Introduction and Main Results

In this paper, we are interested in the hydrodynamical limit of the Boltzman-Monge-Ampere system (BMA)

\begin{equation}
    f + \xi V f + V \psi f, f = Q(f, f) \quad \text{(1.1)}
    \end{equation}

\begin{equation}
    \text{det}(I_j + \varepsilon^2 D\Psi) = \rho \quad \text{(1.2)}
    \end{equation}

where \( f(t, x, \xi) \geq 0 \) the electronic density at time \( t \geq 0 \) point \( x \in [0, 1]^3 = Td \), and with a velocity \( \xi \in \mathbb{R}^d \), and \( I_d \) is the identity matrix defined by

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}
\]

The spatially periodic electric potential is coupled with \( \rho \) through the nonlinear Monge-Ampere equation (1.2). The quantities \( \varepsilon > 0 \) and \( \rho(t, x) \geq 0 \) denote respectively the vacuum electric permittivity and the electronic density at time \( t \geq 0 \).

\[ Q(f, f) \text{ is the Boltzman collision integral. This integral operates only on the } \xi \text{-argument of the distribution } f \text{ and is given by}
\]

\[ Q(f, f) = \int_{\mathbb{R}^d} Q(f'(t)), f'(t, x, \xi) \, d\sigma \, d\xi, \]

where the terms \( f'(t), f'(t, x, \xi) \) defines, respectively the values \( f(t, x, \xi) \) and \( f'(t, x, \xi) \) with \( \xi \) and \( \xi' \) given in terms of \( \xi, \xi' \in \mathbb{R}^d \), and \( \sigma = S^{t-1} = \{ \sigma \in S^{d-1} | \sigma \xi = \sigma \xi' \} \) by

\[
\begin{bmatrix}
\xi' \\
\xi
\end{bmatrix} = \frac{\xi + \xi'}{2} + \frac{\xi - \xi'}{2} \alpha.
\]

The aim of this work is to investigate the hydrodynamic limit of the (BMA) system with optimal transport techniques.

Remark 1

- Existence and uniqueness of \( \Phi \) is due to the polar factorization theorem.

- By setting the change of variables \( y = \nabla \Psi(x) \), we get \( dy = \det D\Psi(x) dx \). So (1.6) can be transformed to:

\[
\varepsilon^2 \Delta \phi \approx \rho - 1
\]
We have
\[
\frac{d}{dt} H_L = \frac{d}{dt} \left[ \frac{1}{2} f(t,x,\xi) \left| \nabla \phi \right|^2 \right] dx dz
\]
\[
= \int f(t,x,\xi) \left( \partial_t \nabla \phi - \nabla \phi \right) dz dx
\]

From the BMA we have
\[
\int \partial_t f \left| \nabla \phi \right|^2 dx dz = - \int \nabla \cdot \left( f(t,x,\xi) \nabla \phi \right) \left| \nabla \phi \right|^2 dx dz
\]
\[
+ \int \nabla \cdot \left( \frac{1}{\epsilon} \nabla \phi \right) f(t,x,\xi) |\nabla \phi|^2 dx dz
\]
\[
+ \int \Omega(f'(u,f^2) |\nabla \phi|^2) dx dz
\]

The last term is equal to zero from the property of Boltzman Operator [1,3,5,7-9,11].

It follows by integrating by party that
\[
\int \partial_t f \left| \nabla \phi \right|^2 dx dz = 2 \int f(t,x,\xi) \nabla \phi \cdot \nabla \phi dx dz + 2 \int f(t,x,\xi) \frac{1}{\epsilon} \nabla \phi \cdot \nabla \phi dx dz.
\]

Thus
\[
\int \partial_t f \left| \nabla \phi \right|^2 dx dz = 2 \int f(t,x,\xi) \nabla \phi \cdot \nabla \phi dx dz + 2 \epsilon \int f(t,x,\xi) \frac{1}{\epsilon} \nabla \phi \cdot \nabla \phi dx dz.
\]

Let us begin with the first term. Use Holder inequality and that \( |\nabla \phi| \leq C \) to decompose
\[
A = \int \frac{1}{\epsilon} f(t,x,\xi) \nabla \phi \cdot \nabla \phi dx dz = 2 \int f(t,x,\xi) \frac{1}{\epsilon} \nabla \phi \cdot \nabla \phi dx dz.
\]

From the second term \( D \), one has
\[
D = \int \frac{1}{\epsilon} f(t,x,\xi) \nabla \phi \cdot \nabla \phi dx dz = \int \nabla \phi \cdot \nabla \phi dx dz.
\]

From the definition of \( \phi \), we have
\[
\nabla \phi \cdot \nabla \phi = |\nabla \phi|^2.
\]

Since \( \nabla \phi \) is divergence free, once gets
\[
\int \nabla \phi dx = 0. \quad \text{Thus form Lemma 4}
\]
 \[
(G = \nabla \phi), \text{ once has}
\]
\[
\int |\nabla \phi|^2 dx = 0.
\]

Since it costs no generality to suppose that for all \( \epsilon \in [0,T] \int f(t,x,\xi) dx = 0, \)
we get from the equation of conservation of mass
\[ \int f(t, x, \xi) \xi \nabla p = -\int \nabla (f(t, x, \xi) \xi) p = \int \frac{d}{dt} \rho \hat{p} \xi - \int \rho \hat{p} \xi \frac{\partial}{\partial t} \]

By Lemma 4 and setting \( Q(t) = -\int \rho \hat{p} \) we can deduce that
\[ \int f(t, x, \xi) \xi \nabla p \leq C(\varepsilon^2 + H(t)) - \frac{dQ}{dt} \]

Thus
\[ D \leq C(H(t) + \varepsilon^2) - \frac{dQ}{dt} \]

We deduce then the following inequality
\[ \frac{d}{dt} (H(t) + Q) \leq C H(t) + \beta(\varepsilon^2) \quad (2.1) \]

Still using 4,
\[ |Q(t)| = |\int \rho \hat{p}| \leq C \varepsilon^2 + \frac{1}{2} H(t) \]

Thus
\[ H(t) + Q \geq \frac{1}{2} H(t) - C \varepsilon^2 \]

So once can transform (2.1) as
\[ \frac{d}{dt} (H(t) + Q) \leq C(H(t) + Q) + C \varepsilon^2 \]

And by Gronwall’s inequality [11] yields
\[ H(t) + Q(t) \leq (H(0) + Q(0)) \exp(Ct) \]

Finally we conclude that
\[ H(t) \leq C(H(0) + C \varepsilon^2) \exp(Ct) \]

Which achieves the proof of the theorem.

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