LOW-DIMENSIONAL SINGULARITIES WITH FREE DIVISORS AS DISCRIMINANTS

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Abstract. We present versal complex analytic families, over a smooth base and of fibre dimension zero, one, or two, where the discriminant constitutes a free divisor. These families include finite flat maps, versal deformations of reduced curve singularities, and versal deformations of Gorenstein surface singularities in \( \mathbb{C}^5 \). It is shown that such free divisors often admit a “fast normalization”, obtained by a single application of the Grauert-Remmert normalization algorithm. For a particular Gorenstein surface singularity in \( \mathbb{C}^5 \), namely the simple elliptic singularity of type \( \tilde{A}_4 \), we exhibit an explicit discriminant matrix and show that the slice of the discriminant for a fixed \( j \)-invariant is the cone over the dual variety of an elliptic curve.

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Introduction

One of the remarkable results in complex singularity theory is that the discriminant in the versal deformation of any isolated complete intersection singularity is a free divisor, a highly singular hypersurface in the ambient smooth base that admits though a compact representation as determinant of any discriminant matrix expressing a basis of the liftable vector fields in terms of the coordinate vector fields; see [34] or Section 2 below.

Variants on this result show the freeness of the discriminant in the base of a versal deformation in a number of further cases, for example space-curve singularities (van Straten [47]), functions on space curves (Goryunov [24], Mond-van Straten [36]), or representation varieties of quivers (Buchweitz-Mond [14]). J. Damon gives in

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a broad survey of why and how free divisors appear frequently in the theory of discriminants and bifurcations.

Here we present further versal complex analytic families, over a smooth base and of fibre dimension zero, one, or two, where the discriminant constitutes yet again a free divisor.

While we consider the question in more generality to deduce sufficient criteria, showing along the way why such free divisors often admit a “fast normalization”, obtained by a single application of the Grauert-Remmert normalization algorithm, the new explicit instances found pertain to

- finite flat maps, thus the case of relative dimension zero, where we characterize freeness of the discriminant completely and show, for example, that the discriminant in the Hilbert scheme or Douady space of points on a smooth complex surface is a free divisor,
- reduced curve singularities, where we recover not only the result on space-curve singularities due to D. van Straten [47], but extend it to include all reduced, smoothable, and unobstructed Gorenstein curve singularities, and
- smoothable Gorenstein surface singularities with reflexive conormal module, thus including all Gorenstein surface singularities in $\mathbb{C}^5$.

This list includes in particular versal deformations of any isolated Gorenstein singularity of dimension at most two that is linked to a complete intersection. More generally, we start with an isolated Cohen-Macaulay singularity, of arbitrary dimension, whose conormal module becomes Cohen-Macaulay when twisted with the canonical module. J. Herzog [28] showed that such a singularity is unobstructed, whence any versal deformation has a smooth base, and that the module of vector fields on the total space of a versal deformation is Cohen-Macaulay. Assuming that the singular locus of the total space is not dense in the critical locus of the versal morphism, the discriminant is necessarily a divisor, and freeness becomes equivalent to sufficient depth of the module of vertical vector fields in the versal deformation. That latter property turns out to be more and more elusive as the dimension of the singularity increases, and it is not quite clear what the actual reach of the criterion is. However, in small enough dimension, the necessary depth can be verified easily, as we show.

Once these rather abstract results of homological nature are presented, we turn to the concrete description of some of the free divisors that arise and point out how they are related to dual varieties of smooth projective varieties. The avatar here is the classical result that the dual variety to the rational normal curve yields the discriminant in the space of univariate polynomials. In particular, building on work by many authors, mainly H. Pinkham [40, 41], E. Looijenga [31, 32, 33], K. Saito [44], K. Wirthmüller [52], J.-Y. Merindol [35], B. Dubrovin [19], and M. Bertola [6, 7, 8], we put a capstone on the structure of the discriminant in the semi-universal deformation of a simple elliptic surface singularity of type $\tilde{A}_4$, exhibiting an explicit discriminant matrix and showing that its slice for a fixed $j$–invariant is the cone over the dual variety of the dual elliptic curve. Note that according to [42, 40, 29] a surface singularity of type $\tilde{A}_4$ is the only instance of a simple elliptic singularity, where the deformation theory is unobstructed but the singularity itself is not a complete intersection, as it is rather given minimally by the Pfaffians of an alternating $(5 \times 5)$–matrix, thus Gorenstein of codimension 3 and therefore (only) linked to a complete intersection.
Specifically, the paper is organized as follows. In Section 1, we recall basic properties of free divisors and indicate how the appearance of free divisors in versal deformations can be explained in terms of the Kodaira-Spencer sequence. Section 2 contains the main result, a general criterion for the discriminant in the base of a versal deformation to be a free divisor. Section 3 investigates the case of finite flat maps with applications to the Hilbert scheme of points, Section 4 applies the main result to families of curves and surfaces. Section 5 contains a review of the classical discriminant of a polynomial and shows that Arnol’d’s description is equivalent to the Bezout representation of the dual variety of the rational normal curve. In Section 6 we show that the analogue of the Bezout formula found by M. Bertola can be used to give a determinantal expression for the dual variety of the elliptic normal curve. After reviewing basic material on the deformation theory of simple elliptic singularities of type $\tilde{A}_4$, we present in Section 7 an explicit discriminant matrix and relate it to the dual variety of an elliptic curve.

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1. Background on Deformations and Free Divisors

We review here the definition and basic properties of free divisors. Further details and a broad survey of possible generalizations are given in [15].

1.1. Let $S$ be a connected complex manifold and $D \subset S$ a reduced hypersurface with defining ideal sheaf $\mathcal{I}_D \subseteq \mathcal{O}_S$. The sheaf $\mathcal{D}er_S(− \log D)$ of logarithmic vector fields along $D$ has as its local sections those vector fields $\chi \in \mathcal{D}er_S$ such that $\chi(\mathcal{I}_D) \subseteq \mathcal{I}_D$, or, equivalently, such that $\chi$ is tangent to $D$ at its regular points. It is clearly an $\mathcal{O}_S$-submodule and a sheaf of complex Lie subalgebras of $\mathcal{D}er_S$.

Definition 1.2. The reduced hypersurface $D \subset S$ is a free divisor at $s \in S$, if $\mathcal{D}er_S(− \log D)$ is a locally free $\mathcal{O}_S$-module at $s$, necessarily then of the same rank as $\mathcal{D}er_S$, equal to the dimension of $S$.

The concept of free divisors was identified by K. Saito in [43] and he stated there the following criterion, nowadays usually named after him:

Proposition 1.3. (Saito’s Criterion) The hypersurface $D \subset S$ is a free divisor at a point $s \in S$ if and only if there are germs of vector fields $\chi_1,\ldots,\chi_n \in \mathcal{D}er_S(− \log D)_s$, such that the determinant of the matrix of coefficients of these germs with respect to some, or any, $\mathcal{O}_{S,s}$-basis of $\mathcal{D}er_{S,s}$, is a reduced equation for $D$ at $s$. In this case, $\chi_1,\ldots,\chi_n$ form a basis for the $\mathcal{O}_{S,s}$-module $\mathcal{D}er_S(− \log D)_s$. □

Any discriminant matrix that describes the inclusion $\mathcal{D}er_S(− \log \Delta) \subseteq \mathcal{D}er_S$, yields thus through its determinant a compact presentation of a defining equation of $D$.

1.4. Free divisors are very special hypersurfaces. If not smooth, they are “maximally singular” in that the singular locus is a Cohen-Macaulay subspace of codimension one in $D$. Equivalently, the Jacobi ideal $\text{jac}_{D,s} \subseteq \mathcal{O}_{D,s}$ of the free divisor, naturally isomorphic to the cokernel $\mathcal{D}er_S/\mathcal{D}er_S(− \log D)$, is a maximal Cohen-Macaulay module of rank one on $D$, and any presentation matrix of it yields a discriminant matrix.
Conversely, as observed by A. G. Alexandrov [1, 2]; see also [15]: a reduced complex hypersurface whose Jacobi ideal constitutes a maximal Cohen-Macaulay module on it is necessarily a free divisor.

These characterizations can be formulated in concrete algebraic terms, without explicit reference to vector fields, just in terms of the Taylor series of a locally defining equation.

**Proposition 1.5.** A (formal) power series \( f \in P := \mathbb{C}[z_1, \ldots, z_n] \) defines a (formal) free divisor if it is reduced, that is, squarefree, and there is an \((n \times n)\)-matrix \( A \) with entries from \( P \) such that

\[
\det A = f \quad \text{and} \quad (\nabla f)A \equiv (0, \ldots, 0) \mod f,
\]

where \( \nabla f = \left( \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right) \) is the gradient of \( f \), and the last condition just expresses that each entry of the (row) vector \((\nabla f)A\) is divisible by \( f \) in \( P \). The columns of \( A \) can then be viewed as the coefficients of a basis, with respect to the partial derivatives \( \partial/\partial z_i \), of the (formal) logarithmic vector fields along the divisor \( f = 0 \), and the cokernel of \( A \) is naturally isomorphic to the Jacobi ideal of \( f \) in \( P/(f) \). \( \square \)

**1.6.** A normal crossing divisor is a free divisor in any dimension, and in the theory of hyperplane arrangements many free arrangements have been constructed by combinatorial means; see e.g. [39, Ch. 4]. We note in passing that it is one of the major outstanding problems in that theory whether freeness is solely a combinatorial property. Going beyond such unions of locally linear spaces, while any reduced plane curve is a free divisor in its ambient plane, the concept becomes quite elusive in higher dimensions. Even free surfaces, in an ambient smooth space of dimension 3, are not classified yet, only a zoo of rather few specimen is known; see [16] for some of those.

We now turn to the appearance of free divisors as discriminants in versal deformations. To this end, we next review some standard notation and results pertaining to (versal) deformations of complex spaces or germs thereof.

**1.7.** Recall the notion of tangent cohomology, the groups that control infinitesimal deformations. For any morphism \( f : X \to S \) of complex analytic germs or spaces, let \( L_{X/S} \in D^{-}(X) \) denote an (analytic) cotangent complex of \( X \) over \( S \), or rather of \( f \). Up to isomorphism, such cotangent complex is a well defined object in the indicated derived category, see [20] or [13, 12].

The cohomology group \( T_{X/S}^i(\mathcal{M}) := H^i(\text{Hom}_{\mathcal{O}_X}(L_{X/S}, \mathcal{M})) \) is the \( i \)-th tangent cohomology of the morphism \( f \) with values in the \( \mathcal{O}_X \)-module \( \mathcal{M} \). We abbreviate as usual \( T_{X/S}^i := T_{X/S}^i(\mathcal{O}_X) \). Similarly, one defines the tangent cohomology sheaves \( T_{X/S}^i(\mathcal{M}) := H^i(\text{Hom}_{\mathcal{O}_X}(L_{X/S}, \mathcal{M})) \). If \( f \) is a morphism of analytic germs and \( \mathcal{M} \) is \( \mathcal{O}_X \)-coherent, then each \( T_{X/S}^i(\mathcal{M}) \) is a coherent \( \mathcal{O}_X \)-module as well. Moreover, tangent cohomology localizes in that \( T_{X/S}^i(\mathcal{M})_x \cong T_{(X,x)/(S,f(x))}^i(\mathcal{M}_x) \) as \( \mathcal{O}_{X,x} \)-modules for any point \( x \) on \( X \). Therefore, we may, and will, use the sheaf or module-theoretic language interchangeably throughout for morphisms of germs.

**1.8.** Note, in particular, that for \( \mathcal{M} \) coherent, \( T_{X/S}^0(\mathcal{M}) \cong \text{Der}_{X/S}(\mathcal{M}) \), the \( \mathcal{O}_X \)-module of \( \mathcal{O}_X \)-linear derivations on \( \mathcal{O}_X \) with values in \( \mathcal{M} \), and that \( T_{X/S}^i(f^*\mathcal{N}) \), for a flat morphism \( f \) and a coherent \( \mathcal{O}_S \)-module \( \mathcal{N} \), parametrizes the isomorphism classes of extensions of \( X \) by \( f^*\mathcal{N} \) over the trivial extension of \( S \) by \( \mathcal{N} \).
1.9. A morphism \( f : X \to S \) is \textit{versal} at \( s \in S \), if it is flat and induces a formally versal deformation of the fibre \( X(s) = f^{-1}(s) \) at \( s \) in \( S \). It is \textit{versal} if it is so at every point in \( S \). Versality is an open property on \( S \) by [20]; see also [12].

If, for a complex germ \( X_0 \), the first tangent cohomology \( T^1_{X_0} \) is a finite dimensional vector space, then \( X_0 \) admits a versal deformation. If \( T^2_{X_0} = 0 \), then the deformation theory of \( X \) is unobstructed, and so the base of any versal deformation, if it exists, is necessarily smooth.

\textbf{Definition 1.10.} Let \( f : (X,0) \to (S,0) \) be a flat morphism of analytic germs\(^1\). For any coherent \( \mathcal{O}_S \)-module \( N \), set

\[
T^0_{X \to S}(N) \cong \{ (D, D') | D \in \text{Der}(\mathcal{O}_S, N), D' \in \text{Der}(\mathcal{O}_X, f^*N), D' \circ f = D \otimes \mathcal{O}_S 1 : \mathcal{O}_S \to N \otimes \mathcal{O}_S \mathcal{O}_X = f^*N \}.
\]

In other words, this \( \mathcal{O}_S \)-module consists of \textit{compatible pairs of vector fields}, one on \( S \) with values in \( N \), the other on \( X \) with values in \( f^*N \).

For \( N = \mathcal{O}_S \), the \( \mathcal{O}_S \)-module \( T^0_{X \to S} := T^0_{X \to S}(\mathcal{O}_S) \) carries naturally a structure of complex Lie algebra with respect to the componentwise bracket of vector fields.

The projection \( p_1 \) from \( T^0_{X \to S} \) to \( T^0_S \cong \text{Der}_S \) is a homomorphism of such Lie algebras. Its image \( \mathcal{L} \subseteq T^0_S \) is a complex Lie subalgebra and \( \mathcal{O}_S \)-submodule, called the submodule of \textit{liftable} vector fields on \( S \).

Similarly, the image of \( T^0_{X \to S}(\mathcal{O}_S) \) under the projection \( p_2 \) to \( T^0_X (f^*\mathcal{O}_S) \cong \text{Der}_X \) consists of what Arnol’d calls the \textit{lowerable} vector fields on \( X \). Those form again a complex Lie subalgebra, but only an \( \mathcal{O}_S \)-submodule of \( T^0_X \).

1.11. The relevance of the \( \mathcal{O}_S \)-module \( T^0_{X \to S}(N) \) of compatible vector fields is its relation to the \textit{Kodaira-Spencer map}, a relation described through the following commutative diagram, with right column the exact \textit{Zariski-Jacobi sequence} of the tangent cohomology \( \mathcal{O}_X \)-modules associated to \( f \) and \( f^*\mathcal{N} \), and left column an exact sequence of \( \mathcal{O}_S \)-modules\(^2\) that essentially reflects the description of \( T^0_{X \to S}(N) \).

\(^1\)Most base points of germs are called “0”. They are usually suppressed from the notation.

\(^2\) To exhibit indeed all modules in the left column as \( \mathcal{O}_S \)-modules, one should write \( f_*T^1_{X/S}(f^*\mathcal{N}) \) there instead of \( T^1_{X/S}(f^*\mathcal{N}) \). However, to keep notation at bay, we allow for this abuse of notation.
as a fibred product. The horizontal arrows represent maps that are linear over $f$.

$$
\begin{array}{c}
0 \\
\downarrow \\
T^0_{X/S}(f^*N) \\
\downarrow \\
T^0_X(f^*N) \\
\downarrow \\
T^0_{X/S}(f^*N) \\
\downarrow \\
T^0_X(f^*N) \\
\downarrow \\
T^0_{X/S}(f^*N) \\
\downarrow \\
T^0_X(f^*N) \\
\downarrow \\
T^1_{X/S}(f^*N) \\
\downarrow \\
T^1_X(f^*N) \\
\downarrow \\
T^1_{X/S}(f^*N) \\
\downarrow \\
\vdots
\end{array}
$$

The map $\delta^N_{X/S}$ is the **Kodaira-Spencer map** associated to $f$ and $N$, while the map labeled $\text{can}$ is induced from the natural $\mathcal{O}_S$-homomorphism $N \to f_*f^*N \cong N \otimes_{\mathcal{O}_S} \mathcal{O}_X, n \mapsto n \otimes 1$, the unit of the adjunction.

1.12. Note the following facts:

1. The **versality criterion**, see [20] or [12], states that for any coherent $\mathcal{O}_S$-module $N$ the support of the cokernel of the Kodaira-Spencer map $\delta^N_{X/S}$ is contained in the locus of $S$ where $f$ is not **versal**. In particular, if $f$ is versal, then $\delta^N_{X/S}$ is surjective for any coherent $\mathcal{O}_S$-module $N$.

2. In particular, for $f$ versal and $N = \mathcal{O}_S$, the left column in (1) above yields the following short exact sequence of $\mathcal{O}_S$-modules that exhibits the liftable vector fields $\mathcal{L}$ from 1.10 as kernel of the Kodaira-Spencer map,

$$
0 \to \mathcal{L} \to \text{Der}_{S} \mathcal{O}_S, \delta^N_{X/S} \to T^1_{X/S} \to 0.
$$

3. If $S$ is **smooth**, the functors $T^i_S(?)$ vanish on any $\mathcal{O}_S$-module for $i \neq 0$.

4. Note as well that $T^1_X(f^*\mathcal{O}_S) \cong T^1_X$ vanishes if, and only if, $X$ is **rigid**.

These facts have the following immediate consequences.

**Corollary 1.13.** If $f : X \to S$ is versal, then the $\mathcal{O}_S$-module $f_*T^1_{X/S}(f^*N)$ is coherent along with $N$. 

Proof. Indeed, as $\mathcal{O}_S$–module, $T^1_{X/S}(f^*\mathcal{N})$ is the homomorphic image of $T^0_S(\mathcal{N})$ under the Kodaira-Spencer map by the versality criterion 1.12(1).

\[ \square \]

Corollary 1.14. If $f : X \to S$ is versal with $S$ smooth, then $T^1_X(f^*\mathcal{N}) = 0$ for any coherent $\mathcal{O}_S$–module $\mathcal{N}$. In particular, $X$ is rigid.

Proof. As $f$ is versal, in the diagram (1) the Kodaira-Spencer map $\delta_X^*f$ is surjective by 1.12(1), whence the map $T^0_S(f_*f^*\mathcal{N}) \to T^1_{X/S}(f^*\mathcal{N})$ must be surjective too. As $T^0_S(f_*f^*\mathcal{N}) = 0$ by smoothness of $S$, it follows that $T^1_{X/S}(f^*\mathcal{N}) = 0$ as claimed. The final assertion follows then from 1.12(4).

\[ \square \]

2. Free Divisors in Versal Deformations

We are now mainly interested in the case where $f$ is a versal morphism of germs with $S$ smooth and where the $\mathcal{O}_S$–module $\mathcal{L}$ of liftable vector fields as in 1.10 is free. From 1.12(2) one obtains immediately the following equivalent characterizations, and much of the subsequent work will be to establish and verify other, more manageable characterizations, such as the one exhibited in 4.1 below.

Lemma 2.1. If $f : X \to S$ is versal with $S$ smooth, then the following conditions are equivalent.

1. The submodule $\mathcal{L} \subseteq T^0_S$ of liftable vector fields is a free $\mathcal{O}_S$–module.
2. The $\mathcal{O}_S$–module $f_*T^1_{X/S}$ is of projective dimension at most 1.
3. The $\mathcal{O}_X$–module $T^1_{X/S}$ is of depth at least $\dim S - 1$.

\[ \square \]

Corollary 2.2. If the equivalent conditions of Lemma 2.1 are satisfied and if furthermore $f|_{\text{Sing } X} : \text{Sing } X \to S$ is not dominant, then

1. The free $\mathcal{O}_S$–module $\mathcal{L}$ of liftable vector fields is of rank equal to $\dim S$.
2. The zeroth Fitting ideal $\mathcal{F}_0 f_*T^1_{X/S} \subseteq \mathcal{O}_S$ of $T^1_{X/S}$ as $\mathcal{O}_S$–module, is principal, generated by the determinant $\Delta$ of any matrix representing the inclusion $\mathcal{L} \subseteq T^0_S$ of free $\mathcal{O}_S$–modules of the same rank.
3. The $\mathcal{O}_S$–module $f_*T^1_{X/S}$ is maximal Cohen-Macaulay on the hypersurface, or divisor, $V(\mathcal{F}_0 f_*T^1_{X/S}) \subseteq S$.
4. The support of $T^1_{X/S}$ is empty, equivalently, $T^1_{X/S} = 0$ or, also, $\mathcal{F}_0 f_*T^1_{X/S} = \mathcal{O}_S$, if, and only if, $f$ is smooth.

Proof. The theorem on generic smoothness implies that over some Zariski-open dense subset $S \setminus f(\text{Sing } X)$, the morphism $f$ is smooth, whence $f_*T^1_{X/S}$ is supported in $S \setminus U$. This implies that $f_*T^1_{X/S}$ is a torsion $\mathcal{O}_S$–module and forces the free $\mathcal{O}_S$–module $\mathcal{L}$ to be of the same rank as $T^0_S = \text{Der}(\mathcal{O}_S)$, which equals $\dim S$. That the support of $f_*T^1_{X/S}$ is then given by the indicated Fitting ideal is nothing but Cramer’s rule. The assertion that $f_*T^1_{X/S}$ is maximal Cohen-Macaulay on its support in $S$ just restates that $T^1_{X/S}$, if not zero, is of projective dimension 1 over $\mathcal{O}_S$.

The module $T^1_{X/S}$ is zero if, and only if, its support is empty, which happens thus if, and only if, every vector field on $S$ can be lifted, and that property in turn is equivalent to $f$ being smooth.

\[ \square \]
Note that $\mathcal{F}_0T^1_{X/S} \subseteq \mathcal{O}_X$ defines the natural analytic structure on the critical locus $C(f)$ of $f$ in $X$, see e.g. [34 4.A].

**Definition 2.3.** With assumptions and notation as in [34], a matrix representing the containment $L \subseteq T^0_S$ is called a discriminant matrix for $f$ and its determinant $\Delta$ constitutes a canonical equation for the resulting divisor. The zero-set $V(\Delta)$ will also be called the (homological) discriminant of $f$. It is not necessarily reduced.

Note that the discriminant might be empty. By 2.2 (4) this will be the case if, and only if, $f$ is smooth.

The case, when $X$ is smooth along with $S$ is classical and at the basis of our results. Indeed, the study of versal maps $f : X \to S$ between smooth spaces with $T^1_{X/S}$ coherent over $S$ is nothing but the study of versal deformations of isolated complete intersection singularities, a situation that is well understood. We recall the pertinent facts, with references to Looijenga’s monograph [34].

**Theorem 2.4.** Let $f : X \to S$ be a versal morphism between smooth spaces with $S$ connected. If the critical locus $C(f)$ is not empty, that is, $f$ itself is not smooth, then one has the following properties:

1. ([34 Thm.2.8]) The dimension of $C(f)$ equals $\dim S - 1$ everywhere and $f$ restricted to $C(f)$ is birational, that is, generically one-to-one on each component, and finite.

2. ([34 Thm.4.7]) $C(f)$ is locally irreducible, reduced, and determinantal, defined by the vanishing of the maximal minors of the Jacobi matrix of a local embedding of $X$ into a smooth space over $S$. In particular, it is locally Cohen-Macaulay. The $\mathcal{O}_X$–module $T^1_{X/S}$ is naturally a maximal Cohen-Macaulay module on $C(f)$.

3. ([34 Proof of Thm.4.7]) The singular locus of $C(f)$ is either empty or equidimensional of codimension

$$\text{codim}_{C(f)} \text{Sing}(C(f)) = \dim X - \dim S + 3 \geq 3.$$ 

In particular, $C(f)$ is normal and normalizes the (reduced) discriminant $f(C(f))$.

4. ([34 4.C]) The critical locus admits a “Springer-like” desingularization, given by the projection onto the first factor of

$$\tilde{C}(f) = \{(x, H) \in C(f) \times \mathbb{P}(T_{f(x)}S) \mid \text{Im}(\text{jac}_x(f)) \subseteq H \subseteq T_{f(x)}S\},$$

where we identify points $H \in \mathbb{P}(T_{f(x)}S)$ with hyperplanes in the tangent space $T_{f(x)}S$ of $S$ at $f(x)$. The same desingularization can be obtained as the Nash blow-up (or “development”) of the discriminant, the closure of the image of the Gauß map of the hypersurface $f(C(f)) \subset S$.

5. ([34 6.D]) The equivalent conditions of [22] are satisfied and so [22, 4] applies. The $\mathcal{O}_S$–module $f_*T^1_{X/S}$ is canonically isomorphic to the Jacobi ideal of the homological discriminant that endows $f(C(f))$ with its reduced structure. The discriminant is in particular a free divisor.

Now we are able to formulate our main results.

**Theorem 2.5.** Let $f : X \to S$ be a versal morphism of analytic germs with $S$ smooth. If $\text{codim}_S f(\text{Sing} X) \geq 2$ and if the submodule $L \subseteq T^0_S$ of liftable vector fields is a free $\mathcal{O}_S$–module, then the discriminant of $f$ is a free divisor, the liftable
vector fields coincide with the logarithmic vector fields along the discriminant, and, as an $\mathcal{O}_S$-module, $f_*T^1_{X/S}$ is isomorphic to the Jacobi ideal of the discriminant.

If even $\text{codim}_S f(\text{Sing } X) \geq 3$, then the algebra $\text{End}_{\mathcal{O}_S}(f_*T^1_{X/S})$ is the normalization both of the critical locus and the discriminant of $f$, unless $f$ is smooth.

**Proof.** The assumptions on $f(\text{Sing } X)$ and $\mathcal{L}$ allow to apply 2.2. If $f$ itself is smooth, then the discriminant is empty and the assertion is vacuously true.

Else, $f_*T^1_{X/S}$ is supported on a proper hypersurface in $S$ by 2.2 and so $f' := f|_{X\setminus f^{-1}(\text{Sing } X)} : S \setminus f(\text{Sing } X)$ is a flat morphism between smooth spaces that is itself not smooth. It follows then from 2.2 that $f_*T^1_{X/S}$ is generically free of rank one on each component of its support. This implies that the defining equation $\Delta$ of the discriminant of $f$ is reduced and the result follows from Saito’s criterion [23], identifying at the same time $\mathcal{L}$ with $\text{Der}_S(-\log \Delta)$ and $f_*T^1_{X/S}$ with the Jacobi ideal of $\Delta$, in view of 1.4.

For the final assertion, let $E := \text{End}_{\mathcal{O}_S}(f_*T^1_{X/S})$. As $f_*T^1_{X/S}$ is coherent, isomorphic to the Jacobi ideal of $\Delta$, the $\mathcal{O}_S$-algebra $E$ is again analytic and the structure morphism $\mathcal{O}_\Delta \rightarrow E$ is finite and generically an isomorphism. Moreover, this morphism of analytic algebras factors through $\mathcal{O}_{C(f)}$. The assumption on the codimension of $f(\text{Sing } X)$ guarantees that $\mathcal{O}_{C(f)} \rightarrow E$ is an isomorphism in codimension 2. There, however, $\mathcal{O}_{C(f)}$ is normal by 2.2, and so $\mathcal{O}_{C(f)} \cong E$. In particular, $E$ satisfies Serre’s condition $R_1$. On the other hand, $f_*T^1_{X/S}$, being maximal Cohen-Macaulay on the hypersurface $\Delta$, is a reflexive $\mathcal{O}_S$-module and so $E$ satisfies Serre’s condition $S_2$. Taken together, $E$ is normal, thus the normalization of both $\mathcal{O}_\Delta$ and $\mathcal{O}_{C(f)}$. \hfill \square

**Remark 2.6.** Let $X$ be any singularity whose local ring satisfies Serre’s condition $S_2$, for example, $X$ a hypersurface, or, more generally, Cohen-Macaulay. With $J = \text{jac}_X$ the Jacobian ideal of $X$ and $J^{-1}$ its $\mathcal{O}_X$-dual, the endomorphism ring $E := \text{End}_X(J^{-1})$ is again an analytic algebra that sits between $\mathcal{O}_X$ and its normalization.

A fundamental fact, established by Vasconcelos [18] in the affine case, and, in slightly different form, earlier by Grauert-Remmert [25, 26]; see also [18]; in the analytic case, says that $E$ coincides with $\mathcal{O}_X$ if, and only if, $X$ is already normal. As $E$ inherits Serre’s property $S_2$, this yields evidently an algorithm for normalization. See also [50] for a detailed discussion.

If the support $\Delta$ of $T^1_{X/S}$ is a free divisor, as in the situation of Theorem 2.5, then it is isomorphic to the Jacobian ideal $J$ of $\Delta$, and its $\mathcal{O}_\Delta$-dual $J^{-1}$ is again a maximal Cohen-Macaulay module on $\Delta$. In particular, transposition defines an isomorphism of analytic algebras, $\text{End}_\Delta(J) \cong \text{End}_\Delta(J^{-1})$.

The second half of 2.5 thus says that, for $\text{codim}_S f(\text{Sing } X) \geq 3$, the discriminant of $f$ is “almost normal”, in the sense that a single step in the algorithm suffices to normalize it.

To apply now the main result, we need thus criteria that guarantee freeness of the module of liftable vector fields and allow to bound the singular locus of $X$. The case of a finite flat map is easy to analyse as we now show.
3. Discriminants of Finite Flat Maps

**Theorem 3.1.** Let \( f : X \to S \) be a finite, flat map of complex spaces with \( X \) normal and \( S \) smooth. The discriminant of \( f \) is a (non empty) free divisor at each point \( s \in S \), at which \( f_*T_X^1 \) is locally free and \( f \) is versal (and ramified).

**Proof.** To begin with, we repeat some of the pertinent arguments from above in the context of sheaves of \( \mathcal{O}_X \)-modules. Applying \( \text{Hom}_X(?, \mathcal{O}_X) \) to the Zariski-Jacobi sequence for \( f \) yields the exact sequence of \( \mathcal{O}_X \)-modules

\[
0 \to T^0_{X/S} \to T^0_X \to T^0_S(\mathcal{O}_X) \to T^1_{X/S} \to T^1_X \to 0,
\]

where the zero at the right end is due to the smoothness of \( S \). Moreover, \( f \) being flat onto the smooth space \( S \), the space \( X \) is locally Cohen-Macaulay. As \( X \) is normal, thus reduced, \( f \) is necessarily generically étale by generic smoothness, and that forces \( T^0_{X/S} = 0 \).

At each point \( s \in S \), at which \( f \) is versal, already the Kodaira-Spencer map \( T^0_S \to T^1_{X/S} \) is surjective, thus so is, as in the proof of 1.14, a fortiori the map \( T^0_S(\mathcal{O}_X) \to T^1_{X/S} \). In all, the (direct image of the) above exact sequence reduces at each point to

\[
0 \to f_*T^0_X \to f_*T^0_S(\mathcal{O}_X) \to f_*T^1_{X/S} \to 0.
\]

Now \( f_*T^0_S(\mathcal{O}_X) = T^0_S(f_*\mathcal{O}_X) \) is locally free as \( \mathcal{O}_S \)-module, because \( S \) is smooth and \( X \) is locally Cohen-Macaulay. Moreover, \( f_*T^1_{X/S} \) is of codimension at least one as \( f \) is generically étale, and it is of projective \( \mathcal{O}_S \)-dimension at most one at \( s \) if and only if \( f_*T^1_X = f_*\Theta_X \) is free at \( s \). If that condition is satisfied, then the sheaf of liftable vector fields \( \mathcal{L} \subset T^0_S \) is locally free at \( s \) by 2.1. As \( X \) is normal, \( \text{Sing} X \) is of codimension at least two in \( X \) and so is \( f(\text{Sing} X) \) in \( S \) as \( f \) is finite and flat. In all, Theorem 2.5 applies. \( \square \)

**Remark 3.2.** The proof shows that conversely, if \( f \) is versal at \( s \in S \), then the homological discriminant is a free divisor at \( s \) if, and only if, \( T^0_X \) is a maximal Cohen-Macaulay \( \mathcal{O}_X \)-module at each point \( x \in X \) over \( s \), equivalently, if \( f_*T^0_X \) is locally free at \( s \).

To produce free divisors from finite flat maps, one may thus start with some unobstructed Artinian scheme (or space) \( X_0 \) and consider a versal deformation \( X \to S \). The only conditions left to be satisfied are then that \( X \) is normal with \( f_*T^0_X \) locally free, equivalently, \( T^0_X \) a maximal Cohen-Macaulay \( \mathcal{O}_X \)-module. Although it is rare that the module of vector fields is maximal Cohen-Macaulay for a normal Cohen-Macaulay singularity, it happens in the following case.

**Proposition 3.3.** Let \( X_0 \) be an artinian space that is (algebraically) linked to a complete intersection. One has then the following facts:

1. The deformation theory of \( X_0 \) is unobstructed, thus, any versal deformation \( f : X \to S \) of \( X_0 \) is a finite flat map onto a smooth base \( S \).
2. The total space \( X \) of any versal deformation of \( X_0 \) is rigid, Cohen-Macaulay and nonsingular in codimension three.
3. The \( \mathcal{O}_X \)-module \( T^0_X \) is maximal Cohen-Macaulay.

**Proof.** This result, except for the statement on the singular locus of \( X \), was first established in [11]. Alternatively, the claims are easily obtained from the work of
Huneke-Ulrich in [30]. Just using [30, Thm 4.2], which gives inter alia the assertion on the singular locus of $X$, the remainder of the statements follows as well from the work of J. Herzog in [28]. □

As an immediate consequence we have the following result.

**Theorem 3.4.** Let $X_0$ be an artinian space that is (algebraically) linked to a complete intersection. For every versal deformation $f : X \to S$ of $X_0$, the cohomological discriminant is a free divisor and $E = \text{End}_S(f_* T^0_{X/S})$ normalizes both the discriminant and the critical locus of $f$. □

An interesting case to which this result applies is the Hilbert scheme of points on a smooth complex surface. Indeed, if $Z$ is a smooth surface, any artinian subscheme of it is Cohen-Macaulay of codimension 2, thus linked to a complete intersection. Without using linkage, Fogarty had already shown in [21] that the Hilbert scheme is smooth in this case, and versality of the Hilbert scheme at each point as a deformation of the associated subscheme follows from its representability, the classical result established by Grothendieck. In summary, we have hence the following application.

**Corollary 3.5.** The Hilbert scheme $\mathcal{H}$ of artinian subschemes of a smooth surface $Z$ is smooth, the universal family $f : X \to \mathcal{H}$ is versal everywhere, and $f_* T^0_X$ is locally free on $S$. The discriminant of $f$ is a free divisor at each point and the endomorphism ring of its Jacobi ideal normalizes both discriminant and critical locus.

**Remark 3.6.** The preceding corollary holds as well, but is simpler, for the Hilbert scheme of artinian subschemes of a smooth curve. Indeed, an artinian subscheme of a smooth curve is a disjoint union of irreducible zero-dimensional schemes that are then necessarily simple of type $A_n$, for appropriate $n$, as the local analytic rings of the curve are just power series rings in one variable. The Hilbert scheme at the corresponding point is the product of versal deformations of the individual singularities. In particular, the occurring singularities are hypersurface singularities to which [23] applies. Arnol’d, in [3, 4], investigated the associated discriminants of those singularities, establishing that they form free divisors and giving explicit closed (local) equations. We review and extend some of that work below.

4. **The Case of Curves and Surfaces**

In this section, $f : X \to S$ will always denote a flat morphism of analytic germs with $S$ smooth.

To apply our main result [25] to versal deformations of curve or surface singularities, we first investigate alternative characterizations of the freeness of the module of liftable vector fields. The key idea, already exploited in [17], is to investigate the depth of $T^0_{X/S}$, the module of “vertical” vector fields with respect to $f$. To this end, one uses the following two results, the first of which simply recalls the behaviour of depth in short exact sequences.

**Lemma 4.1.** If $X$ is rigid, Cohen-Macaulay, and if the module of vector fields on $X$ satisfies

$$\text{depth}_X T^0_X \geq \min\{\dim X, \text{depth}_X T^1_{X/S} + 2\},$$

then the depth of the module of vertical vector fields is given by

$$\text{depth}_X T^0_{X/S} = \min\{\dim X, \text{depth}_X T^1_{X/S} + 2\}.$$
Proof. As $X$ is rigid, the initial segment of the Zariski-Jacobi sequence for the tangent cohomology of $f$ takes the following form; see \[111\] and \[114\]
\[
0 \to T^0_{X/S} \to T^0_X \to T^0_S(O_X) \to T^1_{X/S} \to 0.
\]
Now use the fact that $T^0_S(O_X) \cong T^0_S \otimes_{O_S} O_X$ is a free $O_X$–module, thus of maximal depth as $O_X$–module, and the mentioned property of depth in short exact sequences.

□

Lemma 4.2. Assume $X$ is Cohen-Macaulay with dualizing module $\omega_X$. If the restriction of $f$ to its critical locus is finite but not dominant, then one has, with $d = \dim X - \dim S$, natural isomorphisms of $O_X$–modules
\[
T^0_{X/S} \cong \text{Hom}_X(\Omega^1_{X/S}, O_X) \cong \text{Hom}_X(\omega_X, (\Omega^{-1}_{X/S})^*),
\]
where ( )* denotes the $O_X$–dual.

Proof. The first isomorphism can be taken as the definition of $T^0_{X/S}$. The $O_X$–module $\Omega^1_{X/S}$ is locally free outside of $C(f)$, for any $i$, and $\Omega^1_{X/S}$ coincides there with $\omega_{X/S} \cong \omega_X$, the last isomorphism due to smoothness of the germ $S$.

If the relative dimension $d := \dim X - \dim S$ of $f$ equals zero, that is, $f$ finite, then $\Omega^1_{X/S}$ is a torsion $O_X$–module because $C(f)$ is a proper closed subspace of $X$ by assumption. In this case, the claimed isomorphism is one of zero modules.

If the relative dimension $d$ is at least 1, the assumptions imply that the codimension of the critical locus of $f$ in $X$ is at least 2; see, for example, [34, Thm.2.5]. With $j : X \setminus C(f) \to X$ the open embedding of the complement of the critical locus into $X$, one has then natural isomorphisms
\[
(\Omega^d_{X/S})^{**} \cong j_* j^* \Omega^d_{X/S} \cong \omega_{X/S} \cong \omega_X
\]
and the $O_X$–module homomorphisms
\[
\text{Hom}_X(\Omega^1_{X/S}, O_X) \xrightarrow{\alpha} \text{Hom}_X(\Omega^d_{X/S}, \Omega^d_{X/S}) \xrightarrow{\beta} \text{Hom}_X((\Omega^d_{X/S})^{**}, (\Omega^{-1}_{X/S})^{**}),
\]
with
\[
\alpha(\varphi)(x_0 dx_1 \wedge \cdots \wedge dx_d) := \sum_{i=1}^d (-1)^{i-1} x_0 \varphi(dx_i) dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_d
\]
and $\beta(\psi) := \psi^{**}$, are isomorphisms outside of $C(f)$, thus outside a closed subset of codimension at least 2. As the first and last term are reflexive $O_X$–modules, the composition $\beta \alpha$ is necessarily an isomorphism of $O_X$–modules.

□

After these preliminaries, we can now formulate the following result for versal families of smoothable, reduced curve singularities, extending the material in [47] that motivates it. The assumptions on the fibres mean precisely that the critical locus is finite but not dominant over the base.

Theorem 4.3. Assume $f : X \to S$ is versal with $S$ smooth, $X$ Cohen-Macaulay, $\dim X - \dim S = 1$, and the restriction of $f$ to its critical locus finite but not dominant.

If $T^0_X$ and $\omega_X^*$ are maximal Cohen-Macaulay $O_X$–modules, then the discriminant of $f$ is a free divisor.
Proof. The assumptions first ensure that \([4,1]\) applies, as here depth \(T^0_X = \dim X = \dim S + 1\), and \(X\) is necessarily rigid by \([1,14]\). In view of \([4,2]\) one has \(T^0_{X/S} \cong \omega^*_X\), and so by \([4,1]\) the depth of \(T^1_{X/S}\) is at least \(\dim X - 2 = \dim S - 1\). As the support of \(T^1_{X/S}\) is contained in the critical locus, the assumption on that locus yields depth \(T^1_{X/S} = \dim S - 1\). The claim follows from \([2,5]\) in view of \([2,1]\). \(\Box\)

As a first application we regain the main result from \([4,7]\):

Example 4.4. Let \(C \subset \mathbb{C}^3\) be a reduced space curve singularity. As first shown by Schaps, \([4,6]\), the deformation theory of such singularities is unobstructed, whence any versal deformation of it is of the form \(f : X \to S\) with smooth base \(S\) and \(\dim X - \dim S = 1\). Furthermore, the total space \(X\) is determinantal, given by the vanishing of the maximal minors of a generic \(n \times (n + 1)\) matrix for a suitable \(n\). In particular, the singular locus of \(X\) is of codimension 4 in \(X\), if not empty. As further the singularity of \(C\) is isolated, \(f\) is finite but not dominant when restricted to its critical locus. That \(T^0_X\) is then maximal Cohen-Macaulay was already established in \([2,8]\), and that \(\omega^*_X\) is maximal Cohen-Macaulay as well is contained in the classification of maximal Cohen-Macaulay modules of rank one on determinantal varieties, as one finds it in \([1,9]\). That the discriminant in this case is a free divisor is the main result in \([4,7]\), and \([3,0]\) yields interesting explicit examples of free divisors so obtained. The additional information we obtain here through\([2,6]\)is that \(\text{End}_{\mathcal{O}_S}(T^1_{X/S})\) already normalizes critical locus and discriminant, as \(\text{codim}_S f(\text{Sing } X) \geq \text{codim}_X \text{Sing } X - (\dim X - \dim S) \geq 4 - 1 = 3\).

A second large class of curve singularities that give rise to free divisors as discriminants in their versal deformations is provided by the following application. The particular case of Gorenstein curve singularities in \(\mathbb{C}^4\) was already mentioned in \([4,7]\).

Proposition 4.5. If \(C\) is a Gorenstein curve singularity that is smoothable and satisfies \(T^2_C = 0\), then the discriminant in the base space of any versal deformation \(f : X \to S\) is a free divisor. If \(C\) can further be deformed into a non-smooth isolated complete intersection singularity, then the endomorphism ring of the \(\mathcal{O}_S\)-module \(T^1_{X/S}\) normalizes again discriminant and critical locus.

All the foregoing assumptions are satisfied for reduced Gorenstein curve singularities that are algebraically linked to a complete intersection singularity, in particular, for reduced Gorenstein curve singularities in \(\mathbb{C}^4\).

Proof. If a reduced Gorenstein curve singularity \(C\) is special fibre of a flat morphism \(f : X \to S\) between complex germs with \(S\) smooth, then \(X\) is Gorenstein as well and so \(\omega_X \cong \mathcal{O}_X \cong \omega^*_X\) is automatically a maximal Cohen-Macaulay module. Moreover, if \(T^2_C = 0\), then any versal deformation of \(C\) is of the form \(f : X \to S\) with \(S\) smooth as the deformation theory of \(C\) is unobstructed. In addition, however, \(T^0_X\) is then maximal Cohen-Macaulay over \(\mathcal{O}_X\) as follows from \([2,8]\) Satz 2.3 and remark before Satz 1.6. Thus, \([4,3]\) applies to yield that any unobstructed and reduced Gorenstein curve singularity exhibits a free divisor as discriminant in any versal deformation. If \(C\) deforms into a non-smooth isolated complete intersection singularity, then \(X\) is generically smooth along the critical locus of \(f\) by \([2,4]\) whence the result on normalization of discriminant and critical locus.

Reduced Gorenstein curve singularities that are algebraically linked to a complete intersection necessarily satisfy \(T^2_C = 0\), as can be deduced from \([2,8]\) Satz 1.4]
in conjunction with [19, Thm 4.2.12], a published account of a result in [11]. Any versal deformation of such a singularity has total space that is nonsingular in codimension 6 by [30]. In particular, such singularities are smoothable and the total space is generically nonsingular along the critical locus, whence all the assumptions of the first part are satisfied.

We now turn to deformations of surface singularities. It is tempting to hope that in analogy to 4.4 any isolated Cohen-Macaulay singularity of codimension 2 will produce a free divisor as discriminant in its versal deformation. However, that is not the case, as the following example shows.

Example 4.6. The cone over the rational normal curve in $\mathbb{P}^3$ provides at its vertex an isolated two-dimensional Cohen-Macaulay singularity in codimension two, to which the results of [46] mentioned in 4.4 apply mutatis mutandis. In particular, the module of vector fields on the total space of a versal deformation is Cohen-Macaulay. The structure of the semi-universal deformation of this singularity was determined by Pinkham in [40]. He showed that the dimension of its smooth base equals 2, but that the original singularity constitutes the only singular fibre. Accordingly, the discriminant consists of a single point and is not even a divisor. The culprit is clearly $T^0_{X/S}$; it has only depth 2.

In light of this example, it is perhaps somewhat surprising that the results of 4.5 for Gorenstein curve singularities indeed extend to a significant class of Gorenstein surface singularities. To formulate it, recall that any germ $X_0$ of a complex singularity can be embedded into a smooth germ, say, $X_0 \subseteq Z$, and that the corresponding conormal module is the $O_{X_0}$–module $I/I^2$, with $I \subseteq O_Z$ the ideal defining the embedding. Conormal modules pertaining to different embeddings into smooth germs are stably isomorphic as $O_X$–modules, whence a property such as reflexivity is shared by all such conormal modules.

**Theorem 4.7.** Let $X_0$ be the germ of an isolated Gorenstein surface singularity whose conormal module is reflexive. If $X_0$ is smoothable, then each versal deformation $f : X \to S$ satisfies the assumptions of 2.5 thus, the discriminant of $f$ is a free divisor. If, moreover, $X$ is generically smooth along the critical locus, then $\text{End}_{O_S}(T_{X/S}^0)$ normalizes critical locus and discriminant of $f$

All the foregoing assumptions are satisfied for isolated Gorenstein surface singularities that are linked to complete intersections, in particular for isolated Gorenstein surface singularities in $\mathbb{C}^5$.

**Proof.** As $X_0$ is reduced Cohen-Macaulay of dimension two, reflexivity is the same as maximal depth for a coherent module. In other words, the conormal module is maximal Cohen-Macaulay as $O_{X_0}$–module by assumption. It follows then from [28, 51] that the reduced Gorenstein singularity $X_0$ satisfies $T^2_{X_0} = 0$, whence the deformation theory of such a surface singularity is unobstructed and so any versal deformation $f : X \to S$ has a smooth base $S$, and that furthermore $T^0_{X}$ is maximal Cohen-Macaulay as $O_X$–module, $X$ being rigid and again Gorenstein. To apply [280], it thus remains to verify that $T_{X/S}^0$ is of depth at least $\dim S + 1 = \dim X - 1$. To this end, we use the isomorphism in [1.2] that reduces here to $T_{X/S}^0 \cong (\Omega^1_{X/S})^{**}$, the reflexive hull of the module of relative differential forms, as $d = \dim X - \dim S = 2$ and as $\omega_X \cong O_X$ because $X$ is Gorenstein.
We establish that \((\Omega_{X/S}^1)^{**}\) is of depth at least \(\dim S + 1 = \dim X - 1\) in two steps, showing first that \(\Omega_{X/S}^1\) itself has the desired depth and then that this module is reflexive.

As concerns the depth of \(\Omega_{X/S}^1\), first note that we may lift any embedding \(X_0 \subseteq Z\) of the original germ into a smooth one to an embedding \(X \subseteq Z \times S\) such that \(f\) factors into this embedding followed by the projection onto \(S\) in the second factor. Using yet again the results from [28], the conormal \(O_X\)–module \(J/J^2\) with respect to the ideal \(J \subseteq O_{Z\times S}\) defining the embedding of \(X\) is a maximal Cohen-Macaulay \(O_X\)–module. As \(X\) is reduced along with \(X_0\), this implies that the Zariski-Jacobi sequence associated to the embedding \(X \subseteq Z \times S\) over \(S\) is exact at the left, that is,

\[
0 \to J/J^2 \xrightarrow{j} \Omega_{Z\times S/S}^1 \otimes_{O_Z} O_X \to \Omega_{X/S}^1 \to 0
\]

(2) is a short exact sequence. Indeed, outside the critical locus of \(f\) this sequence is even split exact, whence the kernel of \(j\) is a torsion \(O_X\)–submodule of the maximal Cohen-Macaulay module \(J/J^2\), thus is zero. Now \(\Omega_{Z\times S/S}^1 \otimes_{O_Z} O_X\) is a free \(O_X\)–module, whence this short exact sequence presents \(\Omega_{X/S}^1\) as the cokernel of a monomorphism between maximal Cohen-Macaulay modules. Its depth is thus at most one less than the dimension of \(X\).

It remains to establish reflexivity of \(\Omega_{X/S}^1\). To this end we use a general criterion due to M.Auslander: Let

\[
\begin{array}{ccc}
F_2 & \xrightarrow{\beta} & F_1 \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
F_0 & \rightarrow & \Omega_{X/S}^1 \\
0 & & 0
\end{array}
\]

be the beginning of a free \(O_X\)–resolution of \(\Omega_{X/S}^1\) and set

\[
Tr := Tr(\Omega_{X/S}^1) := \text{coker } \alpha^*,
\]

the Auslander transpose of \(\Omega_{X/S}^1\). One has then an exact sequence

\[
0 \to \text{Ext}_{O_X}^1(Tr, O_X) \to \Omega_{X/S}^1 \to (\Omega_{X/S}^1)^{**} \to \text{Ext}_{O_X}^2(Tr, O_X) \to 0,
\]

with the morphism in the middle the canonical map into the reflexive hull. Thus, \(\Omega_{X/S}^1\) is reflexive if, and only if, \(\text{Ext}_{O_X}^i(Tr, O_X) = 0\) for \(i = 1, 2\). As \(\Omega_{X/S}^1\) is cokernel of a monomorphism of reflexive modules in view of the short exact sequence (2) above, it follows immediately, say, from the snake lemma, that \(\Omega_{X/S}^1\) embeds into its reflexive hull, thus, \(\text{Ext}_{O_X}^1(Tr, O_X) = 0\). To establish vanishing of \(\text{Ext}_{O_X}^2(Tr, O_X)\), we may choose \(F_0 = \Omega_{Z\times S/S}^1 \otimes_{O_Z} O_X\) and identify coker \(\beta\) with \(J/J^2\), thus ker \(\beta^* \cong N_{X/Z\times S} := \text{Hom}_{O_X}(J/J^2, O_X)\), the normal module of the embedding of \(X\) into \(Z \times S\). Dualizing the displayed initial segment of the free resolution into \(O_X\) yields then short exact sequences

\[
0 \to \text{Im } \alpha^* \to \ker \beta^* \cong N_{X/Z\times S} \to \text{Ext}_{O_X}^1(\Omega_{X/S}^1, O_X) \to 0,
\]

\[
0 \to \text{Im } \alpha^* \to F_1^* \xrightarrow{T r} 0
\]

that imply

\[
\text{Ext}_{O_X}^2(Tr, O_X) \cong \text{Ext}_{O_X}^1(\text{Im } \alpha^*, O_X)
\]
as \(F_1^*\) is free, and

\[
\text{Ext}_{O_X}^1(\text{Im } \alpha^*, O_X) \cong \text{Ext}_{O_X}^1(N_{X/Z\times S}, O_X)
\]
as long as
\[(*) \quad \text{Ext}^i_{O_X}(\text{Ext}^1_{X}(\Omega^1_{X/S}, O_X), O_X) = 0 \]
for \(i = 1, 2\). Now \(X_0\) is an isolated smoothable surface singularity, whence the critical locus of \(f\) is of codimension at least 3 in \(X\). As \(\text{Ext}^1_{X}(\Omega^1_{X/S}, O_X)\) is concentrated on \(C(f)\), the required vanishing in \(\text{Ext}^*_{O_X}(\Omega^1_{X/S}, O_X) = 0\) follows. To conclude the argument, we finally use that with \(J^1/J^2\) also its \((\omega_X = O_X)\)–dual module \(N_{X/Z \times S}\) is maximal Cohen-Macaulay, thus satisfies \(\text{Ext}^i_{O_X}(N_{X/Z \times S}, O_X) = 0\) for \(i \neq 0\).

The final assertion on linkage follows as for curves, it simply exploits that we know additionally that \(X\) is smooth in codimension 6. \(\Box\)

5. The Classical Discriminant of a Polynomial

We first note a classical relation between discriminants and dual varieties. For this consider the incidence variety
\[ I = \{(p, H) \mid p \in H\} \subset \mathbb{P}^n \times \mathbb{P}^n \]
together with the two natural projections \(p : I \to \mathbb{P}^n\) and \(\hat{p} : I \to \hat{\mathbb{P}}^n\).

**Proposition 5.1.** Let \(C \subset \mathbb{P}^n\) be a projective curve and
\[ I_C := \hat{p}^{-1}(C) = \{(p, H) \mid p \in C \cap H\} \subset C \times \hat{\mathbb{P}}^n \subset \mathbb{P}^n \times \mathbb{P}^n \]
the corresponding incidence variety. Then the dual variety \(D \subset \mathbb{P}^n\) of \(C\) is the discriminant of the morphism \(\hat{p}|_{I_C} : I_C \to \hat{\mathbb{P}}^n\).

**Proof.** The fiber of \(\hat{p}|_{I_C}\) over a point \(H \in \mathbb{P}^n\) is the intersection \(C \cap H \subset \mathbb{P}^n\). It contains a point of multiplicity at least two if and only if \(H\) is tangent to \(C\). \(\Box\)

This gives the following well known description of the classical discriminant of a polynomial:

**Corollary 5.2.** The discriminant of the universal polynomial
\[ F(u, v) := s_0u^n + \cdots + s_nv^n \]
is isomorphic to the dual variety of the rational normal curve \(\mathbb{P}^1 \hookrightarrow \mathbb{P}^n\) of degree \(n\).

**Proof.** Choosing coordinates \(y_i\) of \(\mathbb{P}^n\) and dual coordinates \(s_i\) of \(\hat{\mathbb{P}}^n\) the incidence variety \(I\) is described by \(\sum_i s_i y_i = 0\) in \(\mathbb{P}^n \times \hat{\mathbb{P}}^n\). If we choose coordinates \((u : v)\) of \(\mathbb{P}^1\), the \(d\)-uple embedding is given by \((u : v) \mapsto (u^n : \cdots : v^n)\). Therefore the equation of \(I_{\mathbb{P}^1} \subset \mathbb{P}^1 \times \mathbb{P}^n\) is
\[ s_0u^n + \cdots + s_nv^n = 0, \]
the universal polynomial. The corollary follows from \(\text{Proposition} 5.1\). \(\Box\)

**Remark 5.3.** Notice that \(\hat{p}|_{I_C} : I_C \to \hat{\mathbb{P}}^n\) is the universal family over the Hilbert scheme \(\text{Hilb}_{n}^{d} \cong \mathbb{P}^n\) of subschemes of length \(n\) on \(\mathbb{P}^1\).

**Remark 5.4.** Note that transverse to the rational normal curve\(^3\)
\[ \mathbb{P}^1 \ni (a, b) \mapsto (a^n : \cdots : \binom{n}{i}a^{n-i}b^i : \cdots : b^n) \in \mathbb{P}^n \]
we find a semi-universal deformation of the \(A_{n-1}\)--singularity \((ax - by)^n = 0\).

\(^3\) We assume here that the binomial coefficients \(\binom{n}{i}\) are invertible.
The following homogeneous equation for the discriminant is due to Bezout (see e.g. [23]).

Proposition 5.5. The coefficients $s_{ij}$ of the generating function

$$\sum_{i,j=1}^{n-1} s_{ij} x^{n-1-i} y^{n-1-j} := \frac{F_u(x,1)F_v(y,1) - F_v(y,1)F_u(x,1)}{y-x}$$

are homogeneous quadratic polynomials in $\mathbb{Z}[s_0, \ldots, s_n]$ and the Bezout determinant of the symmetric $((n-1) \times (n-1))$–matrix $B := (s_{ij})_{i,j=1,\ldots,n-1}$,

$$B(s_0, \ldots, s_n) := \det B$$

yields an equation of the discriminant in $\widetilde{\mathbb{P}}^n$. It is homogeneous of degree $2n-2$, as well as weighted homogeneous of degree $n(n-1)$ with respect to the weights $w(s_i) = i$.

The specialization $B|_{s_0=1, s_1=0}$ constitutes a discriminant matrix for the discriminant in the semi-universal deformation $f(x) = x^n + s_2 x^{n-2} + \cdots + s_n$ of $x^n = 0$ over $S = \mathbb{C}_{s_2, \ldots, s_n}$.

While the above yields a description of the (homogeneous) discriminant in the Hilbert scheme as the determinant of a square matrix of size $n-1$, one may also consider the following slightly less economical version that yields the discriminant of the versal deformation.

To this end, recall the Euler relation $nF(u,v) = uF_u(u,v) + vF_v(u,v)$, or, in its dehomogenized form, $F_v(x,1) = nF(x,1) - xF_u(x,1)$. If one replaces the partial derivative $F_v$ by $F$ in the above expression for the Bezout form, one finds the following result.

Theorem 5.6. The coefficients $s_{ij}'$ of the generating function

$$\sum_{i,j=1}^{n} s_{ij}' x^{n-i} y^{n-j} := \frac{F(x,1)F_v(y,1) - F(y,1)F_u(x,1)}{y-x}$$

are again homogeneous quadratic polynomials in $\mathbb{Z}[s_0, \ldots, s_n]$ and the determinant of the symmetric $(n \times n)$–matrix $B' := (s_{ij}')_{i,j=1,\ldots,n}$ satisfies

$$\det B' = s_0^2 B(s_0, \ldots, s_n).$$

The matrix $B'|_{s_0=1}$ is a discriminant matrix for the discriminant in the versal deformation $f(x) = x^n + s_1 x^{n-1} + \cdots + s_n$ of $x^n = 0$ over $S = \mathbb{C}_{s_1, \ldots, s_n}$.

Moreover, in the case $s_0 = 1$, if we split $F(x,1) = f(x) = (x-r_1) \cdots (x-r_n)$, then the entries $s_{ij}'$ satisfy

$$(*) \quad s_{ij}' = \langle \text{grad}_{s_i} s_j, \text{grad}_{s_j} s_j \rangle = \sum_{k=1}^{n} \frac{\partial s_i}{\partial r_k} \cdot \frac{\partial s_j}{\partial r_k}.$$
Proof. Formula (7) follows from the following elementary calculation (see also [45, (2.4.5) Lemma]):

\[
\frac{F(x,1)F_u(y,1) - F(y,1)F_u(x,1)}{y - x} = \frac{1}{y - x} \sum_{k=1}^{n} \left( -\frac{f(x)f(y)}{y - r_k} + \frac{f(x)f(y)}{x - r_k} \right) = \sum_{k=1}^{n} \frac{f(x)}{x - r_k} \frac{f(y)}{y - r_k} = \sum_{k=1}^{n} \frac{\partial f}{\partial r_k}(x) \frac{\partial f}{\partial r_k}(y) = \sum_{i,j=1}^{n} \left( \sum_{k=1}^{n} \frac{\partial s_i}{\partial r_k} \cdot \frac{\partial s_j}{\partial r_k} \right) \cdot x^{n-i} \cdot y^{n-j}.
\]

Here we have used

\[
\frac{\partial f}{\partial r_i}(x) = -\prod_{j \neq i}(x - r_j).
\]

Let \( M \) be the \((n \times n)\)-matrix

\[
M = (\frac{\partial s_i}{\partial r_j})_{i,j=1,\ldots,n}.
\]

Then we have

\[
B'|_{s_0 = 1} = MM^T.
\]

Let

\[
V = \begin{pmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1
\end{pmatrix}
\]

be the (reversed) Vandermonde matrix. It has determinant

\[
\det V = (-1)^n \prod_{n \geq i > j \geq 1} (r_i - r_j).
\]

Then we have

\[
V^T M = \begin{pmatrix}
\frac{\partial f}{\partial r_1}(r_1) & \cdots & \frac{\partial f}{\partial r_n}(r_1) \\
\vdots & \ddots & \vdots \\
\frac{\partial f}{\partial r_1}(r_n) & \cdots & \frac{\partial f}{\partial r_n}(r_n)
\end{pmatrix},
\]

whence

\[
\det V^T M = (-1)^n \prod_{n \geq i > j \geq 1} (r_i - r_j)^2.
\]

From this we obtain

\[
\det B'|_{s_0 = 1} = \det MM^T = \prod_{n \geq i > j \geq 1} (r_i - r_j)^2 =: \Delta.
\]

By the chain rule

\[
\text{grad}_s(\log \Delta) MM^T = \text{grad}_s(\log \Delta) M^T.
\]

Since the components of the latter vector are symmetric polynomials in \( r_1, \ldots, r_n \), it follows that

\[
\text{grad}_s(\log \Delta) B'|_{s_0 = 1} \in \mathbb{Q}[s_1, \ldots, s_n]^n.
\]

This shows that \( B'|_{s_0 = 1} \) is a discriminant matrix for the discriminant in the versal deformation \( f(x) = x^n + s_1 x^{n-1} + \cdots + s_n \) of \( x^n = 0 \) over \( S = \mathbb{C}[s_1, \ldots, s_n] \). \( \square \)

The description of the entries of the discriminant matrix in terms of the derivatives of the elementary symmetric functions \( s_i \) with respect to the roots \( r_k \) is precisely the form of the discriminant matrix given by Arnol’d in [3, 4], and its relation to the Bezout form can be found, at least implicitly, in [45]. The form (7) of the
entries of the discriminant matrix generalizes both to simple hypersurface singularities and to the simple elliptic surface singularities, in that it uses the action of the associated Coxeter or Weyl group and the fact that the discriminant is precisely the image of the union of the reflection hyperplanes under the orbit map.

**Remark 5.7.** Recently C. D’Andrea and J. V. Chipalkatti [17] proved that even the cone over the dual variety of the rational normal curve of degree \( n \geq 3 \) is a free divisor.

6. The Dual Variety of an Elliptic Normal Curve

Let \( D_\tau \) be the dual variety of the elliptic normal curve \( E_\tau \subset \mathbb{P}^n \). In this section we find a new determinantal expression for \( D_\tau \subset \mathbb{P}^n \).

For this we turn the method of the previous section around and identify \( D_\tau \) with the discriminant of the universal meromorphic function of degree \( n + 1 \) with only one pole at 0 on \( E_\tau \). Recall that the space of such meromorphic functions is generated by the Weierstrass \( \wp \) function, its derivatives and the constant function.

**Proposition 6.1.** Let \( \tau \) be a point in the upper half plane, \( E_\tau \) the corresponding elliptic curve and \( \Phi : E_\tau \to \mathbb{P}^n \)

\[
    z \mapsto (1 : \wp(z) : \wp'(z) : \vdots : \wp^{(n-1)}(z))
\]

its \( n \)-th Weierstrass embedding. Then the dual variety of \( E_\tau \) is isomorphic to the discriminant of the universal meromorphic function

\[
    \lambda(z) = \frac{(-1)^{n-1}}{n!} s_{n+1} \wp^{(n-1)}(z) \pm \cdots - \frac{1}{2} s_3 \wp'(z) + s_2 \wp(z) + s_0
\]

**Proof.** If we denote by \( \frac{(-1)^{i'}}{i'} s_i \) the dual coordinates of \( \mathbb{P}^n \), then the claim follows from 5.1. \qed

Up to a constant \( u \), a meromorphic function \( \lambda \) as above is determined by its zeros \( z_0, \ldots, z_n \). The condition that the only pole should lie at 0 implies that \( \sum_{j=0}^n z_j = 0 \). Using the coordinates \( v = (v_1, \ldots, v_n) \) with

\[
    v_k = \sum_{j=0}^{k-1} z_j = -\sum_{j=k}^n z_j, \quad k = 1, \ldots, n,
\]

we obtain

**Proposition 6.2.** The coefficient \( s_i = s_i(\tau, u, v) \) of the universal meromorphic function

\[
    \lambda(z) = \frac{(-1)^{n-1}}{n!} s_{n+1} \wp^{(n-1)}(z) \pm \cdots - \frac{1}{2} s_3 \wp'(z) + s_2 \wp(z) + s_0
\]

is a Jacobi form of weight \( -i \) and index 1. If we set \( s_{-1}(\tau, u, v) := \tau \) then the algebra \( J \) of Jacobi forms is freely generated by \( s_{-1}, s_0, s_2, \ldots, s_{n+1} \).

**Proof.** This follows from a theorem of K. Wirthmüller [52] (3.6 Theorem [see also [6] Section 1.4, in particular Theorem 1.4]). \qed
Definition 6.3. Let $\Omega^1_J$ be the $J$-module of 1-forms and 
\[ I : \Omega^1_J \times \Omega^1_J \to J \]
be the symmetric bilinear form defined by
\[ I(du, du) = I(d\tau, d\tau) = 0, \quad I(du, dr) = I(dr, du) = 1, \]
\[ I(dv_i, dv_j) = \delta_{ij}, \quad I(du, dv_i) = I(dr, dv_i) = 0, \quad i, j \in \{1, \ldots, n\}. \]

Proposition 6.4. The matrix 
\[ B := (I(ds_i, ds_j))_{i,j=-1,0,2,\ldots,n+1} \]
is a discriminant matrix for the discriminant of the universal meromorphic function
\[ (4) \quad \lambda(z) = \frac{(-1)^{n-1}}{n!} s_{n+1} \psi^{(n-1)}(z) \pm \cdots - \frac{1}{2} s_3 \psi'(z) + s_2 \psi(z) + s_0 \]

Proof. Since $s_{-1}, s_0, s_2, \ldots, s_{n+1}$ generate $J$ it follows from \cite{44}, Formula (5.2.5)]. That $f = \det B$ is an equation for the discriminant. By \cite{44}, (5.3) Assertion we have
\[ B \cdot (\partial_1, \ldots, \partial_{n+2})^t \in \Theta J(-\log f)^{n+2}. \]

To calculate $B$ explicitly one needs to express in terms of the $s_i$ their derivatives. Here we pursue a different approach. For a slight variant of $I$, Bertola has given a Bezout type formula expressing the generating function for the entries of the matrix $B$ in terms of the $s_i$ \cite[Theorem 1.5]. The matrix $B$ for $n = 2$ is computed in \cite[Example 1.2]. Retracing his steps with MAPLE for $n = 4$ and dividing the first row and the first column of the resulting matrix by $-2\pi \sqrt{-1}$ and the other rows and columns by $e^{-2\pi \sqrt{-1}w}$ we obtain:

Example 6.5. Let $\tau \in \mathbb{H}$ be a point in the upper half-plane and $E_{\tau} \subset \mathbb{P}^4$ the corresponding elliptic curve in its 4-th Weierstrass embedding. Then the dual variety of $E_{\tau}$ is defined by the determinant of the matrix

\[
A := \begin{pmatrix}
0 & s_0 & s_2 & s_3 & s_4 & s_5 \\
-1 & s_0 & a_02 & a_03 & a_04 & a_05 \\
0 & -1 & s_2 & a_02 & a_23 & a_24 & a_25 \\
0 & 0 & -1 & s_4 & a_24 & a_34 & a_35 \\
0 & 0 & 0 & -1 & s_5 & a_35 & a_45 & a_55 \\
\end{pmatrix}
\]

where
\[
a_{00} := -\frac{1}{6} g_2 s_0 s_2 - \frac{1}{2} g_3 s_2^2 - \frac{1}{18} g_2^2 s_2 s_4 + \frac{1}{24} g_2^2 s_3 + \frac{1}{8} g_2 g_3 s_3 s_5 \]
\[= -\frac{1}{12} g_2 g_3 s_4^2 + \left(\frac{1}{288} g_2^3 + \frac{3}{40} g_3^2\right) s_5^2 \]
\[a_{02} := -\frac{1}{3} g_2 s_2^2 - g_3 s_2 s_4 + \frac{3}{4} g_3 s_2^3 + \frac{1}{12} g_2^2 s_3 s_5 - \frac{1}{18} g_2^2 s_4 + \frac{9}{80} g_2 g_3 s_5^2 \]
\[a_{03} := -\frac{5}{12} g_2 s_2 s_3 + \frac{1}{2} g_3 s_3 s_4 - \frac{4}{5} g_3 s_2 s_5 - \frac{1}{36} g_2^2 s_4 s_5 \]
\[a_{04} := -\frac{1}{2} g_2 s_2 s_4 + \frac{21}{20} g_3 s_3 s_5 - \frac{1}{2} g_3 s_4^2 + \frac{1}{24} g_2^2 s_5 \]
\[a_{05} := -\frac{7}{12} g_2 s_2 s_5 - \frac{3}{5} g_3 s_4 s_5 \]
FREE DIVISORS

\[ a_{22} := -2s_0s_2 - \frac{2}{3} g_2 s_2 s_4 + \frac{1}{2} g_2 s_3^2 + \frac{3}{2} g_3 s_3 s_5 - g_3 s_4^2 + \frac{3}{40} g_2 s_5^2 \]

\[ a_{23} := -3s_0s_3 - \frac{8}{15} g_2 s_2 s_5 + \frac{1}{3} g_2 s_3 s_4 - \frac{1}{2} g_3 s_4 s_5 \]

\[ a_{24} := -4s_0s_4 + \frac{7}{10} g_2 s_3 s_5 - \frac{1}{3} g_2 s_4^2 + \frac{3}{4} g_3 s_5^2 \]

\[ a_{25} := -5s_0s_5 - \frac{2}{5} g_2 s_4 s_5 \]

\[ a_{33} := -4s_0s_4 + \frac{6}{5} s_2^2 - \frac{1}{6} g_2 s_3 s_5 + \frac{1}{6} g_2 s_4^2 - \frac{1}{2} g_3 s_5^2 \]

\[ a_{34} := \frac{4}{5} s_2 s_3 - 5s_0s_5 + \frac{1}{12} g_2 s_4 s_5 \]

\[ a_{35} := \frac{2}{5} s_2 s_4 - \frac{1}{3} g_2 s_5^2 \]

\[ a_{44} := -2s_2 s_4 + \frac{6}{5} s_3^2 + \frac{1}{2} g_2 s_5^2 \]

\[ a_{45} := -3s_2 s_5 + \frac{3}{5} s_3 s_4 \]

\[ a_{55} := -2s_3 s_5 + \frac{4}{5} s_4^2 \]

with \( g_2 \) and \( g_3 \) the well-known functions

\[ g_2(\tau) = 60 \sum_{m,n}^\prime \frac{1}{(m+n\tau)^4}, \]

\[ g_3(\tau) = 140 \sum_{m,n}^\prime \frac{1}{(m+n\tau)^6}. \]

As usual the symbol \( \sum^\prime \) indicates summation over the nonzero elements of \( \mathbb{Z} \times \mathbb{Z} \).

7. THE DISCRIMINANT OF THE SIMPLE ELLIPTIC SINGULARITY \( \tilde{A}_4 \)

Consider the cone \((X,0)\) over an elliptic normal curve \( E_\tau \subset \mathbb{P}^4 \). The ideal of \( E_\tau \subset \mathbb{P}^4 \) is described by the \( 4 \times 4 \)-Pfaffians of a skew symmetric \( 5 \times 5 \)-matrix \( M \). The germ \((X,0)\) is therefore a Gorenstein surface singularity of codimension 3, namely the simple elliptic singularity of type \( \tilde{A}_4 \). By 4.7 the discriminant of the semi-universal deformation of \((X,0)\) is a free divisor.

In this section we will show that the matrix \( A \) from 6.5 is a discriminant matrix for this discriminant. More precisely, the discriminant is isomorphic to the affine cone over the dual variety of the elliptic normal curve \( \text{Jac}_2 E_\tau \). For this we make extensive use of the fact that the pfaffian description of \( E_\tau \) exhibits \( E_\tau \) as a linear section of the Grassmannian variety \( G(5,2) \), that the dual variety of \( G(5,2) \) is again a Grassmannian of the same type \( G(5,2) \), and that the deformations of \((X,0)\) can be obtained by perturbing the entries of \( M \).

We start by recalling some facts about dual varieties and linear sections. Let \( \mathbb{P}^n = \mathbb{P}(A) \) be a projective space with \( A = H^0(\mathcal{O}(1)) \) and \( \mathbb{P}^n := \mathbb{P}(A^*) \) its dual space. For every linear subspace \( \mathbb{P}^m = \mathbb{P}(B) \subset \mathbb{P}(A) = \mathbb{P}^n \) we obtain quotient \( A \overset{\phi}{\to} B \to 0 \) and consider \( B_\perp = (\text{ker } \phi)^* \).
By construction this gives a quotient \( A^* \to B_\perp \to 0 \) and therefore a linear subspace \( \mathbb{P}_m' := \mathbb{P}(B_\perp) \subset \mathbb{P}(A^*) := \mathbb{P}^n \) which is called the orthogonal space of \( \mathbb{P}^m \subset \mathbb{P}^n \).

**Lemma 7.1.** Let \( X \subset \mathbb{P}^n \) be a smooth variety, \( \bar{X} \subset \mathbb{P}^n \) the dual variety, \( \mathbb{P}^m_\perp \subset \mathbb{P}^n \) a linear subspace and \( \mathbb{P}^m_\perp' \subset \mathbb{P}^n \) the orthogonal space. Assume that \( Y = X \cap \mathbb{P}^m_\perp \) contains a point \( y \) with \( \text{codim} T(y,Y) < \text{codim} X \). Then there exists a point \( y_\perp \) in \( Y_\perp = \bar{X} \cap \mathbb{P}^m_\perp \) with \( \text{codim} T(y_\perp,Y_\perp) < \text{codim} \bar{X} \).

**Proof.** If \( y \in Y \) is a point as above there exists a hyperplane in \( \mathbb{P}^m \) that contains \( \mathbb{P}^m_\perp \) and is tangent to \( y \in X \). This hyperplane represents a point \( y_\perp \in \bar{X} \) because of the tangency condition. Also \( y_\perp \) is in \( \mathbb{P}^m_\perp \) since it contains \( \mathbb{P}^m_\perp \). We have \( y_\perp \in Y_\perp \). By the symmetry of the duality correspondence the hyperplane in \( \mathbb{P}^n \) represented by \( y \) is tangent to \( \bar{X} \) in \( y_\perp \). It also contains \( \mathbb{P}^m_\perp \) since it is an element of \( \mathbb{P}^m \). Therefore \( \text{codim} T(y_\perp,Y_\perp) < \text{codim} \bar{X} \) as claimed.

We now specialize to the case of a Grassmannian in its Pukier embedding. For this let \( V \) be a 5-dimensional vector space and \( \mathbb{P}^9 := \mathbb{P}(\wedge^2 V) \) the projective space of 2-forms with coordinates \( v_{ij} = v_i \wedge v_j \). The Grassmannian \( G := G(V,2) \subset \mathbb{P}^9 \) is defined by the 4 × 4 pfaffians of the generic skew symmetric 5 × 5 matrix

\[
\begin{pmatrix}
0 & v_{12} & v_{13} & v_{14} & v_{15} \\
v_{12} & 0 & v_{23} & v_{24} & v_{25} \\
v_{13} & v_{23} & 0 & v_{34} & v_{35} \\
v_{14} & v_{24} & v_{34} & 0 & v_{45} \\
v_{15} & v_{25} & v_{35} & v_{45} & 0
\end{pmatrix}
\]

The dual variety of \( G \) is again a Grassmannian of the same type

\[
\bar{G} \cong G(2,V) \subset \mathbb{P}(\bigwedge^2 V^*) := \mathbb{P}^9.
\]

We denote the coordinates of \( \mathbb{P}^9 \) by \( v_{ij}^* = v_i^* \wedge v_j^* \). The incidence variety \( H \subset \mathbb{P}^9 \times \mathbb{P}^9 \) is defined by the equation

\[
\sum_{1 \leq i < j \leq 5} v_{ij} v_{ji}^* = 0.
\]

Let now \( W \) be a 5-dimensional quotient space of \( \wedge^2 V \), \( \mathbb{P}^4 := \mathbb{P}(W) \subset \mathbb{P}^9 \) its projectivization and \( \mathbb{P}^4_\perp := \mathbb{P}(W_\perp) \subset \mathbb{P}^9 \) the corresponding orthogonal space.

**Proposition 7.2.** If \( E := \mathbb{P}^4 \cap G \) is smooth of dimension 1 then \( E \subset \mathbb{P}^4 \) is an elliptic normal curve and \( E_\perp := \mathbb{P}^4_\perp \cap G \) is naturally isomorphic to \( \text{Jac}_2 E \).

**Proof.** Since \( E \) is of expected codimension we have \( \deg E = \deg G = 5 \). Adjunction shows that the arithmetic genus of \( E \) is 1. By Lemma 7.1 above \( E_\perp \) must also be smooth and of expected codimension, i.e. an elliptic normal curve.

For the identification of \( E_\perp \) with \( \text{Jac}_2 E \) consider the universal quotient bundle \( Q \) on \( G \) and its restriction \( Q_E \) to \( E \). As the intersection \( \mathbb{P}^4 \cap G \) is transversal, a locally free resolution of \( Q_E \) is obtained by tensoring the Koszul complex associated to \( W \) with \( Q \)

\[
0 \to Q(-5) \to 5Q(-4) \to 10Q(-3) \to 10Q(-2) \to 5Q(-1) \to Q \to Q_E \to 0.
\]

By the Theorem of Bott [9] the cohomology of \( Q(-n) \) vanishes for \( 1 \leq n \leq 5 \). This shows that

\[
H^0(Q_E) = H^0(Q) = V.
\]
Note that $\mathcal{G}$ corresponds to decomposable 2-forms $v \wedge v'$ and such a point lies on $E_\perp$ if it is in the kernel of the map

$$\bigwedge^2 V = \bigwedge^2 H^0(Q_E) \to H^0\left(\bigwedge^2 Q_E\right) = H^0(O_E(1)) = W.$$ 

In particular $Q_E$ can not have any subbundles $\mathcal{L}$ of degree $\deg \mathcal{L} \geq 3$ since in this case $h^0(\mathcal{L}) \geq 3$ by Riemann-Roch and $\bigwedge^2 H^0(\mathcal{L}) \subset \bigwedge^2 V$ spans a projective space of dimension $(\deg \mathcal{L}) - 1 \geq 2$ that is contained in $E_\perp$. This contradicts $\dim E_\perp = 1$.

From $\deg Q_E = 5$ it follows that $Q_E$ is stable. By Atiyah’s classification of stable vector bundles on elliptic curves $[5]$, $Q_E$ is the unique irreducible rank 2 vector bundle with determinant $\det Q_E = O_E(H)$, where $H$ is a hyperplane section of $E \subset \mathbb{P}^4$.

Let now $\mathcal{L} \in \text{Jac}_2 E$ be a line bundle of degree 2. Then there exists a unique nontrivial extension

$$0 \to \mathcal{L} \to \mathcal{F} \to O_E(H) \otimes \mathcal{L}^{-1} \to 0.$$ 

By Atiyah’s classification we must have $\mathcal{F} \cong Q_E$. Taking cohomology we obtain a two dimensional subspace $H^0(\mathcal{L}) \subset V$ and a one dimensional subspace

$$\bigwedge^2 H^0(\mathcal{L}) \subset \bigwedge^2 V$$

that is mapped to $H^0(\bigwedge^2 \mathcal{L}) = 0$ by the map $\bigwedge^2 V \to W$. It therefore represents a projective point on $E_\perp$.

If on the other hand $v \wedge v'$ represents a point on $E_\perp$ then $v \wedge v'$ is mapped to zero in $W$. This means that $v$ and $v'$ are dependent on $E$ and that the image of

$$O \oplus O \xrightarrow{(v,v')} Q_E$$

is a line bundle $\mathcal{L}$ on $E$ with at least 2 sections. Therefore the degree of $\mathcal{L}$ is at least 2. Since a subbundle of $Q_E$ has degree at most 2 we have $\mathcal{L} \in \text{Jac}_2 E$. □

Let now $W \to U$ be a 4-dimensional quotient space of $W$ and $\mathbb{P}^3 := \mathbb{P}(U) \subset \mathbb{P}^4$ the corresponding hypersurface. The associated orthogonal space

$$\mathbb{P}^5 := \mathbb{P}_\perp := \mathbb{P}(U_\perp) \subset \mathbb{P}^9$$

contains $\mathbb{P}^4$. We set $Z := E \cap \mathbb{P}^3 = \mathcal{G} \cap \mathbb{P}^3$ and $Y := \mathcal{G} \cap \mathbb{P}^5$. If $E = \mathcal{G} \cap \mathbb{P}^4$ is a smooth elliptic curve, then $Z$ is a scheme of length 5 on $E$ and $Y$ is a possibly singular Del Pezzo surface of degree 5 that contains $E_\perp$.

**Proposition 7.3.** In the situation just described $Z$ contains a multiple point if, and only if, $Y$ is singular.

**Proof.** This follows from [7,1]. □

**Proposition 7.4.** Let $\hat{\mathbb{P}}^9$ be the blowup of $\mathbb{P}^9$ in $\mathbb{P}^4_\perp$ and $\hat{\mathcal{G}}$ the strict transform of $\mathcal{G}$. Then there exists a natural map $\hat{\mathcal{G}} \to \mathbb{P}^4 := \mathbb{P}(W^*)$ of fiber dimension 2 whose discriminant is equal to the dual variety $E$ of $E \subset \mathbb{P}^4$.

**Proof.** The exact sequence

$$0 \to W^* \to \bigwedge^2 V^* \to W_\perp \to 0$$

...
yields under projection from \( \mathbb{P}^4 = \mathbb{P}(W^*) \) a morphism

\[
\begin{array}{c}\tilde{\mathbb{G}} \xrightarrow{\pi} \mathbb{P}^5 \\
\downarrow \quad \downarrow \pi \\
\mathbb{P}^5 \xrightarrow{\pi} \mathbb{P}(W^*)
\end{array}
\]

The fiber over a point \( \mathbb{P}^3 \in \mathbb{P}^4 = \mathbb{P}(W^*) \) is then \( \mathbb{P}^3 = \mathbb{P}^5 \) and \( \mathbb{P}^5 \cap \tilde{\mathbb{G}} = \mathbb{P}^5 \cap \tilde{\mathbb{G}} = Y \) is a singular surface if and only if \( Z = E \cap \mathbb{P}^3 \) contains a double point by Proposition 7.4. This is the case if and only if \( \mathbb{P}^3 \) is tangent to \( E \) so \( \mathbb{P}^3 \) is a point of the dual variety \( \tilde{E} \) of \( E \).

If we choose a splitting \( \bigwedge^2 V^* \cong W_{+} \oplus W^* \) we can write every decomposable form as

\[ u_i^* \wedge u_j^* = w_{ij}^* + w_{ij}^*. \]

**Corollary 7.5.** There exists a flat deformation of the projective closure of the cone over \( E_\perp \)

\[
\begin{array}{c}X \xrightarrow{\pi} \mathbb{P}^5 \times W^* \\
\downarrow \downarrow \downarrow \pi \downarrow \\
W^* \end{array}
\]

that is described by the \( 4 \times 4 \)-pfaffians of the skew symmetric matrix

\[
\begin{pmatrix}
0 & w_{12}^* & w_{13}^* & w_{14}^* & w_{15}^* \\
-w_{12}^* & 0 & w_{23}^* & w_{24}^* & w_{25}^* \\
-w_{13}^* & -w_{23}^* & 0 & w_{34}^* & w_{35}^* \\
-w_{14}^* & -w_{24}^* & -w_{34}^* & 0 & w_{45}^* \\
-w_{15}^* & -w_{25}^* & -w_{35}^* & -w_{45}^* & 0
\end{pmatrix} + t
\begin{pmatrix}
0 & w_{12}^* & w_{13}^* & w_{14}^* & w_{15}^* \\
-w_{12}^* & 0 & w_{23}^* & w_{24}^* & w_{25}^* \\
-w_{13}^* & -w_{23}^* & 0 & w_{34}^* & w_{35}^* \\
-w_{14}^* & -w_{24}^* & -w_{34}^* & 0 & w_{45}^* \\
-w_{15}^* & -w_{25}^* & -w_{35}^* & -w_{45}^* & 0
\end{pmatrix}
\]

and whose discriminant is isomorphic to the cone over the dual variety \( \tilde{E} \) of \( E \). Homogeneous coordinates of the \( \mathbb{P}^5 \) are given by a basis of \( W_{+} \) and \( t \).

**Proof.** Let \( w_{ij}^* \) be an nonzero element. Over the affine chart given by \( w_{ij}^* = 1 \) the family \( \tilde{\mathbb{G}} \to \mathbb{C}^4 \) is described by

\[
\begin{pmatrix}
0 & w_{12}^* & w_{13}^* & w_{14}^* & w_{15}^* \\
-w_{12}^* & 0 & w_{23}^* & w_{24}^* & w_{25}^* \\
-w_{13}^* & -w_{23}^* & 0 & w_{34}^* & w_{35}^* \\
-w_{14}^* & -w_{24}^* & -w_{34}^* & 0 & w_{45}^* \\
-w_{15}^* & -w_{25}^* & -w_{35}^* & -w_{45}^* & 0
\end{pmatrix} + t
\begin{pmatrix}
0 & w_{12}^* & w_{13}^* & w_{14}^* & w_{15}^* \\
-w_{12}^* & 0 & w_{23}^* & w_{24}^* & w_{25}^* \\
-w_{13}^* & -w_{23}^* & 0 & w_{34}^* & w_{35}^* \\
-w_{14}^* & -w_{24}^* & -w_{34}^* & 0 & w_{45}^* \\
-w_{15}^* & -w_{25}^* & -w_{35}^* & -w_{45}^* & 0
\end{pmatrix}
\]

where \( t = 0 \) is the equation of the exceptional divisor that intersects all fibers in the elliptic curve \( E_{\perp} \). Omitting the condition \( w_{ij}^* = 1 \) we obtain a family as in the proposition. By construction this is independent of our choice of affine chart. Over \( 0 \in W^* \) its fiber is the projective closure of the cone over \( E_{\perp} \). If \( w \in W^* \) is a point different from \( 0 \), \( Y_w \) is the fiber over this point, and \( x = (w_{ij}^*, t) \in Y_w \) is a singular point, then \( x_\lambda = (w_{ij}^*, \lambda^{-1}t) \) will be a singular point of \( Y_{\lambda w} \) for \( \lambda \neq 0 \).

This proves that the discriminant of \( X \to W^* \) is a cone. With Proposition 7.4 we obtain that it is isomorphic to the cone over the dual variety \( \tilde{E} \) of \( E \).

\[ \square \]
Theorem 7.6. Let $\Gamma_\tau = \mathbb{Z} + \tau \mathbb{Z}$ be a lattice, $E = E_\tau = \mathbb{C}/\Gamma_\tau$ the corresponding elliptic curve and $L(\tau)$ the matrix

$$L(\tau) := \begin{pmatrix}
0 & 0 & -x_4 & -2x_3 \\
-x_4 & 0 & -\frac{2}{3}x_4 & -4x_3 \\
0 & \frac{2}{3}x_4 & 0 & -\frac{2}{3}x_4 - \frac{1}{18}g_2(\tau)x_6 \\
x_4 & 4x_3 & \frac{4}{3}x_2 + \frac{1}{12}g_2(\tau)x_6 & 0 \\
x_4 & 8x_2 & \frac{4}{3}x_1 & -\frac{4}{3}g_2(\tau)x_4 - 2g_3(\tau)x_6 \\
2x_3 & 8x_2 & \frac{4}{3}x_1 & 0 \\
\end{pmatrix}. $$

For a fixed $\tau$ the $4 \times 4$ Pfaffians of $L$ yield the cone $(X,0)$ over $E_\tau$ in its fifth Weierstrass embedding as elliptic normal curve of degree 5. If $N$ is the matrix

$$N := \begin{pmatrix}
0 & 0 & s_5 & -3s_4 & 0 \\
0 & 0 & 2s_4 & -24s_4 & 0 \\
-s_5 & -2s_4 & 0 & -4s_2 & 0 \\
3s_4 & 24s_4 & 4s_2 & 0 & 48s_6 \\
0 & 0 & 0 & -48s_6 & 0 \\
\end{pmatrix},$$

the family $L + N$ parametrized by $(\tau, s_0, s_1, s_2, s_3, s_4, s_5)$ constitutes a versal deformation of $(X,0)$ and the matrix $A$ from [63A] is a discriminant matrix for the discriminant of this deformation.

Proof. Let $(y_0, y_2, y_3, y_4, y_5)$ be the coordinates of the vector space $W$, $(x_1, x_2, x_3, x_4, x_6)$ the coordinates of $W_\perp$, and denote by $(y_0^*, y_2^*, y_3^*, y_4^*, y_5^*)$ and $(x_1^*, x_2^*, x_3^*, x_4^*, x_6^*)$ the dual coordinates of $W^*$ and $(W_\perp)^*$ respectively. The matrix

$$M := \begin{pmatrix}
0 & g_3y_0 + \frac{1}{2}g_2y_2 & y_5 & \frac{1}{2}g_2y_0 + \frac{2}{3}y_4 & y_3 \\
-g_3y_0 - \frac{1}{4}g_2y_2 & 0 & -\frac{1}{2}g_2y_0 - y_4 & \frac{1}{2}y_3 & -\frac{1}{2}y_2 \\
y_5 & \frac{1}{2}g_2y_0 + y_4 & 0 & 6y_2 & 0 \\
\frac{1}{3}g_2y_0 - \frac{2}{3}y_4 & \frac{1}{2}y_3 & -6y_2 & 0 & \frac{1}{2}y_0 \\
y_3 & \frac{1}{2}g_2 & 0 & \frac{1}{2}y_0 & 0 \\
\end{pmatrix} = (w_{ij}) $$

gives a mapping $\tilde{M} : \wedge^2 V \to W$, $v_i \wedge v_j \mapsto w_{ij}$, and the matrix $L = (w_{ij}^*)$ a mapping $\tilde{L} : \wedge^2 V^* \to W_\perp$, $v_i^* \wedge v_j^* \mapsto w_{ij}^*$. The sequence

$$0 \to W^* \xrightarrow{\tilde{M}^T} \wedge^2 V^* \xrightarrow{\tilde{L}} W_\perp \to 0$$

is exact. Indeed $y_0^*$ is mapped under $\tilde{M}^T$ to

$$\begin{pmatrix}
0 & g_3 & 0 & -\frac{1}{2}g_2 & 0 \\
-g_3 & 0 & -\frac{1}{2}g_2 & 0 & 0 \\
0 & \frac{1}{2}g_2 & 0 & 0 & 0 \\
\frac{1}{2}g_2 & 0 & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & \frac{1}{2} & 0 \\
\end{pmatrix}$$
and evidently this is mapped to zero by $\tilde{L}$. A similar calculation for the other basis elements shows that the sequence above is a complex. Furthermore the entries of $M$ and $L$ generate $W$ and $W_\perp$ respectively. Consequently both $M$ and $\tilde{L}$ have full rank $5$ and the sequence is exact.

We now claim that the elliptic curve $E = \mathbb{P}(W) \cap G$ is parametrized by the Weierstrass embedding

$$\Phi : E \to \mathbb{P}(W)$$

$$z \mapsto (1 : \psi(z) : \psi'(z) : \psi''(z) : \psi'''(z)).$$

Evaluating the $4 \times 4$-Pfaffians of $\Phi(E)$, that is setting $y_0 = 1$ and $y_k = \psi^{(k-1)}$, $k = 2, 3, 4, 5$, we must show:

$$\frac{1}{2} \psi'(z)\psi'''(z) - \frac{2}{3} \psi''(z)^2 + 2 \psi_2 \psi(z)^2 + 6 \psi_3 \psi(z) + \frac{1}{6} \psi_2^2 = 0,$$

$$\psi'(z)\psi'''(z) + \frac{1}{2} \psi_2 \psi'(z) - \frac{1}{2} \psi(z)\psi'''(z), = 0,$$

$$\frac{1}{2} \psi'(z)^2 - \frac{1}{3} \psi(z)\psi''(z) + \frac{1}{3} \psi_2 \psi(z) + \frac{1}{2} \psi_3 = 0,$$

$$6 \psi(z)\psi'(z) - \frac{1}{2} \psi'''(z) = 0,$$

$$\frac{1}{2} \psi''(z) - 3 \psi(z)^2 + \frac{1}{4} \psi_2 = 0.$$

These relations are a consequence of the following classical relations between the Weierstrass function and its derivatives:

$$\left(\psi'\right)^2 = 4 \psi^3 - g_2 \psi - g_3,$$

$$\psi'' = 6 \psi^2 - \frac{1}{2} g_2,$$

$$\psi''' = 12 \psi \psi'.$$

Note that [29] presents the corresponding calculation for a different, but projectively equivalent embedding of an elliptic curve in $\mathbb{P}^4$.

On the other hand, we consider the embedding $\Psi$ of the elliptic curve $E_\perp = \mathbb{P}(W_\perp) \cap G$ into $\mathbb{P}(W_\perp)$ through

$$\Psi(z) = (-\frac{1}{24} \psi'''(z) : \frac{1}{6} \psi''(z) : -\frac{1}{2} \psi'(z) : \psi(z) : 1).$$

Setting

$$x_k = \frac{(-1)^{4-k}}{(5-k)!} \psi^{(4-k)}, \quad k = 1, 2, 3, 4, 6;$$

with $\psi^{-2} = 1$; this embedding is described by the $4 \times 4$ Pfaffians of the matrix $L$.

For a point $(y_0, y_2, y_3, y_4, y_5) \in \mathbb{P}(W^*)$ the corresponding $\mathbb{P}^3 \subset \mathbb{P}^4 = \mathbb{P}(W)$ is defined by

$$y_0 y_0 + y_2 y_2 + y_3 y_3 + y_4 y_4 + y_5 y_5 = 0.$$

Substituting $y_k = \frac{(-1)^{k}}{(k-1)!} s_k$, for $k = 0, 2, 3, 4, 5$, the equation of $\mathbb{P}^3 \cap E$ becomes the equation [6,1,3].
The matrix
\[
N = \begin{pmatrix}
0 & 0 & s_5 & -3s_4 & 0 \\
0 & 0 & 2s_4 & -24s_3 & 0 \\
-s_5 & -2s_4 & 0 & -4s_2 & 0 \\
3s_4 & 24s_3 & 4s_2 & 0 & 48s_0 \\
0 & 0 & 0 & -48s_0 & 0
\end{pmatrix} = (w^*_ij)
\]
defines a mapping \( \tilde{N} : \bigwedge^2 V^* \to W^* \), \( v^*_i \wedge v^*_j \mapsto w^*_ij \). One easily verifies that this is a left inverse of \( \tilde{M}^T : W^* \to \bigwedge^2 V^* \) and defines a splitting \( \bigwedge^2 V^* \cong W_v \oplus W^* \).

According to results of H. Pinkham (see also [35]), deformations of the cone \( C(E_v) \) over the elliptic curve \( E_v \) lift to projective deformations of the projective closure \( \overline{C}(E_v) \) in \( \mathbb{P}^5 \). Therefore, [5.6] follows from [6.4] and [7.5].

**Remark 7.7.** The deformation of [7.6] is topologically trivial along the \( \tau \)-axis. This follows from the fact that the vector field \( \partial / \partial \tau \) on \( \mathbb{H} \times \mathbb{C}^5 \) can be locally lifted to a vector field on \( \mathbb{C}^5 \times \mathbb{H} \times \mathbb{C}^5 \). To see this, note that
\[
\frac{\partial g_2}{\partial \tau} = \frac{3}{\pi \sqrt{-1}} g_3, \quad \frac{\partial g_3}{\partial \tau} = \frac{1}{6\pi \sqrt{-1}} g_2,
\]
according to [22], and that \( g_2 \) and \( g_3 \) do not vanish at the same time. From [38] it follows that the complement of the discriminant in \( \mathbb{H} \times \mathbb{C}^5 \) is a \( K(\pi, 1) \)-space.

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