PADDED SCHUBERT POLYNOMIALS AND WEIGHTED ENUMERATION OF BRUHAT CHAINS

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Abstract. We prove a common generalization of the fact that the weighted number of maximal chains in the strong Bruhat order on the symmetric group is \( \binom{n}{2}! \) for both the code weights and the Chevalley weights. We also define weights which give a one-parameter family of strong order analogues of Macdonald’s reduced word identity for Schubert polynomials.

1. Introduction

Let \( S_n \) denote the (strong) Bruhat order on the symmetric group \( S_n \) (see Section 2 for background and definitions). Given a function \( wt : Cov(S_n) \to R \) from the set of covering relations of \( S_n \) to a ring \( R \), and a saturated chain \( c = (u_1 < u_2 < \cdots < u_k) \), we define the weight of \( c \) multiplicatively:

\[
wt(c) := 
\prod_{i=1}^{k-1} wt(u_i < u_{i+1}).
\]

For \( v \leq w \) in \( S_n \) we let

\[
m_{wt}(v, w) := 
\sum_{c : v \to w} wt(c)
\]

denote the total weighted number of chains over all saturated chains \( c \) from \( v \) to \( w \).

In this paper, we study several classes of weights which generalize the previously studied code weights [3] and Chevalley weights [8, 10]. Some building blocks for these new weights are given in Definition 1.1.

Definition 1.1. For \( v < w = vt_{ij} \) a covering relation in \( S_n \) with \( i < j \), let \( a_{v < w}, b_{v < w}, c_{v < w}, \) and \( d_{v < w} \) denote the number of dots in the regions \( A, B, C \), and \( D \) respectively in Figure 1. That is,

\[
a_{v < w} = \# \{ k < i \mid v_i < v_k < v_j \}
\]

\[
b_{v < w} = \# \{ i < k < j \mid v_k > v_j \}
\]

\[
c_{v < w} = \# \{ k > j \mid v_i < v_k < v_j \}
\]

\[
d_{v < w} = \# \{ i < k < j \mid v_k < v_i \}.
\]

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Note that we always have $b_{v \leq w} + d_{v \leq w} = j - i - 1$ and $a_{v \leq w} + c_{v \leq w} = v_j - v_i - 1$.

The Chevalley weights $\text{wt}_{\text{Chev}}(v \leq w) : \text{Cov}(S_n) \to \mathbb{Z}[\alpha_1, ..., \alpha_{n-1}]$ assign weight $\alpha_i + \cdots + \alpha_{j-1}$ to the covering relation $v \leq w = vt_{ij}$, where $t_{ij} = (i j)$ is a transposition. It was shown by Stembridge [10] that:

\begin{equation}
\text{m}_{\text{Chev}}(e, w_0)(\alpha_1, ..., \alpha_{n-1}) = \begin{pmatrix} n \\ 2 \end{pmatrix}! \cdot \prod_{1 \leq k < \ell \leq n-1} \frac{\alpha_k + \cdots + \alpha_{\ell-1}}{\ell - k}
\end{equation}

where $w_0 = n(n-1) \cdots 21$ denotes the longest permutation. Specializing all $\alpha_i = 1$ recovers the classical fact:

\begin{equation}
\text{m}_{\text{Chev}}(e, w_0)(1, ..., 1) = \begin{pmatrix} n \\ 2 \end{pmatrix}!.
\end{equation}

Recently, a new set of weights, the code weights $\text{wt}_{\text{code}} : \text{Cov}(S_n) \to \mathbb{N}$ were defined in the course of proving the Sperner property for the weak Bruhat order [4]. In the notation of Definition 1.1, the code weights are defined by $\text{wt}_{\text{code}}(v \leq w) = 1 + 2b_{v \leq w}$. In [3], it was shown that

\begin{equation}
\text{m}_{\text{code}}(w, w_0) = \begin{pmatrix} n \\ 2 \end{pmatrix} - \ell(w) ! \cdot \mathcal{S}_w(1, ..., 1)
\end{equation}

where $\mathcal{S}_w$ is the Schubert polynomial (see Section 2), providing a strong Bruhat order analogue of Macdonald’s well known identity for $\mathcal{S}_w(1, ..., 1)$. 

**Figure 1.** For $v \leq w$ a covering relation in the strong order, the permutation matrices for $v$ and $w$ agree, except that the black dots in $v$ are replaced with the white dots in $w$. No dots may occupy the central region; the numbers of dots in the four labeled regions $A$, $B$, $C$ and $D$ are used in Definition 1.1.
as a weighted enumeration of chains in the weak Bruhat order \([6]\). Letting \(w = e\) in (3) gives:

\[
m_{\text{code}}(e, w_0) = \binom{n}{2}!.
\]

One motivation of this work is to understand and generalize the coincidence between (2) and (4); this is done in Theorem 1.2.

**Theorem 1.2.** Let \(f : \text{Cov}(S_n) \to \mathbb{Z}[z, z^2, z^3, z^4] \) be the weight function defined by

\[
f(v \prec w) := 1 + a_{v \prec w}z_A + b_{v \prec w}z_B + c_{v \prec w}z_C + d_{v \prec w}z_D.
\]

Let \(w^t : \text{Cov}(S_n) \to \mathbb{Z}[z]\) be any weight function obtained from \(f\) by specializing the variables so that \(\{z_A, z_B, z_C, z_D\} = \{0, 0, 2z, z\}\) as multisets, then:

\[
m_{\text{wt}}(e, w_0) = \binom{n}{2}!.
\]

In particular, \(m_{\text{wt}}(e, w_0)\) does not depend on \(z\).

\[\text{Figure 2. The weights considered in Theorem 1.4 (left) and in Theorem 1.3 (right) for } S_3. \text{ Unlabelled edges have weight 1.}\]

Theorem 1.3 provides a common generalization of (1) and (4); see Example 1.5.

**Theorem 1.3.** Let \(w^t : \text{Cov}(S_n) \to \mathbb{Z}[\alpha_1, ..., \alpha_{n-1}, z] \) be defined by

\[
w^t(v \prec vt_{ij}) = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + (b_{v \prec vt_{ij}} - d_{v \prec vt_{ij}})z.
\]

Then

\[
m_{\text{wt}}(e, w_0) = \binom{n}{2}! \cdot \prod_{k < \ell} \frac{\alpha_k + \cdots + \alpha_{\ell-1}}{\ell - k}.
\]

In particular, \(m_{\text{wt}}(e, w_0)\) does not depend on \(z\).
Theorem 1.4 extends (3) to a one-parameter family of strong Bruhat analogues of Macdonald’s identity.

**Theorem 1.4.** Let \( \text{wt} : \text{Cov}(S_n) \to \mathbb{Z}[z] \) be defined by

\[
\text{wt}(v \lessdot w) = 1 + b_{v \lessdot w}(2 - z) + c_{v \lessdot w}z.
\]

Then for any \( w \in S_n \) we have

\[
m_{\text{wt}}(w, w_0) = \left( \binom{n}{2} - \ell(w) \right)! \cdot \mathcal{G}_w(1, \ldots, 1).
\]

In particular, \( m_{\text{wt}}(w, w_0) \) does not depend on \( z \).

**Example 1.5.** Various specializations of the above Theorems give previously known results:

1. Letting \( z_B = 2 \) and \( z_A = z_C = z_D = 0 \) in Theorem 1.2 recovers (4), while letting \( z_B = z_D = 1 \) and \( z_A = z_C = 0 \) recovers (2).
2. Letting all \( z = \alpha_1 = \cdots = \alpha_{n-1} = 1 \) in Theorem 1.3 the weight becomes:

\[
\text{wt}(v \lessdot w = vt_{ij}) = (j - i) + (b_{v \lessdot w} - d_{v \lessdot w})
\]

\[
= (b_{v \lessdot w} + d_{v \lessdot w} + 1) + (b_{v \lessdot w} - d_{v \lessdot w})
\]

\[
= 1 + 2b_{v \lessdot w}.
\]

This recovers the identity (4) for the code weights.

3. Letting \( z = 0 \) in Theorem 1.3 recovers Stembridge’s identity (1) for the Chevalley weights.

4. Letting \( z = 0 \) in Theorem 1.4 recovers the strong order Macdonald identity (3).

Section 2 covers background and definitions. Theorems 1.3 and 1.4 are proven in Sections 3 and 4 respectively. Finally, Section 5 discusses symmetries of the weights from Definition 1.1 and completes the proof of Theorem 1.2.

### 2. Background and definitions

2.1. **Bruhat order.** Let \( s_1, \ldots, s_{n-1} \) denote the adjacent transpositions in the symmetric group \( S_n \). For any permutation \( w \in S_n \), its length \( \ell(w) \) is the minimal number of simple transpositions needed to write \( w = s_{i_1} \cdots s_{i_{\ell}} \) as a product.

The (strong) Bruhat order \( S_n = (S_n, \leq) \) is defined by its covering relations: \( v \lessdot w \) whenever \( w = vt_{ij} \) for some \( i, j \) and \( \ell(w) = \ell(v) + 1 \). The Bruhat order has unique minimal element the identity permutation \( e \), and unique maximal element \( w_0 = n(n-1)\ldots21 \) of length \( \binom{n}{2} \), called the longest element. The Hasse diagram of \( S_3 \) is shown in Figure 2.
2.2. Schubert polynomials and padded Schubert polynomials. For $w \in S_n$, the Schubert polynomials $\mathcal{G}_w(x_1,...,x_n)$, introduced by Lascoux and Schützenberger [5], represent the classes of Schubert varieties in the cohomology $H^*(G/B)$ of the flag variety. They can be defined recursively as follows:

- $\mathcal{G}_{w_0}(x_1,...,x_n) = x_1^{n-1}x_2^{n-1}...x_{n-2}^2x_1 = x^\rho$, where $\rho = (n-1,n-2,...,1)$ denotes the staircase composition, and
- $\mathcal{G}_{w}s_i = N_i \cdot \mathcal{G}_w$ when $\ell(ws_i) < \ell(w)$.

Here $N_i$ denotes the $i$-th Newton divided difference operator:

$$N_i \cdot g(x_1,...,x_n) := \frac{g(x_1,...,x_n) - g(x_1,...,x_{i+1},x_i,...,x_n)}{x_i - x_{i+1}}.$$

The Schubert polynomials $\{\mathcal{G}_w\}_{w \in S_n}$ form a basis for the vector space $V_n = \text{span}_Q\{x^\gamma | \gamma \leq \rho\}$, where here $\leq$ denotes component-wise comparison.

Let $\tilde{V}_n = \text{span}_Q\{x^\gamma y^{\rho - \gamma}\}$, then the padded Schubert polynomials $\tilde{\mathcal{G}}_w$, introduced in [3], are defined as the images of the $\tilde{\mathcal{G}}_w$ under the natural map $x^\gamma \mapsto x^\gamma y^{\rho - \gamma}$ from $V_n \rightarrow \tilde{V}_n$. Define a differential operator $\Delta : \tilde{V}_n \rightarrow \tilde{V}_n$ by

$$\Delta = \sum_{i=1}^{n} x_i \frac{\partial}{\partial y_i}.$$

**Proposition 2.1** ([3]). For any $w \in S_n$ we have:

$$\Delta \tilde{\mathcal{G}}_w = \sum_{u : w \leq u} (1 + 2b_{w,u})\tilde{\mathcal{G}}_u.$$

3. Proof of Theorem 1.3

We will modify a proof idea for [11] due to Stanley [9]. Let’s define some linear operators on the cohomology ring of the flag variety

$$H^*(G/B) \simeq \mathbb{C}[x_1,...,x_n]/I,$$

where $I$ is the ideal generated by all symmetric polynomials in $x_1,...,x_n$ with vanishing constant terms. The core of the argument comes from interpreting the operator $\Delta$ with respect to two different bases of $H^*(G/B)$: one is $\{\mathcal{G}_w | w \in S_n\}$ and the other one is $\{x^\gamma | \gamma \leq \rho\}$.

Recall that we have defined $\Delta$ on $\tilde{V}_n$. We can define it naturally on $V_n$ since $V_n \rightarrow \tilde{V}_n$ is an isomorphism. Namely, it can be seen from definition that $\Delta x^\gamma = (\sum_{i=1}^{n}(n-i - \gamma_i)x_i)x^\gamma$ for $\gamma \leq \rho$ (in which case $\gamma_i = 0$). Moreover, we can extend this definition of $\Delta$ to $\mathbb{C}[x_1,...,x_n]$ by the same formula. We claim that such definition is in fact well-defined on $\mathbb{C}[x_1,...,x_n]/I$. This is formulated in the following technical lemma, which is necessary for the correctness of the main proof but is not related to the key idea of the proof.

**Lemma 3.1.** The linear operator $\Delta : x^\gamma \mapsto (\sum_{i=1}^{n}(n-i - \gamma_i)x_i)x^\gamma$ is well-defined on $\mathbb{C}[x_1,...,x_n]/I$ and coincides with $\sum_{i=1}^{n} x_i \frac{\partial}{\partial y_i}$ on $\tilde{V}_n$. 
Proof. We need to check that if $f \in I$, then $\Delta f \in I$. For convenience, we will first pad every monomial $x^\gamma$ to $x^\gamma y^{\rho - \gamma}$, allowing negative exponents on $y$-variables, so that we can use $\Delta = \sum_i \frac{\partial}{\partial y_i} x_i$, and then specialize $y_i$'s to 1. This is compatible with the definition as in the statement of the lemma. This means $\Delta(fg) = f\Delta(g) + g\Delta(f)$. As a result, it suffices to check if $f$ is a generator of $I$, then $\Delta f \in I$.

Let's pick the power sum symmetric functions $f = x_1^k + \cdots + x_n^k$ as generators, for $k \geq 1$. After padding, we get $\sum_j \left(\frac{x_j}{y_j}\right)^k y^\rho$. Then

$$
\Delta \left(\sum_{j=1}^n \left(\frac{x_j}{y_j}\right)^k y^\rho\right) = \left(\sum_{i=1}^n \frac{\partial}{\partial y_i} x_i\right) \left(\sum_{j=1}^n \left(\frac{x_j}{y_j}\right)^k y^\rho\right) = \sum_{i,j=1}^n \frac{x_i^k}{y_j^k} \left(\frac{\partial}{\partial y_i} y^\rho\right) + \sum_{i=1}^n x_i y^\rho \left(\frac{\partial}{\partial y_i} y_i^k\right) = \left(\sum_{i=1}^n x_i \frac{\partial}{\partial y_i} y^\rho\right) \left(\sum_{j=1}^n \frac{x_j^k}{y_j^k}\right) - (k+1)y^{\rho}\sum_{i=1}^n \frac{x_i^{k+1}}{y_i^{k+1}}.
$$

It is clear that both terms belong to $I$ after specializing $y_i$'s to 1. So we are done. $\square$

Now let $\alpha_1, \ldots, \alpha_{n-1}$ be as in Theorem 1.3 and define a linear operator $M$ as multiplication by

$$
\alpha_1 x_1 + \alpha_2 (x_1 + x_2) + \alpha_3 (x_1 + x_2 + x_3) + \cdots + \alpha_{n-1} (x_1 + \cdots + x_{n-1}) = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_{n-1} x_{n-1}
$$

where $\beta_i = \alpha_i + \cdots + \alpha_{n-1}$. By Monk's rule (see e.g. [7]),

$$
M \mathcal{S}_w = \sum_{w' \equiv w} (\alpha_i + \cdots + \alpha_{j-1}) \mathcal{S}_{w'_{ij}}.
$$

Note that Monk’s rule only holds modulo the ideal $I$, and not as an identity of polynomials. Define another linear operator $R$ by

$$
R \mathcal{S}_w = \sum_{w: w \equiv u} (b_{w \equiv u} - d_{w \equiv u}) \mathcal{S}_u.
$$

Write $M_1 = M$ and define $M_{k+1} = [M_k, R] := M_k R - R M_k$ for $k \geq 1$. Here, $[,]$ is the standard Lie bracket.

Lemma 3.2. The operator $M_k$ is the same as multiplication by the element $(k-1)!((\beta_1 x_1^k + \beta_2 x_2^k + \cdots + \beta_{n-1} x_{n-1}^k)).$
Proof. Let’s analyze $R$ a bit more. We have
\[
R\mathfrak{S}_w = \sum_{u: w \leq u} (b_{w \leq u} - d_{w \leq u})\mathfrak{S}_u
\]
\[
= \sum_{u: w \leq u} (1 + 2b_{w \leq u})\mathfrak{S}_u - \sum_{u: w \leq u} (1 + b_{w \leq u} + d_{w \leq u})\mathfrak{S}_u
\]
\[
= \sum_{u: w \leq u} (1 + 2b_{w \leq u})\mathfrak{S}_u - \sum_{u: w \leq u} (j - i)\mathfrak{S}_{u_{ij}}
\]
\[
= \Delta\mathfrak{S}_w - ((n - 1)x_1 + (n - 2)x_2 + \cdots + x_{n-1})\mathfrak{S}_w
\]
where the last equality follows from Proposition 2.1 and Monk’s rule (as a
special case of $M$ by assigning $\alpha_1 = \cdots = \alpha_{n-1} = 1$).

We use induction on $k$. Since multiplications by polynomials commute
with each other, we have $M_kR - RM_k = M_k\Delta - \Delta M_k$. Let’s compute what
it does on monomials $x^\gamma$:
\[
x^\gamma = M_k\left(\sum_{i=1}^{n-1} (n - i - \gamma_i)x_i\right) x^\gamma
\]
\[
= (k - 1)! \left(\sum_{i,j=1}^{n-1} (n - i - \gamma_i)\beta_j x_i x_j^k\right) x^\gamma
\]
while on the other hand,
\[
\Delta M_k x^\gamma = (k - 1)! \Delta \sum_j \beta_j x_1^{\gamma_j} \cdots x_{j-1}^{\gamma_{j-1}} x_j^{\gamma_j + 1} x_{j+1}^{\gamma_{j+1}} \cdots x_{n-1}^{\gamma_{n-1}}
\]
\[
= (k - 1)! \left(\sum_{i \neq j} (n - i - \gamma_i)\beta_j x_i x_j^k\right) x^\gamma
\]
\[
+ (k - 1)! \left(\sum_i (n - i - \gamma_i - k)\beta_i x_i^{k+1}\right) x^\gamma.
\]
Here, the calculation of $\Delta M x^\gamma$ uses the fact that $\Delta$ is defined on all of
$\mathbb{C}[x_1, \ldots, x_n]/I$ (Lemma 3.1), since the coefficient of $x_j$ may exceed $n - j$.
As a result, we see that $(M_k\Delta - \Delta M_k)x^\gamma = k!(\sum_{i=1}^{n-1} \beta_i x_i^{k+1})x^\gamma$. So the
induction step goes through.

Remark. In fact, the operator $R$ can be more elegantly written as
\[
Rf = y^p \cdot \Delta(f/y^p),
\]
when $f \in \mathbb{W}_n$ is already padded.

Lemma 3.3. View $M_k = (k - 1)! \sum_{i=1}^{n-1} \beta_i x_i^k$ as polynomials. Then $\Pi_k M_k$
lies in the ideal $I$ if $\sum kp_k = \binom{n}{2}$ and $p_1 < \binom{n}{2}$.

Proof. Write $M_k = (k - 1)! \sum_{i=1}^{n} \beta_i x_i^k$ with $\beta_n = 0$. As $\Pi_k M_k$ is
homogeneous of degree $\binom{n}{2}$, we can write it as $f\mathfrak{S}_w$ modulo $I$, where $f$ depends
only on $\beta_i$’s. In fact, we can obtain $f \mathcal{G}_{w_0}$ by first multiplying out $\prod_k M_{p_k}^k$ and then performing subtraction with respect to the homogeneous part of degree $\binom{n}{2}$ in $\mathbb{C}[x_1, \ldots, x_n]/I$. This shows that $f$ is a polynomial of degree at most $\binom{n}{2} - 2\ell + \ell = \binom{n}{2} - \ell$.

On the other hand, if $\beta_i = \beta_i + 1$, then $\prod_{k \geq 1} M_{p_k}^k$ is symmetric in $x_i$ and $x_{i+1}$. Consequently, $N_i(\prod_k M_{p_k}^k) = 0$, where $N_i$ is the $i$-th divided difference operator introduced in Section 2. But $0 = N_i(f \mathcal{G}_{w_0}) = f(N_i \mathcal{G}_{w_0}) = f \mathcal{G}_{w_0 s_i}$. As $\mathcal{G}_{w_0 s_i} \neq 0$, we deduce that $f$ is a polynomial of degree at most $\binom{n}{2} - \ell$.

Lemma 3.4. With $M, R$ as above, $(M + zR)^{\binom{n}{2}} \cdot 1 = M^{\binom{n}{2}} \cdot 1$.

Proof. Notice that $R \cdot 1 = R \cdot \mathcal{G}_c = 0$ as $b_{w \leq s_i} = d_{w \leq s_i} = 0$. The rest is a simple consequence of Lemma 3.2 and Lemma 3.3. Namely, expand $(M + zR)^{\binom{n}{2}}$ and move $R$’s towards the right such that in each step, we replace $\cdots RM_k \cdots$ by $\cdots M_k R \cdots - \cdots M_{k+1} \cdots$, keeping the total degree. In the end when no such moves are possible, either $R$ appears on the right side, resulting in a term equal to 0 (since $R \cdot 1 = 0$), or $\prod_{k \geq 1} M_{p_k}^k$ appears with $\sum kp_k = \binom{n}{2}$, which is also 0 except the single term $M^{\binom{n}{2}}$.

Theorem 1.3 now follows easily.

Proof of Theorem 1.3. Recall that we have

\[ M \cdot \mathcal{G}_w = \sum_{w \leq w_{t_{ij}}} (\alpha_i + \cdots + \alpha_{j-1}) \mathcal{G}_{w_{t_{ij}}} \]

\[ R \cdot \mathcal{G}_w = \sum_{w \leq u} (b_{w \leq u} - d_{w \leq u}) \mathcal{G}_u \]

so putting them together,

\[ (M + zR) \cdot \mathcal{G}_w = \sum_{w \leq u} \text{wt}(w \leq u) \mathcal{G}_u. \]

An iteration (or induction) immediately gives

\[ (M + zR)^\ell \cdot \mathcal{G}_w = \sum_{w \leq u, \ell(w) = \ell(u) - \ell} m_{\text{wt}}(w, u) \cdot \mathcal{G}_u. \]

Taking $w = e$ and $\ell = \binom{n}{2}$ in the above setting, we obtain that $m_{\text{wt}}(e, w_0)$ is the coefficient of $\mathcal{G}_{w_0}$ in $(M + zR)^{\binom{n}{2}}$, modulo $I$. By Lemma 3.4, such coefficient does not depend on $z$. When $z = 0$, our result is given by Stembridge [10] (see also Stanley [9]).
4. Proof of Theorem 1.4

We first note a simple fact about the specialization of $\tilde{S}_w$: since $\tilde{S}_w$ has total $x$-degree $\ell(w)$ and total $y$-degree $(n^2) - \ell(w)$, we have

$$(\Delta \tilde{S}_w)(1, \ldots, 1) = \left( \binom{n}{2} - \ell(w) \right) \tilde{S}_w(1, \ldots, 1).$$

We then have the following lemma.

**Lemma 4.1.** Fix $w \in S_n$. Then

$$\sum_{u: w \preceq u} \tilde{S}_u(1, \ldots, 1) (b_{w \preceq u} - c_{w \preceq u}) = 0.$$

**Proof.** Let’s recall some classical facts about $\tilde{S}_u$ and $\tilde{S}_{u^{-1}}$. Since there is a simple bijection (transpose) between RC-graphs of $u$ and $u^{-1}$ (see for example [1]), the number of monomials appearing in the expansion of $\tilde{S}_u$ is the same as in $\tilde{S}_{u^{-1}}$. This says $\tilde{S}_u(1, \ldots, 1) = \tilde{S}_{u^{-1}}(1, \ldots, 1)$. Moreover, as $\ell(u) = \ell(u^{-1})$, $(\Delta \tilde{S}_u)(1, \ldots, 1) = (\Delta \tilde{S}_{u^{-1}})(1, \ldots, 1)$. In addition, notice that $w \preceq u$ if and only if $w^{-1} \preceq u^{-1}$ and that $b_{w \preceq u} = c_{w^{-1} \preceq u^{-1}}$ via a reflection symmetry of permutation diagrams.

Apply Proposition 2.1 to $w$ and $w^{-1}$ separately. We have

$$\Delta \tilde{S}_{w^{-1}} = \sum_{u: w^{-1} \preceq u} (1 + 2b_{w^{-1} \preceq u^{-1}}) \tilde{S}_u$$

$$= \sum_{u^{-1}: w^{-1} \preceq u^{-1}} (1 + 2b_{w^{-1} \preceq u^{-1}}) \tilde{S}_{u^{-1}}$$

$$= \sum_{u: w \preceq u} (1 + 2c_{w \preceq u}) \tilde{S}_{u^{-1}},$$

$$\Delta \tilde{S}_w = \sum_{u: w \preceq u} (1 + 2b_{w \preceq u}) \tilde{S}_u.$$

Now take the principal specialization and subtract these two equations. The left-hand side becomes zero as explained above. Recalling from above that $\tilde{S}_{u^{-1}}(1, \ldots, 1) = \tilde{S}_u(1, \ldots, 1)$, we obtain the desired equality. \hfill \square

Now we are ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** We proceed by induction on $(\binom{n}{2} - \ell(w))$. The base case $w = w_0$ is trivial as both sides equal 1. Now fix $w$ and assume that the statement is true for all $u$ with $\ell(u) > \ell(w)$. The following calculation
is straightforward:

\[ m_{\text{wt}}(w, w_0) = \sum_{u: w \preceq u} (1 + b_{w \preceq u}(2 - z) + c_{w \preceq u}z)m_{\text{wt}}(u, w_0) \]

\[ = \sum_{u: w \preceq u} (1 + b_{w \preceq u}(2 - z) + c_{w \preceq u}z) \left( \binom{n}{2} - \ell(u) \right)! \tilde{\mathcal{S}}_u(1, \ldots, 1) \]

\[ = \left( \binom{n}{2} - \ell(w) - 1 \right)! \sum_{u: w \preceq u} (1 + 2b_{w \preceq u}) \tilde{\mathcal{S}}_u(1, \ldots, 1) \]

\[ - \left( \binom{n}{2} - \ell(w) - 1 \right)!z \sum_{u: w \preceq u} (b_{w \preceq u} - c_{w \preceq u}) \tilde{\mathcal{S}}_u(1, \ldots, 1). \]

By Lemma 4.1, the second term in the above expression becomes 0. And by the principal specialization of Proposition 2.1, we have that

\[ \sum_{u: w \preceq u} (1 + 2b_{w \preceq u}) \tilde{\mathcal{S}}_u(1, \ldots, 1) = \left( \binom{n}{2} - \ell(w) \right)! \tilde{\mathcal{S}}_w(1, \ldots, 1). \]

Thus the first term in the above expression becomes \( \left( \binom{n}{2} - \ell(w) \right)! \tilde{\mathcal{S}}_w(1, \ldots, 1) \), which is what we want.

\[ \square \]

5. Weight symmetries and the proof of Theorem 1.2

It is well known that the maps \( v \mapsto w_0v \) and \( v \mapsto vw_0 \) are antiautomorphisms of the Bruhat order \( S_n \) and that \( v \mapsto v^{-1} \) is an automorphism \([2]\); Proposition 5.1 determines the effect of these maps on the quantities \( a, b, c, \) and \( d \) from Definition 1.1.

**Proposition 5.1.** Let \( v \preceq w \) be a covering relation in \( S_n \).

1. \( a_{v \preceq w} = d_{v^{-1} \preceq w^{-1}} \) and \( b_{v \preceq w} = c_{v^{-1} \preceq w^{-1}} \),
2. \( b_{v \preceq w} = d_{w_0w \preceq u_0v} \), and
3. \( a_{v \preceq w} = c_{w_0w \preceq v_0w} \).

**Proof.** These are clear from Figure 1 after observing that inversion corresponds to reflecting the permutation matrix across the main (top-left to bottom-right) diagonal, that left multiplication by \( w_0 \) corresponds to reflecting across the vertical axis, and that right multiplication by \( w_0 \) corresponds to reflecting across the horizontal axis.

We can now complete the proof of Theorem 1.2.

**Proof of Theorem 1.2.** There are six cases to consider, depending on which pair of \( z_A, z_B, z_C \), and \( z_D \) are equal to \( z \) and \( 2 - z \) (the others being zero); which element of the pair is sent to \( z \) or \( 2 - z \) does not matter, since the claimed result is independent of \( z \).

For the pair \( \{ z_B, z_C \} \), letting \( w = e \) in Theorem 1.4 proves the result. For \( \{ z_B, z_D \} \), letting \( \alpha_1 = \cdots = \alpha_{n-1} \) in Theorem 1.3 gives weights

\[ \text{wt}(v \preceq w) = (1 + b_{v \preceq u} + d_{v \preceq w}) + (b_{v \preceq w} - d_{v \preceq w})z \]
which clearly give all of the desired linear combinations of $b_{v < w}$ and $d_{v < w}$.

Applying the symmetries from Proposition 5.1 then yields the remaining pairs. □

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