ABJ Wilson loops and Seiberg duality

Hirano Shinji¹,*, Nii Keita², and Shigemori Masaki³,⁴

¹School of Physics and Center for Theoretical Physics, University of the Witwatersrand, WITS 2050, Johannesburg, South Africa
²Department of Physics, Nagoya University, Nagoya 464-8602, Japan
³Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan
⁴Hakubi Center, Kyoto University, Kyoto 606-8501, Japan
*E-mail: shinji.hirano@wits.ac.za

Received September 13, 2014; Accepted October 17, 2014; Published November 27, 2014

We study supersymmetric Wilson loops in the $\mathcal{N} = 6$ supersymmetric $U(N_1)\times U(N_2)\times U(k)$ Chern–Simons-matter (CSM) theory, the ABJ theory, at finite $N_1$, $N_2$, and $k$. This generalizes our previous study on the ABJ partition function. First computing the Wilson loops in the $U(N_1)\times U(N_2)$ lens space matrix model exactly, we perform an analytic continuation, $N_2$ to $-N_2$, to obtain the Wilson loops in the ABJ theory that is given in terms of a formal series and is only valid in perturbation theory. Via a Sommerfeld–Watson-type transform, we provide a non-perturbative completion that renders the formal series well defined at all couplings. This is given by $\min(N_1, N_2)$-dimensional integrals that generalize the “mirror description” of the partition function of the ABJM theory. Using our results, we find the maps between the Wilson loops in the original and Seiberg dual theories and prove the duality. In our approach we can explicitly see how the perturbative and nonperturbative contributions to the Wilson loops are exchanged under the duality. The duality maps are further supported by a heuristic yet very useful argument based on the brane configuration as well as an alternative derivation based on that of Kapustin and Willett (arXiv:1302.2164 [hep-th]).

Subject Index  B04, B06, B16, B21, B83

1. Introduction

Duality is one of the most fascinating phenomena in quantum field theory. It provides an alternative, often nonperturbative, understanding of the theory that is not accessible in the original description. Seiberg duality [1], under which weakly coupled gauge theory is mapped into the strongly coupled one with different rank, and vice versa, is a prominent example. Although the mapping of local operators under Seiberg duality is known to be rather simple [1], the transformation properties of non-local operators, such as Wilson loops, are more nontrivial and less studied.

By its strong–weak nature, any checks of Seiberg duality must involve a nonperturbative approach. The localization method [2–4] is a powerful technique applicable to supersymmetric field theory and reduces the infinite-dimensional path integral of quantum field theory to a finite integral, which can often be regarded as a matrix model integral. This method allows for exact computation of quantities such as the partition function and Wilson loops at strong coupling, and is an ideal tool for studying nonperturbative physics such as Seiberg duality.

Among various supersymmetric gauge theories calculable by the localization method, we focus on the $3D,\mathcal{N} = 6$ supersymmetric $U(N_1)\times U(N_2)\times U(k)$ Chern–Simons-matter (CSM) theory, known as the ABJ theory [5], where $k \in \mathbb{Z}_{\neq 0}$ is the Chern–Simons level. In the special case with $N_1 = N_2$,
this theory is called the ABJM theory [6]. It is believed that the $U(N)_k \times U(N + M)_{-k}$ ABJ theory gives the low-energy description of the system of $N$ M2-branes and $M$ fractional M2-branes probing $\mathbb{C}^4/\mathbb{Z}_k$, and its holographic dual geometry is $\text{AdS}_4 \times S^7/\mathbb{Z}_k$ in M-theory or $\text{AdS}_4 \times \mathbb{CP}^3$ in type IIA superstring [5,6]. It is also conjectured that this theory has another dual description in terms of the $\mathcal{N} = 6$ supersymmetric, parity-violating version of the Vasiliev higher spin theory in four dimensions [7].

Applying the localization method to the ABJ(M) theory on $S^3$ yields [8] the so-called ABJ(M) matrix model, which has been extensively studied recently, leading to much insight into the non-perturbative effects in the theory. For instance, applying the standard large $N$ technique to the ABJ matrix model reproduced the $N^{3/2}$ scaling of the free energy expected from the gravity dual [4,9,10]. For the ABJM theory, the $1/N$ corrections were summed to all orders using the holomorphic anomaly equation at large 't Hooft coupling $\lambda = N/k$ in the type IIA regime $k \gg 1$ and an Airy function behavior was found [11]. Furthermore, the powerful Fermi gas approach was developed and reproduced the Airy function in the M-theory regime (large $N_c$, finite $k$) [12]. Based on the Fermi gas approach, instanton corrections were examined [13,14] and a cancellation mechanism between worldsheet and membrane (D2) instantons was discovered. More recently, the Fermi gas approach was generalized to the ABJ theory in Ref. [15] and further studied in Ref. [16].

It was conjectured in Ref. [5] that the ABJ theory has Seiberg duality, which states that the following two theories are equivalent:

$$U(N_1)_k \times U(N_2)_{-k} = U(2N_1 - N_2 + k)_k \times U(N_1)_{-k},$$

(1.1)

where we assumed $N_1 \leq N_2$. This duality can be understood in the brane realization of the ABJ theory [5] as moving 5-branes past each other and creating/annihilating D3-branes between them by the Hanany–Witten effect [23]. This is a special case of the Givon–Kutasov duality [24] for more general $\mathcal{N} = 2$ CSM theories, which can be regarded as the 3D analog of the 4D Seiberg duality [38]. The equality of the partition function (up to a phase) between the dual theories in (1.1) was proven in Ref. [39], although one mathematical relation was assumed. On the other hand, in the “mirror” framework of Ref. [17], the equality of the dual partition function is more or less obvious [17,22] and, moreover, it was observed that the perturbative and nonperturbative contributions get exchanged into each other under the duality.

Wilson loops are the only observables in pure CS theory [40] and, even in CSM theory, they are very natural objects to consider. It is known that there are two basic circular supersymmetric Wilson loops for the ABJ(M) theory on $S^3$; the Wilson loop carrying a nontrivial representation with respect to one
of the two gauge groups preserves $\frac{1}{6}$ of the supersymmetry \[41–43\] while an appropriate combination of $\frac{1}{6}$-BPS Wilson loops for two gauge groups preserves $\frac{1}{2}$ of the supersymmetry \[44\]. The bulk dual of the $\frac{1}{2}$-BPS Wilson loop is a fundamental string while the bulk dual of the $\frac{1}{6}$-BPS Wilson loop is not completely understood \[41–43\]. By the localization method, these Wilson loops can be computed using the ABJ(M) matrix model \[8\], and the techniques developed for the partition function to study nonperturbative effects can be generalized to Wilson loops, such as the large $N$ analysis \[9,45,46\], the Fermi gas approach \[47\], and the cancellation between worldsheet and membrane instantons \[48,49\].

In this paper we study the $\frac{1}{6}$ and $\frac{1}{2}$-BPS Wilson loops and their Seiberg duality in the ABJ theory, based on the approach developed in Ref. \[17\] for the partition function. We mainly focus on the representations with winding number $n$. Starting with Wilson loops in the lens space matrix model and analytically continuing it, we obtain a new expression for the supersymmetric Wilson loops in the ABJ theory in terms of $\min(N_1, N_2)$-dimensional contour integrals. As an application and as a way to check the consistency of our formula, we study Seiberg duality on the ABJ Wilson loops. The duality rule for Wilson loops is highly nontrivial because they are non-local operators and not charged under the global symmetry. A technique for finding Giveon–Kutasov duality relations for Wilson loops in general $\mathcal{N} = 2$ CSM theories, of which the ABJ theory is a special case, was proposed in Ref. \[50\] based on quantum algebraic relations. We derive the duality rule of the supersymmetric Wilson loops using our integral expression and confirm that it is consistent with the proposal of Ref. \[50\]. We also discuss how the perturbative and nonperturbative effects are mapped under Seiberg duality. Moreover, we provide a heuristic explanation of the duality rule from the brane realization of the ABJ theory. In an appropriate duality frame, the Wilson loop is interpreted as the position of the branes and, carefully following how the Hanany–Witten effect acts on it, we reproduce the correct duality rule.

This paper is organized as follows. In Sect. 2, we present the main results of the paper without proof: the integral expression for the $\frac{1}{6}$-BPS and $\frac{1}{2}$-BPS Wilson loops and their transformation rules under Seiberg duality. In Sect. 3, we explain how to derive the integral expression for the ABJ Wilson loops by analytically continuing the lens space ones. In Sect. 4, we first give a heuristic, brane picture for the Seiberg duality rule, and then present a rigorous proof using the integral expression. Section 5 is devoted to a summary and discussion. The appendices contain further details of the computations in the main text, as well as a discussion of Wilson loops with general representation in Appendix C and an alternative derivation of the Seiberg duality rule using the algebraic approach of Ref. \[50\] in Appendix F.

### 2. Main results

The Wilson loops we are concerned with are those preserving fractions of supersymmetries in the $\mathcal{N} = 6U(N_1)_k \times U(N_2)_{-k}$ CSM theory, also known as the ABJ(M) theory, saturating the Bogomol’nyi–Prasad–Sommerfield (BPS) bound. More specifically, we consider two types of circular BPS Wilson loops on $S^3$, one preserving $\frac{1}{6}$ and the other preserving half of the supersymmetries \[41–44\]. In the main text of this paper, we restrict ourselves to the Wilson loops with winding number $n$, where $n = 1$ (or $n = -1$) corresponds to the fundamental (or anti-fundamental) representation of $U(N_1)$ and/or $U(N_2)$ gauge groups, and the Wilson loops in more general representations will be discussed in Appendix C.

---

\[2\] See Sect. 5 for further discussion.
The ABJ(M) theory consists of two 3D $\mathcal{N} = 2$ vector multiplets $(A^A_{\mu}, \sigma^A, \lambda^A, \tilde{\lambda}^A, D^A)$ with $A = 1, 2$, which are the dimensional reduction of 4D $\mathcal{N} = 1$ vector multiplets and four bifundamental chiral multiplets, an $SU(4)_R$ vector, $(C_I, \psi_I, F_I)$ with $I = 1, \ldots, 4$ in the representation $(N_1, \tilde{N}_2)$, and their conjugates. The $\frac{1}{6}$-BPS Wilson loops of our interest are constructed as [41–43,51]

$$\mathcal{W}^A_{\frac{1}{6}}(N_1, N_2; R)_{k} := \left\{ \text{Tr}_R P \exp \left( i \int A^{1}_{\mu} \dot{x}^{\mu} + \frac{2\pi}{k} |\dot{x}| M^I_{J} C_I \bar{C}^J \right) ds \right\},$$

(2.1)

where the path of the loop specified by the vector $x^\mu(s)$ is a circle, and the matrix $M^I_{J}$ is determined by the supersymmetries preserved; one can choose it to be $M = \text{diag}(-1, -1, 1, 1)$. These are the Wilson loops on the first gauge group $U(N_1)$, but the $\mathcal{W}^A_{\frac{1}{6}}(N_1, N_2; R)_{k}$, those on the second gauge group $U(N_2)$, can be constructed similarly by replacing $N_1 \rightarrow N_2$, $A^1_{\mu} \rightarrow A^2_{\mu}$, and $C_I \rightarrow \bar{C}^J$ ($\bar{C}^J \rightarrow C_J$). The $\frac{1}{6}$-BPS Wilson loop is constructed in Ref. [44] and can be conveniently expressed in terms of the supergroup $U(N_1|N_2)$ as

$$\mathcal{W}^A_{\frac{1}{6}}(N_1, N_2; R)_{k} := \left\{ \text{Tr}_R P \exp \left( i \int A ds \right) \right\},$$

(2.2)

where $R$ is a super-representation of $U(N_1|N_2)$ and $A$ is the super-connection

$$A = \left( \begin{array}{cc} A^{1}_{\mu} \dot{x}^{\mu} - i \frac{2\pi}{k} |\dot{x}| N^I_{J} C_I \bar{C}^J & \sqrt{\frac{2\pi}{k}} |\dot{x}| \psi_I \bar{\eta}_I \\ \sqrt{\frac{2\pi}{k}} |\dot{x}| \eta_I \bar{\psi}^J & A^{2}_{\mu} \dot{x}^{\mu} - i \frac{2\pi}{k} |\dot{x}| N^I_{J} C_I \bar{C}^J \end{array} \right)$$

(2.3)

with the matrix $N^I_{J} = \text{diag}(-1, 1, 1, 1)$, the (super-)circular path $(x^1, x^2) = (\cos s, \sin s)$, and $\eta_I = (e_{2i/2}, -i e_{-2i/2})\delta^I_j$.

As mentioned, we are concerned with Wilson loops with winding number $n$ rather than in generic representations in this paper. Focusing on this class of Wilson loops, the application of the localization technique [3] for the $\frac{1}{6}$-BPS Wilson loops with winding $n$ reduces to the finite-dimensional integrals of the matrix model type [8]

$$\mathcal{W}^A_{\frac{1}{6}}(N_1, N_2; n)_{k} := \left\{ \sum_{j=1}^{N_1} e^{n\mu_j} \right\}$$

(2.4)

where the vacuum expectation value (vev) is with respect to the eigenvalue integrals

$$\langle \mathcal{O} \rangle := \mathcal{N}_{\text{ABJ}} \int \prod_{i=1}^{N_1} \frac{d\mu_i}{2\pi} \prod_{a=1}^{N_2} \frac{d\nu_a}{2\pi} \frac{\Delta_{\text{sh}}(\mu)^2 \Delta_{\text{sh}}(\nu)^2}{\Delta_{\text{ch}}(\mu, \nu)^2} \mathcal{O} e^{-\frac{1}{2k^2} \left( \sum_{i=1}^{N_1} \mu_i^2 - \sum_{a=1}^{N_2} \nu_a^2 \right)}$$

(2.5)

with the shorthand notations for the one-loop determinant factors defined by

$$\Delta_{\text{sh}}(\mu) = \prod_{1 \leq i < j \leq N_1} \left( 2 \sinh \left( \frac{\mu_i - \mu_j}{2} \right) \right), \quad \Delta_{\text{sh}}(\nu) = \prod_{1 \leq a < b \leq N_2} \left( 2 \sinh \left( \frac{\nu_a - \nu_b}{2} \right) \right)$$

(2.6)

and

$$\Delta_{\text{ch}}(\mu, \nu) = \prod_{i=1}^{N_1} \prod_{a=1}^{N_2} \left( 2 \cosh \left( \frac{\mu_i - \nu_a}{2} \right) \right).$$

(2.7)

The coupling constant\(^3\) $g_s$ is related to the CS level $k$ by

$$g_s := \frac{2\pi i}{k},$$

(2.8)

---

\(^3\) Note that this is not the physical string coupling constant. The physical string coupling constant of the dual type IIA string theory in $\text{AdS}_4 \times \mathbb{CP}^3$ is $(g_s)_{\text{physical}} \sim N^{1/4} k^{-5/4}$. 

---

4/62
while the factor $\mathcal{N}_{\text{ABJ}}$ in front is the normalization factor [4]

$$
\mathcal{N}_{\text{ABJ}} := \frac{i^{-\frac{k}{2}(N_1^2 - N_2^2)}}{N_1! N_2!}, \quad \kappa := \text{sign } k.
$$

(2.9)

Meanwhile, the $\frac{1}{2}$-BPS Wilson loop localizes to the supertrace [44]

$$
\mathcal{W}_1^0(N_1, N_2; \mathcal{R})_k = \left( \text{str}_{\mathcal{R}} \left( \begin{array}{cc} e^{\mu_i} & 0 \\ 0 & -e^{\nu_j} \end{array} \right) \right)
$$

(2.10)

that yields, for the $n$ winding Wilson loop, a linear combination of $\frac{1}{2}$-BPS Wilson loops [47]:

$$
\mathcal{W}_1^0(N_1, N_2; n)_k = \mathcal{W}_1^0(N_1, N_2; n)_k - (-1)^{nq_0} \mathcal{W}_1^0(N_1, N_2; n)_k.
$$

(2.11)

Note that the integral (2.5) is a well defined Fresnel integral even for non-integral (but real) $k$ and thus gives a continuous function $k$, although in the physical ABJ theory the CS level $k$ is quantized to an integer.

We now present the results of our analysis of these matrix eigenvalue integrals.

### 2.1. The Wilson loops in ABJ theory

The $\frac{1}{2}$-BPS Wilson loops are only on the first $U(N_1)$ or the second $U(N_2)$ gauge group. Depending on whether $N_1 \leq N_2$ or $N_1 \geq N_2$, their formula takes rather different forms.

- **The $\frac{1}{6}$-BPS $U(N_1)$ Wilson loop with $N_1 \leq N_2$:** In the case of $N_1 \leq N_2$, introducing the normalized Wilson loop $W^1_6(N_1, N_2; n)_k$, we find the $\frac{1}{6}$-BPS Wilson loop with winding number $n$ on the first gauge group $U(N_1)$ to be

$$
W^1_6(N_1, N_2; n)_k := \frac{\mathcal{W}_1^0(N_1, N_2; n)_k}{\mathcal{W}_1^0(N_1, N_2; 0)_k} = q^{-\frac{n^2}{2} + n} I(N_1, N_2; n)_k
$$

(2.12)

where

$$
q := e^{-g_s} = e^{-\frac{2\pi i}{\kappa}}
$$

(2.13)

and

$$
I(N_1, N_2; n)_k := \frac{1}{N_1!} \sum_{l=1}^{N_1} \prod_{i=1}^{N_1} \left[ \frac{-1}{2\pi i} \int_C \frac{\pi ds_i}{\sin(\pi s_i)} \right] q^{-ns_i + n(l-2)} \prod_{i=1}^{N_1} (q^{s_i-n}_1)^1 \prod_{i \neq l} (q^{s_i-n}_1)
$$

\times \prod_{i=1}^{N_1} \left[ \frac{(-1)^{M} (q^{s_i+1}_1)}{1 + q^{n\delta_l}_1} (-q^{s_i+1+n\delta_l}_1) \right] \prod_{j=1}^{n-1} (q^{s_j-s_j}_1) \prod_{j=i+1}^{N_1} (q^{s_j-s_j+n\delta_l}_1)
$$

(2.14)

with $M := |N_2 - N_1| = N_2 - N_1$ and the symbol $(a)_z := (a; q)_z$ is a shorthand notation for the $q$-Pochhammer symbol defined in Appendix A. The choice of the integration contour $C$ will be discussed in detail in Sect. 3.3. We note that the integral expression $I(N_1, N_2; 0)_k$ without winding agrees with that of the partition function in Ref. [17].

---

4 The normalization is essentially by the partition function.

5 The precise relation to the quantity $\Psi$ defined in Ref. [17] is $I(N_1, N_2; 0)_k = N_1 \Psi(N_1, N_2)_k$. 

5/62
• The \( \frac{1}{6} \)-BPS \( U(N_1) \) Wilson loop with \( N_1 \geq N_2 \): In the case of \( N_1 \geq N_2 \), the formula turns out to be slightly more involved and takes the form

\[
W^I_{\frac{1}{6}}(N_1, N_2; n)_k := \frac{W^I_{\frac{1}{6}}(N_1, N_2; n)_k}{W^I_{\frac{1}{6}}(N_2, N_1; 0)_k} = q^{-\frac{n^2}{2} + n} \frac{I^{(1)}(N_1, N_2; n)_k + I^{(2)}(N_1, N_2; n)_k}{I^{(3)}(N_1, N_2; 0)_k},
\]

which is only valid for \(|n| \geq 1\), as will be elaborated in the comments below\(^6\) and we defined

\[
I^{(1)}(N_1, N_2; n)_k := \frac{1}{N_2!} \sum_{c=0}^{n-1} N_2 \prod_{a=1}^{N_2} \left[ -\frac{1}{2\pi i} \int_{C_1[a]} \frac{\pi ds_a}{\sin(\pi s_a)} \right]
\times q^{n(2c-M)} \frac{(q^{1-n}) \cdot (q^{1+n})^{M-1-c}}{(q_1 q)_{M-1-c}} \prod_{a=1}^{N_2} \left( \frac{q^{s_a+1}}{2(-q^{s_a+1})^M} \right)
\times \prod_{a=1}^{N_2} \left( \frac{q^{-s_a+c+1}}{(q^{-s_a+c+1})^1} \cdot \frac{q^{-s_a+c-n}}{(q^{-s_a+c-n})^1} \right) \prod_{b=1}^{N_2} \left( \frac{q^{-s_a-s_b}}{(q^{-s_a-s_b})^1} \right) \prod_{b=a+1}^{N_2} \left( \frac{q^{s_b-s_a}}{(q^{s_b-s_a})^1} \right),
\]

and

\[
I^{(2)}(N_1, N_2; n)_k := \frac{1}{N_2!} \sum_{d=1}^{N_2} \prod_{a=1}^{N_2} \left[ -\frac{1}{2\pi i} \int_{C_2} \frac{\pi ds_a}{\sin(\pi s_a)} \right] q^{-n s_d + n(d-M-2)} \prod_{a=1}^{N_2} \left( \frac{q^{s_a-s_d-n}}{(q^{s_a-s_d})^1} \right)
\times \prod_{a=1}^{N_2} \left( \frac{q^{s_a+1+n\delta_{a}}}{{(1+q^n\delta_{a})}^M} \right) \prod_{b=1}^{N_2} \left( \frac{q^{s_a-s_b+n\delta_{a}}}{(q^{s_a-s_b+n\delta_{a}})^1} \right) \prod_{b=a+1}^{N_2} \left( \frac{q^{s_b-s_a+n\delta_{a}}}{(q^{s_b-s_a+n\delta_{a}})^1} \right),
\]

with \( M := |N_2 - N_1| = N_1 - N_2 \). Note that \( I^{(2)}(N_1, N_2; 0)_k = I(N_2, N_1; 0)_k \) and the normalization in (2.15) differs from that in (2.12) in that the ranks of the gauge groups \( N_1 \) and \( N_2 \) are exchanged. The choice of the integration contours \( C_1 = \{ C_1[a] \} (a = 0, \ldots, n - 1) \) and \( C_2 \) will be discussed in detail in Sect. 3.3. We would, however, like to make a remark concerning the contour \( C_1[a] \): There are subtleties in evaluating the integrals with the contour \( C_1[a] \). In order to properly deal with them, we shall adopt \( \epsilon \)-prescription, shifting the parameter \( M \to M + \epsilon \) with \( \epsilon > 0 \), and the contour is placed between \( s_a = -1 - c \) and \( -1 - c - \epsilon \). Related comments will be made in Sect. 3.4 below (3.37) and Appendix D.

• The \( \frac{1}{6} \)-BPS \( U(N_2) \) Wilson loops: It follows from the definition and symmetry that the \( \frac{1}{6} \)-BPS Wilson loop on the second gauge group \( U(N_2) \) with winding number \( n \) is related to that on the first gauge group \( U(N_1) \) in a simple manner:

\[
W^{II}_{\frac{1}{6}}(N_1, N_2; n)_k = W^{I}_{\frac{1}{6}}(N_2, N_1; n)_{-k}.
\]

• The \( \frac{1}{2} \)-BPS Wilson loop: Meanwhile, the (normalized) \( \frac{1}{2} \)-BPS Wilson loop is given by a linear combination of two (normalized) \( \frac{1}{6} \)-BPS Wilson loops, one on the first and the other on the

\[6\] The expression (2.12), on the other hand, does not have this restriction.
second gauge group, and turns out to take a rather simple form:

\[
W_2(N_1, N_2; n)_k := W^{I}_6(N_1, N_2; n)_k - (-1)^n W^{II}_{6}(N_1, N_2; n)_k
\]

\[
= \begin{cases} 
(-1)^{n+1} q^{\frac{1}{2} n^2 + n} \frac{I^{(1)}(N_2, N_1; n)_{-k}}{I^{(2)}(N_2, N_1; 0)_{-k}} & \text{for } N_1 \leq N_2 \\
q^{-\frac{3}{2} n^2 + n} \frac{I^{(1)}(N_1, N_2; n)_{k}}{I^{(2)}(N_1, N_2; 0)_{k}} & \text{for } N_1 \geq N_2,
\end{cases}
\]

where the two terms \( q^{-\frac{1}{2} n^2 + n} I^{(2)}(N_1, N_2; n)_k \) and \((-1)^n q^{\frac{1}{2} n^2 - n} I(N_2, N_1; n)_{-k}\) cancel out, as we will show later in Sect. 3.5.

- **Comments on the zero winding limit \( n \to 0\):** There are a few subtleties to be addressed in the expressions \( I^{(1)} \) in (2.16) and \( I^{(2)} \) in (2.17). They are linked to the comments made below (3.10) and (3.11) concerning the ranges of the summations and the discussions in Appendix D. Here we focus on the subtlety in the range of the sum \( \sum_{c=0}^{n-1} \) in (2.16). In its original form, the sum over \( c \) is taken from 0 to \( M - 1 \), as derived in (B37) in Appendix B2. However, when \( n \geq 1 \), we can rewrite the sum (B37) + (B38), after passing it to the integral representation discussed in Sect. 3.3, by the sum (2.16) + (2.17). In particular, the upper limit \( M - 1 \) of the sum over \( c \) can be replaced by \( n - 1 \). For \( n < M \) this relies on the fact that the factor \( (q^{1-n})_c = 0 \) when \( n \geq 1 \) and \( c \geq n \). Since the factor \( (q^{1-n})_c \) does not vanish when \( n < 1 \), this implies that the \( n = 0 \) limit of (2.16) that violates the bound \( n \geq 1 \) would not coincide with \( (M \text{ times}) \) the integral representation for the partition function in Ref. [17]. In fact, (2.16) in the \( n = 0 \) limit simply vanishes, whereas (2.17) reduces to \( (N_2 \text{ times}) \) the integral representation for the partition function.

- **Comments on the bound on winding number \( n \):** We observe that the integrals (2.14) and (2.17) diverge when \( |n| \geq \frac{k}{2} \). As we will elaborate later, the contour \( C \) for the integrals goes to imaginary infinity. For large imaginary \( s_l \) and \( s_c \), the integrands become asymptotically \( \exp(2 \pi n i s_l/k) / \sin(\pi s_l) \) and \( \exp(2 \pi n i s_c/k) / \sin(\pi s_c) \), respectively, which grow exponentially when \( |n| > \frac{k}{2} \) and approach a constant when \( |n| = \frac{k}{2} \). This implies that the \( \frac{1}{6} \)-BPS Wilson loops are only well defined for \( |n| < \frac{k}{2} \). In the \( \frac{1}{2} \)-BPS case, it might appear that the Wilson loop is well defined for all values of \( n \), since the integral (2.16) converges for any \( n \). Note, however, that since (2.16) is periodic under the shift \( n \to n + k \) owing to the properties \( q^k = 1 \) and \( (q^{1-n})_{c\geq n} = 0 \), the winding \( n \) is restricted to the range \( |n| \leq \frac{k}{2} \). Moreover, as indicated in (3.38), a closer inspection shows that the \( \frac{1}{2} \)-BPS Wilson loop, in the ABJM limit, diverges at the winding \( n = \frac{k}{2} \) due to the factor \( (1 + q^n)^{-1} \) that comes from the pole at \( s_a = -1 - c \). This implies that, in the case of \( k = 1, 2 \), the only allowed winding is \( n = 0 \) and thus the \( \frac{1}{2} \)-BPS Wilson loop does not exist.\(^8\)

- **Comments on the ABJM limit:** In the ABJM limit the ranks of two gauge groups are equal, i.e., \( N_1 = N_2 \iff M = 0 \), and the two results (2.12) and (2.15) coincide as they should. However, the way they coincide turns out to be very subtle. Naively, it may look that \( I^{(1)}(N_1, N_2; n)_k \)

\(^7\) We thank Nadav Drukker for discussions on this point. This is also in accord with the singularities at \( k = 1, 2 \) observed in Ref. [47]. See further comments on the \( \frac{1}{7} \)-BPS case.

\(^8\) We are indebted to Yasuyuki Hatsuda for discussions on this point, correcting our statements in the previous version of this paper.
vanishes and \( I(N_1, N_2; n)_k \) and \( I^{(2)}(N_1, N_2; n)_k \) coincide in the ABJM limit. However, a careful analysis of the ABJM limit reveals that \( \lim_{N_1 \to N_2} I^{(1)}(N_1, N_2; n)_k \) remains finite, but the sum \( I^{(1)}(N_1, N_2; n)_k + I^{(2)}(N_1, N_2; n)_k \) coincides with \( I(N_1, N_2; n)_k \) in the limit. As will be elaborated in Sect. 3.4, this is rooted in the difference of the integration contours \( C \) in (2.12) and \( C_1, C_2 \) in (2.15). Since \( C \neq C_2 \), the integral \( I^{(2)}(N_1, N_2; n)_k \) differs from \( I(N_1, N_2; n)_k \) even in the ABJM limit, but this difference is canceled by \( I^{(1)}(N_1, N_2; n)_k \). Note that it is very important that \( \lim_{N_1 \to N_2} I^{(1)}(N_1, N_2; n)_k \neq 0 \), since the \( \frac{1}{2} \)-BPS Wilson loop (2.19) exists as nonvanishing in the ABJM limit.

2.2. Seiberg duality of the Wilson loops

As we discussed in the introduction, the ABJ theory is conjectured to possess Seiberg duality as given in (1.1). In our previous paper [17], we have explicitly checked that the partition functions of a dual pair are equal to each other up to a phase factor. The precise form of the phase factor was first conjectured by Ref. [39] and later derived in Ref. [30,52]. The difference of the phase factors in dual pairs is now understood as an anomaly in large gauge transformations [53]. We would like to emphasize that not only does our formula for the partition function in Ref. [17] confirm the proof in Refs. [39,52] but it also allows us to understand how Seiberg duality of the ABJ theory works in detail. In particular, it was observed explicitly in Ref. [17] that the perturbative and nonperturbative contributions to the partition function are exchanged under the duality. In Sect. 4 we will see the same property in the duality of Wilson loops.

- The duality map of \( \frac{1}{6} \)-BPS Wilson loops: Using our formulas, we find the maps between Wilson loops in the original and Seiberg dual theories:

\[
W^{\Pi}_{\frac{1}{6}}(N_1, N_2; n)_k = -W^{\Pi}_{\frac{1}{6}}(\tilde{N}_2, N_1; n)_k - 2(-1)^{n+1} W^{I}_{\frac{1}{6}}(\tilde{N}_2, N_1; n)_k, \tag{2.20}
\]

\[
W^{I}_{\frac{1}{6}}(N_1, N_2; n)_k = W^{\Pi}_{\frac{1}{6}}(\tilde{N}_2, N_1; n)_k \tag{2.21}
\]

for the \( \frac{1}{6} \)-BPS Wilson loops, and the rank of the dual gauge group is denoted by \( \tilde{N}_2 = 2N_1 - N_2 + k \).

- The duality map of \( \frac{1}{2} \)-BPS Wilson loops: For the \( \frac{1}{2} \)-BPS Wilson loops, the duality map turns out to be very simple:

\[
W^{I}_{\frac{1}{2}}(N_1, N_2; n)_k = (-1)^n W^{I}_{\frac{1}{2}}(\tilde{N}_2, N_1; n)_k. \tag{2.22}
\]

Note that these three relations are not independent, but one of them can be derived from the other two. In the following sections, we will refer to (2.20), (2.21), and (2.22) as the \( \frac{1}{6} \)-BPS Wilson loop duality, the flavor Wilson loop duality, and the \( \frac{1}{2} \)-BPS Wilson loop duality, respectively.\(^9\)

We will vindicate these maps by a heuristic yet very useful argument based on the brane configuration in Sect. 4.1 and an alternative proof that is an application of the proof by Kapustin and Willett [50] in Appendix F.

---

\(^9\) Under the duality, the \( U(N_1) \) sector is inert and can be regarded as a "flavor group." The Wilson loops in (2.21) are only on the \( U(N_1) \) "flavor group," hence the name, the flavor Wilson loop duality.
3. The derivation of the results

We follow the same strategy as that employed in the computation of the ABJ partition function [17]. The outline of the derivation is as follows:

(1) We first compute Wilson loops in the $U(N_1) \times U(N_2)$ lens space matrix model, where the Wilson loops are defined in analogy to those in the ABJ theory. The matrix integrals are simply Gaussian and can be done exactly.

(2) We then analytically continue the rank of one of the gauge groups from $N_2$ to $-N_2$. This maps the lens space matrix model to the ABJ matrix model [4,9]. As in the case of the partition function, the result so obtained is expressed in terms of a formal series that is not well defined in the regime of the strong coupling that we are concerned with and is only sensible as perturbative expansion with the generalized $\zeta$-function (polylogarithm) regularization assumed.

(3) Similar to the case of the partition function, we can render the formal series perfectly well defined by means of the Sommerfeld–Watson transform; the resultant expression is given in terms of $\min(N_1, N_2)$-dimensional integrals. This is a nonperturbative completion in the sense that what renders the formal series well defined is an inclusion of nonperturbative contributions, as will be elaborated later.

3.1. The lens space Wilson loop

We define the (unnormalized) Wilson loop on the first gauge group $U(N_1)$ with $n$ windings in the lens space matrix model by

\[ W^l_{\text{lens}}(N_1, N_2; n) := \langle \sum_{j=1}^{N_1} e^{in\mu_j} \rangle, \tag{3.1} \]

where we have defined the expectation value of $\mathcal{O}$ by

\[ \langle \mathcal{O} \rangle := N_{\text{lens}} \int \prod_{i=1}^{N_1} \frac{d\mu_i}{2\pi} \prod_{a=1}^{N_2} \frac{dV_a}{2\pi} \Delta_{\text{sh}}(\mu)^2 \Delta_{\text{sh}}(v)^2 \Delta_{\text{ch}}(\mu, v)^2 e^{-\frac{1}{2\pi i} \left( \sum_{i=1}^{N_1} \mu_i^2 + \sum_{a=1}^{N_2} v_a^2 \right)} \mathcal{O}, \tag{3.2} \]

where the normalization constant $N_{\text{lens}}$ is given by

\[ N_{\text{lens}} = \frac{i^{-\frac{\gamma}{2}}(N_1^2 + N_2^2)}{N_1!N_2!}. \tag{3.3} \]

As will be shown in detail in Appendix B1, the eigenvalue integrals are simply Gaussian and can be carried out exactly. Introducing the normalized Wilson loop, the end result takes the form

\[ W^l_{\text{lens}}(N_1, N_2; n)_k := \frac{W^l_{\text{lens}}(N_1, N_2; n)_k}{W^l_{\text{lens}}(N_1, N_2; 0)_k} = q^{-\frac{N(k^2 + 1)}{2} + n} S(N_1, N_2; n)_k, \tag{3.4} \]

where the function $S(N_1, N_2; n)$ is given by

\[ S(N_1, N_2; n)_k = \frac{1}{N_1!} \sum_{l=1}^{N_1} \sum_{1 \leq C_1, \ldots, C_{N_1}} \frac{q^{-2nC_l + n(N+l-1)}}{q^{C_l - C_{l+1}}} \prod_{k \neq l} \left( \frac{q^{C_l - C_{l+1}}}{q^{C_k - C_l}} \right)_1 \]

\[ \times \prod_{i=1}^{N_1} \left[ \left( q^{C_i - C_{i+1}} \right)_1 \left( q^{C_{i+1} - C_i} \right)_1 \prod_{j=1}^{l-1} \left( q^{C_{i+1} - C_j} \right)_1 \prod_{j=l+1}^{N_1} \left( q^{C_{i+1} - C_j} \right)_1 \right], \tag{3.5} \]

\[ \left( q^{C_{i+1} - C_j} \right)_1 := \sum_{k=0}^{\infty} \frac{(q^{C_{i+1} - C_j})^k}{k!}. \]
where $N = N_1 + N_2$. We note that this is actually a finite sum: There is no contribution from $C_i > N$, since the factor $1/(q)_{N-C_i} = (q^{1-(C_i-N)})_{C_i-N}$ for $N - C_i < 0$ vanishes. We have deliberately rewritten the result in the form of an infinite sum that is suitable for the analytic continuation in the next section.

### 3.2. The analytic continuation

The $\frac{1}{6}$-BPS Wilson loop in ABJ theory can be obtained from the lens space Wilson loop (3.4) by means of the analytic continuation $N_2 \to -N_2$. A little care is needed for the analytic continuation. Namely, the analytic continuation requires a regularization: We first replace $N_2$ by $-N_2 + \epsilon$ and send $\epsilon \to 0$ in the end. As mentioned in Sect. 2.1, we need to treat the cases $N_1 \leq N_2$ and $N_1 \geq N_2$ separately. We leave most of the computational details to Appendix B2.

- **The $\frac{1}{6}$-BPS $U(N_1)$ Wilson loop with $N_1 \leq N_2$:** The function $S(N_1, N_2; n)_k$ in (3.5) is continued as

$$S(N_1, -N_2 + \epsilon, n)_k = (\epsilon \ln q)^{N_1} S^{ABJ}(N_1, N_2, n)_k + O(\epsilon^{N_1+1}), \quad (3.6)$$

where we have defined

$$S^{ABJ}(N_1, N_2, n)_k$$

$$:= \frac{1}{N_1!} \sum_{l=1}^{N_1} \sum_{1 \leq C_1, \ldots, C_{N_1}} q^{-nC_l+n(l-1)} \prod_{k, l=1}^{N_1} \left( \frac{q^{C_l-C_l-n}}{q^{C_l-C_l}} \right)$$

$$\times \prod_{i=1}^{N_1} \left[ \frac{(-1)^{C_i+M} (q^{C_i})_M}{(1+q^{n\delta_l}) (-q^{C_i+n\delta_l})_M} \prod_{j=1}^{i-1} \left( \frac{q^{C_j-C_j-n\delta_l}}{-q^{C_j-C_j-n\delta_l}} \right) \prod_{j=i+1}^{N_1} \left( \frac{q^{C_j-C_j-n\delta_l}}{-q^{C_j-C_j-n\delta_l}} \right) \right] \quad (3.7)$$

with $M := |N_2 - N_1| = N_2 - N_1$. Note that, in contrast to the lens space case, there is no truncation of summations and this sum is really an infinite sum. In fact, this is not a convergent sum and becomes ill defined for the value of $q = e^{-2\pi i/k}$ that is of our actual interest. Thus this expression is at best a formal series and we shall render it well defined on the entire $q$-plane by means of a type of Sommerfeld–Watson transform in the next section.

Hence the analytic continuation yields an expression in terms of formal series

$$\left[ W_{ABJ}^I(N_1, N_2; n)_k \right]_{\text{naive}} := \lim_{\epsilon \to 0} W_{\text{lens}}^I(N_1, -N_2 + \epsilon; n)_k = q^{-\frac{n}{2} + n} \frac{S^{ABJ}(N_1, N_2; n)_k}{S^{ABJ}(N_1, N_2; 0)_k} \quad (3.8)$$

for $N_1 \leq N_2$. After implementing the generalized $\zeta$-function (polylogarithm) regularization, this result agrees with the final result (2.12) only in perturbative expansion in the coupling constant $g_s(\approx -\log q)$. Until the Sommerfeld–Watson-like transform is performed, this result is not nonperturbatively complete.

- **The $\frac{1}{6}$-BPS $U(N_1)$ Wilson loop with $N_1 \geq N_2$:** This case is slightly more involved than the previous case. The analytic continuation of $S(N_1, N_2; n)_k$ consists of two terms

$$S(N_1, -N_2 + \epsilon; n)_k = (\epsilon \ln q)^{N_1} \left( S^{ABJ}_{(1)}(N_1, N_2; n)_k + S^{ABJ}_{(2)}(N_1, N_2; n)_k \right) + O(\epsilon^{N_2+1}), \quad (3.9)$$
where we have defined

\[
S^{ABJ}_{(1)}(N_1, N_2; n)_k := \frac{1}{N_2!} \sum_{c=0}^{n-1} \sum_{c \leq D_1, \ldots, D_{N_2}} q^{n(2c-M)} \frac{(q^{1-n})_c}{(q)_c(q^{1+n})_{M-1-c}} \prod_{a=1}^{N_2} \frac{(q^{D_a})_M}{2(-q^{D_a})_M} \left( \prod_{a=1}^{N_2} \frac{(q^{D_a+c+n})_1}{(q^{D_a+c+n})_1} \prod_{b=1}^{a-1} \frac{(q^{D_a-D_b})_1}{(-q^{D_a-D_b})_1} \prod_{b=a+1}^{N_2} \frac{(q^{D_b-D_c})_1}{(-q^{D_b-D_a})_1} \right)
\]

(3.10)

\[
S^{ABJ}_{(2)}(N_1, N_2; n)_k := \frac{1}{N_2!} \sum_{d=1}^{N_2} \sum_{-n+1 \leq D_d \leq d-1, D_{d-1}, \ldots, D_{N_2}} q^{-nD_d+n(d-M-1)} \prod_{a=1}^{N_2} \frac{(q^{D_a-D_d-n})_1}{(q^{D_a-D_d})_1} \left( \prod_{a=1}^{N_2} \frac{(q^{D_a+n\delta_{ad}})_M}{(1+q^{n\delta_{ad}})(-q^{D_a})_M} \prod_{b=1}^{a-1} \frac{(q^{D_a-D_b})_1}{(-q^{D_a-D_b+n\delta_{ad}})_1} \prod_{b=a+1}^{N_2} \frac{(q^{D_b-D_c})_1}{(-q^{D_b-D_d+n\delta_{ad}})_1} \right)
\]

(3.11)

with \( M := |N_2 - N_1| = N_1 - N_2 \); we replaced the sum \( \sum_{\epsilon=0}^{M-1} \) by \( \sum_{\epsilon=0}^{n-1} \) provided that \( n > 0 \) in the first line of (3.10). See the remark below (2.17) for the explanation. The caveat on the convergence and well definedness of the sum noted in the previous case applies to this case as well. Note that when \( n = 0 \), (3.10) vanishes by definition.

An important remark is in order: In the first line of (3.10), we replaced the sum \( \sum_{\epsilon=0}^{M-1} \) by \( \sum_{\epsilon=0}^{n-1} \) provided that \( n > 0 \) and extended the range of \( D_a \) from \( \sum_{1 \leq D_1, \ldots, D_{N_2}} \) to \( \sum_{c \leq D_1, \ldots, D_{N_2}} \). In sync with this replacement and extension, we extended the range of \( D_d \) from \( \sum_{1 \leq D_d} \) to \( \sum_{-n+1 \leq D_d} \) in the first line of (3.11). As shown in Appendix D, the added contributions conspire to cancel out in the sum \( S^{ABJ}_{(1)} + S^{ABJ}_{(2)} \), justifying the replacement and extensions we have made in (3.10) and (3.11).

Similar to the previous case, the analytic continuation yields an expression in terms of formal series

\[
W^{I}_{ABJ}(N_1, N_2; n)_k \,_{\text{naive}} := \lim_{\epsilon \to 0} W^{I}_{\text{lens}}(N_1, -N_2 + \epsilon; n)_k = q^{-\frac{n^2}{2} + n} \frac{S^{ABJ}_{(1)}(N_1, N_2; n)_k + S^{ABJ}_{(2)}(N_1, N_2; n)_k}{S^{ABJ}_{(2)}(N_1, N_2; 0)_k}
\]

(3.12)

for \( N_1 \geq N_2 \). Note that \( S^{ABJ}_{(1)}(N_1, N_2; 0) = 0 \) as inferred from (3.10) and the denominator \( S^{ABJ}_{(2)}(N_1, N_2; 0) = \bar{S}^{ABJ}(N_2, N_1; 0) \). Again this result is not complete as yet but agrees, after the generalized \( \zeta \)-function regularization, with the final result (2.15) in perturbative expansion in the coupling constant \( g_s (= - \log q) \).

- The 1/8-BPS \( U(N_2) \) Wilson loops: As stated in the summary of the main results, it follows from the definition and symmetry that the Wilson loop on the second gauge group \( U(N_2) \) is obtained from that on the first gauge group \( U(N_1) \) as

\[
W^{II}_{ABJ}(N_1, N_2; n)_k \,_{\text{naive}} = \left[ W^{I}_{ABJ}(N_2, N_1; n) \right] \,_{\text{naive}} (1 - n, N_2; n)_k
\]

(3.13)
• The $\frac{1}{2}$-BPS Wilson loop: The $\frac{1}{2}$-BPS Wilson loop is given by a linear combination of two $\frac{1}{5}$-BPS Wilson loops, one on the first and the other on the second gauge group:

$$W_{\text{ABJ}}^I(N_1, N_2; n)_{k} = W_{\text{ABJ}}^I(N_1, N_2; n)_k - (-1)^n W_{\text{ABJ}}^{II}(N_1, N_2; n)_k = q^{-\frac{n}{2} + n} S_{\text{ABJ}}^{(1)}(N_1, N_2; n)_k S_{\text{ABJ}}^{(2)}(N_1, N_2; 0)_k$$

(3.14)

for $N_1 \geq N_2$, where we used the fact $q^{-\frac{n}{2} + n} S_{\text{ABJ}}^{(1)}(N_1, N_2; n)_k = (-1)^n q^{-\frac{n}{2} - n} S_{\text{ABJ}}^{(2)}(N_2, N_1; n)_{-k}$, which we will show in Sect. 3.5.

In the next subsection we discuss a nonperturbative completion of the above naive results that were given in terms of the formal series $S_{\text{ABJ}}^{(1)}$, $S_{\text{ABJ}}^{(2)}$, and $S_{\text{ABJ}}^{(2)}$.

### 3.3. The integral representation—a nonperturbative completion

The analytic continuation in the previous section yielded tentative results for the ABJ Wilson loops that involve formal series (3.7), (3.10), and (3.11). These are non-convergent formal series, since the summands do not vanish as $g_s$ run to infinity. If we are only interested in perturbative expansion in the coupling $g_s = -\log q$, implementing the generalized $\zeta$-function regularization

$$\sum_{n=0}^{\infty} (-1)^n s^n = \begin{cases} \text{Li}_n(-1) = (2^{n+1} - 1)\zeta(-n) = -\frac{2^{n+1} - 1}{n+1} B_{n+1} & \text{(for } n \geq 1), \\ 1 + \text{Li}_0(-1) = \frac{1}{2} = -B_1 & \text{(for } n = 0), \end{cases}$$

(3.15)

where $\text{Li}_n(z)$ is the polylogarithm and $B_n$ are the Bernoulli numbers, the formal series can be rendered convergent as in the case of the partition function [17]. The $\zeta$-function regularized Wilson loops so obtained indeed reproduce the correct perturbative expansions in $g_s$. However, when $q = e^{-\frac{g_s}{k}}$ is a root of unity with $g_s = 2\pi i / k$, which is the value we are actually interested in and is beyond the perturbative regime, the sums (3.7), (3.10), and (3.11) diverge and require a nonperturbative completion.

Fortunately, as we have done for the partition function [17], these problems can be circumvented by introducing an integral representation similar to the Sommerfeld–Watson transform:\textsuperscript{10}

$$\sum_{1 \leq C_1, \ldots, C_{N_1}} \prod_{i=1}^{N_1} (-1)^{C_i+1}(\ldots) \longrightarrow \prod_{i=1}^{N_1} \left[ -\frac{1}{2\pi i} \int_C \frac{\pi ds_i}{\sin(\pi s_i)} \right](\ldots),$$

(3.16)

$$\sum_{1 \leq D_1, \ldots, D_{N_2}} \prod_{a=1}^{N_2} (-1)^{D_a+1}(\ldots) \longrightarrow \prod_{a=1}^{N_2} \left[ -\frac{1}{2\pi i} \int_C \frac{\pi ds_a}{\sin(\pi s_a)} \right](\ldots),$$

(3.17)

where $C_i$ is replaced by $s_i + 1$ and $D_a$ by $s_a + 1$. As we will see shortly, this is a transformation that adds nonperturbative contributions missed in the formal series. Note, however, that this is a prescription that lacks a first-principles derivation and needs to be justified. In the case of the partition function, this prescription has passed both perturbative and nonperturbative checks. In particular, the latter has confirmed the equivalence of Seiberg dual pairs up to the aforementioned phase factors [17].

\textsuperscript{10} We thank Yoichi Kazama and Tamiaki Yoneya for pointing out the similarity of this prescription to the Sommerfeld–Watson transformation.
Fig. 1. The integration contour $C$ for the partition function: (A) The left figure corresponds to the large $k$ limit where only perturbative (P) poles indicated by red “+” are present. (B) The right figure is an example of the finite $k$ case ($k = 3$ and $M = 2$) where nonperturbative (NP) poles indicated by blue “×” are also present. The green dotted line corresponds to $s_j = s_i - \frac{k}{2} \mod k$ with some $s_i$. Note that some of the P poles and zeros of the integrands coalesce and cancel out for integer $k$.

More recently, a direct proof of this prescription for the partition function was given by Honda by utilizing a generalization of Cauchy identity [22]. In the Wilson loop case, although we are missing a similar derivation, the proof of Seiberg duality in Sect. 4 provides convincing evidence for this prescription.

The contour $C$ of integration in (3.16) and (3.17) is chosen in order that (1) the perturbative expansion is correctly reproduced for small $g_s$ (corresponding to large $k$) and, (2) as we decrease $k$ continuously, the values of integrals remain continuous as functions of $k$. These requirements yield the contour $C$ parallel to and left of the imaginary axis. To elaborate on this, we look into the pole structure of integrands:

- **The pole structure for $n = 0$:** It is illustrative to first review the pole structure for the partition function, i.e., the $n = 0$ case. In this case, without loss of generality, we can assume $N_1 \leq N_2$.

  It is very useful to classify the poles into two classes, perturbative (P) and nonperturbative (NP) poles. We are interested in the summand in (3.7) with $n = 0$. By multiplying the factor $\prod_{i=1}^{N_1} \left[ 1 - 1/(2i \sin(\pi s_i)) \right]$, this becomes the integrand with the replacement $C_i \rightarrow s_i + 1$. The P poles come from the factor $\prod_{j=1}^{N_1} (q^{s_j+1})_M / \sin(\pi s_j)$, whereas the NP poles are from the factors $\prod_{j=1}^{N_1} 1/(-q^{s_j+1})_M$ and $\prod_{j \neq i} 1/(-q^{s_j-s_i})_M$ for generic (real non-integral) values of $k$.

  Hence the P poles are at

$$s_j = \ldots, -M - 2, -M - 1; \quad 0, 1, 2, \ldots$$

(3.18)

with a gap between $s_j = -M - 1$ and 0, as shown in the left part of Fig. 1. In (3.18), we organized poles into groups separated by a semicolon to clarify this gap structure. In the large $k$ limit, these are the only poles. The contour $C$ parallel to the imaginary axis can be placed anywhere in the gap. Indeed, enclosing the contour with an infinite semicircle to the right in the complex $s_j$ plane, the residue integral reduces to the sum (3.7) with the generalized $\zeta$-function regularization (3.15) implemented automatically by the integral formula

$$-\frac{1}{2\pi i} \int_C \frac{\pi ds}{\sin(\pi s)} s^n = -\frac{2n+1}{n+1} B_{n+1} \quad (n \geq 0),$$

(3.19)

and the perturbative expansion in $g_s$ is correctly reproduced. It should now be clear why this class of poles are called P poles.
The NP poles are at
\[ s_j = \frac{k}{2} - M, \frac{k}{2} - (M - 1), \ldots, \frac{k}{2} - 1 \mod k \]  
(3.20)
\[ s_j = s_i + \frac{k}{2} \mod k. \]  
(3.21)

This class of poles are called NP poles because \( k \propto 1/g_s \) and thus the residues are of order \( \exp(-1/g_s) \). Shown in the right panel of Fig. 1 is the case \( k = 3 \) and \( M = 2 \). Note that for integer \( k \) as opposed to generic (non-integral) values of \( k \), there are extra cancellations of the zeros and poles in the factor \( \prod_{j=1}^{N_1} [(q^{sj+1})_M / \sin(\pi s_j)] \), since the zeros
\[ s_j = k - M, k - (M - 1), \ldots, k - 1 \mod k \]  
(3.22)
can coincide with some of the P poles in (3.18). More precisely, the gap between \( s_j = -M - 1 \) and 0 repeats itself periodically modulo \( k \) and thus the P poles for an integer \( k \) appear at
\[ s_j = 0, 1, \ldots, k - M - 1 \mod k. \]  
(3.23)

Now more important is the fact that, for a given \( M < k \) as we decrease \( k \) continuously, the NP poles on the positive real axis move to the left. For a sufficiently large \( k > 2M \), the NP pole closest to the origin is at \( s_j = \frac{k}{2} - M > 0 \). As \( k \) is decreased from \( k > 2M \) to \( k < 2M \), this pole crosses the imaginary axis to the left. As we decrease \( k \) further, more NP poles cross the imaginary axis. For the partition function to be continuous in \( k \), these poles should not cross the contour \( C \) and therefore we need to shift the contour \( C \) to the left so as to avoid the crossing of these NP poles that invade into the real negative region. More precisely, the contour \( C \) has to be placed between \( s_j = -\frac{k}{2} - 1 (> -M - 1) \) and \( \frac{k}{2} - M \) when \( (M \leq k < 2M) \). This is a prescription that needs to be justified. In Ref. [17] it was checked that Seiberg duality holds with this contour prescription, vindicating our integral representation as a nonperturbative completion.

**The pole structure for \( n > 0 \) and \( N_1 \leq N_2 \):** It is straightforward to generalize the contour prescription to the case of Wilson loops. In this case, however, we need to discuss the two cases \( N_1 \leq N_2 \) and \( N_1 > N_2 \) separately. We start with the former, which is simpler than the latter. We are interested in the summand in (3.7). Again by multiplying the factor \( \prod_{j=1}^{N_1} [-1/(2i \sin(\pi s_j))] \), this becomes the integrand with the replacement \( C_i \rightarrow s_i + 1 \). For the P poles the relevant factor is the same as in the partition function, \( \prod_{j=1}^{N_1} [(q^{sj+1})_M / \sin(\pi s_j)] \). This yields the P poles for generic (non-integral) \( k \) at
\[ s_j = \ldots, -M - 2, -M - 1; 0, 1, 2, \ldots. \]  
(3.24)
A similar remark on the integer \( k \) case applies to this case, and the gap between \( s_j = -M - 1 \) and 0 repeats itself periodically modulo \( k \). This implies that the P poles for an integer \( k \) appear at
\[ s_j = 0, 1, \ldots, k - M - 1 \mod k. \]  
(3.25)

For the NP poles the relevant factors are \( \prod_{j=1}^{N_1} 1/(-q^{sj+1+n\delta sj})_M \) and \( \prod_{j \neq i} 1/(-q^{sj-s_i+n\delta sj})_1 \) where \( l \) runs from 1 to \( N_1 \). The NP poles are thus at
\[ s_j = \frac{k}{2} - M - n\delta sj, \frac{k}{2} - (M - 1) - n\delta sj, \ldots, \frac{k}{2} - 1 - n\delta sj \mod k \]  
(3.26)
\[ s_j = s_i + \frac{k}{2} - n\delta sj \mod k. \]  
(3.27)
The pole structure for $\bullet$  

The pole structure for $n > 0$ and $N_1 \geq N_2$: In this case we are interested in the summands (3.10) and (3.11). Again by multiplying the factor $\prod_{\alpha=1}^{N_2}[-1/(2i \sin(\pi s_{\alpha}))]$, these summands become the integrands with the replacement $D_{\alpha} \rightarrow s_{\alpha} + 1$. We first discuss the pole structure of (3.10). This time the relevant factor for the P poles is different from the previous cases, $\prod_{\alpha=1}^{N_2}[(q^{s_{\alpha}+1})_M/\sin(\pi s_{\alpha}) \times (q^{s_{\alpha}+1+c-n})_1/(q^{s_{\alpha}+1+c})_1]$ where $c$ runs from 0 to $n - 1$. This yields the P poles for generic $k$ at

$$s_{\alpha} = \ldots, -M - 2, -M - 1; \quad -1 - c; \quad 0, \ldots, -2 - c + n; \quad -c + n, \ldots$$ (3.28)

Note that there is an additional pole at $s_{\alpha} = -1 - c$ in the gap between $s_{\alpha} = -M - 1$ and 0 and a hole at $s_{\alpha} = -1 - c + n$. In the case of integer $k$, the gap, the additional point $s_{\alpha} = -1 - c$, and the hole $s_{\alpha} = -1 - c + n$ repeat themselves modulo $k$ and the P poles appear at

$$s_{\alpha} = -1 - c; \quad 0, 1, \ldots, -2 - c + n; \quad -c + n, \ldots, k - M - 1 \mod k.$$ (3.29)

To be more precise, if $c$ is small enough and the hole at $s_{\alpha} = -1 - c + n$ falls into a gap, the hole is absent.
Fig. 3. The integration contour \( C_1[c] \) for Wilson loops with \( N_1 \geq N_2 \): The figure is for the integrals (2.16) associated with the sum (3.10). The green dotted line corresponds to \( s_a = s_b + \frac{k}{2} \mod k \) with some \( s_b \). Shown is the case \( k = 3, M = 2, n = 1, \) and \( c = 0 \).

For the NP poles the relevant factors are \( \prod_{a=1}^{N_2} 1/(-q^{s_a+1})_M \times (-q^{s_a+1+c})_1/(-q^{s_a+1+c-n})_1 \) and \( \prod_{a \neq b} 1/(-q^{s_a-s_b})_1 \). The NP poles are thus at

\[
s_a = \frac{k}{2} - M, \ldots, \frac{k}{2} - 2 - c; \quad \frac{k}{2} - c, \ldots, \frac{k}{2} - 1; \quad \frac{k}{2} + (n - 1) - c \mod k
\]

(3.30)

\[
s_a = s_b + \frac{k}{2} \mod k.
\]

(3.31)

The choice of integration contour \( C_1 \) is more involved than the previous cases: (1) In addition to the P poles at \( s_a = 0, 1, \ldots \) for a large \( k \) (i.e., in the perturbative regime), we need to include, for a given \( c \), the P pole at \( s_a = -1 - c \). This means that we place the integration contour for a given \( c \) to the left of the P pole at \( s_a = -1 - c \). (2) As in the previous cases, we require continuity with respect to \( k \). The NP pole at \( s_a = \frac{k}{2} - M \) invades into the negative real axis, as \( k \) is decreased, whereas the NP pole at \( s_a = -\frac{k}{2} + (n - 1) - c \) moves to the right. Hence the contour has to be placed to the left of \( s_a = \min(-1 - c, \frac{k}{2} - M) \) and the right of \( s_a = \max(-M - 1, -\frac{k}{2} + (n - 1) - c) \) for \( c < M \) and to the left of \( s_a = -1 - c \) and the right of \( s_a = -\frac{k}{2} + (n - 1) - c \) for \( c \geq M \).

The pole structure and the contour \( C_1 \) are illustrated in Fig. 3. Note that \( C_1 \) depends on \( c \) and may thus be denoted as \( C_1 = \{C_1[c]\} \) \((c = 0, \ldots, n - 1)\).

Next we turn to the pole structure of (3.11). For the P pole the relevant factor is \( \prod_{a=1}^{N_2} [(q^{s_a+1+n\delta_{ad}})_M / \sin(\pi s_a)] \) where \( d \) runs from 1 to \( N_2 \). This yields for generic \( k \) the P poles at

\[
s_a = \ldots, -M - 2 - n\delta_{ad}, -M - 1 - n\delta_{ad}; \quad -n\delta_{ad}, 1 - n\delta_{ad}, \ldots.
\]

(3.32)

Note that the P poles are shifted by \(-n\delta_{ad}\) as compared to those in the \( N_1 < N_2 \) case. For an integer \( k \) the gap between \( s_a = -M - 1 - n\delta_{ad} \) and \(-n\delta_{ad}\) repeats itself periodically modulo \( k \).

---

\[\text{Note:} \quad (\text{3.31})\]

\[\text{Note:} \quad (\text{3.32})\]

---

\[\text{Note:} \quad (\text{3.33})\]
Fig. 4. The integration contour $C_2$ for Wilson loops with $N_1 \geq N_2$: (a) The left figure is the pole structure and the contour for $a \neq d$. (b) The right figure is for $a = d$ and the contour is shifted by $-n$ as compared to that in (a). The green dotted line corresponds to $s_a = s_b - \frac{k}{2} - n \delta_{ad} \mod k$ with some $s_b$. The contours are the same. Shown is the case $k = 3$, $M = 2$, and $n = 1$.

and the P poles appear at

$$s_a = -n \delta_{ad}, 1 - n \delta_{ad}, \ldots, k - M - 1 - n \delta_{ad} \mod k.$$  \hfill (3.33)

It could happen that, if the winding $n$ is sufficiently large, $k - M - 1 - n$ becomes negative.

For the NP poles the relevant factors are $\prod_{a=1}^{N_2} 1/(q^{s_a + 1})_M$ and $\prod_{a \neq b} 1/(q^{s_a - s_b + n \delta_{ad}})_1$. The NP poles are thus at

$$s_a = \frac{k}{2} - M, \frac{k}{2} - (M - 1), \ldots, \frac{k}{2} - 1 \mod k$$  \hfill (3.34)

$$s_a = s_b + \frac{k}{2} - n \delta_{ad} \mod k.$$  \hfill (3.35)

The choice of integration contour $C_2$ is similar to the $N_1 \leq N_2$ case except that the contour for the variable $s_d$ is to the left of $s_d = \min(-n, \frac{k}{2} - M)$ and the right of $s_d = \max(-M - 1, -\frac{k}{2} - 1)$ and picks up, in particular, the residues from the P poles at $s_d = -1, \ldots, -n$. The pole structure and the contour $C_2$ are illustrated in Fig. 4.

Again this is a prescription that lacks a first-principles derivation. In the case of $N_1 \leq N_2$, similar to the partition function, for a large $k$, enclosing the contour with an infinite semicircle to the right in the complex $s_j$ plane, the residue integral reduces to the sum (3.7), with the generalized $\zeta$-function regularization (3.15) implemented automatically, and the perturbative expansion in $g_s$ is correctly reproduced. In the case of $N_1 \geq N_2$, however, the way this prescription works is more subtle even for a large $k$. Each of the two integral expressions (2.16) and (2.17) picks up extra perturbative contributions from the poles at $s = -1, -2, \ldots, -n$. For $\frac{1}{6}$-BPS Wilson loops these extra contributions cancel out in the sum of (2.16) and (2.17), as shown in Appendix D, thereby reproducing the correct perturbative expansion in $g_s$. As the nonperturbative test, we show in the next section that Seiberg duality holds with our prescription, where it becomes clear that the inclusion of the P poles at $s = -1, -2, \ldots, -n$ is necessary.

\textsuperscript{13} The perturbative equivalence of $\frac{1}{6}$-BPS Wilson loops of the lens space and ABJ matrix models, via the analytic continuation, has been established and checked by direct perturbative calculations.
3.4. Remarks on the ABJM limit

As noted in Sect. 2.1, there are subtleties in taking the ABJM limit $M \to 0$, in particular, in the formula (2.16) for the case $N_1 \geq N_2$. There are two points to be addressed; (1) the agreement of the two formulas (2.12) and (2.15), and (2) the $\frac{1}{2}$-BPS Wilson loop (2.19) in the ABJM limit.

To address the first point, notice that the integrands of (2.14) and (2.17) in the limit $M \to 0$ become identical. However, as remarked before, the contours $C$ and $C_2$ are different. Now since the factors $\prod_{i=1}^{N_1}[1/(q^{s_i+1+n\delta_i})]_M$ in (2.14) and $\prod_{a=1}^{N_2}[1/(-q^{s_a+1})]_M$ in (2.17) are absent, there are no NP poles on the real axis. Hence the difference due to the contours $C$ and $C_2$ only comes from the residues at the P poles $s_d = -1, -2, \ldots, -n$ in (2.17). Therefore, in order for (2.12) and (2.15) to agree, these residues have to be canceled by (2.16). To see this, we carefully take the $M \to 0$ limit of (2.16):

$$I^{(1)}(N_1, N_2; n)_{\kappa} = \frac{1}{N_2!} \sum_{c=0}^{n-1} \prod_{a=1}^{N_2} \left[ -\frac{1}{2\pi i} \int_{C_1[c]} \frac{\pi ds_a}{\sin(\pi s_a)} \right] \times q^{n(2c-M)} \frac{(q^{1-n})_c(q^{1+n})_{M-1-c}}{(q)_c(q)_{M-1-c}} \prod_{a=1}^{N_2} \frac{(q^{s_a+1})_M}{2(-q^{s_a+1})_M} \times \prod_{a=1}^{N_2} \left[ \frac{(-q^{s_a+1+c})_1(q^{s_a+1+c-n})_1}{(q^{s_a+1+c})_1(-q^{s_a+1+c-n})_1} \prod_{b=1}^{a-1} \frac{(q^{s_a-s_b})_1}{(-q^{s_a-s_b})_1} \prod_{b=a+1}^{N_2} \frac{(q^{s_b-s_a})_1}{(-q^{s_b-s_a})_1} \right].$$

(3.36)

In particular, we focus on the factors near the P pole for a selected variable $s_d$ at $s_d = -1 - c$ as $M \to 0+$,

$$\lim_{s_d \to -1-c} \frac{(q^{1-n})_c(q^{1+n})_{M-1-c}}{(q)_c(q)_{M-1-c}} \frac{(q^{s_d+1})_1}{(q^{s_d+1+c})_1} = \frac{q^{n-c}(q^{n-c})_M}{(q^n)_1(q^{-c})_c(q)_{M-1-c}} \quad \lim_{s_d \to -1-c} \frac{(q^{s_d+1})_M}{(q^{s_d+1+c})_1} = \frac{q^{-nc}(q^{n-c})_M}{(q^n)_1} \xrightarrow{M \to 0} \frac{q^{-nc}}{1-q^n}$$

(3.37)

where we used $(q^{1-n})_c(q^{1+n})_{M-1-c} = (-1)^c q^{-nc+\frac{c(c+1)}{2}(q^{n-c})_M/(q^n)_1}$ and $\lim_{\epsilon \to 0}(q^{\epsilon-c})_c(q)_{M-1-c}$. Note that in the $M \to 0_+$ limit these factors vanish away from the P pole $s_d = -1 - c$, i.e., if the limit $s_d \to -1 - c$ is not taken in the first line. Therefore, in the ABJM limit the only contribution comes from the residues at the P poles $s_d = -1 - c$ where $c = 0, \ldots, n - 1$.

An important remark is in order: When $c < M$, the pole at $s_d = -1 - c$ is clearly a simple pole, since the factor $(q^{s_d+1+c})_1$ that appears in the first line of (3.37) is canceled by the same factor in the $q$-Pochhammer symbol $(q^{s_d+1})_M$. However, when $c \geq M$, it is subtle, because there may not seem to be any apparent cancellation of these factors. Nevertheless, we treat the pole at $s_d = -1 - c$ as a simple pole. As it turns out, the proper way to deal with this subtlety is to adopt an $\epsilon$-prescription for the parameter $M$. Namely, $M$ is always kept off an integral value by the shift $M \to M + \epsilon$ with $\epsilon > 0$.

The factor $(q^{s_d+1})_M$ is always assumed to be $(q^{s_d+1})_{M+\epsilon}$ with a non-integral index in our calculations and is defined by (A3) for a non-integral $M + \epsilon$. With this prescription, the factor $(q^{s_d+1+c})_1$ is always canceled by the same factor in $(q^{s_d+1})_{M+\epsilon}$, making the pole at $s_d = -1 - c$ a simple pole. However, there is one more subtlety in this prescription to be clarified. Namely, there appears a pole at $s_d = -1 - c - \epsilon$ even with this prescription when $c \geq M$. If this pole were included within the contour, the contribution (3.37) would have been canceled. In other words, it would have been the
same as treating the pole at $s_d = -1 - c$ as a double pole. Thus our $\epsilon$-prescription involves a particular choice of the contour $C_1[c]$, i.e., to place it between $s_d = -1 - c$ and $-1 - c - \epsilon$ so as to avoid the latter pole. This is a very subtle point and so much of a detail but is absolutely necessary for getting sensible results.

Without loss of generality, we can choose the index $d$ to be 1, since the expression is invariant under permutations of $s_d$. This yields

$$I^{(1)}(N, N; n)_k = \frac{1}{2^{N-1}(N-1)!} \sum_{c=0}^{n-1} \left(-q^n\right)^c \frac{1}{1+q^n} \times \prod_{a=2}^{N} \left[ \left(-\frac{1}{2\pi i} \int_{C_1[c]} \frac{\pi ds_a}{\sin(\pi s_a)} \right) \left(\frac{q^{s_a+1+c}}{(-q^{s_a+1+c-n})}\right)_1 \right] \times \prod_{b=2}^{a-1} \left(\frac{\left(q^{s_b-s_b}\right)_1}{(-q^{s_b-s_b})}\right) \times \prod_{b=a+1}^{N} \left(\frac{\left(q^{s_b-s_a}\right)_1}{(-q^{s_b-s_a})}\right). \quad (3.38)$$

As shown in Appendix D, this is exactly canceled out by the sum of residues in $I^{(2)}(N, N; n)_k$ at the P poles $s_1 = -1, \ldots, -n$. Hence the formula (2.15) in the ABJM limit reduces to $I^{(2)}(N, N; n)_k$ with the contour $C_2$ being replaced by the contour $C$. This proves the agreement of (2.12) and (2.15) in the ABJM limit. We also note that the formula (3.38), when multiplied by $q^{-\frac{1}{2}n^2+n}$, yields the $\frac{1}{2}$-BPS Wilson loop (2.19) in the ABJM limit (up to a normalization). We discuss the $\frac{1}{2}$-BPS Wilson loop further in the next subsection.

### 3.5. The $\frac{1}{2}$-BPS Wilson loop

The $\frac{1}{2}$-BPS Wilson loop is given by (2.19), which follows from the equality

$$q^{-\frac{1}{2}n^2+n}I^{(2)}(N_1, N_2; n)_k = (-1)^n q^{\frac{1}{2}n^2-n}I(N_2, N_1; n)_{-k}, \quad (3.39)$$

where $N_1 \geq N_2$. In order to show this identity, we recall that

$$I^{(2)}(N_1, N_2; n)_k = \frac{1}{N_2} \sum_{d=1}^{N_2} \prod_{a=1}^{N_2} \left[ \left(-\frac{1}{2\pi i} \int_{C_2[a]} \frac{\pi ds_a}{\sin(\pi s_a)} \right) q^{-n_{sad}+(d-M-2)} \prod_{a=1}^{N_2} \left(\frac{q^{s_a-s_d-1}}{(-q^{s_a-s_d})}\right)_1 \right] \times \prod_{a=1}^{N_2} \left[ (1 + q^{-n_{bda}}) \left(q^{n_{sad}}\right)_M \prod_{b=1}^{a-1} \left(\frac{\left(q^{s_a-s_b}\right)_1}{(-q^{s_a-s_b})}\right)_1 \prod_{b=a+1}^{N_2} \left(\frac{\left(q^{s_b-s_a}\right)_1}{(-q^{s_b-s_a})}\right)_1 \right] \quad (3.40)$$

and

$$I(N_2, N_1; n)_{-k} = \frac{1}{N_1} \sum_{d=1}^{N_1} \prod_{a=1}^{N_1} \left[ \left(-\frac{1}{2\pi i} \int_{C[a]} \frac{\pi ds_a}{\sin(\pi s_a)} \right) q^{n_{sad}-(d-2)} \prod_{a=1}^{N_2} \left(\frac{q^{s_a-s_d-1}}{(-q^{s_a-s_d})}\right)_1 \right] \times \prod_{a=1}^{N_2} \left[ \left(-1\right)_M (q^{s_a-1} q^{-1})_M \right] \left(1 + q^{-n_{bda}}\right) q^{-n_{sad}-(d-1)-n_{sad}+q^{-1}} \times \prod_{b=1}^{a-1} \left(\frac{\left(q^{s_a-s_b}\right)_1}{(-q^{s_a-s_b})}\right)_1 \times \prod_{b=a+1}^{N_2} \left(\frac{\left(q^{s_b-s_a}\right)_1}{(-q^{s_b-s_a})}\right)_1, \quad (3.41)$$
where $M := |N_2 - N_1| = N_1 - N_2$. In fact, it is straightforward to check that these two are related, precisely as in (3.39), by the change of variables,

$$t_a = -s_a - 1 - M - n\delta_{ad},$$

(3.42)

for a given $d$, where $s_a$ are the variables in the latter (3.41) and $t_a$ are identified with those in the former (3.40). The contour $C$ is placed in the intervals, $-\frac{k}{2} - 1 < s_a < \min(0, \frac{k}{2} - M)$ for $a \neq d$ and $\max(-M - 1, -\frac{k}{2} - 1 - n) < s_d < \min(0, \frac{k}{2} - M - n)$. By the above change of variables, this becomes precisely the contour $C_2$, where $\max(-M - 1, -\frac{k}{2} - 1) < t_a < \frac{k}{2} - M$ for $a \neq d$ and $\max(-M - 1 - n, -\frac{k}{2} - 1) < t_d < \min(-n, \frac{k}{2} - M)$.

This proves that the $\frac{1}{2}$-BPS Wilson loop is given by $q^{-\frac{1}{2} + n} I^{(1)}(N_1, N_2; n)_k$ up to the normalization.

4. Seiberg duality—derivations and a proof

There is a duality between two ABJ theories [5]. Schematically, when $N_2 > N_1$, the following ABJ theories are equivalent:

$$U(N_1)_k \times U(N_2)_{-k} = U(2N_1 - N_2 + k)_k \times U(N_1)_{-k}.$$  

(4.1)

The partition functions of the two theories agree up to a phase [17,39,52,53]. It was further understood in Ref. [17] how the perturbative and nonperturbative contributions to the partition function are exchanged under the duality map.

The Wilson loops, in contrast, are not invariant under the duality. The mapping rule for $\frac{1}{2}$-BPS Wilson loops in general representations in $\mathcal{N} = 2$ CSM theories with a simple gauge group has been studied by Kapustin and Willett [50]. These Wilson loops correspond to $\frac{1}{6}$-BPS Wilson loops in the ABJ theory. Our results are consistent with their rule and slightly generalize it to the case where the flavor group is gauged. Similar to the case of the partition function [17], our formulas for the Wilson loops allow us to understand an important aspect of the duality, namely, how the perturbative and nonperturbative contributions are exchanged under the duality map.

In this section we provide a proof of the duality map by analyzing our expressions (2.12), (2.15), and (2.19) for the Wilson loops. To this end, let us recall our results for the duality map of the Wilson loops, (2.20), (2.21), and (2.22).

- **The $\frac{1}{6}$-BPS Wilson loop duality:**

$$W^\frac{1}{6}_1(N_1, N_2; n)_k = -W^\frac{1}{6}_1(\tilde{N}_2, N_1; n)_k - 2(-1)^{n+1}W^\frac{1}{6}_1(\tilde{N}_2, N_1; n)_k,$$

(4.2)

- **The flavor Wilson loop duality:**

$$W^1_6(N_1, N_2; n)_k = W^\frac{1}{6}_1(\tilde{N}_2, N_1; n)_k,$$

(4.3)

- **The $\frac{1}{2}$-BPS Wilson loop duality:**

$$W^\frac{1}{2}_1(N_1, N_2; n)_k = (-1)^n W^\frac{1}{2}_1(\tilde{N}_2, N_1; n)_k,$$

(4.4)

where we denoted the rank of the dual gauge group by $\tilde{N}_2 = 2N_1 - N_2 + k$. These three relations are not independent, but one of them can be derived from the other two by using the relation between the $\frac{1}{2}$- and $\frac{1}{6}$-BPS Wilson loops

$$W^\frac{1}{2}_1(N_1, N_2; n)_k = W^\frac{1}{6}_1(N_1, N_2; n)_k - (-1)^n W^\frac{1}{6}_1(N_1, N_2; n)_k.$$  

(4.5)

\footnote{The $-1$ in (3.42) compensates for the orientation flip of the contour.}
Before going into a rigorous technical proof, we provide a heuristic yet very useful and intuitive way to understand how the Wilson loops would be mapped.

4.1. *The brane picture—a heuristic derivation*15

In Refs. [5,6], the brane realization of the ABJ(M) theory was proposed. The brane content of this configuration is given by the following:

\[
\begin{aligned}
\text{D3-brane} & : 0126 \\
\text{NS5-brane} & : 012345 \\
(1, k) \text{ 5-brane} & : 012 \left[ \begin{array}{l}
\frac{3}{7}
\end{array} \right]_\theta \left[ \begin{array}{l}
\frac{4}{8}
\end{array} \right]_\theta \left[ \begin{array}{l}
\frac{5}{9}
\end{array} \right]_\theta.
\end{aligned}
\]

Here, \(x^6\) is periodically identified, and \(\left[ \frac{3}{7}\right]_\theta\) means the direction on the 3–7 plane with an angle \(\theta\) with the \(x^3\) axis, where \(\tan \theta = k\). For \(U(N_1) \times U(N_2)\) theory with \(N_2 - N_1 = M > 0\), there are \(N_1\) D3-branes between an NS5-brane and a \((1, k)\) 5-brane, and \(N_2 = N_1 + M\) D3-branes between the \((1, k)\) and the NS5. This system realizes 3D supersymmetric field theory that lives in the 012 directions and flows in the IR to the ABJ SCFT. Seiberg duality corresponds to moving the NS5 and the \((1, k)\) branes past each other and, during the process, \(k\) D3-branes are created by the Hanany–Witten effect [23] while \(M\) D3-branes are annihilated, in the end leaving D3-branes realizing the dual theory (4.1).

For our purpose, it is convenient to consider the following M-theory lift of the configuration (4.6), in which Wilson loops are geometrically realized [54,55]. Assume that we have a nontrivial Wilson loop along e.g. \(x^2\), namely \(\int dx^2 A_2 \neq 0\).16 If we T-dualize the configuration (4.6) along \(x^2\) and further lift it to 11 dimensions, we obtain

\[
\begin{aligned}
\text{M2} & : 016 \\
\text{M5} & : 012345 \\
\text{M5}' & : 01 \left[ \begin{array}{l}
A
\end{array} \right]_\theta \left[ \begin{array}{l}
\frac{3}{7}
\end{array} \right]_\theta \left[ \begin{array}{l}
\frac{4}{8}
\end{array} \right]_\theta \left[ \begin{array}{l}
\frac{5}{9}
\end{array} \right]_\theta.
\end{aligned}
\]

where “\(A\)” denotes the 11th direction. Note that the \((1, k)\) 5-brane has lifted to an M5-brane (denoted by M5’) that is tilted in four 2-planes with the same angle \(\theta\). In Fig. 5, we schematically describe this configuration. Because M5’-branes are tilted in the \(x^2-x^A\) plane, there are only \(k\) places in which “fractional” M2-branes can stretch between M5’ and M5. Note that only one fractional M2-brane can exist in one place because of the s-rule [54,55]. \(M = N_2 - N_1\) fractional M2-branes are distributed among these \(k\) places. On the other hand, \(N_1\) “entire” M2-branes are going around the \(x^6\) direction and they do not have to sit in these places but can be anywhere.

Because of the Wilson loop, different M2-branes are located at different positions along the \(x^2\) direction. Let the \(x^2\) coordinate of the \(N_1\) M2-branes between M5 and M5’ be \(\mu_j, j = 1, \ldots, N_1\), and that of the \(N_2 = N_1 + M\) M2-branes (both fractional and entire) between M5’ and M5 be \(v_a, a = 1, \ldots, N_2\) (see Fig. 5).17 Furthermore, let the \(x^2\) coordinate of the \(k\) places in which fractional

---

15 We thank Kazutoshi Ohta for explaining to us the brane picture in Refs. [54,55] and for very useful discussions.

16 The relation between the Wilson loop for ABJ theory on flat space and that for ABJ theory on \(S^3\) is not clear. This is one of the reasons why the argument presented here is heuristic.

17 Note that there is no direct relation between the \(\mu, v\) here and the ones that appear in the ABJ matrix model (2.4). In the computation of Wilson loops using localization [8], at saddle points, gauge fields (which correspond to \(\mu, v\) here) vanish and Wilson loops get contribution only from auxiliary fields (which correspond to \(\mu, v\) in the ABJ matrix model).
Fig. 5. M-theory representation of ABJ theory with Wilson loops (presented is the case with $k = 3$, $N_1 = 2$, $N_2 = 4$, $M = 2$). The left and right ends of the configuration are identified. The position of the M2-branes in the $x^2$ direction corresponds to the Wilson loop. Among $k$ places in which fractional M2-branes can end, the occupied ones are denoted by $\bullet$ and the unoccupied ones by $\circ$.

Fig. 6. The Seiberg dual configuration of the configuration in Fig. 5.

M2-branes can end be $y_\alpha$, $\alpha = 1, \ldots, k$. If the radius of the $x^2$ direction is $2\pi$, we have $y_\alpha = \frac{2\pi\alpha}{k} + \text{const}$ and, for $n \in \mathbb{Z}$,

$$
\sum_{\alpha=1}^{k} e^{iny_\alpha} = \sum_{\alpha=1}^{k} e^{i2\pi n\alpha/k} = \begin{cases} 0 & (n \neq 0 \mod k) \\ k & (n = 0 \mod k). \end{cases} \quad (4.8)
$$

As mentioned above, Seiberg duality corresponds to moving M5 and M5$'$ past each other. In this process, $M$ fractional M2-branes get annihilated, and $k$ fractional M2-branes are created, leaving $\tilde{N}_2 = N_1 - M + k = 2N_1 - N_2 + k$. The resulting configuration is shown in Fig. 6. Let the $x^2$ coordinate of the $\tilde{N}_2$ M2-branes (both fractional and entire) between M5 and M5$'$ be $\tilde{\mu}_1, \ldots, \tilde{\mu}_{\tilde{N}_2}$, and that of the $N$ M2-branes between M5$'$ and M5 be $\tilde{\nu}_1, \ldots, \tilde{\nu}_N$ (see Fig. 6).

In the original configuration in Fig. 5, among the $k$ spots $\{1, \ldots, k\}$ at which fractional M2-branes can end, let the occupied ones be $O_1, \ldots, O_M$ and the unoccupied ones be $U_1, \ldots, U_{k-M}$. Of course, $\{O_1, \ldots, O_M\} + \{U_1, \ldots, U_{k-M}\} = \{1, \ldots, k\}$. Clearly,

$$
\{v_1, \ldots, v_{N_2}\} = \{\mu_1, \ldots, \mu_{N_1}\} + \{y_{O_1}, \ldots, y_{O_M}\}. \quad (4.9)
$$

In the dual theory, the positions of the entire M2-branes are unchanged, while the occupied and unoccupied spots for fractional M2-branes are interchanged. Therefore,

$$
\{\tilde{\mu}_1, \ldots, \tilde{\mu}_{\tilde{N}_2}\} = \{\mu_1, \ldots, \mu_{N_1}\} + \{y_{U_1}, \ldots, y_{U_{k-M}}\} \\
= \{\mu_1, \ldots, \mu_{N_1}\} + \{y_1, \ldots, y_k\} - \{y_{O_1}, \ldots, y_{O_M}\}, \quad (4.10)
$$

$$
\{\tilde{\nu}_1, \ldots, \tilde{\nu}_{N_1}\} = \{\mu_1, \ldots, \mu_{N_1}\}. \quad (4.11)
$$
4.1.1. *Fundamental representation*  

Now we want to use this picture to give a very heuristic explanation of Seiberg duality (4.1). Consider the original configuration in Fig. 5. The Wilson loop in the fundamental representation simply measures the position of the M2-branes. Therefore, naively, we have \(^{18}\)

\[
W^I_6(N_1, N_2; n) \sim \sum_{j=1}^{N_1} e^{in\mu_j},
\]

\[
W^{II}_6(N_1, N_2; n) \sim \sum_{a=1}^{N_2} e^{inv_a} = \sum_{j=1}^{N_1} e^{in\mu_j} + \sum_{a=1}^{M} e^{in\nu_0a}.
\]

where in the second equation we used (4.9). Actually, it turns out that, in order to reproduce the explicit results obtained in the current paper, we must set \(v_a \rightarrow v_a + \pi\) by hand so that (4.13) is replaced by

\[
W^{II}_6(N_1, N_2; n) \sim \sum_{a=1}^{N_2} e^{in(v_a+\pi)} = (-1)^n \left( \sum_{j=1}^{N_1} e^{in\mu_j} + \sum_{a=1}^{M} e^{in\nu_0a} \right). \tag{4.14}
\]

In the dual theory, using (4.10) and (4.11), we obtain

\[
W^I_6(\tilde{N}_2, N_1; n) \sim (-1)^{\tilde{N}_2} \sum_{j=1}^{\tilde{N}_2} e^{in\tilde{\mu}_j} = (-1)^n \left( \sum_{j=1}^{N_1} e^{in\mu_j} + \sum_{a=1}^{M} e^{in\nu_0a} \right)
\]

\[
= (-1)^n \left( \sum_{j=1}^{N_1} e^{in\mu_j} - \sum_{a=1}^{M} e^{in\nu_0a} \right), \tag{4.15}
\]

\[
W^{II}_6(\tilde{N}_2, N_1; n) \sim \sum_{a=1}^{N_1} e^{in\nu_a} = \sum_{j=1}^{N_1} e^{in\mu_j}, \tag{4.16}
\]

where we introduced another ad hoc rule \(\tilde{\mu}_j \rightarrow \tilde{\mu}_j + \pi\) just as we did in (4.14). Also, in the second line of (4.15), we used (4.8), assuming that \(n \neq 0 \mod k\). Therefore, comparing (4.12), (4.14) and (4.15), (4.16), we “derived” the following duality relations:

\[
W^I_6(N_1, N_2; n) = W^{II}_6(\tilde{N}_2, N_1; n), \tag{4.17}
\]

\[
W^{II}_6(N_1, N_2; n) + W^I_6(\tilde{N}_2, N_1; n) = 2(-1)^n W^{II}_6(\tilde{N}_2, N_1; n). \tag{4.18}
\]

This means that the \(\frac{1}{2}\)-BPS Wilson loop defined by

\[
W_2(N_1, N_2; n) := W^I_6(N_1, N_2; n) - (-1)^n W^{II}_6(N_1, N_2; n)
\]

is expected to have the following simple transformation rule:

\[
W_2(N_1, N_2; n) = (-1)^n W_2(\tilde{N}_2, N_1; n). \tag{4.20}
\]

\(^{18}\)Note that this is very rough and heuristic; thus “~”. In reality, \(v_a\) is a variable to be integrated over and is not localized at \(y_a\). Even if there is a sense in which they are localized at \(y_a\), we should sum over all possible ways to distribute \(M\) fractional M2-branes over \(k\) positions.
Note that the above arguments are based on the identity (4.8) and are valid only for \( n \neq 0 \mod k \).
We do not expect to get correct equations by setting \( n = 0 \) in the above duality relations, as we commented in Sect. 2.1.\(^{19}\)

Although the ad hoc rule \( v_a \rightarrow \nu_a + \pi, \tilde{\mu}_j \rightarrow \tilde{\mu}_j + \pi \) was crucial to reproduce the correct transformation rule for Wilson loops, its physical meaning is unclear. It is somewhat reminiscent of the fact that, in the ABJ matrix model at large \( N_1, N_2 \) \(^9\), the eigenvalue distribution for \( U(N_2) \) is offset relative to the \( U(N_1) \) eigenvalue distribution on the complex eigenvalue plane, but further investigations are left for future research. Since the arguments given in this subsection are meant to be only heuristic, we simply accept the rule as a working assumption and proceed. In passing, we note that, with the above ad hoc rule, the \( \frac{1}{2} \)-BPS Wilson loop (4.19) can be understood simply as a supertrace as follows:

\[
W_{\frac{1}{2}}(N_1, N_2; n) \sim \sum_j e^{in\mu_j} - \sum_a e^{in\nu_a} = \text{tr} X^n - \text{tr} Y^n = \text{str} Z^n, \tag{4.21}
\]

where \( X = \text{diag}(e^{i\mu}), Y = \text{diag}(e^{i\nu}), \) and \( Z = \begin{pmatrix} X & Y \end{pmatrix} \).

4.1.2. More general representations

The above heuristic method to guess the Seiberg duality relation can be generalized to more general representations.\(^{20}\) For example, in the original \( U(N_1)_k \times U(N_2)_{-k} \) theory, consider the following \( \frac{1}{6} \)-BPS Wilson loops:

\[
W_{\square} \sim \sum_{i \leq j} e^{i(\mu_i + \mu_j)} = \frac{1}{2} \left( \sum_{i=1}^{N_1} e^{i\mu_i} \right)^2 + \frac{1}{2} \sum_{i=1}^{N_1} e^{2i\mu_i} =: \frac{1}{2} (e^{i\mu})^2 + \frac{1}{2} e^{2i\mu}
\]

\[
W_{\square} \sim \sum_{i < j} e^{i(\mu_i + \mu_j)} = \frac{1}{2} \left( \sum_{i=1}^{N_1} e^{i\mu_i} \right)^2 - \frac{1}{2} \sum_{i=1}^{N_1} e^{2i\mu_i} =: \frac{1}{2} (e^{i\mu})^2 - \frac{1}{2} e^{2i\mu}
\]

\[
W_{\square} \sim \sum_{a < b} e^{i(\nu_a + \nu_b)} = \frac{1}{2} \left( \sum_{a=1}^{N_2} e^{i\nu_a} \right)^2 + \frac{1}{2} \sum_{a=1}^{N_2} e^{2i\nu_a} =: \frac{1}{2} (e^{i\nu})^2 + \frac{1}{2} e^{2i\nu}
\]

\[
W_{\square} \sim \sum_{a < b} e^{i(\nu_a + \nu_b)} =: \frac{1}{2} (e^{i\nu})^2 + \frac{1}{2} e^{2i\nu}
\]

\[
W_{\square} \sim - \sum_{i=1}^{N_1} e^{i\mu_i} \sum_{a=1}^{N_2} e^{i\nu_a} =: - (e^{i\mu})^2 - e^{i\mu} e^{i\nu},
\]

where \( W_R \tilde{R} \) denotes the Wilson loop in the representations \( R \) and \( \tilde{R} \) for \( U(N_1) \) and \( U(N_2) \), respectively, and \( \bullet \) means the trivial representation. Also, in the last expression of each line, we used a

\(^{19}\) Actually, Eqs. (4.17) and (4.20) still give correct equations if we set \( n = 0 \), but (4.18) does not.

\(^{20}\) For a \( U(N) \) representation with a Young diagram \( \lambda \), the Wilson loop is \( S_\lambda(e^{i\mu_1}, \ldots, e^{i\mu_N}) \), where \( S_\lambda(x_1, \ldots, x_N) \) is the Schur polynomial [44]. For example, \( S_\Box = \sum_i x_i, S_{\square} = \sum_{i \leq j} x_i x_j, S_{\tilde{R}} = \sum_{i < j} x_i x_j \).
schematic notation, whose meaning is defined by the immediately preceding expression. In the dual
\( U(\tilde{N}_2)_k \times U(N_1)_{-k} \) theory, we have

\[
\tilde{W}_{\square} \sim \frac{1}{2} (e^{i\mu})^2 + \frac{1}{2} (e^{iyO})^2 - e^{i\mu} e^{iyO} + \frac{1}{2} e^{2i\mu} - \frac{1}{2} e^{2iyO},
\]

\[
\tilde{W}_{\bullet} \sim \frac{1}{2} (e^{i\mu})^2 + \frac{1}{2} (e^{iyO})^2 - e^{i\mu} e^{iyO} - \frac{1}{2} e^{2i\mu} + \frac{1}{2} e^{2iyO},
\]

\[
\tilde{W}_{\Box} \sim \frac{1}{2} (e^{i\mu})^2 + \frac{1}{2} (e^{iyO})^2 - e^{i\mu} e^{iyO} - \frac{1}{2} e^{2i\mu} + \frac{1}{2} e^{2iyO},
\]

\[
\tilde{W}_{\square} \sim - (e^{i\mu})^2 + e^{i\mu} e^{iyO}.
\]

(4.23)

The duality relation between \( W \) and \( \tilde{W} \) is readily found to be

\[
W = S \tilde{W}, \quad W = \begin{pmatrix}
W_{\square} \\
W_{\bullet} \\
W_{\Box} \\
W_{\Box}
\end{pmatrix}, \quad S = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 3 & 1 & 2 \\
1 & 0 & 1 & 3 & 2 \\
0 & 0 & -2 & -2 & -1
\end{pmatrix}.
\]

(4.24)

The combinations that have the simple transformation rule are

\[
W_{1/2} := W_{\square} + W_{\bullet} + W_{\Box}, \quad W_{1/2} := W_{\bullet} + W_{\Box} + W_{\Box}.
\]

(4.25)

which transform as

\[
W_{1/2} = \tilde{W}_{1/2}, \quad W_{1/2} = \tilde{W}_{1/2}.
\]

(4.26)

These are precisely the \( 1/2 \)-BPS Wilson loops derived in Ref. [44].

Just as in (4.21), we can write these in terms of a supertrace as follows:

\[
W_{1/2} = \frac{1}{2} (\text{str } Z)^2 + \frac{1}{2} \text{str}(Z^2) = \text{str}_{\Box} Z,
\]

\[
W_{1/2} = \frac{1}{2} (\text{str } Z)^2 - \frac{1}{2} \text{str}(Z^2) = \text{str}_{\Box} Z.
\]

(4.27)

Note that the right-hand side is nothing but a supertrace in the respective representations. More

generally, the \( 1/2 \)-BPS Wilson loop for general representation \( R \) is given by

\[
W_{1/2} = \text{str}_R Z = P_R(\text{str}(Z^n)),
\]

(4.30)

where \( P_R \) is a polynomial in \( \text{str}(Z^n) \) \((n = 1, \ldots, |R|)\) obtained from the Schur polynomial. Each term

in the polynomial contains a product of \(|R|\) \( Z \). The results from the previous section, more specifically

\[\text{(4.28)}\]

\[\text{(4.29)}\]

Note that the \( \frac{1}{2} \)-BPS Wilson loops derived in Ref. [44] are

\[
W_{1/2} = \text{str}_R \left( e^{i\mu} \begin{pmatrix} e^{i\nu} \\ -e^{i\nu} \end{pmatrix} \right)
\]

(4.28)

where \( \mu, \nu \) are the \( \mu, \nu \) that appear in the matrix model (see footnote 17). Our ad hoc rule is to replace

this with

\[
W_{1/2} = \text{str}_R \left( e^{i\mu} \begin{pmatrix} e^{i\nu} \\ e^{i\nu} \end{pmatrix} \right) = \text{str}_R Z
\]

(4.29)

where \( \mu, \nu \) are the positions of the M2-branes.
say that $\text{str}(Z^n) \to (-1)^n \text{str}(Z^n)$ under duality. Therefore, the transformation law for the general $\frac{1}{2}$-BPS Wilson loop is

$$W_R^{1/2} \to (-1)^{|R|} W_R^{1/2}. \quad (4.31)$$

None of the above is a derivation of Seiberg duality for $\frac{1}{6}$-BPS Wilson loops but is merely a motivation for it. However, the fact that it predicts a simple duality law for $\frac{1}{2}$-BPS Wilson loops is evidence that the $\frac{1}{6}$-BPS duality relation is also correct.

### 4.2. A rigorous derivation and proof

Since the mapping (4.3) for the flavor Wilson loop and (4.4) for the $\frac{1}{2}$-BPS Wilson loop are simple as compared to the mapping (4.2) for $\frac{1}{6}$-BPS Wilson loops, it is the best strategy to prove (4.3) and (4.4) and then infer (4.2) from them. In due course, we will also see manifestly the exchange of perturbative and nonperturbative contributions under the duality.

- **The flavor Wilson loop duality:** We first prove the flavor Wilson loop duality:

$$W^1_{\frac{1}{6}}(N_1, N_2; n)_k = W^1_{\frac{1}{6}}(\tilde{N}_2, N_1; n)_k = W^1_{\frac{1}{6}}(N_1, \tilde{N}_2; n)_{-k}, \quad (4.32)$$

which amounts to the equality

$$\frac{q^{-n_2+n}I(N_1, N_2; n)_k}{I(N_1, N_2; 0)_k} = \frac{q^{-n}I(N_1, \tilde{N}_2; n)_{-k}}{I(N_1, \tilde{N}_2; 0)_{-k}}. \quad (4.33)$$

The basic idea for the proof is to show that (1) the integrands in the numerators are identical, i.e., they share exactly the same zeros and poles and the same asymptotics up to the normalization, and (2) the contours are equivalent. The explicit forms of $I(N_1, N_2; n)_k$ and $I(N_1, \tilde{N}_2; n)_{-k}$ are given by

$$I(N_1, N_2; n)_k = \frac{1}{N_1!} \sum_{l=1}^{N_1} \prod_{i=1}^{N_1} \left[ -\frac{1}{2\pi i} \int C \frac{\pi ds_i}{\sin(\pi s_i)} \right] q^{-n s_i + n l - 2} \prod_{i=1}^{N_1} \left( \frac{q^{s_i-n} - 1}{q^{s_i-n_1} - 1} \right) \prod_{i \neq l} \left( \frac{q^{s_i-n} - 1}{q^{s_i-n_1} - 1} \right) \prod_{i=1}^{N_1} \left( \frac{q^{s_i-s_j} - 1}{q^{s_i-s_j} - 1} \right) \prod_{j=1}^{N_1} \left( \frac{q^{s_j-s_i} - 1}{q^{s_j-s_i} - 1} \right) \quad (4.34)$$

$$I(N_1, \tilde{N}_2; n)_{-k} = \frac{1}{N_1!} \sum_{l=1}^{N_1} \prod_{i=1}^{N_1} \left[ -\frac{1}{2\pi i} \int C \frac{\pi ds_i}{\sin(\pi s_i)} \right] q^{-n s_i + n l + M - 2} \prod_{i=1}^{N_1} \left( \frac{q^{s_i-n} - 1}{q^{s_i-n_1} - 1} \right) \prod_{i \neq l} \left( \frac{q^{s_i-n} - 1}{q^{s_i-n_1} - 1} \right) \prod_{i=1}^{N_1} \left( \frac{q^{s_i-s_j} - 1}{q^{s_i-s_j} - 1} \right) \prod_{j=1}^{N_1} \left( \frac{q^{s_j-s_i} - 1}{q^{s_j-s_i} - 1} \right) \quad (4.35)$$

where $M := |N_2 - N_1| = N_2 - N_1$. At first glance the zeros and poles of (4.34) and (4.35) differ in the $M$-dependence, since $M$ is replaced by $k - M$ in the latter. However, introducing
The dual variables in the latter,
\[ \tilde{s}_i = -s_i + \frac{k}{2} - 1 - n\delta_{il}, \quad (4.36) \]
we can easily see that they actually agree. As simple as it may look, we stress that this is a very important map and can be regarded as the duality transformation, as we now justify. In terms of the original variables \( s_i \), the poles of the integrand in (4.35) appear at
\[ P: \quad s_i = 0, 1, \ldots, M - 1 \mod k, \quad (4.37) \]
\[ NP: \quad s_i = -\frac{k}{2} + M - n\delta_{il}, \ldots, \frac{k}{2} - 1 - n\delta_{il} \mod k, \quad (4.38) \]
\[ NP: \quad s_i = s_j + \frac{k}{2} - n\delta_{il} \mod k. \quad (4.39) \]
In terms of the dual variables \( \tilde{s}_i \), these poles are mapped to
\[ NP: \quad \tilde{s}_i = \frac{k}{2} - M - n\delta_{il}, \ldots, \frac{k}{2} - 2 - n\delta_{il}, \frac{k}{2} - 1 - n\delta_{il} \mod k, \quad (4.40) \]
\[ P: \quad \tilde{s}_i = 0, 1, \ldots, k - M - 1 \mod k, \quad (4.41) \]
\[ NP: \quad \tilde{s}_i = \tilde{s}_j + \frac{k}{2} - n\delta_{il} \mod k. \quad (4.42) \]
These are precisely the same as the poles of the integrand in (4.34) and thus the two integrands share exactly the same poles. We emphasize that the P and NP poles are exchanged by the duality transformation (4.36). In terms of the original variables, the contour \( \tilde{C} \) is placed in the intervals, \( \max(M - k - 1, -\frac{k}{2} - 1) < s_i < \min(0, -\frac{k}{2} + M) \) for \( i \neq l \) and \( \max(-M - 1, -\frac{k}{2} - 1 - n) < s_i < \min(0, -\frac{k}{2} + M - n) \). In terms of the dual variables, this becomes the intervals, \( \max(-\frac{k}{2} - 1, -M - 1) < \tilde{s}_i < \min(\frac{k}{2} - M, 0) \) for \( i \neq l \) and \( \max(-\frac{k}{2} - 1, -M - 1 - n) < s_i < \min(\frac{k}{2} - M, -n) \). Hence, the contours \( C \) and \( \tilde{C} \) are equivalent.\(^{22}\)

Meanwhile, the zeros appear only from the factors \( (q^{\tilde{s}_i - s_i})_1 \) and \( (q^{\tilde{s}_i - s_i - n})_1 \) and do not depend explicitly on \( M \). It is easy to check that the zeros are invariant under the duality transformation (4.36). Indeed, the factors that depend on the differences \( s_i - s_j \) in (4.34) and \( \tilde{s}_i - \tilde{s}_j \) in (4.35) only differ from each other by the factor \( q^{2n(l-1)} \) multiplying the latter.

It remains to find the asymptotics of the integrands. The asymptotics to be compared with are those at \( s_i \to i\infty \) in (4.34) and \( \tilde{s}_i \to i\infty \) in (4.35), and for the factors that depend on the differences of the variables we only need to care about the factor \( q^{2n(l-1)} \). Collecting various factors together and taking into account the orientations of the contours, it is straightforward to find that the latter asymptotics is \( i^k q^{-n^2 + 2nl} \) times the former. Since the factor \( i^k \) is canceled by the same factor coming from the normalization, this completes the proof of the equality (4.33).

- **The \( \frac{1}{2} \)-BPS Wilson loop duality**: We next prove the duality for the \( \frac{1}{2} \)-BPS Wilson loop:
\[ W_{\frac{1}{2}}(N_1, N_2; n)_k = (-1)^n W_{\frac{1}{2}}(\tilde{N}_2, N_1; n)_k, \quad (4.43) \]
which from (2.19) amounts to the equality
\[ \frac{q^{\frac{n^2}{2} - n} I^{(1)}(N_2, N_1; n)_k}{I^{(2)}(N_2, N_1; 0)_k} = -\frac{q^{\frac{n^2}{2} + n} I^{(1)}(\tilde{N}_2, N_1; n)_k}{I^{(2)}(\tilde{N}_2, N_1; 0)_k}. \quad (4.44) \]

\(^{22}\)The \(-1 \) in (4.36) compensates for the orientation flip of the contour.
The explicit forms of $I^{(1)}(N_2, N_1; n)_{-k}$ and $I^{(1)}(\tilde{N}_2, N_1; n)_{k}$ are given by

$$I^{(1)}(N_2, N_1; n)_{-k}$$

$$= \frac{1}{N_1!} \sum_{c=0}^{n-1} \prod_{i=1}^{N_1} \left[ -\frac{1}{2\pi i} \int_{C_{[c]}} \frac{\pi \mathrm{d}s_i}{\sin(\pi s_i)} \right] q^n \frac{(q^{1-n})_c (q^{1+n})_{M-1-c}}{(q)_c (q)_M^{1-c}} \prod_{i=1}^{N_1} \frac{(q^{-s_i-1}; q^{-1})_M}{2 (-q^{-s_i-1}; q^{-1})_M} \prod_{j=1}^{N_1} \frac{(q^{-s_i+s_j})_1}{(-q^{-s_i+s_j})_1} \right].$$

$$I^{(1)}(\tilde{N}_2, N_1; n)_{k}$$

$$= \frac{1}{N_1!} \sum_{c=0}^{n-1} \prod_{i=1}^{N_1} \left[ -\frac{1}{2\pi i} \int_{C_{\tilde{C}[c]}} \frac{\pi \mathrm{d}s_i}{\sin(\pi s_i)} \right] q^{n(2c+M)} \frac{(q^{1-n})_c (q^{1+n})_{k-M-1-c}}{(q)_c (q)_{k-M-1-c}} \prod_{i=1}^{N_1} \frac{(q^{s_i+1})_{k-M}}{2 (-q^{s_i+1})_{k-M}} \prod_{j=1}^{N_1} \frac{(q^{s_i-s_j})_1}{(-q^{s_i-s_j})_1} \right].$$

Similar to the flavor Wilson loop duality, introducing the duality transformation in (4.45)

$$\tilde{s}_i = -s_i - \frac{k}{2} - 1,$$

it becomes evident that the two integrands share the same zeros and poles. In terms of the original variables $s_i$, the poles of the integrand in (4.45) appear modulo $k$ at

**P:** $s_i = -1 - c; \quad 0, \ldots, -2 - c + n; \quad -c + n, \ldots, k - M - 1,$

**NP:** $s_i = \frac{k}{2} - M, \ldots, \frac{k}{2} - 2 - c; \quad \frac{k}{2} - c, \ldots, \frac{k}{2} - 1; \quad \frac{k}{2} + n - 1 - c,$

**NP:** $s_i = s_j + \frac{k}{2}.$

In terms of the dual variables $\tilde{s}_i$, these poles are mapped, after shifting by +$k$, to

**NP:** $\tilde{s}_i = -\frac{k}{2} + M, \ldots, \frac{k}{2} - 2 - \tilde{c}; \quad \frac{k}{2} - \tilde{c}, \ldots, \frac{k}{2} - 1; \quad \frac{k}{2} + n - 1 - \tilde{c},$

**P:** $\tilde{s}_i = -1 - \tilde{c}; \quad 0, \ldots, -2 - \tilde{c} + n; \quad -\tilde{c} + n, \ldots, M - 1,$

**NP:** $\tilde{s}_i = \tilde{s}_j + \frac{k}{2},$

where $\tilde{c} = (n - 1) - c$. Indeed, these are exactly the poles of the integrand in (4.46). We again stress that the P and NP poles are exchanged by the duality transformation (4.47). The contour $C_1$ is placed in the interval, max($-M - 1, -\frac{k}{2} + n - 1 - c$) $< s_i < \min(-1 - c, \frac{k}{2} - M)$, that is mapped to max($-\frac{k}{2} + n - 1 - \tilde{c}, -(k - M) - 1$) $< \tilde{s}_i < \min(\frac{k}{2} - (k - M), -1 - \tilde{c}).$ 23 Hence the contour in terms of the variables $\tilde{s}_i$ is equivalent to $C_1$.

As for the zeros, similar to the flavor Wilson loop case, they appear only from the factors that depend on the differences $\tilde{s}_i - \tilde{s}_j$ in (4.45) and $s_i - s_j$ in (4.46). In fact, these factors are exactly the same in (4.45) and (4.46).

23 More precisely speaking, these ranges are for $c < M$, as discussed in Sect. 3.3, but for $c \geq M$ the intervals are not conditional, $-\frac{k}{2} + n - 1 - c < s_i < -1 - c$ and $-\frac{k}{2} + n - 1 - \tilde{c} < \tilde{s}_i < -1 - \tilde{c}$. 

28/62
In order to examine the asymptotics, we first note that
\[
q^n(2\hat{c}+M)\frac{(q^{1-n})\hat{c}(q^{1+n})k-M-\hat{c}}{(q)\hat{c}(q)k-M-\hat{c}} = q^n(n-1)+n(M-c) \lim_{\epsilon \to 0} \frac{(q^\epsilon)_{1}(q^{n-\epsilon})k-M}{(q)\hat{c}(q)k-M-\hat{c}}
\]
\[
= -q^n(n-1)\frac{(q^{1-n})\hat{c}(q^{1+n})M-\hat{c}}{(q)\hat{c}(q)M-\hat{c}}. \quad (4.55)
\]

The asymptotics at \( s_i \to i\infty \) in (4.45) and \( s_j \to i\infty \) in (4.46) then differ only by the factor
\(-i^k q^{-n^2+2n}\) and the factor \( i^k \) is canceled by the same factor from the normalization. This completes the proof of (4.44).

- **The \( \frac{1}{6} \)-BPS Wilson loop duality:** Having proved two of the duality maps (4.3) and (4.4), the definition of the \( \frac{1}{2} \)-BPS Wilson loop (4.5) implies the duality map (4.2) for the \( \frac{1}{6} \)-BPS Wilson loops: The easiest way to see it is to add \((-1)^{n+1}W_1^1(N_1, N_2; n)\) to both sides of (4.2):

\[
W_1(N_1, N_2; n) = -\left( W_1^1(\tilde{N}_2, N_1; n) - (-1)^n W_1^1(N_1, N_2; n) \right)
\]

\[
- 2(-1)^{n+1} \left( W_1^2(\tilde{N}_2, N_1; n) - W_1^2(N_1, N_2; n) \right). \quad (4.56)
\]

Using the flavor Wilson loop duality (4.3) and the relation (4.5), this yields the \( \frac{1}{2} \)-BPS Wilson loop duality:

\[
W_1(N_1, N_2; n) = (-1)^n W_2(\tilde{N}_2, N_1; n). \quad (4.57)
\]

This thereby proves the \( \frac{1}{6} \)-BPS Wilson loop duality (4.2).

### 4.3. The simplest example—the \( U(1)_k \times U(N)_{-k} \) theory

It is illustrative to work out the simplest case, the duality between the \( U(1)_k \times U(N)_{-k} \) and \( U(2 + k - N)_k \times U(1)_{-k} \) ABJ theories, since the integral is 1D and the integration can be explicitly carried out. Apart from its simplicity, \( U(1)_k \times U(N)_{-k} \) may also be relevant to the study of Vasiliev’s higher spin theory with \( U(1) \) symmetry in the ’t Hooft limit, \( N, k \to \infty \) with \( N/k \) fixed [7]. Here we perform the integrals analytically and explicitly check Seiberg duality for some small values of \( N, k, \) and \( n \). In this connection, in the simplest case of \( N = 1 \), a check against the direct integral is provided for arbitrary \( k \) and \( n \) in Appendix E.
Fig. 7. The integration contour: Shown is a deformed contour where $C_2$ is shifted to the left from the vertical line running from $s = \frac{k}{2} - n - (N - 1) + i \infty - \epsilon$ to $\frac{k}{2} - n - (N - 1) - i \infty - \epsilon$, exploiting the absence of poles on the strip between $s = k - N$ and $\frac{k}{2} - n - (N - 1)$.

4.3.1. The flavor Wilson loop

We first consider the duality of flavor Wilson loops that are $\frac{1}{6}$-BPS Wilson loops on the “flavor group” $U(1)$. The $\frac{1}{6}$-BPS flavor Wilson loops are given by

$$W^{I}_{\frac{1}{6}}(1, N; n)_k = q^{-\frac{1}{2}n^2+n} I(1, N; n)_k I(1, N; 0)_k, \quad (4.58)$$

$$W^{II}_{\frac{1}{6}}(\tilde{N}, 1; n)_k = W^{I}_{\frac{1}{6}}(1, \tilde{N}; n)_k = q^{\frac{n^2}{2} - n} I(1, \tilde{N}; n - k) I(1, \tilde{N}; 0 - k), \quad (4.59)$$

where the dual gauge group $\tilde{N} = 2 + k - N$ and the integral expression takes the form

$$I(1, N; n)_k = \frac{(-1)^{N-1} q^{n-1}}{1 + q^n} - \frac{1}{2\pi i} \int_{C} \frac{\pi ds}{\sin(\pi s)} q^{-ns} \frac{(q^{s+1})^{N-1}}{(-q^{s+1+n})^{N-1}}. \quad (4.60)$$

The function $I(1, N; 0)_k$ appearing in the denominators is essentially the partition function and has the following property under Seiberg duality [17,39]:

$$I(1, N; 0)_k = i^{-k} I(1, \tilde{N}; 0 - k). \quad (4.61)$$

Meanwhile, the flavor Wilson loop duality is given by (2.21)

$$W^{I}_{\frac{1}{6}}(1, N; n)_k = W^{II}_{\frac{1}{6}}(\tilde{N}, 1; n)_k, \quad (4.62)$$

which implies

$$I(1, N; n)_k = i^{-k} q^{n^2-2n} I(1, \tilde{N}; n - k). \quad (4.63)$$

This generalizes the relation (4.61) for the partition function; we are going to check this relation explicitly for some small values of $N$, $k$, and $n$.

We first calculate the odd-$k$ case, since it is simpler than the even-$k$ case. The integrand flips the sign under the shift of integration variable, $s \rightarrow s + k$, in this case. Thanks to this property, the integrals can be evaluated by considering the closed contour depicted in Fig. 7: Let us denote the original contour by $C$ and the closed contour by $C'$, which consists of $C + C_1 + C_2 + C_3$ where $C_2$ is parallel
to \( C \) and shifted by \( k \). It is easy to see that there are no contributions from the contours \( C_1 \) and \( C_3 \), and the integral along \( C_2 \) is precisely the same as that along the original contour \( C \) because the integrand only flips the sign under the shift \( s \rightarrow s + k \). It is then clear that the integral along the closed contour \( C' \) is twice that along \( C \):

\[
\int_C (\cdots) = \frac{1}{2} \int_{C'} (\cdots). \tag{4.64}
\]

Thus the integrals can be evaluated by residue calculations. It is straightforward to carry out the calculation and we find that

\[
I(1, N; n)_{\text{odd } k} = \frac{(-1)^{N-1} q^{-n}}{2(1 + q^n)} \left[ \sum_{s=0}^{k-N} (-1)^s q^{-ns} \frac{(q^{s+1})_{N-1}}{(-q^{s+1+n})_{N-1}} \right.
\]

\[
+ \left. \frac{i k}{2} \sum_{c=1}^{N-1} (-1)^{c+1} q^{c(n+c)} \prod_{b=1, b \neq c}^{N-1} \frac{(-q^{b-c-n})_{1}}{(q^{b-c})_{1}} \right], \tag{4.65}
\]

where the first term is the contribution from the P poles at \( s = 0, 1, \ldots, k - N \), while the second term is from the NP poles at \( s = \frac{k}{2} - n - b, (b = 1, \ldots, N - 1) \).

The even-\( k \) case requires more considerations. In contrast to the odd-\( k \) case, the integrand is periodic under the shift \( s \rightarrow s + k \). In addition, some of the P and NP poles merge into double poles, since \( \frac{k}{2} \) is an integer. We can, however, apply a similar trick to that used in Refs. \[17,56\]. For an illustration of this trick, let us consider a generic integral of the form \( \int_C ds \ f(s) \) with \( f(s) \) being periodic under the shift \( s \rightarrow s + k \). The trick is instead to consider the integrand \( g(s) = f(s)(s + a) \) with \( a \) being an arbitrary constant. This integrand shifts as \( g(s + k) = g(s) + kf(s) \) when \( s \) is shifted by \( k \). Thus the integral \( \int_{C'} dsg(s) \) along the closed contour \( C' = C + C_1 + C_2 + C_3 \) yields \( k \int_C dsf(s) \), provided that there are no contributions from \( C_1 \) and \( C_3 \). This way the even-\( k \) case can also be evaluated by residue calculations. Note that the result does not depend on the choice of an arbitrary constant \( a \).

In the following we only show the case when \( \frac{k}{2} - (N - 1) - n > 0 \). The other case, \( \frac{k}{2} - (N - 1) - n < 0 \), however, can be easily derived in a similar manner. The poles encircled by the contour appear at

\[
s = \begin{cases} 
0, \ldots, \frac{k}{2} - N - n & \text{(simple)} \\
\frac{k}{2} - (N - 1) - n, \ldots, \frac{k}{2} - 1 - n & \text{(double).} \\
\frac{k}{2} - n, \ldots, k - N & \text{(simple)}
\end{cases} \tag{4.66}
\]

The residue evaluation then yields

\[
I(1, N; n)_{\text{even } k} = -\frac{(-1)^{N-1} q^{-n}}{1 + q^n} \left[ \sum_{s=0}^{\frac{k}{2} - N - n} (-1)^s q^{-ns} \frac{(q^{s+1})_{N-1}}{(-q^{s+1+n})_{N-1}}(s + a) \right.
\]

\[
+ \left. \sum_{b=1, \ldots, N-1} \lim_{s \to \frac{k}{2} - b - n} \frac{d}{ds} \left( s - \left( \frac{k}{2} - b - n \right) \right)^{2 \pi q^{-ns}} \frac{(q^{s+1})_{N-1}}{\sin \pi s (-q^{s+1+n})_{N-1}}(s + a) \right], \tag{4.67}
\]
where the first line is the contribution from simple poles and the second line that from double poles. These simple poles are a subgroup of P poles in (3.25), while the double poles are composed of P and NP poles, i.e., those P poles in (3.25) coalescing NP poles in (3.26).

On the dual side, the integral \( I(1, \tilde{N}; n)_{-odd_k} \) can be calculated similarly as

\[
I(1, \tilde{N}; n)_{-odd_k} = \frac{q^{n(\tilde{N}+1)}}{2(1+q^n)} \left[ \sum_{s=0}^{k-\tilde{N}} (-1)^s q^{ns} \frac{(q^{s+1})_{\tilde{N}-1}}{(-q^{s+1+n})_{\tilde{N}-1}} \right.
\]

\[
- \frac{ik}{2} \sum_{b=1}^{\tilde{N}-1} (-1)^{\tilde{N}-1+b} q^{-bn} \prod_{a=1, a \neq b}^{\tilde{N}-1} \frac{(-q^{a-b-n})_{1}}{(q^{a-b})_{1}} \].
\]

(4.68)

In the even-\( k \) case, we only show the result for the case when \( \frac{k}{2} - (\tilde{N} - 1) - n > 0 \):

\[
I(1, \tilde{N}; n)_{even_k} = \frac{q^{n(\tilde{N}+1)}}{2k(1+q^n)} \left[ \sum_{s=0, \ldots, \frac{k}{2}-\tilde{N}-n} \sum_{\frac{k}{2}-n, \ldots, k-\tilde{N}} (-1)^s q^{ns} \frac{(q^{s+1})_{\tilde{N}-1}}{(-q^{s+1+n})_{\tilde{N}-1}} (s+a) \right.
\]

\[
+ \sum_{s=\frac{k}{2}-\tilde{N}-n+1, \ldots, \frac{k}{2}-n-1} \lim_{s' \to s} \frac{d}{ds'} \left\{ \frac{\pi(s'-s)^2}{\sin(\pi s')} \frac{q^{n(s')}}{(-q^{s'+1+n})_{\tilde{N}-1}} (s'+a) \right\} \].
\]

(4.69)

We have now collected the necessary data to explicitly check the flavor Wilson loop duality.

- **Numerical checks**

  Let us check (4.63) explicitly for the duality \( U(1)_5 \times U(2)_{-5} = U(5)_5 \times U(1)_{-5} \) with winding \( n = 2 \). The results are

\[
I(1, 2; 2)_5 = (1.46 + 0.47i) + (0.90 - 0.29i) + (0.45 - 0.62i) + (-0.58i)
\]

\[
+ (-0.73 + 1.01i), \quad \text{P}
\]

\[
i^{-5}I(1, 5; 2)_{-5} = (-0.73 + 1.01i)
\]

\[
+ (1.46 + 0.47i) + (0.90 - 0.29i) + (0.45 - 0.62i) + (-0.58i). \quad \text{NP}
\]

(4.70)

(4.71)

Both in the original and dual theories, the first line is the perturbative contribution, while the second line is the nonperturbative contribution. We can explicitly see that the perturbative and nonperturbative contributions are exchanged under Seiberg duality.
Next we consider the even-\(k\) case, \(U(1)_4 \times U(2)_{-4} = U(4)_{-4} \times U(1)_{-4}\) with winding \(n = 1\). For the original theory we have

\[
I(1, 2; 1)_4 = \frac{i - 1}{16} \left[ \frac{(4 - 4i) + (3 + i)a\pi}{2\pi_{\text{P+NP}}} + (-1 - i)(1 + a) \left( \frac{i - 1}{2} \right) (2 + a) \right]
\]

\[
= \frac{i - 1}{16} \left[ -2 + \frac{2 - 2i}{\pi} \right]
\]

(4.72)

where the first term in the first line comes from the double pole at \(s = 0\) and the rest are from the simple P poles at \(s = 1, 2\). Note that the \(a\)-dependence is canceled out, as it should. For the dual theory we have

\[
iI(1, 4, 1)_{-4} = \frac{i - 1}{16} \left[ \frac{(4 - 4i) + (3 + i)(-a')\pi}{2\pi_{\text{P+NP}}} + (-1 - i)(1 - a') \left( \frac{i - 1}{2} \right) (2 - a') \right]
\]

\[
= \frac{i - 1}{16} \left[ -2 + \frac{2 - 2i}{\pi} \right]
\]

(4.74)

where the first term in the first line comes from the double pole at \(s = 0\) and the rest are from the simple NP poles at \(s = -1, -2\). The \(a'\)-dependence is canceled out similar to the previous case. However, we observe that the pole-by-pole maps agree if we identify \(a = -a'\) even though the constants \(a\) and \(a'\) do not seem to carry any physical meaning. As indicated, the P and NP poles are exchanged in the original (4.72) and dual (4.74) Wilson loops. These examples provide evidence for the flavor Wilson loop duality (2.21).

4.3.2. The \(\frac{1}{2}\)-BPS Wilson loop

We next turn to the \(\frac{1}{2}\)-BPS Wilson loops with winding \(n\) (2.19):

\[
W_1(1, N; n)_k = (-1)^{n+1} q^{\frac{n^2}{2} - n} \frac{I^{(1)}(N, 1; n)_{-k}}{I^{(2)}(N, 1; 0)_{-k}},
\]

(4.76)

\[
W_2(\tilde{N}, 1; n)_k = q^{-\frac{n^2}{2} + n} \frac{I^{(1)}(\tilde{N}, 1; n)_k}{I^{(2)}(\tilde{N}, 1; 0)_k}
\]

(4.77)

where the integral expression takes the form

\[
I^{(1)}(N, 1; n)_k = \sum_{c=0}^{n-1} q^{n(2c-N+1)} \frac{(q^{1-n})_c(q^{1+n})_{N-2-c}}{(q)_c(q)_{N-2-c}} \left( \begin{array}{l}
\pi ds
\end{array} \right) \sin(\pi s) \frac{(q^{s+1})_{N-1}}{2(-q^{s+1})_{N-1}}
\]

\[
\times \left( \frac{q^{s+1+c}}{q^{s+1+c+n}} \right)_1 \left( \frac{q^{s+1+c+n}}{q^{s+1+c+n}} \right)_1.
\]

(4.78)

The normalization factor has the relations \(I^{(2)}(N, 1; 0)_{-k} = (-1)^{N-1} I^{(2)}(N, 1; 0)_k = I(1, N; 0)_k\) and obeys the duality relation

\[
I^{(2)}(N, 1; 0)_{-k} = i^{-k} I^{(2)}(\tilde{N}, 1; 0)_k.
\]

(4.79)
Meanwhile, the $\frac{1}{2}$-BPS Wilson loop duality is given by (4.4)

$$W_{\frac{1}{2}}(1, N; n)_k = (-1)^n W_{\frac{1}{2}}(\tilde{N}, 1; n)_k$$

(4.80)

that, together with (4.79), implies

$$I^{(1)}(N, 1, n)_k = -i^{-k} q^{-n^2 + 2n} I^{(1)}(\tilde{N}, 1; n)_k.$$  

(4.81)

This is the relation we are going to check explicitly for some small values of $N, k,$ and $n$.

Similar to the previous case, the integrand for odd $k$ is anti-periodic under the shift $s \to s + k$, whereas it is periodic for even $k$. In the latter case, some of the P and NP poles merge into double poles. We can thus apply the same technique to that used in the previous case to this case.

In the odd-$k$ case, the poles encircled by the contour appear at

$$P: \quad s = -1 - c; \quad 0, 1, \ldots, -2 - c + n; \quad -c + n, \ldots, k - N \mod k$$

(4.82)

$$NP: \quad s = \frac{k}{2} - b + n\delta_{b,1+c} \mod k \quad (b = 1, \ldots, N - 1)$$

(4.83)

for $I^{(1)}(N, 1; n)_k$ and a given $c$ and

$$P: \quad s = -n + c; \quad 0, \ldots, c - 1; \quad c + 2, \ldots, N - 2 \mod k$$

(4.84)

$$NP: \quad s = \frac{k}{2} - b - n\delta_{b,n-c} \mod k \quad (b = 1, \ldots, k - N + 1)$$

(4.85)

for $I^{(1)}(\tilde{N}, 1; n)_k$ and a given $\tilde{c} = (n - 1) - c$. As discussed, the change of the integration variable, $\tilde{s} = -s + \frac{k}{2} - 1$, precisely exchanges P and NP poles in the two theories. It is straightforward to carry out the residue integrals and we find

$$I^{(1)}(N, 1; n)_{\pm \text{odd } k} = \frac{1}{2} \sum_{c=0}^{n-1} \left( \sum_{s=-c}^{k-N} (-1)^s q^{(2c-N+1)} \frac{(q^{1-n})_c (q^{1+n})_{N-2-c}}{(q)_c (q^N_{N-2-c})} \right)$$

$$\times \frac{(q^{s+1})_1 (q^{s+1+c})_1}{2(q^{s+1+c})_1 (q^{s+1+c-n})_1} + \left( (-1)^{c+1} q^{n(c-N+1)} \frac{(q^{a-c})_{N-1} (q^n_1)}{(-q^{-c})_{N-1} (q^n_1)} \right)$$

$$\sum_{b=1}^{N-1} \frac{1}{\mp k} (-1)^\frac{k+i+b-n\delta_{b,1+c}}{2} q^{(2C-N+1)} (q^{1-n})_c (q^{1+n})_{N-2-c} \prod_{a=1}^{N-1} \frac{(-q^{a-b+n(\delta_{b,1+c}-\delta_{b,1+c})+1})_1}{\prod_{a=1}^{N-1} (q^{a-b+n(\delta_{b,1+c}-\delta_{b,1+c})+1})_1}$$

(4.86)

where $q = e^{\mp 2\pi i / k}$ for $\mp k$ with an abuse of notation. The first two lines are the contributions from P poles and the last line is those from NP poles.

In the even-$k$ case we only show the case when $\frac{k}{2} \geq N - 1 + n$. The other case can be calculated in a similar manner. The poles encircled by the contour appear at

$$s = \begin{cases} 
-1 - c; & 0, \ldots, \frac{k}{2} - N \\
\frac{k}{2} - c + n, \ldots, k - N \\
\frac{k}{2} - (N - 1), \ldots, -2 - c - n; & -c + n, \ldots, \frac{k}{2} - 1 - c + n 
\end{cases}$$

(4.87)
Similar to the previous case (4.67), we find that

\[ I^{(1)}(N, 1; n)_{\text{even}} = \frac{-1}{2k} \sum_{c=0}^{n-1} \sum_{s=0, \frac{k}{2} + 1, \ldots, k-N-\frac{1}{2}} (-1)^s q^{n(2c-N+1)} \frac{q^{1-n}s}{(q) c(q) N-2-c} \frac{(q^{1+s}) N-1}{2(-q^{s+1}) N-1} \]

\[ \times \left( \frac{(-q^{s+1}) (q^{s+1}+n)}{(q^{s+1}+c-n) (q^{s+1}+c-n) N-1} \right)(s+a) + \sum_{s=\frac{k}{2}-(N-1), \ldots, -c+n, -c+n, \ldots, -2-c+n} \lim_{s' \to s} \frac{d}{ds'} \left( (s' - s)^2 \pi \right) \frac{q^{n(2c-N+1)}}{\sin \pi s} \left( \frac{-q^{s+1}}{N-1} \right) \left( \frac{-q^{s+1}+c-n}{N-1} \right)(s+a) \right] \]

(4.88)

where \( a \) is an arbitrary constant.

- **A numerical check**

As an explicit check of (4.81), we consider the example of the duality \( U(1)_5 \times U(2)_{-5} = U(5)_5 \times U(1)_{-5} \) with winding \( n = 2 \). For the original theory we have

\[ I^{(1)}(2, 1; 2)_{-5} = \frac{1}{4} \left( \sum_{s=0, 2, 3, 4} (-1)^s q^{2} \frac{q^{1-s}}{(q^{1-s})} - 5iq^2 + \frac{2q(1-q)}{(1+q)(1+q^2)} \right) \]

(4.89)

\[ = \frac{-1}{4} \left[ (1.80 - 2.48i) + (-0.42 + 0.58i) + (0.42 - 0.58i) + (-1.80 + 2.48i) \right. \]

\[ + \frac{(2.93 - 4.04i) + (-2.35i)}{P} \]

(4.90)

where \( q = e^{-\frac{2\pi i}{5}} \). In (4.90) the first line is the contributions from P poles for \( c = 0 \) at \( s = -1, 0, 2, 3 \), the first term in the second line those from NP poles for \( c = 0 \) at \( s = \frac{k}{2} + 1 \), and the last term those from the P pole for \( c = 1 \) at \( s = -2 \). For the dual theory we have

\[ iI^{(1)}(5, 1; 2)_{5} = \frac{i}{4} \left( \frac{-q^{3}}{(q^{3})^2(q^3)} + 5iq^2 \frac{q^3}{(q^3)} + \sum_{b=0, 1, 3, 4} \frac{5i(q^3)^2}{2(q)} \frac{(-1)^b q^{2-b}b}{(-q^{2-b})_1} \frac{\prod_{a=0}^{5}(-q^{a-b}b)_1}{\prod_{a=0}^{5}(-q^{a-b}b)_1} \right) \]

(4.91)

\[ = 1 \left[ \frac{2.93 - 4.04i}{{P}} + \frac{(-2.35i)}{NP} \right. \]

\[ + \frac{(1.80 - 2.48i) + (-0.42 + 0.58i) + (0.42 - 0.58i) + (-1.80 + 2.48i)}{NP} \left. ] \right) \]

(4.92)

The first term is the contributions from P poles for \( c = 1 \) at \( s = -2 \), the second term those from the NP pole for \( c = 0 \) at \( s = \frac{k}{2} + 1 \), and the second line those from NP poles for \( c = 1 \) at
s = \frac{k}{2} - b(b = 0, 1, 3, 4). It is clear that in (4.90) and (4.92) the P and NP poles are interchanged under the duality as expected. Thus this provides evidence for the $\frac{1}{2}$-BPS Wilson loop duality (4.4). By deduction, this result, combined with the flavor Wilson loop duality between (4.70) and (4.71), also constitutes evidence for the $\frac{1}{6}$-BPS Wilson loop duality (4.2).

5. Summary and discussions

In the current paper, we discussed the Wilson loops of the ABJ theory and studied their properties, generalizing the techniques developed in Ref. [17] for the partition function. In more detail, the objects of our interest were the circular $\frac{1}{6}$- and $\frac{1}{2}$-BPS Wilson loops with winding number $n$ in the $U(N_1)_k \times U(N_2)_{-k}$ ABJ theory on $S^3$. By the localization technique, the Wilson loop can be represented as an ordinary integral with $N_1$ variables $\mu_i$ and $N_2$ variables $\nu_{ij}$, corresponding to the eigenvalues of $U(N_1)$ and $U(N_2)$ adjoint matrices. Rather than directly evaluating this ABJ matrix integral, we followed Ref. [17] and started instead with the Wilson loop in the lens space matrix integral, which is related to the ABJ one by the analytic continuation $N_2 \rightarrow -N_2$. Because the lens space Wilson loop can be computed exactly, by continuing it back to the ABJ theory by setting $N_2 \rightarrow -N_2$, we arrived at an infinite sum expression for the ABJ Wilson loop. Actually, this infinite sum is only formal and does not converge. However, by means of a Sommerfeld–Watson transformation, we turned it into a convergent integral of $\min(N_1, N_2)$ variables and successfully obtained the “mirror” description of the ABJ Wilson loop, generalizing that for the partition function. The final expressions are given in (2.12) for the $\frac{1}{6}$-BPS Wilson loop with $N_1 \geq N_2$, in (2.15) for the $\frac{1}{6}$-BPS Wilson loop with $N_1 \leq N_2$, and in (2.19) for the $\frac{1}{2}$-BPS Wilson loop.

The ABJ theory is conjectured to possess a Seiberg-like duality, given in (1.1). Based on the integral expressions for the ABJ Wilson loops, we showed that the Wilson loops have a nontrivial transformation rule, given by (2.20)–(2.22). This result is consistent with the result by Kapustin and Willett [50] and slightly generalized to the case where the flavor group is gauged. We also presented a heuristic explanation of the Seiberg-like duality based on the brane construction of the ABJ theory, followed by a rigorous proof based on the integral representation of Wilson loops. The brane picture is heuristic but quite powerful and can be used to predict the duality rule for Wilson loops with general representations. We also presented another derivation of the duality in Appendix F.

Our method to start from the lens space theory and analytically continue it to ABJ theory involves subtleties associated with a Sommerfeld–Watson transformation to rewrite a divergent sum in terms of a well defined contour integral. In particular, this rewriting has possible ambiguities in the choice of integration contours, including the $\epsilon$-prescription for the parameter $M$. It is necessary to keep $M$ slightly away from an integral value by the shift $M \rightarrow M + \epsilon$ with $\epsilon > 0$ in the course of the calculations. In sync with this shift, the contour $C_1[\epsilon]$ has to be placed between $s = -1 - c$ and $-1 - c - \epsilon$ so as to avoid the pole at $s = -1 - c - \epsilon$. Although the choice we made is well motivated by the continuity in $k$ and necessary to obtain sensible results, and Seiberg duality provides strong evidence in support for it, a direct derivation is certainly desirable. The approaches taken in Refs. [15,22] presumably provide promising directions for that purpose.

Actually, however, this weakness of our approach can be turned around and regarded as its strength. The infinite sum we encounter in the intermediate stage can be understood as giving a perturbative expansion of a gauge theory quantity that, by itself, is incomplete and divergent. Rewriting it in terms of a finite contour integral can be thought of as supplementing it with nonperturbative corrections to make it well defined and complete; more precisely, summing over P poles corresponds to summing up perturbative expansion and including NP poles corresponds to adding nonperturbative corrections.
We emphasize that it is very rare that we can carry out this nonperturbative completion in nontrivial field theories and ABJ Wilson loops provide explicit and highly nontrivial examples for it.

In Ref. [13], the partition function of ABJM theory was evaluated using the Fermi gas approach in detail and a cancellation mechanism was found between nonperturbative contributions. Namely, for certain values of $k$, the contribution from worldsheet instantons diverges but, when that happens, the contribution from membrane instantons also diverges and they cancel each other to produce a finite result. This was generalized to $1/2$-BPS Wilson loops in ABJM theory in Ref. [48] and to ABJ theory in Refs. [15,16]. This phenomenon is reminiscent of what is happening in our formulation, in which the partition function and Wilson loops are expressed as contour integrals. The integrals can be evaluated by summing over the residue of P and NP poles, which are generically simple poles. As we change $k$ continuously, at some integral values of $k$, two such simple poles can collide and become a double pole. For this to happen, the residue of each simple pole must diverge but their sum must remain finite.\footnote{For example, consider $\frac{1}{2\epsilon} \left( \frac{1}{x+\epsilon} - \frac{1}{x-\epsilon} \right) = \frac{1}{x^2 - \epsilon^2}$ where $\epsilon \to 0$.} It is reasonable to conjecture that this cancellation of residues is closely related to the cancellation mechanism of Ref. [13]. We leave this fascinating possibility for future study.

We observed that the $1/6$-BPS Wilson loop diverges for $n \geq \frac{k}{2}$. Actually, because the lens space Wilson loop (3.5) is invariant under $n \to n + k$, we can define the analytically continued $1/6$-BPS ABJ Wilson loops to have this periodicity in $n$ as $n \cong n + k$. Then the $1/6$-BPS ABJ Wilson loop is divergent only for $n = \frac{k}{2} \mod k$, which can be checked in explicit expressions such as (E3). On the other hand, the $1/2$-BPS Wilson loop is finite for all $n$. In the type IIA bulk dual, in $\text{AdS}_4 \times \mathbb{CP}^3$, the $1/2$-BPS Wilson loop corresponds to a fundamental string extending along $\text{AdS}_2$ inside $\text{AdS}_4$ and sitting at a point inside $\mathbb{CP}^3$ [41–43]. There is no problem having $n$ such fundamental strings, which must correspond to the $1/2$-BPS Wilson loop with arbitrary winding $n$. On the other hand, for generic $n$, the $1/2$-BPS Wilson loop has been argued to correspond to smearing the above fundamental string over $\mathbb{CP}^1 \subset \mathbb{CP}^3$ [41], which seems a bit unnatural for an object as fundamental as a Wilson loop. However, particularly for $n = \frac{k}{2} \mod k$, there is a $1/6$-BPS configuration in which a D2-brane is along $S^1 \subset \mathbb{CP}^3$ and carries fundamental string charge dissolved in worldvolume flux [41]. So, it is tempting to conjecture that, for $n \neq \frac{k}{2} \mod k$, there is some different configuration dual to the $1/6$-BPS Wilson loop, which becomes the D2-brane configuration at $n = \frac{k}{2}$. The divergence is presumably related to this phase transition. It would be interesting to actually find such a brane configuration.

In Ref. [7], it was conjectured that the $U(N_1)_k \times U(N_2)_{-k}$ ABJ theory in the fixed $N_1$, large $N_2$, $k$ limit is dual to the $\mathcal{N} = 6$ supersymmetric, parity-violating version of the Vasiliev higher spin theory (where we assumed $N_1 \ll N_2$). Being based on an $N_1$-dimensional integral, our formulation is particularly suited to studying this limit. So, it is very interesting to use our results to evaluate Wilson loops in the higher spin limit and compare them with predictions from the Vasiliev side. It is also interesting to see if our approach can be applied to more general CSM theories with less supersymmetry, such as the necklace quiver [57].

Acknowledgements

We would like to thank Nadav Drukker, Masazumi Honda, Sanefumi Moriyama, Kazutoshi Ohta, and Kazumi Okuyama for useful discussions. S.M. thanks the IPht, CEA-Saclay, where part of this work was done, for hospitality. H.S. would like to thank YITP, Nagoya University and KIAS, where part of this work was done, for their hospitality. The work of S.M. was supported in part by Grant-in-Aid for Young Scientists (B) 24740159 from the Japan Society for the Promotion of Science (JSPS).
Appendix A.  The $q$-analogs

The results in the main text are given in terms of $q$-Pochhammer symbols. In this appendix we provide the definitions and some useful formulas and properties of related quantities. Roughly, a $q$-analog is a generalization of a quantity to include a new parameter $q$, such that it reduces to the original version in the $q \to 1$ limit. In this appendix, we will summarize definitions of various $q$-analogs and their properties relevant to the main text.

- **$q$-number**: For $z \in \mathbb{C}$, the $q$-number of $z$ is defined by
  \begin{equation}
  [z]_q := \frac{1 - q^z}{1 - q}.
  \end{equation}

- **$q$-Pochhammer symbol**: For $a \in \mathbb{C}$, $n \in \mathbb{Z}_{\geq 0}$, the $q$-Pochhammer symbol $(a; q)_n$ is defined by
  \begin{equation}
  (a; q)_n := \prod_{k=0}^{n-1}(1 - aq^k) = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}) = \frac{(a; q)_\infty}{(aq^n; q)_\infty}.
  \end{equation}

For $z \in \mathbb{C}$, $(a; q)_z$ is defined by the last expression:
\begin{equation}
(a; q)_z := \frac{(a; q)_\infty}{(aq^z; q)_\infty} = \prod_{k=0}^{\infty} \frac{1 - aq^k}{1 - aq^{z+k}}.
\end{equation}
This in particular means
\begin{equation}
(a; q)_{-z} = \frac{1}{(aq^{-z}; q)_z}.
\end{equation}

For $n \in \mathbb{Z}_{\geq 0}$,
\begin{equation}
(a; q)_{-n} = \frac{1}{(aq^{-n}; q)_n} = \prod_{k=0}^{n} \frac{1}{1 - a/q^k}.
\end{equation}

Note that the $q \to 1$ limit of the $q$-Pochhammer symbol is not the usual Pochhammer symbol but only up to factors of $(1 - q)$:
\begin{equation}
\lim_{q \to 1} (a^q; q)_n = a(a + 1) \cdots (a + n - 1).
\end{equation}

We often omit the base $q$ and simply write $(a; q)_v$ as $(a)_v$.\textsuperscript{26}

Some useful relations involving $q$-Pochhammer symbols are
\begin{align}
(a)_v &= \frac{(a)_z}{(aq^v)_z-v} = (a)_z(q^z)_v, \\
(q)_v &= (1-q)^v \Gamma_q(v+1), \\
(q^\mu)_v &= \frac{(q^\mu)_\mu v}{(q^\mu)_{\mu-1}} = (1-q)^v \frac{\Gamma_q(\mu+v)}{\Gamma_q(\mu)}, \\
(aq^\mu)_v &= (aq^\mu)_z \mu (aq^z)_{\mu+v-z} = \frac{(aq^\mu)_z}{(aq^\mu)_z-v} = \frac{(aq^z)_{\mu+v-z}}{(aq^z)_{\mu-z}},
\end{align}

\textsuperscript{26} We will not use the symbol $(a)_v$ to denote the usual Pochhammer symbol.
where $\mu, \nu, z \in \mathbb{C}$ and $\Gamma_q(z)$ is the $q$-Gamma function defined below. For $n \in \mathbb{Z}$, we have the following formulas, which “reverse” the order of the product in the $q$-Pochhammer symbol:

$$
(aq^n)_n = (-a)^n q^{n+\frac{1}{2}n(n-1)}(a^{-1}q^{1-n-z})_n, \quad (A11)
$$

$$
(±q^{-n})_n = (±1)^n q^{-\frac{1}{2}n(n+1)}(±q)_n. \quad (A12)
$$

If $n = n + \epsilon$ with $|\epsilon| \ll 1$, the correction to this is of order $\mathcal{O}(\epsilon)$:

$$
(aq^n)_{n+\epsilon} = (-a)^n q^{n+\frac{1}{2}n(n-1)}(a^{-1}q^{1-n-z})_n(1 + \mathcal{O}(\epsilon)), \quad a \neq 1. \quad (A13)
$$

Here we assumed that $a \neq 1$ and $a - 1 \gg \mathcal{O}(\epsilon)$.

- **$q$-factorials:** For $n \in \mathbb{Z}_{\geq 0}$, the $q$-factorial is given by

$$
[n]_q! := [1]_q[2]_q \cdots [n]_q = \frac{(q)_n}{(1-q)^n}, \quad [0]_q! = 1, \quad [n+1]_q! = [n]_q[n-1]_q!. \quad (A14)
$$

- **$q$-Gamma function:** For $z \in \mathbb{C}$, the $q$-Gamma function $\Gamma_q(z)$ is defined by

$$
\Gamma_q(z) := (1-q)^{-z} \prod_{k=1}^{\infty} \frac{1-q^k}{1-q^{z+k}}. \quad (A15)
$$

The $q$-Gamma function satisfies the following relations:

$$
\Gamma_q(z) = (1-q)^{1-z} \frac{(q)_\infty}{(q^z)_\infty} = (1-q)^{1-z} (q)_z^{-1}, \quad (A16)
$$

$$
\Gamma_q(z+1) = [z]_q \Gamma_q(z), \quad (A17)
$$

$$
\Gamma_q(1) = \Gamma_q(2) = 1, \quad \Gamma_q(n) = [n-1]_q! \quad (n \geq 1). \quad (A18)
$$

The behavior of $\Gamma_q(z)$ near non-positive integers is

$$
\Gamma_q(-n+\epsilon) = \frac{(-1)^n q^{\frac{1}{2}n(n+1)}}{\Gamma_q(n+1) \log q} \frac{1}{\epsilon} + \cdots, \quad \Gamma_q(n+1) = [n]_q!, \quad (A19)
$$

where $n \in \mathbb{Z}_{\geq 0}$, and $\epsilon \to 0$. As $q \to 1$, this reduces to the formula for the ordinary $\Gamma(z)$,

$$
\Gamma(-n+\epsilon) = \frac{(-1)^n}{\Gamma(n+1) \epsilon} + \cdots, \quad \Gamma(n+1) = n!. \quad (A20)
$$

- **$q$-Barnes $G$ function:** For $z \in \mathbb{C}$, the $q$-Barnes $G$ function is defined by [58]

$$
G_2(z+1; q) := (1-q)^{-\frac{1}{2}z(z-1)} \prod_{k=1}^{\infty} \left[ \frac{1-q^{z+k}}{1-q^k} \right]^{k} (1-q^k)^{z}. \quad (A21)
$$

Some of its properties are

$$
G_2(1; q) = 1, \quad G_2(z+1; q) = \Gamma_q(z)G_2(z). \quad (A22)
$$

$$
G_2(n; q) = \prod_{k=1}^{n-1} \Gamma_q(k) = \prod_{k=1}^{n-2} [k]_q! = (1-q)^{-\frac{1}{2}n(n-1)} \prod_{j=1}^{n-2} (q)_j = \prod_{k=1}^{n-2} [k]_q^k, \quad (A23)
$$

$$
\prod_{1 \leq A < B \leq n} (q^A - q^B) = q^{\frac{1}{2}n^2(n^2-1)}(1-q)^{\frac{1}{2}n(n-1)}G_2(n+1; q). \quad (A24)
$$

39/62
The behavior of \( G_2(z; q) \) near non-positive integers is
\[
G_2(-n + \epsilon; q) = \frac{(-1)^{\frac{1}{2}(n+1)(n+2)} G_2(n+2; q)}{q^{\frac{1}{2}(n+1)(n+2)}(1-q)^n} e^{n+1} + \cdots ,
\]
(\text{A25})
where \( n \in \mathbb{Z}_{\geq 0} \), and \( \epsilon \to 0 \). As \( q \to 1 \), this reduces to the formula for the ordinary \( G_2(z) \),
\[
G_2(-n + \epsilon) = (-1)^{\frac{1}{2}n(n+1)} G_2(n+2) e^{n+1} + \cdots .
\]
(\text{A26})

We provide computational details of the derivation of (3.4). We first recall the definition of the lens space Wilson loop presented here is a streamlined version of that given for the partition function in Ref. [17].

Appendix B. The computational details

Some details of the calculations in the main text are given in this appendix. In particular, we provide relevant details in the calculation of the Wilson loops in the lens space matrix model and those of their analytic continuation to the Wilson loops in the ABJ(M) matrix model. The analytic continuation presented here is a streamlined version of that given for the partition function in Ref. [17].

B.1. The calculation of the lens space Wilson loop

We provide computational details of the derivation of (3.4). We first recall the definition of the (unnormalized) Wilson loop in the lens space matrix model:
\[
\mathcal{W}^I_{\text{lens}}(N_1, N_2; n) := \left\langle \sum_{j=1}^{N_1} e^{n \mu_j} \right\rangle ,
\]
(\text{B1})
where we have defined the expectation value of \( \mathcal{O} \) by
\[
\langle \mathcal{O} \rangle := N_{\text{lens}} \int \prod_{i=1}^{N_1} \frac{d \mu_{ij}}{2\pi} \prod_{a=1}^{N_2} \frac{d v_a}{2\pi} \Delta_{\text{sh}}(\mu)^2 \Delta_{\text{sh}}(v)^2 \Delta_{\text{ch}}(\mu, v)^2 \mathcal{O} e^{-\frac{1}{2\pi} \left( \sum_{i=1}^{N_1} \mu_{ij}^2 + \sum_{a=1}^{N_2} v_a^2 \right)} .
\]
(\text{B2})
The integrals (\text{B1}) are actually Gaussian and can be performed exactly. To see this, it is convenient to shift the eigenvalues as
\[
\mu_i \to \mu_i - \frac{i\pi}{2}, \quad v_a \to v_a + \frac{i\pi}{2} .
\]
(\text{B3})
These yield
\[
\Delta_{\text{sh}}(\mu) \Delta_{\text{sh}}(v) \Delta_{\text{ch}}(\mu, v) = e^{-\frac{i\pi}{2} N_1 N_2 - \frac{N-1}{2} (\sum_{j=1}^{N_1} \mu_j + \sum_{a=1}^{N_2} v_a)} \Delta(\mu, v) ,
\]
(\text{B4})
where \( N := N_1 + N_2 \). The Vandermonde determinant \( \Delta(\mu, v) \) takes the following form and can be expanded as
\[
\Delta(\mu, v) := \prod_{j<k} (e^{\mu_j} - e^{\mu_k}) \prod_{a<b} (e^{v_a} - e^{v_b}) \prod_{j,a} (e^{\mu_j} - e^{v_a})
\]
\[
= \sum_{\sigma \in S_N} (-1)^\sigma e^{\sum_{j=1}^{N_1} (\sigma(j)-1) \mu_j + \sum_{a=1}^{N_2} (\sigma(N_1+a)-1) v_a} ,
\]
(\text{B5})
where \( S_N \) is the permutation group of length \( N \) and \( (-1)^\sigma \) is the signature of an element \( \sigma \in S_N \). Because each term in (\text{B5}) is an exponential whose exponent is linear in \( \mu_j, v_a \), the integrals (\text{B2})
are all Gaussian. To proceed, we define

$$W(j) := \int \prod_{i=1}^{N_1} \frac{d\mu_i}{2\pi} \prod_{a=1}^{N_2} \frac{dv_a}{2\pi} \Delta_{sh}(\mu_i^2) \Delta_{sh}(v_a^2) \Delta_{ch}(\mu_i, v_a^2) e^{i\mu_i} e^{-\frac{1}{2\pi} \left( \sum_{i=1}^{N_1} \mu_i^2 + \sum_{a=1}^{N_2} v_a^2 \right)}, \quad \text{(B6)}$$

so that $W^I_{\text{lens}}(N_1, N_2; n)_k = N_{\text{lens}} \sum_{j=1}^{N_1} W(j)$. By using the expansion of the Vandermonde determinants, this becomes

$$W(j) = e^{-i2N_1N_2} \int \prod_{i=1}^{N_1} \frac{d\mu_i}{2\pi} \prod_{a=1}^{N_2} \frac{dv_a}{2\pi} \prod_{i=1}^{N_1} \prod_{a=1}^{N_2} e^{i(\mu_i - iz_j)} - (N-1)(\mu_i + v_a) - z_j^2 \sum_{\sigma, \tau \in S_N} (-1)^{\sigma + \tau} \times e^{\sum_{A=1}^{N_1} (\sigma(A) + \tau(A) - 2)\mu_A + \sum_{B=1}^{N_2} \sigma(N_1 + B) + \tau(N_1 + B) - 2)v_B - \mu_A - v_B} \left[ \sum_{i=1}^{N_1} \mu_i^2 + \sum_{a=1}^{N_2} v_a^2 - iz_j \sum_{i=1}^{N_1} \mu_i - \sum_{a=1}^{N_2} v_a \right]. \quad \text{(B7)}$$

It is then straightforward to perform the Gaussian integrals and find that

$$W(j) = (-1)^{N_1N_2} e^{\frac{N_1^2}{2\pi} \sum_{\sigma, \tau \in S_N} (-1)^{\sigma + \tau} e^{\sum_{A=1}^{N_1} \frac{1}{2\pi} \left( \frac{g_s}{2} \sigma(A) \tau(A) + i\pi \sum_{A=1}^{N_1} (\sigma(A) + \tau(A)) + g_s n(\sigma(1) + \tau(1)) \right)} \times e^{-\frac{i\pi N_1}{2} \sum_{B=1}^{N_1} \frac{1}{2\pi} \left( \frac{g_s}{2} \sum_{A=1}^{N_1} \sigma(A) \tau(A) + i\pi \sum_{A=1}^{N_1} (\sigma(A) + \tau(A)) + g_s n(\sigma(1) + \tau(1)) \right)}.$$

where in the second equality we used $i^{-2N_1N_2} e^{-\frac{i\pi N_1}{2} (N+1)(N_1-N_2+N)} = 1$ and

$$\sum_{A=1}^{N_1} \sigma(A) = \sum_{A=1}^{N_1} \tau(A) = \frac{1}{2} N(N + 1), \quad \sum_{A=1}^{N_1} \sigma(A)^2 = \sum_{A=1}^{N_1} \tau(A)^2 = \frac{1}{6} N(N + 1)(2N + 1).$$

Notice that the sum over the permutation $\tau$ in (B8) is the determinant of the Vandermonde matrix

$$(\chi_A)^B := \begin{cases} (-q^{\sigma(A)+n\delta_{A\ell}})^B & (A = 1, \ldots, j, \ldots, N_1) \\ (q^{-\sigma(A)})^B & (A = N_1 + 1, \ldots, N) \end{cases}, \quad \text{(B9)}$$

where we introduced $q = e^{-g_s}$. Thus (B8) yields

$$W(j) = q^{-\frac{N_1^2}{2}} \sum_{\sigma, \tau \in S_N} (-1)^{\sigma + \tau} e^{\sum_{A=1}^{N_1} \frac{1}{2\pi} \left( \frac{g_s}{2} \sigma(A) \tau(A) + i\pi \sum_{A=1}^{N_1} (\sigma(A) + \tau(A)) + g_s n(\sigma(1) + \tau(1)) \right)} \times \left( -1 \right)^{\frac{N_1(N_1+1)}{2}} \prod_{1 \leq A < B \leq N_1} \left( q^{\sigma(A)+n\delta_{A\ell}} - q^{\sigma(B)+n\delta_{B\ell}} \right) \times \prod_{N_1+1 \leq A < B \leq N} \left( q^{\sigma(A)} - q^{\sigma(B)} \right) \prod_{A=1}^{N_1} \prod_{B=N_1+1}^{N} \left( q^{\sigma(B)} + q^{\sigma(A)+n\delta_{A\ell}} \right). \quad \text{(B10)}$$

We now rewrite the sum over the permutation $\sigma$ as the sum over the ways of partitioning $N$ numbers $\{1, 2, \ldots, N\}$ into two groups of ordered numbers $\{(C_1, C_2, \ldots, C_{N_1}), (D_1, D_2, \ldots, D_{N_2})\}$.
where $C_1 \leq C_2 \leq \cdots \leq C_{N_1}$ and $D_1 \leq D_2 \leq \cdots \leq D_{N_2}$. In rewriting, we start with the sequence of numbers $\{\sigma(1), \sigma(2), \ldots, \sigma(N)\}$ and then reorder it into

$$\{D_1, \ldots, D_{D_1}, C_1, D_{a_1+1}, \ldots, D_{a_2}, C_2, D_{a_2+1}, \ldots, D_{a_{N_1}}, C_{N_1}, D_{a_{N_1}+1}, \ldots, D_{N_2}\}, \quad (B11)$$

which is just a way of expressing $\{1, 2, \ldots, N\}$. This obviously yields the sign $(-1)^{\sigma}$. We further reorder it into the partition

$$\{C_1, \ldots, C_{N_1}, D_1, \ldots, D_{N_2}\}. \quad (B12)$$

We can find the sign picked up by this reordering as follows. We first move $C_1$ farthest to the left. This gives the sign $(-1)^{C_1-1}$. We next move $C_2$ next to and to the right of $C_1$. This gives the sign $(-1)^{C_2-2}$. In repeating this process, the move of $C_i$ picks up the sign $(-1)^{C_i-i}$. Thus the sign picked up in the end of all the moves is

$$(-1)^{\sigma}(-1)^{\sum_{i=1}^{N_1} C_i - \frac{1}{2} N_1(N_1+1)} = (-1)^{\sigma + \sum_{i=1}^{N_1} \sigma(A) - \frac{1}{2} N_1(N_1+1)}. \quad (B13)$$

This is exactly the same sign factor as in (B10) and is thus canceled out. Hence we obtain

$$W(j) = (N_1 - 1)!N_2! q^{-\frac{n^2}{2} + n} \left(\frac{g_s}{2\pi}\right)^N q^{-\frac{1}{2} N(N^2-1)} \sum_{l=1}^{N_1} \sum_{\{N_1, N_2\}} q^{-nC_l}$$

$$\times \prod_{1 \leq i < k \leq N_1} \left(q^{C_i+n\delta_{ii}} - q^{C_k+n\delta_{kl}}\right) \prod_{N_1+1 \leq a < b \leq N} \left(q^{D_a} - q^{D_b}\right)$$

$$\times \prod_{i=1}^{N_1} \prod_{a=N_1+1}^{N} \left(q^{C_i+n\delta_{ii}} + q^{D_a}\right). \quad (B14)$$

where $\{N_1, N_2\}$ is the partition that we have just discussed. The factor $(N_1 - 1)!N_2!$ arises for a fixed $l$ since there are so many numbers of the sequence that yield a given partition. Note that, as anticipated from the definition of the Wilson loop, this is $j$-independent and thus yields

$$W_{lens}(N_1, N_2; n)_k = i^{-\frac{N}{2}(N_1^2+N_2^2)} q^{-\frac{n^2}{2} + n} \left(\frac{g_s}{2\pi}\right)^N q^{-\frac{1}{2} N(N^2-1)} \sum_{l=1}^{N_1} \sum_{\{N_1, N_2\}} q^{-nC_l}$$

$$\times \prod_{1 \leq i < k \leq N_1} \left(q^{C_i+n\delta_{ii}} - q^{C_k+n\delta_{kl}}\right) \prod_{N_1+1 \leq a < b \leq N} \left(q^{D_a} - q^{D_b}\right)$$

$$\times \prod_{i=1}^{N_1} \prod_{a=N_1+1}^{N} \left(q^{C_i+n\delta_{ii}} + q^{D_a}\right). \quad (B15)$$

This agrees with the partition function in Ref. [17] (up to the difference in normalizations) when the winding $n = 0$. As we have done for the partition function, using (A27), we further rewrite the Wilson loop (B14) as a product of the $q$-Barnes $G$-function and a “generalization of multiple $q$-hypergeometric function”:

$$W_{lens}(N_1, N_2; n)_k = i^{-\frac{N}{2}(N_1^2+N_2^2)} q^{-\frac{n^2}{2} + n} \left(\frac{g_s}{2\pi}\right)^N q^{-\frac{N(N^2-1)}{6}}$$

$$\times (1-q)^{\frac{N(N-1)}{2}} G_2(N+1; q)S(N_1, N_2; n)_k. \quad (B16)$$
where the special function \( S_n(N_1, N_2) \) is defined by

\[
S(N_1, N_2; n)_k = \sum_{l=1}^{N_1} \sum_{1 \leq k < l \leq N_1} q^{-nC_l} \prod_{l<k\leq N_1} \frac{q^{C_l+n} - q^{C_k}}{q^{C_l} - q^{C_k}} \prod_{1 \leq i < l} \frac{q^{C_i} - q^{C_i+n}}{q^{C_i} - q^{C_i}} \\
\times \prod_{C_i < D_a} \frac{q^{C_i+n\delta_{il}} + q^{D_a}}{q^{C_i} - q^{D_a}} \prod_{C_i > D_a} \frac{q^{D_a} + q^{C_i+n\delta_{il}}}{q^{D_a} - q^{C_i}}.
\]

We now define the normalized Wilson loop by \( W^{\text{lens}}_1(N_1, N_2; n)_k := \frac{\nu^{\text{lens}}_1(N_1, N_2; n)_k}{\nu^{\text{lens}}_1(N_1, N_2; 0)_k} \), which takes the following simpler form:

\[
W^{\text{lens}}_1(N_1, N_2; n)_k = q^{\frac{2}{N} n} S(N_1, N_2; n)_k S(N_1, N_2; 0)_k^{-1}.
\]

In the rest of this appendix we massage (B17) into a more convenient form: Notice first that once the set of \( C_i \) is selected out of the numbers \( \{1, 2, \ldots, N\} \), the \( D_a \) simply fill in the rest of the numbers. Taking this fact into account, we can rewrite the special function (B17) as

\[
S(N_1, N_2; n)_k = \sum_{l=1}^{N_1} \sum_{1 \leq k < l \leq N_1} q^{-nC_l} \prod_{l<k\leq N_1} \frac{q^{C_l+n} - q^{C_k}}{q^{C_l} - q^{C_k}} \prod_{1 \leq i < l} \frac{q^{C_i} - q^{C_i+n}}{q^{C_i} - q^{C_i}} \\
\times \prod_{C_i < D_a} \frac{q^{C_i+n\delta_{il}} + q^{D_a}}{q^{C_i} - q^{D_a}} \prod_{C_i > D_a} \frac{q^{D_a} + q^{C_i+n\delta_{il}}}{q^{D_a} - q^{C_i}}.
\]

In going from (B19) to (B20), we passed the expression

\[
S(N_1, N_2; n)_k
\]

\[
= \sum_{l=1}^{N_1} \sum_{1 \leq k < l \leq N_1} q^{-nC_l} \prod_{k=1}^{l-1} \frac{1 - q^{C_k-C_l+n}}{1 - q^{C_k-C_l}} \\
\times \prod_{i=1}^{N_1-1} \left( \prod_{j=i}^{N_1-1} \frac{-q^{(C_j-C_l-n\delta_{il})+1}}{(q^{C_j-C_l})^{C_{j+1}-C_j-1}} \prod_{j=1}^{i-1} \frac{-q^{C_j-C_l+n\delta_{il}(1+n\delta_{il})+1}}{(q^{C_j-C_l})^{C_{j+1}-C_j-1}} \right) \\
\times \prod_{i=1}^{N_1} \left( \prod_{j=1}^{C_j-N_{C_{N_j}}} \frac{-q^{C_{N_j}+C_j-n\delta_{il}}}{(q^{C_{N_j}})^{C_{j+1}-C_j-1}} \prod_{b=1}^{N_{C_{N_j}}} \frac{1 + q^{C_j-C_{j+b}+b\delta_{il}}}{1 - q^{C_j-C_{j+b}}} \right).
\]
Note that we can extend the range of sums from \(1 \leq C_1 < \cdots < C_{N_1} \leq N\) to the semi-infinite one \(1 \leq C_1 < \cdots < C_{N_1}\), since, by using \((q^a)_m = 1/(q^{a-m})_m\), the factor \(1/(q^{C_{N_1}-C_{j+1}})_{N-C_{N_1}} = (q^{N-C_{j+1}})_{C_{N_1}-N}\) for \(C_{N_1} > N\) vanishes when \(j = N_1\). By repeatedly applying the formula \((a)_n = (a)_m (aq^m)_{n-m}\), we can simplify (B20) to (3.5):

\[
S(N_1, N_2; n)_k \rightarrow_{N_2 \to -N_2} S_{\text{ABJ}}(N_1, N_2; n)_k. \tag{B23}
\]

What we are going to do is simply to replace \(N_2\) in (B22) by \(-N_2 + \epsilon\) and take the \(\epsilon \to 0\) limit. The basic formula to use is

\[
(q)^{-|z|+\epsilon} = \prod_{k=0}^{\infty} \frac{1 - q^{k+1}}{1 - q^{k+1}} - \frac{1}{\epsilon \ln q} \prod_{k=0}^{\infty} \frac{1}{1 - q^{k+1}} + O(\epsilon^0)
\]

\[
= -\frac{1}{\epsilon \ln q (q^{1-|z|})_{|z|-1}} + O(\epsilon^0) = -\frac{(-1)^{|z|-1} q^{\frac{1}{2} |z| (|z|-1)}}{\epsilon \ln q (q^{1-|z|})_{|z|-1}} + O(\epsilon^0), \tag{B24}
\]

where we used (A3) in the first equality. The nontrivial part is the denominator in the factor

\[
\prod_{i=1}^{N_1} \frac{(-q^{1+n_i|z|})_{C_{i-1}} (-q^{1-n_i|z|})_{N-C_i}}{(q)_{C_{i-1}} (q)_{N-C_i}} \tag{B25}
\]

in the second line of (B22). However, we need to treat the two cases \(N_1 \leq N_2\) and \(N_1 \geq N_2\) separately, and the latter turns out to be more involved than the former.

**B.2.1. The \(N_1 \leq N_2\) case**

In this case it is straightforward to apply the formula (B24). The denominator of (B25) yields

\[
(q)_{C_{i-1}} (q)_{N-C_i} \rightarrow (q)_{C_{i-1}} (q)_{-M-C_i + \epsilon} = -\frac{(-1)^{M+C_{i-1}} q^{\frac{1}{2} (M+C_{i-1})(M+C_{i-1})}}{\epsilon \ln q (q^{C_i})_M}, \tag{B26}
\]
where $N = N_1 + N_2$ and $M := |N_2 - N_1| = N_2 - N_1$ and we used $(q)_{C_{-1}/(q)_{M+C_{-1}} = 1/(q_{C_1})_M}$. Meanwhile, we can rewrite the numerator of (B25) after the analytic continuation as

$$
\prod_{i=1}^{N_1} \left( -q^{1+n\delta_{il}} \right)_{C_{-1}} \left( -q^{-n\delta_{il}} \right)_{M-C_i} = \prod_{i=1}^{N_1} \frac{q^{(C_{i}+M)n\delta_{il} + \frac{1}{2}(C_{i}+M)(C_{i}+M-1)}}{(1 + q^{n\delta_{il}})(-q^{C_{i}+n\delta_{il}})_M}.
$$

Putting these together yields

$$
\prod_{i=1}^{N_1} \frac{\left( -q^{1+n\delta_{il}} \right)_{C_{-1}} \left( -q^{-n\delta_{il}} \right)_{N-C_i}}{(q)_{C_{-1}}(q)_{N-C_i}} \to (\epsilon \ln q)^{N_1} \prod_{i=1}^{N_1} \left( -1 \right)^{C_{i}+M} \frac{q^{(C_{i}+M)n\delta_{il}} \left( q_{C_{i}} \right)_M}{\left( 1 + q^{n\delta_{il}} \right) \left( -q^{C_{i}+n\delta_{il}} \right)_{M}}.
$$

We thus find the analytic continuation (3.6)

$$
S(N_1, -N_2 + \epsilon; n)_k \sim (\epsilon \ln q)^{N_1} S^{ABJ}(N_1, N_2; n)_k,
$$

with the special function (3.7) for the ABJ theory

$$
S^{ABJ}(N_1, N_2; n)_k
$$

$$
= \frac{1}{N_1!} \sum_{l=1}^{N_1} \sum_{1 \leq c_1, \ldots, c_{N_1}} q^{-2nC_i + n(-M+l-1)} \prod_{k \neq l} \frac{1 - q_{C_k-C_l}}{1 - q_{C_k-C_l}} \prod_{i=1}^{N_1} \left[ \left( -1 \right)^{C_{i}+M} \frac{q^{(C_{i}+M)n\delta_{il}} \left( q_{C_{i}} \right)_M}{\left( 1 + q^{n\delta_{il}} \right) \left( -q^{C_{i}+n\delta_{il}} \right)_{M}} \prod_{j=1}^{i-1} \frac{q_{C_{j}-C_{1}}}{\left( 1 - q^{C_{j}-C_{1}} \right)} \prod_{j=i+1}^{N_1} \frac{1}{\left( -q^{C_{j}-C_{i}+n\delta_{il}} \right)} \right] .
$$

\[\text{(B30)}\]

**B.2.2. The } N_1 \geq N_2 \text{ case}

As mentioned, this case is more involved than the previous case. The major difference stems from the fact that the factor $(q)_{N-C_i}$ in the denominator of (B25), when analytically continued to $(q)_{N_1-N_2-C_i+\epsilon}$, has the index $N_1 - N_2 - C_i + \epsilon = M - C_i + \epsilon$, which is not always negative, in contrast to the previous case. This index becomes negative when $C_i > M$. When the index is negative, the factor $(q)_{M-C_i+\epsilon}$ is singular and of order $\epsilon^{-1}$ as in (B24). This means that the factor (B25), when analytically continued, vanishes with some power of $\epsilon$. For the purposes of analytic continuation, we are only concerned with the leading vanishing term. In the previous case the leading vanishing term was of order $\epsilon^{N_1}$ as in (B28).

To extract the leading vanishing term of the factor (B25), we need to find in which case the factor $\prod_{i=1}^{N_1} \left( q_{M-C_i+\epsilon} \right)$ is least singular. Now the summand in (B24) is identically zero whenever any of the $C_i$ coincide. In other words, it is only nonzero when all of the $C_i$ are different from each other. Thus we can focus on the case where none of the $C_i$ are equal. Clearly, the factor $\prod_{i=1}^{N_1} \left( q_{M-C_i+\epsilon} \right)$ is least singular when $M$ of the $C_i$ being all different, take values in \{1, \ldots, M\}. Since the order does not matter, without loss of generality, we can choose \{C_1, C_2, \ldots, C_M\} = \{1, 2, \ldots, M\} by taking into account the combinatorial factor $N_1 C_M M! = N_1! / N_2!$. This also implies that the leading vanishing term is of order $\epsilon^{N_1-M} = \epsilon^{N_2}$.

Having understood this point, the remaining task is to (1) plug \{C_1, C_2, \ldots, C_M\} = \{1, 2, \ldots, M\} into (B22) and (2) apply the formula (B24) to the factors $(q)_{M-C_i+\epsilon}$ with $C_i \geq M+1 \geq M + 1$, while taking care of the sum over $l$ by splitting the sum $\sum_{l=1}^{N_1} \text{ into }\sum_{l=1}^{M} + \sum_{l=M+1}^{N_1}$. 

\[\text{(45/62)}\]
We first deal with the factors
\[
\prod_{i=1}^{N_1} \left( \prod_{j=1}^{i-1} \frac{(l)^i}{(-q^{C_l-C_j})} \right) \prod_{j=i+1}^{N_1} \frac{(l)^i}{(-q^{C_j-C_l-n_{hl}})}
\]  
(B31)
in the second line of (B22). The factors with all the indices less than or equal to \(M\) yield
\[
\prod_{i=1}^{M} \left( \prod_{j=1}^{i-1} \frac{(l)^i}{(-q^{C_l-C_j-n_{hl}})} \right) \prod_{j=M+1}^{M} \frac{(l)^i}{(-q^{C_j-C_l-n_{hl}})} = \prod_{i=1}^{M} \frac{(l)_i}{(-q^{1+n_{hl}})_i} \frac{(l-M)^{i-M}}{(-q^{1-n_{hl}})^{i-M}}.
\]  
(B32)
The factors with one of the indices less than or equal to \(M\) and the other one greater than \(M\) yield
\[
\left( \prod_{i=M+1}^{M} \frac{(l)^i}{(-q^{C_l-C_j-n_{hl}})} \right) \left( \prod_{i=M+1}^{M} \frac{(l)^i}{(-q^{C_j-C_l-n_{hl}})} \right)
\]  
(B33)
where we relabeled \(D_a = C_a-M\) with \(a = 1, \ldots, N_2; c = M-l\) for \(l \leq M\) and \(d = l-M\) for \(l > M\). The factors with all the indices greater than \(M\) are trivially of a similar form to (B31).

We next examine the factor in the first line of (B22):
\[
\prod_{k \neq l} \frac{1-q^{C_k-C_l-n}}{1-q^{C_k-C_l}} = \begin{cases} \prod_{k=1, k \neq l}^{M} \frac{1-q^{l-n}}{1-q^{l}} \prod_{k=M+1}^{N_1} \frac{1-q^{C_k-l-n}}{1-q^{C_k-l}} & (l \leq M) \\
\prod_{k=1, k \neq l}^{M} \frac{1-q^{C_k-C_l-n}}{1-q^{C_k-C_l}} \prod_{k=M+1}^{N_1} \frac{1-q^{C_k-C_l-n}}{1-q^{C_k-C_l}} & (l > M) \\
q^{-n(M-1-c)}(q^{1-n}) \prod_{a=1}^{N_2} \frac{1-q^{D_a+c-n}}{1-q^{D_a+c}} & (l \leq M) \\
q^{-nM}(q^{D_d+n}) \prod_{a=1}^{N_2} \frac{1-q^{D_a-D_d-n}}{1-q^{D_a-D_d}} & (l > M) \end{cases}
\]  
(B34)

Finally, we look into the factor (B25). The factors with \(i \leq M\) are exactly the inverse of (B32) and are canceled out, whereas the analytic continuation of the rest of the factors yields
\[
\prod_{i=M+1}^{N_1} \frac{(-q^{1+n_{hl}})}{(q)_{i-1}}(-q^{1-n_{hl}})_{M+1} = (\varepsilon \ln q)^{N_2} \prod_{a=1}^{N_2} \frac{(-1)^{D_a} q^{nD_a \delta_a + D_M l} (-q^{D_a+n_{hl}D_M l})}{(1+q^{n_{hl}D_M l}) (q^{D_a})_M}.
\]  
(B35)

Putting all factors together, we find the analytic continuation (3.9)
\[
S(N_1, \, -N_2 + \varepsilon; n)_k \sim (\varepsilon \ln q)^{N_2} \left( S_{(1)}(N_1, N_2; n)_k + S_{(2)}(N_1, N_2; n)_k \right),
\]  
(B36)
where the special functions for the ABJ theory are given by

\[
S_{(1)}^{ABJ}(N_1, N_2; n) = \frac{1}{N_2!} \sum_{c=0}^{M-1} \sum_{1 \leq D_1, \ldots, D_{N_2}} q^{n(2c-M)} \frac{(q^{1-n})^c (q^{1+n})^{M-1-c}}{(q)_c (q)_M-1-c} \prod_{a=1}^{N_2} \frac{1 - q^{D_a+c-n}}{1 - q^{D_a+c}}
\]

\[
\times \prod_{a=1}^{N_2} \left[ \frac{(-1)^{D_a} (q^{D_a})_M}{2 (-q^{D_a})_M} \frac{(-q^{D_a+c-n})_M}{(-q^{D_a+c-n})_M} \prod_{b=1}^{a-1} \frac{(q^{D_a-D_b})_1}{(-q^{D_a-D_b})_1} \prod_{b=a+1}^{N_2} \frac{(q^{D_b-D_a})_1}{(-q^{D_b-D_a})_1} \right] \tag{B37}
\]

and

\[
S_{(2)}^{ABJ}(N_1, N_2; n) = \frac{1}{N_2!} \sum_{d=1}^{N_2} \sum_{1 \leq D_1, \ldots, D_{N_2}} q^{-nD_d+n(d-M-1)} \prod_{a=1}^{N_2} \frac{(q^{D_a-D_d-n})_1}{(q^{D_a-D_d})_1} \prod_{b=1}^{a-1} \frac{(q^{D_d-D_b})_1}{(-q^{D_d-D_b})_1} \prod_{b=a+1}^{N_2} \frac{(q^{D_b-D_d})_1}{(-q^{D_b-D_d})_1} \tag{B38}
\]

**Appendix C. Wilson loops in general representations**

In this appendix, we present the expressions for Wilson loops in general representations. We will be very brief in explaining how to derive these results, because it is similar to that for the partition function (Ref. [17]) and for Wilson loops with winding number \( n \) (Appendix B).

### C.1. Lens space Wilson loops

Let us start with Wilson loops in the \( U(N_1) \times U(N_2) \) lens space matrix model. In (3.1), we considered representations with winding number \( n \), but here we would like to consider general representations. For the \( U(N_1) \) representation with Young diagram \( \lambda \), the Wilson loop can be computed by inserting \( S_\lambda(e^{\mu_1}, \ldots, e^{\mu_{N_1}}) \) in the matrix integral, where \( S_\lambda(x_1, \ldots, x_{N_1}) \) is the Schur polynomial for \( \lambda \) [44]. Because each term in the polynomial \( S_\lambda(e^{\mu_1}, \ldots, e^{\mu_{N_1}}) \) has the form \( e^{\sum_j m_j \mu_j} \) with \( m_j \in \mathbb{Z} \), all we have to compute in principle is the matrix integral with \( e^{\sum_j m_j \mu_j} \) inserted. If the Wilson line carries a nontrivial representation for \( U(N_2) \), we must also insert a similar factor for \( v_a \).

Therefore, given \( \{m_j\}_{j=1}^{N_1} \), \( \{n_a\}_{a=1}^{N_2} \), the object of our interest here is the following matrix integral:

\[
\gamma^\text{lens}_{[m], [n]}(N_1, N_2) := \langle e^{m \cdot \mu + n \cdot v} \rangle
\]

\[
:= N_1 \text{lens} \int \prod_{j=1}^{N_1} \frac{d\mu j}{2\pi} \int \prod_{a=1}^{N_2} \frac{dv_a}{2\pi} A_{sh}(\mu)^2 A_{sh}(v)^2 A_{ch}(\mu, v)^2 e^{-\frac{1}{2\pi}(\sum_j m_j^2 + \sum_a v_a^2)} e^{m \cdot \mu + n \cdot v}, \tag{C1}
\]

where \( m \cdot \mu = \sum_j m_j \mu_j, n \cdot v = \sum_a n_a v_a \), with \( m_j, n_a \in \mathbb{Z} \). By symmetry, it is clear that this is invariant under \( m_j \leftrightarrow m_l \) and \( n_a \leftrightarrow n_b \). If \( m = n = 0 \), this reduces to the partition function while,
for the case with a single winding number, e.g., \( m_j = m \delta_{1j}, n_a = 0 \), this reduces to the Wilson loop studied in the main text, (3.1), up to a factor:

\[
W_{\{m,0,...,1,0,...\}}(N_1, N_2)_k = \frac{1}{N_1!} W_{\{m\}}^\text{lens}(N_1, N_2; m)_k. \tag{C2}
\]

By carrying out the Gauss integration and doing manipulations similar to (B7)–(B17), we arrive at the following simple combinatorial expression:

\[
W_{\{m\}}^\text{lens}(N_1, N_2)_k = i^{-\frac{z}{2}(N_1^2+N_2^2)} \left( \frac{gs}{2\pi} \right)^{\frac{z}{2}} q^{-\frac{1}{6}N(N^2-1)+\sum_{A=1}^{N}(\frac{p_A^2}{2}+P_A)} \times (1-q)\frac{1}{6}N(N-1) G_2(N+1, q) S_{\{m,n\}}(N_1, N_2),
\]

\[
S_{\{m,n\}}(N_1, N_2) := \sum_{(N_1,N_2)} \frac{1}{N_1!} \sum_{\sigma \in S_{N_1}} \frac{1}{N_2!} \sum_{\tau \in S_{N_2}} q^{-\sum_{j=1}^{N_1} m_{\sigma(j)} C_j - \sum_{a=1}^{N_2} n_{\tau(a)} D_a} \times \prod_{C_j < C_k} \frac{q^{C_j+m_{\sigma(j)} - C_k+m_{\tau(k)}}}{q^{C_j} - q^{C_k}} q^{D_a+n_{\tau(a)} - D_b+n_{\tau(b)}} \prod_{D_a < D_b} \frac{q^{D_a+n_{\tau(a)} - D_b+n_{\tau(b)}}}{q^{D_a} - q^{D_b}} \times \prod_{C_j < D_a} \frac{q^{C_j+m_{\sigma(j)} + D_a+n_{\tau(a)}}}{q^{C_j} - q^{D_a}} q^{D_a+n_{\tau(a)} + C_j+m_{\sigma(j)}-D_b+n_{\tau(b)}} \prod_{D_a < C_j} \frac{q^{D_a+n_{\tau(a)} + C_j+m_{\sigma(j)}-D_b+n_{\tau(b)}}}{q^{D_a} - q^{C_j}}, \tag{C3}
\]

where \( N := N_1 + N_2 \) and \( P_A = (m_j, n_a), A = 1, \ldots, N \). The symbol \( (N_1, N_2) \) denotes the partition of the numbers \( (1, 2, \ldots, N) \) into two groups \( N_1 = (C_1, C_2, \ldots, C_{N_1}) \) and \( N_2 = (D_1, D_2, \ldots, D_{N_2}) \) where \( C_1 \) and \( D_1 \) are ordered as \( C_1 < \cdots < C_{N_1} \) and \( D_1 < \cdots < D_{N_2} \).

To proceed, let us focus on the case with \( n_a = 0 \), namely on the \( U(N_1) \) Wilson loop henceforth. In this case, we can rewrite the product in \( S \) in favor of the \( q \)-Pochhammer symbol, just as we did in (B19)–(B22). After using various formulas for the \( q \)-Pochhammer symbol in Appendix A, the final result can be written as

\[
S_{\{m_j,0\}}(N_1, N_2) = \frac{1}{N_1!} \sum_{\sigma \in S_{N_1}} \prod_{j=1}^{N_1} \left( -q^{N_1+1-j-m_{\sigma(j)}} (q^j)^{N_2} \right) \frac{(q^j)_{N_2}}{(q^j)_{N_1-N_2}} \prod_{j \neq k} \frac{1 + (q^{k-j-m_{\sigma(j)}}) q^{k-j}}{1 - q^{k-j}} \times \sum_{1 \leq C_1 < \cdots < C_{N_1} \leq N_1 + N_2} (-1)^{\sum_{j=1}^{N_1} (C_j-1)} q^{\sum_{j=1}^{N_1} (N_1+N_2-1-C_j) m_{\sigma(j)}} \times \prod_{j=1}^{N_1} \left( -q^{C_j} (q^{C_j+m_{\sigma(j)}})^{N_1-N_2} \right) \prod_{j \neq k} \frac{q^{C_k-C_j+m_{\sigma(k)}-m_{\sigma(j)}} (q^{C_k-C_j}) (q^{C_k-C_j})}{-q^{C_k-C_j-m_{\sigma(j)}} (q^{C_k-C_j}) (q^{C_k-C_j})}, \tag{C5}
\]

In the above, we treated \( N_2 \) as a continuous variable. To obtain an expression for integral \( N_2 \), let us shift \( N_2 \to N_2 + \epsilon \) with \( N_2 \in \mathbb{Z} \) where it is understood that \( \epsilon \) will be taken to zero at the end of computation. By extracting powers of \( \epsilon \) from the \( q \)-Pochhammer symbols in the first line of (C5),
we find, to leading order in small $\epsilon$ expansion,

\[
S_{[m_j],[0]}(N_1, N_2 + \epsilon) = \frac{(-1)^{N_1 N_2 (\epsilon \log q)^{N_1} N_2}}{\prod_{j=1}^{N_1} (1 + q^{m_j})} \times \frac{1}{N_1!} \sum_{\sigma \in S_{N_1}} \sum_{1 \leq C_1 < \cdots < C_{N_1}} (-1)^{\sum_{j=1}^{N_1} (C_j - 1)} q^{\sum_{j=1}^{N_1} C_j m_{\sigma(j)}} \times \prod_{j=1}^{N_1} \frac{(q^{C_j} - 1 - N_2 - \epsilon)}{-(q^{C_j + m_{\sigma(j)}})} \prod_{j < k} \frac{(q^{C_k - C_j + m_{\sigma(j)}})_{1}(q^{C_k - C_j})_1}{-(q^{C_k - C_j + m_{\sigma(k)}})_{1}(q^{C_k - C_j + m_{\sigma(k)}})}.
\]

(C6)

We will drop subleading terms in small $\epsilon$ expansion henceforth. For $N_2 \in \mathbb{Z}_{>0}$, of course, the factor $\epsilon^{N_1}$ in front of (C6) must be canceled by the $q$-Pochhammer symbol $(q^{C_j} - 1 - N_2 - \epsilon)$, because the original expression (C4) was finite to begin with. This $q$-Pochhammer symbol can be rewritten as $(q^{C_j} - 1 - N_2 - \epsilon) = 1/(q^{C_j} - 1 - N_2 - \epsilon)_{N_1 + N_2 + \epsilon} \approx 1/(q^{C_j} - 1 - N_2 - \epsilon)_{N_1 + N_2} = 1/[(1 - q^{C_j} - 1 - N_2 - \epsilon) \cdots (1 - q^{C_j} - 1 - N_2 - \epsilon)]$, where “$\approx$” means up to subleading terms in powers of $\epsilon$. Indeed, for $C_j \leq N_1 + N_2$, this always contains the factor $1/(1 - q^{-\epsilon}) = 1/(\epsilon \log q)$ and, collecting contributions from $C_1, \ldots, C_{N_1}$, we see that this completely cancels the $\epsilon^{N_1}$. Actually, this also means that, if $C_j > N_1 + N_2$, we have fewer powers of $\epsilon$ in the denominator and, as a result, the summand vanishes as $\epsilon \to 0$. Therefore, we are free to remove the upper bound in the $C_j$-sum, as we have already done in (C6).

In (C6), summing over permutations $\sigma \in S_{N_1}$ acting on $m_j$ is the same as summing over permutations acting on $C_j$ (it is easy to show that, if we set $j' = \sigma(j), k' = \sigma(k)$, summing over $j, k, \sigma$ is the same as summing over $j', k', \sigma' = \sigma^{-1}$, with $C_j, C_k$ replaced by $C_{\sigma^{-1}(j')}, C_{\sigma^{-1}(k')}$). Therefore, we can relax the ordering constraint on $C_j$ and forget about the summation over permutations:

\[
S_{[m_j],[0]}(N_1, N_2 + \epsilon) = \frac{(-1)^{N_1 N_2 (\epsilon \log q)^{N_1} N_2}}{\prod_{j=1}^{N_1} (1 + q^{m_j})} \times \frac{1}{N_1!} \sum_{C_1, \ldots, C_{N_1}} (-1)^{\sum_{j=1}^{N_1} (C_j - 1)} q^{\sum_{j=1}^{N_1} C_j m_{j}} \times \prod_{j=1}^{N_1} \frac{(q^{C_j} - 1 - N_2 - \epsilon)}{-(q^{C_j + m_{j}})} \prod_{j < k} \frac{(q^{C_k - C_j + m_{j}})_{1}(q^{C_k - C_j})_1}{-(q^{C_k - C_j + m_{k}})_{1}(q^{C_k - C_j + m_{k}})}.
\]

(C7)

Note that the sum over $C_1, \ldots, C_{N_1}$ is completely unconstrained, because the summand in (C6) vanishes if $C_j = C_k$ with $j \neq k$. We can easily see that, upon setting $m_j = 0$, this reduces to the $S$ function for the partition function [17]. We will refer to this as the unordered formula. The formula (C6) is referred to as the ordered formula.

C.2. Explicit expression for ABJ Wilson loop ($m_j \neq 0, n_a = 0$)

The ABJ Wilson loop is obtained by setting $N_2 \to -N'_2$ with $N'_2 \in \mathbb{Z}_{>0}$ in the above expressions. More precisely, according to Ref. [17], the formula for analytic continuation is

\[
\mathcal{W}_{[m],[0],k}^{ABJ}(N_1, N'_2) = \lim_{\epsilon \to 0} \left(2\pi \right)^{-N'_2} \frac{G_2(N'_2 + 1)}{G_2(-N'_2 + 1 + \epsilon)} \mathcal{W}_{[m],[0]}^{\text{len}}(N_1, -N'_2 + \epsilon)_k.
\]

(C8)
First, consider the case with $N_1 \leq N_2'$. Using the unordered expression (C7), we straightforwardly obtain

$$S_{(m_j),0}(N_1, -N_2'+\epsilon) = \frac{(-1)^{N_1} \epsilon \log q^{N_1}}{\prod_{j=1}^{N_1} (1+ q^{m_j})} \times \frac{1}{N_1!} \sum_{C_j \geq 1} (-1)^{\sum_j (C_j-1)} q^{-\sum_j C_j m_j} \times \prod_{j=1}^{N_1} \left( -q^{C_j + m_j} \right)_{N_2'-N_1} \prod_{j<k} \left( -q^{C_k - C_j + m_k - m_j} \right)_{1} \left( -q^{C_k - C_j} \right). \tag{C9}$$

We have the $\epsilon^{N_1}$ as a prefactor, which remains uncanceled by the $q$-Pochhammer. It is not difficult to show that, when $m_j = m_\delta 1_j$, this reduces to the formula (B30), up to normalization; namely, $S_{(m,0,...),0}(N_1, -N_2'+\epsilon) = (1/N_1) S^{ABJ}(N_1, N_2', m_k)$. The integral representation is

$$S_{(m_j),0}(N_1, -N_2'+\epsilon) = \frac{(-1)^{N_1} \epsilon \log q^{N_1}}{\prod_{j=1}^{N_1} (1+ q^{m_j})} \times \frac{1}{N_1!} \left[ \prod_{j=1}^{N_1} \frac{1}{2\pi i} \int \frac{\pi ds_j}{\sin(\pi s_j)} \right] q^{-\sum_j (s_j+1)m_j} \times \prod_{j=1}^{N_1} \left( -q^{s_j+1+m_j} \right)_{M} \prod_{j<k} \left( -q^{s_k-s_j+m_k-m_j} \right)_{1} \left( -q^{s_k-s_j} \right), \tag{C10}$$

where $C_j \leftrightarrow s_j + 1$ and $M := N_2' - N_1$. Using the formula (C8), the full expression including the prefactors in (C3) is

$$W_{(m_j),0}^{ABJ}(N_1, N_2')_k = i^{\frac{N_1}{2}} \left( \frac{\epsilon}{N_1} \log q^{N_1} \right)^{\frac{N_1}{2}} \left( N_1 - 1 \right)^{\frac{N_1}{2}} \times G_2(M + 1, q) S_{(m_j),0}(N_1, -N_2'+\epsilon) \frac{(1-q)^{\frac{1}{2}M(M-1)}}{\epsilon \ln q^{N_1}}. \tag{C11}$$

The case with $N_1 > N_2'$ is more nontrivial, as is the case for the partition function [17]. In this case, there are some powers of $\epsilon$ coming from $(q^{C_j})_{-(N_1-N_2')} = (q^{C_j})_{-M'} = 1/(q^{C_j-M'})$ in the summand, where we set $M' := N_1 - N_2' > 0$. One choice to get the most singular contribution is

$$C_1 = 1, \quad C_2 = 2, \quad \ldots, \quad C_{M'} = M', \quad \quad C_{M'+1} = M' + C', \quad \ldots, \quad C_{N_1} = M' + C_{N_2}. \tag{C12}$$

For this particular choice, by extracting powers of $\epsilon$ from the $q$-Pochhammer symbol, we can show that (C7) can be rewritten as

$$\frac{(-\epsilon \ln q)^{N_2'}}{N_1! \prod_{j=1}^{N_1} (1+ q^{M'+j})} \sum_{C_j \geq 1} (-1)^{\sum_j (C_j-1)} q^{\sum_j (C_j+M') m_j - \sum_j (C_j+M') m_{M'+j}} \times \prod_{j=1}^{N_2'} \frac{-q^{C_j+M'+j} + q^{M'+j} + q^{C_j-M'} - q^{C_j-M'+j}}{1 \leq j < k \leq M'} \times \prod_{j=1}^{M'} \prod_{k=1}^{N_2'} \frac{-q^{M'+C'_j-j+m_{M'+k}} + q^{M'+C'_j-j} - q^{M'+C'_j-j+m_{M'+k}}}{1 \leq j < k \leq M'}. \tag{C13}$$
The choice (C12) is only one possibility and there are more; there are \( \binom{N_1}{M'} \) ways to choose \( M' \) special \( C_j \) out of \( N_1 \). Furthermore, there are \( M'! \) ways to permute those \( M' \) special \( C_j \). So, in total, we have \( \binom{N_1}{M'} M'! \) ways. We should sum over all these choices, or, equivalently, as we have seen in going between (C6) and (C7), we can fix the order of \( C_j \) to be \( C_1 < C_2 < \cdots < C_{N_1} \), consider only (C12), and sum over the permutations of \( m_j \). So, (C5) can be expressed as

\[
S_{[m,j],[0]}(N_1, -N'_2 + \epsilon) = \frac{(-\epsilon \ln q)^{N'_2}}{N_1! \prod_{j=1}^{M'-1} (q_j)} \sum_{\sigma \in S_{N'_2}} q^{\sum_{j=1}^{M'} (-2j + M') m_{\sigma(j)}} \prod_{1 \leq j < k \leq M'} (q^{k-j + m_{\sigma(k)} - m_{\sigma(j)})_1} \prod_{j=1}^{N'_2} (1 + q^{m_{\sigma(M'+j)}}) \\
\times \sum_{1 \leq C'_1 < \cdots < C'_{N'_2}} \left( -1 \right)^{\sum_{j=1}^{N'_2} (C'_j-1)} q - \sum_{j=1}^{N'_2} (C'_j + M') m_{\sigma(M'+j)} \\
\times \prod_{j=1}^{N'_2} (-q^{C'_j + m_{\sigma(M'+j)}})^{M'} \prod_{j=1}^{M'} \prod_{k=1}^{N'_2} \left( \frac{q^{M'+C'_k - j + m_{\sigma(M'+k)} - m_{\sigma(j)}}_1}{(-q^{C'_k - C_j + m_{\sigma(M'+j)}}_1)} \left( q^{C'_j - C'_k + m_{\sigma(M'+k)} - m_{\sigma(j)}}_1 \right) \right). \tag{C14}
\]

If we want to relax the ordering constraint and let \( C'_j \) run over all positive integers, then the summation will be over \( S_{N'_2} \), meaning that two permutations \( \sigma_1, \sigma_2 \in S_{N_1} \) are identified if \( \{\sigma_1(M'+1), \ldots, \sigma_1(N'_2)\} \) and \( \{\sigma_2(M'+1), \ldots, \sigma_2(N'_2)\} \) are permutations of each other. One can show that, when \( m_j = m \delta_{j,1} \), Eq. (C14) reduces to (B37) plus (B38), up to normalization. Namely, \( S_{[m,0,0,\ldots],[0]}(N_1, -N'_2 + \epsilon) = (1/N_1)[S_{(1)}^{ABJ}(N_1, N'_2, m)_k + S_{(2)}^{ABJ}(N_1, N'_2, m)_k] \). The integral representation is

\[
S_{[m,j],[0]}(N_1, -N'_2 + \epsilon) = \frac{(-\epsilon \ln q)^{N'_2}}{N_1! \prod_{j=1}^{M'-1} (q_j)} \sum_{\sigma \in S_{N'_2}} q^{\sum_{j=1}^{M'} (-2j + M') m_{\sigma(j)}} \prod_{1 \leq j < k \leq M'} (q^{k-j + m_{\sigma(k)} - m_{\sigma(j)})_1} \prod_{j=1}^{N'_2} (1 + q^{m_{\sigma(M'+j)}}) \\
\times \int \left[ \frac{1}{2\pi i} \int_{\pi s_j} q - \sum_{j=1}^{N'_2} (s_j + 1 + M') m_{\sigma(M'+j)} \right] \prod_{j=1}^{N'_2} (-q^{s_j + 1 + m_{\sigma(M'+j)}})^{M'} \prod_{j=1}^{M'} \prod_{k=1}^{N'_2} \left( \frac{q^{M'+s_j + 1 - j + m_{\sigma(M'+k)} - m_{\sigma(j)}}_1}{(-q^{s_j + M'+1 + j - m_{\sigma(M'+k)} + m_{\sigma(j)}}_1)} \right) \\
\times \prod_{1 \leq j < k \leq N'_2} \left( \frac{q^{s_j - s_k + m_{\sigma(M'+k)} - m_{\sigma(M'+j)}))_1}{(-q^{s_j - s_k + m_{\sigma(M'+j)} - m_{\sigma(M'+k))}_1)} \right). \tag{C15}
\]

The full expression including the prefactors in (C3) is

\[
\mathcal{W}^{ABJ}_{[m,j],[0]}(N_1, N'_2)_k = i^{-\frac{1}{2} \binom{N'_1 + N'_2}{2}} (1 - \frac{1}{2} \binom{N'_2 - 1}{N'_2}) \left( \frac{g_8}{2\pi} \right)^{N_1 + N'_2} \frac{1}{\epsilon} \left( M' \right)^{M' - 1} + \sum_{j=1}^{N'_2} (-\frac{1}{2} \mu_j + \mu_j) \\
\times (1 - q)^{\frac{1}{2} M' \left( M' - 1 \right)} G_2(M' + 1, q) S_{[m,j],[0]}(N_1, -N'_2 + \epsilon) \frac{(-\epsilon \ln q)^{N'_2}}{N_1! \prod_{j=1}^{M'-1} (q_j)}. \tag{C16}
\]
Appendix D. The cancellation of residues at $s_a = -1, -2, \ldots, -n$

In this appendix we show that, in our expression (2.15) for $1/6$-BPS Wilson loops in the $N_1 \geq N_2$ case, the contributions from the P poles at $s_a = -1, \ldots, -n$ are absent. Namely, these contributions are canceled between those from $I^{(1)}(N_1, N_2; n)_k$ and $I^{(2)}(N_1, N_2; n)_k$. This is necessary, in particular, for the $1/6$-BPS Wilson loops (2.15) to correctly reproduce the perturbative expansion in $g_s$ and also fills the gap of a proof that the two expressions (2.12) and (2.15) agree in the ABJM limit $N_1 = N_2$.

We first recall the expressions of our interest:

$$I^{(1)}(N_1, N_2; n)_k := \frac{1}{N_2!} \sum_{c=0}^{n-1} \prod_{a=1}^{N_2} \left[ \frac{-1}{2\pi i} \int_{C_1[c]} \frac{\pi ds_a}{\sin(\pi s_a)} \right] \times q^n(2c-M) \left( \frac{q^{1-n}c}{q^{1+n}} \right)_{M-1-c} \prod_{a=1}^{N_2} \left( \frac{q^{s_a+1}}{q^{s_a-1}} \right)_{M} \prod_{a=1}^{N_2} \left( \frac{q^{s_a}}{q^{s_a+1}} \right)_{M} \prod_{a=b+1}^{N_2} \left( \frac{q^{s_a}}{q^{s_a+1}} \right)_{M} \left[ \prod_{a=1}^{N_2} \frac{(-q^{s_a+1})}{(-q^{s_a})} \right],$$

(D1)

and

$$I^{(2)}(N_1, N_2; n)_k := \frac{1}{N_2!} \sum_{d=1}^{n} \prod_{a=1}^{N_2} \left[ \frac{-1}{2\pi i} \int_{C_2} \frac{\pi ds_a}{\sin(\pi s_a)} \right] q^{-n s_d + n(d-M-2)} \prod_{a=1}^{N_2} \left( \frac{q^{s_a-s_d-n}}{q^{s_a-s_d}} \right)_{M} \prod_{a=d}^{N_2} \left( \frac{q^{s_a}}{q^{s_a+1}} \right)_{M} \prod_{a=b+1}^{N_2} \left( \frac{q^{s_a}}{q^{s_a+1}} \right)_{M} \left[ \prod_{a=1}^{N_2} \frac{(-q^{s_a+1})}{(-q^{s_a})} \right].$$

(D2)

In Sect. 3.3 we discussed the pole structures of the integrands and the integration contours $C_1$ and $C_2$ in detail. The contour $C_1$ for a given $c$ is placed to the left of $s_a = \min(-1, -c, -\frac{k}{2} - M)$ and to the right of $s_a = \max(-M - 1, -\frac{k}{2} - n - 1 - c)$, whereas the contour $C_2$ is placed, for $a \neq d$, to the left of $s_a = \min(0, -\frac{k}{2} - M)$ and to the right of $s_a = \max(-M - 1, -\frac{k}{2} - 1)$ and, for $a = d$, to the left of $s_d = \min(-n, -\frac{k}{2} - M)$ and to the right of $s_d = \max(-M - 1, -\frac{k}{2} - 1)$. In particular, there are residues from the P poles at $s_d = -1 - c$, $c = 0, \ldots, n - 1$, for (D1) and $s_d = -1, \ldots, -n$ for (D2). Recall (3.37) for generic (including non-integral) $M$

$$\lim_{s_d \to -1 - c} \left( \frac{q^{1-n}c}{q^{1+n}} \right)_{M-1-c} \left( \frac{q^s}{q^{1+c+1}} \right)_{M} = q^{-nc} \left( \frac{q^{n-c}}{q^n} \right) \left( \frac{q^{s/1+c+1}}{q^{s/1-c}} \right)_{M},$$

(D3)

where we used $(q^{1-n}c)(q^{1+n})_{M-1-c} = (-1)^c q^{-nc+\frac{c}{2}+1}(q^{n-c})_{M}/(q^n)_1$, $\lim_{c \to 0} (q^{n-c})_{M}/(q^n)_1 = (q^{-c})_{M-1-c}$ and (A.3). As remarked below (3.37) in Sect. 3.4, implicit in this calculation is the $\epsilon$-prescription that always enables us to regard the P pole at $s_d = -1 - c$ as a simple pole. Namely, the integer $M$ is shifted to a non-integral value $M + \epsilon$ and the contour $C_1[c]$ is placed between $s_d = -1 - c$ and $-1 - c - \epsilon$ so as to avoid the latter pole.

$^{27}$ A remark similar to footnote 23 applies.
This yields

\[
-2\pi i \text{Res}_{s_d=-1-c} \left[ I^{(1)}(N_1, N_2; n)_k \right] = \frac{(-1)^c}{N_2!} \prod_{a=1 \atop a \neq d}^{N_2} \left[ \frac{-1}{2\pi i} \int_{C_{[c]}} \frac{\pi ds_a}{\sin(\pi s_a)} \right] q^{n(c-M)}(q^{n-c})_M \frac{q^{n(c-M)}(q^{n-c})_M}{(-q^{n-c})_M(-q^n)_1} \times \prod_{a=1 \atop a \neq d}^{N_2} \left[ \frac{(q^{s_a+1})_M}{2(-q^{s_a+1})_M} \frac{(q^{s_a+1+c})_1}{(-q^{s_a+1+c})_1} \right] \times \prod_{b=1 \atop b \neq d}^{a-1} \left[ \frac{(q^{s_b-s_a})_1}{(-q^{s_b-s_a})_1} \right] \prod_{b=a+1 \atop b \neq d}^{N_2} \left[ \frac{(q^{s_b-s_a})_1}{(-q^{s_b-s_a})_1} \right].
\]  

(D4)

Similarly, it is straightforward to find that

\[
-2\pi i \text{Res}_{s_d=-1-c} \left[ I^{(2)}(N_1, N_2; n)_k \right] = \frac{(-1)^{c+1}}{N_2!} \prod_{a=1 \atop a \neq d}^{N_2} \left[ \frac{-1}{2\pi i} \int_{C_2} \frac{\pi ds_a}{\sin(\pi s_a)} \right] q^{n(c-M)}(q^{n-c})_M \frac{q^{n(c-M)}(q^{n-c})_M}{(-q^n)_1(-q^{n-c})_M} \times \prod_{a=1 \atop a \neq d}^{N_2} \left[ \frac{(q^{s_a+1})_M}{2(-q^{s_a+1})_M} \frac{(q^{s_a+1+c})_1}{(-q^{s_a+1+c})_1} \right] \times \prod_{b=1 \atop b \neq d}^{a-1} \left[ \frac{(q^{s_b-s_a})_1}{(-q^{s_b-s_a})_1} \right] \prod_{b=a+1 \atop b \neq d}^{N_2} \left[ \frac{(q^{s_b-s_a})_1}{(-q^{s_b-s_a})_1} \right].
\]  

(D5)

Hence the sum of the two residues \( \text{Res}_{s_d=-1-c} \left[ I^{(1)}(N_1, N_2; n)_k + I^{(2)}(N_1, N_2; n)_k \right] \) exactly cancels out. As \( c \) runs from 0 to \( n-1 \), this implies the cancellation of the residues in (2.15) at the P poles \( s_d = -1, \ldots, -n, (d = 1, \ldots, N_2) \). More precisely, the cancellation requires the equivalence of the contours \( C_1 \) and \( C_2 \). In this regard, note that the integrand of the residue (D4), in particular, does not have a pole at \( s_a = -1-c \) and the contour \( C_1 \) can thus be shifted, past \( s_a = -1-c \), to the left of \( s_a = \min(0, \frac{k}{2}-M) \) so that \( C_1 \) becomes identical to \( C_2 \).

We would like to address the subtlety remarked on below (3.10) and (3.11) concerning the range, in particular, of the sum over \( c \) in (3.10) and (2.16). In its original form, the sum over \( c \) ran from 0 to \( M-1 \). When \( n \) is less than \( M \), the sum simply terminates at \( c = n \), as the factors (D3) vanish when \( c \geq n \). When \( n \) is greater than \( M \), however, in order to replace the upper limit \( M-1 \) by \( n-1 \), we need to show that the contribution from \( c = M \) to \( n-1 \) is absent in (2.15) and (3.12). Note that when \( c \geq M \) in the first line of (D3) the factor \((q)_M^{-1-c}\) in the denominator diverges. Thus, for \( c \geq M \), (D3) might appear to vanish. However, there are nonvanishing contributions coming from the poles at \( s_d = -1-c \) as the second line of (D3) indicates. This implies that the cancellation we have shown above is all we need to ensure the absence of the contribution from \( c = M \) to \( n-1 \) in (2.15) and (3.12). The cancellation of the residues at \( s_a = -1, \ldots, -n \) also justifies the extension of the lower limits of the sum over \( D_a \) in (3.10) and (3.11).

\[ \text{With an abuse of notation, we denote the integrands of } I^{(1)}(N_1, N_2; n)_k \text{ and } I^{(2)}(N_1, N_2; n)_k \text{ by the same symbols.} \]
Appendix E. The $U(1)^k \times U(1)^{-k}$ ABJM theory

As a simplest check of our prescriptions that lack first-principles derivations, we compare the result from the integral representation (2.12)–(2.14) with that from the direct calculation in the case of the $U(1)^k \times U(1)^{-k}$ ABJM theory.

The direct integral of (2.4) yields

$$W_{1/6}^I(1, 1; n)_k = \int \frac{d\mu}{2\pi} \frac{d\nu}{2\pi} e^{-q\nu} e^{\frac{1}{2\pi}(\mu^2 - \nu^2)} = \frac{1}{|k|} \frac{q^{-\frac{1}{2}n^2 + n}}{(1 + q^n)^2} \quad \text{for } |n| < \frac{k}{2},$$  \hspace{1cm} (E1)

where we used the Fourier transform of $1/\cosh x$:

$$\frac{1}{2 \cosh \frac{x}{2}} = \int \frac{dp}{2\pi} e^{\frac{ip}{2\pi}}.$$  \hspace{1cm} (E2)

As noted in Sect. 2.1, the restriction on winding number, $|n| < \frac{k}{2}$, is necessary for the convergence of integrals. Hence the direct calculation gives the normalized $1/6$-BPS Wilson loop

$$W_{1/6}^I(1, 1; n)_k = \frac{4q^{-\frac{1}{2}n^2 + n}}{(1 + q^n)^2}. \hspace{1cm} (E3)$$

On the other hand, as calculated in Sect. 4.3, the result from our integral representation yields\(^{29}\)

$$I(1, 1; n)_k = \frac{q^{-n}}{1 + q^n} \begin{cases} \frac{1}{2} \sum_{s=0}^{k-1} (-1)^s q^{-ns} & \text{for odd } k \\ \frac{1}{2k} \sum_{s=0}^{k-1} (-1)^s q^{-ns}(s + a) & \text{for even } k \end{cases} = \frac{1}{(1 + q^n)^2}. \hspace{1cm} (E4)$$

Thus the normalized $1/6$-BPS Wilson loop (2.12) indeed agrees with (E3), providing evidence for our prescriptions.

Appendix F. An alternative derivation of Seiberg duality

In this appendix we provide an alternative derivation of the duality transformations of ABJ Wilson loops by following Kapustin and Willett [50]. In their approach Seiberg duality is understood as an isomorphism of the algebras that BPS Wilson loops generate.

Before discussing the Wilson loops in the ABJ theory, we first give a brief sketch of their construction of Wilson loop algebras and derivation of duality transformations for generic 3D supersymmetric gauge theories that admit localization. In order to construct the Wilson loop algebras, we

---

\(^{29}\) For even $k$, since $(-q^{-n})^k = 1$, the sum $\sum_{s=0}^{k-1} (-q^{-n})^s = 0$ for an integer $n$. Meanwhile, this sum equals $\frac{1-q^{-n}}{1+q^{-n}} = f(n)$ for a non-integral $n$. To find the sum $\sum_{s=0}^{k-1} (-q^{-n})^s$ for even $k$, we differentiate $f(n)$ w.r.t. $n$ and then send $n$ to an integral value.
first define

\[ \Phi(t) = \prod_{j=1}^{N} (1 + tx_j) \]
\[ = 1 + t \left( \sum_{j=1}^{N} x_j \right) + t^2 \left( \sum_{i<j} x_i x_j \right) + \cdots + t^N \prod_{j=1}^{N} x_j \]
\[ = 1 + t \left[ 1 + t \left( \sum_{j=1}^{N} x_j \right) + t^2 \left( \sum_{i<j} x_i x_j \right) + \cdots \right] + t^N \]
\[ =: \sum_{i=0}^{N} t^i \phi_i \quad \text{(F1)} \]

and

\[ \Psi(t) = \prod_{j=1}^{N} (1 - tx_j)^{-1} \]
\[ = 1 + t \sum_{j=1}^{N} x_j + t^2 \left( \sum_{j=1}^{N} x_j^2 + \sum_{i<j} x_i x_j \right) + \cdots \]
\[ = 1 + t \left[ 1 + t \sum_{j=1}^{N} x_j + t^2 \left( \sum_{j=1}^{N} x_j^2 + \sum_{i<j} x_i x_j \right) + \cdots \right] \]
\[ =: \sum_{i=0}^{N} t^i \psi_i \quad \text{(F2)} \]

where we denoted symmetric polynomials by the corresponding Young diagrams. Each Young diagram corresponds to a specific representation of the Wilson loop. The variables \( x_i \) \((i = 1, \ldots, N)\) will be identified with the integration variables of the matrix models derived from localization. The \( \phi_i \) and \( \psi_i \) generate the ring of symmetric polynomials. Notice, however, that \( \Phi(-t) \Psi(t) = 1 \) by definition and thus the \( \phi_i \) and \( \psi_i \) are not independent.

Next, to find the duality transformations for BPS Wilson loops, we construct an algebra for a given theory from the ring of symmetric polynomials that quantum Wilson loops generate. As it turns out, there is an isomorphism between the algebras for the original and dual theories that can be regarded as the duality transformations. Here we only outline the derivation of the duality transformations:

1. Using the matrix model obtained by localization, the invariance under the shift of an integration variable yields the following identity:

\[ \langle p(x) \rangle = \left( x^M + a_1 x^{M-1} + \cdots \right) = 0, \quad \text{(F3)} \]

where \( x \) is an integration variable and \( \langle \cdots \rangle \) means the vev of the matrix model. The polynomial \( p(x) \) is at most of \( M \)th order in \( x \), where \( M \) is a constant determined by the rank and level of the gauge theory. This is a quantum constraint on the BPS Wilson loops that the Wilson loop algebra is endowed with.

2. From the polynomial \( p(x) \) we construct the following quantities \( \tilde{p}(t) \) and \( \Psi_p(t) \):

\[ \tilde{p}(t) := t^M p(t^{-1}) \quad \text{(F4)} \]
and

\[ \Psi_p(t) := \tilde{p}(t) \Psi(t) \big|_{\text{truncated at } O(t^{M-N})} \]

\[ = \sum_{j=0}^{M-N} t^i \psi_{pi} \]

\[ = 1 + (\Box + a_1)t + O(t^2). \]  

(F5)

In the algebra that Kapustin and Willett identify with a certain quotient of the ring, the classical constraint \( \Phi(-t)\Psi(t) = 1 \) is deformed to

\[ \Phi(-t)\Psi_p(t) = \tilde{p}(t), \]  

(F6)

which the elements, \( \phi_i \) and \( \psi_{pi} \), of the algebra obey in quantum theory. The quantum constraint (F3) is crucial, and it is important that the left-hand side of the constraint (F6) is truncated to a finite polynomial in \( t \), in contrast to the classical constraint \( \Phi(-t)\Psi(t) = 1 \).

3. Owing to the truncation in (F5) and thus in (F6), there exists an isomorphism under the following transformations:

\[ N \leftrightarrow M - N, \]  

(F7)

\[ \phi_i \leftrightarrow (-1)^i \psi_{pi}. \]  

(F8)

That is, the quantum constraint (F6) is invariant under these transformations. The first transformation can be identified with the map of the ranks of the two gauge groups in a dual pair and the second transformations with the maps of the BPS Wilson loops. Thus the duality transformations of the Wilson loops can be extracted order by order in \( t \) from (F8). At \( O(t) \), for instance, we find for the fundamental representation

\[ \Box \rightarrow -\Box - a_1, \]  

(F9)

where the tilde indicates that the Wilson loop is that in the dual theory.

F1. ABJ Wilson loop duality

We now apply the above method to ABJ Wilson loops. The original theory is the \( U(N_1)_k \times U(N_2)_{-k} \) ABJ theory and the dual theory is the \( U(\tilde{N}_2)_k \times U(N_1)_{-k} \) theory with the dual gauge group \( \tilde{N}_2 = k + 2N_1 - N_2 \). In implementing the above procedure it is useful to regard the original theory as the \( U(N_2)_{-k} \) Chern–Simons-matter theory and the \( U(N_1) \) part as flavor. Let us first recall that the \( \frac{1}{6} \)-BPS Wilson loop on the gauge group \( U(N_2) \) with winding \( n \) is given by the eigenvalue integrals

\[ W_{\frac{1}{6}}(N_1, N_2; n) \propto \int \prod_{i=1}^{N_1} \frac{d\mu_i}{2\pi} \prod_{a=1}^{N_2} \frac{dv_a}{2\pi} \frac{\Delta_{sh}(\mu)^2 \Delta_{sh}(v)^2}{\Delta_{ch}(\mu, v)^2} e^{-\frac{1}{2s_g} \left( \sum_{i=1}^{N_1} \mu_i^2 - \sum_{a=1}^{N_2} v_a^2 \right)} \sum_{a=1}^{N_2} e^{-n v_a}. \]  

(F10)
where we omitted the normalization factor as it is not relevant in the following discussion. For later convenience, we introduce the following notations:

\[
\langle \ldots \rangle_{\mu} := \int \prod_{j=1}^{N_1} \frac{d\mu_j}{2\pi} \frac{\Delta_{sh}(\mu)^2}{\Delta_{ch}(\mu, \nu)^2} e^{-\frac{1}{2\pi} \sum_{i=1}^{N_1} \mu_i^2} (\ldots), \tag{F11}
\]

\[
\langle \ldots \rangle_{\nu} := \int \prod_{a=1}^{N_2} \frac{d\nu_a}{2\pi} \frac{\Delta_{sh}(\nu)^2}{\Delta_{ch}(\mu, \nu)^2} e^{\frac{1}{2\pi} \sum_{a=1}^{N_2} \nu_a^2} (\ldots), \tag{F12}
\]

\[
\langle \ldots \rangle_{\mu, \nu} := \int \prod_{j=1}^{N_1} \frac{d\mu_j}{2\pi} \prod_{a=1}^{N_2} \frac{d\nu_a}{2\pi} \frac{\Delta_{sh}(\mu)^2}{\Delta_{ch}(\mu, \nu)^2} \frac{\Delta_{sh}(\nu)^2}{e^{-\frac{1}{2\pi} \sum_{i=1}^{N_1} \mu_i^2 - \sum_{a=1}^{N_2} \nu_a^2}} (\ldots). \tag{F13}
\]

Note that in (F11) only \( \mu_i \) are integrated and \( \nu_a \) are regarded as parameters of the theory. In (F12) the roles of \( \mu_i \) and \( \nu_a \) are interchanged. In this notation the \( \frac{1}{\alpha'} \)-BPS Wilson loop on the \( U(N_2) \) is expressed as

\[
W_{\frac{1}{\alpha'}}^H(N_1, N_2; n)_k \propto \left( \sum_{a=1}^{N_2} e^{\eta_a} \right)_{\mu, \nu}. \tag{F14}
\]

In the expression (F12) we can regard the \( U(N_1)_k \times U(N_2)_{-k} \) ABJ theory as the \( U(N_2) \) Chern–Simons-matter theory at level \(-k\) with \( 2N_1 \) hypermultiplets without Fayet–Iliopoulos terms. From this viewpoint the flavor group is \( U(2N_1) \) instead of \( U(N_1) \). In (F12), \( \mu_i (i = 1, \ldots, N_1) \) can be thought of as mass parameters for the hypermultiplets. The Seiberg dual is then the \( U(N_2) \) Chern–Simons-matter theory at level \( k \) with \( 2N_1 \) hypermultiplets of masses \( \nu_a (a = 1, \ldots, N_1) \) without Fayet–Iliopoulos terms [24]. These parameters are to be integrated in the end.

In Step 1, to find (F3) for the ABJ theory, we consider the following quantity:

\[
\left\langle e^{\eta_a} \prod_{i=1}^{N_1} \left( e^{-\left( \mu_i - \nu_a \right)} + 1 \right)^2 \right\rangle_{\nu}. \tag{F15}
\]

Integrals are invariant under shifts of integration variables. For our purpose, we consider, in particular, the shift

\[
\nu_a \to \nu_a + 2\pi i, \tag{F16}
\]

where \( a \) is a selected index but can be chosen arbitrarily. Note that the inserted factor \( \prod_{a=1}^{N_2} (e^{-\left( \mu_i - \nu_a \right)} + 1)^2 \) exactly cancels the poles that appear in the integrand (F12) and allows us to make the above shift, keeping the range of integration intact. Under this shift (F15) becomes

\[
\left\langle e^{k\pi i + k\nu_a} e^{\eta_a} \prod_{i=1}^{N_1} \left( e^{-\left( \mu_i - \nu_a \right)} + 1 \right)^2 \right\rangle_{\nu}. \tag{F17}
\]

Since this is identical to (F15), we obtain the identity:

\[
\left\langle \left( 1 - e^{k\pi i + k\nu_a} \right) e^{\eta_a} \prod_{i=1}^{N_1} \left( e^{-\left( \mu_i - \nu_a \right)} + 1 \right)^2 \right\rangle_{\nu} = 0. \tag{F18}
\]

\[\text{[30] Precisely speaking, the ABJ theory has a superpotential, whereas the class of theories considered in the Giveon–Kutasov duality [24] do not. Furthermore, the dual theories in the latter case have singlet meson fields that together with dual quarks generate a superpotential. However, this difference can be safely disregarded since the superpotential has no effect on the results of localization and only affects the R-charges of the matter fields.}\]
This equation must hold for any representation of BPS Wilson loops and then yields the operator identity
\[
(1 - (-1)^k x^k) \prod_{l=1}^{N_1} (x e^{-\mu_l} + 1)^2 = 0.
\] (F19)
where we introduced \( x = e^{\nu_a} \). For later convenience, we rewrite this as
\[
0 = p(x) := (x^k - (-1)^k) \prod_{l=1}^{N_1} (x + e^{\mu_l})^2
\] (F20)
\[
= x^{k+2N_1} + \left( 2 \sum_{i=1}^{N_1} e^{\mu_i} \right) x^{k+2N_1-1} + \left( \sum_{i=1}^{N_1} e^{2\mu_i} \right) x^{k+2N_1-2} + \ldots.
\] (F21)
Note that this polynomial is at most of order \( O(x^{k+2N_1}) \).
In Step 2, we introduce the following quantities:
\[
\tilde{p}(t) = t^{k+2N_1} p(t^{-1})
\]
\[
= \left( 1 + (-1)^{k+1} t^k \right) \left[ 1 + t \left( 2 \sum_{i=1}^{N_1} e^{\mu_i} \right) + t^2 \left( \sum_{i=1}^{N_1} e^{2\mu_i} + 4 \sum_{i<j} e^{\mu_i+\mu_j} \right) + O(t^3) \right]
\] (F22)
and
\[
\Psi_{\tilde{p}}(t) := \tilde{p}(t) \Psi(t) = \sum_{i=1}^{t^i \psi_{\tilde{p}i}}
\]
\[
= 1 + t \left( \square + 2 \sum_{i=1}^{N_1} e^{\mu_i} \right) + t^2 \left( \square + 2 \square \sum_{i=1}^{N_1} e^{\mu_i} + \sum_{i=1}^{N_1} e^{2\mu_i} + 4 \sum_{i<j} e^{\mu_i+\mu_j} \right) + O(t^3)
\]
\[
+ (-1)^{k+1} t^k \left( 1 + t \left( 2 \sum e^{\mu_i} + \square \right) + O(t^2) \right).
\] (F23)
The Wilson loop algebra is generated by \( \phi_i \) and \( \psi_{\tilde{p}i} \), which are constrained by (F6).
In Step 3, the Seiberg duality of the Wilson loops is extracted from the transformation (F8). At \( O(t) \) we obtain
\[
\square \Pi \rightarrow -\tilde{\Pi} - 2 \sum_i e^{\mu_i},
\] (F24)
where the subscripts I and II indicate that the Wilson loops are on the first and second gauge groups, respectively. In this example, the BPS Wilson loop of the original \( U(N_2)_{-k} \) theory in the fundamental representation maps to minus that of the dual \( U(\tilde{N}_2)_k \) theory in the fundamental representation shifted by a singlet that depends on the masses \( \mu_i \) in the dual theory.\(^{31}\)

\(^{31}\)In the \( k = 1 \) case it might look naively as if there is an additional identity operator coming from the \( O(t^k) \) term that contributes to the duality map. However, the \( \frac{1}{2} \)-BPS Wilson loop with winding \( n \) is only well defined for \( n < \frac{k}{2} \) and thus the \( n = 1 \) fundamental Wilson loop is ill defined for \( k = 1 \). This method is therefore not applicable to the ABJ theory with \( k = 1 \). For the same reason the \( O(t^{k/2}) \) terms in (F23) are meaningless because the \( \frac{1}{2} \)-BPS Wilson loops with winding \( n < \frac{k}{2} \) are involved at this order.
In order to find the duality maps for the ABJ theory, we need to integrate over the mass parameters \( \mu_i \) \((i = 1, \ldots, N)\). The map (F24) then becomes

\[ \Box \rightarrow -\Box - 2\Box. \quad (F25) \]

In our notation this reads

\[ W_{\Box}(N_1, N_2; 1)_k \rightarrow -W_{\Box}(\tilde{N}_2, N_1; 1)_k - 2W_{\Box}(\tilde{N}_2, N_1; 1)_k. \quad (F26) \]

This indeed agrees with (2.20).

We can run the same procedure for the flavor Wilson loops. Since the flavor group is unchanged under Seiberg duality, we simply obtain the following map:

\[ \Box \rightarrow \Box. \quad (F27) \]

which reads

\[ W_{\Box}(N_1, N_2; 1)_k \rightarrow W_{\Box}(\tilde{N}_2, N_1; 1)_k. \quad (F28) \]

This agrees with (2.21). Finally, the maps (F26) and (F28) imply that

\[ W_{\Box}(N_1, N_2; 1)_k \rightarrow -W_{\Box}(\tilde{N}_2, N_1; 1)_k. \quad (F29) \]

This agrees with (2.22). We have thus succeeded in reproducing Seiberg duality for the ABJ Wilson loops with the winding \( n = 1 \).

**F.2. Wilson loop duality in more general representations**

From the isomorphism of the Wilson loop algebras, we can also extract the duality transformations for the Wilson loops in higher-dimensional representations in the ABJ theory. Here we consider the Wilson loops that involve two boxes in Young diagrams corresponding to the \( \mathcal{O}(t^2) \) terms in (F23):

\[
\psi_{p2} = \Box + 2\Box \sum_i e^{\mu_i} + \sum_i e^{2\mu_i} + 4\sum_{i<j} e^{\mu_i+\mu_j}. \quad (F30)
\]

Integrating over the mass parameters \( \mu_i \), we find the duality transformation of the antisymmetric Wilson loop:

\[
W_{\Box} \rightarrow \tilde{W}_{\Box} + 2\tilde{W}_{\Box} + \tilde{W}_{\Box} + 4\tilde{W}_{\Box}. \quad (F31)
\]

where \( \tilde{W}_{\Box}^{n=2} \) is a fundamental Wilson loop with winding \( n = 2 \) and is decomposed into

\[
\tilde{W}_{\Box}^{n=2} = \tilde{W}_{\Box} - \tilde{W}_{\Box}. \quad (F32)
\]

Using this relation, (F31) yields

\[
W_{\Box} \rightarrow \tilde{W}_{\Box} + \tilde{W}_{\Box} + 3\tilde{W}_{\Box} + 2\tilde{W}_{\Box}. \quad (F33)
\]
This agrees with the result from the heuristic derivation in the brane picture in Sect. 4.1. For the flavor Wilson loops, we simply have

\[
W_{\bullet} \rightarrow \tilde{W}_{\bullet}, \quad (F34)
\]

\[
W_{\bullet \bullet} \rightarrow \tilde{W}_{\bullet \bullet}. \quad (F35)
\]

Lastly, we consider the Wilson loop that is on both gauge groups:

\[
W_{\square \square}. \quad (F36)
\]

The map of this Wilson loop is obtained from the duality transformation for the fundamental Wilson loops:

\[
W_{\square \square} = \square \square \rightarrow \tilde{\square} \tilde{\square} \left( - \tilde{\square} - 2 \tilde{\square} \right)
\]

\[
= - \tilde{W}_{\square \square} - 2 \left( \sum_a \epsilon^{\nu_a} \right)^2
\]

\[
= - \tilde{W}_{\square \square} - 2 \tilde{W}_{\bullet \bullet} - 2 \tilde{W}_{\bullet \square}, \quad (F37)
\]

where integration over \( \nu_a \) is implied. Combining (F33), (F35), and (F37), we find the map of the \( \frac{1}{2} \)-BPS Wilson loop:

\[
W_{\frac{1}{2} \frac{1}{2}} \rightarrow \tilde{W}_{\frac{1}{2} \frac{1}{2}}. \quad (F38)
\]

This again agrees with the result in Sect. 4.1. Although it becomes increasingly cumbersome, it is straightforward to generalize this procedure to more general higher-dimensional representations.

References

[1] N. Seiberg, Nucl. Phys. B 435, 129 (1995) [arXiv:9411149 [hep-th]] [Search inSPIRE].
[2] E. Witten, Commun. Math. Phys. 117, 353 (1988).
[3] V. Pestun, Commun. Math. Phys. 313, 71 (2012) [arXiv:0712.2824 [hep-th]] [Search inSPIRE].
[4] M. Marino, J. Phys. A 44, 463001 (2011) [arXiv:1104.0783 [hep-th]] [Search inSPIRE].
[5] O. Aharony, O. Bergman, and D. L. Jafferis, J. High Energy Phys. 0811, 043 (2008) [arXiv:0807.4924 [hep-th]] [Search inSPIRE].
[6] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, J. High Energy Phys. 0810, 091 (2008) [arXiv:0806.1218 [hep-th]] [Search inSPIRE].
[7] C.-M. Chang, S. Minwalla, T. Sharma, and X. Yin, [arXiv:1207.4485 [hep-th]] [Search inSPIRE].
[8] A. Kapustin, B. Willett, and I. Yaakov, J. High Energy Phys. 1003, 089 (2010) [arXiv:0909.4559 [hep-th]] [Search inSPIRE].
[9] N. Drukker, M. Marino, and P. Putrov, Commun. Math. Phys. 306, 511 (2011) [arXiv:1007.3837 [hep-th]] [Search inSPIRE].
[10] C. P. Herzog, I. R. Klebanov, S. S. Pufu, and T. Tesileanu, Phys. Rev. D 83, 046001 (2011) [arXiv:1011.5487 [hep-th]] [Search inSPIRE].
[11] H. Fuji, S. Hirano, and S. Moriyama, J. High Energy Phys. 1108, 001 (2011) [arXiv:1106.4631 [hep-th]] [Search inSPIRE].
[12] M. Marino and P. Putrov, J. Stat. Mech. 1203, P03001 (2012) [arXiv:1110.4066 [hep-th]] [Search inSPIRE].
[13] Y. Hatsuda, S. Moriyama, and K. Okuyama, J. High Energy Phys. 1301, 158 (2013) [arXiv:1211.1251 [hep-th]] [Search inSPIRE].
[14] Y. Hatsuda, S. Moriyama, and K. Okuyama, J. High Energy Phys. 1305, 054 (2013) [arXiv:1301.5184 [hep-th]] [Search inSPIRE].
[15] S. Matsumoto and S. Moriyama, J. High Energy Phys. 1403, 079 (2014) [arXiv:1310.8051 [hep-th]] [Search inSPIRE].
[16] M. Honda and K. Okuyama, [arXiv:1405.3653 [hep-th]] [Search inSPIRE].
[17] H. Awata, S. Hirano, and M. Shigemori, [arXiv:1212.2966 [hep-th]] [Search INSPIRE].
[18] F. Benini, C. Closset, and S. Cremonesi, J. High Energy Phys. 1002, 036 (2010) [arXiv:0911.4127 [hep-th]] [Search INSPIRE].
[19] A. Kapustin, B. Willett, and I. Yaakov, J. High Energy Phys. 1010, 013 (2010) [arXiv:1003.5694 [hep-th]] [Search INSPIRE].
[20] M. Marino, Commun. Math. Phys. 253, 25 (2004) [arXiv:0207096 [hep-th]] [Search INSPIRE].
[21] M. Aganagic, A. Klemm, M. Marino, and C. Vafa, J. High Energy Phys. 0402, 010 (2004) [arXiv:0211098 [hep-th]] [Search INSPIRE].
[22] M. Honda, J. High Energy Phys. 1312, 046 (2013) [arXiv:1310.3126 [hep-th]] [Search INSPIRE].
[23] A. Hanany and E. Witten, Nucl. Phys. B 492, 152 (1997) [arXiv:9611230 [hep-th]] [Search INSPIRE].
[24] A. Amariti, D. Forcella, L. Girardello, and A. Mariotti, J. High Energy Phys. 1005, 025 (2010) [arXiv:0903.3222 [hep-th]] [Search INSPIRE].
[25] A. Armoni, A. Giveon, D. Israel, and V. Niarchos, J. High Energy Phys. 0907, 061 (2009) [arXiv:0905.3195 [hep-th]] [Search INSPIRE].
[26] J. Evslin and S. Kuperstein, J. High Energy Phys. 0912, 016 (2009) [arXiv:0906.2703 [hep-th]] [Search INSPIRE].
[27] K. Jensen and A. Karch, J. High Energy Phys. 0909, 004 (2009) [arXiv:0906.3013 [hep-th]] [Search INSPIRE].
[28] F. Benini, C. Closset, and S. Cremonesi, J. High Energy Phys. 1110, 075 (2011) [arXiv:1108.5373 [hep-th]] [Search INSPIRE].
[29] A. Amariti, C. Klare, and M. Siani, J. High Energy Phys. 1210, 019 (2012) [arXiv:1111.1723 [hep-th]] [Search INSPIRE].
[30] C. Closset, J. High Energy Phys. 1203, 056 (2012) [arXiv:1201.2432 [hep-th]] [Search INSPIRE].
[31] H. C. Kim, J. Kim, S. Kim, and K. Lee, [arXiv:1204.3895 [hep-th]] [Search INSPIRE].
[32] D. Xie, [arXiv:1311.0889 [hep-th]] [Search INSPIRE].
[33] S. Dwivedi and P. Ramadevi, J. High Energy Phys. 1407, 084 (2014) [arXiv:1401.2767 [hep-th]] [Search INSPIRE].
[34] A. Amariti and D. Forcella, J. High Energy Phys. 1407, 082 (2014) [arXiv:1404.4052 [hep-th]] [Search INSPIRE].
[35] A. Amariti and C. Klare, [arXiv:1405.2312 [hep-th]] [Search INSPIRE].
[36] A. Giveon and D. Kutasov, Rev. Mod. Phys. 71, 983 (1999) [arXiv:9802067 [hep-th]] [Search INSPIRE].
[37] A. Kapustin, B. Willett, and I. Yaakov, [arXiv:1012.4021 [hep-th]] [Search INSPIRE].
[38] E. Witten, Commun. Math. Phys. 121, 351 (1989).
[39] N. Drukker, J. Plefka, and D. Young, J. High Energy Phys. 0811, 019 (2008) [arXiv:0809.2787 [hep-th]] [Search INSPIRE].
[40] B. Chen and J.-B. Wu, Nucl. Phys. B 825, 38 (2010) [arXiv:0809.2863 [hep-th]] [Search INSPIRE].
[41] S.-J. Rey, T. Suyama, and S. Yamaguchi, J. High Energy Phys. 0903, 127 (2009) [arXiv:0809.3786 [hep-th]] [Search INSPIRE].
[42] N. Drukker and D. Trancanelli, J. High Energy Phys. 1002, 058 (2010) [arXiv:0912.3006 [hep-th]] [Search INSPIRE].
[43] M. Marino and P. Putrov, J. High Energy Phys. 1006, 011 (2010) [arXiv:0912.3074 [hep-th]] [Search INSPIRE].
[44] N. Drukker, M. Marino, and P. Putrov, J. High Energy Phys. 1111, 141 (2011) [arXiv:1103.4844 [hep-th]] [Search INSPIRE].
[45] A. Klemm, M. Marino, M. Schierereck, and M. Sorosh, [arXiv:1207.0611 [hep-th]] [Search INSPIRE].
[46] Y. Hatsuda, M. Honda, S. Moriyama, and K. Okuyama, J. High Energy Phys. 1310, 168 (2013) [arXiv:1306.4297 [hep-th]] [Search INSPIRE].
[47] Y. Hatsuda, M. Marino, S. Moriyama, and K. Okuyama, [arXiv:1306.1734 [hep-th]] [Search INSPIRE].
[48] A. Kapustin and B. Willett, [arXiv:1302.2164 [hep-th]] [Search INSPIRE].
[49] D. Berenstein and D. Trancanelli, Phys. Rev. D 78, 106009 (2008) [arXiv:0808.2503 [hep-th]] [Search INSPIRE].
[52] B. Willett and I. Yaakov, [arXiv:1104.0487 [hep-th]] [Search inSPIRE].
[53] C. Closset, T. T. Dumitrescu, G. Festuccia, Z. Komargodski, and N. Seiberg, J. High Energy Phys. 1209, 091 (2012) [arXiv:1206.5218 [hep-th]] [Search inSPIRE].
[54] T. Kitao, K. Ohta, and N. Ohta, Nucl. Phys. B 539, 79 (1999) [arXiv:9808111 [hep-th]] [Search inSPIRE].
[55] K. Ohta and Y. Yoshida, Phys. Rev. D 86, 105018 (2012) [arXiv:1205.0046 [hep-th]] [Search inSPIRE].
[56] K. Okuyama, Prog. Theor. Phys. 127, 229 (2012) [arXiv:1110.3555 [hep-th]] [Search inSPIRE].
[57] M. Honda and S. Moriyama, [arXiv:1404.0676 [hep-th]] [Search inSPIRE].
[58] M. Nishizawa, Lett. Math. Phys. 37, 201 (1996) [arXiv:9505086 [q-alg]] [Search inSPIRE].