On the global solvability of the vacuum Einstein equation in four dimensions

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Abstract
Let $M$ be a connected, simply connected, oriented, closed, smooth four-manifold which is spin (or equivalently having even intersection form) and put $M^\times := M \setminus \{\text{point}\}$. In this paper we prove that if $X^\times$ is a smooth four-manifold homeomorphic but not necessarily diffeomorphic to $M^\times$ (more precisely, it carries a smooth structure à la Gompf) then $X^\times$ can be equipped with a complete Ricci-flat Riemannian metric. As a byproduct of the construction it follows that this metric is self-dual as well consequently $X^\times$ with this metric is in fact a hyper-Kähler manifold. In particular we find that the largest member of the Gompf–Taubes radial family of large exotic $\mathbb{R}^4$’s admits a complete Ricci-flat metric (and in fact is a hyper-Kähler manifold).

The construction is based on a successive application of results of Gompf, Penrose, Taubes and Uhlenbeck on exotic smooth structures, twistor theory, self-dual spaces and singularity removal in Yang–Mills fields, respectively.

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1 Introduction and summary

Solving the vacuum Einstein equation globally, or in other words: Finding a (pseudo-)Riemannian Ricci-flat metric along a differentiable manifold i.e., a metric $g$ which satisfies the second order non-linear partial differential equation

$$\text{Ric}_g = 0$$

over a differentiable manifold $M$, is a century-old evergreen problem dwelling in the heart of modern differential geometry [2] and theoretical physics [34]. The problem of solvability naturally splits up into local and global solvability and also depends on the signature of the metric. Let us first consider the Riemannian case. Thanks to its non-linearity, solvability of the Ricci-flatness condition is already locally problematic; nevertheless exploiting its elliptic character various kinds of local existence results

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(e.g. [5, 11]) are known at least for the related but in some sense complementary equation $\text{Ric}_g = \Lambda g$ (whose solutions are called Einstein metrics) with $\Lambda \neq 0$. As one expects, in these local existence problems the dimension of $M$ plays no special role. However dimensionality issues characteristically enter the game when one considers global solvability. Finding global solutions in four dimensions i.e., when $\dim_\mathbb{R} M = 4$ is particularly important from a physical point of view and quite interestingly, from the mathematical viewpoint, precisely this is the dimension where global solvability is the most subtle. As it is well-known, if $\dim_\mathbb{R} M < 4$ the vacuum Einstein equation reduces to a full flatness condition on the metric hence it admits only a “few” global solutions; on the contrary, if $\dim_\mathbb{R} M > 4$ there are no (known) obstructions for global solvability hence apparently there are “too many” global solutions. A delicate balance is achieved if $\dim_\mathbb{R} M = 4$: For instance by a classical result [19, 32] we know that a Riemannian Einstein (hence in particular a Ricci-flat) metric on a compact $M$ can exist only if its Euler characteristic $\chi(M)$ and signature $\sigma(M)$ obey the inequality $\chi(M) \geq \frac{2}{3} |\sigma(M)|$. This implies for example that the connected sum of at least five copies of complex projective spaces cannot be Einstein. However even in four dimensions if $M$ is non-compact there are no (known) obstruction against the solvability of the vacuum Einstein equation.

The most productive—and both mathematically and physically extraordinary important—presently known method to find global solutions of the Lorentzian vacuum Einstein equation is based on the initial value formulation [34, Chapter 10] which exploits the hyperbolic character of the Ricci-flatness condition (far from being complete, just for recent results cf. eg. [3, 4, 21, 23]) and the references therein). Restricting attention to the four dimensional situation from now on, in this approach one starts with an appropriate initial value data set, subject to the (simpler) vacuum constraint equations, on a three dimensional manifold $S$ and obtains solutions of the original vacuum Einstein equation on a four dimensional manifold $M$ which is always diffeomorphic to the smooth product $S \times \mathbb{R}$ (with the unique smooth structures on the factors) [1]. It is worth calling attention that even if the initial value formulation produces an abundance of solutions from the viewpoint of global analysis and theoretical physics, it is quite inproductive from the viewpoint of (low dimensional) differential topology. To illustrate this, suppose we want to find spaces $(M, g)$ satisfying $\text{Ric}_g = 0$ over a connected and simply connected, open four-manifold $M$. If the initial value formulation is applied, and if in this case we impose a further condition that the corresponding Cauchy surface $S$ be compact, then by the Poincaré–Hamilton–Perelman theorem $S$ must be homeomorphic hence diffeomorphic to the three-sphere $S^3$ consequently $M$ is uniquely fixed to be $S^3 \times \mathbb{R}$ up to diffeomorphisms (but of course this unique $M$ still can carry lot of non-isometric Ricci-flat metrics $g$).

However, switching back to the Riemannian signature for a moment, in the simply connected situation we can apply the main result of this paper as well—considered as a substantially improved and technically revised version of our earlier results [7, 8]—summarized in the following

**Theorem 1.1.** Let $M$ be a connected, simply connected, oriented, closed (i.e., compact without boundary), smooth four-manifold which is spin (or equivalently having even intersection form) and put $M^\times := M \setminus \{\text{point}\}$. If $X^\times$ is a smooth four-manifold homeomorphic but not necessarily diffeomorphic to $M^\times$ such that it carries a smooth structure à la Gompf then $X^\times$ can be equipped with a complete Ricci-flat Riemannian metric.

To make a comparison with the aforementioned differentio-topological rigidity of initial value formulation in the simply connected setting, let us indicate the “size” of the set of non-isometric solutions to the Riemannian vacuum Einstein equation provided by Theorem 1.1. By the fundamental classification result of Freedman [10], connected and simply connected, oriented, closed topological four-manifolds are topologically classified by their intersection form $Q_M : H_2(M; \mathbb{Z}) \times H_2(M; \mathbb{Z}) \to H_4(M; \mathbb{Z}) \cong \mathbb{Z}$. By assumptions in our theorem here, $M$ is spin and smooth hence $Q_M$ must be even hence indefinite taking...
into account the other fundamental result in this field by Donaldson [6]. Therefore if $\sigma(M)$ denotes its signature and $b_2(M)$ its second Betti number then the intersection form of $M$ looks like

$$Q_M = \frac{1}{8} \sigma(M) \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \oplus \frac{1}{2} (b_2(M) - \sigma(M)) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

consequently, unlike the initial value formulation in the simply connected case, the set of solutions provided by Theorem 1.1 already contains many topologically different underlying spaces. But even more, most of these compact $M$’s themselves carry countable infinitely many different smooth structures, too. Finally, passing to the non-compact punctured spaces $M^\times$, the cardinality of the inequivalent smooth structures underlying the Ricci-flat solutions in Theorem 1.1 already reaches that of the continuum in ZFC set theory by a theorem of Gompf [14] (exhibited as Theorem 2.4 here). Therefore the set of non-isometric Ricci-flat spaces exhibited in Theorem 1.1 is huge. As an extreme but important application of Theorem 1.1 we obtain a result whose Lorentzian analogue is not accessible by the initial value formulation:

**Corollary 1.1.** Let $R^4$ be the largest member of the Gomp–Taubes radial family of large exotic $\mathbb{R}^4$’s. Then $R^4$ carries a complete Ricci-flat Riemannian metric.

The proof of Theorem 1.1 is based on a successive application of basic results by Gompf [12, 13, 14], Penrose [26], Taubes [30, 31] and Uhlenbeck [33] on exotic smooth structures, twistor theory, self-dual spaces and singularity removal in Yang–Mills fields, respectively and in principle is simple and works as follows. The idea in the spirit of twistor theory is to convert the real-analytic problem of solving $\text{Ric}_g = 0$ on the real 4-space $M^\times$ into a complex-analytic problem on a complex 3-space $\mathbb{C}$ associated to $M^\times$. Take an arbitrary oriented and closed smooth 4-manifold $M$. In the first step, following Taubes, by connected summing sufficiently (but finitely) many complex projective spaces to $M$, we construct a space $\overline{X}_M \cong M \# \mathbb{CP}^2 \# \ldots \# \mathbb{CP}^2$ which (with respect to its induced orientation) carries a self-dual metric $\overline{\gamma}$. Then, in the second step following Penrose, we observe that $\overline{Z}$, the twistor space of $(\overline{X}_M, \overline{\gamma})$, is a complex 3-manifold. Let $\overline{X}_M \subset \overline{X}_M$ be the open space obtained by deleting a carefully chosen closed subset homeomorphic to a projective line in one $\mathbb{CP}^2$ factor of $\overline{X}_M$ and put $\gamma := \overline{\gamma}|_{\overline{X}_M}$ and $Z := \overline{Z}|_{\overline{X}_M}$. By the aid of $Z$ we can conformally rescale the incomplete self-dual space $(X_M, \gamma)$ to a complete Ricci-flat one $(X_M, g)$ if $M$ is connected and simply connected. In the third and last step, we remove the extra “unused” $\mathbb{CP}^2$’s from $X_M$ to obtain an open smooth space $X^\times$ which is homeomorphic to the punctured space $M^\times$ however is not necessarily diffeomorphic to it by results of Gompf. A closer inspection based on Uhlenbeck’s singularity removal theorem shows that the Ricci-flat complete metric $g$ on $X_M$ also descends to a Ricci-flat complete metric on $X^\times$ if $M$ is in addition spin. The result is therefore a connected, simply connected, open, complete, Ricci-flat Riemannian spin 4-manifold $(X^\times, g)$.

By the conformal invariance of self-duality this technical condition in fact survives the whole procedure. Taking into account that a complete Ricci-flat and self-dual metric on a simply connected 4-manifold always induces a hyper–Kähler structure on it [2, Chapter 13], we can re-formulate the result of our construction as

**Theorem 1.2.** The complete Ricci-flat metric of Theorem 1.1 on $X^\times$ with its fixed orientation is self-dual as well consequently $X^\times$ carries a hyper–Kähler structure, too.

In this way we obtain
Corollary 1.2. The space $\mathbb{R}^4$ of Corollary 1.1 carries a hyper-Kähler structure.

The whole construction indicates that—upon sacrificing self-duality but not Ricci-flatness by replacing Uhlenbeck’s singularity removal theorem at the third step with a much more complicated gluing technique akin to that of [4] for example—the spin condition can be dropped:

**Conjecture 1.1.** Let $M$ be a connected, simply connected, oriented, closed (i.e., compact without boundary), smooth four-manifold and put $M^\times := M \setminus \{\text{point}\}$. If $X^\times$ is a smooth four-manifold homeomorphic but not necessarily diffeomorphic to $M^\times$ such that it carries a smooth structure à la Gompf then $X^\times$ can be equipped with a complete Ricci-flat Riemannian metric.

To close the Introduction we mention that all the Riemannian Ricci-flat solutions of Theorem 1.1 can be converted into Lorentzian ones by virtue of [8, Lemmata 4.1 and 4.2] thereby exhibiting lot of new four dimensional Lorentzian vacuum solutions. These solutions are not accessible with the initial value formulation because they, compared with the time evolution of typical initial data sets, are “too long” in an appropriate sense (cf. [8, Section 5]). Informally speaking, the vacuum Einstein equation is more tractable in Riemannian signature because of the elliptic nature of the Ricci-flatness condition in contrast to its hyperbolic character in Lorentzian signature: Meanwhile solutions in Riemannian signature are protected by elliptic regularity hence “extend well”, the regularity profiles of Lorentzian initial data sets quickly get destroyed during their hyperbolic time evolution.

The paper is organized as follows. Section 2 contains the collection of the required background material with rapid discussions of these results from our viewpoint. Section 3 serves as a warm-up and contains the construction in the simpler non-exoting setting i.e., when $X^\times$ is not only homeomorphic but even diffeomorphic to $M^\times \subset M$. Section 4 is then devoted to the construction in the exotic setting with appropriate modifications. The most important but purely technical difference, compared to the non-exotic situation worked out in the previous section, is an application of Lebesgue integration of functions taking values in the algebraic function field $\mathbb{C}(z)$ of rational functions in one complex variable. This is summarized in Section 5. Finally, in Section 6, for completeness we reproduced Lemma [8, Lemma 4.2] on how to convert the Riemannian solutions into Lorentzian ones.

2 Background material

Let us begin with recalling all the powerful results, techniques, tools to be used in the construction of Ricci-flat metrics in this paper.

(i) *Construction of self-dual spaces.* It is well-known that the Fubini–Study metric on the complex projective space $\mathbb{C}P^2$ with orientation inherited from its complex structure is self-dual (or half-conformally flat) i.e., the anti-self-dual part $W^-$ of its Weyl tensor vanishes; consequently the oppositely oriented complex projective plane $(\mathbb{C}P^2)^{op}$ is anti-self-dual. A powerful generalization of this latter classical fact is Taubes’ construction of an abundance of anti-self-dual 4-manifolds; firstly we exhibit his result but now in an orientation-reversed form:

**Theorem 2.1.** (Taubes [31, Theorem 1.1]) Let $M$ be a connected, compact, oriented smooth 4-manifold. Let $\mathbb{C}P^2$ denote the complex projective plane with its usual orientation and let # denote the operation of taking the connected sum of manifolds. Then there exists a natural number $k_M \geq 0$ such that for all $k \geq k_M$ the modified compact manifold

$$M^\# \mathbb{C}P^2 \# \ldots \# \mathbb{C}P^2$$

admits a self-dual Riemannian metric. \(\diamond\)
Let us very roughly summarize how Taubes’ construction works (see [31, Section 2]). Take an arbitrary connected, oriented, closed Riemannian 4-manifold \((M, h)\) and consider the density of the anti-self-dual part of the Weyl curvature of \(h\) i.e., the pointwise norm \(|W^-|\) along \(M\). If it happens that somewhere around a point \(p \in M\) this curvature density is large then take a \(\mathbb{C}P^2\) with its usual Fubini–Study metric having zero anti-self-dual Weyl tensor and glue it to a ball \(B^4_p\) of sufficiently small radius about the point. The result is a Riemannian metric on \(M\#\mathbb{C}P^2\) having a bit smaller anti-self-dual Weyl tensor. Repeating this procedure probably very (but surely finitely) many times one comes up with a metric on \(\overline{X}_M := M\#\mathbb{C}P^2\# \ldots \# \mathbb{C}P^2\) whose \(W^-\) has already arbitrarily small \(L^2\)-norm. Then, by the aid of the implicit function theorem one perturbs this metric into a new one \(\mathcal{F}\) which is already self-dual i.e., having \(W^- = 0\). It is a crucial feature of Taubes’ construction and from the point of view of our application it is also important that \(\mathcal{F}\), when restricted to any of the \(\mathbb{C}P^2\) summands, will coincide with the standard Fubini–Study metric.

(ii) Tools from twistor theory. Let us now recall Penrose’ twistor method [26] to solve the Riemannian vacuum Einstein equation (for a very clear introduction cf. [2, Chapter 13], [19, 20]). Consider the bundle of unit-length anti-self-dual 2-forms \(S(\wedge^2 \overline{X}_M)\) over a compact oriented space \((\overline{X}_M, \mathcal{F})\) which is self-dual with respect to its orientation (regarding the specific notation below cf. Sections 3 and 4).

Since in 4 dimensions \(\wedge^2 \overline{X}_M\) is a rank 3 real vector bundle over \(\overline{X}_M\), its unit-sphere bundle \(S(\wedge^2 \overline{X}_M)\) is the total space of a smooth \(S^2\)-fibration \(\overline{\pi}: S(\wedge^2 \overline{X}_M) \to \overline{X}_M\). The Levi–Civita connection of the metric \(\mathcal{F}\) on \(\overline{X}_M\) can be used to furnish the real 6-manifold \(S(\wedge^2 \overline{X}_M)\) with a canonical almost complex structure; the fundamental observation of twistor theory is that this almost complex structure is integrable because \(\mathcal{F}\) is self-dual [2, Theorem 13.46]. The resulting complex 3-manifold \(Z \cong S(\wedge^2 \overline{X}_M)\) is called the twistor space while the smooth fibration \(\overline{\pi}: Z \to \overline{X}_M\) the twistor fibration of \((\overline{X}_M, \mathcal{F})\). The most important property of a twistor space of this kind is that its twistor fibers \(\overline{\pi}^{-1}(x) \subset Z\) for all \(x \in \overline{X}_M\) fit into a locally complete complex 4-parameter family \(\overline{X}^C_M\) of projective lines \(Y \subset \overline{Z}\) each with normal bundle \(H \oplus H\), with \(H\) being the dual of the tautological line bundle over \(Y \cong \mathbb{C}P^1\). Moreover, there exists a real structure \(\tau: \overline{Z} \to \overline{Z}\) defined by taking the antipodal maps along the twistor fibers \(Y_x := \overline{\pi}^{-1}(x) \subset \overline{Z}\) for all \(x \in \overline{X}_M \subset \overline{X}^C_M\) which are therefore called real lines among all the lines \(Y\) in \(\overline{Z}\). In other words, \(\overline{Z}\) is fibered exactly by the real lines \(Y_x\) for all \(x \in \overline{X}_M\). Hence the real 4 dimensional self-dual geometry has been encoded into a 3 dimensional complex analytic structure in the sense that one can recover \((\overline{X}_M, \mathcal{F})\) just from \(\overline{Z}\) up to conformal equivalence.

One can go further and raise the question how to recover precisely \((\overline{X}_M, \mathcal{F})\) itself from its conformal class, or more interestingly to us: How to get a Ricci-flat Riemannian 4-manifold \((X_M, g)\) i.e., a solution of the (self-dual) Riemannian vacuum Einstein equation. Not surprisingly, to get the latter stronger structure, one has to specify further data on the twistor space. A fundamental result of twistor theory [26] is that a solution of the 4 dimensional (self-dual) Riemannian vacuum Einstein equation is equivalent to the following set of data (cf. [19, 20]):

* A complex 3-manifold \(Z\), the total space of a holomorphic fibration \(\pi: Z \to \mathbb{C}P^1\);

* A complex 4-parameter family of holomorphically embedded complex projective lines \(Y \subset Z\), each with normal bundle \(H \oplus H\) (here \(H\) is the dual of the tautological bundle i.e., the unique holomorphic line bundle on \(Y \cong \mathbb{C}P^1\) with \(\langle c_1(H), [Y] \rangle = 1\));

* A non-vanishing holomorphic section \(s\) of \(K_Z \otimes \pi^* H^4\) (here \(K_Z\) is the canonical bundle of \(Z\));

* A real structure \(\tau: Z \to Z\) such that it coincides with the antipodal map \(u \mapsto -\pi^{-1}\) upon restricting to the \(\tau\)-invariant elements \(Y \subset Z\) (called real lines) from the family; moreover these real lines are both sections of \(\pi\) and comprise a fibration of \(Z\) (note that \(s\) is constant along each real line).
These data allow one to construct a Ricci-flat and self-dual (i.e., the Ricci and the anti-self-dual Weyl part of the curvature tensor vanishes) solution \((X_M, g)\) of the Riemanian Einstein’s vacuum equation with vanishing cosmological constant as follows. The holomorphic lines \(Y \subset Z\) form a locally complete family and fit together into a complex 4-manifold \(X^4\). This space carries a natural complex conformal structure by declaring two nearby points \(y_1, y_2 \in X^4\) to be null-separated if the corresponding lines intersect i.e., \(Y_1 \cap Y_2 \neq \emptyset\) in \(Z\). Infinitesimally this means that on every tangent space \(T_y X^4 = \mathbb{C}^4\) a null cone is specified. Restricting this to the real lines singled out by \(\tau\) and parameterized by an embedded real 4-manifold \(X_M \subset X^4\) we obtain the real conformal class \([g]\) of a Riemannian metric on \(X_M\). The isomorphism \(s : \pi^* H^4 \cong K_Z\) is essentially uniquely fixed by its compatibility with \(\tau\) and gives rise to a volume form on \(X_M\) this way fixing the metric \(g\) in the conformal class. Given the conformal class, it is already meaningful to talk about the unit-sphere bundle of anti-self-dual 2-forms \(S(\wedge^2 X^4)\) over \(X_M\) with its induced orientation from the twistor space and \(Z\) can be identified with the total space of \(S(\wedge^2 X^4)\). This way we obtain a smooth twistor fibration \(p : Z \to X_M\) whose fibers are \(\mathbb{C}P^1\)s hence \(\pi : Z \to \mathbb{C}P^1\) can be regarded as a parallel translation along this bundle over \(X_M\) with respect to a flat connection which is nothing but the induced connection of \(g\) on \(\wedge^2 X^4\), cf. [22]. Knowing the decomposition of the Riemannian curvature into irreducible components over an oriented Riemannian 4-manifold [28], this partial flatness of \(S(\wedge^2 X^4)\) implies that \(g\) is Ricci-flat and self-dual. Finally note that, compared to the bare twistor space \(Z\) of a flat self-dual manifold \((\overline{X}_M, \overline{\tau})\) above, the essential new requirement for constructing a self-dual Ricci-flat space \((X_M, g)\) is the existence of a holomorphic map \(\pi\) from the twistor space \(Z\) into \(\mathbb{C}P^1\). We conclude our summary of the non-linear graviton construction by referring to [19, 20, 22] or [2, Chapter 13] for further details.

(iii) Removable singularities in Yang–Mills fields. Next let us refresh Uhlenbeck’s by-now classical singularity removal theorem:

**Theorem 2.2.** (Uhlenbeck [33, Theorem 4.1] or [9, Appendix D])

(i) **Local version:** Let \(\nabla^\times\) be a solution of the SU(2) Yang–Mills equations in the open punctured 4-ball \(B^4 \setminus \{0\}\) with \(\|F^\times\|_{L^2(B^4)}^2 = \int_{B^4} |F^\times|^2 < +\infty\) i.e., having finite energy and \(\nabla^\times = d + A^\times\) such that \(A^\times \in L^2_{1, \text{loc}}(B^4 \setminus \{0\})\). Then \(\nabla^\times\) is gauge equivalent to a connection \(\nabla\) which extends smoothly across the singularity to a smooth connection.

(ii) **Global version:** Let \((M, g)\) be a connected, closed, oriented Riemannian 4-manifold and let \(\nabla^\times\) be an SU(2) connection on a vector bundle \(E^\times\) over \(M^\times := M \setminus \{\text{point}\}\) which is a solution of the SU(2) Yang–Mills equations and satisfies \(\|F^\times\|_{L^2(M)}^2 < +\infty\) and there is an \(L^2_{1, \text{loc}}\) gauge for \(\nabla^\times\) around the puncturing of \(M\). Then \(\nabla^\times\) is SU(2) gauge equivalent to a connection \(\nabla\) on a vector bundle \(E\) over \(M\) i.e., to a connection which extends across the pointlike singularity of the original connection. 

Note that if \(\nabla^\times\) is indeed singular while \(\nabla\) is by construction smooth then the connecting gauge transformation cannot be continuous; consequently the resulting extended vector bundle \(E\) can be topologically different from the original one \(E^\times\). Its isomorphism class is however fully determined by the smooth connection \(\nabla\) via the numerical value of the integral \(-\infty < \frac{1}{8\pi} \int_M \text{tr}(F^\nabla \wedge F^\nabla) < +\infty\), the second Chern number of the bundle \(E\).

(iv) **Exotic stuff.** Finally we evoke some results which provide us with a sort of summary of what is so special in four dimensions (i.e., absent in any other ones). First we recall a special class of large exotic (or fake) \(\mathbb{R}^4\)s whose properties we will need here are summarized as follows:

**Theorem 2.3.** (Gompf–Taubes, cf. [15, Lemma 9.4.2, Addendum 9.4.4 and Theorem 9.4.10]) There exists a pair \((\mathbb{R}^4, K)\) consisting of a differentiable 4-manifold \(\mathbb{R}^4\) homeomorphic but not diffeomorphic to the standard \(\mathbb{R}^4\) and a compact oriented smooth 4-manifold \(K \subset \mathbb{R}^4\) such that
(i) $\mathbb{R}^4$ cannot be smoothly embedded into the standard $\mathbb{R}^4$ i.e., $\mathbb{R}^4 \not\subseteq \mathbb{R}^4$ but it can be smoothly embedded as a proper open subset into the complex projective plane i.e., $\mathbb{R}^4 \subsetneq \mathbb{C}P^2$;

(ii) Take a homeomorphism $f : \mathbb{R}^4 \to \mathbb{R}^4$, let $0 \in B^4_t \subset \mathbb{R}^4$ be the standard open 4-ball of radius $t \in \mathbb{R}_+$ centered at the origin and put $R^4_t := f(B^4_t)$ and $R^4_{+\infty} := \mathbb{R}^4$. Then

$$\{ R^4_t \mid r \leq t \leq +\infty \text{ such that } 0 < r < +\infty \text{ satisfies } K \subset R^4_t \}$$

is an uncountable family of nondiffeomorphic exotic $\mathbb{R}^4$’s none of them admitting a smooth embedding into $\mathbb{R}^4$ i.e., $R^4_t \not\subseteq \mathbb{R}^4$ for all $r \leq t \leq +\infty$.

This class of manifolds is called the Gompf–Taubes large radial family. ◊

The fact that any member $R^4_t$ in this family is not diffeomorphic to $\mathbb{R}^4$ implies the counterintuitive phenomenon that $R^4_t \not\cong W \times \mathbb{R}$ i.e., $\mathbb{R}^4_t$ does not admit any smooth splitting into a 3-manifold $W$ and $\mathbb{R}$ (with their unique smooth structures) in spite of the fact that such continuous splittings obviously exist. Indeed, from the contractibility of $R^4_t$ we can see that $W$ must be a contractible open 3-manifold (a so-called Whitehead continuum) however, by an early result of McMillen [24] spaces of this kind always satisfy $W \times \mathbb{R} \cong \mathbb{R}^4$ i.e., their product with a line is always diffeomorphic to the standard $\mathbb{R}^4$. We will call this property of (any) exotic $\mathbb{R}^4$ occasionally below as “creased”.

From Theorem 2.3 we deduce that for all $r < t < +\infty$ there is a sequence of smooth proper embeddings

$$R^4_r \subsetneq R^4_t \subsetneq R^4_{+\infty} = \mathbb{R}^4 \subsetneq \mathbb{C}P^2$$

which are very wild in the following sense. The complement $\mathbb{C}P^2 \setminus R^4$ of the largest member $R^4$ of this family is homeomorphic to $S^2$ regarded as an only “continuously embedded projective line” in $\mathbb{C}P^2$; therefore we shall denote this complement as $S^2 := \mathbb{C}P^2 \setminus R^4 \subset \mathbb{C}P^2$ in order to distinguish it from the ordinary projective lines $\mathbb{C}P^1 = \mathbb{C}P^2 \setminus \mathbb{R}^4 \subset \mathbb{C}P^2$. If $\mathbb{C}P^2 = \mathbb{R}^4 \cup \mathbb{C}P^1 = \mathbb{C}^2 \cup \mathbb{C}P^1$ is any holomorphic decomposition then $R^4 \cap \mathbb{C}P^1 \not\subseteq 0$ (because otherwise $R^4 \not\subseteq \mathbb{R}^4$ would hold, a contradiction) as well as $S^2 \cap \mathbb{C}P^1 \not= 0$ (because otherwise $H_2(R^4;\mathbb{Z}) \cong 0$ would hold since $\mathbb{C}P^1 \subset \mathbb{C}P^2$ represents a generator of $\mathbb{H}2(\mathbb{C}P^2,\mathbb{Z}) \cong \mathbb{Z}$, a contradiction again). Hence an ordinary projective line $\mathbb{C}P^1$ is always intersected by both $R^4$ and $S^2$ such that, up to a countable subset, $S^2 \cap \mathbb{C}P^1$ is at most a Cantor set. These demonstrate that the members of the large radial family “live somewhere between” $\mathbb{R}^4$ and its complex projective closure $\mathbb{C}P^2$. However a more precise identification or location of them is a difficult task because these large exotic $\mathbb{R}^4$’s—although being honest differentiable 4-manifolds—are very transcendental objects [15, p. 366]: they require infinitely many 3-handles in any handle decomposition (like any other known large exotic $\mathbb{R}^4$) and there is presently\(^1\) no clue as how one might draw explicit handle diagrams of them (even after removing their 3-handles). We note that the structure of small exotic $\mathbb{R}^4$’s i.e., which admit smooth embeddings into $\mathbb{R}^4$, is better understood, cf. [15, Chapter 9].

Our last ingredient is the following ménagerie result of Gompf.

**Theorem 2.4.** (Gompf [14, Theorem 2.1]) Let $X$ be a connected (possibly non-compact, possibly with boundary) topological 4-manifold and let $X^\times := X \setminus \{ \text{point} \}$ be the punctured manifold with a single point removed. Then the non-compact space $X^\times$ admits noncountable many (with the cardinality of the continuum in ZFC set theory) pairwise non-diffeomorphic smooth structures. ◊

If for instance $M$ is a connected compact smooth 4-manifold then Gompf’s construction goes as follows: Take the maximal large $R^4$ from Theorem 2.3 and put $X^\times := M\# R^4$. This smooth 4-manifold is obviously homeomorphic to the punctured $M^\times$; more generally, $X^\times_t := M\# R^4_t$ will produce uncountable many smooth structures on the unique topological 4-manifold underlying $X^\times_t$.

\(^1\)More precisely in the year 1999, cf. [15].
3 The construction

In this section, which serves as a warming-up for the next one, we construct solutions of the vacuum Einstein equation on punctured 4-manifolds carrying their standard smooth structure. We begin with an application of Theorem 2.1 as follows.

**Lemma 3.1.** Out of any connected, closed (i.e., compact without boundary) oriented smooth 4-manifold \( M \) one can construct a connected, open (i.e., non-compact without boundary) oriented smooth Riemannian 4-manifold \((X_M, \gamma)\) which is self-dual but incomplete in general.

**Proof.** Pick any connected, oriented, closed, smooth 4-manifold \( M \). Referring to Theorem 2.1 let \( k := \max(1, k_M) \in \mathbb{N} \) be a positive integer, put

\[
\overline{X}_M := M \# \mathbb{C}P^2 \# \ldots \# \mathbb{C}P^2_{k}
\]

and let \( \overline{\gamma} \) be a self-dual metric on it. Then \((\overline{X}_M, \overline{\gamma})\) is a compact self-dual manifold. Pick any \( \mathbb{C}P^2 \) factor within \( \overline{X}_M \) and any (holomorphically embedded) projective line \( \mathbb{C}P^1 \subset \mathbb{C}P^2 \) in that factor (avoiding its attaching point to \( M \)); then \( \mathbb{C}P^1 = \mathbb{C}P^2 \setminus \mathbb{C} \cong \mathbb{C}P^2 \setminus \mathbb{R}^4 \) i.e., the line arises as the complement of an \( \mathbb{R}^4 \) in \( \mathbb{C}P^2 \). Let \( K \subset \mathbb{R}^4 \) be any connected compact subset and put

\[
X_M := M \# \mathbb{C}P^2 \# \ldots \# \mathbb{C}P^2_{k}(\mathbb{C}P^2 \setminus \mathbb{C}P^1) \cong M \# \mathbb{C}P^2 \# \ldots \# \mathbb{C}P^2_{k} \mathbb{R}^4 \cong M^\times \# \mathbb{C}P^2 \# \ldots \# \mathbb{C}P^2_{k-1}
\]

where the operation \#_K means that the attaching point \( y_0 \in \mathbb{R}^4 \) taken to glue \( \mathbb{R}^4 \) with \( M \# \mathbb{C}P^2 \# \ldots \# \mathbb{C}P^2 \) satisfies \( y_0 \in K \in \mathbb{R}^4 \) and \( M^\times := M \#_K \mathbb{R}^4 \cong M \setminus \{ \text{point} \} \) is the punctured space with its inherited smooth structure from the smooth embedding \( M^\times \subset M \). The result is a connected, open 4-manifold \( X_M \) (see Figure 1).

![Figure 1](image-url)  
Figure 1. Construction of \( X_M \) out of \( M \). The gray ellipse represents a single end diffeomorphic to the complement of a connected compact subset \( K \) in \( \mathbb{R}^4 \).

From the proper smooth embedding \( X_M \cong \overline{X}_M \setminus \mathbb{C}P^1 \subset \overline{X}_M \) there exists a restricted self-dual Riemannian metric \( \gamma := \overline{\gamma}|_{X_M} \) on \( X_M \) which is however in general non-complete. ◇

Next we improve the incomplete self-dual space \((X_M, \gamma)\) of Lemma 3.1 to a complete Ricci-flat space \((X_M, g)\) by conformally rescaling \( \gamma \) with a suitable positive smooth function \( \varphi : X_M \to \mathbb{R}^+ \) which is a “multi-task” function in the sense that it kills both the scalar curvature and the traceless Ricci tensor of \( \gamma \) moreover blows up sufficiently fast along the \( \mathbb{R}^4 \)-end of \( X_M \) to render the rescaled metric \( g \) complete.

Two classical examples serve as a motivation. First, let \( S^4 \subset \mathbb{R}^5 \) be the standard 4-sphere equipped with the standard orientation and round metric inherited from the embedding. Put \( \overline{X}_M := S^4 \) and \( \overline{\gamma} := \) the
standard round metric. It is well-known that \((\mathcal{X}_M, \bar{\gamma}) = (S^4, \bar{\gamma})\) is self-dual and Einstein with non-zero cosmological constant i.e., not Ricci-flat. Put \(X_M := S^4 \setminus \{ \infty \} = \mathbb{R}^4\); then \(\gamma = \bar{\gamma}|_{\mathbb{R}^4}\) and \((X_M, \gamma) = (\mathbb{R}^4, \gamma)\) is an incomplete self-dual space. But setting \(\phi : \mathbb{R}^4 \to \mathbb{R}_+\) to be \(\phi(x) := (1 + |x|^2)^{-1}\), then \(\gamma := \phi^{-2} \cdot \gamma\) is nothing but the standard flat metric \(\eta\) on \(\mathbb{R}^4\) which is of course complete and Ricci-flat. Hence \((X_M, \gamma) = (\mathbb{R}^4, \eta)\), the conformal rescaling of \((X_M, \gamma) = (\mathbb{R}^4, \gamma)\), is the desired complete Ricci-flat space in this simple case. Note that \((\mathbb{R}^4, \eta)\) is a trivial hyper-Kähler space, too.

The second example which is more relevant to us and deals with \(\mathbb{C}P^2\) with its Fubini–Study metric. Put \(\overline{\mathcal{X}}_M := \mathbb{C}P^2\) and \(\bar{\gamma} :=\)Fubini–Study metric. It is well-known that \((\overline{\mathcal{X}}_M, \bar{\gamma}) = (\mathbb{C}P^2, \bar{\gamma})\) is self-dual and Einstein with non-zero cosmological constant i.e., not Ricci-flat. Now let \(X_M := \mathbb{C}P^2 \setminus \mathbb{C}P^1 = \mathbb{R}^4\); then \(\gamma = \bar{\gamma}|_{\mathbb{R}^4}\) and \((X_M, \gamma) = (\mathbb{R}^4, \gamma)\) is an incomplete self-dual space. If \(0 \neq (z^0, z^1, z^2) \in \mathbb{C}^3\) and \([z^0 : z^1 : z^2] \in \mathbb{C}P^2\) then take the projective line \(\mathbb{C}P^1 \subset \mathbb{C}P^2\) defined by \(z^0 = 0\). Introducing \(w_i = \frac{z^i}{z^0}\) \((i = 1, 2)\) and \(w = (w_1, w_2) \in \mathbb{C}^2 = \mathbb{R}^4\), on the complementum \(\mathbb{C}P^2 \setminus \mathbb{C}P^1\) the restricted Fubini–Study metric \(\gamma\) looks like

\[\gamma_{ij}(w) = (1 + |w|^2)^{-1} \delta_{ij} - (1 + |w|^2)^{-2} \overline{\omega}_iw_j\]

and along this local part it already possesses a Kähler potential \(K(w) = \log(1 + |w|^2)\). This time define \(\phi : \mathbb{R}^4 \to \mathbb{R}_+\) as

\[\phi(w) := e^{-\frac{3}{4}K(w)} = (1 + |w|^2)^{-\frac{3}{4}}\]  \(\text{(2)}\)

which is a non-holomorphic function and consider the conformally rescaled (real) metric \(g := \phi^{-2} \cdot \gamma\). One can check that this is a complete Ricci-flat metric on \(\mathbb{R}^4\). Hence \((X_M, g) = (\mathbb{R}^4, g)\), the conformal rescaling of \((X_M, \gamma) = (\mathbb{R}^4, \gamma)\) is a complete Ricci-flat space. It is already not flat but note again that nevertheless \(g\) includes a (not asymptotically flat in any sense) hyper-Kähler structure on \(\mathbb{R}^4\) because \(g\) is a complete, self-dual, Ricci-flat metric on the simply connected space \(\mathbb{R}^4\). In our much more general situation set up in Lemma 3.1 we shall use Penrose’ non-linear graviton construction (i.e., twistor theory) [26] to find conformal rescalings. Consider the compact self-dual space \((\overline{\mathcal{X}}_M, \bar{\gamma})\) from Lemma 3.1, take its twistor fibration \(\overline{\pi} : \mathcal{Z} \to \overline{\mathcal{X}}_M\) and let

\[p : Z \longrightarrow X_M\]

be its restriction induced by the smooth embedding \(X_M \subset \overline{X}_M\) i.e., \(Z := \mathcal{Z}|_{X_M}\) and \(p := \overline{\pi}|_{X_M}\). Then \(Z\) is a non-compact complex 3-manifold already obviously possessing all the required twistor data except the existence of a holomorphic mapping \(\pi : Z \to \mathbb{C}P^1\).

**Lemma 3.2.** Consider the connected, open, oriented, incomplete, self-dual space \((X_M, \gamma)\) as in Lemma 3.1 with its twistor fibration \(p : Z \to X_M\) constructed above. If \(\pi_1(M) = 1\) then there exists a holomorphic mapping \(\pi : Z \to \mathbb{C}P^1\).

**Proof.** Let \(x_0 \in X_M\) be a fixed point belonging to the \(\mathbb{R}^4\) factor of \(X_M\) in its decomposition \((1)\). Our aim is to construct a holomorphic map

\[\pi : Z \longrightarrow p^{-1}(x_0) \cong \mathbb{C}P^1\]  \(\text{(3)}\)

that we carry out in two steps.

**First step:** We begin with constructing \(\pi\) over \(\mathbb{R}^4 = \mathbb{C}P^2 \setminus \mathbb{C}P^1\) by classical means. Consider the smooth twistor fibration \(\overline{\pi} : \mathcal{Z}(\mathbb{C}P^2) \to \mathbb{C}P^2\). Since \(\mathbb{R}^4 \subset \mathbb{C}P^2\), writing \(Z(\mathbb{R}^4) := \mathcal{Z}(\mathbb{C}P^2)|_{\mathbb{R}^4}\) and \(p := \overline{\pi}|_{\mathbb{R}^4}\) we obtain a restricted fibration \(p : Z(\mathbb{R}^4) \to \mathbb{R}^4\). Unlike the full twistor fibration over \(\mathbb{C}P^2\), this restricted one is topologically trivial i.e., \(Z(\mathbb{R}^4)\) is homeomorphic to \(\mathbb{R}^4 \times S^2\) since \(\mathbb{R}^4\) is contractible; consequently \(Z(\mathbb{R}^4)\) admits a continuous trivialization over \(\mathbb{R}^4\). This is a necessary topological condition for the existence of the map \(\pi\). It is known that \(\mathcal{Z}(\mathbb{C}P^2) \cong P(T\mathbb{C}P^2)\) i.e., the twistor space of
the complex projective space can be identified with its projective holomorphic tangent bundle. Consequently \( \mathbb{Z}(\mathbb{C}P^2) \) admits a very classical description namely can be identified with the flag manifold \( F_{12}(\mathbb{C}^3) \) consisting of pairs \((l, p)\) where \( 0 \in l \subset \mathbb{C}^3 \) is a line (i.e., a point \([l] \in \mathbb{C}P^2\)) and \( 0 \in l \subset p \subset \mathbb{C}^3 \) is a plane containing the line (i.e., a line \([l] \in [p] \subset \mathbb{C}P^2 \) containing the point). Then in the twistor fibration \( \overline{p} : \mathbb{Z}(\mathbb{C}P^2) \to \mathbb{C}P^2 \) of the complex projective space \( \overline{p} \) sends \((l, p) \in F_{12}(\mathbb{C}^3) \) into the point \([m] := [l] \cap [p] \in \mathbb{C}P^2 \) where \([l] \) is the plane perpendicular to the line \( l \) in \( \mathbb{C}^3 \) with respect to the standard Hermitian scalar product. This is a smooth but not holomorphic fibration over \( \mathbb{C}P^2 \) with \( \mathbb{C}P^1 \)'s as fibers since \( \overline{p}^{-1}([m]) = \{(l, p) \mid [l] \in [m], [m] \in [p]\} \) i.e., it consists of all lines \([p]\) through \([m]\) in \( \mathbb{C}P^2 \) (a copy of \( \mathbb{C}P^1 \)) and a distinguished point \([l]\) on each given by its intersection with \([m]\). Consider now the restricted twistor fibration \( p : Z(\mathbb{R}^4) \to \mathbb{R}^4 \). Fix a point \([m_0]\) in \( \mathbb{C}P^2 \setminus [m_0] = \mathbb{R}^4 \) with target space \( p^{-1}([m_0]) \cong \mathbb{C}P^1 \) consisting of terminating pairs \((l_0, p_0) \in p^{-1}([m_0]) \subset Z(\mathbb{C}P^2 \setminus [m_0])\). Take a starting pair \((l, p) \in Z(\mathbb{C}P^2 \setminus [m_0])\) over a running point \([m] = [l] \cap [p] \in \mathbb{C}P^2 \setminus [m_0]\). Our aim is to construct a holomorphic map which associates to \((l, p)\) another pair \((l_0, p_0)\). We construct this \( \pi : Z(\mathbb{C}P^2 \setminus [m_0]) \to p^{-1}([m_0]) \) very simply as follows. Consider a starting pair \((l, p)\) and take its line component \([p] \subset \mathbb{C}P^2 \). This line has a unique intersection \([l_0] := [m_0] \cap [p]\) with the infinitely distant projective line. Then, given the target space \( p^{-1}([m_0]) \), define the projective line component \([p_0] \subset \mathbb{C}P^2 \) in the terminating pair \((l_0, p_0) \in p^{-1}([m_0]) \) by taking the unique projective line \([p_0]\) connecting \([l_0]\) with \([m_0]\). In short,

\[
\pi((l, p)) := (l_0, p_0) \quad \text{where} \quad l_0 \subset \mathbb{C}^3 \text{ satisfies } [l_0] := [m_0] \cap [p] \quad \text{and} \\
p_0 \subset \mathbb{C}^3 \text{ satisfies that } [p_0] \text{ connects } [l_0] \text{ with } [m_0] \text{ in } \mathbb{C}P^2
\]  

(see Figure 2 for a construction of this map in projective geometry).

It is a classical observation that this map is well-defined on \( Z(\mathbb{C}P^2 \setminus [m_0]) \) and holomorphic; in particular it is the identity on the target space \( p^{-1}([m_0]) \) i.e., \( \pi((l_0, p_0)) = (l_0, p_0) \).

**Second step:** Finally we extend the map \( \pi \), constructed over the \( \mathbb{R}^4 \)-summand of \( X_M \) in (1) above, over the whole space by analytic continuation. Let \( y_0 \in K \subset \mathbb{R}^4 \) be the attaching point taken to glue \( \mathbb{R}^4 = \mathbb{C}P^2 \setminus [m_0] \) with the rest of \( X_M \) as in Lemma 3.1; suppose \( y_0 \neq [m_0] \in \mathbb{R}^4 \). Let \( j : \mathbb{R}^4 \setminus \{y_0\} \to X_M \) be a smooth embedding which identifies \( \mathbb{R}^4 \) with the \( \mathbb{R}^4 \)-end of \( X_M \) in its decomposition (1) such that \( j([m_0]) = x_0 \) where \( x_0 \in X_M \) is the distinguished point of the map (3) to be constructed. Also write \( J : Z(\mathbb{R}^4 \setminus \{y_0\}) \to Z \) for the induced inclusion (i.e., satisfying \( p \circ J = j \)) where now \( p : Z \to X_M \) of the

---

Figure 2. Construction of the map \( \pi \) satisfying \( \pi((l, p)) = (l_0, p_0) \).
twistor space into that of \(X_M\). Then

\[
\pi' := (J^{-1})^*(\pi|_{\mathbb{R}^4 \setminus \{y_0\}}) : V \to p^{-1}(x_0)
\]

is a partially defined holomorphic map on a connected open subset \(V := p^{-1}(j(\mathbb{R}^4 \setminus \{y_0\})) \subset Z\) of the twistor space of \(X_M\). Note that \(Z \setminus V\) is compact. We now extend \(\pi'\) holomorphically over the whole \(Z\) to be the map (3) as follows. Consider an open covering \(X_M = \bigcup_{k \geq 1} U_k\) such that \(U_1 := j(\mathbb{R}^4 \setminus \{y_0\})\) is the whole \(\mathbb{R}^4\)-end in this covering; take the induced open covering \(Z = \bigcup_{k \geq 1} p^{-1}(U_k)\) of the twistor space, too with \(p^{-1}(U_1) = V\). If \(x \in U_1 \subset X_M\) is any point of the \(\mathbb{R}^4\)-end of \(X_M\) then we know that the holomorphic map \(\pi'|_{p^{-1}(x)} : p^{-1}(x) \to p^{-1}(x_0)\) extends to \(\pi' : V \to p^{-1}(x_0)\) above. This straightforward reformulation of the existence of the partial holomorphic map \(\pi'\) as solution of an extendibility problem from the complex submanifold \(p^{-1}(x)\) to \(V \supset p^{-1}(x)\) fits well into the general pattern: Extendibility of analytical objects from complex submanifolds having positive normal bundles.

More precisely, by referring at this step to an important extendibility result of Griffiths [17, Proposition 1.3], this extendibility depends only on two holomorphic data: The pullback tangent bundle \((\pi'|_{p^{-1}(x)})^*(Tp^{-1}(x_0))\) over \(p^{-1}(x)\) and the normal bundle of it as a complex submanifold \(p^{-1}(x) \subset Z\). But the former bundle cannot locally depend on \(x\) because holomorphic line bundles over \(p^{-1}(x) \cong \mathbb{C}P^1\) form a discrete set. Regarding the latter bundle, \(p^{-1}(x) \subset Z\) as a submanifold is a twistor line in \(Z\) and all twistor lines in the twistor fibration have isomorphic normal bundles (which are positive bundles over the twistor fibers, see the summary on twistor theory in Section 2). Since these twistor lines fulfill the whole \(Z\) these arguments convince us that for every twistor line of \(Z\) along which \(\pi\) has already been defined we can suppose that there exists an open subset containing the twistor line and having shape \(p^{-1}(U_k)\) i.e., is a member from the covering \(\bigcup_{k \geq 1} p^{-1}(U_k)\), such that \(\pi\) extends over it. Exploiting on the one hand the compactness of \(Z \setminus V\) we can choose a finite subcovering from this covering, and on the other hand the connectedness and simply connectedness of \(Z\) provided by that of \(X_M\), we can in finitely many steps and in a unique way analytically continue the partial map \(\pi'\) above from the connected open subset \(V \subset Z\) to a holomorphic map (3) over the whole connected \(Z\) as desired.

It also follows that \(\pi : Z \to \mathbb{C}P^1\) i.e., the map (3) constructed in Lemma 3.2 is compatible with the real structure \(\tau : Z \to Z\) already fixed by the self-dual structure in Theorem 2.1 therefore twistor theory provides us with a Ricci-flat (and self-dual) Riemannian metric \(g\) on \(X_M\). We proceed further and demonstrate that, unlike \((X_M, \gamma)\), the space \((X_M, g)\) is complete.

**Lemma 3.3.** The connected and simply connected, open, oriented, Ricci-flat Riemannian manifold \((X_M, g)\) is complete.

**Proof.** Since both \(\gamma\) and this Ricci-flat metric \(g\) arise from the same complex structure on the same twistor space \(Z\) we know from twistor theory that these metrics are in fact conformally equivalent. That is, there exists a smooth non-constant strictly positive function \(\phi : X_M \to \mathbb{R}_+\) such that \(\phi^{-2} \cdot \gamma = g\). Our strategy to prove completeness is to follow Gordon [16] i.e., to demonstrate that an appropriate real-valued function on \(X_M\), in our case \(\log \phi^{-1} : X_M \to \mathbb{R}\), is proper (i.e., the preimages of compact subsets are compact) with bounded gradient in modulus with respect to \(g\) implying the completeness.

Referring to (1) the open space \(X_M\) arises by deleting one projective line from a \(\mathbb{C}P^2\) factor of the closed space \(\overline{X}_M\). First we observe that \(\phi^{-1} : X_M \to \mathbb{R}_+\) is uniformly divergent along this projective line as follows. It is clear that the potential singularities of \(\phi^{-1}\) originate from those of the map (3). But the **First step** map (4) on \(\mathbb{C}P^2\) has an obvious singularity along the line \(\mathbb{C}P^1 = \mathbb{C}P^2 \setminus \mathbb{R}^4\) to be deleted from this \(\mathbb{C}P^2\) factor of \(\overline{X}_M\) consequently \(\phi^{-1}\) is expected to be somehow singular along this line, too. Moreover, the **First step** of the construction of \(\pi\) in Lemma 3.2 deals with a single \(\mathbb{C}P^2\) factor.
in (1) only hence is universal in the sense that it is independent of the remaining $M$ factor in (1). In other words, for all $X_M$ the map (3) arises by analytically continuing the same $\pi$ on $\mathbb{R}^4 = CP^2 \setminus CP^1$ constructed in the First step in Lemma 3.2. So we anticipate $\phi^{-1}: X_M \to \mathbb{R}_+$ with $X_M \subset X_M$ to exhibit a uniform and universal singular behaviour along $X_M \setminus X_M \cong CP^1$ what we analyze now further.\footnote{Actually (2) shows $\phi^{-1}$ explicitly around its singularity.}

The conformal scaling function satisfies with respect to $\gamma$ the following equations on $X_M$:

\[
\begin{cases}
\Delta_\gamma \phi^{-1} + \frac{1}{6} \phi^{-1} \text{Scal}_\gamma &= 0 \text{ (vanishing of the scalar curvature of } g \text{ on } X_M); \\
\nabla_\gamma^2 \phi - \frac{1}{4} \Delta_\gamma \phi \cdot \gamma + \frac{1}{2} \phi \text{Ric}_\gamma^0 &= 0 \text{ (vanishing of the traceless Ricci tensor of } g \text{ on } X_M).
\end{cases}
\tag{5}
\]

The Ricci tensor $\text{Ric}_\gamma$ of $\gamma$ extends smoothly over $X_M$ because it is just the restriction of the Ricci tensor of the self-dual metric $\gamma$ on $X_M$. Therefore both its scalar curvature $\text{Scal}_\gamma$ and traceless Ricci part $\text{Ric}_\gamma^0$ extend. Consequently from the first equation of (5) we can see that $\phi \Delta_\gamma \phi^{-1}$ extends smoothly over $X_M$. Likewise, adding the tracial part to the second equation of (5) we get $\phi^{-1} \nabla_\gamma^2 \phi = -\frac{1}{2} \text{Ric}_\gamma^\gamma$, hence we conclude that the symmetric tensor field $\phi^{-1} \nabla_\gamma^2 \phi$ extends smoothly over $X_M$ so its trace $\phi^{-1} \Delta_\gamma \phi$ as well. The equation $\Delta_\gamma (\phi \phi^{-1}) = 0$ gives $(\Delta_\gamma \phi) \phi^{-1} + 2\gamma (d\phi, d\phi^{-1}) + \phi \Delta_\gamma \phi^{-1} = 0$ and adjusting this a bit we get

\[\phi^2 |d\phi^{-1}|^2_\gamma = \frac{1}{2} (\phi \Delta_\gamma \phi^{-1} + \phi^{-1} \Delta_\gamma \phi) \tag{6}\]

consequently the function $\phi |d\phi^{-1}|_\gamma$ extends smoothly over $X_M$, too. Assume now that $\overline{\phi}^{-1} \in C^0 (\overline{X}_M)$ i.e., $\phi^{-1}$ is extendible over $\overline{X}_M$ at least continuously. Then, by the aforementioned universal behaviour of $\phi^{-1}$ around its interesting part, we can take $\overline{X}_M \setminus \mathcal{F}: = CP^2$ and $\gamma :=$ Fubini–Study metric. However this metric has constant scalar curvature consequently, by the aid of the first equation of (5) and the maximum principle, we could conclude that $\overline{\phi}^{-1}$ is constant on $CP^2$, a contradiction. Assume now that $\overline{\phi}^{-1} \in L^\infty (\overline{X}_M) \setminus C^0(\overline{X}_M)$ i.e., $\phi^{-1}$ extends at least as a discontinuous bounded function. Then its gradient $d\phi^{-1}$ is divergent along $CP^2 \setminus \mathbb{R}^4$ hence from the extendibility of $\phi |d\phi^{-1}|_\gamma$ we obtain that $\phi$ vanishes hence $\phi^{-1}$ is unbounded along $CP^2 \setminus \mathbb{R}^4$, a contradiction again. Therefore $\phi^{-1}: X_M \to \mathbb{R}_+$ with $X_M \subset X_M$ is uniformly divergent along the whole $X_M \setminus X_M$ yielding, on the one hand, that the function $\log \phi^{-1}: X_M \to \mathbb{R}$ is proper.

As a byproduct the inverse function $\phi$ is bounded on $X_M$ i.e., $|\phi| \leq c_1$ with a finite constant. We already know that $|\phi \Delta_\gamma \phi^{-1}| \leq c_2$ and $|\phi^{-1} \Delta_\gamma \phi| \leq c_3$ with other finite constants as well. Since $\phi |d\phi^{-1}|_\gamma = |d(\log \phi^{-1})|_\gamma$ and carefully noticing that $|\xi|_g = \phi |\xi|_\gamma$ on 1-forms we can use (6) and the estimates above to come up with

\[|d(\log \phi^{-1})|^2_g \leq c_1^2 |d(\log \phi^{-1})|^2_\gamma \leq c_1^2 \left( |\phi \Delta_\gamma \phi^{-1}| + |\phi^{-1} \Delta_\gamma \phi| \right) \leq c_1^2 (c_2 + c_3) < +\infty\]

and conclude, on the other hand, that $\log \phi^{-1}: X_M \to \mathbb{R}$ has bounded gradient in modulus with respect to $g$. Therefore, in light of Gordon’s theorem [16], the Ricci-flat space $(X_M, g)$ is complete. ◇

To complete our construction, we remove the extra “unused” $CP^2$’s from $X_M$ in its decomposition (1) without destroying completeness and Ricci flatness.

**Lemma 3.4.** Consider the space $(X_M, g)$ as in Lemma 3.3. If $M$ is moreover spin (or equivalently, having even intersection form) then the orientation and the complete Ricci-flat metric $g$ on $X_M$ descend to the punctured space $M^\times \subset M$ with its inherited smooth structure, rendering it a connected and simply connected, open, oriented, complete, Ricci-flat Riemannian spin 4-manifold $(M^\times, g)$.\footnote{Actually (2) shows $\phi^{-1}$ explicitly around its singularity.}
Proof. To begin with the \(\mathbb{CP}^2\) removal procedure, take an embedded 2-sphere, respectively, in each “unused” \(\mathbb{CP}^2\) summand in the decomposition (1) of \(X_M\), representing a generator of its second homology \(H_2(\mathbb{CP}^2; \mathbb{Z}) \cong \mathbb{Z}\); pick for instance any projective line \(\mathbb{CP}^1 \subset \mathbb{CP}^2\), and cut them off from \(X_M\). Taking into account that \(\mathbb{CP}^2 \setminus \mathbb{CP}^1 \cong \mathbb{R}^4\) we obtain an intermediate manifold \(M^\times \# \mathbb{R}^4 \# \ldots \# \mathbb{R}^4\) out of \(X_M\). It is clear that \(M^\times\) arises from this intermediate manifold by filling in the “centers” of the finitely many \(\mathbb{R}^4\) summands with one-one point, respectively (see Figure 3).

Given this set-up, our strategy to prove the lemma is as follows: First apply Uhlenbeck’s singularity removal theorem at each \(\mathbb{R}^4\) summand to get rid of the corresponding singularity of the Levi–Civita connection of \(g\)—which is certainly an obstacle against the extension of the metric over the “center” of this \(\mathbb{R}^4\) summand in the intermediate manifold—and in this way extend the connection to \(M^\times\). Finally around each former singular point use a geodesic normal coordinate system adapted to this extended smooth connection on \(M^\times\) to conclude that the metric \(g\) restricted to \(M^\times \# \mathbb{R}^4 \# \ldots \# \mathbb{R}^4 \subset X_M\) smoothly extends over the singularities, too. If this procedure works then the result is a smooth complete Ricci-flat metric on \(M^\times\). However, as we shall see shortly, the non-existence of a spin structure on the original compact \(M\) plays the role of an (and the only one) obstruction against the feasibility of this procedure.

So let us take a fixed \(\mathbb{R}^4\) summand in the intermediate manifold. Since \(M^\times \# \mathbb{R}^4 \# \ldots \# \mathbb{R}^4\) locally looks like a punctured \(M\) around this summand i.e., a point \(p \in M\) removed, we can diffeomorphically model \(M^\times \# \mathbb{R}^4 \# \ldots \# \mathbb{R}^4\) around this \(\mathbb{R}^4\)-summand by an open punctured ball. More precisely let \(p \in M\) be a point, \(p \in U \subset M\) a neighbourhood containing the point and consider a local coordinate system \((U, y^1, \ldots, y^4)\) centered at \(p\) i.e., satisfying \(y^1(p) = 0, \ldots, y^4(p) = 0\). Identifying this local coordinate system with one \((x^1, \ldots, x^4)\) about the origin of \(\mathbb{R}^4\) implies that \(p\) is mapped to \(0 \in \mathbb{R}^4\) having coordinates \((x^1, \ldots, x^4) = (0, \ldots, 0)\) and our model for the vicinity of the given \(\mathbb{R}^4\) summand in \(M^\times \# \mathbb{R}^4 \# \ldots \# \mathbb{R}^4\) then looks like

\[
(B^4_r \times (0), x^1, \ldots, x^4)
\]  

(7)
i.e., a coordinatized open 4-ball \(B^4_r(0)\) of (Euclidean) radius \(r > 0\) about \(0 \in \mathbb{R}^4\) but with center removed: \(B^4_r \times (0) := B^4_r(0) \setminus \{0\}\). Consider the restricted tangent bundle \(TB_r^4 \times (0) := T(M^\times \# \mathbb{R}^4 \# \ldots \# \mathbb{R}^4)|_{B^4_r \times (0)}\); using the restrictions of the orientation on \(X_M\) and the metric \(g\), we can render \(TB_r^4 \times (0)\) a real four-rank \(\text{SO}(4)\) vector bundle over the punctured ball \(B^4_r \times (0)\). We claim that \(TB_r^4 \times (0)\) in fact can be reduced to a complex two-rank \(\text{SU}(2) \subset \text{SO}(4)\) vector bundle over the punctured ball. We can see this by exploiting the so far unmentioned feature of our construction namely that as a “byproduct” the space \((X_M, g)\) of Lemma 3.3 carries a compatible hyper-Kähler structure, too. Since the original compact space \((\overline{X}_M, \overline{g})\) of Lemma 3.1 was oriented and self-dual with both properties being conformally invariant, \((X_M, g)\) is in fact a connected, simply connected, oriented, complete self-dual and Ricci-flat space or in other words: A hyper-Kähler 4-manifold [2, Chapter 13]. This implies among other things that the holonomy group of the Levi–Civita connection of \(g\) hence the structure group of \(TX_M\) reduces to \(\text{SU}(2) \subset \text{SO}(4)\). Consider the Levi–Civita connection of \((X_M, g)\). We can therefore suppose that its restriction to \(TB_r^4 \times (0) \subset TX_M\) is an \(\text{SU}(2)\) connection \(\nabla^r\) suffering from a singularity at the origin. We
know moreover that being $\nabla^*$ self-dual, it solves the SU(2) Yang–Mills equations moreover obviously has finite local energy and admits a bounded gauge on $B^4_r(0)$. Consequently, by Uhlenbeck’s singularity removal theorem (see Theorem 2.2) there exists an SU(2) gauge transformation on $T B^4_r(0)$ such that the gauge transformed connection extends across the singularity to a smooth SU(2) connection $\nabla$ on the trivial bundle $T B^4_r(0)$. Consequentially, switching to the global picture, the singularity of the Levi–Civita connection around the fixed $\mathbb{R}^4$ summand of the intermediate space can be removed hence the corresponding $\mathbb{C}P^2$ summand can be deleted from (1) according to our original plan. Repeating this procedure around all the finitely many $\mathbb{R}^4$ summands of $M^x \# \mathbb{R}^4 \# \ldots \# \mathbb{R}^4$ we finally come up with a smooth SU(2) connection over $M^x$.

However there is an important topological subtlety here. For notational simplicity suppose that the intermediate space looks like $M^x \# \mathbb{R}^4$ i.e., possesses one $\mathbb{R}^4$ summand only. Then the singularity removal procedure carried out above convinces us that the original singular Levi–Civita connection defined on the tangent bundle $T (M^x \# \mathbb{R}^4)$, regarded as an SU(2) bundle, indeed extends to a non-singular SU(2) connection on some SU(2) bundle $E^x$ over $M^x$ i.e., it indeed smoothly exists somewhere which is however not necessarily the tangent bundle of $M^x$. This is because, as we emphasized in the discussion after Theorem 2.2, the singularity-removing-gauge-transformation is not continuous in general hence the original global vector bundle carrying the singular connection can change topology during the singularity removal procedure. However, we know the following two things. On the one hand complex two-rank SU(2) vector bundles over $M^x$, like the $E^x$ above carrying the non-singular connection, are classified by various characteristic classes taking values in the groups $H^i(M^x; \pi_{i-1}(SU(2))$ with $i = 1, \ldots, 4$. Knowing the first three homotopy groups of SU(2) and taking into account the non-compactness of $M^x$ these cohomology groups are all trivial consequently we know that $E^x$ is isomorphic to the trivial bundle over $M^x$. On the other hand, real rank-four SO(4) vector bundles over $M^x$, like the tangent bundle $TM^x$ carrying an orientation and a Riemann metric, are classified by various characteristic classes taking values in similar cohomology groups $H^i(M^x; \pi_{i-1}(SO(4))$. Again recalling the first three homotopy groups of the non-simply connected group SO(4) and still keeping in mind that $M^x$ is non-compact, the only potentially non-trivial group here is $H^2(M^x; \mathbb{Z}_2)$ demonstrating that vector bundles of this type over $M^x$ are classified by a single element and this is nothing but their second Stiefel–Whitney class. Consequently if $M$ is spin or equivalently $w_2(TM) = 0 \in H^2(M; \mathbb{Z}_2)$ then by the injection $M^x \subset M$ we find $w_2(TM^x) = 0 \in H^2(M^x; \mathbb{Z}_2)$ as well showing that $TM^x$ is isomorphic to the trivial bundle, too. Therefore we conclude that whenever $M$ is spin, we can identify the vector bundle $E^x$ carrying the non-singular SU(2) connection over $M^x$ with its tangent bundle $TM^x$.

Having understood this, we can finish the proof by extending the metric itself through the singularities. Fortunately this is simple. Consider the restricted connection $\nabla$ about one singular point $p$. This is now an overall (i.e., including the singular point) smooth connection. There exists a $\delta(p) > 0$ such that if $\delta(p) > r > 0$ we can suppose without loss of generality that the coordinate system (7) we take about this singular point is a geodesic normal coordinate system with respect to $\nabla$ implying that the Christoffel symbols $\nabla g_{ij} = \sum_k \Gamma^k \partial_k$ all vanish in the center i.e., $\Gamma^k_{ij}(0, \ldots, 0) = 0$ for all $i, j, k = 1, \ldots, 4$. Then the well-known compatibility equations

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{l=1}^4 (\partial_l g_{ij} + \partial_j g_{il} - \partial_i g_{lj}) \, g^{lk}$$

on $B^4_r(0)$ imply in a well-known way that $g$ extends over the origin, too, such that $g_{ij}(0, \ldots, 0) = \delta_{ij}$ and $\partial_k g_{ij}(0, \ldots, 0) = 0$ for all $i, j, k = 1, \ldots, 4$. The further differentiability i.e., the smoothness of $g$ at the origin follows from the smoothness of the Christoffel symbols there. That $g$ is Ricci-flat is a trivial consequence of the same property of the original metric. ◇
4 Construction in the exotic setting

In this section we shall sink into the bottomless sea of four dimensionality, called Exotica, and repeat the procedure performed in Section 3. That is, we shall construct solutions of the vacuum Einstein equation on smooth 4-manifolds $X^+$ which are only homeomorphic but not diffeomorphic to the punctured manifolds $M^+$ appeared in Section 3. This construction basically goes along the lines of that presented in Section 3 however with essential technical differences. Consequently, those steps which require new tools will be worked out in detail while those which are basically the same as the corresponding ones in Section 3 will be sketched only.

To begin with, we compose Theorems 2.1, 2.3 and 2.4 together as follows.

Lemma 4.1. Out of any connected, closed (i.e., compact without boundary) oriented smooth 4-manifold $M$ one can construct a connected, open (i.e., non-compact without boundary) oriented smooth Riemannian 4-manifold $(X_M, \gamma)$ which is self-dual but incomplete in general.

Proof. Pick any connected, oriented, closed, smooth 4-manifold $M$. Referring to Theorem 2.1 let $k := \max(1, k_M) \in \mathbb{N}$ be a positive integer, put

$$X_M := M \# \mathbb{C}P^2 \#_k \mathbb{C}P^2$$

and let $\gamma$ be a self-dual metric on it. Then $(X_M, \gamma)$ is a compact self-dual manifold. Pick one $\mathbb{C}P^2$ factor within $X_M$ and denote by $S^2 := \mathbb{C}P^2 \setminus R^4$ the complement of the largest exotic $\mathbb{R}^4$-space $R^4 \subset \mathbb{C}P^2$, considered as an only “continously embedded projective line” in that factor, as in the discussion after Theorem 2.3 (we can suppose that the closed subspace $S^2 \subset \mathbb{C}P^2$ avoids the attaching point of $\mathbb{C}P^2$ to $M$). Let $K \subset R^4$ be the connected compact subset as in part (ii) of Theorem 2.3 and put

$$X_M := M \# \mathbb{C}P^2 \#_k (\mathbb{C}P^2 \setminus S^2) \cong M \# \mathbb{C}P^2 \#_k R^4 \cong X^+ \# R^4 \cong X_M$$

(8)

where the operation $\#_k$ means that the attaching point $y_0 \in R^4$ used to glue $R^4$ with $M \# \mathbb{C}P^2 \#_k R^2$ satisfies $y_0 \in K \subset R^4$ and $X^+ := M \#_k R^4$ is a smooth manifold homeomorphic but not diffeomorphic to the puncturation $M^+$ of the original manifold (see Theorem 2.4). The result is a connected, open 4-manifold $X_M$ (see Figure 4).

![Figure 4. Construction of $X_M$ out of $M$ in the exotic setting. The gray zig-zag represents a “creased end” diffeomorphic to the complement of a connected compact subset $K$ in the exotic $R^4$.](image)

From the proper smooth embedding $X_M \cong X_M \setminus S^2 \subset X_M$ there exists a restricted self-dual Riemannian metric $\gamma := \gamma|X_M$ on $X_M$ which is however in general non-complete. ◇
In the case of our situation set up in Lemma 4.1 twistor theory works as follows. Consider the compact self-dual space \((X_M, \gamma)\) from Lemma 4.1, take its twistor fibration \(\overline{p} : \overline{X} \to X_M\) and let
\[
p : Z \to X_M
\]
be its restriction induced by the smooth embedding \(X_M \subset \overline{X}_M\) i.e., \(Z := \overline{Z}|_{X_M}\) and \(p := \overline{p}|_{X_M}\). Then \(Z\) is a non-compact complex 3-manifold already obviously possessing all the required twistor data except the existence of a holomorphic mapping \(\pi : Z \to \mathbb{C}P^1\).

**Lemma 4.2.** Consider the connected, open, oriented, incomplete, self-dual space \((X_M, \gamma)\) as in Lemma 4.1 with its twistor fibration \(p : Z \to X_M\) constructed above. If \(\pi_1(M) = 1\) then there exists a holomorphic mapping \(\pi : Z \to \mathbb{C}P^1\).

**Proof.** Let \(x_0 \in X_M\) be a fixed point belonging to the exotic \(\mathbb{R}^4\) factor \(R^4\) of \(X_M\) in its decomposition (1). Our aim is to construct a holomorphic map
\[
\pi : Z \to p^{-1}(x_0) \cong \mathbb{C}P^1
\]
that we carry out now in three steps very similar to those taken in the proof of Lemma 3.2. Compared with the situation in Lemma 3.2 the main difficulty in the exotic setting is that, unlike the standard \(\mathbb{R}^4 \subset \mathbb{C}P^2\), the exotic space \(R^4 \subset \mathbb{C}P^2\) has a non-empty intersection with all the projective lines in \(\mathbb{C}P^2\) (see the discussion after Theorem 2.3) hence the simple geometric construction used in the First step of Lemma 3.2 to construct (4) does not work here.

**First step:** We begin with constructing by classical means a collection of holomorphic maps called \(\pi_x\), parameterized by the ideal points \(x \in \mathbb{C}P^2 \setminus \mathbb{R}^4\), over the exotic \(\mathbb{R}^4\) factor \(R^4\) of (8). Theorem 2.3 tells us that \(R^4 \subset \mathbb{C}P^2\). Writing \(Z(R^4) := \overline{Z}(\mathbb{C}P^2)|_{R^4}\) and \(p := \overline{p}|_{R^4}\); the restricted fibration \(p : Z(R^4) \to R^4\) is again topologically trivial i.e., \(Z(R^4)\) is homeomorphic to \(R^4 \times S^2 \cong \mathbb{R}^4 \times S^2\) hence admits a continuous trivialization because \(R^4\) is contractible. This is again a necessary condition for the existence of (9). Consider the classical flag manifold description \(F_{12}(\mathbb{C}^3)\) of the twistor space \(\mathbb{Z}(\mathbb{C}P^2)\) together with the notation as introduced in the course of the proof Lemma 3.2. Fix a point \([m] \in R^4\) with a starting pair \((l, p) \in p^{-1}([m]) \subset Z(R^4)\) as well as fix a target point \([m_0] \in R^4\) with \(p^{-1}([m_0]) \subset Z(R^4)\) consisting of terminating pairs \((l_0, p_0)\). Our aim is again in some but surely holomorphic way to assign to every \((l, p)\) another pair \((l_0, p_0)\). Fix an ideal point \(x \in \mathbb{C}P^2 \setminus \mathbb{R}^4\) hence certainly \(x \neq [m]\) and \(x \neq [m_0]\). Assume that \([m] \neq [m_0]\) and \(x\) does not lie on the unique projective line connecting \([m]\) and \([m_0]\). We construct \(\pi_x : p^{-1}([m]) \to p^{-1}([m_0])\) as follows. Viewing the distinct points \([m]\) and \(x \in \mathbb{C}P^2\) as two linearly independent vectors in \(\mathbb{C}^3\) let \(x \times [m] \in \mathbb{C}P^2\) be the point corresponding to the unique line through (any of) their complex vectorial product in \(\mathbb{C}^3\). Then \(x \times [m]\) depends holomorphically on \([m]\). Likewise, consider a point \(x \times [m_0] \in \mathbb{C}P^2\) constructed from \(x\) and \([m_0]\) in a similar way. By the assumption made on the ideal point \(x\) we know that \(x \times [m] \neq x \times [m_0]\). Let \(\ell_x \subset \mathbb{C}P^2\) be the unique projective line passing through \(x \times [m]\) and \(x \times [m_0]\). It is clear that \([m], [m_0] \notin \ell_x\). Consider a starting pair \((l, p)\) and take its line component \([p] \subset \mathbb{C}P^2\). This line has a unique intersection \(\ell_x \cap [p]\). Then, given the target space \(p^{-1}([m_0])\), define the line component \([p_0] \subset \mathbb{C}P^2\) in the terminating pair \((l_0, p_0) \in p^{-1}([m_0])\) by taking the unique projective line connecting \(\ell_x \cap [p]\) with \([m_0]\) and then its point component \([l_0] \subset \mathbb{C}P^2\) by \([l_0] := [p_0] \cap [m_0]\) as usual. Assume now that either \([m] \neq [m_0]\) and \(x\) lies on their connecting line or \([m] = [m_0]\); then we leave \(\pi_x\) unspecified.

In short, \(\pi_x : p^{-1}([m]) \to p^{-1}([m_0])\) is defined as follows:
\[
\pi_x((l, p)) := (l_0, p_0) \quad \text{where} \quad p_0 \subset \mathbb{C}^3 \text{ is the plane such that } [p_0] \subset \mathbb{C}P^2 \text{ connects } \ell_x \cap [p] \text{ and } [m_0] \text{ and } l_0 \subset \mathbb{C}^3 \text{ is the line such that } [l_0] := [p_0] \cap [m_0]\]
whenever \([m] \neq [m_0]\) and \(x\) is not on the line connecting them.
This map is well-defined, holomorphic and onto. Note that letting \([m]\) vary while keeping \(x\) and \([m_0]\) fixed in the definition (10), we can in fact extend \(\pi_x\) above to a map

\[
\pi_x : z(R^4) \setminus p^{-1}(\{\text{the line in } CP^2 \text{ through } x \text{ and } [m_0]\}) \rightarrow p^{-1}([m_0])
\]

(11)
in a holomorphic way.

Second step: Next we fuse all the maps in (11), when \(x\) runs through the complementum of the exotic \(R^4\), into a single-valued holomorphic map \(\pi : z(R^4) \rightarrow p^{-1}([m_0])\) by applying the concept of Lebesgue integration of algebraic-function-field-valued functions summarized in Appendix A. Fix distinct points \([m],[m_0] \in R^4\) and take an ideal point \(x \in CP^2 \setminus R^4\), not lying on the line connecting \([m]\) and \([m_0]\), and consider the corresponding holomorphic (11); letting the ideal point move along \(CP^2 \setminus R^4\) take now a different ideal point \(y \in CP^2 \setminus R^4\), again not lying on the line connecting \([m]\) and \([m_0]\), with corresponding holomorphic map \(\pi_y\) into the same target space. Then along the twistor fiber \(p^{-1}([m]) \subset Z(R^4)\) there exists a restricted commutative diagram

\[
p^{-1}([m]) \xrightarrow{\pi_x|_{p^{-1}([m])}} p^{-1}([m_0]) \quad \pi_y|_{p^{-1}([m])} \xrightarrow{f_{[m]|x}} p^{-1}([m_0])
\]

with \(f_{[m]|x}\) being a holomorphic map. This map is unique because both \(\pi_x\) and \(\pi_y\) are onto. Pick an affine coordinate system \((u,v)\) on a coordinate ball \(U \subset CP^2\) centered about \([m_0] \in U\) i.e., satisfies \((u([m_0]),v([m_0]) = (0,0)\). In this coordinate system any affine line \([p_0] \cap U\) passing through \([m_0]\) looks like \((u([p_0]),v([p_0])) = (u,zu)\) with \(z \in \mathbb{C} \cup \{\infty\} = CP^1\) hence \((l_0,p_0) = z\) provides us with an identification \(p^{-1}([m_0]) \cong CP^1\). However it is known that a holomorphic map from \(CP^1\) into itself is a rational function in a single variable; consequently under this identification \(f_{[m]|x}\) can be described by a unique element \(R_{[m]|x} \in \mathbb{C}(z)\), the field of complex rational functions in one variable \(z\). That is, there exist complex polynomials \(P_{[m]|x}\) and \(Q_{[m]|x}\) such that \(R_{[m]|x}(z) = \frac{P_{[m]|x}(z)}{Q_{[m]|x}(z)}\). In this context for a fixed \((l,p) \in Z(R^4)\) it is worth regarding \(\pi_x((l,p))\) as a particular choice for \(z\) in the abstractly given algebraic function field \(\mathbb{C}(z)\) and denoting this coordinatized \((\mathbb{C}(z),\pi_x)\) simply as \(\mathbb{C}(\pi_x)\). Writing \((l,p)\) in place of \([m]\) whenever \((l,p) \in p^{-1}([m])\) we eventually come up with

\[
\pi_x((l,p)) = R_{(l,p),yx}(\pi_x((l,p))) = \frac{P_{(l,p),yx}(\pi_x((l,p)))}{Q_{(l,p),yx}(\pi_x((l,p)))} \in \mathbb{C}(\pi_x)
\]
and the coefficients of $P_{(l,p),yx}$ and $Q_{(l,p),yx}$ are at least locally continuous functions of $y \in \mathbb{C}P^2 \setminus R^4$ assuming perhaps zero values, therefore the degrees of $P_{(l,p),yx}$ and $Q_{(l,p),yx}$ can in principle jump as $y$ runs through the ideal points (however this is not the case as explained in Lemma 5.3 in Appendix A).

Let $S^2 \subset \mathbb{R}^3$ denote the standard 2-sphere with its inherited orientation, smooth structure and round metric and let $i : S^2 \rightarrow \mathbb{C}P^2$ be a continuous embedding such that $i(S^2) = \mathbb{C}P^2 \setminus R^4$ that is, it is a homeomorphism onto the complement regarded as a “continuously embedded projective line” in $\mathbb{C}P^2$. The subspace topology on $\mathbb{C}P^2 \setminus R^4 \subset \mathbb{C}P^2$ induces a measurable space structure on it such that $i$ is measurable. If $y \in \mathbb{C}P^2 \setminus R^4$ then write $dy$ for the pushforward measure by $i$ on $\mathbb{C}P^2 \setminus R^4$ of the usual normalized volume-form on $S^2$ with respect to its orientation and round metric i.e., the standard Lebesgue measure on $S^2$. Choosing the embedding $i$ carefully, we can suppose that $y \mapsto R_{(l,p),yx}(\pi_x((l,p)))$ is a function from $\mathbb{C}P^2 \setminus R^4$ into $\mathbb{C}(\pi_x)$ which is defined almost everywhere.\(^3\) Now we are in a position to define a map via integrating all the maps so far together i.e.,

$$\pi((l,p)) := \int_{\mathbb{C}P^2 \setminus R^4} R_{(l,p),yx}(\pi_x((l,p))) dy. \quad (12)$$

As explained in Appendix A, the expression on the right hand side as an algebraic-function-field-valued Lebesgue integral over $\mathbb{C}P^2 \setminus R^4$ exists where $\pi_x$ is defined i.e., for all

$$(l,p) \in Z(R^4) \setminus p^{-1}(\{\text{the line in } \mathbb{C}P^2 \text{ through } x \text{ and } |m_0|\}).$$

However it is in fact independent of $x$ hence gives rise to a map $\pi : Z(R^4) \rightarrow p^{-1}(\{m_0\})$ such that the assignment $(l,p) \mapsto \pi((l,p))$ in (12) is holomorphic and in particular is the identity on $p^{-1}(\{m_0\})$. All of these are proved in Lemma 5.3 (also cf. Lemma 5.4).

**Third step:** Now we extend the partial map, just constructed in the First step and the Second step above over the exotic $\mathbb{R}^4$-end of $X_M$ in (8) and having shape (12), over the whole $X_M$ by analytic continuation exactly the same way as in the analogous Second step in Lemma 3.2. Consequently we do not repeat that procedure here again. ◇

It also follows that $\pi : Z \rightarrow \mathbb{C}P^1$ i.e., the map (9) constructed in Lemma 4.2 is compatible with the real structure $\tau : Z \rightarrow Z$ already fixed by the self-dual structure in Theorem 2.1 therefore twistor theory provides us with a Ricci-flat (and self-dual) Riemannian metric $g$ on $X_M$. We proceed further and demonstrate that, unlike $(X_M, \gamma)$, the space $(X_M, g)$ is complete.

**Lemma 4.3.** The connected and simply connected, open, oriented, Ricci-flat Riemannian manifold $(X_M, g)$ is complete.

**Proof.** The metrics $\gamma$ and $g$ are again conformally equivalent i.e., there exists a smooth $\varphi : X_M \rightarrow \mathbb{R}_+$ satisfying $g = \varphi^{-2} \cdot \gamma$ and blowing up again uniformly along the whole $\overline{X_M} \setminus X_M$ i.e., the “continuously embedded projective line” $\mathbb{C}P^2 \setminus R^4$ homeomorphic to $S^2$. This follows from the integral representation (12) of $\pi$ because by construction each individual $\pi_x$ in (10) is singular at $x \in \mathbb{C}P^2 \setminus R^4$ hence their integral is singular along the whole complementum. Consequently replacing the set $\overline{X_M} \setminus X_M$ of Lemma 3.3, diffeomorphic to the holomorphic projective line $\mathbb{C}P^1 = \mathbb{C}P^2 \setminus R^4$, with this continuous one, the proof is the same as that of Lemma 3.3 hence is omitted. ◇

Finally we cut down the “unused” $\mathbb{C}P^2$‘s from $X_M$ to obtain $X^\times$ as in Lemma 3.4.

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\(^3\) Recall that if $|m| \neq |m_0|$ then $y \mapsto R_{(l,p),yx}(\pi_x((l,p))) = \pi_{\ast}(\pi_\ast)$ is not defined for those ideal points $y \in \mathbb{C}P^2 \setminus R^4$ which are contained in the projective line connecting $|m|$ and $|m_0|$ in $\mathbb{C}P^2$; however even in the worst situation the set of these intersection points form, up to countably many points, a Cantor subset $C \subset \mathbb{C}P^2 \setminus R^4$. Hence we can suppose upon selecting $i : S^2 \rightarrow \mathbb{C}P^2$ carefully that $C$ has zero measure with respect to the pushforward measure dy. For more details see the discussion after Theorem 2.3 and the proof of Lemma 5.3 in Appendix A.
Lemma 4.4. Consider the space $(X_M, g)$ as in Lemma 4.3. If $M$ is moreover spin (or equivalently, having even intersection form) then the orientation and the complete Ricci-flat metric $g$ on $X_M$ descend to the punctured space $X^\times$ (which is homeomorphic but not diffeomorphic to the corresponding space $M^\times$ of Lemma 3.4) with its inherited smooth structure, rendering it a connected and simply connected, open, oriented, complete, Ricci-flat Riemannian spin 4-manifold $(X^\times, g)$.

Proof. Taking into account that cutting off the unused $\mathbb{CP}^2$'s in the decomposition (8) of $X_M$ is a completely local procedure (see Figure 6)

![Figure 6. Construction of $X^\times$ out of $X_M$ by removing the extra $\mathbb{CP}^2$'s.](image)

the proof of this lemma is verbatim the same as the proof of Lemma 3.4 hence is omitted. ◇

Collecting finally all the results of Sections 3 and 4 together, the proof of Theorem 1.1 is now complete.

5 Appendix A: Lebesgue integration in algebraic function fields

Here we work out how to integrate functions on manifolds but taking values in the algebraic function field of complex rational functions. The construction of this integral is straightforward and is fully based on the by-now classical approach of Riesz–Szőkefalvi-Nagy (cf. [27, §16, §17]).

Let $P(z), Q(z)$ be two complex polynomials in the single variable $z \in \mathbb{C}$ and let $\mathbb{C}(z)$ denote the (commutative) algebraic function field of fractions $R(z) := \frac{P(z)}{Q(z)}$. A norm of $R$ is defined by the formula $|R|_{c,z_0} := c^{\text{ord}_{z_0}(R)}$ where $c \in (0, 1)$ is a fixed real number and $\text{ord}_{z_0}(R) \in \mathbb{Z}$ is the lowest one among the indices $k \in \mathbb{Z}$ of the non-zero coefficients $a_k \in \mathbb{C}$ in the Laurent expansion

$$R(z) = \sum_{k \gg -\infty}^{+\infty} a_k(z-z_0)^k$$

of $R$ about a fixed point $z_0 \in \mathbb{C}$; note that the number $\text{ord}_{z_0}(R)$ is independent of the particular coordinate system used for the expansion hence is well-defined and this definition makes sense for $R = 0$ if we put $\text{ord}_{z_0}(0) := +\infty$ yielding $0 |_{c,z_0} = 0$. It is known [29, Theorem 1.11] that, being $\mathbb{C}$ algebraically closed, $| \cdot |_{c,z_0}$ with $c \in (0, 1)$ and $z_0 \in \mathbb{C} \cup \{\infty\}$ is the complete list of norms on $\mathbb{C}(z)$ which are trivial on $\mathbb{C}$ and $\mathbb{C}(\infty)$. Then $\mathbb{C}(z)$ can be completed with respect to $| \cdot |_{c,z_0}$ which is $\mathbb{C}(z-z_0)$, the field of formal Laurent series in $z-z_0$. There is an embedding of fields $\mathbb{C}(z) \subset \mathbb{C}(z-z_0)$ for all $c,z_0$ but up to isomorphisms of topological fields, these completions are independent of the norms used [29].

Remark. Unlike the usual norms on $\mathbb{R}$ or $\mathbb{C}$, all the norms of this kind on $\mathbb{C}(z)$ are non-Archimedean hence $\mathbb{C}(z)$ does not admit any norm-compatible embedding into $\mathbb{C}$ consequently its analytical properties are quite different from those of the real or complex numbers. Moreover the spectra of our norms
here are discrete, more precisely \(|C((z - z_0))|_{c, z_0} = c^2 \subset [0, +\infty]\) consequently the spectrum of \(|\cdot|_{c, z_0}\) for all \(c, z_0\) has only one accumulation point \(0 \in \mathbb{R}\). Another essential difference is that, unlike \(\mathbb{R}\), the algebraic function field \(C(z)\) is not ordered.

Let \((M, g)\) be an oriented Riemannian \(m\)-manifold. Then \(M\) is equipped with a measure \(\text{vol}_g\) coming from the volume-form \(d\text{vol}_g := *_{g, 1}\) provided by the orientation and the metric; the corresponding measure of a measurable subset \(\emptyset \subseteq A \subseteq M\) is \(\text{vol}_g(A) := \int_M \chi_A \text{vol}_g = \int_A \text{vol}_g\) where \(\chi_A : M \to (0, 1)\) is the characteristic function of any subset \(\emptyset \subseteq B \subseteq M\). Clearly \(0 \leq \text{vol}_g(A) \leq +\infty\) is a non-negative (extended) real number. Take finitely many measurable subsets \(U_1, \ldots, U_n \subset M\) whose closures are coordinate balls of finite volume but are pairwise almost non-intersecting; that is, \(U_i \subset M\) has the property that \(\overline{U_i}\) is diffeomorphic to the standard closed ball \(\overline{B}^{m} \subset \mathbb{R}^m\) moreover \(0 < \text{vol}_g(U_i) < +\infty\) for all \(i\) but \(\text{vol}_g(U_i \cap U_j) = 0\) for all \(i \neq j\). Also take elements \(R_1, \ldots, R_n \in C((z - z_0))\). A function \(\varphi : M \to C((z - z_0))\) of the form

\[
\varphi := \sum_{j=1}^{n} R_j \chi_{U_j}
\]

is called an elementary step function. This definition makes sense since \(\mathbb{R}\) acts on \(C((z - z_0))\) by multiplication; nevertheless these functions might be ill-defined in boundary points however, as we already anticipate from Lebesgue theory, ambiguities of this sort will be negligible concerning their integrals. The integral of an elementary step function against the measure induced by \(d\text{vol}_g\) is defined as

\[
\int_{M} \varphi \text{vol}_g := \sum_{j=1}^{n} R_j \text{vol}_g(U_j) \in C((z - z_0))
\]

(\(\varphi\) is written sometimes as \(R : M \to C((z - z_0))\) and its integral as \(\int_M R \, dx\), too) in full analogy with the usual case.

Next let us recall the two elementary but fundamental lemmata from [27] what we state here in appropriately modified forms and prove as follows.

**Lemma 5.1.** (cf. [27, Lemme A, p. 30]) Let \(\{\varphi_i\}_{i \in \mathbb{N}}\) be a sequence of elementary step functions from a compact oriented Riemannian manifold \((M, g)\) into \(C((z - z_0))\). If \(\{|\varphi_i|_{c, z_0}\}_{i \in \mathbb{N}}\) is strictly decreasing almost everywhere (a.e.), then the integrals of these functions converge to zero in \(C((z - z_0))\).

**Proof.** As mentioned above the spectrum of \(|\cdot|_{c, z_0}\) has only one limit point \(0 \in \mathbb{R}\) therefore if \(\{|\varphi_i|_{c, z_0}\}_{i \in \mathbb{N}}\) strictly decreases a.e. then in fact \(\varphi_i(x)\big|_{c, z_0} \to 0\) if \(x \in M \setminus B\) as \(i \to +\infty\) where \(B\) is a subset of measure zero i.e., for any \(\delta \geq 0\) there exist open subsets \(V_\delta \subset M\) satisfying \(\text{vol}_g(V_\delta) \leq \delta\) such that \(\emptyset \subseteq B \subset V_\delta\). This means on the one hand that for every \(\varepsilon \geq 0\) there exists an index \(i_{\varepsilon}\) such that for all \(i \geq i_{\varepsilon}\) one finds

\[
0 \leq \left| \int_{M \setminus B} \varphi_i \text{vol}_g \right|_{c, z_0} \leq \sup_{x \in M \setminus B} |\varphi_i(x)|_{c, z_0} \leq \varepsilon.
\]

On the other hand, if for any fixed \(i\) and \(x \in B\) the lowest non-zero coefficient of \(\varphi_i(x)\) in (13) is \(a_{iK}(x) \in C\) then the same coefficient of \(\int_B \varphi_i \text{vol}_g\) can be estimated from above by

\[
\sup_{x \in B} |a_{iK}(x)| \text{vol}_g(V_\delta) \leq \sup_{x \in B} |a_{iK}(x)| \delta
\]

and exploiting the compactness of \(M\) we can assume that the number of the different leading coefficients \(a_{iK}(x)\) is finite as \(x\) runs over \(B\) hence surely \(\sup_{x \in B} |a_{iK}(x)| < +\infty\). It then follows that the leading
coefficient of the integral is arbitrary small hence $| \int_B \psi_i \, d\text{vol}_g \big|_{\mathbb{C}, \varepsilon_0} = 0$ i.e., the integral over $B$ vanishes for every fixed index $i$. Consequently $0 \leq \int_M \psi_i \, d\text{vol}_g \big|_{\mathbb{C}, \varepsilon_0} \leq \varepsilon$ for all $i \geq i_\varepsilon$. That is, the sequence of integrals converges to zero as stated. \( \diamond \)

**Lemma 5.2.** (cf. [27, Lemme B, p. 30]) Let $\{ \psi_i \}_{i \in \mathbb{N}}$ be a sequence of elementary step functions from an oriented Riemannian manifold $(M, g)$ into $\mathbb{C}((z-\varepsilon_0))$. If $\{ |\psi_i|_{z, \varepsilon_0} \}_{i \in \mathbb{N}}$ is increasing and the sequence $\{ \int_M \psi_i \, d\text{vol}_g \}_{i \in \mathbb{N}}$ of the corresponding integrals converges to an element in $\mathbb{C}((z-\varepsilon_0))$, then $\{ \psi_i \}_{i \in \mathbb{N}}$ converges to a finite limit in $\mathbb{C}((z-\varepsilon_0))$ a.e.

**Proof.** Let $\emptyset \subseteq B \subseteq M$ denote the collection of all of those points $x \in M$ where $\phi_i(x)$ is divergent in $\mathbb{C}((z-\varepsilon_0))$ as $i \to +\infty$. This can mean two (not necessarily mutually exclusive) things: either a sequence $\{ a_{iK}(x) \}_{i \in \mathbb{N}}$ of coefficients in the expansions (13) of the $\phi_i(x)$’s is divergent or $\{ |\phi_i(x)|_{z, \varepsilon_0} \}_{i \in \mathbb{N}}$ is divergent i.e., the index set $\{ K_i \}_{i \in \mathbb{N}}$ of the lowest non-zero $a_{iK_i}(x)$’s in the expansions of the $\phi_i(x)$’s with $x \in B$ is unbounded from below. In either cases, since the sequence of the corresponding integrals $\int_M \phi_i \, d\text{vol}_g$ converges to a well-defined element $R \in \mathbb{C}((z-\varepsilon_0))$ with a well-defined expansion (13) whose coefficients are of the form $a_{iK}(x)$, these divergences can be absent from the integral if and only if for every $\delta \geq 0$ there exist open subsets $\emptyset \subseteq B \subset V_\delta \subseteq M$ such that $\text{vol}_g(V_\delta) \leq \delta$. In other words $B$ is of measure zero as stated. \( \diamond \)

From here we proceed in the standard way (cf. [27, §17]) hence we only quickly summarize the main steps. Let $(M, g)$ be a compact oriented Riemannian manifold. If $C_0(M; \mathbb{C}((z-\varepsilon_0)))$ is the set of elementary step functions from $M$ to $\mathbb{C}((z-\varepsilon_0))$ then let $C_1(M; \mathbb{C}((z-\varepsilon_0)))$ denote the set of those functions $f : M \to \mathbb{C}((z-\varepsilon_0))$ which arise as limits of sequences of functions $\{ \phi_i \}_{i \in \mathbb{N}}$ in Lemma 5.2 i.e., arise a.e. as the limits $f(x) := \lim \phi_i(x)$ of increasing elementary step functions with a convergent sequence of corresponding integrals. Define their *integrals*, which therefore exist, to be

$$\int_M f \, d\text{vol}_g := \lim_{i \to +\infty} \int_M \phi_i \, d\text{vol}_g \in \mathbb{C}((z-\varepsilon_0))$$

(again $f$ is written sometimes as $R : M \to \mathbb{C}((z-\varepsilon_0))$ and its integral as $\int_M R \, dx$, too). This definition is correct because, by referring to Lemma 5.1, the integral does not depend on the particular choice of the sequence $\{ \phi_i \}_{i \in \mathbb{N}}$ converging a.e. to a given $f$. The set $C_1(M; \mathbb{C}((z-\varepsilon_0)))$ has already the structure of a vector space over $\mathbb{C}((z-\varepsilon_0))$ and is closed and complete in an appropriate sense; it is more commonly denoted as $L^1(M; \mathbb{C}((z-\varepsilon_0)))$ and called the *space of $\mathbb{C}((z-\varepsilon_0))$-valued Lebesgue integrable functions on $M$* (with respect to a measure coming from the orientation and metric on $M$).

Finally we note that if $f : M \to \mathbb{C}((z-\varepsilon_0))$ or rather $R : M \to \mathbb{C}((z-\varepsilon_0))$ is an integrable function and is expanded as in (13) then its integral reduces to an (in)finite collection of ordinary Lebesgue integrals

$$\int_M R(x) \, dx = \sum_{k \geq -\infty} \left( \int_M a_k(x) \, dx \right) (z-\varepsilon_0)^k \quad (14)$$

as one expects. This integral in general takes value in $\mathbb{C}((z-\varepsilon_0))$ i.e., is a formal power series in $z-\varepsilon_0$ only hence the right hand side does not have to converge. If it does then $\int_M R(x) \, dx \in \mathbb{C}(z) \subset \mathbb{C}((z-\varepsilon_0))$ i.e., is an honest rational function. Consequently the type of integrals we introduced can be calculated in the standard way.

The main purpose of these investigations is to complete the proof of Lemma 4.2 by demonstrating
Lemma 5.3. Using the notations of Lemma 4.2, for a fixed \( x \in \mathbb{C}P^2 \setminus \mathbb{R}^4 \) and
\[
(l, p) \in \mathbb{Z}(\mathbb{R}^4) \setminus p^{-1}(\{\text{the line in } \mathbb{C}P^2 \text{ through } x \text{ and } [m_0]\})
\]
the integral (12) exists. Moreover it is independent of \( x \) hence (12) gives rise to a well-defined map
\[
\pi : \mathbb{Z}(\mathbb{R}^4) \to p^{-1}(\{m_0\}) \text{ which is holomorphic such that } \pi|_{p^{-1}(\{m_0\})} = \text{Id}_{p^{-1}(\{m_0\})}.
\]
Moreover the rational function in the integrand of (12) is a strictly first order polynomial
\[
R_{(l,p),3x}(\pi_x((l,p))) = a_{(l,p),3x} \pi_x((l,p)) + b_{(l,p),3x}
\]
with almost everywhere (a.e.) continuous, bounded, complex-valued coefficients along \( \mathbb{C}P^2 \setminus \mathbb{R}^4 \) such that \( a_{(l,p),3x} \neq 0 \).

Proof. First we prove that for any fixed \( (l,p) \in \mathbb{Z}(\mathbb{R}^4) \setminus p^{-1}(\{m_0\}) \) an element \( \pi((l,p)) \in p^{-1}(\{m_0\}) \) defined by (12) exists. Take the fixed distinct points \( p((l,p)) = [m] \) and \( [m_0] \) in \( \mathbb{R}^4 \) and define
\[
C := (\mathbb{C}P^2 \setminus \mathbb{R}^4) \cap \{\text{the line in } \mathbb{C}P^2 \text{ through } [m] \text{ and } [m_0]\}.
\]

By the discussion after Theorem 2.3, always \( \emptyset \not\subset C \supseteq \mathbb{C}P^2 \setminus \mathbb{R}^4 \). However being \( \mathbb{C}P^2 \) four dimensional and both the complementum of \( \mathbb{R}^4 \) and the line connecting \([m]\) with \([m_0]\) two dimensional, we can suppose that within \( \mathbb{C}P^2 \) they meet in points only hence their intersection \( C \) is a totally disconnected, closed subset of \( \mathbb{C}P^2 \setminus \mathbb{R}^4 \). Consider a measurable space structure on \( \mathbb{C}P^2 \setminus \mathbb{R}^4 \subset \mathbb{C}P^2 \) induced by its subspace topology. Then \( C \) is measurable. Similarly let \( \lambda \) be the standard Lebesgue measure on the target twistor fiber \( p^{-1}(\{m_0\}) \) by identifying it with \( S^2 \subset \mathbb{R}^3 \) carrying the standard unit-volume round metric and orientation from the embedding. Let \( i : p^{-1}(\{m_0\}) \to \mathbb{C}P^2 \) be a continuous embedding which is homeomorphism hence a measurable map between the the target twistor fiber and the complementum. The Lebesgue measure \( \lambda \) gives rise to a pushforward measure \( i_* \lambda \) on the latter space. The standard action of \( \text{SO}(3) \) on \( p^{-1}(\{m_0\}) \cong S^2 \subset \mathbb{R}^3 \) can be transferred by \( i \) to a continuous action on \( \mathbb{C}P^2 \setminus \mathbb{R}^4 \).

Taking into account that \( \lambda \) is \( \text{SO}(3) \)-invariant the same is true for \( i_* \lambda \). Being all continuous \( \text{SO}(3) \) actions on the two-sphere equivalent, the transferred \( \text{SO}(3) \) action on \( \mathbb{C}P^2 \setminus \mathbb{R}^4 \) is equivalent, by a homeomorphism \( h : \mathbb{C}P^2 \setminus \mathbb{R}^4 \to \mathbb{C}P^2 \setminus \mathbb{R}^4 \), to that one on \( \mathbb{C}P^2 \setminus \mathbb{R}^4 \) which arises by rotating the projective lines through \([m_0]\), regarded as elements of \( p^{-1}(\{m_0\}) \), into each other. However this latter action obviously carries the different intersection sets into each other. Consequently, if \( C' := (\mathbb{C}P^2 \setminus \mathbb{R}^4) \cap \{\text{the line in } \mathbb{C}P^2 \text{ through } [m'] \text{ and } [m_0]\} \) is another set of intersection points then, modifying \( i \) to \( h \circ i \) if necessary, we can suppose that \( i_* \lambda(C) = i_* \lambda(C') \). Proceeding further, if \( C \) is countable then \( i_* \lambda(C) = 0 \).

If \( C \) is non-countable then up to a countable subset we can assume that \( C \subset \mathbb{C}P^2 \setminus \mathbb{R}^4 \) is a totally disconnected, perfect, compact, metrizable space i.e., a Cantor set. Let \( K \subset p^{-1}(\{m_0\}) \) be a Cantor set such that \( \lambda(K) = 0 \). Then by [25, Theorem 12.8] there exists a homeomorphism \( h : K \to i^*C \) which by [25, Theorem 13.7] can be extended to a homeomorphism \( h : p^{-1}(\{m_0\}) \to p^{-1}(\{m_0\}) \). Consequently, replacing \( i \) with \( i \circ h \) if necessary, we can suppose for all intersections that \( i_* \lambda(C) = \lambda(i^*C) = \lambda(K) = 0 \).

To summarize, we can suppose that all intersections \( C \subset \mathbb{C}P^2 \setminus \mathbb{R}^4 \) with projective lines through \([m_0]\) are measure-zero subsets within \( \mathbb{C}P^2 \setminus \mathbb{R}^4 \) with respect to a carefully chosen measure \( i_* \lambda \) on it given by the pushforward of the standard Lebesgue measure \( \lambda \) on \( S^2 \) by a once and for all fixed continuous embedding \( i : S^2 \to \mathbb{C}P^2 \) such that \( i(S^2) = \mathbb{C}P^2 \setminus \mathbb{R}^4 \).

If \( y \notin C \) is an ideal point with corresponding map \( \pi_y \), then from its construction (10) it follows that \( y \mapsto \pi_y((l,p)) \) is a continuous function from \( (\mathbb{C}P^2 \setminus \mathbb{R}^4) \setminus C \) into \( p^{-1}(\{m_0\}) \cong \mathbb{C}P^1 \). Take a fixed ideal point \( x \notin C \) and consider its associated map \( \pi_x \). Then, via \( \pi_x((l,p)) = R_{(l,p),3x}(\pi_x((l,p))) \), we find that the coefficients of the map \( y \mapsto R_{(l,p),3x}(\pi_x((l,p))) \) (considered as a rational function with \( y \)-dependent...
coefficients) are also continuous. Now let us analyze the degeneration of \( \pi_y \) as \( y \to C \). It follows from the construction (10) that in this limit \( y \times [m] \) and \( y \times [m_0] \), used to define the line \( \ell_y \subset \mathbb{C}P^2 \), are approaching each other consequently \( \ell_y \) gets ill-defined when \( y \in C \). This implies that the coefficients of \( y \mapsto R_{(l,p),y}(\pi_x((l,p))) \) get ill-defined as well but at least remain bounded for a fixed \( (l,p) \). In other words the coefficients of the map \( y \mapsto R_{(l,p),y}(\pi_x((l,p))) \) for all fixed \( (l,p) \) are a.e. continuous bounded functions from \( \mathbb{C}P^2 \setminus R^4 \) into \( \mathbb{C} \). Consequently their integrals exist yielding, by referring to (14), that for every \( (l,p) \in Z(R^4) \setminus p^{-1}([m_0]) \) with some \( x \notin C \) or equivalently for every \( x \in \mathbb{C}P^2 \setminus R^4 \) and \( (l,p) \in Z(R^4) \setminus p^{-1}([\{\text{the line in } \mathbb{C}P^2 \text{ through } x \text{ and } [m_0]}) \), an integral (12) defined by

\[
\int_{\mathbb{C}P^2 \setminus R^4} R_{(l,p),y}(\pi_x((l,p))) \, dy := \int_{S^2} R_{(l,p),i^y,\pi}(\pi_x((l,p))) \, d(i^y) ,
\]

where \( d(i^y) \) denotes the Lebesgue measure provided by the unit-volume round metric on \( S^2 \), exists.

Secondly we prove that the map \( (l,p) \mapsto \pi_x((l,p)) \) is holomorphic. Regarding this note that (12) is nothing else than a limit of the integrals of step functions approximating \( y \mapsto R_{(l,p),y}(\pi_x((l,p))) \):

\[
\pi((l,p)) = \lim_{n \to +\infty} \sum_{j=1}^{n} R_{(l,p),y_j}(\pi_x((l,p))) \text{vol}(U_j)
\]

where \( y_j \in \mathbb{C}P^2 \setminus R^3 \) such that \( y_j \notin C \) and \( i^y_j \in U_j \subset S^2 \). Putting \( \pi_{y_j}((l,p)) := R_{(l,p),y_j}(\pi_x((l,p))) \) we know from (11) that \( \pi_{y_j} \) is holomorphic in the vicinity of any point \( (l,p) \in p^{-1}([m]) \subset Z(R^4) \) such that \( y_j \notin C \) that is, on \( Z(R^4) \setminus p^{-1}([\{\text{the line in } \mathbb{C}P^2 \text{ connecting } y_j \text{ and } [m_0]}) \). Holomorphicity here means that if \( l_{(l,p)} \) is the induced almost complex operator of \( Z(R^4) \) at \( (l,p) \) and \( \pi_{y_j}((l,p))_* \) is the derivative of \( \pi_{y_j} \) at \( (l,p) \) and \( J_{\pi_{y_j}((l,p))} \) is the induced almost complex operator of \( p^{-1}([m_0]) \cong \mathbb{C}P^1 \) at \( \pi_{y_j}((l,p)) \) then

\[
(\pi_{y_j}((l,p)))_* \circ I_{(l,p)} = J_{\pi_{y_j}((l,p))} \circ (\pi_{y_j}((l,p)))_*,
\]

for all \( j = 1,2,\ldots,n \). Multiplying both sides with the real scalar \( \text{vol}(U_j) \), summing over \( j \) we obtain

\[
\left( \sum_{j=1}^{n} \left( R_{(l,p),y_j,x}(\pi_x((l,p))) \text{vol}(U_j) \right) \right)_* \circ I_{(l,p)} = \sum_{j=1}^{n} \left( J_{\pi_{y_j}((l,p))} \circ (\pi_{y_j}((l,p)))_* \text{vol}(U_j) \right)_*,
\]

where \( \pi_{n((l,p))} = \sum_{j=1}^{n} R_{(l,p),y_j,x}(\pi_x((l,p))) \text{vol}(U_j) \). Taking \( n \to +\infty \) and by the continuity of \( J \) we get

\[
(\pi((l,p)))_* \circ I_{(l,p)} = \left( \int_{\mathbb{C}P^2 \setminus R^4} R_{(l,p),x}(\pi_x((l,p))) \, dy \right)_* = J_{\pi((l,p))}(\int_{\mathbb{C}P^2 \setminus R^4} R_{(l,p),x}(\pi_x((l,p))) \, dy)_*,
\]

demonstrating that \( \pi \) is holomorphic along \( Z(R^4) \setminus p^{-1}(\{\text{the line in } \mathbb{C}P^2 \text{ through } x \text{ and } [m_0]}) \).
Regarding the independence of (12) of \( \pi \), the map \( \pi_x : Z(R^4) \setminus p^{-1}(\{m_0\}) \rightarrow p^{-1}(\{m_0\}) \) constructed in (10) and (11) is well-defined for every fixed \( y \in (\mathbb{C}P^2 \setminus R^4) \cap C \) consequently if we pick two points \( x_1, x_2 \in (\mathbb{C}P^2 \setminus R^4) \cap C \) then \( R_{(l,p),x_1}(\pi_{x_1}((l,p))) = \pi_y((l,p)) = R_{(l,p),x_2}(\pi_{x_2}((l,p))) \) demonstrating the equality of the corresponding integrals i.e., \( \pi \) does not depend on \( \pi_x \). (In fact, taking into account

\[
\pi_{x_2}((l,p)) = \frac{P_{x_2,x_1}(\pi_{x_1}((l,p)))}{Q_{x_2,x_1}(\pi_{x_1}((l,p)))},
\]

the change of the reference point \( x \) in (12) can be regarded as an algebraic change of variables in the coordinatized algebraic function field \( \mathbb{C}(\pi_x) \) of rational functions.) In particular the \( x \)-independence holds true around the target point \( [m_0] \in R^4 \) as well. However, it readily follows from (10) that taking the limit \( [m] \rightarrow [m_0] \) in \( \mathbb{C}P^2 \setminus \{ \text{the line in } \mathbb{C}P^2 \text{ connecting } x \text{ and } [m_0] \} \) any specific \( \pi_x \) approaches the identity of the target twistor fiber. Since \( S^2 \) has unit volume with respect to \( d(y^2) \) the same remains valid for \( \pi \). These demonstrate that \( \pi \) defined via (12) extends to a well-defined holomorphic map \( \pi : Z(R^4) \rightarrow p^{-1}(\{m_0\}) \) satisfying \( \pi|_{p^{-1}(\{m_0\})} = \mathrm{Id}|_{p^{-1}(\{m_0\})} \) as stated.

Last but not least note that for every \( x \in (\mathbb{C}P^2 \setminus R^4) \cap C \) and \( [m] \in R^4 \) the map \( \pi_x \) as defined in (10) is a holomorphic bijection between \( p^{-1}(\{m\}) \) and \( p^{-1}(\{m_0\}) \) and depends continuously on \( x \). Consequently this property cannot change along \( (\mathbb{C}P^2 \setminus R^4) \cap C \) i.e., the same is true for \( \pi_y \). Therefore

\[
R_{(l,p),xy}(\pi_{x}((l,p))) = \pi_{y}((l,p)) = a_{(l,p),xy} \pi_{x}((l,p)) + b_{(l,p),xy}
\]

whose coefficients are a.e. continuous bounded complex-valued functions along \( \mathbb{C}P^2 \setminus R^4 \) such that the former nowhere vanishes and \( a_{(l,p),xx} = 1 \) moreover \( b_{(l,p),xx} = 0 \) holds for the latter. \( \diamond \)

Finally we observe that the more advanced method used in the proof of Lemma 4.2 to construct (9) is a generalization of the simpler one used in the proof of Lemma 3.2 to construct (3) as follows:

**Lemma 5.4.** Let \( (X_M, \gamma) \) be the incomplete self-dual space as in Lemma 3.1 and let \( Z \) denote its twistor space. Then, regarding the construction of a holomorphic map from \( Z \) to \( \mathbb{C}P^1 \), both constructions used in the proofs of Lemmata 3.2 and 4.2 apply and are equivalent in the sense that the resulting holomorphic maps coincide.

**Proof.** Let \( x_0 \in X_M \) be a fixed point in the \( R^4 \) factor of \( X_M \) in its decomposition (1) and consider the holomorphic map (3) constructed in the proof of Lemma 3.2. It is clear that in addition a similar holomorphic map \( \rho : Z \rightarrow p^{-1}(x_0) \) can also be constructed by the technique used in the proof of Lemma 4.2 simply by replacing the “continuously embedded projective line” \( S^2 = \mathbb{C}P^2 \setminus R^4 \) with a standard holomorphic one \( \mathbb{C}P^1 = \mathbb{C}P^2 \setminus R^4 \) in **Step one** and **Step two** in the proof of Lemma 4.2.

Take the maps (3) and \( \rho : Z \rightarrow p^{-1}(x_0) \) and note first of all that \( \pi|_{p^{-1}(x_0)} = \mathrm{Id}|_{p^{-1}(x_0)} = \rho|_{p^{-1}(x_0)} \).

Take any point \( x \in X_M \) then exploiting the connectivity of \( X_M \) we can run a continuous curve \( \{x(t)\}_{t \in [0,1]} \) within \( X_M \) from \( x \) to \( x_0 \) and perform a parallel transport along it from \( p^{-1}(x) \) to \( p^{-1}(x_0) \) via \( \pi \) and then do the same backwards via \( \rho \). By the aid of the corresponding commutative diagram

\[
\begin{array}{ccc}
p^{-1}(x) & \xrightarrow{\pi|_{p^{-1}(x)}} & p^{-1}(x_0) \\
\downarrow{\rho|_{p^{-1}(x)}} & & \downarrow{\rho|_{p^{-1}(x_0)}} \\
p^{-1}(x_0) & \xrightarrow{p^{-1}(x_0)} & p^{-1}(x_0)
\end{array}
\]

we obtain a map \( (\rho|_{p^{-1}(x)})^{-1} \circ \pi|_{p^{-1}(x)} : p^{-1}(x) \rightarrow p^{-1}(x) \) which is holomorphic and represents the homotopy class of the identity mapping in \( \pi_2(p^{-1}(x)) \cong \pi_2(S^2) \cong \mathbb{Z} \). (A homotopy with the identity
arises by moving \( x \) continuously toward \( x_0 \). Then under an identification \( p^{-1}(x) \cong \mathbb{C}P^1 \) it is therefore of the form \( z \mapsto a(x)z + b(x) \) with \( a(x), b(x) \in \mathbb{C} \). It easily follows that the coefficients must be constant over the whole \( Z \) since the twistor fibration \( p : Z \to X_M \) is not holomorphic. Consequently \( \rho = a\pi + b \) with \( a, b \in \mathbb{C} \) hence putting \( x := x_0 \) we find \( a = 1 \) and \( b = 0 \) hence \( \rho = \pi \) as claimed. \( \diamond \)

6 Appendix B: Lorentzian solutions

In this closing section we convert all the Ricci-flat Riemannian metrics of Theorem 1.1 into Ricci-flat Lorentzian ones by (essentially verbatim) recalling [8, Lemma 4.2].

**Theorem 6.1.** Consider the space \( X^\times \) as in Theorem 1.1. Then there exists a smooth Lorentzian metric \( g_L \) on \( X^\times \) such that \( (X^\times, g_L) \) is a, perhaps incomplete, Ricci-flat Lorentzian 4-manifold.

**Proof.** Take the complexification \( T^\mathbb{C}X^\times := T^\mathbb{C}X^\times \otimes_{\mathbb{R}} \mathbb{C} \) of the real tangent bundle as well as the complex bilinear extension of the Riemannian Ricci-flat metric \( g \) found on \( T^\mathbb{C}X^\times \) to a Ricci-flat metric \( g^\mathbb{C} \) on \( T^\mathbb{C}X^\times \). This means that if \( v^\mathbb{C} \) is a complexified tangent vector then both \( v^\mathbb{C} \mapsto g^\mathbb{C}(v^\mathbb{C}, \cdot) := g(v^\mathbb{C}, \cdot) \) and \( v^\mathbb{C} \mapsto g^\mathbb{C}(\cdot, v^\mathbb{C}) := g(\cdot, v^\mathbb{C}) \) are declared to be \( \mathbb{C} \)-linear and \( \text{Ric}_{g^\mathbb{C}} = \text{Ric}_g = 0 \). By virtue of its global triviality (cf. Lemma 3.4), the tangent bundle admits a nowhere vanishing smooth section yielding an orthogonal splitting \( T^\mathbb{C}X^\times = L \oplus L^\perp \) into a real line bundle \( L \subset T^\mathbb{C}X^\times \) spanned the section and its \( g \)-orthogonal complement \( L^\perp \subset T^\mathbb{C}X^\times \). This induces a splitting

\[
T^\mathbb{C}X^\times = L \oplus L^\perp \oplus \sqrt{-1}L \oplus \sqrt{-1}L^\perp \tag{15}
\]

over \( \mathbb{R} \) of the complexification i.e., if \( T^\mathbb{C}X^\times \) is considered as a real rank-8 bundle over \( X^\times \). Define a metric on the real rank-4 sub-bundle \( L^\perp \oplus \sqrt{-1}L \subset T^\mathbb{C}X^\times \) by taking the restriction \( g^\mathbb{C}|_{L^\perp \oplus \sqrt{-1}L} \). It readily follows from the orthogonality and reality of the splitting that this is a non-degenerate real-valued \( \mathbb{R} \)-bilinear form of Lorentzian type on this real sub-bundle. To see this, we simply have to observe that taking real vector fields \( v_1, v_2 : X^\times \to L \) and \( w_1, w_2 : X^\times \to L^\perp \) we can exploit the \( \mathbb{C} \)-bilinearity of \( g^\mathbb{C} \) to write

\[
g^\mathbb{C}|_{L^\perp \oplus \sqrt{-1}L}(\sqrt{-1}v_1, \sqrt{-1}v_1) = g^\mathbb{C}(\sqrt{-1}v_1, \sqrt{-1}v_1) = -g^\mathbb{C}(v_1, v_1) = -g(v_1, v_1) \]

and

\[
g^\mathbb{C}|_{L^\perp \oplus \sqrt{-1}L}(\sqrt{-1}v_1, w_1) = g^\mathbb{C}(\sqrt{-1}v_1, w_1) = \sqrt{-1}g^\mathbb{C}(v_1, w_1) = \sqrt{-1}g(v_1, w_1) = 0
\]

and finally

\[
g^\mathbb{C}|_{L^\perp \oplus \sqrt{-1}L}(w_1, w_2) = g^\mathbb{C}(w_1, w_2) = g(w_1, w_2) .
\]

Consider the \( \mathbb{R} \)-linear bundle isomorphism \( W_L : T^\mathbb{C}X^\times \to T^\mathbb{C}X^\times \) of the complexified tangent bundle defined by, with respect to the splitting (15), as

\[
W_L(v_1, w_1, \sqrt{-1}v_2, \sqrt{-1}w_2) := (v_2, w_1, \sqrt{-1}v_1, \sqrt{-1}w_2) .
\]

It satisfies \( W_L^2 = \text{Id}_{T^\mathbb{C}X^\times} \) or more precisely it is a real reflection with respect to \( g^\mathbb{C} \) making the diagram

\[
\begin{array}{ccc}
T^\mathbb{C}X^\times & \xrightarrow{W_L} & T^\mathbb{C}X^\times \\
\downarrow & & \downarrow \\
X^\times & \xrightarrow{\text{Id}_X} & X^\times \\
\end{array}
\]
commutative. In particular it maps the real tangent bundle $TX^\times = L \oplus L^\perp \subset T^CLX^\times$ onto the real bundle $L^\perp \oplus \sqrt{-1}L \subset T^CLX^\times$ and vice versa. Consequently with arbitrary two tangent vectors $v, w : X^\times \to TX^\times$ putting

$$g_L(v, w) := g^C(WLv, WLw) = g^C|_{L^\perp \oplus \sqrt{-1}L}(WLv, WLw)$$

we obtain a non-degenerate real-valued $\mathbb{R}$-bilinear form of Lorentzian type hence a smooth Lorentzian metric $g_L$ on the original real tangent bundle $TX^\times$.

Concerning the Ricci tensor of $g_L$, the Levi–Civita connections $\nabla^L$ of $g_L$ and $\nabla^C$ of $g^C$ satisfy

$$g_L(\nabla^L_u v, w) + g_L(v, \nabla^L_u w) = d g_L(v, w) u = d g^C(WLv, WLw)u = g^C(\nabla^C_u (WLv), WLw) + g^C(WLv, \nabla^C_u (WLw)) = g^C(WL^2 \nabla^C_u WLv, WLw) + g^C(WLv, WL^2 \nabla^C_u WLw) = g_L((WL^2 \nabla^C_u WLv), w) + g_L(v, (WL^2 \nabla^C_u WLw))$$

yielding $\nabla^L = W\nabla^CWL$ (this is an $\mathbb{R}$-linear operator). Consequendy the curvature $\text{Riem}_{g_L}$ of $g_L$ takes the shape

$$\text{Riem}_{g_L}(v, w) u = [\nabla^L_v, \nabla^L_w] u - \nabla^L_{[v, w]} u = W_L (\text{Riem}^C_{g_L}(v, w) W_L u).$$

Let $\{e_0, e_1, e_2, e_3\}$ be a real orthonormal frame for $g_L$ at $T_pX^\times$ satisfying $g_L(e_0, e_0) = -1$ and $+1$ for the rest; then $W_L e_0 = \sqrt{-1} e_0$ and $W_L e_j = e_j$ for $j = 1, 2, 3$ together with the definition of $g_L$ imply that

$$g_L(\text{Riem}_{g_L}(e_0, v) w, e_0) = g^C(WL(\text{Riem}_{g_L}(e_0, v) w), WLe_0) = g^C(\text{Riem}^C_{g_L}(e_0, v) W_L w, \sqrt{-1}e_0)$$

and likewise

$$g_L(\text{Riem}_{g_L}(e_j, v) w, e_j) = g^C(WL(\text{Riem}_{g_L}(e_j, v) w), WLe_j) = g^C(\text{Riem}^C_{g_L}(e_j, v) W_L w, e_j).$$

Using an orthonormal frame $\{f_1, \ldots, f_m\}$ for a metric $h$ of any signature, its Ricci tensor looks like

$$\text{Ric}_h(v, w) = \sum_{k=1}^m h(f_k, f_k) h(\text{Riem}_h(f_k, v) w, f_k);$$

hence

$$\text{Ric}_{g_L}(v, w) = g_L(e_0, e_0) g_L(\text{Riem}_{g_L}(e_0, v) w, e_0) + \sum_{j=1}^3 g_L(e_j, e_j) g_L(\text{Riem}_{g_L}(e_j, v) w, e_j)$$

$$= g^C(\sqrt{-1}e_0, \sqrt{-1}e_0) g^C(\text{Riem}^C_{g_L}(e_0, v) W_L w, \sqrt{-1}e_0) + \sum_{j=1}^3 g^C(e_j, e_j) g^C(\text{Riem}^C_{g_L}(e_j, v) W_L w, e_j)$$

$$= (\sqrt{-1} - 1) g^C(e_0, e_0) g^C(\text{Riem}^C_{g_L}(e_0, v) W_L w, e_0) + \text{Ric}^C_{g_L}(v, W_L w)$$

$$= (1 - \sqrt{-1}) g_L(\text{Riem}_{g_L}(e_0, v) w, e_0)$$

and we also used $\{e_0, e_1, e_2, e_3\}$ as a complex orthonormal basis for $g^C$ on $T^p_\mathbb{C}X^\times$ to write

$$\sum_{j=0}^3 g^C(e_j, e_j) g^C(\text{Riem}^C_{g_L}(e_j, v) W_L w, e_j) = \text{Ric}^C_{g_L}(v, W_L w) = 0.$$
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