Expansiveness for the geodesic and horocycle flows on compact Riemann surfaces of constant negative curvature

HUYNH MINH HIEN
Department of Mathematics,
Quy Nhon University,
170 An Duong Vuong, Quy Nhon, Vietnam;
e-mail: huynhminhhien@qnu.edu.vn

Abstract

We study expansive properties for the geodesic and horocycle flows on Riemann surfaces of constant negative curvature. It is well-known that the geodesic flow is expansive in the sense of Bowen-Walters and the horocycle flow is positive and negative separating in the sense of Gura. In this paper, we give a new proof for the expansiveness of the geodesic flow and show that the horocycle flow is positive and negative kinematic expansive in the sense of Artigue as well as expansive in the sense of Katok/Hasselblatt but not expansive in the sense of Bowen-Walters. We also point out that the geodesic flow is neither positive nor negative separating.

Keywords: expansiveness; geodesic flow; horocycle flow

1 Introduction

The study of expansive flows started in 1972 with the works of Bowen/Walters [8] and Flinn [10]. In [8], the authors generalized the definition of expansive homeomorphisms to introduce a reasonable definition of expansiveness for flows that is called ‘expansive in the sense of Bowen and Walters’ (or shortly
BW–expansive). Since then, there have been different varieties of expansive flows introduced. In 1984, Komuro [14] gave the notions ‘C–expansive’ (as the same to ‘BW–expansive’), ‘K–expansive’ (as the same to ‘expansive’ in the same of Flinn) and ‘K∗–expansive’ to investigate geometric Lorenz attractors. In general, ‘K–expansive’ is weaker than ‘BW–expansive’ but stronger than ‘K∗–expansive’. In the case of fixed-point-free flows on compact metric spaces, the three notions are equivalent (see [5, 16]).

A different and very interesting kind of expansiveness called ‘separating’ was discovered by Gura [11] in 1984. The author showed that the horocycle flow on a compact surface with negative curvature is positive and negative separating. His definition in [11] requires to separate every pair of points in different orbits. The author also proved a remarkable result: every global time change of such flow is positive and negative separating. In 1995, Katok/Hasselblatt [13] gave another kind of expansiveness (also called KH–expansiveness) which is weaker than BW–expansiveness but implies separation. It then was showed by Artigue [5] that a flow is KH–expansive if and only if it is separating and the set of its fixed points is open.

Recently, in 2016, Artigue [4] used the term ‘geometric expansive’ as K–expansive and introduced the term ‘kinematic expansive’ which is a stronger property than separation and weaker than BW–expansiveness. The author also considered the forms of ‘strong kinematic expansive’, ‘geometric separating’, ‘strong separating’ and ‘separating’ flows. Examples are given to analyze the relationships among the above definitions. Some interesting properties are proved in different contexts: surfaces, suspension flows and compact metric spaces.

Regarding properties of the geodesic flow, in 1967, Anosov [1] showed that the geodesic flow on compact Riemannian manifolds with negative curvature is hyperbolic. In 1972, it was proved by Bowen [7] that hyperbolic flows are BW–expansive and consequently the geodesic flow on compact Riemannian manifolds of negative curvature is BW–expansive. As mentioned above, in 1984, Gura [11] showed that the horocycle on compact surface of negative curvature is positive and negative separating.

The paper is organized as follows. In the next section, we introduce the necessary background material which is well-known in principle [6, 9, 17]. Section 3 is devoted to consider expansive properties mentioned above for the geodesic and horocycle flows on compact factors of the hyperbolic plane. A new detailed proof for BW–expansiveness of the geodesic flow (Theorem 3.2) via a property of the injectivity radius is given. The horocycle flow on
this model is not only positive/negative separating but also positive/negative kinematic expansive (Theorem 3.5) as well as KH–expansive (Theorem 3.9).

In the end, we point out that the horocycle flow is not BW–expansive and the geodesic flow is neither positive nor negative separating; see Remark 3.6.

2 Preliminaries

We consider the geodesic and horocycle flows on compact Riemann surfaces of constant negative curvature. It is well-known that any compact orientable surface with constant negative curvature is isometric to a factor \( \Gamma \setminus \mathbb{H}^2 \) = \( \{ \Gamma z, z \in \mathbb{H}^2 \} \), where \( \mathbb{H}^2 = \{ z = x + iy \in \mathbb{C} : y > 0 \} \) is the hyperbolic plane endowed with the hyperbolic metric \( ds^2 = \frac{dx^2 + dy^2}{y^2} \) and \( \Gamma \) is a Fuchsian group that is discrete subgroup of the group \( \text{PSL}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R}) / \{ \pm E_2 \} \); where \( \text{SL}(2, \mathbb{R}) \) is the group of all real \( 2 \times 2 \) matrices with unity determinant, and \( E_2 \) denotes the unit matrix. In the hyperbolic plane model, geodesics are vertical lines and semi-circles centered on the real axis. The group \( \text{PSL}(2, \mathbb{R}) \) acts transitively on \( \mathbb{H}^2 \) by Möbius transformations \( z \mapsto \frac{az + b}{cz + d} \). If the action is free of fixed points, then the factor \( \Gamma \setminus \mathbb{H}^2 \) has a Riemann surface structure that is a closed Riemann surface of genus at least 2 and has the hyperbolic plane \( \mathbb{H}^2 \) as the universal covering. The unit tangent bundle \( T^1 \mathbb{H}^2 \) is isometric to the group \( \text{PSL}(2, \mathbb{R}) \) and as a consequence, the unit tangent bundle \( T^1 (\Gamma \setminus \mathbb{H}^2) \) is isometric to the quotient space \( \Gamma \setminus \text{PSL}(2, \mathbb{R}) = \{ \Gamma g, g \in \text{PSL}(2, \mathbb{R}) \} \), which is the system of right co-sets of \( \Gamma \) in \( \text{PSL}(2, \mathbb{R}) \), by an isometry \( \Xi \). Since \( \text{PSL}(2, \mathbb{R}) \) is connected, also \( \Gamma \setminus \text{PSL}(2, \mathbb{R}) \) is connected. Furthermore, \( X = \Gamma \setminus \text{PSL}(2, \mathbb{R}) \) is a three-dimensional real analytic manifold.

The geodesic flow on \( T^1 \mathbb{H}^2 \) can be described as the flow \( \varphi_t^G(g) = g a_t \) on \( G := \text{PSL}(2, \mathbb{R}) \), where \( a_t \in G \) denotes the equivalence class obtained from the matrix \( A_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \), and whence the geodesic flow \( (\varphi_t^X)_{t \in \mathbb{R}} \) on \( X = T^1 (\Gamma \setminus \mathbb{H}^2) \) can be described as the ‘quotient flow’

\[
\varphi_t^X(\Gamma g) = \Gamma g a_t
\]

on \( X = \Gamma \setminus \text{PSL}(2, \mathbb{R}) \) by the conjugate relation

\[
\varphi_t^X = \Xi^{-1} \circ \varphi_t^X \circ \Xi.
\] (2.1)

A horocycle is a (euclidean) circle tangent to real axis or a horizontal line. The stable and unstable horocycle flows on \( T^1 \mathbb{H}^2 \) can be described as
the flows: $\theta_t^g = g b_t$, $\eta_t^g = g c_t$ on $G$; where $b_t, c_t \in \text{PSL}(2, \mathbb{R})$ denote the equivalence classes obtained from the matrices $B_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, C_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \in \text{SL}(2, \mathbb{R})$. Therefore the stable and unstable horocycle flows $(\theta_t^X)_{t \in \mathbb{R}}, (\eta_t^X)_{t \in \mathbb{R}}$ on $X = T^1(\Gamma \backslash \mathbb{H}^2)$ can be equivalently described as the flows

$$
\theta_t^X(\Gamma g) = \Gamma g b_t, \quad \eta_t^X(\Gamma g) = \Gamma g c_t
$$
on $X = \Gamma \backslash \text{PSL}(2, \mathbb{R})$ by the conjugate relations

$$
\theta_t^X = \Xi^{-1} \circ \theta_t^X \circ \Xi, \quad \eta_t^X = \Xi^{-1} \circ \eta_t^X \circ \Xi \quad \text{for all} \quad t \in \mathbb{R}.
$$

(2.2)

There are some more advantages to work on $X = \Gamma \backslash \text{PSL}(2, \mathbb{R})$ rather than on $X = T^1(\Gamma \backslash \mathbb{H}^2)$. For example, one can calculate explicitly the stable and unstable manifolds at a point $x$ to be

$$
W_s^X(x) = \left\{ \theta_t^X(x), t \in \mathbb{R} \right\} \quad \text{and} \quad W_u^X(x) = \left\{ \eta_t^X(x), t \in \mathbb{R} \right\}.
$$

The flow $(\varphi_t^X)_{t \in \mathbb{R}}$ is hyperbolic, that is, for every $x \in X$ there exists an orthogonal and $(\varphi_t^X)_{t \in \mathbb{R}}$-stable splitting of the tangent space $T_x X$

$$
T_x X = E^0(x) \oplus E^s(x) \oplus E^u(x)
$$
such that the differential of the flow $(\varphi_t^X)_{t \in \mathbb{R}}$ is uniformly expanding on $E^u(x)$, uniformly contracting on $E^s(x)$ and isometric on $E^0(x) = \left\langle \frac{d}{dt} \varphi_t^X(x) \big|_{t=0} \right\rangle$. One can choose

$$
E^s(x) = \left\langle \frac{d}{dt} \theta_t^X(x) \big|_{t=0} \right\rangle \quad \text{and} \quad E^u(x) = \left\langle \frac{d}{dt} \eta_t^X(x) \big|_{t=0} \right\rangle.
$$

The horocycle flows $(\theta_t^X)_{t \in \mathbb{R}}$ and $(\eta_t^X)_{t \in \mathbb{R}}$ are ergodic [15]. If the space $\Gamma \backslash \mathbb{H}^2$ has a finite volume, each orbit is either periodic or dense. In the case that the space $\Gamma \backslash \mathbb{H}^2$ is compact, there are no periodic orbits for the horocycle flows.

General references for this section are [6, 9], and these works may be consulted for the proofs to all results which are stated above. In what follows, we will drop the superscript $X$ from $(\varphi_t^X)_{t \in \mathbb{R}}, (\theta_t^X)_{t \in \mathbb{R}}, (\eta_t^X)_{t \in \mathbb{R}}$ to simplify notation. We consider the stable horocycle flow only and use the term ‘horocycle flow’ for it. In the whole present paper, we always assume the action of $\Gamma$ on $\mathbb{H}^2$ to be free (of fixed points) and the factor $\Gamma \backslash \mathbb{H}^2$ to be compact. Note that $\Gamma \backslash \mathbb{H}^2$ is compact if and only if $\Gamma \backslash \text{PSL}(2, \mathbb{R})$ is compact.

In the rest of this section we collect some notions and useful technical results.
Lemma 2.1 There is a natural Riemannian metric on $G = \text{PSL}(2, \mathbb{R})$ such that the induced metric function $d_G$ is left-invariant under $G$ and

$$d_G(a_t, e) = \frac{1}{\sqrt{2}}|t|, \quad d_G(b_t, e) \leq |t|, \quad d_G(c_t, e) < |t| \quad \text{for all} \quad t \in \mathbb{R}$$

where $e = \pi(E_2)$ is the unity of $G$.

We define a metric function $d_X$ on $X = \Gamma \backslash \text{PSL}(2, \mathbb{R})$ by

$$d_X(x_1, x_2) = \inf_{\gamma_1, \gamma_2 \in \Gamma} d_G(\gamma_1 g_1, \gamma_2 g_2) = \inf_{\gamma \in \Gamma} d_G(g_1, \gamma g_2),$$

where $x_1 = \Gamma g_1, x_2 = \Gamma g_2$. In fact, if $X$ is compact, one can prove that the infimum is a minimum:

$$d_X(x_1, x_2) = \min_{\gamma \in \Gamma} d_G(g_1, \gamma g_2).$$

It is possible to derive a uniform lower bound on $d_G(g, \gamma g)$ for $g \in \text{PSL}(2, \mathbb{R})$ and $\gamma \in \Gamma \setminus \{e\}$.

**Lemma 2.2** If the space $X = \Gamma \backslash \text{PSL}(2, \mathbb{R})$ is compact, then there exists $\sigma_0 > 0$ such that

$$d_G(\gamma g, g) > \sigma_0 \quad \text{for all} \quad \gamma \in \Gamma \setminus \{e\}.$$

The number $\sigma_0$ is called an injectivity radius. See [17, Lemma 1, p. 237] for a similar result.

For $g = \pi(G) \in \text{PSL}(2, \mathbb{R}), G = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the trace of $g$ is defined by

$$\text{tr}(g) = |a + d|.$$ 

If the action of $\Gamma$ on $\mathbb{H}^2$ is free and the factor $\Gamma \backslash \mathbb{H}^2$ is compact then all elements $g \in \Gamma \setminus \{e\}$ are hyperbolic [17, Theorem 6.6.6], i.e. $\text{tr}(g) > 2$. Furthermore, one gets a stronger result:

**Lemma 2.3** If the factor $\Gamma \backslash \mathbb{H}^2$ is compact, then there exists $\varepsilon_* > 0$ such that

$$\text{tr}(g) \geq 2 + \varepsilon_* \quad \text{for all} \quad g \in \Gamma \setminus \{e\}.$$

Here are some more auxiliary results.
Lemma 2.4 (a) For every $\delta > 0$ there is $\rho > 0$ with the following property. If $G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in \text{SL}(2, \mathbb{R})$ satisfies $|g_{11} - 1| + |g_{12}| + |g_{21}| + |g_{22} - 1| < \rho$ then $d_G(g, e) < \delta$ for $g = \pi(G)$, where $\pi : \text{SL}(2, \mathbb{R}) \to \text{PSL}(2, \mathbb{R})$ is the natural projection.

(b) For every $\varepsilon > 0$ there is $\delta > 0$ with the following property. If $g, h \in G$ satisfying $d_G(g, e) < \delta$ then there are $G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ such that $g = \pi(G), h = \pi(H)$ and

$$|g_{11} - 1| + |g_{12}| + |g_{21}| + |g_{22} - 1| < \varepsilon.$$ 

Proof: (a) See [12, Lemma 2.17 (a)] for a proof.

(b) Indeed, suppose on contrary that

$$|g_{11} - 1| + |g_{12}| + |g_{21}| + |g_{22} - 1| \geq \varepsilon_0$$

(2.3)

for some sequence $d_G(g^j, e) \to 0$ and all $G^j \in \text{SL}(2, \mathbb{R})$ such that $g^j = \pi(G^j)$. For $j \in \mathbb{N}$ take any $G^j \in \text{SL}(2, \mathbb{R})$ so that $g^j = \pi(G^j)$. From (a) we deduce that $|g_{12}^j| + |g_{21}^j| \to 0$, $|g_{11}^j| \to 1$, $|g_{22}^j| \to 1$, and $g_{11}^j g_{22}^j \to 1$. Thus, along a subsequence which is not renamed, either $g_{11}^j \to 1, g_{22}^j \to 1$ or $g_{11}^j \to -1, g_{22}^j \to -1$. The first case is impossible in view of (2.3). In the second case we consider $\tilde{G}^j = -G^j$ which also has $g^j = \pi(\tilde{G}^j)$. But then (2.3) implies

$$|g_{11}^j + 1| + |g_{12}^j| + |g_{21}^j| + |g_{22}^j + 1| \geq \varepsilon_0,$$

and once more this is impossible.

Definition 2.5 Let $\phi : \mathbb{R} \times M \to M$ be a flow.

(a) A point $x \in M$ is called a fixed point (or singular point) if

$$\phi_t(x) = x \quad \text{for all} \quad t \in \mathbb{R}.$$ 

(b) A point $x \in M$ is called a periodic point if there is $T > 0$ such that

$$\phi_T(x) = x.$$
Proposition 2.6 Assume that \( X = \Gamma \backslash \text{PSL}(2, \mathbb{R}) \) is compact. Then the flow \((\theta_t)_{t \in \mathbb{R}}\) does not have a periodic point. In particular, it has no fixed points.

Proof: Suppose in contrary that \( x = \Gamma g \) is a periodic point of \((\theta_t)_{t \in \mathbb{R}}\), i.e. \( \theta_T(x) = x \) for some \( T > 0 \). Then \( g^{-1} \gamma g = b_T \) for some \( \gamma \in \Gamma \) implies \( \text{tr}(\gamma) = \text{tr}(b_T) = 2 \). It follows from Lemma 2.3 that \( \gamma = e \). Therefore \( g = gb_T \) yields \( T = 0 \) which is a contradiction. The latter assertion is obvious. \( \square \)

Definition 2.7 Two continuous flows \( \phi : \mathbb{R} \times X \to X \) and \( \psi : \mathbb{R} \times Y \to Y \) is said to be equivalent if there is a homeomorphism \( h : X \to Y \) such that \( \phi_t = h^{-1} \psi_t h \) for all \( t \in \mathbb{R} \).

Via (2.1) and (2.2), the flows \( (\varphi_t)_{t \in \mathbb{R}} \) and \( (\varphi^X_t)_{t \in \mathbb{R}} \) are equivalent, and so are \( (\theta^X_t)_{t \in \mathbb{R}} \) and \( (\theta_t)_{t \in \mathbb{R}} \). It is easy to see that all the expansive properties introduced in the next section are invariant under equivalence.

3 Expansive properties

In this section we study \( \text{BW} \)-expansive, kinematic expansive, separating, and \( \text{KH} \)-expansive properties for the geodesic flow \( (\varphi^X_t)_{t \in \mathbb{R}} \) and the horocycle flow \( (\theta^X_t)_{t \in \mathbb{R}} \) on \( \mathcal{X} = T^1(\Gamma \backslash \mathbb{H}^2) \). We reprove that the geodesic flow is \( \text{BW} \)-expansive. The horocycle flow is positive/negative kinematic expansive as well as \( \text{KH} \)-expansive but not \( \text{BW} \)-expansive.

3.1 \( \text{BW} \)-expansiveness

This subsection provides a new detailed proof of the expansiveness in the sense of Bowen-Walters for the geodesic flow \( (\varphi^X_t)_{t \in \mathbb{R}} \) owing to a characteristic property of the injectivity radius.

Definition 3.1 (\cite{R}, \( \text{BW} \)-expansive) Let \( (M,d) \) be a compact metric space. A continuous flow \( \phi : \mathbb{R} \times M \to M \) is called \( \text{BW} \)-expansive if for each \( \varepsilon > 0 \), there exists \( \delta > 0 \) with the following property. If \( s : \mathbb{R} \to \mathbb{R} \) is a continuous function with \( s(0) = 0 \) and

\[
d(\phi_t(x), \phi_{s(t)}(y)) < \delta \quad \text{for all} \quad t \in \mathbb{R}
\]

then \( y = \phi_{\tau}(x) \) for some \( \tau \in (-\varepsilon, \varepsilon) \).
It was showed in [8, Theorem 3] that ‘continuous function’ in the above definition can be replaced by ‘increasing homeomorphism’ in the case of fixed-point-free flows. Then this definition becomes the one in [10], ‘K-expansive’ [14], and ‘geometric expansive’ [4].

**Theorem 3.2** The geodesic flow $(\varphi^t_\gamma)_{t \in \mathbb{R}}$ is BW-expansive.

**Proof:** Since BW-expansiveness is an invariant under equivalence [4], it follows from (2.1) that it suffices to show that the flow $(\varphi_t)_{t \in \mathbb{R}}$ is BW-expansive. Let $\varepsilon > 0$ be given, $\varepsilon_0 = e^{\varepsilon/2} - e^{-\varepsilon/2} > 0$ and set $\delta = \delta(\varepsilon_0) < \sigma_0/4$ as in Lemma 2.1(b); here $\sigma_0$ is from Lemma 2.2. Let $x, y \in X$ and $s : \mathbb{R} \to \mathbb{R}$ be continuous with $s(0) = 0$ such that

$$d_X(\varphi_{s(t)}(y), \varphi_t(x)) < \delta \quad \text{for all} \quad t \in \mathbb{R}.$$ 

Write $x = \Gamma g, y = \Gamma h$ for $g, h \in G$. For every $t \in \mathbb{R}$, there is $\gamma(t) \in \Gamma$ so that

$$d_X(\varphi_{s(t)}(y), \varphi_t(x)) = d_X(\Gamma ha_{s(t)}, \Gamma ga_t) = d_G(ha_{s(t)}, \gamma(t)ga_t) < \delta. \quad (3.4)$$

We claim that $\gamma(t) = \gamma(0) =: \gamma$ for all $t \in \mathbb{R}$. For any $t_1, t_2 \in \mathbb{R}$, we have

$$d_G(\gamma(t_2)^{-1}\gamma(t_1)ga_{t_1}, ga_{t_2})$$

$$= d_G(\gamma(t_1)ga_{t_1}, \gamma(t_2)ga_{t_1})$$

$$\leq d_G(\gamma(t_1)ga_{t_1}, ha_{s(t_1)}) + d_G(ha_{s(t_1)}, ha_{s(t_2)}) + d_G(ha_{s(t_2)}, \gamma(t_2)ga_{t_2})$$

$$\quad + d_G(\gamma(t_2)ga_{t_2}, \gamma(t_2)ga_{t_1})$$

$$= d_G(\gamma(t_1)ga_{t_1}, ha_{s(t_1)}) + d_G(\gamma(t_1)ga_{t_1}, \gamma(t_2)ga_{t_2}) + d_G(\gamma(t_2)ga_{t_2}, \gamma(t_2)ga_{t_1}) + d_G(\gamma(t_2)ga_{t_2}, \gamma(t_2)ga_{t_1})$$

$$\leq 2\delta + \frac{1}{\sqrt{2}}|s(t_1) - s(t_2)| + \frac{1}{\sqrt{2}}|t_1 - t_2|,$$

due to Lemma 2.1. For given $L > 0$, we verify that $\gamma(t) = \gamma(0)$ for all $t \in [-L, L]$. Indeed, since $s : [-L, L] \to \mathbb{R}$ is uniformly continuous, there is $0 < \rho = \rho(L, \delta) < \delta$ such that if $t_1, t_2 \in [-L, L]$ and $|t_1 - t_2| < \rho$ then $|s(t_1) - s(t_2)| < \delta$. For $t_1, t_2 \in [0, \rho/2]$, then $|t_1 - t_2| < \rho$ implies $|s(t_1) - s(t_2)| < \delta$. This yields

$$d_G(\gamma(t_2)^{-1}\gamma(t_1)c_{t_1}(t_1), c_{t_1}(t_1)) < 4\delta < \sigma_0.$$ 

\footnote{It is showed in [8, Corollary 4] that BW-expansiveness is an invariant under conjugacy that is weaker than equivalence. Recall that the flows $(\phi_t)_{t \in \mathbb{R}}$ on $X$ and $(\psi_t)_{t \in \mathbb{R}}$ on $Y$ is said to be conjugate if there is a homeomorphism from $X$ to $Y$ mapping the orbits of $(\phi_t)_{t \in \mathbb{R}}$ onto orbits of $(\psi_t)_{t \in \mathbb{R}}$.}
From the property of $\sigma_0$ in Lemma 2.2, it follows that $\gamma(t_2) = \gamma(t_1)$ for $|t_1 - t_2| < \rho$. Here if we specialize this to $t_1 = 0$ and $t_2 \in [0, \rho/2]$, then $\gamma(t_2) = \gamma(0)$ for all $t_2 \in [0, \rho/2]$. Then we repeat the argument for $t_1 = \rho/2$ and $t_2 \in [\rho/2, \rho]$, we deduce that $\gamma(t) = \gamma(0)$ for all $t \in [0, \rho]$, which upon further iteration leads to $\gamma(t) = \gamma(0)$ for all $t \in [0, L]$ and similarly $\gamma(t) = \gamma(0)$ for all $t \in [-L, 0]$. Therefore,

$$d_X(\varphi_s(t)(y), \varphi_t(x)) = d_G(a_{-t}g^{-1}\gamma h a_{s(t)}, e) < \delta \quad \text{for all} \quad t \in \mathbb{R}. \quad (3.5)$$

Write $g^{-1}\gamma h = \pi(K)$ for $K = (\begin{array}{cc} a & b \\ c & d \end{array}) \in \text{SL}(2, \mathbb{R})$. Thus

$$A_{-t}KA_{s(t)} = \begin{pmatrix} ae^{s(t) - t} & be^{-s(t) - t} \\ ce^{s(t) + t} & de^{t - s(t)} \end{pmatrix}$$

Together with (3.5) implies

$$||a|e^{s(t) - t} - 1| + |b|e^{-s(t) - t} + |c|e^{s(t) + t} + ||d|e^{t - s(t)} - 1| < \varepsilon_0 \quad \text{for all} \quad t \in \mathbb{R}, \quad (3.6)$$

using Lemma 2.4(b). Then there is $M > 0$ such that $|s(t) - t| \leq M$ for all $t \in \mathbb{R}$ and hence $s(t) + t \to +\infty$ as $t \to +\infty$ and $s(t) + t \to -\infty$ as $t \to -\infty$. Together with (3.6) this yields $b = c = 0$. Since $ad = 1$ we can assume that $a > 0, d > 0$ and $a = e^{\tau/2}, d = e^{-\tau/2}$ for some $\tau \in \mathbb{R}$. This implies that $g^{-1}\gamma h = a_{\tau}$ or $y = \varphi_{\tau}(x)$. Finally, using $||a| - 1| + ||d| - 1| < \varepsilon_0$, we have $e^{\tau/2} - e^{-\tau/2} < \varepsilon_0 = e^{\tau/2} - e^{-\tau/2}$, consequently $|\tau| < \varepsilon$ which completes the proof.

\[ \square \]

### 3.2 Kinematic expansiveness, separation

This subsection is devoted to demonstrate the kinematic expansiveness for the horocycle flow. It is also showed that the horocycle flow is not BW-expansive while the geodesic flow is not positive/negative separating.

**Definition 3.3 ([4], Kinematic expansive)** Let $(M, d)$ be a compact metric space. A continuous flow $\phi : \mathbb{R} \times M \to M$ is called kinematic expansive if for each $\varepsilon > 0$, there exists $\delta > 0$ with the following property. If

$$d(\phi_t(x), \phi_t(y)) < \delta \quad \text{for all} \quad t \in \mathbb{R} \quad (3.7)$$

then $y = \phi_{\tau}(x)$ for some $\tau \in (-\varepsilon, \varepsilon)$.
If the inequality in (3.7) holds for $t \in [0, \infty)$ (resp. $t \in (-\infty, 0]$) then the flow is called ‘positive kinematic expansive’ (resp. ‘negative kinematic expansive’). If the condition $\tau \in (-\varepsilon, \varepsilon)$ is ignored, the flow is called separating in the sense of Gura.

**Definition 3.4 ([11], Separating)** Let $(M,d)$ be a compact metric space. A continuous flow $\phi : \mathbb{R} \times M \rightarrow M$ is called separating if there exists $\delta > 0$ with the following property. If

$$d(\phi_t(x), \phi_t(y)) < \delta \quad \text{for all} \quad t \in \mathbb{R}$$

then $y = \phi_\tau(x)$ for some $\tau \in \mathbb{R}$; i.e. $x$ and $y$ lie on the same orbit.

The number $\delta$ is called a ‘separating constant’. If the inequality in (3.8) holds only for $t \in [0, \infty)$ (resp. $t \in (-\infty, 0]$) then the flow is called ‘positive separating’ (resp. ‘negative separating’).

It is showed in [11] that the horocycle flow on a compact surface of negative curvature is positive and negative separating. The next result gives a stronger property of the horocycle flow on $\Gamma \backslash \mathbb{H}^2$ which is a compact Riemann surface with constant negative curvature.

**Theorem 3.5** The horocycle flow $(\theta^X_t)_{t \in \mathbb{R}}$ is positive and negative kinematic expansive.

**Proof:** We consider the positive kinematic expansiveness only. Since the positive kinematic expansiveness is invariant under equivalence, it follows from (2.2) that it suffices to show that the flow $(\theta_t)_{t \in \mathbb{R}}$ is positive expansive. Let $\varepsilon > 0$ be given and set $\delta = \delta(\varepsilon)$ as in Lemma 2.4(b). Let $x, y \in X$ be such that

$$d_X(\theta_t(x), \theta_t(y)) < \delta \quad \text{for all} \quad t \geq 0.$$ (3.9)

Write $x = \Gamma g, y = \Gamma h$ for $g, h \in \text{PSL}(2, \mathbb{R})$. For every $t \geq 0$, there is $\gamma(t) \in \Gamma$ so that

$$d_X(\theta_t(y), \theta_t(x)) = d_X(\Gamma h b_t, \Gamma g b_t) = d_\mathcal{G}(\gamma(t) h b_t, g b_t) \leq \delta.$$ (3.10)

Analogously to the proof of Theorem 3.2, we can check that $\gamma(t) = \gamma(0) =: \gamma$ for all $t \in \mathbb{R}$ (here we do not need the uniform continuity of $s(t) = t$). It follows from (3.4) that

$$d_X(\theta_t(y), \theta_t(x)) = d_\mathcal{G}(b_{-tg^{-1}} \gamma h b_t, e) < \delta \quad \text{for all} \quad t \geq 0.$$ (3.10)
Write \( g^{-1} \gamma h = \pi(K) \) for \( K = \left( \begin{array}{cc} g_{11} & g_{12} \\ g_{21} & g_{22} \end{array} \right) \in \text{SL}(2, \mathbb{R}) \). Then

\[
B_{-t} K B_{t} = \left( \begin{array}{cc} g_{11} - t g_{21} & g_{12} - t(g_{21} t + g_{22}) \\ g_{21} & g_{22} + t(g_{21} t + g_{22}) \end{array} \right)
\]

together with (3.10) imply that for all \( t \geq 0 \),

\[
\left| g_{11} - t g_{21} \right| - 1 + \left| (g_{11} - g_{22}) t + g_{21} t^2 + g_{12} \right| + \left| g_{21} t + g_{22} \right| - 1 < \varepsilon_0; \tag{3.11}
\]

here \( \varepsilon_0 = \varepsilon_0(\delta) < \varepsilon \) obtained from Lemma 2.4(b). Letting \( t \to +\infty \) yields \( g_{21} = 0, g_{11} = g_{22} \) and \( |g_{12}| < \varepsilon_0 \). Since \( g_{11} g_{22} - g_{12} g_{21} = 1 \), we get \( g_{11} = g_{22} = 1 \) or \( g_{11} = g_{22} = -1 \) that leads to \( K = \left( \begin{array}{cc} 1 & g_{12} \\ 0 & 1 \end{array} \right) \) or \( K = \left( \begin{array}{cc} -1 & g_{12} \\ 0 & -1 \end{array} \right) \)

and hence \( g^{-1} \gamma h = b_\tau \) with \( \gamma = g_{12}, |\tau| < \varepsilon \). This completes the proof. \( \square \)

**Remark 3.6** (a) The horocycle flow is not BW-expansive. Indeed, for any \( \delta > 0 \), we need to find \( x, y \in X \) and \( s : \mathbb{R} \to \mathbb{R} \) continuous with \( s(0) = 0 \) such that \( d_X(\theta_t(x), \theta_s(y)) < \delta \) for all \( t \in \mathbb{R} \) but the orbits of \( x \) and \( y \) do not coincide. Take \( \rho = \rho(\delta) \) as in Lemma 2.4(a) and choose any \( x = \Gamma g \) and \( y = \Gamma h \) with \( h, g \in \text{PSL}(2, \mathbb{R}), h \neq g, h^{-1} g = \pi(K) \), \( K = \left( \begin{array}{cc} a & 0 \\ 0 & d \end{array} \right) \) such that \( ad = 1, |a - 1| < \rho, |d - 1| < \rho \) and \( \text{tr}(h^{-1} g) = |a + d| < 2 + \varepsilon \); recall \( \varepsilon > 0 \) in Lemma 2.3 we have \( d_G(h^{-1} g, e) < \delta \) due to Lemma 2.4(a). Setting \( s(t) = \frac{d}{a} t \), we have

\[
d_X(\theta_s(t)(x), \theta_t(y)) = d_X(\Gamma g s(t), \Gamma h b_t) \leq d_G(g s(t), h b_t) = d_G(b_{-t} h^{-1} g s(t), e) = d_G(h^{-1} g, e) < \delta \quad \text{for all} \quad t \in \mathbb{R};
\]

using \( b_{-t} h^{-1} g s(t) = h^{-1} g \). It remains to verify that \( x \) and \( y \) are not in the same orbit. Indeed, otherwise there would exist \( \tau \in \mathbb{R} \) such that \( y = \theta_\tau(x) \), then \( \gamma h = g b_\tau \) for some \( \gamma \in \Gamma \) implies \( \text{tr}(\gamma) = \text{tr}(g b_\tau h^{-1}) = \text{tr}(b_\tau h^{-1} g) = |a + d| < 2 + \varepsilon \). It follows from Lemma 2.3 that \( \gamma = e \). This yields \( b_\tau = h^{-1} g = \pi(K) \) and hence \( \tau = 0, h = g \) which contradicts to \( h \neq g \).

(b) The flow geodesic flow is neither positive nor negative separating. Indeed, we consider the equivalent flow \( (\varphi_t)_{t \in \mathbb{R}} \). Since the group \( \Gamma \) is discrete, for every \( \delta > 0 \), there is an \( s \in (-\delta, \delta) \) such that \( a_t b_{-s} \notin \Gamma \) for all \( t \in \mathbb{R} \). Set \( x = \Gamma e \) and \( y = \Gamma b_s \) to have

\[
d_X(\varphi_t(x), \varphi_t(y)) = d_X(\Gamma a_t, \Gamma b_s a_t) \leq d_G(a_t, b_s a_t) \leq |s| e^{-t} < \delta \quad \text{for all} \quad t \geq 0.
\]
However, if \( y = \varphi_r(x) \) then \( \Gamma b_s = \Gamma a_r \) implies that there is \( \gamma = a_r b_{-s} \in \Gamma \) which is a contradiction, whence \( (\varphi_t)_{t \in \mathbb{R}} \) is not positive separating. In the same manner one obtains that the flow \( (\varphi_t)_{t \in \mathbb{R}} \) is not negative separating.

(c) It is worth to recall that the geodesic flow is BW-expansive but neither positive nor negative kinematic expansive while the horocycle flows are positive and negative kinematic expansive but not BW-expansive.

\[ \Diamond \]

### 3.3 KH-expansiveness

In [13] Katok and Hasselblatt introduce the following expansiveness:

**Definition 3.7 ([13], KH-expansive)** Let \((M, d)\) be a compact space. A continuous flow \( \phi_t : M \to M \) is called KH-expansive if there exists \( \delta > 0 \) with the following property. If \( x \in X, s : \mathbb{R} \to \mathbb{R} \) is continuous, \( s(0) = 0 \) and \( d(\varphi_t(x), \varphi_{s(t)}(x)) < \delta \) for all \( t \in \mathbb{R}, y \in X \) is such that \( d(\varphi_t(x), \varphi_{s(t)}(y)) < \delta \) for all \( t \in \mathbb{R} \) then \( x \) and \( y \) lie on the same orbit.

It is clear that KH-expansiveness is weaker than BW-expansiveness but implies separation. Furthermore, one has the following result:

**Proposition 3.8 ([5])** A flow on a compact metric space is KH-expansive if and only if it is separating and the set of its fixed points is open.

It follows immediately from propositions 2.6 and 3.8 that the flow \((\theta_t)_{t \in \mathbb{R}}\) is KH-expansive, and hence the horocycle flow \((\theta^X_t)_{t \in \mathbb{R}}\) is KH-expansive owing to (2.2). Nevertheless we can verify it directly.

**Theorem 3.9** The flow horocycle flow \((\theta^X_t)_{t \in \mathbb{R}}\) is KH-expansive.

**Proof:** If \( x, y \in X, s : \mathbb{R} \to \mathbb{R} \) is continuous, \( s(0) = 0 \) and

\[
d_X(\theta_t(x), \theta_{s(t)}(x)) < \delta \quad \text{for all } \ t \in \mathbb{R} \tag{3.12}
\]

and

\[
d_X(\theta_t(x), \theta_{s(t)}(y)) < \delta \quad \text{for all } \ t \in \mathbb{R}. \tag{3.13}
\]

Analogously to the proof of Theorem 3.2 using (3.12) we can show that there is \( M > 0 \) such that

\[
|s(t) - t| < M \quad \text{for all } \ t \in \mathbb{R}.
\]
This means that

\[ s(t) \to +\infty \quad \text{as} \quad t \to +\infty. \quad (3.14) \]

It follows from (3.12) and (3.13) that

\[ d_X(\theta_{s(t)}(x), \theta_{s(t)}(y)) < 2\delta \quad \text{for all} \quad t \in \mathbb{R}. \]

Together with (3.14), this follows in the same manner of the proof of Theorem 3.5. \qed

Acknowledgments: This work is supported by Vietnam National Foundation for Science and Technology Development (Grant No. 101.02-2017.304). I would like to thank an anonymous referee for carefully reading my paper and useful suggestions. I enjoyed many fruitful discussions with Alfonso Artigue.

References

[1] D. V. Anosov, *Geodesic flows on closed Riemannian manifolds of negative curvature*, Trudy Mat. Inst. Steklov. 90 (1967)

[2] A. Artigue, *Expansive flows on surfaces*, Discrete Contin. Dyn. Syst. Ser. A 33(2) (2013), 505-525

[3] A. Artigue, *Positive expansive flows*, Topol. Appl. 165 (2014), 121-132.

[4] A. Artigue, *Kinematic expansive flows*, Ergod. Th. & Dynam. Sys. 36 (2016), 390-421.

[5] A. Artigue, *Rescaled expansivity and separating flows*, Discrete Contin. Dyn. Syst. Ser. A 38(9) (2018), 4433-4447.

[6] T. Bedford, M. Keane and C. Series (Eds.): *Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces*, Oxford University Press, Oxford 1991

[7] R. Bowen, *Periodic orbits for hyperbolic flows*, Amer. J. Math. Vol. 94 (1972) 1-30.

[8] R. Bowen and P. Walters, *Expansive one-parameter flows*, J. Differential equations 12 (1972), 180-193.

[9] M. Einsiedler and T. Ward, *Ergodic Theory with a View towards Number Theory*, Springer, Berlin-New York 2011
[10] L. Flinn, *Expansive Flows*, PhD thesis, Warwick University 1972

[11] A. Gura, Horocycle flow on a surface of negative curvature is separating, *Mat. Zametki* **36** (1984), 279-284.

[12] H. Huynh and M. Kunze, *Partner orbits and action differences on compact factors of the hyperbolic plane. I: Sieber-Richter pairs*, Nonlinearity **28** (2015), 593-623.

[13] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, Cambridge 1995.

[14] M. Komuro, *Expansive properties of Lorenz attractors*, The Theory of Dynamical Systems and its Applications to Nonlinear Problems (Toyoto, 1984), World Scientific, Singapore (1984), 4-26

[15] B. Marcus, *Unique ergodicity on the horocycle flows: the variable curvature case*, Israel J. Math **21** (1975), 133-144.

[16] M. Oka, *Expansiveness of real flows*, Tsukuba J. Math **14** (1990), no. 1, 18.

[17] J. Ratcliff, *Foundations of Hyperbolic Manifolds*, 2nd edition, Springer, Berlin-Heidelberg-New York 2006