Packing and coloring $r$-bounded axis-parallel rectangles

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Abstract

Let $R$ be a family of axis-parallel rectangles in the plane. The transversal number $\tau(R)$ is the minimum number of points needed to pierce all the rectangles. The independence number $\nu(R)$ is the maximum number of pairwise disjoint rectangles. Given a positive real number $r$, we say that $R$ is an $r$-bounded family if, for any rectangle in $R$, the aspect ratio of the longer side over the shorter side is at most $r$.

Gyárfás and Lehel asked if it is possible to bound the transversal number $\tau(R)$ with a linear function of the independence number $\nu(R)$. Ahlswede and Karapetyan claimed a positive answer for the particular case of $r$-bounded families, but without providing a proof. Chudnovsky et al. confirmed the result proving the bound $\tau \leq (14 + 2r^2)\nu$.

This note aims at giving a simple proof of $\tau \leq 2(r + 1)(\nu - 1) + 1$, slightly improving the previous results. As a consequence of this new approach, we also deduce a constant factor bound for the ratio $\frac{\chi}{\omega}$ in the case of $r$-bounded family.

1 Introduction

1.1 Basic definitions

Let $R$ be a family of axis-parallel rectangles in the plane and $r$ a positive real number. We say that $R$ is an $r$-bounded family if, for any rectangle in $R$, the aspect ratio of the longer side over the shorter side is at most $r$.

We define some general parameters: the transversal number $\tau(R)$ is the minimum number of points needed to pierce all the rectangles in $R$; the independence number $\nu(R)$ is the maximum number of pairwise disjoint rectangles in $R$; the chromatic number $\chi(R)$ is the minimum number of classes in a partition of $R$ into pairwise disjoint rectangles, and the clique number $\omega(R)$ is the maximum number of pairwise intersecting rectangles in $R$. It is straightforward to see that $\tau(R) \geq \nu(R)$, since if we have a set of pairwise disjoint rectangles we need at least one point for each, and $\chi(R) \geq \omega(R)$, since if we have a set of pairwise intersecting rectangles we need a different color for each.

The intersection graph $G(R)$ is the graph having the set of axis-parallel rectangles as its vertex set and an edge for each pair of intersecting rectangles. It is easy to see the relations between $\nu(R)$, $\chi(R)$, $\omega(R)$ and usual graph parameters $\alpha(G(R)), \chi(G(R)), \omega(G(R))$ (see [2]). Furthermore, due to the Helly property,
1.2 Packing results

In 1965, Wegner \cite{15} conjectured that, for a family $\mathcal{R}$ of axis-parallel rectangles, we have $\tau(\mathcal{R}) \leq 2\nu(\mathcal{R}) - 1$. Gyárfás and Lehel \cite{10} asked whether there exists a constant $c$ such that $\tau(\mathcal{R}) \leq c\nu(\mathcal{R})$.

Almost sixty years later, both problems are still open. Jelínek \cite{8} constructed a family of rectangles which shows that $c \geq 2$. Moreover, Chen and Dumitrescu \cite{6} and Sebő \cite{14} recently found examples which prove that, if confirmed, Wegner's conjecture is tight for $\nu \leq 4$.

Besides these, there are several partial results in the literature. Kim et al. \cite{11} give an answer to Gyárfás and Lehel's question in general for any set of homothetic images of a fixed convex body with a constant of 16. Moreover, Chan and Har-Peled \cite{5} proved the corresponding statement for cross-free families of rectangles (i.e., families of rectangles in which the intersection of two rectangles is either empty or contains at least a corner of one of them). Complementing this, Asplund and Grünbaum \cite{2} observed that the intersection graph of a crossing family of rectangles $\mathcal{R}_c$ is perfect, so $\nu(\mathcal{R}_c) = \tau(\mathcal{R}_c)$.

Ahlswede and Karapetyan \cite{1} published a note with some results without providing the proofs. In Statement 1, they claimed that for $r$-bounded families $\tau \leq 2(r + 1)\nu$ holds, offering a particular solution of Gyárfás and Lehel's question. Lately, this statement was confirmed in a paper of Chudnovsky et al. \cite{7} but with a larger factor $(14 + 2r^2)$. In this note we give a simple proof of $\tau \leq 2(r + 1)(\nu - 1) + 1$, slightly improving the previous results.

1.3 Coloring results

Another challenging problem in Geometric Combinatorics is relating the chromatic number and the clique number of family of convex bodies. In 1948, Bielecki \cite{3} asked if it is possible to bound the chromatic number of a family of axis-parallel rectangles using a function of the clique number. Asplund and Grünbaum \cite{2} solved the problem providing a quadratic bound which remained asymptotically best for exactly sixty years. Only recently Chamelsook and Walczak \cite{4} improved the result proving $\chi(\mathcal{R}) = O(\omega(\mathcal{R}) \log(\omega(\mathcal{R})))$. For a lower bound, Krawczyk and Walczak \cite{12} constructed a family $\mathcal{R}$ of axis-parallel rectangles with $\chi(\mathcal{R}) = 3\omega(\mathcal{R}) - 2$ providing the best known lower bound for the problem.

According to the current state of the art and our personal experience, it seems reasonable to ask whether $\frac{\chi}{\omega}$ can be bounded by a constant for families of axis-parallel rectangles. As an addition to our main theorem, we observe that our proof technique implies a positive answer for $r$-bounded families.

2 Results

Let $\mathcal{R}$ be a family of axis-parallel rectangles. Denote for brevity $\tau = \tau(\mathcal{R})$ and $\nu = \nu(\mathcal{R})$. 

\begin{proof}
\end{proof}
Theorem 1. Let $\mathcal{R}$ be a set of axis-parallel rectangles and $r$ be the maximum aspect ratio of a rectangle in $\mathcal{R}$. Then, $\tau \leq 2(r + 1)(\nu - 1) + 1$.

Proof. We prove the result by induction on $\nu$. If $\nu = 1$, then by the Helly property $\tau = 1$, so the inequality holds.

Assume now that $\nu \geq 2$ and the result is true for every family having $\nu' < \nu$. Let $\epsilon := \min_{R \in \mathcal{R}} \{a_R\}$ where $a_R$ is the length of the smaller side of $R$ and let $R_\epsilon$ be the rectangle which realizes this minimum. We assume, without loss of generality, that $a_{R_\epsilon} = \epsilon$ is the height of $R_\epsilon$ and $b_{R_\epsilon}$ is its width. Define $P := \{p_1, \ldots, p_{r + 1}, q_1, \ldots, q_{r + 1}\}$ as the set of $2(r + 1)$ points which subdivide the upper and the lower sides of $R_\epsilon$ into segments of length at most $\frac{b_{R_\epsilon}}{r} \leq \epsilon$, as showed in Figure 1.

Claim 1. If $R \in \mathcal{R}$ and $R \cap R_\epsilon \neq \emptyset$, then $R \cap P \neq \emptyset$.

Indeed, since $R \cap R_\epsilon \neq \emptyset$, then $R$ intersects at least one of the $r$ rectangles with corner points in $P$ partitioning $\mathcal{R}_\epsilon$ (Figure 1). We denote this rectangle as $P(R) := \{p_1, \ldots, p_{r + 1}, q_1, \ldots, q_{r + 1}\}$ as the set of $2(r + 1)$ points which subdivide the upper and the lower sides of $R_\epsilon$ into segments of length at most $\frac{b_{R_\epsilon}}{r} \leq \epsilon$, as showed in Figure 1.

Claim 2. Let $\mathcal{R}' := \{R \in \mathcal{R}, R \cap R_\epsilon = \emptyset\}$, then:

a) $\nu(\mathcal{R}) \geq \nu(\mathcal{R}') + 1$;

b) $\tau(\mathcal{R}) \leq \tau(\mathcal{R}') + 2(r + 1)$.

Indeed, $a)$ follows by noting that adding $R_\epsilon$ to any independent set of $\mathcal{R}'$, we get an independent set again; $b)$ is an immediate consequence of Claim 1, since $|P| = 2r + 1$.

Claim 2 finishes the proof of Theorem 1 by induction.

Corollary 2. Let $\mathcal{S}$ be a family of squares in the plane, then $\tau(\mathcal{S}) \leq 4\nu(\mathcal{S}) - 3$.

In the proof of Theorem 1, Claim 1 shows that every rectangle intersecting $R_\epsilon$ contains a point of $P$. Since any point can be contained in at most $\omega(\mathcal{R})$ elements of $\mathcal{R}$, $R_\epsilon$ has at most $2(r + 1)(\omega(\mathcal{R}) - 1)$ neighbors. We can conclude the following:

Lemma 3. Let $G$ be the intersection graph of an $r$-bounded family of axis-parallel rectangles. Then, $G$ is $2(r + 1)(\omega(G) - 1)$-degenerate.

Theorem 4. Let $\mathcal{R}$ be a set of axis-parallel rectangles and $r$ be the maximum aspect ratio of a rectangle in $\mathcal{R}$. Then, $\chi(\mathcal{R}) \leq 2(r + 1)(\omega(\mathcal{R}) - 1) + 1$. 

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Proof. It is enough to use Lemma 3 and notice that any \( k \)-degenerate graph has chromatic number at most \( k + 1 \).

Theorem 4 together with the following result of Perepelitsa [13], directly implies another result for squares.

**Theorem 5.** (Theorem 7, Corollary 8 [13]) Let \( G \) be the triangle-free intersection graph of a finite number of rectangles. If the intersection of any pair of rectangles is either empty or contains at least a corner of one of them, then \( G \) is a plane graph. Moreover, \( G \) is 3-colorable.

**Corollary 6** (Statement 3 [1]). Let \( S \) be a family of squares in the plane, then \( \chi(S) \leq 4\omega(S) - 3 \). Moreover, if \( G(S) \) is triangle-free, then \( \chi(S) \leq 3 \).

## 3 Conclusion

In this note we offer a simple proof and a slight improvement of a result often cited in the literature but so far without proof. In addition, we state some new results about the chromatic number of \( r \)-bounded families.

Finally, we would like to point out that a further improvement of the bound proposed by Theorem 4, either replacing \( r + 1 \) by \( r \) or removing the multiplicative factor 2, would lead to an affirmative answer of Wegner’s conjecture for families of squares.

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