Uncertainty Inference with Applications to Control Systems

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Abstract

Control systems are usually designed based on nominal values of relevant physical parameters. To ensure that a control system will work properly when the relevant physical parameters vary within certain range, it is crucial to investigate how the performance measure is affected by the variation of system parameters. In this paper, we demonstrate that such issue boils down to the study of the variation of functions of uncertainty. Motivated by this vision, we propose a general theory for inferring function of uncertainties. By virtue of such theory, we investigate concentration phenomenon of random vectors. We derive new multidimensional probabilistic inequalities for random vectors, which are substantially tighter as compared to existing ones. The probabilistic inequalities are applied to investigate the performance of control systems with real parametric uncertainty. It is demonstrated much more useful insights of control systems can be obtained. Moreover, the probabilistic inequalities offer performance analysis in a significantly less conservative way as compared to the classical deterministic worst-case method.

1 Introduction

In control engineering, in order to avoid system failure, it is an essential task to evaluate the performance of the systems affected by uncertainty [12, 13]. Existing methods for performance evaluation of uncertain systems are based on two completely different paradigms. The first paradigm is to treat uncertainty as deterministic bounded parameters [3, 20]. The performance analysis is to seek the worst-case scenario. This approach can be unduly conservative. Moreover, the resultant computational complexity can be NP hard. The second paradigm is to evaluate system performance by assuming some typical distribution for the underlying uncertainty [1, 11]. This approach can be conducted with Monte Carlo simulation. The computational complexity can be shown to be independent of the problem size. The major issue of such paradigm is that the assumed distribution may be significantly different from the actual distribution of the underlying uncertainty. Consequently, the resultant insight from the Monte Carlo simulation can be fairly misleading.

Actually, in the analysis and design of control systems, due to experimental or cognitive limitations, we only have limited information about the uncertainty affecting the systems [10, 19]. Motivated by this situation, we advocate to analyze system performance based on the limited available information. Specifically, we represent such information by constraints of the mathematical expectation of functions of uncertainty. The performance measure of systems is expressed as the mathematical expectation of certain functions of uncertainty. Consequently, the range of such expected value is a good indicator of the performance of the associated system. In this way, we establish a close connection between multidimensional
probabilistic inequalities and the analysis and design of control systems. More formally, the general problem can formulated as follows. Let $X$ be a random vector representing uncertainty affecting the systems. Let $f(.)$ be a function of the uncertainty and $D$ be a domain in the Euclidean space such that $E[f(X)] \in D$. Let $g(X)$ denote the performance of the system. It is desirable to determine the range of $E[g(X)]$. This formulation accommodate a wide range of problems on performance analysis of control systems as special cases. A familiar problem is the robust stability of uncertain system. Within this general framework, we derive tight bounds for $E[g(X)]$, which can be evaluated by computational techniques such as linear programming embedded with gradient search [2] and global optimization techniques such as branch and bound algorithm [18].

The remainder of the paper is organized as follows. In Section 2, we propose a general approach for inferring uncertainty. Such approach is based on a probabilistic characterization of convex sets. In Section 3, we apply the proposed theory of inferring uncertainty to investigate concentration phenomena frequently encountered in uncertain systems. New multidimensional probabilistic inequalities are developed which are useful for analysis of control systems. In Section 4, we apply the probabilistic theory to analyze the stability of control systems affected by parametric uncertainty. Section 5 is the conclusion. Most proofs are given in Appendices.

In this paper, we shall use the following notations. The set of real numbers is denoted by $\mathbb{R}$. The set of nonnegative real numbers is denoted by $\mathbb{R}^+$. The $d$-dimensional Euclidean space is denoted by $\mathbb{R}^d$. The Euclidean norm is denoted by $|\cdot|$. The diameter of $S \subseteq \mathbb{R}^d$ is defined as $\sup\{||x - y|| : x \in S, y \in S\}$. The supremum of an empty set is defined as 0. The set minus operation is denoted by \$. Let $(\Omega, \mathcal{F}, \Pr)$ denote the probability space. The mathematical expectation of random vector $X$ is denoted by $E[X]$. A zero-mean random vector is a random vector such that all the elements of its expected value are zero.

Let $X$ be a discrete random vector in $\mathbb{R}^d$. A vector $x$ in $\mathbb{R}^d$ is said to be a possible value of the discrete random vector if $\Pr\{X = x\} > 0$. That is, a vector $x$ is said to be a possible value of a discrete random vector if $x$ is assumed by the discrete random vector with a positive probability.

The support of a random variable $X$ in $\mathbb{R}^d$ is defined as the set whose complement consists of points in $\mathbb{R}^d$ with zero probability density. We use the abbreviation “i.i.d.” for “independent and identically distributed”. The other notations will be made clear as we proceed.

## 2 A General Theory for Inferring Uncertainty

In this section, we shall develop a general theory for inferring uncertainty. To make the inference more realistic, we avoid the assumption that the exact distribution of uncertainty is known. We shall demonstrate that a unified theory of inference can be established upon a stochastic characteristic of convex sets.

### 2.1 A Stochastic Characteristic of Convex Sets

Our investigation indicates that if a set in a finite-dimensional Euclidean space is convex, then the set contains the expectation of any random vector almost surely contained by the set. More formally, we have established the following result.

**Theorem 1** If $\mathcal{D}$ is a convex set in $\mathbb{R}^n$, then $E[X] \in \mathcal{D}$ holds for any random vector $X$ such that $\Pr\{X \in \mathcal{D}\} = 1$ and that $E[X]$ exists.

Theorem 1 is established in [8]. See Appendix A for a proof. The converse of Theorem 1 asserts that if $\mathcal{D}$ is a set in $\mathbb{R}^n$ such that $E[X] \in \mathcal{D}$ holds for any random vector $X$ such that $\Pr\{X \in \mathcal{D}\} = 1$ and that $E[X]$ exists, then $\mathcal{D}$ is convex. This assertions is well known and is a direct consequence of the definition of a convex set.
Theorem 1 immediately implies Jensen’s inequality. To see this, note that if a function is convex, then its epigraph, the region above its graph, is a convex set. Hence, if \( f \) is a convex function, then for any random variable \( X \), since \((X, f(X))\) is contained by the epigraph of \( f \), it follows from Theorem 1 that \((E[X], E[f(X)])\) is contained by its epigraph. This implies that \( E[f(X)] \geq f(E[X]) \) by the notion of epigraph.

The following result is due to Isii [17].

**Theorem 2** Let \( \mathcal{X} \) be a family of random vectors in \( \mathbb{R}^d \) such that
\[
\Pr\{X \in \mathcal{A}\} = 1, \quad E[f(X)] = \mu \in \mathbb{R}^k \quad \text{for each} \quad X \in \mathcal{X},
\]
where \( \mathcal{A} \) is a subset of \( \mathbb{R}^d \) and \( f(x) \) is a function assuming values in \( \mathbb{R}^k \) for \( x \in \mathcal{A} \). Let \( g(x) \) be real-valued function of \( x \in \mathcal{A} \) such that \( E[g(X)] \) exists for each \( X \in \mathcal{X} \). Then,
\[
\sup_{X \in \mathcal{X}} E[g(X)] = \sup_{Y \in \mathcal{Y}} E[g(Y)],
\]
where
\[
\mathcal{Y} = \{Y \in \mathcal{X} : Y \text{ is a discrete random vector with at most } k+1 \text{ distinct possible values}\}.
\]

This result is correct. However, in his original proof, Isii made a mistake by using an incorrect probability measure in mathematical induction (see, [17, Lemma 2, page 191–192]).

In many applications, because of incomplete information, the equality \( E[f(X)] = \mu \) is hard to satisfy. For example, in many cases, we may not know the exact value of the moment of a random variable. We only have its range. Hence, to infer uncertainty in the most general setting, we propose to represent the incomplete information by the constraint
\[
E[f(X)] \in \mathcal{B},
\]
where \( \mathcal{B} \) is a subset of \( \mathbb{R}^k \). In this framework, we have the following result.

**Theorem 3** Let \( X \) be a random vector in \( \mathbb{R}^d \) such that \( \Pr\{X \in \mathcal{A}\} = 1 \) and \( E[f(X)] \in \mathcal{B} \), where \( \mathcal{A} \) is a subset of \( \mathbb{R}^d \), \( \mathcal{B} \) is a subset of \( \mathbb{R}^k \), and \( f(x) \) is a function assuming values in \( \mathbb{R}^k \) for \( x \in \mathcal{A} \). Let \( g(x) \) be a real-valued function of \( x \in \mathcal{A} \) such that \( E[g(X)] \) exists. Then,
\[
E[g(X)] \leq \sup_{Y \in \mathcal{Y}} E[g(Y)],
\]
where \( \mathcal{Y} \) is the family of discrete random vectors in \( \mathbb{R}^d \) such that for each \( Y \in \mathcal{Y} \),
\[
\Pr\{Y \in \mathcal{A}\} = 1, \quad E[f(Y)] \in \mathcal{B},
\]
and \( Y \) has at most \( k+1 \) distinct possible values.

See Appendix B for a proof. Making use of Theorem 3, we have the following result.

**Theorem 4** Let \( \mathcal{X} \) be a family of random vectors in \( \mathbb{R}^d \) such that
\[
\Pr\{X \in \mathcal{A}\} = 1, \quad E[f(X)] \in \mathcal{B} \quad \text{for each} \quad X \in \mathcal{X},
\]
where \( \mathcal{A} \) is a subset of \( \mathbb{R}^d \), \( \mathcal{B} \) is a subset of \( \mathbb{R}^k \), and \( f(x) \) is a function assuming values in \( \mathbb{R}^k \) for \( x \in \mathcal{A} \). Let \( g(x) \) be a real-valued function of \( x \in \mathcal{A} \) such that \( E[g(X)] \) exists for each \( X \in \mathcal{X} \). Then,
\[
\sup_{X \in \mathcal{X}} E[g(X)] = \sup_{Y \in \mathcal{Y}} E[g(Y)],
\]
where
\[
\mathcal{Y} = \{Y \in \mathcal{X} : Y \text{ is a discrete random vector with at most } k+1 \text{ distinct possible values}\}.
\]
Theorem 4 can be shown as follows. By the assumption that $E[g(X)]$ exists for each $X \in \mathcal{X}$, according to Theorem 3, we have that

$$E[g(X)] \leq \sup_{Y \in \mathcal{Y}} E[g(Y)]$$

for each $X \in \mathcal{X}$. Thus,

$$\sup_{X \in \mathcal{X}} E[g(X)] \leq \sup_{Y \in \mathcal{Y}} E[g(Y)],$$

On the other hand, since $\mathcal{Y}$ is a subset of $\mathcal{X}$, it must be true that

$$\sup_{X \in \mathcal{X}} E[g(X)] \geq \sup_{Y \in \mathcal{Y}} E[g(Y)].$$

So, the theorem must be true.

According to Theorem 4, we have

$$\sup_{Y \in \mathcal{Y}} E[g(Y)] = \sup \left\{ \sum_{\ell=1}^{k+1} \theta_\ell g(y_\ell) : \theta_\ell \geq 0 \text{ for } \ell = 1, \ldots, k+1, \sum_{\ell=1}^{k+1} \theta_\ell = 1, \sum_{\ell=1}^{k+1} \theta_\ell f(y_\ell) \in \mathcal{B} \right\},$$

which can be computed by linear programming embedded with gradient search [2], and branch and bound method [18].

For the important case that $g(.)$ is an indicator function, we have the following result.

**Theorem 5** Let $\mathcal{X}$ be a family of random vectors in $\mathbb{R}^d$ such that $\Pr\{X \in \mathcal{A}\} = 1$ and $E[f(X)] \in \mathcal{B}$ for each $X \in \mathcal{X}$, where $\mathcal{A}$ is a subset of $\mathbb{R}^d$, $\mathcal{B}$ is a subset of $\mathbb{R}^k$, and $f(x)$ is a function assuming values in $\mathbb{R}^k$ for $x \in \mathcal{A}$. Then, $\sup_{X \in \mathcal{X}} \Pr\{X \in \mathcal{C}\} = \max\{P_i : 1 \leq i \leq k+1\}$ for any subset $\mathcal{C}$ of $\mathcal{A}$, where

$$P_i = \sup \left\{ \sum_{\ell=1}^{i} \theta_\ell : \theta_\ell \geq 0 \text{ for } 1 \leq \ell \leq k+1, \ y_\ell \in \mathcal{C} \text{ for } 1 \leq \ell \leq i, \ y_\ell \in \mathcal{A} \setminus \mathcal{C} \text{ for } i < \ell \leq k+1, \sum_{\ell=1}^{k+1} \theta_\ell = 1, \sum_{\ell=1}^{k+1} \theta_\ell f(y_\ell) \in \mathcal{B} \right\}$$

for $i = 1, \ldots, k+1$.

See Appendix C for a proof.

Theorem 5 can be applied to compute bounds for the probability that a systems fails to satisfy pre-specified requirements based on limited information of uncertainty. The bounds can be obtained by Linear programming embedded with gradient search, and the branch and bound method. A demonstration of the application of this theorem is given in Section 4.

### 2.2 Minimum-Range Random Variable Under Moment Constraints

Making use of Theorem 4, we have the following result.

**Theorem 6** Let $Z$ be a zero-mean random variable in $\mathbb{R}$ such that

$$E[Z^k] \geq 1 \text{ for } k \geq 2.$$  \hspace{1cm} (1)

Define $L_Z = \sup\{u \in \mathbb{R} : \Pr\{Z \geq u\} = 1\}$ and $U_Z = \inf\{v \in \mathbb{R} : \Pr\{Z \leq v\} = 1\}$. Then, $U_Z - L_Z \geq \sqrt{5}$. In particular, (1) holds and $U_Z - L_Z = \sqrt{5}$ if $Z$ is a random variable such that $\Pr\{Z = \varphi\} = \frac{1}{\sqrt{5}}$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.
Making use of Theorem 4, we have the following result.

**Theorem 7** Let $Z$ be a zero-mean random variable in $\mathbb{R}$ such that
\[
\mathbb{E}[Z^2] = 1, \quad \mathbb{E}[Z^k] \geq 1 \quad \text{for} \quad k \geq 3.
\] (2)
Define $L_Z = \sup\{u \in \mathbb{R} : \Pr\{Z \geq u\} = 1\}$ and $U_Z = \inf\{v \in \mathbb{R} : \Pr\{Z \leq v\} = 1\}$. Then, $\max(U_Z, |L_Z|) \geq \varphi$, where $\varphi = \frac{1 + \sqrt{5}}{2}$ is the golden ratio. In particular, (2) holds and $\max(U_Z, |L_Z|) = \varphi$ if $Z$ is a random variable such that $\Pr\{Z = \varphi\} = \frac{1}{\sqrt{5}} \varphi$ and $\Pr\{Z = -\frac{1}{\varphi}\} = \frac{1}{\sqrt{5}}$.

3 Concentration Phenomena in Euclidean Space

In many applications, uncertainties can be represented as random vectors in Euclidean space. Consequently, useful insight of the impact of uncertainty to control systems may be obtained by investigating the concentration phenomena of the relevant random vectors. In the sequel, we shall develop multivariate concentration inequalities for random vectors, which generalize Chernoff-Hoeffding inequalities [9, 15]. For that purpose, we shall first propose a unified approach for deriving exponential inequalities which uniformly hold for all values of time for stochastic processes.

3.1 Uniform Exponential Inequalities

The following results provide a unified method for deriving uniform exponential inequalities for real-valued stochastic processes.

**Theorem 8** [Chen (2012)] Let $\mathcal{V}$ be a non-negative, right-continuous function of $t \in [0, \infty)$. Let $\{X_t, t \in \mathbb{R}^+\}$ be a right-continuous stochastic process such that $\mathbb{E}[\exp(s(X_{t'} - X_t)) | \mathcal{F}_t] \leq \exp((\mathcal{V}_t - \mathcal{V}_s)\varphi(s))$ almost surely for arbitrary $t' \geq t \geq 0$ and $s \in (0, b)$, where $b$ is a positive number or infinity, $\varphi(s)$ is a non-negative function of $s \in (0, b)$, and $\mathcal{F}_t$ is the $\sigma$-algebra generated by $\{X_{t'}, 0 \leq t' \leq t\}$. Let $\tau \geq 0$ and $\gamma > 0$. Then,
\[
\Pr\left\{\sup_{t \geq 0} \left[ X_t - X_0 - \gamma \mathcal{V}_t - \frac{\varphi(s)}{s} (\mathcal{V}_t - \mathcal{V}_s) \right] \geq 0 \right\} \leq \exp(\varphi(s) - \tau s) \quad \forall s \in (0, b). \tag{3}
\]
In particular, if $\{s \in (0, b) : \varphi(s) \leq \gamma s\}$ is nonempty and the infimum of $\varphi(s) - \gamma s$ with respect to $s \in (0, b)$ is attained at $s^* \in (0, b)$, then
\[
\Pr\left\{\sup_{t \geq 0} \left[ X_t - X_0 - \gamma \mathcal{V}_t - \frac{\varphi(s^*)}{s^*} (\mathcal{V}_t - \mathcal{V}_s) \right] \geq 0 \right\} \leq \exp(\varphi(s^*) - \gamma s^*) \leq 1, \tag{4}
\]
and $0 \leq \frac{\varphi(s^*)}{s^*} \leq \gamma$.

Theorem 8 is established in [6, 7]. A proof is reproduced in Appendix D. More generally, we have the following results.

**Theorem 9** Let $\{\mathcal{V}(s, t), t \in \mathbb{R}^+\}$ be a real-valued stochastic process parameterized by $s \in (0, b)$, where $b$ is a positive number or infinity. Let $\{X_t, t \in \mathbb{R}^+\}$ be a real-valued stochastic process with $X_0 = 0$. Let $\{\mathcal{Z}(s, t), t \in \mathbb{R}^+\}$ be a right-continuous supermartingale, which is parameterized by $s \in (0, b)$ and adapted to the natural filtration generated by $\{\mathcal{V}(s, t), t \in \mathbb{R}^+\}$ and $\{X_t, t \in \mathbb{R}^+\}$ such that for all $s \in (0, b)$,
\[
\mathbb{E}[\mathcal{Z}(s, 0)] \leq C \quad \text{and} \quad \exp(sX_t - \mathcal{V}(s, t)) \leq \mathcal{Z}(s, t) \quad \text{almost surely for all} \quad t \in \mathbb{R}^+.
\]

Let $\lambda$ be a real number and $g(s)$ be a function of $s \in (0, b)$. Then,
\[
\Pr\left\{\sup_{t > 0} \left[ X_t - \lambda - \frac{\mathcal{V}(s, t) - g(s)}{s} \right] \geq 0 \right\} \leq C \exp(g(s) - \lambda s) \quad \text{for all} \quad s \in (0, b). \tag{5}
\]
In particular, the following assertions hold:

1. If the infimum of \( g(s) \) with respect to \( s \in (0, b) \) is attained at \( s^* \in (0, b) \), then

\[
\text{Pr}\left\{ \sup_{t > 0} \left[ X_t - \lambda s - \frac{\sqrt{s}}{2} g\left( \frac{s}{s} \right) \right] \geq 0 \right\} \leq C \exp(g(s^*) - \lambda s^*).
\]

2. If \( V(s, t) \) is a deterministic function of \( s \in (0, b) \) and \( t \in \mathbb{R}^+ \), then

\[
\text{Pr}\left\{ \sup_{t > 0} \left[ X_t - \lambda s - \frac{\sqrt{s}}{2} V(s, t) \right] \geq 0 \right\} \leq C \exp(V(s, \tau) - \lambda s) \text{ for all } s \in (0, b) \text{ and } \tau \in \mathbb{R}^+.
\]

3. If \( V(s, t) = \varphi(s) V_t \), where \( \varphi(s) \) is a deterministic function of \( s \in (0, b) \) and \( \{ V_t, t \in \mathbb{R}^+ \} \) is a deterministic or stochastic process, then

\[
\text{Pr}\left\{ \sup_{t > 0} \left[ X_t - \lambda s - \frac{\sqrt{s}}{2} (V_t - m) \right] \geq 0 \right\} \leq C \exp(m \varphi(s) - \lambda s) \text{ for all } s \in (0, b) \text{ and } m \in \mathbb{R}.
\]

**Proof.** To prove Theorem 9, note that for all \( s \in (0, b) \),

\[
\text{Pr}\left\{ \sup_{t > 0} \left[ X_t - \lambda s - \frac{\sqrt{s}}{2} g(s) \right] \geq 0 \right\} = \text{Pr}\left\{ \sup_{t > 0} \left[ X_t - \lambda s - \frac{\sqrt{s}}{2} \right] \geq 0 \right\} = \text{Pr}\left\{ \sup_{t > 0} [sX_t - V(s, t)] \geq \lambda s - g(s) \right\} = \text{Pr}\left\{ \sup_{t > 0} \exp(sX_t - V(s, t)) \geq \exp(\lambda s - g(s)) \right\} \leq \text{Pr}\left\{ \sup_{t > 0} Z(s, t) \geq \exp(\lambda s - g(s)) \right\}.
\]

By the supremum inequality, we have

\[
\text{Pr}\left\{ \sup_{t > 0} \left[ X_t - \lambda s - \frac{\sqrt{s}}{2} g(s) \right] \geq 0 \right\} \leq \frac{E[Z(s, 0)]}{E[\exp(\lambda s - g(s))]} \leq \frac{C}{E[\exp(\lambda s - g(s))]} = C \exp(g(s) - \lambda s)
\]

for all \( s \in (0, b) \). This proves (5), from which the particular assertions immediately follow. \( \square \)

It should be noted that if \( \varphi(s) \) has the characteristic of a cumulant-generating function, then the assertion (III) of Theorem 9 can be applied to deduce Theorem 1(b) of [16].

### 3.2 Using Moment Generating Functions

Making use of moment generating functions pertaining to vector magnitude of random vectors, we have obtained the following results.

**Theorem 10** Let \( X_1, X_2, \ldots, X_n \) be i.i.d. zero-mean random vectors. Let \( Z \) be a zero-mean random variable in \( \mathbb{R} \) such that \( E[Z^k] \geq 1 \) for \( k \geq 2 \). Assume that there exists a function \( \mathcal{M}(s) \) such that \( E[e^{s|Z|}] \leq \mathcal{M}(s) \) for all \( s \in (-\tau, \tau) \), where \( \tau > 0 \). Then, for any \( \varepsilon > 0 \),

\[
\text{Pr}\left\{ \left\| \frac{1}{n} \sum_{i=1}^{n} X_i \right\| \geq \varepsilon \right\} \leq \text{Pr}\left\{ \max_{1 \leq i \leq n} \left\| \sum_{i=1}^{t} X_i \right\| \geq n\varepsilon \right\} \leq \inf_{t \in (0, \tau)} e^{-nt\varphi} \left( [\mathcal{M}(t)]^n + [\mathcal{M}(-t)]^n \right).
\]

In particular, (6) holds if the associated random variable \( Z \) has a distribution such that \( \text{Pr}\{Z = \varphi\} = \frac{1}{\sqrt{\pi}} \varphi \) and \( \text{Pr}\{Z = -\frac{1}{\varphi}\} = \frac{1}{\sqrt{\pi}} \varphi \), where \( \varphi = \frac{1 + \sqrt{2}}{2} \) is the golden ratio.

In the case that the moment generating function of the magnitude of a random vector exists, we have the following result.

**Theorem 11** Let \( X_1, X_2, \ldots, X_n \) be i.i.d. zero-mean random vectors such that \( E[e^{s|X|}] = g(s) \) for all \( s \in (-\tau, \tau) \), where \( \tau > 0 \). Let \( \varphi = \frac{1 + \sqrt{2}}{2} \) be the golden ratio. Define

\[
h(t, \varepsilon, n) = e^{-nt\varphi} \left( \left[ g(\varphi t) + \varphi g\left( -\frac{t}{\varphi} \right) \right]^n + \left[ g(-\varphi t) + \varphi g\left( \frac{t}{\varphi} \right) \right]^n \right)
\]

for \( \varepsilon > 0 \) and \( t \in (0, \tau) \). Then, for any \( \varepsilon > 0 \),

\[
\text{Pr}\left\{ \left\| \frac{1}{n} \sum_{i=1}^{n} X_i \right\| \geq \varepsilon \right\} \leq \text{Pr}\left\{ \max_{1 \leq i \leq n} \left\| \sum_{i=1}^{t} X_i \right\| \geq n\varepsilon \right\} \leq \frac{1}{\sqrt{5^n}} \times \inf_{t \in (0, \tau)} h(t, \varepsilon, n),
\]

where \( h(t, \varepsilon, n) \) is a convex function of \( t \in (0, \tau) \) for fixed \( \varepsilon > 0 \) and \( n \).
3.3 Bounded Random Vectors

Because of physical limitations, the magnitude of uncertainty affecting systems are actually bounded. Hence, it is of particular importance to investigate the concentration phenomena of bounded random vectors.

3.3.1 Using Information of Support

In the case that the bounds on the magnitude of random vectors are available, we have the following result.

**Theorem 12** Let \( X_1, \cdots, X_n \) be independent zero-mean random vectors such that \( \Pr\{||X_i|| \leq r_i\} = 1 \) for \( i = 1, \cdots, n \). Then, for all \( \varepsilon > 0 \),

\[
\Pr \left\{ \left| \frac{1}{n} \sum_{i=1}^{n} X_i \right| \geq \varepsilon \right\} \leq \Pr \left\{ \max_{1 \leq \ell \leq n} \left| \sum_{i=1}^{\ell} X_i \right| \geq n\varepsilon \right\} \leq 2 \exp \left( -\frac{2n\varepsilon^2}{5V} \right),
\]

where \( V = \frac{1}{n} \sum_{i=1}^{n} r_i^2 \).

If the diameters of the domain containing random vectors are known, we have the following result.

**Theorem 13** Let \( X_1, \cdots, X_n \) be independent zero-mean random vectors such that \( X_i \) has a support of diameter \( D_i \) for \( i = 1, \cdots, n \). Then, for all \( \varepsilon > 0 \),

\[
\Pr \left\{ \left| \frac{1}{n} \sum_{i=1}^{n} X_i \right| \geq \varepsilon \right\} \leq \Pr \left\{ \max_{1 \leq \ell \leq n} \left| \sum_{i=1}^{\ell} X_i \right| \geq n\varepsilon \right\} \leq 2 \exp \left( -\frac{2n\varepsilon^2}{5V} \right),
\]

where \( V = \frac{1}{n} \sum_{i=1}^{n} D_i^2 \).

For vector-valued martingales of bounded increments, we have derived maximal inequalities as follows.

**Theorem 14** Suppose \( \{X_k : k = 0, 1, 2, 3, \cdots\} \) is a vector-valued martingale and \( \Pr\{||X_k - X_{k-1}|| \leq c_k\} = 1 \) for \( k \in \mathbb{N} \). Then,

\[
\Pr \{||X_n - X_0|| \geq \varepsilon\} \leq 2 \exp \left( -\frac{2\varepsilon^2}{5 \sum_{k=1}^{n} c_k^2} \right)
\]

for all positive integers \( n \) and all positive reals \( \varepsilon \).

3.3.2 Using Information of Support and Variance

To make use of the information of each component of random vectors, we have the following results.

**Theorem 15** Let \( X = [x_1, \cdots, x_d] \) be a zero-mean random vector such that \( \mathbb{E}[||X||^2] \leq \sigma^2 \), the components \( x_1, \cdots, x_d \) are mutually independent, and \( \Pr\{|x_i| \leq r_i\} = 1 \) for \( i = 1, \cdots, d \). Then,

\[
\Pr\{||X|| \geq \varepsilon\} \leq \exp \left( -\frac{2(\varepsilon^2 - \sigma^2)^2}{\sum_{i=1}^{d} r_i^4} \right)
\]

for \( \varepsilon > \sigma \).

If we know the range of each component of random vectors, we have the following result.
Theorem 16 Let $X = [x_1, \cdots, x_d]$ be a zero-mean random vector such that the components $x_1, \cdots, x_d$ are mutually independent and that $\Pr \{ a_i \leq x_i \leq b_i \} = 1$ for $i = 1, \cdots, d$. Define $\sigma^2 = \sum_{i=1}^{d} |a_i b_i|$. Then,

$$\Pr \{|X - \mu| \geq \varepsilon \} \leq \exp \left( -\frac{2(\varepsilon^2 - \sigma^2)^2}{\sum_{i=1}^{d} (b_i - a_i)^4} \right)$$

for $\varepsilon > \sigma$.

Making use of the variance information of random vectors, we have derived simple exponential inequalities as follows.

Theorem 17 Let $X_1, X_2, \cdots$ be independent zero-mean random vectors such that for $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} \Pr \{|X_i| \leq \varepsilon \} \leq s^2_n, \quad \Pr \{|X_i| \leq c_n s_n \text{ for } i = 1, \cdots, n \} = 1,$$

where $c_n > 0$ and $s_n > 0$. Let $\varphi = \frac{1 + \sqrt{5}}{2}$ be the golden ratio. Then,

$$\Pr \left\{ \max_{1 \leq i \leq n} \left| \sum_{i=1}^{\ell} X_i \right| \geq x s_n \right\} \leq 2 \exp \left( -\frac{x^2}{2} \left( 1 - \frac{x \varphi c_n}{2} \right) \right)$$

for $0 < x < \frac{1}{\varphi c_n}$.

Making use of the variance and range information of random vectors, we have derived tight inequalities as follows.

Theorem 18 Let $X_1, \cdots, X_n$ be independent zero-mean random vectors such that $\sum_{i=1}^{n} \Pr \{|X_i| \leq r \} = 1$ for $i = 1, \cdots, n$, where $\sigma \geq 0$ and $r > 0$. Let $\varphi = \frac{1 + \sqrt{5}}{2}$ be the golden ratio. Then, $Pr\left\{ \left| \frac{1}{n} \sum_{i=1}^{n} X_i \right| > r \} = 0$ and

$$\Pr \left\{ \left| \frac{1}{n} \sum_{i=1}^{n} X_i \right| > \varepsilon \right\} \leq \Pr \left\{ \max_{1 \leq i \leq n} \left| \sum_{i=1}^{\ell} X_i \right| \geq n \varepsilon \right\}$$

$$= \inf_{t > 0} e^{-nt} \left\{ \left[ \frac{t \varphi c_n}{\sigma^2 + \varphi^2 c_n} \exp \left( -\frac{t \sigma^2}{\varphi c_n} \right) + \frac{t \varphi^2 c_n}{\sigma^2 + \varphi^2 c_n} \exp \left( \frac{-t \varphi^2 c_n}{\sigma^2 + \varphi^2 c_n} \right) \right]^n \right\}$$

$$\leq 2 \left[ \frac{\varphi^2 c_n}{\sigma^2 + \varphi^2 c_n} \right] \exp \left( \frac{-n \varphi c_n}{\sigma^2 + \varphi^2 c_n} \right) \leq 2 \exp \left( -\frac{n \varphi c_n}{\sigma^2 + \varphi^2 c_n} \right)$$

for $0 < \varepsilon \leq r$.

To apply Theorem 18, we need to bound $||X - \mu||$ and $\mathbb{E}[||X - \mu||^2]$. For this purpose, we have the following result.

Theorem 19 Let $X$ be a random vector with mean $\mu = \mathbb{E}[X]$ and a support of diameter $D$. Then, $||X - \mu|| \leq D$ and $\mathbb{E}[||X - \mu||^2] \leq \frac{D^2}{2}$.

If random vector $X$ is bounded within an ellipse, we have the following result.

Theorem 20 Let $X$ be a random vector such that $||AX + b|| \leq c$, where $A$ is an invertible matrix. Then,

$$||X - \mu|| \leq ||A^{-1}|| \times [c + ||A\mu + b||], \quad \mathbb{E}[||X - \mu||^2] \leq ||A^{-1}|| \times [c^2 - ||A\mu + b||^2],$$

where $\mu = \mathbb{E}[X]$.

It should be noted that Theorem 20 is an extension of Bhatia-Davis inequality [4].
4 Stability of Uncertain Dynamic Systems

In this section, we shall apply the proposed theory of inferencing function of uncertainties to study the stability of uncertain systems. Consider a system which has been studied in [14] by a deterministic approach. The system is as shown in Figure 1.

The system is unstable if

\[ \Pr\{\text{The system is unstable}\} = \Pr\{X \in \mathcal{A}\}, \]

subject to

\[ \Pr\{X \in \mathcal{A}\} = 1, \quad \mathbb{E}[f(X)] \in \mathcal{B}. \]

Therefore, we can apply Theorem 5 to compute a deterministic bound for \(\Pr\{\text{The system is unstable}\}\). With less than 0.05 second, we obtained such upper bound as 0.00031 by a computer program which implements linear programming embedded with the gradient search and the branch and bound algorithms.

5 Conclusion

In this paper, we have developed a general theory for inferring uncertainty. We have applied the general theory to investigate concentration phenomena of random vectors. Multidimensional probabilistic inequalities have been developed which can be useful for the analysis of control systems affected by uncertainty. We
have derived computable tight bounds for the expected values of functions of uncertainty which represent performance of systems. The applications of such results are illustrated by an investigation of the stability of an uncertain system.

A Proof of Theorem 1

We need some preliminary results. If $X$ is a random variable such that $\Pr\{X < c\} = 1$, then it is intuitive that $\mathbb{E}[X] < c$. However, there exists no proof in the literature for such intuition. Since the strictness of the inequality plays a crucial role in our proof of the theorem, we provide a rigorous proof in the sequel.

**Lemma 1** If $X$ is a scalar random variable such that $\Pr\{X < c\} = 1$, then $\mathbb{E}[X] < c$. Similarly, if $X$ is a scalar random variable such that $\Pr\{X > c\} = 1$, then $\mathbb{E}[X] > c$.

**Proof.** We claim that there exists a positive number $\varepsilon > 0$ such that $\Pr\{X \leq c - \varepsilon\} > 0$. To prove the claim, we use a contradiction method. Suppose that the claim is not true. Then, $\Pr\{X \leq c - \varepsilon\} = 0$ for any $\varepsilon > 0$. It follows from the continuity of probability measure that $\Pr\{X < c\} = \lim_{\varepsilon \to 0} \Pr\{X \leq c - \varepsilon\} = 0$. This contradicts to the assumption that $\Pr\{X < c\} = 1$. So, we have proved the claim.

Now let $\varepsilon > 0$ be a positive number such that $\Pr\{X \leq c - \varepsilon\} > 0$. Since $\Pr\{X < c\} = 1$, we have

$$\mathbb{E}[X] = \mathbb{E}\left[X \mathbb{1}_{\{X \leq c - \varepsilon\}}\right] + \mathbb{E}\left[X \mathbb{1}_{\{c - \varepsilon < X < c\}}\right]$$

$$\leq (c - \varepsilon) \Pr\{X \leq c - \varepsilon\} + c \Pr\{c - \varepsilon < X < c\}$$

$$= (c - \varepsilon) \Pr\{X \leq c - \varepsilon\} + c(1 - \Pr\{X \leq c - \varepsilon\})$$

$$= -\varepsilon \Pr\{X \leq c - \varepsilon\} + c < c.$$ 

This proves the first assertion. The second assertion can be shown in a similar way.

**Lemma 2** Assume that $D$ is a closed convex set and $\mathcal{X}$ is a random vector such that $\Pr\{\mathcal{X} \in D\} = 1$, then $\mathbb{E}[\mathcal{X}] \in D$.

**Proof.** We shall use a contradiction method. Denote $\mu = \mathbb{E}[\mathcal{X}]$. Suppose $\mu \notin D$, i.e., $\mu$ is an exterior point of $D$. By the hyperplane separation theorem [5, Theorem 4.11, page 170], there exists a row vector $\alpha$ such that $\alpha\mu < \alpha Z$ for all $Z \in D$. Since $\Pr\{\mathcal{X} \in D\} = 1$, it must be true that $\Pr\{\alpha\mu < \alpha\mathcal{X}\} = 1$. Hence, $\Pr\{\alpha\mu - \alpha\mathcal{X} < 0\} = 1$. It follows from Lemma 1 that $\mathbb{E}[\alpha\mu - \alpha\mathcal{X}] < 0$, which implies that

$$\alpha\mu < \mathbb{E}[\alpha\mathcal{X}] = \alpha\mathbb{E}[\mathcal{X}] = \alpha\mu.$$ 

This is a contradiction. Therefore, it must be true that $\mu \in D$. The proof of the lemma is thus completed.

**Lemma 3** If $X$ is a scalar random variable such that $0 < \Pr\{X < 0\} \leq \Pr\{X \leq 0\} = 1$. Then, $\mathbb{E}[X] < 0$. 

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Proof. We claim that there exists a positive number \( \varepsilon > 0 \) such that \( \Pr\{X \leq -\varepsilon\} > 0 \). To prove the claim, we use a contradiction method. Suppose that the claim is not true. Then, \( \Pr\{X \leq -\varepsilon\} = 0 \) for any \( \varepsilon > 0 \). It follows from the continuity of probability measure that \( \Pr\{X < 0\} = \lim_{\varepsilon \downarrow 0} \Pr\{X \leq -\varepsilon\} = 0 \). This contradicts to the assumption that \( \Pr\{X < 0\} > 0 \). So, we have proved the claim.

Now let \( \varepsilon > 0 \) be a positive number such that \( \Pr\{X \leq -\varepsilon\} > 0 \). Since \( \Pr\{X \leq 0\} = 1 \), we have

\[
\mathbb{E}[X] = \mathbb{E}[X 1_{\{X \leq -\varepsilon\}}] + \mathbb{E}[X 1_{\{-\varepsilon<X\leq 0\}}] \\
\leq -\varepsilon \Pr\{X \leq -\varepsilon\} + 0 \times \Pr\{-\varepsilon < X \leq 0\} \\
\leq -\varepsilon \Pr\{X \leq -\varepsilon\} < 0.
\]

This completes the proof of the lemma.

Lemma 4 Assume that \( D \) is a closed convex set and \( X \) is a random vector such that \( \Pr\{X \in D\} = 1 \) and \( \mu = \mathbb{E}[X] \in \partial D \), then there exist a nonzero row vector \( \alpha \) and a constant \( \beta \) such that \( \Pr\{\alpha X + \beta = 0\} = 1 \).

Proof. As a consequence of the convexity of \( D \) and the assumption that \( \mu = \mathbb{E}[X] \in \partial D \), it is possible to construct a supporting hyperplane \( \alpha Z + \beta = 0 \) through \( \mu \), where \( \alpha \) is a nonzero row vector and \( \beta \) is a constant, such that \( \alpha Z + \beta \leq 0 \) for all \( Z \in D \). By the assumption that \( \Pr\{X \in D\} = 1 \), we have

\[ \Pr\{\alpha X + \beta = 0\} = 1. \]

Since \( \mu \) is in the supporting hyperplane, we have \( \alpha \mathbb{E}[X] + \beta = 0 \). We claim that \( \Pr\{\alpha X + \beta = 0\} = 1 \). To prove this claim, we use a contradiction method. Suppose the claim is not true. Then,

\[ 0 < \Pr\{\alpha X + \beta < 0\} \leq \Pr\{\alpha X + \beta \leq 0\} = 1. \]

It follows from Lemma 3 that \( \mathbb{E}[\alpha X] + \beta < 0 \). This implies that \( \alpha \mathbb{E}[X] + \beta = \mathbb{E}[\alpha X] + \beta < 0 \), which contradicts to the fact that \( \alpha \mathbb{E}[X] + \beta = 0 \). The claim is thus established. Hence, it must be true that \( \Pr\{\alpha X + \beta = 0\} = 1 \). This completes the proof of the lemma.

We are now in a position to prove the theorem. We shall argue by a mathematical induction on the dimension \( n \) of \( \mathcal{D} \). For the dimension \( n = 1 \), the convex set \( \mathcal{D} \) must be an interval of the form \( \mathcal{D} = [a, b] \), or \( \mathcal{D} = (a, b) \), or \( \mathcal{D} = [a, b] \), or \( \mathcal{D} = (a, b] \). Making use of Lemma 1, it is easy to see \( \mathbb{E}[X] \in \mathcal{D} \) as a consequence of \( \Pr\{X \in \mathcal{D}\} = 1 \). Suppose the conclusion \( \mathbb{E}[X] \in \mathcal{D} \) holds for dimension \( n - 1 \). To complete the induction process, we need to show, based on such hypothesis, that the inclusion relationship \( \mathbb{E}[X] \in \mathcal{D} \) holds for dimension \( n \). Let \( \overline{\mathcal{D}} \) denotes the closure of \( \mathcal{D} \). By Lemma 2, we have shown \( \mathbb{E}[X] \in \overline{\mathcal{D}} \). If \( \mu = \mathbb{E}[X] \) is not contained in the boundary of \( \overline{\mathcal{D}} \), then it must be true that \( \mu \in \mathcal{D} \). Hence, to show \( \mathbb{E}[X] \in \mathcal{D} \) for dimension \( n \), it suffices to show it under the assumption that \( \mu = \mathbb{E}[X] \) is contained in the boundary of \( \overline{\mathcal{D}} \). We proceed as follows. Making use of Lemma 4 and the assumption that \( \mu = \mathbb{E}[X] \) is contained in the boundary of \( \overline{\mathcal{D}} \), we conclude that there exist a nonzero row vector \( \alpha \) and a constant \( \beta \) such that \( \Pr\{\alpha X + \beta = 0\} = 1 \).

Define

\[ \mathcal{I} = \{Z \in \mathcal{D} : \alpha Z + \beta = 0\}. \]

Then, \( \mathcal{I} \) is convex and \( \Pr\{X \in \mathcal{I}\} = 1 \). Without loss of any generality, assume that the \( i \)-th element of \( \alpha \), denoted by \( \alpha_i \), is nonzero. Define a linear transform \( \mathcal{T} : \mathcal{I} \rightarrow D \) such that for every element \( Z = [z_1, \ldots, z_n]^\top \) in \( \mathcal{I} \), there exists a corresponding vector \( U = [u_1, \ldots, u_n]^\top = \mathcal{T}(Z) \) such that

\[ u_i = \alpha Z + \beta, \quad u_\ell = z_\ell, \quad \ell \in \{1, \ldots, n\} \setminus \{i\} \]
or equivalently,
\[ U = (I + e_i\alpha - e_i e_i^\top)Z + \beta e_i, \]  
(7)
where \( I \) is an identity matrix of size \( n \times n \) and \( e_i \) is a column matrix with all elements being 0 except the \( i \)-th element being 1. Note that \( D = \{ \mathcal{T}(Z) : Z \in \mathcal{S} \} \) must be convex because the transform \( \mathcal{T} \) is linear and \( \mathcal{S} \) is convex. Define \( Y = [y_1, \ldots, y_n]^\top = \mathcal{T}(X) \). Then,
\[ \Pr\{Y \in D\} = 1, \quad \Pr\{y_i = 0\} = \Pr\{\alpha X + \beta = 0\} = 1 \]
and \( \mathbb{E}[y_i] = 0 \). Define
\[ D^* = \{[u_1, \ldots, u_i, u_{i+1}, \ldots, u_n]^\top : [u_1, \ldots, u_n]^\top \in D\}. \]
Then, \( D^* \) is convex because \( D \) is convex. Define random vector \( V = [v_1, \ldots, v_{n-1}]^\top \) such that \( v_\ell = y_\ell, \ell = 1, \ldots, i-1 \) and \( v_\ell = y_{\ell+1}, \ell = i, \ldots, n-1 \). Then, \( \Pr\{V \in D^*\} = 1 \). Since \( D^* \) is a convex set of \((n-1)\) dimension and \( \Pr\{V \in D^*\} = 1 \), it follows from the induction hypothesis that \( \mathbb{E}[V] \in D^* \). This implies that \( \mathbb{E}[Y] \in D \).

It can be checked that the determinant of the matrix \( I + e_i\alpha - e_i e_i^\top \) in (7) is equal to \( \alpha_i \), which is nonzero. Hence, \( I + e_i\alpha - e_i e_i^\top \) is invertible, and it follows that \( Z = (I + e_i\alpha - e_i e_i^\top)^{-1}(U - \beta e_i) \). This implies that the transform \( \mathcal{T} \) is a one-to-one mapping from \( \mathcal{S} \) to \( D \) and thus the transform is invertible.

Note that \( \mathbb{E}[Y] = \mathcal{T}(\mathbb{E}[X]) \) and the transform \( \mathcal{T} \) maps \( \mathcal{S} \) into \( D \). Now, we have \( \mathbb{E}[Y] \in D \). Taking the inverse transform of \( \mathcal{T} \) yields \( \mathbb{E}[X] \in \mathcal{S} \subseteq \mathcal{D} \). This completes the process of the mathematical induction and the theorem is thus established.

B Proof of Theorem 3

Note that since all elements in \( \mathcal{Y} \) are discrete random vectors, the associated expectation \( \mathbb{E}[g(Y)] \) of any \( Y \in \mathcal{Y} \) must exist. Hence, \( \sup_{Y \in \mathcal{Y}} \mathbb{E}[g(Y)] \) is well-defined provided that \( \mathcal{Y} \) has at least one element. Hence, it suffices to show that the family \( \mathcal{Y} \) contains at least one element \( Y \) with \( \mathbb{E}[g(Y)] \geq \mathbb{E}[g(X)] \). Define \( S = \{(u, v) : u = f(x), v = g(x), x \in \mathcal{A}\} \). Then, \( \Pr\{(f(X), g(X)) \in S\} = 1 \). Note that the convex hull of \( S \), denoted by \( \text{conv}(S) \), is convex. By assumption, both \( \mathbb{E}[f(X)] \) and \( \mathbb{E}[g(X)] \) exist. Hence, by Theorem 1,
\[ (\mathbb{E}[f(X)], \mathbb{E}[g(X)]) \in \text{conv}(S). \]

Note that \( S \) is a subset of \((k+1)\)-dimensional vector space. According to Carathodory’s theorem, there exists \( m \leq k + 2 \) points, \( x_1, \ldots, x_m \) in \( S \) such that \( (\mathbb{E}[f(X)], \mathbb{E}[g(X)]) \) is a convex combination of \( x_1, \ldots, x_m \). The points \( x_1, \ldots, x_m \) are vertexes of the simplex which consists of all convex combinations of \( x_1, \ldots, x_m \). Consider half-line \( \{(u, v) : u = \mathbb{E}[f(X)], v \geq \mathbb{E}[g(X)]\} \). There must exist \( w \geq \mathbb{E}[g(X)] \) such that \( (\mathbb{E}[f(X)], w) \) lie in a proper face of the simplex. Without loss of generality, let \( x_1, \ldots, x_{m-1} \) be the vertex of such proper face. Then, there exist nonnegative numbers \( p_1, \ldots, p_{m-1} \) such that \( \sum_{i=1}^{m-1} p_i = 1 \) and that
\[ \mathbb{E}[f(X)] = \sum_{i=1}^{m-1} p_i f(x_i), \quad w = \sum_{i=1}^{m-1} p_i g(x_i). \]

Hence, we can define a discrete random vector \( Y \) of \((m-1) \leq k+1 \) possible values such that \( \Pr\{Y = x_i\} = p_i \) for \( i = 1, \ldots, m-1 \). Clearly,
\[ \Pr\{Y \in \mathcal{A}\} = 1, \quad \mathbb{E}[f(Y)] = \sum_{i=1}^{m-1} p_i f(x_i) = \mathbb{E}[f(X)] \in \mathcal{B}, \quad \mathbb{E}[g(Y)] = \sum_{i=1}^{m-1} p_i g(x_i) = w \geq \mathbb{E}[g(X)]. \]

This shows that the family \( \mathcal{Y} \) contains at least one element \( Y \) with \( \mathbb{E}[g(Y)] \geq \mathbb{E}[g(X)] \). The proof of the theorem is thus complete.
C Proof of Theorem 5

For $y \in \mathcal{A}$, define $g(y)$ such that $g(y) = 1$ if $y \in C$ and $g(y) = 0$ if $y \in \mathcal{A} \setminus C$. According to Theorem 4, we have

$$
\sup_{X \in \mathcal{X}} \Pr\{X \in \mathcal{C}\} = \sup \left\{ \sum_{\ell=1}^{k+1} \theta_{\ell} g(y_{\ell}) : \theta_{\ell} \geq 0 \text{ for } \ell = 1, \ldots, k+1, \sum_{\ell=1}^{k+1} \theta_{\ell} = 1, \sum_{\ell=1}^{k+1} \theta_{\ell} f(y_{\ell}) \in \mathcal{B} \right\}
$$

which is the same as $P_i$. Hence, we have established that $h(b_1, \ldots, b_{k+1}) = P_i$ holds for all $(b_1, \ldots, b_{k+1}) \in E_i$ for $1 \leq i \leq k+1$. It follows that $Q_i = P_i$ for $1 \leq i \leq k+1$. Therefore, $\sup_{X \in \mathcal{X}} \Pr\{X \in \mathcal{C}\} = \max\{Q_i : 1 \leq i \leq k+1\}$. This completes the proof of the theorem.

D Proof of Theorem 8

Define $W_t = \exp(s(X_t - X_0) - \varphi(s)V_t)$ for $t \geq 0$ and $s \in (0, b)$. Then, for all $s \in (0, b)$ and arbitrary $t' \geq t \geq 0$, we have

$$
\mathbb{E}[W_{t'} | \mathcal{F}_t] = \mathbb{E}[\exp(s(X_{t'} - X_0) - \varphi(s)V_{t'}) | \mathcal{F}_t] = \mathbb{E}[\exp(s(X_{t'} - X_t) - \varphi(s)(V_{t'} - V_t)) W_t | \mathcal{F}_t] = W_t \exp(-\varphi(s)(V_{t'} - V_t)) \mathbb{E}[\exp(s(X_{t'} - X_t)) | \mathcal{F}_t] \leq W_t.
$$

Hence, for any $s \in (0, b)$, $(W_t, \mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a super-martingale with $\mathbb{E}[W_0] = \mathbb{E}[\exp(-\varphi(s)V_0)] \leq 1$. By the assumption on the continuity of the sample paths of $(s(X_t - x_0) - \varphi(s)V_t)_{t \geq 0}$, we have that almost all sample paths of $(W_t)_{t \in \mathbb{R}^+}$ is right-continuous.

To prove (3), note that for any $s \in (0, b)$ and real number $\gamma > 0$,

$$
\Pr\left\{ \sup_{t \geq 0} \left[ X_t - X_0 - \gamma V_t - \frac{\varphi(s)}{s}(V_t - V_0) \right] \geq 0 \right\} = \Pr\left\{ \sup_{t \geq 0} \left[ X_t - X_0 - \gamma V_t - \frac{\varphi(s)}{s}(V_t - V_0) \right] \geq 0 \right\}
$$

$$
= \Pr\left\{ \sup_{t \geq 0} s(X_t - X_0) - \varphi(s)V_t - \gamma s V_t \geq 0 \right\} = \Pr\left\{ \sup_{t \geq 0} s(X_t - X_0) - \varphi(s)V_t \geq \gamma s V_t \right\}
$$

$$
= \Pr\left\{ \sup_{t \geq 0} W_t \geq \exp(\gamma s V_t - \varphi(s)V_t) \right\}
$$

$$
\leq \exp(\varphi(s)V_0 - \gamma s V_t) \leq \exp(\varphi(s) - \gamma s)^V
$$

Here, we have used the definition of $W_t$ in (8). The inequality (9) follows from the super-martingale inequality. This proves (3) and thus (4) immediately follows. This concludes the proof of the theorem.
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