A Note on Generalized Jordan Derivations in Semiprime Rings

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Abstract The main purpose of this paper is to study and investigate some results concerning generalized Jordan derivation and generalized derivation G:R→R on semiprime ring R, where D an additive mapping on R such that D(x^n)=∑_{j=0}^{n} x^{n-j} D(x^{j}) x^{j-1} for all x ∈ R and D acts as left centralizer.

Keywords Semiprime Rings, Derivations, Generalized Derivation, Generalized Jordan Derivation

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1. Introduction

The notion of generalized Jordan derivations on rings was introduced by Nakajima in [15]. It is a unified and generalized description of generalized Jordan derivations and generalized derivations. A classical result of Herstein [6] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein’s result can be found in [1]. Cusack [5] generalized Herstein’s result to 2-torsion free semiprime rings (see also [2] for an alternative proof). An additive mapping D:R→R is called Jordan triple derivation in case D(xyx)=D(x)yx+xD(y)x+xyD(x) holds for all x,y ∈ R. Bresar [3] has proved that any Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. One can easily prove that any Jordan derivation of arbitrary ring is Jordan triple derivation (see for example [1] for the details) which means that the result we have just mentioned generalized Cusack’s generalization of Herstein’s theorem. An additive mapping T: R→R is called a left centralizer in case T(xy)=T(x)y holds for all pairs x, y ∈ R. An additive mapping T: R→R is called left Jordan centralizer in case T(x^n)=∑_{j=0}^{n} x^{n-j} T(x^{j}) x^{j-1} holds for all x ∈ R. The definition of a right centralizer and a right Jordan centralizer should be self-explanatory. Obviously, any left centralizer is a left Jordan centralizer. Molnar [8] has proved the following result. Let R be a 2-torsion free prime ring and let T:R→R be an additive mapping. If T(xyx)=T(x)yx holds for every x, y ∈ R, then T is a left centralizer. The concept of generalized derivation has been introduced by Bresar [4]. It is easy to see that F: R→R is a generalized derivation iff F is of the form F = D+ T, where D is a derivation and T a left centralizer. Jing and Lu [7] introduced a concept of generalized Jordan derivation and generalized Jordan triple derivation. An additive mapping F:R→R is generalized Jordan derivation if F(x^n)=F(x)x^n+xD(x) holds for all x ∈ R where D:R→R is a Jordan derivation. Kun-Shan Liu [13] proved that R is a 2-torsion free ring with identity and let n ≥ 2. Then (i) any Jordan left derivation (hence, any left derivation) D on the ring M_n(R) is identically zero; (ii) any generalized left derivation on M_n(R) is a right centralizer, the full matrix ring M_n(R) (n ≥ 2) is identically zero. The main purpose of this paper is to study and investigate some results concerning generalized Jordan derivation and generalized derivation G:R→R on semiprime ring R.

2. Preliminaries

Throughout, R will represent an associative ring. Given an integer n > 1, a ring R is said to be n-torsion-free if for x ∈ R, nx= 0 implies that x = 0. Recall that a ring R is prime if for a,b ∈ R, aRb = (0) implies that either a = 0 or b = 0, and is semiprime in case aRb = (0) implies that n= 0. Every prime is semiprime but the converse is not true always. An additive mapping D: R→R is called derivation if D(xy)=D(x)y+xD(y) holds for all pairs x,y ∈ R and is called a Jordan derivation in case D(x^n)=D(x)x^n+xD(x) is fulfilled for all x ∈ R. Every derivation is a Jordan derivation. The converse is in general not true. An additive mapping F:R→R is generalized Jordan derivation if F(x^n)=F(x)x^n+xD(x) holds for all x ∈ R where D:R→R is a Jordan derivation, and is called generalized derivation if F(xy)= F(x)y+xD(y) holds for all x,y ∈ R, where D:R→R is a derivation.

The following Lemma are necessary for the paper.
Lemma A [12: Proposition 1.4]

Let \( R \) be a semiprime ring of characteristic not two and \( T:R \to R \) an additive mapping which satisfies \( T(x^n) = T(x)x^n \) for all \( x \in R \). Then \( T \) is a left centralizer.

3. The Main Results

Theorem 3.1

Let \( n > 1 \) be an integer and let \( R \) be a \( n! \)-torsion-free semiprime ring with identity element. Suppose that there exists an additive mappings \( D,G:R \to R \) such that \( D(x^n) = \sum_{j=1}^{n} x^{n-j}D(x^n)x^{j-1} \) for all \( x \in R \) and \( D \) acts as left centralizer if \( G(x) = G(x)x + D(x^{n+1}) \) for all \( x \in R \), then \( G \) is Jordan generalized derivation on \( R \).

Proof: From the relation

\[
G(x^n) = G(x)x + D(x^{n+1})
\]

for all \( x \in R \), with using that \( D,G \) acts as right and left centralizer respectively, we obtain

\[
G(x) = G(x)x + D(x^n)x^{n-1}
\]

for all \( x \in R \). For complete our proof, we must prove that \( D \) is derivation. According our hypothesis, we assume that \( D(x^n) = \sum_{i=0}^{n} \left( \frac{n}{i} \right) x^{n-i}c_i \)

\[
D(x^n) = \left( \sum_{i=0}^{n} \left( \frac{n}{i} \right) x^{n-i}c_i \right) = 0
\]

We adopt the convention that \( x^0 = e \) for all \( x \in R \).

Where \( D:R \to R \) such that \( D(x^n) = \sum_{i=0}^{n} x^{n-i}D(x^n)x^{i-1} \) for all \( x \in R \), by using this relation and rearranging (1) in the sense of collecting together terms involving equal number of factors of \( c \), we obtain

\[
\sum_{i=0}^{n} f_i(x,c) = 0,
\]

where \( f_i(x,c) \) stands for the expression of terms involving \( i \) factors of \( c \). We replace \( c \) by \( e, 2e, 3e, \ldots, (n-1)e \) in turn in (2). Expressing the resulting system of \( n \) homogeneous equations, we see that the coefficient matrix of the system is a van der Monde matrix

\[
\begin{pmatrix}
1 & 1 & & \\
\vdots & \vdots & \ddots & \\
(n-1) & (n-1)(n-2)
\end{pmatrix}
\]

Since the determinant of the matrix is different from zero, it follows that the system has only a trivial solution. In particular,
exists an additive mappings $D,G: R \to R$ such that $D(x^n) = \sum_{j=1}^{n} x^{n-j} D(x^j)$ for all $x \in R$ and $D$ acts as left centralizer if $G(x^n) = G(x)x^n + D(x^{n+1})$ for all $x \in R$, then $G$ is generalized derivation on $R$.

**Proof:** We have from Theorem 3.1, that $G$ is generalized Jordan derivation, therefore, the relation $G(x^2) = G(x)x + x G(x)x$ for all $x \in R$, where $G$ is a Jordan derivation of $R$. Since $R$ is a semiprime ring one can conclude that $D$ is a derivation. Let us denote $G-D$ by $T$. Then we have $T(x^2) = G(x^2) - D(x^2) = G(x)x + x D(x) - D(x)x = (G(x)-D(x))x = T(x)x$. We have therefore $T(x^2) = T(x)x$, for all $x \in R$. In other words, $T$ is a left Jordan centralizer of $R$. Since $R$ is a 2-torsion free semiprime ring one can conclude that $T$ is a left centralizer by Lemma A. Hence $G$ is of the form $G = D + T$, where $D$ is a derivation and $T$ is a left centralizer of $R$, which means that $G$ is a generalized derivation. The proof is complete.

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