Hardy spaces associated with Schrödinger operators on the Heisenberg group

Chin-Cheng Lin*,†, Heping Liu‡, and Yu Liu

Abstract

Let \( L = -\Delta_{\mathbb{H}^n} + V \) be a Schrödinger operator on the Heisenberg group \( \mathbb{H}^n \), where \( \Delta_{\mathbb{H}^n} \) is the sub-Laplacian and the nonnegative potential \( V \) belongs to the reverse Hölder class \( B_{\frac{Q}{Q+2}} \) and \( Q \) is the homogeneous dimension of \( \mathbb{H}^n \). The Riesz transforms associated with the Schrödinger operator \( L \) are bounded from \( L^1(\mathbb{H}^n) \) to \( L^{1,\infty}(\mathbb{H}^n) \). The \( L^1 \) integrability of the Riesz transforms associated with \( L \) characterizes a certain Hardy type space denoted by \( H^1_L(\mathbb{H}^n) \) which is larger than the usual Hardy space \( H^1(\mathbb{H}^n) \). We define \( H^1_L(\mathbb{H}^n) \) in terms of the maximal function with respect to the semigroup \( \{ e^{-sL} : s > 0 \} \), and give the atomic decomposition of \( H^1_L(\mathbb{H}^n) \). As an application of the atomic decomposition theorem, we prove that \( H^1_L(\mathbb{H}^n) \) can be characterized by the Riesz transforms associated with \( L \). All results hold for stratified groups as well.

Key words and phrases. Atomic decomposition, Hardy spaces, Heisenberg group, local Hardy spaces, Riesz transforms, Schrödinger operators, stratified groups.

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1 Introduction

The Schrödinger operators with a potential satisfying the reverse Hölder inequality have been studied by various authors. Some basic results on the Euclidean spaces were established by Fefferman [5], Shen [18], and Zhong [20]. The extension to a more general setting was given by Lu [15] and Li [13]. In this article we consider the Schrödinger operator \( L = -\Delta_{\mathbb{H}^n} + V \) on the Heisenberg group \( \mathbb{H}^n \), where \( \Delta_{\mathbb{H}^n} \) is the sub-Laplacian and the nonnegative potential \( V \) belongs to the reverse Hölder class \( B_{\frac{Q}{Q+2}} \). Here \( Q \) is the homogeneous dimension of \( \mathbb{H}^n \). Let \( R^L_j = X_j L^{-\frac{1}{2}}, j = 1, \ldots, 2n \), be the Riesz transforms

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associated with the Schrödinger operator $L$, where $X_j$’s are left-invariant vector fields generating the Lie algebra of $\mathbb{H}^n$. If $q \geq Q$, $R_j^L$ are Calderón–Zygmund operators (cf. [15]). When $\frac{Q}{2} \leq q < Q$, $R_j^L$ are bounded on $L^p(\mathbb{H}^n)$ for $1 < p \leq \frac{Qq}{Q-q}$ (cf. [13]). In the current paper, we prove, by providing a counterexample, that the above estimate of the range of $p$ is sharp.

It is well known that $R_j^L$ might not be Calderón–Zygmund operators. However, these operators $R_j^L$ do observe some boundedness conditions: they are bounded from $L^1(\mathbb{H}^n)$ to $L^{1,\infty}(\mathbb{H}^n)$ (see Theorem 2 below), and bounded from $H^1(\mathbb{H}^n)$ to $L^1(\mathbb{H}^n)$ (see Remark 6 below). We should remark that, unlike the classical case, the conditions $f \in L^1(\mathbb{H}^n)$ and $R_j^L f \in L^1(\mathbb{H}^n)$ $(j = 1, \cdots, 2n)$ do not ensure $f \in H^1(\mathbb{H}^n)$. In view of these new complication, we will introduce the notion of the Hardy space $H^1_1(\mathbb{H}^n)$ associated with $L$ in terms of the maximal function with respect to the semigroup $\{e^{-sL} : s > 0\}$. The atomic decomposition of $H^1_1(\mathbb{H}^n)$ will be given and, as an application, we prove that $H^1_1(\mathbb{H}^n)$ can be characterized by the Riesz transforms $R_j^L$ associated with the Schrödinger operator $L$. We also remark that the Heisenberg group is a typical case of stratified groups. Our all results can be established for stratified groups by the same arguments.

The current work is inspired by the pioneering work of Dziubański and Zienkiewicz [4], in which the Hardy space associated with the Schrödinger operator on the Euclidean spaces was studied. Equipped with some enhanced technique, we will establish some extended results in the setting of the Heisenberg group. For example, we give some descriptions of kernels (Section 3) and develop the theory of local Hardy spaces (Section 4). The corresponding results to these on Euclidean spaces have been already known. On Euclidean spaces the local Riesz transforms characterization of local Hardy spaces is obtained via subharmonicity (cf. [9]). However, this kind of approach fails on the Heisenberg group as pointed out in [2], so we have to develop a new method. Our approach is to decompose a function $f = \tilde{f} + (f - \tilde{f})$ such that $\tilde{f}$ is in the local Hardy space $h^1(\mathbb{H}^n)$ and the Riesz transforms of $(f - \tilde{f})$ are controlled by the local Riesz transforms of $f$. We then can use the Riesz transforms characterization of Hardy space $H^1$ on stratified groups given by Christ and Geller (cf. [2]). We establish the atomic decomposition theorem for general $H^1_{1,q}$-atoms rather than $H^1_{1,\infty}$-atoms as in [4], because it is more convenient to study the dual space of $H^1_{1}(\mathbb{H}^n)$, which will be dealt with in the forthcoming paper [14]. We also prove the weak $(1, 1)$ boundedness of Riesz transforms $R_j^L$, which is useful to establish the $H^1_1 - L^1$ boundedness of $R_j^L$.

This article is organized as follows. In Section 2, we set notations and state our main results. In Section 3 we give estimates of kernels of the semigroup $\{e^{-sL} : s > 0\}$ and the Riesz transforms, which will be used in the sequel. Most proofs in this section are inspired from [2] and [18]; however, they are new in our setting. In Section 4 we discuss local Hardy spaces $h^1(\mathbb{H}^n)$. Note that an $H^1_{1}$-function is locally equal to a function in a certain scaled local Hardy space. Specifically, $H^1_{1}(\mathbb{H}^n)$ coincides with the local Hardy space $h^1(\mathbb{H}^n)$ if there exists a positive number $C$ such that $\frac{1}{C} \leq V \leq C$ (see Remark
5 below). In Section 5 we establish the atomic decomposition of $H^1_L(\mathbb{H}^n)$. Section 6 is devoted to the Riesz transforms $R_j L$. We prove that $R_j L$ are bounded from $L^1(\mathbb{H}^n)$ to $L^{1,\infty}(\mathbb{H}^n)$, and they characterize $H^1_L(\mathbb{H}^n)$. The counterexample mentioned above is also given in Section 6. Finally, we include in Section 7 a brief discussion the corresponding results for stratified groups without proofs.

Throughout this article, we will use $C$ to denote a positive constant, which is independent of main parameters and not necessarily the same at each occurrence. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $\frac{1}{C} \leq A/B \leq C$. Moreover, we denote the conjugate exponent of $q > 1$ by $q' = q/(q-1)$.

## 2 Notations and main results

We recall some basic facts on the Heisenberg group, which are easily found in many references. The $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$ is the Lie group with underlying manifold $\mathbb{R}^{2n} \times \mathbb{R}$ and multiplication

$$(x, t)(y, s) = (x + y, t + s + 2 \sum_{j=1}^{n} (x_{n+j}y_j - x_jy_{n+j})).$$

A basis for the Lie algebra of left-invariant vector fields on $\mathbb{H}^n$ is given by

$$X_{2n+1} = \frac{\partial}{\partial t}, \quad X_j = \frac{\partial}{\partial x_j} + 2x_{n+j} \frac{\partial}{\partial t}, \quad X_{n+j} = \frac{\partial}{\partial x_{n+j}} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \ldots, n.$$  

All non-trivial commutators are $[X_j, X_{n+j}] = -4X_{2n+1}, \ j = 1, \ldots, n$. The sub-Laplacian $\Delta_{\mathbb{H}^n}$ and the gradient $\nabla_{\mathbb{H}^n}$ are defined respectively by

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{2n} X_j^2 \quad \text{and} \quad \nabla_{\mathbb{H}^n} = (X_1, \ldots, X_{2n}).$$

The dilations on $\mathbb{H}^n$ have the form

$$\delta_r(x, t) = (rx, r^2 t), \quad r > 0.$$  

The Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure on $\mathbb{R}^{2n} \times \mathbb{R}$. The measure of any measurable set $E$ is denoted by $|E|$. We define a homogeneous norm on $\mathbb{H}^n$ by

$$|g| = \left(|x|^4 + |t|^2\right)^{\frac{1}{4}}, \quad g = (x, t) \in \mathbb{H}^n.$$  

This norm satisfies the triangle inequality and leads to a left-invariant distance $d(g, h) = |g^{-1}h|$. Then the ball of radius $r$ centered at $g$ is given by

$$B(g, r) = \{h \in \mathbb{H}^n : |g^{-1}h| < r\}.$$
There is a positive constant $b_1$ such that

$$|B(g,r)| = b_1 r^Q$$

where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$. Exactly,

$$b_1 = |B(0,1)| = \frac{2\pi^{n+\frac{1}{2}} \Gamma(\frac{Q}{2})}{(n+1)\Gamma(n)\Gamma(\frac{n+1}{2})},$$

but it is not important for us.

Now we turn to the Schrödinger operator

$$L = -\Delta_{\mathbb{H}^n} + V.$$  

A nonnegative locally $L^q$ integrable function $V$ on $\mathbb{H}^n$ is said to belong to $B^q$ ($1 < q < \infty$) if there exists $C > 0$ such that the reverse Hölder inequality

$$\left( \frac{1}{|B|} \int_B V(g)^q \, dg \right)^{\frac{1}{q}} \leq C \left( \frac{1}{|B|} \int_B V(g) \, dg \right)$$

holds for every ball $B$ in $\mathbb{H}^n$. Obviously, $B_{q_1} \subset B_{q_2}$ if $q_1 > q_2$. But it is important that the $B^q$ class has a property of “self improvement”: that is, if $V \in B_q$, then $V \in B_{q+\varepsilon}$ for some $\varepsilon > 0$. In this article we always assume that $0 \neq V \in B_{\frac{Q}{2}}$ and then $V \in B_{q_0}$ for some $q_0 > \frac{Q}{2}$. Of course we may assume that $q_0 < Q$.

Let $\{T^s : s > 0\} = \{e^{s\Delta_{\mathbb{H}^n}} : s > 0\}$ be the heat semigroup with the convolution kernel $H_s(g)$. The heat kernel $H_s(g)$ satisfies the estimate

$$0 < H_s(g) \leq Cs^{-\frac{Q}{2}} e^{-A_0 s^{-1}|g|^2},$$

where $A_0$ is a positive constant (cf. [12]). An explicit expression of $H_s(g)$ in terms of the Fourier transform with respect to central variable was given by Hulanicki [11]. Because $V \geq 0$ and $V \in L^q_{\text{loc}}(\mathbb{H}^n)$, the Schrödinger operator $L$ generates a ($C_0$) contraction semigroup $\{T^L_s : s > 0\} = \{e^{-sL} : s > 0\}$. Let $K^L_s(g,h)$ denote the kernel of $T^L_s$. By the Trotter product formula (cf. [10]),

$$0 \leq K^L_s(g,h) \leq H_s(g,h) = H_s(h^{-1}g).$$

Let us consider the maximal functions with respect to the semigroups $\{T^s : s > 0\}$ and $\{T^L_s : s > 0\}$ defined by

$$Mf(g) = \sup_{s > 0} |T^s f(g)|,$$

$$M^Lf(g) = \sup_{s > 0} |T^L_s f(g)|.$$  

It is well known that the maximal function $Mf$ characterizes the Hardy space $H^1(\mathbb{H}^n)$; that is, $f \in H^1(\mathbb{H}^n)$ if and only if $Mf \in L^1(\mathbb{H}^n)$, and $\|f\|_{H^1} \sim \|Mf\|_{L^1}$ (cf. [7]).

We define the Hardy space $H^1_L(\mathbb{H}^n)$ associated with the Schrödinger operator $L$ as follows.
Definition 1. A function \( f \in L^1(\mathbb{H}^n) \) is said to be in \( H^1_L(\mathbb{H}^n) \) if the maximal function \( M^Lf \) belongs to \( L^1(\mathbb{H}^n) \). The norm of such a function is defined by \( \| f \|_{H^1_L} = \| M^Lf \|_{L^1} \).

It is visible from (2) that the space \( H^1_L(\mathbb{H}^n) \) is larger than the usual Hardy space \( H^1(\mathbb{H}^n) \). This fact will be seen from the atomic decompositions of \( H^1_L(\mathbb{H}^n) \).

We define the auxiliary function \( \rho(g, V) = \rho(g) \) by

\[
\rho(g) = \sup_{r>0} \left\{ r : \frac{1}{r^{Q-2}} \int_{B(g, r)} V(h) \, dh \leq 1 \right\}, \quad g \in \mathbb{H}^n.
\]

This kind of auxiliary function was introduced by Shen in [17] for the potential \( V \) satisfying \( \max_{x \in B} V(x) \leq \frac{C}{|B|} \int_{B} V(x) \, dx \). Properties of the auxiliary function \( \rho(g, V) \) are given by Shen [18] on Euclidean spaces and by Lu [15] on homogeneous spaces (their auxiliary function \( m(g, V) = \frac{1}{\rho(g, V)} \) in practice). It is known that \( 0 < \rho(g) < \infty \) for any \( g \in \mathbb{H}^n \) (from Lemma 2 in Section 3).

Definition 2. Let \( 1 < q \leq \infty \). A function \( a \in L^q(\mathbb{H}^n) \) is called an \( H^1_{L, q} \)-atom if the following conditions hold:

(i) \( \text{supp } a \subset B(g_0, r) \),

(ii) \( \| a \|_{L^q} \leq |B(g_0, r)|^{\frac{1}{q}-1} \),

(iii) if \( r < \rho(g_0) \), then \( \int_{B(g_0, r)} a(g) \, dg = 0 \).

Theorem 1. Let \( f \in L^1(\mathbb{H}^n) \) and \( 1 < q \leq \infty \). Then \( f \in H^1_{L, q}(\mathbb{H}^n) \) if and only if \( f \) can be written as \( f = \sum_j \lambda_j a_j \), where \( a_j \) are \( H^1_{L, q} \)-atoms, \( \sum_j |\lambda_j| < \infty \), and the sum converges in \( H^1_{L, q}(\mathbb{H}^n) \) norm. Moreover,

\[
\| f \|_{H^1_{L, q}} \sim \inf \left\{ \sum_j |\lambda_j| \right\},
\]

where the infimum is taken over all atomic decompositions of \( f \) into \( H^1_{L, q} \)-atoms.

The Riesz transforms \( R^L_j \) associated with the Schrödinger operator \( L \) are defined by

\[
R^L_j = X_j L^{-\frac{1}{2}}, \quad j = 1, \cdots, 2n.
\]

As pointed out at the beginning, \( R^L_j \) are bounded on \( L^p(\mathbb{H}^n) \) for \( 1 < p \leq p_0 \) where \( \frac{1}{p_0} = \frac{1}{q_0} - \frac{1}{Q} \). A counterexample will be given in Section 6 to show that the above range of \( p \) is optimal. For \( p = 1 \), we have the following weak type estimate.

Theorem 2. The Riesz transforms \( R^L_j \) are bounded from \( L^1(\mathbb{H}^n) \) to \( L^{1, \infty}(\mathbb{H}^n) \).
The Riesz transforms $R_j^L$ are also bounded from $H^1(\mathbb{H}^n)$ to $L^1(\mathbb{H}^n)$ (see Remark 6 below). However, these operators do not characterize the usual Hardy space $H^1(\mathbb{H}^n)$. They characterize $H^1_L(\mathbb{H}^n)$ which is larger than $H^1(\mathbb{H}^n)$.

**Theorem 3.** A function $f \in H^1_L(\mathbb{H}^n)$ if and only if $f \in L^1(\mathbb{H}^n)$ and $R_j^L f \in L^1(\mathbb{H}^n)$, $j = 1, \cdots, 2n$. Moreover,

$$
\|f\|_{H^1_L} \sim \|f\|_{L^1} + \sum_{j=1}^{2n} \|R_j^L f\|_{L^1}.
$$

The proof of Theorem 3 will be given at the end of Section 5, while the proofs of Theorems 2 and 3 will be given in Section 6.

### 3 Estimates of the kernels

In this section we give some estimates of kernels of the semigroup $\{T^L_s\}$ and the Riesz transforms $R_j^L$, which will be used in the sequel. The proofs of Lemmas 6–9 will closely follow the arguments on $\mathbb{R}^n$ as presented by Shen in [18]. First we collect some basic facts about the potential $V$ satisfying the reverse Hölder inequality. We assume that $V \in B_{q_0}$ for some $q_0 > \frac{2}{Q}$. We may add a restriction that $q_0 < Q$ when necessary. In this case, we assume the relation $\frac{1}{p_0} = \frac{1}{q_0} - \frac{1}{Q}$.

**Lemma 1.** The measure $V(h) \, dh$ satisfies the doubling condition; that is, there exists $C > 0$ such that

$$
\int_{B(g,2r)} V(h) \, dh \leq C \int_{B(g,r)} V(h) \, dh
$$

for all balls $B(g,r)$ in $\mathbb{H}^n$.

**Lemma 2.** For $0 < r < R < \infty$,

$$
\frac{1}{r^{Q-2}} \int_{B(g,r)} V(h) \, dh \leq C \left(\frac{r}{R}\right)^{2-\frac{q}{q_0}} \frac{1}{R^{Q-2}} \int_{B(g,R)} V(h) \, dh.
$$

**Lemma 3.** If $r = \rho(g)$, then

$$
\frac{1}{r^{Q-2}} \int_{B(g,r)} V(h) \, dh = 1.
$$

Moreover,

$$
\frac{1}{r^{Q-2}} \int_{B(g,r)} V(h) \, dh \sim 1 \quad \text{if and only if} \quad r \sim \rho(g).
$$

**Lemma 4.** There exists $l_0 > 0$ such that, for any $g$ and $h$ in $\mathbb{H}^n$,

$$
\frac{1}{C} \left(1 + \frac{|h^{-1}g|}{\rho(g)}\right)^{-l_0} \leq \frac{\rho(h)}{\rho(g)} \leq C \left(1 + \frac{|h^{-1}g|}{\rho(g)}\right)^{l_0}.
$$

In particular, $\rho(h) \sim \rho(g)$ if $|h^{-1}g| < C \rho(g)$.
Lemma 5. There exists $l_1 > 0$ such that, for any $g \in \mathbb{H}^n$,  
\[
\int_{B(g, R)} \frac{V(h)}{|h^{-1} g|^Q} \, dh \leq \frac{C}{R^{Q-2}} \int_{B(g, R)} V(h) \, dh \leq C \left(1 + \frac{R}{\rho(g)}\right)^{l_1}.
\]

For the proofs of Lemmas 1-4 we refer readers to [15].

Now we turn to the estimates of kernels. Let $\Gamma(g, h, \tau)$ denote the fundamental solution for the operator $-\Delta_{\mathbb{H}^n} + i\tau$, where $\tau \in \mathbb{R}$. For any $l > 0$, there exists $C_l > 0$ such that
\[
|\Gamma(g, h, \tau)| \leq \frac{C_l}{\left(1 + |h^{-1} g| |\tau|^{\frac{1}{2}}\right)^l \left(1 + |h^{-1} g| \rho(g)^{-1}\right)^l |h^{-1} g|^{Q-2}},
\]
(3) \[
|\nabla_{\mathbb{H}^n, g} \Gamma(g, h, \tau)| \leq \frac{C_l}{\left(1 + |h^{-1} g| |\tau|^{\frac{1}{2}}\right)^l |h^{-1} g|^{Q-1}},
\]
(4) where $\nabla_{\mathbb{H}^n, g}$ denote the gradient for variable $g$. The estimate (3) still holds for $\nabla_{\mathbb{H}^n, H}$ instead of $\nabla_{\mathbb{H}^n, g}$. The estimates (3) and (4) are easily reduced from the corresponding estimates of heat kernel (cf. [12]). We remark that the explicit expression of $\Gamma(g, h)$ is obtained by Folland [6]:
\[
\Gamma(g, h) = \frac{2^{n-2} \Gamma\left(\frac{n}{2}\right)^2}{\pi^{n+1}} \frac{1}{|h^{-1} g|^{Q-2}}.
\]
Let $\Gamma^L(g, h, \tau)$ denote the fundamental solution for the operator $L + i\tau$, where $\tau \in \mathbb{R}$. For any $l > 0$, there exists $C_l > 0$ such that
\[
|\Gamma^L(g, h,\tau)| \leq \frac{C_l}{\left(1 + |h^{-1} g| |\tau|^{\frac{1}{2}}\right)^l \left(1 + |h^{-1} g| |\rho(g)|\right)^l |h^{-1} g|^{Q-2}}.
\]
(5) Since $\Gamma^L(g, h, \tau) = \Gamma^L(h, g, -\tau)$, we have
\[
|\Gamma^L(g, h, \tau)| \leq \frac{C_l}{\left(1 + |h^{-1} g| |\tau|^{\frac{1}{2}}\right)^l \left(1 + |h^{-1} g| |\rho(g)| + |\rho(h)|\right)^l |h^{-1} g|^{Q-2}}.
\]

Lemma 6. For any $l > 0$, there exists $C_l > 0$ such that if $|h^{-1} g| \leq \rho(g)$, then
\[
|\Gamma^L(g, h, \tau) - \Gamma(g, h, \tau)| \leq \frac{C_l}{\left(1 + |h^{-1} g| |\tau|^{\frac{1}{2}}\right)^l \rho(g)^\delta |h^{-1} g|^{Q-2-\delta}},
\]
where $\delta = 2 - \frac{Q}{n_0} > 0$.

Proof. Note that
\[-\Delta_{\mathbb{H}^n, g}(\Gamma^L(g, h, \tau) - \Gamma(g, h, \tau)) + i\tau(\Gamma^L(g, h, \tau) - \Gamma(g, h, \tau)) = -V(g)\Gamma^L(g, h, \tau),
\]
where $-\Delta_{\mathbb{H}^n, g}$ denotes the sub-Laplacian for variable $g$. Since $\Gamma(g, h, \tau)$ is the fundamental solution for $-\Delta_{\mathbb{H}^n} + i\tau$, we have
\[
\Gamma^L(g, h, \tau) - \Gamma(g, h, \tau) = -\int_{\mathbb{H}^n} \Gamma(g, w, \tau)V(w)\Gamma^L(w, h, \tau) \, dw.
\]
Let \( R = |h^{-1}g| \leq \rho(g) \). By (3) and (5),
\[
|\Gamma^L(g, h, \tau) - \Gamma(g, h, \tau)| \leq \int_{\mathbb{R}^n} \frac{C_l}{(1 + |g^{-1}w| |\tau|^\frac{1}{2})^t |g^{-1}w|^Q-2} V(w) \, dw
\]
\[
\cdot (1 + |h^{-1}w| |\tau|^\frac{1}{2})^t (1 + |h^{-1}w| \rho(h)^{-1})^t |h^{-1}w|^Q-2
\]
\[
= \int_{|g^{-1}w| < \frac{R}{2}} + \int_{|h^{-1}w| < \frac{R}{2}} + \int_{|g^{-1}w| \geq \frac{R}{2}} |h^{-1}w| \geq \frac{R}{2}
\]
\[
= I_1 + I_2 + I_3.
\]

By Lemma 5 and Lemma 2 we have
\[
I_1 \leq \frac{C_l}{(1 + R |\tau|^\frac{1}{2})^t R^2-2} \int_{B(g, \frac{R}{2})} \frac{V(w)}{|g^{-1}w|^Q-2} \, dw
\]
\[
\leq \frac{C_l}{(1 + R |\tau|^\frac{1}{2})^t R^2-2} \int_{B(g, \frac{R}{2})} \frac{1}{R^2-2} V(w) \, dw
\]
\[
\leq \frac{C_l}{(1 + R |\tau|^\frac{1}{2})^t R^2-2} \left( \frac{R}{\rho(g)} \right)^{2-\frac{Q}{6}}.
\]

Similarly,
\[
I_2 \leq \frac{C_l}{(1 + R |\tau|^\frac{1}{2})^t R^2-2} \left( \frac{R}{\rho(g)} \right)^{2-\frac{Q}{6}}.
\]

Note that \( |g^{-1}w| \sim |h^{-1}w| \) when \( |g^{-1}w| \geq \frac{R}{2} \) and \( |h^{-1}w| \geq \frac{R}{2} \). It yields
\[
I_3 \leq \frac{C_l}{(1 + R |\tau|^\frac{1}{2})^t} \int_{|h^{-1}w| \geq \frac{R}{2}} V(w) \, dw
\]
\[
\leq \frac{C_l}{(1 + R |\tau|^\frac{1}{2})^t} \left( \int_{\rho(h) > |h^{-1}w| \geq \frac{R}{2}} \frac{V(w) \, dw}{|h^{-1}w|^{2(Q-2)}} + \rho(h) \int_{|h^{-1}w| \geq \rho(h)} \frac{V(w) \, dw}{|h^{-1}w|^{2(Q-2)} + t} \right).
\]

We may assume \( \rho(h) > \frac{R}{2} \). Otherwise, \( \rho(g) \sim \rho(h) \sim |h^{-1}g| \) and Lemma 3 is obviously true. By Hölder’s inequality, \( B_{\rho_0} \) condition and Lemma 3 we obtain
\[
\int_{\rho(h) > |h^{-1}w| \geq \frac{R}{2}} \frac{V(w) \, dw}{|h^{-1}w|^{2(Q-2)}}
\]
\[
\leq C \left( \int_{B(h, \rho(h))} V(w)^{\rho_0} \, dw \right)^{\frac{1}{\rho_0}} \left( \int_{\frac{R}{2}}^{\rho(h)} t^{-2(Q-2)\rho_0} + Q-1 \, dt \right)^{\frac{1}{\rho_0}}
\]
\[
\leq C \rho(h)^{\frac{Q}{6} - 2} R^{-2(Q-2) + \frac{Q}{6}}
\]
\[
= \frac{C}{R^2-2} \left( \frac{R}{\rho(h)} \right)^{2-\frac{Q}{6}}.
\]
Using Lemma 5 and taking $l$ sufficiently large, we obtain

$$
\rho(h)^l \int_{|w| \geq \rho(h)} \frac{V(w) \, dw}{|h^{-1}w|^{2(Q-2)+l}} \leq C l \rho(h)^l \sum_{j=1}^{\infty} (2^j \rho(h))^{-2(Q-2)-l} \int_{B(h,2^j \rho(h))} V(w) \, dw \\
\leq \frac{C l}{\rho(h)Q-2} \sum_{j=1}^{\infty} 2^{-(l-l_1+Q-2)j} \\
\leq \frac{C l}{\rho(h)Q-2} \\
\leq \frac{C l}{RQ-2} \left( \frac{R}{\rho(h)} \right)^{2-\frac{Q}{q_0}}.
$$

By Lemma 4, $\rho(h) \sim \rho(g)$ when $|h^{-1}g| \leq \rho(g)$. Lemma 6 is proved.

**Lemma 7.** For any $l > 0$, there exists $C_l > 0$ such that

$$
K_s^L(g,h) \leq \frac{C_l}{(1+|h^{-1}g|(|\rho(g)^{-1}+\rho(h)^{-1}))^l} \left| h^{-1}g \right|^{Q}.
$$

Let $A \geq 1$ be a fixed constant. If $|h^{-1}g| \leq A \rho(g)$, then

$$
|K_s^L(g,h) - H_s(g,h)| \leq \frac{C}{\rho(g)^\delta |h^{-1}g|^{Q-\delta}},
$$

where $\delta = 2 - \frac{Q}{q_0} > 0$.

**Proof.** Note that

$$
K_s^L(g,h) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{is\tau} \Gamma^L(g,h,\tau) \, d\tau, \\
H_s(g,h) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{is\tau} \Gamma(g,h,\tau) \, d\tau.
$$

Then (7) follows from (5) by integration. By the same way, (8) follows from Lemma 6 when $A = 1$. Since

$$
|K_s^L(g,h) - H_s(g,h)| \leq 2H_s(g,h) \leq \frac{C}{|h^{-1}g|^{Q}},
$$

(8) still holds for $A > 1$.}

Let

$$
R_j = X_j(-\Delta_{H^n})^{-\frac{1}{2}}, \quad j = 1, \cdots, 2n,
$$

be the usual Riesz transforms with the convolution kernels $R_j(g)$. We denote the kernel of $R_j^L$ by $R_j^L(g,h)$ and write $R_j(g,h) = R_j(h^{-1}g)$. Let

$$
\mathcal{R}^L = (R_1^L, \cdots, R_{2n}^L) = \nabla_{H^n} L^{-\frac{1}{2}}, \quad \mathcal{R} = (R_1, \cdots, R_{2n}) = \nabla_{H^n} (-\Delta_{H^n})^{-\frac{1}{2}}.
$$
The kernels of \(R^L\) and \(R\) are denoted by \(R^L(g, h)\) and \(R(g, h)\), respectively. By functional calculus, we have

\[
L^{-\frac{1}{2}} = \frac{1}{2\pi} \int_{\mathbb{R}} (-i\tau)^{-\frac{1}{2}}(L + i\tau)^{-1} d\tau.
\]

Thus,

\[
R^L(g, h) = \frac{1}{2\pi} \int_{\mathbb{R}} (-i\tau)^{-\frac{1}{2}}\nabla_{\mathbb{H}^n,g} \Gamma^L(g, h, \tau) d\tau.
\]

Similarly,

\[
R(g, h) = \frac{1}{2\pi} \int_{\mathbb{R}} (-i\tau)^{-\frac{1}{2}}\nabla_{\mathbb{H}^n,g} \Gamma(g, h, \tau) d\tau.
\]

**Lemma 8.** Let \(A > 0\) be a fixed constant. Then

\[
\int_{|h^{-1}g| \geq A\rho(h)} |R^L(g, h)| \, dg \leq C.
\]

**Proof.** Let us fix \(g_0\) and \(h\) such that \(R = \frac{|h^{-1}g_0|}{1} > 0\). Let \(u(g) = \Gamma^L(g, h, \tau)\). Then \(u\) satisfies the equation \(-\Delta_{\mathbb{H}^n} u + (V + i\tau)u = 0\) in \(B(g_0, 2R)\). Take \(\phi \in C_{C_{\infty}}(B(g_0, 2R))\) such that \(\phi \equiv 1\) on \(B(g_0, R)\), \(0 \leq \phi \leq 1\), \(|\nabla_{\mathbb{H}^n} \phi| \leq \frac{C}{R}\), and \(|\nabla_{\mathbb{H}^n}^2 \phi| \leq \frac{C}{R^2}\). For \(g' \in B(g_0, R)\), we have

\[
u(g') = \int_{\mathbb{H}^n} \Gamma(g', g, \tau)(-\Delta_{\mathbb{H}^n} + i\tau)(u\phi)(g) \, dg
\]

\[
= \int_{\mathbb{H}^n} \Gamma(g', g, \tau)(-V(g)u(g)\phi(g) - 2\nabla_{\mathbb{H}^n} u(g) \cdot \nabla_{\mathbb{H}^n} \phi(g) - u(g)\Delta_{\mathbb{H}^n} \phi(g)) \, dg
\]

\[
+ 2\int_{\mathbb{H}^n} u(g)\nabla_{\mathbb{H}^n} \Gamma(g', g, \tau) \cdot \nabla_{\mathbb{H}^n} \phi(g) \, dg.
\]

By [4],

\[
|\nabla_{\mathbb{H}^n} u(g_0)| \leq C \int_{B(g_0, 2R)} \frac{V(g) |u(g)|}{|g_0^{-1}|g^{Q-1}} \, dg + \frac{C}{R^{Q+1}} \int_{B(g_0, 2R)} |u(g)| \, dg
\]

\[
\leq C \sup_{B(g_0, 2R)} |u(g)| \left( \int_{B(g_0, 2R)} \frac{V(g) \, dg}{|g_0^{-1}|g^{Q-1}} + \frac{1}{R} \right).
\]

Since \(|h^{-1}g| \sim R\) for any \(g \in B(g_0, 2R)\), by [5],

\[
|\nabla_{\mathbb{H}^n} \Gamma^L(g_0, h, \tau)| \leq \frac{C_l}{(1 + R|\tau|^{\frac{1}{2}})^l} \left( \frac{1}{R^{Q-2}} \int_{B(g_0, 2R)} \frac{V(g) \, dg}{|g_0^{-1}|g^{Q-1}} + \frac{1}{R^{Q-1}} \right).
\]

In view of [10], by integration we get

\[
|R^L(g_0, h)| \leq \frac{C_l}{(1 + R\rho(h)^{-1})^l} \left( \frac{1}{R^{Q-1}} \int_{B(g_0, 2R)} \frac{V(g) \, dg}{|g_0^{-1}|g^{Q-1}} + \frac{1}{R^Q} \right),
\]
which means

\[(12) \quad |R_j^L(g, h)| \leq \frac{C_l}{(1 + |h^{-1}g| \rho(h)^{-1})^{\frac{1}{2}}} \left( \frac{1}{|h^{-1}g|^{Q-1}} \int_{B(g, \frac{|h^{-1}g|}{2})} V(w) \, dw \right) ^{\frac{1}{q_0}} + \frac{1}{|h^{-1}g|^{Q-1}}.
\]

Set \( r = A \rho(h) \). By (12) and the boundedness of fractional integrals (cf. [7, Proposition 6.2]), for \( k \geq 1 \) and \( \frac{1}{p_0} = \frac{1}{q_0} - \frac{1}{Q} \), we get

\[
\left( \int_{2^k r \leq |h^{-1}g| < 2^{k+1} r} |R_j^L(g, h)|^{p_0} \, dg \right)^{\frac{1}{p_0}} \leq C_l 2^{-k l} \left( \frac{1}{(2^k r)^{Q-1}} \left( \int_{|h^{-1}g| < 2^{k+1} r} V(g)^{q_0} \, dg \right)^{\frac{1}{q_0}} + (2^k r)^{-\frac{q_0}{Q}} \right)
\]

\[
\leq C_l 2^{-k l} (2^k r)^{-\frac{q_0}{Q}} \left( \frac{1}{(2^{k+1} r)^{Q-2}} \int_{|h^{-1}g| < 2^{k+1} r} V(g) \, dg + 1 \right)
\]

\[
\leq C_l 2^{-k l} (2^k r)^{-\frac{q_0}{Q}} (2^{k l_1} + 1) \leq C 2^{-k} (2^k r)^{-\frac{q_0}{Q}}
\]

provided \( l \) large enough, where we used the \( B_{q_0} \) condition for the second inequality and Lemma 5 for the third inequality.

Then by Hölder’s inequality,

\[
\int_{|h^{-1}g| \geq r} |R_j^L(g, h)| \, dg \leq C \sum_{k=1}^{\infty} \left( \int_{2^{k-1} r \leq |h^{-1}g| < 2^k r} |R_j^L(g, h)|^{p_0} \, dg \right)^{\frac{1}{p_0}} (2^k r)^{-\frac{q_0}{Q}}
\]

\[
\leq C \sum_{k=1}^{\infty} 2^{-k} = C;
\]

the lemma is proved.

\[\square\]

**Lemma 9.** Let \( A > 0 \) be a fixed constant. Then

\[
\int_{B(h, A \rho(h))} |R_j^L(g, h) - R_j(g, h)| \, dg \leq C.
\]

**Proof.** Set \( r = A \rho(h) \). Suppose \( |h^{-1}g| \leq r \) and let \( R = \frac{|h^{-1}g|}{4} \). By (12), (13) and (6),

\[
|\nabla_{H^n, g} \Gamma^L(g, h, \tau) - \nabla_{H^n, g} \Gamma(g, h, \tau)|
\]

\[
\leq \int_{H^n} |\nabla_{H^n, g} \Gamma(g, w, \tau)| \, V(w) \, |\Gamma^L(w, h, \tau)| \, dw
\]

\[
\leq \int_{H^n} \frac{C_l}{(1 + |g^{-1}w| |\tau|^{\frac{1}{2}})^{\frac{1}{2}}} |g^{-1}w|^{Q-1} \, V(w) \, dw
\]

\[
= \int_{|g^{-1}w| < R} + \int_{|h^{-1}w| < R} + \int_{|h^{-1}w| \geq R |g^{-1}w| \geq R}
\]

\[
= J_1 + J_2 + J_3.
\]

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It is easy to see that
\[
J_1 \leq \frac{C_l}{(1 + R |\tau|^\frac{1}{2})^l} \left( R^{Q-2} \right) \int_{B(g,R)} \frac{V(w) \, dw}{|g^{-1}w|^Q}
\]
and
\[
J_2 \leq \frac{C_l}{(1 + R |\tau|^\frac{1}{2})^l} \left( R^{Q-1} \right) \int_{B(h,R)} \frac{V(w) \, dw}{|h^{-1}w|^Q}.
\]

The above three estimates give
\[
\int_{|h^{-1}w| \geq R} \frac{V(w) \, dw}{|h^{-1}w|^Q}
\]
where we used Lemma 5 for the second inequality and Lemma 2 together with Lemma 3 for the last inequality.

Note that $|g^{-1}w| \sim |h^{-1}w|$ when $|g^{-1}w| \geq R$ and $|h^{-1}w| \geq R$. We have
\[
J_3 \leq \frac{C_l}{(1 + R |\tau|^\frac{1}{2})^l} \int_{|h^{-1}w| \geq R} \frac{V(w) \, dw}{\left( 1 + |h^{-1}w| \rho(h)^{-1} \right)^l |h^{-1}w|^{2Q-3}}
\]
\[
\leq \frac{C_l}{(1 + R |\tau|^\frac{1}{2})^l} \left( \int_{r>|h^{-1}w| \geq R} \frac{V(w) \, dw}{|h^{-1}w|^{2Q-3}} + r^l \int_{|h^{-1}w| \geq 2r} \frac{V(w) \, dw}{|h^{-1}w|^{2Q-3+1}} \right).
\]

It follows from the Hölder inequality and $B_{q_0}$ condition that
\[
\int_{r>|h^{-1}w| \geq R} \frac{V(w) \, dw}{|h^{-1}w|^{2Q-3}} \leq C \left( \int_{B(h,r)} V(w)^{q_0} \, dw \right)^\frac{1}{q_0} \left( \int_{R} t^{-(2Q+3)q_0^{-1}+Q-1} \, dt \right)^\frac{1}{q_0}
\]
\[
\leq C \frac{q_0^{-2}}{R^{-2Q+3+\frac{2}{q_0}}}
\]
\[
= \frac{C}{R^{Q-1}} \left( \frac{R}{r} \right)^{2-\frac{Q}{q_0}}.
\]

Using Lemma 5 and taking $l$ sufficiently large, we obtain
\[
r^l \int_{|h^{-1}w| \geq 2r} \frac{V(w) \, dw}{|h^{-1}w|^{2Q-3+1}} \leq C r^l \sum_{j=1}^{\infty} \left( 2^j r \right)^{-2Q+3-l} \int_{B(h,2^j r)} V(w) \, dw
\]
\[
\leq \frac{C}{r^{Q-1}} \sum_{j=1}^{\infty} 2^{-(l+l_1+Q-1)j}
\]
\[
\leq \frac{C}{r^{Q-1}} \left( \frac{R}{r} \right)^{2-\frac{Q}{q_0}}.
\]

The above three estimates give
\[
J_3 \leq \frac{C_l}{(1 + R |\tau|^\frac{1}{2})^l} \left( R^{Q-1} \right) \left( \frac{R}{r} \right)^{2-\frac{Q}{q_0}}.
\]
Therefore, for $|h^{-1}g| < r \sim \rho(h)$,
\[
\left| \nabla_{H^n,g} \Gamma^L(g,h,\tau) - \nabla_{H^n,g} \Gamma(g,h,\tau) \right| 
\leq C \left( 1 + |h^{-1}g| \frac{1}{|\tau|^{\frac{1}{2}}} \right) \left( \frac{1}{|h^{-1}g|^{Q-2}} \int_{B(g,|h^{-1}g|^{\frac{1}{2}})} V(w) \, dw + \frac{1}{|h^{-1}g|^{Q-1}} \left( \frac{|h^{-1}g|}{r} \right)^{2-\frac{Q}{p_0}} \right).
\]

In view of [(10)] and [(11)], by integration we obtain
\[
(13) \quad |R^L_j(g,h) - R_j(g,h)| \leq C \frac{1}{|h^{-1}g|^{Q-1}} \int_{B(h,2r)} V(g)^{\frac{1}{p_0}} \, dg + C \frac{1}{|h^{-1}g|^{Q-1}} \left( \frac{|h^{-1}g|}{r} \right)^{2-\frac{Q}{p_0}}.
\]

It follows from the boundedness of fractional integrals and (13) that, for $k \leq 0$,
\[
\left( \int_{2^{k-1}r \leq |h^{-1}g| < 2^k r} |R^L_j(g,h) - R_j(g,h)|^{p_0} \, dg \right)^{\frac{1}{p_0}} 
\leq \frac{C}{(2^{k}r)^{Q-1}} \left( \int_{B(h,2r)} V(g)^{q_0} \, dg \right)^{\frac{1}{q_0}} + C 2^{k(2-\frac{Q}{p_0})(2k)^{\frac{Q}{p_0}-Q}} 
\leq C 2^{k(2-\frac{Q}{p_0})(2k)^{\frac{Q}{p_0}} - \frac{Q}{p_0}}.
\]

By Hölder’s inequality, we obtain
\[
\int_{|g^{-1}h| < r} |R^L_j(g,h) - R_j(g,h)| \, dg 
\leq \sum_{k=-\infty}^{0} \left( \int_{2^{k-1}r \leq |g^{-1}h| < 2^k r} |R^L_j(g,h) - R_j(g,h)|^{p_0} \, dg \right)^{\frac{1}{p_0}} (2^{k}r)^{\frac{Q}{p_0}} 
\leq C \sum_{k=-\infty}^{0} 2^{k(2-\frac{Q}{p_0})} = C.
\]

The proof of Lemma 9 is completed. \hfill \Box

4 Local Hardy spaces

An $H^1_L$-function is locally equal to a function in a certain scaled local Hardy space. Before dealing with $H^1_L(\mathbb{H}^n)$, we first discuss local Hardy spaces on the Heisenberg group. The theory of local Hardy spaces on $\mathbb{R}^n$ was studied by Goldberg [9]. It is sure that the local version of Hardy spaces can be extended to more general setting such as homogeneous groups. However, there is no reference about it as far as the authors know.

We recall some results of Hardy spaces on the Heisenberg group (cf. [7]). Let $\mathscr{S}(\mathbb{H}^n)$ denote the Schwartz class and $\mathscr{S}'(\mathbb{H}^n)$ be the space of tempered distributions. For $f \in \mathscr{S}(\mathbb{H}^n)$, we have
\[
\int_{|z^{-1}w| < r} \left| f(z^{-1}w) - f(w) \right| \, dw 
\leq C \int_{B(0,r)} \left| f(z^{-1}w) - f(w) \right| \, dw 
\leq C \int_{B(0,2r)} \left| f(z^{-1}w) - f(w) \right| \, dw 
\leq C \int_{B(0,3r)} \left| f(z^{-1}w) - f(w) \right| \, dw 
\leq C \int_{B(0,4r)} \left| f(z^{-1}w) - f(w) \right| \, dw 
\leq \sum_{k=-\infty}^{0} \left( \int_{2^{k-1}r \leq |z^{-1}w| < 2^k r} \left| f(z^{-1}w) - f(w) \right|^{p_0} \, dw \right)^{\frac{1}{p_0}} (2^{k}r)^{\frac{Q}{p_0}} 
\leq C \sum_{k=-\infty}^{0} 2^{k(2-\frac{Q}{p_0})} = C.
\]
and \( \phi \in \mathcal{S}'(\mathbb{H}^n) \), we define the nontangential maximal function \( M_\phi f \) and the radial maximal function \( M_\phi^+ f \) of \( f \) with respect to \( \phi \) by

\[
M_\phi f(g) = \sup_{|g^{-1}h| < r < \infty} \left| f \ast \phi_r(h) \right|,
\]

\[
M_\phi^+ f(g) = \sup_{r > 0} \left| f \ast \phi_r(g) \right|,
\]

where

\[
\phi_r(g) = r^{-Q} \phi(\delta_1 g).
\]

Denote by \( \{Y_j : j = 1, \cdots, 2n + 1\} \) the basis for the right-invariant vector fields on \( \mathbb{H}^n \) corresponding to \( \{X_j : j = 1, \cdots, 2n + 1\} \); that is,

\[
Y_{2n+1} = X_{2n+1}, \quad Y_j = \frac{\partial}{\partial x_j} - 2r x_{n+j} \frac{\partial}{\partial t}, \quad Y_{n+j} = \frac{\partial}{\partial x_{n+j}} + 2r \frac{\partial}{\partial t}, \quad j = 1, \cdots, n.
\]

Let \( N \in \mathbb{N} \) and \( \phi \in \mathcal{S}(\mathbb{H}^n) \). We define a seminorm on \( \mathcal{S}(\mathbb{H}^n) \) by

\[
\|\phi\|_{(N)} = \sup_{g \in \mathbb{H}^n} \| (1 + |g|)^{(N+1)(Q+1)} Y^I \phi(g) \|,
\]

where

\[
Y = (Y_1, \cdots, Y_{2n+1}), \quad I = (i_1, \cdots, i_{2n+1}), \quad Y^I = Y_1^{i_1} \cdots Y_{2n+1}^{i_{2n+1}}, \quad \text{and} \quad |I| = \sum_{j=1}^{2n+1} i_j.
\]

For \( f \in \mathcal{S}'(\mathbb{H}^n) \), we define the nontangential grand maximal function \( M_{(N)} f \) and the radial grand maximal function \( M_{(N)}^+ f \) by

\[
M_{(N)} f(g) = \sup_{\phi \in \mathcal{S}, |\phi|_{(N)} \leq 1} M_\phi f(g),
\]

\[
M_{(N)}^+ f(g) = \sup_{\phi \in \mathcal{S}, |\phi|_{(N)} \leq 1} M_\phi^+ f(g).
\]

For \( 0 < p \leq 1 \), we set \( N_p = \lfloor Q(\frac{1}{p} - 1) \rfloor + 1 \). The Hardy space \( H^p(\mathbb{H}^n) \) is defined by

\[
H^p(\mathbb{H}^n) = \{ f \in \mathcal{S}'(\mathbb{H}^n) : M_{(N_p)} f \in L^p(\mathbb{H}^n) \}
\]

with

\[
\|f\|_{H^p} = \| M_{(N_p)} f \|_{L^p}.
\]

We say that a function \( \phi \in \mathcal{S}(\mathbb{H}^n) \) is a \textit{commutative approximate identity} if \( \phi \) satisfies

\[
\int_{\mathbb{H}^n} \phi(g) \, dg = 1 \quad \text{and} \quad \phi_r \ast \phi_s = \phi_s \ast \phi_r \quad \text{for all} \quad r, s > 0.
\]

We note that \( \phi_r \ast \phi_s = \phi_s \ast \phi_r \) if \( \phi(x,t) \) is radial or polyradial (cf. \[7\]) with respect to \( x \).
Proposition 1 (\[7, Chapter 4\]). Suppose \( f \in S(H^n) \) and \( 0 < p \leq 1 \). Let \( \phi \) be a commutative approximate identity and \( N \geq N_p \) be fixed. Then

\[
\| M(N)f \|_{L^p} \sim \| M^+_{(N)}f \|_{L^p} \sim \| M_\phi f \|_{L^p} \sim \| M^+_\phi f \|_{L^p}.
\]

Proposition \([7]\) tells us that all these maximal functions give equivalent characterizations of the Hardy space \( H^p(H^n) \). Specifically, the heat kernel \( H_s(g) = \varphi(g) (g) \) where \( \varphi(g) = H_1(g) \) is a commutative approximate identity. Therefore the radial heat maximal function \( Mf = M^+_\phi f \) characterizes the Hardy space \( H^p(H^n) \).

A significant fact of the Hardy space \( H^p(H^n) \) is that an \( H^p \) distribution admits an atomic decomposition. Let \( 0 < p \leq 1 < q \leq \infty \). A function \( a \in L^q(H^n) \) is called an \( H^{p,q} \)-atom if the following conditions hold:

(i) \( \text{supp } a \subset B(g_0, r) \),

(ii) \( \| a \|_{L^q} \leq |B(g_0, r)|^{\frac{1}{q} - \frac{1}{p}} \),

(iii) \( \int_{B(g_0, r)} a(g) g^I dg = 0 \) for \( d(I) < N_p \),

where \( g^I \) denotes the monomial

\[
(x, t)^I = x_1^{i_1} \cdots x_{2n}^{i_{2n+1}}
\]

and \( d(I) \) is the homogeneous degree of \( g^I \) defined by

\[
d(I) = 2i_{2n+1} + \sum_{j=1}^{2n} i_j.
\]

Proposition 2 (\[7, Chapter 3\]). Let \( 0 < p \leq 1 < q \leq \infty \). A distribution \( f \in H^p(H^n) \) if and only if \( f \) can be written as \( f = \sum_j \lambda_j a_j \), where \( a_j \) are \( H^{p,q} \)-atoms, \( \sum_j |\lambda_j|^p < \infty \), and the sum converges in the sense of distributions. Moreover,

\[
\| f \|_{H^p}^p \sim \inf \left\{ \sum_j |\lambda_j|^p \right\},
\]

where the infimum is taken over all atomic decompositions of \( f \) into \( H^{p,q} \)-atoms.

Now we go to the local version of Hardy spaces. We define the local maximal functions \( \widetilde{M}f \)'s by taking supremum over \( 0 < r \leq 1 \) instead of \( 0 < r < \infty \) as follows.

\[
\widetilde{M}_\phi f(g) = \sup_{|g^{-1}h| < r \leq 1} |f * \phi_r(h)|, \quad \widetilde{M}^+_\phi f(g) = \sup_{0 < r \leq 1} |f * \phi_r(g)|,
\]

\[
\widetilde{M}(N)f(g) = \sup_{\|\phi\|_{(N)} \leq 1} \widetilde{M}_\phi f(g), \quad \widetilde{M}^+(N)f(g) = \sup_{\|\phi\|_{(N)} \leq 1} \widetilde{M}^+_\phi f(g).
\]
Definition 3. Let $0 < p \leq 1$. The local Hardy space $h^p(\mathbb{H}^n)$ is defined by

$$h^p(\mathbb{H}^n) = \{ f \in \mathcal{S}'(\mathbb{H}^n) : \tilde{M}_{(N_p)} f \in L^p(\mathbb{H}^n) \}$$

with

$$\|f\|_{h^p} = \|\tilde{M}_{(N_p)} f\|_{L^p}.$$

Similar to Proposition 1, by the same argument as in [7], we have

Proposition 3. Suppose $f \in \mathcal{S}'(\mathbb{H}^n)$ and $0 < p \leq 1$. Let $\phi$ be a commutative approximate identity and $N \geq N_p$ be fixed. Then

$$\|\tilde{M}_{(N)} f\|_{L^p} \sim \|\tilde{M}_{(N)}^+ f\|_{L^p} \sim \|\tilde{M}_\phi f\|_{L^p} \sim \|\tilde{M}_\phi^+ f\|_{L^p}.$$

All these local maximal functions give equivalent characterizations of the local Hardy space $h^p(\mathbb{H}^n)$.

Lemma 10. Let $0 < p \leq 1$ and $f \in h^p(\mathbb{H}^n)$. If $\psi \in \mathcal{S}(\mathbb{H}^n)$ satisfies

$$\int_{\mathbb{H}^n} \psi(g) dg = 1 \quad \text{and} \quad \int_{\mathbb{H}^n} \psi(g)g^I \ dg = 0 \quad \text{for} \quad 0 < d(I) < m = (N_p + 1)(Q + 2)$$

(the cancellation conditions of $\psi$ can be relaxed to an extent, but it is not important for us), then $f - f * \psi \in h^p(\mathbb{H}^n)$ and there exists $C > 0$ such that

$$\|f - f * \psi\|_{H^p} \leq C \|f\|_{h^p}.$$

Proof. Let $\phi$ be a commutative approximate identity. We have

$$M^+_{\phi}(f - f * \psi)(g) \leq \tilde{M}^+_{\phi} f(g) + \sup_{0 < r \leq 1} |f * (\psi * \phi_r)(g)|$$

(14)

$$+ \sup_{1 < r < \infty} |f * (\phi_r - \psi * \phi_r)(g)|.$$

We will prove that there exists $C > 0$ such that

$$\sup_{0 < r \leq 1} |f * (\psi * \phi_r)(g)| \leq C \tilde{M}^+_{(N_p)} f(g),$$

(15)

$$\sup_{1 < r < \infty} |f * (\phi_r - \psi * \phi_r)(g)| \leq C \tilde{M}^+_{(N_p)} f(g).$$

(16)

Then Lemma 10 follows from (14), (15), (16) and Proposition 3.

By [7, Proposition 1.47], the estimate (15) follows from

$$\sup_{0 < r \leq 1} \|\psi * \phi_r\|_{(N_p)} \leq C \|\psi\|_{(N_p+1)} \|\phi\|_{(N_p+1)}.$$
Therefore
\[
\int \psi(h) \frac{d}{dh} (\phi(h) - \phi_r(h)) dh \leq C \int |h|^m \left( \frac{|h|}{|h|} \right) \left( \frac{|h|}{|h|} \right) dh
\]

where \( P_{dJ} \) are homogeneous polynomials of degree \( d(J) - d(I) \). Hence
\[
\|\phi\|_{(N)} \leq C \sup_{\phi \in \mathbb{H}^n, |I| \leq N} (1 + |g|)^{(N+1)(Q+2)} |X^I \phi(g)|.
\]
To obtain the estimate (16), we only need to show
\[
(17) \quad \sup_{1 < |I| \leq |g|} \left| X^I (\phi_r - \psi * \phi_r)(g) \right| \leq C (1 + |g|)^{-m}.
\]
For \( f = X^I \phi \), we use the following Taylor inequality (cf. [7, Corollary 1.44])
\[
|f(hg) - P_g(h)| \leq C |h|^m \sup_{d(J) = m} \left| Y^J f(h_1 g) \right|
\]
where \( P_g(h) \) is the right Taylor polynomial of \( f \) at the point \( g \) of degree \( m - 1 \) and \( b > 1 \) is a constant. From the cancellation conditions of \( \psi \) we get
\[
\left| X^I (\phi_r - \psi * \phi_r)(g) \right| = \left| \int \psi(h) \frac{d}{dh} (X^I \phi - (X^I \phi)(h^{-1} g)) dh \right|
\]
\[
\leq C \int \left| \psi(h) \right| |h|^m \sup_{d(J) = m} \left| Y^J X^I \phi_r(h_1 g) \right| dh
\]
\[
\leq C \left( \int_{|h| \leq \frac{1}{2} b^{-m} |g|} |\psi(h)| |h|^m dh + \int_{|h| > \frac{1}{2} b^{-m} |g|} |\psi(h)| |h|^m dh \right).
\]
For \( |h| \leq \frac{1}{2} b^{-m} |g| \),
\[
\sup_{d(J) = m} \left| Y^J X^I \phi_r(h_1 g) \right| \leq C (1 + |g|)^{-d(I) - m - Q} \leq C (1 + |g|)^{-m}.
\]
Therefore
\[
\int_{|h| \leq \frac{1}{2} b^{-m} |g|} |\psi(h)| |h|^m \sup_{d(J) = m} \left| Y^J X^I \phi_r(h_1 g) \right| dh
\]
\[
\leq C (1 + |g|)^{-m} \int_{|h| \leq \frac{1}{2} b^{-m} |g|} |\psi(h)| |h|^m dh
\]
\[
\leq C (1 + |g|)^{-m}.
\]
On the other hand,
\[
\int_{|h| > \frac{1}{2} b^{-m} |g|} |\psi(h)| |h|^m \sup_{d(J) = m} \left| Y^J X^I \phi_r(h_1 g) \right| dh
\]
\[
\leq C \int_{|h| > \frac{1}{2} b^{-m} |g|} |\psi(h)| |h|^m dh
\]
\[
\leq C (1 + |g|)^{-m}.
\]
Thus we obtain (17) and complete the proof of Lemma 10.
Although the ball $B(g, r)$ is the left translation by $g$ of $B(0, r)$, the shape of $B(g, r)$, as a set of points in $\mathbb{R}^{2n+1}$, much varies with the position of the center $g$. For $g = (x, t)$ and $h = (y, s)$, let $d_E(g, h) = (|x - y|^2 + |t - s|^2)^{\frac{1}{2}}$ denote the Euclidean distance between two points $g$ and $h$. Given a large positive number $A$, there exist points $g$ and $h$ such that $d(g, h) = 1$ and $d_E(g, h) \geq A$, and vice versa. However, we still have the following covering lemma for $\mathbb{H}^n$.

**Lemma 11.** Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{2n+1}) \in \mathbb{Z}^{2n+1}$, $g_\alpha = \frac{1}{2\sqrt{q}} (\alpha_1, \alpha_2, \cdots, \alpha_{2n+1})$, and $B_\alpha = B(g_\alpha, \frac{1}{2})$. Then $\mathbb{H}^n = \bigcup_\alpha B_\alpha$ and $\{ B_\alpha \}$ has finite overlaps property; that is, $1 \leq \sum_\alpha \chi_{B_\alpha}(g) \leq C$.

**Proof.** Lemma 11 is almost obvious. In fact, for any $g \in \mathbb{H}^n$, there exists some $g_\alpha$ such that $|g_\alpha^{-1}g| < \frac{1}{2}$. On the other hand, for any $g_\alpha \neq g_\alpha'$, $|g_\alpha^{-1}g_\alpha'| \geq \frac{1}{2\sqrt{q}}$. \qed

Now we give the atomic decomposition of $h^p(\mathbb{H}^n)$.

**Definition 4.** Let $0 < p \leq 1 < q \leq \infty$. A function $a \in L^q(\mathbb{H}^n)$ is called an $h^{p,q}$-atom if the following conditions hold:

(i) $\text{supp } a \subset B(g_0, r)$,

(ii) $\| a \|_{L^q} \leq |B(g_0, r)|^{\frac{1}{q} - \frac{1}{p}}$,

(iii) if $r < \frac{1}{2}$, then $\int_{B(g_0, r)} a(g) g^I dg = 0$ for $d(I) < N_p$.

**Theorem 4.** Let $0 < p \leq 1 < q \leq \infty$. A distribution $f \in h^p(\mathbb{H}^n)$ if and only if $f$ can be written as $f = \sum_j \lambda_j a_j$ converging in the sense of distributions and in $h^p(\mathbb{H}^n)$ norm, where $a_j$ are $h^{p,q}$-atoms and $\sum_j |\lambda_j|^p < \infty$. Moreover,

$$\| f \|_{h^p} \sim \inf \left\{ \sum_j |\lambda_j|^p \right\},$$

where the infimum is taken over all atomic decompositions of $f$ into $h^{p,q}$-atoms.

**Proof.** Let $f \in h^p(\mathbb{H}^n)$ and $\psi$ satisfy the conditions in Lemma 11. Thus, $f - f \ast \psi \in H^p(\mathbb{H}^n)$ and $\| f - f \ast \psi \|_{H^p} \leq C \| f \|_{h^p}$. By the atomic decomposition of $H^p(\mathbb{H}^n)$, $f - f \ast \psi = \sum_j \lambda_j a_j$, where $a_j$ are $H^{p,q}$-atoms and $\sum_j |\lambda_j|^p \leq C \| f - f \ast \psi \|_{H^p}$. It is clear that an $H^{p,q}$-atom is also an $h^{p,q}$-atom.

Let $\{ B_\alpha \}$ be the collection of balls given in Lemma 11 and $\chi_\alpha$ denote the characteristic function of $B_\alpha$. Set

$$\xi_\alpha(g) = \frac{\chi_\alpha(g)}{\sum_\alpha \chi_\alpha(g)}$$

(18)
and write
\[ (f \ast \psi) \xi_\alpha (g) = \lambda_\alpha a_\alpha (g), \]
where
\[ \lambda_\alpha = |B_\alpha|^{-\frac{1}{p} - \frac{1}{q}} \| (f \ast \psi) \xi_\alpha \|_{L^q}. \]
Then all \( a_\alpha \) are \( h^{p,q} \)-atoms and \( f \ast \psi = \sum_\alpha \lambda_\alpha a_\alpha \). Moreover,
\[
\sum_\alpha |\lambda_\alpha|^p = \sum_\alpha |B_\alpha|^{1 - \frac{p}{q}} \| (f \ast \psi) \xi_\alpha \|^p_{L^q} \leq \sum_\alpha \int_{B_\alpha} (\tilde{M}_\psi f)(g))^p \, dg \leq C \| f \|^p_{h^{p,q}}.
\]
Conversely, let \( a \) be an \( h^{p,q} \)-atom supported on a ball \( B(g_0, r) \). If \( r < \frac{1}{2} \), then \( a \) is also an \( H^{p,q} \)-atom and \( \| a \|^p_{h^{p,q}} \leq \| a \|^p_{H^{p,q}} \leq C \) by Proposition [2]. If \( r \geq \frac{1}{2} \), we choose a commutative approximate identity \( \phi \) such that \( \text{supp} \phi \subset B(0, \frac{1}{2}) \). Then \( \tilde{M}_\phi^+ a \) is supported in \( B(g_0, 2r) \) and \( \| \tilde{M}_\phi^+ a \|_{L^q} \leq C \| a \|_{L^q} \). By Proposition [3]
\[
\| a \|^p_{h^{p,q}} \leq C \| \tilde{M}_\phi^+ a \|^p_{L^q} \leq C \| B(g_0, 2r) \|^{1 - \frac{q}{p}} \| \tilde{M}_\phi^+ a \|^p_{L^q} \leq C \| B(g_0, r) \|^{1 - \frac{q}{p}} \| a \|^p_{L^q} \leq C.
\]
If \( f = \sum_j \lambda_j a_j \), then we have
\[
\| f \|^p_{h^{p,q}} \leq \sum_j |\lambda_j|^p \| a_j \|^p_{h^{p,q}} \leq C \sum_j |\lambda_j|^p.
\]
The proof of Theorem [4] is completed. \( \square \)

**Remark 1.** The restriction \( r < \frac{1}{2} \) in the condition (iii) of Definition [4] is not essential. By a dilation argument, it is easy to know that Theorem [4] still holds if we replace \( \frac{1}{2} \) by a fixed positive constant \( A \).

**Remark 2.** We should pay attention to the \( h^{p,q} \)-atoms appearing in the atomic decomposition of \( f \in h^p(\mathbb{H}^n) \) without the cancellation conditions, which come from \( f \ast \psi \). If \( f \in h^p(\mathbb{H}^n) \) is supported in a ball \( B(g_0, 1) \), we can choose \( \psi \) satisfying the conditions in Lemma [10] and \( \text{supp} \psi \subset B(0, 1) \). Then \( f \ast \psi \) itself is multiple of \( h^{p,q} \)-atom supported on the ball \( B(g_0, 2) \).

We are mainly interested in the case \( p = 1 \). The Hardy space \( H^1(\mathbb{H}^n) \) can be characterized by the Riesz transforms \( R_j \) (cf. [2]). That is, \( f \in H^1(\mathbb{H}^n) \) if and only if \( f \in L^1(\mathbb{H}^n) \) and \( R_j f \in L^1(\mathbb{H}^n), \ j = 1, \cdots, 2n \). Moreover,
\[
(19) \quad \| f \|_{H^1} \sim \| f \|_{L^1} + \sum_{j=1}^{2n} \| R_j f \|_{L^1}.
\]
The local Hardy space \( h^1(\mathbb{H}^n) \) has a similar characterization. On the Euclidean spaces the local Riesz transforms characterization of \( h^1(\mathbb{R}^n) \) is obtained via subharmonicity (cf.


[9]. However, this kind of approach fails on the Heisenberg group as pointed out in [2], so we will use a different method. Let $\zeta \in C^{\infty}(\mathbb{H}^n)$ satisfy $0 \leq \zeta(g) \leq 1$, $\zeta(g) = 1$ for $|g| < \frac{1}{2}$ and $\zeta(g) = 0$ for $|g| > 1$. We also assume that $\zeta$ is radial; that is, $\zeta(g)$ depends only on $|g|$. Set $R_j^{[0]}(g) = \zeta(g) R_j(g)$ and $R_j^{[\infty]}(g) = (1 - \zeta(g)) R_j(g)$. Then we define the local Riesz transforms by $\tilde{R}_j f = f * R_j^{[0]}$, $j = 1, \ldots, 2n$.

**Theorem 5.** A function $f \in h^1(\mathbb{H}^n)$ if and only if $f \in L^1(\mathbb{H}^n)$ and $\tilde{R}_j f \in L^1(\mathbb{H}^n)$, $j = 1, \ldots, 2n$. Moreover,

$$\|f\|_{h^1} \sim \|f\|_{L^1} + \sum_{j=1}^{2n} \|\tilde{R}_j f\|_{L^1}.$$ 

**Proof.** Suppose $f \in L^1(\mathbb{H}^n)$ and $\tilde{R}_j f \in L^1(\mathbb{H}^n)$, $j = 1, \ldots, 2n$. Let $\{B_{\alpha}\} = \{B(g_\alpha, \frac{1}{2})\}$ be the collection of balls given in Lemma 11. Let $B_{\alpha}^{**} = B(g_\alpha, 2)$ and $\chi_{\alpha}^{**}$ is the characteristic function of $B_{\alpha}^{**}$. Write

$$f_\alpha(g) = f(g) \xi_\alpha(g),$$

where $\xi_\alpha$ is defined by (18). Also set

$$\tilde{f}(g) = \sum_{\alpha} \lambda_{\alpha} a_\alpha(g),$$

where

$$a_\alpha(g) = \frac{1}{|B_{\alpha}^{**}|} \chi_{\alpha}^{**}(g) \quad \text{and} \quad \lambda_{\alpha} = \int_{B_{\alpha}} f_\alpha(g) \, dg.$$ 

It is obvious that all $a_\alpha$ are $h^{1,\infty}$-atoms and

$$\sum_{\alpha} |\lambda_{\alpha}| \leq \sum_{\alpha} \int_{B_{\alpha}} |f(g)| \, dg \leq C \|f\|_{L^1}.$$ 

By Theorem 4, $\tilde{f} \in h^1(\mathbb{H}^n)$ and

$$\|\tilde{f}\|_{h^1} \leq C \|f\|_{L^1}.$$ 

We will prove $f - \tilde{f} \in H^1(\mathbb{H}^n)$ and

$$\|f - \tilde{f}\|_{H^1} \leq C \left( \|f\|_{L^1} + \sum_{j=1}^{2n} \|\tilde{R}_j f\|_{L^1} \right),$$

which imply $f \in h^1(\mathbb{H}^n)$ and

$$\|f\|_{h^1} \leq \|f - \tilde{f}\|_{H^1} + \|\tilde{f}\|_{h^1} \leq C \left( \|f\|_{L^1} + \sum_{j=1}^{2n} \|\tilde{R}_j f\|_{L^1} \right).$$

Write

$$R_j(f - \tilde{f})(g) = f * R_j^{[0]}(g) - \tilde{f} * R_j^{[0]}(g) + (f - \tilde{f}) * R_j^{[\infty]}(g).$$

It is well known that $R_j(g)$ is a Calderón-Zygmund kernel satisfying
(a) \(|R_j(g)| \leq \frac{C}{|g|^\alpha}\),

(b) \(|R_j(hg) - R_j(g)| \leq \frac{C|h|}{|g|^{\alpha + 1}}\) for \(|h| < \frac{|g|}{2}\),

(c) \(\int_{a<|g|<b} R_j(g) \, dg = 0\) for any \(0 < a < b < \infty\).

If \(|g^{-1}_\alpha g| \leq 1\) or \(|g^{-1}_\alpha g| \geq 3\), then \(a_\alpha * R_j^{(0)}(g) = 0\). When \(1 < |g^{-1}_\alpha g| < 3\), it is not difficult to get

\[|a_\alpha * R_j^{(0)}(g)| \leq C \log \frac{1}{2 - |g^{-1}_\alpha g|},\]

and hence

\[\|a_\alpha * R_j^{(0)}\|_{L^1} \leq C.\]

It follows that

\[(22) \quad \|\tilde{f} * R_j^{(0)}\|_{L^1} \leq \sum_\alpha |\lambda_\alpha| \|a_\alpha * R_j^{(0)}\|_{L^1} \leq C \sum_\alpha |\lambda_\alpha| \leq C \|f\|_{L^1}.\]

Since \(R_j^{\infty}(g) \leq C\), we have

\[|(f_\alpha - \lambda_\alpha a_\alpha) * R_j^{\infty}(g)| \leq C \|f_\alpha - \lambda_\alpha a_\alpha\|_{L^1} \leq C \|f_\alpha\|_{L^1}.\]

Moreover, when \(|g^{-1}_\alpha g| > 4\), we have

\[|(f_\alpha - \lambda_\alpha a_\alpha) * R_j^{\infty}(g)| = \left| \int_{B_{\alpha}^*} (f_\alpha - \lambda_\alpha a_\alpha)(h) \left( R_j^{\infty}(h^{-1}g) - R_j^{\infty}(g^{-1}_\alpha g) \right) \, dh \right| \leq \int_{B_{\alpha}^*} |(f_\alpha - \lambda_\alpha a_\alpha)(h)| \left| R_j^{\infty}(h^{-1}g) - R_j^{\infty}(g^{-1}_\alpha g) \right| \, dh \leq \frac{C}{|g^{-1}_\alpha g|^{Q+1}} \int_{B_{\alpha}^*} |(f_\alpha - \lambda_\alpha a_\alpha)(h)| \, dh \leq \frac{C}{|g^{-1}_\alpha g|^{Q+1}} \|f_\alpha\|_{L^1}\]

that implies

\[\|f_\alpha - \lambda_\alpha a_\alpha\|_{L^1} \leq C \|f_\alpha\|_{L^1}.\]

Thus,

\[(23) \quad \|f - \tilde{f}\|_{L^1} \leq \sum_\alpha \|f_\alpha - \lambda_\alpha a_\alpha\|_{L^1} \leq C \sum_\alpha \|f_\alpha\|_{L^1} \leq C \|f\|_{L^1}.\]
Thus \( \|R_j(f - \tilde{f})\|_{L^1} \leq C \left( \|f\|_{L^1} + \|\tilde{R}_j f\|_{L^1} \right) \).

Then \(20\) follows from \(19\) and \(24\).

Conversely, suppose \( f \in h^1(\mathbb{H}^n) \). We choose \( \psi \) satisfying the conditions in Lemma 10 and \( \text{supp} \psi \subset B(0, 1) \). Using \(19\) and Lemma 10, we get
\[
\|R_j(f - f \ast \psi)\|_{L^1} \leq C \|f - f \ast \psi\|_{H^1} \leq C \|f\|_{h^1}.
\]

Note that
\[
R_j f = R_j(f - f \ast \psi) + f \ast \phi,
\]
where
\[
\phi = \psi \ast R_j - R_j^\infty \in C^\infty(\mathbb{H}^n).
\]

If \( |g| > 2 \), by the same argument as \(17\), we have
\[
|\phi(g)| = |\psi \ast R_j(g) - R_j^\infty(g)| \leq C (1 + |g|)^{-Q-1}.
\]

Thus \( \phi \in L^1(\mathbb{H}^n) \) and
\[
\|f \ast \phi\|_{L^1} \leq C \|f\|_{L^1} \leq C \|f\|_{h^1}.
\]

From \(25\), \(26\) and \(27\), we obtain
\[
\|\tilde{R}_j f\|_{L^1} \leq \|R_j(f - f \ast \psi)\|_{L^1} + \|f \ast \phi\|_{L^1} \leq C \|f\|_{h^1}.
\]

The proof of Theorem 5 is completed.

\[ \Box \]

**Remark 3.** By a dilation argument, for \( 0 < p \leq 1 \) and \( k \in \mathbb{Z} \), we define the scaled local Hardy space \( h^p_k(\mathbb{H}^n) \) to be
\[
h^p_k(\mathbb{H}^n) = \{ f \in \mathcal{S}'(\mathbb{H}^n) : \tilde{M}_{(N_p), k} f \in L^p(\mathbb{H}^n) \}
\]
with
\[
\|f\|_{h^p_k} = \|\tilde{M}_{(N_p), k} f\|_{L^p},
\]
where the scaled local maximal functions \( \tilde{M}_k f \)'s are defined by taking supremum over \( 0 < r \leq 2^k \) instead of \( 0 < r < 1 \):
\[
\tilde{M}_{\phi, k} f(g) = \sup_{|g|^{-1} r < r \leq 2^k} |f \ast \phi_r(h)|, \quad \tilde{M}_{\phi, k}^+ f(g) = \sup_{0 < r \leq 2^k} |f \ast \phi_r(g)|,
\]
\[
\tilde{M}_{(N), k} f(g) = \sup_{\|\phi\|_{(N)}} \tilde{M}_{\phi, k} f(g), \quad \tilde{M}_{(N), k}^+ f(g) = \sup_{\|\phi\|_{(N)}} \tilde{M}_{\phi, k}^+ f(g).
\]
In a similar way, we have
\[ \|\tilde{M}(N), kf\|_{L^p} \sim \|\tilde{M}^+(N), kf\|_{L^p} \sim \|\tilde{M}_\phi, kf\|_{L^p}, \]
where \( \phi \) is a commutative approximate identity and \( N \geq N_p \) is fixed.

We also have the atomic decomposition of \( h^p_k(\mathbb{H}^n) \) as follows. Let \( 0 < p \leq 1 < q \leq \infty \). A function \( a \in L^q(\mathbb{H}^n) \) is called an \( h^{p,q}_k - \)atom if the following conditions hold:

(i) \( \text{supp} \ a \subset B(g_0, r) \),
(ii) \( \|a\|_{L^q} \leq |B(g_0, r)|^{\frac{1}{q} - \frac{1}{p}} \),
(iii) if \( r < 2^k \), then \( \int_{B(g_0, r)} a(g) g^I \ dg = 0 \) for \( d(I) < N_p \).

Then \( f \in h^p_k(\mathbb{H}^n) \) if and only if \( f \) can be written as \( f = \sum_j \lambda_j a_j \) converging in the sense of distributions and in \( h^{p,q}_k(\mathbb{H}^n) \) norm, where \( a_j \) are \( h^{p,q}_k - \)atoms and \( \sum_j |\lambda_j|^p < \infty \). Moreover,
\[ \|f\|_{h^p_k} \sim \inf \left\{ \sum_j |\lambda_j|^p \right\}, \]
where the infimum is taken over all atomic decompositions of \( f \) into \( h^{p,q}_k - \)atoms. Furthermore, if \( f \in h^1_k(\mathbb{H}^n) \) is supported in a ball \( B(g_0, 2^k) \), then the \( h^{p,q}_k - \)atom appearing in the atomic decomposition of \( f \) without the cancellation condition is supported on the ball \( B(g_0, 2^{k+1}) \).

In the case \( p = 1, h^1_k(\mathbb{H}^n) \) is also characterized by the local Riesz transforms as follows. A function \( f \in h^1_k(\mathbb{H}^n) \) if and only if \( f \in L^1(\mathbb{H}^n) \) and \( \tilde{R}_j^k f \in L^1(\mathbb{H}^n) \), \( j = 1, \cdots, 2n \), where the local Riesz transforms are defined by \( \tilde{R}_j^k f = f * R_j^k \) and \( R_j^k(g) = \zeta(2^{-k}g) R_j(g) \). Moreover,
\[ \|f\|_{h^1_k} \sim \|f\|_{L^1} + \sum_{j=1}^{2n} \|\tilde{R}_j^k f\|_{L^1}. \]

## 5 Proof of atomic decomposition

In this section we prove Theorem 1. First we give a partition of unity related to the auxiliary function \( \rho(g) \). It follows from Lemma 2 that \( 0 < \rho(g) < \infty \) for any \( g \in \mathbb{H}^n \). Therefore,
\[ \mathbb{H}^n = \bigcup_{k=-\infty}^{\infty} \Omega_k \]
where
\[ \Omega_k = \{ g \in \mathbb{H}^n : 2^{k-1} < \rho(g) \leq 2^k \}. \]
From Lemma 3 we get
Lemma 12. For every $R \geq 2$, if $g \in \Omega_k$, $h \in \Omega_j$ and $B(g, 2^k R) \cap B(h, 2^j R) \neq \varnothing$, then $|k - j| \leq C \log_2 R$.

Proof. Choose $g_0 \in B(g, 2^k R) \cap B(h, 2^j R)$ and $g_0 \in \Omega_{k_0}$. By Lemma 4,

$$\frac{\rho(g_0)}{\rho(g)} \leq C \left(1 + \frac{2^k R}{\rho(g)} \right)^{l_0} \leq 2^{C \log_2 R},$$

$$\frac{\rho(g_0)}{\rho(g)} \geq \frac{1}{C} \left(1 + \frac{2^k R}{\rho(g)} \right)^{-l_0} \geq 2^{-C \log_2 R}.$$  

Thus $|k_0 - k| \leq C \log_2 R$. Similarly, $|k_0 - j| \leq C \log_2 R$. Therefore $|k - j| \leq C \log_2 R$. □

We choose $g(k, \alpha) \in \Omega_k$ such that

$$\Omega_k \subset \bigcup_{\alpha} B(g(k, \alpha), 2^{k-1})$$

and

$$B(g(k, \alpha), 2^{k-2}) \cap B(g(k, \alpha'), 2^{k-2}) = \varnothing \quad \text{for} \quad \alpha \neq \alpha'.$$

From Lemma 12, we get

Lemma 13. For every $(k', \alpha')$ and $R \geq 2$,

$$\# \left\{ (k, \alpha) : B(g(k, \alpha), 2^k R) \cap B(g(k', \alpha'), 2^{k'} R) \neq \varnothing \right\} \leq R^C.$$  

Then we have

Lemma 14. There exists $l_2 > 0$ such that, for every $g(k', \alpha')$ and $l \geq l_2$,

$$\sum_{(k, \alpha)} \left(1 + 2^{-k} |g^{-1}_{(k', \alpha')} g(k, \alpha)| \right)^{-l} + \sum_{(k, \alpha)} \left(1 + 2^{-k'} |g^{-1}_{(k', \alpha')} g(k, \alpha)| \right)^{-l} \leq C.$$  

Let $B_{(k, \alpha)} = B(g(k, \alpha), 2^k)$ and $B^*_{(k, \alpha)} = B(g(k, \alpha), 2^{k+1})$, and we use these two notations through the article. We now have the partition of unity.

Lemma 15. There are functions $\xi_{(k, \alpha)} \in C^\infty_c (\mathbb{H}^n)$ such that

(a) supp $\xi_{(k, \alpha)} \subset B_{(k, \alpha)}$;

(b) $0 \leq \xi_{(k, \alpha)}(g) \leq 1$ and $\sum_{(k, \alpha)} \xi_{(k, \alpha)}(g) = 1$ for any $g \in \mathbb{H}^n$;

(c) $|\nabla_{\mathbb{H}^n} \xi_{(k, \alpha)}(g)| \leq C 2^{-k}$ for any $g \in \mathbb{H}^n$.  

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Now we define the local maximal functions with respect to the semigroups \(\{T^L_s\}_{s>0}\) and \(\{T^*_s\}_{s>0}\) by
\[
\tilde{M}^L_k f(g) = \sup_{0<s \leq 4^k} |T^L_s f(g)|,
\]
\[
\tilde{M}_k f(g) = \sup_{0<s \leq 4^k} |T^*_s f(g)|.
\]
We investigate the relation between \(\tilde{M}^L_k f\) and \(\tilde{M}_k f\). In practice, we study the local maximal function \(M_k f\) defined by
\[
M_k f(g) = \sup_{0<s \leq 4^k} |T^L_s f(g) - T^*_s f(g)|.
\]

**Lemma 16.** For every \((k,\alpha)\),
\[
\|M_k(\xi_{(k,\alpha)} f)\|_{L^1} \leq C \|\xi_{(k,\alpha)} f\|_{L^1}.
\]

**Proof.** By Lemma 7, we have
\[
\int_{B^*_k(\alpha)} |M_k(\xi_{(k,\alpha)} f)(g)| \, dg \leq C \int_{B^*_k(\alpha)} \int_{B_k(\alpha)} \frac{|(\xi_{(k,\alpha)} f)(h)|}{\rho(g)^k|h^{-1}g|^{Q-\delta}} \, dh \, dg
\]
(28)
By (1) and (2),
\[
\sup_{0<s \leq 4^k} |K^L_s(g,h) - H_s(g,h)| \leq C 2^{-kQ} e^{-A_0 4^{-k}|h^{-1}g|^2} \quad \text{if} \quad |h^{-1}g| > 2^k.
\]
Therefore,
\[
\int_{|g|_{(k,\alpha)} \geq 2^k+1} |M_k(\xi_{(k,\alpha)} f)(g)| \, dg
\]
\[
\leq \int_{|g|_{(k,\alpha)} \geq 2^k+1} \int_{B_k(\alpha)} |(\xi_{(k,\alpha)} f)(h)| \sup_{0<s \leq 4^k} |K^L_s(g,h) - H_s(g,h)| \, dh \, dg
\]
(29)
\[
\leq C \|\xi_{(k,\alpha)} f\|_{L^1}.
\]
Lemma 16 follows from (28) and (29).

Define
\[
M^L_{(k,\alpha)} f(g) = \sup_{0<s \leq 4^k} |T^L_s(\xi_{(k,\alpha)} f)(g) - \xi_{(k,\alpha)}(g) T^L_s f(g)|.
\]

**Lemma 17.**
\[
\sum_{(k,\alpha)} \|M^L_{(k,\alpha)} f\|_{L^1} \leq C \|f\|_{L^1}.
\]

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Proof. Write
\[
\mathcal{P}_{s,(k,\alpha), (k',\alpha')}^L f(g) = T_s^L(\xi_{(k,\alpha)} \xi_{(k',\alpha')}) f(g) - \xi_{(k,\alpha)}(g) T_s^L(\xi_{(k',\alpha')}) f(g)
\]
\[
= \int_{B_{(k',\alpha')}} f(h) \xi_{(k',\alpha')}(h) (\xi_{(k,\alpha)}(h) - \xi_{(k,\alpha)}(g)) K_s^L(g,h) dh
\]
and
\[
\mathcal{M}_{s,(k,\alpha), (k',\alpha')}^L f(g) = \sup_{0<s\leq 4^k} |\mathcal{P}_{s,(k,\alpha), (k',\alpha')}^L f(g)|.
\]
Set
\[
\Theta_{(k,\alpha)} = \{(k',\alpha') : |g_{(k',\alpha')}^{-1} g_{(k,\alpha)}| \leq A 2^k\},
\]
\[
\Xi_{(k,\alpha)} = \{(k',\alpha') : |g_{(k',\alpha')}^{-1} g_{(k,\alpha)}| > A 2^k\},
\]
where \(A > 0\) is a fixed constant. By Lemma 13, the number of elements in \(\Theta_{(k,\alpha)}\) is bounded by a constant independent of \((k,\alpha)\). By Lemma 12, there exists a constant \(A_1 > 0\) such that \(B_{(k',\alpha')} \subset B(g_{(k,\alpha)}), A_1 2^k\) if \((k',\alpha') \in \Theta_{(k,\alpha)}\). Also by Lemma 12, we can take \(A\) large enough such that \(B_{(k,\alpha)}^* \cap B_{(k',\alpha')}^* = \emptyset\) when \((k',\alpha') \in \Xi_{(k,\alpha)}\).

Suppose \((k',\alpha') \in \Theta_{(k,\alpha)}\). Using Lemma 15 (c) and the mean value theorem (cf. Theorem 1.41), together with (11) and (22), we obtain that
\[
\sup_{0<s\leq 4^k} |(\xi_{(k,\alpha)}(h) - \xi_{(k,\alpha)}(g)) K_s^L(g,h)|
\]
\[
\leq C 2^{-k|h^{-1}g|} \sup_{0<s\leq 4^k} H_s(g,h)
\]
\[
\leq \begin{cases} C 2^{-k|h^{-1}g|^{1-Q}}, & \text{if } |h^{-1}g| \leq 2^k, \\ C 2^{-k(Q+1)} |h^{-1}g| e^{-A_0 4^{-k}|h^{-1}g|^2}, & \text{if } |h^{-1}g| > 2^k. \end{cases}
\]
Therefore,
\[
\int_{\mathbb{H}^n} \mathcal{M}_{s,(k,\alpha), (k',\alpha')}^L f(g) dg
\]
\[
\leq \int_{\mathbb{H}^n} \int_{B_{(k',\alpha')}} |f(h) \xi_{(k',\alpha')}(h)| \sup_{0<s\leq 4^k} |(\xi_{(k,\alpha)}(h) - \xi_{(k,\alpha)}(g)) K_s^L(g,h)| dh dg
\]
\[
\leq C \|f\|_{L^1(B(g_{(k,\alpha)}, A_1 2^k))}. \tag{30}
\]

Let \((k',\alpha') \in \Xi_{(k,\alpha)}\). It is easy to see that \(\xi_{(k',\alpha')}(h) \xi_{(k,\alpha)}(h) = 0, \rho(g) \sim 2^k\) and
\[ |h^{-1}g| \sim |g_{(k',\alpha')}(k,\alpha)| \text{ for } g \in B_{(k,\alpha)} \text{ and } h \in B_{(k',\alpha')} \]. By Lemma \([13]\) we get

\[
\int_{\mathbb{H}^n} M_{(k,\alpha)} f(g) dg \\
\leq \int_{B_{(k,\alpha)}} \left| f(h) \right| \frac{C_l \xi_{(k',\alpha')}(h) \xi_{(k,\alpha)}}{(1 + |h^{-1}g| \rho(g)^{-1}) |h^{-1}g|^{-1}} dh \tag{31} \\
\leq \frac{C}{(1 + 2^{-k} |g_{(k',\alpha')}(k,\alpha)|)} \int_{B_{(k',\alpha')}} \left| f(h) \right| \xi_{(k',\alpha')}(h) \left( \int_{B_{(k,\alpha)}} \frac{dg}{|h^{-1}g|^2} \right) dh
\]

\[ J \]

Taking \( l \geq l_2 \), by (31) and Lemma \([13]\) we get

\[
J_1 \leq C \sum_{(k,\alpha)} \left\| f \right\|_{L^1(B_{(k,\alpha)}, A_{l_22})} \leq C \left\| f \right\|_{L^1} \tag{33}.
\]

Combination of (32), (33) and (34) gives Lemma \([17]\) \( \square \).

We are ready to prove Theorem \([1]\).

**Proof of Theorem \([1]\)** Given \( f \in H^1_1(\mathbb{H}^n) \), since

\[
\left\| \tilde{M}_k(\xi_{(k,\alpha)} f) \right\|_{L^1} \leq \left\| \tilde{M}_k(\xi_{(k,\alpha)} f) \right\|_{L^1} + \left\| M_k(\xi_{(k,\alpha)} f) \right\|_{L^1} \\
\leq \left\| (\xi_{(k,\alpha)} \tilde{M}_k f) \right\|_{L^1} + \left\| M_{(k,\alpha)} f \right\|_{L^1} + \left\| M_k(\xi_{(k,\alpha)} f) \right\|_{L^1},
\]

by Lemma \([16]\) and Lemma \([17]\)

\[
\sum_{(k,\alpha)} \left\| \tilde{M}_k(\xi_{(k,\alpha)} f) \right\|_{L^1} \leq C(\left\| M^L f \right\|_{L^1} + \left\| f \right\|_{L^1}) \leq C \left\| f \right\|_{H^1_L} \tag{35}.
\]

As pointed out in Remark \([3]\) this means every \( \xi_{(k,\alpha)} f \in h^1_1(\mathbb{H}^n) \) and

\[
\xi_{(k,\alpha)} f = \sum_i \Lambda_i^{(k,\alpha)} a_i^{(k,\alpha)} \quad \text{in } L^1(\mathbb{H}^n),
\]
where \( a_i^{(k,\alpha)} \) are \( H^1_{L} \)-atoms and

\[
\sum_i |\lambda_i^{(k,\alpha)}| \leq C \| M_k(\xi_{(k,\alpha)}f) \|_{L^1}.
\]

We also note that each \( h_{(k,\alpha)} \)-atom \( a_i^{(k,\alpha)} \) without the cancellation condition is supported on the ball \( B^*_k = B(g(k,\alpha), 2^{k+1}) \). Therefore all \( h_{(k,\alpha)} \)-atoms \( a_i^{(k,\alpha)} \) are \( H^1_{L} \)-atoms. By (36), (37) and (35), we get

\[
f = \sum_{(k,\alpha)} \sum_i \lambda_i^{(k,\alpha)} a_i^{(k,\alpha)}
\]

and

\[
\sum_{(k,\alpha)} \sum_i |\lambda_i^{(k,\alpha)}| \leq C \sum_{(k,\alpha)} \| M_k(\xi_{(k,\alpha)}f) \|_{L^1} \leq C \| f \|_{H^1_{L}}.
\]

To prove the converse, we only have to show that, for every \( H^1_{L} \)-atom \( a \),

\[
\| M^L a \|_{L^1} \leq C.
\]

Let \( a \) be an \( H^1_{L} \)-atom supported on a ball \( B(g_0, r) \). If \( r > \rho(g_0) \), we further decompose \( a \) as follows. Set

\[
\Theta_a = \{ (k, \alpha) : B(g(k,\alpha), \rho(g(k,\alpha))) \cap B(g_0, r) \neq \emptyset \}.
\]

Then we have

\[
\sum_{(k,\alpha) \in \Theta_a} |B(g(k,\alpha), \rho(g(k,\alpha)))| \leq C |B(g_0, r)|.
\]

In fact, if \( \rho(g(k,\alpha)) > r \) and \( (k, \alpha) \in \Theta_a \), then \( g_0 \in B(g(k,\alpha), 2\rho(g(k,\alpha))) \). By Lemma 4,

\[
\rho(g(k,\alpha)) \leq C \rho(g_0) \leq C r.
\]

Therefore, \( B(g(k,\alpha), \rho(g(k,\alpha))) \subset B(g_0, C r) \), and (39) follows from the finite overlaps property of \( B(g(k,\alpha), \rho(g(k,\alpha))) \). Let \( \chi_{(k,\alpha)} \) be the characteristic function of \( B(g(k,\alpha), \rho(g(k,\alpha))) \). Set

\[
\zeta_{(k,\alpha)}(g) = \frac{\chi_{(k,\alpha)}(g)}{\sum_{(k',\alpha') \in \Theta_a} \chi_{(k',\alpha')}(g)}
\]

and write

\[
\zeta_{(k,\alpha)}(g) a(g) = \lambda_{(k,\alpha)} a_{(k,\alpha)}(g),
\]

where

\[
\lambda_{(k,\alpha)} = \| B(g(k,\alpha), \rho(g(k,\alpha))) \|_{L^1} \| \zeta_{(k,\alpha)} a \|_{L^1}.
\]
Then \( a_{(k,\alpha)} \) is an \( H^1_{L} \)-atom supported on \( B(g_{(k,\alpha)}, \rho(g_{(k,\alpha)})) \) and
\[
a(g) = \sum_{(k,\alpha) \in \Theta_n} \lambda_{(k,\alpha)} a_{(k,\alpha)}(g).
\]

By the Hölder inequality and (41),
\[
\sum_{(k,\alpha) \in \Theta_n} |\lambda_{(k,\alpha)}| \leq \left( \sum_{(k,\alpha) \in \Theta_n} |B(g_{(k,\alpha)}, \rho(g_{(k,\alpha)}))| \right)^{\frac{1}{p}} \left( \sum_{(k,\alpha) \in \Theta_n} \|\zeta_{(k,\alpha)}a\|_{L^q}^q \right)^{\frac{1}{q}}
\leq C \left| B(g_0, r) \right|^{\frac{1}{p}} \| a \|_{L^q} \leq C.
\]

Thus, without loss of generality, we may assume that \( a \) is an \( H^1_{L} \)-atom supported on a ball \( B(g_0, r) \) satisfying \( r \leq \rho(g_0) \). Let \( g_0 \in \Omega_k \). By the same argument of Lemma 10, we have \( \| \mathcal{M} a \|_{L^1} \leq C \| a \|_{L^1} \leq C \). It is clear that \( \| \widetilde{M} a \|_{L^1} \leq C \). Hence
\[
\| \widetilde{M} a \|_{L^1} \leq C.
\]

By (1) and (2),
\[
\sup_{4^k < s < \infty} K^L_s(g, h) \leq C 2^{-kQ},
\]
which yields
\[
\sup_{4^k < s < \infty} |T^L_s a(g)| \leq C 2^{-kQ}
\]
and hence
\[
\int_{B(g_0, 2^{k+1})} \sup_{4^k < s < \infty} |T^L_s a(g)| \, dg \leq C.
\]

Note that \( r \leq 2^k \) and \( \rho(h) \sim 2^k \) whenever \( h \in B(g_0, r) \). By Lemma 7, we have
\[
\int_{|g_0^{-1}g| \geq 2^{k+1}} \sup_{4^k < s < \infty} |T^L_s a(g)| \, dg \leq C \int \sup_{|g_0^{-1}g| \geq 2^{k+1}} \int_{B(g_0, r)} \frac{|a(h)|}{\rho(h) \ |h^{-1}g|} \, dh \, dg \leq C.
\]

We obtain (38) from (10), (11) and (42). This completes the proof of Theorem 11.

**Remark 4.** As we showed above, we may add a restriction in the definition of \( H^1_{L} \)-atom as follows. If \( a \) is an \( H^1_{L} \)-atom supported on a ball \( B(g_0, r) \), then \( r \leq \rho(g_0) \).

**Remark 5.** We have seen that \( H^1_{L}(\mathbb{H}^n) \) is closely related to the local Hardy space \( h^1(\mathbb{H}^n) \). From Theorem 1, Theorem 4 and Remark 1, it is easy to get the following inclusion relations. If \( V \leq C \) then \( H^1_{L}(\mathbb{H}^n) \subset h^1(\mathbb{H}^n) \). If \( V \geq \frac{1}{C} \) then \( h^1(\mathbb{H}^n) \subset H^1_{L}(\mathbb{H}^n) \). Specifically, \( H^1_{L}(\mathbb{H}^n) = h^1(\mathbb{H}^n) \) when \( V \sim 1 \).
6 Riesz transforms characterization

In this section we deal with the Riesz transforms $R^L_j$ associated with the Schrödinger operator $L$. The atomic decomposition is useful to establish the boundedness of operators on Hardy spaces. But in general, as showed in [1] (also see [16]), it is not enough to conclude that an operator extends to a bounded operator on the whole Hardy space by verifying its uniform boundedness on all atoms. The key point is that two quasi-norms corresponding to finite and infinite atomic decompositions are not equivalent. We give the following lemma.

**Lemma 18.** Suppose that $T$ is a bounded sublinear operator from $L^1(H^n)$ to $L^{1,\infty}(H^n)$ such that $\|Ta\|_{L^1} \leq C$ for any $H^1_{L^q}$-atom $a$. Then $T$ is bounded from $H^1_{L^1}(H^n)$ to $L^1(H^n)$.

**Proof.** Suppose $f \in H^1_{L^1}(H^n)$. By Theorem 1, we write
\[
f = \sum_{j=1}^{\infty} \lambda_j a_j \quad \text{with} \quad \sum_{j=1}^{\infty} |\lambda_j| < C \|f\|_{H^1_{L^1}},\]
where $a_j$ are $H^1_{L^q}$-atoms. Set
\[
f_m = \sum_{j=1}^{m} \lambda_j a_j.
\]
It is clear that $f_m$ converges to $f$ in $L^1(H^n)$. The sublinearity of $T$ yields $\|Tf_m - Tf\| \leq |T(f_m - f)|$, and hence $|Tf_m|$ converges to $|Tf|$ in $L^{1,\infty}(H^n)$. There exists a subsequence $\{|Tf_{m_k}\}|$ such that $\lim_{k \to \infty} |Tf_{m_k}(g)| = |Tf(g)|$ almost everywhere $g \in H^n$. Since
\[
\|Tf_{m_k}\|_{L^1} \leq \sum_{j=1}^{m_k} |\lambda_j| \|Ta_j\|_{L^1} \leq C \|f\|_{H^1_{L^1}},
\]
by Fatou’s lemma we get
\[
\|Tf\|_{L^1} \leq C \|f\|_{H^1_{L^1}}.
\]
This proves Lemma 18. \(\square\)

We are going to prove Theorem 2.

**Proof of Theorem 2.** By the Calderón–Zygmund decomposition (cf. [7, 19]), given $f \in L^1(H^n)$ and $\lambda > 0$, we have the decomposition $f = f_1 + f_2$ with $f_2 = \sum_k b_k$, such that

(i) $|f_1(g)| \leq C \lambda$ for almost everywhere $g \in H^n$;

(ii) each $b_k$ is supported in a ball $B_k$, $\int_{B_k} |b_k(g)| \, dg \leq C \lambda |B_k|$, and $\int_{B_k} b_k(g) \, dg = 0$;
(iii) \( \{ B_k \} \) is a finitely overlapping family and \( \sum_k |B_k| \leq \frac{C}{\lambda} \| f \|_{L^1} \).

Since \( R_j^L \) is bounded on \( L^2(\mathbb{H}^n) \), it is clear that

\[
| \{ g \in \mathbb{H}^n : |R_j^L f_1(g)| > \frac{\lambda}{2} \} | \leq \frac{C}{\lambda^2} \| f_1 \|_{L^2}^2 \leq \frac{C}{\lambda} \| f \|_{L^1}.
\]

Let \( B_k = B(g_k, r_k) \). Set \( B^*_k = B(g_k, 2r_k) \) and \( \Omega = \bigcup_k B^*_k \). Then

\[
|\Omega| \leq C \sum_k |B_k| \leq \frac{C}{\lambda} \| f \|_{L^1}.
\]

We only need to consider \( R_j^L f_2(g) \) for \( g \in \Omega^c \). If \( r_k \geq \rho(g_k) \), then \( \rho(h) \leq Cr_k \) for any \( h \in B_k \). By Lemma 8 we get

\[
\int_{|g_k^{-1}g| \geq \rho(g_k)} |R_j^L b_k(g)| \, dg \leq \int_{|g_k^{-1}g| \geq 2\rho(g_k)} \int_{B_k} |R_j^L(g, h)| \, |b_k(h)| \, dh \, dg \leq C \| b_k \|_{L^1}.
\]

If \( r_k < \rho(g_k) \), then \( \rho(h) \sim \rho(g_k) \) for any \( h \in B_k \). Since \( R_j(g) \) is a Calderón–Zygmund kernel, by Lemmas 8 and 9 we obtain

\[
\int_{|g_k^{-1}g| \geq 2r_k} |R_j^L b_k(g)| \, dg \leq \int_{2r_k \leq |g_k^{-1}g| < 2\rho(g_k)} |R_j^L b_k(g)| \, dg + \int_{|g_k^{-1}g| \geq 2\rho(g_k)} |R_j^L b_k(g)| \, dg
\]

\[
\leq \int_{2r_k \leq |g_k^{-1}g| < 2\rho(g_k)} \int_{B_k} |R_j^L(g, h) - R_j(g, h)| \, |b_k(h)| \, dh \, dg
\]

\[
+ \int_{2r_k \leq |g_k^{-1}g| < 2\rho(g_k)} \int_{B_k} |R_j(g, h) - R_j(g, g_k)| \, |b_k(h)| \, dh \, dg
\]

\[
+ \int_{|g_k^{-1}g| \geq 2\rho(g_k)} \int_{B_k} |R_j^L(g, h)| \, |b_k(h)| \, dh \, dg
\]

\[
\leq C \| b_k \|_{L^1}.
\]

In any case we have

\[
\| R_j^L b_k \|_{L^1((B^*_k)^c)} \leq C \| b_k \|_{L^1}.
\]

Then

\[
\int_{\Omega^c} |R_j^L f_2(g)| \, dg \leq \sum_k \| R_j^L b_k \|_{L^1((B^*_k)^c)} \leq C \sum_k \| b_k \|_{L^1} \leq C\lambda \sum_k |B_k| \leq C \| f \|_{L^1}.
\]

Therefore

\[
| \{ g \in \Omega^c : |R_j^L f_2(g)| > \frac{\lambda}{2} \} | \leq \frac{C}{\lambda} \| f \|_{L^1}.
\]

Theorem 2 follows from the combination of (33), (34) and (36). \( \Box \)

The proof of Theorem 3 is similar to the one in Theorem 4. We will keep the notations used in former sections. As consequences of Lemmas 8 and 9 we have
Lemma 19. For every \((k, \alpha)\),
\[
\| R_j^L(\xi_{(k,\alpha)}f) \|_{L^1(B_{(k,\alpha)}^c)} \leq C \| \xi_{(k,\alpha)}f \|_{L^1}.
\]

Lemma 20. For every \((k, \alpha)\),
\[
\| R_j^L(\xi_{(k,\alpha)}f) - R_j(\xi_{(k,\alpha)}f) \|_{L^1(B_{(k,\alpha)}^c)} \leq C \| \xi_{(k,\alpha)}f \|_{L^1}.
\]

Similar to Lemma 17, we have

Lemma 21.
\[
\sum_{(k, \alpha)} \| R_j^L(\xi_{(k,\alpha)}f) - \xi_{(k,\alpha)} R_j^L f \|_{L^1} \leq C \| f \|_{L^1}.
\]

Proof. Given \((k, \alpha)\), we write
\[
Q_j^L(\xi_{(k,\alpha)}, (k', \alpha')) f(g) = R_j^L(\xi_{(k,\alpha)} \xi_{(k',\alpha')}(g)) - \xi_{(k,\alpha)}(g) R_j^L(\xi_{(k',\alpha')}(g))(g)
= \int_{B_{(k', \alpha')}} f(h) \xi_{(k',\alpha')}(h) (\xi_{(k,\alpha)}(h) - \xi_{(k,\alpha)}(g)) R_j^L(g, h) dh.
\]

Suppose \((k', \alpha') \in \Theta_{(k,\alpha)}\) and \(h \in B_{(k', \alpha')}\). Then \(\rho(h) \sim 2^{k'} \sim 2^k\). By Lemma 8
\[
\int_{B^n} |(\xi_{(k,\alpha)}(h) - \xi_{(k,\alpha)}(g)) R_j^L(g, h)| dg
\leq 2 \int_{\{|h^{-1}g| \geq 2^{k'}\}} |R_j^L(g, h)| dg + C \int_{B(h, 2^{k'})} 2^{-k'} |h^{-1}g| |R_j^L(g, h)| dg
\leq C + C \int_{B(h, 2^{k'})} 2^{-k'} |h^{-1}g| |R_j^L(g, h)| dg.
\]

Using (12) with \(l = 1\), we get
\[
\int_{B(h, 2^{k'})} 2^{-k'} |h^{-1}g| |R_j^L(g, h)| dg
\leq C \int_{B(h, 2^{k'})} 2^{-k'} dg + C \int_{B(h, 2^{k'})} \left( \frac{1}{|h^{-1}g|^{Q-1}} \int_{B(g, |h^{-1}g|)} \frac{V(w) \, dw}{|g^{-1}w|^{Q-1}} \right) \, dg
\leq C + C \int_{B(h, 2^{k'})} \left( \frac{1}{|h^{-1}g|^{Q-1}} \int_{B(g, |h^{-1}g|)} \frac{V(w) \, dw}{|g^{-1}w|^{Q-1}} \right) \, dg.
\]

Note that \(p'_a(Q - 1) < Q\) as \(q_0 > \frac{Q}{2}\). By the Hölder inequality and the boundedness of
fractional integrals, we obtain

\[
\int_{B(h,2^{k'})} \left( \frac{1}{|h^{-1}g|^{Q-1}} \int_{B(g,\frac{|h^{-1}w|}{2})} \frac{V(w) \, dw}{|g^{-1}w|^{Q-1}} \right) \, dg
\]

\[
\leq \left( \int_{B(h,2^{k'})} \frac{dg}{|h^{-1}g|^{p_0(Q-1)}} \right)^{\frac{1}{p_0}} \left( \int_{B(h,2^{k'})} \left( \int_{B(g,\frac{|h^{-1}w|}{2})} \frac{V(w) \, dw}{|g^{-1}w|^{Q-1}} \right) \, dg \right)^{\frac{1}{p_0}}
\]

\[
\leq C 2^{-k'(Q-2)} \left( \int_{B(h,2^{k'+1})} V(g) \, dg \right)\
\]

(47)

\[
\leq C,
\]

where we used the $B_{q_0}$ condition and Lemma 3 in the last two inequalities. The above three estimates yield

\[
\int_{H^n} |(\xi(k, \alpha)(h) - \xi(k, \alpha)(g)) R_j^L (g, h) | \, dg \leq C.
\]

Note that $B(k', \alpha') \subset B(g(k, \alpha), A_1 2^{k})$ when $(k', \alpha') \in \Theta(k, \alpha)$. Hence,

(48) \[ \| Q_{j, (k, \alpha), (k', \alpha')} f \|_{L^1} \leq C \| \xi(k', \alpha') f \|_{L^1} \leq C \| f \|_{L^1(B(g(k, \alpha), A_1 2^{k}))}. \]

Let $(k', \alpha') \in \Xi(k, \alpha)$ and $h \in B(k', \alpha')$. We have $\xi(k, \alpha)(h) = 0$, $\rho(h) \sim 2^{k'}$ and $|h^{-1}g| \sim |g^{-1}(k', \alpha') g(k, \alpha)|$ for $g \in B(k, \alpha)$. By (12), we get

\[
\int_{H^n} |(\xi(k, \alpha)(h) - \xi(k, \alpha)(g)) R_j^L (g, h) | \, dg
\]

\[
\leq \int_{B(k, \alpha)} \left( \frac{C_1}{(1 + 2^{-k'} |g(k', \alpha') g(k, \alpha)|)^{\frac{1}{2}}} \left( \frac{1}{|h^{-1}g|^{Q-1}} + \frac{1}{|h^{-1}g|^{Q-1}} \int_{B(g,\frac{|h^{-1}w|}{2})} \frac{V(w) \, dw}{|g^{-1}w|^{Q-1}} \right) \right) \, dg
\]

\[
\leq \frac{C}{(1 + 2^{-k'} |g(k', \alpha') g(k, \alpha)|)^{\frac{1}{2}}} \left( \frac{1}{|g(k', \alpha') g(k, \alpha)|^{Q-1}} \int_{B(k, \alpha)} \int_{B(g,\frac{|h^{-1}w|}{2})} \frac{V(w) \, dw}{|g^{-1}w|^{Q-1}} \, dg \right).
\]

Similar to (17), by the Hölder inequality, the boundedness of fractional integrals and the $B_{q_0}$ condition, we obtain

\[
\frac{1}{|g(k', \alpha') g(k, \alpha)|^{Q-1}} \int_{B(k, \alpha)} \int_{B(g,\frac{|h^{-1}w|}{2})} \frac{V(w) \, dw}{|g^{-1}w|^{Q-1}} \, dg
\]

\[
\leq C |g(k', \alpha') g(k, \alpha)|^{2-Q} \int_{B(g(k', \alpha'), \frac{2}{3} |g(k', \alpha') g(k, \alpha)|}) V(g) \, dg
\]

\[
\leq C (1 + 2^{-k'} |g(k', \alpha') g(k, \alpha)|)^{l_1},
\]

where we used Lemma 3 for the last inequality. Thus,

\[
\int_{H^n} |(\xi(k, \alpha)(h) - \xi(k, \alpha)(g)) R_j^L (g, h) | \, dg \leq C (1 + 2^{-k'} |g(k', \alpha') g(k, \alpha)|)^{-l_0 + l_1},
\]

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and hence

\[(49) \quad \|Q_{j,(k,\alpha)}^{L}(x)\|_{L^1} \leq C(1 + 2^{-k'}|g_{(k',\alpha')}^{-1} g_{(k,\alpha)}|)^{-l+1} \|\xi(k',\alpha')f\|_{L^1}.

We have

\[
\sum_{(k,\alpha)}\|R_j^{L}(\xi(k,\alpha)f) - \xi(k,\alpha)R_j^{L}f\|_{L^1} \leq \sum_{(k,\alpha)}\sum_{(k',\alpha')}\|Q_{j,(k,\alpha)}^{L}(x)\|_{L^1} + \sum_{(k,\alpha)}\sum_{(k',\alpha')\in\mathcal{E}(k,\alpha)}\|Q_{j,(k,\alpha)}^{L}(x)\|_{L^1} = J_1 + J_2.
\]

By (48) and Lemma 13 we get

\[(50) \quad J_1 \leq C \sum_{(k,\alpha)}\|f\|_{L^1(B(g_{(k,\alpha)}; A_1; 2^{-l}))} \leq C \|f\|_{L^1}.

Taking \(l \geq l_1 + l_2\), by \(49\) and Lemma 14 we get

\[(51) \quad J_2 \leq C \sum_{(k,\alpha)}\sum_{(k',\alpha')\in\mathcal{E}(k,\alpha)}(1 + 2^{-k'}|g_{(k',\alpha')}^{-1} g_{(k,\alpha)}|)^{-l+l_0} \|\xi(k',\alpha')f\|_{L^1} \leq C \|f\|_{L^1}.

Lemma 21 follows from the combination of \(50\), \(51\) and \(52\).

Now we give the proof of Theorem 3.

**Proof of Theorem 3** Assume first that

\[\|f\|_{L^1} + \sum_{j=1}^{2n} \|R_j^{L}f\|_{L^1} < \infty.\]

Since

\[
\|R_j(\xi(k,\alpha)f)\|_{L^1(B_{(k,\alpha)}^*)} \leq \|R_j^{L}(\xi(k,\alpha)f) - R_j(\xi(k,\alpha)f)\|_{L^1(B_{(k,\alpha)}^*)} + \|R_j^{L}(\xi(k,\alpha)f)\|_{L^1(B_{(k,\alpha)}^*)} + \|R_j^{L}(\xi(k,\alpha)f)\|_{L^1((B_{(k,\alpha)}^*)^c)} + \|R_j^{L}(\xi(k,\alpha)f) - R_j^{L}(\xi(k,\alpha)f)\|_{L^1} + \|\xi(k,\alpha)R_j^{L}f\|_{L^1},
\]

Lemmas 19 − 21 give

\[
\sum_{(k,\alpha)}\|R_j(\xi(k,\alpha)f)\|_{L^1(B_{(k,\alpha)}^*)} \leq C(\|f\|_{L^1} + \|R_j^{L}f\|_{L^1}).
\]
Note that
\[ \| R_j(\xi_{(k,\alpha)}f) \|_{L^1(B^*_j(k,\alpha))} = \| \tilde{R}_j^{[k]}(\xi_{(k,\alpha)}f) \|_{L^1(B^*_j(k,\alpha))}, \]
\[ \| \tilde{R}_j^{[k]}(\xi_{(k,\alpha)}f) \|_{L^1} = \| \tilde{R}_j^{[k]}(\xi_{(k,\alpha)}f) \|_{L^1(B^*_j(k,\alpha))}. \]

Since \( \| R_j^{[k]}(\cdot) - R_j^{[k+3]}(\cdot) \|_{L^1} \leq C \), we have
\[ \| \tilde{R}_j^{[k]}(\xi_{(k,\alpha)}f) - \tilde{R}_j^{[k+3]}(\xi_{(k,\alpha)}f) \|_{L^1(B^*_j(k,\alpha))} \leq C \| \xi_{(k,\alpha)}f \|_{L^1}, \]
which yields
\[
\sum_{(k,\alpha)} \| \tilde{R}_j^{[k]}(\xi_{(k,\alpha)}f) \|_{L^1} \leq \sum_{(k,\alpha)} \| \tilde{R}_j^{[k]}(\xi_{(k,\alpha)}f) - \tilde{R}_j^{[k+3]}(\xi_{(k,\alpha)}f) \|_{L^1(B^*_j(k,\alpha))} + \sum_{(k,\alpha)} \| R_j(\xi_{(k,\alpha)}f) \|_{L^1(B^*_j(k,\alpha))} \leq C(\| f \|_{L^1} + \| R_j f \|_{L^1}).
\]
(53)

As pointed out in Remark 3, (53) implies that every \( \xi_{(k,\alpha)}f \) is in \( h^1_k(\mathbb{H}^n) \) and allows the atomic decomposition
\[ \xi_{(k,\alpha)}f = \sum_i \lambda_i^{(k,\alpha)} a_i^{(k,\alpha)} \]
with
\[ \sum_i |\lambda_i^{(k,\alpha)}| \leq C \left( \| \xi_{(k,\alpha)}f \|_{L^1} + \sum_{j=1}^{2n} \| \tilde{R}_j^{[k]}(\xi_{(k,\alpha)}f) \|_{L^1} \right). \]

Therefore we get
\[ f = \sum_{(k,\alpha)} \sum_i \lambda_i^{(k,\alpha)} a_i^{(k,\alpha)} \]
and
\[ \sum_{(k,\alpha)} \sum_i |\lambda_i^{(k,\alpha)}| \leq C \left( \| f \|_{L^1} + \sum_{j=1}^{2n} \| R_j f \|_{L^1} \right). \]

For the reverse inequality, by Theorem 2 and Lemma 18, we have to check
(54)
\[ \| R_j f \|_{L^1} \leq C \]
for every \( H^1_{L^q} \)-atom \( a \). Let \( a \) be an \( H^1_{L^q} \)-atom supported on a ball \( B(g_0, r) \). Of course, we may assume that \( q \leq 2 \). Since \( R_j f \) is bounded on \( L^q(\mathbb{H}^n) \),
\[ \| R_j f \|_{L^1(B(g_0, 2r))} \leq |B(g_0, 2r)|^{-\frac{1}{q}} \| R_j f \|_{L^q} \leq C |B(g_0, r)|^{-\frac{1}{q}} \| f \|_{L^q} \leq C. \]

By the same argument as (45), we have
\[ \| R_j f \|_{L^1(B(g_0, 2r)^c)} \leq C \| a \|_{L^1} \leq C. \]
This proves (54), and the proof of Theorem 3 is finished. \( \square \)
Remark 6. It follows from Theorem 2, Lemma 18, and (51) that the Riesz transforms $R_j^L$ are bounded from $H^1_0(\mathbb{H}^n)$ to $L^1(\mathbb{H}^n)$, and hence bounded from $H^1(\mathbb{H}^n)$ to $L^1(\mathbb{H}^n)$.

Finally we construct a counterexample which shows that the range of $p$ ensuring the boundedness of the Riesz transforms $R_j^L$ cannot be improved. The counterexample is similar to the one on $\mathbb{R}^n$ given by Shen [18]. The main difference is the appearance of the function $\psi(g)$. Define the function $\psi(g)$ by
\[
\psi(g) = \frac{|x|^2}{|x(t)|^2}, \quad 0 \neq g = (x, t) \in \mathbb{H}^n.
\]
This function was introduced in [8]. We simply remark that $\psi$ is homogeneous of degree zero and $0 \leq \psi \leq 1$.

Lemma 22. Let $1 < q < \infty$. If $-\frac{Q}{q} < \beta < \infty$, then $|g|^\beta \psi(g)$ belongs to $B_q$ class.

Proof. Given a ball $B(h, r)$ with $|h| \leq 2r$, we have
\[
\left( \frac{1}{|B(h, r)|} \int_{B(h, r)} (|g|^\beta \psi(g))^q dg \right)^{\frac{1}{q}} \leq C \left( r^{-Q} \int_{B(0, 4r)} |g|^q dg \right)^{\frac{1}{q}} \leq Cr^\beta.
\]
On the other hand, there exists $h_0 = (x_0, t_0) \in B(h, \frac{r}{2})$ such that $|x_0|^2 \geq \frac{r^2}{16}$. If $g \in B(h_0, \frac{r}{8})$, then $|g| \sim r$ and $\psi(g) \sim 1$. Hence
\[
\frac{1}{|B(h, r)|} \int_{B(h, r)} |g|^\beta \psi(g) dg \geq \frac{1}{C} \frac{1}{|B(h, r)|} \int_{B(h_0, \frac{r}{8})} |g|^\beta \psi(g) dg \geq \frac{1}{C} r^\beta.
\]
It follows that
\[
\left( \frac{1}{|B(h, r)|} \int_{B(h, r)} |g|^\beta \psi(g) dg \right)^{\frac{1}{q}} \leq \frac{C}{|B(h, r)|} \int_{B(h, r)} |g|^\beta \psi(g) dg.
\]
In the case of $|h| > 2r$, we have $|g| \sim |h|$ when $g \in B(h, r)$. Note that $P(g) = |g|^2 \psi(g)$ is a nonnegative polynomial of homogeneous degree two and satisfies
\[
\max_{g \in B} P(g) \leq \frac{C}{|B|} \int_B P(g) dg \quad \text{for every ball } B \subset \mathbb{H}^n.
\]
Therefore,
\[
\left( \frac{1}{|B(h, r)|} \int_{B(h, r)} (|g|^\beta \psi(g))^q dg \right)^{\frac{1}{q}} \leq C|h|^{\beta - 2} \max_{g \in B(h, r)} P(g)
\]
\[
\leq C|h|^{\beta - 2} \left( \frac{1}{|B(h, r)|} \int_{B(h, r)} P(g) dg \right)
\]
\[
\leq C \left( \frac{1}{|B(h, r)|} \int_{B(h, r)} |g|^\beta \psi(g) dg \right)
\]
and Lemma 22 is proved. \qed
Now we consider the nonnegative potential
\[ V(g) = |g|^{\beta - 2} \psi(g), \quad 0 < \beta < 2. \]
We look for a radial solution \( u(g) = f(|g|) \) of the equation
\[ (55) \quad - \Delta_{\mathbb{H}^n} u + |g|^{\beta - 2} \psi u = 0. \]
By the following facts (cf. [8]):
\[ (56) \quad \Delta_{\mathbb{H}^n}(|g|) = \frac{Q - 1}{|g|} \psi(g) \quad \text{and} \quad |\nabla_{\mathbb{H}^n}(|g|)|^2 = \psi(g) \quad \text{for} \quad |g| \neq 0, \]
it turns out that \( f \) must satisfy
\[ f''(|g|) + \frac{Q - 1}{|g|} f'(|g|) - |g|^{\beta - 2} f(|g|) = 0 \quad \text{for} \quad |g| \neq 0. \]
It is easy to verify that
\[ v(g) = \sum_{m=0}^{\infty} \frac{|g|^{\beta m}}{m! \beta^2 m \Gamma(\frac{Q - 2}{\beta} + m + 1)} \]
is a radial solution of the equation (55).

Let \( \phi \in C^\infty_c(\mathbb{H}^n) \) such that \( \phi = 1 \) for \( |g| \leq 1 \). Set \( u = \phi v \). Then
\[ -\Delta_{\mathbb{H}^n} u + |g|^{\beta - 2} \psi u = \eta, \]
where \( \eta = -2 \nabla_{\mathbb{H}^n} v \cdot \nabla_{\mathbb{H}^n} \psi - v \Delta_{\mathbb{H}^n} \phi \in C^\infty_c(\mathbb{H}^n) \).

Given \( \frac{Q}{2} < q_1 < Q \), let \( \beta = 2 - \frac{Q}{q_1} \). By Lemma 22, \( V(g) = |g|^{-\frac{Q}{q_1}} \psi(g) \in B_\eta \) for any \( q < q_1 \). If \( R_\eta^p \) is bounded on \( L^p(\mathbb{H}^n) \) for \( p = p_1 \) where \( \frac{1}{p_1} = \frac{1}{q_1} - \frac{1}{Q} \), then
\[ \|\nabla_{\mathbb{H}^n} u\|_{L^{p_1}} \leq \|\nabla_{\mathbb{H}^n}(-\Delta_{\mathbb{H}^n} + V)^{-\frac{1}{2}} \eta\|_{L^{p_1}} \leq C \|(-\Delta_{\mathbb{H}^n} + V)^{-\frac{1}{2}} \eta\|_{L^{p_1}}. \]
It follows from (3) and (2) that
\[ \left|(-\Delta_{\mathbb{H}^n} + V)^{-\frac{1}{2}} \eta(g)\right| \leq C \int_{\mathbb{H}^n} \frac{|\eta(h)|}{|g|^{-1} h^{Q - 1}} \, dh \leq \frac{C}{(1 + |g|)Q - 1}. \]
Thus
\[ \|\nabla_{\mathbb{H}^n} u\|_{L^{p_1}} \leq C \|(-\Delta_{\mathbb{H}^n} + V)^{-\frac{1}{2}} \eta\|_{L^{p_1}} < \infty. \]
On the other hand, by (56) we have
\[ |\nabla_{\mathbb{H}^n} u| \sim |g|^{-\frac{Q}{q_1}} \psi \frac{1}{2} = |g|^{-\frac{Q}{q_1}} \psi \frac{1}{2}, \quad \text{as} \quad g \to 0. \]
By the following formula about changing variables
\[ \int_{\mathbb{H}^n} f(x,t) \, dx \, dt = \int_{S^{2n-1}} \int_{-\pi}^{\pi} \int_{0}^{\infty} f(r \cos \theta)^{\frac{1}{2}} \, x', r^2 \sin \theta \, (\cos \theta)^{n-1} r^{Q-1} \, dr \, d\theta \, dx', \]
where \( S^{2n-1} \) is the unit sphere in \( \mathbb{R}^{2n} \) (cf. [3]), it is easy to see that \( \nabla_{\mathbb{H}^n} u \notin L^{p_1}(\mathbb{H}^n) \). We have a contradiction.
7 Results for stratified groups

In this section, we state results for stratified groups. We consistently use the same notations and terminologies as those in Folland and Stein’s book [7].

Let $G$ be a stratified group of dimension $d$ with the Lie algebra $\mathfrak{g}$. This means that $\mathfrak{g}$ is equipped with a family of dilations $\{\delta_r : r > 0\}$ and $\mathfrak{g}$ is a direct sum $\bigoplus_{j=1}^m \mathfrak{g}_j$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, $\mathfrak{g}_1$ generates $\mathfrak{g}$, and $\delta_r(X) = r^jX$ for $X \in \mathfrak{g}_j$. $Q = \sum_{j=1}^m j d_j$ is called the homogeneous dimension of $G$, where $d_j = \dim \mathfrak{g}_j$. $G$ is topologically identified with $\mathfrak{g}$ via the exponential map $\exp : \mathfrak{g} \to G$ and $\delta_r$ is also viewed as an automorphism of $G$. We fix a homogeneous norm of $G$, which satisfies the generalized triangle inequalities

\[ |xy| \leq \gamma(|x| + |y|) \quad \text{for all } x, y \in G, \]
\[ ||xy| - |x|| \leq \gamma|y| \quad \text{for all } x, y \in G \text{ with } |y| \leq \frac{|x|}{2}, \]

where $\gamma \geq 1$ is a constant. The ball of radius $r$ centered at $x$ is written by

\[ B(x, r) = \{y \in G : |x^{-1}y| < r\}. \]

The Haar measure on $G$ is simply the Lebesgue measure on $\mathbb{R}^d$ under the identification of $G$ with $\mathfrak{g}$ and the identification of $\mathfrak{g}$ with $\mathbb{R}^d$, where $d = \sum_{j=1}^m d_j$. The measure of $B(x, r)$ is

\[ |B(x, r)| = b r^Q, \]

where $b$ is a constant.

We identify $\mathfrak{g}$ with $\mathfrak{g}_L$, the Lie algebra of left-invariant vector fields on $G$. Let $\{X_j : j = 1, \cdots, d_1\}$ be a basis of $\mathfrak{g}_1$. The sub-Laplacian $\Delta_G$ is defined by

\[ \Delta_G = \sum_{j=1}^{d_1} X_j^2. \]

The theory of Hardy spaces on homogeneous groups was studied by Folland and Stein [7]. Christ and Geller [2] gave the Riesz transforms characterization of the Hardy space $H^1$ on stratified groups. The theory of local Hardy spaces on stratified groups can be established as in Section 4. In detail, we define the scaled local maximal functions $\widetilde{M}_{\delta r} f$’s by

\[ \widetilde{M}_{\delta r} f(x) = \sup_{|x^{-1}y| < r \leq 2^k} |f \ast \phi_r(y)|, \]
\[ \widetilde{M}_{\delta r} f(x) = \sup_{0 < r \leq 2^k} |f \ast \phi_r(x)|, \]
\[ \widetilde{M}_{(N), \delta r} f(x) = \sup_{\phi \in \mathcal{F}(G) \atop \|\phi\|_{(N)} \leq 1} \widetilde{M}_{\delta r} f(x), \]
\[ \widetilde{M}_{(N), \delta r} f(x) = \sup_{\phi \in \mathcal{F}(G) \atop \|\phi\|_{(N)} \leq 1} \widetilde{M}_{\delta r} f(x). \]

The scaled local Hardy space $h^p_k(G)$ ($p \leq 1$) is defined by

\[ h^p_k(G) = \left\{ f \in \mathcal{S}'(G) : \widetilde{M}_{(N_p), k} f \in L^p(G), N_p = [Q(1 - \frac{1}{p})] + 1 \right\}. \]
with
\[ \|f\|_{h^p_k} = \|\widetilde{M}_{(N_p), k}f\|_{L^p}. \]

Then we have
\[ \|\widetilde{M}(N), k f\|_{L^p} \approx \|\widetilde{M}^{+}_{(N), k}f\|_{L^p} \approx \|\widetilde{M}_{\phi, k}f\|_{L^p} \approx \|\widetilde{M}^+_{\phi, k}f\|_{L^p}, \]

where \( \phi \) is a commutative approximate identity and \( N \geq N_p \) is fixed. We have the atomic decomposition of \( h^p_k(G) \) as follows. Let \( 0 < p \leq 1 < q \leq \infty \). A function \( a \in L^q(G) \) is called an \( h^p_k \)-atom if the following conditions hold:

(i) \( \text{supp} a \subset B(x_0, r) \),

(ii) \( \|a\|_{L^q} \leq |B(x_0, r)|^{\frac{1}{q} - \frac{1}{p}} \),

(iii) if \( r < 2^k \), then \( \int_{B(x_0, r)} a(x) x^I \, dx = 0 \) for \( d(I) < N_p \),

where \( d(I) \) is the homogeneous degree of the monomial \( x^I \). Then \( f \in h^p_k(G) \) if and only if \( f \) can be written as \( f = \sum_j \lambda_j a_j \) converging in the sense of distributions and in \( h^p_k(G) \) norm, where \( a_j \) are \( h^p_k \)-atoms and \( \sum_j |\lambda_j|^p < \infty \). Moreover,
\[ \|f\|_{h^p_k}^p \approx \inf\left\{ \sum_j |\lambda_j|^p \right\}, \]

where the infimum is taken over all atomic decompositions of \( f \) into \( h^p_k \)-atoms. For \( p = 1 \), \( h^1_k(G) \) is also characterized by the local Riesz transforms. Let
\[ R_j = X_j(-\Delta_G)^{-\frac{1}{2}}, \quad j = 1, \ldots, d_1, \]

be the Riesz transforms with the convolution kernel \( R_j(x) \). A function \( f \in h^1_k(G) \) if and only if \( f \in L^1(G) \) and \( \widetilde{R}^j[k]f \in L^1(G), j = 1, \ldots, d_1 \), where the local Riesz transforms are defined by \( \widetilde{R}^j[k]f = f \ast R^j[k] \) and \( R^j[k](x) = \zeta(2^{-k}x)R_j(x) \) with \( \zeta \in C^\infty(G) \) satisfying \( 0 \leq \zeta(x) \leq 1 \), \( \zeta(x) = 1 \) for \( |x| < \frac{1}{2^k} \), and \( \zeta(x) = 0 \) for \( |x| > 1 \). Moreover,
\[ \|f\|_{h^1_k} \approx \|f\|_{L^1} + \sum_{j=1}^{d_1} \|\widetilde{R}^j[k]f\|_{L^1}. \]

Let us consider the Schrödinger operator \( L = -\Delta_G + V \), where the potential \( V \) is nonnegative and belongs to the reverse Hölder class \( B_2^* \). We define the Hardy space \( H^1_L(G) \) associated with the Schrödinger operator \( L \) by the maximal function with respect to the semigroup \( \{ T^s_L : s > 0 \} = \{ e^{-sL} : s > 0 \} \). A function \( f \in L^1(G) \) is said to be in \( H^1_L(G) \) if the maximal function \( M^L f \) belongs to \( L^1(G) \), where \( M^L f(x) = \sup_{s > 0} |T^s_l f(x)| \). The norm of such a function is defined by \( \|f\|_{H^1_L} = \|M^L f\|_{L^1} \). The atomic decomposition of \( H^1_L(G) \) is as follows. Let \( 1 < q \leq \infty \). A function \( a \in L^q(G) \) is called an \( H^1_L \)-atom if the following conditions hold:
(i) supp $a \subset B(x_0, r)$,

(ii) $\|a\|_{L^q} \leq |B(x_0, r)|^{\frac{1}{q} - 1}$,

(iii) if $r < \rho(x_0)$, then $\int_{B(x_0, r)} a(x) \, dx = 0$,

where the auxiliary function $\rho(x) = \rho(x, V)$ is defined as before; that is,

$$\rho(x) = \sup_{r > 0} \left\{ r : \frac{1}{r^{q-2}} \int_{B(x, r)} V(y) \, dy \leq 1 \right\}, \quad x \in G.$$ 

Let $f \in L^1(G)$ and $1 < q \leq \infty$. Then $f \in H^1_{L^q}(G)$ if and only if $f$ can be written as $f = \sum_j \lambda_j a_j$, where $a_j$ are $H^1_{L^q}$-atoms, $\sum_j |\lambda_j| < \infty$, and the sum converges in $H^1_{L^q}(G)$ norm. Moreover,

$$\|f\|_{H^1_{L^q}} \sim \inf \left\{ \sum_j |\lambda_j| \right\},$$

where the infimum is taken over all atomic decompositions of $f$ into $H^1_{L^q}$-atoms. The Hardy space $H^1_{L^q}(G)$ is also characterized by the Riesz transforms $R^L_j$ associated with the Schrödinger operator $L$. These Riesz transforms are defined by

$$R^L_j = X_j L^{-\frac{1}{2}}, \quad j = 1, \ldots, d_1.$$ 

Each $R^L_j$ is bounded on $L^p(G)$ for $1 < p \leq Q$ and bounded from $L^1(G)$ to $L^{1,\infty}(G)$. A function $f \in H^1_{L^q}(G)$ if and only if $f \in L^1(G)$ and $R^L_j f \in L^1(G)$, $j = 1, \ldots, d_1$. Moreover,

$$\|f\|_{H^1_{L^q}} \sim \|f\|_{L^1} + \sum_{j=1}^{d_1} \|R^L_j f\|_{L^1}.$$ 

These results for the Hardy space $H^1_{L^q}$ on stratified groups can be proved by the same argument as for the Heisenberg group. In fact, the estimates in Section 3 keep true for stratified groups.

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