \mu^{1/\gamma} \prod_{i=1}^{n} E \left[ \mu^{-X_i/\gamma} \right] \leq \exp \left( \frac{1}{\gamma} (1 + \ln(\mu) - \mu) \right).
\]

Proof. By (a) scaling up and each \( X_i \) by a factor of \( 1/\gamma \), (b) applying Lemma B.3 (with \( \mu \) scaled up by a factor of \( 1/\gamma \) and \( \beta = 1/\gamma \)), and (c) canceling terms, we have

\[
P \left[ \sum_{i=1}^{n} X_i < 1 \right] \overset{(a)}{=} P \left[ \sum_{i=1}^{n} \frac{X_i}{\gamma} < \frac{1}{\gamma} \right] \overset{(b)}{=} \exp \left( \frac{1}{\gamma} + \frac{1}{\gamma} \ln \frac{\mu}{\gamma} - \frac{\mu}{\gamma} \right) \overset{(c)}{=} \exp \left( \frac{1}{\gamma} (1 + \ln \mu - \mu) \right),
\]

as desired. \[\blacksquare\]

Lemma B.5. Let \( X_1, \ldots, X_n \in [0, \gamma] \) be independent random variables for \( \gamma > 0 \), let \( \mu = E \left[ \sum_{i=1}^{n} X_i \right] \), and \( \beta \in \mathbb{R}_{>0} \). If \( \beta < \mu \), then

\[
P \left[ \sum_{i=1}^{n} X_i < \beta \right] \leq \left( \frac{\mu}{\beta} \right)^{\beta/\gamma} \prod_{i=1}^{n} E \left[ \left( \frac{\beta}{\mu} \right)^{X_i/\gamma} \right] \leq \exp \left( \frac{\beta - \mu + \beta \ln(\mu/\beta)}{\gamma} \right).
\]

Proof. For each \( i \), we have \( X_i/\gamma \leq 1 \). By (a) Lemma B.3 with \( \beta, \mu \), and the \( X_i \)'s scaled by a factor of \( 1/\gamma \), we have

\[
P \left[ \sum_{i=1}^{n} X_i < \beta \right] = P \left[ \frac{1}{\gamma} \sum_{i=1}^{n} X_i < \frac{\beta}{\gamma} \right] \overset{(a)}{\leq} \exp \left( \frac{\beta}{\gamma} + \frac{\beta}{\gamma} \ln \frac{\mu}{\beta} - \frac{\mu}{\gamma} \right) = \exp \left( \frac{1}{\gamma} \left( \beta + \beta \ln \frac{\mu}{\beta} - \mu \right) \right),
\]

as desired. \[\blacksquare\]
Lemma B.6. Let $X_1,\ldots,X_n \in [0,\gamma]$ be independent with $\mu = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right]$, and $\delta \in (0,1)$. If $\mu > \gamma$ and $\frac{\mu}{\gamma} \geq \ln \frac{1}{\delta} + \ln \frac{1}{\delta} + 1 + \frac{1 + \ln(1/\delta)}{\ln(1/\delta) - 1}$, then $\mathbb{P}\left[\sum_{i=1}^{n} X_i < 1\right] \leq \delta$.

Proof. We set $\frac{\mu}{\gamma} = \ln \frac{1}{\delta} + \ln \frac{1}{\delta} + 1 + \epsilon$ for a variable $\epsilon \in \mathbb{R}$, and find the minimum the choice of $\epsilon$ that obtains the desired tail inequality. By Lemma B.4 it suffices to choose $\epsilon$ such that $\frac{\mu}{\gamma} - \ln \frac{\mu}{\gamma} - 1 \geq \ln(1/\delta)$. We have

$$\frac{\mu}{\gamma} - \ln \frac{\mu}{\gamma} - 1 = \ln \frac{1}{\delta} + \ln \frac{1}{\delta} + 1 + \epsilon - \ln \left(\ln \frac{1}{\delta} + \ln \frac{1}{\delta} + 1 + \epsilon\right) - 1$$

$$= \ln \frac{1}{\delta} + \ln \frac{1}{\delta} + \epsilon - \ln \left(\ln \frac{1}{\delta} + \ln \frac{1}{\delta} + 1 + \epsilon\right).$$

Set $\epsilon$ such that $1 + \epsilon + \ln(1/\delta) = \epsilon \ln(1/\delta)$; namely, set $\epsilon = \frac{1 + \ln(1/\delta)}{\ln(1/\delta) - 1}$. (Note that the requirement that $\delta \in (0,1)$ ensures that $\ln(1/\delta) > 1$.) Then, continuing the above, we have

$$\frac{\mu}{\gamma} - \ln \frac{\mu}{\gamma} - 1 = \cdots = \ln \frac{1}{\delta} + \ln \frac{1}{\delta} + \epsilon - \ln \left(\left(1 + \epsilon\right) \ln \left(\frac{1}{\delta}\right)\right)$$

$$= \ln \left(\frac{1}{\delta}\right) + \epsilon - \ln(1 + \epsilon)^{(a)} \geq \ln \left(\frac{1}{\delta}\right),$$

where (a) uses the inequality $1 + \epsilon \leq \exp(\epsilon)$.

Lemma B.7. Let $X_1,\ldots,X_n \in [0,1]$ be independent with $\mu = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right]$, $\delta \in (0,1)$, and $\beta \in [0,\mu]$.

If $\frac{\mu}{\gamma} \geq \ln \frac{1}{\delta} + \ln \frac{1}{\delta} + 1 + \frac{1 + \ln(1/\delta)}{\ln(1/\delta) - 1}$, then $\mathbb{P}\left[\sum_{i=1}^{n} X_i < \beta\right] \leq \delta$.

Proof. By Lemma B.3 we have $\mathbb{P}\left[\sum_{i=1}^{n} X_i < \beta\right] \leq \delta$ if

$$\beta + \beta \ln \frac{\mu}{\beta} - \mu \leq -\ln \left(\frac{1}{\delta}\right) \iff \frac{\mu}{\beta} - \ln \frac{\mu}{\beta} - 1 \geq \frac{\ln(1/\delta)}{\beta}. $$

Consider setting $\frac{\mu}{\beta} = \frac{\ln(1/\delta)}{\beta} + \ln \left(\frac{\ln(1/\delta)}{\beta}\right) + 1 + \epsilon$ for a value $\epsilon$ to be determined. Then

$$\frac{\mu}{\beta} - \ln \frac{\mu}{\beta} - 1 - \frac{\ln(1/\delta)}{\beta} = \ln \left(\frac{\ln(1/\delta)}{\beta}\right) + \epsilon - \ln \left(\frac{\ln(1/\delta)}{\beta} + \ln \left(\frac{\ln(1/\delta)}{\beta}\right) + 1 + \epsilon\right).$$

Set $\epsilon$ such that

$$\ln \left(\frac{\ln(1/\delta)}{\beta}\right) + 1 + \epsilon = \frac{\epsilon \ln(1/\delta)}{\beta};$$
namely,
\[ \epsilon = \frac{\ln(\ln(1/\delta)/\beta) + 1}{\ln(1/\delta)/\beta - 1}. \]

Then
\[ (3) = \ln\left(\frac{\ln(1/\delta)}{\beta}\right) + \epsilon - \ln\left((1 + \epsilon)\frac{\ln(1/\delta)}{\beta}\right) \overset{(a)}{=} \epsilon - \ln(1 + \epsilon) \geq 0, \]

where (a) uses the inequality \( 1 + \epsilon \leq e^\epsilon \). Plugging back in, we have
\[
\frac{\mu}{\beta} = \ln\left(\frac{\ln(1/\delta)}{\beta}\right) + 1 + \frac{\ln(\ln(1/\delta)/\beta) + 1}{\ln(1/\delta)/\beta - 1} = \frac{\ln(1/\delta)/\beta}{\ln(1/\delta)/\beta - 1} \left(\ln\left(\frac{\ln(1/\delta)}{\beta}\right) + 1\right)
\]

\[ \blacksquare \]