BLOWING UP SOLUTIONS FOR SUPERCRITICAL YAMABE PROBLEMS ON MANIFOLDS WITH NON UMBILIC BOUNDARY

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Abstract. We build blowing-up solutions for a supercritical perturbation of the Yamabe problem on manifolds with boundary, provided the dimension of the manifold is $n \geq 7$ and the trace-free part of the second fundamental form is non-zero everywhere on the boundary.

1. Introduction

In this paper we are interested in the existence of blowing-up solutions to problems which are supercritical perturbations of the boundary Yamabe problem, that is we are interested in finding a family $u_\varepsilon$ of solutions for the problem

\[
\begin{aligned}
L_g u &= 0 &\quad &\text{on } M \\
\frac{\partial}{\partial \nu} u + \frac{n-2}{2} h_g(x) u &= (n-2) u^{\frac{n}{n-2}} + \varepsilon &\quad &\text{on } \partial M
\end{aligned}
\]

where $\varepsilon > 0$, $L_g := \Delta_g - \frac{n-2}{4(n-1)} R_g$ is the conformal Laplacian, $R_g$ is the scalar curvature of $M$, $h_g$ is the mean curvature on $\partial M$ and $\nu$ is the outward normal.

Our main result is the following

Theorem 1. Let $M$ be a manifold of positive type. Assume $n \geq 7$, and the trace-free second fundamental form of $\partial M$ is non zero everywhere. Then there exists a solution $u_\varepsilon$ of (1) such that $u_\varepsilon$ blows up when $\varepsilon \to 0^+$.

This result can be read in a threefold way. First, it is an existence result for a supercritical Yamabe type problem in manifolds with boundary. Secondly, it says that the family of solutions of this supercritical problem is not a $C^2$ compact set, in fact it is not possible find an uniform $C^2$ bound to the set $\{u_\varepsilon \in H^1_g(M) : u_\varepsilon \text{ solution of (1)}\}$, This represents an obstruction to the extension of the compactness result of [3] to the supercritical case. Finally, Theorem 1 has an interpretation in the sense of stability of Yamabe boundary problem. Following Druet [13], we say that the Yamabe boundary problem

\[
\begin{aligned}
L_g u &= 0 &\quad &\text{on } M \\
\frac{\partial}{\partial \nu} u + \frac{n-2}{2} h_g(x) u &= (n-2) u^{\frac{n}{n-2}} &\quad &\text{on } \partial M
\end{aligned}
\]

is stable with respect to perturbation from above if, for any sequence $\varepsilon_n \to 0$, and any sequence of $u_{\varepsilon_n}$ solution of (1) (with $\varepsilon_n$ as a parameter) then $u_{\varepsilon_n}$ converges, up to subsequence, to a solution $u_0$ of (2) in $C^2(M)$. Since by our result we showed a family of solutions which blows up while $\varepsilon \to 0$, then the Yamabe boundary problem is not stable from perturbation of the critical exponent from above. Notice that in the same spirit the result of Almaraz in [3] states also that the Yamabe boundary problem is stable with respect of perturbation from below of the critical exponent.

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We give some recall of the classical Yamabe problem and to the Yamabe boundary problem which allow us to give a framework to our result and to clarify the previous considerations.

1.1. The Yamabe problem. In 1960 Yamabe [41] raised the following question: Given \((M, g)\) a compact Riemannian manifold of dimension \(n \geq 3\), without boundary, it is possible to find in the conformal class of \(g\) a metric \(\tilde{g}\) of constant scalar curvature?

Analytically this problem is equivalent to find positive solution of the critical problem

\[
- \Delta_g u + \frac{n-2}{4(n-1)} R_g u = c u^{\frac{n+2}{n-2}} \text{ in } M.
\]

Here \(-\Delta_g\) is the Laplace Beltrami operator and \(R_g\) is the scalar curvature of the original metric. In this case the new metric \(\tilde{g} := u^{\frac{4}{n-2}} g\) has scalar curvature \(R_{\tilde{g}} \equiv \frac{4(n-1)}{n-2} R_g\). A positive answer to this problem was given by Yamabe [41], Trudinger [40], Aubin [6] and Schoen [39].

Problem (3) has a variational structure and a solution could be found as a critical point of the quotient

\[
Q_M(u) := \frac{\int_M \left( |\nabla u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) d\mu_g}{\left( \int_M |u|^{\frac{2(n+2)}{n-2}} d\sigma_g \right)^{\frac{n-2}{2}}}, \quad u \in H^1_g(M).
\]

It is well known that if

\[
Q(M) = \inf_{u \in H^1_g(M)} Q_M(u) > 0
\]

(otherwise said the manifold is of positive type) the solution of (3) is not unique, while uniqueness holds, up to symmetries, if \(Q(M) \leq 0\). At this point people start asking whether if, given a manifold of positive type, the family of solutions of (3) is \(C^2\) compact, that is there is a \(C^2\) uniform estimate on the set of solutions. The question has been solved by S. Brendle, M. A. Khuri, F. C. Marques and R. Schoen in a series of works [7, 9, 31]. Their result can be summarized as follows: compactness holds for dimensions \(3 \leq n \leq 24\) if the manifold is not conformally equivalent to a round sphere, while for \(n \geq 25\) it is possible to construct manifolds for which the set of solutions is not a compact set.

In a series of paper Druet, Hebey and Robert ([13, 14, 16] and the reference therein) studied the compactness of the solutions of Yamabe type problems in which the linear term has the form \(a(x)u\) where \(a\) is a smooth function on \(M\). As a further result they proved the following stability result. Given a sequence a sequence of smooth functions \(a_j\) converging in \(H^2(M)\) to \(R_g\) for \(j \to \infty\), with \(a_j(x) \leq R_g\) for all \(x \in M\) and for all \(j\), then the sequence of \(\{u_j\}\) solutions of

\[
- \Delta_g u + \frac{n-2}{4(n-1)} a_j u = c u^{\frac{n+2}{n-2}} \text{ in } M.
\]

converges, up to subsequences, to a solution \(u_{\infty}\) of (3) while \(j \to \infty\).

In this sense they proved that the Yamabe problem (3) is stable with respect to the perturbations from below of the linear term. Also, they showed blow up phenomena -and so instability- with other choices of perturbation of linear term. After them, the stability vs. instability of (3) has been studied by several authors. We limit ourselves to cite [21, 38, and 15]. In the last paper a sharp description of the blow up profile for solutions of perturbed problem is given.
In a similar spirit, Micheletti, Pistoia, and Vetois \cite{37} proved blow up for a class of Yamabe type problems with slightly subcritical and slightly supercritical nonlinearity.

1.2. Boundary Yamabe problem. In 1992, Escobar \cite{18} generalized the classical Yamabe problem to compact Riemannian manifolds with regular boundary. In this case, one can ask if there exists a conformal metric \( \tilde{g} \) which has both constant scalar curvature and constant mean curvature of the boundary. In this case, the analytic formulation for the Yamabe problem takes the form

\[
\begin{align*}
-\Delta_{\tilde{g}} u + \frac{n-2}{4(n-1)} R_{\tilde{g}} u &= c_1 u^{\frac{n+2}{n-2}} \quad \text{on } M, \\
\frac{\partial}{\partial \nu} u + \frac{n-2}{2} H_{\tilde{g}}(x) u &= c_2 u^{\frac{n}{n-2}} \quad \text{on } \partial M.
\end{align*}
\]

Typically, people addressing this problem fix one among the constant \( c_1, c_2 \) to be zero. In the case \( c_1 \neq 0 \) we limit ourselves to cite, besides Escobar-Han and Li \cite{30}, Ambrosetti, Li, and Malchiodi \cite{5}, Djadli, Malchiodi, Ould Ahmedou \cite{11, 12}, and the recent paper of Disconzi and Khuri which studied also compactness of solutions for problem (5). The other case, in which the target manifold is scalar flat while \( c_2 \neq 0 \), is interesting since it can be interpreted also as a multidimensional version of the Riemann mapping problem. In addition, in this case the nonlinearity appears in the boundary condition and it is critical for the Sobolev immersion of \( H^1(M) \) in \( L^p(\partial M) \). The analytical version of this problem is (2) and its variational formulation consists in finding critical points of the quotient

\[
Q_{\partial}(u) := \frac{\int_M \left( |\nabla u|^2 + \frac{n-2}{4(n-1)} R_{\tilde{g}} u^2 \right) d\mu_{\tilde{g}} + \int_{\partial M} \frac{n-2}{2} H_{\tilde{g}} u^2 d\sigma_{\tilde{g}}}{\left( \int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}} \right)^{\frac{n-2}{n-1}}}, \quad u \in H^1_{\tilde{g}}(M)
\]

and in complete analogy with the Yamabe problem when

\[
Q_{\partial}(M) := \inf_{u \in H^1_{\tilde{g}}(M)} Q_{\partial}(u)
\]

is positive, the solution is no more unique. For Problem (2), the principal existence results are due to Escobar \cite{18, 19, 20}, Marques \cite{34, 35} and Almaraz \cite{1}. Recently, Brendle and Chen \cite{8} and Mayer and Ndiaye \cite{36} covered all the remaining cases. Concerning compactness of solutions in manifold of positive type, the first result for (2) is given by Felli and Ould Ahmedou which in \cite{22} proved compactness for scalar flat manifolds not conformally equivalent to the closed disk, using the Positive Mass Theorem. Compactness has been proved also for manifolds whose trace-free part of the second fundamental form is non-zero everywhere on the boundary by Almaraz \cite{3} for dimensions \( n \geq 7 \) and recently extended to dimension \( n = 5,6 \) by Kim, Musso and Wey \cite{32}, and to dimensions \( n = 3 \) \cite{5} and \( n = 4 \) \cite{32} without any further assumption on the second fundamental form. When the trace-free part of the second fundamental form is non-zero everywhere on the boundary, the boundary of the manifold is said non umbilic and Almaraz exploited this condition to bypass the Positive Mass Theorem. This strategy has been recently adapted to manifold with umbilic boundary, that is when the tensor of the second fundamental form vanishes on the boundary, by the authors, provided that the Weyl tensor is always different from zero on the boundary and \( n \geq 6 \) in \cite{23, 24}.

Dimension \( n = 24 \) appears to be relevant also in boundary Yamabe problems, in fact Almaraz in proved that for \( n \geq 25 \) there exists manifold with umbilic boundary for which compactness of solutions fails. Compactness for manifold with umbilic boundary for \( n \leq 24 \) with no further assumption is still an open problem.
There is another strong analogy between classical and boundary Yamabe problems about stability. In fact, in the aforementioned works of Druet Hebey and Robert [13, 14, 16] Yamabe problem (3) appears to be stable for perturbation of the scalar curvature $R_g$ from below, while there are several examples of instability when $R_e - R_g$ is somewhere positive. This phenomenon appears also in boundary Yamabe problem when one try to perturb the mean curvature term: stability depend on the sign of $h_e - h_g$. Indeed, in [28, 29] the authors with Pistoia proved that the perturbed problem

$$
\begin{cases}
L_g u = 0 & \text{in } M \\
\partial_\nu u + \frac{n-2}{2} h_g u + \varepsilon \gamma u = u^{\frac{2(n-1)}{n-2}-1} & \text{on } \partial M.
\end{cases}
$$

when $\gamma \in C^2(M)$ is strictly positive on $\partial M$ then there is a family of solution which blows up as $\varepsilon \to 0$ when $n \geq 7$ and the boundary is non umbilic or $n \geq 11$, the manifold has umbilic boundary and the Weyl tensor never vanishes od $\partial M$ (this second case has been recently extended up to dimension 8 in [26]). In these two papers the authors also construct examples of $\gamma$ which changes sign for which the Yamabe boundary problem is not stable. On the other hand, when $\gamma$ is everywhere negative on $\partial M$ the Yamabe boundary problem appears to be stable for perturbation of mean curvature as proved in [25], in the umbilic boundary case for $n \geq 7$ and in non umbilic case when $n > 8$ and the Weyl tensor does not vanish on $\partial M$ and when $n = 8$ with a slighter restrictive assumption on the Weyl tensor, so the analogy with the role of scalar curvature for classical Yamabe problem is complete.

In both classical and boundary Yamabe problems, as a corollary of the compactness of solutions, people get that the problem is also stable for perturbation from below of the critical exponent. So stability is proved for scalar flat manifolds in [22], for manifolds with non umbilic boundary in [3, 32], and for umbilic boundary manifolds whose Weyl tensor never vanishes on $\partial M$ in [23, 24]. On the other hand, in the present paper (and in [26] for umbilic boundary manifolds) we prove that Yamabe boundary problem is unstable with respect of perturbation form above of the critical exponent.

Recently, in [17], The Yamabe type problem

$$
\begin{cases}
-\Delta_g u + A(x)u = 0 & \text{on } M; \\
\frac{\partial u}{\partial n} - B(x)u = (n-2)(u^+)^{\frac{n}{n-2}} & \text{on } \partial M.
\end{cases}
$$

is studied and there are a series of compactness results which depend on the sign of $A - R_g$ and $B - h_g$.

2. Preliminaries.

It is well known that there exists a global conformal transformation which maps the manifold $M$ in a manifold for which the mean curvature of the boundary is identically zero. In order to simplify our problem, we choose a metric $(M, g)$ such that $h_g \equiv 0$. We also set $a = \frac{n-2}{4(n-1)} R_g$, so Problem (1) reads as

$$
\begin{cases}
-\Delta_g u + au = 0 & \text{on } M; \\
\frac{\partial u}{\partial n} = (n-2)(u^+)^{\frac{n}{n-2}} + \varepsilon & \text{on } \partial M.
\end{cases}
$$

Since the manifold is of positive type, then

$$
\langle (u, v) \rangle_H = \int_M (\nabla_g u \nabla_g v + au v) d\mu_g
$$

is an equivalent scalar product in $H^1_0$, which induces to the equivalent norm $\| \cdot \|_H$.
We define the exponent 

\[ s_\varepsilon = \frac{2(n-1)}{n-2} + n\varepsilon \]

and the Banach space \( \mathcal{H} := H^1(M) \cap L^s(\partial M) \) endowed with norm \( \|u\|_\mathcal{H} = \|u\|_H + \|u\|_{L^s(\partial M)} \). By trace theorems, we have the following inclusion \( W^{1,s}(M) \subset L^t(\partial M) \) for \( t \leq \frac{n}{n-2} \).

We recall the following result, by Nittka [33, Th. 3.14]

**Remark 2.** Let \( \frac{2n}{n+2} \leq q < \frac{n}{2} \), \( r > 0 \). Then there exists a constant \( c \) such that if \( f_0 \in L^{q+r}(\Omega), \beta \) bounded and measurable and \( g \in L^{\frac{(n-1)q}{n-2} + r}(\partial \Omega) \) and \( u \in H^1(\Omega) \) is the unique weak solution of

\[
\begin{cases}
Lu = f_0 & \text{on } \Omega \\
\frac{\partial u}{\partial \nu} + \beta u = g & \text{on } \partial \Omega
\end{cases}
\]

where \( L \) is a strictly elliptic second order operator, then

\[
|u|_{L^{\frac{2n}{n-q}}(\Omega)} + |u|_{L^{\frac{(n-1)q}{n-2} + r}(\partial \Omega)} \leq |f_0|_{L^{q+r}(\Omega)} + |g|_{L^{\frac{(n-1)q}{n-2} + r}(\partial \Omega)}
\]

We consider \( i : H^1(M) \to L^{\frac{2(n-1)}{n-2}}(\partial M) \) and its adjoint with respect to \( \langle \cdot , \cdot \rangle_\mathcal{H} \) \( i^* : L^{\frac{2(n-1)}{n-2}}(\partial M) \to H^1(M) \)

defined by

\[
\langle \varphi , i^*(g) \rangle = \int_{\partial M} \varphi g \, d\sigma \text{ for all } \varphi \in H^1
\]

so that \( u = i^*(g) \) is the weak solution of the problem

\[
\begin{cases}
-\Delta u + a(x)u = 0 & \text{on } M \\
\frac{\partial u}{\partial \nu} = g & \text{on } \partial M.
\end{cases}
\]

By [33] Th. 3.14 (see Remark 2) we have that, if \( u \in H^1 \) is a solution of (6), then for \( \frac{2n}{n+2} \leq q < \frac{n}{2} \) and \( r > 0 \) it holds

\[
|u|_{L^{\frac{(n-1)q}{n-2}}(\partial M)} = |i^*(g)|_{L^{\frac{(n-1)q}{n-2}}(\partial M)} \leq |g|_{L^{\frac{(n-1)q}{n-2} + r}(\partial \Omega)}.
\]

By this result, we can choose \( q, r \) such that

\[
\frac{(n-1)q}{n-2q} = \frac{2(n-1)}{n-2} + n\varepsilon \text{ and } \frac{(n-1)q}{n-q} + r = \frac{2(n-1) + n(n-2)\varepsilon}{n + (n-2)\varepsilon}
\]

that is

\[
q = \frac{2n + n^2 \left( \frac{n-2}{n-1} \right) \varepsilon}{n + 2 + 2n \left( \frac{n-2}{n-1} \right) \varepsilon} \text{ and } r = \frac{2(n-1) + n(n-2)\varepsilon}{n + (n-2)\varepsilon} = \frac{2(n-1) + n(n-2)\varepsilon}{n + (n-2)\varepsilon} \left( \frac{n}{n-1} \right) \varepsilon;
\]

so we have that, if \( u \in L^{\frac{2(n-1)}{n-2} + n\varepsilon}(\partial M) \), then \( f_\varepsilon(u) \in L^{\frac{2(n-1) + n(n-2)\varepsilon}{n} + n\varepsilon}(\partial M) \) and, in light of (11), that also \( i^*(f_\varepsilon(u)) \in L^{\frac{2(n-1)}{n} + n\varepsilon}(\partial M) \). Here \( f_\varepsilon(u) = (n-2)(u^+)^{\frac{n-2}{n-1}} \).

At this point we are allowed to rewrite Problem (8) as

\[
u = i^*(f_\varepsilon(u)), \ u \in \mathcal{H}.
\]

We recall now some properties of the Fermi coordinates: since we choose a conformal metric for which \( H_\varphi \equiv 0 \), we have the following expansions in a neighborhood of \( y = 0 \) (we use the notation \( y = (z,t), \ z \in \mathbb{R}^n \) and \( t \geq 0 \)). We use the convention
that \(a, b, c, d = 1, \ldots, n\) and \(i, j, k, l = 1, \ldots, n - 1\) and the einstein convention on repeated indices.

\[
|g(y)|^{1/2} = 1 - \frac{1}{2} \left[ |\pi|^2 + \text{Ric}_\eta(0) \right] t^2 - \frac{1}{6} \bar{R}_{ij}(0) z_i z_j + O(|y|^3)
\]

\[
g^{ij}(y) = \delta_{ij} + 2h_{ij}(0)t + \frac{1}{3} \bar{R}_{ikjl}(0) z_k z_l + 2 \frac{\partial h_{ij}(0)}{\partial z_k} t z_k
\]

\[
+ [R_{mnj}(0) + 3h_{ik}(0)h_{kj}(0)] t^2 + O(|y|^3)
\]

\[
g^{\alpha\beta}(y) = \delta_{\alpha\beta}
\]

where \(\pi\) is the second fundamental form and \(h_{ij}(0)\) are its coefficients, \(\bar{R}_{ikjl}(0)\) and \(R_{abcd}(0)\) are the curvature tensor of \(\partial M\) and \(M\), respectively, \(\bar{R}_{ij}(0) = R_{ikjk}(0)\) are the coefficients of the Ricci tensor, and \(\text{Ric}_\eta(0) = R_{n\alpha\alpha}(0) = R_{\alpha\alpha}(0)\) (see [13]).

We conclude these preliminaries introducing the integral quantity

\[
I_m^a = \int_0^\infty \frac{\rho^a}{(1 + \rho^2)^m} d\rho
\]

and the following identities which are obtained by direct computation.

\[
I_m^a = \frac{2m-\alpha}{2m-\alpha+1} I_m^{a+1} \quad \text{for} \quad \alpha + 1 < 2m
\]

\[
I_m^a = \frac{2m-\alpha-1}{2m-\alpha} I_m^{a+1} \quad \text{for} \quad \alpha + 1 < 2m
\]

\[
I_m^a = \frac{2m-\alpha-3}{\alpha+1} I_m^{a+2} \quad \text{for} \quad \alpha + 3 < 2m.
\]

3. The finite dimensional reduction.

Given \(q \in \partial M\) and \(\psi^\delta_q : \mathbb{R}^n_+ \rightarrow M\) the Fermi coordinates in a neighborhood of \(q\), we define

\[
W_{\delta,q}(\xi) = U_\delta \left( \left( \psi^\delta_q \right)^{-1} (\xi) \right) \chi \left( \left( \psi^\delta_q \right)^{-1} (\xi) \right) = \frac{1}{\delta^{n-2}} U \left( \frac{\xi}{\delta} \right) \chi(y) = \frac{1}{\delta^{n-2}} U \left( x \right) \chi(\delta x)
\]

where \(y = (z, t)\), with \(z \in \mathbb{R}^{n-1}\) and \(t \geq 0\), \(\delta x = y = \left( \psi^\delta_q \right)^{-1} (\xi)\) and \(\chi\) is a radial cut off function, with support in ball of radius \(R\).

Here \(U_\delta(y) = \frac{1}{\delta^{n-2}} U \left( \frac{y}{\delta} \right)\) is the one parameter family of solution of the problem

\[
\begin{cases}
-\Delta U_\delta = 0 & \text{on } \mathbb{R}^n_+; \\
\frac{\partial U_\delta}{\partial t} = -(n-2) U_\delta^{n-2} & \text{on } \partial \mathbb{R}^n_+.
\end{cases}
\]

and \(U(z, t) := \frac{1}{\left( [1 + t]^2 + |z|^2 \right)^{n/2}}\) is the standard bubble in \(\mathbb{R}^n_+\).

Now, let us consider the linearized problem

\[
\begin{cases}
-\Delta \phi = 0 & \text{on } \mathbb{R}^n_+; \\
\frac{\partial \phi}{\partial \nu} + n U \pi_\nu^{n-2} \phi = 0 & \text{on } \partial \mathbb{R}^n_+;
\phi \in H^1(\mathbb{R}^n_+).
\end{cases}
\]

and it is well know (see, for instance, [28 Lemma 6]) that every solution of \((18)\) is a linear combination of the functions \(j_1, \ldots, j_n\) defined by

\[
\frac{\partial U}{\partial x_i}, \quad i = 1, \ldots, n - 1 \quad \text{and} \quad j_n = \frac{n-2}{2} U + \sum_{i=1}^{n} y_i \frac{\partial U}{\partial y_i}.
\]
Given \( q \in \partial M \) we define, for \( b = 1, \ldots, n \)
\[
Z^b_{\delta,q}(\xi) = \frac{1}{\delta^{b-\frac{n}{2}}} j_b \left( \frac{1}{\delta} (\psi_q^b)^{-1}(\xi) \right) \chi \left( (\psi_q^b)^{-1}(\xi) \right)
\]
and we decompose \( H^1(M) \) in the direct sum of the following two subspaces
\[
K_{\delta,q} = \text{Span} \{ Z^1_{\delta,q}, \ldots, Z^n_{\delta,q} \}
\]
\[
K^+_{\delta,q} = \{ \varphi \in H^1(M) : \langle \varphi, Z^b_{\delta,q} \rangle_H = 0, \ b = 1, \ldots, n \}
\]
and we define the projections
\[
\Pi = H^1(M) \rightarrow K_{\delta,q} \quad \Pi^+ = H^1(M) \rightarrow K^+_{\delta,q}.
\]
Given \( q \in \partial M \) we also define in a similar way
\[
V_{\delta,q}(\xi) = \frac{1}{\delta^{b-\frac{n}{2}}} v_q \left( \frac{1}{\delta} (\psi_q^b)^{-1}(\xi) \right) \chi \left( (\psi_q^b)^{-1}(\xi) \right),
\]
and
\[
(v_q)_y(y) = \frac{1}{\delta^{b-\frac{n}{2}}} v_q \left( \frac{y}{\delta} \right); \quad \text{here } v_q : \mathbb{R}^n_+ \rightarrow \mathbb{R} \text{ is the unique solution of the problem}
\]
\[
\begin{cases}
-\Delta v = 2h_{ij}(y)t\partial^2_{ij}U & \text{on } \mathbb{R}^n_+; \\
\frac{1}{\delta^{n/2}} + nu^{\frac{2}{n}}v = 0 & \text{on } \partial \mathbb{R}^n_+.
\end{cases}
\]
such that \( v_q \) is \( L^2(\mathbb{R}^n_+) \)-ortogonal to \( j_b \) for all \( b = 1, \ldots, n \) Here \( h_{ij} \) is the second fundamental form and we use the Einstein convention of repeated indices. We remark that
\[
|\nabla^r v_q(y)| \leq C(1 + |y|)^{3-r-n} \text{ for } r = 0, 1, 2,
\]
\[
\int_{\partial \mathbb{R}^n_+} U \nabla v_q = 0
\]
and
\[
\int_{\partial \mathbb{R}^n_+} \Delta v_q v_q dz dt \leq 0.
\]
In addition, the map \( q \mapsto v_q \) is in \( C^2(\partial M) \). (see [3] Proposition 5.1 and estimate (5.9)) and [28] for the last claim).

The function \( v_q \) is related to the first order expansion of the metric tensor \( g_{ij} \)
(see eq. [14]) and provides a sharp correction term of the bubble in order to give a good ansatz for a solution. Indeed, recasting Problem \([5]\) as \( u = i^* (f_c(u)) \), \( u \in \mathcal{H} \), we look for solution of \([12]\) having the form
\[
u = W_{\delta,q} + \delta V_{\delta,q} + \phi, \text{ with } \phi \in K^+_{\delta,q} \cap \mathcal{H}.
\]
or, in other terms, we want to solve the following couple of equation
\[
\Pi^+ \{ W_{\delta,q} + \delta V_{\delta,q} + \phi - i^* (f_c(W_{\delta,q} + \delta V_{\delta,q} + \phi)) \} = 0;
\]
\[
\Pi \{ W_{\delta,q} + \delta V_{\delta,q} + \phi - i^* (f_c(W_{\delta,q} + \delta V_{\delta,q} + \phi)) \} = 0.
\]
We rewrite \([25]\) as
\[
L(\phi) = N(\phi) + R
\]
where \( L := L_{\delta,q} \) is the linear operator
\[
L : K^+_{\delta,q} \cap \mathcal{H} \rightarrow K^+_{\delta,q} \cap \mathcal{H}
\]
\[
L(\phi) := \Pi^+ \{ \phi - i^* (f'_c(W_{\delta,q} + \delta V_{\delta,q})(\phi)) \}
\]
and the nonlinear term \( N(\phi) \) and the remainder term \( R \) are
\[
N(\phi) := \Pi^1 \{ i^* (f(W_{\delta,q} + \delta V_{\delta,q}) - f(W_{\delta,q} + \delta V_{\delta,q})) - f'(W_{\delta,q} + \delta V_{\delta,q}) [\phi] \};
\]
\[
R := \Pi^1 \{ i^* (f(W_{\delta,q} + \delta V_{\delta,q})) - W_{\delta,q} - \delta V_{\delta,q} \}.
\]

The rest of this section is devoted to show that for any choice of \( \delta, q \) a solution \( \phi \) of the \((29)\) exists. We remark that the choice of \( r_0 \) is crucial to obtain a good estimate on the size of the remainder term \( R \), which allows us to prove the main result of this section, Proposition 5.

**Lemma 3.** Assume \( n \geq 7 \) and let \( \delta = \sqrt{\lambda} \). For \( a, b \in \mathbb{R} \), \( 0 < a < b \) there exists a positive constant \( C_0 = C_0(a, b) \) such that, for \( \varepsilon \) small, for any \( q \in \partial M \), for any \( \lambda \in [a, b] \) and for any \( \phi \in K_{\lambda,q}^{\perp} \cap H \) there holds
\[
\| L_{\delta,q}(\phi) \|_H \geq C_0 \| \phi \|_H.
\]

**Proof.** The proof of this Lemma is very similar to the proof of \([27] \) Lemma 2 and will be omitted. \( \square \)

**Lemma 4.** Assume \( n \geq 7 \) and let \( \delta = \sqrt{\lambda} \). For \( a, b \in \mathbb{R} \), \( 0 < a < b \) there exists a positive constant \( C_1 = C_1(a, b) \) such that, for \( \varepsilon \) small, for any \( q \in \partial M \) and for any \( \lambda \in [a, b] \) there holds
\[
\| R \|_H \leq C_1 \varepsilon \ln \varepsilon
\]

**Proof.** We estimate firstly \( \| R \|_H \). We have
\[
\| R \|_H \leq \| i^* (f_*(W_{\delta,q} + \delta V_{\delta,q})) - i^* (f_0(W_{\delta,q} + \delta V_{\delta,q})) \|_H + \| i^* (f_0(W_{\delta,q} + \delta V_{\delta,q})) - W_{\delta,q} - \delta V_{\delta,q} \|_H.
\]

We start by estimating the second term. By definiton of \( i^* \) there exists \( \Gamma = i^* (f_0(W_{\delta,q} + \delta V_{\delta,q})) \), that is a function \( \Gamma \) solving
\[
\begin{cases}
- \Delta \gamma \Gamma + a(x) \Gamma = 0 & \text{on } M \\
\frac{\partial \gamma}{\partial \nu} \Gamma = f_0(W_{\delta,q} + \delta V_{\delta,q}) & \text{on } \partial M.
\end{cases}
\]

So we have
\[
\| i^* (f_0(W_{\delta,q} + \delta V_{\delta,q})) - W_{\delta,q} - \delta V_{\delta,q} \|_H^2 = \| \Gamma - W_{\delta,q} - \delta V_{\delta,q} \|_H^2
\]
\[
= \int_M [- \Delta \gamma (\Gamma - W_{\delta,q} - \delta V_{\delta,q}) + a(\Gamma - W_{\delta,q} - \delta V_{\delta,q}) (\Gamma - W_{\delta,q} - \delta V_{\delta,q})] d\mu_g
\]
\[
+ \int_{\partial M} \left[ \frac{\partial}{\partial \nu} (\Gamma - W_{\delta,q} - \delta V_{\delta,q}) (\Gamma - W_{\delta,q} - \delta V_{\delta,q}) \right] d\sigma
\]
\[
= \int_M [- \Delta \gamma (W_{\delta,q} + \delta V_{\delta,q}) - a(W_{\delta,q} + \delta V_{\delta,q}) (\Gamma - W_{\delta,q} - \delta V_{\delta,q})] d\mu_g
\]
\[
+ \int_{\partial M} \left[ f_0(W_{\delta,q} + \delta V_{\delta,q}) - \frac{\partial}{\partial \nu} (W_{\delta,q} + \delta V_{\delta,q}) \right] (\Gamma - W_{\delta,q} - \delta V_{\delta,q}) d\sigma
\]
\[
=: I_1 + I_2.
\]

We have
\[
I_1 \leq C \| \Delta \gamma (W_{\delta,q} + \delta V_{\delta,q}) - a(W_{\delta,q} + \delta V_{\delta,q}) \|_{L_{\lambda,q}^\infty(M)} \| \Gamma - W_{\delta,q} - \delta V_{\delta,q} \|_H.
\]

By direct computation we have immediately that \( \| (W_{\delta,q} + \delta V_{\delta,q}) \|_{L_{\lambda,q}^\infty(M)} = O(\delta^2) \). Then we proceed as in \([28] \) eq. (35). Recalling that in local charts the Laplace
Beltrami operator is

$$\Delta_y W_{\delta,q} = \Delta_{\text{euc}} (U_\delta(u) \chi(y)) + [g^{ij}(y) - \delta_{ij}] \partial_{ij}^2 (U_\delta(u) \chi(y)) - g^{ij}(y) \Gamma_{ij}^k (y) \partial_k (U_\delta(u) \chi(y))$$

and noticing that by (13) and (14) for the Christoffel symbols holds $\Gamma_{ij}^k (y) = O(|y|)$, we have, in variables $y = \delta x$,

$$\Delta_y W_{\delta,q} = U_\delta(u) \Delta_{\text{euc}} (\chi(y)) + 2 \nabla U_\delta(u) \nabla \chi(y) + [g^{ij}(y) - \delta_{ij}] \partial_{ij}^2 (U_\delta(u) \chi(y)) - g^{ij}(y) \Gamma_{ij}^k (y) \partial_k (U_\delta(u) \chi(y))$$

$$= \frac{1}{\delta^{n+2}} \left( 2 h_{ij}(0) \delta x_n \frac{1}{\delta^2} \partial_{ij} U(x) + g^{ij}(x) \Gamma_{ij}^k (x) \frac{1}{\delta} \partial_k U + o(\delta)c(x) \right)$$

$$= \frac{1}{\delta^2} \left( 2 h_{ij}(0) x_n \partial_{ij}^2 U(y) + O(\delta)c(y) \right)$$

(31)

where, with abuse of notation, we call $c(x)$ function with $|\int_{R^n} c(x) dx| \leq C$. In a similar way, by (21) and by (14) we have

$$\delta \Delta_y V_{\delta,q} = \frac{1}{\delta^2} \left( -2 h_{ij}(0) x_n \partial_{ij}^2 U(y) + O(\delta)c(y) \right)$$

(32)

Thus, in local chart by (31) and (32) we get

$$|\Delta_y (W_{\delta,q} + \delta V_{\delta,q})|_{L^\infty(M)} = \delta^{\frac{n+2}{n}} \frac{1}{\delta^2} O(\delta) = O(\delta^2)$$

(33)

and we conclude that

$$I_1 = O(\delta^2) \|\Gamma - W_{\delta,q}\|_H.$$  

Notice that the estimate (33) is possible since we carefully choose the function $v_q$ as a solution of (21).

For the second integral $I_2$ we proceed in a similar way, getting

$$I_2 \leq C \left| f_0(W_{\delta,q} + \delta V_{\delta,q}) - \frac{\partial}{\partial \nu} (W_{\delta,q} - \delta V_{\delta,q}) \right|_{L^2(n-1)_{\nu}(\partial M)} \|\Gamma - W_{\delta,q} - \delta V_{\delta,q}\|_H$$

and again, arguing similarly to (19) Lemma 9], we have, since $U$ is a solution of (11),

$$\int_{\partial M} \left( (n-2) W_{\delta,q} - \frac{\partial}{\partial \nu} W_{\delta,q} \right) R \leq \int_{\partial M} \left( (n-2) W_{\delta,q} - \frac{\partial}{\partial \nu} W_{\delta,q} \right) R d\sigma$$

$$= O(\delta^2) \|R\|_H.$$

Now we estimate

$$\int_{\partial M} \left\{ (n-2) \left[ (W_{\delta,q} + \delta V_{\delta,q})^\frac{n+2}{n} - W_{\delta,q}^\frac{n+2}{n} \right] - \delta \frac{\partial V_{\delta,q}}{\partial \nu} \right\} R d\sigma$$

$$\leq c (n-2) \left[ ((W_{\delta,q} + \delta V_{\delta,q})^\frac{n+2}{n} - W_{\delta,q}^\frac{n+2}{n}) \right] - \delta \frac{\partial V_{\delta,q}}{\partial \nu} \|R\|_{L^2(n-1)_{\nu}(\partial M)}$$

$$= O(\delta^2) \|R\|_H.$$
and, by Taylor expansion and by definition of the function $v_q$ (see (21)
\[
\left| n - 2 \left[ \left( W_{\delta,q} + \delta V_{\delta,q} \right)^{n_{\delta}} - W_{\delta,q}^{n_{\delta}} \right] - \delta \frac{\partial V_{\delta,q}}{\partial \nu} \right|_{L^{2(n-1)}(\partial M)}
\]
\[
\leq \left( \int_{|z|<\frac{1}{2}} \left| - 2 \left[ \left( (U + \delta v_q)^{n_{\delta}} - U^{n_{\delta}} \right) + \delta \frac{\partial v_q}{\partial t} \right] \right|^{2(n-1)} \right) + o(\delta^2)
\]
\[
\leq \epsilon \left( \int_{|z|<\frac{1}{2}} \left( n \left( (U + \delta V_{\delta,q})^{n_{\delta}} - U^{n_{\delta}} - \delta V_{\delta,q} \right) \right) \right) + o(\delta^2)
\]
Notice that, for $\delta$ small enough, $U + \delta v_q > 0$ if $|y| \leq 1/\delta$ by the decay estimates (22). At this point, using again Taylor expansion, we have
\[
\int_{|z|<\frac{1}{2}} \left| \left( (U + \delta v_q)^{n_{\delta}} - U^{n_{\delta}} \right) \right|^{2(n-1)}
\]
where the last integral is bounded since $n \geq 7$.
Thus
\[
\left| n - 2 \left[ \left( (W_{\delta,q} + \delta V_{\delta,q})^{n_{\delta}} - W_{\delta,q}^{n_{\delta}} \right) - \delta \frac{\partial V_{\delta,q}}{\partial \nu} \right|_{L^{2(n-1)}(\partial M)} = O(\delta^2),
\]
so
\[
I_2 = O(\delta^2) \frac{\|v_q\|}{\|V_q\|} H + \frac{\|v_q\|}{\|V_q\|} H \text{ and, consequently,}
\]
\[
\|v_q^* (f_0(W_{\delta,q} + \delta V_{\delta,q})) - W_{\delta,q} - \delta V_{\delta,q}\| = O(\delta^2).
\]
To conclude the first part of the proof we estimate the term
\[
\|v_q^* (f_0(W_{\delta,q} + \delta V_{\delta,q})) - i^* (f_0(W_{\delta,q} + \delta V_{\delta,q}))\|_H.
\]
It is useful to recall the following Taylor expansions with respect to $\epsilon$
\[
U^2 = 1 + \epsilon \ln U + \frac{1}{2} \epsilon^2 \ln^2 U + o(\epsilon^2)
\]
\[
\delta^{1-\epsilon^{n-2}} = 1 - \epsilon \frac{n-2}{2} \ln \delta + \epsilon^2 \frac{(n-2)^2}{8} \ln^2 \delta + o(\epsilon^2 \ln^2 \delta)
\]
We have that
\[
\|v_q^* (f_0(W_{\delta,q} + \delta V_{\delta,q})) - i^* (f_0(W_{\delta,q} + \delta V_{\delta,q}))\|_H
\]
\[
= \left\{ \int_{|z|<\frac{1}{2}} \left| \left( \delta^{1-\epsilon^{n-2}} (U + \delta v_q)^{n_{\delta}} - 1 \right) \right|^{2(n-1)} \right\} + O(\delta^2)
\]
\[
\leq \left\{ \int_{|z|<\frac{1}{2}} \left| \left( \frac{1}{\delta^{1-\epsilon^{n-2}}} (U + \delta v_q)^{n_{\delta}} - 1 \right) \right|^{2(n-1)} \right\} + O(\delta^2)
\]
\[
= O(\epsilon \ln |\delta|) + O(\epsilon) + O(\delta^2)
\]
Remembering that $\delta = \lambda \sqrt{\epsilon}$ we get the required estimate in $H$-norm.
To conclude the proof, we have to control $|R|_{L^2(\partial M)}$. As in the previous case we consider
\[
|R|_{L^2(\partial M)} \leq |v_q^* (f_0(W_{\delta,q} + \delta V_{\delta,q})) - i^* (f_0(W_{\delta,q} + \delta V_{\delta,q}))\|_{L^2(\partial M)}
\]
\[
+ |i^* (f_0(W_{\delta,q} + \delta V_{\delta,q})) - W_{\delta,q} - \delta V_{\delta,q}|_{L^2(\partial M)}
\]
and we start estimating the second term. Taken $\Gamma = i^* (f_0(W_{\delta,q} + \delta V_{\delta,q}))$ the solution of \((28)\), we have that the function $\Gamma - W_{\delta,q} - \delta V_{\delta,q}$ solves the problem

\[
\begin{align*}
-\Delta_g(\Gamma - W_{\delta,q} - \delta V_{\delta,q}) + a(x)(\Gamma - W_{\delta,q} - \delta V_{\delta,q}) \\
eq -\Delta_g(W_{\delta,q} + \delta V_{\delta,q}) + a(x)(W_{\delta,q} + \delta V_{\delta,q}) \\
\frac{\partial}{\partial \nu}(\Gamma - W_{\delta,q} - \delta V_{\delta,q}) = f_0(W_{\delta,q} + \delta V_{\delta,q}) - \frac{\partial}{\partial \nu}(W_{\delta,q} + \delta V_{\delta,q}) \\
on M
\end{align*}
\]

\(\Gamma = \partial\) and let $\Gamma = \partial\), so, by Remark $2$, we get

\[
|\Gamma - W_{\delta,q} - \delta V_{\delta,q}|_{L^{1+n}(\partial M)} \leq | - \Delta_g(W_{\delta,q} + \delta V_{\delta,q}) + a(x)(W_{\delta,q} + \delta V_{\delta,q})|_{L^{1+n}(M)}
\]

Moreover the map $\phi_{\delta,q}$ exists a positive constant $C$.

Remark 2. Assume $n \geq 7$ and let $\delta = \lambda \sqrt{\varepsilon}$. For $a, b \in \mathbb{R}$, $0 < a < b$ there exists a positive constant $C = C(a, b)$ such that, for $\varepsilon$ small, for any $q \in \partial M$ and for any $\lambda \in [a, b]$ there exists a unique $\phi_{\delta,q}$ which solves \((28)\) with

\[
\|\phi_{\delta,q}\|_{H^2} \leq C\varepsilon |\ln \varepsilon|
\]

Moreover the map $q \mapsto \phi_{\delta,q}$ is a $C^1(\partial M, H)$ map.

Proposition 5. Assume $n \geq 7$ and let $\delta = \lambda \sqrt{\varepsilon}$. For $a, b \in \mathbb{R}$, $0 < a < b$ there exists a positive constant $C = C(a, b)$ such that, for $\varepsilon$ small, for any $q \in \partial M$ and for any $\lambda \in [a, b]$ there exists a unique $\phi_{\delta,q}$ which solves \((28)\) with

\[
\|\phi_{\delta,q}\|_{H^2} \leq C\varepsilon |\ln \varepsilon|
\]

Moreover the map $q \mapsto \phi_{\delta,q}$ is a $C^1(\partial M, H)$ map.

Proof. First we prove that the nonlinear operator $N$ defined \((28)\) is a contraction on a suitable ball of $H$. Recalling that

\[
\|\phi_{\delta,q}\|_{H^2} \leq C\varepsilon |\ln \varepsilon|
\]

we estimate the two right hand side terms separately.
By the continuity of $i^*: L^{\frac{2(n-1)}{n-2}}(\partial M) \to H$, and by Lagrange theorem we have
\[ \|N(\phi_1) - N(\phi_2)\|_H \]
\[ \leq \|[f'_c(W_{\delta,q} + \theta \phi_1 + (1 - \theta)\phi_2 + \delta V_{\delta,q}) - f'_c(W_{\delta,q} + \phi_2)]\|_{L^{\frac{2(n-1)}{n-2}}(\partial M)} \]
and, since $|\phi_1 - \phi_2|^{\frac{2(n-1)}{n-2}} \in L^{\frac{2(n-1)}{n-2}}(\partial M)$ and $|f'_c(\cdot)|^{\frac{2(n-1)}{n-2}} \in L^{\frac{2(n-1)}{n-2}}(\partial M)$, we have
\[ \|N(\phi_1) - N(\phi_2)\|_H \]
\[ \leq \|[f'_c(W_{\delta,q} + \theta \phi_1 + (1 - \theta)\phi_2 + \delta V_{\delta,q}) - f'_c(W_{\delta,q} + \phi_2)]\|_{L^{\frac{2(n-1)}{n-2}}(\partial M)} \]
\[ = \gamma \|\phi_1 - \phi_2\|_H \]
where we can choose
\[ \gamma := \|[f'_c(W_{\delta,q} + \theta \phi_1 + (1 - \theta)\phi_2 + \delta V_{\delta,q}) - f'_c(W_{\delta,q} + \phi_2)]\|_{L^{\frac{2(n-1)}{n-2}}(\partial M)} < 1, \]
provided $\|\phi_1\|_H$ and $\|\phi_2\|_H$ sufficiently small.

For the second term we argue in a similar way and, recalling that, by (10), $|i^*(g)|_{L^{\infty}(\partial M)} \leq |g| \frac{2^{(n-1)}}{(n-1)^{n-1}}$, we have
\[ \|N(\phi_1) - N(\phi_2)\|_{L^{\infty}(\partial M)} \]
\[ \leq \|[f'_c(W_{\delta,q} + \theta \phi_1 + (1 - \theta)\phi_2 + \delta V_{\delta,q}) - f'_c(W_{\delta,q} + \phi_2)]\|_{L^{\frac{2(n-1)}{n-2}}(\partial M)} \]
Since $\phi_1, \phi_2, W_{\delta,q} \phi_1 V_{\delta,q} \in L^{\infty}$ we have that $|\phi_1 - \phi_2|^{\frac{2(n-1)}{n-2} + n(n-2)} \in L^{\frac{n(n-2)}{n-2}}(\partial M)$ and $|f'_c(\cdot)|^{\frac{2(n-1)}{n-2} + n(n-2)} \in L^{\frac{n(n-2)}{n-2}}(\partial M)$. So we conclude as above that we can choose $\|\phi_1\|_{L^{\infty}(\partial M)}$, $\|\phi_2\|_{L^{\infty}(\partial M)}$ sufficiently small in order to get
\[ \|N(\phi_1) - N(\phi_2)\|_{L^{\infty}(\partial M)} \leq \gamma \|\phi_1 - \phi_2\|_H \cdot \]
So
\[ \|N(\phi_1) - N(\phi_2)\|_H \leq \gamma \|\phi_1 - \phi_2\|_H \]
with $\gamma < 1$, provided $\|\phi_1\|_H$, $\|\phi_2\|_H$ small enough.

With the same strategy it is possible to prove that if $\|\phi\|_H$ is sufficiently small there exists $\gamma < 1$ such that $\|N(\phi)\|_H \leq \gamma \|\phi\|_H$.

At this point, recalling Lemma 3 and Lemma 4, it is not difficult to prove that there exists a constant $C > 0$ such that, if $\|\phi\|_H \leq C \varepsilon \ln \varepsilon$ then the map
\[ T(\phi) := L^{-1}(N(\phi) + R_{\varepsilon, \delta,q}) \]
is a contraction from the ball $\|\phi\|_H \leq C \varepsilon \ln \varepsilon$ in itself, and we get the first claim by the Contraction Mapping Theorem. The regularity claim can be proven by means of the Implicit Function Theorem.

\[ \square \]

4. The reduced problem

For any given $(\delta, q)$, we are able to solve the infinite dimensional problem (25) by Proposition 3. Now, set $\delta = \lambda \sqrt{\varepsilon}$, we try to solve (38) finding a critical point of the functional
\[ (39) \quad J_\varepsilon(u) := \frac{1}{2} \int_M |\nabla u|^2 + au^2 d\mu - \frac{(n-2)^2}{2(n-1) + \varepsilon(n-2)} \int_{\partial M} \langle u^* \rangle^{\frac{2(n-1)+\varepsilon}{n-1}} d\sigma, \]
evaluated in $W_{\lambda \sqrt{\varepsilon}, q} + \lambda \sqrt{\varepsilon} V_{\lambda \sqrt{\varepsilon}, q} + \phi_{\lambda \sqrt{\varepsilon}, q}$. We observe that, by Remark 2 the functional $J_\varepsilon$ is well defined on $H$. Since $J_\varepsilon(W_{\lambda \sqrt{\varepsilon}, q} + \lambda \sqrt{\varepsilon} V_{\lambda \sqrt{\varepsilon}, q} + \phi_{\lambda \sqrt{\varepsilon}, q})$ depends only, given $\varepsilon$, on $(\lambda, q) \in [a, b] \times \partial M$, we set $I_\varepsilon(\lambda, q) := J_\varepsilon(W_{\lambda \sqrt{\varepsilon}, q} + \delta V_{\lambda \sqrt{\varepsilon}, q} + \phi_{\lambda \sqrt{\varepsilon}, q}).$
Lemma 6. Assume $n \geq 7$ and $\delta = \lambda \sqrt{\varepsilon}$. It holds

$$|I_{\varepsilon}(\lambda, q) - J_{\varepsilon}(W_{\delta, q} + \delta V_{\delta, q})| = o(\varepsilon)$$

$C^0$-uniformly for $q \in \partial M$ and $\lambda$ in a compact set of $(0, +\infty)$.

This result can be obtained following the lines of [28] Lemma 6, so we postpone the proof in the Appendix, and we proceed to the main result of this section.

Proposition 7. Assume $n \geq 7$ and $\delta = \lambda \sqrt{\varepsilon}$. It holds

$$J_{\varepsilon}(W_{\lambda, \pi, q} + \lambda \sqrt{\varepsilon} V_{\lambda, \pi, q}) = A + B(\varepsilon) + \varepsilon \lambda \varphi(q) + C \varepsilon \ln \lambda + o(\varepsilon),$$

$C^0$-uniformly for $q \in \partial M$ and $\lambda$ in a compact set of $(0, +\infty)$, where

$$A = \frac{(n-2)(n-3)}{2(n-1)^2} \omega_{n-1} L^2_{n-1}$$

$$B(\varepsilon) = \varepsilon \left[ \frac{(n-2)^3}{2(n-1)} \int_{\mathbb{R}^{n-1}} U^{\frac{2(n-1)}{n-2}}(z, 0) dz - \frac{n-2}{2(n-1)} \int_{\mathbb{R}^{n-1}} U^{\frac{2(n-1)}{n-2}}(z, 0) \ln U(z, 0) dz \right]$$

$$\varphi(q) = \frac{1}{2} \int_{\mathbb{R}^n} \Delta \varphi \varphi dt - \frac{(n-6)(n-2)\omega_{n-1} L^2_{n-1}}{4(n-1)^2(n-4)} \|\pi(q)\|^2 \leq 0.$$

$$C = \frac{(n-2)^2(n-3)}{4(n-1)^2} \omega_{n-1} L^2_{n-1} > 0.$$

Proof. Since $\frac{(n-2)^2}{2(n-1)^2} = \frac{C(n-2)^2}{2(n-1)} - \frac{(n-2)^3}{2(n-1)} + o(\varepsilon)$, we can write

$$J_{\varepsilon}(W_{\delta, q} + \delta V_{\delta, q})$$

$$= \frac{1}{2} \int_M |\nabla_g(W_{\delta, q} + \delta V_{\delta, q})|^2 + a(W_{\delta, q} + \delta V_{\delta, q})^2 d\mu_g$$

$$- \left[ \frac{(n-2)^3}{2(n-1)} + o(\varepsilon) \right] \int_{\partial M} ((W_{\delta, q} + \delta V_{\delta, q})^+) \frac{2(n-1)}{n-2} d\sigma$$

$$= \frac{1}{2} \int_M |\nabla_g(W_{\delta, q} + \delta V_{\delta, q})|^2 + a(W_{\delta, q} + \delta V_{\delta, q})^2 d\mu_g$$

$$- \left[ \frac{(n-2)^3}{2(n-1)} + o(\varepsilon) \right] \int_{\partial M} ((W_{\delta, q} + \delta V_{\delta, q})^+) \frac{2(n-1)}{n-2} d\sigma$$

$$= \frac{1}{2} \int_M |\nabla_g(W_{\delta, q} + \delta V_{\delta, q})|^2 + a(W_{\delta, q} + \delta V_{\delta, q})^2 d\mu_g$$

$$- \left[ \frac{(n-2)^3}{2(n-1)} + o(\varepsilon) \right] \int_{\partial M} ((W_{\delta, q} + \delta V_{\delta, q})^+) \frac{2(n-1)}{n-2} d\sigma$$

$$+ \left[ \frac{(n-2)^3}{2(n-1)} + o(\varepsilon) \right] \int_{\partial M} ((W_{\delta, q} + \delta V_{\delta, q})^+) \frac{2(n-1)}{n-2} d\sigma.$$

For the first part we proceed as in [28] Proposition 13 (which we refer to for the proof) obtaining that

$$\frac{1}{2} \int_M |\nabla_g(W_{\delta, q} + \delta V_{\delta, q})|^2 + a(W_{\delta, q} + \delta V_{\delta, q})^2 d\mu_g$$

$$- \left[ \frac{(n-2)^3}{2(n-1)} \right] \int_{\partial M} ((W_{\delta, q} + \delta V_{\delta, q})^+) \frac{2(n-1)}{n-2} d\sigma$$

$$= A + \varepsilon \lambda^2 \varphi(q) + o(\varepsilon).$$

Using again [36] and [37], proceeding similarly to [35], and recalling that $\delta = \lambda \sqrt{\varepsilon}$ we have
\[
\int_{\partial M} ((W_{\delta,q} + \delta V_{\delta,q})^+)^{\frac{2(n-1)}{n-2} + \epsilon} - ((W_{\delta,q} + \delta V_{\delta,q})^+)^{\frac{2(n-1)}{n-2}} d\sigma
\]
\[
\leq \int_{|z|<\frac{1}{4}} \frac{1}{\delta^2 + \epsilon^2} ((U + \delta v_q)^\epsilon - 1) (U + \delta v_q)^{\frac{2(n-1)}{n-2}} dz + o(\delta^2)
\]
\[
= \int_{\mathbb{R}^{n-1}} \left( -\frac{n-2}{4} \epsilon \ln \epsilon - \frac{n-2}{2} \epsilon \ln \lambda + \epsilon \ln (U) + O(\epsilon^2 \ln \epsilon) \right) U^{\frac{2(n-1)}{n-2}} dz + o(\epsilon)
\]
\[
= \frac{n-2}{4} \epsilon |\ln\epsilon| \int_{\mathbb{R}^{n-1}} U^{\frac{2(n-1)}{n-2}} dz + \epsilon \int_{\mathbb{R}^{n-1}} U^{\frac{2(n-1)}{n-2}} \ln(U) dz
\]
\[
- \frac{n-2}{4} \epsilon \ln \lambda \int_{\mathbb{R}^{n-1}} U^{\frac{2(n-1)}{n-2}} dz + o(\epsilon).
\]
Finally, with the same technique,
\[
\left[ \epsilon \left( \frac{n-2}{2(n-1)} + o(\epsilon) \right) \int_{\partial M} ((W_{\delta,q} + \delta V_{\delta,q})^+)^{\frac{2(n-1)}{n-2} + \epsilon} d\sigma \right]
\]
\[
= \left[ \epsilon \left( \frac{n-2}{2(n-1)} + o(\epsilon) \right) \int_{|z|<\frac{1}{4}} \frac{1}{\delta^2 + \epsilon^2} (U + \delta v_q)^{\frac{2(n-1)}{n-2}} (U + \delta v_q)^\epsilon dz + o(\delta^2) \right]
\]
\[
= \frac{\epsilon(n-2)^3}{2(n-1)} U^{\frac{2(n-1)}{n-2}} + o(\epsilon),
\]
and the proof follows easily taking in account that
\[
\frac{(n-2)^2}{4(n-1)} \int_{\mathbb{R}^{n-1}} U^{\frac{2(n-1)}{n-2}} dz = \frac{(n-2)^2}{4(n-1)} \omega_{n-1} I_{n-1} = \frac{(n-2)^2(n-3)}{4(n-1)^2} \omega_{n-1} I_{n-1}.
\]

5. Proof of Theorem [1]

Once we have a critical point of the reduced functional
\[
I_\epsilon(\lambda, q) := J_\epsilon(W_{\lambda\sqrt{\tau,q} + \lambda\sqrt{\tau}V_{\lambda\sqrt{\tau,q} + \phi_{\lambda\sqrt{\tau,q}}}^+}
\]
we solve Problem [3]. In fact it holds the following result.

Lemma 8. If \((\lambda, q) \in (0, +\infty) \times \partial M\) is a critical point for the reduced functional \(I_\epsilon(\lambda, q)\), then the function \(W_{\lambda\sqrt{\tau,q} + \lambda\sqrt{\tau}V_{\lambda\sqrt{\tau,q} + \phi_{\lambda\sqrt{\tau,q}}}}\) is a solution of [3]. Here \(\phi_{\lambda\sqrt{\tau,q}}\) is defined in Proposition [5].

Proof. The proof is similar to the proofs of [28] Lemma 15 and [27] Proposition 5, so we sketch only the main steps. Set \(q = q(y) = v_{\lambda q}^b(y)\). Since \((\lambda, q)\) is a critical point for the \(I_\epsilon(\lambda, q)\), and since \(W_{\lambda\sqrt{\tau,q} + \lambda\sqrt{\tau}V_{\lambda\sqrt{\tau,q} + \phi_{\lambda\sqrt{\tau,q}}}}\) solves [25], we have, for \(h = 1, \ldots, n - 1\),
0 = \frac{\partial}{\partial y_h} I_c(\lambda, q(y)) \bigg|_{y=0}
= \langle \lambda \sqrt{\varepsilon} W_{\lambda \sqrt{\varepsilon}, q(y)} + \lambda \sqrt{\varepsilon} V_{\lambda \sqrt{\varepsilon}, q(y)} + \phi_{\lambda \sqrt{\varepsilon}, q(y)} \rangle - i^* \left( f_c(\lambda \sqrt{\varepsilon} W_{\lambda \sqrt{\varepsilon}, q(y)} + \lambda \sqrt{\varepsilon} V_{\lambda \sqrt{\varepsilon}, q(y)} + \phi_{\lambda \sqrt{\varepsilon}, q(y)}) \right),
= \frac{\partial}{\partial y_h} (\lambda \sqrt{\varepsilon} W_{\lambda \sqrt{\varepsilon}, q(y)} + \lambda \sqrt{\varepsilon} V_{\lambda \sqrt{\varepsilon}, q(y)} + \phi_{\lambda \sqrt{\varepsilon}, q(y)}) \rangle |_{y=0}
= \sum_{i=1}^{n} c_i \langle \lambda \sqrt{\varepsilon} W_{\lambda \sqrt{\varepsilon}, q(y)} + \lambda \sqrt{\varepsilon} V_{\lambda \sqrt{\varepsilon}, q(y)} + \phi_{\lambda \sqrt{\varepsilon}, q(y)} \rangle |_{y=0}
= \sum_{i=1}^{n} c_i \langle \lambda \sqrt{\varepsilon} W_{\lambda \sqrt{\varepsilon}, q(y)} + \lambda \sqrt{\varepsilon} V_{\lambda \sqrt{\varepsilon}, q(y)} + \phi_{\lambda \sqrt{\varepsilon}, q(y)} \rangle |_{y=0}
= \sum_{i=1}^{n} c_i \langle \lambda \sqrt{\varepsilon} W_{\lambda \sqrt{\varepsilon}, q(y)} + \lambda \sqrt{\varepsilon} V_{\lambda \sqrt{\varepsilon}, q(y)} + \phi_{\lambda \sqrt{\varepsilon}, q(y)} \rangle |_{y=0}

Arguing as in Lemma 6.1 and Lemma 6.2 of [7] we have:

\langle \lambda \sqrt{\varepsilon} W_{\lambda \sqrt{\varepsilon}, q(y)} + \lambda \sqrt{\varepsilon} V_{\lambda \sqrt{\varepsilon}, q(y)} + \phi_{\lambda \sqrt{\varepsilon}, q(y)} \rangle |_{y=0} = \frac{\partial}{\partial y_h} W_{\lambda \sqrt{\varepsilon}, q(y)} + \lambda \sqrt{\varepsilon} V_{\lambda \sqrt{\varepsilon}, q(y)} + \phi_{\lambda \sqrt{\varepsilon}, q(y)} = O\left( \frac{1}{\sqrt{\varepsilon}} \right)

\langle \lambda \sqrt{\varepsilon} W_{\lambda \sqrt{\varepsilon}, q(y)} + \lambda \sqrt{\varepsilon} V_{\lambda \sqrt{\varepsilon}, q(y)} + \phi_{\lambda \sqrt{\varepsilon}, q(y)} \rangle |_{y=0} = \frac{\partial}{\partial y_h} W_{\lambda \sqrt{\varepsilon}, q(y)} + \lambda \sqrt{\varepsilon} V_{\lambda \sqrt{\varepsilon}, q(y)} + \phi_{\lambda \sqrt{\varepsilon}, q(y)} = O(1).\]

We conclude that

0 = \frac{1}{\lambda \sqrt{\varepsilon}} \sum_{i=1}^{n} c_i \left( \delta_h + O(1) \right)

and so \( c_i \) = 0, where \( h = 1, \ldots, n - 1 \).

Arguing analogously for \( \frac{\partial}{\partial I_c(\lambda, q)} |_{\lambda=\lambda} \) we can prove that \( c_i = 0 \) for all \( i = 1, \ldots, n \), obtaining that \( W_{\lambda \sqrt{\varepsilon}, q} + \lambda \sqrt{\varepsilon} V_{\lambda \sqrt{\varepsilon}, q} + \phi_{\lambda \sqrt{\varepsilon}, q} \) solves also \( (26) \), and so the proof is complete.

**Proof of Theorem 1** By our assumption on the second fundamental form and by \( (24) \), we have that the function \( \varphi(q) \) defined in Proposition 4 is strictly negative on \( \partial M \). We recall as well, that the number \( C \) defined in the same proposition is positive. Then, defined

\[ I : [a, b] \times \partial M \to \mathbb{R} \]
\[ I(\lambda, q) = \lambda \varphi(q) + C \ln d \]

we have that for any \( M < 0 \) there exist \( a, b \) such that

\[ I(\lambda, q) < M \text{ for any } q \in \partial M, \lambda \notin [a, b] \]

and

\[ \frac{\partial I}{\partial \lambda} (a, q) \neq 0, \quad \frac{\partial I}{\partial \lambda} (a, q) \neq 0 \quad \forall q \in \partial M. \]

Then the function \( I \) admits a absolute maximum on \([a, b] \times \partial M\). This maximum is also \( C^2 \)-stable. In other words, if \((\lambda_0, q_0)\) is the maximum point for \( I \), for any function \( f \in C^2([a, b] \times \partial M) \) with \( \| f \|_{C^0} \) sufficiently small, then the function \( I + f \) on \([a, b] \times \partial M\) admits a maximum point \((\lambda, q)\) close to \((\lambda_0, q_0)\).

Then, taken an \( \varepsilon \) sufficiently small, in light of Proposition 4 and Proposition 7 there exists a pair \((\lambda_\varepsilon, q_\varepsilon)\) maximum point for \( I_\varepsilon(\lambda, q) \). Thus, by Lemma 8...
Proof of Lemma 6. We have, for some \( \varepsilon \):

\[
\int_M \left( \nabla_g W_{\delta,q} + \delta \nabla g V_{\delta,q} \right) \nabla \phi_{\delta,q} + a(x) \left( W_{\delta,q} + \delta V_{\delta,q} \right) \phi_{\delta,q} d\mu_g
\]

By integration by parts we have

\[
= \int_M \left( \nabla_g W_{\delta,q} + \delta \nabla g V_{\delta,q} \right) \nabla \phi_{\delta,q} + a(x) \left( W_{\delta,q} + \delta V_{\delta,q} \right) \phi_{\delta,q} d\mu_g
\]

By Holder inequality one can easily obtain

\[
\int_M \left( \nabla_g W_{\delta,q} + \delta \nabla g V_{\delta,q} \right) \nabla \phi_{\delta,q} + a(x) \left( W_{\delta,q} + \delta V_{\delta,q} \right) \phi_{\delta,q} d\mu_g
\]

By definition of \( \| \cdot \|_{L^2} \),

\[
\int_M \left( \nabla_g W_{\delta,q} + \delta \nabla g V_{\delta,q} \right) \nabla \phi_{\delta,q} + a(x) \left( W_{\delta,q} + \delta V_{\delta,q} \right) \phi_{\delta,q} d\mu_g = \| \phi_{\delta,q} \|^2_{L^2} = o(\varepsilon).
\]

By Holder inequality one can easily obtain

\[
\int_M a \left( W_{\delta,q} + \delta V_{\delta,q} \right) \phi_{\delta,q} d\mu_g \leq C \left| W_{\delta,q} \right|_{L^\frac{2(n+1)}{n}} \left| \phi_{\delta,q} \right|_{L^\frac{2(n-1)}{n-2}(\partial M)} = O(\delta^2) \left\| \phi_{\delta,q} \right\|_H = o(\varepsilon)
\]

Since \( \left( W_{\delta,q} + \delta V_{\delta,q} + \theta \phi_{\delta,q} \right) \) belongs to \( L^\frac{2(n-1)+n(n-2)}{2(n-2)(n-1)} \) and since \( 2 \left( \frac{2(n-1)+n(n-2)}{2(n-2)(n-1)} \right) = \frac{4(n-1)+2n(n-2)}{2(n-2)^2} < s_\varepsilon \), by Holder inequality we obtain

\[
\int_{\partial M} \left( W_{\delta,q} + \delta V_{\delta,q} + \theta \phi_{\delta,q} \right) \nabla \phi_{\delta,q} d\sigma_g \leq C \left( \left| W_{\delta,q} + \delta V_{\delta,q} + \theta \phi_{\delta,q} \right|_{L^s(\partial M)} \right) \left\| \phi_{\delta,q} \right\|_{H} = o(\varepsilon).
\]

By integration by parts we have

\[
\int_M \left( \nabla_g W_{\delta,q} + \delta \nabla g V_{\delta,q} \right) \nabla \phi_{\delta,q} d\mu_g = -\int_M \Delta_g (W_{\delta,q} + \delta V_{\delta,q}) \phi_{\delta,q} d\mu_g + \int_{\partial M} \left( \frac{\partial}{\partial \nu} W_{\delta,q} + \delta \frac{\partial}{\partial \nu} V_{\delta,q} \right) \phi_{\delta,q} d\sigma_g.
\]

and, as in (33), we get

\[
\int_M \Delta_g (W_{\delta,q} + \delta V_{\delta,q}) \phi_{\delta,q} d\mu_g \leq |\Delta_g (W_{\delta,q} + \delta V_{\delta,q})|_{L^\frac{2n}{n+2}(M)} \left\| \phi_{\delta,q} \right\|_H = O(\delta^2) \left\| \phi_{\delta,q} \right\|_H = o(\varepsilon),
\]
and for the boundary term in (10), in light of (35) and the following formulas, we get

$$\int_{\partial M} \left[ \left( \frac{\partial}{\partial v} W_{\delta,q} + \delta \frac{\partial}{\partial v} V_{\delta,q} \right) - \left( W_{\delta,q} + \delta V_{\delta,q} \right)^{\frac{n-2}{2}} \right] \phi_{\delta,q} d\sigma_g$$

$$\leq \left( \left( W_{\delta,q} + \delta V_{\delta,q} \right)^{\frac{n-2}{2}} - \left( W_{\delta,q} + \delta V_{\delta,q} \right)^{\frac{n-2}{2}} \right) \phi_{\delta,q} d\sigma_g$$

At this point it remains to estimate

$$\int_{\partial M} \left[ \left( W_{\delta,q} + \delta V_{\delta,q} \right)^{\frac{n-2}{2}} - \left( W_{\delta,q} + \delta V_{\delta,q} \right)^{\frac{n-2}{2}} \right] \phi_{\delta,q} d\sigma_g$$

and we proceed as in (38) to get

$$\int_{\partial M} \left[ \left( W_{\delta,q} + \delta V_{\delta,q} \right)^{\frac{n-2}{2}} - \left( W_{\delta,q} + \delta V_{\delta,q} \right)^{\frac{n-2}{2}} \right] \phi_{\delta,q} d\sigma_g$$

and we conclude the proof.

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