A NOTE ON COMMUTATIVE ALGEBRAS AND THEIR MODULES IN QUASICATEGORIES

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The purpose of this document is to develop a neat combinatorial theory of modules over commutative algebras in ∞-categories in the vein of the theory of modules over associative algebras set out in [Lur12 §4.2, 4.3]. In fact, we’ll describe a cute commutative analog \(CM\otimes\) of the operads \(LM\otimes\) and \(BM\otimes\) of [Lur12 §4.2.1] and [Lur12 §4.3.1]. This should ease the task of constructing and manipulating such modules. In particular, we prove the following theorem, which plays a role in [Gla14]. It’s tautological for 1-categories but sort of subtle for ∞-categories, and it’s just generally nice to know:

**Theorem 1.** Let \(C\otimes\) be a symmetric monoidal ∞-category. We’ll denote the category of finite sets by \(F\) and the category of finite pointed sets by \(F^*\). A datum comprising a commutative algebra \(E\) in \(C\) and a module \(M\) over it - that is, an object of \(\text{Mod}^F(C)^\otimes\) [Lur12 Definition 3.3.3.8] - gives rise functorially to a functor

\[A_{E,M} : F^* \to C\]

such that

\[A_{E,M}(S) \simeq E^{\otimes S^o} \otimes M.\]

We’ll prove Theorem 1 by first describing the ∞-operad \(CM\otimes\) that parametrizes this data and then giving a map from \(F_*\) to the symmetric monoidal envelope of \(CM\otimes\). We’ll work extensively with the model category of ∞-preoperads [Lur12 §2.1.4], which we’ll denote \(PO\).

**Definition 2.** We define the operad \(CM\otimes\) as (the nerve of) the 1-category in which

- an object is a pair \((S, U)\) consisting of an object \(S\) of \(F_*\) and a subset \(U\) of \(S^o\);
- a morphism from \((S, U)\) to \((T, V)\) is a morphism \(f : S \to T\) in \(F_*\) such that for each \(v \in V\), the set \(U \cap f^{-1}(v)\) has cardinality exactly 1.

It’s easily checked that the functor \(CM\otimes \to F_*\) that maps \((S, U)\) to \(S\) makes \(CM\otimes\) into an ∞-operad. The following result isolates the hard work involved in proving Theorem 1:

**Proposition 3.** We give \((F_*)_\langle 1 \rangle/\) the structure of an ∞-preoperad by letting the target map \(t : (F_*)_\langle 1 \rangle/ \to F_*\) create marked edges. Define a map of ∞-preoperads

\[\phi : (F_*)\langle 1 \rangle/ \to CM\otimes\]

by

\[\phi(j : 1 \to S) = \begin{cases} (S, \{j(1)\}) & \text{if } j(1) \in S^o \\ (S, \emptyset) & \text{otherwise.} \end{cases}\]
Thus we embed \((\mathcal{F}_s)_{(1)}\) as the full subcategory of \(\mathcal{CM}^\circ\) spanned by those objects \((S,U)\) for which the cardinality of \(U\) is at most 1.

Then \(\phi\) is an trivial cofibration in \(\mathcal{PO}\).

The proof will take the form of a series of lemmas.

**Lemma 4.** Suppose \(q : E \to B\) is an inner fibration of \(\infty\)-categories, \(K\) a simplicial set and \(r : K^{\leq} \to B\) a map such that for each edge \(e : k_1 \to k_2\) of \(K\) and for each \(l \in E\) with \(q(l) = r(k_1)\), there is a cocartesian lift of \(e\) to \(E\) with source \(l\). Denote the cone point of \(K^{\leq}\) by \(c\) and suppose \(d \in E\) is such that \(q(d) = r(c)\). Then there is a map \(r' : K^{\leq} \to E\) lifting \(r\) and taking every edge of \(K^{\leq}\) to a cocartesian edge of \(E\).

**Proof.** Clearly we can lift in such a way that the image of every edge of \(K^{\leq}\) with source \(c\) is cocartesian; let \(r'\) be such a lift. We claim that \(r'\) already has the desired property. Indeed, \(r'\) can be viewed as a section of the cocartesian fibration \(q_K : E \times_B K^{\leq} \to K^{\leq}\), and then the result follows from [Lur09, Proposition 2.4.2.7]. \(\square\)

**Lemma 5.** Let \(p : \mathcal{O}^\circ \to \mathcal{F}_s\) be any \(\infty\)-operad. For any set \(T\), let \(\mathcal{P}_T\) be the poset of subsets of \(T\) ordered by reverse inclusion, and let \(\mathcal{P}_T' = \mathcal{P}_T \setminus \{T\}\), so that \(\mathcal{P}_T \cong (\mathcal{P}_T')^\circ\). For each \(S \in \mathcal{F}_s\), let \(r_S : \mathcal{P}_{S^o} \to \mathcal{F}_s\) denote the obvious diagram of inert morphisms. Let \(X \in \mathcal{O}_S^\circ\), let \(\rho : \mathcal{P}_{S^o} \to \mathcal{O}_S^\circ\) be the cocartesian lift of \(r_S\) with \(\rho(S^o) = X\) whose existence is guaranteed by Lemma 4 and suppose \(|S^o| > 1\). Then \(\rho\) is a limit diagram relative to \(p\).

**Proof.** We work by induction on the size of \(S^o\); the case \(|S^o| = 2\) is an immediate consequence of the \(\infty\)-operad axioms. For each \(k \in \mathbb{N}\), let \(\mathcal{P}_{S^o}^{\leq k}\) denote the poset of subsets of \(T\) of cardinality at most \(k\). Then the restriction of \(\rho\) to \((\mathcal{P}_{S^o}^{\leq k-1})^\circ\) is a \(p\)-limit diagram by the \(\infty\)-operad axioms. We now argue by induction on \(k\) that for each \(k\) with \(1 \leq k \leq |S^o| - 1\), the restriction of \(\rho\) to \((\mathcal{P}_{S^o}^{\leq k})^\circ\) is a limit diagram. Indeed, for each such \(k > 1\), the \(|S^o| - k\) edition of the lemma implies that \(\rho|_{\mathcal{P}_{S^o}^{\leq k-1}}\) is \(p\)-right Kan extended from \(\rho|_{\mathcal{P}_{S^o}^{\leq k-1}}\). Comparing the \(p\)-right Kan extensions along both paths across the commutative diagram

\[
\begin{array}{ccc}
\mathcal{P}_{S^o}^{\leq k-1} & \to & \mathcal{P}_{S^o}^{\leq k} \\
\downarrow & & \downarrow \\
(\mathcal{P}_{S^o}^{\leq k-1})^\circ & \to & (\mathcal{P}_{S^o}^{\leq k})^\circ
\end{array}
\]

gives the induction step, and thence the result. \(\square\)

**Corollary 6.** Retaining the notation of Lemma 5 any nontrivial subcube of \(\rho\) is a \(p\)-limit diagram. That is, if \(U_1\) and \(U_2\) are two subsets of \(S^o\) with \(U_1 \subseteq U_2\) and \(|U_2 \setminus U_1| > 1\), then the restriction of \(\rho\) to the subposet \(\mathcal{P}_{U_1,U_2}\) spanned by those subsets \(V\) of \(S^o\) with \(U_1 \subseteq V \subseteq U_2\) is a \(p\)-limit diagram.

**Proof.** By restricting to \(\mathcal{P}_{U_2}\), we may assume \(U_2 = S^o\). Let

\[\mathcal{P}'_{U_1,U_2} = \mathcal{P}_{U_1,U_2} \setminus U_2\]

and let \(\mathcal{Q}\) be the closure of \(\mathcal{P}'_{U_1,U_2}\) under downward inclusion. Then \(\mathcal{Q}\) contains \(\mathcal{P}_{S^o}^{\leq 1}\), so by the above discussion, \(\rho\) is \(p\)-right Kan extended from \(\mathcal{Q}\). But \(\mathcal{P}'_{U_1,U_2}\) is coinitial in \(\mathcal{Q} = \mathcal{Q}_{U_2}\), so we’re good. \(\square\)
Proof of Proposition 3. Now let $O^\otimes$ be any $\infty$-operad, and let $F : (\mathcal{F}_s)_{(1)}/ \to O^\otimes$ be a morphism of $\infty$-preoperads. Consider the diagram

$$
\begin{array}{ccc}
(F_s)_{(1)}/ & \xrightarrow{\phi} & O^\otimes \\
\downarrow{\phi} & & \downarrow{p} \\
\mathcal{CM}^\otimes & \xrightarrow{\rho} & F_s
\end{array}
$$

We claim that the $F^\ast$-relative right Kan extension $\phi, F$ along the dotted line exists [Lur09 §4.3.2]. Indeed, let $(S, U)$ be an object of $\mathcal{CM}^\otimes$, and let $Q_{(S,U)}$ be the subposet of $\mathcal{P}_{S^0}$ spanned by subsets $T$ such that

- $S^0 \setminus U \subseteq T$, and
- $|T \cap U| \leq 1$.

Then the natural map

$$
j_{(S,U)} : Q^q_{(S,U)} \to \mathcal{CM}^\otimes
$$

which takes all edges of $Q^q_{(S,U)}$ to inert edges of $\mathcal{CM}^\otimes$ gives rise to a map

$$
k_{(S,U)} : Q_{(S,U)} \to (F_s)_{(1)}/ \times_{\mathcal{CM}^\otimes} (\mathcal{CM}^\otimes)_{(S,U)}/
$$

and $k_{(S,U)}$ is easily observed to be coninitial. Thus it suffices to show that $F \circ k_{(S,U)}$ admits a $p$-limit. But $F \circ k_{(S,U)}$ can be embedded, up to equivalence, into the cube of inert edges

$$
\rho : \mathcal{P}_{S^0} \to O^\otimes
$$

such that

$$
\rho_{S^0} = \prod_U F((1), \{1\}) \times \prod_{S^0 \setminus U} F((1), \emptyset)
$$

where $\prod$ denotes a product relative to $p$. By Corollary 6 together with the induction argument used in the proof of Lemma 5, we see that $\rho_{S^0}$ is a $p$-limit of $F \circ k_{(S,U)}$. So $\phi, F$ exists [Lur09 Lemma 4.3.2.13], and it is clear that $\phi, F$ is a morphism of $\infty$-operads. Since any morphism of preoperads from $(F_s)_{(1)}/$ to an $\infty$-operad extends over $\mathcal{CM}^\otimes$, $\phi$ must be a trivial cofibration. □

Now we’ll relate our construction to Lurie’s category of modules. Let $C^\otimes$ be an $\infty$-operad. Employing the notation of [Lur12 §3.3.3], we define

$$
\widehat{\text{Mod}}(C) := \text{Mod}^{\mathcal{F}_s}(C^\otimes)_{(1)}
$$

with analogous definitions of $\tilde{\text{Mod}}(C)$ and $\overline{\text{Mod}}(C)$.

**Proposition 7.** There is an equivalence of $\infty$-categories

$$
\text{Mod}(C) \cong \text{Fun}^\otimes(\mathcal{CM}^\otimes, C^\otimes).
$$

**Proof.** Let $X$ be a simplicial set equipped with the constant map $1_X : X \to \mathcal{F}_s$ with image $(1)$. One then has set bijections

$$
\text{Hom}(X, \text{Mod}(C)) \cong \text{Hom}_{\mathcal{F}_s}(X \times (\mathcal{F}_s)_{(1)/}, C^\otimes)
$$

and

$$
\text{Hom}(X, \overline{\text{Mod}}(C)) \cong \text{Hom}^\otimes(X \times (\mathcal{F}_s)_{(1)/}, C^\otimes)
$$

where $\text{Hom}^\otimes$ denotes the set of $\infty$-preoperad maps; this is to say that we have an isomorphism of categories between $\text{Mod}(C)$ and the category $\text{Fun}^\otimes((\mathcal{F}_s)_{(1)/}, C^\otimes)$ of $\infty$-preoperad maps from $(\mathcal{F}_s)_{(1)/}$ to $C^\otimes$. Moreover, when $O^\otimes = F_s$, the trivial
Kan fibration $\theta$ of \cite[Lemma 3.3.3.3]{Lur12} is an isomorphism, so there is no difference between $\Mod(C)$ and $\Mod(C)$.

Finally, we claim that the restriction map

$$\phi^* : \Fun^\otimes((\mathcal{F}_+)^{\otimes^\perp}, C^\otimes) \to \Fun^\otimes((\mathcal{F}_+)^{\otimes^\perp}/, C^\otimes)$$

is a trivial Kan fibration. Noting that the categorical pattern $P_0$ of \cite[Lemma B.1.13]{Lur12} serves as a unit for the product of categorical patterns, we deduce from \cite[Remark B.2.5]{Lur12} that the product map

$$\mathcal{P}\mathcal{O} \times \mathcal{S}et^\perp_\Delta \to \mathcal{P}\mathcal{O}$$

is a left Quillen bifunctor. This means, in particular, that for each $n$, the morphism of $\infty$-preoperads

$$(\mathcal{F}_+)^{\otimes^\perp}/ \times (\Delta^n)^\perp \cup (\mathcal{F}_+)^{\otimes^\perp} \times (\partial\Delta^n)^\perp$$

is a trivial cofibration in $\mathcal{P}\mathcal{O}$, which gives the result. \hfill $\Box$

We now wish to characterize the symmetric monoidal envelope of $\mathcal{C}M^\otimes$.

**Definition 8.** Let $\mathcal{F}^+$ be the category whose objects are pairs $(S, U)$, with $S$ a finite set and $U \subseteq S$ a subset, and in which a morphism from $(S, U) \to (T, V)$ is a morphism $f : S \to T$ of finite sets inducing a bijection $f|_U : U \cong V$. The disjoint union (which, mind you, is definitely not a coproduct) makes $\mathcal{F}^+$ into a symmetric monoidal category $(\mathcal{F}^+)^\otimes$.

**Proposition 9.** The symmetric monoidal envelope $\Env(\mathcal{C}M^\otimes)$ is isomorphic to $(\mathcal{F}^+)^\otimes$.

**Proof.** We briefly sketch the proof, which is a routine 1-categorical exercise. By definition, the symmetric monoidal envelope of $\Env(\mathcal{C}M^\otimes)$ has objects

$$(S, U, f : S^n \to T)$$

where $T \in \mathcal{F}$. Our isomorphism will map this object to the object

$$(T, (f^{-1}(t), f^{-1}(t) \cap U)_{t \in T})$$

of $(\mathcal{F}^+)^\otimes$. \hfill $\Box$

**Corollary 10.** A $\mathcal{C}M^\otimes$-algebra parametrizing a commutative monoid $E$ and a module $M$ in $C^\otimes$ gives rise to a functor $A_{E, M} : \mathcal{F}_+ \to C$ such that

$$A_{E, M}(S) \simeq E^{\otimes S^n} \otimes M.$$  

**Proof.** We construct $A_{E, M}$ by embedding $\mathcal{F}_+$ as the full subcategory of $\mathcal{F}^+$ consisting of objects $(S, U)$ for which $|U| = 1$. \hfill $\Box$

**References**

[Gla14] Saul Glasman. A spectrum-level Hodge filtration on topological Hochschild homology. arXiv preprint arXiv:1408.3065, 2014.

[Lur09] Jacob Lurie. *Higher Topos Theory*. Princeton University Press, 2009.

[Lur12] Jacob Lurie. *Higher Algebra*. 2012.