INTERSECTION OF SOLVABLE HALL SUBGROUPS IN FINITE GROUPS

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INTRODUCTION

Throughout the paper the term “group” we always use in the meaning “finite group”. We use symbols $A \leq G$ and $A \trianglelefteq G$ if $A$ is a subgroup of $G$ and $A$ is a normal subgroup of $G$ respectively. Given $H \leq G$ by $H_G = \cap_{g \in G} H^g$ we denote the kernel of $H$.

Assume that $G$ acts on $\Omega$. An element $x \in \Omega$ is called a $G$-regular point, if $|xG| = |G|$, i.e., if the stabilizer of $x$ is trivial. We define the action of $G$ on $\Omega^k$ by

$$g : (i_1, \ldots, i_k) \mapsto (i_1 g, \ldots, i_k g).$$

If $G$ acts faithfully and transitively on $\Omega$, then the minimal $k$ such that $\Omega^k$ possesses a $G$-regular point is called the base size of $G$ and is denoted by $\text{Base}(G)$. For every natural $m$ the number of $G$-regular orbits on $\Omega^m$ is denoted by $\text{Reg}(G, m)$ (this number equals 0 if $m < \text{Base}(G)$). If $H$ is a subgroup of $G$ and $G$ acts on the set $\Omega$ of right cosets of $H$ by right multiplications, then $G/H_G$ acts faithfully and transitively on $\Omega$. In this case we denote $\text{Base}(G/H_G)$ and $\text{Reg}(G/H_G, m)$ by $\text{Base}_H(G)$ and $\text{Reg}_H(G, m)$ respectively. We also say that $\text{Base}_H(G)$ is the base size of $G$ with respect to $H$. Clearly, $\text{Base}_H(G)$ is the minimal $k$ such that there exist elements $x_1, \ldots, x_k \in G$ with $H^{x_1} \cap \ldots \cap H^{x_k} = H_G$. Thus, the base size of $G$ with respect to $H$ is the minimal $k$ such that there exist $k$ conjugates of $H$ with intersection equals $H_G$.

The following results were obtained in this direction. In 1966 D.S.Passman proved (see \cite{10}) that a $p$-solvable group possesses three Sylow $p$-subgroups whose intersection equals the $p$-radical of $G$. Later in 1996 V.I.Zenkov proved (see \cite{18}) that the same conclusion holds for arbitrary finite group $G$. In \cite{4} S.Dolfi proved that in every $\pi$-solvable group $G$ there exist three conjugate $\pi$-Hall subgroups whose intersection equals $O_{\pi}(G)$ (see also \cite{13}). Notice also that V.I.Zenkov in \cite{19} constructed an example of a group $G$ possessing a solvable $\pi$-Hall subgroup $H$ such that the intersection of five conjugates of $H$ equals $O_{\pi}(G)$, while the intersection of every four conjugates of $H$ is greater than $O_{\pi}(G)$.

In \cite{19} It was conjectured that if $H$ is a solvable Hall $\pi$-subgroup of a finite group $G$, then $\text{Base}_H(G) \leq 5$. The following theorem allows to reduce the conjecture to the case of almost simple groups.

\textbf{Theorem 1.} \cite{16} Theorem 1 \cite{16} Let $G$ be a finite group possessing a solvable $\pi$-Hall subgroup $H$. Assume that for every simple component $S$ of $E(G)$ of the factor group $G = G/S(G)$, where $S(G)$ is the solvable radical of $G$, the following condition holds:

for every $L$ such that $S \leq L \leq \text{Aut}(S)$ and contains a solvable $\pi$ – Hall subgroup $M$,

$$\text{Base}_M(L) \leq 5 \quad \text{and} \quad \text{Reg}_M(L, 5) \geq 5,$$

Then $\text{Base}_H(G) \leq 5$ and $\text{Reg}_H(G, 5) \geq 5$.

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Later in [14, Theorem 2] it was shown that the inequality \( \text{Reg}_H(G, 5) \geq 5 \) holds if \( H \) is a solvable Hall \( \pi \)-subgroup of an almost simple group \( G \), whose socle is either alternating, or sporadic, or an exceptional group of Lie type.

We prove the following theorem in the paper.

**Theorem 2.** Let \( S \) be a simple classical group and \( G \) is chosen so that \( S \leq G \leq \hat{S} \), where \( S \) is a group of inner-diagonal automorphisms of \( S \). Assume also that \( G \) possesses a solvable Hall subgroup \( H \). Then \( \text{Reg}_H(G, 5) \leq 5 \).

In view of [14, Theorem 3], if \( G \) is a classical group over a field of characteristic \( p \) and \( H \) is a Hall \( \pi \)-subgroup of \( G \) with \( p \in \pi \), then \( \text{Reg}_H(G, 5) \leq 5 \), i.e. Theorem 2 holds in this case. So we need to prove Theorem 2 in case \( p \notin \pi \), and we assume that \( p \notin \pi \) below.

1. **Preliminaries**

Let \( \mathbb{G} \) be a connected reductive algebraic group over algebraically closed field \( \mathbb{F}_p \) of positive characteristic \( p \) and let \( \sigma : \mathbb{G} \to G \) be a Frobenius morphism. If \( \overline{H} \) is a \( \sigma \)-stable subgroup of \( \mathbb{G} \) (so \( (\overline{H})^\sigma = \overline{H} \)), then \( \overline{H}_\sigma \) denotes the subgroup of \( \sigma \)-invariant elements of \( \overline{H} \).

Let \( G \) be a finite group such that \( G_0 = O^\sigma(\mathbb{G}_\sigma) \leq G \leq \mathbb{G}_\sigma \) (Note that all classical groups can be obtained in this way). Here \( O^\sigma(\mathbb{G}_\sigma) \) is the subgroup of \( \mathbb{G}_\sigma \) generated by all \( \sigma \)-elements of \( \mathbb{G}_\sigma \). Then \( T = \overline{T} \cap G \) is a maximal torus of \( G \) and \( N(G, T) = \overline{N} \cap G \) is the algebraic normaliser of \( T \) in \( G \).

In our notation for finite classical groups we follow [9]. In particular, \( p \) is prime, \( q = p^f \) for some positive integer \( f \) and \( u \) is 2 in unitary case and 1 otherwise, so the natural module for a classical group is over \( \mathbb{F}_q^u \). For unification of some formulations we use \( GL_n^+(q) \) and \( GL_n^-(q) \) for \( GL_n(q) \) and \( GU_n(q) \) respectively.

If \( n \) is a positive integer, \( r \) is an odd prime and \( (r, n) = 1 \), then \( e(r, n) \) is minimal positive integer \( e \) such that \( n^e \equiv 1 \mod r \). If \( n \) is an odd integer, then let \( e(2, n) = 1 \) if \( n \equiv 1 \mod 4 \) and \( e(2, n) = 2 \) if \( n \equiv -1 \mod 4 \).

**Lemma 1.1 ([7, Lemma 1]).** Let \( G \) be a finite group and \( A \) its normal subgroup. If \( H \) is some Hall \( \pi \)-subgroup of \( G \) then \( H \cap A \) is a Hall \( \pi \)-subgroup of \( A \) and \( HA/A \) is one in \( G/A \).

Following P. Hall [7], we say that a group \( G \) is an \( E_\pi \)-group, if \( G \) possesses a Hall \( \pi \)-subgroup.

**Lemma 1.2.** Let \( H \leq GSp_4(q) \) such that \( H \) stabilises a decomposition
\[
V = V_1 \downarrow V_2
\]
with \( \dim V_i = 2 \) and \( V_i \) non-degenerate for both \( i = 1, 2 \). Then there exist \( x, y, z \in Sp_4(q) \) such that \( H \cap H^x \cap H^y \cap H^z \leq Z(GSp_4(q)) \).

**Proof.** Let \( e_1, f_1, e_2, f_2 \) be a basis of \( V \) such that \( V_i = \langle e_i, v_i \rangle \) and \( (e_i, f_i) = 1 \). Let \( x, y, \) and \( z \) be matrices
\[
\begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
respectively in this basis. It is routine to check that \( x, y, z \in Sp_4(q) \).

Denote \( (V_i)_x \) by \( W_i \) and \( (V_i)_y \) by \( U_i \) for \( i = 1, 2 \). We claim that if \( g \in S \cap S^y \cap S^y \), then \( g \) stabilises \( V_i, i = 1, 2 \). Assume the opposite, so \( (V_1)g = (V_2) \). Therefore, \( (W_1)g = W_2 \) and \( (U_1)g = U_2 \). Thus,
\[
(V_1 \cap W_1)g = (V_1)g \cap (W_1)g = (V_2 \cap W_2)
\]
and
\[
(V_1 \cap U_1)g = (V_1)g \cap (U_1)g = (V_2 \cap U_2).
\]
Notice that \((V_2 \cap W_2) = (V_2 \cap U_2) \) but \((V_1 \cap W_1) \neq (V_1 \cap U_1)\) which is a contradiction since \(g\) is invertible. Therefore, \(g = \text{diag}[g_1, g_2]\), \(g_i \in GL_2(q)\). Also, \(g = h^2\) where \(h \in S^{s-1} \cap S\), \(g = t^u\) where \(t \in S^{s-1} \cap S\) and \(g = t^z\) where \(s \in S^{s-1} \cap S\). It is routine to check that \(h = \text{diag}[h_1, h_2]\), \(t = \text{diag}[t_1, t_2]\) and \(s = \text{diag}[s_1, s_2]\) with \(h_i, t_i, s_i \in GL_2(q)\).

Now calculations show that

\[
g = \begin{pmatrix}
    h_{(1,1)} & h_{(1,2)} & 0 & 0 \\
    0 & h_{(1,4)} & 0 & 0 \\
    0 & 0 & h_{(2,1)} & 0 \\
    0 & 0 & h_{(2,3)} & h_{(2,4)}
\end{pmatrix}
= \begin{pmatrix}
    t_{(1,1)} & 0 & 0 & 0 \\
    t_{(1,3)} & t_{(1,4)} & 0 & 0 \\
    0 & 0 & t_{(2,1)} & 0 \\
    0 & 0 & t_{(2,3)} & t_{(2,4)}
\end{pmatrix}
= \begin{pmatrix}
    s_{(1,1)} & 0 & 0 & 0 \\
    s_{(1,3)} & s_{(1,4)} & 0 & 0 \\
    0 & 0 & s_{(2,1)} & s_{(2,2)} \\
    0 & 0 & 0 & s_{(2,4)}
\end{pmatrix}
\]

for some \(h_{(i,j)}, t_{(i,j)}, s_{(i,j)} \in \mathbb{F}_q\) with

\[
h_{(1,1)} = h_{(2,1)}; \quad t_{(1,1)} = t_{(2,4)}; \\
h_{(1,4)} = h_{(2,4)}; \quad t_{(1,4)} = t_{(2,1)}.
\]

So \(g\) is scalar and \(g \in Z(GSp_4(q))\).

\[
\]

2. Hall subgroups of odd order

In this section we assume \(2, p \notin \pi\), where \(p\) is the characteristic of the base field of a classical group \(G\).

**Lemma 2.1** ([Theorem A]). Suppose the finite group \(G\) has a Hall \(\pi\)-subgroup where \(\pi\) is a set of primes not containing 2. Then all Hall \(\pi\)-subgroups of \(G\) are conjugate.

Let \(\mathcal{G}\) be a simple classical algebraic group of adjoint type, \(\sigma\) be a Frobenius morphism such that \(G_0\) is a finite simple group. Let \(G_0 \leq G \leq \mathcal{G}_\sigma\), so \(G\) is an almost simple group. It follows from [5] that the group \(G\) has a \(\pi\)-Hall subgroup if, and only if, every composition factor of \(G\) has a \(\pi\)-Hall subgroup. Therefore, we can assume \(G = H_1G_0\), where \(H_1 \in \text{Hall}_\pi(\mathcal{G}_\sigma)\). Indeed, if \(H \in \text{Hall}_\pi(G)\), then there exists \(H_1 \in \text{Hall}_\pi(\mathcal{G}_\sigma)\) such that \(H = H_1 \cap G\) by Lemma 1.1 and Theorem 2.1. So, if

\[ H_1^{g_1} \cap \ldots \cap H_1^{g_k} = 1 \]

for some \(k\) with \(g_i \in H_1G_0\), then \(g_i = h_i \cdot s_i\) with \(h_i \in H_1\) and \(s_i \in G_0\). Therefore

\[ H^{s_1} \cap \ldots \cap H^{s_k} \leq H_1^{g_1} \cap \ldots \cap H_1^{g_k} = 1. \]

Moreover, by Lemma 1.1 and [11] Lemma 2.1(e), we can assume that \(H\) is a Hall \(\pi\)-subgroup of \(\mathcal{G} \in \{GL_n(q), GU_n(q), GSp_{2n}(q), GO_n^+(q)\}\). and \(G = H \cdot (\mathcal{G} \cap SL_n(q^n))\).

Criteria for existence and structure of odd order Hall subgroups of classical groups is studied in [6]. It is explicitly shown in [15] that, if exists, \(\pi\)-Hall subgroup of a classical group \(G\) of Lie type lies in \(N(G, T)\) for some maximal torus \(T\).

**Lemma 2.2.** Let \(\mathcal{G} \in \{GL_n(q), GU_n(q), GSp_{2n}(q), GO_n^+(q)\}\) with \(n \geq 2, 3, 4, 7\) in linear, unitary, symplectic and orthogonal cases respectively. Let \(q\) be such that \(\mathcal{G}\) is not solvable. Let \(\pi\) be a set of primes such that \(2, p \notin \pi\) and \(|\pi \cap \pi(G)| \geq 2\), let \(r\) be the smallest prime in \(\pi \cap \pi(G)\), and let \(\tau = \pi \setminus \{r\}\). Let \(H\) be a Hall \(\pi\)-subgroup of \(\mathcal{G}\). If \(G = H \cdot (\mathcal{G} \cap SL_n(q^n))\), then there exist \(x, y, z \in G\) such that

\[ H \cap H^x \cap H^y \cap H^z \leq Z(\mathcal{G}). \]
Proof. Denote by \( r \) the minimal number in \( \pi \cap \pi(G) \), and \( (\pi \cap \pi(G)) \setminus \{ r \} \) by \( \tau \). Recall, that, by [6] Theorem 4.9, \( \hat{G} \) is a \( E_\pi \) subgroup if, and only if, \( \hat{G} \) is \( E_{\{t,s\}} \) for all \( t, s \in \pi \). By Theorem [6] Theorem 4.6, if \( \hat{G} \in \{ GL_n(q), GU_n(q), GSp_{2n}(q) \} \), then \( H \) has a normal abelian Hall \( \tau \)-subgroup, \( \hat{G} \) satisfies \( D_\tau \), all \( \tau \)-subgroups of \( \hat{G} \) are abelian and \( e(q, t) = e(q, s) \) for all \( t, s \in \pi \). By [6] Theorem 4.8, if \( \hat{G} = GO_n^+(q) \), then \( H \) has a normal abelian Hall \( \tau \)-subgroup, \( \hat{G} \) satisfies \( D_\tau \), all \( \tau \)-subgroups of \( \hat{G} \) are abelian and either \( H \) is cyclic or \( e(q, t) = e(q, s) \) for all \( t, s \in \pi \).

Let \( \hat{G} = GL_n(q) \). By [6] Theorems 4.2 and 4.6, \( \hat{G} \) is a \( E_\pi \) group if, and only if, \( n < b \) for every \( s \in \pi \), and one of the following is true:

\((A)\) \( a = b; \)
\((B)\) \( a = r - 1, b = r, (q^{r-1} - 1)_r = r, \) and \( [\frac{n}{r-t}] = [n/r]; \)
\((C)\) \( a = r - 1, b = t, (q^{r-1} - 1)_t = r, \) and \( [\frac{n}{r-t}] = [n/r]; \)
\((D)\) \( a = r - 1, b = 1, (q^{r-1} - 1)_r = r, \) and \( [\frac{n}{r-t}] = [n/r]. \)

If \( H \) is Abelian, then there exists \( x \in G \) such that

\[ H \cap H^x \leq Z(GL_n(q)) \]

by [17] Theorem 1. So we assume that \( H \) is not abelian, so, by the proof of [15] Theorem 4, a Sylow \( r \)-subgroup of \( \hat{G} \) is not abelian.

Assume that \((A)\) is realised. By the proof of [15] Theorem 4, \( H \) lies in the subgroup \( G_1 = GL_{[n/a]}(q^a) \) of \( \hat{G} \). Precisely, \( H \) lies in the group of monomial matrices of \( G_1 \). So

\[ V = V_1 \oplus \ldots \oplus V_{[n/a]} \oplus W \]

where \( \dim V_i = a \) for \( i \in \{1, \ldots, [n/a]\} \), \( \dim W = n - [n/a] \cdot a, W \subseteq C_H(V) \) and \( H \) permutes \( V_i \). Therefore \( H \) lies in a maximal irreducible group of \( H \cdot SL_{[n/a]}.a(q) \) (if \( [n/a] > 1 \), then \( H \) lies in a maximal imprimitive subgroup \( M \subseteq C_H; \) if \( [n/a] = 1 \), then \( H \) is abelian) and there exist \( x, y, z \in SL_{[n/a]}.a(q) \leq SL_n(q) \) such that

\[ H \cap H^x \cap H^y \cap H^z \leq Z(GL_{[n/a]}.a(q)) \times I_{n-[n/a]-a} \]

by [2] Theorem 1.1. Notice, that if \( a > 1 \), then \( H \cap (Z(GL_{[n/a]}.a(q)) \times I_{n-[n/a]-a}) = 1 \); if \( a = 1 \), then \( [n/a] = n \), so the statement follows in both cases.

Assume that \((B)\) or \((C)\) is realised. By the proof of [15] Theorem 4, \( H \) lies in \( G_1 = GL_{[n/r]}(q^r) \times GL_{r-1}(q) \leq \hat{G} \)

and

\[ (q^{r-1})_r = |G| = |GL_{r-1}(q)t| = |G_1| = r; \]

Also, Hall \( \tau \)-subgroup of \( \hat{G} \) lies in the subgroup of diagonal matrices of \( GL_{[n/r]}(q^r) \). Let \( V = U \oplus W \) where \( U \) is the natural module for \( GL_{[n/r]}.r(q) \) and \( W \) is the natural module for \( GL_{r-1}(q) \). So

\[ H = H_\tau \times R \]

where \( H_\tau \leq GL_{[n/r]}.r(q) \) stabilises the decomposition

\[ U = V_1 \oplus \ldots \oplus V_{[n/r]} \]

with \( \dim V_i = r; \)

and \( R \leq GL_{r-1}(q) \) is a cyclic \( r \)-subgroup. Therefore, as \( H \) in the previous case, \( H_\tau \) lies in the maximal irreducible subgroup of \( H_\tau \cdot SL_{[n/r]}.r(q) \) and there exist \( x_1, y_1, z_1 \in SL_{[n/r]}.r(q) \) such that

\[ H \cap H_\tau^x \cap H_\tau^y \cap H_\tau^z \leq Z(GL_{[n/a]}.a(q)) \times I_{n-[n/a]-a} \]
by \cite[Theorem 1.1]{2}. By \cite[Theorem 1]{17}, there exist \( x_2 \in R \cdot SL_{r-1}(q) \) (so we can assume \( x_2 \in SL_{r-1}(q) \)) such that \( R \cap R^{x_2} = 1 \), since \( a = r - 1 \geq 1 \), so \( R \cap Z(GL_{r-1}(q)) = 1 \). Let \( x = \text{diag}[x_1, x_2] \), \( y = \text{diag}[y_1, I_{r-1}] \), \( z = \text{diag}[z_1, I_{r-1}] \). It is easy to see that

\[
H \cap H^x \cap H^y \cap H^z = 1.
\]

Assume that (D) is realised. By the proof of \cite[Theorem 4]{15}, \( H \) lies in the group of monomial matrices of \( \hat{G} \), so \( H \) lies in the maximal imprimitive group of \( H \cdot SL_n(q) \) and there exist \( x, y, z \in SL_n(q) \) such that

\[
H \cap H^x \cap H^y \cap H^z \leq Z(GL_n(q))
\]

by \cite[Theorem 1.1]{2}. Let \( \hat{G} = GU_n(q) \). By \cite[Theorems 4.3 and 4.6]{6}, \( \hat{G} \) is a \( E_\pi \) group if, and only if, \( n < bs \) for all \( s \in \tau \), and one of the following is true:

(A) \( a = b \equiv 0 \mod 4 \);

(B) \( a = b \equiv 2 \mod 4 \) and \( 2n < bs \) for all \( s \in \tau \);

(C) \( a = b \equiv 1 \mod 2 \);

(D) \( r \equiv 1 \mod 4 \), \( a = r - 1 \), \( b = 2r \), \((q^n - 1)_{r} = r \), and \( [\frac{n}{r-1}] = [\frac{n}{r}] \);

(E) \( r \equiv 3 \mod 4 \), \( a = \frac{r-1}{2} \), \( b = 2r \), \((q^n - 1)_{r} = r \), and \( [\frac{n}{r-1}] = [\frac{n}{r}] \);

(F) \( r \equiv 1 \mod 4 \), \( a = r - 1 \), \( b = 2r \), \((q^n - 1)_{r} = r \), and \( [\frac{n}{r}] = [\frac{n}{r}] + 1 \) and \( n \equiv r - 1 \) (mod \( r \));

(G) \( r \equiv 3 \mod 4 \), \( a = \frac{r-1}{2} \), \( b = 2r \), \((q^n - 1)_{r} = r \), and \( [\frac{n}{r}] = [\frac{n}{r}] + 1 \) and \( n \equiv r - 1 \) (mod \( r \));

(H) \( r \equiv 1 \mod 4 \), \( a = r - 1 \), \( b = 2 \), \((q^n - 1)_{r} = r \), and \( [\frac{n}{r}] = [\frac{n}{r}] + 1 \) and \( n \equiv r - 1 \) (mod \( r \));

(I) \( r \equiv 3 \mod 4 \), \( a = \frac{r-1}{2} \), \( b = 2 \), \((q^n - 1)_{r} = r \), and \( [\frac{n}{r}] = [\frac{n}{r}] + 1 \) and \( n \equiv r - 1 \) (mod \( r \));

If \( H \) is abelian, then there exists \( x \in G \) such that

\[
H \cap H^x \leq Z(GU_n(q))
\]

by \cite[Theorem 1]{17}. So let \( H \) be non-abelian.

In cases (A)–(C), by the proof of \cite[Theorem 4]{15}, \( H \) lies in subgroup \( G_1 = GL_{[n/a]}(q^a) \) of \( \hat{G} \) and the statement follows as in case (A) for \( \hat{G} = GL_n(q) \).

In cases (D)–(G), by the proof of \cite[Theorem 4]{15}, \( H \) is abelian.

In cases (H) and (I), by the proof of \cite[Theorem 4]{15}, \( H \) lies in the group of monomial matrices of \( \hat{G} \) so \( H \) lies in the maximal imprimitive group of \( G \) and there exist \( x, y, z \in SU_n(q) \) such that

\[
H \cap H^x \cap H^y \cap H^z \leq Z(GU_n(q))
\]

by \cite[Theorem 1.1]{2}. Let \( \hat{G} = GO_n^c \). By \cite[Theorems 4.4 and 4.6]{6}, \( \hat{G} \) is a \( E_\pi \) group if, and only if, \( n < bs \) for all \( s \in \tau \), and one of the following is true:

(A) \( \varepsilon = + \), \( a = b \equiv 0 \mod 2 \) and \( n < bs \);

(B) \( \varepsilon = + \), \( a = b \equiv 1 \mod 2 \) and \( n < 2bs \);

(C) \( \varepsilon = - \), \( a = b \equiv 0 \mod 2 \) and \( n < bs \);

(D) \( \varepsilon = - \), \( a = b \equiv 1 \mod 2 \) and \( n < bs \);

(E) \( \varepsilon = - \), \( a \equiv 1 \mod 2 \), \( b = 2a \) and \( n = 4a \);

(F) \( \varepsilon = - \), \( b \equiv 1 \mod 2 \), \( a = 2b \) and \( n = 4b \);

The proof in cases (A)–(D) is analogous to the proof for \( \hat{G} = GL_n(q) \) in case (A) and for \( \hat{G} = GU_n(q) \) in cases (A)–(C). In cases (E) and (D), by the proof of \cite[Theorem 4]{15}, \( H \) is abelian.

Let \( \hat{G} = GSp_{2n}(q) \). By \cite[Theorem 4.5]{6}, \( \hat{G} \) is a \( E_\pi \) group if, and only if, one of the following is true:
(A) \( a = b \equiv 0 \) mod 2 and \( 2n < bs \) for all \( s \in \tau \);
(B) \( a = b \equiv 1 \) mod 2 and \( n < bs \) for all \( s \in \tau \);

In both cases the proof is analogous to the proof for \( \hat{G} = GL_n(q) \) in case (A) and for
\( \hat{G} = GU_n(q) \) in cases (A)–(C) unless \( G \leq GSp_4(q) \) and \( a = 2 \), so \( H \) lies in maximal subgroup \( M \) stabilising a decomposition of \( V \) into two non-degenerate subspaces. In this case \( M \) can be a standard subgroup in terms of [2]. If it is the case, then the statement follows by Lemma 1.2.

\[ 3. \text{ Hall subgroups of even order} \]

In this section we assume \( 2 \in \pi \) and \( p \notin \pi \), where \( p \) is the characteristic of the base field of a classical group \( G \).

Let \( \overline{G} \) be a simple classical algebraic group of adjoint type, \( \sigma \) be a Frobenius morphism such that \( G_0 \) is a finite simple group. Let \( G_0 \leq \overline{G} \leq \overline{G}_{\sigma} \), so \( G \) is an almost simple group.

Assume that \( 3 \notin \pi \). It follows from [15] Conjectures 1.2 and 1.3 (this Conjectures follows from the results of [15]) that if \( G \) has a Hall \( \pi \)-subgroup \( H \), then \( H \) is solvable and all such subgroups are conjugate in \( G \). Also, a finite group \( R \) has a \( \pi \)-Hall subgroup if, and only if, every composition factor of \( R \) has a \( \pi \)-Hall subgroup.

Therefore, we can assume \( G = H \cdot G_0 \), where \( H \in \text{Hall}_\pi(\overline{G}_{\sigma}) \) as in previous section. Moreover, by Lemma [11] and [11] Lemma 2.1, we can assume that \( H \) is a Hall \( \pi \)-subgroup of \( \hat{G} \in \{GL_{n}(q), GU_{n}(q), GSp_{2n}(q), GO_{n}^{\varepsilon}(q)\} \) and \( G = H \cdot \hat{G} \cap SL_n(q^n) \).

**Lemma 3.1.** Let \( 3 \notin \pi \) and \( 2 \in \pi \). Let \( H \) be a solvable Hall \( \pi \)-subgroup of
\( \hat{G} \in \{GL_{n}(q), GU_{n}(q), GSp_{2n}(q), GO_{n}^{\varepsilon}(q)\} \)
with \( n \geq 2, 3, 4, 7 \) in linear, unitary, symplectic and orthogonal cases respectively. Let \( q \) be such that \( \hat{G} \) is not solvable. Let \( G_0 = SL_{n}(q^n) \cap \hat{G} \). If \( G = H \cdot G_0 \), then there exist \( x, y, z \in G \) such that
\( H \cap H^x \cap H^y \cap H^z \leq Z(\hat{G}) \).

**Proof.** Let \( H_0 = H \cap G_0 \). By [12] Theorem 5.2, \( H_0 \) lies on \( N(G_0, T_0) \) where \( T_0 \) is a maximal torus of \( G_0 \) such that \( N(G_0, T_0) \) contains a Sylow 2-subgroup of \( G_0 \) (all such tori are conjugate in \( G_0 \) by [12] Lemma 3.10) and one of the following is realised
- \( e(2, q) = 1 \) and \( \pi \cap \pi(G_0) \subseteq \pi(q - 1) \);
- \( e(2, q) = 2 \) and \( \pi \cap \pi(G_0) \subseteq \pi(q + 1) \).

It is easy to see that, if \( T \geq T_0 \) is a maximal torus of \( \hat{G} \) containing a Sylow 2-subgroup, then \( H \leq N(\hat{G}, T) \), since \( |N(\hat{G}, T)|_{\pi} = |\hat{G}|_{\pi} \). By [3] Theorem 1 (or the proof of [12] Lemma 3.10), \( N(G, T) \), and hence \( H \), stabilises a decomposition
\[ V = V_1 \perp \ldots \perp V_{[k]} \perp W \quad (3.1) \]
where \( \dim V_i = 2 \) and \( \dim W \in \{0, 1, 2\} \). By that we mean that \( H \) stabilises \( W \) and permutes \( V_i \). If \( \hat{G} \) is unitary, symplectic or orthogonal, then \( V_i \)'s are pairwise isometric non-degenerate subspace and \( W \) is a non-degenerate subspace. In particular, if \( \hat{G} \) is orthogonal and \( \dim W = 2 \), then we assume that \( W \) is not of the same type as \( V_i \) since otherwise we can take \( V_{k+1} := W \).

If \( n = 2 \), so \( \hat{G} = GL_2(q) \), then \( H \) lies in a maximal \( C_3 \)-subgroup \( M \) of \( G \) and the statement follows by [2] Theorem 1.1.

Assume \( n > 2 \) and \( \hat{G} \) is not orthogonal. If \( n \) is even, then \( H \) lies in a maximal imprimitive (stabilising the decomposition (3.1)) subgroup \( M \) of \( G \), so the statement follows by [2] Theorem 1.1] unless \( G \leq GSp_4(q) \) and the statement follows by Lemma 1.2.
Let $n \geq 3$ is odd, so $\hat{G}$ is $GL_n(q)$ or $GU_n(q)$. Let $\{v_1, \ldots, v_n\}$ be a basis (orthonormal if $\hat{G} = GU_n(q)$) such that $V_i = \langle v_{2i-1}, v_{2i} \rangle$ for $i \in \{1, \ldots, [n/2]\}$ and $W = \langle v_n \rangle$. Let $\sigma \in \text{Sym}(n)$ be $(1, 2, \ldots, n)$ and

$$x = \text{PermMat}(\sigma) \cdot \text{diag}(\text{sgn}(\sigma), 1, \ldots, 1) \in SL_n^+(q).$$

Therefore, $H \cap H^x$ stabilises decompositions (3.1) and

$$\langle v_2, v_3 \rangle \perp \langle v_4, v_5 \rangle \perp \ldots \perp \langle v_{n-1}, v_n \rangle \perp \langle v_1 \rangle.$$

It is easy to see that $H \cap H^x$ consists of diagonal matrices, so $H \cap H^x$ is abelian. Therefore, by [17, Theorem 1], there exists $y \in G$ such that

$$(H \cap H^x) \cap (H \cap H^y)^y \leq Z(G).$$

Assume now that $\hat{G}$ is orthogonal, so $n \geq 7$. If $\dim W = 0$, then $H$ lies in a maximal imprimitive (stabilising the decomposition (3.1)) subgroup $M$ of $G$, so the statement follows by [2] Theorem 1.1.

Let $\dim W = 1$, so $n$ is odd and $\hat{G} = GO_n(q)$. Let $Q$ be the quadratic form associated with $\hat{G}$ and let $Q(v_n) = \lambda \in \mathbb{F}_q^*$ where $\langle v_n \rangle = W$. Since $q$ is odd, $Q : V_i \to \mathbb{F}_q$ is surjective (see [9] §2.5), we can choose a basis $\beta_i = \{v_{2i-1}, v_{2i}\}$ of $V_i$ such that $Q(v_{2i-1}) = \lambda$ and $f(v_{2i-1}, v_{2i}) = 0$ where $f$ is the bilinear form associated with $Q$. Let $\sigma \in \text{Sym}(n)$ be $(1, 3, 5, \ldots, n-2, n)$ and

$$x = \text{PermMat}(\sigma) \cdot \text{diag}(\text{sgn}(\sigma), 1, \ldots, 1) \in SO_n(q).$$

Therefore, $H \cap H^x$ stabilises decompositions (3.1) and

$$\langle v_3, v_2 \rangle \perp \langle v_5, v_4 \rangle \perp \ldots \perp \langle v_{n-1}, v_n \rangle \perp \langle v_1 \rangle.$$

It is easy to see that $H \cap H^x$ consists of diagonal matrices, so $H \cap H^x$ is abelian. Therefore, by [17, Theorem 1], there exists $y \in G$ such that

$$(H \cap H^x) \cap (H \cap H^y)^y \leq Z(G).$$

Let $\dim W = 2$, so $n$ is even and $\hat{G} = GO_n^+(q)$. By [2] Lemma 2.5.12, we can choose a basis $\beta_i = \{v_{2i-1}, v_{2i}\}$ of $V_i$ and a basis $\{v_{n-1}, v_n\}$ of $W$ such that $Q(v_{2i-1}) = 1$ and $f(v_{2i-1}, v_{2i}) = 0$. Let $\sigma \in \text{Sym}(n)$ be $(1, 3, 5, \ldots, n-1)$ and

$$x = \text{PermMat}(\sigma) \cdot \text{diag}(\text{sgn}(\sigma), 1, \ldots, 1) \in SO_n^+(q).$$

Therefore, $H \cap H^x$ stabilises decompositions (3.1) and

$$\langle v_3, v_2 \rangle \perp \langle v_5, v_4 \rangle \perp \ldots \perp \langle v_{n-1}, v_n \rangle \perp \langle v_1, v_n \rangle.$$

It is easy to see that $H \cap H^x$ consists of diagonal matrices, so $H \cap H^x$ is abelian. Therefore, by [17, Theorem 1], there exists $y \in G$ such that

$$(H \cap H^x) \cap (H \cap H^y)^y \leq Z(G).$$

\[ \text{Remark 3.2.} \] Let $\hat{G} = GL_n(q)$ and let $H$ be as in Lemma 3.1. If $n$ even, then, by [3], there almost always exists just two conjugates of $H$ whose intersection lies in $Z(\hat{G})$. If $n \geq 5$ is odd, then one can show that $H \cap H^x \leq Z(\hat{G})$ where

$$x = \begin{pmatrix} 1 & 1 & 0 & \ldots & 0 \\ 0 & 1 & 1 & 0 & \ldots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ldots & 0 & 1 & 1 \\ 1 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$
Using a similar technique, Baykalov in [1] show that, if \( R \) is a solvable imprimitive subgroup in \( \hat{G} = GU_n(q) \) (\( GSp_n(q) \) respectively), then in almost all cases there exist \( x \) and \( y \) in \( SU_n(q) \) (\( Sp_n(q) \) respectively) such that \( S \cap S^x \cap S^y \leq Z(\hat{G}) \).

**Lemma 3.3.** Let \( p \notin \pi \) and \( 2, 3 \in \pi \). Let \( H \) be a solvable Hall \( \pi \)-subgroup of \( \hat{G} \in \{GL_n(q), GU_n(q), GSp_n(q), GO^+_n(q)\} \) with \( n \geq 2, 3, 4, 7 \) in linear, unitary, symplectic and orthogonal cases respectively. Let \( q \) be such that \( \hat{G} \) is not solvable. Let \( G_0 = SL_n(q^u) \cap \hat{G} \). If \( G = H \cdot G_0 \), then there exist \( x, y, z \in G \) such that

\[
H \cap H^x \cap H^y \cap H^z \leq Z(\hat{G}).
\]

**Proof.** Assume that \( \hat{G} \) is not orthogonal. By [11, Lemma 4.1], \( H \) stabilises a decomposition

\[
V = V_1 \perp \ldots \perp V_k,
\]

into a direct sum of pairwise orthogonal non-degenerate (arbitrary if \( V \) is linear) subspaces \( V_i \) where \( \dim(V_i) \leq 2 \) for \( i \in \{1, \ldots, k\} \). If \( \hat{G} = GL_n(q) \), then, by the proof of [11, Lemma 4.3], we can assume that either \( \dim V_i = 1 \) for all \( i \) or \( \dim V_i = 2 \) for \( i < k \) and \( \dim V_k \in \{1, 2\} \).

If \( \hat{G} = GSp_n(q) \), then \( \dim V_i = 2 \) for all \( i \) since all one-dimensional subspaces are singular in this case. The rest of the proof is as in Lemma 3.1.

Assume now \( \hat{G} = GSp^+_n(q) \). Since \( H \) is solvable, one of \((a)-(e)\) holds in [11, Lemma 6.7]. In cases \((a)-(c)\), \( H \) stabilises a decomposition of \( V \) as in Lemma 3.1 and the proof as in Lemma 3.1 works. In cases \((d)\) and \((e)\) we have \( n = 11 \) and \( n = 12 \), \( H \) stabilises decompositions

\[
V = (V_1 \perp V_2 \perp V_3 \perp V_4) \perp (W_1 \perp W_2 \perp W_3)
\]

and

\[
V = (V_1 \perp V_2 \perp V_3 \perp V_4) \perp (W_1 \perp W_2 \perp W_3 \perp W_4)
\]

respectively. By that we mean that \( H \) permutes \( V_i \)'s and \( W_i \)'s between and stabilises \( \sum_{i=1}^4 V_i \), \( \sum_{i=1}^3 W_i \) and \( W_4 \). Here \( V_i, W_i \) are non-degenerate, \( \dim V_i = 2 \) and \( \dim W_i = 1 \). As in Lemma 3.1 we can choose the basis \( \{v_1, \ldots, v_n\} \) of \( V \) such that \( V_i = \langle v_{2i-1}, v_{2i} \rangle \), \( W_i = \langle v_{8+i} \rangle \),

\[
Q(v_1) = Q(v_3) = Q(v_5) = Q(v_7) = Q(v_9) = Q(v_{10}) = Q(v_{11}) = Q(v_{12})
\]

and \( f(v_i, v_j) = 0 \) for \( i \neq j \). Let \( \sigma \in \text{Sym}(n) \) be \((1, 3, 5, 9)(7, 10)\) and

\[
x = \text{PermMat}(\sigma) \in SO^+_n(q).
\]

Therefore, \( H \cap H^x \) stabilises the decomposition above and

\[
(\langle v_3, v_2 \rangle \perp \langle v_5, v_4 \rangle \perp \langle v_9, v_8 \rangle \perp \langle v_{10}, v_{12} \rangle) \perp (\langle v_1, v_7 \rangle \langle v_{11} \rangle) \perp \langle v_{12} \rangle).
\]

It is easy to see that \( H \cap H^x \) consists of diagonal matrices, so \( H \cap H^x \) is abelian. Therefore, by [17, Theorem 1], there exists \( y \in G \) such that

\[
(H \cap H^x) \cap (H \cap H^y) \leq Z(G).
\]

\( \square \)

Now Theorem 2 follows by Lemmas 2.2, 3.1 and 3.3.
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