Graphical representation of invariants and covariants in general relativity†

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Received 4 July 1994, in final form 19 September 1994

Abstract. We present a graphical way of describing invariants and covariants in (four-dimensional) general relativity. This frees us from the complexity of treating many suffixes. Two new off-shell relations between \((\text{mass})^6\) invariants are obtained. These are important for 2-loop off-shell calculations in perturbative quantum gravity. We list all independent invariants with dimensions of \((\text{mass})^4\) and \((\text{mass})^6\). Furthermore, the six-dimensional Gauss–Bonnet identity is expressed in terms of the independent invariants.

PACS numbers: 0270, 0420, 04Z0C, 0460

1. Introduction

In relation to the development of the unified theory and the investigation of the initial stage of the universe, the physical importance of quantum gravity is growing increasingly. For several years the statistical aspect of (Euclidean) quantum gravity (critical dimension, etc) has been vigorously investigated and has been clarified using lower dimensional models. The renormalizability problem, however, does not go beyond the pioneering work of 't Hooft and Veltman [TV]. The problem seems crucial for approaching quantum gravity perturbatively [I1,12]. In such research we must treat invariants with higher mass dimensions: \((\text{mass})^{2+2n}\) for \(n\)-loop perturbation order (in the ordinary gauge). In this circumstance it seems important to develop a systematic and convenient way of dealing with those invariants. We present a graphical way, which enables us visually to discriminate between various invariants. It sets us free from the complexity of suffix-contraction and of algebraic computation. Mathematically this is a representation of invariants (covariants) in terms of graphs.

As is well known, the symmetry of the Riemann curvature tensor is not so simple (see section 2). It is an important problem [P, G, VW] to find the independent products of Riemann tensors. In the recent intensive work of [FKWC] the problem is tackled from the representation theory of the symmetric group. We present a new method using graphs. In particular, practical usefulness is stressed. We examine closely the \((\text{mass})^4\) invariants and the \((\text{mass})^6\) invariants and find all relations, at the off-shell level (that is, with no use of

† The content of this paper was in part reported at the annual meetings of the Physical Society of Japan in the autumn of 1992 and in the spring of 1993.
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§ [BOS] is a recent extensive review.
the field equation), between them. It is known that those relations are very important for investigating quantum gravity [VW,11,12].

In section 2 we list all well known symmetries and relations which hold for invariants and covariants in general relativity. A graphical representation of Riemann tensors and the covariant derivatives is defined in section 3. Some rules, which are basic relations in the graphical calculation, are presented. In section 4 all invariants with dimension \( (\text{mass})^4 \) and with dimension \( (\text{mass})^6 \) are expressed graphically. We reduce, in section 5, the number of \( (\text{mass})^6 \) invariants to eight by using the rules above and two new off-shell relations. In section 6 we comment on the independence of the finally listed invariants. Furthermore, the Gauss–Bonnet identity in six-dimensional spacetime is explicitly written in terms of the independent invariants.

Some appendices are given in order to complement the content of the text. In appendix A some important graphical rules, which are skipped in the text, are presented. All graphical rules appearing in the paper are listed in appendix B in such a way that the relation between the rules is manifest. In appendix C a way of classifying all graphs is presented. There we introduce some indices for the classification. \( M^6 \)-invariants of the type \( \nabla R \times \nabla R \) are classified completely in appendix D. This result is used in the text (subsection 4.2(i)). Finally, in appendix E, we give the standard transformation procedures needed to ensure one-to-one correspondence between a graph and a literal mathematical expression.

2. Preliminaries

Before the main text we summarize the present notation and list all well known symmetry properties of covariants.

(1) Notation

\[
R_{\mu \nu \alpha \beta} = \partial_\alpha \Gamma^\mu_{\nu \beta} + \Gamma^\gamma_{\alpha \beta} \Gamma^\mu_{\nu \gamma} - (\alpha \leftrightarrow \beta)
\]

\[
\Gamma^\mu_{\nu \rho} = \frac{1}{2} \delta^{\alpha\beta} (\partial_\nu g_{\mu\beta} + \partial_\mu g_{\nu\beta} - \partial_\beta g_{\mu\nu})
\]

\[
R_{\mu \nu} = R^\sigma_{\mu \nu \sigma} \quad R = R^{\mu}_{\mu}.
\]

From these definitions we know that mass dimension of the Riemann curvature tensor, \( [R_{\mu \nu \alpha \beta}] = (\text{mass})^2 \), under the definition \( [g^{\alpha \beta}] \equiv (\text{mass})^0, \ [\partial_\alpha] \equiv (\text{mass})^1 \).

(2) Symmetry

\[
R_{\mu \nu \lambda \sigma} = R_{\lambda \sigma \mu \nu} = - R_{\nu \lambda \mu \sigma} = - R_{\mu \sigma \nu \lambda}. \quad (2.2a)
\]

\[
R_{\mu \nu \lambda \sigma} + R_{\nu \lambda \mu \sigma} + R_{\lambda \mu \nu \sigma} = 0. \quad (2.2b)
\]

\[
R_{\mu \nu} = R_{\nu \mu}. \quad (2.2c)
\]

\[
\nabla_\delta R_{\alpha \beta \mu \nu} + \nabla_\mu R_{\delta \beta \nu \sigma} + \nabla_\nu R_{\alpha \beta \delta \mu} = 0 \quad \text{(Bianchi identity)} \quad (2.2d)
\]

\[
\nabla^\sigma R_{\mu \nu \lambda \sigma} = \nabla_\mu R_{\nu \lambda} - \nabla_\nu R_{\mu \lambda}. \quad (2.2e)
\]

\[
\nabla^\mu (R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R) = 0. \quad (2.2f)
\]

In addition to the above symmetries which are derived by the manipulation of local quantities, there exists one relation, between \( (\text{mass})^4 \) invariants, which is related to a global (topological) quantity.
(3) Gauss–Bonnet identity (in four dimensions)

\[
\int d^4x \sqrt{g} R_{\mu\nu\alpha\beta} R_{\lambda\sigma\gamma\delta} \varepsilon^{\mu\nu\lambda\sigma} \varepsilon^{\alpha\beta\gamma\delta}
\]

\[
= 4 \int d^4x \sqrt{g} (R^2 + R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4R_{\mu\nu} R^{\mu\nu})
\]

(2.3)

where \( \varepsilon^{\mu\nu\lambda\sigma} \) is the totally antisymmetric tensor.

(4) Covariant ingredients. In table 1 we list all covariant ingredients that make up invariants.

| Type          | Description                                      |
|---------------|--------------------------------------------------|
| Type 1        | (2.2a) and (2.2c). The symmetries of this type are automatically built into the graphical representation. |
| Type 2        | (2.2b) and (2.2d)–(2.2f) and some others. We treat type 2 symmetries as graphical rules. (2.2b) is expressed by the rules defined in subsection 3.2, and (2.2d)–(2.2f) are expressed by the rules defined in subsection 3.3. Some other symmetries are relegated to appendix A. |
| Type 3        | (2.3) and some others. The forms of type 3 symmetries depend on the spacetime dimension. We treat them cases by case. We consider mainly the dimension four. The new identities we find are of this type. |

3. A graphical way of representing covariants and invariants

3.1. Basic definitions

Let us define a graphical representation for the Riemann tensor:

![Graphical representation for the Riemann tensor](image)

Figure 1. Graphical representation for the Riemann tensor \( R_{\mu\nu\lambda\sigma} \).
Definition 1. **Graph of Riemann tensor.** The Riemann tensor, $R_{\mu\nu\lambda\sigma}$, is graphically expressed as in figure 1 with the following items (A1–A4).

(A1) The graph is expressed by two kinds of lines (dotted and solid lines) and two vertex points which connect two different kinds of lines. We call the dotted line the *suffix line* because it represents the suffix flow. The arrow indicates the direction of the suffix flow. The suffix which flows in (out) is defined to be the left (right) suffix of each doublet $(\mu\nu)$ or $(\lambda\sigma)$ appearing in $R_{\mu\nu\lambda\sigma}$. The solid line represents the Riemann tensor itself.

(A2) We need not distinguish between upper (contravariant) and lower (covariant) suffixes because we are, in the end event, interested only in invariants. ($R_{\nu\lambda\sigma}^\mu$, $R_{\mu\nu\lambda\sigma}^\sigma$, $R_{\mu\nu\lambda\sigma}^\nu\lambda\sigma$ are also represented by figure 1.)

(A3) We may write the graph in any way we like unless we change the graph topologically and change the suffix flow (figure 2(a)).

(A4) The graph of figure 1 satisfies the symmetries (2.2a) which the Riemann tensor $R_{\mu\nu\lambda\sigma}$ has (figure 3).

† The ordinary mathematical symbols (=, +, −, ×) are used with their ordinary meanings.
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Figure 3. The symmetries of the Riemann tensor.

Figure 4. Contraction of two suffixes. (a) No 'tadpole'; (b) Ricci tensor; (c) scalar curvature.

We add the following notes.

(N1) We may freely move the vertex point unless it jumps another vertex point (figure 2(b))†.
(N2) Different lines may freely cross each other (figures 2(c), (d))‡.

Definition 2. Graphical method of contraction. We represent the contraction of two suffixes simply by connecting them by a suffix line. Examples are given in figures 4(a), (b) and (c) which express $R_{\mu,\lambda\alpha}^{\nu} = 0$, $R_{\nu,\lambda\mu}^{\mu} = R_{\nu\lambda}$, $R_{\nu}^{\nu} = R$, respectively. Before the contraction, in general we must adjust the flow directions based on the symmetry of figure 3, in such a way that the flow-in suffix (flow-out suffix) is connected to the flow-out suffix (flow-in suffix) so as to guarantee continuous suffix flow.

† The closed suffix line will soon be explained.
‡ The omission of arrows will be explained soon.
We can easily see the following theorems about graphs.

**Theorem Th1.** From the symmetry of figure 3 and the continuous flow definition of the graph, generally a graph changes its sign by \((-1)^n\) under a change of suffix flow, where \(n\) is the number of vertices which the suffix line has (figure 5(a)).

![Diagram of graph changes](image)

Figure 5. Changes of sign of a graph caused by a change of suffix flow.

**Theorem Th2.** In general all suffix lines are closed for the invariants involving no covariant derivatives (figure 4(c)). (Figure 4(c), which represents \(R\), is the simplest case of the graphical representation of invariants: the \((\text{mass})^2\) invariant.)

**Theorem Th3.** The arrow on a suffix line (closed or non-closed) may be omitted if the line has an even number of vertices, because there exists no ambiguity in the direction of the arrow for such a case (figures 4(b), (c), 5(b)).
Theorem Th4. Figure 4(a) says that the 'tadpole' graph is prohibited.

We add here a note.

(N3) The right-hand side of figure 4(b), where the arrow is omitted, clearly shows the symmetry (2.2c).

In order to show one-to-one correspondence between a graph and a literal mathematical expression modulo the symmetries of type 1, we describe the standard transformation procedures in appendix E. From A4 and N3 we see that the symmetries of type 1 are automatically satisfied in the above definition of graphs. Let us build in the symmetries of type 2 in the following subsections.

3.2. Graphical rules (1): the case with no covariant derivatives

We can express the relation (2.2b) by a graphical rule: Cyc 1 (figure 6(a)). If we make use of the graphical representation, various relations between products of Riemann tensors are easily obtained. We demonstrate here a simple example: the relation $R_{\gamma \delta \xi \omega}^{\alpha} r_{\omega}^{\beta} = \frac{1}{2} R_{\gamma \delta \xi \omega}^{\alpha} r_{\omega}^{\beta} (\text{figure 7(a)})$ and its proof using Cyc 1 (figure 8). We call the graphical relation of figure 7(a) Cyc 2, since it will be used as one of the basic relations in the following. By contracting $\alpha$ and $\beta$ in figure 7(a) we obtain the relation figure 7(b) (Cyc 2a). The numbers 1, 2 in Cyc 2 distinguish the two solid lines and they are useful when we will later use the rule for terms with covariant derivatives. Similarly we can obtain another useful graphical rule: Cyc 3 (figure 9(a)), using Cyc 1,

By contracting the suffixes $\mu$ and $\lambda$ in figure 9(a) we obtain figure 9(b) (Cyc 3a). Furthermore, figure 9(c) (Cyc 3b) is obtained by contracting $\nu$ and $\sigma$ in figure 9(b). Cyc 3a and 3b are also obtained from Cyc 2 (see appendix B). We add here some notes.

(N4) Cyc 2 is useful when we want to increase or decrease the number of suffix loops.

(N5) Cyc 3 is useful when we want to change the flow of a suffix line.

3.3. Graphical rules (2): the case with covariant derivatives

In addition to the graphical representation for the curvature tensors, we introduce that for the covariant derivative. (See figure 10.)

Definition 3. Graph of a covariant derivative. Covariant derivatives are graphically represented as in figure 10(a) ($\nabla_{\alpha} R_{\mu \nu \lambda \sigma}$) and figure 10(b) ($\nabla_{\beta} \nabla_{\alpha} R_{\mu \nu \lambda \sigma}$) with the following items (B1–B3). The symbol $\triangleright$ represents the covariant derivative.
Figure 7. (a) Cyc 2. $R_{\gamma\delta\tau\alpha}R_{\alpha\tau\gamma\beta} = \frac{1}{2} R_{\gamma\delta\tau\alpha}R_{\alpha\tau\gamma\beta}$. (b) Cyc 2a.

Figure 8. Proof of Cyc 2.

(B1) The order of the covariant derivatives is definitively specified by the graphical definition.
Figure 9. (a) Cyc 3. (b) Cyc 3a. (c) Cyc 3b.

Figure 10. Graphical representation of the covariant derivative. (a) $\nabla_\rho R_{\mu\nu\lambda\sigma}$, (b) $\nabla_\beta \nabla_\sigma R_{\mu\nu\lambda\sigma}$.

(B2) There is no arrow for the suffix line of a covariant derivative. When a suffix of a covariant derivative is contracted with that of a vertex, the flow-in or flow-out does not have a meaning for the covariant derivative.

(B3) The covariant derivative symbol ($\leftarrow\rightarrow$) has one suffix line and the line may connect anywhere in the symbol.
We can graphically represent the Bianchi relations (2.24)–(2.26) as in figure 11(a) (Bian 1), figure 11(b) (Bian 1a) and figure 11(c) (Bian 1b), respectively. Bian 1a is obtained by contracting $\beta$ and $\delta$ in Bian 1. Bian 1b is obtained by contracting $\mu$ and $\lambda$ in Bian 1a.

\[
\begin{align*}
\beta & \quad \delta & \quad \nu \\
\alpha & \quad \mu & \quad \alpha & \quad \nu & \quad \alpha & \quad \delta \\
\end{align*}
\]

(a)

\[
\begin{align*}
\nu & \\
\mu & \quad \lambda \\
\end{align*}
\]

(b)

\[
\begin{align*}
& = \frac{1}{2}
\end{align*}
\]

(c)

Figure 11. The Bianchi relations (a) Bian 1; (b) Bian 1a; (c) Bian 1b.

Note that the rules presented in subsection 3.2 hold true for the case with covariant derivatives. In particular figure 6(b) (Cyc 1'), figure 7(c) (Cyc 2') and figure 9(d) (Cyc 3') hold true from the relations figure 6(a) (Cyc 1), figure 7(a) (Cyc 2) and figure 9(a) (Cyc 3), respectively.

\[
\begin{align*}
\begin{align*}
\nu & \quad \alpha & \quad \sigma \\
\mu & \quad \lambda \\
\end{align*}
\end{align*} + \begin{align*}
\begin{align*}
\nu & \quad \sigma \\
\mu & \quad \lambda \\
\end{align*} + \begin{align*}
\begin{align*}
\nu & \quad \alpha & \quad \sigma \\
\mu & \quad \lambda \\
\end{align*} = 0
\end{align*}
\]

Figure 6. (b) Cyc 1'.

In order to make the presentation simple, we make here arguments, related to covariant derivatives, minimal and relegate the full examination to appendices A and D. There exist some other important graphical rules related to the commutator of covariant derivatives.
Figure 7. (c) Cyc 2'.

Figure 9. (d) Cyc 3'.

\([\nabla, \nabla] \sim R\) (see figure A1(a)-(d)). There also exist some other important rules between \(\nabla R \times \nabla R\)-covariants derived from Bian 1 (see figure A2(a)-(g) and figure A3(a)-(c)).

We may ignore the total derivative terms (such as \(\nabla_{\mu} \nabla_{\nu}(R^{\mu} R^{\nu})\)) in the action. We could express the arbitrariness as graphical rules. We do not, however, adopt those relations as the rules, because the arbitrariness of the total derivatives is easily taken into account when we allocate the covariant derivatives in an expression (see appendix D as an example of such treatment). In the calculation of the counter-terms\(\dagger\), the total derivatives may be treated as arbitrary quantities because they always appear in the action. In some cases, however, such as the calculation of the Weyl anomaly, the total derivative terms are also important.

The remaining symmetries which we do not yet touch on are type 3 symmetries such as the Gauss–Bonnet identity (2.3). There also exist some other relations which will be treated in section 5. We take into account those symmetries (or relations) case by case.

We list, in appendix B, the ‘tree’ structure of all graphical rules presented in this paper and show clearly the relation between the ‘primary’ rules and the ‘descendant’ rules.

4. Graphical representation of invariants

Now we list all invariants with a fixed mass dimension and represent them graphically,

4.1. Invariants with dimension \((\text{mass})^4\)

We can easily find the three invariants, with the mass dimension \((\text{mass})^4\), as shown in figures 12(a), (b) and (c) which represent \(R^2\), \(R_{\mu\nu} R^{\mu\nu}\) and \(R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma}\), respectively.

\(\dagger\) Counter-terms appear in the renormalization theory of the quantum field theory. They are added in advance in the (classical) Lagrangian in order not to cause ultraviolet divergences in the perturbative calculation.
Note that we have the relation Cyc 3b (figure 9(c)) and we need not consider the total derivative $\nabla_\mu \nabla^\mu R$.

It is well known, in four-dimensional quantum gravity, that there exists one relation between the three invariants above, the Gauss–Bonnet identity (2.3). Here we note that the integrant of (2.3), $R_{\mu\nu\rho\sigma} R_{\lambda\gamma\delta} e^{\mu\nu\rho\sigma} e^{\gamma\delta\lambda}$, can be expressed graphically as in figure 13. One (say, the upper suffix) of each pair of contracted suffixes is explicitly written in figure 13 in order to specify which suffixes are anti-symmetrized. We will compare figure 13 with its six-dimensional counter-part in section 6.

Therefore two out of three invariants above are (locally) independent invariants, say,

$$R^2, \quad R_{\mu\nu} R^{\mu\nu}. \quad (4.1)$$

We will comment on their independence in section 6.

4.2. Invariants with dimension (mass)$^6$

Invariants with the mass dimension (mass)$^6$ have two types: $\nabla R \times \nabla R$ and $R \times R \times R$.

(i) $\nabla R \times \nabla R$ (figures 14(a), (b)). The full treatment of this type of invariant is relegated to appendix D, where the classification index is introduced. The result says that the independent invariants are $O_1 \equiv \nabla^\mu R \cdot \nabla_\mu R$ (figure 14(a)) and $O_2 \equiv \nabla^\mu R_{\sigma\delta} \cdot \nabla_\mu R^{\sigma\delta}$ (figure 14(b)).

(ii) $R \times R \times R$. Products of three curvature tensors can be listed as in figures 15(a)–(f) and figures 16(a)–(f). We have three kinds of curvature tensors: the scalar curvature $R$, Ricci curvature tensor $R_{\mu\nu}$, and Riemann curvature tensor $R_{\mu\nu\lambda\sigma}$. Here we call all these
Figure 14. (a) \( O_1 = \nabla^\mu R \cdot \nabla_\mu R \); (b) \( O_2 = \nabla^\mu R_{\alpha\beta} \cdot \nabla_\mu R_{\alpha\beta} \).

quantities curvature tensors. Figures 15(a)--(f) represent \( P_1 = R R R, \ P_2 = R R \mu \nu R_{\mu \nu} \), \( P_3 = R R_{\mu \nu \lambda \sigma} R^{\mu \nu \lambda \sigma}, \ P_4 = R_{\mu} R^{\nu \lambda} R_{\lambda}^{\mu}, \ P_5 = R_{\mu \nu \lambda \sigma} R^{\mu \lambda} R^{\nu \sigma}, \ P_6 = R_{\mu \nu \lambda \sigma} R_{\tau}^{\nu \lambda \sigma} R_{\tau}^{\mu} \), respectively. We can easily check that the invariants \( \{ P_i \} \) are independent so far as the symmetries of type 1 and 2 are concerned. They will turn out, in section 5, to be dependent when the symmetries of type 3 are taken into account.

Figure 15. The invariants: (a) \( P_1 = R R R \); (b) \( P_2 = R R_{\mu} R^{\mu} \); (c) \( P_3 = R R_{\mu \nu \lambda \sigma} R^{\mu \nu \lambda \sigma}, \) (d) \( P_4 = R_{\mu} R^{\nu \lambda} R_{\lambda}^{\mu} \); (e) \( P_5 = R_{\mu \nu \lambda \sigma} R^{\mu \lambda} R^{\nu \sigma}, \) (f) \( P_6 = R_{\mu \nu \lambda \sigma} R_{\tau}^{\nu \lambda \sigma} R_{\tau}^{\mu} \).

Note that \( \{ O_i \} \) and \( \{ P_i \} \) are those invariants which vanish on-shell, \( R_{\mu \nu} = 0 \): the field equation of the Einstein–Hilbert Lagrangian \( \mathcal{L} = (1/\kappa) \sqrt{-g} \ R \). The remaining kind are different products of three Riemann curvature tensors which do not vanish on-shell. We can list them as in figures 16(a)--(f). Figures 16(a)--(f) represent \( A_1 = R_{\mu \nu \lambda \sigma} R^{\alpha \lambda} \tau_{\alpha \nu \omega} R^{\omega \tau \mu} \),
\[ B_1 = R_{\mu
u\sigma} R^{\nu}_{\lambda\omega} R^\lambda_{\mu\sigma\omega}, \quad B_2 = R_{\mu\nu\sigma\tau} R_{\lambda\omega} R^{\nu\mu\sigma\tau}, \quad B_3 = R_{\mu\nu\omega\tau} R^\nu_{\lambda\sigma} R^\lambda_{\mu\sigma\omega}, \quad C_1 = R_{\mu\nu\sigma\tau} R^{\nu}_{\lambda\omega} R^\lambda_{\mu\sigma\omega}, \quad C_2 = R_{\mu\nu\sigma\tau} R^{\nu}_{\lambda\omega} R^\lambda_{\mu\sigma\omega}, \]

respectively.

In appendix C we check, by use of the classification index, that we have no missing graph in the above list.

We will derive all relations between \( \{O_i, P_i, A_i, B_i, C_i\} \) and find independent invariants in section 5. (Here we list the correspondence between invariants defined in other references and those defined in the present paper. As for [VW], \( A_1^{\text{VW}} = A_1, A_2^{\text{VW}} = -B_1, A_3^{\text{VW}} = C_2 \). As for [11], \( O_1^1 = P_1, O_2^1 = P_2, O_3^1 = P_4, O_4^1 = P_5, O_5^1 = P_3, O_6^1 = P_6, O_7^1 = -A_1, O_8^1 = O_1, O_9^1 = O_2 \).

5. Application of graphical rules and new off-shell relations

We can prove the following relations between \( \{A_i, B_i, C_i\} \) in a way similar to the proof of Cyc 2 (figure 8) in section 3. We list the relations with the names of rules which are
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\[
\begin{align*}
\text{totally} & \quad \text{anti-sym.} \\
\text{+ w.r.t.} & \quad = 0 \\
(\alpha,\beta,\gamma,\delta,\varepsilon)
\end{align*}
\]

**Figure 17.** \(R^{\alpha}_{\beta\gamma\delta\epsilon} R^{\gamma\delta\epsilon}_{\alpha} R^{\beta}_{\gamma\epsilon} = 0\) where \([\ldots]\) means anti-symmetrization.

\[
\begin{align*}
\text{totally} & \quad \text{anti-sym.} \\
\text{+ w.r.t.} & \quad = 0 \\
(\alpha,\beta,\gamma,\delta,\varepsilon,\theta)
\end{align*}
\]

**Figure 18.** \(R^{\alpha\beta}_{\gamma\delta} R^{\gamma\delta}_{\alpha} R^{\alpha}_{\beta} = 0\).

necessary for their proofs:

\[
\begin{align*}
B_2 &= \frac{1}{2} A_1 (\text{Cyc 2}), \\
B_3 &= C_2 - C_1 (\text{Cyc 3}) = -B_1 \text{(relation following)}, \\
C_1 &= \frac{1}{4} A_1 (\text{Cyc 2}, \text{Cyc 2}), \\
C_2 &= C_1 - B_1 (\text{Cyc 1}) = \frac{1}{4} A_1 - B_1 \text{(previous relation)}.
\end{align*}
\]

(5.1)

Therefore we see that all invariants made of three Riemann curvature tensors are described only by \(A_1\) and \(B_1\) [VW].

There exist relations of a different kind, which come not from the rules in section 3 but from the fact that the dimension of the spacetime is four.

**Off-shell relation 1.** Let us consider the identity of figure 17. This idea was explicitly noticed in [GS,FKWC]. The identity figure 17 holds true because each greek suffix runs from 0 to 3 (or from 1 to 4 for Euclidean gravity) in four-dimensional spacetime. This identity turns out to be, by use of a computer,\(^\dagger\)

\[
-P_2 + \frac{1}{2} P_3 + 2 P_4 - 4 P_5 - 5 P_6 + A_1 - 2 B_1 = 0.
\]

(5.2)

The on-shell case of (5.2), \(A_1 = +2B_1\), was obtained in [VW] by use of the spinor formalism.

**Off-shell relation 2.** Similarly we can consider the identity of figure 18. This identity turns out to be, by use of a computer,

\[
I = 8(-P_1 + 12 P_2 - 3 P_3 - 16 P_4 + 24 P_5 + 24 P_6 - 4A_1 + 8B_1) = 0.
\]

(5.3)

The on-shell case of (5.3) again gives \(A_1 = +2B_1\). At the off-shell level, however, (5.2) and (5.3) are independent relations.

\(^\dagger\) The computer algorithm can be roughly explained as follows. The anti-symmetrization produces 5! graphs. Each of them corresponds to some invariant of \(R \times R \times R\) type in subsection 4.2(ii). The 'computer' can identify it, and can replace it by some reduced invariants using the relation (5.1).
Off-shell relation 3. From the relations (5.2) and (5.3) we obtain the relation between \{P_i\} as follows,

\[-P_1 + 8P_2 - P_3 - 8P_4 + 8P_5 + 4P_6 = 0.\]  

(5.4)

This relation can also be directly derived by the identity of figure 19. The independent relations are two out of the above three relations (5.2)–(5.4). We have checked any other choice of anti-symmetrized suffixes and of an initial graph, in the above procedure, does not lead to another independent relation.

\[
\text{Figure 19. } R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = 0.
\]

Therefore we can list eight (= 2(O_1) + 6(P_1) + 2(A_1, B_1) - 2(off-shell relations)) independent (mass)^6 invariants, say, as follows†:

\[O_1, O_2, P_1, P_2, P_3, P_4, P_5, A_1.\]  

(5.5)

As for their independence we make comment in section 6. The importance of \(A_1\), which is the unique non-vanishing term on-shell in (5.5), was first pointed out in [K].

6. Independence of invariants, six-dimensional Gauss–Bonnet identity and conclusion

The final proof of the independence of the listed invariants (4.1) and (5.5) can be done using the weak gravity expansion approach. Let us consider the case of weak gravity and introduce the ‘linear field’ \(h^{\mu\nu}\) instead of the metric \(g^{\mu\nu}\):

\[g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1,\]  

(6.1)

where \(\delta_{\mu\nu}\) is the Minkowski metric (the flat spacetime). Each invariant can be expanded around the flat metric and can be expressed by an infinite series of powers of \(h_{\mu\nu}\) (with derivatives). We have checked the independence of the listed invariants (4.1) and (5.5) by looking at the first few orders of \(h_{\mu\nu}\) by use of a computer.

The procedure presented in this paper is valid for any spacetime dimension. All relations in this paper, except (2.3) (figure 13), (5.2) (figure 17), (5.3) (figure 18) and (5.4) (figure 19), are valid irrespective of the spacetime dimension. Particularly in six-dimensional spacetime, we can obtain the explicit form of the Gauss–Bonnet identity as follows,

\[
\int d^6x \sqrt{|g|} R_{\mu\nu\alpha\beta} R_{\lambda\sigma\gamma\delta} R_{\tau\omega\delta\epsilon} \varepsilon^{\mu\nu\lambda\sigma\omega\gamma} \varepsilon^{\alpha\beta\delta\epsilon} \theta
\]

\[= \int d^6x \sqrt{|g|} \text{(left-hand side of figure 18)}
\]

\[= \int d^6x \sqrt{|g|} I = \text{topological invariant},\]  

(6.2)

† In [GS] nine terms are listed. They missed the relation (5.4).
where $I$ has appeared in (5.3) and the above formula is explicitly rewritten as

$$8 \times \int d^5 x \sqrt{g} \left( - R^3 + 12 R R_{\mu \nu} R^{\mu \nu} - 3 R R_{\mu \nu \lambda \sigma} R^{\mu \nu \lambda \sigma} - 16 R_{\mu \nu} R^{\mu \nu} R_{\lambda \mu} \right. $$

$$+ 24 R_{\mu \nu \lambda \sigma} R^{\mu \lambda} R^{\nu \sigma} + 24 R_{\mu \nu \lambda \sigma} R_{\tau}^{\nu \lambda \sigma} R^{\mu \tau} - 4 R_{\mu \nu \lambda \sigma} R^{\sigma \tau} R^{\nu \mu \lambda \sigma} \left. \right)$$

$$+ 8 R_{\mu \nu \tau \sigma} R^{\nu \lambda \sigma \tau} R^{\lambda \mu \nu \sigma} = \text{topological invariant.} \quad (6.3)$$

It is known, in six-dimensional quantum gravity, that 1-loop counter-terms are given by (mass)$^6$ invariants in the ordinary gauge [VW]. They are given by the linear combination of the nine $(= 2(O_1) + 6(P_i) + 2(A_1, B_1) - 1 \text{ (Gauss–Bonnet identity)})$ independent invariants, say, as follows,

$$O_1, O_2, P_1, P_2, P_3, P_4, P_5, P_6, A_1. \quad (6.4)$$

Because $A_1$ does not vanish on-shell, 1-loop counter-terms, in six-dimensional quantum gravity, do not trivially vanish on-shell. The importance of this fact was stressed in [VN,VW]. It must be compared with the four-dimensional case (4.1), where 1-loop counter-terms trivially vanish on-shell. The relation (6.3) and the left-hand side of figure 18 are the six-dimensional counterparts of the relation (2.3) and figure 13 in four dimensions, respectively.

Finally, we comment on those invariants which have not been considered in the present paper.

1. The extension to invariants with higher mass dimension ((mass)$^8,\ldots$), in four spacetime dimensions, is straightforward. They are important in higher-order perturbative quantum gravity.

2. Pseudoscalars: $e^{\mu \nu \lambda \sigma} R_{\mu \nu \sigma \tau} R_{\lambda \sigma \tau} , e^{\mu \nu \lambda \sigma} R_{\mu \nu \sigma \tau} R_{\lambda \sigma \tau} R_{\lambda \sigma \tau} \ldots$. As far as the pure gravity (Einstein–Hilbert) theory or Bose-matter (scalar, vector)-gravity coupled theories, in four spacetime dimensions, are concerned, pseudoscalars might not appear as a quantum effect. As for Fermi-matter-gravity coupled theories, it is well known that they appear through the chiral anomaly.

3. There are some non-local invariants which become local in a specific gauge. The famous example is $R(1/\Delta)R$ in two-dimensional gravity, which is local in the conformal gauge [FP]. It was exploited as the solvable model of two-dimensional Euclidean quantum gravity. Similar non-local invariants are discussed in four dimensions [AM]. At present, however, the role of those non-local invariants remains obscure in four-dimensional spacetime.

We recall that Feynman diagrams (graphs), which represent various scattering processes of particles, have played a very important role in the practical and formal development of QED, QCD and other renormalizable quantum field theories. We hope the present approach will play an analogous role in further progress in quantum gravity.

The results (5.2) and (5.3) and the proof of the independence of terms in (4.1) and (5.5) by the weak gravity expansion are obtained by the algebraic software FORM [V,VV].

† The symbolic manipulation program FORM was written by J A M Vermaseren. Version 1.0 of the program and the manual are available via anonymous ftp from nikhef.nikhef.nl.
Acknowledgments

The author thanks Professor J A M Vermaseren (NIKHEF-H, Amsterdam) for kindly explaining the usage of his original and excellent algebraic software FORM in the spring of 1992 at KEK, Japan. The author also thanks Dr T Watanabe (School of Food and Nutritional Sciences, University of Shizuoka) for kind help in drawing diagrams, and Professor N Nakanishi (RIMS, Kyoto University) for reading the manuscript carefully.

Appendix A. Other important rules

As for the relations connected with the commutator of covariant derivatives, we have the following ones,

\[ [\nabla_\lambda, \nabla_\mu] R^\lambda_{\sigma \tau \omega} = R^\lambda_{\kappa \lambda \mu} R^\kappa_{\sigma \tau \omega} - R^\kappa_{\sigma \lambda \mu} R^\lambda_{\kappa \tau \omega} - R^\kappa_{\tau \lambda \mu} R^\lambda_{\omega \lambda \mu} R^\lambda_{\sigma \tau \kappa} \]  
\[ - [\nabla^\alpha, \nabla_\mu] R_{\alpha \tau} = R_{\alpha \tau} R_{\alpha \mu} - R_{\tau \alpha \beta} R_{\nu \alpha \beta \mu} \]  
\[ [\nabla_\alpha, \nabla_\beta] R^\alpha_{\beta \tau \omega} = 0 \]  
\[ - [\nabla_\alpha, \nabla_\beta] R^\alpha_{\tau \beta \tau} = 0. \]  
\[ (A.1a) \]  
\[ (A.1b) \]  
\[ (A.1c) \]  
\[ (A.1d) \]

These relations are expressed graphically in figures A1(a)–(d), respectively.

Another set of important relations is that between \( \nabla R \times \nabla R \) covariants. Figure A2(a) (Bian 2) can be verified by figure 11(a) (Bian 1). By connecting \( \sigma \) and \( \gamma \) in figure A2(a), we obtain figure A2(b) (Bian 2.1). By connecting \( \nu \) and \( \gamma \) in figure A2(a) we obtain figure A2(c) (Bian 2.2). Similarly we obtain figure A2(d) (Bian 2.1a) and figure A2(e) (Bian 2.1b) from figure A2(b), and obtain figure A2(f) (Bian 2.2a) and figure A2(g) (Bian 2.2b) from figure A2(c).

There exists another type of relations between \( \nabla R \times \nabla R \) covariants. See figure A3(a) (Bian 3). It can be verified by figure 11(a) (Bian 1). By contraction of some indices we obtain figure A3(b) (Bian 3.1) and figure A3(c) (Bian 3.2).

The relations in this appendix are exploited in appendix D, where all \( \nabla R \times \nabla R \) invariants are systematically treated.
Figure A1. The relations: (a) $\text{Com} 1, [\nabla_\gamma, \nabla_\mu] R^\alpha_{\rho\sigma\tau\omega}$; (b) $\text{Com} 1.1, -[\nabla^\omega, \nabla_\mu] R_{\alpha\tau}$; (c) $\text{Com} 1.2, [\nabla_\alpha, \nabla_\beta] R^\alpha_{\rho\sigma\tau\omega}$; (d) $\text{Com} 1.3, -[\nabla_\alpha, \nabla_\beta] R^\alpha_{\rho\sigma\tau\omega}$. 
Figure A2. Various relations derived from the Bianchi identity (2.2d). (a) Bian 2; (b) Bian 2.1; (c) Bian 2.2; (d) Bian 2.1a; (e) Bian 2.1b; (f) Bian 2.2a; (g) Bian 2.2b.
Figure A3. Various relations derived from the Bianchi identity (2.2d). (a) Bian 3; (b) Bian 3.1; (c) Bian 3.2.

Appendix B. Tower of rules

We list all graphical rules presented in this paper with the relation between them. For each item we list the name of the rule, the mass dimension, the number of external suffixes and the figure number.

(i) Cycle relation:
(ii) Bianchi identity:

\[
\begin{align*}
\text{Bian 1b} & \quad \text{M}^3,1 \\
\text{Fig.11c} & \\
\text{Bian 1a} & \quad \text{M}^3,3 \\
\text{Fig.11b} & \\
\text{Bian 1} & \quad \text{M}^3,6 \\
\text{Fig.11a} & \\
\downarrow & \\
\text{Bian 2} & \quad \text{M}^6,6 \\
\text{Fig.1Aa} & \\
\downarrow & \\
\text{Bian 3} & \quad \text{M}^6,4 \\
\text{Fig.1Ac} & \\
\downarrow & \\
\text{Bian 3.1} & \quad \text{M}^6,4 \\
\text{Fig.1Bb} & \\
\downarrow & \\
\text{Bian 3.2} & \quad \text{M}^6,4 \\
\text{Fig.1Cc} & \\
\downarrow & \\
\text{Bian 2.1} & \quad \text{M}^6,4 \\
\text{Fig.1Ad} & \\
\downarrow & \\
\text{Bian 2.1a} & \quad \text{M}^6,2 \\
\text{Fig.1Ae} & \\
\downarrow & \\
\text{Bian 2.1b} & \quad \text{M}^6,0 \\
\text{Fig.1Af} & \\
\downarrow & \\
\text{Bian 2.2} & \quad \text{M}^6,4 \\
\text{Fig.1Ag} & \\
\downarrow & \\
\text{Bian 2.2a} & \quad \text{M}^6,2 \\
\text{Fig.1Ah} & \\
\downarrow & \\
\text{Bian 2.2b} & \quad \text{M}^6,0 \\
\text{Fig.1Ai} & \\
\end{align*}
\]

(iii) Commutator relation:

\[
\begin{align*}
\text{Com 1} & \quad \text{M}^4,4 \\
\text{Fig.1Aa} & \\
\downarrow & \\
\text{Com 1.1} & \quad \text{M}^4,2 \\
\text{Fig.1Ab} & \\
\downarrow & \\
\text{Com 1.2} & \quad \text{M}^4,2 \\
\text{Fig.1Ac} & \\
\downarrow & \\
\text{Com 1.3} & \quad \text{M}^4,2 \\
\text{Fig.1Ad} & \\
\end{align*}
\]

Appendix C. Classification of graphs

As the mass dimension increases, the number of graphs to deal with increases rapidly and the graphs of invariants and covariants become complicated. In order to systematically treat such graphs we must find an index for classifying them. This index is useful particularly for listing all possible graphs without the risk of missing any. (This procedure is also important for the development of the computer program algorithm for such a calculation.)

Figure C1. A graph representing a general covariant.
Figure C2. Simple covariants with no covariant derivative.

Let us consider a graph, which represents a covariant, with $N_R$ solid lines, $N_E$ external (suffix) lines, and $N_V$ covariant derivatives (see figure C1 for an example). Its mass dimension is $M^{2N_R+N_V}$. The number of vertices is $V = 2N_R$. Let $d_i$ ($i = 1 \sim N_R$) be the number of covariant derivatives on the $i$th solid line. This set $\{d_i\}$ is the classification index for the part of covariant derivatives and we denote this situation symbolically as $(d_1, d_2, \ldots , d_{N_R})_V$. Then

\[ (d_1, d_2, \ldots , d_{N_R})_V : \quad N_V = \sum_{i=1}^{N_R} d_i, \quad d_i \geq 0. \quad (C.1) \]

We call such a suffix line that connects between vertex points or $>$ within the graph, the internal (suffix) line. There are three types of internal lines depending on the kinds of endpoints: (1) vertex-vertex; (2) $\triangledown$-vertex; (3) $\triangledown-\triangledown$. Let $N_I$ be the number of internal lines of all types, which is equal to the number of contractions in the covariant:

\[ 2N_I = \sum_{i=1}^{N_R} (4 + d_i) - N_E = 4N_R + N_V - N_E. \quad (C.2) \]

The number of total (internal and external) suffix lines is given by $S = N_I + N_E = 2N_R + \frac{1}{2} N_R + \frac{1}{2} N_E$. Furthermore, the following relation\(\S\) holds true:

\[ 3(V + N_V) = 2 \left( N_I + \sum_{i=1}^{N_R} (d_i + 1) \right) + N_E. \quad (C.3) \]

In the following we explain the four cases of covariants separately: (ia) $N_V = 0, N_E = 0$; (ib) $N_V = 0, N_E \neq 0$; (iia) $N_V \neq 0, N_E = 0$; (iib) $N_V \neq 0, N_E \neq 0$.

(i) $N_V = 0$. This is the case of covariants with no covariant derivative. The covariants are composed only of some curvature tensors. Some simple examples are listed in table C1 and figure C2.

| Graph       | $N_R$ | $N_E$ | $V$ | $N_I$ | $S$ |
|-------------|------|------|----|------|----|
| Figure C2(a)| 1    | 4    | 2  | 0    | 4  |
| Figure C2(b)| 1    | 2    | 2  | 1    | 3  |
| Figure C2(c)| 1    | 0    | 2  | 2    | 2  |

\(\S\) This relation is an analogue of the relation, in field theory, between the number of vertices, the number of propagators and the number of external lines.
\( (a) \) \( N_E = 0 \). This is the case of invariants. All suffix lines are internal and make up a number of closed loops. Let \( L \) be the number of closed loops. Let \( v_i \) be the number of vertices in the \( i \)th closed loop. The set \( \{ v_i \} \) is the classification index for the closed loop part. We denote it symbolically as \( \{ v_1, v_2, \ldots, v_L \} \). Because there cannot exist the graphs of purely suffix lines (no vertex) from the definition, we can say \( v_i \neq 0 \). No tadpole graphs (Th4 in section 3) means \( v_i \neq 1 \). Then the following relation holds true:

\[
(v_1, v_2, \ldots, v_L) : \quad V = \sum_{i=1}^{L} v_i = 2N_E, \quad v_i \geq 2. \tag{C.4}
\]

We can use this formula to classify all invariants with no covariant derivatives. Three cases \( (N_R = 1, 2, 3) \) are classified in table C2 and corresponding graphs in the text are given.

| \( N_R \) | \( N_I = 2N_R \) | Index | Corresponding graphs |
|---|---|---|---|
| 1 | 2 | \( (2)_l \) | Figure 4(c) |
| 2 | 4 | \( (2, 2)_l \) | Figure 12(a) (disconnected) |
| | | | Figure 12(c) |
| | | \( (4)_l \) | Figures 12(b), 9(c) |
| 3 | 6 | \( (2, 2, 2)_l \) | Figure 15(a) (disconnected) |
| | | | Figure 15(c) (disconnected) |
| | | | Figure 16(a) |
| | | \( (3, 3)_l \) | Figures 16(b), (d) another one |
| | | \( (2, 4)_l \) | Figure 15(b) (disconnected) |
| | | | Figures 15(f'), 16(c) |
| | | \( (6)_l \) | Figures 15(d), (e) |
| | | | Figures 16(e), (f') |

(The graph ‘another one’ in \( (3, 3)_l \) vanishes due to Cyc 2a (figure 7(b)).)

\( (b) \) \( N_E \neq 0 \). The graphs under consideration are covariants (or contravariants), which are made only of curvature tensors, with \( N_E \) suffixes. See table C3 and figure C3 for simple examples.

| Graph | \( N_R \) | \( N_E \) | Index | \( N_I \) |
|---|---|---|---|---|
| Figure C3(a) | 2 | 2 | \( (2)_l (2)_o \) | 3 |
| Figure C3(b) | 3 | 2 | \( (3)_l (3)_o \) | 5 |
| Figure C3(c) | 4 | 2 | \( (4)_l (4)_o \) | 7 |
| Figure C3(d) | 2 | 4 | \( (2)_l (1, 1)_o \) | 2 |

The suffix lines make up line strings of two kinds. One kind is the closed loop, which appeared in the previous case \( (a) \). The other kind is a string of lines that starts from an external suffix line and, through some vertices, ends up with some other external suffix line. Let us call such a string of lines the open line. Let \( K \) be the number of open lines which appears in the graph considered. \( K \) is given by \( K = \frac{1}{2}N_E \). Let \( u_i \) \( (i = 1 \sim K) \) be the number of vertices within the \( i \)th open line \( (u_i \geq 1) \). The set \( \{ u_i \} \) is the classification index.
for the open line part. We denote it symbolically as \((u_1, u_2, \ldots, u_K)_o\). Then the following relation holds true:\(^\dagger\):

\[
(u_1, v_2, \ldots, v_L)_o(u_1, u_2, \ldots, u_K)_o : \quad V = \sum_{i=1}^{L} v_i + \sum_{i=1}^{K} u_i = 2N_R,
\]

\[v_i \geq 2, \quad L \geq 0, \quad u_i \geq 1, \quad K = \frac{1}{2}N_E.\tag{C.5}\]

Furthermore we have the relation:

\[
N_1 = 2N_R - \frac{1}{2}N_E = \sum_{i=1}^{L} v_i + \sum_{i=1}^{K} (u_i - 1).
\]

This formula can be used to classify all covariants with no covariant derivatives. As an example, let us classify the case \(N_R = 2, \ N_E = 4\). (C.5) can be written, in this case, as

\[
4 = \sum_{i=1}^{L} v_i + \sum_{i=1}^{K} u_i, \quad v_i \geq 2, \quad L \geq 0, \quad u_i \geq 1, \quad K = 2. \tag{C.6}
\]

All possible cases can be listed as follows:\(^\ddagger\):

1. \((2)_l(1, 1)_o : L = 1, \quad (v_i) = (2), \quad K = 2, \quad (u_i) = (1, 1)\)
2. \((\emptyset)_l(1, 3)_o : L = 0, \quad (v_i) = (\emptyset), \quad K = 2, \quad (u_i) = (1, 3)\)
3. \((\emptyset)_l(2, 2)_o : L = 0, \quad (v_i) = (\emptyset), \quad K = 2, \quad (u_i) = (2, 2)\).

They are listed in table C4 and figure C4.

(ii) \(N_V \neq 0\). Now we consider covariants involving some covariant derivatives.

\(^\dagger\) \(L\) (the number of closed loops) and \(v_i\) \((i = 1 \sim L, \ v_i \geq 2)\), the number of vertices within the \(i\)th closed loop\) can be defined as in the previous case (ia). This note is valid in the following cases.

\(^\ddagger\) \(\emptyset\) means the empty set.
(iia) $N_V \neq 0$, $N_E = 0$. First we consider invariants with some covariant derivatives. See figure C5 ($N_R = 3$, $N_E = 0$, $N_V = 4$, $(1, 1, 2)_v(2, 2)_v(0, 2)_v$) as an example. In this case all suffix lines are internal and they make up line strings of two kinds. One kind is the closed loop as before. The other is such a string of lines that starts from one covariant derivative ($\triangleright$) and, through some vertices, ends at some other covariant derivative. We call such a string of lines the $\nabla \nabla$ line. Let $J$ be the number of $\nabla \nabla$ lines. We easily see that $J = \frac{1}{2} N_V$ in the present case of $N_E = 0$. Let $t_i (\geq 0, i = 1 \sim J)$ be the number of vertices in the $i$th $\nabla \nabla$ line. The set $\{t_i\}$ is the classification index for the $\nabla \nabla$ line part. We denote it symbolically as $(t_1, t_2, \ldots, t_J)_{\nabla \nabla}$. Then the following relation holds true:

$$(u_1, u_2, \ldots, u_L)_i(t_1, t_2, \ldots, t_J)_{\nabla \nabla} : \quad V = \sum_{i=1}^{L} u_i + \sum_{i=1}^{J} t_i = 2N_R, \quad (C.7)$$

$$u_i \geq 2, \quad L \geq 0, \quad t_i \geq 0, \quad J = \frac{1}{2} N_V.$$ 

Furthermore, we have the relation:

$$N_I = 2N_R + \frac{1}{2} N_V = \sum_{i=1}^{L} u_i + \sum_{i=1}^{J} (t_i + 1).$$

We can use (C.7) as the classification index for invariants involving covariant derivatives. See appendix D.

Table C4. Classification of $N_R = 2$, $N_E = 4$, $N_V = 0$.

| Index | Connected graphs | Disconnected graphs |
|-------|------------------|---------------------|
| $(2)_v(1,1)_v$ | Figure C4(a) | Figure C4(b) |
| $(3)_v(1,1)_v$ | Figure C4(c) | no graph |
| $(3)_v(2,2)_v$ | Figure C4(d), (e) | Figure C4(f) |

Figure C4. All covariants for the case: $N_D = 0$, $N_R = 2$, $N_E = 4$.

(iib) $N_V \neq 0$, $N_E \neq 0$. This is the most general case: covariants with some covariant derivatives. See again figure C1. Then we notice a new kind of string line besides the previous ones (closed loop, open line, $\nabla \nabla$ line). It starts at a covariant derivative and, through some vertices, ends at an external suffix line. We call this string of lines the $\nabla$
**open line.** Let $H$ ($\geq 0$) be the number of $\nabla$ open lines in the graph considered. Let $r_i$ ($\geq 0$, $i = 1 \sim H$) be the number of vertices within the $i$th $\nabla$ open line. This set $\{r_i\}$ is the classification index for the $\nabla$ open line part. We denote it symbolically as $(r_1, r_2, \ldots, r_H)_{\nu\nu}$. Then the following relation holds true:

$$(v_1, v_2, \ldots, v_L)(u_1, u_2, \ldots, u_K)_{\nu}(t_1, t_2, \ldots, t_H)_{\nu\nu}(r_1, r_2, \ldots, r_H)_{\nu\nu} :$$

$$V = \sum_{i=1}^{L} v_i + \sum_{i=1}^{K} u_i + \sum_{i=1}^{J} t_i + \sum_{i=1}^{H} r_i = 2N_R,$$

$$2K + H = N_E, \quad 2J + H = N_V, \quad L \geq 0, \quad K \geq 0, \quad J \geq 0, \quad H \geq 0,$$

$$v_i \geq 2, \quad u_i \geq 1, \quad t_i \geq 0, \quad r_i \geq 0.$$

Note that the following relation can be checked:

$$N_1 (= 2N_R + \frac{1}{2}N_V - \frac{1}{2}N_E) = \sum_{i=1}^{L} v_i + \sum_{i=1}^{K} (u_i - 1) + \sum_{i=1}^{J} (t_i + 1) + \sum_{i=1}^{H} r_i.$$  

We can use (C.8) as the classification index for the general covariants. We can check the special cases of (C.8): (1) $N_E = 0$; (2) $N_V = 0$; (3) $N_E = 0, N_V = 0$; these reduce to (C.7), (C.5) and (C.4), respectively.

**Appendix D. Classification of $\Delta R \times \Delta R$ invariants and independent terms**

We classify $\nabla R \times \nabla R$ invariants $(N_R = 2, N_E = 0, N_V = 2)$ by use of the classification index introduced in appendix C. The contents of this appendix complement subsection 4.2(i), where $O_1$ and $O_2$ are given as independent terms without detailed calculation. The necessary equations for its classification are, from (C.1),

$$(d_1, d_2)_{\nu} : \quad 2 = \sum_{i=1}^{2} d_i, \quad d_i \geq 0 \quad (D.1)$$

and, from (C.7),

$$(v_1, v_2, \ldots, v_L)(t_1)_{\nu\nu} : \quad 4 = \sum_{i=1}^{L} v_i + t_1, \quad L \geq 0, \quad v_i \geq 2, \quad J = 1, \quad t_1 \geq 1. \quad (D.2)$$
(D.1) says there are two possibilities: (i) \((2, 0)\); (ii) \((1, 1)\). Case (i) always reduces to case (ii) by the arbitrariness of total derivatives. Therefore we may consider only case (ii). From (D.2) all possibilities are given as follows:

(a1) \((4)_l(0)_{\nu\nu} : L = 1, v_1 = 4, J = 1, t_1 = 0\)

(a2) \((2, 2)_l(0)_{\nu\nu} : L = 2, (v_1, v_2) = (2, 2), J = 1, t_1 = 0\)

(b) \((3)_l(1)_{\nu\nu} : L = 1, v_1 = 3, J = 1, t_1 = 0\)

(c) \((2)_l(2)_{\nu\nu} : L = 1, v_1 = 2, J = 1, t_1 = 2\)

(d) \((0)_l(4)_{\nu\nu} : L = 0, J = 1, t_1 = 4\).

Note that the case \(t_1 = 3\) does not exist, because of the condition \(v_i \geq 2\). We can list all invariants graphically as follows.

(a1) \((4)_l(0)_{\nu\nu}\)

\[
\begin{array}{c}
\begin{tikzpicture}
\draw[thick] (0,0) -- (1,1); \\
\draw[thick] (0,0) -- (-1,1); \\
\draw[thick] (1,0) -- (0,1); \\
\draw[thick] (-1,0) -- (-0,1); \\
\end{tikzpicture}
\end{array}
\]

(b) \((2, 2)_l(0)_{\nu\nu}\)

\[
\begin{array}{c}
\begin{tikzpicture}
\draw[thick] (0,0) -- (1,1); \\
\draw[thick] (0,0) -- (-1,1); \\
\draw[thick] (1,0) -- (0,1); \\
\draw[thick] (-1,0) -- (-0,1); \\
\end{tikzpicture}
\end{array}
\]

(b) \((3)_l(1)_{\nu\nu}\)

\[
\begin{array}{c}
\begin{tikzpicture}
\draw[thick] (0,0) -- (1,1); \\
\draw[thick] (0,0) -- (-1,1); \\
\draw[thick] (1,0) -- (0,1); \\
\draw[thick] (-1,0) -- (-0,1); \\
\end{tikzpicture}
\end{array}
\]
Now we examine graph by graph, and obtain the independent invariants. $O_1$ and $O_2$ are the graphs given in subsection 4.2. Among the above graphs there are the following relations. We list them with the names of the rules necessary for their proofs

$Q_3 = 0$ (Cyc 2a), \quad $Q_5 = \frac{1}{2}O_1$ (Bian 1b), \quad $Q_7 = \frac{1}{4}O_1 ((\text{Bian 1b})^2)$,

$Q_2 = 2Q_5$ (Cyc 1), \quad $Q_1 = \frac{1}{2}Q_2$ (Cyc 3b), \quad $Q_1 = 2Q_{11}$ (Bian 2.2b),

$Q_9 = 4Q_4$ (CYC 2), \quad $Q_{10} = \frac{1}{2}Q_4$ (CYC 3a), \quad $Q_9 = 2Q_8$ (Bian 1a), \quad $Q_9 = 2Q_{11}$ (Bian 2.2b),

$Q_{10} = \nabla^\mu L_\mu + P_4 - P_3 + Q_7$ (tot deri, Com 1.1, tot deri), \quad $Q_9 = \frac{1}{2}Q_4$ (CYC 2),

$Q_{11} = \nabla^\mu K_\mu + \frac{1}{2}P_6 - 2C_1 + C_2 + Q_9$ (tot deri, Com 1, Cyc 3a),

$Q_2 = \nabla^\mu J_\mu + 2P_6 - 4P_4 + 4P_5 - 4B_1 - 2B_2 + 4O_2 - O_1$

(Bian 1, tot deri, Com 1, tot deri, Bian 1a, Bian 1a, tot deri, Com 1.1, tot deri),

where $(P_i, B_i, C_i)$ are defined in subsection 4.2, 'tot deri' means equivalent modification using a total derivative term, and
The above result reduces to

\[ Q_3 = 0, \quad Q_6 = \frac{1}{4} Q_1, \quad Q_7 = \frac{1}{4} Q_1, \]

\[ Q_2 = 2 Q_1 = 4 Q_{11} = 2 Q_5 = \nabla^\mu J_\mu + 2 P_6 - 4 P_4 + 4 P_5 - 2 B_2 - 4 B_1 + 4 O_2 - O_1, \]

\[ Q_9 = \frac{1}{2} Q_4 = Q_8 = \nabla^\mu (\frac{1}{4} J_\mu - K_\mu) - P_4 + P_5 - \frac{1}{2} B_2 - B_1 + 2 C_1 - C_2 + O_2 - \frac{1}{4} O_1, \]

\[ Q_{10} = \nabla^\mu L_\mu + P_4 - P_5 + \frac{1}{4} O_1. \]

Therefore we see all \( R \times R \) invariants are expressed, in the action, by \( O_1, O_2 \) and some \( R \times R \times R \) invariants.

**Appendix E. Map between a literal mathematical expression and its graph**

In the text we have introduced the graphical representation in a practical way. In order to treat it more rigorously it must be defined by the map \( f \) from the set of all covariants to the set of all graphs (defined below).

\[ f : C \to G \]

\[ C = \{ \text{all covariants} \}/\sim \]

\[ G = \bigcup_{n=1}^{\infty} (\bar{G} \otimes \bar{G} \otimes \cdots \otimes \bar{G}) \]

\[ \bar{G} = \{ \text{all connected graphs composed of} \]
where the equivalence relations $\sim$ and $\approx$ are defined as follows.

(i) $c_1 \sim c_2$, $c_1, c_2 \in C$.

- A covariant $c_1$ can be transformed to another covariant $c_2$ by raising or lowering the suffixes (cf A2 in subsection 3.1); or
- $c_1$ and $c_2$ are related by type 1 symmetries, (2.2a) or (2.2c).

(ii) $g_1 \approx g_2$, $g_1, g_2 \in G$.

- A graph $g_1$ can be continuously deformed to another graph $g_2$ without changing the relative connection between vertex points and $\geq$s within $g_1$ (cf A3, N1 and N2); or
- $g_1$ can be transformed to $g_2$, except the sign, by changing the suffix flow following the rule Th1 in subsection 3.1.

The sets $C$ and $G$ are well defined by the above definitions.

We define the map $f$ through the following three definitions.

**Definition 1.** The map $f$ is defined as follows:

\[
\begin{align*}
\mathcal{V} & \quad \mathcal{S} \\
\mu & \quad \lambda \\
\end{align*}
\]

\[
f([R_{\mu\nu\lambda\sigma}]) \overset{\text{def}}{=} \mathcal{V} \quad \mathcal{S}
\]

\[
f([\nabla_{\tau}R_{\mu\nu\lambda\sigma}]) \overset{\text{def}}{=} \mathcal{V} \quad \mathcal{S}
\]

\[
f([\nabla_{\omega} \nabla_{\tau}R_{\mu\nu\lambda\sigma}]) \overset{\text{def}}{=} \mathcal{V} \quad \mathcal{S}
\]

\[
f([g^{\mu\nu}]) \overset{\text{def}}{=} \mathcal{V}
\]
where the notation \([c_1]\) means the element of \(C\) whose representative is \(c_1\), and the higher derivative covariants are defined similarly.

**Definition 2.** 'Homomorphism' is satisfied:

\[
f(c_1 \cdot c_2) = f(c_1) \circ f(c_2) = f(c_2) \circ f(c_1)
\]

(E.3)

where the symbols \(\cdot\) and \(\circ\) are the binary operators ('products') in \(C\) and \(G\), respectively. \(\cdot\) is the suffix contraction in the usual expression.

**Definition 3.** The definition of the binary operator \(\circ\) in \(f(c_1) \circ f(c_2)\) is: connect the lines with the same suffix name in accordance with definition 2 of subsection 3.1.

For simple examples we can easily obtain their representation from the above definition:

\[
f(R_{\mu \nu \lambda \sigma}) = f(R_{\mu \nu \lambda \sigma} \cdot g^{\mu \alpha})
\]

\[
= f(R_{\mu \nu \lambda \sigma}) \circ f(g^{\mu \alpha})
\]

(E.4a)

\[
f(R_{\mu \nu \lambda \sigma} \cdot R^{\mu \nu \lambda \sigma}) = f(R_{\mu \nu \lambda \sigma}) \circ f(R^{\mu \nu \lambda \sigma})
\]

(E.4b)

For the general expression, the above definition is equivalent to the following procedure. We describe it by taking an example.

**Standard procedure 1.** Literal mathematical expression (E.5) \(\rightarrow\) graph \(A_1\) (figure 16(a)).

For a given expression:

\[
- R^{\lambda \sigma}_{\mu \nu} R^{\tau \omega}_{\lambda \sigma} R^\mu_{\tau \omega}
\]

(E.5)

we obtain its graphical expression as follows:
(1) Replace each component of an expression by a graph following the definition (E.2) (figure E1).

(2) Change the direction of arrows, following the definitions of subsection 3.1, in such a way that the suffix flows 'continuously' when the same suffix name lines are connected (figure E2).

(3) Connect all pairs of the same suffix name lines. We obtain the graph $A_1$ (figure 16(a)).

We can obtain easily the inverse map $f^{-1} : G \rightarrow C$:

$$f^{-1} \left( \begin{array}{cc} \nu & \sigma \\ \mu & \lambda \end{array} \right) = [R_{\mu\nu\lambda\sigma}],$$

$$f^{-1} \left( \begin{array}{cc} \nu & \tau \\ \mu & \lambda \end{array} \right) = [\nabla_\tau R_{\mu\nu\lambda\sigma}] \quad \text{etc.,}$$

(E.6)

$$f^{-1}(g_1 \circ g_2) = f^{-1}(g_1) \cdot f^{-1}(g_2) = f^{-1}(g_2) \cdot f^{-1}(g_1).$$

As for the general graph, the inverse map $f^{-1}$ can be obtained by inverting the standard procedure 1.

**Standard procedure 2.** Graph $A_1$ (figure 16(a)) $\rightarrow$ literal mathematical expression (E.7).

For a given graph $A_1$ (figure 16(a)) we can obtain its expression as follows:

(1) Write in arrows (their directions are arbitrary) at all non-arrowed suffix lines as in figure E3. Write in suffixes (their symbols are arbitrary) at all suffix lines as in figure E3.
(2) By 'cutting' the graph, figure E3, into pieces in such a way that each piece is the simple
form of (E.6) (figure E4).

(3) Then replace each piece of the graph by a literal mathematical expression following
(E.6). Upper and lower suffixes are approximately adjusted. Then we obtain

\[ [R_{\mu\nu}^{\sigma\lambda} R_{\sigma\omega}^{\tau\omega} R_{\lambda\mu}^{\nu\mu}] \]  

(E.7)

which is equivalent to (E.5).

The inverseness above can be confirmed by noting the fact \( f^{-1}(c) = c \) for all \( c \in C \)
or \( ff^{-1}(g) = g \) for all \( g \in G \). We see the present map (representation is one-to-one.

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