NONDIVERGENCE ELLIPTIC EQUATIONS
WITH UNBOUNDED COEFFICIENTS

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Abstract. We study the nonvariational equation
\[ \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f \]
in domains of \( \mathbb{R}^n \). We assume that the coefficients \( a_{ij} \) are in \( \text{BMO} \) and the equation is elliptic, but not uniformly, and consider \( f \) in \( L^2(\mathbb{R}^n) \), or even in the Zygmund class \( L^2 \log^\alpha L(\mathbb{R}^n) \). We also solve Dirichlet problem.

1. Introduction. We consider the second order linear partial differential operator
\[ \mathcal{L} = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \] (1)
where the coefficients \( a_{ij} = a_{ji} \) are measurable functions defined on a domain \( \Omega \) of \( \mathbb{R}^n \). The operator \( \mathcal{L} \) is elliptic in the sense that the following inequality
\[ \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq |\xi|^2 \] (2)
holds, \( \forall \xi \in \mathbb{R}^n \), for a.e. \( x \in \Omega \).

Our basic assumption is that the coefficients \( a_{ij} \) belong to the John-Nirenberg space \( \text{BMO} \) of functions with bounded mean oscillation. We denote by \( A = (a_{ij}) \) the coefficient matrix, and write
\[ \|A\|_{\text{BMO}} = \max_{i,j=1,...,n} \|a_{ij}\|_{\text{BMO}}. \]

We shall study the equation
\[ \mathcal{L} u = f, \] (3)
mainly for \( f \in L^2(\Omega) \), and consider strong solutions, namely, functions in Sobolev classes satisfying (3) a.e. Thus, we deal with equation (3) for a class of operators possibly not uniformly elliptic (even not locally).

One of our main results reads as follows.

2000 Mathematics Subject Classification. Primary: 35J15; Secondary: 35J99.

Key words and phrases. Nondivergence elliptic equations, unbounded coefficients, a priori estimates, Dirichlet problem.

Partially supported by PRIN MIUR (2006): “Equazioni e sistemi ellittici e parabolici: stime a priori, esistenza e regolarità”, and by GNAMPA.
Theorem 1.1. There exists a constant $\varepsilon_0 > 0$ depending only on the dimension $n$ such that, assuming
\[
\|A\|_{BMO(\Omega)} < \varepsilon_0 ,
\] the following holds. For every $u \in W^{2,2}_{\text{loc}}(\Omega)$ solution to (3), if $f \in L^2 \log^\alpha L_{\text{loc}}(\Omega)$, $\alpha > 0$, then actually
\[
D^2 u \in L^2 \log^\alpha L_{\text{loc}}(\Omega; \mathbb{R}^{n \times n})
\] and for any cube $Q \subset 2Q \subset \Omega$ we have
\[
\|D^2 u\|_{L^2 \log^\alpha L(Q; \mathbb{R}^{n \times n})} \leq C \left( \|f\|_{L^2 \log^\alpha L(2Q)} + \|D^2 u\|_{L^2(2Q; \mathbb{R}^{n \times n})} \right). \tag{5}
\]

Even though under the assumption (4) the operator $\mathcal{L}$ might be thought of as a perturbation of an operator with constant coefficients, we emphasize that the coefficients can be unbounded. Notice also that, due to the local nature of the above result, we can replace the global assumption (4) by a local version. This means, as an example, that the same result follows under the assumption $A \in VMO$. (The space $VMO$ is defined as the closure in $BMO$ of the subspace of uniformly continuous functions.) On the other hand, as is well known, functions in $VMO$ have a number of attractive properties which are not shared by general $BMO$-functions.

To achieve Theorem 1.1, we make a study of the equation (3) in the entire space $\mathbb{R}^n$. Among other things, we prove the following.

Theorem 1.2. Under assumption (4), for every $f \in L^2(\mathbb{R}^n)$ equation (3) has a solution $u$ satisfying
\[
\|D^2 u\|_{L^2(\mathbb{R}^n; \mathbb{R}^{n \times n})} \leq C \|f\|_{L^2(\mathbb{R}^n)}. \tag{6}
\]
The solution is unique modulo affine functions.

Our arguments will work also for $n = 2$, but here we are concerned mainly with the case of dimension $n > 2$, since in the plane the theory of elliptic equations is in many respects simpler and more general. We only mention that recent and fundamental results are largely due to the interplay with quasiconformal mappings, see [2], [3].

For a uniformly elliptic operator $\mathcal{L}$ uniqueness for the Dirichlet problem in a bounded domain $\Omega$ has been studied in a number of papers, mainly by Aleksandrov, Bakel’man, Pucci (we refer to [13] and the references therein). Concerning the existence, it is well-known that the uniform ellipticity condition is generally not enough, additional conditions being needed. A well-known “cone” condition was introduced by Cordes. It deals with the scattering of the eigenvalues of the coefficient matrix $A$; in the two-dimensional case, it is a consequence of the ellipticity condition, so it is redundant. In [19] Miranda proved that the problem is well posed in $W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ if the coefficients of $\mathcal{L}$ lie in $W^{1,2}(\Omega)$. See also [12]. A deep result due to Alvino-Trombetti [1] improved Miranda’s result. They assumed that the distributional derivatives of the coefficients $a_{ij}$ belong to the Marcinkiewicz! space $L^{n,\infty}$ and the constants in the weak-type inequality for each $\frac{\partial u}{\partial x_i}$ are sufficiently small. However, we notice that $BMO$ functions need not be differentiable and in this sense assumption (4) is qualitatively more general than the smallness condition for the $L^{n,\infty}$-norms of the derivatives of the coefficients. We mention [21], where assumption (4) is considered for bounded coefficients.

The study of uniformly elliptic equations with $VMO$-coefficients originated in the works of Chiarenza-Frasca-Longo. In [4] and [5] the authors prove a higher integrability result for local solutions and solve the Dirichlet problem, respectively.
We shall not relay on any representation formula for the derivatives of the solution, but prove our estimates from the ground, using as main tools that second order minors of the Hessian matrix lie in the Hardy space \(\mathcal{H}^1\) and the celebrated \(\mathcal{H}^1\)-BMO duality of C.Fefferman [10], [11].

2. Preliminary results.

2.1. Zygmund classes. We shall need to consider Zygmund classes \(L^p \log^\alpha L(\mathbb{R}^n)\), \(1 < p < \infty, \alpha \in \mathbb{R}\) (\(\alpha \geq 0\) for \(p = 1\)): these are Orlicz spaces defined by the functions \(\Phi(t) = t^p \log^\alpha(e + t)\), that is,

\[
L^p \log^\alpha L(\mathbb{R}^n) = \left\{ f : \int_{\mathbb{R}^n} |f|^p \log^\alpha(e + |f|) \, dx < +\infty \right\}.
\]

There is a technical difficulty in considering general \(\alpha\), since we need \(\Phi\) increasing and convex; this is clearly the case for \(p + \alpha \geq 1\). Actually, almost completely we shall only consider the cases \(p = 2\) and \(\alpha = \pm 1\). Here we limit to the essential matter and refer to [22] for further information on Orlicz spaces.

The spaces \(L^p \log^\alpha L(\mathbb{R}^n)\) are endowed with the Luxemburg norm

\[\|f\|_{L^p \log^\alpha L(\mathbb{R}^n)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} |f|^{p \lambda} \log^{\alpha} \left( e + \frac{|f|}{\lambda} \right) \, dx \leq 1 \right\}.
\]

The following inclusions trivially hold

\[L^p \log^\alpha L(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \subset L^p \log^{-1} L(\mathbb{R}^n)\]

with continuous imbeddings

\[\|f\|_{L^p \log^{-1} L} \leq \|f\|_{L^p} \leq \|f\|_{L^p \log L} .\]

It is often useful to note that the integral expression

\[\|f\|_{L^p \log L} = \left\{ \int_{\mathbb{R}^n} |f|^{p \lambda} \log \left( e + \frac{|f|}{\lambda} \right) \, dx \right\}^{1/p} \]

is comparable with the norm ([18])

\[\|f\|_{L^p \log L} \leq \|f\|_{L^p \log L} \leq 2 \|f\|_{L^p \log L} .\]

We shall use the following elementary inequality; for all \(s, t \geq 0\), we have

\[s t \leq s \log(1 + s) + e^t - 1 .\]

We have the H"older-type inequalities

\[\|fg\|_{L^c \log^\gamma L} \leq C_{\alpha\beta}(a, b) \|f\|_{L^a \log^\alpha L} \cdot \|g\|_{L^b \log^\beta L} \]

whenever \(a, b > 1\) and \(\alpha, \beta \in \mathbb{R}\) are coupled by the relationships

\[\frac{1}{c} = \frac{1}{a} + \frac{1}{b}, \quad \frac{\gamma}{c} = \frac{\alpha}{a} + \frac{\beta}{b} .\]

We refer to [14] and [15] for a sharp version of H"older inequality in Orlicz spaces.

The following lemma (see [17]) establishes boundedness of singular integral operators in the Zygmund classes.

**Lemma 2.1.** Let \(T\) be a singular integral operator in \(\mathbb{R}^n\). For each \(1 < p < \infty\) and \(\alpha \in \mathbb{R}\), there exists a constant \(C = C(p, \alpha)\) such that

\[\| T f \|_{L^p \log^\alpha L} \leq C \| f \|_{L^p \log^\alpha L} , \quad \forall f \in L^p \log^\alpha L .\]

We shall use the following lemmas, which can be easily deduced from [16]. We include the statements for reader’s convenience.
Lemma 2.2. For $\beta \geq 0$, $\alpha \leq 0$, $p \geq 1$ with $p\beta + \alpha \geq 0$, there exists a constant $C = C(\alpha, \beta, p) > 0$ such that
\[
\left\| f \log^\beta \left( e + \frac{|f|}{\|f\|_p} \right) \right\|_{L^p \log^n L} \leq C \|f\|_{L^p \log^{p\beta + \alpha} L} \tag{14}
\]
for all $f \in L^p \log^{p\beta + \alpha} L$.

Lemma 2.3. Let $T$ be a singular integral operator in $\mathbb{R}^n$, $m \in \mathbb{N}_0$ and $\alpha + pm > 0$, $1 < p < \infty$. Then there exists a constant $C = C_p(m, \alpha)$ such that for any $f \in \text{ker } T$, we have
\[
\left\| T \left[ f \log^{m+1} \left( e + \frac{|f|}{\|f\|_p} \right) \right] \right\|_{L^p \log^n L} \leq C \|f\|_{L^p \log^{pm + \alpha} L}. \tag{15}
\]

2.2. $BMO$ and Hardy spaces. Our main source here will be [24].

For a measurable set $E$ of positive and finite measure $|E|$ and an integrable function $a$ on $E$, we denote by
\[
a_E = \int_E a(x) \, dx = \frac{1}{|E|} \int_E a(x) \, dx
\]
the mean of $a$ over $E$. A locally integrable function $a$ on $\mathbb{R}^n$ is said to have bounded mean oscillation, or to belong to $BMO$ if
\[
\|a\|_{BMO} = \sup_Q \int_Q |a(x) - a_Q| \, dx \tag{16}
\]
is finite, where $Q$ varies in the family of cubes of $\mathbb{R}^n$ with edges parallel to co-ordinate axes. Modulo constant functions, $\| \cdot \|_{BMO}$ is a norm and $BMO$ is a Banach space. Functions in $BMO$ are in $L^p_{\text{loc}}(\mathbb{R}^n)$, for any finite $p > 1$; in fact, they are locally exponentially integrable, as shown by the well-known John-Nirenberg lemma: there exists a constant $\Theta = \Theta(n)$ such that for every $a \in BMO(\mathbb{R}^n)$ and every cube $Q$, we have
\[
\int_Q \exp \left( \frac{|a(x)|}{\|a\|_{BMO}} \right) \, dx \leq 2. \tag{17}
\]

Clearly, bounded functions have bounded mean oscillation. On the contrary, $BMO$-functions need not to be bounded. The usual example is $x \mapsto \log |x|$.

As further examples, we recall that functions with distributional derivatives in the Marcinkiewicz space $L^{n, \infty}$ belong to $BMO$ and
\[
\|a\|_{BMO} \leq C(n) \|D^a\|_{n, \infty}.
\]

We also recall [7] that, if $f$ is a locally integrable function on $\mathbb{R}^n$ whose Hardy-Littlewood maximal function
\[
Mf(x) = \sup \{ |f|_Q : x \in Q \} \tag{18}
\]
is finite for a.e. $x \in \mathbb{R}^n$, then $\log Mf \in BMO(\mathbb{R}^n)$ with norm controlled by a constant $C = C(n)$ depending only on $n$.

Composition with a Lipschitz function preserves $BMO$: if $\Phi$ is a Lipschitz function, then
\[
\|\Phi(a)\|_{BMO} \leq 2L \|a\|_{BMO},
\]
where $L$ is the Lipschitz constant of $\Phi$.

We say that a $BMO$ function $a$ is in $VMO$ (or that $a$ has vanishing mean oscillation), if
\[
\lim_{\text{diam } Q \to 0} \int_Q |a(x) - a_Q| \, dx = 0, \tag{19}
\]
uniformly with respect to the cube $Q$. Obviously, uniformly continuous functions and functions of class $W^{1,n}$ belong to $VMO$.

We shall also consider functions of bounded mean oscillation on a domain $\Omega$ of $\mathbb{R}^n$. These are locally integrable functions on $\Omega$ for which the supremum at (16) is finite, the cube $Q$ being contained in $\Omega$.

Functions which are $BMO$ in a cube $\Omega$ can be easily extended to the whole space $\mathbb{R}^n$. For later reference, we give an elementary argument which provides an extension. For simplicity, we assume that $a$ is initially defined on $\Omega = [0,1]^n$. By reflection through the co-ordinate hyperplanes, we extend $a$ to $[-1,1]^n$ and then to $\mathbb{R}^n$ periodically. It is easily seen that

$$\|a\|_{BMO} \leq C(n) \|a\|_{BMO(\Omega)}.$$  

The Hardy space is defined as

$$\mathcal{H}^1(\mathbb{R}^n) = \{ g \in L^1(\mathbb{R}^n) : R_j g \in L^1(\mathbb{R}^n), j = 1, \ldots , n \}$$  

equipped with the norm

$$\|g\|_{\mathcal{H}^1} = \|g\|_1 + \sum_{j=1}^n \|R_j g\|_1,$$

where $R_j$ are the Riesz transforms.

The following result is a particular case of the $\mathcal{H}^1$-$BMO$ duality of C.Fefferman.

**Lemma 2.4.** There exists a constant $C = C(n)$ such that, if $a \in L^\infty$, $g \in \mathcal{H}^1$, then

$$\int_{\mathbb{R}^n} a g \, dx \leq C \|a\|_{BMO} \|g\|_{\mathcal{H}^1}.$$  

**Remark 1.** The essence of this estimate is that in the right hand side we have the $BMO$-norm of $a$.

In the sequel we shall need the following result.

**Lemma 2.5.** If the Hessian matrix $D^2 u$ belongs to $L^2(\mathbb{R}^n; \mathbb{R}^{n \times n})$, then its second order minors are in the Hardy space $\mathcal{H}^1$ with norms controlled by $\|D^2 u\|^2_{L^2}$.

It suffices to note that each minor can be written as a div-curl product ([20], [25]) and apply the theory of [6]. As an example,

$$\begin{vmatrix} D_{ik} u & D_{ij} u \\ D_{hk} u & D_{hj} u \end{vmatrix} = \nabla (D_k u \cdot (D_{hj} u e_i - D_{ij} u e_h)).$$

We shall also use a variant of the above lemma.

**Lemma 2.6 ([9]).** If $B \in L^2 \log L(\mathbb{R}^n; \mathbb{R}^n)$, $E \in L^2 \log^{-1} L(\mathbb{R}^n; \mathbb{R}^n)$ verify $\text{div} B = 0$ and $\text{curl} E = 0$ in the sense of distributions, then $B \cdot E \in \mathcal{H}^1(\mathbb{R}^n)$ and

$$\|B \cdot E\|_{\mathcal{H}^1} \leq C(n) \|B\|_{L^2 \log L} \|E\|_{L^2 \log^{-1} L}.$$

3. **Invertibility of the operator $L$.** We start by proving the a priori estimate (6).

By ellipticity condition (2) we have

$$|D^2 u|^2 \leq \sum_{i,j,k=1}^n a_{ij} (D_{ik} u D_{jk} u)$$

$$= \sum_{i,j,k=1}^n a_{ij} (D_{ik} u D_{jk} u - D_{kk} u D_{ij} u) + (L u)(\Delta u).$$
and hence
\[ \int_{\mathbb{R}^n} |D^2 u|^2 \leq \sum_{i,j,k=1}^n \int_{\mathbb{R}^n} a_{ij}(D_{ik}u D_{jk}u - D_{kk}u D_{ij}u) + \int_{\mathbb{R}^n} (\mathcal{L} u)(\Delta u) \tag{24} \]

We first assume that the coefficients are bounded, that is, $\mathcal{L}$ is uniformly elliptic. By the $H^1$-BMO duality we get
\[ \int_{\mathbb{R}^n} a_{ij}(D_{ik}u D_{jk}u - D_{kk}u D_{ij}u) \leq C_1(n) \|a_{ij}\|_{BMO} \|D_{ik}u D_{jk}u - D_{kk}u D_{ij}u\|_{H^1} \]
and by Lemma 2.5
\[ \sum_{i,j,k=1}^n \|D_{ik}u D_{jk}u - D_{kk}u D_{ij}u\|_{H^1} \leq C_2(n) \|D^2 u\|_2^2. \]

Therefore, from (24) we have
\[ \|D^2 u\|_2^2 \leq C_1(n) C_2(n) \|A\|_{BMO} \|D^2 u\|_2^2 + \int_{\mathbb{R}^n} (\mathcal{L} u)(\Delta u). \tag{25} \]

If $\varepsilon_0$ is any constant such that
\[ C_1(n) C_2(n) \varepsilon_0 < 1 \]
and the coefficient matrix $A$ verifies (4), then the first term in the right hand side of (25) is absorbed in the left hand side giving
\[ \|D^2 u\|_2^2 \leq C \int_{\mathbb{R}^n} (\mathcal{L} u)(\Delta u). \tag{26} \]

By Hölder inequality, this clearly implies the estimate.

Now we want to get rid of the assumption of boundedness on the coefficients. To this end, we consider the operator
\[ \mathcal{L}^\sigma = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \]
whose coefficient matrix $A^\sigma = (a_{ij}^\sigma)$ is defined for $\sigma > 0$ by
\[ A^\sigma = I + \frac{A - I}{(1 + \sigma |A - I|^2)^{1/2}}, \tag{27} \]
where $I$ is the identity matrix. The operator $\mathcal{L}^\sigma$ is clearly elliptic, as the inequality
\[ \sum_{i,j=1}^n a_{ij}^\sigma(x) \xi_i \xi_j \geq |\xi|^2 \]
holds, $\forall \xi \in \mathbb{R}^n$, for a.e. $x \in \mathbb{R}^n$. Moreover, the map $A \mapsto (1 + |A|^2)^{-1/2}A$ is Lipschitz with constant 1 on the space of matrices. As remarked in Subsection 2.2, this implies that $\|A^\sigma\|_{BMO} \leq 2 \|A\|_{BMO}$. Hence, we have the estimate for $\mathcal{L}^\sigma$:
\[ \|D^2 u\|_{L^2(\mathbb{R}^n; \mathbb{R}^{n \times n})} \leq C \| \mathcal{L}^\sigma u \|_{L^2(\mathbb{R}^n)}, \]
with $C$ independent of $\sigma$. We deduce (6) letting $\sigma \to 0$, since if $D^2 u \in L^2(\mathbb{R}^n; \mathbb{R}^{n \times n})$ and $\mathcal{L} u \in L^2(\mathbb{R}^n)$, then by Lebesgue dominated convergence theorem $\mathcal{L}^\sigma u \to \mathcal{L} u$ in $L^2(\mathbb{R}^n)$. 
Proof of Theorem 1.2. Estimate (6) clearly implies uniqueness. Actually, if $D^2u \in L^2(\mathbb{R}^n; \mathbb{R}^{n \times n})$ and $\mathcal{L} u = 0$, then (6) yields $D^2u = 0$, hence $u$ is an affine function.

Existence can be easily proved for the uniformly elliptic operator $\mathcal{L}^\sigma$, $\sigma > 0$. It follows from the case of the Laplacian, by the well known continuity method, see [13]. Let $u^\sigma$ be the solution to the equation
\[
\mathcal{L}^\sigma u^\sigma = f.
\]
Then we have the estimate
\[
\|D^2u^\sigma\|_2 \leq C \|f\|_2, \tag{28}
\]
the constant $C$ being independent of $\sigma$. Hence, there are $u$ with $D^2u \in L^2(\mathbb{R}^n; \mathbb{R}^{n \times n})$ and a subsequence $(u^\sigma)$ such that $D^2u^\sigma \rightharpoonup D^2u$ weakly in $L^2(\mathbb{R}^n; \mathbb{R}^{n \times n})$, as $\sigma \to 0$. (See [23] for more details.) To conclude, we observe that $A^\sigma \to A$ strongly in $L^p_{\text{loc}}(\mathbb{R}^n)$ for any finite $p \geq 1$, thus passing to the limit into the equation (28) we find
\[
\mathcal{L} u = f,
\]
that is, $u$ solves (3).

We end this Section showing that estimate (6) implies existence and uniqueness of solution to Dirichlet problem in a cube $\Omega$, if the coefficient matrix $A$ is diagonal.

**Theorem 3.1.** Let $A$ be a diagonal matrix verifying (2) and (4). Then, for every $f \in L^2(\Omega)$, the problem
\[
\begin{aligned}
\mathcal{L} u &= f \text{ in } \Omega \\
u &\in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)
\end{aligned} \tag{30}
\]
has a unique solution. Moreover, the following estimate holds
\[
\|D^2u\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \leq C \|f\|_{L^2(\Omega)}. \tag{31}
\]

**Proof.** For simplicity, we assume that $\Omega = [0,1]^n$ and then consider the extension of the coefficients presented in Subsection 2.2. In a similar way we extend the right hand side $f$.

We also extend $u$, first to $[-1,1]^n$ by an odd reflection through the co-ordinate hyperplanes. Indeed, it is easily seen that, if $u \in W^{2,2}([0,1]^n) \cap W^{1,2}_0([0,1]^n)$, then the function
\[
\text{sgn}(x_1 \cdots x_n) u(|x_1|, \ldots, |x_n|), \quad x = (x_1, \ldots, x_n) \in [-1,1], \tag{32}
\]
belongs to $W^{2,2}([-1,1]^n) \cap W^{1,2}_0([-1,1]^n)$. Next, we extend $u$ to $\mathbb{R}^n$ periodically; we denote by $u$ also the extended function.

Here we use that $A$ is diagonal and get
\[
|\mathcal{L} u| = |f| \quad \text{a.e. in } \mathbb{R}^n. \tag{33}
\]
For fixed $k \in \mathbb{N}$, we set $Q_k = [-k,k]^n$ and choose $\varphi \in C^\infty_0(Q_{k+1})$ such that
\[
0 \leq \varphi \leq 1, \quad \varphi \equiv 1 \text{ on } Q_k, \quad |D\varphi| \leq 2, \quad |D^2\varphi| \leq 2,
\]
and set $v = \varphi u$. Clearly $v \in W^{2,2}(\mathbb{R}^n)$ and
\[
\mathcal{L} v = \varphi f + h,
\]
where $h = 2 (ADu, D\varphi) + u \mathcal{L} \varphi \in L^2(\mathbb{R}^n)$ is supported in $Q_{k+1} \setminus Q_k$. Applying (6), we find
\[
\|D^2u\|_{L^2(Q_k; \mathbb{R}^{n \times n})} \leq C \|f\|_{L^2(Q_{k+1})} + C \|Du\|_2 + |u|_{L^2(Q_{k+1}\setminus Q_k)},
\]
with $C$ depending only on $n$, but not on $k$. Hence, due to the way we made the extension from $\Omega = [0, 1]^n$, we have

$$
\|D^2 u\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \leq C (1 + 1/k)^n \|f\|_{L^2(\Omega)} + C [(1 + 1/k)^n - 1]|\|Du\| + |u\|_{L^2(\Omega)}
$$

and letting $k \to +\infty$ we get (31). Using estimate (31) in place of (6) and arguing as in the proof of Theorem 1.2, yields the proof of Theorem 3.1.

\[ \square \]

4. The $L^2 \log^\alpha L$ regularity. Our aim in this Section is to study the problem

\[
\begin{aligned}
\mathcal{L} u &= f \text{ in } \mathbb{R}^n \\
D^2 u &= \in L^2 \log^\alpha L(\mathbb{R}^n; \mathbb{R}^{n \times n})
\end{aligned}
\tag{34}
\]

for a fixed exponent $\alpha \geq 0$. We assume that the coefficient matrix $A = A(x)$ of $\mathcal{L}$ satisfies (2) and (4) for $\varepsilon_0 > 0$ small enough. We assume also that there exist a constant matrix $A_0$ and a compact set $K$ such that $A(x) = A_0$ for a.e. $x \in \mathbb{R}^n - K$. Our main result is

**Theorem 4.1.** Under the above assumptions, problem (34) has a unique solution, modulo affine functions, for every $f \in L^2 \log^\alpha L(\mathbb{R}^n)$.

The result can be deduced as a standard consequence of an a priori estimate.

**Proposition 1.** There exists a constant $C = C(n, K, A_0, \varepsilon_0, \alpha) > 0$ such that, if $D^2 u \in L^2 \log^\alpha L(\mathbb{R}^n; \mathbb{R}^{n \times n})$ and $\mathcal{L} u \in L^2 \log^\alpha L(\mathbb{R}^n)$, then

\[
\|D^2 u\|_{L^2 \log^\alpha L(\mathbb{R}^n; \mathbb{R}^{n \times n})} \leq C \|\mathcal{L} u\|_{L^2 \log^\alpha L(\mathbb{R}^n)}.
\tag{35}
\]

**Proof.** As in Section 3, it is enough to prove the result assuming that the coefficients are bounded, with a constant in (35) independent of $\|A\|_{\infty}$. First, we consider the case $\alpha$ integer. Arguing by induction, we assume that (35) holds for a fixed $\alpha = m \in \mathbb{N}_0$, that is,

\[
\|D^2 u\|_{L^2 \log^m L(\mathbb{R}^n; \mathbb{R}^{n \times n})} \leq C \|\mathcal{L} u\|_{L^2 \log^m L(\mathbb{R}^n)},
\tag{36}
\]

and prove the estimate for $\alpha = m + 1$. By ellipticity condition (2) we have

\[
|DF|^2 \leq \sum_{i,j,k=1}^n a_{ij} (D_i F^k D_k F^j - D_k F^k D_i F^j) + (\mathcal{L} u)(\Delta u),
\tag{37}
\]

where for simplicity we denoted by $F = Du$ the gradient map. By homogeneity, we can assume that $\|DF\|_2 = 1$. Using Hodge decomposition, we write

\[
DF \log^{m+1} (e + |DF|) = DG + H,
\tag{38}
\]

where both components in the right hand side are explicitly given by means of Riesz transforms in terms of the left hand side

\[
H = \mathcal{T} [DF \log^{m+1} (e + |DF|)], \quad DG = (I - \mathcal{T}) [DF \log^{m+1} (e + |DF|)].
\]

(See [15], [16], [17] for more details.) Notice also that, by uniqueness of Hodge decomposition, we have

\[
\mathcal{T} DF = 0.
\tag{39}
\]

By boundedness of singular integral operators in Zygmund classes, see Lemma 2.1, for a constant $C = C(n, m)$ we have

\[
\|DG\|_{L^2 \log^{m-1} L} \leq C \|DF \log^{m+1} (e + |DF|)\|_{L^2 \log^{m-1} L}.
\]

Moreover, by Lemma 2.2,

\[
\|DF \log^{m+1} (e + |DF|)\|_{L^2 \log^{m-1} L} \leq C \|DF\|_{L^2 \log^{m+1} L}
\tag{40}
\]
and hence
\[
\|DG\|_{L^2 \log^{-m-1} L} \leq C \|DF\|_{L^2 \log^{m+1} L}.
\] (41)
On the other hand, in view of (39), by Lemma 2.3, we have
\[
\|H\|_{L^2 \log^{-m} L} \leq C \|DF\|_{L^2 \log^{m} L}
\] (42)
and similarly
\[
\|H\|_{L^2 \log^{-m+1} L} \leq C \|DF\|_{L^2 \log^{m+1} L}.
\] (43)
Multiplying both sides of (37) by \(\log^{m+1} (e + |DF|)\) and using (38), we end up with the pointwise inequality
\[
|DF|^2 \log^{m+1} (e + |DF|) \leq \sum_{i,j,k=1}^n a_{ij} (D_i G_k D^k F^j - D_k G^k D^j F)
\] (44)
\[+ \sum_{i,j,k=1}^n a_{ij} (H^k_i D^k F^j - H^k_i D^j F)
\] (45)
\[+ (Q u) (\Delta u) \log^{m+1} (e + |DF|).
\]
Integrating over \(\mathbb{R}^n\) we find
\[
\|DF\|_{L^2 \log^{m+1} L}^2 \leq I_1 + I_2 + I_3,
\] (46)
where
\[
I_1 = \sum_{i,j,k=1}^n \int_{\mathbb{R}^n} a_{ij} (D_i G_k D^k F^j - D_k G^k D^j F) \, dx,
\]
\[
I_2 = \sum_{i,j,k=1}^n \int_{\mathbb{R}^n} a_{ij} (H^k_i D^k F^j - H^k_i D^j F) \, dx,
\]
\[
I_3 = \int_{\mathbb{R}^n} (Q u) (\Delta u) \log^{m+1} (e + |DF|) \, dx.
\]
To estimate \(I_1\), we use Lemma 2.4 and Lemma 2.6 and find
\[
I_1 \leq C(n) \|A\|_{BMO} \|DG\|_{L^2 \log^{-m-1} L} \|DF\|_{L^2 \log^{m+1} L}
\] (47)
and recalling (41)
\[
I_1 \leq C(n) \|A\|_{BMO} \|DF\|^2_{L^2 \log^{m+1} L}.
\]
Concerning \(I_2\), we easily find
\[
I_2 \leq C(n) \int_{\mathbb{R}^n} (|A - A_0| + |A_0|) |DF| |H| \, dx.
\] (48)
Inequality (11) gives for \(N > 0\)
\[
|A - A_0| |DF| |H| \leq \frac{|DF| |H|}{N} \log \left(1 + \frac{|DF| |H|}{\|DF| |H|_1}\right)
\] (49)
\[+ \frac{|DF| |H|_1}{N} (e^{N|A - A_0|} - 1).
\]
Integrating over \(\mathbb{R}^n\) and recalling (10), we obtain
\[
\int_{\mathbb{R}^n} |A - A_0| |DF| |H| \, dx \leq \frac{2}{N} \|DF| |H| \|_{L^\log L}
\] (50)
\[+ \frac{|DF| |H|_1}{N} \int_Q (e^{N|A - A_0|} - 1) \, dx,
\]
where \( Q \) is any fixed cube of \( \mathbb{R}^n \) such that \( A(x) = A_0 \) for a.e. \( x \in \mathbb{R}^n - Q \). Comparing with John-Nirenberg inequality (17), for large \( N \) we can clearly make the last integral in (50) finite:

\[
\int_Q (e^{N|A - A_0|} - 1) \, dx < +\infty.
\]

On the other hand, using Hölder inequality (12) and (43), we get

\[
\|DF| |H|\|_{L^2 \log \lambda - 1} \leq C \|DF\|_{L^2 \log \lambda + 1} \|H\|_{L^2 \log - \lambda} \leq C \|DF\|_{L^2 \log \lambda + 1} \tag{51}
\]

and in a similar way, using (42),

\[
\|DF| |H|\|_1 \leq C \|DF\|_{L^2 \log \lambda} \|H\|_{L^2 \log - \lambda} \leq C \|DF\|_{L^2 \log \lambda} \tag{52}
\]

By (48), (50), (51) and (52), we get

\[
I_2 \leq \frac{C}{N} \|DF\|_{L^2 \log \lambda + 1}^2 + C |A_0| \|DF\|_{L^2 \log \lambda}^2.
\]

Using our induction assumption (36), and (10), we find

\[
I_2 \leq C \|\mathcal{L} u\|_{L^2 \log \lambda + 1}^2 + \frac{C}{N} \|DF\|_{L^2 \log \lambda + 1}^2 \tag{53}
\]

For \( I_3 \), with the aid of Hölder inequality (12) and of (40),

\[
I_3 \leq \int_{\mathbb{R}^n} |\mathcal{L} u| |DF| \log^{m+1} (e + |DF|) \, dx
\]

\[
\leq C \|\mathcal{L} u\|_{L^2 \log \lambda + 1} \|DF\| \log^{m+1} (e + |DF|) \|DF\|_{L^2 \log - \lambda} \leq C \|\mathcal{L} u\|_{L^2 \log \lambda + 1} \|DF\|_{L^2 \log \lambda} \tag{54}
\]

Putting together the above estimates (47), (53) and (54) for \( I_1, I_2 \) and \( I_3 \), we deduce

\[
\|DF\|_{L^2 \log \lambda + 1} \leq C (\|A\|_{BMO} + 1/N) \|DF\|_{L^2 \log \lambda}^2 + C \|\mathcal{L} u\|_{L^2 \log \lambda}^2 \tag{55}
\]

As \( \|A\|_{BMO} \) is small, we can take \( N \) large and the first term in the right hand side can be absorbed in the left hand side, hence

\[
\|DF\|_{L^2 \log \lambda}^2 \leq C \|\mathcal{L} u\|_{L^2 \log \lambda}^2 \|DF\|_{L^2 \log \lambda}^2
\]

which clearly implies estimate (35) with \( \alpha = m + 1. \) The case of a general \( \alpha \) follows by interpolation.

**Remark 2.** For the estimate (35), we can dispense with the assumption that \( A \) is constant outside a compact set if \( u \) has compact support, the constant then depending on the support.

We deduce the following regularity result.
Corollary 1. If $D^2 u \in L^2(\mathbb{R}^n; \mathbb{R}^{n \times n})$ and $\mathcal{L} u \in L^2 \log^\alpha L(\mathbb{R}^n)$, $\alpha > 0$, then $D^2 u \in L^2 \log^\alpha L(\mathbb{R}^n; \mathbb{R}^{n \times n})$ and
\[
\|D^2 u\|_{L^2 \log^\alpha L} \leq C \|\mathcal{L} u\|_{L^2 \log^\alpha L}.
\] (56)

5. Local solutions. In this Section, we prove Theorem 1.1.

Fix a cube $Q \subset 2Q \subset \Omega$ and a test function $\varphi \in C_0^\infty(2Q)$ satisfying
\[
0 \leq \varphi \leq 1 \quad \text{on} \ 2Q, \quad \varphi \equiv 1 \quad \text{on} \ Q,
\]

\[
|D\varphi| \leq \frac{1}{\text{diam} \ Q}, \quad |D^2\varphi| \leq \frac{1}{(\text{diam} \ Q)^2}.
\]

We may assume that
\[
u_{2Q} = (D_1 u)|_{2Q} = \cdots = (D_n u)|_{2Q} = 0,
\] (57) otherwise we replace $u$ by $u - p$, for a suitable polynomial $p$ of degree at most 1. We define $v = \varphi u$. Hence $v \in W^{2,2}(\mathbb{R}^n)$ and the partial derivatives of $v$ are computed as
\[
Dv = \varphi Du + u D\varphi; \quad D^2 v = \varphi D^2 u + D\varphi \otimes Du + Du \otimes D\varphi + u D^2 \varphi.
\]

Without loss of generality, we can assume that the matrix $A$ is defined on the entire space $\mathbb{R}^n$. Clearly
\[
\mathcal{L} v = \varphi \mathcal{L} u + 2 \langle ADu, D\varphi \rangle + u \mathcal{L} \varphi = \varphi f + h,
\]
where we set $h = 2 \langle ADu, D\varphi \rangle + u \mathcal{L} \varphi$. Since $\varphi$ has compact support and the coefficients $a_{ij}$ are locally integrable with any finite exponent, by Sobolev embedding theorem we clearly see that the function $\varphi f + h$ belongs to $L^2 \log^\alpha L(\mathbb{R}^n)$. Indeed, we have
\[
|h| \leq |A| \left(\frac{|Du|}{\text{diam} \ Q} + \frac{|u|}{(\text{diam} \ Q)^2}\right)
\]
and thus, in view of (57), by Poincaré inequality,
\[
\|h\|_{L^2 \log^\alpha L(\mathbb{R}^n)} \leq C(u, \alpha) \|D^2 u\|_{L^2(2Q; \mathbb{R}^{n \times n})}.
\] (58)

Therefore, by Corollary 1, we get $D^2 v \in L^2 \log^\alpha L(\mathbb{R}^n; \mathbb{R}^{n \times n})$ and
\[
\|D^2 v\|_{L^2 \log^\alpha L} \leq C \|\varphi f + h\|_{L^2 \log^\alpha L}.
\] (59)

Hence $D^2 u \in L^2 \log^\alpha L(Q; \mathbb{R}^{n \times n})$ and estimate (5) follows from (59) and (58).

Now we discuss briefly the case of VMO coefficients.

Corollary 2. Assume that the coefficients of $\mathcal{L}$ belong to VMO. For every $u \in W^{2,2}_0(\Omega)$ solution to (3), if $f \in L^2 \log^\alpha L_{\text{loc}}(\Omega)$, $\alpha > 0$, then actually
\[
D^2 u \in L^2 \log^\alpha L_{\text{loc}}(\Omega; \mathbb{R}^{n \times n})
\]
and for any cube $Q \subset 2Q \subset \Omega$ estimate (5) holds.

Clearly, it is enough to prove (5) for cubes $Q$ of sufficiently small diameter. By definition, for fixed $\varepsilon > 0$, we find $\delta > 0$ such that
\[
\text{diam} \ Q < \delta \quad \Rightarrow \quad \int_Q |a(x) - a_Q| \, dx < \varepsilon,
\]
see (19). We now fix a cube $Q$ such that $\text{diam} \ Q < \delta/2$, restrict the coefficient matrix $A$ to $2Q$, and then extend it to the entire space as indicated in Subsection 2.2. This
provides us with a new coefficient matrix $\tilde{A}$, which coincides with $A$ on $2Q$, and has globally small $BMO$-norm. Applying Theorem 1.1 to $\tilde{A}$ yields the result.

We conclude this Section by presenting an example of an operator satisfying our assumptions. Let us consider the matrix field $A = (a_{ij})$ defined on $\mathbb{R}^n$ as follows:

$$a_{nn}(x) = 1 + \lambda \log^+ \frac{1}{|x|}$$

and, for $(i, j) \neq (n, n)$,

$$a_{ij} = \delta_{ij}.$$

Clearly, $A(x) = I$ the identity matrix for $|x| > 1$, and ellipticity condition (2) holds if $\lambda > 0$. Cordes’ condition is not satisfied. Moreover, the entries of $A$ belong to $BMO$ and $\|A\|_{BMO} = C(n)\lambda$, hence for small $\lambda$ assumption (4) holds. Finally, we remark that $a_{nn} \notin L^\infty \cup VMO$.

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