ON THE REPRESENTATION THEORY OF NON-SEMISIMPLE (GRADED) DEFORMED FOMIN-KIRILLOV ALGEBRAS.

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Abstract. This work is motivated to study the representation theory of the non-semisimple deformed Fomin-Kirillov algebras $\mathcal{D}_4(\alpha_1, \alpha_2)$. In particular, we consider Gabriel’s theorem applications in regard of constructing algebraic presentations.

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Introduction

Here and throughout this paper we assume $K$ to be an algebraically closed field of zero characteristic, we denote by $S_n$ the symmetric group on $n$ letters.

In the context of studying Schubert calculus, Fomin and Kirillov introduced a family of quadratic $K$-algebras $\mathcal{E}_n$, that contains a commutative subalgebra isomorphic to the cohomology ring of the flag manifold. Commonly known as Fomin-Kirillov algebras, we present:

Definition 0.1 (\cite{1} Definition 2.1). Given a positive integer $n \geq 3$. $\mathcal{E}_n$ is the quadratic $K$-algebra generated by $x_{ij} = -x_{ji}$ for $1 \leq i \neq j \leq n$
subject the following relations:

(1a) \[ x_{ij}^2 = 0 \quad | \quad 1 \leq i, j \leq n \text{ distinct}, \]

(1b) \[ x_{ij}x_{kl} - x_{kl}x_{ij} = 0 \quad | \quad 1 \leq i, j, k, l \leq n \text{ distinct}, \]

(1c) \[ x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij} = 0 \quad | \quad 1 \leq i, j, k \leq n \text{ distinct}. \]

While on the surface, it is rather straightforward to present \( \mathcal{E}_n \), some of the structure’s elementary properties remain challenging to approach:

For example, it is well-known that \( \mathcal{E}_n \) is finite-dimensional for \( n \leq 5 \) and the opposite is conjectured to true otherwise.

Moreover, the nature of \( \mathcal{E}_n \) as a braided Hopf algebra over the symmetric group \( S_n \) indicates a strong connection to Nichols-algebras over braided vector spaces:

For \( n \leq 4 \) in [2], and \( n = 5 \) in [3] it was showed that the \( \mathcal{E}_n \) is a Nichols-algebra. A statement that is conjectured to be true for \( n \geq 6 \) in [2], [5].

It is also of worth to mention that the algebra \( \mathcal{E}_n \) appears to share some distinctive properties with other types of algebras, most famous of which is that of preprojective type of \( A_{n-1} \), which shares the same number of indecomposable modules with \( \mathcal{E}_n \) for \( n \leq 5 \) and is known to be of infinite representation-type otherwise; This is recognized as Majid’s conjecture, which does not have a precise expression, nonetheless highlights that the numerology is not accidental, further details can be explored in [6] and in [7].

With that in mind, we remark that from the point of view of graded algebras, \( \mathcal{E}_n \) remains an approachable candidate of an algebraic structure that is both naturally and symmetrically graded, moreover, the unique action of the symmetric group \( S_n \) on \( \mathcal{E}_n \) defined as:

\[ \sigma(xy) = (\sigma x)(\sigma y) : x, y \in \mathcal{E}_n, \sigma \in S_n \sigma(x_{ij}) = x_{\sigma(i)\sigma(j)} \quad | \quad i \neq j. \]

validates- above other reasons- the study of \( \mathcal{E}_n \) from the viewpoint of PBW-deformations. We recall that a PBW deformation of a graded algebra \( A \) is a filtered algebra \( D \) such that the associated graded algebra of \( D \) is isomorphic to \( A \), we formalize a definition:

**Definition 0.2.** Given \( \alpha_1, \alpha_2 \in K \). The deformed Fomin-Kirillov algebra, denoted by \( D_n(\alpha_1, \alpha_2) \), is the quadratic \( K \)-algebra generated by
\( x_{ij} = -x_{ji} \) for \( 1 \leq i \neq j \leq n \) subject the following relations:

\[ (2a) \quad x_{ij}^2 = \alpha_1 \quad | \quad 1 \leq i, j \leq n \text{ distinct}, \]

\[ (2b) \quad x_{ij}x_{kl} - x_{kl}x_{ij} = 0 \quad | \quad 1 \leq i, j, k, l \leq n \text{ distinct}, \]

\[ (2c) \quad x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij} = \alpha_2 \quad | \quad 1 \leq i, j, k \leq n \text{ distinct}. \]

In 2018, motivated by understanding Nichols and Fomin-Kirillov algebras by means of PBW-deformations. Heckenberger and Vendramin established a framework objected to the classification and the study of representation theory of non-semisimple deformations of Fomin-Kirillov algebras. In particular, the authors theorized:

**Theorem 0.1.** \([4]\) Theorem 2.11] The algebra \( D_3(\alpha_1, \alpha_2) \) is semi-simple if and only if:

\[(3\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \neq 0\]

In this case \( D_3(\alpha_1, \alpha_2) \cong (K^{2,2})^3 \)

Further, proposed:

**Proposition 0.2.** \([4]\) Proposition 2.15, 2.16] The following hold:

1. The algebra \( D_3(\alpha_1, -\alpha_1) \) is isomorphic to the product of three copies of the preprojective algebra of type \( A_2 \).

2. The algebra \( D_3(\alpha_1, 3\alpha_1) \) is isomorphic to the path algebra of the double Kronecker quiver bounded by the relations of the coinvariant ring of \( S_3 \)

Later that year, Wolf in \([8]\) continued the study by examining the case of \( D_4(\alpha_1, \alpha_2) \), where it was proved that the algebra \( D_4(\alpha_1, \alpha_2) \) is semisimple if:

\[ \alpha_1(3\alpha_1 - \alpha_2)(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \neq 0 \]

Further, conjectured that:

**Conjecture 0.1.** \([8]\) Corollary 2.32] The algebra \( D_4(\alpha_1, \alpha_2) \) is semisimple if and only if:

\[ (\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \neq 0 \]

Moreover, it has been calculated that the radical of \( D_4(\alpha, -\alpha) \) is generated by the commutator, that is, \( \sigma[x_{12}, x_{13}] \) for all \( \sigma \in S_4 \), while that of \( D_4(\alpha, \alpha) \) is generated by:

\[ \sigma(x_{12}x_{13} + x_{12}x_{14} + x_{12}x_{23} + x_{13}x_{23} + x_{14}x_{12} + \alpha_1) \mid \sigma \in S_4 \]

where both ideals are of 552-dimensional, and their corresponding quotient algebras are of 24-dimension.
This paper is motivated by Conjecture 0.1 and mainly aim to address the representation theory of the non-semisimple deformation of $D_4(\alpha_1, \alpha_2)$ as follows:

In Section 1 we present some preliminaries by setting up common terminology, notation and elementary results. Section 2 is dedicated to the study of $D_n(\alpha, -\alpha)$ where we start by proving that the algebra $D_n(\alpha, -\alpha)$ admits a quiver-presentation that coincides with the graph of the nil-Coxeter group associated with $S_n$, we then consider the special case of $n = 4$ where we prove that generic indecomposable projective $D_4(\alpha, -\alpha)$-modules are radically graded and are isomorphic to the nil-Coxeter algebra of $S_4$. Finally we consider the algebra $D_4(\alpha, \alpha)$ in Section 3, which we show to be non-basic, implying the existence of a basic associated Morita equivalent algebra, we construct an algebraic presentation of its graded generic indecomposable projective modules and then proceed to propose a family of which the none-graded version belongs to.

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1. Preliminaries

By a $K$-algebra, we mean an associative unital algebra over $K$. We say that a $K$-algebra $A$ is connected if it is not isomorphic to a direct product of two non-trivial algebras. We also denote $A$’s Jacobson’s radical, that is, the intersection of all maximal ideals of $A$ by $radA$.

We further understand $Ext^1_A(\rho, \rho')$ the space of extensions between two $A$-representations $\rho, \rho'$ as equivalence classes $Z^1(\rho, \rho')/B^1(\rho, \rho')$, where: $Z^1(\rho, \rho')$ denotes the space of $(1)$-cocycles, that is:

$$Z^1(\rho, \rho') = \{ f : A \to Hom_K(\rho, \rho') \mid f(xy) = \rho(x)f(y) + f(x)\rho'(y) \}$$

and $B^1(\rho, \rho')$ denotes the space of coboundaries.

1.1. Graded and filtered algebras.

**Definition 1.1.** Given $A$ a filtered algebra, that is, an algebra with a family of subspaces $\{F_{i+1} \subseteq F_i, i \geq 0\}$ such that: $1 \in F_0$, $F_iF_j \subseteq F_{i+j}$ and $\bigcup F_n = A$. We define the associated graded algebra of $A$, denoted by $grA$ by setting $(grA)_n = F_n/F_{n+1}$ and $grA = \oplus (grA)_n$. 
Example 1.1. Given $A$ a finite-dimensional $K$-algebra. Then $A$ is radically filtered as follows:

$$F_r = (\text{rad}A)^r \subset \cdots F_2 = (\text{rad}A)^2 \subset F = \text{rad}A \subset F_0 = A$$

where $r$ is the minimal positive integer such that $F_{r+1} = 0$.

Note 1. Unless otherwise mentioned, $grA$ denotes the associated graded algebra of a $K$-algebra $A$ with respect to the radical filtration.

Definition 1.2. Given $\phi : M \to N$ a filtered homomorphism, that is, $\phi(M_j) \subseteq \phi(M) \cap N_j$. If it happens that $\phi(M_j) = \phi(M) \cap N_j$ for each $j$ applicable, then $\phi$ is called strict.

Example 1.2. If $\alpha : M \to N$ is an arbitrary homomorphism and $M$ is given the induced filtration $M_j = \alpha^{-1}(\alpha(M) \cap N_j)$ then $\alpha$ is a strict filtered homomorphism. Similarly, for $\alpha$ surjective and if $N$ is given the induced filtration $N_j = \alpha(M_j)$, then $\alpha$ is strict as well.

Corollary 1 ([9] Corollary 6.14). For $\phi : M \to N$ a filtered homomorphism. Then $gr\phi$ is injective (surjective) if and only if $\phi$ is injective (surjective) and $\phi$ is strict.

1.2. Basic algebras.

Definition 1.3. A finite-dimensional $K$-algebra $A$ is said to be basic if and only if the quotient algebra $A/\text{rad}(A)$ is isomorphic to a product of copies of $K$.

Remark 1. Every simple module over a basic $K$-algebra is one-dimensional.

Definition 1.4 ([10] Corollary 6.10). Let $A$ be a (not necessarily basic) $K$-algebra, then there exists a basic $K$-algebra $A^b$ associated with $A$ such that is Morita equivalent to $A$, that is, there exists a $K$-linear equivalence of the modules categories $modA^b$ and $modA$.

1.3. The quiver of a finite dimensional algebra.

Definition 1.5. A quiver is a quadruple $Q = (Q_0, Q_1, s, t)$ with $Q_0$ and $Q_1$ finite sets and two maps $s, t : Q_1 \to Q_0$. The elements of $Q_0$ and $Q_1$ are called vertices and arrows of $Q$ respectively. We say an arrow $\alpha$ in $Q_1$ starts in $s(\alpha)$ and terminates in $t(\alpha)$.

Example 1.3. Let $A$ be a basic and connected finite-dimensional $K$-algebra and $\{S_1, \cdots, S_n\}$ a complete set of simple $A$-modules. The (ordinary) quiver of $A$, denoted by $Q_A$, is defined as follows:

\[\text{such } r \text{ exists for Jacobson radical are nilpotent} \]
(1) The vertices of $Q_A$ are numbers $1, \ldots, n$ which are in bijective correspondence with the simples $S_1, \ldots, S_n$.

(2) Given two points $i, j \in Q_A$, the arrows $\alpha : i \to j$ are in bijective correspondence with the vectors in a basis of the $K$-vector space $\text{Ext}^1_A(S_i, S_j)$.

Remark 2. A path of length $m \geq 1$ in $Q$ is a tuple $(\alpha_1, \ldots, \alpha_m)$ of arrows of $Q$ such that $s(\alpha_i) = t(\alpha_{i+1})$ for all $1 \leq i \leq m - 1$, we write such path as $\alpha_1 \cdots \alpha_m$ if no misunderstanding occurs. Additionally, for each vertex $i$ of $Q$ there exists a path $e_i$ of trivial length such that $s(e_i) = t(e_i) = i$.

Definition 1.6. Let $Q$ be a quiver. The path algebra $KQ$ of $Q$ is the $K$-algebra whose underlying $K$-vector space has as its basis the set of all $Q$-paths of length $l \geq 0$ in $Q$ such that the product of two basis vectors $\alpha_1 \cdots \alpha_m$ and $\beta_1 \cdots \beta_{m'}$ is trivial if $t(\alpha_m) \neq s(\beta_1)$ and equal to the composed path $\alpha_1 \cdots \alpha_m \beta_1 \cdots \beta_{m'}$ otherwise.

Theorem 1.1 (Gabriel [10] Theorem 3.7). Let $A$ be a basic and connected finite-dimensional $K$-algebra. There exists an admissible ideal $I$ of $KQ_A$ such that $A \cong KQ_A/I$.

2. Part 01: Representation theory of $D_n(\alpha, -\alpha)$.

For convenience, we denote $\Lambda := D_n(\alpha, -\alpha)$, we further normalize the $K$ parameter $\alpha$ to $1_K$.

Lemma 2.1. Given $\sigma \in S_n$. The algebra homomorphism: $\rho_\sigma : A \to K$ defined by mapping a generator $x_{ij}$ - with $1 \leq i \neq j \leq n$- to:

$$\rho_\sigma(x_{ij}) = \begin{cases} +1 & | \sigma(i) < \sigma(j), \\ -1 & | \sigma(i) > \sigma(j), \end{cases}$$

is a well defined one-dimensional representation of $\Lambda$.

Proof. The proof follows verifying that subjecting $\rho_\sigma$ to the defining relations of $\Lambda$ yields valid equations in $K$: Indeed as (2a) and (2b) hold trivially, one only has to check that for $1 \leq i, j, k \leq n$ distinct, then:

$$\rho_\sigma(x_{ij})\rho_\sigma(x_{jk}) + \rho_\sigma(x_{jk})\rho_\sigma(x_{ki}) + \rho_\sigma(x_{ki})\rho_\sigma(x_{ij}) = -1$$
Lemma 2.2. Given $\rho$ a one-dimensional $\mathcal{D}_n(\alpha,-\alpha)$-representation. Then there exists $\sigma \in S_n$ such that $\rho = \rho_\sigma$.

Proof. Let $\rho$ be a one-dimensional $\Lambda$-representation, (2a) implies that $\rho(x_{ij}) = \pm 1$, furthermore, for distinct $1 \leq i, j, k \leq n$, then (2c) can hold if one of the following possibilities occurs:

$$
+y_{ij} = +y_{jk} = +y_{il} = +1.
+y_{ij} = -y_{jk} = +y_{ik} = +1.
-y_{ij} = +y_{jk} = +y_{ik} = +1.
+y_{ij} = +y_{jk} = -y_{ik} = +1.
+y_{ij} = -y_{jk} = +y_{ik} = +1.
-y_{ij} = -y_{jk} = -y_{ik} = +1.
$$

Which in and of itself yield the claim.

Remark 3. The previous lemma can be alternatively proven by setting:

$$
L_i := \{1 \leq j \leq i - 1 \mid \rho(x_{ji}) = -1\} \subseteq \{0, \ldots, i - 1\}
$$

$$
R_i := \{i + 1 \leq j \leq n \mid \rho(x_{ij}) = -1\} \subseteq \{0, \ldots, n - i\}
$$

and defining the mapping $\sigma$ on $\{1, \ldots, n\}$ where $\sigma(i) = i + r_i - l_i$, the claim follows by showing that $\sigma$ is indeed a permutation such that $\rho = \rho_\sigma$.

Lemma 2.3. Given $\sigma, \tau \in S_n$, then for all $x \in A$ we have:

$$
\rho_\sigma(\tau(x)) = \rho_{\sigma \tau}(x).
$$

Proof. Since $\rho_\sigma, \rho_\tau$ and the group action of $S_n$ is multiplicative, it is enough to verify the claim for a generator $x_{ij}$ with $1 \leq i \neq j \leq n$ which hold directly since:

$$
\rho_\sigma(\tau(x_{ij})) = \rho_\sigma(x_{\tau(i)\tau(j)}) = \rho_{\sigma \tau}(x_{ij}).
$$

Question: Computing the radical of $\Lambda$ for a positive integer $n \geq 5$ remains open for the moment. We highlight that our results would imply the basicness of the algebra $\Lambda$ once we were able to verify that $\{\rho_\sigma \mid \sigma \in S_n\}$ gives a complete system of simple $\Lambda$-representations.
Our aim in at this point is to prove the following theorem:

**Theorem 2.4.** Given $\sigma, \tau \in S_n$. If $\tau = g\sigma$ where $g$ denotes a non-simple transposition of $S_n$, then $\dim K \operatorname{Ext}^1_{\Lambda}(\rho_\sigma, \rho_\tau) = 1$, and 0 in any other case.

**Remark 4.** In the purpose of proving Theorem 2.4, we start by utilizing Lemma 2.3, which implies that we may set $\sigma = e$ with no further restrictions. Furthermore, for $\tau \in S_n$, the space of extensions $\operatorname{Ext}^1_{\Lambda}(\rho_e, \rho_\tau)$ has a generating set of the form:

$$\{ f_{ij} + B^1(\rho_e, \rho_\tau) \mid 1 \leq i < j \leq n \mid f_{ij} \in K \}$$

such that the following hold:

(3a) $f_{ij}(1 + \rho_\tau(x_{ij})) = 0$

(3b) $f_{ij}(1 - \rho_\tau(x_{kl})) - f_{kl}(1 - \rho_\tau(x_{ij})) = 0$

(3c) $f_{ij}(\rho_\tau(x_{jk}) - 1) + f_{jk}(1 - \rho_\tau(x_{ik})) - f_{ik}(1 + \rho_\tau(x_{ij})) = 0$

(3d) $f_{ij}(1 - \rho_\tau(x_{ik})) + f_{jk}(\rho_\tau(x_{ij}) - 1) - f_{ik}(\rho_\tau(x_{jk}) + 1) = 0$

where $1 \leq i < j \leq n$ in (3a), $1 \leq i, j, k, l \leq n$ distinct in (3b), and $1 \leq i < j < k \leq n$ in both (3c) and (3d).

We now verify 2.4 by showcasing the following set of propositions:

**Proposition 2.5.** If $\tau$ is a non-simple transposition. Then:

$$\dim K \operatorname{Ext}^1_{\Lambda}(\rho_e, \rho_\tau) = 1$$

**Proof.** Assuming that $\tau = (s, t)$ a non-simple transposition, then $\rho_\tau(x_{ij}) = -1$ if and only if:

$$(i = s \text{ and } j \leq t) \text{ or } (i \geq s \text{ and } j = t)$$

Therefore, for all $i < j$, (3a) implies that $f_{ij} = 0$ except those of the form:

$$f_{s(s+1)} \cdots f_{st} \text{ and } f_{s+1(t-1)t}$$

Further, we have:

$$f_{s(s+r)} = f_{s+r} \text{ via (3c) } (i = s, j = s + r, k = t) \quad \mid 1 \leq r \leq s - 1.$$  $$f_{s(s+1)} = f_{s+r} \text{ via (3d) } (i = s, j = s + 1, k = s + r) \quad \mid 2 \leq r \leq s - 1.$$  

Therefore, we are in the situation where:

$$f_{s(s+1)} = \cdots = f_{s(t-1)} = f_{s+1(t-1)t} = \cdots = f_{t-1(t-1)t}.$$  

Which with a proper choice of basis assert the claim. \qed
Remark 5. Given $\tau \in S_n$ such that $\tau \neq \mathbf{7}$. Then $\tau$ has one of the following form:

\[
\begin{aligned}
\tau &= e \\
\text{\tau is a simple transposition} \\
\text{\tau is a cycle of length } p \geq 3 \\
\text{\tau has at least two commutative cycles of length } p, q \geq 2
\end{aligned}
\]

Proposition 2.6. If $\tau = e$. Then $\dim_k \text{Ext}^1_{\Lambda}(\rho_e, \rho_\tau) = 0$.

Proof. Assuming that $\tau = e$, this would imply that $\rho_\tau(x_{ij}) = 1$ for all $1 \leq i < j \leq n$. Now we have $f_{ij} = 0$ directly via $(3a)$ which in and of itself assures the claim. $\square$

Proposition 2.7. If $\tau$ is a simple transposition. Then $\dim_k \text{Ext}^1_{\Lambda}(\rho_{id}, \rho_\tau) = 0$.

Proof. Assuming that $\tau = (s, s+1)$ for $s = 1, 2, \ldots, n$, this would imply that $\rho_\tau(x_{ij}) = 1$ for all $1 \leq i < j \leq n$ except for $(i, j) = (s, s+1)$. Now we have $f_{ij} = 0$ for all $1 \leq i < j \leq n$ except for $(i, j) = (s, s+1)$ directly via $(3a)$ which hold the claim with a proper change of basis. $\square$

Proposition 2.8. If $\tau$ is a cycle of length $p \geq 3$. Then $\dim_k \text{Ext}^1_{\Lambda}(\rho_{id}, \rho_\tau) = 0$.

Proof. Assuming that $\tau = (a_1, \ldots, a_p)$ is a cycle of length $p \geq 3$ where $1 \leq a_1, \ldots, a_p \leq n$, $a_i \neq a_j$ for all $i \neq j$ ordered such that $a_1 < a_j$ for all $2 \leq j \leq p$.

Since $a_1 < a_p$ and $\tau(a_p) = a_1 < a_2 = \tau(a_1) \implies \rho_\tau(x_{a_1 a_p}) = -1$.

We make a bases change such that $f_{a_1 a_p} = 0$.

As for the case of $i \leq a_1$, we have:

\[
\begin{aligned}
\tau(i) &= i < a_2 = \tau(a_1) \implies \rho_\tau(x_{ia_1}) = 1. & \implies & f_{ia_1} = 0 \text{ via } (3a) \\
\tau(i) &= i < a_1 = \tau(a_p) \implies \rho_\tau(x_{a_p i}) = 1. & \implies & f_{a_p i} = 0 \text{ via } (3a)
\end{aligned}
\]

As for the case of $a_p < i$, we have:

\[
\begin{aligned}
\tau(a_p) &= a_1 < i = \tau(i) \implies \rho_\tau(x_{a_p i}) = +1 \implies f_{a_p i} = 0 \text{ via } (3a) \\
\tau(a_p) &= a_1 < a_2 = \tau(a_p) \implies \rho_\tau(x_{a_1 a_p}) = -1 \implies f_{a_1 i} = 0 \text{ via } (3a)
\end{aligned}
\]

Now if $a_p = a_1 + 1$ then all possible cases for $i$ has been considered and the claim follows. If not, then for all $a_1 < i < a_p$ we have $\tau(i) > \tau(a_p)$ and hence $f_{a_1 i} = f_{i a_p}$ via $(3a)$ and we have two cases to consider here as well:
If $\rho_\tau(x_{a1i}) = 1$ then $f_{a1i} = 0$ via 3a. Otherwise, we are in the situation where:

\[ a_1 < i < a_p \text{ and } \tau(a_1) > \tau(i) > \tau(a_p) \]

which implies that $p > 3^2$ and hence $\tau$ is of the form:

\[ \tau = (a_1, \ldots, r, \ldots, k, \ldots, a_p) \mid r > k > a_1 \]

which implies that $f_{a1i} = f_{ia_p} = 0$ via 3b for $(a_1 < k), (r < a_p)$. Therefore, with all possible cases considered for $\tau$ a cycle of length $p \geq 3$ we have $\dim_k \text{Ext}_1^A(\rho_{id}, \rho_\tau) = 0$ as claimed. \hfill \Box

**Proposition 2.9.** If $\tau$ has at least two commutative cycles of length $p, q \geq 2$. Then $\dim_k \text{Ext}_1^A(\rho_{id}, \rho_\tau) = 0$.

**Proof.** Assume that $\tau$ has commutative cycles $(a_1, \ldots, a_p), (b_1, \ldots, b_q)$ for $p, q \geq 2$ such that $a_1 < a_j$ for $2 \leq j \leq p$, $b_1 < b_j$ for $2 \leq j \leq q$ and $a_1 < b_1$. We start with:

\[ a_1 < a_p \text{ and } \tau(a_p) = a_1 < a_2 = \tau(a_1) \quad \Longrightarrow \quad \rho_\tau(x_{a1a_p}) = -1 \]

\[ b_1 < b_q \text{ and } \tau(b_q) = b_1 < b_2 = \tau(b_1) \quad \Longrightarrow \quad \rho_\tau(x_{b1b_q}) = -1 \]

For convenience of reference, we denote $a_1 = s, a_p = t, b_1 = s', b_q = t'$, and consider a change of basis such that $f_{st} = 0$, further:

\[ \rho_\tau(x_{st}) = -1 \quad \Longrightarrow \quad f_{kl} = 0 \quad | \quad 1 \leq k < l \leq n; s \neq t \neq k \neq l \quad \text{via 3b} \]

\[ \rho_\tau(x_{s't'}) = -1 \quad \Longrightarrow \quad f_{kl} = 0 \quad | \quad 1 \leq k < l \leq n; s' \neq t' \neq k \neq l \quad \text{via 3b} \]

Note that the first line implies in particular $f_{s't'} = 0$. And we are left with the following cases of $f_{ss'}, f_{st'}, f_{\min(s',t')\max(s',t')}$ and $f_{\min(t',t)\max(t',t)}$. On one hand, for $f_{ss'}$, we have:

\[ \begin{align*}
\text{if } \rho_\tau(x_{ss'}) = +1 & \quad \Longrightarrow \quad f_{ss'} = 0 \text{ vie 3a} \\
\text{if } \rho_\tau(x_{ss'}) = -1 & \quad \Longrightarrow \quad f_{ss'} = 0 \text{ vie 3c} \quad (i = s, j = s', k = t')
\end{align*} \]

While on the other hand, the remaining cases are processed by differentiating possible orderings of $t, s', t'$:

If $s < s' < t < t'$, then:

\[ f_{s't} = 0 \text{ via 3c} (i = s, j = s', k = t), \]

\[ f_{t't} = 0 \text{ via 3d} (i = s', j = t', k = t), \]

\[ f_{st'} = 0 \text{ via 3d} (i = s, j = t', k = t). \]

If $s < s' < t < t'$, then we have:

\[ \tau(t) = \tau(a_p) = a_1 < b_1 = \tau(b_q) = \tau(t') \quad \Longrightarrow \quad \rho_\tau(x_{tt'}) = +1 \]

\[ \text{otherwise we get } i < a_p \text{ and } i > a_p \text{ a clear contradiction.} \]
and then:
\[ f_{\tau} = 0 \text{ via } \delta(i = t, j = t'), \]
\[ f_{s\tau} = 0 \text{ via } \delta(i = s, j = s', k = t), \]
\[ f_{st'} = 0 \text{ via } \delta(i = s, j = t, k = t'). \]

Finally, for the case of \( s < t < s' < t' \), then we have:
\[ \rho_\tau(x_{tt'}) = +1 \text{ and: } \]
\[ f_{\tau} = 0 \text{ via } \delta(i = t, j = t'), \]
\[ f_{ts'} = 0 \text{ via } \delta(i = s, j = t, k = s'), \]
\[ f_{st'} = 0 \text{ via } \delta(i = s, j = t, k = t'). \]

Therefore, with all possible cases considered for \( \tau \) an \( S_n \)-element such that it has at least two commutative cycles of length \( p, q \geq 2 \), we have \( \dim_k \text{Ext}_1^\Lambda(\rho_{id}, \rho_\tau) = 0 \) as claimed. \( \square \)

Note 2. Given \( \sigma \in S_n \), let \( \tau = \overline{g}\sigma \) for \( \overline{g} \) some non-simple transposition of \( S_n \). Then the space of extensions \( \text{Ext}_1^\Lambda(\rho_\sigma, \rho_\tau) \) is one-dimensional by Theorem 2.4, we denote the single generator of such space by \( x(\sigma; \tau) \).

Remark 6. We conclude the connectedness of the algebra \( \Lambda \) by Theorem 2.4 and the fact that the group \( S_n \) can be generated by non-simple transpositions. In other words, for all positive integers \( n \geq 3 \), the algebra \( D_n(\alpha, -\alpha) \) cannot be written as direct product of two non-trivial algebras.

The special case of \( n = 4 \). We consider the special case of \( \Lambda = D_4(\alpha, -\alpha) \) where the \( K \) parameter \( \alpha \) remain normalized to \( 1_K \). Furthermore, we set \( t_1 = (1, 3), t_2 = (1, 4) \) and \( t_3 = (2, 4) \) the non-simple transpositions of \( S_4 \).

2.0.1. Quiver-presentation of \( \Lambda \). As the Jacobson radical of the finite-dimensional algebra \( \Lambda \) is generated by the commutator, we deduce that the algebra \( \Lambda \) is basic with a complete system of simple representations: \( \{ \rho_\sigma \mid \sigma \in S_4 \} \), moreover, Remark 6 implies that the algebra \( \Lambda \) is connected. Therefore, by Theorem 2.4 we conclude that the ordinary quiver of \( \Lambda \) denoted by \( Q_\Lambda \) has a vertices set of the form \( \{ \sigma \mid \sigma \in S_4 \} \), and there exists an arrow from \( \sigma \) to \( \tau \) labeled by \( \alpha(\sigma; \tau) \) if and only if \( \tau = t_i \sigma \) for \( i = 1, 2, 3 \).

Denote by \( \phi \) Gabriel’s theorem morphism associated with \( \Lambda \). By Gabriel’s Theorem we deduce that \( \Lambda \cong KQ_\Lambda/\ker(\phi) \), furthermore, we observe that:
\[ KQ_\Lambda = \bigoplus_{\sigma \in S_4} e_\sigma KQ_\Lambda \]
This in particular implies that from the viewpoint of representation theory, the study of $\Lambda$ can be reduced to that of $\Gamma$ the indecomposable projective $\Lambda$-representation understood to be a quotient of $e_\nu KQ_\Lambda$ by the kernel of $\pi = \phi_{\nu e}$.

**Remark 7.** When considered in terms of Gabriel’s theorem, the action of the symmetric group $S_4$ on $\Lambda$ is understood for any two arrows $\alpha_1, \alpha_2$ as $\sigma(\alpha_1 \alpha_2) = \sigma \alpha_1 \sigma \alpha_2$ where:

$$\sigma \alpha(\tau_1; \tau_2) = \alpha(\sigma \tau_1; \sigma \tau_2) \quad | \quad \tau_1, \tau_2, \sigma \in S_4$$

2.0.2. *Quiver-representation of $gr \Gamma$.*

**Proposition 2.10.**

$$r_i := \alpha(e; t_i) \alpha(t_i; e) \in ker(gr \pi) \quad | \quad i = 1, 2, 3.$$  

**Proof.** Consider the $e_\nu KQ_\Lambda$-module $M$ given as the quotient by the (two-sided) ideal generated by:

$$\{e_\nu, rad^3 KQ_\Lambda, rad^2 KQ_\Lambda e_\sigma \mid e \neq \sigma \in S_4\}$$

we remark here that $M$ exists as a $e_\nu KQ_\Lambda/ker(gr \pi)$-module if and only if the following (graded) algebra map:

$$\rho : \Lambda \rightarrow End(K^5)$$

$$x_{st} \mapsto \rho(x_{st}) = \begin{bmatrix}
\rho_c(x_{st}) & x_{st}(e; t_1) & x_{st}(e; t_2) & x_{st}(e; t_3) & g_{st}(e; e) \\
0 & \rho_1(x_{st}) & 0 & 0 & x_{st}(t_1; e) \\
0 & 0 & \rho_2(x_{st}) & 0 & x_{st}(t_2; e) \\
0 & 0 & 0 & \rho_3(x_{st}) & x_{st}(t_3; e) \\
0 & 0 & 0 & 0 & \rho_e(x_{st})
\end{bmatrix}$$

exists up to the third power of the radical as a $\Lambda$-representation, that is, if and only if:

$$x_{ij}(e; t_1) x_{ij}(t_i; e) + rad^3 \Lambda = 0 \mid i = 1, 2, 3.$$  

that is, if and only if:

$$gr \pi(\alpha(e; t_i) \alpha(t_i; e)) = 0 \mid i = 1, 2, 3.$$  

asserting the claim. \hfill \Box

An identical method of argument as before proposes the following:

**Proposition 2.11.** *The following is of $ker(gr \pi)$*

$$r_3 := \alpha(e; t_1) \alpha(t_1; t_1 t_3) - \alpha(e; t_3) \alpha(t_3; t_1 t_3)$$

$$r_4 := \alpha(e; t_1) \alpha(t_1; t_1 t_2) \alpha(t_1 t_2; t_1 t_2 t_1) - \alpha(e; t_2) \alpha(g_2; t_2 t_1) \alpha(t_2 t_1; t_2 t_1)$$

where $g_{st}(e; e)$ a set of $K$-parameters determined by the defining relations of $\Lambda$.  

---

3 where $g_{st}(e; e)$ a set of $K$-parameters determined by the defining relations of $\Lambda$
Remark 8. Our computations suggest that there exists no further non-trivial relations of length \( n \geq 4 \). In other words, we conclude that:

\[
\ker(gr \pi) = \{ \sigma r_i, \ | \ \sigma \in S_4, i = 1, \ldots, 4 \}.
\]

For \( i = 1, 2, 3 \), we rename paths \( \alpha(\sigma; t_i \sigma) \) to \( s_i \). This, along corresponding paths composition to the obvious multiplication implies that we may further identify the algebra \( gr \Gamma \) with that of the bounded free associative algebra \( K \langle s_1, s_2, s_3 \rangle \). In other words:

**Corollary 2.** Up to a higher power of the radical, the algebra \( gr \Gamma \) is isomorphic to the nil-Coxeter algebra associated with \( S_4 \), that is:

\[
gr \Gamma \cong K \langle s_1, s_2, s_3 \rangle / ker(gr \pi) \mid ker(gr \pi) = \begin{cases} s_i^2 & | i = 1, 2, 3 \\ s_1s_3 - s_3s_1 & \end{cases}
\]

2.0.3. Quiver-representation of \( \Gamma \).

**Proposition 2.12.** The algebra \( \Gamma \) is isomorphic to the nil-Coxeter algebra associated with \( S_4 \).

**Proof.** This is a direct consequence of the deletion property of Coxeter systems combined with the fact that elements of the ideal \( ker(gr \pi) \) are minimal and maximal in the precise length sense. \( \square \)

3. Part 02: Representation theory of \( D_4(\alpha, +\alpha) \).

For convenience, we denote \( \Lambda := D_4(\alpha, \alpha) \), the \( K \)-parameter \( \alpha \) remains normalized to \( 1_K \).

**Lemma 3.1.** The algebra \( \Lambda \) is not basic.

**Proof.** Assume for a contradiction that the algebra \( \Lambda \) is basic, that is, every simple \( \Lambda \)-module is one-dimensional say of the form:

\[
\rho : \Lambda \to K \\
x_{ij} \mapsto \rho(x_{ij}) = y_{ij}
\]

On one hand, \([2a]\) implies that \( y_{ij}^2 = 1 \), that is, \( y_{ij} = \pm 1 \) for all \( 1 \leq i < j \leq 4 \). On the other hand \([2c]\) implies that \( y_{ij}y_{jk} - (y_{jk}y_{ik} + y_{ik}y_{ij}) = 1 \) for \( 1 \leq i < j < k < 4 \). We discuss:

As for the case of \( y_{ij}y_{jk} = +1 \) we get that:

\[
y_{ij} = y_{jk} \text{ and } y_{jk}y_{ik} + y_{ik}y_{ij} = 0
\]

While the case of \( y_{ij}y_{jk} = -1 \) implies:

\[
y_{ij} = -y_{jk} \text{ and } y_{jk}y_{ik} + y_{ik}y_{ij} = -2
\]

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A clear contradiction on both cases proving that indeed, the algebra $\Lambda$ is no basic. □

Remark 9. Lemma 3.1 implies the existence of a basic $K$-algebra $\Lambda^b$ such that Morita equivalent to $\Lambda$, that is:

$$\text{mod}\Lambda \xrightarrow{\cong_{\mathcal{M}}} \text{mod}\Lambda^b \xrightarrow{\cong_{\mathcal{M}}} \text{mod}\Lambda$$

In particular, such equivalence preserves simplicity and exactness.

**Proposition 3.2.** Given $\sigma \in S_4$. The algebra homomorphism $\rho_\sigma : \Lambda \to K^{2\times 2}$ defined by mapping a generator $x_{ij}$ for $1 \leq i \neq j \leq 4$ to:

$$\rho_\sigma(x_{ij}) = \begin{cases}
+1 & | \sigma(i, j) \in \{(1, 3), (4, 2)\} \\
0 & | \sigma(i, j) \in \{(4, 1), (s, s+1) \mid s = 1, 2, 3\} \\
0 & | \sigma(i, j) \in \{(1, 3), (4, 2)\}
\end{cases}$$

is a well-defined two-dimensional simple $\Lambda$-representation.

**Proof.** For the claim to hold, one must verify that subjecting $\rho_\sigma$ to the defining relations does not yield any contradictions for all $\sigma \in S_4$. As 2a and 2b holds directly, one remark that 2c holds by observing that for $1 \leq \sigma(i) \neq \sigma(j) \neq \sigma(k) \leq 4$, then one of the following situations occurs:

$$\rho_\sigma(x_{ij}x_{jk}) = 1 \quad \rho_\sigma(x_{jk}x_{ki}) = 1 \quad \rho_\sigma(x_{ki}x_{ij}) = 1$$

which in and of itself assert the claim. □

**Lemma 3.3.** Let $\tau, \sigma \in S_4$. Then for all $x \in \Lambda$ we have:

$$(\tau \cdot \rho_\sigma)(x) = (\rho_\sigma(\tau^{-1}x)) = \rho_{\sigma \tau^{-1}}(x).$$

**Proof.** As $\rho_\tau$, $\rho_{\tau \sigma}$ and the group action by $S_4$ is multiplicative, the claim is asserted by remarking that:

$$(\tau \cdot \rho_\sigma)(x_{ij}) = (\rho_\sigma(\tau^{-1}x_{ij})) = \rho_{\sigma \tau^{-1}}(x_{ij})$$

for all $x_{ij}$ generating $\Lambda$. □

**Question:** On the existence of $\mu$ a 2-dimensional simple $D_5(+1, +1)$-representation such that $\mu(x_{12}) = \rho_c(x_{12})$. We deduce that $\mu(x_{45}) = \pm \mu(x_{12})$, which in and of itself implies that $\mu(x_{23}) = \pm \mu(x_{13}) = \pm \mu(x_{12})$ by 2a and 2b contradicting 2c. In other words $\{\rho_\sigma \mid \sigma \in S_4\}$ does not extents to a complete system of simple 2-dimensional
\(D_5(\pm 1, \pm 1)\)-representations, and the question of computing simple \(D_5(\pm 1, \pm 1)\)-representations remains open for the moment.

\textbf{Note 3.} Denote by \(V\) the Klein four-subgroup of the symmetric group \(S_4\), that is, the subgroup generated by the permutations \(\nu_1 = (13)(24)\) and \(\nu_2 = (14)(23)\), further, we set \(\nu_3 = \nu_1 \nu_2\). One may assume with no loss of generality that \(V\) fixes 1 to realize the associated quotient group as the symmetric group on three letters \(\{2, 3, 4\}\) which we denote by \(S_3\).

\textbf{Proposition 3.4.} Let \(\sigma \in S_4\), then \(\rho_\sigma \cong \nu_i \rho_\sigma\) via conjugation with \(m_{\sigma \nu_\sigma}^{-}\) for \(i = 1, 2, 3\) where:

\[
\begin{align*}
    m_{\nu_1} := & \begin{bmatrix} 0 & +1 \\ +1 & 0 \end{bmatrix} & \quad m_{\nu_2} := & \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix} & \quad m_{\nu_3} := & \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}
\end{align*}
\]

\textbf{Proof.} Given any \(x \in \Lambda\), then:

\[
\nu_i \rho_\sigma(x) = \rho_{\sigma \nu_i^{-}}(x) = \rho_{\sigma \nu_\sigma - \sigma}(x) = \sigma^{-} \rho_{\sigma \nu_\sigma}(x).
\]

Now \(\sigma \nu_\sigma^{-} \in V\) since \(V\) is a normal subgroup of \(S_4\), in other words, \(\sigma \nu_\sigma^{-} = \nu_j\) for some \(j = 1, 2, 3\). The claim is then asserted since \(\rho_{\nu_j} \cong \rho_e\) via \(m_{\nu_j}\), that is:

\[
\rho_{\nu_j}(x) = m_{\nu_j} \rho_e(x) m_{\nu_j}^{-} \quad \mid j = 1, 2, 3.
\]

\[\Box\]

Our aim in at this point is to prove the following theorem:

\textbf{Theorem 3.5.} Let \(\sigma, \tau \in \overline{S_3}\), then:

\[
Ext_{\Lambda}^1(\rho_\sigma, \rho_\tau) \cong Ext_{\Lambda}^1(\mu \rho_\sigma, \mu \rho_\tau) \quad \mid \mu \in S_4.
\]

Furthermore, we have:

\[
\dim_K Ext_{\Lambda}^1(\rho_\sigma, \rho_\tau) = \begin{cases} 
2 & \tau = \sigma \cdot s_2^- \\
1 & \tau = \sigma \cdot s_3^- \\
0 & \text{otherwise}. 
\end{cases}
\]

We discuss the first part of the theorem as follows:

\textbf{Proposition 3.6.} Let \(\sigma, \tau \in \overline{S_3}\). Then:

\[
Ext_{\Lambda}^1(\rho_\sigma, \rho_\tau) \cong Ext_{\Lambda}^1(\mu \rho_\sigma, \mu \rho_\tau) \quad \mid \mu \in S_4.
\]

\textbf{Proof.} The claim is a natural consequence of Lemma 3.3 induced by the group action. In details, given \(f \in Z^1(\rho_\sigma, \rho_\tau)\) and \(\mu \in S_4\) then for all \(x_{ij}\) generating \(\Lambda\) we have:

\[
\mu(f) \in Z^1(\mu \rho_\sigma, \mu \rho_\tau) \mid \mu f(x_{ij}) = f(\mu^{-1} x_{ij})
\]
a well-defined (1-)cocycles, that is a coboundary if and only if \( f \) is.

**Remark 10.** The special case of \( \mu \in V \) in Proposition 3.6 implies that the isomorphism as described is a self-inverse. Furthermore, such isomorphism can be realized by means of Proposition 3.4, which implies that for an arbitrary \( f \in Z^1(\rho_\sigma, \rho_\tau) \), then for all \( x_{ij} \) generating \( \Lambda \) we have:

\[
v(f) \in Z^1(\mu_\rho_\sigma, \mu_\rho_\tau) \mid v(f)(x_{ij}) = m_{\sigma\rho_\sigma}^{-1} f(x_{ij}) m_{\tau\rho_\tau}^{-1}
\]

a well-defined (1-)cocycles that is a coboundary if and only if \( f \) is.

**Note 4.** We understand arbitrary \( \overline{f} \in Ext^1_\Lambda(\rho_e, \sigma\rho_e) \), as a class of the form \( f + B^1(\rho_e, \sigma\rho_e) \) where \( f : \Lambda \to K^{2,2} \) maps a generator \( x_{ij} \) to:

\[
f(x_{ij}) = \begin{bmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{bmatrix} \mid f(xy) = \rho_e(x)f(y) + f(x)\sigma\rho_e(y).
\]

Furthermore, we denote basis changing matrices of \( K^{2,2} \) by:

\[
\lambda = \begin{bmatrix} \lambda_1 & \lambda_3 \\ \lambda_2 & \lambda_4 \end{bmatrix} \mid \lambda_1, \cdots, \lambda_4 \in K.
\]

**Proposition 3.7.** Given \( \sigma \in S_3 \) a non-Coxeter generator. Then the space of extensions \( Ext^1_\Lambda(\rho_e, \sigma\rho_e) \) is of null-dimension.

**Proof.** Given an arbitrary \( \overline{f} \in Ext^1_\Lambda(\rho_e, \sigma\rho_e) \):

The case of \( \sigma = e \) is then resolved by a change of basis of which we set:

\[
2\lambda_1 = 0 \quad 2\lambda_2 = +c_{13} \quad 2\lambda_3 = +c_{13} - 2a_{12} \quad 2\lambda_4 = +2c_{12}
\]

which would imply that:

\[
\begin{align*}
f(x_{12}) & = f(x_{34}) = 0 \quad \text{via } 2a, 2b \\
f(x_{13}) & = f(x_{24}) = 0 \quad \text{via } 2a, 2c : (i = 1, j = 2, k = 3), \\
f(x_{14}) & = f(x_{23}) = 0 \quad \text{via } 2a, 2b
\end{align*}
\]

The case of \( \sigma = s_2s_3s_2 \) is resolved by setting:

\[
2\lambda_1 = +a_{24} \quad 2\lambda_2 = +c_{13} \quad 2\lambda_3 = -b_{13} \quad 2\lambda_4 = -d_{24}
\]

which would then imply that:

\[
\begin{align*}
f(x_{13}) & = f(x_{24}) = 0 \quad \text{via } 2a \\
f(x_{12}) & = f(x_{23}) = 0 \quad \text{via } 2a, 2c : (i = 1, j = 2, k = 3, 4), \\
f(x_{14}) & = f(x_{34}) = 0 \quad \text{via } 2a, 2b
\end{align*}
\]

The case of \( \sigma = s_3s_2 \) is resolved by setting:

\[
2\lambda_1 = +a_{14} + a_{23} \quad 2\lambda_2 = +a_{14} - a_{23} \\
2\lambda_3 = (+a_{14} + a_{23}) + 2a_{13} \quad 2\lambda_4 = (-a_{14} + a_{23}) + 2c_{13}
\]
which would then imply that:
\[
\begin{align*}
  f(x_{13}) &= f(x_{24}) = 0 & \text{via } 2a, \\
  f(x_{14}) &= f(x_{23}) = 0 & \text{via } 2a, 2c: (i = 1, j = 2, k = 3, 4), \\
  f(x_{34}) &= 0 & \text{via } 2a, 2c: (i = 1, 2, j = 3, k = 4), \\
  f(x_{12}) &= 0 & \text{via } 2a, 2c: (i = 1, j = 4, k = 2).
\end{align*}
\]

Finally, the case of \( \sigma = s_2 s_3 \) is resolved by setting:
\[
\begin{align*}
  2\lambda_1 &= -a_{13} + a_{24} & 2\lambda_2 &= +c_{13} - c_{24} \\
  2\lambda_3 &= -a_{13} - a_{24} & 2\lambda_4 &= -c_{13} - c_{24}
\end{align*}
\]
which would imply:
\[
\begin{align*}
  f(x_{13}) &= f(x_{24}) = 0 & \text{via } 2a, \\
  f(x_{12}) &= 0 & \text{via } 2a, 2c: (i = 1, j = 2, 4, k = 3, 2), \\
  f(x_{34}) &= 0 & \text{via } 2a, 2b, \\
  f(x_{14}) &= 0 & \text{via } 2a, 2c: (i = 1, j = 2, 4, k = 4, 3), \\
  f(x_{23}) &= 0 & \text{via } 2a, 2c: (i = 2, 1, j = 3, k = 4, 2).
\end{align*}
\]

In other words, up to an isomorphism, generic elements of the space \( Ext^1_\Lambda(\rho_e, \rho_\sigma) \) are trivial and therefore, the space in and of itself is of null-dimension as claimed for all \( \sigma \) non-Coxeter generator. \( \square \)

**Proposition 3.8.** The space of extensions \( Ext^1_\Lambda(\rho_e, \sigma \rho_e) \) is of one-dimensional for \( \sigma = s_3 \), while is two-dimensional for \( \tau = s_2 \).

**Proof.** Given an arbitrary \( f \in Ext^1_\Lambda(\rho_e, \sigma \rho_e) \): The case of \( \sigma = s_3 \) is resolved by a change of basis of which we configure:
\[
\begin{align*}
  \lambda_2 &= -\lambda_1 + a_{14} & \lambda_3 &= -\lambda_1 - a_{13} & \lambda_4 &= -\lambda_1 + a_{14} - c_{13}
\end{align*}
\]
where for \( 2\beta_2 = -b_{12} - a_{13} + c_{13} + a_{34} \), we set:
\[
2\lambda_1 = +b_{12} + a_{14} - c_{13} + \beta_2 = -a_{13} + a_{14} + a_{34} - \beta_2
\]

This would imply that:
\[
\begin{align*}
  f(x_{13}) &= f(x_{24}) = 0 & \text{via } 2a, 2b, \\
  f(x_{14}) &= f(x_{23}) = 0 & \text{via } 2a, 2b, 2c: (i = 1, j = 2, k = 3).
\end{align*}
\]

Furthermore, \( 2a \) conclude that:
\[
\begin{align*}
  f(x_{12}) &= \beta_2, \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} e(x_{34}) &= \beta_2, \begin{bmatrix} +1 & 0 \\ 0 & +1 \end{bmatrix}
\end{align*}
\]

Where all the other defining relations of \( \Lambda \) are satisfied. In other words, up to an isomorphism, the space \( Ext^1_\Lambda(\rho_e, s_3 \rho_e) \) has a generating set of
the form: \( \{ \overline{f}_t = f_t + B^1(\rho_e, s_3\rho_e) \} \) where \( f_t \) is defined by mapping the generators of \( \Lambda \) as follows:

\[
f_{2}(x_{12}) = \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}, \quad f_{2}(x_{34}) = \begin{bmatrix} +1 & 0 \\ 0 & +1 \end{bmatrix}, \quad f_{2}(x_{ij}) = 0 \mid \text{otherwise.}
\]

As for the case of \( \sigma = s_2 \), we set:

\[
2\lambda_1 = +a_{12} - a_{34} \quad \quad 2\lambda_2 = -a_{12} - a_{34} \\
2\lambda_3 = -b_{12} + b_{34} \quad \quad 2\lambda_4 = -b_{12} - b_{34}
\]

which would imply that:

\[
\begin{align*}
f(x_{12}) &= f(x_{34}) = 0 \mid \text{via 2a} \\
f(x_{13}) &= 0 \mid \text{via 2a 2b: } (i = 1, j = 2, k = 3, 4), \\
f(x_{24}) &= 0 \mid \text{via 2a 2c: } (i = 1, 2, j = 4, k = 2),
\end{align*}
\]

further:

\[
\begin{align*}
f(x_{14}) &= \beta_1.\text{diag}(+1, -1) \mid \text{via 2a 2c: } (i = 1, j = 2, k = 4), \\
f(x_{23}) &= \beta_3.\text{diag}(+1, +1) \mid \text{via 2a 2c: } (i = 2, j = 3, k = 4).
\end{align*}
\]

where the \( K \)-parameters \( \beta_1, \beta_3 \) are generically given as:

\[
2\beta_1 = a_{12} - b_{12} + a_{34} + b_{34} + 2a_{14}, \quad 2\beta_3 = -a_{12} - b_{12} - a_{34} + b_{34} + 2a_{23}
\]

and all the other defining relations of \( \Lambda \) are satisfied. In other words, up to an isomorphism, the space \( \text{Ext}^1_\Lambda(\rho_e, s_2\rho_e) \) has a generating set of the form: \( \{ \overline{f}_t = f_t + B^1(\rho_e, s_2\rho_e) \mid t = 1, 3. \} \) where \( f_t \) are defined by mapping the generators of \( \Lambda \) as follows:

\[
\begin{align*}
f_1(x_{14}) &= \text{diag}(+1, -1) \\
f_3(x_{23}) &= \text{diag}(+1, +1) \\
f_t(x_{ij}) &= 0 \mid \text{otherwise.}
\end{align*}
\]

\[\square\]

**Corollary 3.** Let \( \sigma, \tau \in \mathfrak{S}_3 \). Then:

\[
\dim_K \text{Ext}^1_\Lambda(\rho_\sigma, \rho_\tau) = \begin{cases} 2 & \tau = \sigma s_2^- \\
1 & \tau = \sigma s_3^- \\
0 & \text{otherwise.}
\end{cases}
\]

**Remark 11.** We set \( \{ \overline{g}_t = g_t + B^1(\nu_1\rho_e, \nu_1\rho_{s_2}) \mid t = 1, 3. \} \) a generating set of the space \( \text{Ext}^1_\Lambda(\nu_1\rho_e, \nu_1\rho_{s_2}) \), where \( g_t \) are defined by mapping the
generators of $\Lambda$ as follows:

\[
\begin{align*}
    g_1(x_{14}) &= \text{diag}(+1, +1) \\
    g_3(x_{23}) &= \text{diag}(+1, -1) \\
    g_t(x_{ij}) &= 0 \quad \text{otherwise.}
\end{align*}
\]

and remark that:

\[
\begin{align*}
    f_1 &\cong \begin{cases} 
    -g_3 & \text{via } \nu_1 \\
    -g_1 & \text{via } v \mid \mu = \nu_1
    \end{cases} \\
    f_3 &\cong \begin{cases} 
    -g_1 & \text{via } \nu_1 \\
    +g_3 & \text{via } v \mid \mu = \nu_1
    \end{cases}
\end{align*}
\]

This, in particular, indicates that the induced action of the $V$ (and that of $S_4$ in general) does not extend to an action onto $\Lambda$-extensions. In other words, the algebra $\Lambda^b$ is not invariant under the action of $V$.

Note 5. The space of extensions $Ext^1_{\Lambda^b}(\mathfrak{F}(\rho_e), \mathfrak{F}(\rho_{s_2}))$ is two-dimensional, we set $\{\mathfrak{F}(f_1), \mathfrak{F}(f_3)\}$ a generating set. Similarly, we set $\{\mathfrak{F}(f_2)\}$ a generating set of the one-dimensional space $Ext^1_{\Lambda^b}(\mathfrak{F}(\rho_e), \mathfrak{F}(\rho_{s_3}))$.

3.0.1. Quiver-presentation of $\Lambda^b$. Now that we have verified that the algebra $\Lambda^b$ is basic, connected and finite-dimensional, then Theorem 3.5 supported by definition implies that $Q_\Lambda$ the ordinary quiver of $\Lambda$ (and that of $\Lambda^b$ since $\Lambda$ and $\Lambda^b$ are Morita equivalent) has the following shape:

\[
\begin{align*}
    s_2 &\xrightarrow{\beta_2(s_3; s_3 s_2)} s_3 s_2 \\
    e &\xrightarrow{\beta_3(s_3; s_3 s_2)} s_3 s_2 s_3 \\
    s_3 &\xrightarrow{\beta_1(s_3; s_3 s_2 s_3)} s_3 s_2 s_3 \\
    s_2 &\xrightarrow{\beta_1(s_3; s_3 s_2 s_3)} s_3 s_2 s_3
\end{align*}
\]

Note 6. We simply write $\beta_1, \beta_2, \beta_3$ if no confusions occurs.

Denote by $\phi$ Gabriel’s theorem morphism associated with $\Lambda^b$. By Gabriel’s Theorem we deduce that $\Lambda \cong KQ_\Lambda/\ker(\phi)$, furthermore, we observe that:

\[
KQ_{\Lambda^b} = \bigoplus_{\sigma \in S_3} e_\sigma KQ_\Lambda
\]

This in particular implies that from the viewpoint of representation theory, the study of $\Lambda^b$ can be reduced to that of $\Gamma$ the indecomposable projective $\Lambda^b$-representation understood to be a quotient of $e_\sigma KQ_\Lambda$ by the kernel of $\pi = \phi|_{e_\sigma}$.
3.0.2. Quiver-representation of $\text{gr} \Gamma$.

**Proposition 3.9.** $\text{gr}(\pi)(r_t) = 0 \mid t = 1 \cdots, 4$, where:

- $r_t = \beta_t^2 \mid t = 1, 2, 3.$
- $r_4 := \beta_1 \beta_3 - \beta_3 \beta_1.$

**Proof.** Consider the $e_e KQ_\Lambda$-module given as the quotient by the (two-sided) ideal generated by:

$$\{ e_{e \text{rad}}^3 KQ_\Lambda, e_{e \text{rad}}^2 KQ_\Lambda e_\sigma \mid e \neq \sigma \in S_3 \}$$

The claim follows by verify that such module exists as an $e_e KQ_\Lambda / \ker(\text{gr} \pi)$-module if and only if up to the third power of the radical we have:

- $\mathcal{F}(f_i^2) = 0 \mid i = 1, 2, 3.$
- $\mathcal{F}(f_1 f_3 - f_3 f_1) = 0$

that is, if and only if up to the third power of the radical, we have:

- $f_i^2 = 0 \mid i = 1, 2, 3.$
- $f_1 f_3 - f_3 f_1 = 0$

that is, if and only if:

$$\text{gr} \pi(r_i) = 0 \mid i = 1, \cdots, 4.$$ 

which asserts the claim. \qed

An identical method of argument as before proposes the following:

**Proposition 3.10.** $\text{gr}(\phi)r_t = 0 \mid t = 5, 6$, where:

- $r_5 := \beta_2 \beta_1 \beta_2 - \beta_1 \beta_2 \beta_3 - \beta_3 \beta_2 \beta_1$
- $r_6 := \beta_2 \beta_3 \beta_2 - \beta_1 \beta_2 \beta_1 + \beta_3 \beta_2 \beta_3$

**Remark 12.** Our computations suggest that there exists no further non-trivial relations of length $n \geq 4$. In other words, we conclude that:

$$\ker(\text{gr} \pi) = \{ \sigma.r_t \mid \sigma \in S_3, t = 1, \cdots, 6. \}$$

identifying the path algebra $e_e KQ_\Lambda$ with the free associative algebra $K\langle s_1, s_2, s_3 \rangle$ by corresponding path composition to the obvious multiplication yield the following:

**Corollary 4.** Up to a higher power of the radical, following hold:

$$\text{gr} \Gamma \cong K\langle s_1, s_2, s_3 \rangle / \ker(\text{gr}(\pi)) = \begin{cases} 
  t_i := s_t^2, & t = 1, 2, 3. \\
  t_4 := s_1 s_3 - s_3 s_1, \\
  t_5 := s_2 s_1 s_2 - s_1 s_2 s_3 - s_3 s_2 s_1, \\
  t_6 := s_2 s_3 s_2 - s_1 s_2 s_1 + s_3 s_2 s_3. 
\end{cases}$$
In particular, we find that the algebra $gr\Gamma$ is 24-dimensional. The following corollary provides a basis:

**Corollary 5.** The following set of polynomials form a basis of the algebra $gr\Gamma$:

\[
\begin{align*}
&l = 1 : (s_3, s_1, s_2) \\
&l = 2 : (s_3s_1, s_3s_2, s_1s_2, s_2s_3, s_2s_1) \\
&l = 3 : \left(\frac{3\pi s_1s_2, s_3s_2s_3, s_3s_2s_1}{s_1s_2s_3, s_1s_2s_1, s_2s_3s_1}\right) \\
&l = 4 : \left(\frac{3\pi s_1s_2s_3s_1s_2, s_3s_2s_3s_1, s_1s_2s_3s_1, s_2s_3s_1s_2}{s_3s_2s_3s_1, s_1s_2s_3s_1, s_2s_3s_1s_2}\right) \\
&l = 5 : \left(\frac{3\pi s_1s_2s_3s_1}{s_3s_2s_3s_1s_2, s_1s_2s_3s_1}\right) \\
&l = 6 : (s_3s_1s_2s_3s_1s_2)
\end{align*}
\]

**Remark 13.** Remark that the basis elements $u_1 := s_2s_3s_1s_2, u_2 := (s_3s_1s_2)^2$ corresponds to paths $p_1, p_2$ (respectively) such that:

\[
s(p_1) = s(p_2) = t(p_1) = t(p_2) \quad \quad l(p_1), l(p_2) > 2
\]

3.0.3. **Quiver-representation of $\Gamma$.** Remark 13 along with the fact that $\Gamma \cong K\langle s_1, s_2, s_3\rangle/\ker(\pi)$ implies the existence of some $K$-polynomials $q_i$ such that:

\[
\Gamma = K\langle s_1, s_2, s_3\rangle/\ker(\pi) = \left\{ \begin{array}{l}
\lambda_1 + q_1u_1 + q_2u_2, \\
\lambda_2 + q_3u_1 + q_4u_2, \\
\lambda_3 + q_5u_1 + q_6u_2, \\
\lambda_4 + q_7u_1 + q_8u_2, \\
\lambda_5 = s_2s_1s_2 - s_1s_2s_3 - s_3s_2s_1, \\
\lambda_6 = s_2s_3s_2 - s_1s_2s_1 + s_3s_2s_3
\end{array} \right\}
\]

**Remark 14.** Corollary 5 implies on one hand that:

\[
\pi(s_1\lambda_4 - \lambda_1s_3) = q_7(s_1s_2s_3s_1s_2) + q_1(s_3s_2s_3s_1s_2) = 0 \\
\pi(s_1\lambda_3 - \lambda_4s_3) = q_5(s_1s_2s_3s_1s_2) + q_7(s_3s_2s_3s_1s_2) = 0
\]

that is, $q_1 = q_5 = q_7 = 0$. Further, we remark for $q$ any non-trivial $K$-polynomials that:

\[
t_1 := s_1 + q_3s_2s_3s_1s_2 \quad \Rightarrow \quad t_1^2 = s_1^2 + 2qu_2 \\
t_2 := s_2 + q_3s_1s_2 \quad \Rightarrow \quad t_2^2 = s_2^2 + qu_1 + q^2u_2
\]

We remark that there exists no basis elements $u$ such that setting:

\[
t_3 := s_3 + qu \quad \Rightarrow \quad t_3^2 = s_3^2 + q'u_2
\]

Therefore, we set $t_3 := s_3$ and propose:
Proposition 3.11. The following hold:

\[ \Gamma = \langle t_1, t_2, t_3 \rangle / \ker(\pi) = \begin{cases} 
  t_1^2, t_2^2, t_3^2 + q_1 u_2, \\
  t_1 t_3 - t_3 t_1 + q_2 u_2, \\
  t_2 t_1 t_2 - t_1 t_2 t_3 - t_3 t_1 t_2, \\
  t_2 t_3 t_2 - t_1 t_2 t_3 + t_3 t_2 t_3 
\end{cases} \]

for \( q_1, q_2 \) two \( K \)-polynomials.

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