Stroboscopic Variation Measurement

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Abstract

A new procedure of the linear position measurement which allows to obtain sensitivity better than the Standard Quantum Limit and close to the Energetic Quantum Limit is proposed and analyzed in detail. Proposed method is based on the principles of stroboscopic quantum measurement and variation quantum measurement and allows to avoid main disadvantages of both these procedures. This method can be considered as a good candidate for use as a local position meter in the “intracavity” topologies of the laser gravitational-wave antennae.

1 Introduction

It has been known for many years that sensitivity of the weak force detectors based on “ordinary” position meters is limited by the Standard Quantum Limit \( F_{\text{SQL}} \) (see discussion on “ordinary” and other position meters in [2]). On the other hand, the scientific society faced with the necessity to overcome this limit due to requirements of the gravitational-wave experiments [3]. In order to detect gravitational force producing small relative displacement of the antenna’s test masses, it is necessary to use methods that allow to overcome this fundamental limit of sensitivity. There are two promising measurement techniques that can be applied to solve this problem. These are stroboscopic measurement and variation measurement methods.

Stroboscopic measurement [4] is the technique based on the fact that harmonic oscillator momentum perturbation at \( t = 0 \) does not influence its coordinate at \( t + n\pi/\omega_m \), where \( \omega_m \) is the the oscillator’s eigenfrequency and \( n \) is an integer. In other words, the back action noise does not influence the measurement result if instantaneous measurements are being made at regular time intervals of \( \Delta t = \pi n/\omega_m \). Using the stroboscopic measurement it is possible to detect force

\[
F \simeq F_{\text{SQL}}\sqrt{\omega_m\theta},
\]

where \( F_{\text{SQL}} \) is the value of force corresponding to the SQL and \( \theta \) is the duration of each measurement.

Variation measurement technique can be divided into two main classes: the time-domain variation measurement [5, 6] and spectral variation measurement [7]. Time-domain variation measurements...
measurement allows to eliminate the back action noise from the meter’s output signal by introducing time-dependent cross-correlation between the meter’s measurement and back-action noise. In the particular case of interferometric position meters being used in gravitational-wave detectors these noises correspond to phase and power shot noises of the optical pumping correspondingly, and this correlation can be introduced by modulation of the local oscillator phase. In the spectral variation measurement frequency-dependent cross-correlation is used.

Unfortunately, all these methods have their own disadvantages. In order to achieve sensitivity substantially better than the SQL using interferometric position meter in stroboscopic regime it is necessary to use short pulses of optical pumping with high power \( W = \pi W_0/\omega_m \theta \gg W_0 \), where \( W_0 \) is the mean power. Moreover, as \( \theta \ll \omega_m^{-1} \) the meter’s output signal contains additional noise due to brownian fluctuations of high-frequency internal modes of the test body. Estimates show that in the real gravitational-wave antennae this noise may be comparable or even larger than the fundamental quantum noise. Another disadvantage of the stroboscopic measurement is that it can be applied only on meters with the harmonic oscillator for the probe mass. At the same time only free masses are used for the test object in the contemporary laser gravitational wave antennae.

On the other hand, in the case of the original form of time-domain variation measurement it is necessary to know shape and arrival time of the signal to be detected. As for the gravitational wave astronomy where the SQL overcoming is crucial, this procedure can not be applied because arrival time as well as exact shape of the signal are unknown. Spectral variation measurement is free from this disadvantage, but in the case of interferometric position meters it requires to use additional optical devices with bandwidth comparable with bandwidth of the main interferometer. For example, in the case of LIGO gravitational-wave observatory it is proposed to use additional Fabry-Perot cavities with the length comparable with the length of the main cavities of the antenna [7].

In the articles [6, 8] the modified version of the time-domain variation measurement which allows to circumvent these disadvantages had been considered. This “discrete sampling variation measurement” (DSVM) is based on approximation of the signal by series of short rectangular “slices” with duration \( \tau < \pi/\omega_{\text{max}} \), where \( \omega_{\text{max}} \) is the upper signal frequency, and periodical applying of the variation measurement procedure fitted for such a rectangular pulse. In accordance with sampling theorem signal shape can be completely restored using signal values on consequent time intervals \( t_n = n\tau \) as sampling coefficients.

In this article we suggest new procedure which combines time-dependent pumping of the stroboscopic measurement, the meter noises time-dependent cross-correlation used in variation measurement, and the discrete sampling method. We propose to call this method “stroboscopic-variation measurement” (SVM). We will show that such a combined procedure allows to obtain sensitivity several times better than the pure DSVM procedure. This sensitivity can be close to traditional forms of the variation measurement and also close to the Energetic Quantum Limit [9, 10], which defines the ultimate sensitivity that can be achieved at given meter’s energy.

In the next section the idealized version of the SVM procedure based on the the instant position measurements is considered. In the section 3 we show that rigorous optimization of the variation measurements gives this procedure as a result. In the section 4 and in the appendix A we consider more realistic version of the SVM procedure based on position measurements with finite duration. In the appendix B the Energetic Quantum Limit for linear position meters is considered in detail and it is also shown that sensitivity of the spectral variation measurement is equal to this limit. Appendix C devoted to the explicit solution of the optimization problem that arises in section 3.
2 The idea of the SVM procedure

Consider a slightly modified version of the triple measurement procedure discussed in the article [11]. Let constant force $F$ with duration $\tau$ to act on a harmonic test oscillator with eigenfrequency $\omega_m$. In order to detect this force three instant measurements of the oscillator’s position are performed at the moments $t = 0, t = \tau/2$, and $t = \tau$. The results of these measurements can be presented as

$$\tilde{x}_j = x_j + x_{j\text{ fluct}},$$

where $x_j\ (j = 1, 2, 3)$ are the “actual” values of the test object position at these moments of time,

$$x_2 = x_1 \cos \frac{\omega_m \tau}{2} + \frac{p_1 + p_{1\text{ fluct}}}{m\omega_m} \sin \frac{\omega_m \tau}{2} + \frac{F}{m\omega_m^2} \left(1 - \cos \frac{\omega_m \tau}{2}\right),$$

$$x_3 = x_1 \cos \omega_m \tau + \frac{p_1 + p_{1\text{ fluct}}}{m\omega_m} \sin \omega_m \tau + \frac{p_{2\text{ fluct}}}{m\omega_m} \sin \frac{\omega_m \tau}{2} + \frac{F}{m\omega_m^2} \left(1 - \cos \omega_m \tau\right),$$

where $p_1$ is the initial momentum of the test object and $p_{1,2\text{ fluct}}$ are the test object momentum perturbations during the first and the second measurement correspondingly.

Estimate of the force $F$ which does not depend on the initial state of the test object can be obtained using the formula

$$\tilde{F} = \frac{m\omega_m^2}{2 \left(1 - \cos \frac{\omega_m \tau}{2}\right)} \left(\tilde{x}_1 - 2\tilde{x}_2 \cos \frac{\omega_m \tau}{2} + \tilde{x}_3\right).$$

Uncertainty of this value $\Delta F$ depends on the mean square errors for each of the measurement $\Delta x_j$, corresponding mean square perturbations of the test object momentum $\Delta p_j$, and the cross-correlations between the measurement errors and perturbations $\Delta x_{p_j}$. They must satisfy the Heisenberg relation

$$\Delta^2 x_j \Delta^2 p_{j\text{ fluct}} - \Delta^2 x_{p_j} = \frac{\hbar^2}{4}.$$  

The experimentalist should optimize measurement by minimizing value of $\Delta F$. Taking into account formula (4) it is easy to show that optimal values of cross-correlation are equal to

$$\Delta x_{p1} = \Delta x_{p3} = 0, \quad \Delta x_{p2} = \frac{\Delta^2 p_2}{2m\omega_m} \tan \frac{\omega_m \tau}{2}.$$  

and in this case

$$\left(\Delta F\right)^2 = \frac{m^2 \omega_m^4}{4 \left(1 - \cos \frac{\omega_m \tau}{2}\right)^2} \left(\frac{\hbar^2}{4\Delta^2 p_1} + \frac{\hbar^2}{\Delta^2 p_2} \cos^2 \frac{\omega_m \tau}{2} + \frac{\hbar^2}{4\Delta^2 p_3}\right).$$
Described procedure is very close to the variation measurement. Really, we eliminated the back-action term in the expression (6) using cross-correlation between $x_{2 \text{fluct}}$ and $p_{2 \text{fluct}}$ (see formulae (5)). In such a procedure it is possible to obtain, in principle, arbitrary high precision by increasing the values of $\Delta_{p,j}$.

On the other hand, if $\omega_m \tau = \pi n$ then the term corresponding to the second measurement in the formulae (5,6) vanishes. It means that only two measurements are necessary in this case, the first and the third ones, which correspond to the stroboscopic procedure. Therefore stroboscopic measurement which uses time-dependent pumping and variation measurement which uses time-dependent cross-correlation of noises should be considered as particular cases of “stroboscopic-variation” procedure described here.

In the real experiment the values of $\Delta_{p,j}$ can not be arbitrary high due to energy limitations in the meter. Therefore value (7) can be further optimized taking into account additional condition that

$$\Delta_{p,1}^2 + \Delta_{p,2}^2 + \Delta_{p,3}^2 = S_F \tau,$$

where $S_F$ is some given value (sense of this peculiar notation will be explained below too). In the case of the interferometric position meter it is proportional to the optical energy stored in the interferometer, averaged over the time $\tau$.

It is easy to show that in the optimal case we will obtain the following expression:

$$\Delta_{p,1}^2 = \Delta_{p,3}^2 = \frac{1}{2} \left| \cos \frac{\omega_m \tau}{2} \right| \Delta_{p,2}^2.$$

(9)

It is convenient to describe sensitivity by the meter effective noise spectral density

$$S_{\text{meter}} = (\Delta F)^2 \tau,$$

as it was done in the article [8]. If the conditions (5) are valid, then the sensitivity is defined by the following formulae:

$$S_{\text{meter}} \over S_{\text{EQL}} = \left\{ \begin{array}{ll}
(\frac{\omega_m \tau}{\pi})^4 \cot^4 \frac{\omega_m \tau}{4}, & \text{if } 0 \leq \omega_m \tau \leq \pi, \\
(\frac{\omega_m \tau}{\pi})^4 & \text{if } \pi \leq \omega_m \tau \leq 2\pi,
\end{array} \right.$$  

(11)

for the harmonic oscillator, and

$$S_{\text{meter}} \over S_{\text{EQL}} = \left( \frac{4}{\pi} \right)^4$$

(12)

for the free mass ($\omega_m \to 0$).

Here

$$S_{\text{EQL}} = \frac{\pi^4 \hbar^2 m^2}{4 S_F \tau^4}.$$  

(13)
is a convenient scale factor for the meter’s noise equal to the Energetic Quantum Limit in the free test mass case taken at frequency (see Appendix [3]).

It should be noted that $S_{\text{meter}} = S_{\text{EQL}}$ only in the case of the pure stroboscopic measurement ($\omega_m \tau = \pi$), and $S_{\text{meter}} > S_{\text{EQL}}$ for all other values of $\omega_m \tau$ (see the lowest curve in the Fig. [3]).

3 Linear position meter with time-dependent parameters

Consider now more realistic situation when the meter which can continuously monitor position $x(t)$ of the test object (for example, interferometric position meter) is used to detect signal force $F_{\text{signal}}(t)$. When the meter is turned on, its output signal can be represented as

$$\tilde{x}(t) = \hat{x}_{\text{init}}(t) + \hat{x}_{\text{fluct}}(t) + D^{-1}[F_{\text{signal}}(t) + \hat{F}_{\text{fluct}}(t)],$$ (14)

where $D$ is the linear differential operator describing the dynamics of the test object\(^1\), $\hat{x}_{\text{fluct}}(t)$ is the noise added by the meter, $\hat{F}_{\text{fluct}}(t)$ is the back-action force, and $\hat{x}_{\text{init}}(t)$ is the “initial” position of the test object which describes its motion when signal force and back action force of the meter are absent. It should be noted that, whereas $\hat{x}_{\text{fluct}}(t), \hat{F}_{\text{fluct}}(t)$ and $\hat{x}_{\text{init}}(t)$ are quantum-mechanical operators, $\tilde{x}(t)$ is a classical observable (see discussion on de-quantization in linear position meters in the article [11]).

Spectral densities of these noises $S_x$ and $S_F$, and their cross spectral density $S_{xF}$ satisfy the uncertainty relation

$$S_x S_F - S_{xF}^2 = \frac{\hbar^2}{4}.$$ (15)

Suppose that the experimentalist can change values of these spectral densities within the boundaries defined by the condition (15). If the interferometric position meter is used, the experimentalist can change the ratio $S_F/S_x$ changing the pumping power (as in the case of the stroboscopic measurement) and also he can change the value of $S_{xF}$ by changing the phase of the local oscillator (as in the case of the variational measurement).

We suppose in this article for simplicity that noises $\hat{F}_{\text{fluct}}(t)$ and $\hat{x}_{\text{init}}(t)$ can be regarded as “white” ones, and that the values of the spectral densities can be changed instantly (without any inertia). It means that the experimentalist have to be able to change the pumping energy in the interferometric position meter very quickly compared to the signal frequency. In the real experiments these approximations are valid only if relaxation time of the interferometric meter cavities is small compared to the signal characteristic time-scale (see brief discussion on this topic in the conclusion).

On the first stage of signal processing information about the test object initial conditions (term $x_{\text{init}}(t)$) is being excluded from the output signal $\tilde{x}(t)$ by applying operator $D$ to (14). As a result one will obtain the following expression for the estimation of the force $F(t)$:

\(^1\)In the case of the harmonic oscillator operator $D$ can be written as

$$D = \frac{d^2}{dt^2} + \omega_0^2.$$
\[ \hat{F}(t) = D\hat{x}(t) = F_{\text{signal}}(t) + \hat{F}_{\text{fluct}}(t) + D\hat{x}_{\text{fluct}}(t). \]  

(16)

The next stage is the optimal estimation of the force \( F(t) \) averaged over short time interval, as it has been suggested in the DSVM method \([6, 8]\). Mean-square error of this estimation is equal to

\[ (\Delta F)^2 = \frac{\hbar^2}{4} \int_{-\tau/2}^{\tau/2} \left( \frac{Dv(t)}{S_F(t)} \right)^2 dt + \int_{-\tau/2}^{\tau/2} S_F(t)[a(t)Dv(t) + v(t)]^2 dt, \]

(17)

where \( a = S_{xF}/S_F \) is the variational factor and \( v(t) \) is the filter function which must satisfy

\[ v(t) \bigg|_{t=\pm\tau/2} = 0, \quad \frac{dv(t)}{dt} \bigg|_{t=\pm\tau/2} = 0, \quad \int_{-\tau/2}^{\tau/2} v(t) dt = 1. \]

(18)

We also suppose that function \( S_F(t) \) is limited by the condition

\[ \int_{-\tau/2}^{\tau/2} S_F(t) dt = S_F\tau. \]

(19)

(compare with condition \((\text{8})\)).

One can readily see that functional \((\text{17})\) is a sum of two non-negative items, and variational coefficient \( a(t) \) appears only in the second one which originates in the back-action of the meter. It is evident that if variational factor \( a(t) \) satisfies the equation:

\[ a(t)Dv(t) + v(t) = 0 \]

(20)

then the second term vanishes and expression \((\text{17})\) takes form

\[ (\Delta F)^2 = \frac{\hbar^2}{4} \int_{-\tau/2}^{\tau/2} \left( \frac{Dv(t)}{S_F(t)} \right)^2 dt. \]

(21)

Expressions \((\text{21}), (\text{18}), \) and \((\text{19})\) form the optimization problem to be solved in order to find characteristics for optimal signal processing.

It should be noted that the standard Lagrange optimization procedure can not be used here, as function under integral in \((\text{21})\) is not a continuously differentiable one, so the Lagrange equations could not be solved. This problem can be solved using the Lyapunov’s problems optimization theory (See \([12]\) for details). The explicit solution is presented in Appendix \([3]\), and here we will provide only the final expression.
In the particular case of arbitrarily short pumping pulses the optimal function $S_F(t)$ is equal to

$$S_F(t) = \begin{cases} \frac{\overline{S_F \tau}}{2(1 + \cos \frac{\omega_m \tau}{2})} \left[ \delta(t - \tau/2) + 2 \cos \frac{\omega_m \tau}{2} \delta(t) + \delta(t + \tau/2) \right], & \text{if } \omega_m \tau < \pi, \\ \frac{\overline{S_F \tau}}{2} \left[ \delta(t - \frac{\pi}{\omega_m}) + \delta(t + \frac{\pi}{\omega_m}) \right], & \text{if } \omega_m \tau > \pi, \end{cases} \quad (22)$$

The measurement error in this case is described by the formulae \((11, 12)\).

So we have shown that the sequence of the three instant measurements described in section \(2\) represents an optimal procedure when the experimentalist is able to change freely the values of $S_F, S_x$ and $S_{xF}$.

### 4 Pumping pulses with finite duration

It is evident, of course, that $\delta$-like pumping pulses could not be obtained in practice. Therefore we consider here pumping pulses with finite duration.

In order to reconstruct the signal force shape, the sequence of triple measurements described above have to be used. The experimentalist has a choice here, whether to use completely independent triads with each of them containing all three pulses (see Fig. 1(a)) or use overlapping triads with common first and third pulses (see Fig. 1(b)). The second variant is more “energy-saving”, but it does not permit to use variational technique for the first and the third pulses. It is possible to show that due to this reason its sensitivity is limited by the same condition \(1\) as sensitivity of the traditional stroboscopic procedure. Here we will consider the first variant only.

In this case sensitivity is described by the formula

$$S_{\text{meter}} = \frac{\hbar^2 m^2 \omega_m^4}{4 \overline{S_F} (1 - a_1^2/a_2)} = \frac{(\omega_m \tau/\pi)^4}{1 - a_1^2/a_2} S_{\text{EQL}}, \quad (23)$$

where

$$a_j = \frac{1}{\overline{S_F \tau}} \int_{-\tau/2}^{\tau/2} S_F(t) \cos^j \omega_m t \, dt, \text{ where } j = 1, 2. \quad (24)$$
It depends on the shape of the pumping pulse, but it is evident that in any case the larger is $S_F$ (i.e. the pumping power), the smaller is noise, and there are no absolute limitations similar to the SQL or the formula (1) here.

Consider now quasi-optimal case when pumping pulses has small but finite duration $\theta \ll \tau$. Suppose that $\theta$ is the shortest pumping pulse duration that can be used in real experiment. In any case duration $\theta$ should be large compared to the interferometer’s cavities relaxation time. Let also the first and the third pulses to be divided between consequent measurement cycles, as it is shown in Fig. 1 (a). In this case function $S_F(t)$ can be presented as

$$S_F(t) = \overline{S}_F \tau \left[ k \Delta(t + \tau/2) + (1 - k) \Delta(t) + k \Delta(t - \tau/2) \right],$$

where $t \in [\tau( j - \frac{1}{2}), \tau ( j + \frac{1}{2})]$. Here factor $k$ describes the relative energy of the pumping pulses, $0 \leq k \leq 1$, and $\Delta(t)$ is the narrow $\delta$–like function that describes pumping pulse shape, and let

$$\int_{-\tau/2}^{\tau/2} \Delta(t) \, dt = 1, \quad \int_{-\tau/2}^{\tau/2} |t| \Delta(t) \, dt = \alpha_1 \theta, \quad \int_{-\tau/2}^{\tau/2} t^2 \Delta(t) \, dt = (\alpha_2 + \alpha_1^2) \theta^2, \quad (26)$$

where $\alpha_1, \alpha_2$ are some numeric factors which depend on the shape of the pulses.

Substituting these expression into the formula (23), neglecting the terms of order of magnitude $\theta^3$ or higher and optimizing the result with respect to $k$, we will obtain that

$$\frac{S_{\text{meter}}}{S_{\text{EQL}}} \approx \begin{cases} \left( \frac{\omega_m T}{\pi} \right)^4 \cot^4 \frac{\omega_m T}{4} \left[ 1 + 2 \alpha_1^2 \cos \frac{\omega_m T}{2} - 2 \alpha_2 (1 + \cos \frac{\omega_m T}{2}) \right] \omega_m^2 \theta^2 \quad & \text{if } \omega_m T < f(\theta), \\ \left( \frac{\omega_m T}{\pi} \right)^4 \cot^4 \frac{\omega_m T}{4} \left[ 1 + 2 \alpha_1^2 \cos \frac{\omega_m T}{2} - 2 \alpha_2 (1 + \cos \frac{\omega_m T}{2}) \right] \omega_m^2 \theta^2 \quad & \text{if } \omega_m T > f(\theta), \end{cases} \quad (27)$$

where $T = \tau - 2 \alpha_1 \theta$ and $f(\theta) \approx \pi + (\alpha_1^2 - \alpha_2) \omega_m^2 \theta^2$.

These expressions correspond to the following optimal values of $k$:

$$k \approx \begin{cases} \frac{1}{2 \cos^2 \frac{\omega_m T}{4}} \left[ 1 + \left( \frac{\alpha_2}{\sin^2 \frac{\omega_m T}{2}} - \frac{\alpha_1^2 \cos \frac{\omega_m T}{2}}{4 \cos^4 \frac{\omega_m T}{4}} \right) \omega_m^2 \theta^2 \right] \quad & \text{if } \omega_m T < f(\theta), \\ 1 \quad & \text{if } f(\theta) < \omega_m T < \pi, \\ \frac{1}{2 \sin^2 \frac{\omega_m T}{4}} \left[ 1 + \frac{\alpha_2^2 \cos \frac{\omega_m T}{2}}{4 \sin^2 \frac{\omega_m T}{4}} \omega_m^2 \theta^2 \right] \quad & \text{if } \omega_m T > \pi. \end{cases} \quad (28)$$

In the appendix A, rectangular pumping pulses with arbitrary duration $\theta$ are considered and explicit formulae for the measurement error are provided for this case. We suppose rectangular pulses to be a good model for real pumping pulses as the main disadvantage of square shaped function, that is infinite derivative on edges, does not influence the final expression for the measurement error, because due to the function $S_F(t)$ that consists of rectangular pulses, is included to $\Delta F$ as integrand only and the final expression does not depend on its derivatives.
5 Conclusion

Stroboscopic variation measurement presented in this article allows to obtain sensitivity better than the Standard Quantum Limit and close to the Energetic Quantum Limit using time-dependent values of meter noises spectral densities. In particular, if interferometric position meter is used then this procedure can be implemented by using both pumping power and phase of the local oscillator modulation. At the same time, it does not require pumping power to be in a non-classical state.

It is important that the stroboscopic variation measurement does not require information about the signal shape and arrival time (as the usual variation measurement does). At the same time, sensitivity of the stroboscopic variation measurement does not depend crucially on the duration of the pumping pulses (as in the case of the usual stroboscopic measurement), see Fig.3.

Evident area of this procedure application is the laser gravitational-wave antennae. However, in authors’ opinion, it can hardly be used in traditional topologies of gravitational-wave antennae due to two reasons. First, in this procedure the optical energy stored in the interferometer must vary with characteristic time \( t \ll \Omega^{-1} \), where \( \Omega \) is the signal frequency. At the same time, relaxation time \( \tau^{*} \) of the Fabry-Perot cavities used in the large-scale gravitational-wave antennae is close to the \( \Omega^{-1} \) which makes the energy modulation very difficult from the technological point of view.

Second, in traditional topologies of gravitational-wave antennae very high values of the optical pumping power are required in order to overcome the Standard Quantum Limit [10] which, in particular, leads to undesirable non-linear effects in the large-scale Fabry-Perot cavities [13].

In the articles [14, 13, 16] a new class of so-called “intracavity” readout schemes for gravitational-wave antennae which allow in principle to increase sensitivity without increasing the pumping power is proposed. In such a schemes it is necessary to measure a small displacement of local test object with very high precision using, for example, a small (table-top scale) interferometric position meter. We think that stroboscopic variation measurement can be used most effectively in these meters.

Acknowledgments

This paper was supported in part by the California Institute of Technology, US National Science Foundation, by the Russian Foundation for Basic Research, and by the Russian Ministry of Industry and Science. Special thanks to V.B.Braginsky and S.P.Vyatchanin for fruitful discussions and valuable remarks.

Appendix
A Rectangular pumping pulses

Suppose pumping pulses we have introduced above to be square shaped. In this case function \( \Delta(t) \) in \( 25 \) is equal to

\[
\Delta(t) = \begin{cases} 
\frac{1}{\theta} & \text{if } |t| < \theta/2, \\
0 & \text{otherwise}.
\end{cases}
\]  

(29)

Taking the above assumptions and formula \( 11 \) into account one can obtain the following expression for the measurement sensitivity:

\[
\frac{S_{\text{meter}}}{S_{\text{EQL}}} = \frac{(\omega_m \tau / \pi)^4}{1 - \frac{32}{\omega_m \theta} \sin^2 \frac{\omega_m \theta}{4} \left[ k \cos \frac{\omega_m T}{2} + (1 - k) \cos \frac{\omega_m \theta}{4} \right]^2}.
\]  

(30)

The following values of \( k \) minimize \( 30 \):

\[
k = \begin{cases} 
\frac{\omega_m \theta + \sin \omega_m \theta}{\sin \frac{\omega_m \theta}{4}(\cos \frac{\omega_m \theta}{4} - \cos \omega_m T)} - \frac{\cos \frac{\omega_m \theta}{4}}{\cos \frac{\omega_m \theta}{4} - \cos \frac{\omega_m T}{2}}, & \text{if } 2\omega_m \theta < \omega_m T < f(\theta), \\
1, & \text{if } f(\theta) < \omega_m T < \pi, \\
\frac{\cos \frac{\omega_m \theta}{4}}{\cos \frac{\omega_m \theta}{4} - \cos \frac{\omega_m T}{2}}, & \text{if } \omega_m T > \pi.
\end{cases}
\]  

(31)

where

\[
f(\theta) = 2 \arccos \left\{ \frac{1}{4} \left( 2 \cos \frac{\omega_m \theta}{4} - \sqrt{\sin \omega_m \theta - \frac{8 \omega_m \theta}{\sin \frac{\omega_m \theta}{2}}} \right) \right\}
\]  

(32)

Values \( 1 - k \) for different \( \theta/\tau \) are plotted in Fig. 2.

Corresponding values of the measurement sensitivities are defined by the following formulae:

\[
\frac{S_{\text{meter}}}{S_{\text{EQL}}} = \left( \frac{4}{\pi} \right)^4 \left( \frac{\tau/T}{4} \right)^4 \left[ 1 + \frac{1}{6} (\theta/T)^2 + \frac{29}{80} (\theta/T)^4 \right],
\]  

(33)

for the free mass, and

\[
\frac{S_{\text{meter}}}{S_{\text{EQL}}} = \begin{cases} 
\frac{(\omega_m \tau / \pi)^4}{1 - \frac{2}{\omega \theta} \frac{2 \sin \frac{\omega_m \theta}{4}(1 + 2 \cos \frac{\omega_m \theta}{4} \cos \frac{\omega_m T}{2}) - \omega_m \theta}{\cos^2 \frac{\omega_m \theta}{4} (\cos \frac{\omega_m \theta}{4} + \cos \omega_m \theta)^2}}, & \text{if } \omega_m T < f(\theta), \\
\frac{(\omega_m \tau / \pi)^4}{1 - \frac{32}{\omega_m \theta} \sin^2 \frac{\omega_m \theta}{4} \cos^2 \frac{\omega_m T}{2}}, & \text{if } f(\theta) < \omega_m T < \pi, \\
\left( \frac{\omega_m \tau}{\pi} \right)^4, & \text{if } \omega_m T > \pi.
\end{cases}
\]  

(34)
for the harmonic oscillator.

These values are plotted in Figure 3. Different curves correspond to several values of the pumping pulse duration $\theta$. The topmost curve represents sensitivity of the DSVM procedure with constant pumping power \[8\].

Here triangle ABC represents the “pure stroboscopic” area where $k = 1$. It should be noted that in the area right to this triangle the value of $S_{\text{meter}}$ does not depend on pulse duration and remains the same as in the case of $\delta$-function like pumping (21).

## B Energetic Quantum Limit

This appendix is devoted to the formulae (13) derivation and also the expression for the energetic quantum limit when the harmonic oscillator is used as a probe body.

In order to detect the external force acting on some test object the following inequality should be fulfilled (see [16]):

$$
\int_{-\infty}^{\infty} |F(\omega)|^2 S_{\text{pos}}(\omega) \frac{d\omega}{2\pi} \geq \frac{\hbar^2}{4},
$$

(35)

where $F(\omega)$ is the signal force spectrum, and $S_{\text{pos}}(\omega)$ is the test object position fluctuations spectral density.

Suppose position meter with back-action force spectral density $S_F(\omega)$ is attached to the test object. Let also this object dynamics to be described by function $D(\omega)$, that is the operator Fourier transform. Then spectral density $S_{\text{pos}}(\omega)$ can be expressed in terms of back-action force spectral density $S_F(\omega)$ as

$$
S_{\text{pos}} = \frac{S_F(\omega)}{|D(\omega)|^2}.
$$

(36)

Therefore, condition (35) can be rewritten as:

$$
\int_{-\infty}^{\infty} \frac{|F(\omega)|^2}{S_{\text{EQL}}(\omega)} \frac{d\omega}{2\pi} \geq 1,
$$

(37)
where

$$S_{\text{EQL}}(\omega) = \frac{\hbar^2}{4} \frac{|D(\omega)|^2}{S_F(\omega)}. \quad (38)$$

Exactly the same condition can be obtained, if the uncertainty relation for linear position meter noises is used. Really, the total net noise of such a meter can be described by its spectral density

$$S_{\text{total}}(\omega) = S_F(\omega) + 2D(\omega)S_{xF}(\omega) + |D(\omega)|^2 S_x(\omega), \quad (39)$$

and spectral densities of the meter noises must satisfy the following condition:

$$S_x(\omega)S_F(\omega) - S_{xF}^2(\omega) \geq \frac{\hbar^2}{4}. \quad (40)$$

Minimizing value $S_{\text{total}}(\omega)$ taking this additional condition into account, one can easily show that $S_{\text{total}}(\omega) \geq S_{\text{EQL}}(\omega)$.

Suppose that (i) $S_F$ does not depend on frequency and (ii) we want to have $S_{\text{EQL}}(\omega)$ below some given value $S_{\text{EQL}}$ in some given spectral range $0 \leq \omega \leq \omega_{\text{max}}$.

Let our test object be a free mass. In this case

$$D(\omega) = -m\omega^2, \quad (41)$$

and

$$S_{\text{EQL}}(\omega) = \frac{\hbar^2 m^2 \omega^4}{4S_F}. \quad (42)$$

It is evident that this spectral density is maximal if $\omega = \omega_{\text{max}}$, and the maximum is equal to

$$S_{\text{EQL}} = S_{\text{EQL}}(\omega_{\text{max}}) = \frac{\hbar^2 m^2 \omega_{\text{max}}^4}{4S_F}. \quad (43)$$

2back-action noise is supposed to be "white"
This is the EQL for a free mass. Let the test object be a harmonic oscillator now. In this case we will obtain that

\[ D(\omega) = m(\omega_m^2 - \omega^2), \]  

(44)

and

\[ S_{\text{EQL}}(\omega) = \frac{\hbar^2 m^2 (\omega_m^2 - \omega^2)^2}{4S_F}. \]  

(45)

Suppose that we can freely tune \( \omega_m \) in order to minimize the maximum value(s) of the \( S_{\text{EQL}}(\omega) \) in the spectral range \( 0 \leq \omega \leq \omega_{\text{max}} \). It is evident that in optimal case

\[ \omega_m^2 = \frac{\omega_{\text{max}}^2}{2}, \]  

(46)

and maximum values of the \( S_{\text{EQL}}(\omega) \) are equal to

\[ S_{\text{EQL}} = S_{\text{EQL}}(0) = S_{\text{EQL}}(\omega_{\text{max}}) = \frac{\hbar^2 m^2 \omega_{\text{max}}^4}{16S_F}. \]  

(47)

This is the EQL for a harmonic oscillator.

\section{C The explicit optimization method}

In this appendix we will pay some attention to the explicit derivation of formulae presented above.

Here we concentrate on the expression \([21]\) method of optimization when conditions \([18]\) and \([19]\) are applied. The harmonic oscillator case is considered, as the free mass case can be obtained simply by assuming \( \omega_m \to 0 \).

In order to reduce the two variables problem \([21]\) into the single variable \( v(t) \) variational problem it is necessary to express function \( S_F(t) \) in terms of \( v(t) \) using the Lagrange equation as follows:

\[ \frac{\hbar^2}{4} \left( \frac{Dv(t)}{S_F} \right)^2 - \lambda = 0, \quad \Rightarrow \quad S_F(t) = \frac{\hbar}{2\sqrt{\lambda}} |Dv(t)|, \]  

(48)

where \( \lambda \) is the Lagrange multiplier. There is no difficulty now to transform the initial optimization problem into the Liapunov’s one (See \([12,17]\) for details) by changing variable \( v(t) \) in accordance with the formula \( Dv(t) = u(t) \):

\[
\begin{align*}
\int_{-\tau/2}^{\tau/2} |u(t)| \, dt &\to \min, \\
\int_{-\tau/2}^{\tau/2} u(t) \, dt &= \omega_m^2, \\
\int_{-\tau/2}^{\tau/2} \sin(\omega_m(\tau/2 - t)) \, u(t) \, dt &= 0, \\
\int_{-\tau/2}^{\tau/2} \cos(\omega_m(\tau/2 - t)) \, u(t) \, dt &= 0.
\end{align*}
\]  

(49)

Using the ideas represented in \([12,17]\) the following minimum value of \([49]\) can be obtained

\[
u(t) = \begin{cases} 
\mu \delta(t - \frac{\tau}{2}) + \nu \delta(t + \frac{\tau}{2}), & \omega_m \tau \in [0, \pi] \\
\beta \delta(t - \vartheta) + \gamma \delta(t + \vartheta), & \omega_m \tau \in (\pi, \infty),
\end{cases}
\]  

(50)
where \{\mu, \nu, \xi\} and \{\beta, \gamma, \vartheta\} are indefinite coefficients that can be defined through substitution of function \(u(t)\) into the supplementary condition equations in (49). The final expression for \(u(t)\) may be represented as

\[
u(t) = \begin{cases} 
\frac{\omega_m^2}{1 - \cos \frac{\omega_m \tau}{2}} \left[ \delta(t - \frac{\pi}{2}) - 2 \cos \frac{\omega_m \tau}{2} \delta(t) + \delta(t + \frac{\pi}{2}) \right], & \omega_m \tau \in [0, \pi] \\
\frac{\omega_m^2}{2} \left[ \delta(t - \frac{\pi}{2 \omega_m}) + \delta(t + \frac{\pi}{2 \omega_m}) \right], & \omega_m \tau \in (\pi, \infty).
\end{cases}
\]

(51)

Filtering function \(v(t)\) is expressed in terms of \(u(t)\) as

\[
v(t) = \frac{1}{\omega_m} \int_{-\frac{\pi}{2 \omega_m}}^{t} \sin(\omega_m(t - t'))u(t') dt'.
\]

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