NEW EXTREMAL BINARY SELF-DUAL CODES FROM A BAUMERT-HALL ARRAY

ABIDIN KAYA AND BAHATTIN YILDIZ

Abstract. In this work, we introduce new construction methods for self-dual codes using a Baumert-Hall array. We apply the constructions over the alphabets $\mathbb{F}_2$ and $\mathbb{F}_4 + u\mathbb{F}_4$ and combine them with extension theorems and neighboring constructions. As a result, we construct 46 new extremal binary self-dual codes of length 68, 26 new best known Type II codes of length 72 and 8 new extremal Type II codes of length 80 that lead to new 3−(80, 16, 665) designs. Among the new codes of length 68 are the examples of codes with the rare $\gamma = 5$ parameter in $W_{68,2}$. All these new codes are tabulated in the paper.

1. Introduction

Finding extremal binary self-dual codes with new weight enumerators has been a topic of considerable interest in the Coding Theory community for decades now. There are motivating factors for this interest that come, in part, from the connection of self-dual codes to structures such as designs, lattices and invariant polynomials. The Assmus-Mattson theorem, for example, establishes a strong connection between self-dual codes and designs. This connection was used in [16] to find new 3-designs from Type II extremal binary self-dual codes of length 80.

To understand some of the construction methods for self-dual codes, we recall that a binary self-dual code of length $2n$ is generated by (up to equivalence) a matrix of the form $[I_n | A]$, where $A$ is an $n \times n$ block matrix. In the absence of any other algorithm, the randomness of the matrix $A$ makes for an impractical search field of $2^{n^2}$. Even when the orthogonality relations are factored in, we still have a search field of size $2^{O(n^2)}$, which is still considerably far from being practical and impractical with the current technology. This is partly the reason why the existence/non-existence of the Type II extremal binary self-dual code of length 72 is still an open problem for coding theorists.

The difficulties in a general search for self-dual codes have led researchers to utilize particular types of matrices in an effort to reduce the search field. Most common techniques in the literature use some variation of a construction that uses circulant matrices. The double-circulant, bordered double circulant, four circulant constructions are all special constructions that use circulant matrices. In most instances of these constructions, the search field is reduced from $2^{O(n^2)}$ to $2^{O(n)}$, which is a big improvement in the implementation of search algorithms.

Another approach that has been used in the literature is combining the constructions mentioned above over rings that are equipped with orthogonality-preserving
Gray maps. The algebraic structure of rings and the nature of the Gray map lead to binary self-dual codes with a particular automorphism group that may have been missed by the previous constructions. This has been successfully applied in works such as [10], [14], [15], [16], [17]. In [13], these ideas were applied for constructing formally self-dual codes of high minimum distances.

In this work, we describe a construction coming from a Baumert-Hall array to find binary self-dual codes. Similar matrices coming from Discrete Mathematics have been used to construct self-dual codes before, e.g. [15], [21]. We apply the constructions over the binary field as well as the rings \( \mathbb{F}_2 + u \mathbb{F}_2 \) and \( \mathbb{F}_4 + u \mathbb{F}_4 \), which are equipped with orthogonality-preserving Gray maps. The constructions turn out to be efficient as we are able to find many new extremal binary self-dual codes. In particular we find 46 extremal binary self-dual codes of length 68 with new weight enumerators, including the examples of codes with the rare \( \gamma = 5 \) parameter in \( W_{68,2} \). The existence of codes with \( \gamma = 8 \) and \( \gamma = 9 \) is still an open problem. We also find 26 new best known Type II codes of length 72 and 8 new extremal Type II codes of length 80 that lead to new \( 3 - (80, 16, 665) \) designs.

The rest of the work is organized as follows. In section 2, we give the preliminaries on circulant matrices, self-dual codes, alphabets we use and Baumert-Hall arrays. In section 3, we describe the construction method for self-dual codes. In section 4, we give the numerical results and tables corresponding to the codes constructed. We finish the paper with concluding remarks and directions for possible future research.

2. Preliminaries

2.1. Matrices. The circulant matrices are a special type of matrices that are used heavily in many constructions for extremal self-dual codes. We recall that a circulant matrix is a square matrix where each row is a right-circular shift of the previous row. In other words, if \( \varpi \) is the first row, a typical circulant matrix is of the form

\[
\begin{bmatrix}
\varpi \\
\sigma(\varpi) \\
\sigma^2(\varpi) \\
\vdots \\
\sigma^{n-1}(\varpi)
\end{bmatrix},
\]

where \( \sigma \) denotes the right circular shift. It is clear that, with \( T \) denoting the permutation matrix corresponding to the \( n \)-cycle \( (123\ldots n) \), a circulant matrix with first row \( (a_1, a_2, \ldots, a_n) \) can be expressed as a polynomial in \( T \) as:

\[
a_1I_n + a_2T + a_3T^2 + \cdots + a_nT^{n-1}.
\]

Since \( T \) satisfies \( T^n = I_n \), this shows that circulant matrices commute. This property of circulant matrices is essential in the four-circulant constructions as well as the constructions that we will be using in subsequent sections.

\( \lambda \)-circulant matrices are similar to circulant matrices, where instead of the right circular shift, the \( \lambda \)-circular shift is used:

\[
\sigma_{\lambda}(a_1, a_2, \ldots, a_n) = (\lambda a_n, a_1, \ldots, a_{n-1}).
\]

Thus a \( \lambda \)-circulant matrix is a matrix of the form
the elements of the upper bound for the minimum distance $d$ weights of all codewords are multiples of 4 and Type I otherwise. Rains finalized the usual binary self-dual codes. Binary self-dual codes are called Type II if the $\langle v, v \rangle = 0$ for all $v \in C$. If $C \subseteq C^\perp$, $C$ is called self-orthogonal, and $C$ is self-dual if $C = C^\perp$.

The main case of interest for us is the case when $R = \mathbb{F}_2$, in which case we obtain the usual binary self-dual codes. Binary self-dual codes are called Type II if the weights of all codewords are multiples of 4 and Type I otherwise. A self-dual binary code is called extremal if it meets the bound. Extremal binary self-dual codes of different lengths have particular weight enumerators as has been described in [6], [7]. However, while for some lengths a complete classification has been completed, for many lengths the existence of codes with a particular weight enumerator is still an open problem. With the constructions that we apply in this work, we have added to the list of known codes.

We will be considering two special rings besides the binary field in constructing our examples, i.e., the ring $\mathbb{F}_2 + u\mathbb{F}_2$ and $\mathbb{F}_4 + u\mathbb{F}_4$. Let $\mathbb{F}_4 = \mathbb{F}_2(\omega)$ be the quadratic field extension of $\mathbb{F}_2$, where $\omega^2 + \omega + 1 = 0$. The ring $\mathbb{F}_4 + u\mathbb{F}_4$ defined via $u^2 = 0$ is a commutative binary ring of size 16. We may easily observe that it is isomorphic to $\mathbb{F}_2[\omega, u]/\langle u^2, \omega^2 + \omega + 1 \rangle$. The ring has a unique non-trivial ideal $\langle u \rangle = \{0, u, u\omega, u + u\omega\}$. Note that $\mathbb{F}_4 + u\mathbb{F}_4$ can be viewed as an extension of $\mathbb{F}_2 + u\mathbb{F}_2$ and so we can describe any element of $\mathbb{F}_4 + u\mathbb{F}_4$ in the form $\omega a + \bar{\omega} b$ uniquely, where $a, b \in \mathbb{F}_2 + u\mathbb{F}_2$.

The maps $\phi_1 : \mathbb{F}_2 + u\mathbb{F}_2 \rightarrow \mathbb{F}_2^2$, given by $\phi_1(a + ub) = (b, a + b)$ and

$$\varphi_{\mathbb{F}_4 + u\mathbb{F}_4} : (\mathbb{F}_4 + u\mathbb{F}_4)^n \rightarrow (\mathbb{F}_2 + u\mathbb{F}_2)^{2n}, a\omega + b\bar{\omega} \mapsto (a, b), \quad a, b \in (\mathbb{F}_2 + u\mathbb{F}_2)^n$$

are orthogonality and distance preserving maps that were described partially in [19] and were fully described and used in [16]. They will be used here to construct binary self-dual codes.

In order to fit the upcoming tables we use hexadecimal number system to describe the elements of $\mathbb{F}_4 + u\mathbb{F}_4$. The one-to-one correspondence between hexadecimals

$$\begin{bmatrix}
\sigma_1^\perp
\
\sigma_2^\perp
\
\vdots
\
\sigma_{n-1}^\perp
\end{bmatrix},$$

where $\sigma_1^\perp$ is the first row. $\lambda$-circulant matrices share the commutativity property of circulant matrices in matrix multiplication.

2.2. Background on Codes. Let $R$ be a finite ring. A linear code $C$ of length $n$ over $R$ is an $R$-submodule of $R^n$. The elements of $C$ are called codewords.

Let $\langle u, v \rangle$ be inner product of two codewords $u$ and $v$ in $R^n$ which is defined as $\langle u, v \rangle = \sum_{i=1}^n u_i v_i$, where the operations are done in $R^n$. The dual code of a code $C$ is $C^\perp = \{ v \in R^n \mid \langle u, v \rangle = 0 \text{ for all } v \in C \}$. If $C \subseteq C^\perp$, $C$ is called self-orthogonal, and $C$ is self-dual if $C = C^\perp$.

Our examples, i.e., the ring $\mathbb{F}_2^\perp = \mathbb{F}_2(\omega)$ be the quadratic field extension of $\mathbb{F}_2$, where $\omega^2 + \omega + 1 = 0$. The ring $\mathbb{F}_4 + u\mathbb{F}_4$ defined via $u^2 = 0$ is a commutative binary ring of size 16. We may easily observe that it is isomorphic to $\mathbb{F}_2[\omega, u]/\langle u^2, \omega^2 + \omega + 1 \rangle$. The ring has a unique non-trivial ideal $\langle u \rangle = \{0, u, u\omega, u + u\omega\}$. Note that $\mathbb{F}_4 + u\mathbb{F}_4$ can be viewed as an extension of $\mathbb{F}_2 + u\mathbb{F}_2$ and so we can describe any element of $\mathbb{F}_4 + u\mathbb{F}_4$ in the form $\omega a + \bar{\omega} b$ uniquely, where $a, b \in \mathbb{F}_2 + u\mathbb{F}_2$.

The maps $\phi_1 : \mathbb{F}_2 + u\mathbb{F}_2 \rightarrow \mathbb{F}_2^2$, given by $\phi_1(a + ub) = (b, a + b)$ and

$$\varphi_{\mathbb{F}_4 + u\mathbb{F}_4} : (\mathbb{F}_4 + u\mathbb{F}_4)^n \rightarrow (\mathbb{F}_2 + u\mathbb{F}_2)^{2n}, a\omega + b\bar{\omega} \mapsto (a, b), \quad a, b \in (\mathbb{F}_2 + u\mathbb{F}_2)^n$$

are orthogonality and distance preserving maps that were described partially in [19] and were fully described and used in [16]. They will be used here to construct binary self-dual codes.

In order to fit the upcoming tables we use hexadecimal number system to describe the elements of $\mathbb{F}_4 + u\mathbb{F}_4$. The one-to-one correspondence between hexadecimals
and binary 4 tuples is as follows:

\[
\begin{align*}
0 & \leftrightarrow 0000, \quad 1 \leftrightarrow 0001, \quad 2 \leftrightarrow 0010, \quad 3 \leftrightarrow 0011, \\
4 & \leftrightarrow 0100, \quad 5 \leftrightarrow 0101, \quad 6 \leftrightarrow 0110, \quad 7 \leftrightarrow 0111, \\
8 & \leftrightarrow 1000, \quad 9 \leftrightarrow 1001, \quad A \leftrightarrow 1010, \quad B \leftrightarrow 1011, \\
C & \leftrightarrow 1100, \quad D \leftrightarrow 1101, \quad E \leftrightarrow 1110, \quad F \leftrightarrow 1111.
\end{align*}
\]

To express elements of \( \mathbb{F}_4 + u\mathbb{F}_4 \), we use the ordered basis \( \{u\omega, \omega, u, 1\} \). For instance, \( 1 + u\omega \) in \( \mathbb{F}_4 + u\mathbb{F}_4 \) is expressed as 1001 which is 9.

2.3. Arrays for orthogonal designs. We begin with the following general definition of an orthogonal design:

**Definition 2.1.** An orthogonal design of order \( n \) and type \((u_1, u_2, \ldots, u_s)\) on variables \( x_1, x_2, \ldots, x_s \) is an \( n \times n \) matrix \( A \) with entries \( \{0, \pm x_1, \pm x_2, \ldots, \pm x_s\} \) where \( x_i \)'s are commuting indeterminates and

\[
AA^T = \sum_{i=1}^n u_i x_i^2 I_n.
\]

It is denoted by \( OD(n; u_1, u_2, \ldots, u_s) \).

For instance

\[
\begin{pmatrix}
a & b & c & d \\
-b & a & -d & c \\
-c & d & a & -b \\
-d & -c & b & a
\end{pmatrix}
\]

is an \( OD(4; 1, 1, 1, 1) \). When we replace \( a, b, c \) and \( d \) respectively with symmetric circulant matrices we get the Williamson array \[23\]

\[
\begin{pmatrix}
A & B & C & D \\
-B & A & -D & C \\
-C & D & A & -B \\
-D & -C & B & A
\end{pmatrix}.
\]

Here, \( A, B, C \) and \( D \) are symmetric circulant matrices.

Baumert-Hall arrays \([3]\) are a generalization of Williamson arrays.

**Definition 2.2.** A \( 4t \times 4t \) array of \( \pm A, \pm B, \pm C, \pm D \) is said to be a Baumert-Hall array if each indeterminate occurs exactly \( t \) times in each row and column and the distinct rows are formally orthogonal.

The Goethals-Seidel array is introduced in \([11]\) and is a special Baumert-Hall array defined as:

\[
\begin{pmatrix}
A & BR & CR & DR \\
-BR & A & D^T R & -C^T R \\
-CR & -D^T R & A & B^T R \\
-DR & C^T R & -B^T R & A
\end{pmatrix}.
\]

where \( A, B, C \) and \( D \) are circulant square matrices and \( R \) is the back diagonal matrix. It is used to construct self-dual codes in \([4]\).
A short Kharaghani array is introduced in [18]:
\[
\begin{pmatrix}
A & B & CR & DR \\
-B & A & DR & -CR \\
-CR & -DR & A & B \\
-DR & CR & -B & A \\
\end{pmatrix}
\]
where \(A, B, C\) and \(D\) are circulant square matrices and \(R\) is the back diagonal matrix. Recently, the array and a variation of it are used to construct self-dual codes in [15].

3. Self-dual codes via Baumert-Hall arrays

In this section, we give constructions for self-dual codes via a Baumert-Hall array. The constructions are applicable over commutative Frobenius rings with various characteristics. Throughout the section let \(\mathcal{R}\) denote a commutative Frobenius ring.

In [9], Gholamiangonabadi and Kharaghani introduced the following Baumert-Hall array:

\[
\begin{pmatrix}
A & B & C & D \\
-B & A & -D & C \\
-C^T & D^T & A^T & -B^T \\
-D^T & -C^T & B^T & A^T \\
\end{pmatrix}
\]

where \(A, B, C\) and \(D\) are circulant matrices which are amicable with the pairing \((A, B)\) and \((C, D)\) (i.e., \(AB^T = BA^T\) and \(CD^T = DC^T\)). The array (3.1) can be used to construct self-dual codes as follows:

**Theorem 3.1.** Let \(\lambda\) be an element of the ring \(\mathcal{R}\) with \(\lambda^2 = 1\). Let \(C\) be the linear code over \(\mathcal{R}\) of length \(8n\) generated by the matrix in the following form:

\[
G := I_{4n} \begin{pmatrix}
A & B & C & D \\
-B & A & -D & C \\
-C^T & D^T & A^T & -B^T \\
-D^T & -C^T & B^T & A^T \\
\end{pmatrix}
\]

where \(A, B, C\) and \(D\) are \(\lambda\)-circulant matrices over the ring \(\mathcal{R}\) satisfying the conditions

\[
\begin{align*}
AA^T + BB^T + CC^T + DD^T &= -I_n \quad \text{and} \\
AB^T - BA^T + CD^T - DC^T &= 0.
\end{align*}
\]

Then \(C\) is self-dual.

**Proof.** Let \(A, B, C\) and \(D\) be \(\lambda\)-circulant matrices and

\[
N := \begin{pmatrix}
A & B & C & D \\
-B & A & -D & C \\
-C^T & D^T & A^T & -B^T \\
-D^T & -C^T & B^T & A^T \\
\end{pmatrix}
\]
then it is enough to show that $NN^T = -I_n$. We observe that

$$NN^T = \begin{pmatrix} X & Y & Z & T \\ -Y & X & -T & -Z \\ Z^T & -T^T & X^T & U \\ T^T & Z^T & -U & X^T \end{pmatrix}$$

where

$$X = AA^T + BB^T + CC^T + DD^T, \quad Y = -AB^T + BA^T - CD^T + DC^T, \quad Z = -AC + BD + CA - BD, \quad T = -AD - BC + CB + DA, \quad U = C^T D - D^T C + A^T B - B^T A.$$  

Circulant matrices commute therefore $Z = 0$ and $T = 0$. By (3.3) $Y = 0$ and $U = 0$. By (3.2) $X = -I_n$. Result follows.

A particular case of Theorem 3.1 is given where the condition (3.2) is split into two conditions, which allows us to search for pairs of matrices $(A, B)$ and $(C, D)$ stepwise.

**Corollary 3.2.** Let $\lambda$ be an element of the ring $\mathcal{R}$ with $\lambda^2 = 1$. Let $C$ be the linear code over $\mathcal{R}$ of length $8n$ generated by the matrix in the following form;

$$G := \begin{pmatrix} A & B & C & D \\ -B & A & -D & C \\ -C^T & D^T & A^T & -B^T \\ -D^T & -C^T & B^T & A^T \end{pmatrix}$$

where $A, B, C$ and $D$ are $\lambda$-circulant matrices over the ring $\mathcal{R}$ satisfying the conditions

(3.4) $AA^T + BB^T + CC^T + DD^T = -I_n,
(3.5) AB^T - BA^T = 0$ and
(3.6) $CD^T - DC^T = 0.$

Then $C$ is self-dual.

It is easily observed that symmetric circulant matrices are amicable. Therefore we have the following result;

**Corollary 3.3.** Let $\lambda$ be an element of the ring $\mathcal{R}$ with $\lambda^2 = 1$. Let $C$ be the linear code over $\mathcal{R}$ of length $8n$ generated by the matrix in the following form;

$$G := \begin{pmatrix} A & B & C & D \\ -B & A & -D & C \\ -C^T & D^T & A^T & -B^T \\ -D^T & -C^T & B^T & A^T \end{pmatrix}$$

where $A, B$ are symmetric circulant matrices and $C$ and $D$ are $\lambda$-circulant matrices over the ring $\mathcal{R}$ satisfying the conditions

(3.7) $AA^T + BB^T + CC^T + DD^T = I_n$ and
(3.8) $CD^T - DC^T = 0.$

Then $C$ is self-dual.
Remark 3.4. Note that if we assume $A,B,C$ and $D$ are all symmetric circulant matrices, then we obtain a special case of Corollary 3.3 which corresponds to Williamson array. Through computational results we observed that this case is not promising in constructing self-dual codes.

4. Computational Results

The constructions introduced in Section 3 are applied over rings of characteristic 2 such as $\mathbb{F}_2$, $\mathbb{F}_2 + u\mathbb{F}_2$ and $\mathbb{F}_4 + u\mathbb{F}_4$. By using the corresponding Gray maps binary self-dual codes of lengths 64 and 80 have been constructed.

The possible weight enumerators of extremal Type I self-dual codes (of parameters $[64,32,12]$) were determined in [6] as:

$$W_{64,1} = 1 + (1312 + 16\beta) y^{12} + (22016 - 64\beta) y^{14} + \cdots; 14 \leq \beta \leq 284,$$

$$W_{64,2} = 1 + (1312 + 16\beta) y^{12} + (23040 - 64\beta) y^{14} + \cdots; 0 \leq \beta \leq 277.$$  

The existence of the codes is unknown for most of the $\beta$ values. Most recently codes with new parameters were constructed in [14] (by a bordered four circulant construction) and [2]. Together with these, codes exist with weight enumerators for $\beta =14, 16, 18, 20, 22, 24, 25, 26, 28, 29, 30, 32, 36, 39, 44, 46, 53, 59, 60, 64$ and 74 in $W_{64,1}$ and for $\beta =0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 32, 33, 35, 36, 37, 38, 40, 41, 44, 48, 51, 52, 56, 58, 64, 72, 80, 88, 96, 104, 108, 112, 114, 118, 120 and 184 in $W_{64,2}$.

In the following tables, we give extremal binary self-dual codes of length 64 obtained from applying constructions described in Theorem 3.1 and Corollary 3.2 over the ring $\mathbb{F}_4 + u\mathbb{F}_4$. The orthogonality-preserving Gray map $\phi_1 \circ \phi_{\mathbb{F}_4 + u\mathbb{F}_4} : (\mathbb{F}_4 + u\mathbb{F}_4)^n \to \mathbb{F}_2^n$ is used in finding the binary codes. Thus, we need self-dual codes over $\mathbb{F}_4 + u\mathbb{F}_4$ of length 16 for such codes. $r_A, r_B, r_C, r_D$ denote the first rows of the matrices $A, B, C, D$ that appear in the constructions.
Table 1. Extremal binary self-dual codes of length 64 from self-dual codes of length 16 over $\mathbb{F}_4 + u\mathbb{F}_4$ by Theorem 3.1

| $D_i$ | $\lambda$ | $r_A$ | $r_B$ | $r_C$ | $r_D$ | $|Aut(D_i)|$ | $\beta$ in $W_{64,2}$ |
|-------|------------|-------|-------|-------|-------|-------------|----------------------|
| $D_1$ | 3          | (1, B) | (7, C) | (6, D) | (4, 5) | $2^4$       | 4                   |
| $D_2$ | 3          | (C, 6) | (D, 9) | (2, 3) | (C, 0) | $2^4$       | 12                  |
| $D_3$ | 3          | (6, E) | (A, 9) | (0, E) | (F, B) | $2^4$       | 24                  |
| $D_4$ | 3          | (F, 8) | (E, 9) | (C, 1) | (A, 4) | $2^4$       | 28                  |
| $D_5$ | 3          | (6, C) | (1, D) | (1, 4) | (F, D) | $2^4$       | 32                  |
| $D_6$ | 3          | (C, 4) | (9, 5) | (F, 5) | (E, 9) | $2^4$       | 36                  |
| $D_7$ | 3          | (5, 5) | (9, 3) | (2, 5) | (7, 1) | $2^4$       | 40                  |
| $D_8$ | 3          | (D, 8) | (0, 3) | (4, 1) | (F, 5) | $2^4 \times 3$ | 44                  |
| $D_9$ | 3          | (E, E) | (1, 5) | (5, 5) | (C, 9) | $2^4$       | 48                  |
| $D_{10}$ | 3        | (E, E) | (1, 7) | (5, 5) | (4, 9) | $2^4$       | 52                  |
| $D_{11}$ | 9         | (1, F) | (6, C) | (1, A) | (C, 2) | $2^4$       | 0                   |
| $D_{12}$ | 9         | (D, 2) | (1, 4) | (7, F) | (B, 8) | $2^4$       | 4                   |
| $D_{13}$ | 9         | (9, C) | (3, 3) | (5, B) | (2, 2) | $2^4$       | 5                   |
| $D_{14}$ | 9         | (B, 3) | (6, E) | (4, 9) | (E, 2) | $2^4$       | 8                   |
| $D_{15}$ | 9         | (1, 7) | (6, 9) | (5, D) | (6, 4) | $2^4$       | 12                  |
| $D_{16}$ | 9         | (B, A) | (7, B) | (6, 4) | (C, 0) | $2^4$       | 13                  |
| $D_{17}$ | 9         | (B, 4) | (D, D) | (4, C) | (B, D) | $2^4$       | 16                  |
| $D_{18}$ | 9         | (0, 2) | (C, 9) | (9, 3) | (B, 7) | $2^4$       | 17                  |
| $D_{19}$ | 9         | (3, D) | (A, F) | (8, 6) | (D, 1) | $2^4$       | 21                  |
| $D_{20}$ | 9         | (7, 3) | (D, 4) | (2, 9) | (1, 6) | $2^4$       | 32                  |
| $D_{21}$ | B         | (C, 3) | (2, 2) | (B, D) | (3, 1) | $2^4$       | 5                   |
| $D_{22}$ | B         | (B, 4) | (9, F) | (E, C) | (D, D) | $2^4$       | 8                   |
| $D_{23}$ | B         | (E, 9) | (8, 2) | (B, F) | (1, 1) | $2^4$       | 9                   |
| $D_{24}$ | B         | (5, B) | (5, 2) | (6, 8) | (3, 5) | $2^4$       | 13                  |
| $D_{25}$ | B         | (3, 6) | (8, 6) | (A, F) | (C, 9) | $2^4$       | 17                  |
| $D_{26}$ | B         | (C, 4) | (F, D) | (9, C) | (F, B) | $2^4$       | 20                  |
| $D_{27}$ | B         | (6, 9) | (A, D) | (8, 6) | (1, 6) | $2^4$       | 21                  |
| $D_{28}$ | B         | (7, 1) | (D, 8) | (C, 8) | (9, 7) | $2^4$       | 25                  |
| $D_{29}$ | B         | (B, 5) | (5, D) | (B, 3) | (0, 7) | $2^4$       | 28                  |
| $D_{30}$ | B         | (3, 3) | (6, 6) | (3, E) | (A, 4) | $2^4$       | 40                  |
Table 2. Extremal binary self-dual codes of length 64 from self-dual codes of length 16 over $\mathbb{F}_4 + u\mathbb{F}_4$ by Corollary 3.2

| $\mathcal{E}_i$ | $\lambda$ | $r_A$ | $r_B$ | $r_C$ | $r_D$ | $|\text{Aut}(\mathcal{E}_i)|$ | $\beta$ in $W_{64,2}$ |
|----------------|-----------|-------|-------|-------|-------|----------------------|------------------|
| $\mathcal{E}_1$ | 3         | $(F, B)$ | $(1, E)$ | $(7, 1)$ | $(0, A)$ | $2^4$ | 0 |
| $\mathcal{E}_2$ | 3         | "     | "     | $(9, 9)$ | $(A, A)$ | $2^5$ | 4 |
| $\mathcal{E}_3$ | 3         | "     | "     | $(7, D)$ | $(E, 4)$ | $2^4$ | 24 |
| $\mathcal{E}_4$ | 3         | "     | "     | $(1, 1)$ | $(8, 8)$ | $2^5$ | 28 |
| $\mathcal{E}_5$ | 3         | "     | "     | $(4, 4)$ | $(7, 7)$ | $2^5$ | 52 |
| $\mathcal{E}_6$ | 3         | $(9, 6)$ | $(5, F)$ | $(4, C)$ | $2^5$ | 12 |
| $\mathcal{E}_7$ | 3         | "     | "     | $(F, 7)$ | $(C, C)$ | $2^4$ | 16 |
| $\mathcal{E}_8$ | 3         | "     | "     | $(E, E)$ | $(7, 7)$ | $2^5$ | 20 |
| $\mathcal{E}_9$ | 3         | "     | "     | $(F, F)$ | $(4, 4)$ | $2^5$ | 36 |
| $\mathcal{E}_{10}$ | 9         | $(F, B)$ | $(B, C)$ | $(2, 2)$ | $(3, 9)$ | $2^3$ | 5 |
| $\mathcal{E}_{11}$ | 9         | $(D, 3)$ | $(3, 4)$ | $(A, 2)$ | $(3, B)$ | $2^4$ | 13 |
| $\mathcal{E}_{12}$ | 9         | $(0, 2)$ | $(B, 3)$ | $(B, D)$ | $(C, B)$ | $2^4$ | 17 |
| $\mathcal{E}_{13}$ | B         | $(3, 1)$ | $(2, 2)$ | $(B, 5)$ | $(C, 9)$ | $2^3$ | 5 |
| $\mathcal{E}_{14}$ | B         | $(9, 6)$ | $(5, 9)$ | $(0, 2)$ | $(B, 9)$ | $2^3$ | 17 |

4.1. New Type II self-dual $[72, 36, 12]_2$-codes. The existence of Type II extremal binary code of length 72 is still an open problem. The best known Type II self-dual codes of length 72 have parameters $[72, 36, 12]$. The possible weight enumerators for these codes are given in [7] as

$$W_{72} = 1 + (4398 + \alpha)y^{12} + (197073 - 12\alpha)y^{16} + \ldots$$

Note that a code with weight enumerator $\alpha = -4398$ would correspond to the above-mentioned extremal binary self-dual code (of parameters $[72, 36, 16]$). For a list of known $\alpha$ values we refer to [22]. We construct codes with new $\alpha$ values by Corollary 3.3 over the binary field $\mathbb{F}_2$. Since $A$ and $B$ are symmetric circulant matrices only first 5 entries of the first rows are given in Table 3.

Table 3. Type II self-dual codes of length 72 by Corollary 3.3

| $C_{72,i}$ | $r_A$ | $r_B$ | $r_C$ | $r_D$ | $\alpha$ |
|------------|-------|-------|-------|-------|---------|
| $C_{72,1}$ | $(11011)$ | $(01010)$ | $(101100101)$ | $(1100010010)$ | $-2736$ |
| $C_{72,2}$ | $(10110)$ | $(00000)$ | $(0011100110)$ | $(0100111110)$ | $-2748$ |
| $C_{72,3}$ | $(11001)$ | $(10101)$ | $(0110100110)$ | $(0010011111)$ | $-2844$ |
| $C_{72,4}$ | $(00111)$ | $(10110)$ | $(0001000011)$ | $(1100101110)$ | $-2964$ |
| $C_{72,5}$ | $(11001)$ | $(10101)$ | $(0011000110)$ | $(0110110111)$ | $-3060$ |
| $C_{72,6}$ | $(00000)$ | $(10100)$ | $(0101110110)$ | $(0001001010)$ | $-3396$ |

Example 4.1. Let $A = circ(0011010000)$, $B = circ(1001001000)$, $C = circ(0001010001)$ and $B = circ(1111110000)$ which are amicable with the pairing $(A, B)$ and $(C, D)$. In other words, the conditions 3.5 and 3.6 are satisfied. Moreover, they satisfy the equation 3.4. Let $C_{73,7}$ be the code obtained via Corollary 3.2. Then it is a Type II self-dual $[72, 36, 12]$ code with weight enumerator $\alpha = -3618$ and automorphism group of order 36.
By using Theorem 3.1 we obtain more codes, which are listed in Table 4.

**Table 4. Type II self-dual codes of length 72 by Theorem 3.1**

| \(C_{72,i} \) | \(r_A \) | \(r_B \) | \(r_C \) | \(r_D \) | \(\alpha \) |
|-------------|-------------|-------------|-------------|-------------|-------------|
| \(C_{72,8} \) | (011111011) | (001100111) | (110110011) | (010011101) | 2682 |
| \(C_{72,9} \) | (000011111) | (100001110) | (111011011) | (010001101) | 2700 |
| \(C_{72,10} \) | (011011111) | (011100100) | (000010110) | (101010011) | 2754 |
| \(C_{72,11} \) | (010011101) | (111000011) | (010000101) | (111101011) | 2790 |
| \(C_{72,12} \) | (110111001) | (110101011) | (000011100) | (000011111) | 2802 |
| \(C_{72,13} \) | (011000100) | (101110111) | (101010100) | (110110100) | 2862 |
| \(C_{72,14} \) | (100111011) | (010011110) | (001110000) | (000110111) | 2982 |
| \(C_{72,15} \) | (100101000) | (111111001) | (000011110) | (011110011) | 2988 |
| \(C_{72,16} \) | (110001000) | (011011001) | (011111001) | (011010101) | 3132 |
| \(C_{72,17} \) | (001000011) | (000111000) | (001101101) | (001101001) | 3150 |
| \(C_{72,18} \) | (000010110) | (001101111) | (010101110) | (111111111) | 3654 |
| \(C_{72,19} \) | (101111011) | (110101101) | (000011111) | (001101011) | 3690 |
| \(C_{72,20} \) | (000001111) | (101101111) | (001110000) | (101011010) | 3774 |
| \(C_{72,21} \) | (000100000) | (001011011) | (001111100) | (001110110) | 3780 |
| \(C_{72,22} \) | (100101101) | (000110110) | (010001011) | (111000000) | 3792 |
| \(C_{72,23} \) | (000010101) | (011110001) | (011011011) | (000111100) | 3906 |
| \(C_{72,24} \) | (010000010) | (000001110) | (110100101) | (011010111) | 3918 |
| \(C_{72,25} \) | (101101111) | (101110101) | (001011000) | (000100110) | 4068 |
| \(C_{72,26} \) | (100010111) | (001000100) | (100110001) | (001010010) | 4086 |

4.2. **New extremal Type II binary self-dual codes of length 80.** The weight enumerator of an extremal Type II binary self-dual code of length 80 (of parameters [80, 40, 16]) is uniquely determined as 1 + 97565y16 + 12882688y20 + · · · [11]. Recently, new such codes were constructed via the four circulant construction over \( \mathbb{F}_2 \oplus \mathbb{F}_2 \) in [16]. Here, we construct 8 new codes by using Theorem 3.1 over the binary alphabet. The nonequivalence of the codes is checked by the invariants. Let \( c_1, c_2, \ldots, c_{97565} \) be the codewords of weight 16 in an extremal Type II self-dual code of length 80 and let \( I_j = \{ (c_k, c_i) \mid d(c_k, c_i) = j, k < l \} \) where \( d \) is the Hamming distance. Two codes are inequivalent if their \( I_{16} \)-values are different since \( I_{16} \) is invariant under a permutation of the coordinates.

**Table 5. Type II self-dual codes of length 80 by Theorem 3.1**

| \(G_i \) | \(r_A \) | \(r_B \) | \(r_C \) | \(r_D \) | \(|Aut(G_i)| \) | \(I_{16} \) |
|---------|---------|---------|---------|---------|-------------|---------|
| \(G_1 \) | (1110100000) | (1101000000) | (1101110000) | (1101110000) | \(2^4 \times 5 \) | 20039280 |
| \(G_2 \) | (1100101000) | (0100111110) | (1110100000) | (1110100000) | \(2^4 \times 5 \) | 20248440 |
| \(G_3 \) | (0000100010) | (1010000000) | (1000000000) | (0111111111) | \(2^4 \times 5 \) | 20062880 |
| \(G_4 \) | (0100000000) | (1001001000) | (1001001000) | (1110000000) | \(2^4 \times 5 \) | 20034200 |
| \(G_5 \) | (0111110010) | (0110101000) | (1010100100) | (0000101000) | \(2^4 \times 5 \) | 20457960 |
| \(G_6 \) | (1110000001) | (0100000010) | (1111000000) | (0110001111) | \(2^4 \times 3 \times 5 \) | 19992780 |
| \(G_7 \) | (1100111110) | (1110000000) | (0001001000) | (0100100110) | \(2^4 \times 3 \times 5 \) | 20008440 |
| \(G_8 \) | (1001100100) | (1001000100) | (1100101111) | (0010100000) | \(2^4 \times 3 \times 5 \) | 20082720 |
Combining this with the last known number of such codes from [16], we obtain the following theorem:

**Theorem 4.2.** There exist at least 44 inequivalent extremal Type II self-dual codes of length 80.

The codewords of weight 16 in an extremal Type II code of length 80 form a 3-design by Assmus-Matson theorem. Hence, we have the following subsequent result:

**Theorem 4.3.** There are at least 44 non-isomorphic 3 - (80, 16, 665) designs.

### 4.3. New extremal binary self-dual codes of length 68 from \( F_2 + uF_2 \)-extensions

In [3], possible weight enumerators of an extremal binary self-dual code of length 68 (of parameters [68, 34, 12]) are characterized as follows:

\[
\begin{align*}
W_{68,1} &= 1 + (442 + 4\beta) y^{12} + (10864 - 8\beta) y^{14} + \cdots, \\
W_{68,2} &= 1 + (442 + 4\beta) y^{12} + (14960 - 8\beta - 256\gamma) y^{14} + \cdots,
\end{align*}
\]

where \( 0 \leq \gamma \leq 9 \) by [12]. The existence of codes is known for various parameters for \( W_{68,1} \); for a list of known such codes we refer to [22]. The existence of codes is known for \( \gamma = 0, 1, 2, 3, 4, 5, 6 \) and \( \gamma = 7 \) in \( W_{68,2} \). Recently, the first examples of codes with \( \gamma = 5 \) were constructed in [10]. Yankov et al. constructed codes with \( \gamma = 7 \) by considering codes with an automorphism group of order 7 in [24].

We construct extremal binary self-dual codes of length 68 with new weight enumerators via the building-up construction over \( F_2 + uF_2 \) applied to the codes of length 64 obtained at the beginning of Section 4. We obtain examples of extremal binary self-dual codes of length 68 with weight enumerator \( \gamma = 5 \) in \( W_{68,2} \).

In the sequel, let \( R \) be a commutative ring of characteristic 2 with identity.

**Theorem 4.4.** (8) Let \( C \) be a self-dual code over \( R \) of length \( n \) and \( G = \langle r_i \rangle \) be a \( k \times n \) generator matrix for \( C \), where \( r_i \) is the \( i \)-th row of \( G \), \( 1 \leq i \leq k \). Let \( x \) be a vector in \( R^n \) such that \( x^2 = 1 \) and \( X \) be a vector in \( R^n \) with \( \langle X, X \rangle = 1 \). Let \( y_i = \langle r_i, X \rangle \) for \( 1 \leq i \leq k \). Then the following matrix

\[
\begin{bmatrix}
1 & 0 & X \\
y_1 & cy_1 & r_1 \\
\vdots & \vdots & \vdots \\
y_k & cy_k & r_k
\end{bmatrix}
\]

generates a self-dual code \( D \) over \( R \) of length \( n + 2 \).

Currently, the existence of codes with weight enumerators for \( \gamma = 0, 1, 2, 3, 4, 6 \) and 7 is known. Recently, new codes in \( W_{68,2} \) have been obtained in [24, 10]. These codes exist for \( \gamma = 3, 4 \) and 5 in \( W_{68,2} \) when

- \( \gamma = 3 \), \( \beta \in \{ 2m + 1 | m = 38, 40, 43, 44, 47, \ldots, 77, 79, 80, 81, 83, 89, 96 \} \) or \( \beta \in \{ 2m | m = 39, \ldots, 92, 94, 95, 97, 98, 101, 102 \} \);
- \( \gamma = 4 \), \( \beta = 103, 105, 107, 113, 115, 117, 119, 121, 129, 139, 141, 143, 145, 149, 157, 161 \) or \( \beta \in \{ 2m | m = 43, 46, 47, 48, 49, 51, 52, 54, 55, 56, 58, 60, \ldots, 90, 92, 97, 98 \} \);
- \( \gamma = 5 \) with \( \beta \in \{ m | m = 158, \ldots, 169 \} \)

We obtain 46 new codes with weight enumerators for \( \gamma = 3 \) and \( \beta = 91 ; \gamma = 4 \) and \( \beta = 90, 106, 109, 112, 114 ; \gamma = 5 \) and \( \beta = 113, 116, \ldots, 153 \) in \( W_{68,2} \) using Theorem 4.3 and neighboring constructions.
The following table contains the new extremal binary self-dual codes of length 68 obtained from applying Theorem 4.4 for $R = \mathbb{F}_2 + u\mathbb{F}_2$ over $\varphi_{\mathbb{F}_4 + u\mathbb{F}_4}(D_{16})$:

| $C_{68,i}$ | $c$ | $X$ | $\gamma$ | $\beta$ |
|------------|-----|-----|----------|--------|
| $C_{68,1}$ | $1 + u$ | $(1131113u3u0110103u0uu003u003u1031)$ | 3 | 91 |
| $C_{68,2}$ | $1 + u$ | $(30u30011131111113313100u11uu10011)$ | 4 | 90 |
| $C_{68,3}$ | $1 + u$ | $(3131331u3u0130300u0u030u103013)$ | 5 | 119 |
| $C_{68,4}$ | $1 + u$ | $(3131331u3u0130300u0u030u103013)$ | 5 | 129 |
| $C_{68,5}$ | $1 + u$ | $(3111333u3u03303u30u001003u0301)$ | 5 | 137 |

4.4. New extremal binary self-dual codes of length 68 via neighboring construction. Two binary self-dual codes of length $2k$ are said to be neighbors if their intersection has dimension $k - 1$. Let $C$ be a binary self-dual code of length $2k$ and $x \in \mathbb{F}_2^{2k} - C$. Then $D = \langle x \rangle^\perp \cap C, x \rangle$ is a neighbor of $C$. We consider the neighbors of the codes in Table 6 and obtain new codes with $\gamma = 4$ and 5 which are listed in Table 8 and 7 respectively. The generator matrix of $C$ is formed into standard form which allows us to fix first 34 entries of $x$ as 0 without loss of generality. The remaining 34 entries of $x$ are given in the corresponding tables.
Table 7. New extremal binary self-dual codes of length 68 with $\gamma = 5$ (36 codes)

| $N_{68,1}$ | $c_{68,1}$ | $x$ | $\beta$ |
|------------|------------|-----|--------|
| $N_{68,1}$ | $c_{68,1}$ | (0111010000101000111110101111110111111) | 113 |
| $N_{68,2}$ | $c_{68,2}$ | (1011100111101011110101100001101001) | 116 |
| $N_{68,3}$ | $c_{68,3}$ | (10110001111101101000101001011111101) | 117 |
| $N_{68,4}$ | $c_{68,3}$ | (0110001010110000000101100010100101) | 118 |
| $N_{68,5}$ | $c_{68,3}$ | (0111000001111000101000110010101111) | 120 |
| $N_{68,6}$ | $c_{68,3}$ | (100010001111100100101011101101101) | 121 |
| $N_{68,7}$ | $c_{68,4}$ | (1100110011111110110100010110111101) | 122 |
| $N_{68,8}$ | $c_{68,4}$ | (0001010111110101100101110001001010) | 123 |
| $N_{68,9}$ | $c_{68,3}$ | (000110101100101111011000011110011) | 124 |
| $N_{68,10}$ | $c_{68,3}$ | (111100111101001001011101101011001) | 125 |
| $N_{68,11}$ | $c_{68,3}$ | (1010001011011011101101010111000110) | 126 |
| $N_{68,12}$ | $c_{68,3}$ | (00001110011011111111110111111011) | 127 |
| $N_{68,13}$ | $c_{68,3}$ | (110101101110110100110011110111111) | 128 |
| $N_{68,14}$ | $c_{68,3}$ | (0011000011111010101111000001001011) | 129 |
| $N_{68,15}$ | $c_{68,4}$ | (111100111100101111011000011110100) | 131 |
| $N_{68,16}$ | $c_{68,4}$ | (100110111110111111111111111111110) | 132 |
| $N_{68,17}$ | $c_{68,3}$ | (100101111111111111111111111010101) | 133 |
| $N_{68,18}$ | $c_{68,4}$ | (111100111101011111111111111111110) | 134 |
| $N_{68,19}$ | $c_{68,4}$ | (001001110111011101110111111111110) | 135 |
| $N_{68,20}$ | $c_{68,4}$ | (110101011011000111110010010101111) | 136 |
| $N_{68,21}$ | $c_{68,5}$ | (010101101111111111111111111111110) | 138 |
| $N_{68,22}$ | $c_{68,5}$ | (010010110111111111111111111111110) | 139 |
| $N_{68,23}$ | $c_{68,3}$ | (001100010101101111111111111111110) | 140 |
| $N_{68,24}$ | $c_{68,5}$ | (0011110001111111101110110111011101) | 141 |
| $N_{68,25}$ | $c_{68,5}$ | (101010100101111111111111111111110) | 142 |
| $N_{68,26}$ | $c_{68,4}$ | (001101010100101111111111111111110) | 143 |
| $N_{68,27}$ | $c_{68,4}$ | (111011110100110111111111111111110) | 144 |
| $N_{68,28}$ | $c_{68,5}$ | (001101011011011101110111111111110) | 145 |
| $N_{68,29}$ | $c_{68,5}$ | (101101101011111111111111111111110) | 146 |
| $N_{68,30}$ | $c_{68,5}$ | (101011101101101110110111111111110) | 147 |
| $N_{68,31}$ | $c_{68,5}$ | (101110101111111111111111111111110) | 148 |
| $N_{68,32}$ | $c_{68,5}$ | (001111111000000111110111111111110) | 149 |
| $N_{68,33}$ | $c_{68,5}$ | (100011010111111111111111111111110) | 150 |
| $N_{68,34}$ | $c_{68,5}$ | (001001000011111111111111111111110) | 151 |
| $N_{68,35}$ | $c_{68,5}$ | (101010000100111111111111111111110) | 152 |
| $N_{68,36}$ | $c_{68,5}$ | (101101101101101110111111111111110) | 153 |
Table 8. New extremal binary self-dual codes of length 68 with $\gamma = 4$ (5 codes)

| $N_{68, 37}$ | $C_{68, 4}$ | $x$ | $\beta$ |
|--------------|-------------|-----|---------|
| $N_{68, 38}$ | $C_{68, 2}$ | (1110011010010101110011101010010100) | 107 |
| $N_{68, 39}$ | $C_{68, 4}$ | (0110001101100010100111011010000110) | 109 |
| $N_{68, 40}$ | $C_{68, 3}$ | (101000001011011000100001101100001) | 112 |
| $N_{68, 41}$ | $C_{68, 3}$ | (01100011011000001000001101100001) | 114 |

Remark 4.5. We construct 39 new codes with the rare parameter for $\gamma = 5$ in $W_{68, 2}$. Together with these, the existence of codes with weight enumerator $\gamma = 5$ in $W_{68, 2}$ is known for 51 different $\beta$ values.

5. Conclusion

The special structure of the matrices of Baumert-Hall arrays and more generally orthogonal designs provide a strong link between discrete structures and self-dual codes. The reduced search field is instrumental in finding extremal self-dual codes. As has been demonstrated in the paper, these constructions can be combined with other methods in the literature such as extensions, search over rings and neighbors. We have been able to find a substantial number of new extremal binary self-dual codes using these techniques, thus filling many gaps in the literature of such codes. Using the Assmus-Mattson theorem we were also able to come up with new designs, establishing a key link between self-dual codes and designs.

The effectiveness of our methods indicate that they can be applied in different settings as well. We envision two possible directions for future research. One is to apply the ideas and methods to different lengths than we have considered. However, it should be noted that higher lengths would require higher computational power and so the complexity might become an issue. Another possible idea is to apply these constructions to other rings than the ones we have considered.

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Sampoerna Academy, L’Avenue Campus, 12780, Jakarta, Indonesia
E-mail address: abidin.kaya@sampoernaacademy.sch.id

Department of Mathematics and Statistics, Northern Arizona University, Flagstaff, AZ 86001, USA
E-mail address: bahattin.yildiz@nau.edu