The heat kernel for deformed spheres

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Abstract

We derive the asymptotic expansion of the heat kernel for a Laplace operator acting on deformed spheres. We calculate the coefficients of the heat kernel expansion on two- and three-dimensional deformed spheres as functions of deformation parameters. We find that under some deformation the conformal anomaly for free scalar fields on $R^4 \times \tilde{S}^2$ and $R^6 \times \tilde{S}^2$ is canceled.

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The asymptotic expansion of the heat kernel, corresponding to the elliptic second-order differential operator acting on an arbitrary manifold \( M \) has been investigated in connection with index theorems [1] and some applications in field theory [2, 3]. The kernel \( K(x, y, t) \) satisfies a heat equation for the some second-order operator \( H = -D^2 + X \) defined on a smooth N-dimensional Riemannian manifold (\( X \) is a scalar function)

\[
(\partial_t + H)K(x, y, t) = 0
\]

with the boundary condition

\[
K(x, y, 0) = \delta(x, y).
\]

The asymptotic expansion of \( K(x, y, t) \) has been derived for various models [4]-[7] in a general form [6] and in a numerical form for some homogeneous spaces [7]. Under \( t \to 0 \) the heat kernel has the following expansion

\[
K(x, y, t) = (4\pi t)^{-N/2} \Delta^{1/2}(x, y) \exp \left( -\frac{\sigma^2}{4t} \right) \sum_{n=0}^{\infty} a_n(x, y)(t)^n
\]

where \( \Delta \) is the invariant Van Vleck-Morette determinant [8], \( 2\sigma(x, y) \) is the square of the geodesic distance between \( x \) and \( y \). In terms of \( K(x, y, t) \), one can write a simple integral representation for the one-loop effective action. If one takes regularization with the short-distance cut-off \( L [9] \), the regularized one-loop effective action \( W^{(1)} \) can be defined as

\[
W^{(1)} = \int_L^{\infty} dt \frac{d}{t} K(t)
\]

Here \( K(t) = tr \int d^N x g^{1/2} K(x, x, t) \) with the asymptotic expansion

\[
K(t) = \sum_{n=0}^{\infty} A_n t^{n-N/2} = \sum_{n=0}^{\infty} tr \int d^N x g^{1/2} a_n(x, x)
\]

The divergent terms in \( W^{(1)} \) are proportional to the first coefficients \( a_n(x, x) \). For even-dimensional spaces the most important is the coefficient \( a_{N/2}(x, x) \), since this single coefficient for a given theory determines various anomalies [10].

In this letter we explicitly calculate the coefficients \( a_n(x, x) \) for two- and three-dimensional spaces obtained from the metric deformation of two- and three-dimensional spheres respectively. We obtain the coefficients \( a_n \) as functions of the deformation parameters and show that under some deformation the conformal anomaly is canceled for free scalar fields defined on \( \tilde{S}^2 \times R^4 \) and \( \tilde{S}^2 \times R^6 \).

Let us begin with the scalar Laplacian eigenvalues on deformed spheres. The metric on the deformed sphere \( \tilde{S}^{d+1} \) can be expressed in the form

\[
ds^2 = dx_0^2 + \sin^2 x_0 d\Omega^2
\]
where \(d\Omega^2\) is the metric on the (deformed) \(\tilde{S}^d\). Any scalar function can be represented as a sum of eigenfunctions \(Y_{(l)}(x_i)\) of the Laplace operator on \(\tilde{S}^d\)

\[
\phi(x_0, x_i) = \sum_{(l)} f_{(l)}(x_0) Y_{(l)}(x_i).
\]  

Substituting the decomposition (1) in the eigenvalue equation

\[
\Delta \phi = \lambda \phi
\]

we obtain the following ordinary differential equation

\[
[\partial_0^2 + d\cot x_0 \partial_0 - \frac{a_{(l)}}{\sin^2 x_0}] f_{(l)} = \lambda f_{(l)}.
\]  

The \(-a_{(l)}\) is the eigenvalue of the Laplace operator on \(\tilde{S}^d\) corresponding to \(Y_{(l)}\). We shall drop the subscripts \((l)\) for a while. Let us make the substitution

\[f = h \sin^b(x_0), \quad b = \frac{1}{2}(1 - d + \sqrt{(1 - d)^2 + 4a})\]

and change the independent variable

\[z = \frac{1}{2}(\cos x_0 + 1).\]

The equation (2) then takes the form

\[z(z - 1)h'' + (1 + c)(z - \frac{1}{2})h' + eh = 0,
\]

\[e = b(b + d) + \lambda, \quad c = 2b + d.
\]  

Prime denotes differentiation with respect to \(z\). According to the general prescription [11] let us express \(h\) as the power series

\[h(z) = \sum_{k=0}^{\infty} \alpha_k z^k.
\]  

Substitution of (4) in (3) gives a recurrent condition on the coefficients \(\alpha_k\)

\[\alpha_{k+1} = \frac{\alpha_k (k-1) + (1+c)k + e}{(k+1)(k+(c+1)/2)}.
\]  

The denominator of (5) is positive for all \(k\). The eigenfunctions \(h_k\) can be found by imposing the condition on the numerator of (5) to be equal to zero. We obtain the eigenvalues

\[
\lambda_{(l)k} = -k^2 - (1 + q)k - \frac{1}{2}(1 - d + q + 2a_{(l)})
\]
where we restored the dependence on the index \((l)\). The eigenvalues \(a(l)\) can be defined using the same formula (6) with \(d \to d - 1\). Repeating these steps we can obtain the spectrum of scalar Laplace operator on \(\tilde{S}^{d+1}\) in terms of \(d + 1\) non-negative integers and \(d + 1\) scale parameters.

For \(d = 3\) equation (6) was obtained in [12] by the same methods.

In the case of the unit round \(d\)-sphere \(\tilde{S}^d\) with \(a(l) = l(l + d - 1)\) we obtain from (6)

\[
\lambda_{(l)k} = -(k + l)(k + l + d) = -n(n + d), \quad n = k + l
\]

Thus equation (6) reproduces the correct eigenvalues of the scalar Laplace operator on the unit round \(S^{d+1}\). One can also verify that the degeneracies have the correct values.

With the deformation of a two-dimensional sphere, we consider rescaling \(l^2 \to \rho l^2\), \((\rho > 0)\) where \(l^2\) are the eigenvalues of a Laplace operator on the unit sphere \(S^1\). The eigenvalues (6) for \(\tilde{S}^2\) can be written as

\[
\lambda_{l,k} = -(k + \rho l + 1/2)^2 + 1/4
\]

The heat kernel for the eigenvalues (7) is defined as

\[
K(t) = K_1(t) + K_2(t) = e^{t/4} (2 \sum_{l=1}^\infty \sum_{k=\rho l + 1/2}^\infty e^{-k^2 t} + \sum_{k=1/2}^\infty e^{-k^2 t})
\]

To derive the asymptotic expansion for the first term in (8) we rewrite the sum over \(k\) by using the Mellin transform

\[
f(s, t) = \int_0^\infty dx x^{s-1} e^{-x^2 t} = \frac{1}{2} \Gamma(\frac{s}{2}) t^{-s/2}.
\]

Performing the inverse transform

\[
\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} ds' k^{-s'} f(s', t)
\]

and summing over \(k\) we obtain

\[
K_1(t) = e^{t/4} \frac{1}{2\pi i} \int_C ds' \sum_{l=1}^\infty \Gamma(\frac{s'}{2}) t^{-s'/2} \zeta(s', \rho l + 1/2) + R(t).
\]

Here the contour \(C\) covers the poles of \(\Gamma(\frac{s'}{2})\) at points \(s' = -2m\) as well as poles of \(g(s') = \sum_{l=1}^\infty \zeta(s', \rho l + 1/2)\) and

\[
R(t) = e^{t/4} \frac{1}{2\pi i} \int_D ds' \Gamma(\frac{s'}{2}) t^{-s'/2} g(s')
\]
where the contour $D$ consists of the semicircumference at infinity on the left. The formula (9) is understood to be exact, but it is difficult to compute $R(t)$ explicitly. However, one can show that $R(t)$ vanishes exponentially as $t \to 0$. Thus, for small $t$, one can discard $R(t)$ relative to the power series, leaving the asymptotic expansion for $K(t)$. (The calculations of $R(t)$ for some series can be found in [13]). Using the Hermite formula [11]

$$
\zeta(z, q) = \frac{q}{n^2} + \frac{q^{1-n}}{n-1} + 2 \int_0^\infty dx \sin(z \arctan(x/q)) \frac{(q^2 + x^2)^{-1/2}}{e^{2px} - 1}
$$

for $\zeta(s', \rho l + 1/2)$ in (9), after summing over $l$ and integrating over $s'$ we obtain the following heat kernel expansion

$$
K_1(t) = e^{t/4} \left( \frac{1}{\rho t} - \frac{1}{2t^{1/2}} + \sum_{m=0}^\infty \frac{(-1)^m}{m!} \left( -\frac{2}{2m+1} \rho^{2m+1} \zeta(-2m - 1, 1 + 1/(2\rho)) \right) + \rho^{2m} \zeta(-2m, 1 + 1/(2\rho)) - \frac{2}{2m+1} \rho^{-2m-1} \zeta(-2m - 1) + F(-2m, \rho) \right)
$$

(10)

where

$$
F(z, \rho) = 2 \sum_{p=0}^\infty (-1)^{p+1} c_p(z) \sum_{n=0}^\infty \frac{\Gamma(n + z/2)}{\Gamma(z/2)n!} \rho^{-2p - 2n - z - 1} \times \zeta(2p + 2n + z, 1 + 1/(2\rho)) \zeta(-2p - 2n - 1), \quad (2p + 2n + z \neq 0).
$$

and the coefficients $c_p$ are determined from

$$
\sin(z \arctan(x)) = \sum_{p=0}^\infty c_p(z) x^{2p+1}
$$

The asymptotic expansion for $K_2$ in (8) can be derived by using the same method. After a little calculation (discarding the exponentially small contribution) we find

$$
K_2(t) = e^{t/4} \frac{\pi^{1/2}}{2t^{1/2}}, \quad (t \to 0).
$$

(11)

Substituting (10) and (11) in (8) and performing a numerical computation we get the following values of some $a_n(\rho)$ $(a_0 = 1)$

| $n$ | $\rho = 0.2$ | $\rho = 0.6$ | $\rho = 1$ | $\rho = 1.8$ |
|-----|--------------|--------------|------------|-------------|
| 1   | 0.1733       | 0.2267       | 0.3333     | 0.7067      |
| 2   | 0.0077       | 0.0263       | 0.0667     | 0.2439      |
| 3   | -0.0016      | 0.0024       | 0.0127     | 0.0902      |
For \( \rho = 1 \) we have from (12) in a numerical form the famous asymptotic expansion for unit round \( S^2 \)

\[
K(t) = \frac{1}{t} + 0.3333 + 0.0667t + 0.0127t^2 + 0.0032t^3 + \ldots
\]

The next space we would like to consider is a three-sphere with another homogeneous deformation which can be represented as \( SU(2) \times U(1)/U(1) \) (the Taub space). The eigenvalues of the Laplace operator can be written as

\[
\lambda_{n,j} = n^2 - 1 + \omega(2j - n + 1)^2
\]

where \( \omega \) is the deformation parameter. The range of \( \omega \) is \(-1 < \omega < \infty\) and \( \omega = 0 \) corresponds to round \( S^3 \). Then the heat kernel takes the form

\[
K(t) = \sum_{n=1}^{\infty} n \sum_{j=0}^{n-1} \exp(-\lambda_{n,j})t
\]

First we rewrite the sum over \( j \) using the identity

\[
\sum_{j=0}^{n-1} \exp(-\omega(2j - n + 1)^2)t = \left( \sum_{j=-(n-1)/2}^{\infty} - \sum_{(n+1)/2}^{\infty} \right) e^{-4\omega j^2 t}
\]

Now it has the form similar to (8) and can be evaluated by means of the Mellin transform. A straightforward calculation gives

\[
K(t) = e^t \sum_{k=0}^{\infty} \frac{\omega^k(-1)^k(2k)!}{k!} \sum_{r=0}^{2k} \frac{B_r 2^r k^{-(r+1)/2}}{r!} \sum_{p=0}^{\infty} \frac{\Gamma(3/2 + p)t^{k-p-3/2}}{(2k - 2p - r)!(2p + 1)!}
\]

Here we used the representation

\[
\zeta(-m, q) = -\sum_{r=0}^{m+1} m!B_r q^{m+1-r} \frac{1}{r!(m-r+1)!}
\]

where \( B_r \) are Bernoulli numbers. After similar manipulations with the sum over \( n \) in (15) we obtain

\[
K(t) = e^t \sum_{k=0}^{\infty} \frac{\omega^k(-1)^k(2k)!}{4k!} \sum_{r=0}^{2k} \frac{B_r 2^r k^{-(r+1)/2}}{r!} \sum_{p=0}^{\infty} \frac{\Gamma(3/2 + p)t^{k-p-3/2}}{(2k - 2p - r)!(2p + 1)!}
\]

\[
= \frac{\pi^{1/2}}{4(1+\omega)^{1/2}} \left( 1 + \frac{3 + 4\omega}{3(1+\omega)} \right)
\]
With $\omega = 0$ the expansion for round $S^3$ is reproduced.

As is known the divergencies in the one-loop effective action for even-dimensional spaces lead to scale symmetry breaking and give rise to a nonvanishing conformal anomaly. The conformal anomaly has a geometrical structure and is expressed by means of $a_N/2$. In our case $a_n$ depend on the deformation parameters and can be equal to zero with the appropriate parametric values.

Let us consider the one-loop effective action for scalar fields on $R^m \times \tilde{S}^2$ where $R^m$ is Euclidean $m$-dimensional space. The conformal anomaly arises when we take the expectation value of the momentum-energy tensor $T_{\mu\nu}$ with the metric as a background classical field

$$< T_{\mu\nu} > = \frac{g_{\mu\nu}}{Z[g]} \frac{\delta Z[g]}{\delta g_{\mu\nu}}$$

where $Z[g]$ is the generating functional of the theory. Zeta-function regularization gives

$$< T_{\mu\nu} > = \frac{1}{(4\pi)^{m+2}/2} a_{(m+2)/2}$$

From (10),(11),(8) one can compute that the anomaly (17) for scalar fields on $R^4 \times \tilde{S}^2$ and $R^6 \times \tilde{S}^2$ is removed with the values $\rho = 0.41$ and $\rho = 0.51$ respectively. The Casimir energy is finite for these spaces and can be computed explicitly. (Now this problem is under consideration). For scalar fields on the 4-dimensional space $R^1 \times SU(2) \times U(1)/U(1)$ the anomaly

$$< T_{\mu\nu} > = \frac{1}{(4\pi)^2} \frac{32\omega^2 + 40\omega + 15}{30(1 + \omega)^2}$$

can not be removed with any value of $\omega$.

It should be noted that different type of deformed spheres have been considered previously in multidimensional models [15]. However, only small one-parameter deformations have been used for the calculation of the one-loop potential. In our case the deformation removing the conformal anomaly can not be considered small.

Note added

The manifolds with singular points were also studied in the context of orbifold factors of spheres, and flat conical spaces. The corresponding references can be found in the papers [16],[17]. One of the us (D.V) is grateful to Guido Cognola for
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