Casimir energy in the gauge/gravity description of Bjorken flow?

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In the AdS/CFT description of four-dimensional QCD matter undergoing Bjorken expansion, does the holographic energy-momentum tensor contain a Casimir-type contribution that should not be attributed to thermal matter? When the bulk isometry ansatz that yielded such a Casimir term for (1+1)-dimensional boundary matter is generalised to a four-dimensional boundary, we show that a Casimir term does not arise, owing to singularities in the five-dimensional bulk solution. The geometric reasons are traced to a difference between the isometries of AdS\textsubscript{3} and AdS\textsubscript{d+1} for \(d \geq 3\).

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1 Introduction

Ultrarelativistic heavy ion collisions can be modelled by the scale-free Bjorken expansion \cite{Bjorken:1982qr} in which the energy density $\epsilon$ and the temperature $T$ depend on the proper time $\tau$ as\footnote{We use the Minkowski coordinates $(t, x, x^2, x^3)$, in which $ds^2 = -dt^2 + dx^2 + (dx^2)^2 + (dx^3)^2$, $x$ is the longitudinal spatial coordinate and $(x^2, x^3)$ are the transversal spatial coordinates. The collision is at $t = 0 = x$. In the quadrant to the future of the collision, $t > |x|$, we use the Milne-like coordinates $(\tau, y, x^2, x^3)$, in which $\tau = \sqrt{t^2 - x^2}$ with $0 < \tau < \infty$, $y = (1/2) \log((t + x)/(t - x))$ with $-\infty < y < \infty$ and the metric reads $ds^2 = -d\tau^2 + \tau^2 dy^2 + (dx^2)^2 + (dx^3)^2$.}

$$
\epsilon(\tau) \sim T^4(\tau) \sim \frac{1}{\tau^{4/3}}, \quad T(\tau) \sim \frac{1}{\tau^{1/3}}.
$$

(1.1)

If also the shear viscosity $\eta$ is scale free, $\eta \sim T^3$, relativistic hydrodynamics predicts \cite{Kovtun:2004de} that $\epsilon$ and $T$ are corrected by terms of order $1/(\tau T) \sim 1/\tau^{2/3}$,

$$
T(\tau) = T_f \left( \frac{\tau_f}{\tau} \right)^{1/3} - \frac{c}{\tau},
$$

(1.2)

where $c$ is a positive constant and the normalisation of $T(\tau)$ at large $\tau$ is encoded in the constants $T_f$ and $\tau_f$. The physical basis of this fluid model is that for large nuclei moving in the $x$ direction one may take the transverse size to be infinite so that there is no dependence on the transverse coordinates. Also the incident energy is taken to be infinite, so that there is no preferred longitudinal frame and the flow is a similarity flow, $v = x/t$, and $\epsilon(\tau)$ does not depend on the rapidity coordinate $y$. Realistic physical conditions violate these assumptions in obvious ways; also, there is a hadronisation transition at some $T = T_c$.

When QCD matter is approximated by a conformal fluid, AdS$_5$/CFT$_4$ duality provides a tool for predicting the thermodynamical and hydrodynamical properties of the matter. Conceptually, the approach is very simple: find the relevant solution of the bulk Einstein theory with a negative cosmological constant, and compute from it the boundary energy-momentum tensor, the temperature $T(\tau)$ and the entropy density $s(\tau)$. What complicates the task is that the relevant bulk solution is known only in terms of asymptotic large $\tau$ expansions \cite{Kovtun:2004de, Son:2007vk, Son:2007yd, 2007PhRvD..76l6008S, 2007PhRvL..98t2302S, 2007PhRvD..76h6002S, 2007PhRvD..76r6009S, 2008PhRvD..77f6008S, 2009PhRvD..79i6004S, 2009PhRvD..79h6003S}.

One finds for $T(\tau)$ an expansion in $1/(\tau T) \sim 1/\tau^{2/3}$, generalising (1.2):

$$
T(\tau) = T_f \left( \frac{\tau_f}{\tau} \right)^{1/3} + \frac{\eta_0}{\tau} - \frac{\eta_0^2}{T_f} \frac{1}{\tau^{5/3}} + \frac{\eta_0^2 A}{T_f \tau_f^{1/3}} \frac{1}{\tau^{7/3}} + \frac{\eta_0^2 B}{(T_f \tau_f^{1/3})^3} \tau^3 + \cdots, \quad (1.3)
$$

where in the second term on the right-hand side we have

$$
\eta_0 = \frac{1}{6\pi}, \quad (1.4)
$$

following from the viscosity prediction $\eta/s = 1/(4\pi)$, and the third term on the right-hand side contains the corresponding quantities from conformal second order hydrodynamics \cite{1998PhRvL..80.2937C, 1999PhRvD..60b6011C, 2007PhRvD..76h6009S}. The unknown dimensionless constants $A$ and $B$ would correspond to third and...
fourth order hydrodynamics. For the energy density we then have, using units in which \( T_f \tau_f^{1/3} = \sqrt{2} / (3^{1/4} \pi) \) \[8, 15\],

\[
\epsilon(\tau) = \frac{3\pi^2}{8} N_c \left( \frac{(T_f \tau_f^{1/3})^4}{\tau^{4/3}} + \frac{4\eta_0(T_f \tau_f^{1/3})^3}{\tau^2} + \frac{2\eta_0^2(1 + \log 4)(T_f \tau_f^{1/3})^2}{\tau^{8/3}} + \right.
\]

\[
\left. + \frac{4\eta_0^3 T_f \tau_f^{1/3}(-2 + \log 8 + A)}{\tau^{10/3}} + \frac{\eta_0^4(-5 + 6 \log^2 2 + 12A + 4B)}{\tau^4} + \cdots \right). \quad (1.5)
\]

A question that these expansions do not directly address, however, is whether the holographic energy-momentum tensor provided by the AdS/CFT correspondence should be attributed in its entirety to excitations of the boundary matter. The holographic energy-momentum tensor could conceivably contain also a Casimir-type vacuum energy term, corresponding to a QFT ‘vacuum’ that is not the conventional Minkowski vacuum but instead a quantum state adapted to the expansion of the plasma.

Such nonzero vacuum expectation values are commonplace in curved spacetime quantum field theory, and they do occur also in flat spacetime, in particular for vacua that are adapted to special families of (possibly noninertial) observers \[18\]. An example that is relevant for us is the Rindler vacuum. This is a state defined in the quadrant \( x > |t| \) of Minkowski space and seen as a no-particle state by the family of the observers given by

\[
t = \xi \sinh \eta, \quad x = \xi \cosh \eta, \quad x^i = b^i, \quad i = 2, 3, \quad \xi > 0 \tag{1.6a}\]

where the constants \( b^i \in \mathbb{R} \) and \( \xi > 0 \) specify the observer’s trajectory and \( \eta \) equals \( 1/\xi \) times the observer’s proper time. Each trajectory follows an orbit of a boost in \((t, x)\) and has uniform linear acceleration of magnitude \( 1/\xi \) \[19\]. For a conformal scalar field, the energy-momentum in the Rindler vacuum has a nonvanishing expectation value \[20\]: using \((\eta, \xi, x^2, x^3)\) as the coordinates, the metric reads

\[
ds_{\text{Rindler}}^2 = -\xi^2 d\eta^2 + d\xi^2 + (dx^2)^2 + (dx^3)^2, \quad (1.7)
\]

and the energy-momentum tensor is given in these coordinates by

\[
T_{\mu}^{\nu} = \frac{1}{1440\pi^2 \xi^4} \text{ diag } (3, -1, -1, -1). \quad (1.8)
\]

Now, the Milne-like coordinates \((\tau, y, x^2, x^3)\) are defined in the quadrant \( t > |x| \) of Minkowski space, and they are adapted to the Bjorken flow in the sense that the velocity vector of the flow is \( \partial_\tau \). The metric in these coordinates reads

\[
ds_{\text{Milne}}^2 = -d\tau^2 + \tau^2 dy^2 + (dx^2)^2 + (dx^3)^2. \quad (1.9)
\]
The Rindler energy-momentum tensor \( (1.8) \) is singular at \( t = x \), but if it is analytically continued across this singularity into the quadrant \( t > |x| \) and expressed in the coordinates \((\tau, y, x^2, x^3)\), it becomes

\[
T_{\mu \nu} = \frac{1}{1440\pi^2\tau^4} \text{diag}(-1, 3, -1, -1).
\]  

(1.10)

This energy-momentum tensor has thus exactly the \( \tau \)-dependence of the last term displayed in \((1.5)\). Could the last term displayed in \((1.5)\) therefore be a vacuum energy contribution that should be subtracted before reading off from \((1.5)\) the energy density due to excitations of the fluid? Note that this term is independent of \( T_f \), and it is the only term in \((1.5)\) that survives in the limit \( T_f \to 0 \).

A case in which such a Casimir-type vacuum energy term is present, and indeed crucial for obtaining consistent scale-free thermodynamics, is the \((1+1)\)-dimensional Bjorken flow \([21]\). The Casimir term in the holographic energy-momentum tensor is identified from the limit of a vanishing bulk black hole and is given by

\[
\epsilon = p = -\frac{\mathcal{L}}{16\pi G_3 \tau^2},
\]

(1.11)

where \( \mathcal{L} \) is the length scale of the bulk cosmological constant. This Casimir term duly has the form of the energy-momentum tensor of a conformal scalar field in the appropriate conformal vacuum adapted to the expanding fluid flow \([21]\). There is also evidence that a similar Casimir term could be present in the spatially isotropic counterpart of the Bjorken flow in \( d \geq 3 \) dimensions, with \( \epsilon \) and \( p \) proportional to \( 1/\tau^d \) \([22]\).

In this paper we attempt to identify the prospective Casimir contribution to the boundary energy-momentum tensor by assuming that the corresponding bulk solution has more symmetry, by one more Killing vector, than what the symmetries of the boundary Bjorken flow require. We find the bulk solution explicitly, and we show that it is locally just an unusual foliation of the Schwarzschild-AdS\(_5\) “bubble of nothing” \([23, 24, 25]\). The holographic energy-momentum tensor turns out to have the form \((1.10)\), with an overall coefficient that is proportional to the mass parameter of the bulk solution. This holographic energy-momentum tensor is a limiting case of the family of boost-invariant energy-momentum tensors considered in \([3]\), and our bulk solution can thus be considered as completing part of the programme initiated in \([3]\). However, we show that the bulk solution has always a singularity, either curvature or conical, except when the solution reduces to pure AdS\(_5\) and the holographic energy-momentum tensor vanishes. Our bulk solution does therefore not provide compelling evidence for a nonvanishing Casimir energy-momentum tensor.

We begin by briefly reviewing in Section \(2\) the Bjorken flow ansatz on the boundary and the corresponding metric ansatz in the bulk. In Section \(3\) we specialise the bulk ansatz in a way that gives the bulk an additional Killing vector, solve the field equations and find the holographic energy-momentum tensor. The global properties of the solution are discussed in Section \(4\) and the possible interpretation of the holographic energy-momentum tensor in terms of quantum field theory is discussed in Section \(5\). Section \(6\) presents a summary and discusses the prospects of identifying a Casimir term under weaker assumptions.
2 Bjorken flow and its dual ansatz

Recall that in Milne-like coordinates \((\tau, y, x^2, x^3)\) Minkowski metric takes the form (1.9), and these coordinates are adapted to the Bjorken flow so that the velocity vector of the flow is \(\partial_\tau\). The boost-invariance and the transverse translational invariance of the flow imply that the hydrodynamic variables are independent of the rapidity \(y\) and the transverse spatial coordinates \((x^2, x^3)\). Assuming the matter to be a conformally invariant perfect fluid, whose energy-momentum tensor satisfies \(T^\mu_{\nu} = 0\) and \(\nabla_\mu T^{\mu\nu} = 0\), and working in the coordinates (1.9), it can be shown that

\[
T^\mu_{\nu} = \begin{pmatrix}
-\epsilon(\tau) & 0 & 0 & 0 \\
0 & -\epsilon(\tau) - \tau \epsilon'(\tau) & 0 & 0 \\
0 & 0 & \epsilon(\tau) + \frac{1}{2} \tau \epsilon'(\tau) & 0 \\
0 & 0 & 0 & \epsilon(\tau) + \frac{1}{2} \tau \epsilon'(\tau)
\end{pmatrix},
\]

where the only undetermined function is the energy density \(\epsilon(\tau) = T_{\tau\tau} = -T^{\tau\tau}\). If the energy-momentum tensor satisfies the weak energy condition, \(T_{\mu\nu} t^\mu t^\nu \geq 0\) for any timelike vector \(t^\mu\), \(\epsilon(\tau)\) must satisfy

\[
\epsilon(\tau) \geq 0, \quad -4\epsilon(\tau) \leq \tau \epsilon'(\tau) \leq 0.
\]

In the special case in which \(\epsilon(\tau)\) has the power-law behaviour \(\tau^{-p}\), the energy-momentum tensor takes the form

\[
T^\mu_{\nu} = \epsilon(\tau) \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & p - 1 & 0 & 0 \\
0 & 0 & 1 - \frac{1}{2}p & 0 \\
0 & 0 & 0 & 1 - \frac{1}{2}p
\end{pmatrix},
\]

and the weak energy condition (2.2) then implies \(0 \leq p \leq 4\).

By the symmetries of the Bjorken flow, one may expect its five-dimensional gravity dual to be of the form [3]

\[
ds^2 = \frac{L^2}{z^2} \left\{ -a(\tau, z)d\tau^2 + \tau^2 b(\tau, z)dy^2 + c(\tau, z) \left[ (dx^2)^2 + (dx^3)^2 \right] + dz^2 \right\},
\]

where the positive functions \(a(\tau, z), b(\tau, z)\) and \(c(\tau, z)\) are such that the metric satisfies five-dimensional Einstein’s equations with the cosmological constant \(-6/L^2\),

\[
R_{MN} = -\frac{4}{L^2} g_{MN},
\]

and \(a(\tau, z), b(\tau, z)\) and \(c(\tau, z)\) all tend to 1 as \(z \to 0\). Once the dual solution is found, its holographic energy-momentum tensor can be computed from the asymptotic small \(z\) expansion of the metric,

\[
ds^2 = \frac{L^2}{z^2} [g_{\mu\nu} dx^\mu dx^\nu + dz^2],
\]

\[
g_{\mu\nu}(\tau, z) = g^{(0)}_{\mu\nu}(\tau) + g^{(2)}_{\mu\nu}(\tau)z^2 + g^{(4)}_{\mu\nu}(\tau)z^4 + \ldots,
\]
where \( g^{(0)}_{\mu\nu} \) is the Milne metric \( \text{(1.9)} \): the result is \( \text{[26]} \)

\[
T_{\mu\nu} = \frac{L^3}{4\pi G_5} \left[ g^{(4)}_{\mu\nu} - \frac{1}{8} g^{(0)}_{\mu\nu} [ (\text{Tr} g_{(2)})^2 - \text{Tr} (g^2_{(2)}) ] - \frac{1}{2} ( g_{(2)} g^{-1}_{(0)})_{\mu\nu} + \frac{1}{4} ( \text{Tr} g_{(2)} ) g_{(2)\mu\nu} \right].
\]

(2.7)

3 Bulk ansatz with an additional isometry

We look for a bulk solution in which the functions \( a(\tau, z) \), \( b(\tau, z) \) and \( c(\tau, z) \) in the ansatz \( \text{(2.4)} \) depend on \( \tau \) and \( z \) solely through the combination \( s \equiv (z/\tau)^2 \). Geometrically, this means that in addition to the Killing vector \( \partial_y \) and the three Killing vectors that generate the \( E_2 \) isometries in \( (x^2, x^3) \), the metric admits also the Killing vector \( \tau \partial_\tau + z \partial_z + x^2 \partial_{x^2} + x^3 \partial_{x^3} \).

Writing \( a(s) = sh^2(s) \), where \( h > 0 \) and \( s > 0 \), Einstein’s equations yield for \( h \) the single ordinary differential equation

\[
h \left[ h^2 - 4s^2(h')^2 \right] = (h^2 - 1) \left( 4s^2h'' + 4sh' - h \right),
\]

(3.1)

where the prime denotes derivative with respect to \( s \). Once \( h \) is found, the functions \( b(s) \) and \( c(s) \) are given in terms of quadratures as

\[
\log \frac{b(s)}{b(0)} = \int_0^s \frac{(sh^2)' \left[ 3h^2 - 1 - (sh^2)' \right]}{s^2(h^2 - 1)(h^2)'},
\]

(3.2)

\[
\log \frac{c(s)}{c(0)} = \int_0^s \frac{(sh^2)'}{s(h^2 - 1)}. \tag{3.3}
\]

To solve \( \text{(3.1)} \), we write it in terms of \( \log s \) as the independent variable. This makes the equation autonomous, and it can integrated by regarding \( dh/d(\log s) \) as a function of \( h \). With the boundary condition \( a(s) = sh^2(s) \to 1 \) as \( s \to 0 \), we find that \( h(s) \) is determined implicitly by

\[
\frac{1}{\sqrt{s}} = h(s) \exp \left\{ \int_{h(s)}^\infty \frac{1}{h} \left[ \frac{1}{h} - \frac{\sqrt{h^2 - 1}}{\sqrt{h^2 - 1} - \mu} \right] \right\}, \tag{3.4}
\]

where \( \mu \) is a dimensionless constant of integration. We take \( \mu \) to be real-valued. Equations \( \text{(3.2)} \) and \( \text{(3.3)} \) and the definition of \( h \) then yield

\[
a = sh^2, \tag{3.5a}
\]

\[
b = s \left[ h^2(h^2 - 1) - \mu \right], \tag{3.5b}
\]

\[
c = s(h^2 - 1) \exp \left[ 2 \int_h^\infty \frac{dh}{\sqrt{h^2 - 1} \sqrt{h^2 - 1 - \mu}} \right], \tag{3.5c}
\]

where we have adopted the boundary conditions \( b \to 1 \) and \( c \to 1 \) as \( s \to 0 \) and the argument \( s \) is being suppressed. Note that the integrals in \( \text{(3.4)} \) and \( \text{(3.5c)} \) converge at infinity.
The functions \( h(s), a(s), b(s) \) and \( c(s) \) are well defined for sufficiently small \( s \), and the small \( s \) expansions of the metric coefficients read

\[
\begin{align*}
    a(s) &= 1 + \frac{1}{4} \mu s^2 + O(s^3), \\
    b(s) &= 1 - \frac{3}{4} \mu s^2 + O(s^3), \\
    c(s) &= 1 + \frac{1}{4} \mu s^2 + O(s^3).
\end{align*}
\]

From (2.6), (2.7) and (3.6) we then find that the holographic energy-momentum tensor in the coordinates of (1.9) is given by

\[
T_{\mu \nu} = \frac{L^3}{4 \pi G_5} \frac{\mu}{4 \tau^4} \text{diag} (1, -3, 1, 1).
\]

Note the similarity with (1.10).

### 4 Global properties of the bulk solution

To analyse the global properties of the bulk solution given by (3.4) and (3.5), we first replace the coordinates \((\tau, z)\) by \((\gamma, h)\), where \( h \) is as in (3.4) and

\[
\tau = L \exp \left[ \gamma + \int_0^\infty \frac{dh}{\sqrt{h^2 - 1} \sqrt{h^2 (h^2 - 1) - \mu}} \right].
\]

We then write \( h = \sqrt{1 + (\rho/L)^2} \), where \( \rho > 0 \). For given \( \mu \), these transformations are well defined for sufficiently small \( s \), and the corresponding regime in the coordinates \((\gamma, y, x^2, x^3, \rho)\) is that of sufficiently large \( \rho \). The metric takes the form

\[
ds^2 = \left( \frac{\rho^2}{L^2} + 1 - \frac{\mu L^2}{\rho^2} \right) L^2 dy^2 + \frac{d\rho^2}{\left( \frac{\rho^2}{L^2} + 1 - \frac{\mu L^2}{\rho^2} \right)} + \rho^2 \left\{ -d\gamma^2 + e^{-2\gamma} L^{-2} \left[ (dx^2)^2 + (dx^3)^2 \right] \right\}.
\]

The metric (4.2) is recognised as a double analytic continuation of Schwarzschild-AdS\(_5\) [27]. \( \rho \) is the usual Schwarzschild radial coordinate, \( L y \) is the Euclidean Schwarzschild time, and the part multiplied by \( \rho^2 \) is the metric on \((2 + 1)\)-dimensional de Sitter space written in the spatially flat coordinate patch \((\gamma, x^2, x^3)\),

\[
ds^2_{dS_3} = -d\gamma^2 + e^{-2\gamma} L^{-2} \left[ (dx^2)^2 + (dx^3)^2 \right],
\]

which is the analytic continuation of the round unit \( S^3 \) to Lorentzian signature. The parameter \( \mu \) is proportional to the Schwarzschild mass. The isometry group is seven-dimensional, consisting of translations in \( y \) and the six-dimensional isometry group of \((2 + 1)\) de Sitter space. Einstein’s equations have therefore resulted into two more Killing vectors than the four that we assumed in the metric ansatz.

The global properties of the solution depend on the sign of \( \mu \):
• When $\mu > 0$, the metric (4.2) is locally the Schwarzschild-AdS “bubble of nothing” [23, 24, 25]. The range of $\rho$ is $\rho_+ < \rho$, where

$$\rho_+ = L \sqrt{\sqrt{\mu + \frac{1}{4}} - \frac{1}{2}},$$

and there is a conical singularity at $\rho \to \rho_+$. If $y$ were periodic with period

$$\frac{2\pi(\rho_+/L)}{2(\rho_+/L)^2 + 1},$$

the conical singularity would be replaced by a bolt-type [28] fixed point set of the Killing vector $\partial_y$, and the metric (4.2) would then be the genuine “bubble of nothing”. Periodicity in $y$ does however not appear physically appropriate for modelling ion collisions on the boundary.

• When $\mu < 0$, the metric (4.2) has a scalar curvature singularity at $\rho \to 0$.

• When $\mu = 0$, the metric (4.2) is locally AdS$_5$, and $\rho \to 0_+$ is a coordinate singularity on a null hypersurface. Tracing back to the form of the metric in (2.4), this solution reads $a(\tau, z) = b(\tau, z) = c(\tau, z) = 1$, and the null hypersurface $\rho = 0$ is at $\tau = z$.

To end this section, we recall that the usual way of attaching to the metric (4.2) a conformal boundary is via hypersurfaces of constant $\rho$. Replacing $\rho$ by the coordinate $\zeta$ by

$$\rho^2 = \frac{L^2}{\zeta^2} - \frac{1}{2} + \frac{(\mu + \frac{1}{4})\zeta^2}{4L^2},$$

the metric takes the usual Fefferman-Graham form,

$$ds^2 = \frac{L^2}{\zeta^2} \left[ 1 - \frac{\zeta^2}{2L^2} + \frac{(\mu + \frac{1}{4})\zeta^4}{4L^4} \right] L^2 ds^2_{dS_3} + \left[ 1 - \frac{(\mu + \frac{1}{4})\zeta^4}{4L^4} \right] L^2 dy^2 + d\zeta^2,$$

and the boundary metric at $\zeta \to 0$ is

$$ds^2_{\text{bubble--b}} = L^2 (dy^2 + ds^2_{dS_3}).$$

In the coordinates of (4.8), the holographic energy-momentum tensor reads [25]

$$T_{\mu \nu} = \frac{L^3}{4\pi G_5} \frac{\mu + \frac{1}{4}}{4L^4} \text{ diag } (-3, 1, 1, 1).$$

As discussed in the context of the spatially isotropic Bjorken flow in [22], the transformation between (3.7) and (4.9) is compatible with the four-dimensional conformal anomaly and with the conformal transformation between the boundary metrics (1.9) and (4.3),

$$ds^2_{\text{Milne}} = (\tau/L)^2 ds^2_{\text{bubble--b}}$$

with $\tau/L = e^\gamma$. 

7
5 Casimir interpretation of the holographic energy-momentum tensor?

We wish to discuss whether the energy-momentum tensor \((3.7)\), for some nonvanishing value of \(\mu\), could be present as a Casimir part in the holographic energy-momentum tensor computed from the less symmetric bulk solution in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. As immediate consistency checks for such an interpretation, we note that 

\[ T_{\mu\nu}(3.7) \]

is traceless, and it is invariant under the longitudinal boosts and under the \(E_2\) isometries in \((x^2, x^3)\). Also, 

\[ T_{\mu\nu} \]

is singular at the collision event and on its light cone. Finally, 

\[ T_{\mu\nu} \]

has the form of the energy-momentum tensor \((1.10)\) that came by analytically continuing the Rindler vacuum energy-momentum tensor \((1.8)\) from the quadrant \(x > |t|\) to the Bjorken flow region, with the overall sign agreeing if \(\mu < 0\).

However, there is both a bulk argument and a boundary argument against such an interpretation.

The bulk argument is that the bulk solution has a singularity for every nonvanishing \(\mu\), as discussed in section 4. Even if such a bulk singularity is regarded as physically acceptable from the boundary viewpoint, the bulk geometry seems not to provide a criterion for fixing a distinguished nonzero value of \(\mu\).

As a preparation for the boundary argument, recall [21] that in the case of the \((1 + 1)\)-dimensional boundary, the Casimir term \((1.11)\) has the form of the energy-momentum tensor of a conformal scalar field in the conformal vacuum state, defined by the massless limit of the mode functions \((5.38)\) of [18]. If the mass in the mode functions \((5.38)\) of [18] is strictly positive, on the other hand, it can be verified that the mode sum expression for the vacuum polarisation \(\langle \phi^2 \rangle\) remains divergent at small spatial momentum even after the Minkowski vacuum contribution is subtracted mode by mode, using \((5.41)\) of [18]. This indicates that the state is not Hadamard and standard techniques do not furnish it with a well-defined energy-momentum tensor [29, 30]. Physically, this state is pathological at small spatial momentum since the mode functions in \((5.38)\) of [18] become in this limit their own complex conjugates.

Now, consider a conformal scalar field on the \((3 + 1)\)-dimensional boundary \((1.9)\). The symmetries suggest that the prospective vacuum state with the energy-momentum tensor \((3.7)\) should be defined in terms of mode functions whose dependence on \((\tau, y)\) is as in \((5.38)\) of [18], with the effective mass coming from the Fourier-momenta in \((x^2, x^3)\). However, the mode sum expression for \(\langle \phi^2 \rangle\) is again divergent even after mode-by-mode subtraction of the Minkowski vacuum contribution, indicating that the state is not Hadamard and does not have a well-defined energy-momentum tensor. This suggests that the energy-momentum tensor \((3.7)\) may not be the vacuum energy-momentum tensor of any state that is regular in the Hadamard sense of [29, 30], even though it is related to the Rindler vacuum energy-momentum tensor by analytic continuation across the Rindler horizon.

Note that both of these objections disappear if \(\mu > 0\) and \(y\) is periodic with the period \((4.5)\). In the bulk the periodicity removes the conical singularity. On the boundary periodicity of \(y\) makes the \(y\)-momentum discrete, and the divergence in the mode sum integral for \(\langle \phi^2 \rangle\) in the limit of small \(y\)-momentum is then no longer present. Periodicity in \(y\) does however not appear physically appropriate in the ion collision setting, as we mentioned after \((4.5)\).
6 Conclusions

We have asked whether the holographic energy-momentum tensor found in the AdS/CFT description of heavy ion collisions in the Bjorken flow approximation should be attributed in its entirety to the expanding matter, or whether part of it should be interpreted as a Casimir-like vacuum energy-momentum term. This question is prompted by the observation that in the corresponding $(1+1)$-dimensional Bjorken flow problem such a Casimir term is present, and this term is indeed crucial for consistency of the scale-free hydrodynamic approximation beyond the high density limit [21].

We postulated a bulk ansatz that assumes one more Killing vector than those of the boundary Bjorken flow. We found the corresponding bulk solution in terms of quadratures, we showed that the holographic energy-momentum has the anticipated form, and we showed that this bulk solution is locally just the Schwarzschild-AdS$_5$ “bubble of nothing” in an unusual foliation. However, this bulk solution has a singularity except when it reduces to pure AdS$_5$. Our bulk solution does therefore not provide compelling evidence for a nonvanishing Casimir part in the energy-momentum tensor of the approximate Bjorken flow bulk solutions analysed in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

We show in the Appendix that our bulk solution can be readily generalised so that the scale factor of the transverse boundary dimensions equals $(\tau/L)^p$ with $-\infty < p \leq 1$, with $p = 0$ being the Bjorken flow case. For $0 < p \leq 1$, the boundary metric is then an expanding cosmology, and one might attempt to interpret the holographic energy-momentum tensor as that of plasma in an expanding cosmology. A problem with such an interpretation is however that the $p \neq 0$ bulk solution has the same singularities as the $p = 0$ solution.

To summarise, our $(4+1)$-dimensional bulk ansatz did not lead to a viable Casimir term in the $(3+1)$-dimensional boundary energy-momentum tensor. While our additional bulk Killing vector is an obvious generalisation of the Killing vector that does lead to a viable Casimir term in the lower-dimensional setting of a $(2+1)$-dimensional bulk and a $(1+1)$-dimensional boundary [21], might one perhaps have fared better by postulating a different Killing vector in the $(4+1)$-dimensional bulk ansatz? We shall now argue from the symmetries of the AdS solution that this is unlikely.

Recall that in coordinates adapted to the $d$-dimensional boundary Bjorken flow, the pure AdS$_{d+1}$ bulk solution reads

\begin{align*}
\frac{L^2}{z^2} \left[ -dt^2 + \tau^2 dy^2 + dz^2 \right], & \quad (\text{for } d = 2) \quad (6.1a) \\
\frac{L^2}{z^2} \left[ -dt^2 + \tau^2 dy^2 + (dx^2)^2 + \cdots + (dx^{d-1})^2 + dz^2 \right], & \quad (\text{for } d \geq 3) \quad (6.1b)
\end{align*}

where $y$ is the longitudinal rapidity coordinate and the transverse coordinates $x^i$ are present only for $d \geq 3$. This is the solution one would a priori expect to be the bulk ground state. Now, the $d = 2$ solution (6.1a) has the Killing vector $\tau \partial_\tau + z \partial_z$, which is timelike near the infinity and commutes with the Bjorken flow Killing vector $\partial_y$. It is this Killing vector, via the temperature and entropy of its Killing horizon, that makes it possible to interpret the solution (6.1a) as giving the boundary Bjorken flow a nonzero temperature and entropy [21]. In the $d \geq 3$ solution (6.1b), by contrast, the only Killing vectors that commute with both
the Bjorken flow longitudinal Killing vector $\partial_y$ and the transversal Killing vectors $\partial_{x^i}$ can be verified to be linear combinations of these Killing vectors themselves. The Killing vector of our ansatz, $\tau \partial_\tau + z \partial_z + \sum_i x^i \partial_{x^i}$, does not commute with the symmetries of the Bjorken flow, nor can it be replaced with one that would. The Killing horizon argument of $d = 2$ does therefore not generalise into a thermodynamical Bjorken flow interpretation of the $d \geq 3$ AdS solution (6.11).

That being said, the $d = 4$ bulk solution that is known in terms of its late time expansion is known to have an event horizon [14]. The possibility of a Casimir contribution in the holographic energy-momentum tensor should perhaps be examined in a formalism that allows genuinely time-dependent notions of temperature and entropy [31].

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**A Appendix: Non-constant transverse dimensions**

In this appendix we show how the bulk solution found in the main text generalises to give a boundary metric in which the scale factor of the transversal dimensions equals $(\tau/L)^p$ with $-\infty < p \leq 1$.

We start by generalising the bulk metric (4.2) to

$$
\begin{align*}
 ds^2 &= \left( \frac{\rho^2}{L^2} + k - \frac{\mu L^2}{\rho^2} \right) L^2 dy^2 + \frac{d\rho^2}{\left( \frac{\rho^2}{L^2} + k - \frac{\mu L^2}{\rho^2} \right)}, \\
 &\quad + \rho^2 \left\{ -d\gamma^2 + e^{-2\sqrt{k} \gamma} L^{-2} \left[ (dx^2)^2 + (dx^3)^2 \right] \right\},
\end{align*}
$$

(A.1)

where $k$ is a non-negative constant. For $k > 0$, the metric (A.1) is obtained from (4.2) by first doing the coordinate transformation

$$(y, \rho, \gamma, x^2, x^3) = (\sqrt{k} \tilde{y}, \tilde{\rho}/\sqrt{k}, \sqrt{k} \tilde{\gamma}, \sqrt{k} \tilde{x}^2, \sqrt{k} \tilde{x}^3)$$

(A.2)

with $\mu = \tilde{\mu}/k^2$, and then dropping the tildes. Taking the limit $k \to 0$ in (A.1) does not bring in qualitative changes for our purposes: the $k = 0$ metric (A.1) is still locally AdS$_5$ when $\mu = 0$, it has a conical singularity when $\mu > 0$ and a scalar curvature singularity when $\mu < 0$. Note that when $\mu > 0$, the $k = 0$ metric is locally the AdS$_5$ soliton [32], and it would be globally the AdS$_5$ soliton if $y$ were periodic with period $\pi/\mu^{1/4}$.

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2We thank Don Marolf for raising this question.
We now generalise the coordinate transformation of Section 4 to (A.1). Working at sufficiently large $\rho$, we replace $\rho$ by $h = \sqrt{1 + (\rho/L)^2}$ and define $s$ and $\tau$ by

$$\frac{1}{\sqrt{s}} = h \exp \left\{ \int_{h}^{\infty} dh \left[ \frac{1}{h - \sqrt{h^2 - 1}} \right] \right\},$$

(A.3)

$$\tau = \mathcal{L} \exp \left[ \gamma + \int_{h}^{\infty} \frac{dh}{\sqrt{h^2 - 1}} \right].$$

(A.4)

We then define $z = \tau \sqrt{s}$, or $s = (z/\tau)^2$, and write the metric in the coordinates $(\tau, y, x^2, x^3, z)$, where $h$ is defined as a function of $s$ by (A.3). We find that the metric takes the form (2.4) with

$$a(s) = 1 + \frac{1}{2}(1 - k)s + \left[ \frac{1}{3\mu} + \frac{1}{16}(k - 1)^2 \right] s^2 + O(s^3),$$

(A.5a)

$$b(s) = 1 + \frac{1}{2}(k - 1)s + \left[ -\frac{3}{2\mu} + \frac{1}{16}(k - 1)^2 \right] s^2 + O(s^3),$$

(A.5b)

$$c(s) = \left( \frac{\tau}{\mathcal{L}} \right)^{2(1-\sqrt{k})} s(h^2 - 1) \exp \left[ 2\sqrt{k} \int_{h}^{\infty} \frac{dh}{\sqrt{h^2 - 1}} \right].$$

(A.5c)

where all three expressions are functions of the suppressed argument $s$, they are well defined for sufficiently small $s$, and their small $s$ expansions read

$$a(s) = 1 + \frac{1}{2}(1 - k)s + \left[ \frac{1}{3\mu} + \frac{1}{16}(k - 1)^2 \right] s^2 + O(s^3),$$

(A.6a)

$$b(s) = 1 + \frac{1}{2}(k - 1)s + \left[ -\frac{3}{2\mu} + \frac{1}{16}(k - 1)^2 \right] s^2 + O(s^3),$$

(A.6b)

$$c(s) = \left( \frac{\tau}{\mathcal{L}} \right)^{2(1-\sqrt{k})} \left\{ 1 - \frac{1}{2}(\sqrt{k} - 1)^2 s + \left[ \frac{1}{4\mu} + \frac{1}{16}(\sqrt{k} - 1)^4 \right] s^2 + O(s^3) \right\}. \tag{A.6c}$$

The conformal boundary metric is therefore

$$ds^2 = -d\tau^2 + \tau^2 dy^2 + \left( \frac{\tau}{\mathcal{L}} \right)^{2(1-\sqrt{k})} [ (dx^2)^2 + (dx^3)^2 ].$$

(A.7)

From (2.7), the holographic energy-momentum tensor reads

$$T_{\mu \nu} = \frac{\mathcal{L}^3}{4\pi G_5} \frac{1}{4\tau^4} \left\{ \mu \text{ diag } (1, -3, 1, 1) \right. \left. + \frac{1}{2} \text{ diag } \left( (\sqrt{k} - 1)^3(\sqrt{k} + 3), -(\sqrt{k} - 1)^3(3\sqrt{k} + 1), (k - 1)^2, (k - 1)^2 \right) \right\}, \tag{A.8}$$

reducing to (3.7) for $k = 1$.

In the boundary metric (A.7), the scale factor of the transverse dimensions equals $(\tau/\mathcal{L})^p$, where $p = 1 - \sqrt{k}$. Note that $-\infty < p \leq 1$. The Ricci scalar of the metric equals $R = 4p^2/\tau^2$:
for $p \neq 0$, the metric hence does not satisfy Einstein’s vacuum equations and has a scalar curvature singularity at $\tau \to 0$. For $0 < p \leq 1$, the metric can thus be understood as an expanding cosmology, with an initial singularity at $\tau = 0$, and the expansion is isotropic when $p = 1$. In the range $0 < p \leq 1$, or $0 \leq k < 1$, one might therefore attempt to interpret the energy-momentum tensor $\mathcal{A}$ as that of plasma in an expanding cosmology. However, as the bulk singularities are qualitatively similar for all $k$, the objections that were discussed in the main text for $k = 1$ are present also for other values of $k$.

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