Review of the time-symmetric hyperincursive discrete harmonic oscillator separable into two incursive harmonic oscillators with the conservation of the constant of motion

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Abstract. This paper deals with a review of the properties of the hyperincursive discrete harmonic oscillator separable into two incursive discrete harmonic oscillators. We begin with a presentation step by step of the second order discrete harmonic oscillator. Then the 4 incursive discrete equations of the hyperincursive discrete harmonic oscillator are presented. The constants of motion of the two incursive discrete harmonic oscillators are analyzed. After that, we give a numerical simulation of the incursive discrete harmonic oscillator. The numerical values correspond exactly to the analytical solutions. Then we present the hyperincursive discrete harmonic oscillator. And we give also a numerical simulation of the hyperincursive discrete harmonic oscillator. The numerical values correspond also to the analytical solutions. Finally, we demonstrate that a rotation on the position and velocity variables of the incursive discrete harmonic oscillators gives rise to a pure quadratic expression of the constant of motion which is an ellipse. This result is fundamental because it gives an explanation of the effect of the discretization of the time in discrete physics. The information obtained from the incursive and hyperincursive discrete equations is richer than the information obtained by continuous physics. In conclusion, we have shown the temporal discretization of the harmonic oscillator produces a rotation similarly to the formalism of the special relativity dealing with rotations.

1. Introduction
This paper deals with a review of the properties of the hyperincursive discrete equation separable into incursive discrete equations.

This paper is a continuation of my two papers presented at the IXth and Xth Symposia on unified field mechanics, honoring noted French mathematical physicist Jean-Pierre Vigier.

The first paper [1] concerns the hyperincursive algorithms of classical harmonic oscillator applied to quantum harmonic oscillator separable into incursive oscillators.

The second paper [2] deals a unified discrete mechanics given by the bifurcation of the hyperincursive discrete harmonic oscillator, the hyperincursive discrete Schrödinger quantum equation, the hyperincursive discrete Klein-Gordon equation and the Dirac quantum relativist equations.
In my second Vigier paper [2], I have demonstrated that the second order hyperincursive discrete Klein-Gordon equation bifurcates to the 4 Dirac first order equations, in one space dimension. The temporal Klein-Gordon is similar in mathematical form to the hyperincursive discrete classical harmonic oscillator.

A good introduction to incursion and hyperincursion is given in the following series of papers on the total incursive control of linear, non-linear and chaotic systems [3], on computing anticipatory systems with incursion and hyperincursion [4], on the computational derivation of quantum and relativist systems with forward-backward space-time shifts [5], on a review of incursive, hyperincursive and anticipatory systems, with the foundation of anticipation in electromagnetism [6], then, on the precision and stability analysis of Euler, Runge-Kutta and incursive algorithms for the harmonic oscillator [7], and finally, on the new concept of deterministic anticipation in natural and artificial systems [8].

I wrote a series of theoretical papers on the discrete physics with Adel Antippa on the harmonic oscillator via the discrete path approach [9], on anticipation, orbital stability, and energy conservation in discrete harmonic oscillators [10], on the dual incursive system of the discrete harmonic oscillator [11], on the superposed hyperincursive system of the discrete harmonic oscillator [12], on the incursive discretization, system bifurcation, and energy conservation [13], on the hyperincursive discrete harmonic oscillator [14], on the synchronous discrete harmonic oscillator [15], on the discrete harmonic oscillator, a short compendium of formulas [16], on the time-symmetric discretization of the harmonic oscillator [17], and finally, on the discrete harmonic oscillator, evolution of notation and cumulative erratum [18]. This discrete physics is based on the fundamental mathematical development of the hyperincursive and incursive discrete harmonic oscillator.

This paper is organized as follows.
Section 2 deals with a presentation step by step of the second order discrete harmonic oscillator. Section 3 develops the 4 incursive discrete equations of the hyperincursive discrete harmonic oscillator. Then section 4 presents the constant of motion of the hyperincursive discrete harmonic oscillator. In section 5, we give a numerical simulation of the incursive discrete harmonic oscillator. Section 6 presents the hyperincursive discrete harmonic oscillator. Section 7 gives a numerical simulation of the hyperincursive discrete harmonic oscillator.

Finally, section 8 deals with a rotation on the position and velocity variables of the incursive discrete harmonic oscillators that gives rise to a pure quadratic expression of the constant of motion.

This result is fundamental because it gives an explanation of the effect of the discretization of the time in discrete physics. The information obtained from the hyperincursive discrete equations is richer than obtained by continuous physics. We have shown the temporal discretization of the harmonic oscillator produces a rotation like in the formalism of the special relativity.

2. Presentation step by step of the second order hyperincursive discrete harmonic oscillator
The harmonic oscillator is represented by the second order temporal ordinary differential equations

\[ \frac{d^2 x(t)}{dt^2} = -\omega^2 x(t) \]  \hspace{1cm} (1a)

with the velocity given by

\[ v(t) = \frac{dx(t)}{dt} \]  \hspace{1cm} (1b)

where \( x(t) \) is the position and \( v(t) \) the velocity as functions of the time \( t \), and where the pulsation \( \omega \) is related to the spring constant \( k \) and the oscillating \( m \) by

\[ \omega^2 = k/m \]  \hspace{1cm} (1c)

The harmonic oscillator can be represented by the two ordinary differential equations:

\[ dx(t)/dt = v(t) \]
\[ dv(t)/dt = -\omega^2 x(t) \]  \hspace{1cm} (2a,b)
The solution is given by
\[
x(t) = x(0) \cos(\omega t) + [v(0)/\omega] \sin(\omega t)
\]
\[
v(t) = -\omega x(0) \sin(\omega t) + v(0) \cos(\omega t)
\]
with the initial conditions \(x(0)\) and \(v(0)\).

In the phase space, given by \(x(t), v(t)\), the solutions are given by closed curves (orbital stability).

The period of oscillations is given by \(T = 2\pi/\omega\).

The energy \(E(t)\) of the harmonic oscillator is constant and is given by
\[
E(t) = \frac{1}{2} m v^2(t)/2 + \frac{1}{2} m v^2(0)/2 = E(0) = e_0
\]

The harmonic oscillator is computable by recursive functions from the discretization of the differential equations. The differential equations of the harmonic oscillator depend on the current time. In the discrete form, there are the discrete current time \(t\) and the interval of time \(\Delta t = h\).

The discrete time is defined as: \(t_k = t_0 + kh, k = 0,1,2,\ldots\)

where \(t_0\) is the initial value of the time and \(k\) is the counter of the number interval of time \(h\).

The discrete position and velocity variables are defined as \(x(k) = x(t_k)\) and \(v(k) = v(t_k)\).

The discrete equations consist in computing firstly the first equation to obtain, \(x(k+1)\), and then compute the second equation in using the just computed, \(x(k+1)\), as follows

\[
x(k+1) = x(k) + h v(k)
\]
\[
v(k+1) = v(k) - h \omega x(k + 1)
\]

In fact, the first equation used the forward derivative and the second equation used the backward derivative,

\[
[v(k+1) - v(k)]/h = -\omega^2 x(k)
\]

The position, \(x(k+1)\), and the velocity, \(v(k)\), are not computed at the same time step. I called such a system, an incursive system, for inclusive or implicit recursive system, e.g. [4].

A second possibility occurs if the second equation is firstly computed, and then the first equation is computed in using the just computed, \(v(k+1)\), as follows

\[
v(k+1) = v(k) - h \omega x(k)
\]
\[
x(k+1) = x(k) + h v(k + 1)
\]

In fact, the first equation used the forward derivative and the second equation used the backward derivative,

\[
[v(k+1) - v(k)]/h = -\omega^2 x(k)
\]

The position, \(v(k+1)\), and the velocity, \(v(k)\), are not computed at the same time step. But in using the two incursive systems, we see that the position in the first incursion, \(x(k+1)\), corresponds to the velocity in the second incursion, \(v(k+1)\), at the same time step, \((k+1)\). And similarly, we see that the velocity in the first incursion, \(v(k)\), corresponds to the position in the second incursion, \(x(k)\), at the same time step \(k\). So, both incursions give two successive positions and velocities at two successive time steps, \(k\) and \(k+1\).

In the following paragraphs, it will be given a generalized equation that integrates both incursions to form a hyperincursive system.

An important difference between the incursive and the recursive discrete systems is the fact that in the incursive system, the order in which the computations are made is important: this is a sequential
computation of equations. In the recursive systems, the order in which the computations are made is without importance: this is a parallel computation of equations.

The two incursive harmonic oscillators are numerically stable, contrary to the classical recursive algorithms like the Euler and Runge-Kutta algorithms [7].

In my paper [3], I defined a generalized forward-backward discrete derivative

\[ D_w = w D_f + (1 - w) D_b \] (6)

where \( w \) is a weight taking the values between 0 and 1, and where the discrete forward and backward derivatives on a function \( f \) are defined by

\[ D_f(f) = \frac{\Delta f}{\Delta t} = \frac{f(k+1) - f(k)}{h} \]
\[ D_b(f) = \frac{\Delta^{-1} f}{\Delta t} = \frac{f(k) - f(k-1)}{h} \] (7a,b)

The generalized incursive discrete harmonic oscillator is given by [Dubois, 1995] as:

\[ (1 - w) x(k+1) + (2w - 1) x(k) - w x(k-1) = h v(k) \]
\[ w v(k+1) + (1 - 2w) v(k) + (w - 1) v(k-1) = -h \omega^2 x(k) \] (8a,b)

When \( w = 0 \), \( D_0 = D_b \), this gives the first incursive equations:

\[ x(k+1) - x(k) = h v(k) \]
\[ v(k) - v(k-1) = -h \omega^2 x(k) \] (9a,b)

When \( w = 1 \), \( D_1 = D_f \), this gives the second incursive equations:

\[ x(k) - x(k-1) = h v(k) \]
\[ v(k+1) - v(k) = -h \omega^2 x(k) \] (10a,b)

When \( w = 1/2 \), \( D_{1/2} = \frac{[D_f + D_b]}{2} \), this gives the hyperincursive equations:

\[ x(k+1) - x(k-1) = +2h v(k) \]
\[ v(k+1) - v(k-1) = -2h \omega^2 x(k) \] (11a,b)

where the discrete derivative is given by

\[ D_s = D_{1/2} = \frac{[D_f + D_b]}{2} \]
\[ D_s(f) = \frac{[f(k+1) - f(k-1)]}{2h} \] (11c)

that defines a time-symmetric derivative noted \( D_s \).

**NB:** the time-symmetric derivative \( D_s \) in hyperincursive discrete equations

\[ D_s(f) = \frac{[f(k+1) - f(k-1)]}{2h} \]

is not the same as the classical central derivative \( D_c \) given in classical difference equations theory,

\[ D_c(f) = \frac{[f(k+1/2) - f(k-1/2)]}{h}. \]

These equations (11a,b) integrate the two incursive equations [4-6].

Let us remark that this first hyperincursive equation (11a) can be also obtained by adding the equation (9a) to the equation (10a), and the second hyperincursive equation (11b) by adding the equation (9b) to the equation (10b).

In putting the velocity, \( v(k) \), of the first equation (11a)

\[ v(k) = \frac{x(k+1) - x(k-1)}{2h} \] (12a)

to the second equation (11b),
\[ x(k+2) - 2x(k) + x(k-2) = -4h^2\omega^2x(k) \]  

(12b)

one obtains what I called the second order hyperincursive discrete harmonic oscillator, corresponding to the second order differential equations of the harmonic oscillator given by equation (1a)

\[ \frac{d^2x(t)}{dt^2} = -\omega^2x(t) \]

with the velocity given by the equation (1b)

\[ v(t) = \frac{dx(t)}{dt} \]

In conclusion of this section, we have presented the second order hyperincursive discrete harmonic oscillator given by the equation (12c)

\[ x(k+2) - 2x(k) + x(k-2) = -4h^2\omega^2x(k) \]  

(12c)

that is separable into 4 first order incursive discrete equations of the harmonic oscillator.

The next section will present the 4 dimensionless incursive discrete equations of the hyperincursive harmonic oscillator.

3. The 4 incursive discrete equations of the hyperincursive discrete harmonic oscillator

For the discrete harmonic oscillator, let us use the dimensionless variables, \( X \) and \( V \), of Antippa and Dubois [16], for the variables, \( x \) and \( v \), as follows:

\[
X(k) = \left[\frac{k}{2}\right]^{1/2} x(k)
\]

\[
V(k) = \left[\frac{m}{2}\right]^{1/2} v(k)
\]

(13a,b)

with the dimensionless time

\[
\tau = \omega t
\]

(14a)

where the pulsation (1c) is given by

\[
\omega = \left[\frac{k}{m}\right]^{1/2}
\]

(14b)

and with the dimensionless interval of time given by

\[
\Delta\tau = \omega \Delta t = \omega h = H
\]

(15)

So, the two incursive dimensionless harmonic oscillators are given by the following 4 first order discrete equations:

First Incursive Oscillator, from the dimensionless equations (4a,b):

\[
X_1(k + 1) = X_1(k) + HV_1(k)
\]

\[
V_1(k + 1) = V_1(k) - HX_1(k + 1)
\]

(16a,b)

Second Incursive Oscillator, from the dimensionless equations (5a,b):

\[
V_2(k + 1) = V_2(k) - HX_2(k)
\]

\[
X_2(k + 1) = X_2(k) + HV_2(k + 1)
\]

(17a,b)

These incursive discrete oscillators are non-recursive computing anticipatory systems.

Indeed, in equation (16b) of the first incursive oscillator, the velocity, \( V_1(k + 1) \), at the future next time step, \( (k + 1) \), is computed from the velocity, \( V_1(k) \), at the current time step, \( k \), and the position, \( X_1(k + 1) \), at the future next time step, \( (k + 1) \), which represents an anticipatory system represented by an anticipation of one time step, \( k \).

Similarly, in equation (17b) of the second incursive oscillator, the position, \( X_2(k + 1) \), at the future next time step, \( (k + 1) \), is computed from the position, \( X_2(k) \), at the current time step, \( k \), and
the velocity, $V_2(k + 1)$, at the future next time step, $(k + 1)$, which represents an anticipatory system represented by an anticipation of one time step, $k$.

These two incursive discrete harmonic oscillators define a discrete hyperincursive harmonic oscillator given by four incursive discrete equations.

A complete mathematical development of incursive and hyperincursive systems was presented in a series of papers by Adel F. Antippa and Daniel M. Dubois on the harmonic oscillator via the discrete path approach [9], on anticipation, orbital stability, and energy conservation in discrete harmonic oscillators [10], on the dual incursive system of the discrete harmonic oscillator [11], on the superposed hyperincursive system of the discrete harmonic oscillator [12], on the incursive discretization, system bifurcation, and energy conservation [13], on the hyperincursive discrete harmonic oscillator [14], on the synchronous discrete harmonic oscillator [15], on the discrete harmonic oscillator, a short compendium of formulas [16], on the time-symmetric discretization of the harmonic oscillator [17], and finally, on the discrete harmonic oscillator, evolution of notation and cumulative erratum [18].

In the next section we will present the constant of motion of the discrete harmonic oscillator.

### 4. The constant of motion of the hyperincursive discrete harmonic oscillator

The constant of motion of the first incursive equations (16a,b) is given by

\[
K_1(k) = X_1^2(k) + V_1^2(k) + H X_1(k)V_1(k) = K_1 = \text{constant} \quad (18a)
\]

**Theorem 1:** The expression $K_1(k) = X_1^2(k) + V_1^2(k) + H X_1(k)V_1(k)$ is a constant of motion of the first incursive equations (16a,b).

**Proof:** Multiply the first equation (16a) by $X_1(k + 1)$ at right and the second equation (16b) by $V_1(k + 1)$ at left, then add the two equations, and one obtains successively

\[
K_1(k + 1) = X_1(k + 1) X_1(k + 1) + V_1(k + 1)V_1(k + 1) + H X_1(k + 1)V_1(k + 1)
\]

\[
= X_1(k) X_1(k) + H X_1(k)V_1(k) + H X_1(k + 1)V_1(k) + V_1(k)V_1(k) - H V_1(k) X_1(k + 1)
\]

\[
= X_1(k) X_1(k) + H X_1(k)V_1(k) + V_1(k)V_1(k) = K_1(k) = K_1 = \text{constant}
\]

So, the expression is constant because the expression is invariant in two successive temporal steps.

In replacing the expression of the velocity $V_1(k)$ from equation (16a) to the $H$ term in equation (18a), the term depending on $H$ disappears, as follows

\[
X_1(k)X_1(k) + V_1(k)V_1(k) = X_1(k)[X_1(k + 1) - X_1(k)] = K_1
\]

or

\[
X_1(k)X_1(k + 1) + V_1(k)V_1(k) = K_1 \quad (18b)
\]

which looks like the conservation of the energy.

The constant of motion of the second incursive oscillator

\[
V_2(k + 1) = V_2(k) - H X_2(k)
\]

\[
X_2(k + 1) = X_2(k) + H V_2(k + 1)
\]

is given by
Theorem 2: The expression
\[ K_2(k) = X_2^2(k) + V_2^2(k) - H X_2(k) V_2(k) = K_2 = \text{constant} \] (19a)

is a constant of motion of the second incursive equations (17a,b).

Proof: Multiply the first equation (17a) by \( V_2(k + 1) \) at right and the second equation (17b) by \( X_2(k + 1) \) at left, then add the two equations, and one obtains successively
\[
K_2(k + 1) = X_2(k + 1) X_2(k + 1) + V_2(k + 1) V_2(k + 1) - H X_2(k + 1) V_2(k + 1)
\]
\[
= X_2(k) X_2(k) - H X_2(k) V_2(k) + V_2(k) V_2(k + 1)
\]
\[
= X_2(k) X_2(k) - H X_2(k) V_2(k) + V_2(k) V_2(k) = K_2(k) = K_2 = \text{constant}
\]

So, the expression is constant because the expression is invariant in two successive temporal steps.

In replacing the expression of the position \( X_2(k) \) from equation (17a) to the \( H \) term in equation (19a), the term depending on \( H \) disappears
\[
X_2(k) X_2(k) + V_2(k) V_2(k) - [V_2(k) - V_2(k + 1)] V_2(k) = K_2
\]
or
\[
X_2(k) X_2(k) + V_2(k + 1) V_2(k) = K_2
\]
that also looks like the conservation of the energy.

These constants of motion (18a) and (19a) differ with the inversion of the sign of \( H \), as follows
\[
+H = +\omega h = +\omega \Delta t,
\]
\[
-H = -\omega h = -\omega \Delta t
\]
because the inversion of the discrete time interval of the first incursion gives the second incursion.

NB: It is very important to notice that there is a fundamental difference between an inversion of the sign of the discrete time, \( \Delta t \), in the discrete equations and an inversion of the sign of the continuous time, \( t \), in the differential equations.

Let us now consider a simple example of the solution of the discrete position and the discrete velocity of the dimensionless discrete harmonic oscillator, given by the following analytical solution (synchronous solution)
\[
X_1(k) = \cos(k\pi/N)
\]
\[
V_1(k) = -\sin((k + 1)\pi/N)
\]
and
\[
X_2(k) = \cos((k + 1)\pi/N)
\]
\[
V_2(k) = -\sin(k\pi/N)
\]
where \( N \) is the number of iterations for a cycle of the oscillator, with the index of iterations \( k = 0, 1, 2, 3, \ldots \).

The interval of discrete time \( H \) depends of \( N \) (for a synchronous solution):
\[
H = 2 \sin(\pi/N)
\]

For \( N = 6 \), for example
\[
H = 2 \sin(\pi/6) = 1
\]

The two constants of motion, with the solutions (21a,b) and (22a,b) are given by
\[
\cos^2(k\pi/N) + \sin^2((k+1)\pi/N) - H \cos(k\pi/N)\sin((k+1)\pi/N) = K_1
\]
\[
\cos^2((k+1)\pi/N) + \sin^2(k\pi/N) + H \cos((k+1)\pi/N)\sin(k\pi/N) = K_2
\]

For \(N = 6, H = 1, k = 0\), one obtains the same constant of motion for the two incursive oscillators:

\[
\begin{align*}
\cos^2(0) + \sin^2(\pi/6) - \cos(0)\sin(\pi/6) &= 1.0 + 0.25 - 0.5 = 0.75 = K_1 \\
\cos^2(\pi/6) + \sin^2(0) + \cos(\pi/6)\sin(0) &= 0.75 + 0.0 + 0.0 = 0.75 = K_2
\end{align*}
\]

And the averaged energy is a constant given by

\[
[E_1(k) + E_2(k)]/2 = [X_1^2(k) + V_1^2(k) + X_2^2(k) + V_2^2(k)]
\]
\[
= [\cos^2((k+1)\pi/N) + \sin^2(k\pi/N) + \cos^2(k\pi/N) + \sin^2((k+1)\pi/N)]/2 = 1
\]

A very interesting and important invariant, \(INV_{12}\), is given by

\[
INV_{12} = X_1(k)X_2(k) + V_1(k)V_2(k) = \text{constant}
\]

With the values of the example, this gives a constant \(INV_{12} = X_1(k)X_2(k) + V_1(k)V_2(k) = \cos(k\pi/N)\cos((k+1)\pi/N) + \sin((k+1)\pi/N)\sin(k\pi/N) = \cos(\pi/N)\)

For \(N = 6\), \(INV_{12} = \cos(\pi/6) = 3^{1/2}/2 = 0.8660\)

For large value of \(N\),

\(INV_{12} \approx 1\)

Now in the next section, we will give a numerical simulation of the two incursive discrete harmonic oscillators in view of comparing with the analytical solutions that we have presented in this section.

5. Numerical simulation of the incursive discrete harmonic oscillator

This section gives the numerical simulation of the two incursive harmonic oscillators.

Firstly, the parameters for the simulation are given as follows.

The number of iterations,

\(N = 6\)

The interval of discrete time is then given by

\(H = 2 \sin(\pi/N) = 2 \sin(\pi/6) = 1\)

The boundary conditions

\(X_1(0) = \cos(0) = 1\)
\(V_1(0) = -\sin(\pi/6) = -0.5\)
The table 1A gives the simulation of the first incursive discrete equations (16a,b) of the harmonic oscillator.

Table 1A. Simulation of the incursive discrete equations (16a,b).

| N | H | k | $X_1(k)$ | $V_1(k)$ | $E_1(k)$ | $E_{F1}(k)$ | $K_1(k)$ |
|---|---|---|----------|----------|----------|-------------|----------|
| 6 | 1 | 0 | 1.000    | −0.500   | 1.25     | −0.50       | 0.75     |
| 1 |   | 1 | 0.500    | −1.000   | 1.25     | −0.50       | 0.75     |
| 2 |   | 2 | −0.500   | −0.500   | 0.50     | 0.25        | 0.75     |
| 3 |   | 3 | −1.000   | 0.500    | 1.25     | −0.50       | 0.75     |
| 4 |   | 4 | −0.500   | 1.000    | 1.25     | −0.50       | 0.75     |
| 5 |   | 5 | 0.500    | 0.500    | 0.50     | 0.25        | 0.75     |
| 6 |   | 6 | 1.000    | −0.500   | 1.25     | −0.50       | 0.75     |
| 7 |   | 7 | 0.500    | −1.000   | 1.25     | −0.50       | 0.75     |

NB: see the correspondence of the variables with the hyperincursive harmonic oscillator at table 3:

\[
X/g_{286} ≈(k) = X(2^k) \quad V/g_{286} ≈(k) = V(2^k + 1)
\]

In the table 1A, we give the energy

\[
E_1(k) = X_1^2(k) + V_1^2(k)
\]

the forward energy

\[
E_{F1}(k) = +HX_1(k)V_1(k)
\]

and the constant of motion

\[
K_1(k) = X_1^2(k) + V_1^2(k) + HX_1(k)V_1(k) = K_1 = \text{constant}
\]

The numerical values correspond exactly to the analytical solutions

\[
X_1(k) = \cos(k\pi/N) \\
V_1(k) = −\sin((k + 1)\pi/N)
\]

Secondly, the parameters for the simulation are given as follows.

The number of iterations,

\[
N = 6
\]

The interval of discrete time is then given by

\[
H = 2 \sin(\pi/N) = 2 \sin(\pi/6) = 1
\]

The boundary conditions,

\[
X_2(0) = \cos(\pi/6) = (3/4)^{1/2} = 0.8660 \\
V_2(0) = −\sin(0) = 0
\]

Table 1B gives the simulation of the second incursive discrete equations (17a,b) of the harmonic oscillator.
Table 1B. Simulation of the discrete incursive equations (17a,b).

| N | H | k | X₂(k) | V₂(k) | E₂(k) | Eₚ₂(k) | K₂(k) |
|---|---|---|-------|-------|-------|---------|-------|
| 6 | 1 | 0 | 0.866 | 0.000 | 0.75  | 0.00    | 0.75  |
| 1 | 0.000 | -0.866 | 0.75  | 0.00    | 0.75  |
| 2 | -0.866 | -0.866 | 1.50  | -0.75  | 0.75  |
| 3 | -0.866 | 0.000 | 0.75  | 0.00    | 0.75  |
| 4 | 0.000 | 0.866 | 0.75  | 0.00    | 0.75  |
| 5 | 0.866 | 0.866 | 1.50  | -0.75  | 0.75  |
| 6 | 0.866 | 0.000 | 0.75  | 0.00    | 0.75  |
| 7 | 0.000 | -0.866 | 0.75  | 0.00    | 0.75  |

NB: see the correspondence of the variables with the hyperincursive harmonic oscillator at table 3:

\[ X_2(k) = X(2k + 1), \ V_2(k) = V(2k) \] (33)

In table 1B, we give the energy

\[ E_2(k) = X_2^2(k) + V_2^2(k) \]

the backward energy

\[ E_{p2}(k) = - H X_2(k) V_2(k) \]

and the constant of motion

\[ K_2(k) = X_2^2(k) + V_2^2(k) - H X_2(k) V_2(k) = K_2 = \text{constant} \] (19a)

The numerical values correspond exactly to the analytical solutions

\[ X_2(k) = \cos((k+1)\pi/N) \]
\[ V_2(k) = - \sin(k\pi/N) \] (22a,b)

In the next section, we define the hyperincursive discrete harmonic oscillator.

6. The hyperincursive discrete harmonic oscillator

For the hyperincursive discrete harmonic oscillator, given by the equations (11a,b), we use the dimensionless variables, \(X\) and \(V\), for the variables, \(x\) and \(v\), as follows:

\[ X(k) = [k/2]^{1/2} x(k) \]
\[ V(k) = [m/2]^{1/2} v(k) \] (13a,b)

with the dimensionless time

\[ \tau = \omega t \] (14a)

where the pulsation (1c) is given by

\[ \omega = [k/m]^{1/2} \] (14b)

and with the dimensionless interval of time given by

\[ \Delta \tau = \omega \Delta t = \omega h = H \] (15)

So, the two equations (11a,b) are then transformed to the following two dimensionless equations of the hyperincursive discrete harmonic oscillator

\[ X(k + 1) = X(k - 1) + 2HV(k) \] (34a)
\[ V(k + 1) = V(k - 1) - 2H X(k) \]

for \( k = 1,2,3,... \), with the 4 even and odd boundary conditions \( X(0), V(1), V(0), X(1) \).

This hyperincursive discrete harmonic oscillator is a recursive computing system that is separable into two independent incursive discrete harmonic oscillators, as shown in table 2A and table 2B.

Table 2A gives the first iterations of the hyperincursive discrete equations (34a,b).

It is well-seen that there are two independent series of iterations defining two incursive discrete harmonic oscillators, as given in table 2B.

Table 2A. This table gives the first iterations of the hyperincursive discrete equations (34a,b).

| HYPERINCURSIVE DISCRETE HARMONIC OSCILLATOR | V(k + 1) = V(k - 1) - 2H X(k) |
|---------------------------------------------|---------------------------------|
| X(k + 1) = X(k - 1) + 2H V(k)               |                                  |
| Boundary conditions: X(0) = C_1, V(1) = C_2, V(0) = C_3, X(1) = C_4 |

| k | Iterations |
|---|------------|
| 1 | X(2) = X(0) + 2H V(1) | V(2) = V(0) - 2H X(1) |
| 2 | X(3) = X(1) + 2H V(2) | V(3) = V(1) - 2H X(2) |
| 3 | X(4) = X(2) + 2H V(3) | V(4) = V(2) - 2H X(3) |
| 4 | X(5) = X(3) + 2H V(4) | V(5) = V(3) - 2H X(4) |
| 5 | X(6) = X(4) + 2H V(5) | V(6) = V(4) - 2H X(5) |
| 6 | X(7) = X(5) + 2H V(6) | V(7) = V(5) - 2H X(6) |
| ... | --- | --- |

Table 2B. This table shows the two independent incursive discrete harmonic oscillators.

| FIRST INCURSIVE DISCRETE HARMONIC OSCILLATOR | SECOND INCURSIVE DISCRETE HARMONIC OSCILLATOR |
|---------------------------------------------|-----------------------------------------------|
| Boundary conditions: X(0) = C_1, V(1) = C_2 | Boundary conditions: V(0) = C_3, X(1) = C_4 |

| k | Iterations | Iterations |
|---|------------|------------|
| 1 | X(2) = X(0) + 2H V(1) | V(2) = V(0) - 2H X(1) |
| 2 | V(3) = V(1) - 2H X(2) | X(3) = X(1) + 2H V(2) |
| 3 | X(4) = X(2) + 2H V(3) | V(4) = V(2) - 2H X(3) |
| 4 | V(5) = V(3) - 2H X(4) | X(5) = X(3) + 2H V(4) |
| 5 | X(6) = X(4) + 2H V(5) | V(6) = V(4) - 2H X(5) |
| 6 | V(7) = V(5) - 2H X(6) | X(7) = X(5) + 2H V(6) |
| ... | --- | --- |

As well-seen in table 2B, the first incursive harmonic oscillator, with the boundary conditions, X(0), V(1), is given by

\[ X(2k) = X(2k - 2) + 2HV(2k - 1) \]
\[ V(2k + 1) = V(2k - 1) - 2HX(2k) \]  

(35a,b)

and the second incursive harmonic oscillator, with the boundary conditions, V(0), X(1), is given by

\[ V(2k) = V(2k - 2) - 2HX(2k - 1) \]
\[ X(2k + 1) = X(2k - 1) + 2HV(2k) \]  

(36a,b)
for \( k = 1, 2, 3, \ldots \)

Let us remark that the difference between the two incursive oscillators, given by eqs. (16a,b; 17a,b) and (35a,b; 36a,b), holds in the labeling of the successive time steps. In the incursive oscillators, (16a,b; 17a,b), the position and velocity are computed at the same time step while in the incursive oscillators, (35a,b; 36a,b), the position and the velocity are computed at successive time steps, but the numerical simulations of both give the same values.

Each incursive oscillator is the discrete time inverse, \(+\Delta t \rightarrow -\Delta t\) and, \(-\Delta t \rightarrow +\Delta t\) of the other incursive oscillator, defined by time forward and time backward derivatives. So the two incursive oscillators are not reversible. But the superposition of the two incursive oscillators given by the hyperincursive discrete oscillator is reversible.

In putting the expression of \( V(k) \) from the equation (34a)
\[
V(k) = \frac{[X(k+1) - X(k-1)]}{2H}
\]  
(37)
to the equation (34b), one obtains the second order hyperincursive discrete harmonic oscillator
\[
X(k+2) - 2X(k) + X(k-2) = -4H^2X(k)
\]  
(38)
With the dimensionless variables, the dimensionless energy is given by
\[
E(k) = X^2(k) + V^2(k)
\]  
(39)
Now, in the next section, we will give an example of numerical simulations of the incursive discrete harmonic oscillators.

7. Numerical simulation of the hyperincursive discrete harmonic oscillator

This section gives the numerical simulation of the hyperincursive discrete harmonic oscillator.

The parameters for the simulation are given as follows. The number of iterations,
\[
N = 12
\]  
(40)
The interval of discrete time is then given by
\[
H = \sin(2\pi/N) = \sin(\pi/6) = 0.5
\]  
(41)
NB: When \( N \) is large,
\[
H = \sin(2\pi/N) \approx 2\pi/N = \omega \Delta t = 2\pi\Delta t/T
\]  
(42)
so, the period \( T \) of the harmonic oscillator is
\[
T = 2\pi/\omega = N\Delta t
\]  
(43)
The boundary conditions are given by
\[
X(0) = C_1 = \cos(0) = 1
\]
\[
V(1) = C_2 = -\sin(\pi/6) = -0.5
\]
\[
V(0) = C_3 = -\sin(0) = 0
\]
\[
X(1) = C_4 = \cos(\pi/6) = (3)^{1/2}/2 = 0.8660
\]  
(44a,b,c,d)

Table 3 gives the simulation of the hyperincursive discrete equations (34a,b) of the harmonic oscillator.
Table 3. Numerical simulation of the equations (34a,b)

| N  | H  | k   | X(k)   | V(k)   | E(k)   |
|----|----|-----|--------|--------|--------|
| 12 | 0.5| 0   | 1.0000 | 0.0000 | 1.0000 |
|    |    | 1   | 0.8660 | -0.5000| 1.0000 |
|    |    | 2   | 0.5000 | -0.8660| 1.0000 |
|    |    | 3   | 0.0000 | -1.0000| 1.0000 |
|    |    | 4   | -0.5000| -0.8660| 1.0000 |
|    |    | 5   | -0.8660| -0.5000| 1.0000 |
|    |    | 6   | -1.0000| 0.0000  | 1.0000 |
|    |    | 7   | -0.8660| 0.5000  | 1.0000 |
|    |    | 8   | -0.5000| 0.8660  | 1.0000 |
|    |    | 9   | 0.0000 | 1.0000  | 1.0000 |
|    |    | 10  | 0.5000 | 0.8660  | 1.0000 |
|    |    | 11  | 0.8660 | 0.5000  | 1.0000 |
|    |    | 12  | 1.0000 | 0.0000  | 1.0000 |
|    |    | 13  | 0.8660 | -0.5000 | 1.0000 |

Let us remark that this hyperincursive discrete harmonic oscillator represents alternatively the values of the two incursive harmonic oscillators, given at table 1A and table 1B, with the following correspondence:

\[ X(k) = \cos\left(\frac{2k\pi}{N}\right) \]
\[ V(k) = -\sin\left(\frac{2k\pi}{N}\right) \]

In table 3, we give also the energy

\[ E(k) = X^2(k) + V^2(k) \]

At table 3, the simulation of the equations (34a,b) of the hyperincursive discrete harmonic oscillator gives exactly the values of the analytical solutions

\[ X(k) = \cos(2k\pi/N) \]
\[ V(k) = -\sin(2k\pi/N) \]

Where the conservation of energy is, \( E(k) = 1 \)

Finally, in the next section, we will present a new derivation of recursive discrete harmonic oscillator based on a rotation of the incursive discrete harmonic oscillator.

8. Rotation of the incursive harmonic oscillators to recursive discrete harmonic oscillators

In the expression of the constant of motion of the first incursive harmonic oscillator, a rotation on the position and velocity variables gives rise to a pure quadratic expression of the constant of motion, similarly to the constant of energy of the classical continuous harmonic oscillator.

The constant of motion (18a)

\[ X_1(k)X_1(k) + H X_1(k)V_1(k) + V_1(k)V_1(k) = K_1 \]

is an expression of a quadratic curve

\[ Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \]

with \( A = 1, B = H, C = 1, D = 0, E = 0, F = -K_1 \)

\[ x = X_1(k), \ y = V_1(k) \]
The quantity
\[ \Delta = B^2 - 4AC = \text{INV} \]  \hspace{1cm} (49)

is an invariant under rotations and is known as the discriminant of equation (47).

The discriminant of the constant of motion is given by
\[ \Delta = B^2 - 4AC = H^2 - 4 < 0 \]  \hspace{1cm} (50)

which defines an ellipse.

NB: This inequality gives the maximum value of the discrete interval of time
\[ H = \omega \Delta t < 2 \]  \hspace{1cm} (51)

and this is exactly the maximum value for the discrete harmonic oscillator:
\[ H = 2 \sin(\pi/N) \]  \hspace{1cm} (52)

The equations for the rotation are given by
\[ X/g_{286}(k) = \cos(\theta) u/g_{286}(k) - \sin(\theta) v/g_{286}(k) \]
\[ V/g_{286}(k) = \sin(\theta) u/g_{286}(k) + \cos(\theta) v/g_{286}(k) \]  \hspace{1cm} (53a,b)

With \( A = C \), the angle \( \theta \) is given by
\[ \theta = \pi/4, \]  \hspace{1cm} (54a)
so
\[ \cos(\pi/4) = 2^{-1/2} = \rho \]  \hspace{1cm} (54b)
and
\[ \sin(\pi/4) = 2^{-1/2} = \rho \]  \hspace{1cm} (54c)

With the equations (54b,c) the equations (53a,b) of the rotation transformed to
\[ X_{1}(k) = \rho(u_{1}(k) - v_{1}(k)) \]
\[ V_{1}(k) = \rho(u_{1}(k) + v_{1}(k)) \]  \hspace{1cm} (55a,b)

So the constant of motion becomes
\[ (u_{1}(k) - v_{1}(k))^2 + H(u_{1}(k) - v_{1}(k))(u_{1}(k) + v_{1}(k)) + (u_{1}(k) + v_{1}(k))^2 = 2K_{1} \]
\[ u_{1}^{2}(k) + v_{1}^{2}(k) - 2u_{1}(k)v_{1}(k) + Hu_{1}^{2}(k) - Hv_{1}^{2}(k) + u_{1}^{2}(k) + v_{1}^{2}(k) + 2u_{1}(k)v_{1}(k) = 2K_{1} \]  \hspace{1cm} (56a)
\[ u_{1}^{2}(k) + v_{1}^{2}(k) + H[u_{1}^{2}(k) - v_{1}^{2}(k)]/2 = K_{1}(k) = K_{1} \]

For the second incursion, the constant of motion is obtained by inversion the sign of \( H \):
\[ u_{2}^{2}(k) + v_{2}^{2}(k) - H[u_{2}^{2}(k) - v_{2}^{2}(k)]/2 = K_{2}(k) = K_{2} \]  \hspace{1cm} (56b)

that is also a pure quadratic function.

Now let us give the discrete equations of the first oscillator,
\[ X_{1}(k + 1) = X_{1}(k) + H V_{1}(k) \]
\[ V_{1}(k + 1) = V_{1}(k) - H X_{1}(k + 1) = V_{1}(k) - HX_{1}(k) - H^{2} V_{1}(k) \]  \hspace{1cm} (16a,b)

Let us make the rotation to the first Incursive Oscillator,
\[ \rho (u_1(k+1) - v_1(k+1)) = \rho (u_1(k) - v_1(k)) + H \rho (u_1(k) + v_1(k)) \]
\[ \rho (u_1(k+1) + v_1(k+1)) = \rho (u_1(k) + v_1(k)) - H \rho (u_1(k) - v_1(k)) - H^2 \rho (u_1(k) + v_1(k)) \]

Let us add the two equations
\[ 2\rho u_1(k+1) = 2\rho u_1(k) + 2H \rho v_1(k) - H^2 \rho (u_1(k) + v_1(k)) \]
and after division by \(2\rho\), we obtain the first rotated equation of the first incursive oscillator:
\[ u_1(k+1) = u_1(k) + H v_1(k) - H^2(u_1(k) + v_1(k))/2 \quad (57a) \]
Let us subtract the two equations
\[ -2\rho v_1(k+1) = -2\rho v_1(k) + 2H \rho u_1(k) - H^2 \rho (u_1(k) + v_1(k)) \]
and after division by \(-2\rho\), we obtain the second rotated equation of the first incursive oscillator:
\[ v_1(k+1) = v_1(k) - H u_1(k) - H^2(u_1(k) + v_1(k))/2 \quad (57b) \]

With a similar rotation, the two equations of the second incursive oscillator
\[ V_2(k+1) = V_2(k) - H X_2(k) \]
\[ X_2(k+1) = X_2(k) + H V_2(k+1) \quad (17a,b) \]
are transformed to
\[ v_2(k+1) = v_2(k) + H u_2(k) - H^2(u_2(k) + v_2(k))/2 \quad (58a) \]
\[ u_2(k+1) = u_2(k) - H v_2(k) - H^2(u_2(k) + v_2(k))/2 \quad (58b) \]

These equations are the same as the equations of the first oscillator by inversion of the sign of \(H\).

In conclusion, the 4 recursive equations of the discrete harmonic oscillator are given by
\[ u_1(k+1) = u_1(k) + H v_1(k) - H^2(u_1(k) + v_1(k))/2 \quad (59a) \]
\[ v_1(k+1) = v_1(k) - H u_1(k) - H^2(u_1(k) + v_1(k))/2 \quad (59b) \]
\[ u_2(k+1) = u_2(k) - H v_2(k) - H^2(u_2(k) + v_2(k))/2 \quad (60a) \]
\[ v_2(k+1) = v_2(k) + H u_2(k) - H^2(u_2(k) + v_2(k))/2 \quad (60b) \]
with the corresponding constant of motion
\[ u_1^2(k) + v_1^2(k) + H[u_1^2(k) - v_1^2(k)]/2 = K_1(k) = K_1 \quad (61) \]
\[ u_2^2(k) + v_2^2(k) - H[u_2^2(k) - v_2^2(k)]/2 = K_2(k) = K_2 \quad (62) \]

This result is fundamental because it gives an explanation of the effect of the discretization of the time in discrete physics. The information obtained from the hyperincursive discrete equations is richer than obtained by continuous physics. We have shown the temporal discretization of the harmonic oscillator produces a rotation like in the formalism of the relativity.

**9. Conclusion**

This paper deals with a review of the properties of the hyperincursive discrete harmonic oscillator separable into two incursive discrete harmonic oscillators.

We begin with a presentation step by step of the second order discrete harmonic oscillator. Then one develops the 4 incursive discrete equations of the hyperincursive discrete harmonic oscillator.

The constant of motion of the hyperincursive discrete harmonic oscillator is analyzed.

After that, we give a numerical simulation of the incursive discrete harmonic oscillator.

Then we present the hyperincursive discrete harmonic oscillator. And we give a numerical simulation of the hyperincursive discrete harmonic oscillator.
Finally, we demonstrate that a rotation on the position and velocity variables of the incursive discrete harmonic oscillators gives rise to a pure quadratic expression of the constant of motion. This result is fundamental because it gives an explanation of the effect of the discretization of the time in discrete physics. The information obtained from the hyperincursive discrete equations is richer than obtained by continuous physics. We have shown the temporal discretization of the harmonic oscillator produces a rotation like in the formalism of the special relativity.

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