The Stability of D-term Cosmic Strings

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Abstract

In this note, which is based on hep-th/0611111, we review the stability of the static, positive deficit angle D-term string solutions of $D = 4$, $N = 1$ supergravity with a constant Fayet-Iliopoulos term. We prove the semi-classical stability of this class of solutions using standard positive energy theorem techniques. In particular, we discuss how the negative deficit angle D-term string, which also solves the Killing spinor equations, violates the dominant energy condition and so is excluded from our arguments.

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Introduction

There has been a remarkable resurgence of interest in cosmic strings in recent years (see [1], for instance). An observation of the double galaxy gravitational lens candidate CSL-1 [2] which was indicative of a cosmic string sparked particular interest in the field. Further observations of CSL-1 by the Hubble space telescope later showed that this was in fact a pair of very similar, but distinct galaxies [3]. Despite this, the interest in cosmic strings remains high, and other, indirect, experimental evidence is suggestive of the existence of these objects [4].

There have also been considerable theoretical advances in our understanding of supersymmetric cosmic strings. String topological defects were found some time ago in globally supersymmetric theories [5], but were only recently embedded into supergravity [6]. The particular class of local cosmic strings we are interested in can be found as solitonic solutions supported by a $D$-term potential in four-dimensional $\mathcal{N} = 1$ supergravity with constant Fayet-Iliopoulos terms [6]. A $D$-term string can also be understood as a $D_{1+q}$-brane wrapping a calibrated $q$-cycle in an internal manifold of a string theory compactification [6]. In fact, these $D$-term string solutions were found previously as point-like solutions in three-dimensional supergravity [7].

Much attention has focused on cosmological aspects of string theory cosmic strings, e.g. string networks [8]. However, it is surprising to note that the stability of a single, isolated supersymmetric string solution of supergravity has not been discussed. Bogomol’nyi bounds for general cosmic strings were constructed originally by Comtet and Gibbons [10] and the energy of local string solutions in current discussions, including the $D$-term strings, is usually defined using such Bogomol’nyi-type arguments [6, 11]. However, as noted in [11], a Bogomol’nyi bound does not prove the stability of such local string solutions, as one is implicitly assuming that the solutions remain axisymmetric. It is therefore possible that non-axisymmetric perturbations or string worldvolume perturbations could lead to instabilities.

In this article, which is a summary of [9], we discuss the semi-classical stability of the $D$-term string solution of $D = 4$, $\mathcal{N} = 1$ supergravity with a constant Fayet-Iliopoulos term. Regardless of the particular theory in which one is interested, the stability of cosmic strings is necessary if we hope to observe them. We apply the spinorial Witten-Nester method to prove a positive energy theorem for the $D$-term cosmic string background with positive deficit angle. We also pay particular attention to the negative deficit angle $D$-term string, which is known to violate the dominant energy condition. Within the class of string solutions we consider, this violation implies that the negative deficit angle $D$-term string must have a naked pathology and therefore the positive energy theorem we prove does not apply to it.
The $D$-term string in $\mathcal{N} = 1$ supergravity

Let us begin by briefly reviewing the relevant aspects of four-dimensional $\mathcal{N} = 1$ supergravity with constant Fayet-Iliopoulos terms [6]. The Lagrangian for the bosonic sector of this theory is

$$ e^{-1} \mathcal{L} = \frac{1}{2} R - \hat{\partial}_\mu \phi \hat{\partial}^\mu \phi^* - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D^2, \quad (1) $$

where $\phi$ is the $U(1)$-charged Higgs field, the Kähler potential is given by $K = \phi^* \phi$ and the superpotential vanishes. The $D$-term potential is defined by $D = g\xi - g\phi^* \phi$, where $\xi$ is a constant that we choose to be positive. $W_\mu$ is an abelian gauge field and we define

$$ F_{\mu\nu} \equiv \partial_\mu W_\nu - \partial_\nu W_\mu, \quad \hat{\partial}_\mu \phi \equiv (\partial_\mu - igW_\mu)\phi. \quad (2) $$

The fermions are Majorana spinors, however it is often convenient to split them into complex parts using left and right projectors $P^L,R = \frac{1}{2}(1 \pm \gamma^5)$. The supersymmetry transformations for the fermions (the Killing spinor equations) can then be written as

$$ \delta \psi_\mu = \hat{\nabla}_\mu \epsilon = \nabla_\mu \epsilon + \frac{i}{2} \gamma_5 A^B_\mu \epsilon, \quad \delta \chi^L = \frac{1}{2} (\partial - igW) \phi \epsilon^R, \quad \delta \lambda = \frac{1}{4} \gamma^{\mu\nu} F_{\mu\nu} \epsilon + \frac{i}{2} \gamma_5 D \epsilon. \quad (4) $$

The covariant derivative on fermions is defined as $\nabla_\mu = \partial_\mu + \frac{1}{4} \omega^{ab}_\mu (\epsilon) \gamma_{ab}$. The gravitino $U(1)$ connection $A^B_\mu$ plays an important role in the gravitino transformations.

$$ A^B_\mu = \frac{1}{2} \left[ \phi \hat{\partial}_\mu \phi^* - \phi^* \hat{\partial}_\mu \phi \right] + gW_\mu \xi. \quad (5) $$

The cosmic string solutions to this theory found in [6] solve the Killing spinor equations (3) - (4) for some non-vanishing $\epsilon$. The metric ansatz in cylindrically symmetric form is

$$ ds^2 = -dt^2 + dz^2 + dr^2 + C^2(r) d\theta^2, \quad (6) $$

where the plane of the string is parametrised by $r$ and $\theta$. We choose vierbein $e^1 = dr$ and $e^2 = C(r) d\theta$, which gives $\omega_{b1} = -C'(r)$ as the only non-vanishing spin connection component.

The Higgs field and gauge potential have the following form

$$ \phi(r, \theta) = f(r) e^{i n \theta}, \quad gW_\mu dx^\mu = n \alpha'(r) d\theta. \quad (7) $$

where $\theta$ is an azimuthal angle, and $f(r)$ is a real function that outside the string core approaches the vacuum value $f^2 = \xi$, for which the $D$-term vanishes. One can

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1We are using natural units, setting $M_P = 1$. We direct the reader to [9] for a full explanation of our conventions and a complete list of references.
solve for the profile functions $\alpha(r)$ and $C(r)$ explicitly in limiting cases, and one sees that the metric describes a spacetime with a conical deficit angle proportional to the Fayet-Iliopoulos constant $\xi$. A globally well-behaved spinor parameter is defined by $\epsilon_L(\theta) = e^{\mp \frac{1}{2}i\theta} \epsilon_{0L}$ where $\epsilon_{0L}$ is a constant spinor parameter satisfying $\gamma^{12} \epsilon_0 = \mp \gamma_5 \epsilon_0$. By demanding that the following condition holds

$$1 - C'(r) = \pm A^B_{\theta}, \quad (8)$$

one can then find solutions to the gravitino Killing spinor equation (3). As noted originally for three-dimensional supergravity [7], the key to solving this Killing spinor equation in a conical spacetime is the $U(1)$ charge of the gravitino. This allows the singular spin connection term to be cancelled precisely because both the $U(1)$ charge and the deficit angle are set by the Fayet-Iliopoulos term $\xi$.

When the distance $r$ from the string core is large, the solution (6) takes the form of an asymptotically locally flat conical metric with an angular deficit angle due to the constant FI term $\xi$:

$$ds^2 = -dt^2 + dz^2 + dr^2 + r^2 (1 \mp n \xi)^2 d\theta^2, \quad (9)$$

with the composite gauge field given by $A^B_{\theta} = n\xi$. Note that in the limit $r \to \infty$ the full supersymmetry is restored as $F_{\mu\nu} = 0$, $D = 0$, $\partial_r \phi = \hat{\partial}_\theta \phi = 0$ and $R_{\mu\nu} ab = 0$, which corresponds to the enhancement of supersymmetry away from the core of the string.

In [6], the string energy density was defined using a Bogomol’nyi style argument. As the solution is time-independent, the ansatz could be directly inserted into the action with Gibbons-Hawking boundary terms included to give an energy functional. The integral was then restricted to only run over directions transverse to the string to ensure it produced a finite result. Using the Bogomol’nyi method, this integral was then written in the following way

$$\mu_{\text{string}} = \int dr d\theta C(r) \left\{(\hat{\partial}_r \phi \pm iC^{-1} \hat{\partial}_\theta) \phi|^2 \right\} + \frac{1}{2} [F_{12} \mp D]^2 + \int dr d\theta \left[ \partial_r (C' \pm A^B_{\theta}) \mp \hat{\partial}_\theta A^B_{r} \right] - \int d\theta C' \bigg|_{r=\infty} + \int d\theta C' \bigg|_{r=0}. \quad (10)$$

The condition arising from the gravitino Killing spinor equation (8) implies that the first term in the second line in (10) vanishes. The first line vanishes by the remaining Killing spinor equations $\delta \lambda = 0 = \delta \chi_L$. The energy density is thus given by the difference between the boundary terms at $r = 0$ and at $r = \infty$ [6]:

$$\mu_{\text{string}} = 2\pi (C'|_{r=0} - C'|_{r=\infty}) = \pm 2\pi n \xi, \quad (11)$$

which agrees with the expected answer for a cosmic string solution [1]. It is interesting to note that as supersymmetry only fixed the metric function $C(r)$ up to a sign [6],
the energy also has a sign ambiguity. It was initially argued that the positive energy theorem implies that the negative energy solution should be ignored [6]. This is not correct, and the role of negative energy solution, i.e. the negative deficit angle solution, must be carefully reconsidered.

**Negative deficit angle D-term strings**

A basic assumption in the proof of any positive energy theorem, which holds for all reasonable matter fields, is that the stress-energy tensor satisfies the dominant energy condition i.e. that for any timelike or null vector \( u^a \), \(-T^b_a u^a \) is non-spacelike, which implies that \( T_{ab}u^a u^b \geq 0 \). It is straightforward to check that the supersymmetric Lagrangian (1) satisfies the dominant energy condition, which can be conveniently restated as saying that matter energy density is non-negative in any orthonormal frame i.e. \( T_{00} \geq 0 \). For a general cylindrically symmetric spacetime, Comtet and Gibbons have shown that it is possible to find a useful rewriting of the metric in which it becomes clear that the sign of the string deficit angle \( \delta \) is completely determined by \( T_{00} \):

\[
\delta \sim +\int_{\Sigma_2} T_{00} + (\ldots)^2,
\]

where \( \Sigma_2 \) is a two-dimensional submanifold transverse to the string (we do not require the specific form of the terms in brackets). Hence, we see that it is only possible to have a solution with \( \delta < 0 \) if the Lagrangian violates the dominant energy condition. This implies that a \( \delta < 0 \) string is not a regular solution to the field equations derived from our Lagrangian (1), and therefore needs a source with negative \( T_{00} \). Although the \( \delta < 0 \) solution is not known in closed form for small radius, the string ansatz we are using (6) does not have a \( g_{tt} \) component, and hence does not allow it to have horizons in the interior of the solution. This means that the defect, which requires the presence of a source, sweeps out a worldvolume over infinite time. In other words, the region of the solution that violates Einstein’s equations is naked, which implies that no spacelike surface will be able to avoid it. Therefore no Cauchy surface exists in the \( \delta < 0 \) spacetime and so the positive energy theorem does not apply to it.

**The positive energy theorem and semi-classical stability**

We shall now apply the standard Witten-Nester technique to prove the positive energy theorem for the positive deficit angle string backgrounds. We begin by defining the generalised Witten-Nester 2-form [12]:

\[
E^\mu\nu = \bar{\eta} \gamma^{\mu\nu\rho} \hat{\nabla}_\rho \eta ,
\]
where we are using the supercovariant derivative defined by the gravitino supersymmetry transformation (3), and \( \eta \) denotes a commuting spinor function that asymptotically tends to a background Killing spinor \( \tilde{\nabla}_\rho \eta = 0 \). We now define the Witten-Nester four-momentum as the integral of the dual of \( E \)

\[
P_{\mu} v^\mu = \int_{\partial M} \ast E = \frac{1}{2} \int_{\partial M} dS_{\mu\nu} E^{\mu\nu} = \int_{M} d\Sigma_\nu \nabla_\mu E^{\mu\nu}
\]

(14)

where \( v^\mu = \bar{\eta} \gamma^\mu \eta \). In the final equality we have assumed that there are no internal boundaries, since the \( \delta > 0 \) D-term is regular [6], and used Gauss’ law to write a volume integral. At this point, one should understand that \( \partial M \) is the two-dimensional boundary of an arbitrary three-dimensional subsurface \( M \). In order to evaluate the Witten-Nester total four-momentum explicitly for a particular solution, the surface charge integral must be regulated [13]. One must wrap the spatial worldvolume of the string such that \( \partial M = \mathbb{R}_r \times S^1_\theta \rightarrow S^1_\theta \times S^1_\theta \), and then integrate out the \( z \)-contribution. The charge integral is then defined only over spatial directions transverse to the string, and it is formally the same as the equivalent three-dimensional expression [7]. By considering linearised perturbations that vanish asymptotically, one then sees that the surface integral form of the Witten-Nester total four-momentum becomes:

\[
P_{\mu} v^\mu = \frac{1}{8} \int_{\partial M} dS_{\mu\nu} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\delta\alpha\beta\sigma} \Delta \omega_{\alpha\beta} \bar{\eta} \gamma^\rho \eta + \frac{1}{4} \int_{\partial M} dS_{\mu\nu} \varepsilon^{\mu\nu\rho\sigma} A^B_{\rho} \bar{\eta} \gamma^\rho \eta .
\]

(15)

The first term in (15), which we shall denote \( P_{\mu} v^\mu \), is Nester’s expression for the gravitational four-momentum, where \( \Delta \omega_{\alpha\beta} \) is the difference of the spin connection with respect to the reference spacetime with \( A^B_{\rho} = 0 \), i.e. Minkowski spacetime. The second term in (15), which we shall denote \( J^R_{\mu} v^\mu \), defines the R-charge of the string, i.e. the holonomy of composite gauge potential.

In order to prove the positivity of the Witten-Nester four-momentum we now turn to the volume integral expression. We want to show that an arbitrary on-shell perturbation of a supersymmetric solution that vanishes asymptotically, but is otherwise unbounded, contributes a positive amount to the total energy. As such, we shall not wrap the spatial direction of the string worldvolume, such that \( M = \mathbb{R}_r \times \mathbb{R}_z \times S^1_\theta \), allowing for the most general perturbations. A lengthy calculation using the standard manipulations [12] then leads to the following expression

\[
P_{\mu} v^\mu = \int_{M} d\Sigma_\nu \left( \tilde{\nabla}_\mu \eta \gamma^\rho \bar{\nabla}_\rho \eta + \delta \lambda \gamma^\nu \delta \lambda + 2 \delta \chi_L \gamma^\nu \delta \chi_L + 2 \delta \chi_R \gamma^\nu \delta \chi_R \right)
\]

(16)

where \( \delta \lambda \) and \( \delta \chi_{L,R} \) are the supersymmetry transformations (4), defined now with a commuting spinor parameter \( \eta \). If we now choose \( \Sigma \) to be an initial hypersurface with simple timelike norm and require that the spinors obey the generalised Witten condition \( \gamma^j \tilde{\nabla}_j \eta = 0 \), we see that (16) is then manifestly positive. As our spinors are
Majorana, it is straightforward to show that the Killing vector $v^\mu$ is non-spacelike and future directed, and thus that positivity of (16) implies that energy is positive.

As our choice of initial hypersurface $\Sigma$ was arbitrary, we can allow for arbitrary variations of it. This means that our expressions get promoted to fully covariant versions, and the Witten condition becomes $\gamma^\mu \nabla_\mu \eta = 0$. If we now use the covariant form of our result that $P_\mu \geq 0$ in conjunction with (15), we reproduce the Bogomol’nyi bound for the $D$-term string:

$$P_\mu v^\mu = \pi_0 \left( P_\nu - J^R_\nu \right) \gamma^\nu \eta_0 \geq 0.$$  \hfill (17)

Looking again at (16), we see that this inequality is saturated when the solution is supersymmetric, i.e. when $\delta \lambda = \delta \chi = \delta \psi_\mu = 0$, or equivalently the condition (8) holds. Here $\delta \psi_\mu$ has been promoted to $\delta \psi_\mu$ by allowing for arbitrary variations of the hypersurface $\Sigma$. It is possible to bring the Bogomol’nyi bound (17) into the more familiar form $P_0 - Q^R \geq 0$ by taking the trace over the basis of spinors.

Our result proves that the positive deficit angle $D$-term string is stable against perturbations that vanish asymptotically, but are arbitrarily large in the bulk. An analogous result holds for point-like sources in three-dimensional supergravity [7], however this is not sufficient to prove the stability of string solutions in four-dimensions, despite the fact that the linearised charge integrals agree. Instabilities in cylindrically symmetric spacetimes usually arise in the massive Kaluza-Klein tensor perturbations in the dimensionally reduced theory [14], and it is precisely this sector that is truncated in the three dimensional theory.

In order to complete the semi-classical proof, one must consider whether a non-perturbative quantum tunnelling effect could arise, i.e. a Coleman-de Luccia bounce solution [15]. In other words, if the $D$-term string is viewed as a false vacuum state of the supergravity theory, is it possible to find a Euclidean bubble solution that would allow decay to a true vacuum state with lower energy? In [16] it was shown that such decay modes via bubble nucleation are inconsistent with ten- and eleven-dimensional supergravity theories, and the same result can be applied here. In short, if the putative true vacuum solution is required to asymptote to the original false vacuum solution, then it must also admit an asymptotically Killing spinor that is well-defined on the whole hypersurface. However, the positive energy theorem then implies that the energy of this solution can only be higher than that of the false vacuum, making it energetically unfavourable for the nucleation to take place. If the energy is equal to that of the false vacuum then the spinor must be globally Killing, which means the solution is no different from the false vacuum solution. In other words, no bubble is being nucleated. This completes the proof of the semi-classical stability of the positive deficit angle $D$-term string in four-dimensional $\mathcal{N} = 1$ supergravity with Fayet-Iliopoulos terms.
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References

[1] A. C. Davis and T. W. B. Kibble, Contemp. Phys. 46 (2005) 313 [hep-th/0505050]; M. Majumdar, [hep-th/0512062]; M. Sakellariadou, “Cosmic strings,” [hep-th/0602276].

[2] M. Sazhin et al., Mon. Not. Roy. Astron. Soc. 343 (2003) 353 [astro-ph/0302547].

[3] E. Agol, C. J. Hogan and R. M. Plotkin, Phys. Rev. D 73 (2006) 087302 [astro-ph/0603838].

[4] R. E. Schild et al, Astron. Astrophys. 422 (2004) 477 [astro-ph/0406434].

[5] S. C. Davis, A. C. Davis and M. Trodden, Phys. Lett. B 405 (1997) 257 [hep-ph/9702360].

[6] G. Dvali, R. Kallosh and A. Van Proeyen, JHEP 0401 (2004) 035 [hep-th/0312005].

[7] K. Becker, M. Becker and A. Strominger, Phys. Rev. D 51 (1995) 6603 [hep-th/9502107]; J. D. Edelstein, C. Núñez and F. A. Schaposnik, Nucl. Phys. B 458 (1996) 165 [hep-th/9506147] and Phys. Lett. B 375 (1996) 163 [hep-th/9512117].

[8] E. J. Copeland, R. C. Myers and J. Polchinski, JHEP 0406 (2004) 013 [hep-th/0312067].

[9] A. Collinucci, P. Smyth and A. Van Proeyen, to appear in JHEP, [hep-th/0611111].

[10] A. Comtet and G. W. Gibbons, Nucl. Phys. B 299 (1988) 719.

[11] A. Achúcarro et al., JHEP 0601 (2006) 102 [hep-th/0511001].
[12] E. Witten, Commun. Math. Phys. 80 (1981) 381; J.M. Nester, Phys. Lett. 83A (1981) 241; G. W. Gibbons and C. M. Hull, Phys. Lett. B 109 (1982) 190; P.K. Townsend, Phys. Lett. B148 (1984) 55.

[13] G. W. Gibbons, G. T. Horowitz and P. K. Townsend, Class. Quant. Grav. 12 (1995) 297 [hep-th/9410073].

[14] R. Gregory and R. Laflamme, Phys. Rev. Lett. 70 (1993) 2837 [hep-th/9301052].

[15] S. R. Coleman and F. De Luccia, Phys. Rev. D21 (1980) 3305.

[16] M. M. Taylor-Robinson, Phys. Rev. D55 (1997) 4822-4838 [hep-th/9609234].