Lifting Automorphisms of Quotients of Adjoint Representations

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Abstract. Let $g_i$ be a simple complex Lie algebra, $1 \leq i \leq d$, and let $G = G_1 \times \cdots \times G_d$ be the corresponding adjoint group. Consider the $G$-module $V = \oplus r_i g_i$ where $r_i \in \mathbb{N}$ for all $i$. We say that $V$ is large if all $r_i \geq 2$ and $r_i \geq 3$ if $G_i$ has rank 1. In Quotients, automorphisms and differential operators, http://arxiv.org/abs/1201.6369 (2012), we showed that when $V$ is large any algebraic automorphism $\psi$ of the quotient $Z := V//G$ lifts to an algebraic mapping $\Psi: V \to V$ which sends the fiber over $z$ to the fiber over $\psi(z)$, $z \in Z$. (Most cases were already handled in J. Kuttler, Lifting automorphisms of generalized adjoint quotients, Transformation Groups 16 (2011), 1115–1135.) We also showed that one can choose a biholomorphic lift $\Psi$ such that $\Psi(gv) = \sigma(g)\Psi(v)$, $g \in G$, $v \in V$, where $\sigma$ is an automorphism of $G$. This leaves open the following questions: Can one lift holomorphic automorphisms of $Z$? Which automorphisms lift if $V$ is not large? We answer the first question in the affirmative and also answer the second question. Part of the proof involves establishing the following result for $V$ large. Any algebraic differential operator of order $k$ on $Z$ lifts to a $G$-invariant algebraic differential operator of order $k$ on $V$. We also consider the analogues of the questions above for actions of compact Lie groups.

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1. Introduction

Our base field is $\mathbb{C}$, the field of complex numbers. Let $G$ be a complex reductive group and $X$ a smooth affine $G$-variety. We denote the algebra of polynomial functions on $X$ by $\mathcal{O}(X)$. For the following, we refer to [Kra84], [Lun73] and [VP89]. By Hilbert, the algebra $\mathcal{O}(X)^G$ is finitely generated, so that we have a quotient variety $Z := X//G$ with coordinate ring $\mathcal{O}(Z) = \mathcal{O}(X)^G$. Let $\pi: X \to Z$ denote the morphism dual to the inclusion $\mathcal{O}(X)^G \subset \mathcal{O}(X)$. Then $\pi$ sets up a bijection between the points of $Z$ and the closed orbits in $X$. If $Gx$ is a closed orbit, then the isotropy group $H = G_x$ is reductive. The slice representation of $H$ at $x$ is its action on $N_x$ where $N_x$ is an $H$-complement to $T_x(Gx)$ in $T_x(X)$. Let $Z_{(H)}$ denote the points of $Z$ such that the isotropy groups of the
corresponding closed orbits are in the conjugacy class \((H)\) of \(H\). The \(Z(H)\) give a finite stratification of \(Z\) by locally closed smooth subvarieties. In particular, there is a unique open stratum \(Z(H)\), the \textit{principal stratum}, which we also denote by \(Z_{pr}\). We call \(H\) a \textit{principal isotropy group} and any associated closed orbit a \textit{principal orbit} of \(G\).

As shorthand for saying that \(X\) has finite principal isotropy groups we say that \(X\) has FPIG. If \(X\) has FPIG, then there is an open set of closed orbits and a closed orbit is principal if and only if the slice representation of its isotropy group is trivial. Set \(X_{pr} := \pi^{-1}(Z_{pr})\). We say that \(X\) is \(k\)-principal if \(X\) has FPIG and \(\text{codim} \ X \setminus X_{pr} \geq k\).

For \(Y\) an affine variety let \(\text{Aut}(Y)\) (resp. \(\text{Aut}_h(Y)\)) denote the automorphisms (resp. biholomorphisms) of \(Y\). If \(y \in Y\), we denote by \(\text{Aut}_h(Y, y)\) the germs of biholomorphisms of \(Y\) which fix \(y\).

Our main interest is in the following case. Let \(V = \bigoplus_{i=1}^d r_i g_i\) where the \(g_i\) are simple complex Lie algebras and \(r_i g_i\) denotes the direct sum of \(r_i\) copies of \(g_i\), \(r_i \in \mathbb{N}\). Let \(G_i\) denote the adjoint group of \(g_i\) and let \(G\) denote the product of the \(G_i\). We denote by \(0 \in Z := V//G\) the image of \(0 \in V\). Let \(\psi \in \text{Aut}_h(Z)\).

We say that \(\Psi \in \text{Aut}_h(V)\) is a lift of \(\psi\) if \(\pi \circ \Psi = \psi \circ \pi\). Equivalently, \(\Psi\) maps the fiber \(\pi^{-1}(z)\) to the fiber \(\pi^{-1}(\psi(z))\), \(z \in Z\). Let \(\sigma\) be an automorphism of \(G\). We say that \(\Psi\) is \(\sigma\)-equivariant if \(\Psi(gv) = \sigma(g)\Psi(v)\) for all \(v \in V\), \(g \in G\). We say that \(V\) is large if \(r_i \geq 2\) for all \(i\) and \(r_i \geq 3\) if \(g_i \cong \mathfrak{sl}_2\).

\textbf{Lemma 1.1.} \textit{The following are equivalent:}

1. \(V\) is large.
2. \(Z\) has no codimension one strata.
3. \(V\) is 2-principal.
4. Any \(\psi \in \text{Aut}_h(Z)\) preserves the stratification of \(Z\).
5. No \((r_i g_i)//G_i\) is smooth.

\textbf{Proof.} By [Sch13, Proposition 3.1], (1) and (2) are equivalent. Since \(G\) is semisimple, [Sch80, Corollary 7.4] shows that (2) implies (3). Since \(V\) is an orthogonal representation, [Sch13, Theorem 1.2] shows that (3) implies (4), which in turn clearly implies (5). Since some \((r_i g_i)//G_i\) is smooth when \(V\) is not large, (5) implies (1). \(\square\)

The lifting problem for \(V\) has been investigated in [Kut11] and [Sch12]. In [Sch12] we showed that \(V\) has the following lifting property.

\textbf{Theorem 1.2.} \textit{Let \(V\) be large. Then for any \(\psi \in \text{Aut}(Z)\) there is a morphism \(\Psi : V \to V\) such that \(\pi \circ \Psi = \psi \circ \pi\).}

Most cases of the theorem were first established by Kuttler [Kut11]. The lift given in the theorem is not necessarily an automorphism of \(V\) nor is it necessarily \(\sigma\)-equivariant for some \(\sigma\). From [Sch12] we get the following
Corollary 1.3. Let $V$ and $\psi$ be as above. Then there is an automorphism $\sigma$ of $G$ and a $\sigma$-equivariant lift $\Psi \in \text{Aut}_h(V)$. Hence $\psi$ sends $Z(H)$ to $Z(\sigma(H))$ for every stratum $Z(H)$ of $Z$.

The action of $\psi$ on the strata is obtained by a different method in [Kut11].

The above results lead to the following

Problems 1.4. 1. Can we lift elements of $\text{Aut}_h(Z)$ to $\sigma$-equivariant elements of $\text{Aut}_h(V)$?

2. Which automorphisms lift if $V$ is not large?

We find necessary and sufficient conditions for a $\psi$ in $\text{Aut}_h(Z)$ to have a holomorphic $\sigma$-equivariant lift. The extra conditions that one needs have to do with the codimension one strata of $Z$.

Suppose that some $r_i$ is 1. Then $Z_i := \mathfrak{g}_i / G_i \simeq \mathfrak{t}_i / \mathcal{W}_i$ where $\mathfrak{t}_i$ is the Lie algebra of a maximal torus of $G_i$ and $\mathcal{W}_i$ is the Weyl group. Now the closures of the codimension one strata of $Z_i$ are the images of the reflection hyperplanes in $\mathfrak{t}_i$. If $\mathfrak{g}_i$ has two root lengths, then we get two strata of codimension one in $Z$, corresponding to the reflection hyperplanes of the short roots and the long roots. Let $D_{i,s}$ and $D_{i,\ell}$ be the corresponding divisors of $Z$. If $\mathfrak{g}_i$ is simply laced, then $D_{i,s} = \emptyset$. Let $D_s$ and $D_\ell$ be the union of the $D_{i,s}$ and $D_{i,\ell}$ for $r_i = 1$.

There is one other way of getting a codimension one stratum, which occurs if $r_i = 2$ and $\mathfrak{g}_i \simeq \mathfrak{sl}_2$. Then $(2\mathfrak{sl}_2, \mathcal{P}\mathcal{S}L_2) \simeq (2\mathbb{C}^3, \mathbb{S}O_3)$. Thus the quotient is $\mathbb{C}^3$ and the closure of the codimension one stratum is the zero set $D_i$ of a non-degenerate quadratic form $xy - z^2$. The corresponding isotropy class is $(\mathbb{S}O_2)$. Let $D_0$ denote the union of the $D_i$ for $r_i = 2$ and $\mathfrak{g}_i \simeq \mathfrak{sl}_2$. Then the closure of the codimension one strata of $Z$ is $D := D_0 \cup D_s \cup D_\ell$.

Here is our main theorem.

Theorem 1.5. Let $V$, $G$, $D$ and $D_s$ be as above. Let $\psi \in \text{Aut}_h(Z)$. Then $\psi$ has a $\sigma$-equivariant biholomorphic lift for some $\sigma$ if and only if $\psi$ preserves $D$ and $D_s$.

If $\Psi : V \rightarrow V$ is a $\sigma$-equivariant biholomorphic lift of $\psi$, then $\psi$ sends $Z(H)$ to $Z_{\sigma(H)}$ for all $(H)$. Thus $\psi$ permutes the strata, in particular the codimension one strata. Moreover, if $(H)$ corresponds to the short roots of some $\mathfrak{g}_i$ (so $r_i = 1$), then $(\sigma(H))$ corresponds to the short roots of some $\mathfrak{g}_j$ with $r_j = 1$. Hence $\psi$ has to stabilize $D_s$ and we see that the conditions of the theorem are necessary. For sufficiency we proceed in several steps. The first step, taken in §2, is to show that if $\psi$ preserves $D$, then $\psi$ preserves the stratification, i.e., it permutes the strata. If $\psi$ also preserves $D_s$, then we say that $\psi$ is strongly stratification preserving.

Next we need to use an action of $\mathbb{C}^*$ on $Z$. For $z = \pi(v)$, $t \cdot z = \pi(tv)$, $t \in \mathbb{C}^*$. One can also see the action as follows. Let $p_1, \ldots, p_r$ be homogeneous generators of $\mathcal{O}(V)^G$ and let $p = (p_1, \ldots, p_r) : V \rightarrow \mathbb{C}^r$. Let $Y$ denote the image of $p$. Then we can identify $Y$ with $Z = V / G$. If $e_i$ is the degree of $p_i$, then we
have a $\mathbb{C}^*$-action on $Y$ where $t \in \mathbb{C}^*$ sends $(y_1, \ldots, y_r) \in Y$ to $(t^{e_1}y_1, \ldots, t^{e_r}y_r)$. The isomorphism $Y \simeq Z$ identifies the two $\mathbb{C}^*$-actions.

We say that $\psi \in \text{Aut}(Z)$ is quasilinear if $\psi(t \cdot z) = t \cdot \psi(z)$ for all $z \in Z$ and $t \in \mathbb{C}^*$. We write $\psi \in \text{Aut}_{\text{ql}}(Z)$. A main idea, as in [Sch12], is to deform a general $\psi$ to one that is quasilinear. Assume that $\psi(0) = 0$. For $t \in \mathbb{C}^*$, let $\psi_t(z)$ denote $t^{-1} \cdot \psi(t \cdot z)$. We say that $\psi \in \text{Aut}_d(Z,0)$ is deformable if $\psi_0(z) := \lim_{t \to 0} \psi_t(z)$ exists for all $z$ sufficiently close to 0. Let $S = \bigoplus S_k$ denote the graded ring $\mathcal{O}(Z)$, where the grading comes from the $\mathbb{C}^*$-action. Let $\mathcal{H}(Z)$ denote the holomorphic functions on $Z$ and $\mathcal{H}(Z,0)$ the germs of holomorphic functions at $0 \in Z$. Then $\psi$ is deformable if and only if $\psi^*S_k \subset S_k \cdot \mathcal{H}(Z,0)$ for all $k$. Equivalently, $\psi^*y_i \in S_{e_i} \cdot \mathcal{H}(Z,0)$ for all $i$. This is a vanishing condition on some of the partial derivatives of $\psi$ at 0. If $\psi$ is deformable, then $\psi^*\psi_t$ sends $y_i$ to the term homogeneous of degree $e_i$ of the Taylor series of $\psi^*y_i$ at 0, considered as an element of $\mathcal{O}(Z)$. Clearly, if $\psi$ is deformable, the family $\psi_t$ (including $t = 0$) is holomorphic in $z$ and $t$. (If $\psi = \psi(x, z)$ also depends holomorphically on parameters $x \in X$, then the family $\psi_t(x, z)$ is holomorphic in $t$, $x$ and $z$.) Each $\psi_t$ preserves the germ of $Z$ at 0 and $\psi_0$ preserves $Z$. If $\psi$ and $\psi^{-1}$ are deformable, then $(\psi^{-1})_0$ is an inverse to $\psi_0$ so that $\psi_0 \in \text{Aut}_{\text{ql}}(Z)$. If $\psi$ is also (strongly) stratification preserving, then so is $\psi_0$ since the strata are invariant under the action of $\mathbb{C}^*$. We say that $V$ is good if every stratification preserving $\psi \in \text{Aut}_d(Z,0)$ is deformable. We say that $V$ has the lifting property if every strongly stratification preserving $\psi \in \text{Aut}_{\text{ql}}(Z)$ has a $\sigma$-equivariant lift $\Psi \in \text{GL}(V)$ for some $\sigma$. Then $\Psi \circ g \circ \Psi^{-1} = \sigma(g)$, so that $\Psi \in N_{\text{GL}(V)}(G)$, the normalizer of $G$ in $\text{GL}(V)$.

**Theorem 1.6.** Let $(V,G)$ be as above. Then

1. $V$ is good.
2. $V$ has the lifting property

We deduce our main theorem from the one above using a result on lifting of isotopies as in [Sch80]. Our proof of Theorem 1.6 breaks up into three parts. Recall that $V = \bigoplus_{i=1}^{3} r_iG_i$. Let $V_1$ be the sum of the submodules $r_iG_i \simeq 2sl_2$, let $V_2$ be the sum of the submodules $r_iG_i$ with $r_i = 1$ and let $V_3$ be the sum of the $r_iG_i$ which are large. Let $\tilde{G}_i$ be the image of $G$ in $\text{GL}(V_i)$, $1 \leq i \leq 3$. We show that Theorem 1.6 holds for each $(V_i, \tilde{G}_i)$. The first two cases are handled in §3 and §4. The third case is a bit more difficult.

Let $W$ be an $H$-module where $H$ is reductive. Set $Y := W//H$ and let $\pi: W \to Y$ be the quotient morphism. Let $\mathcal{D}^k(W)$ (resp. $\mathcal{D}^k(Y)$) denote the algebraic differential operators on $W$ (resp. $Y$) of order at most $k$ (see [Sch95, §3]). Then we have a morphism $\pi_*: \mathcal{D}^k(W)^H \to \mathcal{D}^k(Y)$ where $\pi_*(P)(f) = P(\pi^*(f))$ for $P \in \mathcal{D}^k(W)^H$ and $f \in \mathcal{O}(Y) \simeq \mathcal{O}(W)^H$.

**Definition 1.7.** We say that the $H$-module $W$ is admissible if

1. $W$ is 2-principal.
2. \( \pi_*: \mathcal{D}^k(W)^H \to \mathcal{D}^k(Y) \) is surjective for all \( k \).

From [Sch12, Theorem 2.2] we have the following result:

**Theorem 1.8.** If \( W \) is admissible, then \( W \) is good.

([Sch12, Theorem 2.2] is actually stated for \( \text{Aut}_h(Y) \), but the proof for \( \text{Aut}_h(Y,0) \) is the same.) In sections §5 and §6 we show that \((V_3, \bar{G}_3)\) is admissible, hence good. From [Sch12, 1.9, 1.16] we see that \( V_3 \) has the lifting property. Thus each \((V_i, \bar{G}_i)\) satisfies the conclusions of Theorem 1.6. In §7 we combine these facts to prove Theorems 1.5 and 1.6. In §8 we consider the analogues of our results for compact Lie groups.

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2. **Preserving the stratification**

Let \( H \) be reductive and \( W \) an \( H \)-module. Let \( Y = W/H \). For \((L)\) and \((M)\) conjugacy classes of subgroups of \( H \) we write \((M) \geq (L)\) if \( L \) is conjugate to a subgroup of \( M \). From [Sch80, Lemma 5.5] one has

**Lemma 2.1.** Let \( S = Y(L) \) be a stratum of \( Y \). Then \( S \) is irreducible and

\[
\overline{S} = \bigcup_{(M) \geq (L)} Y(M)
\]

Let \( \mathfrak{X}(Y) \) denote the strata preserving vector fields on \( Y \). These are the derivations of \( \mathcal{O}(Y) \) which are tangent to \( S \) along \( S \) for every stratum \( S \) of \( Y \). Equivalently, they are the derivations preserving the ideals vanishing on the closures of the strata of \( Y \). Let \( \mathfrak{X}_h(Y) \) denote the holomorphic strata preserving vector fields on \( Y \) and let \( \mathfrak{X}_h(Y,0) \) denote the corresponding germs at \( 0 \). Let \( \mathfrak{X}(W) \) denote the vector fields on \( W \). Then it is easy to see that any element of \( \mathfrak{X}(W)^H \) preserves the pull-backs of the ideals vanishing on the closures of the strata of \( Z \) [Sch80, Corollary 1.3]. Thus we have a natural morphism \( \pi_*: \mathfrak{X}(W)^H \to \mathfrak{X}(Y) \). We also have \( \pi_*: \mathfrak{X}_h(W)^H \to \mathfrak{X}_h(Y) \) and similarly for \( \mathfrak{X}_h(W,0)^H \) and \( \mathfrak{X}_h(Y,0) \). Now we assume that \( W \) is orthogonal.

**Proposition 2.2.** Let \( H \) be reductive and \( W \) an orthogonal \( H \)-module.

1. Let \( A \) be a derivation of \( \mathcal{O}(Y) \) which preserves the codimension one strata of \( Y \). Then \( A \) preserves all the strata of \( Y \). The same result holds for holomorphic derivations of \( \mathcal{H}(Y) \) and germs of holomorphic derivations at \( 0 \in Y \) [Sch80, 3.5, 5.8, 6.1].

2. Let \( A \in \mathfrak{X}(Y) \) (resp. \( \mathfrak{X}_h(Y) \), resp. \( \mathfrak{X}_h(Y,0) \)). Then there is a \( B \in \mathfrak{X}(W)^H \) (resp. \( \mathfrak{X}_h(W)^H \), resp. \( \mathfrak{X}_h(W,0)^H \)) such that \( \pi_*B = A \) [Sch80, 3.7, 6.7, 6.9].
Corollary 2.3. Let $\psi \in \text{Aut}(Y)$ such that $\psi$ preserves the union of the codimension one strata. Then $\psi$ preserves the stratification of $Y$.

Proof. We follow the argument of [Sch13]. Let $A \in \mathfrak{X}(Y)$ and define $\psi^*(A) = \psi^* \circ A \circ (\psi^*)^{-1}$. This is a derivation of $\mathcal{O}(Y)$ which, by Proposition 2.2, again lies in $\mathfrak{X}(Y)$. Thus $\psi^*$ preserves $\mathfrak{X}(Y)$. One computes that, in terms of tangent vectors, $(\psi^* A)(y) = d(\psi^{-1})_{\psi(y)} A(\psi(y)), \ y \in Y$. Let $s \in S$ where $S$ is a stratum of $Y$. Then the evaluation at $s$ of all elements of $\pi_* \mathfrak{X}(W)^H = \mathfrak{X}(Y)$ is precisely $T_s S$. This is established in [Sch13, Proposition 2.2]. Now the dimension of $\mathfrak{X}(Y) = \psi^* \mathfrak{X}(Y)$ evaluated at $s$ is the same as the dimension of $\mathfrak{X}(Y)$ evaluated at $\psi(s)$. Hence the stratum $S'$ containing $\psi(s)$ has the same dimension as $S$. Since there are finitely many strata and the strata are irreducible, we see that $\psi(S) = S'$. Hence $\psi$ permutes the strata of $Y$.

Remark 2.4. The same argument works to show that if $\psi \in \text{Aut}_{h}(Y)$ or $\text{Aut}_{h}(Y,0)$ and $\psi$ preserves the union of the codimension one strata, then it preserves the stratification.

3. Copies of $2\mathfrak{sl}_2$

Let $(W_i, H_i) = (2\mathfrak{sl}_2, \text{PSL}_2) \simeq (2\mathbb{C}^3, \text{SO}_3), \ i = 1, \ldots, n$. Let $W = \oplus W_i$ have the product action of $H = \prod_i H_i$ with quotient $Y$. Let $Y_i$ denote $W_i//H_i$. Then $Y = Y_1 \times \cdots \times Y_n$ where each $Y_i$ is isomorphic to $\mathbb{C}^3$, and the non principal strata of $Y_i$ are the nonzero points in a cone $\text{Var}(xy - z^2)$ (denoted $S_i$) and the origin of $\mathbb{C}^3$. Let $S'_i$ denote the product of $S_i$ and the principal strata of the other $Y_j$. The $S'_i$ are the codimension one strata of $Y$.

Proposition 3.1. The $H$-module $W$ is good and has the lifting property.

Proof. Let $\psi \in \text{Aut}_h(Y)$ preserve the codimension one strata. Then $\psi$ preserves the stratification. The point 0 is the lowest dimensional stratum, hence it is preserved. Since each $\mathcal{O}(W_i)^H$ is generated by quadratic invariants, $\psi_0$ exists and is just $d\psi(0)$, considered as a mapping of $Y \simeq T_0(Y)$ to itself. The same argument works in case that $\psi \in \text{Aut}_h(Y,0)$. Hence $W$ is good.

Now let $\psi$ be quasilinear and strata preserving. Then $\psi$ permutes the $S'_i$ and modulo a permutation of the $W_i$ (which induces an automorphism $\sigma$ of $H$), we may reduce to the case that $\psi$ preserves each $S'_i$. Now $S'_i$ is the product of $S_i$ with the other $Y_j$. The singular points of this space are the product of $0 \in Y_i$ with the product of the other $Y_j$. By taking intersections we see that $\psi$ preserves the product of $Y_i$ with the origin of the other $Y_j$. Thus $\psi = \text{diag}(A_1, \ldots, A_n)$ where $A_i \in \text{GL}(3)$ preserves the cone $\overline{S_i}, \ i = 1, \ldots, n$. But then $A_i$ is in $\mathbb{C}^*$ times the orthogonal group of $xy - z^2$. This is the image of the group $\text{GL}_2$ which acts on $2\mathfrak{sl}_2 \simeq \mathfrak{sl}_2 \otimes \mathbb{C}^2$ via its action on $\mathbb{C}^2$, commuting with the action of $\text{PSL}_2$. Hence $\psi$ lifts and $W$ has the lifting property.
4. The adjoint representation

We now consider the case where \((W, H) = (h, H)\) and \(H\) is semisimple. We have \(h = \bigoplus_{i=1}^n h_i\) where \(h_i\) is simple. Let \(H_i\) denote the corresponding adjoint group so that \(H = \prod_i H_i\). Let \(t_i\) be a maximal toral subalgebra of \(h_i\) and let \(W_i\) be the corresponding Weyl group. Set \(t = \oplus_i t_i\) and \(W = \prod_i W_i\). Let \(Y_i\) denote \(h_i / H_i \simeq t_i / W_i\). Define \(D_s\) and \(D_t\) as in the introduction.

Let \(\psi \in \text{Aut}_h(Y, 0)\) preserve \(D_s\) and \(D_t\). First we only use the information that \(\psi\) preserves \(D_s \cup D_t\). Let \(\tilde{\tau}\) denote the quotient mapping \(t \to Y = Y_1 \times \cdots \times Y_n\).

**Proposition 4.1.** Let \(\psi\) be as above. Then

1. \(\psi\) is stratification preserving. In particular, \(\psi(Y_{pr}) \subset Y_{pr}\).
2. \(\psi\) has a lift \(\tau \in \text{Aut}_h(t, 0)\) which is \(\sigma\)-equivariant for some \(\sigma \in \text{Aut}(W)\).
3. \(\psi\) is deformable and \(\psi_0\) lifts to \(\tau'(0) \in N_{GL(0)}(W)\).
4. \(W\) is good.

**Proof.** Part (1) follows from Corollary 2.3. Then [Lya83], [KLM03, Theorem 5.4] or [Sch09, Theorem 3.1] show that \(\psi\) has a \(\sigma\)-equivariant lift \(\tau\) to \(t\), giving (2). Clearly \(\tau(0) = 0\). Let \(x \in t\) and set \(y = \tilde{\tau}(x)\). Then \(\lim_{t\to 0} \psi_t(y) = \tilde{\tau}(\lim_{t\to 0} \tau(t(x))) = \tilde{\tau}(\tau'(0)x)\). Thus \(\psi\) is deformable and we have (3) and (4).

We are not done yet. If \(\psi\) is quasilinear, then we have a linear lift \(\tau\) which is \(\sigma\)-equivariant. We have to lift \(\tau\) to \(h\). We have not used yet that \(\psi\) preserves \(D_s\). Let \(\Phi \subset t^*\) be the roots of \(h\). For \(\alpha \in \Phi\), let \(\tau_\alpha\) denote the corresponding reflection in \(t\). We have the Killing form \(B_i\) on \(t_i^*\). Since \(W_i\) acts irreducibly on \(t_i^*\) for each \(i\), any \(W\)-invariant non degenerate bilinear form on \(t^*\) has to be of the form \(\oplus c_i B_i\) where \(c_i \in \mathbb{C}^*\). Let \(\tau \in N_{GL(0)}(W)\). Then \(\tau\) permutes the factors \(t_i\) so we have an action of \(\tau\) on \(\{1, \ldots, n\}\) where \(\tau(t_i) = t_{\tau(i)}\). Let \(\tau^*: t^* \to t^*\) be composition with \(\tau\). We have the push-forward \(\tau_* (B_i)\) where \(\tau_* (B_i)(\xi, \eta) = B_i(\tau^* \xi, \tau^* \eta)\) for \(\xi, \eta \in t^*_{\tau(i)}\). Let \(\tau_* B = \oplus_i \tau_* B_i\) where \(B = \oplus_i B_i\) is the Killing form. We will also denote \(B\) by \((, , )\).

**Lemma 4.2.** We can modify \(\tau|_{t_i}\) by a scalar \(d_i\), \(1 \leq i \leq n\), such that \(\tau_* B = B\).

**Proof.** Since \(\tau\) normalizes \(W\), \(\tau_* B\) is invariant under \(W\), hence is of the form \(\oplus c_i B_i\). Since \(\tau\) acts on \(\{1, \ldots, n\}\) by a product of cycles, one easily sees that one can modify the restriction of \(\tau\) to \(t_i\) by a scalar \(d_i\) such that \(\tau_* B_i = B_{\tau(i)}\), \(1 \leq i \leq n\). Note that this modification lifts to \(GL(h)^H \simeq (\mathbb{C}^*)^n\).

**Lemma 4.3.** Assume that \(\tau\) preserves the Killing form. Then for all \(\alpha \in \Phi\), \(\tau^* (\alpha) = c(\alpha)\varphi(\alpha)\) where \(\varphi: \Phi \to \Phi\) and \(c(\alpha) > 0\).

**Proof.** The composition \(\tau^{-1} \circ \tau\alpha \circ \tau\) is a reflection in \(W\), hence it is \(\tau_\beta\) for some \(\beta \in \Phi\). Thus \(\tau^* \alpha = c\beta\) for some \(c \in \mathbb{C}^*\). Since \(\tau\) preserves the Killing form,
that $c^2(\beta, \beta)$ is positive, hence $c \in \mathbb{R}^*$. Perhaps changing $\beta$ to $-\beta$, we can arrange that $c > 0$. Set $\varphi(\alpha) = \beta$ and $c(\alpha) = c$.

\textbf{Theorem 4.4.} Let $\tau \in N_{GL(n)}(W)$ be a lift of $\psi \in \text{Aut}_q(Y)$ where $\psi$ is strongly stratification preserving. Then $\tau$ lifts to $N_{GL(n)}(H)$. Hence $W$ has the lifting property.

\textbf{Proof.} We may assume that $\tau$ preserves the Killing form and induces a bijection $\varphi$ of $\Phi$ where $\tau^*(\alpha) = c(\alpha)\varphi(\alpha)$ and $c(\alpha) > 0$, $\alpha \in \Phi$. Thus $\tau^*$ permutes the reflection hyperplanes in $t_\mathbb{R}^*$, the real span of the roots. Hence $\tau^*$ permutes the chambers and there is an element $w \in W$ such that $w\tau^*$ preserves the Weyl chamber $C(\Delta)$ where $\Delta$ is a base of $\Phi$. Hence we may assume that $\tau^*$ preserves $C(\Delta)$ and it follows that $\varphi$ preserves $\Delta$.

Let $i \in \{1, \ldots, n\}$ and set $j = \tau(i)$. Let $\alpha$, $\beta \in \Delta_j$ where $\Delta_j$ is the base of $\Phi_j \subset t^*_\mathbb{R}$. Let $\langle \beta, \alpha \rangle$ denote $2(\beta, \alpha)/(|\alpha|, \alpha)$. Since $\tau^*$ is an isometry we have

$$\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = \langle \tau^*(\beta), \tau^*(\alpha) \rangle \langle \tau^*(\alpha), \tau^*(\beta) \rangle \langle \varphi(\beta), \varphi(\alpha) \rangle \langle \varphi(\alpha), \varphi(\beta) \rangle.$$  

Thus $\varphi: \Delta_j \rightarrow \Delta_i$ preserves the product $\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle$. Since $\tau$ preserves the collection of reflection hyperplanes for the short roots, we have that $|\alpha| < |\beta|$ if and only if $|\varphi(\alpha)| < |\varphi(\beta)|$ where $|\alpha|$ is the norm of $\alpha$, etc. Now the $\langle \beta, \alpha \rangle$ are negative (or zero) and one can see that the numbers $\langle \beta, \alpha \rangle$ are preserved by $\varphi$ (see [Hum72, Ch. III, Table 1]). Hence the restriction of $\varphi$ to $\Delta$ is a diagram automorphism. Now changing $\tau$ by a diagram automorphism, which lifts to $N_{GL(n)}(H)$, we can reduce to the case that $\varphi$ is the identity on $\Delta$. Thus $\tau^*$ preserves the $t^*_i$ and sends $\alpha$ to $c(\alpha)\alpha$, $\alpha \in \Delta_i$, where $c(\alpha) > 0$. Hence $\tau^*$ commutes with the simple reflections on $t^*$, so it centralizes $W$. Since each $W_i$ acts irreducibly on $t^*_i$, $\tau^*$ must be multiplication by a (positive) scalar on each $t^*_i$. Since $\tau^*$ is an isometry, we see that we have reduced to the case that $\tau^*$ is the identity.

\section{5. Modularity}

We want to prove that $(V, G)$ is admissible if it is large. We first need to recall some notions and results from [Sch94, Sch95]. Let $H$ be a linear algebraic group and $X$ an $H$-variety. Let $X(n)$ or $(X, H)(n)$ denote the set of points of $X$ with isotropy group of dimension $n$. Then $X(n)$ is constructible and we define the \textit{modularity} of $X$, mod $(X, H)$, by

$$\text{mod} \ (X, H) = \sup_{n \geq 0} (\dim X(n) - \dim H + n)$$

where, as usual, the dimension of the empty set is $-\infty$. If $X(n)$ is irreducible, then $\dim X(n) - \dim H + n$ is the transcendence degree of $\mathbb{C}(X(n))^H$. We have that

$$\text{mod} \ (X, H) = \dim X - \dim H + \sup_{n \geq 0} (n - \text{codim} \ X(n)).$$

Now suppose that $X(0) \neq \emptyset$. We say that $X$ is \textit{k-modular} if
mod \( (X \setminus X_{(0)}, H) \leq \dim X - \dim H - k \).

Equivalently, \( \text{codim} X_{(n)} \geq n + k \) for all \( n \geq 1 \). Let us assume from now on that \( H \) is reductive and that \( X \) is an affine \( H \)-variety with FPIG. Then \( X_{(0)} \neq \emptyset \) and \( \dim X - \dim H = \dim X/\!/H \) so that \( X \) is \( k \)-modular if and only if \( \text{mod} (X \setminus X_{(0)}, H) \leq \dim X/\!/H - k \).

There are geometric interpretations of \( X \) being \( 0 \)-modular. From now on assume that \( X \) is smooth. Let \( A_1, \ldots, A_r \) be a basis of the Lie algebra of \( H \). We may consider the \( A_i \) as vector fields \( \tilde{A}_i \) on \( X \), hence as functions \( f_{A_i} \) on \( T^*X \).

Then \( X \) is \( 0 \)-modular if and only if the \( f_{A_i} \) are a regular sequence in \( \mathcal{O}(T^*X) \), i.e., their set of common zeroes has codimension \( r \). Another interpretation is the following. Let \( \mu : T^*X \to \mathfrak{h}^* \) be the moment mapping which sends \( \xi \in T^*_xX \) to the element of \( \mathfrak{h}^* \) whose value on \( A \in \mathfrak{h} \) is \( \tilde{A}(x)(\xi) \). Then \( X \) is \( 0 \)-modular if and only if all the fibers of \( \mu \) have codimension \( r \) [Sch95, Remark 8.6].

**Remark 5.1.** Note that if \( X \) is \( k \)-modular for any \( k \geq 0 \), then \( \text{mod} (X, H) = \dim X - \dim H \).

Suppose that \( X \) is \( 2 \)-principal. Then we can find invariant functions \( h_1 \) and \( h_2 \) which vanish on \( X \setminus X_{\text{pr}} \) and are a regular sequence in \( \mathcal{O}(X) \). If \( X \) is also \( 2 \)-modular, then \( h_1, h_2, f_{A_1}, \ldots, f_{A_r} \) is a regular sequence in \( \mathcal{O}(T^*X) \) [Sch95, Lemma 9.7]. Then we can apply [Sch95, Proposition 8.15]:

**Proposition 5.2.** Suppose that there are \( h_1, h_2 \) as above such that \( h_1, h_2 \) and the \( f_{A_i} \) are a regular sequence in \( \mathcal{O}(T^*X) \). Then \( \pi_* : \mathcal{D}^k(X)^H \to \mathcal{D}^k(X/\!/H) \) is surjective for every \( k \geq 0 \).

**Corollary 5.3.** Suppose that \( X \) is \( 2 \)-principal and \( 2 \)-modular. Then \( \pi_* : \mathcal{D}^k(X)^H \to \mathcal{D}^k(X/\!/H) \) is surjective for every \( k \geq 0 \).

In the next section we will show that our \( V \) of interest is \( 2 \)-modular. We will need the following result. Recall that a group action is *almost effective* if the kernel of the action is finite.

**Lemma 5.4.** Let \( H \) be connected reductive. Let \( T \) denote the connected center of \( H \) and let \( H_s \) denote \([H, H]\). Let \( W := W' \oplus W'' \) be a direct sum of \( H \)-modules where \( T \) acts trivially on \( W' \). Assume that the action of \( T \) on \( W'' \) is almost effective. If \( (W', H_s) \) is \( 0 \)-modular, then so is \( (W, H) \).

**Proof.** By Vinberg [Vin86], for every \( q \geq 0 \), \( \text{codim}(W'', T)_{(q)} \geq q \). Let \( w = (w', w'') \) where \( w' \in W' \) and \( w'' \in (W'', T)_{(q)} \). If \( \dim H_w = p \), then \( p \geq q \) and \( \dim(H_s)_{w'} = p' - q \) for \( p' \geq p \). Thus

\[
\sup_{p' \geq q} (p - \text{codim}((W' \times (W'', T)_{(q)}), H)_{(p)}) \\
\leq \sup_{p' \geq q} (p' - q - \text{codim}((W', H_s)_{(p' - q)}) + q - \text{codim}(W'', T)_{(q)} \\
\leq \sup_{p' \geq 0} (p' - \text{codim}(W', H_s)_{(p')} \leq 0.
\]
Since \( q \leq p \) is arbitrary, we see that \((W, H)\) is 0-modular.

\[\text{6. Large representations.}\]

We assume that \( W = \bigoplus_{i=1}^{n} r_{i}\mathfrak{h}_{i} \) is large. Recall that \( W \) is 2-principal. By Corollary 1.3 a quasilinear \( \psi \in \text{Aut}_{q}(W/\mathcal{H}) \) has a biholomorphic \( \sigma \)-equivariant lift \( \Psi \). Then \( \Psi'(0) \) is also a lift, so that \( W \) has the lifting property. Thus, by Corollary 5.3, we only need to show that \( W \) is 2-modular. It is easy to see that \( W \) is 2-modular if and only if each \( r_{i}\mathfrak{h}_{i} \) is 2-modular, so we may reduce to the case that \( H \) is simple. The case \( H = \text{PSL}_{2} \) follows from [Sch95, Theorem 11.9], so we only have to consider the case where the rank of \( H \) is at least two. Moreover, we need only consider the case of two copies of \( \mathfrak{h} \) since increasing the number of copies can only increase the modularity [Sch95, Proposition 11.5].

Let \( v \in W \setminus W(0) \). Then \( \dim H_{v} > 0 \) and there is an injective homomorphism \( \lambda : \mathfrak{L} \to H_{v} \), where \( \mathfrak{L} \) is a copy of the multiplicative group \( \mathbb{C}^{*} \) or the additive group \( \mathbb{C}^{+} \). First consider the case where \( \mathfrak{L} = \mathbb{C}^{*} \). Acting by an element of \( H \), we may assume that \( \text{Im} \lambda \subseteq T \), where \( T \) denotes a maximal torus of \( H \). Identify \( \mathfrak{L} \) with its image in \( T \). Since the action of \( T \) on \( W \) is diagonalizable and \( \mathfrak{L} \subset T \), there are only finitely many possible \( W_{\mathfrak{L}} \). Let \( C_{\mathfrak{L}} \) denote \( C_{H}(\mathfrak{L}) \). Then \( H \cdot W_{\mathfrak{L}} \) is constructible and the \( H \)-orbits there all intersect \( W_{\mathfrak{L}} \) in a union of \( C_{\mathfrak{L}} \) orbits. Hence \( \text{mod} (H \cdot W_{\mathfrak{L}}, H) \leq \text{mod} (W_{\mathfrak{L}}, C_{\mathfrak{L}}) \).

Now suppose that \( \mathfrak{L} = \mathbb{C}^{+} \). By the Jacobson-Morozov theorem, \( \lambda \) extends to an injective homomorphism, also called \( \lambda \), from \( \text{SL}_{2} \) or \( \text{PSL}_{2} \) to \( H \). We identify \( \mathfrak{L} \) with its image in \( H \) and we let \( S \) denote \( \lambda(\text{SL}_{2}) \) or \( \lambda(\text{PSL}_{2}) \) as the case may be. Up to conjugation in \( H \), there are only finitely many possible \( \lambda \). Let \( C_{\mathfrak{L}} \) denote \( C_{H}(S) \times T' \), where \( T' \) is the maximal torus of \( S \). Then again we have the estimate that \( \text{mod} (H \cdot W_{\mathfrak{L}}, H) \leq \text{mod} (W_{\mathfrak{L}}, C_{\mathfrak{L}}) \).

Since the orbit of every point in \( W \setminus W(0) \) intersects one of our finite collection of sets \( W_{\mathfrak{L}} \), and since \( \dim W/\mathcal{H} = \dim H \), our discussion above gives the following.

**Lemma 6.1.** Suppose that \( \text{mod} (W_{\mathfrak{L}}, C_{\mathfrak{L}}) \leq \dim H - 2 \) for all \( \mathfrak{L} \) of dimension one. Then \( W \) is 2-modular.

The next two propositions show that we always have \( \text{mod} (W_{\mathfrak{L}}, C_{\mathfrak{L}}) \leq \dim H - 2 \), hence they complete our proof that \( W \) is admissible, hence good.

**Proposition 6.2.** Suppose that \( \mathfrak{L} = \mathbb{C}^{*} \). Then \( \text{mod} (W_{\mathfrak{L}}, C_{\mathfrak{L}}) \leq \dim H - 2 \).

**Proof.** As above, we may assume that \( \mathfrak{L} \) is contained in a maximal torus \( T \) of \( H \). The action of \( C_{\mathfrak{L}} \) on \( W_{\mathfrak{L}} \) is twice the adjoint representation of \( C_{\mathfrak{L}} \). A maximal torus of \( C_{\mathfrak{L}} \) is \( T \), and the connected center of \( C_{\mathfrak{L}} \) is a subtorus \( T' \) of \( T \) containing \( \mathfrak{L} \). Then \( C_{\mathfrak{L}}/T' \) is semisimple, so its action on \( W_{\mathfrak{L}} \) is a trivial representation plus the sum of twice the adjoint representation of each simple component \( M_{i} \) of \( C_{\mathfrak{L}}/T' \). By induction on rank, \( 2\mathfrak{m}_{i} \) is 2-modular if \( M_{i} \) has rank at least two. If
\( m_i \simeq \mathfrak{sl}_2 \), then \( 2m_i \) is 1-modular. It then follows from Remark 5.1 that
\[
\text{mod} (W^L, C_L) = \dim W^L - \dim C_L + \dim T' = \dim h^L + \dim T'.
\]
Let \( \alpha_1, \ldots, \alpha_k \) be the positive roots of \( h \) which are nontrivial when restricted to \( T' \). As \( T' \)-module,
\[
h = \theta_m + \bigoplus_{i=1}^k (h_{\alpha_i} + h_{-\alpha_i})
\]
where \( m = \dim h^L \) and \( \theta_m \) denotes the trivial module of dimension \( m \). We need to show that \( \dim h - 2 - (m + \dim T') \) is nonnegative, i.e., that \( -2 - \dim T' + 2k \geq 0 \). Since the action of \( T' \) on \( h \) is effective, \( k \geq \dim T' \). Thus, if \( \dim T' \geq 2 \), we obtain the desired estimate. The only problem is the case that \( T' = L \) has dimension one. Suppose that only one positive root \( \alpha \) of \( H \) acts nontrivially on \( L \). We may assume that \( \alpha \) is simple. Then, since \( H \) has rank at least two, there is a simple root \( \beta \) (acting trivially on \( L \)) such that \( \alpha + \beta \) is a root. Then \( \alpha + \beta \) acts nontrivially on \( L \) and we have a contradiction. Hence \( k \geq 2 \) and \( \text{mod} (W^L, C_L) \leq \dim H - 2 \).

**Proposition 6.3.** Suppose that \( L = \mathbb{C}^+ \). Then \( \text{mod} (W^L, C_L) \leq \dim W/\mathbb{C} - 2 \).

**Proof.** Let \( L \subset S \subset H \) be as discussed above. As \( S \)-module, \( h = \sum_i m_iR_i \) where \( R_i \) denotes the space of binary forms of degree \( i \) and \( m_i \geq 0 \) is the multiplicity of \( R_i \). The fixed points of \( L \) in each \( R_i \) have dimension one, so that \( W^L \) has dimension \( 2 \sum_i m_i \). The Lie algebra of \( C_L \) has dimension \( m_0 + 1 \) and the action of \( C_H(S) \) on \( 2h^S \) is twice its adjoint representation. Write \( C_H(S)^0 = C_sT' \) where \( C_s \) is semisimple and \( T' \) is the connected center. As we saw above, if \( C_s \) is nontrivial, its action on \( 2h^S \) is 1-modular. Recall that \( T'' \) is the maximal torus of \( S \). Now \( T'' \) acts on \( m_iR_i \) via a homomorphism to \( \text{GL}_{m_i} \), hence \( T'' \) acts almost effectively on \( \bigoplus_{i \geq 1} m_iR_i^L \). Since \( T'' \) commutes with \( S \), it acts trivially on \( s \subset m_2R_2 \). However, \( T'' \) acts nontrivially on \( s^L \). Thus \( T' \times T'' \) acts almost effectively on \( W^L \). Using Lemma 5.4, we get that \( \text{mod} (W^L, C_L) \) is \( \dim W^L - \dim C_H(S) - 1 \). Now \( \dim W^L \) is \( 2(m_0 + \sum_{i \geq 1} m_i) \) and \( \dim C_H(S) = m_0 \). We need that \( \dim W^L - m_0 - 1 \leq \dim h - 2 \). Hence we need
\[
2(m_0 + \sum_{i \geq 1} m_i) - m_0 - 1 \leq \sum_{i \geq 0} (i+1)m_i - 2, \text{ i.e., } 0 \leq -1 + \sum_{i \geq 2} (i-1)m_i.
\]
Since \( s \simeq R_2 \subset h \), we have that \( m_2 \geq 1 \), giving us the desired inequality.

### 7. Proofs of the main theorems

**Lemma 7.1.** Let \( H \) be a simple adjoint group of rank at least two. Let \( S \) be a codimension one stratum of \( Y := h/\mathbb{C} \). Then \( \overline{S} \) contains a codimension two stratum of \( Y \).

**Proof.** Let \( t \) be a maximal toral subalgebra of \( h \). Then there is a simple root \( \alpha \) such that \( \overline{S} \) is the image in \( Y \) of the corresponding root hyperplane \( t_\alpha \). Let \( \beta \) be another simple root and let \( S' \) denote the corresponding stratum of \( Y \). Then \( \overline{S} \cap \overline{S'} \) is the image of \( t_\alpha \cap t_\beta \) and contains a stratum of codimension two.
Let $V = \oplus r_i g_i$ and $G$ be as in Theorem 1.6. We have decomposed $V$ as $V_1 \oplus V_2 \oplus V_3$ and $G$ as $G_1 \times G_2 \times G_3$ where $(V_1, G_1)$ is a direct sum of representations isomorphic to $(2sl_2, PSL_2)$, $(V_2, G_2)$ is the adjoint representation of a semisimple group and $(V_3, G_3)$ is large. Then $Z = Z_1 \times Z_2 \times Z_3$ where $Z = \mathcal{G}/G$ and $Z_i = V_i \mathcal{G}$, $1 \leq i \leq 3$. We have shown that each $(V_i, G_i)$ is good and has the lifting property. Now we have to show the same for $(V, G)$.

For $S_i$ a stratum of $Z_1$, let $S'_i$ denote the product of $S_i$ and the principal strata of the other two factors of $Z$, $1 \leq i \leq 3$. Let $\psi \in \text{Aut}_3(Z, 0)$ be stratification preserving and write $\psi = (\psi_1, \psi_2, \psi_3)$ where $\psi_1$ is a germ sending $Z$ to $Z_i$, $1 \leq i \leq 3$.

**Lemma 7.2.**

1. $\psi$ preserves the strata of the form $R'_3$ and the sets of the form $Z_1 \times Z_2 \times R_3$ where $R_3$ is a stratum of $Z_3$.

2. The analogues of (1) hold for strata $R_1$ of $Z_1$ and $R_2$ of $Z_2$.

3. $d\psi(0)$ has the form $\text{diag}(A_1, A_2, A_3)$ where $A_i: T_0(Z_i) \to T_0(Z_i)$ is an isomorphism, $1 \leq i \leq 3$.

4. For $z$ near 0, $\psi_3(z_1, z_2, z_3)$ is a strata preserving automorphism of $Z_3$, fixing 0, parameterized by $(z_1, z_2) \in Z_1 \times Z_2$.

5. $\psi_{3,0}(z) = \lim_{t \to 0} t^{-1} \cdot \psi_3(t \cdot z) = \lim_{t \to 0} t^{-1} \cdot \psi_3(0, 0, t \cdot z)$ exists. Thus $\psi_{3,0}(z) = \psi_{3,0}(0, 0, z_3)$ can be considered as an element of $\text{Aut}_{qE}(Z_3)$.

6. The analogues of (4)–(5) hold for $\psi_1$ and $\psi_2$.

7. $\psi$ is deformable and $\psi_0 = (\psi_{1,0}, \psi_{2,0}, \psi_{3,0})$ where $\psi_{i,0} \in \text{Aut}_{qE}(Z_i)$ is strata preserving, $1 \leq i \leq 3$.

**Proof.** Let $R_3$ be a stratum of $Z_3$. Then $\psi(R'_3)$ is of the form $S_1 \times S_2 \times S_3$ where $S_i$ is a stratum of $Z_i$, $1 \leq i \leq 3$. If $S_1$ is not $Z_1$, then $\psi(R'_3)$ is in the closure of a codimension one stratum, hence so is $R'_3$. Since $Z_3$ has no codimension one strata, we get a contradiction. Thus $S_1 = Z_1$ and similarly $S_2 = Z_2$. Hence $\psi$ permutes the strata of the form $R'_3$ and all sets of the form $Z_1 \times Z_2 \times R_3$. By Lemma 2.1 we see that $\psi$ preserves sets of the form $Z_1 \times Z_2 \times R_3$, giving (1).

Let $S_1$ be a codimension one stratum of $Z_1$. Then $S'_1$ is the codimension one stratum $R$ of a copy of $Y := (2sl_2)/\mathcal{G}$ times the principal stratum of the other factors of $Z$. Moreover, $R$ is $R$ union a point which has codimension three in $Y \simeq \mathbb{C}^3$. Now consider the case where we have a codimension one stratum $S_2$ of $Z_2$. Then $S'_2$ is the codimension one stratum $R$ of a copy of $Y := I/L$, where $L$ is simple, times the principal stratum of the other factors of $Z$. If $\psi$ sends $S'_1$ to $S'_2$, then the corresponding space $I/L$ has dimension at least three and contains no codimension two strata. Thus rank $L > 1$ and we get a contradiction to Lemma 7.1. It follows that $\psi$ permutes all strata of the form $S'_1$ so $\psi$ permutes the sets of the form $S_1 \times Z_2 \times Z_3$. The analogous results hold for strata $S_2$ of $Z_2$ and we have (2).
Clearly \(d\psi_0\) has the form given in (3). Thus for \((z_1, z_2)\) near 0, \(\psi_3(z_1, z_2, z_3)\) is a germ of an automorphism of \(Z_3\) which fixes 0. For this one uses the inverse function theorem for varieties, see, for example, [HS07, Lemma 14.15]. Here we don’t need to worry about preserving strata since \(V_3\) is large. Hence we have (4) and since \(V_3\) is good, we have (5). In the case of \(\psi_1\), we note that \(\psi\) respects the stratification by strata \(S_1 \times Z_2 \times Z_3\) where \(S_1\) is a stratum of \(Z_1\). Thus \(\psi_1(z_1, z_2, z_3)\), near 0, is a family of strata preserving automorphisms of \(Z_1\), fixing \(0 \in Z_1\), with parameters \((z_2, z_3) \in Z_2 \times Z_3\). The analogous result holds for \(\psi_2\) and since \(V_1\) and \(V_2\) are good, we have (6). Part (7) is immediate.

**Proof of Theorem 1.6.** It follows from Lemma 7.2 that \(V\) is good, and we also have that any \(\psi \in \text{Aut}_{qe}(Z)\) is a product of elements \(\psi_i \in \text{Aut}_{qe}(Z_i)\), \(1 \leq i \leq 3\), where \(\psi_2\) preserves the closures of the strata corresponding to short roots. Then it follows from Proposition 3.1, Theorem 4.4 and Corollary 1.3 that \(V\) has the lifting property.

**Proof of Theorem 1.5.** By Theorem 1.6 we are in the following situation. We have \(\psi \in \text{Aut}_h(Z)\) and \(\psi\) is a deformation \(\psi_t\) with \(\psi_1 = \psi\) and \(\psi_0\) lifts to a \(\sigma\)-equivariant \(\Psi_0 \in \text{GL}(V)\). Let \(\tilde{\psi}_t = \psi_t \circ (\psi_0^{-1})\). Then \(\tilde{\psi}_t\) is an isotopy starting at the identity consisting of strata preserving automorphisms. Thus it is obtained by integrating a holomorphic (complex) time dependent strata preserving vector field \(A(t, z)\). Since \(\pi_1: X_h(V)^G \rightarrow X_h(Z)\) is surjective, we can lift \(A\) to a time dependent invariant holomorphic vector field \(B(t, v)\). By [Sch12, Theorem 3.4] we can integrate \(B\) for \(0 \leq t \leq 1\) to get \(G\)-equivariant biholomorphic lifts \(\Psi_t\) of \(\psi_t\). Then \(\Psi_1 \circ \Psi_0\) is a \(\sigma\)-equivariant biholomorphic lift of \(\psi\).

8. **Compact Lie groups**

Let \(K_i, i = 1, \ldots, d\), be simple compact adjoint Lie groups and consider the natural action of \(K := \prod_i K_i\) on \(W := \bigoplus_i t_i\), where \(r_i \in \mathbb{N}, 1 \leq i \leq d\). Let \(\pi: W \rightarrow W/K\) denote the quotient mapping. We can put a smooth structure on \(W/K\) (see [Sch75]), as follows. A function on \(W/K\) is smooth if it pulls back to a (necessarily \(K\)-invariant) smooth function on \(W\). We can make this more concrete, as follows. Let \(p_1, \ldots, p_r\) be homogeneous generators of \(R[W]^K\) and let \(p = (p_1, \ldots, p_r): W \rightarrow \mathbb{R}^r\). Now \(p\) is proper and separates the \(K\)-orbits in \(W\). Let \(X\) denote \(p(W) \subset \mathbb{R}^r\). Then \(X\) is a closed semialgebraic set and \(\bar{p}\) induces a homeomorphism \(\bar{p}: W/K \rightarrow X\). The main theorem of [Sch75] says that \(p^*C^\infty(X) = C^\infty(W^K)\) where \(f\) on \(X\) is smooth if it extends to a smooth function in a neighborhood of \(X\). Now we can define the notion of a smooth mapping \(X \rightarrow X\) (or \(W/K \rightarrow W/K\)) in the obvious way. We also see an \(\mathbb{R}^r\)-action on \(X\) where \(t \cdot (y_1, \ldots, y_r) = (t^{e_1}y_1, \ldots, t^{e_r}y_r)\) for \(t \in \mathbb{R}^r\), \((y_1, \ldots, y_r) \in X\) and \(e_j\) the degree of \(p_j, j = 1, \ldots, r\). We denote by \(\text{Aut}_{qe}(W/K)\) the quasilinear automorphisms of \(W/K\), i.e., the automorphisms which commute with the \(\mathbb{R}^r\)-action.

We have the usual stratification of \(W/K\) determined by the conjugacy classes of isotropy groups of orbits. There is a unique open stratum, the principal stratum, which consists of the orbits with trivial slice representation. It is
connected and dense in $W/K$. Let $E_s$ be the union of the closures of the codimension one strata of $W/K$ corresponding to the short roots. We call a diffeomorphism $\Psi: W \to W$ $\sigma$-equivariant if $\Psi \circ k = \sigma(k) \circ \Psi$, $k \in K$, where $\sigma \in \text{Aut}(K)$.

**Theorem 8.1.** Let $\psi: W/K \to W/K$ be a diffeomorphism. Then $\psi$ lifts to a $\sigma$-equivariant diffeomorphism $\Psi: W \to W$ if and only if $\psi$ preserves $E_s$.

**Proof.** Necessity of the condition on $\psi$ is established as before. Now assume that $\psi$ preserves $E_s$. By [Los01, Theorem 2.2], $\psi_0(x) = \lim_{t \to 0} t^{-1} \cdot \psi(t \cdot x)$ exists for every $x \in W/K$ where, of course, $\psi_0$ is quasilinear. By [Bie75, Theorem A], $\psi_0$ (and $\psi$) necessarily preserve the principal stratum and the union of the codimension one strata. We want to show that $\psi_0$ has a $\sigma$-equivariant lift to $\text{GL}(W)$. Complexifying and applying Lemma 7.2 we see that it is enough to treat the following three cases:

1. $W$ is a direct sum of representations isomorphic to $(2\mathbb{R}^3, \text{SO}(3, \mathbb{R}))$.
2. $W = \mathfrak{t}$ where $K$ is compact semisimple. Here we need that $\psi_0$ preserves $E_s$.
3. $W$ is large (defined as before).

The proof of lifting in case (1) is as in Proposition 3.1. For case (3) one can use [Sch12, Proposition 8.5] which allows one to deduce the lifting property in (3) from Corollary 1.3. Note that (3) is not a trivial consequence of Corollary 1.3, since we have to show that the lift preserves $W \subset W \otimes \mathbb{C}$. For case (2), a theorem of Strub [Str82] (see also [Sch09, Theorem 2.11]) shows that there is a lift of $\psi_0$ to $\tau \in N_{\text{GL}(\mathfrak{t})}(W)$ where $\mathfrak{t}$ is the Lie algebra of a maximal torus of $K$ and $W$ is the Weyl group. Then the argument of Lemma 4.2 through Theorem 4.4 shows that $\tau$ lifts to an element of $N_{\text{GL}(\mathfrak{t})}(K)$. Thus, if $\psi$ preserves $E_s$, there is a smooth deformation $\psi_t$ with $\psi_1 = \psi$ such that $\psi_0$ lifts to $\Psi_0 \in N_{\text{GL}(W)}(K)$. By the isotopy lifting theorem of [Sch80] there is an equivariant diffeomorphism lifting $\psi \circ \psi_0^{-1}$, hence there is a $\sigma$-equivariant diffeomorphism $\Psi$ lifting $\psi$. ■

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