THE STANDARD LAPLACE OPERATOR

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Abstract. The standard Laplace operator is a generalization of the Hodge Laplace operator on differential forms to arbitrary geometric vector bundles, alternatively it can be seen as generalization of the Casimir operator acting on sections of homogeneous vector bundles over symmetric spaces to general Riemannian manifolds. Stressing the functorial aspects of the standard Laplace operator $\Delta$ with respect to the category of geometric vector bundles we show that the standard Laplace operator commutes not only with all homomorphisms, but also with a large class of natural first order differential operators between geometric vector bundles. Several examples are included to highlight the conclusions of this article.

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1. Introduction

The standard Laplace operator on geometric vector bundles can be seen as the quintessence of and motivation for the work of the authors on special holonomy in general and Weitzenböck formulas in particular. Its mere existence is some kind of miracle, because it generalizes both the Casimir operator on symmetric spaces to general Riemannian manifolds and the Hodge Laplace operator on differential forms to more general vector bundles.

Underlying the construction of the standard Laplace operator is the concept of geometric vector bundles on a Riemannian manifold $(M, g)$ well adapted to the geometry arising from a reduction of its holonomy group. Intuitively speaking a geometric vector bundle is a vector bundle on $M$ associated to the principal subbundle $\text{Hol}_M \subset \text{SO}_M$ of the bundle of oriented orthonormal frames encoding this holonomy reduction, a precise axiomatic definition avoiding all the subtleties of holonomy reduction and principal bundles is given below. By construction geometric vector bundles form a category of vector bundles with connection on the manifold $M$. The characteristic properties of geometric vector bundles among all vector bundles with connection allows us however to construct a curvature endomorphism

$$q(R) \in \Gamma(\text{End} VM)$$

for every geometric vector bundle $VM$ parametrized by the curvature tensor $R$ of a metric connection $\nabla$ on the tangent bundle of $M$, which commutes with all homomorphisms of
geometric vector bundles. Needless to say this curvature endomorphism has been considered in numerous special cases before and we will recall several interesting examples below, however we believe that only the category of geometric vector bundles brings the construction of the curvature endomorphism \( q(R) \) to the point. The standard Laplace operator associated to a geometric vector bundle is the sum of the rough Laplacian with the curvature endomorphism:

\[
\Delta := \nabla^* \nabla + q(R).
\]

By construction \( \Delta \) is a Laplace type operator which commutes with all homomorphisms of geometric vector bundles. Of course the same statement is true for every linear combination of \( \nabla^* \nabla \) and \( q(R) \), the variation of the curvature tensor \( R \) under the Ricci flow \( \delta g = 2 \text{Ric} \) for example can be written neatly in the form

\[
\delta R = (\nabla^* \nabla + \frac{1}{2} q(R)) R + \frac{1}{2} \text{Der}_\text{Ric} R.
\]

The decisive advantage of the linear combination (1) is however that \( \Delta \) commutes not only with all homomorphisms of geometric vector bundles, but also with many natural first order differential operators. More precisely the commutator is calculated in Theorem 3.1

\[
[\Delta, D] = -\sigma_\circ(q(\bar{\nabla} R) - \delta R),
\]

and depends only on the principal symbol \( \sigma_\circ \) of the natural first order differential operator \( D \) and the covariant derivative \( \bar{\nabla} R \) of the curvature tensor \( R \) of the given metric connection \( \bar{\nabla} \) on \( TM \). On a symmetric space for example \( \bar{\nabla} R = 0 \) and thus \( \Delta \) commutes with all natural differential operators between geometric vector bundles, showing a priori that it agrees with the Casimir operator on homogeneous vector bundles.

Illustrating the intrinsic beauty of the construction of the standard Laplace operator in many explicit and rather well–known examples makes this article somewhat reminiscent of a review article, nevertheless it contains a couple of new ideas besides establishing the general commutator formula (1) and discussing its applications; both the underlying concept of geometric vector bundles and the presentation of the curvature endomorphism \( q(R) \) as an integral over sectional curvatures certainly deserve a more detailed analysis in the future.

In the second section of this article we introduce the concept of geometric vector bundles and discuss several constructions of the curvature endomorphism \( q(R) \), in particular its integral presentation. In the third section we prove the commutator formula (2) in Theorem 3.1 and Corollary 3.3 and in the fourth section we sketch some direct applications of this commutator formula to the geometry of normal homogeneous spaces, nearly Kähler manifolds in dimension six, manifolds with \( G_2 \)– and Spin(7)–holonomy and quaternion–Kähler manifolds.

2. Preliminaries

2.1. Geometric Vector Bundles. In differential geometry it is well–known that many interesting flavors of geometry come along with a corresponding reduction of the holonomy group, in fact this observation is one of the basic tenets of Cartan’s generalization of the Erlangen program nowadays called Cartan geometry. In Riemannian geometry such a holonomy reduction can be thought of as a subbundle \( \text{Hol} M \subset \text{SO} M \) of the orthonormal frame
bundle of a manifold $M$ which is a principal bundle itself under the induced action of a closed subgroup $\text{Hol} \subset \text{SO}(T)$ and tangent to a metric connection $\nabla$ on the tangent bundle $TM$. The most direct implication of such a holonomy reduction is that the metric connection $\nabla$ induces a principal connection on $\text{Hol}M$. The association functor, which turns a representation $V$ of $\text{Hol}$ into the associated vector bundle $\text{Hol}M \times_{\text{Hol}} V$ on $M$, thus becomes a functor $\text{As}_{\text{Hol}M} : \text{Rep}_{\text{Hol}} \rightsquigarrow \text{Vect}_M$ from the category of representations of $\text{Hol}$ to the category of vector bundles with connection over $M$ under parallel homomorphisms.

In essence a geometric vector bundle on $M$ is a vector bundle in the image of this association functor. In order to avoid the subtleties of holonomy reductions and the association functor we will provide a more direct axiomatic definition of a geometric vector bundle though, which characterizes a geometric vector bundle $VM$ by an infinitesimal representation of the holonomy algebra subbundle $\mathfrak{hol}M \subset \mathfrak{so}(TM) \cong \Lambda^2 TM$ for the given metric connection $\nabla$:

**Definition 2.1 (Geometric Vector Bundles).**

A geometric vector bundle on a Riemannian manifold $(M, g)$ endowed with a metric connection $\nabla$ on its tangent bundle is a vector bundle $VM$ endowed with a connection $\nabla$ and a parallel infinitesimal representation $\star : \mathfrak{hol}M \otimes VM \to VM$ of the holonomy algebra bundle $\mathfrak{hol}M \subset \Lambda^2 TM$ such that the curvature $R^\nabla$ of the connection $\nabla$ is determined via

$$R^\nabla_{X,Y} = R_{X,Y}\star$$

for all $X, Y \in \Gamma(TM)$ by the curvature tensor $R$ of the metric connection $\nabla$ on $TM$.

Needless to say geometric vector bundles form a category, the morphisms in this category are parallel homomorphisms $F : VM \to \tilde{VM}$ of vector bundles with connections which commute with the infinitesimal representation of the holonomy algebra bundle $\mathfrak{hol}M$. Evidently this category is closed under taking duals, direct sums, tensor products and exterior and symmetric powers, to name only a few constructions of linear algebra. The relevance of geometric vector bundles for this article is that the curvature endomorphism and the standard Laplace operator constructed below are essentially functors in the sense that they commute with all morphisms in the category of geometric vector bundles.

**Example 1:** A classical example of a geometry related to a holonomy reduction is Kähler geometry: On a Kähler manifold $(M, g, J)$ the bundle of orthonormal frames reduces to complex linear orthonormal frames with its unitary structure group $U(n) \cong \text{Hol} \subset \text{SO}(T)$. In turn the holonomy algebra bundle equals the bundle $\mathfrak{hol}M \subset \mathfrak{so}(TM)$ of skew symmetric endomorphisms of $TM$ commuting with $J$ with projection $\text{pr}_{\mathfrak{hol}} : \Lambda^2 TM \to \mathfrak{hol}M$ given by:

$$\text{pr}_{\mathfrak{hol}}(X \wedge Y) = \text{pr}^{1,0}X \wedge \text{pr}^{0,1}Y + \text{pr}^{0,1}X \wedge \text{pr}^{1,0}Y.$$

It is well known in this case that the geometric vector bundle of antiholomorphic differential forms $VM = \Lambda^{0,\bullet} T^* M$ is a Clifford bundle with parallel Clifford multiplication defined as

$$X \bullet := \sqrt{2}( \text{pr}^{1,0}X^\flat \wedge - \text{pr}^{0,1}X_\perp),$$
moreover its fiber $V_xM$ is a spinor module for the Clifford algebra $\text{Cl}(T_xM, g_x)$ in every point $x \in M$. Nevertheless the vector bundle $VM$ is not considered to be the spinor bundle $\Sigma M$ of a Kähler manifold $M$, and Kähler manifolds as simple as $CP^2$ are not even spin. The concept of geometric vector bundles clarifies this apparent contradiction: The infinitesimal representation $\bigstar : \mathfrak{hol} M \otimes VM \longrightarrow VM$ of the geometric vector bundle $VM$ is the restriction of the standard representation of $\mathfrak{so} TM$ on differential forms to $\mathfrak{hol} M$ and $VM$, for the spinor bundle $\Sigma M$ however the infinitesimal representation $\bigstar_{\Sigma} : \mathfrak{so} TM \otimes \Sigma M \longrightarrow \Sigma M$ is declared by way of axiom to be induced by Clifford multiplication with bivectors:

$$(X \wedge Y) \bigstar_{\Sigma} := \frac{1}{2} ( X \bullet Y \bullet + g(X, Y) \text{id}_{\Sigma M} ).$$

Of course we may simply redefine the infinitesimal representation of $VM$ to be the restriction of $\bigstar_{\Sigma}$ to the actual holonomy bundle $\mathfrak{hol} M \subset \mathfrak{so} TM$ of Kähler geometry. This replacement however will not result in a geometric vector bundle $(VM, \nabla, \bigstar_{\Sigma})$ isomorphic to the spinor bundle $\Sigma M$ as a geometric vector bundle, because the difference between the representations $\text{pr}_{\mathfrak{hol}}(X \wedge Y) \bigstar_{\Sigma} - \text{pr}_{\mathfrak{hol}}(X \wedge Y) \bigstar = -i g(JX, Y) \text{id}_{\Sigma M}$ implies that the action of the curvature tensor $R_{X,Y} \bigstar_{\Sigma} = R_{X,Y} \bigstar + i \text{Ric}(JX,Y) \text{id}_{\Sigma M}$ under $\bigstar_{\Sigma}$ does not agree with the curvature $R_{X,Y} = R_{X,Y} \bigstar$ of the given connection $\nabla$ on $VM$ unless the Kähler manifold $M$ is actually Ricci flat and thus a Calabi–Yau manifold.

2.2. Generalized Gradients. On sections of a geometric vector bundle $VM$ we have a distinguished set of natural first order differential operators called generalized gradients defined by projecting the covariant derivative $\nabla$ to its isotypical components. More precisely we consider the canonical decomposition $T \otimes V = \bigoplus V_{\varepsilon}$ of the Hol–representation $T \otimes V$ into isotypical components with corresponding projections $\text{pr}_{\varepsilon} : T \otimes V \longrightarrow V_{\varepsilon} \subset T \otimes V$, which become parallel projections $\text{pr}_{\varepsilon} : TM \otimes VM \longrightarrow V_{\varepsilon}M$ between the corresponding geometric vector bundles. The generalized gradient $P_{\varepsilon}$ is defined for all $\varepsilon$ as the composition:

$$P_{\varepsilon} : \Gamma(VM) \xrightarrow{\nabla} \Gamma(T^*M \otimes VM) \xrightarrow{\bigstar} \Gamma(TM \otimes VM) \xrightarrow{\text{pr}_{\varepsilon}} \Gamma(V_{\varepsilon}M).$$

In a similar vein we may define natural second order differential operators on sections of $VM$ by taking constant linear combinations of the operators $P_{\varepsilon}^* P_{\varepsilon}$. Certainly the most important example of this kind of second order differential operators is the so called rough Laplacian

$$\nabla^* \nabla := -\sum_{\mu} \left( \nabla_{E_{\mu}} \nabla_{E_{\mu}} - \nabla_{E_{\mu}} E_{\mu} \right) = \sum_{\varepsilon} P_{\varepsilon}^* P_{\varepsilon},$$

where the sum is over some local orthonormal frame $\{E_{\mu}\}$ of the tangent bundle $TM$. Besides generalized gradients we will consider natural first order differential operators arising from an arbitrary Hol–equivariant homomorphism $\sigma_{\tilde{V}} : T \otimes \tilde{V} \longrightarrow \tilde{V}$ for two representations $V$ and $\tilde{V}$ as well, if only to simplify notation. More precisely we compose $\nabla$ with the parallel
extension of \( \sigma \) to vector bundles to define the natural first order differential operator:

\[
D_\sigma : \Gamma(VM) \xrightarrow{\nabla} \Gamma(T^*M \otimes VM) \xrightarrow{\sigma} \Gamma(TM \otimes VM) \xrightarrow{\sigma_0} \Gamma(\tilde{V}M) .
\]

Alternatively we may write \( D_\sigma \) as a sum over a local orthonormal basis \( \{ E_\mu \} \) of the tangent bundle with respect to the Riemannian metric \( g \) in order to make this definition resemble the definition of the Dirac operator on Clifford bundles with the Clifford multiplication \( \bullet \) replaced by the parallel bilinear operation \( \sigma_0 : TM \otimes VM \to \tilde{V}M, X \otimes v \mapsto X \cdot v \):

\[
D_\sigma := \sum_\mu E_\mu \cdot \nabla E_\mu .
\]

2.3. Metric Connections with Torsion. Working with metric, but not necessarily torsion free connections on Riemannian manifolds requires some care even for specialists in Riemannian geometry to ensure that torsion freeness is not assumed implicitly in some innocuously looking argument or other. Throughout this article we will consider metric connections \( \bar{\nabla} \) with parallel skew–symmetric torsion only, their curvature tensors share most of the classical symmetries of the curvature tensor of the Levi–Civita connection. Working through a proof of the first Bianchi identity for torsion free connections we easily see that it extends to arbitrary connections \( \bar{\nabla} \) on \( TM \) with torsion tensor \( T(X,Y) := \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] \):

\[
R_{X,Y,Z} + R_{Y,Z,X} + R_{Z,X,Y} = (\bar{\nabla}_X T)(Y,Z) + (\bar{\nabla}_Y T)(Z,X) + (\bar{\nabla}_Z T)(X,Y) + T(T(X,Y),Z) + T(T(Y,Z),X) + T(T(Z,X),Y) .
\]

In order to derive some meaningful conclusions from this rather tautological version of the first Bianchi identity let us first restrict the class of connections considered in this article:

**Definition 2.2 (Connections with Parallel Skew–Symmetric Torsion).**

A metric connection \( \bar{\nabla} \) on the tangent bundle \( TM \) of a Riemannian manifold \((M, g)\) with torsion tensor \( T \) is called a metric connection with parallel skew–symmetric torsion provided the expression \( \theta(X,Y,Z) := g(T(X,Y),Z) \) defines a parallel 3–form \( \theta \in \Gamma(\Lambda^3 T^*M) \).

Evidently the torsion tensor itself is parallel \( \bar{\nabla} T = 0 \) for a metric connection \( \bar{\nabla} \) with parallel skew–symmetric torsion, moreover the endomorphism \( Y \mapsto T(X,Y) \) is skew–symmetric with respect to the Riemannian metric \( g \) for every \( X \in \Gamma(TM) \). Due to \( \bar{\nabla} T = 0 \) the tautological version of the first Bianchi identity reduces to the identity

\[
R(X,Y,Z,W) + R(X,Y,Z,W) + R(X,Y,Z,W) = \frac{1}{2} g(T \wedge T)(X,Y,Z,W)
\]

valid for the curvature tensor \( R(X,Y,Z,W) := g(R_{X,Y,Z,W}) \) of a connection \( \bar{\nabla} \) with parallel skew–symmetric torsion, where \( \frac{1}{2} g(T \wedge T) \in \Gamma(\Lambda^4 T^*M) \) denotes the parallel 4–form:

\[
\frac{1}{2} g(T \wedge T)(X,Y,Z,W) := g(T(X,Y), T(Z,W)) + g(T(Y,Z), T(X,W)) + g(T(Z,X), T(Y,W))
\]
In light of the modified first Bianchi identity \( \text{(4)} \) a standard proof for the well–known symmetry \( R(X, Y, Z, W) = R(Z, W, X, Y) \) of the curvature tensor of the Levi–Civita connection goes through more or less verbatim to prove \( R(X, Y, Z, W) = R(Z, W, X, Y) \) for the curvature tensor \( R \) of a metric connection \( \nabla \) with parallel skew–symmetric torsion (cf. \([9]\)). In particular the Ricci tensor \( \text{Ric}(X, Y) := \sum_{\mu} \mathcal{R}(X, E_\mu, E_\mu, Y) \) is symmetric as well.

2.4. The Curvature Endomorphism. The standard identification of the Lie algebra bundle \( \mathfrak{so}TM \) of skew symmetric endomorphism on a Riemannian manifold with the bivector bundle \( \mathfrak{so}TM = \Lambda^2TM \) realizes the holonomy bundle \( \mathfrak{hol}M \) as a subbundle of \( \Lambda^2TM \) endowed with a scalar product induced by the Riemannian metric on \( M \). The parallel orthogonal projection map \( \text{pr}_{\mathfrak{hol}} : \Lambda^2TM \longrightarrow \mathfrak{hol}M \subset \Lambda^2TM \) to the holonomy subbundle allows us to define the standard curvature endomorphism for every geometric vector bundle:

**Definition 2.3 (Curvature Endomorphism).** Let \( \{E_\mu\} \) be a local orthonormal frame of the tangent bundle \( TM \). The curvature endomorphism \( q(R) \in \text{End} VM \) is defined for every geometric vector bundle \( VM \) as the sum:

\[
q(R) := \frac{1}{2} \sum_{\mu\nu} \text{pr}_{\mathfrak{hol}}(E_\mu \wedge E_\nu) \star \mathcal{R}_{E_\mu, E_\nu}.
\]

Of course the definition of \( q(R) \) is independent of the choice \( \{E_\mu\} \) of a local orthonormal frame for the tangent bundle \( TM \). It is more important to observe however that this standard argument of linear algebra can be applied as well to the induced local orthonormal frame \( \{E_\mu \wedge E_\nu\} \) of the Lie algebra bundle \( \mathfrak{so}TM = \Lambda^2TM \). Considering the curvature tensor as an operator \( R : \Lambda^2TM \longrightarrow \mathfrak{hol}M, X \wedge Y \longmapsto R_{X,Y} \), of vector bundles we may write

\[
q(R) = \frac{1}{2} \sum_{\mu\nu} \text{pr}_{\mathfrak{hol}}(E_\mu \wedge E_\nu) \star \mathcal{R}_{E_\mu, E_\nu} = \sum_{\alpha} (\text{pr}_{\mathfrak{hol}}X_\alpha) \star \mathcal{R}(X_\alpha) \star
\]

with an arbitrary local orthonormal basis \( \{X_\alpha\} \) of \( \mathfrak{so}TM \). For a local orthonormal basis \( \{X_\alpha, X_\beta^\perp\} \) adapted to the decomposition \( \mathfrak{so}TM = \mathfrak{hol}M \oplus \mathfrak{hol}^\perp M \) we find in particular

\[
q(R) = \sum_{\alpha} X_\alpha \star \mathcal{R}(X_\alpha) \star,
\]

because the sum over \( \{X_\beta^\perp\} \) vanishes. In particular \( q(R) \) is a symmetric endomorphism for every geometric vector bundle and for every metric connection \( \nabla \) on the tangent bundle \( TM \) with parallel skew–symmetric torsion, because \( R \in \text{Sym}^2\mathfrak{hol}M \subset \text{Sym}^2\Lambda^2TM \) is symmetric for such connections according to Subsection 2.3. In particular we can choose the local orthonormal basis \( \{X_\alpha\} \) of \( \mathfrak{hol}M \) to be a basis of eigenvectors of the curvature operator \( R : \Lambda^2TM \longrightarrow \mathfrak{hol}M \), hence \( q(R) \geq 0 \) is a non–negative operator provided all eigenvalues of the curvature operator are non–positive and vice versa.

A similar argument shows that \( q(R) \) is hereditary under successive holonomy reductions in the sense that every geometric vector bundle \( VM \) adapted to a holonomy reduction \( \mathfrak{hol}M \)
remains a geometric vector bundle under a further reduction $\mathfrak{hol}M \subset \mathfrak{hol}M$ of the holonomy algebra bundle; in this situation the curvature endomorphism $q(R)$ does not depend on which projection $pr_{\mathfrak{hol}}$ or $pr_{\mathfrak{hol}}$ is chosen in Definition 2.3. The projection $pr_{\mathfrak{hol}}$ for example makes no difference at all for a geometric vector bundle $VM$, whose infinitesimal representation $\star: \mathfrak{hol}M \otimes VM \rightarrow VM$ is actually the restriction of an infinitesimal representation of the generic holonomy bundle $\mathfrak{so}TM \supset \mathfrak{hol}M$. In particular the curvature endomorphism $q(R)$ equals the Ricci endomorphism for every geometric subbundle of the tangent bundle:

$$q(R)X = \frac{1}{2} \sum_{\mu \nu} pr_{\mathfrak{hol}}(E_\mu \wedge E_\nu) \star R_{E_\mu, E_\nu}X \equiv \frac{1}{2} \sum_{\mu \nu} (E_\mu \wedge E_\nu) \star R_{E_\mu, E_\nu}X$$

$$= \frac{1}{2} \sum_{\mu \nu} \left( g(E_\mu, R_{E_\mu, E_\nu}X) E_\nu - g(E_\nu, R_{E_\mu, E_\nu}X) E_\mu \right) = \sum_\nu \text{Ric}(E_\nu, X) E_\nu.$$

**Example 1:** The example motivating Definition 2.3 is certainly the forms representation $V = \Lambda^*T^*$ in generic holonomy $\mathfrak{hol} = \mathfrak{so}T$, in which a bivector $X \wedge Y \in \Lambda^2T \cong \mathfrak{so}T$ acts by $(X \wedge Y)\star := Y^b \wedge X^\perp - X^b \wedge Y^\perp$. In this example the curvature endomorphism becomes:

$$q(R) = \frac{1}{2} \sum_{\mu \nu} (E_\mu \wedge E_\nu) \star R_{E_\mu, E_\nu} = - \sum_{\mu \nu} E_\nu^b \wedge E_\perp \lambda R_{E_\mu, E_\nu}.$$

The curvature term on the right hand side is well–known, because it appears in the classical Weitzenböck formula for the Hodge–Laplace operator $\Delta_{\text{Hodge}} := dd^* + d^*d$ acting on $p$–forms, the classical Weitzenböck formula $\Delta_{\text{Hodge}} = \nabla^*\nabla + q(R)$ thus forms the blue print for the definition (1) of the standard Laplace operator $\Delta$. As a direct consequence of this definition the standard Laplace operator $\Delta = \Delta_{\text{Hodge}}$ equals the Hodge–Laplace operator for every geometric vector bundle occurring as a subbundle in the bundle of differential forms.

**Example 2:** In the generic case $\mathfrak{hol} = \mathfrak{so}T$ we consider the representation $V = \text{Sym}^2T^*$ of trace–free symmetric 2–tensors. In the notation of (1) the curvature endomorphism $q(R)$ acting on sections of the corresponding geometric vector bundle $\text{Sym}^2T^*M$ becomes the sum

$$q(R) = 2 \hat{R} + \text{Der}_{\text{Ric}},$$

where $(\text{Der}_{\text{Ric}}h)(X, Y) := h(\text{Ric}X, Y) + h(\text{Ric}Y, X)$ denotes the standard derivative extension of the Ricci endomorphism to bilinear forms, whereas the curvature operator $\hat{R}$ is defined specifically for symmetric 2–tensors $h \in \Gamma(\text{Sym}^2T^*M)$ by:

$$(\hat{R}h)(X, Y) := \sum_\mu h(R_{X, E_\mu}Y, E_\mu).$$

Given this ad hoc definition it is not even clear that $\hat{R}h$ is actually symmetric, although this can be shown by using the first Bianchi identity and the skew–symmetry of the endomorphism $Z \mapsto R_{X,Y}Z$ for every metric connection on the tangent bundle $TM$. 
Example 3: Consider a Riemann symmetric space $M = G/K$. The isometry group $G$ of $M$ can be thought of as a principal bundle over $M$ with structure group structure equal to the holonomy reduction $G \cong \text{Hol} M$, in consequence geometric vector bundles on $M$ are homogeneous vector bundles and vice versa. There exists a unique non-degenerate invariant scalar product $g^{\text{ext}}$ on the Lie algebra $\mathfrak{g}$ of $G$, which restricts to the Riemannian metric $g$ on the symmetric complement $p = T_{eK}M \subset \mathfrak{g}$ of the isotropy subalgebra $\mathfrak{k} \subset \mathfrak{g}$. The curvature endomorphism $q(R)$ acts on every geometric vector bundle with the Casimir operator of $\mathfrak{k}$ defined as a sum over an orthonormal basis $\{X_\alpha\}$ of $\mathfrak{k}$ with respect to $g^{\text{ext}}$:

$$q(R) = \text{Cas}_{g^{\text{ext}}} := -\sum_\alpha X_\alpha \ast X_\alpha \ast$$

(cf. Lemma 5.2 in [12]).

Example 4: Let $(M, g)$ be a spin manifold with the spin structure defined by a principal Spin$^T$–bundle $\text{Spin} M$ lifting the bundle $\text{SO} M$ of oriented orthonormal frames. For the geometric vector bundle $\Sigma M$ corresponding to the spinor representation $\Sigma$ of Spin$^T$ the curvature endomorphism acts as multiplication with the scalar curvature $\kappa \in C^\infty(M)$ of $g$:

$$q(R) = \frac{\kappa}{8}.$$

This assertion is essentially equivalent to the well-known Lichnerowicz–Schrödinger formula $D^2 = \nabla^* \nabla + \frac{\kappa}{4}$ for the square of the Dirac operator $D$ acting on sections of the so called spinor bundle $\Sigma M$ (cf. [14]). In consequence $q(R) = \frac{\kappa}{8}$ for every geometric vector bundle occurring in the spinor bundle with respect to some holonomy reduction $\text{Hol} M \subset \text{SO} M$.

In applications it is certainly useful to know several different presentations of the curvature endomorphism. A particularly elegant definition of $q(R)$ can be given by using the concept of conformal weights arising in conformal geometry. Consider for this purpose the conformal weight operator $B \in \text{Hom}_{\text{Hol}}(T \otimes T, \text{End} V)$ defined for all $X, Y \in T$ and $v \in V$ by:

$$B_{X \otimes Y} v := \text{pr}_{\text{hol}}(X \wedge Y) \ast v ,$$

Using the conformal weight operator we may write the curvature endomorphism in the form:

$$q(R) v = B(\nabla^2 v) = \sum_{\mu \nu} B_{E_\mu \otimes E_\nu} \nabla^2_{E_\mu, E_\nu} v .$$

for every section $v \in \Gamma(VM)$ of $VM$. Alternatively the conformal weight operator can be interpreted as a Hol–equivariant endomorphism of the representation $T^* \otimes V$, as such it can be written in the form $B = \sum_\varepsilon b_\varepsilon \text{pr}_\varepsilon$, where $\text{pr}_\varepsilon$ are the projections to the isotypical components $V_\varepsilon \subset T^* \otimes V$. The eigenvalues or conformal weights $b_\varepsilon$ of $B$ can be computed easily by a very simple formula (cf. [15]). Via the relation $\text{pr}_\varepsilon(\nabla^2 \psi) = -P_\varepsilon P_\varepsilon \psi$ between
the projections $p_{\varepsilon}$ and the hermitian squares of the generalized gradients $P_{\varepsilon}$ this calculation leads to the following universal Weitzenböck formula studied extensively in [15]:

$$q(R) = -\sum b_{\varepsilon} P_{\varepsilon}^* P_{\varepsilon}$$

For the generic Riemannian holonomy group $SO^T$ the conformal weight operator and the universal Weitzenböck formula were first considered by P. Gauduchon in [4], other uses of the conformal weight operator $B$ besides its use in conformal geometry can be seen in [2] and [7].

2.5. Integral Representation. Yet another presentation of the curvature endomorphism particularly useful under additional assumptions on the sectional curvatures of the Riemannian manifold $(M, g)$ writes $q(R)$ at a point $x \in M$ as an integral over the Grassmannian of 2–planes of $T_x M$ with respect to the Fubini–Study volume density. For a given cut off parameter $\Lambda \in \mathbb{R}$ this integral representation of the curvature endomorphism is readily established by using the integration techniques for the sectional curvature discussed in [13] and reads

$$\frac{1}{\text{Vol}(Gr_2 T)} \left(\frac{m + 2}{4}\right) \int_{Gr_2 T_x M} \left( g(R_{X,Y}, X) - \Lambda \right) p_{\text{hol}}(X \wedge Y) \ast p_{\text{hol}}(X \wedge Y) \ast |\text{vol}|$$

$$= \left(\frac{\Lambda}{12}(m + 2)(m + 1) - \frac{\kappa}{12} \frac{m + 4}{m}\right) \text{Cas}_{\Lambda^2}^{\text{hol}} + \sum_{\mu\nu} p_{\text{hol}}(\text{Ric}^o E_\mu \wedge E_\nu) \ast p_{\text{hol}}(E_\mu \wedge E_\nu) \ast$$

$$- \frac{1}{2} q(R) + \frac{1}{48} \sum_{\lambda\mu\nu\rho} g(T \wedge T)(E_\lambda, E_\mu, E_\nu, E_\rho) p_{\text{hol}}(E_\lambda \wedge E_\mu) \ast p_{\text{hol}}(E_\nu \wedge E_\rho) \ast,$$

where $\kappa$ denotes the scalar curvature, $\text{Ric}^o := \text{Ric} - \frac{\kappa}{m} \text{id}$ the trace free part of the Ricci endomorphism and $\text{Cas}_{\Lambda^2}^{\text{hol}}$ the Casimir operator of the Lie algebra bundle $\mathfrak{hol} M$

$$\text{Cas}_{\Lambda^2}^{\text{hol}} : = -\frac{1}{2} \sum_{\mu\nu} p_{\text{hol}}(E_\mu \wedge E_\nu) \ast p_{\text{hol}}(E_\mu \wedge E_\nu) \ast$$

in $\Lambda^2$–normalization. With respect to a Hol–invariant scalar product or hermitian form on the representation $V$ corresponding to a geometric vector bundle $VM$ the endomorphism $p_{\text{hol}}(X \wedge Y) \ast$ is skew symmetric or skew hermitian respectively with non–positive square in the sense of operators. Every upper bound $\text{sec} \leq \Lambda$ on the sectional curvatures of the Riemannian manifold $M$ thus leads in the torsion free case to a pointwise upper bound

$$\frac{1}{2} q(R) \leq \left(\frac{\Lambda}{12}(m + 2)(m + 1) - \frac{\kappa}{12} \frac{m + 4}{m}\right) \text{Cas}_{\Lambda^2}^{\text{hol}} + \sum_{\mu\nu} p_{\text{hol}}(\text{Ric}^o E_\mu \wedge E_\nu) \ast p_{\text{hol}}(E_\mu \wedge E_\nu) \ast$$

for the curvature endomorphism $q(R)$, similarly every lower bound $\text{sec} \geq \Lambda$ implies

$$\frac{1}{2} q(R) \geq \left(\frac{\Lambda}{12}(m + 2)(m + 1) - \frac{\kappa}{12} \frac{m + 4}{m}\right) \text{Cas}_{\Lambda^2}^{\text{hol}} + \sum_{\mu\nu} p_{\text{hol}}(\text{Ric}^o E_\mu \wedge E_\nu) \ast p_{\text{hol}}(E_\mu \wedge E_\nu) \ast$$
in the sense of operators acting on the fiber $V_x M$ of a geometric vector bundles over a point $x \in M$. In [6] this argument was used in the special case of geometric vector bundles $\text{Sym}^p T^* M$ of trace–free symmetric $p$–tensors on a Riemannian manifold $M$ with sectional curvature $\text{sec} \leq 0$ to show that the curvature endomorphism $q(R)$ is non–positive.

2.6. The Standard Laplacian. The curvature term $q(R)$ is clearly functorial in the sense that it commutes with all morphism between geometric vector bundles. In order to obtain a second order differential operator enjoying the same functoriality property we simply add the rough Laplacian $\nabla^* \nabla$ and obtain the standard Laplace operator on geometric vector bundles:

**Definition 2.4 (Standard Laplace Operator).**

The standard Laplace operator acting on sections of a geometric vector bundle $VM$ is defined as the sum $\Delta = \nabla^* \nabla + q(R)$ of the rough Laplacian and the curvature endomorphism $q(R)$.

The functorial nature of the standard Laplace operator for geometric vector bundles explains in a sense the work of many an author in differential geometry working for example in analogues of the Hodge decomposition of differential forms on Kähler manifolds: The standard Laplace operator commutes by definition with all morphisms of geometric vector bundles. Similar Laplace type operators with no or at most a limited functoriality have been present in the literature for a long time say in [1], section 1. I, and in [11] in the special case of tensor fields on Riemannian manifolds.

Example 1: In the Riemannian case $\text{Hol} = \text{SO} T$ and the geometric vector bundle $\Lambda^p T^* M$ of differential forms of degree $p$ corresponding to the representation $V = \Lambda^p T^*$ the standard Laplace operator $\Delta$ coincides with the Hodge Laplace operator $\Delta_{\text{Hodge}} := d^* d + dd^*$. In fact the definition just reflects the classical Weitzenböck formula for $\Delta_{\text{Hodge}}$. However the functorial point of view is a decisive advantage in this example, because $\Delta_{\text{Hodge}}|_{VM} = \Delta$ for every geometric subbundle of the bundle of differential forms (cf. [14]) immediately implies the generalized Hodge decomposition of de Rham cohomology

$$H_{dR}^*(M) = \bigoplus_V \text{Hom}_{\text{Hol}}(V, \Lambda^* T^*) \otimes \ker \Delta_V,$$

under arbitrary holonomy reductions $\text{Hol} M \subset \text{SO} M$, analogous decompositions hold true for every eigenvalue of the Hodge–Laplace operator $\Delta_{\text{Hodge}}$ on the bundle of differential forms. The important point in this decomposition is that the restriction of $\Delta_{\text{Hodge}}$ to a parallel subbundle $VM$ only depends on the representation $V$ and not on its embedding $V \subset \Lambda^p T^*$. We note that a similar decomposition of the de Rham cohomology is discussed in the book of D. Joyce (cf. Theorem 3.5.3 in [10]).

Example 2: In the generic holonomy case $\text{Hol} = \text{SO} T$ the standard Laplace operator $\Delta$ coincides with the Lichnerowicz Laplacian $\Delta_L$ (cf. [1], 1.143, [11]) on the geometric vector bundle $VM = \bigotimes^p T^* M$ of $p$–tensors. Especially interesting is the case of trace–free symmetric 2–tensors with associated representation $V = \text{Sym}^2 T^*$. In particular the space of
The standard Laplace operator 11

infinitesimal Einstein deformations of an Einstein metric $g$ with Ricci curvature $\text{Ric} = \frac{\kappa}{m} g$ and scalar curvature $\kappa$ can be identified with the space of symmetric, trace and divergence free endomorphisms $H$ of $TM$ satisfying the eigenvalue equation $\Delta_L H = \frac{2m}{\kappa} H$ (cf. [1]).

Example 3: On a Riemannian symmetric space $M = G/K$ with Riemannian metric $g$ induced by an invariant scalar product $g^{\text{ext}}$ on the Lie algebra $\mathfrak{g}$ geometric vector bundles are homogeneous and vice versa, moreover the standard curvature endomorphism $q(R)$ agrees with the Casimir operator of the isotropy algebra $\mathfrak{k}$. An easy calculation based on this fact mentioned above shows that the standard Laplace operator $\Delta$ is actually the Casimir operator of the isometry group $G$ on sections of a homogeneous vector bundle (cf. Lemma 5.2 in [12]).

Concerning the presentation of the curvature endomorphism $q(R)$ as a sum of hermitian squares of generalized gradients we remark that the universal Weitzenböck formula (5) extends directly to the standard Laplace operator, writing it as a linear combination of hermitian squares of generalized gradients with coefficients determined by the conformal weights $b_\varepsilon$

\begin{equation}
\Delta = \sum_\varepsilon (1 - b_\varepsilon) P_\varepsilon^* P_\varepsilon,
\end{equation}

which are as we have said before very easy to compute. This way to write the standard Laplace operators has been used in [15] together with zero curvature term Weitzenböck formulas to characterize for example all harmonic forms on $G_2$ and Spin(7)--manifolds.

3. The Commutator Formula

In this section we will calculate the commutator of the standard Laplacian and a generalized gradient $D_\diamond$ from sections of a geometric vector bundle $VM$ to sections of a geometric vector bundle $\tilde{V}M$ over $M$. In order to compute the commutator of $\nabla^* \nabla$ and $D_\diamond$ it is convenient to recall the concept of iterated covariant derivatives for sections of a general vector bundle $VM$ over $M$ endowed with a connection $\nabla$ with respect to an auxiliary, not necessarily torsion free connection $\bar{\nabla}$ on the tangent bundle $TM$. Specifically the second iterated covariant derivative is defined for every section $\psi \in \Gamma(VM)$ by

\[ \nabla^2 X, Y \psi := \nabla_X (\nabla_Y \psi) - \nabla_{\nabla_X Y} \psi \]

and much in the same spirit the third iterated covariant derivative reads:

\[ \nabla^3 X, Y, Z \psi := \nabla_X \nabla_Y \nabla_Z \psi - \nabla_X (\nabla_Y \nabla_Z \psi) + \nabla_{\nabla_X Y} \nabla_Z \psi - \nabla_{\nabla_X Z} \nabla_Y \psi - \nabla_{\nabla_Y Z} \nabla_X \psi. \]

Our calculation of the commutator $[\nabla^* \nabla, D_\diamond]$ relies on the following two identities for third iterated covariant derivatives known collectively as the Ricci identities (cf. [1], Corollary 1.22):

\[ \nabla^3 X, Y, Z - \nabla^3 Y, X, Z = R_{X,Y} \nabla_Z - \nabla_{R_{X,Y} Z} - \nabla^2_{T(X,Y),Z}, \]

\[ \nabla^3 Y, X, Z - \nabla^3 Y, Z, X = (\nabla_Y R)_{X,Z} + R_{X,Z} \nabla_Y - \nabla_{(\nabla_Y T)(X,Z)} - \nabla^2_{Y,T(X,Z)}. \]
Adding these two Ricci identities together while setting \( Z = Y \) results in the identity:

\[
\nabla^3_{X,Y,Y} - \nabla^3_{Y,Y,X} = \sum_{\lambda} (\nabla_{\nabla_X E_\lambda} R)_{Y,Y} = 2R_{X,Y} \nabla_Y - \nabla^2_{T(X,Y),Y} - \nabla^2_{Y,T(X,Y)} - \nabla_{(\nabla_Y T)(X,Y)}.
\]

This is the key identity in the proof of the commutator formula for the standard Laplacian:

**Theorem 3.1 (Commutator Formula).**

Let \((M, g)\) be a Riemannian manifold and let \(\nabla\) be a metric connection with parallel skew-symmetric torsion \(T\) in the sense that the expression \(\theta(X, Y, Z) := g(T(X, Y), Z)\) defines a parallel differential form \(\theta \in \Gamma(\Lambda^3 T^* M)\). Consider a generalized gradient \(D_\diamond : \Gamma(V M) \to \Gamma(T^* M \otimes VM) \xrightarrow{\sigma_\diamond} \Gamma(TM \otimes VM) \xrightarrow{\sigma_\diamond} \Gamma(\tilde V M)\) from a geometric vector bundle \(VM\) to a geometric vector bundle \(\tilde V M\) associated to a \(\text{Hol}\)-equivariant bilinear operation \(\diamond : T \otimes V \to \tilde V\) between the associated representations. The commutator of the standard Laplace operator \(\Delta\) acting on both \(VM\) and \(\tilde V M\) with \(D_\diamond\) reads:

\[
[D, D_\diamond] = \Delta_\nabla \circ D_\diamond - D_\diamond \circ \Delta_\nabla = -\sigma_\diamond(\theta(\nabla R) - \delta R^*)
\]

\[
= -\sum_{\lambda} E_{\lambda} \diamond (\theta(\nabla_{E_{\lambda}} R) - (\delta R)_{E_{\lambda}}^*)
\]

**Proof.** In terms of the parallel extension \(\diamond : TM \otimes VM \to \tilde V M\) of \(\diamond\) we may write \(D_\diamond\) as

\[
D_\diamond := \sigma_\diamond \circ \nabla = \sum_{\lambda} E_{\lambda} \circ \nabla_{E_{\lambda}}
\]

with a local orthonormal basis \(\{E_{\lambda}\}\). Using the parallelity of \(\diamond\) we find for all \(X \in \Gamma(TM)\)

\[
\nabla_X \circ D_\diamond = \sum_{\lambda} \left( (\nabla_{X} E_{\lambda}) \circ \nabla_{E_{\lambda}} + E_{\lambda} \circ \nabla_X \nabla_{E_{\lambda}} \right)
\]

\[
= \sum_{\lambda} E_{\lambda} \circ \left( -\nabla_{\nabla_X E_{\lambda}} + \nabla_X \nabla_{E_{\lambda}} \right) = \sum_{\mu} E_{\lambda} \circ \nabla^2_{X,E_{\lambda}}
\]

where in the second line we used the parallelity \(\sum_{\lambda}(\nabla_{X} E_{\lambda}) \otimes E_{\lambda} = -\sum_{\lambda} E_{\lambda} \otimes (\nabla_{X} E_{\lambda})\) of the cometric tensor with respect to a metric connection. Applying this identity twice we find:

\[
\nabla^* \circ D_\diamond = -\sum_{\lambda\mu} E_{\lambda} \circ \nabla^3_{E_{\mu},E_{\mu},E_{\lambda}}
\]

\[
D_\diamond \circ \nabla^* = -\sum_{\lambda\mu} E_{\lambda} \circ \nabla^3_{E_{\lambda},E_{\mu},E_{\mu}}
\]
In consequence the commutator \([\nabla^*\nabla, D_\circ]\) equals the sum of the key identity (7)
\[
[\nabla^*\nabla, D_\circ] = \sum_{\lambda\mu} E_\lambda \diamond \left( \nabla^3_{E_\lambda, E_{\mu}, E_{\mu}} - \nabla^2_{E_\lambda, E_{\mu}, E_{\mu}} \right)
= 2 \sum_{\lambda\mu} E_\lambda \diamond R_{E_\lambda, E_{\mu}} \nabla_{E_{\mu}} - \sum_{\lambda\mu} E_\lambda \diamond \nabla_{R_{E_\lambda, E_{\mu}} E_{\mu}} - \sum_{\lambda\mu} E_\lambda \diamond (\nabla_{E_{\mu}} R)_{E_{\mu}, E_{\lambda}}.
\]
over a local orthonormal basis, because the torsion \(\nabla T = 0\) is parallel by assumption and the other two terms involving \(T\) cancel out due to \(\sum_{\mu}(T(X, E_{\mu}) \otimes E_{\mu} + E_{\mu} \otimes T(X, E_{\mu})) = 0\) for all \(X\), after all \(Y \mapsto T(X,Y)\) is a skew symmetric endomorphism of \(TM\) for skew–symmetric torsion \(T\). In terms of the Ricci endomorphism \(\text{Ric} X := \sum_{\mu} R_{X, E_{\mu}} E_{\mu}\) and the divergence \((\delta R)_X := -\sum_{\mu}(\nabla_{E_{\mu}} R)_{E_{\mu}, X}\) of the curvature tensor this result can be written:
\[
(8) \ [\nabla^*\nabla, D_\circ] = 2 \sum_{\lambda\mu} E_\lambda \diamond R_{E_\lambda, E_{\mu}} \nabla_{E_{\mu}} - \sum_{\lambda} E_\lambda \diamond \nabla_{\text{Ric} E_\lambda} + \sum_{\lambda} E_\lambda \diamond (\delta R)_{E_\lambda} \ast .
\]
In a second step we calculate the commutator of the curvature endomorphism \(q(R)\) characteristic for the standard Laplace operator \(\Delta\) with the generalized gradient \(D_\circ\). Recalling the alternative definition of the curvature term \(q(R) = \sum_\alpha \chi_\alpha \ast R(\chi_\alpha) \ast\) as a sum over a local orthonormal basis \(\{X_\alpha\}\) of the holonomy algebra bundle \(\mathfrak{ho} \subset \Lambda^2 TM\) we find
\[
[q(R), D_\circ] = \sum_{\lambda} \left( q(R) E_\lambda \right) \diamond \nabla_{E_\lambda} + \sum_{\lambda} E_\lambda \diamond \left( q(R) \nabla_{E_\lambda} \right)
+ 2 \sum_{\lambda\mu\alpha} g(\chi_\alpha \ast E_\lambda, E_{\mu}) E_{\mu} \diamond R(\chi_{\alpha}) \ast \nabla_{E_\lambda} - \sum_{\lambda} E_\lambda \diamond \nabla_{E_\lambda} q(R)
= \sum_{\lambda} \text{Ric} E_\lambda \diamond \nabla_{E_\lambda} - \sum_{\lambda} E_\lambda \diamond q(\nabla_{E_\lambda} R)
+ 2 \sum_{\lambda\mu} E_{\mu} \diamond R \left( \sum_{\alpha} g(\chi_\alpha, E_\lambda \wedge E_{\mu}) \chi_\alpha \right) \ast \nabla_{E_\lambda}
= \sum_{\lambda} \text{Ric} E_\lambda \diamond \nabla_{E_\lambda} - \sum_{\lambda} E_\lambda \diamond q(\nabla_{E_\lambda} R) + 2 \sum_{\lambda\nu} E_{\nu} \diamond R_{E_{\lambda}, E_{\nu}} \nabla_{E_\lambda},
\]
because the curvature endomorphism \(q(R)\) equals the Ricci endomorphism on the tangent bundle and \(R(\text{pr}_{\mathfrak{ho}}(E_\lambda \wedge E_{\mu})) \ast = R(E_\lambda \wedge E_{\mu}) \ast = R_{E_{\lambda}, E_{\mu}}\) as the curvature endomorphism factorizes over the orthogonal projection to the holonomy algebra bundle \(\mathfrak{ho} \subset \Lambda^2 TM\). Adding the commutator \([\nabla^*\nabla, D_\circ]\) from equation (8) to the commutator \([q(R), D_\circ]\) calculated above and using the symmetry \(\sum_{\lambda}(\text{Ric} E_{\lambda}) \otimes E_{\lambda} = \sum_{\lambda} E_{\lambda} \otimes (\text{Ric} E_{\lambda})\) of the Ricci curvature endomorphism we eventually obtain the commutator formula:
\[
[\Delta, D_\circ] = - \sum_{\lambda} E_\lambda \diamond q(\nabla_{E_\lambda} R) + \sum_{\lambda} E_\lambda \diamond (\delta R)_{E_\lambda} \ast .
\]
Corollary 3.2 (Commutator Formula for First Order Operators).

Every natural first order differential operator $D : \Gamma(VM) \to \Gamma(\tilde{VM})$ between geometric vector bundles can be written as a sum $D = D_\diamond + F$ of the generalized gradient associated to its principal symbol $\sigma_\diamond : T^*M \otimes VM \to \tilde{VM}$ and a homomorphism $F : VM \to \tilde{VM}$ of geometric vector bundles. In consequence we find $[\Delta, D] = [\Delta, D_\diamond]$.

Because both error terms $\sum \lambda E_\lambda \circ q(\nabla_{E_\lambda} R)$ and $\sum \lambda E_\lambda \circ (\delta R)_{E_\lambda} \ast$ on the right hand side of the commutator formula of Theorem 3.1 are the images of the covariant derivative $\nabla R$ of the curvature tensor under homomorphisms of geometric vector bundles, we conclude:

Corollary 3.3 (Simple Vanishing Criterion).

Let Hol be the holonomy group of the metric connection $\nabla$ and suppose that $\nabla R$ is a section of a geometric vector bundle CM with corresponding Hol–representation $C$. A sufficient condition for the vanishing of the commutator $[\Delta, D] = 0$ of the standard Laplace operator with a natural first order differential operator $D$ from a geometric vector bundle $VM$ to a geometric vector bundle $\tilde{VM}$ is the vanishing of the space of Hol–equivariant homomorphisms:

$$\text{Hom}_{\text{Hol}}(C, \text{Hom}(V, \tilde{V})) = \{0\}.$$ 

For later uses it may be helpful to recall the well–known fact that for the Levi–Civita connection $\nabla$ for the Riemannian metric $g$ the divergence $\delta R$ of the Riemannian curvature tensor $R$ relates via $\delta R = d^c \text{Ric}$ to the covariant derivative of the Ricci curvature. In consequence $\delta R$ vanishes on all manifolds with parallel Ricci tensor and the error term in the commutator formula involves only $q(\nabla R)$. According to an observation of Gray [5] manifolds with parallel Ricci tensor are locally the Cartesian product of Einstein manifolds and conversely every local product of Einstein manifolds has parallel Ricci curvature provided all 2–dimensional Einstein factors have constant scalar curvature.

4. Examples

4.1. Differential and Codifferential. Consider a Riemannian manifold $(M, g)$ and the Hodge–Laplace operator $\Delta_{\text{Hodge}} := d^*d + dd^*$ acting on the space of $p$–forms $\Gamma(\Lambda^pT^*M)$. Because both $d$ and $d^*$ are boundary operators $d^2 = 0 = d^{*2}$ the Hodge–Laplace operator commutes with both $d$ and $d^*$ due to the rather trivial argument:

$$[\Delta_{\text{Hodge}}, d] = d^*d^2 + dd^*d - dd^*d - d^2d^* = 0.$$

The commutator formula of Theorem 3.1 can be interpreted as a vast generalization of this simple observation forming the philosophical underpinning of Hodge theory. Taking $\nabla$ to the Levi–Civita connection on $TM$ and $\nabla$ to be the induced connection on the geometric vector bundle $\Lambda^pT^*M$ of $p$–forms we may write the exterior derivative $d = \sigma_\lambda \circ \nabla$ as a generalized gradient corresponding to the wedge product $\sigma_\lambda : T \otimes \Lambda^pT^* \to \Lambda^{p+1}T^*$, $X \otimes \omega \mapsto X^\flat \wedge \omega$. The commutator $[\Delta_{\text{Hodge}}, d]$ thus falls under the ambit of Theorem 3.1 because the standard
The standard Laplace operator $\Delta = \Delta_{\text{Hodge}}$ agrees with the Hodge–Laplace operator in this setup as we remarked previously. In order to determine this commutator we verify first of all the equation:

$$\sigma \wedge (q(\bar{\nabla} R)) = -\sum_{\lambda \mu \nu} E^\lambda_\mu \wedge E^\nu_\mu \wedge (\nabla_{E^\lambda_\mu} R)_{E^\mu_\nu, E^\nu_\mu}$$

$$= -\sum_{\lambda \mu \nu} E^\mu_\nu \wedge E^\lambda_\mu \wedge (\nabla_{E^\lambda_\mu} R)_{E^\mu_\nu, E^\nu_\mu} + \sum_{\lambda \mu} E^\mu_\mu \wedge (\nabla_{E^\lambda_\mu} R)_{E^\mu_\mu, E^\lambda_\mu} = \sigma \wedge (\delta R^*) .$$

The omission of the first sum is justified by using the second Bianchi identity in the calculation

$$\sum_{\lambda \mu} E^\mu_\mu \wedge (\nabla_{E^\lambda_\mu} R)_{E^\mu_\mu, X} = +\frac{1}{2} \sum_{\lambda \mu} E^\mu_\mu \wedge E^\lambda_\mu \wedge ( (\nabla_{E^\lambda_\mu} R)_{E^\mu_\mu, X} - (\nabla_{E^\mu_\mu} R)_{E^\lambda_\mu, X} )$$

$$= -\frac{1}{2} \sum_{\lambda \mu} E^\mu_\mu \wedge E^\lambda_\mu \wedge (\nabla_X R)_{E^\lambda_\mu, E^\mu_\mu} = 0$$

for all $X \in \Gamma(TM)$, where $\sum_{\lambda \mu} E^\lambda_\mu \wedge E^\mu_\mu \wedge R^\nu_{E^\lambda_\mu, E^\mu_\nu} = 0$ holds true for every algebraic curvature tensor and thus in particular for $R' = \nabla_X R$ due to the first Bianchi identity. In passing we remark that this consequence of the first Bianchi identity arises from the proof of the classical Weitzenböck formula for the Hodge–Laplace operator $\Delta_{\text{Hodge}}$. The error term $\sigma \wedge (q(\nabla R) - \delta R^*) = 0$ in the commutator formula of Theorem 3.1 thus vanishes for the exterior derivative and $\Delta_{\text{Hodge}} = \Delta$ commutes with $d$. Mutatis mutandis this argument establishes the vanishing of the commutator $[\Delta_{\text{Hodge}}, d^*] = 0$ as well.

4.2. Symmetric and Normal Homogeneous Spaces. Symmetric spaces are essentially Riemannian manifolds $(M, g)$ with parallel Riemannian curvature tensor $R$ with respect to the Levi–Civita connection $\nabla$. The error terms in the commutator formula of Theorem 3.1 corresponding to $q(\nabla R)$ and $\delta R$ thus both vanish, in consequence the standard Laplacian $\Delta$ commutes with all equivariant first order differential operators between geometric, i.e. homogeneous vector bundles on $M$ vindicating the identification of $\Delta$ with the Casimir operator of the isometry group of a symmetric space $M$.

Interestingly the same statement is true on compact normal homogeneous spaces $(M, g)$ characterized by a Riemannian metric induced from an invariant scalar product on the Lie algebra of the isometry group. The metric connection of choice under this assumption is not the Levi–Civita connection $\nabla$ however, but the reductive connection $\bar{\nabla}$ characterized by having parallel skew symmetric torsion and parallel curvature tensor $R$. With this proviso the standard Laplace operator on geometric vector bundles can be identified with the Casimir operator of the isometry group of a normal homogeneous space $M$ (cf. Lemma 5.2 in [12]).

4.3. The Rarita–Schwinger Operator. Let $(M, g)$ be a Riemannian spin manifold with the spin structure defined by a lift Spin $M$ of the principal $\text{SO}(T)$–bundle $\text{SO}(M)$ of oriented orthonormal frames to the structure group Spin $T$ as before. Because the Clifford multiplication $\bullet : T \otimes \Sigma \longrightarrow \Sigma$ for the spinor representation $\Sigma$ is equivariant under Spin $T$, its kernel
\[ \Sigma_3^+ \subset T \otimes \Sigma \text{ is a Spin } T \text{-representation with associated geometric vector bundle } \Sigma_3^+ M \text{ on } M. \]

In passing we remark that \( \Sigma_3^+ \) equals the Cartan summand in the tensor product \( T \otimes \Sigma \) in odd dimensions, in even dimensions \( \Sigma_3^+ = \Sigma_3^+ \oplus \Sigma_3^- \) decomposes like the spinor representation itself. The \textit{Rarita–Schwinger operator} is defined as the generalized gradient

\[ D_\circ : \Gamma(\Sigma_3^+ M) \rightarrow \Gamma(\Sigma_3^+ M) \]

associated to the Spin \( T \)-equivariant homomorphism \( \sigma_\circ : T \otimes \Sigma_3^+ \rightarrow \Sigma_3^+ \) given by Clifford multiplication in the \( \Sigma \)-factor followed by projection \( X \circ \psi := \text{pr}_{\Sigma_3^+}((\text{id} \otimes X)\psi) \). It is well–known, compare for example [8] or [13], that the covariant derivative of the curvature tensor of an Einstein manifold is a section \( \nabla R \in \Gamma(CM) \) of the geometric vector bundle corresponding to the Cartan summand \( C := \text{Sym}_0^2 T^* \subset \text{Sym}_0^2 T^* \otimes \text{Sym}_0^2 T^* \) in the tensor product of harmonic cubic and quadratic polynomials on \( T \); and it is easily verified that this Cartan summand does not occur in the endomorphisms of \( \Sigma_3^+ \). In consequence \( \Delta \) commutes \([\Delta, D_\circ] = 0\) with the Rarita–Schwinger operator \( D_\circ \) on Einstein spin manifolds due to Corollary 3.3. A direct proof of this commutator formula can be found in [8].

### 4.4. Nearly Kähler manifolds.

A six dimensional nearly Kähler manifold \( (M, g, J) \) is a Riemannian manifold of dimension 6 endowed with an orthogonal almost complex structure \( J \) satisfying the integrability condition \( (\nabla_X J)X = 0 \) for the Levi–Civita connection \( \nabla \) and all vector fields \( X \in \Gamma(TM) \). In this situation it is better to replace the Levi–Civita connection by the \textit{canonical hermitian connection} \( \nabla_X := \nabla_X + \frac{1}{2} J(\nabla_X J) \) on the tangent bundle, which is a metric connection with parallel, skew–symmetric torsion in the sense of Definition 2.2 and makes \( J \) parallel \( \nabla J = 0 \). In consequence the holonomy group of \( \nabla \) is a subgroup of \( \text{Hol} = \text{SU}(T, J, \theta) \subset \text{SO} \ T \) unless \( M \) is actually Kähler with \( \nabla J = 0 \).

Considering only the case of strictly nearly Kähler manifolds with \( \nabla J \neq 0 \) we conclude that the defining 3–dimensional representation of \( \text{SU}(T, J, \theta) \cong \text{SU}(3) \) gives rise to a geometric vector bundle \( EM \) on \( M \) such that \( TM \otimes \mathbb{C} = EM \oplus \bar{EM} \). Due to \( \Lambda^* \mathbb{C} = \mathbb{C} \oplus \bar{E} \oplus \bar{E} \oplus \mathbb{C} \) etc. the geometric vector bundle of complex–valued differential forms on \( M \) decomposes

\[ \Lambda^* T^* M \otimes \mathbb{C} \cong (\mathbb{C}M \oplus EM \oplus \bar{E}M \oplus CM) \otimes (\mathbb{C}M \oplus \bar{E}M \oplus EM \oplus CM) \]

into copies of the trivial bundle \( \mathbb{C}M \), \( EM \) and \( \bar{EM} \), the complexified holonomy algebra bundle \( \mathfrak{su} M \otimes \mathbb{C} \) and \( \text{Sym}^2EM \) as well as \( \text{Sym}^2 \bar{EM} \). On the other hand it is well–known that the curvature tensor of the canonical hermitian connection \( \nabla \) can be written as a sum

\[ R = \frac{\omega}{10} R^c + R^{\text{CY}} \]

of a parallel standard curvature tensor \( R^c \) and a curvature tensor of Calabi–Yau type, i. e. a real section \( R^{\text{CY}} \in \Gamma(\text{Sym}^2 EM \otimes \text{Sym}^2 \bar{EM}) \) of the vector bundle \( \text{Sym}^2 EM \otimes \text{Sym}^2 \bar{EM} \). Its covariant derivative then is a real section \( \nabla R \in \Gamma(CM) \) of the vector bundle corresponding to the representation \( C = \text{Sym}^3 E \otimes \text{Sym}^3 \bar{E} \oplus \text{Sym}^3 \bar{E} \otimes \text{Sym}^3 E \), which is simply too complex to occur in the endomorphisms of \( \Lambda^* T^* \otimes \mathbb{C} \).
T splits on the other hand into copies of the trivial representation $R^4$. Quaternion–Kähler Manifolds. Spin(7)–holonomy, the details of this argument are left to the reader.

Possible to check that the Riemannian curvature tensor $\nabla V$ on the geometric vector bundle $V\rightarrow M$ with holonomy group contained in $\text{Sp}(1)$ visible by 4 with holonomy group contained in $\text{Sp}(1)$ and $\text{Sp}(n)$ respectively by $H$ and $E$. In general $H$ and $E$ do not give rise to geometric vector bundles on $M$, because neither representation extends to a representation of $\text{Hol} = \text{Sp}(1) \cdot \text{Sp}(n)$, however all totally even powers of $H$

**Proposition 4.1** (Commutator Formulas for Nearly Kähler Manifolds).

Let $(M, g, J)$ be a six dimensional nearly Kähler manifold with canonical hermitian connection $\nabla$. The standard Laplacian $\Delta = \nabla^* \nabla + q(R)$ acting on differential forms commutes with the exterior derivative $d$ and the codifferential $d^*$ and thus with the Hodge–Laplace operator:

$$[\Delta, d] = 0 \quad [\Delta, d^*] = 0 \quad [\Delta, \Delta_{\text{Hodge}}] = 0.$$  

In fact $\Delta$ commutes with all natural first order differential operators on differential forms.

4.5. $G_2$– and Spin(7)–Manifolds. Consider a Riemannian manifold $(M, g)$ of dimension seven with holonomy $G_2$. The irreducible geometric vector bundles associated to this holonomy reduction to $\text{Hol} = G_2 \subset \text{SO} T$ correspond to the irreducible representations $V_{[a,b]}$ of $G_2$, which are parametrized by their highest weight, a linear combination $a\omega_1 + b\omega_2$ with integer coefficients $a, b \geq 0$ of the two fundamental weights $\omega_1$ and $\omega_2$ corresponding respectively to the 7–dimensional isotropy representation $T$ and the adjoint representation $\text{hol} = g_2$. It is possible to check that the Riemannian curvature tensor $R$ is a section of the geometric vector bundle $V_{[0,2]}M$ and that its covariant derivative $\nabla R$ is a section of $V_{[1,2]}M$, compare [13], page 162. The representation of $G_2$ on the exterior algebra $\Lambda^* T^*$ of alternating forms on $T$ splits on the other hand into copies of the trivial representation $\mathbb{R}$, the isotropy representation $T$, the holonomy representation $\text{hol} = g_2$ and $\Lambda^3_{27} := V_{[2,0]}$. Considering the decompositions

$$T \otimes g_2 = V_{[1,0]} \oplus V_{[2,0]} \oplus V_{[1,1]},$$

$$T \otimes \Lambda^3_{27} = V_{[1,0]} \oplus V_{[2,0]} \oplus V_{[3,0]} \oplus V_{[0,1]} \oplus V_{[1,1]},$$

$$g_2 \otimes g_2 = V_{[0,0]} \oplus V_{[2,0]} \oplus V_{[3,0]} \oplus V_{[0,1]} \oplus V_{[0,2]},$$

$$g_2 \otimes \Lambda^3_{27} = V_{[1,0]} \oplus V_{[2,0]} \oplus V_{[3,0]} \oplus V_{[0,1]} \oplus V_{[1,1]} \oplus V_{[2,1]},$$

$$\Lambda^3_{27} \otimes \Lambda^3_{27} = V_{[0,0]} \oplus V_{[1,0]} \oplus 2 V_{[2,0]} \oplus V_{[3,0]} \oplus V_{[4,0]} \oplus V_{[0,1]} \oplus 2 V_{[1,1]} \oplus V_{[2,1]} \oplus V_{[0,2]},$$

we conclude that $\nabla R$ cannot result in a homomorphism between any two irreducible components of the exterior algebra $\Lambda^* T^*$ of alternating forms on $T$, in turn Corollary 3.3 tells us that the standard Laplace operator $\Delta$ commutes with every natural first order differential operator on differential forms. In particular $\Delta$ commutes $[\Delta, d_c] = 0$ with the modified differential $d_c$ introduced by Verbitsky in [16]. A very similar result holds true in the case of Spin(7)–holonomy, the details of this argument are left to the reader.

4.6. Quaternion–Kähler Manifolds. A Riemannian manifold $(M, g)$ of dimension $4n$ divisible by 4 with holonomy group contained in $\text{Sp}(1) \cdot \text{Sp}(n) \subset \text{SO}(4n)$ is called a quaternion–Kähler manifold. In order to describe the geometric vector bundles on a quaternion–Kähler manifold $M$ associated to this holonomy reduction we will denote the defining 2– and 2n–dimensional complex representations of $\text{Sp}(1)$ and $\text{Sp}(n)$ respectively by $H$ and $E$. In general $H$ and $E$ do not give rise to geometric vector bundles on $M$, because neither representation extends to a representation of $\text{Hol} = \text{Sp}(1) \cdot \text{Sp}(n)$, however all totally even powers of $H$
and $E$ do. The complexified tangent bundle for example corresponds to the geometric vector bundle $TM \otimes_{\mathbb{R}} \mathbb{C} = HM \otimes EM$.

The Riemannian curvature tensor $R$ of a quaternion–Kähler manifold $M$ can be written as the sum of the curvature tensor of the quaternionic projective space $\mathbb{H}P^n$ with the same scalar curvature and a curvature tensor of hyperkähler type $R = R^{\mathbb{H}P^n} + R^{\text{hyper}}$, i.e. a real section $R^{\text{hyper}} \in \Gamma(\text{Sym}^4 EM)$ of the geometric vector bundle $\text{Sym}^4 EM$. Working out the details of the second Bianchi identity we observe that the covariant derivative $\nabla R$ of the curvature tensor is a real section of the geometric vector bundle $HM \otimes \text{Sym}^5 EM$ (cf. [14]). The geometric vector bundle $\Lambda^* T^* M \otimes_{\mathbb{R}} \mathbb{C}$ of complex–valued alternating forms on $M$ decomposes in a rather complicated way into a sum of geometric vector bundles of the form $\text{Sym}^k H M \otimes \Lambda^{a,b} EM$ with $k \geq 0$ and $n \geq a \geq b \geq 0$ (cf. [17]), where $\Lambda^{a,b} E \subset \Lambda^a E \otimes \Lambda^b E$ is the Cartan summand in the tensor product of the kernels $\Lambda^a E \subset \Lambda^a E$ and $\Lambda^b E \subset \Lambda^b E$ of the contraction with the symplectic form. Generically there are ten generalized gradients defined for sections of the geometric vector bundles $\text{Sym}^k H M \otimes \Lambda^{a,b} EM$ and the standard Laplace operator $\Delta$ commutes with at least eight of these generalized gradients:

**Proposition 4.2** (Commutator Formula for Quaternion–Kähler Manifolds).

Every generalized gradient $D_\sigma : \Gamma(\text{Sym}^k H M \otimes \Lambda^{a,b} EM) \longrightarrow \Gamma(VM)$ to sections of a geometric vector bundle $VM$ on a quaternion–Kähler manifold $M$ corresponding to a representation of the form $V = \text{Sym}^{k+1} H \otimes \Lambda^{a,b} E$ or $V = \text{Sym}^{k+1} H \otimes \Lambda^{a,b} E$ commutes with $\Delta$.

**Proof.** Because all quaternion–Kähler manifolds are automatically Einstein, we only need to consider the error term $\sigma_\circ(q(\nabla R))$ in the commutator formula of Theorem 3.1, where $\sigma_\circ$ is the parallel vector bundle extension of the $\text{Sp}(1) \cdot \text{Sp}(n)$–equivariant isotypical projection corresponding to $D_\sigma$. Since $\sigma_\circ(q(\nabla R))$ is a homomorphism of vector bundles parametrized by the section $\nabla R \in \Gamma(HM \otimes \text{Sym}^4 EM)$, we may replace all geometric vector bundles by their respective representations reducing the problem to the problem to show that $\sigma_\circ(q(R')) = 0$ for all $R' \in H \otimes \text{Sym}^5 E$ or equivalently for all linear generators $R' = h \otimes \frac{1}{\sqrt{e}} e^5$ with arbitrary $h \in H$, $e \in E$. In a similar vein $q(\frac{1}{\sqrt{e}} e^4) \sim e \wedge e^5 \otimes e \wedge e^5$ according to [14] for every subrepresentation of $\Lambda E \otimes \Lambda E$ and so in particular for $\Lambda^{a,b} E$. In turn the homomorphism

$$\sigma_\circ(q(h \otimes \frac{1}{\sqrt{e}} e^5)) : \text{Sym}^k H \otimes \Lambda^{a,b} E \longrightarrow \text{Sym}^{k+1} H \otimes \Lambda^{a+1,b} E$$

for example equals the trivial homomorphism $\sigma_\circ(q(h \otimes \frac{1}{\sqrt{e}} e^5)) = 0$, because

$$\sigma_\circ \left( q(h \otimes \frac{1}{\sqrt{e}} e^5) \right)(\alpha \otimes \eta) := (h^\perp \alpha) \otimes \text{pr}_{\Lambda^{a+1,b} E} \left( (e \wedge \text{id}) q \left( \frac{1}{\sqrt{e}} e^4 \right) \eta \right) \sim (h^\perp \alpha) \otimes \text{pr}_{\Lambda^{a+1,b} E} \left( (e \wedge e \wedge e^5 \otimes e \wedge e^5) \eta \right) = 0$$

for all $\alpha \in \text{Sym}^k H$ and $\eta \in \Lambda^{a,b} E$, where $\text{pr}_{\Lambda^{a+1,b} E}$ denotes the isotypical projection to the Cartan summand $\Lambda^{a+1,b} E \subset \Lambda^{a+1} E \otimes \Lambda^b E$ in the tensor product of $\Lambda^{a+1} E$ and $\Lambda^b E$. Evidently the argument hinges on the statement that the operator $e \wedge e \wedge e^5 \otimes e \wedge e^5 = 0$ is
trivial; replacing it by the analogous statements $e^b \wedge e \wedge e^b \otimes e \wedge e^b = 0$ etc. the other seven cases are extremely similar and pose no further difficulties.

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