Several Classes of Plain Dynamic Systems Qualitative Investigation

I A Andreeva
Peter the Great St.Petersburg Polytechnic University,
Polytechnicheskaya, 29, St. Petersburg, 195251, Russia
E-mail: irandr@inbox.ru

Abstract

Dynamic systems in applications are useful as mathematical models of those processes and phenomena, where statistical events, or fluctuations, may be disregarded. Dynamic systems may be divided into the two main categories – the systems with continuous time (the flows) and systems with discrete time (the cascades). During the investigations of flows normal autonomous systems of ordinary differential equations are used. The present work is devoted to the original rigorous research of some important family of dynamic systems having reciprocal polynomial right parts, which are the forms of theirs phase variables. The whole wide family under consideration is being split into numeric subfamilies belonging to different hierarchical levels, and is subjected to the first and second Poincare transformations, or mappings. As a result, the full qualitative pattern of trajectories is constructed – using the Poincare sphere – in the Poincare disk. A series of new special investigation methods developed, useful for further investigations of similar dynamic systems’ classes.

1. Introduction.

Some natural events enforce their investigator to omit fluctuations (or the so-called “statistical phenomena”). In the mathematical modeling of such cases dynamic systems are used, and those dynamic systems are described with autonomous systems of ordinary differential equations (o.d.e.’s), which are defined in some domains and satisfying in those domains the conditions of the Cauchy theorem (of their solutions’ existence and uniqueness).

The first task of such a research appears to be to reveal curves, defined by a suitable o.d.e.’s system. A phase space of the dynamic system is to be split into trajectories reaching the goal of further research of their limit behavior, classification of equilibrium positions, and describing of its attracting and repulsive manifolds.

Each and every normal autonomous differential system of the second order with polynomial right parts allows its total qualitative investigation, according to J.H. Poincare [1]. Some kinds of such systems are successfully studied [2 - 4], i.e. systems having nonzero linear terms, systems with nonlinear homogeneous terms having odd degrees such as 3, 5, 7, revealing a center (or a focus) in theirs singular point O (0,0), etc.

In this paper a class (or a family) of dynamic systems is investigated on a real plane x, y

\[
\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y),
\]

(1)

here X(x, y), Y(x, y) are considered as reciprocal polynomials of phase variables x and y, such as X is taken as a cubic, and Y as a square form, \(X(0, I) > 0, Y(0, I) > 0\). An aim is to depict in a Poincaré
disk a totality of existing for the (1)-systems different under the topological understanding phase portraits with quasi-coefficient criteria of each one. A consecutive mappings method of Poincare, considerably supplemented with author’s new methods, especially developed for the goals of this precise research, as well as methods invented by the famous Russian specialist in the field of the Qualitative theory of ordinary differential equations and dynamic systems, St. Petersburg mathematician Professor Alexey F. Andreev (1923 – 2017), allows to solve the established task. Firstly a central mapping is used of the plane $x$, $y$ (from a center $(0, 0, 1)$ of a Poincare sphere $\Sigma$), supplemented by a line at infinity $\left( R^2_{x,y} \right)$ onto a Poincare sphere $\Sigma: x^2 + y^2 + z^2 = 1$ with diametrically opposite points identified. Secondly we use the orthogonal mapping of a lower enclosed semi sphere of a Poincare sphere $\Sigma$ to a Poincare disk $\mathbb{D}: x^2 + y^2 \leq 1$ with diametrically opposite points of its boundary $I$ identified [1].

2. Some core definitions

- $\varphi(t, p), \ p = (x, y)$ – a fixed point := a solution (a motion) of an Eq. (1) – system with initial data $(0, p)$.
- $L_p := \varphi = \varphi(t, p), t \in \mathbb{R}_{\max}$, a trajectory of a motion $\varphi(t, p)$.
- $L^s_p(\cdot) := +(-\cdot)$- a semi trajectory of a trajectory $L_p$.
- $O$-curve of a system := its semi trajectory $L^s_p (p \neq O, s \in \{+,-\})$, adjoining to a point $O$ under a condition that $\varphi(t) \to +\infty$.
- $O^s(-\cdot)$ curve of a system := its $O$-curve $L^s_p(\cdot)$.
- $O_s(-\cdot)$-curve of a system := its $O$-curve, adjoining to a point $O$ from a domain $x > 0$ $(x < 0)$.
- $TO$-curve of a system := its $O$-curve, which, being supplemented by a point $O$, touches some ray in it.

A nodal bundle of $NO$-curves of a system := an open continuous family of its $TO$-curves $L^s_p$, where $s \in \{+,-\}$ is a fixed index, $p \in \Delta$, $\Delta$ - a simple open arc, $L^s_p \cap \Delta = \{p\}$.

A saddle bundle of $SO$-curves of a system, a separatrix of the point $O$ := a fixed $TO$-curve, which isn’t included into some bundle of $NO$-curves of a system.

$H, P, E$ - $O$-sectors of a system: a hyperbolic, a parabolic, an elliptical sectors.

A topological type (a T-type) of a singular point $O$ of a system := a word $A_0$ constructed of letters $S, N$ (as well as a word $B_0$ constructed of letters $H, P, E$), describes a circular order of bundles $S, N$ of its O-curves (of its O-sectors $H, P, E$ correspondingly) when traversing the singular point $O$ in the $\langle+\rangle$-direction (that means counter clockwise), beginning from someone.

$$P(u) := X(1, u) \equiv p_0 + p_1 u + p_2 u^2 + p_3 u^3,$$

$$Q(u) := Y(1, u) \equiv a + bu + cu^2.$$  

Note 1.
Considered per each (1)-system:
1) The topological type of a singular point $O$ in its form $B_0$ can be easily understood and restored using its T-type (a topological type) in the form $A_0$, and back words (we use in our research work both forms);
2) Real roots of a polynomial $P(u)$ (or of a polynomial $Q(u)$) in fact appear to be angular coefficients of isoclines of the infinity (isoclines of a zero, correspondingly);
3) We always number the polynomials’ $P(u), Q(u)$ real roots, using the ascending order.

3. Systems characterized with existing of 3 and 2 different multipliers in right parts: (3.2) – systems.
Now we make a step to the lower (the first) hierarchical level of a broad list of subsystems. Let’s at first consider (1)-systems, for which the decompositions of \( X(x, y), Y(x, y) \) into real forms of lower degrees will contain 3 and 2 multipliers, i.e.:
\[
X(x, y) = p_2 (y - u_1 x)(y - u_2 x)(y - u_3 x), \quad Y(x, y) = c(y - q_1 x)(y - q_2 x),
\]
and \( p_2 > 0, \quad c > 0, \quad u_1 < u_2 < u_3, \quad q_1 < q_2, \quad u_i \neq q_j \) for every \( i \) and \( j \).

The investigation includes some main steps, common for all subsystems belonging to the first hierarchical level.

\[
P(u) := X(1, u) \equiv p_2(u - u_1)(u - u_2)(u - u_3), \quad Q(u) := Y(1, u) \equiv c(u - q_1)(u - q_2).
\]

\( \text{RSP} (\text{RSQ}) \) – denotes an ascending sequence of real roots of polynomial \( P(u), Q(u) \), \( \text{RSPQ} \) – means the same for the both \( P(u), Q(u) \).

A Double Change (DC)-transformation means the variables change: \((t, y) \rightarrow (-t, -y)\). This change converts the considered system to another one, where order of numbering, as well as signs of polynomial roots for \( P(u), Q(u) \), and the direction of motion along its trajectories \((t \rightarrow t)\) become reversed. Further we name a pair of different (2)-systems mutually reversed according to the DC-transformation, if after such a change they convert one into the other, and independent of the otherwise.

10 types of \( \text{RSPQ} \) appear to be for some (2) - system, because \( C_{2}^2 = \frac{2!}{2!0!} = 10 \).

As it has become clear from the investigation with the DC-transformation of (2) - systems, six of them were found to be independent in pairs. And additional four dynamic systems have some mutually inverted pair among the first six systems.

Thus it makes sense to assign the special unique number \( r \in \{1, ..., 10\} \) to every one amongst the different \( \text{RSPQ} \)’s belonging to (2) – systems, using the following principle: \( \text{RSPQ} r = \frac{7}{10} \) must appear to be independent pairwise, but \( \text{RSPQ} \)’s having numbers \( r = \frac{7}{10} \) must turn out to be mutually inverted with regard to \( \text{RSPQ} \)’s obtained numbers \( r = \frac{7}{10} \).

An \( r \)-family of Eq. (2) – systems \( := \) a set of systems of (2) – family with the \( \text{RSPQ} \) number equals to \( r \).

After this, keeping in mind the common course of action and investigating plan, let’s research sub subfamilies of (2)-systems marked by the numbers \( r = \frac{7}{10} \). For those sub subfamilies, where we meet numbers \( r = \frac{7}{10} \), we are going to receive results via the DC-transformation of sub subfamilies, \( r = \frac{7}{10} \).

The necessary items of a common research plan are enlisted below.

1) To select and list singular points existing for the systems belonging to the interesting for us at this time subfamily in a Poincare disk \( \Omega \). As the research shows, those are the finite singular point \( O(0, 0) \in \Omega \) and infinite singular points \( Q_{\pm}^{i}(u_{i}, 0) \in \Gamma \), \( i = 0, 3, u_{0} = 0 \). Per each of them we apply notions of a saddle (S) and node (N) bundles of adjacent semi trajectories, of a separatrix, and a topo dynamical type of this point (a TD – type).

2) To split the subfamily into sub subfamilies corresponding to the numbers \( s = \frac{17}{17} \). Per each sub subfamily we obtain topo dynamical types of their singular points, as well as separatrices which belong to those singular points.

3) Research separatrices’ pattern and behavior per each one among singular points belong to dynamic systems of the taken sub subfamily \( \forall s \in \{1, ..., 7\} \). Very urgent at this step of the investigation appear to be the following questions: a question of uniqueness of the continuation per each separatrix from a tiny neighborhood, surrounding a taken singular point, to all the lengths of investigated separatrix, and also a question of the details of a mutual arrangement of different
separatrices in a Poincare disk $\Omega$. Those topics were actually fully clarified for all subfamilies of dynamic systems under investigation.

4) Finally, to construct all possible phase portraits characteristic of dynamic systems belonging to a taken subfamily, adding (quasi coefficient) criteria of each special topologically different type of a phase portrait existence.

As far as it could be obtained for the (3.2) – systems, the research results look like the follows. Dynamic systems, belonging to the sub subfamily characterized by the number $r=1$, have 25 topologically different possible types of phase portraits in the Poincare disk. Per numbers 2 and 3: there appeared to be revealed 9 types of possible phase portraits per each sub subfamily. Dynamic systems from the sub subfamilies with numbers 4 and 5: 7 types of portraits exist per each sub subfamily. The number $r=6$ brings us as much as 36 topologically different portraits’ types. So, as the thorough investigation of all sub subfamilies helped us to understand, as much as 93 different types of different in the topological meaning types of possible phase portraits do exist in the Poincare disk for dynamic systems of (3.2) – type. This amount is not large in fact, since every taken subfamily contains the uncountable number of systems.

4. Two categories of dynamic systems with combinations of two multipliers in their right parts. The (2.2) – systems belonging to the A-class.

Let’s now consider the (2.2) – systems’ subfamily of (1) – systems. Now their decompositions of forms $X(x, y), Y(x, y)$ into the lowest degrees real multipliers will contain two multipliers per each right part:

\[ x(x, y) = p(y - u_1 x)^k_1 (y - u_2 x)^k_2, \quad y(x, y) = q(y - q_1 x)(y - q_2 x). \]  

(3)

here $p, q, u_1, u_2, q_1, q_2 \in \mathbb{R}$, $p > 0$, $q > 0$, $u_1 < u_2$, $q_1 < q_2$, $u_i \neq q_j$ for each $i, j \in \{1, 2\}$, $k_1, k_2 \in \mathbb{N}$, $k_1 + k_2 = \delta$.

We are going to split their whole totality into the following two categories (or classes) of (3) - systems. The A class describes dynamic systems having $k_1 = 1, k_2 = 2$, while the same time the B class describes dynamic systems, for which $k_1 = 2, k_2 = 1$.

So, the A class of the (3) – subfamily looks like:

\[ \frac{dx}{dt} = p(y - u_1 x)(y - u_2 x)^2, \quad \frac{dy}{dt} = q(y - q_1 x)(y - q_2 x) \]

\[ P(u) := X(1, u) \equiv p(u - u_1)(u - u_2)^2, \quad Q(u) := Y(1, u) \equiv q(u - q_1)(u - q_2). \]  

(4)

For them will appear 6 different sequences for their $RSPQ$:

\[ C_4^1 = \frac{4!}{2!2!} = 6. \]  

It is natural to give them the numbers from 1 to 6.

And $r$-family of Eq. (4) – systems we now will name a set of (4) – systems characterized by the $RSPQ$ number $r$ taken from the six possible their variants.

The necessary investigation steps per each taken subfamily of dynamic systems from the (4) – systems family we enlist here.

1) For their singular points we introduce the notions of S (saddle) and N (node) bundles of semi trajectories, appear to be adjacent to a given studying point; and additionally the notion of the separatix together with the notion of the topo dynamical type (TD-type) of the taken singular point, as it was done during the previous investigation of (3.2)-systems subfamily.
2) Divide the taken subfamily under consideration into sub subfamilies with the corresponding numbers \( s \in \{1, \ldots, 5\} \). Determine the topo dynamical (TD)-types of all their singular points, together with the separatrices of those singular points under investigation \( \mathcal{S}_2 = \mathbb{S}^5 \).

3) For each one among the five obtained sub subfamilies study the behavior of separatrices of their revealed singularities with the goal to answer the two important questions: firstly, a question about the uniqueness or not of those separatrices’ global continuation from the little neighborhood of a taken singular point to all the lengths inside the Poincare disk \( \Omega \), and secondly, a question about different separatrices’ arrangement in the Poincare disk \( \Omega \) too.

In the Poincare disk the whole pattern of a mutual arrangement of separatrices becomes invariant in the case if for a given number \( s \) the revealed global continuation of each separate separatrix, belonging to every singular point of the dynamic system from the sub subfamily with the same number \( s \), appears to be unique. Thus, all dynamic systems belong to the chosen sub subfamily, having the number \( s \), show in a Poincare disk one and the same common type of a phase portrait.

Oppositely, in the case if for a taken number \( s \), dynamic systems belong to the corresponding sub subfamily have, for instance, \( m \) separatrices, for which their global continuations aren’t unique, such a sub subfamily has to be split into \( m \) additional ”sub sub subfamilies”, situated already at the next hierarchical level.

According to the further conducted research, for all ones among those sub sub subfamilies, and for each separatrix of their singular points the global continuation appears to be unique, and as a result the mutual arrangement of all their separatrices is invariant inside the Poincare disk.

Consequently, the topological type of the phase portrait of all dynamic systems from that sub sub subfamily in the Poincare disk \( \Omega \) appears to be common for the whole taken into consideration sub sub subfamily.

4) Finally it becomes possible to construct strictly and describe all possible, different in the topological understanding, phase portraits in the Poincare disk \( \Omega \) for the dynamic systems belonging to Eq. (4) – families, in the two possible forms (that means in the form of some table and in the graphic form as well), and outline for every appearing phase portrait quasi-coefficient criteria of its actual appearance.

Dynamic (2.2)-systems belonging to the A special class show 45 types of phase portraits, topologically different, in a Poincare disk.

5. (2.2)- systems, the B-class.

\[
\frac{dx}{ds} = p(y - u_1 x)^2(y - u_2 x), \quad \frac{dy}{ds} = q(y - q_1 x)(y - q_2 x),
\]

\( P(u) := X(1, u) \equiv p(u - u_1)^2(u - u_2), \quad Q(u) := Y(1, u) \equiv q(u - q_1)(u - q_2) \).

RSPQ may take here six variants, since \( C_2^2 = 6 \).

All the (5) – subfamily of dynamic systems during its rigorous study (following the common way of investigation, the steps of which were enlisted upper), was subdivided to 52 different sub subfamilies of the lower hierarchical level, and all the dynamic systems belonging to every taken sub subfamily have in a Poincare disk \( \Omega \) one possible and common phase portrait for every given subfamily. There were obtained and outlined 52 of them for this (2.2)-systems, class B.

6. Systems, including in their right parts 3 and 1 different multipliers correspondingly: (3.1) – systems.

\[
\frac{dx}{ds} = p_2(y - u_1 x)(y - u_2 x)(y - u_3 x), \quad \frac{dy}{ds} = c(y - q_1 x)^2,
\]
\( p_2 > 0, \; c > 0, \; u_1 < u_2 < u_3, \; q \in \mathbb{R}, \; q \neq u_i, \; i = 1, 3. \)

We now are splitting the \((6)\) – subfamily, or the subfamily of \((3.1)\)-systems, introducing sub subfamilies with numbers \( r = \frac{1}{4} \).

They appear to be the sets of dynamic systems, characterized with some \( RSPQ \) number \( r \), where \( r \) be theirs numbers in the following list of the admissible for them \( RSPQs \).

1. \( u_1, u_2, u_3, q \).
2. \( u_1, u_2, q, u_3 \).
3. \( u_1, q, u_2, u_3 \).
4. \( q, u_1, u_2, u_3 \).

Using the double change of variables (DC): \((t, y) \rightarrow (-t, y)\), we can conclude as a result: this change translates subfamilies of such \((3.1)\)-systems, corresponding to numbers \( r = 1, 2, 3, 4 \), into the subfamilies having the numbers \( r = 4, 3, 2, 1 \) (correspondingly - and also backwards). Thus, the subfamilies of \((6)\) – systems, corresponding to the numbers 1 and 2, actually appear not to be connected one to another via the DC-change, and subfamilies, which are corresponding to the numbers 3 and 4 - are not connected also; but the subfamily with the number 3 appears to be mutually inversed via the DC-change to the subfamily 2, as well as the subfamily 4 shows its mutual inversion to the subfamily number 1, correspondingly. This idea is obvious from a look to their \( RSPQ \) sequences [6 - 9].

1) Now we research the subfamilies of \((3.1)\)-systems, \( r = 1, 2 \), taking into consideration our common plan for all \((1)\) – systems’ investigation [5], and:

1. Take \( r \in \{1, 2\} \), then we divide the given subfamily into sub subfamilies [5 - 10], \( s = \frac{1}{9} \), and reveal the TD-types of their singularities.
2. Find out an \( \alpha(\omega) \) – limit set of every \( \alpha(\omega) \) – separatrix, describe for all separatrices their mutual arrangement in the Poincare disk.
3. Construct possible topologically different phase portraits \((6)\) –subsystems.

2) Consistently study sub subfamilies of \((6)\) –systems, numbered with \( r = 3, 4 \), with the useful help of the DC-change for results revealed for subfamilies, numbered by \( r = 2, 1 \). As a result we construct all existing phase portraits for all subfamilies, including 3 and 4.

Finally, it became clear, that for sub subfamilies of the \((6)\) – systems with numbers 1, 2, 3 and 4 there exist

\[ 15 + 11 + 11 + 15 = 52 \]

types of phase portraits in \( \Omega \).

7. Conclusions

The present paper describes the original investigation.

Its core goal is to strictly find out and describe all topologically different phase portraits, possible in a Poincare disk, for the broad family of the differential dynamic systems, containing numerous subfamilies of different hierarchical levels and orders. Those phase portraits were constructed during the fundamental theoretical research in the two ways – as tables and in a graphical form also [24 – 26]. Each table contains several \((5 - 6)\) lines. Each its line describes some invariant cell of a phase portrait (i.e. its boundary, a source and a sink of a phase flow) [32 – 34]. Another important aim of the present research work is to develop and apply a whole series of new and effective research methods [8-10].

The work has a theoretical character, but those new methods may be rather helpful and useful for a broad scale of applied studies in the field of dynamic systems [11 – 14, 22, 23, 27 – 29], especially the systems having polynomial right parts. The work may be interesting for students, post-graduate students and researchers [16 – 21, 31].
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