A Coalgebraic View on Reachability

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Abstract
Coalgebras for an endofunctor provide a category-theoretic framework for modeling a wide range of state-based systems of various types. We provide an iterative construction of the reachable part of a given pointed coalgebra that is inspired and resembles the standard breadth-first search procedure to compute the reachable part of a graph. We also study coalgebras in Kleisli categories. For a functor extending a functor on the case category we show that the reachable part of a given pointed coalgebra can be computed in that base category.

1 Introduction
Coalgebras provide a convenient category theoretic framework in which to model state-based systems and automata whose transition type is described by an endofunctor. For example, classical deterministic and non-deterministic automata, labelled transition systems as well as their weighted and probabilistic variants arise as instances of coalgebras.

A key notion in the theories of state-based systems of various types is reachability, i.e. the construction of a subsystem of a given system containing precisely those states that can be reached from (a set of) initial states along a path in the transition graph of the system. For example, in automata theory, computing the reachable part of a given deterministic automaton is the first step in every minimization procedure. It is well-known that reachability has a simple formulation on
the level of coalgebras. In fact, a pointed coalgebra, i.e. one with a given initial state, is called reachable if it does not contain any proper subcoalgebra containing the initial state [4]. Moreover, for a functor preserving intersections, the reachable part of a given pointed coalgebra is obtained by taking the intersection of all the subcoalgebras containing the initial state. The purpose of the present paper is a more thorough study of reachable coalgebras and, in particular, a new iterative construction of the reachable part of a given pointed coalgebra.

After recalling some preliminaries in Section 2, we discuss some background material on endofunctors on Set preserving intersections in Section 3 and on the canonical graph of a coalgebra in Set in Section 4.

In Section 5 we present a new iterative construction of the reachable part of a given pointed coalgebra that is inspired and closely resembles the standard breadth-first search in graphs. Our construction works for coalgebras over every well-powered category \( \mathcal{C} \) having coproducts and a factorization system \((\mathcal{E}, \mathcal{M})\), where \( \mathcal{M} \) consists of monomorphisms. Moreover, the coalgebraic type functor \( F : \mathcal{C} \to \mathcal{C} \) is assumed to have least bounds, a notion previously introduced by Block [10]. Extending a result by Gumm [13] for set functors, we prove in Proposition 5.7 that a functor has least bounds if and only if it preserves intersections. Moreover, this is equivalent, to the existence of a left-adjoint to the operator \( \circ_f : \text{Sub}(Y) \to \text{Sub}(X) \) for every \( f : X \to FY \), which assigns to every subobject \( m \) of \( Y \) the pullback of \( Fm \) along \( f \). Note that, for a coalgebra \( c : C \to FC \), this operator is Jacobs’ “next time” operator [15]. In our iterative construction of the reachable part we use its left-adjoint \( \ominus_c \) on \( \text{Sub}(C) \), which corresponds to the “previous time” operator of classical linear temporal logic [19]. In fact, we consider a coalgebra \( c : C \to FC \) together with an \( I \)-pointing, i.e. a morphism \( i_C : I \to C \), where \( I \) is some object, and we prove in our main result Theorem 5.18 that the reachable part of the given \( I \)-pointed coalgebra is given by the union of all \( \ominus_c(m_0) \), where \( m_0 \) is given by \((\mathcal{E}, \mathcal{M})\)-factorizing the given \( I \)-pointing \( i_C \). Moreover, we prove that, whenever \( F \) preserves inverse images, the reachable part is a coreflection of \((C, c, i_C)\) into the category of \( I \)-pointed reachable \( F \)-coalgebras (Theorem 5.21). We also show that for an \( I \)-pointed coalgebra in Set the above iterative construction of the reachable part can be performed as a standard breadth-first search on the canonical graph (Corollary 5.24).

Finally, we study in Section 6 coalgebras for a functor \( \bar{F} \) on a Kleisli category over \( \mathcal{C} \), which is an extension of an endofunctor \( F \) on \( \mathcal{C} \). Here we show that the reachable part of a given \( I \)-pointed \( \bar{F} \)-coalgebra can be constructed as the reachable part of a related coalgebra in \( \mathcal{C} \).

Related work. Our results are based on the notion of reachable coalgebras introduced by Adámek et al. [4]. Our construction of the reachable part appears in work by Wißmann, Dubut, Katsumata, and Hasuo [27] (see Lemma A.5 of the full version), where it is used as an auxiliary construction in order to give a characterization of the reachability of a coalgebras in terms of paths [27, Section 3.5]. However, that work does not connect the construction with the “previous time”
The “previous time” operator is also studied by Barlocco, Kupke, and Rot [8]. They work with a complete and well-powered category \( \mathcal{C} \), and, like us, they show that the reachable part of a given pointed coalgebra can be obtained by an iterative construction using the previous time operator. Their results were obtained independently and almost at the same time as ours.

2 Pointed and Reachable Coalgebras

In this section we recall some preliminaries on pointed and reachable coalgebras for an endofunctor. A coalgebra for an endofunctor \( F: \mathcal{C} \to \mathcal{C} \) (or \( F \)-coalgebra, for short) is a pair \( (C, c) \) where \( C \) is a object of \( \mathcal{C} \) called the carrier of the coalgebra and \( c: C \to FC \) a morphism called the structure of the coalgebra. A coalgebra homomorphism \( h: (C, c) \to (D, d) \) is a morphism \( h: C \to D \) of \( \mathcal{C} \) that commutes with the structures on \( C \) and \( D \), i.e. the following square commutes:

\[
\begin{array}{ccc}
C & \xrightarrow{c} & FC \\
h \downarrow & & \downarrow Fh \\
D & \xrightarrow{d} & FD
\end{array}
\]

**Definition 2.1.** Given an endofunctor \( F: \mathcal{C} \to \mathcal{C} \) and an object \( I \) of \( \mathcal{C} \), an \( I \)-pointed \( F \)-coalgebra is a triple \( (C, c, i_C) \) where \( (C, c) \) is an \( F \)-coalgebra and \( i_C: I \to C \) a morphism of \( \mathcal{C} \). A homomorphisms of pointed coalgebras from \( (C, c, i_C) \) to \( (D, d, i_D) \) is a coalgebra homomorphism \( h: (C, c) \to (D, d) \) that commutes with the pointings, i.e. \( h \cdot i_C = i_D \). We denote by

\[
\text{Coalg}_I(F)
\]

the category of \( I \)-pointed \( F \)-coalgebra and their homomorphisms.

**Example 2.2.** Pointed coalgebras allow to capture many kinds of state-based systems categorically. We just recall a couple of examples; for further examples, see e.g. [23].

1. Deterministic automata are 5-tuples \( (S, \Sigma, \delta, s_0, F) \), with a set \( S \) of states, an input alphabet \( \Sigma \), a next-state function \( \delta: S \times \Sigma \to S \), an initial state \( s_0 \in S \) and a set \( F \subseteq S \) of final states. Here we fix the input alphabet \( \Sigma \). Representing the subset \( F \) by its characteristic function \( f: S \to \{0, 1\} \), and currying \( \delta \) we see that a deterministic automaton is, equivalently, a pointed coalgebra for \( FX = \{0, 1\} \times X^\Sigma \) on \( \text{Set} \) with the pointing \( s_0: 1 \to S \) given by the initial state.

2. Non-deterministic automata are similar to deterministic ones, except that in lieu of a next-state function one has a next-state relation \( \delta \subseteq S \times \Sigma \times S \) and a set of initial states \( I \subseteq S \). These data can be represented as two functions \( i: 1 \to \mathcal{P}S \) and \( c: S \to \mathcal{P}(1+\Sigma \times S) \), where \( \mathcal{P} \) denotes the power-set. That means
that a non-deterministic automaton is, equivalently, a coalgebra for the functor $F X = 1 + \Sigma \times X$ on the Kleisli category of the monad $\mathcal{P}$, i.e. the category $\text{Rel}$ of sets and relations.

(3) Pointed graphs are, equivalently, coalgebras for the power-set functor $\mathcal{P} : \text{Set} \rightarrow \text{Set}$. Indeed, a pointed coalgebra

$$1 \xrightarrow{v_0} V \xrightarrow{e} \mathcal{P} V$$

consists of a set of vertices $V$ with directed edges given by a binary relation, represented by $e$, and a distinguished node $v_0 \in S$.

(4) The category of nominal sets provides a framework where freshness of names or resources in systems can be modelled or where systems can store values from infinite data types. We briefly recall the definition of the category $\text{Nom}$ of nominal sets (see e.g. Pitt [22]). We fix a countably infinite set $A$ of atomic names. Let $\mathcal{S}_f(A)$ denote the group of all finite permutations on $A$ (which is generated by all transpositions $(a \ b)$ for $a, b \in A$). Let $X$ be a set with an action of this group, denoted by $\pi \cdot x$ for a finite permutation $\pi$ and $x \in X$. A subset $A \subseteq A$ is called a support of an element $x \in X$ provided that every permutation $\pi \in \mathcal{S}_f(A)$ that fixes all elements of $A$ also fixes $x$:

$$\pi(a) = a \text{ for all } a \in A \implies \pi \cdot x = x.$$ 

A nominal set is a set with an action of the group $\mathcal{S}_f(A)$ such that every element has a finite support. The category $\text{Nom}$ is formed by nominal sets and equivariant maps, i.e. maps preserving the given group action. Each nominal set $X$ is thus equipped with an equivariant map $\text{supp} : X \rightarrow \mathcal{P}_f(A)$ that assigns to each element its support. For example, the set of terms of the $\lambda$-calculus modulo renaming of bound variables is a nominal set, where the support of a $\lambda$-term is the set of its free variables. Variable binding can be modelled by the binding functor on $\text{Nom}$. This functor maps a nominal set $X$ to the nominal set $[A](X) = ([A] \times X)/\sim$ where $(a, x) \sim (b, y)$ iff $(c \ a) \cdot x = (c \ b) \cdot y$ for any fresh $c$, i.e. $c \notin \text{supp}(x) \cap \text{supp}(y)$. That means that $\sim$ abstracts $\alpha$-equivalence known from calculi with name binding such as the $\lambda$-calculus. In fact, the set of $\lambda$-expressions modulo $\alpha$-equivalence is the initial algebra for the endofunctor $F X = A + X \times X + [A]X$ on $\text{Nom}$ [12].

Coalgebras for functors on $\text{Nom}$ have been studied e.g. in [17, 20, 21].

There are a number of different notions of automata featuring a nominal set of states and which process words over the infinite input alphabet $A$. One example of a coalgebraic notion of automata are regular nondeterministic nominal automata (RNNA) [24]; they are precisely that coalgebras for the functor on nominal sets given by

$$F X = 2 \times \mathcal{P}_{ufs}(A \times X) \times \mathcal{P}_{ufs}([A]X),$$

where $\mathcal{P}_{ufs}$ is a variant of the finite power-set functor on $\text{Nom}$ – it maps a nominal set $X$ to the nominal set of all of its uniformly supported subsets $S$, i.e. $S$ is an equivariant subset of $X$ such that $\bigcup_{x \in S} \text{supp}(x)$ is finite.
Intuitively, in a coalgebra $C \rightarrow 2 \times \mathcal{P}_{ufs}(A \times C) \times \mathcal{P}_{ufs}([A]C)$, 2 marks whether a state is final; $\mathcal{P}_{ufs}([A]X)$ is the set of binding transitions from the state $x$, i.e. where the input character is stored for later use; and $\mathcal{P}_{ufs}(A \times X)$ is the set of transitions that compare the input character to a character stored earlier. Let $A^\#_n$ be the nominal set of $n$-tuples with distinct components, i.e. $A^\#_n = \{(a_1, \ldots, a_n) \in A^n \mid |\{a_1, \ldots, a_n\}| = n\}$. Then $A^\#_n$-pointed RNNA accept nominal languages, i.e. equivariant maps $L: A^* \rightarrow 2$, whose support has a cardinality of at most $n$ [24, Corollary 5.5], under both language semantics considered in loc.cit. Note that it is important not to restrict $I$ to be the terminal object; in fact, that would restrict initial objects to have empty support, which may not be desirable in applications.

(5) An alternative approach to bisimulation of transition systems via so called open maps was introduced by Joyal, Nielsen, and Winskel [16]. There, one considers functors of type $J: \mathcal{P} \rightarrow \mathcal{M}$ from a small category of $\mathcal{P}$ “paths” or “linear systems” to the category $\mathcal{M}$ of “all systems” under consideration. This functor $J$ defines a notion of open map – we do not recall the definition as it is irrelevant here; for details and the definition of open map see loc. cit. The objects in $\mathcal{M}$ are usually defined as systems with an initial state, and morphisms in $\mathcal{M}$ are maps between systems that preserve (but not necessarily reflect) outgoing transitions of states, whereas the open maps in $\mathcal{M}$ are morphisms that do reflect the outgoing transitions of states that are in the reachable part of the system. Let $|\mathcal{P}|$ denote the set of objects of $\mathcal{P}$. It was shown by Lasota [18] that the canonical functor $\mathcal{M}(J(-), (-)): \mathcal{M} \rightarrow \text{Set}^{|\mathcal{P}|}$ sends open maps in $\mathcal{M}$ to $F$-coalgebra homomorphisms for the following functor $F$ on $\text{Set}^{|\mathcal{P}|}$:

$$F: \text{Set}^{|\mathcal{P}|} \rightarrow \text{Set}^{|\mathcal{P}|} \quad (X_P)_{P \in \mathcal{P}} \mapsto \left( \prod_{Q \in |\mathcal{P}|} \mathcal{P}(X_Q)^{\mathcal{P}(P, Q)} \right)_{P \in \mathcal{P}}$$

If $\mathcal{P}$ has an initial object $0_\mathcal{P}$ that is preserved by $J$, then the subcategory of $\mathcal{M}$ formed by open maps in fact embeds into the category of $I$-pointed $F$-coalgebras [27], where

$$I \in \text{Set}^{|\mathcal{P}|} \quad \text{with} \quad I_P = \begin{cases} 1 & \text{if } P = 0_\mathcal{P} \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that once again $I$ is not the terminal object of $\text{Set}^{|\mathcal{P}|}$.

Our overall setting is that of a category $\mathcal{C}$ equipped with a factorization system $(\mathcal{E}, \mathcal{M})$, i.e. (1) $\mathcal{E}$ and $\mathcal{M}$ are classes of morphisms of $\mathcal{C}$ that are closed under composition with isomorphisms, (2) every morphism $f$ of $\mathcal{C}$ has a factorization $f = m \cdot e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$, and (3) the following unique diagonal fill-in
property holds: for every commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{e} & B \\
\downarrow{f} & & \downarrow{g} \\
C & \xrightarrow{m} & D
\end{array}
\]

with \( e \in \mathcal{E} \) and \( m \in \mathcal{M} \) there exists a unique morphism \( d: B \to C \) such that \( m \cdot d = g \) and \( d \cdot e = f \). We will denote morphisms in \( \mathcal{M} \) by \( \rightarrow \) and those in \( \mathcal{E} \) by \( \Rightarrow \). Typical examples are given with \( \mathcal{E} \) being a subclass of epimorphisms, and \( \mathcal{M} \) a subclass of monomorphisms: (regular epi, mono) in regular categories, (epi, strong mono) in quasitoposes, (epi, mono) in toposes, etc.

We shall later assume that \( \mathcal{M} \) is a class of monomorphisms. Whenever we speak of a subobject of some object \( X \) we mean one that is represented by a morphism \( m: S \Rightarrow X \) in \( \mathcal{M} \). Moreover, we shall speak of “the subobject \( m \)”, i.e. we use representatives to refer to subobjects. The subobjects of an object \( X \) form a partially ordered class

\[
\text{Sub}(X)
\]

in the usual way: for \( m: S \Rightarrow X \) and \( m': S' \Rightarrow X \) we have \( m \leq m' \) if there exists \( i: S \to S' \) with \( m' \cdot i = m \).

**Remark 2.3.** \((\mathcal{E}, \mathcal{M})\)-factorization systems have many properties known from surjective and injective maps on \( \text{Set} \) (see [2, Chapter 14]):

1. \( \mathcal{E} \cap \mathcal{M} \) is the class of isomorphisms of \( \mathcal{C} \).
2. \( \mathcal{M} \) is stable under pullbacks.
3. If \( f \cdot g \in \mathcal{M} \) and \( f \in \mathcal{M} \), then \( g \in \mathcal{M} \).
4. \( \mathcal{E} \) and \( \mathcal{M} \) are closed under composition.

**Remark 2.4.** (1) Subcoalgebras of pointed coalgebras are understood to be formed w.r.t. the class \( \mathcal{M} \), i.e. a subcoalgebra is represented by a homomorphism \( m: (S,s,i_S) \Rightarrow (C,c,i_C) \) with \( m \in \mathcal{M} \). Similarly, a quotient coalgebra is represented by a coalgebra homomorphism \( q: (C,c,i_C) \Rightarrow (Q,q,i_Q) \) with \( q \in \mathcal{E} \).

2. Suppose that \( F: \mathcal{C} \to \mathcal{C} \) preserves \( \mathcal{M} \)-morphisms. Then the factorization system \((\mathcal{E}, \mathcal{M})\) lifts to \( \text{Coalg}_I(F) \) as follows. For every homomorphism \( h: (C,c,i_C) \Rightarrow (D,d,i_D) \) one takes its factorization \( h = m \cdot e \) in \( \mathcal{C} \) and then obtains a unique coalgebra structure such that \( e \) and \( m \) are coalgebra homomorphisms between pointed coalgebras using the unique diagonal fill-in property:
Definition 2.5 (Reachable coalgebra [4]). An $I$-pointed coalgebra $(C, c, i_0)$ is called reachable if it has no proper pointed subcoalgebra, i.e. every homomorphism $m: (C', c', i_{C'}) \to (C, c, i_C)$ of $I$-pointed coalgebras with $m \in \mathcal{M}$ is an isomorphism.

Remark 2.6. When $I = 0$ is the initial object and $\mathcal{M}$ is a class of monomorphisms, then a coalgebra is reachable if and only if it is quotient coalgebra of $(0, u, \text{id}_0)$ where $u: 0 \to F0$ is the unique morphism. Indeed, if $(C, c, i_C)$ is reachable let $h: (0, u, \text{id}_0) \to (C, c, i_C)$ be the unique coalgebra homomorphism and take the $(\mathcal{E}, \mathcal{M})$-factorization $h = m \cdot e$. Then $m$ represents an $I$-pointed subcoalgebra $(C, c, i_C)$ and thus is an isomorphism.

Conversely, if $e: (0, u, \text{id}_0) \to (C, c, i_C)$ is a quotient coalgebra and $m: (S, s, i_S) \to (C, c, i_C)$ is any subcoalgebra then $e = m \cdot h$ where $h: 0 \to S$ is the unique morphism. By the unique diagonalization property, we obtain $d: C \to S$ such that $m \cdot d = \text{id}_C$. Thus $m$ is a split epimorphism and a monomorphism, whence an isomorphism. Consequently, $(C, c, i_C)$ is reachable.

Finally, it follows that if the unique morphisms $0 \to X$ are in $\mathcal{M}$ for every object $X$, then $(0, u, \text{id}_0)$ is the only reachable coalgebra.

Example 2.7. (1) For a pointed graph, reachability is clearly the usual graph theoretic concept: a pointed coalgebra $(V, a, v_0)$ for $\mathcal{P}$ is reachable if and only if every of its nodes can be reached by a directed path from the distinguished node $i_S$.

(2) A deterministic automaton regarded as a pointed coalgebra for $\mathcal{F}X = \{0, 1\} \times X^\Sigma$ on $\text{Set}$ is reachable if and only if every of its states is reachable in finitely many steps from its initial state. This is not difficult to see directly, but it follows immediately from Theorem 4.6.

3 Functors preserving intersections

We shall see in Section 5 that the central assumption for our constructions of the reachable part of a pointed $\mathcal{F}$-coalgebra is equivalent to the functor $\mathcal{F}$ preserving all intersections. For set functors we discuss this condition in the present section. Indeed, it is an extremely mild condition satisfied by many set functors of interest:

Example 3.1. The collection of set functors which preserve intersections is closed under products, coproducts, and composition. Consequently, every polynomial endofunctor on $\text{Set}$ preserves all intersections. Moreover it is easy to see that the power set functor $\mathcal{P}$, the bag functor $\mathcal{B}$ mapping every set $X$ to the set of finite multisets on $X$, as well as the functor $\mathcal{D}$ mapping $X$ to the set of (countably supported) probability measures on $X$ preserve intersections.

Among the finitary set functors “essentially” all functors preserve intersections. This follows from the results of Trnková on set functors as we shall now explain. First, recall that a functor is called finitary, if it preserves filtered colimits. For a
set functor $F$ this is equivalent to being \textit{finitely bounded} \cite[Corollary 3.11]{set-theory}, which is the following condition: for every element $x \in FX$ there exists a finite subset $M \subseteq X$ such that $x \in Fi[FM]$, where $i: M \hookrightarrow X$ is the inclusion map.

Secondly, as shown by Trnková \cite{trnkova}, every set functor preserves finite non-empty intersections. Moreover, she proved that one can turn every set functor into one that preserves all finite intersections by a simple modification at the empty set:

\textbf{Proposition 3.2 (Trnková \cite{trnkova}).} For every set functor $F$ there exists an essentially unique set functor $\bar{F}$ which coincides with $F$ on nonempty sets and functions and preserves finite intersections (whence monomorphisms).

For the proof see \cite[Propositions III.5 and II.4]{trnkova}; for a more direct proof see Adámek and Trnková \cite[III.4.5]{set-theory}. We call the functor $\bar{F}$ the \textit{Trnková hull} of $F$.

\textbf{Remark 3.3.} (1) In fact, Trnková gave a construction of $\bar{F}$: she defined $\bar{F}\emptyset$ as the set of all natural transformations $C_{01} \to F$, where $C_{01}$ is the set functor with $C_{01}\emptyset = \emptyset$ and $C_{01}X = 1$ for all non-empty sets $X$, and $\bar{F}e$, for the empty map $e: \emptyset \to X$ with $X \neq \emptyset$, maps a natural transformation $\tau : C_{01} \to F$ to the element given by $\tau_X : 1 \to FX$.

(2) There is also a different construction of $\bar{F}$ due to Barr \cite{barr}: consider the two functions $t, f : 1 \hookrightarrow 2$. Their intersection is the empty function $e : \emptyset \to 1$. Since $\bar{F}$ must preserve this intersection it follows that $\bar{F}e$ is monic and forms (not only a pullback but also) an equalizer of $\bar{F}t = Ft$ and $\bar{F}f = Ff$. Thus $\bar{F}$ must be defined on $\emptyset$ (and $e$) as the equalizer

$$
\begin{array}{ccc}
\bar{F}\emptyset & \xrightarrow{\bar{F}e} & \bar{F}1 = F1 \\
\downarrow & & \downarrow Ff \\
F2,
\end{array}
$$

and on all nonempty functions $f$, one defines $\bar{F}f = Ff$.

(3) Trnková proved that $\bar{F}$ defines a set functor preserving finite intersections. From the proof in \textit{op. cit.} it also follows that if $F$ is finitary, so if $\bar{F}$.

(4) Furthermore, $\bar{F}$ is a reflection of $F$ into the full subcategory of the category of all endofunctors on $\text{Set}$ given by those endofunctors preserving finite intersections. That means there is a natural transformation $r : F \to \bar{F}$ such that for every natural transformation $s : F \to G$ where $G : \text{Set} \to \text{Set}$ preserves finite intersections there exists a unique natural transformation $s^2 : \bar{F} \to G$ such that $s^2 \cdot r = s$ (see \cite[Corollary VII.2]{categorical-algebra} for details).

(5) Finally, note that the categories of coalgebras for $F$ and its Trnková hull $\bar{F}$ are clearly isomorphic.

For the following fact, see e.g. Adámek et al. \cite[Proof of Lemma 8.8]{coalgebras}; we include the proof for the convenience of the reader.

\textbf{Corollary 3.4.} \textit{The Trnková hull of a finitary set functor preserves all intersections.}
Proof. Let $F$ be a finitary set functor. Since $\bar{F}$ is finitary and preserves finite intersections, for every element $x \in \bar{F}X$, there exists a least finite set $m : Y \hookrightarrow X$ with $x$ contained in $\bar{F}m$. Preservation of all intersections now follows easily: given subsets $v_i : V_i \hookrightarrow X$, $i \in I$, with $x$ contained in the image of $\bar{F}v_i$ for each $i$, then $x$ also lies in the image of the finite set $v_i \cap m$, hence $m \subseteq v_i$ by minimality. This proves $m \subseteq \bigcap_{i \in I} v_i$, thus, $x$ lies in the image of $\bar{F}(\bigcap_{i \in I} v_i)$, as required. \qed

Remark 3.5. Note that the argument in Corollary 3.4 can be generalized to locally finitely presentable categories, see e.g. Adámek and Rosický [6] for the definition. In fact, let $\mathcal{C}$ be a locally finitely presentable category in which every finitely generated object only has a finite number of subobjects; for example, Set or the categories of nominal sets (see Example 5.3(4)), of posets, and of graphs.

Then every finitary endofunctor $F$ on $\mathcal{C}$ preserving finite intersections preserves all intersections. Indeed, since $F$ is finitary, for every monomorphism $m : X \rightarrow FY$ with $X$ finitely generated there exists a subobject $z : Z \hookrightarrow Y$ with $Z$ finitely generated such that $m$ factorizes through $Fz$, i.e. there exists some $g : X \rightarrow FZ$ such that $Fm \cdot g = f$ (see e.g. [5]). Since $F$ preserves finite intersections it follows that there is a least subobject $z' : Z' \hookrightarrow Y$ such that $m$ factorizes through $Fz'$. Indeed, take the intersection $m' : Y' \hookrightarrow Y$ of all such subobjects $m'$, which is equal to the finite intersection of all $m' \cap m \leq m$ by hypothesis. Since $F$ preserves this finite intersection we obtain a morphism $g' : X \rightarrow FY'$ such that $Fm' \cdot g' = m$.

Preservation of all intersections now follows easily. Given subobjects $v_i : V_i \hookrightarrow X$, $i \in I$, and $m : Y \rightarrow FX$, with $m \leq Fv_i$ for all $i \in I$. We first assume that $Y$ is finitely generated. Take the least $m' : Y' \hookrightarrow X$ such that $m$ factorizes through $Fm'$, i.e. $m \leq Fm'$. Then we have $m \leq F(v_i \cap m') \leq Fv_i$ for all $i \in I$, where the first inclusion uses that $F$ preserves finite intersections. Thus, $m$ factorizes through $Fv_i$, and therefore $m' \leq v_i$ by minimality for every $i \in I$. Thus, $m \leq Fm' \leq F(\bigcap_{i \in I} v_i)$ as desired.

For arbitrary $m : X \rightarrow FY$ write $X$ as the directed union of all its subobjects $s_j : X_j \rightarrow X$ with $X_j$ finitely generated. Then every $s_j$ is contained in $F(\bigcap_{i \in I} v_i)$ by the previous argument, and therefore so is their union $m$.

Let us conclude this section by coming back to coalgebras to note that the condition that $F$ preserves intersection is significant for us because it entails that every $F$-coalgebra has a reachable part, i.e. a unique reachable subcoalgebra. Indeed, recall [4] that for an intersection preserving endofunctor $F$ on a category $\mathcal{C}$ with intersections a reachable subcoalgebra can be obtained as the intersection of all subcoalgebras of $(C, c, i_C)$. Moreover, this intersection is the unique reachable subcoalgebra of $(C, c, i_C)$, for given two reachable subcoalgebras $S_1$ and $S_2$ of $(C, c, i_C)$ their intersection forms an $I$-pointed subcoalgebra of $S_1$ and $S_2$ and so must be isomorphic to both, thus $S_1 \cong S_2$. 

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Note that for a given functor $F : \text{Set} \to \text{Set}$ one may define for every set $X$ a map $\tau_X : FX \to PX$ by
\[
\tau_X(t) = \{ x \in X \mid 1 \xrightarrow{t} FX \text{ does not factor through } F(X \setminus \{x\}) \xrightarrow{Fi} FX \},
\]
where $i : X \setminus \{x\} \hookrightarrow X$ denotes the inclusion map.

Intuitively, $\tau_X(t)$ is the set of elements of $X$ used to define $t$.

**Definition 4.1** [13]. The canonical graph of a coalgebra $c : C \to FC$ is the graph given by
\[
C \xleftarrow{c} FC \xrightarrow{\tau_C} PC.
\]

**Example 4.2.** For the functor $FX = \{0,1\} \times X^\Sigma$, we have for every $i = 0, 1$ and $t : \Sigma \to X$ that
\[
\tau_X(i, t) = \{ t(s) \mid s \in \Sigma \}.
\]
Hence, the canonical graph of a deterministic automaton considered as an $F$-coalgebra is precisely its usual state transition graph (forgetting the labels of transitions and the finality of states).

**Lemma 4.3** [13, Theorem 7.4]. If $F : \text{Set} \to \text{Set}$ preserves all intersections, then the above maps $\tau_X : FX \to PX$ form a sub-cartesian transformation, i.e. for every injective map $m : X \to Y$ the following diagram is a pullback square:
\[
\begin{array}{ccc}
FX & \xrightarrow{\tau_X} & PX \\
\downarrow{Fm} & & \downarrow{Pm} \\
FX & \xrightarrow{\tau_Y} & PX
\end{array}
\]

Conversely, if $\tau$ is a sub-cartesian transformation, then $F$ preserves all intersections.$^1$

**Theorem 4.4** (Gumm [13, Theorem 8.1]). Assume that $F$ preserves inverse images and all intersections. Then $\tau : F \to P$ is a natural transformation.

**Example 4.5.** To see that $\tau$ is not a natural transformation in general, one may consider the functor $R : \text{Set} \to \text{Set}$ defined by $RX = \{(x, y) \in X \times X : x \neq y\} + \{\ast\}$ on sets $X$ and for a function $f : X \to Y$ put
\[
Rf(\ast) = \ast \quad \text{and} \quad Rf(x, y) = \begin{cases} 
\ast & \text{if } f(x) \neq f(y) \\
(f(x), f(y)) & \text{else.}
\end{cases}
\]
Now let $X = \{0,1\}$, $Y = \{0\}$, and $f : X \to Y$ the evident function. Then $(0, 1) \in FX$, and $\tau_X(0, 1) = X$. Further, $\mathcal{P}f(X) = Y$. But $Rf(0, 1) = \ast$, and $\tau_Y(\ast) = \emptyset$.$^1$

$^1$For this converse, loc. cit. assumed that $F$ preserves monomorphisms; however, this is not needed since $\mathcal{P}$ preserves monomorphisms and monomorphisms are stable under pullback.
Our observation in this section is that reachability of a coalgebra and its canonical graph are equivalent concepts:

**Theorem 4.6.** Let \( F : \text{Set} \to \text{Set} \) preserve all intersections. Then a coalgebra for \( F \) is reachable if and only if so is its canonical graph.

**Proof.** Let \((C, c, i_C)\) be an \( I \)-pointed \( F \)-coalgebra. Then we see that subcoalgebras of \((C, c, i_C)\) are in one-to-one correspondence with subgraphs of the canonical graph. Indeed, given any subcoalgebra \( m : (S, s, i_S) \to (C, c, i_C) \), we have that \((S, \tau_S \cdot s, i_S)\) is an \( I \)-pointed subgraph of \((C, \tau_C \cdot c, i_C)\) via \( m \) due to the commutativity of (4.2). Conversely, let \((S, s, i_S)\) be an \( I \)-pointed subgraph of the canonical graph \((C, \tau_C \cdot c, i_C)\) via the monomorphism \( m : S \to C \), say. Then, using that (4.2) is a pullback, we obtain an \( F \)-coalgebra structure on \( S \) turning it into a subcoalgebra of \((C, c, i_C)\):

\[
\begin{array}{ccc}
S & \xrightarrow{s} & FS \\
\downarrow{m} & & \downarrow{\tau_S} \\
C & \xrightarrow{c} & FC
\end{array}
\xrightarrow{\text{m}}
\begin{array}{ccc}
P S & \xrightarrow{\text{Fm}} & PC \\
\downarrow{\text{Pm}} & & \\
C & \xrightarrow{\text{c}} & FC
\end{array}
\]

We conclude that \((C, c, i_C)\) does not have any proper subcoalgebra w.r.t. \( F \) if and only if its canonical pointed graph \((C, \tau_C \cdot c, i_C)\) does not have a proper subcoalgebra w.r.t \( P \). As we saw in Example 2.7(1), the latter is equivalent to that pointed graph being reachable, which completes the proof. \( \square \)

## 5 Iterative Construction

This section is devoted to a new iterative construction of the reachable part of a given \( I \)-pointed coalgebra \((C, c, i_C)\), the unique reachable subcoalgebra of \((C, c, i_C)\), reminiscent of breath first search for graphs.

**Assumption 5.1.** For the remainder of this section we assume that the base category \( \mathcal{C} \) has arbitrary (small) coproducts, is well-powered and is equipped with an \((\mathcal{E}, \mathcal{M})\)-factorization system, where \( \mathcal{M} \) is a class of monomorphisms.

**Remark 5.2.** We collect a number of easy consequence of Assumption 5.1.

(1) Note that the above assumptions imply that \( \mathcal{C} \) has all unions, i.e. for every object \( C \) of \( \mathcal{C} \) the partially ordered set \( \text{Sub}(C) \) of its subobjects has all joins. In fact, given a family \((m_i : C_i \to C)_{i \in I}\), their union \( m \) is given by the following \((\mathcal{E}, \mathcal{M})\)-factorization:

\[
\bigcup_{i \in I} C_i \xrightarrow{[m_i]_{i \in I}} \bigcup_{i \in I} m_i \xrightarrow{m} C.
\]
(2) It follows that Sub(C) is a complete lattice, and therefore that C has all intersections. Moreover, we show that intersections are given by pullbacks (even though we did not assume their existence). In fact, given the family \((m_i: C_i \to C)_{i \in I}\) take their meet \(m: M \to C\) in Sub(C). The morphisms \(p_i: M \to C_i\) witnessing \(m \leq m_i\) yield the projections of the (wide) pullback. Moreover, given any compatible cone \(f_i: X \to C_i\) such that \(m_i \cdot f_i = m_j \cdot f_j\) for all \(i, j \in I\) take the \((\mathcal{E}, \mathcal{M})\)-factorization \(n \cdot e\) of that morphism and use the diagonal fill-in property

\[
\begin{array}{ccc}
X & \xrightarrow{e} & X' \\
\downarrow f_i & & \downarrow n \\
C_i & \xrightarrow{m_i} & C
\end{array}
\]

in order to see that \(n \leq m_i\) for all \(i \in I\). Thus, we have \(n \leq m\), which is witnessed by a (necessarily unique) morphism \(h: X' \to M\) such that \(n \cdot h = m\). Then \(h \cdot e\) is the desired unique factorizing morphism showing that \(M\) is a wide pullback of the \(m_i\).

(3) In addition, using the well-poweredness of \(C\) we see that it has preimages, i.e. pullbacks along morphisms in \(\mathcal{M}\). Indeed, suppose we are given a morphism \(f: X \to Y\) and a subobject \(m: S \hookrightarrow Y\). Then we form the family of all subobjects \(m_i: M_i \hookrightarrow X\) for which there exists a restricting morphism \(f_i: M_i \to M\), i.e. \(f \cdot m_i = m \cdot f_i\), and we take their union:

\[
(p: P \hookrightarrow X) := \bigcup \{m_i: M_i \hookrightarrow X \mid \exists f_i: M_i \to M \text{ with } f \cdot m_i = m \cdot f_i\}
\]

Using the diagonal fill-in property, we obtain a morphism \(f': P \to M\) such that the diagram below commutes:

\[
\begin{array}{ccc}
\bigcup_{i \in I} M_i & \xrightarrow{(f_i)_{i \in I}} & M \\
\downarrow P & & \downarrow m \\
\bigcup m_i & \xrightarrow{f} & Y
\end{array}
\]

In order to show that the lower square is a pullback, suppose that we have morphisms \(p: Z \to X\) and \(q: Z \to M\) with \(f \cdot p = m \cdot q\). Take the \((\mathcal{E}, \mathcal{M})\)-factorization \(p = (Z \xrightarrow{e'} I \xrightarrow{m'} X)\). By the unique diagonal fill-in property we obtain some \(d: I \to M\) such that \(m \cdot d = f \cdot m'\) and \(d \cdot e' = q\). Thus, \(m': I \hookrightarrow X\) is one of the subobjects \(m_i\) above, and therefore \(m' \leq \bigcup m_i\), i.e. we have a morphism \(s: I \to P\) with \((\bigcup m_i) \cdot s = m'\). Then \(h := s \cdot e': Z \to P\) is the desired factorization of \(p, q\). Indeed, we have

\[(\bigcup m_i) \cdot (s \cdot e') = m' \cdot e' = p,\]
and to see that $f' \cdot h = q$ we use that $m$ is a monomorphism and compute

$$m \cdot f' \cdot h = f \cdot (\bigcup m_i) \cdot h = f \cdot p = m \cdot q.$$  

**Example 5.3.** (1) Recall that every complete category $\mathcal{C}$ is equipped with a (strong epi, mono)-factorization system and with an (epi, strong mono)-factorization system [2, Theorems 14.17 and 14.19].

Hence, every complete and well-powered category $\mathcal{C}$ with coproducts meets Assumption 5.1.

(2) The category $\text{Rel}$ of sets and relations has all coproducts and a factorization system given by

$$\mathcal{E} = \text{all surjective relations}, \quad \text{and} \quad \mathcal{M} = \text{all injective maps}.$$  

(3) Let us consider the distribution monad $\mathcal{D}$ on Set given as a Kleisly triple $(D, \eta, (-)^*)$ as follows: for every set $X$ we have

$$\mathcal{D}X = \{f : X \to [0, 1] | \sum_{x \in X} f(x) = 1\}$$

(note that the above sum necessarily only has countably many non-zero summands) and $\eta_X : X \to \mathcal{D}X$ given by the Dirac distribution

$$\eta_X(x)(y) = \begin{cases} 1 & x = y, \\ 0 & \text{else.} \end{cases}$$

The Kleisli extension of a map $h : X \to \mathcal{D}Y$ is the map $h^* : \mathcal{D}X \to \mathcal{D}Y$ given by

$$h^*(f)(y) = \sum_{x \in X} f(x) \cdot h(x)(y).$$

The Kleisli category $\mathcal{Kl}(\mathcal{D})$ has all coproducts and a factorization system given by the following classes of morphisms:

$$\mathcal{E} = \{e : X \to \mathcal{D}Y | \forall y \in Y \exists x \in X : e(x)(y) \neq 0\}, \text{and}$$

$$\mathcal{M} = \{m : X \to \mathcal{D}Y | m = \eta_Y \cdot m' \text{ for some injective map } m' : X \to Y\}.$$  

Hence, the class $\mathcal{M}$ consists essentially of injective maps considered as morphisms in $\mathcal{Kl}(\mathcal{D})$. It is easy to see that the two classes of morphisms are closed under composition, and that every morphism in $\mathcal{Kl}(\mathcal{D})$ has an essentially unique $(\mathcal{E}, \mathcal{M})$-factorization, given by the least bounds of $\mathcal{D}$. Moreover, it is not difficult to see that $\mathcal{E}$ and $\mathcal{M}$ contain all isomorphisms of $\mathcal{Kl}(\mathcal{D})$. In fact, we show below that those isomorphisms correspond precisely to bijective maps; more precisely, a morphism $h : X \to \mathcal{D}Y$ is an isomorphism if and only if $h = \eta_Y \cdot h'$ for some bijective map $h' : X \to Y$. It follows that $(\mathcal{E}, \mathcal{M})$ is a factorization system [2, Theorem 14.7].
We proceed to prove the above characterization of isomorphisms in $\mathcal{Kl}(\mathcal{D})$, i.e. we show that $h: X \to \mathcal{D}(Y)$ is an isomorphism if and only if for every $x \in Y$ there exists a $y \in Y$ with $h(x)(y) = 1$ and for every $x \in X$ there is at most one $y \in Y$ with $h(x)(y) > 0$ (and hence $h(x)(y) = 1$).

The ‘if’ direction clearly holds. For the ‘only if’ direction suppose that $g: Y \to X$ is inverse to $h$. Firstly, for every $y \in Y$ there is some $x \in X$ with $h(x)(y) > 0$, because $1 = (h \circ g)(y)(y) = \sum_{x \in X} g(y)(x) \cdot h(x)(y)$. For the same reason there is some $y \in Y$ with $h(x)(y) > 0$ for every $x \in X$, and similarly for $g$. Now, it remains to show that for every $x \in X$, there is at most one $y \in Y$ with $h(x)(y) > 0$. Let $y_1, y_2 \in Y$ with $h(x)(y_1) \neq 0 \neq h(x)(y_2)$, and let $y \in Y$ with $g(y)(x) \neq 0$. Then $(h \circ g)(y)(y_1) \neq 0 \neq (h \circ g)(y)(y_2)$, and hence $y_1 = y = y_2$ since $(h \circ g)$ is the identity morphism $\eta_Y$ in $\mathcal{Kl}(\mathcal{D})$.

(4) Let $S = (S, +, \cdot, 0, 1)$ be a semiring. Then similarly as in point (3) we obtain a monad $S(-)$ on Set given as a Kleisli triple as follows: for every set $X$, we have

$$S(X) = \{ f: X \to S \mid f(x) \neq 0 \text{ for finitely many } x \in X \},$$

and the unit $\eta_X: X \to S(X)$ and the Kleisli lifting are defined precisely as in point (3).

We also consider the same classes $\mathcal{C}$ and $\mathcal{M}$ as in the previous point (3), and they can be shown to form a factorization system provided that the given semiring fulfills the following conditions: $(S, +, 0)$ and $(S, \cdot, 1)$ are positive monoids, i.e. whenever $a + b = 0$ then $a = 0$ or $b = 0$, and similarly for the multiplication $\cdot$ and 1, and the semiring is zero-divisor-free, i.e. whenever $a \cdot b = 0$ then $a = 0$ or $b = 0$.

Our construction of the reachable part is based on the following notion capturing the part of an object $Y$ that is actually used by a morphism $f: X \to FY$. For the class of all monomorphisms this notion was introduced by Alwin Block [10] under the name “base”:

**Definition 5.4.** Given a functor $F: \mathcal{C} \to \mathcal{D}$ and a class $\mathcal{M}$ of monomorphisms of $\mathcal{C}$. We say that $F$ has least bounds (w.r.t. $\mathcal{M}$) if for every morphism $f: X \to FY$ there is a least morphism $m: Z \to Y$ in $\mathcal{M}$ such that $f$ factors through $Fm$. This means, there exists some $g: X \to FZ$ with

$$
\begin{array}{ccc}
X & \xrightarrow{f} & FY \\
\downarrow{g} & & \uparrow{Fm} \\
FZ & & \text{in } \mathcal{D},
\end{array}
$$

and for every $m': Z' \to Y$ in $\mathcal{M}$ and $g': X \to FZ'$ with $Fm' \cdot g' = f$ there exists a (necessarily unique) $h: Z \to Z'$ with $m' \cdot h = m$, i.e. $m \leq m'$ in $\text{Sub}(Y)$.

The triple $(Z, g, m)$ is called the bound of $f$, and the above triple $(Z', g', m')$ is said to compete with the bound.

**Proposition 5.5.** Functors having least bounds are closed under composition.
Proof. Let \( F : \mathcal{C} \to \mathcal{D} \) and \( G : \mathcal{D} \to \mathcal{D}' \) have least bounds w.r.t the classes \( \mathcal{M} \) and \( \mathcal{M}' \) of \( \mathcal{C} \) and \( \mathcal{D} \), respectively. We will prove that \( GF \) has least bounds w.r.t. \( \mathcal{M} \).

In order to see this consider a morphism \( f : X \to GFY \). Then first take its bound w.r.t. \( G \) to obtain \( g : X \to GFZ \) and \( m : Z \to FY \) in \( \mathcal{M}' \) such that \( Gm \cdot g = f \), and then take the bound of \( m \) w.r.t. \( F \) to obtain \( g' : Z \to F'Z' \) and \( m' : Z' \to Y \) in \( \mathcal{M} \) such that \( Fm' \cdot g' = g \):

\[
\begin{array}{c}
X \xrightarrow{f} GFY \\
\downarrow{g} \quad \downarrow{Gm} \\
GFZ \xrightarrow{Gg'} GFZ'
\end{array}
\]

Then \( Gg' \cdot g \) and \( m' \) form the desired bound of \( f \) w.r.t. \( GF \). Indeed, given any \( g'' : X \to GFZ'' \) and \( m'' : Z'' \to GFY \) with \( GFm'' \cdot g'' = f \) one first uses minimality of the bound \((Z, g, m)\) to obtain some \( h : Z \to FZ'' \) with \( Gm'' \cdot h = m \), and then one uses the minimality of \((Z', g', m')\) w.r.t. \( F \) to obtain \( h' : Z' \to Z'' \) such that \( m'' \cdot h' = m' \) as required.

Gumm [13, Corollary 4.8] proved that, in the case where \( F \) is an endofunctor on \( \text{Set} \) and \( \mathcal{M} \) is the class of all monomorphisms, \( F \) has least bounds if and only if it preserves intersections. We now provide the proof in our setting, and we slightly extend the result by a statement involving the following operator, which extends the “next time” operator of Jacobs [15] for coalgebras to arbitrary morphisms:

**Definition 5.6.** For every morphism \( f : X \to FY \) we define the operator

\[
\bigcirc_f : \text{Sub}(Y) \to \text{Sub}(X)
\]

as follows (we drop the subscript \( f \) whenever this morphism is clear from the context): given a subobject \( m : S \to Y \) we form the preimage of \( Fm \) under \( f \), i.e. we form the pullback below:

\[
\begin{array}{c}
\bigcirc_S \\ \downarrow{m}
\end{array}
\xymatrix{
S \ar@{->}[r]^{f[m]} & FS \\
X \ar@{->}[u]^{f} \ar@{->}[r]_{Fm} & FY
\end{array}
\]

**Proposition 5.7.** Let \( F : \mathcal{C} \to \mathcal{C} \) preserve \( \mathcal{M} \)-morphism, i.e. \( Fm \) lies in \( \mathcal{M} \) for every \( m \) in \( \mathcal{M} \). Then the following are equivalent:

(1) \( F \) preserves intersections.

(2) \( F \) has least bounds w.r.t. \( \mathcal{M} \).

(3) For every \( f : X \to FY \), the operator \( \bigcirc_f \) has a left-adjoint.

Note that since \( F \) preserves \( \mathcal{M} \)-morphisms, \( g \) of Definition 5.4 is unique.
**Proof.** For (3) $\Rightarrow$ (1) choose $f = \text{id}_Y$ then $\bigcirc: m \mapsto Fm$ is a right-adjoint and so preserves all meets, i.e. $F$ preserves intersections.

The converse (1) $\Rightarrow$ (3) follows from the easily established fact that intersections are stable under preimage, i.e. for every morphism $f: X \to Y$ and every family $m_i: S_i \to Y$ of subobjects the intersection $m: P \to X$ of the preimages of the $m_i$ under $f$ yields a pullback

$$
\begin{array}{ccc}
P & \xrightarrow{m} & \bigcap S_i \\
f \downarrow & & \downarrow \bigcap m_i \\
X & \xrightarrow{f} & Y \\
\end{array}
$$

For (1) $\Rightarrow$ (2), consider $f: X \to FY$ and define $Y'$ to be the intersection of all subobjects with the desired factorization property:

$$(m: Y' \to Y) := \bigcap \{m_i: Y_i \to Y \mid \exists f_i: X \to FY_i \text{ with } Fm_i \cdot f_i = f \}$$  (5.1)

This intersection exists since $\mathcal{C}$ is well-powered, and it is preserved by $F$. The witnessing morphisms $f_i: X \to FY_i$ from (5.1) form a cone for this intersection, inducing a unique $f': X \to FY'$ such that $Ff_i \cdot f' = f_i$ for all $i$, where $s_i: Y' \to Y_i$ are the morphisms witnessing $m \leq m_i$, i.e. we have $m_i \cdot s_i = m$. It follows that we have

$$Fm \cdot f' = Fm_i \cdot Fs_i \cdot f' = Fm_i \cdot f_i = f,$$

whence $(Y, f', m)$ is the desired bound of $f$. In fact, minimality clearly holds: whenever $(\bar{Y}, g, \bar{m})$ competes with that triple, we see that $\bar{m}$ is contained in the set in (5.1), thus $m \leq \bar{m}$.

For (2) $\Rightarrow$ (1), consider an intersection

$$(w: W \to Z) = \bigcap \{y_i: Y_i \to Z \mid i \in I\},$$

and let $w_i: W \to Y_i$, $i \in I$, denote the corresponding pullback projections. Suppose we have a competing cone

$$(c_i: C \to FY_i)_{i \in I} \text{ with } Fy_i \cdot c_i = Fy_j \cdot c_j \text{ for all } i, j \in I.$$ We can assume wlog that $I \neq \emptyset$, because for $I = \emptyset$, the intersection $w = \text{id}_Z$ is preserved by every functor. We need to prove that there is a unique morphism $u: C \to FW$ such that $c_i \cdot u = w_i$ for all $i \in I$. Using (2), we take the bound $(Z', f', z)$ of $f := Fy_i \cdot c_i$ for some $i \in I$. Hence, the following diagrams commute for all $i \in I$:

$$
\begin{array}{ccc}
FY_i & \xrightarrow{Fy_i} & FZ \\
\downarrow \quad & & \downarrow \quad \\
FZ' & \leftarrow & \quad
\end{array}
$$
For every $i \in I$, the triple $(Y_i, c_i, y_i)$ competes with the bound of $f$. Hence, we have, for every $i \in I$, a unique $z_i: Z' \to Y_i$ with $y_i \cdot z_i = z$. Thus, $(z_i: Z' \to Y_i)_{i \in I}$ is a competing cone for the intersection of all $y_i$, and so we obtain a unique morphism $v: Z' \to W$ such that the following triangles commute:

\[
\begin{array}{ccc}
Y_i & \xrightarrow{z_i} & W \\
\downarrow{w_i} & & \\
Z' & \xrightarrow{v} & W
\end{array}
\]

for every $i \in I$.

Furthermore, since every $F y_i$ is monomorphic, the following diagrams commute:

\[
\begin{array}{ccc}
FY_i & \xrightarrow{F y_i} & FZ \\
\uparrow{c_i} & & \uparrow{F z_i} \\
C & \xrightarrow{f'} & FZ'
\end{array}
\]

for every $i \in I$.

Now let $u := Fv \cdot f': C \to FW$. Then the following diagram commutes:

\[
\begin{array}{ccc}
FY_i & \xrightarrow{F y_i} & FW \\
\uparrow{F w_i} & & \uparrow{F v} \\
C & \xrightarrow{f'} & FW
\end{array}
\]

for all $i \in I$ as desired. Since $F w_i$ is monomorphic for every $i \in I$ and $I \neq \emptyset$, $u$ is the unique morphism such that $F w_i \cdot u = c_i$ for every $i \in I$.

Assumption 5.8. In addition to Assumption 5.1 we now assume that $F: C \to C$ is a functor preserving $\mathcal{M}$-morphisms and preserving intersections (equivalently, $F$ has least bounds).

Remark 5.9. Note that for $C = \text{Set}$ with the usual factorizations system given by surjective and injective maps we may drop the assumption that $F$ preserves monomorphism. In fact, we may always work with the Trnková hull $\bar{F}$ recalling that the category of $\bar{F}$-coalgebras is isomorphic to the category of $F$-coalgebras (see Section 3).

Example 5.10. Let us continue Example 5.3.

(1) Every intersection-preserving functor on a complete and well-powered category has least bounds. Note that this does not need the existence of coproducts; in fact, the proof of Proposition 5.7 just needs the existence of intersections in $C$.

(2) It is easy to see that every functor $\bar{F}$ on $\text{Rel}$ extending an intersection-preserving set functor $F$ satisfies our assumptions, i.e. $\bar{F}$ preserves maps and injective ones. Moreover, given a morphism $f: X \to FY$ in $\text{Rel}$, we write it as a
map \( f : X \to \mathcal{P}FY \). Using that \( \mathcal{P} : \text{Set} \to \text{Set} \) preserves all intersection and thus has least bounds, the argument of Proposition 5.5 instantiated to show that \( \mathcal{P}F \)
has least bounds also shows how to obtain the bound of \( f \) w.r.t. \( \bar{F} \) on \( \text{Rel} \).

A similar argument holds for extended functors \( \bar{F} \) on \( \mathcal{KL}(D) \) and \( \mathcal{KL}(S(-)) \) for a semiring satisfying the previously mentioned conditions (see Example 5.3(4)).

However, we will see in Section 6 that one can construct the reachable part of an \( \bar{F} \)-coalgebra even if one does not have a factorization system on the Kleisli category, i.e. without any further assumption on the semiring \( S \).

From the fact that \( F \) has least bounds we obtain for every morphism \( f : X \to FY \) the following operator from the subobjects of \( X \) to those of \( Y \):

**Definition 5.11.** Let \( f : X \to FY \) be a morphism. The operator

\[
\ominus f : \text{Sub}(X) \to \text{Sub}(Y)
\]

takes a subobject \( m : S \to X \) to the bound of \( f \cdot m \). In particular, we have the commutative square below (we omit the subscript whenever \( f \) is clear from the context):

\[
\begin{array}{ccc}
S & \xrightarrow{g} & F(S) \\
\downarrow{m} & & \downarrow{F(m)} \\
X & \xrightarrow{f} & FY
\end{array}
\]

**Proposition 5.12.** For every morphism \( f : X \to FY \), the operator \( \ominus \) is the left-adjoint of the “next time” operator \( \odot \) from Definition 5.6.

Consequently, \( \ominus \) preserves all unions, and in particular it is monotone.

**Proof.** Let \( f : X \to FY \) be any morphism and assume that \( m \leq \odot m' \) for some subobjects \( m : S \to X \) and \( m' : S' \to Y \). Then we have a commutative diagram

\[
\begin{array}{ccc}
S & \xrightarrow{i} & \odot S' & \xrightarrow{f[m']} & FS' \\
\downarrow{m} & & \downarrow{\odot m'} & & \downarrow{Fm'} \\
X & \xrightarrow{f} & FY
\end{array}
\]

and therefore \((S', f[m'] \cdot s, m')\) is competing with the bound of \( f \cdot m \). Thus, \( \ominus m \leq m' \). Conversely, suppose that \( \ominus m \leq m' \), witnessed by \( j : \ominus S \to S' \). Then consider the following diagram, where \( g : S \to F(\ominus S) \) comes from the bound of \( f \cdot m \):

\[
\begin{array}{ccc}
S & \xrightarrow{m} \odot S' & \xrightarrow{f[m']} \odot S' & \xrightarrow{f[m']} FS' \\
\downarrow{\ominus m'} & \downarrow{Fj} & \downarrow{Fm'} & \downarrow{F(m')} \\
X & \xrightarrow{f} FY & \xrightarrow{Fm'} FY
\end{array}
\]

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Since its outside and its right-hand part commute, we obtain the dashed arrow using the universal property of the pullback forming $\circ m$. This proves that $m \leq \circ m'$ as desired.

**Remark 5.13.** For a coalgebra $c: C \to FC$, the operators $\circ$ and $\ominus$ on $\text{Sub}(C)$ are a generalized semantic counterpart of the next time and previous time operators, respectively, of classical linear temporal logic, see e.g. Manna and Pnueli [19]. In fact, consider the functor $FX = \mathcal{P}(A \times X)$ on $\text{Set}$ whose coalgebras are labelled transition systems. For a transition system $c: C \to \mathcal{P}(A \times C)$ and a subset $m: S \hookrightarrow C$ we have

$\ominus S = \{ x \in C \mid c(a, x) \in S \text{ for all } a \in A \}$,

$\bigcirc S = \{ x \in C \mid x \in c(a, s) \text{ for some } a \in A \text{ and } s \in S \}$.

**Proposition 5.14.** For an intersection preserving functor $F$ on $\text{Set}$, $\ominus$ may be computed on the canonical graph of a given coalgebra.

**Proof.** Let $c: C \to FC$ be a coalgebra and $s: S \hookrightarrow C$ be any subset of states. We will show that $\ominus s$ computed w.r.t. $(C, c)$ or its canonical graph $(C, \tau_C \cdot c)$ agree. Indeed, this follows from the fact that any triple $(Z, g, m)$ competing to the bound of $c \cdot m$ yields the triple $(Z, \tau_C \cdot g, m)$ competing to the bound of $\tau_C \cdot c \cdot s$, and moreover, every triple $(Z, g', m)$ competing to the latter bound must satisfy $g' = \tau_C \cdot g$ where $(Z, g, m)$ competes to the former bound:

In fact, one uses that (4.2) is a pullback. Now if $(Z, g, m)$ is the bound of $c \cdot s$ then $(Z, \tau_Z \cdot g, m)$ is the bound of $\tau_C \cdot c \cdot s$, for if $(\hat{Z}, \hat{g}, \hat{m})$ is any competing triple (w.r.t. $\mathcal{P}$) we see that $\hat{g}' = \tau_Z \cdot \hat{g}$ such that $(\hat{Z}, \hat{g}, \hat{m})$ is a competing triple (w.r.t. $F$), and thus $m \leq \hat{m}$. Conversely, if $(Z, g', m)$ is the bound of $\tau_C \cdot m \cdot s$, then $g' = \tau_Z \cdot g$ so that $(Z, g, m)$ is the bound of $c \cdot s$, for if $(\hat{Z}, \hat{g}, \hat{m})$ is a competing triple (w.r.t. $F$), then $(Z, \tau_Z \cdot \hat{g}, \hat{m})$ is a competing triple (w.r.t. $\mathcal{P}$), and thus $m \leq \hat{m}$. \hfill $\square$

**Remark 5.15.** In connection with reachable coalgebras, the operator $\ominus$ was recently used in the work of Barlocco, Kupke, and Rot [8]. Their results were obtained independently from ours but almost at the same time. They work with a complete and well-powered category $\mathcal{C}$, so $\mathcal{M}$ is the class of all monomorphisms (cf. Example 5.3(1)). First they show that every intersection preserving endofunctor $F$ on $\mathcal{C}$ has least bounds (i.e. the implication $(1) \implies (2)$ in Proposition 5.7).

Furthermore, it is easy to see that, for every $F$-coalgebra $(C, c), \ominus$ is a monotone operator on $\text{Sub}(C)$. In addition, we see that $\ominus$ preserves all unions. Indeed,
in the setting of a complete and well-powered category, every \( \text{Sub}(C) \) is a complete lattice having all intersections. Thus (the proof of) Proposition 5.7 shows that \( \ominus \) has the left-adjoint \( \ominus \).

Moreover, it is shown in loc. cit. that for every point \( i_0 : 1 \to C \) the reachable part of \( (C, c, i_0) \) is the least fixed point of \( i_0 \lor \ominus(-) \).

However, note that the assumption of completeness may be limiting applications, e.g. the category \( \text{Rel} \) in Example 5.3(2) is not complete.

Barlocco at al. also prove the following fact. The proof is the same in our setting, and we present it here for the convenience of the reader.

**Proposition 5.16.** Let \( c : C \to FC \) be a coalgebra and \( m : S \implies C \) a subobject. Then \( S \) carries a subcoalgebra of \( (C, c) \) if and only if \( m \) is a prefixed point of \( \ominus \).

**Proof.** Suppose first that we have \( \ominus m \leq m \), i.e. we have some \( i : \ominus S \implies S \) such that \( m \cdot i = \ominus m \). Then the following diagram commutes:

\[
\begin{array}{ccc}
S & \xrightarrow{g} & F(\ominus S) & \xrightarrow{F_i} & FS \\
\downarrow{m} & & \swarrow{F(\ominus m)} & & \downarrow{Fm} \\
C & \xrightarrow{c} & FC & & \\
\end{array}
\]

This shows that \( (S, Fi \cdot g) \) is a subcoalgebra of \( (C, c) \).

Conversely, suppose that \( m : (S, s) \implies (C, c) \) is a subcoalgebra. Then \( (S, s, m) \) competes with the bound \( (\ominus S, g, \ominus m) \) of \( c \cdot m \) and therefore we have \( \ominus m \leq m \). \( \square \)

We now present our new construction of the reachable part of an \( I \)-pointed coalgebra as the union of all iterated applications of \( \ominus \) on the given \( I \)-pointing.

**Construction 5.17.** Given an \( I \)-pointed \( F \)-coalgebra \( I \xrightarrow{i_C} C \xrightarrow{c} FC \), define subobjects \( m_k : C_k \implies C \), \( k \in \mathbb{N} \), inductively:

1. Let \( C_0 \) be the \((\mathcal{E}, \mathcal{M})\)-factorization of \( i_C \):
   \[
i_C \\
I \xrightarrow{i_C} C_0 \xrightarrow{m_0} C.
   \]
   \( \text{(5.2)} \)

2. Given \( m_k : C_k \to C \), let \( m_{k+1} = \ominus m_k : C_{k+1} = \ominus C_k \implies C \), i.e. \( m_{k+1} \) is given by the bound of \( c \cdot m_k \):
   \[
   \begin{array}{ccc}
   C_k & \xrightarrow{c_k} & FC_{k+1} \\
   \downarrow{m_k} & & \downarrow{Fm_{k+1}} \\
   C & \xrightarrow{c} & FC \\
   \end{array}
   \]
   \( \text{(5.3)} \)

We define the subobject \( m : R \implies C \) to be the union of all \( m_k \), \( k \in \mathbb{N} \):

\[
m := \bigcup_{k \in \mathbb{N}} \ominus^k(m_0).
\]
Theorem 5.18. For every $I$-pointed coalgebra $(C, c, i_C)$, the union $m: R \rightharpoonup C$ is a reachable subcoalgebra of $(C, c, i_C)$.

Proof. We will prove that $R$ carries the structure of an $I$-pointed coalgebra, which is reachable, and that $m$ is a homomorphism of $I$-pointed coalgebras.

By Remark 2.3(3), $e \cdot \text{in}_k: C_k \rightarrow R$ is in $\mathcal{M}$ for all $k \in \mathbb{N}$, because $m_k = m \cdot e \cdot \text{in}_k$ and $m$ is in $\mathcal{M}$.

(1) The pointing $i_R := e \cdot \text{in}_0 \cdot i'_C: I \rightarrow R$ is preserved by $m$ because

\[
\begin{array}{ccc}
I & \xrightarrow{i_C} & C \\
\downarrow & & \downarrow m \\
\downarrow & & \downarrow m \\
C_0 & \xrightarrow{e \cdot \text{in}_0} & R
\end{array}
\]

commutes. For the coalgebra structure on $R$, first note that the outside of the following diagram commutes for every $\ell \in \mathbb{N}$ using (5.3):

\[
\begin{array}{cccc}
C_\ell & \xrightarrow{\Pi_{k \in \mathbb{N}} C_k} & \xrightarrow{\Pi_{k \geq 1} FC_k} & \xrightarrow{\Pi_{k \in \mathbb{N}} C_k} \\
\downarrow m_\ell & \downarrow \Pi_{k \in \mathbb{N}} \text{in}_k & \downarrow F \text{in}_{\ell+1} & \downarrow F[m_{\ell+1}] \\
\downarrow [m_k]_{k \in \mathbb{N}} & \downarrow \Pi_{k \geq 1} F \text{in}_k & \downarrow F[m_k]_{k \in \mathbb{N}} \\
C & \xrightarrow{\Pi_{k \in \mathbb{N}} F C_k} & \xrightarrow{\Pi_{k \in \mathbb{N}} C_k} & \xrightarrow{\Pi_{k \in \mathbb{N}} C_k} \\
\downarrow c & \downarrow \text{in}_\ell & \downarrow \text{in}_{\ell+1} & \downarrow \text{in}_\ell
\end{array}
\]

Since the coproduct injections $(\text{in}_\ell: C_\ell \rightarrow \Pi_{\ell \in \mathbb{N}} C_k)_{\ell \in \mathbb{N}}$ are jointly epic, it follows that the lower part commutes. We take the $(\mathcal{E}, \mathcal{M})$-factorization $[m_k]_{k \in \mathbb{N}} = e \cdot m$ to obtain the union $m: R \rightharpoonup C$ (see Remark 5.2(1)). Using that $F$ preserves $\mathcal{M}$ and the diagonal fill-in property (see Remark 2.4(2)) we obtain a coalgebra structure $r: R \rightarrow FR$ such that $e$ and $m$ are $F$-coalgebra homomorphisms.

(2) For reachability, let $h: (S, s, i_S) \rightarrow (R, r, i_R)$ with $h \in \mathcal{M}$ be a pointed subcoalgebra. In the following we will define morphisms $d_k: C_k \rightharpoonup S$ (in $\mathcal{M}$) satisfying

\[
\begin{array}{ccc}
C_k & \xrightarrow{m_k} & C \\
\downarrow d_k & & \downarrow m \\
S & \xrightarrow{h} & R
\end{array}
\]

for all $k \in \mathbb{N}$. (5.4)

We define $d_0: C_0 \rightharpoonup S$ using the diagonal fill-in property; in fact, in the diagram below the outside commutes since $m_0 \cdot i_C' = i_C$ is the pointing of $C$ and $m \cdot h$...
I preserves pointings:

\[ I \xrightarrow{i_C} C_0 \]

\[ S \xrightarrow{h} R \xrightarrow{m} C \]

Given \( d_k : C_k \to S \), note that the following diagram commutes:

\[ C_k \xrightarrow{d_k} S \xrightarrow{s} FS \]

\[ S \xrightarrow{h} R \xrightarrow{r} FR \]

\[ C \xrightarrow{c} FC \]

The commutativity of its outside means that \((S, s \cdot d_k, m \cdot h)\) competes with the bound \( m_{k+1} : C_{k+1} \to C \) of \( c \cdot m_k \). Thus, there exists a morphism \( d_{k+1} : C_{k+1} \to S \) with \( m \cdot h \cdot d_{k+1} = m_{k+1} \).

Putting all squares (5.4) together, we see that the diagram on the left below commutes:

\[ \coprod_{k \in \mathbb{N}} C_k \xrightarrow{e} R \xrightarrow{m} \coprod_{k \in \mathbb{N}} C_k \]

\[ S \xrightarrow{h} R \xrightarrow{[d_k]_{k \in \mathbb{N}}} \]

\[ \coprod_{k \in \mathbb{N}} C_k \xrightarrow{e} R \xrightarrow{h} \coprod_{k \in \mathbb{N}} C_k \]

Since \( m \in \mathcal{M} \) is monomorphic the outside of the diagram on the right above commutes, and we apply the diagonal fill-in property again to see that \( h \) is a split epimorphism. Since we also know that \( h \in \mathcal{M} \) is a monomorphism, it is an isomorphism, whence \( (R, r, i_R) \) is reachable as desired.

**Definition 5.19.** We call the above \( I \)-pointed coalgebra \((R, r, i_R)\) the reachable part of \((C, c, i_C)\).

**Remark 5.20.** Note that it follows from an easy lattice theoretic argument that for every join-preserving map \( \varphi : L \to L \) on a complete lattice \( L \), and every \( \ell \in L \) the least fixed point of \( \ell \lor \varphi(-) \) is given by the join

\[ \bigvee_{i \in \mathbb{N}} \varphi^i(\ell). \]  

Indeed, to see this recall that, by Kleene’s fixed point theorem, the least fixed point of \( \ell \lor \varphi(-) \) is the join of the \( \omega \)-chain given by \( x_0 = \bot \), the least element of \( L \), and \( x_{n+1} = \ell \lor \varphi(x_n) \). We will show by induction that

\[ x_n = \bigvee_{i < n} \varphi^i(\ell) \quad \text{for all } n < \omega. \]
The base case $n = 0$ is clear since the empty join is $\perp$, and for the induction step we use that $\varphi$ preserves joins to compute:

$$x_{n+1} = \ell \lor \varphi(x_n) = \ell \lor \varphi \left( \bigvee_{i < n} \varphi^i(\ell) \right) = \ell \lor \bigvee_{i < n} \varphi^{i+1}(\ell) = \bigvee_{i < n+1} \varphi^i(\ell).$$

Applying this to $\varphi = \ominus$ we see that our Construction 5.17 of the reachable part coincides with the least fixed point of $i_0 \lor \ominus(-)$ considered by Barlocco et al. (cf. Remark 5.15).

**Theorem 5.21.** Suppose that $F$ preserves inverse images. Then the full subcategory of $\text{Coalg}_I(F)$ given by all reachable coalgebras is coreflective.

**Proof.** Let $(C, c, i_C)$ be an $I$-pointed $F$-coalgebra and let, $m: (R, r, i_R) \rightarrow (C, c, i_C)$ be its reachable part. We will show that the this is a coreflection by verifying the corresponding universal property.

Suppose we have a homomorphism $h: (S, s, i_S) \rightarrow (C, c, i_C)$ where $(S, s, i_S)$ is reachable. We need to show that $h$ factors uniquely through $(R, r, i_R)$. Uniqueness is clear since $m$ is monic. For existence, we form the inverse image of $m$ under $h$, i.e. we form the following pullback:

$$\begin{array}{ccc}
P & \xrightarrow{h'} & R \\
m' \downarrow & & \downarrow m \\
S & \xleftarrow{h} & C
\end{array} \quad (5.6)$$

We will equip $P$ with a pointed coalgebra structure making $m'$ and $h'$ pointed coalgebra homomorphisms. Since $m \cdot i_R = i_C = h \cdot i_S$, we obtain a pointing $i_P$ on $P$, and since $F$ preserves inverse images we also obtain a coalgebra structure $p$:

$$\begin{array}{ccc}
P & \xrightarrow{m'} & \exists p \rightarrow R \\
\exists p & \xrightarrow{h'} & h' \\
S & \xleftarrow{h} & C
\end{array}$$

This definition of $(P, p, i_P)$ makes $m'$ and $h'$ pointed coalgebra homomorphisms. Since $(S, s, i_S)$ is reachable and $m'$ is monomorphic, the latter must be an isomorphism. Thus, $h' \cdot \phi^{-1}$ is the desired factorization of $h$ through $(R, r, i_R)$, cf. (5.6).

**Corollary 5.22.** If $F$ preserves inverse images, then reachable $F$-coalgebras are closed under quotients.
Proof. Given a reachable coalgebra \((R, r, i_R)\) and \(e: (R, r, i_R) \twoheadrightarrow (Q, q, i_Q)\) in \(\text{Coalg}_I(F)\) carried by an \(\varepsilon\)-morphism. Suppose that \(m: (Q', q', i'_Q) \rightarrow (Q, q, i_Q)\), \(m \in \mathcal{M}\), is the inclusion of the reachable part of \(Q\). By Theorem 5.21, \(e\) factors through \(m\):

\[
\begin{array}{ccc}
(Q', q', i'_Q) & \xrightarrow{m} & (Q, q, i_Q) \\
\text{\_\_\_} & \uparrow{h} & \downarrow{\_\_\_} \\
(R, r, i_R) & \xrightarrow{\_\_\_} & \text{\_\_\_}
\end{array}
\]

Using the diagonal fill-in property, we obtain a unique coalgebra homomorphism \(d: (Q, q, i_Q) \rightarrow (Q', q', i'_Q)\) with \(h = d \cdot e\) and \(m \cdot d = \text{id}\). This implies that \(m\) is a split epimorphism, and since \(m \in \mathcal{M}\) is a monomorphism, it is an isomorphism.

Example 5.23. For functors not preserving inverse images, reachable coalgebras need not be closed under quotient. For example, recall the functor \(R: \text{Set} \rightarrow \text{Set}\) from Example 4.5 and consider the coalgebras \(c: C \rightarrow RC\) with \(C = \{x, y, z\}\) and \(c(x) = (y, z)\) and \(c(y) = c(z) = *\) and \(d: D \rightarrow RD\) with \(D = \{x', y'\}\) and \(d(x') = d(y') = *\). Then \((D, d)\) is a quotient of \((C, c)\) via the coalgebra homomorphism \(q\) with \(q(x) = x'\) and \(q(y) = q(z) = y'\). However, \((C, c, x)\) is reachable whereas \((D, d, x')\) is not.

Note that, in the light of the proof of Corollary 5.22, this example also shows that reachable \(F\)-coalgebras need not form a coreflective subcategory if \(F\) does not preserves inverse images.

Finally, let us come back to \(\mathcal{C} = \text{Set}\) and canonical graphs. We call the subset \(m_k: C_k \hookrightarrow C\) of Construction 5.17 the \(n^{th}\) step of the construction of the reachable part.

**Corollary 5.24.** Let \(F: \text{Set} \rightarrow \text{Set}\) preserve all intersections. Then the \(n^{th}\) steps of the constructions of the reachable parts of an \(I\)-pointed coalgebra and its canonical graph are the same.

Indeed, this follows by an easy induction from the fact that for a coalgebra \(c: C \rightarrow FC\) and a subset \(s: S \hookrightarrow C, \varnothing s\) may be computed on the canonical graph of \((C, c)\) (see Proposition 5.14).

**Remark 5.25.** (1) From Corollary 5.24 we can conclude that for an intersection preserving set functor \(F\), the reachable part of a given \(I\)-pointed \(F\)-coalgebra \((C, c, i_C)\) may be computed by a standard graph algorithm such as breadth-first-search. We thus obtain an efficient and generic algorithm for reachability of coalgebras for intersection-preserving set functors.

(2) Moreover, the \(n^{th}\) subset \(C_n \hookrightarrow C\) contains precisely all the states of \(C\) that are reachable along a path of length precisely \(n\) in the canonical graph of \(C\) from a state in the image \(C_0\) of the \(I\)-pointing \(i_C: I \rightarrow C\).
6  Reachability in a Kleisli Category

In the following, we present a reduction from the reachability construction in a Kleisli category for a monad on \( \mathcal{C} \) to the reachability construction in the base category \( \mathcal{C} \). This makes the reachability construction applicable in Kleisli categories that fail to have an \((\mathcal{E}, \mathcal{M})\)-factorization system for the desired class \( \mathcal{M} \) of monomorphisms that determines the notion of subcoalgebra. Coalgebras over Kleisli categories are used to study the trace semantics of various kinds of systems, see e.g. Hasuo, Jacobs, and Sokolova [14].

For our reduction, we need that finite coproducts in \( \mathcal{C} \) are well behaved. In fact, recall [11] that a category is called \emph{extensive} if it has finite coproducts and for every pair \( A, B \) of objects the canonical functor \( \mathcal{C}/A \times \mathcal{C}/B \to \mathcal{C}/(A + B) \) is an equivalence of categories.

**Remark 6.1.** We further recall a few properties of extensive categories from [11].
(1) First note that a category with finite coproducts is extensive if and only if it has pullbacks along coproduct injections and for every commutative diagram

\[
\begin{array}{ccc}
A_1 & \xrightarrow{a_1} & A \\
\downarrow{h_1} & & \downarrow{h} \\
B & \xrightarrow{\text{inl}} & B + C \\
\downarrow{\text{inr}} & & \downarrow{h_2} \\
& A_2 & C
\end{array}
\]

we have that \( A_1 \xrightarrow{a_1} A \xleftarrow{a_2} A_2 \) is a coproduct iff both squares are pullbacks.

(2) In an extensive category \( \mathcal{C} \), the coproduct injections are monomorphisms, and coproducts are \emph{disjoint}, i.e. the intersection of the subobjects \( \text{inl}: A \hookrightarrow A + B \) and \( \text{inr}: B \hookrightarrow A + B \) is the initial object \( 0 \).

**Example 6.2.** Many categories with set-like coproducts are extensive, for example the category \( \text{Set} \) itself, as well as the categories of partially ordered sets, nominal sets, and graphs as well as every category of presheaves. In addition, the categories of unary algebras and of Jónsson-Tarski algebras (i.e. algebras \( A \) with one binary operation \( A \times A \to A \) that is an isomorphism) are extensive. More generally, every topos is extensive.

The category of monoids is not extensive.

**Remark 6.3.** Recall that the Kleisli category \( \mathcal{K}(T) \) for a monad \( (T, \mu, \eta) \) on \( \mathcal{C} \), has the same objects as \( \mathcal{C} \) and hom sets \( \mathcal{K}(T)(X,Y) = \mathcal{C}(X,TX) \). We use the notation \( f: X \to Y \) to denote a morphism \( f \in \mathcal{K}(T)(X,Y) \), and we call such morphisms \emph{Kleisli morphisms}. The composition of Kleisli morphisms \( f: X \to Y \) and \( g: Y \to Z \) is denoted by \( g \circ f \) and defined by

\[
(g \circ f) = (X \xrightarrow{f} TY \xrightarrow{Tg} TTZ \xrightarrow{\mu_Z} TZ).
\]

The identity morphism on \( X \) is given by the unit \( \eta_X: X \to TX \) of the monad.
Recall [1, Definition IV.2] that a monad \((T, \mu, \eta)\) is called consistent if \(\eta_X\) is a monomorphism for every object \(X\) of \(\mathcal{C}\). Note that on \(\text{Set}\) all but two monads are consistent. In fact, only the monad \(C_1\) mapping all sets to 1 and \(C_{01}\) mapping non-empty sets to 1 and the empty set to itself are inconsistent (see e.g. [1, Lemma IV.3]).

The following terminology is borrowed from functional programming:

**Definition 6.4.** A Kleisli morphism \(f: X \to Y\) is called pure if \(f: X \to TY\) factorizes through \(\eta_Y: Y \to TY\). The pure morphisms form a subcategory of \(\mathcal{K}(T)\), and if \(T\) is consistent, then this subcategory can be identified with the base category \(\mathcal{C}\) via the canonical functor \(J: \mathcal{C} \hookrightarrow \mathcal{K}(T)\) given by

\[
J(f: X \to Y) = (\eta_Z \cdot f): X \to Y.
\]

Consequently, we write \(g: X \to Y\) for pure morphisms in diagrams in \(\mathcal{K}(T)\) and omit the explicit application of the functor \(J\).

Finally, a coalgebra homomorphism between coalgebras for a functor on \(\mathcal{K}(T)\) which is pure is called pure coalgebra homomorphism.

Recall that an extension of a functor \(F: \mathcal{C} \to \mathcal{C}\) to \(\mathcal{K}(T)\) is a functor \(\bar{F}: \mathcal{K}(T) \to \mathcal{K}(T)\) such that the square below commutes:

\[
\begin{array}{ccc}
\mathcal{K}(T) & \xrightarrow{F} & \mathcal{K}(T) \\
J \uparrow & & \uparrow J \\
\mathcal{C} & \xrightarrow{F} & \mathcal{C}
\end{array}
\]

If \(T\) is consistent, then a functor on \(\mathcal{K}(T)\) is an extension iff it preserves pure morphisms.

**Example 6.5.** (1) We have already mentioned that the category \(\text{Rel}\) is the Kleisli category for the power-set monad \(\mathcal{P}\). Coalgebras over \(\mathcal{K}(\mathcal{P})\) are system with non-deterministic branching. For example, non-deterministic automata with input alphabet \(\Sigma\) are coalgebras for \(FX = 1 + \Sigma \times X\) (see Example 2.2(2)).

(2) For the finite power-set monad \(\mathcal{P}_f X = \{S \subseteq X \mid S \text{ finite}\}\), coalgebras on \(\mathcal{K}(\mathcal{P}_f)\) are finitely branching non-deterministic systems. For example, finitely branching transition systems with label alphabet \(\Sigma\) are coalgebras for the functor \(FX = \Sigma \times X\) on \(\mathcal{K}(\mathcal{P}_f)\).

(3) Consider the Kleisli category of the monad \(S(-)\) given by a semiring \(S\) (see Example 5.3(4)). The 1-pointed coalgebras for the functor \(\bar{F}X = 1 + \Sigma \times X\) are weighted automata with the input alphabet \(\Sigma\). Indeed, a coalgebra structure

\[
c: C \to S(1+\Sigma \times X) \cong S \times S(\Sigma \times C)
\]

assigns to each state \(x\) an output value in \(S\), and for every pair of states \(x, y \in C\), \(c(x)(y) = (a, s)\) means that there is a transition from \(x\) to \(y\) with label \(a \in \Sigma\) and weight \(s \in S\).
For the distribution monad \( D \), coalgebras on \( K\ell(D) \) have probabilistic branching. For example, for the functor \( F : X = \Sigma \times X \) are labelled Markov chains. Indeed, a coalgebra \( c : C \to D(\Sigma \times C) \) assigns to a state \( x \) a distribution over pairs of labels and next states.

**Assumption 6.6.** We assume that \( C \) is an extensive category, that \( (T, \mu, \eta) \) is a consistent monad on \( C \), and that \( \bar{F} : K\ell(T) \to K\ell(T) \) is an extension of \( F : C \to C \). Further we consider the class \( \mathcal{M} \) of all pure monomorphisms:

\[
\mathcal{M} := \{ m : X \to Y \mid m \text{ is a monomorphism in } C \}.
\]

**Remark 6.7.** (1) The notions of a subobject and of subcoalgebras in \( K\ell(T) \) are understood w.r.t. the class \( \mathcal{M} \), i.e. a subobject is represented by a Kleisli morphism \( m : S \rightarrow X \) in \( \mathcal{M} \) and a subcoalgebra by a coalgebra homomorphism \( h : (S, s) \to (C, c) \) carried by a Kleisli morphism in \( \mathcal{M} \).

**Notation 6.8.** We write \( \text{Coalg}_I(\bar{F}) \) for the subcategory of \( \text{Coalg}_I(\bar{F}) \) given by pure coalgebra homomorphisms.

**Construction 6.9.** For every \( I \)-pointed coalgebra \( I \xrightarrow{i_C} C \xrightarrow{c} FC \) in \( K\ell(T) \) we form following \( TF + T \)-coalgebra in \( C \):

\[
C + I \xrightarrow{c+i_C} TFC + TC \xrightarrow{TF \text{inl} + F \text{inl}} (TF + T)(C + I)
\]

together with the \( I \)-pointing \( \text{inr} : I \to C + I \). This defines the object assignment of a functor \( G : \text{Coalg}_I(\bar{F}) \to \text{Coalg}_I(TF + T) \) that maps a pure coalgebra homomorphism \( h : (C, c, i_C) \to (D, d, i_D) \) to \( Gh = h + \text{id}_I \).

In fact, \( Gh \) is a homomorphism of \( I \)-pointed coalgebras for \( TF + T \) on \( C \) as shown by the following commutative diagram:

\[
\begin{array}{ccc}
I & \xrightarrow{\text{inr}} & C + I \\
\text{inr} & & \downarrow h + I \\
D + I & \xrightarrow{d + i_D} & TFD + TD \\
\end{array}
\]

The fact that \( \eta \) is monic is used here to ensure that we may regard \( h \) as morphism of \( C \).

**Proposition 6.10.** The functor \( G \) reflects isomorphisms and it preserves and reflects subcoalgebras.

**Proof.** Let \( h : (C, c, i_C) \to (D, d, i_D) \) be a morphism in \( \text{Coalg}_I(\bar{F}) \). If \( h \) is in \( \mathcal{M} \), then it is a monomorphism in \( C \) and, moreover, so is \( Gh = h + \text{id}_I \) since monomorphisms are closed under coproducts in the extensive category \( C \). This shows that \( G \) preserves subcoalgebras.

To see that it reflects them assume that \( h + \text{id}_I \) is a monomorphism in \( C \). Then we have that \( \text{inl} \cdot h = (h + \text{id}_I) \cdot \text{inl} \) is monomorphic since \( \text{inl} \) is so. Thus \( h \) is a monomorphism in \( C \), whence it is a pure monomorphism in \( K\ell(T) \).

\[\text{inl} \cdot h = (h + \text{id}_I) \cdot \text{inl} \]

The fact that \( \eta \) is monic is used here to ensure that we may regard \( h \) as morphism of \( C \).
We proceed to proving that $G$ reflects isomorphisms. Consider $h : (C, c, i_C) \to (D, d, i_D)$ in $\text{Coalg}^p_I(\bar{F})$ such that $Gh = h + \text{id}_I$ is an isomorphism in $\mathcal{C}$. By extensivity, we have the pullback

$$
\begin{array}{ccc}
C & \xrightarrow{\text{inl}} & C + I \\
\downarrow h & & \downarrow \text{h+I} \\
D & \xrightarrow{\text{inl}} & D + I \\
\end{array}
$$

Thus, $h$ is an isomorphism in $\mathcal{C}$, whence in $\mathcal{K}(T)$.

**Lemma 6.11.** Suppose that $T$ and $F$ preserve finite intersections, and let $h : (D, d, i_D) \to G(C, c, i_C)$ be a morphism in $\text{Coalg}_I(T(F + T))$ carried by a monomorphism in $\mathcal{C}$.

Then there exists a morphism $g : (E, e, i_E) \to (C, c, i_C)$ in $\text{Coalg}^p_I(\bar{F})$ such that $(D, d, i_D) = G(E, e, i_E)$ and $h = Gg$.

**Proof.** Consider a morphism $h$ in $\text{Coalg}_I(T(F + T))$ which is monomorphic in $\mathcal{C}$, i.e. the following diagram commutes

$$
\begin{array}{ccc}
I & \xrightarrow{i_D} & D & \xrightarrow{d} & TFD + TD \\
\downarrow \text{inr} & & \downarrow h & & \downarrow \text{TFh+Th} \\
C + I & \xrightarrow{c+i_C} & TFC + TC & \xrightarrow{TF\text{inl}+T\text{inl}} & TF(C + I) + T(C + I) \\
\end{array}
$$

Form the pullbacks of the coproducts injections of $C + I$ along $h$:

$$
\begin{array}{ccc}
E & \xrightarrow{m} & D & \xleftarrow{i_D} & I \\
\downarrow g & & \downarrow h & & \downarrow \text{inr} \\
C & \xrightarrow{\text{inl}} & C + I & \xleftarrow{\text{inr}} & I \\
\end{array}
$$

In order to see that the right-hand square is indeed a pullback, suppose we are given morphisms $p : X \to D$ and $q : X \to I$ such that $h \cdot p = \text{inr} \cdot q$. Then we have

$$
h \cdot i_D \cdot q = \text{inr} \cdot q = h \cdot p
$$

and therefore we have $i_D \cdot q = p$ since $h$ is monomorphic. It follows that $q$ is the unique mediating morphism.

Thus, by extensivity, we have $D = E + I$ with coproduct injections $m$ and $i_D$ and hence, $h = g + \text{id}_I$.

Note that the two pullbacks in (6.2) are in fact intersections. Since $TT$ and $F$ preserve finite intersections, the middle square in the next diagram is a pullback, and so is the right-hand one, by extensivity. By combining (6.1) and (6.2) we
see that the outside of the following diagram commutes, and so we obtain the morphism \( e : E \to TFE \) as indicated:

\[
\begin{array}{ccc}
E & \xrightarrow{d \cdot m} & TFE & \xrightarrow{TFm} & TFD & \xrightarrow{\text{inl}} & TFD + TD \\
g & \downarrow & \downarrow{TFg} & \downarrow{TFh} & & \downarrow \text{inl} & \downarrow{TFh+Th} \\
C & \xrightarrow{c} & TFC & \xrightarrow{TF_{\text{inl}}} & TF(C + I) & \xrightarrow{\text{inl}} & TF(C + I) + T(C + I)
\end{array}
\] (6.3)

Similarly, we obtain \( i_E \):

\[
\begin{array}{ccc}
I & \xrightarrow{d \cdot i_D} & TE & \xrightarrow{Tm} & TD & \xrightarrow{\text{inr}} & TFD + TD \\
i_C & \downarrow & \downarrow{Th} & \downarrow \text{inr} & & \downarrow \text{inr} & \downarrow \text{inr} \\
TC & \xrightarrow{T_{\text{inl}}} & T(C + I) & \xrightarrow{\text{inr}} & TF(C + I) + T(C + I)
\end{array}
\] (6.4)

Thus, we have seen that \( g : (E, e, i_E) \to (C, c, i_C) \) is a morphism in \( \text{Coalg}_I^{\mathcal{F}}(\bar{F}) \), i.e. the diagram below commutes in \( \mathcal{K}(T) \):

\[
\begin{array}{ccc}
I & \xrightarrow{i_E} & E & \xrightarrow{c} & \bar{F}E \\
i_C & \downarrow & \downarrow{g} & \downarrow{\bar{F}g} \\
C & \xrightarrow{c} & \bar{F}C
\end{array}
\]

Finally, we establish that \( G(E, e, i_E) = (D, d, i_D) \) by showing that the isomorphism \([m, i_D]\) (in \( \mathcal{C} \)) is a morphism \( G(E, e, i_E) \) to \( (D, d, i_D) \) in \( \text{Coalg}_I(TF + T) \):

\[
\begin{array}{ccc}
I & \xrightarrow{\text{inr}} & E + I & \xrightarrow{\epsilon + i_E} & TFE + TE & \xrightarrow{TF_{\text{inl}} + T_{\text{inl}}} & TF(E + I) + T(E + I) \\
\downarrow & \downarrow & \downarrow{[m, i_D]} & \downarrow & \downarrow{T_{\text{inl}}} & \downarrow & \downarrow{T_{[m, i_D]}} \\
D & \xrightarrow{d} & TFD + TD
\end{array}
\]

Indeed, the two triangles trivially commute, and for the middle part consider the coproduct components separately. In fact, the left- and right-hand components are the upper parts of (6.3) and (6.4), respectively. This completes the proof.

**Corollary 6.12.** Let \( T \) and \( F \) preserve finite intersections, then:

1. The functor \( G \) preserves and reflects reachable coalgebras. That is, \( G(C, c, i_C) \) is reachable iff \( (C, c, i_C) \) is.

2. The reachable part of an \( I \)-pointed \( \bar{F} \)-coalgebra \( (C, c, i_C) \) is (up to isomorphism) given by the reachable part of \( G(C, c, i_C) \).

More precisely, in order to construct the reachable part of an \( I \)-pointed \( \bar{F} \)-coalgebra \( (C, c, i_C) \) one proceeds as follows:
(1) Construct the reachable part of $G(C, c, i_C)$, call the carrier $D$.
(2) Then $D = E + I$ for some subobject $E$ of $C$.
(3) $E$ is the carrier of the reachable subcoalgebra of $(C, c, i_C)$.

Note that if $\mathcal{K}(T)$ and $\mathcal{F}$ fulfill Assumption 5.8, then this gives the same result as Construction 5.17 because the reachable part of a coalgebra is unique.

Proof. (1) For reflection, let $G(C, c, i_C)$ be reachable. By Proposition 6.10, every subcoalgebra $m: (S, s, i_S) \rightarrow (C, c, i_C)$ is preserved by $G$, thus $Gm$ is an isomorphism, whence $m$ is one.

For preservation, consider a subcoalgebra $m: (D, d, i_D) \rightarrow G(C, c, i_C)$. By Lemma 6.11, there exists $m': (E, e, i_E) \rightarrow (C, c, i_C)$ in $\text{Coalg}_I^p(\mathcal{F})$ such that $m = Gm'$. Since $G$ reflects subcoalgebras by Proposition 6.10, $m'$ is a subcoalgebra. Finally, since $(C, c, i_C)$ is reachable, $m'$ is an isomorphism, thus so is $m$.

(2) This follows from point (1) noting that $(C, c, i_C)$ has a unique reachable subcoalgebra since $G$ reflects isomorphisms by Proposition 6.10.

Remark 6.13. Observe that in the case where the base category is $\text{Set}$ one may drop the assumption that $T$ and $F$ preserves finite intersections. If $I$ is the empty set then the reachable part of every coalgebra is the empty subcoalgebra (in both $\mathcal{C}$ and $\mathcal{K}(T)$), and so the statement is trivial (cf. Remark 2.6).

If $I$ is non-empty, the intersections computed in the above proof are non-empty and thus preserves by every $T$ and $F$, see Trnková [25].

7 Conclusions and Future Work

We have presented a new iterative construction of the reachable part of a given $I$-pointed coalgebra. Our constructions work for coalgebras for intersection-preserving endofunctors over a category $\mathcal{C}$ which has coproducts and a factorization system $((\mathcal{E}, \mathcal{M}))$ where $\mathcal{M}$ consists of monomorphisms.\(^3\) For coalgebras over $\text{Set}$ we saw that their reachable part can be constructed by running the standard breadth-first search algorithm on their canonical graph. Finally, we have considered coalgebras over Kleisli categories for a consistent finite-intersection-preserving monad $T$. We have shown that for a functor $\tilde{F}$ on $\mathcal{K}(T)$ extending a finite-intersection-preserving functor $F$ on $\mathcal{C}$, the reachable part of a given $I$-pointed coalgebra can be obtained from the reachable part of an $I$-pointed $(TF + T)$-coalgebra canonically constructed from the given one.

There remains a number of questions for future work. First, it should be interesting to see whether our results still hold if we drop our assumption that $\mathcal{M}$ is a class of monomorphisms. Secondly, it seems that the reachability construction can be further generalized from working with (operators on) subcoalgebras to working with fibrations, with the subobject fibration yielding the present level of

\(^3\)Note that it actually suffices that every object has a complete lattice of subobjects and preimages (such as in every complete and well-powered category).
generality. Finally, we have seen that breadth-first search is an instance of our reachability construction. A fibrational approach might provide other breadth-first search based algorithms such as Dijkstra’s algorithm for shortest paths and Prim’s algorithm for minimum spanning trees as special instances.

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