A New Kind of Graded Lie Algebra and Parastatistical Supersymmetry

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Abstract

In this paper the usual \( Z_2 \) graded Lie algebra is generalized to a new form, which may be called \( Z_{2,2} \) graded Lie algebra. It is shown that there exists close connections between the \( Z_{2,2} \) graded Lie algebra and parastatistics, so the \( Z_{2,2} \) can be used to study and analyse various symmetries and supersymmetries of the paraparticle systems.

Key Words: Graded Lie Algebra, Parastatistics, Supersymmetry.

1 Introduction

It is well known that the symmetry plays a fundamental role in theoretical physics research. When a new symmetry of investigating system is revealed, it will lead not only to better understanding of the system but sometimes give rise to establishing unexpected and important relations between different theories and lines of research. Parastatistics first was introduced as an exotic possibility extending the Bose and Fermi statistics \(^1\), and for the long period of time the interest to it was rather academic. Recently it finds applications in the physics of the quantum Hall effect and it probably is relevant to high temperature superconductivity \(^2\), so it draws more and more attentions from both theoretical and experimental physicists. Supersymmetry, established in the early 70’s, unifies Bose and Fermi symmetry and leads to deeply developments of field and string theories which become the cornerstones of modern theoretical physics. A natural question is, can we unify parabosons and parafermions into some kind of supersymmetric theories? Though some recent researches indicated that the so-called parasupersymmetry (paraSUSY) maybe unifies these paraparticles \(^3\), nevertheless, the two concepts, i.e., the parastatistics and the

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supersymmetry, seem to be still independent from the point of view of algebraic structure. In fact, the mathematical basis of usual supersymmetry is $\mathbb{Z}_2$ graded Lie algebra [4], in which not only commutators but also anticommutators are involved. However, the characteristic of algebraic relations of parastatistics is trilinear commutation relations, or double commutation relations. Thus two paraparticles neither commute nor anticommute with each other, and in place of this, three paraparticles satisfy some complicated commutation relations, which certainly bring some intrinsic difficulties into the research of paraSUSY.

A new kind of graded Lie algebra (we call it $\mathbb{Z}_{2,2}$ graded Lie algebra) is introduced in this paper to provide a suitable mathematical basis for the paraSUSY theories. This $\mathbb{Z}_{2,2}$ graded Lie algebra is a natural extension or generalization of the usual $\mathbb{Z}_2$ one. Especially, the trilinear commutation relations of parastatistics can automatically appear in the structure of $\mathbb{Z}_{2,2}$ graded Lie algebra, therefore, supersymmetric theories based on the $\mathbb{Z}_{2,2}$ graded Lie algebra may unify the concepts of parastatistics and supersymmetry. It can be shown that the algebraic structure of paraSUSY systems indeed is the $\mathbb{Z}_{2,2}$ graded Lie algebra. So we can classify and analyse the possible symmetries and supersymmetries of paraparticle systems more systematically and more effectively from the point of view of $\mathbb{Z}_{2,2}$ graded Lie algebra.

The paper is arranged as follows. In section 2 the mathematical definition of the $\mathbb{Z}_{2,2}$ is introduced with some examples showing how to produce a $\mathbb{Z}_{2,2}$ graded Lie algebra from a usual Lie algebra, especially how to derive the characteristic trilinear commutation relations of parastatistics. In section 3 we demonstrate that the algebraic structure of systems consisting of parabosons and parafermions is nothing but the $\mathbb{Z}_{2,2}$ graded Lie algebra. We analyse the various symmetries and supersymmetries of paraparticle systems in section 4, and give some summary and discussion in the last section.

## 2 $\mathbb{Z}_{2,2}$ graded Lie algebra

Let vector space $L$ over a field $K$ be a direct sum of four subspaces $L_{ij}$ ($i,j=0,1$), i.e.,

$$L = L_{00} \oplus L_{01} \oplus L_{10} \oplus L_{11}. \quad (1)$$

For any two elements in $L$, we define a composition (or product) rule, written $\circ$, with the following properties

(i) Closure: For $\forall u, v \in L$, we have $u \circ v = w \in L$, i.e.,

$$L \circ L \rightarrow L. \quad (2)$$

(ii) Bilinearity: For $\forall u, v, w \in L$, $c_1, c_2 \in K$, we have

$$(c_1 u + c_2 v) \circ w = c_1 u \circ w + c_2 v \circ w, \quad w \circ (c_1 u + c_2 v) = c_1 w \circ u + c_2 w \circ v. \quad (3)$$

(iii) Grading: For $\forall u \in L_{ij}, v \in L_{mn}, (i,j,m,n = 0,1)$, we have

$$u \circ v = w \in L_{(i+m)\text{mod}2,(j+n)\text{mod}2}. \quad (4)$$
i.e.
\[ L_{ij} \circ L_{mn} \to L_{(i+m) \bmod 2,(j+n) \bmod 2}, \]  
where \( \bmod 2 \) means cutting 2 when \( i + m = 2 \). For instance, \( L_{00} \circ L_{00} \to L_{00}, L_{10} \circ L_{11} \to L_{01}, L_{11} \circ L_{11} \to L_{00}, \cdots \).

(iv) Supersymmetrization: For \( \forall u \in L_{ij}, v \in L_{mn} \), we have
\[ u \circ v = -(-1)^{g(u) \cdot g(v)} v \circ u, \tag{5} \]
here we assign to any \( u \in L_{ij} \) a degree \( g(u) = (i, j) \) which satisfies
\[ g(u) \cdot g(v) = (i, j) \cdot (m, n) = im + jn, \]
where \( v \in L_{mn} \). Obviously, the \( g(u) \) looks like a two-dimensional vector and the above two expressions are exactly dot product and additive operations of the two-dimensional vectors.

(v) Generalized Jacobi identities: For \( \forall u \in L_{ij}, v \in L_{kl}, w \in L_{mn}, (i, j, k, l, m, n = 0, 1) \), we have
\[ u \circ (v \circ w)(-1)^{g(u) \cdot g(w)} + v \circ (w \circ u)(-1)^{g(v) \cdot g(u)} + w \circ (u \circ v)(-1)^{g(w) \cdot g(v)} = 0. \tag{6} \]
It is easily to know that there are totally 20 different possibilities for constructing the generalized Jacobi identities from 4 subspaces of \( L \) \( (C_4^1 + C_4^1C_3^2 + C_4^3 = 20) \).

**Definition:** A linear space satisfying the above conditions (i)-(v) is called the \( Z_2^2 \) graded Lie algebra.

For instance, one can define the product rule on \( L \) as
\[ u \circ v = uv - (-1)^{g(u) \cdot g(v)} vu, \tag{7} \]
for \( \forall u, v \in L \), where the elements \( u, v \) can be treated as operators in some Hilbert space and the expression \( uv \) can be understood as a product of the two operators, it is straightforward to check this product rule satisfying the above conditions (ii), (iv) and (v). Furthermore, if one impose the closure and the grading on the product rule (7), the all elements in \( L \) will form a \( Z_2^2 \) graded Lie algebra according to commutation or anticommutation relations, and the generalized Jacobi identities will take the usual trilinear (or double brackets) form.

Thus we give the full definition of the \( Z_2^2 \) graded Lie algebra. It is easy to see that the \( Z_{2,2} \) graded Lie algebra is a directly generalization or extension of the usual \( Z_2 \) graded Lie algebra in the following several aspects: their product rules are the same, and both of them have the three basic characters of graded Lie algebra, i.e., grading, supersymmetrization and generalized Jacobi identities. The only difference between them is, for \( Z_2 \) case, which is a direct sum of two subspaces, the grading is one-dimensional, and for \( Z_{2,2} \) case, which is a direct sum of four subspaces, the grading is two-dimensional. Correspondingly, the degree of elements in \( Z_2 \) is only a number, and that in \( Z_{2,2} \) is a two-dimensional vector.
It is worth mentioning that there are only four possibilities to construct the generalized Jacobi identities in terms of trilinear or double brackets mathematically, i.e.,

\[
[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0,
\]

\[
[A, \{B, C\}] + [B, \{C, A\}] + [C, \{A, B\}] = 0,
\]

\[
[A, \{B, C\}] + \{B, [C, A]\} - \{C, [A, B]\} = 0,
\]

\[
[A, [B, C]] + \{B, [C, A]\} - \{C, [A, B]\} = 0.
\]

Here we want to point out that only the first three expressions of (8) appear in the generalized Jacobi identities of the usual $Z_2$ graded Lie algebra, however, all the four expressions of (8) will appear in the generalized Jacobi identities of $Z_{2,2}$. This fact indicates that the $Z_{2,2}$ is more complete in the algebraic structure, and has higher symmetries than $Z_2$. Their further connection will become clearer later in the symmetry analysis.

Now we take the Lie algebra $su(1, 1)$ as an example to show how to produce the $Z_{2,2}$ extension of $su(1, 1)$. We take $su(1, 1)$ as the $L_{00}$ subspace which is three-dimensional, however, the dimensions of subspaces $L_{01}$, $L_{10}$ and $L_{11}$ are restricted by the generalized Jacobi identities. Simple calculations show that in a nontrivial maximum-dimensional extension, the above three subspaces can only contain 2, 2 and 1 elements respectively. Using the notations of elements in different subspaces showing in the Table 1, we have

\[
[\tau_1, \tau_2] = -\tau_3, \quad [\tau_2, \tau_3] = \tau_1, \quad [\tau_3, \tau_1] = \tau_2,
\]

\[
[\tau_i, Q_\alpha] = -(t_i)_{\alpha\beta}Q_\beta, \quad [\tau_i, a_m] = -(c_{i})_{mn}a_n, \quad [\tau_i, f] = 0,
\]

\[
\{Q_\alpha, Q_\beta\} = (h_{i})_{\alpha\beta}\tau_i, \quad \{Q_\alpha, a_m\} = \lambda_1(\sigma_2)_{am}f, \quad \{Q_\alpha, f\} = \lambda_2(I)_{am}a_m,
\]

\[
\{a_m, a_n\} = (d_{i})_{mn}\tau_i, \quad \{a_m, f\} = \lambda_3(I)_{ma}Q_\alpha,
\]

where $i = 1, 2, 3$, $m, n = 1, 2$ and $\alpha, \beta = 1, 2$. The first line in (9) is the Lie algebraic relations in $L_{00}$ subspace, and the others on (9) are the algebraic relations of the $Z_{2,2}$ extension. All the structure constants such as $t_i, c_i, h_i, d_i$ are determined by the generalized Jacobi identities as

\[
t_1 = -\sigma_1/2, \quad t_2 = -\sigma_2/2, \quad t_3 = -i\sigma_3/2, \quad c_i = t_i,
\]

\[
h_1 = -2i\lambda_1\lambda_2\sigma_3, \quad h_2 = -2\lambda_1\lambda_2 I, \quad h_3 = 2\lambda_1\lambda_3\sigma_1,
\]

\[
d_1 = -2i\lambda_1\lambda_3\sigma_3, \quad d_2 = -2\lambda_1\lambda_3 I, \quad d_3 = 2\lambda_1\lambda_3\sigma_1,
\]

where $\sigma_i$ and $I$ are the Pauli matrices and (2 $\times$ 2) unit matrix respectively. It needs to be explained that the three undetermined constants $\lambda_i$ may be absorbed into redefinitions of the elements in $L_{01}$, $L_{10}$ and $L_{11}$ subspaces.

If we consider a unitary representation of the $Z_{2,2}$ extension of the $su(1, 1)$, and take the constants as $\lambda_1 = -i$, $\lambda_2 = 0$ and $\lambda_3 = 2$, then under mappings $-2i\tau_3 \rightarrow M(a)$, $2(-\tau_1 + i\tau_2) \rightarrow B^\dagger$, $2(\tau_1 + i\tau_2) \rightarrow B$, $a_1 \rightarrow a^\dagger$ and $a_2 \rightarrow a$, we can derive from (9) the following relations

\[
[M(a), a] = -a, \quad [B, a] = 0, \quad [B, a^\dagger] = 2a,
\]

\[
\{a^\dagger, a\} = 2M(a), \quad \{a, a\} = 2B,
\]

(10)
together with the adjoint ones. Obviously, these relations are exactly equivalent to the single mode paraboson algebraic relations

\[ \{a^\dagger, a\} = -2a, \quad \{a, a\} = 0, \quad \{a, a^\dagger\} = 4a. \quad (11) \]

Thus we see that the parastatistical algebraic relations can be automatically derived from the \(Z_{2,2}\) algebraic relations.

### 3 Algebraic structure of parastatistics

Let us consider a paraparticle system consisting of \(M\)-mode parabosons (whose creation and annihilation operators are denoted by \(a_k^\dagger\) and \(a_k\) respectively) and \(N\)-mode parafermions (denoted by \(f_k^\dagger\) and \(f_k\)), whose algebraic structure can be expressed in terms of the following 12 independent relations (the Latin indices take values from 1 to \(M\), and the Greek ones from 1 to \(N\))

\[
\begin{align*}
[a_k, \{a_k^\dagger, a_m\}] &= 2\delta_{kl}a_m, \quad [f_\alpha, [f_\beta^\dagger, f_\gamma]] = 2\delta_{\alpha\beta}f_\gamma,
[a_k, \{a_l, a_m\}] = 0, \quad [f_\alpha, [f_\beta, f_\gamma]] = 0, \\
[a_k, [f_\beta^\dagger, f_\gamma]] = 0, \quad [f_\alpha, \{a_k^\dagger, a_l\}] = 0, \\
[a_k, \{a_l, f_\alpha\}] = 0, \quad \{f_\alpha, \{a_k, f_\beta\}\} = 0, \\
[a_k, \{a_l^\dagger, f_\alpha\}] = 2\delta_{kl}f_\alpha, \quad \{f_\alpha, \{a_k^\dagger, f_\beta\}\} = 0, \\
[a_k, \{a_l, f_\alpha\}] = 0, \quad \{f_\alpha, \{a_k, f_\beta^\dagger\}\} = 2\delta_{\alpha\beta}a_k. \quad (12)
\end{align*}
\]

Other parastatistical relations of this system can be derived from the above 12 basic relations by taking Hermitian conjugate or by using the generalized Jacobi identities (8). Now we define the following 6 new operators

\[
\begin{align*}
\{a_k^\dagger, a_l\} &= 2M_{kl}(a), \quad \{a_k, a_l\} = 2B_{kl}(a), \quad \{a_k, f_\alpha\} = 2F_{k\alpha}, \\
[f_\alpha^\dagger, f_\beta] &= 2M_{\alpha\beta}(f), \quad [f_\alpha, f_\beta] = 2B_{\alpha\beta}(f), \quad \{a_k^\dagger, f_\alpha\} = 2Q_{k\alpha}, \quad (13)
\end{align*}
\]

with their Hermitian conjugate operators. Since \(M_{kl}^\dagger(a) = M_{lk}(a)\) and \(M_{\alpha\beta}^\dagger(f) = M_{\beta\alpha}(f)\), actually, there are totally 10 kinds of new operators independently, i.e.,

\[\begin{align*}
M_{kl}(a), B_{kl}(a), B_{kl}^\dagger(a), M_{\alpha\beta}(f), B_{\alpha\beta}(f), B_{\alpha\beta}^\dagger(f), F_{k\alpha}, F_{k\alpha}^\dagger, Q_{k\alpha}, Q_{k\alpha}^\dagger.
\end{align*}\]

Using these new operators one can rewrite (12) as

\[
\begin{align*}
[a_k, M_{lm}(a)] &= \delta_{kl}a_m, \quad [a_k, M_{\alpha\beta}(f)] = 0, \quad [a_k, B_{lm}(a)] = 0, \\
[f_\alpha, M_{kl}(a)] = 0, \quad [f_\alpha, M_{\beta\gamma}(f)] = \delta_{\alpha\beta}f_\gamma, \quad [f_\alpha, B_{\beta\gamma}(f)] = 0, \\
[a_k, F_{l\alpha}] = 0, \quad [a_k, Q_{l\alpha}] = \delta_{kl}f_\alpha, \quad [a_k, Q_{l\alpha}^\dagger] = 0, \\
\{f_\alpha, F_{k\beta}\} = 0, \quad \{f_\alpha, Q_{k\beta}\} = 0, \quad \{f_\alpha, Q_{k\beta}^\dagger\} = \delta_{\alpha\beta}a_k. \quad (14)
\end{align*}
\]

5
Furthermore, by virtue of (13), (14) and (8), one can derive the following closed algebraic relations satisfied by the new operators

\[
\begin{align*}
[M_{kl}(a), M_{mn}(a)] &= \delta_{ml}M_{kn}(a) - \delta_{kn}M_{ml}(a), \\
[M_{kl}(a), B_{mn}(a)] &= -\delta_{km}B_{ln}(a) - \delta_{kn}B_{ml}(a), \\
[B_{kl}(a), B_{mn}^{\dagger}(a)] &= \delta_{mk}M_{nl}(a) + \delta_{nl}M_{mk}(a) + \delta_{ml}M_{nk}(a) + \delta_{nk}M_{ml}(a), \\
[B_{kl}(a), B_{mn}(a)] &= 0; \quad (15)
\end{align*}
\]

\[
\begin{align*}
[M_{\alpha\beta}(f), M_{\sigma\rho}(f)] &= \delta_{\sigma\beta}M_{\alpha\rho}(f) - \delta_{\alpha\rho}M_{\sigma\beta}(f), \\
[M_{\alpha\beta}(f), B_{\sigma\rho}(f)] &= -\delta_{\alpha\sigma}B_{\beta\rho}(f) - \delta_{\alpha\rho}B_{\sigma\beta}(f), \\
[B_{\alpha\beta}(f), B_{\sigma\rho}^{\dagger}(f)] &= -\delta_{\sigma\alpha}M_{\rho\beta}(f) - \delta_{\rho\beta}M_{\sigma\alpha}(f) + \delta_{\sigma\beta}M_{\rho\alpha}(f) + \delta_{\rho\alpha}M_{\sigma\beta}(f), \\
[B_{\alpha\beta}(f), B_{\sigma\rho}(f)] &= 0; \quad (16)
\end{align*}
\]

\[
\begin{align*}
[M_{kl}(a), M_{\alpha\beta}(f)] &= 0, \quad [M_{kl}(a), B_{\alpha\beta}(f)] = 0, \quad [M_{\alpha\beta}(f), B_{kl}(a)] = 0, \\
[B_{kl}(a), B_{\alpha\beta}(f)] &= 0, \quad [B_{kl}(a), B_{\alpha\beta}^{\dagger}(f)] = 0; \quad (17)
\end{align*}
\]

\[
\begin{align*}
[M_{kl}(a), F_{m\alpha}] &= -\delta_{km}F_{l\alpha}, \quad [M_{\alpha\beta}(f), F_{k\gamma}] = -\delta_{\alpha\gamma}F_{k\beta}, \\
[B_{kl}(a), F_{m\alpha}] &= 0, \quad [B_{\alpha\beta}^{\dagger}(f), F_{k\alpha}] = -\delta_{k\gamma}Q_{l\alpha} - \delta_{lm}Q_{k\alpha}, \\
[B_{\alpha\beta}(f), F_{k\gamma}] &= 0, \quad [B_{\alpha\beta}^{\dagger}(f), F_{k\gamma}] = -\delta_{k\gamma}Q_{l\alpha}^{\dagger} + \delta_{\alpha\gamma}Q_{k\beta}^{\dagger}, \\
[M_{kl}(a), Q_{m\alpha}] &= \delta_{lm}Q_{k\alpha}, \quad [M_{\alpha\beta}(f), Q_{k\gamma}] = -\delta_{\alpha\gamma}Q_{k\beta}, \\
[B_{kl}(a), Q_{m\alpha}] &= \delta_{km}F_{l\alpha} + \delta_{lm}F_{k\alpha}, \quad [B_{\alpha\beta}^{\dagger}(f), Q_{m\alpha}] = 0, \\
[B_{\alpha\beta}(f), Q_{k\gamma}] &= \delta_{\alpha\gamma}F_{k\beta}^{\dagger} - \delta_{\beta\gamma}F_{k\alpha}^{\dagger}, \quad [B_{\alpha\beta}(f), Q_{k\gamma}] = 0; \quad (18)
\end{align*}
\]

\[
\begin{align*}
\{F_{ka}, F_{l\beta}\} &= 0, \quad \{Q_{ka}, Q_{l\beta}\} = 0, \quad \{F_{ka}, F_{l\beta}^{\dagger}\} = \delta_{\beta\alpha}M_{lk}(a) - \delta_{lk}M_{\beta\alpha}(f), \\
\{Q_{ka}, Q_{l\beta}^{\dagger}\} &= 0, \quad \{Q_{ka}, Q_{l\beta}\} = \delta_{\beta\alpha}M_{kl}(a) + \delta_{kl}M_{\beta\alpha}(f), \\
\{F_{ka}, Q_{l\beta}\} &= \delta_{kl}B_{\alpha\beta}(f), \quad \{F_{ka}, Q_{l\beta}^{\dagger}\} = \delta_{\alpha\beta}B_{kl}(a), \quad (19)
\end{align*}
\]

with their Hermitian conjugate ones. After observing these relations carefully, one can see that the 10 new operators form the usual \(Z_2\) graded Lie algebra, whose Bose subspace includes the operators \(M_{kl}(a), B_{kl}(a), B_{kl}^{\dagger}(a), M_{\alpha\beta}(f), B_{\alpha\beta}(f), B_{\alpha\beta}^{\dagger}(f)\), and whose Fermi subspace includes the operators \(F_{ka}, F_{ka}^{\dagger}, Q_{ka}, Q_{ka}^{\dagger}\). Moreover, if considering the whole algebraic relations (13-19) together, one can find that actually the following 14 operators...
\( (a_k, a_k^\dagger, f_\alpha, f_\alpha^\dagger, M_{kl}(a), B_{kl}(a), B_{kl}^\dagger(a), M_{\alpha\beta}(f), B_{\alpha\beta}(f), B_{\alpha\beta}^\dagger(f), F_{ka}, F_{ka}^\dagger, Q_{ka}, Q_{ka}^\dagger) \) form a \(Z_{2,2}\) graded Lie algebraic system according to the product rule (7) (we may call it para-Lie superalgebraic system), the operators of whose four subspaces are showing in the Table 2. Obviously, from the point of view of the algebraic structure, this para-Lie superalgebraic system is exactly equivalent to the parastatistical algebraic relations (12). Thus we arrive at a conclusion: The parastatistical algebraic relations (12) are equivalent to the para-Lie superalgebraic relations (13-19) on the algebraic structure, and the latter is the algebraic relations of \(Z_{2,2}\) graded Lie algebra mathematically. This conclusion can be written as following theorem

**Theorem:** The algebraic structure of parastatistics is a \(Z_{2,2}\) graded Lie algebra.

## 4 Symmetries and supersymmetries of a para-Lie superalgebraic system

From the point of view of the para-Lie superalgebra or the \(Z_{2,2}\) graded Lie algebra, it is easily to analyse the symmetries and supersymmetries of a paraparticle system as follows:

1. From (15) and (16) it is clear that the two subset \((M_{kl}(a), B_{kl}(a), B_{kl}^\dagger(a))\) and \((M_{\alpha\beta}(f), B_{\alpha\beta}(f), B_{\alpha\beta}^\dagger(f))\) form the dynamical symmetry algebras of a pure parabose and a pure parafermi subsystems \((sp(2M, R)\) and \(so(2N, R)\) respectively. So the whole subspace \(L_{\text{bose}}\) (see the Table 2) forms a Lie algebra \(sp(2M, R) \oplus so(2N, R)\).

2. From (15-19) it is clear that the whole subspace \(L_{\text{bose}} \oplus L_{\text{fermi}}\) form a \(Z_2\) graded Lie algebra, in which \((M_{kl}(a), M_{\alpha\beta}(f), F_{ka}, F_{ka}^\dagger)\) and \((M_{kl}(a), M_{\alpha\beta}(f), Q_{ka}, Q_{ka}^\dagger)\) form two dynamical supersymmetric algebras of the paraparticle system respectively. It should be pointed out that for the former there is no way to construct a dynamical model with a positive definite Hamiltonian, so it is a unaccepted supersymmetry in physics, however, for the latter it is indeed possible to realize dynamical supersymmetric models with definite physical meanings.

3. From (13-19) it is clear that \((a_k, a_k^\dagger, M_{kl}(a), B_{kl}(a), B_{kl}^\dagger(a))\) form a pure parabose statistical algebra \(osp(1/2M)\), so the whole subspace \(L_{\text{bose}} \oplus L_{\text{parabose}}\) form a \(Z_2\) graded Lie algebra \(osp(1/2M) \oplus so(2N, R)\).

4. Similarly, \((f_\alpha, f_\alpha^\dagger, M_{\alpha\beta}(f), B_{\alpha\beta}(f), B_{\alpha\beta}^\dagger(f))\) form a pure parafermi statistical algebra \(so(2N + 1, R)\), so the whole subspace \(L_{\text{bose}} \oplus L_{\text{parafermi}}\) form a Lie algebra \(so(2N + 1, R) \oplus sp(2M, R)\).

Furthermore, besides the parabosons and parafermions (with \(p > 1\) where \(p\) is the parastatistics order), one can include ordinary bosons and fermions \((p = 1)\) in the considering system, or more detailed, put creation and annihilation operators of the bosons (fermions) into the subspace \(L_{\text{bose}}\) (\(L_{\text{fermi}}\)). Since according to the product rule of \(Z_{2,2}\), either the commutators or anticommutators between para- and non-para- particles will be zero, the whole algebraic structure of the enlarging system is not changed. In other words, for the system consisting of bosons, fermions, parabosons and parafermions, the
algebraic structure of its statistical relations is still the \( Z_{2,2} \) graded Lie algebra. Therefore, we can analyse the various potential supersymmetries of such a system unifiedly within the framework of \( Z_{2,2} \). Further research shows that only the following three kinds of supersymmetries (including six different cases) are possible mathematically between the four kinds of particles:

(i) The supersymmetries between boson and fermion or paraboson and parafermion, which are realized by some fermi-like supercharges \( Q_F \);
(ii) The supersymmetries between boson and parafermion or paraboson and fermion, which are realized by some parafermi-like supercharges \( Q_{PF} \);
(iii) The supersymmetries between boson and paraboson or fermion and parafermion, which are realized by some parabose-like supercharges \( Q_{PB} \).

After detailed analysis we find that it is not possible to write out a positive definite Hamiltonian in the case (iii), so we do not consider the case (iii) further. To our knowledge, so far only part of the above six possible supersymmetric cases has been studied in the literature. For instance, the supersymmetry of boson and fermion in the case (i) is studied by ordinary supersymmetric quantum mechanics [5], and the supersymmetry of boson and parafermion in the case (ii) is studied in the paraSUSY quantum mechanics [6]. Other supersymmetric cases need to be studied further. Comparing with the ordinary statistics in which there is only one kind of supersymmetry, i.e., boson-fermion supersymmetry, the parastatistics allows existence of more supersymmetries. This also indicates that the paraSUSY can be analysed and studied more conveniently within the framework of \( Z_{2,2} \) graded Lie algebra.

5 Conclusion

The concept of \( Z_{2,2} \) graded Lie algebra is introduced in this paper, which has intrinsic connection with the parastatistics and the paraSUSY. The \( Z_{2,2} \) graded Lie algebra can unify not only paraboson and parafermion, but also boson, fermion, paraboson and parafermion within one algebraic structure. It is well-known that when the order \( p \) of parastatistics goes to 1, the parastatistics will reduce to the ordinary statistics, as well as the parabose and the parafermi subspaces will reduce to the ordinary bose and fermi subspaces, so the \( Z_{2,2} \) graded Lie algebra will reduce to the ordinary \( Z_2 \) one. Therefore for the ordinary statistics only the \( Z_2 \) graded Lie algebra is needed. However, when \( p > 1 \), except original bose and fermi subspaces, one has to introduce two extra parabose and parafermi subspaces into the algebraic structure. In this sense we say that the \( Z_{2,2} \) graded Lie algebra is more complete in the structure and with higher symmetry than the \( Z_2 \) one. Considering the discussion in section 2, we can also say that the \( Z_2 \) graded Lie algebra (or orsinary statistics) is an one-dimensional reduction of the \( Z_{2,2} \) one (or parastatistics), and the \( Z_{2,2} \) (or parastatistics) is a two-dimensional generalization of the \( Z_2 \) (or ordinary statistics).

A high-dimensional system has higher symmetry and is more complete constructionally than a low-dimensional system, this is a common fact. Also in this sense, we may call
the $Z_{2,2}$ graded Lie algebra (para-Lie superalgebra) as "two-dimensional" $Z_2$ graded Lie algebra (ordinary Lie superalgebra). Of course, searching Fock space representations of the $Z_{2,2}$ graded Lie algebra and constructing concrete paraSUSY dynamical models are very important and urgent problems. We will present the Fock representation for a system with one-mode parabose and one-mode parafermi degrees of freedom and discuss relevant paraSUSY dynamical model in a separate paper.

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