Appropriateness of dynamical systems for the comparison of different embedding methods via calculation of the maximum Lyapunov exponent

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Abstract. The embedding of time series provides a valuable, and sometimes indispensable, tool in order to analyze the dynamical properties of a chaotic system. To this purpose, the choice of the embedding dimension and lag is decisive. The scientific literature describes several methods for selecting the most appropriate parameter pairs. Unfortunately, no conclusive criterion to decide which method – and thus which embedding pair – is the best has been so far devised. A widely employed quantity to compare different methods is the maximum Lyapunov exponent (MLE) because, for chaotic systems that have explicit analytic representations, MLE can be numerically evaluated independently of the embedding dimension and lag. Within this framework, we investigated the dependence on the calculated MLE on the embedding dimension and lag in the case of three dynamical systems that are also widespreadly used as reference systems, namely the Lorenz, Rössler and Mackey–Glass attractors. By also taking into account the statistical fluctuations of the calculated MLE, we propose a new method to assess which systems provide suitable test benches for the comparison of different embedding methods via MLE calculation. For example we found that, despite of its popularity in this scientific context, the Rössler attractor is not a reliable workbench to test the validity of an embedding method.

1. Introduction
Embedding experimental data provides an indispensable tool in order to analyze the dynamical properties of a chaotic system. Given a time series \( x_i, i = 0, \ldots, n \), an embedding vector \( i \) is defined as \( \mathbf{X}_i = (x_i, x_{i+L}, \ldots, x_{i+(m-1)L}) \), \( m \in \mathbb{N} \) is the embedding dimension and \( L \in \mathbb{N} \) is the lag. Provided that proper values of \( m \) and \( L \) are chosen, a dynamics \( F: \mathbf{X}_i \rightarrow \mathbf{X}_{i+1} \) can be defined, which is representative of the original chaotic system; this is the essential statement of Takens-Mañé embedding theorem [1]. Regrettably, Takens-Mañé theorem says nothing on how to choose the two parameters \((m, L)\).

Because the choice of the embedding parameters is crucial to the subsequent analysis, many embedding methods were developed. The most cited are: Gao-Zheng’s method [2], Schuster’s method [3], the method of characteristic length (CL), global false neighbours and the autocorrelation function (GFNN–A) and global false neighbours and mutual information (GFNN–MI) [4]. Each motivates its choice of \((m, L)\) by addressing one or more particular features of the dynamics of the system.

One of the basic features of chaos is the peculiar dependence on initial conditions. A quantitative measure of the response sensitivity to small changes in initial conditions is provided
by the Lyapunov exponents of the dynamical system. Lyapunov exponents are asymptotic measures that characterize the average rate of growth (or, conversely, “shrinking”) of small perturbations in the phase-space. In the case of a chaotic orbit, at least one Lyapunov exponent is positive: initially nearby orbits exponentially diverge from each other.

Cellucci et al. compared different embedding methods by considering the difference between the maximum – and therefore most significant – Lyapunov exponent (MLE) \( \lambda(\alpha, m, L) \) calculated for the embedded system and the MLE \( \chi(\alpha) \) calculated by using the algorithm developed by Benettin et al. [5]: the best embedding method gives an embedding pair \((m, L)\) that minimizes this difference. In their paper, Cellucci et al. used different chaotic systems, and it was tacitly assumed that all chaotic systems were equally effective as test benches. However, a chaotic system, for which the calculated MLE is essentially embedding–independent, is de facto useless to test an embedding method. Conversely, a stronger dependence is linked to a higher appropriateness of the dynamical system for testing and comparing different embedding methods via MLE calculation. In this paper, we consider the dependence of \( \lambda(\alpha) \) on the embedding pairs \((m, L)\) for three chaotic systems, namely the Lorenz, Rössler and Mackey–Glass attractors.

The paper is organized as follows: Section 2 presents a short overview of the concepts regarding the embedding, as well as Gao–Zheng’s method to determine MLE; the reference dynamical systems are the topic of Section 3; the results are discussed in Section 4.

2. Theoretical background

For each system \( \alpha \), we calculated a reference maximum Lyapunov exponent \( \chi(\alpha) \) by applying the so–called standard method [6]. This method, developed by Benettin et al. [7, 8], evaluates the maximum Lyapunov exponent \( \chi(\alpha) \) directly from the differential equations that characterize the dynamical system. Consequently, \( \chi(\alpha) \) is independent of the embedding pair \((m, L)\).

As mentioned at the beginning of Section 1, an embedding procedure with dimension \( m \) and lag \( L \) maps a sample time sequence \( \{x_i\} \) of a scalar, real variable \( x \), into a sequence of embedding vector \( \{X_i\} \), with \( X_i \in \mathbb{R}^m \). The MLE \( \lambda(\alpha, m, L) \) can be then evaluated by using the method proposed by Gao and Zheng [2]. Let us consider all pairs \( X_i, X_j \) that satisfy the following two conditions [5]: 1) \( \|X_i - X_j\| \leq r \), with \( r \) corresponding to the 0.1 percentile of the distribution of all distances \( \|X_i - X_j\| \); 2) \( |i - j| \geq c_0 \), with \( c_0 \) being the location of the first minimum of the autocorrelation function. Let us also define the quantity \( \Lambda(k) \) as follows:

\[
\Lambda(k) = \frac{1}{N_{ref}} \sum_{i,j} \log \frac{\|X_{i+k} - X_{j+k}\|}{\|X_i - X_j\|},
\]

where \( k \) is an non–negative, integer delay, \( N_{ref} \) is the number of pairs and \( \| \ldots \| \) represents the Euclidean norm. Then, for small values of \( k \), the dependence of \( \Lambda \) on \( k \) turns out to be linearly growing and independent of the boundary conditions. In this linear regime, the MLE can be estimated by the slope of \( \Lambda(k) \), i.e. \( [\Lambda(k_2) - \Lambda(k_1)]/([k_2 - k_1] \delta t) \), where \( \delta t \) is the time step of the sample time sequence.

3. Reference dynamical systems

Three of the most common continuous dynamical systems that are considered in the literature are [2, 9]: the Lorenz attractor, the Rössler attractor and the Mackey–Glass attractor. In the present work, to compute the evolution of these dynamical systems, a fourth–order Runge–Kutta
algorithm was used. The system of differential equations of the Lorenz attractor is [10]:

\[
\begin{align*}
\frac{dx}{dt} &= \sigma(y - x) \\
\frac{dy}{dt} &= x(r - z) - y \\
\frac{dz}{dt} &= xy - bz
\end{align*}
\]

To simulate this system, we used \( \sigma = 10, r = 45.92, b = \frac{8}{3} \) and a sampling interval \( \delta t \) equal to 0.03.

The Rössler attractor is governed by the following system of differential equations [11]:

\[
\begin{align*}
\frac{dx}{dt} &= -y - z \\
\frac{dy}{dt} &= x + ay \\
\frac{dz}{dt} &= b + z(x - c)
\end{align*}
\]

To simulate this system, we used \( a = 0.15, b = 0.2, c = 10 \) and a sampling interval \( \delta t \) equal to 0.125.

The Mackey–Glass attractor is governed by the following equation [12]:

\[
\frac{dx(t)}{dt} = a \frac{x(t-\tau)}{1 + [x(t-\tau)]^c} - bx(t), \quad a, b, c > 0.
\]

To simulate this system, we used \( a = 0.2, b = 0.1, c = 10, \tau = 30 \) and a sampling interval \( \delta t \) equal to 1.5.

For each dynamical system, only the \( x \)–coordinate was considered in the embedding process.

4. Results
Considering the dynamical systems \( \alpha \) in Section 3, we calculated the reference MLE \( \chi_\alpha \) provided by the standard method and the MLE \( \lambda_\alpha(m, L) \) estimated by using Gao–Zheng’s method with the embedding parameters \((m, L)\).

“Good” \((m, L)\) pairs are ideally characterized by equal values of \( \lambda_\alpha(m, L) \) and \( \chi_\alpha \). However, because \( \chi_\alpha \) and \( \lambda_\alpha(m, L) \) were calculated on finite samples, a perfect assessment of the potential equality of the two quantities is hindered by statistical fluctuations. To tackle this issue, given a dynamical system \( \alpha \) and for each point \((m, L)\) of the lattice \(1 \leq m \leq 9, 0 \leq L \leq 18\), we carried out a number of 10 independent calculations of \( \lambda_{i,\alpha}(m, L) \) \((i = 1, \ldots, 10)\). Then, on each 10–fold set, the average \( \lambda_\alpha(m, L) \) and the standard deviation \( \sigma_\alpha(m, L) \) was calculated. Finally, we calculated the mean \( \Sigma_\alpha \) of the set of \( \sigma_\alpha(m, L) \); the quantity \( \Sigma_\alpha \) is therefore representative of the dynamical system \( \alpha \) only.

We propose to deem an embedding pair \((m, L)\) to be “good” if \( \lambda_\alpha(m, L) \) is in the interval \([\chi_\alpha - \Sigma_\alpha, \chi_\alpha + \Sigma_\alpha]\). For the three dynamical systems under consideration here, Tab. 1 reports \( \chi_\alpha, \Sigma_\alpha \) and finally the statistics of the “good” \((m, L)\) pairs. To compare the dynamical system we considered the percentage of “good” pairs: this parameter gives a quantitative value of the difficulty to select a \((m, L)\) pair compatible with the respective standard value \( \chi_\alpha \). Rössler and Mackey–Glass attractors have a value of 81% and 68%, respectively. This means that it is not difficult to randomly select a “good” pair. On the other hand, the Lorenz attractor has a value of 25% only. Our conclusion is that the Rössler and Mackey–Glass attractors, at least with the boundary conditions described in Section 3, are not as useful to compare embedding methods as the Lorenz attractor is.
Table 1. Statistic of the “good” embedding pairs for each of the three dynamical system of Section 3. The standard value $\chi_\alpha$ and confidence interval $\Sigma_\alpha$ are also reported.

| Dynamical system (\(\alpha\)) | $\chi_\alpha$ | $\Sigma_\alpha$ | # of “good” pairs | total # of pairs | % “good” pairs |
|-------------------------------|---------------|-----------------|-------------------|-----------------|---------------|
| Lorenz                        | 2.09          | 0.10            | 36                | 144             | 25            |
| Rössler                       | 0.129         | 0.009           | 117               | 144             | 81            |
| Mackey–Glass                  | 0.0066        | 0.0003          | 98                | 144             | 68            |

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