An accurate algorithm for solving biological population model by the variational iteration method using He’s polynomials

Mohamed Zellal\textsuperscript{a,b} and Kacem Belghaba\textsuperscript{b}

\textsuperscript{a}Hassiba Benbouali University of Chlef. BP 151, Hay Essalem, Algeria; \textsuperscript{b}Laboratory of Mathematic and Its Applications (LAMAP), University of Oran1, Oran, Algeria

ABSTRACT
In the present work, we apply the variational iteration method using He’s polynomials (VIMHP) for solving four examples of the biological population model (BPM). The proposed method is a combination of He’s variational iteration and the homotopy perturbation methods. The suggested algorithm is practically more reliable, highly efficient for use in such problems. The proposed method finds the solution without any restrictive assumptions, discretization and linearization. The approximate solution converges very rapidly to the exact solution which confirms the accuracy of this method as an easy algorithm for computing the solution for wider classes of linear and nonlinear differential equations.

1. Introduction
Variational iteration method (VIM) and homotopy perturbation method (HPM) were first proposed by the Chinese mathematician Ji-Huan He (He, 1999a, 1999b). Both methods were used to solve a large variety of scientific fields such as partial differential equations, with approximations converging rapidly to accurate solutions. The VIM technique is used in Abdou and Soliman (2005); Biazar & Aminikhah (2009) for solving Burgers and coupled Burgers equation. In Wazwaz (2007a) the applications of the present method to linear and nonlinear systems of PDEs are provided. The variational iteration technique is employed to solve the nonlinear diffusion equations, linear and nonlinear ODEs (Wazwaz, 2007b, 2009). Also this method is applied in Mohyud-Din & Yildirim (2010) for solving delay differential equations, and in Yusufoglu (2008) for studying the Klein–Gordon equation. AL-Jawary (2016) employed VIM for Fokker–Planck equation. In Siddiqi & Muzammal (2015) VIM is employed to solve seventh order boundary value problems. This technique modified is also employed in AL-Jawary, Radhi, & Ravnik (2018) to solve nonlinear thin film flow problem, etc. The convergence of variational iteration method is investigated in Odibat (2010); Tatari & Dehghan (2007).

HPM has been used by many mathematicians and engineers to solve various functional equations. This method was further developed and improved by He and applied to nonlinear oscillators with discontinuities (He, 2004), nonlinear wave equations (He, 2005a), and boundary value problems (He, 2005b). It can be said that homotopy perturbation method is a universal one and is able to solve various kinds of nonlinear functional equations. For example it was applied to hyperbolic partial differential equations (Biazar & Ghazvini, 2008), to the reaction-diffusion Brusselator system (Ayati, Biazar, & Ebrahimi, 2015), to Helmholtz equation and fifth-order KdV equation (Rafei & Ganji, 2006) and to other equations (Biazar & Aminikhah, 2009; He, 2003; Sweilam & Khader, 2009), etc. This method continuously deforms the difficult equation under study into a simple equation, easy to solve. In this method the solution is considered as the summation of an infinite series which usually converges rapidly to exact solutions. Ayati and Biazar (2015); Biazar and Ghazvini (2009) investigated the convergence of homotopy perturbation method.

The biological population model (BPM) which arise in many physical important phenomena (Bear, 1972; Gurtin & Maccamy, 1977; Okubo, 1980) was solved by many numerical methods such as VIM and Adomian decomposition method (ADM) (Shakeri & Dehghan, 2007), the HPM (Roul, 2010), homotopy analysis method (HAM) (Arafa, Rida, & Mohamed, 2011), homotopy perturbation transform method (HPTM) (Kumar, Singh, & Kumar, 2013).
In this paper, we will implement the variational iteration method using He’s polynomials (VIMHP), (Daga & Pradhan, 2013; Matinfar, Mahdavi, & Raeisi, 2010; Matinfar & Mahdavi, 2012), to obtain the approximate solution for the biological population equation as a nonlinear model as follows:

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2(u^2)}{\partial x^2} + \frac{\partial^2(u^2)}{\partial y^2} + f(u), \quad t \geq 0, (x, y) \in \mathbb{R}^2, \\
f(u) &= hu^a(1 - ru^b), \quad u(x, y, 0) = h(x, y),
\end{aligned}
\]

where \( u \) denotes population density, \( f \) represents the population supply due to births and deaths, \( h, a, b, r \) are real numbers, \( h \) is the initial condition (Alara et al., 2011; Roul, 2010; Shakeri & Dehghan, 2007).

This paper is organized as follows: Section 2 is devoted to a short description of VIM, HPM and VIMHP. In Section 3, four examples and any numerical results are presented to study the performance of the proposed method, followed by the comparison of the approximate solutions and the solutions obtained respectively by the variational iteration method and Adomian decomposition method. Finally in Section 4, a conclusion finishes this work.

2. Description of numerical methods

2.1. He’s variational iteration method

To clarify the VIM, we begin by considering a differential equation in the formal form: (AL-Jawary, 2016; AL-Jawary et al., 2018; He, 1999a; Odibat, 2010; Siddiqi & Muzzammal, 2015)

\[ Lu(x, t) + Nu(x, t) = g(x, t), \]

where \( L \) is a linear operator defined by \( L = \sum_{i=1}^{m} m_i \), \( m_i \in \mathbb{N} \), \( N \) is a nonlinear operator and \( g \) is a known analytical function. The VIM allows us to write a correct functional of the following type:

\[
u_{m+1}(x, t) = u_0(x, t) + \int_0^t \frac{1}{\lambda(s)} \left[ Lu_0(x, s) + Nu_0(x, s) - g(x, s) \right] ds,
\]

where \( \lambda \) is a general Lagrange’s multiplier (He, 1999a; Inokuti, Sekine, & Mura, 1978), which can be identified optimally via the variational theory, and \( \tilde{u}_0 \) is a restricted variation which means \( \delta \tilde{u}_0 = 0 \), yields the following Lagrange multipliers,

\[
\lambda = -1, \quad \text{for } m = 1,
\]

\[
\lambda = s-t, \quad \text{for } m = 2,
\]

and in general, for \( m \geq 1 \)

\[
\lambda = \frac{(-1)^m (s-t)^{m-1}}{(m-1)!}.
\]

Therefore, substituting (4) into functional (3) we obtain the following iteration formula,

\[
u_{m+1}(x, t) = u_0(x, t) + \int_0^t \frac{1}{\lambda(s)} \left[ Lu_0(x, s) + Nu_0(x, s) - g(x, s) \right] ds,
\]

\[
\text{where } \lambda = \frac{(-1)^m (s-t)^{m-1}}{(m-1)!}.
\]

2.2. Basic idea of He’s homotopy perturbation method

We illustrate the following nonlinear differential equation (Biazar & Ghazvini, 2009; He, 1999b):

\[ A(u) - f(r) = 0, \quad r \in \Omega, \]

with the boundary conditions

\[ B(u) = 0, \quad r \in \Gamma, \]

where \( A \) is a general differential operator, \( B \) is a boundary function, \( f(r) \) is an analytic function, and \( \Gamma \) is the boundary of the domain \( \Omega \). Generally speaking, the operator \( A \) can be divided into two parts \( L \) and \( N \), where \( L \) is linear but \( N \) is nonlinear. Therefore, Equation (6) can be rewritten in the following form:

\[ L(u) + N(u) - f(r) = 0. \]

By using the homotopy technique, we construct a homotopy:

\[ V(r; p) : \Omega \times [0, 1] \rightarrow \mathbb{R}, \]

which satisfies:

\[ H(V; p) = (1-p)[L(V) - L(u_0)] + p[A(V) - f(r)] = 0, \quad r \in \Omega, \]

or

\[ H(V; p) = L(V) - L(u_0) + pL(u_0) + p[N(V) - f(r)] = 0, \quad r \in \Omega, \]

where \( p \in [0, 1] \) is an embedding parameter, \( u_0 \) is an initial approximation for the solution of Equation (6), which satisfies the boundary conditions. Obviously, from Equations (10) and (11) we have

\[ \begin{align*}
H(V; 0) &= L(V) - L(u_0) = 0, \\
H(V; 1) &= A(V) - f(r) = 0
\end{align*} \]

The changing process of \( p \) forms zero to unity is just that of \( V(r, p) \) from \( u_0(r) \) to \( u(r) \). In topology, this is called homotopy. According to the HPM, we can first use the embedding parameter \( p \) as a small parameter, and assume that the solution of Equations (10) and (11) can be written as a power series in \( p \):

\[ V = V_0 + pV_1 + p^2V_2 + \cdots \]
Setting \( p = 1 \), gives the solution of (6)
\[
  u = V_0 + V_1 + V_2 + \cdots
\]  

(14)

The method considers the nonlinear term \( Nu \) as
\[
  Nu(x, t) = \sum_{n=0}^{\infty} p^n H_n(u),
\]  

(15)

using where the He’s polynomials \( H_n \) (Ghorbani & Nadjafi, 2007; Ghorbani, 2009) shown below:
\[
  H_n(u_0, \cdots, u_n) = \frac{1}{n! c^n} \left[ N \left( \sum_{l=0}^{n} (p^l u_l) \right) \right]_{p=0}^n, n = 0, 1, 2, 3, \cdots
\]  

(16)

2.3. Variational iteration method using He’s polynomials

In this section, we highlight briefly the main point of the VIMHP, where more details can be found in Matinfar & Pradhan (2013); Matinfar et al. (2010); Matinfar & Mahdavi (2011); Mohyud-Din & Yildirim (2010). We consider the following equation:
\[
  Lu(x, t) + Nu(x, t) = g(x, t).
\]  

(17)

The VIM allows us to write a correct functional of the following type:
\[
  u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda (s) \left[ Lu_n(x, s) + Nu_n(x, s) - g(x, s) \right] ds,
\]  

(18)

where \( \lambda \) is a general Lagrange’s multiplier. Now, by using the homotopy perturbation method (Biazar & Ghazvini, 2009; He, 1999b) we can construct an equation as follows:
\[
  \sum_{i=0}^{\infty} p^i u_i(x, t) = u_0(x, t) + \int_0^t \lambda (s) \left[ N \left( \sum_{i=0}^{\infty} p^i u_i(x, s) \right) - g(x, s) \right] ds.
\]  

(19)

As it is seen, the procedure is constructed by coupling of VIM and HPM methods. A comparison of like powers of \( p \) gives solutions of various orders. By equating the terms of (19) with identical powers of \( p \), and taking the limit as \( p \) tends to 1, we obtain the formula.
\[
  u(x, t) = \lim_{p \to 1} \sum_{i=0}^{\infty} p^i u_i(x, t)
\]  

(20)

So that we consider the variational iteration method and He’s polynomials to calculate the approximate solutions. For later numerical computation, the expression of \( \psi_n \) for \( n \geq 1 \) below denotes the \( n \)th-order approximation to \( u(x, t) \) obtained by VIMHP.
\[
  \psi_n(x, t) = \sum_{i=0}^{n-1} u_i(x, t).
\]  

(21)

Absolute error = \( |u(x, t) - \psi_n(x, t)| \).

3. Numerical results

Here we present some examples, with an available analytical solution to illustrate the efficiency of the method described in the previous section to solve the problem (1). Then a comparison is made with analytical results obtained in Shakeri & Dehghan (2007) using the VIM and ADM methods and in Roul (2010) using HPM, to demonstrate the accuracy and the effectiveness of the present method VIMHP.

Example 1 We consider BPM (1) with \( a = 1, r = 0: \)
\[
  \frac{\partial u(x, y, t)}{\partial t} = \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} + h u(x, y, t),
\]  

(22)

subjected to the following initial condition:
\[
  u_0(x, y, t) = u(x, y, 0) = \sqrt{x + y + xy},
\]  

(23)

the correct functional is given by:
\[
  u_{n+1}(x, y, t) = u_n(x, y, t) + \int_0^t \lambda (s) \left[ \frac{\partial u_n}{\partial s} - \frac{\partial^2 u_n}{\partial x^2} - \frac{\partial^2 u_n}{\partial y^2} + h \bar{u}_n \right] ds.
\]  

(24)

yields the following stationary conditions:
\[
  \left\{ \begin{array}{l}
  1 + \lambda(s)_{|s=\infty} = 0 \\
  \lambda'(s) = 0.
  \end{array} \right.
\]  

(25)

The Lagrange multiplier can be identified as \( \lambda(s) = -1 \), consequently, we obtain the following iteration
\[
  u_{n+1}(x, y, t) = u_n(x, y, t) - \int_0^t \left[ \frac{\partial u_n}{\partial s} - \frac{\partial^2 u_n}{\partial x^2} - \frac{\partial^2 u_n}{\partial y^2} + h \bar{u}_n \right] ds.
\]  

(26)

Applying the VIMHP, we obtain:
\[
  \left\{ \begin{array}{l}
  u_0 + pu_1 + p^2 u_2 + \cdots = \sqrt{x + y + xy} \\
  + p\int_0^t \left\{ \frac{\partial^2}{\partial x^2} (u_0 + pu_1 + p^2 u_2 + \cdots)^2 \right\} ds. \\
  \end{array} \right.
\]  

(27)

To find \( u_i, i = 0, 1, 2, \ldots \) of this problem, the equating coefficients of corresponding power of \( p \) on both sides are used. Furthermore, we obtain the recurrence relation as given below:
$$p^0 : u_0(x, y, t) = \sqrt{x + y + xy}. $$

$$p^1 : u_1(x, y, t) = \int \left[ \frac{\partial^2 p^0}{\partial x^2} + \frac{\partial^2 p^0}{\partial y^2} + hu_0 \right] ds = \sqrt{x + y + xy} ht, $$

$$p^2 : u_2(x, y, t) = \int \left[ \frac{\partial^2 (2u_1u_1 + u_1^2)}{\partial x^2} + \frac{\partial^2 (2u_1u_1 + u_1^2)}{\partial y^2} + hu_1 \right] ds = \sqrt{x + y + xy} \frac{ht^2}{2!}, $$

$$p^3 : u_3(x, y, t) = \int \left[ \frac{\partial^2 (2u_2u_2 + u_1^2) + \partial^2 u_2u_2 + u_1^2}{\partial x^2} + \frac{\partial^2 (2u_2u_2 + u_1^2) + \partial^2 u_2u_2 + u_1^2}{\partial y^2} + hu_2 \right] ds = \sqrt{x + y + xy} \frac{ht^3}{3!}, $$

The approximate solution of the problem ((22)–(23)) is given by:

$$u(x, y, t) = \sqrt{x + y + xy} \left( 1 + ht + \frac{(ht)^2}{2!} + \frac{(ht)^3}{3!} + \cdots \right),$$

and in the closed form as:

$$u(x, y, t) = \sqrt{x + y + xy} e^{ht}. $$

If we change the initial condition as $$u(x, y, 0) = \sqrt{ax + by + cxy},$$ the approximate solution is given by

$$u(x, y, t) = \sqrt{ax + by + cxy} \left( 1 + ht + \frac{(ht)^2}{2!} + \frac{(ht)^3}{3!} + \cdots \right),$$

the exact solution will be obtained

$$u(x, y, t) = \sqrt{ax + by + cxy} e^{ht}, $$

where $$a, b, c$$ are real numbers. The same result was obtained by Roul (2010); Shakeri & Dehghan (2007) with parameter $$\alpha = 0, \gamma = 1$$ and $$h = \frac{1}{2}.$$ 

**Example 2** We consider BPM (1) with $$a = 1, r = 0$$ and $$h = 1:

$$\frac{\partial u(x, y, t)}{\partial t} = \frac{\partial^2 u^2(x, y, t)}{\partial x^2} + \frac{\partial^2 u^2(x, y, t)}{\partial y^2} + w(x, y, t), $$

subjected to the following initial condition:

$$u_0(x, y, t) = \sqrt{\sin(0x)\cosh(0y)}, \quad \theta \in \mathbb{R}, $$

and as it is shown before, $$\lambda(s) = -1,$$ which yields to the following iteration formula:

$$u_{n+1}(x, y, t) = u_0(x, y, t) + \int \left[ \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + u_n \right] ds. $$

Applying the variational iteration method using He’s polynomials gives:

$$u_0 + pu_1 + p^2u_2 + \cdots = \sqrt{\sin(0x)\cosh(0y)} + \int \left[ \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + u_0 \right] ds + \cdots $$

Similarly that in previous Example 1, we have:

$$p^0 : u_0(x, y, t) = \sqrt{\sin(0x)\cosh(0y)}, $$

$$p^1 : u_1(x, y, t) = \int \left[ \frac{\partial^2 (2u_2u_2 + u_1^2) + \partial^2 u_2u_2 + u_1^2}{\partial x^2} + \frac{\partial^2 (2u_2u_2 + u_1^2) + \partial^2 u_2u_2 + u_1^2}{\partial y^2} + hu_1 \right] ds = \sqrt{\sin(0x)\cosh(0y)} t, $$

$$p^2 : u_2(x, y, t) = \int \left[ \frac{\partial^2 (2u_2u_2 + u_1^2) + \partial^2 u_2u_2 + u_1^2}{\partial x^2} + \frac{\partial^2 (2u_2u_2 + u_1^2) + \partial^2 u_2u_2 + u_1^2}{\partial y^2} + hu_2 \right] ds = \sqrt{\sin(0x)\cosh(0y)} \frac{t^2}{2!}, $$

$$p^3 : u_3(x, y, t) = \int \left[ \frac{\partial^2 (2u_2u_2 + u_1^2) + \partial^2 u_2u_2 + u_1^2}{\partial x^2} + \frac{\partial^2 (2u_2u_2 + u_1^2) + \partial^2 u_2u_2 + u_1^2}{\partial y^2} + hu_2 \right] ds = \sqrt{\sin(0x)\cosh(0y)} \frac{t^3}{3!}, $$

And so on. Therefore, the approximate solution of the problems ((29)–(30)) is given by:

$$u(x, y, t) = \sum_{i=0}^{\infty} u_i = \sqrt{\sin(0x)\cosh(0y)} \left( 1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots \right), $$

which converges very rapidly to the exact solution

$$u(x, y, t) = \sqrt{\sin(0x)\cosh(0y)} e^t. $$

Changing the initial condition as $$u(x, y, 0) = \sqrt{\sin(0x)\sinh(0y)},$$ gives an exact solution of the following type:

$$u(x, y, t) = \sqrt{\sin(0x)\sinh(0y)} e^t, \quad \theta \in \mathbb{R}. $$

**Example 3** The numerical results from examples 3 and 4 are shown in Figures (1–4). The comparison between the exact and (VIMHP) solution graphs reveals the accuracy of our used method. We consider BPM (1) with $$a = 1, b = 1:

$$\frac{\partial u(x, y, t)}{\partial t} = \frac{\partial^2 u^2(x, y, t)}{\partial x^2} + \frac{\partial^2 u^2(x, y, t)}{\partial y^2} + \lambda(s) \left( 1 - ru(x, y, t) \right), $$

with the initial condition:
\[ u_0(x, y, t) = e^{\sqrt{p} (x+y)}. \]  \hspace{1cm} (35)

By applying aforesaid method subject to the initial condition and \( \lambda(s) = -1 \), we have:

\[
\begin{align*}
  u_0 + pu_1 + \cdots &= e^{\sqrt{p} (x+y)} + p \int_0^t \left\{ \frac{\partial^2 (u_0^2)}{\partial x^2} + \frac{\partial^2 (u_0^2)}{\partial y^2} + h(u_0 - n^2_0) \right\} ds \\
  + &p \int_0^t h(u_0 + pu_1 + \cdots)(1-r(u_0 + pu_1 + \cdots))ds
\end{align*}
\]

Comparison of the coefficients of like powers of \( p \) gives

\[
\begin{align*}
  p^0 : u_0(x, y, t) &= e^{\sqrt{p} (x+y)}, \\
  p^1 : u_1(x, y, t) &= \int_0^t \left[ \frac{\partial^2 (u_0^2)}{\partial x^2} + \frac{\partial^2 (u_0^2)}{\partial y^2} + h(u_0 - n^2_0) \right] ds \\
  &= e^{\sqrt{p} (x+y)}ht, \\
  p^2 : u_2(x, y, t) &= \int_0^t \left[ \frac{\partial^2 (2u_0u_1)}{\partial x^2} + \frac{\partial^2 (2u_0u_1)}{\partial y^2} + h(u_1 - 2nu_0u_1) \right] ds \\
  &= e^{\sqrt{p} (x+y)} \frac{h^2t^2}{2!}, \\
  p^3 : u_3(x, y, t) &= \int_0^t \left[ \frac{\partial^2 (2u_0u_2 + u_1^2)}{\partial x^2} + \frac{\partial^2 (2u_0u_2 + u_1^2)}{\partial y^2} + h(u_2 - 2nu_0u_2 - nu_1^2) \right] ds \\
  &= e^{\sqrt{p} (x+y)} \frac{h^3t^3}{3!},
\end{align*}
\]

\( \vdots \)

In a similar way, we can determine the other components of the VIMHP. Finally, the approximate solution of the problem (Equations (34)–(35)) in a series form is given as:

\[ u(x, y, t) = e^{\sqrt{p} (x+y)} \sum_{k=0}^{+\infty} \frac{(ht)^k}{k!}, \]  \hspace{1cm} (36)

and in closed form by,

\[ u(x, y, t) = e^{\sqrt{p} (x+y) + ht}. \]  \hspace{1cm} (37)
Table 1. The absolute error $|\text{Exact} - \psi_1|$ for different values of $(x, y, t)$ when $h = -1$ and $r = -8/9$ in Example 3.

| $(x, y)$ | $|\text{Exact} - \psi_1|$ |
|----------|-------------------|
| $(-10, -10)$ | 5.1983E-9 |
| $(-8, -8)$ | 7.4813E-8 |
| $(-4, -4)$ | 2.8382E-7 |
| $(-2, -2)$ | 1.0767E-6 |
| $(0, 0)$ | 4.0847E-6 |
| $(2, 2)$ | 1.5497E-5 |
| $(4, 4)$ | 5.8790E-5 |
| $(6, 6)$ | 2.2320E-4 |
| $(8, 8)$ | 8.4605E-4 |
| $(10, 10)$ | 3.2096E-3 |

Table 2. Comparison between exact solution and $\psi_4$ at $t = 0.2$ when $h = -1$ and $r = -8/9$ in Example 3.

| $(x, y)$ | Exact value $\psi_4$ | $|\text{Exact} - \psi_4|$ |
|----------|-------------------|-------------------|
| $(-10, -10)$ | 1.01949E-3 | 1.01486E-3 |
| $(-8, -8)$ | 3.9529E-3 | 3.95248E-3 |
| $(-4, -4)$ | 1.49956E-2 | 1.49944E-2 |
| $(-2, -2)$ | 5.68882E-2 | 5.68838E-2 |
| $(0, 0)$ | 2.15815E-1 | 2.15798E-1 |
| $(2, 2)$ | 3.10599 | 3.10575 |
| $(4, 4)$ | 11.78310 | 11.78218 |
| $(6, 6)$ | 44.70118 | 44.69769 |
| $(8, 8)$ | 169.58145 | 169.56817 |
| $(10, 10)$ | 643.33567 | 643.28534 |

Table 3. Comparison between exact solution and $\psi_3$ obtained by VIMHP at $t = 20$, in Example 4.

| $(x, y)$ | Exact value $\psi_3$ |
|----------|-------------------|
| $(-450, -450)$ | 224.93991121318537 |
| $(-400, -400)$ | 199.9393137096020 |
| $(-300, -300)$ | 149.9399185229425 |
| $(-250, -250)$ | 124.940024591990 |
| $(0, 0)$ | 8.593912638296976 |
| $(50, 50)$ | 25.07696486153311 |
| $(100, 100)$ | 50.060974302577023 |
| $(200, 200)$ | 100.0612139714349 |
| $(350, 350)$ | 175.0645714205666 |
| $(500, 500)$ | 250.0639501540889 |

Table 4. The absolute error by VIMHP and VIM, ADM (Shakeri & Dehghan, 2007) for different values of $(x, y)$ at $t = 10$ in Example 4.

| $(x, y)$ | $|\text{Exact} - \psi_1|$ | $|\text{Exact} - \text{VIM}|$ | $|\text{Exact} - \text{ADM}|$ |
|----------|-------------------|-------------------|-------------------|
| $(-450, -450)$ | 9.8139E-1 | 7.2722E-1 | 2.5734E-1 |
| $(-400, -400)$ | 1.7668E-15 | 1.1070E-11 | 3.2571E-6 |
| $(-300, -300)$ | 7.4574E-2 | 3.5787E-11 | 3.8410E-6 |
| $(0, 0)$ | 7.3883E-3 | 7.7787E-1 | 8.0947E-1 |
| $(50, 50)$ | 5.7070E-11 | 6.7882E-8 | 2.0689E-8 |
| $(100, 100)$ | 1.7965E-12 | 8.8701E-9 | 5.1928E-5 |
| $(200, 200)$ | 5.6333E-14 | 1.8040E-10 | 1.3003E-5 |
| $(350, 350)$ | 3.4370E-15 | 9.0969E-11 | 4.2485E-6 |
| $(500, 500)$ | 7.7977E-16 | 4.5626E-12 | 2.0823E-6 |

which is in full agreement with Roul (2010); Shakeri & Dehghan (2007) when $h = -1$ and $r = -8/9$.

Example 4 (Figure 3)

We consider BPM (1) with $a = -1, b = 1, h = -\frac{1}{9}$ and $r = 48$:

$$\frac{\partial u(x, y, t)}{\partial t} + \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} = \frac{1}{96} u(x, y, t) - \frac{1}{2},$$  

subjected to following initial condition:

$$u_0(x, y, t) = u(x, y, 0) = \frac{1}{4} \sqrt{2(x^2 + y^2)} + y + 5,$$  

as it is shown before, $\lambda(s) = -1$, applying the present method VIMHP, we have:

$$u_0 + pu_1 + \cdots = \frac{1}{4} \sqrt{2(x^2 + y^2)} + y + 5 + p \int_0^t \left( \frac{\partial^2}{\partial x^2} u_0 + \frac{\partial^2}{\partial y^2} u_0 + \cdots \right)^2 \, ds.$$

According to VIMHP procedures, we now successively obtain

$$a^0 : u_0(x, y, t) = \frac{1}{4} \sqrt{2(x^2 + y^2)} + y + 5,$$

$$a^1 : u_1(x, y, t) = \frac{1}{24} \left( 2(x^2 + y^2) + y + 5 \right)^2 t,$$

$$a^2 : u_2(x, y, t) = \frac{1}{288} \left( 2(x^2 + y^2) + y + 5 \right)^3 t^2,$$

$$a^3 : u_3(x, y, t) = \frac{1}{1728} \left( 2(x^2 + y^2) + y + 5 \right)^4 t^3 + \cdots,$$

So we obtain the components which constitute $u(x, y, t)$, thus we will have:

$$u(x, y, t) = \frac{1}{4} \sqrt{2(x^2 + y^2)} + y + 5$$

$$+ \frac{1}{24} \left( 2(x^2 + y^2) + y + 5 \right)^2 t$$

$$- \frac{1}{288} \left( 2(x^2 + y^2) + y + 5 \right)^3 t^2$$

$$+ \frac{1}{1728} \left( 2(x^2 + y^2) + y + 5 \right)^4 t^3 + \cdots,$$

and so we have the exact solution of the problem ((38)–(39)) as presented in Shakeri & Dehghan (2007):

$$u(x, y, t) = \frac{1}{4} \sqrt{2(x^2 + y^2)} + y + \frac{t}{3} + 5.$$  

From Table (1–4) the comparison reveals that our approximate solutions are in very good agreement with the solutions given in Shakeri & Dehghan (2007). Moreover, the findings strongly demonstrate that the proposed method is more accurate and convenient for solving the nonlinear differential equations.
4. Conclusion

In this paper we have shown that the VIMHP was successfully employed to solve the biological population model. This method is based on VIM and HPM. For our equations, the results are exactly the same as those obtained by the VIM, ADM and HPM methods. As for the advantages of VIMHP, it is a clear advantage of this technique over the HPM method. We do not need to solve a differential equation at each iteration, it reduces the size of computation without the restrictive assumption. The approximate solutions given by VIMHP converge to its exact solution faster in the studied cases. Finally, we concluded that the proposed scheme is very powerful, efficient and reliable in finding the analytical solutions for a wider class of linear and nonlinear differential equations.

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