A NOTE ON THE $p$-PARABOLICITY OF SUBMANIFOLDS

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Abstract. We give a geometric criterion which shows $p$-parabolicity of a class of submanifolds in a Riemannian manifold, with controlled second fundamental form, for $p \geq 2$.

1. Introduction

A Riemannian manifold $M^n$ is called parabolic if it does not admit a nonconstant positive superharmonic function (referred to the 2-Laplacian). Otherwise it is called hyperbolic. In the paper [LS], T. Lyons and D. Sullivan established a list of equivalent conditions to check the 2-hyperbolicity, (and hence, the 2-parabolicity) of an oriented Riemannian manifold, under the name of the Kelvin-Nevanlinna-Royden criterion. Following this research, in the paper [GT], V. Gol’dshtein and M. Troyanov extended this criterion to the $p$-Laplace operator, ($1 < p < \infty$).

Among the characterizations given by the Kelvin-Nevanlinna-Royden criterion for $p$-parabolicity of a Riemannian manifold, we shall use in this paper those that states that a manifold $M$ is $p$-parabolic, (with $1 < p < \infty$), if and only if it has vanishing $p$-capacity, namely, there exists a compact set $D \subseteq M$ such that $\text{Cap}_p(D, M) = 0$.

Here, the $p$-capacity of $D$ is defined by

$$\text{Cap}_p(D, M) = \inf u \int_M \| \nabla u \|^p d\mu,$$

where the infimum is taken over all real-valued functions $u \in C_0^\infty(M)$, with $u \geq 1$ in $D$, (see [HKM]).

On the other hand, in [GT] and [T1] it is possible to find a set of geometric criteria for this so-called type problem for the $p$-Laplacian in Riemannian

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manifolds, namely, to decide when a Riemannian manifold is $p$-parabolic of $p$-hyperbolic.

In particular, in the Corollary 5.2 of [T1], (see too Proposition 4 in [GT]), it is presented a characterization of $p$-parabolicity for Riemannian manifolds with a warped cylindrical end, and in Corollary 5.4 of [T1], (see too Proposition 3 in [GT]), we have a sufficient condition for $p$-parabolicity in terms of the volume growth of the manifold.

While these two criteria are intrinsic, we are going to present in this paper a geometric criterion to decide if a submanifold $S^m$ properly immersed in an ambient manifold $N^n$ with a pole is $p$-parabolic, which involves (lower) bounds for the mean curvature and the second fundamental form of $S$.

This criterion is based, (as in [MP1] and, specially, [HMP] from which this paper can be considered a spin-off), in the Hessian-Index analysis of the (restricted to $S$) extrinsic distance function from the pole, (see [GreW]).

1.1. **Outline of the paper.** We shall present the basic definitions concerning the $p$-Laplacian in Section 2. Section 3 is devoted to the study of the curvature setting where our results hold, together with the Hessian and Laplacian analysis needed. Main results are stated and proved in Sections 4, 5 and 6.

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2. **The $p$-Laplacian**

Let $M$ be a non-compact Riemannian manifold, with the Riemannian metric $\langle \cdot, \cdot \rangle$ and the Riemannian volume form $d\mu$.

The $p$-Laplacian of a $C^2$ function $u$ is defined as

$$\Delta_p u = \text{div}(\|\nabla u\|^{p-2}\nabla u).$$

When $p = 2$, we have the usual Laplacian, and the classical potential theory developed from the study of the solutions of the Laplace equation

$$\Delta_2 u = 0.$$  

However, when $p \neq 2$, equation

$$(2.1) \quad \Delta_p u = 0,$$  

is nonlinear and degenerates at the zeroes of the gradient of $u$.

Then, the solutions of (2.1), known usually as $p$-harmonic functions, need not be smooth, nor even $C^2$ and equation (2.1) must be interpreted in a weak sense.
In this way, and given $1 < p < \infty$, we say that a function $u \in W^{1,p}_{\text{loc}}(M)$ is a (weak) solution to the $p$-Laplace equation

\begin{equation}
- \text{div} \left( \|\nabla u\|^{p-2} \nabla u \right) = 0
\end{equation}

in $M$ if

\begin{equation}
\int_M \langle \|\nabla u\|^{p-2} \nabla u, \nabla \phi \rangle \, d\mu = 0
\end{equation}

for all $\phi \in C^\infty_0(M)$. Here, $\nabla u \in L^1_{\text{loc}}(M)$ is the distributional gradient of $u \in L^1_{\text{loc}}(M)$. Furthermore, $L^1(M)$ denotes the space of measurable functions $f : M \rightarrow \mathbb{R}$ with finite norm $\|f\|_1 < \infty$, and $L^1_{\text{loc}}(M)$ is its corresponding local space defined through the open sets in $M$ with compact closure, (see [HKM] p. 13). In its turn, the space $W^{1,p}(M)$, $1 \leq p < \infty$ is the Sobolev space of all functions $u \in L^p(M)$ whose distributional gradient $\nabla u$ belongs to $L^p(M)$, equipped with the norm $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$. The corresponding local space $W^{1,p}_{\text{loc}}(M)$ is defined as in [HKM].

Continuous solutions of (2.2) are called $p$-harmonic. Here the continuity assumption makes no restriction since every solution of (2.2) has a continuous representative. The extension of regularity results of this kind, (see [E] and [Li]), from the Euclidean setting to the Riemannian setting is detailed in [HMP], Remark 9.2.

A function $u \in W^{1,p}_{\text{loc}}(M)$ is called a $p$-supersolution in $M$ if

\begin{equation}
\int_M \langle \|\nabla u\|^{p-2} \nabla u, \nabla \phi \rangle \, d\mu \geq 0,
\end{equation}

for all non-negative $\phi \in C^\infty_0(M)$. If, moreover, $u$ is lower semicontinuous, then $u$ is $p$-superharmonic, (and we denote $\Delta_p u \leq 0$).

Similarly, a function $v \in W^{1,p}_{\text{loc}}(M)$ is called a $p$-subsolution in $M$ if $-v$ is a $p$-supersolution. If, moreover, $v$ is lower semicontinuous, then $v$ is $p$-subharmonic, ($\Delta_p v \geq 0$).

A fundamental feature of solutions of (2.2) is the following well-known maximum (or comparison) principle which will be instrumental for the comparison technique presented below in Sections 5 and 6. If $u \in W^{1,p}(M)$ is a $p$-supersolution, $v \in W^{1,p}(M)$ is a $p$-subsolution, and $\max(v-u, 0) \in W^{1,p}_{\text{loc}}(M)$, then $u \geq v$ a.e. in $M$. In particular, if $D \subset M$ is a precompact open set, $u \in C(\overline{D})$ is a $p$-supersolution, $v \in C(\overline{D})$ is a $p$-subsolution, and $u \geq v$ in $\partial D$, then $u \geq v$ in $D$. We refer to [HKM] 3.18 for a short proof of the comparison principle.

3. Comparison Constellations

We assume throughout the paper that $S^m$ is a non-compact, properly immersed, and connected Riemannian submanifold of a complete Riemannian
manifold $N^n$. Furthermore, we assume that $N^n$ possesses at least one pole. Recall that a pole is a point $o$ such that the exponential map $\exp_o: T_o N^n \to N^n$ is a diffeomorphism. For every $x \in N^n \setminus \{o\}$ we define $r(x) = \dist_N(o, x)$, and this distance is realized by the length of a unique geodesic from $o$ to $x$, which is the radial geodesic from $o$. We also denote by $r$ the restriction $r|: S \to \mathbb{R}_+ \cup \{0\}$. This restriction is called the extrinsic distance function from $o$ in $S^m$. The gradients of $r$ in $N$ and $S$ are denoted by $\nabla^N r$ and $\nabla^S r$, respectively. Let us remark that $\nabla^S r(x)$ is just the tangential component in $S$ of $\nabla^N r(x)$, for all $x \in S$. Then we have the following basic relation:

$$\nabla^N r = \nabla^S r + (\nabla^N r)^\perp,$$

where $(\nabla^N r)^\perp(x)$ is perpendicular to $T_x S$ for all $x \in S$.

With the extrinsic distance at hand, we define the following domains:

**Definition 3.1.** Given a connected and complete $m$-dimensional submanifold $S^m$ in a complete Riemannian manifold $N^n$ with a pole $o$, we denote the extrinsic metric balls of (sufficiently large) radius $R$ and center $o$ by $D^R(o)$. They are defined as any connected component of the intersection

$$B^R(o) \cap S = \{x \in S: r(x) < R\},$$

where $B^R(o)$ denotes the open geodesic ball of radius $R$ centered at the pole $o$ in $N^n$. Using these extrinsic balls we define the $o$-centered extrinsic annuli

$$A^R(o) = D^R(o) \setminus \overline{D^\rho(o)}$$

in $S^m$ for $\rho < R$, where $D^R(o)$ is the component of $B^R(o) \cap S$ containing $D^\rho(o)$.

**Remark 3.2.** We must to point out that these extrinsic domains are pre-compact, (because the submanifold $S$ is properly immersed), and that the radii $R$ that produce smooth boundaries $\partial D^R(o)$ are dense in $\mathbb{R}$ by Sard’s theorem and the Regular Level Set Theorem, because the distance function $r$ is smooth in $N^n \setminus \{o\}$, and hence, its restriction to $S$, $r|: S$.

3.1. **Curvature restrictions.**

**Definition 3.3.** Let $o$ be a point in a Riemannian manifold $M$ and let $x \in M \setminus \{o\}$. The sectional curvature $K_M(\sigma_x)$ of the two-plane $\sigma_x \in T_x M$ is then called an $o$-radial sectional curvature of $M$ at $x$ if $\sigma_x$ contains the tangent vector to a minimal geodesic from $o$ to $x$. We denote these curvatures by $K^o_M(\sigma_x)$.

**Definition 3.4.** The $o$-radial mean convexity $\mathcal{C}(x)$ of $S$ in $N$, is defined as follows:

$$\mathcal{C}(x) = -\langle \nabla^N r(x), H_S(x) \rangle, \quad x \in S,$$

where $H_S(x)$ denotes the mean curvature vector of $S$ in $N$.

Moreover, for $p > 2$ we define:
Definition 3.5. The o-radial component $B(x)$ of the second fundamental form of $S$ in $N$, is defined as:

$$B(x) = -(\nabla^N r(x), \alpha_x(U_r, U_r)),$$

where

$$U_r = \nabla^S(r(x))/\|\nabla^S r(x)\| \in T_xS \subset T_xN$$

is the unit tangent vector to $S$ in the direction of $\nabla^S r(x)$ (resp. tacitly assumed to be 0 in case $\nabla^S r(x) = 0$).

Finally,

Definition 3.6. The o-radial tangency $T(x)$ of $S$ in $N$ is defined as follows:

$$T(x) = \|\nabla^S r(x)\|$$

for all $x \in S$.

Upper and lower bounds of $C(x)$, $B(x)$ and $T(x)$ together with a suitable control on the o-radial sectional curvatures of the ambient space will eventually control the $p$-Laplacian of restricted radial functions on $S$.

3.2. Model spaces.

Definition 3.7 (See [Gri], [GreW]). A $w$-model $M^m_w$ is a smooth warped product with base $B^1 = [0, \Lambda] \subset \mathbb{R}$ (where $0 < \Lambda \leq \infty$), fiber $F^{m-1} = S^{m-1}_{1}$ (i.e. the unit $(m - 1)$-sphere with standard metric), and warping function $w: [0, \Lambda] \rightarrow \mathbb{R}_+ \cup \{0\}$, with $w(0) = 0$, $w'(0) = 1$, and $w(r) > 0$ for all $r > 0$. The point $o_w = \pi^{-1}(0)$, where $\pi$ denotes the projection onto $B^1$, is called the center point of the model space. If $\Lambda = \infty$, then $o_w$ is a pole of $M^m_w$.

Remark 3.8. The simply connected space forms $K^m(b)$ of constant curvature $b$ are $w$-models as we mentioned in [HMP], (see [GreW] and [Gri]).

Proposition 3.9 (See [GreW] and [Gri]). Let $M^m_w$ be a $w$-model with warping function $w(r)$ and center $o_w$. The distance sphere of radius $r$ and center $o_w$ in $M^m_w$ is the fiber $\pi^{-1}(r)$. This distance sphere has the constant mean curvature $\eta_w(r) = \frac{w''(r)}{w(r)}$. On the other hand, the $o_w$-radial sectional curvatures of $M^m_w$ at every $x \in \pi^{-1}(r)$ (for $r > 0$) are all identical and determined by

$$K_{o_w,M^m_w}(\sigma_x) = -\frac{w''(r)}{w(r)}.$$

3.3. Comparison constellations. We now collect the previous ingredients and formulate the general framework for our $p$-parabolicity comparison result, which results a dual setting with respect the curvature assumptions stated in [HMP] to obtain $p$-hyperbolicity.
Definition 3.10. Let $N^n$ denote a Riemannian manifold with a pole $o$. Let $S^m$ denote a connected complete submanifold properly immersed in $N^n$. Let $M^m_w$ denote a $w$-model with center $o_w$; see Definition 3.7. We shall assume that the $o$-radial sectional curvatures of $N$ are bounded from below by the $o_w$-radial sectional curvatures of $M^m_w$:

$$K_{o,N}(\sigma_x) \geq -\frac{w''(r)}{w(r)}$$

for all $x$ with $r = r(x) \in [0, R]$.

Then the triple $\{N^n, S^m, M^m_w\}$ is called a *comparison constellation with lower tangency* on the interval $[0, R]$ if the radial tangency $T$ and the radial convexity functions $B$ and $C$ of the submanifold $S^m$ are all bounded from below by smooth radial functions $g(r)$, $\lambda(r)$, and $h(r)$, respectively:

$$T(x) \geq g(r(x)),$$

$$B(x) \geq \lambda(r(x)),$$

$$C(x) \geq h(r(x))$$

for all $x \in S^m$ with $r(x) \in [0, R]$.

Note that the radial tangency is, in a natural way, bounded from above by 1. This fact motivates the following

Definition 3.11. We assume the same general hypothesis on $S$ and $N$ than in the definition above. The triple $\{N^n, S^m, M^m_w\}$ is called a *comparison constellation with upper tangency* on the interval $[0, R]$ when the radial convexity functions $B$ and $C$ of the submanifold $S^m$ are all bounded from below by smooth radial functions $\lambda(r)$, and $h(r)$, respectively:

$$B(x) \geq \lambda(r(x)),$$

$$C(x) \geq h(r(x))$$

for all $x \in S^m$ with $r(x) \in [0, R]$.

3.4. Hessian and Laplacian comparison analysis. The 2nd order analysis of the restricted distance function $r|_P$, defined on manifolds with a pole is firstly and foremost governed by the Hessian comparison Theorem A in [GreW]:

**Theorem 3.12** (See [GreW], Theorem A). Let $N = N^n$ be a manifold with a pole $o$, let $M = M^m_w$ denote a $w$-model with center $o_w$, and $m \leq n$. Suppose that every $o$-radial sectional curvature at $x \in N \setminus \{o\}$ is bounded from below by the $o_w$-radial sectional curvatures in $M^m_w$ as follows:

$$K_{o,N}(\sigma_x) \geq -\frac{w''(r)}{w(r)}$$
for every radial two-plane $\sigma_x \in T_xN$ at distance $r = r(x) = \text{dist}_N(o,x)$ from $o$ in $N$. Then the Hessian of the distance function in $N$ satisfies
\[
\text{Hess}^N(r(x))(X,X) \leq \text{Hess}^M(r(y))(Y,Y)
\]
for every unit vector $X$ in $T_xN$ and for every unit vector $Y$ in $T_yM$ with $r(y) = r(x) = r$ and $\langle \nabla^M r(y), Y \rangle_M = \langle \nabla^N r(x), X \rangle_N$.

As a consequence of this result, we have the following Laplacian inequality:

**Proposition 3.13.** Let $N^n$ be a manifold with a pole $o$, let $M^m_w$ denote a $w-$model with center $o_w$. Suppose that every $p-$radial sectional curvature at $x \in N - \{o\}$ is bounded from below by the $o_w-$radial sectional curvatures in $M^m_w$ as follows:
\[
K(\sigma(x)) = K_{o,N}(\sigma_x) \geq -\frac{w''(r)}{w(r)}
\]
for every radial two-plane $\sigma_x \in T_xN$ at distance $r = r(x) = \text{dist}_N(o,x)$ from $o$ in $N$. Then we have for every smooth function $f(r)$ with $f'(r) \leq 0$ for all $r$, (respectively $f'(r) \geq 0$ for all $r$):
\[
\Delta^S(f \circ r) \geq (\leq) \left( f''(r) - f'(r)\eta_w(r) \right) \|\nabla^S r\|^2
\]
\[
+ m f'(r) \left( \eta_w(r) + \langle \nabla^N r, H_S \rangle \right)
\]
where $H_S$ denotes the mean curvature vector of $S$ in $N$.

### 4. Main results

Applying the notion of a comparison constellation as defined in the previous section, we now formulate our main $p$-parabolicity results. The proofs are developed through the following sections.

**Theorem 4.1.** Consider a comparison constellation with lower tangency \{\$N^n, S^m, M^m_w\} on the interval $[0, \infty[$. Assume further that the functions $h(r)$ and $\lambda(r)$ are balanced with respect to the warping function $w(r)$ by the following inequality:
\[
\mathcal{M}_p(r) := (m + p - 2) \eta_w(r) - m h(r) - (p - 2) \lambda(r) \geq 0.
\]
Let $\Lambda_{g,p}(r)$ denote the function
\[
\Lambda_{g,p}(r) = w(r) \exp \left( - \int_{\rho}^{r} \frac{\mathcal{M}_p(t)}{(p - 1)g^2(t)} dt \right).
\]
Suppose finally that $p \geq 2$ and that
\[
\int_{\rho}^{\infty} \Lambda_{g,p}(t) dt = \infty.
\]
Then $S^m$ is $p$-parabolic.

**Theorem 4.2.** Consider a comparison constellation with upper tangency 
\{\(N^n, S^m, M^m_w\)\} on the interval \([0, \infty[\). Assume further that the functions 
\(h(r)\) and \(\lambda(r)\) are balanced with respect to the warping function \(w(r)\) by the 
following inequality:

\[(4.3) \quad M_p(r) := (m + p - 2) \eta_w(r) - m h(r) - (p - 2) \lambda(r) \leq 0.\]

Let \(\Lambda_p(r)\) denote the function

\[\Lambda_p(r) = w(r) \exp \left( - \int_r^\infty \frac{M_p(t)}{(p - 1)} \, dt \right).\]

Suppose finally that \(p \geq 2\) and that

\[(4.4) \quad \int_r^\infty \Lambda_p(t) \, dt = \infty.\]

Then $S^m$ is $p$-parabolic.

**Remark 4.3.** It is easy to check that, when \(p = 2\), the lower bound \(\lambda(r(x))\) for the \(o\)-radial component of the second fundamental form \(B(x)\) is obsolete when we consider a comparison constellation, (with upper or lower tangency), \{\(N^n, S^m, M^m_w\)\} on the interval \([0, \infty[\). Moreover, the function \(M_2(r)\) becomes in this case

\[M_2(r) = m(\eta_w(r) - h(r)).\]

With this consideration at hand, we can find a version of Theorem 4.1 for \(p = 2\) in the paper [MP1], where it is used a more restrictive balance condition which implies condition (4.1). On the other hand, we have, based on the same consideration, a version of Theorem 4.2 for \(p = 2\) in the paper [EP], where we can find a direct proof and some consequences in connection with [MP1].

**Corollary 4.4.** Consider a comparison constellation with upper tangency 
\{\(N^n, S^m, M^m_w\)\} on the interval \([0, \infty[\). Assume that \(M_p(r) \leq 0\) for all 
\(r > 0\) and that the warping function \(w(r)\) is bounded from below by a positive 
constant on \([r_0, \infty[\), for some \(r_0 > 0\). Then, $S^m$ is $p$-parabolic.

**Proof.** To obtain the result it suffices to apply Theorem 4.2 taking into 
account that, under the hypothesis, \(\Lambda_p(r) \geq w(r)\) for all \(r > 0\). \(\square\)

**Corollary 4.5.** Consider a comparison constellation with upper tangency 
\{\(N^n, S^m, M^m_w\)\} on the interval \([0, \infty[\). Assume further that, given \(q \geq 2\),
\(M_q(r) \leq 0\) for all \(r > 0\) and \(h(r) \leq \eta_w(r) \leq \lambda(r)\) for all \(r > 0\).

Assume furthermore that

\[(4.5) \quad \int_r^\infty \Lambda_q(t) \, dt = \infty.\]
Then $S^m$ is $p$-parabolic, for all $p \geq q$.

Proof. It is straightforward to see that, if $p \geq q$,

\begin{equation}
M_p(r) = M_q(r) + (p - q)(\eta_w(r) - \lambda(r)) \leq M_q(r) \leq 0,
\end{equation}

since $\eta_w(r) - h(r) < 0$ for all $r > 0$. Then, it is easy to check that, under the hypothesis,

\begin{equation}
\frac{M_p(r)}{p - 1} \leq \frac{M_q(r)}{q - 1},
\end{equation}

so $\int_{\rho}^{\infty} \Lambda_p(t) dt \geq \int_{\rho}^{\infty} \Lambda_q(t) dt = \infty$.

Applying Theorem 4.2, the results follows. \hfill \Box

5. Proof of Theorem 4.1

We define on model spaces $M^m_w$, the modified Laplacian

\begin{equation}
L \psi(x) = \Delta^M_w \psi(x) + \psi'(r(x)) \left( \frac{M(r(x))}{(p - 1) g^2(r(x))} - m \eta_w(r(x)) \right),
\end{equation}

for smooth functions $\psi$ on $M^m_w$. If $\psi = \psi(r)$ only depends on the radial distance $r$, then

\begin{equation}
L \psi(r) = \psi''(r) + \psi'(r) \left( \frac{M(r)}{(p - 1) g^2(r)} - \eta_w(r) \right).
\end{equation}

Consider now the following Dirichlet-Poisson problem associated to $L$:

\begin{equation}
\begin{cases}
L \psi = 0 \text{ on } A^w_{\rho,R}, \\
\psi = 0 \text{ on } \partial B^w_\rho, \\
\psi = 1 \text{ on } \partial B^w_R,
\end{cases}
\end{equation}

where $A^w_{\rho,R}$ is the annular domain in the model space defined as $A^w_{\rho,R} = B^w_R - B^w_\rho$.

The explicit solution to the Dirichlet problem (5.2) is given in the following Proposition which is straightforward,

Proposition 5.1. The solution to the Dirichlet problem (5.2) only depends on $r$ and is given explicitly - via the function $\Lambda_{g,p}(r)$ introduced in Theorem 4.1 by:

\begin{equation}
\psi_{\rho,R}(r) = \frac{\int_{\rho}^{r} \Lambda_{g,p}(t) dt}{\int_{\rho}^{R} \Lambda_{g,p}(t) dt}.
\end{equation}
The corresponding ‘drifted’ 2-capacity is
\[
\text{Cap}_L(A^w_{\rho,R}) = \int_{\partial D^w_{\rho}} \langle \nabla^M \psi_{\rho,R}, \nu \rangle \, dA
\]
\[
= \text{Vol}(\partial D^w_{\rho}) \Lambda_{g,p}(\rho) \left( \int_{\rho}^R \Lambda_{g,p}(t) \, dt \right)^{-1}.
\]

It is easy to see, using equation (5.3) and the balance condition (4.1) that
\[
\psi'_{\rho,R}(r) \geq 0,
\]
\[
\psi''_{\rho,R}(r) - \psi'_{\rho,R}(r) \eta_w(r) = -\psi'_{\rho,R}(r) \frac{M(r)}{(p-1) g^2(r)} \leq 0.
\]

Now, we need the following result, which relates the p-Laplacian of a radial function \(f(r)\) with the operator \(L\).

**Lemma 5.2.** Let \(\{N^n, S^n, M^n\}\) be a comparison constellation with lower tangency on \([0, R]\) for \(R > 0\). Let \(f \circ r\) be a smooth real-valued function with \(f' \geq 0\), and suppose now that \(f(r)\) satisfies the following condition:
\[
f''(r) - f'(r) \eta_w(r) \leq 0.
\]

Then, for all \(x \in S\),
\[
\Delta^S f(r(x)) \leq (p - 1) F^{p-2}(x) g^2(r(x)) L(f(r(x))),
\]
where \(L\) is the second order differential operator defined by equation (5.1) and \(F\) is given by equation (5.7)
\[
F(x) = f'(r(x)) \| \nabla^S r(x) \|.
\]

**Proof.** Computing as in [HMP], we have
\[
\Delta^S f(r(x)) = F^{p-2}(x) \left( (p - 2) \left( f''(r(x)) \| \nabla^S r(x) \| \right)^2 
\]
\[
+ f'(r(x)) \left( \frac{< \nabla^S r(x), \nabla^S \| \nabla^S r(x) \| >}{\| \nabla^S r(x) \|} \right) + \Delta^S f(r(x)) \right).
\]

This partial ‘isolation’ of the factor \((p - 2)\) is the reason behind the general assumption \(p \geq 2\) in this work. Once we have equation (5.8), we argue as follows:

First, it is easy to see that
\[
\frac{< \nabla^S r(x), \nabla^S \| \nabla^S r(x) \| >}{\| \nabla^S r(x) \|} = \text{Hess}^N(r(x)) (U_r, U_r) + < \nabla^N r(x), \alpha_x (U_r, U_r) >.
\]

This quantity is bounded from above using Theorem 3.12 and the lower bound of \(B(x)\). On the other hand, since the \(o\)-radial mean convexity of \(S\),
\( C(x) \) is bounded from below by the function \( h(r(x)) \), we obtain the following estimate using Proposition 3.13 (recall that \( f'(r) \geq 0 \))

\[
(5.10) \quad \Delta^S(f \circ r) \leq (f''(r) - f'(r)\eta_w(r)) \|\nabla^S r\|^2 + m f'(r) (\eta_w(r) - h(r)).
\]

So, using the fact that \( f(r) \) satisfies inequality (5.6) and that \( \|\nabla^S (r)\| \geq g(r) \), we have

\[
(5.11) \quad \Delta^S(f(r(x))) \leq (p - 1)F^{p-2}(x)g^2(r) L(f(r)),
\]

as claimed in the lemma. \( \square \)

Now we transplant the model space solutions \( \psi_{\rho,R}(r) \) of equation (5.2) into the extrinsic annulus \( A_{\rho,R} = D_R(o) \setminus \bar{D}_\rho(o) \) in \( S \) by defining

\[
\Psi_{\rho,R}: A_{\rho,R} \rightarrow \mathbb{R}, \quad \Psi_{\rho,R}(x) = \psi_{\rho,R}(r(x)).
\]

Since \( \psi_{\rho,R}'(r) \geq 0 \) and \( L \psi_{\rho,R} = 0 \) in \( A_{w,\rho,R} \), we obtain, applying Lemma 5.2 to the function \( \Psi_{\rho,R} \),

\[
(5.12) \quad \|\nabla^S u(x)\| \leq \|\nabla^S \Psi_{\rho,R}(x)\|
\]

for all \( x \in \partial D_\rho(o) \).

With same arguments than in [HMP], but inverting all inequalities, we obtain

\[
\text{Cap}_p(\bar{D}_\rho(o), D_R(o)) \leq \left( \frac{\text{Cap}_L(A_{w,\rho,R}^w)}{\text{Vol}(\partial D_\rho^w)} \right)^{p-1} \int_{\partial D_\rho} \|\nabla^S r\|^{p-1} d\mathcal{H}^{m-1}.
\]
As, on the other hand, $D_\rho(o)$ is precompact with a smooth boundary thence,
\[
\int_{\partial D_\rho} \| \nabla^S r \|^{p-1} d\mathcal{H}^{m-1} > 0.
\]

So finally we have
\[
\text{Cap}_p(\bar{D}_\rho(o), S^m) = 
\lim_{R \to \infty} \text{Cap}_p(\bar{D}_\rho(o), D_R(o)) 
\leq \left( \int_{\partial D_\rho} \| \nabla^S r \|^{p-1} d\mathcal{H}^{m-1} \right) \left( \lim_{R \to \infty} \frac{\text{Cap}_L(A_{\rho,R}^w)}{\text{Vol}(\partial D_\rho^w)} \right)^{p-1} = 0,
\]

since $\lim_{R \to \infty} \text{Cap}_L(A_{\rho,R}^w) = 0$ by hypothesis (4.2) and equality (5.4). Thus $\bar{D}_\rho(o)$ is a compact subset with zero $p$-capacity in $S^m$, and $p$-parabolicity of that submanifold follows.

6. Proof of Theorem 4.2

We define the following modified Laplacian $L$ on model spaces $M_w^m$,
\[
L \phi(x) = \Delta^{M_w^m} \phi(x) + \phi'(r(x)) \left( \frac{M(r(x))}{(p-1)} - m \eta_w(r(x)) \right),
\]
for smooth functions $\phi$ on $M_w^m$. As before, if $\phi = \phi(r)$ only depends on the radial distance $r$, then
\[
L \phi(r) = \phi''(r) + \phi'(r) \left( \frac{M(r)}{(p-1)} - \eta_w(r) \right).
\]

Consider now the smooth radial solution $\phi_{\rho,R}(r)$ of the Dirichlet-Poisson problem associated to $L$ and defined on the annulus $A_{\rho,R}^w = B_R^w - B_\rho^w$.

Now we transplant the model space solutions $\phi_{\rho,R}(r)$ of this problem into the extrinsic annulus $A_{\rho,R} = D_R(o) \setminus \bar{D}_\rho(o)$ in $S$ as in the proof of Theorem 4.1 so we have
\[
\Phi_{\rho,R} : A_{\rho,R} \to \mathbb{R}, \quad \Phi_{\rho,R}(x) = \phi_{\rho,R}(r(x)).
\]

Using the lower bounds of $\mathcal{B}(x)$, and $\mathcal{C}(x)$, the fact that $\Phi_{\rho,R}'(r) \geq 0$ and applying Theorem 3.12 and Proposition 3.13 we obtain, as we did for any radial function $f(r)$ satisfying $f'(r) \geq 0$ in the proof of Lemma 5.2,
\[
\Delta^S \Phi_{\rho,R} \leq \left( \Phi_{\rho,R}''(r) - \Phi_{\rho,R}'(r) \eta_w(r) \right) \| \nabla^S r \|^2 + m \Phi_{\rho,R}'(r) (\eta_w(r) - h(r)).
\]

Hence, as
\[
\Phi_{\rho,R}''(r) - \Phi_{\rho,R}'(r) \eta_w(r) \geq 0 \quad \forall r > 0
\]
because the balance condition (4.3) and \(\|\nabla S_r\| \leq 1\), we obtain

\[
\Delta^S_p(\Phi_{\rho,R}(r(x))) \leq (p - 1)F^{p-2}(x)g^2(r) L(\Phi_{\rho,R}(r)) = 0,
\]

where \(F = \Phi'_{\rho,R}(r(x))\|\nabla S_r(x)\|\). The rest of the proof follows in the same way than in Section 3.

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