In this paper, we study convex optimization using a very general formulation called BSGD (Block Stochastic Gradient Descent). At each iteration, some but not necessary all components of the argument are updated. The direction of the update can be one of two possibilities: (i) A noise-corrupted measurement of the true gradient, or (ii) an approximate gradient computed using a first-order approximation, using function values that might themselves be corrupted by noise. This formulation embraces most of the currently used stochastic gradient methods. We establish conditions for BSGD to converge to the global minimum, based on stochastic approximation theory. Then we verify the predicted convergence through numerical experiments. Our results show that when approximate gradients are used, BSGD converges while momentum-based methods can diverge. However, not just our BSGD, but also standard (full-update) gradient descent, and various momentum-based methods, all converge, even with noisy gradients.

1 Introduction

1.1 Brief Literature Review

Suppose $J : \mathbb{R}^n \to \mathbb{R}$ is a $C^2$ strictly convex function, and assume without loss of generality that its global minimum is at $\theta^* = 0$. We discuss two types of iterative methods for computing the minimum. In the first type, every component of the current guess is updated at each time step, which might be called “synchronous updating” for want of a better phrase. In this highly structured situation, it is possible to analyze the behavior of the algorithms to great depth using results from convex analysis. In the second, the role of measurement errors / noise is explicitly taken into account, and the analysis is based on stochastic approximation theory. In this subsection, we give a brief summary of the relevant literature for each type of method. Given that the literature in convex optimization is vast, our review is quite narrowly focused on the papers that are directly relevant to the specific class of problems studied here.
Since $J(\cdot)$ is strictly convex, a simple gradient descent algorithm applied to error-free measurements of the gradient $f(\theta) = -\nabla J(\theta)$ will converge to $\theta^*$. Further, momentum-based accelerated algorithms such as those of Nesterov [1], Sutskever et al. [2], and Bengio [3] will achieve still faster convergence compared to steepest descent. In [4], the performance of these algorithms is analyzed for the case where the gradients are corrupted by noise. It is shown that the allowed bound on the noise depends on the condition number $K$ for the case where the gradients are corrupted by noise. It is shown that the allowed bound on the noise depends on the condition number $K$ for the case where the gradients are corrupted by noise. It is shown that the allowed bound on the noise depends on the condition number $K$ for the case where the gradients are corrupted by noise. It is shown that the allowed bound on the noise depends on the condition number $K$ for the case where the gradients are corrupted by noise.

In Deep Neural Networks, computing the gradient $\nabla J(\cdot)$ can be computationally intensive. Therefore another approach is to approximate the gradient by first-order differences, and then invoke stochastic approximation theory initiated in [7]. As proposed in [8] and improved in [9], one chooses the index $i$ to be updated, and then computes the quantity

$$y_{t+1,i} := \frac{J(\theta_t - c_t e_i) + \xi^{-}_{t+1,i} - (J(\theta_t + c_t e_i) + \xi^{+}_{t+1,i})}{2c_t}$$

as an approximation to $-\nabla J(\theta_t)_i$. Here $c_t$ is called the “increment,” and $\xi^{-}_{t+1,i}, \xi^{+}_{t+1,i}$ are measurement errors that corrupt in (1) can be corrupted by noise. In [9] it is shown that, under suitable conditions, the sequence of iterations $\{\theta_t\}$ converges almost surely to the unique stationary point of $J(\cdot)$. It is worth noting that the analysis in [9] does not assume that $J(\cdot)$ is convex, only that there is a unique stationary point. The above approach is sharpened in [10] to the case where the function $J(\cdot)$ is not necessarily bounded below, and/or there are multiple stationary points. However, the iterations use a noise-corrupted exact gradient, not an approximate gradient as in (2). Note that computing this approximate gradient requires either $2n$ [8] or $n + 1$ [9] function evaluations at each iteration. An alternative that requires just a few function evaluations at each iteration (perhaps just one evaluation), consists of choosing a random vector $\phi_t \in \mathbb{R}^n$ at each iteration, and setting

$$\nabla J(\theta) \approx \frac{J(\theta + c_t \phi) - J(\theta)}{c_t} \phi.$$  

This approach was (apparently) first proposed in [11], and studied further in [12, 13]. In (2), the probability distribution of $\phi$ is chosen to ensure that every component is non-zero with probability one. The analysis in [13] assumes that there are no measurement errors in computing $J(\cdot)$. In contrast, the analysis in [12] follows earlier approaches of [8, 9], as in (1), in permitting noisy measurements of the function. Moreover, the convergence analysis is based on stochastic approximation theory.

In all of the above and related papers, every component of the current guess is updated at each time instant. In recent years, when the dimension $n$ is very large (which it is in problems of deep learning and deep reinforcement learning), some researchers have proposed the use of coordinate gradient descent. In this approach, at each instant of time $t$, an index $i \in [n]$ is chosen at random (or in some deterministic manner), and only the $i$-th component of $\theta_t$ is updated, using either the true gradient $\nabla J(\theta_t)_i$, or a noisy version thereof. See [14] for a good survey, and [15] for further analysis. It is noteworthy that in all these papers, the measurements of the gradient are assumed to be error-free. The update rule can be expressed as

$$\theta_{t+1} = \theta_t + \beta_t e_i \circ [-\nabla J(\theta_t) + \xi_{t+1}],$$

(3)
where $\xi_{t+1}$ denotes the measurement error if any, $\circ$ denotes the Hadamard, or componentwise, product, $\mathbf{e}_i$ denotes the elementary unit vector in row $i$ and zeros elsewhere, and $\{\beta_t\}$ is a predetermined sequence of step sizes. Observe that, if $i$ assumes each value in $[n]$ with an equal probability $1/n$, then

$$E[\mathbf{e}_i \circ \nabla J(\theta_t)] = \frac{1}{n} \nabla J(\theta_t).$$

In some circles, this approach is known as “stochastic gradient descent” (SGD); see for example [14, 15]. However, if $\xi_t = 0$ for all $t$, then randomness arises only in choosing the index $i$ to be updated, and everything else is deterministic. With noise-free measurements, it is possible to carry out very detailed analysis of the algorithm in (3). Indeed, in [15] the analysis encompasses even some nonconvex functions.

1.2 Contributions of the Present Paper

When the dimension $n$ is very large, and only one component of $\theta_t$ is updated at each $t$, then in essence one is trading off the savings in memory calls against an increase in the number of iterations. To illustrate, suppose that one wishes that on average, each component of $\theta_t$ is updated $T$ times. Then, on average, a total of $nT$ iterations would be required. On the other side, if the entire $n$-dimensional vector is updated at each $t$, then only $T$ iterations would be required, but at every iteration an $n$-dimensional vector would have to be updated. An intermediate approach, which could be called Batch Stochastic Gradient Descent (BSGD), would be to update some but not necessarily all components of $\theta_t$ at each time instant. Specifically, at each instant $t$, one selects a set of indices $S(t) \leq [n]$ to be updated. If the available data consists of noisy measurements of the gradient, we define

$$y_{t+1} = -\nabla J(\theta_t) + \xi_{t+1}.$$  \hfill (4)

If the available data consists of approximate gradients based on noisy function evaluations, then we define $y_{t+1}$ as in (1). In either case, the updating rule is

$$\theta_{t+1} = \theta_t + \beta_t \mathbf{e}_{S(t)} \circ y_{t+1},$$  \hfill (5)

where

$$[\mathbf{e}_{S(t)}]_i = \begin{cases} 1, & \text{if } i \in S(t), \\ 0, & \text{if } i \notin S(t). \end{cases}$$

In component form, the updating rule in (5) can be expressed as

$$\theta_{t+1,j} = \begin{cases} \theta_{t,j} + \beta_t y_{t+1,i}, & \text{if } i \in S(t), \\ \theta_{t,j}, & \text{if } i \notin S(t). \end{cases}$$  \hfill (6)

The choice of the update set $S(t)$ can be random; more details are given in subsequent sections.

As mentioned above, previous surveys such as [14, 15] study the case where $S(t)$ is a singleton set (possibly random) for each $t$. The vector $y_{t+1}$ can be either the exact gradient of a noise-corrupted exact gradient. The case where $y_{t+1}$ is defined as in (1), and $S(t) = [n]$ for all $t$ (that is, all components are updated at each $t$), is analyzed in [9], and a simpler analysis is given in [16]. Attempts to extend either approach to the case of batch updating face two technical challenges: First, while $-\nabla J(\theta_t)$ is always a descent direction, at least locally near $\theta_t$, batch-sampled gradients of the form $\mathbf{e}_{S(t)} \circ -\nabla J(\theta_t)$ are not descent directions in general. In the analysis of “coordinate
gradient descent” (where \(|S(t)| = 1\) for all \(t\)), this difficulty is handled by restricting \(J(\cdot)\) to be convex. The second difficulty is that when \(y_{t+1}\) is given by (1), we can write

\[
y_{t+1,i} = J(\theta_t - c_t e_i) - J(\theta_t + c_t e_i) + \frac{\xi_{t+1,i}^- - \xi_{t+1,i}^+}{2c_t}.
\]

Thus, if we let \(c_t \rightarrow 0\) as \(t \rightarrow \infty\), then variance of the noise term approaches infinity. To handle this difficulty, we adapt the approach of [16] to batch updating.

In addition to the above-mentioned theoretical contributions, we have also carried out extensive numerical computations to support the theoretical claims. We include as many computations as we can, subject to space limitations.

2 Convergence Theorems for Batch Stochastic Gradient Descent

Throughout, we study strictly convex \(C^2\) functions \(J : \mathbb{R}^n \rightarrow \mathbb{R}\). By translating coordinates, it can be assumed that the global minimum occurs at \(\theta^* = 0\), and \(J(0) = 0\). Further, it is assumed that there exist constants \(0 < \underline{c} < \overline{c} < \infty\) such that

\[
c I_n \leq \nabla^2 J(\theta) \leq \overline{c} I_n, \tag{7}
\]

where \(I_n\) denotes the \(n \times n\) identity matrix, and the notation \(A \leq B\) (when \(A, B\) are symmetric matrices) denotes that \(B - A\) is positive semidefinite.

Until now we have not specified how the set \(S(t)\) of indices to be updated is chosen. For this purpose, we introduce \(n\) different \(\{0,1\}\)-valued stochastic processes \(\{\kappa_{t,i}\}\) for \(t \geq 0\) and \(i \in [n]\).

No assumptions are made regarding the relationship between \(\{\kappa_{t,i}\}\) and \(\{\kappa_{t,j}\}\) for \(i \neq j\). The significance of these processes, which we refer to as the “update” processes, is that at time \(t\), the \(i\)-th component of \(\theta_t\) is updated if and only if \(\kappa_{t,i} = 1\). For each \(t\), define

\[
S(t) = \{i \in [n] : \kappa_{t,i} = 1\}.
\]

Now we state our two main theorems.

**Theorem 1.** Let \(\{\mathcal{F}_t\}\) be any filtration with respect to which the processes \(\{\theta_t\}, \{\kappa_{t,i}\}\) and \(\{\xi_t\}\) are adapted. Suppose the error process \(\{\xi_t\}\) satisfies

\[
E(\xi_{t+1} | \mathcal{F}_t) = 0 \text{ a.s., } \forall t \geq 0, \tag{8}
\]

\[
E(\xi_{t+1}^2 | \mathcal{F}_t) \leq d(1 + \|\theta_t\|_2^2) \text{ a.s., } \forall t, \tag{9}
\]

for some constant \(d\). Suppose \(\{\beta_t\}\) is a deterministic sequence of step sizes satisfying

\[
\sum_{t=0}^{\infty} \beta_t^2 < \infty. \tag{10}
\]

Then the sequence of iterations \(\{\theta_t\}\) is bounded almost surely. If, in addition, we have

\[
\sum_{t=0}^{\infty} \beta_{t,i} = \infty \text{ a.s., } \forall i \in [n], \tag{11}
\]

where \(\beta_{t,i} := \beta_t I_{\{i \in S(t)\}}\), then \(\theta_t \rightarrow 0\) almost surely as \(t \rightarrow \infty\).
Theorem 2. Suppose (8) and (9) hold, and suppose further that
\[ \sum_{t=0}^{\infty} \beta_t c_t < \infty, \sum_{t=0}^{\infty} (\beta_t/c_t)^2 < \infty. \] (12)
Then the sequence of iterations \( \{\theta_t\} \) is bounded almost surely. If, in addition, we have
\[ \sum_{t=0}^{\infty} \beta_{t,i} = \infty \quad \text{a.s., \forall } i \in [n], \] (13)
then \( \theta_t \to 0 \) almost surely as \( t \to \infty \).

Note that (10) and (11) are analogous to the well-known Robbins-Monro conditions introduced in [7], adjusted for random step sizes. Similarly, (10) and (12) are analogous to the conditions introduced by Blum in [9].

3 Proofs
3.1 Some Preliminaries on Convex Functions
We begin with a purely technical (and rather obvious) result that facilitates the proofs.

Lemma 1. Suppose \( J \) is \( C^2 \) and satisfies (7). Then

(i) There exists a \( C^0 \) function \( H : \mathbb{R}^n \to \mathbb{R}^{n \times n} \) such that \( H(\theta) \) is symmetric for all \( \theta \in \mathbb{R}^n \), and
\[ \nabla J(\theta) = H(\theta) \theta, \forall \theta \in \mathbb{R}^n, \] (14)
\[ cI_n \leq H(\theta) \leq \bar{c}I_n, \forall \theta \in \mathbb{R}^n. \] (15)

(ii) Suppose \( S \subseteq [n] \). Then
\[ \langle \theta, e_S \circ \nabla J(\theta) \rangle \geq c\|e_S \circ \theta\|_2^2, \forall \theta \in \mathbb{R}^n. \] (16)
In particular
\[ \theta_i[\nabla J(\theta)]_i \geq c\theta_i^2, \forall i \in [n], \theta \in \mathbb{R}^n. \] (17)

(iii) We also have that
\[ \|\nabla J(\theta)\|_2 \leq \bar{c}\|\theta\|_2 \] (18)

Proof. Since \( \nabla J(0) = 0 \) (because \( 0 \) is the global minimum of \( J(\cdot) \)), we can write
\[ \nabla J(\theta) = \int_0^1 \frac{d}{d\lambda} \nabla J(\lambda \theta) \ d\lambda = \int_0^1 \nabla^2 J(\lambda \theta) \theta \ d\lambda. \]
Hence (14) holds with
\[ H(\theta) = \int_0^1 \nabla^2 J(\lambda \theta) \ d\lambda. \]
Therefore
\[ H(\theta) - cI_n = \int_0^1 [\nabla^2 J(\lambda \theta) - cI_n] \ d\lambda \geq 0. \]
The proof that $H(\theta) \leq \bar{c}I_n$ is similar. To prove (16), rearrange indices and partition so that

$$e_s \circ \theta = \begin{bmatrix} \theta_1 \\ 0 \end{bmatrix}, \ H(\theta) = \begin{bmatrix} H_{11}(\theta) & H_{12}(\theta) \\ H_{21}(\theta) & H_{22}(\theta) \end{bmatrix}. $$

Then

$$\langle \theta, e_s \circ \nabla J(\theta) \rangle = \langle e_s \circ \theta, e_s \circ \nabla J(\theta) \rangle $$

$$= \begin{bmatrix} \theta_1^\top \\ 0 \end{bmatrix} \begin{bmatrix} H_{11}(\theta) \theta_1 \\ H_{21}(\theta) \theta_1 \end{bmatrix} $$

$$= \theta_1^\top H_{11}(\theta) \theta_1 \geq \bar{c} \| \theta_1 \|^2_2 = \bar{c} \| e_s \circ \theta \|^2_2, $$

where the last step follows from the fact that $H_{11}(\theta)$ is a principal submatrix of $H(\theta)$, and $H(\theta) \geq \bar{c}I_n$. Now (17) follows by choosing $S = \{i\}$. Finally, (18) is a consequence of the right inequality in (15).

3.2 Proof of Theorem 1

Our proof is based on the “almost supermartingale lemma” of Robbins & Siegmund [17, Theorem 1]. That paper is rather difficult to locate, but a summary of it is also found in a recent survey paper as [18, Lemma 4.1]. The result states the following:

**Lemma 2.** Suppose $\{z_t\}, \{\delta_t\}, \{\gamma_t\}, \{\psi_t\}$ are stochastic processes taking values in $[0, \infty)$, adapted to some filtration $\{F_t\}$, satisfying

$$E(z_{t+1}|F_t) \leq (1 + \delta_t)z_t + \gamma_t - \psi_t \ a.s., \ \forall t. \quad (19)$$

Define

$$\Omega_0 := \{\omega : \sum_{t=0}^{\infty} \delta_t(\omega) < \infty\} \cap \{\omega : \sum_{t=0}^{\infty} \gamma_t(\omega) < \infty\}. \quad (20)$$

Then for all $\omega \in \Omega_0$, $\lim_{t \to \infty} z_t(\omega)$ exists, and in addition,

$$\sum_{t=0}^{\infty} \psi_t(\omega) < \infty, \ \forall \omega \in \Omega_0. \quad (21)$$

In particular, if $P(\Omega_0) = 1$, then $\{z_t\}$ is bounded almost surely.

**Proof.** First it is shown that, when (10) holds, then $\{\theta_t\}$ is bounded almost surely. We have from (5), (8) and (9) that

$$E(\|\theta_{t+1}\|^2_2|F_t) = \|\theta_t\|^2_2 + \beta_t^2 \| e_{S(t)} \circ \nabla J(\theta_t) \|^2_2 $$

$$+ \beta_t^2 E(\| \xi_{t+1} \|^2_2|F_t) $$

$$- 2\beta_t \langle \theta_t, e_{S(t)} \circ \nabla J(\theta_t) \rangle $$

$$\leq \|\theta_t\|^2_2 + \beta_t^2 [\bar{c}^2 \| \theta_t \|^2_2 + d(1 + \| \theta_t \|^2_2)], $$

because

$$\langle \theta_t, e_{S(t)} \circ \nabla J(\theta_t) \rangle \geq 0. $$

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Now by Lemma 2, whenever (10) holds, it follows that $\|\theta_t\|_2$ is bounded almost surely. Define

$$\Omega_0 := \{\omega : \sup_t \|\theta_t\|_2 < \infty\},$$

and observe that $P(\Omega_0) = 1$.

Next, we analyze $\theta_{t+1,i}$ for each index $i \in [n]$. We have, from the update rule (6), that

$$E(\theta_{t+1,i}^2 | F_t) = \theta_{t,i}^2 + \beta_{t,i}^2 \|\nabla J(\theta_t)\|_i^2$$

$$+ E(\xi_{t+1,i}^2 | F_t) - 2\beta_{t,i}^2 \theta_{t,i} \|\nabla J(\theta_t)\|_i$$

$$\leq \theta_{t,i}^2 + \beta_{t,i}^2 [c^2 \|\theta_t\|_2^2 + d(1 + \|\theta_t\|_2^2)]$$

$$- 2\epsilon \beta_{t,i} \theta_{t,i}^2.$$

Here we use (16) and (14). Now apply Lemma 2 with $z_t = \theta_{t,i}^2, \delta_t = 0, \gamma_t = \beta_{t,i}^2 [c^2 \|\theta_t\|_2^2 + d(1 + \|\theta_t\|_2^2)],$

$$\psi_t = 2\epsilon \beta_{t,i} \theta_{t,i}^2.$$

The summability of $\{\delta_t\}$ is obvious. As for $\{\gamma_t\}$, observe that

$$\sum_{t=0}^{\infty} \gamma_t = \sum_{t=0}^{\infty} \beta_{t,i}^2 [c^2 \|\theta_t\|_2^2 + d(1 + \|\theta_t\|_2^2)]$$

$$\leq \left( \sum_{t=0}^{\infty} \beta_t^2 \right) \left( d + (c^2 + d) \sup_t \|\theta_t(\omega)\|_2^2 \right)$$

$$< \infty, \forall \omega \in \Omega_0.$$

Define

$$\Omega_1 := \{\omega \in \Omega : \sum_{t=0}^{\infty} \beta_{t,i}(\omega) = \infty, \forall i \in [n]\},$$

and observe that $P(\Omega_1) = 1$ due to (11). Then, for $\omega \in \Omega_0 \cap \Omega_1$, Lemma 2 gives us that there exists a $\eta_i(\omega) \geq 0$ such that $\theta_{t,i}^2(\omega) \to \eta_i(\omega)$ as $t \to \infty$, and

$$\sum_{t=0}^{\infty} \beta_{t,i}(\omega) \theta_{t,i}^2(\omega) < \infty.$$ 

Now suppose by way of contradiction that there exists an $\omega^* \in \Omega_0 \cap \Omega_1$ such that $\eta_i(\omega^*) \geq 2\epsilon > 0$. Pick a finite $T = T_i$ such that $\theta_{t,i}^2(\omega) > \epsilon$ for all $t \geq T$. Then

$$\sum_{t=T}^{\infty} \beta_{t,i}(\omega) \theta_{t,i}^2(\omega) \geq \epsilon \sum_{t=T}^{\infty} \beta_{t,i}(\omega) = \infty,$$

which is a contradiction. Hence no such $\omega^*$ exists, and $\theta_{t,i}^2 \to 0$ for all $\omega \in \Omega_0 \cap \Omega_1$. The proof is concluded by observing that $P(\Omega_0 \cap \Omega_1) = 1$. \qed
3.3 Proof of Theorem 2

The proof of Theorem 2 draws upon the approach put forth in [16], which is considerably simpler than that in [9].

Proof. Recall that $y_{t+1,i}$ as defined in (1) can be equivalently rewritten as

$$y_{t+1,i} = \frac{J(\theta_t - c_t e_i) - J(\theta_t + c_t e_i)}{2c_t} + \frac{\xi_{t+1}^- - \xi_{t+1}^+}{2c_t}.$$  

Let us examine the first term. Using Taylor’s theorem, one can conclude that

$$y_{t+1,i} = -[\nabla J(\theta_t)]_i + c_t r_{t,i} + \frac{1}{c_t} \xi_{t+1,i},$$

for some $\lambda_t \in [-1, 1]$. Hence we can write $y_{t+1,i}$ of (1) as

$$y_{t+1,i} = -[\nabla J(\theta_t)]_i + c_t r_{t,i} + \frac{1}{c_t} \xi_{t+1,i},$$

so that

$$y_{t+1} = -\nabla J(\theta_t) + c_t r_t + \frac{1}{c_t} \xi_{t+1}.$$  

where $\xi_{t+1} = (\xi_{t+1}^- - \xi_{t+1}^+)/2$, and

$$r_{t,i} = 2[\nabla^2 J(\theta + \lambda_t e_i)]_{ii}.$$  

Though $r_t$ is random (because $\theta_t$ is random), the upper bound in (7) guarantees that

$$|r_{t,i}| \leq 2\bar{c}, \forall t.$$  

(Here we use the fact that for any matrix $A$, we have that $A_{ii} \leq \|A\|_S$ for all $i \in [d]$, where $\| \cdot \|_S$ denotes the spectral norm.) Consequently $\|r_t\|^2 \leq 2\sqrt{\bar{c}} =: M_0$, for all $t$.

Now let analyze the recursion (6), using the hypotheses (8) and (9). This gives

$$\|\theta_{t+1}\|^2 = \|\theta_t + \beta_t e_{S(t)} \circ y_{t+1}\|^2$$

$$= \|\theta_t + e_{S(t)} \circ [-\beta_t \nabla J(\theta_t)] + \beta_t c_t r_t + (\beta_t/c_t) \xi_{t+1}\|^2.$$  

If we expand the Euclidean norm squared on the right side, we get four square terms and six cross product terms. Out of these six cross product terms, three become zero due to (8). The conditional expectation of the remaining seven terms can be bounded using (9), and the following bounds:

$$\|r_t\|^2 \leq M_0, \|\nabla J(\theta_t)\|_2 \leq \bar{c}\|\theta_t\|_2,$$

$$\langle \theta_t, e_{S(t)} \circ \nabla J(\theta_t) \rangle \geq \underline{c}\|e_{S(t)} \circ \theta_t\|_2^2.$$  

These substitutions lead to the following complicated expression which we will simplify in stages.

$$E(\|\theta_{t+1}\|^2 | F_t) \leq \|\theta_t\|^2 [1 + \beta_t^2 \bar{c}^2 + (\beta_t/c_t)^2 d]$$

$$+ \beta_t^2 \bar{c}^2 M_0^2 + (\beta_t/c_t)^2 d$$

$$+ 2\|\theta_t\|_2 (\beta_t c_t M_0 + \beta_t^2 c_t \bar{c} M_0)$$

$$- 2\beta_t \langle \theta_t, e_{S(t)} \circ \nabla J(\theta_t) \rangle.$$  

(24)
On the right side, there are constant terms, linear terms, and quadratic terms in \( \|\theta_t\|_2 \). To make things uniform, we replace \( \|\theta_t\|_2 \) by its upper bound\(^1\) \( 1 + \|\theta_t\|_2^2 \), and \( \beta_t(\theta_t, e_{S(t)} \circ \nabla J(\theta_t)) \) by its lower bound \( \beta_t \| e_{S(t)} \circ \theta_t \|_2^2 \). This gives

\[
E(\|\theta_{t+1}\|_2^2|\mathcal{F}_t) \leq \|\theta_t\|_2^2(1 + p_{t,1}) + p_{t,2} - 2\beta_t \| e_{S(t)} \circ \theta_t \|_2^2,
\]

where

\[
p_{t,1} = \beta_t^2 c^2 + 2(\beta_t c_t M_0 + \beta_t^2 c_t \bar{c} M_0) + (\beta_t/c_t)^2 d,
\]

\[
p_{t,2} = 2(\beta_t c_t M_0 + \beta_t^2 c_t \bar{c} M_0) + \beta_t^2 c_t^2 M_0^2.
\]

Define \( p_t = \max\{p_{t,1}, p_{t,2}\} \). Then we can replace (25) by

\[
E(\|\theta_{t+1}\|_2^2|\mathcal{F}_t) \leq \|\theta_t\|_2^2(1 + p_t) + p_t - 2\beta_t \| e_{S(t)} \circ \theta_t \|_2^2.
\]

To show that \( \|\theta_t\|_2 \) is bounded almost surely, we neglect the term \(-2\beta_t \| e_{S(t)} \circ \theta_t \|_2^2\), leading to

\[
E(\|\theta_{t+1}\|_2^2|\mathcal{F}_t) \leq \|\theta_t\|_2^2(1 + p_t) + p_t.
\]

So if we can establish that \( \{p_t\} \) is summable, then the desired conclusion would follow from Lemma 2. Recall the two hypotheses from (12), namely

\[
\sum_{t=0}^{\infty} \beta_t c_t < \infty, \sum_{t=0}^{\infty} (\beta_t/c_t)^2 < \infty.
\]

So, if \( \{\beta_t c_t\} \) is summable, then it is also square-summable. Moreover, since \( \beta_t \) is bounded, the sequence \( \{\beta_t^2 c_t\} \) is also summable. With these observations, we see that every term in the definitions of \( p_{t,1} \) and \( p_{t,2} \) is summable, whence so is \( p_{t,i} \). Therefore \( \|\theta_t\|_2 \) is bounded almost surely when (12) holds.

Next, it is shown that \( \theta_t \to 0 \) almost surely as \( t \to \infty \) if, in addition, (13) holds. For this purpose, we write out the update rule componentwise using (22), namely

\[
\theta_{t+1,i} = \theta_{t,i} - \beta_{t,i} [\nabla J(\theta)]_i + \beta_{t,i} c_t \xi_{t,i} + (\beta_{t,i}/c_t) \xi_{t+1,i}.
\]

Therefore

\[
E(\theta_{t+1,i}^2|\mathcal{F}_t) \leq \theta_{t,i}^2 + \beta_{t,i}^2 c_t^2 \|\theta_t\|_2^2 + \beta_{t,i}^2 c_t^2 M_0^2
\]

\[
+ (\beta_{t,i}/c_t)^2 d(1 + \|\theta_t\|_2^2)
\]

\[
+ \beta_{t,i} c_t M_0 \|\theta_t\|_1 + \beta_{t,i}^2 c_t \|\theta_t\|_2
\]

\[
- 2\beta_{t,i} c_t \theta_{t,i}^2.
\]

Since \( \beta_{t,i} \) either equals \( \beta_t \) or 0, we can replace (28) by

\[
E(\theta_{t+1,i}^2|\mathcal{F}_t) \leq \theta_{t,i}^2 + \beta_{t,i}^2 c_t^2 \|\theta_t\|_2^2 + \beta_{t,i}^2 c_t^2 M_0^2
\]

\[
+ (\beta_t/c_t)^2 d(1 + \|\theta_t\|_2^2)
\]

\[
+ \beta_t c_t M_0 \|\theta_t\|_1 + \beta_t^2 c_t \|\theta_t\|_2 - 2\beta_{t,i} c_t \theta_{t,i}^2.
\]

\(^1\)Actually \( \|\theta_t\|_2 \leq (1 + \|\theta_t\|_2^2)/2 \). However, we drop the factor 2 to keep the formulas less cluttered.
Define
\[ \Omega_0 := \{ \omega : \sup_t \| \theta_t (\omega) \|_2 < \infty \}, \]
and observe that \( P(\Omega_0) = 1 \). Now apply Lemma 2 with
\[ z_t = \theta^2_t, \delta_t = 0, \psi_t = 2\beta_t \bar{c}_t T^2, \gamma_t = \beta_t c_t M_0^2 + \alpha(1 + \| \theta_t \|_2^2), \]
So long as \( \omega \in \Omega_0 \), all of these sequences are bounded. Next, as before, define
\[ \Omega_1 := \{ \omega \in \Omega : \sum_{t=0}^{\infty} \beta_{t,i} (\omega) = \infty, \ \forall i \in [n] \}, \]
Then, for \( \omega \in \Omega_0 \cap \Omega_1 \), we have that there exists an \( \eta_i \) such that \( \theta^2_{t,i} (\omega) \rightarrow \eta_i (\omega) \) as \( t \rightarrow \infty \), and
\[ \sum_{t=0}^{\infty} \beta_{t,i} \theta^2_{t,i} < \infty. \]
As before, this implies that \( \eta_i (\omega) = 0 \) for all \( \omega \in \Omega_0 \cap \Omega_1 \).

4 Computational Results

To validate the theory, we have conducted the following numerical experiment: We set \( J(\theta) = \theta^\top A \theta \), where \( A \) is a \( 1,000 \times 1,000 \) positive definite matrix with a condition number of 100, whose eigenvectors are not aligned closely with the elementary basis vectors; that is, \( A \) is far from being a diagonal matrix. The computations were conducted in python using NumPy and PyTorch (for GPU Acceleration) libraries. BSGD was implemented using approximate gradients with noisy function valuations. The conditions in (12) (Blum’s conditions) were satisfied by choosing
\[ c_t = \frac{c_0}{(1 + (t/\tau))^p}, \beta_t = \frac{\beta_0}{(1 + (t/\tau))^q}, \]
where \( \tau \) controls the rate of decay with respect to \( t \). To satisfy (12), the exponents \( p, q \) must satisfy the conditions
\[ p \leq 1, p + q > 1, p - q > 0.5. \]
Here we chose \( \tau = 200, c_0 = 0.1, \beta_0 = 0.01, p = 1, q = 0.02. \) To implement BSGD, we chose the processes \( \kappa_{t,i} \) to be independent binary random variables with \( \Pr \{ \kappa_{t,i} = 1 \} = \rho \) for all \( t, i \). The “rate” \( \rho \) was assigned various values to measure performance. The noise was additive white Gaussian noise with various SNR. The source code for the implementation is available here.²

Figure 1 shows that when measurements are noise-free, and \( \rho = 0.2 \), that is, at each time instant, an average of 20% of the components are updated. All momentum-based methods converge, which is not surprising in view of [13] (though in that paper all components are updated at each time). BSGD also converges, albeit more slowly. The surprise is that at a SNR of just 50db, all momentum-based methods diverge.
Figure 1: Comparison of Momentum-Based Methods and BSGD with no noise and with SNR = 50 dB.

Figure 2 shows the convergence of BSGD with various values of the rate $\rho$, ranging from 0.05 to 0.20. Even with a fairly high SNR of 50dB, BSGD converged for all values of $\rho$ studied.

In the final experiment, we compared BSGD with a constant increment $c_t = 0.1$ for all $t$. As expected, with a constant increment the iterations did not converge, but oscillated. In contrast, BSGD converged, at both 10dB noise and 20dB noise. The results are shown in Figure 3.

5 Conclusions

In this paper, we have introduced a very general class of Batch Stochastic Gradient Descent (BSGD) algorithms which include most existing stochastic gradient methods as special cases. The update direction can either be a noisy measurement of the exact gradient, or a first-order approximation to the gradient based on noisy function measurements. The convergence of BSGD is established in both cases using stochastic approximation theory. Numerical experiments show that when as little as 5% of the coordinates are updated at each iteration, and approximate gradients are used, BSGD still converges, while batch momentum-based convex optimization methods diverge at quite low noise levels. However, when noise-corrupted exact gradients are used, both BSGD and batch momentum-based methods converge. Existing theory of momentum-based algorithms does not apply to the case of batch updating. Hence this is a suitable topic for further research.

\footnote{\url{https://www.github.com/TUdayKiranReddy/BSGD}}
Figure 2: Convergence of BSGD at Various Rates $\rho$ with no noise and with SNR = 50 dB.

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Figure 3: Comparison of BSGD and Constant Step Size with SNR = 10 dB and SNR = 20 dB.

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