Quarkonium spectra with a Klein-Gordon Oscillator as a confining potential

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Abstract

A quantum relativistic model is proposed to represent the confining potential of $c\bar{c}$ and $b\bar{b}$ mesons by using the Klein Gordon oscillator. We have solved the two-body problem obtaining the eigenvalues and the corresponding mass spectra, which fit very well with experimental data. We compare the results of our model with others obtained from different quark binding potentials.

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I. INTRODUCTION

The study of bound states of quark antiquark systems has been performed with different formalisms in the last two decades. These systems are a profitable testing ground for new ideas and methods to explain confinement and strong interaction dynamics. In particular, the mass spectrum for charmed mesons has largely been measured and modeled. Some time ago, Kang and Schnitzer [1] have proposed a model to simulate confinement between the constituents of mesons. These authors considered a linear potential (LP) in the Klein Gordon equation. In this way they obtained the mass spectrum for charmonium. Later on, Ram and Hasala [2] followed this procedure with a quadratic potential (QP). There are also some works in the literature, devoted to the study of quarkonium systems from a non-relativistic point of view, with different potentials. As an example of that Eichten et al. [3] have studied the charmonium family with linear plus Coulomb interaction as a confining potential. Furthermore, Quigg and Rosner [4] have studied the bound states of quarkonium with logarithmic, and power law potentials.

In this work we introduce an alternative relativistic quantum mechanical model which includes the Klein Gordon Oscillator in order to obtain the mass spectra for bound states of $\bar{b}b$ and $c\bar{c}$ mesons. We present a scheme that considers quarks as spinless particles, in the same spirit of ref. [1] and [2], which reproduces with high precision the charmonium and bottomonium mass spectra. The harmonic oscillator (HO) is a natural candidate to simulate the quark binding. It is one of the simplest potentials, well behaved, which takes into account asymptotic freedom and confinement. The Klein-Gordon oscillator (KGO) was introduced by Bruce and Minning [5]. This interaction gives the usual HO in the non-relativistic limit.

In this frame we have solved the two-body problem. This alternative relativistic treatment is very suitable to study bound states due to its simplicity. We compare our results for charmonium, with those obtained for linear and quadratic potentials in the Klein Gordon equation [1] [2]. Moreover, we have included in our analysis the values for the mass spectra for $c\bar{c}$ and $\bar{b}b$ mesons that we have obtained by considering the treatment for two
body systems in a quantum relativistic frame. This treatment was presented in a paper by Arshansky and Horwitz [6]. There, it is derived the mass spectra and the wave functions for the 4-D harmonic oscillator (4D) as a quantum relativistic two-body bound state, but the authors have not fit their mass spectrum with the systems we are interested on, something we shall do below.

The paper is organized as follows. In Section II we comment on the Klein-Gordon oscillator formalism. Section III is devoted to describe the model we propose. In Sect. IV we compare our results for charmonium with those obtained in refs. [1] [2], and with those that we have obtained considering the formalism in ref. [6]. We also present a table for the results of bottomonium masses within KGO and 4D models. Finally, we sketch our conclusions.

II. THE FORMALISM

In 1989 Moshinsky and Szczepaniak [7] proposed a new type of interaction in the Dirac equation. This potential is linear in both, momenta and coordinates, and the corresponding equation was named “Dirac Oscillator” because in the non-relativistic limit, the harmonic oscillator is obtained. Later on, this kind of interaction was also introduced in the Klein-Gordon (K-G) equation [5] by making the minimal substitution:

\[ \vec{P} \rightarrow \vec{P} - im \hat{\gamma} \hat{\Omega}.\vec{Q} \]

where \( m \) is the mass of the particle and \( \vec{P} = \hat{\eta}\vec{p}, \vec{Q} = \hat{\eta}\vec{q} \) are respectively, coordinate and momentum operators. The matrix \( \hat{\Omega} \) is chosen to be 3x3 with coefficients \( \hat{\Omega}_{ij} = \omega_i \delta_{ij} \). The constants \( \omega_i \) (with \( i = 1, 2, 3 \)) are the oscillator frequencies along the \( x, y, z \) axes. The matrix \( \hat{\gamma} \) satisfies the anticommutation relations:

\[ \{ \hat{\gamma}, \hat{\eta} \} = 0 \quad \hat{\gamma}^2 - \hat{\eta}^2 = 1. \]

Thus, the corresponding K-G equation (in natural units) for the wave function \( \Psi(\vec{q}, t) \) has the following form:
\[ \frac{\partial^2}{\partial t^2} \Psi(\vec{q}, t) = \left( \vec{p}^2 + m^2 \vec{q} \cdot \vec{r}^2 \vec{q} + m \gamma_3 r \Omega + m^2 \right) \Psi(\vec{q}, t). \]

In order to solve it, it is convenient to recast the K-G equation, and some ways to do this are shown in ref. [8]. One of these possibilities is the Sakata-Taketani approach [9] characterized by the equation:

\[ i \frac{\partial}{\partial t} \Phi = \left\{ \frac{\vec{p}(\tau_3 + i \tau_2)\vec{p}}{2m} + m \tau_3 \right\} \Phi \quad (2) \]

where \( \tau_i \) are Pauli matrices and \( \Phi \) has the form

\[ \Phi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}. \quad (3) \]

This two component wave function is related with \( \Psi \) as follows

\[ \begin{cases} \phi = \left( \Psi + \frac{i}{m} \partial_t \Psi \right) / 2 \\ \chi = \left( \Psi - \frac{i}{m} \partial_t \Psi \right) / 2. \end{cases} \quad (4) \]

In order to obtain the oscillator-like equation, in the isotropic case where \( \omega_1 = \omega_2 = \omega_3 \), the minimal coupling

\[ \vec{p} \rightarrow \vec{p} - im \omega \tau_1 \vec{r} \quad (5) \]

must be introduced in the K-G equation (2) giving:

\[ i \frac{\partial}{\partial t} \Phi = \left\{ \frac{(\vec{p} - im \omega \tau_1 \vec{r})(\tau_3 + i \tau_2)(\vec{p} - im \omega \tau_1 \vec{r})}{2m} + m \tau_3 \right\} \Phi \quad (6) \]

which yields

\[ [\Box + 2m \omega^2 \vec{r}^2 - 3m \omega + m^2] \Psi = 0. \quad (7) \]

The corresponding energy spectrum [8] is

\[ E^2 = 2m \omega (N_1 + N_2 + N_3) + m^2 \quad (8) \]

with \( N_1, N_2, N_3 = 0, 1... \)

On the basis of the results presented in this summary, our model can be introduced. Our purpose is to work out the two-body problem in the frame of the formalism we have commented.
III. THE MODEL

We propose that the above KG oscillator could model a quark antiquark confining potential taking into account some further considerations. In the case of one body KGO, we note from (5) that the oscillator interaction has its origin at 0. If one desires to change the origin to $r_0$, the replacement $r - r_0$ must be done. We are interested in studying the case of two interacting particles with the same mass. We call with subindex 1 and 2 the magnitudes corresponding to each particle. We propose a slight different ansatz when compared to (5). It reads,

$$\vec{p}_1 \rightarrow \vec{p}_1' = \vec{p}_1 - im\frac{\omega}{2}\tau_1(\vec{r}_1 - \vec{r}_2)$$

$$\vec{p}_2 \rightarrow \vec{p}_2' = \vec{p}_2 - im\frac{\omega}{2}\tau_1(\vec{r}_2 - \vec{r}_1),$$

giving rise to two coupled equations:

$$i \frac{\partial}{\partial t_1} \Phi = \left[ \frac{\vec{p}_1'(\tau_3 + i\tau_2)\vec{p}_1'}{2m} + m\tau_3 \right] \Phi$$

and

$$i \frac{\partial}{\partial t_2} \Phi = \left[ \frac{\vec{p}_2'(\tau_3 + i\tau_2)\vec{p}_2'}{2m} + m\tau_3 \right] \Phi.$$  

Summing the last two expressions, one obtains an equation which corresponds to the case of two particles:

$$i \frac{\partial}{\partial t_1} \Phi + i \frac{\partial}{\partial t_2} \Phi = \left[ \frac{\vec{p}_1'(\tau_3 + i\tau_2)\vec{p}_1'}{2m} + \frac{\vec{p}_2'(\tau_3 + i\tau_2)\vec{p}_2'}{2m} + 2m\tau_3 \right] \Phi.$$  

In quantum mechanics, the study of a two body system could be thought of as a free particle plus a particle in a central potential, which depends on the relative coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$. In order to solve our system we have made the following change of variables:

$$T = \frac{t_1 + t_2}{2}, \quad t = t_1 - t_2$$

and
\[ \vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2}, \quad \vec{r} = \vec{r}_1 - \vec{r}_2. \]  

(15)

The momenta \( p_1 \) and \( p_2 \) can be written as follows:

\[ p_1 = \frac{\vec{P}}{2} + \vec{p} \]  

(16)

\[ p_2 = \frac{\vec{P}}{2} - \vec{p} \]  

(17)

Shifting now to these new variables, eq.(13) gives:

\[ \left( \frac{\partial^2}{\partial T^2} + P^2 + 4p^2 + m^2 \omega^2 r^2 - 6m\omega + 4m^2 \right) \Phi(\vec{P}, \vec{p}, T). \]  

(18)

This equation can be solved by separation:

\[ \Phi(\vec{R}, \vec{r}, T) = \phi_1(\vec{R}, T) \phi_2(\vec{r}) \]  

(19)

giving rise to the following two equations

\[ \left( \frac{\partial^2}{\partial T^2} + P^2 + E^2 \right) \phi_1(\vec{R}, T) = (\Box + E^2) \phi_1(\vec{R}, T) \]  

(20)

and

\[ \left( \vec{p}^2 + \mu^2 \omega^2 r^2 - 3\mu \omega + 4\mu^2 - \frac{E^2}{4} \right) \phi_2(\vec{r}) = 0 \]  

(21)

where \( m = 2\mu \). We are particularly interested in solving the last equation, since the first one, has of trivial solution. In spherical coordinates this equation takes the form:

\[ \left( p_r^2 + \frac{L^2}{r^2} + \mu^2 \omega^2 r^2 - 3\mu \omega + 4\mu^2 - \frac{E^2}{4} \right) \phi_2(\vec{r}) = 0. \]  

(22)

The first term can be written as:

\[ p_r = -i \left( \frac{\partial}{\partial r} + \frac{1}{r} \right). \]  

(23)

This Hermitian operator is the radial component of the linear momentum. There are two independent eigenfunctions. In analogy with non-relativistic quantum harmonic oscillator,
only one of them is physically acceptable because the other diverges at the origin. So, the
eigenvalues and eigenfunctions are:

\[ E^2 - 16\mu^2 = 8\mu\omega(2n + l) \quad n = 0, 1... \quad l = 0, 1... \] (24)

\[ \Phi(r, \theta, \phi)_{n,l,m} = A_{n,l}r^l e^{-\frac{\mu\omega r^2}{2}} L^{l+\frac{1}{2}}(\mu\omega r^2) Y_{l,m}(\theta, \phi) \] (25)

and \( A_{n,l} \) is the normalization constant:

\[ A_{n,l} = \left[ \frac{2(\mu\omega)^{l+\frac{3}{2}}\Gamma(n + 1)}{\gamma(n + l + \frac{3}{2})} \right]^{1/2}. \] (26)

Note that the states are degenerated for particular values of \( n \) and \( l \). The same situation
appears in the 4D model.

The solution we have previously derived enable us to compute the mass spectrum for
quarkonium, as we will show below.

**IV. RESULTS AND CONCLUSIONS**

Let us now compute some specific values for the mass spectra for charmonium and bot-
tonium. We have considered a two free parameter set: \( m, \omega \). For the \( c\bar{c} \) family \( m_c = 1.5 \text{ GeV} \) and \( \omega \simeq 0.385 \text{ GeV} \). Instead, for bottonium, \( m_b = 5.17 \text{ Gev} \) and \( \omega \simeq 0.12 \text{ GeV} \).

The values for \( \omega \) were obtained by adjusting the rms radius of the ground state \( \text{[3.10]} \).

Our results are shown in Tables I and II. Table I displays the values for the masses of
carmonium mesons within different relativistic models, for particular quantum numbers \( n \)
and \( l \). The explicit values for the 4D model arise from the computation of charmonium
spectrum in such a general scheme. The other two known values, LP and QP, are taken
from ref. \( \text{[1,2]} \). With bold typos (KGO) are presented the values we have obtained with our
model. From Table I, we can see that our scheme produces acceptable results, without any
further assumption.

In Table II, results for \( b\bar{b} \) mesons are presented. Here, only the 4D model can be con-
fronted with our one, because, as far as we know, the LP and QP treatments were not
applied to this mesons. To do such a comparison we choose the mass for bottom quark to be 5.16 GeV, which is bigger than the experimental value. This value for the mass was used in \cite{3} yielding $\omega \simeq 0.113$ GeV. Thus, we can expect an overestimation of the resulting mass spectrum, due to such a choice. Anyway, taken the $b$ mass heavier than the experimental value, the masses for $b\bar{b}$ mesons predicted from these two models are very close to the experimental data. We again get agreement between 4D and KGO.

Summing up, we have presented a simple quantum relativistic scheme to take account the confining potential of quarkonium. In this frame we have solved the two-body problem obtaining the corresponding eigenfunctions and eigenvalues. We have made a comparison between our model and other relativistic models. This family of mesons entails for a restrictive arena where comparison can be made, because plenty of models exist with agreement among them and experimental data. Thus, this can be thought of as a first test of any alternative setting. Our results are in good agreement with previous theoretical predictions and with experimental data (available in \cite{11}). This fact allows us to conclude that our treatment reproduces with a high degree of precision the masses of charmonium and bottomonium family. In addition, we stress that our model is entirely analytical.

We expect that our simple model could be extended to other situations in the framework of quark models.

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TABLES

TABLE I. Particular values for $c\bar{c}$ mass spectrum are shown. In KGO and 4D models, we have considered $m_c = 1.5$ GeV and $\omega = 0.385$ GeV.

| Relativistic Model | $l=0$ | $l=1$ | $l=2$ | $l=3$ | $l=4$ |
|--------------------|-------|-------|-------|-------|-------|
| n=0                |       |       |       |       |       |
| LP                 | 3.105 | 3.456 | 3.761 | 4.034 | 4.286 |
| QP                 | 3.179 | 3.946 | 4.193 | 4.436 | 4.675 |
| 4D                 | 3.531 | 3.844 | 4.133 | 4.404 | 4.659 |
| KGO                | 3.000 | 3.363 | 3.691 | 3.991 | 4.271 |
| n=1                |       |       |       |       |       |
| LP                 | 3.696 | 3.964 | 4.215 | 4.450 | 4.673 |
| QP                 | 3.695 | 3.946 | 4.193 | 4.436 | 4.675 |
| 4D                 | 4.133 | 4.404 | 4.659 | 4.901 | 5.131 |
| KGO                | 3.691 | 3.991 | 4.271 | 4.533 | 4.781 |
| n=2                |       |       |       |       |       |
| LP                 | 4.169 | 4.397 | 4.616 | 4.826 |       |
| QP                 | 4.186 | 4.426 | 4.662 | 4.895 |       |
| 4D                 | 4.659 | 4.901 | 5.131 | 5.351 | 5.563 |
| KGO                | 4.271 | 4.533 | 4.781 | 5.017 | 5.242 |
| n=3                |       |       |       |       |       |
| LP                 | 4.580 | 4.783 | 4.980 |       |       |
| QP                 | 4.656 | 4.886 | 5.114 |       |       |
| 4D                 | 5.131 | 5.351 | 5.563 | 5.767 | 5.964 |
| KGO                | 4.781 | 5.017 | 5.242 | 5.458 | 5.666 |
TABLE II. Particular values for $b\bar{b}$ mass spectrum are shown. We considered $m_b = 5.17$ GeV and $\omega = 0.113$ GeV.

|      | Relativistic Model | $l=0$ | $l=1$ | $l=2$ | $l=3$ | $l=4$ |
|------|-------------------|------|------|------|------|------|
| n=0  | 4D                | 10.358 | 10.667 | 10.795 | 10.922 | 11.047 |
|      | KGO               | 10.340 | 10.452 | 10.564 | 10.674 | 10.782 |
| n=1  | 4D                | 10.795 | 10.922 | 11.047 | 11.171 | 11.293 |
|      | KGO               | 10.564 | 10.674 | 10.782 | 10.890 | 10.997 |
| n=2  | 4D                | 10.047 | 11.171 | 11.293 | 11.415 | 11.534 |
|      | KGO               | 10.782 | 10.890 | 10.997 | 11.103 | 11.208 |
| n=3  | 4D                | 11.293 | 11.415 | 11.534 | 11.653 | 11.770 |
|      | KGO               | 10.997 | 11.103 | 11.208 | 11.311 | 11.414 |
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