Dissipative cosmological solutions

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Abstract

The exact general solution to the Einstein equations in a homogeneous Universe with a full causal viscous fluid source for the bulk viscosity index $m = 1/2$ is found. We have investigated the asymptotic stability of Friedmann and de Sitter solutions, the former is stable for $m \geq 1/2$ and the latter for $m \leq 1/2$. The comparison with results of the truncated theory is made. For $m = 1/2$, it was found that families of solutions with extrema no longer remain in the full case, and they are replaced by asymptotically Minkowski evolutions. These solutions are monotonic.

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1 Introduction

It is believed that quantum effects played a fundamental role in the early Universe. For instance, vacuum polarisation and particle production arise from a quantum description of matter. However it is known that both of them can be modeled in terms of a classical bulk viscosity [1]. Other processes capable of producing important dissipative stresses include interactions between matter and radiation [2], quarks and gluons [3], different components of dark matter [4], and those mediated by massive particles. It happens also due to the decay of massive superstrings modes [5], gravitational string production [6] [7] and phase transitions.

Cosmological models with a viscous fluid have been studied by several authors. Some of the interesting subjects addressed by them were the effects of viscous stresses on the avoidance of the initial singularity [8], the dissipation of a primordial anisotropy [9], the production of entropy [2], inflation and deflation [10]. The evolution of a homogeneous isotropic spatially-flat universe filled with a causal viscous fluid, whose bulk viscosity coefficient is related to the energy density by the power law $\zeta \sim \rho^{\alpha}$, and a truncated version of the transport equation for the viscous pressure, has been investigated thoroughly [11] [12] [13]. This model was based in the relativistic second-order theory of non-equilibrium thermodynamics developed by Israel and Steward [14]. Their formulation employs fiducial local equilibrium magnitudes like thermodynamical pressure and temperature, related by an equilibrium Gibbs equation, but generalizes the expression of the entropy flux by the inclusion of quadratic terms in dissipative non-equilibrium magnitudes like bulk-viscosity pressure and heat flux. An alternative formulation of this theory, called Extended Irreversible Thermodynamics has been made by Pavón, Jou and co-workers [15] [16]. Here, non-equilibrium magnitudes like temperature and pressure are introduced instead, and a generalized Gibbs equation including the dissipative magnitudes is employed. The equivalence between both formalisms up to second-order departure from equilibrium has been demonstrated [17].

Recently, it was shown that the truncated version of the transport equation leads to pathological behaviour in the late universe, as the thermodynamical temperature increases with the expansion, leading to an unphysical heating of the Universe [18]. Then, some authors have begun to employ the full version of the transport equation for the bulk viscous pressure, and made
comparisons with the results obtained with the truncated one \[19\] \[20\] \[21\] \[18\] \[22\] \[23\] \[24\] \[25\] \[26\]. One important issue that arises in this context is the choice of the equations of state. For a dissipative Boltzmann gas it was shown that inflationary solutions appear not to arise, and it was conjectured that the inflationary solutions of the truncated causal theory are spurious and an artificial consequence of truncation \[19\]. In \[27\] we have adopted also the full version of the transport equation but we have taken the simple forms of the equations of state used in \[24\] and \[18\], and we have classified the possible behaviors near a singularity. In the present paper we complete the analysis of this cosmological model. We present the set of equations that describe the model in section 2. In section 3 we solve them when \(m = 1/2\) and give a detailed analysis of their solutions, including an analysis of the energy conditions \[28\]. In section 4 we study of stability of those asymptotic solutions that appear for \(m \neq 1/2\) in the truncated causal and noncausal theories by means of the Lyapunov method. The conclusions are stated in section 5.

2 The Model

In the case of the homogeneous, isotropic, spatially flat Robertson-Walker metric

\[
ds^2 = dt^2 - a^2(t)(dx_1^2 + dx_2^2 + dx_3^2) \tag{1}
\]

only the bulk viscosity needs to be considered. Thus we replace in the Einstein equations the equilibrium pressure \(p\) by an effective pressure \[2\]

\[
H^2 = \frac{1}{3} \rho \tag{2}
\]

\[
\dot{H} + 3H^2 = \frac{1}{2}(\rho - p - \sigma) \tag{3}
\]

where \(H = \dot{a}/a, \dot{\cdot} = d/dt, \rho\) is the energy density, \(\sigma\) is the viscous pressure, and we use units \(c = 8\pi G = 1\). As equation of state we take

\[
p = (\gamma - 1)\rho \tag{4}
\]
with a constant adiabatic index $0 \leq \gamma \leq 2$, and for $\sigma$ we assume the transport equation,

$$\sigma + \tau \dot{\sigma} = -3\zeta H - \frac{1}{2} \tau \sigma \left(3H + \frac{\dot{H}}{\tau} - \frac{\dot{\zeta}}{\zeta} - \frac{\dot{T}}{T}\right)$$

(5)

which arises from the causal irreversible thermodynamics theory, as the simplest way (linear in $\sigma$) to guarantee positive entropy production [18] [20]. Here $\zeta$ is the bulk viscous coefficient, and following [29]

$$\zeta = \alpha \rho^m$$

(6)

where $\alpha > 0$ and $m$ are constants. Also, $\tau$ is the bulk relaxation time and we choose the law

$$\tau = \frac{\zeta}{\rho}$$

(7)

which ensures causal propagation of viscous signals [31]. Finally, $T$ is the equilibrium temperature and following [21] we choose

$$T = \kappa \rho^r$$

(8)

where $\kappa > 0$ and $r \geq 0$ are constants, which is the simplest way to guarantee a positive heat capacity. In the context of standard irreversible thermodynamics, $p$, $\rho$, $T$ and the number density $n$ are equilibrium magnitudes, which are related by equations of state of the form $\rho = \rho(T, n)$ and $p = p(T, n)$. Further the thermodynamic relation holds [4]

$$\left(\frac{\partial \rho}{\partial n}\right)_T = \frac{\rho + p}{n} - \frac{T}{n} \left(\frac{\partial p}{\partial T}\right)_n,$$

(9)

which follows from the requirement that the entropy is a state function. In our model, this relation imposes the constrain

$$r = \frac{\gamma - 1}{\gamma}$$

(10)

so that $0 \leq r \leq 1/2$ for $1 \leq \gamma \leq 2$. Using (2)-(8), we obtain
\[
\frac{9}{4} (\gamma - 2) H^3 + \frac{3^{2-m}}{2\alpha} \gamma H^4 |H|^{-2m} = 0 \quad (11)
\]
for \( m \neq 1/2 \), and

\[
H \ddot{H} - (1 + r) \dot{H}^2 + AH^2 \dot{H} - BH^4 = 0 \quad (12)
\]
for \( m = 1/2 \), where \( A \equiv 3/2 \left[ 1 + (1 - r) \gamma + 2/\gamma_0 \right] \), \( B \equiv 9/4 \left( 2 - \gamma - 2\gamma/\gamma_0 \right) \), and \( \gamma_0 = \sqrt{3\alpha \text{ sgn}H} \). It is easy to check that solutions of (12) never change sign, and we will consider in the following only expanding evolutions, that is \( H > 0 \). Thus, taking into account (10), we note that \( A > 0 \).

3 The case \( m = 1/2 \)

For the integration of (12), provided that \( r \neq 0 \), it is convenient to make the change of variable \( H = y^{-1/r} \). It turns this equation into

\[
\ddot{y} + Ay^{-\frac{2}{r}} \dot{y} + Bry^{1-\frac{2}{r}} = 0 \quad (13)
\]
which is a particular instance of the class of equations

\[
\ddot{y} + \alpha_1 f(y) \dot{y} + \alpha_2 f(y) \int dy f(y) + \alpha_3 f(y) = 0 \quad (14)
\]
that linearises through the transformation \[13\] \[27\] \[32\] \[33\]

\[
z = \int dy f(y), \quad \eta = \int dt f(y) \quad (15)
\]
becoming

\[
\frac{d^2z}{d\eta^2} + \alpha_1 \frac{dz}{d\eta} + \alpha_2 z + \alpha_3 = 0 \quad (16)
\]
Thus, for \( \alpha_1 = 1 \), \( \alpha_2 = (r-1)B/A^2 \) and \( \alpha_3 = 0 \) we obtain the general solution of (12) in parametric form

\[
H(\eta) = [Rz(\eta)]^{1/r} \quad (17)
\]
\[
\Delta t(\eta) = \frac{1}{A} \int d\eta H(\eta)^{-1}
\]
where $\Delta t = t - t_0$, and $t_0$ is an arbitrary constant and $R$ is a constant. For the case $r = 0$, that is isothermal matter, it is easy to verify that (17) is also the general solution. The constrain (10) implies that

$$ z(\eta) = C_+ \exp (\lambda_+ \eta) + C_- \exp (\lambda_- \eta) \tag{18} $$

where $\lambda_{\pm} = (1/2)[-1 \pm (1 - 4\alpha_0)^{1/2}]$ are real, and $C_{\pm}$ are arbitrary integration constants. When any one of them vanishes, we obtain the one-parameter families of solutions, which have explicit form:

$$ H_\pm(t) = \nu_{\pm}/\Delta t, \quad B \neq 0 \tag{19} $$

$$ H_+ = \nu_0/\Delta t, \quad H_- = H_0, \quad B = 0 \tag{20} $$

$$ \nu_\pm = \frac{1}{2B} \left\{-A \pm \left[A^2 - 4B(r - 1]\right]^{1/2}\right\} \quad \nu_0 = \frac{1-r}{A} \tag{21} $$

and $H_0$ is an arbitrary positive constant. As $0 \leq \nu_+ \leq (2/3)(\sqrt{2} - 1)$ and $1/6 \leq \nu_0 \leq 2/9$, the behavior of $a_+(t)$ is Friedmann. On the other hand, the de Sitter behavior for $B = 0$ has already been considered in [18] [22].

### 3.1 Two-parameter families of solutions

As shown in appendix A, for some values of $\gamma$ and $\gamma_0$, $a(t)$ can be written in closed form in terms of known functions. However, for arbitrary values of $\gamma$ and $\gamma_0$, we need to study the solution in the parametric form (17). Depending on the case, the behavior of $H(t)$ for small or large time is given by the behavior of $z(\eta)$ near a zero point or for $\pm \infty$. In the first case, $z \sim \eta$ for $\eta \to 0$, $H \sim 1/(\Delta t)^{1/r}$, provided $r \neq 0$, or $H \sim \exp(H\Delta t)$ if $r = 0$. Thus

$$ a(t) \sim a_0 \exp \left(\frac{K}{r} \Delta t^{1/r}\right), \quad r > 0 \tag{22} $$

$$ a(t) \sim a_0 \exp \left[\frac{1}{K} \exp (K\Delta t)\right], \quad r = 0 $$

where $a_0$ and $K$ are arbitrary integration constants. In the second case, either the first or the second term in (18) dominates, and the leading behavior of $H(t)$ is given by (19) (or (20), if $B = 0$). Thus, we obtain the following classification of the two-parameter families of solutions:
A. The evolution begins at a singularity with a Friedmann leading behavior as \((19^+)\) for \(B \neq 0\) or \((20^+)\) for \(B = 0\), and so there are particle horizons. Then the scale factor expands with an asymptotically Minkowski behavior for large time like \((22)\).

B. The evolution is asymptotically Minkowski in the far past with behavior \((22)\), then it expands and its behavior in the future is either:

1. asymptotically Friedmann, as \((19^-)\) for \(B < 0\);
2. asymptotically de Sitter, as \((20^-)\) for \(B = 0\);
3. divergent at finite time, with leading behavior \((18^-)\) for \(B > 0\).

C. The evolution begins at a singularity with a Friedmann leading behavior as \((19^+)\) for \(B \neq 0\) or \((20^+)\) for \(B = 0\); and so there are particle horizons. Then the scale factor expands, and its behavior in the future is like B.

3.2 Energy conditions

Let us consider whether this viscous fluid obeys the energy conditions when \(m = 1/2\) \([28]\). Using \((3)(3)\), the dominant energy condition (DEC) \(\rho \geq \max(\rho, p + \sigma)\) implies \(-3H^2 \leq \dot{H} \leq 0\), while the strong energy condition (SEC) \(\rho + 3p + 3\sigma \geq 0\) implies \(\dot{H} + H^2 \leq 0\). We find that DEC is violated part of the time in the following families: B1, in the far past as well as for large times (if \(\nu^- < 1/3\)); near the singularity in C1, C2 and C3; C3, near the ”explosion”. DEC is violated always in families A, B2 and B3. SEC is satisfied always in families A and C1 (if \(\nu^- \leq 1\)). It is violated part of the time in the families: B1, in the far past (if \(\nu^- \leq 1\)); for large times in C1 (if \(\nu^- > 1\)), C2 and C3. SEC is violated always in families B1 (if \(\nu^- > 1\), B2 and B3. Thus, we find that all these two-parameter solutions violate the energy conditions sometime.

4 The case \(m \neq 1/2\)

We will make use of the method of the Lyapunov function \([34]\) to investigate the asymptotically stability of the de Sitter and asymptotically Friedmann solutions that occur in the noncausal and truncated causal models.
4.1 Stability of the de Sitter solution

Equation (11) admits a de Sitter solution for $\gamma < 2$ [18]

$$H_0 = \left[3^m \alpha \left(\frac{2 - \gamma}{2\gamma}\right)\right]^{\frac{1}{1 - 2m}}$$

(23)

To study its stability, if $r > 0$, we make first the change of variable $H = y^{-1/r}$ in (11)

$$\ddot{y} + \frac{3}{2} \left[1 + (1 - r) \gamma\right] y^{-\frac{1}{r}} y + \frac{3^{1-m}}{\alpha} y^{\frac{m-1}{r}} \dot{y} +$$

$$\frac{9}{4} (2 - \gamma) r y^{1+\frac{2}{r}} - \frac{3^{2-m} \gamma r y^{1+\frac{2}{r}+2m}}{2\alpha} = 0$$

(24)

and then we rewrite it as

$$\frac{d}{dt} \left[\frac{1}{2} y^2 + V(y)\right] = - \left\{\frac{3}{2} \left[1 + (1 - r) \gamma\right] y^{-\frac{1}{r}} + \frac{3^{1-m}}{\alpha} y^{\frac{2(m-1)}{r}}\right\} \dot{y}^2$$

(25)

where

$$V(y) = \frac{9 (2 - \gamma) r^2 y^{2(r-1)}}{8 (r - 1)} - \frac{3^{2-m} \gamma r^2}{2\alpha(2r - 3 + 2m)} y^{2\frac{r-3+2m}{r}}$$

(26)

for $2r - 3 + 2m \neq 0$, and

$$V(y) = \frac{9 (2 - \gamma) r^2 y^{2(r-1)}}{8 (r - 1)} - \frac{3^{\frac{3}{2}+r}\gamma r}{2\alpha} \ln \frac{y}{y_0}$$

(27)

for $2r - 3 + 2m = 0$. We see that this potential has a unique minimum for $y > 0$ at $y_0 = H_0^{-r}$ provided $m < 1/2$, while it has a maximum for $m > 1/2$. Also, taking into account (10), the right hand side of (25) is negative definite.

On the other hand, if $r = 0$, we make the change of variable $dt = dx/H$ in (11), and we obtain

$$\frac{d}{dx} \left[\frac{1}{2} H'^2 + V(H)\right] = -H'^2 \left[3 + \frac{3^{1-m}}{\alpha} H^{1-2m}\right]$$

(28)

where $' \equiv d/dx$, and
\[ V(H) = -\frac{9}{8}H^2 + \frac{3^{2-m}}{2\alpha(3-2m)}H^{3-2m}, \quad m \neq \frac{3}{2} \]  
\[ V(H) = -\frac{9}{8}H^2 + \sqrt{\frac{3}{2\alpha}} \ln \frac{H}{H_0}, \quad m = \frac{3}{2} \]  

This potential has a unique minimum for \( H > 0 \) at \( H_0 \) provided \( m < 1/2 \), while it has a maximum for \( m > 1/2 \). Also the right hand side of (28) is negative definite.

Thus we find that an exponential inflationary regime is asymptotically stable for \( t \to \infty \) for any initial condition \( H > 0 \) provided that \( m < 1/2 \), but this regime becomes unstable for \( m > 1/2 \). This result improves over previous studies based in small perturbations about solution (23) [18] [22].

4.2 Stability of the asymptotically Friedmann solution

For \( m > 1/2 \) it is easy to check that (11) admits a solution whose leading term is \( 2/(3\gamma t) \). To study its stability, if \( r > 0 \), we make the change of variables

\[ H = \frac{[u(z)]^{-\frac{1}{2}}}{t}, \quad t^{2m-1} = z \]  
in (11) which takes the form

\[ \frac{d}{dz} \left[ \frac{1}{2} u'^2 + U(u, z) \right] = D(u, u', z) \]  

where \( ' \equiv d/dz \). We consider that \( u \) lies in a neighbourhood of \( u_0 \equiv (3\gamma/2)^{-1/r} \). Thus, when \( z \to \infty \)

\[ U(u, z) = \frac{3^{1-m}r^2}{2\alpha(2m-1)^2} \left[ u^{2(r+m-1) \over r} - {3\gamma u^{2r+2m-3} \over r} \right] \frac{1}{z} + O \left( \frac{1}{z^2} \right) \]  
for \( 2r + 2m \neq 2, 3 \),

\[ U(u, z) = \frac{3^r}{\alpha(1-2r)^2} \left( \ln \frac{u}{u_0} + \frac{3\gamma r}{2} u^{-1} \right) \frac{1}{z} + O \left( \frac{1}{z^2} \right), \quad r + m = 1 \]
\[ U(u, z) = \frac{3^{r-\frac{1}{2}}}{4\alpha(1-r)^2} \left( ru^\frac{1}{2} - \frac{3\gamma}{2} \ln \frac{u}{u_0} \right) \frac{1}{z} + O \left( \frac{1}{z^2} \right), \quad 2r + 2m = 3 \] (35)

and

\[ D(u, u', z) = -\frac{3^{1-m}}{\alpha(2m-1)} u^{2(m-1)} u'^2 + O \left( \frac{1}{z} \right) \] (36)

On the other hand, if \( r = 0 \), we make the change of variables

\[ H = \exp \left[ u(z) \right], \quad z = t^{2m-1} \] (37)

in (14) which takes again the form (32), but now we consider that \( u \) lies in a neighbourhood of \( u_0 \equiv \ln(2/3) \), and when \( z \to \infty \)

\[ U(u, z) = \frac{3^{1-m}}{2\alpha (2m-1)^2} \left[ \frac{3e^{(3-2m)u}}{3 - 2m} - \frac{e^{(2-2m)u}}{1 - m} \right] \frac{1}{z} + O \left( \frac{1}{z^2} \right) \] (38)

for \( 2m \neq 2, 3, \)

\[ U(u, z) = \frac{1}{\alpha} \left( \frac{3}{2} e^u - u \right) \frac{1}{z} + O \left( \frac{1}{z^2} \right), \quad m = 1 \] (39)

\[ U(u, z) = \frac{1}{4\sqrt{3}\alpha} \left( \frac{3}{2} u + e^u \right) \frac{1}{z} + O \left( \frac{1}{z^2} \right), \quad 2m = 3 \] (40)

and

\[ D(u, u', z) = -\frac{3^{1-m}}{\alpha(2m-1)} e^{(2-2m)u} u'^2 + O \left( \frac{1}{z^2} \right) \] (41)

As \( U(u, z) \) has a unique minimum at \( u_0 \) for any \( m \neq 1/2 \), and \( D(u, u', z) \) is negative definite for \( m > 1/2 \), we find that solutions with leading Friedmann behaviour \( a \sim t^{2/(3\gamma)} \) when \( t \to \infty \) are asymptotically stable for \( m > 1/2 \), but become unstable for \( m < 1/2 \).
5 Conclusions

We have improved our previous analysis of a cosmological model with a causal viscous fluid by considering the full transport equation for the bulk viscosity instead of the truncated one \cite{13}. We have kept the equation of state that relates the equilibrium pressure and energy density, and we have added a power-law relationship between the equilibrium temperature and energy density.

When \( m = 1/2 \), the splitting of the large time asymptotic behavior of solutions in terms of \( \text{sgn}(-B) \) follows closely that of the truncated model in terms of \( \text{sgn}(\gamma - \gamma_0) \), which in its turn resembles the classification for the noncausal solutions. However, families of solutions with extrema no longer remain in the full case, and we find instead asymptotically Minkowski evolutions. Moreover, all solutions are monotonic.

The singular behavior for \( m = 1/2 \) is also very similar to that of the truncated model because, excluding (19–), singular solutions have particle horizons. The only difference is that \( \nu_+ \) in (21+) is smaller than the corresponding magnitude in \cite{13}. Similar result has already been verified for \( m > 1/2 \) \cite{27}.

As in the truncated model, we demonstrate that there is no value of \( m \) for which there is a stable expanding de Sitter period in the far past. This further support the conclusion of that causality avoids the deflationary behavior proposed by Barrow \cite{7}.

If \( m < 1/2 \), a stable exponential inflationary phase occurs in the far future for \( 1 \leq \gamma < 2 \); the condition \( B = 0 \) is required if \( m = 1/2 \), and such a behavior is unstable for \( m > 1/2 \). This is the same structural behavior already found in the causal truncated and noncausal models. It has been noted however that the inflation rate in the full causal theory is different, lower for \( m < 1/2 \) and greater for \( m > 1/2 \). Also, there is no stiff-fluid viscous exponential inflation, unlike in the truncated or non-causal theories \cite{18}.

It has been conjectured that viscosity-driven exponential inflationary solutions are spurious and an artificial consequence of use of causal truncated or noncausal theories \cite{13}. This suggestion arises from study of a cosmological model containing a dissipative Boltzmann gas. Our results show however that this kind of conclusions does not generalize to other models where the viscous fluid is described by a different equation of state. Moreover, they rather
suggest that some features of the solutions, like their asymptotic behavior depend more strongly on the equation of state than on the thermodynamical theory employed.

If \( m > 1/2 \), we find that the perfect fluid behavior \( a \sim t^{2/(3\gamma)} \) for \( t \to \infty \) is asymptotically stable. This arises because the viscous pressure decays faster than the thermodynamical pressure. However, if \( m = 1/2 \) and \( B < 0 \), both pressures decay asymptotically as \( t^{-2} \) and the exponent becomes \( \nu_{-} \). The perfect fluid behavior becomes unstable if \( m < 1/2 \). Here also, no structural change occurs with respect to the truncated causal or noncausal models.

6 Appendix A

The equation (12) has a two-dimensional Lie group of point symmetries, whose generators are \( \partial/\partial t \) and \( t \partial/\partial t - H \partial/\partial H \). We look for a simple transformation to a new dependent variable \( v \) that is linear in \( \dot{v} \), preserves this group of symmetries and keeps the order of the differential equation. This technique can be extended to some polynomial autonomous differential equations arising when we consider quantum effects due to vacuum polarisation terms [35]. Thus, we are led to the transformation

\[
\dot{H} = f H^2 + g H \frac{\dot{v}}{v}
\]

(42)

where \( f \) and \( g \) are two constants determined by the requirement that the differential equation in \( v(t) \) is linear. Inserting (12) in (12) we obtain

\[
[(1 - r) f^2 + A f - B] H^4 - g [(2r - 1) f - A] \frac{\dot{v}}{v} H^3 -
\]

\[
g \left[ -\frac{\ddot{v}}{v} + (gr + 1) \frac{\dot{v}^2}{v^2} \right] H^2 = 0
\]

(43)

Then, provided we choose \( f = A / (2r - 1) \), \( g = -1/r \), and we impose the constrain

\[
\frac{Br}{A^2} - \frac{1}{(2 - \frac{1}{r})^2} = 0
\]

(44)
equation (12) turns into $\ddot{v} = 0$. To integrate (12), we make the change of variable $H = -\dot{u}/(fu)$, and it becomes

$$\frac{\ddot{u}}{u} + \frac{1}{r} \frac{\dot{v}}{v} = 0$$

which has the first integral $\dot{u}v^{-1/r} = C$, with some arbitrary integration constant $C$. Thus we obtain

$$H = (1 - 2r) \frac{v^{-\frac{1}{r}}}{A \int dtv^{-\frac{1}{r}}}$$

and the scale factor

$$a(t) = a_0 \left| \Delta t \right|^{-\frac{1}{r}} + K$$

where $\nu = (1 - 2r)/A$ and $K$ is an arbitrary integration constant. We note that, using (10), (44) becomes a curve on the parameter plane $(\gamma, \gamma_0)$. This explicit general solution can also be derived from a more comprehensive mathematical framework, as shown in [32].

For $B = 0$, an implicit expression for the solution of (12) exists. In effect, equation (13) admits a first integral

$$\dot{y} + \frac{Ar}{r - 1} y^{-\frac{1}{r}} = E$$

and we obtain

$$\Delta t = \frac{1}{H_0} \left( \frac{H_0}{H} \right)^r 2F_1 \left( 1, \frac{r}{r - 1}, \frac{2r - 1}{r - 1} \frac{Ar}{r - 1} \left( \frac{H}{H_0} \right)^{1-r} \right)$$

$$\Delta t = \frac{(1 - r)}{AH_0} \left( \frac{a}{a_0} \right)^{\frac{A}{r}} 2F_1 \left( \frac{1}{1 - r}, \frac{1}{1 - r}, \frac{2 - r}{1 - r} \left( \frac{a}{a_0} \right)^{A} \right)$$

where $H_0, a_0$ are arbitrary integration constants.
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Figure Captions

Fig. 1. Selected solutions for $\gamma = 1.1$ ($B > 0$). The curves of class A and C3 correspond to (47) with $K = -1$. The curve of class B3 corresponds to (47) with $K = 1$. 