Abstract

In all spatial dimensions $d$, we study the static and dynamical properties of a general-
ized Smoluchowski equation which describes the evolution of a gas obeying a logotropic 
equation of state, $p = A \ln \rho$. A logotrope can be viewed as a limiting form of polytrope 
($p = K \rho^\gamma$, $\gamma = 1 + 1/n$), with index $\gamma = 0$ or $n = -1$. In the language of generalized 
thermodynamics, it corresponds to a Tsallis distribution with index $q = 0$. We solve 
the dynamical logotropic Smoluchowski equation in the presence of a fixed external force 
deriving from a quadratic potential, and for a gas of particles subjected to their mutual 
gravitational force. In the latter case, the collapse dynamics is studied for any negative 
index $n$, and the density scaling function is found to decay as $r^{-\alpha}$, with $\alpha = \frac{2n}{n-1}$ for 
$n < -\frac{d}{2}$, and $\alpha = \frac{2d}{d+2}$ for $-\frac{d}{2} \leq n < 0$.

Key words: Nonlinear meanfield Fokker-Planck equations, generalized thermodynam-
ics, polytropic equation of state, self-gravitating systems, chemotaxis.

1 Introduction

Recently, several researchers have studied generalized forms of Fokker-Planck equations [1, 2]. 
These equations arise when the coefficients of diffusion, mobility and friction in the usual 
Fokker-Planck equations (Kramers, Smoluchowski,...) explicitly depend on the concentration 
of particles. This can take into account phenomenologically “hidden constraints” that are not 
directly accessible to the observer [3]. These nonlinear Fokker-Planck equations are associated 
with generalized free energy functionals and non-standard distributions. For example, Tsallis 
$q$-distributions constitute an important class of non-Boltzmannian distributions [4]. Plastino 
& Plastino [5] have shown that they could be obtained as stationary solutions of a nonlinear 
Fokker-Planck equation, with potential applications to the context of porous media. Recently, 
Kaniadakis [6], Frank [7] and Chavanis [8] have shown that it was possible to construct more 
general Fokker-Planck equations leading to other forms of non-standard distributions. For 
example, Chavanis [8, 9, 10] has introduced a generalized Smoluchowski equation involving 
an arbitrary barotropic equation of state $p = p(\rho)$. This can be derived from a generalized 
Kramers equation in a strong friction limit [8, 11]. The usual Smoluchowski equation leading 
to the Boltzmann distribution corresponds to an isothermal equation of state $p = \rho T$ while the 
Tsallis distributions are associated with a polytropic equation of state $p = K \rho^\gamma$ where $\gamma = 1 + \frac{1}{n}$ 
plays the role of the Tsallis $q$-parameter.
In this paper, we consider the case of a logotropic equation of state
\[ p = A \ln \rho \]
where \( A \) is a positive constant. This equation of state sometimes appears in astrophysics \[12\], but we consider it here at a more general level. In Sec. 2, we show that a logtrope can be viewed as a limiting form of a polytrope with \( \gamma = 0 \) (or as a Tsallis distribution with \( q = 0 \)). In Sec. 3, we solve the logotropic Smoluchowski equation with a fixed quadratic potential. This is a particular case of the general analytical solution found by Plastino \& Plastino \[5\] and Tsallis \& Buckman \[13\] for the nonlinear Fokker-Planck equation (associated with polytropes). In Sec. 4, we consider a coupling with the gravitational potential \[14, 15\] and study the logotropic Smoluchowski-Poisson (LSP) system. This is a particular case of the nonlinear (polytropic) Smoluchowski-Poisson system studied in \[15\], but with a negative index \( n = -1 \).

We find that the collapse of polytropes with negative index is peculiar: (i) for \( n < -d/2 \), the scaling exponent is \( \alpha = 2n/(n - 1) \) like for positive index (ii) for \( -d/2 < n < 0 \), the scaling exponent is \( \alpha = 2d/(d + 2) \) corresponding to the collapse of a cold (pressureless) system. In particular, the scaling exponent of logotropes is \( \alpha = 2d/(d + 2) \) (\( d > 2 \)).

2 Nonlinear mean-field Fokker-Planck equations

2.1 The generalized Smoluchowski equation

In a recent series of papers \[15, 16, 17, 18, 19, 20, 21, 22, 23\], we have studied a general class of nonlinear mean-field Fokker-Planck equations of the form

\[ \frac{\partial \rho}{\partial t} = \nabla \cdot \left[ \frac{1}{\xi} (\nabla p + \rho \nabla \Phi) \right], \]

\[ \Phi(r, t) = \int u(|r - r'|)\rho(r', t)dr', \]

introduced in Chavanis \[8, 9, 10, 24\]. This type of equations arises in different domains of physics, chemistry, astrophysics and biology \[25\]. The drift-diffusion equation (1) can be viewed as a generalized Smoluchowski equation where \( p = p(\rho) \) is a barotropic equation of state and \( \Phi \) is a potential. This potential can be imposed by an external medium (in which case \( \Phi = \Phi_{ext}(r) \) is assumed given) or generated by the particles themselves through the mean-field equation (2) where \( u(|r - r'|) \) is a binary potential of interaction. The generalized Smoluchowski equation (1) can be derived from a generalized Kramers equation in a strong friction limit \( \xi \to +\infty \) by performing a Chapman-Enskog expansion in terms of the small parameter \( 1/\xi \) \[11\] or by using a method of moments \[10\]. It can also be directly obtained from the damped barotropic Euler equations by neglecting the inertial term in the equation for the velocity \[8\]. Thus, the generalized Smoluchowski equation can be interpreted as an overdamped limit of kinetic or hydrodynamic equations in which a friction force (real or effective) is present \[22\].

The generalized mean-field Smoluchowski system (1)-(2) decreases the Lyapunov functional

\[ F[\rho] = \int \rho \int_\rho^{\rho'} \frac{p(\rho')}{\rho'^2} d\rho' dr + \frac{1}{2} \int \rho \Phi dr, \]

which can be interpreted as a generalized free energy \[8\]. Indeed, \( \dot{F} = -\int \frac{1}{\xi \rho}(\nabla p + \rho \nabla \Phi)^2 dr \leq 0 \). The stationary solutions of Eq. (1) satisfy the condition

\[ \nabla p + \rho \nabla \Phi = 0, \]
which can be viewed as a condition of hydrostatic equilibrium. Using Eq. (4) and the equation of state \( p = p(\rho) \), we find that the equilibrium density field is of the form \( \rho = \rho(\Phi(r)) \). When substituted in Eq. (2), this yields an integro-differential equation for the potential \( \Phi(r) \). The steady states of Eqs. (1)-(2) are extrema of the free energy (3) at fixed mass \( M = \int \rho \, dr \). They satisfy the first order variations \( \delta F - \alpha \delta M = 0 \) which directly return Eq. (4). Furthermore, the steady states are linearly dynamically stable if, and only if, they are minima of the free energy (3) at fixed mass \( M = \int \rho \, dr \). 

When \( \Phi \) is interpreted as the gravitational potential determined by the Poisson equation, the stationary solutions of the generalized Smoluchowski-Poisson system (GSP) are the same as the stationary solutions of the Euler-Poisson system

\[
\frac{\partial \rho}{\partial t} + \nabla (\rho \mathbf{u}) = 0, \tag{5}
\]

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p + \nabla \Phi, \tag{6}
\]

\[
\Delta \Phi = S_d G \rho, \tag{7}
\]

describing barotropic stars [26]. However, the dynamics of these two systems is different as the Euler-Poisson system describes a fluid without friction \( \xi = 0 \) while the generalized Smoluchowski-Poisson system describes a Brownian gas in an overdamped limit \( \xi \to +\infty \) [22]. With these analogies and differences in mind, we can use many results obtained in astrophysics [26] when studying nonlinear mean-field Fokker-Planck equations of the form (1)-(2). We think therefore that it is enlightening to use a similar vocabulary and similar notations.

We note, finally, that drift-diffusion equations of the form (1)-(2) also arise in biology, in the context of chemotaxis [18, 27]. They can be written in the form

\[
\frac{\partial \rho}{\partial t} = \nabla \cdot [\chi (\nabla p - \rho \nabla c)], \tag{8}
\]

\[
\frac{\partial c}{\partial t} = -k(c)c + f(c)\rho + D_c \Delta c. \tag{9}
\]

They describe the collective motion of cells (usually bacteria or amoebae) that diffuse and that are attracted by a chemical substance that they emit. Here, \( \rho(\mathbf{r}, t) \) denotes the cell density and \( c(\mathbf{r}, t) \) is the concentration of chemo-attractant which induces the drift force. This model has been introduced by Keller & Segel [28]. In its most studied version [29], \( p(\rho) = \frac{D_c}{\chi} \rho \) is a linear function of the density which is similar to an isothermal equation of state. However, the general Keller-Segel model [28] allows the possibility that the diffusion be anomalous so that the function \( p = p(\rho) \) can be nonlinear. Note also that in the limit of large diffusivity of the chemical \( D_c \to +\infty \) (and for \( k(c) = k \) and \( f(c) = f \)), the time derivative \( \partial c/\partial t \) can be neglected [29] so that Eq. (9) reduces to a Poisson equation (like in gravity) with a screening term \(-k c\).

Therefore, the general study of equations of the form (1)-(2) connects several active areas of research in theoretical physics: nonlinear Fokker-Planck equations, self-gravitating systems, chemotaxis and non-extensive thermostatistics. Here, we shall focus on the case where the equation of state is that of a logotrope \( p = A \ln \rho \) and we shall make the connection with the previously studied polytropes \( p = K \rho^\gamma \).
2.2 The polytropic Smoluchowski equation

For the polytropic equation of state \( p = K \rho^\gamma \) (with \( \gamma = 1 + 1/n \)), we obtain the polytropic Smoluchowski equation

\[
\frac{\partial \rho}{\partial t} = \nabla \cdot \left[ \frac{1}{\xi} (K \nabla \rho^\gamma + \rho \nabla \Phi) \right].
\]

The free energy \( F \) corresponding to a polytropic equation of state can be conveniently written

\[
F[\rho] = \frac{K}{\gamma - 1} \int (\rho^\gamma - \rho) \, d\mathbf{r} + \frac{1}{2} \int \rho \Phi \, d\mathbf{r},
\]

and the stationary solutions of Eq. (10) are given by

\[
\rho = \left[ \lambda - \frac{\gamma - 1}{K \gamma} \Phi \right]^{\frac{1}{\gamma - 1}}.
\]

This polytropic distribution reproduces the statistics introduced by Tsallis in his generalized thermodynamics [4]. When \( \Phi \) is a fixed external potential, the polytropic Smoluchowski equation (10) is equivalent to the nonlinear Fokker-Planck equation introduced by Plastino & Plastino [5]. When \( \Phi = 0 \), it returns the equation of porous media. We note that the free energy (11) can be written \( F = E - T_{\text{eff}} S \) with \( E = \frac{1}{2} \int \rho \Phi \, d\mathbf{r} \), \( T_{\text{eff}} = K \) and \( S = -\frac{1}{\gamma - 1} \int (\rho^\gamma - \rho) \, d\mathbf{r} \). In the language of generalized thermodynamics, this can be viewed as a Tsallis free energy with index \( \gamma \) where \( K \) plays the role of a generalized temperature [8]. For \( \gamma = 1 \), i.e. \( n = +\infty \), we recover the isothermal equation of state \( p = \rho T \) and the Boltzmann free energy \( F = \frac{1}{2} \int \rho \Phi \, d\mathbf{r} + T \int \rho \ln \rho \, d\mathbf{r} \). As we shall see, the case \( \gamma = 0 \), i.e. \( n = -1 \), is also special and corresponds to what have been called logotropes in astrophysics [12].

2.3 The logotropic Smoluchowski equation

Let us consider the logotropic equation of state

\[
p = A \ln \rho,
\]

where \( A \) is a constant. This equation of state has been introduced in astrophysics to account for certain properties of molecular clouds that could not be understood in terms of isothermal distributions [12]. However, this equation of state can have application in more general situations, beyond the realm of astrophysics, so we shall consider it here on a general footing. Inserting Eq. (13) in Eq. (11), we get the logotropic Smoluchowski equation

\[
\frac{\partial \rho}{\partial t} = \nabla \cdot \left[ \frac{1}{\xi} (A \nabla \ln \rho + \rho \nabla \Phi) \right].
\]

The free energy \( F \) of a logotrope takes the form

\[
F[\rho] = -A \int \ln \rho \, d\mathbf{r} + \frac{1}{2} \int \rho \Phi \, d\mathbf{r}.
\]

It can be written \( F = E - T_{\text{eff}} S \) with \( T_{\text{eff}} = A \) and

\[
S[\rho] = \int \ln \rho \, d\mathbf{r}.
\]
In a previous paper [30], this functional has been called the log-entropy. The stationary states of the logotropic Smoluchowski equation are given by

\begin{equation}
\rho = \frac{1}{\lambda + \frac{1}{A} \Phi}.
\end{equation}

Note that when the potential is quadratic, i.e. \( \Phi = \frac{r^2}{2} \), the equilibrium distribution is the Lorentzian as noted in [30]. Thus, the log-entropy (16) can be seen as the functional associated with the Lorentzian distribution.

### 2.4 The polytropic index \( \gamma = 0 \)

Returning to the polytropic Smoluchowski equation (10), we note that the index \( \gamma = 0 \) is special because the polytropic Smoluchowski equation (10) and the Tsallis free energy (11) become trivial for this index except if \( K \to +\infty \) [30]. This type of indetermination is often characteristic of a logarithmic behavior as can be seen with a simple change of variables. If we set \( K = A/\gamma \), the polytropic Smoluchowski equation (10) can be rewritten

\begin{equation}
\frac{\partial \rho}{\partial t} = \nabla \cdot \left[ \frac{1}{\xi} \left( A \rho^{\gamma-1} \nabla \rho + \rho \nabla \Phi \right) \right].
\end{equation}

Under this form, this equation describes polytropes for \( \gamma \neq 0 \) and logotropes for \( \gamma = 0 \). Therefore, a logotrope can be viewed as a polytrope with index \( \gamma = 0 \), i.e. \( n = -1 \). The stationary solution of equation (18) is given by

\begin{equation}
\rho = \left[ \lambda - \frac{\gamma - 1}{A} \Phi \right]^{\frac{1}{\gamma-1}},
\end{equation}

and this distribution passes to the limit for \( \gamma \to 0 \). Indeed, for \( \gamma \neq 0 \) we can put Eq. (19) in the form of Eq. (12) and for \( \gamma = 0 \), we recover Eq. (17). The free energy associated with Eq. (18) can be written

\begin{equation}
F[\rho] = \frac{A}{\gamma(\gamma-1)} \int (\rho^\gamma - \rho) \, d\mathbf{r} + \frac{1}{2} \int \rho \Phi \, d\mathbf{r}.
\end{equation}

For \( \gamma \neq 0 \), we can put Eq. (20) in the form of Eq. (11) and for \( \gamma = 0 \), we recover Eq. (15) up to additive terms that do not depend on the density. Since only the variations of \( F \) matter, these terms (that can be infinite!) do not play any role [30]. In the same spirit, the general polytropic equation of state can be written \( p = \frac{A}{\gamma} \rho^\gamma \) with the convention that \( \lim_{\gamma \to 0} \rho^\gamma = \ln \rho + \text{Cst} \). Therefore, \textit{in the language of generalized thermodynamics, logotropes correspond to Tsallis statistics with index } \( \gamma = 0 \).

### 2.5 Other formulations

We can write the generalized Smoluchowski equation in the form

\begin{equation}
\frac{\partial \rho}{\partial t} = \nabla \cdot \left[ D \rho C''(\rho) \nabla \rho + \beta \rho \nabla \Phi \right],
\end{equation}

where \( C(\rho) \) is a convex function, \( \beta \) is a constant which can be interpreted as a generalized inverse temperature and \( D \) is a positive constant which can be interpreted as a generalized
diffusion coefficient. The mobility \( \mu = 1/\xi \) satisfies a form of Einstein relation \( \mu = D\beta \). From the convex function \( C \) we define a generalized entropic functional

\[
S[\rho] = -\int C(\rho) \, d\mathbf{r},
\]

and a generalized free energy \( J = S - \beta E \). This functional increases monotonically with time \( \dot{J} = \int \frac{D}{\rho}(\rho C''(\rho)\nabla \rho + \beta \rho \nabla \Phi)^2 \, d\mathbf{r} \geq 0 \). This can be viewed as a form of \( H \)-theorem in a canonical description where the inverse temperature \( \beta \) is fixed \[8\]. The steady solutions of Eq. (21) are determined by the integro-differential equation

\[
C'(\rho) = -\beta \Phi - \alpha,
\]

where \( \Phi \) is related to \( \rho \) by Eq. (2) in the general case. They are critical points of \( J \) at fixed mass \( M \); indeed, the first order variations satisfy \( \delta J - \alpha \delta M = 0 \) which returns Eq. (23). Furthermore, the linearly dynamically stable stationary solutions of Eq. (21) are maxima of \( J \) at fixed mass \( M \).

For the Tsallis entropic functional written in the form

\[
S[\rho] = -\frac{1}{\gamma(1-\gamma)} \int (\rho^\gamma - \rho) \, d\mathbf{r},
\]

the generalized Smoluchowski equation (21) reads

\[
\frac{\partial \rho}{\partial t} = \nabla \cdot \left[ D \left( \rho^{\gamma-1} \nabla \rho + \beta \rho \nabla \Phi \right) \right].
\]

Comparing with Eq. (18), we find that \( \beta = 1/A \). This is consistent with the previous formalism where \( A = T_{\text{eff}} \) is interpreted as an effective temperature. The stationary solution of Eq. (25) can be written

\[
\rho = \left[ \lambda - \beta(\gamma - 1)\Phi \right]^{1/\gamma}.
\]

We note that the parameter \( \beta \) has not the dimension of an inverse temperature, although this is the quantity which naturally enters in the free energy functional \( J \) and in the variational principle \( \delta S - \beta \delta E - \alpha \delta M = 0 \) determining the stationary solution (26). As discussed in [31], there are different notions of temperature in the case of polytropic distributions. We can write the distribution (26) in the alternative form

\[
\rho = Q \left[ 1 - b(\gamma - 1)\Phi \right]^{\frac{1}{\gamma-1}},
\]

where \( Q = \lambda^{\frac{1}{\gamma-1}} \) and \( b = \beta/\lambda \) now has the dimension of an inverse temperature.

### 2.6 The logotropic Kramers and Landau equations

The generalized Fokker-Planck equations (1) and (21) are written in position space. We can also introduce generalized Fokker-Planck equations in velocity space. When the friction term is linear in \( v \), they are referred to as generalized Kramers equations. Some examples are given in [8] for different forms of entropic functionals \( S = -\int C(f) \, d\mathbf{r} \). For the log-entropy \( C(f) = -\ln f \) considered in this paper, we obtain the logotropic Kramers equation

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} \cdot \left[ D \left( \frac{\partial \ln f}{\partial v} + \beta f v \right) \right].
\]
The generalized Kramers equation is associated with a canonical description where the temperature $\beta$ is fixed and the free energy $J = S - \beta E$ increases \[8, 24\]. Alternatively, we can consider the generalized Landau equation that is associated with a microcanonical description where the energy $E = \int f^2 \, dv$ is conserved and the generalized entropy $S$ increases \[3, 24\]. For the log-entropy $C(f) = -\ln f$, we obtain the logotropic Landau equation

\[
\frac{\partial f}{\partial t} = A \frac{\partial}{\partial v} \int u^2 \delta^{\mu \nu} - u^\mu u^\nu \left[ f' \frac{\partial \ln f}{\partial v^\nu} - f \frac{\partial \ln f'}{\partial v'^\nu} \right] \, dv',
\]

where $f = f(v, t)$ and $f' = f(v', t)$. The steady solution of the logotropic Kramers and Landau equations is the Lorentzian

\[
f = \frac{1}{\lambda + \beta v^2}.
\]

Note that if the velocity is not bounded, this distribution is not normalizable in $d \geq 2$.

### 3 The logotropic Smoluchowski equation with a fixed quadratic potential

We shall here provide an analytical time-dependant solution of the logotropic Smoluchowski equation in $d = 1$ for a fixed quadratic potential $\Phi = x^2/2 + \lambda x$. This solution is a straightforward extension of the solution found by Plastino & Plastino \[5\] and Tsallis & Bukman \[13\] for the polytropic Smoluchowski equation with $\gamma \neq 0$. We cannot directly make $\gamma = 0$ in their solution for the reason discussed in Sec. \[24\]. However, if we write the polytropic Smoluchowski equation in the form

\[
\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left[ \rho^{\gamma-1} \frac{\partial \rho}{\partial x} + \beta \rho x + \lambda \rho \right],
\]

we can follow exactly the reasoning of \[5, 13\] and finally let $\gamma \to 0$ since this limit is now well-behaved. Note that Eq. (31) with $\lambda = 0$ coincides with the polytropic and logotropic Kramers equation if $x$ is replaced by $v$ \[8\].

The stationary solution of Eq. (31) can be written

\[
\rho = Q \left[ 1 - \frac{1}{2} b(\gamma - 1) \left( x + \frac{\lambda}{2} \right)^2 \right]^{\frac{1}{\gamma - 1}},
\]

where $b = \beta/Q^{\gamma-1}$ and $Q$ is determined by the normalization condition (see below). For $\gamma > 1$, the density goes to zero at finite values of $x = x_1, x_2$ (say) and it is implicitly understood that $\rho = 0$ for $x < x_1$ and $x > x_2$. For $\gamma < 1$, the density distribution is defined for all $x$ and it decreases algebraically as $x^{-2/(1-\gamma)}$. Following the original idea of Plastino & Plastino \[5\], we look for time-dependant solutions of the polytropic Smoluchowski equation in the form

\[
\rho(x, t) = Q(t) \left[ 1 - \frac{1}{2} b(t)(\gamma - 1)(x - m(t))^2 \right]^{\frac{1}{\gamma - 1}},
\]

where $Q(t), b(t)$ and $m(t)$ are functions of time. The normalization condition $\int_{-\infty}^{\infty} \rho \, dx = 1$ requires that $\gamma > -1$ and leads to the relation

\[
b(t) = 2Q(t)^2 \mu^2,
\]
where we have defined

\begin{equation}
\mu \equiv \int_{-\infty}^{+\infty} \left[ 1 - (\gamma - 1)y^2 \right]^\frac{1}{2-\gamma} dy.
\end{equation}

Substituting the Ansatz (33) in Eq. (31), we find after straightforward calculations that

\begin{equation}
\dot{m} = -\beta m - \lambda,
\end{equation}

\begin{equation}
\dot{Q} = -2\mu^2 Q^{\gamma+2} + \beta Q.
\end{equation}

Starting from a Dirac distribution \( \rho(x, 0) = \delta(x - x_0) \), these equations are readily integrated and we obtain

\begin{equation}
Q(t) = \left[ \frac{\beta}{2\mu^2} \frac{1}{1 - e^{-\beta(\gamma+1)t}} \right]^\frac{1}{\gamma+1},
\end{equation}

\begin{equation}
m(t) = x_0 e^{-\beta t} + \frac{\lambda}{\beta} (e^{-\beta t} - 1),
\end{equation}

\begin{equation}
b(t) = 2\mu^2 Q(t)^2.
\end{equation}

For \( \gamma \neq 0 \), this returns the results obtained in [5, 13]. However, we can now pass to the logotropic limit \( \gamma \to 0 \) and get

\begin{equation}
\rho(x, t) = \frac{Q(t)}{1 + \frac{1}{2} b(t)(x - m(t))^2},
\end{equation}

\begin{equation}
Q(t) = \frac{\beta}{2\pi^2} \frac{1}{1 - e^{-\beta t}},
\end{equation}

\begin{equation}
m(t) = x_0 e^{-\beta t} + \frac{\lambda}{\beta} (e^{-\beta t} - 1),
\end{equation}

\begin{equation}
b(t) = 2\pi^2 Q(t)^2.
\end{equation}

In particular for \( \beta = 0 \), we obtain

\begin{equation}
Q(t) = \frac{1}{2\pi^2 t}, \quad m(t) = x_0 - \lambda t, \quad b(t) = \frac{1}{2\pi^2 t^2}.
\end{equation}

Hence

\begin{equation}
\rho(x, t) = \frac{1}{2\pi^2 t} \frac{1}{1 + \frac{(x-x_0)^2}{4\pi^2 t^2}}.
\end{equation}
4 The logotropic Smoluchowski-Poisson system

We shall consider here the coupling between the logotropic Smoluchowski equation and gravity. Thus, we consider the logotropic Smoluchowski-Poisson (LSP) system

\[ \frac{\partial \rho}{\partial t} = \nabla \cdot \left[ \frac{1}{\xi} (A \nabla \ln \rho + \rho \nabla \Phi) \right], \]

in \( d \)-dimensions \(^1\). This can be viewed as a particular case of the polytropic Smoluchowski-Poisson system studied in \[15\], corresponding to the index \( \gamma = 0 \) \((n = -1)\). This case must be treated separately (i) because the study of \[15\] is restricted to \( n \geq 0 \) and (ii) because the index \( \gamma = 0 \) needs a special discussion as we have seen previously. As indicated in Sec. \[2.1\] the steady solutions of the LSP system are the same as the steady solutions of the Euler-Poisson system with an equation of state \( p = A \ln \rho \) describing logotropic stars and logotropic clusters in astrophysics \[12\]. Therefore, the results of Secs. \[4.1-4.7\] for the steady states also apply to logotropic stars and logotropic clusters, while the results of Sec. \[4.8\] for the dynamics only apply to the LSP system.

4.1 The logotropic Lane-Emden equation

For a spherically symmetric distribution, the equation of hydrostatic equilibrium \[4\] combined with the Gauss theorem \( d\Phi/dr = GM(r)/r^{d-1} \) can be written

\[ \frac{dp}{dr} = -\rho \frac{GM(r)}{r^{d-1}}, \]

where \( M(r) = \int \rho(r', t)S_d r^{d-1} dr' \) is the mass within the sphere of radius \( r \). The foregoing equation can be put in the form

\[ \frac{1}{r^{d-1}} \frac{d}{dr} \left( \frac{r^{d-1} dp}{\rho} \right) = -S_d G \rho. \]

For the logotropic equation of state \( p = A \ln \rho \), we define

\[ \rho = \rho_0/\theta, \quad \xi = \left( \frac{S_d G \rho_0^2}{A} \right)^{1/2} r, \]

where \( \rho_0 \) is the central density and \( \xi \) a scaled distance. Substituting these relations in Eq. \[50\], we obtain what might be called the logotropic Lane-Emden equation

\[ \frac{1}{\xi^{d-1}} \frac{d}{d\xi} \left( \xi^{d-1} \frac{d\theta}{d\xi} \right) = \frac{1}{\theta}, \]

with \( \theta(0) = 1 \) and \( \theta'(0) = 0 \).

\(^1\)It should be made clear that, in the present work, we consider a self-gravitating Langevin gas with a \textit{prescribed} polytropic or logotropic equation of state. We furthermore consider a strong friction limit in which the dynamics is governed by the (generalized) Smoluchowski equation. This context is very different from the works of Taruya & Sakagami \[14, 32\] who consider the Hamiltonian \( N \)-stars system described by the orbit-averaged-Fokker-Planck equation and show, by means of direct numerical simulations, that the transient phases of the dynamics (quasi-equilibrium states) can be fitted by a sequence of polytropic distributions (Tsallis) with a time dependant index \( n(t) \) or \( q(t) \).
4.2 The singular logotropic sphere

We note that for \( d > 1 \), the function

\[
\theta_s = \frac{\xi}{\sqrt{d - 1}},
\]

is solution of the logotropic Lane-Emden equation. This is, however, not a regular solution since it does not satisfy the boundary conditions at \( \xi = 0 \). In terms of the density, this corresponds to a singular sphere

\[
\rho_s = \left[ \frac{A}{S_d(d - 1)G} \right]^{1/2} \frac{1}{r},
\]

whose density diverges at the origin. It is interesting to compare this singular logotropic sphere whose density profile decreases as \( r^{-1} \) for \( d > 1 \) to the singular isothermal sphere \( \rho_s = [2(d - 2)/(S_dG\beta)]r^{-2} \) whose density profile decreases as \( r^{-2} \) for \( d > 2 \) \([17]\). We note that the total mass of the system diverges for \( r \to +\infty \). However, if we consider box-confined configurations (in a sphere of radius \( R \)), the density distribution of the singular logotropic sphere can be written

\[
\rho_s = \left[ \frac{M(d - 1)}{S_dR^{d-1}} \right] \frac{1}{r}.
\]

It should be finally recalled that such singular solutions are not limited to isothermal and logotropic distributions: polytropic spheres with index \( n > n_3 = d/(d - 2) \) (for \( d > 2 \)) and \( n < -1 \) (for \( d > 1 \)) also admit singular solutions scaling like \( r^{-\alpha} \) with \( \alpha = 2n/(n - 1) \) \([15]\).

4.3 Logotropic profile in \( d = 1 \)

In \( d = 1 \), the logotropic Lane-Emden equation becomes

\[
\frac{d^2 \theta}{d\xi^2} = \frac{1}{\theta},
\]

with \( \theta(0) = 1 \) and \( \theta'(0) = 0 \). This is similar to the equation of motion of a particle in a potential \( V(\theta) = -\ln \theta \) where \( \theta \) plays the role of position and \( \xi \) the role of time. Using the boundary conditions at \( \xi = 0 \), the first integral is

\[
\frac{1}{2} \left( \frac{d\theta}{d\xi} \right)^2 - \ln \theta = 0,
\]

leading to

\[
\int_1^\theta \frac{d\phi}{\sqrt{\ln \phi}} = \sqrt{2}\xi.
\]

Performing the change of variable \( \ln \phi = y^2 \), we obtain the equivalent expression

\[
\sqrt{2} \int_0^{\sqrt{\ln \theta}} e^{y^2} dy = \xi.
\]

For \( \xi \to +\infty \), we get

\[
\theta \sim \xi \sqrt{2 \ln \xi}.
\]

Therefore, the density profile decreases like \( 1/(r\sqrt{\ln r}) \). The mass within a domain of size \( r \) behaves as \( M(r) \sim (\ln r)^{1/2} \) for \( r \to +\infty \) so that the total mass in infinite. The density profile of a one-dimensional logotrope is represented in Fig. [10].
4.4 Asymptotic behaviors

We shall determine the asymptotic behaviors of the solutions of the logotropic Lane-Emden equation (52). For $\xi \to 0$, using a Taylor expansion, we obtain

$$\theta = 1 + \frac{1}{2d} \xi^2 - \frac{1}{8d(d+2)} \xi^4 + ... \quad (\xi \to 0).$$

(61)

To investigate the limit $\xi \to +\infty$, we first perform the change of variables $t = \ln \xi$ and $\theta = \xi z$. Equation (52) then becomes

$$\frac{d^2 z}{dt^2} + d \frac{dz}{dt} + \left( \frac{d^2 - 2d + 2}{d-1} \right) z = 0.$$

(62)

This is similar to the damped motion of a particle in a potential $V(z) = -\ln z + (d-1)z^2/2$. For $d > 1$, the potential has a minimum at $z_c = 1/\sqrt{d-1}$. Thus, for $t \to +\infty$ the “particle” will reach this minimum, i.e. $z \to z_c$. Returning to original variables, this implies that $\theta \to \theta_s = \xi/\sqrt{d-1}$ for $\xi \to +\infty$. To get the next order correction, we set $z = z_c + z'$ where $z' \ll z_c$. Keeping only terms that are linear in $z'$, we obtain

$$\frac{d^2 z'}{dt^2} + d \frac{dz'}{dt} + \left( \frac{d^2 - 2d + 2}{d-1} \right)^2 z' = 0.$$

(63)

Noting that the discriminant $\Delta(d) = d^2 - 4(d^2 - 2d + 2)^2/(d-1)^2$ of the associated quadratic equation is always negative for $d > 1$, we find that

$$z' = e^{-dt/2} \cos \left( \frac{\sqrt{-\Delta}}{2} t + \delta \right).$$

(64)

Returning to original variables, we obtain the asymptotic behavior

$$\theta \sim \theta_s(\xi) \left\{ 1 + \frac{C}{\xi d/2} \cos \left( \frac{\sqrt{-\Delta}}{2} \ln \xi + \delta \right) \right\}, \quad (\xi \to +\infty).$$

(65)
As in the case of isothermal and polytropic distributions [17, 15], the density profile of a logotrope presents damped oscillations around the singular solution. The density profiles of a logotrope in \(d = 2\) and \(d = 3\) are represented in Fig. 2. For these dimensions \(\Delta(2) = -12\) and \(\Delta(3) = -16\).

### 4.5 The Milne variables

As in the case of polytropes [15], it is convenient to introduce the variables

\[
\begin{align*}
u & = \frac{\xi}{\theta' \	heta}, \\
v & = \frac{\xi \theta''}{\theta},
\end{align*}
\]

which are the appropriate forms of the Milne variables [33] in the present context. Using the logotropic Lane-Emden equation (52) we easily derive

\[
\begin{align*}
\frac{1}{u} \frac{d u}{d \xi} & = \frac{1}{\xi}(d - v - u), \\
\frac{1}{v} \frac{d v}{d \xi} & = \frac{1}{\xi}(2 - d + u - v),
\end{align*}
\]

so that the function \(v(u)\) satisfies the first order differential equation

\[
\begin{align*}
\frac{u}{v} \frac{d v}{d u} & = -\frac{u - v - d + 2}{u + v - d}.
\end{align*}
\]

As for isothermal and polytropic spheres, the reduction of a second order differential equation (52) to a first order differential equation (69) is a consequence of the homology theorem [33]. The
solution curve in the Milne plane is parameterized by \( \xi \) going from 0 to \(+\infty\). It is represented in Fig. 3 and forms a spiral for \( d > 1 \). The points of horizontal tangent correspond to \( 2 - d + u - v = 0 \) and the points of vertical tangent to \( d - v - u = 0 \). They coincide for \( u_s = d - 1 \) and \( v_s = 1 \) which corresponds to the singular logotropic sphere (53). On the other hand, the curve starts at \( (d, 0) \) for \( \xi = 0 \) with the slope \( (dv/du)_0 = -(d + 2)/d \) and rolls up to the singular logotropic sphere (when \( d > 1 \)) for \( \xi \to +\infty \). For \( d = 1 \), the solution curve tends to \((0, 1)\) for \( \xi \to +\infty \). It does not make a spiral but it presents a maximum at some point.

### 4.6 The series of equilibria

Since the density profile of logotropic distributions in \( d > 1 \) decreases as \( r^{-1} \) at large distances, the total mass is infinite (the total mass is also infinite in \( d = 1 \)). Therefore, we shall enclose these distributions in a spherical box of radius \( R \) as in the case of isothermal and polytropic spheres \[17, 15\]. For box-confined logotropes, the solution of Eq. (52) is terminated by the box at a normalized radius

\[
\alpha = \left( \frac{S_d G \rho_0^2}{A} \right)^{1/2} R.
\]

The total mass of the configuration can be written

\[
M = \int_0^R \rho S_d d^{d-1} dr = S_d \rho_0 \left( \frac{A}{S_d G \rho_0^2} \right)^{d/2} \int_0^\alpha \frac{1}{\theta} \xi^{d-1} d\xi
\]

\[
= S_d \rho_0 \left( \frac{A}{S_d G \rho_0^2} \right)^{d/2} \int_0^\alpha d \xi \left( \xi^{d-1} \frac{d\theta}{d\xi} \right) d\xi = \left( \frac{S_d A}{G} \right)^{1/2} R^{d-1} \theta'(\alpha).
\]

It is therefore natural to introduce the dimensionless parameter

\[
\eta = M \left( \frac{G}{S_d A} \right)^{1/2} \frac{1}{R^{d-1}}.
\]

Figure 3: The solutions of the logotropic Lane-Emden equation in the Milne plane \((u, v)\) for \( d = 1, d = 2 \) and \( d = 3 \).
Then, the series of equilibria is defined by

\[ \eta = \theta'(\alpha). \]  

(73)

This relation can be interpreted in different manners. For example, if we fix \( A \) and \( R \), the parameter \( \alpha \) is proportional to the central density \( \rho_0 \) and the parameter \( \eta \) to the mass \( M \). Therefore, Eq. (73) can be interpreted as the mass-central density relation. Alternatively, if we fix \( M \) and \( R \), then the parameter \( \eta \) is related to \( A = T_{\text{eff}} \) which is interpreted as a “generalized temperature”. Thus Eq. (73) gives the relation between the central density and the generalized temperature. In terms of the Milne variables, the relation (73) can be rewritten

\[ \eta = \left( \frac{v_0}{u_0} \right)^{1/2}, \]

(74)

where, by definition, \( u_0 = u(\alpha) \) and \( v_0 = v(\alpha) \) are the values of the Milne variables at the box radius. The series of equilibria (73) is plotted in Fig. 4 and it presents damped oscillations for \( d > 1 \) (for \( d = 1 \) the curve is monotonic). Similar oscillations are encountered for isothermal spheres in Newtonian gravity [34] and general relativity [35]. For \( \alpha \to +\infty \), we recover the singular sphere with \( \eta_s = 1/\sqrt{d - 1} \). The control parameter \( \eta(\alpha) \) is extremum for \( \alpha_c \) such that \( d\eta/d\alpha = 0 \). Using Eqs. (67)-(68), we find that this condition is equivalent to

\[ u_0 = d - 1 = u_s. \]

(75)

The number of extrema can be obtained by a simple graphical construction in the Milne plane. They are determined by the intersection between the \((u, v)\) curve and the line \( u = u_s \). Since the straight line \( u = u_s \) passes by the center of the spiral, there is an infinite number of extrema. The critical value of \( \eta \) corresponding to the first maximum is \( \eta_c = 0.715657... \) in \( d = 3 \) and \( \eta_c = 1.028728... \) in \( d = 2 \). Clearly, there is no steady state for \( \eta > \eta_c \). Similar results are found for isothermal systems [34, 17]. Note, however, that the series of equilibria \( \eta(\alpha) \) for logotropes already presents oscillations in \( d = 2 \) unlike isothermal spheres for which oscillations appear for \( d > 2 \).

### 4.7 Stability analysis

We shall now study the dynamical stability of logotropic spheres by determining whether they are \textit{minima} of the generalized free energy functional [15] at fixed mass. As shown in [8], a stationary solution of the generalized Smoluchowski-Poisson system (1)-(2) is linearly dynamically stable if, and only if, it is a minimum of the functional \( F \) defined by Eq. (3). In that Brownian context, \( F \) is interpreted as a Lyapunov functional or as a (generalized) free energy, and its minimization corresponds to an effective thermodynamical stability criterion in the canonical ensemble. On the other hand, a stationary solution of the barotropic Euler-Poisson system [15]-[17] is nonlinearly dynamically stable if it is a minimum of \( F \). In that Euler context, \( F \) is related to the energy functional of the barotropic gas \( W \) (see [15, 31, 36] for details). Therefore, our stability analysis has applications for these two different systems: a Brownian gas described by nonlinear Fokker-Planck equations and a “normal” gas described by the Euler equations. Since the following stability analysis is very similar to that developed in [15] for polytropic distributions, we shall only give the main lines of the analysis and refer to [15] for more details.

The second order variations of \( F \) are given by

\[ \delta^2 F = \frac{A}{2} \int \frac{(\delta \rho)^2}{\rho^2} \, dr + \frac{1}{2} \int \delta \rho \delta \Phi \, dr. \]

(76)
Figure 4: Series of equilibria for box-confined logotropes in $d = 1$, $d = 2$ and $d = 3$.

Figure 5: Perturbation profile corresponding to the first mode of instability at $\eta_c$ in $d = 2$ and $d = 3$. 
Restricting ourselves to spherically symmetric perturbations, introducing the quantity $q(r)$ through the defining relation

$$\delta \rho = \frac{1}{S_d r^{d-1}} \frac{d q}{dr},$$

and using integrations by parts, we can put the second order variations of $F$ in the quadratic form

$$\delta^2 F = -\frac{1}{2} \int_0^R q \frac{d}{dr} \left( \frac{A}{S_d r^{d-1}} \frac{d}{dr} + \frac{G}{r^{d-1}} \right) q \, dr.$$

We are thus led to considering the eigenvalue equation

$$\frac{d}{dr} \left( \frac{A}{S_d r^{d-1}} \frac{d}{dr} + \frac{G}{r^{d-1}} \right) q(r) = \lambda q(r),$$

and determine when, in the series of equilibria, the eigenvalues pass from positive (stable) to negative (unstable) values. To that purpose, it is sufficient to determine the point of marginal stability corresponding to $\lambda = 0$. Introducing the dimensionless quantities of Sec. 4.1, we have to solve

$$\mathcal{L} F \equiv \frac{d}{d \xi} \left( \frac{\theta^2}{\xi^{d-1}} \frac{d F}{d \xi} \right) + \frac{F}{\xi^{d-1}} = 0,$$

with the boundary conditions $F(0) = 0$ and $F(\alpha) = 0$ implied by the conservation of mass. Noting that

$$\mathcal{L}(\xi^{d-1} \theta') = 2 \theta', \quad \mathcal{L} \left( \frac{\xi^d}{\theta} \right) = 2(d-1) \theta',$$

we find that the solution of Eq. (80) satisfying $F(0) = 0$ is

$$F = a \left[ \frac{\xi^d}{\theta} - (d-1) \xi^{d-1} \theta' \right],$$

where $a$ is an arbitrary constant (due to the linearity of the eigenvalue equation). The critical values of $\alpha$ where an eigenvalue becomes zero are determined by the condition $F(\alpha) = 0$. Using Eq. (82) and introducing the Milne variables (66), we find that this is equivalent to $u_0 = d - 1 = u_s$. Comparing with Eq. (73), we find that a new eigenvalue becomes zero each time that $\eta$ is extremum. As a result, the series of equilibria becomes unstable at the point of maximum $\eta$, previously denoted $\eta_c$. This result is consistent with the turning point argument of Poincaré (see [15] for details). The perturbation triggering the instability at the marginal point is given by

$$\frac{\delta \rho}{\rho_0} = \frac{1}{S_d \xi^{d-1}} \frac{d F}{d \xi}.$$

In terms of the Milne variables, this can be rewritten

$$\frac{\delta \rho}{\rho} = \frac{a}{S_d} (u_s - v).$$

Hence, the number of oscillations of the perturbation profile $\delta \rho(\xi)$ can be easily determined by graphical constructions. The values of $\xi$ where $\delta \rho(\xi)$ vanishes are determined by the intersection
between the \((u, v)\) curve and the line \(v = v_s\). Of course, we have to consider only values of \(\xi \leq \alpha\). At the point of instability \(\eta_c\), the perturbation has only one node (see Fig. 5).

We can also study the linear dynamical stability of a stationary solution of the LSP system and show the equivalence with the stability criterion based on the minimization of the free energy. Linearizing the generalized SP system around a stationary solution and writing the perturbation in the form \(\delta \rho \sim e^{\lambda t}\) we finally arrive at the eigenvalue equation [8]:

\[
\frac{d}{dr} \left( \frac{p' (\rho)}{S_d \rho r^{d-1}} \frac{dq}{dr} \right) + \frac{G q}{r^{d-1}} = \frac{\lambda \xi}{S_d \rho r^{d-1}} q.
\]

Specializing to the case of a logotropic equation of state \(p = A \ln \rho\), we obtain

\[
\frac{d}{dr} \left( \frac{A}{S_d \rho^2 r^{d-1}} \frac{dq}{dr} \right) + \frac{G q}{r^{d-1}} = \frac{\lambda \xi}{S_d \rho r^{d-1}} q.
\]

This equation is similar to Eq. (79) and they coincide at the point of marginal stability. Therefore, the stability threshold is the same in the two approaches and the above results concerning the form of the perturbation apply. Finally, if we consider the linear dynamical stability of a steady solution of the logotropic Euler-Poisson system (5)-(7), we obtain an eigenvalue equation of the form [86] where \(\lambda \xi\) is replaced by \(\lambda^2\) [22]. Once again, the point of marginal dynamical stability coincides with the point where the energy functional ceases to be a minimum and becomes a saddle point. Therefore, the conditions of linear and nonlinear dynamical stability for the Euler-Poisson system coincide.

### 4.8 Self-similar solutions of the polytropic and logotropic Smoluchowski-Poisson system

When no stable equilibrium state exists, the polytropic Smoluchowski-Poisson system admits self-similar solutions describing a gravitational collapse. These solutions have been given in [15] for a polytropic index \(n \geq 0\). As we shall see, the situation is different, and richer, when \(n \leq 0\) (this case includes in particular the logotropes \(n = -1\)). We restrict our analysis to a space of dimension \(d > 2\) (the dimension \(d = 2\) is critical and requires a particular and intricate treatment; see [17] for isothermal systems).

If we introduce the function \(s(r, t) = M(r, t)/r^d\), which scales like the density \(\rho(r, t)\), we can rewrite the polytropic Smoluchowski-Poisson system (10)-(48) as a single differential equation [15]:

\[
\frac{\partial s}{\partial t} = \Theta \left( r \frac{\partial s}{\partial r} + ds \right)^{1/n} \left( \frac{\partial^2 s}{\partial r^2} + \frac{d + 1}{r} \frac{\partial s}{\partial r} \right) + \left( r \frac{\partial s}{\partial r} + ds \right) s,
\]

where we have defined \(A = K \gamma\) and introduced an effective temperature \(\Theta = A/S_d^{1/n}\). We note that this equation passes to the limit for \(n = -1\). We look for self-similar solutions of the form

\[
s(r, t) = \rho_0(t) S \left( \frac{r}{r_0(t)} \right),
\]

where \(\rho_0(t)\) is proportional to the central density. The scaling solution [88] only holds in the central region of the cluster. If we assume that all the terms scale the same in Eq. (87), which needs not be the case (see below), we find that

\[
\Theta \rho_0^{1/n+1} \sim \rho_0^2.
\]
We can then define the scaling radius by
\[ r_0 = \left( \frac{\Theta}{\rho_0^{1-1/n}} \right)^{1/2}. \]
In that case, the relation between the typical central density and the typical core radius is
\[ \rho_0 \sim r_0^{-\alpha}, \quad \text{with} \quad \alpha = \frac{2n}{n-1}. \]
Substituting these relations in Eq. (87), we find that
\[ \frac{d\rho_0}{dt} \left( S + \frac{1}{\alpha} xS' \right) = \rho_0^2 \left[ (xS' + dS)^{1/n} \left( S'' + \frac{d+1}{x} S' + (xS' + dS)S \right) \right], \]
where we have set \( x = r/r_0 \). This implies that \( (1/\rho_0^2)(d\rho_0/dt) \) is a constant that we arbitrarily set equal to \( \alpha \). This leads to
\[ \rho_0(t) = \frac{1}{\alpha}(t_{\text{coll}} - t)^{-1}, \]
so that the central density becomes infinite in a finite time \( t_{\text{coll}} \). The scaling equation now reads
\[ \alpha S + xS' = (xS' + dS)^{1/n} \left( S'' + \frac{d+1}{x} S' + (xS' + dS)S \right). \]
For \( x \to +\infty \), we have the scaling \( S(x) \sim x^{-\alpha} \) so that the density behaves as \( \rho \sim r^{-\alpha} \) for \( r \to +\infty \). At \( t = t_{\text{coll}} \) the density profile is proportional to \( 1/r^\alpha \).

However, we could be in a situation in which the collapse is dominated by the gravitational drift like in the case of cold systems where \( \Theta = 0 \). In that case, the dynamical equation (87) reduces to
\[ \frac{\partial s}{\partial t} \simeq \left( r \frac{\partial s}{\partial r} + ds \right) s. \]
This equation with \( \Theta = 0 \) has been solved in [17]. The collapse evolution is self-similar and the relation between the typical central density and the typical core radius is
\[ \rho_0 \sim r_0^{-\alpha}, \quad \text{with} \quad \alpha = \frac{2d}{d+2}. \]
We expect that this “cold regime” will prevail over the regime where the diffusion and the gravitational terms scale the same way if \( (\rho_0)_{\Theta=0} \gg (\rho_0)_{\Theta \neq 0} \), i.e. if
\[ \frac{2d}{d+2} > \frac{2n}{n-1}. \]
Indeed, one expects on the basis of free energy arguments that the most natural evolution is the one which leads to the most efficient collapse of the core. This is obtained by selecting the largest value of \( \alpha \) between Eqs. (91) and (96).

Let us consider the range of validity of this inequality:
(i) The case \( n \geq 0 \) has been considered in [14]. It is shown that self-similar solutions exist only for \( n > n_3 = d/(d-2) \) (otherwise the system tends to a complete polytrope). We note
that inequality (97) is never satisfied in that case so that the scaling exponent is $\alpha = 2n/(n-1)$ independent on the dimension. The results of [15] are therefore unaltered.

(ii) For $n < 0$, the inequality (97) is equivalent to $n_c = -d/2 \leq n < 0$ where $n_c = -d/2$ is a new critical index which does not seem to have been introduced before. In that case, the collapse of the polytropic SP system is similar to the case $\Theta = 0$ (cold system) and the scaling exponent is $\alpha = 2d/(d+2)$. The corresponding scaling profile is known analytically in an implicit form [17]. Finally, for $n < n_c = -d/2$, the scaling exponent is $\alpha = 2n/(n - 1)$. In particular, for the logotropes with $n = -1$ in $d > 2$, the scaling exponent is $\alpha = 2d/(d+2)$. Therefore, the density profile of a collapsed logotrope scales as $\rho \sim r^{-2d/(d+2)}$ while the density profile of an equilibrium logotrope scales as $\rho \sim r^{-1}$ for $r \to +\infty$ (see Sec. 4.4). This situation differs from the case of isothermal and polytropic systems with $n \geq 0$ where the collapse exponent $\alpha = 2n/(n-1)$ is the same as the exponent controlling the decay of the equilibrium density profile at large distances [17, 15].

The above results are summarized in Fig. 6. In Fig. 7, we show the data collapse obtained by solving the dynamical equation (87) with $n = -1$ (logotrope) in dimensions $d = 3$ and $d = 5$. The scaling exponent agrees with its predicted theoretical value (96) corresponding to a pressureless collapse at $\Theta = 0$. The scaling profile is also in good agreement with the analytical profile at $\Theta = 0$ obtained in [15] in implicit form. In Fig. 8, we show the time evolution of the density profile during the collapse obtained by solving the dynamical equation (87) in dimension $d = 3$ for $n = -1 > n_c = -3/2$ and $n = -4 < n_c = -3/2$. The exponents are in good agreement with the theoretical predictions (96) and (91) respectively.

5 Conclusion

In this paper, we have studied a special class of nonlinear mean-field Fokker-Planck equations introduced in [8], corresponding to a logotropic equation of state. This can be viewed as
Figure 7: Data collapse for $n = -1$ obtained by solving the dynamical equation (87). Upper curve: $d = 3$, $\alpha = 2d/(d+2) = 6/5$. Lower curve: $d = 5$, $\alpha = 2d/(d+2) = 10/7$; the curve has been shifted by an arbitrary factor. The invariant profile (dashed line) corresponds to $\Theta = 0$. It is known analytically in implicit form [17].

Figure 8: The density profile (divided by $S_d$) is plotted for successive times so that the central density increases by a factor 2 at each time, starting from a central density 300. The dimension is $d = 3$. Upper curve: $n = -4 < n_c = -d/2$ so that $\alpha = 2n/(n - 1) = 8/5$ (regular scaling); the figure has been shifted by a factor 100 for clarity. Lower curve: $n = -1 > n_c$ so that $\alpha = 2d/(d+2) = 6/5$ (scaling at $\Theta = 0$). The dashed lines are the theoretical slopes.
a limiting form of polytropic equation of state with $\gamma = 0$. In the language of generalized thermodynamics, this corresponds to a Tsallis distribution with $q = 0$. Special attention has been given to the logotropic Smoluchowski-Poisson system where we have studied the steady states, their stability and, when unstable, the resulting gravitational collapse. In particular, we have shown that for polytropic indices $-d/2 \leq n < 0$ in $d > 2$ (including the logotropes $n = -1$), the dynamics below the critical effective temperature $\Theta < \Theta_c$ coincides with the dynamics at $\Theta = 0$. This is similar to the coarsening dynamics of spin systems strictly below the ferromagnetic critical temperature, which is controlled by the zero temperature fixed point \[37\]. This work completes previous investigations of the authors who studied the generalized Smoluchowski-Poisson system for an isothermal equation of state \[16, 17\], a polytropic equation of state with $n \geq 0$ \[15\], a Fermi-Dirac equation of state \[18\] and a Bose statistics in velocity space \[21\]. Our results can also have applications for the chemotactic aggregation of bacterial populations in biology that is governed by similar drift-diffusion equations \[18, 27, 23\].

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