The Shortest Connection Game

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Abstract

We introduce Shortest Connection Game, a two-player game played on a directed graph with edge costs. Given two designated vertices in which they start, the players take turns in choosing edges emanating from the vertex they are currently located at. In this way, each of the players forms a path that originates from its respective starting vertex. The game ends as soon as the two paths meet, i.e., a connection between the players is established. Each player has to carry the cost of its chosen edges and thus aims at minimizing its own total cost.

In this work we analyze the computational complexity of Shortest Connection Game. On the negative side, Shortest Connection Game turns out to be computationally hard even on restricted graph classes such as bipartite, acyclic and cactus graphs. On the positive side, we can give a polynomial time algorithm for cactus graphs when the game is restricted to simple paths.

Keywords: shortest path problem, game theory, computational complexity, cactus graph

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1. Introduction

We consider the following game on a directed graph $G = (V, E)$ with vertex set $V$ and a set of directed edges $E$ with nonnegative edge costs $c(u, v)$ for each edge $(u, v) \in E$ and two designated vertices $s, t \in V$.

In Shortest Connection Game two players $A$ and $B$ start from their respective homebases ($A$ in $s$ and $B$ in $t$). The aim of the game is to establish a connection between $s$ and $t$ in the following sense: The players take turns in moving along an edge and thus each of them constructs a directed path. The game ends as soon as one player reaches a vertex $m$, i.e. a meeting point, which was already visited by the other player. This means that at the end of the game one player, say $A$, has selected a path from $s$ to $m$, while $B$ has selected a path from $t$ to $m$ and possibly further on to additional vertices (or vice versa). Each player has to carry the cost of its chosen edges and wants to minimize the total costs it has to pay.

Note that it is not always beneficial for both players to move closer to each other. Instead, one player may take advantage of cheap edges and move away from the other player, who then has to bear the burden of building the connection.

To avoid unnecessary technicalities we assume that for every graph considered a solution of the game does exist. This could be checked in a preprocessing step e.g. by executing a breadth-first-search starting in parallel from $s$ and $t$.

As a motivational example consider the decision problem of two persons $A$ and $B$ (let’s say a married couple) who want to decide on a joint holiday trip. Each such trip consists of a number of components such as transportation by car, train, or plane, accommodation, sight-seeing trips, special events etc. Thus, it requires a certain effort to organize such a trip (inquiries, reservations, etc.) The two persons have different ideas of such a trip but want to reach a consensus, i.e. a trip they both agree with. Their decision procedure runs as follows:

Person $A$ makes a first offer, i.e. a detailed plan of a trip containing all its components. Naturally, putting together such a trip requires a certain effort (e.g. measured in time). Then Person $B$ makes an offer corresponding to its own preferences. Most likely the two offered trip plans will not match. Thus, it is $A$’s turn again to modify its previous offer, e.g. by changing an excursion, moving to a different hotel or changing the mode of transportation. This change of plans again requires a certain effort. In this way, the decision process goes on with a sequence of offers alternating between $A$ and $B$. A final decision is reached as soon as one person proposes a trip (= offer) which is identical to an offer proposed by the other person in one of the previous rounds. It means that a trip was identified that both persons agree with.

The total cost incurred by the decision process for each person consists of the sum of efforts spent by that person until the final decision was reached. Note that the effort to generate an offer depends on the sequence of offers proposed
before since the effort of changing from one offer to another may differ a lot and thus different sequences of offers give rise to different total efforts.

Formally, we can represent each offer by a vertex $v \in V$ in a graph $G = (V, E)$. Each edge $(u, v) \in E$ represent the change from the offer associated to $u$ to offer $v$ and incurs costs $c(u, v)$. The two persons $A, B$ start with initial offers $s \in V$ and $t \in V$ respectively. The sequence of trips offered by each person corresponds to a simple path in the graph (it does not make sense in this setting to make the same offer twice). The final decision is reached as soon as a vertex is offered that lies on both paths. Note that this is not necessarily the last offer proposed by both persons. Assuming total knowledge (an old married couple) each person will try to minimize its own total cost taking into account the rational optimal decisions by the other person.

We will impose two restrictions on the problem setting to ensure that the two players actually meet in some vertex and to guarantee finiteness of the game. To enable a feasible outcome of the game we restrict the players in every decision to choose only edges which still permit a meeting point of the two paths.

\begin{itemize}
  \item[(R1)] The players can not select an edge which does not permit the paths of the players to meet.
\end{itemize}

Note that this aspect may lead to interesting strategic behavior, since we also require that each player must be able to choose an edge in every round. Thus, each player has to take care that the other player does not get stuck.

Secondly, we want to guarantee that the game remains finite. This raises the question of how to handle cycles. It could be argued that moving in a cycle makes sense for a player who wants to avoid an edge of large costs. However, for the sake of finiteness each cycle should be traversed at most once as guaranteed by the following restriction:

\begin{itemize}
  \item[(R2)] Each edge may be used at most once.
\end{itemize}

Note that (R2) does not rule out the possibility to visit a vertex more than once. However, it is also interesting to restrict the game to simple paths and thus exclude cycles completely by the following condition.

\begin{itemize}
  \item[(R3)] Each vertex may be used at most once by each player.
\end{itemize}

While (R1) and (R2) will be strictly enforced throughout the paper, we will consider the variants with and without (R3).

We will study Shortest Connection Game on general directed graphs (also the case of an undirected graph is interesting, but this is beyond the scope of the
current paper) and on relevant special graph classes namely bipartite graphs, acyclic graphs, cactus graphs and trees. Each of these special cases (with the trivial exception of acyclic graphs) arises if the associated undirected graph implied by removing the orientation from all edges has the stated property.

1.1. Game Theoretic Setting

The game described above can be represented as a finite game in extensive form. All feasible decisions for the players can be represented in a game tree, where each node corresponds to the decision of a certain player in a vertex of the graph $G$. A similar representation was given in [4] for Shortest Path Game (see also Section 1.2).

The standard procedure to determine equilibria in a game tree is backward induction (see Osborne [10, ch. 5]). This means that for each node in the game tree whose child nodes are all leaves, the associated player can reach a decision by choosing the best of all child nodes w.r.t. the cost resulting from its corresponding path in $G$. Then these leaf nodes can be deleted and the costs accrued by each of the two players is moved to its parent node. In this way, we can move upwards in the game tree towards the root and settle all decisions along the way.

This backward induction procedure implies a strategy for each player given by the decisions in the game tree: Always choose the edge according to the result of backward induction. This strategy for both players is a Nash equilibrium and also a so-called subgame perfect equilibrium (a slightly stronger property), since the decisions made in the underlying backward induction procedure are also optimal for every subtree.

The outcome, if both players follow this strategy, is a set of two directed paths, namely from $s$ to some vertex $v_A$ and from $t$ to some vertex $v_B$, where both paths contain the unique common vertex $m$ which coincides with at least one of the vertices $v_A, v_B$. These paths make up the unique subgame perfect equilibrium and will be denoted by $spe$-conn. A $spe$-conn for Shortest Connection Game is the particular solution in the game tree with minimal cost for both selfish players under the assumption that they have complete and perfect information of the game and know that the opponent will also strive for its own selfish optimal value. An illustration of Shortest Connection Game is given in the following example.

Example 1. Consider the following graph where numbers refer to the actual costs of the corresponding edges and the vertices are labelled by letters. In the

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1 This means that two directed edges $(u, v)$ and $(v, u)$ imply two parallel edges $(u, v)$ in the associated undirected graph.

2 As a tie-breaking rule we will use the “optimistic case”, where in case of indifference a player chooses the option with lowest possible cost for the other player.
associated game tree (depicted below) spe-conn is determined by backward induction. Thick arrows indicate the optimal decision in each node. Each node of the game tree contains the current positions of both players and the cost until the end of the game from the current position as it is computed by backward induction; e.g. \((a, 4|c, 2)\) says that if \(A\) is in \(a\) and \(B\) in \(c\), in the subgame perfect equilibrium the respective total cost for establishing a meeting point is 4 for \(A\) and 2 for \(B\). The number depicted for each edge of the game tree gives the cost of the edge in the graph associated with the corresponding decision. In our example spe-conn is given by the sequence \(s, d, b, e\) for \(A\) (cost 4), and \(t, c, e\) (cost 5) for \(B\). Note that in general the feasible moves from the current position might be restricted by the paths chosen for reaching this position from the starting point \((s, t)\), e.g. because of \((R2)\); in the given acyclic graph, however, this is not the case.

In this setting finding the spe-conn for the two players is not an optimization problem as dealt with in combinatorial optimization but rather the identification of two sequences of decisions for the two players fulfilling a certain property in the game tree. Clearly, such a spe-conn solution can be computed in exponential time by exploring the full game tree.
It is the main goal of this paper to study the complexity status of finding this \textit{spe-conn}. In particular, we want to establish the hardness of computation for general graphs and identify special graph classes where a \textit{spe-conn} is either still hard to determine or can be found in polynomial time without exploring the exponential size game tree.

Comparing the outcome of our game, i.e. the total cost of \textit{spe-conn}, with the cost of the shortest connection obtained by a cooperative strategy we can consider the \textit{Price of Anarchy} (cf. [9]) defined by the ratio between these two values. However, as illustrated by the following example it is easy to see that the Price of Anarchy can become arbitrarily large.

**Example 2.** In the graph depicted below there are only two feasible solutions with meeting points $v_1$ or $v_2$ and player $A$ can decide between the two. The cheapest connection would use $v_1$ with total cost 3 while the selfish optimal solution of $A$ would use $v_2$ with total cost $1 + M$, i.e. the Price of Anarchy is $\frac{M + 1}{3}$ which can become arbitrarily large.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{example_graph.png}
\caption{Example graph for Price of Anarchy}
\end{figure}

An intuitive approach to the problem may suggest that each agent should reach the meeting point via a \textit{shortest path} from its origin using a certain number of edges resulting from the iterative selection process. However, the following example shows that the restriction to shortest paths from both origins to every possible meeting point under any possible number of rounds may miss the optimal strategy.

**Example 3.** Consider the graph represented in the following figure. All unlabelled edges have cost 0. For sake of clarity we number only some of the vertices. Edges with two arrows represent two edges in either direction.

$A$ starts the game by moving from $s$ to $v_1$. If $B$ chooses the edge $(t, v_7)$, $A$ can move to $v_2$ and to $v_3$. This forces $B$ to move via $v_4$ to the meeting point $v_2$ implying cost $M$ for both $A$ and $B$. However, $B$ can do better by starting with $(t, v_6)$. If $A$ still moves to $v_2$, then $B$ enters the detour via $v_8$, which forbids $A$ to take the path to $v_3$ but forces $A$ to go via $v_4$ and $v_5$ to the meeting point $v_6$ at a cost of $2M + 4$, while $B$ has cost 6. A better option for $A$ would be the more expensive edge to $v_9$. Now there is no need for $B$ to use the detour as a
“deterrent”. Instead, B can go via $v_5$ to the meeting point $v_4$ with cost $2M + 2$ for A and cost 4 for B which is the spe-conn of the game.

Note that neither A nor B can use their shortest paths with three edges from $s$ (resp. $t$) via $v_2$ (resp. $v_7$) to $v_4$.

In Section 2 we will show that this failure of the straightforward idea can not be overcome at all since the underlying decision problem is PSPACE-complete on general graphs. Note that by using depth-first search the problem can be seen to be in PSPACE since the height of the game tree is bounded by $2|E|$. In every node currently under consideration we keep a list of decisions still remaining to be explored and among all previously explored options starting in this node we keep the cost of the currently preferred subpath. By proceeding in a depth-first manner there are at most $2|E|$ vertices on the path from the root of the game tree to the current vertex for which the information is kept.

1.2. Related Literature and Our Contribution

Recently, Darmann et al. [4] (see also Darmann et al. [3]) considered the closely related Shortest Path Game where two players move together from a joint origin to a joint destination and take turns in selecting the next edge. The player choosing an edge has to pay its cost, and each of the players is aiming at minimizing its own total cost. Several complexity results are given for Shortest Path Game including PSPACE-completeness for directed bipartite graphs and a linear time algorithm for directed acyclic graphs.

In contrast, the results of our work show that not only Shortest Connection Game is PSPACE-complete in directed bipartite graphs, but also remains strongly NP-hard in the case of directed acyclic bipartite graphs, irrespective of whether or not the game is restricted to simple paths, i.e. (R3) is imposed (see Sections 2.1 and 2.2). Searching for polynomially solvable special cases in Section 3 we move on from trees, where Shortest Connection Game is
trivial to solve, to a slightly more general graph class, namely cactus graphs, i.e. graphs where each edge is contained in at most one simple cycle. On the positive side, we provide a polynomial time algorithm for the case of cactus graphs when (R3) is imposed (see Section 3.2), while we can show that Shortest Connection Game is already strongly NP-hard on cactus graphs without condition (R3) (see Section 3.1). Thus, we can describe a rather precise boundary between polynomially solvable and hard graph classes. Note that the latter result is somewhat surprising given the fact that the related Shortest Path Game is easy to solve on cactus graphs (see Darmann et al. [4]).

Another related game is Geography (see Schaefer [11]), and in particular its variants Partizan Arc Geography and Partizan Vertex Geography (see Fraenkel and Simonson [5]). Partizan Arc Geography and Partizan Vertex Geography both are two-player games played on a directed graph (there are no costs involved). In these games, starting from its designated homebase vertex, each of the two players forms a path by alternately moving along edges (in Partizan Arc Geography only not-yet traversed edges can be taken, in Partizan Vertex Geography only not-yet visited vertices can be moved to). The first player unable to perform another move is the loser of the game. Note that the restrictions defining the two games share the spirit of (R2) and (R3). Both of these games are known to be PSPACE-complete for directed bipartite graphs and NP-hard for directed acyclic graphs, while they are solvable in polynomial time for trees (see Fraenkel and Simonson [5]). Recall that our problem Shortest Connection Game turns out to share this complexity behavior and, in particular, is also polynomial time solvable for trees (see Proposition 5). However, not only do we enrich the analysis of Shortest Connection Game by considering cactus graphs, but we also show that imposing or relaxing (R3) yields in fact different computational complexity results on this graph class.

In the classical game Geography, the two players move together along unused edges starting from the same designated starting vertex and take turns in selecting the next edge. Again, the first player unable to choose an edge loses the game. Besides the well-known PSPACE-completeness result of Schaefer [11] several complexity results have been provided for Geography and its variants such as Vertex Geography (each vertex can be visited at most once); see, e.g., Lichtenstein and Sipser [8], Fraenkel and Simonson [5], Fraenkel et al. [7], and Bodlaender [1]. In particular, both Geography and Vertex Geography are known to be solvable in polynomial time when played on cactus graphs (cf. Bodlaender [1]).

Another variant of two players taking turns in their decision on a discrete optimization problem and each of them optimizing its own objective function was considered for the Subset Sum problem in Darmann et al. [2].
2. Hardness Results for Shortest Connection Game

Let us first define formally the decision problem associated to Shortest Connection Game.

**Shortest Connection Game**

**Input:** A directed graph $G = (V, E)$ with nonnegative edge lengths $c_e$ for $e \in E$, two dedicated vertices $s, t$ in $V$, and nonnegative values $C_A, C_B$.

**Question:** Does $spe\text{-}conn$ induce a path for agent $A$ with total cost $\leq C_A$ and a path for $B$ with total cost $\leq C_B$?

2.1. PSPACE-completeness for directed bipartite graphs

Our proof is based on a reduction from Quantified $\leq 4$-Sat. By Quantified $\leq k$-Sat, we refer to the variation of Quantified 3-Sat where each clause consists of at most $k$ literals (instead of exactly three).

**Theorem 1.** Quantified $\leq 4$-Sat is PSPACE-complete even when each clause contains exactly one universal literal.

**Proof.** It can be concluded from \cite{5} that Quantified $\leq 3$-Sat is PSPACE-complete when restricted to quantified Boolean formulas with at most one universal literal per clause. Let $F$ be such a formula, i.e.,

$$F = \exists x_1 \forall x_2 \ldots \exists x_{n-1} \forall x_n : \phi(x_1, \ldots, x_n)$$

with $\phi(x_1, \ldots, x_n) = C_1 \land C_2 \ldots \land C_m$ such that $C_j$ contains at most three literals of which at most one is universal.

Let $\gamma$ be the set of clauses among $C_1, \ldots, C_m$ that contain existential literals only. If $\gamma = \emptyset$, there is nothing to show. Otherwise, w.l.o.g. let $\gamma = \{C_1, C_2, \ldots, C_g\}$, for some $g \in \mathbb{N}$. For each $C_j \in \gamma$ introduce a new (dummy) existential variable $e_j$ and a new universal variable $u_j$, and define new clauses $C_{j,1} := C_j \lor u_j$ and $C_{j,2} := C_j \lor \bar{u}_j$. Let

$$F' = \exists x_1 \forall x_2 \ldots \exists x_{n-1} \forall x_n \exists e_1 \forall u_1 \exists e_2 \forall u_2 \ldots \exists e_g \forall u_g : \phi'(x_1, \ldots, x_n)$$

where $\phi'(x_1, \ldots, x_n)$ is derived from $\phi(x_1, \ldots, x_n)$ by replacing each $C_j \in \gamma$ by the clauses $C_{j,1}$ and $C_{j,2}$.

Clearly, this transformation is polynomial, and $F$ is true if and only if $F'$ is true. Note that each clause contains at most 4 literals exactly one of which is universal. \hfill \Box

**Theorem 2.** Shortest Connection Game is PSPACE-complete for directed bipartite graphs even if all costs are bounded by a constant.
In instance $M \in \text{Quantified \!-\! 4\!-\!Sat}$ of variables and set $C = \{C_1, \ldots, C_m\}$ of clauses, with exactly one universal literal per clause. We reduce $I$ to an instance $M$ of Shortest Connection Game as follows.

**Part 1: Construction of Graph $G$.**
First, we introduce a directed graph $G = (V, E)$, which again will be 2-colored in order to verify that $G$ is bipartite. An illustration of $G$ is given in Fig. [1] where, for the sake of readability, the attention is restricted to the clause $C_1 = (x_1 \lor \bar{x}_2 \lor x_3)$. To formally create $G$ from $I$, we introduce

- for each $i$, $1 \leq i \leq n$, a “hexagon”, i.e.,
  - green vertices $v_{i,0}, v_{i,2}, v_{i,4}$
  - red vertices $v_{i,1}, v_{i,3}, v_{i,5}$
  - edges $x_i := (v_{i,0}, v_{i,1}), (v_{i,1}, v_{i,2}), (v_{i,2}, v_{i,3})$
  - edges $\bar{x}_i := (v_{i,0}, v_{i,5}), (v_{i,5}, v_{i,4})$ and $(v_{i,4}, v_{i,3})$

- for each $i$, $1 \leq i \leq n - 1$, an edge $(v_{i,3}, v_{i+1,0})$

- green vertices $y, d_B$ and red vertex $d_A$

- edges $(y, d_A)$ and $(v_{n,3}, d_B)$

- for each $j$, $1 \leq j \leq m$
  - green vertices $C_{j,A}, q_{j,B}, w_j$ and red vertices $C_{j,B}, q_{j,A}, z_j$
  - edges $(d_A, C_{j,A}), (C_{j,A}, q_{j,B}), (q_{j,B}, w_j), (w_j, z_j)$
  - edges $(d_B, C_{j,B}), (C_{j,B}, q_{j,A}), (q_{j,B}, z_j), (z_j, w_j)$
  - edges $(C_{j,A}, v_{i,1})$ and $(v_{i+2}, q_{j,A})$ if $i$ is odd and $\bar{x}_i \in C_j$
  - edges $(C_{j,A}, v_{i,5})$ and $(v_{i,4}, q_{j,A})$ if $i$ is odd and $x_i \in C_j$
  - edge $(q_{j,B}, v_{i,5})$ if $i$ is even and $\bar{x}_i \in C_j$
  - edge $(q_{j,B}, v_{i,1})$ if $i$ is even and $x_i \in C_j$

The edge costs are given by, $c(w_j, z_j) = c(z_j, w_j) = 1$, $c(z_j, q_{j,B}) = 2.1$ for $1 \leq j \leq m$. Finally, $c(e) = \varepsilon$ with $\varepsilon < \frac{1}{2n+6}$ for each of the remaining edges $e \in E$.

Note that $G$ is bipartite, since each edge links a green vertex with a red vertex. In instance $M$ of Shortest Connection Game, player $A$ starts in vertex $s := v_{1,0}$ and $B$ starts in vertex $t := v_{2,0}$. It is not hard to see that, by construction, in order to establish a meeting point at least one of the players has to traverse an edge of cost at $1$. In other words, $c(P) < 1$ implies $c(Q) > 1$ for $P, Q \in \{A, B\}$, $P \neq Q$.

**Part 2: The Reduction.**
In what follows, we show that $I$ is a “yes” instance of Quantified \!-\! 4\!-\!Sat.
Figure 1: Directed graph $G$ in instance $\mathcal{M}$
if and only if the outcome of instance $M$ of Shortest Connection Game yields $c(A) < 1$.

In instance $M$, player $A$ starts the game with moving an edge emanating from $s = v_{1.0}$. I.e., $A$ moves along one of the edges $x_1$ and $x_2$. In the next step, $B$ needs to decide between moving along $x_2$ and $x_3$. In the next step ($A$ is either in $v_{1.1}$ or $v_{1.5}$, depending on $A$’s choice in the first step), there is only one edge $A$ can move along; after that move, $A$ is either in $v_{1.2}$ or $v_{1.4}$. Similarly, $B$ has no choice but to follow the only available edge, and ends up in either $v_{2.2}$ or $v_{2.4}$. In the following step $A$ possibly has the choice between moving (i) along an edge to $q_{j,A}$ for some $1 \leq j \leq m$ or (ii) along the edge to $v_{1.3}$.

(i) Assume that $A$ chooses the former. Clearly, $B$ cannot reach a meeting point with $A$ in less than four moves, because $B$ would have to traverse $d_B$ in order to do so. Hence, irrespective of $B$’s decisions in the next three steps, $A$’s decision to move to $q_{j,A}$ either causes $A$ to move along the edges $(q_{j,A}, w_j)$, $(w_j, z_j)$, $(z_j, q_{j,B})$, resulting in $c(A) = 2\varepsilon + 1 + 2.1 = 3.1 + 2\varepsilon$, or is even infeasible because a meeting point is no longer possible.

(ii) Assume that $A$ moves to $v_{1.3}$: $B$ necessarily moves to $v_{2.3}$. In the next steps, first $A$ moves to $v_{3.0}$ and $B$ to $v_{4.0}$. The above situation repeats: First, $A$ chooses between $x_3$ and $x_4$, then $B$ chooses between $x_4$ and $x_5$. After each of the players makes another move, $A$ again needs to decide between moving along an edge to $q_{j,A}$ for some $1 \leq j \leq m$ or along the edge to $v_{3.3}$. Again, the former move is either infeasible, or the game would end up with $A$ having to carry at least the cost of $c(A) = 3.1 + 6\varepsilon$. This leads us to the following simple observation.

**Observation 1.** If, for some $1 \leq j \leq m$, $A$ reaches $q_{j,A}$ via a feasible move without traversing $y$, then the game ends with $A$ having to carry a cost of at least $3.1 + 2\varepsilon$.

We will argue that using the optimal strategy, $A$ will always traverse $y$. Note that if the meeting point is not one of the vertices $w_j$, then $A$ has to traverse at least one edge of cost at least one. Together with the above observation, this means that the best possible outcome for $A$ results from meeting in one of these vertices such that $A$ uses edges of cost $\varepsilon$ only. It is not hard to verify that the number of edges $B$ needs to traverse to reach such a vertex $w_j$ is $4\frac{\varepsilon}{2} + 4$. Hence, we get the following observation.

**Observation 2.** In instance $M$, we have $c(A) \geq \varepsilon(2n + 4)$.

Assume that $A$ reaches vertex $y$. By construction of $G$, this means that $A$ has not moved along an edge with endpoint $q_{j,A}$ for some $1 \leq j \leq m$ yet. In particular, $A$ has used exactly $4\frac{\varepsilon}{2} = 2n$ edges. After $2n$ edges, $B$’s position is in $d_B$ in any case. Let $\tau$ be the truth assignment that sets to true exactly the set of edges (i.e., literals) of $X$’s choice so far.

After each player has chosen $2n$ edges, it is $A$’s turn, moving along the only edge $(y, d_A)$ emanating from $y$. Now, it is $B$’s turn to choose the next vertex to be visited, i.e., to pick a vertex $C_{j,B}$, $1 \leq j \leq m$. We will show that $B$ can choose a vertex $C_{j,B}$ with the result that the game ends with $c(B) < 1$.
(and, as a consequence, $c(A) > 1$) if and only if there is a clause $C_j$ which is not satisfied by the produced truth assignment $\tau$. Putting things the other way round, we show that $I$ is a “yes”-instance if and only if in instance $M$ of \textsc{Shortest Connection Game} the outcome is $c(A) \leq 1$. For the sake of readability, w.l.o.g. we assume that $B$ chooses vertex $C_{1,B}$. By construction of $G$, this implies that there are exactly three possible meeting points: $w_1, z_1$ and $q_{1,B}$.

\textbf{Case I:} $I$ is a “yes”-instance. I.e., assume that $C_1$ is satisfied by $\tau$.

\textbf{Case Ia:} $C_1$ is satisfied by its unique universal literal, i.e., $\bar{x}_2$. We show that for $A$ moving along the edge $(d_A, C_{1,A})$ results in the lowest possible total cost for $A$. If $A$ does so, in the next step, $B$ necessarily moves along $(C_{1,B}, q_{1,B})$. Next, $A$ moves along $(C_{1,A}, q_{1,A})$. We distinguish the following cases (see also Table \ref{tab:1}).

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
\textbf{CASE Ia} & $A$ & $C_{1,A}$ & $q_{1,A}$ & $w_1$ & infeasible \\
\hline
$B$ & $q_{1,B}$ & $v_{2,5}$ &  \\
\hline
\textbf{CASE Ib} & $A$ & $C_{1,A}$ & $q_{1,A}$ & $w_1$ & $c(A) = \varepsilon(2n+4)$ \\
$B$ & $q_{1,B}$ & $z_1$ & $z_1$ & $c(B) = 1 + \varepsilon(2n+3)$ &  \\
\hline
\end{tabular}
\caption{Moves and resulting outcomes in subcase I of Case 1 in instance $M$.}
\end{table}

\textbf{CASE Ia:} $B$ moves along $(q_{1,B}, v_{2,5})$. Then, $A$ necessarily moves along $(q_{1,A}, w_1)$, and $B$ is not able to move along an edge, because $B$ has used the edge $\bar{x}_2$ already. Hence, in contradiction with (R1), a meeting point is no longer possible, because the players need to move alternately.

\textbf{CASE Ib:} $B$ needs to move along $(q_{1,B}, z_1)$. $A$ moves along $(q_{1,A}, w_1)$. The game ends with $B$ moving along $(z_1, w_1)$, because moving along $(z_1, q_{1,B})$ would be more expensive for $B$. The corresponding costs are $c(A) = \varepsilon(4n^2 + 4) = \varepsilon(2n+4)$ (which is the lowest possible cost for $A$, see Observation 2) and $c(B) = \varepsilon(4n^2 + 3) + 1 = 1 + \varepsilon(2n+3)$.

To summarize these cases, $B$ moves along $(q_{1,B}, z_1)$, i.e., Case Ib applies. Thus, in Case I, $A$ has a strategy such that the game ends with $c(A) = \varepsilon(2n+4) < 1$.

\textbf{Case II:} $C_1$ is not satisfied by its universal literal $\bar{x}_2$. I.e., $C_1$ is satisfied by at least one of its existential literals $x_1, x_3$. W.l.o.g. assume that $x_1$ is set to true. Now, $A$ has to choose between the edges $(d_A, C_{1,A})$ and $(d_A, C_{3,A})$ for some $j > 1$.

\textbf{Case IIa:} $A$ moves along the edge $(d_A, C_{1,A})$. In the next step, $B$ has to move along $(C_{1,B}, q_{1,B})$. $A$ may decide to (see also Table \ref{tab:2})

(i) move along $(C_{1,A}, q_{1,A})$. Then, $B$ moves along $(q_{1,B}, v_{2,5})$, because otherwise the game ends with $c(B) > 1$. In the next step, $A$ has to move along $(q_{1,A}, w_1)$. Since $C_1$ is not satisfied by $\bar{x}_2$, i.e., $\bar{x}_2$ has not been used yet, $B$ can move along $(v_{2,5}, v_{2,4})$ in the next step. The following moves must be $A$ moving along $(w_1, z_1)$ and $B$ along $(v_{2,4}, v_{2,3})$. In the final step, $A$ has
to end the game using the edge \((z_1, q_{1,B})\), because otherwise a meeting point is no longer possible. Thus, the game ends with \(c(A) > 3.1\) and \(c(B) = \varepsilon(2n + 5)\).

(ii) move along \((C_{1,A}, v_{1,5})\). \(B\) has the choice between moving along (*) \((q_{1,B}, v_{2,5})\) or (**) \((q_{1,B}, z_1)\).

(*) Assume \(B\) chooses the former. In the following steps, \(A\) has to move along \((v_{1,5}, v_{1,4})\) and \(B\) along \((v_{2,5}, v_{2,4})\). Now, if \(A\) moves along \((v_{1,4}, v_{1,3})\), \(B\) has to move along \((v_{2,4}, q_{2,3})\), and \(A\) cannot perform another move, i.e., a meeting point is no longer possible. If \(A\) chooses \((v_{1,4}, q_{1,A})\) instead of \((v_{1,4}, v_{1,3})\), again \(B\) has to move along \((v_{2,4}, q_{2,3})\), leaving \(A\) with the only possible move along \((q_{1,A}, w_1)\).

Now, \(B\) is unable to move, and a meeting point is no longer possible. In either case, we hence get a contradiction with (R1).

(**) Hence, \(B\) moves along \((q_{1,B}, z_1)\), followed by \(A\) moving along \((v_{1,5}, v_{1,4})\).

If \(B\) moves along \((z_1, q_{1,B})\), the cost of \(B\) increases by \(c(z_1, q_{1,B}) = 2.1\); i.e., in the end, \(B\) will have to carry a cost of \(c(B) \geq 2.1 + \varepsilon(4\frac{n}{2} + 2)\). If \(B\) moves along \((z_1, w_3)\), \(A\) has to move along \((v_{1,4}, q_{1,A})\) to ensure a meeting point is still possible. The final two moves necessarily are that \(B\) moves along \((w_1, z_1)\), followed by \(A\) moving along \((q_{1,A}, w_1)\). The corresponding costs are \(c(A) = \varepsilon(2n + 6)\) and \(c(B) = \varepsilon(2n + 3) + 2\).

As a consequence, in Case IIa, player \(A\) chooses to move along \((C_{1,A}, v_{1,5})\), which finally leads to an outcome with \(c(A) = \varepsilon(2n + 6)\) and \(c(B) = \varepsilon(2n + 3) + 2\).

**Case IIIa:** \(A\) moves along the edge \((d_A, C_{j,A})\) for some \(j > 1\). By construction, it follows that this admits a meeting point if and only if \(C_{j,A}\) is made up of one the existential literals of \(C_1\) that have already been selected by \(A\), i.e., set to true under \(\tau\). Again, \(B\) has no choice but to move along \((C_{1,B}, q_{1,B})\). Now, it is easy to see that \(A\) moving along \((C_{j,A}, q_{j,A})\) would not admit a meeting point since from \(q_{j,A}\) \((j > 1)\) none of the vertices \(w_1, z_1, q_{1,B}\) can be reached. The same is true if \(A\) moves towards the endpoint of an edge, i.e., literal, whose negation is not contained in \(C_1\). Thus, \(A\) needs to move towards the endpoint of a literal, whose negation is contained in \(C_1\). Analogously to Case IIa above,
it follows that the game ends with \( c(A) = \varepsilon(2n + 6) \) and \( c(B) = \varepsilon(2n + 3) + 2 \) if \( A \) moves along \((C_j,A, v_{1,5})\).

Summing up Case I and Case II it follows that, if \( C_1 \) is satisfied, then Shortest Connection Game ends with \( c(A) \leq \min\{\varepsilon(2n + 4), \varepsilon(2n + 6)\} < 1 \) and \( c(B) \geq \min\{1 + \varepsilon(2n + 3), 2 + \varepsilon(2n + 3)\} > 1 \).

Case 2: \( I \) is a “no”-instance. I.e., assume that \( C_1 \) is not satisfied by \( \tau \).

Assume \( A \) does not move along the edge \((d_A, C_{1,A})\). Hence, \( A \) must move along an edge \((d_A, C_{j,A})\), for some \( j > 1 \). Recall that \( w_1, z_1, q_{1,B} \) are the only possible meeting points. However, from \( C_{j,A} \) it is impossible for \( A \) to reach one of these points, because otherwise it must be reached via the endpoint of the negation of an existential literal that make up clause \( C_1 \), i.e., \( C_1 \) must be satisfied by \( \tau \).

Thus, \( A \) moves along \((d_A, C_{1,A})\). As it turns out, the same moves (and hence the same outcome) as in Case Ia(i) of Case I results. Next, \( B \) moves along \((C_{1,B}, q_{1,B})\). Since \( C_1 \) is not satisfied, \( A \) is not able to move along \((C_{1,A}, v_{1,5})\) or \((C_{1,A}, v_{3,5})\), because otherwise a meeting point is no longer possible. Thus, \( A \) needs to move along \((C_{1,A}, q_{1,A})\). Now, assume \( B \) moves along \((q_{1,B}, v_{2,5})\), which is possible because \( \overline{x}_2 \), and hence the edges \((v_{2,5}, v_{2,4})\) and \((v_{2,4}, v_{2,3})\) have not been used yet. In the next steps, \( A \) has to move along \((q_{1,A}, w_1)\), \( B \) along \((v_{2,5}, v_{2,4})\), \( A \) along \((w_1, z_1)\), and \( B \) along \((v_{2,4}, v_{2,3})\). Finally, \( A \) has no choice but to move along \((z_1, q_{1,B})\) since otherwise a meeting point is no longer possible, because \( B \) cannot select any further edge because \( B \) is stuck in vertex \( v_{2,3} \). Consequently, the game ends with \( c(A) = 1 + 2.1 + \varepsilon(2n + 4) = 3.1 + \varepsilon(2n + 4) \) and \( c(B) = \varepsilon(2n + 5) \).

To summarize, instance \( I \) is a “yes”-instance of Quantified \( \leq 4 \)-Sat if and only if instance \( \mathcal{M} \) of Shortest Connection Game ends with \( c(A) < 1 \). \( \Box \)

Note that the above \( \text{PSPACE} \)-completeness result holds also if Shortest Connection Game is restricted to simple paths, i.e., (R3) is imposed.

**Theorem 3.** Imposing (R3), Shortest Connection Game is \( \text{PSPACE} \)-complete for directed bipartite graphs.

**Proof.** The proof works analogously to the proof of Theorem 2 when the considered instance of Shortest Connection Game is slightly modified. In particular, we replace instance \( \mathcal{M} \) with graph \( G \) by instance \( \mathcal{M}' \) with graph \( G' \), where \( G' \) is created from \( G \) by, for each \( 1 \leq j \leq m \),

- removing the vertices \( w_j, z_j \) and all edges to which \( w_j \) or \( z_j \) are incident
- introducing the edges \((q_{j,A}, q_{j,B})\) and \((q_{j,B}, q_{j,A})\) of cost 1 each. \( \Box \)

### 2.2. Directed Acyclic Bipartite Graphs

In this section, we consider Shortest Connection Game in directed acyclic bipartite graphs. Note that due to the acyclicity, (R2) (and also (R3)) is obviously satisfied. It turns out that Shortest Connection Game remains
intractable when restricted to directed acyclic bipartite graphs. In particular, we provide an NP-hardness result by giving a reduction from Vertex Cover.

**Vertex Cover**

**Input:** Undirected graph \( H = (V_H, E_H) \), integer \( k \in \mathbb{N} \).

**Question:** Is there a vertex cover of size at most \( k \), i.e., a subset \( V' \subseteq V_H \) with \( |V'| \leq k \) such that for each edge \( \{i, j\} \in E_H \) at least one of \( i, j \) belongs to \( V' \)?

**Remark.** Clearly, a vertex cover of size at most \( k \) exists if and only if there is a vertex cover of size exactly \( k \).

**Theorem 4.** Shortest Connection Game is strongly NP-hard for directed acyclic bipartite graphs.

**Proof.** For the sake of readability, we provide a detailed proof for the case of directed acyclic graphs; finally, however, we will describe how the proof can easily be extended to directed acyclic bipartite graphs.

Given an instance \( V \) of Vertex Cover with an undirected graph \( H = (V_H, E_H) \) and an integer \( k \in \mathbb{N} \), we construct an instance \( W \) of Shortest Connection Game with directed graph \( G \) as follows. We assume \( k \geq 3 \). Within this proof, let \( n := |V_H| \) and \( m := |E_H| \). W.l.o.g. let \( V_H = \{1, \ldots, n\} \) and \( E_H = \{e_1, \ldots, e_m\} \). W.l.o.g., we assume that \( \{1, 2\} \in E_H \).

**Part 1: Construction of Graph \( G \).**

We derive a directed graph \( G = (V, E) \) as follows (see also Fig. 2):

- we introduce vertices \( s, t, d, d', f, g \) and \( u_1, \ldots, u_{k+1} \)
- edges \( (d, d'), (g, d), (g, f), (t, u_1), (u_{k+1}, d), \) and \( (u_h, u_{h+1}) \) for \( 1 \leq h \leq k \)
- for each vertex \( i \in V_H \) introduce
  - a vertex \( s_i \) and the edges \( (s, s_i) \) and \( (s_i, g) \)
  - edge \( (s_i, s_j) \) for all \( j > i \)
- for each edge \( e = \{i, j\} \in E_H \) with \( i < j \) we introduce
  - vertices \( t_{i,j}, b_{i,j}, \ell_{i,j}, h_{i,j}, v_{i,j}, w_{i,j}, y_{i,j}, z_{i,j} \)
  - vertices \( s_{i,j}, \alpha_{i,j}, \beta_{i,j}, \gamma_{i,j}, \eta_{i,j}, \delta_{i,j}, \kappa_{i,j}, \lambda_{i,j}, \mu_{i,j} \)
  - edges \( (u_{h+1}, t_{i,j}), (t_{i,j}, b_{i,j}), (b_{i,j}, \ell_{i,j}), (b_{i,j}, h_{i,j}) \)
  - edges \( (\ell_{i,j}, v_{i,j}), (\ell_{i,j}, w_{i,j}), (h_{i,j}, y_{i,j}), (h_{i,j}, z_{i,j}) \)
  - edges \( (v_{i,j}, s_i), (y_{i,j}, s_j), (f, s_{i,j}) \)
Figure 2: Directed graph $G = (V, E)$ in instance $W$
- edges \((s_{i,j}, \alpha_{i,j}), (\alpha_{i,j}, \beta_{i,j}), (\beta_{i,j}, \gamma_{i,j}), (\gamma_{i,j}, \eta_{i,j})\)
- edges \((s_{i,j}, \delta_{i,j}), (\delta_{i,j}, \kappa_{i,j}), (\kappa_{i,j}, \lambda_{i,j}), (\lambda_{i,j}, \mu_{i,j})\)
- edges \((\alpha_{i,j}, z_{i,j}), (\delta_{i,j}, w_{i,j})\)

For the sake of readability, Fig. 2 displays the detailed construction of the graph for the edge \(e = \{1, 2\} \in E_H\). Note that \(G\) is an acyclic directed graph. The costs of the edges in graph \(G = (V, E)\) are defined as follows. Let \(c(g, d) = 3\).

For each \(e = \{i, j\} \in E_H\) let \(c(\ell_{i,j}, v_{i,j}) = c(h_{i,j}, y_{i,j}) = 1\), \(c(\delta_{i,j}, w_{i,j}) = c(\alpha_{i,j}, z_{i,j}) = 2\), and \(c(\gamma_{i,j}, \eta_{i,j}) = c(\lambda_{i,j}, \mu_{i,j}) = 2\). For the remaining edges \(e' \in E\) let \(c(e') = \varepsilon\) with \(\varepsilon = \frac{1}{20}\). Note that in any instance of \(\text{VERTEX COVER}\) obviously \(k \leq n\) must hold. Hence, the size of \(G\) is polynomial in the size of \(H\), since

\[
|V| = 6 + (k + 1) + n + 17m \in \mathcal{O}(n + m)
\]

and

\[
|E| = 5 + k + 2n + \frac{(n - 1)n}{2} + 21m \in \mathcal{O}(n^2 + m)
\]

In instance \(W\) of \(\text{SHORTEST CONNECTION GAME}\), we are given the directed graph \(G\) and the two players \(A, B\). \(A\) starts in vertex \(s\), while \(B\) starts in vertex \(t\). \(A\) is the first player to move along an edge.

**Part 2: The Reduction.** We show that the following claim holds:

Instance \(W\) of \(\text{SHORTEST CONNECTION GAME}\) ends with \(c(A) < 1\) if and only if in instance \(V\) of \(\text{VERTEX COVER}\), graph \(H = (V_H, E_H)\) admits a vertex cover of size at most \(k\).

By construction of \(G\), the first \((k + 1)\) edges player \(B\) selects obviously form the path \((t, u_1, u_2, \ldots, u_{k+1})\).

**Observation 1.** If the head of the \((k + 1)\)-th edge selected by player \(A\) is different from \(g\), then the game ends with \(c(A) > 3\).

**Proof of Observation 1.**

**Case a:** Assume that \(A\) has already visited \(g\) before selecting the \((k + 1)\)-th edge. I.e., after selecting the \((k + 1)\)-th edge, player \(A\) is either in one of the vertices \(d, d'\) or in a vertex of the set \(E' \cup \{f\}\), where \(E' := \{s_{i,j}, \alpha_{i,j}, \beta_{i,j}, \gamma_{i,j}, \delta_{i,j}, \kappa_{i,j}, \lambda_{i,j}, z_{i,j}, w_{i,j} \mid \{i, j\} \in E_H\}\).

If \(A\) is in \(d\) or \(d'\), \(A\) must have moved along the edge \((g, d)\) of cost 3 because this is the only way \(A\) can reach the vertices \(d, d'\). Thus, observation 1 is obviously true.

Assume that \(A\) is in \(E' \cup \{f\}\).

- First, suppose \(A\) is in one of the vertices in \(E'\). We will show that this in fact is not feasible, because otherwise a meeting point is not possible. W.l.o.g., assume \(A\) to be in one of the vertices of \(E'\) with index \(\{i, j\} = \{1, 2\}\). Then, it is not hard to verify that the possible meeting points are either (i) \(w_{1,2}\) or \(z_{1,2}\), or (ii) the vertices \(s, s_1, \ldots, s_n, g\).
To reach one of the vertices \( w_{1,2} \) or \( z_{1,2} \) (if reachable for \( A \)), \( A \) has at most 2 more edges to select before \( A \) gets stuck in one of these vertices; however, \( B \), who is currently in \( u_{k+1} \), still needs to move along at least four edges to reach one of \( w_{1,2} \), \( z_{1,2} \). Hence, meeting in one of these vertices is impossible, because the players need to take turns at selecting edges. For the vertices listed in (ii), \( B \) needs to move along at least 5 edges to reach one of them; however, \( A \) has at most four edges to select before \( A \) is stuck in a vertex. Hence, meeting in one of these vertices is impossible as well.

- Second, suppose that \( A \) is in \( \{f\} \), i.e., has chosen edge \((g, f)\) as her \((k + 1)\)-th edge. Recall that \( B \) is in \( u_{k+1} \) after selecting her \((k + 1)\)-th edge. Now, \( A \) has to move along an edge \((f, s_{1,i})\). W.l.o.g., assume that \( A \) moves along \((f, s_{1,2})\). Clearly, the best strategy for \( B \) is to connect with \( A \) as soon as possible, only moving along edges of cost \( \varepsilon \). Moving to \( d \) is infeasible, since \( A \) cannot reach \( d \) anymore. Hence, \( B \) moves along \((u_{k+1}, t_{1,2})\).

Now \( A \) has to choose between moving to \( \alpha_{1,2} \) and \( \delta_{1,2} \). Assume that \( A \) moves to \( \alpha_{1,2} \) (the case \( A \) moves to \( \delta_{1,2} \) follows in analogous manner). Next, \( B \) necessarily selects edge \((t_{1,2}, b_{1,2})\).

- If \( A \) moves along the edge \((\alpha_{1,2}, z_{1,2})\), \( A \) is stuck in \( z_{1,2} \). However, \( B \) cannot end the game with the selection of just one additional edge, hence a meeting point is not possible.
- Hence, \( A \) needs to move along the edge \((\alpha_{1,2}, \beta_{1,2})\) instead. After \( B \)'s next move (either to \( h_{1,2} \) or \( t_{1,2} \), \( A \) necessarily moves along \((\beta_{1,2}, \gamma_{1,2})\). Clearly, irrespective of whether \( B \) is in \( h_{1,2} \) or \( t_{1,2} \), the game does not end after \( B \)'s next move. Hence, \( A \) needs to select an additional edge, which necessarily is the edge \((\gamma_{1,2}, \eta_{1,2})\) of cost 3.

**Case b:** Assume that \( A \) has not visited \( g \) yet. Since \( g \) is not the head of the \((k + 1)\)-th edge selected by \( A \), by construction of \( G \) that means that after both players have selected \((k + 1)\) edges, \( A \) is in some vertex \( s_i \), \( i \in V_H \), while \( B \) is located in \( u_{k+1} \).

Irrespective of \( A \)'s next move (after which \( A \) is in some \( s_i \), \( i \in V_H \) or in \( g \)), it is \( B \)'s best choice to move to \( d \): As a consequence, \( A \) is forced to move along \((g, d)\) of cost 3 either immediately or after each of the players have moved along another edge of cost \( \varepsilon \).

Summing up, we have shown that instance \( \mathcal{W} \) of Shortest Connection Game ends with \( c(A) > 3 \) if \( g \) is not the head of the \((k + 1)\)-th edge selected by player \( A \).

Assume that the \((k + 1)\)-th edge selected by \( A \) ends in \( g \), i.e., after both players have selected \((k + 1)\) edges, \( A \) is in \( g \) and \( B \) in \( u_{k+1} \). Let \( V' := \{i \in V_H \mid A \text{ has visited } s_i\} \). Note that \( |V'| = k \).
**Table 3:** Moves and outcomes in case IIa(1) in instance $W$.

| Case IIa(1)(i)(*) | $A$  | $B$          | $c(A) = 3 + \epsilon(k+6)$ | $c(B) = 1 + \epsilon(k+6)$ |
|-------------------|------|--------------|----------------------------|-----------------------------|
| $A$               | $f$  | $s_{1,2}$    | $\alpha_{1,2}$            | $\beta_{1,2}$ $\gamma_{1,2}$ $\eta_{1,2}$ |
| $B$               | $t_{1,2}$ | $b_{1,2}$ | $h_{1,2}$ | $y_{1,2}$ | $s_2$ | $g$ |

| Case IIa(1)(i)(**) | $A$  | $B$          | $c(A) = 2 + \epsilon(k+4)$ | $c(B) = \epsilon(k+5)$ |
|-------------------|------|--------------|----------------------------|-----------------------------|
| $A$               | $f$  | $s_{1,2}$    | $\alpha_{1,2}$ $z_{1,2}$  | $c(A) = 2 + \epsilon(k+4)$ |
| $B$               | $t_{1,2}$ | $b_{1,2}$ | $h_{1,2}$ | $z_{1,2}$ | $c(B) = \epsilon(k+5)$ |

**Case I:** $A$ moves along $(g,d)$. Clearly, then $B$ moves along $(u_{k+1},d)$; the game hence ends with $c(A) > 3$ and $c(B) = (k+2)\epsilon$.

**Case II:** $A$ moves along $(g,f)$. Now, it is $B$’s turn. In what follows, we argue that $c(B) > 1$ or, equivalently, $c(A) < 1$ if and only if $V'$ is a vertex cover of graph $H$ in instance $V$. In other words, we show that $c(B) < 1$ if and only if $B$ is able to move along an edge $(u_{k+1},t_{1,j})$ such that $\{i,j\} \cap V' = \emptyset$. We illustrate this by assuming that, w.l.o.g., $B$ moves along $(u_{k+1},t_{1,2})$.

**Case IIa:** $\{1,2\} \cap V' = \emptyset$.

1. $A$ moves along $(f,s_{1,2})$. $B$’s next move is along $(t_{1,2},b_{1,2})$. Now, $A$ needs to decide between moving to $\alpha_{1,2}$ or $\delta_{1,2}$. Assume $A$ chooses the former. We show that for $B$, the optimal behaviour is to move to $h_{1,2}$, leading to an outcome\(^3\) of $c(A) = 2 + \epsilon((k+1)+3)$ and $c(B) = ((k+1)+4)\epsilon$ (see also Table 3).

   i) Assume $B$ moves to $h_{1,2}$.
   
      (* If $A$ moves along $(\alpha_{1,2},\beta_{1,2})$, $B$ must not move along $(h_{1,2},z_{1,2})$, since otherwise a meeting point is no longer possible. Hence, $B$ has to move along $(h_{1,2},y_{1,2})$, imposing additional costs of 1 on player $B$. Next, $A$ moves along the only available edge $(\beta_{1,2},\gamma_{1,2})$. Now, $B$ has no other choice but to move along $(y_{1,2},s_2)$. Recall that, by assumption, $s_2$ has not been visited yet. I.e., the game is not over yet. Hence, $A$ needs to move along one more edge, i.e., the edge $(\gamma_{1,2},\eta_{1,2})$ of cost 3. In the next move, $B$ ends the game by moving, e.g., along edge $(s_2,g)$. Summing up, $A$’s decision to move along $(\alpha_{1,2},\beta_{1,2})$ results in an outcome of $c(A) > 3$.

   (**) If $A$ moves along $(\alpha_{1,2},z_{1,2})$ of cost 2 instead, in the next step the game ends with $B$’s move along $(h_{1,2},z_{1,2})$. The respective costs are $c(A) = 2 + \epsilon((k+1)+3)$ and $c(B) = ((k+1)+4)\epsilon$.

   Hence, $A$ will move along $(\alpha_{1,2},z_{1,2})$ instead of $(\alpha_{1,2},\beta_{1,2})$ after $B$’s move to $h_{1,2}$. The game ends with $c(A) = 2 + \epsilon(k+4)$ and $c(B) = \epsilon(k+5)$.

\(^3\)It is not hard to see that by analogous arguments we can conclude the game also ends with $c(A) = 2 + \epsilon((k+1)+3)$ and $c(B) = ((k+1)+4)\epsilon$ in the case that $A$ moves to $\delta_{1,2}$ instead of $\alpha_{1,2}$.  

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(ii) Assume $B$ moves to $\ell_{1,2}$. Irrespective of $A$’s next move, $B$ must not move along $(\ell_{1,2}, w_{1,2})$, because there is no edge emanating from $w_{1,2}$ and $A$ is unable to reach $w_{1,2}$ from $\alpha_{1,2}$. Hence, $B$ would have to move along $(\ell_{1,2}, v_{1,2})$ of cost 1, which exceeds the total cost $B$ would have to carry if $B$ moved along $h_{1,2}$ instead of $\ell_{1,2}$.

(2) $A$ moves along $(f, s_{i,j})$ for some $\{i, j\} \in E_B \setminus \{1, 2\}$. $B$ moves along $(t_{1,2}, b_{1,2})$. Clearly, $A$ and $B$ cannot meet in one of the vertices $w_{i,j}, z_{i,j}$ by construction of $G$. Hence, $B$ must travel one of the vertices $s_{1}, s_{2}$ in order to enable a meeting point. Now, $A$ has exactly four more edges to select before $A$ gets stuck in a vertex (i.e., in $\eta_{i,j}$ or $\mu_{i,j}$), while $B$ needs to select exactly three more edges to reach one of the vertices $s_{1}, s_{2}$. However, since none of the vertices have been visited yet, that means that $B$ would have to select a further edge in order to end the game. Hence, $A$ needs to move along either $(\gamma_{i,j}, \eta_{i,j})$ or $(\lambda_{i,j}, \mu_{i,j})$, i.e., along an edge of cost 3.

Consequently, $A$’s optimal strategy is to move along $(f, s_{1,2})$.

Summing up, in Case IIa the game ends with $c(A) = 2 + \varepsilon(k + 4)$ and $c(B) = \varepsilon(k + 5)$.

**Case IIIa:** $\{1, 2\} \cap V' \neq \emptyset$. Assume that $1 \in V'$ (the case $2 \in V'$ follows by analogous arguments).

(1) $A$ moves along $(f, s_{1,2})$. $B$ moves along $(t_{1,2}, b_{1,2})$. Assume that $A$ moves along $(s_{1,2}, \delta_{1,2})$ (the case that $A$ moves along $(s_{1,2}, \alpha_{1,2})$ follows analogously).

(i) $B$ moves along $(b_{1,2}, h_{1,2})$. Now, if $A$ moves along $(\delta_{1,2}, w_{1,2})$, a meeting point is no longer possible. Hence, $A$ needs to move along $(\delta_{1,2}, \kappa_{1,2})$. Clearly, $B$ has to move along $(h_{1,2}, y_{1,2})$ with cost 1, otherwise a meeting point is not possible. Now, $A$ needs to move along $(\kappa_{1,2}, \lambda_{1,2})$, followed by $B$ moving along the edge $(y_{1,2}, s_{2})$.

Table 4 summarizes the moves performed.

(*) If $s_{2}$ has been visited already by $A$, i.e., if $2 \in V'$, then the game ends, the total costs being $c(A) = \varepsilon((k + 1) + 5)$ and $c(B) = 1 + \varepsilon(k + 5)$.

(**) Otherwise, $A$ needs to move along one more edge, i.e., the edge $(\lambda_{1,2}, \mu_{1,2})$ of cost 3, before the game ends with $B$’s last move, e.g., along the edge $(s_{2}, q)$. In this case the respective costs are $c(A) = 3 + \varepsilon((k + 1) + 5)$ and $c(B) = 1 + \varepsilon(k + 6)$.

| Case IIb(1)(i) | A | f | s_{1,2} | \delta_{1,2} | \kappa_{1,2} | \lambda_{1,2} |
|----------------|---|---|---------|------------|------------|------------|
| B             | t_{1,2} | b_{1,2} | h_{1,2} | y_{1,2} | s_{2}          |

Table 4: Situation in Case IIb(1)(i) in instance $W$. 

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(ii) $B$ moves along $(b_{1,2}, \ell_{1,2})$. Now, $A$ has the choice between $(\delta_{1,2}, w_{1,2})$ and $(\delta_{1,2}, \kappa_{1,2})$ (see also Table 5).

(*): If $A$ chooses $(\delta_{1,2}, w_{1,2})$ with cost 2, then $B$ obviously moves along $(\ell_{1,2}, v_{1,2})$ and the game ends. In this case, the costs of $A$ exceed 2, i.e., $c(A) > 2$ holds.

(**): If $A$ chooses the cheaper edge $(\delta_{1,2}, \kappa_{1,2})$ with cost $\varepsilon$, then $B$ needs to move along $(\ell_{1,2}, v_{1,2})$ of cost 1 to enable a meeting point. In the next step, $A$ moves along $(\kappa_{1,2}, \lambda_{1,2})$, followed by $B$ moving along $(v_{1,2}, s_1)$. This last move ends the game, because by assumption $s_1$ has been visited by $A$, i.e., $1 \in V'$ holds. The respective costs are $c(A) = ((k+1)+5)\varepsilon$ and $c(B) = 1 + \varepsilon(k + 5)$.

Hence, if $B$ moves along $(b_{1,2}, \ell_{1,2})$, then $A$ moves along $(\delta_{1,2}, \kappa_{1,2})$, which finally leads to an outcome of $c(A) = ((k+1)+5)\varepsilon$ and $c(B) = 1 + \varepsilon(k + 5)$.

The same outcome is achieved if $B$ moves along $(b_{1,2}, h_{1,2})$ instead and $2 \in V'$ holds; if the latter is not the case, $B$ does not move along $(b_{1,2}, h_{1,2})$ because this would lead to a worse outcome for $B$.

To sum up, if $A$ moves along $(f, s_{1,2})$, then the game ends with $c(A) = ((k+1)+5)\varepsilon$ and $c(B) = 1 + \varepsilon(k + 5)$.

(2) $A$ moves along $(f, s_{i,j})$ for some $\{i, j\} \in E_H \setminus \{1, 2\}$. After $B$’s move along $(\ell_{1,2}, b_{1,2})$, assume that $A$ moves along $(s_{i,j}, \delta_{i,j})$ (the case $A$ moves along $(s_{i,j}, \alpha_{i,j})$ follows by analogous arguments). As above, $B$ has the choice between $(b_{1,2}, h_{1,2})$ and $(b_{1,2}, \ell_{1,2})$.

(i) $B$ moves along $(b_{1,2}, h_{1,2})$. With exactly the same reasoning as above, it is not hard to see that the game ends with (i) $c(A) = ((k+1)+5)\varepsilon$ and $c(B) = 1 + \varepsilon(k + 5)$ if $2 \in V'$ or (ii) $c(A) = 3 + \varepsilon((k+1)+5)$ and $c(B) = 1 + \varepsilon(k + 6)$ if $2 \not\in V'$.

(ii) $B$ moves along $(b_{1,2}, \ell_{1,2})$. Now, $A$ has no choice but to move along $(\delta_{i,j}, \kappa_{i,j})$, since moving to $w_{i,j}$ makes $A$ getting stuck in $w_{i,j}$, and $B$ is unable to reach a meeting point by moving along a single edge emanating from $\ell_{1,2}$. Exactly as above, it follows that the game ends with $c(A) = ((k+1)+5)\varepsilon$ and $c(B) = 1 + \varepsilon(k + 5)$.

Hence, analogously to above it follows that the game ends with $c(A) = ((k+1)+5)\varepsilon$ and $c(B) = c(B) = 1 + \varepsilon(k + 5)$ if $A$ moves along $(f, s_{i,j})$ for some $\{i, j\} \in E_H \setminus \{1, 2\}$.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
Case IIb(1)(ii) (*) & $A$ & $f$ & $s_{1,2}$ & $\delta_{1,2}$ & $w_{1,2}$ & $c(A) = 2 + \varepsilon(k + 4)$ \\
& $B$ & $t_{1,2}$ & $b_{1,2}$ & $\ell_{1,2}$ & $w_{1,2}$ & $c(B) = \varepsilon(k + 5)$ \\
\hline
Case IIb(1)(ii) (**) & $A$ & $f$ & $s_{1,2}$ & $\delta_{1,2}$ & $\kappa_{1,2}$ & $\lambda_{1,2}$ & $c(A) = \varepsilon(k + 6)$ \\
& $B$ & $t_{1,2}$ & $b_{1,2}$ & $\ell_{1,2}$ & $v_{1,2}$ & $s_1$ & $c(B) = 1 + \varepsilon(k + 5)$ \\
\hline
\end{tabular}
\caption{Moves and outcomes in Case IIb(1)(ii) in instance $W$.}
\end{table}
As a consequence, Case IIb ends with $c(A) = ((k + 1) + 5)\varepsilon$ and $c(B) = 1 + \varepsilon(k + 5)$.

**Conclusion:** If (i) the head of the $(k+1)$-th edge selected by $A$ is not $g$, or (ii) $A$ moves along $(g, d)$, then the game ends with $c(A) > 3$ (Observation 1 resp. Case I). On the other hand, in Case II $A$ is guaranteed an outcome of less than 3.

As a consequence, in our considered instance $W$ player $A$ is in vertex $g$ after selecting her $(k+1)$-th edge and moves along $(g, f)$ in the next step. Thus, Case IIa ($V'$ is not a vertex cover of size $k$ in the graph $H$) or Case IIb ($V'$ is a vertex cover of size $k$ in the graph $H$) applies. In particular, it follows that instance $W$ of Shortest Connection Game ends with $c(A) = ((k + 1) + 5)\varepsilon < 1$ (or equivalently, $c(B) > 1$) if and only if there is a vertex cover of size $k$ in instance $V$ of Vertex Cover. Therewith, Shortest Connection Game is NP-hard for directed acyclic graphs.

Finally, we extend this NP-hardness result to directed acyclic graphs that are bipartite. In order to do so, we modify instance $W$ with graph $G = (V, E)$ into instance $W'$ with graph $G'$ by “splitting” each edge of $E$; i.e., $G'$ is created from $G$ by, for each $e = (u, v) \in E$ of cost $c$, introducing vertex $m_e$ and replacing $e$ with the two edges $(u, m_e)$ and $(m_e, v)$ of cost $\frac{c}{2}$ each. It is easy to see that the resulting graph is bipartite. It is also not hard to verify that the nature of the game is preserved and arguing analogously to above yields the desired result. □

3. **Shortest Connection Game on Cactus Graphs**

After the negative complexity results of Section 2 we try to identify polynomially solvable special cases and give some indications on the boundary between NP-hard and polynomially solvable cases. As an easy first step we state a trivial positive result for trees. Clearly (R3) is satisfied in this case.

**Proposition 5.** If $G$ is a tree, then $\text{spe-conn}$ of Shortest Connection Game can be determined in $O(n)$ time.

As a graph class which is only slightly more complex than trees, many authors consider cactus graphs for optimization problems. A cactus graph is a graph where each edge is contained in at most one simple cycle. Equivalently, any two simple cycles have at most one vertex in common. This means that one could contract each cycle into a vertex in a unique way and obtain a tree. Note that cactus graphs are a subclass of series-parallel graphs and thus have treewidth at most 2.

Rather surprisingly, we will show in the section below that Shortest Connection Game is already NP-hard on cactus graphs without restriction (R3). This is remarkable since it contrasts related games such as Geography or Shortest Path Game (cf. Bodlaender [1] and Darmann et al. [4]). On the other hand, imposing (R3) makes the problem polynomially solvable on cacti.
3.1. Hardness for Shortest Connection Game on Cactus Graphs

To show the NP-hardness of Shortest Connection Game on cactus graphs in its general form without condition (R3) we use a reduction from the well known strongly NP-complete problem 3-Partition:

3-Partition

**Input:** A multiset \( C = \{\tilde{c}_1, \tilde{c}_2, ..., \tilde{c}_{3n-1}, \tilde{c}_{3n}\} \) of \( 3n \) integers such that \( \sum_{i=1}^{3n} = n\tilde{K} \).

**Question:** Is there a partition of \( C \) into \( n \) triples such that the elements of each triple sum up to exactly \( \tilde{K} \).

**Theorem 6.** Shortest Connection Game is strongly NP-hard for directed cactus graphs.

**Proof.** Given a 3-Partition instance \( I \), we consider an equivalent 3-Partition instance \( J \) which is defined as follows. Let \( M = 2n\tilde{K} \), \( c_i = M + 2\tilde{c}_i \) and \( K = 3M + 2\tilde{K} \). Then \( J \) asks for a partition of the \( c_i \) into triples such that each triple sums to exactly \( K \). Clearly \( J \) is polynomially bounded in the size of \( I \).

Given \( J \), we will construct an instance of Shortest Connection Game for players \( A \) and \( B \) by means of a cactus graph \( G_{3P} \) as illustrated by Figure 3.1. All edge costs are 0 with the exception of two edges emanating in \( a_4 \) resp. \( b_4 \) with cost 1 resp. 2 (marked in red). The black integers describe the number of edges of the corresponding cycles. Player \( A \) and \( B \) start at vertices \( a_1 \) and \( b_1 \) respectively; \( A \) is first to choose an edge.

The high-level idea of the construction can be described as follows: Whenever \( J \) is a "yes" instance, players \( A \) and \( B \) reach a spe-conn with cost \((0, 1)\), whereas in a "no" instance spe-conn has cost \((1, 0)\). In case of a "yes" instance player \( B \) can dictate a path starting at \( b_1 \) going to \( b_2 \), \( b_3 \) and \( b_4 \). At \( b_4 \) player \( B \) moves through the cycles of length \( K \) and returns back to \( b_4 \) while avoiding the cycle of length \( 3M - 2 \) with cost 2. From there the path continues via \( b_5 \), \( b_3 \) and \( b_2 \) to the unique meeting point. Note that at \( b_5 \) some of the red edges might be used. In case of a "no" instance \( A \) can force \( B \) to use the cycle of length \( 3M - 2 \) with cost 2 at \( b_4 \) whenever \( B \) starts by moving to \( b_2 \) (and onwards to \( b_4 \)). Therefore, \( B \) chooses the direct path of length 3 to \( t \) with cost 1.
Assume that $B$’s first move is to go to vertex $b_2$. At this point of the game, player $A$ has no choice but to move towards $a_3$ since otherwise a feasible solution is not possible because $B$ would get stuck. Analogously, player $B$ now cannot move towards $t$ from $b_2$ (via the path of length four), since this does not allow for a feasible solution. Hence $B$ has to go towards $b_3$. When the players are in $a_3$ and $b_3$ respectively, $A$ cannot go towards $t$ by going “back” towards $a_2$ since again $B$ would get stuck. Hence, $A$ moves towards $a_4$. It is not difficult to see that this means that $B$ has to move towards $b_4$ because otherwise a feasible outcome is not possible anymore. As a consequence, $A$ and $B$ end up in $a_4$ and $b_4$ respectively, with $B$ having to choose the next edge.

Going directly to $t$ (via $b_5$ and $b_3$) is not feasible for $B$ since this requires $B$ to traverse at least 8 edges; $A$ however needs only 6 moves directly or at least $6 + \frac{AM}{2}$ moves via the shortest cycle at $a_4$. Also, going to $t$ via $b_5$ and some of the red cycles at $b_5$ do not allow for a feasible solution because in that way, $B$ can traverse at most $8 + \frac{M}{2} + 5 \max_i\{c_i\}$ edges.

Now, $B$ tries to avoid the cycle of length $3M - 2$ since it contains the most expensive edge in the graph of cost 2. So $B$ can choose either a cycle of length $K$ or the cycle of length $\frac{AM}{2}$ (and optionally the second one of the same length).

In the following we say that $B$ and $A$ are synchronized whenever $B$ is the one to choose the next move starting from $b_4$ and $A$ at the same time is located at $a_4$. Assume that $B$ chooses a cycle of length $K$: since $J$ is a “yes” instance $A$ can choose three cycles corresponding to three of the $c_i$ summing up to exactly $K$, so that $B$ and $A$ stay synchronized. When $B$ chooses to traverse a cycle of length $\frac{AM}{2}$ (or both of them), $A$ can choose to traverse a cycle of the same length (or both of them). By the same argument as above as long as $B$ and $A$ are synchronized $B$ cannot go to $t$. Hence, no matter which cycle $B$ chooses $A$ can choose cycles so that they stay synchronized until all cycles of length $K$ are consumed by $B$ and $A$ has only three of the cycles corresponding to three of the $c_i$ left. So $B$ has to take the cycle of length $3M - 2$ which results in a suboptimal cost of 2 for $B$ (see remarks at the beginning of this proof). Note that this move is feasible since $A$ can use the remaining cycles corresponding to the $c_i$. By the length of these cycles, $A$ is still at a vertex of its third chosen cycle when $B$ reaches $b_4$. Due to the choice of the $c_i$, $A$ needs an even number of edges until $a_4$ is reached. By construction, now $B$ can go to $t$ by using as many red edges at $b_5$ as needed in order to guarantee a feasible path for $A$ (in particular, note that the number of red edges used by $B$ is even too). Realizing this outcome with cost 2 resulting from the move to $b_2$, $B$ will instead choose the direct path from $b_1$ to $t$ of length three with cost 1, and $A$ can choose the path via $a_2$ with zero cost.

**Let $J$ be a ”no” instance of $3$-Partition.** We will argue that in this case $B$ can always choose a path with cost 0. Now $B$ avoids the direct path from $b_1$ to $t$ which would have cost 1. By the same arguments as in the ”yes” case $B$ goes to $b_4$ and $A$ to $a_4$ and they are synchronized. $B$ can choose cycles of length $K$ and $A$ is not able to stay synchronized with $B$ by only choosing cycles
corresponding to $c_i$ until all cycles of length $K$ are consumed by $B$. This follows by the fact that the corresponding 3-PARTITION instance is a "no" instance. In the following we will analyze all possible moves by $A$ which lead to a loss of synchronization. Assume that at stage $r_0$ (i.e., after player $B$ has traversed exactly $r_0$ edges and $A$ has traversed $r_0 + 1$ edges, i.e., it is player $B$’s turn to choose an edge) the players are synchronized. Assume $B$ chooses a cycle of length $K$ and $A$ chooses cycles such that at stage $r_1 = r_0 + 2K$ they are not synchronized any more. Note that $B$ is the one to choose the next edge at $r_1$. At stage $r_1$, $B$ has at least one cycle of length $K$ and $A$ at least six cycles corresponding to six $c_i$ left to choose because $J$ is a "no" instance.

**Case 1:** $A$ answers with three cycles corresponding to three $c_i$ such that at $r_1$ they are not synchronized any more. Recall that $K = 3M + 2\tilde{K}$ and $c_i = M + 2\tilde{c}_i$, hence there are two possibilities for being not synchronized at $r_1$.

**Case 1.1:** $A$ did not yet finish its third cycle corresponding to some $c_i$ at $r_1$.

In this case, $A$ has still less than

$$3(M + 2\max_i{\tilde{c}_i}) - (3M + 2K) \leq 4\max_i{\tilde{c}_i}$$

edges to go in order to reach $a_4$ for the next time. Here, $3(M + 2\max_i{\tilde{c}_i})$ is the total sum of edges that $A$ needs for three large cycles corresponding to $c_i$; $3M + 2\tilde{K} = K$ is the number of edges that $B$ needs in order to reach $b_4$ at $r_1$. Note that $\tilde{K} > \tilde{c}_i$ for all $i$. But now $B$ can feasibly go to $t$ by using as many of the red cycles at $b_5$ as necessary in order to feasibly force $A$ to $t$ via the edge of cost 1 at $a_4$ (see Figure 3). The game thus ends with a total cost of 1 for $A$ and 0 for $B$.

**Case 1.2:** $A$ has finished its third cycle (and reached $a_4$ again) before stage $r_1$ is reached. Following $B$’s next move after $A$ has reached $a_4$, $A$ has the following three options:

**Case 1.2.1:** $A$ chooses to traverse both cycles of length $\frac{3M}{4}$. But now $B$ can simply choose a cycle of length $K$ and go to $t$ via enough red cycles at $b_5$.

This is feasible since after the cycles of length $\frac{3M}{4}$ $A$ can still choose two cycles
corresponding to two \( c_i \) and then has a “delay” (with respect to the moment when \( B \) reaches \( b_4 \) after the cycle of length \( K \)) which lies in the following interval:

\[
\left[ \frac{M}{2} - 4\tilde{K}, \frac{M}{2} + 4 \max_i \{\tilde{c}_i\} \right]
\]

(1)

The lower bound comes from the following estimate. At \( r_0 \) \( A \) and \( B \) are synchronized, hence at \( r_1 \) \( A \) has used at most

\[
(3M + 2\tilde{K}) - 3(M + 2 \min_i \{\tilde{c}_i\}) \leq 2\tilde{K}
\]

edges of its first cycle of length \( \frac{3M}{4} \). Hence, at the moment when \( B \) reaches \( b_4 \) after \( r_1 \) for the next time, \( A \) has at least

\[
\left( \frac{3M}{2} + 2(M + 2 \min_i \{\tilde{c}_i\}) - 2\tilde{K} \right) - \left( 3M + 2\tilde{K} \right) \geq \frac{M}{2} - 4\tilde{K}
\]

edges to traverse on the cycle corresponding to the second \( c_i \) (cf. Figure 4).

Hence, the game ends with a total cost of 1 for \( A \) and 0 for \( B \).

The upper bound of (1) can be derived by the following argument. At \( r_0 \) \( A \) and \( B \) are synchronized, hence at \( r_1 \) \( A \) has used at least two edges of its first cycle of length \( \frac{3M}{4} \). Therefore, when \( B \) reaches \( b_4 \) for the first time after stage \( r_1 \), \( A \) has at most

\[
\left( \frac{3M}{2} + 2(M + 2 \max_i \{\tilde{c}_i\}) - 2 \right) - \left( 3M + 2\tilde{K} \right) \leq \frac{M}{2} + 4 \max_i \{\tilde{c}_i\}
\]

edges left to traverse on the cycle corresponding to the last \( c_i \).

**Case 1.2.2:** \( A \) chooses one cycle of length \( \frac{3M}{4} \). \( B \) now has to choose the cycle of length \( K \) again (\( B \) cannot foresee whether \( A \) chooses only one of the cycles of length \( \frac{3M}{4} \)). Now \( A \) chooses 3 cycles corresponding to three \( c_i \). When \( B \) finishes the cycle of length \( K \), the number of edges which \( A \) needs for finishing the last cycle lies in the interval:

\[
\left[ \frac{3M}{4} - 4\tilde{K}, \frac{3M}{4} + 6 \max_i \{\tilde{c}_i\} \right]
\]
If $A$ has more than $\frac{3M}{4}$ moves left, $B$ chooses one cycle of length $\frac{3M}{4}$ and goes to $t$ via red cycles at $b_5$. Otherwise, $B$ chooses both cycles of length $\frac{3M}{4}$ forcing $A$ to choose another cycle corresponding to a $c_i$ (note that at least three of them are still available). Finally, the number of edges which $A$ needs for finishing this cycle lies in the interval:

\[
\left[ \frac{M}{4} - 4K, \frac{M}{4} + 2 \max_i \{c_i\} \right]
\]

Now again $B$ can reach $t$ via red edges at $b_5$; the final outcome hence yields a cost of 1 for $A$ and 0 for $B$.

Case 1.2.3: $A$ chooses a cycle corresponding to a $c_i$. But now $B$ can choose the cycle of length $\frac{3M}{4}$. When $B$ reaches $b_4$ after traversing this cycle, $A$ hast at most $\frac{M}{4} + 2 \max_i \{c_i\}$ edges left on its cycle. Hence $B$ can again proceed to $t$ via red edges at $b_5$, resulting in a total cost of 1 for $A$ and 0 for $B$.

Case 2: $A$ answers with three cycles containing at least one cycle of length $\frac{3M}{4}$ such that at $r_1$ they are not synchronized any more. Recall that at $r_0$ both players were synchronized and $B$ chose a cycle of length $K$. Now there are five possibilities for $A$ in Case 2: (1) $A$ chooses one or both cycles of length $\frac{3M}{4}$ followed by cycles corresponding to $c_i$. (2) $A$ answers with one cycle corresponding to a $c_i$, one cycle of length $\frac{3M}{4}$ followed by two cycles corresponding to two $c_i$. (3) $A$ answers with one cycle corresponding to a $c_i$, both cycles of length $\frac{3M}{4}$ followed by one cycle corresponding to $c_i$. (4) $A$ answers with two cycles corresponding to two $c_i$, one cycle of length $\frac{3M}{4}$ followed by one cycle corresponding to $c_i$. (5) $A$ answers with two cycles corresponding to two $c_i$ and two cycles of length $\frac{3M}{4}$.

In any of these cases arguments similar to the arguments of Case 1.2 show that $B$ can feasibly reach $t$ without using the cycle of length $3M - 2$ with cost 2. Hence, in this case for spe-conn we have a cost of 1 for $A$ and 0 for $B$. □

3.2. A Polynomial Algorithm for Shortest Connection Game restricted to Simple Paths on Cactus Graphs

It turns out that in contrast to the negative result of Theorem 6 imposing (R3) and thus eliminating cycles allows a polynomial time algorithm for Shortest Connection Game on a cactus graph. It is based on a dynamic programming approach which mimics the backward induction process. However, because of the strong structural property of a cactus graph, we can reduce the feasible domain for one player, say $A$, to an acyclic directed graph, which allows a linear time backtracking process. For each vertex, which $A$ has currently reached, we consider all (that is $n$) possible vertices as current positions of $B$ and determine recursively the “best” (in the spe-conn sense) paths for both players to a potential meeting point. The cactus property allows to reduce the options for these meeting points.
Figure 5: The Cactus Graph $G_{3P}$
To construct the acyclic digraph for $A$ we first introduce an auxiliary edge set $E' \subseteq E$. Initially, $E'$ contains all edges of $E$ that might be used by player $A$. More formally, $(u, v) \in E$ is contained in $E'$ if there exists a directed path from $s$ to $u$. Next, consider all directed cycles $C$ in $E'$. By definition of a cactus, there can be only one path from $s$ entering $C$ at some vertex $v$: Otherwise, a second path entering $C$ at $v'$ would imply that either the edges between $v$ and $v'$ or between $v'$ and $v$ are contained in two cycles. Clearly, $A$ cannot use the “last” edge of cycle $C$, i.e. the edge $(u, v)$ entering $v$, since $A$ already must have passed through $v$ to reach $C$. Therefore, we eliminate $(u, v)$ from $E'$ which makes $E'$ acyclic and connected. $E'$ will remain constant throughout the execution of the algorithm.

For every vertex $a \in V$ define $M(a) \subseteq V$ as the set of vertices which player $A$ must visit in order to reach $a$ from $s$, i.e. $M(a)$ consists of those vertices that are contained in every directed path from $s$ to $a$. $M(a)$ also contains $a$.

**Proposition 7.** If in spe-conn player $A$ moves from $s$ until $a \in V$, then the meeting point $m$ is contained in $M(a)$ w.l.o.g.

**Proof.** Consider a possible meeting point $m \notin M(a)$. This means that there exists a cycle $C$ which $A$ has to traverse on the way from $s$ to $a$ and whose edges are oriented in such a way that $A$ has two possible ways to pass through $C$, one of them containing $m$. There can exist at most one such cycle: Otherwise, assume that there is a second cycle $C'$ with a potential meeting point $m'$. Now, there is a path from $t$ to $m$ and from $t$ to $m'$. Since there is also path between $m$ and $m'$, one of the two edges in $C$ emanating from $m$ must be contained in two cycles.

Since only one such configuration can exist, we can exchange the roles of $A$ and $B$ (and add a dummy edge $(t', t)$ to preserve the role of the first mover). □

In the following we describe a dynamic programming scheme to determine spe-conn.

Assume that $A$ is currently in $a \in V$ and $B$ in $b \in V$ with $A$ making the next move. We introduce arrays $\text{Cost}_A(a, b)$ resp. $\text{Cost}_B(a, b)$ for $A$ resp. $B$ containing the costs of the optimal subgame perfect equilibrium paths from the current vertices to the end of the game.

If a configuration $(a, b)$ is found out to be infeasible (e.g. $a$ is a leaf of $E'$ and $b \neq a$), then we set $\text{Cost}_A(a, b) = \text{Cost}_B(a, b) = \infty$. Note that this does not concern the aspect whether $(a, b)$ can be reached from the starting configuration $(s, t)$, but only whether an endpoint of the game could be reached from $(a, b)$. The cost of spe-conn will finally be reported in $\text{Cost}_A(s, t)$ and $\text{Cost}_B(s, t)$.

The recursive computations of these arrays will be performed in a bottom up way guided by an auxiliary edge set $F \subseteq E$ initialized by $F = E'$. Recall that $F$
is acyclic. Throughout the computation $F$ contains all edges leading to vertices which were not yet considered as positions of $A$. In the main computation we determine for each vertex $a \in V$ with outdegree 0 w.r.t. $F$ (i.e. for vertices whose successors were all dealt with) the entries $Cost(a, b)$ for all $b \in V$. Note that each entry of $Cost(a, \cdot)$ is computed only once and never updated. After completion of this task all incoming edges $(u, a)$ are eliminated from $F$ possibly generating new vertices with outdegree 0 w.r.t. $F$. This process is continued until $F = \emptyset$ and $Cost(s, b)$ is determined for all $b$.

Processing a vertex $a$ with no successors w.r.t. $F$ works as follows: For all $b \in M(a)$ player $B$ has reached a meeting point. Hence, $Cost_A(a, b) = Cost_B(a, b) = 0$. (This includes the case $a = b$.) For all $b \notin M(a)$, we consider all possible decisions for $A$ in $a$ and all possible reactions of $B$ in $b$. Therefore, let $\text{succ}(a) = \{v \in V \mid (a, v) \in E'\}$ and $\text{succ}(b) = \{v \in V \mid (b, v) \in E\}$. If $\text{succ}(a) = \emptyset$ then player $A$ “got stuck” and no feasible solution can be reached. Thus, $Cost_A(a, b) = Cost_B(a, b) = \infty$. Otherwise, if $\text{succ}(b) = \emptyset$ then player $B$ has no feasible move left. If player $A$ can still reach $b$ by traversing one edge, $A$ reaches a meeting point. Therefore, if $(a, b) \in E'$ then $Cost_A(a, b) = c(a, b)$, $Cost_B(a, b) = 0$. Otherwise, if $(a, b) \notin E'$ then $Cost_A(a, b) = Cost_B(a, b) = \infty$. There remains the general case where both $\text{succ}(a)$ and $\text{succ}(b)$ are non-empty.

For all possible decisions taken by $A$ in $a$, i.e. for all $a' \in \text{succ}(a)$, we determine the best reaction by $B$. If $a' = b$ then no move is necessary for $B$ and we set $b(a') = b$. Otherwise, there is

$$b(a') = \arg \min_{b' \in \text{succ}(b)} \{c(b, b') + Cost_B(a', b')\}. \quad (2)$$

This implies the cost of $A$ resulting from choosing $a'$. It remains to select the best decision $\bar{a}$ for $A$:

$$\bar{a} = \arg \min_{a' \in \text{succ}(a)} \{c(a, a') + Cost_A(a', b(a'))\} \quad (3)$$

Finally, we set

$$Cost_A(a, b) = c(a, \bar{a}) + Cost_A(\bar{a}, b(\bar{a}))$$
$$Cost_B(a, b) = c(b, b(\bar{a})) + Cost_B(\bar{a}, b(\bar{a})).$$

**Theorem 8.** *Imposing (R3), spec-corn of Shortest Connection Game on cactus graphs can be computed in $O(n^2)$ time.*

**Proof.** Throughout the execution of the algorithm, the total number of vertices $a'$ considered in (2) as a successor of some $a$ is bounded by $|E'|$, since each edge is used only once (although a vertex may well be considered multiple times). For each candidate $a'$ we have to consider all possible combinations of some vertex with a successor $b'$. The number of such pairs $(b, b')$ is trivially bounded by $|E|$. Thus, the running time is bounded by $O(|E|^2)$. The statement follows since it is well-known that the number of edges of a cactus graph is $O(n)$.
4. Conclusion

In this work, we have shown that *Shortest Connection Game* is computationally difficult not only on general graphs, which follows from our PSPACE-completeness-proof for bipartite graphs, but still *NP*-hard even for more restricted graph classes such as directed acyclic graphs or cactus graphs. Since *Shortest Connection Game* is trivial on trees, this hardness result for cactus graphs establishes rather clearly the boundary between tractable and hard cases.

On the other hand, imposing the restriction that each vertex can be visited only once by each player, i.e. a restriction to simple paths, makes *Shortest Connection Game* polynomially solvable on cactus graphs. Under this restriction, we believe that it is a challenging problem to determine the computational complexity status of *Shortest Connection Game* and possibly find polynomial time algorithms for classical generalizations of cactus graphs such as, for instance, outerplanar graphs or series parallel graphs. At least for a dynamic programming based approach it seems to be necessary to have some knowledge about possible meeting points on the way to a certain configuration, as given by Proposition 7 for cactus graphs. Unfortunately, even for outerplanar graphs no direct generalization of this statement could be found. Thus, we believe that *Shortest Connection Game* might be computationally hard even for minor generalizations of cactus graphs.

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