COMMUTATORS OF SMALL RANK AND REDUCIBILITY OF OPERATOR SEMIGROUPS

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Abstract. It is easy to see that if $G$ is a non-abelian group of unitary matrices, then for no members $A$ and $B$ of $G$ can the rank of $AB - BA$ be one. We examine the consequences of the assumption that this rank is at most two for a general semigroup $S$ of linear operators. Our conclusion is that under obviously necessary, but trivial, size conditions, $S$ is reducible. In the case of a unitary group satisfying the hypothesis, we show that it is contained in the direct sum $G_1 \oplus G_2$, where $G_1$ is at most $3 \times 3$ and $G_2$ is abelian.

1. Introduction

It is easy to see that if $G$ is a non-abelian group of unitary matrices, then for no members $A$ and $B$ of $G$ can the rank of $AB - BA$ be one. Indeed, suppose that $A, B \in G$ be such that $AB \neq BA$. Then $ABA^{-1}B^{-1} - I = (AB - BA)A^{-1}B^{-1}$. Since $ABA^{-1}B^{-1}$ is a member of $G$, it is a unitary matrix; hence it is diagonalizable via a unitary similarity. If the rank of $AB - BA$ were equal to one, exactly one diagonal entry of $ABA^{-1}B^{-1}$ would be different from one, so that $\det(ABA^{-1}B^{-1})$ would be different from one, which is, clearly, a contradiction. In particular, this shows that the condition rank$(AB - BA) \leq 1$ for all $A, B$ in a unitary group $G$ implies that $G$ is abelian.

For semigroups of matrices and, more generally, linear operators on Banach spaces, the corresponding problem is more difficult. The following result was obtained in [6, Corollary 2].

Theorem 1.1 ([6]). Let $S$ be a semigroup of Schatten $p$-class operators on a Hilbert space. If rank$(AB - BA) \leq 1$ for all $A, B \in S$, then $S$ is triangularizable.

This was generalized to compact operators on arbitrary Banach spaces in [7, Theorem 9.2.10]. For non-compact operators, this question was studied in a series of papers. In [2, Lemma 5], the authors showed that the same conclusion holds for semigroups of algebraic operators, and in [3], it was shown that every non-commutative doubly generated semigroup $S$ with the condition that $\text{rank}(AB - BA) \leq 1$
for all \(A, B \in S\) has a hyperinvariant subspace. Finally, it was generalized to arbitrary operators on Banach spaces in [4] as follows:

**Theorem 1.2** ([4]). Let \(X\) be a Banach space of dimension at least two. Let \(S\) be a non-commutative semigroup of operators on \(X\). If \(\text{rank}(AB - BA) \leq 1\) for all \(A, B \in S\), then \(S\) is reducible.

It is natural to try to replace the rank-one condition in the above statements with the condition \(\text{rank}(AB - BA) \leq r\), where \(r \in \mathbb{N}\) is fixed. The following quick example shows that one cannot expect the same answer as in Theorem 1.2, even for semigroups of finite-rank operators.

**Example 1.3.** Let \(H\) be a finite- or infinite-dimensional Hilbert space. For all \(i, j = 1, 2, \ldots\), denote the \(i, j\)-matrix unit by \(E_{ij}\). That is, for a fixed orthonormal basis \((e_i)\), we have \(E_{ij}(e_k) = \delta_{jk}e_i\). The semigroup

\[ S = \{E_{ij} : i, j \in \mathbb{N}\} \cup \{0\} \]

is an irreducible semigroup of operators of rank \(\leq 1\) such that \(\text{rank}(AB - BA) \leq 2\) for all \(A, B \in S\).

In the present paper, we obtain results regarding the following question: when does the assumption \(\text{rank}(AB - BA) \leq 2\) for all operators \(A\) and \(B\) in a semigroup \(S\) imply reducibility of \(S\)? Our main argument uses special unitary groups whose structure is also of some independent interest and is a subject of study in the last section of this paper.

Throughout the paper, the linear space \(\mathbb{C}^n\) is considered as a Hilbert space with the standard inner product \(\langle \cdot, \cdot \rangle\). In the case of infinite-dimensional spaces, the term *operator* is reserved for the bounded linear operators. The set of operators on a Banach space \(X\) is denoted by \(B(X)\). The term *invariant subspace* means a non-trivial invariant subspace. A *semigroup* is a set \(S\) of operators on \(X\) such that \(AB \in S\) for all \(A, B \in S\). A semigroup \(S \subseteq B(X)\) is *reducible* if it admits an invariant subspace, and it is *triangularizable* if there exists a chain \(C\) that is maximal as a chain of subspaces of \(X\) and that has the property that every member of \(C\) is \(S\)-invariant (see [7, Definition 7.1.1]). A semigroup \(S \subseteq B(X)\) is *irreducible* if it is not reducible. The symbol \(\text{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_n\}\) denotes the \(n \times n\) diagonal matrix with \(\alpha_1, \alpha_2, \ldots, \alpha_n\) on the diagonal. The symbol \(\text{nul}(A)\) denotes the dimension of \(\ker A\). Finally, we will write \(A \equiv B\) if the matrices \(A\) and \(B\) are unitarily similar.

### 2. Reducibility of Semigroups

We will start by investigating the structure of certain very special groups of unitaries.

**Definition 2.1.** Let \(p\) and \(q\) be two prime numbers. The symbol \(\mathcal{G}(p, q, A)\) will denote the group of unitaries generated by the \(p \times p\) matrices

\[
S = \begin{bmatrix}
0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots \\
0 & 0 & \ldots & 1 & 0
\end{bmatrix}
\quad \text{and} \quad
A = \begin{bmatrix}
\omega_1 & 0 & \ldots & 0 & 0 \\
0 & \omega_2 & \ldots & 0 & 0 \\
0 & 0 & \ldots & \omega_{p-1} & 0 \\
0 & 0 & \ldots & 0 & \omega_p
\end{bmatrix},
\]

where \(A\) is not a scalar multiple of the identity and \(\omega_i^q = 1\) for all \(i = 1, 2, \ldots, p\).
Our interest in these groups stems from the fact that if \( \mathcal{G} \) is a minimal non-abelian group of matrices, then \( \mathcal{G} \) admits a subgroup \( \mathcal{G}_0 \) whose restriction to a \( \mathcal{G}_0 \)-invariant subspace is closely related to a group of the form \( \mathcal{G}(p,q,A) \) (see [7, Lemma 4.2.9]).

**Proposition 2.2.** Let \( p, q \) be two prime numbers and \( A \) be a \( p \times p \) matrix as in Definition 2.1. If \( \text{rank}(XY - YX) \leq 2 \) for all \( X, Y \in \mathcal{G}(p,q,A) \), with the equality achieved on some members of it, then either

1. \( p = 2 \) or
2. \( p = 3 \) and \( q = 2 \).

**Proof.** Denote the group \( \mathcal{G}(p,q,A) \) by \( \mathcal{G} \), for simplicity of notation. It is not hard to see that every member of \( \mathcal{G} \) can be written in the form \( DS^k \), where \( D \) is a diagonal matrix whose diagonal entries are \( q \)-roots of unity, \( S \) is the cyclic permutation as in Definition 2.1, and \( 0 \leq k < p \). Moreover, if \( X_1 = D_1 S^{k_1} \) and \( X_2 = D_2 S^{k_2} \), then \( X_1 X_2 = D_3 S^{k_1+k_2} \), for some diagonal matrix \( D_3 \).

Let \( X \) and \( Y \) be arbitrary members of \( \mathcal{G} \). It follows from the above observation that \( XYX^{-1}Y^{-1} \) is a diagonal matrix. It is clear that if

\[
\text{rank}(XY - YX) = 2,
\]

then exactly two eigenvalues of \( XYX^{-1}Y^{-1} \) are not equal to one. Since

\[
\det(XYX^{-1}Y^{-1}) = 1,
\]

we conclude that \( XYX^{-1}Y^{-1} \) is of the form

\[
\text{diag}(1,\ldots,1, \omega, 1,\ldots,1,\bar{\omega},1,\ldots,1),
\]

where \( \omega \neq 1 \) and \( \omega^q = 1 \), and each of the series of ones between \( \omega \) and \( \bar{\omega} \) could be absent.

Observe that if \( D = \text{diag}(d_1,\ldots,d_{p-1},d_p) \), then \( SDS^{-1} = \text{diag}(d_p,d_1,\ldots,d_{p-1}) \). It follows that \( \mathcal{G} \) has a member of the form

\[
A_0 = \text{diag}(\omega,1,\ldots,1,\bar{\omega},1,\ldots,1),
\]

where \( \omega \neq 1, \omega^q = 1 \), and the series of ones between \( \omega \) and \( \bar{\omega} \) is shorter than the series of ones following \( \bar{\omega} \).

Suppose that \( p > 3 \), so that \( p \geq 5 \). If the series of ones between \( \omega \) and \( \bar{\omega} \) is not absent, then consider \( B = SA_0^{-1}S^{-1} \). It follows that

\[
A_0B = A_0SA_0^{-1}S^{-1} = \text{diag}(\omega,\bar{\omega},1,\ldots,1,\bar{\omega},\omega,1,\ldots,1),
\]

so that \( \text{rank}(A_0S - SA_0) = \text{rank}(A_0SA_0^{-1}S^{-1} - I) = 4 \), contrary to the assumptions. So, the series of ones between \( \omega \) and \( \bar{\omega} \) must be absent, and

\[
A_0 = \text{diag}(\omega,\bar{\omega},1,\ldots,1).
\]

However, in this case we may consider \( C = S^2A_0^{-1}S^{-2} \). We get

\[
A_0C = A_0S^2A_0^{-1}S^{-2} = \text{diag}(\omega,\bar{\omega},\bar{\omega},\omega,1,\ldots,1),
\]

so that \( \text{rank}(A_0S^2 - S^2A_0) = 4 \).

This shows that either \( p = 2 \) or \( p = 3 \). Suppose that \( p = 3 \). We claim that, necessarily, \( q = 2 \). Assume that \( q > 2 \). Then, by the same argument as above,

\[
A_0 = \text{diag}(\omega,\bar{\omega},1) \in \mathcal{G},
\]

where \( \omega \neq 1 \) and \( \omega^q = 1 \). Clearly, \( SA_0^{-1}S^{-1} = \text{diag}(1,\bar{\omega},\omega) \), so that

\[
A_0SA_0^{-1}S^{-1} = \text{diag}(\omega,\omega^2,\omega).
\]
If $q > 2$, then all the diagonal entries of this matrix are different from 1, so that $\text{rank}(A_0 S - SA_0) = 3$, a contradiction.

The next proposition records certain observations about the groups $G$ satisfying $\text{rank}(XY - YX) \leq 2$ for all $X, Y \in G$. We will need the following notation.

**Definition 2.3.** Let $S$ be a set of $n \times n$ matrices and $M$ be a linear subspace of $\mathbb{C}^n$. Then we put

$$S(M) = \{ T \in S : TM \subseteq M \}.$$  

**Proposition 2.4.** Let $G$ be a non-abelian group of unitary $n \times n$ matrices, and assume $\text{rank}(AB - BA) \leq 2$ for all $A, B \in G$. If $M$ is a linear subspace of $\mathbb{C}^n$, then $G(M) = G(M^\perp)$ is a subgroup of $G$ and at least one of the unitary groups $G(M)|_M$ or $G(M)|_{M^\perp}$ is abelian.

**Proof.** If $n \leq 2$, then the conclusions of the proposition are evident. Therefore, we will assume in the proof that $n \geq 3$.

Let $A, B \in G$. Since $ABA^{-1}B^{-1}$ is a unitary and $\text{rank}(ABA^{-1}B^{-1} - I) = \text{rank}(AB - BA) \leq 2$, it follows from the first paragraph of the introduction that $\text{rank}(AB - BA)$ is 0 or 2, and hence $ABA^{-1}B^{-1} = \text{diag}(\omega, \omega', 1, 1, \ldots, 1) \neq I$ for some $\omega \neq 1 \neq \omega'$. Also, since $1 = \det(ABA^{-1}B^{-1}) = \omega \omega'$, it follows that $\omega' = \bar{\omega}$.

Next, assume $M$ is a linear subspace of $\mathbb{C}^n$. Clearly, $G(M) = G(M^\perp)$. Assume, if possible, that both $G(M)|_M$ and $G(M)|_{M^\perp}$ are non-abelian. For $i = 1, 2$, choose $A_i = C_i \oplus D_i \in G(M)$ decomposed according to $\mathbb{C}^n = M \oplus M^\perp$, such that $C_1 C_2 \neq C_2 C_1$. Notice that the condition $D_1 D_2 \neq D_2 D_1$ would imply

$$\text{rank}(A_1 A_2 - A_2 A_1) = 4;$$

hence $D_1 D_2 = D_2 D_1$. Assume, if possible, that $D_1$ is not in the centre of $G(M)|_{M^\perp}$. In this case, choose $A_3 = C_3 \oplus D_3 \in G(M)$ such that $D_1 D_3 \neq D_3 D_1$ and, consequently, $C_3 C_1 = C_1 C_3$. Then $C_1 (C_2 C_3) \neq C_2 C_1 C_3 = (C_2 C_3) C_1$ and $D_1 (D_2 D_3) = D_2 D_1 D_3 \neq (D_2 D_3) D_1$, which is a contradiction. Thus $D_1$ and, by symmetry, $D_2$ belong to the centre of $G(M)|_{M^\perp}$. Now, since $G(M)|_{M^\perp}$ is not abelian, there exist $A_3 = C_3 \oplus D_3$ and $A_4 = C_4 \oplus D_4$ in $G(M)$ such that $C_3 C_4 = C_4 C_3$ and $D_3 D_4 \neq D_4 D_3$. Another symmetrical argument reveals that $C_3, C_4$ belong to the centre of $G(M)|_M$. Then $(C_1 C_3)(C_2 C_4) = C_1 C_2 C_3 C_4 \neq C_2 C_1 C_3 C_4 = (C_2 C_4)(C_1 C_3)$ and $(D_1 D_3)(D_2 D_4) = D_2 D_3 D_4 D_1 \neq D_2 D_1 D_3 D_4 = (D_2 D_4)(D_1 D_3)$, and, hence, $\text{rank}[(A_1 A_3)(A_2 A_4) - (A_2 A_4)(A_1 A_3)] = 4$, a contradiction.

Before we state our main theorem, we need two lemmas.

**Lemma 2.5.** Let $G$ be a non-abelian unitary group on $\mathbb{C}^n$ and $N \subseteq \mathbb{C}^n$ be a 3-dimensional subspace. Assume that $G$ has a subgroup $G_0$ such that $N$ is $G_0$-invariant and, in some basis $\{e_1, e_2, e_3\}$ of $N$, $G_0|_N = G(3, q, A)$, where $q$ is a prime number and $A$ is a diagonal matrix as in Definition 2.1. If $\text{rank}(XY - YX) \leq 2$ for all $X, Y \in G$, then $N$ is $G$-invariant. Moreover, $G|_N$ is irreducible and $G|_{N^\perp}$ is abelian.

**Proof.** Since $G$ is non-abelian, in view of the observation made at the beginning of the introduction, $\text{rank}(XY - YX) = 2$, for some $X, Y \in G$. By Proposition 2.2, $q$ must be equal to 2. Considering matrices of the form $XY X^{-1} Y^{-1}$, as in the proof of Proposition 2.2 we conclude that $G(3, 2, A)$ admits a diagonal matrix $B$ with...
eigenvalues \{1, -1, -1\}. Considering \(SBS^{-1}\) and \(S^2BS^{-2}\), where \(S\) is the cyclic permutation as in Definition 2.1, we conclude that the matrices
\[
\text{diag}(1, -1, -1), \quad \text{diag}(-1, 1, -1), \quad \text{and} \quad \text{diag}(-1, -1, 1)
\]
all belong to \(G(3, 2, A)\).

Pick an arbitrary \(Z \in G\) and assume that \(\mathcal{N}\) is not \(Z\)-invariant. Fix a matrix \(\widetilde{S} \in G\) such that \(\widetilde{S}|_{\mathcal{N}} = S\). Choose a basis \(\{e_4, e_5, \cdots, e_n\}\) for \(\mathcal{N}^\perp\) consisting of eigenvectors of \(\widetilde{S}\). Since \(\mathcal{N}\) (and, hence, \(\mathcal{N}^\perp\)) is not \(Z\)-invariant, there exist \(i \leq 3\) and \(j \geq 4\) such that \(\langle Ze_j, e_i \rangle \neq 0\). Due to the cyclic nature of the conditions of the theorem with respect to the ordered triple \((e_1, e_2, e_3)\), we may and shall assume without loss of generality that \(i = 1\). Let \(M\) be the 2-dimensional subspace of \(\mathbb{C}^n\) spanned by \(\{e_1, e_2\}\) and write
\[
Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \quad \text{with respect to} \quad \mathbb{C}^n = M \oplus M^\perp.
\]
The matrix \(\text{diag}(-1, -1, 1) \in G(3, 2, A)\) can be obtained as \(CSC^{-1}S^{-1}\), where \(C = \text{diag}(1, -1, -1) \in G(3, 2, A)\). This shows that if \(\widetilde{C} \in G\) is such that \(\widetilde{C}|_{\mathcal{N}} = C\), then the matrix
\[
T = \widetilde{C}S\widetilde{C}^{-1}\widetilde{S}^{-1} = \text{diag}(-1, -1, 1, \ldots, 1) \in G.
\]
With respect to \(\mathbb{C}^n = M \oplus M^\perp\), this matrix has the form
\[
T = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}.
\]
Then
\[
TZ - ZT = \begin{bmatrix} 0 & 2Z_{12} \\ -2Z_{21} & 0 \end{bmatrix}.
\]
Since \(\text{rank}(TZ - ZT) \leq 2\), we conclude that \(\text{rank}(Z_{12}) = \text{rank}(Z_{21}) = 1\), for none of \(Z_{12}\) and \(Z_{21}\) are zero. It follows that \(Z_{12}e_k (k = 3, 4, \cdots, n)\) are multiples of \(Z_{12}e_j\). Replacing \(Z\) by \(Z\widetilde{S}\) changes the first column \(Z_{12}e_3\) of \(Z_{12}\) to \(Z_{11}e_1\) and its \((j - 2)\text{nd}\) column \(Z_{12}e_j\) to \(\lambda_j Z_{12}e_j\), where \(\lambda_j\) is the eigenvalue of \(\widetilde{S}\) corresponding to \(e_j\). Thus, again, \(Z_{11}e_2\) is a multiple of \(Z_{12}e_j\). Another replacement of \(Z\) by \(Z\widetilde{S}^2\) reveals that the first two rows of \(Z\) are linearly dependent, a contradiction. This shows that \(\mathcal{N}\) is \(G\)-invariant.

Finally, the irreducibility of \(G|_{\mathcal{N}}\) follows from the fact that \(G(p, q, A)\) is irreducible (see, e.g., [7, Lemma 4.2.8]), and the commutativity of \(G|_{\mathcal{N}^\perp}\) was established in Proposition 2.4.1.

Lemma 2.6. Let \(S\) be a semigroup of \(n \times n\) matrices and \(\mathcal{N}\) be a subspace of \(\mathbb{C}^n\) such that, with respect to the decomposition \(\mathbb{C}^n = \mathcal{N} \oplus \mathcal{N}^\perp\), the representation of every member \(Z \in S\),
\[
Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},
\]
has the property that \(\text{rank}(Z_{21}) \leq 1\). Then each \(Z \in S\) admits an invariant subspace \(\mathcal{N}_Z\) such that either \(\mathcal{N}_Z \subseteq \mathcal{N}\), in which case \(\dim(\mathcal{N}/\mathcal{N}_Z) \leq 1\), or \(\mathcal{N} \subseteq \mathcal{N}_Z\), in which case \(\dim(\mathcal{N}_Z/\mathcal{N}) \leq 1\) and \(\mathcal{N}_Z = \text{span}\{\mathcal{N}, Z\mathcal{N}\}\).
Proof. Denote the dimension of $\mathcal{N}$ by $k$. Clearly, there is no loss of generality in assuming that $2 \leq k \leq n - 2$.

Let $Z \in S$ be such that $\mathcal{N}$ is not $Z$-invariant. Since $\text{rank}(Z_{21}) = 1$, by choosing appropriate bases $\{e_1, \ldots, e_k\}$ for $\mathcal{N}$ and $\{e_{k+1}, \ldots, e_n\}$ for $\mathcal{N}^\perp$, we may assume that only the $(1,1)$-entry of $Z_{21}$ is non-zero.

Consider the matrix $Z^2 \in S$. Its $(2,1)$-block is equal to $Z_{21}Z_{11} + Z_{22}Z_{21}$. Notice that only the first row of the matrix $Z_{21}Z_{11}$ may contain non-zero entries and only the first column of the matrix $Z_{22}Z_{21}$ may contain non-zero entries. Since the rank of $Z_{21}Z_{11} + Z_{22}Z_{21}$ is assumed to be at most one, we conclude that one of the matrices $Z_{21}Z_{11}$ or $Z_{22}Z_{21}$ must satisfy the property that all its entries, except perhaps the $(1,1)$-entry, are equal to zero. If all but the $(1,1)$-entry of $Z_{21}Z_{11}$ are zero, then $Z_{11}$ (and, hence, $Z$) leaves invariant the space span$\{e_2, \ldots, e_k\}$. If all but the $(1,1)$-entry of $Z_{22}Z_{21}$ are zero, then in the first column of $Z_{22}$ only the first entry may be non-zero, so that $Z$ leaves invariant the space span$\{e_1, \ldots, e_k, e_{k+1}\}$. \qed

Now we are ready to prove the main theorem of the paper.

**Theorem 2.7.** Let $\mathcal{G}$ be a group of unitary $n \times n$ matrices. If $\text{rank}(AB - BA) \leq 2$ for all $A, B \in \mathcal{G}$, then there is a subspace $\mathcal{M}$ of $\mathbb{C}^n$ such that $1 \leq \dim \mathcal{M} \leq 3$ and $\mathcal{G} \subseteq \mathcal{G}_1 \oplus \mathcal{G}_2$ with $\mathcal{G}_2$ abelian, where the direct sum is with respect to the decomposition $\mathbb{C}^n = \mathcal{M} \oplus \mathcal{M}^\perp$.

**Proof.** Clearly, there is no loss of generality in assuming that $\mathcal{G}$ is not abelian and $n \geq 4$. Moreover, we may also assume that $\mathcal{G} = \mathbb{T}\mathcal{G}$, where $\mathbb{T}$ is the unit circle on the complex plane.

Since $\mathcal{G} = \mathbb{T}\mathcal{G}$, it is a compact Lie group, so [4, Theorem 5] implies that $\mathcal{G}$ contains a finite non-abelian subgroup. It follows that $\mathcal{G}$ contains a minimal non-abelian subgroup. By [4, Lemma 4.2.9], every minimal non-abelian finite group admits an invariant subspace $\mathcal{N}$ such that the restriction of the group to $\mathcal{N}$ is, after a similarity, generated by two matrices $\alpha A$ and $\beta S$, where $A$ is a non-scalar diagonal matrix, $S$ is the cyclic permutation, and $\alpha, \beta \in \mathbb{T}$. Since $\mathcal{G} = \mathbb{T}\mathcal{G}$, we conclude that $\mathcal{G}$ contains a subgroup $\mathcal{G}_0$ whose restriction to $\mathcal{N}$ is equal (in an appropriate basis) to the group $\mathcal{G}(p, q, A)$.

It follows from Proposition 2.2 and Lemma 2.5 that, without loss of generality, $p = 2$. Since $\mathcal{G}(2, q, A)$ is not abelian, it contains a matrix $C$ of the form $XYX^{-1}Y^{-1}$ different from the identity. By the properties of $\mathcal{G}(p, q, A)$, this matrix is necessarily diagonal, and its diagonal entries are $q$-roots of the unity. Since $\det(C) = 1$, we have $C = \text{diag}(\omega, \bar{\omega})$, for some $\omega \neq 1$, $\omega^q = 1$.

If $Z \in \mathcal{G}$ is an arbitrary matrix, then, considering the rank of $ZC - CZ$, we conclude that, with respect to the decomposition $\mathbb{C}^n = \mathcal{N} \oplus \mathcal{N}^\perp$, $Z$ is represented as

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},$$

where $\text{rank}(Z_{21}) \leq 1$. By Lemma 2.10, either $Z$ admits an eigenvector in $\mathcal{N}$ or the space span{$\mathcal{N}, Z\mathcal{N}$} has dimension 3 and is $Z$-invariant. Notice that this space contains $\mathcal{N}$ as a subspace of codimension one.

First, we claim that, assuming $\mathcal{N}$ is not $\mathcal{G}$-invariant, $\mathcal{G}$ admits a matrix without an eigenvector in $\mathcal{N}$.
Indeed, let $V \in \mathcal{G}$ be such that $\mathcal{N}$ is not $V$-invariant. If $V$ does not have eigenvectors in $\mathcal{N}$, we are done. Suppose that $V$ has an eigenvector in $\mathcal{N}$. Write $V$ as

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}.$$ 

Let $f \in \mathcal{N}$ be an eigenvector of $V$. Clearly, $\text{span}\{f\} = \ker(V_{21})$ and $f$ is an eigenvector for $V_{11}$. Since $G(2,q,A)$ is irreducible (see, e.g., [7, Lemma 4.2.8]), there exists $U \in G(2,q,A)$ such that $f$ is not an eigenvector of $UV_{11}$. Then there exists a matrix $Z \in \mathcal{G}$ of the form $U \oplus D$, where $D$ is a unitary $(n - 2) \times (n - 2)$ matrix. Since $\ker(DV_{21}) = \ker(V_{21}) = \text{span}\{f\}$, the matrix $ZV$ does not admit eigenvectors in $\mathcal{N}$.

Let $T \in \mathcal{G}$ be a matrix without eigenvectors in $\mathcal{N}$. Since $T$ is a unitary matrix, every invariant subspace of it is reducing. By Lemma 2.6, there exist an orthonormal basis $\{e_1, e_2\}$ of $\mathcal{N}$ and a unit vector $e_3$ in $\mathcal{N}^\perp$ such that, relative to the decomposition $\mathbb{C}^n = \text{span}\{e_1\} \oplus \text{span}\{e_2\} \oplus \text{span}\{e_3\} \oplus (\mathcal{N} \ominus \text{span}\{e_3\})$, $T$ is written in the form

$$T = \begin{bmatrix} p & q & w & 0 \\ r & s & u & 0 \\ 0 & t & v & 0 \\ 0 & 0 & 0 & U \end{bmatrix},$$

where $r \neq 0$, $t \neq 0$, and $U$ is an $(n - 3) \times (n - 3)$ unitary matrix.

Let $S \in \mathcal{G}$ be arbitrary. Write, relative to the same decomposition,

$$S = \begin{bmatrix} a & b & * & * \\ c & d & * & * \\ e & f & * & * \\ g & h & * & * \end{bmatrix},$$

where $a, b, c, d, e, f$ are complex numbers, $g$ and $h$ are $(n-3)$-vectors, and the symbol $*$ stands for a number or a matrix whose value does not concern us. Multiplying $T$ by $S$, we get

$$TS = \begin{bmatrix} * & * & * & * \\ ar + cs + eu & br + ds + fu & * & * \\ ct + ev & dt + fv & * & * \\ Ug & Uh & * & * \end{bmatrix}.$$ 

Recall that

$$\text{rank}\left(\begin{bmatrix} c \\ g \\ h \end{bmatrix}\right) \leq 1 \quad \text{and} \quad \text{rank}\left(\begin{bmatrix} ct & dt + fv \\ Ug & Uh \end{bmatrix}\right) \leq 1.$$ 

Suppose that one of the vectors $g$ and $h$ is not zero, say, $g \neq 0$. Then there exists $\alpha \in \mathbb{C}$ such that $h = \alpha g$, $f = \alpha e$ and $dt + fv = \alpha(ct + ev)$. Since $t \neq 0$, we conclude that $d = \alpha c$. It follows that

$$\text{rank}\left(\begin{bmatrix} c & d \\ e & f \\ g & h \end{bmatrix}\right) = 1.$$ 

Repeating the same argument with the matrix $TS$ replacing the matrix $S$, we obtain

$$\text{rank}\left(\begin{bmatrix} ar + cs + eu & br + ds + fu \\ ct + ev & dt + fv \\ Ug & Uh \end{bmatrix}\right) = 1.$$
It follows that $br + ds + fu = \alpha (ar + cs + eu)$. Since $r \neq 0$, the only possibility is that $b = \alpha a$. However, this implies that

$$\begin{pmatrix} a & b \\ c & d \\ e & f \\ g & h \end{pmatrix}$$

This is impossible since the matrix $S$ is unitary, hence invertible.

The case $h \neq 0$ brings us to the same conclusion. Therefore $g = h = 0$. Since $S$ was chosen arbitrarily, this implies that $M = \mathcal{GN}$ is $\mathcal{G}$-invariant and $M \perp (N \perp \cap \mathop{\text{span}} \{e_3\})$. Under the assumption that $N$ is not $\mathcal{G}$-invariant, this means that $M = \mathop{\text{span}} \{e_1, e_2, e_3\}$, a 3-dimensional $\mathcal{G}$-invariant subspace. The rest of the conclusions of the theorem follow from Proposition 2.4. □

**Corollary 2.8.** Let $X$ be a Banach space and $S = \mathbb{R}^+ S$ be a semigroup of operators on $X$ containing a non-zero compact operator such that the minimal rank of non-zero operators in $S$ is at least 4. If $\text{rank}(AB - BA) \leq 2$ for all $A, B \in S$, then $S$ is reducible.

**Proof.** Suppose that $S$ is irreducible. It is well-known that a non-trivial ideal of an irreducible semigroup is irreducible. Thus, there is no loss of generality in assuming that $S$ consists of compact operators.

Denote the minimal non-zero rank of operators in $S$ by $r$. By [7, Lemma 8.1.15], $r$ is finite and there exists an idempotent $E \in S$ of rank $r$. Let $S_0 = ESE\mid \text{Range } E$. Then $S_0$ is represented as a semigroup of $r \times r$ matrices. Moreover, every member of this semigroup is either invertible or zero, by the minimality of the rank $r$ in $S$. Also, as a compression of an irreducible semigroup, the semigroup $S_0$ must be irreducible. By [7, Lemma 3.1.6], $S_0 \setminus \{0\}$ is a group of matrices. Moreover, there exists a group $\mathcal{G}$ of unitary matrices such that, after a similarity, $S_0 \setminus \{0\} \subseteq \mathbb{R}^+ \mathcal{G}$. Clearly, $\mathcal{G}$ must be irreducible, too. Also, the proof of [7, Lemma 3.1.6] shows that the group $\mathcal{G}$ is, in fact, similar to the group $\{e^{i\theta(T)} : T \in \mathcal{G}_0\}$. Hence, the condition $\text{rank}(AB - BA) \leq 2$ holds for all $A, B \in \mathcal{G}$. This, obviously, contradicts the conclusion of Theorem 2.7. □

We remark that the condition about the rank in Corollary 2.8 cannot be improved. This is clear if the minimal rank is allowed to be equal to 2 (take, for example, the group of $2 \times 2$ unitaries). The following proposition exhibits an example of an irreducible group of $3 \times 3$ unitary matrices with the property $\text{rank}(AB - BA) \leq 2$ for all $A, B \in \mathcal{G}$.

**Proposition 2.9.** Let

$$T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \quad \text{and} \quad \quad S = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$ 

Then the group $\mathcal{G} = \langle T, S \rangle$ is irreducible and satisfies the condition that $\text{rank}(AB - BA) \leq 2$ for all $A, B \in \mathcal{G}$. 

Proof. By [7, Lemma 4.2.8], the group $\mathcal{G}$ is irreducible. Let us show that
\[
\text{rank } (AB - BA) \leq 2
\]
for all $A, B \in \mathcal{G}$.

Observe that every member of $\mathcal{G}$, being a finite product of matrices $T, S, T^{-1}$ and $S^{-1}$, can be written in one of the following three forms:
\[
\begin{bmatrix}
\alpha & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & \gamma
\end{bmatrix}, \quad
\begin{bmatrix}
0 & \alpha & 0 \\
0 & 0 & \beta \\
\gamma & 0 & 0
\end{bmatrix}, \quad \text{or } \begin{bmatrix}
0 & 0 & \alpha \\
\beta & 0 & 0 \\
0 & \gamma & 0
\end{bmatrix},
\]
with $\alpha, \beta, \gamma \in \{1, -1\}$. Moreover, among the numbers $\alpha, \beta, \gamma$ exactly two or none are equal to $-1$, the rest being equal to $1$. For $A \in \mathcal{G}$, let us refer to the particular form of $A$ among the three forms above as the pattern of $A$.

A routine check shows that for all matrices $A$ and $B \in \mathcal{G}$, the patterns of $AB$ and $BA$ are the same. Hence, the difference $AB - BA$ must have the same pattern, too. Now, since there are exactly zero or two elements equal to $-1$ among the non-zero elements of $AB$ and $BA$, a quick check shows that there is at least one entry $(i,j)$ such that $(AB)_{ij}$ and $(BA)_{ij}$ are both equal to $1$ or to $-1$ simultaneously. But this means that the difference $AB - BA$ has at most two non-zero entries, so that $\text{rank } (AB - BA) \leq 2$. \hfill \square

3. On the structure of the group $\mathcal{G}(p, q, A)$

For prime numbers $p$ and $q$, let $\mathcal{G} = \mathcal{G}(p, q, A)$ be the irreducible group with generators $A$ and $S$ as defined before. These groups played a central role in our arguments from Section 2. In the present section, we will further study the structure of these groups in terms of the following parameters:

\begin{align}
(3.1) \quad \rho &= \min \{\text{rank}(D - I) \neq 0 : D \in \mathcal{G} ; D \text{ diagonal} \}, \\
(3.2) \quad r &= \max \{\text{rank}(XYX^{-1}Y^{-1} - I) : X, Y \in \mathcal{G} \}.
\end{align}

Note. Clearly, $1 \leq \rho \leq r \leq p$.

Throughout the remainder of the paper, $\mathcal{G} = \mathcal{G}(p, q, A)$ for some $p, q, A$. If $p, q$ are fixed, we may also write $\mathcal{G}_A, \rho_A$ and $r_A$ to denote $\mathcal{G}(p, q, A)$, $\rho$ and $r$, respectively.

**Theorem 3.1.** Let $\mathcal{D}_A$ be the collection of all diagonal matrices in $\mathcal{G}_A = \mathcal{G}(p, q, A)$ and let $\mathcal{S}$ be the subgroup generated by $S$. Also, let $\mathcal{C}_A$ be the commutator subgroup of $\mathcal{G}_A$. Then $\mathcal{G}_A = \mathcal{D}_A \mathcal{S} = \mathcal{S} \mathcal{D}_A$ and $\mathcal{C}_A \subset \mathcal{D}_A$. Moreover, if $\mathcal{C}_A \neq \mathcal{D}_A$, then one of the following cases holds:

(i) $\mathcal{C}_A$ contains no non-scalar matrix. Then $p/2 \leq \rho_A \leq r_A = p = q$ and $\mathcal{C}_A = \{\eta I : \eta^p = 1\}$.

(ii) $\mathcal{C}_A$ contains non-scalar matrices, and for any non-scalar $B \in \mathcal{C}_A$, $\mathcal{D}_B = \{\epsilon I : \epsilon^p = 1\}$ and $\rho_B \leq \rho_B \leq r_B \leq r_A$ and $\rho_A \leq 2\rho_B$.

Proof. For convenience, we drop the subscript $A$ and will only maintain the subscript $B$ to avoid confusion. Consider the general word
\[
G = A^\alpha S^\beta A^{\alpha_2} S^{\beta_2} \cdots A^{\alpha_m} S^{\beta_m} \in \mathcal{G}
\]
for some integers $m, \alpha_1, \beta_1, \cdots, \alpha_m, \beta_m$. Since
\[
S^\beta A^{\alpha} S^{-\beta} \in \mathcal{D}, \ \forall \alpha, \beta \in \mathbb{Z},
\]

it follows that every word of the form (3.3) can be rewritten as
\[(3.5) \quad G = DS^\gamma, \text{ for some } D \in \mathcal{D},\]
where \( \gamma = \beta_1 + \beta_2 + \cdots + \beta_m \). Now, \( G \) is diagonal if and only if \( \gamma = 0 \) (mod \( p \)). Then \( \mathcal{G} = \mathcal{D} \mathcal{S} \) and \( \mathcal{C} \subset \mathcal{D} \). Since \( \mathcal{G}^{-1} = \mathcal{G} \), it follows that \( \mathcal{G} = \mathcal{G} \mathcal{D} \).

To prove (i), assume \( \mathcal{C} \) contains no non-scalar matrix. Since \( \mathcal{C} \neq \{I\} \), there exists \( C = \eta I \) for some complex number \( \eta \neq 1 \) and some \( C \in \mathcal{C} \). It is easy to see that \( \omega = 1 \). Also, \( \omega^p = \det(C) = 1 \). Hence, \( q \mid p \), and thus \( q = p = r \). Since \( C \) is a group, \( \mathcal{C} = \{\eta I : \eta^p = 1\} \). Now, if \( \text{rank}(D - I) = \rho < p/2 \), then \( D \) and \( SD^{-1}S^{-1} \) each have at most \( \rho \) entries different from 1 and, hence, \( DSD^{-1}S^{-1} \neq \eta I \) for some \( \eta \in \mathbb{C} \), a contradiction.

For (ii), assume there exists a non-scalar \( B \in \mathcal{C} \). Then the subgroup \( \mathcal{G}_B \) of \( \mathcal{G} \) is non-abelian and the relations (3.4) and (3.5) can be sharpened as follows:
\[(3.6) \quad S^\beta B^\alpha S^{-\beta} = B^\alpha B^{-\alpha} S^\beta B^\alpha S^{-\beta} \in \mathcal{C}_B, \forall \alpha, \beta \in \mathbb{Z}, \]
\[(3.7) \quad G = DS^\gamma, \text{ for some } D \in \mathcal{C}_B.\]
This shows that \( \mathcal{D}_B \subset \mathcal{C}_B \), which proves \( \mathcal{D}_B = \mathcal{C}_B \). Since \( \det(C) = 1 \) for all \( C \in \mathcal{C} \), it follows that \( \text{rank}(D - I) \geq 2 \) whenever \( I \neq D \in \mathcal{D} \). The inequality \( \rho_B \leq 2 \rho_A \) follows from the fact that if \( \text{rank}(D - I) = \rho \), then \( \text{rank}(DSD^{-1}S^{-1} - I) \leq 2\rho \) and the rest of (ii) is clear.

The next corollary studies the case \( \rho = 1 \). We continue to use the notation established in the previous paragraphs.

**Corollary 3.2.** It is always true that \( 2 \leq r \leq p \) and, if \( I \neq C \in \mathcal{C} \), then
\[ \text{rank}(C - I) \geq 2. \]
In particular, if \( \rho = 1 \), then one of the following cases holds:
(i) In this case, \( \mathcal{C} = \{I, -I\} \subset \mathcal{D} = \{I, -I, \text{diag}(1, -1), \text{diag}(-1, 1)\} \).
(ii) \( \mathcal{C} \) contains non-scalar matrices, and for any non-scalar \( B \in \mathcal{C} \), \( \rho_B \geq 2 \) and \( \mathcal{C}_B = \mathcal{D}_B \). The lower bound 2 is attained for some \( B \).

If \( p = q = 2 \), then \( \mathcal{D} = \{I, -I, \text{diag}(1, -1), -\text{diag}(1, -1)\} \) and \( \mathcal{C} = \{I, -I\} \).

**Proof.** Observe that if \( \text{rank}(X^{-1}Y^{-1}XY - I) = 1 \), then \( 1 \neq \det(X^{-1}Y^{-1}XY) = 1 \), a contradiction. Thus, \( 2 \leq r \leq p \). Now, if \( D \in \mathcal{D} \) and \( \text{rank}(D - I) = 1 \), then \( \det(D) \neq 1 \) and, hence, \( D \notin \mathcal{C} \). In particular, if \( \rho = 1 \), then \( \mathcal{D} \neq \mathcal{C} \) and, in view of Theorem 3.1 one of the following cases holds.

*Case 1.* \( p/2 \leq 1 \leq r = p = q \), which implies that \( r = p = q = 2 \) and \( \mathcal{D} \neq \mathcal{C} = \{I, -I\} \). Thus \( \mathcal{D} = \{I, -I, \text{diag}(1, -1), \text{diag}(-1, 1)\} \) is the only choice left.

*Case 2.* There exists a non-scalar \( B \in \mathcal{C} \), and for any such \( B, \mathcal{C}_B = \mathcal{D}_B \) and \( \rho_B \geq 2 \). Now, if \( D \in \mathcal{D} \) has exactly one diagonal entry different from 1, then \( DSD^{-1}S^{-1} \) is a commutator with exactly two diagonal entries different from 1.

Conversely, if \( p = q = 2 \), then \( \text{rank}(C - I) = 2 \) whenever \( I \neq C \in \mathcal{C} \), which implies that \( \mathcal{D} \neq \{I, -I\} = \mathcal{C} \). Thus, \( \mathcal{D} = \mathcal{C} \cup \{\text{diag}(1, -1), \text{diag}(-1, 1)\} \) and, hence, \( \rho = 1 \).

The following theorem studies the case \( \rho = 2 \).
Theorem 3.3. If \( p = 2 \), then either

(i) \( r = p \) and \( q > 2 \) or
(ii) \( r = p - 1, \ q = 2 \).

Proof. If \( p = 2 \), then \( r = 2 \). Also, \( q > 2 = p \) by Corollary 3.2.

So, we assume \( p \geq 3 \). Let \( \mathcal{D}_2 \) be the (non-empty) collection of all matrices \( D \in \mathcal{D} \) such that exactly \( p - 2 \) entries on the main diagonal of \( D \) are equal to 1. We claim there exists \( \Delta \in \mathcal{D}_2 \) for which exactly the first two diagonal entries are different from 1. Let \( s \) be the minimal positive integer for which there exist a positive integer \( h \) and a matrix \( D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p) \in \mathcal{D}_2 \) such that \( \lambda_h \neq 1 \) and \( \lambda_{h+1} \neq 1 \).

Examining \( S^{-h+1}DS^{h-1} \) and \( S^{-h-s+1}DS^{h+s-1} \) reveals that \( 1 \leq s < p/2 \) and allows us to assume without loss of generality that \( h = 1 \). Let \( p - 1 = ms + t \) for some non-negative integers \( m, t \) with \( 0 \leq t \leq s - 1 \), and, in fact, since \( p \) is an odd prime, it follows that either \( s = 1 \) or \( 0 \leq t \leq s - 2 \). Let \( \lambda_1 = \omega \) and \( \lambda_{s+1} = \omega^a \neq 1 \) for some primitive \( q \)th root \( \omega \) of 1 and some positive integer \( a < q \).

For \( 1 \leq k \leq m - 1 \), assume \( \Delta_1, \Delta_2, \ldots, \Delta_k \in \mathcal{D}_2 \) are constructed such that \( \Delta_1 = D \) and the first and the \((ks+1)\)th diagonal entries of \( \Delta_k \) are \( \omega^k \) and \( \omega^{a^k} \), respectively, where \( \epsilon_k := (-1)^{k+1} \). Define \( \Delta_{k+1} = S^{ks}D^{s^k}S^{-ks} \Delta_k^{-1} \). This finite induction yields \( \Delta_m \in \mathcal{D}_2 \), whose first diagonal entry is \( \omega^{t^m} \) and whose \((ms+1)\)th diagonal entry is \( \omega^{a^m} \) (necessarily, \( \neq 1 \)). Now, observe that the first and the \((t+2)\)nd diagonal entries of \( S^{t+1} \Delta_m S^{-t-1} \in \mathcal{D}_2 \) are \( \omega^m \) and \( \omega^{t^m} \), respectively. Since all other entries are equal to 1 and \( \omega^{t^m} \neq 1 \), it follows that \( \omega^{a^m} \neq 1 \). By minimality, \( t + 2 \geq s + 1 \); hence, \( s = 1 \) and \( t = 0 \).

Thus, there exists \( k \in \{1, 2, \ldots, q - 1\} \) such that

\[
\Delta = \text{diag}(\omega, \omega^k, 1, 1, \cdots, 1) \in \mathcal{D}_2.
\]

Let \( \Omega := \Gamma S^{t-1} \in \mathcal{C} \), where

\[
\Gamma = \text{diag}(\omega, \omega^k, \omega, \omega^k, \cdots, \omega, \omega^k, 1) = \Pi_{j=0}^{(p-3)/2} S^{2j} \Delta S^{-2j} \in \mathcal{D}.
\]

Hence

\[
\Omega = \text{diag}(\omega, \omega^{k-1}, \omega^{1-k}, \omega^{k-1}, \cdots, \omega^{1-k}, \omega^k, \omega^{-k}).
\]

Let us assume \( q \geq 3 \) and settle the problem in this case. We claim \( k \geq 2 \); otherwise,

\[
S^{-1} \Delta S \Gamma^p \Pi_{i=1}^{p-2} S_i \Delta^{(-1)^i} S^{-i} = \text{diag}(\omega^2, 1, 1, \cdots, 1) \in \mathcal{D}
\]

and \( \text{rank}(\Delta - I) = 1 \), a contradiction. Therefore, \( k \geq 2 \) and the proof of part (i) follows from the fact that \( r = \text{rank}(\Omega - I) = p \).

All we have to do now is settle the case \( p > q = 2 \). In (3.9), \( \omega = \omega^k = -1 \), and one can deduce that

\[
\Delta' := \Delta S \Delta S^{-1} = \text{diag}(-1, 1, -1, 1, 1, \cdots, 1) \in \mathcal{D}_2.
\]

Choose a positive integer \( u \) such that \( p = 4u \pm 1 \). Define \( \Omega' := \Gamma' S (\Gamma')^{-1} S^{-1} \in \mathcal{C} \), where

\[
\Gamma' = \text{diag}(-1, 1, -1, 1, \cdots, -1, 1, -1) = \Pi_{j=0}^{u-1} S^{4j} \Delta' S^{-4j} \in \mathcal{D}.
\]

Hence,

\[
\Omega' = \text{diag}(1, -1, -1, -1, \cdots, -1, -1, -1).
\]

Since \( r \geq \text{rank}(\Omega' - I) = p - 1 \), it follows that \( p - 1 \leq r \leq p \). Also, since \( \det(C) = 1 \) for all \( C \in \mathcal{C} \), it follows that \( \text{rank}(C) \neq p \), and we are done. \( \square \)
Based on Theorem 3.3, we can sharpen Corollary 3.2 as follows.

**Corollary 3.4.** If $\rho = 1$, then one of the following cases holds:

(i) $r = p = q = 2$. In this case,
\[ C = \{I, -I\} \subset D = \{I, -I, \text{diag}(1, -1), \text{diag}(-1, 1)\}. \]

(ii) $p = r$ and $q > 2$.

(iii) $r = p - 1$ and $q = 2$.

**Proof.** Part (i) is the same as part (i) of Corollary 3.2. Let $B \in C$ be as in part (ii) of Corollary 3.2 such that $\rho_B = 2$. By Theorem 3.3, we have one of the following cases.

Case 1. $r_B = p$ and $q > 2$. Then $p \leq r \leq p$, which proves (ii).

Case 2. $r_B = p - 1$ and $q = 2$. Then $r_B$ is even and, hence, $p$ is odd. If $r$ were equal to $p$, we would have $-I \in C$, which is impossible since the determinant of every member of $C$ is equal to one. This proves (iii). $\square$

The following corollary studies the case $r = 2$; its easy proof is left to the interested reader.

**Corollary 3.5.** If $r = 2$, then one of the following cases holds:

(i) $\rho = 1$ and $p = q = 2$. In this case,
\[ C = \{I, -I\} \subset D = \{I, -I, \text{diag}(1, -1), \text{diag}(-1, 1)\}. \]

(ii) $\rho = 1$, $p = 2$ and $q > 2$. In this case,
\[ C = \{\text{diag}(\omega, \bar{\omega}) : \omega^q = 1\} \subset D = \{\text{diag}(\omega, \eta) : \omega^q = \eta^q = 1\}. \]

(iii) $\rho = 1$, $p = 3$ and $q = 2$. In this case,
\[ D = C \cup \{-I, \text{diag}(-1, 1, 1), \text{diag}(1, -1, 1), \text{diag}(1, 1, -1)\}. \]

(iv) $\rho = 2$, $p = 2$ and $q > 2$. In this case, $C = D = \{\text{diag}(\omega, \eta) : \omega^q = \eta^q = 1\}$.

(v) $\rho = 2$, $p = 3$ and $q = 2$. In this case,
\[ C = D = \{\text{diag}(\omega, \bar{\omega}) : \omega^q = 1\}. \]

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