Super graphs on groups

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Graphs on groups

By this title, I mean (to a first approximation) graphs whose vertex set is the set of elements in a group $G$, and which are defined in terms of the structure of $G$. This means that they are invariant under the automorphism group of $G$. (I am not considering Cayley graphs here.) The simplest example is the commuting graph of $G$, in which $x$ and $y$ are joined if and only if $xy = yx$. This was used by Brauer and Fowler in their seminal 1955 paper on centralizers of involutions in simple groups of even order. (Brauer and Fowler don’t use the word “graph”, but make extensive use of the distance function in the graph after the identity is deleted. In the commuting graph, elements of the centre are joined to everything; so for considerations of connectedness, diameter, etc., it makes sense to remove the centre. But for other questions it is better to leave it in, and I shall do so. I will denote this graph by $\text{Com}(G)$.

When are two of the graphs equal?

It is interesting first to find the classes of groups for which the hierarchy is not strict. We obtain known and important classes:

**Theorem**

- The power graph of $G$ is equal to the enhanced power graph if and only if every element of $G$ has prime power order.
- The enhanced power graph of $G$ is equal to the commuting graph if and only if every Sylow subgroup of $G$ is cyclic or generalized quaternion.

In each case, all groups satisfying the condition have been determined. See my paper in the *International Journal of Group Theory*, 11 (2022), 43–124.

Into the second dimension

Let $A$ be a graph on the group $G$, and $B$ a partition of $G$.

- The *$B$ super$A$ graph* on $G$ has vertex set $G$, with $x$ joined to $y$ if there exist elements $x'$ and $y'$, $B$-equivalent to $x$ and $y$ respectively, such that $x'$ and $y'$ are $A$-adjacent. (By convention, we take vertices in the same $B$-class to be adjacent.)
- The *condensed $B$ super$A$ graph* on $G$ has vertex set the set of $B$-classes, two classes $C$ and $D$ joined if and only if there exist $x \in C$ and $y \in D$ such that $x$ and $y$ are $A$-adjacent.

A condensed supergraph can be expanded by blowing each vertex up to a clique of the appropriate size.

A research discussion group in south India

From March to August 2021, Ambat Vijayakumar and Aparna Lakshmanan at CUSAT in Kochi, Kerala, ran an on-line discussion group on graphs and groups, which led to many new results and ideas.

One of the participants was Lavanya Selvaganesh, who defined a graph she called the *superpower graph* of $G$, in which $x$ and $y$ are joined if there exist $x'$ and $y'$, having the same order as $x$ and $y$ respectively, which are joined in the power graph. This led us (G. Arunkumar, Rajat Kanti Singh, Lavanya Selvaganesh and me) to the following generalization (arXiv 2112.02396) . . .

Some equivalence relations

I will consider three equivalence relations:

- equality;
- conjugacy;
- same order.

Others can be imagined. With equality, we just get the original graph $A$. I will denote the conjugacy and order superpower graphs by $\text{Conj}_G(Pow(G))$ and $\text{Ord}_G(Pow(G))$, with similar notation for the superfineenhanced power graphs and supercommuting graphs.
The 2-dimensional hierarchy

Here is the resulting 2-dimensional hierarchy:

\[
\begin{array}{ccc}
\text{Conj} \text{Com}(G) & \text{Ord} \text{EPow}(G) & \text{Ord} \text{Pow}(G) \\
\text{EPow}(G) & \text{Conj} \text{Pow}(G) & \\
\text{Pow}(G) & \\
\end{array}
\]

Motivation

To convince you that this is not just generalization for its own sake, I need to show you two things: the supergraphs are closely connected to the group structure; and there are some interesting results, or results with interesting proofs, concerning these graphs. This I hope to do in the rest of this talk.

The first result I will show you was done as a "proof of concept".

Completeness

The following table describes groups whose power graph, enhanced power graph, commuting graph, or their conjugacy or order supergraph is complete.

|          | power graph | enhanced power graph | commuting graph |
|----------|-------------|----------------------|-----------------|
| equality | cyclic      | cyclic               | abelian         |
| conjugacy| cyclic      | cyclic               | abelian         |
| order    | p-group     | (+)                  | (+)             |

Here (+) means that the group G has an element whose order is the exponent m of G; equivalently, the spectrum of G (the set of orders of elements of G) is the set of all divisors of m.

Coincidences

I have cheated you slightly:

Theorem
For any finite group G, the graphs \text{Ord} \text{Com}(G) and \text{Ord} \text{EPow}(G) coincide.

So there are only eight different graphs, not nine. There are no further cases in which two of the graphs coincide for all groups. For other pairs, we know something:

Theorem
- The conjugacy supercommuting graph of G is equal to the commuting graph if and only if G is a 2-Engel group, that is, satisfies the identity \([x, y, y] = 1\);
- the conjugacy superpower graph of G is equal to the power graph if and only if G is a Dedekind group, that is, one in which every subgroup is normal.

Comments

Dedekind groups are all known. Such a group is either abelian or of the form \(A \times B \times C\) where A is a quaternion group, B an elementary abelian 2-group, and C an abelian group of odd order.

Engel groups have had a lot of attention. Any nilpotent group of class 2 is 2-Engel, and every 2-Engel group is nilpotent of class at most 3 (shown independently by Hopkins and Levi).

The first part uses the following result:

Theorem
A group G satisfies the 2-Engel identity if and only if every centralizer is a normal subgroup.

The only proof we found in the literature was a StackExchange post by Korhonen, using a result of Kappe. Information on earlier proofs welcome!

A problem

Problem
Determine the class of groups for which equality holds between two adjacent graphs in the 2-dimensional hierarchy in the remaining cases.

There are twelve edges in the diagram I showed you earlier, and we have only dealt with five of them, so plenty remains to be done.
Dominant vertices
Finding the dominant vertices in these graphs (those joined to all others) is an extension of the problem of determining when they are complete. Here are some results:

Theorem
- The set of dominant vertices in the power graph of $G$ is the whole of $G$, if $G$ is a cyclic $p$-group; the identity and the generators of $G$, if $G$ is cyclic but not a $p$-group; the centre, if $G$ is a generalized quaternion group; and only the identity in all other cases.
- The set of dominant vertices in the enhanced power graph of $G$ is the cyclicizer of $G$, the product of the Sylow $p$-subgroups of $Z(G)$ for those primes $p$ for which the Sylow subgroups of $G$ are cyclic or generalized quaternion.
- The set of dominant vertices in the commuting graph is the centre of $G$.

And for the super graphs . . .

Theorem
- If $A$ is the power graph, enhanced power graph, or commuting graph, then the set of dominant vertices in the conjugacy super $A$ graph of $G$ is the same as the set of dominant vertices in the $A$ graph.
- Let $G$ be a group not or prime power order, having exponent $m$. Then the set of dominant vertices in the order superpower graph consists of the identity and the elements of order $m$ (if any).

Moving up
Some of the condensed supergraphs had been looked at earlier. We move up in the hierarchy and examine the partition into conjugacy classes. Thus the condensed conjugacy super $A$ graph has vertices the conjugacy classes, two classes $C$ and $D$ adjacent if there are elements $x \in C$, $y \in D$ which are adjacent in the graph $A$.

The SCC-graph
The condensed conjugacy supergraphs for the commuting and nilpotent graphs were studied by Herzog, Longobardi and Maj and by Mohammadian and Erfanian respectively, under the names commuting conjugacy class graph (CCC-graph) and nilpotent conjugacy class graph (NCC-graph) respectively. We examined the analogous soluble conjugacy class graph (SCC-graph). Recall that the vertices are the conjugacy classes, two classes $C$ and $C'$ joined if there exist $x \in C$ and $y \in C'$ such that $(x,y)$ is soluble.

Completeness
An old and well-known result of Landau in 1903 states that there are only finitely many finite groups with any given number of conjugacy classes.

Clique number
We were able to extend this as follows.

Theorem
Given a positive integer $d$, there are only finitely many finite groups $G$ such that the clique number of the SCC-graph of $G$ is equal to $d$.

In particular, the finite groups in which the clique number of the graph is at most 3 are the cyclic groups of orders 1, 2 and 3 and the symmetric group of degree 3.

We have not examined the growth rate for the number or largest order of such a group. Also, we have not tried to find all groups whose SCC-graph has clique number 4 (these include the alternating group $A_5$).
Proof sketch

The proof proceeds as follows. By the results on the previous slide we may assume that $G$ is not soluble. Then we can reduce to the case where the soluble radical is trivial, and further to the case where the socle is almost simple; then the Classification of Finite Simple Groups gives the result.

An alternative proof when we know the group is simple is the following. If the clique number of the SCC-graph is bounded, then the number of prime divisors of an element order in $G$ is bounded. A recent result of Hung and Yang then bounds the number of prime divisors of $G$. Then we can bound these prime divisors, and hence the exponent of $G$. But there are only finitely many finite simple groups of given exponent (an old result of Gareth Jones).

Universality

I will finish with some remarks on the question: Given a type $A$ of graph on groups, which finite graphs can be embedded in $A(G)$ as an induced subgraph for some group $G$?

Theorem

Given a finite complete graph with edges coloured red, green and blue in any manner, there is a group $G$ such that red edges belong to $EPow(G)$, green edges to $Com(G)$ but not $EPow(G)$, and blue edges not to $Com(G)$.

This theorem shows that enhanced power graphs are universal (ignoring the blue-green distinction), and commuting graphs are universal (ignoring the red-green distinction), but also graphs which are the differences between the edge sets of these two are universal (ignoring the red-blue distinction).

Threshold graphs

I am going to give you a very weak result on SCC-graphs (but it is the best I can do at the moment). First I define the class of graphs involved.

A threshold graph is one in which each vertex $x$ has a real number weight $w(x)$, and there is a threshold $t$, such that $x$ and $y$ are joined if and only if $w(x) + w(y) > t$.

Threshold graphs form the class defined by forbidding three induced subgraphs on four vertices: the cycle, the path, and two disjoint edges.

They are also the class of graphs obtained by adding vertices one at a time, each new vertex joined to either all or none of the existing vertices.

SCC-graphs

Proposition

For any threshold graph $\Gamma$, there is a finite group $G$ such that the SCC-graph of $G$ contains $\Gamma$ as an induced subgraph.

This is unlikely to be best possible. I don’t know any graph which can’t be so embedded. But it is the best I can do at present.

Even more embarrassingly, this rather weak result uses the celebrated theorem of Green and Tao about primes in arithmetic progression!

Certainly not all SCC-graphs are threshold. Taking $G = S_7$, we can use the classes of a 7-cycle, two disjoint 3-cycles, a 4-cycle, and a 5-cycle to get a path on four vertices.

Sketch proof

Here is a brief sketch. We are given a threshold graph, with vertex weights and threshold. We adjust these slightly so that they are rational, and multiply up to make them integers.

Now we apply Green and Tao to scale up further so that the weights are primes chosen from an arithmetic progression, while the threshold is greater than all the weights.

Now let $G$ be the symmetric group whose degree is the threshold, and take the vertices to be conjugacy classes of cycles of the appropriate prime length. If the sum of two primes is below the threshold, there are conjugates with disjoint support, generating an abelian group; but if it is above, then any two supports intersect, and so the prime cycles generate the alternating group.
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