Dust ball physics and the Schwarzschild metric

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A physics-first derivation of the Schwarzschild metric is given. Gravitation is described in terms of the effects of tidal forces (or of spacetime curvature) on the volume of a small ball of test particles (a dust ball), freely falling after all particles were at rest with respect to each other initially. The possibility to express Einstein’s equation this way and some of its ramifications have been enjoyably discussed by Baez and Bunn [Am. J. Phys. 73, 644 (2005)]. Since the formulation avoids the use of tensors, neither advanced tensor calculus nor sophisticated differential geometry are needed in the calculation. The derivation is not lengthy and it has visual appeal, so it may be useful in teaching.

I. INTRODUCTION

With the increasing scope of its applications, general relativity (GR) has become a subject that is taught more and more frequently already in (upper-level) undergraduate courses. A physics-first approach has been developed and advocated in which the physical consequences of interesting metrics are explored before Einstein’s field equations. Similar goals are pursued in the so-called intertwined + active-learning approach. The Schwarzschild metric, in particular, is a useful tool in such a program, as it permits the quantitative discussion of the four classical tests of GR.

While this strategy is viable as a means leading to increased interest in, and deeper understanding of, general relativistic phenomena, it has the disadvantage that the metric will appear out of nowhere and its justification must wait until a lot of mathematics has been learned. It would then seem desirable to be able to obtain the metric from simple arguments avoiding the full glory and difficulty of tensor calculus and differential geometry.

These arguments must somehow replace the field equations in a derivation of the metric, as it is well-known that a metric describing curved spacetime cannot be derived without some ingredient going beyond the Einstein equivalence principle (EEP) combined with the Newtonian limit (NL). Detailed discussions why this is true for the Schwarzschild metric have been given in Refs. 7 and 8. However, deriving the perihelion shift with a slightly more general metric than the Schwarzschild one proves to be a bit lengthy. Some may consider it preferable to do the perihelion shift calculation only with the final form of the metric, which has fewer terms than the expansion used in Ref. 9.

Using two postulates, it is possible to derive the exact form of the Schwarzschild metric with little effort. Yet, the approach discussed in Ref. 10 requires a profound understanding of wave phenomena and is the outcome of a research project rather than a classroom method. It might be used in a class of exceptionally gifted students, demanding a flexibility of thinking that even some teachers in the field of relativity may not muster. This suggests to look for yet another approach that is both less computationally demanding than the one from Ref. 9 and not as radically innovative as the one from Ref. 10.

As it turns out, there is an additional way. It produces the Schwarzschild metric on the basis of the physical contents of the field equations but neither requires advanced tensor calculus nor mastery of differential geometry. The author became aware of this possibility through the beautiful paper The meaning of Einstein’s equation by J. C. Baez and E. F. Bunn. They explain the physics of the field equations in terms of the volume dynamics of a dust ball falling freely in the gravitational field. The physical law embodied in the field equations may be described in a single sentence that deserves to be cited: Given a small ball of freely falling test particles initially at rest with respect to each other, the rate at which it begins to shrink is proportional to its volume times: the energy density at the center of the ball, plus the pressure in the x direction at that point, plus the pressure in the y direction, plus the pressure in the z direction.” To distinguish it from the field equations proper, I will call this the dust ball (DB) law (of gravitation) in the following 12.

In the case of vacuum, there is no energy density nor pressure, so we may state the essence of the vacuum field equations in a tensor-free formula

\[ \frac{\ddot{V}}{V} \bigg|_{\tau=0} = 0, \]   (1)

where a dot signifies a derivative with respect to the proper time \( \tau \) of the center particle of the ball and \( V \) is the ball’s volume as measured by the center particle. Let us term this the DB vacuum equation.

Naturally, particles are assumed to be so small that the attraction between them due to their own gravitational field is negligible. Baez and Bunn warn that the DB law is not the formulation of Einstein’s equation that is most easily applied in various general relativistic settings. Often the field equations in tensorial form are easier to use.
and better suited to doing calculations. But it is certainly true that Eq. (1) is a physics-first formulation expressing the essence of the vacuum field equations (with the fine print added by Ref. 11 that it is fully equivalent to the field equations only, if the equation is required to hold for small balls of arbitrary initial velocities). To state and apply the law no advanced mathematical tools such as covariant derivatives, higher-rank differential forms or the curvature tensor are needed.

Clearly, Eq. (1) could be used in a GR course at an early stage, with the promise that its equivalence to the (vacuum) field equations would be proven later. This proof will not be given here, it is contained in Ref. 11. The purpose of this paper, then, is to use the DB vacuum equation in addition to the EEP [incorporating special relativity (SR)] and the NL to derive the metric outside a spherically symmetric mass distribution. We shall see that this is not entirely trivial but it is an exercise worth doing and it might be of use in classroom, yielding a true physics-first derivation of the Schwarzschild metric.

The remainder of this paper is organized as follows. Section II introduces the general form of a stationary spherically symmetric metric compatible with the choice of the circumferential radius as a radial coordinate and with time orthogonality. Then, the strategy for the calculation of the rate of volume change (1) based on the geodesic equations is discussed. A subtlety in the evaluation of the rate of volume change (1) based on the geodesic equations is discussed. A subtlety in the evaluation of the rate of volume change (1) based on the geodesic equations is discussed. A subtlety in the evaluation of the rate of volume change (1) based on the geodesic equations is discussed. A subtlety in the evaluation of the rate of volume change (1) based on the geodesic equations is discussed. A subtlety in the evaluation of the rate of volume change (1) based on the geodesic equations is discussed. A subtlety in the evaluation of the rate of volume change (1) based on the geodesic equations is discussed. A subtlety in the evaluation of the rate of volume change (1) based on the geodesic equations is discussed. A subtlety in the evaluation of the rate of volume change (1) based on the geodesic equations is discussed. A subtlety in the evaluation of the rate of volume change (1) based on the geodesic equations is discussed. A subtlety in the evaluation of the rate of volume change (1) based on the geodesic equations is discussed. A subtlety in the evaluation of the rate of volume change (1) based on the geodesic equations is discussed. A subtlety in the evaluation of the rate of volume change (1) based on the geodesic equations is discussed. A subtlety in the evaluation of the rate of volume change (1) based on the geodesic equations is discussed. A subtlety in the evaluation of the rate of volume change (1) based on the geodesic equations is discussed. A subtlety in the evaluation of the rate of volume change (1) based on the geodesic equations is discussed.

Equations (3) through (7) constitute five equations for four variables, but only four of them are independent. It is often useful to replace one of the four equations of motion by (3), a first integral, therefore all five equations have been written out here.

We may assume our initial dust ball to be spherical. During a sufficiently short time interval, it will then deform into an ellipsoid under tidal forces. By choosing a sufficiently symmetric initial state, we are able to predict the directions of the semi-axes. The volume of an ellipsoid with semi-axes \( \ell_a(\tau) \), \( \ell_b(\tau) \), and \( \ell_c(\tau) \) is given by \( V(\tau) = \frac{2}{3} \ell_a(\tau) \ell_b(\tau) \ell_c(\tau) \), and using that the dust particles are at rest with respect to each other at \( \tau = 0 \), we find for the rate of volume change described by Eq. (1)

\[
\frac{\dot{V}}{V} |_{\tau=0} = \frac{\dot{\ell}_a}{\ell_a} |_{\tau=0} + \frac{\dot{\ell}_b}{\ell_b} |_{\tau=0} + \frac{\dot{\ell}_c}{\ell_c} |_{\tau=0} .
\]

(The denominators on the r.h.s. are all equal, if our initial ellipsoid is a sphere.) What we have to do, essentially, is to calculate, for a short time interval, the trajectories of the center particle and of three particles at the ends of the three semi-axes, in order to obtain the relative rate of volume change, and to set this equal to zero at \( \tau = 0 \).

With the two highly symmetric configurations indicated, a dust ball that falls radially as shown in Fig. II and one that is in a circular orbit, Fig. II the effort can be even reduced further.

For a radially falling DB with initial velocity zero, the direction of motion of all the falling particles is towards the center of symmetry. Therefore, we need to know only by how much the central particle and the particle at the end of a radial semi-axis fall during a small time interval. The length changes of the two semi-axes in the angular directions follow simply from the radial interval fallen
by the center particle multiplied by the angular separation between that particle and the particle terminating the considered semiaxis, because this angle remains unchanged during radial fall of the particles. – In the case of a circular orbit of a DB, we know that the semiaxis along the direction of motion will not change its length, so we have to calculate only the change of two semiaxes.

Note that we cannot conclude from the DB vacuum equation that the volume \( V \) remains constant at all times. At first sight, it might be suggestive that if the volume of our ball does not change for an infinitesimal time interval, we may infer its constancy for arbitrary finite times. The reason this does not work is the side condition that the test particles in the ball must be at rest with respect to each other, initially. After the first deformation of the ball the particles are no longer at rest with respect to each other. They are moving apart or closing in, so the argument cannot be extended.

Before embarking on the calculation, let us do a practice example. That is, we try to calculate the terms of the right-hand side of Eq. (8) in a case where we know what the result has to be. This may reveal technical subtleties that have to be heeded.

### A. Dust ball falling freely in the Rindler metric

In this subsection, we will develop the correct procedure of calculating the rate of change of a dust ball’s diameter in the frame of its center particle, considering such a ball in an accelerating frame of reference.

The Rindler metric describes a set of accelerated observers, each of which has a constant (in time) proper acceleration, while the whole ensemble performs Born rigid motion. This means that observers at different positions along the direction parallel to the motion must experience different proper accelerations, so that their distance shrinks, from the point of view of an inertial observer, just in the right amount required by Lorentz contraction. Then, in the comoving frame of each Rindler observer distances between them remain constant. The inertio-gravitational field given by the Rindler metric is the closest analogy to a uniform gravitational field that is possible in general relativity. Since the metric can be obtained from the Minkowski metric by a global coordinate transformation, it corresponds to flat spacetime and is within the realm of SR.

This implies that we immediately know Eq. (1) to be satisfied in the Rindler metric for a freely falling dust ball because the local inertial system in which its center particle is at rest is even global, being described by the Minkowski metric in all of spacetime. Hence, if the other dust particles are at rest with respect to it at some initial time (and hence at rest with respect to each other), they will stay at rest with respect to it forever. In fact, this is even true for a dust ball of arbitrary size. Nevertheless, we will assume it to be small, because for a large ball the condition of all particles being at rest with respect to the center particle will not translate to being at rest in terms of Rindler coordinates; this only holds for objects in a sufficiently local system. And of course, the purpose of the practice calculation is to see whether we can get the result known to be true on physical grounds using
the untransformed coordinates of the Rindler metric.

The line element reads
\[ ds^2 = -\frac{g^2}{c^2}x^2 dt^2 + dx^2 + dy^2 + dz^2, \]

hence the Lagrangian is
\[ -\frac{c^2}{2} = \frac{1}{2} \left[ \frac{\ddot{g}^2}{c^2}x^2 \dot{t}^2 + \ddot{x}^2 + \dot{y}^2 + \dot{z}^2 \right], \]

leading to the equations of motion for freely falling particles
\[ \frac{d}{dt}(x^2\dot{t}) = 0, \]
\[ \ddot{x} = \frac{g^2}{c^2}x, \quad \ddot{y} = 0, \quad \ddot{z} = 0. \]

Let the center particle satisfy the initial conditions \( x(0) = x_0, \ y(0) = y_0 = 0 \) and all particles from the ball \( \dot{x}(0) = \dot{y}(0) = z(0) = 0 \), then we obviously have for all particles \( y(\tau) = \text{const.} \) and \( z(\tau) = \text{const.} \). For the center particle, we find \( x^2\dot{t} = \varepsilon = \text{const.} \) and inserting this into the definition of the Lagrangian, we get
\[ \frac{-g^2}{c^2}x^2 + \dot{x}^2 = -c^2. \]

This determines \( \varepsilon = c^2x_0/y_0. \) Next, we need an equation of motion for a particle located initially at \( x = x_0 + \varepsilon \dot{x}(0) \).

Note that because this satisfies the initial condition \( \delta\dot{x}|_{\tau=0} = 0 \), its constant of motion (energy integral) following from (11) will not be the same value \( \varepsilon \) as for the center particle, so we cannot evaluate \( \delta\dot{x}(\tau) \) by direct application of (13). However, multiplying through with \( x^2 \) and removing the constant of integration by taking the time derivative, we obtain
\[ \ddot{x} = \frac{c^2}{x} - \frac{\dot{x}^2}{x}, \]

an equation that holds for all particles from the DB. The motion of \( \delta\dot{x} \) is described by the so-called variational equation, obtainable from (13) by replacing \( x \) with \( x + \delta x \), \( \dot{x} \) with \( \dot{x} + \delta\dot{x} \), etc. and subtracting the equation for \( x \) from that for \( x + \delta x \).

This is the pedestrian’s way. A route that is a bit faster is to use \( \delta f(x, \dot{x}, \ddot{x}) = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial \dot{x}} \delta \dot{x} + \frac{\partial f}{\partial \ddot{x}} \delta \ddot{x}, \)

based on the fact that the proper time \( \tau \) is not varied along with the dependent variables. We then find
\[ \delta\dot{x} = \frac{c^2}{x^2} + \frac{\dot{x}^2}{x^2} \delta x - \frac{2\dot{x}}{x} \delta \dot{x}, \]
\[ \left. \frac{\delta \dot{y}}{\delta \dot{x}} \right|_{\tau=0} = \frac{c^2}{x^2} \times 0 > 0, \]

with the second line following from the fact that \( \dot{x}(0) = 0 \).

Since the \( y \) and \( z \) coordinates of our particles remain constant, we have \( \delta\dot{y} = \delta \dot{z} = 0 \). Hence, if we calculate the rate of volume change according to (8) (replacing \( \dot{\ell}_x \) with \( \delta\dot{x} \), \( \ell_t \) with \( \delta y \), etc.), we will not get zero!

Clearly, we must have missed something. The point to be observed here is that after a short time interval \( \tau \), the center particle will not be at rest, i.e., coordinate stationary in Rindler coordinates anymore, it will have acquired a small velocity \( -v = \dot{x}(0) \tau = -c^2\tau/x(0) \). Correspondingly, the Rindler frame accelerates with respect to the freely falling frame, reaching velocity \( v \) at time \( \tau \).

To obtain the spatial interval \( \delta x_c \) in the frame of the center particle, we have to apply a local Lorentz transformation
\[ \delta x_c = \gamma (\delta \dot{x} + v\delta t), \quad \delta t_c = \gamma (\delta \dot{t} + \frac{v}{c^2} \delta \dot{x}), \]

where \( \delta \dot{x} = \delta \dot{x} \) and \( \delta \dot{t} = \frac{\delta \dot{x}}{c^2} \delta t \) are the proper space and time intervals of a fiducial Rindler observer and \( \gamma = 1/\sqrt{1 - \frac{v^2}{c^2}} \). \( \delta t_c \) is the variation of the time interval in the frame of the center particle. But this is its proper time and we consider variations at constant proper time, hence \( \delta t_c = \delta \tau = 0 \). Then we have \( \delta \dot{t} = -(v/c^2)\delta \dot{x} \)

\[ \delta x_c = \gamma \delta x \left( 1 - \frac{v^2}{c^2} \right) \]

Thus the spatial interval in the frame of the center particle is found multiplying the Rindler frame interval by an appropriate Lorentz factor. The formula turns out to be the same as that for standard length contraction. We can make \( v \) as small as we like by considering arbitrarily small time intervals \( \tau \). But we must take a second derivative which renders the effect non-negligible. Expanding
\[ \delta x_c = \left( 1 - \frac{v^2}{2c^2} \right)\delta x = \left( 1 - \frac{c^2}{2x^2\tau^2} \right) \delta x, \]

we obtain the second derivative as \( \delta \ddot{x} - c^2/x^2 \delta x|_{\tau=0} \) for small \( \tau \), and this becomes exact for \( \tau \to 0 \), whence
\[ \left. \frac{\delta \ddot{x}_c}{\delta \dot{x}_c} \right|_{\tau=0} = \left. \frac{\delta \ddot{x}}{\delta \dot{x}} \right|_{\tau=0} - \left. \frac{c^2}{x^2} \right|_{\tau=0} = 0, \]

which looks right. Plugging the expressions for \( \delta \ddot{x}_c/\delta x_c, \delta \dot{y}_c/\delta y_c, \) and \( \delta \dot{z}_c/\delta z_c, \) the latter two being unchanged between the center particle frame and the Rindler frame, into formula (5) for the rate of volume change, we find zero, as we must.

What our practice calculation reveals is that we have to take into account velocity changes (referring to the coordinate stationary frame) of the center particle between the beginning and the end of the small proper time interval considered. Equipped with this cautionary cue, we now proceed to the central calculation.

**B. Distance rates of change**

In this subsection, we will obtain expressions for the rates of change of the semi-axes of our dust balls, when they are aligned with the axes of the spherical coordinate system.

We assume that the motion of the center particle remains in the plane \( \vartheta = \pi/2 \), which simplifies equations...
A spherical dust ball in radial free fall will elongate in the radial direction and shrink in the two angular ones, due to the convergence of the particle trajectories towards the center. Figure 1 tries to convey this qualitatively.

For a dust ball orbiting the center of symmetry of the system \((r = 0)\) on a circular trajectory, tidal forces will also tend to elongate the ball in the radial direction, because particles farther from that center will be too fast to be kept at the radius of the orbit, so the will move outwards, and particles closer to the center will be too slow for their orbit, so they are drawn inwards. Particles ahead of or behind, the center particle of the DB on its equilibrium trajectory will neither move away nor get closer to it. Finally, particles above and below the equatorial plane will be drawn towards it, so there is shrinkage along the \(\vartheta\) direction. An attempt was made to depict this behavior approximately in Fig. 2.

In both configurations, we may assume the semiaxes of the ellipsoid to be oriented along the coordinate directions for symmetry reasons. If the radial diameter of the ball is \(2\delta r\) and its angular extensions are \(2\delta\vartheta\) and \(2\delta\varphi\), respectively, the semiaxes of the ellipsoid are given by

\[
\delta r = \sqrt{h(r)}\delta r, \\
\delta\vartheta = r\delta\vartheta, \\
\delta\varphi = r\sin \vartheta \delta\varphi.
\]

(20)

(21)

(22)

Initial conditions may be obtained from

\[
\delta\ell_r = \frac{h'}{2\sqrt{h}}\vartheta\delta\vartheta + \sqrt{h}\delta\varphi, \\
\delta\ell_\vartheta = i\delta\vartheta + r\delta\varphi, \\
\delta\ell_\varphi = i\delta\varphi + r\delta\varphi,
\]

(23)

(24)

(25)

where for brevity we drop the argument of the functions of \(r\) and \(\vartheta\) and where in the last equation, we already have used that \(\vartheta = \pi/2\). If the particles of the ball are initially at rest with respect to the center particle, \(\delta\ell_r, \delta\ell_\vartheta, \) and \(\delta\ell_\varphi\) must all be zero at \(\tau = 0\). Therefore, if \(\vartheta|_{\tau=0} = 0\), we obtain as initial condition for the variation of \(r\) \(\delta\vartheta|_{\tau=0} = 0\), from Eq. (23). This is true for both configurations that we consider. Equally, we may conclude \(\delta\vartheta|_{\tau=0} = 0\) and \(\delta\varphi|_{\tau=0} = 0\) for particles aligned along the \(\vartheta\) and \(\varphi\) semiaxes. The treatment of, say, a DB falling inward on a radial trajectory with an initially nonzero radial velocity would be much more complicated, because then the initial first-order time derivatives of the variations of the three variables would also be nonzero.

Because we need to compute second derivatives at \(\tau = 0\) only, we immediately drop the vanishing first-order derivative terms in the following expressions

\[
\frac{\delta\ell_r}{\delta r} = \frac{h'}{2h} \frac{\delta r}{\delta r}, \\
\frac{\delta\ell_\vartheta}{\delta\ell_\vartheta} = \frac{\vartheta}{r} + \frac{\delta\vartheta}{\delta\vartheta}, \\
\frac{\delta\ell_\varphi}{\delta\ell_\varphi} = \frac{\varphi}{r} + \frac{\delta\varphi}{\delta\varphi},
\]

(26)

(27)

\[
\frac{\delta\ell_r}{\delta\ell_\vartheta} = \frac{\delta\ell_r}{\delta\ell_\varphi} = \frac{h^2}{c^2} \frac{\delta r}{\delta\varphi} = \frac{h^2}{c^2} \frac{\delta r}{\delta\vartheta} + \frac{h^2}{c^2} \frac{\delta r}{\delta\varphi},
\]

(28)

To evaluate these formulas, we have to obtain \(\dot{r}\) from the equations of motion for the center particle and \(\delta\varphi, \delta\vartheta, \) and \(\delta\varphi\) from the equations of motion for a particle displaced in one of the coordinate directions with respect to the center particle by either \(\delta r, \delta\vartheta\) or \(\delta\varphi\). Since these displacements are small, we may obtain the needed equations by linearization about the trajectory of the center particle.

The sum of the three terms given by Eqs. (26) through (28), taken at \(\tau = 0\), may not yet be what we need as the rate of volume change appearing in Eq. (11), because the result refers to the coordinate stationary frame of our metric. If the velocity of the center particle of the dust ball changes during the small time interval considered, we have to apply a correction using a Lorentz factor such as the one necessary in the Rindler metric.

For the DB in a circular orbit, neither the radial coordinate nor the polar angle \(\vartheta\) of the center particle change by free fall. The azimuthal angle \(\varphi\) changes in time, but its coordinate velocity remains constant. We need not calculate any correction for Eq. (28) anyway, because we know by symmetry that \(\delta\ell_r/\delta\ell_\varphi = 0\). The volume change is calculable using Eqs. (26) and (27) only.

In the case of the DB falling along a radial trajectory, the \(\vartheta\) and \(\varphi\) coordinates of the center particle remain unchanged, so we do not need any correction either. Moreover, we know that \(\delta\vartheta = \delta\varphi = 0\) in that case, because all particles (starting with initial velocity zero) will fall towards the center and hence, their angular coordinates will remain unchanged. Tidal effects in the \(\vartheta\) and \(\varphi\) directions are trivial here.

However, we have to calculate a correction for the \(r\) direction. Multiplying \(\delta\ell_r\) by the appropriate Lorentz factor \(\sqrt{1 - \frac{\vartheta^2}{c^2}} = \sqrt{1 - \frac{1}{c^2} h(x)^2}\), we obtain for the relative rate of length change in the frame of the center particle

\[
\frac{\delta\ell_\vartheta}{\delta\ell_r} = \frac{h^2}{c^2} \frac{\delta r}{\delta\vartheta} = \frac{h^2}{c^2} \frac{\delta r}{\delta\varphi} + \frac{h^2}{c^2} \frac{\delta r}{\delta\varphi},
\]

(29)

everything to be evaluated at \(\tau = 0\).

C. Radial fall of a dust ball

In this subsection, we will find a constraint on the metric by imposing Eq. (11) on the volume rate of change of a dust ball falling radially, as in Fig. 1 from an initial “rest” state.

We put the center of our ball of test particles at the initial radius \(r_0\), take \(\vartheta = \pi/2\) and may take, without restriction of generality, \(\varphi = 0\) as initial azimuthal angle. Then the ball is dropped with zero initial velocity with respect to a local coordinate stationary observer. This defines the initial rest state.

Equations (1) and (2) give us \(\dot{\vartheta} = \dot{\varphi} = 0\). To completely determine the motion we consider Eqs. (1) and (2), which
provide
\[ \dot{t} = \frac{\varepsilon_r}{f(r)} , \quad \varepsilon_r = \text{const.} \quad (30) \]
\[ \dot{r}^2 = \frac{c^2}{h(r)} \left( \frac{\dot{r}^2}{f(r)} - 1 \right) , \quad (31) \]
and the condition \( \dot{r} = 0 \) at \( r = r_0 \) determines the constant: \( \varepsilon_r = \sqrt{f(r_0)} \). Since the equation contains only a single constant of motion, we may obtain a second-order equation of motion for all dust particles involving \( r \) alone simply by isolating the constant [multiplying Eq. (31) by \( f(r)h(r) \)] and taking the proper time derivative:
\[ \ddot{r} = -\frac{c^2 f'}{2fh} \left( \frac{f h'}{2f} \right) \dot{r}^2 . \quad (32) \]
From this emerges one of the quantities required in the evaluation of (26):
\[ \ddot{r}(0) = -\frac{c^2 f'}{2fh} \bigg|_{r=r_0} . \quad (33) \]
Taking the variation of (32), we get
\[ \delta \ddot{r} = -\frac{c^2}{2fh} \left( \frac{f'}{f} \right)^\prime \delta r - \frac{\left( \frac{f h'}{2f} \right)^\prime}{2fh} \delta r - 2 \frac{(f h')'}{2fh} \dot{r} \delta \dot{r} . \quad (34) \]
We need this only at \( \tau = 0 \), where \( \dot{r} = 0 \), so we drop the second and third terms, which leads to
\[ \delta \ddot{r}|_{\tau=0} = -\frac{c^2}{2h} \left[ \frac{f f''}{f^2} - \frac{f'}{f} \left( \frac{f'}{f} + \frac{h'}{h} \right) \right] \delta r . \quad (35) \]
Since particles from the ends of the \( \vartheta \) and \( \varphi \) semiaxes of our ellipsoid fall at constant angle \( \delta \vartheta \) and \( \delta \varphi \), respectively, we infer from (27) and (28)
\[ \frac{\delta \ddot{\vartheta}}{\delta \ell_{\vartheta}} = \frac{\delta \ddot{\varphi}}{\delta \ell_{\varphi}} = \ddot{r} = \frac{\dot{r}}{r} . \quad (36) \]
Putting everything together, we find
\[ \frac{\dot{V}}{V} \bigg|_{\tau=0} = \frac{\delta \ell_{\vartheta}}{\delta \ell_{\vartheta}} \bigg|_0 + \frac{\delta \ell_{\varphi}}{\delta \ell_{\varphi}} \bigg|_0 + \frac{\delta \ell_{\varphi}}{\delta \ell_{\varphi}} \bigg|_0 \\
= -\frac{c^2}{2h} \left[ \frac{f f''}{f^2} - \frac{f'}{f} \left( \frac{f'}{f} + \frac{h'}{h} \right) \right] + \frac{h'}{2h} \dot{r} \ddot{r} - \frac{h^2}{c^2} \ddot{r} + 2 \ddot{r} \bigg|_0 = \frac{c^2}{2h} \frac{f''}{f} + \frac{c^2 f f'}{4h} \frac{f'}{f} + \frac{h'}{h} - \frac{c^2 f'}{fh} \frac{1}{r^2} \ddot{r} \bigg|_0 = 0 . \quad (37) \]
Of course, the value of \( r \) is \( r_0 \) in the last equation, but since \( r_0 \) can be taken arbitrarily, we may drop the subscript. The last line of Eq. (37) is a differential equation in \( r \) that the metric functions must satisfy in order for the DB vacuum equation (11) to hold. It can be simplified a bit (multiplying by \( -2fh/c^2 f' \)), which yields
\[ \frac{f f''}{f'} - \frac{1}{2} \left( \frac{f'}{f} + \frac{h'}{h} \right) + \frac{2}{r} = 0 . \quad (38) \]
In principle, this completes the analysis of a dust ball falling radially. However, we realize that our differential equation is second order for \( f \), so we will need at least two boundary conditions for \( f - \) and one for \( h - \) when finally trying to solve it. One boundary condition for each of the two functions is trivial: we require the line element (2) to become Minkowskian as \( r \to \infty \). Then we have \( \lim_{r \to \infty} f(r) = 1 \) and \( \lim_{r \to \infty} h(r) = 1 \).

A second boundary condition for \( f \) may be obtained as follows: Eq. (31) has the form of a one-dimensional law of energy conservation in Newtonian mechanics. Setting \( r_0 = \infty \) (\( \Rightarrow \varepsilon_r = 1 \)) and multiplying by \( m/2 \), we get
\[ \frac{m}{2} \frac{d^2 r}{dt^2} - \frac{GmM}{r} = 0 . \quad (39) \]
which we require to become identical, for sufficiently large \( r \), to the corresponding law of energy conservation obtained from Newton’s law of gravitation (for a particle of mass \( m \) the velocity of which becomes zero at infinity):
\[ \frac{m}{2} \frac{d^2 r}{dt^2} - \frac{GmM}{r} = 0 . \quad (40) \]
\( G \) is Newton’s gravitational constant. Using \( h(r) \sim 1 \) (\( r \to \infty \)), we have \( 1/f(r) - 1 \sim 2GM/c^2 r (r \to \infty) \) and
\[ f(r) \sim \frac{1}{1 + \frac{2GM}{c^2 r}} \sim 1 - \frac{2GM}{c^2 r} \quad (r \to \infty) . \quad (41) \]
This second boundary condition that \( f(r) \) must satisfy at large \( r \) introduces the Schwarzschild radius \( r_S = 2GM/c^2 \).

**D. Dust ball in circular orbit**

In this subsection, we repeat the procedure of Sec. 1C for a dust ball circling the mass distribution, as in Fig. 2, with all dust grains initially having the same velocity as the center particle (a notion that makes sense because of the closeness of the particles). A second constraint on the metric will be obtained.

The motion of a particle in a circular orbit \( r = r_0 \) is characterized by two constants of motion, arising via integration of Eqs. (4) and (7)
\[ \dot{t} = \frac{\varepsilon_c}{f(r_0)} , \quad \varepsilon_c = \text{const.} \quad (42) \]
\[ r_0^2 \dot{\phi} = n = \text{const.} \quad (43) \]
They describe conservation of energy and of angular momentum, respectively. The first equation implies \( \dot{t} = \text{const.} \), the second \( \dot{\phi} = \omega = \text{const.} \) is the angular frequency of the particle referred to its proper time. It is then convenient to use these constants in the definition of the Lagrangian (3) and in the radial geodesic equation (5) to determine their values in terms of the radius \( r_0 \) of the orbit (using \( \dot{r} = \dot{\varphi} = 0 \)):
\[ -\frac{c^2}{f(r_0)} + r_0^2 \omega^2 = -c^2 , \]
\[ f'(r_0) c^2 + \varepsilon_c^2 \frac{\varepsilon_c^2}{f(r_0)^2} - r_0 \omega^2 = 0, \]  \hspace{1cm} (44)

which yields (we drop radial arguments again)

\[ \varepsilon_c^2 = \frac{f'^2}{f - r_0 f'/2}, \quad \omega^2 = \frac{c^2 f'}{2 r_0 f - r_0 f'/2}. \]  \hspace{1cm} (45)

This completes the description of the trajectory of the dust ball’s center particle.

As to the variations giving the trajectories of slightly displaced particles, we start with the equation that is simplest to treat. This is Eq. (40). Because \( \dot{\vartheta} = 0 \) and \( \vartheta = \pi/2 \), terms of the equation in which neither \( \vartheta \) nor its derivative are varied must vanish. Thus we have

\[ \frac{d}{dt} \left[ r^2 \dot{\vartheta} \right] - \varphi^2 \dot{\varphi} \frac{d}{d\vartheta} \left( \sin \vartheta \cos \vartheta \right) \delta \vartheta = 0, \]

\[ \dot{\vartheta} + \omega^2 \delta \vartheta = 0. \]  \hspace{1cm} (46)

This equation can be integrated exactly and shows particles that are slightly displaced in the direction of the polar angle to oscillate about the equatorial plane, as long as they remain close enough to the center particle. We are interested only in the short-time behavior. From Eq. (27) and the fact that \( \dot{r} = r_0 = 0 \), we note

\[ \frac{\delta \ddot{\vartheta}}{\delta \ell_0} \frac{\delta \ddot{\varphi}}{\delta \vartheta} = -\omega^2. \]  \hspace{1cm} (47)

To arrive at the corresponding result for the radial direction is a bit more tedious than in the case of purely radial fall, because the variational equations for all variables but \( \vartheta \) are coupled. Taking the variations of Eqs. (5), (3), and (7) and dropping all terms containing a factor \( \dot{\vartheta} = 0, \dot{\varphi} = 0 \), we get

\[ \delta \vec{r} = -\frac{1}{2} \left( \frac{f'}{h} \right) c^2 \dot{\vartheta}^2 \delta r - \frac{f'}{h^2} c^2 \dot{\vartheta} \delta t \right] + \left( \frac{r'}{h} \right) \varphi^2 \delta r + \frac{2 r'}{h} \delta \varphi, \]

\[ = 0 - f' c^2 \dot{\vartheta}^2 \delta r - 2 f' c^2 \dot{\vartheta} \delta t + 2 r' \varphi^2 \delta r + 2 r^2 \delta \varphi, \]

\[ = \delta \varphi \frac{2 r}{r^2} \delta \dot{r}, \]  \hspace{1cm} (48)

\[ \delta \varphi = \frac{2}{r^2} \dot{r} \delta \varphi. \]  \hspace{1cm} (49)

After having varied the coordinates, we may use the integrals of motion for the variables referring to the center particle (and we set \( r = r_0 \)). Eq. (49) then simplifies to

\[ \varepsilon_c c^2 \delta t = \varepsilon_0 \omega \delta \varphi, \]  \hspace{1cm} (50)

i.e., the \( \delta r \) term vanishes. We may use this equation to remove \( \delta \varphi \) from Eq. (18), which then reduces to

\[ \delta \vec{r} = \delta \varphi \left[ \frac{1}{2} \left( \frac{f''}{h} - \frac{f'h'}{h^2} \right) c^2 \varepsilon_c^2 + \left( \frac{1}{h} \frac{r_0 h'}{h^2} \right) \omega^2 \right] \]

\[ + \delta \varphi \left( -\frac{f'}{h} c^2 \varepsilon_c^2 r_0 + \frac{2 r_0}{h} \omega \right). \]  \hspace{1cm} (52)

We would like to eliminate the factor \( \delta \varphi \) from this equation, too. It looks as if this might be achieved via integration of Eq. (50), but to do so, we need an initial condition for \( \delta \varphi \). Note that this cannot be taken from Eq. (25), which tells us that \( \delta \varphi = 0 \) along the \( \varphi \) direction. The initial conditions discussed in Sec. [11] referred to particles displaced from the center particle along a local coordinate axis, i.e., the other two coordinates were kept constant. However, here we need an initial condition for \( \delta \varphi \) as \( r \) changes. Fortunately, the situation is simple enough to infer the correct condition easily. The velocity component along the \( \varphi \) direction of a selected particle relative to the center particle is given, as long as the relative motion is slow, by \( r \varphi - r_0 \omega \). A condition for initial relative rest is then \( r \varphi \big|_{\tau=0} = r_0 \omega \) or, if expressed in terms of the displacement coordinates \( \delta r \) and \( \delta \varphi \),

\[ \delta r \varphi \big|_{\tau=0} + r_0 \omega \delta \varphi \big|_{\tau=0} = 0. \]  \hspace{1cm} (53)

Since we have to solve (52) only for an infinitesimal time interval, this initial condition may be directly used to express \( \delta \varphi \) in the equation. It is not necessary to first solve (50). Essentially, all the coefficient functions of the differential equation (52) are only required at \( \tau = 0 \). Setting \( \delta \varphi = -\delta r \omega /r_0 \), we obtain

\[ \frac{d}{dr} \left[ f''(r) - \frac{f'}{h} \right] c^2 \frac{\varepsilon_c^2}{f} + \omega^2 \left( \frac{1}{h} \frac{r_0 h'}{h^2} + \frac{r_0 f'}{h} - \frac{2}{h} \right). \]  \hspace{1cm} (54)

Finally, we insert the definitions (5) of \( \varepsilon_c^2 \) and \( \omega^2 \) into Eq. (54) to find

\[ \frac{\delta \vec{r}}{\delta \ell} = \frac{\delta \varepsilon}{\delta \ell} \frac{\delta \varphi}{\delta \vartheta} = \frac{2}{h(f - r_0 f'/2)} \left( -f'' - \frac{f'}{r_0} + \frac{2 f'^2}{f} \right), \]  \hspace{1cm} (55)

wherefrom, after combination with (17), we arrive at

\[ \frac{\dot{V}}{V} \bigg|_{\tau=0} = \frac{\delta \vec{r} \big|_{\tau=0}}{\delta \ell_0} = \frac{\delta \vec{r} \big|_{\tau=0}}{\delta \ell_0} \bigg|_{\tau=0} \]

\[ = \frac{2}{2(h(f - r_0 f'/2))} \left( \frac{1}{h} \left( -f'' - \frac{f'}{r_0} + \frac{2 f'^2}{f} \right) - \frac{f'}{r_0} \right) \]

\[ = \frac{1}{r^2 - \frac{h}{r} \left( f'' - \frac{f'}{r_0} + \frac{2 f'^2}{f} \right) - \frac{f'}{r_0} - \frac{1}{r}. \]  \hspace{1cm} (56)

Again, we may simplify a bit by removing common prefactors, and we rename \( r_0 \) to \( r \). Our second equation for the functions \( f \) and \( h \) then reads

\[ \frac{h}{r} = \frac{f''}{f} - \frac{f'}{f} + \frac{1}{r}. \]  \hspace{1cm} (57)

### III. SOLUTION FOR THE METRIC

All that remains to be done is to solve the system of ordinary differential equations (38) and (57). The fact that in both equations the explicit appearance of the independent variable is in the inverse form \( 1/r \) suggests that a simplification may arise, if we transform to the new variable \( u = 1/r \). Setting

\[ f(r) = F(u), \quad h(r) = H(u), \quad f'(r) = -u^2 F'(u), \quad h'(r) = -u^2 H'(u), \]
\[ f''(r) = 2u^3 F'(u) + u^4 F''(u), \]

the equations become

\[ \frac{F''}{F'} = \frac{1}{2}\left(\frac{F'}{F} + H'\right), \]

\[ H = 1 + u\left(\frac{F''}{F'} - \frac{F'}{F}\right), \]

where we have again suppressed the argument for brevity. Equation (60) can be directly integrated once:

\[
\ln F' = \frac{1}{2} \ln FH + \hat{\alpha} \quad \Rightarrow \quad F'^2 = \alpha^2 FH.
\]

Herein, \( \hat{\alpha} \) and \( \alpha \) are constants of integration \( (\alpha = e^{\hat{\alpha}}) \). Inserting \( H \) from (60), we obtain an equation for \( F \) alone:

\[
F'^2 = \alpha^2 \left[ F + u\left(\frac{F''F}{F'} - F'\right)\right].
\]

To find the general solution to this equation is difficult.\(^{22}\) On the other hand, a particular solution may be found by inspection: obviously, there must be a solution of the form \( F' = \text{const.}; \) the \( F'' \) term vanishes in this case, and the product of \( u \) and \( -F' \) just cancels the linear term of \( F \). The remainder of the equation is a relationship between constants. Let us then set \( F = Au + B \) and plug it into Eq. (62). This gives

\[
A^2 = \alpha^2 B \quad \Rightarrow \quad F(u) = \alpha\sqrt{B}u + B
\]

\[
\Rightarrow \quad f(r) = \frac{\alpha\sqrt{B}}{r} + B
\]

and using the asymptotics (31), we can read off both constants \( (B = 1, \alpha = -rs) \). As it turns out, the solution obtained by inspection is precisely the one we need, it satisfies the physical boundary conditions.

Therefore, \( F(u) = 1 - rsu \), and from (61), we find

\[
H(u) = \frac{F'^2}{\alpha^2 F} = \frac{1}{1 - rsu}.
\]

We end up with both metric functions, and hence the Schwarzschild metric, being determined:

\[
f(r) = 1 - \frac{rs}{r}, \quad h(r) = \frac{1}{1 - rs/r}.
\]

IV. CONCLUSIONS

What we have accomplished is a derivation of the Schwarzschild metric in a way I would consider completely physics first. The fundamental law Eq. (11), on which the derivation is based, does not have the form of a field equation. It is expressed in terms of physical objects, balls of test particles that can be easily visualized.

The derivation is shorter than calculations based on the traditional tensor calculus, such as Schwarzschild’s original one.\(^{22}\) When antisymmetry is built into the tensor formalism as in the theory of differential forms, the resulting gain in efficiency permits derivations that are more concise. In Grøn and Hervik’s book\(^{22}\) the calculation is done in four pages – but not before page 215. A lot of mathematics has to be learned up to that point, whereas the approach given here uses standard calculus.

Its main advantage lies, however, in its transparency and visual appeal. What happens physically can be easily imagined. The only purely mathematical step is the final solution of the two differential equations in Sec. III.

There are two ways to present the approach. In the first, no reference to the field equations is needed at all. The DB vacuum equation may be taken as the postulate of a new physical law. Indeed, this postulate is easy to motivate, because it also holds in Newtonian physics, with proper time replaced by the absolute Newtonian time and without restriction to a particularly moving frame – volumes are frame independent in Newtonian physics. Given this law, the derivation of Newton’s universal law of gravitation is a three-liner, assuming spherical symmetry of the potential to be determined.

Once it is accepted as the physical meaning underlying Newton’s law of gravitation outside a mass distribution, its generalization to the relativistic case is straightforward and follows a standard scheme when going from classical mechanics to SR: replace time by proper time and specify the (inertial) frame, in which the law holds. This gives the form (1) of the DB vacuum equation.

To motivate the full DB law of gravitation starting from a Newtonian version is less straightforward. It is clear that the mass density should be replaced by something involving energy density and it is plausible due to the special relativistic relationships in which energy is only part of a four vector that momentum flow must enter the equations as well, which leads to the appearance of pressure.\(^{24}\) However, it takes a leap of faith to be sure that energy and pressure (or stresses) will appear in the DB law precisely as stated. On the other hand, if we wish to just derive the Schwarzschild metric, we need only the DB vacuum equation, for the validity of which simple arguments are available. If cosmological problems are to be treated by the method as well, as is done in Ref. 11, then the full DB law is needed and, instead of making it a postulate, it may be preferable to present it as a lookahead to Einstein’s equation.

The postulational approach would be particularly useful in an undergraduate course in which the field equations were to be omitted completely.

Alternatively, if the field equations are presented in the course anyway, which will certainly be true in most graduate courses, a second way of presentation may be more appropriate. Spell out the DB law of gravitation, state it to be a particular formulation of the physical law that will be expressed in terms of partial differential equations later, promise a rigorous derivation of the DB law then, and proceed with physical applications. The Schwarzschild metric will thus not appear out of the blue but find some justification from an underlying law. Cosmological models may be discussed without first in-
Introducing the Riemann curvature tensor.

It might be added that a difference in philosophy between the DB law and the Einstein equation is that the former is a Lagrangian description, working in the frame of the dust ball, whereas the latter is Eulerian in nature. Lagrangian descriptions tend to be simpler locally, but their extension to all of space is not as natural as that of Eulerian ones. In complicated cases, global solutions will be more easily obtained within a Eulerian framework. The Schwarzschild case is simple enough to be solved in a Lagrangian scheme as well, here even in one that gets by without partial differential equations.

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2. T. Chen, *Relativity, Gravitation and Cosmology* (Oxford University Press, Oxford, 2005).
3. J. B. Hartle, “General relativity in the undergraduate physics curriculum,” Am. J. Phys. 74, 14–21 (2006).
4. I will use the terms *Einstein’s field equations, Einstein’s equation*, and *field equations* interchangeably. They all mean the same thing.
5. N. Christensen and T. Moore, “Teaching general relativity to undergraduates,” Physics Today 65(6), 41–47 (2012).
6. T. A. Moore, *A General Relativity Workbook* (University Science Books, Sausalito, 2013).
7. A. Schild, “Equivalence Principle and Red-Shift Measurements,” Am. J. Phys. 28, 778–780 (1960).
8. R. P. Gruber, R. H. Price, S. M. Matthews, W. R. Cordwell, and L. F. Wagner, “The impossibility of a simple derivation of the Schwarzschild metric,” Am. J. Phys. 56, 265–269 (1988).
9. K. Kassner, “Classroom reconstruction of the Schwarzschild metric,” Eur. J. Phys. 36, 065031 (1–20), (2015).
10. K. Kassner, “How to obtain the Schwarzschild metric before Einstein’s field equations,” (2016), arXiv:1602.08309v2 [gr-qc].
11. J. C. Baez and E. F. Bunn, “The meaning of Einstein’s equation.” Am. J. Phys. 73, 644–652 (2005).
12. Note that in general, the condition of the dust particles in the ball initially being at rest with respect to each other makes sense only for a sufficiently small ball. General relativity does not allow us to assign a physical meaning to relative velocities, hence a relative state of rest, for objects that are not very close to each other.
13. In standard relativistic Lagrangian mechanics, the action of a free particle between two events is, up to a prefactor, the integral of the line element \( \int ds \). The Lagrangian is then proportional to \( ds/dt \). The Lagrangian is, apart from a prefactor, the square of this, but with the arbitrary time coordinate \( t \) replaced by the proper time \( \tau \). It can be shown that, if an affine parameter \( \tau \) provided by the proper time for massive particles, is chosen as time coordinate in the action integral, extremalisation of \( \int (ds/d\tau)^2 d\tau \) yields the same equations of motion as extremalisation of \( \int ds/dt d\tau \). Therefore, our Lagrangian produces the correct equations of motion – and it is easier to use than the standard Lagrangian, avoiding the appearance of certain square roots. The factor \( \frac{1}{2} \) has been introduced for convenience, to cancel out some factors of 2, appearing in taking derivatives. Finally, that \( L \) is constant is of course due to the fact that \( L = \frac{1}{2}(ds/d\tau)^2 \) and that \( ds^2 = -c^2d\tau^2 \) for massive particles.
14. I use this notion that is close in spirit to Einstein’s original ideas about gravity, because a number of contemporary authors would object to calling the field experienced by Rindler observers a gravitational one – the spacetime of the Rindler metric is flat.
15. The metric is a vacuum solution to Einstein’s equation.
16. R. J. Cook, “Physical time and physical space in general relativity,” Am. J. Phys. 72, 214–219 (2004).
17. At the beginning of the dust ball’s free fall, \( t \) may be identified with \( \tau \) – that is why it is correct to calculate the velocity as \( \dot{x}(0) \). As soon as the dust ball center is not coordinate stationary anymore, \( t \) and \( \tau \) become different.
18. Nevertheless, this is not standard length contraction. The semiaxis of our dust ball in its direction of motion is maximum in its rest frame, i.e., in the frame of the center particle. In fact, \( dx \) is not the length of any object, it is the spatial interval between two events (or dust particles) at different Rindler times, the time interval between them being \( dt, dx \), is the corresponding interval at a fixed proper time in the center particle frame and may therefore be interpreted as an extension of the dust ball at that time.
19. But see the discussion of initial conditions for \( \delta \phi \) in Sec. 7.
20. The angular frequency as seen by a coordinate stationary observer at \( r_0 \) will be smaller.
21. It is, however, not impossible. On presentation of the equation to the computer algebra system MAPLE 17, the latter spat out a general solution dependent on two constants of integration, in an implicit form. Obtaining an explicit form would require the solution of a transcendental equation to invert a function.
22. K. Schwarzschild, “Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie,” in *Sitzungsberichte der Königlich-Preußischen Akademie der Wissenschaften* (Reimer, Berlin, 1916) pp. 189–196, English translation: *On the Gravitational Field of a Mass Point According to Einstein’s Theory*, S. Antoci and A. Loiner, arXiv:physics/9905030v1.
23. Ø. Grøn and S. Hervik, *Einstein’s General Theory of Relativity: With Modern Applications in Cosmology* (Springer Science & Business Media, Springer, Berlin, 2007).
24. M. Janssen, “Of pots and holes: Einstein’s bumpy road to general relativity,” Ann. Phys. (Leipzig) 14, 58–85 (2005).