BOUNDING THE COLLAPSBILITY NUMBER OF SIMPLICIAL COMPLEXES AND GRAPHS

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Abstract. We introduce and study a new combinatorial invariant the theta-number \( \theta(X) \) of simplicial complexes, and prove that the inequality \( C(X) \leq \theta(X) \) holds for every simplicial complex \( X \), where \( C(X) \) denotes the collapsibility number of \( X \). We display the advantages of working with the theta-number. Its purely combinatorial formulation enables us to verify the validity of the existing bounds on both Leray and collapsibility numbers as well as provide new bounds involving other parameters.

We show that the theta-number, collapsibility and Leray numbers of a vertex decomposable simplicial complex are all equal. Moreover, we prove that the theta-number of the independence complex of a graph \( G \) is closely related to its induced matching number \( \text{im}(G) \) as it happens to the Leray number of such complexes. We identify graph classes where they are equal, and otherwise provide upper bounds involving it. In particular, we prove that the theta-number is bounded from above by \( 2^{\text{im}(G)} \) for every \( n \)-vertex graph \( G \), and in the case of \( 2K_2 \)-free graphs, we lower this bound to \( 2 \log n \). Furthermore, we verify that the theta-number is contraction minor monotone on the underlying graph.

1. Introduction

A (generalized) simplicial complex \( X \) on a vertex set \( V \) is simply a family of subsets of \( V \), closed under inclusion. In particular, \( \text{vert}(X) := \{ x \in V : \{ x \} \in X \} \) is called the set of actual vertices of \( X \). If \( v \in V \setminus \text{vert}(X) \), we say that \( v \) is a ghost vertex of \( X \). A set \( A \subseteq X \) is said to be a face of \( X \), and the dimension of a face \( A \) is \( \dim(A) = |A| - 1 \). The dimension \( \dim(X) \) of a complex \( X \) is the maximum dimension of a face in \( X \). For a given vertex \( v \in \text{vert}(X) \), the deletion and link subcomplexes of \( X \) at the vertex \( v \) is defined by \( \text{del}(X; v) := \{ S \subseteq X : v \notin S \} \) and \( \text{lk}(X; v) := \{ T \subseteq X : v \notin T \text{ and } T \cup \{ v \} \in X \} \). For convenience, if \( u \) is a ghost vertex, we set \( \text{lk}(X; u) := \text{del}(X; u) = \emptyset \). A vertex \( v \in V \) is said to be a cone vertex of \( X \) if \( \text{lk}(X; v) = \text{del}(X; v) \), and denote by \( V^\circ \), the set of all non-cone vertices in \( X \). Unless stated otherwise, by a vertex of a simplicial complex, we mean an actual vertex of it.

We introduce and study a new combinatorial invariant on simplicial complexes.

Definition 1. We define the theta-number of a simplicial complex \( X \) by \( \theta(X) := 0 \) if \( V(X)^\circ = \emptyset \), and

\[
\theta(X) := \min_{v \in V(X)^\circ} \{ \max\{ \theta(\text{del}(X; v)), \theta(\text{lk}(X; v)) + 1 \} \}
\]

whenever \( V(X)^\circ \neq \emptyset \).

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Our main motivation to study such an invariant comes from the fact that it is closely related to two other well-known parameters of simplicial complexes that we recall next.

A (finite) simplicial complex $X$ is $k$-collapsible if it can be reduced to the void complex by repeatedly removing a face of size at most $k$ that is contained in a unique maximal face of $X$. The collapsibility number $C_p(X)$ of $X$ is the smallest integer $k$ such that it is $k$-collapsible. On the other hand, a simplicial complex $X$ is $k$-Leray (over a field $k$, mostly chosen to be rationals), if the homology of every induced subcomplex $X[S] := \{ A \in X : A \subseteq S \}$ of $X$ with $S \subseteq V(X)$ vanishes in dimension $k$ and larger (with coefficients in $k$). The Leray number $L_p(X)$ of $X$ is the least integer $k$ for which $X$ is $k$-Leray. This coincides with the (Castelnuovo-Mumford) regularity of the Stanley-Reisner ring of $X$ (see [1] for details). Wegner proved that the inequality $L_p(X) \leq C_p(X)$ always holds [14].

We initiate a detail analyze on the properties of theta-number of simplicial complexes. In particular, we verify that the inequality $C_p(X) \leq \theta(X)$ holds for every simplicial complex $X$. In fact, this is rather an easy consequence of a known upper bound on the collapsibility number due to Tancer [13]. By taking the advantages of its recursive formulation, we exhibit that the theta-number may overcome some technical difficulties caused by homology/geometric dependencies of Leray and collapsibility numbers. We prove that the theta-number is additive under the simplicial join operation. Meanwhile, we note that the additivity of the collapsibility number is currently unknown.

In an analogy with Leray numbers [1], we show that $\theta(X)$ always equals to either $\theta(\text{del}(X; v))$ or $\theta(\text{lk}(X; v)) + 1$ for every vertex $v$ in $X$. This brings the use of a reduction process that may ease calculations in specific cases as well as bounding the theta-number of simplicial complexes in general. We say that a simplicial complex $X$ is a $\theta$-prime complex if $\theta(\text{del}(X; v)) < \theta(X)$ for each vertex $v \in V(X)$. We prove that a simplicial complex $X$ is either itself $\theta$-prime or else it can be decomposed into its induced subcomplexes $X_1, \ldots, X_n$ for which each subcomplex $X_i$ is $\theta$-prime with $\theta(X) = \sum_{i=1}^n \theta(X_i)$. It would be an intriguing question to figure out what makes a simplicial complex $\theta$-prime.

In the case of independence complexes of graphs, we show that the theta-number is closely related to the induced matching number. We describe graph classes where they are equal, and otherwise provide upper bounds involving it. We prove that the theta-number is bounded from above by $2\sqrt{n \cdot \text{im}(G)}$ for every $n$-vertex graph $G$, and in the case of $2K_2$-free graphs, we lower this bound to $2 \log n$. Furthermore, we verify that the theta-number is monotone decreasing under edge contractions on the underlying graph.

Matoušek and Tancer [12] show that the gap $C(X) - L(X)$ between the collapsibility and Leray numbers could be arbitrarily large. On the other hand, we verify that the theta-number, collapsibility and Leray numbers of a vertex decomposable simplicial complex are equal.

It would be important to point out that when the characteristic of the coefficient field is irrelevant, the existing calculations or known bounds on the Leray number of simplicial complexes are combinatorial in nature [1, 5]. In other words, all these methodologies used to calculate or bound the Leray number are in fact applicable for the theta-number. Our present work supports this assertion in generality.
2. Properties of theta-number of simplicial complexes

This section exposes some properties satisfied by our new parameter. We prove that the theta-number is additive on simplicial join operation, and characterize those simplicial complexes for which $\theta(X) = 1$. We initially note that in many cases, we will apply to an induction on the number of vertices of a simplicial complex. So, wherever it is convenient, we will not repeat to mention this formally.

Let $X$ be a simplicial complex and $S \subseteq V$ a subset. Then the subcomplex of $X$ induced by the set $S$ is defined by $X[S] := \{A \in X: A \subseteq S\}$.

**Proposition 2.** $\theta(X[S]) \leq \theta(X)$ for any subset $S \subseteq V$.

**Proof.** We may assume that $X$ contains no cone vertex. So, let $x \in V = V^\circ$ be a vertex such that $\theta(X) = \max\{\theta(\text{del}(X; x)), \theta(\text{lk}(X; x)) + 1\}$. There are two cases to consider.

- **Case 1:** $x \in S$. It then follows from the Definition 1 that
  \[
  \theta(X[S]) \leq \max\{\theta(\text{del}(X[S]; x)), \theta(\text{lk}(X[S]; x)) + 1\} = \max\{\theta(\text{del}(X; x)[S]), \theta(\text{lk}(X; x)[S]) + 1\} \\
  \leq \max\{\theta(\text{del}(X; x)), \theta(\text{lk}(X; x)) + 1\} = \theta(X),
  \]
  where the second inequality is by the induction.

- **Case 2:** $x \notin S$. Notice that $X[S] \cong \text{del}(X; x)[S]$ so that
  \[
  \theta(X[S]) \leq \theta(\text{del}(X; x)) \\
  \leq \max\{\theta(\text{del}(X; x)), \theta(\text{lk}(X; x)) + 1\} = \theta(X),
  \]
  where the first inequality is by the induction. \hfill \Box

If we denote by $X^n$, the $n$-dimensional simplex on a $(n + 1)$-element set, and by $\partial X^n$ its boundary, then $\theta(\partial X^n) = n$ for each $n \geq 1$. This follows from the obvious facts that $\text{del}(\partial X^n; x) \cong X^{n-1}$ and $\text{lk}(\partial X^n; x) \cong \partial X^{n-1}$ for each vertex $x$ together with the induction on $n$.

**Lemma 3.** Let $u$ and $v$ be two vertices of a simplicial complex $X$. Then the followings hold:

1. $\text{lk}(\text{lk}(X; v); u) = \text{lk}(\text{lk}(X; u); v)$,
2. $\text{del}(\text{del}(X; v); u) = \text{del}(\text{del}(X; u); v)$,
3. $\text{del}(\text{lk}(X; v); u) = \text{lk}(\text{del}(X; u); v)$.

**Proof.** We only verify (3), since the proofs of (1) and (2) are routine. So, consider a face $A \in \text{del}(\text{lk}(X; v); u)$. This means that $A \in \text{lk}(X; v)$ and $u \notin A$. The former implies that $v \notin A$ and $A \cup \{v\} \in X$, and the latter forces that $A \in \text{del}(X; u)$. Combining these, we obtain that $A \in \text{lk}(\text{del}(X; u); v)$. For the opposite direction, let $B \in \text{lk}(\text{del}(X; u); v)$ be given. It follows that $B \in \text{del}(X; u)$, $v \notin B$ and $B \cup \{v\} \in \text{del}(X; u)$. Notice that $B \in \text{lk}(X; v)$ while $u \notin B$. In other words, $B \in \text{del}(\text{lk}(X; v); u)$. \hfill \Box
Theorem 4. \( \theta(\text{lk}(X; v)) \leq \theta(\text{del}(X; v)) \) for every vertex \( v \) in \( X \).

Proof. Note first that the claimed inequality holds if \( v \notin V^\circ \) by our previous convention. So, assume that \( v \in V^\circ \). Then there exists a vertex \( z \in V \setminus \{v\} \) such that

\[
\theta(\text{del}(X; v)) = \max\{\theta(\text{del}(\text{del}(X; v); z)), \theta(\text{lk}(\text{del}(X; v); z)) + 1\}.
\]

If such a vertex \( z \) is a ghost vertex of the complex \( \text{lk}(X; v) \), then \( \text{lk}(X; v) = \text{lk}(\text{del}(X; z); v) \).

So, we conclude by the induction that

\[
\theta(\text{lk}(X; v)) = \theta(\text{del}(\text{del}(X; z); v)) \leq \theta(\text{del}(\text{del}(X; z); v)) \leq \theta(\text{del}(X; v)).
\]

Otherwise, by employing Lemma 5 together with the induction, we have that

\[
\theta(\text{lk}(X; v)) \leq \max\{\theta(\text{del}(\text{lk}(X; v); z)), \theta(\text{lk}(\text{lk}(X; v); z)) + 1\} = \max\{\theta(\text{lk}(\text{del}(X; z); v)), \theta(\text{lk}(\text{lk}(X; z); v)) + 1\} \leq \max\{\theta(\text{del}(\text{del}(X; z); v)), \theta(\text{lk}(\text{del}(X; v); z)) + 1\} = \theta(\text{del}(X; v)).
\]

\[\square\]

Corollary 5. \( \theta(X) \) always equals to one of \( \theta(\text{del}(X; v)) \) or \( \theta(\text{lk}(X; v)) + 1 \) for every vertex \( v \) in \( X \).

Definition 6. We call a vertex \( v \) of \( X \), a \( \theta \)-prime vertex if \( \theta(\text{del}(X; v)) < \theta(X) \). A simplicial complex \( X \) is called a \( \theta \)-prime complex if every vertex of it is \( \theta \)-prime.

We remark that if \( v \) is a \( \theta \)-prime vertex of \( X \), then \( \theta(X) = \theta(\text{lk}(X; v)) + 1 \) by Corollary 5. Furthermore, we count the complex \( \{\emptyset\} \), as a \( \theta \)-prime complex with \( \theta(\{\emptyset\}) = 0 \).

The notion of a \( \theta \)-prime vertex allows us to formulate a \( \theta \)-prime reduction process. Let \( X \) be a simplicial complex and \( F = \{v_1, v_2, \ldots, v_n\} \) be an ordered subset of \( V(X) \).

We repeatedly apply Corollary 5 to the vertices of \( F \) and associate an integer prime \( \text{prime}_i \) at each state \( i \geq 0 \). We start with the complex \( X_0 := X \), and set \( F_0 := F \) and \( \text{prime}_0 := 0 \). Pick a vertex \( v_{i+1} \in F_i \) for some \( i \geq 0 \). If \( v_{i+1} \) is a \( \theta \)-prime vertex of \( X_i \), we define \( X_{i+1} := \text{lk}(X_i; v_{i+1}) \), \( F_{i+1} := F_i \cap V(\text{lk}(X_i; v_{i+1})) \) and \( \text{prime}_{i+1} := \text{prime}_i + 1 \). On the other hand, if \( v_{i+1} \) is not a \( \theta \)-prime vertex of \( X_i \), we then set \( X_{i+1} := \text{del}(X_i; v_{i+1}) \), \( F_{i+1} := F_i \setminus \{v_{i+1}\} \) and \( \text{prime}_{i+1} := \text{prime}_i \).

The reduction process terminates when \( F_k = \emptyset \) for some \( k \geq 0 \), in which case we denote by \( X_F \) and \( \text{prime}_F \), the resulting complex and the count of in how many steps the counter is incremented by one. Observe that for any subset \( F \subseteq V(X) \), the inequality \( \theta(X) \leq \theta(X_F) + \text{prime}_F \) holds.

We call an ordered subset \( \{v_1, \ldots, v_k\} \) of vertices of \( X \), a \( \theta \)-prime set if it is the set of those vertices at which the counter is incremented by one under the reduction process on \( X \) with respect to \( V \). Notice that every \( \theta \)-prime set for \( X \) has the same order, namely \( \theta(X) \).

Proposition 7. Given a simplicial complex \( X \) and \( F \subseteq V \). If \( A = \{v_1, \ldots, v_k\} \subseteq F \) is the subset of vertices at which the counter is incremented by one under the reduction process on \( X \) with respect to \( F \), then \( A \in X \).
Proof. We proceed by the induction on \( k \). If \( k = 1 \), there is nothing to show. Denote by \( l_i \), the stage of the reduction process that applies to the vertex \( v_i \). In other words, \( l_i \) is the integer so that \( X_{l_i} = \text{lk}(X_{(l_i-1)}; v_i) \) for each \( i \in [k] \). If we consider the complex \( X_{l_i} = \text{lk}(X_{(l_i-1)}; v_i) \), we may assume that \( \{v_2, \ldots, v_k\} \in X_{l_i} \) by the induction hypothesis. However, this implies that \( \{v_1, \ldots, v_k\} \in X_{(l_i-1)} \). Since \( X_{(l_i-1)} \) is a subcomplex of \( X \), the claim follows. \( \square \)

Corollary 8. \( \theta(X) \leq \dim(X) + 1 \) for every simplicial complex \( X \).

We may further improve the bound of Corollary 8. Recall that a subset \( S \subseteq V \) is said to be a circuit (minimal non-face) of \( X \) if \( S \) is not a face of \( X \) while any proper subset of \( S \) is.

Definition 9. We call a subset \( C \subseteq V \), a circuit cover for \( X \) provided that \( |S \cap C| \geq |S| - 1 \) for every circuit \( S \) in \( X \). We define the circuit cover number of a simplicial complex \( X \) by

\[
\text{ccn}(X) := \min\{\dim(X[C]) : C \text{ is a circuit cover of } X\} + 1.
\]

Theorem 10. \( \theta(X) \leq \text{ccn}(X) \) for every simplicial complex \( X \).

Proof. Suppose that \( C \) is a circuit cover of \( X \) for which \( \text{ccn}(X) = \dim(X[C]) + 1 \). We run the \( \theta \)-prime reduction on \( X \) with respect to the set \( C \), and let \( U = \{u_1, \ldots, u_k\} \subseteq C \) be the subset of vertices at which the counter is incremented by one. Since \( C \) is a circuit cover, the complex \( X_C \) is a simplex. Therefore, we conclude that \( \theta(X) \leq k \leq \text{ccn}(X) \), since \( U \in X[C] \) by Proposition 7. \( \square \)

We next prove that the theta-number is additive on simplicial join operation. Meanwhile, we note that whether a similar property holds for the collapsibility number is currently unknown.

If \( X_1 \) and \( X_2 \) are two simplicial complexes on disjoint sets \( V_1 \) and \( V_2 \), then their join is the simplicial complex \( X_1 \ast X_2 := \{A \cup B : A \in X_1, B \in X_2\} \).

Theorem 11. If \( X_1 \) and \( X_2 \) are two simplicial complexes on disjoint sets \( V_1 \) and \( V_2 \), then

\[
\theta(X_1 \ast X_2) = \theta(X_1) + \theta(X_2).
\]

Proof. We first note that the equality \( (V_1 \cup V_2)^c = V_1^c \cup V_2^c \) holds. Furthermore, if \( x \in V_1^c \), then \( \text{del}(X_1 \ast X_2; x) \cong \text{del}(X_1; x) \ast X_2 \) and \( \text{lk}(X_1 \ast X_2; x) \cong \text{lk}(X_1; x) \ast X_2 \). Thus, the equalities

\[
\theta(\text{del}(X_1 \ast X_2; x)) = \theta(\text{del}(X_1; x)) + \theta(X_2)
\]

\[
\theta(\text{lk}(X_1 \ast X_2; x)) = \theta(\text{lk}(X_1; x)) + \theta(X_2)
\]

hold by the induction. Assume now that \( a \in (V_1 \cup V_2)^c \) and \( b \in V_1^c \) are vertices such that

\[
\theta(X_1 \ast X_2) = \max(\theta(\text{del}(X_1 \ast X_2; a)), \theta(\text{lk}(X_1 \ast X_2; a)) + 1)
\]

\[
\theta(X_1) = \max(\theta(\text{del}(X_1; b)), \theta(\text{lk}(X_1; b)) + 1).
\]

It then follows that

\[
\theta(X_1 \ast X_2) \leq \max(\theta(\text{del}(X_1 \ast X_2; b)), \theta(\text{lk}(X_1 \ast X_2; b)) + 1)
\]

\[
= \max(\theta(\text{del}(X_1; b)), \theta(\text{lk}(X_1; b)) + 1) + \theta(X_2)
\]

\[
= \theta(X_1) + \theta(X_2),
\]
and similarly
\[
\theta(X_1 \ast X_2) = \max\{\theta(\text{del}(X_1 \ast X_2; a)), \theta(\text{lk}(X_1 \ast X_2; a)) + 1\}
\]
\[
= \max\{\theta(\text{del}(X_1; a)), \theta(\text{lk}(X_1; a)) + 1\} + \theta(X_2)
\]
\[
\geq \theta(X_1) + \theta(X_2),
\]
which completes the proof. \(\Box\)

**Definition 12.** Let \(X\) be a simplicial complex and let \(R = \{R_1, \ldots, R_r\}\) be a set of pairwise vertex disjoint subsets of \(V\) such that \(|R_i| \geq 2\) for each \(1 \leq i \leq r\). Then \(R\) is said to be an *induced decomposition* of \(X\) if \(X[\bigcup_{i=1}^{r} R_i] \cong X[R_1] \ast \cdots \ast X[R_r]\), and \(R\) is maximal with this property. The set of induced decompositions of a complex \(X\) is denoted by \(\mathcal{ID}(X)\).

Let \(R = \{R_1, \ldots, R_r\}\) be an induced decomposition of a complex \(X\). If each \(X[R_i]\) is a \(\theta\)-prime complex, then we call \(R\) as a *prime decomposition* of \(X\), and the set of prime decompositions of a complex \(X\) is denoted by \(\mathcal{PD}(X)\).

**Theorem 13.** For any simplicial complex \(X\), we have
\[
\theta(X) = \max\{\sum_{i=1}^{r} \theta(X[R_i]): \{R_1, \ldots, R_r\} \in \mathcal{PD}(X)\}.
\]

*Proof.* If \(X\) is itself a \(\theta\)-prime complex, there is nothing to prove. Otherwise there exists a vertex \(v \in V\) such that \(\theta(X) = \theta(\text{del}(X; v))\). If \(\text{del}(X; v)\) is a \(\theta\)-prime complex, then \(\{\text{del}(X; v)\} \in \mathcal{PD}(X)\) so that the result follows. Otherwise, we have \(\theta(\text{del}(X; v)) = \max\{\sum_{i=1}^{r} \theta(\text{del}(X; v)[S_i]): \{S_1, \ldots, S_l\} \in \mathcal{PD}(\text{del}(X; v))\}\) by the induction. However, since \(\mathcal{PD}(\text{del}(X; v)) \subseteq \mathcal{PD}(X)\) for such a vertex, the claim follows. \(\Box\)

Our final aim in this section is to identify those simplicial complexes \(X\) for which \(\theta(X) = 1\). Having this in mind, we shortly review graph’s terminology first.

When \(G = (V, E)\) is a (finite and simple) graph, we denote by \(N_G(x) := \{y \in V: xy \in E\}\), the (open) neighborhood of \(x\) in \(G\), whereas \(N_G[x] := N_G(x) \cup \{x\}\) is its closed neighborhood. The degree of a vertex \(x\) and the maximum degree of a graph \(G\) are denoted by \(\text{deg}_G(x)\) and \(\Delta(G)\) respectively. Furthermore, \(\overline{G}\) denotes the complement of the graph \(G\). For a given subset \(S \subseteq V\), the subgraph \(G[S]\) of \(G\) induced by the set \(S\) is the graph on \(S\) with \(E(G[S]) = E \cap (S \times S)\). Note that \(\text{Ind}(G)[S] = \text{Ind}(G[S])\) for every subset \(S \subseteq V\).

A set \(A \subseteq V\) is said to be an *independent set* of \(G\) provided that it contains no edge of \(G\). The maximum size \(\alpha(G)\) of an independent set in \(G\) is the independence number of a graph \(G\). The family of all independent sets forms a simplicial complex, the *independence complex* \(\text{Ind}(G)\) of \(G\). Note also that a subset \(C \subseteq V\) is a *vertex cover* of \(G\) provided that \(V \setminus C\) is an independent set.

**Definition 14.** For a given graph \(G\), we define \(\theta(G) := \theta(\text{Ind}(G))\). Furthermore, we say that a graph \(G\) is *\(\theta\)-prime* if \(\text{Ind}(G)\) is a \(\theta\)-prime complex.
Since \( \text{del}(\text{Ind}(G); v) = \text{Ind}(G - v) \) and \( \text{lk}(\text{Ind}(G); v) = \text{Ind}(G - N_G[v]) \) for every vertex \( v \) in \( G \), we may naturally think of the theta-number of a graph as a purely combinatorial graph invariant without any emphasis on its independence complex. Note also that \( V^\circ = V(\text{Ind}(G))^\circ \) is the set of all non-isolated vertices in \( G \). We may therefore write \( \theta(G) = 0 \) if \( E = \emptyset \), and

\[
\theta(G) = \min_{v \in V^\circ} \{ \max\{\theta(G - v), \theta(G - N_G[v]) + 1\} \}
\]

otherwise. In the guise, Corollary \ref{corollary:theta} can be rewritten as in the following form.

**Corollary 15.** \( \theta(G) \) always equals to one of \( \theta(G - v) \) or \( \theta(G - N_G[v]) + 1 \).

We remark that the circuits of the complex \( \text{Ind}(G) \) are exactly the edges of the graph \( G \); hence, a circuit cover is nothing but a vertex cover in \( G \). Therefore, we may rewrite the circuit cover number as

\[
\text{ccn}(G) = \min\{\alpha(G[F]) \colon F \text{ is a vertex cover for } G\}.
\]

Notice that the gap between \( \text{ccn}(G) \) and \( \theta(G) \) could be arbitrarily large. For instance, if we denote by \( K_{n,n} \), the complete bipartite graph with \( n \geq 1 \), then \( \text{ccn}(K_{n,n}) = n \) while \( \theta(K_{n,n}) = 1 \).

We denote by \( K_n, P_n \) and \( C_k \), the complete, path and cycle graphs on \( n \geq 1 \) and \( k \geq 3 \) vertices respectively.

**Example 16.** \( \theta(C_n) = 2 \) for each \( n \geq 4 \). Firstly, we note that \( \theta(C_n) \leq \alpha(C_n) = 2 \). On the other hand, \( \theta(C_n - N_{C_n}[v]) = 1 \) for each vertex \( v \in V(C_n) \) as the graph \( C_n - N_{C_n}[v] \) is isomorphic to \( K_2 \). This implies that \( 2 \geq \theta(C_n - N_{C_n}[v]) + 1 \geq 2 \) so that \( \theta(C_n - N_{C_n}[v]) + 1 = 2 \). Furthermore, since \( \theta(C_n - v) \leq \theta(C_n) \leq 2 \), we conclude that \( \max\{\theta(C_n - v), \theta(C_n - N_{C_n}[v]) + 1\} = 2 \) for every vertex \( v \in V(C_n) \), which proves that \( \theta(C_n) = 2 \). Finally, we point out that one may easily verify that the graph \( C_n \) is \( \theta \)-prime for each \( n \geq 4 \).

**Proposition 17.** If \( u \) is a vertex of a graph \( H \) such that the set \( N_H[u] \) is a vertex cover for \( H \), then either \( \theta(H) = \theta(H - u) \) or else \( \theta(H) = 1 \).

**Proof.** If \( \theta(H) = \theta(H - u) \), there is nothing to prove. Otherwise, it follows that \( \theta(H) = \theta(H - N_H[u]) + 1 \) by Corollary \ref{corollary:theta}. However, since \( V \setminus N_H[u] \) is an independent set, we conclude that \( \theta(H - N_H[u]) = 0 \).

We say that \( G \) is \( H \)-free if no induced subgraph of \( G \) is isomorphic to \( H \). A graph \( G \) is called **chordal** if it is \( C_r \)-free for every \( r > 3 \). Moreover, a graph \( G \) is said to be **co-chordal** if its complement \( \overline{G} \) is a chordal graph.

**Lemma 18.** If \( G \) is a co-chordal graph, then there exists a vertex \( x \) such that \( N_G[x] \) is a vertex cover for \( G \).

**Proof.** Consider the complement \( \overline{G} \) of \( G \). Since \( \overline{G} \) is a chordal graph, it has a simplicial vertex \( \overline{x} \), say \( x \in V \). This means that the set \( N_{\overline{G}}(x) \) induces a complete subgraph of \( \overline{G} \). Now, if \( u, v \in V \) are two vertices with \( uv \notin E(G) \), then \( V(\overline{G} - N_{\overline{G}}(x)) \cap \{u, v\} \neq \emptyset \). However, this latter fact is equivalent to saying that \( N_G[x] \cap \{u, v\} \neq \emptyset \) whenever \( uv \in E(G) \), i.e., the set \( N_G[x] \) is a vertex cover for \( G \).
Proposition 19. Let $G$ be a graph with $E(G) \neq \emptyset$. Then $G$ is co-chordal if and only if $\theta(G) = 1$.

Proof. By Example [16] we may easily conclude that $G$ is co-chordal if $\theta(G) = 1$. Therefore, we only need to verify that $\theta(G) = 1$ provided that $G$ is co-chordal.

Now, suppose that $G$ is a co-chordal graph. If $G$ contains a vertex $v$ such that $G - v$ is an edgeless graph, then $ccn(G) = 1$ so that $\theta(G) = 1$. Otherwise, we may apply to an induction on the order of $G$ to conclude that $\theta(G - v) = 1$ for each vertex $v \in V$. In order to complete the proof, we are left to prove that $\theta(G - N_C[x]) = 0$ for some vertex $x \in V$. However, this is exactly what Lemma [18] shows.

Theorem 20. Let $X$ be a simplicial complex. Then $\theta(X) = 1$ if and only if $X = \text{Ind}(G)$ for some co-chordal graph $G$.

Proof. This follows from Proposition [19] together with the fact that $X$ must be a flag simplicial complex, i.e., if $S$ is a circuit in $X$, then $|S| = 2$ by Proposition [2].

3. Collapsibility of simplicial complexes

In this section, we prove an analog of Corollary [4] for the collapsibility number of simplicial complexes. It implies that $C(X)$ always equals to either $C(\text{del}(X;\{v\}))$ or $C(\text{lk}(X;\{v\}))+1$ for every vertex $v$ in $X$. We begin with recalling some of the definitions mentioned in the introduction more formally.

A face $A \in X$ is a free face of $X$ if there exists a unique facet (maximal face) containing it. When $A$ is a free face and $B$ is the unique facet containing $A$, we denote by $[A, B]$, the interval $\{C \in X: A \subseteq C \subseteq B\}$. This defines an operation $X \xrightarrow{[A, B]} X - [A, B]$, which is called an elementary $k$-collapse provided that $|A| \leq k$. For two simplicial complexes $X$ and $Y$, we say that $X$ is $k$-collapsible to $Y$, if there exists a sequence of elementary $k$-collapses

$$X = X_0 \xrightarrow{[A_1, B_1]} X_1 \xrightarrow{[A_2, B_2]} \cdots \xrightarrow{[A_d, B_d]} X_d = Y,$$

in which case we write $X \xleftarrow{k} Y$. In particular, a simplicial complex $X$ is called $k$-collapsible if $X \xleftarrow{k} \emptyset$. The collapsibility number of a simplicial complex $X$ is defined by

$$C(X) := \min\{k: X \text{ is } k\text{-collapsible}\}.$$

The family of $k$-collapsible simplicial complexes were introduced by Wegner [14].

We next recall a crucial result of Tancer [13].

Lemma 21. [13, Proposition 1.2] $C(X) \leq \max\{C(\text{del}(X;\{v\})), C(\text{lk}(X;\{v\}))+1\}$ holds for every vertex $v$ of $X$.

Theorem 22. $C(\text{lk}(X;\{v\})) \leq C(\text{del}(X;\{v\}))$ for every vertex $v$ in $X$.

Proof. Assume that $C(\text{del}(X;\{v\})) = k \geq 1$ and it requires $d$ elementary $k$-collapses in order to reduce $\text{del}(X;\{v\})$ to the void complex, that is, there exists a sequence

$$\text{del}(X;\{v\}) = Y_0 \xrightarrow{[A_1, B_1]} Y_1 \xrightarrow{[A_2, B_2]} \cdots \xrightarrow{[A_d, B_d]} Y_d = \emptyset$$

of elementary $k$-collapses. We then appeal to an induction on the number of faces of $X$, that is, on $|X|$ to show that $\text{lk}(X;\{v\})$ is $k$-collapsible.
**Case 1:** $B_1 \in \text{lk}(X; v)$. We note that the face $B_1 \cup \{v\}$ must be a facet of $X$. However, this in turn forces that $A_1 \in X$ is a free face of $X$ and $B_1 \cup \{v\}$ is the unique facet containing it. Consider the elementary $k$-collapse $X \xrightarrow{[A_1, B_1 \cup \{v\}]} X_1 := X - [A_1, B_1 \cup \{v\}]$.

**Claim 1.1:** $\text{del}(X_1; v) = Y_1$.

**Proof of Claim 1.1:** Suppose that $U \in \text{del}(X_1; v)$. Then, $U \in X_1$, $v \notin U$ and $U \cup \{v\} \in X_1$. This implies that neither of faces $U$ and $U \cup \{v\}$ is contained in the interval $[A_1, B_1 \cup \{v\}]$. In other words, $U \notin [A_1, B_1]$ so that $U \in \text{del}(X; v) - [A_1, B_1] = Y_1$. For the other direction, consider a face $S \in Y_1$. Then, $S \in \text{del}(X; v)$ and $S \notin [A_1, B_1]$; hence, $A_1 \notin S$. However, we then have that $S \notin [A_1, B_1 \cup \{v\}]$ so that $S \in \text{del}(X_1; v)$.

**Claim 1.2:** $\text{lk}(X; v) \xrightarrow{[A_1, B_1]} \text{lk}(X_1; v)$ is an elementary $k$-collapse.

**Proof of Claim 1.2:** Notice first that $A_1$ is a free face of $\text{lk}(X; v)$ and $B_1$ is the unique facet containing it. So, it remains to show that $\text{lk}(X_1; v) = \text{lk}(X; v) - [A_1, B_1]$. Assume that $D \in \text{lk}(X; v) - [A_1, B_1]$ is a face. This means that $D \in X$, $v \notin D$ and $D \cup \{v\} \in X$. Since $A_1$ is a free face, we conclude that $A_1 \notin D$. In other words, we have that $D \notin [A_1, B_1 \cup \{v\}]$. This also implies that $D \cup \{v\} \notin [A_1, B_1 \cup \{v\}]$, since $v \notin A_1$. However, this means that $D \in \text{lk}(X_1; v)$. Next, suppose that $F \in \text{lk}(X_1; v)$ is a face, and assume to the contrary that $F \in [A_1, B_1]$. However, since $v \notin F$, we would have $F \in [A_1, B_1 \cup \{v\}]$ so that $F \notin X_1$, a contradiction.

Now, $C(\text{del}(X_1; v)) \leq k$ by Claim 1.1 together with the fact that $Y_1$ is $k$-collapsible. This implies that $C(\text{lk}(X_1; v)) \leq k$ by the induction. Finally, we conclude that $C(\text{lk}(X; v)) \leq k$ by Claim 1.2.

**Case 2:** $B_1 \notin \text{lk}(X; v)$. Since $B_1 \in \text{del}(X; v)$, then it follows that $B_1 \cup \{v\} \notin X$. In other words, $A_1$ is a free face of $X$ and $B_1$ is the unique facet containing it. So, there is an elementary $k$-collapse $X \xrightarrow{[A_1, B_1]} X' := X - [A_1, B_1]$.

**Claim 2.1:** $\text{del}(X'; v) = Y_1$.

**Proof of Claim 2.1:** If $S \in \text{del}(X'; v)$, then $v \notin S$, $S \in X$ and $S \notin [A_1, B_1]$. In other words, $S \in \text{del}(X; v) - [A_1, B_1] = Y_1$. On the other hand, if $T \in Y_1$, then $v \notin T$, $T \in X$ and $T \notin [A_1, B_1]$. Therefore, we have that $T \in \text{del}(X'; v)$.

**Claim 2.2:** $\text{lk}(X; v) = \text{lk}(X'; v)$.

**Proof of Claim 2.2:** Let $L \in \text{lk}(X; v)$ be given. If $L \in [A_1, B_1]$, then it follows that $A_1 \cup \{v\} \in X$, since $A_1 \cup \{v\} \leq L \cup \{v\} \in X$. However, since $B_1$ is the unique facet in $X$ containing $A_1$, we must have that $v \in B_1 \in \text{del}(X; v)$, a contradiction. Thus, we conclude that $L \in X'$; hence, $L \in \text{lk}(X'; v)$. For the other direction, suppose that $P \in \text{lk}(X'; v)$ is a face. This means that $v \notin P$, $P \in X'$ and $P \cup \{v\} \in X'$. However, since $X'$ is a subcomplex of $X$, we have that $P \in X$ and $P \cup \{v\} \in X$. In other words, $P \in \text{lk}(X; v)$.

Now, $C(\text{del}(X'; v)) \leq k$ by Claim 2.1 together with the fact that $Y_1$ is $k$-collapsible; hence $C(\text{lk}(X'; v)) = C(\text{lk}(X; v)) \leq k$ by Claim 2.2 and the induction.

**Corollary 23.** $C(X)$ always equals to one of $C(\text{del}(X; v))$ or $C(\text{lk}(X; v)) + 1$ for every vertex $v$ in a simplicial complex $X$.

**Proof.** Recall first that the collapsibility number satisfies that $C(X[S]) \leq C(X)$ for any subset $S \subseteq V [X]$. Corollary 4.1 so that $C(\text{del}(X; v)) \leq C(X)$ holds for every vertex $v$. Next, assume to the contrary that the claim is not true. However it then follows that $C(\text{del}(X; v)) < C(X) < C(\text{lk}(X; v)) + 1 \leq C(\text{del}(X; v)) + 1$. □
a contradiction, where the inequalities follow from Lemma \[21\] and Theorem \[22\] respectively. □

**Remark 24.** Following Corollary \[23\] one may define the notion of a $C$-prime vertex as well as a $C$-prime reduction process on simplicial complexes as they are discussed for the theta-number in Section \[2\].

4. **Theta-number and collapsibility of simplicial complexes**

In this section, we prove that our new parameter provides an upper bound to the collapsibility number. We also verify that an already known bound on the collapsibility number due to Lew \[11\] is a valid upper bound to the theta-number of every simplicial complex. Finally, we show that the theta-number, the collapsibility and Leray numbers of a vertex decomposable simplicial complex are all equal.

**Theorem 25.** $C(X) \leq \theta(X)$ for every simplicial complex $X$.

*Proof.* Note first that $C(X) = \theta(X) = 0$ whenever $V^\circ = \emptyset$. On the other hand, if $v$ is a vertex for which $\theta(X) = \max\{\theta(\text{del}(X; v)), \theta(\text{lk}(X; v)) + 1\}$ holds, then

$$C(X) \leq \max\{C(\text{del}(X; v)), C(\text{lk}(X; v)) + 1\} \leq \max\{\theta(\text{del}(X; v)), \theta(\text{lk}(X; v)) + 1\} = \theta(X),$$

where the first inequality is due to Lemma \[21\] while the second is by the induction. □

The characterization of simplicial complexes satisfying $C(X) = 1$ seems to first appear in the work of Lekkerkerker and Boland \[8\]. In our language, they show that $C(X) = 1$ if and only if $X = \text{Ind}(G)$ for some co-chordal graph $G$ with $E(G) \neq \emptyset$ (see also \[14\]). Therefore, combining their result with Theorem \[20\] we obtain the following.

**Corollary 26.** $C(X) = 1$ if and only if $\theta(X) = 1$.

**Example 27.** The inequality $C(X) \leq \theta(X)$ could be strict. Consider the simplicial complex $X_6$ on \{1, 2, 3, 4, 5, 6\} (compare to \[6\], Table 1) with the following facets

$$\{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 4, 5\}, \{1, 3, 6\}, \{2, 3, 5\}, \{2, 3, 6\}, \{2, 5, 6\}, \{3, 4, 6\}, \{3, 5, 6\}.$$

The complex $X_6$ is 2-collapsible, while the link of each vertex contains an induced subcomplex isomorphic to a triangulation of a one-dimensional sphere. In other words, we have $C(X_6) = 2 < 3 = \theta(X_6)$. However, we do not know whether the gap $\theta(X) - C(X)$ could be arbitrarily large.

Our next aim is to show that an existing upper bound on the collapsibility number of simplicial complexes also bounds the theta-number.

Denote by $S(X)$, the collection of all (ordered) subsets $\{x_1, x_2, \ldots, x_k\}$ of vertices in $X$ satisfying the following condition:

There exist facets $A_1, A_2, \ldots, A_{k+1}$ of $X$ such that

- $x_i \notin A_i$ for each $1 \leq i \leq k$. 

x_i \in A_j \text{ for all } 1 \leq i < j \leq k + 1.

**Theorem 28.** If \{v_1, \ldots, v_k\} is a \(\theta\)-prime set of \(X\), then \{v_1, \ldots, v_k\} \in S(X).

**Proof.** As in the proof of Proposition \[7\] denote by \(l_i\), the stage of the reduction process that applies to the vertex \(v_i\). Since \{v_2, \ldots, v_k\} is a \(\theta\)-prime set of \(X_{l_1} = \text{lk}(X_{(l_1-1)}; v_1)\), we may appeal to an induction in order to assume that \{v_2, \ldots, v_k\} \in S(X_{l_1}). So, there exist facets \(A_2, \ldots, A_{k+1}\) of \(X_{l_1}\) such that

- \(v_i \not\in A_j^i\) for \(2 \leq i \leq k\),
- \(v_i \in A_j^i\) for \(2 \leq i < j \leq k + 1\).

Now, if we define \(A_i := A_i^i \cup \{v_1\}\) for each \(2 \leq i \leq k + 1\), then \(A_i\) is a facet of \(X_{l_1-1}\).

Finally, since \(v_1\) is the first \(\theta\)-prime vertex of the reduction process, the complex \(X_{l_1-1}\) is obtained from \(X\) by the deletions of some vertices \(u_1, \ldots, u_r\) consecutively, that is, \(X_{l_1-1} = X[V - \{u_1, \ldots, u_r\}]\) for some \(r \geq 0\). It then follows that either \(A_i\) is a facet of \(X\) or there exists a subset \(U_i \subseteq \{u_1, \ldots, u_r\}\) such that \(A_i \cup U_i\) is a facet of \(X\) for each \(1 \leq i \leq k + 1\). In either case, the resulting set of facets satisfies the requirements. \(\square\)

**Corollary 29.** If \(k(X)\) denotes the maximum size of a set in \(S(X)\), then \(\theta(X) \leq k(X)\).

We remark that the bound of Corollary \[29\] is proved by Lew [11, Theorem 4] for the collapsibility number, which is adapted from the work of Matoušek and Tancer [12]. In the same paper, Matoušek and Tancer show that the gap \(C(X) - \mathcal{L}(X)\) between the collapsibility and Leray numbers could be arbitrarily large. On the other hand, we next verify that these three invariants coincide on a distinguished class of simplicial complexes.

A simplicial complex \(X\) is said to be vertex decomposable if it is either a simplex or else there exists a vertex \(v\) such that \(\text{del}(X; v)\) and \(\text{lk}(X; v)\) are vertex decomposable, and every facet of \(\text{del}(X; v)\) is a facet of \(X\). In the latter, the vertex \(v\) is called a shedding vertex of \(X\).

**Theorem 30.** \(\theta(X) = C(X) = L(X)\) for every vertex decomposable simplicial complex.

**Proof.** Since the inequalities \(\mathcal{L}(X) \leq C(X) \leq \theta(X)\) hold, we are only left to verify that \(\theta(X) \leq \mathcal{L}(X)\) whenever \(X\) is vertex decomposable. If \(X\) is a simplex, there is nothing to prove. Otherwise, let \(v\) be a shedding vertex of \(X\). It then follows from [3, Theorem 1.5] that \(\mathcal{L}(X) = \max\{\mathcal{L}(\text{del}(X; v)), \mathcal{L}(\text{lk}(X; v)) + 1\}\). Therefore, once we apply to an induction, we conclude that

\[
\theta(X) \leq \max\{\theta(\text{del}(X; v)), \theta(\text{lk}(X; v)) + 1\} \\
\leq \max\{\mathcal{L}(\text{del}(X; v)), \mathcal{L}(\text{lk}(X; v)) + 1\} \\
= \mathcal{L}(X). \tag*{\square}
\]

5. THE THETA-NUMBER OF INDEPENDENCE COMPLEXES OF GRAPHS

In this section, we only consider the theta-number of independence complexes of graphs. In an analogy with the Leray number of such complexes [1], we first prove that the theta-number is monotone decreasing under edge contractions on the underlying graph.
then show that the theta-number of graphs is closely related to the induced matching number. In particular, we provide upper bounds involving the induced matching number. Notice that any possible upper bound on the theta-number of a graph also bounds the collapsibility number of its independence complex. Finally, we describe a family of \( \theta \)-prime graphs such that their independence complexes are weak pseudo-manifolds.

5.1. **Theta-number under edge contractions.** We recall that if \( e = xy \) is an edge of a graph \( G \), the contraction of \( e \) on \( G \) is the graph \( G/e \) defined by \( V(G/e) = (V(G)\{x, y\}) \cup \{w\} \) and \( E(G/e) = E(G - \{x, y\}) \cup \{wz: z \in N_G(x) \cup N_G(y)\} \). A graph \( H \) is a contraction minor of \( G \) if \( H \) is obtained from \( G \) by a sequence of edge contraction.

**Theorem 31.** If \( H \) is a contraction minor of \( G \), then \( \theta(H) \leq \theta(G) \).

We divide the proof of Theorem 31 into several steps. We abbreviate \( N_G[x] \cup N_G[y] \) to \( N_G[e] \) when \( e = xy \) is an edge of \( G \).

**Lemma 32.** If \( e = xy \) is an edge of a graph \( G \), then \( \theta(G - N_G[e]) + 1 \leq \theta(G) \).

**Proof.** Observe that \( \{G - N_G[e], K_2\} \) is an induced decomposition of \( G \), where \( K_2 \) corresponds to the edge \( e = xy \). In other words, \( \theta(G - N_G[e]) + 1 \leq \theta(G) \) by Theorem 13. \( \square \)

**Proposition 33.** If \( e = xy \) is an edge of a graph \( G \), then \( \theta(G - \{x, y\}) \leq \theta(G) \leq \theta(G - \{x, y\}) + 1 \).

**Proof.** The first inequality directly follows from Proposition 2. A similar reasoning together with Corollary 13 also implies the validity of the following inequalities:

\[
\begin{align*}
\theta(G - N_G[x]) & \leq \theta(G - \{x, y\}), \\
\theta(G - x) & \leq \max\{\theta(G - \{x, y\}), \theta((G - x) - N_{G-x}[y]) + 1\}, \\
\theta((G - x) - N_{G-x}[y]) & \leq \theta(G - \{x, y\}).
\end{align*}
\]

Now, if \( x \) is a \( \theta \)-prime vertex of \( G \), then it follows that \( \theta(G) = \theta(G - N_G[x]) + 1 \leq \theta(G - \{x, y\}) + 1 \). So, assume that \( x \) is not a \( \theta \)-prime vertex; hence, \( \theta(G) = \theta(G - x) \). If \( y \) is a \( \theta \)-prime vertex of \( G - x \), we conclude that \( \theta(G) = \theta(G - x) = \theta((G - x) - N_{G-x}[y]) + 1 \leq \theta(G - \{x, y\}) + 1 \). On the other hand, if \( y \) is not a \( \theta \)-prime vertex of \( G - x \), we would have that \( \theta(G) = \theta(G - x) = \theta((G - x) - y) = \theta(G - \{x, y\}) \leq \theta(G - \{x, y\}) + 1 \), which completes the proof. \( \square \)

**Proposition 34.** If \( e = xy \) is an edge of a graph \( G \), then \( \theta(G - \{x, y\}) \leq \theta(G/e) \leq \theta(G - \{x, y\}) + 1 \).

**Proof.** Assume that \( w_e \) is the vertex of \( G/e \) obtained by contracting the edge \( e = xy \) in \( G \). Now, \( \theta(G/e) \leq \max\{\theta(G/e - w_e), \theta(G/e - N_{G/e}[w_e]) + 1\} \) by Corollary 13. Since the graphs \( G/e - w_e \) and \( G - \{x, y\} \) are isomorphic, we have that \( \theta(G/e - w_e) = \theta(G - \{x, y\}) \). Similarly, \( \theta(G/e - N_{G/e}[w_e]) = \theta(G - N_G[e]) \leq \theta(G - \{x, y\}) \), since \( G/e - N_{G/e}[w_e] \cong G - N_G[e] \). \( \square \)

**Theorem 35.** The inequality \( \theta(G/e) \leq \theta(G) \leq \theta(G/e) + 1 \) holds for the contraction of any edge \( e \) of \( G \).

**Proof.** Suppose that \( w_e \) is the vertex of \( G/e \) obtained by contracting the edge \( e = xy \) in \( G \). Observe that \( (G/e) - w_e \cong G - \{x, y\} \) and \( (G/e) - N_{G/e}[w_e] \cong G - N_G[e] \). By
Remark 39. There are two cases. Now, if \( \theta(G/e) = \theta((G/e) - w_e) = \theta(G - \{x, y\}) \leq \theta(G) \). On the other hand, if \( \theta(G/e) = \theta((G/e) - N(G/e)(w_e)) + 1 = \theta(G - N_G[e]) + 1 \leq \theta(G) \) by Lemma 32. For the second inequality, we note that \( \theta(G) \leq \theta(G - \{x, y\}) + 1 \leq \theta(G/e) + 1 \) by Propositions 33 and 34.

The proof of Theorem 31 follows from Theorem 35. We remark that we do not know whether the collapsibility number is monotone decreasing under edge contractions.

5.2. Bounds on the \( \theta \)-number of graphs. Recall that a matching in a graph is a subset of edges no two of which share a vertex. The minimum size of a maximal matching in \( G \) is denoted by \( \text{min-m}(G) \). An induced matching is a matching \( M \) if no two vertices belonging to different edges of \( M \) are adjacent. The maximum size of an induced matching of \( G \) is known as the \textit{induced matching number} \( \text{im}(G) \) of \( G \).

**Proposition 36.** \( \text{im}(G) \leq \theta(G) \leq \text{ccn}(G) \leq \text{min-m}(G) \) for every graph \( G \).

**Proof.** If we denote by \( nK_2 \), the disjoint union of \( n \) copies of \( K_2 \), the first inequality follows from the fact that \( \theta(nK_2) = n \) for each \( n \geq 1 \) together with Proposition 2. For the last one, if \( M \) is a maximal matching in \( G \), then \( V(M) \), the set of vertices incident to edges in \( M \), is a vertex cover. Therefore, the inequality \( \text{ccn}(G) \leq \alpha(G[V(M)]) \leq |M| \) holds.

**Corollary 37.** If a graph \( G \) contains an induced matching \( M \) for which \( V(M) \) is a vertex cover, then \( \theta(G) = \text{im}(G) \).

Graphs satisfying the requirement of Corollary 37 are commonly known as \textit{graphs with an efficient edge dominating set} (or \textit{graphs with a dominating induced matching}) [1]. We demonstrate an application of Corollary 37 in the following example. Let \( n \) and \( k \) be two positive integers with \( n \geq 2k \). Recall that the Kneser graph \( K(n; k) \) is the graph whose vertices are all \( k \)-subsets of an \( n \)-set, where two vertices are adjacent if and only if the corresponding \( k \)-subsets are disjoint.

**Example 38.** \( \theta(K(2n + 1; n)) = \frac{1}{2} \binom{2n}{n} \) for every \( n \geq 1 \). Indeed, if we define \( M_n := E(K(2n; n)) \), then \( M_n \) forms an induced matching of size \( \frac{1}{2} \binom{2n}{n} \) in \( K(2n + 1; n) \). On the other hand, \( V(M_n) \) is a vertex cover for \( K(2n + 1; n) \), since \( V(K(2n + 1; n)) \setminus V(M_n) \) is an independent set from which the claim follows.

**Remark 39.** The equality \( \text{im}(H) = \text{ccn}(H) \) could be possible even if the graph \( H \) admits no dominating induced matchings. For instance, the graph \( H \) depicted in Figure 1 satisfies \( \text{im}(H) = \text{ccn}(H) = \alpha(H) = 3 \), while it has no dominating induced matchings.

Graphs with \( \theta(G) = \text{im}(G) \) are not limited to those of Corollary 37. Let \( \mathcal{G} \) be a graph class with the property that if \( G \in \mathcal{G} \), then both subgraphs \( G - x \) and \( G - N_G[x] \) belong to \( \mathcal{G} \) for each vertex \( x \in V(G) \). Notice that if there exists a vertex \( v \in V(G) \) such that \( \text{im}(G - N_G[v]) < \text{im}(G) \) for every graph \( G \in \mathcal{G} \), then \( \theta(G) = \text{im}(G) \). The class of chordal graphs is an example.

We recall that the \textit{maximum privacy degree} of a graph \( G \) is defined by

\[
\Gamma(G) = \max\{|N_G[x] \setminus N_G[y]| : xy \in E(G)\}.
\]

The proof of the following result is almost identical to that of [1, Theorem 1.3].
Theorem 40. \( \theta(G) \leq (\Gamma(G) + 1) \text{im}(G) \) holds for every graph \( G \). In particular, the inequality \( \theta(G) \leq \Delta(G) \text{im}(G) \) holds for every graph \( G \).

Theorem 41. If \( G \) is an \( n \)-vertex graph, then \( \theta(G) \leq 2 \sqrt{n \cdot \text{im}(G)} \).

Proof. If \( E(G) = \emptyset \), there is nothing to prove. So, we may assume that \( G \) contains at least one edge. We first prove that \( \theta(G) \leq d \cdot \text{im}(G) + \frac{n}{d+1} \) for every integer \( 1 \leq d < n \). If \( \Delta(G) \leq d \), then the required inequality is the consequence of Theorem 40. We may therefore assume that \( 1 \leq d < \Delta(G) \). We define \( W_d(G) = \{ v \in V : \text{deg}_G(v) > d \} \), and apply to the \( \theta \)-prime reduction process on \( G \) with respect to the set \( W_d(G) \). If we denote by \( G_d \) the resulting graph and \( \text{prime}_d \), the counter of the process, then \( \theta(G_d) \leq \text{im}(G_d) \).

Since \( \Delta(G_d) \leq d \), we have \( \theta(G_d) \leq d \cdot \text{im}(G_d) \leq d \cdot \text{im}(G) \). On the other hand, the inequality \( \text{prime}_d \leq \frac{n}{d+1} \) holds, since if a vertex \( v \in W_d(G) \) turns out to be a \( \theta \)-prime vertex at some stage of the process, then we delete \( v \) and all of its neighbors, that is, we delete at least \( (d + 1) \) vertices. Therefore, we conclude that \( \theta(G) \leq d \cdot \text{im}(G) + \frac{n}{d+1} \).

Finally, if we set \( d := \left\lfloor \sqrt{\frac{n}{\text{im}(G)}} \right\rfloor - 1 \), the claim follows. \( \Box \)

Remark 42. One may naturally generalize the notion of the induced matching number of a graph to arbitrary simplicial complexes, possibly by using the concept of the circuit. So, it would be interesting to decide whether a similar bound as in Theorem 41 would be valid under such a generalization.

We may improve the upper bound in the absence of an induced subgraph. We have two candidates to forbid. We denote by \( K_{n,m} \), the complete bipartite graph for any \( n, m \geq 1 \). In particular, the graph \( K_{1,3} \) is known as the claw. Once again, the proof of the following is almost identical to that of [1, Theorem 1.2].

Theorem 43. \( \theta(G) \leq 2 \text{im}(G) \) if \( G \) is a claw-free graph.

We next consider \( 2K_2 \)-free graphs, and begin with recalling a technical property that they must satisfy. Let \( G = (V, E) \) be a graph and \( S,T \subseteq V \) be two subsets. We denote by \( G(S,T) \), the subgraph of \( G \) on \( S \cup T \) consisting of those edges of \( G \) whose one end in \( S \) and the other end in \( T \). We say that a subset \( L \subseteq V \) meets \( G(S,T) \) if \( L \cap \{ x, y \} \neq \emptyset \) for each edge \( xy \) in \( G(S,T) \).

Theorem 44. [2, Theorem 1] If \( G \) is a \( 2K_2 \)-free graph for which \( S \subseteq V \) is an independent set, then there exists \( x \in T := V - S \) such that \( N_G(x) \) meets \( G(S,T) \).
Lemma 45. Let $G$ be a $2K_2$-free graph and $S$ be an independent set of $G$. If $x \in T = V - S$ is a $\theta$-prime vertex such that $N_G(x)$ meets $G(S,T)$, then $\theta(G) \leq \theta(G[T]) + 1$.

Proof. Since $x$ is $\theta$-prime, we have $\theta(G) = \theta(G - N_G[x]) + 1$. However, if $e = xy \in E(G - N_G[x])$, then we must have $x, y \in T$. In other words, $E(G - N_G[x]) \subseteq E(G[T])$; hence, $\theta(G - N_G[x]) \leq \theta(G[T])$. □

Theorem 46. $\theta(G) \leq 2 \log n$ for every $n$-vertex $2K_2$-free graph $G$.

Proof. We proceed by induction on the order of $G$. We may therefore assume that $G$ is a $\theta$-prime graph.

If $G$ has a vertex $v \in V$ such that $\deg_G(v) \geq n$, then $\theta(G) = \theta(G - N_G[v]) + 1 \leq 2 \log \left(\frac{n}{2}\right) + 1 \leq 2 \log n$, where the first inequality is due to the induction. As a result we may further suppose that $\Delta(G) < \frac{n}{2}$. We pick a vertex $x \in V$ and a neighbor $y \in N_G(x)$. We let $S := G - (N_G[x] \cup N_G[y])$ and $T := N_G(y) \setminus N_G[x]$. Note that $S$ is an independent set of $G[S \cup T] = G - N_G[x]$. It then follows that there exists a vertex $z \in T$ such that $N_G(z)$ meets $G(S,T)$. If $z$ is a $\theta$-prime vertex, then

$$
\theta(G) = \theta(G - N_G[x]) + 1 \leq \theta(G[T]) + 2 \leq 2 \log \left(\frac{n}{2}\right) + 2 = 2 \log n,
$$

where the second inequality is due to the fact that $|T| < \frac{n}{2}$.

If $z$ is not a $\theta$-prime vertex of $G - N_G[x]$, we look at the graph $G[S \cup T \setminus \{z\}]$, and apply once again Theorem 44 together with Lemma 45. It then follows that there exists a subset $T' \subset T$, and vertex $z' \in T'$ such that $\theta(G[S \cup T]) = \theta(G[S \cup T'])$, the set $N_G(z')$ meets $G(S,T')$ and $z'$ is a $\theta$-prime vertex of $G[S \cup T']$. Since $|T'| < |T| < \frac{n}{2}$, the claim follows. □

5.3. Strongly $\theta$-prime graphs. In this final subsection, we describe a family of $\theta$-prime graphs such that their independence complexes are weak pseudo-manifolds. Recall that a simplicial complex $X$ is called a weak pseudo-manifold if all facets of $X$ have the same dimension, say $d$, and each $(d-1)$-dimensional face of $X$ is contained in exactly two facets of $X$.

We note that as in the case of the Leray number of graphs [1], we are far from a convincing description of $\theta$-prime graphs. However, we may at least reveal the existence of a property preventing a graph to be $\theta$-prime.

Proposition 47. If $x, y \in V$ are two vertices in $G$ satisfying $N_G[x] \subseteq N_G[y]$ and $\deg_G(y) \geq 2$, then $G$ is not a $\theta$-prime graph.

Proof. Assume to the contrary that $G$ is a $\theta$-prime graph. If we define $A := V \setminus N_G[y]$, then $\{G[A], K_2\}$ is an induced decomposition of $G$, where $V(K_2) = \{x, y\}$. Since $x$ and $y$ are $\theta$-primes, it follows that

$$
\theta(G - y) = \theta(G - x) = \theta(G - N_G[y]) = \theta(G - N_G[x]) < \theta(G) = \theta(G - N_G[y]) + 1 = \theta(G[A]) + 1 = \theta(G[A] \cup K_2).
$$

In other words, $\theta(G) = \theta(G - (N_G(y) \setminus \{x\}))$ so that $G$ is not a $\theta$-prime graph. □
We call a \( \theta \)-prime graph \( G \), a strongly \( \theta \)-prime graph if either \( G = \emptyset \) or else \( G - N_G[x] \) is a strongly \( \theta \)-prime graph for every vertex \( x \in V \). If \( G \) is a strongly \( \theta \)-prime graph, then there exists a sequence \( \{x_1, \ldots, x_k\} \) of some vertices of \( G \) with \( \theta(G) = k \) such that if we set \( G_0 := G \) and \( G_{i+1} := G_i - N_{G_i}[x_{i+1}] \) for each \( 0 \leq i < k \), then \( G_k = \emptyset \). We call such a sequence a strongly \( \theta \)-prime sequence of \( G \). We remark that strongly \( \theta \)-prime sequences of a strongly \( \theta \)-prime graph are of the same length, namely, \( \theta(G) \).

**Corollary 48.** If \( G \) is a strongly \( \theta \)-prime graph, so is \( G - N_G[S] \) for every independent set \( S \) in \( G \).

We recall that a graph is well-covered if all its maximal independent sets are of the same size, and a well-covered graph \( G \) is 1-well-covered if \( G - x \) is well-covered for each vertex \( x \in V \).

**Lemma 49.** If \( G \) is a strongly \( \theta \)-prime graph, then \( G \) is 1-well-covered.

*Proof.* We initially verify that \( G \) is well-covered. Assume to the contrary that there exist two maximal independent sets \( A = \{x_1, \ldots, x_n\} \) and \( B = \{y_1, \ldots, y_m\} \) with \( n \neq m \). Now, if we define \( G_0 := G \) and \( G_{i+1} := G_i - N_{G_i}[x_{i+1}] \) for each \( 1 \leq i < n \), and similarly, \( G'_0 := G \), \( G'_{j+1} := G'_j - N_{G'_j}[y_{j+1}] \) for each \( 1 \leq j < m \), then each of the graphs in the sequences \( G_0, G_1, \ldots, G_n \) and \( G'_0, G'_1, \ldots, G'_m \) is a strongly \( \theta \)-prime graph. However, it then follows that both \( \{x_1, \ldots, x_n\} \) and \( \{y_1, \ldots, y_m\} \) form strongly \( \theta \)-prime sequences for \( G \). This implies that \( \theta(G) = n = m \), a contradiction.

For the 1-well-covered property, it is sufficient to show that every vertex in \( G \) is a shedding vertex of \( \text{Ind}(G) \) (compare to [10, Corollary 3.4]). Since \( G \) is well-covered, so is \( G - N_G[x] \). In particular, we have \( \alpha(G - N_G[x]) = \alpha(G) - 1 \). If \( S \) is an independent set in \( G - N_G[x] \), there exists a maximum independent set \( S' \) in \( G - N_G[x] \) with \( S \subseteq S' \). However, since \( |S'| = \alpha(G) - 1 \), we would have that \( G - N_G[S'] \cong K_2 \) as \( G \) is a strongly \( \theta \)-prime graph. This means that there exists \( y \in N_G(S') \) such that \( V(G - N_G[S']) = \{x, y\} \). It then follows that \( S \cup \{y\} \) is an independent set in \( G \).

**Corollary 50.** If \( G \) is a strongly \( \theta \)-prime graph, then \( \text{Ind}(G) \) is a weak pseudo-manifold.

*Proof.* Assume that \( \alpha(G) = n \), and consider an independent set \( I \) in \( G \) of size \( (n - 1) \). Since \( G - N_G[I] \cong K_2 \), the set \( I \) is contained in exactly two maximum independent sets.

We remark that \( \theta(G) = \alpha(G) \) whenever \( \text{Ind}(G) \) is a weak pseudo-manifold, which follows easily from the fact that \( C(\text{Ind}(G)) = \alpha(G) \). However, we note that when \( \text{Ind}(H) \) is a weak pseudo-manifold, the graph \( H \) does not need to be even a \( \theta \)-prime graph. For instance, consider a graph whose complement is the disjoint union of two 5-cycles. We close our discussion with an example of a strongly \( \theta \)-prime graph.

**Example 51** (Associahedral graphs). For \( n \geq 4 \), we consider a (convex) realization of the \( n \)-cycle graph \( C_n \) (an \( n \)-gon) in the plane. Then, the associahedral graph \( A(n) \) is defined to be the graph whose vertices are \( \frac{n(n-3)}{2} \)-diagonals of \( C_n \), that is, the edge set of the complement graph \( \overline{C_n} \), such that two vertices form an edge in \( A(n) \) if and only if the corresponding diagonals intersect at some interior point of \( C_n \). The independence
complex of $A(n)$ is known as the *simplicial associahedron* (see [9] for more details), and it is a simplicial manifold of dimension $(n-4)$.

Observe that $A(4) \cong K_2$ and $A(5) \cong C_5$, while we depict the graph $A(6)$ in Figure 2. Notice that the graph $A(6)$ is not vertex-transitive.

Note also that the graph $A(n)$ is a strongly $\theta$-prime graph with $\theta(A(n)) = n-3$ and $\text{im}(A(n)) = \lfloor \frac{n}{2} \rfloor - 1$ for each $n \geq 4$.

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