GENERIC ELEMENTS IN ZARISKI-DENSE SUBGROUPS AND ISOSPECTRAL LOCALLY SYMMETRIC SPACES

GOPAL PRASAD AND ANDREI S. RAPINCHUK

Abstract. The article contains a survey of our results on length-commensurable and isospectral locally symmetric spaces and of related problems in the theory of semi-simple algebraic groups. We discuss some of the techniques involved in this work (in particular, the existence of generic tori in semi-simple algebraic groups over finitely generated fields and of generic elements in finitely generated Zariski-dense subgroups) as well as some open problems. The article is an expanded version of the talk given at the workshop by the first-named author.

1. Introduction

The object of the talk delivered at the workshop by the first-named author was, and of this article is, to give an exposition of recent results on isospectral and length-commensurable locally symmetric spaces associated with simple real algebraic groups ([40], [43]) and related problems in the theory of semi-simple algebraic groups ([17], [18], and [41]). One of the goals of our paper [40] was to study the problem beautifully formulated by Mark Kac in [25] as “Can one hear the shape of a drum?” for the quotients of symmetric spaces of the groups of real points of absolutely simple real algebraic groups by cocompact arithmetic subgroups. A precise mathematical formulation of Kac’s question is whether two compact Riemannian manifolds which are isospectral (i.e., have equal spectra – eigenvalues and multiplicities – for the Laplace-Beltrami operator) are necessarily isometric. In general, the answer to this question is in the negative as was shown by John Milnor [35] already in 1964 by constructing two non-isometric isospectral flat tori of dimension 16. Later M.-F. Vigneras [50] used arithmetic properties of quaternion algebras to produce examples of arithmetically defined isospectral, but not isometric, Riemann surfaces. On the other hand, T. Sunada [49], inspired by a construction of nonisomorphic number fields with the same Dedekind zeta-function, proposed a general and basically purely group-theoretic method of producing nonisometric isospectral Riemannian manifolds which has since then been used in various ways. It is important to note, however, that the nonisometric isospectral manifolds constructed by Vigneras and Sunada are commensurable, i.e. have a common finite-sheeted cover. This suggests that one should probably settle for the following weaker version of Kac’s original question: Are any two isospectral compact Riemannian manifolds necessarily commensurable? The answer to this modified question is still negative in the general case: Lubotzky, Samuels and Vishne [32], using the Langlands correspondence, have constructed examples of noncommensurable isospectral locally symmetric spaces associated with absolutely simple real groups of type $\text{A}_n$ (cf. Problem
Nevertheless, it turned out that the answer is actually in the affirmative for several classes of locally symmetric spaces. Prior to our paper \cite{40}, this was known to be the case only for the following two classes: arithmetically defined Riemann surfaces \cite{48} and arithmetically defined hyperbolic 3-manifolds \cite{12}.

In \cite{40}, we used Schanuel’s conjecture from transcendental number theory (for more about this conjecture, and how it comes up in our work, see below) and the results of \cite{17}, \cite{41} to prove that any two compact isospectral arithmetically defined locally symmetric spaces associated with absolutely simple real algebraic groups of type other than \(A_n\) (\(n > 1\)), \(D_{2n+1}\) (\(n > 1\)), or \(E_6\) are necessarily commensurable. One of the important ingredients of the proof is the connection between isospectrality and another property of Riemannian manifolds called iso-length-spectrality. More precisely, for a Riemannian manifold \(M\) we let \(L(M)\) denote the weak length spectrum of \(M\), i.e. the collection of the lengths of all closed geodesics in \(M\) (note that for the existence of a “nice” Laplace spectrum, \(M\) is required to be compact, but the weak length spectrum \(L(M)\) can be considered for any \(M\) (i.e., we do not need to assume that \(M\) is compact)). Then two Riemannian manifolds \(M_1\) and \(M_2\) are called iso-length spectral if \(L(M_1) = L(M_2)\). It was first proved by Gangolli \cite{15} in the rank one case, and then by Duistermaat and Guillemin \cite{9} and Duistermaat, Kolk and Varadarajan \cite{10} in the general case (see Theorem 10.1 in \cite{40}) that any two compact isospectral locally symmetric spaces are iso-length spectral. So, the emphasis in \cite{40} is really on the analysis of iso-length spectral locally symmetric spaces \(M_1\) and \(M_2\). In fact, we prove our results under the much weaker assumption of length-commensurability, which means that \(Q \cdot L(M_1) = Q \cdot L(M_2)\). (The set \(Q \cdot L(M)\) is sometimes called the rational length spectrum of \(M\); its advantage, particularly in the analysis of questions involving commensurable manifolds, is that it is invariant under passing to a finite-sheeted cover – this property fails for the Laplace spectrum or the length spectrum. At the same time, \(Q \cdot L(M)\) can actually be computed in at least some cases, while precise computation of \(L(M)\) or the Laplace spectrum is not available for any compact locally symmetric space at this point.) The notion of length-commensurability was introduced in \cite{40}, and the investigation of its qualitative and quantitative consequences for general locally symmetric spaces is an ongoing project. For arithmetically defined spaces, however, the main questions were answered in \cite{40}, and we would like to complete this introduction by showcasing the results for arithmetic hyperbolic spaces.

Let \(\mathbb{H}^n\) be the real hyperbolic \(n\)-space. By an arithmetically defined real hyperbolic \(n\)-manifold we mean the quotient \(\mathbb{H}^n/\Gamma\), where \(\Gamma\) is an arithmetic subgroup of \(\text{PSO}(n, 1)\) (which is the isometry group of \(\mathbb{H}^n\)); see \cite{43} regarding the notion of arithmeticity.

**Theorem 1.1.** (cf. \cite{40} Corollary 8.17 and Remark 8.18) Let \(M_1\) and \(M_2\) be arithmetically defined real hyperbolic \(n\)-manifolds.

If \(n \not\equiv 1 \pmod{4}\), then in case \(M_1\) and \(M_2\) are not commensurable, after a possible interchange of \(M_1\) and \(M_2\), there exists \(\lambda_1 \in L(M_1)\) such that for any \(\lambda_2 \in L(M_2)\), the ratio \(\lambda_1/\lambda_2\) is transcendental over \(\mathbb{Q}\). (Thus, for such \(n\) the length-commensurability, and hence isospectrality, of \(M_1\) and \(M_2\) implies their commensurability.)
On the contrary, for any \( n \equiv 1 \pmod{4} \), there exist \( M_1 \) and \( M_2 \) as above that are length-commensurable, but not commensurable.

What is noteworthy is that there is no apparent geometric reason for this dramatic distinction between the length-commensurability of hyperbolic \( n \)-manifolds when \( n \not\equiv 1 \pmod{4} \) and \( n \equiv 1 \pmod{4} \) – in our argument the difference comes from considerations involving Galois cohomology – see Theorem 4.2 and subsequent comments.

Our general results for arithmetically defined length-commensurable locally symmetric spaces (cf. §5) imply similar (but not identical!) assertions for complex and quaternionic hyperbolic manifolds. At the same time, one can ask about possible relations between \( \mathbb{Q} \cdot L(M_1) \) and \( \mathbb{Q} \cdot L(M_2) \) (or between \( \mathbb{Q} \cdot L(M_1) \) and \( \mathbb{Q} \cdot L(M_2) \)) if \( M_1 \) and \( M_2 \) are not length-commensurable. The results we will describe in §8 assert that if \( \mathbb{Q} \cdot L(M_1) \neq \mathbb{Q} \cdot L(M_2) \), then no polynomial-type relation between \( L(M_1) \) and \( L(M_2) \) can ever exist; in other words, these sets are very different. This is, for example, the case if \( M_1 \) and \( M_2 \) are hyperbolic manifolds of finite volume having different dimensions!

### 2. Length-commensurable locally symmetric spaces and weakly commensurable subgroups

#### 2.1. Riemann surfaces

Our analysis of length-commensurability of locally symmetric spaces relies on a purely algebraic relation between their fundamental groups which we termed weak commensurability. It is easiest to motivate this notion by looking at the length-commensurability of Riemann surfaces. In this discussion we will be using the realization of \( \mathbb{H}^2 \) as the complex upper half-plane with the standard hyperbolic metric \( ds^2 = y^{-2}(dx^2 + dy^2) \). The usual action of \( \text{SL}_2(\mathbb{R}) \) on \( \mathbb{H}^2 \) by fractional linear transformations is isometric and allows us to identify \( \mathbb{H}^2 \) with the symmetric space \( \text{SO}_2(\mathbb{R}) \backslash \text{SL}_2(\mathbb{R}) \). It is well-known that any compact Riemann surface \( M \) of genus > 1 can be obtained as a quotient of \( \mathbb{H}^2 \) by a discrete subgroup \( \Gamma \subset \text{SL}_2(\mathbb{R}) \) with torsion-free image in \( \text{PSL}_2(\mathbb{R}) \). Now, given any such subgroup \( \Gamma \), we let \( \pi: \mathbb{H}^2 \to \mathbb{H}^2/\Gamma =: M \) denote the canonical projection. It is easy to see that

\[
t \mapsto e^{ti} = i \cdot \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \quad t \in \mathbb{R},
\]

is a unit-velocity parametrization of a geodesic \( c \) in \( \mathbb{H}^2 \). So, if \( \gamma = \text{diag}(t_\gamma, t_\gamma^{-1}) \in \Gamma \) then the image \( \pi(c) \) is a closed geodesic \( c_\gamma \) in \( M \) whose length is given by the formula

\[
\ell_\Gamma(c_\gamma) = \frac{2}{n_\gamma} \cdot \log t_\gamma
\]

(assuming that \( t_\gamma > 1 \), where \( n_\gamma \) is an integer \( \geq 1 \) (winding number in case \( c_\gamma \) is not primitive). Generalizing this construction, one shows that every semi-simple element \( \gamma \in \Gamma \setminus \{\pm 1\} \) gives rise to a closed geodesic \( c_\gamma \) in \( M \) whose length is given by \( (1) \) where \( t_\gamma \) is the eigenvalue of \( \pm \gamma \) which is > 1, and conversely, any closed geodesic in \( M \) is obtained this way. As a result,

\[
\mathbb{Q} \cdot L(M) = \mathbb{Q} \cdot \{\log t_\gamma \mid \gamma \in \Gamma \setminus \{\pm 1\} \text{ semi-simple}\}.
\]
Now, suppose we have two quotients $M_1 = \mathbb{H}^2/\Gamma_1$ and $M_2 = \mathbb{H}^2/\Gamma_2$ as above, and let $c_{\gamma_i}$ be a closed geodesic in $M_i$ for $i = 1, 2$. Then

$$\ell_{\Gamma_1}(c_{\gamma_1})/\ell_{\Gamma_2}(c_{\gamma_2}) \in \mathbb{Q} \iff \exists m, n \in \mathbb{N} \text{ such that } t_{\gamma_i}^m = t_{\gamma_2}^n,$$

or equivalently, the subgroups generated by the eigenvalues of $\gamma_1$ and $\gamma_2$ have nontrivial intersection. This leads us to the following.

2.2. Definition. Let $G_1 \subset \text{GL}_{N_1}$ and $G_2 \subset \text{GL}_{N_2}$ be two semi-simple algebraic groups defined over a field $F$ of characteristic zero.

(a) Semi-simple elements $\gamma_1 \in G_1(F)$ and $\gamma_2 \in G_2(F)$ are said to be weakly commensurable if the subgroups of $F^\times$ generated by their eigenvalues intersect nontrivially.

(b) (Zariski-dense) subgroups $\Gamma_1 \subset G_1(F)$ and $\Gamma_2 \subset G_2(F)$ are weakly commensurable if every semi-simple element $\gamma_1 \in \Gamma_1$ of infinite order is weakly commensurable to some semi-simple element $\gamma_2 \in \Gamma_2$ of infinite order, and vice versa.

It should be noted that in [40] we gave a more technical, but equivalent, definition of weakly commensurable elements, viz. we required that there should exist maximal $F$-tori $T_i$ of $G_i$ for $i = 1, 2$ such that $\gamma_i \in T_i(F)$ and for some characters $\chi_i \in X(T_i)$ we have

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1.$$

This (equivalent) reformulation of (a) immediately demonstrates that the notion of weak commensurability does not depend on the choice of matrix realizations of the $G_i$'s, and more importantly, is more convenient for the proofs of our results.

The above discussion of Riemann surfaces implies that if two Riemann surfaces $M_1 = \mathbb{H}^2/\Gamma_1$ and $M_2 = \mathbb{H}^2/\Gamma_2$ are length-commensurable, then the corresponding fundamental groups $\Gamma_1$ and $\Gamma_2$ are weakly commensurable. Our next goal is to explain why this implication remains valid for general locally symmetric spaces.

2.3. Length-commensurability and weak commensurability: the general case.

First, we need to fix some notations related to general locally symmetric spaces. Let $G$ be a connected adjoint real semi-simple algebraic group, let $\mathcal{G} = G(\mathbb{R})$ considered as a real Lie group, and let $\mathfrak{X} = \mathcal{K} \backslash \mathcal{G}$, where $\mathcal{K}$ is a maximal compact subgroup of $\mathcal{G}$, be the associated symmetric space endowed with the Riemannian metric coming from the Killing form on the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$. Furthermore, given a torsion-free discrete subgroup $\Gamma$ of $\mathcal{G}$, we let $\mathfrak{X}_\Gamma = \mathfrak{X}/\Gamma$ denote the corresponding locally symmetric space. Just as in the case of Riemann surfaces, to any nontrivial semi-simple element $\gamma \in \Gamma$ there corresponds a closed geodesic $c_\gamma$, whose length is given by

$$\ell_\Gamma(c_\gamma) = \frac{1}{n_\gamma} \cdot \lambda_\Gamma(\gamma),$$

where $n_\gamma$ is an integer $\geq 1$ and

$$\lambda_\Gamma(\gamma)^2 := \sum_\alpha (\log |\alpha(\gamma)|)^2,$$

with the summation running over all roots of $G$ with respect to a fixed maximal $\mathbb{R}$-torus $T$ of $G$ whose group of $\mathbb{R}$-points contains $\gamma$ ($\ell_\Gamma(c_\gamma)$ is thus a submultiple of $\lambda_\Gamma(\gamma)$). This
formula looks much more intimidating than (1), so in order to make it more manageable we first make the following observation. Of course, the \( \mathbb{R} \)-torus \( T \) may not be \( \mathbb{R} \)-split, so not every root \( \alpha \) may be defined over \( \mathbb{R} \). However,

\[
|\alpha(\gamma)|^2 = \chi(\gamma)
\]

where \( \chi = \alpha + \overline{\alpha} \) (with \( \overline{\alpha} \) being the conjugate character in terms of the natural action of Gal(\( \mathbb{C}/\mathbb{R} \)) on \( X(T) \), and, as usual, \( X(T) \) is viewed as an additive group) is a character defined over \( \mathbb{R} \) and which takes positive values on \( T(\mathbb{R}) \). Such characters will be called positive. So, we can now re-write (2) in the form

\[
(3) \quad \lambda_T(\gamma)^2 = \sum_{i=1}^{p} s_i (\log \chi_i(\gamma))^2
\]

where \( \chi_1, \ldots, \chi_p \) are certain positive characters of \( T \) and \( s_1, \ldots, s_p \) are positive rational numbers whose denominators are divisors of 4. The point to be made here is that the subgroup \( P(T) \subset X(T) \) of positive characters may be rather small. More precisely, \( T \) is an almost direct product of an \( \mathbb{R} \)-anisotropic subtorus \( A \) and an \( \mathbb{R} \)-split subtorus \( S \). Then any character of \( T \) which is defined over \( \mathbb{R} \) vanishes on \( A \). This easily implies that the restriction map yields an embedding \( P(T) \hookrightarrow X(S)_{\mathbb{R}} = X(S) \) with finite cokernel; in particular, the rank of \( P(T) \) as an abelian group coincides with the \( \mathbb{R} \)-rank \( \text{rk}_\mathbb{R} T \) of \( T \).

Before formulating our results, we define the following property. Let \( G \subset \text{GL}_N \) be a semi-simple algebraic group defined over a field \( F \) of characteristic zero. We say that a (Zariski-dense) subgroup \( \Gamma \subset G(F) \) has property (A) if for any semi-simple element \( \gamma \in \Gamma \), all the eigenvalues of \( \gamma \) lie in the field of algebraic numbers \( \overline{\mathbb{Q}} \) (note that the latter is equivalent to the fact that for any maximal \( F \)-torus \( T \) of \( G \) containing \( \gamma \) and any character \( \chi \in X(T) \), we have \( \chi(\gamma) \in \overline{\mathbb{Q}}^\times \) – this reformulation shows, in particular, that this property does not depend on the choice of a matrix realization of \( G \)). Of course, this property automatically holds if \( \Gamma \) is arithmetic, or more generally, if \( \Gamma \) can be conjugated into \( \text{SL}_N(K) \) for some number field \( K \).

Let us now consider the rank one case first.

The rank one case. Suppose \( \text{rk}_\mathbb{R} G = 1 \) (the examples include the adjoint groups of \( \text{SO}(n,1) \), \( \text{SU}(n,1) \) and \( \text{Sp}(n,1) \); the corresponding symmetric spaces are respectively the real, complex and quaternionic hyperbolic \( n \)-spaces). Then given a nontrivial semi-simple element \( \gamma \in \Gamma \), for any maximal \( \mathbb{R} \)-torus \( T \) of \( G \) containing \( \gamma \) we have \( \text{rk}_\mathbb{R} T = 1 \), which implies that the group \( P(T) \) of positive characters is cyclic and is generated, say, by \( \chi \). Then it follows from (3) that

\[
(4) \quad \lambda_\Gamma(\gamma) = \frac{\sqrt{m}}{2} \cdot |\log \chi(t)|
\]

where \( m \) is some integer \( \geq 1 \) depending only on \( G \); note that this formula is still in the spirit of (1), but potentially involves some irrationality which can complicate the analysis of length-commensurability.

Now, suppose that \( G_1 \) and \( G_2 \) are two simple algebraic \( \mathbb{R} \)-groups of \( \mathbb{R} \)-rank one. For \( i = 1, 2 \), let \( \Gamma_i \subset G_i(\mathbb{R}) = \mathcal{G}_i \) be a discrete torsion-free subgroup having property (A).
Given a nontrivial semi-simple element \(\gamma_i \in \Gamma_i\), we pick a maximal \(\mathbb{R}\)-torus \(T_i\) of \(G_i\) whose group of \(\mathbb{R}\)-points contains \(\gamma_i\) and let \(\chi_i\) be a generator of the group of positive characters \(P(T_i)\). Then according to \([4]\),

\[
\lambda_{\Gamma_1}(\gamma_1) = \frac{\sqrt{m_1}}{2} \cdot |\log \chi_1(\gamma_1)| \quad \text{and} \quad \lambda_{\Gamma_2}(\gamma_2) = \frac{\sqrt{m_2}}{2} \cdot |\log \chi_2(\gamma_2)|
\]

for some integers \(m_1, m_2 \geq 1\). By a theorem proved independently by Gel'fond and Schneider in 1934 (which settled Hilbert's seventh problem – cf. \([3]\)), the ratio

\[
\frac{\log \chi_1(\gamma_1)}{\log \chi_2(\gamma_2)}
\]

is either rational or transcendental. This result implies that the ratio \(\ell_{\Gamma_1}(\gamma_1)/\ell_{\Gamma_2}(\gamma_2)\), or equivalently, the ratio \(\lambda_{\Gamma_1}(\gamma_1)/\lambda_{\Gamma_2}(\gamma_2)\) can be rational only if

\[
\chi_1(\gamma_1)^{n_1} = \chi_2(\gamma_2)^{n_2}
\]

for some nonzero integers \(n_1, n_2\), which makes the elements \(\gamma_1\) and \(\gamma_2\) weakly commensurable. (Of course, we get this conclusion without using the theorem of Gel'fond-Schneider if \(G_1 = G_2\), hence \(m_1 = m_2\).) This argument shows that the length-commensurability of \(X_{\Gamma_1}\) and \(X_{\Gamma_2}\) implies the weak commensurability of \(\Gamma_1\) and \(\Gamma_2\).

Finally, we recall that if \(G \subset GL_N\) is an absolutely simple real algebraic group not isomorphic to \(PGL_2\) then any lattice \(\Gamma \subset G(\mathbb{R})\) can be conjugated into \(SL_N(K)\) for some number field \(K\) (cf. \([44\ \text{7.67 and 7.68}])\), hence possesses property (A). This implies that if \(X_{\Gamma_1}\) and \(X_{\Gamma_2}\) are rank one locally symmetric spaces of finite volume then their length-commensurability always implies the weak commensurability of \(\Gamma_1\) and \(\Gamma_2\) except possibly in the following situation: \(G_1 = PGL_2\) and \(\Gamma_1\) cannot be conjugated into \(PGL_2(K)\) for any number field \(K \subset \mathbb{R}\) while \(G_2 \neq PGL_2\) (in \([40\ \text{this was called the exceptional case (E)}])\). Nevertheless, the conclusion remains valid also in this case if one assumes the truth of Schanuel's conjecture (see below) – this follows from our recent results \([43\ \text{which we will discuss in \([38\ \text{cf. Theorem \[8.1]}])\].

**The general case.** If \(\text{rk}_{\mathbb{R}} G > 1\) then \(p\) may be \(> 1\) in \([3]\), hence \(\lambda_{\Gamma}(\gamma)\), generally speaking, is not a multiple of the logarithm of the value of a positive character. Consequently, the fact that the ratio \(\lambda_{\Gamma_1}(\gamma_1)/\lambda_{\Gamma_2}(\gamma_2)\) is a rational number does not imply directly that \(\gamma_1\) and \(\gamma_2\) are weakly commensurable. While the implication nevertheless is valid (under some natural technical assumptions), it is hardly surprising now that the proof requires some nontrivial information about the logarithms of the character values. More precisely, our arguments in \([40\ \text{and 43]}\) assume the truth of the following famous conjecture in transcendental number theory (cf. \([2]\)).

**2.4. Schanuel's conjecture.** If \(z_1, \ldots, z_n \in \mathbb{C}\) are linearly independent over \(\mathbb{Q}\), then the transcendence degree (over \(\mathbb{Q}\)) of the field generated by

\[
z_1, \ldots, z_n; \ e^{z_1}, \ldots, e^{z_n}
\]

is \(\geq n\). In fact, we will only need the consequence of this conjecture that for nonzero algebraic numbers \(z_1, \ldots, z_n\), (any values of) their logarithms \(\log z_1, \ldots, \log z_n\) are algebraically
independent once they are linearly independent (over \( \mathbb{Q} \)). In order to apply this statement in our situation, we first prove the following elementary lemma.

**Lemma 2.5.** Let \( G_1 \) and \( G_2 \) be two connected semi-simple real algebraic groups. For \( i = 1, 2 \), let \( T_i \) be a maximal \( \mathbb{R} \)-torus of \( G_i \), \( \gamma_i \in T_i(\mathbb{R}) \) and let \( \chi_{d_i}^{(1)}, \ldots, \chi_{d_i}^{(r)} \) be positive characters of \( T_i \) such that the set

\[
S_{i} = \{ \log \chi_{d_i}^{(1)}(\gamma_i), \ldots, \log \chi_{d_i}^{(r)}(\gamma_i) \} \subset \mathbb{R}
\]

is linearly independent over \( \mathbb{Q} \). If \( \gamma_1 \) and \( \gamma_2 \) are not weakly commensurable then the set \( S_1 \cup S_2 \) is also linearly independent.

**Proof.** Assume the contrary. Then there exist integers \( s_1, \ldots, s_{d_1}, t_1, \ldots, t_{d_2} \), not all zero, such that

\[
s_1 \log \chi_1^{(1)}(\gamma_1) + \cdots + s_{d_1} \log \chi_{d_1}^{(1)}(\gamma_1) - t_1 \log \chi_1^{(2)}(\gamma_2) - \cdots - t_{d_2} \log \chi_{d_2}^{(2)}(\gamma_2) = 0.
\]

Consider the following characters

\[
\psi^{(1)} := s_1 \chi_1^{(1)} + \cdots + s_{d_1} \chi_{d_1}^{(1)} \quad \text{and} \quad \psi^{(2)} := t_1 \chi_1^{(2)} + \cdots + t_{d_2} \chi_{d_2}^{(2)}
\]

of \( T_1 \) and \( T_2 \) respectively. Then \( \psi^{(1)}(\gamma_1) = \psi^{(2)}(\gamma_2) \), and hence

\[
\psi^{(1)}(\gamma_1) = 1 = \psi^{(2)}(\gamma_2)
\]

because \( \gamma_1 \) and \( \gamma_2 \) are not weakly commensurable. This means that

\[
s_1 \log \chi_1^{(1)}(\gamma_1) + \cdots + s_{d_1} \log \chi_{d_1}^{(1)}(\gamma_1) = t_1 \log \chi_1^{(2)}(\gamma_2) + \cdots + t_{d_2} \log \chi_{d_2}^{(2)}(\gamma_2),
\]

and therefore all the coefficients are zero because the sets \( S_1 \) and \( S_2 \) are linearly independent. A contradiction. \( \square \)

We are now ready to connect length-commensurability with weak commensurability.

**Proposition 2.6.** Let \( G_1 \) and \( G_2 \) be two connected semi-simple real algebraic groups. For \( i = 1, 2 \), let \( \Gamma_i \subset G_i(\mathbb{R}) \) be a subgroup satisfying property (A). Assume that Schanuel’s conjecture holds. If semi-simple elements \( \gamma_1 \in \Gamma_1 \) and \( \gamma_2 \in \Gamma_2 \) are not weakly commensurable then \( \lambda_{\Gamma_1}(\gamma_1) \) and \( \lambda_{\Gamma_2}(\gamma_2) \) are algebraically independent over \( \mathbb{Q} \).

**Proof.** It follows from (3) that

\[
\lambda_{\Gamma_1}(\gamma_1)^2 = \sum_{i=1}^{p} s_i (\log \chi_i^{(1)}(\gamma_1))^2 \quad \text{and} \quad \lambda_{\Gamma_2}(\gamma_2)^2 = \sum_{i=1}^{q} t_i (\log \chi_i^{(2)}(\gamma_2))^2,
\]

where \( s_i \) and \( t_i \) are positive rational numbers, and \( \chi_i^{(1)} \) and \( \chi_i^{(2)} \) are positive characters on maximal \( \mathbb{R} \)-tori \( T_1 \) and \( T_2 \) of \( G_1 \) and \( G_2 \) whose groups of \( \mathbb{R} \)-points contain the elements \( \gamma_1 \) and \( \gamma_2 \), respectively. After renumbering the characters, we can assume that

\[
a_1 := \log \chi_1^{(1)}(\gamma_1), \ldots, a_m := \log \chi_m^{(1)}(\gamma_1)
\]

(respectively, \( b_1 := \log \chi_1^{(2)}(\gamma_2), \ldots, b_n := \log \chi_n^{(2)}(\gamma_2) \)) form a basis of the \( \mathbb{Q} \)-subspace of \( \mathbb{R} \) spanned by \( \log \chi_i^{(1)}(\gamma_1) \) for \( i \leq p \) (respectively, by \( \log \chi_i^{(2)}(\gamma_2) \) for \( i \leq q \)). It follows from Lemma 2.5 that the numbers

\[
a_1, \ldots, a_m; b_1, \ldots, b_n
\]
are linearly independent. By our assumption, $\Gamma_1$ and $\Gamma_2$ possess property (A), so the character values $\chi_i^{(j)}(\gamma_j)$ are all algebraic numbers. So, it follows from Schanuel’s conjecture that the numbers in (6) are algebraically independent over $\mathbb{Q}$. As is seen from (5), $\lambda \Gamma_1(\gamma_1)$ and $\lambda \Gamma_2(\gamma_2)$ are represented by nonzero homogeneous polynomials of degree two, with rational coefficients, in $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$, respectively, and therefore they are algebraically independent. □

This proposition leads us to the following.

**Theorem 2.7.** Let $G_1$ and $G_2$ be two connected semi-simple real algebraic groups. For $i = 1, 2$, let $\Gamma_i \subset G_i(\mathbb{R})$ be a discrete torsion-free subgroup having property (A). Assume that Schanuel’s conjecture holds. If $\Gamma_1$ and $\Gamma_2$ are not weakly commensurable, then, possibly after reindexing, we can find $\lambda_1 \in L(X_{\Gamma_1})$ which is algebraically independent from any $\lambda_2 \in L(X_{\Gamma_2})$. In particular, $X_{\Gamma_1}$ and $X_{\Gamma_2}$ are not length-commensurable.

Combining this with the discussion above of property (A) for lattices and of the exceptional case $(E)$, we obtain the following.

**Corollary 2.8.** Let $G_1$ and $G_2$ be two absolutely simple real algebraic groups, and for $i = 1, 2$ let $\Gamma_i$ be a lattice in $G_i(\mathbb{R})$ (so that the locally symmetric space $X_{\Gamma_i}$ has finite volume). If $X_{\Gamma_1}$ and $X_{\Gamma_2}$ are length-commensurable then $\Gamma_1$ and $\Gamma_2$ are weakly commensurable.

The results we discussed in this section shift the focus in the analysis of length-commensurability and/or isospectrality of locally symmetric spaces to that of weak commensurability of finitely generated Zariski-dense subgroups of simple (or semi-simple) algebraic groups. In §3 we will first present some basic results dealing with the weak commensurability of such subgroups in a completely general situation, one of which states that the mere existence of such subgroups implies that the ambient algebraic groups either are of the same type, or one of them is of type $B_n$ and the other of type $C_n$ for some $n \geq 3$ (cf. Theorem 3.1). We then turn to much more precise results in the case where the algebraic groups are of the same type and the subgroups are $S$-arithmetic (see §4), and finally derive some geometric consequences of these results (see §5). Next, §7 contains an exposition of the recent results of Skip Garibaldi and the second-named author [18] that completely characterize weakly commensurable $S$-arithmetic subgroups in the case where one of the two groups is of type $B_n$ and the other is of type $C_n$ ($n \geq 3$). In §8 we discuss a more technical version of the notion of weak commensurability, which enabled us to show in [43] (under mild technical assumptions) that if two arithmetically defined locally symmetric spaces $M_1 = X_{\Gamma_1}$ and $M_2 = X_{\Gamma_2}$ are not length-commensurable then the sets $L(M_1)$ and $L(M_2)$ (or $\mathbb{Q} \cdot L(M_1)$ and $\mathbb{Q} \cdot L(M_2)$) are very different. The proofs of all these results use the existence (first established in [39]) of special elements, which we call generic elements, in arbitrary finitely generated Zariski-dense subgroups; we briefly review these and more recent results in this direction in §9 along with the results that relate the analysis of weak commensurability with a problem of independent interest in the theory of semi-simple algebraic groups of characterizing simple $K$-groups having the same isomorphism classes of maximal $K$-tori (cf. §10). Finally, in §11 we discuss some open problems.
3. Two basic results implied by weak commensurability and the definition of arithmeticity

Our next goal is to give an account of the results from [40] concerning weakly commensurable subgroups of semi-simple algebraic groups. We begin with the following two theorems that provide the basic results about weak commensurability of arbitrary finitely generated Zariski-dense subgroups of semi-simple groups.

**Theorem 3.1.** Let $G_1$ and $G_2$ be two connected absolutely almost simple algebraic groups defined over a field $F$ of characteristic zero. Assume that there exist finitely generated Zariski-dense subgroups $\Gamma_i$ of $G_i(F)$ which are weakly commensurable. Then either $G_1$ and $G_2$ are of the same Killing-Cartan type, or one of them is of type $B_n$ and the other is of type $C_n$ for some $n \geq 3$.

The way we prove this theorem is by showing that the Weyl groups of $G_1$ and $G_2$ have the same order, as it is well-known that the order of the Weyl group uniquely determines the type of the root system, except for the ambiguity between $B_n$ and $C_n$. On the other hand, groups $G_1$ and $G_2$ of types $B_n$ and $C_n$ with $n > 2$ respectively, may indeed contain weakly commensurable subgroups. This was first shown in [40, Example 6.7] using a cohomological construction which we will briefly recall in §7. Recently in [18] another explanation was given using commutative étale subalgebras of simple algebras with involution. We refer the reader to §7 for this argument as well as a complete characterization of weakly commensurable $S$-arithmetic subgroups in the algebraic groups of types $B_n$ and $C_n$ (see Theorem 7.2).

**Theorem 3.2.** Let $G_1$ and $G_2$ be two connected absolutely almost simple algebraic groups defined over a field $F$ of characteristic zero. For $i = 1, 2$, let $\Gamma_i$ be a finitely generated Zariski-dense subgroup of $G_i(F)$, and $K_{\Gamma_i}$ be the subfield of $F$ generated by the traces $\text{Tr} \text{Ad} \gamma$, in the adjoint representation, of $\gamma \in \Gamma_i$. If $\Gamma_1$ and $\Gamma_2$ are weakly commensurable, then $K_{\Gamma_1} = K_{\Gamma_2}$.

We now turn to the results concerning weakly commensurable Zariski-dense $S$-arithmetic subgroups, which are surprisingly strong. In §4 we will discuss the weak commensurability of $S$-arithmetic subgroups in absolutely almost simple algebraic groups $G_1$ and $G_2$ of the same type, postponing the case where one of the groups is of type $B_n$ and the other of type $C_n$ to §7. Since our results rely on a specific way of describing $S$-arithmetic subgroups in absolutely almost simple groups, we will discuss this issue first.

3.3. The definition of arithmeticity: Let $G$ be an algebraic group defined over a number field $K$, and let $S$ be a finite subset of the set $V^K$ of all places of $K$ containing the set $V^K_\infty$ of archimedean places. Fix a $K$-embedding $G \subset \text{GL}_N$, and consider the group of $S$-integral points

$$G(\mathcal{O}_K(S)) := G \cap \text{GL}_N(\mathcal{O}_K(S)).$$
Then, for any field extension $F/K$, the subgroups of $G(F)$ that are commensurable with $G(\mathcal{O}_K(S))$ are called $S$-arithmetic, and in the case where $S = V^K_\infty$ simply arithmetic (note that $\mathcal{O}_K(V^K_\infty) = \mathcal{O}_K$, the ring of algebraic integers in $K$). It is well-known that the resulting class of $S$-arithmetic subgroups does not depend on the choice of $K$-embedding $G \subset \text{GL}_N$ (cf. [36]). The question, however, is what we should mean by an arithmetic subgroup of $G(F)$ when $G$ is an algebraic group defined over a field $F$ of characteristic zero that is not equipped with a structure of $K$-group over some number field $K \subset F$. For example, what is an arithmetic subgroup of $G(\mathbb{R})$ where $G = \text{SO}_3(f)$ and $f = x^2 + ey^2 - \pi z^2$? For absolutely almost simple groups the “right” concept that we will formalize below is given in terms of the forms of $G$ over the subfields $K \subset F$ that are number fields. In our example, we can consider the following rational quadratic forms that are equivalent to $f$ over $\mathbb{R}$:

$$f_1 = x^2 + y^2 - 3z^2 \quad \text{and} \quad f_2 = x^2 + 2y^2 - 7z^2,$$

and set $G_i = \text{SO}_3(f_i)$. Then for each $i = 1, 2$, we have an $\mathbb{R}$-isomorphism $G_i \simeq G$, so the natural arithmetic subgroup $G_i(\mathbb{Z}) \subset G_i(\mathbb{R})$ can be thought of as an “arithmetic” subgroup of $G(\mathbb{R})$. Furthermore, one can consider quadratic forms over other number subfields $K \subset \mathbb{R}$ that again become equivalent to $f$ over $\mathbb{R}$; for example,

$$K = \mathbb{Q}(\sqrt{2}) \quad \text{and} \quad f_3 = x^2 + y^2 - \sqrt{2}z^2.$$

Then for $G_3 = \text{SO}_3(f_3)$, there is an $\mathbb{R}$-isomorphism $G_3 \simeq G$ which allows us to view the natural arithmetic subgroup $G_3(\mathbb{Z}) \subset G_3(\mathbb{R})$, where $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$, as an “arithmetic” subgroup of $G(\mathbb{R})$. One can easily generalize such constructions from arithmetic to $S$-arithmetic groups by replacing the rings of integers with the rings of $S$-integers. So, generally speaking, by an $S$-arithmetic subgroup of $G(\mathbb{R})$ we mean a subgroup which is commensurable to one of the subgroups obtained through this construction for some choice of a number subfield $K \subset \mathbb{R}$, a finite set $S$ of places of $K$ containing all the archimedean ones, and a quadratic form $\bar{f}$ over $K$ that becomes equivalent to $f$ over $\mathbb{R}$. The technical definition is as follows.

Let $G$ be a connected absolutely almost simple algebraic group defined over a field $F$ of characteristic zero, $\overline{G}$ be its adjoint group, and $\pi: G \to \overline{G}$ be the natural isogeny. Suppose we are given the following data:

- a number field $K$ with a fixed embedding $K \hookrightarrow F$;
- an $F/K$-form $\mathcal{G}$ of $\overline{G}$, which is an algebraic $K$-group such that there exists an $F$-isomorphism $\psi_\mathcal{G} \simeq \overline{G}$, where $\psi_\mathcal{G}$ is the group obtained from $\mathcal{G}$ by the extension of scalars from $K$ to $F$;
- a finite set $S$ of places of $K$ containing $V^K_\infty$ but not containing any nonarchimedean places $v$ such that $\mathcal{G}$ is $K_v$-anisotropic.

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1. We recall that two subgroups $\mathcal{H}_1$ and $\mathcal{H}_2$ of an abstract group $\mathcal{G}$ are called commensurable if their intersection $\mathcal{H}_1 \cap \mathcal{H}_2$ is of finite index in each of the subgroups.

2. We note that if $\mathcal{G}$ is $K_v$-anisotropic then $\psi(\mathcal{O}_K(S))$ and $\psi(\mathcal{O}_K(S \cup \{v\}))$ are commensurable, and therefore the classes of $S$- and $(S \cup \{v\})$-arithmetic subgroups coincide. Thus, this assumption on $S$ is necessary if we want to recover it from a given $S$-arithmetic subgroup.
We then have an embedding \( \iota : \mathcal{G}(K) \rightarrow \mathcal{G}(F) \) which is well-defined up to an \( F \)-automorphism of \( \mathcal{G} \) (note that we do not fix an isomorphism \( F\mathcal{G} \cong \mathcal{G} \)). A subgroup \( \Gamma \) of \( G(F) \) such that \( \pi(\Gamma) \) is commensurable with \( \sigma(\iota(\mathcal{O}_K(S))) \), for some \( F \)-automorphism \( \sigma \) of \( \mathcal{G} \), will be called a \((\mathcal{G}, K, S)\)-arithmetic subgroup or an \( S \)-arithmetic subgroup described in terms of the triple \((\mathcal{G}, K, S)\). As usual, \((\mathcal{G}, K, V^\infty_K)\)-arithmetic subgroups will simply be called \((\mathcal{G}, K)\)-arithmetic.

We also need to introduce a more general notion of commensurability. The point is that since weak commensurability is defined in terms of eigenvalues, a subgroup \( \Gamma \subset G(F) \) is weakly commensurable with any conjugate subgroup, while the latter may not be commensurable with the former in the usual sense. So, to make theorems asserting that in certain situations “weak commensurability implies commensurability” possible (and such theorems are in fact one of the goals of our analysis) one definitely needs to modify the notion of commensurability. The following notion works well in geometric applications. Let \( G_i \), for \( i = 1, 2 \), be a connected absolutely almost simple \( F \)-group, and let \( \pi_i : G_i \rightarrow G_i \) be the isogeny onto the corresponding adjoint group. We will say that the subgroups \( \Gamma_i \) of \( G_i(F) \) are commensurable up to an \( F \)-isomorphism between \( G_1 \) and \( G_2 \) if there exists an \( F \)-isomorphism \( \sigma : G_1 \rightarrow G_2 \) such that \( \sigma(\pi_1(\Gamma_1)) \) is commensurable with \( \pi(\Gamma_2) \) in the usual sense. The key observation is that the description of \( S \)-arithmetic subgroups in terms of triples \((\mathcal{G}, K, S)\) is very convenient for determining when two such subgroups are commensurable in the new generalized sense.

\[\text{Proposition 3.4.} \] Let \( G_1 \) and \( G_2 \) be connected absolutely almost simple algebraic groups defined over a field \( F \) of characteristic zero, and for \( i = 1, 2 \), let \( \Gamma_i \) be a Zariski-dense \((\mathcal{G}_i, K_i, S_i)\)-arithmetic subgroup of \( G_i(F) \). Then \( \Gamma_1 \) and \( \Gamma_2 \) are commensurable up to an \( F \)-isomorphism between \( G_1 \) and \( G_2 \) if and only if \( K_1 = K_2 =: K \), \( S_1 = S_2 \), and \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are \( K \)-isomorphic.

It follows from the above proposition that the arithmetic subgroups \( \Gamma_1 \), \( \Gamma_2 \), and \( \Gamma_3 \) constructed above, of \( G(\mathbb{R}) \), where \( G = \text{SO}_3(f) \), are pairwise noncommensurable: indeed, \( \Gamma_3 \), being defined over \( \mathbb{Q}(\sqrt{2}) \), cannot possibly be commensurable to \( \Gamma_1 \) or \( \Gamma_2 \) as these two groups are defined over \( \mathbb{Q} \); at the same time, the non-commensurability of \( \Gamma_1 \) and \( \Gamma_2 \) is a consequence of the fact that \( \text{SO}_3(f_1) \) and \( \text{SO}_3(f_2) \) are not \( \mathbb{Q} \)-isomorphic since the quadratic forms \( f_1 \) and \( f_2 \) are not equivalent over \( \mathbb{Q} \).

4. Results on weakly commensurable \( S \)-arithmetic subgroups

In view of Proposition 3.4, the central question in the analysis of weak commensurability of \( S \)-arithmetic subgroups is the following: Suppose we are given two Zariski-dense \( S \)-arithmetic subgroups that are described in terms of triples. Which components of these triples coincide given the fact that the subgroups are weakly commensurable? As the following result demonstrates, two of these components must coincide.

\[\text{This notion of arithmetic subgroups coincides with that in Margulis' book} \] for absolutely simple adjoint groups.
Theorem 4.1. Let $G_1$ and $G_2$ be two connected absolutely almost simple algebraic groups defined over a field $F$ of characteristic zero. If Zariski-dense $(G_i, K_i, S_i)$-arithmetic subgroups $\Gamma_i$ of $G_i(F)$, where $i = 1, 2$, are weakly commensurable for $i = 1, 2$, then $K_1 = K_2$ and $S_1 = S_2$.

In general, the forms $G_1$ and $G_2$ do not have to be $K$-isomorphic (see [40], Examples 6.5 and 6.6 as well as the general construction in §9). In the next theorem we list the cases where it can nevertheless be asserted that $G_1$ and $G_2$ are necessarily $K$-isomorphic, and then give a general finiteness result for the number of $K$-isomorphism classes.

Theorem 4.2. Let $G_1$ and $G_2$ be two connected absolutely almost simple algebraic groups defined over a field $F$ of characteristic zero, of the same type different from $A_n$, $D_{2n+1}$, with $n > 1$, or $E_6$. If for $i = 1, 2$, $G_i(F)$ contain Zariski-dense weakly commensurable $(G_i, K, S)$-arithmetic subgroups $\Gamma_i$, then $G_1 \simeq G_2$ over $K$, and hence $\Gamma_1$ and $\Gamma_2$ are commensurable up to an $F$-isomorphism between $G_1$ and $G_2$.

In this theorem, type $D_{2n}$ ($n \geq 2$) required special consideration. The case $n > 2$ was settled in [41] using the techniques of [40] in conjunction with results on embeddings of fields with involutive automorphisms into simple algebras with involution. The remaining case of type $D_4$ was treated by Skip Garibaldi [17], whose argument actually applies to all $n$ and explains the result from the perspective of Galois cohomology, providing thereby a cohomological insight into the difference between the types $D_{2n}$ and $D_{2n+1}$. We note that the types excluded in the theorem are precisely the types for which the automorphism $\alpha \mapsto -\alpha$ of the corresponding root system is not in the Weyl group. More importantly, all these types are honest exceptions to the theorem – a general Galois-cohomological construction of weakly commensurable, but not commensurable, Zariski-dense $S$-arithmetic subgroups for all of these types is given in [40, §9].

Theorem 4.3. Let $G_1$ and $G_2$ be two connected absolutely almost simple groups defined over a field $F$ of characteristic zero. Let $\Gamma_1$ be a Zariski-dense $(G_1, K, S)$-arithmetic subgroup of $G_1(F)$. Then the set of $K$-isomorphism classes of $K$-forms $\mathfrak{G}_2$ of $G_2$ such that $G_2(F)$ contains a Zariski-dense $(\mathfrak{G}_2, K, S)$-arithmetic subgroup weakly commensurable to $\Gamma_1$ is finite.

In other words, the set of all Zariski-dense $(K, S)$-arithmetic subgroups of $G_2(F)$ which are weakly commensurable to a given Zariski-dense $(K, S)$-arithmetic subgroup is a union of finitely many commensurability classes.

A noteworthy fact about weak commensurability is that it has the following implication for the existence of unipotent elements in arithmetic subgroups (even though it is formulated entirely in terms of semi-simple ones). We recall that a semi-simple $K$-group is called $K$-isotropic if $\text{rk}_K G > 0$; in characteristic zero, this is equivalent to the existence of nontrivial unipotent elements in $G(K)$. Moreover, if $K$ is a number field then $G$ is $K$-isotropic if and only if every $S$-arithmetic subgroup contains unipotent elements, for any $S$. 
Theorem 4.4. Let $G_1$ and $G_2$ be two connected absolutely almost simple algebraic groups defined over a field $F$ of characteristic zero. For $i = 1, 2$, let $\Gamma_i$ be a Zariski-dense $(\mathcal{H}_i, K, S)$-arithmetic subgroup of $G_i(F)$. If $\Gamma_1$ and $\Gamma_2$ are weakly commensurable then $\text{rk}_K \mathcal{H}_1 = \text{rk}_K \mathcal{H}_2$; in particular, if $\mathcal{G}_1$ is $K$-isotropic, then so is $\mathcal{G}_2$.

We note that in [40, §7] we prove a somewhat more precise result, viz. that if $G_1$ and $G_2$ are of the same type, then the Tits indices of $\mathcal{G}_1/K$ and $\mathcal{G}_2/K$ are isomorphic, but we will not get into these technical details here.

The following result asserts that a lattice\footnote{A discrete subgroup $\Gamma$ of a locally compact topological group $\mathcal{G}$ is said to be a lattice in $\mathcal{G}$ if $\mathcal{G}/\Gamma$ carries a finite $\mathcal{G}$-invariant Borel measure.} which is weakly commensurable with an $S$-arithmetic group is itself $S$-arithmetic.

Theorem 4.5. Let $G_1$ and $G_2$ be two connected absolutely almost simple algebraic groups defined over a nondiscrete locally compact field $F$ of characteristic zero, and for $i = 1, 2$, let $\Gamma_i$ be a Zariski-dense lattice in $G_i(F)$. Assume that $\Gamma_1$ is a $(K, S)$-arithmetic subgroup of $G_1(F)$. If $\Gamma_1$ and $\Gamma_2$ are weakly commensurable, then $\Gamma_2$ is a $(K, S)$-arithmetic subgroup of $G_2(F)$.

5. Geometric applications

We are now in a position to give the precise statements of our results on isospectral and length-commensurable locally symmetric spaces. Throughout this subsection, for $i = 1, 2$, $G_i$ will denote an absolutely simple real algebraic group and $X_i$ the symmetric space of $G_i = G_i(\mathbb{R})$. Furthermore, given a discrete torsion-free subgroup $\Gamma_i \subset G_i$, we let $X_{\Gamma_i} = X_i/\Gamma_i$ denote the corresponding locally symmetric space. The geometric results are basically obtained by combining Theorem 2.7 and Corollary 2.8 with the results on weakly commensurable subgroups from the previous section. It should be emphasized that when $X_{\Gamma_1}$ and $X_{\Gamma_2}$ are both rank one spaces and we are not in the exceptional case (E) (which is the case, for example, for all hyperbolic $n$-manifolds with $n \geq 4$) our results are unconditional, while in all other cases they depend on the validity of Schanuel’s conjecture.

Now, applying Theorems 3.1 and 3.2 we obtain the following.

Theorem 5.1. Let $G_1$ and $G_2$ be connected absolutely simple real algebraic groups, and let $X_{\Gamma_i}$ be a locally symmetric space of finite volume, of $G_i$, for $i = 1, 2$. If $X_{\Gamma_1}$ and $X_{\Gamma_2}$ are length-commensurable, then (i) either $G_1$ and $G_2$ are of same Killing-Cartan type, or one of them is of type $B_n$ and the other is of type $C_n$ for some $n \geq 3$, (ii) $K_{\Gamma_1} = K_{\Gamma_2}$.

It should be pointed out that assuming Schanuel’s conjecture in all cases, one can prove this theorem (in fact, a much stronger statement – see Theorem 8.1) assuming only that $\Gamma_1$ and $\Gamma_2$ are finitely generated and Zariski-dense.

Next, using Theorems 4.2 and 4.3 we obtain
Theorem 5.2. Let $G_1$ and $G_2$ be connected absolutely simple real algebraic groups, and let $G_i = G_i(\mathbb{R})$, for $i = 1, 2$. Then the set of arithmetically defined locally symmetric spaces $X_{\Gamma_2}$ of $G_2$, which are length-commensurable to a given arithmetically defined locally symmetric space $X_{\Gamma_1}$ of $G_1$, is a union of finitely many commensurability classes. It in fact consists of a single commensurability class if $G_1$ and $G_2$ have the same type different from $A_n$, $D_{2n+1}^+$ with $n > 1$, or $E_6$.

Furthermore, Theorems 4.4 and 4.5 imply the following rather surprising result which has so far defied all attempts of a purely geometric proof.

Theorem 5.3. Let $G_1$ and $G_2$ be connected absolutely simple real algebraic groups, and let $X_{\Gamma_1}$ and $X_{\Gamma_2}$ be length-commensurable locally symmetric spaces of $G_1$ and $G_2$ respectively, of finite volume. Assume that at least one of the spaces is arithmetically defined. Then the other space is also arithmetically defined, and the compactness of one of the spaces implies the compactness of the other.

In fact, if one of the spaces is compact and the other is not, the weak length spectra $L(X_{\Gamma_1})$ and $L(X_{\Gamma_2})$ are quite different – see Theorem 8.6 for a precise statement (we note that the proof of this result uses Schanuel’s conjecture in all cases).

Finally, we will describe some applications to isospectral compact locally symmetric spaces. So, in the remainder of this section, the locally symmetric spaces $X_{\Gamma_1}$ and $X_{\Gamma_2}$ as above will be assumed to be compact. Then, as we discussed in §1 the fact that $X_{\Gamma_1}$ and $X_{\Gamma_2}$ are isospectral implies that $L(X_{\Gamma_1}) = L(X_{\Gamma_2})$, so we can use our results on length-commensurable spaces. Thus, in particular we obtain the following.

Theorem 5.4. If $X_{\Gamma_1}$ and $X_{\Gamma_2}$ are isospectral, and $\Gamma_1$ is arithmetic, then so is $\Gamma_2$.

Thus, the Laplace spectrum can see if the fundamental group is arithmetic or not – to our knowledge, no results of this kind, particularly for general locally symmetric spaces, were previously known in spectral theory.

The following theorem settles the question “Can one hear the shape of a drum?” for arithmeticly defined compact locally symmetric spaces.

Theorem 5.5. Let $X_{\Gamma_1}$ and $X_{\Gamma_2}$ be compact locally symmetric spaces associated with absolutely simple real algebraic groups $G_1$ and $G_2$, and assume that at least one of the spaces is arithmetically defined. If $X_{\Gamma_1}$ and $X_{\Gamma_2}$ are isospectral then $G_1 = G_2 := G$. Moreover, unless $G$ is of type $A_n$, $D_{2n+1}$ ($n > 1$), or $E_6$, the spaces $X_{\Gamma_1}$ and $X_{\Gamma_2}$ are commensurable.

It should be noted that our methods based on length-commensurability or weak commensurability leave room for the following ambiguity in the proof of Theorem 5.5: either $G_1 = G_2$ or $G_1$ and $G_2$ are $\mathbb{R}$-split forms of types $B_n$ and $C_n$ for some $n \geq 3$ - and this ambiguity is unavoidable, cf. the end of §7. The fact that in the latter case the locally symmetric spaces cannot be isospectral was shown by Sai-Kee Yeung [51] by comparing the traces of the heat operator (without using Schanuel’s conjecture), which leads to the statement of the theorem given above.
6. Absolutely almost simple algebraic groups having the same maximal tori

The analysis of weak commensurability is related to another natural problem in the theory of algebraic groups of characterizing absolutely almost simple \( K \)-groups having the same isomorphism/isogeny classes of maximal \( K \)-tori – the exact nature of this connection will be clarified in Theorem 9.8 and the subsequent discussion. Some aspects of this problem over local and global fields were considered in [16] and [26]. Another direction of research, which has already generated a number of results (cf. [4], [17], [31], [41]) is the investigation of local-global principles for embedding tori into absolutely almost simple algebraic groups as maximal tori (in particular, for embedding of commutative étale algebras with involutive automorphisms into simple algebras with involution); some number-theoretic applications of these results can be found, for example, in [14]. A detailed discussion of these issues would be an independent undertaking, so we will limit ourselves here to the following theorem (cf. [40, Theorem 7.5] and [18, Proposition 1.3]).

**Theorem 6.1.** (1) Let \( G_1 \) and \( G_2 \) be connected absolutely almost simple algebraic groups defined over a number field \( K \), and let \( L_i \) be the smallest Galois extension of \( K \) over which \( G_i \) becomes an inner form of a split group. If \( G_1 \) and \( G_2 \) have the same \( K \)-isogeny classes of maximal \( K \)-tori then either \( G_1 \) and \( G_2 \) are of the same Killing-Cartan type, or one of them is of type \( B_n \) and the other is of type \( C_n \), and moreover, \( L_1 = L_2 \).

(2) Fix an absolutely almost simple \( K \)-group \( G \). Then the set of isomorphism classes of all absolutely almost simple \( K \)-groups \( G' \) having the same \( K \)-isogeny classes of maximal \( K \)-tori is finite.

(3) Fix an absolutely almost simple simply connected \( K \)-group \( G \) whose Killing-Cartan type is different from \( A_n, D_{2n+1} \) \((n > 1)\) or \( E_6 \). Then any \( K \)-form \( G' \) of \( G \) (in other words, any absolutely almost simple simply connected \( K \)-group \( G' \) of the same type as \( G \) that has the same \( K \)-isogeny classes of maximal \( K \)-tori as \( G \)), is isomorphic to \( G \).

The construction described in [40, §9] shows that the types excluded in (3) are honest exceptions, i.e., for each of those types one can construct non-isomorphic absolutely almost simple simply connected \( K \)-groups \( G_1 \) and \( G_2 \) of this type over a number field \( K \) that have the same isomorphism classes of maximal \( K \)-tori. On the other hand, the analysis of the situation where \( G_1 \) and \( G_2 \) are of types \( B_n \) and \( C_n \), respectively, over a number field \( K \) and have the same isomorphism/isogeny classes of maximal \( K \)-tori is given in Theorem 7.3 below (cf. [18, Theorem 1.4 and 1.5]).

Of course, the question about determining absolutely almost simple algebraic \( K \)-groups by their maximal \( K \)-tori makes sense over general fields. It is particularly interesting for division algebra where it can be reformulated as the following question which is somewhat reminiscent of Amitsur’s famous theorem on generic splitting fields (cf. [1], [21]): What can one say about two finite-dimensional central division algebras \( D_1 \) and \( D_2 \) over the same field \( K \) given the fact that they have the same isomorphism classes of maximal subfields? For recent results on this problem see [11], [19], [29], [47].
7. Weakly commensurable subgroups in groups of types B and C

Let $G_1$ and $G_2$ be absolutely almost simple algebraic groups over a field $K$ of characteristic zero. According to Theorem 3.1 finitely generated weakly commensurable Zariski-dense subgroups $\Gamma_1 \subset G_1(K)$ and $\Gamma_2 \subset G_2(K)$ can exist only if $G_1$ and $G_2$ are of the same Killing-Cartan type or one of them is of type $B_n$ and the other is of type $C_n$ for some $n \geq 3$. Moreover, the results we described in [13] provide virtually complete answers to the key questions about weakly commensurable $S$-arithmetic subgroups in the case where $G_1$ and $G_2$ are of the same type. In this section, we will discuss recent results [18] that determine weakly commensurable arithmetic subgroups when $G_1$ is of type $B_n$ and $G_2$ is of type $C_n$ ($n \geq 3$).

First of all, it should be pointed out that $S$-arithmetic subgroups in groups of types $B_n$ and $C_n$ can indeed be weakly commensurable. The underlying reason is that if $G_1$ is a split adjoint group of type $B_n$ and $G_2$ is a split simply connected group of type $C_n$ ($n \geq 2$) over any field $K$ of characteristic $\neq 2$, then $G_1$ and $G_2$ have the same isomorphism classes of maximal $K$-tori. For the reader’s convenience we briefly recall the Galois-cohomological proof of this fact given in [40, Example 6.7].

It is well-known that for any semi-simple $K$-group $G$ there is a natural bijection between the set of $G(K)$-conjugacy classes of maximal $K$-tori of $G$ and the set

$$\mathcal{C}_K := \text{Ker} \left( H^1(K, N) \to H^1(K, G) \right),$$

where $T$ is a maximal $K$-torus of $G$ and $N$ is the normalizer of $T$ in $G$ (cf. [40, Lemma 9.1]). Let $W = N/T$ be the corresponding Weyl group and introduce the following natural maps in Galois cohomology:

$$\varphi : H^1(K, N) \to H^1(K, W) \quad \text{and} \quad \psi : H^1(K, W) \to H^1(K, \text{Aut } T).$$

To apply these considerations to the groups $G_1$ and $G_2$, we will denote by $T_i$ a fixed maximal $K$-split torus of $G_i$ and let $N_i$, $W_i$, $\mathcal{C}_K^{(i)}$, $\theta_K^{(i)}$ and $\nu_K^{(i)}$ be the corresponding objects attached to $G_i$. It follows from an explicit description of the root systems of types $B_n$ and $C_n$ that there exist $K$-isomorphisms $\varphi : T_1 \to T_2$ and $\psi : W_1 \to W_2$ such that for the natural action of $W_i$ on $T_i$ we have

$$\varphi(w \cdot t) = \psi(w) \cdot \varphi(t) \quad \text{for all} \quad t \in T_i, \ w \in W_i.$$

Since $G_i$ is $K$-split, we have $\theta_K^{(i)}(\varphi^{(i)}_K) = H^1(K, W_i)$ (cf. [20, 28, 15]). So, $\psi$ induces a natural bijection between $\theta_K^{(1)}(\varphi^{(1)}_K)$ and $\theta_K^{(2)}(\varphi^{(2)}_K)$. Finally, we observe that if $S_i$ is a maximal $K$-torus of $G_i$ in the $G_i(K)$-conjugacy class corresponding to $c_i \in \mathcal{C}_K^{(i)}$, then the $K$-isomorphism class of $S_i$ is determined by $\nu_K^{(i)}(\theta_K^{(i)}(c_i))$, and if $\psi(\theta_K^{(1)}(c_1)) = \theta_K^{(2)}(c_2)$ then $S_1$ and $S_2$ are $K$-isomorphic. It follows that $G_1$ and $G_2$ have the same classes of maximal $K$-tori, as required.

Subsequently, in [18] a more explicit explanation of this fact was given. More precisely, let $A$ be a central simple algebra over $K$ with a $K$-linear involution $\tau$ (involution of the first kind). We recall that $\tau$ is called orthogonal if $\dim_A A^\tau = n(n + 1)/2$ and symplectic if $\dim_A A^\tau = n(n - 1)/2$. Furthermore, if $\tau$ is orthogonal and $n = 2m + 1$
(m \geq 2) then A = M_n(K) and the corresponding algebraic group G = SU(A, \tau) coincides with the orthogonal group SO_n(q) of a nondegenerate n-dimensional quadratic form q = q_\tau over K, hence is a simple adjoint algebraic K-group of type B_m (note that the K-rank of G equals the Witt index of q). If \tau is symplectic then necessarily n = 2m and G = SU(A, \tau) is an almost simple simply connected K-group of type C_m; moreover G is K-split if and only if A = M_n(K), in which case G is of course isomorphic to Sp_{2m}. Next, in all cases, any maximal K-torus T of G has the form T = SU(E, \sigma) where E is a \tau-invariant n-dimensional commutative étale K-subalgebra of A such that for \sigma = \tau|E we have

$$\dim E^\sigma = \left[\frac{n + 1}{2}\right].$$

So, the question whether G = SU(A, \tau), with A and \tau as above, has a maximal K-torus of a specific type can be reformulated as follows: Let (E, \sigma) be an n-dimensional commutative étale K-algebra with an involutive K-automorphism \sigma satisfying (7).

When does there exist an embedding (E, \sigma) \hookrightarrow (A, \tau) as algebras with involution? While in the general case this question is nontrivial (cf. [41]), the answer in the case where the group G splits over K is quite straightforward.

**Proposition 7.1.** ([18, 2.3 and 2.5]) Let A = M_n(K) with a K-linear involution \tau, and let (E, \sigma) be an n-dimensional commutative étale K-algebra with involution satisfying (7). In each of the following situations

1. \tau is symplectic,
2. \tau is orthogonal such that the corresponding quadratic form q_\tau has Witt index m,

there exists a K-embedding (E, \sigma) \hookrightarrow (A, \tau).

(This proposition should be viewed as an analogue for algebras with involution of the following result of Steinberg [52]: Let G_0 be a quasi-split simply connected almost simple algebraic group over a field K. Then given an inner form G of G_0, any maximal K-torus T of G admits a K-embedding into G_0. While the proof of this result is rather technical, the proof of Proposition 7.1 as well as of the corresponding assertion for the algebras with involution involved in the description of algebraic groups of type A and D is completely elementary – see [18, §2].)

Now, fix n \geq 2 and let A_1 = M_{n_1}(K), where n_1 = 2n + 1, with an orthogonal involution \tau_1 such that the Witt index of the corresponding quadratic form q_{\tau_1} is n, and let A_2 = M_{n_2}(K), where n_2 = 2n, with a symplectic involution \tau_2. According to Proposition 7.1 for i = 1, 2, the maximal K-tori of G_i = SU(A_i, \tau_i) are of the form T_i = SU(E_i, \sigma_i) for a commutative étale K-algebra E_i of dimension n_i with an involution \sigma_i satisfying (7). On the other hand, the correspondence

$$(E_2, \sigma_2) \mapsto (E_1, \sigma_1) := (E_2 \times K, \sigma_2 \times \text{id}_K)$$

defines a natural bijection between the isomorphism classes of commutative étale K-algebras with involution satisfying (7), of dimension n_2 and n_1, respectively. Since SU(E_1, \sigma_1) = SU(E_2, \sigma_2) in these notations, we again obtain that G_1 and G_2 have the same isomorphism classes of maximal K-tori (cf. [18], Remark 2.6).
Now, let $K$ be a number field and $S$ be any finite set of places of $K$ containing the set $V^K_\infty$ of archimedean places. Furthermore, let $G_1$ be a split adjoint $K$-group of type $B_n$, and $G_2$ be a split simply connected $K$-group of type $C_n$ ($n \geq 2$). Then the fact, discussed above, that $G_1$ and $G_2$ have the same isomorphism classes of maximal $K$-tori immediately implies that the $S$-arithmetic subgroups in $G_1$ and $G_2$ are weakly commensurable (cf. [10, Examples 6.5 and 6.7]).

A complete determination of weakly commensurable $S$-arithmetic subgroups in algebraic groups $G_1$ and $G_2$ of types $B_n$ and $C_n$ ($n \geq 2$) respectively was recently obtained by Skip Garibaldi and the second-named author [18]. To formulate the result we need the following definition. Let $G_1$ and $G_2$ be absolutely almost simple algebraic groups of types $B_n$ and $C_n$ with $n \geq 2$, respectively, over a number field $K$. We say that $G_1$ and $G_2$ are twins (over $K$) if for each place $v$ of $K$, either both groups are split or both are anisotropic over the completion $K_v$. (We note that since groups of these types cannot be anisotropic over $K_v$ when $v$ is nonarchimedean, our condition effectively says that $G_1$ and $G_2$ must be $K_v$-split for all nonarchimedean $v$.)

**Theorem 7.2.** ([18, Theorem 1.2]) Let $G_1$ and $G_2$ be absolutely almost simple algebraic groups over a field $F$ of characteristic zero having Killing-Cartan types $B_n$ and $C_n$ ($n \geq 3$) respectively, and let $\Gamma_i$ be a Zariski-dense $(G_i, K, S_i)$-arithmetic subgroup of $G_i(F)$ for $i = 1, 2$. Then $\Gamma_1$ and $\Gamma_2$ are weakly commensurable if and only if the groups $G_1$ and $G_2$ are twins.

(We recall that according to Theorem 1.1 if Zariski-dense $(G_1, K_1, S_1)$- and $(G_2, K_2, S_2)$-arithmetic subgroups are weakly commensurable then necessarily $K_1 = K_2$ and $S_1 = S_2$, so Theorem 7.2 in fact treats the most general situation.)

The necessity is proved using generic tori (cf. §9) in conjunction with the analysis of maximal tori in real groups of types $B_n$ and $C_n$ (which can also be found in [13]). The proof of sufficiency is obtained using the above description of maximal $K$-tori in terms of commutative étale $K$-subalgebras with involution and the local-global results for the existence of an embedding of commutative étale algebras with involution into simple algebras with involution established in [11]; an alternative argument along the lines outlined in the beginning of this section can be given using Galois cohomology of algebraic groups (cf. [18, §9]).

As we already mentioned in §6, the analysis of weak commensurability involved in the proof of Theorem 7.2 leads to, and at the same time depends on, the following result describing when groups of types $B_n$ and $C_n$ have the same isogeny/isomorphism classes of maximal $K$-tori.

**Theorem 7.3.** ([18, Theorem 1.4]) Let $G_1$ and $G_2$ be absolutely almost simple algebraic groups over a number field $K$ of types $B_n$ and $C_n$ respectively for some $n \geq 3$.

1. The groups $G_1$ and $G_2$ have the same isogeny classes of maximal $K$-tori if and only if they are twins.
2. The groups $G_1$ and $G_2$ have the same isomorphism classes of maximal $K$-tori if and only if they are twins, $G_1$ is adjoint, and $G_2$ is simply connected.
Theorem 7.2 has the following interesting geometric applications [43]. Again, let $G_1$ and $G_2$ be simple real algebraic groups of types $B_n$ and $C_n$, respectively. For $i = 1, 2$, let $\Gamma_i$ be a discrete torsion-free ($\mathcal{G}_i, K$)-arithmetic subgroup of $G_i = G_i(\mathbb{R})$, and let $X_{\Gamma_i}$ be the corresponding locally symmetric space of $G_i$. Then if $\mathcal{G}_1$ and $\mathcal{G}_2$ are not twins, the locally symmetric spaces $X_{\Gamma_1}$ and $X_{\Gamma_2}$ are not length-commensurable. As one application of this result, we would like to point out the following assertion: Let $M_1$ be an arithmetic quotient of the real hyperbolic space $\mathbb{H}^p (p \geq 5)$ and $M_2$ be an arithmetic quotient of the quaternionic hyperbolic space $\mathbb{H}^q _Q (q \geq 2)$. Then $M_1$ and $M_2$ are not length-commensurable. The results of [18] are used to handle the case $p = 2n$ and $q = n - 1$ for some $n \geq 3$; for other values of $p$ and $q$, the claim follows from Theorem 5.1.

On the other hand, suppose $G_1 = \text{SO}(n+1,n)$ and $G_2 = \text{Sp}_{2n}$ over $\mathbb{R}$ (i.e., $G_1$ and $G_2$ are the $\mathbb{R}$-split forms of types $B_n$ and $C_n$, respectively) for some $n \geq 3$. Furthermore, let $\Gamma_i$ be a discrete torsion-free ($\mathcal{G}_i, K$)-arithmetic subgroup of $G_i$ for $i = 1, 2$, and let $M_i = X_{\Gamma_i}$. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are twins then

$$\mathbb{Q} \cdot L(M_2) = \lambda \cdot \mathbb{Q} \cdot L(M_1) \quad \text{where} \quad \lambda = \sqrt{\frac{2n+2}{2n-1}}.$$  

Thus, despite the fact that they are associated with groups of different types, the locally symmetric spaces $M_1$ and $M_2$ can be made length-commensurable by scaling the metric on one of them; this, however, will never make them isospectral [51]. What is interesting is that so far this is the only situation in our analysis commensurability of isospectral and length-commensurable locally symmetric spaces where isospectrality manifests itself as an essentially stronger condition.

8. On the fields generated by the lengths of closed geodesics

In [8] (and also at the end of [17]) we discussed the consequences of length-commensurability of two locally symmetric spaces $M_1$ and $M_2$; our focus in this section will be on the consequences of non-length-commensurability of $M_1$ and $M_2$. More precisely, we will explore how different in this case the sets $L(M_1)$ and $L(M_2)$ (or $\mathbb{Q} \cdot L(M_1)$ and $\mathbb{Q} \cdot L(M_2)$) are and whether they can in fact be related in any reasonable way? Of course, one can ask a number of specific questions that fit this general perspective: for example, can $L(M_1)$ and $L(M_2)$ differ only in a finite number of elements, in other words, can the symmetric difference $L(M_1) \triangle L(M_2)$ be finite? Or can it happen that $\overline{\mathbb{Q}} \cdot L(M_1) = \overline{\mathbb{Q}} \cdot L(M_2)$, where $\overline{\mathbb{Q}}$ is the field of all algebraic numbers; in other words, can the use of the field $\overline{\mathbb{Q}}$ in place of $\mathbb{Q}$ in the definition of length-commensurability essentially change this relation? One relation between $L(M_1)$ and $L(M_2)$ that would make a lot of sense geometrically is that of similarity, requiring that there be a real number $\alpha > 0$ such that

$$L(M_2) = \alpha \cdot L(M_1) \quad \text{(or \quad } \mathbb{Q} \cdot L(M_2) = \alpha \cdot \mathbb{Q} \cdot L(M_1) ),$$  

which means that $M_1$ and $M_2$ can be made iso-length-spectral (resp., length-commensurable) by scaling the metric on one of them. From the algebraic standpoint, one can generalize this relation by considering arbitrary polynomial relations between $L(M_1)$.
and $L(M_2)$ instead of just linear relations although this perhaps does not have a clear geometric interpretation. To formalize this general idea, we need to introduce some additional notations and definitions.

For a Riemannian manifold $M$, we let $\mathcal{F}(M)$ denote the subfield of $\mathbb{R}$ generated by the set $L(M)$. Given two Riemannian manifolds $M_1$ and $M_2$, we set $\mathcal{F}_i = \mathcal{F}(M_i)$ for $i \in \{1, 2\}$ and consider the following condition

$(T_i)$ the compositum $\mathcal{F}_1 \mathcal{F}_2$ has infinite transcendence degree over the field $\mathcal{F}_{3-i}$.

Informally, this condition means that $L(M_i)$ contains “many” elements which are algebraically independent from all the elements of $L(M_{3-i})$, implying the nonexistence of any nontrivial polynomial dependence between $L(M_1)$ and $L(M_2)$. In particular, $(T_i)$ implies the following condition

$(N_i)$ $L(M_i) \not\subset A \cdot \mathbb{Q} \cdot L(M_{3-i})$ for any finite set $A$ of real numbers.

In [43], we have proved a series of results asserting that if $M_i = X_{G_i}$ for $i = 1, 2$, are the quotients of symmetric spaces $X_i$ associated with absolutely simple real algebraic groups $G_i$ by Zariski-dense discrete torsion-free subgroups $\Gamma_i \subset G_i(\mathbb{R})$, then in many situations the fact that $M_1$ and $M_2$ are not length-commensurable implies that conditions $(T_i)$ and $(N_i)$ hold for at least one $i \in \{1, 2\}$. To give precise formulations, in addition to the standard notations used earlier, we let $w_i$ denote the order of the (absolute) Weyl group of $G_i$. We also need to emphasize that all geometric results in [43] assume the validity of Shanuel’s conjecture. This assumption, however, enables one to establish results that are somewhat stronger than the corresponding results in [5] and do not require that $\Gamma_1$ and $\Gamma_2$ have property $(A)$ (2.2). We begin with the following result which strengthens Theorem 5.1.

**Theorem 8.1.** Assume that the Zariski-dense subgroups $\Gamma_1$ and $\Gamma_2$ are finitely generated (which is automatically the case if these subgroups are lattices).

1. If $w_1 > w_2$ then $(T_1)$ holds;
2. If $w_1 = w_2$ but $K_{\Gamma_1} \not\subset K_{\Gamma_2}$ then again $(T_1)$ holds.

Thus, unless $w_1 = w_2$ and $K_{\Gamma_1} = K_{\Gamma_2}$, the condition $(T_1)$ holds for at least one $i \in \{1, 2\}$; in particular, $M_1$ and $M_2$ are not length-commensurable.

As follows from Theorem 8.1 we only need to consider the case where $w_1 = w_2$, which we will assume – recall that this entails that either $G_1$ and $G_2$ are of the same Killing-Cartan type, or one of them is of type $B_n$ and the other is of type $C_n$ ($n \geq 3$). Then it is convenient to divide our results for arithmetic subgroups $\Gamma_1$ and $\Gamma_2$ into three theorems: the first one will treat the case where $G_1$ and $G_2$ are of the same type which is different from $A_n$, $D_{2n+1}$ ($n > 1$) and $E_6$, the second one – the case where both $G_1$ and $G_2$ are one of the types $A_n$, $D_{2n+1}$ ($n > 1$) and $E_6$, and the third one – the case where $G_1$ is of type $B_n$ and $G_2$ is of type $C_n$ for some $n \geq 3$.

**Theorem 8.2.** With notations as above, assume that $G_1$ and $G_2$ are of the same Killing-Cartan type which is different from $A_n$, $D_{2n+1}$ ($n > 1$) and $E_6$ and that the
subgroups $\Gamma_1$ and $\Gamma_2$ are arithmetic. Then either $M_1 = X_{\Gamma_1}$ and $M_2 = X_{\Gamma_2}$ are commensurable (hence also length-commensurable), or conditions ($T_i$) and ($N_i$) hold for at least one $i \in \{1, 2\}$.

(This theorem strengthens part of Theorem 5.2. We also note that ($T_i$) and ($N_i$) may not hold for both $i = 1$ and 2; in fact, it is possible that one of $L(M_1)$ and $L(M_2)$ is contained in the other.)

**Theorem 8.3.** Again, keep the above notations and assume that the common Killing-Cartan type of $G_1$ and $G_2$ is one of the following: $A_n$, $D_{2n+1}$ ($n > 1$) or $E_6$ and that the subgroups $\Gamma_1$ and $\Gamma_2$ are arithmetic. Assume in addition that $K_{\Gamma_i} \neq \mathbb{Q}$ for at least one $i \in \{1, 2\}$. Then either $M_1$ and $M_2$ are length-commensurable (although not necessarily commensurable), or conditions ($T_i$) and ($N_i$) hold for at least one $i \in \{1, 2\}$.

To illustrate possible applications of these theorems, we will now give explicit statements for real hyperbolic manifolds – cf. Theorem 1.1 similar results are available for complex and quaternionic hyperbolic spaces.

**Corollary 8.4.** Let $M_i$ ($i = 1, 2$) be the quotients of the real hyperbolic space $\mathbb{H}^d_i$ with $d_i \neq 3$ by a torsion-free Zariski-dense discrete subgroup $\Gamma_i$ of $G_i(\mathbb{R})$, where $G_i = \text{PSO}(d_i, 1)$.

(i) If $d_1 > d_2$ then conditions ($T_1$) and ($N_1$) hold.

(ii) If $d_1 = d_2$ but $K_{\Gamma_1} \not\subset K_{\Gamma_2}$ then again conditions ($T_1$) and ($N_1$) hold.

Thus, unless $d_1 = d_2$ and $K_{\Gamma_1} = K_{\Gamma_2}$, conditions ($T_i$) and ($N_i$) hold for at least one $i \in \{1, 2\}$.

Assume now that $d_1 = d_2 =: d$ and the subgroups $\Gamma_1$ and $\Gamma_2$ are arithmetic.

(iii) If $d$ is either even or is congruent to 3(mod 4), then either $M_1$ and $M_2$ are commensurable, hence length-commensurable, or ($T_i$) and ($N_i$) hold for at least one $i \in \{1, 2\}$.

(iv) If $d \equiv 1$ (mod 4) and in addition $K_{\Gamma_i} \neq \mathbb{Q}$ for at least one $i \in \{1, 2\}$ then either $M_1$ and $M_2$ are length-commensurable (although not necessarily commensurable), or conditions ($T_i$) and ($N_i$) hold for at least one $i \in \{1, 2\}$.

Now, we consider the case where one of the groups is of type $B_n$ and the other of type $C_n$ ($n \geq 3$). The theorem below strengthens the results of [71].

**Theorem 8.5.** Notations as above, assume that $G_1$ is of type $B_n$ and $G_2$ is of type $C_n$ for some $n \geq 3$ and the subgroups $\Gamma_1$ and $\Gamma_2$ are arithmetic. Then either ($T_i$) and ($N_i$) hold for at least one $i \in \{1, 2\}$, or

$$\mathbb{Q} \cdot L(M_2) = \lambda \cdot \mathbb{Q} \cdot L(M_1) \quad \text{where} \quad \lambda = \sqrt{\frac{2n+2}{2n-1}}.$$

The following interesting result holds for all types (cf. Theorem 5.3).

**Theorem 8.6.** For $i = 1, 2$, let $M_i = X_{\Gamma_i}$ be an arithmetically defined locally symmetric space, and assume that $w_1 = w_2$. If $M_2$ is compact and $M_1$ is not, then conditions ($T_i$) and ($N_i$) hold.
Finally, we have the following result which shows that the notion of “length-similarity” for arithmetically defined locally symmetric spaces is redundant if $G_1$ and $G_2$ are of the same type (cf., however, Theorem 8.3 regarding the case where $G_1$ and $G_2$ are of types $B_n$ and $C_n$).

**Corollary 8.7.** Let $M_i = X_{\Gamma_i}$, for $i = 1, 2$, be arithmetically defined locally symmetric spaces. Assume that there exists $\lambda \in \mathbb{R}_{>0}$ such that

$$Q \cdot L(M_1) = \lambda \cdot Q \cdot L(M_2).$$

Then

(i) if $G_1$ and $G_2$ are of the same type which is different from $A_n$, $D_{2n+1}$ $(n > 1)$ and $E_6$, then $M_1$ and $M_2$ are commensurable, hence length-commensurable;

(ii) if $G_1$ and $G_2$ are of the same type which is one of the following: $A_n$, $D_{2n+1}$ $(n > 1)$ or $E_6$, then, provided that $K_{\Gamma_i} \neq \mathbb{Q}$ for at least one $i \in \{1, 2\}$, the spaces $M_1$ and $M_2$ are length-commensurable (although not necessarily commensurable).

The proofs of the results in this section use a generalization of the notion of weak commensurability which we termed *weak containment*. To give a precise definition, we temporarily return to the general set-up where $G_1$ and $G_2$ are semi-simple algebraic groups defined over a field $F$ of characteristic zero, and $\Gamma_i$ is a Zariski-dense subgroup of $G_i(F)$ for $i = 1, 2$.

**8.8. Definition.** (a) Semi-simple elements $\gamma_1^{(1)}, \ldots, \gamma_{m_1}^{(1)} \in \Gamma_1$ are *weakly contained* in $\Gamma_2$ if there are semi-simple elements $\gamma_1^{(2)}, \ldots, \gamma_{m_2}^{(2)}$ such that

$$\chi_1^{(1)}(\gamma_1^{(1)}) \cdots \chi_{m_1}^{(1)}(\gamma_{m_1}^{(1)}) = \chi_1^{(2)}(\gamma_1^{(2)}) \cdots \chi_{m_2}^{(2)}(\gamma_{m_2}^{(2)}) \neq 1$$

for some maximal $F$-tori $T_k^{(j)}$ of $G_j$ whose group of $F$-rational points contains $\gamma_k^{(j)}$ and some characters $\chi_k^{(j)}$ of $T_k^{(j)}$ for $j \in \{1, 2\}$ and $k \leq m_j$.

(b) Semi-simple elements $\gamma_1^{(1)}, \ldots, \gamma_{m_1}^{(1)} \in \Gamma_1$ are *multiplicatively independent* if for some (equivalently, any) choice of maximal $F$-tori $T_i^{(1)}$ of $G_1$ such that $\gamma_i^{(1)} \in T_i(F)$ for $i \leq m_1$, a relation of the form

$$\chi_1^{(1)}(\gamma_1^{(1)}) \cdots \chi_{m_1}^{(1)}(\gamma_{m_1}^{(1)}) = 1,$$

where $\chi_i \in X(T_i)$ implies that

$$\chi_1^{(1)}(\gamma_1^{(1)}) = \cdots = \chi_{m_1}^{(1)}(\gamma_{m_1}) = 1.$$

(c) We say that $\Gamma_1$ and $\Gamma_2$ as above satisfy *property* $(G_i)$, where $i = 1$ or $2$, if for any $m \geq 1$ there exist semi-simple elements $\gamma_1^{(i)}, \ldots, \gamma_m^{(i)} \in \Gamma_i$ of infinite order that are multiplicatively independent and are *not* weakly contained in $\Gamma_{3-i}$.

---

5We note that [8] means that the subgroups of $\mathbb{F}^\times$ generated by the eigenvalues of $\gamma_1^{(1)}, \ldots, \gamma_{m_1}^{(1)}$ and by those of $\gamma_1^{(2)}, \ldots, \gamma_{m_2}^{(2)}$ for some (equivalently, any) matrix realizations of $G_1 \subset \text{GL}_{N_1}$ and $G_2 \subset \text{GL}_{N_2}$, intersect nontrivially.
Using Schanuel’s conjecture, we prove (cf. [43, Corollary 7.3]) that if \( X_{\Gamma_1} \) and \( X_{\Gamma_2} \) are locally symmetric spaces as above with finitely generated Zariski-dense fundamental groups \( \Gamma_1 \) and \( \Gamma_2 \), then the fact that these groups satisfy property \((C_i)\) for some \( i \in \{1, 2\} \) implies that the locally symmetric spaces satisfy conditions \((T_i)\) and \((N_i)\) for the same \( i \). So, the way we prove Theorems 8.1-8.3 and 8.5-8.6 is by showing that condition \((C_i)\) holds in the respective situations. For example, Theorem 8.1 is a consequence of the following algebraic result.

**Theorem 8.9.** Assume that \( \Gamma_1 \) and \( \Gamma_2 \) are finitely generated (and Zariski-dense).

(i) If \( w_1 > w_2 \) then condition \((C_1)\) holds;  
(ii) If \( w_1 = w_2 \) but \( K_{\Gamma_1} \not\subset K_{\Gamma_2} \) then again \((C_1)\) holds.

Thus, unless \( w_1 = w_2 \) and \( K_{\Gamma_1} = K_{\Gamma_2} \), condition \((C_i)\) holds for at least one \( i \in \{1, 2\} \).

In [43] we prove much more precise results in the case where the \( \Gamma_i \) are arithmetic, which leads to the geometric applications described above. We refer the interested reader to [43] for the technical formulations of these results; a point we would like to make here, however, is that our “algebraic” results (i.e., those asserting that condition \((C_i)\) holds in certain situation) do not depend on Schanuel’s conjecture.

### 9. Generic elements and tori

The analysis of weak commensurability and its variations in [18], [40] and [43] relies on the remarkable fact, first established in [39], that any Zariski-dense subgroup of the group of rational points of a semi-simple group over a finitely generated field of characteristic zero contains special elements, to be called **generic elements** here. It is convenient to begin our discussion of these elements with the definition of **generic tori**.

Let \( G \) be a connected semi-simple algebraic group defined over an infinite field \( K \). Fix a maximal \( K \)-torus \( T \) of \( G \), and, as usual, let \( \Phi = \Phi(G, T) \) denote the corresponding root system, and let \( W(G, T) \) be its Weyl group. Furthermore, we let \( K_T \) denote the (minimal) splitting field of \( T \) in a fixed separable closure \( \overline{K} \) of \( K \). Then the natural action of the Galois group \( \text{Gal}(K_T/K) \) on the character group \( X(T) \) of \( T \) induces an injective homomorphism

\[
\theta_T : \text{Gal}(K_T/K) \to \text{Aut}(\Phi(G, T)).
\]

We say that \( T \) is **generic** (over \( K \)) if

\[
(9) \quad \theta_T(\text{Gal}(K_T/K)) \supset W(G, T).
\]

For example, any maximal \( K \)-torus of \( G = \text{SL}_n/K \) is of the form \( T = \text{R}_{E/K}^{(1)}(\text{GL}_1) \) for some \( n \)-dimensional commutative étale \( K \)-algebra \( E \). Then such a torus is generic over \( K \) if and only if \( E \) is a separable field extension of \( K \) and the Galois group of the normal closure \( L \) of \( E \) over \( K \) is isomorphic to the symmetric group \( S_n \). It is well-known that for each \( n \geq 2 \) one can write down a system of congruences such that any monic polynomial \( f(t) \in \mathbb{Z}[t] \) satisfying this system of congruences has Galois group \( S_n \). It turns out that one can prove a similar statement for maximal tori of an arbitrary semi-simple algebraic group \( G \) over a finitely generated field \( K \) of characteristic zero.
classes of \( W \). In order to avoid technical details, we will restrict ourselves here to the case of absolutely almost simple groups.

**Theorem 9.1.** (\cite{40}, Theorem 3.1) Let \( G \) be a connected absolutely almost simple algebraic group over a finitely generated field \( K \) of characteristic zero, and let \( r \) be the number of nontrivial conjugacy classes of the Weyl group of \( G \). Then

1. There exist \( r \) inequivalent nontrivial discrete valuations \( v_1, \ldots, v_r \) of \( K \) such that the completion \( K_{v_i} \) is locally compact and \( G \) splits over \( K_{v_i} \) for all \( i = 1, \ldots, r \);
2. For any choice of discrete valuations \( v_1, \ldots, v_r \) as in (1), one can find maximal \( K_{v_i} \)-tori \( T(v_i) \) of \( G \), one for each \( i \in \{1, \ldots, r\} \), with the property that any maximal \( K \)-torus \( T \) of \( G \) which is conjugate to \( T(v_i) \) by an element of \( G(K_{v_i}) \), for all \( i = 1, \ldots, r \), is generic (i.e., the inclusion \( (9) \) holds).

The first assertion is an immediate consequence of the following, which actually shows that we can find the \( v_j \)'s so that \( K_{v_j} = \mathbb{Q}_{p_j} \), where \( p_1, \ldots, p_r \) are distinct primes.

**Proposition 9.2.** (\cite{38}, \cite{39}) Let \( K \) be a finitely generated field of characteristic zero and \( R \subset K \) a finitely generated subring. Then there exists an infinite set \( \Pi \) of primes such that for each \( p \in \Pi \) there exists an embedding \( \varepsilon_p : K \hookrightarrow \mathbb{Q}_p \) with the property \( \varepsilon_p(R) \subset \mathbb{Z}_p \).

To sketch a proof of the second assertion of Theorem \ref{thm:9.1} we fix a maximal \( K \)-torus \( T_0 \) of \( G \). Given any other maximal torus \( T \) of \( G \) defined over an extension \( F \) of \( K \) there exists \( g \in G(F) \) such that \( T = \iota_g(T_0) \), where \( \iota_g(x) = gxg^{-1} \). Then \( \iota_g \) induces an isomorphism between the Weyl groups \( W(G,T_0) \) and \( W(G,T) \). A different choice of \( g \) will change this isomorphism by an inner automorphism of the Weyl group, implying that there is a canonical bijection between the sets \( [W(G,T_0)] \) and \( [W(G,T)] \) of conjugacy classes in the respective groups; we will denote this bijection by \( \iota_{T_0,T} \).

Now, let \( v \) be a nontrivial discrete valuation of \( K \) such that the completion \( K_v \) is locally compact and splits \( T_0 \). Using the Frobenius automorphism of the maximal unramified extension \( K_v^{ur} \) in conjunction with the fact that \( H^1(K_v,G) \), where \( G \) is the simply connected cover of \( G \), vanishes (cf. \cite{8}, \cite{27}), one shows that given a nontrivial conjugacy class \( c \in [W(G,T_0)] \), one can find a maximal \( K_v \)-torus \( T(v,c) \) such that given any maximal \( K_v \)-torus \( T \) of \( G \) that is conjugate to \( T(v,c) \) by an element of \( G(K_v) \), for its splitting field \( K_vT \) we have

\[ \text{rk}_T(\text{Gal}(K_vT/K_v)) \cap \iota_{T_0,T}(c) \neq \emptyset. \]  

Now, if \( v_1, \ldots, v_r \) are as in part (1), then using the weak approximation property of the variety of maximal tori (cf. \cite{36}, Corollary 7.3), one can pick a maximal \( K \)-torus \( T_0 \) which splits over \( K_{v_i} \) for all \( i = 1, \ldots, r \). Let \( c_1, \ldots, c_r \) be the nontrivial conjugacy classes of \( W(G,T_0) \). Set \( T(v_i) = T(v_i,c_i) \) for \( i = 1, \ldots, r \) in the above notation. Then it is not difficult to show that the tori \( T(v_1), \ldots, T(v_r) \) are as required.

\footnote{One can alternatively use the fact that if we endow \( \widetilde{G} \) with the structure of a group scheme over \( O_v \) as a Chevalley group, then \( H^1(K_v^{ur}/K_v, \widetilde{G}(O_v^{ur})) \), where \( K_v^{ur} \) is the maximal unramified extension of \( K_v \) with the valuation ring \( O_v^{ur} \), vanishes, which follows from Lang’s theorem \cite{39} or its generalization due to Steinberg \cite{52}, see \cite{36} Theorem 6.8}
The method described above enables one to construct generic tori with various additional properties, in particular, having prescribed local behavior.

**Corollary 9.3.** (Corollary 3.2 in [40]). Let $G$ and $K$ be as in Theorem 9.1, and let $V$ be a finite set of inequivalent nontrivial rank 1 valuations of $K$. Suppose that for each $v \in V$ we are given a maximal $K_v$-torus $T(v)$ of $G$. Then there exists a maximal $K$-torus $T$ of $G$ for which (9) holds and which is conjugate to $T(v)$ by an element of $G(K_v)$, for all $v \in V$.

It should be noted that the method of $p$-adic embeddings that we used in the proof of Theorem 9.1, and which is based on Proposition 9.2, has many other applications – see [42].

We are now prepared to discuss generic elements whose existence in any finitely generated Zariski-dense subgroup is the core issue in this section.

**9.4. Definition.** Let $G$ be a connected semi-simple algebraic group defined over a field $K$. A regular semi-simple element $g \in G(K)$ is called generic (over $K$) if the maximal torus $T := Z_G(g)^\circ$ is generic (over $K$). We shall refer to $T$ as the torus associated with $g$.

Before proceeding with the discussion of generic elements, we would like to point out that some authors adopt a slight variant of this definition by requiring that the extension $K_g$ of $K$ generated by the eigenvalues of $g$ be “generic,” which is more consistent with the notion of a “generic polynomial” in Galois theory. We note that $K_g \subset K_T$ making the Galois group $\text{Gal}(K_g/K)$ a quotient of the group $\text{Gal}(K_T/K)$. Then the requirement that the order of $\text{Gal}(K_g/K)$ be divisible by $|W(G,T)|$ (which is probably one of the most natural ways to express the “genericity” of $K_g/K$) a priori may not imply the inclusion (9), which is most commonly used in applications (although the former does imply the latter if $G$ is an inner form of a split group – cf. [40, Lemma 4.1], in particular, if $G$ is absolutely almost simple of type different from $A_n$ $(n > 1)$, $D_n$ $(n \geq 4)$ and $E_6$). On the other hand, even for a regular element $g \in T(K)$, where $T$ is a maximal generic $K$-torus of $G$, the field $K_g$ may be strictly smaller than $K_T$. (Example: Let $G = \text{PSL}_2$ over $K$, and let $T$ be a maximal $K$-torus of $G$ of the form $\mathbb{R}_{L/K}^{(1)}(\text{GL}_1)$ where $L/K$ is a quadratic extension; then an element $g \in T(K)$ of order two is regular with $K_g = K$, while $K_T = L$). This problem, however, does not arise if $G$ is absolutely almost simple, $T$ is generic and $g \in T(K)$ has infinite order. Indeed, then the $K$-torus $T$ is irreducible, i.e., it does not contain any proper $K$-subtori since $W(G,T)$ acts irreducibly on $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. It follows that every element $g \in T(K)$ of infinite order generates a Zariski-dense subgroup of $T$, hence $K_g = K_T$, so the order of $\text{Gal}(K_g/K)$ is divisible by $|W(G,T)|$. Thus, the above variant of the definition of a generic element leads to the same concept for elements of infinite order. Recall that according to a famous result due to Selberg (cf. [44] Theorem 6.11), any finitely generated subgroup $\Gamma$ of $G(K)$ contains a torsion-free subgroup $\Gamma'$ of finite index, which therefore is also Zariski-dense. So, the following theorem (Theorem 9.6) that asserts the existence of generic elements in an arbitrary Zariski-dense subgroup in the sense
of our definition also implies the existence of generic elements in the sense of the other definition.

9.5. Let \( K \) be a field and \( G \) a connected absolutely almost simple \( K \)-group. Let \( g \in G(K) \) be a generic element and let \( T = Z_G(g)^0 \). Then \( g \in T(K) \) (see \cite{3}, Corollary 11.12). As \( T \) does not contain proper \( K \)-subtori, the cyclic group generated by any \( t \in T(K) \) of infinite order is Zariski-dense in \( T \). So \( Z_G(t) = Z_G(T) = T \); which implies that \( t \) is generic (over \( K \)) and \( Z_G(t) \) is connected; in particular, if \( g \) is of infinite order, then \( Z_G(g) = T \) is connected. If \( n \in G(K) \) is such that \( ntn^{-1} \) commutes with \( t \), then \( n \) normalizes \( T \), i.e., \( n \) lies in the normalizer \( N_G(T)(K) \) of \( T \) in \( G(K) \). If \( x \) is an element of \( G(K) \) of infinite order such that for some nonzero integer \( a \), \( t := x^a \) lies in \( T(K) \), then as \( x \) commutes with \( t \), it commutes with \( T \) and hence it lies in \( T(K) \).

It is also clear that the tori associated to two generic elements are equal if and only if the elements commute.

It is easily seen that the natural action of \( N_G(T)(K) \) on the character group \( X(T) \) commutes with the natural action of \( \text{Gal}(K_T/K) \). Now since \( T \) is generic, \( \theta_T(\text{Gal}(K_T/K)) \supset W(G,T) \), and as \( W(G,T) \) acts irreducibly on \( X(T) \otimes \mathbb{C} \), we see that the elements of \( N_G(T)(K) \) act by \( \pm 1 \) on \( X(T) \). Therefore, for \( n \in N_G(T)(K) \), \( n^2 \) commutes with \( T \) and hence lies in \( T(K) \). Now if \( n \in G(K) \) is such that \( ngn^{-1} \) commutes with \( g \), then \( n \) belongs to \( N_G(T)(K) \) and \( n^2 \in T(K) \). So, if moreover \( n \) is of infinite order, then it actually lies in \( T(K) \). Thus an element of \( G(K) \) of infinite order which does not belong to \( T(K) \) cannot normalize \( T \).

**Theorem 9.6.** Let \( G \) be a connected absolutely almost simple algebraic group over a finitely generated field \( K \), and let \( \Gamma \) be a finitely generated Zariski-dense subgroup of \( G(K) \). Then \( \Gamma \) contains a generic element (over \( K \)) of infinite order.

(It is not difficult to show, e.g. using Burnside’s characterization of absolutely irreducible linear groups, that any Zariski-dense subgroup \( \Gamma \subset G(K) \) contains a finitely generated Zariski-dense subgroup, so the assumption in the theorem that \( \Gamma \) be finitely generated can actually be omitted.)

**Sketch of the proof.** Fix a matrix \( K \)-realization \( G \subset \text{GL}_N \), and pick a finitely generated subring \( R \subset K \) so that \( \Gamma \subset G(R) := G \cap \text{GL}_N(R) \). Let \( r \) be the number of nontrivial conjugacy classes in the Weyl group \( W(G,T) \). Using Proposition 9.2, we can find \( r \) distinct primes \( p_1, \ldots, p_r \) such that for each \( i \leq r \) there exists an embedding \( \varepsilon_i : K \hookrightarrow \mathbb{Q}_{p_i} \) such that \( \varepsilon_i(R) \subset \mathbb{Z}_{p_i} \) and \( G \) splits over \( \mathbb{Q}_{p_i} \). Let \( v_i \) be the discrete valuation of \( K \) obtained as the pullback of the \( p_i \)-adic valuation of \( \mathbb{Q}_{p_i} \) so that \( K_{v_i} = \mathbb{Q}_{p_i} \). Pick maximal \( K_{v_i} \)-tori \( T(v_1), \ldots, T(v_r) \) as in part (2) of Theorem 9.1. Let \( \Sigma_i \) be the Zariski-open \( K_{v_i} \)-subvariety of regular elements in \( T(v_i) \). It follows from the Implicit Function Theorem that the image \( \Omega_i \) of the map

\[
G(K_{v_i}) \times \Sigma_i(K_{v_i}) \to G(K_{v_i}), \quad (g, t) \mapsto gtg^{-1},
\]

is open in \( G(K_{v_i}) \) and intersects every open subgroup of the latter. On the other hand, as explained in \cite{46}, §3, the closure of the image of the diagonal embedding \( \Gamma \hookrightarrow \prod_{i=1}^r G(K_{v_i}) \) is open, hence contains some \( U = \prod_{i=1}^r U_i \) where \( U_i \subset G(K_{v_i}) \) is a
Zariski-open torsion-free subgroup. Then

\[ U_0 := \prod_{i=1}^{r}(U_i \cap \Omega_i) \]

is an open set that intersects \( \Gamma \), and it follows from our construction that any element \( g \in \Gamma \cap U_0 \) is a generic element of infinite order.

Basically, our proof shows that given a finitely generated Zariski-dense subgroup \( \Gamma \) of \( G(K) \), one can produce a finite system of congruences (defined in terms of suitable valuations of \( K \)) such that the set of elements \( \gamma \in \Gamma \) satisfying this system of congruences consists entirely of generic elements (and additionally this set is in fact a coset of a finite index subgroup in \( \Gamma \), in particular, it is Zariski-dense in \( G \)). Recently, Jouve-Kowalski-Zywina [24], Gorodnik-Nevo [22] and Lubotzky-Rosenzweig [33] developed different quantitative ways of showing that generic elements exist in abundance (in fact, these results demonstrate that “most” elements in \( \Gamma \) are generic). More precisely, the result of [22] gives the asymptotics of the number of generic elements of a given height in an arithmetic group, while the results of [33], generalizing earlier results of [24], are formulated in terms of random walks on groups and apply to arbitrary Zariski-dense subgroups in not necessarily connected semi-simple groups. These papers introduce several new ideas and techniques, but at the same time employ the elements of the argument from [39] we outlined above.

The proofs of the results in [40], [43] use not only Theorem 9.6 itself but also its different variants that provide generic elements with additional properties, e.g. having prescribed local behavior (cf. Corollary 9.3). We refer the interested reader to these papers for precise formulations (which are rather technical), and will only indicate here the basic “multidimensional” version of Theorem 9.6 that was developed in [33].

**Theorem 9.7.** (cf. [43, Theorem 3.4]) Let \( G, K \) and \( \Gamma \subset G(K) \) be as in Theorem 9.6. Then for any \( m \geq 1 \) one can find generic semisimple elements \( \gamma_1, \ldots, \gamma_m \in \Gamma \) of infinite order that are multiplicatively independent.

Finally, we would like to formulate a result that enables one to pass from the weak commensurability of two generic semi-simple elements to an isogeny, and in most cases even to an isomorphism, of the ambient tori. This result relates the analysis of weak commensurability to the problem of characterizing algebraic group having the same isomorphism/isogeny classes of maximal tori.

**Theorem 9.8.** (Isogeny Theorem, [40, Theorem 4.2]) Let \( G_1 \) and \( G_2 \) be two connected absolutely almost simple algebraic groups defined over an infinite field \( K \), and let \( L_i \) be the minimal Galois extension of \( K \) over which \( G_i \) becomes an inner form of a split group. Suppose that for \( i = 1, 2 \), we are given a semi-simple element \( \gamma_i \in G_i(K) \) contained in a maximal \( K \)-torus \( T_i \) of \( G_i \). Assume that (i) \( G_1 \) and \( G_2 \) are either of the same Killing-Cartan type, or one of them is of type \( B_n \) and the other is of type \( C_n \), (ii) \( \gamma_1 \) has infinite order, (iii) \( T_1 \) is \( K \)-irreducible, and (iv) \( \gamma_1 \) and \( \gamma_2 \) are weakly commensurable. Then
(1) there exists a $K$-isogeny $\pi: T_2 \to T_1$ which carries $\gamma_2^{m_2}$ to $\gamma_1^{m_1}$ for some integers $m_1, m_2 \geq 1$;
(2) if $L_1 = L_2 =: L$ and $\theta_{T_1}(\text{Gal}(L_{T_1}/L)) \supset W(G_1, T_1)$, then $\pi^*: X(T_1) \otimes_{\mathbb{Z}} \mathbb{Q} \to X(T_2) \otimes_{\mathbb{Z}} \mathbb{Q}$ has the property that $\pi^* (\mathbb{Q} \cdot \Phi(G_1, T_1)) = \mathbb{Q} \cdot \Phi(G_2, T_2)$. Moreover, if $G_1$ and $G_2$ are of the same Killing-Cartan type different from $B_2 = C_2$, $F_4$ or $G_2$, then a suitable rational multiple of $\pi^*$ maps $\Phi(G_1, T_1)$ onto $\Phi(G_2, T_2)$, and if $G_1$ is of type $B_n$ and $G_2$ is of type $C_n$, with $n > 2$, then a suitable rational multiple $\lambda$ of $\pi^*$ takes the long roots in $\Phi(G_1, T_1)$ to the short roots in $\Phi(G_2, T_2)$ while $2\lambda$ takes the short roots in $\Phi(G_1, T_1)$ to the long roots in $\Phi(G_2, T_2)$.

It follows that in the situations where $\pi^*$ can be, and has been, scaled so that $\pi^* (\Phi(G_1, T_1)) = \Phi(G_2, T_2)$, it induces $K$-isomorphisms $\overline{\pi}: \overline{T}_2 \to \overline{T}_1$ and $\overline{\pi}: \overline{T}_2 \to \overline{T}_1$ between the corresponding tori in the simply connected and adjoint groups $G_i$ and $\overline{G_i}$, respectively, that extend to $\overline{K}$-isomorphisms $\overline{G}_2 \to \overline{G}_1$ and $\overline{G}_2 \to \overline{G}_1$. Thus, the fact that Zariski-dense torsion-free subgroups $\Gamma_1 \subset G_1(K)$ and $\Gamma_2 \subset G_2(K)$ are weakly commensurable implies (under some minor technical assumptions) that $G_1$ and $G_2$ have the same $K$-isogeny classes (and under some additional assumptions, even the same $K$-isomorphism classes) of generic maximal $K$-tori that nontrivially intersect $\Gamma_1$ and $\Gamma_2$, respectively.

For a “multidimensional” version of Theorem 9.8 which is formulated using the notion of weak containment (see [8] in place of weak commensurability, see [43] Theorem 2.3).

9.9. We would like to conclude this section with one new observation (Theorem 9.10) which is directly related to the main theme of the workshop – thin groups. This observation was inspired by the conversations of the first-named author with Igor Rivin at the Institute for Advanced Study.

Let $G$ be a connected absolutely almost simple algebraic group over a field $K$ of characteristic zero, and let $T$ be a maximal $K$-torus of $G$. We let $\Phi_>(G, T)$ (resp., $\Phi_<(G, T)$) denote the set of all long (resp., short) roots in the root system $\Phi(G, T)$; by convention,

$$\Phi_>(G, T) = \Phi_<(G, T) = \Phi(G, T)$$

if all roots have the same length. Furthermore, we let $G_T^\gamma$ denote the $K$-subgroup of $G$ generated by $T$ and the 1-parameter unipotent subgroups $U_a$ for $a \in \Phi_>(G, T)$. Then $G_T^\gamma$ is a connected semi-simple subgroup of $G$ of maximal absolute rank (so, in fact, just the $U_a$’s for $a \in \Phi_>(G, T)$ generate $G_T^\gamma$). By direct inspection, one verifies that $G_T^\gamma \neq G$ precisely when $\Phi(G, T)$ has roots of different lengths, and then $G_T^\gamma$ is a semi-simple group of type $(A_1)^n$ if $G$ is of type $C_n$, and an absolutely almost simple group of type $D_n$, $D_4$ and $A_2$ if $G$ is of type $B_n$, $F_4$ and $G_2$, respectively. On the other hand, the subgroups $U_a$ for $a \in \Phi_<(G, T)$ generate $G$ in all cases. Finally, for any connected subgroup of $G$ containing $T$ there exists a subset $\Psi \subset \Phi(G, T)$ such that $G$ is generated by $T$ and $U_a$ for all $a \in \Psi$.

Theorem 9.10. Let $g$ be a generic element of infinite order and $T := Z_G(g)$ be the associated maximal torus. Let $x \in G(K)$ be any element of infinite order not contained
Furthermore, let $\Gamma$ be the (abstract) subgroup of $G(K)$ generated by $g$ and $x$, and let $H$ be the identity component of the Zariski-closure of $\Gamma$. Then either $H = G$ or $H = G_T^\sigma$. Consequently, $g$ and $x$ generate a Zariski-dense subgroup of $G$ if all roots in the root system $\Phi(G,T)$ are of same length.

**Proof.** As $g$ is a generic element of infinite order, the cyclic group generated by it is Zariski-dense in $T$, and so the cyclic group generated by $xg^{-1}$ is Zariski-dense in the conjugate torus $xTx^{-1}$. Since $x \notin T(K)$ and is of infinite order, it cannot normalize $T$ (see 9.5). Thus $H$ contains at least two different (generic) maximal $K$-tori, namely $T$ and $xTx^{-1}$. Assume that $H \neq G$. Since $H$ is connected and properly contains $T$, it must contain a 1-parameter subgroup $U_a$ for some $a \in \Phi(G,T)$. Then being defined over $K$, $H$ also contains $U_b$ for all $b$ of the form $b = \sigma(a)$ with $\sigma \in \text{Gal}(K_T/K)$. Now since $T$ is generic, using the fact that the Weyl group $W(G,T)$ acts transitively on the subsets of roots of same length (cf. [7, Ch. VI, Prop. 11]), we see that $H$ contains $U_b$ for all roots $b \in \Phi(G,T)$ of same length as $a$. If $a$ were a short root then the above remarks would imply that $H = G$, which is not the case. Thus, $a$ must be long, and therefore $H$ contains $G_T^\sigma$ but does not contain $U_b$ for any short root $b$. This clearly implies that $H = G_T^\sigma$. \[\square\]

**Remark 9.11.** It is worth noting that the types with roots of different lengths are honest exceptions in Theorem 9.10 in the sense that for any absolutely almost simple algebraic group $G$ of one of those types over a finitely generated field $K$ one can find two generic elements $\gamma_1, \gamma_2 \in G(K)$ that generate $G_T^\sigma \neq G$ for a generic maximal $K$-torus $T$. To see this, we first pick an arbitrary generic element $\gamma_1 \in G(K)$ of infinite order provided by Theorem 9.6 and let $T = Z_G(\gamma_1)$ be the corresponding torus. Since $H := G_T^\sigma$ is semi-simple, the group $H(K)$ is Zariski-dense in $H$ (cf. [8, Corollary 18.3]). So, there exists $h \in H(K)$ such that $\gamma_2 := h\gamma_1h^{-1} \notin T(K)$. Then $\gamma_2 \in H(K)$ is also generic over $K$, and the Zariski-closure of the subgroup generated by $\gamma_1$ and $\gamma_2$ is contained in (in fact, is equal to) $H$.

### 10. Some open problems

The analysis of weak commensurability has led to a number of interesting problems in the theory of algebraic and Lie groups (cf., for example, [6]) and its applications to locally symmetric spaces, and we would like to conclude this article with a brief discussion of some of these problems.

According to Theorem 10.10 if two lattices in the groups of rational points of connected absolutely almost simple groups over a nondiscrete locally compact field are weakly commensurable and one of the lattices is arithmetic, then so is the other. At the same time, it has been shown by means of an example (cf. [40, Remark 5.5]) that a Zariski-dense subgroup weakly commensurable to a rank one arithmetic subgroup need not be arithmetic. It would be interesting, however, to understand what happens with higher rank $S$-arithmetic subgroups.
Problem 10.1. Let $G_1$ and $G_2$ be two connected absolutely almost simple algebraic groups defined over a field $F$ of characteristic zero, and let $\Gamma_1 \subset G_1(F)$ be a Zariski-dense $(K,S)$-arithmetic subgroup whose $S$-rank\footnote{We recall that if $\Gamma$ is $(\mathcal{G},K,S)$-arithmetic, then the $S$-rank of $\Gamma$ is defined to be $\sum_{v \in S} \text{rk}_K \mathcal{G}$, where $\text{rk}_K \mathcal{G}$ denotes the rank of $\mathcal{G}$ over a field $F \supseteq K$.} is $\geq 2$. If $\Gamma_2 \subset G_2(F)$ is a Zariski-dense subgroup weakly commensurable to $\Gamma_1$, then is $\Gamma_2$ necessarily $S$-arithmetic?

This problem appears to be very challenging; the answer is not known even in the cases where $\Gamma_1$ is $\text{SL}_3(\mathbb{Z})$ or $\text{SL}_2(\mathbb{Z}[1/p])$. One should probably start by considering Problem 10.1 in a more specialized situation, e.g., assuming that $F$ is a nondiscrete locally compact field, $\Gamma_1 \subset G_1(F)$ is a discrete Zariski-dense (higher rank) $S$-arithmetic subgroup, and $\Gamma_2 \subset G_2(F)$ is a (finitely generated) discrete Zariski-dense subgroup weakly commensurable to $\Gamma_1$ (these restrictions would eliminate $\text{SL}_2(\mathbb{Z}[1/p])$ as a possibility for $\Gamma_1$, but many interesting groups such as $\text{SL}_3(\mathbb{Z})$ would still be included). The nature of these assumptions brings up another question of independent interest.

Problem 10.2. Let $G_1$ and $G_2$ be connected absolutely almost simple algebraic groups over a nondiscrete locally compact field $F$, and let $\Gamma_i$ be a finitely generated Zariski-dense subgroup of $G_i(F)$ for $i = 1, 2$. Assume that $\Gamma_1$ and $\Gamma_2$ are weakly commensurable. Does the discreteness of $\Gamma_1$ imply the discreteness of $\Gamma_2$?\footnote{It is well-known that for a semi-simple algebraic group $G$ over a nondiscrete nonarchimedean locally compact field $F$ of characteristic zero and a discrete subgroup $\Gamma \subset G(F)$, the quotient $G(F)/\Gamma$ has finite measure if and only if it is compact, so the problem in this case becomes vacuous.}

An affirmative answer to Problem 10.2 was given in [40, Proposition 5.6] for the case where $F$ is a nonarchimedean local field, but the case $F = \mathbb{R}$ or $\mathbb{C}$ remains open. Another interesting question is whether weak commensurability preserves cocompactness of lattices.

Problem 10.3. Let $G_1$ and $G_2$ be connected absolutely almost simple algebraic groups over $F = \mathbb{R}$ or $\mathbb{C}$, and let $\Gamma_i \subset G_i(F)$ be a lattice for $i = 1, 2$. Assume that $\Gamma_1$ and $\Gamma_2$ are weakly commensurable. Does the compactness of $G_1(F)/\Gamma_1$ imply the compactness of $G_2(F)/\Gamma_2$?\footnote{We recall that the co-compactness of a lattice in a semi-simple real Lie group is equivalent to the absence of nontrivial unipotents in it (cf. [44, Corollary 11.13]). So, Problem 10.3 can be rephrased as the question whether for two weakly commensurable lattices $\Gamma_1$ and $\Gamma_2$, the existence of nontrivial unipotent elements in one of them implies their existence in the other; in this form the question is meaningful for arbitrary Zariski-dense subgroups (not necessarily discrete or of finite covolume). The combination of Theorems 4.4 and 10.5 implies the affirmative answer to Problem 10.3 in the case where one of the lattices is arithmetic, but no other cases have been considered so far.}

We recall that the co-compactness of a lattice in a semi-simple real Lie group is equivalent to the absence of nontrivial unipotents in it (cf. [44, Corollary 11.13]). So, Problem 10.3 can be rephrased as the question whether for two weakly commensurable lattices $\Gamma_1$ and $\Gamma_2$, the existence of nontrivial unipotent elements in one of them implies their existence in the other; in this form the question is meaningful for arbitrary Zariski-dense subgroups (not necessarily discrete or of finite covolume). The combination of Theorems 4.4 and 10.5 implies the affirmative answer to Problem 10.3 in the case where one of the lattices is arithmetic, but no other cases have been considered so far.

From the general perspective, one important problem is to try to generalize our results on length-commensurable and/or isospectral arithmetically defined locally symmetric spaces of absolutely simple real Lie groups to arithmetically defined locally symmetric spaces of arbitrary semi-simple Lie groups, or at least those of $\mathbb{R}$-simple Lie groups. To highlight the difficulty, we will make some comments about the latter case.
An $\mathbb{R}$-simple adjoint group $G$ can be written in the form $G = R_{C/\mathbb{R}}(H)$ (restriction of scalars) where $H$ is an absolutely simple complex algebraic group. Arithmetic lattices in $G(\mathbb{R}) \simeq H(\mathbb{C})$ come from the forms of $H$ over a number field admitting exactly one complex embedding. The analysis of weak commensurability of even arithmetic lattices $\Gamma_1 \subset G_1(\mathbb{R})$ and $\Gamma_2 \subset G_2(\mathbb{R})$, where $G_i = R_{C/\mathbb{R}}(H_i)$ for $i = 1, 2$, cannot be implemented via the study of the forms of the $G_i$’s, forcing us to study directly the forms of the $H_i$’s. But the relation of weak commensurability of semi-simple elements $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ in terms of $G_1$ and $G_2$, i.e. the fact that $\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1$ for some characters $\chi_i$ of maximal $\mathbb{R}$-tori $T_i$ of $G_i$ such that $\gamma_i \in T_i(\mathbb{R})$, translates into a significantly more complicated relation in terms of $H_1$ and $H_2$. Indeed, pick maximal $\mathbb{C}$-tori $S_i$ of $H_i$ so that $T_i = R_{C/\mathbb{R}}(S_i)$, and let $\delta_i \in S_i(\mathbb{C})$ be the element corresponding to $\gamma_i$ under the identification $T_i(\mathbb{R}) \simeq S_i(\mathbb{C})$. Then there exist characters $\chi_i', \chi_i''$ of $S_i$ such that $\chi_i(\gamma_i) = \chi_i'(\delta_i)\chi_i''(\delta_i)$. So, the relation of weak commensurability of $\gamma_1$ and $\gamma_2$ assumes the following form in terms of $\delta_1$ and $\delta_2$:

$$\chi_1'(\delta_1)\overline{\chi_1'\!(\delta_1)} = \chi_2'(\delta_2)\overline{\chi_2'\!(\delta_2)}.$$

It is not clear if this type of relation would lead to the results similar to those we described in this article for the weakly commensurable arithmetic subgroups of absolutely almost simple groups. So, the general problem at this stage is to formulate for general semi-simple groups (or at least $\mathbb{R}$-simple groups) the “right” notion of weak commensurability and explore its consequences. We will now formulate a particular case of this general program that would be interesting for geometric applications.

**Problem 10.4.** Let $G_1$ and $G_2$ be almost simple complex algebraic groups. Two semisimple elements $\gamma_i \in G_i(\mathbb{C})$ are called $\mathbb{R}$-weakly commensurable if there exist complex maximal tori $T_i$ of $G_i$ for $i = 1, 2$ such that $\gamma_i \in T_i(\mathbb{C})$ and for suitable characters $\chi_i$ of $T_i$ we have

$$|\chi_1(\gamma_1)| = |\chi_2(\gamma_2)| \neq 1.$$

Furthermore, Zariski-dense (discrete) subgroups $\Gamma_i \subset G_i(\mathbb{C})$ are $\mathbb{R}$-weakly commensurable if every semisimple element $\gamma_1 \in \Gamma_1$ of infinite order is $\mathbb{R}$-weakly commensurable to some semisimple element $\gamma_2 \in \Gamma_2$ of infinite order, and vice versa. Under what conditions does the $\mathbb{R}$-weak commensurability of Zariski-dense (arithmetic) lattices $\Gamma_i \subset G_i(\mathbb{C})$ ($i = 1, 2$) imply their commensurability?

The result of [12] seems to imply that the $\mathbb{R}$-weak commensurability of arithmetic lattices in $SL_2(\mathbb{C})$ does imply their commensurability, but no other results in this direction are available.

Turning now to the geometric aspect, we would like to reiterate that most of our results deal with the analysis of the new relation of length-commensurability, which eventually implies the results about isospectral locally symmetric spaces. At the same time, the general consequences of isospectrality and iso-length-spectrality are much better understood than those of length-commensurability. So, as an overarching problem, we would like to propose the following.

**Problem 10.5.** Understand consequences (qualitative and quantitative) of length-commensurability for locally symmetric spaces.
(Here by *quantitative consequences* we mean results stating that in certain situations a family of length-commensurable locally symmetric spaces consists either of a single commensurability class or of a certain bounded number of commensurability classes, and by *qualitative consequences* - results guaranteeing that the number of commensurability classes in a given class of length-commensurable locally symmetric spaces is finite.)

There are various concrete questions within the framework provided by Problem 10.5 that were resolved in [40] for arithmetically defined locally symmetric spaces but remain open for locally symmetric spaces which are not arithmetically defined. For example, according to Theorem 5.3, if $X_{\Gamma_1}$ and $X_{\Gamma_2}$ are length-commensurable and at least one of the spaces is arithmetically defined, then the compactness of one of them implies the compactness of the other. It is natural to ask if this can be proved for locally symmetric spaces which are not arithmetically defined.

**Problem 10.6.** (Geometric version of Problem 10.3) Let $X_{\Gamma_1}$ and $X_{\Gamma_2}$ be length-commensurable locally symmetric spaces of finite volume. Does the compactness of $X_{\Gamma_1}$ always imply the compactness of $X_{\Gamma_2}$?

In [40], §9, for each of the exceptional types $A_n$, $D_{2n+1}$ ($n > 1$) and $E_6$, we have constructed examples of length-commensurable, but not commensurable, compact arithmetically defined locally symmetric spaces associated with a simple real algebraic group of this type. It would be interesting to see if this construction can be refined to provide examples of iso-length spectral or even isospectral compact arithmetically defined locally symmetric spaces that are not commensurable.

**Problem 10.7.** For each of the types $A_n$ ($n > 1$), $D_{2n+1}$ ($n > 1$) and $E_6$, construct examples of isospectral compact arithmetically defined locally symmetric spaces that are not commensurable.

Currently, such a construction is known only for *inner forms* of type $A_n$ (cf. [32]); it relies on some delicate results from the theory of automorphic forms [23], the analogues of which are not yet available for groups of other types.

As we already mentioned, in [40] we focused on the case where $G_1$ and $G_2$ are absolutely (almost) simple real algebraic groups. From the geometric perspective, however, it would be desirable to consider a more general situation where $G_1$ and $G_2$ are allowed to be either arbitrary real semi-simple groups (without compact factors), or at least arbitrary $\mathbb{R}$-simple groups. This problem is intimately related to the problem, discussed above, of generalizing our results on weak commensurability from absolutely almost simple to arbitrary semi-simple groups. In particular, a successful resolution of Problem 10.4 would enable us to extend our results to the (arithmetically defined) locally symmetric spaces associated with $\mathbb{R}$-simple groups providing thereby a significant generalization of the result of [12] where the case $G_1 = G_2 = \mathbb{R}_C/\mathbb{R}(SL_2)$ (that leads to arithmetically defined hyperbolic 3-manifolds) was considered.

Finally, the proof of the result that connects the length-commensurability of $X_{\Gamma_1}$ and $X_{\Gamma_2}$ to the weak commensurability of $\Gamma_1$ and $\Gamma_2$ relies (at least in the higher rank case)
on Schanuel’s conjecture. It would be interesting to see if our geometric results can be made independent of Schanuel’s conjecture.

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Department of Mathematics, University of Virginia, Charlottesville, VA 22904
E-mail address: asr3x@virginia.edu