Abstract—A network of locally interacting agents can be thought of as performing a distributed computation. But not all computations can be faithfully distributed. This paper investigates which global, linear transformations can be computed using local rules, i.e., rules which rely solely on information from adjacent nodes in a network. The main result states that a linear transformation is computable in finite time using local rules if and only if the transformation has positive determinant. An optimal control problem is solved for finding the local interaction rules, and simulations are performed to elucidate how optimal solutions can be obtained.

I. INTRODUCTION

One common theme when designing control and coordination mechanisms for distributed, multi-agent systems is that the information, on which decisions are based, is restricted to be shared among agents that are adjacent in the underlying information-exchange network, e.g., [1], [2], [3], [4]. As a result, local rules are needed for processing the information and coordinating the agents in the network in such a way that some global objective is achieved. Problems that fit this description can be found in a variety of applications, including power systems [5], [6], [7], formation control [8], [9], [10], [11], [12], distributed sensor networks [13], [14], smart textiles [15], and distributed optimization [16], [17]. In this paper we take initial steps towards developing a general theory of local implementability/computability of such global behaviors.

As such, one key aspect of algorithm design is the definition of local interaction rules that produce desired global behaviors. An example of this are consensus algorithms for computing averages in a distributed manner. In fact, consensus plays a role in many different applications, including multi-agent robotics, distributed sensor fusion, and power network control, e.g., [3], [6], [18]. To this end, let the scalar state of each node in a network be \( x_i \in \mathbb{R} \), with initial condition \( x_i(t_0) = \xi_i \), \( i = 1, \ldots, n \), where \( n \) is the number of nodes in the network. By stacking the states together in \( x \in \mathbb{R}^n \), we implicitly perform an asymptotic, global computation through the so-called consensus equation

\[
\dot{x}_i = - \sum_{j \in N_i} (x_i - x_j),
\]

where \( N_i \) encodes a neighborhood relationship in the underlying information-exchange network. And, as long as the network is connected and undirected, all node values will converge to the same value, namely the average of the initial conditions, e.g., [2]. In other words,

\[
\lim_{t \to \infty} x(t) = \frac{1}{n} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \xi,
\]

where \( \xi \) is the vector containing all the initial node values. As such, the consensus equation is asymptotically computing the average, which is a global property since it relies on the state of every node in the network.

In this work, we are interested in problems where networks are tasked with computing arbitrary linear transformations of the initial node states. In particular, we answer two fundamental questions: What global, linear transformations can be computed using local rules? How do we find the local rules that would compute a given linear transformation? This paper answers both questions and presents existence guarantees of local rules for a given computation.

Some work has been done in the general area of obtaining global information with local interactions. In [19], a fixed weighting scheme was used to compute linear transformations on networks. That work focused on cases where each agent computes the same linear transformation, as is the case with the consensus computation where each node computes the average, while we, in this paper, do not wish to restrict ourselves to this special case. In a certain sense, the investigation in [20] follows this line of inquiry as well. There, quadratic invariance was used to establish whether or not a convex optimization problem exists whose solution is a decentralized implementation of a centralized feedback controller. [21] further expounds on this idea and provides a practical, graph theoretic method for finding this distributed controller. Our work distinguishes itself from this body of work by using a time varying weighting method, which admits the computation of global, linear transformations in finite time.

In fact, in this paper, we consider computations that are to be performed using local rules over a static and undirected information-exchange network. The local rules, once obtained, admits a decentralized implementation, where “decentralized” in this context means that each node in the network only needs to communicate state information among adjacent nodes in the network. In particular, we ask if it is possible to define local interaction laws such that \( x(t_f) = T \xi \), given the linear transformation \( T \) and the initial conditions \( x(t_0) = \xi \). Necessary and sufficient
conditions are given for this to be possible, and they state that local interaction rules exist if and only if $T$ has positive determinant.

The remainder of this paper is organized as follows: In Section III the problem under consideration is introduced and the general class of admissible, local interaction rules is established. In Section III necessary and sufficient conditions are presented under which global, linear computations can be performed in a decentralized manner. In Section IV an optimal control problem is formulated, which provides a way to find interaction rules, the time varying weighting functions numerically, and in Section V two instantiations of the method are presented together with simulation results.

II. PROBLEM DEFINITION

To formalize what is meant by local interactions, we first need to discuss the information-exchange network over which the interactions are defined. To this end, let $V$ be a vertex set with cardinality $n$, and $E \subset V \times V$ be an edge set with cardinality $m$, where we insist on $(i, i) \in E$, $\forall i \in V$, as well as $(i, j) \in E \Rightarrow (j, i) \in E$. Let $G$ be the graph $G = (V, E)$, where the assumptions on $E$ imply that $G$ is undirected and contains self-loops. We moreover assume that $G$ is connected. As the main purpose with $G$ is to encode adjacency information in the information-exchange network, we introduce the operator $\text{sparse}(G)$ to capture these adjacencies, and we say that an $n \times n$ matrix $M \in \text{sparse}(G)$ if $(i, j) \notin E \Rightarrow M_{ij} = 0$.

There are a number of different ways in which local interactions can be defined. In this paper, we assume that they are given by time-varying, piecewise continuous weights associated with the edges in the network. If $x_i \in \mathbb{R}$ is the scalar state associated with node $i \in V$, we define a local interaction as a continuous-time process

$$\dot{x}_i(t) = \sum_{j (i, j) \in E} w_{ij}(t) x_j(t).$$

(3)

Note that we do not insist on $w_{ij} = w_{ji}$ even though $G$ is undirected, as shown in Figure II.

If we stack the states together in $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$, what we mean by local interactions is thus

$$\dot{x}(t) = W(t)x(t), \; W(t) \in \text{sparse}(G),$$

(4)

with solution

$$x(t) = \Phi(t, t_0)x(t_0),$$

(5)

where $\Phi$ is the state transition matrix associated with the system in Equation 4 e.g., [22].

Now, the purpose of the local interactions is to perform a global, linear computation. In other words, given the $n \times n$ matrix $T$ and the initial condition $x(t_0) = \xi$, what we would like to do is find $W(t) \in \text{sparse}(G)$, $t \in [t_0, t_f]$, such that

$$x(t_f) = T\xi.$$  

(6)

But, comparing this expression to Equation 5 this simply means that what we would like is

$$\Phi(t_f, t_0) = T.$$  

(7)

If this was indeed the case, then the local interactions, as defined through $W(t)$, would indeed compute $T\xi$ over the interval $[t_0, t_f]$ for all possible values of $\xi$, i.e., one can think of the network as a black box that takes $\xi$ as the input at time $t_0$ and, at time $t_f$, returns $T\xi$ as the output.

As a final observation before we can formulate the general problem of performing global, linear computations using local interactions, we note that state transition matrix satisfies the same dynamics as Equation 4 i.e.,

$$\frac{d\Phi(t, t_0)}{dt} = W(t)\Phi(t, t_0),$$

(8)

with initial condition $\Phi(t_0, t_0) = I$, where $I$ is the $n \times n$ identity matrix.

Problem 1 [Local Computation]

Given a linear transformation $T$ and a connected graph $G$, find $W(t) \in \text{sparse}(G)$, $t \in [t_0, t_f]$, such that

$$X(t) = W(t)X(t),$$

(9)

with boundary conditions $X(t_0) = I$, $X(t_f) = T$.

III. ON THE EXISTENCE OF SOLUTIONS

The main point with this paper is an exploration of what linear transformations $T$ admit a local implementation, i.e., for what $T$ Problem 1 has a solution. In this section, we develop necessary and sufficient conditions for this to be the case.

We start by observing that since $X(t)$ is really the state transition matrix $\Phi(t, t_0)$, it is always invertible,

$$X(t)^{-1} = \Phi(t, t_0)^{-1} = \Phi(t_0, t).$$

(10)

As a direct consequence of this, $T$ has to be invertible for a solution to Problem 1 to exist, i.e., we need that $\det(T) \neq 0$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{sparse_graph.png}
\caption{An example of the sparsity structure and the local interaction rules used in this paper.}
\end{figure}
have thus established the following necessary condition for local rules, as understood in this paper, to achieve consensus in finite time. This follows directly from the fact that the set of all GL turns out to be both necessary and sufficient for a solution. And, surprisingly enough, Problem 1 to have a solution, \( T \) has to satisfy \( \det(T) > 0 \).

To state this fact more compactly, let \( GL_n^+ (\mathbb{R}) \) denote the set of all \( n \times n \) real matrices with positive determinant. We have thus established the following necessary condition for the existence of a solution:

**Lemma 1.** A solution to Problem 1 exists only if \( T \in GL_n^+ (\mathbb{R}) \).

One consequence of Lemma 1 is that it is impossible to use local rules, as understood in this paper, to achieve consensus in finite time. This follows directly from the fact that the consensus computation is given by the linear map

\[
T_{cons} = \frac{1}{n} I^T 1, \tag{11}
\]

where 1 is a vector of length \( n \), with all entries equal to one. And,

\[ \text{rank}(T_{cons}) = 1, \]

i.e., \( \det(T_{cons}) = 0 \). We state this fact as a corollary:

**Corollary 1.** There is no solution to Problem 1 which admits finite time consensus.\(^1\)

Now that we have established necessary conditions for Problem 1 to have a solution, we turn our attention to sufficient conditions. And, surprisingly enough, \( T \in GL_n^+ (\mathbb{R}) \) turns out to be both necessary and sufficient for a solution to exist, which constitutes the main result in this paper:

**Theorem 1.** A solution to Problem 1 exists if and only if \( T \in GL_n^+ (\mathbb{R}) \).

As we have already established sufficiency, what must be shown is that whenever \( \det(T) > 0 \), there is a \( W(t) \in \text{sparse}(G) \) that drives \( X \) from \( I \) to \( T \). The remainder of this section is devoted to the establishment of this fact. However, before we can give the proof to Theorem 1 a number of supporting results are needed, involving the controllability of nonlinear, drift-free systems, i.e., systems of the form

\[
\dot{x} = \sum_{i=1}^p g_i(x) u_i, \tag{12}
\]

where \( x \in \mathbb{R}^n \) is the state of the system, and \( u_1, \ldots, u_p \in \mathbb{R} \) are the control inputs. For the sake of easy reference, we start by recalling Chow’s Theorem, as formulated in [23], for such drift-free systems:

**Theorem 2** (Chow’s Theorem, e.g. [23]). The system in Equation 12 is locally controllable about a point \( x_0 \) if and only if

\[
dim(\Delta(x_0)) = n, \tag{13}
\]

where \( \Delta \) is the involutive closure of the distribution \( \text{span}\{g_1, \ldots, g_p\} \).

The system is moreover controllable if it is locally controllable everywhere. And, the proof that \( T \in GL_n^+ (\mathbb{R}) \) is sufficient for Problem 1 to have a solution will hinge on showing that the dynamics, as defined through the local interaction rules in Equation 13, is indeed controllable everywhere on \( GL_n^+ (\mathbb{R}) \). To this end, we first must rewrite the dynamics in Problem 1 on the appropriate form. For this, we need the index matrix \( I_{ij} \in \mathbb{R}^{n \times n} \), which has a one at the \( i \)th row and \( j \)th column, and zeros everywhere else. The index matrix allows us to rewrite

\[
\dot{X} = WX
\]

as

\[
\dot{X} = \left( \sum_{i=1}^n \sum_{j=1}^n W \odot I_{ij} \right) X, \tag{14}
\]

where the \( \odot \) symbol represents element-wise matrix product, i.e.,

\[
\dot{X} = \left( \begin{bmatrix} w_{11} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{bmatrix} + \ldots + \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & w_{nn} \end{bmatrix} \right) X, \tag{15}
\]

where we have suppressed the explicit dependence on \( t \) for the sake of notational ease.

Rearranging the terms and letting

\[
g_{ij}(X) = I_{ij} X, \tag{16}
\]

we get the drift-free matrix formulation

\[
\dot{X} = \sum_{i=1}^n \sum_{j(i,j) \in E} g_{ij}(X) w_{ij}. \tag{17}
\]

To clarify, \( g_{ij}(X) \) is a matrix whose \( i \)th row contains the \( j \)th row of \( X \), with the rest of the elements in the matrix equal to 0,
by \( \vec{g}_{ij} = \text{vec}(g_{ij}) \), resulting in the vectorized version of Equation \([17]\)

\[
\text{vec}(X) = \sum_{i=1}^{n} \sum_{j(i,j) \in E} \vec{g}_{ij}(X)w_{ij}.
\]  

(19)

The first order of business towards establishing controllability of this system is the derivation of the Lie brackets for the system in Equation \([19]\).

**Lemma 2.**

\[
[g_{ij}(X), \vec{g}_{kl}(X)] = \begin{cases} 
-\vec{g}_{il}(X) & \text{if } j = k, \ i \neq l \\
\vec{g}_{kj}(X) & \text{if } i = l, \ j \neq k \\
0 & \text{otherwise}
\end{cases}
\]  

(20)

**Proof.** The Lie bracket \([g_{ij}, \vec{g}_{kl}]\) is given by

\[
\frac{\partial \vec{g}_{kl}}{\partial \text{vec}(X)} \vec{g}_{ij} = \frac{\partial \vec{g}_{ij}}{\partial \text{vec}(X)} \vec{g}_{kl},
\]  

(21)

where we have suppressed the explicit dependence on \(X\). Substitution of Equation \([16]\) into Equation \([21]\) the above expression yields

\[
\frac{\partial (\text{vec}(I_{ij}X))}{\partial \text{vec}(X)} \text{vec}(I_{ij}X) - \frac{\partial (\text{vec}(I_{ik}X))}{\partial \text{vec}(X)} \text{vec}(I_{kl}X),
\]  

(22)

which can be rewritten, using the Kronecker product, as

\[
\frac{\partial ((I \otimes I_{ik}) \text{vec}(X))}{\partial \text{vec}(X)} (I \otimes I_{ij}) \text{vec}(X) - \frac{\partial ((I \otimes I_{ij}) \text{vec}(X))}{\partial \text{vec}(X)} (I \otimes I_{ik}) \text{vec}(X)
\]  

(23)

Taking the derivatives yields

\[
(I \otimes I_{ik})(I \otimes I_{ij}) \text{vec}(X) - (I \otimes I_{ij})(I \otimes I_{ik}) \text{vec}(X).
\]  

(24)

Using the mixed product property of the Kronecker product, Equation \([21]\) can be further simplified as

\[
(I \otimes I_{ij}I_{ik}) \text{vec}(X) - (I \otimes I_{ik}I_{ij}) \text{vec}(X)
\]  

(25)

i.e., the Lie bracket in Equation \([21]\) becomes

\[
[g_{ij}(X), \vec{g}_{kl}(X)] = \text{vec}(I_{ik}I_{ij}X) - \text{vec}(I_{ij}I_{ik}X).
\]  

(26)

Now, using the fact that, \(I_{ij}I_{kl} = I_{il} \) if \( j = k \) and \( I_{ij}I_{kl} = 0 \) otherwise, we can break down Equation \([26]\) into 3 cases: First if \( j = k \) and \( i \neq j \) we get \([\vec{g}_{ij}, \vec{g}_{ij}] = -\text{vec}(I_{il}X) = -\vec{g}_{il} \). The second case occurs when \( i = l \) and \( j \neq k \), in which case \([\vec{g}_{ij}, \vec{g}_{kl}] = \text{vec}(I_{kj}X) = \vec{g}_{kj} \). Otherwise, the Lie bracket is 0, and the lemma follows. □

Now that Lie brackets can be computed in general for this problem, we must determine if the involutive closure of the distribution associated with the system in Equation \([19]\) contains enough independent vector fields for local controllability. To help with this determination, we provide the following lemma.

**Lemma 3.** If node \( i \) is path-connected to node \( j \), then \( \vec{g}_{ij}(X) \) is in the distribution \( \overline{\Delta}(X) \).

**Proof.** That node \( i \) is path-connected to node \( j \) means that there is a path through adjacent nodes in the graph \( G \) that starts at node \( i \) and ends at node \( j \). Assume that the path goes through the nodes \( N_1, \ldots, N_q \), i.e., \( N_1 \) is adjacent to \( N_2 \), \( N_2 \) is adjacent to \( N_3 \), and so forth, while \( N_1 = i \) and \( N_q = j \). Since these nodes are adjacent, we, by definition, have that \( \vec{g}_{N_1N_2}, \vec{g}_{N_2N_3}, \ldots, \vec{g}_{N_{q-1}N_q} \in \overline{\Delta}(X) \).

The involutive closure contains every possible Lie bracket that can be recursively created from elements \( \overline{\Delta}(X) \), which implies that the problem is to create \( \vec{g}_{ij} \) from some combination of Lie brackets from elements in \( \overline{\Delta}(X) \). And, from Lemma \([2]\) we know that \([\vec{g}_{N_1N_2}, \vec{g}_{N_2N_3}] \) is equal to \(-\vec{g}_{N_1N_4} \). Applying Lemma \([2]\) again gives \([-\vec{g}_{N_1N_4}, \vec{g}_{N_2N_3}] = \vec{g}_{N_1N_3} \). This procedure can be repeated until we arrive at one of two possible cases. If \( q \) is even, the result is \([-\vec{g}_{N_1N_{q-1}}, \vec{g}_{N_{q-1}N_q}] = \vec{g}_{N_1N_q} \). If \( q \) is odd we get \([\vec{g}_{N_1N_{q-1}}, \vec{g}_{N_{q-1}N_q}] = -\vec{g}_{N_1N_q} \). In either case, we are able to construct \( \vec{g}_{N_1N_q} \) from previous Lie brackets, as shown in Figure \([2]\). And, as \( N_1 = i \) and \( N_q = j \), we have \( \vec{g}_{ij} \in \overline{\Delta}(X) \). □
which simplifies to
\[ \sum_{i=1}^{n} g_{ii} = X. \]  
(28)

And, since \( X \) is a linear combination of \( n \) linearly independent matrices, \( \text{rank}(X) = n \), and the first implication follows.

Next, we must show that
\[ \text{rank}(X) = n \Rightarrow \dim(\Delta(X)) = n^2, \]  
which we do by contradiction. Using the expression \( g_{ij} = \mathbb{I}_{ij}X, \) \( n^2 \) matrices can be formed from \( X \). Let us assume that they are not linearly independent. This implies that there exists a set of coefficients \( \alpha_{ij} \) such that, for some \((k,l)\),
\[ \sum_{(i,j) \neq (k,l)} g_{ij} \alpha_{ij} = g_{kl}. \]  
(30)

Since \( X \) has full rank, \( X \) can be removed from Equation 30 based on the fact that \( g_{ij} = \mathbb{I}_{ij}X \), yielding
\[ \sum_{(i,j) \neq (k,l)} \mathbb{I}_{ij} \alpha_{ij} = \mathbb{I}_{kl}. \]  
(31)

By definition of the index matrix, Equation 31 cannot be true, since every matrix in the sum on the left has a value of zero where \( \mathbb{I}_{kl} \) has value of 1. Therefore, \( g_{ij} \) is really a state transition matrix, i.e., it is indeed invertible (with rank \( n \)), \( \exists I \) such that \( g_{ij} = \mathbb{I}_{ij}X \).

\( \Box \)

Since \( X \) is really a state transition matrix, i.e., it is indeed invertible (with \( \text{rank}(X) = n \)), the system in Equation 17 is locally controllable everywhere on \( \text{GL}_n^+(\mathbb{R}) \) as long as the underlying graph \( G \) is connected:

**Theorem 3. The system**
\[ \dot{X} = WX, \quad W \in \text{sparse}(G) \]

**is locally controllable everywhere on \( \text{GL}_n^+(\mathbb{R}) \) if \( G \) is connected.**

Theorem 3 and Lemma 1 give us all the ammunition needed to prove the main result in this paper, namely

**Theorem 1.**
\[ T_{cons}2 = \begin{bmatrix} \frac{1}{n} & 1/n & \cdots & 1/n \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}. \]  
(32)

We have
\[ \det(T_{cons2}) = \frac{1}{n} \]  
(33)
and, as such, it is computable using local rules. In this case, the network average is only computed by a single node (node 1 in this case), while the remaining nodes return to their initial values at the end of the computation. This can in fact be generalized to any scalar, non-zero, linear map \( \ell : \mathbb{R}^n \to \mathbb{R} \) through
\[ T_\ell \xi = \begin{bmatrix} \ell(\xi) \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix}, \]
where we have assumed that \( \ell(\xi) \) depends on \( \xi. \)

The point with this is that it is possible to compute any scalar, non-zero, linear map as long as the computation only has to take place at a single node.

**IV. OPTIMAL LOCAL INTERACTIONS**

Just because we know that a computation \( T\xi \) can be done using local rules does it not follow that we can (easily) find these rules, encoded through \( W(t) \in \text{sparse}(G) \), such that \( X = WX, \quad X(t_0) = I, \quad X(t_f) = T \). In this section, we address this problem in the context of optimal control.

Let the cost be given by
\[ J(W) = \int_0^{t_f} \frac{1}{2} ||W(t)||^2_F \]  
(34)
where \( ||\cdot||_F \) is the Frobenius norm. The resulting constrained minimization problem becomes

**Problem 2 [Optimal Local Interactions]**
\[ \min_W J(W) = \int_0^{t_f} \frac{1}{2} ||W(t)||^2_F \]  
(35)
such that
\[ X = WX, \quad W(t) \in \text{sparse}(G), \quad \forall t \in [t_0, t_f] \]
\[ X(t_0) = I, \quad X(t_f) = T. \]  
(36)

The Hamiltonian associated with Problem 2 (e.g., [26]), with costate matrix \( \lambda \), is given by
\[ H = \text{vec}(\lambda)^T \text{vec}(WX) + \frac{1}{2} ||W||^2_F. \]  
(37)

If not, simply pick another node in the network that where the computation takes place.
We can rewrite the Hamiltonian as
\[ H = \sum_{i=1}^{n} \sum_{j=(i,j) \in E} \sum_{k=1}^{n} \lambda_{ik} w_{ij} x_{jk} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=(i,j) \in E} w_{ij}^2. \] (38)

The optimality conditions are
\[ 0 = \frac{\partial H}{\partial w_{ij}} = \sum_{k=1}^{n} \lambda_{ik} x_{jk} + w_{ij}, \] i.e., the optimal weights are given by
\[ w_{ij} = -\sum_{k=1}^{n} \lambda_{ik} x_{jk}, \] (40)
which yields \( m + n \) optimality conditions. This is also the number of nonzero values in the \( W \) matrix.

We get the costate equations from the derivative of the Hamiltonian with respect to \( X \):
\[ \dot{\lambda}_{ij} = -\frac{\partial H}{\partial X_{ij}} = -\sum_{\langle i,j,k \rangle \in E} w_{ki} \lambda_{kj}. \] (41)

By substituting the optimality conditions into both the state and costate equations, we get \( 2n \) equations with initial and final conditions on \( X_{ij} \). The resulting, two-boundary point problem becomes
\[
\begin{align*}
\dot{X}_{ij} &= -\sum_{\langle k \rangle \in E} X_{kj} \sum_{l=1}^{n} \lambda_{il} X_{kl} \\
X(t_0) &= I, \quad X(t_f) = T \\
\dot{\lambda}_{ij} &= \sum_{\langle k \rangle \in E} \lambda_{kj} \sum_{l=1}^{n} \lambda_{kl} X_{il},
\end{align*}
\] (42)
which can be solved numerically, as will be seen in the next section.

V. SIMULATIONS AND EXAMPLES

Computing linear transformation of states can be useful in a variety of network applications. In this section we explore two concrete examples. The first involves improving the convergence rates in distributed computations and the second involves information exchange among non-local agents.

A. Improving Convergence Rates

Consider, again, the consensus equation,
\[ \dot{x} = -L_s x, \] (43)
where \( x \in \mathbb{R}^n \) is the state of the system and \( L_s \) is the graph Laplacian associated with a given, sparse yet connected graph \( G_s \). These dynamics are known to converge exponentially in the algebraic connectivity of the graph \( G_s \), e.g.,[3], where the algebraic connectivity is given by the second smallest eigenvalue of \( L_s \).

For some applications this convergence rate may not be fast enough. Instead, one might want to make the system behave as if the graph was more dense; thus improve the convergence rate. Let \( G_d \) be a dense graph, with corresponding consensus dynamics
\[ \dot{z} = -L_d z, \] (44)
where \( z \) represents a “desired” state (how we would like \( x \) to behave) and \( L_d \) is the graph Laplacian of \( G_d \). The desired system in Equation 44 has state transition matrix \( \Phi_d(t, t_0) \), with \( \Phi_d(t_0, t_0) = I \), and
\[ \Phi_d(t_f, t_0) = e^{-L_d(t_f - t_0)}, \] (45)
which we thus set as the linear transformation we would like to compute by the original, sparse network.

In other words, let \( T = e^{-L_d(t_f - t_0)} \). The goal is to compute \( T \) using local time varying weights on the graph \( G_s \). And, since matrix exponentials are invertible, \( T \in \text{GL}_n^+(\mathbb{R}) \), and a solution does indeed exist.

As an instantiation of this, consider a 5 node system. We can solve Problem 2 numerically, using test-shooting, for this system in order to find \( W \) such that
\[ \dot{X} = WX, \quad W \in \text{sp}(G_s) \]
\[ X(t_0) = I, \quad X(t_f) = e^{-L_d(t_f - t_0)}. \] (46)

The solution to this problem yields a set of time varying weights \( W \) and matrices \( X \). The weights can be executed in a decentralized manner, once they have been obtained (using centralized computations). The result is a sparse network that acts at a higher rate, as if it was indeed dense. This is shown in Figures 5-7. In Figure 5 the agreement error,
\[ ||x(t) - 1/n1^T 1 x(0)|| \]
is shown both for the original (sparse) consensus dynamics (Equation 43) and for the optimal, “densified” version.

![Optimal State Trajectories](image)

Fig. 3. This plot shows the evolution of each element of the state transition matrix \( \Phi(t, t_0) \) over the time interval from \( t_0 \) to \( t_f \). Since \( \Phi(t_0, t_0) = I \) all elements are either 0 or 1 initially.

B. Swapping Node Values

As another example, consider the situation when the linear transformation represents a reordering (or swapping) of states. For a 4 node case, where agents 1 and 2 and agents...
This sequential approach avoids the numerical ill-conditioning, and the solution is shown in Figures 6-8.

Fig. 6. The evolution of the state transition matrix for the 4-node swapping problem, with \( \Phi(t_0, t_0) = I \), \( \Phi((t_f - t_0)/2, t_0) = T_1 \), and \( \Phi(t_f, t_0) = T_{\text{swap}} \).

Fig. 7. The weight functions define the local interactions needed to achieve the swap in the 4-node case.

VI. CONCLUSIONS

In this paper, a step was taken towards computing arbitrary global functions on networks with local interaction rules. In particular, it presented a method which allows a networked system to compute global, linear transformations using only local rules.

We derived necessary and sufficient conditions under which it is possible to use a distributed, time-varying weighting scheme to compute the transformation \( T \) for undirected, connected networks with fixed topology. Specifically, we showed that the necessary and sufficient condition for \( T \) to be locally computable is that it has positive determinant, i.e., \( T \in \text{GL}_n^\times(\mathbb{R}) \).

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