Quantum correlations for arbitrarily high-dimensional Bell inequality

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(Dated: February 2, 2008)

We analyze the correlation structure of bipartite arbitrary-dimensional Bell inequalities via novel conditions of correlations in terms of differences of joint probabilities called correlators. The conditions of correlations are shown to be necessary for the multi-level Bell state. In particular, we find that the bipartite arbitrary-dimensional Bell-type inequalities introduced by Collins-Gisin-Linden-Massar-Popescu [Phys. Rev. Lett. 88, 040404 (2002)] and Son-Lee-Kim [Phys. Rev. Lett. 96, 060406 (2006)] are composed of correlators, and we reveal that the maximal violations by the Bell state just fulfill the conditions of quantum correlations. Correlators can be considered as essential elements of Bell inequalities.

PACS numbers: 03.65.Ud,03.67.Mn

\section{I. INTRODUCTION}

The remarkable properties of entanglement goes essentially beyond the classical correlation constrained by two plausible assumptions, namely \textit{locality} and \textit{realism} (local realism) \cite{footnote1}. Local realism is also the central view of Einstein, Podolsky, and Rosen (EPR) \cite{EPR} on the quantum mechanics. The assumption of \textit{realism} states that outcomes of measurements are predeterministic, and the one of locality says that a measurement performed by one party of a system does not influence the result of the measurement performed by another party. By the assumptions of local realism, the \textit{Bell inequalities} \cite{Bell1, Bell2, Bell3} for two-level systems have been proposed to experimentally invalidate the point of view of EPR and to show that quantum mechanics is not locally realistic. For the aspect of quantum information processing \cite{footnote2}, the nonlocal features of quantum correlations enable people to perform high-security and novel quantum communication \cite{footnote3, footnote4}. Moreover, it helps to solve the problems that have no solutions in classical information theory \cite{footnote5}.

In addition to entanglement for quantum two-level systems (qubits), entangled quantum multi-level systems (qudits) attract much attention for their nonlocal characters \cite{footnote6, footnote7, footnote8} and advantages in quantum information processing \cite{footnote9}. Collins \textit{et al.} \cite{footnote9} have reformulate Bell inequalities to construct a large family of multi-level inequalities in terms of a novel constraint for local-realistic theories called Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequality. Recently, Son, Lee, and Kim (SLK) \cite{footnote10} presented generic Bell inequalities and their variants for arbitrary high-dimensional systems through the generalized GHZ nonlocality \cite{footnote11}.

In this work, we adopt a different approach to multi-level Bell inequalities. We address the following question. What are the essential properties of quantum correlations of entangled qudits that can be defined concretely and be detected efficiently. We wonder whether these essential features can be detected by Bell inequalities, i.e., whether the kernels of Bell inequalities consist some correlation conditions that are necessary for the entangled qudits. In order to attain this aim, we use novel conditions of correlations in terms of pairs of difference of joint probabilities called correlators to investigate correlation structure of multi-level Bell state and Bell inequalities. These correlators can be measured locally by each party, and by which the dependent properties of qudits can be revealed in different directions of measurements. In particular, we show that the CGLMP and SLK inequalities are comprised of conditions of quantum correlations in terms of correlators. In the following, an introduction to the conditions of correlations will be given as a preliminary to further discussions and results.

\section{II. CORRELATION CONDITION}

Before proceeding further, let us revisit the scenario of a two-party Bell-type experiment for identifying the correlations between outcomes of measurements. Therein, measurements on each spatially-separated particle are assumed to be performed with two distinct results from different observables. In each run of the experiment, the first observer chooses $V_1$ and the second one chooses $V_2$ for their local measurements on their particles respectively. After measurements, a set of results $v_1$ and $v_2$, which can be either 0 or 1, is acquired. If sufficient runs of such measurements have been made under the chosen local measurement setting, the correlation between experimental outcomes can be revealed through the analytical analysis of experimental records. In analogy, the multi-level Bell type experiments work in the same way as mentioned above. The key idea of this work is to utilize the correlation relations in terms of differences of joint probabilities, i.e. correlators. We define two correlators in terms of the differences of joint probabilities, which are given by

$$C_0 = P(0, 0) - P(1, 0) \quad \text{and} \quad C_1 = P(1, 1) - P(0, 1),$$

(1)
where \( P(v_1, v_2) \) denote the joint probabilities for obtaining the sets of results \((v_1, v_2)\) under a given local measurement setting. We can show that outcomes of measurements performed on a system composed of two uncorrelated parts must satisfy the following criteria:

\[
C_0 \geq 0 \quad \text{and} \quad C_1 \leq 0 \quad \text{or} \quad C_0 \leq 0 \quad \text{and} \quad C_1 \geq 0. \tag{2}
\]

To prove it, note that, for two uncorrelated parts, \( C_0 \) and \( C_1 \) can be recast as:

\[
C_0 = |P(0) - P(1)| P_2(0), \quad C_1 = |P(1) - P(0)| P_2(1),
\]

where \( P_k(v_k) \) represent the probabilities to get a specific measurement result for party \( k = 1, 2 \). Since \( P_k(v_k) \geq 0 \), thus we always get \( C_0 C_1 \leq 0 \), hence it ends the proof.

In analogy, we can formulate another set of correlators which is dual to the former one by \( C_m = P(0, 0) - P(0, 1) \) and \( C_1 = P(1, 1) - P(1, 0) \). Using these equations, we can also set a condition for correlation between measurements on pairs.

It is clear that, if the value of the product, \( C_0 C_1 \), is positive, one can assert that there must be correlations between the outcomes of the measurements in the composite system in some way. In the quantum regime, we consider \( C_0 \) and \( C_1 \) for a two-qubit pure entangled state:

\[
|\psi\rangle = \sin(\xi) |01\rangle_z \otimes |01\rangle_z + \cos(\xi) |11\rangle_z \otimes |11\rangle_z,
\]

where \( |v_1 v_2\rangle_z = |v_1\rangle_1 \otimes |v_2\rangle_2 \) is the eigenstate of Pauli-operator \( \sigma_z \) with eigenvalue \((-1)^{v_1}\) for the \( k\)th party. If the local measurement setting is chosen as \((\sigma_x, \sigma_z)\), we obtain a violation of the criterion (2) by

\[
C_0 = C_1 = \sin(2\xi)/2,
\]

and that \( C_m = 1 \) under the setting \((\sigma_x, \sigma_z)\). Thus, we have, \( C_0 = C_{(x)} + C_{(z)} = 1 + \sin(2\xi) \), for the state \( |\psi\rangle \). On the other hand, for the corresponding \( C_m \) for the separable state \( \rho_{ab} = \sin^2(\xi) |00\rangle_{zz} \otimes |00\rangle_{zz} + \cos^2(\xi) |11\rangle_{zz} \otimes |11\rangle_{zz} \), it also exhibits the same correlation in the results under the setting \((\sigma_x, \sigma_z)\) and is also equal to one. However, since the corresponding \( C_0 \) and \( C_1 \) are both zero under the local setting \((\sigma_x, \sigma_z)\), the correlation is erased.

The above scenario for telling entanglement involves the summation of criteria, \( C_0 \) and \( C_1 \), under two measurement settings. Since entanglement manifests itself via quantum correlation in different directions of measurements, it makes \( C_0 > C_0 \). For general cases, as will be discussed, we can prepare more settings of local measurements and introduce more terms with the same meanings as \( C_0 \) and \( C_1 \) to investigate quantum correlations embedded in entangled states and perform identification of their nonlocal properties further.

### III. Quantum Correlation of Bipartite Arbitrary-Dimensional Bell State

The necessary conditions constructed by Eq. (1) for a system composed of two-level independent pair can be extended to general ones. We give a set of correlators for a system composed of two \( d \)-level parts,

\[
C_m^{(12)} = P(v_1^{(1)} = -m, v_2^{(2)} = m) - P(v_1^{(1)} = 1 - m, v_2^{(2)} = m), \tag{3}
\]

for \( m = 0, 1, ..., d - 1 \), where \( \doteq \) denotes equality modulus \( d \). The superscripts, \((12)\), \((1)\), and \((2)\), mean that the first measurement, \( P_1^{(1)} \), and the second measurement, \( P_2^{(2)} \), have been selected from two choices of each party. For a system composed of two independent \( d \)-level parts, it must not satisfy the following condition: either \( C_m > 0 \) or \( C_m < 0 \) for all \( m \)'s. If either of the above conditions is satisfied by a bipartite system, there must be correlations between the measurement outcomes. Let us give a concrete example with \( d = 3 \) for above statement. From Eq. (3), we represent the correlators explicitly by

\[
C_m^{(12)} = P(0, 0) - P(1, 0), \quad C_m^{(1)} = P(2, 1) - P(0, 1),
\]

and furthermore if the particles composed of the bipartite system are independent we have the following relations between probabilities

\[
C_m^{(12)} = |P_1(0) - P_1(1)| P_2(0), \quad C_m^{(1)} = |P_2(2) - P_2(1)| P_1(1),
\]

If we have the results: \( P_1(1) > P_1(2) \) and \( P_1(2) > P_1(0) \), it turns out that \( P_1(0) < P_1(1) \) which means that it is impossible to have \( C_m^{(12)} > 0 \) for all \( m \)'s. Thus we can prove the statement for arbitrary \( d \) by the same way proposed above.

Other types of correlators similar to Eq. (3) can be readily formulated:

\[
C_m^{(21)} = P(v_1^{(2)} = -d - m, v_2^{(1)} = m) - P(v_1^{(2)} = m, v_2^{(1)} = m), \tag{4}
\]

\[
C_m^{(ii)} = P(v_1^{(i)} = -d - m, v_2^{(i)} = m) - P(v_1^{(i)} = m, v_2^{(i)} = m), \tag{5}
\]

for \( i = 1, 2 \).

Now, through the derived correlators, let us progress towards analysis of the correlation structure of the bipartite arbitrary-dimensional Bell state:

\[
|\psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{v=0}^{d-1} |v\rangle_{1z} \otimes |v\rangle_{2z}. \tag{6}
\]
We represent the state $|\psi_d\rangle$ in the following eigenbasis of some observable $V_{k}^{(q)}$,

$$
|l\rangle_{kq} = \frac{1}{d} \sum_{m=0}^{d-1} \exp[i \frac{2\pi m}{d} (d + n_k^{(q)})] |m\rangle_{kz}, \tag{7}
$$

and then the joint probabilities for obtaining the measured outcome $(v_1^{(i)}, v_2^{(j)})$ for the state $|\psi_d\rangle$ are given by

$$
P_{\psi_d}(v_1^{(i)}, v_2^{(j)}) = \frac{1}{2d^3 \sin^2 \left(\frac{\pi}{d} \left( v_1^{(i)} + v_2^{(j)} + n_1^{(i)} + n_2^{(j)} \right) \right)}, \tag{8}
$$

where $(n_1^{(i)}, n_2^{(j)})$ denote the local parameters of the local measurement settings $(V_1^{(i)}, V_2^{(j)})$. For the set of local parameters given by

$$
n_1^{(1)} = 0, n_2^{(1)} = 1/4, n_1^{(2)} = 1/2, n_2^{(2)} = -1/4, \tag{9}
$$

$C_m^{(ij)}$ can be evaluated analytically, and we arrive at

$$
C_{m,\psi_d}^{(ij)} = \frac{1}{2d^2} \left[ \csc^2 \left( \frac{\pi}{4d} \right) - \csc^2 \left( \frac{3\pi}{4d} \right) \right], \tag{10}
$$

for $i, j = 1, 2$. Since $C_m^{(ij)} > 0$ for all $m$’s with any finite value of $d$, we ensure that outcomes of measurements performed on the particles of the state $|\psi_d\rangle$ are dependent under four different local measurement settings. Thus, we can consider each set of the condition $C_m^{(ij)} > 0$ as a necessary one of the Bell state $|\psi_d\rangle$, and hence the corresponding correlation structure of $|\psi_d\rangle$ could be specified concretely and analytically via the correlators.

Furthermore, to compare the correlation embedded in $|\psi_d\rangle$ predicted by quantum mechanics with the one by local-realistic theories, we could utilize the necessary conditions proposed above to achieve this aim. First, we combine all of the correlators involved in the necessary conditions of $|\psi_d\rangle$ and evaluate the summation of all $C_m^{(ij)}$’s,

$$
C_d = C^{(11)} + C^{(12)} + C^{(21)} + C^{(22)}, \tag{11}
$$

where $C^{(ij)} = \sum_{m=0}^{d-1} C_m^{(ij)}$. Then we have

$$
C_{d,\psi_d} = \frac{2}{d^2} \left[ \csc^2 \left( \frac{\pi}{4d} \right) - \csc^2 \left( \frac{3\pi}{4d} \right) \right]. \tag{12}
$$

One can find that $C_{d,\psi_d}$ is an increasing function of $d$. For instance, if $d = 3$, one has $C_{3,\psi_d} \approx 2.87293$. In the limit of large $d$, we obtain, $\lim_{d \to \infty} C_{d,\psi_d} = (16/3\pi)^2 \approx 2.88209$.

We proceed to consider the maximum value of $C_d$ by local-realistic theories. The following derivation is based on deterministic local models, since any probabilistic model can be converted into a deterministic one. We substitute a chosen set, $(v_1^{(1)}, v_2^{(1)}, v_1^{(2)}, v_2^{(2)})$, into $C^{(ij)}$, and $C_d$ turns into

$$
C_{d,LR} = \delta[(v_1^{(1)} + v_2^{(1)}) \mod d, 0] - \delta[-(v_1^{(1)} + v_2^{(1)}) \mod d, 1] + \delta[(v_1^{(2)} + v_2^{(2)}) \mod d, 0] - \delta[-(v_1^{(2)} + v_2^{(2)}) \mod d, 1] + \delta[-(v_1^{(1)} + v_2^{(1)}) \mod d, 1] - \delta[(v_1^{(2)} + v_2^{(1)}) \mod d, 0], \tag{13}
$$

where $\delta[x, y]$ represents the Kronecker delta symbol. It is apparent that there are three non-vanishing terms at most among the four positive delta functions under some specific condition for $(v_1^{(1)}, v_2^{(1)}, v_1^{(2)}, v_2^{(2)})$. We also know that there must exist one non-vanishing negative delta function in $C_{d,LR}$ under the same condition. Therefore, in the regime governed by local-realistic theories, the value of $C_{d,LR}$ is bounded by 2, i.e., $C_{d,LR} \leq 2$.

From the above discussions, we realize that $C_{d,\psi_d} > C_{d,LR}$. Therefore, the quantum correlations are stronger than the ones predicted by the local-realistic theories. With this fact, the derived equation $C_d$ can be utilized to tell quantum correlations from classical ones. For $d = 2$, $C_{2,\psi_2} = 2\sqrt{2}$ and the equation $C_d$ is the same as that in the CHSH $[3]$ inequality. Moreover, the $C^{(ij)}$ terms are just the expectation values of the outcome products which appear in the CHSH inequality. Then, we can reinterpret the correlation functions as a summation of all $C_0^{(ij)}$ and $C_1^{(ij)}$ which formulate correlation criteria for measurements on pairs. This idea can be applied to arbitrary high-dimensional systems and to construct new types of correlation functions, $C^{(ij)}$. Although the values of maximal quantum violation are slightly smaller than the ones derived by Collins et al. $[5]$ and Fu $[14]$, the total number of joint probabilities required by each of the presented correlation functions $C^{(ij)}$ is only $2d$, which is much smaller than that in Fu’s general correlation function, which is about $O(d^2)$. It implies that the proposed correlation functions include the essential parts of quantum correlation of the state $|\psi_d\rangle$.

Another feature of the sum of all correlators will be discussed here is its robustness to noise. If the state $|\psi_d\rangle$ suffered from white noise and turns into a mixed one in the form

$$
\rho = p_{\text{noise}}/d^2 \mathbb{I} + (1 - p_{\text{noise}}) |\psi_d\rangle \langle \psi_d|, \tag{14}
$$

where $\rho$ describes the noise fraction, the value of $C_d$ for state $\rho$ becomes $C_{d,\rho} = (1 - p_{\text{noise}})C_{d,\psi_d}$. If the criterion, $C_{d,\rho} > 2$, i.e., $p_{\text{noise}} < 1 - 2/C_{d,\psi_d}$, is imposed on the system, one ensures that the mixed state still exhibits quantum correlations in outcomes of measurements. For instance, to maintain the quantum correlation for the limit of large $d$, the system must have $p_{\text{noise}} < 0.30604$.

Through the work by Masanes about tightness of Bell inequality from a geometric point of view $[15]$, we have examined our Bell-type inequality. The result shows that
the inequality is non-tight, i.e., it is not an optimal detector of non-local-realistic correlation. The detailed proof and discussions are given in the appendix.

IV. CORRELATION STRUCTURE OF CGLMP INEQUALITY

Let us introduce more correlators like \( C^{(ij)} \) to describe the quantum correlations of \(|\psi_d\rangle\). The first four sets of correlators could be the one with the following form:

\[
\begin{align*}
C^{(ii)}_{m0} &= P(v_1^i = m, v_2^i = m) \\
C^{(12)}_{m0} &= P(v_1^i = m, v_2^i = m) \\
C^{(21)}_{m0} &= P(v_1^i = m, v_2^i = m) \\
C^{(i)}_{m0, \psi_d} &= \frac{1}{2d^3} \left[ \csc^2 \left( \frac{\pi}{4d} \right) - \csc^2 \left( \frac{3\pi}{4d} \right) \right],
\end{align*}
\]

which is the same with \( C^{(ij)}_{m, \psi_d} \) and \( C^{(ij)}_{m0, \psi_d} > 0 \) for all \( m \)’s with any finite \( d \). Thus we know that the particles of the pair are dependent on each other. The second four sets of correlators are introduced by

\[
\begin{align*}
C^{(ii)}_{m1} &= P(v_1 = m + 1, v_2^i = m) \\
C^{(12)}_{m1} &= P(v_1^i = m + 1, v_2^i = m) \\
C^{(21)}_{m1} &= P(v_1^i = m + 1, v_2^i = m) \\
C^{(i)}_{m1, \psi_d} &= \frac{1}{2d^3} \left[ \csc^2 \left( \frac{5\pi}{4d} \right) - \csc^2 \left( \frac{7\pi}{4d} \right) \right],
\end{align*}
\]

and the corresponding expectation values for the state \(|\psi_d\rangle\) are

\[
C^{(ij)}_{m1, \psi_d} = \frac{1}{2d^3} [\csc^2 \left( \frac{5\pi}{4d} \right) - \csc^2 \left( \frac{7\pi}{4d} \right)],
\]

and are strictly greater than zero for all \( m \)’s with any finite \( d \). Then we obtain another four sets of correlators which could be utilized to describe the dependence of the entangled pair.

Furthermore, let us progress towards to general sets of correlators which are formulated by

\[
\begin{align*}
C^{(ii)}_{mk} &= P(v_1^i = m + k, v_2^i = m) \\
C^{(12)}_{mk} &= P(v_1^i = m + k - 1, v_2^i = m) \\
C^{(21)}_{mk} &= P(v_1^i = m + k + 1, v_2^i = m) \\
C^{(i)}_{mk, \psi_d} &= \frac{1}{2d^3} \left[ \csc^2 \left( \frac{(1 + 4k)\pi}{4d} \right) - \csc^2 \left( \frac{(3 + 4k)\pi}{4d} \right) \right].
\end{align*}
\]

for \( k = 0, \ldots, \lfloor d/2 \rfloor \). We deduce that the particles composed of the Bell state are indeed dependent from the positive expectation values of correlators with the following general forms:

\[
C^{(ij)}_{mk, \psi_d} = \frac{1}{2d^3} \left[ \csc^2 \left( \frac{(1 + 4k)\pi}{4d} \right) - \csc^2 \left( \frac{(3 + 4k)\pi}{4d} \right) \right].
\]

Thus we can feature the quantum correlations embedded in the bipartite \( d \)-level Bell state in the \( 4\lfloor d/2 \rfloor \) sets of correlators. Thus we could take a linear combination of all of these sets of correlators as a mean of identification:

\[
C_d = \sum_{k=0}^{\lfloor d/2 \rfloor} \sum_{i,j=1}^{d-1} f(k) C^{(ij)}_{mk, \psi_d},
\]

where \( f(k) \) denotes the coefficient of combination which is function of \( k \).

If we let \( f(k) \) be

\[
f(k) = 1 - \frac{2k}{d - 1},
\]

the summation of all of the correlators \( C_d \) becomes the kernel of the CGLMP inequality [3]:

\[
C_{CGLMP} = \sum_{k=0}^{\lfloor d/2 \rfloor} \sum_{i,j=1}^{d-1} (1 - \frac{2k}{d - 1}) C^{(ij)}_{mk, \psi_d},
\]

where \( C^{(ij)}_{k} = \sum_{m=0}^{d-1} C^{(ij)}_{mk, \psi_d} \). The local realistic constraint proposed by Collins et al. [9] specifies that the correlations exhibited by local realistic theories have to satisfy the condition:

\[
C_{CGLMP,LR} \leq 2.
\]

On the other hand, by Eq. (20), quantum correlations of the Bell state will give a violation of the CGLMP inequality for arbitrary high-dimensional systems. Thus, through Eqs. (20) and (21) and the related discussions, we realize that the CGLMP inequality is composed of correlators for correlations and know that the corresponding violations for the Bell state just fulfill the conditions of quantum correlations of the entangled pair.
V. CORRELATION STRUCTURE OF SLK INEQUALITY

From the discussions in the previous sections, we could realize that the features of entanglement of the Bell state are described by sets of correlators with positive expectation values. Hence, we could generalize the formulations of correlators for describing quantum correlations by the following specification. The entanglement of the bipartite $d$-level Bell state is featured in the correlators under different measurement settings:

$$ C_{m}^{(l)}(\alpha, \beta) = P(v_1 \doteq m + \alpha, v_2 = m) - P(v_1 \doteq m + \beta, v_2 = m), $$

for $m = 0, ..., d - 1$, where $l = [i, j]$ stands for measurement setting and $\alpha$ and $\beta$ are real numbers, and, most importantly, the values of correlators for $|\psi_d\rangle$ strictly fulfill the criterion

$$ C_{m,\psi_d}^{(l)}(\alpha, \beta) > 0, \text{ for } m = 0, ..., d - 1, $$

or

$$ C_{m,\psi_d}^{(l)}(\alpha, \beta) < 0, \text{ for } m = 0, ..., d - 1. $$

(26)

(27)

To have a compact form, it should be noted that we have omitted the denotations of measurement setting $i, j$ from the measured outcomes $v_k$’s. A linear combination of these correlators is utilized to identify the state $|\psi_d\rangle$:

$$ \tilde{C}_d = \sum_{l} \sum_{\alpha, \beta} f_l(\alpha, \beta) C^{(l)}(\alpha, \beta) $$

(28)

where $C^{(l)}(\alpha, \beta) = \sum_{m=0}^{d-1} C_{m}^{(l)}(\alpha, \beta)$ and the coefficient of combination, $f_l(\alpha, \beta)$, depends on $\alpha$, $\beta$, and $l$.

Let us give a concrete example to show above formulation by the following sum of correlators:

$$ \sum_{l} \sum_{\alpha=0}^{d-1} \sum_{m=0}^{d-1} f_l(\alpha) \sum_{m=0}^{d-1} P(v_1 \doteq m + \alpha, v_2 = m), $$

(29)

where

$$ f_l(\alpha) = \sin(2\alpha l \pi)[\cot(\alpha l \pi/d) - \cot(\alpha l \pi)]/4, $$

$$ \alpha = \nu + \nu_l, $$

(30)

(31)

$\nu$, and $\nu_l$ are constants. It is worth to note that

$$ \sum_{\alpha=0}^{d-1} f_l(\alpha) = 0, $$

(32)

which indicates that the sum of positive $f_l$’s and negative ones is zero and implies that one can always have the following relation:

$$ \sum_{\alpha=0}^{d-1} f_l(\alpha) \sum_{m=0}^{d-1} P(v_1^{(i)} \doteq m + \alpha, v_2^{(j)} = m) = (d - 1)/4, $$

for the Bell state, the value of the SLK kernel is

$$ C_{SLK,\psi_d} = d - 1. $$

(36)

(37)

Son et al. [10] have shown that local-realistic theories predict the value of the kernel by

$$ C_{SLK,LR} \leq \frac{1}{4} \left[ 3 \cot(\frac{\pi}{4d}) - \cot(\frac{\pi}{3d}) \right] - 1, $$

(38)

which is called the SLK inequality. Thus the SLK inequality can be violated by the Bell state by a factor:

$$ \lim_{d \rightarrow \infty} \frac{C_{SLK,LR}}{C_{SLK,\psi_d}} = \frac{8}{3\pi}. $$

(39)
for arbitrary-high dimension.

VI. SUMMARY

In this work, we have analyzed the structures of Bell inequalities for bipartite multi-level systems by conditions of correlations in terms of correlators. We start with an investigation into the correlation properties of the multi-level Bell state, and then we give specifications of the correlation structure in terms of correlators. Through these correlators for the Bell state, we construct Bell inequalities with fewer analyses of measured outcomes. We also show that the CGLMP [9] and SLK [10] inequalities are composed of correlation conditions in terms of correlators. From the quantum mechanical point of view, we reveal that correlators are the essential elements of the Bell inequalities for arbitrarily high-dimentional systems.

Note added.—During preparation of our manuscript, we were aware of one related structure of Bell inequalities for $d$-level bipartite systems [13].

APPENDIX: TIGHTNESS OF BELL INEQUALITIES

Every tight Bell inequality fulfills the following conditions [13]:

**Condition 1.** All the generators of the convex polytope must belong either to the half-space or to the hyperplane.

**Condition 2.** There must be $4d(d-1)$ linear independent generators among the ones that belong to the hyperplane.

On the other hand, non-tight Bell inequalities satisfy only Condition 1. Then, we will examine the proposed BI by these conditions for tightness.

Firstly, we discuss condition 1 for the inequality. Although the same result has been shown in our paper, i.e., the derivation of the bound of the propose Bell inequality, we follow the approach presented by Masanes [15] for completeness. The summation of all correlators of quantum correlation can be written as:

$$C_d = P(v^{(1)}_1 + v^{(1)}_2 = 0) - P(v^{(1)}_1 + v^{(1)}_2 = -1) + P(v^{(1)}_1 + v^{(2)}_2 = 0) - P(v^{(1)}_1 + v^{(2)}_2 = 1) + P(v^{(2)}_1 + v^{(2)}_2 = 0) - P(v^{(2)}_1 + v^{(2)}_2 = -1) + P(v^{(2)}_1 + v^{(2)}_2 = -1) - P(v^{(2)}_1 + v^{(2)}_2 = 0),$$

(A.1)

To have an explicit form of $C_d$ for further discussion, we define the following variables:

$$\chi_{11} = v^{(1)}_1 + v^{(1)}_2 + \hat{d}_{11},$$
$$\chi_{12} = -v^{(1)}_1 - v^{(2)}_2 + \hat{d}_{12},$$
$$\chi_{22} = v^{(2)}_1 + v^{(2)}_2 + \hat{d}_{22},$$
$$\chi_{21} = -v^{(2)}_1 - v^{(1)}_2 - 1 + \hat{d}_{21},$$

(A.2)

where $\hat{d}_{ij}$ denotes a multiple of $d$ and $\chi_{ij} \in \{-1,0\}$ for $i,j=1,2$. In particular, the sum of the variables satisfies the constrain:

$$\sum_{i,j=1}^{2} \chi_{ij} = -1.$$  

(A.3)

With the defined variables, $C_d$ is written as:

$$C_d = \sum_{ij=1}^{2} P(\chi_{ij} = 0) - P(\chi_{ij} = -1).$$  

(A.4)

Next, we proceed to consider the extreme values of $C_d$ under the local realistic theories. The all possible sets of $(\chi_{11}, \chi_{12}, \chi_{22}, \chi_{21})$ which fulfill the constraint of the sum of the variables are as the following:

(i) three of the variables are 0 and the rest is $-1$;
(ii) all of the variables are $-1$.

The first class can be applied to arbitrary $d$, and, however, the second one is only applicable for $d = 3$. Thus, we have $C_{d,LHV} = 2$ for the class (i) and $C_{3,LHV} = -4$ for (ii), which mean that for all the generators of the convex polytope for $C_{d,LHV}$ the value $C_{d,LHV}$ is equal or less than 2. Thus the proposed Bell inequality fulfills the first condition.

Second, we consider the second condition for the Bell inequality. All the generators of the convex polytope are written as:

$$G = \left| v^{(1)}_1, v^{(2)}_1 \right\rangle \oplus \left| v^{(1)}_1, v^{(2)}_1 + v^{(1)}_2 \right\rangle \oplus \left| v^{(2)}_1, v^{(1)}_2 \right\rangle \oplus \left| v^{(2)}_1, v^{(2)}_2 \right\rangle,$$

(A.5)

which, with the defined variables, can also be represented as the following:

$$\left| v^{(1)}_1, \chi_{11} - v^{(1)}_1 \right\rangle \oplus \left| v^{(1)}_1, -\chi_{12} - v^{(1)}_1 \right\rangle$$
$$\oplus \left| v^{(1)}_1 - \chi_{11} - \chi_{21} - 1, \chi_{11} - v^{(1)}_1 \right\rangle$$
$$\oplus \left| v^{(1)}_1 + \chi_{12} + \chi_{22}, -\chi_{12} - v^{(1)}_1 \right\rangle,$$

(A.6)

where $\left| v^{(i)}_1, v^{(j)}_2 \right\rangle = \left| v^{(i)} \mod d, v^{(j)} \mod d \right\rangle$. Through a linear transformation with the preservation of orthogonality which is given by:

$$\sum_{v,k=0}^{d-1} (|v, \chi_{11} \rangle \langle v, \chi_{11} - v| \oplus |v, \chi_{12} \rangle \langle v, -\chi_{12} - v|$$
$$\oplus |v - \chi_{11}, \chi_{21} \rangle \langle v - \chi_{11} - \chi_{21} - 1, \chi_{11} - v|$$
$$\oplus |v + \chi_{12}, \chi_{22} \rangle \langle v + \chi_{12} + \chi_{22}, -\chi_{12} - v|),$$

(A.7)
\[ G \] can be transformed into
\[ |v_1^{(1)} \rangle \chi_{11} \oplus |v_1^{(1)} \rangle \chi_{12} \]
\[ \oplus |v_1^{(1)} - \chi_{11}, \chi_{21} \rangle \oplus |v_1^{(1)} + \chi_{12}, \chi_{22} \rangle. \] \( \text{(A.8)} \)

The generators which satisfy \( C_{d, LHV} = 2 \) are the ones with variables belonging to (i). Thus, the generators contained in the hyperplane are shown as:
\[ |v, -1 \rangle \oplus |v, 0 \rangle \oplus |v + 1, 0 \rangle \oplus |v, 0 \rangle, \] \( \text{(A.9)} \)
\[ |v, 0 \rangle \oplus |v, -1 \rangle \oplus |v, 0 \rangle \oplus |v - 1, 0 \rangle, \] \( \text{(A.10)} \)
\[ |v, 0 \rangle \oplus |v, 0 \rangle \oplus |v, -1 \rangle \oplus |v, 0 \rangle, \] \( \text{(A.11)} \)
\[ |v, 0 \rangle \oplus |v, 0 \rangle \oplus |v, 0 \rangle \oplus |v, -1 \rangle, \] \( \text{(A.12)} \)

for \( v \in \{0, 1, ..., d - 1\} \). The total number of linear independent generators is \( 4d \) which is smaller than \( 4d(d - 1) \) involved in the condition of tightness. Then the proposed Bell inequality is non-tight.

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