Hamiltonian and gradient properties of certain type of dynamical systems

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Abstract

From the standpoint of neural network dynamics we consider dynamical system of special type possesses gradient (symmetric) and Hamiltonian (antisymmetric) flows. The conditions when Hamiltonian flow properties are dominant in the system are considered. A simple Hamiltonian has been studied for establishing oscillatory pattern conditions in system under consideration.

Keywords: Neural network, Hamiltonian dynamical systems, gradient dynamical system.

1 Introduction and setting the problem.

It is well known [1] that synaptic connections in biological neural networks are seldom symmetric since the signal sent by neurons along their axons are sharp spikes and the relevant information is not contained in the spikes themselves but in the so called firing rates, which depends on the magnitude of the membrane potentials which governs all the process. On the other hand it should be pointed out that the recent neurophysiological observation of extremely low firing rates [2] without some doubt on the general usefulness of this notion as really the relevant neural variable. Thereby one can use some natural continuous variables to describe neural networks as dynamical systems of special structure like gradient (symmetric) and Hamiltonian skew-symmetrical flows. This gives rise to making use of a lot of methods and techniques for studying the structural stability of the networks and the existence of so called coherent temporal structures fitting for learning process.

Based on the considerations above one can introduce a class of nonlinear dynamical systems

\[
\frac{du}{dt} = K(u)
\]  

(1)

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where $M \ni u$ is some smooth finite-dimensional metrizible manifold $K : M \to T(M)$ is a vector field on $M$, modeling the information transfer process in a biological neural network under regard. The question is what conditions should be involved on the flow (1) for it to be represented as follows:

$$K(u) = - \text{grad} V(u) - \vartheta(u) \nabla H(u),$$

that is as a mixed sum of a gradient flow and a Hamiltonian flow on $M$. Here $V : M \to \mathbb{R}$ is the potential function and $H : M \to \mathbb{R}$ is the Hamiltonian function relevant to the flow (1), $\nabla := \{ \frac{\partial}{\partial u} : u \in M \}$, $g : T(M) \times T(M) \to \mathbb{R}^+$ is a Riemannian metrics and $\vartheta : T^*(M) \to T(M)$ is a Poisson structure on $M$.

Thus we need to find the corresponding metrics and Poisson structure on $M$ subject to which the representation (2) holds on $M$. We shall dwell on this topics in the proceeding chapter.

## 2 Poissonian structure analysis

Assume first that the representation (2) holds, that is

$$- \vartheta(u) \nabla H(u) = K(u) + \text{grad} V(u) := K_V(u)$$

for all $u \in M$ and some $\vartheta$ and $g$ structures on $M$. This means therefore that the constructed vector field (3) is exactly Hamiltonian. Thereby one has (4) the expression

$$\vartheta^{-1}(u) = \varphi'(u) - \varphi'^*(u),$$

where $\varphi \in T^*(M)$ is some nonsymmetric solution to the linear determining equation

$$\frac{d\varphi}{dt} + K_V^* \varphi = \nabla \mathcal{L}. \quad (5)$$

Here, by definition, the flow $K_V$ is defined as

$$\frac{du}{d\tau} = K_V(u)$$

and $\mathcal{L} : M \to \mathbb{R}$ is a suitable smooth function chosen for convenience when solving (5). It is clear (see [3]) that the symplectic structure (4) doesn’t depend on the choice of the function $\mathcal{L} : M \to \mathbb{R}$. As the second step, assume that the metrics and Poisson structures on $M$ are given a priori. Then due to (4) the following equation for determining the potential function $V : M \to \mathbb{R}$ holds:

$$\varphi' \cdot K + \varphi' \cdot \text{grad} V + K'^* \cdot \varphi + (\text{grad} V)'^* \cdot \varphi = \nabla \mathcal{L}, \quad (7)$$

where the element $\varphi \in T^*(M)$ has been assumed also to be known a priori as a solution to the equation (5). The expression (7) is a linear second order equation.
in partial derivatives on the potential function $V : M \to R$. If this equation is compatible, then its solution exists and the decomposition (2) holds.

As one can check, the equation (7) almost everywhere possesses a solution for the vector $\psi = \nabla V$, that is the following expression

$$\nabla V = \psi = g^{-1} \nabla V$$

holds on $M$ for some $\psi \in T(M)$. Thereby, one gets

$$\nabla V = g \psi.$$  \hspace{1cm} (9)

Making use now of the well-known Volterra condition (see [3]), $(\nabla V)'^* \equiv (\nabla V)'$, we obtain the following criterion on the metrics $g : T(M) \times T(M) \to R_+$ :

$$(g \psi)'^* = (g \psi)'.$$  \hspace{1cm} (10)

Since from (9) also one follows that

$$\langle g \psi, u_x \rangle = \langle \nabla V, u_x \rangle = dV/dx,$$  \hspace{1cm} (11)

the condition (11) is evidently equivalent to such one:

$$(g \psi)'^* u_x - \frac{d}{dx} (g \psi) = 0.$$  \hspace{1cm} (12)

Calculating the left handside expression of (12) one gets the following final result:

$$(g' \psi u_x - g' \psi x) = g \psi' u_x - \psi' (g u_x),$$  \hspace{1cm} (13)

which is feasible at check, if the metrics is given. Otherwise, if this is not the case, the linear expression (13) determines a suitable metrics as its solution subject to the mapping $g : T(M) \times T(M) \to R_+$. As soon as the equation (13) is compatible, its solution exists defining a suitable metrics on the manifold $M$.

The results delivered above can be successfully applied to many interesting dynamical systems modeling information processes in neural networks, mentioned in introduction. Below we shall demonstrate some of them having applications at studying coherent temporal structures.

3 Coherent temporal structures formation.

One considers a network with two groups of neuron $\{x_i \in R : i=1,n\}$ and $\{y_j \in R: j=1,m\}$, connected in such a way, that inside both groups the synaptic strengths are symmetric, whereas between groups they are antisymmetric. That is, neurons $\{x\}$ are excitatory to $\{y\}$ and neurons $\{y\}$ are inhibitory to $\{x\}$.
This model is expressed in the form (2), where

\[ V = \sum_{i=1}^{n} \left( -\frac{1}{2} \beta_1 x_i^2 + \beta_2 \frac{x_i^4}{4} \right) + \frac{1}{2} \sum_{i,j=1}^{n} \beta_{i,j} x_i x_j \]  

\[ + \sum_{j=1}^{m} \left( -\frac{1}{2} \beta_4 y_j^2 + \beta_5 \frac{y_j^4}{4} \right) + \frac{1}{2} \sum_{i,j=1}^{m} \beta_{i,j} y_i y_j, \]

\[ H = \frac{1}{2} \left( \sum_{i=1}^{n} x_i^2 + \sum_{j=1}^{m} y_j^2 + \sum_{i=1,j=1}^{n,m} w_{i,j} x_i y_j \right) \]  

with the standard metrics \( g = 1 \), a skew-symmetric Poisson structure \( \vartheta = J \in \text{Sp}(R^{n+m}), u = \{(x, y) \in R^n \times R^m\} \),

\[ g = \begin{pmatrix} 1 & 0 \\ \vdots & \ddots \\ 0 & \ldots & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I_{(n,m)} \\ -I_{(n,m)} & 0 \end{pmatrix} \]  

or

\[ J = \begin{pmatrix} J_{(n)} & 0 \\ 0 & J_{(m)} \end{pmatrix} \]

with constant \( \beta \) and elements \( w_{i,j} \) being parameters, \( I_{(n,m)} = \{\delta_{ij} : i = 1, n, j = 1, m\} \), \( J_{(n)} \) and \( J_{(m)} \) being some skew-symmetric matrices.

It is worth to mention here that the representation (2) with structures (16) is not unique and some other solutions to the equation (13) can be found.

This system (14) as we shall demonstrate below possesses a so called coherent-temporal structure important for studying learning processes in biological neural networks.

Assume for simplicity that all \( \beta \) - parameters are proportional to a small enough parameter \( \varepsilon > 0 \), that is \( \{\beta\} \simeq \{\varepsilon \beta\} \) and consider first our flow (2) at \( \varepsilon = 0 \). It is easy to see that our model then possesses a closed orbit in the space of \( \{x\} \) and \( \{y\} \) - parameters, say \( \sigma : S^1 \to M = R^n \times R^m \), satisfying the equation

\[ d\sigma/d\tau = -J\nabla H(\sigma) \]  

for all \( \tau \in S^1 \). Moreover, the Hamiltonian function \( H : M \to R \) in (15) is a conservation law of (18). Take now \( \varepsilon \neq 0 \); then one can state (4) that there exists a function \( H_\varepsilon : M \to R \), such, that for some closed orbit \( \sigma_\varepsilon : S^1 \to M \) this function \( H_\varepsilon : M \to R \) be a constant of motion (not a conservative quantity), that is for all small enough \( \varepsilon > 0 \)

\[ dH_\varepsilon(\sigma_\varepsilon)/dt = O(\varepsilon^2) \]  

as \( \varepsilon \to 0 \). Then one can formulate the following proposition about the existence of a limiting cycle in our model at \( \varepsilon > 0 \) small enough.
**Proposition.** Let our model possess at small enough $\varepsilon > 0$ a smooth constant of motion $H_\varepsilon : M \to \mathbb{R}$ and a closed $\varepsilon$-deformed orbit $\sigma_\varepsilon : S^1 \to \mathbb{R}$. Moreover, at $\varepsilon = 0$ the constant of motion $H_0 : M \to \mathbb{R}$ is a first integral of the model in the neighborhood of the orbit $\sigma_0$. Then a necessary condition for the existence of a limiting cycle at $\varepsilon > 0$ small enough is vanishing the following circular integral:

$$\oint_{S^1} \langle \nabla H_0(\sigma_0), \text{grad} V(\sigma_0) \rangle \, dt = 0. \quad (20)$$

Having substituted expression (14) into (20), one finds numerical constraints on the parameters locating our closed orbit $\sigma_0 : S^1 \to \mathbb{R}$ in the phase space $M$. Thereby, we can localize this way possible coherent temporal patterns available in our neuron network under study.

Using this approach let us consider the equation of motion on the variables $(x, y) \in \mathbb{R}^{n+m}$. The Lagrangian equation corresponding to potential (14), Hamiltonian (15) and matrix (17) can be represented as

$$\left( \begin{array}{c} \dot{x} \\ \dot{y} \end{array} \right) + W \left( \begin{array}{c} x \\ y \end{array} \right) = 0, \quad (21)$$

where columns $x = \{x_1, ..., x_n\}^T$, $y = \{y_1, ..., y_m\}^T$, and matrix $W = \left( \begin{array}{cc} A_1 & B_1 \\ A_2 & B_2 \end{array} \right)$.

$$A_1 = J_n^2 + J_n w J_m w^T, \quad A_2 = J_m^2 + J_m w^T J_n w, \quad B_1 = J_n^2 w + J_n w J_m, \quad B_2 = J_m^2 w + J_m w^T J_n, \quad w = \{w_{ik}\}.$$ A solution to the matrix equation (21) can be represented as $(x, y)^T = \mathbf{a} \exp(i\lambda t)$, with $\lambda \in \mathbb{C}$ being nontrivial only if the following determinant

$$| -\lambda^2 g + W | = 0 \quad (22)$$

is equal to zero.

Equation (22) is one of the degree $n+m$ subject to $\lambda^2 \in \mathbb{C}$ and determines $2(n+m)$ eigen frequencies $\omega_s = \{\pm \omega_1, ..., \pm \omega_{m+n}\}$. In this case the solution gets the form

$$\left( \begin{array}{c} x_s \\ y_s \end{array} \right) = \mathbf{a}_s \exp(i\omega_s t) + \mathbf{a}_s^* \exp(-i\omega_s t). \quad (23)$$

Amplitudes $\mathbf{a}_s = \{a_{s,1}, ..., a_{s,n+m}\}^T$ must satisfy the matrix equation

$$(-\omega_s^2 g + W) \mathbf{a}_s = 0. \quad (24)$$

If we take that amplitudes $a_{s,1} = a_{s,1}^* = 1$ for any $s \in \{1, n+m\}$ solve (24), we can get coefficients $K_{s,i}$ of the distribution of amplitudes relative to any frequency $\omega_s$. For this case the solution (23) can be represented as

$$x_{sj} = a_{s,1} K_{s,j} \exp(i\omega_s t) + a_{s,1}^* K_{s,j} \exp(-i\omega_s t)$$

$$= K_{s,j} \exp(i\omega_s t) + K_{s,j} \exp(-i\omega_s t), \quad (25)$$

$$y_{sj} = K_{s,j+n} \exp(i\omega_s t) + K_{s,j+n} \exp(-i\omega_s t). \quad (26)$$
Here $K_{s,i}, K_{s,j+n}$ are given constants depending on the frequencies $\omega_s$. Consider now the scalar product

$$\langle \nabla H_0(\sigma_0), \nabla V(\sigma_0) \rangle = \left( x_1 + \sum_{j=1}^{m} w_{1j} y_j \right) \left( -\beta_1 x_1 + \beta_2 x_3 + \frac{1}{2} \sum_{j=1}^{n} \beta^{(1)}_{1,j} x_j \right) + \ldots$$

$$+ \left( x_n + \sum_{j=1}^{n} w_{nj} y_j \right) \left( -\beta_1 x_n + \beta_2 x_n + \frac{1}{2} \sum_{j=1}^{n} \beta^{(1)}_{n,j} x_j \right) + \ldots$$

$$+ \left( y_1 + \sum_{i=1}^{n} w_{1i} x_i \right) \left( -\beta_1 y_1 + \beta_2 y_1 + \frac{1}{2} \sum_{j=1}^{n} \beta^{(2)}_{1,j} y_j \right) + \ldots$$

$$+ \left( y_n + \sum_{j=1}^{n} w_{ni} x_i \right) \left( -\beta_1 y_n + \beta_2 y_n + \frac{1}{2} \sum_{j=1}^{n} \beta^{(2)}_{n,j} y_j \right).$$

Having substituted solution into last expression and integrated it along the period $T = \frac{2\pi}{\omega_s}$ we get a hyperplane which determines the parameters of our neural network model:

$$\left( x_1 + \sum_{j=1}^{m} w_{1j} K_{s,n+j} \right) \left( -\beta_1 + \frac{3}{4} \beta_2 + \frac{1}{2} \sum_{j=1}^{n} \beta^{(1)}_{1,j} K_{s,j} \right) + \ldots$$

$$+ \left( K_{s,n} + \sum_{j=1}^{m} w_{nj} K_{s,n+j} \right) \left( -\beta_1 K_{s,n} + \frac{3}{4} \beta_2 K_{s,n}^{3} + \frac{1}{2} \sum_{j=1}^{n} \beta^{(1)}_{n,j} K_{s,j} \right) + \ldots$$

$$+ \left( K_{s,n+1} + \sum_{i=1}^{n} w_{1i} K_{s,i} \right) \left( -\beta_3 K_{s,1+n} + \frac{3}{4} \beta_4 K_{s,1+n}^{3} + \frac{1}{2} \sum_{j=1}^{n} \beta^{(2)}_{1,j} K_{s,j+n} \right) + \ldots$$

$$+ \left( K_{s,n+m} + \sum_{j=1}^{n} w_{ni} K_{s,i} \right) \left( -\beta_3 K_{s,m+n} + \frac{3}{4} \beta_4 K_{s,m+n}^{3} + \frac{1}{2} \sum_{j=1}^{n} \beta^{(2)}_{n,j} K_{s,j+n} \right) = 0$$

Thus, as the index $s$ changes from $s = 1$ to $n + m$, we can get in the general case $n + m$ dimensional submanifolds determining parameters $\{\beta_1, \beta_2, \beta_3, \beta_4, \beta^{(1)}_{n,j}, \beta^{(2)}_{n,j}\}$, at which the chosen oscillatory structure will persist for all $t \in R_+$, thereby realizing a stable neural network and related with it the temporal pattern under study.

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