Solutions and continuum limits to nonlocal discrete sine-Gordon equations: Bilinearization reduction method

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Abstract
In this paper, we investigate local and nonlocal reductions of a discrete negative order Ablowitz–Kaup–Newell–Segur equation. By the bilinearization reduction method, we construct exact solutions in double Casoratian form to the reduced nonlocal discrete sine-Gordon equations. Then, nonlocal semidiscrete sine-Gordon equations and their solutions are obtained through the continuum limits. The dynamics of soliton solutions are analyzed and illustrated by asymptotic analysis. The research ideas and methods in this paper can be generalized to other nonlocal discrete integrable systems.

KEYWORDS
bilinearization reduction approach, discrete negative order AKNS equation, dynamics, nonlocal reductions, solutions

1 | INTRODUCTION

The nonlocal integrable systems, dating back to the appearance of nonlocal $PT$-symmetric nonlinear Schrödinger (NLS) equation, exert important roles in mathematics and physics. In mathematics, they usually possess many interesting properties, such as multisoliton solutions, Lax integrability, and an infinite number of conservation laws. As for physics, the nonlocal integrable systems have possible applications in multiplace events and other areas of physics, such as quantum chromodynamics, electric circuits, optics, Bose-Einstein condensates, and so forth. Following the establishment of nonlocal $PT$-symmetric NLS equation, many other new nonlocal integrable equations have been proposed. Examples include the nonlocal Korteweg-de
Vries equation,\textsuperscript{10} the reverse space-time modified Korteweg–de Vries (mKdV) equation,\textsuperscript{11} the reverse space-time sine-Gordon (sG) equation,\textsuperscript{12} the reverse space-time NLS equation,\textsuperscript{13} the fully $\mathcal{P}\mathcal{T}$-symmetric and partially $\mathcal{P}\mathcal{T}$-symmetric Davey–Stewartson equations,\textsuperscript{15,16} and many others. In addition, many researchers have successfully extended certain known effective approaches to solve the nonlocal integrable systems, such as the inverse scattering transform,\textsuperscript{13,17} Darboux transformations,\textsuperscript{18,19} the bilinear method,\textsuperscript{20–22} and the Cauchy matrix approach.\textsuperscript{23}

To date, many interesting results on nonlocal integrable systems have been obtained. Inspired by Ablowitz and Musslimani’s work, for example, a new unified two-parameter wave model, connecting integrable local and nonlocal vector NLS equations, was investigated. This model is in possession of a Lax pair and an infinite number of conservation laws and is $\mathcal{P}\mathcal{T}$-symmetric.\textsuperscript{24} In light of the relations between nonlocal and local integrable equations, not only the integrability of nonlocal equations was established immediately but also their Lax pairs and analytical solutions were constructed from those of the local equations.\textsuperscript{25} With the assumption of stationary solution, general Jacobi elliptic-function and hyperbolic-function solutions were obtained for a nonlocal NLS equation,\textsuperscript{26} in which the bounded cases obey either the $\mathcal{P}\mathcal{T}$ - or anti-$\mathcal{P}\mathcal{T}$-symmetric relation. It turns out that the focusing nonlocal NLS equation has four types of bounded Jacobi elliptic-function solutions, as well as the bright- and dark-soliton solutions. Besides the nonlocal continuous integrable systems, there is also some progress in the nonlocal semidiscrete integrable systems. For instance, by discretizing the transverse coordinate $x$ into the discrete lattice sites, a semidiscrete version of the $\mathcal{P}\mathcal{T}$-symmetric NLS equation was introduced in Ref. 27, where its integrability and $\mathcal{P}\mathcal{T}$-symmetry breaking conditions were identified. By the Ablowitz–Ladik scattering problem, Ablowitz and Musslimani proposed an integrable nonlocal semidiscrete NLS equation,\textsuperscript{28} whose linear Lax pair formulation, Hamiltonian properties, and action-angle variables were investigated in Ref. 29. Furthermore, various aspects of nonlocal semidiscrete integrable models on exact solutions and gauge equivalence have been performed.\textsuperscript{30–32}

Since the information of the whole hierarchy of integrable partial differential equations can be encoded into an integrable partial difference equation in an implicit way, the integrable partial difference equations (namely, discrete integrable systems) have much richer structures (see Ref. 33). Most of the mathematical methods in the theory of integrable systems are invented based on the differential operators, so usually, they are no longer directly effective for the studies of discrete equations (relating to difference operators). Therefore, to deeply understand the intrinsic structure of nonlocal discrete systems, it is essential to develop classic methods. For the first paper on nonlocal discrete integrable systems, one can refer to Ref. 34, where two types of nonlocal discrete integrable equations were investigated as reductions of two-component Adler–Bobenko–Suris systems (cf. Ref. 35). In particular, a reverse-$(n, m)$ nonlocal H1 equation and a reverse-$n$ nonlocal H1 equation were shown explicitly. Quite recently, nonlocal complex reduction of a discrete negative order Ablowitz–Kaup–Newell–Segur (AKNS) equation was studied in Ref. 36, where a nonlocal complex discrete sG equation, as well as its exact solutions in Cauchy matrix type, was constructed. We refer the reader to the recent survey\textsuperscript{37} and the references therein for more details on the Cauchy matrix solutions of nonlocal complex integrable equations.

The AKNS system is usually indispensable to the studies of nonlocal integrable systems. Among all discrete versions of the AKNS system, one is the following discrete (nonpotential) negative order AKNS (dnAKNS) system

\begin{equation}
4(u + \hat{u} - (\bar{u} + \hat{\bar{u}})\omega) = \delta\varepsilon(u + \hat{u} + (\bar{u} + \hat{\bar{u}})\omega), \tag{1a}
\end{equation}
The notations adopted in (1) are as follows: all dependent variables $u$, $v$, and $w$ are functions of discrete coordinates $(n, m) \in \mathbb{Z}^2$, for example, $u = u_{n,m}$; the difference operations $u \mapsto \tilde{u}$ and $u \mapsto \hat{u}$ denote elementary shifts in the two directions of the lattice, that is, $\tilde{u} = u_{n+1,m}$, $\hat{u} = u_{n,m+1}$, whereas for the down and combined shifts, we have $\bar{u} = u_{n-1,m}$ and $\hat{\tilde{u}} = u_{n+1,m+1}$. Aside from that, $\varepsilon$ and $\delta$ are continuous lattice parameters associated with the grid size in the directions of the lattice given by the independent variables $n$ and $m$, respectively. System (1)\textsuperscript{38} was established by discretizing the bilinear operators of the usual negative order AKNS equation,\textsuperscript{39,40} which can be transformed back to the negative order AKNS equation under appropriate continuum limits.

Motivated by the understanding of relations between discrete and continuous integrable systems, we shall explore nonlocal reductions of the dnAKNS equation (1) through the so-called bilinearization reduction method, which was originally proposed in Refs. \textsuperscript{30} and \textsuperscript{21} to solve the nonlocal integrable models reduced from the AKNS hierarchy.\textsuperscript{41,42} This method involves, first taking appropriate reductions to get the nonlocal integrable systems, and second solving the matrix equation algebraically to derive the exact solutions. By employing the bilinearization reduction method, the present paper will be devoted to deriving some nonlocal discrete sG equations, as well as their exact solutions in double Casoratian form, from the dnAKNS equation (1).

The paper is organized as follows. Section 2 presents double Casoratian solutions to the dnAKNS equation (1). In Section 3, we first consider local and nonlocal reduction of the dnAKNS equation (1) in the real case. Next, multisoliton solutions and Jordan-block solutions for the resulting real nonlocal discrete sG equation along with their basic analytical features are exhibited. And then, continuum limits, including semicontinuous limits and continuous limits, are discussed emphatically. In Section 4, complex nonlocal discrete sG equation reduced from the dnAKNS equation (1) is investigated, as well as the corresponding one-soliton solution, dynamics, and continuum limits. Section 5 is for conclusions and some remarks.

\section{DOUBLE CASORATIAN SOLUTIONS TO THE DNAKNS EQUATION (1)}

In this section, we show how to construct double Casoratian solutions for the dnAKNS equation (1). To begin we introduce some notations on the double Wronskian/Casoratian, which will be used in the rest part of the present paper.

For the basic column vectors $\phi = (\phi_1, \phi_2, \ldots, \phi_{N+M+2})^T$ and $\psi = (\psi_1, \psi_2, \ldots, \psi_{N+M+2})^T$ with continuous spatial variable $x$, the $(N + M + 2) \times (N + M + 2)$ double Wronskian is defined as

\begin{align*}
W^{(N,M)}(\phi, \psi) &= |\phi, \phi^{(1)}, \ldots, \phi^{(N)}; \psi, \psi^{(1)}, \ldots, \psi^{(M)}| \\
&= |0, 1, \ldots, N; 0, 1, \ldots, M| = |\hat{N}; \hat{M}|,
\end{align*}

(2)

where $\phi^{(i)} = \frac{\partial \phi}{\partial x^i}$, $\psi^{(i)} = \frac{\partial \psi}{\partial x^i}$, and we use the compact form as given in Ref. \textsuperscript{43}. Here $\hat{N}$ indicates the set of consecutive columns 0, 1, $\ldots$, $N$. The Casoratian is a discrete version of the Wronskian.
Let $E$ be a shift operator defined as
\[ E^j f_{n,m} = f_{n+j,m}, \quad (j \in \mathbb{Z}). \] (3)

For the basic column vectors $\Phi = (\Phi_1, \Phi_2, ..., \Phi_{N+M+2})^T$ and $\Psi = (\Psi_1, \Psi_2, ..., \Psi_{N+M+2})^T$ with the discrete independent variable $n$, the double Casoratian is defined as
\[ C^{(N,M)}(\Phi, \Psi) = |\Phi, E^2 \Phi, ..., E^{2N} \Phi; \Psi, E^2 \Psi, ..., E^{2M} \Psi| \]
\[ = |0, 1, ..., N; 0, 1, ..., M| = |\hat{N}; \hat{M}|. \] (4)

### 2.1 Bilinearization and double Casoratian solutions

To proceed, we first recall bilinearization of the system (1) as shown in Ref. 38. Through the transformation of dependent variables
\[ u = g/f, \quad v = h/f, \quad w = \frac{\hat{f}\hat{f}}{f\hat{f}}, \] (5)
the system (1) can be transformed into
\[ 4(\hat{g}f - \hat{g}\hat{f} - \hat{g}\hat{f} + g\hat{f}) = \delta\epsilon(\hat{g}f + \hat{g}\hat{f} + \hat{g}\hat{f} + g\hat{f}), \] (6a)
\[ 4(\hat{h}f - \hat{h}\hat{f} - \hat{h}\hat{f} + h\hat{f}) = \delta\epsilon(\hat{h}f + \hat{h}\hat{f} + \hat{h}\hat{f} + h\hat{f}), \] (6b)
\[ \hat{f}f - f^2 = \epsilon^2 gh. \] (6c)

And then, it can be directly checked that the bilinear equations (6) are invariant under the gauge transformation
\[ f \rightarrow f \exp(\alpha_0 n + \beta_0 m), \quad g \rightarrow g \exp(\alpha_0 n + \beta_0 m), \quad h \rightarrow h \exp(\alpha_0 n + \beta_0 m), \] (7)
where $\alpha_0$ and $\beta_0$ are two constants.

The crucial point of applying the Casoratian technique is that the condition equations hold for each Casoratian entry. Thus, let us focus on the following condition equation set (CES):
\[ \tilde{\Phi} = A\Phi, \quad \hat{\Phi} = [((4 + \delta\epsilon)E^2 - (4 - \delta\epsilon))(4 - \delta\epsilon)E^2 - (4 + \delta\epsilon))^{-1}]^{1/2} \Phi, \] (8a)
\[ \tilde{\Psi} = A^{-1}\Psi, \quad \hat{\Psi} = [((4 + \delta\epsilon)E^2 - (4 - \delta\epsilon))(4 - \delta\epsilon)E^2 - (4 + \delta\epsilon))^{-1}]^{1/2} \Psi, \] (8b)
where $A$ is an arbitrary invertible complex constant matrix of order $N + M + 2$ and $E$ is the shift operator defined by (3). Then double Casoratian solutions of the bilinear equations (6) can be described in the following theorem.

**Theorem 1.** The double Casoratian determinants
\[ f = |\hat{N}; \hat{M}|, \quad g = (1/\epsilon)|\hat{N} + 1; \hat{M} - 1|, \quad h = (1/\epsilon)|\hat{N} - 1; \hat{M} + 1|, \] (9)
solve the bilinear system (6), provided that the basic column vectors $\Phi$ and $\Psi$ are given by the CES (8).

By solving the CES (8), we know that the basic column vectors $\Phi$ and $\Psi$ are expressed as

$$
\Phi = A^n[((4 + \delta \varepsilon)A^2 - (4 - \delta \varepsilon)I)((4 - \delta \varepsilon)A^2 - (4 + \delta \varepsilon)I)^{-1}]^{m/2}C^+, \quad (10a)
$$

$$
\Psi = A^{-n}[((4 + \delta \varepsilon)A^2 - (4 - \delta \varepsilon)I)((4 - \delta \varepsilon)A^2 - (4 + \delta \varepsilon)I)^{-1}]^{-m/2}C^-, \quad (10b)
$$

where $C^\pm = (c_1^\pm, c_2^\pm, \ldots, c_{N+1}^\pm; d_1^\pm, d_2^\pm, \ldots, d_{M+1}^\pm)^T$ are constant column vectors. Here and hereafter $I$ is the unit matrix whose index indicating the size is omitted. We denote (9) by $f(\Phi, \Psi)$, $g(\Phi, \Psi)$ and $h(\Phi, \Psi)$ when their entries are taken as (10). For the convenience of calculation, Equation (10) is written by replacing $A$ by $e^K$ as

$$
\Phi = e^{Kn}[(4 + \delta \varepsilon \coth K)(4 - \delta \varepsilon \coth K)^{-1}]^{m/2}C^+, \quad (11a)
$$

$$
\Psi = e^{-Kn}[(4 + \delta \varepsilon \coth K)(4 - \delta \varepsilon \coth K)^{-1}]^{-m/2}C^-.
$$

2.2 Similarity invariance of exact solutions

Note that in the basic column vectors (11), $K$ is an arbitrary constant matrix. To show the similarity invariance of exact solutions, we suppose $\overline{K}$ is any matrix that is similar to $K$, that is,

$$
\overline{K} = TKT^{-1}, \quad (12)
$$

where $T$ is the transform matrix. By taking (12) into (11), and denoting $\overline{C}^\pm = TC^\pm$, we naturally get the following two new basic column vectors:

$$
\overline{\Phi} = T\Phi = e^{\overline{K}n}[(4 + \delta \varepsilon \coth \overline{K})(4 - \delta \varepsilon \coth \overline{K})^{-1}]^{m/2}\overline{C}^+, \quad (13a)
$$

$$
\overline{\Psi} = T\Psi = e^{-\overline{K}n}[(4 + \delta \varepsilon \coth \overline{K})(4 - \delta \varepsilon \coth \overline{K})^{-1}]^{-m/2}\overline{C}^-.
$$

Since for the arbitrariness of matrix $A$, it is apparent that $\Phi$ and $\Psi$ still satisfy the CES (8) with $A = e^K$. Noticing the relations $f(\overline{\Phi}, \overline{\Psi}) = |T|f(\Phi, \Psi)$, $g(\overline{\Phi}, \overline{\Psi}) = |T|g(\Phi, \Psi)$ and $h(\overline{\Phi}, \overline{\Psi}) = |T|h(\Phi, \Psi)$, as well as the transformation (5), one can easily find that $(f(\Phi, \Psi), g(\Phi, \Psi), h(\Phi, \Psi))$ lead to same solutions for the dnAKNS equation (1). Now let us start from the following $\Phi$ and $\Psi$:

$$
\Phi = e^{\Gamma n}[(4 + \delta \varepsilon \coth \Gamma)(4 - \delta \varepsilon \coth \Gamma)^{-1}]^{m/2}C^+, \quad (14a)
$$

\footnote{The exponential of a matrix is always an invertible matrix. The inverse matrix of $e^X$ is given by $e^{-X}$. The matrix exponential then gives a map 

$$
\exp : M_s(\mathbb{C}) \rightarrow GL(s, \mathbb{C})
$$

from the space of all $s \times s$ matrices to the general linear group of degree $s$, that is, the group of all $s \times s$ invertible matrices. In fact, this map is surjective, which means that every invertible complex matrix can be written as the exponential of some other matrix (cf. Ref. 44).}
\[
\Psi = e^{-\Gamma n}[ (4 + \delta \varepsilon \coth \Gamma)(4 - \delta \varepsilon \coth \Gamma)^{-1}]^{-\frac{m}{2}} C^-, \tag{14b}
\]

where \( K \) in (11) is replaced by its Jordan canonical form \( \Gamma \). Then various exact solutions, including soliton solutions, Jordan-block solutions, rational solutions, and mixed solutions, for the dnAKNS system (1) can be derived in terms of different eigenvalue structures of matrix \( \Gamma \).

### 3 | REAL REDUCTION: SOLUTIONS AND CONTINUUM LIMITS

In this section, we adopt the bilinearization reduction method to reduce the dnAKNS equation (1) in the real sense. By taking appropriate reduction, the real local and nonlocal discrete sG (rndsG) equation along with some exact solutions is derived. Moreover, the continuum limits are investigated to construct two real local and nonlocal semidiscrete sG equations as well as a real local and nonlocal continuous sG equation. In what follows, for the function \( f = f(x_1, x_2) \), notation \( f_\sigma \) means \( f_\sigma = f(\sigma x_1, \sigma x_2) \), where \( \sigma = \pm 1 \). For \( \sigma = 1 \), the notation \( f_\sigma = f \) should not be confused with the component \( f_1 \) in matrix. When both the independent variables \( x_1 \) and \( x_2 \) are discrete, referred to as the discrete case, it is necessary to figure out \( \tilde{f}_\sigma = f(\sigma x_1 + \sigma, \sigma x_2) \), \( \hat{f}_\sigma = f(\sigma x_1, \sigma x_2 + \sigma) \) and \( \tilde{\hat{f}}_\sigma = f(\sigma x_1 + \sigma, \sigma x_2 + \sigma) \), respectively.

#### 3.1 | Real reduction

The system (1) admits real reduction

\[
v = \eta u, \quad \eta, \sigma = \pm 1. \tag{15}
\]

It is worth noting that Equation (1c) implies

\[
w(n, m) = \prod_{j=n_0}^{n-1} \frac{1 + \varepsilon^2 u(j, m)\bar{u}(j, m)}{1 + \varepsilon^2 \hat{u}(j, m)\hat{\bar{u}}(j, m)}, \quad n_0 \in \mathbb{Z}, \tag{16}
\]

which possesses the nonautonomous structure (see Refs. 45 and 46). Imposing (15) into (16) gives rise to \( w = w_\sigma \). Thus, the rndsG equation reads

\[
4(u + \hat{u} - (\bar{u} + \bar{\hat{u}})w) = \delta \varepsilon (u + \hat{u} + (\bar{u} + \bar{\hat{u}})w), \tag{17a}
\]

\[
(1 + \eta \varepsilon^2 \hat{u}\bar{u})w = (1 + \eta \varepsilon^2 \bar{u}u)w. \tag{17b}
\]

When \( \sigma = 1 \), Equation (17) is the real local discrete sG equation, while when \( \sigma = -1 \), Equation (17) is the real nonlocal discrete sG equation. It is obvious that Equation (17) is preserved under transformation \( u \to -u \). Besides, Equation (17) with \( (\sigma, \eta) = (\pm 1, 1) \) and with \( (\sigma, \eta) = (\pm 1, -1) \) can be transformed into each other by taking \( u \to iu \).

For the sake of constructing exact solutions to Equation (17), it is necessary to impose suitable constraint on the pair \((\Phi, \Psi)\) in double Casoratian (9) so that the transformation (5) coincides with the reduction (15). To this end, we take \( M = N \). Due to arbitrariness of vectors \( C^\pm \), we replace \( C^\pm \)
by \(e^{\pm NT_c}\) in (14) and pay attention to the following double Casorati determinants:

\[
f = \begin{vmatrix} e^{-NT_c}\Phi(N); e^{NT_c}\Psi(N) \end{vmatrix}, \quad g = \frac{1}{\varepsilon} \begin{vmatrix} e^{-NT_c}\Phi(N+1); e^{NT_c}\Psi(N-1) \end{vmatrix}, \quad h = \frac{1}{\varepsilon} \begin{vmatrix} e^{-NT_c}\Phi(N-1); e^{NT_c}\Psi(N+1) \end{vmatrix}.
\]

The following theorem shows double Casoratian solutions to the rndsG equation (17).

**Theorem 2.** The functions \(u = g/f\) and \(w = \tilde{f}/(\tilde{f}g)\) with

\[
f = \begin{vmatrix} e^{-NT_c}\Phi(N); e^{NT_c}\Psi(N) \end{vmatrix}, \quad g = (1/\varepsilon) \begin{vmatrix} e^{-NT_c}\Phi(N+1); e^{NT_c}\Psi(N-1) \end{vmatrix}, \quad h = \begin{vmatrix} e^{-NT_c}\Phi(N-1); e^{NT_c}\Psi(N+1) \end{vmatrix},
\]

solve the rndsG equation (17), if the \((2N + 2)\)th-order column vectors \(\Phi\) and \(\Psi\) defined by (14) satisfy the following relation:

\[
\Psi = T\Phi_\sigma,
\]

where \(T \in \mathbb{C}^{(2N+2) \times (2N+2)}\) is a constant matrix satisfying

\[
\Gamma T + \sigma T \Gamma = 0, T^2 = \begin{cases} -\eta I, & \text{with } \sigma = 1, \\ \eta |e^\Gamma|^2 I, & \text{with } \sigma = -1, \end{cases}
\]

and \(C^- = TC^+\).

The first equation in (20) is nothing but the famous Sylvester equation (cf. Refs. 47 and 48). The idea of the verification for Theorem 2 is using the constraints (19) and (20) to determine the relationship between \((f, h)\) and \((f_\sigma, g_\sigma)\), leading to (15). Here we just present the verification for the case of \(\sigma = -1\).

**Proof.** To begin, we substitute (19) into \(f\), yielding

\[
f = \begin{vmatrix} e^{-NT_c}\Phi, e^{(-N+2)T_c}\Phi, \ldots, e^{NT_c}\Phi; e^{NT_c}T\Phi_1, e^{(N-2)T_c}T\Phi_1, \ldots, e^{-NT_c}T\Phi_1 \end{vmatrix}.
\]

In terms of (20), one can rearrange \(f\) as

\[
f = \begin{vmatrix} e^{-NT_c}\Phi, e^{(-N+2)T_c}\Phi, \ldots, e^{NT_c}\Phi; T e^{-NT_c}\Phi, T e^{(N-2)T_c}\Phi, \ldots, T e^{-NT_c}\Phi \end{vmatrix}
\]

\[
= (\eta |e^\Gamma|^2)^{-N+1} |T||e^{-NT_c}\Phi, T e^{(-N+2)T_c}\Phi, \ldots, T e^{NT_c}\Phi, e^{(-N+2)T_c}\Phi, \ldots, e^{-NT_c}\Phi| \]

\[
= (-\eta |e^\Gamma|^2)^{-N+1} |T||e^{NT_c}\Phi_1, e^{(-N-2)T_c}\Phi_1, \ldots, e^{-NT_c}\Phi_1; T e^{NT_c}\Phi, T e^{(-N+2)T_c}\Phi, \ldots, T e^{NT_c}\Phi| \]

\[
= (-\eta |e^\Gamma|^2)^{-N+1} |T||e^{NT_c}\Phi_1, e^{(-N-2)T_c}\Phi_1, \ldots, e^{-NT_c}\Phi_1; e^{-NT_c}T\Phi, e^{(N+2)T_c}T\Phi, \ldots, e^{NT_c}T\Phi| \]

\[
= (-\eta |e^\Gamma|^2)^{-N+1} |T|f_{-1}.
\]
A similar calculation leads to $h = \eta^{-1}(-\eta|e^\Gamma|^{-2})^{N+1}|T|g_{-1}$. With the help of transformation (5), we arrive at

$$v = \frac{h}{f} = \frac{\eta^{-1}(-\eta|e^\Gamma|^{-2})^{N+1}|T|g_{-1}}{\eta f_{-1}} = \frac{g_{-1}}{\eta f_{-1}} = \eta u_{-1},$$

(22)

which coincides with the reduction (15) for the rndsG equation (17).

Inserting (19) into (18) manifest that exact solutions to Equation (17) can then be described as

$$u = g/f \quad \text{and} \quad w = \frac{\tilde{f}}{\tilde{f}}$$

where $T$ and $\Gamma$ satisfy matrix equations (20).

### 3.2 Some examples of solutions

The aim of this part is to give some types of exact solutions to the nonlocal rndsG equation ($\sigma = -1$). The key step is to solve the matrix equations (20). Without loss of generality, we just consider the case of $\eta = 1$. For simplicity, solutions $\Gamma$ and $T$ are taken as block matrices

$$\Gamma = \begin{pmatrix} L & 0 \\ 0 & -L \end{pmatrix}, \quad T = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

(24)

where $L$ is Jordan canonical matrix. For further analysis, we need to distinguish two forms of matrix $L$: diagonal form and Jordan-block form, leading to multisoliton solutions and Jordan-block solutions, respectively. Notations

$$\xi_j = k_j n + \tau_j m, \quad e^\tau_j = \left(\frac{4 + \delta \epsilon \coth k_j}{4 - \delta \epsilon \coth k_j}\right)^{\frac{1}{2}}, \quad j = 1, 2, \ldots, N + 1$$

(25)

are introduced, where we assume $(\delta \epsilon \coth k_j)^2 < 16$ to guarantee the real property of $\tau_j$.

Due to the block structure of matrix $T$, we note that $C^+$ can be gauged to be $\bar{I} = (1, 1, \ldots, 1; 1, 1, \ldots, 1)^T$ or $\bar{I} = (1, 0, \ldots, 0; 1, 0, \ldots, 0)^T$. It implies that the solutions for Equation (17) are independent of phase parameters $C^+$, that is, the initial phase has always to be 0.

**Soliton solutions:** Let $L$ be a diagonal matrix composed by distinct real nonzero eigenvalues $k_j$, given by

$$L = \text{Diag}(k_1, k_2, \ldots, k_{N+1}), \quad |L| \neq 0, \quad k_i \neq k_j, \quad (i \neq j).$$

(26)

Here we take $C^+ = \bar{I}$, so that $\Phi$ is obtained as

$$\Phi_j = \begin{cases} e^{\xi_j}, & j = 1, 2, \ldots, N + 1, \\ e^{-\xi_j}, & j = N + 1 + s, \quad s = 1, 2, \ldots, N + 1. \end{cases}$$

In the case of $N = 0$, we have one-soliton solution
Solution $u$ in (27a) describes a stable wave with the traveling velocity $-\tau_1/k_1$, which is single-peaked and unidirectional. Its height relative to $u = 0$ is approximately equal to $\sinh^2 k_1/\epsilon$, and its width is proportional to $(2k_1)^{-1}$. In view of the sign of the parameter $k_1\epsilon$, the “amplitude” of the wave can be positive or negative, which corresponds to soliton or antisoliton, as depicted in Figure 1.

In the case of $N = 1$, we get two-soliton solutions

$$u = g/f, \quad w = \frac{\tilde{f}\tilde{f}}{(f\tilde{f})},$$

with

$$f = \cosh 2(k_1 + k_2) \sinh^2(\xi_1 - \xi_2) + \cosh 2(k_1 - k_2) \cosh^2(\xi_1 + \xi_2) - \cosh 2\xi_1 \cosh 2\xi_2,$$

$$g = (\cosh 2k_1 - \cosh 2k_2)(\sinh 2k_1 \cosh 2\xi_2 - \sinh 2k_2 \cosh 2\xi_1)/\epsilon.$$

We now examine the asymptotic form of the two-soliton solutions $u$ in (28) as $m \to \pm\infty$, which can be derived by the analysis of moving-coordinate expansions. For convenience, we call the asymptotic solitons as $k_1$-soliton and $k_2$-soliton, respectively. Without the loss of generality, we set

$$0 < \delta \epsilon < 4, \quad \frac{1}{2}\ln\frac{4 + \delta \epsilon}{4 - \delta \epsilon} < k_1 < k_2,$$

(30)

to guarantee $\tau_1 > \tau_2 > 0$.

To proceed, we consider $\xi_2 = \text{const.}$, and express $\xi_2 = k_2 k_1 \xi_1 + \xi_1$ in terms of $\xi_1$ and $\xi_1 = m\Delta c_1$, where $\Delta c_1 = \tau_2 - \frac{k_2}{k_1} \tau_1 < 0$ is the relative speed of the moving coordinates. Note that $m \to \pm\infty$ corresponds to $\xi_1 \to \mp\infty$. We then asymptotically expand $f$ and $g$ for large $\xi_1$ with fixed $\xi_1$. This yields, after neglecting subdominant exponential terms,

$$f \simeq \frac{1}{2}(\sinh^2(k_2 \pm k_1)e^{2\xi_1} + \sinh^2(k_2 \mp k_1)e^{-2\xi_1})e^{\mp 2\xi_2},$$

$$g \simeq \frac{1}{2\epsilon} \sinh 2k_1 (\cosh 2k_1 - \cosh 2k_2)e^{\mp 2\xi_2},$$

(31a) (31b)
and hence, the $k_1$-soliton appears
\[ u \approx \frac{\sinh 2k_1 (\cosh 2k_1 - \cosh 2k_2)}{\varepsilon (\sinh^2 (k_2 \pm k_1)e^{2\xi_1} + \sinh^2 (k_2 \mp k_1)e^{-2\xi_1})}, \quad m \to \pm \infty, \quad (32) \]
whose asymptotical behavior follows:

\begin{align*}
\text{top point traces} : & \quad n(m) = \pm \frac{1}{2k_1} \ln \frac{\sinh(k_2 - k_1)}{\sinh(k_1 + k_2)} - \frac{\tau_1}{k_1} m, \\
\text{amplitude} : & \quad u = \frac{\sinh 2k_1 (\cosh 2k_1 - \cosh 2k_2)}{2\varepsilon \sinh(k_1 + k_2) \sinh(k_2 - k_1)}, \\
\text{speed} : & \quad -\frac{\tau_1}{k_1}, \\
\text{phase shift} : & \quad -\frac{1}{k_1} \ln \frac{\sinh(k_2 - k_1)}{\sinh(k_1 + k_2)}. \quad (33a-d)
\end{align*}

To continue, we next consider $\xi_2 =$-const., and express $\xi_1 = \frac{k_1}{k_2} \xi_2 + \zeta_2$ in terms of $\xi_2$ and $\zeta_2 = m\Delta c_2$ with $\Delta c_2 = \tau_1 - \frac{k_1}{k_2} \tau_2 > 0$. By asymptotically expanding $f$ and $g$ for large $\xi_2$ with fixed $\xi_2$, and neglecting subdominant exponential terms, we obtain
\begin{align*}
\bar{f} & \approx \frac{1}{2}(\sinh^2 (k_2 \mp k_1)e^{2\xi_2} + \sinh^2 (k_2 \pm k_1)e^{-2\xi_2})e^{\pm 2\xi_1}, \quad m \to \pm \infty, \quad (34a) \\
\bar{g} & \approx \frac{1}{2\varepsilon} \sinh 2k_2 (\cosh 2k_2 - \cosh 2k_1)e^{\pm 2\xi_1}, \quad m \to \pm \infty, \quad (34b)
\end{align*}
and thus, the $k_2$-soliton reads
\[ u \approx \frac{\sinh 2k_2 (\cosh 2k_2 - \cosh 2k_1)}{\varepsilon (\sinh^2 (k_2 \mp k_1)e^{2\xi_2} + \sinh^2 (k_2 \pm k_1)e^{-2\xi_2})}, \quad m \to \pm \infty, \quad (35) \]
whose asymptotical behavior follows:

\begin{align*}
\text{top point traces} : & \quad n(m) = \pm \frac{1}{2k_2} \ln \frac{\sinh(k_2 - k_1)}{\sinh(k_1 + k_2)} - \frac{\tau_2}{k_2} m, \\
\text{amplitude} : & \quad u = \frac{\sinh 2k_2 (\cosh 2k_2 - \cosh 2k_1)}{2\varepsilon \sinh(k_1 + k_2) \sinh(k_2 - k_1)}, \\
\text{speed} : & \quad -\frac{\tau_2}{k_2}, \\
\text{phase shift} : & \quad -\frac{1}{k_2} \ln \frac{\sinh(k_2 - k_1)}{\sinh(k_1 + k_2)}. \quad (36a-d)
\end{align*}

The solution $u$ in (28) is illustrated in Figure 2.

Remark 1. The eigenvalues in (26) can be extended to complex numbers. For example, if taking $k_{2j} = k_{2j-1}^*$ with $j = 1, 2, \ldots, N + 1$, then we obtain the breather solutions (see Figure 2C).
Solution $u$ given by (28) with $\epsilon = \delta = 1$: (A) two-soliton solutions with $k_1 = 0.5$ and $k_2 = 2$; (B) 2D plot of (a) at $m = 3$; and (C) breather solution with $k_1 = 0.05 + i$ and $k_2 = 0.05 - i$ at $m = 1$.

Jordan-block solution: Now let $L$ be a Jordan-block matrix

$$L = \begin{pmatrix}
    k_1 & 0 & \cdots & 0 \\
    1 & k_1 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & \cdots & 1 & k_1
\end{pmatrix}_{(N+1) \times (N+1)}, \quad k_1 \neq 0 \in \mathbb{R},$$

and $C^+ = \mathbb{I}$. In principle, the Jordan-block solutions can be obtained from the multisoliton solutions by taking limit. Here we omit this procedure and directly give the result. For (37), solution $\Phi$ is composed by

$$\Phi_j = \begin{cases}
    \frac{\partial^{j-1} e^{i \xi}}{(j-1)!}, & j = 1, 2, \ldots, N + 1, \\
    \frac{\partial^{s-1} e^{-i \xi}}{(s-1)!}, & j = N + 1 + s, \quad s = 1, 2, \ldots, N + 1.
\end{cases}$$

When $N = 1$, the simplest Jordan-block solution is

$$u = \frac{\sinh 2k_1 [(1 + \xi) \cosh 2(k_1 - \xi) + (1 - \xi) \cosh 2(k_1 + \xi)]}{e(\xi^2 \sinh^2 2k_1 + \cosh^2 2\xi)},$$

$$w = \frac{\tilde{f} \tilde{f}}{(f \tilde{f})}, \quad f = \xi^2 \sinh^2 2k_1 + \cosh^2 2\xi,$$

where $\xi = n - \frac{4m\delta\epsilon}{16 \sinh^2 k_1 - (\delta\epsilon)^2 \cosh^2 k_1}$. This solution is depicted in Figure 3.

3.3 Continuum limits

We will now carry out (semi-)continuous limits of the rndsG equation (17). Through the semicontinuous limits in the $m$-direction and the $n$-direction, respectively, we derive two real local and nonlocal semidiscrete sG equations. Furthermore, by continuous limits, we arrive at the real local and nonlocal continuous sG equation.
Together with the connections between the various parameters in Equation (17) and the lattice spacing, we use the formula

$$\lim_{m \to \infty} (1 + k/m)^m = e^k,$$

in the purpose to give the discrete-continuous relation, where the discrete exponential functions

$$e^{\tilde{\varepsilon}_j} := e^{k_j n} \left( \frac{4 + \delta \varepsilon \coth k_j}{4 - \delta \varepsilon \coth k_j} \right)^{m/2}, \quad j = 1, 2, \ldots, N + 1$$

should be considered.

**Semicontinuous limit in \( m \)-direction:** The semicontinuous limit in \( m \)-direction used here is the so-called straight continuum limit, namely, the discrete variable \( m = 0 \) corresponds to the continuous variable \( t = 0 \). To use (39) as a semicontinuous limit in the \( m \)-direction, we write

$$\left( \frac{4 + \delta \varepsilon \coth k_j}{4 - \delta \varepsilon \coth k_j} \right)^{m/2} = \left( 1 + \frac{2\delta \varepsilon \coth k_j}{4 - \delta \varepsilon \coth k_j} \right)^{m/2}, \quad j = 1, 2, \ldots, N + 1,$$

and therefore,

$$\delta \text{ must approach zero as } m \to \infty, \text{ that is, } m\delta = t.$$  

(42)

As a consequence, the discrete exponential functions become

$$e^{\tilde{\varepsilon}_j} \to e^{\lambda_j}, \text{ with } \lambda_j := k_j n + \frac{\varepsilon \coth k_j}{4} t, \quad j = 1, 2, \ldots, N + 1.$$  

(43)

Interpreting the dependent variables \( u(n, m) := \mu(n, t), \ w(n, m) := \omega(n, t) \) and substituting the Taylor expansions\(^2 \)

$$\tilde{u} = \mu(t + \delta) = \mu + \delta \mu' + \cdots,$$

(44a)

---

\(^2\)Noticing transformation \( w = \tilde{f} \tilde{f}' / (\tilde{f} \tilde{f}') \) and limit (42), we know \( w = (f + \delta f' + \cdots) \tilde{f} / f \tilde{f}' \to (f + \delta f') \tilde{f} / f \tilde{f}' = 1 - \delta \frac{\tilde{f} - f \tilde{f}'}{f \tilde{f}} = 1 - \delta \omega' \), where \( \omega = e^{-1} \ln(\tilde{f} / \tilde{f}) \).
\[\hat{u} = \bar{\mu}(t + \delta) = \bar{\mu} + \delta \bar{\mu}' + \cdots,\]  
\[\hat{u}_\sigma = \mu(\sigma t + \sigma \delta) = \mu_\sigma + \delta (\mu_\sigma)' + \cdots,\]  
\[w \to 1 - \delta \epsilon \omega'.\]

into the rndsG equation (17), we obtain as coefficient of the leading term of order \(O(\delta)\) the real local and nonlocal semidiscrete sG (rnsdsG-t) equation

\[2(\bar{\mu} - \mu)' = \epsilon (\mu + \bar{\mu})(1 - 2\omega'),\]  
\[e^{\epsilon(\omega - \omega)} - 1 = \eta \epsilon^2 \mu \mu_\sigma,\]

where the prime denotes the derivative with respect to \(t\) and \(\omega = \omega_\sigma\). Similar to the discrete case, Equation (45) is preserved under transformation \(\mu \to -\mu\) and Equation (45) with \((\sigma, \eta) = (\pm 1, 1)\) and \((\sigma, \epsilon) = (\pm 1, -1)\) can be transformed into each other by taking \(\mu \to i\mu\).

Next, we show double Casoratian solutions to the rnsdsG-t equation (45).

**Theorem 3.** The functions \(\mu = g/f\) and \(\omega = \epsilon^{-1} \ln(f/g)\) with

\[f = |e^{-NT\tilde{\Phi}(N)}; e^{NTT\tilde{\Phi}(N)}|, \quad g = (1/\epsilon)|e^{-NT\tilde{\Phi}(N+1)}; e^{NTT\tilde{\Phi}(N+1)}|,\]

solve the rnsdsG-t equation (45), where \(\Phi = e^{\Gamma T/4} e^{\Gamma_t/2} C^+\) and \(T\) is a constant matrix of order \(2(N + 1)\) satisfying

\[\Gamma T + \sigma TT = 0, T^2 = \begin{cases} -\eta I, \text{ with } \sigma = 1, \\ \eta |e^I|^2 I, \text{ with } \sigma = -1. \end{cases}\]

In what follows, taking \(\Gamma\) and \(T\) as (24), we discuss explicit solutions for Equation (45) with \((\eta, \sigma) = (1, -1)\).

**Soliton solutions:** Let \(L\) be the diagonal matrix (26). The basic column vector \(\Phi\) is thus given by

\[\Phi_j = \begin{cases} e^{\lambda_j}, & j = 1, 2, \ldots, N + 1, \\ e^{-\lambda_s}, & j = N + 1 + s, \quad s = 1, 2, \ldots, N + 1. \end{cases}\]

In the case of \(N = 0\), Equation (45) has one-soliton solution

\[\mu = (\sinh 2k_1 \operatorname{sech} 2\lambda_1)/\epsilon,\]  
\[\omega = [\ln(\cosh 2(k_1 + \lambda_1) \operatorname{sech} 2\lambda_1)]/\epsilon.\]

The solution \(\mu\) in (49a) still describes a stable traveling wave with velocity \(-\epsilon \coth k_1/(4k_1)\) and amplitude \(\sinh 2k_1/\epsilon\), where the width is proportional to \((2k_1)^{-1}\). We depict soliton and antisoliton in Figure 4 in terms of the sign of the parameter \(k_1\).
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FIGURE 4 One-soliton solution \( \mu \) given by (49a) with \( \varepsilon = 1 \): (A) shape and movement with \( k_1 = 0.6 \); (B) soliton for \( k_1 = 0.6 \) at \( t = 1 \); and (C) antisoliton for \( k_1 = -0.6 \) at \( t = 1 \).

FIGURE 5 Solution \( \mu \) given by (50) with \( \varepsilon = 1 \): (A) two-soliton solutions with \( k_1 = 1 \) and \( k_2 = 1.5 \); (B) 2D plot of (a) at \( t = 1 \); and (C) breather solution with \( k_1 = 0.6 + i \) and \( k_2 = 0.6 - i \) at \( t = 1 \).

In the case of \( N = 1 \), Equation (45) has two-soliton solutions

\[
\mu = \frac{g}{f}, \quad \omega = \varepsilon^{-1} \ln \left( \frac{\tilde{f}}{f} \right),
\]

where \( f \) and \( g \) are given by (29) up to a replacement of \( \xi_i \) by \( \lambda_i \), \( (i = 1, 2) \). Solution \( \mu \) in (50) has a similar asymptotical behavior as \( u \) in (28), which is sketched in Figure 5.

**Theorem 4.** Suppose that \( 0 < k_1 < k_2 \) and \( \varepsilon > 0 \). Then, when \( t \to \pm \infty \), the \( k_1 \)-soliton asymptotically follows:

- **Top point traces**: \( n(t) = \pm \frac{1}{2k_1} \ln \frac{\sinh(k_2 - k_1)}{\sinh(k_1 + k_2)} - \frac{\varepsilon \coth k_1}{4k_1} t \),

- **Amplitude**: \( \mu = \frac{\sinh 2k_1 (\cosh 2k_1 - \cosh 2k_2)}{2 \varepsilon \sinh(k_2 - k_1) \sinh(k_1 + k_2)} \),

- **Speed**: \( -\frac{\varepsilon}{4k_1} \coth k_1 \),

- **Phase shift**: \( \frac{1}{k_1} \ln \frac{\sinh(k_2 - k_1)}{\sinh(k_1 + k_2)} \),

and the \( k_2 \)-soliton asymptotically follows:

- **Top point traces**: \( n(t) = \mp \frac{1}{2k_2} \ln \frac{\sinh(k_2 - k_1)}{\sinh(k_1 + k_2)} - \frac{\varepsilon \coth k_2}{4k_2} t \),
Jordan-block solution $\mu$ given by (54a) with $k_1 = 0.5$ and $\epsilon = 1$: (A) shape and motion and (B) 2D-plot of (a) at $t = 1$.

$$\textit{amplitude} : \mu = \frac{\sinh 2k_2 (\cosh 2k_2 - \cosh 2k_1)}{2\epsilon \sinh(k_2 - k_1) \sinh(k_1 + k_2)},$$ (52b)

$$\textit{speed} : -\frac{\epsilon}{4k_2} \coth k_2,$$ (52c)

$$\textit{phase shift} : -\frac{1}{k_2} \ln \frac{\sinh (k_2 - k_1)}{\sinh(k_1 + k_2)}.$$ (52d)

Jordan-block solutions: Let $L$ be the Jordan-block matrix (37), the basic entries $\{\Phi_j\}$ are thereby of the form

$$\Phi_j = \begin{cases} \frac{\partial_j e^{\lambda_1}}{(j-1)!}, & j = 1, 2, ..., N+1, \\ \frac{\partial_s e^{-\lambda_1}}{(s-1)!}, & j = N+1+s, s = 1, 2, ..., N+1. \end{cases}$$ (53)

In the case of $N = 1$, the simplest Jordan-block solution is

$$\mu = \frac{4[\sinh 4k_1 \cosh 2\lambda_1 + 2 \cosh^2 k_1 (\epsilon - 4n \sinh^2 k_1) \sinh 2\lambda_1]}{\epsilon [4 \cosh^2 2\lambda_1 + (\epsilon \coth k_1 - 2n \sinh 2k_1)^2]},$$ (54a)

$$\omega = \epsilon^{-1} \ln(\tilde{f}/f), \quad f = 4 \cosh^2 2\lambda_1 + (\epsilon \coth k_1 - 2n \sinh 2k_1)^2,$$ (54b)

where the behavior of $\mu$ is depicted in Figure 6.

Semicontinuous limit in $n$-direction
The straight continuum limit cannot be applied to determine the semicontinuous limit in $n$-direction for the discrete exponential function (39). This difficulty can be bypassed, if we introduce

$$e^{2l_j} := \frac{4 + \delta \epsilon \coth k_j}{4 - \delta \epsilon \coth k_j}, \quad j = 1, 2, ..., N+1,$$ (55)
and rewrite the discrete exponential functions \( \{ e^{\xi_j} \} \) as

\[
e^{\xi_j} = e^{\delta_j} := e^{l_j m + \frac{\delta \coth l_j}{4} x}, \quad j = 1, 2, \ldots, N + 1.
\]

(56)

For the functions \( \{ e^{\delta_j} \} \), we set

\[
n \to \infty, \quad \epsilon \to 0, \quad x = n \epsilon \sim O(1),
\]

(57)

and then, get the limit

\[
e^{\delta_j} \to e^{\theta_j} := e^{l_j m + \frac{\delta \coth l_j}{4} x}, \quad j = 1, 2, \ldots, N + 1.
\]

(58)

To proceed, we reinterpret the variables \( u \) and \( w \) as \( u(n, m) := \chi(x, m) \) and \( w(n, m) = \varpi(x, m) \). Then, by substituting the Taylor expansions

\[
\bar{u} = \chi(x + \epsilon) = \chi + \epsilon \dot{\chi} + \cdots,
\]

\[
\hat{u} = \hat{\chi}(x + \epsilon) = \hat{\chi} + \epsilon \hat{\chi} + \cdots,
\]

(59)

\[
w \to 1 + \epsilon(\varpi - \hat{\varpi})
\]

(59c)

into the rndsG equation (17), we immediately obtain another real local and nonlocal semidiscrete sG (rnsdsG-x) equation

\[
2(\hat{\chi} - \dot{\chi}) + (\hat{\chi} + \chi)[2(\hat{\varpi} - \varpi) - \delta] = 0,
\]

(60a)

\[
\hat{\varpi} = \eta \chi \chi, \quad \eta = \eta(x, m)
\]

(60b)

where \( \varpi = \varpi(x, m) \) and \( \hat{f} \) denotes the derivative of \( f \) with respect to \( x \). Unsurprisingly, Equation (60) is preserved under transformation \( \chi \to -\chi \), and Equation (60) with \( (\sigma, \eta) = (\pm 1, 1) \) and \( (\sigma, \eta) = (-1, -1) \) can be transformed into each other by taking \( \chi \to i\chi \).

Now we absorb \( e^{l_N} \) in \( \Phi(n, m) \) and focus on the basic column vector \( \Phi(n, m) := \phi(x, m) \). Under the continuum limit (57), the Taylor expansion of \( E^{2j} \Phi(n, m) \) is given by

\[
E^{2j} \Phi(n, m) = \Phi(n + 2j, m) = \phi(x + 2j \epsilon, m) = \phi + 2j\epsilon \partial_x \phi + \frac{(2j\epsilon)^2}{2!} \partial_x^2 \phi + \cdots
\]

with \( j = 1, 2, \ldots, N + 1 \). Simultaneously, we have

\[
f \to |\Delta_N \phi^{(N)}; T \phi^{(N)}|, \quad g \to \frac{1}{\epsilon} |\Delta_{N+1} \Delta_{N-1} \phi^{(N+1)}; T \phi^{(N-1)}|
\]

(61)

\[\text{3 Under the limit (57), we have } w = \frac{(f + \epsilon \dot{f} + \cdots) \hat{f}}{f \hat{f} + \epsilon \hat{f} + \cdots} \to \frac{(f + \epsilon \dot{f}) \hat{f}}{f \hat{f} + \epsilon \hat{f}} = 1 + \epsilon \frac{\hat{f} f \ddot{f} - f \hat{f} \dot{f}}{f \hat{f}} = 1 + \epsilon(\varpi - \hat{\varpi}), \text{ where } \varpi = \hat{f} / f.\]
where
\[
\Delta_N = \begin{pmatrix}
2\varepsilon & \frac{(2\varepsilon)^2}{2!} & \cdots & \frac{(2\varepsilon)^N}{N!} \\
4\varepsilon & \frac{(4\varepsilon)^2}{2!} & \cdots & \frac{(4\varepsilon)^N}{N!} \\
\vdots & \vdots & \ddots & \vdots \\
2N\varepsilon & \frac{(2N\varepsilon)^2}{2!} & \cdots & \frac{(2N\varepsilon)^N}{N!}
\end{pmatrix}.
\]

(62)

Consequently, for the dependent variables \(u\) and \(w\), we have
\[
u = \frac{g}{f} \rightarrow \chi = \frac{\Delta_{N+1} \Delta_{N-1}}{\varepsilon |\Delta_N^2|} \frac{|\hat{\phi}^{(N+1)}; T\hat{\phi}^{(N-1)}_\sigma|}{|\hat{\phi}^{(N)}; T\hat{\phi}^{(N)}_\sigma|} = 2 \frac{|\hat{\phi}^{(N+1)}; T\hat{\phi}^{(N-1)}_\sigma|}{|\hat{\phi}^{(N)}; T\hat{\phi}^{(N)}_\sigma|},
\]

(63a)

\[
w = \frac{\hat{f} \hat{f}}{\hat{f} \hat{f}} \rightarrow \omega = \frac{\partial_x |\hat{\phi}^{(N)}; T\hat{\phi}^{(N)}_\sigma|}{|\hat{\phi}^{(N)}; T\hat{\phi}^{(N)}_\sigma|}.
\]

(63b)

Double Casoratian solutions to the rnsdsG-x equation (60) can be summarized by the following theorem.

**Theorem 5.** The functions \(\chi = g/f\) and \(\omega = \dot{f}/f\) with
\[
f = |\hat{\phi}^{(N)}; T\hat{\phi}^{(N)}_\sigma|, \quad g = 2|\hat{\phi}^{(N+1)}; T\hat{\phi}^{(N-1)}_\sigma|,
\]

(64)

where \(\phi = e^{\Omega m + \frac{\delta \coth \Omega}{4} C^+}\), solve the rnsdsG-x equation (60), in which \(T\) is a constant matrix of order \(2(N + 1)\), satisfying matrix equations
\[
\Omega T + \sigma T \Omega = 0, \quad T^2 = -\sigma \eta I.
\]

(65)

We take \((\sigma, \eta) = (-1, 1)\) and set
\[
\Omega = \begin{pmatrix}
S & 0 \\
0 & -S
\end{pmatrix}, \quad T = \begin{pmatrix}
I & 0 \\
0 & -I
\end{pmatrix}.
\]

(66)

**Soliton solutions:** Let \(S\) be a diagonal matrix
\[
S = \text{Diag}(l_1, l_2, \ldots, l_{N+1}),
\]

(67)

and \(C^+ = I\). The one-soliton solution is thus given by
\[
\chi = \delta \coth l_1 \text{ sech } 2\vartheta_1/2,
\]

(68a)

\[
\omega = \delta \coth l_1 \tanh 2\vartheta_1/2.
\]

(68b)

Solution \(\chi\) appears as a stable traveling wave with amplitude \(\delta \coth l_1/2\) and traveling velocity \(-4l_1/(\delta \coth l_1)\). The width is proportional to \(2/(\delta \coth l_1)\). Soliton and antisoliton determined by the sign of the parameter \(l_1\delta\) are depicted in Figure 7.
FIGURE 7  One-soliton solution $\chi$ given by (68a) with $\delta = 1$: (A) shape and movement with $k_1 = 1$; (B) soliton for $k_1 = 1$ at $m = 1$; and (C) antisoliton for $k_1 = -1$ at $m = 1$.

FIGURE 8  Solution $\chi$ given by (69) with $\epsilon = 1$: (A) two-soliton solutions with $k_1 = 1$ and $k_2 = 2$; (B) 2D plot of (a) at $m = 1$; and (C) breather solution with $k_1 = 0.2 + i$ and $k_2 = 0.2 - i$ at $m = 1$.

The two-soliton solutions are of form

$$\chi = \frac{g}{f}, \quad \varpi = \frac{\dot{f}}{f}, \quad (69)$$

in which

$$f = (\text{csch} \ l_1 \text{csch} \ l_2)^2 [ (\cosh 2l_1 \cosh 2l_2 - 1) \cosh 2\vartheta_1 \cosh 2\vartheta_2$$

$$- \sinh 2l_1 \sinh 2l_2 (1 + \sinh 2\vartheta_1 \sinh 2\vartheta_2)], \quad (70a)$$

$$g = \delta (\coth^2 l_1 - \coth^2 l_2)(\coth l_1 \cosh 2\vartheta_2 - \coth l_2 \cosh 2\vartheta_1). \quad (70b)$$

Figure 8 displays the movement of solution $\chi$, whose dynamic behavior is listed in Theorem 6.

**Theorem 6.** Suppose that $0 < l_1 < l_2$ and $\delta > 0$. Then, when $m \to \pm \infty$, the $l_1$-soliton asymptotically follows:

- **top point traces** : $x(m) = \frac{2}{\delta \coth l_1} \left( \mp \frac{\sinh (l_2 - l_1)}{\sinh (l_1 + l_2)} 2l_1 m \right), \quad (71a)$

- **amplitude** : $\chi = \frac{\coth l_1 (\coth^2 l_1 - \coth^2 l_2)}{2(\csc h l_1 \csc h l_2)^2 \sinh (l_2 - l_1) \sinh (l_1 + l_2)}, \quad (71b)$

- **speed** : $-\frac{4l_1}{\delta \coth l_1}, \quad (71c)$
phase shift : \[-\frac{4}{\delta \coth l_1} \ln \frac{\sinh(l_2 - l_1)}{\sinh(l_1 + l_2)}\], \hspace{1cm} (71d)

and the \(l_2\)-soliton asymptotically follows:

\begin{align*}
\text{top point traces} : \quad & x(m) = \frac{2}{\delta \coth l_2} \left( \pm \ln \frac{\sinh(l_2 - l_1)}{\sinh(l_1 + l_2)} - 2l_2m \right), \\
\text{amplitude} : \quad & \chi = \frac{\coth l_2 (\coth^2 l_2 - \coth^2 l_1)}{2(\text{csch} l_1 \text{csch} l_2)^2 \sinh(l_2 - l_1) \sinh(l_1 + l_2)}, \\
\text{speed} : \quad & -\frac{4l_2}{\delta \coth l_2}, \\
\text{phase shift} : \quad & \frac{4}{\delta \coth l_2} \ln \frac{\sinh(l_2 - l_1)}{\sinh(l_1 + l_2)}.
\end{align*} \hspace{1cm} (72a-d)

**Jordan-block solutions:** Let \(S\) be the Jordan-block matrix of order \(N + 1\) given by

\[
S = \begin{pmatrix}
l_1 & 0 & \cdots & 0 \\
1 & l_1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & l_1
\end{pmatrix},
\] \hspace{1cm} (73)

and \(C^+ = I\). Then \(\phi\) is composed by

\[
\phi_j = \begin{cases}
\frac{\delta_j^{j-1} e^{\delta_1}}{(j-1)!}, & j = 1, 2, \ldots, N + 1, \\
\frac{\delta_j^{j-1} e^{-\delta_1}}{(s-1)!}, & j = N + 1 + s, \quad s = 1, 2, \ldots, N + 1.
\end{cases}
\] \hspace{1cm} (74)

The simplest Jordan-block solution is expressed as

\[
\chi = \frac{\delta \coth l_1 \text{csch}^2 l_1 [4 \text{csch}^2 l_1 \cosh 2\delta_1 + 2(4m - \delta x \text{csch}^2 l_1) \coth l_1 \sinh 2\delta_1]}{4 \text{csch}^4 l_1 \cosh^2 2\delta_1 + (4m - \delta x \text{csch}^2 l_1)^2 \coth^2 l_1}, \hspace{1cm} (75a)
\]

\[
\omega = \frac{2\delta \coth l_1 \text{csch}^4 l_1 (x \delta \coth l_1 - 4m \cosh l_1 \sinh l_1 + \sinh 4\delta_1)}{4 \text{csch}^4 l_1 \cosh^2 2\delta_1 + (4m - \delta x \text{csch}^2 l_1)^2 \coth^2 l_1}, \hspace{1cm} (75b)
\]

where the behavior of \(\chi\) is depicted in Figure 9.

**Continuous limits**

Through continuous limits one can recover the rnsdsG-\(t\) equation (45) or the rnsdsG-\(x\) equation (60) to the real local and nonlocal continuous sG (rncsG) equation.

To derive the rncsG equation from the rnsdsG-\(t\) equation (45), we start from its semidiscrete exponential functions \(\{e^{k_j}\}\). With new spectral parameters

\[
a_j = (\epsilon \coth k_j)^{-1}, \quad j = 1, 2, \ldots, N + 1 \hspace{1cm} (76)
\]
FIGURE 9  Jordan-block solution $\chi$ given by (75a) with $k_1 = 1$ and $\delta = 1$: (A) shape and motion and (B) 2D plot of (a) at $m = 1$.

and limit (57), we find $\{e^{\lambda_j} \rightarrow e^{\psi_j} = e^{a_j x + \frac{1}{4a_j}}\}$. Proceeding as before, we reinterpret the variables $\mu$ and $\omega$ as $\mu(n, t) := \alpha(x, t)$ and $\omega(n, t) := \beta(x, t)$. Inserting the Taylor expansions $\bar{\mu} = \alpha(x + \epsilon) = \alpha + \epsilon \alpha' + \cdots$,

$\bar{\mu}' = \alpha'(x + \epsilon) = \alpha' + \epsilon \alpha'' + \cdots$,

$e^{\epsilon(\omega - \tilde{\omega})} - 1 = \epsilon(\omega - \tilde{\omega}) + \epsilon^2 (\omega - \tilde{\omega})^2 / 2! + \cdots = \epsilon^2 \tilde{\beta} + \cdots$ (77c)

into Equation (45) and then to leading order, we obtain the rncsG equation

$\dot{\alpha}' + 2\alpha \beta' = \alpha, \quad \dot{\beta} = \eta \alpha \alpha'_\sigma$, (78)

where $\beta = \beta_\sigma$. Equation (78) is preserved under transformation $\alpha \rightarrow -\alpha$, and Equation (78) with $(\sigma, \eta) = (\pm 1, 1)$ and $(\sigma, \eta) = (\pm 1, -1)$ can be transformed into each other by taking $\alpha \rightarrow i\alpha$. The rncsG equation (78), first proposed in Ref. 50, has been studied by many researchers from various methods, such as inverse scattering transform 12 and bilinearization reduction method. 21

Similar to earlier case, for the rnsdsG-x equation (60), we introduce

$b_j = \frac{\delta}{4} \coth l_j, \quad j = 1, 2, \ldots, N + 1$. (79)

It is readily to see that by limit (42), the semidiscrete exponential functions $\{e^{\delta_j}\}$ satisfy $\{e^{\delta_j} \rightarrow e^{\psi_j} := e^{b_j x + \frac{l_j}{4b_j}}\}$. We reinterpret the variables $\chi$ and $\varpi$ as $\chi(x, m) := \alpha(x, t)$ and $\varpi(x, m) := \beta(x, t)$. Then, taking the Taylor expansions

$\hat{\chi} = \alpha(t + \delta) = \alpha + \delta \alpha' + \cdots$, (80a)

$\hat{\chi}' = \dot{\alpha}(t + \delta) = \dot{\alpha} + \delta \dot{\alpha}' + \cdots$, (80b)

$\hat{\varpi} = \beta(t + \delta) = \beta + \delta \beta' + \cdots$ (80c)

into Equation (60), we get the rncsG equation (78).
As indicated in Ref. 21 as well as Theorem 5, we can make a clear description of exact solutions for the rnccsG equation (78).

**Theorem 7.** The functions $\alpha = g/f$ and $\beta = \hat{f}/f$ with

$$f = |\phi^{(N)}; T\phi^{(N)}_\sigma|, \quad g = 2|\phi^{(N+1)}; T\phi^{(N-1)}_\sigma|,$$

solve the rnccsG equation (78), where $\phi = e^{Bx+(4B)^{-1}t}C$ and $T$ is a constant matrix of order $2(N+1)$ satisfying

$$BT + \sigma TB = 0, \quad T^2 = -\sigma \eta I.$$

(82)

4 | COMPLEX REDUCTION: SOLUTIONS AND CONTINUUM LIMITS

In the previous section, we have applied the bilinearization reduction technique to construct exact solutions for the real local and nonlocal sG-type equations. In this section, we consider complex reduction of the dnAKNS equation (1), which yields the complex local and nonlocal discrete sG equation. By solving the matrix equation set, one-soliton solution and dynamical behaviors are presented. We then inspect continuum limits of the resulting complex local and nonlocal discrete sG equation, as well as exact solutions and dynamics. For convenience, we call the resulting complex local and nonlocal sG-type equations as cnndS, cnndsG-t, cnndsG-x, and cnncS, respectively.

4.1 | Complex reduction

Imposing complex reduction

$$v = \eta u_\sigma^*,$$  \quad $\eta, \sigma = \pm 1,$

(83)

where asterisk denotes the complex conjugate, into the dnAKNS equation (1), we arrive at the cnndS equation

$$4(u + \hat{u} - (\hat{u} + \hat{u})w) = \delta \epsilon (u + \hat{u} + (\hat{u} + \hat{u})w),$$

(84a)

$$\left(1 + \eta \epsilon^2 \hat{u} \hat{u}^* \sigma\right)w = \left(1 + \eta \epsilon^2 uu^* \sigma\right)w,$$

(84b)

where $w = w_\sigma^*.$ Equation (84) is a local equation as $\sigma = 1$ and a nonlocal equation as $\sigma = -1$. This equation is preserved under transformations $u \rightarrow -u$ and $u \rightarrow \pm iu$.

To implement reduction on the level of double Casoratian solutions, we take $M = N$ and then have the following result.

**Theorem 8.** The functions $u = g/f$ and $w = \hat{f}/(\hat{f})$ with

$$f = |e^{-NT\Phi^{(N)}_\sigma}; e^{NT\Phi^{(N)}_\sigma}|, \quad g = (1/\epsilon)|e^{-NT\Phi^{(N+1)}_\sigma}; e^{NT\Phi^{(N+1)}_\sigma}|,$$

(85)
solve the cnndsG equation (84), if the \((2N + 2)\)th-order column vector \(\Phi\) is given by (14), where \(T \in \mathbb{C}^{(2N+2) \times (2N+2)}\) in (85) is a constant matrix, satisfying

\[
\Gamma T + \sigma T \Gamma^* = 0, \quad TT^* = \begin{cases} -\eta I, & \text{with } \sigma = 1, \\ \eta|e^{i\theta}|^2 I, & \text{with } \sigma = -1. \end{cases} \tag{86}
\]

### 4.2 One-soliton solution

We note that, compared with the double Casoratian solutions of the rndsG equation (17), the solutions for the cnndsG equation (84) are more complicated. Hence, we just construct one-soliton solution of equation (84) in the case of \((\eta, \sigma) = (1, -1)\). Let matrices \(\Gamma\) and \(T\) be of form

\[
\Gamma = \begin{pmatrix} L & 0 \\ 0 & L^* \end{pmatrix}, \quad T = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} |e^{i\theta}|. \tag{87}
\]

When the discrete spectral parameters \(\{\kappa_j\}\) in diagonal matrix (26) are different nonzero complex numbers, the basic column vector \(\Phi\) is made up of

\[
\Phi_j = \begin{cases} c_j e^{\xi_j}, & j = 1, 2, \ldots, N + 1, \\ d_s e^{\xi_s}, & j = N + 1 + s, \quad s = 1, 2, \ldots, N + 1, \end{cases} \tag{88}
\]

where and whereafter \(\{c_j, d_s\}\) are complex constants.

The one-soliton solution of Equation (84) is

\[
u = \frac{2c_1 d_1 \sinh(k_1^*-k_1)}{\epsilon(|c_1|^2 e^{-2\xi_1} - |d_1|^2 e^{-2\xi_1})}, \tag{89a}
\]

\[
w = \frac{\tilde{f} \hat{f}}{(\tilde{f} \hat{f})}, \quad |f| = |c_1|^2 e^{\xi_1} - |d_1|^2 e^{\xi_1}, \tag{89b}
\]

where \(\xi_1\) is defined as (25). To demonstrate the dynamics of solution \(u\) in an analytic way, we take \(k_1 = k_{11} + ik_{12}\), with which (89a) becomes

\[
\epsilon_1 = \frac{[(\delta \epsilon)^2 - 16](1 + e^{4k_{11}}) + 2((\delta \epsilon)^2 + 16)e^{2k_{11}} \cos 2k_{12})^2 + (16\delta \epsilon e^{2k_{11}} \sin 2k_{12})^2}{(4 + \delta \epsilon + (\delta \epsilon - 4)e^{2k_{11}} \cos 2k_{12})^2 + ((\delta \epsilon - 4)e^{2k_{11}} \sin 2k_{12})^2}, \tag{90b}
\]

\[
\epsilon_2 = \arctan \frac{16\delta \epsilon e^{2k_{11}} \sin 2k_{12}}{(4 + \delta \epsilon + (\delta \epsilon - 4)e^{2k_{11}} \cos 2k_{12})^2 + ((\delta \epsilon - 4)e^{2k_{11}} \sin 2k_{12})^2}, \tag{90c}
\]

and the corresponding envelop reads

\[
|u|^2 = \frac{4|c_1 d_1|^2 \epsilon_1^m e^{4nk_{11}} \sin^2 2k_{12}}{\epsilon^2 (|c_1|^4 + |d_1|^4 - 2|c_1 d_1|^2 \cos(4nk_{12} + 2m\epsilon_2))}. \tag{91}
\]
The carrier wave \((91)\) is oscillatory due to the cosine in the denominator. When \(|c_1| = |d_1|\), wave \((91)\) has singularities along points

\[
n(m) = \frac{\ell \pi - m \varepsilon_2}{2k_{12}}, \quad \ell \in \mathbb{Z}.
\]  

While when \(|c_1| \neq |d_1|\), wave \((91)\) is nonsingular and reaches its extrema along points

\[
n(m) = -\frac{m \varepsilon_2}{2k_{12}} + \frac{1}{4k_{12}} \left( \arcsin \left( \frac{|c_1|^4 + |d_1|^4)k_{11}}{2|c_1d_1|^2\sqrt{k_{11}^2 + k_{12}^2}} - j + 2\ell \pi \right) \right), \quad \ell \in \mathbb{Z},
\]  

where \(\sin j = k_{11}/\sqrt{k_{11}^2 + k_{12}^2}\). The velocity is \(-\varepsilon_2/(2k_{12})\). It is easy to see that for any \(m\), \(|u|^2\) approaches to 0 as either \((k_{11} < 0, n \to +\infty)\) or \((k_{11} > 0, n \to -\infty)\). We depict such a wave in Figure 10.

### 4.3 | Continuum limits

Applying the two semicontinuous limits introduced in Subsection 3.3 to the cndsG equation \((84)\), we can obtain the cnsdsG-\(t\) and cnsdsG-\(x\) equations, respectively. Moreover, the cncsG equation can be derived by performing the continuous limits.

**Semicontinuous limit in \(m\)-direction**

Under the semicontinuous limit in \(m\)-direction \((42)\), inserting the Taylor expansions \((44)\) into Equation \((84)\), we realize that the coefficient of the leading order \(O(\delta)\) is exactly the cnsdsG-\(t\) equation

\[
2(\bar{\mu} - \mu)' = \epsilon(\mu + \bar{\mu})(1 - 2\omega'),
\]

\[
e^{\epsilon(\omega - \bar{\omega})} - 1 = \eta \varepsilon^2 \mu \bar{\mu}^n.
\]

It can be checked that this equation is preserved under transformations \(\mu \to -\mu\) and \(\mu \to \pm i\mu\).
Double Casoratian solutions of the cnGdsG-t equation (94) are presented in the following theorem.

**Theorem 9.** The functions \( \mu = g/f \) and \( \omega = \varepsilon^{-1} \ln(\bar{f}/f) \) with

\[
    f = |e^{-NT\hat{\Phi}(N)}; e^{NT\Phi_T(N)}|, \quad g = (1/\varepsilon)|e^{-NT\hat{\Phi}(N+1)}; e^{NT\Phi_T(N+1)}|,
\]

solve the cnGdsG-t equation (94), where \( \Phi = e^{\Gamma_t + c} \coth \Gamma t/4 C^+ \) and \( T \) is a constant matrix of order \( 2(N + 1) \), satisfying

\[
    \Gamma T + \sigma T^* = 0, \quad T^* = \begin{cases} -\eta I, & \text{with } \sigma = 1, \\ \eta |e^{\Gamma^*}|^2 I, & \text{with } \sigma = -1. \end{cases}
\]

For \((\sigma, \eta) = (-1, 1)\), we take matrices \( \Gamma \) and \( T \) as block form (87). To the diagonal matrix \( L \) defined as (26) with distinct complex nonzero eigenvalues \( \{k_j\} \), the column vector \( \Phi \) with entries

\[
    \Phi_j = \begin{cases} c_j e^{\lambda_j}, & j = 1, 2, \ldots, N + 1, \\ d_s e^{\lambda_s^*}, & j = N + 1 + s, \quad s = 1, 2, \ldots, N + 1, \end{cases}
\]

yields the multisoliton solutions, where \( \{\lambda_j\} \) are defined as (43).

In particular, the one-soliton solution is described as

\[
    \mu = \frac{2c_1d_1 \sinh(k_1 - k_1^*)}{\varepsilon(|d_1|^2 e^{-2k_1} - |c_1|^2 e^{-2k_1^*})},
\]

\[
    \omega = \frac{1}{\varepsilon} \ln \frac{|c_1|^2 e^{2(k_1 + \lambda_1)} - |d_1|^2 e^{2(k_1^* + \lambda_1^*)}}{e^{k_1 + k_1^*}(|c_1|^2 e^{2\lambda_1} - |d_1|^2 e^{2\lambda_1^*})}.
\]

For \( k_1 = k_{11} + ik_{12} \), we have

\[
    \mu = \frac{2ic_1d_1 e^{2k_{11}n + Y_1 t} \sin 2k_{12}}{\varepsilon(|d_1|^2 e^{-i(2k_{12} - Y_2 t)} - |c_1|^2 e^{i(2k_{12} - Y_2 t)})},
\]

where \( Y_1 = \frac{\varepsilon \sinh 2k_{11}}{2(\cosh 2k_{11} - \cos 2k_{12})} \) and \( Y_2 = \frac{\varepsilon \sin 2k_{12}}{2(\cosh 2k_{11} - \cos 2k_{12})} \). The wave package

\[
    |\mu|^2 = \frac{4|c_1d_1|^2 e^{4k_{11}n + 2Y_1 t} \sin^2 2k_{12}}{\varepsilon^2(|c_1|^4 + |d_1|^4 - 2|c_1d_1|^2 \cos(4nk_{12} - 2Y_2 t))},
\]

is still quasi-periodic. When \( |c_1| = |d_1| \), wave (100) has singularities along points

\[
    n(t) = \frac{\ell \pi + Y_2 t}{2k_{12}}, \quad \ell \in \mathbb{Z}.
\]

While when \( |c_1| \neq |d_1| \), this wave is nonsingular, and reaches its extrema along points

\[
    n(t) = \frac{\ell Y_2}{2k_{12}} + \frac{1}{4k_{12}} \left( \arcsin \frac{(|c_1|^4 + |d_1|^4)k_{11}}{2|c_1d_1|^2 \sqrt{k_{11}^2 + k_{12}^2}} - j + 2\ell \pi \right), \quad \ell \in \mathbb{Z}.
\]
The traveling speed is $\frac{\gamma_2}{2k_1^2}$. For any $t$, $|\mu|^2$ approaches to zero as either ($k_{11} < 0, n \to +\infty$) or ($k_{11} > 0, n \to -\infty$). The dynamic of (100) is depicted in Figure 11.

Semicontinuous limit in $n$-direction

Employing the semicontinuous limit in $n$-direction, that is, (57), we find that the leading order of Taylor expansion for Equation (84) gives rise to the cnsdsG-x equation

$$2(\hat{\chi} - \chi) + (\hat{\chi} + \chi)[2(\hat{\omega} - \omega) - \delta] = 0,$$

$$\hat{\omega} = \eta \chi \chi^*,$$

where $\omega = \omega^*$. This equation is preserved under transformations $\chi \to -\chi$ and $\chi \to \pm i\chi$.

**Theorem 10.** The functions $\chi = g/f$ and $\omega = f/f$ with

$$f = |\phi^{(N)}; T\phi^{(N)}|, \quad g = 2|\phi^{(N+1)}; T\phi^{(N)}|,$$

solve the cnsdsG-x equation (103), where $\phi = e^{\Omega t + \frac{\delta \coth \Omega}{4} N} C^+$ and $T$ is a constant matrix of order $2(N + 1)$, satisfying

$$\Omega T + \sigma T \Omega^* = 0, \quad TT^* = -\sigma \eta I.$$  

We still take $(\sigma, \eta) = (-1, 1)$. In the case of block matrices (66) with diagonal matrix $S$ in (67), the column vector $\phi$ composed by

$$\phi_j = \begin{cases} c_j e^{\theta_j}, & j = 1, 2, \ldots, N + 1, \\ d_s e^{\theta_s}, & j = N + 1 + s, \quad s = 1, 2, \ldots, N + 1, \end{cases}$$

produces the multisoliton solutions, where $\{\theta_j\}$ are defined as (58).

In the present case, the one-soliton solution is of form

$$\chi = \frac{\delta c_1 d_1 (\coth l^*_1 - \coth l_1)}{2(|c_1|^2 e^{-2\theta_1} - |d_1|^2 e^{-2\theta_1})}.  \quad (107a)$$
One-soliton solution $|\chi|^2$ given by (108) with $\delta = 1$ and $k_1 = 0.005 + 8i$: (A) shape and movement with $c_1 = 0.5 + 0.4i$ and $d_1 = 0.3 + 0.5i$; (B) 2D plot of (a) at $m = 1$; and (C) 2D plot with $c_1 = d_1 = 1 + i$ at $m = 1$.

\[
\varpi = \frac{\delta (\coth l_1 - \coth l_1^*) (|c_1|^2 e^{2\delta_1} + |d_1|^2 e^{2\delta_1^*})}{4(|c_1|^2 e^{2\delta_1} - |d_1|^2 e^{2\delta_1^*})}.
\] (107b)

Setting $l_1 = l_{11} + il_{12}$ leads to the modulus of $\chi$, given by

\[
|\chi|^2 = \frac{|\delta c_1 d_1|^2 e^{4ik_1 l_{11} + 2h_1 x} (\cosh 2l_{11} - \cos 2l_{12})^2 \sin^2 2l_{12}}{|c_1|^4 + |d_1|^4 - 2|c_1 d_1|^2 \cos (4ml_{12} - 2h_2 x)}.
\] (108)

where $h_1 = \frac{\delta \sinh 2l_{11}}{2(\cosh 2l_{11} - \cos 2l_{12})}$ and $h_2 = \frac{\delta \sin 2l_{12}}{2(\cosh 2l_{11} - \cos 2l_{12})}$.

Wave $|\chi|^2$ still has a quasi-periodic phenomenon. When $|c_1| = |d_1|$, it has singularities along points

\[
x(m) = \frac{2ml_{12} + \ell \pi}{h_2}, \quad \ell \in \mathbb{Z}.
\] (109)

While when $|c_1| \neq |d_1|$, this wave is nonsingular, and reaches its extrema along points

\[
x(m) = \frac{2ml_{12}}{h_2} + \frac{1}{2h_2} \left\{ \arcsin \frac{(|c_1|^4 + |d_1|^4)h_1}{2|c_1 d_1|^2 \sqrt{h_1^2 + h_2^2}} - \varphi + 2\ell \pi \right\}, \quad \ell \in \mathbb{Z},
\] (110)

where $\sin \varphi = h_1 / \sqrt{h_1^2 + h_2^2}$. This wave travels with speed $2l_{12}/h_2$. For any $m$, $|\chi|^2$ approaches to zero as either ($h_1 < 0, x \to +\infty$) or ($h_1 > 0, x \to -\infty$). The dynamic of (108) is depicted in Figure 12.

**Continuous limits**

Performing the continuous limits (42) and (57), respectively, to the cnssdsG-x equation (103) and the cnssdsG-t equation (94), we get the cnssG equation

\[
\dot{\alpha}' + 2\alpha \beta' = \alpha, \quad \dot{\beta} = \eta \alpha \alpha^* \beta, 
\] (111)

where $\beta = \beta_0^*$. This equation is preserved under transformations $\alpha \to -\alpha$ and $\alpha \to \pm i\alpha$. 
Theorem 11. The functions $\alpha = g/f$ and $\beta = \dot{f}/f$ with
\[
f = |\phi^{(N)}; T\phi^{*(N)}|, \quad g = 2|\phi^{(N+1)}; T\phi^{*(N-1)}|, \tag{112}
\]
solve the cnsg equation (111), where $\phi = e^{Bx+(4B)^{-1}tC}$ and $T$ is a constant matrix of order $2(N + 1)$, satisfying
\[
BT + \sigma TB^* = 0, \quad TT^* = -\sigma\eta I. \tag{113}
\]

5 | CONCLUSIONS

In this paper, local and nonlocal reductions of the dnAKNS equation (1) are investigated. As a result, real/complex local and nonlocal discrete sG equations are derived, where exact solutions to the nonlocal discrete sG equation are determined by applying the so-called bilinearization reduction method. In the real case, by solving the matrix equations (20), we obtain one-, two-soliton solutions and the simplest Jordan-block solution for the rndsG equation (17). Dynamics of variable $u$ are given by asymptotic analysis as a demonstration. Through the semicontinuous limit in $m$-direction, respectively, $n$-direction, two real nonlocal semidiscrete sG equations (Equations (45) and (60) with $\sigma = -1$) are derived. For these two equations, we present their one-, two-soliton solutions and the simplest Jordan-block solution, as well as the dynamical properties. Furthermore, by the continuous limits, we retrieve the real nonlocal sG equation (78). In the complex case, we focus on the matrix equations (86). In the assumption of $L$ being a diagonal matrix with different complex nonzero discrete spectral parameters $\{k_j\}$, the one-soliton solution and its dynamics behavior are discussed. By using the same semicontinuous limits and continuous limits as the real case, the construction of the cnssdsG-$t$, cnssdsG-$x$, and cnsgsG equations, that is, Equations (94), (103), and (111), are presented. One-soliton solution and corresponding dynamics to the first two equations are studied emphatically. The one-soliton solutions of the rndsG/rndsG-$t$/rndsG-$x$ equations exhibit the usual bell-type structure, whereas of the cnssG/cnssdsG-$t$/cnssdsG-$x$ equations behave quasi-periodically.

We end the paper with the following remarks. First of all, for solutions to the matrix equations (20) with $\sigma = -1$, we have other choices, one of which reads (cf. Refs. 21 and 51)
\[
\Gamma = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}, \quad T = |e^\Gamma| \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{114}
\]
In the simplest case, we take $\Lambda_1 = k_1$ and $\Lambda_2 = \bar{k}_1$ and have the one-soliton solution
\[
u = (e^{\xi_1 + \bar{\xi}_1} \sinh(k_1 - \bar{k}_1) \sech (\xi_1 - \bar{\xi}_1))/\epsilon, \tag{115a}
\]
\[
\begin{align*}
\omega &= \cosh(k_1 - \bar{k}_1 + \xi_1 - \bar{\xi}_1) \cosh(\tau_1 - \bar{\tau}_1 + \xi_1 - \bar{\xi}_1) \sech(\xi_1 - \bar{\xi}_1) \\
& \quad \sech(k_1 - \bar{k}_1 + \tau_1 - \bar{\tau}_1 + \xi_1 - \bar{\xi}_1), \tag{115b}
\end{align*}
\]
where $\tilde{\xi}_1 = \xi_1|_{k_1 \to \bar{k}_1}$ and $\tilde{\tau}_1 = \tau_1|_{k_1 \to \bar{k}_1}$. Under the assumption $k_1 \neq -\bar{k}_1$, solution $u$ in (115a) behaves totally different from the one (27a). In fact, for a given $m$, $u \to \pm(\mp)\infty$ under the limit $n \to \pm\infty$ with assumption $\epsilon(k_1 - \bar{k}_1) > (\prec)0$.

What is more, we have shown that the bilinearization reduction method is valid in the study of nonlocal discrete integrable systems. Compared with the Cauchy matrix reduction approach, there is a great advantage of the bilinearization reduction method. The latter method allows one to construct exact solutions of the real reduced equations. While in the Cauchy matrix reduction scheme, one cannot achieve that. This is because, in the Cauchy matrix reduction approach, solutions of the original before-reduction AKNS system should satisfy two Sylvester equations. In the real reduction case, the solvability of these two Sylvester equations usually conflicts with the solvability of the Sylvester equation in the matrix equation set (for more detailed explanations, one can refer to the conclusions in Ref. 37).

In addition, let us go back to the sG equation. Through suitable reciprocal transformation, the sG equation can be transformed into the short pulse, which describes the propagation of short optical pulses in nonlinear media. In Ref. 54, the hodograph transformations between nonlocal short pulse models and nonlocal sG system are revealed. It is shown that the independent variables of the short pulse models and sG equation that are connected via hodograph transformation are covariant in nonlocal reductions. This gives us the motivation to consider the hodograph transformations between the discrete short pulse models and the discrete sG equations as well as their nonlocal cases.

To summarize, from the dnAKNS equation (1), we consider its local and nonlocal reductions by utilizing the bilinearization reduction method. This approach can also be used to discuss nonlocal reductions of the positive-order discrete AKNS system, which admits double Casoratiian solutions. Very recently, we employ this method to study the nonlocal reductions of a discrete positive-order AKNS equation. Consequently, the real/complex local and nonlocal discrete mKdV equations are obtained. We also investigated solutions and continuum limits of the resulting nonlocal discrete mKdV equations (cf. Ref. 57).

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Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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