Self-sustained oscillations in homogeneous shear flow.

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Abstract. Generation of the large-scale coherent vortical structures in homogeneous shear flow couples dynamical processes of energy and enstrophy production. In the large rate of strain limit, the simple estimates of the contributions to the energy and enstrophy equations result in a dynamical system, describing experimentally and numerically observed self-sustained non-linear oscillations of energy and enstrophy. It is shown that the period of these oscillations is independent upon the box size and the energy and enstrophy fluctuations are strongly correlated.
Due to its seeming simplicity, the problem of homogeneous shear flow has widely been used as a benchmark for numerical and experimental tests of various closures for turbulence modelling. All early closures were based on the Kolmogorov ideas developed for statistically steady isotropic and homogeneous small-scale turbulence interacting with the non-universal large-scale flow-field. It became clear that to validate this physically appealing concept, one had to verify and understand the symmetries and other statistical properties of the small-scale velocity fluctuations in the real-life flows. This was the main focus of the experimental studies of homogeneous shear flow [1]-[6]. The interest in this model flow is also related to the recent numerical investigations which revealed coherent structures resembling those, responsible for turbulence production, in the wall- sheared flows [7]-[13]. This system has also often been used for calibration of various constants in semi-empirical turbulence models [15],[16].

The problem is formulated as follows: consider a flow in a cube of a side $a$, so that $-a < x_i < a$. The velocity field

$$\mathbf{v}(x,t) = U(y)\mathbf{e}_1 + \mathbf{u}(x,t)$$

with the imposed mean velocity $\langle \mathbf{v} \rangle = U(y)\mathbf{e}_1 = Sy\mathbf{e}_1$ (the definition of the averaging operation will be introduced below). The vorticity is defined then:

$$\Omega = -S\mathbf{e}_3 + \omega$$

The equations of motion for the fluctuating components of velocity and vorticity are (density $\rho = 1$):

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -S\mathbf{e}_1 - \nabla p - U(y)\partial_x \mathbf{u} + \nu \nabla^2 \mathbf{u},$$

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = \omega \cdot \nabla \mathbf{u} - S(-\partial_z \mathbf{e}_3 + \partial_x \mathbf{e}_1 + \partial_y \mathbf{e}_2) - Sy\partial_x \omega$$

and

$$\nabla \cdot \mathbf{u} = 0$$
The $x$, $y$ and $z$-components of velocity field are denoted hereafter as $u$, $v$ and $w$, respectively. Let us define the averaging operations:

$$F = \langle F(x,t) \rangle = \frac{1}{TV} \int_0^T \int_V \xi dx dt F(x,t)$$

in the limit $V = a^3 \to \infty; T \to \infty$. The statistically steady state is assumed here. The spacial averaging is defined as

$$F(t) = \overline{F(x,t)} = \frac{1}{V} \int_V dxF(x,t)$$

The kinetic energy equation is ($\nu \to 0$):

$$\partial_t K + \frac{1}{2} u_i \nabla_i u_j^2 = -\tau_{uv} S - \nabla_i p u_i - \mathcal{E}$$

with $\tau_{uv} = \overline{uv}$. The contribution $U(y) \frac{\partial u_j^2}{\partial x} = 0$ due to the symmetry of the problem.

Since in a homogeneous flow all spacial derivatives of the mean properties are equal to zero, the modelling is reduced to investigation of the time evolution of turbulent kinetic energy $K = \frac{u_i^2}{2}$ and dissipation rate $\mathcal{E} = \nu (\partial_t u_j)^2$.

For small perturbations from isotropic and homogeneous state ($S \to 0$ and $\overline{pu} = \overline{pv} = 0$), the typical turbulence models, based on the equilibrium ideas are [15], [16]:

$$\partial_t K = -\tau_{ij} S_{ij} - \mathcal{E}$$

(7)

and

$$\partial_t \mathcal{E} = -C_{\epsilon 1} \tau_{ij} S_{ij} \frac{\mathcal{E}}{K} - C_{\epsilon 2} \frac{\mathcal{E}^2}{K}$$

(8)

where the Reynolds stress $\tau_{ij} = \overline{uv}$. The coefficients $C_{\epsilon i} = O(1)$. A simple expression, valid at the long times $t > K/\mathcal{E}$,

$$\tau_{ij} \approx -\nu_T S_{ij}$$

(9)

with turbulent viscosity $\nu_T \propto K^2/\mathcal{E}$, closes the set of equations (7)-(9) and defines the so called $K - \mathcal{E}$ model, widely used in engineering for modelling the not-too strongly sheared
flows. The unknown magnitudes of the proportionality coefficients are typically determined in a following way. Consider a flow with $S = 0$. The unknown coefficient $C_{e2}$ can be found from comparing the analytic solution of the simple equations (7)-(9) with experimental and numerical data. The same flow can also be used to test the results of analytic theories [15]-[16]. If $S \neq 0$, solution of (7)-(9) is not easy and the coefficient $C_{e1}$ can be found from comparison with the data. The solution of equation (7)-(9) with the fixed values of the coefficients showed a close to exponential long-time growth of turbulent kinetic energy in a good agreement with the outcome of direct numerical simulations [15]-[16]. If the shear is imposed on a decaying isotropic turbulence at $t = 0$, the observed [16] initial, short-time decay of kinetic energy is readily explained by the fact that turbulent viscosity

$$
\nu_T \approx \int_0^t v(0)v(\tau)d\tau
$$

which is small at short times.

Recent numerical experiments revealed a much more complicated picture. Driven by a very strong shear (the criterion is derived below), in the long-time limit, the system developed a limit cycle-like strong fluctuations of the total kinetic energy about the mean value $<\overline{K}>$ [7]-[13]. The amplitude of these fluctuations was up to two-three times that of $<\overline{K}>$. Similar effect was observed by Borue et. al. [14] in a three-dimensional Kolmogorov flow driven by a steady forcing $f = (0, 0, \cos(x))$. Elucidation of the origin of these oscillations is the goal of this paper.

The physical process observed in both homogeneous shear and Kolmogorov flows can be described in two steps: first, the shear generates both kinetic energy and vortical structures leading to the access of the energy production. Then, the structures become unstable and rapidly disappear with the energy dissipation taking over. The process repeats itself. The evolution of kinetic energy and enstrophy fluctuations in 3D Kolmogorov flow, conducted by Borue et al [14], revealed extremely strong correlation: the sharp spikes in the enstrophy and energy time-signals were almost simultaneous with a slight time-lag, thus supporting the importance of coherent vortical structures in the process.

At the long times the numerical homogeneous shear flow problem (1)-(3) has two very
important features. We can see from the equation of motion that the flow, defined on a cube, cannot be periodic in space. Second, the integral scale $L$ in this situation is not a dynamic variable which is a function of $K$ and $E$, but prescribed by the box size, so that $L \approx a$. This puts strong constrains on the modelling of various contributions to the equations (1)-(3).

Now, we would like to establish the main characteristic length-scales. The non-universal velocity fluctuations belong to the range of scales $a \approx L < r < r_c$ with the cross-over scale $r_c \approx \sqrt{E}/S^{3/4}$ are dominated by powerful anisortopic coherent structures (vortices). The universal range, populated by the more or less isotropic excitations, spreads over the interval $r_c < r < r_d \approx (\nu^3/E)^{1/4}$. The Kolmogorov spectrum can be expected in the range with the total energy of quasy-isotropic fluctuations

$$q \approx \int_{r_d}^{r_c} E(k) dk \propto r_c^{\frac{4}{3}} - r_d^{\frac{4}{3}} > 0$$ (10)

We can see that the inertial range shrinks to zero when the strain rate becomes large. This fact, noticed in Ref. [13], defines the strong shear regime. In strongly anisortopic flow, the simple expression (9), is invalid.

First, let us consider the equation for $\omega^2$:

$$\frac{1}{2} \partial_t \omega^2 + \frac{1}{2} \partial_i u_i \omega^2 = \omega \cdot \omega \cdot \nabla u - S(-\omega_z \partial_z w + \omega_1 \partial_x w + \omega_2 \partial_z v) - \nu (\partial_i \omega_j)^2$$ (11)

Due to powerful, shear-generated coherent vortical structures, $\omega_y \omega_x = O(\omega^2)$ and the contribution involving longitudinal derivative $\partial_z w$ can be neglected. The simple dimensional considerations lead to:

$$\bar{\omega} \cdot \omega \cdot \nabla u = O(u_{rms} \omega^2 / L); \quad S(\omega_z \partial_z w + \omega_y \partial_x v) = O(S \omega^2)$$

and denoting $A(t) = \omega^2/S^2$ we have:

$$\partial_t A(t) = \gamma (\frac{a}{LS} \sqrt{K} - \alpha) S A(t)$$ (12)

with all coefficients $\gamma$, $a$ and $\alpha = O(1)$. 


To estimate $\tau_{uv}$, we observe that in the limit of interest (see below) the only relevant time scale is $\tau_c \approx 1/S$. From the equation (3) we have an estimate for the stress:

$$\tau_{uv} = \frac{u(t)v(t)}{t_{t-\tau_c}} \approx u(t) \int_{t-\tau_c}^{t} d\lambda(-u(\lambda) \cdot \nabla v(\lambda) - \nabla y p(\lambda))$$  \hspace{1cm} (13)

where the “initial condition $\tau_{uv}(t-\tau_c)$ was neglected for simplicity (see below). The dimensional estimate gives:

$$\tau_{uv} = B(t) \frac{K^{\frac{3}{2}}}{S L}$$  \hspace{1cm} (14)

where $B(t)$ is an “anisotropy factor” or “order parameter” characterizing the varying in time strength of the coherent vortical structures. The appearance of this factor is natural (see below) since in an isotropic, non-sheared flow lacking coherent vortices $B(t) = \text{const} = 0$, while in the strongly sheared flow these structures contribute to the energy production. The dissipation rate in shear flows is estimated as $\mathcal{E} = bK^{\frac{3}{2}}/L$ with the coefficient $b \approx 1$ leading to a model equation:

$$K_t(t) \approx (B(t) - b) \frac{K^{\frac{3}{2}}}{L}$$  \hspace{1cm} (15)

or, introducing $y = \sqrt{K}$:

$$2\partial_t y(t) = \frac{B(t) - b}{L} y^2$$  \hspace{1cm} (16)

By the virtue of (2), the mean value of vorticity in homogeneous shear flow is $<\Omega> = S$. Thus, the natural measure of the strength of the strongly anisotropic fluctuating coherent vortical structures is the ratio $v^2/S^2$. Based on these considerations, we set $A(t) = B(t)$. This result can be derived in the third-order of the iteration procedure of the expression (13). Indeed, inserting an unknown initial condition into (13), one can use $\tau_{uv}(t-\tau_c) = \tau_0$ as a zero order solution. Then, after simple resummation (neglecting the first -order contributions) we obtain:

$$\tau_{uv}(t) = \tau_0 + \int_{t-\tau_c}^{t} u(t) \cdot \nabla v(\lambda) + \int \int \int u(t) \cdot \nabla u(\lambda') \cdot \nabla u(\lambda'') \cdot \nabla v(\lambda''') d\lambda d\lambda' d\lambda'' + \cdots$$
This expression immediately gives (14) with \( B(\omega) \approx \frac{\omega^2}{S^2} \), provided the derivatives \( \partial_i u_j \approx \omega \). This approximate derivation is given here to demonstrate the mechanism of vorticity appearance in the expression for the Reynolds stress \( \tau_{uv} \). It will become clear below that the power of vorticity in the expression \( B(\omega) \propto \omega^n \) is unimportant.

The fact that the anisotropic ordered structures can influence the magnitude and even sign of the energy fluxes is known for a long time. The rigorous linear stability analysis developed in \([17]-[18]\), showed that even in three-dimensional flows the strongly anisotropic structures (basic flows), are capable of reversing the sign of the energy flux, due to the “negative viscosity” effects and lead to substantial growth of a small large-scale perturbation. In the opposite limit of the isotropic basic flows, the theory showed generation of positive effective viscosity and acceleration of the energy dissipation. These features are incorporated in the model (12),(15), (16): indeed, we see that if \( A(t) > b \), the energy grows, while, when \( A < b \), it decays. The model equation (12) includes the well-known process of the vortex break-down: when \( \nu^2 >> \omega^2 L^2 \), the instability leads to the vortex disappearance.

Defining dimensionless variables \( Z = \frac{y}{S^2} \), \( Z_0 = \frac{\alpha}{a} \) and \( T = St \) gives:

\[
2\partial_T Z(T) = a(A(T) - b)Z^2(T) \tag{17}
\]

and

\[
\partial_T A(T) = -\gamma(Z(T) - Z_0)A(T) \tag{18}
\]

where \( Z_0 > 0 = \text{const} \), related to the mean amplitude of \( Z(t) \).

The equations (17),(18) have a steady-state solution \( Z(T) = Z_0 \) and \( A(T) = b \). A simple linear stability analysis shows periodic solution when the amplitude of the perturbation is very small. The numerical solutions of quations (17),(18), presented on Figs. 1-6, revealed strong non-linear oscillations. All calculations were performed with Mathematica\(^TM\). In a wide range of parameter variation, the system generates non-linear oscillations with the shape depending upon initial values \( Z(0) \) and \( A(0) \). For a given set of parameters the frequency of oscillations is proportional to the strain rate \( S \).
For the initial values of $Z(0)$ and $A(0) \approx 1$, the solution shows reasonably smooth oscillations with $Z$ and $A$ being somewhat out of phase (see Figs. 1,2). The result supports a general physical picture of the anisotropy $A(t)$ (order parameter) and energy growing (decaying) together with some time-lag. The energy fluctuations are by a factor 2-3 larger than $S^2 L^2$. When the initial energy was doubled to $Z(0) = 3$, the oscillations became much less symmetric with the steeper energy grows (Figs. 3,4). The crucial role of the “order parameter” $A(0)$ is demonstrated on Figs. 5,6 corresponding to $Z(0) = 2$ and $A(0) = 0.1$. We can see the formation steep shock-like structures, somewhat resembling turbulence-production bursts.

In the range of large $Z(0)$ and very small $A(0) << 1$, the solution blows up, indicating the unphysicality of these initial conditions corresponding to the large energy fluctuations ($u_{rms}(0) >> SL$) and small anisotropy (order) parameter ($\bar{\omega}^2 << S^2$).

To discuss the above results, let us look at this work from a somewhat different angle. The Kolmogorov relation $S_{3,0}(r) = (u(x + r) - u(x))^3 \propto r$, is a statement about constancy of the energy flux for inertial range wave numbers $k \approx 1/r >> 1/L$ of isotropic and homogeneous turbulence. As $r \to L$, the structure function $S_{3,0}(r) \to 0$. In strongly anisotropic flows with the integral scale $L \approx a$, this is not so: depending on the spacial distribution of velocity (vorticity), the moment $S_3(L) \neq 0$. If vorticity (enstrophy) is an “order” parameter, characterizing deviations from isotropy, then $S_3(L) \approx B(\omega) K^2 / L$ where $B(\omega) \to 0$ when the strength of the structures diminishes. This qualitative statement is supported by the well-known fact that the velocity field, generated by the vortex $v(r) \propto \Gamma \phi(r)$, where the circulation $\Gamma = O(\omega L^2)$. Combined with the equation for the enstrophy, the two relations (17),(18) form a dynamical system leading to strong fluctuations of both energy and enstrophy. The shape of the function $B(\omega)$ does not seem to influence the qualitative aspects of the process: the model (17),(18) is invariant under transformation $A(t) \to A^a$ with a simple rescaling of time.

All this is valid when $L \approx a$. If this is not so, the magnitude of the fluctuations must substantially decrease. Indeed, if $a >> L$, then we are dealing with $N = (a/L)^d$ independent systems. Here $d$ is the force dimensionalty. Since the phases are crucially important , we expect the amplitude of the fluctuations to decrease as $1/\sqrt{N}$. This can easily be tested on an
example of 3D Kolmogorov flow in a box with the side $a$ driven by the force $\mathbf{f} = (0, 0, \cos(\frac{x}{L}))$ by varying the forcing scale.

To conclude: based on the equations of motion and some physical considerations, we propose a dynamic model, coupling vorticity (enstrophy) and energy fluctuations in a homogeneous shear flow. This model generates strongly correlated self-sustained oscillations of both enstrophy and energy similar to those observed in experiments and direct numerical simulations. The calculated time-lag is similar to that observed in a numerical study of 3D Kolomogorov flow by Borue et al [14].

It is not yet clear if, properly parametrized, this simple model can mimic turbulent bursts which are at the core of the energy production in turbulent wall flows. In case of a positive answer, the model of this kind can serve as a boundary condition (“wall function”) for turbulence simulations, neglecting the detailed consideration of dynamics of the viscous sublayer. The achieved computational economy makes this aspect of the work worth pursuing.

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Figure 1: Time-evolution of $Z^2(T) \propto K(T)$ (higher amplitude curve) and $A(T)$ vs $T$. 
$Z(0) = 1.41, \ A(0) = 1.3$
Figure 2: Parametric plot $Z^2(T)$ (horizontal) vs. $A(T)$
Figure 3: Time -evolution of $Z^2(T) \propto K(T)$ (higher amplitude curve) and $A(T)$ vs $T$. $Z_0 = 3$; $A(0) = 1.3$
Figure 4: Parametric plot $Z^2(T)$ (horizontal) vs $A(T)$
Figure 5: Time-evolution of $Z^2(T) \propto K(T)$ (higher amplitude curve) and $A(T)$ vs $T$. $Z_0 = 2$; $A(0) = 0.1$
Figure 6: Parametric plot $Z^2(T)$ (horizontal) vs $A(T)$