Certain new unified integrals associated with the product of
generalized Bessel functions

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Abstract
Our focus to presenting two very general integral formulas whose integrands are the integrand given in the Oberhettinger’s integral formula and a finite product of the generalized Bessel function of the first kind, which are expressed in terms of the generalized Lauricella functions. Among a large number of interesting and potentially useful special cases of our main results, some integral formulas involving such elementary functions are also considered.

Keywords: Gamma function, Generalized hypergeometric function \( _pF_q \), generalized Lauricella series in several variables, Generalized Bessel function, Oberhettinger’s integral formula.

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1 Introduction and Preliminaries
Throughout this paper, \( \mathbb{N}, \mathbb{C}, \text{ and } \mathbb{Z}_0^- \) denote the sets of positive integers, real numbers, complex numbers, and nonpositive integers, respectively, and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).

Recently, Baricz introduced and studied some fundamental properties and characteristics of the generalized Bessel function of the first kind, \( w_v(z) \) which are defined by (see, for example, [3, p.10, Eqn. (1.15)]; for a very recent work, see also [4, 5, 6]):

\[
w_{v,b,c}(z) = \sum_{n \geq 0} \frac{(-1)^n c^n (\frac{z}{2})^{v+2n}}{n! \Gamma(v + n + \frac{b+1}{2})},
\]

where \( \Gamma(z) \) is the familiar Gamma function (see [14, Section 1.1], see also [1]).

Here, it is worth mentioning that, Bessel function of the first kind \( J_v(z) \) and \( I_v(z) \) are frequently used in studying solutions of differential equations, and they are associated with a wide range of problems in important areas of mathematical physics, like problems of acoustics, radio physics, hydrodynamics, and atomic, nuclear physics, probability

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theory and statistics. These considerations have led various workers in the field of special functions for exploring the possible extensions and applications for the Bessel functions (see, [2, 7, 9, 10, 11, 13, 17]). Here we prove two very general integral formulas whose integrands are the integrand given in the Oberhettingers integral formula and a finite product of the generalized Bessel function of the first kind, which are expressed in terms of the generalized Lauricella functions. Among a large number of interesting and potentially useful special cases of our main results, some integral formulas involving such elementary functions are also considered.

The generalized Lauricella functions (see, for example, [16, p. 36, Eq. (19)]) is (cf. Srivastava and Daoust [15, p. 454]; see also [16, p. 37])

\[
F_{A,B(1),\ldots,B(n)}^{C,D(1),\ldots,D(n)}\left(\begin{array}{c}
\frac{z_1}{z_n}
\vdots
\frac{z_i}{z_n}
\vdots
\frac{z_n}{z_n}
\end{array}\right) = F_{A,B^{(1)},\ldots,B^{(n)}}^{C,D^{(1)},\ldots,D^{(n)}}\left(\begin{array}{c}
\left(\begin{array}{c}
\theta^{(1)}
\vdots
\theta^{(n)}
\end{array}\right);
\left(\begin{array}{c}
\psi^{(1)}
\vdots
\psi^{(n)}
\end{array}\right)
\end{array}\right);
\left(\begin{array}{c}
\phi^{(1)}
\vdots
\phi^{(n)}
\end{array}\right);\left(\begin{array}{c}
\delta^{(1)}
\vdots
\delta^{(n)}
\end{array}\right);z_1,\ldots,z_n
\right)
\]

(1.2)

where, for convenience,

\[
\Omega(k_1,\ldots,k_n) = \prod_{j=1}^{A} (a_j)^{k_1 \theta^{(1)} + \cdots + k_n \theta^{(n)}},
\]

(1.3)

the coefficients

\[
\begin{align*}
\theta_j^{(m)} & \quad (j = 1, \ldots, A); \\
\phi_j^{(m)} & \quad (j = 1, \ldots, B^{(m)}); \\
\psi_j^{(m)} & \quad (j = 1, \ldots, C); \\
\delta_j^{(m)} & \quad (j = 1, \ldots, D^{(m)}); \forall m \in \{1, \ldots, n\}
\end{align*}
\]

(1.4)

are real and positive, and \(A\) abbreviates the array of \(A\) parameters \(a_1,\ldots,a_A\), \(B^{(m)}\) abbreviates the array of \(B^{(m)}\) parameters

\[b_j^{(m)} \quad (j = 1, \ldots, B^{(m)}); \forall m \in \{1, \ldots, n\},\]

with similar interpretations for \(C\) and \(D^{(m)}\) \((m = 1, \ldots, n)\); et cetera.

The multiple series (1.2) converges absolutely either

(i) \(\Delta_t > 0 \quad (i = 1, \ldots, n), \quad \forall z_1,\ldots,z_n \in \mathbb{C},\)

or

(ii) \(\Delta_t = 0 \quad (i = 1, \ldots, n), \quad \forall z_1,\ldots,z_n \in \mathbb{C}, \quad |z_i| < \rho_i \quad (i = 1, \ldots, n).\)

The multiple series (1.2) is divergent when \(\Delta_t < 0 \quad (i = 1, \cdot \cdot \cdot, n)\) except for the trivial case \(z_1 = 0, \ldots, z_n = 0\). Here

\[
\Delta_t \equiv 1 + \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} = \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} \quad (i = 1, \ldots, n),
\]

(1.5)

\[
\rho_i = \min_{\mu_1,\ldots,\mu_\kappa > 0} \{E_i\} \quad (i = 1, \cdot \cdot \cdot, n),
\]

(1.6)
with
\[
E_i = (\mu, \eta) \left( 1 + \sum_{j=1}^{B(i)} \sigma_j^{(i)} - \sum_{j=1}^{A(i)} \varphi_j^{(i)} \right) \frac{\left\{ \prod_{j=1}^{A(i)} (\delta_j^{(i)})^{\theta_j^{(i)}} \right\} \left\{ \prod_{j=1}^{B(i)} (\delta_j^{(i)})^{\varphi_j^{(i)}} \right\}}{\left\{ \prod_{j=1}^{D(i)} (\delta_j^{(i)})^{\varphi_j^{(i)}} \right\} \left\{ \prod_{j=1}^{C(i)} (\delta_j^{(i)})^{\varphi_j^{(i)}} \right\}}.
\]

(1.7)

Also, for interested researchers see [15] and [16, pp. 39-40]). We also required (see [14, Section 1.5])

\[
\rho F_q \left[ \begin{array}{c}
\alpha_1, \ldots, \alpha_p; \\
\beta_1, \ldots, \beta_q;
\end{array} \right| z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!}
= \rho F_q (\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z),
\]

where \((\lambda)_n\) is the Pochhammer symbol defined (for \(\lambda \in \mathbb{C}\)) by (see [14, p. 2 and pp. 4-6]):

\[
(\lambda)_n := \begin{cases}
1 & (n = 0) \\
\lambda (\lambda + 1) \ldots (\lambda + n - 1) & (n \in \mathbb{N} := \{1, 2, 3, \ldots\})
\end{cases}
= \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-)
\]

and \(\mathbb{Z}_0^-\) denotes the set of nonpositive integers.

The Oberhettinger’s integral formula [12]:

\[
\int_0^{\infty} x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} \frac{dx}{x} = 2^{\lambda - \mu - 1} \frac{\Gamma(2\mu) \Gamma(\lambda - \mu)}{\Gamma(1 + \mu + \mu)} \frac{a^{\lambda - \mu}}{\Gamma(\lambda + \mu + \mu)},
\]

(1.10)

provided \(0 < \Re(\mu) < \Re(\lambda)\).

2 Unified Integrals Involving Generalized Bessel Functions

**Theorem 2.1.** If \(x > 0; \lambda, \mu, v_j, b, c \in \mathbb{C}\) with \(\Re(v_j) > -1, 0 < \Re(\mu) < \Re(\lambda + v_j) \quad (j = 1, \ldots, n)\). Then there holds the following integral formula:

\[
\int_0^{\infty} x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} \prod_{j=1}^{n} \frac{\left( v_j + \frac{y_j}{2} \right)}{\Gamma(v_j + \frac{y_j}{2})} \Gamma(2\mu) \Gamma(1 + \lambda + \sum_{j=1}^{n} v_j) \Gamma(\lambda - \mu + \sum_{j=1}^{n} v_j) \\
\frac{\Gamma(\lambda + \mu + \sum_{j=1}^{n} v_j)}{\Gamma(1 + \lambda + \mu + \sum_{j=1}^{n} v_j)} \\
\cdot F_{2;1+\ldots+1}^{2;0;0;0;\ldots;\ldots;\ldots;\ldots} \left[ \\
\begin{array}{c}
1 + \lambda + \sum_{j=1}^{n} v_j : 2, \ldots, 2 \\
1 + \lambda + \mu + \sum_{j=1}^{n} v_j : 2, \ldots, 2
\end{array} \right] \\
\left[ \\
\begin{array}{c}
\lambda - \mu + \sum_{j=1}^{n} v_j : 2, \ldots, 2 \\
\lambda + \sum_{j=1}^{n} v_j : 2, \ldots, 2
\end{array} \right] ; \frac{y_1^2}{4a^2}, \ldots, \frac{y_n^2}{4a^2} \\
\left[ v_1 + \frac{1+b}{2} : 1 \right] ; \ldots ; \left[ v_n + \frac{1+b}{2} : 1 \right] ; -c \frac{y_1^2}{4a^2}, \ldots, -c \frac{y_n^2}{4a^2}
\]

(2.1)
Theorem 2.2. For \( \lambda, \mu, v_j, b, c \in \mathbb{C} \) with \( \Re(v_j) > -1 \), \( 0 < \Re(\mu) < \Re(\lambda + v_j) \) (\( j = 1, \ldots, n \)) and \( x > 0 \). Then there holds the following integral formula:

\[
\int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} \prod_{j=1}^n w_{v_j} \left( \frac{x y_j}{x + a + \sqrt{x^2 + 2ax}} \right) \, dx
\]

\[= 2^{1-\mu} a^{\mu-\lambda} \left( \prod_{j=1}^n \left( \frac{y_j}{\Gamma(v_j + \frac{b+1}{2})} \right) \right) \frac{\Gamma(\lambda - \mu) \Gamma(1 + \lambda + \sum_{j=1}^n v_j) \Gamma(2\mu + 2\sum_{j=1}^n v_j)}{\Gamma(\lambda + \sum_{j=1}^n v_j) \Gamma(1 + \lambda + \mu + 2\sum_{j=1}^n v_j)}
\]

\[
\cdot \left[ \begin{array}{c}
1 + \lambda + \sum_{j=1}^n v_j : 2, \ldots, 2, \\
2 + \lambda + 2\sum_{j=1}^n v_j : 4, \ldots, 4,
\end{array} \right] ; \quad \left[ \begin{array}{c}
\lambda + \sum_{j=1}^n v_j : 2, \ldots, 2,
\end{array} \right] ;
\]

(2.2)

Proof. By applying (1.1) to the integrand of (2.1) and then interchanging the order of integral sign and summation, we get

\[
\mathcal{J} = \int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} \prod_{k_1=0}^\infty \left( -c \right)^{k_1} \frac{y_1}{k_1} \frac{1}{\Gamma(v_1 + k_1 + \frac{b+1}{2})} \ldots \prod_{k_n=0}^\infty \left( -c \right)^{k_n} \frac{y_n}{k_n} \frac{1}{\Gamma(v_n + k_n + \frac{b+1}{2})} \, dx
\]

and

\[
\mathcal{J} = \sum_{k_1=0}^\infty \ldots \sum_{k_n=0}^\infty \left( -c \right)^{k_1} \frac{y_1}{k_1} \frac{1}{\Gamma(v_1 + k_1 + \frac{b+1}{2})} \ldots \left( -c \right)^{k_n} \frac{y_n}{k_n} \frac{1}{\Gamma(v_n + k_n + \frac{b+1}{2})} \frac{y_{v_1 + \ldots + v_n + 2k_1 + \ldots + 2k_n}}{\Gamma(1 + \lambda + \mu + v_1 + \ldots + v_n + 2k_1 + \ldots + 2k_n)} \frac{y_{v_1 + \ldots + v_n + 2k_1 + \ldots + 2k_n}}{\Gamma(1 + \lambda + \mu + v_1 + \ldots + v_n + 2k_1 + \ldots + 2k_n)}
\]

(2.3)

In view of the conditions given in Theorem 2.1, since

\[
\Re(v_j) > -1, \quad 0 < \Re(\mu) < \Re(\lambda + v_j) \leq \Re(\lambda + v_j + 2k_j)
\]

\[
(k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad \text{and} \quad j = 1, \ldots, n),
\]

we can apply the integral formula (1.10) to the integral in (2.3) and obtain the following expression:

\[
\mathcal{J} = 2^{1-\mu} a^{\mu-\lambda} \sum_{k_1=0}^\infty \ldots \sum_{k_n=0}^\infty \left( -c \right)^{k_1} \frac{y_1}{k_1} \frac{1}{\Gamma(v_1 + k_1 + \frac{b+1}{2})} \ldots \left( -c \right)^{k_n} \frac{y_n}{k_n} \frac{1}{\Gamma(v_n + k_n + \frac{b+1}{2})} \frac{y_{v_1 + \ldots + v_n + 2k_1 + \ldots + 2k_n}}{\Gamma(1 + \lambda + \mu + v_1 + \ldots + v_n + 2k_1 + \ldots + 2k_n)}
\]

And we have

\[
\mathcal{J} = 2^{1-\mu} a^{\mu-\lambda} \sum_{k_1=0}^\infty \ldots \sum_{k_n=0}^\infty \left( -c \right)^{k_1} \frac{y_1}{k_1} \frac{1}{\Gamma(v_1 + k_1 + \frac{b+1}{2})} \ldots \left( -c \right)^{k_n} \frac{y_n}{k_n} \frac{1}{\Gamma(v_n + k_n + \frac{b+1}{2})} \frac{y_{v_1 + \ldots + v_n + 2k_1 + \ldots + 2k_n}}{\Gamma(1 + \lambda + \mu + v_1 + \ldots + v_n + 2k_1 + \ldots + 2k_n)}
\]
For (2.1) and (2.2), we obtain two further pairs of integral formulae, reads as follows.

\[ J = 2^{-1-\mu}d^{2-\lambda} \left( \prod_{j=1}^{n} \frac{(-x)^{y_j}}{z_j^{2+y_j}} \right) \frac{\Gamma(2\mu)\Gamma(\lambda - \mu + \sum_{j=1}^{n} y_j)\Gamma(1 + \lambda + \sum_{j=1}^{n} y_j)}{\Gamma(1 + \lambda + \mu + \sum_{j=1}^{n} y_j)\Gamma(\lambda + \sum_{j=1}^{n} y_j)} \]

Finally, we interpret the multiple series in (2.4) as a special case of the general hypergeometric series in several variables defined by (1.2). We are thus conclude to the (2.1). The assertion (2.2) of the Theorem 2.2 can be proved by a similar argument.

**Remark 2.1.** If we set \( n = 1 \) in (2.1) and (2.2) we can arrive at the (2.1) and (2.2)et al. [8] and for \( c = b = 1 \) in (2.1) and (2.2) we can arrive at (2.1) and (2.2) in Choi and Agarwal [8].

### 3 Special Cases

Here we give only two example. For our purpose the particular cases of the function \( w_{\nu,b,c}(z) \) given by (1.1), is worthy to mention here.

For \( b = c = 1 \) in (1.1), we obtain the familiar Bessel function \( J_\nu(z) \) defined by (see [3])

\[ J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{\nu+2n}}{n! \Gamma(\nu + n + 1)}, \quad (z \in \mathbb{C}). \]  

(3.1)

For \( b = -c = 1 \) in (1.1), we obtain the modified Bessel function \( I_\nu(z) \) defined by (see [3])

\[ I_\nu(z) = \sum_{n=0}^{\infty} \frac{1 (\frac{z}{2})^{\nu+2n}}{n! \Gamma(\nu + n + 1)}, \quad (z \in \mathbb{C}). \]  

(3.2)

For \( b - 1 = c = 1 \) in (1.1), we obtain the spherical Bessel function \( K_\nu(z) \) defined by

\[ K_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{\nu+2n}}{n! \Gamma(\nu + n + \frac{3}{2})}, \quad (z \in \mathbb{C}). \]  

(3.3)

We now present the new unified integrals in terms of the spherical Bessel function \( K_\nu(z) \) by setting \( b = 2 \) and \( c = 1 \) in (2.1) and (2.2), we obtain two further pairs of integral formulae, reads as follows.

**Corollary 3.1.** Let the condition of Theorem 2.2 be satisfied. Then the following integral formula holds true.

\[ \int_{0}^{\infty} x^{a-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} \frac{1}{x + a + \sqrt{x^2 + 2ax}} \left( \prod_{j=1}^{n} \frac{y_j}{\nu_j} \right) dx = 2^{1-\mu}d^{2-\lambda} \left( \prod_{j=1}^{n} \frac{(-x)^{y_j}}{z_j^{2+y_j}} \right) \frac{\Gamma(2\mu)\Gamma(\lambda - \mu + \sum_{j=1}^{n} y_j)\Gamma(1 + \lambda + \mu + \sum_{j=1}^{n} y_j)}{\Gamma(1 + \lambda + \mu + \sum_{j=1}^{n} y_j)\Gamma(\lambda + \sum_{j=1}^{n} y_j)} \]

\[ \left[ 1 + \lambda + \sum_{j=1}^{n} y_j ; 2, \ldots, 2 \right] ; \left[ \lambda - \mu + \sum_{j=1}^{n} y_j ; 2, \ldots, 2 \right] : \frac{1}{[v_1 + \frac{3}{2} : 1]; \ldots; [v_n + \frac{3}{2} : 1]}; - \frac{\nu_1^2}{4a^2}; \ldots; - \frac{\nu_n^2}{4a^2} \right]. \]
Corollary 3.2. Let the condition of Theorem 2.2 be satisfied. Then the following integral formula holds true.

\[
\int_0^\infty x^{\mu-1} \left(x + a + \sqrt{x^2 + 2ax}\right)^{-\lambda} \prod_{j=1}^n K_{\nu_j} \left(\frac{x\nu_j}{x + a + \sqrt{x^2 + 2ax}}\right) \, dx
\]

\[= 2^{1-\mu} a^{\mu-\lambda} \left(\prod_{j=1}^n \frac{(\frac{\lambda}{2})^{\nu_j}}{\Gamma(\nu_j + \frac{1}{2})}\right) \Gamma(\lambda - \mu) \Gamma(1 + \lambda + \sum_j^n \nu_j) \frac{\Gamma(\lambda + \sum_j^n \nu_j)}{\Gamma(1 + \lambda + \mu + \sum_j^n \nu_j)}
\]

\[
\frac{1 + \lambda + \sum_j^n \nu_j : 2, \ldots, 2}{\begin{array}{c}1 + \lambda + \mu + 2\sum_j^n \nu_j : 4, \ldots, 4; \\
\vdots \end{array}}
\]

\[
\frac{\lambda + \sum_j^n \nu_j : 2, \ldots, 2}{\begin{array}{c}[v_1 + \frac{3}{2} : 1]; \ldots; [v_n + \frac{3}{2} : 1]; \\
-\frac{v_1^2}{16}; \ldots; -\frac{v_n^2}{16}\end{array}}
\]

4 Concluding Remarks

The results provided in this paper are easily converted in terms of the various Bessel functions after suitable parametric replacements, and further, trigonometric functions, hyperbolic functions and exponential function. The explicit details of those special cases are left to the interested reader. We would like to emphasize that the results derived in this paper may be potentially useful due mainly to the demonstrated production of a variety of specialized integral formulas, in particular, associated with such elementary functions as trigonometric functions, hyperbolic functions and exponential function as well as various Bessel functions.

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