Translating Labels to Hypersequents for Intermediate Logics with Geometric Kripke Semantics

Robert Rothenberg
School of Computer Science
University of St Andrews
St Andrews, Fife KY16 9SX
Scotland, UK
rr@cs.st-andrews.ac.uk

Abstract. We give a procedure for translating geometric Kripke frame axioms into structural hypersequent rules for the corresponding intermediate logics in $\text{Int}^\ast/\text{Geo}$ that admit weakening, contraction and in some cases, cut. We give a procedure for translating labelled sequents in the corresponding logic to hypersequents that share the same linear models (which correspond to Gödel-Dummett logic). We prove that labelled proofs $\text{Int}^\ast/\text{Geo}$ can be translated into hypersequent proofs that may use the linearity rule, which corresponds to the well-known communication rule for Gödel-Dummett logic.

1 Introduction

The syntactic elements of Gentzen-style sequent calculi can be extended so as give cut-free calculi for various non-classical logics. We show how to translate proofs in one common extension, labelled sequent calculi, into another another common extension, hypersequent calculi, for a subset of intermediate logics. Labelled sequent calculi, apparently introduced in [?], contain formulae which are annotated with labels, and often the sequents themselves are annotated with terms that indicate the relationships between labels. Hypersequents, which are generally attributed to [?] (though they have occurred earlier, e.g. [?] and [?]), are lists or multisets of sequents.

Developing a formal translation between proof systems is a topic of interest. The obvious reasons for doing so are to allow one to separate interface from implementation in automated proof assistants (especially where one formalism is more conducive to automation), and to translate proofs of meta properties such as interpolation into alternative formalisms. A less obvious reason for developing translations are to gain a better understanding of the meaning of particular syntactic features that proof systems extend with respect to sequent calculi. This is useful for developing new notations which can combine multiple syntactic features. Such a notation can be used to conceive of new extensions to sequent calculi, or develop of formal hierarchy of the relative strength of proof systems.

Labelled calculi can be seen as an alternative notation for other formalisms, where the locations of formulae in a structure are encoded as labels, and relational formulae to indicate how these locations relate to one another. Labelled calculi also incorporate
information about data structure in the object language of the calculus. For example, labels can be associated with the components of the hypersequent-like structure,

$$\begin{array}{c}
A, \ldots \mid x \\
B, \ldots \mid y \\
C, \ldots \mid z
\end{array}$$

that can be translated into a kind of labelled sequent: $A^{x \rightarrow y \rightarrow z}, B^{y \rightarrow z}, C^{z \rightarrow z}$. Relations can be added to encode relationships (such as subset relations) between components: $x \leq y, y \leq z; A^{x \rightarrow y \rightarrow z}, B^{y \rightarrow z}, C^{z \rightarrow z}$. Labels and relations can be used to reduce the complexity of data structures, e.g. the above structure may be easier to search for formulae in than the original structure. (However, it may be easier to reason about than the original data structure proof theoretically, or it may turn out that the original structure is more conducive to searching for formulae by parallel algorithms.)

1.1 Related Work

The relationship between labelled sequents and hypersequents has been a folkloric one in proof theory, with no published formal comparisons that we are aware of, beyond specific calculi. Most of the work has been for systems based on the modal logic $S5$ [?], $\Box$ and $\Diamond$ (the latter work also connects systems for the logic $N3$). Work connecting specific hypersequent and labelled calculi for $\mathbf{A}$ and $\mathbf{L}$ is given in [?].

General work on deriving a relational semantics, which can be used as the basis for labelled calculi, from Hilbert- or Gentzen-style calculi (which presumably can be extended to hypersequents) is given in [?].

Work on translating some Kripke frame axioms (that we call “geometric Kripke frames”) into structural rules that admit weakening, contraction and cut for a $G3$-style labelled calculus is discussed in [?]. That work was extended to general work on translating between hypersequents and labelled sequents for logics in $\mathbf{Int}/\mathbf{Geo}$ in [?], and is used as a basis for parts of this paper—in particular, a method was given for translating labelled sequent proofs for logics in $\mathbf{Int}/\mathbf{Geo}$ into simply labelled proofs for a corresponding calculus augmented with a form of the communication rule from [?].

2 Preliminaries

We give a brief overview of the class of logics, $\mathbf{Int}/\mathbf{Geo}$, along with labelled sequents, hypersequents, and simply labelled sequents, which will be used to give calculi for logics in that class.

2.1 Intermediate Logics with Geometric Kripke Semantics

Intermediate Logics ($\mathbf{Int}$) are (propositional) logics between Intuitionistic ($\mathbf{Int}$) and Classical ($\mathbf{Class}$) Logics that are obtained by adding additional axioms to $\mathbf{Int}$. Below, we give a semantic characterisation of a subclass of them, $\mathbf{Int}/\mathbf{Geo}$, that we call Intermediate Logics with Geometric Kripke Semantics.
**Definition 1.** An Intuitionistic Kripke Frame is a structure $\langle W, R \rangle$ where $W$ is a set of atomic points, $R$ is pre-ordered binary relation on $W$. An Intuitionistic Kripke Model $M$ is an Intuitionistic Kripke Frame extended with $D$, a function from points to sets of atomic formulae, and is monotonic w.r.t. $R$—i.e., for all $x, y \in W$, if $R_{xy}$, then $D(x) \subseteq D(y)$. A forcing relation $M, x \models A$ for propositional formulae is defined as follows:

1. $M, x \models A$ iff $A \in D(x)$ for all $x \in W$;
2. $M, x \not\models \bot$, i.e. $\bot \notin D(x)$ for all $x \in W$;
3. $M, x \models A \land B$ iff $M, x \models A$ and $M, x \models B$;
4. $M, x \models A \lor B$ iff either $M, x \models A$ or $M, x \models B$;
5. $M, x \models A \supset B$ iff for all $y$ such that $R_{xy}$, $M, y \models A$ implies $M, y \models B$.

If $M, x \models A$ for all $x \in W$, then we write simply that $M \models A$.

Models for many logics in $\text{Int}^*$ can be obtained by extending the frames of an Intuitionistic Kripke Model $M = \langle W, R, D \rangle$ with additional axioms on $R$. For many well-known logics, such as those in Table 1, the frame axioms are geometric implications—that is, they are of the form $\forall \bar{x}.(A \supset B)$, where $A$ and $B$ do not contain implications (other than $\top$) or universal quantifiers as subformulae. The logics that correspond to such models are said to be in the class $\text{Int}^*/\text{Geo}$. These logics are of interest because the structural rules which correspond to the characteristic frame axioms can be added to $\text{G3}$-style labelled sequent calculi without affecting the admissibility of the standard structural rules, as will be discussed below.

**Remark 1.** Reflexivity and transitivity axioms are geometric implications.

**Table 1** Some well-known logics with their characteristic axioms and frame axioms.

| Logic                  | Axiom                  | Frame Axiom               |
|-----------------------|------------------------|---------------------------|
| Jankov-De Morgan      | $\neg A \lor \neg \neg A$ | $\forall xy \in W . \exists z \in W . R_{xz} \lor R_{zx}$ |
| Gödel-Dummet (GD)     | $(A \supset B) \lor (B \supset A)$ | $\forall xy \in W . R_{xy} \lor R_{yx}$ |
| Bounded-Depth of 2    | $B \lor (B \supset (A \lor \neg A))$ | $\forall xyz \in W . R_{xy} \land R_{yz} \supset R_{yx} \lor R_{zy}$ |
| Classical (Class)     | $A \lor \neg A$        | $\forall xy \in W . R_{xy} \supset R_{yx}$ |

**Proposition 1 (Pointed Models [? §7.2]).** Let $M = \langle W, R, D \rangle$ and $M' = \langle W', R', D' \rangle$ be Kripke models for a logic in $\text{Int}^*$, such that $M \models A$ iff $M' \models A$. Then if $M'$ is pointed, i.e. $\exists x \in W'$ (a distinguished point) s.t. $\forall y \in W'$, $R_{xy}'$, then $R'$ is a partial order, i.e. it is also anti-symmetric.

**Remark 2.** In [?], the frame axiom for Jankov-De Morgan logic (Table 1) is given for pointed models, i.e. $\forall xy \in W. R_{wx} \land R_{wy} \supset \exists z \in W. R_{xz} \lor R_{zx}$. However, both versions are interderivable by Proposition 1.
2.2 Labelled Sequent Calculi

Labelled sequents are an extension of Gentzen-style sequents, where the logical formulae are annotated with (atomic) labels, e.g. \((A \lor B)^{\cdot}\). It is common in contemporary systems, such as [7], that the sequents are also annotated with a collection of (binary) relations, called relational formulae, between labels, e.g. \(x \leq y\). (Such systems are generally used to reason about a logic’s corresponding relational models.) Further kinds of labelled calculi are discussed in [7,7].

We denote labelled sequents as \(\Sigma; \Gamma \Rightarrow A\), where \(\Sigma\) is an arbitrary multiset of relational formulae, and underlined multiset variables are multisets of formulae with arbitrary labels. (Labelled multiset variables, e.g. \(\Gamma^{\cdot}\), denote multisets of formulae with the same label.) The semantics for labelled sequents is given in Definition 2. The vocabulary describing sequents from [7] is extended naturally for labelled sequents. A calculus \(\text{G3I}\) [7,7] for \(\text{Int}\) is given in Fig. 1.

**Definition 2 (Semantics of Labelled Sequents).** Let \(\mathcal{M} = (W, R, D)\) be a Kripke model for a logic in \(\text{Int}^*/\text{Geo}\). Then \(\mathcal{M} \models \Sigma; \Gamma \Rightarrow A\) iff for each \(w \in \text{Lab}(\Sigma, \Gamma, A)\), there exists a (not necessarily unique) \(\hat{w} \in W\), such that the consistency of \(\Sigma\) with \(R\)—i.e., for all \(x \leq y \in \Sigma\), \(R \hat{x}\hat{y}\)—implies either \(\mathcal{M} \not\models \hat{\Sigma}\) or \(\mathcal{M} \models \hat{\Sigma} \supset \hat{A}\), where \(\mathcal{M} \not\models A\) iff \(\mathcal{M}, \hat{x} \not\models A\).

\[
\begin{array}{cccc}
\Sigma; \Gamma \Rightarrow A \quad & \Sigma; \Gamma \Rightarrow \Gamma & \Sigma; \Gamma \Rightarrow \Gamma & \Sigma; \Gamma \Rightarrow \Gamma \\
\Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot} & \Sigma; \Gamma \Rightarrow (A \land B)^{\cdot} & \Sigma; \Gamma \Rightarrow A^{\cdot} & \Sigma; \Gamma \Rightarrow B^{\cdot} \\
\Sigma; \Gamma \Rightarrow (A \land B)^{\cdot} & \Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot} & \Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot} & \Sigma; \Gamma \Rightarrow (A \land B)^{\cdot} \\
\Sigma; \Gamma \Rightarrow A^{\cdot} & \Sigma; \Gamma \Rightarrow B^{\cdot} & \Sigma; \Gamma \Rightarrow A^{\cdot} & \Sigma; \Gamma \Rightarrow B^{\cdot} \\
\Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot} & \Sigma; \Gamma \Rightarrow (A \land B)^{\cdot} & \Sigma; \Gamma \Rightarrow (A \land B)^{\cdot} & \Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot} \\
\Sigma; \Gamma \Rightarrow A^{\cdot} & \Sigma; \Gamma \Rightarrow B^{\cdot} & \Sigma; \Gamma \Rightarrow A^{\cdot} & \Sigma; \Gamma \Rightarrow B^{\cdot} \\
\Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot} & \Sigma; \Gamma \Rightarrow (A \land B)^{\cdot} & \Sigma; \Gamma \Rightarrow (A \land B)^{\cdot} & \Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot} \\
\Sigma; \Gamma \Rightarrow A^{\cdot} & \Sigma; \Gamma \Rightarrow B^{\cdot} & \Sigma; \Gamma \Rightarrow A^{\cdot} & \Sigma; \Gamma \Rightarrow B^{\cdot} \\
\Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot} & \Sigma; \Gamma \Rightarrow (A \land B)^{\cdot} & \Sigma; \Gamma \Rightarrow (A \land B)^{\cdot} & \Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot} \\
\Sigma; \Gamma \Rightarrow A^{\cdot} & \Sigma; \Gamma \Rightarrow B^{\cdot} & \Sigma; \Gamma \Rightarrow A^{\cdot} & \Sigma; \Gamma \Rightarrow B^{\cdot} \\
\Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot} & \Sigma; \Gamma \Rightarrow (A \land B)^{\cdot} & \Sigma; \Gamma \Rightarrow (A \land B)^{\cdot} & \Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot} \\
\end{array}
\]

| \(x \leq y, \Sigma; P^{\cdot}, \Gamma \Rightarrow P^{\cdot}, A\) | \(\Sigma; \Gamma \Rightarrow A^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow B^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow (A \land B)^{\cdot}\) |
| \(\Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow A^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow B^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot}\) |
| \(\Sigma; \Gamma \Rightarrow A^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow B^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow (A \land B)^{\cdot}\) |
| \(\Sigma; \Gamma \Rightarrow (A \land B)^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow (A \land B)^{\cdot}\) |
| \(\Sigma; \Gamma \Rightarrow A^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow B^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow (A \land B)^{\cdot}\) |
| \(\Sigma; \Gamma \Rightarrow (A \land B)^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow (A \land B)^{\cdot}\) |
| \(\Sigma; \Gamma \Rightarrow A^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow B^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow (A \land B)^{\cdot}\) |
| \(\Sigma; \Gamma \Rightarrow (A \land B)^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow (A \land B)^{\cdot}\) |
| \(\Sigma; \Gamma \Rightarrow A^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow B^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow (A \lor B)^{\cdot}\) | \(\Sigma; \Gamma \Rightarrow (A \land B)^{\cdot}\) |

Fig. 1: The labelled calculus \(\text{G3I}\). \(P\) is atomic, and \(y\) is fresh for \(R \supseteq\).

**Proposition 2 ([7,7]).** The weakening, contraction and cut rules
are admissible in \( \text{G3I}^* \).

**Proposition 3.** The rules

\[
\frac{x \leq y, \Sigma; \Gamma \Rightarrow A}{x \leq y, \Sigma; \Gamma \Rightarrow A} \quad \text{LW}_x \\
\frac{x \leq y, \Sigma; \Gamma \Rightarrow A}{x \leq y, \Sigma; \Gamma \Rightarrow A} \quad \text{LW} \\
\frac{x \leq y, \Sigma; \Gamma \Rightarrow A}{x \leq y, \Sigma; \Gamma \Rightarrow A} \quad \text{RW}
\]

are admissible in \( \text{G3I} \).

**Proof.** Using cut.

**Remark 3.** The \( \text{L} \leq \) and \( \text{R} \leq \) rules are primitive in the system \( \text{L} \) for \( \text{BiInt} [?] \).

In \([?]\), it was shown that any set of geometric implications is constructively equivalent to a set consisting of formulae of the form \( \forall \bar{x}.(A_0 \supset \exists \bar{y}.(A_1 \lor \ldots \lor A_n)) \), where each \( A_i \) is a conjunction of atomic formulae, such as relational formulae. Formulae in that form, such as the frame axioms from Table 1, can be translated into rules of the form:

\[
\frac{\overline{A_1}, \overline{A_0}, \Sigma; \Gamma \Rightarrow A}{\overline{A_0}, \Sigma; \Gamma \Rightarrow A} \\
\frac{\overline{A_1}, \overline{A_0}, \Sigma; \Gamma \Rightarrow A}{\overline{A_0}, \Sigma; \Gamma \Rightarrow A} \\
\frac{\overline{A_1}, \overline{A_0}, \Sigma; \Gamma \Rightarrow A}{\overline{A_0}, \Sigma; \Gamma \Rightarrow A} \\
\frac{\overline{A_1}, \overline{A_0}, \Sigma; \Gamma \Rightarrow A}{\overline{A_0}, \Sigma; \Gamma \Rightarrow A}
\]

where (in an abuse of notation) \( \overline{A}_i \) is the multiset of relational formulae in \( A_i \), and the variables correspond to labels. A translation method is given in Definition 3:

**Definition 3.** Given a geometric implication of the form \( \forall \bar{x}.(A_0 \supset \exists \bar{y}.(A_1 \lor \ldots \lor A_n)) \), the corresponding geometric rule can be obtained by straightforward analysis of the sequent \( \forall \bar{x}.(A_0 \supset \exists \bar{y}.(A_1 \lor \ldots \lor A_n)), \overline{A_0}, \Sigma; \Gamma \Rightarrow A \) in a \( \text{G3} \)-style sequent calculus for \( \text{Int} \), such as \( \text{G3I} [?] \).

In \([?]\) it was shown that geometric rules can be added to \( \text{G3} \)-style calculi without losing the admissibility of weakening, contraction and cut. This allowed the development of labelled sequent frameworks for various non-classical logics in \([?]\). The corresponding rules to frame axioms from Table 1 are in Fig. 2. We denote \( \text{G3I} \) augmented with arbitrary geometric rules such as those from Table 1 as \( \text{G3I}^* \).

### 2.3 Hypersequent Calculi

A hypersequent is a non-empty multiset of sequents, called its *components*, and is written as \( \Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_n \Rightarrow A_n \). The semantics are given below:
Definition 4 (Semantics of Hypersequents). Let $\mathcal{M}$ be a model for a logic in $\text{Int}^*/\text{Geo}$. Then $\mathcal{M} \models \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ if there exists $1 \leq i \leq n$ such that $\mathcal{M} \models \Gamma_i \Rightarrow \Delta_i$, i.e. either $\mathcal{M} \not\models \land \Gamma_i$, or $\mathcal{M} \models \lor \Delta_i$. ($\Gamma_i \Rightarrow \Delta_i$ is called the distinguished component.)

The vocabulary describing sequents from $\mathcal{M}$ is extended naturally for hypersequents. Rules of hypersequent calculi can be classified as either *internal rules* (rules which have only one active component in each premiss and one principal component in the conclusion), and *external rules*, which are rules that are not internal rules. The *standard external rules* are $\text{EW}$ and $\text{EC}$. (For brevity, multiple instances of $\text{LW}$, $\text{RW}$ or $\text{RW}$ will be combined in proofs as $\text{W}$, and multiple instances of $\text{LC}$, $\text{RC}$ or $\text{EC}$ will be combined as $\text{C}$.)

The hypersequent calculus $\text{HG3ipm}$ $\mathcal{M}$ for $\text{Int}$ given in Fig. 3 was obtained from a multisuccedent variant of $\text{G3ip} \mathcal{M}$ $\mathcal{M}$ by adding side components to the rules and the standard external rules to the calculus.

---

**Fig. 2:** Extension rules to $\text{G3I}$ some well-known logics. ($z$ is fresh for $\text{dir}$.)

| Rule | Description |
|------|-------------|
| $x \leq z, y \leq z, \Sigma; \Gamma \Rightarrow A$ | $\Sigma; \Gamma \Rightarrow A$ (dir) |
| $x \leq y, \Sigma; \Gamma \Rightarrow A$ | $y \leq x, \Sigma; \Gamma \Rightarrow A$ (lin) |
| $y \leq x, x \leq y \leq z, \Sigma; \Gamma \Rightarrow A$ | $z \leq y, x \leq y \leq z, \Sigma; \Gamma \Rightarrow A$ (bd) |
| $x \leq y \leq z, \Sigma; \Gamma \Rightarrow A$ | $x \leq y, \Sigma; \Gamma \Rightarrow A$ (sym) |

**Fig. 3:** The hypersequent calculus $\text{HG3ipm}$. $P$ is atomic.
Proposition 4 ([?]). The standard internal weakening and contraction rules

\[
\begin{align*}
\frac{\mathcal{H} \mid \Gamma \Rightarrow A}{\mathcal{H} \mid A, \Gamma \Rightarrow A} & \quad \text{LW} \\
\frac{\mathcal{H} \mid \Gamma \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow A, A} & \quad \text{RW} \\
\frac{\mathcal{H} \mid A, A, \Gamma \Rightarrow A}{\mathcal{H} \mid A \Rightarrow A, A} & \quad \text{LC} \\
\frac{\mathcal{H} \mid \Gamma \Rightarrow A, A, A}{\mathcal{H} \mid \Gamma \Rightarrow A \Rightarrow A} & \quad \text{RC}
\end{align*}
\]

are admissible in $HG_3ipm$.

We note that the many hypersequent calculi treat the components as corresponding to points in the Kripke semantics of a logic, e.g. [?,?] or [?], and use this as a motivation for translating geometric frame axioms into hypersequent rules. We do this by using monotonicity to encode relations between points as subset relations between components in the following procedure:

**Definition 5 (Translation of Geometric Axioms to Hypersequent Rules).** Structural hypersequent rules are obtained from geometric frame axioms by the following method:

1. Translate the frame axiom into a geometric rule using the procedure from Definition 3, and expand the sets of relations in the conclusion and premisses to their transitive closures.

2. Create a base schematic hypersequent by associating each principal label $x$ from the conclusion with a component containing a unique pair of multiset variables $\Gamma_x$ and $\Delta_x$, e.g. for a rule with principal labels $x, y$, the base schematic hypersequent is $\mathcal{H} \mid \Gamma_x \Rightarrow \Delta_x \mid \Gamma_y \Rightarrow \Delta_y$.

3. Take the base schematic hypersequent: for each relation $x \leq y$ in the conclusion, add $\Gamma_x$ to the antecedent of the component associated with $y$, and add $\Delta_x$ to the succedent of the component associated with $x$. (When there is a symmetric relation between components, they can be merged into a single component.) In the previous example, $x \leq y$ would be $\mathcal{H} \mid \Gamma_x \Rightarrow \Delta_x \mid \Gamma_y \Rightarrow \Delta_y$.

4. Using the result of step 3, repeat the same process for each premiss. For fresh labels, add new components, but do not add new variables.

5. Remove multiple occurrences of the same variable in the antecedent or succedent, as well as duplicate schematic components, e.g. $\Gamma_x, \Gamma_x, \Gamma_y \Rightarrow \Gamma_y$ can be changed to $\Gamma_x, \Gamma_y \Rightarrow \Gamma_y$.

Remark 4. We note that the treatment the components as corresponding to points in the Kripke semantics of a logic appears to be at odds with the Definition 4. However, by Proposition 1, we can assume that a model has a distinguished point which the distinguished component corresponds to. From monotonicity, that component is true in all points of the model. Hence the hypersequent is true by addition.

**Lemma 1.** The method from Definition 5 yields sound rules for the corresponding logics in $Int^* / Geo$.

**Proof.** Note that step 3 constructs components which satisfy the monotonicity property with respect to subsets of formulae in the corresponding points in a model in accordance with the frame axiom. Note Remark 4 w.r.t. connecting the relational semantics with hypersequent semantics.
Lemma 2. The method from Definition 5 yields rules which admit internal weakening and contraction in a HG3ipm-like calculus.

Proof. Note that the rules have the subformula property, i.e., each multiset variable in the premisses occurs in the conclusion. Thus instances of internal weakening can be permuted to lower derivation depths. Note also that the antecedents (and succedents) of components in the conclusions are subsets of the antecedents (and succedents) of corresponding components in each premiss, and that the rules are context sharing. Thus instances of internal contraction can be permuted to lower derivation depths.

Applying the method from Definition 5 to the axioms in Table 1 yields the rules in Fig. 4. Note that the dir, lin and sym rules are interderivable with the rules LQ, Com and S from the literature, e.g., [7]. HG3ipm* is the system HG3ipm augmented by these rules.

In [7] are given syntactic conditions for structural hypersequent rules to admit cut: linear conclusion—multiset variables must not occur more than once in the conclusion; The method in Definition 5 yields rules which meet these conditions, in cases where the original frame axiom is of the form ∀ x. (T ⊃ B), e.g., dir and lin. Otherwise, the rules do not have linear conclusions, and cut is not necessarily admissible. (This limitation can be overcome in cases where the components in the conclusion are linearly ordered, by treating hypersequents as lists with restrictions on permutation, where the accessibility relation between components determined by their relative order, as in [7].)

In [7], a procedure is given that transforms hypersequent rules based on Hilbert-style axioms whose forms are in parts of a “substructural hierarchy” so that they meet the syntactic conditions of cut-admissibility, possibly with additional premisses. That procedure is not applicable to all rules, e.g., bd2, which has a characteristic axiom which is in a different part of the hierarchy. (A discussion of that procedure is beyond the scope of this paper, however.) Note also that the conclusion of bd2 is not linear.

In Theorem 1 below, we show how to translate cut-free labelled proofs into cut-free simply labelled (an alternative notation for hypersequents) proofs using these rules along with the lin rule. Hence, the following conjecture:

Conjecture 1. The method in Definition 5 yields rules which admit the cut rule in a HG3ipm-like calculus augmented with the lin rule (see Fig. 4).

Remark 5. Parallel variants of the L∨ and R∧ rules are admissible using the lin rule (e.g., see Proposition 8 below), and can allow cut to be permuted above rules with non-linear conclusions.

2.4 Simply Labelled Calculi

Simply labelled calculi such as [7] or [8] are (syntactically) labelled calculi without relational formulae, but with a similar semantics to hypersequents (see Definition 6). They can be treated as an alternative notation for hypersequents, where formulae are annotated with a name for the component that they occur in. Translation between the two formalisms is straightforward, and will be omitted for brevity. The only issues with
translation are in regards to a notion similar to alpha-equivalence on labels (which is addressed in Definition 7), and hypersequents with an empty component, i.e. hypersequents of the form $H|\Rightarrow$. Since the empty component is never true in any interpretation, the latter issue can be safely ignored for logics in $\text{Int}^*/\text{Geo}$.

A simply labelled calculus $\text{LG3ipm}^*$ [?] is given in Fig. 5 as a translation from $\text{HG3ipm}^*$. 

**Definition 6.** Let $\Gamma/x =_{\text{def}} \{ A^x | A^x \in \Gamma \}$. Let $M = (W, R, D)$ be a Kripke model for a logic in $\text{Int}^*/\text{Geo}$. Then $M \models \Gamma \Rightarrow \Delta$ iff there exists a label $x \in \text{Lab}(\Gamma, \Delta)$ such that $M \models \Gamma/x \Rightarrow \Delta/x$, i.e. either $M \not\models \Gamma/x$ or $M \models \Delta/x$.

**Definition 7 (Subset Modulo Permutation).** Let $\Gamma \approx \Delta$ mean that two multisets of labelled formulae are identical, modulo permutation of labels. Then $\Gamma \subset\sim \Delta$ iff there exists $\Gamma'$ such that $\Gamma' \approx \Gamma$ and $\Gamma' \subseteq \Delta$. This notion is extended naturally for sequents.

**Proposition 5 (Label Substitution).** Let $\Gamma \Rightarrow \Delta$ be a simply labelled sequent, and $x, y$ be labels. If $\text{LG3ipm}^* \vdash \Gamma \Rightarrow \Delta$, then $\text{LG3ipm}^* \vdash [y/x]\Gamma \Rightarrow [y/x]\Delta$.

**Proof.** Straightforward.

**Proposition 6.** Weakening and contraction are admissible in $\text{LG3ipm}^*$.

**Proof.** Straightforward.

## 3 Translation of Labelled Proofs to Simply Labelled Proofs

Labelled sequents are more expressive than hypersequents. It is not obvious what hypersequent that an arbitrary labelled sequent with relational formulae, e.g. $x \leq y; A^x \Rightarrow A^y$, corresponds to. Here we use the same idea for translating frame axioms into hypersequent rules, and use monotonicity to encode relational formulae as subset relations between the components. A translation from labelled sequents to simply labelled sequents is given below. (The translation from simply labelled sequents to hypersequents is straightforward, and is omitted for brevity.)

| $H[\Gamma_1 \Rightarrow A_1 | \Gamma_2 \Rightarrow A_2]; \Gamma_1, \Gamma_2 \Rightarrow \Delta_2$ | $H[\Gamma_1 \Rightarrow A_1 | \Gamma_2 \Rightarrow A_2]$ | $\text{dir}$ |
| $H[\Gamma_1 \Rightarrow A_1 | \Gamma_2 \Rightarrow A_2]$ | $H[\Gamma_1 \Rightarrow A_1 | \Gamma_2 \Rightarrow A_1, A_2]$ | $\text{lin}$ |
| $H[\Gamma_1 \Rightarrow A_1; \Gamma_2, \Gamma_3 \Rightarrow A_2]$ | $H[\Gamma_1 \Rightarrow A_1, A_2 | \Gamma_2, \Gamma_3 \Rightarrow A_2, A_3]$ | $\text{bd}_2$ |
| $H[\Gamma_1 \Rightarrow A_1, A_2 | \Gamma_2, \Gamma_3 \Rightarrow A_2, A_3]$ | $H[\Gamma_1 \Rightarrow A_1, A_2 | \Gamma_2, \Gamma_3 \Rightarrow A_2, A_3]$ | $\text{sym}$ |

**Fig. 4: Hypersequent rules of well-known logics, obtained from the rules in Fig. 2.**
Fig. 5: The calculus LG3ipm*. $P$ is atomic, $x \# A'$ in $R \supset$ and $x, y \# \Gamma', A'$ in the structural rules.
Definition 8 (Transitive Unfolding). Let $\Sigma^+$ be the transitive closure of $\Sigma$, so that

$$\overline{\text{lab}}_x(\Sigma^+) = \{ y \mid x \leq y \in \Sigma^+ \}$$

Let $\overline{\Sigma}$ be the list of labels constructed from the multiset of relational formulae $\Sigma$ [?], and let $\overline{\Sigma}$ be the reversed list from $\overline{\Sigma}$. Then we define the functions

$$\overline{\text{TU}} \overline{\Sigma} \Gamma = \begin{cases} \overline{\text{TU}} \overline{\Sigma} \Gamma \cup \cup (\text{dy}:[y/x] \Gamma \parallel x) & \text{where } \overline{\Sigma} = x :: \overline{\Sigma}' \text{ otherwise} \\ \Gamma & \text{otherwise} \end{cases}$$

$$\overline{\text{TU}} \overline{\Sigma} \Delta = \begin{cases} \overline{\text{TU}} \overline{\Sigma} \Delta \cup \cup (\text{dx}:[x/y] \Delta \parallel y) & \text{where } \overline{\Sigma} = x :: \overline{\Sigma}' \text{ otherwise} \\ \Delta & \text{otherwise} \end{cases}$$

where $x :: \overline{\Sigma}$ and $x :: \overline{\Sigma}'$ denote lists of labels, with $x$ as the head, and $\parallel$ is used as an alternative for the map function. Then $(\Sigma; \Gamma \Rightarrow \Delta)^* = \overline{\text{TU}} \overline{\Sigma} \Gamma \Rightarrow (\overline{\text{TU}} \overline{\Sigma} \Delta)$.

Note that there is no 1-1 relation between a labelled sequent and its transitive unfolding, e.g. $(x \leq y, x \leq z; A^i \Rightarrow B^i)^* = (x \leq y, y \leq z; A^i \Rightarrow B^i)^*$. Furthermore, despite encoding relations between labels as subset relations between components, there are no rules in the corresponding simply labelled sequent (or hypersequent) calculus to preserve this relation. For example, take a labelled sequent that is derivable in $\text{G3I}$ without any extension rules, $x \leq y; (A \lor B)^i, (B \supset C)^i \Rightarrow A^i, C^i$. It’s transitive unfolding, $(A \lor B)^{ii}, (B \supset C)^{iii} \Rightarrow A^i, C^{ii}$, cannot be derived in $\text{LG3ipm}$. The occurrences of $A \lor B$ in two different slices must be analysed in parallel using a rule such as Proposition 8 below, which requires linearity. This is unsurprising, as the slices (or components) correspond to chains through points in a model, rather than points in a model.

We now show that proofs in a labelled calculus based on $\text{G3I}^*$ can be translated into proofs in a simply labelled calculus based on $\text{LG3ipm}^*$ for logics in $\text{Int}^*/\text{Geo}$ augmented with the lin rule. (We use the notation $\rho_i$ to indicate a “trivially invertible” form of the rule $\rho$ with the principal formula in all premisses, e.g. $\Gamma \supset$. Note that the rules are interderivable using weakening and contraction.) The translation of proofs from the simply labelled to hypersequent calculus $\text{HG3ipm}$ is straightforward, and is omitted for brevity.

Proposition 7. The rule

$$\Gamma \Rightarrow A, \bot^i \quad \frac{\Gamma \Rightarrow A \quad \bot^i} \Gamma \Rightarrow A \quad \text{R}_\bot$$

is admissible in $\text{LG3ipm}^*$.

Proof. By induction on the derivation depth.

Proposition 8. The rule

$$A^i, A^i, \Gamma \Rightarrow A, B^i, B^i, \Gamma \Rightarrow A \quad \frac{(A \lor B)^i, (A \lor B)^i \Rightarrow A \lor B} \text{lv}^*$$

is derivable in $\text{LG3ipm}^*$ using lin.
Proof. We use \( \Gamma^1 \) as shorthand for \( \Gamma_1 \) below:

\[
\begin{align*}
A^x, A^y, \Gamma^1, \Gamma^2, \Gamma' \Rightarrow A', A_1^x, A_2^y & \quad \text{W} & B^x, B^y, \Gamma^1, \Gamma^2, \Gamma' \Rightarrow A', A_1^x, A_2^y & \quad \text{W} \\
A^x, A^y, B^y, \Gamma^1, \Gamma^2, \Gamma' \Rightarrow A', A_1^x, A_2^y & \quad \text{lin} & A^x, B^y, \Gamma^1, \Gamma^2, \Gamma' \Rightarrow A', A_1^x, A_2^y & \quad \text{lin}
\end{align*}
\]

(1)

\[
\begin{align*}
B^x, B^y, \Gamma^1, \Gamma^2, \Gamma' \Rightarrow A', A_1^x, A_2^y & \quad \text{W} & A^x, A^y, \Gamma^1, \Gamma^2, \Gamma' \Rightarrow A', A_1^x, A_2^y & \quad \text{W} \\
B^x, A^y, \Gamma^1, \Gamma^2, \Gamma' \Rightarrow A', A_1^x, A_2^y & \quad \text{lin} & A^x, B^y, \Gamma^1, \Gamma^2, \Gamma' \Rightarrow A', A_1^x, A_2^y & \quad \text{lin}
\end{align*}
\]

(2)

where \( x, y \not\in \Gamma^1, A' \) in (1) and (2).

\[
\begin{align*}
A^x, A^y, \Gamma \Rightarrow A & \quad \text{L} & A^x, B^y, \Gamma \Rightarrow A & \quad \text{L} \\
A^x, (A \vee B)^y, \Gamma \Rightarrow A & \quad \text{L} & B^x, A^y, \Gamma \Rightarrow A & \quad \text{L} \\
B^x, (A \vee B)^y, \Gamma \Rightarrow A & \quad \text{L} & (A \vee B)^y, (A \vee B)^y, \Gamma \Rightarrow A & \quad \text{L}
\end{align*}
\]

Lemma 3. The rule

\[
\frac{\Gamma \Rightarrow A, A^x, A^y}{\Gamma \Rightarrow A^x, A^y} \quad \text{R} \subseteq
\]

where \( \Gamma \parallel x \subseteq \Gamma \parallel y \), is admissible in \( \text{LG3ipm}^+ \text{lin} \).

Proof. By induction on the structure of \( A \), and the derivation depth of the premiss.

1. For the base case, \( A \) is atomic, and the premiss is an axiom (with derivation depth of 0). There are two subcases:
   (a) Suppose \( A^x \) is the principal formula, \( A^x \in \Gamma \), but by the constraints on side formulae, so is \( A^y \in \Gamma \). Therefore the conclusion of the rule is also an axiom.
   (b) Otherwise, the conclusion is also an axiom.

   Note that this case applies to generalised axioms of greater derivation depth.

2. Suppose \( A \) is atomic, but at a derivation depth greater than 0. Then there are two subcases:
   (a) If \( A = \bot \), then the conclusion is derivable by Proposition 7.
   (b) Otherwise the \( \text{R} \subseteq \) rule can be permuted to lower derivation depth.

3. Suppose the premiss is the conclusion of an instance of \( \text{L} \\bot \). There are the following subcases:
   (a) Suppose the principal label is \( x \):

   \[
   \begin{align*}
   C^x, D^x, (C \wedge D)^y, \Gamma' \Rightarrow A, A^x, A^y & \quad \text{L} \wedge \\
   (C \wedge D)^y, (C \wedge D)^y, \Gamma' \Rightarrow A, A^x, A^y & \quad \text{R} \subseteq \\
   (C \wedge D)^y, (C \wedge D)^y, \Gamma' \Rightarrow A, A^y & \quad \text{R} \subseteq
   \end{align*}
   \]

   Then the following can be derived, where \( \text{R} \subseteq \) is permuted to a lower depth:

   \[
   \begin{align*}
   C^x, D^x, (C \wedge D)^y, \Gamma' \Rightarrow A, A^y & \quad \text{L} \wedge^{-1} \\
   C^x, D^x, C^x, D^y, \Gamma' \Rightarrow A, A^y & \quad \text{R} \subseteq \\
   C^x, D^x, C^x, D^y, \Gamma' \Rightarrow A, A^y & \quad \text{L} \wedge^2
   \end{align*}
   \]
(b) For all other cases, the constraint on the antecedent is not affected, so the $R_C$ instance can be permuted upwards.

4. Suppose the premiss is derived by an instance of $R_C \land$. There are two subcases:
   (a) The principal formula of $R_C \land$ is one of the active formulae of $R_C$, such that $A = C \land D$. Without loss of generality, we assume that $A^x$ is the principal formula. The following is derivable, by permuting the $R_C$ rule to smaller formulae and a lower derivation depth:
   \[
   \begin{align*}
   \Gamma &\Rightarrow A, C^x, (C \land D)^y & R_C \land_1^1 \\
   \Gamma &\Rightarrow A, C^x, C^y & R_C \\
   \Gamma &\Rightarrow A, C^y & R_C \\
   \Gamma &\Rightarrow A, (C \land D)^y & R_C \land_1^1 \\
   \end{align*}
   \]
   (b) Otherwise the $R_C$ rule can be permuted to lower derivation depth.

5. Suppose the premiss is the conclusion of an instance of $L_V$. There are the following subcases:
   (a) Suppose the principal label is $x$:
   \[
   \begin{align*}
   C^x, (C \lor D)^y, \Gamma' &\Rightarrow A, A^x, A^y & L_V \\
   (C \lor D)^y, (C \lor D)^y, \Gamma' &\Rightarrow A, A^x, A^y & R_C \\
   \end{align*}
   \]
   Then the following can be derived, where $R_C$ is permuted to a lower depth:
   \[
   \begin{align*}
   C^x, (C \lor D)^y, \Gamma' &\Rightarrow A, A^x, A^y & L_V \lor_1^1 \\
   C^x, C^y, \Gamma' &\Rightarrow A, A^x, A^y & R_C \\
   C^x, \Gamma' &\Rightarrow A, A^y & R_C \\
   \end{align*}
   \]
   Recall that $L_V \lor_1$ requires the lin rule.
   (b) For all other cases, the constraint on the antecedent is not affected, so the $R_C$ instance can be permuted upwards.

6. Suppose the premiss is derived by an instance of $R_V$. There are two subcases:
   (a) The principal formula of $R_V$ is one of the active formulae of $R_C$, such that $A = C \lor B$. Without loss of generality, we assume that $A^x$ is the principal formula. The following is derivable, by permuting the $R_C$ rule to smaller formulae and a lower derivation depth:
   \[
   \begin{align*}
   \Gamma &\Rightarrow A, C^x, D^x, (C \lor D)^y & R_V^{-1} \\
   \Gamma &\Rightarrow A, C^x, D^y, C^y & R_C^{-1} \\
   \Gamma &\Rightarrow A, C^y, D^y & R_C^{-1} \\
   \Gamma &\Rightarrow A, (C \lor D)^y & R_V \\
   \end{align*}
   \]
   (b) Otherwise the $R_C$ rule can be permuted to lower derivation depth.

7. Suppose the premiss is the conclusion of an instance of $L_S$. There are the following subcases:
(a) Suppose the principal label is $x$:

\[
(C \supset D)^x, (C \supset D)^y, \Gamma' \Rightarrow \Delta, A^x, A^y, (C \supset D)^z, D^x, \Gamma' \Rightarrow \Delta, A^x, A^y \\
(C \supset D)^x, (C \supset D)^y, \Gamma' \Rightarrow \Delta, A^y \\
(C \supset D)^x, (C \supset D)^y, \Gamma' \Rightarrow \Delta, A^x \\
\]

Then the following can be derived, where $R_\subseteq$ is permuted to a lower depth, abbreviating $C \supset D$ as $CD$:

\[
\begin{align*}
CD^x, CD^y, \Gamma' &\Rightarrow \Delta, A^x, A^y, C^x & R_\subseteq & W \\
\end{align*}
\]

(b) For all other cases, the constraint on the antecedent is not affected, so the $R_\subseteq$ instance can be permuted upwards.

8. Suppose the premiss is derived by an instance of $R_\supset$. There are two subcases:
   (a) The principal formula of $R_\supset$ is one of the active formulae of $R_\subseteq$, such that $A = C \supset B$. Without loss of generality, we assume that $A^x$ is the principal formula.
   The following is derivable, by permuting the $R_\subseteq$ rule to smaller formulae and a lower derivation depth, abbreviating $C \supset D$ as $CD$:

\[
\begin{align*}
C^x, \Gamma &\Rightarrow \Delta', D^x, CD^x & W \\
\end{align*}
\]

(b) Otherwise the $R_\subseteq$ rule can be permuted to lower derivation depth.

The inverted form of $R_\subseteq$ is admissible using $RW$. 
Remark 6. Note that the $R \subseteq$ rule corresponds to the $R \leq$ rule in $RG3ipm$. However, the dual $L \subseteq$ rule

\[
\begin{align*}
A^x, A^y, \Gamma \Rightarrow A^\omega \quad \frac{A^x, \Gamma \Rightarrow A^\omega}{L^\omega}
\end{align*}
\]

where $A^\omega / y \subseteq A^\omega / x$ (that would correspond to $L \leq$ in $RG3ipm$) is not admissible in $LG3ipm^* + lin$. Suppose the premiss is $A^x, A^y, (A \supset B)^y \Rightarrow B^x, B^y$. The conclusion $A^x, (A \supset B)^y \Rightarrow B^x, B^y$ is not derivable.

Corollary 1. The rule

\[
\begin{align*}
\frac{\Gamma \Rightarrow A^\omega}{\Gamma \Rightarrow (A \land B)^\omega, A^\omega, \Gamma \Rightarrow (A \land B)^\omega}
\end{align*}
\]

where $\Gamma \parallel x_i \parallel \Gamma \parallel x_n$ (for $1 \leq i \leq n$), is admissible in $LG3ipm^*$.

Proof. Straightforward, using $R \subseteq$ and $RW$.

Proposition 9. The rule

\[
\begin{align*}
\frac{\Gamma \Rightarrow (A \supset B)^\omega, A}{\Gamma \Rightarrow B^\omega, (A \supset B)^\omega}
\end{align*}
\]

is admissible in $LG3ipm^*$.

Proof. By induction on the derivation depth.

Theorem 1. Let $\Sigma; \Gamma \Rightarrow A$ be a labelled sequent. If $G3I^* \vdash \Sigma; \Gamma \Rightarrow A$ then $LG3ipm^* + lin \vdash (\Sigma; \Gamma \Rightarrow A)^*$. Proof. By induction on the derivation depth.

1. Suppose $\Sigma; \Gamma \Rightarrow A$ is an axiom. Then $(\Sigma; \Gamma \Rightarrow A)^*$ is also an axiom.
2. Suppose $\Sigma; \Gamma \Rightarrow A$ is the conclusion of an instance of refl. We apply an instance of contraction to remove duplicate formulae labelled with $x$ in the antecedent and succedent.
3. Suppose $\Sigma; \Gamma \Rightarrow A$ is the conclusion of an instance of trans. We apply an instance of contraction to remove duplicate formulae labelled with $z$ (unfolded from $x$) in the antecedent, and to remove duplicate formulae labelled with $x$ (unfolded from $z$) in the succedent.
4. Suppose $\Sigma; \Gamma \Rightarrow A$ is the conclusion of an instance of $L \land$:

\[
\begin{align*}
&\Sigma; \Gamma, A^{x_1}, B^{y_1} \Rightarrow A^\omega \\
&\Sigma; \Gamma, (A \land B)^{y_1} \Rightarrow A^\omega
\end{align*}
\]

Let $(\Sigma; \Gamma, A^{x_1}, B^{y_1} \Rightarrow A)$ = $\Sigma^*, A^{x_1}, B^{y_1}, \ldots, A^{x_n}, B^{y_n} \Rightarrow A^*$, where $x_1 \leq x_i \in \Sigma^+$ (for $1 \leq i \leq n$). The corresponding proof in $LG3ipm^*$ is derived using $n$ instances of $L \land$:

\[
\begin{align*}
&\Sigma^*, A^{x_1}, B^{y_1}, \ldots, A^{x_n}, B^{y_n} \Rightarrow A^* \\
&\Sigma^*, (A \land B)^{y_1}, \ldots, (A \land B)^{y_n} \Rightarrow A^*
\end{align*}
\]
5. Suppose $\Sigma; \Gamma \Rightarrow A$ is the conclusion of an instance of $R\land$:

$$\frac{\Sigma; \Gamma \Rightarrow A^x, A}{\Sigma; \Gamma \Rightarrow (A \land B)^y, A} \quad R\land$$

where $x_2 \leq x_1, \ldots, x_n \leq x_1 \in \Sigma^*$ for $n \geq 1$. Let

$$(\Sigma; \Gamma \Rightarrow A^x, A^*) = (\Gamma^* \Rightarrow A^x, \Delta^*)$$

where $\Gamma^* \parallel x_i \parallel \Gamma$ and $\Delta^* \parallel x_i \parallel \Delta$ for $1 \leq i \leq n$. The corresponding proof in LG3ipm* is derived using the $R\land^*$ rule from Proposition 1:

$$\frac{\Gamma^* \Rightarrow A^x, \ldots, A^x, A^*, \Gamma^* \Rightarrow B^x, \ldots, B^x, A^*}{\Gamma^* \Rightarrow (A \land B)^y, \ldots, (A \land B)^y, A^*} \quad R\land^*$$

Note that $(\Sigma; \Gamma \Rightarrow (A \land B)^y, A^*) = (\Gamma^* \Rightarrow (A \land B)^y, \ldots, (A \land B)^y, A^*)$.

6. Suppose $\Sigma; \Gamma \Rightarrow A$ is the conclusion of an instance of $L\lor$. The case is the dual of case 5 above, using the $L\lor^*$ rule from Proposition 8.

7. Suppose $\Sigma; \Gamma \Rightarrow A$ is the conclusion of an instance of $R\lor$. The case is the dual of case 4 above.

8. Suppose $\Sigma; \Gamma \Rightarrow A$ is the conclusion of an instance of $L\supset$:

$$\frac{x_1 \leq y_1 \Sigma; (A \supset B)^y, \Gamma \Rightarrow A^y, A^y \quad x_1 \leq y_1 \Sigma; (A \supset B)^y, B^y, \Gamma \Rightarrow A}{x_1 \leq y_1 \Sigma; (A \supset B)^y, \Gamma \Rightarrow A} \quad L\supset$$

where $x_1 \leq y_1, \ldots, x_m \leq y_1, y_2, \ldots, y_1 \leq y_n$ for $m, n \geq 1$. Let

$$(x_1 \leq y_1 \Sigma; (A \supset B)^y, \Gamma \Rightarrow A^y, A^y) = (A \supset B), \Gamma^* \Rightarrow A^*, A^*, \ldots, A^*, A^* \quad (4)$$

$$(x_1 \leq y_1 \Sigma; (A \supset B)^y, B^y, \Gamma \Rightarrow A^y, A^y) = (A \supset B), \Gamma^* \Rightarrow A^*, A^*, \ldots, A^*, \Delta^* \quad (5)$$

where $(A \supset B) = (A \supset B)^y, (A \supset B)^y, \Delta^*$. We can derive the following from (4), for $1 \leq i \leq n$:

$$\frac{(A \supset B), \Gamma^* \Rightarrow A^*, A^*, \ldots, A^*, A^*}{R \subseteq}$$

$$\frac{(A \supset B), \Gamma^* \Rightarrow A^*, A^*}{R W}$$

$$\frac{(A \supset B), \Gamma^* \Rightarrow A^*, A^*}{R \subseteq}$$

(Clearly the last two inference steps are omitted for $i = 1$.) We first derive the following:

$$\frac{(A \supset B), \Gamma^* \Rightarrow A^*, A^*}{(A \supset B), B^y, \ldots, B^y, \Gamma \Rightarrow A^*, A^*} \quad W \quad (5)$$

$$\frac{(A \supset B), \Gamma \Rightarrow A^*}{(A \supset B), B^y, \ldots, B^y, \Gamma \Rightarrow A^*} \quad L \supset$$

8. Suppose $\Sigma; \Gamma \Rightarrow A$ is the conclusion of an instance of $R\lor$. The case is the dual of case 4 above.
10. Suppose \( \Gamma \vdash \Delta \) is the conclusion of an instance of ordering rules such as \( \text{dir} \), \( \text{lin} \) or \( \text{sym} \). The corresponding proof in \( \text{LG3ipm}^* \) is derived using the simply labelled form of that rule, with weakening and contraction as appropriate. (Recall the method for deriving the hypersequent rule from the corresponding geometric rule.)

Example 1. Take the following proof in \( \text{G3I} \) (using context-splitting rules for brevity):

\[
\begin{align*}
\frac{\frac{A^x \Rightarrow A^x}{A^x \Rightarrow A^x} \quad \frac{x \leq y; B^y, (B \supset C)^y \Rightarrow B^y \quad y \leq y; C^y \Rightarrow C^y}{C^y \Rightarrow C^y} \quad \text{refl}}{x \leq y; (A \lor B)^y, (B \supset C)^y \Rightarrow A^x, C^y} & \quad \Gamma \ni \vdash \Delta
\end{align*}
\]

From Theorem 1, we can construct a proof of \( (x \leq y; (A \lor B)^y, (B \supset C)^y \Rightarrow A^x, C^y)^* \) in \( \text{LG3ipm}^* \):

\[
\begin{align*}
\frac{A^{xy} \Rightarrow A^x \quad C \quad B^{xy}, (B \supset C)^y \Rightarrow B^{xy} \quad B^{xy}, (B \supset C)^y \Rightarrow C^{xy}}{
\frac{C^{xy} \Rightarrow C^{xy} \quad C^{xy} \Rightarrow C^{xy}}{B^{xy}, (B \supset C)^y \Rightarrow C^{xy} \quad (A \lor B)^y, (B \supset C)^y \Rightarrow A^x, C^{xy}}
\end{align*}
\]

Note that the contractions are superfluous for this example.
4 Discussion

We gave a method of translating frame axioms for logics in $\text{Int}^*/\text{Geo}$ into structural rules for hypersequent calculi that admit weakening, contraction. In some cases, such as when the frame axioms consist only of a disjunction, these rules also admit cut. The translation method makes use of monotonicity to encode relations between points in Kripke frames as subset relations between components in hypersequents.

We introduced “transitive unfolding” as a method for obtaining simply labelled sequents (an alternative notation for hypersequents) from arbitrary labelled sequents. We also gave a method (as a proof) for translating proofs in labelled sequent calculi into proofs in the corresponding simply labelled calculi, augmented by the $\text{lin}$ rule. In other words, an arbitrary labelled proof—even for a logic weaker than $\text{GD}$—can be translated into a hypersequent proof that may be for a logic based on $\text{GD}$. It also justifies the conjecture that the hypersequent rules obtained from geometric frame axioms admit cut when the linearity rule is present.

The requirement that the translation of a labelled proof to a hypersequent proof may require a stronger logic than the original labelled proof is in itself not surprising, as labelled sequents are more expressive than hypersequents. What is surprising is that the translation does not require $\text{Class}$. We note that the equivalence

$$\left( \bigwedge_{i=1}^{n} A_i \supset \bigvee_{i=1}^{n} B_i \right) \equiv \bigvee_{i=1}^{n} \left( A_i \supset B_i \right)$$

is classical (from left-to-right). Furthermore, the semantics of hypersequents, when components correspond to points in an intuitionistic model, seem to be classical, as $\exists x \in W \models x \vdash A$ implies $\exists x \models A$ holds in $\text{Class}$. This suggests that hypersequent calculi where the components correspond to points in the Kripke semantics of a non-classical logic, e.g. $[?,?]$ or $[?;?]$, is strange, and worthy of further investigation.

We note that this work can be adapted for similar labelled calculi, such as the intuitionistic fragment of $[?]$. This work also can be easily adapted to calculi for other families of logics, such as modal logics, so long as they are normal logics—that is, they have a pre-ordered and monotonic relational semantics.

We have omitted an explicit discussion on translating labelled calculi into hypersequent calculi, although we believe that the method for translating geometric rules into structural hypersequent rules can also be adapted to logical rules as well. It is an area for future investigation.

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