Properties of Fourier spectrum of the signal, generated at the accumulation point of period-tripling bifurcations

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Abstract

Universal regularities of the Fourier spectrum of signal, generated by complex analytic map at the period-tripling bifurcations accumulation point are considered. The difference between intensities of the subharmonics at the values of frequency corresponding to the neighbor hierarchical levels of the spectrum is characterized by a constant \( \gamma = 21.9 \text{ dB} \), which is an analogue of the known value \( \gamma_F = 13.4 \text{ dB} \), intrinsic to the Feigenbaum critical point. Data of the physical experiment, directed to the observation of the spectrum at period-tripling accumulation point, are represented.

One of the simplest objects, demonstrating non-trivial dynamics, is the system with discrete time, defined by the quadratic map. In particular, the period-doubling bifurcations cascade, characterized by the universal Feigenbaum properties, occurs in this system \( \square \). By generalization to the case of complex variable the quadratic map

\[
z_{n+1} = \lambda - z_n^2, \quad \lambda, z \in \mathbb{C}
\]
demonstrates the fractal object, known as Mandelbrot set, on the parameter plane (see fig. 1) [2].

This object is an aggregate of "leaves", inside which the periodic dynamics takes place in a restricted domain of phase space, and fractal pattern corresponding to the restricted chaotic dynamics (restriction means, that orbits of the map doesn’t escape to infinity). It is possible to define the paths at the complex parameter plane, which traverse the Mandelbrot set "leaves" according the special order. Following these paths one can obtain different period-multiplication bifurcation cascades [3, 4], for example period-tripling cascade. In the work of Golberg-Sinai-Khanin [4] the critical point of the period-tripling bifurcations accumulation has been considered. Let us denote it as GSK point:

\[ \lambda_c = 0.0236411685377 + 0.7836606508052i. \]  

(2)

If one separate real and imaginary parts in the 1D complex map, then one obtain the equivalent description of the dynamics by the 2D map of the pair of real variables. Notice, that this 2D map is of special form, because the corresponded functions must satisfy to the Cauchy-Rieman conditions (analyticity conditions). As it is shown in [5, 6], even the small non-analytic perturbation in the map leads to the drastic distortion of the dynamics, in particular, to the destruction of the universal properties of the period-tripling and other period-multiplication cascades. In view of this fact the problem of possibility of realisation of the phenomena of complex analytic dynamics in the physical systems get a fundamental significance (see, for example [7]). In paper [8] the problem of realization of the Mandelbrot set on the parameter plane of the coupled systems is discussed. The coupling must have a special form and provide the special symmetry of the system, necessary to the analyticity conditions implementation.

Based on this idea, in the paper [9] the electronic analog device, modelling the dynamics of the coupled logistic maps is proposed. In this system the first observation of the Mandelbrot set in physical experiment is carried out.

With the help of this real experimental device one can observe the bifurcations, intrinsic to
Studying the dynamical regimes in different systems, it is convenient to use the spectral properties of the generated signal. Thereupon, let us address to the problem of the Fourier spectrum of the signal, generated by the complex quadratic map at the critical point GSK of the accumulation of the period-tripling bifurcations. From the data of the renormalization group analysis \[3, 4\] one can conclude, that the infinite number of unstable cycles of periods \( N = 3^k \), where \( k = 1, 2, 3, ... \), exist at the GSK point. With \( N \to \infty \) these cycles approximate the critical attractor more and more precisely. Direct and reverse discrete Fourier transformations can be defined as follows:

\[
\begin{align*}
    z_n &= \sum_{m=0}^{N-1} c_m \exp \left( \frac{2\pi i}{N} mn \right), \\
    c_m &= \frac{1}{N} \sum_{m=0}^{N-1} z_n \exp \left( -\frac{2\pi i}{N} mn \right),
\end{align*}
\]

where \( z_n \) is the sequence of the values of dynamical variable, generated at the GSK critical point. The starting element of this sequence is the extremum of the map, i.e. \( z_0 = 0 \). Value \( f = m/N \) corresponds to the frequency of the component \( c_m \). Let us introduce the designation for the squared amplitude of the \( m \)-th component \( S(f) = S(m/N) = |c_m|^2 \).

At figure 2 (a) the Fourier spectrum of the signal, generated at the GSK point is shown. It is represented as dependence of the harmonic’s intensities in dB units, i.e. \( 10 \log S \), on the frequency \( f \). Notice that spectrum demonstrates fractal nature: there are infinite number of peaks at the frequencies corresponding to the periods of the existing in GSK point unstable cycles. Structure of the spectrum between two neighbor peaks are repeated. As it is visible at the figure, the difference of intensities of harmonics at the frequencies \( 1/3^k \) and \( 1/3^{k+1} \) is approximately \( \gamma = 21.9 \) dB. This value is an analog of the known constant \( \gamma = 13.4 \) dB intrinsic to the Feigenbaum critical point \( \Omega \).

At the figure 2 (b) the same spectrum of the map (1) at the GSK point, but in double logarithmic scale is shown. Moreover, the spectral intensities are rationed to the corresponded frequencies in power \( \kappa = 6.15 \) (this constant is selected empirically). With such representation
the self-similar structure of the spectrum looks more clear. It is easy to see, that amplitude of the peaks and there positional relationship are periodic.

Obtained constants $\gamma$ and $\kappa$ demonstrate the universal nature, they are an attributes of the critical GSK point. For example, in paper [10] with the help of constant $\kappa$ the self-similar hierarchical structure of the complexified Hénon map at the GSK point is demonstrated.

Furthermore, the universal properties of the spectrum at the GSK point are demonstrated in mentioned physical experiment. With the aid of the described in [9] device, several first period-tripling bifurcations have been successively observed in physical experiment (to get immediately the GSK point is impossible problem because of noise and experimental error). Figure 2 (c) demonstrates the Fourier spectrum of the signal generated by experimental device with the values of driving parameters, corresponding to the existence of the cycle of period 9 (following period tripled cycles are difficultly observable in experimental conditions). One can see the high peaks at the corresponded frequencies. The realization of typical difference between the amplitudes of intensities of harmonics at the tripling frequencies in order to constant $\gamma = 21.9$ dB is confirmed.

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Figure 1: Mandelbrot set of the complex map (1). Gray color designates the regions of periodic dynamics (periods of cycles are marked by respective numbers). Black pattern corresponds to the restricted in the phase space chaotic dynamics. White color means escape of the orbits to infinity. The line \( L_2 \) and \( L_3 \) show the pathes, following by which one can obtain period-doubling or period-tripling bifurcation cascades, respectively.
Figure 2: Fourier spectra: (a) – spectrum of signal of the complex map (1) at the critical point (2); (b) – the same spectrum in double-logarithmic scale; (c) – spectrum of the experimental signal with period 9.