NEwton polytopes of rank 3 cluster variables

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Abstract. We characterize the cluster variables of skew-symmetrizable cluster algebras of rank 3 by their Newton polytopes. The Newton polytope of the cluster variable $z$ is the convex hull of the set of all $p \in \mathbb{Z}^3$ such that the Laurent monomial $x^p$ appears with nonzero coefficient in the Laurent expansion of $z$ in the cluster $x$. We give an explicit construction of the Newton polytope in terms of the exchange matrix and the denominator vector of the cluster variable.

Along the way, we give a new proof of the fact that denominator vectors of non-initial cluster variables are non-negative in a cluster algebra of arbitrary rank.

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2010 Mathematics Subject Classification. Primary 13F60, Secondary 52B20.

The first author was supported by the University of Alabama, University of Nebraska–Lincoln, Korea Institute for Advanced Study, and the NSF grant DMS 1800207. The second author was supported by Oakland University, and by the NSA grant H98230-16-1-0303. The third author was supported by the NSF-CAREER grant DMS-1254567, the NSF grant DMS-1800860 and by the University of Connecticut.
1. Introduction

Cluster algebras were discovered by Fomin and Zelevinsky in 2001. Since then, it has been shown that they are related to diverse areas of mathematics such as algebraic geometry, total positivity, quiver representations, string theory, statistical physics models, non-commutative geometry, Teichmüller theory, tropical geometry, KP solitons, discrete integrable systems, quantum mechanics, Lie theory, algebraic combinatorics, WKB analysis, knot theory, number theory, symplectic geometry, and Poisson geometry.

A cluster algebra is equipped with a set of distinguished generators called cluster variables. These generators are very far from being fully understood. Explicit combinatorial formulas that are manifestly positive are known for cluster variables in cluster algebras from surfaces \[16\] and for cluster algebras of rank 2 \[14\]. For skew-symmetric cluster algebras, there is the cluster character formula for the cluster variables \[17\] as well as the \(F\)-polynomial formula \[6\] and for skew-symmetrizable cluster algebras there is the scattering diagram approach \[7\], but none of these provide computable formulas.

For a general cluster algebra, we do know that cluster variables satisfy the Laurent phenomenon \[8\] and positivity \[15, 7\], namely, every cluster variable \(z\) can be written as 

\[
  z = \sum_{\mathbf{p} \in \mathbb{Z}^n} e(\mathbf{p}) x^\mathbf{p},
\]

where \(e(\mathbf{p}) \geq 0\) for all \(\mathbf{p}\), and \(e(\mathbf{p}) > 0\) for finitely many \(\mathbf{p}\).

A natural question is how to describe the set \(S(z) := \{\mathbf{p} : e(\mathbf{p}) > 0\}\). However this can be very hard in general (see Remark 5.3). More feasible questions would be the following.

(a) Describe the Newton polytope of \(z\) (which is the convex hull of \(S(z)\) by definition).

(b) Find a subset \(U(z) \subset \mathbb{R}^n\) such that the condition \(S(z) \subset U(z)\) uniquely detects the cluster variable \(z\) (up to a scalar) among all elements in the cluster algebra.

These problems have been solved in \[13\] for the rank 2 case. In fact, it turns out that the Newton polytope is a solution to (b). The paper \[13\] also introduced a so-called greedy basis, which includes all cluster variables, and found a certain support condition that uniquely detects each greedy basis element (up to a scalar) among all elements in the cluster algebra. An alternative characterization of greedy elements using a support condition (SC) plays an essential role in the construction of quantum greedy bases of rank 2 cluster algebras \[12\]. This support condition was a key ingredient in \[5\], where it was shown that, in rank 2, the greedy basis coincides with the theta basis defined in \[7\].

In this paper, we consider the rank 3 case. We solve problems (a) and (b), and prove that the Newton polytope of \(z\) is a solution to (b). We also generalize the result to quantum cluster variables. The step from rank 2 to rank 3 is known to be difficult, since one has to add the dynamics of the exchange matrix to the problem. In rank 2, the mutation is trivial on the level of the exchange matrix. In rank 3 however, except for a few small cases, the mutation class of the matrix is infinite and the representation theory of the quiver is wild.

Along the way, we give a new elementary proof of the fact that denominator vectors of non-initial cluster variables are non-negative in a cluster algebra of arbitrary rank. This was conjectured by Fomin and Zelevinsky in \[9\] and recently proved by Cao and Li in \[3\] using the positivity theorem.
Recently, Fei has studied combinatorics of $F$-polynomials using a representation-theoretic approach [10]. In that paper it was shown that the $F$-polynomial of every cluster variable of an acyclic skew-symmetric cluster algebra has saturated support, which means that all lattice points in the Newton polytope of the $F$-polynomial are in the support of the $F$-polynomial. Briefly speaking, in the case of skew-symmetric rank 3 cluster algebras, Fei’s result is related to our work in the following sense. The Newton polytope of a cluster variable (which lies in a plane inside $\mathbb{R}^3$) is a projection of the Newton polytope of the corresponding $F$-polynomial (which is usually 3-dimensional) under a linear map. The supports of $F$-polynomials are expected to be saturated but cluster variables are not saturated in general. On the other hand, the Newton polytopes of the $F$-polynomials are difficult to determine but the Newton polytopes of the cluster variables can be explicitly determined. Please see Corollary 5.2, Remark 5.3 and Remark 7.2 for more details.

The paper is organized as follows. In Section 2, we briefly review the solutions for (a) and (b) for rank 2 cluster algebras and in Section 3, we explain our notation and recall several basic facts about cluster algebras. We prove the non-negativity of denominator vectors in Section 4. Our main theorem is presented in Section 5 and proved in Section 6. We then give an example in Section 7. Finally, in Section 8 we prove a quantum analogue of the main theorem.

2. Rank 2

In this section, we let $B$ be a $2 \times 2$ skew-symmetrizable matrix and $\mathcal{A}(B)$ the corresponding cluster algebra with principal coefficients.

2.1. Greedy basis. It is proved in [13] that for each rank 2 cluster algebra there exists a so-called greedy basis defined as follows. Let $B = \begin{bmatrix} 0 & h \\ c & 0 \end{bmatrix}$ denote the exchange matrix. Then for $(a_1, a_2) \in \mathbb{Z}^2$, define $c(p, q)$ for $(p, q) \in \mathbb{Z}^2_{\geq 0}$ recursively by $c(0, 0) = 1$,

$$c(p, q) = \max \left( \sum_{k=1}^{p} (-1)^{k-1} c(p-k, q) \binom{a_2-cq+k-1}{k}, \sum_{k=1}^{q} (-1)^{k-1} c(p, q-k) \binom{a_1-bp+k-1}{k} \right)$$

and define the greedy element at $(a_1, a_2)$ as

$$x[a_1, a_2] = \sum \frac{c(p, q) x_1^{bp} x_2^{cq}}{x_1^{a_1} x_2^{a_2}}.$$

Recall that an element of $\mathcal{A}(B)$ is called positive if its Laurent expansion is positive in every seed. A positive element is indecomposable if it cannot be written as a sum of two positive elements. Finally, a basis $\mathcal{B}$ is called strongly positive if any product of elements from $\mathcal{B}$ can be expanded as a positive linear combination of elements of $\mathcal{B}$.

**Theorem 2.1.** [13] The set $\mathcal{B} = \{ x[a_1, a_2] \mid (a_1, a_2) \in \mathbb{Z}^2 \}$ is a strongly positive basis for the cluster algebra $\mathcal{A}(B)$. Moreover $\mathcal{B}$ contains all cluster monomials and all elements of $\mathcal{B}$ are indecomposable positive elements of $\mathcal{A}(B)$. $\mathcal{B}$ is called the greedy basis.

Here the fact that $\mathcal{B}$ is strongly positive follows from [5], where it is shown that the greedy basis coincides with the theta function basis defined in [7].
2.2. Characterization using support conditions. The following alternative characterization of greedy elements using a support condition (SC) plays an essential role in the construction of greedy bases of rank 2 quantum cluster algebras [12].

**Theorem 2.2.** The coefficients \( c(p,q) \) of \( x[a_1,a_2] \) are determined by:

1. **Normalization condition (NC)**: \( c(0,0) = 1 \).
2. **Divisibility condition (DC)**: if \( a_2 > cq \), then \( (1 + x)^{a_2-cq} \sum_i c(i,q)x^i \).
3. **Support condition (SC)**: \( c(p,q) = 0 \) outside the region given in [12, Figure 1].

Moreover, if \( x[a_1,a_2] \) is a cluster variable then condition (SC) becomes \( c(p,q) = 0 \) outside the closed triangle with vertices \((0,0), (a_2,0), (0,a_1)\), as shown in Figure 1.

The main theorem of this paper, Theorem 5.1, gives a similar characterization for cluster variables for every rank 3 cluster algebra.

3. Preparation

3.1. Definition, notations, and facts in cluster algebras. We recall the definition of skew-symmetrizable cluster algebras with principal coefficient.

A square matrix \( B \) is called skew-symmetrizable if there exists a positive integer diagonal matrix \( D \) such that \( DB \) is skew-symmetric.

Let \( n \) be a positive integer. Let \( \mathcal{T} \) denote the \( n \)-regular tree whose edges are labeled by integers in \( \{1,\ldots,n\} \) so that each vertex is incident on \( n \) edges with distinct labels. The notation \( t \xrightarrow{k} t' \) means that the edge joining \( t \) and \( t' \) is labeled by \( k \).

Denote by \( \mathcal{F} \) the field of rational functions \( \mathbb{Q}(x_1,\ldots,x_{2n}) \). To distinguish between mutable variables \( x_1,\ldots,x_n \) and coefficient variables \( x_{n+1},\ldots,x_{2n} \), we also use the notation \( y_i = x_{n+i} \), for \( i = 1,\ldots,n \). For \( p = (p_1,\ldots,p_n) \in \mathbb{Z}^n \), let \( x^p = x_1^{p_1}\cdots x_n^{p_n} \), \( y^p = y_1^{p_1}\cdots y_n^{p_n} \). For \( \tilde{p} = (p_1,\ldots,p_{2n}) \in \mathbb{Z}^{2n} \), let \( x^\tilde{p} = x_1^{p_1}\cdots x_{2n}^{p_{2n}} \).

Each vertex \( t \in \mathcal{T} \) is attached with a seed \( \Sigma_t = (x(t), \tilde{B}(t)) \) where:

- \( \tilde{B}(t) = [b_{ij}^{(t)}] \) is a \( 2n \times n \) integer matrix such that the submatrix \( B(t) \) formed by the top \( n \) rows of \( \tilde{B}(t) \) is skew-symmetrizable.
- \( x(t) = \{x_1(t),\ldots,x_n(t)\} \) is an \( n \)-tuple of elements of \( \mathcal{F} \).
Theorem 3.1. Proved by Gross-Hacking-Keel-Kontsevich in \[7, \text{Corollary 5.5}\].

For any real number \(a\), let \([a]_+ := \max(a, 0)\). Given a seed \(\Sigma_t = (x(t), \tilde{B}(t))\) and an edge \(t \xrightarrow{k} t’\), we define the mutation of \(\Sigma_t\) to be \(\mu_k(\Sigma_t) = \Sigma_{t'} = (x(t’), \tilde{B}(t’))\), where

\[
b_{ij}^{(t')} = \begin{cases} 
-b_{ij}^{(t)} & \text{if } i = k \text{ or } j = k, \\
b_{ij}^{(t)} + \text{sgn}(b_{ik}^{(t)}\mathbb{b}_{jk}^{(t)}) & \text{otherwise.}
\end{cases}
\]

\[
x_i(t’) = \begin{cases} 
x_k(t)^{-1} \left( \prod_{j=1}^{2n} x_j(t)^{b_{jk}^{(t)}} + \prod_{j=1}^{2n} x_j(t)^{-b_{jk}^{(t)}} \right) & \text{if } i = k, \\
x_i(t) & \text{otherwise.}
\end{cases}
\]

Each \(x_i(t)\) is called a cluster variable. The cluster algebra \(\mathcal{A}\) is the \(\mathbb{Q}^{[x_1^\pm, \ldots, x_{2n}^\pm]}\)-subalgebra of \(\mathcal{F}\) generated by all cluster variables.

For each seed \(\Sigma_t\), let \(C(t)\) be the \(n \times n\) submatrix of \(\tilde{B}(t)\) formed by the bottom \(n\) rows of \(\tilde{B}(t)\). Its columns, \(c_1(t), \ldots, c_n(t)\), are called c-vectors. We need the following theorem proved by Gross-Hacking-Keel-Kontsevich in [7, Corollary 5.5].

**Theorem 3.1.** (Sign-coherence of c-vectors) [7] In a skew-symmetrizable cluster algebra, every c-vector \(c_k^{(t)} = [b_{ik}^{(t)}]_{i=n+1}^{2n}\) is in \(\mathbb{Z}_{\geq 0}^n \cup \mathbb{Z}_{\leq 0}^n\).

The following lemma is known but we could not find a precise reference, so we give a proof here.

**Lemma 3.2.** The determinant of \(C(t)\) is 1 or \(-1\). As a consequence, the c-vectors are linearly independent, and all c-vectors are nonzero.

**Proof.** We prove the statement by induction. The initial \(C\)-matrix is the identity matrix, so its determinant is 1. Now assume the lemma is true for \(t\). Consider an edge \(t \xrightarrow{k} t’\) in \(\mathbb{T}\). We need to show the lemma holds for \(t’\). As observed in [1, (3,2)], for a fixed index \(1 \leq k \leq n\) and a fixed sign \(\varepsilon \in \{\pm 1\}\), the mutation relation for \(\tilde{B}\) is can be rewritten as

\[
\tilde{B}(t’') = (J_{2n,k} + E_k)\tilde{B}(t)(J_{n,k} + F_k)
\]

where \(J_{2n,k}\) (resp. \(J_{n,k}\)) is the diagonal \((2n \times 2n)\)- (resp. \((n \times n)\)-) matrix whose diagonal entries are all 1’s except for \(-1\) in the \(k\)-th position, \(E_k\) is the \((2n \times 2n)\)-matrix whose only nonzero entries are \(\max(0, -\varepsilon b_{ik}^{(t)})\) at \((i, k)\)-entries for all \(i\), \(F_k\) is the \((n \times n)\)-matrix whose only nonzero entries are \(\max(0, \varepsilon b_{ik}^{(t)})\) at \((k, j)\)-entries for all \(j\).

By Theorem 3.1, the c-vector \(c_k = [b_{ik}^{(t)}]_{i=n+1}^{2n}\) is either in \(\mathbb{Z}_{\geq 0}^n \cup \mathbb{Z}_{\leq 0}^n\). We choose \(\varepsilon\) to be 1 (resp. \(-1\)) if the \(k\)-th c-vector \(c_k\) is in \(\mathbb{Z}_{\geq 0}^n\) (resp. \(\mathbb{Z}_{\leq 0}^n\)). Then \(E_k\) is the zero matrix, implying that \(C(t’’) = C(t)(J_{n,k} + F_k)\). Since both \(C(t)\) and \((J_{n,k} + F_k)\) have determinant \(\pm 1\), \(C(t)\) also has determinant \(\pm 1\). \(\Box\)
We also need the following fact, which is shown for skew-symmetric cluster algebras in [6], but we could not find a reference in the skew-symmetrizable setting. A close reference is [11, Lemma 5.1] which describes a similar idea in the skew-symmetric case.

**Lemma 3.3.** The $F$-polynomial of every cluster variable of a skew-symmetrizable cluster algebra has constant term 1.

**Proof.** We prove that the conclusion holds for cluster variables in every seed, by induction on the distance from the current seed to the initial seed in the $n$-regular tree $T$.

The statement is true for the initial seed since the constant term of $\mu_k(t)$ is 1 since $B$ is acyclic.

Assume the conclusion is true for the seed $t$ and unknown for $t' = \mu_k(t)$. The rule of change of $F$-polynomials under mutation is given in [8, Proposition 5.1], where the only $F$-polynomial that changes under mutation $\mu_k$ is:

$$F_k^{(t')} = \frac{y[c_k^{(t)}]^+ \prod_{i=1}^n (F_i^{(t)})[b_i^{(t)}]^+ + y[-c_k^{(t)}]^+ \prod_{i=1}^n (F_i^{(t)})[-b_i^{(t)}]^+}{F_k^{(t)}}$$

By sign-coherence of $c$-vectors (Theorem 3.1), $[c_k^{(t)}]^+ \in \mathbb{Z}_{\geq 0} \cup \mathbb{Z}_{\leq 0}$. Assuming $c_k^{(t)} \in \mathbb{Z}_{\geq 0}$ (the other case can be proved similarly), we have an equality in $\mathbb{Z}[y_1, \ldots, y_n]$:

$$F_k^{(t')} F_k^{(t)} = y[c_k^{(t)}]^+ \prod_{i=1}^n (F_i^{(t)})[b_i^{(t)}]^+ + \prod_{i=1}^n (F_i^{(t)})[-b_i^{(t)}]^+$$

Moreover $y[c_k^{(t)}]^+ \neq 1$ since $c_k^{(t)} \neq 0$. Letting $y_1 = \cdots = y_n = 0$, we immediately conclude that the constant term of $F_k^{(t')}$ is 1. So the conclusion is true for $t'$.

□

For convenience, we introduce simpler notations for rank 3 cluster algebras. Let

$$B = [b_{ij}] = \begin{bmatrix} 0 & a & -c' \\ -a' & 0 & b \\ c & -b' & 0 \end{bmatrix}, \quad \tilde{B} = [b_{ij}] = \begin{bmatrix} 0 & a & -c' \\ -a' & 0 & b \\ c & -b' & 0 \end{bmatrix}$$

The assumption that $\tilde{B}$ be skew-symmetrizable implies the existence of positive integers $\delta_1, \delta_2, \delta_3$ such that $\delta_i b_{ij} = -\delta_j b_{ji}$ for all $i, j$. So we can define

$$\bar{a} := \delta_1 a = \delta_2 a', \quad \bar{b} := \delta_2 b = \delta_3 b', \quad \bar{c} := \delta_3 c = \delta_1 c'$$

thus $DB = \begin{bmatrix} 0 & \bar{a} & -\bar{c} \\ -\bar{a} & 0 & \bar{b} \\ \bar{c} & -\bar{b} & 0 \end{bmatrix}$

We say $B$ is cyclic if $a, b, c$ are either all strictly positive or all strictly negative, otherwise $B$ is acyclic.

Note that $aa', bb', cc' \geq 0$. Denote the $i$-th column of $B$ by $B_i$. Then

$$\bar{b}B_1 + \bar{c}B_2 + \bar{a}B_3 = \mathbf{0}.$$ 

In particular, the vectors $B_1, B_2, B_3 \in \mathbb{R}^3$ are coplanar, which is a fact essential for this paper.

In this paper, we assume the non-degeneracy condition that at most one of $a, b, c$ is zero.
Remark 3.4. In the degenerate case when at least two of $a, b, c$ are zero, some of the proofs in this paper may not longer work. On the other hand, the degenerate case is essentially a rank 2 cluster algebra and our main theorems follows directly from [12]. Note that if the initial $B$-matrix satisfies the non-degeneracy condition, then all the $B$-matrices obtained by mutations satisfy the non-degeneracy condition.

3.2. Circular order.

Definition 3.5. We say a sequence of coplanar vectors $v_1, \ldots, v_n$ is in circular order if there is an $\mathbb{R}$-linear isomorphism $\phi$ from a plane containing these vectors to the complex plane $\mathbb{C}$ such that

\[ \phi(v_k) = r_k e^{\sqrt{-1} \theta_k}, \quad r_k \geq 0 \quad (1 \leq k \leq n), \quad \text{and} \quad \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n \leq \theta_1 + 2\pi. \]

We introduce the following notation. Given $(d_1, d_2, d_3) \in \mathbb{Z}^3$, define

\[ (3.1) \quad v_i = d_i B_i \quad (\text{for } i = 1, 2, 3) \quad \text{and} \quad v_4 = -v_1 - v_2 - v_3. \]

The following easy observation is very useful for proving the circular order condition.

Lemma 3.6. Assume $(d_1, d_2, d_3) \in \mathbb{Z}^3 \setminus \{(0, 0, 0)\}$. Then $B_1, B_2, B_3, v_4$ is in circular order if and only if the following conditions hold:

1. If $B_1, B_2, B_3$ are not in the same half plane, then $v_4 = \lambda_1 B_1 + \lambda_2 B_2$ for some $\lambda_1, \lambda_2 \geq 0$.
2. If $B_1, B_2, B_3$ are strictly in the same half plane (so no two are in opposite directions), then $B_j = \eta_j B_i + \eta_k B_k$ for some $\eta_1, \eta_2 \geq 0$. In particular, if two of $B_1, B_2, B_3$ are in the same direction, then one of them is $B_j$.
3. If two of $B_1, B_2, B_3$ are in opposite directions, then either $B_i, B_k$ are in opposite directions, or $B_i, B_j$ are in opposite directions, $d_k = 0$, and $v_4, B_i$ are in the same direction”, or “$B_j, B_k$ are in opposite directions, $d_i = 0$, and $v_4, B_k$ are in the same direction”.

Proof. This is an easy observation using Figure 2 as reference. \qed

3.3. Weakly convex quadrilaterals.

Definition 3.7. Assume four points $P_1, P_2, P_3, P_4 \in \mathbb{R}^3$, not necessarily distinct, are coplanar. We call the polygon $P = P_1 P_2 P_3 P_4$ a weakly convex quadrilateral if the four vectors $\overrightarrow{P_1 P_2}, \overrightarrow{P_2 P_3}, \overrightarrow{P_3 P_4}, \overrightarrow{P_4 P_1}$ are in circular order.

We use convention that $P_{i+4k} = P_i$ for $1 \leq i \leq 4$ and $k \in \mathbb{Z}$.

If $P = P_1 P_2 P_3 P_4$ is a weakly convex quadrilateral, we denote by $|P| \subset \mathbb{R}^3$ the convex hull of $\{P_1, P_2, P_3, P_4\}$.

Remark 3.8. By definition, $|P|$ is just a bounded convex subset of a real plane inside $\mathbb{R}^3$, while $P$ “remembers” four points in the polygon which are not necessarily distinct and not necessarily the vertices of the polygon. Nevertheless, the set of vertices of $|P|$ is a subset of $\{P_1, \ldots, P_4\}$. See Figure 3 for some examples of $P$. When we talk about the physical features of a weakly convex quadrilateral $P$, where we do not have to pay attention to the four special points $P_1, \ldots, P_4$ of $P$, we would for simplicity identify $P$ with $|P|$, the underlying convex set. This applies to the phrases like “a point is contained in $P$”, “the Newton polytope of a cluster variable is $P$”, “$P$ is a line segment”, “$P$ is a triangle”, or “$\dim P = 1$”. If we need to use the actual coordinates of the points $P_1, \ldots, P_4$, then we distinguish $P$ from $|P|$. This includes Lemma 3.9, Lemma 3.11, and Section 6.3.
In the following, we shall give a more explicit description of weakly convex quadrilaterals. Recall that $\mathbf{P}$ is a (usual) convex quadrilateral if the four vectors $\overrightarrow{P_1P_2}$, $\overrightarrow{P_2P_3}$, $\overrightarrow{P_3P_4}$, $\overrightarrow{P_4P_1}$ are in circular order, all nonzero, and that no two are in the same direction. In terms of complex numbers, that is to say: if there is an $\mathbb{R}$-linear isomorphism $\phi$ such that
\[ \phi(\overrightarrow{P_kP_{k+1}}) = r_k e^{\sqrt{-1} \theta_k}, \quad r_k > 0 \quad (1 \leq k \leq 4), \quad \text{and} \quad \theta_1 < \theta_2 < \theta_3 < \theta_4 < \theta_1 + 2\pi. \]

**Lemma 3.9.** Let $\mathbf{P} = P_1P_2P_3P_4$. The following are equivalent:
1. $\mathbf{P}$ is weakly convex.
2. $\mathbf{P}$ is the limit of a sequence of convex quadrilaterals.
3. $\mathbf{P}$ is one of the following:
   - $\dim \mathbf{P} = 0$ (that is, $P_1 = P_2 = P_3 = P_4$, so $\mathbf{P}$ degenerates to a point);
   - $\dim \mathbf{P} = 1$, $\mathbf{P}$ is a line segment $P_iP_{i+3}$ and $P_i, P_{i+1}, P_{i+2}, P_{i+3}$ are arranged in order, for some $1 \leq i \leq 4$;
   - $\dim \mathbf{P} = 1$, $\mathbf{P}$ is a line segment $P_iP_{i+2}$ which contains $P_{i+1}$ and $P_{i+3}$, for some $1 \leq i \leq 4$;
   - $\dim \mathbf{P} = 2$, $\mathbf{P}$ is a triangle $P_iP_{i+1}P_{i+2}$ whose side $P_{i+2}P_i$ contains the point $P_{i+3}$, for some $1 \leq i \leq 4$;
   - $\dim \mathbf{P} = 2$, $P_1P_2P_3P_4$ is a (usual) convex quadrilateral.

**Proof.** (1)$\iff$(2): Assume the sequence of $\mathbf{P}^{(j)} = P_1^{(j)}P_2^{(j)}P_3^{(j)}P_4^{(j)}$ has limit $\mathbf{P}$ and
\[ \phi(\overrightarrow{P_k^{(j)}P_{k+1}^{(j)}}) = r_k^{(j)} e^{\sqrt{-1} \theta_k^{(j)}}, \quad r_k^{(j)} > 0 \quad (1 \leq k \leq 4), \quad \text{and} \quad \theta_1^{(j)} < \theta_2^{(j)} < \theta_3^{(j)} < \theta_4^{(j)} < \theta_1^{(j)} + 2\pi. \]

By choosing appropriate angles $\theta_k^{(j)}$ and replace the sequence by a subsequence if necessary, we can assume
\[ \lim_{j \to \infty} r_k^{(j)} = r_k, \quad \lim_{j \to \infty} \theta_k^{(j)} = \theta_k. \]
Then by the property of limits we conclude that \( r_k \geq 0 \) and \( \theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4 \leq \theta_1 + 2\pi \). So \( P_1 P_2, \ldots, P_4 P_1 \) are in circular order.

(2)\(\Leftarrow\)(3): Obvious from the Figure 3.

\[ P_1 = \cdots = P_4 \]

**Figure 3.** Bottom figures are the limit of the corresponding top convex quadrilaterals for the five cases in Lemma 3.9 (3)

(1)\(\Rightarrow\)(3): we show the contrapositive. If \( P \) is not listed in (3), then it must be one of those listed below, all of which are obviously not weakly convex (see Figure 4).

- \( \dim P = 1 \), \( P \) is a line segment \( P_i P_{i+3} \) and \( P_i, P_{i+2}, P_{i+1}, P_{i+3} \) are arranged in order, for some \( 1 \leq i \leq 4 \);
- \( \dim P = 2 \), \( P \) is a triangle \( P_i P_{i+1} P_{i+2} \) for some \( 1 \leq i \leq 4 \), and the line segment \( P_{i+2} P_i \) does not contain the point \( P_{i+3} \).

\[ P_i \]

**Figure 4.** Quadrilaterals that are not weakly convex

**Definition 3.10.** Given \( r \) vectors \( P_1, P_2, \ldots, P_r \) in \( \mathbb{R}^3 \) with coordinates \( P_i = \left[ \begin{array}{c} p_{i1} \\ p_{i2} \\ p_{i3} \end{array} \right] \), define their minimum vector by

\[ \overrightarrow{\text{min}}(P_1, \ldots, P_r) = \left[ \begin{array}{c} m_1 \\ m_2 \\ m_3 \end{array} \right], \quad \text{with } m_i = \min(p_{1,i}, p_{2,i}, \ldots, p_{r,i}) \]

For example, if \( P_1 = \left[ \begin{array}{c} 1/3 \\ 2/3 \end{array} \right] \) and \( P_2 = \left[ \begin{array}{c} 3/6 \\ 4/6 \end{array} \right] \), then \( \overrightarrow{\text{min}}(P_1, P_2) = \left[ \begin{array}{c} 1/3 \\ 1/6 \end{array} \right] \).
3.4. Weakly convex quadrilaterals in rank 3 cluster algebras. We are going to construct a weakly convex quadrilateral $P_d$ for all positive integer vectors $d \in \mathbb{Z}^3$. Later the vector $d$ will be the denominator vector of a cluster variable. For the initial denominator vectors $d \in \{[-1, 0, 0], [0, -1, 0], [0, 0, -1]\}$, we let $P_d$ be the degenerate quadrilateral consisting of the point $d$. For all other $d$ we have the following lemma. Recall that $v_i = d_i B_i$.

Lemma 3.11. For all $d \in \mathbb{Z}_0^3 \setminus (0,0,0)$ there exists a weakly convex quadrilateral $P_d = P_1 P_2 P_3 P_4$ (We use $P_d$ when there is no confusion of which matrix $B$ we refer to.) such that

1. There is a permutation $(i,j,k)$ of $(1,2,3)$ such that
   1a) $b_{ij} \geq 0, b_{jk} \geq 0$.
   1b) $\overrightarrow{P_1P_2} = v_i, \overrightarrow{P_2P_3} = v_j, \overrightarrow{P_3P_4} = v_k$.
   1c) the four vectors $B_i, B_j, B_k, v_4 = -v_i - v_j - v_k$ are in circular order.

2. $\min(P_1, P_2, P_3, P_4) = -d$.

Moreover the convex hull $|P_d|$ of $P_d$ is unique.

Proof. We define $P_d = P_1 P_2 P_3 P_4$ as follows. For an illustration see Example 7.1. First define a quadrilateral $\tilde{P}$ by the vertices $\tilde{P}_1 = (0,0,0), \tilde{P}_2 = \tilde{P}_1 + v_i, \tilde{P}_3 = \tilde{P}_2 + v_j, \tilde{P}_4 = \tilde{P}_3 + v_k$. Clearly this quadrilateral satisfies condition (1b), but it does not necessarily satisfy condition (2). Let $d' = -\min(\tilde{P}_1, \tilde{P}_2, \tilde{P}_3, \tilde{P}_4)$. Then define the quadrilateral $P_d$ as the translation of the quadrilateral $\tilde{P}$ by $d' - d$. Then $P_d$ satisfies conditions (1b) and (2).

Also note that condition (1c) implies the weaker condition that the vectors $\overrightarrow{P_1P_2} = v_i, \overrightarrow{P_2P_3} = v_j, \overrightarrow{P_3P_4} = v_k, \overrightarrow{P_4P_1} = v_4$ are in circular order. Thus (1c) implies that the quadrilateral is weakly convex.

Thus it remains to show conditions (1a) and (1c). We prove these in five separate cases. See Figures 5–9.

(Case 1) Suppose $Q$ is acyclic and $abc \neq 0$. We may assume without loss of generality that $a, b > 0$, and $c < 0$. Let $(i,j,k) = (1,2,3)$. Thus condition (1a) holds. Furthermore, in this case, $B_2 = (-b/c)B_1 + (-a/c)B_3$ is a positive linear combination of $B_1$ and $B_3$. Hence all three vectors $B_1, B_2, B_3$ lie in the same half plane. Thus Lemma 3.6(b) implies that $B_1, B_2, B_3, v_4$ are in circular order. This proves (1c).

![Figure 5](Case 1)

(Case 2) Suppose $Q$ is acyclic, one of $a, b, c$ is zero and the other two have the same sign. That is, $Q$ has two arrows forming a length-2 directed path. Without loss of generality, we may assume that $Q$ is $1 \rightarrow 2 \rightarrow 3$, that is, $a, b > 0$ and $c = 0$. The vectors $B_1$ and $B_3$ are in opposite directions.
If $d_2 > 0$, we let $(i, j, k) = (1, 2, 3)$. Then condition (1a) holds and condition (1c) holds by Lemma 3.6(c). Note that condition (1a) would also hold for $(i, j, k) = (3, 1, 2)$, or $(2, 3, 1)$; however, condition (1c) would fail for both. Thus the permutation of $(i, j, k)$, and hence the quadrilateral $P_d$, are unique in this case.

If $d_2 = 0$, conditions (1a) and (1c) hold for $(i, j, k) = (1, 2, 3)$ and $(3, 1, 2)$. In both cases, the quadrilateral degenerates to a line segment in the direction of $B_1$, of length $\max(|v_1|, |v_3|)$. The case $(1,2,3)$ is illustrated in the second picture of Figure 6. In particular, even though $P_d$ may not be uniquely determined by the conditions of the lemma, the convex hull $|P_d|$ is unique.

**Figure 6.** (Case 2)

(Case 3) Suppose $Q$ is acyclic, one of $a, b, c$ is zero and the other two have the opposite sign. Then exactly one vertex is adjacent to both the other two, and this vertex is either a sink or a source.

(Case 3a) If this vertex is a sink. Without loss of generality, assume $Q$ is $1 \rightarrow 3 \leftarrow 2$, that is, $a = 0$, $b > 0 > c$. The vectors $B_1$ and $B_2$ are in the same direction. Condition (1a) is satisfied for the two permutations $(i, j, k) = (1, 2, 3)$ or $(2, 1, 3)$, and both satisfy condition (1c) by Lemma 3.6(a). Both cases give the same $|P_d|$, which is a triangle with edges $v_1 + v_2, v_3, v_4$, in that order.

(Case 3b) If this vertex is a source. Without loss of generality, assume $Q$ is $2 \leftarrow 1 \rightarrow 3$, that is, $b = 0$, $a > 0 > c$. The vectors $B_2$ and $B_3$ are in the same direction. Condition (1a) is satisfied for the two permutations $(i, j, k) = (1, 2, 3)$ or $(1, 3, 2)$, and both satisfy condition (1c) by Lemma 3.6(a). Both cases give the same $|P_d|$, which is a triangle with edges $v_1, v_2 + v_3, v_4$, in that order.
(Case 4) Suppose $Q$ is cyclic. Without loss of generality we may assume $a, b, c > 0$. Condition (1a) narrows down the choices of $(i, j, k)$ to $(1, 2, 3), (2, 3, 1),$ or $(3, 1, 2)$.
If $v_4 = 0$, then for all of the above three choices of $(i, j, k)$, we get the same $|P|$ which is a triangle with edges $v_1, v_2, v_3$, in that order.

In the following we assume $v_4 \neq 0$. If $v_4$ is strictly between $v_1$ and $v_2$ (respectively, $v_2$ and $v_3$, $v_3$ and $v_1$), then the circular order condition implies the unique choice $(i, j, k) = (2, 3, 1)$ (resp. $(3, 1, 2), (1, 2, 3)$). If $v_4$ is in the same direction as $v_1$, then $(i, j, k) = (1, 2, 3)$ or $(2, 3, 1)$. But they give the same quadrilateral which degenerates to a (possibly degenerated) triangle with edges $v_1 + v_4, v_2, v_3$, in that order. Similar argument holds for $v_4$ being in the same direction as $v_2$ or $v_3$.

\[\square\]

3.5. A substitution lemma. In the following lemma, we describe the effect of replacing the variable $x_i$ by its mutation $x_i'$. We use the following notation.

$$
\alpha_1 = \begin{bmatrix}
-1 & 0 & 0 \\
[\alpha']_+ & 1 & 0 \\
[-c]_+ & 0 & 1 
\end{bmatrix}
\quad \alpha_2 = \begin{bmatrix}
1 & [-a]_+ & 0 \\
0 & -1 & 0 \\
0 & [b]_+ & 1 
\end{bmatrix}
\quad \alpha_3 = \begin{bmatrix}
1 & 0 & [c]_+ \\
0 & 1 & [-b]_+ \\
0 & 0 & -1 
\end{bmatrix}
$$

$$
\beta_1 = \begin{bmatrix}
-1 & 0 & 0 \\
[-a']_+ & 1 & 0 \\
[c]_+ & 0 & 1 
\end{bmatrix}
\quad \beta_2 = \begin{bmatrix}
1 & [a]_+ & 0 \\
0 & -1 & 0 \\
0 & [-b']_+ & 1 
\end{bmatrix}
\quad \beta_3 = \begin{bmatrix}
1 & 0 & [-c']_+ \\
0 & 1 & [b]_+ \\
0 & 0 & -1 
\end{bmatrix}
$$

Lemma 3.12. Let $p = (p_1, p_2, p_3)$ and $r \geq 0$. For $i = 1, 2, 3$ let $f_i$ be a Laurent polynomial of the form

$$
f_i = a_{i0}x^{p+b_0B_i} + a_{i1}x^{p+b_1B_i} + \ldots + a_{in}x^{p+b_nB_i} = \sum_{j=0}^{n} a_{ij}x^{p+b_jB_i},
$$

where $a_0, \ldots, a_n \in \mathbb{Q}^\mathbb{P}$, $0 = b_0 < b_1 < \ldots < b_n = r$, so that the exponents in $f_i$ are points on the line segment from $p$ to $p + rB_i =: q_i$.

Let $g_i$ be the rational function obtained from $f_i$ by substituting

$$
x_1 \text{ by } (p^a x_1^{3} + p^b x_2^{3} + p^c x_3^{3})/x_1 \quad \text{if } i = 1;
$$

$$
x_2 \text{ by } (p^a x_1^{b} + p^b x_2^{b} + p^c x_3^{b})/x_2 \quad \text{if } i = 2;
$$

$$
x_3 \text{ by } (p^a x_1^{c} + p^b x_2^{c} + p^c x_3^{c})/x_3 \quad \text{if } i = 3.
$$

where $p^-, p^+ \in \mathbb{P}$.

If $g_i$ is a Laurent polynomial, then

$$
g_i = a'_{i0}x^{p' + b'_0B_i} + a'_{i1}x^{p' + b'_1B_i} + \ldots + a'_{in}x^{p' + b'_nB_i} = \sum_{j=0}^{n} a'_{ij}x^{p' + b'_jB_i},
$$

where $0 = b'_0 < b'_1 < \ldots < b'_n = r'$, so that the exponents in $g_i$ are points on the line segment from $p'$ to $p' + r'B_i =: q'$. Moreover $r' = r + p_i$, $a'_{0} = (p^+)a_0$, $a'_{n} = (p^-a_n$, and

$$
p' = \alpha_i(p), \quad q' = \beta_i(q).
$$

Proof. By symmetry, it suffices to show the case $i = 1$. We simply write $f, g$ instead of $f_1, g_1$. First assume $a \geq 0$ and $c \leq 0$. Then
\[ f = \sum_{j=0}^{n} a_j x^{p_j} b_j = x^P \sum_{j=0}^{n} a_j x^{b_j} B_1 \]

Note that the last sum does not depend on \( x_1 \) since \( B_1 = \left[ \begin{smallmatrix} 0 \\ -a' \end{smallmatrix} \right] \). Therefore

\[
g = \left[ x_1^{-1} x_2^a x_3^{-c} (p^+ + p^+ B_1) \right] x_1^{p_1} x_2^{p_2} x_3^{p_3} \sum_{j=0}^{n} a_j x^{b_j} B_1
\]

\[ = x_1^{-p_1} x_2^{p_2+a_1} x_3^{p_3-c_1} (p^+ + p^+ B_1) \sum_{j=0}^{n} a_j x^{b_j} B_1 \]

\[ = x_1^{-p_1} x_2^{p_2+a_1} x_3^{p_3-c_1} (p^- p_0 + \ldots + (p^+) a_n x^{(p^+ B_1)}. \]

So \( r' = r + p_1, a'_0 = (p^- p_0 a_0, a'_n = (p^+) a_n, \) and \( p' = (-p_1, p_2 + a_1 p_1, p_3 - c_1) = \alpha_1(p) \).

Moreover, since \( q = p + r B_1 \), we have \( q' = (p, p_2 - a_1 r, p_3 + c r) = (-q_1, q_2, q_3) = \beta_1(q) \).

This completes the proof in the case \( a \geq 0 \) and \( c \leq 0 \).

The remaining cases \( "c \geq 0, a \leq 0", "a, c \leq 0", "a, c \geq 0" \) are proved similarly.

3.6. A Newton polytope change lemma. The following lemma describes how the Newton polytopes change under mutation. It will be used in the proof of the support condition in our main result Theorem 5.1.

Lemma 3.13. Assume that \( F \) is a Laurent polynomial in \( x_1, x_2, x_3 \), and after substituting

\[ x_1 \mapsto \frac{(p^+ x_2^{a_1} x_3^{c_1} + p^- x_2^{a_2} x_3^{c_2})}{x_1^{r'}} \]

in \( F \), we get a Laurent polynomial \( G \) in \( x_1', x_2, x_3 \). Assume the Newton polytope \( R \) of a Laurent polynomial \( F \) lies in a plane \( S \) parallel to the plane span(\( B_1, B_2, B_3 \)), and \( R \) satisfies the following condition:

\( R \) is bounded by two (possibly length zero) line segments \( q_1q_2, q_3q_4 \), as well as an “inflow” boundary \( T_{in} \) joining \( q_1 \) and \( q_3 \), and an “outflow” boundary \( T_{out} \) joining \( q_2 \) and \( q_4 \). Each line \( \ell \) parallel to \( B_1 \) intersect at most once with \( T_{in} \) and at most once with \( T_{out} \), and \( \ell \) intersects \( T_{in} \) if and only if it intersects \( T_{out} \). So it induces a bijection \( \phi : T_{in} \to T_{out} \). Moreover, \( \phi(q_1) = q_2 \), \( \phi(q_3) = q_4 \), and for all \( t \in T_{in}, \phi(t) \in t + \mathbb{R}_{\geq 0} B_1 \).

Now define

\[ q_1' = \beta_1(q_2), \quad q_2' = \alpha_1(q_1), \quad q_3' = \beta_1(q_4), \quad q_4' = \alpha_1(q_3), \]

\[ T'_{out} = \alpha_1(T_{in}), \quad T'_{in} = \beta_1(T_{out}). \]

Then the convex hull of the support of \( G \) is the region \( R' \subset S' \), where \( S' \) is a plane parallel to \( \text{span}(B'_1, B'_2, B'_3) \), such that the following condition holds:

\( R' \) is bounded by two line segments \( q'_1q'_2, q'_3q'_4 \), as well as an “inflow” boundary \( T'_{in} \) joining \( q'_1 \) and \( q'_3 \), and an “outflow” boundary \( T'_{out} \) joining \( q'_2 \) and \( q'_4 \). Each line \( \ell \) parallel to \( B'_1 \) intersect at most once with \( T'_{in} \) and at most once with \( T'_{out} \), and \( \ell \) intersects \( T'_{in} \) if and only if it intersects \( T'_{out} \). So it induces a bijection \( \phi' : T'_{in} \to T'_{out} \). Moreover, \( \phi'(q'_1) = q'_2, \)

\( \phi(q'_3) = q'_4 \), and for all \( t \in T'_{in}, \phi'(t) \in t + \mathbb{R}_{\geq 0} B'_1 \).
Proof. First note that by the linearity of $\alpha_1$ and $\beta_1$, $R$ is convex if and only if $R'$ is convex.

Denote $F = \sum e(p_1, p_2, p_3)x_1^{p_1}x_2^{p_2}x_3^{p_3}$. For a fixed integer $t$, let

$$F_t = \sum e(t, p_2, p_3)x_1^{p_2}x_2^{p_3} = a_0x^p + \cdots + a_nx^q, \quad q = p + rB_1$$

and let

$$G_{-t} = a_0x^{p'} + \cdots + a_nx^{q'}, \quad q' = p' + r'B_1$$

be obtained from $F_t$ by substitution (3.2). It suffices to show that if the support of $F_t$ (which is the segment $pq$) is in $R$ if and only if the support of $G_{-t}$ (which is the segment $p'q'$) is in $R'$. See Figure 10.

Assume the line through $p$ parallel to $B_1$ intersects with $T_{in}$ and $T_{out}$ at $p_0$ and $q_0$, respectively. Assume the line through $p'$ parallel to $B_1$ intersects with $T'_{out}$ and $T'_{in}$ at $p'_0$ and $q'_0$, respectively. Then

$$\alpha_1(p) = p', \quad \alpha_1(p_0) = p'_0, \quad \beta_1(q) = q', \quad \beta_1(q_0) = q'_0.$$ 

It suffices to show

(i) $p \in p_0 + \mathbb{R}_{>0}B_1$ if and only if $p' \in p'_0 + \mathbb{R}_{>0}B_1$.

(ii) $q \in q_0 + \mathbb{R}_{\leq 0}B_1$ if and only if $q' \in q'_0 + \mathbb{R}_{\leq 0}B_1$.

Indeed, For (i): since $\alpha_1$ is linear and fixes $B_1$, and $p' - p_0 \in \mathbb{R}B_1$, we see that $p' - p_0 = \alpha_1(p - p_0) = p - p_0$. This implies (i). And (ii) can be proved similarly.

4. Denominator vectors of non-initial cluster variables are non-negative

If $f$ is an element of the ambient field we shall use the notation $f|t$ for the expansion of $f$ in the variables in the seed $\Sigma_t$. For $t = t_0$, we simply denote $f|t_0$ by $f$.

Recall that the $d$-vector of a cluster variable $z$ is $d \in \mathbb{Z}^n$ such that

$$z = \frac{N(x_1, \ldots, x_n)}{x^d}$$

where $N(x_1, \ldots, x_n)$ is a polynomial with coefficients in $\mathbb{Z}[y_i^{\pm}]$ which is not divisible by any cluster variable $x_i$ ($1 \leq i \leq n$). Equivalently, we can describe $d$ as follows. Write $z$ as a sum of Laurent monomials as $z = \sum_{p \in \mathbb{Z}^n} e(p)x^p$, and define the support of $z$ as the set

$$\text{supp}(z) = \{p \mid e(p) \neq 0\}.$$
Let $P_1, \ldots, P_m$ be the vertices of the convex hull of $\text{supp}(z)$. Then
\begin{equation}
\mathbf{d} = -\min_{\mathbf{p} \in \text{supp}(z)} \langle \mathbf{p} \rangle = -\min(P_1, \ldots, P_m)
\end{equation}

It was conjectured in [9, Conjecture 7.4 (1)] that the d-vector of any non-initial cluster variable is nonnegative, and this conjecture was proved recently in [3] using positivity. Below, we give an alternative short proof by an elementary argument.

**Theorem 4.1.** Let $\mathcal{A}$ be a skew-symmetrizable cluster algebra of arbitrary rank. The d-vector of any non-initial cluster variable is nonnegative.

For the proof of the theorem we need the following lemma. For skew-symmetric cluster algebras, this lemma was proved in [4, Lemma 3.7]. We say that a rank of any non-initial cluster variable is nonnegative.

**Lemma 4.2.** Let $\mathcal{A}$ be a skew-symmetrizable cluster algebra of arbitrary rank, without isolated mutable variables. If a cluster variable is of the form $cx^{\tilde{a}}$, where $c \in \mathbb{Q} \setminus \{0\}$ and $\tilde{a} = (a_1, \ldots, a_{2n})$, then
\begin{enumerate}
  
  \item $a_1, \ldots, a_n$ are all nonnegative.

  \item $c = 1$ and $\tilde{a} = e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (with 1 at the $i$-th coordinate) for some $1 \leq i \leq n$.

\end{enumerate}

As a conclusion, the Laurent expansion (in $\mathbb{Z}[x_1^\pm, \ldots, x_{2n}^\pm]$) of a non-initial cluster variable has more than one term.

**Proof.** (1) If false, we assume without loss of generality that $a_1 < 0$. Then expanding $cx^{\tilde{a}}$ in the seed $\mu_1(\Sigma_0)$, we get $x_1^{\mu_1-a_1}cx^{(0,0,\ldots,0)}(M_1 + M_2)^{-a_1}$, for some monomials $M_1 \neq M_2$, and this cannot be a Laurent polynomial.

(2) Apparently $c$ is a nonzero integer, by the Laurent phenomenon [8]. Since all cluster variables can be written as subtraction-free expressions, by specializing the initial variables $x_1 = \cdots = x_{2n} = 1$, we see that $c$ is positive. Next, choose any seed $\Sigma_i$ that contains the cluster variable $cx^{\tilde{a}}$; denote the cluster of this seed by $\{x_1' = cx^{\tilde{a}}, x_2', \ldots, x_n'\}$. For $i = 1, \ldots, n$, let $f_i$ be the Laurent expansion of $x_i$ in $\{x_1' = cx^{\tilde{a}}, x_2', \ldots, x_n', x_{n+1}', \ldots, x_{2n}'\}$ (so $f_i = x_i$, only writing in the Laurent expansion form to remind us). Then
\begin{equation}
x_1' = cx^{\tilde{a}} = cf_1^{a_1} \cdots f_n^{a_n} x_{n+1}' x_{n+2}' \cdots x_{2n}'
\end{equation}

This has the following two consequences.

(a) $c = 1$. Indeed, substituting $x_1' = \cdots = x_n' = x_{n+1} = \cdots = x_{2n} = 1$, we get $1 = c \prod_{i=1}^n f_i^{a_i}(x_1' = \cdots = x_{2n} = 1)$. Since all factors in the right hand side are positive integers, we must have $c = 1$, and $f_i(x_1' = \cdots = x_{2n} = 1) = 1$ for every $1 \leq i \leq n$ with $a_i > 0$.

(b) For every $1 \leq i \leq n$ such that $a_i > 0$, the Laurent expansion $f_i$ must be a Laurent monomial in $\mathbb{Z}[x_1^\pm, \ldots, x_n^\pm, x_{n+1}^\pm, \ldots, x_{2n}^\pm]$ with coefficient 1. To see this, first observe that this Laurent expansion cannot have more than one term, otherwise the right hand side of (4.2) must have more than one term, so cannot equal to $x_1'$, a contradiction. So we can write $f_i = ux^{\tilde{b}(i)}$ for some $u \in \mathbb{Z}$ and $\tilde{b}(i) \in \mathbb{Z}^{2n}$. Next, since $f_i(x_1' = \cdots = x_{2n} = 1) = 1$ as in the proof of (a), we must have $u = 1$. This proves (b).
Now combine (b) and part (1), we see that for every \( 1 \leq i \leq n \) such that \( a_i > 0 \), we must have \( f_i = x^\mathbf{b}^{(i)} \), where \( \mathbf{b}^{(i)} \in \mathbb{Z}^n_{>0} \times \mathbb{Z}^n \).

Thus

\[
(4.3) \quad x_1' = f_1^{a_1} \cdots f_n^{a_n} x_{n+1}^{a_{n+1}} \cdots x_{2n}^{a_{2n}} = \left( \prod_{1 \leq i \leq n, a_i > 0} x^{t(a_i \mathbf{b}^{(i)})} \right) x_{n+1}^{a_{n+1}} \cdots x_{2n}^{a_{2n}}.
\]

Since \( x_1', \ldots, x_n, x_{n+1}, \ldots, x_{2n} \) are algebraically independent, the exponents on both sides of (4.3) must match, that is,

\[
(1, 0, \ldots, 0) = \left( \sum_{1 \leq i \leq n} a_i \mathbf{b}^{(i)} \right) + (0, \ldots, 0, a_{n+1}, \ldots, a_{2n}).
\]

Only looking at the first \( n \) coordinates of the above equality, and letting \( \mathbf{b}^{(i)} \in \mathbb{Z}^n \) be the first \( n \) coordinates of \( \mathbf{b}^{(i)} \), we have

\[
(4.4) \quad (1, 0, \ldots, 0) = \left( \sum_{1 \leq i \leq n} a_i \mathbf{b}^{(i)} \right).
\]

Next we observe that \( \mathbf{b}^{(i)} \neq (0, \ldots, 0) \), since otherwise, \( x_i = f_i \in \mathbb{Z}[x_{n+1}^\pm, \ldots, x_{2n}^\pm] \), which contradicts the assumption that \( x_1, \ldots, x_{2n} \) are algebraically independent. This observation together with (4.4) implies that there is exactly one \( 1 \leq i \leq n \) such that \( a_i > 0 \), and for this \( i \) we have \( a_i = 1 \) and \( \mathbf{b}^{(i)} = (1, 0, \ldots, 0) \). Then \( x_1' = cx^\mathbf{a} = x_i x_{n+1}^{a_{n+1}} \cdots x_{2n}^{a_{2n}} \), so its \( F \)-polynomial is \( x_{n+1}^{a_{n+1}} \cdots x_{2n}^{a_{2n}} \). On the other hand, by [8, Proposition 5.2], the \( F \)-polynomial is not divisible by any of \( x_{n+1}, \ldots, x_{2n} \). This forces \( a_{n+1} = \cdots = a_{2n} = 0 \), therefore \( x_1' = x_i \) is indeed an initial cluster variable, and \( \mathbf{a} = e_i \).

\[ \square \]

**Proof of Theorem 4.1.** By [9, (7.7)], \( d \)-vectors do not depend on the coefficients. So we assume the cluster algebra \( \mathcal{A} \) has principal coefficient, with rank \( n \). In particular, there is no isolated mutable variable.

Assume that \( x[\mathbf{d}] \) is a non-initial cluster variable of \( \mathcal{A} \), and \( d_k < 0 \) for some \( 1 \leq k \leq n \). Write

\[
(4.5) \quad x[\mathbf{d}] = x_k^{-d_k} f.
\]

Throughout the proof we use the notation \( f|_{\mu_i(t_0)} \) for the expansion of \( f \) in the cluster obtained from the initial cluster by mutation in direction \( i \). In other words, \( f|_{\mu_i(t_0)} \) is obtained from \( f \) by replacing \( x_i \) by an expression of the form \((M_1 + M_2)/x_i\), where \( M_1, M_2 \) are monomials. We claim that, \( f \) and \( f|_{\mu_i(t_0)} \) (for every \( i = 1, \ldots, n \)) are Laurent polynomials; that is, \( f \) is in the upper bound \( \mathcal{U} \) of \( \mathcal{A} \) associated with the initial seed (see [1, Definition 1.1]). Indeed, \( f = x_k^{d_k} x[\mathbf{d}] \) is Laurent because it is a product of two Laurent polynomials. For the same reason, \( f|_{\mu_i(t_0)} = x_k^{d_k} (x[\mathbf{d}]|_{\mu_i(t_0)}) \) is also Laurent for \( i \neq k \), and \( f|_{\mu_k(t_0)} \) is Laurent because \( f \) does not contain negative powers of \( x_k \), that is, \( f = \sum_{d \geq 0} x_k^d h_d \) where \( h_d \) is a Laurent polynomial in \( x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \), thus substituting \( x_k \) by (binomial)/(monomial) still gives a Laurent polynomial.

Since our cluster algebra has principal coefficients, the matrix \( \mathbf{B} \) is of full rank, and therefore [1, Corollary 1.9] implies that the upper bound \( \mathcal{U} \) is equal to the upper cluster algebra \( \mathcal{A} \).
Let $\Sigma_t = (x'_1, \ldots, x'_n, \tilde{B}')$ be a seed that contains $x[d] = x'_\ell$ with $1 \leq \ell \leq n$. Rewriting (4.5) using $\Sigma_t$ as the initial seed, we get

$$x'_\ell = (x_k|_t)^{-d_k}(f|_t)$$

(4.6)

Since $f$ is in the upper cluster algebra $\mathcal{A}$, $f|_t$ is Laurent in $x'_1, \ldots, x'_n, x_{n+1}, \ldots, x_{2n}$.

We assert that $x_k|_t$ is equal to some $x'_\ell$. Otherwise, Lemma 4.2 implies that the Laurent expansion of $x_k|_t$ in the seed $\Sigma_t$ must have more than one term, so cannot be equal to the left hand side, which leads to a contradiction.

If $i \neq \ell$, then (4.6) gives $f|_t = x'_\ell/(x'_i)^{-d_k}$. But this cannot be in the upper cluster algebra. In fact, even $x'_\ell/x'_i$ is not in the upper cluster algebra, because if we rewrite it using the seed $\mu_i(\Sigma_t)$, then $x'_i$ is replaced by $(M_1 + M_2)/x'_i''$ whose numerator is some binomial, and it is obvious that $x'_i x''_i/(M_1 + M_2)$ is not a Laurent polynomial in $\mu_i(t)$. So we get a contradiction.

Therefore $i = \ell$ and (4.6) gives $f|_t = (x'_\ell)^{1+d_k}$, where $d_k < 0$. If $d_k \leq -2$, then a similar argument as above gives a contradiction. So $d_k = -1$, and $f = 1$, thus $x[d] = x_k$ is an initial cluster variable, contradicting the assumption. □

5. Main Theorem

In this section we state our main result. It gives a characterization of the cluster variables of an arbitrary rank 3 cluster algebra in terms of support, normalization and divisibility conditions.

**Theorem 5.1.** Let $\mathcal{A}$ be a cluster algebra of rank 3 with principal coefficients and let $x[d]$ be a cluster variable of $\mathcal{A}$ with $d$-vector $d$. Let $P_d$ be a weakly convex quadrilateral constructed in Lemma 3.11. Then

$$x[d] = \sum_{p \in \mathbb{Z}^3} e(p)x^p = \sum_{p_1, p_2, p_3} e(p_1, p_2, p_3)x_{p_1}^{p_1}x_{p_2}^{p_2}x_{p_3}^{p_3}$$

where $e(p) \in \mathbb{Z}[y_1, y_2, y_3]$ is uniquely characterized by the following conditions.

(S) (Support condition) The coefficient $e(p) = 0$ unless $p \in P_d$. Equivalently, the Newton polytope of $x[d]$ is contained in $P_d$.

(N) (Normalize condition) There is precisely one $e(p)$ that has a nonzero constant term, which must be 1. Moreover, the greatest common divisor of all $e(p)$ is 1.

(D) (Divisibility condition) For each $k = 1, 2, 3$ and $m < 0$,

$$\left( \prod_{i=1}^3 x_i^{-|b_k|} + y_k \prod_{i=1}^3 x_i^{|b_k|} \right)^{-m} \text{ divides } \sum_{p \in \mathbb{Z}^3; y_k = m} e(p)x^p,$$

in the sense that the quotient is in $\mathbb{Z}[x_1^\pm, x_2^\pm, x_3^\pm, y_k]$.

Moreover, (NC) can be replaced by

(NC') There exists a $p \in \mathbb{Z}^3$ such that $e(p)$ has a nonzero constant term. Moreover for each vertex $p$ of the convex hull $|P_d|$, $e(p)$ is a monomial $y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3}$ for some $\alpha_i \in \mathbb{Z}_{\geq 0}$.

And (SC) can be replaced by the following stronger condition.

(SC') The Newton polytope of $x[d]$ (which, by definition, is the convex hull of the set $\{p | e(p) \neq 0\}$) is $P_d$. 

Moreover, (NC) can be replaced by

(NC') There exists a $p \in \mathbb{Z}^3$ such that $e(p)$ has a nonzero constant term. Moreover for each vertex $p$ of the convex hull $|P_d|$, $e(p)$ is a monomial $y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3}$ for some $\alpha_i \in \mathbb{Z}_{\geq 0}$.

And (SC) can be replaced by the following stronger condition.

(SC') The Newton polytope of $x[d]$ (which, by definition, is the convex hull of the set $\{p | e(p) \neq 0\}$) is $P_d$. 

The proof of the theorem is given in the next section. For an example see Section 7. The theorem has the following consequence on the support of $F$-polynomials:

**Corollary 5.2.** Using the same notation as in Theorem 5.1, let $g$ be the $g$-vector of $x[d]$. The support of the $F$-polynomial is contained in the following (possibly unbounded) polyhedron:

$$F_d := \mathbb{R}_{\geq 0} \cap \varphi_B^{-1}(\{p - g \mid p \in P_d\})$$

where $\varphi_B : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear map $q \mapsto Bq$, and $\varphi_B^{-1}$ sends a set to its preimage.

**Proof.** The support is in $\mathbb{R}_{\geq 0}^3$ because the $F$-polynomial is in $\mathbb{Z}[y_1, y_2, y_3]$. Next, by equation (6.1) (and using the notation therein), $e(p) \neq 0$ if and only if $f_{ijk} \neq 0$ for some $q = (i, j, k)$ satisfying $Bq + g = p$, that is $Bq = p - g$. This implies that the support of the $F$-polynomial is in $\varphi_B^{-1}(\{p - g \mid p \in P_d\})$. □

**Remark 5.3.** (1) It has been conjectured that the support of the $F$-polynomial of a cluster variable is always saturated, which is proved in [10] for acyclic skew-symmetric cluster algebras. We say that a non-initial cluster variable is always saturated, which is proved in [10] for acyclic skew-symmetric cluster algebras. We say that a non-initial cluster variable is always saturated, which is proved in [10] for acyclic skew-symmetric cluster algebras.

(2) As computed at the end of §7, the convex polyhedron $F_d$ is often not equal to the Newton polytope of the $F$-polynomial. In fact, $\varphi_B^{-1}(\{p - g \mid p \in P_d\})$ is a union of parallel lines in the direction of $(\bar{b}, \bar{a}, \bar{c})$ (the vector that spans the kernel of $\varphi_B$). If the intersection of such a line with $\mathbb{R}_{\geq 0}^3$ is nonempty, then the intersection is unbounded if and only if $\bar{a}, \bar{b}, \bar{c}$ are either all in $\mathbb{R}_{\geq 0}$ or all in $\mathbb{R}_{\geq 0}$. As a consequence, $F_d$ is unbounded for non-acyclic cluster algebras, so in general it only gives a rough upper bound of the Newton polytope of the $F$-polynomial.

### 6. Proof of Theorem 5.1

We first show uniqueness. Thus we prove that, given $d \in \mathbb{Z}_{\geq 0}^3 \setminus \{(0, 0, 0)\}$ which is a $d$-vector of a non-initial cluster variable, then there is only one Laurent polynomial satisfying the condition $(SC) + (NC) + (DC)$ (respectively $(SC) + (NC') + (DC)$).

Indeed, let $f$ be such a Laurent polynomial. Then the Newton polytope of $f$ and $x[d]$ are both contained in $P_d$. After changing the initial seed to some seed $\Sigma_t$ that contains $x[d]$ as a cluster variable (for simplicity, assume it to be $x_1(t)$), we see that the Newton polytope of $f|_t$ is equal to the Newton polytope of $x[d]|_t$, which is a one point set $P_d = \{(1, 0, 0)\}$. But then there is obviously only one Laurent polynomial satisfying $(SC) + (NC) + (DC)$ (resp. $(SC) + (NC') + (DC)$), namely $x_1(t)$. Thus $f|_t = x_1(t) = x[d]|_t$, which implies $f = x[d]$. 

\[ \begin{array}{c}
1 \\
\end{array} \begin{array}{c}
2 \\
\end{array} \begin{array}{c}
3 \\
\end{array} \]
It remains to show that a cluster variable \( x[d] \) satisfies all the conditions given in the theorem. This is obviously true for initial cluster variables. So by Lemma 4.1 we can assume that it is non-initial, i.e., \( d \in \mathbb{Z}^3_{\geq 0} \). We show each condition in a separate subsection.

6.1. Proof of (DC). To prove (DC), we use the universal Laurent phenomenon. We only show (DC) for \( k = 1 \) because the other cases are similar. Define \( h(x_2, x_3) = x_1 x_1' \). Then

\[
h(x_2, x_3) = \prod x_i^{-b_{ik}^+} + y_k \prod x_i^{b_{ik}^+} = x_2^{[a_i^+]} x_3^{[c_i^+]} + x_2^{[-a_i^+]} x_3^{[-c_i^+]} y_1 = x_2^{[-a_i^+]} x_3^{[-c_i^+]} (x_2^c + x_3 y_1),
\]

where the last identity holds because \( [m]_+ = m + [-m]_+ \).

Denote the Laurent expansion of \( x[d]|_{\mu_1(t_0)} \) by

\[
x[d]|_{\mu_1(t_0)} = \sum_{p_1', p_2', p_3'} e(p_1', p_2', p_3')(x_1')^{p_1'} x_2^{p_2'} x_3^{p_3'}, \quad \text{where } e'(p_1', p_2', p_3') \in \mathbb{Z}[y_1', y_2', y_3']
\]

and for each \( j = 1, 2, 3 \), \( y_j' = \prod_{i=1}^{6} x_i^{b_{ij}^+} \) is a Laurent polynomial in \( x_4 = y_1, x_5 = y_2, x_6 = y_3 \), where we denote by \( B' = [b_{ij}'] \) the B-matrix of the seed \( \mu_1(\Sigma_{t_0}) \). Then we have

\[
\sum_{p_1, p_2, p_3} e(p_1, p_2, p_3)x_1^{p_1} x_2^{p_2} x_3^{p_3} = \sum_{p_1', p_2', p_3'} e(p_1', p_2', p_3')(x_1')^{p_1'} x_2^{p_2'} x_3^{p_3'}
\]

\[
= \sum_{p_1', p_2', p_3'} e(p_1', p_2', p_3') \left( \frac{h(x_2, x_3)}{x_1} \right)^{p_1'} x_2^{p_2'} x_3^{p_3'} = \sum_{p_1', p_2', p_3'} e(p_1', p_2', p_3') x_1^{-p_1'} \left( h(x_2, x_3)^{p_1'} x_2^{p_2'} x_3^{p_3'} \right)
\]

For the fixed \( p_1 < 0 \), taking the coefficient of \( x_1^{p_1} \) (so we should take \( p_1' = -p_1 \) on the right hand side), we get an equality

\[
\sum_{p_2, p_3} e(p_1, p_2, p_3)x_2^{p_2} x_3^{p_3} = h(x_2, x_3)^{-p_1} \sum_{p_2', p_3'} e(-p_1, p_2', p_3') x_2^{p_2'} x_3^{p_3'}
\]

Thus the left hand side is divisible by \( h(x_2, x_3)^{-p_1} \), which is equivalent to the condition (DC) for \( k = 1 \).

From the proof we can also conclude that, if a Laurent polynomial satisfies (DC), then it is in the upper bound/upper cluster algebra \( \mathcal{U} = \mathcal{A} \). Indeed, (DC) implies that, \( x[d] \) is in the upper bound \( U(\Sigma_{t_0}) \), where \( \Sigma_{t_0} \) is the initial seed. Since we assume \( B \) is skew-symmetrizable, \( \Sigma_{t_0} \) is totally mutable. Moreover, \( \Sigma_{t_0} \) is coprime because \( \tilde{B} \) is full rank [1, Proposition 1.8]. Therefore [1, Corollary 1.7] implies that \( U(\Sigma_{t_0}) \) is equal to the upper cluster algebra \( \mathcal{A}(\Sigma_{t_0}) \).

6.2. Proof of (NC). It is shown in [9] that \( x[d] \) can be expressed by its F-polynomial

\[
F(y_1, y_2, y_3) = \sum_{i, j, k \geq 0} f_{ijk} y_1^i y_2^j y_3^k,
\]

as follows (where \( g = (g_1, g_2, g_3) \) is its g-vector)

\[
x[d] = x^g F(y_1 x_2^{-a_i^c} x_3^{-c_i^c}, y_2 x_1^{-a_i^c} x_3^{-c_i^c}, y_3 x_1^{-a_i^c} x_2^{-c_i^c}) = x_1^{g_1} x_2^{g_2} x_3^{g_3} \sum_{i, j, k \geq 0} f_{ijk} (y_1 x_2^{-a_i^c} x_3^{-c_i^c})^i (y_2 x_1^{-a_i^c} x_3^{-c_i^c})^j (y_3 x_1^{-a_i^c} x_2^{-c_i^c})^k.
\]

So

\[
e(p) = \sum_{i, j, k \geq 0} f_{ijk} y_1^i y_2^j y_3^k, \quad \text{where } i, j, k \geq 0 \text{ satisfy } B \begin{bmatrix} i \\ j \\ k \end{bmatrix} + g = p
\]
Since the constant term of the F-polynomial of any cluster variable is 1 (Lemma 3.3), there is only one $e(p)$ which has a nonzero constant term, which must be 1.

Now assume the greatest common divisor of all $e(p)$, which exists uniquely up to sign, is $h \in \mathbb{Z}[y_1, y_2, y_3]$ and $h \neq \pm 1$. Since one of $e(p)$ has constant term 1, we can choose $h$ to have constant term 1. Thus $h$ has at least two terms. Define

$$X = x[d]/h.$$  

We observe that $X$ still satisfies (DC), thus is in the upper cluster algebra $\mathcal{A}$. Indeed, for each $m < 0$, denote

$$Y = \left( \sum_{p \in \mathbb{Z}^3: p_k = m} e(p) x^p \right) / \left( \prod_{i=1}^{3} x_i^{-b_{ik}+} + y_k \prod_{i=1}^{3} x_i^{b_{ik}+} \right)^{-m}.$$  

Then $Y \in \mathbb{Z}[x_1^\pm, x_2^\pm, x_3^\pm, y_k]$ since $x[d]$ satisfies (DC). We need to show that $Y/h$ is also in $\mathbb{Z}[x_1^\pm, x_2^\pm, x_3^\pm, y_k]$. Since $\mathbb{Z}[x_1^\pm, x_2^\pm, x_3^\pm, y_k]$ is a UFD and $h$ divides the numerator of $Y$, it suffices to show that $h$ is relatively prime to the denominator of $Y$, or equivalently, show that $h$ is relatively prime to $\prod_{i=1}^{3} x_i^{-b_{ik}+} + y_k \prod_{i=1}^{3} x_i^{b_{ik}+}$. This is false only if $h = 1 + y_k$ and $b_{1k} = b_{2k} = b_{3k} = 0$, which will not happen because we assume the $B$-matrix is non-degenerate (see Remark 3.4).

Similar to the proof of Lemma 4.1, let $\Sigma = (x_1', \ldots, x_n', y_1', \ldots, y_n', B')$ be a seed that contains $x[d] = x_\ell'$. Then

$$x_\ell' = (X|_t)(h|_t) .$$  

We claim that $h|_t$, written as a Laurent polynomial in $y_1', y_2', y_3'$, has the same number of terms as $h$ written as a Laurent polynomial in $y_1, y_2, y_3$. Indeed, $y_i' = y_i c_i$ where $c_i$ are the $c$-vectors. By Lemma 3.2, the $c$-vectors $c_1, c_2, c_3 \in \mathbb{Z}^3$ are linearly independent, so distinct monomials in $y_1', y_2', y_3'$ convert to distinct monomials in $y_1, y_2, y_3$.

By the above claim, $h|_t$ is a Laurent polynomial with at least two terms. So the right hand side of (6.2) has at least two terms, but the left hand side has only one term, a contradiction. Therefore the greatest common divisor of all $e(p)$ is 1.

### 6.3. Proof of (SC').

Proving (SC') also proves (SC). Let $x = x[d]$ be a cluster variable, expressed as a Laurent polynomial in the initial seed $\Sigma_0$. Assume that $\Sigma_0, \Sigma_1, \Sigma_2, \ldots, \Sigma_m$ is a sequence of mutations of seeds, and that $x$ is a cluster variable in $\Sigma_m$. We want to show that the condition (SC') holds for $x$ over the seed $\Sigma_m$. We use induction on $m$.

If $m = 0$ then $x$ is an initial cluster variable and (SC') holds by definition of $P_d$. (Recall that as exceptional cases, $P_d$ is defined for $d = (-1, 0, 0), (0, -1, 0), (0, 0, 1)$ at the beginning of §3.4). For the induction step, we need to show that the quadrilateral $P_d$ is compatible with the mutation. This is the longest part of the proof, consisting of a case-by-case computation of the boundary of the quadrilaterals. Without loss of generality, we only need to discuss the cases described in Lemma 3.11.

#### 6.3.1. Proof of (SC') Case 1.

Assume $a, b > 0$ and $c < 0$ Using Lemma 3.11 we obtain the quadrilateral $P_d$ having the following vertices:
More precisely, by changing the initial seed from $t_0$ to $\mu_1(t_0)$, we substitute $x_1$ by $(p_1^+ x_2 x_3^{-c} P_1^-)/x_1'$ in $x[d]$, and get a cluster variable $x'[d'] = \sum e'(p)x'^p = \sum e'(p_1, p_2, p_3)(x_1')^{p_1} x_2^{p_2} x_3^{p_3}$ with $d$-vector $d'$. Then we need to show that the convex hull of the set $\{p|e'(p) \neq 0\}$ is $|P_{d'}|$, where $B'$ is as follows (note that $a, a' > 0$ and $c, c' < 0$ by assumption):

$$B' := \mu_1(B) = \begin{bmatrix} 0 & -a & c' \\ a' & b + \text{sgn}(a')((-a')(-c')) & 0 \\ -c & -b' + \text{sgn}(c)[ca] & 0 \end{bmatrix}.$$ 

First, we use Lemma 3.13 to determine the convex hull of $\{p|e'(p) \neq 0\}$. Let $T_{in}$ be the segment $P_1P_4$, $T_{out}$ be the broken line $P_2P_3P_4$, and define points $P'_1, \ldots, P'_4$ to satisfy $\alpha_1(P_1P_4) = P'_1P_4'$ and $\beta_1(P_2P_3P_4) = P'_1P'_2P'_3$, that is,

$$P'_1 = \alpha_1(P_1) = \beta_1(P_2) = (d_1, -d_2, -d_3 + b'd_2),$$

$$P'_2 = \beta_1(P_3) = (d_1 - ad_2, -d_2, -d_3),$$

$$P'_3 = \beta_1(P_4) = (d_1 - ad_2 + c'd_3, -d_2 + bd_3, -d_3),$$

$$P'_4 = \alpha_1(P_4) = (d_1 - ad_2 + c'd_3, -a'd_1 + (a'a' - 1)d_2 + (b - a'c')d_3, cd_1 - acd_2 + (cc' - 1)d_3).$$

Then Lemma 3.13 guarantees that convex hull of the set $\{p|e'(p) \neq 0\}$ is $|P'_1P'_2P'_3P'_4|$. (See Figure 11.)

![Figure 11](image_url)

**Figure 11.** (SC'), Case 1, $\mu_1$. (Left: projection to $xy$-plane. Right: projection to $xy$-plane followed by reflection about $y$-axis.)

Next, we explicitly determine $d'$. By (4.1), $d'$ is equal to the $-\min$ of the vertices of the convex hull $|P'_1P'_2P'_3P'_4|$, therefore

$$d' = -\min(P'_1, \ldots, P'_4)$$

(6.3)
Thus
\[
d'_{1} = - \min(d_1, d_1 - ad_2, d_1 - ad_2 + c'd_3) = ad_2 - c'd_3 - d_1,
d'_{2} = - \min(-d_2, -d_2 + bd_3, -ad_1 + (aa' - 1)d_2 + (b - a'c')d_3) \\
= - \min(-d_2, -d_2 + bd_3, -d_2 + ad_1 + bd_3) = 2d_2, \\
d'_{3} = - \min(-d_3 + b'd_2, d_3, cd_1 - acd_2 + (cc' - 1)d_3) \\
= - \min(-d_3 + b'd_2, d_3, -d_3 - cd_1') = 2d_3
\]

Lastly, we show that \( x'[d'] \) satisfies (SC'), that is, the convex hull of the set \( \{P|e'(P) \neq 0\} \) is equal to \( |P'_{d'}| \), or equivalently,
\[
|P'_1P'_2P'_3P'_4| = |P'_{d'}|
\]
If \( d'_1 < 0 \), then by Lemma 4.1, \( d' = (-1, 0, 0) \), and (SC') is trivially true. So in the following we assume \( d'_1 \geq 0 \).

We shall show that we can actually take \( P'_{d'} = P'_1P'_2P'_3P'_4 \), that is, \( P'_1, \ldots, P'_4 \) satisfy the two conditions in Lemma 3.11 (recall that \( P'_{d'} \) may not be unique but its convex hull \( |P'_{d'}| \) is).

The condition (2) follows from (6.3).

The condition (1) holds for \((i, j, k) = (2, 3, 1)\). Indeed:

For (1a), we have
\[
\begin{align*}
\mu & = b_{23} = b \geq 0 \text{ and } b_{31} = -c \geq 0. \\
& \text{For (1b), we have } v_i = d_iB_i \text{ for } i = 1, 2, 3, v_4 = -v_1 - v_2 - v_3 = (ad_2 - c'd_3, ad_1 - ad_2 + a'c' - b)d_3, -cd_1 + (b' + ac)d_2 - cc'd_3. \text{ It is straightforward to check } P'_1P'_2 = v'_2, P'_2P'_3 = v'_3, P'_3P'_4 = v'_4.
\end{align*}
\]

For (1c), we have \( B'_2, B'_3, B'_1, v'_4 \) are in circular order because \( B'_2, B'_3, B'_1 \) are strictly in the same half plane, and
\[
DB' = \begin{bmatrix} -b \\ c \\ a \end{bmatrix} = \begin{bmatrix} 0 & -\bar{a} & c \\ \bar{a} & 0 & \bar{b} \\ -\bar{c} & -\bar{b} & 0 \end{bmatrix} \Rightarrow B' = \begin{bmatrix} -b \\ c \\ a \end{bmatrix} = 0 = -\bar{b}B'_1 + \bar{c}B'_2 + \bar{a}B'_3 = 0
\]
implies that \( B'_3 = (-\bar{c}/\bar{a})B'_2 + (\bar{b}/\bar{a})B'_1 \) where both coefficients are positive. So (1c) follows from Lemma 3.6.

This completes the proof that the quadrilateral changes as expected under the mutation \( \mu_1 \).

The rest of the proof is similar to the above discussion of the quadrilateral change after \( \mu_1 \). For this reason we simply point out the difference.

To show that the quadrilateral changes as expected under the mutation \( \mu_2 \):

Substitute \( x_2 \) by \((p_2^2x_3 + p_2^2 x_1^3)/x_2 \) in \( x[d] \), and get \( x'[d'] \). Define
\[
B' := \mu_2(B) = \begin{bmatrix} 0 & -a & ab - c' \\ a' & 0 & -b \\ c - a'b' & b' & 0 \end{bmatrix}
\]

Note that
\[
DB' = \begin{bmatrix} a\bar{b}/\delta_2 - \bar{c} \\ \bar{a} \\ \bar{a} \end{bmatrix} = \begin{bmatrix} 0 & -a & a\bar{b}/\delta_2 - \bar{c} \\ \bar{a} & 0 & -\bar{b} \\ \bar{a} & 0 & a\bar{b}/\delta_2 - \bar{c} \end{bmatrix} = 0
\]
implies \( bB'_1 + (a\bar{b}/\delta_2 - c)B'_2 + \bar{a}B'_3 = 0 \).
The $y$-coordinate of $P_1$ is $-d_2 + ad_1$ and the $y$-coordinate of $P_4$ is $-d_2 + bd_3$; thus their difference is $-a'd_1 + bd_3$. We will distinguish three cases according to the sign of this difference.

(i) Suppose $-a'd_1 + bd_3 < 0$. Geometrically, it means that $P_4$ is strictly lower than $P_1$ after projection to $xy$-plane; see Figure 12. Since $T_{in}$ and $T_{out}$ are taken relative to $B_2$ in this case, we have $T_{in} = P_1P_2$ and $T_{out} = P_1P_3$. Thus in the notation of Lemma 3.13, we have $q_1 = P_1$, $q_2 = P_1$, $q_3 = P_2$, $q_4 = P_3$, $T'_{out} = \alpha_2(T_{in})$, $T'_{in} = \beta_2(T_{out})$. Therefore Lemma 3.13 implies

$$P'_1 = \beta_2(P_1) = ((aa' - 1)d_1 - ad_2, d_2 - a'd_1, -d_3 - cd_1 + b'd_2),$$
$$P'_2 = \alpha_2(P_1) = (-d_1, d_2 - a'd_1, -d_3 + (a'b' - c)d_1),$$
$$P'_3 = \alpha_2(P_2) = \beta_2(P_3) = (-d_1, d_2, -d_3),$$
$$P'_4 = \beta_2(P_4) = (-d_1 + (ab - c')d_3, d_2 - bd_3, -d_3).$$

![Figure 12. (SC'), Case 1, $\mu_2$, (i).](image)

We claim that $x'[d']$ also satisfies (SC'). If $d'_2 < 0$, then $d' = (0, -1, 0)$, and (SC') is trivially true. So we assume $d'_2 \geq 0$. The condition (2) of Lemma 3.11 determines the vector $d' = (d'_1, d'_2, d'_3)$:

$$d'_2 = -\min(d_2 - a'd_1, d_2 - bd_3),$$
$$d'_3 = -\min(-d_3 - cd_1 + b'd_2, -d_3 + (a'b' - c)d_1, -d_3) = d_3,$$
$$d'_1 = -\min((aa' - 1)d_1 - ad_2, -d_1, -d_1 + (ab - c')d_3),$$
$$d'_1 = -\min(-d_1 + ad_2, -d_1, -d_1 + (ab - c')d_3) = d_1.$$

We show that the three conditions in Lemma 3.11 (1) hold for $(i, j, k) = (2, 1, 3)$:

(1a) $b'_{21} = a' \geq 0$ and $b'_{13} = ab - c' \geq 0.$

(1b) Use $v'_4 = (aa'd_1 - ad_2 - (ab - c')d_3, -a'd_1 + bd_3, -cd_1 + b'd_2).$

(1c) $B'_2, B'_1, B'_3$ are not strictly in the same half plane, and $v'_4 = \lambda_2 B'_2 + \lambda_3 B'_3$ with $\lambda_2 = (-cd_1 + b'd_2)/b' \geq 0$, $\lambda_3 = (-a'd_1 + bd_3)/(b) > 0$. So $B'_2, B'_1, B'_3, v'_4$ are in circular order by Lemma 3.6.

(ii) Suppose $-a'd_1 + bd_3 > 0$. Geometrically, it means that $P_4$ is strictly higher than $P_1$ after projection to $xy$-plane; see Figure 13. So $T_{in} = P_1P_3P_2$ and $T_{out} = P_3P_4$. Therefore Lemma 3.13 implies
Lemma 3.11 determines the vector $b_a$ after projection to $P$ and $\beta$. To show that the quadrilateral changes as expected under the mutation $\beta$, we claim that $x$.

Substitute $d$ (1b) Use $b$ (1a) We show that the three conditions in Lemma 3.11 (1) hold for $(i, j, k) = (1, 3, 2)$:

(1a) $b_{13} = ab - c' \geq 0$ and $b_{32} = b' \geq 0$.

(1b) Use $v'_4 = (-ad_2 + c'd_3, bd_3 - a'd_1, (a'b' - c)d_1 + b'd_2 - bb'd_3).

(1c) $B'_1, B'_3, B'_2$ are not in the same half plane, and $v'_4 = \lambda_1 B'_1 + \lambda_2 B'_2$ with $\lambda_1 = (bd_3 - a'd_1)/a' > 0$, $\lambda_2 = -(ad_2 + c'd_3)/(a) \geq 0$. So $B'_1, B'_3, B'_2, v'_4$ are in circular order by Lemma 3.6.

(iii) Suppose $-a'd_1 + bd_3 = 0$. Geometrically, it means that $P_4$ is at the same height at $P_1$ after projection to $xy$-plane; see Figure 14. So $T_{in} = P_1P_2$ and $T_{out} = P_3P_4$.

The Newton polytope of $x'[d']$ is in a triangle $P'_1P'_2P'_3$, determined by $\alpha_2(P_1P_2) = P'_1P'_2$ and $\beta_2(P_3P_4) = P'_3P'_4$. We can view this as a degenerate case of either (i) or (ii), and the proof of (SC') still works. Note that these two $P_d$ gives the same triangle convex hull $|P_d|$.

To show that the quadrilateral changes as expected under the mutation $\mu_3$:

Substitute $x_3$ by $(p_3^+ + p_1^- x_2 x_1 c')/x_3$ in $x[d]$, and get $x'[d']$.

$$B' := \mu_3(B) = [b'_{ij}] = \begin{bmatrix} 0 & a & c' \\ -a' & 0 & -b \\ -c & b' & 0 \end{bmatrix}$$
Apply Lemma 3.12 to $T_{in} = P_1 P_2 P_3$, $T_{out} = P_1 P_4$, we obtain the quadrilateral $P_1' P_2' P_3' P_4'$, determined by $\alpha_3(P_1 P_2 P_3) = P_2' P_3' P_4'$ and $\beta_3(P_1 P_4) = P_1' P_4'$. See Figure 15. Thus

$P_1' = \beta_3(P_1) = (-d_1 - c(-cd_1 + b'd_2 - d_3), -d_2 + a'd_1 + b(-d_3 - cd_1 + b'd_2), d_3 + cd_1 - b'd_2)$,

$P_2' = \alpha_3(P_1) = (-d_1, -d_2 + a'd_1, d_3 + cd_1 - b'd_2)$,

$P_3' = \alpha_3(P_2) = (-d_1, -d_2, d_3 - b'd_2)$,

$P_4' = \alpha_3(P_3) = \beta_3(P_4) = (-d_1 + ad_2, -d_2, d_3)$.

Figure 15. (SC’), Case 1, $\mu_3$. (Left: projection to $yz$-plane. Right: projection to $yz$-plane followed by reflection about $y$-axis.)

Figure 14. (SC’), Case 1, $\mu_2$, (iii). (Left: projection to $xy$-plane. Middle and Right: projection to $xy$-plane followed by reflection about $x$-axis.)

To show $x'[d']$ satisfies (SC’). We can assume $d_3' \geq 0$ as before. Then $d_1' = d_1$, $d_2' = d_2$,

$d_3' = -cd_1 + b'd_2 - d_3$, $v'_4 = (-c(-cd_1 + b'd_2 - d_3) - ad_2, a'd_1 + b(-d_3 - cd_1 + b'd_2), cd_1 - b'd_2)$.

We claim the conditions in Lemma 3.11 (1) hold for $(i, j, k) = (3, 1, 2)$. Indeed:

For (1a): $b'_{31} = -c \geq 0$, $b'_{12} = a > 0$.

For (1b): straightforward check.

For (1c): $B_3', B_1', B_2'$ are strictly in the same half-plane, and $B_1' = (a'/b')B_3' + (-c/b')B_2'$

where both coefficients are nonnegative.

6.3.2. Proof of (SC’) Case 2. Assume $Q$ is of the form $1 \rightarrow 2 \rightarrow 3$, that is, $a, b > 0$ and $c = 0$.

This is a degenerated case of Case 1. We shall only explain the difference in the argument.

For $\mu_1$: assume $d_3' \geq 0$. The vectors $B_1$ and $B_3$ are in opposite direction, so $P_1 P_2$ is parallel to $P_3 P_4$. The point $P_3'$ is on the line segment $P_2' P_4'$, so $|P_a|$ is the triangle $P_1' P_2' P_4'$. The proof is same as Case 1; the circular order condition (1c) trivially holds (where $(i, j, k) = (2, 3, 1)$) because $B_1'$ and $B_3'$ are in the same direction. See Figure 16.
For $\mu_2$: same argument as in Case 1.

For $\mu_3$: assume $d_3' \geq 0$. The point $P_2'$ is on the line segment $P_1'P_3'$, so $|P_d|$ is the triangle $P_1'P_3'P_4'$. The proof is same as Case 1; the circular order condition trivially holds (where $(i,j,k) = (3,1,2)$) because $B_1'$ and $B_3'$ are in the same direction. See Figure 17.

6.3.3. Proof of (SC') Case 3. (a) Assume $Q$ is of the form $1 \rightarrow 3 \leftarrow 2$, that is, $a = 0$, $b > 0 > c$. Thus $B_1$ and $B_2$ are in the same direction.

This is degenerated from Case 1. We shall explain the difference.

For $\mu_1$: Note $B_1' = (0,0,-c), B_2' = (0,0,-b'), B_3' = (c',b,0), (i,j,k) = (2,3,1)$. To show that $B_2', B_3', B_1', v_4'$ are in circular order, we use the fact that $B_2'$ and $B_1'$ are in opposite directions. See Figure 18.

For $\mu_2$: Note $B_1' = (0,0,c), B_2' = (0,0,b'), B_3' = (-c',-b,0)$. The three cases described in (Case 1, $\mu_2$) degenerate to:
(i): If \( bd_3 < 0 \). Impossible since we assume \( d_3 \geq 0 \).

(ii) and (iii): If \( bd_3 \geq 0 \). Take \((i,j,k) = (1,3,2)\). Vectors \( B'_3, B'_1, B'_2, v'_4 \) are in circular order because \( B'_1, B'_2 \) are in opposite directions. Also see Figure 18.

For \( \mu_3 \): This is the same argument as Case 1.

(b) Assume \( Q \) is of the form \( 2 \leftrightarrow 1 \rightarrow 3 \), that is, \( a > 0, b = 0, c < 0 \). Then \( B_2 \) and \( B_3 \) are in the same direction.

This case is also degenerated from Case 1. We shall explain the difference.

For \( \mu_1 \): This is the same argument as in Case 1.

For \( \mu_2 \): As before, assume \( d'_2 = bd_3 - d_2 = -d_2 \geq 0 \). Since we assume \( d_2 \geq 0 \), we must have \( d'_2 = d_2 = 0 \). Note \( B'_1 = (0, a', c), B'_2 = (-a, 0, 0), B'_3 = (-c', 0, 0) \). The three cases described in (Case 1, \( \mu_2 \)) degenerate to:

(i) and (iii): If \( -a'd_1 \leq 0 \). Then take \((i,j,k) = (2,1,3)\). Vectors \( B'_2, B'_1, B'_3, v'_4 \) are in circular order because \( B'_2, B'_3 \) are in opposite directions.

(ii): If \( -a'd_1 > 0 \). Then \( d_1 < 0, d = (-1,0,0) \) and \( x[d] = x_1 \), which is a trivial case.

\[ P_1 \rightarrow P_2 \rightarrow P_3 \rightarrow P_4 \]

**Figure 19.** (SC’), Case 4, \( \mu_2 \) or \( \mu_3 \). (Left: projection to \( xz \)-plane. Right: projection to \( xz \)-plane followed by reflection about \( z \)-axis.)

For \( \mu_3 \): Take \((i,j,k) = (3,1,2)\). Vectors \( B'_3, B'_1, B'_2, v'_4 \) are in circular order because \( B'_3 = (c', 0, 0) \) and \( B'_2 = (a, 0, 0) \) are in opposite directions.

6.3.4. **Proof of (SC’) Case 4.** Suppose \( a, b, c > 0 \). Renumbering vertices \((1,2,3)\) as \((2,3,1)\) or \((3,1,2)\) if necessary, we can assume that \( B_1, B_2, B_3, v_4 \) are in circular order (and still satisfy \( b_{21}, b_{23}, b_{31} > 0 \)); thus

\[
(6.4) \quad (-ad_2 + c'd_3, a'd_1 - bd_3, -cd_1 + b'd_2) = v_4 = \lambda_1 B_1 + \lambda_3 B_3 = (-c'\lambda_3, -a'\lambda_1 + b\lambda_3, c\lambda_1)
\]

for some real numbers \( \lambda_1, \lambda_3 \geq 0 \). So by Lemma 3.11, we get the same expression for \( P_1, \ldots, P_4 \) as in Case 1:

\[
\begin{align*}
P_1 &= (-d_1, -d_2 + a'd_1, -d_3 - cd_1 + b'd_2), \\
P_2 &= (-d_1, -d_2, -d_3 + b'd_2), \\
P_3 &= (-d_1 + ad_2, -d_3), \\
P_4 &= (-d_1 + ad_2 - c'd_3, -d_2 + bd_3, -d_3).
\end{align*}
\]

It is easy to check that \( d = \text{min}(P_1, \ldots, P_4) \) by observing \( ad_2 - c'd_3 = c'\lambda_3 \geq 0 \) and \(-cd_1 + b'd_2 = c\lambda_1 \geq 0\).

To show that the quadrilateral changes as expected under the mutation \( \mu_1 \):
If \(\mu_1(Q)\) is acyclic, then we can apply the previous argument for \(\mu_1(Q)\) to conclude that the quadrilateral is compatible with the mutation. So in below we assume that \(\mu_1(Q)\) is still cyclic, i.e., \(ac - b' > 0\), or equivalently, \(a'c' - b > 0\).

By changing the initial seed from \(\Sigma_0\) to \(\mu_1(\Sigma_0)\), we substitute \(x_1\) by \((p_1 x_2' + p_2 x_3')/x_1\) in \(x'[d']\), and get \(x'[d']\). Using Lemma 3.12, we obtain that the support of \(x'[d']\) lies in the quadrilateral \(P_1'P_2'P_3'P_4'\), determined by \(\alpha_1(P_1P_4P_3) = P_1'P_4'P_3'\) and \(\beta_1(P_2P_3) = P_1'P_2'\). Thus \(P_1' = \alpha_1(P_1) = \beta_1(P_2), P_2' = \beta_1(P_3), P_3' = \alpha_1(P_3), P_4' = \alpha_1(P_4),\) and more explicitly

\[
\begin{align*}
P_1' &= (d_1, -d_2, -d_3 + b'd_2 - cd_1), \\
P_2' &= (d_1 - ad_2, -d_2, -d_3 - cd_1 + acd_2), \\
P_3' &= (d_1 - ad_2, (aa' - 1)d_2 - a'd_1, -d_3), \\
P_4' &= (d_1 - ad_2 + c'd_3, -a'd_1 + (aa' - 1)d_2 + (b - a'c)d_3, -d_3).
\end{align*}
\]

See Figure 20.

![Figure 20](image)

**Figure 20.** (SC'), Case 4, \(\mu_1\). (Left: projection to \(xy\)-plane. Right: projection to \(xy\)-plane followed by reflection about \(y\)-axis.)

We claim that this \(x'[d']\) also satisfies (SC). If \(d_1' < 0\), it is trivially true. So we assume \(d_1' \geq 0\). Denote

\[
B' = \mu_1(B) = \begin{bmatrix} 0 & -a & c' \\ a' & 0 & b - a'c' \\ -c & ac - b' & 0 \end{bmatrix}
\]

By (4.1), we have \(d = \min(P_1', P_2', P_3', P_4')\), thus \(d_1' = ad_2 - d_1, d_2' = d_2, d_3' = d_3\), because \(d_1' = ad_2 = d_1 \geq 0\). The vector \(v_4'\) is equal to \(v_4' = -v_1' - v_2' - v_3' = (ad_2 - c'd_3, a'd_1 - aa'd_2 + (a'c' - b)d_3, b'd_2 - cd_1)\).

We show that the conditions in Lemma 3.11 (1) are all satisfied for \((i, j, k) = (2, 1, 3)\):

For (1a): We have \(b_{21}' = a' \geq 0\) and \(b_{13}' = c' \geq 0\).

For (1b): This is straightforward.

For (1c): We have \(B_2', B_1', B_3', v_4'\) are in circular order because \(B_2', B_1', B_3'\) are not in the same half plane, and \(v_4' = (b'd_3 - cd_1)/(ac - b')B_2' - (a'd_1 - aa'd_2 + (a'c' - b)d_3)/(a'c' - b)B_3'\) where the first coefficient is \(c\lambda_1/(ac - b') \geq 0\), the second coefficient is \((a'\lambda_1 + (a'c' - b)\lambda_3)/(a'c' - b) \geq 0\).

To show that the quadrilateral changes as expected under the mutation \(\mu_2\):

Like above, we can assume \(\mu_2(Q)\) is cyclic, i.e., \(ab - c' > 0, a'b' - c > 0\). Define

\[
B' = [b'_{ij}] = \begin{bmatrix} 0 & -a & ab - c' \\ a' & 0 & -b \\ c - a'b' & b' & 0 \end{bmatrix}
\]

There are three cases to consider:
(i) Suppose \(-a'd_1 + bd_3 < 0\). See Figure 21.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure21}
\caption{(SC’), Case 4, \(\mu_2\), (i). (Left: projection to \(xy\)-plane. Right: projection to \(xy\)-plane followed by reflection about \(x\)-axis.)}
\end{figure}

The quadrilateral \(P_d\) is determined by \(\alpha_2(P_1P_2) = P_2'P_3'\) and \(\beta_2(P_3P_4) = P_1'P_3'P_4'\). Thus

\[
\begin{align*}
P_1' &= \beta_2(P_1) = ((aa' - 1)d_1 - ad_2, d_2 - a'd_1, -d_3 - cd_1 + b'd_2), \\
P_2' &= \alpha_2(P_1) = (-d_1, d_2 - a'd_1, -d_3 + (a'b' - c)d_1), \\
P_3' &= \alpha_2(P_2) = \beta_2(P_3) = (-d_1, d_2, -d_3), \\
P_4' &= \beta_2(P_4) = (-d_1 + (ab - c')d_3, d_2 - bd_3, -d_3).
\end{align*}
\]

We claim that \(x'[d']\) also satisfies (SC’). Like before, assume \(d_2' \geq 0\). First compute:

\[
\begin{align*}
d_1' &= d_1, \\
d_2' &= a'd_1 - d_2 \text{ (because of the assumption \(-a'd_1 + bd_3 < 0\)}, \\
d_3' &= d_3 \text{ (recall \(-cd_1 + b'd_2 = c\lambda_1 \geq 0\)),} \\
v_4' &= (aa'd_1 - ad_2 - (ab - c')d_3, -a'd_1 + bd_3, -cd_1 + b'd_2).
\end{align*}
\]

Then show that the conditions in Lemma 3.11 (1) hold for \((i, j, k) = (2, 1, 3)\):

(1a) We have \(b_{21}' = a \geq 0\) and \(b_{13}' = ab - c \geq 0\).

(1b) This is straightforward.

(1c) We have \(B_2', B_1', B_3'\) are not strictly in the same half plane, and \(v_4' = \lambda_2'B_2' + \lambda_3'B_3'\) with \(\lambda_2' = (-cd_1 + b'd_2)/b' = c\lambda_1/b \geq 0\), \(\lambda_3' = (a'd_1 - bd_3)/b > 0\), so \(B_2', B_1', B_3', v_4'\) are in circular order.

(ii) Suppose \(-a'd_1 + bd_3 > 0\). See Figure 22.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure22}
\caption{(SC’), Case 4, \(\mu_2\), (ii). (Left: projection to \(xy\)-plane. Right: projection to \(xy\)-plane followed by reflection about \(x\)-axis.)}
\end{figure}
The quadrilateral $P_d$ is determined by $\alpha_2(P_4P_1P_2) = P'_4P'_1P'_2$ and $\beta_2(P_3P_4) = P'_2P'_3$. Thus

$$P'_1 = \alpha_2(P_1) = (-d_1, d_2 - a'd_1, -d_3 + (a'b' - c)d_1),$$
$$P'_2 = \alpha_2(P_2) = \beta_2(P_3) = (-d_1, d_2, -d_3),$$
$$P'_3 = \beta_2(P_4) = (-d_1 + (ab - c')d_3, d_2 - bd_3, -d_3),$$
$$P'_4 = \alpha_2(P_4) = (-d_1 + ad_2 - c'd_3, d_2 - bd_3, -b'd_2 + (bb' - 1)d_3).$$

We claim that, this $x'[d']$ also satisfies (SC). Like before, assume $d'_2 \geq 0$. We have

$$d'_1 = -\min(-d_1, -d_1 + (ab - c')d_3, -d_1 + ad_2 - c'd_3),$$
$$d'_2 = -\min(-d_2 - a'd_1, d_2, -d_2) = bd_3 - d_2$$

because $ab - c' > 0$ and $ad_2 - c'd_3 = c'\lambda_3 \geq 0$,

$$d'_3 = -\min(-d_3 + (a'b' - c)d_1, -d_3, -b'd_2 + (bb' - 1)d_3),$$
$$d'_4 = d_3$$

because $a'b' - c > 0$ and $d'_2 \geq 0$,

$$v'_4 = (-ad_2 + c'd_3, bd_3 - a'd_1, (a'b' - c)d_1 + b'd_2 - bb'd_3).$$

To show that the conditions in Lemma 3.11 (1) hold for $(i, j, k) = (1, 3, 2)$, the only nontrivial condition is (1c) $B'_1, B'_3, B'_2, v'_4$ are in circular order. To see this, note that $B'_1, B'_3, B'_2$ are not in the same half plane, and $v'_4 = \lambda'_1B'_1 + \lambda'_2B'_2$ with $\lambda'_1 = (bd_3 - a'd_1)/a' > 0$, $\lambda'_2 = (ad_2 - c'd_3)/a' = c'\lambda_3/a' \geq 0$.

(iii) Suppose $-a'd_1 + bd_3 = 0$. See Figure 23. The Newton polytope of $x'[d']$ is a triangle. We can view this as a degenerate case of either (i) or (ii), and the proof of (SC') therein still holds.

![Figure 23](image)

**Figure 23.** (SC'), Case 4, $\mu_2$, (iii). (Left: projection to $xy$-plane. Middle and Right: projection to $xy$-plane followed by reflection about $x$-axis.)

To show that the quadrilateral changes as expected under the mutation $\mu_3$:

Assume $\mu_3(Q)$ is cyclic, i.e. $bc - a' > 0$, $b'c' - a > 0$. See Figure 24. We have

$$B' := \mu_3(B) = \begin{bmatrix} 0 & a - b'c' & c' \\ bc - a' & 0 & -b \\ -c & b' & 0 \end{bmatrix}$$
The quadrilateral $\mathbf{P}_d$ is determined by $\alpha_3(P_2P_3) = P_2'P_3'$ and $\beta_3(P_2P_1P_4) = P_1'P_4'P_3'$. See Figure 24. Thus

$$
P_1' = \beta_3(P_1) = (-d_1, (a' - bc)d_1 + (bb' - 1)d_2 - bd_3, cd_1 - b'd_2 + d_3),
$$
$$
P_2' = \beta_3(P_2) = (-d_1, (bb' - 1)d_2 - bd_3, -b'd_2 + d_3),
$$
$$
P_3' = \alpha_3(P_2) = (-d_1 + b'c'd_2 - c'd_3, -d_2, -b'd_2 + d_3),
$$
$$
P_4' = \alpha_3(P_3) = \beta_3(P_4) = (-d_1 + ad_2 - c'd_3, -d_2, d_3).
$$

![Figure 24. (SC'), Case 4, $\mu_3$. (Left: projection to $yz$-plane. Right: projection to $yz$-plane followed by reflection about $y$-axis.)](image)

To show $x'[\mathbf{d}']$ satisfies (SC'). We can assume $d_3' \geq 0$ as before. Compute

$$
d_4' = -\min(cd_1 - b'd_2 + d_3, -b'd_2 + d_3) = b'd_2 - d_3,
$$
$$
d_2' = -\min((a' - bc)d_1 + (bb' - 1)d_2 - bd_3, (bb' - 1)d_2 - bd_3, -d_2)
$$
$$
= -\min(-d_2 + (bc - a')\lambda_1 + b\lambda_3, bd_3 - d_2 - d_2) = d_2,
$$
$$
d_1' = -\min(-d_1, -d_1 + b'c'd_2 - c'd_3, -d_1 + ad_2 - c'd_3)
$$
$$
= -\min(-d_1, -d_1 + c'd_3, -d_1 + c\lambda_3) = d_1,
$$
$$
v_4' = (-ad_2 + c'd_3, (a' - bc)d_1 + b'd_2 - bd_3, cd_1 - b'd_2).
$$

We claim the conditions in Lemma 3.11 (1) holds for $(i, j, k) = (1, 3, 2)$. Indeed:

For (1a): We have $b_{13}' = c' \geq 0, b_{32}' = b' > 0$.

For (1b): This is a straightforward check.

For (1c): We have $B_1', B_3', B_2'$ are not in the same half-plane, and $v_4' = \lambda_1' B_1' + \lambda_2' B_2'$ with coefficients $\lambda_1' = (((a' - bc)d_1 + b'd_2 - bd_3)/(bc - a')) = ((bc - a')\lambda_1 + b\lambda_3)/(bc - a') \geq 0$, $\lambda_2' = (-ad_2 + c'd_3)/(a' - b'c') = c'\lambda_3/(b'c' - a') \geq 0$.

This completes the proof of (SC') for all four cases.

6.4. **Proof of (NC').** The existence of $e(\mathbf{p})$ with nonzero constant term follows from (NC). To show the second part of (NC'), it suffices to show that the property that $e(\mathbf{p})$ is a monomial for each vertex of $\mathbf{P}_d$ is invariant under mutation. Suppose therefore that $\mathbf{p} \in \mathbb{Z}^3$ is a vertex of the weakly convex quadrilateral $\mathbf{P}_d$ of the cluster variable $x[\mathbf{d}]$ with respect to the initial seed $\Sigma_{t_0}$, and suppose that $e(\mathbf{p}) = y_1^{t_1} y_2^{t_2} y_3^{t_3}$. Let $\mathbf{P}'$ be the weakly convex quadrilateral of the same cluster variable but with respect to the seed $\Sigma_{\mu_3(t_0)}$. Thus $\mathbf{P}'$ is obtained from $\mathbf{P}$ by substituting $x_1$ by $(M_1 + M_2)/x_1$ (substituting $x_2, x_3$ can be argued similarly). By Lemma 3.13, the vertices of $|\mathbf{P}'|$ are obtained as either

(a) $\alpha_1(\mathbf{p})$, where $\mathbf{p}$ is a vertex of $|\mathbf{P}_d|$, the intersection of the line $\mathbf{p} + R B_1$ with the quadrilateral $\mathbf{P}_d$ is a line segment $|\mathbf{pq}|$ with $\mathbf{q} = \mathbf{p} + r B_1$ ($r \geq 0$), or

(b) $\beta_1(\mathbf{q})$, where $\mathbf{q}$ is a vertex of $|\mathbf{P}_d|$, the intersection of the line $\mathbf{q} + R B_1$ with the quadrilateral $\mathbf{P}_d$ is a line segment $|\mathbf{pq}|$ with $\mathbf{p} = \mathbf{q} - r B_1$ ($r \geq 0$).
We only need to consider (a) because (b) can be argued similarly. We use Lemma 3.12 with \( f = \sum e(p)x^p \) the Laurent expansion of \( x[d] \) in the seed \( \Sigma_0 \) and \( g \) its Laurent expansion in the seed \( \Sigma_{\mu_1(t_0)} \). The vertex \( p \) corresponds to the term \( a_0x^p \) (with \( b_0 = 0 \)) in the lemma and it transforms to the new vertex \( p' \) yielding the term \( a_0'x^{p'} \) (with \( b_0' = 0 \)) in \( g \). The lemma implies \( a_0' = (p^+)^{p_1}a_0 \) and thus

\[
e'(p') = (p^+)^{p_1}e(p) = \left( \prod_i y_i^{[e_i]+} \right)^{p_1}e(p) = \prod_i y_i^{r_i+p_1[e_i]+}\]

which is a Laurent monomial in \( y_1, y_2, y_3 \), so it is also a Laurent monomial in \( y_1', y_2', y_3' \). By induction on the number of mutations it follows that \( e(p) \) is a monomial for all vertices \( p \) of all \( |P_d| \). This completes the proof of condition (NC') and of Theorem 5.1.

7. Example

Example 7.1. Consider \( a = b = -c = 2 \):

\[
Q = \begin{array}{c}
1 \\
2 \\
3
\end{array}
\quad B = \begin{bmatrix}
0 & 2 & 2 \\
-2 & 0 & 2 \\
-2 & -2 & 0
\end{bmatrix}
\]

Consider the cluster variable \( x[6,2,1] \), obtained by the mutation sequence \( \{1,2,3\} \). The quadrilateral \( P_d \) is computed as in Lemma 3.11 as follows. Start with vertices

\[
\begin{align*}
P_1 &= (0,0,0) \\
P_3 &= P_1 + dB_2 = (4,-12,-16) \\
P_4 &= P_3 + dB_3 = (6,-10,-16)
\end{align*}
\]

and then shift by the vector

\[
-\min(P_1, P_2, P_3, P_4) - d = -(0,-12,-16) - (6,2,1) = (-6,10,15)
\]

to obtain the vertices of \( P_d \) as follows

\[
P_1 = (-6,10,15), P_2 = (-6,-2,3), P_3 = (-2,-2,-1), P_4 = (0,0,-1).
\]

On the other hand, the cluster has the following Laurent expansion.

\[
x[6,2,1] = x_1^{-6}x_2^{-2}x_3^{-1} (x_2^{12}x_3^{16} + 6x_1^{10}x_3^{14}y_1 + 2x_1^2x_2^{8}x_3^{10}y_2 + 15x_2^4x_3^8y_2 + 8x_1^2x_2^6x_3^8y_3)\]

\[
+20x_2^6x_3^{10}y_1 + x_1^4x_2^4x_3^4y_1^2 + 12x_2^2x_3^2y_2 + x_1^2x_2^6y_1^2 y_3 + 15x_2^4x_3^4y_1^4\]

\[
+2x_2^2x_3^2y_1^2 + 8x_2^2x_3^2x_1^4y_2 + 6x_2^2x_3^2y_1^2 + x_1^4y_1^2 + 2x_1^2x_3^6y_2 + x_3^4y_1^6).
\]

We project the support to 2nd and 3rd exponents of \( x \) (that is, draw a point of coordinate \((i,j)\) if \( x_i^2x_j \) appear in \( x[6,2,1] \)). We obtain the picture in Figure 25.

The monomials in corresponding positions are:

\[
\begin{bmatrix}
6x_1^{-6}x_3^{-1}y_1 & 15x_1^{-6}x_3^{-1}x_1^{-6}x_2^{13}y_1 & 6x_1^{-6}x_2^{13}y_1 \\
x_1^{-6}x_2^{-2}x_3^6y_2 & 15x_1^{-6}x_2^{-2}x_2^{13}y_1 & 6x_1^{-6}x_2^{13}y_1 \\
x_1^{-1}x_2^3x_3y_2 & 15x_1^{-1}x_2^3x_2^3y_1 & 6x_1^{-1}x_2^3x_2^3y_1 \\
x_1^{-1}x_2^3x_3y_2 & 15x_1^{-1}x_2^3x_2^3y_1 & 6x_1^{-1}x_2^3x_2^3y_1
\end{bmatrix}
\]
So after the substitution, we get $p_1^+ x_2^0 x_3^c + p_1^- / x_1 = (x_2^2 x_3^5 + y_1) / x_1$ in $x[6, 2, 1]$ (because $p_1^+ = 1$, $p_1^- = y_1$). The 7 terms with $x_1^{-6}$ (that is, the line segment $|P_1 P_2|$) adds up to be

$$f = x_1^{-6} x_2^{10} x_3^{15} + 6 x_1^{-6} x_2^{8} x_3^{13} y_1 + \cdots + x_1^{-6} x_2^{-2} x_3^{3} y_1^6 = x_2^{-2} x_3^{3} (\frac{x_2^2 x_3^2 + y_1}{x_1})^6$$

So after the substitution, we get $x_1^6 x_2^{-2} x_3^3$, which also follows in general using Lemma 3.12 (1), with $p = (-6, 10, 15)$, $q = (-6, -2, 3) = p + 6(0, -2, -2)$: since the first and last term of $q$ in that lemma will have exponents $p' = \alpha_1(p) = q' = \beta_1(q) = (6, -2, 3)$, which is $P'_1$.

In general, let $f$ be the sum of the terms containing $x_1^{-constant}$. Then after substitution the two endpoints are mapped by $\alpha_1$ and $\beta_1$, both being linear maps. Consider the line segments parallel to line $P_1 P_2$ and in the quadrilateral $P_1 P_2 P_3 P_4$. The two ends of each line segment are mapped by maps $\alpha_1$ and $\beta_1$. So the segment $P_1 P_4$ (i.e. the set of right endpoints) is mapped by $\alpha_1$, $P_2 P_3 P_4$ (i.e. the set of left endpoints) is mapped by $\beta_1$. Their image encloses a new quadrilateral $P_{a'} = P'_1 P'_2 P'_3 P'_4$ (which actually degenerates to a triangle) with

$$P'_1 = (6, -2, 3), \quad P'_2 = (2, -2, -1), \quad P'_3 = (0, 0, -1), \quad P'_4 = (0, 0, -1).$$

**Remark 7.2.** Note that the $F$-polynomial of $x[6, 2, 1]$ is $1 + 6y_1 + 2y_1^2 y_2 + 15y_1^2 + 8y_1^3 y_2 + 20y_1^3 + y_1^4 y_2^2 + 12y_1^4 y_2 + y_1^6 y_2^3 + 15y_1^4 + 2y_1^5 y_2^2 + 8y_1^5 y_2 + 6y_1^5 + y_1^6 y_2 + 2y_1^6 y_2 + y_1^6$. Its Newton polytope has vertices $(0, 0, 0), (6, 2, 1), (6, 2, 0), (6, 0, 0), (4, 2, 0)$. In contrast, the region $F_{d}$ as defined in Corollary 5.2 is a convex polyhedron with vertices $(0, 0, 0), (6, 0, 0), (8, 0, 2), (6, 2, 0), (5, 3, 0), (8, 0, 3)$, which contains the Newton polytope of the $F$-polynomial as a proper subset.

### 8. Quantum analogue

In this section, we prove that Theorem 5.1 generalizes to the quantum cluster algebras introduced in [2]. We consider here only principal coefficient. The statement with non-principal coefficients should follow easily from this.
First we fix some notation. For a nonzero integer $\delta$, define \([n]_\delta = (v^{\delta n} - v^{-\delta n}) / (v^{\delta} - v^{-\delta})\) (note that \([n]_\delta = [n]_{-\delta}\)) and define the quantum binomial coefficient (where $k, n \in \mathbb{Z}$, $k \geq 0$)
\[
\binom{n}{k}_\delta = \frac{[n]_\delta [n-1]_\delta \cdots [n-k+1]_\delta}{[k]_\delta [k-1]_\delta \cdots [1]_\delta}
\]
Define
\[
(x + y)_\delta^n = \sum_{k \geq 0} \binom{n}{k}_\delta x^k y^{n-k}
\]
For example,
\[
(x + y)_3^2 = y^3 + (v^5 + 1 + v^{-5})xy^2 + (v^5 + 1 + v^{-5})x^2y + y^3.
\]

**Remark 8.1.** To see the motivation of the above definition: consider two quasi-commuting variables $X, Y$ with $YX = v^{2\delta} XY$. Denote $X^{(i,j)} := v^{i\delta} X^i Y^j$. Then the above quantum binomial coefficients satisfy
\[
(X + Y)_\delta^n = \sum_{k \geq 0} \binom{n}{k}_\delta v^{k(n-k)\delta} X^k Y^{n-k} = \sum_{k \geq 0} \binom{n}{k}_\delta X^{(k,n-k)}.
\]

Let $B$ be a skew-symmetrizable matrix, $D$ a positive diagonal matrix such that $DB$ is skew-symmetric. Let
\[
\Lambda = \begin{bmatrix} 0 & -D \\ D & -DB \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ I \end{bmatrix}.
\]
Recall that the base quantum torus $\mathcal{T}(\Lambda)$ is the $\mathbb{Z}[v^\pm]$-algebra with a distinguished $\mathbb{Z}[v^\pm]$-basis \(\{X^e : e \in \mathbb{Z}^{2n}\}\) and the multiplication is given by
\[
X^e X^f = v^{\Lambda(e,f)} X^{e+f} \quad (e, f \in \mathbb{Z}^{2n})
\]
where $\Lambda(e, f) = e^T \Lambda f$.

We introduce the following convention to represent a quantum Laurent polynomial using a commutative Laurent polynomial: namely, we define a function
\[
\varphi : \mathbb{Z}[v^\pm][x_1, \ldots, x_6] \to \mathcal{T}(\Lambda)
\]
\[
\sum a_{i_1, \ldots, i_6} x_1^{i_1} \cdots x_6^{i_6} \mapsto \sum a_{i_1, \ldots, i_6} X^{(i_1, \ldots, i_6)}
\]
Recall that we denote $y_1 = x_1, y_2 = x_5, y_3 = x_6$.

We have the following generalization of our main result.

**Theorem 8.2.** A quantum cluster variable $x[d]$ with $d$-vector $d$ can be written as
\[
x[d] = \varphi \sum_{p \in \mathbb{Z}^3} e(p) x^p = \varphi \sum_{p_1, p_2, p_3} e(p_1, p_2, p_3) x_1^{p_1} x_2^{p_2} x_3^{p_3}
\]
where $e(p) \in \mathbb{Z}[v^\pm][y_1, y_2, y_3]$ is uniquely characterized by the following conditions:

- **(SC)** (Support condition) The coefficient $e(p) = 0$ unless $p \in P_d$. Equivalently, the Newton polytope of $x[d]$ is contained in $P_d$.
- **(NC)** (Normalize condition) There is only one $e(p)$ has a nonzero constant term, which must be 1. Moreover, the greatest common divisor of all $e(p)$ is 1.
(DC) (Divisibility condition) For each $k = 1, 2, 3$, if $p_k < 0$, then
\[
\left( \prod_{i=1}^{3} x_i^{[b_{ik}]+} + y_k \prod_{i=1}^{3} x_i^{[b_{ik}]+} \right)^{-p_k} \text{ divides } \sum_{p_1, \ldots, p_k, \ldots, p_3} e(p_1, p_2, p_3) x_1^{p_1} x_2^{p_2} x_3^{p_3}
\]

where the notation $\hat{p}_k$ under the sum means that we have $p_k$ fixed and the other two $p_i$ run over all integers.

Moreover, (NC) can be replaced by:

(NC') There is a coefficient $e(\mathbf{p})$ with nonzero constant term, and for each vertex $\mathbf{p}$ of the convex hull $[\mathbf{P}_d]$, $e(\mathbf{p})$ is a monomial in $y_1, y_2, y_3$.

And (SC) can be replaced by a stronger condition:

(SC') The Newton polytope of $x[\mathbf{d}]$ is equal to $\mathbf{P}_d$.

Proof. The proof is similar to Theorem 5.1. The main difference is the quantum version of the divisibility condition (DC), which follows easily from Lemma 8.3 below. □

Lemma 8.3. Fix $1 \leq k \leq 3$ and fix $p_k \in \mathbb{Z}$. Let $f$ be a Laurent polynomial
\[
f = \sum_{p_1, \ldots, p_k, \ldots, p_3} e(p_1, p_2, p_3) x_1^{p_1} x_2^{p_2} x_3^{p_3}
\]

where $e(p_1, p_2, p_3) \in \mathbb{Z}[v^\pm][y_1, y_2, y_3]$. Then $\varphi(f) \in \mathcal{T}(\Lambda)$ is a Laurent polynomial in $\mathbb{Z}[v^\pm][X^\pm_1, \ldots, (X_k^\pm', \ldots, X_3^\pm, X_4^\pm, X_5^\pm, X_6^\pm]$ if and only if
\[
\left( \prod_{i=1}^{3} x_i^{[b_{ik}]+} + y_k \prod_{i=1}^{3} x_i^{[b_{ik}]+} \right)^{[-p_k]} \text{ divides } f.
\]

Proof. Without loss of generality assume $k = 1$. So $p_1$ is fixed throughout the proof.

Let $\bar{B}' = \mu_1(\bar{B})$. By definition of mutation of quantum cluster variables,
\[
X_1 = X_1^{r-e_1+[-\bar{B}'_1]+} + X_1^{r-e_1+[-\bar{B}_1]+} = X_1^{r-e_1+[-\bar{B}_1]+} + X_1^{r-e_1+[-\bar{B}_1]+},
\]

where the second equality holds because $\bar{B}_1' = -\bar{B}_1$. We introduce the following notation: for $p, q \in \mathbb{Z}^3$, let
\[
X^{[p]}_{[q]} = X^{(p_1,p_2,p_3,q_1,q_2,q_3)}, \quad X^{[p]} = X^{(p_1,p_2,p_3,q_1,q_2,q_3)}
\]

We denote the Laurent expansion of $\varphi(f)$ in the original cluster (respectively in the cluster $\{X_1', X_2, X_3, X_4, X_5, X_6\}$) as follows
\[
\varphi(f) = \sum_{p_2, p_3, q} e_{pq} X^{[p]}_{[q]} \quad (\text{respectively } \varphi(f) = \sum_{p_2, p_3, q} e'_{pq} X^{[p']}_{[q']}),
\]

where $p = (p_1, p_2, p_3)$ and $p' = (p_1', p_2', p_3')$ and $p_1' = -p_1$. So $e(p_1, p_2, p_3) = \sum_{q} e_{pq} X^{[p]}_{[q]}$. For convenience of notation, we let $\bar{p} = (0, p_2, p_3)$, and similar for $\bar{p}'$. Then for fixed $p_1$,
\[
\sum_{p, q} e_{pq} v^{\varphi_1([p], p_1)} X^{[p]}_{[q]} X_1^{p_1} = \sum_{p', q'} e'_{pq'} X^{[p']}_{[q']}
\]

Now we prove the lemma in two cases: $p_1 < 0$ and $p_1 \geq 0$. 
(Case 1) $p_1 < 0$. We shall show that

$$
\begin{align*}
(8.1) \quad e_{pq} &= \sum_k e'_{p'q'} \left[ \frac{-p_1}{k} \right]_{\delta_1} \\

Indeed,

$$
\begin{align*}
\sum_{p,q} e_{pq} u^{-\Lambda([p]_{q}, p_1 e_1)} X_{[q]}^{[p]} &= \sum_{p',q'} e'_{p'q'} X'_{[q]}^{[p']} X^{p_1 e_1} \\

=& \sum_{p',q'} e'_{p'q'} X'_{[q]}^{[p']} (X'_{e_1} X^{p_1 e_1} + X^{e_1} X^{p_1 e_1}) \\

=& \sum_{p',q',k} e'_{p'q'} \left[ \frac{-p_1}{k} \right]_{\delta_1} X^{p_1 e_1 + k [-B_1] + (-p_1 - k) [B_1] +} \\

=& \sum_{p',q',k} e'_{p'q'} \left[ \frac{-p_1}{k} \right]_{\delta_1} u^{\Lambda([p]_{q}, p_1 e_1 + k [-B_1] + (-p_1 - k) [B_1] +)} \sum_k X^{[p']}_{[q]} \\

=& \sum_{p',q',k} e'_{p'q'} \left[ \frac{-p_1}{k} \right]_{\delta_1} u^{\Lambda([p]_{q}, p_1 e_1 + k [-B_1] + (-p_1 - k) [B_1] +)} \\

(The equality "\(\simeq\)" is because

$$
X^{e_1} X^{p_1 e_1} = u^{2\Lambda(-e_1 + e_1 + [B_1] +)} X^{e_1} X^{p_1 e_1} X^{e_1} X^{p_1 e_1}
$$

where $\Lambda'(e_1 + [B_1] + , e_1 + [B_1] +) = \Lambda'(e_1 + [B_1] + , B_1) = \Lambda(e_1, B_1) = -\delta_1$. The equality "\(\simeq\)" is because the exponent of $X'$ has zero in the first coordinate, so we can replace $X'$ by $X$). Now compare the coefficients of $X_{[q]}^{[p]}$ on both sides, we get

$$
\begin{align*}
(8.2) \quad e_{pq} u^{-\Lambda([p]_{q}, p_1 e_1)} &= \sum_k e'_{p'q'} \left[ \frac{-p_1}{k} \right]_{\delta_1} u^{\Lambda([p]_{q}, p_1 e_1 + k [-B_1] + (-p_1 - k) [B_1] +)} \\

\end{align*}
$$

where $p'$ and $q'$ are determined by

$$
\begin{align*}
\begin{bmatrix} \bar{p}' \\ \bar{q}' \end{bmatrix} &+ k [-B_1] + (-p_1 - k) [B_1] + = \begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix}
\end{align*}
$$

which can be rewritten as

$$
\begin{align*}
\begin{bmatrix} \bar{p}' \\ \bar{q}' \end{bmatrix} &+ (-p_1) [-B_1] + (-p_1 - k) B_1 = \begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix}
\end{align*}
$$

We claim that the exponents of $u$ on both sides of (8.2) are equal. Indeed,

$$
\begin{align*}
\Lambda'([\bar{p}]_{\bar{q}}, p_1 e_1 + k [-B_1] + (-p_1 - k) [B_1] +)

=& \Lambda([\bar{p}]_{\bar{q}}, -p_1 e_1 + p_1 [B_1] + k [-B_1] + (-p_1 - k) [B_1] +)

=& \Lambda([\bar{p}]_{\bar{q}}, -p_1 e_1 + k [-B_1] + k [B_1] +) = -\Lambda([\bar{p}]_{\bar{q}}, p_1 e_1) - k \Lambda([\bar{p}]_{\bar{q}}, B_1)
\end{align*}
$$
Moreover, $\Lambda([p\atop q], \tilde{B}_1)$ is the 1st coordinate of the following row vector, so is equal to 0:

$$\begin{bmatrix} p \\ q \end{bmatrix}^T \Lambda \tilde{B} = \begin{bmatrix} p \\ q \end{bmatrix}^T \begin{bmatrix} 0 & -D \\ D & -DB \end{bmatrix} \begin{bmatrix} B \\ I \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}^T \begin{bmatrix} -D \\ 0 \end{bmatrix} = -p^T D$$

$$= - \begin{bmatrix} 0 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{bmatrix} = - \begin{bmatrix} 0 & p_2\delta_2 & p_3\delta_3 \end{bmatrix}$$

Thus we can cancel out the exponents of $v$ on both sides of (8.2), and obtain (8.1).

The lemma then follows easily from (8.1):

$$f = \sum_{p,q} e_{pq} x^{[p\atop q]} = \sum_{p,q,k} e'_{p'q'} \begin{bmatrix} -p_1 \\ k \end{bmatrix} \begin{bmatrix} x^{[p\atop q]} \end{bmatrix}_{\delta_1}$$

$$= \sum_{p',q',k} e'_{p'q'} \begin{bmatrix} -p_1 \\ k \end{bmatrix} \begin{bmatrix} x^{[p'\atop q']} + p_1 e_1 + (-p_1)[-\tilde{B}_1] + (-p_1-k)\tilde{B}_1 \end{bmatrix}_{\delta_1}$$

$$= \sum_{p',q'} e'_{p'q'} x^{[p'\atop q']} \begin{bmatrix} x^{[p'\atop q']} + p_1 e_1 + (-p_1)[\tilde{B}_1] + (1 + x^{\tilde{B}_1})^{-p_1} \end{bmatrix}_{\delta_1}$$

$$= \sum_{p',q'} e'_{p'q'} x^{[p'\atop q']} \begin{bmatrix} \prod x_i^{[-b_{i1}]+} + y_1 \prod x_i^{[b_{i1}]+} \end{bmatrix}_{\delta_1}$$

therefore $f$ is divisible by $(\prod x_i^{[-b_{i1}]+} + y_1 \prod x_i^{[b_{i1}]+})^{-p_1}$ if and only if finite many $e'_{p'q'}$ are nonzero.

(Case 2) $p_1 \geq 0$. Similar to above, we have

$$e'_{p'q'} = \sum_{k} e_{pq} \begin{bmatrix} p_1 \\ k \end{bmatrix}_{\delta_1}$$

So it is always true that only finite many $e'_{p'q'}$ are nonzero. Meanwhile, the divisibility condition becomes “1 divides $f$” which is also always true. \qed

Example 8.4. The coefficients of the quantum cluster variable $x[6,2,1]$ corresponding to Example 7.1 are shown in the following matrix, where $[n] := [n]_1$:

$$\begin{bmatrix}
1 & 2 & [4] & [6] & [1] \\
[6][5][4] & 2 & [6][5] & [3][2] & [2] \\
[6][5] & [3][2] & [2][4] & [2] & [2] \\
1 & [2][4] & [6] & [3] & [1] \\
[2] & [2] & 1 & 1 & \end{bmatrix}$$
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