A HIGHER DISPERSION KdV EQUATION ON THE HALF-LINE

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Abstract. The initial-boundary value problem (ibvp) for the \(m\)-th order dispersion Korteweg-de Vries (KdV) equation on the half-line with rough data and solution in restricted Bourgain spaces is studied using the Fokas Unified Transform Method (UTM). Thus, this work advances the implementation of the Fokas method, used earlier for the KdV on the half-line with smooth data and solution in the classical Hadamard space, consisting of function that are continuous in time and Sobolev in the spatial variable, to the more general Bourgain spaces framework of dispersive equations with rough data on the half-line. The spaces needed and the estimates required arise at the linear level and in particular in the estimation of the linear pure ibvp, which has forcing and initial data zero but non-zero boundary data. Using the iteration map defined by the Fokas solution formula of the forced linear ibvp in combination with the bilinear estimates in modified Bourgain spaces introduced by this map, well-posedness of the nonlinear ibvp is established for rough initial and boundary data belonging in Sobolev spaces of the same optimal regularity as in the case of the initial value problem for this equation on the whole line.

1. Introduction and Results

In this work, we study the initial-boundary value problem (ibvp) for the \(m\)-th order dispersion Korteweg-de Vries (KdVm) equation on the half-line, that is

\[
\begin{align*}
\partial_t u + (-1)^j x^m u + uu_x = 0, & \quad 0 < x < \infty, \quad 0 < t < T, \\
u(x,0) &= u_0(x), \quad 0 < x < \infty, \\
u(0,t) &= g_0(t), \ldots, \partial_t^{j-1} u(0,t) = g_{j-1}(t), \quad 0 < t < T,
\end{align*}
\]

where \(m = 2j + 1, j \in \mathbb{N}\). Note that \(m = 3 (j = 1)\) gives the celebrated Korteweg-de Vries (KdV) equation ([48], [9]), and \(m = 5 (j = 2)\) gives a Kawahara equation [41]. The initial data belong in the Sobolev spaces \(H^s(0,\infty)\) and the boundary data \(g_\ell(t) \in H_\ell^{(s+j-\ell)/m}(0,T), \ell = 0,1,\ldots,j-1,\) arise in the estimation of the linear pure ibvp and also reflect the time regularity of the solution to the linear KdVm initial value problem (ivp). Here, we show that if \(s > -j + \frac{1}{4}\), then ibvp (1.1) is well-posed with solution in an appropriately modified Bourgain space restricted to \((0,\infty) \times (0,T)\) for some lifespan \(T > 0\) depending on the size of the data. Thus, the optimality \(s > -j + \frac{1}{4}\) of our ibvp well-posedness result here is exactly the same with the one obtained in [18] for the Cauchy problem of this equation on the whole line. The starting point is to define the iteration map via the Fokas solution formula of the forced linear ibvp and then after deriving appropriate bilinear estimates for the nonlinearity in the modified Bourgain spaces we show that this map is a contraction in a ball. For the KdV equation with smooth data \((s > 3/4)\) well-posedness on the half-line was proved in [22], and for KdV with smooth data \((s > m/4)\) well-posedness of
the ibvp was proved in [60]. Both works are based on the Fokas method but the solution spaces used, which are subsets of the space of functions that are Sobolev in \( x \) and continuous in \( t \), are motivated by the work on the KdV Cauchy problem by Kenig, Ponce and Vega in [43]. Our work here advances the implementation of the Fokas method for solving ibvp with smooth data (see [22, 23, 34, 35, 36, 37, 60]) to the Bourgain spaces framework of dispersive equations with rough data on the half-line.

Next, we define the spaces needed for stating our main result precisely. We recall that for any \( s \) and \( b \) real numbers, the Bourgain space \( X^{s,b}(\mathbb{R}^2) \) corresponding to the linear part of KdV\( m \) is defined by the norm

\[
\|u\|^2_{X^{s,b}} = \|\hat{u}\|^2_{L^2} + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 + |\xi|)^{2s} (1 + |\tau - \xi^m|)^{2b} |\hat{u}(\xi, \tau)|^2 d\xi d\tau,
\]

(1.2)

where \( \hat{u} \) denotes the space-time Fourier transform

\[
\hat{u}(\xi, \tau) = \int_{-\infty}^{\infty} e^{-i(\xi x + \tau t)} u(x, t) dx dt.
\]

(1.3)

Also, we shall need the following modification of the Bourgain norm

\[
\|u\|^2_{X^{s,b,\alpha}(\mathbb{R}^2)} = \|\hat{u}\|^2_{L^2} + \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 + |\xi|)^{2\alpha} |\hat{u}(\xi, \tau)|^2 d\xi d\tau \right] \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ (1 + |\xi|)^{s} (1 + |\tau - \xi^m|)^{b} + \chi_{|\xi| < 1} (1 + |\tau|^\alpha) \right] |\hat{u}(\xi, \tau)|^2 d\xi d\tau.
\]

(1.4)

A similar modification was introduced first by Bourgain [7], where the norm used for the periodic case with \( b = 1/2 \) was modified by the \( \alpha \)-part in order to prove well-posedness of the KdV equation on the line in \( H^s, s \geq 0 \). This idea was also utilized later by Colliander, Keel, Staffilani, Takaoka and Tao in [11] for the global well-posedness of KdV on the circle for \( s \geq -1/2 \). The Bourgain norm with \( b > 1/2 \) was used by Kenig, Ponce and Vega in [44] in order to extend the well-posedness of KdV in \( H^s(\mathbb{R}) \) to \( s > -3/4 \).

Finally, we shall need the restriction space \( X^{s,b,\alpha}_{\mathbb{R}^+ \times (0,T)} \), which is defined as follows

\[
X^{s,b,\alpha}_{\mathbb{R}^+ \times (0,T)} = \{ u; u(x,t) = v(x,t) \text{ on } \mathbb{R}^+ \times (0,T) \} \text{ with } v \in X^{s,b,\alpha}(\mathbb{R}^2),
\]

(1.5)

and which is equipped with the norm

\[
\|u\|_{X^{s,b,\alpha}_{\mathbb{R}^+ \times (0,T)}} = \inf_{v \in X^{s,b,\alpha}} \left\{ \|v\|_{s,b,\alpha}; v(x,t) = u(x,t) \text{ on } \mathbb{R}^+ \times (0,T) \right\}.
\]

(1.6)

Now, we are ready to state our first result precisely. It reads as follows.

**Theorem 1.1 (Well-posedness of KdV\( m \) on the half-line).** If \(-j + \frac{1}{4} < s \leq j + 1, s \neq \frac{1}{2}, \frac{3}{2}, \ldots, j - \frac{1}{2}\), then there exists \( b \in (0, \frac{1}{2}) \) depending on \( s \) such that for any initial data \( u_0 \in H^s(0, \infty) \) and boundary data \( g_\ell \in H^{\frac{1}{m}(s+j-\ell)}(0,T), \ell = 0, 1, \ldots, j - 1 \), there is a lifespan \( T_0 > 0 \) such that the ibvp (1.1) with compatibility condition:

\[
\frac{\partial^r}{\partial \tau^r} u_0(0) = g_\ell(0), \text{ for } \ell \text{ such that } \frac{1}{m}(s+j-\ell) > \frac{1}{2} \text{ or } s > \ell + \frac{1}{2},
\]

(1.7)

admits a unique solution \( u \in X^{s,b,\alpha}_{\mathbb{R}^+ \times (0,T_0)} \), satisfying the size estimate

\[
\|u\|_{X^{s,b,\alpha}_{\mathbb{R}^+ \times (0,T_0)}} \leq C \left( \|u_0\|_{H^s(\mathbb{R}^+)} + \sum_{\ell=0}^{j-1} \|g_\ell\|_{H_{\ell}^{s+j-\ell}(0,T)} \right),
\]

(1.8)
for some \( \alpha \in (\frac{1}{2}, 1) \). Furthermore, an estimate for the lifespan is given by

\[
T_0 = c_0 \left( 1 + \|u_0\|_{H^s(\mathbb{R})} + \sum_{\ell=0}^{j-1} \|g_\ell\|_{H^{\frac{4}{\beta}+\frac{j-\ell}{\beta}}(0,T)} \right)^{-4/\beta}, \quad c_0 = c_0(s, b, \alpha),
\]

for some \( \beta > 0 \) depending on \( s \) and \( m \) (see (1.22)). Finally, the solution depends \( \text{Lip} \)-continuously on the data \( u_0 \) and \( g_\ell \), \( \ell = 0, 1, \ldots, j - 1 \).

As we have mentioned earlier, the optimality \( s > -j + \frac{1}{2} \) for the data in our well-posedness Theorem 1.1 on the half-line is exactly the same with the optimality of the data in the well-posedness result on the whole line of KdVm obtained in [18].

We prove Theorem 1.1 by showing that the iteration map defined via the solution formula of the forced linear KdVm ibvp, which is obtained by the Fokas method, is a contraction in the solution space \( X^{s, b, \alpha} \). Therefore, we begin with the linear KdVm ibvp with forcing, that is

\[
\begin{align*}
\partial_t u + (-1)^{j+1} \partial_x^{j+1} u &= f(x, t), \quad 0 < x < \infty, \ 0 < t < T, \\
u(x, 0) &= u_0(x), \quad 0 < x < \infty, \quad (1.10a) \\
u(0, t) &= g_0(t), \cdots, \partial_x^{j-1} u(0, t) = g_{j-1}(t), \quad 0 < t < T, \quad (1.10c)
\end{align*}
\]

where \( T > 0 \) is any given time. Using the Fokas method, also referred in the literature as the Unified Transform Method (UTM), we get the following solution formula to the problem (1.10) (see [60] or Section 7 for an outline of the derivation):

\[
u(x, t) = S[u_0, g_0, \ldots, g_{j-1}; f] \doteq \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi + it\xi^m} [\hat{u}_0(\xi) + F(\xi, t)] d\xi \]

\[
+ \sum_{p=1}^{j} \sum_{n=1}^{j+1} C_{p,n} \int_{\partial D_{2p}^+} e^{ix\xi + it\xi^m} [\hat{u}_0(\alpha_{p,n}\xi) + F(\alpha_{p,n}\xi, t)] d\xi
\]

\[
+ \sum_{p=1}^{j} \sum_{\ell=0}^{j-1} C_{p,\ell} \int_{\partial D_{2p}^+} e^{ix\xi + it\xi^m} [i\xi^{2j-\ell} \tilde{g}_\ell(\xi^m, T)] d\xi,
\]

where \( C_{p,n} \) and \( C_{p,\ell} \) are constants and the rotation numbers \( \alpha_{p,n} \) are given by

\[
\alpha_{p,n} = e^{i[m-(2p+1)+2n]\frac{\pi}{m}}, \quad p = 1, 2, \ldots, j \quad n = 1, 2, \ldots, j + 1.
\]

Also, \( \hat{u}_0(\xi) \) is the Fourier transform of \( u_0(x) \) on the half-line, which is defined by the formula

\[
\hat{u}_0(\xi) \doteq \int_0^\infty e^{-ix\xi} u_0(x) dx, \quad \text{Im}(\xi) \leq 0,
\]

and \( F(\xi, t) \) is the following time integral of the half-line Fourier transform of the forcing \( f(\cdot, t) \)

\[
F(\xi, t) \doteq \int_0^t e^{-i\xi^m \tau} f(\xi, \tau) d\tau = \int_0^t e^{-i\xi^m \tau} \int_0^\infty e^{-i\xi x} f(x, \tau) dx d\tau, \quad \text{Im}(\xi) \leq 0.
\]

Furthermore, \( \tilde{g}_\ell \) is the temporal Fourier transform of \( g_\ell \) over the interval \([0, t]\)

\[
\tilde{g}_\ell(\xi, t) \doteq \int_0^t e^{-i\xi \tau} g_\ell(\tau) d\tau.
\]

Finally, the domains \( D_{2p}^+ \) in the upper half-plane are as in the two figures below:
Below we show the Fokas solution formula (1.11) for the KdV \((j = 1)\) and its domain \(D^+ = D_2^+\)

\[
\frac{1}{2\pi} \int_{\infty}^{\infty} e^{i\xi x + i\xi^3 t} [\hat{u}_0(\xi) + F(\xi, t)] d\xi 
+ \frac{1}{2\pi} \int_{\partial D^+} e^{i\xi x + i\xi^3 t} \{e^{i\frac{2\pi}{3}} [\hat{u}_0(e^{i\frac{2\pi}{3}} \xi) + F(e^{i\frac{2\pi}{3}} \xi, t)] + e^{i\frac{4\pi}{3}} [\hat{u}_0(e^{i\frac{4\pi}{3}} \xi) + F(e^{i\frac{4\pi}{3}} \xi, t)]\} d\xi 
- \frac{3}{2\pi} \int_{\partial D^+} e^{i\xi x + i\xi^3 t} \xi^2 \tilde{g}_0(\xi^3, t) d\xi.
\] (1.16)

Also, here we show the solution formula (1.11) for KdV5 (Kawahara equation) together with its domains \(D_2^+\) and \(D_4^+\)

\[
u(x, t) = \frac{1}{2\pi} \int_{\infty}^{\infty} e^{i\xi x + i\xi^5 t} [\hat{u}_0(\xi) + F(\xi, t)] d\xi 
+ \sum_{p=1}^{2} \sum_{n=1}^{3} C_{p,n} \int_{\partial D_2^+} e^{i\xi x + i\xi^5 t} [\hat{u}_0(\alpha_{p,n} \xi) + F(\alpha_{p,n} \xi, t)] d\xi 
+ \sum_{\ell=0}^{1} \sum_{p=1}^{2} C_{p,\ell} \int_{\partial D_2^+} e^{i\xi x + i\xi^5 t} (i\xi)^{4-\ell} \tilde{g}_\ell(\xi^6, t) d\xi.
\] (1.17)

For the solution (1.11) to ibvp (1.10), we have the following basic estimate.

**Theorem 1.2** (Forced linear KdVm estimate on the half-line). Suppose that \(-j - \frac{1}{2} < s \leq j + 1, s \neq \frac{1}{2}, \frac{3}{2}, \ldots, j - \frac{1}{2}, 0 < T < \frac{1}{2}\). Then for some \(0 < b < \frac{1}{2}\) and \(\alpha > \frac{1}{2}\) the Fokas formula (1.11) defines a solution to the forced linear KdVm ibvp (1.10) with compatibility condition (1.7), which
satisfies the estimate
\[
\|S[u_0, g_0, \ldots, g_j, \ldots; f]\|_{X^{s,b,\alpha}(\mathbb{R}^+ \times (0,T))} 
\leq \begin{cases}
  c_{s,b,\alpha} \left[ \|u_0\|_{H^s_x(\mathbb{R}^+ \times (0,T))} + \sum_{\ell=0}^{j-1} \|g\|_{H^{s+\ell-b/2}_t L^2_x(\mathbb{R}^+ \times (0,T))} \right], & -1 \leq s \leq 1/2, \\
  c_{s,b,\alpha} \left[ \|u_0\|_{H^s_x(\mathbb{R}^+ \times (0,T))} + \sum_{\ell=0}^{j-1} \|g\|_{H^{s+\ell-b/2}_t L^2_x(\mathbb{R}^+ \times (0,T))} \right], & s \not\in [-1, 1/2],
\end{cases}
\]  
(1.18)

where \( Y^{s,b} \) is a “temporal” Bourgain space defined by the norm
\[
\|u\|^2_{Y^{s,b}} = \int_{\mathbb{R}} \int_{\mathbb{R}} (1 + |\tau|)^{2b/3} (1 + |\tau - \xi|^m)^{2b} |\tilde{a}(\xi, \tau)|^2 d\xi d\tau.
\]  
(1.19)

We will prove our well-posedness result by showing that the iteration map defined by the Fokas solution formula is a contraction on a ball of the space \( X^{s,b,\alpha}(\mathbb{R}^+ \times \mathbb{R}) \). The key ingredient is the basic linear estimate (1.18) where the forcing \( f \) is replaced by the KdVm nonlinearity \( \partial_x(u^2) \), which is quadratic. To apply this linear estimate we will extend each one of its two factors from \( \mathbb{R}^+ \times \mathbb{R} \) to \( \mathbb{R}^2 \) appropriately, and to show that the iteration map is a contraction we shall need the following bilinear estimate.

**Theorem 1.3 (Bilinear estimates).** If \( s > -j + 1/4 \), then there is \( 0 < b < 1/2 \) and \( \frac{1}{2} < \alpha \leq 1 - b \) such that the following estimate holds
\[
\|
\partial_x (f \cdot g)
\|_{X^{s,-b,\alpha-1}(\mathbb{R}^2)} 
\leq c_{s,b,\alpha} \|f\|_{X^{s,b',\alpha'}(\mathbb{R}^2)} \|g\|_{X^{s,b',\alpha'}(\mathbb{R}^2)}, \quad f, g \in X^{s,b',\alpha'}(\mathbb{R}^2),
\]  
(1.20)

where \( b' \) and \( \alpha' \) are chosen as follows
\[
\frac{1}{2} - \beta \leq b' \leq \frac{1}{2} < \alpha' \leq \alpha \leq \frac{1}{2} + \beta,
\]  
(1.21)

and \( \beta \) is given by
\[
\beta = \begin{cases}
  \min\left\{ \frac{1}{12m}, \frac{m-s}{3m} \right\}, & s \geq 0, \\
  \min\left\{ \frac{j - \frac{3}{4}}{2m}, \frac{s + 1/2}{2m} \right\}, & -\frac{1}{2} < s < 0, \\
  \frac{1}{32m} \left[ s - (-j + 1/4) \right], & -j + 1/4 < s \leq -\frac{1}{2}.
\end{cases}
\]  
(1.22)

We note that this is a more general result than is needed here and is of interest in its own right. Furthermore, it can be shown that for any \( s > -j + 1/4 \) and \( b < 1/4 \) the estimates (1.20) fails. A similar estimate in \( X^{s,b} \) with \( b > 1/2 \) for the KdVm equation on the line was proved in [18]. For the KdV bilinear estimates in the Bourgain space \( X^{s,\frac{1}{2},\alpha} \) were first proved in [7], and in \( X^{s,b} \) with \( b > 1/2 \) were proved in [44].

Also, we shall need the following bilinear estimate in “temporal” Bourgain space, which is used when we estimate the Sobolev norm of the solution of the forced ivp in the time variable if \( s > 1/2 \) or \( s < -1 \) (see (3.23)). For the KdV equation, estimate (1.23) was proved by Holmer in [38].

**Theorem 1.4 (Bilinear estimate in \( Y^{s,b} \)).** If \( b, b' \) and \( \alpha' \) satisfies (1.21), then we have
\[
\|\partial_x (fg)\|_{Y^{s,-b}(\mathbb{R}^2)} \leq c_{s,b,\alpha} \|f\|_{X^{s,b',\alpha'}(\mathbb{R}^2)} \|g\|_{X^{s,b',\alpha'}(\mathbb{R}^2)}, \quad -j + 1/4 < s < m,
\]  
(1.23)

\[
\|\partial_x (fg)\|_{Y^{s,-b}} \leq \|\partial_x (fg)\|_{Y^{s,-b}} + c_{s,b} \|f\|_{X^{s,b'}} \|g\|_{X^{s,b'}}, \quad -j + 1/4 < s < m.
\]  
(1.24)
Besides the method we follow here, there are two other approaches to the study of initial-boundary value problems for KdV type equations. In the first approach, which has been initiated by Colliander and Kenig in [12] and by Holmer in [38] the forced linear ibvp is written as a superposition of ivps on the line. Then, the modern harmonic analysis techniques developed earlier for proving well-posedness of the nonlinear equation in Bourgain spaces are utilized. In the second approach, which for KdV has been initiated by Bona, Sun and Zhang in [3, 4, 5], the forced linear ibvp is solved via a Laplace transform in the time variable and then by deriving appropriate estimates well-posedness of the ibvp is established.

As we have mentioned before KdVm includes the KdV equation ($m = 3$), which is integrable, and the Kawahara equation ($m = 5$), which is not integrable. The literature about the KdV is very extensive. It begins with Scott Russell’s observation of the “great wave of translation” [56] and continues with the derivation of the KdV model by Boussinesq in 1877 [9] and Korteweg and de Vries in 1895 [48]. Then, in 1965, Zabusky and Kruskal [61] observed numerically that soliton solutions of KdV interact almost linearly by preserving their shape and speed after a collision. Soon after the KdV initial value problem on the line with data of sufficient smoothness and decay was solved by Gardner, Greene, Kruskal and Miura [30] via the inverse scattering transform (IST) which is based on its integrability, that is its Lax pair formulation [49]. The KdV ivp in Sobolev spaces $H^s$ using methods from partial differential equations has been studied extensively by many authors. For well-posedness results on the line using Bourgain spaces we refer the reader to Bourgain [7] when $s \geq 0$, Kenig, Ponce and Vega [44] when $s > -3/4$, Guo [32] when $s = -3/4$, and to Colliander, Keel, Staffilani, Takaoka, Tao [11] for its global well-posedness when $s > -3/4$. For additional results we refer the reader to [2, 55, 13, 14, 8, 40, 45, 46, 47, 57, 1, 51, 10, 31] and the references therein.

Concerning the Fokas method for solving ibvp, whose initial motivation came from integrable equations (in particular the KdV and the cubic NLS equations), we refer the reader [15, 19, 20, 27, 24, 25, 26, 50, 28, 29, 54] and the references therein. Also, for a detailed introduction to this method we refer to the book [21]. For additional work on solving ibvp we refer the reader to [16, 39, 17, 6, 59] and the references therein.

Our work is structured as follows. In Section 2, we study the reduced pure ibvp, which is the homogeneous linear problem with initial data zero but non-zero boundary data and derive a key estimate for its solution in modified Bourgain spaces. In Section 3, guided by the reduced pure ibvp, we decompose the forced linear ibvp into four simpler linear sub-problems and derive appropriate estimates for their solutions. Then, combining these estimates we prove Theorem 1.2, which provides the basic estimate for the Fokas solution formula and the iteration map of the nonlinear problem. In Section 4, we prove the needed bilinear estimate for the nonlinearity in modified Bourgain spaces $X^{s,b,\alpha}$. Then, in section 5, we prove the temporal bilinear estimate in $Y^{s,b}$ spaces. In Section 6, we prove our KdVm well-posedness result, Theorem 1.1, using estimates (1.18) for the forced linear ibvp and the bilinear estimates (1.20). Finally, in Section 7 we provide a brief outline of the Fokas solution formula for the forced linear ibvp on the half-line.

2. Reduced pure ibvp

We begin with the most basic linear KdVm ibvp on the half-line. This is the homogeneous ibvp with zero initial data and nonzero boundary data. Furthermore, we assume that the boundary data $h_\ell$ are test functions of time which are compactly supported in the interval $[0,2]$. This problem,
which we call the **reduced pure ibvp**, reads as follows:

\[
\begin{align*}
\partial_t v + (-1)^{j+1}\partial_x^{2j+1} v &= 0, & 0 < x < \infty, & 0 < t < 2, \\
v(x, 0) &= 0, \\
v(0, t) &= h_0(t), \ldots, \partial_x^{j-1} v(0, t) = h_{j-1}(t), & 0 < t < 2.
\end{align*}
\] (2.1)

In this situation we have that

\[
\tilde{h}_\ell(\xi, 2) = \int_0^2 e^{-i\xi \tau} h_\ell(\tau) d\tau = \int_\mathbb{R} e^{-i\xi \tau} h_\ell(\tau) d\tau = \tilde{h}_\ell(\xi),
\] (2.2)

and the Fokas solution formula of our reduced pure ibvp (2.1) takes the simple form

\[
v(x, t) = S[0, h_0, \cdots, h_{j-1}; 0] = \sum_{p=1}^j \sum_{\ell=0}^{j-1} C_{p, \ell} \int_{\partial D_{2p}^+} e^{i\xi x + i\xi^m t} (i \xi)^{2j-\ell} \tilde{h}_\ell(\xi^m) d\xi = \sum_{p=1}^j \sum_{\ell=0}^{j-1} C_{p, \ell} v_{p \ell},
\] (2.3)

where

\[
v_{p \ell}(x, t) = \int_{\partial D_{2p}^+} e^{i\xi x + i\xi^m t} (i \xi)^{2j-\ell} \tilde{h}_\ell(\xi^m) d\xi, \quad x \in \mathbb{R}^+, \quad t \in [0, 2].
\] (2.4)

In the next result we estimate this solution in the Hadamard and the Bourgain spaces. Our objective is to obtain the optimal bounds in temporal Sobolev spaces. In fact, in both cases the bounds that arise naturally are the ones suggested by the time regularity of the solution to the linear homogeneous Cauchy problem with data in \(H^s(\mathbb{R})\) (for KdV, see [42], [38], [22]) More precisely, we have the following result.

**Theorem 2.1** (Estimates for pure ibvp on the half-line). For boundary data test functions that are compactly supported in the interval \([0, 2]\), the solution for the reduced pure ibvp (2.1) satisfies the following Hadamard space estimate

\[
\sup_{t \in \mathbb{R}} \|S[0, h_0, \cdots, h_{j-1}; 0]\|_{H^s_x(0, \infty)} \leq c_s \sum_{\ell=0}^{j-1} \|h_\ell\|_{H^{s+j-\ell}^m(\mathbb{R})}, \quad s \geq 0.
\] (2.5)

In addition, for \(b \in [0, \frac{1}{2}]\) and \(\frac{1}{2} < \alpha \leq \frac{1}{2} + \frac{1}{m}(s + j + \frac{1}{2})\) it satisfies the Bourgain spaces estimate

\[
\|S[0, h_0, \cdots, h_{j-1}; 0]\|_{X^{s, b, \alpha}(\mathbb{R}^+ \times (0, 2))} \leq c_{s, b, \alpha} \sum_{\ell=0}^{j-1} \|h_\ell\|_{H^{s+j-\ell}^m(\mathbb{R})}, \quad s > -j - \frac{1}{2}.
\] (2.6)

The proof of the Hadamard space estimate (2.5) can be found in [60] for KdVm and in [22] for KdV. The restriction \(s \geq 0\) comes from the use of the physical space description of the Sobolev norm. Here, we focus on the Bourgain spaces estimate (2.6) which is new and useful. It is the basic ingredient in the proof of well-posedness of KdVm on the half-line.

**Proof of Theorem 2.1.** Here we present the proof of the estimate (2.6). Using the parametrization \([0, \infty) \ni \xi \rightarrow \gamma \xi\) for the right side of the domain \(D_{2p}^+\), and the parametrization \([0, \infty) \ni \xi \rightarrow \gamma' \xi\) for the left side of \(D_{2p}^+\), we obtain the following decomposition \(v_{p \ell}(x, t) = V_1(x, t) + V_2(x, t)\), where

\[
V_1(x, t) = \int_0^\infty e^{i\gamma' \xi x - i\xi^m t} (\gamma \xi)^{2j-\ell} \tilde{h}_\ell(-\xi^m) d\xi \simeq \int_0^\infty e^{-i\xi^m t} e^{i\gamma' \xi x} e^{-\gamma' \xi^m \xi^{2j-\ell}} \tilde{h}_\ell(-\xi^m) d\xi,
\] (2.7)

\[
V_2(x, t) = \int_0^\infty e^{i\gamma \xi x + i\xi^m t} (\gamma' \xi)^{2j-\ell} \tilde{h}_\ell(\xi^m) d\xi \simeq \int_0^\infty e^{i\xi^m t} e^{i\gamma' \xi x} e^{-\gamma' \xi^m \xi^{2j-\ell}} \tilde{h}_\ell(\xi^m) d\xi,
\] (2.8)
and
\[ \gamma = e^{i(2p-1)\frac{\pi}{m}} = \cos\left(\frac{2p-1}{m} \pi\right) + i \sin\left(\frac{2p-1}{m} \pi\right), \quad \gamma' = e^{i2p\frac{\pi}{m}} = \cos\left(\frac{2p}{m} \pi\right) + i \sin\left(\frac{2p}{m} \pi\right). \]

Note, that the imaginary parts of \( \gamma \) and \( \gamma' \) are positive, which is crucial for our estimates.

Here we only estimate \( V_1 \). The estimation of \( V_2 \) is similar. Also, we split \( V_1 \) as the sum of two functions, one for \( \xi \) near 0 and the other away from 0, that is \( V_1 = v_0 + v_1 \), where
\[ v_0(x, t) = \int_0^1 e^{-i\xi^m} e^{i\gamma R \xi x} e^{-\gamma \psi} e^{2j-\ell} h_{\ell}(\xi x) d\xi, \quad x \in \mathbb{R}^+, \quad t \in [0, 2], \]
and
\[ v_1(x, t) = \int_1^\infty e^{-i\xi^m} e^{i\gamma R \xi x} e^{-\gamma \psi} e^{2j-\ell} h_{\ell}(\xi x) d\xi, \quad x \in \mathbb{R}^+, \quad t \in [0, 2]. \]
The estimate of the Bourgain norm for \( v_0 \) follows from the boundedness of the Laplace transform in \( L^2 \) and we will do it later. Next, we estimate the Bourgain norm for \( v_1 \). Using the identity
\[ e^{i\gamma \psi} = \frac{\partial_x^j [e^{i\gamma R \xi x} e^{-\gamma \psi}]}{i\gamma^j} \]
and the fact that \( e^{-\gamma \psi} \) is exponentially decaying in \( \xi \) for \( x > 0 \) we can take the \( \partial_x^j \)-derivative outside the integral sign in (2.10) to rewrite \( v_1(x, t) \) as follows
\[ v_1(x, t) = \frac{1}{i\gamma^j} \left( \frac{\partial_x^j e^{i\gamma R \xi x} e^{-\gamma \psi}}{i\gamma^j} \right) \int_1^\infty e^{-i\xi^m} e^{i\gamma R \xi x} e^{-\gamma \psi} e^{2j-\ell} h_{\ell}(\xi x) d\xi, \quad x \in \mathbb{R}^+, \quad t \in [0, 2]. \]

Next, we extend \( v_1 \) from \( \mathbb{R}^+ \times [0, 2] \) to \( \mathbb{R} \times \mathbb{R} \) by using the one-sided cutoff function \( \rho(x) \), which satisfies \( 0 \leq \rho(x) \leq 1, \ x \in \mathbb{R} \), and is as follows
\[ \rho(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x \leq -1. \end{cases} \]

Using it, we extend \( v_1 \) via the formula below (keeping the same notation \( v_1 \) for it)
\[ v_1(x, t) = \frac{1}{i\gamma^j} \left( \frac{\partial_x^j e^{i\gamma R \xi x} e^{-\gamma \psi} \rho(\gamma R \xi x)}{i\gamma^j} \right) \int_1^\infty e^{-i\xi^m} e^{i\gamma R \xi x} e^{-\gamma \psi} e^{2j-\ell} h_{\ell}(\xi x) d\xi, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \]
where \( \gamma_I \) is the imaginary part of \( \gamma \) and \( \gamma_R \) is its real part. Also, we could have localized \( v_1 \) in \( t \) further by multiplying it by the standard cutoff function \( \psi(t) \) in \( C_0^\infty(-1, 1) \) such that \( 0 \leq \psi \leq 1 \) and \( \psi(t) = 1 \) for \( |t| \leq 1/2 \). Then using the estimate
\[ \|\psi(t)v_1\|_{X^{s,b,0}_{x,b,a}(\mathbb{R}^2)} \leq c_\psi \|v_1\|_{X^{s,b,0}_{x,b,a}(\mathbb{R}^2)}, \]
we are reduced in estimating \( \|\cdot\|_{X^{s,b,0}_{x,b,a}(\mathbb{R}^2)} \), which we do next. Notice that the quantities under the integral defining \( v_1 \) make sense for all \( t \) since \( t \) appears in oscillatory terms.

Extension (2.13) is good since \( \rho(\gamma_I \xi x) = 1 \) for \( x > 0 \). Also, \( e^{-\gamma \psi} \rho(\gamma_I \xi x) \) is bounded for all \( x \) and \( t \) since \( e^{-\gamma \psi} \leq e \) and \( \rho \leq 1 \), that is \( |e^{-\gamma \psi} \rho(\gamma_I \xi x)| \leq e^1 \cdot 1, \ x \in \mathbb{R}, \ t \in \mathbb{R} \). Making the change of variables \( \tau = -\xi^m \) and defining
\[ \eta(x) = e^{i(2R)\frac{\pi}{m}} e^{-x} \rho(x), \]
we write \( v_1 \) in the form
\[ v_1(x, t) \simeq \partial_x^j \int_{-\infty}^1 e^{i\tau t} \eta(-\gamma_I \tau^{1/m} x) \tau^{-(\ell+j)/m} \widehat{h}_{\ell}(\tau) d\tau, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \]
where
and prove the following result for it.

**Lemma 2.1** (Bourgain space estimate for reduced ibvp). For any \( \varepsilon > 0 \), if \( s \geq -j - \frac{1}{2} - \varepsilon \), and \( b \geq 0 \), then the function \( v_1(x,t) \), which is part of the solution \( v = S[0,h_0,\ldots,h_{j-1};0] \) to pure ibvp (2.1) and defined by (2.13) satisfies the space estimate

\[
\|v_1\|_{X^{s,b}} \leq c_{s,b} \|h_{\ell}\|_{H^{s+mb+\frac{1}{2}+\varepsilon} (\mathbb{R})}. \tag{2.17}
\]

Moreover, if we chose \( \varepsilon \) such that \( \varepsilon \leq m(\frac{1}{2} - b) \), which is possible if \( b < 1/2 \), then we have the estimate (needed in our well-posedness theorem)

\[
\|v_1\|_{X^{s,b}} \leq c_{s,b} \|h_{\ell}\|_{H^{s+mb} (\mathbb{R})}. \tag{2.18}
\]

**Proof of Lemma 2.1.** In order to estimate the \( \|v_1\|_{s,b} \), we need to calculate the Fourier transform of \( v_1(x,t) \). Using the inverse Fourier transform, we get

\[
\widehat{v}_1(x,\tau) \simeq \begin{cases} 
\partial_x^j \eta(-\gamma_1 \tau^{1/m}x) \gamma^{-\ell}/m \widehat{h}_{\ell}(\tau), & \tau < -1, \\
0, & \tau \geq -1.
\end{cases} \tag{2.19}
\]

In addition, taking the Fourier transform with respect to \( x \), we get

\[
\widehat{v}_1(\xi,\tau) \simeq \begin{cases} 
\xi^j F(\xi,\tau) \gamma^{-\ell}/m \widehat{h}_{\ell}(\tau), & \tau < -1, \\
0, & \tau \geq -1,
\end{cases} \tag{2.20}
\]

where \( F(\xi,\tau) \) is given by \( F(\xi,\tau) \doteq \int_{x \in \mathbb{R}} e^{-i\xi x} \eta(-\gamma_1 \tau^{1/m}x) dx \). Also, using the fact that \( \eta \) is a Schwarz function and making a change of variables, we get the following result.

**Lemma 2.2.** For any \( n \geq 0, \tau < -1 \) and \( \xi \in \mathbb{R} \), we have

\[
|F(\xi,\tau)| \leq c_{\rho,\gamma,n} \cdot \frac{1}{|\tau|^{1/m}} \cdot \left( \frac{|\tau|^{1/m}}{\xi + |\tau|^{1/m}} \right)^n, \tag{2.21}
\]

where \( c_{\rho,\gamma,n} \) is a constant depending on \( \gamma, n \) and \( \rho \), which is described in (2.12).

Furthermore, using (2.20) we get

\[
\|v_1\|_{X^{s,b}}^2 \simeq \int_{-\infty}^{-1} \int_{-\infty}^{\infty} (1 + |\xi|)^{2s} (1 + |\tau - \xi^m|)^{2b} |\xi^j F(\xi,\tau)|^2 d\xi \cdot |\tau^{-\ell}/m \widehat{h}_{\ell}(\tau)|^2 d\tau. \tag{2.22}
\]

Next we will estimate the \( d\xi \) integral in (2.22). In fact, we have the following estimate:

**Lemma 2.3.** For any \( \varepsilon > 0 \), if \( s \geq -j - \frac{1}{2} - \varepsilon \), \( b \geq 0 \) then we have

\[
\int_{-\infty}^{\infty} (1 + |\xi|)^{2s} (1 + |\tau - \xi^m|)^{2b} |\xi^j F(\xi,\tau)|^2 d\xi \leq c_{s,b,m} |\tau|^{2(s+\varepsilon)+2mb-1+2\varepsilon}/m, \tag{2.23}
\]

where \( c_{s,b,m} \) is a constant depending on \( s, b \) and \( m \).

We prove (2.23) below. Next, combining it with (2.22), we obtain

\[
\|v_1\|_{X^{s,b}}^2 \leq c_{s,b,m} \int_{-\infty}^{-1} \tau^{s+mb+\frac{1}{2}+\ell+\varepsilon} \cdot \widehat{h}_{\ell}(\tau)^2 d\tau \leq c_{s,b,m} \|h_{\ell}\|_{H^{s+mb+\frac{1}{2}+\ell+\varepsilon}}^2\|\tau^{s+mb+\frac{1}{2}+\ell+\varepsilon}\|_{H^{s+mb+\frac{1}{2}+\ell+\varepsilon}},
\]

which is the desired estimate (2.17) for \( v_1 \). \( \square \)
Proof of Lemma 2.3. Using the estimate (2.21), denoting the integrand by
\[ I(\xi, \tau) = (1 + |\xi|)^{2s}(1 + |\tau - \xi^m|)^{2b} |\xi|^2 |F(\xi, \tau)|^2 \lesssim (1 + |\xi|)^{2s}(1 + |\tau - \xi^m|)^{2b} \frac{\xi^{2j} |\tau|^{2n - 2}}{|\xi|^{2n} + |\tau|^{2n/m}}, \]
and using our assumption \( b \geq 0, |\tau| > 1 \), we have
\[ 1 \leq |\xi| + |\tau|^\frac{1}{m} \Rightarrow 1 + |\tau - \xi^m| \leq 2(|\tau| + |\xi|^m) \Rightarrow (1 + |\tau - \xi^m|)^{b} \leq c_{b,m}(|\tau|^b + |\xi|^{mb}), \tag{2.24} \]
where \( c_{b,m} \) is a constant depending on \( b \) and \( m \). Hence we obtain
\[ I(\xi, \tau) \leq c_{s,b,m} \left[ (1 + |\xi|)^{2s} |\tau|^{2b} + (1 + |\xi|)^{2s} |\xi|^{2mb} \right] \cdot \frac{\xi^{2j} |\tau|^{2n - 2}}{|\xi|^{2n} + |\tau|^{2n/m}}, \tag{2.25} \]
where \( c_{s,b,m} \) is a constant depending on \( s, b \) and \( m \). Now we shall consider the following cases:
- \( |\xi| > |\tau|^\frac{1}{m} \) and \( |\xi| \leq |\tau|^\frac{1}{m} \)

Case \( |\xi| > |\tau|^\frac{1}{m} \). Here we have \( |\xi| \geq 1 \) and choosing \( n = (s + j) + mb + \frac{1}{2} + \varepsilon \), from (2.25) we have
\[ I(\xi, \tau) \lesssim c_{s,b,m} (1 + |\xi|)^{2s} |\xi|^{2mb} \frac{\xi^{2j} |\tau|^{2n - 2}}{|\xi|^{2n} + |\tau|^{2n/m}} \lesssim c_{s,b,m} |\xi|^{-1 - 2\varepsilon} |\tau|^{\frac{2(s+j)+2mb-1+2\varepsilon}{m}}. \]
Thus, for the integral (2.23) we have the following inequality \( \int_{|\xi| \geq 1} I(\xi, \tau) d\xi \leq c_{s,b,m} |\tau|^{\frac{2(s+j)+2mb-1+2\varepsilon}{m}} \), which is the desired estimate (2.23) in this case.

Case \( |\xi| \leq |\tau|^\frac{1}{m} \). Then \( 1 + |\xi| \lesssim |\tau|^{1/m} \). Choosing \( n = (s + j) + \frac{1}{2} + \varepsilon \), from (2.25) we have
\[ I(\xi, \tau) \lesssim c_{s,b,m} (1 + |\xi|)^{2s} |\tau|^{2b} \frac{\xi^{2j} |\tau|^{2n - 2}}{|\xi|^{2n} + 1} \lesssim (1 + |\xi|)^{2s} |\tau|^{2b} \frac{|\tau|^{2n - 2}}{(|\xi| + 1)^{2n}} \lesssim (1 + |\xi|)^{-1 - 2\varepsilon} |\tau|^{\frac{2(s+j)+2mb-1+2\varepsilon}{m}}. \]
Therefore, integrating \( \xi \) we get \( \int_{|\xi| \leq |\tau|^{1/m}} I(\xi, \tau) d\xi \lesssim |\tau|^{\frac{2(s+j)+2mb-1+2\varepsilon}{m}} \), which is the desired estimate (2.23) in this case. This completes the proof of Lemma 2.3. □

Estimation of \( D^\alpha \) norm. Now we estimate the second part in the modified Bourgain norm \( \| \cdot \|_{s,b,\alpha} \). More precisely we have the result.

Lemma 2.4. If \( s > -j - \frac{1}{2} \) and \( \frac{1}{2} < \alpha \leq \frac{1}{m}(s + j + \frac{1}{2}) \), then we have
\[ \| v_1 \|_{\partial D^\alpha}^2 \leq \int_{-\infty}^{\infty} \int_{-1}^{1} (1 + |\tau|)^{2\alpha} |\widehat{v_1}(\xi, \tau)|^2 d\xi d\tau \lesssim \| h_{\ell} \|^2_{H^\frac{1}{m+\frac{1}{\alpha}-\frac{1}{\ell}}(\mathbb{R})}. \tag{2.26} \]

Proof of Lemma 2.4. First, we recall that the Fourier transform of \( v_1 \) is given by (2.20). Thus we have
\[ \| v_1 \|_{\partial D^\alpha}^2 = \int_{-\infty}^{-1} \int_{-1}^{1} (1 + |\tau|)^{2\alpha} |\xi|^2 F(\xi, \tau)^{-\ell+j/m} |\widehat{h_{\ell}}(\tau)|^2 d\xi d\tau, \]
where \( F(\xi, \tau) = \int_{\mathbb{R}} e^{-i\xi x} \eta(-\gamma_{\ell} \frac{1}{m} x) dx \). Also, applying Lemma 2.2 with \( n = 0 \), we get the following bound for \( F \), that is \( |F(\xi, \tau)| \leq |\tau|^{-1/m} \). Hence, after integrating \( \xi \), we have
\[ \| v_1 \|_{\partial D^\alpha}^2 \lesssim \int_{-\infty}^{-1} (1 + |\tau|)^{2\alpha} |\tau|^{\frac{1}{m} + \frac{1}{\alpha}} |\widehat{h_{\ell}}(\tau)|^2 d\tau \lesssim \int_{-\infty}^{-1} (1 + |\tau|)^{\frac{2\alpha n - 2 - 2j - 2l}{m}} |\widehat{h_{\ell}}(\tau)|^2 d\tau \leq \| h_{\ell} \|^2_{H^\frac{1}{m+\frac{1}{\alpha}-\frac{1}{\ell}}(\mathbb{R})}. \]
Choosing \( \alpha \) such that \( \alpha \leq \frac{1}{2} + \frac{1}{m}(s + j + \frac{1}{2}) \), we get the desired estimate (2.26). □
Bound near $\xi = 0$. Next, we estimate the Bourgain norm for $v_0$. We begin with extending it from $\mathbb{R}^+ \times (0, 2)$ to $\mathbb{R} \times \mathbb{R}$ (keeping the same notation)

$$v_0(x, t) = \int_0^1 e^{i\xi \varphi_1(x)} e^{-i \xi^m t} \xi^{2j-\ell} \hat{h}_t(-\xi^m) d\xi, \quad x \in \mathbb{R}, \ t \in \mathbb{R},$$

(2.27)

where $\varphi_1(x)$ is a smooth version of $|x|$. More precisely

$$\varphi_1(x) = \begin{cases} x, & x \geq 0 \\ -x, & x \leq -1 \\ \text{smooth on } \mathbb{R}. \end{cases}$$

(2.28)

For $v_0$, we have the $L^2$ estimate

$$\|\psi(t)v_0\|_{L^2}^2 \lesssim \|\partial_x \varphi_1(x)\|_{L^2}^2 + \|\partial_x^n \varphi_1(x)\|_{L^2}^2 + \|\partial_t^{n_2} [\varphi(t)v_0]\|_{L^2}^2 + \|\partial_x^{n_1} \partial_t^{n_2} [\varphi(t)v_0]\|_{L^2}^2,$$  

(2.29)

where $n_1 = n_2(s, b) = 2m [b] + 2[s] + 2$ and $n_2 = n_3(b, \alpha) = 2[b] + 2[\alpha] + 2$. Using the $L^2$ boundedness of Laplace transform (see Lemma 2.3 in [22] or [33]) we get

$$\|\partial_x^{n_1} \partial_t^{n_2} [\varphi \cdot v_0]\|_{L^2}^2 \leq C_{n_1, n_2} \int_0^\infty |\hat{h}_t(\tau)|^2 d\tau \lesssim \|h_\ell\|_{H_1}^2 \quad \forall n_1, n_2 \in \mathbb{N}_0.$$  

(2.30)

**End of Proof for Theorem 2.1.** Combining Lemma 2.1, Lemma 2.4 and estimate (2.29) with (2.30), we get estimate (2.6). This completes the proof of Theorem 2.1. \(\square\)

### 3. Proof of Forced Linear IBVP Estimates

In this section, we prove the basic linear estimate (1.18). We begin by decomposing the forced linear ibvp (1.10) into a homogeneous ibvp (A) and an inhomogeneous ibvp with zero data (B). Then, we decompose both problems further in a convenient way simplifying both their Fokas solution formula and its estimation.

**A. The homogeneous linear ibvp:**

$$\begin{align*}
\partial_t u + (-1)^{j+1} \partial_x^{2j+1} u &= 0, \\
u(x, 0) &= u_0(x) \in H_x^s(0, \infty), \\
\partial_x^\ell u(0, t) &= g_\ell(t) \in H_1^{\frac{2j+\ell}{m}}(0, T),
\end{align*}$$

(3.1a-c)

with solution denoted by $u(x, t) \doteq S[u_0, g_0, \ldots, g_{j-1}] (x, t)$ and which is defined in (1.11). We decompose it further into the following two problems.

**A$_1$. The homogeneous linear ivp:**

$$\begin{align*}
\partial_t U + (-1)^{j+1} \partial_x^{2j+1} U &= 0, \\
U(x, 0) &= U_0(x) \in H_x^s(\mathbb{R}),
\end{align*}$$

(3.2a-b)

where $U_0 \in H_x^s(\mathbb{R})$ is an extension of the initial datum $u_0 \in H_x^s(0, \infty)$ such that

$$\|U_0\|_{H_x^s(\mathbb{R})} \leq 2 \|u_0\|_{H_x^s(0, \infty)}$$

(3.3)

with its solution given by

$$U(x, t) = S[U_0; 0] (x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x + i \xi^m t} \hat{U}_0(\xi) d\xi,$$

(3.4)
where $\widetilde{U}_0(\xi) = \int_\mathbb{R} e^{-i\xi x} U_0(x) dx$, $\xi \in \mathbb{R}$.

A2. The homogeneous linear ibvp with zero initial data:

\begin{align*}
\partial_t u + (-1)^{j+1} \partial_x^{2j+1} u &= 0, \\
u(x, 0) &= 0, \\
\partial_x^\ell u(0, t) &= g_\ell(t) - \partial_x^\ell U(0, t) = \mathcal{G}_\ell(t) \in H_t^{\frac{s+1-j}{2}}(0, T), \quad \ell = 0, 1, \ldots, j - 1,
\end{align*}

with solution $u(x, t) = \mathcal{S}[0, G_0, \ldots, G_{j-1}; 0](x, t)$, which is defined in (1.11).

B. The forced linear ibvp with zero initial data:

\begin{align*}
\partial_t u + (-1)^{j+1} \partial_x^{2j+1} u &= f(x, t), \\
u(x, 0) &= 0, \\
\partial_x^\ell u(0, t) &= 0, \quad \ell = 0, 1, \ldots, j - 1,
\end{align*}

whose solution $u(x, t) = \mathcal{S}[0, 0, \ldots, 0; f](x, t)$ is defined in (1.11). This problem can be further decomposed into the following two problems (B1) and (B2).

B1. The forced linear ivp with zero initial data:

\begin{align*}
\partial_t W + (-1)^{j+1} \partial_x^{2j+1} W &= w(x, t), \\
W(x, 0) &= 0,
\end{align*}

where $w$ is an extension of the forcing $f$ such that

\begin{align*}
\|w\|_{X^{s, -b, \alpha}(-\mathbb{R}^2)} &\leq 2\|f\|_{X^{s, -b, \alpha}(-\mathbb{R}^+ \times (0, T))}, \quad -1 \leq s \leq \frac{1}{2}, \\
\|w\|_{X^{s, -b, \alpha}(-\mathbb{R}^2)} + \|w\|_{Y^{s, -b}(-\mathbb{R}^2)} &\leq 2(\|f\|_{X^{s, -b, \alpha}(-\mathbb{R}^+ \times (0, T))} + \|f\|_{Y^{s, -b}(-\mathbb{R}^+ \times (0, T))}), s \not\in [-1, 1/2],
\end{align*}

where $Y^{s, b}$ is defined in (1.19). The solution of this problem is given by Duhamel’s formula

\begin{align*}
W(x, t) &= \mathcal{S}[0; w](x, t) = -\frac{i}{2\pi} \int_0^t \int_{\xi}^{t} e^{i\xi x + i\xi m(t-t')} \hat{w}(\xi, t') dt' d\xi, \\
&= -i \int_0^t \mathcal{S}[w(\cdot, t'); 0](x, t-t') dt',
\end{align*}

where $\hat{w}$ is the Fourier transform of $w$ with respect to $x$, and $\mathcal{S}[w(\cdot, t'); 0]$ in the Duhamel representation (3.10) denotes the solution (3.4) of ivp (3.2) (that is Problem A1) with $w(x, t')$ in place of the initial data and zero forcing.

B2. The homogeneous linear ibvp with zero initial data:

\begin{align*}
\partial_t v + (-1)^{j+1} \partial_x^{2j+1} v &= 0, \\
v(x, 0) &= 0, \\
\partial_x^\ell v(0, t) &= -\partial_x^\ell W(0, t) = -W_\ell(t) \quad \ell = 0, 1, \ldots, j - 1,
\end{align*}

whose solution $v(x, t) = \mathcal{S}[0, -W_0, \ldots, -W_{j-1}; 0](x, t)$ is defined in (1.11).

Next we describe the estimates for each one of the above sub-problems.
Theorem 3.1 (Estimates for homogeneous ivp $A_1$). The solution $U = S[U_0; 0]$ to ivp (3.2) defined by formula (3.4) satisfies the space estimate
\[
\sup_{t \in [0,T]} \|S[U_0; 0](t)\|_{H^s_x(\mathbb{R})} = \|U_0\|_{H^s_x(\mathbb{R})}, \quad s \in \mathbb{R},
\] (3.12)
and the time estimate for its $\ell$-th derivative (needed to have boundary data in desired space)
\[
\sup_{x \in \mathbb{R}} \|\psi(t)\partial_x^\ell S[U_0; 0](x)\|_{H^s_x(\mathbb{R})} \leq c_\ell \|U_0\|_{H^s_x(\mathbb{R})}, \quad s \in \mathbb{R},
\] (3.13)
where $\ell = 0, 1, \ldots, j - 1$ and $\mu_\ell = \frac{1}{m}(s + j - \ell)$. Also, it satisfies the following estimate in modified Bourgain spaces
\[
\|\psi(t)S[U_0; 0](x,t)\|_{X^{s,b,\alpha}} \leq c_\psi \|U_0\|_{H^s_x(\mathbb{R})}, \quad \forall s,b,\alpha \in \mathbb{R},
\] (3.14)
where $c_\psi$ is a constant depending only on $\psi$. Here and elsewhere in this paper $\psi$ is a cutoff function in $C^\infty_0(-1,1)$ such that $0 \leq \psi \leq 1$ and $\psi(t) = 1$ for $|t| \leq 1/2$.

Proof of Theorem 3.1. The proof of the space estimate (3.12) is straightforward. The proof of the time estimate (3.13) is similar to that for KdV, which can be found in Holmer [38], and Colliander and Kenig [12]. Finally, estimate (3.14) follows from inequality
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_{|\xi|<1}(1 + |\tau|)^{2\alpha} |\hat{u}(\xi, \tau)|^2 d\xi d\tau \lesssim \|u\|_{X^{0,\alpha}}^2,
\]
and the estimate
\[
\|\psi(t)\partial_x^\ell S[U_0; 0](x,t)\|_{X^{s,b}} \leq c_\ell \|U_0\|_{H^s_x(\mathbb{R})}, \quad \text{where } c_\ell = \|\psi\|_{H^\ell}, \quad s,b \in \mathbb{R},
\] (3.15)
whose proof can be found in [18].

Theorem 3.2 (Estimates for pure ibvp on the half-line). Let $s \neq \frac{1}{2}, \frac{3}{2}, \ldots, j - \frac{1}{2}$. The solution of the pure ibvp (3.5) satisfies the space estimate
\[
\sup_{t \in \mathbb{R}} \|S[0, G_0, \ldots, G_{j-1}; 0]\|_{H^s_x(\mathbb{R}^+; \mathbb{R})} \leq c_{s,m} \sum_{\ell=0}^{j-1} \|G_\ell\|_{H^\ell_{s,m}(0,T)}, \quad s \geq 0.
\] (3.16)
Also, for $b \in [0, \frac{1}{2})$, $\frac{1}{2} < \alpha \leq \frac{1}{2} + \frac{1}{m}(s + j + \frac{1}{2})$ its solution satisfies the estimate in Bourgain spaces
\[
\|S[0, G_0, \ldots, G_{j-1}; 0]\|_{X^{s,b,\alpha}(\mathbb{R}^+; \mathbb{R})} \leq c_{s,b,\alpha} \sum_{\ell=0}^{j-1} \|G_\ell\|_{H^\ell_{s,m}(0,T)}, \quad s > -j - \frac{1}{2}.
\] (3.17)

Proof of Theorem 3.2. The proof of the space estimate (3.16) can be found in [60] for KdV, and in [22] for KdV. Here, we prove estimate (3.17), which is new. We do this by transforming problem $A_2$ to the reduced pure ibvp (2.1). For this, we extend $G_\ell$ from $(0, T)$ to a function $h_\ell$ on $\mathbb{R}$ supported in $[0, 2]$ and such that $\|h_\ell\|_{H^\ell_{s,m}(\mathbb{R})} \lesssim \|G_\ell\|_{H^\ell_{s,m}(0,T)}$ for any $\ell = 0, 1, \ldots, j - 1$, via the following result, whose proof can be found in [52, 53, 23, 60].

Lemma 3.1. For a general function $h^*(t) \in H^s_x(0,2)$, $s \geq 0$, let the extension
\[
\tilde{h}^*(t) = \begin{cases} h^*(t), & t \in (0,2), \\ 0, & \text{elsewhere}. \end{cases}
\]
If $0 \leq s < \frac{1}{2}$, then the extension $\tilde{h}^* \in H^s_x(\mathbb{R})$ and for some $c_s > 0$ we have
\[
\|\tilde{h}^*\|_{H^s_x(\mathbb{R})} \leq c_s \|h^*\|_{H^s_x(0,2)}.
\] (3.18)
If \( \frac{1}{2} < s \leq 1 \), then for estimate (3.18) to hold we must have the condition
\[
h^*(0) = h^*(2) = 0. \tag{3.19}
\]
Also, we shall need the following multiplier by a characteristic estimate from Holmer [38].
\[
\|\chi_{(0,\infty)}g\|_{H^s(\mathbb{R})} \leq c_s\|g\|_{H^s(\mathbb{R})}, \quad g \in H^s(\mathbb{R}), \quad -\frac{1}{2} < s < \frac{1}{2}, \tag{3.20}
\]
Since \( h_\ell \) extends \( G_\ell \) from \( (0, T) \) to \( \mathbb{R} \), we have \( \mathcal{S}[0, G_0, \ldots, G_{j-1}; 0] (x, t) = \mathcal{S}[0, h_0, \ldots, h_{j-1}; 0] (x, t) \), for \( x \in \mathbb{R}_+, t \in (0, T) \). In fact, by Theorem 2.1, for \( \frac{1}{2} > s > -\frac{1}{2} - \frac{1}{m}, b \in [0, \frac{1}{2}) \) and \( \frac{1}{2} < \alpha \leq \frac{1}{2} + \frac{1}{m}(s + j + \frac{1}{2}) \) we get
\[
\|\mathcal{S}[0, G_0, \ldots, G_{j-1}; 0]\|_{X_{s,b,\alpha}(\mathbb{R}^+ \times (0,T))} = \|\mathcal{S}[0, h_0, \ldots, h_{j-1}; 0]\|_{X_{s,b,\alpha}(\mathbb{R}^+ \times (0,T))} \tag{3.21}
\]
which is the desired estimate (3.17). This completes the proof of Theorem 3.2. \( \Box \)

**Theorem 3.3** (Estimates for forced ivp \( B_1 \)). The solution \( W = \mathcal{S}[0; w] \) of the forced ivp (3.7) defined by equations (3.9)–(3.10) satisfies the following estimate in modified Bourgain spaces
\[
\|\psi(t)\mathcal{S}[0; w] (x, t)\|_{X_{s,b,\alpha}} \lesssim c_{s,b,\alpha} \|w\|_{X_{s,b,\alpha}}^{1-b}, \quad s \in \mathbb{R}, \quad 0 < b < \frac{1}{2} < \alpha < 1, \tag{3.22}
\]
and the time estimate (needed to have boundary data in desired space)
\[
\sup_{x \in \mathbb{R}} \|\psi(t)\partial_x^\ell \mathcal{S}[0; w]\|_{H^s(\mathbb{R})} \lesssim \left\{ \begin{array}{ll}
\frac{c_{s,m,\ell,b}}{1+|\xi|} \|w\|_{X_{s,b}}^{1-b}, & 1 \leq s \leq \frac{1}{2}, \\
c_{s,m,\ell,b} \|w\|_{X_{s,b}} + \|w\|_{Y_{s,b}}, & s \in \mathbb{R},
\end{array} \right. \tag{3.23}
\]
where \( X_{s,b} \) is the Bourgain space (1.2) and \( Y_{s,b} \) is a “temporal” Bourgain space defined by (1.19).

**Proof of Theorem 3.3.** First, we prove estimate (3.22). For the \( X_{s,b} \) part of \( \|\psi \cdot \mathcal{S}[0; w]\|_{X_{s,b,\alpha}} \), i.e. \( \|\psi(t)\mathcal{S}[0; w](x, t)\|_{X_{s,b,\alpha}} \), we have the next basic estimate, whose proof can be found in [18]
\[
\|\psi(t)\mathcal{S}[0; w](x, t)\|_{X_{s,b,\alpha}}^2 \lesssim \|w\|_{X_{s,b,\alpha}}^2 + \int_\mathbb{R} (1 + |\xi|)^2 \left( \int_\mathbb{R} \frac{|\tilde{\psi}(\xi, \tau)|}{1 + |\tau - \xi^m|} d\tau \right)^2 d\xi, \quad 0 < b < 1. \tag{3.24}
\]
Since \( b - 1 < -\frac{1}{2} < -b \) we get \( \|w\|_{X_{s,b,\alpha}} \lesssim \|w\|_{X_{s,b}} \). For the second term in (3.24), writing \( 1 + |\tau - \xi^m| = (1 + |\tau - \xi^m|)^{1-b}(1 + |\tau - \xi^m|^b) \) and applying the Cauchy-Schwartz inequality for the \( \tau \)-integral we obtain:
\[
\int_\mathbb{R} (1 + |\xi|)^{2\alpha} \left( \int_\mathbb{R} \frac{|\tilde{\psi}(\xi, \tau)|}{1 + |\tau - \xi^m|} d\tau \right)^2 d\xi \leq c_\delta \int_\mathbb{R} (1 + |\xi|)^{2\alpha} \int_\mathbb{R} \frac{|\tilde{\psi}(\xi, \tau)|^2}{(1 + |\tau - \xi^m|)^{2\alpha}} d\tau d\xi \lesssim \|w\|_{X_{s,b}}^{2\alpha}. \tag{3.25}
\]
For the \( \alpha \)-part of the norm \( \|\psi \cdot \mathcal{S}[0; w]\|_{X_{s,b,\alpha}} \), using the fact that \( |\xi| \leq 1 \), which gives \( (1 + |\tau|)^{2\alpha} \simeq (1 + |\tau - \xi^m|)^{2\alpha}, \) we get
\[
\int_{-1}^1 (1 + |\tau|)^{2\alpha} |\psi \mathcal{S}[0; w](\xi, \tau)|^2 d\xi d\tau \simeq \int_{-1}^1 (1 + |\tau - \xi^m|)^{2\alpha} |\chi_{|\xi| \leq 1}\psi \mathcal{S}[0; w](\xi, \tau)|^2 d\xi d\tau. \tag{3.26}
\]
Also, since \( \psi \mathcal{S}[0; w]^x(\xi, t) = -i\psi(t) \int_0^t e^{i\xi^m(t-t')\xi} \tilde{\psi}(\xi, t') dt' \), we obtain
\[
\chi_{|\xi| \leq 1}\psi \mathcal{S}[0; w](\xi, \tau) = \overline{\psi \mathcal{S}[0; w]}(\xi, \tau),
\]
where \( \tilde{w}_1^x(\xi, t) = \chi_{|\xi| \leq 1} \hat{w}^x(\xi, t) \). Using \( w_1 \) notation, from (3.26) we have

\[
\int_{\mathbb{R}} \int_{-1}^{1} (1 + |\tau|)^{2\alpha} |\hat{w}(0; w)(\xi, \tau)|^2 d\xi d\tau = \| \hat{w}(0; w_1)(x, t) \|_{X^{0, \alpha}}^2.
\]  

(3.27)

For \( \| \psi(t)S[0; w_1](x, t) \|_{X^{0, \alpha}} \), applying estimate (3.24) with \( s = 0 \) and \( b = \alpha > \frac{1}{3} \), we get

\[
\| \psi(t)S[0; w_1](x, t) \|_{X^{0, \alpha}}^2 \lesssim \int_{\mathbb{R}} \int_{-1}^{1} (1 + |\tau|)^{2\alpha-2} |\hat{w}(\xi, \tau)|^2 d\xi d\tau.
\]  

(3.28)

Using estimates (3.27), (3.28) and the fact that \( \| \xi \| \leq 1 \) again, we obtain

\[
\int_{\mathbb{R}} \int_{-1}^{1} (1 + |\tau|)^{2\alpha} |\hat{w}(0; w)(\xi, \tau)|^2 d\xi d\tau \lesssim \int_{\mathbb{R}} \int_{-1}^{1} (1 + |\tau|)^{2\alpha-2} |\hat{w}(\xi, \tau)|^2 d\xi d\tau.
\]  

(3.29)

Combining (3.24) and (3.25) with (3.29) and taking into consideration that \( b - 1 < -b \), we get

\[
\| \psi(t)S[0; w](x, t) \|_{X^{b, -b}}^2 \lesssim \| w \|_{X^{b, -b}}^2 + \| w \|_{X^{b, -b}}^2 + \int_{\mathbb{R}} \int_{-1}^{1} (1 + |\tau|)^{2\alpha-2} |\hat{w}(\xi, \tau)|^2 d\xi d\tau
\]

\[
\lesssim \| w \|_{X^{s, -s}}^2, \quad s \in \mathbb{R}, \quad 0 \leq b < \frac{1}{2} < \alpha < 1.
\]

This completes the proof of estimate (3.22).

**Proof of estimate (3.23).** For the KdV equation, this estimate was proved in [38] (see Lemma 5.6). Also a similar estimate (for \( s = 0 \)) was proved in [12] (see Lemma 5.5). Differentiating the solution formula (3.9), i.e. \( S[0; w](x, t) = \mathcal{F}^{-1}_{\mathbb{R}} L_{t=0} e^{i\xi x + i\xi \tau(t-t')} \hat{w}(\xi, t') dt' \) \( \xi \) times with respect to \( x \) and decomposing it (like in [18]), we obtain the Bourgain writing

\[
\psi(t) \partial_x^\ell S[0, w](x, t) \simeq \psi(t) \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(\xi x + \tau \xi)} \frac{1 - \psi(\tau - \xi m)}{\tau - \xi m} \xi^\ell \hat{w}(\xi, \tau) d\xi d\tau
\]

(3.30)

\[
-\psi(t) \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(\xi x + \tau \xi m)} \frac{1 - \psi(\tau - \xi m)}{\tau - \xi m} \xi^\ell \hat{w}(\xi, \tau) d\xi d\tau
\]

(3.31)

\[
+\psi(t) \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(\xi x + \tau \xi m)} \frac{[e^{i(\tau - \xi m) t} - 1]}{\tau - \xi m} \xi^\ell \hat{w}(\xi, \tau) d\xi d\tau.
\]

(3.32)

Next, we estimate each term above separately. We start by estimating (3.31).

**Estimate for (3.31).** For this term, we have

(3.31) \( \simeq \psi(t) \partial_x^\ell S[F_1, 0] \), where \( F_1(\xi) \simeq \int_{\mathbb{R}} \frac{1 - \psi(\tau - \xi m)}{\tau - \xi m} \hat{w}(\xi, \tau) d\tau \).

Using estimate (3.13), we get

\[
\sup_{x \in \mathbb{R}} \big\| (3.31) \big\|_{H^{2 + \frac{1}{2} - \frac{\ell}{m}}(\mathbb{R})} \lesssim \sup_{x \in \mathbb{R}} \big\| \psi(t) \partial_x^\ell S[F_1, 0] \big\|_{H^{2 + \frac{1}{2} - \frac{\ell}{m}}(\mathbb{R})} \lesssim \big\| F_1 \big\|_{H^2(\mathbb{R})}^2
\]

\[
= \int_{\mathbb{R}} (1 + |\xi|)^{2\alpha} \int_{\mathbb{R}} \frac{1 - \psi(\tau - \xi m)}{\tau - \xi m} \hat{w}(\xi, \tau) d\tau \bigg| d\xi
\]

\[
\lesssim \int_{\mathbb{R}} (1 + |\xi|)^{2\alpha} \left( \int_{\mathbb{R}} \frac{|\hat{w}(\xi, \tau)|}{1 + |\tau - \xi m|} d\tau \right)^2 d\xi \lesssim \| w \|_{X^{s, -s}}^2.
\]

where in the last step we used estimate (3.25). This gives the desired estimate (3.23) for (3.31).

**Estimate for (3.32).** For this term, using Taylor’s expansion we have

(3.32) \( \simeq \sum_{k=1}^{\infty} \frac{1}{k!} t^k \partial_x^k \psi(t) S[c_k, 0] \), where \( c_k(\xi) \simeq \int_{\mathbb{R}} \psi(\tau - \xi m) \cdot (\tau - \xi m)^{k-1} \hat{w}(\xi, \tau) d\tau \).
Letting $\psi_k(t) \doteq t^k[\psi(t)]^{1/2}$ and using estimate (3.13), we get

$$
\sup_{x \in \mathbb{R}} \| (3.32) \|_{H^{-\frac{s+1-j-\ell}{m-1}}(\mathbb{R})} \lesssim \sum_{k=1}^{\infty} \frac{1}{k!} \sup_{x \in \mathbb{R}} \| \psi_k(t) \cdot [\psi(t)]^{1/2} \partial_x^s S[c_k, 0] \|_{H^{-\frac{s+1-j-\ell}{m-1}}(\mathbb{R})} \lesssim \sum_{k=1}^{\infty} \frac{c\psi_k}{k!} \| c_k \|_{H^s},
$$

where $c\psi_k \doteq \| \hat{\psi}_k(\tau) \|_{L^1}$ (like (3.33) below). Since the $\tau$-integration is over $|\tau - \xi^m| \leq 1$ and $|\psi(\tau - \xi^m)| \leq 1$ from the last relation we obtain that

$$
\sup_{x \in \mathbb{R}} \| (3.32) \|_{H^{-\frac{s+1-j-\ell}{m-1}}(\mathbb{R})} \lesssim \left( \int_{\mathbb{R}} (1 + |\xi|)^{2s} \left( \int_{|\tau - \xi^m| \leq 1} |\hat{w}(\xi, \tau)| d\tau \right)^2 d\xi \right)^{1/2} \cdot \sum_{k=1}^{\infty} \frac{1}{k!} \lesssim \| w \|_{s,-b},
$$

where last inequality follows from (3.25). This gives the desired estimate (3.23) for term (3.32).

**Estimate for (3.30).** We rewrite this term as

$$(3.30) \simeq \psi(t)h(x, t), \quad \text{where} \quad h(x, t) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(\xi x + \tau t)} \frac{1 - \psi(\tau - \xi^m)}{\tau - \xi^m} \xi^\ell \hat{w}(\xi, \tau) d\tau d\xi.
$$

Using the property $\hat{f} \cdot g \simeq \hat{f} \ast \hat{g}$ we get $\hat{\psi} \cdot \hat{h}(x, \tau) \simeq \int_{\mathbb{R}} \hat{\psi}(\tau - \tau_1) \hat{h}(x, \tau_1) d\tau_1$, which combined with the inequality $(1 + |\tau|)^\mu \leq (1 + |\tau_1|)^\mu (1 + |\tau - \tau_1|)^{|\mu|}$, for $\mu = \frac{s+1-j-\ell}{m}$, we get

$$
\| \psi h \|_{H^{-\frac{s+1-j-\ell}{m-1}}(\mathbb{R})} \lesssim \left\| \int_{\mathbb{R}} (1 + |\tau|)^{\frac{s+1-j-\ell}{m}} (1 + |\tau^m - \tau|)^{\frac{s+1-j-\ell}{m}} \hat{\psi}(\tau - \tau_1) \hat{h}(x, \tau_1) d\tau_1 \right\|_{L^2(\mathbb{R})}
\lesssim \| \hat{\psi}(\tau)(1 + |\tau|)^{\frac{s+1-j-\ell}{m}} \|_{L^1(\mathbb{R})} \cdot \left\| (1 + |\tau|)^{\frac{s+1-j-\ell}{m}} \hat{h}(x, \tau) \right\|_{L^2(\mathbb{R})}
= c_{\psi} \| (1 + |\tau|)^{\frac{s+1-j-\ell}{m}} \hat{h}(x, \tau) \|_{L^2(\mathbb{R})},
$$

where in the second step we use the Young’s inequality for $r = q = 2$, and $p = 1$ and and we bound the constant $c_{\psi}^2 \doteq \| \hat{\psi}(\tau)(1 + |\tau|)^{\frac{s+1-j-\ell}{m}} \|_{L^1}^2$ as follows

$$
c_{\psi}^2 \leq \int_{\mathbb{R}} |\hat{\psi}(\tau)|^2 (1 + |\tau|)^{2\frac{s+1-j-\ell}{m}} \xi^\ell \hat{w}(\xi, \tau) d\xi \lesssim \| \psi \|_{H^{-\frac{s+1-j-\ell}{m-1}}(\mathbb{R})}^2.
$$

Since $\hat{h}(x, \tau) \simeq \int_{\mathbb{R}} e^{i\xi x - i(\tau - \xi^m)} \xi^\ell \hat{w}(\xi, \tau) d\xi$, we have

$$
\| \psi h \|^2_{H^{-\frac{s+1-j-\ell}{m-1}}(\mathbb{R})} \leq c_{\psi}^2 \int_{\mathbb{R}} (1 + |\tau|)^{\frac{2(s+1-j-\ell)}{m}} \left\| \int_{\mathbb{R}} e^{i\xi x - i(\tau - \xi^m)} \xi^\ell \hat{w}(\xi, \tau) d\xi \right\|^2 d\tau
\leq c_{\psi}^2 \int_{\mathbb{R}} (1 + |\tau|)^{\frac{2(s+1-j-\ell)}{m}} \left( \int_{\mathbb{R}} \frac{1}{1 + |\tau - \xi^m|} \xi^\ell |\hat{w}(\xi, \tau)| d\xi \right)^2 d\tau.
$$

Now we consider the following two cases:

- $-1 \leq s \leq \frac{1}{2}$ and $s \notin [-1, \frac{1}{2}]$. 
Case $-1 \leq s \leq \frac{1}{2}$. For this case, multiplying and dividing $|\tilde{w}(\xi, \tau)|$ by $(1+|\tau|)^{2s/m}$ and using Cauchy-Schwartz inequality for the integral of $d\xi$, we get
\[
\left\| \psi h \right\|_{H^{s+j-\ell}_t \cap L^\infty_x} \lesssim c_s \int_{\mathbb{R}} (1 + |\tau|) \frac{(s+j-\ell)}{m} G_1(\tau) \int_{\mathbb{R}} \frac{(1 + |\xi|)^{2s}}{(1 + |\tau - \xi|^m)^{2b}} d\xi d\tau,
\]
where $G_1(\tau) = \int_{\mathbb{R}} \frac{\varepsilon^{2\ell}}{(1+|\tau - \xi|^m)^{2b} (1+|\xi|)^2} d\xi$. Taking the supremum in $\tau$ for $(1 + |\tau|)^{2(s+j-\ell)/m} G_1(\tau)$, we get
\[
\left\| \psi h \right\|_{H^{s+j-\ell}_t \cap L^\infty_x} \lesssim c_s \left\| (1 + |\tau|)^{2(s+j-\ell)/m} G_1(\tau) \right\|_{L^\infty_{\tau}} \left\| w \right\|_{Y_{s,b}}^2.
\]
For $G_1(\tau)$, we have the following result:
\[
\left\| (1 + |\tau|)^{2(s+j-\ell)/m} G_1(\tau) \right\|_{L^\infty_{\tau}} \lesssim c_{s,b}, \quad -1 \leq s \leq \frac{1}{2}, \quad 0 \leq b < \frac{1}{2}, \tag{3.34}
\]
where $c_{s,b}$ is a constant depending on $s$ and $b$. We shall prove estimate (3.34) later. Now using it we get the desired estimate (3.30) in the case $s \leq \frac{1}{2}$.

Case $s \not\in [-1, \frac{1}{2}]$. For this case, multiplying and dividing $|\tilde{w}(\xi, \tau)|$ by $(1+|\tau|)^{s/m}$ and using the Cauchy-Schwartz inequality for the integral of $d\xi$, we get
\[
\left\| \psi h \right\|_{H^{s+j-\ell}_t \cap L^\infty_x} \lesssim c_s \int_{\mathbb{R}} (1 + |\tau|) \frac{(s+j-\ell)}{m} G_2(\tau) \int_{\mathbb{R}} \frac{(1 + |\xi|)^{2s/m}}{(1 + |\tau - \xi|^m)^{2b}} d\xi d\tau,
\]
where $G_2(\tau) = \int_{\mathbb{R}} \frac{\varepsilon^{2\ell}}{(1+|\tau - \xi|^m)^{2b} (1+|\xi|)^2} d\xi$. Like in the case $s \in [-1, \frac{1}{2}]$, we get
\[
\left\| \psi h \right\|_{H^{s+j-\ell}_t \cap L^\infty_x} \lesssim c_s \left\| (1 + |\tau|)^{2(s+j-\ell)/m} G_2(\tau) \right\|_{L^\infty_{\tau}} \left\| w \right\|_{Y_{s,b}}^2.
\]
For $G_2(\tau)$, we have the following result:
\[
\left\| (1 + |\tau|)^{2(s+j-\ell)/m} G_2(\tau) \right\|_{L^\infty_{\tau}} \lesssim c_{s,b}, \quad s \in \mathbb{R}, \quad 0 < b < \frac{1}{2}, \tag{3.35}
\]
Hence, we complete the proof of Theorem 3.3 once we prove estimates (3.34) and (3.35). The proof of estimate (3.35) is similar to the proof of estimate (3.34). Here we prove only estimate (3.34).

**Proof of estimate** (3.34). To prove this estimate, we consider the following two cases:

- $|\xi| \leq 1$ and $|\xi| > 1$

**Case** $|\xi| \leq 1$. Since $G_1(\tau) = \int_{-1}^1 \frac{\varepsilon^{2\ell}}{(1+|\tau - \xi|^m)^{2b} (1+|\xi|)^2} d\xi$, we have
\[
(1 + |\tau|) \frac{(s+j-\ell)}{m} G_1(\tau) \lesssim (1 + |\tau|)^{2(s+j-\ell)/m} = (1 + |\tau|)^{2s-2\ell+2j-2m+2mb/m},
\]
which is bounded since $\frac{1}{m} \cdot (2s-2\ell+2j-2m+2mb) \leq 0$ when $s \leq \frac{1}{2}, \ell \geq 0$ and $0 \leq b < \frac{1}{2}$.

**Case** $|\xi| > 1$. Since $(1 + |\xi|)^{2s} \approx |\xi|^{2s}$, after making the change of variables $\xi_1 = \xi^m$, we get $G_1(\tau) \lesssim \int_{|\xi_1| > 1} \frac{1}{(1+|\tau - \xi|^m)^{2b} (1+|\xi_1|)^2} d\xi_1$. Next, we consider the following two subcases:

- $|\xi_1| \leq \frac{1}{2} |\tau|$ and $|\xi_1| > \frac{1}{2} |\tau|$
Subcase $|\xi_1| \leq \frac{1}{2} |\tau|$. Then, we have $(1 + |\tau - \xi_1|)^{2 - 2b} \simeq (1 + |\tau|)^{2 - 2b}$ and $|\tau| \geq 2$. Thus,

$$G_1(\tau) \simeq \frac{1}{|\xi_1| = 1} \int_{|\xi_1| = 1} \frac{1}{(1 + |\tau - \xi_1|)^{2 - 2b}|\xi_1|^{\frac{2 + 2j - 2\ell}{m}}} d\xi_1 \leq 2(1 + |\tau|)^{2b - 2} \int_{|\xi_1| = 1} \frac{1}{|\xi_1|^{\frac{2 + 2j - 2\ell}{m}}} d\xi_1,$$

which implies that $(1 + |\tau|)^{\frac{2 + j - \ell - 0}{m}} G_1(\tau)$ is bounded if $s \leq 1/2$ and $b < 1/2$.

Subcase $|\xi_1| \geq \frac{1}{2} |\tau|$. Since $|\xi_1| \geq 1 + |\tau|$, we get $\frac{2 + 2j - 2\ell}{m}$, which implies that

$$G_1(\tau) \lesssim (1 + |\tau|)^{-\frac{2 + 2j - 2\ell}{m}} \int_{|\xi_1| = \frac{1}{2} |\tau|} \frac{1}{(1 + |\tau - \xi_1|)^{2 - 2b}} d\xi_1 \lesssim (1 + |\tau|)^{-\frac{2 + 2j - 2\ell}{m}} \int_0^\infty \frac{1}{(1 + x)^{2 - 2b}} dx,$$

where in the last step we make the change of variables $x = \xi_1 - \tau$. Therefore, we get $(1 + |\tau|)^{\frac{2 + j - \ell - 0}{m}} G_1(\tau) \lesssim \int_0^\infty \frac{1}{(1 + x)^{2 - 2b}} dx$, which is bounded if $b < \frac{1}{2}$. □

Estimates for pure ibvp $B_2$. By the time estimate (3.23) we have $-W_t(t) \in H^\mu_x(0, T)$. Thus, the solution of problem $B_2$ is like that of problem $A_2$ and is estimated by using Theorem 3.2.

Proof of Theorem 1.2. Now using the results above we can estimate the solution of the forced linear ibvp. For $x \geq 0$ and $0 \leq t < T < \frac{1}{2}$ we have

$$S[u_0, g_0, \ldots, g_{j-1}; f] = \psi(t) S[u_0; 0] + S[0, G_0, \ldots, G_{j-1}; 0] + \psi(t) S[0; w] + S[0, -W_0, \ldots, -W_{j-1}; 0].$$

This together with estimates (3.13), (3.14) and (3.17)-(3.23) gives the desired result (1.18). □

4. Proof of Bilinear Estimate in modified Bourgain spaces $X^{s,b,\alpha}$

In this section, we prove the bilinear estimate in Bourgain spaces $X^{s,b,\alpha}$. Following [7] and [44] we begin the proof of the bilinear estimate (1.20) by first providing an equivalent $L^2$ formulation. For this, using the fact that $a^2 + b^2 \simeq (|a| + |b|)^2$, we get

$$\|h\|_{s,b,\alpha}^2 \simeq \int_{\mathbb{R}^2} \left(1 + |\xi|\right)^s(1 + |\tau - \xi|)^b + \chi_{|\xi| < 1}(1 + |\tau|)^\alpha)^2 \hat{h}(\xi, \tau)^2 d\xi d\tau,$$

and if for a function $h$ we use the Bourgain type combination

$$c_h(\xi, \tau) \doteq \left(1 + |\xi|\right)^s(1 + |\tau - \xi|)^b + \chi_{|\xi| < 1}(1 + |\tau|)^\alpha \hat{h}(\xi, \tau),$$

then the modified Bourgain norm of $h$ is equivalent to the $L^2$ norm of $c_h$, that is

$$\|h\|_{s,b,\alpha} = \|c_h(\xi, \tau)\|_{L^2_x L^2_\tau}.$$  \hspace{1cm} (4.3)

Next, we form the $\| \cdot \|_{s,-b,-\alpha-1}$-norm of

$$w_{fg} \doteq \frac{1}{2} \partial_x [f \cdot g].$$  \hspace{1cm} (4.4)

Using the definition of convolution and the relation (4.2) we have

$$|\hat{w}_{fg}(\xi, \tau)| \simeq \left|\xi \int_{\mathbb{R}^2} \hat{f}(\xi - \xi_1, \tau - \tau_1) \hat{g}(\xi_1, \tau_1) d\xi_1 d\tau_1\right|$$

$$\leq |\xi| \int_{\mathbb{R}^2} \frac{c_f(\xi - \xi_1, \tau - \tau_1)}{(1 + |\xi - \xi_1|)^s(1 + |\tau - \tau_1|)^b + \chi_{|\xi - \xi_1| < 1}(1 + |\tau - \tau_1|)^\alpha}$$

$$\times \frac{c_g(\xi_1, \tau_1)}{(1 + |\xi_1|)^s(1 + |\tau_1 - \xi_1^3|)^b + \chi_{|\xi_1| < 1}(1 + |\tau_1|)^\alpha} d\xi_1 d\tau_1.$$
Thus, collecting all multipliers together to form the important quantity $Q$, we see that to prove bilinear estimate (1.20) it suffices to show the $L^2$ inequality
\[
\left\| \int \int_{\mathbb{R}^2} Q(\xi, \xi_1, \tau, \tau_1) c_f(\xi - \xi, \tau - \tau_1) \, d\xi d\tau \right\|_{L^2_{\xi}L^2_{\tau}} \lesssim \|c_f\|_{L^\infty_{\xi}L^1_{\tau}} \|c_g\|_{L^\infty_{\xi}L^1_{\tau}}.
\] (4.6)

Next, we bound the term (4.5a) from above. Since
\[
1 - \alpha \geq b,
\] (4.7)
the second term that has numerator $\chi_{|\xi|<1}$ (and so $\xi$ is bounded) is absorbed by the first term. Thus,
\[
(4.5a) = \frac{(1 + |\xi|)^s}{(1 + |\tau - \xi^m|)^b} + \frac{\chi_{|\xi|<1}}{(1 + |\tau|)^{1-\alpha}} \lesssim \frac{(1 + |\xi|)^s}{(1 + |\tau - \xi^m|)^b}.
\] (4.8)

Combining estimate (4.8) with estimate (4.5) we get the following form of $Q$, which still involves $\alpha$ terms
\[
Q(\xi, \xi_1, \tau, \tau_1) \lesssim |\xi| \times \frac{(1 + |\xi|)^s}{(1 + |\tau - \xi^m|)^b}
\times \frac{1}{(1 + |\xi_1|)^s(1 + |\tau_1 - \xi^m_1|)^b + \chi_{|\xi_1|\leq1}(1 + |\tau_1|)^{\alpha'}}
\times \frac{1}{(1 + |\xi - \xi_1|)^s(1 + |\tau - \tau_1 - (\xi - \xi_1)^m|)^b + \chi_{|\xi - \xi_1|\leq1}(1 + |\tau - \tau_1|)^{\alpha'}}.
\] (4.9a, 4.9b, 4.9c)

So it suffices to prove the bilinear estimates by replacing $Q$ with the right-hand side of the above inequality. Note that all of the above is valid for any $s$. To make further reduction for $Q$, we need to consider the following two cases:

- $s \geq 0$ and $-\frac{1}{2} < s < 0$
Case $s \geq 0$. Collecting all the factors with $s$ power we rewrite the above bound for $Q$ as

$$Q(\xi, \xi_1, \tau, \tau_1) \lesssim \frac{|\xi|}{(1 + |\tau - \xi_1|^s)}$$

(4.10a)

$$\times \frac{(1 + |\xi|)^s}{(1 + |\xi_1|)^s(1 + |\xi - \xi_1|)^s}$$

(4.10b)

$$\times \frac{1}{(1 + |\tau_1 - \xi_1|^m)^{\alpha'} + \chi_{|\xi_1| \leq 1}(1 + |\tau_1|)^{\alpha'}(1 + |\xi_1|)^{-s}}$$

(4.10c)

$$\times \frac{1}{(1 + |\tau - \tau_1 - (\xi - \xi_1)^m|)^{\alpha'} + \chi_{|\xi - \xi_1| \leq 1}(1 + |\tau - \tau_1|)^{\alpha'}(1 + |\xi - \xi_1|)^{-s}}$$

(4.10d)

Since $s \geq 0$, like on the line, we have the estimate

$$(1 + |\xi|)^s \leq (1 + |\xi - \xi_1|)^s(1 + |\xi_1|)^s \Leftrightarrow \frac{(1 + |\xi|)^s}{(1 + |\xi - \xi_1|)^s(1 + |\xi_1|)^s} \lesssim 1,$$

(4.11)

which helps us remove term (4.10b). Also, since $|\xi_1|$ and $|\xi - \xi_1|$ are bounded we can remove $(1 + |\xi_1|)^{-s}$ and $(1 + |\xi - \xi_1|)^{-s}$. Thus, for any $s \geq 0$, we have $Q(\xi, \xi_1, \tau, \tau_1) \leq Q_0(\xi, \xi_1, \tau, \tau_1)$, where

$$Q_0(\xi, \xi_1, \tau, \tau_1) = \frac{|\xi|}{(1 + |\tau - \xi_1|^m)^{\alpha'} + \chi_{|\xi_1| \leq 1}(1 + |\tau_1|)^{\alpha'}(1 + |\xi_1|)^{-s}}$$

(4.12)

Thus, for $s \geq 0$ to prove our bilinear estimate (4.6), it suffices to prove the following simpler one

$$\left\| \int_\mathbb{R}^2 Q_0(\xi, \xi_1, \tau, \tau_1) c_f(\xi - \xi_1, \tau - \tau_1) c_g(\xi_1, \tau_1) d\xi_1 d\tau_1 \right\|_{L^2_x L^2_\tau} \lesssim \|c_f\|_{L^2_x L^2_\tau} \|c_g\|_{L^2_x L^2_\tau},$$

(4.13)

which corresponds to proving the bilinear estimate when $s = 0$. Moreover, by symmetry (in convolution writing), we may assume that

$$|\xi - \xi_1| \leq |\xi_1|.$$

(4.14)

Then, we have $1 = \chi_{|\xi_1| > 1} \cdot \chi_{|\xi - \xi_1| > 1} + \chi_{|\xi_1| \leq 1} \cdot \chi_{|\xi - \xi_1| \leq 1} + \chi_{|\xi_1| > 1} \cdot \chi_{|\xi - \xi_1| \leq 1}$. Therefore, we can rewrite $Q_0(\xi, \tau, \xi_1, \tau_1)$ as: $Q_0(\xi, \tau, \xi_1, \tau_1) = Q_1(\xi, \tau, \xi_1, \tau_1) + Q_2(\xi, \tau, \xi_1, \tau_1) + Q_3(\xi, \tau, \xi_1, \tau_1)$, where

$$Q_1(\xi, \tau, \xi_1, \tau_1) = \chi_{|\xi_1| > 1} \cdot \chi_{|\xi - \xi_1| > 1} \cdot Q_0(\xi, \tau, \xi_1, \tau_1)$$

(4.15)

$$= \frac{|\xi|}{(1 + |\tau - \xi_1|^m)^{\alpha'} + \chi_{|\xi_1| \leq 1}(1 + |\tau_1|)^{\alpha'}(1 + |\tau - \tau_1 - (\xi - \xi_1)^m|)^{\alpha'}} \cdot \chi_{|\xi - \xi_1| > 1}$$

and

$$Q_2(\xi, \tau, \xi_1, \tau_1) = \chi_{|\xi_1| \leq 1} \cdot \chi_{|\xi - \xi_1| \leq 1} \cdot Q_0(\xi, \tau, \xi_1, \tau_1)$$

(4.16)

$$= \frac{|\xi|}{(1 + |\tau - \xi_1|^m)^{\alpha'} + \chi_{|\xi_1| \leq 1}(1 + |\tau_1|)^{\alpha'}(1 + |\tau - \tau_1 - (\xi - \xi_1)^m|)^{\alpha'} + (1 + |\tau - \tau_1|)^{\alpha'}} \cdot \chi_{|\xi - \xi_1| \leq 1}$$

(4.17)

Note that $Q_1$ is like the regular Bourgain norm, because we do not have any term related to $\alpha$. 
To prove the bilinear estimate (4.13), it suffices to prove that for $\ell = 1, 2, 3$
\[
\left\| \iint_{\mathbb{R}^2} Q_\ell(\xi, \tau) c_f(\xi - \xi, \tau - \tau) c_g(\xi, \tau) d\xi d\tau \right\|_{L^2_{\xi, \tau}} \lesssim \|c_f\|_{L^2_{\xi}} \|c_g\|_{L^2_{\xi}}.
\] (4.18)

**Estimation when the multiplier is $Q_1$.** This is similar to the whole line case (see [18]). We do it by considering the following two possibilities:

- $|\xi| \leq 1$ and $|\xi| > 1$

**Case $|\xi| \leq 1$.** In this case, we need to prove (4.18) with $Q_1$ replaced by $\chi_{|\xi| \leq 1} Q_1$, where $\chi_{|\xi| \leq 1}$ is the characteristic function of the region $\{(\xi, \xi, \tau, \tau) : |\xi| \leq 1\}$. This is done by applying duality and the Cauchy-Schwarz inequality first in $(\xi, \tau)$ and then in $(\xi, \tau)$. Doing so, and after some manipulations, the desired estimate takes the form
\[
\left\| \iint_{\mathbb{R}^2} (\chi_{|\xi| \leq 1} Q_1)(\xi, \tau, \tau) c_f(\xi - \xi, \tau - \tau) c_g(\xi, \tau) d\xi d\tau \right\|_{L^2_{\xi, \tau}} \leq \|\Theta_1\|_{L^1_{\xi, \tau}} \|c_f\|_{L^2_{\xi}} \|c_g\|_{L^2_{\xi}},
\]
where $\Theta_1$ is as in the following lemma, which provides its estimate

**Lemma 4.1.** If $0 < b < \frac{1}{2}$, and $0 < b' < \frac{1}{2}$, then there exists $c > 0$ such that for $\xi, \tau, \xi_1 \in \mathbb{R}$
\[
\Theta_1(\xi, \tau) \doteq \frac{1}{(1 + |\xi - \xi_1|^2)^{2b}} \int_{|\xi| \leq 1} \int_{\xi_1} (1 + |\tau - \tau - (\xi - \xi_1)|)^{2b'} (1 + |\tau - \xi|^2)^{2b} \lesssim 1.
\]
The proof is similar to the proof of Lemma 7.1 in [18] and we omit it.

**Case $|\xi| > 1$.** In this case, by symmetry (in convolution writing), we may assume that
\[
|\tau - \tau - (\xi - \xi_1)| \leq |\tau - \tau - \xi_1|.
\] (4.19)
Therefore, following [7] and [44], to prove the bilinear estimate above we consider the following two microlocalizations:

**Microlocalization I.** $|\tau - \xi_1| \leq |\tau - \xi|$. In this case we define the domain $B_I$ to be
\[
B_I \doteq \{(\xi, \tau, \xi_1, \tau_1) \in \mathbb{R}^4 : |\tau - \tau - (\xi - \xi_1)| \leq |\tau - \tau - \xi_1| \leq |\tau - \xi|,
|\xi_1| > 1, |\tau - \tau - \xi_1| > 1, |\xi| > 1\}.
\] (4.20)

**Microlocalization II.** $|\tau - \xi| \leq |\tau - \xi_1|$. In this case we define the domain $B_{II}$ to be
\[
B_{II} \doteq \{(\xi, \tau, \xi_1, \tau_1) \in \mathbb{R}^4 : |\tau - \tau - (\xi - \xi_1)| \leq |\tau - \tau - \xi_1|,
|\tau - \xi| \leq |\tau - \tau - \xi_1|, |\xi_1| > 1, |\tau - \tau - \xi_1| > 1, |\xi| > 1\}.
\] (4.21)

**Proof of bilinear estimate in Microlocalization I:** In this case, $Q_1$ is replaced by the set $\chi_{B_I} Q_1$. Like before, using the Cauchy-Schwarz inequality with respect to $(\xi_1, \tau_1)$ and taking the supremum in $(\xi, \tau)$ we arrive at
\[
\left\| \iint_{\mathbb{R}^2} (\chi_{B_I} Q_1)(\xi, \tau, \tau_1) c_f(\xi - \xi, \tau - \tau) c_g(\xi, \tau_1) d\xi d\tau \right\|_{L^2_{\xi, \tau}} \leq \|\Theta_2\|_{L^1_{\xi, \tau}} \|c_f\|_{L^2_{\xi}} \|c_g\|_{L^2_{\xi}}.
\]
Thus, to prove our bilinear estimate in microlocalization I, it suffices to show the following result.

**Lemma 4.2.** If $\frac{6 + 3m}{12m} \leq b' \leq 1/2$, then for $|\xi| > 1$, and $\tau \in \mathbb{R}$ we have
\[
\Theta_2(\xi, \tau) \doteq \frac{\xi^2}{(1 + |\tau - \xi|^2)^{2b'}} \int_{\mathbb{R}^2} \chi_{B_I}(\xi, \tau, \xi_1, \tau_1) d\tau_1 d\xi_1 \lesssim 1.
\] (4.22)
Proof of bilinear estimate in Microlocalization II: Using duality and applying the Cauchy-Schwarz inequality twice, first in \((\xi, \tau_1)\) and then in \((\xi, \tau)\), we get

\[
\left\| \int_{\mathbb{R}^2} (\chi_{B_1}Q_1)(\xi, \xi, \tau, \tau_1) \phi_j(\xi - \xi, \tau - \tau_1)c_g(\xi, \tau_1)d\xi_1d\tau_1 \right\|_{L_{\xi, \tau}^2} \leq \|\Theta_3(\xi, \tau_1)\|_{L_{\xi, \tau}^{2/3}} \|\phi_j\|_{L_{\xi}^2} \|c_g\|_{L_{\tau}^2}.
\]

Thus, to prove our bilinear estimate in microlocalization II, it suffices to show the following result.

**Lemma 4.3.** If \(\max\left\{ \frac{4+3(m-1)}{12(m-1)}, \frac{6+3m}{12m} \right\} \leq b' \leq b < 1/2\), then for \(\xi_1, \tau_1 \in \mathbb{R}\) we have

\[
\Theta_3(\xi_1, \tau_1) \leq \frac{1}{(1 + |\tau_1 - \xi_1|^m)^{2(b' - 1)}} \int_{\mathbb{R}^2} (1 + |\tau - \tau_1 - (\xi_1)^m|)^{2b'(1 + |\tau - \xi_1|^m)^{2b'}} \lesssim 1. \tag{4.23}
\]

Next, we shall prove the Lemmas 4.2 and 4.3. We begin with the first one.

**Proof of Lemma 4.2.** To estimate the quantity \(\Theta_2(\xi, \tau)\) (and \(\Theta_3(\xi_1, \tau_1)\) later), we shall need the following calculus estimates, whose proof can be found in [44], [38] and [18].

**Lemma 4.4.** If \(1 > \ell > 1/2, b' > 1/2\) then

\[
\int_{\mathbb{R}} \frac{dx}{(1 + |x-a|)^{2\ell}(1 + |x-c|)^{2b'}} \lesssim \frac{1}{(1 + |a-c|)^{2\ell}}, \tag{4.24}
\]

\[
\int_{\mathbb{R}} \frac{dx}{(1 + |x|)^{2\ell}|a - x|^2} \lesssim \frac{1}{(1 + |a|)^{2\ell}}, \tag{4.25}
\]

\[
\int_{\mathbb{R}} \frac{dx}{(1 + |x-a|)^{2(1-\ell)}(1 + |x-c|)^{2b'}} \lesssim \frac{1}{(1 + |a-c|)^{2(1-\ell)}}, \tag{4.26}
\]

\[
\int_{|x|\leq c} \frac{dx}{(1 + |x|)^{2(1-\ell)}|\sqrt{a - x}|} \lesssim \frac{(1 + c)^{2(\ell-1/2)}}{(1 + |a|)^{1/2}}. \tag{4.27}
\]

And

\[
\int_{\mathbb{R}} \frac{dx}{(1 + |x-a|)^{2\ell}(1 + |x-c|)^{2b'}} \lesssim \frac{1}{(1 + |a-c|)^{2\min\{\ell', \ell\}}} \tag{4.28}
\]

In addition, if \(\frac{1}{2} < \ell' \leq \ell < \frac{1}{2}\), then

\[
\int_{\mathbb{R}} \frac{dx}{(1 + |x-a|)^{2\ell}(1 + |x-c|)^{2b'}} \lesssim \frac{1}{(1 + |a-c|)^{2\ell+2\ell'-1}}. \tag{4.29}
\]

Also, in this case using the triangle inequality we have

\[
|\tau - (\xi - \xi_1)^m - \xi_1^m| = |\tau - \tau_1 - (\xi - \xi_1)^m + \tau_1 - \xi_1^m| \leq |\tau - \tau_1 - (\xi - \xi_1)^m| + |\tau_1 - \xi_1^m| \leq 2|\tau - \xi_1^m|.
\]

Furthermore, integrating with respect to \(\tau_1\) and applying estimate (4.29) with \(\ell = \ell' = b', a = \tau - (\xi - \xi_1)^m\) and \(c = \xi_1^m\), we get

\[
\Theta_2(\xi, \tau) \lesssim \frac{\xi^2}{(1 + |\tau - \xi_1^m|)^{2b'}} \int_{\mathbb{R}} \frac{d\xi_1}{(1 + |\tau - (\xi - \xi_1)^m - \xi_1^m|)^{4b'-1}} = \frac{\xi^2}{(1 + |\tau - \xi_1^m|)^{2b'}} I(\xi, \tau), \tag{4.30}
\]

where \(I\) is defined in the lemma below, where it is also estimated.

**Lemma 4.5.** Let \(m = 2j + 1 \geq 3\) and \(\frac{1}{4} < b' < 1/2\). If \(|\xi^m - (\xi - \xi_1)^m - \xi_1^m| \lesssim |\tau - \xi^m|\), then for all \(|\xi| > 1\) and \(\tau \in \mathbb{R}\) we have

\[
I(\xi, \tau) \lesssim \int_{\mathbb{R}} \frac{d\xi_1}{(1 + |\tau - (\xi - \xi_1)^m - \xi_1^m|)^{4b'-1}} \lesssim \frac{|\xi|^{-\frac{1}{2}(m-2)}(1 + |\tau - \xi_1^m|)^{2-4b'}}{(1 + |\tau - 2^{1-m}\xi_1^m|)^{\frac{1}{2}}}. \tag{4.31}
\]
The proof of lemma 4.5 is similar to the proof of Lemma 7.3 in [18]. Combining (4.31) with (4.30), we get the desired estimate (4.22) for \( \Theta_2 \) and this completes the proof of Lemma 4.2. □

**Proof of Lemma 4.3.** We recall that

\[
\Theta_3(\xi_1, \tau_1) = \frac{1}{(1 + |\tau_1 - \xi_1^m|)^{2\beta}} \int_{\mathbb{R}^2} \chi_{B_{12}}(\xi, \tau, \xi_1, \tau_1) \frac{\xi_2}{(1 + |\tau - \tau_1 - (\xi - \xi_1^m)|)^{2\beta}} \frac{d\tau d\xi}{(1 + |\tau - \xi_1^m|)^{2\beta}}.
\]

As before, using estimate (4.29) with \( x = \tau, \ell = b, \ell' = b', a = \xi_2^m \) and \( c = \tau_1 - (\xi_1 - \xi)^m \), we get

\[
\Theta_3(\xi_1, \tau_1) \lesssim \frac{1}{(1 + |\tau_1 - \xi_1^m|)^{2\beta}} \int_{|\xi| > 1} \frac{\xi_2}{(1 + |\tau_1 - (\xi_1 - \xi)^m - \xi_2^m|)^{2\beta + 2\beta - 1}} \frac{d\xi}{(1 + |\tau_1 - \xi_1^m|)^{2\beta}} I(\xi_1, \tau_1),
\]

where \( d_m(\xi_1, \xi) \) is defined as follows

\[
d_m(\xi_1, \xi) = -\xi_2^m + \xi_2^m - (\xi - \xi_1^m).
\]

In order to show that \( \Theta_3 \) is bounded, we need to consider the following two cases:

- \(|\xi| \leq 10|\xi_1|\)
- \(|\xi| > 10|\xi_1|\)

**Case** \(|\xi| \leq 10|\xi_1|\). Then,

\[
\Theta_3(\xi_1, \tau_1) \lesssim \frac{\xi_2^2}{(1 + |\tau_1 - \xi_1^m|)^{2\beta}} \int_{|\xi| > 1} \frac{d\xi}{(1 + |\tau_1 - \xi_1^m|)^{2\beta - 1}} \lesssim \frac{\xi_2^2}{(1 + |\tau_1 - \xi_1^m|)^{2\beta}} I(\xi_1, \tau_1),
\]

where \( I \) is defined in (4.31). Applying Lemma 4.5, we get

\[
\Theta_3(\xi_1, \tau_1) \lesssim \frac{1}{(1 + |\tau_1 - \xi_1^m|)^{2\beta}} \frac{|\xi_1|^2}{(1 + |\tau_1 - \xi_1^m|)^{2\beta}} \frac{|\xi_1| - \frac{1}{2} (1 + |\tau_1 - \xi_1^m|)^{2 \log 2}}{1 + |\tau_1 - \xi_1^m|^{2\beta} - (1 + |\tau_1 - \xi_1^m|)^{2\beta}}.
\]

Like KdVm on the line (see [18]), we consider the following two subcases:

- \(|\tau_1 - \xi_1^m| \leq \frac{1}{2} |\xi_1|^m\)
- \(|\tau_1 - \xi_1^m| > \frac{1}{2} |\xi_1|^m\)

**Subcase** \(|\tau_1 - \xi_1^m| \leq \frac{1}{2} |\xi_1|^m\). Then, using the triangle inequality, we have

\[
|\tau_1 - 2^{1-m} \xi_1^m| = |(\tau_1 - \xi_1^m) + (1 - 2^{1-m}) \xi_1^m| \geq \frac{3}{4} |\xi_1|^m - |\tau_1 - \xi_1^m| \geq \frac{3}{4} |\xi_1|^m - \frac{1}{2} |\xi_1|^m = \frac{1}{4} |\xi_1|^m.
\]

Hence, from (4.34) we get \( \Theta_3(\xi_1, \tau_1) \lesssim \frac{|\xi_1|^3}{|\xi_1|^m} \frac{1}{|\xi_1|^m} \frac{1}{|\xi_1|^m} = |\xi_1|^{-m+3} \frac{1}{|\xi_1|^m} \lesssim 1 \).

**Subcase** \(|\tau_1 - \xi_1^m| > \frac{1}{2} |\xi_1|^m\). Then \(|\tau_1 - \xi_1^m| \gtrsim |\xi_1|^m\) and therefore from (4.34) we get

\[
\Theta_3(\xi_1, \tau_1) \lesssim \frac{|\xi_1|^3}{|\xi_1|^m} \frac{1}{|\xi_1|^m} \frac{1}{|\xi_1|^m} = \frac{1}{|\xi_1|^m (6\beta - 2) - (3 - \frac{m}{2})}.
\]

Since \(|\xi_1| > 1\), the above quantity is bounded if \( m(6\beta - 2) - (3 - \frac{m}{2}) \geq 0\), which implies that

\[
b' \geq \frac{6 + 3m}{12m}.
\]

This completes the proof of Lemma 4.3 in this case.
Case $|\xi| > 10|\xi_1|$. Then, using the triangle inequality $|\tau_1 - \xi_1^m|$ is bounded from below as follows
\[
|\tau_1 - \xi_1^m| \geq \frac{1}{3} \left[ |\tau - \xi_1^m| + |(\tau_1 - \xi_1^m)| + |\tau - \tau_1 - (\xi - \xi_1)^m| \right] \geq |\tau - \xi_1^m + (\tau_1 - \xi_1^m) + (\tau - \tau_1 - (\xi - \xi_1)^m)| = |d_m(\xi, \xi_1)|, \tag{4.36}
\]
which can be bounded by the following result.

**Lemma 4.6.** If $m$ is an odd positive integer, then there is a positive constant $c_m$ such that
\[
|d_m(\xi, \xi_1)| \geq c_m|\xi|^{m-3}|\xi_1^m(\xi - \xi_1)|, \tag{4.37}
\]
and
\[
|d_m(\xi, \xi_1)| \geq c_m|\xi|^{m-3}|\xi_1^m(\xi - \xi_1)|. \tag{4.38}
\]
Also if $|\xi| \geq 1$, $|\xi_1| \geq 1$ and $|\xi - \xi_1| \geq 1$, then
\[
|\xi_1^m(\xi - \xi_1)| \geq \frac{1}{3}|\xi| \quad \text{and} \quad |\xi_1^m(\xi - \xi_1)| \geq \frac{1}{3}|\xi|. \tag{4.39}
\]

The proof of Lemma 4.6 can be found in [18], now we complete the proof in this case. Since $|\xi| > 10|\xi_1|$, we have $|\xi - \xi_1| \geq |\xi| - |\xi_1| \geq \frac{1}{10}|\xi|$ and $|\xi - \xi_1| \leq |\xi| + |\xi_1| \geq \frac{1}{10}|\xi|$, which gives us that
\[
|\xi - \xi_1| \simeq |\xi|. \tag{4.40}
\]
Combining estimates (4.36) and (4.37) with (4.40), we get $|\tau_1 - \xi_1^m| \gtrsim |\xi|^{m-1}|\xi_1|$ or $|\xi| \leq (|\tau_1 - \xi_1^m||\xi_1|^{-1})^{\frac{1}{m-1}}$. In addition, using estimate (4.32) we have
\[
\Theta_3(\xi_1, \tau_1) \lesssim |\xi_1|^{-\frac{1}{m-1}}|\tau_1 - \xi_1^m|^{-\frac{1}{m-1}} \int_{|\xi| > 1} \frac{d\xi}{(1 + |\tau_1 - \xi_1^m|)^{2b'}} \leq |\xi_1|^{-\frac{2}{m-1}}(1 + |\tau_1 - \xi_1^m|)^{-\frac{1}{m-1} - 2b'} I(\xi_1, \tau_1),
\tag{4.41}
\]
where $I$ is defined in (4.31). Applying Lemma 4.5, we get
\[
\Theta_3(\xi_1, \tau_1) \lesssim |\xi_1|^{-\frac{1}{m-1}}(1 + |\tau_1 - \xi_1^m|)^{-\frac{1}{m-1} + \frac{1}{2}b} |\xi_1|^{-\frac{1}{2}(m-2)}(1 + |\tau_1 - \xi_1^m|)^{2b'} \leq |\tau_1 - \xi_1^m|^{-\frac{1}{m-1} + \frac{1}{2}b} \frac{|\xi_1|^{-\frac{1}{2}(m-2)}(1 + |\tau_1 - \xi_1^m|)^{2b'}}{(1 + |\tau_1 - \xi_1^m|)^{\frac{1}{2}}}. \tag{4.42}
\]
Now using the triangle inequality, we can bound $|\tau_1 - \xi_1^m|$ from below, that is
\[
|\tau_1 - \xi_1^m| = |\tau_1 - \xi_1^m + (1 - 2^{1-m})\xi_1| \geq |\tau_1 - \xi_1^m| - |\xi_1^m| = \frac{1}{2}|\tau_1 - \xi_1^m| + \left(\frac{1}{2}|\tau_1 - \xi_1^m| - |\xi_1^m|\right) \geq \frac{1}{2}|\tau_1 - \xi_1^m|. \tag{4.36}
\]
Combining the above estimate with (4.42), we obtain
\[
\Theta_3(\xi_1, \tau_1) \lesssim |\xi_1|^{-\frac{1}{m-1} - \frac{1}{2}(m-2)}(1 + |\tau_1 - \xi_1^m|)^{\frac{1}{m-1} + \frac{1}{2}b}. \tag{4.43}
\]
Since $|\xi_1| > 1$, the above quantity is bounded if and only if
\[
b' \geq \frac{4 + 3(m - 1)}{12(m - 1)}. \tag{4.44}
\]
This completes the proof of Lemma 4.3. □
**Estimation when the multiplier is** $Q_2$. In this case, applying Cauchy-Schwarz inequality with respect to $\xi_1, \tau_1$, and taking the super norm over $(\xi, \tau)$ we get

$$\left\| \int_{\mathbb{R}^2} Q_2(\xi, \xi_1, \tau, \tau_1) c_f(\xi - \xi_1, \tau - \tau_1) c_g(\xi_1, \tau_1) d\xi_1 d\tau_1 \right\|_{L^2_{\xi_1} L^2_{\tau_1}} \lesssim \|\Theta_4\|_{L^\infty_{\xi, \tau}}^{1/2} \|c_f\|_{L^2_{\xi, \tau}^2} \|c_g\|_{L^2_{\xi, \tau}^2}, \quad (4.45)$$

which shows that the proof of the bilinear estimate follows from the next result.

**Lemma 4.7.** If $b \geq 0$ and $\alpha' > \frac{1}{2}$, then for $\xi, \tau \in \mathbb{R}$, we have

$$\Theta_4(\xi, \tau) \lesssim \frac{\xi^2}{(1 + |\tau - \xi|^m)^{2b}} \int_{\mathbb{R}^2} \frac{\chi_{|\xi_1| \leq 1} \chi_{|\xi - \xi_1| \leq 1} d\xi_1 d\tau_1}{[(1 + |\tau - \tau_1 - (\xi - \xi_1)|^m)^{2b} + (1 + |\tau - \tau_1|)^{\alpha'}]^2} \lesssim 1. \quad (4.46)$$

The proof of this lemma is straightforward. Using the fact that $|\xi_1|$ and $|\xi|$ are bounded and applying estimate (4.24) with $\ell = \alpha'$, $x = \tau_1$, $a = 0$ and $c = \tau_1$, we get the desired estimate (4.46).

**Estimation when the multiplier is** $Q_3$. To prove the estimate (4.18) for $Q_3$, we will consider two possible microlocalizations:

**Microlocalization III.** $|\tau_1 - \xi_1^m| \leq |\tau - \xi^m|$. In this case we define the domain $B_{III}$ to be

$$B_{III} \doteq \{(\xi, \tau, \xi_1, \tau_1) \in \mathbb{R}^4 : |\tau_1 - \xi_1^m| \leq |\tau - \xi^m|, |\xi_1| > 1, |\xi - \xi_1| \leq 1\}. \quad (4.47)$$

**Microlocalization IV.** $|\tau - \xi^m| \leq |\tau_1 - \xi_1^m|$. In this case we define the domain $B_{IV}$ to be

$$B_{IV} \doteq \{(\xi, \tau, \xi_1, \tau_1) \in \mathbb{R}^4 : |\tau - \xi^m| \leq |\tau_1 - \xi_1^m|, |\xi_1| > 1, |\xi - \xi_1| \leq 1\}. \quad (4.48)$$

**Proof of bilinear estimate in Microlocalization III.** As before, using the Cauchy-Schwarz inequality with respect to $(\xi_1, \tau_1)$ and taking the super norm over $(\xi, \tau)$ we arrive at

$$\left\| \int_{\mathbb{R}^2} \chi_{B_{III}} Q_3(\xi, \xi_1, \tau, \tau_1) c_f(\xi - \xi_1, \tau - \tau_1) c_g(\xi_1, \tau_1) d\xi_1 d\tau_1 \right\|_{L^2_{\xi_1} L^2_{\tau_1}} \lesssim \|\Theta_5\|_{L^\infty_{\xi, \tau}}^{1/2} \|c_f\|_{L^2_{\xi, \tau}^2} \|c_g\|_{L^2_{\xi, \tau}^2}^{1/2}. \quad (4.49)$$

Thus, to prove our bilinear estimate in microlocalization III, it suffices to show the following result.

**Lemma 4.8.** If $\frac{1}{3} \leq b' \leq b < \frac{1}{2} < \alpha'$, then for $\xi, \tau \in \mathbb{R}$ we have

$$\Theta_5(\xi, \tau) \lesssim \frac{\xi^2}{(1 + |\tau - \xi^m|)^{2b}} \int_{\mathbb{R}^2} \frac{\chi_{|\xi_1| > 1} \chi_{B_{III}}(\xi, \tau, \xi_1, \tau_1) \chi_{|\xi - \xi_1| < 1} \chi_{|\tau_1 - \xi_1^m|} d\xi_1 d\tau_1}{[(1 + |\tau - \tau_1 - (\xi - \xi_1)|^m)^{2b} + (1 + |\tau - \tau_1|)^{\alpha'}]^2} \lesssim 1. \quad (4.49)$$

**Proof of bilinear estimate in Microlocalization IV.** As before, using duality and the Cauchy-Schwarz inequality twice, first in $(\xi_1, \tau_1)$ and then in $(\xi, \tau)$, we get

$$\left\| \int_{\mathbb{R}^2} \chi_{B_{IV}} Q_3(\xi, \xi_1, \tau, \tau_1) c_f(\xi - \xi_1, \tau - \tau_1) c_g(\xi_1, \tau_1) d\xi_1 d\tau_1 \right\|_{L^2_{\xi_1} L^2_{\tau_1}} \lesssim \|\Theta_6\|_{L^\infty_{\xi_1, \tau_1}}^{1/2} \|c_f\|_{L^2_{\xi, \tau}^2} \|c_g\|_{L^2_{\xi, \tau}^2}. \quad (4.49)$$

Thus, to prove our bilinear estimate in microlocalization IV, it suffices to show the following result.
Lemma 4.9. If $\frac{1}{2} \leq b' \leq b < \frac{1}{2} < \alpha'$, then for $\xi, \tau \in \mathbb{R}$ we have

\[
\Theta_0(\xi_1, \tau_1) \approx \frac{\chi_{|\xi_1| > 1}}{(1 + |\tau_1 - \xi_1|^m)^{2b'} - \beta} \int_{\mathbb{R}^2} \frac{\chi_{B_{m/2}}(\xi, \tau, \xi_1, \tau_1) \xi^2}{(1 + |\tau - \xi|^m)^{2b}} \chi_{|\xi - \xi_1| \leq 1} d\xi d\tau / [(1 + |\tau - \tau_1 - (\xi - \xi_1)^m)|b'(1 + |\tau - \tau_1|)^{\alpha'}|^2] \lesssim 1. \tag{4.50}
\]

The proof of Lemma 4.9 is similar to the proof of Lemma 4.8. So, here we provide only the proof of Lemma 4.8.

Proof of Lemma 4.8. Since $|\tau - \xi|^m \geq |\tau_1 - \xi_1|^m$, and using $\alpha' > \frac{1}{2} > b'$, $|\xi - \xi_1| \leq 1$ we get

\[
\Theta_0(\xi, \tau) \lesssim \frac{\xi^2}{(1 + |\tau - \xi|^m)^{4b + 2\beta - 3}} \int_{\mathbb{R}^2} \frac{\chi_{|\xi_1| > 1}}{(1 + |\tau_1 - \xi_1|^m)^{2 - 2b'} - \beta} \chi_{|\xi - \xi_1| \leq 1} d\xi d\tau. \tag{4.51}
\]

Now, using estimate (4.28) with $\ell' = \alpha'$, $\ell = 1 - b$, $x = \tau_1$, $a = \tau$ and $c = \xi_1^m$

\[
\Theta_0(\xi, \tau) \lesssim \frac{\xi^2}{(1 + |\tau - \xi|^m)^{4b + 2\beta - 2}} \int_{\mathbb{R}} \frac{\chi_{|\xi_1| > 1}}{(1 + |\tau - \xi_1|^m)^{\min\{2\alpha', 2\beta - 2\}} - \beta} d\xi_1. \tag{4.52}
\]

For $|\xi| \leq 20$, it is obvious that $Q_5 \lesssim 1$. For $|\xi| \geq 20$, using the fact that $\xi_1 \simeq \xi$ and making the change of variables $\mu = \xi^m$ for the integral of $d\xi$, for $4b + 2\beta - 2 > 0$ we also get $Q_5 \lesssim 1$. Thus, we complete the proof of Lemma 4.8. $\square$

Case $-j + \frac{1}{4} < s < 0$. We recall that in order to prove the bilinear estimate (1.20), it suffices to prove $L^2$ inequality (4.6) with $Q$ is estimated in (4.9). Also, similar to the case $s \geq 0$, by further reduction we get

\[
Q(\xi, \xi_1, \tau, \tau_1) \lesssim \frac{|\xi|}{(1 + |\tau - \xi|^m)^{\beta}(1 + |\tau_1 - \xi_1|^m)^{\beta'}} \chi_{|\xi_1| \leq 1}(1 + |\tau_1|)^{\alpha'} \times \frac{1}{(1 + |\xi_1|^\alpha(1 + |\xi - \xi_1|^\beta))}. \tag{4.53}
\]

Furthermore, like KdVm on the line $\mathbb{R}$ we can restrict our estimations into the set

\[
E = \{ (\xi, \xi_1, \tau, \tau_1) \in \mathbb{R}^4 : |\xi_1 - \xi| > 1 \text{ and } |\xi_1| > 1 \}. \tag{4.54}
\]

Hence, the $\alpha$ terms in the denominators of $Q$ can be dropped and and the quantity $Q$ given in (4.53) becomes

\[
Q_4(\xi, \xi_1, \tau, \tau_1) \lesssim \frac{|\xi|(1 + |\xi|^\alpha)|\xi_1(\xi - \xi_1)|^{-s}}{(1 + |\tau - \xi|^m)^{\beta}(1 + |\tau_1 - \xi_1|^m)^{\beta'} - \beta} \frac{1}{(1 + |\tau_1 - (\xi - \xi_1)^m|^b(1 + |\tau - \tau_1|)^{\alpha'})}. \tag{4.55}
\]

Moreover, by symmetry (in convolution writing), we may assume that

\[
|\tau - \tau_1 - (\xi - \xi_1)^m| \leq |\tau_1 - \xi_1|^m. \tag{4.56}
\]

Finally, following [7], [44] and [18], in order to prove (4.6) we distinguish two cases (microlocalization):

**Microlocalization I.** $|\tau_1 - \xi_1|^m \leq |\tau - \xi|^m$. In this case we define the domain $E_I$ to be

\[
E_I = \{ (\xi, \xi_1, \tau, \tau_1) \in \mathbb{R}^4 : |\tau - \tau_1 - (\xi - \xi_1)|^m \leq |\tau_1 - \xi_1|^m \leq |\tau - \xi|^m, |\xi_1| > 1, |\xi - \xi_1| > 1 \}. \tag{4.57}
\]
Microlocalization II. $|\tau - \xi^m| \leq |\tau_1 - \xi_1^m|$. In this case we define the domain $E_{II}$ to be

$$E_{II} = \{ (\xi, \tau, \xi_1, \tau_1) \in \mathbb{R}^4 : |\tau - \tau_1 - (\xi - \xi^m)| \leq |\tau_1 - \xi_1^m|, |\tau - \xi^m| \leq |\tau_1 - \xi_1^m|, |\xi_1| > 1, |\xi - \xi_1| > 1 \}.$$ (4.58)

Proof of bilinear estimate in Microlocalization I. Here $Q$ is replaced with the $\chi_{E_I} Q$ and our $L^2$ inequality (4.6) reads as

$$\left\| \int \int_{\mathbb{R}^2} (\chi_{E_I} Q_4)(\xi, \xi_1, \tau, \tau_1) c_f(\xi - \xi_1, \tau - \tau_1) c_g(\xi_1, \tau_1) d\xi_1 d\tau_1 \right\|_{L^2} \lesssim \|c_f\|_{L^2} \|c_g\|_{L^2}.$$ (4.59)

As before using the Cauchy-Schwarz inequality with respect to $(\xi_1, \tau_1)$ and taking the supremum in $(\xi, \tau)$ we get

$$\left\| \int \int_{\mathbb{R}^2} (\chi_{E_I} Q_4)(\xi, \xi_1, \tau, \tau_1) c_f(\xi - \xi_1, \tau - \tau_1) c_g(\xi_1, \tau_1) d\xi_1 d\tau_1 \right\|_{L^2} \lesssim \|\Theta_I\|_{L^2}^{1/2} \|c_f\|_{L^2} \|c_g\|_{L^2},$$

where $\Theta_I$ is defined and estimated in the following lemma.

Lemma 4.10. If $\max\{\frac{1}{2} - \frac{s - (j + \frac{1}{2})}{2s}, \frac{5}{12}, \frac{2s + 2s}{12s}\} \leq b' < \frac{1}{2}$ and $-j + \frac{1}{4} < s < 0$, then for $\xi, \tau \in \mathbb{R}$

$$\Theta_I(\xi, \tau) \equiv \frac{\xi^2(1 + |\xi|)^{2s}}{(1 + |\tau - \xi^m|)^{2b'}} \int \int_{\mathbb{R}^2} \chi_{E_I}(\xi, \tau, \xi_1, \tau_1) \xi^2(1 + |\xi|)^{2s} \xi_1(\xi - \xi_1)^{-2s} d\tau_1 d\xi_1 \lesssim 1.$$ (4.60)

The proof of Lemma 4.10 is omitted since it is similar to the proof of Lemma 7.4 in [18]. In fact, if we choose $\frac{1}{2} - \frac{1}{4} \beta_1 \leq b' < b < \frac{1}{2}$, where $\beta_1 = \beta$, which is defined in Theorem 2.1 in [18], then Lemma 4.10 is reduced to the Lemma 7.4 in [18].

Proof of bilinear estimate in Microlocalization II. Using duality and Cauchy-Schwarz inequality twice, first in $(\xi_1, \tau_1)$ and then in $(\xi, \tau)$, as before, we have

$$\left\| \int \int_{\mathbb{R}^2} (\chi_{E_{II}} Q_4)(\xi, \xi_1, \tau, \tau_1) c_f(\xi - \xi_1, \tau - \tau_1) c_g(\xi_1, \tau_1) d\xi_1 d\tau_1 \right\|_{L^2} \lesssim \|\Theta_{II}\|_{L^2}^{1/2} \|c_f\|_{L^2} \|c_g\|_{L^2}.$$where $\Theta_{II}$ is defined and estimated in the following lemma.

Lemma 4.11. If $\max\{\frac{1}{2} - \frac{s - (j + \frac{1}{2})}{2s}, \frac{5}{12}, \frac{2s + 2s}{6m}\} \leq b' < \frac{1}{2}$ and $-j + \frac{1}{4} < s < 0$, then for $\xi, \tau \in \mathbb{R}$

$$\Theta_{II}(\xi_1, \tau_1) \equiv \frac{1}{(1 + |\tau_1 - \xi_1^m|)^{2b'}} \int \int_{\mathbb{R}^2} \chi_{E_{II}}(\xi, \tau_1, \xi_1, \tau) \xi^2(1 + |\xi|)^{2s} \xi_1(\xi - \xi_1)^{-2s} d\tau d\xi \lesssim 1.$$ (4.61)

Proof of Lemma 4.11. Since $0 < b' < b < \frac{1}{2}$, by applying calculus estimate (4.29) with $\ell = b, \ell' = b', \alpha = \tau_1 - (\xi - \xi_1)^m, \beta = \xi^m$ and $x = \tau, \tau_1$, we get

\[
\Theta_{II}(\xi_1, \tau_1) \leq \frac{1}{(1 + |\tau_1 - \xi_1^m|)^{2b'}} \int_{\mathbb{R}} \frac{|\xi|^2 (1 + |\xi|)^{2s} (\xi_1(\xi - \xi_1))^{-2s}}{(1 + |\tau_1| + (\xi - \xi_1)^m - \xi^m)^{2b + 2b'} - 1} d\xi
\]

\[
= \frac{1}{(1 + |\tau_1 - \xi_1^m|)^{2b'}} \int_{\mathbb{R}} \frac{|\xi|^2 (1 + |\xi|)^{2s} (\xi_1(\xi - \xi_1))^{-2s}}{(1 + |\tau_1| + (\xi - \xi_1)^m - \xi^m + \xi^m)^{2b + 2b'} - 1} d\xi
\]

\[
= \frac{1}{(1 + |\tau_1 - \xi_1^m|)^{2b'}} \int_{\mathbb{R}} \frac{|\xi|^2 (1 + |\xi|)^{2s} (\xi_1(\xi - \xi_1))^{-2s}}{(1 + |\tau_1| + d_m(\xi, \xi_1))^{2b + 2b'} - 1} d\xi. \quad (4.62)
\]

Then, we complete the proof by following argument similar to those used in the proof of Lemma 7.5 in [18]. \qed
5. Proof of Bilinear estimates in temporal $Y^{s,b}$ spaces

In this section we prove Theorem 1.4, that are the bilinear estimates in the spaces $Y^{s,b}$. These appears in the basic linear estimate via the time estimate of the forced ivp with zero data, i.e. estimate (3.23). Since the proof of estimates (1.24) is similar to that of estimate (1.23) and for $m = 3$ estimate (1.23) is proved in [38]. Here we only provide an outline of the proof for estimate (1.23) with $s \geq 0$. For $s \geq 0$ we have the following inequality

$$
\|wfg\|_{Y^{s,-b}} \lesssim \iint_{\mathbb{R}^2} \chi_{|\tau|>10^m|\xi|^m}(1 + |\tau|)^{\frac{a}{m}} (1 + |\tau - \xi|)^{-2b} |\hat{w}_{fg}(\xi, \tau)|^2 d\xi d\tau + \|wfg\|_{X^{s,-b}},
$$

(5.1)

where $wfg = \partial_\tau (f \cdot g)$. So, to prove the “temporal” bilinear estimate (1.23) it suffices to show that

$$
\left( \iint_{\mathbb{R}^2} \chi_{|\tau|>10^m|\xi|^m}(1 + |\tau|)^{\frac{a}{m}} (1 + |\tau - \xi|)^{-2b} |\hat{w}_{fg}(\xi, \tau)|^2 d\xi d\tau \right)^{1/2} \lesssim \|f\|_{X^{s',b',a'}} \|g\|_{X^{s',b',a'}}.
$$

(5.2)

Like before, writting the $\| \cdot \|_{s,b,a'}$-norm of $h$ as the $L^2$ norm of $c_h$, that is $\|h\|_{s,b,a'} \simeq \|c_h(\xi, \tau)\|_{L^2_{\xi,\tau}}$, where $c_h$ is defined in (4.2), the estimate (5.2) reads as follows

$$
\left( \iint_{\mathbb{R}^2} \chi_{|\tau|>10^m|\xi|^m}(1 + |\tau|)^{\frac{a}{m}} (1 + |\tau - \xi|)^{-2b} |\hat{w}_{fg}(\xi, \tau)|^2 d\xi d\tau \right)^{1/2} \lesssim \|c_f\|_{L^2_{\xi,\tau}} \|c_g\|_{L^2_{\xi,\tau}}.
$$

(5.3)

Next, expressing $\hat{w}_{fg}(\xi, \tau) \simeq \xi \iint_{\mathbb{R}^2} \hat{f}(\xi - \xi_1, \tau - \tau_1)\hat{g}(\xi_1, \tau_1)d\xi_1 d\tau_1$ in terms of $c_f$ and $c_g$ we see that the inequality (5.3) takes the following $L^2$ formulation

$$
\left\| \iint_{\mathbb{R}^2} Q(\xi, \xi_1, \tau, \tau_1)c_f(\xi - \xi_1, \tau - \tau_1)c_g(\xi_1, \tau_1)d\xi_1 d\tau_1 \right\|_{L^2_{\xi,\tau}} \lesssim \|c_f\|_{L^2_{\xi,\tau}} \|c_g\|_{L^2_{\xi,\tau}},
$$

(5.4)

where

$$
Q(\xi, \xi_1, \tau, \tau_1) = \chi_{|\tau|>10^m|\xi|^m} \frac{|\xi|}{(1 + |\tau - \xi|)^b} \times (1 + |\tau|)^{\frac{a}{m}} \chi_{|\xi_1|\leq 10^m|\xi|^m} \chi_{|\xi - \xi_1|\leq 10^m|\xi|^m} \chi_{|\tau_1|\leq 10^m|\xi|^m} \chi_{|\tau - \tau_1|\leq 10^m|\xi|^m}.
$$

(5.5a)

$$
\times \frac{1}{(1 + |\xi|)^s(1 + |\tau_1 - \xi|)^{b'} + \chi_{|\xi_1|\leq 10^m|\xi|^m}(1 + |\tau_1|)^{b'}}
$$

(5.5b)

$$
\times \frac{1}{(1 + |\xi - \xi_1|)^s(1 + |\tau - \tau_1 - (\xi - \xi_1)|)^{b'} + \chi_{|\xi - \xi_1|\leq 10^m|\xi|^m}(1 + |\tau - \tau_1|)^{b'}}.
$$

(5.5c)

In additon, like the bilinear estimate in the space $X^{s,b,a}$, collecting all the factors with $s$ power, making further reduction, and observing that if $|\tau| \leq 10^m|\xi|^m$, then

$$
\frac{(1 + |\tau|)^{s/m}}{(1 + |\xi_1|)^s(1 + |\xi - \xi_1|)^{s/m}} \lesssim \frac{(1 + 10^m|\xi|^m)^{s/m}}{(1 + |\xi|)^s(1 + |\xi - \xi_1|)^{s/m}} \lesssim 1,
$$

which implies $Q(\xi, \xi_1, \tau, \tau_1) \lesssim Q_0(\xi, \xi_1, \tau, \tau_1)$, where $Q_0$ is given by (4.12), we reduce bilinear estimate (5.4) to the bilinear estimate in $X^{s,b,a}$ with $s = 0$, i.e. estimate (4.13). Thus, we assume $|\tau| > 10^m|\xi|^m$ and $Q(\xi, \xi_1, \tau, \tau_1)$ becomes $Q_1(\xi, \xi_1, \tau, \tau_1)$, which is given by

$$
Q_1(\xi, \xi_1, \tau, \tau_1) = \chi_{|\tau|>10^m|\xi|^m} \chi_{|\tau|>10^m|\xi_1|^m} \frac{(1 + |\tau|)^{s/m}}{(1 + |\tau - \xi|)^b(1 + |\xi|)^s(1 + |\xi - \xi_1|)^{s/m}} \frac{1}{(1 + |\tau - \tau_1 - \xi|)^{b'}(1 + |\tau_1 - \xi_1|)^{b'}}.
$$

(5.6)

Now, in order to prove estimate (5.4), it suffices to show that

$$
\left\| \iint_{\mathbb{R}^2} Q_1(\xi, \xi_1, \tau, \tau_1)c_f(\xi - \xi_1, \tau - \tau_1)c_g(\xi_1, \tau_1)d\xi_1 d\tau_1 \right\|_{L^2_{\xi,\tau}} \lesssim \|c_f\|_{L^2_{\xi,\tau}} \|c_g\|_{L^2_{\xi,\tau}}.
$$

(5.7)
To show this, like before using the Cauchy-Schwarz inequality with respect to \((\xi_1, \tau_1)\) and taking the supremum over \((\xi, \tau)\) we get
\[
\left\| \int \int_{\mathbb{R}^2} Q_1(\xi, \xi_1, \tau, \tau_1) c_f(\xi-\xi_1, \tau-\tau_1) c_g(\xi_1, \tau_1) d\xi_1 d\tau_1 \right\|_{L^2_{\xi, \tau}} \leq \left\| \Theta_1 \right\|_{L^\infty_{\xi, \tau}}^{1/2} \left\| c_f \right\|_{L^2_{\xi, \tau}} \left\| c_g \right\|_{L^2_{\xi, \tau}},
\] (5.8)
where \(\Theta_1\) is defined and estimated in the following result:

**Lemma 5.1.** If \(0 \leq s < m\) and \(\max\left\{ \frac{2s+m}{6m}, \frac{m+2}{6m} \right\} \leq b' < b < \frac{1}{2}\) satisfy , then we have
\[
\Theta_1(\xi, \tau) \leq \frac{\chi_{|\tau|>10^m|\xi|^m} |\xi|^2 (1 + |\tau|)^{2s/m}}{(1 + |\tau - \xi|^m)^{2b}} \int_{\mathbb{R}^2} \frac{\chi_{|\tau|>10^m|\xi|}}{(1 + |\tau - \xi|^m)^{2b}} \left(1 + |\tau - \xi|^m\right)^{2b'} (1 + |\tau - \xi|^m)^{2b'} d\xi_1, \quad (5.9)
\]

**Proof of Lemma 5.1.** Applying calculus estimate (4.29) with \(\ell = \ell' = b'\), \(a = \tau - (\xi - \xi_1)^m\), \(c = \xi_1^m\) and \(x = \tau_1\) we arrive at the following estimate
\[
\Theta_1(\xi, \tau) \leq \frac{\chi_{|\tau|>10^m|\xi|^m} |\xi|^2 (1 + |\tau|)^{2s/m}}{(1 + |\tau - \xi|^m)^{2b}} \int_{\mathbb{R}} \frac{\chi_{|\tau|>10^m|\xi|^m}}{(1 + |\tau - \xi|^m)^{2b}} \left(1 + |\tau - \xi|^m\right)^{2b'} (1 + |\tau - \xi|^m)^{2b'} d\xi_1, \quad (5.10)
\]
Then, for \(0 \leq s < m\) and \(\max\left\{ \frac{2s+m}{6m}, \frac{m+2}{6m} \right\} \leq b' < b < \frac{1}{2}\), using \(|\tau| > 10^m|\xi|^m\) and \(|\tau| > 10^m|\xi|^m\) we get the desired estimate (5.9). Here we omit the detail of the proof. □

6. Well-posedness in modified Bourgain spaces – Proof of Theorem 1.1

We only prove well-posedness for \(-1 \leq s < \frac{1}{2}\). The proof for \(s \in (-\frac{1}{2}, -1) \cup (\frac{1}{2}, j + 1)\), \(s \neq \frac{3}{2}, \frac{5}{2}, \ldots, j - \frac{1}{2}\), is similar. Also, we assume that
\[
0 < T < 1/2.
\]

**Small data.** First, we prove Theorem 1.1 for initial and boundary data such that
\[
\|u_0\|_{H^s(\mathbb{R}^+)} + \sum_{\ell=0}^{j-1} \|g_\ell\|_{H^{s+j-\ell} \cap (0,T)} \leq \frac{1}{144C^2}, \quad \text{with} \quad C = c_{s,b,a} + \frac{1}{2} c_{s,b,a}^2, \quad (6.1)
\]
where \(c_{s,b,a}\) is the constant appearing in the estimate (1.18) and the bilinear estimates (1.20). Under the above smallness condition (6.1), we prove that the integral equation
\[
u = \Phi \nu = S \left[ u_0, g_0, \ldots, g_{j-1}; -\frac{1}{2} \partial_x (u^2) \right], \quad (6.2)
\]
has a unique solution in the space \(X^{s,b,a}(\mathbb{R}^+ \times (0,T))\). For this, we shall prove that the iteration map \(\Phi\) has a fixed point in \(X^{s,b,a}(\mathbb{R}^+ \times (0,T))\). In fact, for any \(u\) in the (closed) ball
\[
B = \left\{ u \in X^{s,b,a}(\mathbb{R}^+ \times (0,T)) : \|u\|_{X^{s,b,a}(\mathbb{R}^+ \times (0,T))} \leq \frac{1}{24C} \right\}, \quad (6.3)
\]
using linear estimate (1.18) with forcing replaced by \(-\frac{1}{2}\partial_x(u^2)\) and bilinear estimates (1.20) we get
\[
\|\Phi u\|_{X^{s,b,\alpha}(\mathbb{R}^+ \times (0,T))} \leq C_{s,b,\alpha} \left( \|u_0\|_{H^s(\mathbb{R}^+)} + \sum_{\ell=0}^{j-1} \|g\|_{H^{s+\ell}(\mathbb{R}^+,0,T)} + \frac{1}{2} \|\partial_x(u^2)\|_{X^{s-b,\alpha-1}(\mathbb{R}^+ \times (0,T))} \right)
\]
\[
\leq C_{s,b,\alpha} \left( \|u_0\|_{H^s(\mathbb{R}^+)} + \sum_{\ell=0}^{j-1} \|g\|_{H^{s+\ell}(\mathbb{R}^+,0,T)} + \frac{1}{2} \|\partial_x(u^2)\|_{X^{s-b,\alpha-1}(\mathbb{R}^2)} \right)
\]
\[
\leq C \left( \|u_0\|_{H^s(\mathbb{R}^+)} + \sum_{\ell=0}^{j-1} \|g\|_{H^{s+\ell}(\mathbb{R}^+,0,T)} + \|\tilde{u}\|^2_{X^{s,b,\alpha}(\mathbb{R}^2)} \right),
\]
where \(\tilde{u}\) is an extension of \(u\) from \(\mathbb{R}^+ \times (0,T)\) to \(\mathbb{R}^2\) such that
\[
\|\tilde{u}\|_{X^{s-b,\alpha-1}(\mathbb{R}^2)} \leq 2\|u\|_{X^{s,b,\alpha}(\mathbb{R}^+ \times (0,T))}. \tag{6.4}
\]
Furthermore, using estimate (6.4) we get
\[
\|\Phi u\|_{X^{s,b,\alpha}(\mathbb{R}^+ \times (0,T))} \leq C \left( \|u_0\|_{H^s(\mathbb{R}^+)} + \sum_{\ell=0}^{j-1} \|g\|_{H^{s+\ell}(\mathbb{R}^+,0,T)} + 4\|u\|^2_{X^{s,b,\alpha}(\mathbb{R}^+ \times (0,T))} \right). \tag{6.5}
\]
And, since \(u \in B\), we have \(\|\Phi u\|_{X^{s,b,\alpha}(\mathbb{R}^+ \times (0,T))} \leq C \left( \frac{1}{144C^2} + \frac{1}{144C^2} \right) \leq \frac{1}{24C}. \) Thus \(\Phi\) maps the ball \(B\) into itself.

To show that \(\Phi\) is a contraction, for any \(u, v \in B\), using linear estimate (1.18) with forcing replaced by \(-\frac{1}{2}\partial_x(u^2 - v^2)\) we get
\[
\|\Phi u - \Phi v\|_{X^{s,b,\alpha}(\mathbb{R}^+ \times \mathbb{R})} \leq \frac{C_{s,b,\alpha}}{2} \|\partial_x(u^2 - v^2)\|_{X^{s-b,\alpha-1}(\mathbb{R}^+ \times \mathbb{R})}
\]
\[
\leq \frac{C_{s,b,\alpha}}{2} \|\partial_x(\tilde{u}^2 - \tilde{v}^2)\|_{X^{s-b,\alpha-1}(\mathbb{R}^2)},
\]
where \(\tilde{u}\) is the extension of \(u\) from \(\mathbb{R}^+ \times (0,T)\) to \(\mathbb{R}^2\) satisfying (6.4). The extension of \(v\) is obtained as follows. First, we extend \(w = v - u\) from \(\mathbb{R}^+ \times (0,T)\) to \(\mathbb{R}^2\) such that
\[
\|\tilde{w}\|_{X^{s,b,\alpha}(\mathbb{R}^2)} \leq 2\|v - u\|_{X^{s,b,\alpha}(\mathbb{R}^+ \times (0,T))}. \tag{6.6}
\]
Then defining \(\tilde{v} = \tilde{w} + \tilde{u}\), we see that \(\tilde{v}\) extends \(v\) from \(\mathbb{R}^+ \times (0,T)\) to \(\mathbb{R}^2\). In addition, using the triangle inequality, we get
\[
\|\tilde{v}\|_{X^{s,b,\alpha}(\mathbb{R}^2)} \leq 2\|v - u\|_{X^{s,b,\alpha}(\mathbb{R}^+ \times (0,T))} + 2\|u\|_{X^{s,b,\alpha}(\mathbb{R}^+ \times (0,T))} \leq \frac{1}{24C}. \tag{6.7}
\]
Combining estimate (6.4) and estimate (6.6) again, we get
\[
\|\Phi u - \Phi v\|_{X^{s,b,\alpha}(\mathbb{R}^+ \times (0,T))} \leq C \left( \|\tilde{u}\|_{s,b,\alpha} + \|\tilde{v}\|_{s,b,\alpha} \right) \cdot \|\tilde{u} - \tilde{v}\|_{s,b,\alpha}
\]
\[
\leq C \cdot \frac{4}{12C} \cdot \|\tilde{u} - \tilde{v}\|_{s,b,\alpha} \leq \frac{2}{3} \|u - v\|_{X^{s,b,\alpha}(\mathbb{R}^+ \times (0,T))}, \tag{6.8}
\]
which shows that \(\Phi\) is a contraction on \(B\). Since \(B\) is a complete Banach space, by the contraction mapping theorem there is a unique \(u \in B\) such that \(\Phi u = u\).

**Lip-continuous dependence on data.** Let \(u_0(x)\), \(g\ell(t)\) and \(v_0(x)\), \(h\ell(t)\) be two sets of data satisfying the smallness condition (6.1). If \(u\) is the solution that corresponds to \(u_0(x)\), \(g\ell(t)\), which we denote
by \( u(x, t) = \psi(t)S[u_0, g_0, \ldots, g_{j-1}; -\frac{1}{2}\partial_x(u^2)] \), and \( v \) is the solution that corresponds to \( v_0(x), h_\ell(t) \), that is \( v(x, t) = \psi(t)S[v_0, h_0, \ldots, h_{j-1}; -\frac{1}{2}\partial_x(u^2)] \) then

\[
    u(x, t) - v(x, t) = \psi(t)S[u_0 - v_0, g_0 - h_0, \ldots, g_{j-1} - h_{j-1}; -\frac{1}{2}\partial_x(u^2 - v^2)].
\]  

(6.9)

Using linear estimate (1.18), extensions \( \tilde{u}, \tilde{v} \) (as above), and bilinear estimate (1.20), we have

\[
    \|u - v\|_{X^{s,b,\alpha}(\mathbb{R}^2 \times (0, T))} \leq C\left(\|u_0 - v_0\|_{H^s(\mathbb{R}^2)} + \sum_{\ell=0}^{j-1} \|g_\ell - h_\ell\|_{H^{s_j+j-\frac{3}{2}}(0,T)} + C\|\tilde{u} - \tilde{v}\|_{s,b,\alpha}\right)
\]

\[
    \leq C\left(\|u_0 - v_0\|_{H^s(\mathbb{R}^2)} + \sum_{\ell=0}^{j-1} \|g_\ell - h_\ell\|_{H^{s_j+j-\frac{3}{2}}(0,T)} + \frac{2}{3}\|u - v\|_{X^{s,b,\alpha}(\mathbb{R}^2 \times (0, T))}\right).
\]

(6.10)

Moving all \( \|u - v\|_{X^{s,b,\alpha}(\mathbb{R}^2 \times (0, T))} \) to the lhs gives

\[
    \|u - v\|_{X^{s,b,\alpha}(\mathbb{R}^2 \times (0, T))} \leq 3C\left(\|u_0 - v_0\|_{H^s(\mathbb{R}^2)} + \sum_{\ell=0}^{j-1} \|g_\ell - h_\ell\|_{H^{s_j+j-\frac{3}{2}}(0,T)}\right)\|u - v\|_{X^{s,b,\alpha}(\mathbb{R}^2 \times (0, T))},
\]

which completes the proof of Lip-continuous dependence on data.

**Large data.** For any size initial data \( u_0 \in H^s \), boundary data \( g \), and for \( T^* \) such that

\[
    0 < T^* \leq T < 1/2,
\]

we replace the integral equation (6.2) with its following localization

\[
    u(x, t) = \Phi_{T^*}u = S[u_0, g_0, \ldots, g_{j-1}; -\frac{1}{2}\partial_x(\psi_{2T^*} \cdot u^2)], \quad |t| \leq T^*,
\]

(6.11)

where \( \psi_{T^*}(t) = \psi(t/T^*) \) with \( \psi(t) \) being our familiar cutoff function in \( C^\infty_0(-1,1) \) with \( 0 \leq \psi(t) \leq 1, \psi(t) = 1 \) for \( |t| \leq 1/2 \). First, we notice that for \( |t| \leq T^* \), the fixed point of the iteration map (6.12) is the solution to the KdVm ibvp (1.1). Thus, \( \Phi_{T^*}(u) = \Phi(u) \) if \( |t| \leq T^* \), i.e. when \( |t| \leq T^* \), then \( \Phi_{T^*}(u) \) becomes the iteration map (6.2). Next, we shall choose appropriate \( T^* \) and use the contraction mapping theorem to show that there is a fixed point of the iteration map (6.12) in the ball \( B(\mathcal{B}) \subseteq X^{s,b,\alpha}(\mathbb{R}^2 \times (0, T)) \). In fact, using the linear estimate (1.18) with forcing replaced by \(-\frac{1}{2}\partial_x(u^2)\), for \( b < b_1 \), which are given below in (6.16), we get

\[
    \|\Phi_{T^*}(u)\|_{X^{s,b,\alpha}(\mathbb{R}^2 \times (0, T))} \leq \|\Phi_{T^*}(u)\|_{X^{s,b_1,\alpha_1}(\mathbb{R}^2 \times (0, T))}
\]

\[
    \leq c_{s,b,\alpha}\left(\|u_0\|_{H^s(\mathbb{R}^2)} + \sum_{\ell=0}^{j-1} \|g_\ell\|_{H^{s_j+j-\frac{3}{2}}(0,T)} + \frac{1}{2}\|\psi_{2T^*}(t)\partial_x(u^2(t))\|_{X^{s_j+j-\frac{3}{2}}(\mathbb{R}^2 \times (0, T))}\right)
\]

\[
    \leq c_{s,b,\alpha}\left(\|u_0\|_{H^s(\mathbb{R}^2)} + \sum_{\ell=0}^{j-1} \|g_\ell\|_{H^{s_j+j-\frac{3}{2}}(0,T)} + \frac{1}{2}\|\psi_{2T^*}(t)\partial_x(u^2(t))\|_{s_j+j-\frac{3}{2}}\right),
\]

where \( \tilde{u} \) is the extension of \( u \) from \( \mathbb{R}^2 \times (0, T) \) to \( \mathbb{R}^2 \), which satisfies (6.4). Now, we estimate the \( \|\cdot\|_{s,-b_1,\alpha_1-1} \). For this we shall need the following result.

**Lemma 6.1.** Let \( \eta(t) \) be a function in the Schwartz space \( S(\mathbb{R}) \). If \(-\frac{1}{2} < b' \leq b \leq \frac{1}{2} \) and \(-\frac{1}{2} < \alpha' - 1 \leq \alpha - 1 < \frac{1}{2} \) (\( \frac{1}{2} < \alpha' \leq \alpha < 1 \) is sufficient condition) then for any \( 0 < T^* \leq 1 \) we have

\[
    \|\eta(t/T^*)u\|_{X^{s,b',\alpha'-1}} \leq c_1(\eta, b, b', \alpha, \alpha') \max\{T^{s-b-b'}, T^{s_\alpha-\alpha'}\}\|u\|_{X^{s,b,\alpha-1}}.
\]

(6.14)
The proof of this result is based on the following multiplier estimate in $X^{s,b}$ spaces, which can be found in [58] (see page 101, Lemma 2.11), i.e.

$$\| \eta(t/T^s)u \|_{X^{s,b}} \leq c_1(\eta, b, b') T^{s-b'} \| u \|_{X^{s,b}}. \quad (6.15)$$

Applying estimate (6.14) with the following choice

$$b = \frac{1}{2} - \beta \text{ (in place of } b') \quad \text{and} \quad b_1 = \frac{1}{2} - \frac{1}{2}\beta \text{ (in place of } b), \quad (6.16)$$

and

$$\alpha = \frac{1}{2} + \frac{1}{2}\beta \text{ (in place of } \alpha') \quad \text{and} \quad \alpha_1 = \frac{1}{2} + \beta \text{ (in place of } \alpha), \quad (6.17)$$

where $\beta$ is defined in (1.22) and it is only depending on $s$ for fixed $m$. From (6.13) we obtain

$$\| \Phi_{T^s}(u) \|_{X^{s,b,\alpha}(R^+ \times (0,T))} \leq c_{s,b,\alpha} \left( \| u_0 \|_{H^s(R^+)} + \sum_{\ell=0}^{j-1} \| g_\ell \|_{H^{s+j-T_{m/\ell}}(0,T)} + \frac{c_1}{2} T^{s+\frac{1}{2}\beta} \| \partial_x(u^2(t)) \|_{s,-b,\alpha_1-1} \right).$$

Then the bilinear estimates (1.20) reads as follows

$$\| \partial_x(f \cdot g) \|_{s,-b,\alpha_1-1} \leq c_{s,b,\alpha} \| f \|_{s,b,\alpha} \| g \|_{s,b,\alpha}, \quad f, g \in X^{s,b,\alpha}, \quad (6.18)$$

and we will use it in this form. Therefore, we get

$$\| \Phi_{T^s}(u) \|_{X^{s,b,\alpha}(R^+ \times (0,T))} \leq c_2 \left( \| u_0 \|_{H^s(R^+)} + \sum_{\ell=0}^{j-1} \| g_\ell \|_{H^{s+j-T_{m/\ell}}(0,T)} + T^{s+\frac{1}{2}\beta} \| \tilde{u} \|_{s,b,\alpha} \right), \quad (6.19)$$

where $c_2 = c_2(s,b,\alpha) = c_{s,b,\alpha} + \frac{1}{2}c_1 \cdot c_{s,b,\alpha}^2$. From (6.19) we see that for the map $\Phi_{T^s}$ (6.12) to be onto, it suffices to have

$$c_2 \left( \| u_0 \|_{H^s(R^+)} + \sum_{\ell=0}^{j-1} \| g_\ell \|_{H^{s+j-T_{m/\ell}}(0,T)} + 4T^{s+\frac{1}{2}\beta} \| u \|_{X^{s,b,\alpha}(R^+ \times (0,T))}^2 \right) \leq r. \quad (6.20)$$

And, since $u \in B(r)$ it suffices to have

$$c_2 \left( \| u_0 \|_{H^s(R^+)} + \sum_{\ell=0}^{j-1} \| g_\ell \|_{H^{s+j-T_{m/\ell}}(0,T)} \right) + 4c_2 T^{s+\frac{1}{2}\beta} r^2 \leq r. \quad (6.20)$$

To show that $\Phi_{T^s}$ is a contraction, again, using linear estimate (1.18) with forcing replaced by $-\frac{1}{2}\partial_x(u^2 - v^2)$, extensions $\tilde{u}, \tilde{v}$ (as above), for $b \leq b_1$ we have

$$\| \Phi_{T^s}(u) - \Phi_{T^s}(v) \|_{X^{s,b,\alpha}(R^+ \times (0,T))} \leq \frac{c_{s,b,\alpha}}{2} \| \psi_{2T^s}(t) \partial_x(u^2(t) - v^2(t)) \|_{X^{s,-b_1,\alpha_1}(R^+ \times (0,T))} \quad (6.21)$$

Applying estimate (6.14) with $b, b_1$ given in (6.16) and $\alpha, \alpha_1$ given by (6.17), we get

$$\| \Phi_{T^s}(u) - \Phi_{T^s}(v) \|_{X^{s,b,\alpha}(R^+ \times (0,T))} \leq \frac{c_1 \cdot c_{s,b,\alpha}}{2} T^{s+\frac{1}{2}\beta} \| \partial_x(\tilde{u}(t) + \tilde{v}(t))(\tilde{u}(t) - \tilde{v}(t)) \|_{s,-b,\alpha_1-1}. \quad (6.22)$$
Next, using the bilinear estimate (6.18), from (6.22) we get

\[ ||\Phi_{T^*}(u) - \Phi_{T^*}(v)||_{X^{2s,b,\alpha}(\mathbb{R}^+ \times (0,T))} \leq c_2 T^{\frac{1}{2} \beta} ||\tilde{u} + \tilde{v}||_{\beta s,b,\alpha} \]

Thus, in order to make the iteration map \( \Phi_{T^*} \) a contraction map, it suffices to have

\[ 16c_2 T^{\frac{1}{2} \beta} r \leq \frac{1}{2}. \]  

Combining conditions (6.20) with (6.24), we see that it suffices to have

\[ c_2 \left( ||u_0||_{H^s(\mathbb{R}^+)} + \sum_{\ell=0}^{j-1} ||g_\ell||_{H^{s+b-\ell}_t} \right) + \frac{1}{8} r \leq r \iff r \geq \frac{8}{2} c_2 \left( ||u_0||_{H^s(\mathbb{R}^+)} + \sum_{\ell=0}^{j-1} ||g_\ell||_{H^{s+b-\ell}_t} \right). \]

So, we choose the radius to be

\[ r = 2c_2 \left( ||u_0||_{H^s(\mathbb{R}^+)} + \sum_{\ell=0}^{j-1} ||g_\ell||_{H^{s+b-\ell}_t} \right). \]  

Then, from (6.24) it suffices to have \( T^{\frac{1}{2} \beta} \leq (32c_2 r)^{-1} \), which follows from choosing

\[ T^* = \frac{1}{2} (1 + 32c_2 r)^{-\frac{1}{2} \beta} < \frac{1}{2}. \]  

Combining this choice of \( T^* \) together with choice (6.25) for \( r \) we get

\[ T^* = \frac{1}{2} \left[ 1 + 64c_2^2 \left( ||u_0||_{H^s(\mathbb{R}^+)} + \sum_{\ell=0}^{j-1} ||g_\ell||_{H^{s+b-\ell}_t} \right) \right]^{-\frac{1}{2} \beta} \]

\[ \geq c_0 \cdot \left( 1 + ||u_0||_{H^s(\mathbb{R}^+)} + \sum_{\ell=0}^{j-1} ||g_\ell||_{H^{s+b-\ell}_t} \right)^{-\frac{4}{3}} = T_0, \]

for some \( c_0 \) depending on \( c_2(s,b,\alpha) \) and \( \beta = \beta(s) \), that is \( c_0 = c_0(s,b,\alpha) \). Thus we choose the lifespan as stated in (1.9). This completes the proof of well-posedness for \(-\frac{1}{2} < s < \frac{1}{2}\). □

Lip continuity of the data to solution map and uniqueness is similar to the well-posedness on the line described in [18].

7. Derivation of the Fokas Solution Formula

Here, we provide an outline of UTM for the solution to the forced linear KdV ibvp in three steps. First, we use the Fourier transform on the half-line to get a solution formula to ibvp (1.10) via the Fourier inversion formula on the real line. Then, we deform the contour via the Cauchy’s Theorem and derive a formula for the solution integrating over the contours \( \partial D_{2p}^+ \), \( p = 1, 2, \ldots, j \) in the upper half of the complex plane. Finally, we eliminate the unknown boundary data and get the desired solution formula (1.11).

**Step 1: Solving KdV ibvp (1.10) via half-line Fourier transform.** If \( \tilde{u} \) is a solution to the LKdV formal adjoint equation

\[ \partial_t \tilde{u} + (-1)^{j+1} \partial_x^{2j+1} \tilde{u} = 0, \]  

then multiplying it by \( u \) and equation (1.10a) by \( \tilde{u} \), and adding the resulting equations gives

\[ \tilde{u} \partial_t u + u \partial_t \tilde{u} + (-1)^{j+1} (\tilde{u} \partial_x^m u + u \partial_x^m \tilde{u}) = \tilde{u} f, \]
or
\[
(\ddot{u}u)_t + (-1)^{j+1}[\ddot{u}\partial_x^{2j}u + \cdots + (-1)^n \partial_x^n \ddot{u}\partial_x^{2j-n}u + \cdots + \partial_x^{2j}\ddot{u}u]_x = \ddot{u}f. \tag{7.2}
\]
Then, choosing as \(\ddot{u}\) the exponential solutions to transpose equation (7.1)
\[
\ddot{u} = e^{-i\xi x-i\xi^m t}, \quad \xi \in \mathbb{C}, \tag{7.3}
\]
and substituting them into identity (7.2) we get the **divergence form**:
\[
(e^{-i\xi x-i\xi^m t}u)_t + (-1)^{j+1}(e^{-i\xi x-i\xi^m t})[\partial_x^{2j}u + \cdots + i^n \xi^n \partial_x^{2j-n}u + \cdots + (-1)^j \xi^{2j}u]_x = \ddot{u}f. \tag{7.4}
\]
Integrating the divergence form (7.4) from \(x = 0\) to \(\infty\) gives the \(t\)-equation
\[
(e^{-i\xi x-i\xi^m t}u)(\xi, t) = e^{-i\xi x-i\xi^m t}f(\xi, t) + (-1)^{j+1}e^{-i\xi^m t}g(\xi, t), \tag{7.5}
\]
where \(\ddot{u}\) and \(\ddot{f}\) are the half-line Fourier transforms of \(u\) and \(f\), which are defined in (1.13), and \(g\) is the following combination of \(m = 2j + 1\) boundary data (some of which are not given)
\[
g(\xi, t) = \partial_x^{2j}u(0, t) + \cdots + i^n \xi^n \partial_x^{2j-n}u(0, t) + \cdots + (-1)^j \xi^{2j}u(0, t). \tag{7.6}
\]
Integrating (7.5) from 0 to \(t\), \(0 \leq t \leq T\), we obtain the so called **global relation**:
\[
e^{-i\xi^m t}\ddot{u}(\xi, t) = \ddot{u}_0(\xi) + F(\xi, t)
+ \left(-1\right)^{j+1}[\ddot{g}_0(\xi^m, t) + \cdots + i^n \xi^n \ddot{g}_{2j-n}(\xi^m, t) + \cdots + (-1)^j \xi^{2j}\ddot{g}_0(\xi^m, t), \quad \text{Im}(\xi) \leq 0, \tag{7.7}
\]
where \(F(\xi, t)\) and \(\ddot{g}_0(\xi, t)\) are given in (1.14) and (1.15) respectively. Now, inverting (7.7) we get
\[
u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x+i\xi^m t}[\ddot{u}_0(\xi) + F(\xi, t)]d\xi
+ \frac{(-1)^{j+1}}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x+i\xi^m t}\ddot{g}_0(\xi^m, t) + \cdots + (-1)^j \xi^{2j}\ddot{g}_0(\xi^m, t)]d\xi. \tag{7.8}
\]

**Step 2:** Deforming integration over the contours \(\partial D^+_{2p}\) in the upper half-pane. Formula (7.8) contains \((j + 1)\) unknown data. To eliminate them, we deform the integration contour from \(\mathbb{R}\) to \(\partial D^+_{2p}\), \(p = 1, 2, \cdots, j\). This is expressed by the following result.

**Lemma 7.1.** *The solution \(u(x, t)\) to ibvp (1.10) can be written in the form*
\[
u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x+i\xi^m t}[\ddot{u}_0(\xi) + F(\xi, t)]d\xi
+ \frac{(-1)^{j+1}}{2\pi} \sum_{p=1}^{j} \int_{\partial D^+_{2p}} e^{i\xi x+i\xi^m t}[\ddot{g}_0(\xi^m, t) + \cdots + (-1)^j \xi^{2j}\ddot{g}_0(\xi^m, t)]d\xi, \tag{7.9}
\]
*where the domains \(D^+_2, \cdots, D^+_j\) are shown in Figure 1.1 (if \(j\) is odd) or Figure 1.2 (if \(j\) is even), and the orientation of the boundary \(\partial D^+_{2p}\) is given by the left-hand rule.*

The proof of above lemma can be found in [60]. Also, a similar lemma for the KdV equation can be found in [22].

**Step 3:** Eliminating the unknown boundary data. For each \(p = 1, 2, \cdots, j\), we construct a linear system with \(j + 1\) equations (as many as the unknown data). For this we apply the invariant
transformations of $\xi^n$, i.e. $\xi \rightarrow \alpha_{p,n} \xi$, $n = 1, 2, \cdots, j + 1$, where $\alpha_{p,n}$ are defined in (1.12), and use the global relation (7.7). Thus, we obtain the linear system of the following $j + 1$ equations

$$
e^{-i\xi^n t} \hat{u}(\alpha_{p,1} \xi, t) = \hat{u}_0(\alpha_{p,1} \xi) + F(\alpha_{p,1} \xi, t)
+ (-1)^{j+1} [g_{2j}^{\alpha}(\xi^n, t) + \cdots + (\alpha_{p,1})^j \hat{g}_{2j-\ell}(\xi^n, t) + \cdots + (\alpha_{p,1})^{2j}(i\xi)^{2j} \hat{g}_0(\xi^n, t)],$$

$$
\cdots
$$

$$
e^{-i\xi^n t} \hat{u}(\alpha_{p,j+1} \xi, t) = \hat{u}_0(\alpha_{p,j+1} \xi) + F(\alpha_{p,j+1} \xi, t)
+ (-1)^{j+1} [g_{2j}^{\alpha}(\xi^n, t) + \cdots + (\alpha_{p,j+1})^j \hat{g}_{2j-\ell}(\xi^n, t) + \cdots + (\alpha_{p,j+1})^{2j}(i\xi)^{2j} \hat{g}_0(\xi^n, t)].$$

Solving these equations for $(i\xi)^\ell \hat{g}_{2j-\ell}(\xi^n, t)$, $\ell = 0, 1, \cdots, j$, and substituting the obtained solutions into formula (7.9), we get the desired Fokas solution formula (1.11) involving only the given data and no unknown data.

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