THE CENTRAL CORRELATIONS OF
HYPERCHARGE, ISOSPIN, COLOUR AND CHIRALITY
IN THE STANDARD MODEL

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Abstract

The correlation of the fractionally represented hypercharge group with the isospin and colour group in the standard model determines as faithfully represented internal group the quotient group \( \frac{U(1) \times SU(2) \times SU(3)}{Z_2 \times Z_3} \). The discrete cyclic central abelian-nonabelian internal correlation involved is considered with respect to its consequences for the representations by the standard model fields, the electroweak mixing angle and the symmetry breakdown. There exists a further discrete \( \mathbb{Z}_2 \)-correlation between chirality and Lorentz properties and also a continuous \( U(1) \)-external-internal one between hyperisospin and chirality.
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1 Nonabelian Synchronization of Hypercharge

The standard model of the electroweak and strong interactions in its minimal form [12] shows a ‘global’ relation between the internal abelian hypercharge properties and the nonabelian isospin-colour ones as described in [7, 9, 1]. In the following, the results of these papers are used and, sometimes for convenience, reformulated.

The fundamental standard model fields transform with irreducible representations $[2J_L|2J_R]$ of the external Lorentz group $\text{SL}(2\mathbb{C})$ and irreducible representations $[6y], [2T]$ and $[2C_1, 2C_2]$ of hypercharge $U(1)$ (rational hypercharge number $y$), isospin $\text{SU}(2)$ (integer or halfinteger isospin $T$) and colour $\text{SU}(3)$ as given in the following table:

| field       | symbol | $\text{SL}(2\mathbb{C})$ | $\text{U}(1)$ | $\text{SU}(2)$ | $\text{SU}(3)$ |
|-------------|--------|--------------------------|---------------|---------------|---------------|
| left lepton | $\Psi$ | $[1|0]$                  | $-\frac{1}{2}$ | $1$           | $0, 0$        |
| right lepton| $\Psi^*$ | $[0|1]$                  | $-1$          | $0$           | $0, 0$        |
| left quark  | $q$    | $[1|0]$                  | $\frac{1}{6}$ | $1$           | $1, 0$        |
| right quarks| $u, d$ | $[0|1]$                  | $\frac{2}{3}, -\frac{1}{3}$ | $0$           | $1, 0$        |
| Higgs       | $H$    | $[0|0]$                  | $-\frac{1}{2}$ | $1$           | $0, 0$        |
| hypercharge gauge | $A$ | $[1|1]$                  | $0$           | $0$           | $0, 0$        |
| isospin gauge | $B$ | $[1|1]$                  | $0$           | $2$           | $0, 0$        |
| colour gauge | $G$  | $[1|1]$                  | $0$           | $0$           | $1, 1$        |

With respect to the Lorentz group, $[0|0]$ designates scalar fields, $[1|0]$ and $[0|1]$ are left and right handed spinor fields resp., $[1|1]$ vector fields. The external and internal multiplicity (singlet, doublet, triplet, quartet, octet, etc.) of the Lorentz-group, isospin and colour representations can be computed from the natural numbers $2J_{L,R}, 2T, 2C_1, 2C_2 \in \mathbb{N} = \{0, 1, \ldots\}$

$$N_{\text{ext}}(\Psi) = (2J_L + 1)(2J_R + 1)$$

$$N_{\text{int}}(\Psi) = N_{\text{iso}}(\Psi)N_{\text{col}}(\Psi), \quad \begin{cases} N_{\text{iso}}(\Psi) = 2T + 1 \\ N_{\text{col}}(\Psi) = \frac{(2C_1 + 1)(2C_2 + 1)(2C_1 + 2C_2 + 2)}{2} \end{cases}$$

Fields and antifields have reflected quantum numbers:

| field | $2J_L|2J_R$ | $y$ | $2T$ | $[2C_1, 2C_2]$ |
|-------|-------------|-----|------|----------------|
| $\Psi$ | $[2J_R|2J_L]$ | $-y$ | $[2T]$ | $[2C_2, 2C_1]$ |

The correlation of the external with the internal properties will be discussed in section 7 after the discussion of the internal ones.

If a nontrivial hypercharge $y$ in the normalization above for the fundamental fields with different antifields $\Psi \neq \Psi^*$ is written as a rational $y = \frac{Z(y)}{N(y)}$ with integer nominator $Z(y) \in \mathbb{Z}$ and natural denominator $N(y) \in \mathbb{N}$, where $Z(y)$ and $N(y)$ have no common nontrivial divisor, the internal multiplicity coincides with the hypercharge fractionality $N(y) = N_{\text{int}}$, e.g. $6 = N_{\text{int}}(q)$. 
Using \(6y\) in the normalization above as elements of the cyclic groups \(\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}\) for \(n = 2\) (isospin) and \(n = 3\) (colour)

|       | \(6y\) | \(6y\mod 2\) | \(6y\mod 3\) |
|-------|--------|--------------|--------------|
| \(I\) | \(-3\) | \(1\)        | \(0\)        |
| \(e\) | \(-6\) | \(0\)        | \(0\)        |
| \(q\) | \(1\)  | \(1\)        | \(1\)        |
| \(u, d\) | \(4, -2\) | \(0\) | \(-1\) |
| \(H\) | \(-3\) | \(1\)        | \(0\)        |
| \(A, B, G\) | \(0\) | \(0\)        | \(0\)        |

the correlation between hypercharge and internal nonabelian properties reads

for isospin \(\text{SU}(2): \quad 6y\mod 2 = 2T\mod 2\)
for colour \(\text{SU}(3): \quad 6y\mod 3 = 2(C_1 - C_2)\mod 3\)

This can be formalized as follows: The fractional hypercharge numbers \(y\) reflect the representations of the centrum \(\mathbb{I}(2) \times \mathbb{I}(3)\) (‘David star group’ as direct product of two cyclic groups with prime order - the real sign group \(\mathbb{I}(2)\) and the complex ‘Mercedes star group’ \(\mathbb{I}(3)\)) of the isospin-colour group \(\text{SU}(2) \times \text{SU}(3)\)

\[
\text{centr } \text{SU}(n) \cong \mathbb{I}(n) = \{ \exp \frac{2\pi i r}{n} \mid r = 0, 1, \ldots, n-1 \} \cong \mathbb{Z}_n, \quad n \geq 1
\]

\[
\text{centr } [\text{SU}(2) \times \text{SU}(3)] \cong \mathbb{I}(2) \times \mathbb{I}(3) \cong \mathbb{I}(6)
\]

The endomorphisms of the cyclic group \(\mathbb{I}(n)\) (for \(n\) prime simple groups, even fields) are determined by the mapping of the cyclic element \(\exp \frac{2\pi i}{n}\)

\[
\nu^r : \mathbb{I}(n) \longrightarrow \mathbb{I}(n), \quad \exp \frac{2\pi i}{n} \longmapsto \exp \frac{2\pi i}{n}r, \quad r = 0, \ldots, n-1
\]

nontrivial for \(r \neq 0\) and faithful (injective), i.e. \(\mathbb{I}(n)\)-automorphisms, if \(r\) and \(n\) are relatively prime, i.e. naturals \(r, n \neq 0\) with only 1 as common divisor. All \(\mathbb{I}(n)\)-subgroups \(\mathbb{I}(d(n))\) arise with as \(n\)-divisor \(d(n)\), for \(n\) prime only \(\mathbb{I}(1) = \{1\}\) and \(\mathbb{I}(n)\). In the endomorphisms \(\nu^r\) they come both as images \(\nu^r[\mathbb{I}(n)]\) and as kernels \(\mathbb{I}(r(n)) \cong \mathbb{I}(n)/\nu^r[\mathbb{I}(n)]\). The irreducible isospin and colour representations, denoted by \(\text{rep } \text{SU}(2)\) and \(\text{rep } \text{SU}(3)\) resp., represent the centrum as follows

\[
[2T] \in \text{rep } \text{SU}(2) : \quad \exp \pi i \longmapsto \exp \pi i \cdot 2T \mod 2
\]

\[
[2C_1, 2C_2] \in \text{rep } \text{SU}(3) : \quad \exp \frac{2\pi i}{3} \longmapsto \exp \frac{2\pi i}{3} \cdot 2(C_1 - C_2) \mod 3
\]

i.e. the nontrivial irreducible ones are faithful as follows

\[
[2T] \in \text{rep } \text{SU}(2) \text{ for } \begin{cases} \text{SU}(2)/\mathbb{I}(2) \cong \text{SO}(3) \iff 2T \mod 2 = 0 \\ \text{SU}(2) \iff 2T \mod 2 = 1 \end{cases}
\]

\[
[2C_1, 2C_2] \in \text{rep } \text{SU}(3) \text{ for } \begin{cases} \text{SU}(3)/\mathbb{I}(3) \iff 2(C_1 - C_2) \mod 3 = 0 \\ \text{SU}(3) \iff 2(C_1 - C_2) \mod 3 = \pm 1 \end{cases}
\]

Therewith, the ‘synchronization’ of the nonabelian isospin-colour center \(\mathbb{I}(2) \times \mathbb{I}(3)\) with the hypercharge property shows that the group, faithfully represented in the standard model, is the quotient group

\[
\text{U}(2 \times 3) \cong \frac{\text{U}(1) \times \text{SU}(2) \times \text{SU}(3)}{\mathbb{I}(2) \times \mathbb{I}(3)}
\]
The universal covering group for the Lie algebra of $U(2 \times 3)$ is $\mathbb{R} \times SU(2) \times SU(3)$ with the maximal compact subgroup $U(1) \times SU(2) \times SU(3)$. Hypercharge is centrally connected with isospin and colour. Similar to the full unitary group $U(n)$, $n \geq 2$, which is a composition product $\circ$ of two normal subgroups, but not a direct product $\times$

$$U(n) = U(1_n) \circ SU(n) \Rightarrow \left\{ \begin{array}{l} U(n)/U(1_n) \cong SU(n)/\mathbb{I}(n) \\ U(n)/SU(n) \cong U(1)/\mathbb{I}(n) \cong U(1) \end{array} \right.$$  

the group $U(2 \times 3)$, defined in $U(6)$ with Pauli and Gell-Mann matrices $\{\tau^a\}_{a=1}^3$ and $\{\lambda^j\}_{j=1}^8$ resp.

$$\exp \left[ \frac{i}{\beta} \alpha_2 \otimes 1_3 + \frac{i}{\beta_2} \tau^a \otimes 1_3 + \frac{i}{\beta_3} \gamma_3 \otimes \lambda^j \right] \in U(2 \times 3)$$

is a product $\circ$ of two normal subgroups, where the nonabelian one is a direct product of two normal subgroups

$$U(2 \times 3) = U(1_6) \circ [SU(2) \otimes 1_3 \times 1_2 \otimes SU(3)]$$

$$U(1_6) \cap SU(2) \otimes 1_3 \cong \mathbb{I}(2)$$

$$U(1_6) \cap 1_2 \otimes SU(3) \cong \mathbb{I}(3)$$

$$U(1_6) \cap [SU(2) \otimes 1_3 \times 1_2 \otimes SU(3)] \cong \mathbb{I}(2) \times \mathbb{I}(3)$$

$$\Rightarrow \left\{ \begin{array}{l} U(2 \times 3)/U(1_6) \cong SU(2)/\mathbb{I}(2) \times SU(3)/\mathbb{I}(3) \\ U(2 \times 3)/SU(2) \otimes 1_3 \times 1_2 \otimes SU(3) \cong U(1) \end{array} \right.$$  

Having nontrivial invariant subgroups, the groups $U(n)$ or $U(2 \times 3)$ are not simple, but they have no nontrivial direct factor.

The common cyclic subgroup of hypercharge and isospin-colour defines a ‘unification’ quite different from the grand unification schemes, which embed $U(1) \times SU(2) \times SU(3)$ or $U(2 \times 3)$ into a larger Lie group, e.g. $SU(5)$ or $SO(10)$. The minimal real 12-dimensional Lie algebra of the standard model remains unchanged by the discrete $\mathbb{I}(6)$-correlation. The global properties of the group - not the Lie algebra local ones are relevant. As known from the bound states of the nonrelativistic hydrogen atom with the angular momentum and perihel conservation indicating a covering symmetry $\mathbb{I}[10] SU(2) \times SU(2)$ and the orthogonality of angular momentum and perihel vector imposing the cyclic centrum correlation $\mathbb{I}[10] SU(2) \times SU(2) \cong SO(4)$, also such discrete ‘unifications’ have strong consequences, as seen in the ‘square degenerated’ representation spectrum of the hydrogen atom bound states.

2 Fundamental and Cyclic Representations

Before the interpretation of the fundamental standard model fields as $U(2 \times 3)$-representations (section 3) it is useful to give the definition of fundamental and cyclic representations.

Any simple Lie algebra $L_r$ of rank $r$, e.g. the rank $(n-1)$ Lie algebra log $SU(n)$ of $SU(n)$, has $r$ fundamental representations. Each of the $r$ vertices in the $L_r$-Dynkin diagram is uniquely associated to a root of the Lie algebra and then also to that fundamental representation whose heighest weight
is not orthogonal to this root. The highest weight of any irreducible $L_r$-representation is a unique positive integer linear combination of the highest weights of the fundamental representations $[\mathbf{1}, \mathbf{3}, \mathbf{8}]$.

E.g. the $(n-1)$ fundamental representations for the Lie algebra $\log \mathbf{SU}(n)$ and, with the same notation, for the group $\mathbf{SU}(n)$

$$\begin{cases} [0, \ldots, 0, 1, 0, \ldots, 0], & r = 1, \ldots, n-1 \end{cases}$$

acting on complex $n\choose r$-dimensional vector spaces, combine all irreducible representations $[2J_1, \ldots, 2J_{n-1}]$ using natural numbers $2J_r$. The $(n-1)$ fundamental representations reflect the $(n-1)$ nontrivial unit roots in $I(n)$. As representation of the covering group $\mathbf{SU}(n)$ they are faithful for $\mathbf{SU}(n)/I(r(n))$ with $I(r(n))$ defined in section 1.

In the additive monoid for the equivalence classes of irreducible representations for the group $\mathbf{SU}(n)$

$$\text{repSU}(n) = \{[2J_1, \ldots, 2J_{n-1}] \mid 2J_r \in \mathbb{N} \} \cong \mathbb{N}^{n-1}$$

there are submonoids for the irreducible representations of the quotient groups $\mathbf{SU}(n)/I(d(n))$. Any representation defines its $n$-ality (triality $[\mathbf{3}]$ for $\mathbf{SU}(3)$) by

$$r = \left(\sum_{s=1}^{n-1} s2J_s\right) \mod n \in \mathbb{Z}_n$$

This leads immediately to the submonoids $\text{repSU}(n)/I(r(n)) \subseteq \text{repSU}(n)$ for the representations of the locally isomorphic quotient groups.

The submonoid with trivial $n$-ality $r = 0$ characterizes the irreducible representations of the adjoint group $\mathbf{SU}(n)/I(n)$ where the isospin and colour gauge fields are members of for $n = 2$ and $n = 3$ resp.

To avoid misunderstandings with respect to the embeddings of the quotient group representation monoids $\text{repSU}(n)/I(r(n)) \subseteq \text{repSU}(n)$, exemplified with the rotation and the spin group $\text{repSO}(3) \subset \text{repSU}(2)$: Any $\text{SO}(3)$-representation can be considered as an $\text{SU}(2)$-representation, trivial for $I(2)$. However, the halfinteger $\text{SU}(2)$-representations, e.g. $[1]$, sometimes called ‘2-valued representations’, are not admitted as $\text{SO}(3)$-representations. By definition, a representation as a mapping has to be unique.

The positive factor $2J_r$ in the combinations of the $r$ fundamental representations of a simple Lie algebra $L_r$ is connected with the totally symmetric power of the $r$-th fundamental representations. Under the fundamental representations there are distinguished cyclic fundamental representations, maximally three independent ones. They generate by totally antisymmetric powers all fundamental representations. E.g. the simple Lie algebra $\log \mathbf{SU}(n)$ has one independent cyclic fundamental representation given by a representation $[0, \ldots, 0, 1, 0, \ldots, 0]$ where $r$ and $n$ are relatively prime, e.g. by the defining complex $n$-dimensional one $[1, 0, \ldots, 0]$. The other fundamental representations are isomorphic to the totally antisymmetrized powers of the defining
one
\[ \wedge^{r}[1,0,\ldots,0] = [0,\ldots,0,1,0,\ldots,0], \quad r = 1,\ldots,n-1 \]
\[ \wedge^n[1,0,\ldots,0] = [0,\ldots,0] \]

The relation to the centrum \( \mathbb{I}(n) \) is obvious, \( \exp \frac{2\pi i}{n} = \left( \exp \frac{2\pi i}{n} \right)^r \).

In general, for any simple Lie algebra \( L_r \), one can take as cyclic fundamental representations a subset of the representations at the maximally three 'loose ends' of its Dynkin diagram. 'Entering' the diagram step by step from one 'loose end', one finds all the other fundamental representations in the totally antisymmetric 1st, 2nd etc. power of the 'loose end' representation. If one encounters a double line - for the exceptional \( F_4 \), the orthogonal \( \log SO(2n+1) \) and symplectic \( \log Sp(2n) \) - or a branching vertex - for \( \log SO(2n) \) and the exceptional \( E_6,7,8 \) - one might have to stop the journey. For \( G_2 \) the 7-dimensional representation is cyclic.

Since the Dynkin diagrams for \( \log SU(n) \) have single lines only and no branching vertex, the diagram can be 'gobbled up' from any of the two loose ends.

The \((n-1)\) fundamental representations for \( SU(n) \) are cyclic representations for the quotient groups \( SU(n)/\mathbb{I}(r(n)) \), e.g.

\[
\begin{align*}
\text{rep } SU(4) : & \left\{ \begin{array}{l}
[1,0,0] \text{ or } [0,0,1] \text{ for } SU(4) \\
[0,1,0] \text{ for } SU(4)/\mathbb{I}(2)
\end{array} \right.
\text{rep } SU(6) : & \left\{ \begin{array}{l}
[1,0,0,0,0] \text{ or } [0,0,0,1,0] \text{ for } SU(6) \\
[0,1,0,0,0] \text{ or } [0,0,0,1,0] \text{ for } SU(6)/\mathbb{I}(2) \\
[0,0,1,0,0] \text{ for } SU(6)/\mathbb{I}(3)
\end{array} \right.
\end{align*}
\]

For the adjoint group \( SU(n)/\mathbb{I}(n) \), the real \((n^2-1)\)-dimensional representation is cyclic

\[
\begin{align*}
cyclic [2] & \in \text{rep } SU(2)/\mathbb{I}(2) \subset \text{rep } SU(2) \\
cyclic [1,0,\ldots,0,1] & \in \text{rep } SU(n)/\mathbb{I}(n) \subset \text{rep } SU(n), \; n \geq 3
\end{align*}
\]

### 3 Fundamental Standard Fermion Fields as Fundamental \( U(2 \times 3) \)-Representations

The equivalence classes of the irreducible representations of the abelian phase group \( U(1) \) are characterized by the integer winding numbers

\[
\text{rep } U(1) = \{ [z] \mid z \in \mathbb{Z} \}
\]

There are two fundamental representations \([\pm 1]\), the faithful defining ones, for the rank 1 group \( U(1) \) which combine all irreducible representations by positive integer multiples \( n[\pm 1] = [\pm n] \). They are realized for hypercharge \( U(1) \) by the lepton fields \( e, e^* \)

\[
\begin{array}{c|c}
 e^* & U(1) \\
\hline
 1 & \end{array} \quad \text{and} \quad \begin{array}{c|c}
 e & U(1) \\
\hline
 -1 & \end{array}
\]

fundamental \( y = \pm 1 \)
Correspondingly, \(2n\) fundamental representations will be defined for the full unitary group \(U(n)\)

\[
\begin{align*}
\{ \psi | 0, \ldots, 0, 1, 0, \ldots, 0 \}, & \quad r = 1, \ldots, n-1, \text{ and } [1]|0, \ldots, 0 \} \\
\{ -\psi | 0, \ldots, 0, 1, 0, \ldots, 0 \}, & \quad r = 1, \ldots, n-1, \text{ and } [-1]|0, \ldots, 0 \}
\end{align*}
\]

They are the antisymmetric powers of two cyclic representations

\[
\begin{align*}
\bigwedge^n [1]|0, \ldots, 0, 1, 0, \ldots, 0 & = [\psi]|0, \ldots, 0, 1, 0, \ldots, 0, r = 1, \ldots, n-1 \\
\bigwedge^n [-\psi]|0, \ldots, 0, 1, 0, \ldots, 0 & = [-1]|0, \ldots, 0, r = 1, \ldots, n-1
\end{align*}
\]

and act on complex \(\binom{n}{r}\)-dimensional vector spaces. All irreducible \(U(n)\)-representations are given by

\[
\begin{align*}
\text{rep} \ U(n) = \{ |y|2J_1, \ldots, 2J_{n-1} \} & \quad |y| = z + \frac{1}{n} \sum_{r=1}^{n-1} r 2J_r, \quad z \in \mathbb{Z}, \quad 2J_r \in \mathbb{N} \\
|y|2J_1, \ldots, 2J_{n-1}^* & = [-y]|2J_{n-1}, \ldots, 2J_1
\end{align*}
\]

with a rational winding number \(y \in \frac{1}{n}\mathbb{Z}\) for \(U(1_n)\) in \(U(n)\).

The \(U(n)\)-representation monoid can be embedded with a hypercharge renormalization (multiplication with \(n\)) as a submonoid of the representation monoid for the direct product group, as a true submonoid for the nonabelian case

\[
\text{rep} \ U(n) \leftrightarrow \text{rep} \ [U(1) \times SU(n)] = \text{rep} \ U(1) \times \text{rep} \ SU(n)
\]

The two cyclic fundamental \(U(n)\)-representations combine the cyclic one for representations of the adjoint group \(U(n)/\text{centr} \ U(n) \cong SU(n)/I(n)\)

for \(SU(2)\) : \([\frac{1}{2}|1] \otimes [-\frac{1}{2}|1] \cong [0|2]\) \\
\(n \geq 3\) : \([\frac{1}{n}|1, 0, \ldots, 0] \otimes [-\frac{1}{n}|0, \ldots, 0, 1] \cong [0||1, 0, \ldots, 0, 1]

E.g. for \(U(2)\), there are four fundamental representations

\[
\exp i\left[\frac{1}{2}\alpha \mathbf{l}_2 + \frac{1}{2}\beta_a \tau^a\right] \quad \text{and} \quad \exp i\left[-\frac{1}{2}\alpha \mathbf{l}_2 - \frac{1}{2}\beta_a \tau^a\right]
\]

which are realized for hyperisospin by the \(2 \cdot 2\) fundamental lepton fields of the
standard model with isospin $T = \frac{1}{2}, 0$

$$\text{rep } U(2) = \{[y||2T] \mid y = z + T, \ z \in \mathbb{Z}, \ 2T \in \mathbb{N}\}$$

| $U(2)$ | $U(2)$ |
|--------|--------|
| $1^*$  | $\frac{1}{3}|1,1]$ |
| $e^*$  | $1|0,0]$ |

fundamental $y = \pm \frac{r}{2}, \ r = 1, 2$

The fermion isosinglet fields of the standard model realize the $2 \cdot 3$ fundamental $U(3)$-representations with colour triplet, antitriplet and singlet

$$\text{rep } U(3) = \{[y||2C_1, 2C_2] \mid y = z + \frac{2(C_1-C_2)}{3}, \ z \in \mathbb{Z}, \ 2C_{1,2} \in \mathbb{N}\}$$

| $U(3)$ | $U(3)$ |
|--------|--------|
| $d$    | $-\frac{1}{3}|1,0]$ |
| $u^*$  | $-\frac{1}{3}|0,1]$ |
| $e$    | $-1|0,0]$ |
| $d^*$  | $\frac{1}{3}|0,1]$ |
| $u$    | $\frac{1}{3}|1,0]$ |
| $e^*$  | $1|0,0]$ |

fundamental $y = \pm \frac{r}{3}, \ r = 1, 2, 3$

Fundamentality for representations of the group $U(2 \times 3)$ coincides with standard model fundamentality: The totally antisymmetric powers of the two defining cyclic fundamental $U(2 \times 3)$-representations

$$[y||2T; 2C_1, 2C_2] = \begin{cases} 
[+\frac{1}{3}|1;1,0] \\
[-\frac{1}{3}|1;0,1] 
\end{cases}$$

give rise to the $2 \cdot 5$ fundamental $U(2 \times 3)$-representations, which are realized by the $2 \cdot 5$ fundamental fermion fields in the standard model (section 1)

$$\text{rep } U(2 \times 3) = \{[y||2T; 2C_1, 2C_2] \mid y = z + T + \frac{2(C_1-C_2)}{3}, \ z \in \mathbb{Z}, \ 2T, 2C_{1,2} \in \mathbb{N}\}$$

| $U(2 \times 3)$ | $U(2 \times 3)$ |
|----------------|----------------|
| $q$            | $-\frac{1}{3}|1;1,0]$ |
| $d^*$          | $-\frac{1}{3}|0;0,1]$ |
| $l^*$          | $\frac{1}{3}|1;0,0]$ |
| $u$            | $\frac{1}{3}|0;0,1]$ |
| $e^*$          | $1|0;0,0]$ |
| $q^*$          | $\frac{1}{3}|1;1,0]$ |
| $d$            | $-\frac{1}{3}|0;0,1]$ |
| $l$            | $\frac{1}{3}|1;0,0]$ |
| $u^*$          | $-\frac{1}{3}|0;0,1]$ |
| $e$            | $-1|0;0,0]$ |

fundamental $y = \pm \frac{r}{6}, \ r = 1, 2, 3, 4, 6$

The 5th power with hypercharge number $\frac{5}{6}$ is not called fundamental since it is a positive linear combination of the 2nd and 3rd one

$$[\frac{5}{6}|1;0,1] = [\frac{1}{2}|0;0,1] + [\frac{1}{3}|1;0,0]$$
The irreducible $U(2 \times 3)$-representations can be embedded by hypercharge renormalization $6y$ as a true submonoid in the representations of the direct product group

$$\text{rep } U(2 \times 3) \leftrightarrow \text{rep } [U(1) \times SU(2) \times SU(3)]$$

which should be seen in parallel to the submonoids for grand unified schemes, e.g. $\text{rep } SU(5) \subset \text{rep } [U(1) \times SU(2) \times SU(3)]$. The hypercharge number, up to an integer determined by the nonabelian properties, leads to the fractionality condition for the standard model fields as given in section 1.

### 4 Integer Charges and Fractional Hypercharges

The irreducible $U(1)$-representations as $U(1)$-endomorphisms

$$U(1) \rightarrow U(1), \quad \exp i\alpha \leftrightarrow \exp iy\alpha \quad [y] \in \text{rep } U(1) \Rightarrow y = z \in \mathbb{Z}$$

have to use integer winding numbers (charge numbers) $y \in \mathbb{Z}$ because of $\exp i(\alpha + z2\pi) = \exp i\alpha$. Since the $U(1)_n$ subgroup in $U(n)$ is synchronized with $SU(n)$, fractional hypercharge numbers are possible for $U(n)$-representations

$$[y][2J_1, \ldots 2J_r-1] \in \text{rep } U(n) \Rightarrow y = z + \frac{1}{n} \sum_{r=1}^{n-1} r2J_r \in \frac{1}{n}\mathbb{Z}$$

$$\exp 2\pi iy \in I(n) \cong U(1)_n \cap SU(n)$$

If, somewhere, the group $U(1)$ arises as a direct factor, e.g. in a direct product standard model group $U(1) \times SU(2) \times SU(3)$ or in an asymptotic particle symmetry Cartan subgroup of $U(2 \times 3)$ (section 6), its winding numbers have to be integer. For a direct product standard model group the hypercharge numbers of section 1 have to be renormalized to integers $6y$. $U(n)$-representations with trivial $n$-ality, like the fundamental right lepton field $e$, have integer $U(1)$-winding numbers from the beginning. To attribute integer charge numbers to fields with fractional hypercharge numbers, like to the fundamental left lepton field $l$ or to the quark fields $q, u, d$ (section 1), these field have to be modified, as done in the Higgs and confinement induced rearrangement of the standard model fields to particle related fields (section 6).

### 5 Normalization of $U(n)$-Lie Algebras

The central connection of the internal groups relates to each other also the normalization of their invariant metrics which arise in the gauge field coupling constants.
All complex representations of a real Lie group come with an invariant conjugation, e.g. the SU(n) and U(n)-representations spaces with a defining definite scalar product

\[ V_n \times V_n \rightarrow \mathbb{C}, \quad \langle g \cdot v | g \cdot w \rangle = \langle v | w \rangle \quad \text{for all } g \in \text{U}(n) \]
e.g. with orthonormal bases \( \{ e^A \}^{A=1}_n \) and \( \{ e^*_A \}^{A=1}_n \) for \( V_n \) and its dual space \( V_n^* \) resp.

\[ \langle e^A | e^B \rangle = \delta^{AB}, \quad \langle e^*_A | e^*_B \rangle = \delta^{AB} \]

A scalar product for a representation defines by its powers a scalar product for the product representations. The scalar product of the cyclic \( \text{U}(n) \)-representations \( \frac{1}{n} | 1, 0, \ldots, 0 \rangle \) on \( V_n \) and \( \frac{1}{n} | 0, 0, \ldots, 1 \rangle \) on \( V_n^* \) induces a scalar product to the product space

\[ (V_n \otimes V_n^*) \times (V_n \otimes V_n^*) \rightarrow \mathbb{C}, \quad \langle e^A \otimes e^*_B | e^C \otimes e^*_D \rangle = \delta^{AC} \delta_{BD} \]

\( V_n \otimes V_n^* \cong \mathbb{C}^{n^2} \) is the \( n^2 \)-dimensional defining space for the group \( \text{U}(n) \) and its Lie algebra \( \text{log} \text{U}(n) \). A reordering gives the metric for both the abelian Lie subalgebra \( \text{log} \text{U}(1)_n \) with basis \( \{ i1_n \} \) and the nonabelian one \( \text{log} \text{SU}(n) \) with a basis \( \{ i\tau(n) \} \) of generalized Pauli matrices (Pauli matrices proper \( \{ i\tau \} \) for \( n = 2 \), Gell-Mann matrices \( \{ i\lambda \} \) for \( n = 3 \) etc.)

\[
V_n \otimes V_n^* \supset \text{log} \text{U}(n) = \text{log} \text{U}(1)_n \oplus \text{log} \text{SU}(n)
\]

\[
\delta^{AC} \delta_{BD} = \begin{cases} 
\delta^{A}_B \delta^{C}_D & \text{for} \ \text{U}(1) \\
\frac{1}{2} \delta^{A}_B \delta^{C}_D + \frac{1}{n} \tau^A_B \tau^C_D & \text{for} \ \text{U}(2) \\
\frac{1}{2} \delta^{A}_B \delta^{C}_D + \frac{1}{12} \lambda^A_B \lambda^C_D & \text{for} \ \text{U}(3) \\
\frac{1}{n} \delta^{A}_B \delta^{C}_D + \frac{1}{n(n+1)} \tau(n)_B^A \tau(n)_D^C & \text{for} \ \text{U}(n), \ n \geq 2
\end{cases}
\]

This involves as relative normalization of both Lie subalgebras

\[ n \geq 2 : \quad \frac{||\text{log SU}(n)||^2}{||\text{log U}(1)_n||^2} = \frac{\langle \tau(n)|\tau(n) \rangle_{(1_n,1_n)}}{\langle 1_n|1_n \rangle} = \frac{n(n+1)}{n} = n + 1 \]

The absolute normalization is not determined.

The coupling constants in the gauge field-current couplings of the standard model

\[
g_1 A J(1) + g_2 B J(2) + g_3 G J(3)
\]

\[
J(1) = \frac{1}{2} [q^*1_6 q - 2d^*1_3 d - 3l^*1_2 l + 4u^*1_3 u - 6e^*e]
\]

\[
J(2) = \frac{1}{2} [q^*\tau \otimes 1_3 q + l^*\lambda]
\]

\[
J(3) = \frac{1}{2} [q^*1_2 \otimes \lambda q + d^*\lambda d + u^*\lambda u]
\]

have as relative normalizations in a \( \text{U}(2 \times 3) \)-formulation

\[ \langle 1|1 \rangle : \langle \tau|\tau \rangle : \langle \lambda|\lambda \rangle = g_1^2 : g_2^2 : g_3^2 = 1 : 3 : 4 \]

leading to the tree value for the Weinberg angle

\[ \tan^2 \varphi = \frac{g_2^2}{g_3^2} = \frac{1}{3}, \quad \sin^2 \varphi = \frac{1}{4} \]
6 Definition of the Electromagnetic Group

In the following, particles are defined with Wigner [13] as irreducible unitary representations of the Poincaré group, which includes the representations of the translations. With such a definition, the quark fields \( q, u, d \), parametrizing strong interactions and hadrons, give not rise to asymptotic quark particles if they are confined and, therewith, if they cannot develop translation degrees of freedom.

The transition from the standard model fields to the standard model particles requires a special basis (ground state) to measure the eigenvalues for the particle properties. It is the definition of a symmetry to distinguish no special basis, e.g. it does not make sense to distinguish an upper and lower component in the left lepton field \( l \) with \( U(2) \)-symmetry. However, a basis with upper and lower component \( \left( \nu_{eL} \right) \) is necessary to define the neutrino and electron particle for a lepton field.

The discrete \( \mathbb{I}(2) \times \mathbb{I}(3) \)-correlation of the hypercharge group with the isospin and colour group influences strongly the symmetry breakdown structure which establishes eigenvector bases for the particles. The transition from the fields with \( U(2 \times 3) \)-properties to particles, e.g. \( l \mapsto \left( \nu_{eL} \right) \), has to take into account a maximal subgroup of diagonalizable operators as subgroup of \( U(2 \times 3) \). For a dynamics with a degenerate ground state as in the standard model, the asymptotic space has to take care, in addition, of the ground state frozen symmetries, parametrized in the standard model by the ground state properties of a Higgs field \( H \) as a fundamental \( U(2 \times 3) \)-representation \( [-\frac{1}{2}||1;0,0]\), trivial for colour \( SU(3) \) and cyclic \( [-\frac{1}{2}||1] \) for \( U(2) \). From the internal real 12-dimensional Lie group operators \( \hat{U}(2 \times 3) \) for interactions, there remains only an abelian electromagnetic \( U(1) \)-symmetry for particles.

In contrast to the fields in the basic dynamics, all particles have trivial isospin symmetry, therefore, the electron and its neutrino can have different mass and different electromagnetic charge, and - if confinement is true - all particles behave trivially with respect to colour transformations. Since the hypercharge \( U(1) \)-subgroup in \( U(2 \times 3) \) is nontrivially synchronized both with isospin \( SU(2) \) using \( \mathbb{I}(2) \) and with colour \( SU(3) \) using \( \mathbb{I}(3) \), the definition of an abelian electromagnetic symmetry group \( U(1) \subset U(2 \times 3) \) with nontrivial hypercharge contributions has to sever the relation to both nonabelian factors.

The distinction of an electromagnetic charge \( U(1) \) in the electroweak hyperisospin \( U(2) \), orthogonal to the Higgs ground state expectation value defined direction

\[
\langle H \rangle = \left( \frac{0}{m_0} \right) \neq 0, \quad \exp i\alpha \frac{1+\tau_3}{2} \in U(1)_+ \subset U(2)
\]
determines uniquely an electromagnetic group \( U(1) \) only for colour singlets \([z+T||2T;0,0]\]

\[
U(2) \cong \frac{U(1) \times SU(2)}{I(2)} \xrightarrow{\langle H \rangle \neq 0} U(1)_+ \cong U(1)
\]

\( \langle H \rangle \neq 0 \) does not cut the \( \mathbb{I}(3) \)-relation between hypercharge \( U(1) \) and colour \( SU(3) \) in \( U(2 \times 3) \). The electroweak breakdown by the Higgs field with the
definition of the electromagnetic charge $\mathbf{U}(1)$ for particles makes sense only together with a colour confinement

\[
\mathbf{U}(2 \times 3) \xrightarrow{(H) \neq 0} \mathbf{U}(1)
\]

A colour confinement serves simultaneously two things: It trivializes $\mathbf{SU}(3)$ and it allows the definition of an electromagnetic $\mathbf{U}(1)$-group. In a $\mathbf{U}(3)$-symmetric dynamics, quark triplet and antitriplet fields have both nontrivial hypercharge and colour properties, to attribute to them a unique electromagnetic charge $\mathbf{U}(1)$ does not make sense.

A Cartan Lie subalgebra defines eigenvectors with eigenvalues for the ‘infinitesimal’ Lie algebra action. The exponential of a Cartan Lie algebra in the Lie group under consideration gives an abelian subgroup. A product of two abelian groups with nontrivial common subgroup allows no independent measurement of both factors. Therefore, a Cartan subgroup of a Lie group will be defined as a maximal subgroup of a Cartan Lie algebra exponential which can be written as a direct product of 1-dimensional Lie groups. This definition of a Cartan subgroup as a maximal diagonalizable direct product group is convenient for the unitary groups, there exist other definitions [11]. The dimension of a Cartan subgroup need not coincide with the rank of a Lie algebra and, therewith, with the dimension of a Cartan Lie algebra.

The exponentials of Cartan Lie algebras for $\mathbf{SU}(n)$ and $\mathbf{U}(n)$ are real $(n-1)$ and $n$-dimensional tori $\mathbf{U}(1)^{n-1}$ and $\mathbf{U}(1)^{n}$ resp. With a diagonal $\mathbf{SU}(2)$-Cartan subgroup

\[
\mathbf{SU}(2) \supset \mathbf{U}(1) \ni \exp \frac{i}{2} \beta_3 \tau^3
\]

one obtains diagonal $\mathbf{SU}(n)$-Cartan subgroups as exponentials

\[
\mathbf{SU}(n) \ni \exp \sum_{r=2}^{n} \beta_r \tau(n, r) = \prod_{r=2}^{n} \mathbf{U}(1)_{0r}, \quad \mathbf{U}(1)_{0r} \cong \mathbf{U}(1)
\]

for $\mathbf{SU}(2)$: $\tau(2, 2) = \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

for $\mathbf{SU}(3)$: $\tau(3, 2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\tau(3, 3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Cartan subgroups of $\mathbf{U}(n) = \mathbf{U}(1_n) \circ \mathbf{SU}(n)$ have to take care of the central $\mathbb{I}(n)$-correlation, e.g. the $\mathbf{U}(1)$-isomorphic subgroups $\exp \frac{i}{2} \alpha_1 \mathbf{1}_3 \in \mathbf{U}(1_2)$ and $\exp \frac{i}{2} \beta_3 \tau^3 \in \mathbf{U}(1)_0$ form a product subgroup $\mathbf{U}(1_2) \circ \mathbf{U}(1)_0 \subset \mathbf{U}(2)$, but with the nontrivial intersection $\exp i \pi \tau^3 = \exp i \pi \tau^3 = -\mathbf{1}_2$, no direct product $\mathbf{U}(1) \times \mathbf{U}(1)$ in $\mathbf{U}(2)$. A $\mathbf{U}(n)$-Cartan subgroup with appropriate Lie parameters is given by

\[
\mathbf{U}(n) \ni \exp i \sum_{r=1}^{n} \alpha_r \mathbf{1}(n, r) = \prod_{r=1}^{n} \mathbf{U}(1)_r, \quad \mathbf{U}(1)_r \cong \mathbf{U}(1)
\]
using a maximal system of \( n \) orthogonal projectors for the Lie algebra \( \log U(n) \)

\[
\{1(n,r) \mid r = 1, \ldots, n\} \quad \text{with} \quad \begin{cases} 
1(n,r) \circ 1(n,s) = \delta_{rs} 1(n,r) \\
\sum_{r=1}^{n} 1(n,r) = 1_n
\end{cases}
\]

e.g. with matrices \( 1(n,r) \) having only one nontrivial diagonal entry

for \( U(2) \):
\[
1(2,1) = \begin{pmatrix} 1+i \tau_3 \\ 0 \end{pmatrix}, \quad 1(2,2) = \begin{pmatrix} 1-i \tau_3 \\ 0 \end{pmatrix}
\]

for \( U(3) \):
\[
1(3,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad 1(3,2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad 1(3,3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

Both for \( SU(n) \) and \( U(n) \) the rank \((n-1)\) and \( n \) resp. gives also the Cartan subgroup dimension.

This is different for the rank 4 standard group \( U(2 \times 3) = U(1_6) \circ [SU(2) \otimes 1_3 \times 1_2 \otimes SU(3)] \) where the Cartan subgroups with direct \( U(1) \)-factors have to be looked for in the real 4-dimensional exponential of a Cartan Lie algebra

\[
\exp \left( \sum \frac{i \alpha_1}{2} \sigma_3 \otimes 1_3 + \frac{i \lambda_3}{2} \gamma_8 \otimes 1_3 \right) \in U(1)^4
\]

In a Cartan subgroup, hypercharge \( \exp \frac{i \theta}{6} 1_6 \) has to be correlated either with isospin or with colour. Correspondingly, there arises two systems - one with two and one with three orthogonal projectors for \( U(2) \) and \( U(3) \) resp.

hyperisospin: \[
\begin{cases}
\text{projectors:} & \{ 1(2,r) \otimes 1_3 \mid r = 1, 2 \} \\
\text{Cartan subgroup:} & \left( \exp i \alpha_3 \right) \otimes 1_3 \\
& \in U(1_+) \times U(1_-)
\end{cases}
\]

hypercharge-colour: \[
\begin{cases}
\text{projectors:} & \{ 1_2 \otimes 1(3,r) \mid r = 1, 2, 3 \} \\
\text{Cartan subgroup:} & 1_2 \otimes \left( \exp i \alpha_1 \right) \\
& \in U(1_1) \times U(1_2) \times U(1_0)
\end{cases}
\]

The two Cartan subgroups have dimension two and three resp.

For the characterization of the asymptotic particle states for the fields in the \( U(2 \times 3) \)-symmetric standard model as eigenvectors with eigenvalues of a maximal diagonal subgroup, one has to decide either for \( U(1)_1 \times U(1)_2 \times U(1)_3 \subset U(3) \) leading to isospin trivial particles with possibly nontrivial hypercharge-colour properties or for \( U(1)_+ \times U(1)_- \subset U(2) \) leading to colour singlet particles with possibly nontrivial hyperisospin properties as done in the standard model. By the Higgs field colour singlet property, the Higgs mechanism decides for the Cartan subgroup in hyperisospin \( U(2) \), therein it establishes a basis with a remaining \( U(1)_+ \times U(1)_- \) and trivializes, in addition, one direct factor \( U(1)_- \) which leaves nontrivially only the electromagnetic \( U(1)_+ \)-symmetry

\[
\begin{align*}
U(2 \times 3) & \xrightarrow{\text{confinement}} 3 \bigwedge U(2 \times 3) \cong U(2) \\
U(2) & \xrightarrow{\langle H \rangle \neq 0} U(1)_+ \times U(1)_- \cong \frac{U(1)_+ \times U(1)_-}{U(1)_-} \cong U(1)
\end{align*}
\]
7 The External-Internal Correlation

In addition to the internal correlation of hypercharge with isospin-colour, the standard dynamics shows in the Yukawa interaction

\[(\mu_e e^\dagger l + \mu_u q^\dagger u + \mu_d d^\dagger q)H \text{ h.c. with Yukawa couplings } \mu_{e,u,d} \in \mathbb{R}\]

an internal-external correlation for chirality (left and right handedness) and hyperisospin. The Higgs field, which connects left and right handed fields, is a hyperisospin doublet.

Also the Lorentz transformations, defined by the covering group \(\text{SL}(\mathfrak{I}^2)\) with center \(\text{centr} \text{SL}(\mathfrak{I}^2) \cong \mathbb{I}(2)\) of the orthochronous group \(\text{SO}^+(1,3)\), and being a subgroup of the unimodular group \(\lambda \in \text{UL}(2) \subset \text{GL}(\mathfrak{I}^2)\) with \(|\det \lambda| = 1\), have an \(\mathbb{I}(2)\)-correlation with the external phase group \(\text{U}(1/2) \subset \text{GL}(\mathfrak{I}^2)\).

\[
\text{UL}(2) = \text{U}(1/2) \circ \text{SL}(\mathfrak{I}^2), \quad \left\{ \begin{array}{l} \text{U}(1/2) \cap \text{SL}(\mathfrak{I}^2) \cong \mathbb{I}(2) \\
\text{UL}(2)/\text{U}(1/2) \cong \text{SL}(\mathfrak{I}^2)/\mathbb{I}(2) \cong \text{SO}^+(1,3)\end{array}\right.
\]

The finite dimensional irreducible \(\text{UL}(2)\)-representations

\[
\text{rep } \text{UL}(2) = \{ |c||2J_L|2J_R\} \mid c = z + J_L - J_R, \ z \in \mathbb{Z}, \ 2J_{L,R} \in \mathbb{N} \}
\]

are characterized by left and right spin winding numbers \(2J_{L,R}\) and, in addition, a chirality number \(c\), characterizing the representation of \(\text{U}(1/2) \subset \text{UL}(2)\). Integer spin fields have integer chirality numbers \(c\), halfinteger spin comes with halfintegral chirality.

From the \(\text{U}(2 \times 3)\) and \(\text{UL}(2)\)-invariant Yukawa interaction, the \(\text{UL}(2)\)-properties can be attributed to the fundamental fields as follows

|  | \(\text{UL}(2)\) |
|---|---|
| l | \(\pm z + z_l | 1\rangle 0\rangle\) |
| e | \(\pm z + z_l | 0\rangle 1\rangle\) |
| q | \(\pm z + z_q | 0\rangle 1\rangle\) |
| d | \(\pm z + z_q | 0\rangle 1\rangle\) |
| u | \(\pm z + z_q | 0\rangle 1\rangle\) |
| H | \((-1) | 0\rangle 0\rangle\) |
| A, B, G | \(0 | 1\rangle 1\rangle\) |

With chirality number \(c = -1\) fixed for the Higgs field, the chirality numbers of quark and lepton fields are determined up to integers.

A first suggestion for a central external-internal correlation could be an isospin-spin correlation as expressed by the quotient group \(\frac{\text{SL}(\mathfrak{I}^2) \times \text{SU}(2)}{\mathbb{I}(2)}\), leading to the correlation half integer spin with half integer isospin and integer spin with integer isospin as seen in the lepton and quark isodoublet fields \(l\) and \(q\) or the gauge fields \(A, B\) and \(G\) resp. However, the integer isospin, half integer spin fermion fields \(e, u\) and \(d\) and the Lorentz scalar isodoublet Higgs field \(H\) contradict such a suggestion. The external-internal correlation is different.
The internal hypercharge $U(1) \subset U(2 \times 3)$ and the external chirality $U(1) \subset UL(2)$, have to use the same phase - both $U(1)$ coincide in the Higgs field with the vanishing combination $c - 2y$. This gives with the choice

$$z_l = -2, \quad z_q = 0$$

as $U(1)$-properties of the standard model fields

| field     | $U(1)_{\text{ext}}$ | $U(1)_{\text{int}}$ | $U(1)_{\text{ferm}}$ |
|-----------|---------------------|---------------------|---------------------|
| $e$       | $-\frac{3}{2}$     | $-1$               | $-\frac{1}{2}$     |
| $l$       | $-\frac{3}{2}$     | $-\frac{5}{2}$    | $-\frac{1}{2}$     |
| $d$       | $-\frac{1}{2}$     | $-\frac{1}{2}$    | $\frac{1}{2}$      |
| $q$       | $\frac{1}{2}$      | $\frac{1}{2}$     | $\frac{1}{2}$      |
| $u$       | $\frac{1}{2}$      | $\frac{1}{2}$     | $\frac{1}{2}$      |
| $H$       | $-1$               | $-\frac{3}{2}$    | $0$                |
| $A, B, G$ | $0$                | $0$                | $0$                |

The correlation of external and internal $U(1)$ defines a fermion number group $U(1)$

$$U(1)_{\text{ext}} \subset UL(2) \quad U(1)_{\text{int}} \subset U(2 \times 3) \quad \Rightarrow U(1)_{\text{ferm}} \simeq \frac{U(1)_{\text{ext}} \times U(1)_{\text{int}}}{U(1)}$$

$$f = c - 2y = \begin{cases} -\frac{1}{2} & \text{for lepton fields } e, l \\ +\frac{1}{6} & \text{for quark fields } d, q, u \end{cases}$$

For the lefthanded fields $l, q$, the fermion number $U(1)$ coincides with the hypercharge $U(1)$, i.e. $c = 3f = 3y$.

Taking into account the identification of the external and internal phase group, the symmetry group, faithfully represented in the standard model, shows three central correlations - the discrete internal one by $I(2) \times I(3)$, the discrete external one by $I(2)$ and, finally, the continuous external-internal one by $U(1)$

$$\frac{U(1)_{\text{ext}} \times U(1)_{\text{int}} \times SL(2) \times SU(2) \times SU(3)}{U(1)_{\text{ext}} \times I(2) \times I(2) \times I(3)} \simeq \frac{UL(2) \times U(2 \times 3)}{U(1)}$$

It’s as complicated!

The representations of the external-internal group by the standard model fields are summarized as follows

| field           | $UL(2) \circ U(2 \times 3)$ |
|-----------------|-----------------------------|
| right lepton    | $e$                         |
| left lepton     | $l$                         |
| right down quark| $d$                         |
| left quark      | $q$                         |
| right up quark  | $u$                         |
| Higgs           | $H$                         |
| hypercharge     | $A$                         |
| isospin gauge   | $B$                         |
| colour gauge    | $G$                         |
8 The Complication of the Standard Model Group

From an esthetical standpoint, debatable of course, the external-internal group \( \mathbb{U}(1) \times \mathbb{SL}(\mathfrak{C}^2) \times \mathbb{U}(1) \times \mathbb{SU}(2) \times \mathbb{SU}(3) \) with five direct factors (chirality, Lorentz symmetry, hypercharge, isospin, colour) may look rather unnatural and complicated, even more its quotient group above with the four central correlations. Grand unified theories hope for a correlation of the direct factors via ‘nondiagonal’ supplements in larger simple groups, like \( \mathbb{SU}(5) \) or \( \mathbb{SO}(10) \) for the internal symmetry, where, however, if no additional benefits arise, the symmetry breakdown mechanism for the asymptotic particles with a leftover electromagnetic \( \mathbb{U}(1) \) destroys the simplification hoped for.

The fundamental standard model fields are rather close to their associated asymptotic particles. With the important exceptions of the hadrons, requiring colour confinement, and the gauge fields related particles, requiring gauge invariance, there is even a bijective correspondence between interaction parametrizing fields and particle fields. This is similar to the quantum mechanical harmonic oscillator where the position-momentum operators \((x,p)\) are linearily and bijectively related to the energy eigenstates defining creation-annihilation pair \((u,u^*)\) with \(u = \frac{x+i\mu}{\sqrt{2}}\). For a quantum mechanical dynamics with a Hamiltonian, not written with energy eigenvectors, like for the non-relativistic hydrogen atom, the connection between the dynamics building operators \((x,p)\) and the energy eigenstates generating operators are nonlinear and may be rather complicated. Could it be that the standard model group complication arises by its close relationship to the asymptotic particles? Is the complicated standard model symmetry a consequence of a linearization with many energy-momentum and electromagnetic eigenfields to approximate a nonlinear dynamics, which can be formulated with a few quantum fields implementing a smaller symmetry? Does the complicated internal group of the standard model express not a subgroup of a larger symmetry group, but a representation spectrum of a smaller group? I will close this paper with some speculations for such a small unification, in some sense orthogonal to those aiming at grand unification schemes.

As seen in the hypercharge-isospin-colour correlation, the full unitary group \( \mathbb{U}(n) \) has a root structure with respect to \( \mathbb{U}(1) \) and the totally antisymmetric product in analogy to the cyclotomic group \( \mathbb{I}(n) \) as its centrum having a root structure with respect to the trivial group \( \{1\} \) and the number product

\[
\mathbb{\wedge}^n \mathbb{U}(n) \cong \mathbb{U}(1), \quad (\mathbb{I}(n))^n \cong \{1\}
\]

Obviously, such a structure is used for the baryonic hadronization \(\mathbb{\wedge}^3 q \cong \mathbb{N}\) with the quarks as ‘cubic roots’ of the nucleon.

For relatively prime \(n, m \in \mathbb{N}\), one has the root structure for \(\mathbb{U}(n \times m)\) as defined as subgroup of \(\mathbb{U}(nm)\)

\[
\mathbb{\wedge}^m \mathbb{U}(n \times m) \cong \mathbb{U}(n), \quad (\mathbb{I}(n) \times \mathbb{I}(m))^m \cong \mathbb{I}(n)
\]
Such a property holds for the centrally connected standard model group $U(2 \times 3)$, not, however, for the direct product group $U(1) \times SU(2) \times SU(3)$ since $\bigwedge m U(1) \cong \{1\}$ for $m \geq 2$.

If there is a dynamics underlying the standard model, given by an invariant of the full unitary group $U(2)$ and built with the defining representation, carried by fermion fields

\[
\psi, \psi^* \cong [\frac{1}{2}||1] \in \text{rep } U(2) \\
\psi \wedge \psi, \psi^* \wedge \psi^* \cong [\frac{1}{2}||0] \in \text{rep } U(1) \\
\psi \wedge \psi \otimes \psi^* \wedge \psi^* \cong [0||0] \in \text{rep } \{1\}
\]

it may give rise to energy-momenta eigenstates in the basic field products, e.g.

\[
\psi \otimes \psi^* \wedge \psi^* \cong [\frac{1}{2}||1] \in \text{rep } U(2)
\]

The occurrence of energy-momentum eigenstates, created by the original cyclic $\psi$ and its product $\psi \otimes \psi^* \wedge \psi^*$, may be parametrized by introducing two asymptotically oriented fields, related to leptons and quarks, the quarks carrying an additional $SU(3)$ as a custodian symmetry which expresses their cubic root origin and distinguishes them from the leptons

\[
\psi \sim 1 \cong [-\frac{1}{2}|1; 0, 0] \in \text{rep } U(2) \\
\psi \otimes \psi^* \wedge \psi^* \sim \bigwedge q \cong [\frac{1}{2}|1; 0, 0] \in \text{rep } U(2) \\
q \cong [\frac{1}{6}|1; 1, 0] \in \text{rep } U(2 \times 3)
\]

With the introduction of a spectrum describing group for the Fermi fields, an asymptotic parametrization for energy-momentum eigenstates also in the bosonic adjoint $U(2)/U(1_2)$-representations might be necessary, introducing hypercharge, isospin and colour fields

\[
\psi \otimes \psi^* \sim A, B \cong [0|0; 0, 0], [0|2; 0, 0] \in \text{rep } U(2) \\
G \cong [0|0; 1, 1] \in \text{rep } U(2 \times 3)
\]

Obviously with $\bigwedge U(2 \times 3) \cong U(2)$, the basic representation structure is embedded, $\text{rep } U(2) \hookrightarrow \text{rep } U(2 \times 3)$. The common origin of all the particle oriented fields, introduced for the linearization of the basic dynamics, remains visible in the hypercharge $U(1)$-relation both to the centrum $\mathbb{I}(2)$ of isospin $SU(2)$ and to the centrum $\mathbb{I}(3)$ of the colour symmetry $SU(3)$.

A substantiation of such speculations, as sketched on the group theoretical representation level only, even without discussing the external-internal correlation, requires the solution of a quantum field theoretical bound state problem which seems to be extremely difficult, as known from the attempts to determine the hadronic spectrum from quantum chromodynamics.
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