Holographic Entropy on the Brane in de Sitter Schwarzschild Space

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ABSTRACT

The relationship between the entropy of de Sitter (dS) Schwarzschild space and that of the CFT, which lives on the brane, is discussed by using Friedmann-Robertson-Walker (FRW) equations and Cardy-Verlinde formula. The cosmological constant appears on the brane with time-like metric in dS Schwarzschild background. On the other hand, in case of the brane with space-like metric in dS Schwarzschild background, the cosmological constant of the brane does not appear because we can choose brane tension to cancel it. We show that when the brane crosses the horizon of dS Schwarzschild black hole, both for time-like and space-like cases, the entropy of the CFT exactly agrees with the black hole entropy of 5-dimensional dS background as it happens in the AdS/CFT correspondence.

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1 Introduction

The holographic duality which connects $d+1$-dimensional gravity in Anti-de Sitter (AdS) background with $d$-dimensional conformal field theory (CFT) has been discussed vigorously for some years\cite{1}. The one of the evidences for the existence of the AdS/CFT correspondence is that the isometry of $d+1$-dimensional AdS space $SO(d,2)$ is identical with the conformal symmetry of $d$-dimensional Minkowski space. Recently much attention has been paid for the duality between de Sitter (dS) gravity and CFT by the analogy of the AdS/CFT correspondence\cite{2,3,4}, because the isometry of $d+1$-dimensional de Sitter space, $SO(d+1,1)$, exactly agrees with the conformal symmetry of $d$-dimensional Euclidean space. Thus it might be natural to expect the correspondence between $d+1$-dimensional gravity in de Sitter space and $d$-dimensional Euclidean CFT (the dS/CFT correspondence). Moreover the holographic principle between the radiation dominated Friedmann-Robertson-Walker (FRW) universe in $d$-dimensions and same dimensional CFT with a dual $d+1$-dimensional AdS description was studied in ref.\cite{5}. Especially, we can see the correspondence between black hole entropy and the entropy of the CFT which is derived by making the appropriate identifications for FRW equation with the generalized Cardy formula\cite{1}. The generalized Cardy formula expresses the entropy formula of the CFT in any dimensions\cite{1}. From the point of brane-world physics\cite{7}, the CFT/FRW relation sheds further light on the study of the brane CFT in the background of AdS Schwarzschild black hole\cite{6}. There was much activity on the studies of related questions\cite{6,11,11,12,13}.

The purpose of this letter is the further study of the CFT in de Sitter (dS) Schwarzschild background guided by the analogy of AdS Schwarzschild background. The investigation of dS brane in dS Schwarzschild background in terms of FRW equations has been initiated in ref.\cite{11}. The important difference between AdS space and dS space is the sign of cosmological constant. In case of the brane with time-like (Minkowski) metric on AdS Schwarzschild background\cite{6}, the cosmological constant does not appear because we can choose brane tension to cancel it. But it is impossible for dS Schwarzschild background with the positive cosmological constant. We will see that the cosmological constant always appears in FRW equations deduced from time-like brane trajectory in dS Schwarzschild background. It is interesting to note that the brane with space-like (Euclidean) metric in dS Schwarzschild background, the cosmological constant of the brane does not appear for the same reason in case of AdS Schwarzschild background. From the point of view of the dS/CFT correspondence, the investigation of space-like brane is more interesting than that of time-like brane.

Furthermore we argue the entropy of the brane CFT which is derived by using generalized Cardy formula for both time-like and space-like branes. We will see that when the brane crosses the horizon of dS Schwarzschild black hole, both for time-like and space-like branes, the entropy of the CFT is identical with the black hole entropy of 5-dimensional dS background as it happens in the AdS/CFT correspondence.

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1 The Cardy formula\cite{6} is originally the entropy formula of the CFT only for 2-dimensions.
2 The work deriving brane cosmological equation in a systematic way for a brane embedded in a bulk with a cosmological constant has been first examined in ref.\cite{12}.
2 FRW equations in the background of de Sitter Schwarzschild black hole

We first consider a 4-dimensional time-like brane in 5-dimensional dS Schwarzschild background. From the analogy of the AdS/CFT correspondence, we can regard that 4-dimensional CFT exists on the brane which is the boundary of the 5-dimensional dS Schwarzschild background. The dynamics of the brane is described by the boundary action:

$$L_b = -\frac{1}{8\pi G_5} \int_{\partial \mathcal{M}} \sqrt{-g} \mathcal{K} + \frac{\kappa}{8\pi G_5} \int_{\partial \mathcal{M}} \sqrt{-g}, \quad \mathcal{K} = K^i_i$$ (1)

Here $G_5$ is 5-dimensional bulk Newton constant, $\partial \mathcal{M}$ denotes the surface of the brane, $g$ is the determinant of the induced metric on $\partial \mathcal{M}$, $K_{ij}$ is the extrinsic curvature, $\kappa$ is a parameter related to tension of the brane.

From this Lagrangian, we can get the equation of motion of the brane as follows:

$$K_{ij} = \frac{\kappa}{2} g_{ij}$$ (2)

which implies that $\partial \mathcal{M}$ is a brane of constant extrinsic curvature. The bulk action is given by 5-dimensional Einstein action with cosmological constant. The dS Schwarzschild space is one of the exact solutions of bulk equations of motion and can be written in the following form,

$$ds^2_5 = \hat{G}_{\mu\nu} dx^\mu dx^\nu = -e^{2\rho} dt^2 + e^{-2\rho} dr^2 + r^2 d\Omega_3^2,$$

$$e^{2\rho} = \frac{1}{l^2} \left( -\mu + r^2 - \frac{r^4}{l^2} \right).$$ (3)

Here $l$ is the curvature radius of dS and $\mu$ is the black hole mass. In case of AdS Schwarzschild gravity, there is a holographic relation between FRW brane universe which is reduction from AdS Schwarzschild background and boundary CFT which lives on the brane[8, 10]. We assume that there are some holographic relations between FRW universe which is reduction from dS Schwarzschild background and boundary CFT. To investigate it, we rewrite dS Schwarzschild metric (3) in the form of FRW metric by using a new time parameter $\tau$. And the parameter $t$ and $r$ in (3) are the function of $\tau$, namely $r = r(\tau), t = t(\tau)$. For the purpose of getting the 4-dimensional FRW metric, we impose the following condition,

$$-e^{2\rho} \left( \frac{\partial t}{\partial \tau} \right)^2 + e^{-2\rho} \left( \frac{\partial r}{\partial \tau} \right)^2 = -1.$$ (4)

Thus we obtain FRW metric:

$$ds^2_4 = g_{ij} dx^i dx^j = -d\tau^2 + r^2 d\Omega_3^2.$$ (5)
The extrinsic curvature, $K_{ij}$, of the brane can be calculated and expressed in term of the function $r(\tau)$ and $t(\tau)$. Thus one rewrites the equations of motion (2) as

$$\frac{dt}{d\tau} = -\frac{\kappa r}{2} e^{-2\rho} .$$

(6)

Using (4) and (6), we can derive FRW equation for a radiation dominated universe, Hubble parameter $H$ which is defined by $H = \frac{\dot{r}}{r}$ is given by

$$H^2 = \frac{1}{l^2} - \frac{1}{r^2} + \frac{\mu}{r^4} + \frac{\kappa^2}{4} .$$

(7)

Following AdS Schwarzschild gravity case, we choose $\kappa = 2/l$ from now on. This equation can be rewritten by using 4-dimensional energy $E_4$ and volume $V$ in the form of the standard FRW equation with the positive cosmological constant $\Lambda$:

$$H^2 = -\frac{1}{r^2} + \frac{8\pi G_4 E_4}{3} + \frac{\Lambda}{3} ,$$

$$E_4 = \frac{3\mu V}{8\pi G_4 r^4} , \quad \Lambda = \frac{6}{l^2} .$$

(8)

Here $G_4$ is the 4-dimensional gravitational coupling, which is defined by

$$G_4 = \frac{2G_5}{l} .$$

(9)

$E_4$ can be regarded as 4-dimensional energy on the brane in dS Schwarzschild background which is identical with AdS Schwarzschild case. The cosmological constant $\Lambda$ does not appear in AdS Schwarzschild background because we can choose brane tension $\kappa$ to cancel the cosmological constant of AdS Schwarzschild background. But it is impossible for dS Schwarzschild case because if we choose brane tension to cancel the cosmological constant of dS Schwarzschild background, the brane tension should be imaginary.

By differentiating eq.(8) with respect to $\tau$, we obtain the second FRW equation:

$$\dot{H} = -4\pi G_4 E_4 \left(\frac{E_4}{V} + p\right) + \frac{1}{r^2} ,$$

$$p = \frac{\mu}{8\pi G_4 r^4} .$$

(10)

Here $p$ is 4-dimensional pressure of the matter on the boundary. From eqs.(8) and (10), we find that the energy-momentum tensor is traceless:

$$T^\text{matter} \mu^\mu = -\frac{E_4}{V} + 3p = 0 .$$

(11)

Therefore the matter on the brane can be regarded as the radiation, which is consistent with ref. This means the field theory on the brane should be CFT as in case of AdS Schwarzschild background.

$^5$ From the point of view of brane-world physics, the tension of brane should be determined without ambiguity. In fact, we can calculate it to cancel the leading divergence of bulk AdS Schwarzschild $^6$ The stress-energy tensor of CFT was calculated in some asymptotically dS space in a sense of dS/CFT correspondence.
Next, we consider space-like brane in 5-dimensional dS Schwarzschild background. Similarly, we impose the following condition to obtain space-like brane metric instead of eq.(4):

\[-e^{2\rho} \left( \frac{\partial t}{\partial \tau} \right)^2 + e^{-2\rho} \left( \frac{\partial r}{\partial \tau} \right)^2 = 1.\]  

(12)

Thus we get following FRW-like metric:

\[ds^2 = g_{ij} dx^i dx^j = d\tau^2 + r^2 d\Omega_3^2.\]  

(13)

We again calculate the equations of motion and the extrinsic curvature of space-like brane instead of (2) and (3). These equations lead FRW like equation as follows:

\[H^2 = \frac{1}{l^2} + \frac{1}{r^2} - \frac{\mu}{r^4} + \frac{\kappa^2}{4}.\]  

(14)

To cancel the cosmological constant, we take \(\kappa = 2/l\) in the same way of AdS Schwarzschild gravity. We assume this equation can be rewritten by using 4-dimensional energy \(E_4\) and volume \(V\) by the analogous form of the standard FRW equations:

\[H^2 = \frac{1}{r^2} - \frac{8\pi G_4 E_4}{3 V}, \quad E_4 = \frac{3\mu V}{8\pi G_4 r^4}.\]  

(15)

\[\dot{H} = 4\pi G_4 \left( \frac{E_4}{V} + p \right) - \frac{1}{r^2}, \quad p = \frac{\mu}{8\pi G_4 r^4}.\]  

(16)

Therefore we find the energy-momentum tensor is traceless from eqs.(13) and (16) again.

We stress again that we can take cosmological constant to zero for FRW-like equation in space-like brane in dS Schwarzschild background as the same way in the AdS/CFT correspondence. This will imply that the dS/CFT correspondence can be valid for space-like brane in dS Schwarzschild background.

3 The Cardy-Verlinde formula for the dS/CFT correspondence

In ref.\[5\], E. Verlinde showed that the \(d\)-dimensional FRW equation can be regarded as an analogue of the Cardy formula of 2-dimensional CFT\[6\].

\[S_4 = 2\pi \sqrt{\frac{c}{6}} \left( L_0 - \frac{c}{24} \right).\]  

(17)

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7This metric is also derived by Wick-rotation \(\tau \rightarrow i\tau\) in eq.(5).

8In ref.\[12, 13\], the similar equations to eqs.(7), (14) were obtained in terms of Ricci scalar of the induced metric of the brane.

9The reason why the sign of FRW equations is different from the standard FRW equations\[8\] results from the condition (12), namely \(\tau \rightarrow i\tau\) in eq.(5).
For time-like brane of 5-dimensional dS Schwarzschild background, identifying

\[
\frac{2\pi}{3} \left( E_4 r + \frac{\Lambda V r}{8\pi G_4} \right) \Rightarrow 2\pi L_0 ,
\]
\[
\frac{V}{8\pi G_4 r} \Rightarrow \frac{c}{24} ,
\]
\[
\frac{HV}{2G_4} \Rightarrow S_4 ,
\]
(18)

FRW equation (8) has the form (17). The effect of the cosmological constant appears in Cardy formula. We included contribution of the cosmological constant in \( L_0 \) because it shifts the vacuum energy. This means the cosmological entropy bound which is discussed in ref. [5] should be changed. The Bekenstein bound [5] in 4-dimensions is

\[
S \leq S_B, \quad S_B \equiv \frac{2\pi}{3} Er .
\]
(19)

Using eq.(18), the Bekenstein entropy bound should be changed as follows:

\[
S \leq S_B, \quad S_B \equiv \frac{2\pi}{3} \left( Er + \frac{\Lambda V r}{8\pi G_4} \right) .
\]
(20)

Then we find out that the effect of the cosmological constant appears in the change of the Bekenstein entropy bound.

For the case of space-like brane, identifying

\[
\frac{2\pi}{3} E_4 r \Rightarrow 2\pi L_0 ,
\]
\[
\frac{V}{8\pi G_4 r} \Rightarrow \frac{c}{24} ,
\]
\[
\frac{HV}{2G_4} \Rightarrow S_4 ,
\]
(21)

which is identical with AdS Schwarzschild case [8] exactly.

For both cases, the moments when the brane crosses the horizon\[r = r_H\] which is derived from \( e^{2\phi(r_H)} = 0 \), the Hubble parameter in (7) becomes

\[
H = \pm \frac{1}{l} .
\]
(22)

Here the plus sign corresponds to the expanding brane universe and the minus one to the contracting universe. We choose the expanding case. Note that eq.(22) is the same form as the case of AdS Schwarzschild black hole [8, 10]. Using eqs.(9),(18), we obtain 4-dimensional entropy \( S_4 \) as follows:

\[
S_4 = \frac{V}{2lG_4} = \frac{V}{4G_5} .
\]
(23)

\[\text{[10]}\]The time-like brane can only cross the black hole horizon, but the space-like brane can cross the black hole and cosmological horizons.
This entropy is nothing but the Bekenstein-Hawking entropy of 5-dimensional dS black hole similar to AdS/CFT correspondence.\cite{8,10} We now understand that discovered relation between FRW equations and entropy formulas in ref.\cite{5} can be also applied to dS Schwarzschild background. If we take time-like brane which is the boundary of dS Schwarzschild background, the cosmological constant of the brane appears in FRW equations. Therefore the effect of cosmological constant contributes to raising the the Bekenstein entropy bound. But if we take space-like brane in dS Schwarzschild background, we obtain the approximately same result of AdS Schwarzschild black hole.\cite{8,10} The difference between space-like brane in dS Schwarzschild background and time-like brane in AdS Schwarzschild background is the sign of FRW equations. When the brane crosses the horizon of dS Schwarzschild black hole, both for time-like and space-like brane, the entropy formula of the CFT exactly agrees with the black hole entropy of 5-dimensional dS background as it happens in the AdS/CFT correspondence. This implies that the holographic principle holds true for dS Schwarzschild background.

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References

[1] J.M. Maldacena, Adv.Theor.Math.Phys. 2 (1998) 231, hep-th/9711200; E. Witten, Adv.Theor.Math.Phys. 2 (1998) 253, hep-th/9802150; S. Gubser, I. Klebanov and A. Polyakov, Phys.Lett. B428 (1998) 105, hep-th/9802109; O. Aharony, S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, Phys.Repts. 323 (2000) 183, hep-th/9905111. Further references are contained therein.

[2] A. Strominger, hep-th/0106113, hep-th/0110087; For review, see M. Spradlin, A. Strominger, A. Volovich, hep-th/0110007.

[3] C.M. Hull, JHEP 9807 (1998) 021, hep-th/9806146; E. Witten, hep-th/0106103; P.O. Mazur, E. Mottola, hep-th/0106151; S. Nojiri, S.D. Odintsov, Phys.Lett.B519 (2001)145, hep-th/0106191; hep-th/0107134, hep-th/0110064; E. Silverstein, hep-th/0106209; D. Klemm, hep-th/0106247; A. Chamblin, N. D. Lambert, hep-th/0107031; J. Bros, H. Epstein, U. Moschella, hep-th/0107091; E. Halyo, hep-th/0107169; A. J. Tolley, N. Turok, hep-th/0108113; T. Shiromizu, D. Ida, T. Torii, hep-th/0109057; C.M. Hull, hep-th/0109213; S. Cacciatori, D. Klemm, hep-th/0110031; B. McInnes, hep-th/0110062; V. Balasubramanian, J. de Boer and D. Minic, hep-th/0110108; Y.S. Myung, hep-th/0110123; Ulf. H. Danielsson, hep-th/0110263; R.-G. Cai, hep-th/0111093.

\(^{11}\)The black hole entropy of AdS background agrees with that of dS background.
[4] R.-G. Cai, Y. S. Myung and Y.-Z. Zhang, hep-th/0110234.

[5] E. Verlinde, hep-th/0008140.

[6] J.L. Cardy, Nucl. Phys. B270 (1986) 967.

[7] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370, hep-th/9905221; Phys. Rev. Lett. 83 (1999) 4690, hep-th/9906064.

[8] I. Savonije and E. Verlinde, Phys. Lett. B507 (2001) 305, hep-th/0102042.

[9] D. Kutasov and F. Larsen, JHEP 0101 (2001) 001, hep-th/0009244; F.-L. Lin, Phys. Rev. D63 (2001) 064026, hep-th/0010127; S. Nojiri and S.D. Odintsov, hep-th/0011113 to appear in Int. J. Mod. Phys. A; hep-th/0103078 to appear in Class. Quant. Grav.; B. Wang, E. Abdalla and R.-K. Su, Phys. Lett. B503 (2001) 394, hep-th/0101073; D. Klemm, A. Petkou and G. Siopsis, Nucl. Phys. B601 (2001) 380, hep-th/0101076; Y.S. Myung, hep-th/0102184; R. Brustein, S. Foffa and G. Veneziano, Phys. Lett. B507 (2001) 270, hep-th/0101083; R.-G. Cai, Phys. Rev. D63 (2001) 124018, hep-th/0102113; R.-G. Cai and Y.-Z. Zhang, hep-th/0105214; A. Biswas and S. Mukherji, JHEP 0103 (2001) 046, hep-th/0102138; D. Birmingham and S. Mokhtari, Phys. Lett. B508 (2001) 365, hep-th/0103108; D. Klemm, A. Petkou, G. Siopsis and D. Zanon, hep-th/0104111; D. Youm, Mod. Phys. Lett. A16 (2001) 1263, hep-th/0105036; S. Nojiri, O. Obregon, S.D. Odintsov, H. Quevedo and M.P. Ryan, Mod. Phys. Lett. A16 (2001) 1181, hep-th/0105052; R.-G. Cai, Y.S. Myung and N. Ohta, hep-th/0105070; B. Wang, E. Abdalla and R.-K. Su, hep-th/0106086; L. Gappiello and W. Muck, hep-th/0107238; I. Brevik, S. D. Odintsov, gr-qc/0110103.

[10] S. Nojiri, S.D. Odintsov and S. Ogushi, hep-th/0105117 to appear in Int. J. Mod. Phys. A; hep-th/0108172 to appear in Phys. Rev. D.

[11] S. Nojiri, S.D. Odintsov, hep-th/0107134.

[12] P. Binetruy, C. Deffayet and D. Langlois, Nucl. Phys. B (2000) 269, hep-th/9905012.

[13] P. Binetruy, C. Deffayet, U. Ellwanger and D. Langlois, Phys. Lett. B 477 (2000) 285, hep-th/9910219. Further references are contained therein.