ON THE CONTINUITY OF THE TOPOLOGICAL ENTROPY OF NON-AUTONOMOUS DYNAMICAL SYSTEMS

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ABSTRACT. Let M be a compact Riemannian manifold. The set \( F(M) = \{ (f_i) \in \mathbb{Z} \mid f_i \in C^r \} \) consisting of sequences \((f_i) \in \mathbb{Z}\) of \(C^r\)-diffeomorphisms on \(M\) can be endowed with the compact topology or with the strong topology. A notion of topological entropy is given for these sequences. I will prove this entropy is discontinuous at each sequence if we consider the compact topology on \(F(M)\). On the other hand, if \(r \geq 1\) and we consider the strong topology on \(F(M)\), this entropy is a continuous map.

1. Introduction

In 1965, R. L. Adler, A. G. Konheim and M. H. McAndrew introduced the topological entropy of a continuous map \(\phi : X \to X\) on a compact topological space \(X\) via open covers of \(X\). Roughly, topological entropy is the exponential growth rate of the number of essentially different orbit segments of length \(n\). In 1971, R. Bowen defined the topological entropy of a uniformly continuous map on an arbitrary metric space via spanning and separated sets, which, when the space is compact, it coincides with the topological entropy as defined by Adler, Konheim and McAndrew. Both definitions can be found in [11].

Let \(M\) be a compact metric space. Let \(f = (f_i) \in \mathbb{Z}\) be a sequence of homeomorphisms defined on \(M\). The \(n\)-th composition is defined, for each \(i \geq 1\), as
\[
    f^n_i = f_{i+n-1} \circ \cdots \circ f_i \quad \text{and} \quad f^{-n}_i = f_{i-n}^{-1} \circ \cdots \circ f_{i-1}^{-1} : M_i \to M_{i-n}, \quad n \geq 0.
\]

This notion is known as non-stationary dynamical systems or non-autonomous dynamical systems (see [3], [6], [7]). S. Kolyada and L. Snoha, in [7], introduced a notion of topological entropy for this type of dynamical systems, which generalizes the notion of entropy for single dynamical systems. They only considered sequences of type \((f_i)_{i \geq 0}\) and the entropy for this sequence is a single number (possibly \(+\infty\)). Naturally, this idea can be extended to two-sided sequences \((f_i)_{i \in \mathbb{Z}}\). We will consider sequences of type \((f_i)_{i \in \mathbb{Z}}\) because, since each \(f_i\) is a homeomorphism, we can compute another entropy for the same sequence by considering the composition of the inverse of each \(f_i\) for \(i \to -\infty\) (see Remark 5.9).

Firstly, the entropy of a non-autonomous dynamical system \((f_i)_{i \in \mathbb{Z}}\) will be defined as a sequence of non-negative numbers \((a_i)_{i \in \mathbb{Z}}\), where each \(a_i\) depends only on \(f_j\) for \(j \geq i\). Then we will see that \((a_i)_{i \in \mathbb{Z}}\) is a constant sequence (see Corollary 5.6). Consequently, this common number will be considered as the entropy of \((f_i)_{i \in \mathbb{Z}}\). As a consequence, we will also see the entropy of a non-autonomous dynamical system can be considered as the topological entropy of a single homeomorphism defined on the union disjoint of infinitely many copies of \(M\) (see Remark 5.7).
Let \( F^r(M) \) be the set consisting of families \((f_i)_{i \in \mathbb{Z}}\) of \( C^r\)-diffeomorphisms on \( M \), where \( M \) is a compact Riemannian manifold. \( F^r(M) \) can be endowed with the compact topology and the strong topology (see Definitions 3.7 and 3.8). In this paper I will show that, if \( r \geq 1 \) and if we consider the strong topology on \( F^r(M) \), the entropy depends continuously on each sequence in \( F^r(M) \). In contrast, with the product topology on \( F^r(M) \), the entropy is discontinuous at any sequence.

Many results are well-known about the continuity of the entropy of single maps. In [8], Newhouse proved that the topological entropy of \( C^\infty\)-diffeomorphisms on a compact Riemannian manifold is an upper semicontinuous map. Furthermore, if \( M \) is a surface, this map is continuous. The entropy for any homeomorphism on the circle \( S^1 \) is zero. Therefore, it depends continuously on homeomorphisms on \( S^1 \). In contrast, the entropy does not depend on continuous maps that are not homeomorphisms on the circle (see [4]). About the continuity of the entropy for flows, readers could see [9].

Next, I will talk about the structure of this work. In the next section I will give a motivation for this work as well as some further generalizations. Section 3 will be devoted to remembering the notions of non-autonomous dynamical systems. Furthermore we will also see the type of conjugacies that work for these systems and the strong and product topologies on \( F^r(M) \). In Section 4 will be introduced the entropy for non-autonomous dynamical systems. This will be given using both open partitions of \( M \) and also separated and spanning sets. These definitions coincide, as in the case of single maps. Some properties of the entropy will be given in Section 5. These properties generalize to the ones of the entropy of single maps. Finally, in Section 6, we will see that the entropy is continuous on \( F^r(M) \) with the strong topology if \( r \geq 1 \). More specifically, it is locally constant. In contrast, it is discontinuous at any sequence if we consider the product topology on \( F^r(M) \).

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2. Motivation and Further Generalizations

Dynamical systems are classified via topological conjugacies. Uniform conjugacies (see Definition 3.3) are very suitable for classifying non-autonomous dynamical systems, time-one maps of flows, discrete time process generated by non-autonomous differential equations, among others systems (see [3], [6], [2], [9]). In that case, the entropy plays a fundamental role, since it is invariant by uniform conjugacies (see Theorem 5.4). In [3], [2] and [1] can be found several properties that are invariant by uniform conjugacies.

Next, considering the product topology on \( F^r(M) \), the entropy for non-autonomous dynamical systems could be a new tool to study the continuity of the topological entropy for some single maps (see Proposition 6.4).

The entropy to be constructed here, it will be fixed a metric space \( M \) and each map \( f_i : M \to M \) will be a homeomorphisms. This notion can be extended considering, for each \( i \in \mathbb{Z} \), a more general metric space \( M_i \) (that is, \( M_i \) must not necessarily be of the form \( M \times \{i\} \), as will be considered in this work) with a fixed metric \( d_i \) and each \( f_i \) being a continuous map on \( M_i \) to \( M_{i+1} \), not necessarily a homeomorphism. An interesting work would be to study the properties of this entropy.

In this work will be proved the continuity of the entropy of non-autonomous dynamical systems as long as each diffeomorphism \( f_i \) is of class \( C^r \) with \( r \geq 1 \). Another very interesting work would be to study the continuity of this entropy for sequences of Hölder continuous homeomorphisms. In this case, \( M \) could be a general metric space, that is, not necessarily a differentiable manifold, and
the continuity could depend on the Hausdorff dimension of $M$. A series of results that could be very useful to work on this problem can be found in [4], [7], [15], among others papers.

3. Non-autonomous Dynamical Systems, Uniform Conjugacy and Strong Topology

Given a metric space $M$ with metric $d$, consider the disjoint union

$$M = \bigsqcup_{i \in \mathbb{Z}} M_i = \bigcup_{i \in \mathbb{Z}} M \times i.$$  

The set $M$ will be called total space and the $M_i$ will be called components. Remember that a subset $A \subseteq M$ is open in $M$ if only if $A \cap M_i$ is open in $M_i$, for all $i \in \mathbb{Z}$. The total space will be equipped with the metric

$$d(x, y) = \begin{cases} \min\{1, d(x, y)\} & \text{if } x, y \in M_i \\ 1 & \text{if } x \in M_i, y \in M_j \text{ and } i \neq j. \end{cases}$$  

(3.1)

Two metrics $\rho_1$ and $\rho_2$ on a topological space $X$ are uniformly equivalent if there exist positive numbers $k$ and $K$ such that $k \rho_1(x, y) \leq \rho_2(x, y) \leq K \rho_1(x, y)$ for all $x, y \in X$. It is clear that if $\hat{d}$ and $\tilde{d}$ are uniformly equivalent metrics on $M$, then, $\mathbf{d}$ and $\mathbf{\hat{d}}$, obtained as in (3.1), generate the same topology on $M$ and, in that case, they are uniformly equivalent on $M$. On the other hand, if $\hat{d}_i$ and $\tilde{d}_i$ are uniformly equivalent metrics on $M_i$ for each $i \in \mathbb{Z}$, then the metrics $\hat{d}$ and $\tilde{d}$, defined similarly as in (3.1), generate the same topology on the total space, but they are not necessarily uniformly equivalent on $M$ (notice that $M$ is not compact). Throughout this work, we fix a metric $d$ on $M$ and we consider the metric on the total space as it was defined in (3.1). Without losing generality, we can suppose that the diameter of $M$ is less than or equal to 1 (in that case, if $x, y \in M$, then $d(x, y) = d(x, y)$).

**Definition 3.1.** A non-autonomous dynamical system $f$ on $M$, which will be denoted by $(M, f)$, is an application $f : M \to M$, such that, for each $i \in \mathbb{Z}$, $f|M_i = f_i : M_i \to M_{i+1}$ is a homeomorphism. Sometimes we use the notation $f = (f_i)_{i \in \mathbb{Z}}$. A $n$-th composition is defined, for each $i \in \mathbb{Z}$, as

$$f^n_i := \begin{cases} f_{i+n-1} \circ \cdots \circ f_i : M_i \to M_{i+n} & \text{if } n > 0 \\ f_{i-n}^{-1} \circ \cdots \circ f_{i-1}^{-1} : M_i \to M_{i-n} & \text{if } n < 0 \\ I_i : M_i \to M_i & \text{if } n = 0, \end{cases}$$

where $I_i$ is the identity on $M_i$.

A simple example of a non-autonomous dynamical systems is the constant family associated to a homeomorphism:

**Example 3.2.** Let $\varphi : M \to M$ be a homeomorphism. The constant family $(M, f)$ associated to $\varphi$ is the sequence $(f_i : M_i \to M_{i+1})_{i \in \mathbb{Z}}$ defined as $f_i(x, i) = (\varphi(x), i + 1)$ for each $x \in M$ and $i \in \mathbb{Z}$.

Next we talk about the morphisms between non-autonomous dynamical systems. Take

$$N = \bigsqcup_{i \in \mathbb{Z}} N \times \{i\},$$

where $N$ is a metric space, and consider a non-autonomous dynamical system $g$ defined on $N$. A topological conjugacy between $(M, f)$ and $(N, g)$ is a map $h : M \to N$, such that, for each $i \in \mathbb{Z}$, $h|_{M_i} = h_i : M_i \to N_i$ is a homeomorphism and $h_{i+1} \circ f_i = g_i \circ h_i : M_i \to N_{i+1}$, that is, the following
diagram commutes:

\[
\begin{array}{cccc}
M_{-1} & \overset{f_{-1}}{\longrightarrow} & M_0 & \overset{f_0}{\longrightarrow} & M_1 & \overset{f_1}{\longrightarrow} & M_2 \\
\downarrow h_{-1} & & \downarrow h_0 & & \downarrow h_1 & & \downarrow h_2 & \\
N_{-1} & \overset{g_{-1}}{\longrightarrow} & N_0 & \overset{g_0}{\longrightarrow} & N_1 & \overset{g_1}{\longrightarrow} & N_2
\end{array}
\]

It is clear that the topological conjugacies define an equivalence relation on the set consisting of the non-autonomous dynamical systems on \( M \). However, if \( M_0 \) and \( N_0 \) are homeomorphic, the partition obtained by this relation is trivial: Indeed, if \( h_0 \) is a homeomorphism between \( M_0 \) and \( N_0 \), the systems \((M,f)\) and \((N,g)\) are conjugate by \( h : M \rightarrow N \) defined as

\[
h_i = \begin{cases} 
  h_0 & \text{if } i = 0 \\
  g_{i-1} \circ \cdots \circ g_0 \circ h_0 \circ f_{-1}^{-1} \circ \cdots \circ f_{i-1}^{-1} & \text{if } i > 0 \\
  g_i^{-1} \circ \cdots \circ g_{-1}^{-1} \circ h_0 \circ f_{-1} \circ \cdots \circ f_i & \text{if } i < 0.
\end{cases}
\]

One type of conjugacy that works for the class of non-autonomous dynamical systems are the uniform conjugacy:

**Definition 3.3.** We say that a topological conjugacy \( h : M \rightarrow N \) between \((M,f)\) and \((N,g)\) is uniform if \((h_i : M_i \rightarrow N_i)_{i \in \mathbb{Z}}\) and \((h_i^{-1} : N_i \rightarrow M_i)_{i \in \mathbb{Z}}\) are equicontinuous sequences (that is, \( h \) and \( h^{-1} \) are uniformly continuous). In that case we will say that the systems are uniformly conjugate.

Since the composition of uniformly continuous applications is uniformly continuous, the class consisting of non-autonomous dynamical systems becomes a category, where the objects are the non-stationary dynamical systems and the morphisms are the uniform conjugacies.

Another notion of conjugacy, which is weaker than the conjugacy given in Definition 3.3, that also works for non-autonomous dynamical systems is the next one:

**Definition 3.4.** A positive (negative) uniform conjugacy between two systems \((M,f)\) and \((N,g)\) is a sequence of homeomorphisms \( h_i : M_i \rightarrow N_i \) for \( i \geq 0 \) (for \( i \leq 0 \)) such that \((h_i)_{i \geq 0}\) and \((h_i^{-1})_{i \geq 0}\) are equicontinuous and \( h_{i+1} \circ f_i = g_i \circ h_i \) for every \( i \geq 0 \) (for every \( i \leq -1 \)). That is, \((f_i)_{i \geq 0}\) and \((g_i)_{i \geq 0}\) are uniformly conjugate.

The following lemma it is clear and therefore we will omit the proof.

**Lemma 3.5.** \((M,f)\) and \((N,g)\) are positive (negative) uniformly conjugate if and only if, for any \( i_0 \in \mathbb{Z} \), there exists a sequence of homeomorphisms \((h_i)_{i \geq i_0}\) and \((h_i^{-1})_{i \leq i_0}\) such that \((h_i)_{i \geq i_0}\) and \((h_i^{-1})_{i \geq i_0}\) are equicontinuous and \( h_{i+1} \circ f_i = g_i \circ h_i \) for every \( i \geq i_0 \) (for every \( i \leq i_0 \)).

Take two homeomorphisms \( g_1 : X_1 \rightarrow X_1 \) and \( g_2 : X_2 \rightarrow X_2 \) defined on two compact metric spaces \( X_1 \) and \( X_2 \). Let \( f_1 \) and \( f_2 \) be the constant families associated, respectively, to \( g_1 \) and to \( g_2 \). It is clear that if \( g_1 \) and \( g_2 \) are topologically conjugate (i.e., there exists a homeomorphism \( h : X_1 \rightarrow X_2 \) such that \( h \circ g_1 = g_2 \circ h \)) then \( f_1 \) and \( f_2 \) are uniformly conjugate. In [11] is proved the reciprocal is not always true, that is, there exist uniformly conjugate constant families \( f_1 \) and \( f_2 \), associated, respectively, to two homeomorphisms \( g_1 \) and \( g_2 \) that are not topologically conjugate.

**Definition 3.6.** Let \((M,f)\) and \((\tilde{M},\tilde{f})\) be non-autonomous dynamical systems. We say that \((\tilde{M},\tilde{f})\) is a gathering of \((M,f)\) if there exists a strictly increasing sequence of integers \((n_i)_{i \in \mathbb{Z}}\) such that \( \tilde{M}_i = M_{n_i} \) and \( \tilde{f}_i = f_{n_{i+1}} \circ \cdots \circ f_{n_{i+1}} \circ f_{n_{i}} \):

\[
\cdots \xrightarrow{f_{n_{i+1}} \circ \cdots \circ f_{n_{i+1}} \circ f_{n_{i}}} M_{n_i} \xrightarrow{f_{n_{i+1}} \circ \cdots \circ f_{n_{i+1}} \circ f_{n_{i}}} M_{n_{i+1}} \cdots
\]
If \((\tilde{M}, \tilde{f})\) is a gathering of \((M, f)\), we say that \((M, f)\) is a dispersal of \((\tilde{M}, \tilde{f})\).

In [3], Proposition 2.5, is proved that any non-autonomous dynamical system has a dispersal, which has a gathering, which is equal to the constant family associated to the identity on \(M\). Notice that, if \((M, f)\) and \((N, g)\) are uniformly conjugate by \(h = (h_i)_{i \in \mathbb{Z}}\), then the gatherings \((\tilde{M}, \tilde{f})\) and \((\tilde{N}, \tilde{g})\) obtained, respectively, of \((M, f)\) and \((N, g)\) by a sequence of integers \((n_i)_{i \in \mathbb{Z}}\), are uniformly conjugate by the family \(h = (\tilde{h}_i)_{i \in \mathbb{Z}}\) :

\[
\begin{align*}
M_{n_{i-1}} \xrightarrow{f_{n_{i-1}}} \cdots \xrightarrow{f_{n_i}} M_{n_i} \xrightarrow{f_{n_{i+1}-1}} \cdots \xrightarrow{f_{n_{i+1}}} M_{n_{i+1}} \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
M_{n_{i-1}} \xrightarrow{g_{n_{i-1}}} \cdots \xrightarrow{g_{n_i}} M_{n_i} \xrightarrow{g_{n_{i+1}-1}} \cdots \xrightarrow{g_{n_{i+1}}} M_{n_{i+1}}
\end{align*}
\]

I will finish this section giving two different topologies to the space consisting of non-autonomous dynamical systems: the product topology and the strong topology. All the results and notions that will be presented in this part of the work can be found in [5]. Let \(r \geq 0\). I will suppose that \(M\) is a compact \(C^r\)-Riemannian manifold with Riemannian norm \(\| \cdot \|\). This norm induces a metric \(d\) on \(M\). Set

\[
F^r(M) = \{ f = (f_i)_{i \in \mathbb{Z}} : f_i : M \to M_{i+1} \text{ is a } C^r\text{-diffeomorphism} \}.
\]

If \(r = 0\), \(F^r(M)\) consists of the sequences of homeomorphisms. Let

\[
\text{Diff}^r(M_i, M_{i+1}) = \{ g : M_i \to M_{i+1} : g \text{ is a } C^r\text{-diffeomorphism} \}.
\]

The Riemannian metric \(\| \cdot \|\) induces a \(C^r\)-metric on \(\text{Diff}^r(M_i, M_{i+1})\), which will be denoted by \(d^r\). Notice that

\[
F^r(M) = \bigsqcup_{i = -\infty}^{+\infty} \text{Diff}^r(M_i, M_{i+1}).
\]

**Definition 3.7.** The product topology on \(F^r(M)\) is generated by the sets

\[
\mathcal{U} = \prod_{i < j} \text{Diff}^r(M_i, M_{i+1}) \times \prod_{i = -j}^{j} \{ U_i \} \times \prod_{i > j} \text{Diff}^r(M_i, M_{i+1}),
\]

where \(U_i\) is an open subset of \(\text{Diff}^r(M_i, M_{i+1})\), for \(-j \leq i \leq j\), for some \(j \in \mathbb{N}\). The space \(F^r(M)\) with the product topology will be denoted by \((F^r(M), \tau_{\text{prod}})\).

**Definition 3.8.** For each \(f \in F^r(M)\) and a sequence of positive numbers \(\varepsilon = (\varepsilon_i)_{i \in \mathbb{Z}}\), a strong basic neighborhood of \(f\) is the set

\[
B^r(f, \varepsilon) = \{ g = (g_i)_{i \in \mathbb{Z}} \in F^r(M) : d^r(f_i, g_i) < \varepsilon_i, \text{ for all } i \in \mathbb{Z} \}.
\]

The \(C^r\)-strong topology (or \(C^r\)-Whitney topology) on \(F^r(M)\) is generated by the strong basic neighborhoods of each \(f \in F^r(M)\). The space \(F^r(M)\) with the strong topology will be denoted by \((F^r(M), \tau_{str})\).

Notice that \(\tau_{str}\) is finer than \(\tau_{prod}\) on \(F^r(M)\), that is,

\[
I : (F^r(M), \tau_{str}) \to (F^r(M), \tau_{prod})
\]

\[
(f_i)_{i \in \mathbb{Z}} \mapsto (f_i)_{i \in \mathbb{Z}}
\]

is continuous.
4. Entropy for Non-stationary Dynamical Systems

In this section we will see how the topological entropy for a non-autonomous dynamical systems \((f_i)_{i \in \mathbb{Z}}\) is constructed. Firstly, this entropy will not be a real number, but a sequence of non-negative numbers (possibly \(+\infty\)) \((a_i)_{i \in \mathbb{Z}}\) where each \(a_i\) depends only on \(f_j\) for \(j \geq i\). In the next section will be proved that this sequence is constant. Consequently, this common value will be considered as the topological entropy of \((f_i)_{i \in \mathbb{Z}}\). The proof of the statements in this section can be found in [11] for the case of a single map. Such proofs can be adapted for non-stationary dynamical systems and, therefore, will be omitted.

In order to define the entropy, consider the following notions: an open cover of \(M\) is a collection of open subsets of \(M\), \(\mathcal{A} = \{A_i\}_{i \in \mathbb{A}}\), such that \(M = \bigcup_i A_i\). In this section, \(\mathcal{A}\) and \(\mathcal{B}\) will denote open covers of \(M\). Since \(M_i = M \times \{i\}\), if \(\mathcal{A}\) is an open cover of \(M\), then \(\mathcal{A}_i = \mathcal{A} \times \{i\}\) is an open cover of \(M_i\). By abuse of notation, I will omit the sub index \(i\) of \(\mathcal{A}_i\) for covers of \(M_i\).

**Definition 4.1.** Let \(N(\mathcal{A})\) be the number of sets in a finite subcover of \(\mathcal{A}\) with smallest cardinality. The entropy of \(\mathcal{A}\) is the number \(H(\mathcal{A}) := \log N(\mathcal{A})\).

For each \(i \in \mathbb{Z}\) and \(n \geq 0\), set \((f_i^n)^{-1}(A) = (f_{i+n-1} \circ \cdots \circ f_i)^{-1}(A) : A \in \mathcal{A}\). Set \(\mathcal{A} \vee \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}\). Inductively we can define \(\bigvee_{m=1}^{k} \mathcal{A}^n\) for a collection of open covers \(\mathcal{A}^1, ..., \mathcal{A}^k\) of \(M\). \(\mathcal{B}\) is a refinement of \(\mathcal{A}\) if each element of \(\mathcal{B}\) is contained in some element of \(\mathcal{A}\).

**Proposition 4.2.** The entropy satisfies the following properties:

1. \(H(\mathcal{A} \vee \mathcal{B}) \leq H(\mathcal{A}) + H(\mathcal{B})\).
2. If \(\mathcal{B}\) is a refinement of \(\mathcal{A}\) then \(H(\mathcal{A}) \leq H(\mathcal{B})\).
3. \(H(\mathcal{A}_i) = H((f_i^n)^{-1}(\mathcal{A}))\) for each \(i \in \mathbb{Z}\) and \(k \geq 0\).
4. \(H(\bigvee_{k=0}^{n-1} (f_i^k)^{-1}(\mathcal{A})) \leq nH(\mathcal{A})\), for each \(i \in \mathbb{Z}\) and \(n \geq 1\).
5. The limit

\[
H_i(f, \mathcal{A}) = \lim_{n \to +\infty} \frac{1}{n} H \left( \bigvee_{k=0}^{n-1} (f_i^k)^{-1}(\mathcal{A}) \right)
\]

exists and is finite, for each \(i \in \mathbb{Z}\).

**Definition 4.3.** We define the entropy of \(f\) relative to \(\mathcal{A}\) as the sequence \(\mathcal{H}(f, \mathcal{A}) = (H_i(f, \mathcal{A}))_{i \in \mathbb{Z}}\). The topological entropy of \(f\) is the sequence \(\mathcal{H}(f) = (H_i(f))_{i \in \mathbb{Z}}\), where

\[
\mathcal{H}(f) = \sup \{H_i(f, \mathcal{A}) : \mathcal{A} \text{ is an open cover of } M\}.
\]

From now on, \(X\) will represent a compact metric space. We recall the topological entropy of a homeomorphism \(g : X \to X\), which we denote by \(h(g)\), is defined considering open covers of \(X\). Definition 4.3 only makes sense when \(\mathcal{A}\) is an open cover of \(M\) instead of a general open cover of \(M\). If we consider arbitrary collections of open covers of each \(M_i\), the limit (4.1) could be infinite (we can take open covers \(\mathcal{A}_i\) of each \(M_i\) with \(N(\mathcal{A}_i)\) arbitrarily large, for each \(i\)).

Now we introduce the definition of topological entropy using spanning and separated subsets. That entropy will be called \(\star\)-topological entropy for differentiate it from the topological entropy. As in the case of a single homeomorphism, the topological entropy coincides with \(\star\)-topological entropy for sequences (see Theorem 5.1).

**Definition 4.4.** Let \(n \in \mathbb{N}\), \(\varepsilon > 0\) and \(i \in \mathbb{Z}\) be given. We say that a compact subset \(K \subseteq M_i\) is a \((n, \varepsilon)\)-span of \(M_i\) with respect \(f\) if for each \(x \in M_i\) there exists \(y \in K\) such that \(\max_{0 \leq j < n} d(f_i^j(x), f_i^j(y)) < \varepsilon\), i.
Definition 4.5. The \( \star \)-topological entropy of \( f \) is the sequence \( H(f) = (H_i(f))_{i \in \mathbb{Z}} \) given by

\[
H_i(f) = \lim_{\varepsilon \to 0} r[i](\varepsilon, f).
\]

Now we define the entropy for families using separated subsets and we will prove that the entropy considering span subsets coincide with the entropy considering separated subsets.

Definition 4.6. Let \( n \in \mathbb{N}, \varepsilon > 0 \) and \( i \in \mathbb{Z} \) be fixed. A subset \( E \subseteq M_i \) is called \( (n, \varepsilon) \)-separated with respect to \( f \) if given \( x, y \in E \), with \( x \neq y \), we have \( \max_{0 \leq j < r} d(f_j(x), f_j(y)) > \varepsilon \), i.e., if for all \( x \in E \), the set \( \bigcap_{k=0}^{n-1}(f_k^j(x), \varepsilon) \) contains no other point of \( E \).

Denote by \( s[n, i](\varepsilon; f) \) the largest cardinality of any \( (n, \varepsilon) \)-separated subset of \( M_i \) with respect to \( f \). Set \( s[i](\varepsilon; f) = \limsup_{n \to +\infty} n \log s[n, i](\varepsilon; f) \).

Proposition 4.7. Given \( \varepsilon > 0 \) and \( i \in \mathbb{Z} \) we have:

1. \( r[n, i](\varepsilon; f) \leq s[n, i](\varepsilon; f) \leq r[n, i](\varepsilon/2; f) \), for all \( n > 0 \).
2. \( r[i](\varepsilon; f) \leq s[i](\varepsilon; f) \leq r[i](\varepsilon/2; f) \), for all \( n > 0 \).

From Proposition 4.7 we have \( H_i(f) = \lim_{\varepsilon \to 0} s[i](\varepsilon; f) \) for all \( i \in \mathbb{Z} \). Consequently, \( H_i(f) \) can be defined using either span or separated subsets.

Notice that if \( f \) is a constant family associated to a homeomorphism \( \phi : X \to X \), then it is clear that

\[
H_i(f) = h(\phi), \quad \text{for all } i \in \mathbb{Z}.
\]

Therefore, \( H \) generalizes the notion of topological entropy for single homeomorphisms.

Some estimations of the topological entropy for non-autonomous dynamical systems can be found in [10], [13] and [14].

5. Some Properties of the Entropy

In this section we will see some properties of the topological entropy. Some of them are analogous to the well-known properties of entropy for single maps. For singular maps, the topological entropy is invariant by topological conjugacies. The main result of this section is to prove the analogous result for non-autonomous dynamical systems, that is, the entropy is invariant by uniformly conjugacies between sequences (see Theorem 5.4). This result will be fundamental to show the continuity of the entropy in Section 6 (see Theorem 6.9).

As we had mentioned, the notions of entropy for families of homeomorphisms, considering either open covers or separated subsets, coincide. This fact can be proved analogously as in the case of single homeomorphisms (see [11], Chapter 7, Section 2):

Proposition 5.1. For each \( i \in \mathbb{Z} \) we have \( H_i(f) = H_i(f) \).

The topological entropy \( h(\phi) \) of a single homeomorphism \( \phi : X \to X \) satisfies \( h(\phi^n) = |n|h(\phi) \), for \( n \in \mathbb{Z} \). For families we have:
Proposition 5.2. Suppose \( f = (f_i)_{i \in \mathbb{Z}} \) is an equicontinuous sequence. Fix \( n \geq 1 \). Let \( (\bar{M}, \bar{f}) \) be the gathering obtained of \( (M, f) \) by the sequence \( (n_i)_{i \in \mathbb{Z}} \), that is, \( \bar{M}_i = M_{n_i} \) and \( \bar{f}_i = f_{n(i+1)-1} \circ \cdots \circ f_{n_i} \):
\[
\cdots \rightarrow M_{n(i-1)} \xrightarrow{f_{n(i+1)-1} \circ \cdots \circ f_{n_i}} M_{n_i} \xrightarrow{f_{n(i+1)-1} \circ \cdots \circ f_{n_{i+1}}} M_{n_{i+1}} \cdots
\]
Thus, for each \( i \in \mathbb{Z} \) we have \( H_i(\bar{f}) = nH_{in}(f) \).

Proof. For \( i \in \mathbb{Z}, x, y \in M_{n_i} \) and \( m > 0 \), we have
\[
\max_{0 \leq k < m} d(f_{n_i}^{k}(x), f_{n_i}^{k}(y)) = \max_{0 \leq k < m} d(f_{n_i}^{k}(x), f_{n_i}^{k}(y)) \leq \max_{0 \leq j \leq m} d(f_{n_i}^{j}(x), f_{n_i}^{j}(y)).
\]
This fact proves that, for all \( \varepsilon > 0 \), each \( (nm, \varepsilon) \)-span subset \( K \) of \( M_{n_i} \) with respect to \( f \) is a \( (m, \varepsilon) \)-span subset of \( M_{n_i} \) with respect to \( \bar{f} \). Consequently, we obtain \( r[m, n_i](\varepsilon, \bar{f}) \leq r[mn, n_i](\varepsilon, f) \). Hence, \( H_i(\bar{f}) \leq nH_{in}(f) \).

On the other hand, since \( f \) is equicontinuous, we can prove that \( (f_{n_i})_{i \in \mathbb{Z}}, (f_{2n_i})_{i \in \mathbb{Z}}, \ldots, (f_{(m-1)n_i})_{i \in \mathbb{Z}} \) is a collection of equicontinuous families. Consequently, given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that
\[
\max_{1 \leq k < n} d(f_{nj+k} \cdots f_{nj+1} f_{nj}(x), f_{nj+k} \cdots f_{nj+1} f_{nj}(y)) : x, y \in M_{nj}, d(x, y) < \delta < \varepsilon.
\]
Now, if \( K \) is a \( (m, \delta) \)-span of \( M_{n_i} \) with respect to \( \bar{f} \), then, for all \( x \in M_{n_i} \), there exists \( y \in K \) such that
\[
\max_{0 \leq k < m} d(f_{nj+k}^{(m-1)n_i}(x), f_{nj+k}^{(m-1)n_i}(y)) < \delta.
\]
Thus,
\[
\max_{0 \leq k < m} d(f_{nj+k}^{(m-1)n_i}(x), f_{nj+k}^{(m-1)n_i}(y)) < \varepsilon,
\]
Consequently, we have
\[
\max_{0 \leq k < m} d(f_{nj+k}^{(m-1)n_i}(x), f_{nj+k}^{(m-1)n_i}(y)) < \varepsilon.
\]
Therefore,
\[
\max_{0 \leq k < m} d(f_{nj+k}^{(m-1)n_i}(x), f_{nj+k}^{(m-1)n_i}(y)) < \varepsilon.
\]
that is, \( K \) is a \( (mn, \varepsilon) \)-span of \( M_{n_i} \) with respect to \( f \). Hence, we have \( r[mn, n_i](\varepsilon, \bar{f}) \geq r[mn, n_i](\varepsilon, f) \) and, therefore, \( H_i(\bar{f}) \geq nH_{in}(f) \), which proves the proposition. \( \square \)

From the proof of Proposition 5.2 we have always the inequality
\[
H_i(\bar{f}) \leq nH_{in}(f).
\]

Proposition 5.3. Suppose \( f = (f_i)_{i \in \mathbb{Z}} \) is a sequence consisting of isometries, that is, \( f_i : M_i \rightarrow M_{i+1} \) is an isometry for all \( i \). Thus \( H_i(f) = 0 \), for all \( i \in \mathbb{Z} \).

Proof. If follows directly from Definition 4.5. \( \square \)

In the following theorem we will see that the entropy for non-autonomous dynamical systems is invariant for uniform conjugacies. This result generalizes the fact that the topological entropy of homeomorphisms defined on compact metric spaces is invariant by topological conjugacies.

Theorem 5.4. If \( (M, f) \) and \( (N, g) \) are uniformly conjugate, then \( H_i(f) = H_i(g) \) for all \( i \in \mathbb{Z} \).
Proof. Fix \( i \in \mathbb{Z} \). Let \( h = (h_i)_{i \in \mathbb{Z}} \) be a uniform conjugacy between \( f \) and \( g \). Since \( h \) is equicontinuous, given \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that, for all \( j \geq i \), if \( x, y \in M_j \) and \( d(x, y) < \delta \), then \( d(h(x), h(y)) < \varepsilon \). Let \( K \) be a \((m, \delta)\)-span of \( M_i \) with respect to \( f \). Thus, for all \( x \in M_i \) there exists \( y \in K \) such that \( \max_{0 \leq j < m} d(f^j(x), f^j(y)) = \max_{0 \leq j < m} d(g^j(x), g^j(y)) \). Consequently, if \( 0 \leq j < m \),

\[
\varepsilon > \max_{0 \leq j < m} d(h_{i+j} \circ f^j(x), h_{i+j} \circ f^j(y)) = \max_{0 \leq j < m} d(g^j \circ h_i(x), g^j \circ h_i(y)).
\]

This fact proves that \( r(m, i)(\varepsilon, f) \geq r(m, i)(\delta, g) \). Hence, \( H_i(f) \geq H_i(g) \). Since \( h^{-1} \) is equicontinuous, analogously we can prove that \( H_i(f) \leq H_i(g) \). \hfill \Box

It follows from the proof of the above theorem that if \( (f_i)_{i \in \mathbb{Z}} \) and \((g_i)_{i \in \mathbb{Z}} \) are uniformly conjugate (see Lemma 3.5) then \( H_{i_0}(f) = H_{i_0}(g) \). Furthermore, the entropy for homeomorphisms depends only on the future:

**Corollary 5.5.** Suppose that there exists \( i_0 \in \mathbb{Z} \) such that \( f_j = g_j \) for all \( j \geq i_0 \). Then for all \( i \in \mathbb{Z} \) we have \( H_i(f) = H_i(g) \).

Proof. It is clear that \((f_i)_{i \in \mathbb{Z}} \) and \((g_i)_{i \in \mathbb{Z}} \) are uniformly conjugate (take \( h_j = \text{Id} \) for each \( j \geq i_0 \)). It follows from Lemma 3.5 that, for any \( i \in \mathbb{Z} \), \((f_i)_{i \geq i} \) and \((g_i)_{i \geq i} \) are uniformly conjugate. By the proof of the Theorem 5.4 we have \( H_i(f) = H_i(g) \) for all \( i \in \mathbb{Z} \). \hfill \Box

**Corollary 5.6.** For all \( i, j \in \mathbb{Z} \) we have \( H_i(f) = H_j(f) \).

Proof. It is sufficient to prove that \( H_i(f) = H_{i+1}(f) \) for all \( i \in \mathbb{Z} \). Fix \( i \in \mathbb{Z} \). Take the family \( g = (g_j)_{j \in \mathbb{Z}} \), where \( g_j = \text{Id} : M_j \to M_{j+1} \) for each \( j \leq i \), the identity on \( M_i \), and \( g_j = f_j \) for \( j > i \). Thus \( H_i(f) = H_i(g) \).

For each \( x, y \in M_i \) and \( n \geq 2 \) we have

\[
\max_{0 \leq j < n} d(g^j(x), g^j(y)) = \max_{0 \leq j < n-1} d(g^j_{i+1}(x), g^j_{i+1}(y)).
\]

Using this fact we can prove that \( H_i(g) = H_{i+1}(g) \). Consequently, we have that \( H_i(f) = H_{i+1}(f) \). \hfill \Box

**Remark 5.7.** We can consider the sequence \( f = (f_i)_{i \in \mathbb{Z}} \) as a homeomorphism \( f : M \to M \) and then calculate the topological entropy of \( h(f) \) via spanning or separated sets of \( M \). It is not difficult to prove that \( h(f) = H_i(f) \) for any \( i \). Hence, from now on we will omit the index \( i \) of \( H_i \) and we will consider the entropy of a non-autonomous dynamical system as a single number, as a consequence of Corollary 5.6.

Remember that I am fixing one metric \( d \) on \( M \) and then considering the metric on the total space as in (3.1). If we consider another metric \( \tilde{d} \) uniformly equivalent to \( d \) on \( M \), then the identity

\[
I : (M, d) \to (M, \tilde{d})
\]

\[p \mapsto p\]

is a uniformly continuous map. It follows from Theorem 7.4 in [11] that the topological entropy of \( f \) considering the metric \( \tilde{d} \) on \( M \) coincides with the topological entropy of \( f \) considering \( d \) on \( M \). If follows that the entropy for a non-autonomous dynamical system on \( M \) does not depend on equivalent metrics on \( M \).

We can define the inverse of \( f \) as \( f^{-1} = (g_i)_{i \in \mathbb{Z}} \), where \( g_i := f^{-1}_i : M_{i+1} \to M_i \) for each \( i \). In this case, \( f^{-1}_{i+1} := I_{i+1} : M_{i+1} \to M_{i+1} \) and \( f^{-1}_0 = g_{1-n+1} \circ \cdots \circ g_i : M_{i+1} \to M_{i+1} \) for \( n > 0 \). In the case of a single homeomorphism \( \phi : X \to X \), we have \( h(\phi) = h(\phi^{-1}) \) (see [11], Theorem 7.3). The following example proves that, in general, we could have \( H(f) \neq H(f^{-1}) \).
Remark 5.9. Let \( f : M \to M \) be a homeomorphism on \( M \) with non-zero topological entropy. Let \( f_i : M_i \to M_{i+1} \) be the diffeomorphisms defined as \( f_i = I \) for \( i \geq 0 \) and \( f_i = \phi \) for \( i < 0 \) and take \( f = (f_i)_{i \in \mathbb{Z}} \). From Corollary 5.5 we have \( H(f) = h(I) = 0 \) and \( H(f^{-1}) = h(\phi) \neq 0 \), for each \( i \in \mathbb{Z} \).

There are dynamical systems defined on a compact metric space that are not topologically conjugate but they have the same topological entropy. Now, from Theorem 5.4 we have that two constant families associated to homeomorphisms with different topological entropies cannot be uniformly conjugate. On the other hand, in \([1]\) is proved that there are constant families, associated to homeomorphisms with the same topological entropy, that can be uniformly topologically conjugate. One natural question that arise from this notion of entropy are as follows: Let \((M, f)\) and \((M, g)\) be constant families. If \( H(f) = H(g) \) then \( f \) and \( g \) are always uniformly conjugate? The answer is negative, as shows the following example:

Example 5.8. Let \( I : M \to M \) be the identity on \( M \) and \( \phi : M \to M \) be a homeomorphism on \( M \)

\[ W^s(x, \psi) = \{ y \in X : \rho(\psi^n(x), \psi^n(y)) \to 0 \text{ as } n \to +\infty \}. \]

This set is called the stable set for \( \psi \) at \( x \). In \([1]\) is proved that, if \( h = (h_i)_{i \in \mathbb{Z}} \) is a uniform conjugacy between \((M, f)\) and \((N, g)\), then, for each \( x \in M \), we have

\[ h_i(W^s(x, f)) = W^s(h_i(x), g) \quad \text{ and } \quad h_i(W^u(x, f)) = W^u(h_i(x), g). \]

Let \( M \) be \( S^1 \), \( p_N \) be the north pole and \( p_S \) be the south pole of \( S^1 \). Suppose that \( \phi : M \to M \) is a homeomorphism with stable set \( W^s(p_N, \phi) = M \setminus \{p_S\} \). Let \( f \) and \( g \) be the constant families associated to \( \phi \) and to the identity on \( M \), respectively. Then \( H(f) = H(g) = 0 \) for all \( i \in \mathbb{Z} \), because all the homeomorphisms on the circle has zero entropy (see \([4,2]\)). On the other hand, we have \( W^s((p_N, 0), f) = [M \setminus \{p_S\}] \times \{0\} \) and \( W^s((p_N, 0), g) = \{(p_N, 0)\} \). Since the uniform conjugacies preserve the stable sets, we have that \( f \) and \( g \) can not be uniformly conjugate.

6. On the Continuity of the Entropy

Finally we will see that the entropy is continuous considering the strong topology on \( F^r(M) \), for \( r \geq 1 \). More specifically, the entropy is locally constant, that is, each \( (f_i)_{i \in \mathbb{Z}} \in F^r(M) \) has a strong basic neighborhood in which the entropy is constant. In contrast, on the continuity of \( H : (F^r(M), \tau_{\text{prod}}) \to \mathbb{R} \cup \{+\infty\} \), we have:

Proposition 6.1. Suppose that \( H(F^r(M)) \) has two or more elements. Then \( H : (F^r(M), \tau_{\text{prod}}) \to \mathbb{R} \cup \{+\infty\} \) is discontinuous at any \( f \in CF^r(M) \).

Proof. Let \( f = (f_i)_{i \in \mathbb{Z}} \in F^r(M) \). Since \( H(F^r(M)) \) has two or more elements, there exists \( g = (g_i)_{i \in \mathbb{Z}} \in F^r(M) \) such that \( H(g) \neq H(f) \). Let \( \mathcal{V} \in \tau_{\text{prod}} \) an open neighborhood of \( f \). For some \( k \in \mathbb{N} \), the family \( h = (h_i)_{i \in \mathbb{Z}} \), defined by

\[ h_i = \begin{cases} f_i & \text{if } -k \leq i \leq k \\ g_i & \text{if } i > k \text{ or } i < -k, \end{cases} \]

belongs to \( \mathcal{V} \), by definition of \( \tau_{\text{prod}} \). It is follow from Corollary 5.5 that

\[ H(h) = H(g), \]

which proves the proposition, since \((F^r(M), \tau_{\text{prod}})\) a metric space. \( \square \)
we denote the constant family associated to $\varphi$.

**Proof.** It is sufficient to prove that each $(f_i)_{i \in \mathbb{Z}}$, with $(f_i)_{i \in \mathbb{Z}} \in \text{CF}'(M)$, is open in $\text{CF}'(M)$. Let $(\varepsilon_i)_{i \in \mathbb{Z}}$ be a sequence of positive numbers with $\varepsilon_i \to 0$ as $|i| \to \pm \infty$. Consider the strong basic neighborhood $B'(f_i)_{i \in \mathbb{Z}}, (\varepsilon_i)_{i \in \mathbb{Z}} \subseteq \text{F}'(M)$ of $(f_i)_{i \in \mathbb{Z}}$. Notice that

$$(f_i)_{i \in \mathbb{Z}} = B'(f_i)_{i \in \mathbb{Z}}, (\varepsilon_i)_{i \in \mathbb{Z}} \cap \text{CF}'(M).$$

Consequently, $(f_i)_{i \in \mathbb{Z}}$ is open in $(\text{CF}'(M), \tilde{\tau}_{\text{str}})$. □

The application

$$\pi_0 : (\text{F}'(M), \tau) \to (\text{Diff}'(M, 0, M_1), d')$$

$$(f_i)_{i \in \mathbb{Z}} \mapsto f_0$$

is continuous for $\tau \in \{\tau_{\text{str}}, \tau_{\text{prod}}\}$. Hence, the restriction

$$\tilde{\pi}_0 = \pi_0|_{\text{CF}'(M)} : (\text{CF}'(M), \tilde{\tau}) \to (\text{Diff}'(M, 0, M_1), d')$$

is continuous for $\tilde{\tau} \in \{\tilde{\tau}_{\text{str}}, \tilde{\tau}_{\text{prod}}\}$. We can identify $(\text{Diff}'(M, 0, M_1), d')$ with the space $(\text{Diff}'(M), d')$, the space consisting of diffeomorphisms on $M$ endowed with the $C'$-metric obtained from the metric $d$ on $M$. From now on we will make use of this identification. For a $C'$-diffeomorphism $\phi : M \to M$, we denote the constant family associated to $\phi$ by $f_{\phi}$. Notice that $\tilde{\pi}_0$ is invertible, in fact,

$$\tilde{\pi}_0^{-1} : (\text{Diff}'(M), d') \to (\text{CF}'(M), \tilde{\tau})$$

$$\phi \mapsto f_{\phi}.$$ 

Clearly, if $\tilde{\tau} = \tilde{\tau}_{\text{str}}$, then $\tilde{\pi}_0^{-1}$ is not continuous (see Proposition 6.2). On the other hand, we have:

**Proposition 6.3.** If $\tilde{\tau} = \tilde{\tau}_{\text{prod}}$, then $\tilde{\pi}_0^{-1}$ is continuous.

**Proof.** All the open subsets of $(\text{CF}'(M), \tilde{\tau}_{\text{prod}})$ are union of sets with the form

$$\mathcal{U} = \left( \prod_{i < -j} \text{Diff}'(M_i, M_{i+1}) \times \prod_{i = -j}^j \{U_i\} \times \prod_{i > j} \text{Diff}'(M_i, M_{i+1}) \right) \cap \text{CF}'(M),$$

where $U_i$ is an open subset of $\text{Diff}'(M_i, M_{i+1})$, for $-j \leq i \leq j$. Notice that

$$(\tilde{\pi}_0^{-1})^{-1}(\mathcal{U}) = \tilde{\pi}_0(\mathcal{U}) = \bigcap_{i = -j}^j U_i,$$

which is an open subset of $\text{Diff}'(M)$. Thus, $\tilde{\pi}_0^{-1}$ is continuous. □

Consequently, we have:

**Proposition 6.4.** $H : (\text{CF}'(M), \tilde{\tau}_{\text{prod}}) \to \mathbb{R}$, is continuous if, and only if, $h : (\text{Diff}'(M), d') \to \mathbb{R}$ is continuous.

**Proof.** It is clear, because $H = h \circ \tilde{\pi}_0$ and $\tilde{\pi}_0$ is a homeomorphism. □
**Remark 6.5.** Proposition 6.4 could be a useful tool to show the continuity of the topological entropy at some $C^r$-diffeomorphisms: to show that $h$ is continuous at $\phi \in \text{Diff}^r(M)$, we could try to prove that $H|_{C^r(M)}$ is continuous at $f_\phi$. In order to prove this fact, we have to find an open neighborhood $U \subseteq \text{Diff}(M)$ of $\phi$, such that each constant family associated to any diffeomorphism in $U$ is uniformly conjugate to $f_\phi$. Therefore, by Theorem 5.4 and (4.2), we had that

$$h(\psi) = H(f_\psi) = H(f_\phi) = h(\phi) \quad \text{for any } \psi \in U.$$

In [1] we will prove that there exist diffeomorphisms $\phi$ and $\psi$ which are not topologically conjugate, however $f_\phi$ and $f_\psi$ could be uniformly conjugate.

Finally, we will prove the continuity of $H : (\mathcal{F}(M), \tau_{str}) \to \mathbb{R} \cup \{+\infty\}$ for any $r \geq 1$. It is sufficient to prove the case when $r = 1$.

Remember we are supposing that $M$ is a compact Riemannian manifold with Riemannian norm $\| \cdot \|$, which induces a metric $d$ on $M$, and then we consider the metric $d$ on $M$ as in (3.1). Let $\varrho > 0$ be such that, for each $x \in M$, the exponential application

$$\exp_x : B(0, \varrho) \to B(x, \varrho)$$

is a diffeomorphism and $\|v\| = d(\exp_x(v), x)$, for all $v \in B(0, \varrho)$, that is, $\varrho$ is the injectivity radius of $M$. We will suppose that $\varrho < 1/2$.

We will fix $f = (f_i)_{i \in \mathbb{Z}} \in \mathcal{F}(M)$. For $\delta > 0$ and $r = 0, 1$, set

$$D^r(I_i, \delta) = \{h \in \text{Hom}(M, M_i) : h \text{ is a } C^r \text{-diffeomorphism and } d^r(h, I_i) \leq \delta\}$$

and

$$D^1(f_i, \delta) = \{g \in \text{Diff}^1(M_i, M_{i+1}) : d^1(g, f_i) \leq \delta\}.$$

The closure of $D^1(I_i, \delta)$ on $D^0(I_i, \delta)$ will be denoted by $\overline{D^1(I_i, \delta)}$.

**Lemma 6.6.** There exist two sequences $(r_i)_{i \geq 0}$ and $(\delta_i)_{i \geq 0}$, with $r_i \to 0$ as $i \to +\infty$, such that, for each $g \in D^1(f_i, \delta_i)$, the map

$$\tilde{G}_{i+1} : D^r(I_{i+1}, r_{i+1}) \to D^r(I_i, r_i)$$

$$h \mapsto g^{-1}hf_i$$

is well-defined for each $i \geq 1$.

**Proof.** Notice that if $g \in \text{Diff}^1(M_i, M_{i+1})$ and $h \in \text{Diff}^1(M_{i+1}, M_{i+1})$, we have

$$d^1(g^{-1}hf_i, I_i) \leq d^1(g^{-1}hf_i, g^{-1}f_i) + d^1(g^{-1}f_i, I_i) \quad \text{for } i \geq 0.$$

If $h$ is $C^1$-close to $I_{i+1}$, then $g^{-1}hf_i$ is $C^1$-close to $g^{-1}f_i$ and if $g$ is $C^1$-close to $f_i$, then $g^{-1}f_i$ is $C^1$-close to $I_i$. Fix $r_0 \in (0, \varrho/4)$. There exist $r_1 \in (0, r_0/2)$ and $\delta_0 > 0$ such that, if $h \in D^1(I_i, r_1)$ and $g \in D^1(f_0, \delta_0)$, then $g^{-1}hf_0 \in D^1(I_0, r_0)$. Take $r_2 \in (0, r_1/2)$ and $\delta_1 > 0$ such that, if $h \in D^1(I_2, r_2)$ and $g \in D^1(f_1, \delta_1)$, then $g^{-1}hf_1 \in D^1(I_1, r_1)$. Hence, inductively, we can build two sequences $(r_i)_{i \geq 0}$ and $(\delta_i)_{i \geq 0}$, with $r_i \in (0, r_{i-1}/2)$ for each $i \geq 1$, such that $g^{-1}hf_i \in D^1(I_i, r_i)$, which proves the lemma.

Analogously, we can find a sequence of positive numbers $(\delta_i)_{i \leq 0}$ and $(r_i)_{i \leq 0}$, with $r_i \to 0$ as $i \to -\infty$, such that for each $g \in D^1(f_{i-1}, \delta_{i-1})$, the map

$$\tilde{G}_{i-1} : D^r(I_{i-1}, r_{i-1}) \to D^r(I_i, r_i)$$

$$h \mapsto f_{i-1}hg^{-1}$$

is well-defined for each $i \leq 0$.

\[\text{Here, } 0_x \text{ is the zero vector in } T_xM, \text{ the tangent space of } M \text{ at } x.\]
Lemma 6.7. There exist two sequences \( \tilde{h} = (\tilde{h}_i)_{i \geq 0} \in \prod_{i \geq 0} D^0(I, r_i) \) and \( \hat{h} = (\hat{h}_i)_{i \leq 0} \in \prod_{i \leq 0} D^0(I, r_i) \) such that
\[
\tilde{G}_{i+1} \tilde{h}_{i+1} = \tilde{h}_i \text{ for all } i \geq 0 \quad \text{and} \quad \tilde{G}_{i-1} \tilde{h}_{i-1} = \tilde{h}_i \text{ for all } i \leq 0.
\]

**Proof.** For each \( i > 0 \), let \( h^i = \tilde{G}_1 \circ \cdots \circ \tilde{G}_i(I) \). It follows from Lemma 6.6 that \( h^i \) belongs to \( D^1(I_0, r_0) \). Consequently, the sequence \( (h^i)_{i \geq 0} \) is equicontinuous, because each \( h^i \) is \( C^1 \) and the sequence has uniformly bounded derivative. Hence, there exist a subsequence \( i_m \to \infty \) and \( \tilde{h}_0 \in D^0(I_0, r_0) \) such that \( h^{i_m} \to \tilde{h}_0 \) as \( m \to \infty \). Notice that \( G_1 \) is invertible and both \( G_1 \) and \( G_1^{-1} \) are continuous. Consequently,
\[
G_1(D^1(I, r_1)) = G_1(D^1(I, r_1)).
\]
Since \( \tilde{h}_0 \in G_1(D^1(I, r_1)) \), we have
\[
\tilde{h}_1 = \tilde{G}_1^{-1}(\tilde{h}_0) \in D^1(I, r_1) \subseteq D^0(I, r_1).
\]
Inductively, we can prove
\[
\tilde{h}_i = \tilde{G}_i^{-1} \circ \cdots \circ \tilde{G}_1^{-1}(\tilde{h}_0) \in D^0(I, r_i) \quad \text{for each } i \geq 1.
\]
Take \( \hat{h} = (\hat{h}_i)_{i \geq 0} \). It is clear that \( \tilde{G}_{i+1} \tilde{h}_{i+1} = \tilde{h}_i \) for all \( i \geq 0 \).

The proof of the existence of \( \hat{h} \) is analogous and therefore we omit it. □

Notice that \( \tilde{h}_0 \) is a limit of \( C^1 \)-diffeomorphisms, which are \( \varrho/4 \)-close to \( I_0 \) in the \( C^1 \)-topology. Consequently, for each \( x \in M_0 \),
\[
[\exp_x^{-1} \circ \tilde{h}_0 \circ \exp_x - \exp_x^{-1} \circ I_0 \circ \exp_x]_{B(0, \varrho)}
\]
is \( \varrho/4 \)-Lipschitz. Since \( \varrho < 1 \), we can prove that \( \tilde{h}_0 \) is injective. Furthermore, for each \( i \geq 0 \) and \( x \in M_i \), we have
\[
d(\tilde{h}_i^{-1}(x), x) = d(\tilde{h}_i^{-1}(x), \tilde{h}_i \tilde{h}_i^{-1}(x)) = d(y, \tilde{h}_i(y)),
\]
where \( y = \tilde{h}_i^{-1}(x) \). Hence \( d^0(\tilde{h}_i, I_i) = d^0(\tilde{h}_i^{-1}, I_i) \) for each \( i \geq 0 \).

Analogously, we can prove that \( \hat{h}_i \) is invertible and \( d^0(\hat{h}_i, I_i) = d^0(\hat{h}_i^{-1}, I_i) \) for each \( i \leq 0 \).

Lemma 6.8. The families \( (\hat{h}_i)_{i \geq 0} \), \( (\hat{h}_i^{-1})_{i \geq 0} \), \( (\tilde{h}_i)_{i \leq 0} \) and \( (\tilde{h}_i^{-1})_{i \leq 0} \) are equicontinuous.

**Proof.** Let \( \varepsilon > 0 \). Since \( \tilde{h}_i, \tilde{h}_i^{-1} \in D^0(I, r_i) \) and \( r_i \to 0 \) as \( i \to +\infty \), there exists \( k > 0 \) such that, for each \( i > k \),
\[
\max\{d^0(\tilde{h}_i, I_i), d^0(\tilde{h}_i^{-1}, I_i)\} < \varepsilon/3.
\]
Hence, if \( i < k \) and \( x, y \in M_i \), with \( d(x, y) < \varepsilon/3 \), then
\[
d(\hat{h}_i(x), \hat{h}_i(y)) \leq d(\hat{h}_i(x), I_i(x)) + d(I_i(x), I_i(y)) + d(I_i(y), \hat{h}_i(y)) < \varepsilon
\]
and
\[
d(\hat{h}_i^{-1}(x), \hat{h}_i^{-1}(y)) \leq d(\hat{h}_i^{-1}(x), I_i(x)) + d(I_i(x), I_i(y)) + d(I_i(y), \hat{h}_i^{-1}(y)) < \varepsilon.
\]
On the other hand, it is clear that there exists \( \delta \in (0, \varepsilon/3) \) such that, if \( 0 \leq i \leq k \), and \( x, y \in M_i \), with \( d(x, y) < \delta \), then
\[
\max\{d(\hat{h}_i(x), \hat{h}_i(y)), d(\tilde{h}_i(x)^{-1}, \tilde{h}_i^{-1}(y))\} < \varepsilon.
\]
The facts above prove that for each \( i \geq 0 \), if \( x, y \in M_i \) and \( d(x, y) < \delta \), then
\[
\max\{d(\hat{h}_i(x), \hat{h}_i(y)), d(\tilde{h}_i^{-1}(x), \tilde{h}_i^{-1}(y))\} < \varepsilon.
\]
Consequently, \( (\hat{h}_i)_{i \geq 0} \) and \( (\hat{h}_i^{-1})_{i \geq 0} \) are equicontinuous. Analogously we can prove that \( (\tilde{h}_i)_{i \leq 0} \) and \( (\tilde{h}_i^{-1})_{i \leq 0} \) are equicontinuous. □

Finally, we have:
Theorem 6.9. For all $r \geq 1$, 

$$H : (F'(M), \tau_{str}) \to \mathbb{R} \cup \{+\infty\} \quad \text{and} \quad H^{(-1)} : (F'(M), \tau_{str}) \to \mathbb{R} \cup \{+\infty\}$$

are locally constants.

Proof. Let $f \in F'(M)$. It follows from Lemmas 6.7 and 6.8 that there exists a strong basic neighborhood $B^1(f, (r_i)_{i \in \mathbb{Z}})$ such that every $g \in B^1(f, (r_i)_{i \in \mathbb{Z}})$ is positively and negatively uniformly conjugate to $f$. Thus, from Theorem 5.4 we have $H(g) = H(f)$ and $H^{(-1)}(g) = H^{(-1)}(f)$ for all $g \in B^1(f, (r_i)_{i \in \mathbb{Z}})$, which proves the theorem. □

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