Four-fermion operators at dimension 6: dispersion relations and UV completions

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Abstract

A major task in phenomenology today is constraining the parameter space of SMEFT and constructing models of fundamental physics that the SM derives from. To this effect, we report an exhaustive list of sum rules for 4-fermion operators of dimension 6, connecting low energy Wilson coefficients to cross sections in the UV. Unlike their dimension 8 counterparts which are amenable to a positivity bound, the discussion here is more involved due to the weaker convergence and indefinite signs of the dispersion integrals. We illustrate this by providing examples with weakly coupled UV completions leading to opposite signs of the Wilson coefficients for both convergent and non convergent dispersion integrals. We further decompose dispersion integrals under weak isospin and color groups which lead to a tighter relation between IR measurements and UV models. These sum rules can become an effective tool for constructing consistent UV completions for SMEFT following the prospective measurement of these Wilson coefficients.
1 Introduction

Testing the Standard Model (SM) and searching for new physics are two essential goals of the current and future experimental programs in particle physics. In this respect, all of the measurements can be classified as low energy (SM scale) and high energy experiments. For low energy observables, the Standard Model Effective Field Theory (SMEFT) provides an excellent tool to consistently parameterize new physical perturbations, classified order by order in the form of non-renormalizable operators with higher dimensions. We expect new physics to kick in above at least the weak scale, and as we approach the regime of high energies greater than this scale, the applicability of EFT techniques becomes successively questionable. Reliable calculations then require a discussion of the explicit UV completions, and thus it’s clear that the connection between UV and IR observables and predictions becomes somewhat model dependent, and explicit matching is required to infer useful information. In this direction, dispersion...
relations provide a model independent way to connect low and high energy measurements, in the form of *sum rules* for low energy Wilson coefficients and high energy cross sections. This provides a consistent way to match the known and measurable low energy, and speculative high energy quantities (for a recent reappraisal see [1] and for a textbook introduction [2, 3]). Their power lies in their generality - they follow from the simple and sacred physical requirements of Poincare invariance, unitary and locality. Recently there has been significant attention directed toward the application of the dispersion relations and sum rules for SMEFT [4, 5, 6, 7]. For the four fermion interactions most of the effort so far has been focused on the dimension 8 operators ([8, 9, 10, 11]) where the sum rules lead to positivity constraints on the Wilson coefficients in a model independent way.

On the other hand, from a phenomenological point of view, dimension eight operators are very hard to measure at experiments; and most likely the new physics will demonstrate itself first via dimension six corrections to the SM. Thus, it becomes crucial to understand similar dispersion relations for the dimension six operators. The situation here is drastically different from the dimension 8 discussion because the relevant dispersion integral, aside from being possibly non-convergent, is of indefinite sign and doesn’t admit any simple model independent positivity bound. However, the situation is far from hopeless, and the dispersion relations turn out to be instructive in a different way: instead of being viewed as a constraint on Wilson coefficients, these sum rules are to be used as a *tool to constrain the UV completions* of these operators, given signs to be measured in the IR. Therefore, in a way, we are approaching the IR-UV relationship from the opposite standpoint to what is customary. Our emphasis is on model building for a full theory by taking IR measurements as our input, instead of trying to predict these measurements from general inputs from the UV theory. We will show that different signs of the Wilson coefficients will be related to the dominance of the particle collision cross sections in the various channels, and decompose these cross sections as explicitly as possible to indicate the quantum numbers of initial states with dominant cross sections. Moreover, it is crucial to emphasize that sum rules can only be written down for a subspace of the dimension 6 basis, namely the effective 4 fermion operators that can generate forward amplitudes. Based on these sum rules, we will report examples of the weakly coupled UV completions which can lead to either sign of the Wilson coefficients. Such information, which we believe was not consistently summarized before, can become a useful guide for the future measurements in case some of the Wilson coefficients are discovered to be non zero. These measurements, supplemented with the sum rules we derive, will bring us closer to an understanding of the fundamental physics which the SMEFT derives from.

The manuscript is organized as follows: in section 2 we briefly review dispersion integrals. In section 3, we study in detail the operator $(\bar{e}_R \gamma^\mu e_R)^2$ and illustrate the relation between UV completions and signs of the effective operator at tree level and 1 loop. In section 4, we present the whole set of the four fermion operators and identify which of them can be constrained by the dispersion relations. Results are summarized in the section 5. Most details of the calculations have been relegated to the appendices.

## 2 Review of dispersion relations

In this section, we will review dispersion relations and their applications to constraints on EFTs following the discussion in [1, 12, 4, 13] (readers familiar with the formalism can proceed directly to section 3). It is a general principle that the non-analyticities associated with scattering amplitudes have a physical origin in the form of poles and branch cuts arising from localized particle states, and thresholds. The positivity of the spectral function in the Kallen-Lehmann decomposition generalizes to more general cross sections, which can be related to elastic forward scattering amplitudes via a dispersion integral, to be reviewed in a moment. What this means
in an EFT context is that, in perturbation theory, one can evaluate the 2 sides of a dispersion integral to a certain order; allowing us to extract information about the effective IR coupling that contributes to that amplitude at low energies on one side of the relation, from general observations about the UV piece of the dispersion relation without any explicit matching.

While unitarity reflects in the positivity of the spectral function and cross sections, we need additional information about the high energy behaviour of the amplitude to control the dispersion integral at the infinite contour. The asymptotics of amplitudes at high energies is a question about the unitarity and locality of the theory. The famous Froissart bound-while technically proved only for theories with a mass gap, but believed to hold true generally-tells us that the behaviour of the amplitude \( A(s) \) is such that \( A(s)/s^2 \to 0 \) as \( s \to \infty \) ([14, 15, 16]). This, in general allows us to write down a dispersion relation with 2 subtractions, i.e. a linear polynomial of the form \( a(t) + b(t)s \) supplemented by a contour integral picking up the nonanalytic structure of the amplitude. \( a(t), b(t) \) cannot be determined by unitarity alone, but the nonanalytic structure can be related to manifestly positive cross sections via the optical theorem. We can then differentiate this relation w.r.t \( s \) twice to get rid of the unknown subtractions, and we’re left with a manifestly positive integral on the right, and the coefficient of \( s^2 \) in \( A(s) \) on the left-therby leading to what are conventionally called ‘positivity bounds’ [1] on EFT parameters.

This prescription, however, cannot be directly applied to dimension 6 operators. Their contribution to 2 \( \to \) 2 amplitudes scales as \( p^2 \), and so \( d^2A(s)/ds^2 \) kills information about their couplings, and we cannot constrain them in any way. The best we can do is to look at \( dA(0)/ds \), and be left with a dispersion integral of indefinite sign as well as an undetermined subtraction constant (which we’ll call \( C_\infty \), as it captures the pole of the amplitude at infinity).

Let us briefly derive this dispersion relation from first principles. Consider a process \( ab \to ab \) with the amplitude \( A_{ab \to ab} \equiv A_{ab}(s,t) \), and in the forward limit \( (t \to 0) \). This amplitude can be expanded as

\[
A_{ab}(s,0) = \sum_n c_n(\mu^2)(s - \mu^2)^n, \quad c_n(\mu^2) = \frac{1}{n!} \frac{\partial^n}{\partial s^n} A_{ab}(s,0)|_{s=\mu^2} \tag{1}
\]

about some arbitrary reference scale \( \mu^2 \) where the amplitude is analytic. We can now use Cauchy’s theorem to write

\[
\frac{1}{2\pi i} \int_{\gamma} ds \frac{A_{ab}(s,0)}{(s - \mu^2)^n+1} = \sum_{s_i, \mu^2} \text{Res} A_{ab}(s,0) \frac{A_{ab}(s,0)}{(s - \mu^2)^n+1} = c_n(\mu^2) + \sum_{s_i} \text{Res} \frac{A_{ab}(s,0)}{(s - \mu^2)^n+1}, \tag{2}
\]

where \( s_i \) are the physical poles associated with IR stable resonance exchanges in the scattering, and the contour of integration is shown on the Fig. 1. The residues at physical poles are IR structures that we will drop henceforth. This can always be done if the scale \( \mu \) is chosen such that \( \mu^2 \gg m_{IR}^2 \), where \( m_{IR}^2 \) corresponds to the scale of the \( s_i \) poles. Indeed, the last term in Eq.2 gives corrections of the order \( \mathcal{O}(m_{IR}^2/\mu^2) \), which can be safely ignored.

The analytic structure of the amplitude allows to decompose the integral as a sum of the contributions along the branch cuts and over infinite circle, so that schematically

\[
\frac{1}{2\pi i} \int_{\gamma} ds \frac{A_{ab}(s,0)}{(s - \mu^2)^n+1} = \text{integrals along cuts} + \text{integral on big circle} = C^m_\infty + I_n
\]

\[
C^m_\infty = \int_0^{2\pi} d\theta \frac{A_{ab}(|s_\lambda|e^{i\theta},0)}{2\pi (|s_\lambda|e^{i\theta} - \mu^2)^n+1} \cdot (|s_\lambda|e^{i\theta}) \tag{3}
\]

The integration over the branch cuts can be written as a sum of the integrals over discontinuities
Figure 1: Analytic structure in complex s plane. The infinite circle is centered at $2m^2$, and will be traversed counterclockwise.

$$2\pi i I_n = \int_{4m^2}^{\infty} \left( \frac{A_{ab}(s + i\epsilon, 0) - A_{ab}(s - i\epsilon, 0)}{(s - \mu^2)^{n+1}} + (-1)^n \frac{A_{ab}(4m^2 - s - i\epsilon, 0) - A_{ab}(4m^2 - s + i\epsilon, 0)}{(s - 4m^2 + \mu^2)^{n+1}} \right).$$

Since $4m^2 - s = u$ for $t = 0$, the second term is just the $u$ channel crossed amplitude for the process $a\bar{b} \rightarrow a\bar{b}$ i.e. $A_{ab}$(instead of $ab \rightarrow ab$). Using the optical theorem, we can rewrite the discontinuity in terms of cross section and in the limit $m \rightarrow 0$, and $\mu \rightarrow 0$ we obtain:

$$I_n = \int \frac{ds}{\pi s^n} \left( \sigma_{ab} + (-1)^n \sigma_{a\bar{b}} \right).$$

(5)

For dimension six operators, we will be interested in dispersion relations of Eq. 2 for the case $n = 1$.

$$c_1(\mu^2) = \int \frac{ds}{\pi s} \left( \sigma_{ab} - \sigma_{a\bar{b}} \right) + C_{(n=1)}.$$  

(6)

Note that the quantity $c_n(\mu^2)$ on the left hand side can be evaluated in IR using the EFT expansion. This introduces an additional source of corrections of the order $\mathcal{O}(\mu^2/\Lambda^2)$, where $\Lambda$ is the scale suppressing higher dimensional operators. We can see that the dispersion relations are valid up to corrections of the order $\mathcal{O}(m_{IR}^2/\mu^2, \mu^2/\Lambda^2)$, and these can be ignored if $\Lambda^2 \gg \mu^2 \gg m_{IR}^2$.

At last, let us mention that the forward limit $t \rightarrow 0$ must be taken with care, and is in principle problematic in the presence of massless particles propagating in the t-channel of the UV amplitude (see for example [4, 13]). In fact, we always have the usual SM Coulomb singularities that lead to the bad behaviour in the forward limit. The way out of this problem is by using IR mass regulators to match the known SM contributions to both sides of the dispersion relation, and subtract them away.

1Crossing relations for particles with spin become more nontrivial (see for example [17, 18]). However, in the case of the massless spin 1/2 particles, which are the interest of this paper, the usual crossing relations for the forward amplitude remain valid [17] and we will not worry about these issues in the rest of the paper.
3 Warm up exercise

We consider the simplest case of a fully right handed operator which is made up of singlet fields \( e_R \), all of the same generation (the dispersion relation for this operator was presented in [6] too),

\[ c_{RR}(\bar{e}_R \gamma \mu e_R)(\bar{e}_R \gamma \mu e_R) . \]  

(7)

Following the strategy outlined in the previous section, we start by considering the amplitude \( A_{\bar{e}e} \) and derive the following dispersion relation

\[ \left. \frac{dA_{\bar{e}e}(s,0)}{ds} \right|_{s=0} = \int \frac{ds}{\pi s} (\sigma_{eR\bar{e}R} - \sigma_{eReR}) + C_\infty , \]  

(8)

where we have omitted the \((n=1)\) subscript for \( C_\infty \). The amplitude in the IR \((s \to 0)\) limit can be safely calculated using the EFT and we find (we use helicity amplitudes; for notations and for the explicit conventions see appendix A):

\[ A_{\bar{e}e}(s,t) = c_{RR} \cdot 2([2\gamma_1\mu][3\gamma_4\mu] - [3\gamma_1\mu][2\gamma_4\mu]) = -8c_{RR}[23] , \]  

(9)

so that we arrive at the following sum rule for the \( c_{RR} \) Wilson coefficient\(^2\)

\[ -8c_{RR} = \int \frac{ds}{\pi s} (\sigma_{eR\bar{e}R} - \sigma_{eReR}) + C_\infty \]  

(10)

Let us see how this equation can be used as guidance for UV completions that lead to the possible signs of the \( c_{RR} \) Wilson coefficient.

3.1 Charge neutral vector exchange

Let us start with the negative sign for \( c_{RR} \). The dispersion relation predicts that this will be generated by the models with resonances in \( e\bar{e} \) channel (apart from the \( C_\infty \) contribution). The simplest model which can enhance the \( \sigma_{e\bar{e}} \) cross section is a simple \( Z' \) with the interaction

\[ \mathcal{L}_{Z'} = \lambda Z'_\mu \bar{e}_R e_R. \]  

(11)

Integrating \( Z' \) at tree level we obtain for the Wilson coefficient

\[ c_{RR} = -\frac{\lambda^2}{2M_{Z'}^2} , \]  

(12)

where the sign follows the prediction of the dispersion relations. However, inspecting the amplitudes carefully, we see that the massive vector exchange in the \( t \)-channel spoils the convergence of the amplitude in the forward region, making the integral over the infinite circle non-vanishing. To this end, let us look at the amplitude \( A_{eReR} \) in detail-

\[ iA = -\lambda^2 \left( [2\gamma_4\mu][3\gamma_1\mu] - [3\gamma_4\mu][2\gamma_1\mu] \right) \left( \frac{-i}{s-M_{Z'}^2} - \frac{-i}{t-M_{Z'}^2} \right) \]  

\[ A(s,t) = -2\lambda^2[23] \left( \frac{1}{s-M_{Z'}^2} + \frac{1}{t-M_{Z'}^2} \right) . \]  

(13)

\(^2\)In this expression we should take the value of the Wilson coefficient at the scale \( \mu \to 0 \). The RGE evolution of the Wilson coefficients from the EFT cut off scale to \( \mu \) can lead to the modification of the Eq.10 (see [19] for a recent discussion). In this paper we will assume that these running effects are subleading and can be safely ignored.
In the forward limit, this amplitude goes as

$$A(s,t)|_{t \to 0} = -2\lambda^2 s \left( \frac{1}{s - M_{Z'}^2} + \frac{1}{-M_{Z'}^2} \right).$$

(14)

We can see that the integral over infinite contour becomes non zero and is equal to

$$C_\infty^{(Z')} = \frac{2\lambda^2}{M_{Z'}^2}.$$  

(15)

We see that even though the contribution from the infinite contour is non-zero, it turns out of the same sign and size as the cross section part of the dispersion relation

$$\left[ \int \frac{ds}{\pi s} (\sigma_{\text{e}e\bar{R}} - \sigma_{\text{e}R\bar{e}}) \right]^{(Z')} = \frac{2\lambda^2}{M_{Z'}^2},$$

(16)

(see appendix B for details of the calculation). The fact that exchange of the elementary vector boson spoils the convergence of the amplitude in the forward limit at large $s$ is not new and was observed for example in [4] in the discussion of the other dimension six operators.

Let us extend the discussion for the operators with two fermion flavours. For example $c_{\text{e}\mu}(\bar{e}_R\gamma^\mu e_R)(\bar{\mu}_R\gamma^\mu \mu_R)$ contributes $e\bar{\mu} \to e\bar{\mu}$ in the IR. This operator can be generated by two kinds of UV completions with a charge neutral vector boson-

$$\mathcal{L}_{UV}^{(1)} = \lambda Z_{(1)}^\mu (\bar{e}_R\gamma^\mu e_R + h.c.) \quad \mathcal{L}_{UV}^{(2)} = (\lambda_1 Z_{(2)}^\mu \bar{e}_R\gamma^\mu e_R + \lambda_2 Z_{(2)}^\mu \bar{\mu}_R\gamma^\mu \mu_R)$$

(17)

The analysis in both cases is very similar to the single flavour discussion; however, in the first case ($\mathcal{L}_{UV}^{(1)}$) the integral over infinite contour vanishes, since there is no amplitude with $Z_{(1)}$ in the $t$-channel. Writing down the dispersion relations for the $e\bar{\mu} \to e\bar{\mu}$ scattering we will obtain (note that there is a different numerical prefactor compared to Eq.10 due to combinatorics):

$$c_{\text{e}\mu} = -\frac{1}{2} \left[ \int \frac{ds}{\pi s} (\sigma_{\text{e}R\bar{R}} - \sigma_{\text{R}e\bar{R}}) \right] = -\frac{|\lambda|^2}{M_{Z'}^2}.$$  

(18)

In the second case ($\mathcal{L}_{UV}^{(2)}$), we are in the opposite situation since both cross sections $\sigma_{e\bar{\mu}(\mu)} = 0$ vanish at leading order in perturbation theory. However there is a forward amplitude for this process, which comes from $t$-channel diagram and it contributes only to $C_\infty$. In other words, the pole at infinity saturates the dispersion relation, and even though no corresponding UV cross section can be measured to constrain this coefficient, it can be nonzero because of this pole. In fact, a simple calculation yields

$$c_{\text{e}\mu} = -\frac{C_\infty}{2} = -\frac{\lambda_1\lambda_2}{M_{Z'}^2}$$

(19)

which can be either positive or negative depending on the values of the $\lambda_1,\lambda_2$ couplings. Let us continue with our examination of the UV completions for the various signs of the $c_{RR}$

3.2 Charge two scalar

What about the positive sign of $c_{RR}$? The dispersion relation in Eq. 10 predicts that this happens for UV completions that generate only $\sigma_{ee}$ cross section. The simplest possibility is a charge two scalar with the interaction

$$\mathcal{L} = \kappa \phi \bar{e}_R e_R + h.c.$$  

(20)
Then at the order $O(\kappa^2)$, only $\sigma_{ee}$ will be non-vanishing, so the Wilson coefficient must be positive. Indeed, integrating out the scalar field at tree level gives

$$c_{RR} = \frac{|\kappa|^2}{2M^2}$$

which is manifestly positive. In this case the forward amplitude converges quickly enough, so that $C_\infty = 0$—this is just the statement that a scalar cannot be exchanged in the $t$ channel when opposite helicity fermions are scattered. We see that the both signs of the Wilson coefficient are possible with a weakly coupled UV completion. One can still wonder whether the negative sign of the $c_{RR}$ interactions in the Eq. 12 is related to the $t$—channel pole and non-convergence of the amplitude in the UV. To quell any doubts, in the next subsection we will build a weakly coupled UV completion without new vector bosons and with convergent forward amplitudes.

### 3.3 UV completion at 1-loop

Let us extend the SM with vector-like fermion $\Psi$ of charge 1 and a charge $-2$ complex scalar $\phi$ with a Yukawa interaction

$$L = |D\phi|^2 + i\bar{\Psi} D\Psi + M_\psi \bar{\Psi} \Psi - M^2_\phi |\phi|^2 + y\bar{\psi} R \phi. \tag{22}$$

This generates an effective operator at the order $O(y^4)$, and at this order the only cross section available is $\sigma_{ee}$. The dispersion relation predicts that the Wilson coefficient must be negative. Moreover, $C_\infty = 0$ here as the amplitude scales slowly enough with $s$. Indeed, integrating out heavy fields at one loop we obtain

$$c_{RR} = -\frac{|y|^4}{128\pi^2 M_\psi M_\phi} f(x), \quad x \equiv \frac{M_\psi}{M_\phi}$$

$$f(x) = \frac{(x + 4x^3 \log x - x^5)}{(1 - x^2)^3}, \quad \lim_{x \to 1} f(x) = 1/3 \tag{23}$$

where one can see that the function $f(x)$ is always positive. See appendix B for explicit verification of the dispersion integral in the case $M_\psi = M_\phi$.

In summary, this warm up exercise shows us that both signs of the Wilson coefficients are possible within weakly coupled theories. Contribution of the infinite contours is important for the $t$-channel exchange of the vector resonances. Interestingly, both signs of the Wilson coefficient are possible even for the weakly coupled models with vanishing $C_\infty$.

In the following, we will derive the set of the dispersion relations for the whole set of four fermion operators and identify the UV completions leading to the various signs of the Wilson coefficients.

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3This result for the Wilson coefficients contradicts the findings of the Ref.[20], where the only possible sign of the Wilson coefficient was found to be positive.
4 Four fermion operators

First of all, let us define a complete basis of the four fermion operators, and we will do this following the notations of the Ref. [21, [22]:

purely left-handed

\[
O_{il}^{ijkm} = \left( \bar{\ell}_L \gamma_\mu \mu_\nu \ell_L \right) \left( \bar{\ell}_L \gamma_\mu \mu_\nu \ell_L \right), \quad O_{qq}^{(1)ijkm} = \left( \bar{q}_L \gamma_\mu \mu_\nu q_L \right) \left( \bar{q}_L \gamma_\mu \mu_\nu q_L \right),
\]
\[
O_{qq}^{(3)ijkm} = \left( \bar{q}_L \gamma_\mu \sigma_\sigma q_L \right) \left( \bar{q}_L \gamma_\mu \sigma_\sigma q_L \right), \quad O_{ql}^{(1)ijkm} = \left( \bar{\ell}_L \gamma_\mu \mu_\nu \ell_L \right) \left( \bar{q}_L \gamma_\mu \mu_\nu q_L \right),
\]
\[
O_{cl}^{(3)ijkm} = \left( \bar{\ell}_L \gamma_\mu \sigma_\sigma \ell_L \right) \left( \bar{q}_L \gamma_\mu \sigma_\sigma q_L \right),
\]

purely right-handed

\[
O_{iijk}^{ijkm} = \left( \bar{e}_R \gamma_\mu e_R \right) \left( \bar{e}_R \gamma_\mu e_R \right), \quad O_{iijk}^{(2)ijkm} = \left( \bar{u}_R \gamma_\mu u_R \right) \left( \bar{u}_R \gamma_\mu u_R \right),
\]
\[
O_{dd}^{ijkm} = \left( \bar{d}_R \gamma_\mu d_R \right) \left( \bar{d}_R \gamma_\mu d_R \right), \quad O_{ed}^{ijkm} = \left( \bar{e}_R \gamma_\mu e_R \right) \left( \bar{d}_R \gamma_\mu d_R \right),
\]
\[
O_{iijk}^{(2)ijkm} = \left( \bar{d}_R \gamma_\mu d_R \right) \left( \bar{d}_R \gamma_\mu d_R \right), \quad O_{ed}^{ijkm} = \left( \bar{e}_R \gamma_\mu e_R \right) \left( \bar{d}_R \gamma_\mu d_R \right),
\]

left-right

\[
O_{le}^{ijkm} = \left( \bar{\ell}_L \gamma_\mu \ell_L \right) \left( \bar{e}_R \gamma_\mu e_R \right), \quad O_{qee}^{ijkm} = \left( \bar{q}_L \gamma_\mu \ell_L \right) \left( \bar{e}_R \gamma_\mu e_R \right),
\]
\[
O_{lu}^{ijkm} = \left( \bar{l}_L \gamma_\mu \ell_L \right) \left( \bar{u}_R \gamma_\mu u_R \right), \quad O_{had}^{ijkm} = \left( \bar{l}_L \gamma_\mu \ell_L \right) \left( \bar{d}_R \gamma_\mu d_R \right),
\]
\[
O_{qle}^{ijkm} = \left( \bar{q}_L \gamma_\mu \ell_L \right) \left( \bar{u}_R \gamma_\mu u_R \right), \quad O_{qee}^{ijkm} = \left( \bar{q}_L \gamma_\mu \ell_L \right) \left( \bar{e}_R \gamma_\mu e_R \right),
\]
\[
O_{iijk}^{(1)ijkm} = \left( \bar{q}_L \gamma_\mu \ell_L \right) \left( \bar{u}_R \gamma_\mu u_R \right), \quad O_{iijk}^{(2)ijkm} = \left( \bar{q}_L \gamma_\mu \ell_L \right) \left( \bar{d}_R \gamma_\mu d_R \right),
\]
\[
O_{iijk}^{(1)ijkm} = \left( \bar{q}_L \gamma_\mu \ell_L \right) \left( \bar{u}_R \gamma_\mu u_R \right), \quad O_{iijk}^{(2)ijkm} = \left( \bar{q}_L \gamma_\mu \ell_L \right) \left( \bar{d}_R \gamma_\mu d_R \right),
\]
\[
O_{iijk}^{(1)ijkm} = \left( \bar{q}_L \gamma_\mu \ell_L \right) \left( \bar{u}_R \gamma_\mu u_R \right), \quad O_{iijk}^{(2)ijkm} = \left( \bar{q}_L \gamma_\mu \ell_L \right) \left( \bar{d}_R \gamma_\mu d_R \right),
\]

baryon number violating

\[
O_{duq} = \epsilon_{ABC} \left( \bar{d}_R A_{u_R} B \right) \left( \bar{q}_L C \right) \left( \bar{e}_R 2 \ell_L \right), \quad O_{quu} = \epsilon_{ABC} \left( \bar{q}_L A_{u_R} B \right) \left( \bar{e}_R 2 \ell_L \right) \left( \bar{e}_R 2 \ell_L \right),
\]
\[
O_{duu} = \epsilon_{ABC} \left( \bar{d}_R A_{u_R} B \right) \left( \bar{u}_R C \right) \left( \bar{e}_R 2 \ell_L \right), \quad O_{qee} \epsilon_{ABC} \left( \bar{q}_L A_{u_R} B \right) \left( \bar{q}_L C \right) \left( \bar{e}_R 2 \ell_L \right) \left( \bar{e}_R 2 \ell_L \right). \quad (27)
\]

The rest of the possible operators can be reduced via some Fierzing to the basis of Eq.24-25-26, using the completeness relations for the SU(2) and SU(3) generators

\[
\sum_{a=1}^{3} (\sigma^a)_{ij} (\sigma^a)_{kl} = 2 \left( \delta_{il} \delta_{kj} - \frac{1}{2} \delta_{ij} \delta_{kl} \right),
\]

\[
\sum_{A=1}^{8} (T^A)_{ij} (T^A)_{kl} = 2 \left( \delta_{il} \delta_{kj} - \frac{1}{3} \delta_{ij} \delta_{kl} \right),
\]

As we have seen in the previous section, the dispersion relations are effective in the case of forward scattering i.e. when the initial and final states are the same 4. Therefore, only the following subspace of operators can be subject to sum rules -

\[
O_{il}^{ijjj}, O_{l}^{ijjj}, O_{qq}^{(1)ijjj}, O_{qq}^{(1)ijjj}, O_{ql}^{(1)ijjj}, O_{ql}^{(1)ijjj}, O_{q}^{(1)ijjj}, O_{q}^{(1)ijjj}, O_{ql}^{(1)ijjj}, O_{ql}^{(1)ijjj},
\]

4Recently it was shown that the scattering of the mixed(entangled) flavour states can lead to the additional constraints in the case of the dimension eight operators [8, 9], where strict positivity bounds can be applied. In the case of dimension six operators the measurements of the cross sections for the mixed states looks almost impossible, so we do not investigate this direction further.
which will be the focus of this paper. For fully right-handed operators, the discussion follows
closely the results reported for $O_{ee}$ above. Therefore, we will henceforth report only the results
and the examples of UV completions leading to various signs.

4.1 Experimental constraints

Having defined the operators which we will consider in our discussion, let us briefly mention
the status of the experimental bounds based on the discussion in [23, 24]. Current bounds on
on four lepton and two lepton two quark operator come from the combinations of the $Z,W$
pole observables, fermion production at LEP, low energy neutrino scatterings, parity violating
electron scatterings, and parity violation in atoms. One of the challenges in deriving these
bounds comes from the modifications of $W,Z$ vertices which too can contribute to the same
low energy observables, so that the global fit including the $W,Z$ pole observables becomes
necessary. For example, for two lepton two quark operators Ref. [24] has found nine flat
directions unbounded experimentally. Current combinations of the low energy experimental
constraints as well as LHC measurements bound the various Wilson coefficients in the range
$10^{-2} - 10^{-3}$ (where the operators are assumed to be suppressed by the $v_{ew}^2$ scale), which means
sensitivity to the scales $O$(few TeV). Just to be specific, for example the four-electron operator
discussed in Eq. 7 is bounded by the Bhabha scattering measurements at LEP-2[25] and SLAC
E158 experiment for the Møller scattering ($e^-e^- \rightarrow e^-e^-$)[26], where both experiments are
testing the complementary combinations of the Wilson coefficients leading to the net sensitivity
of $\sim 4 \times 10^{-3}v_{ew}^{-2}$ on the value of the Wilson coefficient. LHC measurements of the dilepton
production in $pp$ scattering leads to additional strong constraints on the two quark-two lepton
operators [27, 28], where for some operators we will become sensitive to new physics up to the
scale of $\sim 50$ TeV. So far, all of the measurements are consistent with SM predictions.

4.2 FULLY RIGHT HANDED

4.2.1 $O_{ee}$

This operator has already been discussed in the section 3 and we would just like to emphasize
that there are no sum rules for more than two flavours of fermions. Following the notations of
Eq. 24-25 the dispersion relations can be summarized as:

\[-8c_{iii}^{ee} = \int \frac{ds}{\pi s} \left( \sigma_{i,\bar{e}_{i}} - \sigma_{e_{i}i} \right) + C_{\infty} \tag{30}\]

\[-2(c_{e}^{jj})_{i \neq j} = \int \frac{ds}{\pi s} \left( \sigma_{j,e_{j}} - \sigma_{e_{j}j} \right) + C_{\infty} \tag{31}\]

Note that in this simple case where the fields are singlets, the operators $O_{e}^{jj}$ and $O_{e}^{ij}$
are identical after Fierzing; and $O_{e}^{ij}$ and $O_{e}^{ji}$ are just trivially identical by symmetrization, so we
report the dispersion relation only in terms of $c_{e}^{jj}$ in order to not double-count. Summarising
the discussion about UV completions in the section 3 we have:

$c_{ee} < 0$ : neutral $Z'$ at tree level; Vectorlike singlet fermion $\Psi$ and a heavy singlet complex
scalar $\Phi$ with $Q[\Phi\Psi] = -1$ at 1 loop.

$c_{ee} > 0$ : Charge 2 scalar; for different flavours ($O_{e}^{ij}$)$_{i \neq j}$, $Z'$ can lead to a possibly positive
sign as well if the couplings to the different flavours of the fermions are of opposite signs (see
Eq.19)
4.2.2 $O_{uu}, O_{dd}$

Let us proceed with our investigation of the four fermion quark operators. The discussion proceeds exactly in the same way as for the leptons, except for new color structure. Fierzing them into the basis of Eq.24-25 there are only six structures of the operators $O_{uu,dd}^{ijij}, O_{uu,dd}^{ijij}$, which are in this case not related by a Fierz identity because of an implicit contraction of color indices. Let us start with the operators where all of the quarks have the same hypercharge, and focus on the operator $O_{uu}^{iii}$. Denoting by $\alpha, \beta$ the color indices and considering same and different color scatterings, we will obtain the following relations:

\[-8c_{uu}^{iii} = \int \frac{ds}{\pi s} (\sigma_{ua,ua} - \sigma_{ua,ua}) + C_{uu}^{(6)} = \int \frac{ds}{\pi s} \left( \frac{2\sigma_{a\bar{u}}^{(8)} + \sigma_{u\bar{u}}^{(1)}}{3} - \sigma_{uu}^{(6)} \right) - C_{uu}^{(6)} \]

\[-4c_{uu}^{iii} = [\int \frac{ds}{\pi s} (\sigma_{ua,u\bar{a}} - \sigma_{ua,u\bar{a}}) + C_{uu}^{(3)}]_{\alpha \neq \beta} = \int \frac{ds}{\pi s} \left( \sigma_{u\bar{u}}^{(8)} - \frac{\sigma_{uu}^{(3)} + \sigma_{uu}^{(6)}}{2} \right) + C_{uu}^{(6)} . \quad (32)\]

In the last step, we have decomposed the various possibilities of the initial state fermions in terms of the $SU(3)$ QCD representations. This is convenient, since the Wigner-Eckart theorem requires the amplitudes to remain the same for all of the components of the irreducible representation. In particular, for the quark antiquark scattering the initial state will always be decomposed as a singlet and octet of $SU(3)$. Even though measuring $\sigma^{(8)}$ and $\sigma^{(1)}$ independently at collider experiment looks practically impossible, such dispersion relations can become very useful for model building if the non-zero values of the Wilson coefficients are found. Note that we can calculate the integral over the infinite contour using amplitude $A_{u\bar{u}}$ or its crossed version $A_{u\bar{u}}$ and the values of this integrals will satisfy (see appendix C for details):

\[-C_{uu}^{(6)}(6) = \frac{2C_{uu}^{(8)}(6)}{3} + \frac{C_{uu}^{(1)}(6)}{3} \]

\[-C_{uu}^{(6)}(6) + C_{uu}^{(3)}(6) = C_{uu}^{(8)}(6) . \quad (33)\]

Re-expressing everything in terms of the color averaged cross section we will obtain

\[-\frac{16}{3} c_{uu}^{iii} = \int \frac{ds}{\pi s} (\sigma_{u\bar{u}} - \sigma_{uu}) - \frac{1}{3} C_{uu}^{(6)}(6) + \frac{2}{3} C_{uu}^{(8)}(6) . \quad (34)\]

Again, $C_{uu}^{(6)}$ can be non-vanishing, for example, in UV models with charge neutral vector resonances exchange in the $t$ channel, but unlike the four electron case here this resonance can be either singlet or octet of $SU(3)$QCD. Extending this analysis to the case of different flavour of the up quarks we will obtain:

\[-2c_{uu}^{ijij} = \int \frac{ds}{\pi s} \left( \frac{2\sigma_{u\bar{u}}^{(8)} + \sigma_{u\bar{u}}^{(1)}}{3} - \sigma_{uu}^{(6)} \right) - C_{uu}^{(6)}(6) \]

\[-2(c_{uu}^{ijij} + c_{uu}^{ijij}) = \int \frac{ds}{\pi s} \left( \sigma_{u\bar{u}}^{(8)} - \frac{\sigma_{uu}^{(3)} + \sigma_{uu}^{(6)}}{2} \right) + C_{uu}^{(6)}(6) . \quad (35)\]

We again mention that the operators $O_{uu}^{ijij}$ and $O_{uu}^{ijij}$ (similarly for $O_{uu}^{ijij}$ and $O_{uu}^{ijij}$) are trivially identical, so it’s important that we don’t double count them. As before, expressing everything in terms of uncolored cross sections, we find

\[-2c_{uu}^{ijij} - \frac{2}{3} c_{uu}^{ijij} = \int \frac{ds}{\pi s} \left( \sigma_{u\bar{u}} - \sigma_{uu} \right) + \frac{8}{9} C_{uu}^{(8)}(6) + \frac{1}{9} C_{uu}^{(1)}(6) . \quad (36)\]
and exactly the same relations hold for the down quarks.

Let us look at the possible UV completions. In the case of $c_{\nu \nu}$, we will have a negative sign of the Wilson coefficient with $Z'$, and a positive sign for the charge $-4/3$ scalar. Similar to the lepton case, we can generate a negative Wilson coefficient by adding vectorlike fermions and a complex scalar with $Q[\Phi \Psi] = 2/3$ and $(\Phi \psi)$ — fundamental of QCD. The discussion of two fermion flavours is almost identical to the lepton case.

To demonstrate an explicit verification of these sum rules, in Appendix (B.2), we provide an example of a UV-completion of the type $gV_A^\mu(\bar{u}_i \gamma^\mu T^A u_j)$. This is a flavor diagonal interaction with a color octet vector and a universal coupling; where the sum rule for the Wilson coefficients is saturated by the pole at infinity since no leading order cross sections are available.

### 4.2.3 $O_{ud}^{(1), (8)}$

Just as in the previous section, we obtain (we will omit here flavour indices as these do not play any role, since of the two up quarks and two down quarks should be the same to form sum rules)

$$-2(c_{ud}^{(1)} - \frac{1}{6} c_{ud}^{(8)} + \frac{1}{2} c_{ud}^{(8)}) = \int \frac{ds}{\pi s} \left( \frac{2\sigma^{(8)}_{ud} + \sigma^{(1)}_{ud}}{3} - \sigma^{(6)}_{ud} \right) + \frac{1}{3} (2C^{ud(8)}_\infty + C^{ud(1)}_\infty)$$  

$$-2(c_{ud}^{(1)} - \frac{1}{6} c_{ud}^{(8)}) = \int \frac{ds}{\pi s} \left( \sigma^{(8)}_{ud} - \frac{\sigma^{(3)}_{ud} + \sigma^{(6)}_{ud}}{2} \right) + C^{ud(8)}_\infty$$  

(37)

Rewriting the result in terms of uncolored cross section, we will obtain

$$-2c_{ud}^{(1)} = \int \frac{ds}{\pi s} (\sigma_{ud} - \sigma_{ud}) + \frac{8}{9} C^{ud(8)}_\infty + \frac{1}{9} C^{ud(1)}_\infty$$  

(38)

Interestingly, we see that no constraints can be obtained for $c_{ud}^{(8)}$ if we don’t have precise information about the color structure of the initial state. Experiments which are sensitive only to the total scattering cross section will be blind to $c_{ud}^{(8)}$.

### 4.2.4 $O_{eu}, O_{ed}$

The only operators with sum rule are of the form

$$(\bar{e}_R^i \gamma^\mu e_R^j)(\bar{u}^i_{RJa} \gamma^\mu u^j_{RJa})$$,  

(39)

where no summation over $i, j$ is assumed. The sum rule is identical one for both $u$ and $d$ quarks and is given by:

$$-2c_{eu}^{ijj} = \int \frac{ds}{\pi s} \left( \sigma_{e_{u},j} - \sigma_{e_{d},i} \right) + C_{\infty}^{eu} \text{ and } u \leftrightarrow d.$$  

(40)

UV completions are as before, with a positive sign for $u(d)$ coming from a charge $1/3(4/3)$ scalar and a negative sign from a charge $5/3(2/3)$ vector field $V$, note that the amplitude is convergent in the forward limit and the infinite integrals do vanish). Neutral charge $Z'$ can lead to the arbitrary sign of the Wilson coefficient; again, in this case the dispersion relations are saturated by the integrals at infinity.

### 4.3 SUM RULES FOR EW DOUBLETS

In the next 2 subsections we study operators that contribute to doublet-singlet scattering.
4.3.1 $O_{le}, O_{lu}, O_{ld}, O_{qe}$

Let us start with the fully leptonic operator and study the forward scattering of $l^p q$ where $p = 1, 2$ is the isospin, in which case the sum rules are of the form

$$-2c_{le}^{ij} = \int \frac{ds}{\pi s} \left( \sigma_{i^p e^j_{R}}^{u} - \sigma_{i^p e^j_{R}}^{d} \right) + C_{\infty}^{l, e_j}$$

$$= \int \frac{ds}{\pi s} \left( \sigma_{i^p e^j_{R}}^{u} - \sigma_{i^p e^j_{R}}^{d} \right) + C_{\infty}^{l, e_j}$$

Similarly, we can write down the sum rules for the the quark lepton operators-

$$-2c_{lu}^{ij} = \int \frac{ds}{\pi s} \left( \sigma_{i^p u^j_{R}}^{u} - \sigma_{i^p u^j_{R}}^{d} \right) + C_{\infty}^{l, u_j} \quad \text{and} \quad u \leftrightarrow d$$

$$-2c_{qe}^{ij} = \int \frac{ds}{\pi s} \left( \sigma_{q^p e^j} - \sigma_{q^p e^j} \right) + C_{\infty}^{q, e_j},$$

where again $p$ stand for the $SU(2)_L$ index. Note that these sum rules hold true for any isospin for the lepton and any color of the quark.

4.3.2 $O_{qd}^{(1),(8)}, O_{qd}^{(1),(8)}$

In this case, the discussion follows closely the one for the quark singlets, and so we arrive at two sum rules (we again suppress the flavour index for brevity)

$$-2(c_{qd}^{(1)} - \frac{1}{6} c_{qd}^{(8)} + \frac{2}{3} c_{qd}^{(8)}) = \int \frac{ds}{\pi s} \left( \frac{2\sigma_q^{(8)}}{3} + \frac{c_{qd}^{(1)}}{3} - \sigma_q^{(6)} \right) + \frac{1}{3} (2c_{qd}^{(8)} + C_{\infty}^{qd}(1))$$

$$-2(c_{qd}^{(1)} - \frac{1}{6} c_{qd}^{(8)}) = \int \frac{ds}{\pi s} \left( \sigma_q^{(8)} - \frac{c_{qd}^{(1)}}{2} \right) + C_{\infty}^{qd}(8).$$

Note that $\sigma_q$ stands for $\sigma_{q^p}$ where $p$ is a $SU(2)$ index and cross sections on the right hand side of the Eq.43 can be taken for any component of the quark doublet. Rewriting the result in terms of uncolored cross section, we will obtain

$$-2c_{qd}^{(1)} = \int \frac{ds}{\pi s} \left( \sigma_{q^d}^{(8)} - \sigma_{qd}^{(8)} \right) + \frac{8}{9} C_{\infty}^{qd}(8) + \frac{1}{9} C_{\infty}^{qd}(1).$$

Finally, we now study the left handed operators that contribute to doublet-doublet scattering, where the doublet is that of weak isospin.

4.3.3 $O_{ll}$

Let us start with the four lepton operator $O_{ll}^{(ij,jj)}$. Expanding in components, the following sum rules can be derived (we assume $i \neq j$ and we do not write the operators obtained by interchange of $i \leftrightarrow j$ which are identical, just as in the discussion for up quarks; see Eq. 75)

$$-2c_{ll}^{ij} - 2c_{ll}^{jj} = \int \frac{ds}{\pi s} \left( \sigma_{i^e e^j} - \sigma_{i^e e^j} \right) + \int \frac{ds}{\pi s} \left( \sigma_{i^e e^j} - \sigma_{i^e e^j} \right) + C_{\infty}^{e, e^j}$$

$$-2c_{ll}^{ij} = \int \frac{ds}{\pi s} \left( \sigma_{i^e e^j} - \sigma_{i^e e^j} \right) + C_{\infty}^{e, e^j}. \quad (45)$$
We can decompose the amplitude into the weak isospin amplitudes (see appendix C for details) to obtain the following dispersion relations

\[-2c^{ijj}_{ll} - 2c^{ijji}_{ll} = \int \frac{ds}{\pi s} \left[ \frac{1}{2} \left( \sigma^{(1)}_{ij} + \sigma^{(3)}_{ij} \right) - \sigma^{(3)}_{ij} \right] - C^{ijj}_{\infty} \]

\[-2c^{ijj}_{ll} = \int \frac{ds}{\pi s} \left[ \sigma^{(3)}_{ij} - \frac{1}{2} \left( \sigma^{(3)}_{ij} + \sigma^{(1)}_{ij} \right) \right] + C^{ijj}_{\infty} \] (46)

where \((i, j)\) and \((i, \bar{j})\) refer to the leptons from \(l_i, l_j (\bar{l}_j)\) doublets and \(\sigma^{(3,1)}_{ij,(ij)}\) refers to cross section from the triplet and singlet initial state formed by \(ij\) or \(i\bar{j}\). In the case of an operator formed by just one lepton family, we will obtain:

\[-8c_{ll} = \int \frac{ds}{\pi s} \left[ \sigma_{e\bar{e}(\nu\bar{\nu})} - \sigma_{e\bar{e}(\nu\bar{\nu})} \right] + C^{ee}_{\infty} = \int \frac{ds}{\pi s} \left[ \frac{1}{2} \left( \sigma^{(1)}_{ll} + \sigma^{(3)}_{ll} \right) - \sigma^{(3)}_{ll} \right] - C^{ll(3)}_{\infty} \]

\[-4c_{ll} = \int \frac{ds}{\pi s} \left[ \sigma_{e\bar{e}(\nu\bar{\nu})} + \sigma_{e\bar{e}(\nu\bar{\nu})} \right] + C^{ee}_{\infty} = \int \frac{ds}{\pi s} \left[ \sigma^{(3)}_{ll} - \frac{1}{2} \left( \sigma^{(3)}_{ll} + \sigma^{(1)}_{ll} \right) \right] + C^{ll(3)}_{\infty} \] (47)

4.3.4 \(O^{(3,1)}_{lq}\)

In this case, only the operators with \(iijj\) flavour structure can contribute; and we arrive at the following dispersion relations

\[-2c^{(1)}_{lq} - 2c^{(3)}_{lq} = \int \frac{ds}{\pi s} \left[ \sigma_{ed(\nu\bar{\nu})} - \sigma_{ed(\nu\bar{\nu})} \right] + C^{ed(\nu\bar{\nu})}_{\infty} \]

\[-2c^{(1)}_{lq} + 2c^{(3)}_{lq} = \int \frac{ds}{\pi s} \left[ \sigma_{ed(\nu\bar{\nu})} + \sigma_{ed(\nu\bar{\nu})} \right] + C^{ed(\nu\bar{\nu})}_{\infty} \] (48)

As before, decomposing cross section under isospin we will obtain

\[-2c^{(1)}_{lq} - 2c^{(3)}_{lq} = \int \frac{ds}{\pi s} \left[ \frac{1}{2} \left( \sigma^{(1)}_{lq} + \sigma^{(3)}_{lq} \right) - \sigma^{(3)}_{lq} \right] - C^{lq(3)}_{\infty} \]

\[-2c^{(1)}_{lq} + 2c^{(3)}_{lq} = \int \frac{ds}{\pi s} \left[ \sigma^{(3)}_{lq} - \frac{1}{2} \left( \sigma^{(1)}_{lq} + \sigma^{(3)}_{lq} \right) \right] + C^{lq(3)}_{\infty} \] (49)

4.3.5 \(O^{(3)}_{qq}\) and \(O^{(3)}_{qq}\)

Let us start with one family, in terms of the octet and singlet cross sections,

\[-8 \left( c^{(1)}_{qq} + c^{(3)}_{qq} \right) = \int \frac{ds}{\pi s} \left[ \frac{2\sigma^{(8)}_{uu} + \sigma^{(1)}_{uu}}{3} - \sigma^{(6)}_{uu} \right] - C^{(6)uu}_{\infty} \]

\[-4 \left( c^{(1)}_{qq} + c^{(3)}_{qq} \right) = \int \frac{ds}{\pi s} \left[ \sigma^{(8)}_{uu} - \frac{\sigma^{3}_{uu} + \sigma^{(6)}_{uu}}{2} \right] + C^{(8)uu}_{\infty} \]

\[-4 \left( c^{(1)}_{qq} + c^{(3)}_{qq} \right) = \int \frac{ds}{\pi s} \left[ \frac{2\sigma^{(8)}_{ud} + \sigma^{(1)}_{ud}}{3} - \sigma^{(6)}_{ud} \right] - C^{ud(6)}_{\infty} \]

\[-4 \left( c^{(1)}_{qq} + c^{(3)}_{qq} \right) = \int \frac{ds}{\pi s} \left[ \sigma^{(8)}_{ud} - \frac{\sigma^{(6)}_{ud} + \sigma^{(3)}_{ud}}{2} \right] + C^{ud(8)}_{\infty} \] (50)
We can proceed further by performing the double decomposition in terms of the $SU(2)_L$ multiplets using the relations
\[
\sigma_{u\bar{u}} = \frac{1}{2} \left( \sigma_{qq}^{(1)} + \sigma_{qq}^{(3)} \right), \quad \sigma_{ud} = \frac{1}{2} \left( \sigma_{qq}^{(1)} + \sigma_{qq}^{(3)} \right).
\]

(51)

Then we will obtain (the first index will refer now to QCD multiplet and the second one to electroweak).
\[
-8 \left( c_{qq}^{(1)} + c_{qq}^{(3)} \right) = \int \frac{ds}{\pi s} \left[ \frac{1}{6} \left( 2\sigma_{qq}^{(8,1)} + \sigma_{qq}^{(1,1)} + 2\sigma_{qq}^{(8,3)} + \sigma_{qq}^{(1,3)} \right) - \sigma_{qq}^{(6,3)} \right] - C_{qq\infty}^{(6,3)}
\]
\[
-4 \left( c_{qq}^{(1)} + c_{qq}^{(3)} \right) = \int \frac{ds}{\pi s} \left[ \frac{1}{2} \left( \sigma_{qq}^{(8,1)} + \sigma_{qq}^{(3,3)} \right) - \frac{1}{2} \left( \sigma_{qq}^{(3,3)} + \sigma_{qq}^{(3,6)} \right) \right] + C_{qq\infty}^{(8,1)} + C_{qq\infty}^{(8,3)} \frac{1}{2}
\]
\[
-4 \left( c_{qq}^{(1)} + c_{qq}^{(3)} \right) = \int \frac{ds}{\pi s} \left[ \frac{1}{3} \left( 2\sigma_{qq}^{(8,3)} + \sigma_{qq}^{(1,3)} \right) - \frac{1}{2} \left( \sigma_{qq}^{(6,1)} + \sigma_{qq}^{(6,3)} \right) \right] - C_{qq\infty}^{(6,1)} + C_{qq\infty}^{(6,3)} \frac{1}{2}
\]
\[
-4 \left( c_{qq}^{(1)} - c_{qq}^{(3)} \right) = \int \frac{ds}{\pi s} \left[ \sigma_{qq}^{(8,3)} - \frac{1}{4} \left( \sigma_{qq}^{(3,1)} + \sigma_{qq}^{(3,6)} + \sigma_{qq}^{(3,3)} + \sigma_{qq}^{(3,6)} \right) \right] + C_{qq\infty}^{(8,3)}
\]

(52)

In terms of the color averaged cross sections,
\[
\frac{16}{3} \left( c_{qq}^{(1)} + c_{qq}^{(3)} \right) = \int \frac{ds}{\pi s} \left[ \frac{1}{2} \sigma_{qq}^{(3)} + \frac{1}{2} \sigma_{qq}^{(1)} - \sigma_{qq}^{(3)} \right] - \frac{C_{qq\infty}^{(6,3)}}{3} + \frac{C_{qq\infty}^{(8,1)}}{3} + \frac{C_{qq\infty}^{(8,3)}}{3}
\]
\[
-4 \left( c_{qq}^{(1)} - c_{qq}^{(3)} \right) = \int \frac{ds}{\pi s} \left[ \frac{1}{2} \sigma_{qq}^{(3)} - \frac{1}{2} \sigma_{qq}^{(1)} + \frac{1}{2} \sigma_{qq}^{(3)} \right] - \frac{C_{qq\infty}^{(6,1)}}{6} + \frac{C_{qq\infty}^{(6,3)}}{6} + \frac{2C_{qq\infty}^{(8,3)}}{3}
\]

(53)

In the case of two flavours, the disperion relations become:
\[
-2 \left( c_{qq}^{ijj} + c_{qq}^{iijj} + c_{qq}^{(3)iijj} + c_{qq}^{(3)iijj} \right) = \int \frac{ds}{\pi s} \left[ \frac{1}{6} \left( 2\sigma_{qq}^{(8,1)} + \sigma_{qq}^{(1,1)} + 2\sigma_{qq}^{(8,3)} + \sigma_{qq}^{(1,3)} \right) - \sigma_{qq}^{(6,3)} \right] - C_{qq\infty}^{(6,3)}
\]
\[
-2 \left( c_{qq}^{ijj} + c_{qq}^{iijj} \right) = \int \frac{ds}{\pi s} \left[ \frac{1}{2} \left( \sigma_{qq}^{(8,1)} + \sigma_{qq}^{(3,3)} \right) - \frac{1}{2} \left( \sigma_{qq}^{(3,3)} + \sigma_{qq}^{(3,6)} \right) \right] + C_{qq\infty}^{(8,1)} + C_{qq\infty}^{(8,3)} \frac{1}{2}
\]
\[
-2 \left( c_{qq}^{ijj} - c_{qq}^{(3)iijj} + 2c_{qq}^{(3)iijj} \right) = \int \frac{ds}{\pi s} \left[ \frac{1}{3} \left( 2\sigma_{qq}^{(8,3)} + \sigma_{qq}^{(1,3)} \right) - \frac{1}{2} \left( \sigma_{qq}^{(6,1)} + \sigma_{qq}^{(6,3)} \right) \right] - C_{qq\infty}^{(6,1)} + C_{qq\infty}^{(6,3)} \frac{1}{2}
\]
\[
-2 \left( c_{qq}^{ijj} - c_{qq}^{(3)iijj} \right) = \int \frac{ds}{\pi s} \left[ \sigma_{qq}^{(8,3)} - \frac{1}{4} \left( \sigma_{qq}^{(3,1)} + \sigma_{qq}^{(3,6)} + \sigma_{qq}^{(3,3)} + \sigma_{qq}^{(3,6)} \right) \right] - C_{qq\infty}^{(8,3)}
\]

(54)

The power of these relations allows to understand immediately the signs of the Wilson coefficients in the various UV completions. For example, for a scalar diquark which is in $(\bar{6}, 1, -1/3)$ representation under $SU(3) \times SU(2) \times U(1)_Y$ we will get:
\[
c_{qq,6}^{ijj} = c_{qq,6}^{(3)iijj} = -c_{qq,6}^{(3)iijj} = -c_{qq,6}^{ijj} > 0.
\]

(55)

Similarly, for a scalar diquark which is in $(3, 1, -1/3)$ will get:
\[
c_{qq,3}^{ijj} = c_{qq,3}^{ijj} = -c_{qq,3}^{(3)iijj} = -c_{qq,3}^{(3)iijj} > 0.
\]

(56)
Finally, we can sum and report these sum rules in terms of color averaged cross sections, which yield 2 equations depending on whether the initial and final state form $SU(2)_L$ triplets or singlets.

$$-2(c_{qq}^{iijj} + c_{qq}^{(3)iijj} + \frac{1}{3}c_{qq}^{jjii} + \frac{1}{3}c_{qq}^{(3)jjii}) = \int \frac{ds}{\pi s} \left[ \frac{\sigma_{q\bar{q}}^{(3)} + \sigma_{q\bar{q}}^{(1)}}{2} - \sigma_{q\bar{q}}^{(3)} \right] - \frac{C_{qq}^{(6,3)} + C_{qq\infty}^{(8,1)}}{3}$$

$$-2(c_{qq}^{iijj} - c_{qq}^{(3)iijj} + \frac{2}{3}c_{qq}^{jjii}) = \int \frac{ds}{\pi s} \left[ \sigma_{q\bar{q}}^{(3)} - \frac{\sigma_{q\bar{q}}^{(1)} + \sigma_{q\bar{q}}^{(3)}}{2} \right] - \frac{C_{qq\infty}^{(6,1)} + C_{qq\infty}^{(6,3)}}{6} + \frac{2C_{qq\infty}^{(8,3)}}{3}.$$  

\(57\)

5 Summary

In this work, we explored the sum rules for four-fermion operators at dimension six level. As expected, the convergence of the dispersion integrals leading to the dimension six Wilson coefficients is not guaranteed, and in particular is spoiled by the t-channel exchange of the vector bosons. This additional feature can modify the predictions of the dispersion relations for sign and strength of IR interactions, and for some UV completions the value of the Wilson coefficients can be even saturated by the pole at infinity. However we find that this ambiguity of IR couplings is not related to the (non)convergence of the dispersion integrals and as an example, we have constructed, in addition to tree level, 1-loop weakly coupled models (see section 3.3) where both signs become available even when the integral over the infinite circle vanishes.

We presented forward dispersion relations for all possible four-fermion dimension six operators. To facilitate the connection between the values of the Wilson coefficients and new physics scenarios, we have performed the decomposition in terms of the $SU(2)$ and $SU(3)$ multiplets. Such relations predict in a model independent way processes with enhanced cross section in the case of discoveries in low energy experiments. We carefully indicate all the relevant quantum numbers of the quantities involved in our dispersion relations in order to provide a convenient dictionary for future measurements, where the precise structure of initial states is often unavailable. This can have interesting consequences; for example, Eq.38 tells us that measuring uncoloured cross sections in the UV clouds any information about $c_{ud,(qu),(qd)}^{(8)}$ Wilson coefficients, despite it contributing formally to sum rules with fixed initial colours.

We emphasize that these sum rules are to be interpreted as a model independent link between UV and IR measurements, as opposed to the usual positivity bounds. Even though less constraining on the EFT parameter space, these relations can instead be used as a powerful tool for model building to unearth the underlying, fundamental physics that is to be explored in the coming years.

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A Massless spinor helicity conventions

We will briefly summarize the key results relevant to us (for a pedagogical introduction see [29]) in the (−, −, −, −) signature (we will follow the conventions discussed in [30, 31, 32]). We have the 2 component spinors $v_{L/R}, u_{L/R}$ and their barred versions. They are related by crossing symmetry, $u_{L/R} = v_{R/L}, \bar{u}_{L/R} = \bar{v}_{R/L}$. It is important to realize that for antiparticles, the spinor has opposite handedness to the field that describes it. For instance, a right chiral field $e_R$ has an antiparticle which has the spinor $v_L$, while the particle carries the spinor $u_R$. In other words, both $u_R, v_L$ correspond to a right chiral field; whereas $v_R, u_L$ correspond to a left chiral field. To be absolutely clear, we will just refer to the handedness of the relevant spinor as opposed to the helicity of a particle/antiparticle wherever necessary.

Operationally, we will assign the brackets

$$\bar{v}_L = \bar{u}_R \equiv [, \bar{v}_R = \bar{u}_L \equiv ), v_L = u_R \equiv ], v_R = u_L \equiv].$$

The inner product is antisymmetric-as is expected for grassman-valued quantities-

$$\langle pq \rangle = -\langle qp \rangle \quad [pq] = -[qp]$$

Note that this also means that $\langle pp \rangle = 0 = [pp]$. Mixed brackets vanish. The formalism encodes a lot of power-for example, it tells us that a $\langle$ and $]$ type spinor cannot occur at a vertex unless there’s a $\gamma^\mu$ involved-a vector connects opposite helicity particles. Similarly, same helicity spinors making up a vertex indicate a scalar is involved.

We will not insist on taking all momenta ingoing/outgoing; in our calculations, the momenta labelled 1, 2 are always incoming and 3, 4 are always outgoing. We can freely work with negative momenta via the standard analytic continuation-

$$| - p \rangle = i|p \rangle \quad | - p] = i|p]$$

These brackets satisfy the property

$$\langle 1|\gamma^\mu2 \rangle = [2|\gamma^\mu1 \rangle$$

Furthermore, we have

$$[i|\gamma\mu|i] = 2p_i \quad \langle ij][ij] = -2p_i \cdot p_j = (p_i - p_j)^2$$

since $p_i^2 = 0$ for massless spinors. We therefore have our mandelstam variables-

$$s = 2p_1 \cdot p_2 = -[12] \langle 12 \rangle \quad t = -2p_3 \cdot p_1 = [13] \langle 13 \rangle \quad u = -2p_4 \cdot p_1 = [14] \langle 14 \rangle$$

Finally, we have the all important Fierz rearrangement-

$$[1|\gamma^\mu2 \rangle[3|\gamma\mu4 \rangle = -2[13] \langle 24 \rangle$$

B Details about cross sections and loop amplitudes

In this appendix we will give details about explicit verification of the dispersion relations presented in the text for various models.
Figure 2: Fermion statistics dictate these diagrams subtract

**B.1 Z' at tree level**

Let us start with neutral vector $Z'$ coupled to right-handed current via $\lambda Z'_\mu \bar{e}_R \gamma^\mu e_R$. It generates $e_R e_R$ scattering through the diagrams. The full amplitude will be given by

$$A_{e\bar{e}} = -2\lambda^2 [14]\langle 23 \rangle \left( \frac{1}{s-m^2} + \frac{1}{t-m^2} \right)$$

(65)

Matching the IR and UV amplitudes at low energies we will obtain

$$-8c_{ee}^{1111} [14] \langle 23 \rangle = -2\lambda^2 [14] \langle 23 \rangle \left( \frac{1}{-m^2} + \frac{1}{-m^2} \right) \implies c_{ee}^{1111} = -\frac{\lambda^2}{2m^2}$$

(66)

Let us verify that this is consistent with our dispersion relation. With a vector $Z'$ at order $O(\lambda^2)$ in perturbation theory we have $\sigma_{e\bar{e}} \neq 0$ and $\sigma_{ee} = 0$. To calculate the cross sections, note that by the optical theorem, we have

$$\text{Im}(e \bar{e} \rightarrow e \bar{e}) = s\sigma_{e\bar{e}}^{tot}$$

(67)

We use the fact that $\text{Im}\left(\frac{1}{p^2-m^2+i\epsilon}\right) = -\pi\delta(p^2-m^2)$ which, when substituted in the amplitude (14) gives us

$$\text{Im}(e^+_Le^-_R \rightarrow e^+_Le^-_R) = 2\lambda^2 \pi s\delta(s-m^2)$$

(68)

Starting from dispersion relation in Eq.10 we will get,

$$-8c_{ee}^{1111} = \int \frac{ds}{\pi s} (\sigma_{e\bar{e}} - 0) + C_\infty = \int \frac{ds}{\pi s^2} \text{Im}(e\bar{e} \rightarrow e\bar{e}) + C_\infty = \frac{2\lambda^2}{m^2} + C_\infty.$$

(69)

Calculating explicitly $C_\infty$ we will obtain:

$$C_\infty = \int_0^{2\pi} d\theta A(|s_\Lambda|e^{i\theta}, 0) \cdot (|s_\Lambda|e^{i\theta}) = \frac{2\lambda^2}{m^2}$$

$$A|_{t \rightarrow 0} = -2\lambda^2 s \left( \frac{1}{s-m^2} + \frac{1}{t-m^2} \right)$$

(70)

Which is of the same sign as the dispersion integral, and therefore we find

$$-8c_{ee}^{1111} = 4\lambda^2/m^2 \implies c_{ee}^{1111} = -\lambda^2/2m^2$$

(71)

as claimed in (12), and our dispersion relation is explicitly verified.
B.2 Integrating out color octet

Very similarly to the charge neutral $Z'$ we can consider effects coming from integrating out color octet $V$ which has zero electric charge. Let us look for example on octet interacting with right-handed up quark current:

$$g_{ij}V^A_{\mu}(\bar{u}_i\gamma^\mu T^A u_j) \implies c_{ijkl}^u = \frac{-g_{kj}g_{il}}{3M_V^2} + \frac{g_{ij}g_{kl}}{3M_V^2}. \quad (72)$$

Let us assume that the octet couplings are universal and flavour diagonal, then $g_{ij} = g\delta_{ij}$, and the Wilson coefficients are equal to

$$c_{iijj}^{uu} = \frac{2}{3}g^2 \frac{M^2_V}{3M^2_V}, \quad c_{ijji}^{uu} = -\frac{2}{3}g^2 \frac{M^2_V}{3M^2_V}. \quad (73)$$

Now let us look at dispersion relations for $i \neq j$, then similar to the discussion in Eq. 19, the cross sections will vanish at $O(g^2)$ and the right hand side of Eq. 75 will be controlled by the contribution of the integrals over infinite contours.

$$C^{(8)}_\infty = 8g^2 \frac{M^2_V}{3M^2_V}, \quad C^{aa}_\infty = -C^{(6)}_\infty = -\frac{4g^2}{3M^2_V} \quad (74)$$

which confirm the dispersion relations

$$-2c_{ijjj}^{uu} = C^{aa}_{\infty, (6)}$$
$$-2(c_{ijjj}^{uu} + c_{ijji}^{uu}) = C^{aa}_{\infty, (8)}. \quad (75)$$

B.3 Charge 2 scalar at tree level

Let us build a model where only $\sigma_{ee(\bar{e} \bar{e})}$ is present at the lowest order in perturbation theory. This can be done with a charge (2) scalar, which interacts as follows $(\lambda \phi \bar{e}_R e^c_R + h.c)$ where the $c$ subscript stands for charge conjugation. Matching the amplitudes in EFT and UV theory we will obtain

$$-8c_{ee}^{1111}[14]\langle 23 \rangle = -2!2!\lambda^2 \frac{[14]\langle 32 \rangle}{-m^2} \implies c_{ee}^{1111} = \frac{\lambda^2}{2m^2}. \quad (76)$$

Then the scattering cross section is equal to:

$$\sigma_{ee}^{tot} = 4\lambda^2 \pi \delta(s - m^2). \quad (77)$$

So that dispersion relation becomes:

$$-8c_{ee}^{1111} = \int \left(0 - \frac{ds}{\pi s} \sigma_{++}\right) = -\frac{4\lambda^2}{m^2} \quad (78)$$

and as expected we find: $c_{ee}^{1111} = +\frac{\lambda^2}{2m^2}$. 

Figure 3: Scalar production in $\bar{e}e$ collision. note that the vertex is $= 2!(-i\lambda)$. 

will obtain
B.4 Dispersion relation at 1-loop

At last let us consider the following UV completion for the $(\bar{e}\gamma_\mu e)(\bar{e}\gamma_\mu e)$ operator. It will demonstrate that it is possible to have a negative Wilson coefficient with vanishing integrals over infinite circles. Let us extend SM with a new heavy scalar and fermion with interactions

$$\lambda(\Phi \bar{e}_R \Psi) + h.c,$$

where electric charges of new fields satisfy $Q[\Phi] + Q[\Psi] = -1$. Let us start by deriving the $c_{ee}$ Wilson coefficient.

We consider $e\bar{e} \rightarrow e\bar{e}$ scattering; then the amplitude will be given by a box diagram and it’s crossed version. In order to match with EFT predictions, we can focus on the processes

![Figure 4: Forward amplitude at $O(\lambda^4)$ order.](image)

where external particles have vanishing momentum, in which case the amplitude will be given by

$$iM = \lambda^4 [1|\gamma_\mu|2][4|\gamma_\nu|3] \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu k^\nu}{(k^2 - m^2)^4} - (2 \leftrightarrow 3).$$

(80)

Now, we have assumed that the masses of the new fields are equal $m[\Phi] = m[\Psi] = m$; the loop function for arbitrary masses is reported in the main text. Performing the integral, which is finite, and doing the Fierz rearrangements we will obtain:

$$M = \frac{1}{3} \frac{\lambda^4}{16\pi^2m^2}[14](23), \Rightarrow c_{ee}^{1111} = -\frac{1}{3} \frac{\lambda^4}{128\pi^2m^2}$$

(81)

So we see that sign of the Wilson coefficient is indeed negative.

By looking at the amplitude at $s \rightarrow \infty$ we can see that $A(s)/s \rightarrow 0$ at infinite circle, so all we need to know is the cross section for $e\bar{e}$ scattering to verify the dispersion relations. The total cross section at the order $O(\lambda^4)$ will be given by the two processes $e\bar{e} \rightarrow \Psi \bar{\Psi}$ and $e\bar{e} \rightarrow \Phi \Phi^*$, and there will be no processes $ee \rightarrow$ anything at $O(\lambda^4)$. Performing the calculation we obtain

$$\sigma(e\bar{e} \rightarrow \Psi \bar{\Psi}) = \frac{\lambda^4}{16\pi s^2} \sqrt{(s - 4m^2)}$$

$$\sigma(e\bar{e} \rightarrow \Phi \Phi^*) = \frac{\lambda^4}{64\pi s^2} \left(-8 \sqrt{s(s - 4m^2)} - 4s \log \left(\frac{s - \sqrt{s(s - 4m^2)}}{s + \sqrt{s(s - 4m^2)}}\right)\right)$$

(82)

Performing the calculation for the dispersion integral we will obtain:

$$\int \frac{ds}{\pi s} (\sigma(e\bar{e} \rightarrow \Psi \bar{\Psi}) + \sigma(e\bar{e} \rightarrow \Phi \Phi^*)) = \frac{\lambda^4}{\pi^2 m^2}(1/96 + 1/96) = \frac{\lambda^4}{48m^2\pi^2} = -8c_{ee}^{1111}$$

(83)

satisfying the identity of Eq.10.
C Decomposition of cross sections in terms of SU(2) and SU(3) irreps

In this section, we will give details of the decomposition of amplitudes in terms of the irreducible representations of the electroweak SU(2) and QCD SU(3) groups. The Wigner-Eckart theorem tells us that the resulting amplitudes and cross sections will depend only on representations of the initial state (see for similar decompositions of the isospin [33, 34], custodial [4, 35] and other groups [5, 13]). Let us start with two lepton doublet scattering $L_1 L_2 \rightarrow L_1 L_2$ where $L_1, L_2$ are $SU(2)_L$ doublet leptons, for eg $(\nu, e)^T$. Then, the initial state can be decomposed as a singlet and a triplet under SU(2): $2 \otimes 2 = 3 \oplus 1$ where the singlet and triplet states are defined as follows:

$$S = \text{singlet} = \frac{1}{\sqrt{2}} (|\nu e\rangle - |e\nu\rangle)$$

$$T = \text{triplet} = \begin{cases} 
|\nu\nu\rangle 
\frac{1}{\sqrt{2}} (|\nu e\rangle + |e\nu\rangle) 
|ee\rangle 
\end{cases},$$  \hspace{1cm} (84)

where $(\nu, e)$ are the components of EW doublet. Similarly, we can decompose the states for the lepton and anti-lepton scattering, where we will find:

$$L_1 = (\nu_1, e_1)^T, \quad \bar{L}_2 = (-\bar{e}_2, \bar{\nu}_2)^T$$

$$\tilde{S} = \text{singlet} = \frac{1}{\sqrt{2}} (|e\bar{e}\rangle + |\nu\bar{\nu}\rangle)$$

$$\tilde{T} = \text{triplet} = \begin{cases} 
-|\nu\bar{e}\rangle 
\frac{1}{\sqrt{2}} (|\nu\bar{\nu}\rangle - |e\bar{e}\rangle) 
|e\bar{\nu}\rangle 
\end{cases}.$$  \hspace{1cm} (87)

Using this decomposition we can immediately see that the amplitude for the forward scatterings of the various components of the doublets will be decomposed as

$$A_{ee} = A^{(3)}_{LL}, \quad A_{\bar{e}e} = \frac{A^{(3)}_{LL} + A^{(1)}_{LL}}{2},$$

$$A_{\nu e} = \frac{A^{(1)}_{LL} + A^{(3)}_{LL}}{2}, \quad A_{\bar{\nu} e} = A^{(3)}_{L\bar{L}},$$

$$A_{\nu\nu} = A^{(3)}_{LL}, \quad A_{\bar{\nu} \bar{\nu}} = \frac{A^{(3)}_{LL} + A^{(1)}_{LL}}{2}.$$  \hspace{1cm} (88)
and similarly, we can decompose the cross sections for quark lepton doublet scatterings. Note that forward amplitudes will satisfy the following crossing relations:

\[ A_{LL}^{(3)}(s, u) = \frac{A_{LL}^{(3)}(u, s) + A_{LL}^{(1)}(u, s)}{2}, \quad A_{LL}^{(3)}(s, u) + A_{LL}^{(1)}(s, u) = A_{LL}^{(3)}(u, s). \] (89)

Since we are looking at the dispersion relations for dimension six operators and the amplitudes in IR scale linearly with \( s \), the integrals over infinite circle contours must satisfy:

\[ C_{LL}^{(3)}(s) \equiv \int_{\text{infinite circle}} \frac{ds}{s^2} A_{LL}^{(3)}(s), \quad -C_{LL}^{(3)} = \frac{C_{LL}^{(3)} + C_{LL}^{(1)}}{2}, \quad -C_{LL}^{(3)} + \frac{C_{LL}^{(1)}}{2} = C_{LL}^{(3)}. \] (90)

The situation is very similar for the quark quark doublet scattering but there we can decompose the initial state in the representations of the color SU(3) as well (see [5] for an example).

### C.1 SU(3) decomposition

Let us consider for simplicity scattering of the quarks which are singlets under SU(2), in which case

\[ 3 \otimes 3 = \bar{3} \oplus 6, \quad 3 \otimes \bar{3} = 1 \oplus 8 \] (91)

In the case of two particle scattering, the only two possibilities are when initial particles have the same, or different colors. For the quark antiquark scattering various initial color states can be decomposed as

\[ |1\bar{1}\rangle = \frac{S}{\sqrt{3}} + \frac{\lambda_8}{\sqrt{6}} + \frac{\lambda_2}{\sqrt{2}}, \quad |2\bar{2}\rangle = \frac{S}{\sqrt{3}} + \frac{\lambda_8}{\sqrt{6}} - \frac{\lambda_2}{\sqrt{2}}, \] \[ |3\bar{3}\rangle = \frac{S}{\sqrt{3}} - \lambda_8 \sqrt{\frac{2}{3}}, \quad |1\bar{2}\rangle = \frac{\lambda_4 + i\lambda_5}{\sqrt{2}}, \quad |2\bar{1}\rangle = \frac{\lambda_4 - i\lambda_5}{\sqrt{2}}, \] \[ |3\bar{1}\rangle = \frac{\lambda_4 + i\lambda_5}{\sqrt{2}}, \quad |31\rangle = \frac{\lambda_4 - i\lambda_5}{\sqrt{2}}, \quad |23\rangle = \frac{\lambda_6 + i\lambda_7}{\sqrt{2}}, \quad |32\rangle = \frac{\lambda_6 - i\lambda_7}{\sqrt{2}}. \] (92)

Where \( S = \frac{|1\bar{1}| + |2\bar{2}| + |3\bar{3}|}{\sqrt{3}} \) is a SU(3) singlet state and \((\lambda_1...\lambda_8)\) are components of an octet, which can be formed using Gell-Mann matrices (our normalization is \( \langle \lambda_i | \lambda_j \rangle = \delta_{ij} \)). Similarly, we can decompose the quark-quark initial state in terms of the 6 and \( \bar{3} \). Note that in this case, the same and different color initial states can be schematically decomposed as

\[ |\alpha\alpha\rangle = 6_{\alpha\alpha}, \quad |\alpha\beta\rangle_{\alpha \neq \beta} = \frac{6_{\alpha\beta} \pm 3_{\alpha\beta}}{\sqrt{2}}. \] (93)

Then, the Wigner Eckart theorem tells us that the total cross sections and forward scattering amplitudes will satisfy the following relations:

\[ \sigma_{\alpha\alpha} = \sigma^{(6)}, \quad \sigma_{\alpha\beta}|_{\alpha \neq \beta} = \frac{1}{2}(\sigma^{(3)} + \sigma^{(6)}), \] \[ \sigma_{\alpha\bar{\alpha}} = \frac{\sigma^{(1)} + 2\sigma^{(8)}}{3}, \quad \sigma_{\alpha\bar{\beta}}|_{\alpha \neq \beta} = \sigma^{(8)}. \] (94)
where $\alpha(\bar{\beta})$ indices indicate whether we are looking at the same or different color scatterings in $qq$, or $q\bar{q}$ channels ($q$ here stands for a quark, which can be either up or down type). In case we are interested in the color averaged cross sections, these will be related to the above as follows

$$
\sigma_{qq} \equiv (\sigma_{qq})_{\text{col.aver.}} = \frac{2}{3}\sigma^{(6)} + \frac{1}{3}\sigma^{(3)}
$$

$$
\sigma_{q\bar{q}} \equiv (\sigma_{q\bar{q}})_{\text{col.aver.}} = \frac{1}{9}\sigma^{(1)} + \frac{8}{9}\sigma^{(8)}
$$

(96)

At last, forward amplitudes decomposed under QCD representations will satisfy the following crossing relations:

$$
A^{(6)}_{qq}(s,u) = \frac{A^{(1)}_{q\bar{q}}(u,s) + 2A^{(8)}_{q\bar{q}}(u,s)}{3}, \quad A^{(3)}_{qq}(s,u) + A^{(6)}_{qq}(s,u) = \frac{A^{(8)}_{q\bar{q}}(u,s)}{2} \quad (97)
$$

Similarly, the contours over the infinite circles will be related as follows:

$$
-C^{qq(6)}_{\infty} = \frac{C^{q\bar{q}(1)}_{\infty} + 2C^{q\bar{q}(8)}_{\infty}}{3}, \quad -C^{qq(3)}_{\infty} + C^{qq(6)}_{\infty} = C^{q\bar{q}(8)}_{\infty}. \quad (98)
$$

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