Computing and Analysis of Multi Server Queueing Model Using Pentagonal Fuzzy Numbers

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Abstract. In the present study, analysed the multiple server queue miniature using Pentagonal Fuzzy number. To cope with a real-world scenario, the input parameters are converted from crisp to fuzzy type by taking into account. The system parameters, rate of arrival and rate of service are taken fuzzy. The system characteristics like expected number of clients waiting in queue and expected number of clients waiting in system are calculated using DSW algorithm and using alpha-cut at various levels, in interval [0,1]. Using pentagonal number arithmetic, the performance measures are calculated directly as PFNs. Different numerical examples with arrival rate and service rate as PFN are demonstrated to show the authenticity implementation of the performance measures. Finally effect of different parameters are shown for validation purpose.

1. Introduction
Queueing models are prepared for deeper understanding of system performance. Using these models, we can predict waiting time and queue length and can make business decision about the assets required to deliver a service. Presently we all are seeing covid-19 pandemic is pushing down the health system due to the lack of resources. Now with the help of these models we can calculate minimum number of resources required at the time of pandemic explosion. Queueing systems also used in the analysis of cable networks, computer systems, and call management system. Queueing theories define characteristics of the queue, like expected number of clients waiting in system and expected waiting time and provides the tools for optimizing queues. Economically, queueing theory describes the building of cost-effective workflow systems.

1.1. Fuzzy Queueing Basics
The word fuzzy involve things that are not quite clear. In the real world, we may get a condition in which we cannot determine whether the declaration is completely true or false. In this situation, fuzzy logic provides very useful suppleness for logic. We can also consider the uncertainties of any situation using fuzzy. Fuzzy logic asserts the shades of grey instead of pure white and pure black. We can express a fuzzy set A in X as a set of ordered pairs of universe or universe of discourse. We characterized fuzzy set by a membership function (MF).

\[ X = \{(x, \mu_A(x)) \mid x \in U\} \]

Alpha cut can be utilized to transform fuzzy into a crisp. The fuzzy set theory was created by Zadeh and then it has been established in a broad range of physical issues. In traditional logic being member...
of a set is regarded as 0 and 1, this suggests that if a member goes or not in the set. Theory of fuzzy queueing is an effective tool used to model queueing systems considering their natural inaccuracy. The usual FQT model supposes a specific fixed number of servers with constant fuzzy arrival and service rates. It is common, though, to encounter circumstances where the number of servers, arrival ratio, and the rate of service change with time. In those cases, the unpredictability of these parameters could substantially affect the behavior of the queue system and confuse its assessment. Numerous organizations like banks, health systems and communications departments routinely use queueing theory models to help determine capacity levels needed to experienced demands in a more efficient way.

A new approach to measure the fuzzy performance of M/M/S queueing systems has been presented utilizing fuzzy estimators by Botzoris et al. [1]. Various models of fuzzy queue using triangular and trapezoidal fuzzy number has been discussed by Shanmugasundaram et al. [2]. The performance measures have been introduced for computation of a single server fuzzy retrial queue with breakdowns and repairs using α-cuts method by Mukeba et al. [3]. Analytical results for M/F/1 and FM/FM/1 systems have presented by Shanmugasundaram & Venkatesh [4]. Study on performance of FM/FM/1 using pentagonal fuzzy number was discussed by Madhuri et al. [5]. Many factors related to fuzzy priority discipline queueing system has been provided by Visalakshi & Suvitha [6] and they also included PFN in their work. Triangular fuzzy numbers in queueing model with the help of α – cut method was used Sujatha et al. [7]. Narayanaamoorthy and Ramya [8] introduces performance measures of multi-server queueing model in hexagonal fuzzy numbers. Ehsanifar et al. [9] built a simulation model to predict the performance indices of fuzzy exponential queueing system. Ritha et al. [10] proposed a common process to achieve the membership function of the performance criteria in three stage tandem queues with the inter-arrival time and service time are in fuzzy. The queueing analysis of a multi-component machine repair system comprising of operating as well as standby machines and a skilled repairman using fuzzy queue has been investigated by Meena et al. [11]. Zaki et al. [12] explained behaviour of fuzzy queue using trapezoidal fuzzy number. Thangaraj [13] proposed fuzzy queue using priority discipline.

In the machine repair model for verification of numerical outcomes using ANFIS is discussed by Ahuja and Jain [14]. The ANFIS technology is also employed by Sanga and Jain [15] for finding the performance measures of the complex queueing systems. The comparative illustration for a FM/FD/1 queueing model is presented by Karupothu and Kumar [16] they have attainability of performance measure and tested utilizing α-cuts. FM/FM/C model is analyzed using pentagonal fuzzy number by Prameela and Kumar [17]. Markovian queueing miniature is analyzed by Usha Prameela and Kumar [18] using pentagonal, heptagonal, and octagonal fuzzy numbers.

In some circumstances, the parameters for number of customers arrived and the rate at which they are being served are often fuzzy and cannot be conveyed in accurate terms. Thus, to express these parameters in linguistic terms such as "the expected arrival rate is almost 5" and "the expected mean service rate is almost 10" are much more reasonable under these conditions. It can reveal fuzzy queues as more realistic than the usual queues using crisp parameters. Queueing model has provided the limited capability in explaining some situation in real life while fuzzy queueing theory has a capability of making decision from multiple inputs or criteria.

2. Research Methodology

2.1. Alpha Cut

Alpha cut set or level is the set of those fuzzy element whose degree of membership is greater than or equal to alpha where alpha is the number between zero and one. Alpha cut set or level set is not a fuzzy set, but it is a normal set because here we only write the element not their degree of membership. It uses several α cut levels intervals defined in membership functions. It avoids abnormality in the results obtained by the application of the discernment reaching on the domain of fuzzy variables.
2.2. DSW (Dong, Shah, Wong) Algorithm

It uses several α cut levels intervals defined in membership functions. It prevents irregularity in the results created by fuzziness of input parameters. We can represent any continuous membership function by a cut α = 0 to α = 1. Here are the steps:
1. Select a α-cut value in interval [0,1].
2. Based on α calculate the intervals in the input.
3. Now using interval arithmetic find the interval output for each level.
4. Follow steps 1 to 3 for each level.

2.3. Standard Interval Arithmetic

Let two interval numbers \( X = [x_1, x_2] \) and \( Y = [y_1, y_2] \) such that \( a_1 < a_2 \) and \( b_1 < b_2 \). We can define the general arithmetic operations like \(+, -, \times, /\) on interval as following:

a) \( X + Y = [x_1 + y_2, x_2 + y_1] \)

b) \( X - Y = [x_1 - y_2, x_2 - y_1] \)

c) \( X \times Y = [\min(x_1 y_1, x_1 y_2, x_2 y_1, x_2 y_2), \max(x_1 y_1, x_1 y_2, x_2 y_1, x_2 y_2)] \)

d) \( X \div Y = [x_1 y_2, y_2 y_1] \)

e) \( \propto [x_1, x_2] = [\propto x_1, \propto x_2] \) for \( \propto > 0 \)

f) \( \propto [x_1, x_2] = [\propto x_2, \propto x_1] \) for \( \propto < 0 \)

2.4. The model description FM/FM/C

The M/M/C queue represents multi server queue with arrival governed by Poisson process and service is exponentially distributed. In everyday queueing challenges are presumed that clients arrive nearly in fixed manner at a service facility. But this supposition is disregarded in many real queueing conditions. There are many congestion situations in which arrivals occur randomly. In this study, we use the fuzzy Markovian queueing model FM/FM/C using pentagonal fuzzy numbers as input and output parameters. In this model \( \lambda \) denotes arrival rate with Poisson distribution. The fuzzy service time is taken as \( \mu \). The proposed fuzzy queueing model gives an example to pentagonal fuzzy numbers can be used to handle fuzziness in real queue system like road traffic congestion.

2.5. Performance Measures

Performance measures decides the threshold value of waiting period within the queue that is acceptable to customers and no one need to wait more than that value. Also, the service time should be decreased to optimise the queue performance. The queue does not get excessively long or full, or entirely backed up. In short, we want that the servers’ capacity should be fully utilised (they are not free for too much of their working time – like cancelled appointments or oversupply of servers, their utilisation should be in high) server utilization can be estimated by dividing the amount of time that the server is busy during a simulation by the amount of time covered by the simulation.
2.6. Probability of No Customer in the System
This represents probability of no customer in the system. To avoid complication in calculation this is taken as a crisp number.

\[ P_0 = \left[ \sum_{n=0}^{c-1} \frac{1}{n!} \left( \frac{\tilde{\lambda}}{\tilde{\mu}} \right)^n + \frac{1}{C!} \left( \frac{\tilde{\lambda}}{\tilde{\mu}} \right)^C \frac{C}{C - \left( \frac{\tilde{\lambda}}{\tilde{\mu}} \right)} \right]^{-1} \]  

If \( C=2 \)

\[ P_0 = \left[ 1 + \frac{\tilde{\lambda}}{\tilde{\mu}} + \frac{\tilde{\lambda}^2}{2 \mu} \right]^{-1} \]  

If \( C=3 \)

\[ P_0 = \left[ 1 + \frac{\tilde{\lambda}}{\tilde{\mu}} + \frac{\tilde{\lambda}^2}{2 \mu} + \frac{\tilde{\lambda}^3}{6 \mu^2} \right]^{-1} \]

Expected number of clients waiting in the system

\[ \tilde{L}_q = \frac{\left( \frac{\tilde{\lambda}}{\tilde{\mu}} \right)^{c+1}}{(C-1)! \left( C - \left( \frac{\tilde{\lambda}}{\tilde{\mu}} \right) \right) P_0} \]
Expected average waiting time in the queue

$$\bar{W}_q = \frac{r_q}{\bar{\lambda}}$$  \hspace{1cm} (6)

Expected average waiting time in the system

$$\bar{W}_s = \bar{W}_q + \frac{1}{\bar{\mu}}$$  \hspace{1cm} (7)

2.7. Pentagonal numbers \([x_1, x_2, x_3, x_4, x_5]\\)

A 5-tuple subset of a real number having specific membership function (shown above) is known as Pentagonal Fuzzy Number (PFN). Basically, it is just representation of membership function of the fuzzy variable that is pentagon in shape. It represented as a 5-tuple subset of a real number. Its membership function of \(X = [x_1, x_2, x_3, x_4, x_5]\\)

$$\mu_a(x, X, r) = \begin{cases} 
\frac{r(x-x_1)}{x_2-x_1}, & \text{if } x_1 \leq x \leq x_2 \\
1 + (1-r)\frac{x-x_2}{x_3-x_2}, & \text{if } x_2 \leq x \leq x_3 \\
1, & \text{if } x = x_3 \\
1 - (1-r)\frac{x-x_3}{x_4-x_3}, & \text{if } x_3 \leq x \leq x_4 \\
\frac{r(x-x_4)}{x_5-x_4}, & \text{if } x_4 \leq x \leq x_5 \\
0, & \text{if } x > x_5 
\end{cases}$$  \hspace{1cm} (8)

2.8. Pentagonal Numbers Arithmetic

Let two pentagonal numbers \(U = [u_1, u_2, u_3, u_4, u_5]\\) and \(V = [v_1, v_2, v_3, v_4, v_5]\\) such that \(u_i < u_{i+1}\\) and \(v_i < v_{i+1}\\), left and weights are taken as equal. We can define the general arithmetic operations like \{+, -, *, /\\} on pentagonal numbers as following:

a) \(U + V = [u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4, u_5 + v_5]\\)

b) \(U - V = [u_1 - v_5, u_2 - v_4, u_3 - v_3, u_4 - v_2, u_5 - v_1]\\)

c) \(U \times V = [u_1v_1, u_2v_2, u_3v_3, u_4v_4, u_5v_5]\\)

d) \(U \div V = [u_1, u_2, u_3, u_4, u_5] \times [\frac{1}{v_5}, \frac{1}{v_4}, \frac{1}{v_3}, \frac{1}{v_2}, \frac{1}{v_1}]\\)

e) \(a [u_1, u_2, u_3, u_4, u_5] = [au_1, au_2, au_3, au_4, au_5]\\) for \(a > 0\\)

f) \(a[v_1, v_2, v_3, v_4, v_5] = [av_5, av_4, av_3, av_2, av_1]\\) for \(a < 0\\)
3. Numerical Example
Consider FM/FM/2 model with rate of appearance of customers $\lambda$ and rate of providing service $\bar{\mu}$ as pentagonal fuzzy numbers represented by $\lambda = [10, 11, 12, 13, 14]$ and $\bar{\mu} = [22, 23, 24, 25, 26]$. First, we define membership function for both arrival rate and service rate using equation (8) as shown below using weight $r=0.6$.

$$\mu_a(x, \lambda, 0.6) = \begin{cases} 
0.6 \times (x - 10), & \text{if } 10 \leq x \leq 11 \\
1 + 0.4 \times (x - 11), & \text{if } 11 \leq x \leq 12 \\
1, & \text{if } x=12 \\
1 - 0.4 \times (13 - x), & \text{if } 12 \leq x \leq 13 \\
0.6 \times (13 - x), & \text{if } 13 \leq x \leq 14 \\
0, & \text{if } x > 14 
\end{cases}$$

$$\mu_s(x, \bar{\mu}, 0.6) = \begin{cases} 
0.6 \times (x - 22), & \text{if } 22 \leq x \leq 23 \\
1 + 0.4 \times (x - 23), & \text{if } 23 \leq x \leq 24 \\
1, & \text{if } x=24 \\
1 - 0.4 \times (24 - x), & \text{if } 24 \leq x \leq 25 \\
0.6 \times (25 - x), & \text{if } 25 \leq x \leq 26 \\
0, & \text{if } x > 26 
\end{cases}$$

After then we perform the calculations using two ways:

(1) Using DSW algorithm, at 10 different values of $\alpha$ level, $\alpha$-cut are calculated for both fuzzy inputs $\lambda$ and $\bar{\mu}$ and then all the performance measure are calculated using interval arithmetic mentioned at equation (1). These outputs are shown below in table 1.

(2) Using pentagonal arithmetic all performance measures are calculated directly and this will give all the measures as pentagonal fuzzy numbers shown below in figures (1-6).

3.1. Example 1

$\lambda = [10, 11, 12, 13, 14]$, $\bar{\mu} = [22, 23, 24, 25, 26]$, C=2
Figure 1. Membership function of service rate as pentagonal number

Figure 2. Membership function of arrival rate as pentagonal number

Figure 3. Membership function of Expected waiting time of customer in queue as pentagonal number

Figure 4. Membership function of Expected waiting time of customer in system as pentagonal number
To examine the effect of increment in service rate and arrival rate using $\lambda = [15,16,17,18,19], \mu = [26,27,28,29,30]$ as input parameters and follow the same procedure as discussed above for example 1. The results obtained are shown in figures (7-12) and table 2.

$\lambda = [15,16,17,18,19], \mu = [26,27,28,29,30], \alpha = 2$
Figure 7. Membership function of arrival rate as pentagonal number

Figure 8. Membership function of service rate as pentagonal number

Figure 9. Membership function of Expected number of customers in queue as pentagonal number

Figure 10. Membership function of Expected number of customers in system as pentagonal number
Figure 11. Membership function of Expected average waiting time for customers in queue as pentagonal number

Figure 12. Membership function of Expected average waiting time for customers in system as pentagonal number

Table 2. Performance Measure Intervals at Different Alpha Level At

| α Level | $L_q(10^4)$ | $L_s(10^4)$ | $W_q(10^4)$ | $W_s(10^4)$ |
|---------|-------------|-------------|-------------|-------------|
| 0.1     | [124.449, 914.258] | [4062.9105, 7159.3182] | [9.0662, 90.5591] | [396.2456, 541.8108] |
| 0.2     | [138.2028, 845.8181] | [4168.9721, 6972.3004] | [10.2255, 75.1375] | [403.6926, 519.8015] |
| 0.3     | [151.9567, 777.3773] | [4275.0336, 6785.2825] | [11.3849, 82.8483] | [407.4160, 508.7962] |
| 0.4     | [165.7105, 708.9366] | [4381.0951, 6598.2646] | [12.5442, 97.4267] | [411.1395, 497.7922] |
| 0.5     | [179.4643, 640.4958] | [4487.1566, 6411.2468] | [13.7036, 113.95] | [414.8629, 486.7876] |
| 0.6     | [193.2181, 572.0550] | [4593.2181, 6224.2289] | [14.8629, 122.050] | [418.2583, 476.2018] |
| 0.7     | [228.2469, 512.3746] | [4778.2469, 6001.5050] | [18.0916, 145.9482] | [422.2583, 476.2018] |
| 0.8     | [263.2757, 452.6942] | [4963.2757, 5778.7811] | [21.3204, 189.8914] | [429.6537, 465.6160] |
| 0.9     | [298.3045, 393.0137] | [5148.3045, 5556.0572] | [24.5491, 233.8346] | [437.0491, 455.0302] |
| 1.0     | [333.3333, 333.3333] | [5333.3333, 5333.3333] | [27.7778, 27.7778] | [444.4444, 444.4444] |

4. Sensitivity Analysis

To examine sensitivity of this miniature we have taken different fuzzy values of arrival rate and service rate by increase and decrease slightly to the original one taken in the example 1 in section 3. For example, first we consider the arrival rate $\lambda = [10,11,12,13,14]$ and then decrease in each member by 0.5 that is $\lambda = [10.5,11.5,12.5,13.5,14.5]$ taken as pentagonal fuzzy number and service rate is taken same as in example 1. The results obtained shows that the performance measures are less then obtained in example 1. Similarly, we analyse the effect of increasing each member by 0.5 of parameter $\lambda$ that is when $\lambda = [9.5, 10.5, 11.5, 12.5, 13.5]$ keeping service rate same, then there is slight increase in the performance measures. The similar results are obtained by increment and decrement of service rate taken arrival rate stationary on increase of service rate performance measures are decreased on increase of service rate and there is decrement in performance measures on increment of service rate. Thus, calculations using pentagonal numbers directly as fuzzy output gives flexible decisions in the context of queueing models. Refer table 3 and 4.
Table 3. Sensitivity Analysis by First Increase in Arrival Rate and then Decrease in Arrival Rate

| Performance measures | $\bar{\lambda} + .5 = [10.5,11.5,12.5,13.5,14.5]$ and $\bar{\lambda} - .5 = [9.5,10.5,11.5,12.5,13.5,14.5]$ | Expected interval | Mid value | Expected interval | Mid value |
|-----------------------|--------------------------------------------------------------------------------------------------|------------------|-------------|------------------|-------------|
| $L_q(0.0333)(10^{-4})$ | [378.9091, 143.8912] | 1027.8512 | [291.7914, 106.8858] | 810.4047 |
| $L_s(0.5333)(10^{-4})$ | [5587.2425, 4275.9425] | 7498.5363 | [5083.4580, 3851.7576] | 6829.8382 |
| $W_q(0.0028)(10^{-4})$ | [30.3127, 10.1122] | 96.9990 | [25.3732, 8.0829] | 84.4627 |
| $W_s(0.0444)(10^{-4})$ | [446.9794, 397.2917] | 548.2507 | [442.0398, 395.2624] | 535.7144 |

Table 4. Sensitivity Analysis by First increase in Service Rate and then Decrease in Service Rate

| Performance measures | $\bar{\mu} = [10,11,12,13,14]$ | Expected interval | Mid value | Expected interval | Mid value |
|-----------------------|--------------------------------------------------------------------------------------------------|------------------|-------------|------------------|-------------|
| $L_q(0.0333)(10^{-4})$ | [117.9805, 844.2111] | 312.4972 | [131.3962, 992.4459] | 356.0868 |
| $L_s(0.5333)(10^{-4})$ | [3981.5888, 6951.3821] | 5210.4564 | [4147.6694, 7381.7655] | 5462.4697 |
| $W_q(0.0028)(10^{-4})$ | [8.5942, 83.6146] | 26.0414 | [9.5731, 98.3113] | 29.6739 |
| $W_s(0.0444)(10^{-4})$ | [388.4191, 524.9070] | 434.2047 | [404.3977, 559.9823] | 455.2058 |

5. Conclusion
To exhibit the use of the model, different parametric values have been stated and clearly demonstrated with the help of graphs. The numerical results provide significant insight to system designers. The proposed model has been tested by considering the crisp and fuzzy value of parameters. The sensitivity analysis of the proposed model in case of crisp parameters is also discussed. The uncertainty involved in parameters has been modelled using the pentagonal fuzzy numbers. Comparison of results using system parameters as crisp value, as interval and as pentagonal fuzzy numbers is presented by using two examples with tables and graphs. The results presented clearly indicate that we can use PFN in place of crisp numbers and handle the real time fuzziness very conveniently. Also, crisp results can also be obtained using the mid value of pentagonal fuzzy numbers. Sensitivity analysis for small changes in input parameters is also presented.

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