Comparing metrics at large: 
harmonic vs quo-harmonic coordinates

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November 2, 2021

Abstract

To compare two space-times on large domains, and in particular 
the global structure of their manifolds, requires using identical frames 
of reference and associated coordinate conditions. In this paper we use 
and compare two classes of time-like congruences and correspon 
ding adapted coordinates: the harmonic and quo-harmonic classes. Besides 
the intrinsic definition and some of their intrinsic properties and dif 
fences we consider with some detail their differences at the level of 
the linearized approximation of the field equations. The hard part of 
this paper is an explicit and general determination of the harmonic 
and quo-harmonic coordinates adapted to the stationary character of 
three well-know metrics, Schwarzschild’s, Curzon’s and Kerr’s, to or 
der five of their asymptotic expansions. It also contains some relevant 
remarks on such problems as defining the multipoles of vacuum solu 
tions or matching interior and exterior solutions.

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1 Introduction

Let us consider three well-known vacuum solutions of Einstein’s equations:

1. The Schwarzschild solution written in Droste-Hilbert coordinates:
   \[
   ds^2_S = -\left(1 - \frac{2M}{\bar{r}}\right)dt^2 + \frac{1}{1 - 2M/\bar{r}}d\bar{r}^2 + \bar{r}^2(d\bar{\theta}^2 + \sin^2\bar{\theta}d\bar{\varphi}^2),
   \]
   with
   \[
   \bar{r} > 2M. \tag{1}
   \]

2. The Curzon solution written in spherical Weyl-related coordinates:
   \[
   ds^2_C = -e^{-2M/\tilde{r}}dt^2 + e^{2M/\tilde{r}}\left[\exp\left(-\frac{M^2\sin^2\tilde{\theta}}{\tilde{r}^2}\right)(d\tilde{r}^2 + \tilde{r}^2d\tilde{\varphi}^2) + \tilde{r}^2\sin^2\tilde{\theta}d\tilde{\varphi}^2\right], \tag{3}
   \]
   with
   \[
   \tilde{r} > 0. \tag{4}
   \]

3. The Kerr solution written in Boyer-Lindquist coordinates,
   \[
   ds^2_K = -\left(1 - \frac{2M\tilde{r}}{\sigma^2}\right)dt^2 + \frac{4M\tilde{r}}{\sigma^2}a\sin\tilde{\theta}dt\ d\tilde{\varphi}
   + \frac{\sigma^2}{\Delta}d\tilde{r}^2 + \sigma^2d\tilde{\theta}^2 + \left(\tilde{r}^2 + a^2 + \frac{2M\tilde{r}}{\sigma^2}a^2\sin\tilde{\theta}^2\right)\sin\tilde{\theta}^2d\tilde{\varphi}^2, \tag{5}
   \]
   with
   \[
   \sigma^2 = \tilde{r}^2 + a^2\cos\tilde{\theta}^2 > 2M\tilde{r}, \quad \Delta = \tilde{r}^2 + a^2 - 2M\tilde{r} > 0. \tag{6}
   \]

We have used different notations to distinguish the radial and angular coordinates to emphasize that there is no connection \textit{a priori} that makes sense across the three solutions.

These three space-time metrics share some intrinsic properties, i.e. which are independent of the system of coordinates being used to describe them. Namely: the three solutions possess a main global time-like Killing vector $\xi^\alpha$ in the domains specified in Eqs. (2), (4) and (6). This vector is actually a generator of a group of isometries which includes as sub-group the group of rotations around an axis, and the specular symmetry across a plane. The intrinsic differences are also well known. Namely: the Schwarzschild and Curzon solutions are static, i.e. $\xi^\alpha$ is integrable, while the Kerr solution is only stationary. The Schwarzschild solution is spherically symmetric while the Curzon and the Kerr solutions are only axially symmetric.
The three systems of coordinates used in Eqs. (1), (3) and (5) share also some common properties:

i) The space coordinates are adapted to the main time-like Killing vector $\xi^\alpha$ as well as to the space-like Killing vector $\zeta^\alpha$ corresponding to the axial symmetry. This is reflected by the fact that the gravitational potentials are independent of $t$ and $\varphi$. They are also adapted to the specular symmetry of the solutions.

ii) They clearly show their Euclidean behavior at spatial infinity in the frame of reference corresponding to the congruence defined by $\xi^\alpha$.

These common properties by no means make these coordinates unique, and therefore it would be even heuristically unjustified to refer to these three metrics as being written using a “common” system of coordinates. This paper deals with the problem of further restricting the systems of coordinates up to the point of making such an assertion acceptable.

We shall see that the solution to this problem sheds new light on the problem of matching them to interior solutions. On the other hand it is obvious that being able to use a “common” system of coordinates is the only available means to compare different solutions on large domains of their manifolds.

Coordinate specifications can be made on different criteria. The Droste-Hilbert coordinates in Eq. (1) are completely determined by the requirement

$$ds^2_{\Sigma} |_{\Sigma} = \bar{r}^2 (d\bar{\theta}^2 + \sin^2 \bar{\theta} d\bar{\varphi}^2),$$ (7)

$\Sigma$ being any of the 2-surfaces $t =$ const., $\bar{r} =$ const. It can be used only for spherically symmetric space-times. The Weyl-like coordinates in Eq. (3) are specially tailored for static axially symmetric metrics. The Schwarzschild solution could be written using such coordinates but we all know how misleading this would be from a physical point of view.

The Boyer-Lindquist coordinates are akin to Droste-Hilbert coordinates in the sense that when the parameter $a$ in the Kerr solution is made zero then Eq. (5) becomes Eq. (1).

It is clear that the connections between the three systems of coordinates we have considered are as yet too loose and partial to be of any interest. Above all because they do not shed any new light on any other problem.

We take in this paper the point of view according to which to determine the physical content of the space-time metric being considered as well as to be able to compare two of them on large domains it is necessary to use systems of coordinates whose definition make sense independently of the metric to which the definition is applied.

In Section 2 we consider the definition and some properties of harmonic and quo-harmonic congruences and the intrinsically related harmonic and
quo-harmonic coordinates. The linear approximation discussed in Section 3 provides a simplified framework to illustrate some interesting properties of those congruences.

Section 4 deals with the problem of writing the Schwarzschild solution, up to terms of order $M^5$ included, both in harmonic and quo-harmonic co-ordinates adapted to the intrinsic symmetry structure described above.

In Section 5 we consider the same problem for the Curzon solution. Our results show in particular that Schwarzschild and Curzon solutions coincide when terms of order $M^3$ and higher can be neglected. This is the case when $M$ is the mass of the Sun and the physical system being considered is the solar system.

The same problem for the Kerr solution is analysed in Section 6. Our results show in particular that when considering asymptotic developments in $1/r$ of this metric using quo-harmonic coordinates it is necessary to include some logarithmic terms behaving as $r^{-5} \ln r$.

2 Harmonic and quo-harmonic coordinates

Whenever the metric

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

of a space-time model is given using an explicit system of coordinates $x^\alpha$, two associated geometric structures are also implicitly given: i) the congruence $C$ of world-lines defined by the parametric equations

$$x^i = \text{constant}, \quad i = 1, 2, 3,$$

which we shall always assume to be time-like in the domain $D$ of interest, and ii) the foliation $F$ defined by the equation

$$x^0 = \text{constant},$$

which we shall always assume to be space-like on $D$.

Two questions can be asked which are relevant in the theory of frames of reference or more generally whenever we are interested in the suitability of the coordinate system to handle a particular problem: i) does the congruence $C$ belong to any particular type, like being for instance a Killing or Born or any other type intrinsically defined? and ii) can the foliation $F$ be intrinsically characterized independently of $C$, or does it have any particular intrinsic properties connected with it?

Systems of coordinates can be divided in two classes: those that are specific to particular models or classes of models and are useful because they take
advantage of particular properties of them, and those that make sense whatever model is being considered. Isotropic, Droste-Hilbert or Schwarzschild coordinates for space-times with spherical symmetry, or Weyl coordinates for static axially symmetric models belong to the first class.

Gauss coordinates belong to the second class, but to our knowledge the single problem where they have played an important role in the development of the theory of General Relativity is in the problem of matching exterior to interior solutions of Einstein’s equations across space-like hypersurfaces. More on that on Section 4.3.

2.1 Harmonic congruences and coordinates

A widely used system of coordinates \( x^\alpha \) belonging to this second class is that of harmonic coordinates, which is characterized by these two groups of equations:

\[
\square x^i = 0 \tag{11}
\]

and

\[
\square x^0 = 0, \tag{12}
\]

where \( \square \) is the intrinsic d’Alembertian of the space-time metric. The splitting above in two groups of the four equations \( \square x^\alpha = 0 \) will be understood in a moment.

Conversely, given a time-like congruence defined by a unit tangent vector field \( u^\alpha \) it may be asked what type of adapted coordinates can be most appropriate to use. For instance, we know that if \( u^\alpha \) is collinear with a Killing vector field then it is always possible to use a system of harmonic coordinates such that the congruence with parametric equations (9) is the same as that defined by \( u^\alpha \). But in general this is not the case, because if we consider the harmonic coordinates \( x^i \) as functions of generic coordinates, \( x^i = f^i(y^\alpha) \), then the functions \( f^i \) must satisfy the system of equations

\[
\square f^i = 0, \quad u^\alpha \partial_\alpha f^i = 0, \quad \text{rank}(\partial_\alpha f^i) = 3, \tag{13}
\]

and this requires integrability conditions on the vector field \( u^\alpha \) that in general are not satisfied. A particularly interesting case in which they do not hold is that of irrotational Born (or rigid) congruences when they are not Killing \( [1] \). As a matter of fact, Eqs. (13) can be considered as the definition of a new intrinsic type of congruences: that of harmonic congruences, i.e. those admitting a set of adapted harmonic coordinates of space.

Notice that the time coordinate can always be required to satisfy Eq. (12) and therefore this condition is, without any supplementary conditions, unre-
lated to any particular congruence: it is enough to find a solution of
\[ \Box f^0 = 0, \quad u^\alpha \partial_\alpha f^0 \neq 0. \] (14)

It is for this reason that we preferred to refer to harmonic coordinates splitting the conditions on the space and time coordinates.

2.2 Quo-harmonic congruences and coordinates

As we have already mentioned these considerations are particularly relevant in any theory of frames of reference that requires to extend this concept beyond the Born congruences which are, generically, notoriously exceptional. As a contribution towards an appropriate generalization it has been introduced \[2, 3, 4\] a new type of congruences which are defined much in the same way as we defined harmonic congruences by Eqs. (13), except for a slight modification of the first group of conditions, which becomes
\[ \Box f^i + \Lambda^\alpha \partial_\alpha f^i = 0, \] (15)
where
\[ \Lambda^\alpha = -u^\rho \nabla_\rho u^\alpha \] (16)
is, up to the sign, the intrinsic curvature of the congruence \( u^\alpha \). We shall also consider later a quo-harmonic time coordinate defined by
\[ \Box f^0 + \Lambda^\alpha \partial_\alpha f^0 = 0, \quad u^\alpha \partial_\alpha f^0 \neq 0. \] (17)

This class of quo-harmonic congruences contains, in contradistinction to the class of harmonic congruences, the whole class of Born congruences. And it provides sufficient generality to be an essential ingredient to define the concept of rigidity without which no useful meaning can be given to the concept of frame of reference. This is why we shall spend some time to justify the consideration of quo-harmonic congruences and quo-harmonic coordinates; two concepts that are related but that by no means are identical.

2.3 Harmonic and quo-harmonic coordinate classes

The next section is dedicated to compare Einstein’s linearized equations using harmonic or quo-harmonic coordinates. In the remaining sections instead we shall restrict ourselves to the consideration of the three stationary metrics mentioned in the Introduction. These metrics are stationary and have been written in a system of coordinates adapted to a main Killing vector. All Killing congruences are both harmonic and quo-harmonic. This can be seen
as follows. In a system of coordinates adapted to a Killing vector field all the potential $g_{\alpha\beta}$ are independent of time and Eqs. (15) become

$$\hat{\Delta} \hat{f}^k = \frac{1}{\sqrt{\hat{g}}} \partial_i \left( \sqrt{\hat{g}} \hat{g}^{ij} \partial_j \hat{f}^k \right) = 0, \quad \hat{g} \equiv \det (\hat{g}_{ij}),$$  \hspace{1cm} (18)

where it has been used the decomposition

$$ds^2 = - \left[ \xi (- dt + \varphi_i dx^i) \right]^2 + \hat{g}_{ij} dx^i dx^j$$  \hspace{1cm} (19)

with

$$\xi \equiv \sqrt{-g_{00}}, \quad \varphi_i \equiv \xi^{-2} g_{0i}, \quad \hat{g}_{ij} \equiv g_{ij} + \xi^2 \varphi_i \varphi_j.$$  \hspace{1cm} (20)

Since Eqs. (18) always have three solutions independent of $t$ it follows that the Killing congruences are quo-harmonic. On the other hand Eqs. (13) become

$$\bar{\Delta} \bar{f}^k = \frac{1}{\sqrt{\bar{g}}} \partial_i \left( \sqrt{\bar{g}} \bar{g}^{ij} \partial_j \bar{f}^k \right) = 0, \quad \bar{g}_{ij} \equiv \xi^2 \hat{g}_{ij}, \quad \bar{g} \equiv \det (\bar{g}_{ij}),$$  \hspace{1cm} (21)

and these equation also have three independent solutions independent of $t$. Therefore Killing congruences are also harmonic.

Notice however that if $\Lambda_i \neq 0$ harmonic coordinates are different from the quo-harmonic ones even for a congruence that is both harmonic and quo-harmonic — a case that includes as we have seen the Killing congruences — in which case they reveal different aspects of the three models being considered.

The harmonic coordinates are derived from those used in Eqs. (1), (3) and (5) by a pure space transformation

$$x^i = x^i (y^j, \lambda, \mu),$$  \hspace{1cm} (22)

where $y^j$ is any of the systems of coordinates used in (1), (3) and (5), $\lambda$ is the set of parameters on which depends the metric ($M$, but also $a$ for the Kerr metric) and $\mu$ is any set of constants coming from integrating the coordinate conditions being demanded. For $x^i$ to be harmonic and Cartesian at space infinity the gravitational potentials $g_{\alpha\beta}(t, x^i)$ have to be solutions of the equations

$$\Gamma^k \equiv g^{\alpha\beta} \Gamma^k_{\alpha\beta} = 0,$$  \hspace{1cm} (23)

which are Eqs. (11) when harmonic coordinates are used. Here $\Gamma^k_{\alpha\beta}$ are the Christoffel symbols. Of course any other system of coordinates $z^i$ derived from $x^i$ by

$$z^i = f^i(x^j)$$  \hspace{1cm} (24)
with functions $f^i$ independent of the parameters $\lambda$ and $\mu$ can be said to belong to the harmonic class and be used to fulfill the requirement of universality mentioned before. In particular, we shall systematically use not harmonic Cartesian-like coordinates but rather polar coordinates that belong to the same harmonic class and are related to them by the familiar formulas:

$$x^1 = r \sin \theta \cos \varphi, \quad x^2 = r \sin \theta \sin \varphi, \quad x^3 = r \cos \theta.$$  \hfill (25)

We will also use quo-harmonic coordinates, which are derived from the original ones by space-like transformations of the type (22) but requiring instead of (23) Eqs. (15), which become

$$\hat{\Gamma}^k \equiv \Gamma^k + g^{kj} \Lambda_j = 0 \hfill (26)$$

when quo-harmonic coordinates are used. For stationary metrics these equations reduce to

$$\Gamma^k - g^{kj} \partial_j \ln \xi = 0. \hfill (27)$$

### 3 The linear approximation

We assume in this section that the metric admits a congruence $u^\alpha$ and a system of adapted coordinates $x^\alpha$ such that it can be written as

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \hfill (28)$$

where $\eta_{\alpha\beta}$ are the Galilean coefficients of the Minkowski metric, and $h_{\alpha\beta}$ are small quantities whose products can be neglected, as well as the products of derivatives of any order, in the domain $D$ of interest.

Adapted coordinate transformations leaving form-invariant (28) are up to a Lorentz transformation

$$x'^0 = x^0 + \zeta^0(x^\alpha), \quad x'^i = x^i + \zeta^i(x^j), \hfill (29)$$

where the $\zeta^\alpha$ are small quantities, and the $\zeta^i$ do not depend on time.

Under such an adapted coordinate transformation the quantities $h_{\alpha\beta}$ transform as follows:

$$h'_{00} = h_{00} + 2\partial_0 \zeta_0, \hfill (30)$$

$$h'_{0i} = h_{0i} + \partial_i \zeta_0, \hfill (31)$$

$$h'_{ij} = h_{ij} + \partial_i \zeta_j + \partial_j \zeta_i. \hfill (32)$$
The differential invariants under these transformations are

\[ \Lambda_i = \frac{1}{2} \partial_i h_{00} - \partial_0 h_{0i}, \]

(33)

\[ \Omega_{ij} = \partial_i h_{0j} - \partial_j h_{0i}, \]

(34)

\[ \Sigma_{ij} = \partial_0 h_{ij}, \]

(35)

\[ \hat{R}_{ijkl} = -\frac{1}{2}(\partial_{ik} h_{jl} + \partial_{jl} h_{ik} - \partial_{il} h_{jk} - \partial_{jk} h_{il}). \]

(36)

These expressions give in this approximation the values of intrinsically well-defined objects associated to the time-like congruence [3, 4]: the sign-reversed acceleration (16), the vorticity field, the deformation rate and the Zel’manov-Cattaneo tensor [7, 8].

3.1 Harmonic congruences

Let us consider the quantities defined in (23), which in this approximation reduce to

\[ \Gamma_i = -\partial_0 h_{0i} + \partial_j h^j_i - \frac{1}{2} \partial_i (-h_{00} + \hat{h}), \]

(37)

where

\[ \hat{h} = \delta^{ij} h_{ij}. \]

(38)

Under an adapted coordinate transformation those quantities and their derivatives with respect to time transform as follows:

\[ \Gamma'_i = \Gamma_i + \Delta \zeta_i, \quad \partial_0 \Gamma'_i = \partial_0 \Gamma_i. \]

(39)

Since at this approximation a congruence is harmonic iff a system of coordinates exists such that \( \Gamma'_i = 0 \), it follows that

\[ \partial_0 \Gamma_j = \partial_i \Sigma^i_j - \frac{1}{2} \partial_j \Sigma + \partial_0 \Lambda_j = 0, \quad \Sigma \equiv \Sigma^i_j \]

(40)

characterizes the harmonic congruences to this approximation. This is in fact an invariant condition that guarantees the existence of a solution of the equation

\[ \Delta \zeta_i = -\Gamma_i \]

(41)

for \( \zeta_i \) not depending of \( x^0 \).

If the harmonic congruence is a Born congruence then \( \Sigma_{ij} = 0 \) and it follows from (10) that \( \Lambda_i \) must be independent of time. If moreover we assume that \( \Omega_{ij} \) is also independent of time then it is easy to show that the congruence is in fact a Killing congruence, i.e. a system of adapted coordinates exist such that \( \partial_0 g_{\alpha\beta} = 0 \).
Let us now consider the linearized Einstein’s field equations,

\[ R_{\alpha\beta} = U_{\alpha\beta}, \quad U_{\alpha\beta} \equiv T_{\alpha\beta} - \frac{1}{2} (\eta^{\rho\sigma} T_{\rho\sigma}) \eta_{\alpha\beta}, \tag{42} \]

where \( R_{\alpha\beta} \) is the linearized Ricci tensor:

\[ R_{\alpha\beta} = -\frac{1}{2} (\Box h_{\alpha\beta} - \partial_\beta \Gamma_\alpha - \partial_\alpha \Gamma_\beta). \tag{43} \]

Unlike in Section 2, here \( \Box \) is the d’Alembertian associated to the Minkowski metric, \( T_{\alpha\beta} \) is some approximation to the energy-momentum tensor, \( \Gamma_i \) are the quantities defined in (37), and we get from Eq. (23):

\[ \Gamma_0 \equiv -\partial_0 h_{00} + \partial_j h^j_0 - \frac{1}{2} \partial_0 (-h_{00} + \hat{h}). \tag{44} \]

Assuming that the congruence \( u^\alpha \) is harmonic, that harmonic coordinates are used, \( \Gamma_i = 0 \), and that a foliation with \( \Gamma_0 = 0 \) is selected, then the familiar equations

\[ R_{\alpha\beta} = -\frac{1}{2} \Box h_{\alpha\beta} = U_{\alpha\beta} \tag{45} \]

are obtained. These show explicitly the hyperbolic type of Einstein’s linearized field equations when space-time harmonic coordinates can be used.

### 3.2 Quo-harmonic congruences

Let us consider now the quantities defined in (26):

\[ \hat{\Gamma}_i = \partial_j h^j_i - \frac{1}{2} \partial_0 \hat{h}. \tag{46} \]

Under an adapted coordinate transformation these quantities and their derivatives with respect to time transform as

\[ \hat{\Gamma}_{\iota'} = \Gamma_i + \Delta \zeta_i, \quad \partial_0 \Gamma_{\iota'} = \partial_0 \Gamma_i. \tag{47} \]

Since at this approximation a congruence is quo-harmonic iff a system of coordinates exist such that \( \hat{\Gamma}_{\iota'} = 0 \), the condition

\[ \partial_0 \hat{\Gamma}_j = \partial_j \Sigma^j_i - \frac{1}{2} \partial_j \Sigma = 0 \tag{48} \]

characterizes the quo-harmonic congruences to this approximation. This is an invariant condition that guarantees the existence of a solution of the equation

\[ \Delta \zeta_i = -\hat{\Gamma}_i \tag{49} \]
with $\zeta_i$ independent of $x^0$.

From (40) and (48) it follows that a congruence can be harmonic and quoharmonic at the same time iff $\Lambda_i$ is independent of time. Notice however that this does not mean that in this case harmonic and quoharmonic coordinates are the same.

Let us assume now that the congruence $u^\alpha$ is quoharmonic, that quoharmonic coordinates are used, $\hat{\Gamma}_i = 0$, and that a foliation is selected —as it can always be done by using (31)— such that $\hat{\Gamma}_0 = 0$, where

$$\hat{\Gamma}_0 \equiv \partial_i h^i_0. \quad (50)$$

The later hypothesis is equivalent to assume that $x^0$ also satisfies the quoharmonicity condition (17). Under these assumptions, Einstein’s field equations become

$$R_{00} = -\frac{1}{2} \left( \triangle h_{00} + \partial_0 \hat{h} \right) = U_{00}, \quad (51)$$

$$R_{0i} = -\frac{1}{2} \left( \triangle h_{0i} + \frac{1}{2} \partial_i \hat{h} \right) = U_{0i}, \quad (52)$$

$$R_{ij} = -\frac{1}{2} \left( \Box h_{ij} + \partial_0 (\partial_i h_{0j} + \partial_j h_{0i} - \partial_{ij} h_{00}) \right) = U_{ij}, \quad (53)$$

where $\triangle$ is the Laplacian constructed with the 3-dimensional Euclidean $\delta_{ij}$ metric. Introducing the traceless tensor

$$k_{ij} \equiv h_{ij} - \frac{1}{3} \hat{h} \delta_{ij}, \quad (54)$$

the last group of equations splits in two groups: the scalar equation

$$\bar{R} = -\frac{1}{2} (\Box \hat{h} - \triangle h_{00}) = \bar{U}, \quad \bar{R} \equiv \delta^{ij} R_{ij}, \quad \bar{U} \equiv \delta^{ij} U_{ij}, \quad (55)$$

and the tensor equation

$$R_{ij} - \frac{1}{3} \bar{R} \delta_{ij} = U_{ij} - \frac{1}{3} \bar{U} \delta_{ij}. \quad (56)$$

Taking into account Eqs. (51)–(53), the scalar Eq. (53) reduces to

$$-\frac{1}{2} \triangle \hat{h} = U_{00} + \bar{U} \quad (57)$$

and the tensor Eq. (56) becomes

$$-\frac{1}{2} \left( \Box k_{ij} + \partial_0 (\partial_i h_{0j} + \partial_j h_{0i} - \partial_{ij} h_{00}) \right) - \frac{1}{6} \triangle h_{00} \delta_{ij} = U_{ij} - \frac{1}{3} \bar{U} \delta_{ij}. \quad (58)$$
Assuming $T_{\alpha\beta}(x^\rho)$ known everywhere and for all times, and satisfying the conservation equations
\[ \partial_\alpha T^\alpha_{\beta} = 0, \] (59)
\( \hat{h} \) can now be obtained integrating the elliptic equation (57). After that $h_{00}$ can be obtained integrating the elliptic equation (51). And then $h_{0i}$ can be obtained integrating the elliptic equation (52). Finally the single part of the metric that necessitates to integrate hyperbolic equations is the traceless piece $k_{ij}$ appearing in Eqs (58). We can say then that neither $h_{00}$, nor $h_{0i}$, nor $\hat{h}$ are propagating quantities in quo-harmonic coordinates. Only the wave-part of the metric may propagate. When solving the aforementioned equations one have to make sure that conditions $\hat{\Gamma}_0 = 0$ and $\hat{\Gamma}_i = 0$ are satisfied, but they are compatible conditions because of conservation law (59) and the fact that from the field equations (51)–(53) one gets
\[ -\frac{1}{2} \Delta \hat{\Gamma}_0 = \partial_\alpha T^\alpha_0 \] (60)
and
\[ -\frac{1}{2} \Box \hat{\Gamma}_i = \partial_\alpha T^\alpha_i + \frac{1}{2} \partial_0 \hat{\Gamma}_0. \] (61)

3.3 Stationary solutions

If the frame of reference in which Eqs. (45) or Eqs. (51)–(53) have been written is that corresponding to a Killing congruence implying that $h_{\alpha\beta}$ are time independent then the preceding equations can be split in two groups. The first group is common to both harmonic and quo-harmonic coordinates:
\[ -\frac{1}{2} \Delta h_{00} = U_{00}, \] (62)
\[ -\frac{1}{2} \Delta h_{0i} = U_{0i}, \quad \partial_i h^i_0 = 0. \] (63)
The second group is
\[ -\frac{1}{2} \Delta h_{ij} = U_{ij}, \quad \partial_j h^j_i - \frac{1}{2} \partial_i \left( -h_{00} + \hat{h} \right) = 0, \] (64)
when using harmonic coordinates and
\[ -\frac{1}{2} \left( \Delta h_{ij} - \partial_i h_{00} \right) = U_{ij}, \quad \partial_j h^j_i - \frac{1}{2} \partial_i \hat{h} = 0, \] (65)
when using quo-harmonic coordinates. The solutions $h_{ij} = \bar{h}_{ij}$ of (64) and $h_{ij} = \tilde{h}_{ij}$ of (65) are related as follows:
\[ \bar{h}_{ij} = \tilde{h}_{ij} + \partial_i \zeta_j + \partial_j \zeta_i, \] (66)
\( \zeta_j \) being a solution of the equation

\[ \Delta \zeta_j = \frac{1}{2} \partial_j h_{00}. \tag{67} \]

If \( U_{\alpha\beta} \) is known, smooth and decreasing fast enough at infinity, or compact with discontinuities of the first kind across the border of the support, then the solution to the preceding field equations are:

\[ h_{00} = \frac{1}{2\pi} \int \frac{U_{00}}{R} \, dV, \quad h_{0i} = \frac{1}{2\pi} \int \frac{U_{0i}}{R} \, dV, \tag{68} \]

and

\[ \bar{h}_{ij} = \frac{1}{2\pi} \int \frac{U_{ij}}{R} \, dV, \quad \tilde{h}_{ij} = \frac{1}{4\pi} \int \frac{2U_{ij} - \partial_{ij} h_{00}}{R} \, dV, \tag{69} \]

As it is well known the solutions thus obtained are at least of class \( C^1 \). This is one of the reasons, among many others, including the analogy with electromagnetism, to require in general that the potential \( g_{\alpha\beta} \) be of class \( C^1 \) across the surface of discontinuities of the first kind in the theory of relativity. Surprisingly this requirement is not respected by many authors because of what appears to be a confusion between two connected but different concepts: that of metrics that can be matched and that of metrics that have been matched.

The multipole structure of the potentials \( h_{00} \) and \( h_{0i} \) can be determined as it is standard in electromagnetic theory. The multipole structures of \( \bar{h}_{ij} \) and \( \tilde{h}_{ij} \) are in general different but related by Eq. (66). Notice also that the multipole structure of a particular solution is independent of any system of adapted and admissible coordinates, harmonic, quo-harmonic or else, if instead of the potentials we consider the invariant quantities (33)–(36). Furthermore, as a consequence of the field equations (64)–(65) for stationary metrics in vacuum, (36) is equivalent to

\[ \hat{R}_{ij} = -\frac{1}{2} \partial_{ij} h_{00} \tag{70} \]

both in harmonic and in quo-harmonic coordinates, so that the invariant multipole structure in these two coordinate classes for this kind of metrics is equal and determined by the common values of \( h_{00} \) and \( h_{0i} \). In the last section of this paper we shall come back to the linear approximation in a different context connected with the problem of defining the multipole structure of a vacuum solution of Einstein’s equations when the source is not known.
4 Schwarzschild metric

To write the Schwarzschild solution we are going to use as basis the following forms
\[ \omega^0 = dt, \quad \omega^1 = dR, \quad \omega^2 = R d\Theta, \quad \omega^3 = R \sin \Theta d\phi \] (71)
when we use harmonic coordinates and
\[ \omega^0 = dt, \quad \omega^1 = dr, \quad \omega^2 = r d\theta, \quad \omega^3 = r \sin \theta d\varphi \] (72)
with quo-harmonic coordinates, so that the metric coefficients \(g_{\mu\nu}\) are defined as follows:
\[ ds^2 = g_{\mu\nu} \omega^\mu \omega^\nu. \] (73)
These functions reduce to the Minkowskian values when \(R\) or \(r\) go to infinity,
\[ \lim_{R \to \infty} g_{\mu\nu} = \eta_{\mu\nu}, \] (74)
and they will be written as
\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \] (75)
where the \(h_{\mu\nu}\) are functions of \(R\) or \(r\).

4.1 Schwarzschild metric in harmonic coordinates

We can obtain a system of harmonic coordinates preserving the spherical symmetry if we perform a coordinate transformation in the form \((\tilde{r}, \tilde{\theta}, \tilde{\varphi}) \to (R, \Theta, \phi)\), where \(R = R(\tilde{r})\), \(\Theta = \tilde{\theta}, \phi = \tilde{\varphi}\). In fact, the Cartesian coordinates associated to the set of spherical coordinates \((R, \Theta, \phi)\) are harmonic and \(\lim_{\tilde{r} \to \infty} R(\tilde{r}) = \tilde{r}\) is satisfied if one chooses [7]
\[ R(\tilde{r}) = \tilde{r} - M + C \left[ \frac{\tilde{r} - M}{2} \ln \left( 1 - \frac{2M}{\tilde{r}} \right) + M \right], \] (76)
where \(C\) is an integration constant, which will be discussed below in Section 4.3.

We can invert the coordinate transformation in series of \(M/R\) and the functions \(h_{\mu\nu}\) of the non-null metric components written in harmonic coordinates to order five are
\[ h_{00} = 2 \frac{M}{R} - 2 \frac{M^2}{R^2} + 2 \frac{M^3}{R^3} - \frac{6 + 2C M^4}{3 R^4} + \frac{6 + 4C M^5}{3 R^5}, \]
\[ h_{11} = 2 \frac{M}{R} + 2 \frac{M^2}{R^2} + \frac{6 - 4C M^3}{3 R^3} + \frac{6 - 10C M^4}{3 R^4} + \frac{10 - 28C M^5}{5 R^5}, \]
\[ h_{22} = h_{33} = 2 \frac{M}{R} + \frac{M^2}{R^2} + \frac{2C M^3}{3 R^3} + \frac{2C M^4}{3 R^4} + \frac{2C M^5}{5 R^5}. \] (77)
4.2 Schwarzschild metric in quo-harmonic coordinates

We can also obtain a system of quo-harmonic coordinates preserving the spherical symmetry if we perform a coordinate transformation in the form $(\bar{r}, \bar{\theta}, \bar{\varphi}) \to (r, \theta, \varphi)$, with $r = r(\bar{r}), \theta = \bar{\theta}, \varphi = \bar{\varphi}$. The Cartesian coordinates associated to $(r, \theta, \varphi)$ are quo-harmonic and the condition $\lim_{\bar{r} \to \infty} r = \bar{r}$ is satisfied if

$$r(\bar{r}) = \bar{r} \left[ A \left( 1 - \frac{3M}{2\bar{r}} \right) + (1 - A) \sqrt{1 - \frac{2M}{\bar{r}}} \left( 1 - \frac{M}{2\bar{r}} \right) \right], \quad (78)$$

where $A$ is an integration constant to be discussed later. Notice the remarkable fact that for $A = 0$ the interval $2M \leq \bar{r} < \infty$ corresponds to $0 \leq r < \infty$. No such thing is possible in harmonic coordinates.

If we invert the coordinate transformation in series of $M/r$, the functions $h_{\mu\nu}$ of the non-null metric components in the quo-harmonic coordinates are to order five

$$h_{00} = 2\frac{M}{r} - 3\frac{M^2}{r^2} + 9\frac{M^3}{2r^3} - \frac{29 - 2AM^4}{4r^4} + \frac{99 - 18A M^5}{8r^5},$$

$$h_{11} = 2\frac{M}{r} + \frac{M^2}{r^2} - \frac{1 - 2AM^3}{2r^3} + \frac{AM^4}{4r^4} - \frac{5 - 6AM^5}{8r^5},$$

$$h_{22} = h_{33} = 3\frac{M}{r} + \frac{9M^2}{4r^2} + \frac{1 - AM^3}{2r^3}. \quad (79)$$

4.3 Matching with an interior solution

To analyse the meaning of the constants $C$ and $A$ appearing in (76)–(77) and (78)–(79) respectively, let us consider the problem of constructing a full model for the gravitational field of a spherically symmetric star. We can use the Schwarzschild metric (1) for the exterior field but we also need an interior solution and then both metrics have to be matched at the star surface. It is well known that the continuity across the matching surface of the first and second fundamental forms is enough to guarantee that the matching can be done without a surface layer [12]. But from our point of view the problem is not completely solved with this: to actually make the matching we also need to found for both metrics a set of “admissible coordinates” in the sense of Lichnerowicz [13] in which all the derivatives of the metric coefficients are continuous across the star surface. Furthermore, we want this common set of coordinates to have the same physical meaning for the interior and exterior solutions: they must be members of a class of systems
of coordinates defined with independence of the particular problem under study. In particular we are interested here in the cases in which the common coordinates are harmonic (or quo-harmonic) for both the interior and the exterior metric.

In fact, Quan-Hui Liu [9] considered a particular interior with uniform density and found that the matching with the Schwarzschild metric (1) in harmonic admissible coordinates is only possible if the constant $C$ in (76) has a given value depending only on the radius of the star surface.

A similar result arises in quo-harmonic coordinates [11]: an interior solution with constant density can be matched in quo-harmonic admissible coordinates only if the constant $A$ in (78) has a particular value determined by the interior solution, the matching radius and the constant $M$.

These two particular examples suggest that the integration constants that appear when solving the differential conditions of definition for harmonic or quo-harmonic coordinates will be fixed (hopefully in a unique way) only when the full problem of finding a common set of such coordinates for both the internal and external gravitational fields is addressed.

## 5 Curzon metric

We are going to write the Curzon metric in the same bases (71)–(72) we have used for the Schwarzschild metric. The metric coefficients $g_{\mu\nu}$ in Eq. (73) satisfy the asymptotic property (74) and we will use again notation (75), but now the $h_{\mu\nu}$ quantities are functions of $R$ and $\Theta$ or of $r$ and $\theta$. Moreover, they have the following structure:

$$h_{\mu\mu} = \sum_l h^{(2l)}_{\mu\mu} P_{2l}, \quad h_{12} = \sum_l h^{(2l)}_{12} P^1_{2l}, \quad (80)$$

where $h^{(n)}_{\mu\nu}$ are only functions of $R$ or $r$ and $P_n$ and $P^1_n$ are the Legendre polynomials and associated functions in $\cos \Theta$ or $\cos \theta$.

### 5.1 Curzon metric in harmonic coordinates

One may obtain a system of harmonic coordinates preserving the axial symmetry and the reflection symmetry with respect to the plane $z = 0$ by using an appropriate coordinate transformation in the form $(\tilde{r}, \tilde{\theta}, \tilde{\varphi}) \rightarrow (R, \Theta, \phi)$, where $R = R(\tilde{r}, \tilde{\theta}), \Theta = \Theta(\tilde{r}, \tilde{\theta}), \phi = \tilde{\varphi}$. By solving the harmonicity equation that the Cartesian coordinates associated to the set of spherical coordinates $(R, \Theta, \phi)$ must satisfy, and demanding that $\lim_{\tilde{r} \to \infty} R = \tilde{r}$, we obtain to order
five in the expansion parameter $M/R$:

\[
\frac{R}{\bar{r}} = 1 + \frac{\sin^2 \tilde{\theta}}{2} \frac{M^2}{\bar{r}^2} - \frac{2 C_1 + 2 C_2 - 3 C_2 \sin^2 \tilde{\theta}}{6} \frac{M^3}{\bar{r}^3} + \frac{4 - 5 \sin^2 \tilde{\theta}}{24} \frac{M^4}{\bar{r}^4} \\
+ \frac{1}{240} \left[ 240 C_1 + (8 C_1 + 200 C_2 - 420 C_3 - 840 C_4) \sin^2 \tilde{\theta} \right. \\
- \left. (240 C_2 - 525 C_3 - 600 C_4) \sin^4 \tilde{\theta} \right] \frac{M^5}{\bar{r}^5}, \quad (81)
\]

\[
\frac{\cos \Theta}{\cos \theta} = 1 - \frac{1}{2} \sin^2 \tilde{\theta} \frac{M^2}{\bar{r}^2} - \frac{C_2}{2} \sin^2 \tilde{\theta} \frac{M^3}{\bar{r}^3} - \frac{4 - 11 \sin^2 \tilde{\theta}}{24} \frac{M^4}{\bar{r}^4} \\
- \frac{1}{80} \left[ 16 C_1 + 80 C_2 - 140 C_3 - 80 C_4 \right. \\
- \left. (120 C_2 - 175 C_3 - 200 C_4) \sin^2 \tilde{\theta} \right] \sin^2 \tilde{\theta} \frac{M^5}{\bar{r}^5}, \quad (82)
\]

where $C_1, C_2, C_3, C_4$ are integration constants.

In this system of harmonic coordinates the functions $h^{(2)}_{\mu\nu}$ for the non-null components of the Curzon metric are, to order five,

\[
h^{(0)}_{00} = \frac{2 M}{R} - \frac{2 M^2}{R^2} + 2 \frac{M^3}{R^3} - \frac{6 + 2 C_1}{3} \frac{M^4}{R^4} + \frac{6 + 4 C_1}{3} \frac{M^5}{R^5},
\]

\[
h^{(2)}_{00} = -\frac{2 M^3}{3 R^3} + \frac{4 - 2 C_2}{3} \frac{M^4}{R^4} - \frac{44 - 28 C_2}{21} \frac{M^5}{R^5},
\]

\[
h^{(4)}_{00} = \frac{38 M^5}{105 R^5}, \quad (83)
\]

\[
h^{(0)}_{11} = \frac{2 M}{R} + \frac{2 M^2}{R^2} + \frac{6 - 4 C_1}{3} \frac{M^3}{R^3} + \frac{6 - 10 C_1}{3} \frac{M^4}{R^4} + \frac{10 - 28 C_1}{5} \frac{M^5}{R^5},
\]

\[
h^{(2)}_{11} = -\frac{2 + 4 C_2}{3} \frac{M^3}{R^3} - \frac{4 + 10 C_2}{3} \frac{M^4}{R^4} \\
- \frac{220 - 168 C_1 + 420 C_2 + 420 C_3 - 360 C_4}{105} \frac{M^5}{R^5},
\]

\[
h^{(4)}_{11} = \frac{38 + 420 C_3 + 480 C_4}{105} \frac{M^5}{R^5}, \quad (84)
\]

\[
h^{(2)}_{12} = -\frac{C_2 M^3}{6} \frac{M^4}{R^4} + \frac{2 - 3 C_2}{9} \frac{M^5}{R^5},
\]
5.2 Curzon metric in quo-harmonic coordinates

To obtain a system of quo-harmonic coordinates preserving the axial and reflection symmetries we will perform a coordinate transformation in the form $(\tilde{r}, \tilde{\theta}, \tilde{\phi}) \rightarrow (r, \theta, \varphi)$, with $r = r(\tilde{r}, \tilde{\theta}), \theta = \theta(\tilde{r}, \tilde{\theta}), \varphi = \tilde{\phi}$. By demanding $\lim_{\tilde{r} \to \infty} r = \tilde{r}$ and that the Cartesian coordinates associated to the spherical $(r, \theta, \varphi)$ satisfy the quo-harmonicity equation we obtain to order five in $M/r$

\[
\frac{r}{\tilde{r}} = 1 - \frac{1}{2} \frac{M}{\tilde{r}} + \frac{\sin^2 \tilde{\theta} M^2}{2 \tilde{r}^2} + \frac{1 + A_1 - 2A_2 - (3 - 3A_2) \sin^2 \tilde{\theta}}{4} \frac{M^3}{\tilde{r}^3} - \frac{3 + 3A_1 - 6A_2 - (13 - 9A_2) \sin^2 \tilde{\theta}}{24} \frac{M^4}{\tilde{r}^4}
\]
\[
+ \frac{1}{240} \left[ 9 + 9A_1 - 18A_2 + 6A_4 
+ (59 - 6A_1 + 327A_2 - 288A_3 - 21A_4) \sin^2 \tilde{\theta} 
- (115 + 360A_2 - 360A_3 - 15A_4) \sin^4 \tilde{\theta} \right] \frac{M^5}{\tilde{r}^5}, \quad (88)
\]

\[
\frac{\cos \theta}{\cos \tilde{\theta}} = 1 - \frac{\sin^2 \tilde{\theta} M^2}{2 \tilde{r}^2} + \frac{2 - 3A_2}{4} \sin^2 \tilde{\theta} \frac{M^3}{\tilde{r}^3} - \frac{7 - 11 \sin^2 \tilde{\theta}}{24} \sin^2 \tilde{\theta} \frac{M^4}{\tilde{r}^4}
- \frac{1}{240} \left[ 64 - 36A_1 + 387A_2 - 288A_3 - 6A_4 
- (20 + 540A_2 - 360A_3 - 15A_4) \sin^2 \tilde{\theta} \right] \frac{M^5}{\tilde{r}^5}, \quad (89)
\]

where \(A_1, A_2, A_3\) and \(A_4\) are integration constants.

In this system of quo-harmonic coordinates the functions \(h_{\mu\nu}^{(2)}\) for the non-null components of Curzon metric are written as follows:

\[
h_{00}^{(0)} = 2 \frac{M}{r} - 3 \frac{M^2}{r^2} + \frac{9 M^3}{2 r^3} - \frac{29 - 2A_1 M^4}{4 r^4} + \frac{99 - 18A_1 M^5}{8 r^5},
\]

\[
h_{00}^{(2)} = - \frac{2 M^3}{3 r^3} + \frac{10 - 3A_2 M^4}{3 r^4} - \frac{431 - 189A_2 M^5}{42 r^5},
\]

\[
h_{00}^{(4)} = \frac{38 M^5}{105 r^5},
\]

\[
h_{11}^{(0)} = 2 \frac{M}{r} + \frac{M^2}{r^2} - \frac{1 - 2A_1 M^3}{2 r^3} + \frac{A_1 M^4}{4 r^4} - \frac{5 - 6A_1 M^5}{8 r^5},
\]

\[
h_{11}^{(2)} = \frac{4 - 6A_2 M^3}{3 r^3} + \frac{1 - 3A_2 M^4}{6 r^4}
+ \frac{733 - 252A_1 + 189A_2 - 576A_3 + 18A_4 M^5}{210 r^5},
\]

\[
h_{11}^{(4)} = - \frac{114 - 96A_3 - 4A_4 M^5}{35 r^5},
\]

\[
h_{12}^{(2)} = - \frac{A_2 M^3}{4 r^3} + \frac{11 - 18A_2 M^4}{36 r^4}
- \frac{2002 - 1008A_1 + 1701A_2 - 2304A_3 + 72A_4 M^5}{2520 r^5},
\]

19
\( h_{12}^{(4)} = \frac{24A_4 + A_4 M^5}{280} \frac{r^5}{r^5}, \) (92)

\( h_{22}^{(0)} = \frac{3 M}{r} + \frac{9 M^2}{4 r^2} + \frac{1 - 3A_1 + 3A_2 M^3}{6} \frac{r^3}{r^3} - \frac{19 - 27A_2 M^4}{36} \frac{r^4}{r^4} + \frac{28 - 9A_1 + 18A_2 - 36A_3 M^5}{90} \frac{r^5}{r^5}, \)

\( h_{22}^{(2)} = \frac{-1 + 3A_2 M^3}{3} \frac{r^3}{r^3} + \frac{13 - 27A_2 M^4}{9} \frac{r^4}{r^4} - \frac{94 - 441A_1 + 882A_2 - 468A_3 + 54A_4 M^5}{630} \frac{r^5}{r^5}, \)

\( h_{22}^{(4)} = \frac{-57 - 72A_3 - 3A_4 M^5}{35} \frac{r^5}{r^5}. \) (93)

\( h_{33}^{(0)} = \frac{3 M}{r} + \frac{9 M^2}{4 r^2} + \frac{5 - 3A_1 - 3A_2 M^3}{6} \frac{r^3}{r^3} + \frac{19 - 27A_2 M^4}{36} \frac{r^4}{r^4} - \frac{28 - 9A_1 + 18A_2 - 36A_3 M^5}{90} \frac{r^5}{r^5}, \)

\( h_{33}^{(2)} = \frac{-M^3}{r^3} + \frac{7 - 27A_2 M^4}{18} \frac{r^4}{r^4} - \frac{214 - 63A_1 + 126A_2 - 252A_3 M^5}{126} \frac{r^5}{r^5}, \)

\( h_{33}^{(4)} = \frac{19 M^5}{35} \frac{r^5}{r^5}. \) (94)

Since now we have two sets of coordinates that can be considered common to the Schwarzschild and Curzon metrics, we can compare the two metrics. From the results in Section 4 we see that both metrics coincide if terms of order \( M^3/R^3 \) (or \( M^3/r^3 \)) and higher are neglected. This means that in most practical cases (such as in relativistic celestial mechanics) these two metrics are indistinguishable if used in harmonic or quo-harmonic coordinates. Notice that if one naively identifies the coordinates used in (I) and (3) the difference between the Schwarzschild and Curzon metrics starts with terms proportional to \( M/\bar{r} = M/\tilde{r} \). These remarks have no special utility except that they demystify the apparent weirdness of the Curzon solution.

6 Kerr metric

To write the Kerr metric we will use again (71)–(73) and the metric coefficients \( g_{\mu\nu} \) will satisfy (74). The \( h_{\mu\nu} \) quantities of (75) are functions of \( R \) and
Θ or of \( r \) and \( \theta \) and may be written as

\[
h_{\mu\nu} = \sum_l h_{\mu\nu}^{(2l)} P_{2l}, \quad h_{12} = \sum_l h_{12}^{(2l)} P_{2l}, \quad h_{03} = \sum_l h_{03}^{(2l+1)} P_{2l+1},
\]

(95)

where \( h_{\mu\nu}^{(n)} \) are only functions of \( R \) or \( r \) and \( P_n \) and \( P_n^l \) are the Legendre polynomials and associated functions in \( \cos \Theta \) or \( \cos \theta \).

We will use the dimensionless quantity

\[
\alpha \equiv \frac{a}{M}
\]

(96)

to make easier the comparison with the results for the Schwarzschild metric, which is recovered in the limit \( \alpha = 0 \).

### 6.1 Kerr metric in harmonic coordinates

To preserve the axial symmetry and the reflection symmetry with respect to the plane \( z = 0 \) we perform a coordinate transformation \((\hat{r}, \hat{\theta}, \hat{\phi}) \rightarrow (R, \Theta, \phi)\), with \( R = R(\hat{r}, \hat{\theta}), \Theta = \Theta(\hat{r}, \hat{\theta}), \phi = \hat{\phi} \). If \( \lim_{\hat{r} \to \infty} R = \hat{r} \) and the Cartesian coordinates associated to the spherical \((R, \Theta, \phi)\) are harmonic, the transformation is given to order five in \( 1/R \) (now we have two dimensionless expansion parameters: \( M/R \) and \( a/R \)) by

\[
\frac{R}{\hat{r}} = 1 - \frac{M}{\hat{r}} + \frac{\alpha^2}{2} \sin^2 \hat{\theta} \frac{M^2}{\hat{r}^2} - \frac{2D_1 + \alpha^2 D_2 \left(2 - 3 \sin^2 \hat{\theta}\right) M^3}{6 \hat{r}^3} - \frac{16D_1 + 4\alpha^2 \left[4D_2 + (3 - 6D_2) \sin^2 \hat{\theta}\right]}{24 \hat{r}^4} - \frac{1}{120} \left[144D_1 + 30\alpha^2 \left(2D_2 + D_3 - 16D_4\right) + 5\alpha^2 \left(156 - 8D_1 - 18D_2 - 9D_3 + 480D_4 - 20\alpha^2 D_2\right) \sin^2 \hat{\theta}\right. \\
\left. - 60\alpha^2 \left(13 + 35D_4 - 2\alpha^2 D_2\right) \sin^4 \hat{\theta}\right] \frac{M^5}{\hat{r}^5},
\]

(97)

\[
\frac{\cos \Theta}{\cos \hat{\theta}} = 1 - \frac{\alpha^2}{2} \sin^2 \hat{\theta} \frac{M^2}{\hat{r}^2} - \frac{\alpha^2 \left(1 + D_2\right) M^3}{2 \hat{r}^3} - \frac{\alpha^2 \left(12D_2 - 3\alpha^2 \sin^2 \hat{\theta}\right) \sin^2 \hat{\theta} M^4}{8 \hat{r}^4} + \frac{\alpha^2}{8} \left[52 - 32D_2 + 2D_3 + 80D_4 - 4\alpha^2 D_2
\right. \\
\left. - \left(52 + 140D_4 - \alpha^2 \left(5 + 12D_2\right)\right) \sin^2 \hat{\theta}\right] \sin^2 \hat{\theta} \frac{M^5}{\hat{r}^5},
\]

(98)
where $D_1, D_2, D_3, D_4$ are integration constants.

In this system of harmonic coordinates the functions $h^{(i)}_{\mu\nu}$ for the non-null components of the Kerr metric are written as follows to order five:

\begin{align*}
h^{(0)}_{00} &= 2 \frac{M}{R} - 2 \frac{M^2}{R^2} + 2 \frac{M^3}{R^3} - \frac{6 + 2D_1 M^4}{3} \frac{R^4}{R^4} + \frac{6 + 4D_1 - 2\alpha^2 M^5}{3} \frac{R^5}{R^5}, \\
\end{align*}

\begin{align*}
h^{(2)}_{00} &= -2\alpha^2 \frac{M^3}{R^3} + \alpha^2 \frac{18 - 2D_2 M^4}{3} \frac{R^4}{R^4} - \alpha^2 \frac{34 - 4D_2 M^5}{3} \frac{R^5}{R^5}, \\
\end{align*}

\begin{align*}
h^{(4)}_{00} &= 2\alpha^4 \frac{M^5}{R^5},
\end{align*}

\begin{align*}
h^{(0)}_{11} &= 2 \frac{M}{R} + 2 \frac{M^2}{R^2} + \frac{6 - 4D_1 - 4\alpha^2 M^3}{3} \frac{R^3}{R^3} + \frac{6 - 10D_1 - 10\alpha^2 M^4}{3} \frac{R^4}{R^4} \\
&\quad + \frac{30 - 84D_1 - 94\alpha^2 M^5}{15} \frac{R^5}{R^5}, \\
\end{align*}

\begin{align*}
h^{(2)}_{11} &= -\alpha^2 \frac{2 + 4D_2 M^3}{3} \frac{R^3}{R^3} + \alpha^2 \frac{4 - 10D_2 M^4}{3} \frac{R^4}{R^4} - \alpha^2 \frac{202 + 42D_3 - 24\alpha^2 M^5}{21} \frac{R^5}{R^5}, \\
\end{align*}

\begin{align*}
h^{(4)}_{11} &= \alpha^4 \frac{416 + 1120D_4 + 30\alpha^2 M^5}{35} \frac{R^5}{R^5},
\end{align*}

\begin{align*}
h^{(2)}_{12} &= \alpha^2 \frac{1 - D_2 M^3}{6} \frac{R^3}{R^3} + \alpha^2 \frac{2 - D_2 M^4}{3} \frac{R^4}{R^4} + \alpha^2 \frac{100 - 42D_2 + 14D_3 - 8\alpha^2 M^5}{21} \frac{R^5}{R^5}, \\
\end{align*}

\begin{align*}
h^{(4)}_{12} &= \alpha^2 \frac{52 + 140D_4 - 5\alpha^2 M^5}{140} \frac{R^5}{R^5},
\end{align*}

\begin{align*}
h^{(0)}_{22} &= 2 \frac{M}{R} + \frac{M^2}{R^2} + \frac{2D_1 + \alpha^2(1 + D_2) M^3}{3} \frac{R^3}{R^3} + \frac{2D_1 + \alpha^2(1 + 2D_2) M^4}{3} \frac{R^4}{R^4} \\
&\quad + \frac{12D_1 - \alpha^2(56 - 30D_2 + 5D_3 + 60D_4) + 5\alpha^4 M^5}{30} \frac{R^5}{R^5}, \\
\end{align*}

\begin{align*}
h^{(2)}_{22} &= -\alpha^2 \frac{4 + 2D_2 M^3}{3} \frac{R^3}{R^3} + \alpha^2 \frac{2 - 6D_2 M^4}{3} \frac{R^4}{R^4} \\
&\quad + \alpha^2 \frac{208 - 210D_2 + 49D_3 - 420D_4 - 13\alpha^2 M^5}{42} \frac{R^5}{R^5}, \\
\end{align*}

\begin{align*}
h^{(4)}_{22} &= \alpha^2 \frac{312 + 840D_4 + 40\alpha^2 M^5}{35} \frac{R^5}{R^5},
\end{align*}
\begin{align*}
h_{33}^{(0)} &= \frac{2M}{R} + \frac{M^2}{R^2} + \frac{2D_1 + \alpha^2(3 - D_2)}{3} \frac{M^3}{R^3} + \frac{2D_1 - \alpha^2(1 + 2D_2)}{3} \frac{M^4}{R^4} \\
&\quad + \frac{12D_1 + 5\alpha^2(24 - 6D_2 + D_3 + 12D_4) - 5\alpha^4 M^5}{30} \frac{M^5}{R^5}, \\
h_{33}^{(2)} &= -2\alpha^2 \frac{M^3}{R^3} + \alpha^2 \frac{4 - 2D_2}{3} \frac{M^4}{R^4} \\
&\quad + \alpha^2 \frac{48 - 18D_2 + 5D_3 + 60D_4 - 5\alpha^2 M^5}{6} \frac{M^5}{R^5}, \\
h_{33}^{(4)} &= 2\alpha^4 \frac{M^5}{R^5}, \\
h_{03}^{(1)} &= 2\alpha \frac{M^2}{R^2} - 2\alpha \frac{M^3}{R^3} + 2\alpha \frac{M^4}{R^4} - \alpha \frac{30 + 10D_1 + \alpha^2(6 + 4D_2)}{15} \frac{M^5}{R^5}, \\
h_{03}^{(3)} &= -\frac{2}{3} \alpha^3 \frac{M^4}{R^4} + \alpha^3 \frac{26 - 6D_2 M^5}{15} \frac{M^5}{R^5}.
\end{align*}

6.2 Kerr metric in quo-harmonic coordinates

We can also obtain a system of quo-harmonic coordinates preserving the axial and specular symmetries by using a change of coordinate in the form $(\tilde{r}, \tilde{\theta}, \tilde{\phi}) \to (r, \theta, \phi)$, with $r = r(\tilde{r}, \tilde{\theta}), \theta = \theta(\tilde{r}, \tilde{\theta}), \phi = \phi$. The Cartesian coordinates associated to $(r, \theta, \phi)$ will be quo-harmonic and $\lim_{r \to \infty} r = \tilde{r}$ will be satisfied if, up to order $1/r^5$, we choose

\begin{align*}
\frac{r}{\tilde{r}} &= 1 - \frac{3}{2} \frac{M}{2\tilde{r}} + \alpha^2 \frac{2}{2} \sin^2 \tilde{\theta} \frac{M^2}{\tilde{r}^2} - \frac{3 - 3B_1 - \alpha^2 B_2 \left(4 - 6 \sin^2 \tilde{\theta}\right)}{12} \frac{M^3}{\tilde{r}^3} \\
&\quad - \frac{3 - 3B_1 - 2\alpha^2 \left[2B_2 + (1 - 3B_2) \sin^2 \tilde{\theta}\right] + \alpha^4 \sin^4 \tilde{\theta}}{12} \frac{M^4}{\tilde{r}^4} \\
&\quad - \frac{6}{35} \alpha^2 \left(2 - 3 \sin^2 \tilde{\theta}\right) \frac{M^5}{\tilde{r}^5} \ln \frac{M}{\tilde{r}} \\
&\quad - \frac{1}{3360} \left[1890 - 1890B_1 - 12\alpha^2 \left(38 + 175B_2 + 96B_3 + 2B_4\right) + 840\alpha^4ight. \\
&\quad \left.\quad - 2\alpha^2 \left(1044 - 420B_1 - 1575B_2 - 864B_3 - 60B_4\right)ight. \\
&\quad \left.\quad \quad + 140\alpha^2 (9 - 10B_2)\right] \sin^2 \tilde{\theta} \\
&\quad - 105\alpha^2 \left(B_4 - \alpha^2 (15 - 32B_2)\right) \sin^4 \tilde{\theta} \frac{M^5}{\tilde{r}^5},
\end{align*}
\[
\cos \theta \cos \theta = 1 - \frac{\alpha^2}{2} \sin^2 \theta \frac{M^2}{r^2} - \alpha^2 \frac{5 - 2B_2}{4} \sin^2 \theta \frac{M^3}{r^3}
\]
\[
- \frac{\alpha^2}{8} \left( 17 - 12B_2 - 3\alpha^2 \sin^2 \theta \right) \sin^2 \theta \frac{M^4}{r^4}
\]
\[
+ \frac{12}{35} \alpha^2 \sin^2 \theta \frac{M^5}{r^5} \ln \frac{M}{r}
\]
\[
- \frac{\alpha^2}{1120} \left( 3694 - 3920B_2 + 384B_3 - 20B_4 - 140\alpha^2 (3 + 4B_2)
\right.
\]
\[
\left. + \left( 35B_4 - 35\alpha^2 (37 - 48B_2) \right) \sin^2 \theta \right) \sin^2 \theta \frac{M^5}{r^5},
\]
(106)

where we have again four integration constants: \( B_1, B_2, B_3 \) and \( B_4 \).

Notice that, unlike in all the preceding examples, here we do not have only powers of \( 1/r \) but also terms of the form \( \ln r/r^5 \).

Functions \( h_{\mu \nu}^{(0)} \) for the non-null components of Kerr metric appear as follows in this system of quo-harmonic coordinates:

\[
h_{00}^{(0)} = 2 \frac{M}{r} - 3 \frac{M^2}{r^2} + \frac{9}{2} \frac{M^3}{r^3} - \frac{29 - 2B_1 M^4}{4} \frac{M^4}{r^4} + \frac{297 - 54B_1 + 8\alpha^2 M^5}{24} \frac{M^5}{r^5},
\]

\[
h_{00}^{(2)} = -2\alpha^2 \frac{M^3}{r^3} + \alpha^2 \frac{27 + 2B_2 M^4}{3} \frac{M^4}{r^4} - \alpha^2 \frac{82 + 9B_2 M^5}{3} \frac{M^5}{r^5},
\]

\[
h_{00}^{(4)} = 2\alpha^4 \frac{M^5}{r^5},
\]

(107)

\[
h_{11}^{(0)} = 2 \frac{M}{r} + \frac{M^2}{r^2} - \frac{3}{6} B_1 + \frac{8\alpha^2 M^3}{12} \frac{M^3}{r^3} + \frac{3B_1 + 120\alpha^2 M^4}{12} \frac{M^4}{r^4}
\]
\[
- \frac{25 - 30B_1 + 104\alpha^2 M^5}{40} \frac{M^5}{r^5},
\]

\[
h_{11}^{(2)} = -\alpha^2 \frac{2 - 4B_2 M^3}{3} \frac{M^3}{r^3} - \alpha^2 \frac{2 - B_2 M^3}{3} \frac{M^4}{r^4} - \frac{96 \alpha^2 M^5}{35} \frac{M^5}{r^5} \ln \frac{M}{r} + \frac{96 \alpha^2 B_3 M^5}{35} \frac{M^5}{r^5},
\]

\[
h_{11}^{(4)} = \frac{2}{35} \alpha^2 B_4 \frac{M^5}{r^5},
\]

(108)

\[
h_{12}^{(2)} = -\alpha^2 \frac{7 - 2B_2 M^3}{12} \frac{M^3}{r^3} + \alpha^2 \frac{5 + 4B_2 M^4}{12} \frac{M^4}{r^4} + \alpha^2 \frac{32 M^5}{35} \frac{M^5}{r^5} \ln \frac{M}{r}
\]
\[
- \alpha^2 \frac{61 - 210B_2 + 768B_3 M^5}{840} \frac{M^5}{r^5},
\]

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We may now compare the Schwarzschild, Curzon and Kerr metrics in a common (harmonic or quo-harmonic) set of coordinates defined with independence of the models being analysed. If we neglect again terms of order \(1/R^3\) (or \(1/r^3\)) and higher, we see that the only difference between the two static cases and the Kerr metric is the term \(h_{03}^{(1)} = 2Ma/R^2\) (or \(h_{03}^{(1)} = 2Ma/r^2\)).

\[
\begin{align*}
h_{12}^{(4)} &= \frac{\alpha^2 B_4 + 315\alpha^2 M^5}{560} \frac{1}{r^5}, \\
h_{22}^{(0)} &= \frac{3M}{r} + \frac{9M^2}{4r^2} + \frac{3 - 3B_1 + \alpha^2(5 - 2B_2)}{6} \frac{M^3}{r^3} - \frac{\alpha^2}{6} \frac{1 + 3B_2 M^4}{r^4} \\
&- \frac{8}{35} \alpha^2 M^5 \frac{\ln M}{r} + \alpha^2 \frac{48 - 70B_2 + 192B_3 - 3B_4 - 105\alpha^2 M^5}{840} \frac{1}{r^5}, \\
h_{22}^{(2)} &= -\frac{\alpha^2}{3} \frac{10 - 2B_2 M^3}{r^3} - \frac{\alpha^2}{3} \frac{8}{r^4} \frac{1 - 6B_2 M^4}{r^3} \\
&+ \frac{8}{5} \alpha^2 M^5 \frac{\ln M}{r} + \alpha^2 \frac{168 + 490B_2 - 1344B_3 - 15B_4 - 525\alpha^2 M^5}{840} \frac{1}{r^5}, \\
h_{22}^{(4)} &= \frac{\alpha^2}{70} \frac{3B_4 + 315\alpha^2 M^5}{r^5}, \\
h_{33}^{(0)} &= \frac{3M}{r} + \frac{9M^2}{4r^2} + \frac{3 - 3B_1 + \alpha^2(3 + 2B_2)}{6} \frac{M^3}{r^3} - \frac{\alpha^2}{2} \frac{5 - B_2 M^4}{r^4} \\
&+ \frac{8}{35} \alpha^2 M^5 \frac{\ln M}{r} + \alpha^2 \frac{2864 + 70B_2 - 192B_3 + 3B_4 + 105\alpha^2 M^5}{840} \frac{1}{r^5}, \\
h_{33}^{(2)} &= -3\alpha^2 \frac{M^3}{r^3} + \alpha^2 \frac{(2 + B_2) M^4}{r^4} + \frac{8}{7} \alpha^2 \frac{M^5}{r^5} \frac{\ln M}{r} \\
&- \alpha^2 \frac{2648 - 350B_2 + 960B_3 - 15B_4 - 525\alpha^2 M^5}{840} \frac{1}{r^5}, \\
h_{33}^{(4)} &= \frac{3\alpha^4}{70} \frac{M^5}{r^5}, \\
h_{03}^{(1)} &= \frac{2\alpha M^2}{r^2} - 3\alpha \frac{M^3}{r^3} + 9 \alpha \frac{2M^4}{r^4} - \frac{\alpha}{6} \frac{435 - 30B_1 + 4\alpha^2(15 - 4B_2) M^5}{r^5}, \\
h_{03}^{(3)} &= -\frac{2}{3} \frac{\alpha^3 M^4}{r^4} + \alpha^3 \frac{35 + 6B_2 M^5}{15} \frac{1}{r^5}.
\end{align*}
\]
7 Final comments

The approximation scheme in Sections 4, 5 and 6 is based on expanding an exact solution of the complete Einstein’s equations about the infinity point of an appropriate radial coordinate. In the so-called linear approximation discussed in Section 3 the exact solutions of the approximated (linearized) field equations were used. Nevertheless, by carefully selecting some terms from the expansions in Sections 4, 5 and 6 one may recover exact solutions of the linearized theory.

For instance, from the expressions in Section 4.1 for the Schwarzschild metric in harmonic coordinates, one may easily find the following solution of the linear approximation:

\[
\begin{align*}
    h_{00} &= \frac{2M}{R}, \\
    h_{0i} &= 0.
\end{align*}
\]  

The same happens (after replacing \( R \) with \( r \)) in the expressions in Section 4.2 for the Schwarzschild metric in quo-harmonic coordinates. Notice that we are here writing only the metric components that give the common invariant multipole structure, as discussed in Section 3.3.

In the case of the expressions in Section 5.1 for the Curzon metric in harmonic coordinates, one may easily find the following solution of the linear approximation with more complex angular structure:

\[
\begin{align*}
    h_{00} &= \frac{2M}{R} - \frac{2M^3}{3R^3}P_2(\cos \Theta) + \frac{38M^5}{105 R^5}P_4(\cos \Theta), \\
    h_{0i} &= 0.
\end{align*}
\]

Substituting \( r \) for \( R \) and \( \theta \) for \( \Theta \) in this result we get the solution of the linearized field equations contained in the approximated solution of Section 5.2 for quo-harmonic coordinates.

In the case of the Kerr metric, the solutions of the linearized theory we can extract from Section 3 are

\[
\begin{align*}
    h_{00} &= \frac{2M}{R} - \frac{2Ma^2}{R^3}P_2(\cos \Theta) + \frac{2Ma^4}{R^5}P_4(\cos \Theta), \\
    h_{01} &= h_{02} = 0, \\
    h_{03} &= \frac{4Ma}{R^2}P_1^1(\cos \Theta) - \frac{4Ma^3}{3 R^4}P_3^1(\cos \Theta)
\end{align*}
\]

in harmonic coordinates, as well as in quo-harmonic coordinates if one uses \((r, \theta)\) instead of \((R, \Theta)\).
Acknowledgments

The work of JMA was supported by the University of the Basque Country through the research project UPV172.310-EB150/98 and the General Research Grant UPV172.310-G02/99. Ll. Bel gratefully acknowledges as visiting professor the hospitality of the UPV/EHU. AM was supported by the research projects CICYT BF2000-0604 and CIRIT 2000SGR-00023. JM and ER were supported by the research project BFM2000 - 1322 (Ministerio de Ciencia y Tecnologia, Spain)

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