Symmetries of the quantum damped harmonic oscillator

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Abstract

For the non-conservative Caldirola–Kanai system, describing a quantum damped harmonic oscillator, a couple of constant-of-motion operators generating the Heisenberg–Weyl algebra can be found. The inclusion of the standard time evolution generator (which is not a symmetry) as a symmetry in this algebra, in a unitary manner, requires a non-trivial extension of this basic algebra and hence of the physical system itself. Surprisingly, this extension leads directly to the so-called Bateman dual system, which now includes a new particle acting as an energy reservoir. In addition, the Caldirola–Kanai dissipative system can be retrieved by imposing constraints. The algebra of symmetries of the dual system is presented, as well as a quantization that implies, in particular, a first-order Schrödinger equation. As opposed to other approaches, where it is claimed that the spectrum of the Bateman Hamiltonian is complex and discrete, we obtain that it is real and continuous, with infinite degeneracy in all regimes.

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1. Introduction

The interest in dissipative systems at the quantum level has remained constant since the early days of quantum mechanics. Within its framework, the attempts to describe damping, which intuitively could be understood as a mesoscopic property, have motivated a huge amount of papers, both from theoretical and more applied points of view.

Applications of quantum dissipation include quantum optics [31] and the study of decoherence phenomena [18]. It is customary to model dissipation by means of the theory of
open systems or the thermal bath approach, in which a damped system is considered to be a subsystem of a bigger one with many, or infinite, degrees of freedom [35]. However, damped systems are interesting in themselves as fundamental ones if one wants to capture the essential dissipative behavior in a simple model. In particular, the quantum damped harmonic oscillator has attracted much attention, as could be considered one of the simplest and paradigmatic examples of non-conservative system.

The quantum damped harmonic oscillator has been described frequently by the Caldirola–Kanai model [9, 19], which includes a time-dependent Hamiltonian. It has been claimed that, in the quantum theory built out of this kind of models, uncertainty relations are not preserved under time evolution and could eventually be violated [8, 27], but this apparent inconsistency is associated with a confusion between canonical momentum and ‘physical’ momentum (see, for instance, [29]; in fact, the same author proposed a logarithmic nonlinear Schrödinger equation to deal with this situation [30]). In contrast, the main drawback of these models might be that they do not describe the interplay between the system and the environment, so that phenomena such as decoherence are not taken into account. It is nevertheless interesting to note that these kinds of models can in fact be derived from a system-plus-reservoir model [36, 32] in the case of Ohmic spectral density.

The analysis of damping from the symmetry point of view has proved to be especially fruitful. In a purely classical context, the symmetries of the equation of the damped harmonic oscillator with time-dependent parameters were found in [22]. For the usual damped harmonic oscillator, finite-dimensional point symmetry groups for the corresponding Lagrangian (the un-extended Schrödinger group [24–26]) and the equations of motion (SL(3, R)) respectively, were found in [12, 13], as well as an infinite contact one for the set of trajectories of the classical equation.

As far as the quantum theory is concerned, in [1] the authors provided a neat framework to study a generalized version of the Caldirola–Kanai model with time-dependent parameters (frequency, damping coefficient and external force), based on a quantum version of the Arnold transformation (QAT) [3]. The integrals of motion and symmetries were identified and exploited to calculate wavefunctions, basic operators and the exact time evolution operator, generalizing some of the results in [16]. Many papers related to the Caldirola–Kanai model keep appearing [34, 4], showing that the debate about fundamental quantum damping is far from being closed (the reader can see [1] for further references on the Caldirola–Kanai model).

For instance, at the quantum level of this model, it remains to address the symmetry role of the time translation generator $i\hbar \frac{\partial}{\partial t}$. In fact, $i\hbar \frac{\partial}{\partial t}$ acting on a solution of the Caldirola–Kanai Schrödinger equation is no longer a solution. As a consequence, the time evolution operator $\hat{U}$ does not constitute a one-parameter group of unitary transformations. Equivalently, the solution of the equation $i\hbar \frac{\partial}{\partial t} \hat{U} = \hat{H}(t) \hat{U}$ is not $e^{-\frac{i}{\hbar} \hat{H}(t)}$, but rather the time ordered product $T e^{-\frac{i}{\hbar} \int \hat{H}(t) dt}$, referring to the Neumann series, or the Magnus series $e^{-\frac{i}{\hbar} \hat{D}(t)}$ [6].

The purpose of this paper is to throw some light on the subject of quantum dissipation with the guide of symmetry. The starting point will be the Caldirola–Kanai system and we shall see that this model contains, somehow, a reminiscence of more general, system-plus-reservoir models. Using the QAT, basic integrals of the motion closing a Heisenberg–Weyl algebra can be found, imported from those of the free particle, together with their squares. Time translations in the damped system do not belong to the imported quadratic, conserved operators. This is to be expected, as the classical equation of motion includes a friction term and the energy in this system is not conserved. The following question immediately arises: is there any finite-dimensional group of symmetries containing time translations and, at least, the basic operators? The answer is ‘yes’, and we shall pay attention to this question in the
case of the damped harmonic oscillator and the surprising consequences of the subsequent calculation: for this symmetry to act properly, it is necessary to enlarge the physical system with a new degree of freedom, corresponding to a new particle with interesting properties.

In his original paper [5], Bateman looked for a variational principle for equations of motion with a friction term linear in velocity, but he allowed the presence of extra equations. This trick effectively doubles the number of degrees of freedom, introducing a time-reversed version of the original damped harmonic oscillator, which acts as an energy reservoir and could be considered as an effective description of a thermal bath, as opposed to the standard approach followed in the theory of open quantum systems, where the thermal bath contains many, or infinite, degrees of freedom. The Hamiltonian that describes this system was later rediscovered by Feschbach and Tikochinsky [33, 17, 23, 15] and the corresponding quantum theory was immediately analyzed.

Some issues regarding Bateman’s dual system arose in the widespread Feschbach–Tikochinsky construction of the corresponding Hilbert space and spectrum. This construction leads to a set of complex eigenvalues of the Hamiltonian (see [14] and references therein), and it has been interpreted that the vacuum of the theory decays with time [11]. In this paper, we shall provide a rigorous construction of the physical Hilbert space and the real spectrum of the Hamiltonian, which are not found by means of the Feschbach–Tikochinsky construction. To do so, we shall take advantage of the adoption of a canonical quantization in which the Schrödinger equation is of first order. This approach is very similar to that presented in [28] and it also has some relationship with the one in [14]; there, the generalized eigenvectors corresponding to the complex eigenvalues of the Hamiltonian are interpreted as resonant states.

Bateman’s dual system is still discussed frequently [7, 21]. Many authors have considered this model as a good starting point for the formulation of the quantum theory of dissipation. One of the aims of this paper will be to show that the study of the symmetries of the Caldirola–Kanai model leads to Bateman’s dual system, so that it should be considered as a natural starting point for the study of quantum dissipation. This could be seen as a ‘step-back-and-forth’ approach (from a model of bath, then the Caldirola–Kanai model and finally Bateman’s dual system), arriving at an idealization which intends to capture some of the features of quantum dissipation. Moreover, we shall show that it is possible to get back to the Caldirola–Kanai system by imposing constraints in the Bateman classical system.

The paper is organized as follows. We begin in section 2 by recalling the results from [1], in the case of the damped harmonic oscillator with constant coefficients in order to import basic operators from the free particle system, which satisfy the condition of being integrals of the motion and close a Heisenberg–Weyl algebra. In section 3, we look for a minimal algebra of operators which includes the basic conserved operators of the Caldirola–Kanai system and the generator of time translations \( \frac{\partial}{\partial t} \). In so doing, we arrive at an algebra of conserved operators of an enlarged physical system, which turns out to be the Bateman dual system. We also show how, at the classical level, the Caldirola–Kanai system can be recovered from the Bateman system by imposing a constraint.

In section 4, we revise the quantization of the Bateman dual system. In particular, we find a first-order Schrödinger equation, from which the wavefunctions and the energy spectrum can be obtained. Finally, an appendix is devoted to the study of a non-minimal, infinite-dimensional symmetry algebra for the damped particle.

\[4\] We feel that the ultimate reason of this interpretation is the lack of a vacuum representation of the relevant symmetry algebra (see section 3).
2. Basic operators in the Caldirola–Kanai model

The classical equation of motion describing a damped harmonic oscillator (DHO) is given by
\[ \ddot{x} + \gamma \dot{x} + \omega^2 x = 0, \]  
where \( \omega \) is a frequency and \( \gamma \) is the damping constant, both defining the system. This equation can be derived, in particular, from the Hamilton principle with a time-dependent Lagrangian. The corresponding, time-dependent, Hamiltonian function, to be named \( H_{CK} \), is given by
\[ H_{CK} = e^{-\gamma t} \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 e^{\gamma t}. \]  
The canonical quantization of \( H_{CK} \) leads to the Caldirola–Kanai equation \([9, 19]\), which is a Schrödinger equation for the DHO:
\[ i\hbar \frac{\partial \hat{\phi}}{\partial t} = H_{CK} \hat{\phi} \equiv -\frac{\hbar^2}{2m} e^{-\gamma t} \frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 e^{\gamma t} \hat{\phi}. \]  
Some comments on the features of this equation have already been made in the introduction. Due to the explicit time dependence of the Hamiltonian function, the corresponding operator \( H_{CK} \) is not a conserved operator and it is not possible to formulate an eigenvalue equation for it (that is, a time-independent Schrödinger equation). Hence, there does not exist a one-parameter group of time evolution. This is directly linked with the fact that the operator \( H_{CK} \) does not preserve the space of solutions of equation (3) (see [1] for a further discussion).

Now we are just interested in identifying basic position and momentum operators associated with classical conserved quantities (Noether invariants). In general, a conserved quantum operator \( \hat{O}(t) \) must satisfy the relation:
\[ \frac{d}{dt} \hat{O}(t) \equiv \frac{\partial}{\partial t} \hat{O}(t) + \frac{i}{\hbar} [\hat{H}(t), \hat{O}(t)] = 0. \]  
These operators are also known as dynamical invariants. Although the particular method to obtain these conserved operators is not fundamental in what follows, we find especially useful for that purpose the QAT technique developed in [1].

The QAT for the DHO maps unitarily solutions of the Schrödinger equation for the Galilean free particle to solutions of (3) and, accordingly, conserved operators in the Hilbert space of solutions of the free Schrödinger equation to operators acting on the Hilbert space of solutions of (3).

In particular, we are going to map the Galilean momentum operator \( \hat{\pi} \), corresponding to the classical conserved quantity ‘momentum’, and the position operator \( \hat{x} \), corresponding to the classical, conserved quantity ‘initial position’, to operators for the Caldirola–Kanai system. These free operators are, explicitly:
\[ \hat{\pi} = -i\hbar \frac{\partial}{\partial \kappa}, \quad \hat{x} = \kappa + \frac{i\hbar}{m} \tau \frac{\partial}{\partial \kappa}, \]  
that is, those basic, canonically commuting operators with constant expectation values for the free evolution.

The only relevant part of the QAT in which we are interested here is how the above operators are transformed (we refer the reader to [1] for further information). The QAT is defined just by two independent, classical solutions of (1), \( u_1(t) \) and \( u_2(t) \), satisfying the initial conditions \( u_1(0) = 0, u_2(0) = 1, \dot{u}_1(0) = 1, \dot{u}_2(0) = 0 \). Their Wronskian \( W(t) \equiv \dot{u}_1(t)u_2(t) - u_1(t)\dot{u}_2(t) \) does not vanish. These classical solutions are
\[ u_1(t) = \frac{1}{\Omega} e^{-\gamma t} \sin \Omega t, \quad u_2(t) = e^{-\gamma t} \cos \Omega t + \frac{\gamma}{2\Omega} e^{-\gamma t} \sin \Omega t, \]  

where \( \Omega = \sqrt{\omega^2 - \gamma^2} \).
for which $W(t) = e^{-\gamma t}$, and
\[
\Omega = \sqrt{\omega^2 - \frac{\gamma^2}{4}}
\] (7)
is the characteristic frequency for the damped oscillator. Note that these solutions have good limit in the case of critical damping $\omega = \frac{\gamma}{2}$ (i.e. $\Omega = 0$).

Performing the QAT for operators (5) results in formulas (see [1]):
\[
\hat{P} = -i\hbar\frac{\partial}{\partial x} - m\frac{\dot{u}_2}{W} x, \quad \hat{X} = \frac{\dot{u}_1}{W} x + i\hbar\frac{\dot{u}_1}{m} \frac{\partial}{\partial x}.
\] (8)

It can be checked by direct computation that these expressions satisfy condition (4) provided that $u_1$ and $u_2$ are solutions of (1). Explicitly:
\[
\hat{P} = -i\hbar\frac{e^{-\gamma t}}{2\Omega} (2\Omega \cos \Omega t + \gamma \sin \Omega t) \frac{\partial}{\partial x} + m\frac{e^{-\gamma t}}{4\Omega} (\gamma^2 + 4\Omega^2) \sin \Omega t x,
\] (9)
\[
\hat{X} = \frac{e^{-\gamma t}}{2\Omega} (2\Omega \cos \Omega t - \gamma \sin \Omega t) x + i\hbar\frac{e^{-\gamma t}}{m\Omega} \sin \Omega t \frac{\partial}{\partial x},
\] (10)
with
\[
[\hat{X}, \hat{P}] = i\hbar.
\] (11)

They are the basic, canonically commuting operators with constant expectation values (or, in other words, the two independent dynamical invariants) for the evolution of the Caldirola–Kanai system describing the quantum DHO.

This way, we have imported the Heisenberg–Weyl algebra of conserved operators, which are symmetry generators, from the free particle system, but we have not done so for a generator of time evolution. The time evolution generator present in the free particle, and which is a constant-of-motion operator, may also be brought into a constant-of-motion operator in the Caldirola–Kanai system and will be realized as a quadratic operator in $\hat{P}$ and $\hat{X}$ (more precisely, $\hat{P}^2$), but this does not represent the actual time evolution generator of the Caldirola–Kanai oscillator. It is desirable that the system be completely characterized by its symmetries, including its time evolution, but so far, this had not been the case.

### 3. Addressing dissipative systems in a symmetry framework

Even though it is possible to set up a clear framework to deal with the quantum Caldirola–Kanai system for the DHO by employing the QAT, this does not provide by itself a well-defined operator (on solutions) generating the actual time evolution. As mentioned previously, this is rooted in the fact that the conventional time evolution generator is not included in the symmetry algebra: the Hamiltonian does not preserve the Hilbert space of solutions of the Caldirola–Kanai Schrödinger equation.

We would like to emphasize that the underlying motivation for constituting a closed algebra of quantum observables is the general consensus that the quantization of any physical system must be a unitary and irreducible representation of some Poisson sub-algebra of classical observables characterizing the system to be quantized, irrespective of the particular method devoted to the achievement of this task (quantization method). The inclusion of the Hamiltonian in the chosen Poisson sub-algebra, and therefore in the quantum algebra, guarantees the characterization of the physical system. One may accordingly wonder what happens if the generator of time evolution for the damped harmonic oscillator is forced to belong to the algebra containing the basic operators. We shall pursue this issue in this section.
3.1. Including time symmetry

In the Caldirola–Kanai model of the damped harmonic oscillator, neither the operator $i\hbar \frac{\partial}{\partial t}$, nor $\hat{H}_{\text{CK}}$ (which coincides with the former on solutions) preserve the space of solutions of the Schrödinger equation. We shall impose the condition of $i\hbar \frac{\partial}{\partial t}$ being a symmetry generator. But it will be done in an elegant way, trying to close with $\hat{X}$ and $\hat{P}$ an enlarged Lie algebra of (conserved, symmetry generating) observables acting on the (possibly enlarged) Hilbert space $\mathcal{H}$. Implicitly, we are forcing the system to become conservative, but this is the only information we are going to provide.

As a first step, we shall take advantage of the explicit expressions obtained for the conserved basic operators: we compute the commutators of $\hat{X}$ and $\hat{P}$ with $\hat{H} = i\hbar \frac{\partial}{\partial t}$, and the resulting expressions are considered as new operators. After that, the commutators of the new operators with the formers are also computed, in the hope that this process ends up closing a Lie algebra at a finite number of steps. In fact, the resulting enlarged algebra is finite-dimensional and includes $\hat{X}, \hat{P}, \hat{H}$ and four more generators\(^5\) (plus the central one $\hat{I}$), denoted by $\hat{Q}, \hat{\Pi}, \hat{G}_1$ and $\hat{G}_2$:

\[
\hat{P} = -i\hbar e^{\frac{\gamma}{\Omega}} \left( \cos \Omega t + \frac{\gamma}{2\Omega} \sin \Omega t \right) \frac{\partial}{\partial x} + m \omega^2 \frac{\partial}{\partial x} e^{\frac{\gamma}{\Omega}} \sin \Omega t x \\
\hat{X} = e^{\frac{\gamma}{\Omega}} \left( \cos \Omega t - \frac{\gamma}{2\Omega} \sin \Omega t \right) x + i\hbar e^{\frac{\gamma}{m\Omega}} \sin \Omega t \frac{\partial}{\partial x} \\
\hat{\Pi} = -i\hbar e^{-\frac{\gamma}{\Omega}} \left( \cos \Omega t - \frac{\gamma}{2\Omega} \sin \Omega t \right) \frac{\partial}{\partial x} + m \omega^2 \frac{\partial}{\partial x} e^{-\frac{\gamma}{\Omega}} \sin \Omega t x \\
\hat{Q} = e^{\frac{\gamma}{\Omega}} \left( \cos \Omega t - \frac{3\gamma}{2\Omega} \sin \Omega t \right) x + i\hbar e^{-\frac{\gamma}{m\Omega}} \sin \Omega t \frac{\partial}{\partial x} \\
\hat{G}_1 = -\frac{1}{4\Omega^2} \left( -4\omega^2 + \gamma^2 \cos 2\Omega t + 2\gamma \Omega \sin 2\Omega t \right) \\
\hat{G}_2 = \frac{\gamma}{\Omega^2} \sin^2 \Omega t,
\]

which close the eight-dimensional algebra $\hat{A}$:

\[
\begin{align*}
[\hat{X}, \hat{P}] &= i\hbar \hat{I} \\
[\hat{X}, \hat{Q}] &= -i\hbar \hat{G}_2 \\
[\hat{X}, \hat{\Pi}] &= i\hbar \hat{G}_1 \\
[\hat{P}, \hat{Q}] &= -i\hbar \hat{G}_1 + i\hbar \gamma \hat{G}_2 \\
[\hat{P}, \hat{\Pi}] &= -i\hbar \omega^2 \hat{G}_2 \\
[\hat{H}, \hat{X}] &= -i\hbar \hat{\Pi} \\
[\hat{H}, \hat{P}] &= 2i\hbar \omega^2 \hat{X} - i\hbar \omega^2 \hat{Q} \\
[\hat{H}, \hat{Q}] &= -2i\hbar \gamma \hat{X} - i\hbar \gamma \hat{\Pi} \\
[\hat{H}, \hat{\Pi}] &= 3i\hbar \omega^2 \hat{X} - 2i\hbar \omega^2 \hat{Q} - i\hbar \gamma \hat{\Pi} \\
[\hat{H}, \hat{G}_1] &= -i\hbar \gamma \hat{G}_1 + 2i\hbar \omega^2 \hat{G}_2 \\
[\hat{H}, \hat{G}_2] &= -2i\hbar \hat{G}_1 + i\hbar \gamma \hat{G}_2 + 2i\hbar \hat{I}.
\end{align*}
\]

The outcome of this process depends, obviously, on the way $\frac{\partial}{\partial t}$ acts on the explicit realization of the Heisenberg–Weyl generators, which in this case encodes somehow the time evolution of the Caldirola–Kanai oscillator ($\hat{X}$ and $\hat{P}$ were conserved operators under the evolution of this system).

Now we have to unveil the physical content of (12). This algebra $\hat{A}$ corresponds to a central extension of an algebra, to be named $\mathcal{A}$, and this particular central extension itself

\(^5\)At least in the simpler case of the damped particle, infinitely many new generators can be included in its Lie algebra. See the appendix for further details.
determines the actual basic conjugated pairs, fixing a specific quantization. The operators \( \hat{Q} \) and \( \hat{\Pi} \) (plus \( \hat{I} \)) expand a Heisenberg–Weyl subalgebra, and \( \hat{H}, \hat{G}_1 \) and \( \hat{G}_2 \) expand a 2D affine algebra (with \( \hat{H} \) acting as dilations). However, in this realization \( \hat{Q} \) and \( \hat{\Pi} \) are not basic operators\(^6\), while \( \hat{H} \) and \( \hat{G}_2 \) are conjugate ones: time proves to be a basic variable, this way contributing to the parametrization of the solution manifold. This result is puzzling: time is not expected to be a coordinate or momentum of any degree of freedom and therefore the corresponding generator should not appear as an element of a conjugate pair. We then wonder whether it is possible or not to take advantage of the information encoded in the algebra \( \mathcal{A} \) to describe a physical system in which we have an evolution with respect to the ordinary time variable. Even the possibility that in the quantum reduction process, in which \( \hat{Q} \) and \( \hat{\Pi} \) cease to be basic operators, \( \hat{H} \) would regain its expected role of time evolution generator could not be discarded. However, this requires a deeper analysis and fortunately this is not the only solution.

In fact, \( \mathcal{A} \) is not the only possible central extension of \( \mathcal{A} \) and our strategy here will be to consider other possible quantizations of the un-extended algebra \( \mathcal{A} \), labeled by its possible central extensions. A detailed study shows that there are three relevant kinds of central extensions, describing systems with different degrees of freedom. Thinking of the algebra above as an abstract Lie algebra, it can be shown that a couple of parameters, in addition to the ordinary mass parameter \( m \), control the central extensions that are allowed by the Jacobi identity of Lie algebras. We shall focus on a one-parameter family \( \mathcal{A}_k \) which contains the most relevant cases from the physical point of view:

\[
[X, \hat{P}] = i\hbar \hat{I} \quad \quad \quad \quad \quad \quad \quad [\hat{Q}, \hat{\Pi}] = 2i\hbar \hat{G}_1 - i\hbar k \hat{I}
\]
\[
[X, \hat{Q}] = -\frac{i\hbar}{m} \hat{G}_2 \quad \quad \quad \quad \quad \quad \quad [X, \hat{\Pi}] = i\hbar \hat{G}_1
\]
\[
[\hat{P}, \hat{Q}] = -i\hbar \hat{G}_1 + i\hbar \gamma \hat{G}_2 + i\hbar (k - 1) \hat{I} \quad \quad \quad \quad [\hat{P}, \hat{\Pi}] = -i\hbar \omega \hat{G}_2
\]
\[
[\hat{H}, \hat{X}] = -\frac{i\hbar}{m} \hat{I} \quad \quad \quad \quad \quad \quad \quad [\hat{H}, \hat{P}] = 2i\hbar \omega \hat{X} - i\hbar \omega \hat{\Pi}
\]
\[
[\hat{H}, \hat{Q}] = -2i\hbar \omega \hat{X} - \frac{i\hbar}{m} \hat{P} + i\hbar \gamma \hat{Q} \quad \quad \quad [\hat{H}, \hat{\Pi}] = 3i\hbar \omega \hat{X} - 2i\hbar \omega \hat{Q} - i\hbar \gamma \hat{\Pi}
\]
\[
[\hat{H}, \hat{G}_1] = -i\hbar \gamma \hat{G}_1 + 2i\hbar \omega \hat{G}_2 \quad \quad \quad [\hat{H}, \hat{G}_2] = -2i\hbar \hat{G}_1 + i\hbar \gamma \hat{G}_2 + i\hbar (1 + k) \hat{I}
\]

It is now convenient to perform the linear shift:

\[
\hat{Q} \equiv \hat{Q} + (k - 1) \hat{X},
\]

so that the actual degrees of freedom diagonalize:

\[
[X, \hat{P}] = i\hbar \hat{I} \quad \quad \quad \quad \quad \quad \quad [\hat{Q}, \hat{\Pi}] = i\hbar (k + 1) \hat{G}_1 - i\hbar k \hat{I}
\]
\[
[X, \hat{Q}] = -\frac{i\hbar}{m} \hat{G}_2 \quad \quad \quad \quad \quad \quad \quad [X, \hat{\Pi}] = i\hbar \hat{G}_1
\]
\[
[\hat{P}, \hat{Q}] = -i\hbar \hat{G}_1 + i\hbar \gamma \hat{G}_2 \quad \quad \quad [\hat{P}, \hat{\Pi}] = -i\hbar \omega \hat{G}_2
\]
\[
[\hat{H}, \hat{X}] = -\frac{i\hbar}{m} \hat{I} \quad \quad \quad \quad \quad \quad \quad [\hat{H}, \hat{P}] = i\hbar \omega (1 + k) \hat{X} - i\hbar \omega \hat{\Pi}
\]
\[
[\hat{H}, \hat{Q}] = -i\hbar \gamma (1 + k) \hat{X} - \frac{i\hbar}{m} \hat{P} + i\hbar \gamma \hat{Q} \quad [\hat{H}, \hat{\Pi}] = i\hbar \omega (2k + 1) \hat{X}
\]

\[
+ \frac{i\hbar}{m} (1 - k) \hat{I} \quad \quad \quad \quad \quad \quad \quad - 2i\hbar \omega \hat{Q} - i\hbar \gamma \hat{\Pi}
\]
\[
[\hat{H}, \hat{G}_1] = -i\hbar \gamma \hat{G}_1 + 2i\hbar \omega \hat{G}_2 \quad \quad \quad [\hat{H}, \hat{G}_2] = -2i\hbar \hat{G}_1 + i\hbar \gamma \hat{G}_2 + i\hbar (1 + k) \hat{I}
\]

By noting the appearance of the central generator \( \hat{I} \) on the right-hand side of these commutation relations, we can see that the centrally extended algebras \( \mathcal{A}_k \) can be classified as follows.

\(^6\) This can be seen as an anomaly, see \([2, 10]\).
- For $k \neq \pm 1$, a generic family $\tilde{A}_i$ describes systems with three degrees of freedom: $(\hat{X}, \hat{P})$, $(\hat{Q}, \hat{\Pi})$ and $(\hat{H}, \hat{G}_2)$, thus time being a conjugate variable.
- For $k = 1$, $\tilde{A}_1 = \tilde{A}$ (already considered) describes an anomalous system with two degrees of freedom: $(\hat{X}, \hat{P})$ and $(\hat{H}, \hat{G}_2)$, time being again a conjugate variable.
- For $k = -1$, $\tilde{A}_{-1}$ describes a system with just two degrees of freedom: $(\hat{X}, \hat{P})$ and $(\hat{Q}, \hat{\Pi})$.

The third case is what we are looking for: although it contains two degrees of freedom, time is not a conjugate variable to any other. We should stress at this point that $\tilde{A}_{-1}$ is in fact the unique central extension, inside the family $\tilde{A}_i$, in which $\hat{H}$ is directly an ordinary time evolution generator\(^7\). This is the reason why the whole process is unambiguously defined.

A complete study of the entire family of central extensions of $\mathcal{A}$ and specially the original case $k = 1$ and the corresponding aforementioned anomalous reduction of the quantum representation, which might eventually end up with a non-conserved Hamiltonian operator associated with $\hat{H}$, will be published elsewhere.

Explicitly, for $\tilde{A}_{-1}$ we have

\[
\begin{align*}
[\hat{X}, \hat{P}] &= \mathrm{i}h\hat{H} & [\hat{Q}, \hat{\Pi}] &= \mathrm{i}h\hat{H} \\
[\hat{X}, \hat{Q}] &= -\frac{\mathrm{i}h}{m}\hat{G}_2 & [\hat{X}, \hat{\Pi}] &= \mathrm{i}h\hat{G}_1 \\
[\hat{P}, \hat{Q}] &= -\mathrm{i}h\hat{G}_1 + \mathrm{i}h\gamma\hat{G}_2 & [\hat{P}, \hat{\Pi}] &= -\mathrm{i}h\omega^2\hat{G}_2 \\
[\hat{H}, \hat{X}] &= -\frac{\mathrm{i}h}{m}\hat{\Pi} & [\hat{H}, \hat{P}] &= -\mathrm{i}h\omega^2\hat{Q} \\
[\hat{H}, \hat{Q}] &= \frac{\mathrm{i}h}{m}(\hat{P} + 2\hat{\Pi}) + \mathrm{i}h\gamma\hat{Q} & [\hat{H}, \hat{\Pi}] &= -\mathrm{i}h\omega^2(\hat{X} + 2\hat{Q}) - \mathrm{i}h\gamma\hat{\Pi}.
\end{align*}
\]

In this case, we note that the operators $\hat{G}_1$ and $\hat{G}_2$ commute with the basic couples $(\hat{X}, \hat{P})$ and $(\hat{Q}, \hat{\Pi})$ and, therefore, will be represented as constants (times the identity operator) in any irreducible representation. Even more, their commutation rules with $\hat{H}$ determine that the value of these constants must be zero ($[\hat{H}, \hat{G}_1] = [\hat{H}, \hat{G}_2] = 0$, as $\hat{G}_1$ and $\hat{G}_2$ are proportional to the identity $\hat{I}$); then, solve for $\hat{G}_1$ and $\hat{G}_2$ using the expressions above\(^8\). Technically, the operators $\hat{G}_1$ and $\hat{G}_2$ are gauge, in the sense that, in the resulting physical system, their symmetry transformation does not produce any change in the parameters associated with basic operators.

In consequence, the effective dimension of the algebra $\tilde{A}_{-1}$ is $5 + 1$. Let us denote by $\hat{B}$ the effective reduced Lie algebra, whose homonymous generators $(\hat{X}, \hat{P})$, $(\hat{Q}, \hat{\Pi})$, $\hat{H}$ and $\hat{I}$ have the commutation relations:

\[
\begin{align*}
[\hat{X}, \hat{P}] &= \mathrm{i}h\hat{H} & [\hat{Q}, \hat{\Pi}] &= \mathrm{i}h\hat{H} \\
[\hat{X}, \hat{Q}] &= 0 & [\hat{X}, \hat{\Pi}] &= 0 \\
[\hat{Q}, \hat{P}] &= 0 & [\hat{P}, \hat{\Pi}] &= 0 \\
[\hat{H}, \hat{X}] &= -\frac{\mathrm{i}h}{m}\hat{\Pi} & [\hat{H}, \hat{P}] &= -\mathrm{i}h\omega^2\hat{Q} \\
[\hat{H}, \hat{Q}] &= \frac{\mathrm{i}h}{m}(\hat{P} + 2\hat{\Pi}) + \mathrm{i}h\gamma\hat{Q} & [\hat{H}, \hat{\Pi}] &= -\mathrm{i}h\omega^2(\hat{X} + 2\hat{Q}) - \mathrm{i}h\gamma\hat{\Pi}.
\end{align*}
\]

This way, we end up with two pairs of independent, canonical operators along with a Hamiltonian. Since we have selected the representation in which $\hat{H}$ is not a basic operator, it can

\(^7\)This is still true for the complete family of central extensions of the algebra $\mathcal{A}$. In addition, for $k = -1$ both degrees of freedom share the same central extension parameters.

\(^8\)To be precise, in the critical-damping case $\Omega = \sqrt{\omega^2 - \frac{\gamma^2}{4}} = 0$ this only implies $\hat{G}_1 = \frac{\omega}{\gamma}\hat{G}_2$, although they still can be represented trivially.
be written in terms of the basic ones in an irreducible representation (note that symmetrization has been imposed to solve the ordering ambiguity, so as to achieve unitarity):

$$\hat{H} = \frac{1}{m} \hat{\Pi} \hat{\Pi} - \frac{\gamma}{2} (\hat{Q} \hat{\Pi} + \hat{\Pi} \hat{Q}) - \frac{\hat{Q}^2}{m} - m \omega^2 \hat{X} \hat{Q} - m \omega^2 \hat{Q}^2. \quad (14)$$

The physical content of this Hamiltonian can be analyzed classically by solving the Hamilton equations of motion. After that, a canonical quantization would also be possible. However, it is interesting to establish the relationship of the physical system at which we have arrived with a known one: the new system, with two degrees of freedom, is actually the Bateman dual system.

3.2. Bateman’s dual system

Let us perform the following linear transformation of operators:

$$\hat{X} = \hat{\gamma} + \frac{1}{m \gamma} \hat{p}_x + \frac{1}{2} \hat{\gamma}$$
$$\hat{P} = \hat{p}_x - m \gamma \hat{\gamma} - m \omega^2 \hat{\gamma}$$
$$\hat{\bar{Q}} = -\hat{\gamma} - \frac{1}{m \gamma} \hat{p}_y + \frac{1}{2} \hat{\gamma}$$
$$\hat{\bar{\Pi}} = \hat{p}_y + m \gamma \hat{\gamma} - m \omega^2 \hat{\gamma}$$

the inverse of which is given by

$$\hat{\gamma} = \hat{X} + \hat{\bar{Q}}$$
$$\hat{p}_x = \frac{1}{2} (\hat{P} + \hat{\bar{\Pi}}) + m \omega^2 (\hat{X} + \hat{\bar{Q}})$$
$$\hat{p}_y = -\frac{1}{m \gamma} (\hat{P} - \hat{\bar{\Pi}})$$
$$\hat{\gamma} = \hat{p}_x + m \gamma (\hat{X} - \hat{\bar{Q}}). \quad (15)$$

This transforms the Hamiltonian (14) into the so-called Bateman’s dual Hamiltonian: [5]

$$\hat{H} = \frac{\hat{p}_x \hat{p}_y}{m} + \frac{\gamma}{2} (\hat{\gamma} \hat{p}_x - \hat{\gamma} \hat{p}_y) + m \omega^2 \hat{\gamma}. \quad (17)$$

$$\hat{H}$$ closes a 5+1 dimensional algebra with \((\hat{\gamma}, \hat{p}_x)\) and \((\hat{\gamma}, \hat{p}_y)\):

$$[\hat{\gamma}, \hat{p}_x] = i \hbar \hat{\gamma} \quad [\hat{\gamma}, \hat{p}_y] = i \hbar \hat{\gamma}$$
$$[\hat{\gamma}, \hat{\gamma}] = 0 \quad [\hat{\gamma}, \hat{\gamma}] = 0$$
$$[\hat{\gamma}, \hat{\gamma}] = 0 \quad [\hat{\gamma}, \hat{\gamma}] = 0$$

$$[\hat{H}, \hat{\gamma}] = \frac{i \hbar}{m} \left( -\hat{p}_x + m \gamma \hat{\gamma} \right) \quad [\hat{H}, \hat{p}_x] = i \hbar \left( -\frac{\gamma}{2} \hat{p}_x + m \Omega^2 \hat{\gamma} \right)$$

$$[\hat{H}, \hat{\gamma}] = \frac{i \hbar}{m} \left( -\hat{p}_y - m \gamma \hat{\gamma} \right) \quad [\hat{H}, \hat{p}_y] = i \hbar \left( \frac{\gamma}{2} \hat{p}_y + m \Omega^2 \hat{\gamma} \right). \quad (18)$$

It can be checked that the corresponding classical Hamiltonian \(H\) describes a damped particle \((x, p_x)\) and its time reversal \((y, p_y)\) by computing the second-order classical equations out of the Hamilton equations:

$$\ddot{x} + \gamma \dot{x} + \omega^2 x = 0, \quad \ddot{y} - \gamma \dot{y} + \omega^2 y = 0. \quad (19)$$
The system is conservative, so that our objective of including the generator of time evolution among the symmetries has been accomplished, though at the cost of including a new degree of freedom.

3.3. Recovering Caldirola–Kanai system from the conservative Bateman system

Before addressing the quantum description of Bateman’s dual system in the following section, let us insist on the classical theory in this subsection. It is interesting to realize how the Bateman dual system actually generalizes the Caldirola–Kanai damped harmonic oscillator in the sense that a Caldirola–Kanai particle and its mirror image can be recovered by a canonical transformation.

In Bateman’s original paper [5], the Caldirola–Kanai Lagrangian was obtained by imposing the simple constraint $y = e^{\gamma t}x$ (Dekker [15] also described an analogous process by using a complex canonical transformation). Here we provide a (real) canonical transformation which takes the Bateman Hamiltonian to the difference of a couple of dual Caldirola–Kanai Hamiltonians. That is:

$$
\begin{align*}
    y' &= \frac{1}{\sqrt{2}} \left( y - e^{\gamma t} x + \frac{\gamma}{2m\Omega^2} p_x \right) \\
    p'_y &= \frac{1}{\sqrt{2}} \left( p_y - \frac{\omega^2}{\Omega^2} e^{-\gamma t} p_x + m \frac{\gamma}{2} x \right) \\
    x' &= \frac{1}{\sqrt{2}} \left( x + e^{-\gamma t} y - \frac{\gamma}{2m\Omega^2} e^{\gamma t} p_x \right) \\
    p'_x &= \frac{1}{\sqrt{2}} \left( \frac{\omega^2}{\Omega^2} p_x + e^{\gamma t} p_y - m \frac{\gamma}{2} e^{\gamma t} x \right) \\
    t' &= t.
\end{align*}
$$

This transformation, although explicitly time dependent, is the lifting of a canonical transformation among initial constants. After performing the transformation, the new Hamiltonian $H'$ (including the time derivative of the corresponding generating function) becomes

$$
H' = e^{-\gamma t} \frac{p_x^2}{2m} + \frac{m\omega^2}{2\Omega^2} x^2 e^{\gamma t} - e^{\gamma t} \frac{p_y^2}{2m} - \frac{1}{2} m\omega^2 y^2 e^{-\gamma t},
$$

which is nothing other than the Caldirola–Kanai Hamiltonian in terms of the variables $(x', p_x')$ minus a dual Caldirola–Kanai Hamiltonian in terms of the variables $(y', p_y')$. The relative minus sign in the new Hamiltonian $H'$ is unavoidable at least with a real canonical transformation.

A constraint to obtain the original Caldirola–Kanai Hamiltonian now reduces simply to the form:

$$
y' = 0, \quad p'_y = 0.
$$

4. Quantization of Bateman’s dual system

In principle, the quantum theory of the Bateman system can be addressed performing the usual canonical quantization in the position representation. Then, the Schrödinger equation for the Bateman Hamiltonian (17) is given by

$$
\begin{align*}
    i\hbar \frac{\partial \phi(x, y, t)}{\partial t} &= \left( -\frac{\hbar^2}{m} \frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} - i\hbar \frac{\gamma}{2} \left( y \frac{\partial}{\partial y} - x \frac{\partial}{\partial x} \right) + m\Omega^2 xy \right) \phi(x, y, t).
\end{align*}
$$
It has been argued that the quantum Bateman system possesses inconsistencies, such as complex eigenvalues and non-normalizable eigenstates [33, 17, 23, 15]. However, it should be noted that this observation is meaningless if the proper Hilbert space of the system is not specified, together with the precise way in which it is constructed. In this respect, Chruściński and Jurkowski [14] showed that $\hat{H}$ has in fact a real, continuous spectrum (we shall provide a proof of this in subsection 4.2), and that the complex eigenvalues are associated with resonances, which in last instance are the responsible of dissipation.

4.1. First-order Schrödinger equation

In order to construct the quantum representation of the Bateman algebra (18), we shall choose a mixed representation of position-momentum on complex functions of $x$ and $p_y$:

$$\hat{x} = x, \quad \hat{p}_x = -i\hbar \frac{\partial}{\partial x}, \quad \hat{y} = i\hbar \frac{\partial}{\partial p_y}, \quad \hat{p}_y = p_y.$$  \hspace{1cm} (24)

With this choice to represent the basic operators, the Hamiltonian becomes

$$\hat{H} = i\hbar \left( \frac{\gamma}{2} x - \frac{p_y}{m} \right) \frac{\partial}{\partial x} + i\hbar \left( \frac{\gamma}{2} p_y + m\Omega^2 x \right) \frac{\partial}{\partial p_y} + i\hbar \frac{\gamma}{2},$$  \hspace{1cm} (25)

that is, a first-order operator which is Hermitian with this particular ordering. Operators (24) and (25) (together with the identity) are Hermitian and represent the algebra (18) on complex, square integrable functions $\phi(x, p_y)$ with integration measure $d\phi$. As a consequence, the time-dependent Schrödinger equation also becomes of first order:

$$i\hbar \frac{\partial \phi}{\partial t} = \hat{H} \phi = i\hbar \left( \frac{\gamma}{2} x - \frac{p_y}{m} \right) \frac{\partial \phi}{\partial x} + i\hbar \left( \frac{\gamma}{2} p_y + m\Omega^2 x \right) \frac{\partial \phi}{\partial p_y} + i\hbar \frac{\gamma}{2}. \hspace{1cm} (26)$$

4.2. Solving the equation: eigenfunctions and spectrum

Before proceeding with the solution of the time-independent Schrödinger equation, let us firstly note that, being the Schrödinger equation of first order, it is possible to find the general form of the solution:

$$\phi(x, p_y, t) = e^{\gamma t} f \left( e^{\frac{\gamma}{2} t} (x \cos(\Omega t) - \frac{p_y}{m\Omega} \sin(\Omega t)) \right) e^{\frac{\gamma}{2} t} (p_y \cos(\Omega t) + m\Omega x \sin(\Omega t)) \right),$$

where $f$ is an arbitrary complex function of two variables. This expression can be written in a more convenient form in order to compare with the solutions of the time-independent equation, by defining

$$z_+ \equiv x + \frac{1}{m\Omega} p_y, \quad z_- \equiv x - \frac{1}{m\Omega} p_y,$$  \hspace{1cm} (27)

which are complex and $z_- = z_+^*$ when $\Omega \in \mathbb{R}$ (the underdamping case). Now, the solution is written:

$$\phi(x, p_y, t) = e^{\frac{\gamma}{2} t} f \left( e^{\frac{\gamma}{2} + i\Omega t} z_+, e^{\frac{\gamma}{2} - i\Omega t} z_- \right)$$  \hspace{1cm} (28)

or even:

$$\phi(x, p_y, t) = e^{\frac{\gamma}{2} t} f \left( e^{2i\Omega t} z_-, e^{2i\Omega t} z_+ \right).$$  \hspace{1cm} (29)

Since this system is autonomous, it makes sense to formulate an eigenvalue problem for the Hamiltonian. It was already mentioned in the introduction that several attempts of constructing the eigenfunctions of the Bateman dual system can be found in the literature. The most widespread is Feschbach–Tikochinsky’s [33, 17]; they in fact looked for creation and
annihilation operators to find the Hamiltonian eigenfunctions. This approach is appealing, as it unveils an SU(1, 1) structure for the system [11], although the fact that the Hamiltonian is not positive-definite leads to undesirable consequences: an unphysical complex, discrete spectrum along with the corresponding (non-normalizable, not even to a delta function) wavefunctions. It should be mentioned that many authors have nevertheless considered that this construction leads to the correct physical description of the quantum Bateman system [15, 11].

Two sets of authors proposed independently a different approach to find the real part of the spectrum (and therefore, the physically meaningful one) in the underdamping case. On the one hand, Rideau et al. [28] performed a canonical quantization in a mixed position-momentum representation (compare (28) with their results). On the other hand, Chrusciński and Jurkowski [14] made use of angular variables to perform a first-order quantization of the Hamiltonian, arriving at the correct spectrum, although it is not clear from their construction to what extent they can give a consistent construction of the basic operators (compare (29) with their results). We shall follow the first approach, although we shall give the corresponding expressions for each regime: underdamping, overdamping and critical damping.

The general solution of the time-independent Schrödinger equation can be found in terms of the variables (27). The eigenfunction of $\hat{H}$ with eigenvalue $E$ is given in terms of an arbitrary function $g$ of a single argument:

$$\phi(z_+, z_-) = \left( \frac{z_+ - z_-}{z_+ z_-} \right)^{\frac{E - n \gamma}{2\Omega}} g\left( \frac{z_-}{z_+} \right) \frac{i}{\Omega} \frac{\partial}{\partial z_-}.$$  

This means that the spectrum is infinitely degenerate. To break the degeneration, we proceed as usual: we select another operator which commutes with the Hamiltonian and find their simultaneous eigenfunctions. In fact, it is possible to split the Hamiltonian into two operators of that kind:

$$\hat{H}_\Omega = -i\hbar \left( \frac{p_x}{m} \frac{\partial}{\partial x} - m\Omega^2 x \frac{\partial}{\partial p_y} \right),$$  

$$\hat{D} = i\hbar \frac{\gamma}{2} \left( x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial p_y} + 1 \right),$$

with:

$$\hat{H} = \hat{H}_\Omega + \hat{D}, \quad [\hat{H}_\Omega, \hat{H}] = 0, \quad [\hat{D}, \hat{H}] = 0.$$

Then, the simultaneous eigenfunctions are simply:

$$\phi_{n, \lambda}(z_+, z_-) = \left( \frac{z_+}{z_-} \right)^{\frac{n}{2}} (z_+ z_-)^{-\frac{1}{2} - i\lambda},$$  

which satisfy:

$$\hat{H}_\Omega \phi_{n, \lambda} = n\hbar \Omega \phi_{n, \lambda}, \quad \hat{D} \phi_{n, \lambda} = \lambda \hbar \gamma \phi_{n, \lambda} \quad \Rightarrow \quad \hat{H} \phi_{n, \lambda} = \hbar(n\Omega + \lambda \gamma) \phi_{n, \lambda},$$

where the allowed values of $n$ and $\lambda$ depend on the regime.

Underdamping. In this case, $\Omega$ is real; therefore, $n$ must be real since the spectrum of $\hat{H}_\Omega$ must be real. Furthermore, $\frac{i}{\Omega}$ is a pure phase, twice the one of $z_-$. So, $n$ is necessarily an integer. This may be seen as a consequence of the fact that $\hat{H}_\Omega$ represents a rotation in the $x, p_y$ plane for $\Omega^2 > 0$.

Regarding the value of $\lambda$, it must be real, given that $\gamma$ is real. No further constraints are required for $\phi_{n, \lambda}$ to be well defined. This is consistent with $\hat{D}$ representing a dilation, with spectrum the real line. So, the spectrum of the Hamiltonian is

$$E = n\hbar \Omega + \lambda \hbar \gamma.$$


That is, we obtain a spectrum which is continuous and countably, infinitely degenerated. These results coincide with those in [14], although they are obtained here in a neater way: in [14], the authors quantize angular variables and hence the basic operator ‘multiply by the angle’ is not defined. We have avoided this problem: our basic operators are just (24).

Overdamping. Now $\Omega$ is pure imaginary, cf (7). In order to keep the spectrum real, $n$ must be pure imaginary as well. Let us then define

$$\tilde{n} \equiv -\text{i}n, \quad \tilde{\Omega} \equiv -\text{i}\Omega.$$

The variables (27) are now real:

$$z_+ \equiv x + \frac{1}{m\tilde{\Omega}}p_y, \quad z_- \equiv x - \frac{1}{m\tilde{\Omega}}p_y,$$

and it can be checked that solutions (32) can be written in the form:

$$\phi_{n,\lambda}(z_+, z_-) = e^{-\text{i}\tilde{n} \text{arccoth}(\frac{n}{\tilde{\Omega}})}(z_+ z_-)^{-\frac{1}{2} - \text{i}\lambda}.$$  \hfill (35)

Given that $\text{arccoth}(\frac{n}{\tilde{\Omega}})$ is not a periodic function, $\tilde{n}$ can take any real value. This is consistent with the fact that the action of $\hat{H}_\Omega$ is analogous to that of a space–time boost in this regime.

For the value of $\lambda$, the same considerations as before lead to the conclusion that it must be real. Therefore, the spectrum of the Hamiltonian is now real with two real indices:

$$E = \tilde{n}\hbar\tilde{\Omega} + \lambda\hbar\gamma.$$  \hfill (36)

In other words, the spectrum in this case is continuous and uncountably, infinitely degenerated.

Critical damping. In order to find the eigenfunctions and the spectrum in the critical damping case, it is necessary to solve the eigensystem for operators (30) and (31) when $\Omega$ is zero. This leads to eigenfunctions of the form:

$$\phi_{k,\lambda}^c(x, p_y) = e^{\text{i}k p_y} (p_y^2)^{-\frac{1}{2} - \text{i}\lambda},$$

satisfying ($\hat{H}_0 = \hat{H}_\Omega = 0$):

$$\hat{H}_0 \phi_{k,\lambda}^c = \hbar\gamma \phi_{k,\lambda}^c, \quad \hat{D} \phi_{k,\lambda}^c = \lambda\hbar\gamma \phi_{k,\lambda}^c \Rightarrow \hat{H}\phi_{k,\lambda}^k = \hbar\gamma(k + \lambda)\phi_{k,\lambda}^c,$$

where $k$ and $\lambda$ must be real, so that the spectrum

$$E = \hbar\gamma(k + \lambda)$$  \hfill (38)

is again continuous and uncountably, infinitely degenerated.

It is now clear that this quantization, which renders the Hamiltonian first order, allows a straightforward construction of the Hilbert space of the quantum Bateman dual system as well as the corresponding physical spectrum, which is not complex but real, as it must be for a Hermitian Hamiltonian.

5. Conclusions and outlook

In this paper we have found, starting with the non-conservative Caldirola–Kanai model for the quantum damped harmonic oscillator, a larger system containing two degrees of freedom (the original system and its time reversal one) by imposing that the time evolution generator $\text{i}\hbar \frac{\partial}{\partial t}$ closes a symmetry algebra with the basic operators $\hat{X}$ and $\hat{P}$. The system obtained is the Bateman dual system which, in some sense, can be considered as the simplest model for a system-reservoir approach to the damped harmonic oscillator.
It should be stressed that the process followed is unambiguous, in the sense that the
symmetry algebra of the Bateman dual system is the only extension of the Caldirola–Kanai
symmetry algebra where the time generator is not associated with a canonical variable. It
would be worth investigating, however, the other possible extensions of the Caldirola–Kanai
symmetry algebra where this is not primarily the case, for both the two and the three degrees
of freedom cases (see section 3.1).

As commented in the introduction, the quantization of Bateman’s dual system has
usually been considered full of inconsistencies since Feschbach and Tikochinsky [33, 17,
23, 15] constructed its Hilbert space made of non-normalizable energy eigenstates with
complex discrete eigenvalues. Many other authors insisted on this construction and suggested
alternatives [11, 7], but in this paper we have shown that, using a first-order Schrödinger
equation, the quantization is well defined and the spectrum is real and continuous, with infinite
degeneracy [28, 14].

The only pathology of the Bateman dual system is that the spectrum of the Hamiltonian
is not bounded from below, but this is not a problem as long as the whole Bateman system is
isolated (i.e. there is no external interaction). This problem can be easily solved by restricting
the initial conditions for the two particles in such a way that the kinetic energy be positive
[5]. This effectively describes a system with just one degree of freedom, as shown in section
3.3 by imposing the constraint (22) after performing the canonical transformation (20), which
is equivalent to a relation among initial constants. It would be interesting to investigate the
resultant one-degree-of-freedom systems after imposing other relations among initial constants
that give rise to positive kinetic energy.

The Bateman dual system solves many of the problems that the Caldirola–Kanai model
for damped harmonic oscillator suffers from. For instance, while the time evolution under
the Liouville equation for the density matrix of the Caldirola–Kanai system does not give
rise to decoherence (one of the main points raised against the Caldirola–Kanai model), the
Bateman dual system suffers from decoherence and classical correlations after the mirror
degree of freedom is traced out [20]. Also, selecting appropriate dense subspaces of the
Bateman Hilbert space, an effective evolution describing damped oscillations with complex
discrete frequencies is obtained [28], and these complex discrete frequencies, appearing as
the eigenvalues of the Bateman dual Hamiltonian in the Feschbach–Tikochinsky quantization,
turn out to be resonant energies corresponding to resonant (Gamow) states for the Bateman
dual system [14]. Even more, an splitting of the Hilbert space can be made in such a way that
in each subspace the originally unitary time evolution falls down to a semigroup, and therefore
irreversible to an evolution [14]. In addition, the Caldirola–Kanai system can be derived from
Bateman’s dual system (see section 3.3).

To end this section, we shall mention that the result that the algebra $\tilde{A}$ (12) is finite-
dimensional depends crucially on the fact that the frequency $\omega$ and the damping coefficient $\gamma$
are constant. For time-dependent frequency $\omega(t)$ and damping coefficient $\gamma(t)$, the procedure
of iteratively commuting the resulting new operators with all other previously computed will
not close to a finite algebra (except, perhaps, for very special cases). In this case, the resulting
extended algebra will probably contain an infinite number of degrees of freedom, a situation
that presumably fits better with the concept of thermal bath. A deeper study of this general
case would be extremely interesting, at least in a perturbative way.

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Appendix. Infinite-dimensional symmetry in the ‘free’ damped particle

In this appendix, we turn our attention to the ‘free’ damped particle as the simplest case of physical system subjected to a dissipative force and perform a similar analysis to that carried out in subsection 3.1 for the damped harmonic oscillator, forcing the introduction of the time generator in the symmetry algebra.

The basic operators for the damped particle (obtained as the $\omega \to 0$ of the damped harmonic oscillator, see equations (9) and (10)) are:

$$
\hat{P} = -i\hbar \frac{\partial}{\partial x}, \quad \hat{X} = x + \frac{i\hbar}{m\gamma} (1 - e^{-\gamma t}) \frac{\partial}{\partial x}.
$$
(A.1)

Renaming $\hat{P} \equiv \hat{P}_0$, we introduce operators $\hat{P}_n$ and $\hat{Y}_n$ ($n$ being a non-negative integer):

$$
\hat{H}_G = i\hbar e^{\gamma t} \frac{\partial}{\partial t} \quad \hat{H}_{DP} = i\hbar \frac{\partial}{\partial t}
$$

$$
\hat{P}_n = -i\hbar e^{-\gamma nt} \frac{\partial}{\partial x} \quad \hat{Y}_n = i e^{-\gamma nt}
$$

$$
\hat{X} = x + \frac{i\hbar}{m\gamma} (1 - e^{\gamma t}) \frac{\partial}{\partial x}.
$$
(A.2)

It is interesting that they close an infinite-dimensional Lie algebra with non-null commutators:

$$
[\hat{H}_G, \hat{P}_n] = -i\hbar \gamma n \hat{P}_{n-1} \quad [\hat{H}_{DP}, \hat{P}_n] = -i\hbar \gamma n \hat{P}_n
$$

$$
[\hat{H}_G, \hat{X}] = -i\hbar \hat{Y}_0 \quad [\hat{H}_{DP}, \hat{X}] = -i\hbar \hat{P}_1
$$

$$
[\hat{X}, \hat{P}_n] = \hbar \hat{Y}_n \quad [\hat{H}_{DP}, \hat{Y}_n] = -i\hbar \gamma n \hat{Y}_n \quad [\hat{H}_G, \hat{Y}_n] = -i\hbar \gamma n \hat{Y}_{n-1}
$$

$$
[\hat{H}_G, \hat{Y}_n] = -i\hbar \gamma n \hat{Y}_{n-1} \quad [\hat{H}_{DP}, \hat{H}_G] = i\hbar \gamma \hat{H}_G,
$$
(A.3)

containing a (centrally extended) Galilei algebra as a subalgebra, for which $\hat{Y}_0 = i$ is the central generator. Even, the generators in the left column of (A.3), for $n = 0, 1$, also close a finite-dimensional subalgebra in which $\hat{H}_G$ is a dynamical generator dual to $\hat{Y}_1$.

It is interesting to look at the algebra (A.3) as some sort of gauge algebra. In fact, we may start from the specific realization $\hat{H}_G, \hat{X}, \hat{P}_0, \hat{Y}_0$ of the centrally extended Galilei algebra and try to turn it into a gauge (local) algebra by allowing the multiplication by certain functions of time (positive integer powers of $e^{-\gamma t}$) of the space and time generators, and accordingly of the central one. This can be seen as the simplest symmetry of the free particle which results as a consequence of requiring the gauge invariance under the generator $e^{-\gamma t} \hat{H}_G$.

It would be worth investigating the possibility of repeating this brief analysis for the case of the symmetry of the harmonic oscillator.

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