No nonminimally coupled massless scalar hair for spherically symmetric neutral black holes

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**Abstract**

We provide a remarkably compact proof that spherically symmetric neutral black holes cannot support static nonminimally coupled massless scalar fields. The theorem is based on causality restrictions imposed on the energy-momentum tensor of the fields near the regular black-hole horizon.
I. INTRODUCTION

The non-linearly coupled Einstein-scalar field equations have attracted the attention of physicists and mathematicians for more than five decades. Interestingly, the composed Einstein-scalar system is characterized by very powerful and elegant no-hair theorems \[1–3\] which rule out the existence of asymptotically flat black-hole solutions with regular event horizons that support various types of scalar (spin-0) matter configurations.

The early no-hair theorems of Chase \[4\] and Bekenstein \[5\] have ruled out the existence of regular black holes supporting static minimally coupled massless scalar field configurations. The no-hair theorems of Bekenstein \[5\] and Teitelboim \[6\] have excluded the existence of black-hole hair made of minimally coupled massive scalar fields \[7–9\]. Later no-hair theorems of Heusler \[10\] and Bekenstein \[11\] have ruled out the existence of neutral black-hole spacetimes supporting static matter configurations made of minimally coupled scalar fields with positive semidefinite self-interaction potentials.

The physically interesting regime of scalar fields nonminimally coupled to gravity has been investigated by several authors. In a very interesting paper, Mayo and Bekenstein \[12\] have proved that spherically symmetric charged black holes cannot support matter configurations made of charged scalar fields nonminimally coupled to gravity with generic values of the dimensionless coupling parameter \(\xi\) [The physical parameter \(\xi\) quantifies the nonminimal coupling of the field to gravity, see Eq. (4) below]. Intriguingly, the rigorous derivation of a no-hair theorem for neutral scalar fields nonminimally coupled to gravity seems to be a mathematically more challenging task. In particular, the important no-hair theorems of \[12, 13\] can be used to rule out the existence of spherically symmetric scalar hairy configurations in the restricted physical regimes \(\xi < 0\) and \(\xi \geq 1/2\) \[14\].

The main goal of the present paper is to present a (remarkably compact) unified no-hair theorem for neutral massless scalar fields nonminimally coupled to gravity which is valid for generic values of the dimensionless coupling parameter \(\xi\) (In particular, below we shall extend the interesting no-scalar hair theorems of \[12\] and \[13\] to the physical regime of nonminimally coupled neutral scalar fields with \(0 < \xi < 1/2\)). Our theorem, to be proved below, is based on simple physical restrictions imposed by causality on the energy-momentum tensor of the fields near the regular horizon of the black-hole spacetime.
II. DESCRIPTION OF THE SYSTEM

We consider a non-linear physical system composed of a neutral black hole of horizon radius $r_H$ and a massless scalar field $\psi$ with nonminimal coupling to gravity. The composed black-hole-scalar-field system is assumed to be static and spherically symmetric, in which case the spacetime outside the black-hole horizon is characterized by the curved line element \[ ds^2 = -e^{\nu}dt^2 + e^{\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \] (We shall use natural units in which $G = c = 1$)

where $\nu = \nu(r)$ and $\lambda = \lambda(r)$ [Here $(t, r, \theta, \phi)$ are the Schwarzschild coordinates]. As explicitly proved in [12], regardless of the matter content of the curved spacetime, a non-extremal regular black hole is characterized by the near-horizon relations \[ e^{-\lambda} = L \cdot x + O(x^2) \] where $x \equiv \frac{r - r_H}{r_H}$ ; $L > 0$ \[ \nu' r_H = -\frac{1}{x} + O(1) \] ; $\nu' r_H = \frac{1}{x} + O(1)$ .

The curved black-hole spacetime is non-linearly and non-minimally coupled to a real massless scalar field $\psi$ whose action is given by \[ S = S_{EH} - \frac{1}{2} \int (\partial_{\alpha} \psi \partial^{\alpha} \psi + \xi R \psi^2) \sqrt{-g} d^4x , \] where the dimensionless physical parameter $\xi$ quantifies the strength of the nonminimal coupling of the field to gravity, $R(r)$ is the scalar curvature of the spacetime, and $S_{EH}$ is the Einstein-Hilbert action. As explicitly shown in [12], in the near-horizon $x \ll 1$ region, $R$ is given by the simple expression \[ R = \frac{4L - 2}{r_H^2} \cdot [1 + O(x)] . \]

From the action (4) one finds the characteristic differential equation \[ \partial_{\alpha} \partial^{\alpha} \psi - \xi R \psi = 0 \] for the nonminimally coupled scalar field. Using the metric components (1) of the curved black-hole spacetime, one can express the scalar radial equation in the form \[ \psi'' + \frac{1}{2} \left( \frac{4}{r} + \nu' - \lambda' \right) \psi' - \xi Re^\lambda \psi = 0 . \]
(Here a prime ′ denotes a spatial derivative with respect to the radial coordinate $r$).

The action (4) also yields the compact expressions

$$T^t_t = e^{-\lambda \xi(4/r - \lambda')\psi\psi' + (2\xi - 1/2)(\psi')^2 + 2\xi\psi\psi''}$$

and

$$T^t_t - T^\phi_\phi = e^{-\lambda \xi(2/r - \nu')\psi\psi'}$$

for the components of the energy-momentum tensor. As explicitly proved in [12], regardless of the matter content of the theory, a regular hairy black-hole spacetime must be characterized by finite mixed components of the energy-momentum tensor:

$$\{|T^t_t|, |T^r_r|, |T^\theta_\theta|, |T^\phi_\phi|\} < \infty .$$

In addition, it was proved in [12] that causality requirements enforce the characteristic inequalities [a]

$$|T^\theta_\theta| = |T^\phi_\phi| \leq |T^t_t| \geq |T^r_r|$$

on the components of the energy-momentum tensors of physically acceptable systems. Note that the relations [12]

$$\text{sgn}(T^t_t) = \text{sgn}(T^t_t - T^r_r) = \text{sgn}(T^t_t - T^\phi_\phi)$$

provide necessary conditions for the validity of the characteristic energy conditions (11).

III. THE NO-HAIR THEOREM FOR STATIC NONMINIMALLY COUPLED MASSLESS SCALAR FIELDS

In the present section we shall explicitly prove that a spherically symmetric non-extremal neutral black hole cannot support non-linear hair made of static nonminimally coupled massless scalar fields.

[a] As explicitly shown by Bekenstein and Mayo [12], for spherically symmetric spacetimes one can write $\epsilon = -T^t_t - \sum_{i=1}^3 c_i^2(T^t_t - T^i_i)$ and $j^\mu j_\mu = -(T^t_t)^2 - \sum_{i=1}^3 c_i^2[(T^t_t)^2 - (T^i_i)^2]$, where $\epsilon \equiv T_{\mu\nu}u^\mu u^\nu$ and $j^\mu \equiv -T^\mu_\nu u^\nu$ are respectively the energy density and the Poynting vector according to a physical observer with a 4-velocity $u^\nu$, and the coefficients $\{c_i\}_{i=0}^3$ are characterized by the normalization condition $-c_0^2 + \sum_{i=1}^3 c_i^2 = -1$ (this relation guarantees that $u^\mu u_\mu = -1$ [12]). For physically acceptable systems in which the transfer of energy is not superluminal, the energy density should be of the same sign as $-T^t_t$ and the Poynting vector should be non-spacelike ($j^\mu j_\mu \leq 0$) for all observers [12] (that is, for all choices of the coefficients $\{c_i\}_{i=0}^3$). These physical requirements yield the characteristic energy conditions (11).
We start our proof with the scalar radial equation (7) which, in the near-horizon $x \ll 1$ region, can be written in the form [see Eqs. (2), (3), and (5)]

$$\frac{d^2 \psi}{dx^2} + \frac{1}{x} \frac{d \psi}{dx} + \frac{\beta}{x} \psi = 0 \quad ; \quad \beta \equiv \xi(2 - 4L)/L . \quad (13)$$

The general mathematical solution of the ordinary differential equation (13) can be expressed in terms of the familiar Bessel functions (see Eq. 9.1.53 of [16])

$$\psi(x) = A \cdot J_0(2\beta^{1/2}x^{1/2}) + B \cdot Y_0(2\beta^{1/2}x^{1/2}) \quad \text{for} \quad x \ll 1 , \quad (14)$$

where \( \{A, B\} \) are constants. Using Eqs. 9.1.8 and 9.1.12 of [16], one finds the asymptotic near-horizon behavior

$$\psi(x \to 0) = A \cdot [1 - \beta x + O(x^2)] + B \cdot [\pi^{-1} \ln(\beta x) + O(1)] \quad (15)$$

of the radial scalar function. Substituting Eqs. (2), (3), and (15) into Eq. (8) and taking cognizance of the energy condition (10) [12], one immediately realizes that the coefficient of the singular term in the asymptotic near-horizon solution (15) should vanish [17]:

$$B = 0 . \quad (16)$$

We therefore find that the nonminimally coupled scalar field is characterized by the near-horizon behavior

$$\psi(x \ll 1) = A \cdot J_0(2\beta^{1/2}x^{1/2}) . \quad (17)$$

Substituting (17) into (8) and (9) and using the near-horizon relations (2) and (3), one obtains the simple expressions

$$T_t^t = \xi \cdot \frac{L\psi\psi'}{r_H(1 - 8\pi \xi \psi^2)} \cdot [1 + O(x)] \quad (18)$$

and

$$T_t^t - T_\phi^\phi = -\xi \cdot \frac{L\psi\psi'}{r_H(1 - 8\pi \xi \psi^2)} \cdot [1 + O(x)] \quad (19)$$

for the components of the energy-momentum tensor in the near-horizon $x \ll 1$ region. We immediately find from (18) and (19) the near-horizon relation

$$T_t^t = -(T_t^t - T_\phi^\phi) , \quad (20)$$

in contradiction with the characteristic relation (12) imposed by causality on the energy-momentum components of physically acceptable systems.
IV. SUMMARY

In this compact analysis, we have proved that if a spherically symmetric neutral black hole can support non-linear configurations made of nonminimally coupled massless scalar fields, then in the near-horizon \((r - r_H)/r_H \ll 1\) region the energy momentum components of the fields must be characterized by the relation \(T^t_t = -(T^t_t - T^\phi_\phi)\) [see Eq. (20)]. However, one realizes that this near-horizon behavior is in contradiction with the characteristic relation \(\text{sgn}(T^t_t) = \text{sgn}(T^t_t - T^\phi_\phi)\) [see Eq. (12)] which, as explicitly proved in [12], is imposed by causality on the energy-momentum components of generic physically acceptable systems. Thus, there are no physically acceptable solutions for the eigenfunction of the external nonminimally coupled massless scalar fields except the trivial one, \(\psi \equiv 0\).

We therefore conclude that spherically symmetric neutral black holes cannot support static configurations made of nonminimally coupled massless scalar fields with generic values of the dimensionless physical parameter \(\xi\).

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generic values of the dimensionless coupling parameter $\xi$ and for generic values of the scalar field eigenfunction $\psi(r)$.

[15] As shown in [12], the expansion coefficient is given by $L \equiv 1 + 8\pi T \frac{r^2}{r_H^2} > 0$, where $-T(r_H)$ is the energy density of the matter fields at the black-hole horizon.

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[17] It is worth noting that, for $\beta = 0$, one finds from (13) the asymptotic near-horizon functional behavior $\psi(x \to 0) = A \cdot \ln(x) + B$, where $\{A, B\}$ are constants. Substituting this expression into (8) and taking cognizance of the energy condition (10) [12], one immediately realizes that, for $A \neq 0$, this near-horizon scalar function is physically unacceptable. One therefore concludes that, for physically acceptable spacetimes, the scalar eigenfunction is strictly constant. Substituting $\psi = B$ into Eqs. (8) and (9), one finds that the components of the energy momentum tensor are identically zero. Thus, the constant $B$ has no influence on physical quantities and one may therefore take $B = 0$ without loss of generality.