Abstract

In this paper, we study the additive complexity \( \rho^+(n) \) of a Thue-Morse like sequence \( t = \sigma^\infty(0) \) with the morphism \( \sigma : 0 \rightarrow 01, 1 \rightarrow 12, 2 \rightarrow 20 \). We show that \( \rho^+(n) = 2 \lfloor \log_2(n) \rfloor + 3 \) for all integers \( n \geq 1 \). Consequently, \( (\rho^+(n))_{n \geq 1} \) is a 2-regular sequence.

Keywords: Thue-Morse like sequence, Additive complexity, \( k \)-regular sequence

2010 MSC: 05D99, 11B85

1. Introduction

Recently the study of the abelian complexity of infinite words was initiated by G. Richomme, K. Saari, and L. Q. Zamboni [16]. For example, the abelian complexity functions of some notable sequences, such as the Thue-Morse sequence and all Sturmian sequences, were studied in [16] and [6] respectively. There are also many other works including the unbounded abelian complexity, see [4, 8, 9, 12, 15] and references therein. At the mean time, many authors had devoted to the generalizations of the abelian complexity. For instance, \( l \)-abelian complexity, cyclic complexity and binomial complexity are first presented in [11], [5] and [14] respectively. In 1994, G. Pirillo and S. Varricchio [13] raised the following question: do there exist infinite words avoiding additive squares or additive cubes? Based on this infamous problem, H. Ardal, T. Brown, V. Jungić and J. Sahasrabudhe proposed the additive complexity for infinite word on a finite subset of \( \mathbb{Z} \) in [2]. It follows from the definition of additive equivalence in Section 2 that the additive complexity \( \{\rho^+(n)\} \) coincides with the abelian complexity \( \{\rho^B(n)\} \) for every infinite word on the alphabet composed of two integers. For every infinite word on the alphabet composed of integers whose cardinality is not less than three, it is easy to know that \( \rho^+(n) \leq \rho^B(n) \) for every \( n \).

Let \( \sigma \) be the morphism \( 0 \mapsto 01, 1 \mapsto 12, 2 \mapsto 20 \) on \( \{0,1,2\} \) and \( t := \sigma^\infty(0) \). The infinite sequence \( t \) is a Thue-Morse like sequence (see [3, 17]). Further, \( t \) is 2-automatic and uniformly recurrent (see [7]). A sequence \( w = w_0w_1w_2 \cdots \) is a \( k \)-automatic sequence if its \( k \)-kernel \( \{(w_{k^n+c})_{n \geq 0} \mid c \geq 0, 0 \leq c < k^e\} \) is finite. If the \( \mathbb{Z} \)-module generated by its \( k \)-kernel is finitely generated, then \( w \) is a \( k \)-regular sequence.

In this paper, we investigate the additive complexity function \( \rho^+(n) \) of \( t \), where \( \rho^+(n) \) is the number of different digit sums of all words (of length \( n \)) that occur in \( t \). We give the explicit value of \( (\rho^+(n))_{n \geq 1} \).
Theorem 1. For all integer \( n \geq 1 \),
\[
\rho^+_t(n) = 2\lfloor \log_2 n \rfloor + 3.
\]
Consequently, we know that the additive complexity function \( (\rho_t(n))_{n \geq 1} \) satisfying the recurrence relations: \( \rho^+_t(1) = 3 \) and for all \( n \geq 1 \),
\[
\rho^+_t(2n) = \rho^+_t(2n + 1) = \rho^+_t(n) + 2.
\]
The above recurrence relations imply the regularity of \( (\rho_t(n))_{n \geq 1} \).

Corollary 1. The additive complexity \( (\rho_t(n))_{n \geq 1} \) of \( t \) is a 2-regular sequence.

From the above corollary, it is natural to have the following conjecture.

Conjecture 1. The additive complexity of any \( k \)-automatic sequence is a \( k \)-regular sequence.

This paper is organized as follows. In Section 2, we give some notations. In Section 3, we prove Theorem 1. The proof is separated into 3 steps. Each step gives a more specific result.

2. Preliminaries

An alphabet \( A \) is a finite and non-empty set (of symbols) whose elements are called letters. A (finite) word over the alphabet \( A \) is a concatenation of letters in \( A \). The concatenation of two words \( u = u_0u_1\cdots u(m) \) and \( v = v_0v_1\cdots v(n) \) is the word \( uv = u_0u_1\cdots u_m v_0v_1\cdots v_n \). The set of all finite words over \( A \) including the empty word \( \varepsilon \) is denoted by \( A^* \). An infinite word \( w \) is an infinite sequence of letters in \( A \). The set of all infinite words over \( A \) is denoted by \( A^\mathbb{N} \).

The length of a finite word \( w \in A^* \), denoted by \( |w| \), is the number of letters contained in \( w \). We set \( |\varepsilon| = 0 \). For any word \( u \in A^* \) and any letter \( a \in A \), let \( |u|_a \) denote the number of occurrences of \( a \) in \( u \).

A word \( w \) is a factor of a finite (or an infinite) word \( v \), written by \( w \prec v \) if there exist a finite word \( x \) and a finite (or an infinite) word \( y \) such that \( v = xwy \). When \( x = \varepsilon \), \( w \) is called a prefix of \( v \), denoted by \( w \triangleleft v \); when \( y = \varepsilon \), \( w \) is called a suffix of \( v \), denoted by \( w \triangleright v \).

For a real number \( x \), let \( \lfloor x \rfloor \) (resp. \( \lceil x \rceil \)) be the integer that is less (resp. larger) than or equal to \( x \). For every natural number \( n \) and some positive integer \( b \geq 2 \), set \( (n)_b \) be the regular \( b \) -ary expansion of \( n \).

2.1. Additive complexity

Now we assume that \( A \subset \mathbb{Z} \). Let
\[
w = w_0w_1w_2\cdots \in A^\mathbb{N}
\]
be an infinite word. Denote by \( \mathcal{F}_w(n) \) the set of all factors of \( w \) of length \( n \), i.e.,
\[
\mathcal{F}_w(n) := \{w_iw_{i+1}\cdots w_{i+n-1} : i \geq 0\}.
\]
Write \( \mathcal{F}_w = \cup_{n \geq 1} \mathcal{F}_w(n) \). The subword complexity function \( \rho_w : \mathbb{N} \to \mathbb{N} \) of \( w \) is defined by
\[
\rho_w(n) := \# \mathcal{F}_w(n).
\]
Denote the digit sum of \( u = u_0\cdots u_{|u|-1} \in A^* \) by
\[
\text{DS}(u) := \sum_{j=0}^{|u|-1} u_j.
\]
Two finite words \( u, v \in A^* \) is additive equivalent if \( \text{DS}(u) = \text{DS}(v) \). The additive equivalent induces an equivalent relation, denoted by \( \sim_+ \).
Definition 1. The additive subword complexity function $\rho_w^+ : \mathbb{N} \to \mathbb{N}$ of $w$ is defined by

$$\rho_w^+(n) := \# \{ \mathcal{F}_w(n) / \sim_+ \}.$$ 

In fact,

$$\rho_w^+(n) = \# \{ DS(u) : u \in \mathcal{F}_w(n) \}. \quad (2.1)$$

The additive complexity can also be obtained from the Parikh vector. Let $u$ be an infinite word on an alphabet $A = \{ a_0, a_1, \cdots, a_{q-1} \}$ and $v$, a factor of $u$. The Parikh vector of $v$ is the $q$-uplet

$$\psi(w) = (|v|_{a_0}, |v|_{a_1}, \cdots, |v|_{a_{q-1}}).$$

Denote by $\Psi_u(n)$, the set of the Parikh vectors of the factors of length $n$ of $u$:

$$\Psi_u(n) = \{ \psi(v) : v \in \mathcal{F}_u(n) \}.$$ 

The abelian complexity of $u$ is defined by:

$$\rho_u^{ab}(n) = \# \Psi_u(n).$$

Given any infinite word $w$ on an alphabet $B = \{ b_0, b_1, \cdots, b_{q-1} \} \in \mathbb{Z}_q$ with $b_0 < b_1 < \cdots < b_{q-1}$. It is easy to verify that the additive complexity of $w$ is

$$\rho_w^+(n) = \# \{ (b_0, b_1, \cdots, b_{q-1}), \psi(v) : v \in \mathcal{F}_w(n) \} \quad (2.2)$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in the Euclid space $\mathbb{R}^q$. In fact,

$$DS(u) = \langle (b_1, \cdots, b_{q-1}), \psi(u) \rangle$$

for every factor $u \in \mathcal{F}_w(n)$.

3. Additive complexity of $t$

In this section, we prove Theorem 1. According to (2.1), the study of the additive complexity function turns out to be the study of digit sums of all factors. Our strategy in the proof of Theorem 1 is the following:

- (Proposition 1) give the upper and lower bounds of $\rho_u^{ab}(n)$ for all $n \geq 1$;
- (Proposition 2) show that the upper and lower bounds can be attained;
- (Proposition 3) study the all the accessible values of digit sums.

Then, Theorem 1 follows from Proposition 1, 2 and 3.

3.1. Upper and lower bounds of digit sums of factors

Proposition 1. For every integer $n \geq 1$,

$$n - \lfloor \log_2 n \rfloor - 1 \leq DS(u) \leq n + \lfloor \log_2 n \rfloor + 1$$

for all $u \in \mathcal{F}_t(n)$. 

3
Note that for every \( u = u_0 u_1 \cdots u_{n-1} \in F_k(n) \),

\[
DS(u) = \sum_{i=0}^{n-1} u_i = 0 \cdot |u_0| + 1 \cdot |u_1| + 2 \cdot |u_2| = |u_0| + |u_1| + 2|u_2| - |u_0| = n + |u_2| - |u_0|.
\]  

(3.1)

To prove Proposition \( \Box \) we only need to show that for all \( u \in F_k(n) \),

\[-|\log_2 n| - 1 \leq |u_2| - |u_0| \leq |\log_2 n| + 1. \]  

(3.2)

The following lemmas are aimed to analysis the quantity \(|u_2| - |u_0|\).

**Lemma 1.** For every \( u \in \{0, 1, 2\}^* \),

\[
|\sigma(u)| - |\sigma(u)|_0 = |u_1| - |u_0|, \\
|\sigma(u)|_1 - |\sigma(u)|_0 = |u_1| - |u_2|.
\]

**Proof.** It follows from the definition of \( \sigma \) that

\[
|\sigma(u)|_0 = |u_0| + |u_2|, \quad |\sigma(u)|_1 = |u_0| + |u_1|, \quad |\sigma(u)|_2 = |u_1| + |u_2|.
\]

The above equations give the required results. \( \square \)

Let \( a, b, c \) be any arrangement of \( 0, 1, 2 \). Define \( \tau_c : a \mapsto b, b \mapsto a, c \mapsto c \). For every finite word \( w = w_0 w_1 \cdots w_{n-1} \in \{0, 1, 2\}^* \), let \( w^R = w_n w_{n-1} \cdots w_1 w_0 \) be the mirror of \( w \). For every \( x \in \{0, 1, 2\} \), write \( x := x - 1 \mod 3 \) and \( \overline{x} := x + 1 \mod 3 \). The morphisms \( \sigma \) and \( \tau \) have the following commutative property.

**Lemma 2.** For every \( u \in F_k \) and every \( c = 0, 1, 2 \),

\[
\sigma(\tau_c(u))^R = \tau_{\overline{c}}(\sigma(u))^R.
\]  

\( (3.3) \)

**Proof.** It is easy to check \( (3.3) \) for all \( u \in F_k(1) = \{0, 1, 2\} \). Assume that \( (3.3) \) holds for all \( u \in \bigcup_{i=1}^{n-1} F_k(i) \). For any \( u \in F_k(n) \), we have \( u = va \) where \( v \in F_k(n-1) \) and \( a \in \{0, 1, 2\} \). Then

\[
\sigma(\tau_c(u))^R = \sigma(\tau_c(va))^R = \sigma(\tau_c(a)^R \tau_c(v)^R)
\]

\[
= \sigma(\tau_c(a))^R \sigma(\tau_c(v))^R = \tau_c(\sigma(a))^R \tau_c(\sigma(v))^R \quad \text{(by the assumption)}
\]

\[
= \tau_{\overline{c}}(\sigma(v))^R = \tau_{\overline{c}}(\sigma(u))^R,
\]

which implies that \( (3.3) \) holds for all \( u \in F_k(n) \) and \( c = 0, 1, 2 \). \( \square \)

While \( \sigma \) maps every factor of \( t \) to a factor of \( t \), the morphism \( \tau_c \) maps every factor of \( t \) to the mirror of some factor of \( t \).

**Lemma 3.** If \( u \in F_k \), then \( \tau_c(u)^R \in F_k \) for \( c = 0, 1, 2 \).

**Proof.** When \( u \in F_k(1) \cup F_k(2) \), the result can be checked directly. Now, suppose the result holds for all \( u \in \bigcup_{i=1}^{n-1} F_k(i) \) (where \( n \geq 3 \)). Let \( u \in F_k(n) \). If \( n \) is odd, then \( u = a\sigma(v) \) or \( \sigma(v)b \) where \( v \in F_k([n/2]) \) and \( a, b \in \{0, 1, 2\} \), which also imply that \( av = \sigma(\overline{a}v) \) or \( \overline{vb} = \sigma(vb) \) with \( \overline{av}, \overline{vb} \in F_k([n/3]) \). By Lemma 2 for \( c = 0, 1, 2 \),

\[
\tau_c(\overline{av})^R = \tau_c(\sigma(\overline{a}v))^R = \sigma(\tau_c(\overline{av}))^R.
\]
Since \( aw \in \mathcal{F}_t(\frac{n}{2}) \), by the inductive hypothesis, \( \tau_c(aw)^R \in \mathcal{F}_t(\frac{n}{2}) \). So \( \tau_c(aw)^R \in \mathcal{F}_t(n+1) \) and \( \tau_c(u)^R \in \mathcal{F}_t(n) \). The same is true for the case \( u = \sigma(v)b \).

If \( n \) is even, then \( u = \sigma(w) \) or \( a\sigma(v)b \) where \( w \in \mathcal{F}_t(n/2) \), \( v \in \mathcal{F}_t(\frac{n}{2} - 1) \) and \( a, b \in \{0, 1, 2\} \). When \( u = a\sigma(v)b \), we have \( awb = \sigma(awb) \) with \( awb \in \mathcal{F}_t(\frac{n}{2} + 1) \). By Lemma 2, for \( c = 0, 1, 2 \),

\[
\tau_c(awb)^R = \tau_c(\sigma(awb))^R = \sigma(\tau_c(awb))^R.
\]

By the inductive hypothesis, \( \tau_c(awb)^R \in \mathcal{F}_t \). So \( \tau_c(awb)^R \in \mathcal{F}_t \) which implies \( \tau_c(u) \in \mathcal{F}_t \). When \( u = \sigma(w) \), the result follows from Lemma 2 and the inductive hypothesis in the same way. □

**Lemma 4.** Let \( n \geq 1 \) be an integer and \( u \in \mathcal{F}_t(n) \).

1. There exists \( x \in \mathcal{F}_t(\lfloor n/2 \rfloor) \) such that
\[
|x_1| - |x_0| - 1 \leq |u_2 - u_0| \leq |x_1| - |x_0| + 1.
\] (3.4)

2. There exists \( y \in \mathcal{F}_t(\lfloor n/2 \rfloor) \) such that
\[
|y_1| - |y_2| - 1 \leq |u_1 - u_0| \leq |y_1| - |y_2| + 1.
\] (3.5)

3. There exists \( z \in \mathcal{F}_t(\lfloor n/2 \rfloor) \) such that
\[
|z_1| - |z_2| - 1 \leq |u_1 - u_2| \leq |z_0| - |z_2| + 1.
\]

**Proof.** (1) If \( n \) is odd, then \( u = a\sigma(v) \) or \( \sigma(v)b \) where \( v \in \mathcal{F}_t(\lfloor n/2 \rfloor) \) and \( a, b \in \{0, 1, 2\} \). In either case,

\[
|u_2 - u_0| = \begin{cases} 
|\sigma(v)|_2 - |\sigma(v)|_0 - 1, & \text{if } a, b = 0, \\
|\sigma(v)|_2 - |\sigma(v)|_0, & \text{if } a, b = 1, \\
|\sigma(v)|_2 - |\sigma(v)|_0 + 1, & \text{if } a, b = 2.
\end{cases}
\]

Letting \( x = v \), the result follows.

If \( n \) is even, then \( u = \sigma(w) \) or \( a\sigma(v)b \) where \( w \in \mathcal{F}_t(n/2) \), \( v \in \mathcal{F}_t(\frac{n}{2} - 1) \) and \( a, b \in \{0, 1, 2\} \). When \( u = \sigma(w) \), by Lemma 1, \( |u_2 - u_0| = |w_1| - |w_0| \). Choosing \( x = w \), we have the desired result. When \( u = a\sigma(v)b \), let \( \alpha = a - 1 \) (mod 3) and \( \beta = b + 1 \) (mod 3). Then \( awb = \sigma(awb) \). By Lemma 1,

\[
|awb|_2 - |awb|_0 = |\sigma(awb)|_2 - |\sigma(awb)|_0 = |awb|_1 - |awb|_0,
\]

which implies

\[
|u_2 - u_0| = |awb|_1 - |awb|_0 + |awb|_0 - |awb|_2.
\]

When \( ab \neq 00 \) and \( 12 \),

\[
|u_2 - u_0| = |awb|_1 - |awb|_0 + \begin{cases} 
-1, & \text{if } ab = 01, 20, \\
0, & \text{if } ab = 02, 10, 21, \\
1, & \text{if } ab = 11, 22.
\end{cases}
\]
After applying Lemma 4, Hence (3.2) holds.

Let \( n \in \mathbb{Z} \), let \( \frac{x}{2} = y \), then we have \( 0 \leq y \leq 1 \). Therefore, for any \( \ell \in \mathbb{Z} \), we have \( 0 \leq \ell y \leq \ell \). For every \( k \in \mathbb{Z} \), let \( \ell = \ell^k \), then we have \( 0 \leq \ell^k y \leq \ell^k \). Now we are ready to prove Proposition 1.

Let \( (d_k)_k \) be any sequence such that \( d_k = 2^k + 1 \). Applying (3.5) to \( \tau_1(\ell) \), we have \( \tau_1(\ell) \in \mathcal{F}_t \) and

\[ |u|_1 - |u|_0 = |\tau_1(\ell)|_2 - |\tau_1(\ell)|_0. \]

Applying (3.3) to \( \tau_1(\ell) \), we have \( x \in \mathcal{F}_t \) such that

\[ |x|_1 - |x|_0 - 1 = |\tau_1(\ell)|_2 - |\tau_1(\ell)|_0 \leq |x|_1 - |x|_0 + 1. \]

Let \( y = \tau_1(\ell) \). Then, \( y \in \mathcal{F}_t \) and \( |y|_1 - |y|_2 = |x|_1 - |x|_0 \). We have the desired result.

(3) Applying (3.5) to \( \tau_2(y) \) and letting \( z = \tau_2(y) \), the result follows.

Now we are ready to prove Proposition 1.

**Proof of Proposition 1.** For every \( n \geq 1 \), there exists \( k \geq 1 \) such that \( 2^k \leq n < 2^{k+1} + 1 \). Suppose \( u \in \mathcal{F}_t(u) \). Let \( n_1 = n \). By Lemma 4, we have \( x(1) \in \mathcal{F}_t([n_1/2]) \) such that

\[ |x(1)|_1 - |x(1)|_0 - 1 = |u|_2 - |u|_0 \leq |x(1)|_1 - |x(1)|_0 + 1. \]

Let \( n_2 = [n_1/2] \). Apply Lemma 4 to \( x(1) \), we have \( x(2) \in \mathcal{F}_t([n_2/2]) \) such that

\[ |x(2)|_1 - |x(2)|_2 - 1 = |x(1)|_1 - |x(1)|_0 \leq |x(2)|_1 - |x(2)|_2 + 1. \]

Therefore,

\[ |x(2)|_1 - |x(2)|_2 - 2 \leq |u|_2 - |u|_0 \leq |x(2)|_1 - |x(2)|_2 + 2. \]

Let \( n_3 = [n_2/2] \) and apply Lemma 4 to \( x(2) \). Then we have \( x(3) \in \mathcal{F}_t([n_3/2]) \) satisfying

\[ |x(3)|_0 - |x(3)|_2 - 3 \leq |u|_2 - |u|_0 \leq |x(3)|_0 - |x(3)|_2 + 3. \]

After applying Lemma 4 \( k \) times as above, we obtain that

\[ -1 - \log_2 n = -1 - k \leq |u|_2 - |u|_0 \leq 1 + k = 1 + |\log_2 n|. \]

Hence (3.2) holds.

3.2. Maximal and minimal digit sums

Let \( (d_k)_k \geq -1 \in \{0, 1, 2\} \infty \) where

\[ d_k = \begin{cases} 0 & \text{if } k \equiv 3, 4 \mod 6, \\ 1 & \text{if } k \equiv 1, 2 \mod 6, \\ 2 & \text{if } k \equiv 0, 5 \mod 6. \end{cases} \]

Let \( (c_\ell)_\ell \geq 1 \in \{0, 1, 2\} \infty \) given by \( c_\ell = \ell + 1 \pmod{3} \). Applying Lemma 1 several times, it follows that for \( k = 0, 1, 2, 3, 4, 5, \)

\[ |\sigma^k(d_k)|_2 - |\sigma^k(d_k)|_0 = 1 \quad \text{and} \quad |\sigma^k(c_\ell)|_2 - |\sigma^k(c_\ell)|_0 = 0. \quad (3.6) \]

In fact, these equalities hold for all \( k \geq 1 \).
Lemma 5. For all $\ell \geq 1$,
\[
|\sigma^\ell(d_{k})|_2 - |\sigma^\ell(d_{k})|_0 = 1 \quad \text{and} \quad |\sigma^\ell(c_{k})|_2 - |\sigma^\ell(c_{k})|_0 = 0.
\]

Proof. Applying Lemma 4 six times, one obtain that for every $u \in \{0, 1, 2\}^*$,
\[
|\sigma^\ell(u)|_2 - |\sigma^\ell(u)|_0 = |u|_2 - |u|_0.
\]
(3.7)

For all $\ell \geq 1$, we have $\ell = 6j + k$ where $j \geq 1$ and $k = 0, 1, 2, 3, 4, 5$. Then $d_\ell = d_k$ and $c_\ell = c_k$. By (3.6) and (3.7),
\[
|\sigma^\ell(d_{k})|_2 - |\sigma^\ell(d_{k})|_0 = |\sigma^{6j+k}(d_{k})|_2 - |\sigma^{6j+k}(d_{k})|_0 = |\sigma^k(d_{k})|_2 - |\sigma^k(d_{k})|_0 = 1
\]
and
\[
|\sigma^\ell(c_{k})|_2 - |\sigma^\ell(c_{k})|_0 = |\sigma^{6j+k}(c_{k})|_2 - |\sigma^{6j+k}(c_{k})|_0 = |\sigma^k(c_{k})|_2 - |\sigma^k(c_{k})|_0 = 0.
\]
We have the desired. 

Now we define a sequence of words $\{W(n)\}_{n \geq 1}$ whose digit sums will attain the upper bound in Proposition 3.

Let $W_1 := 2$. For $n \geq 2$, $W_n$ is defined as follows: suppose $2^k \leq n < 2^{k+1}$ for some $k$ and the 2-adic expansion of $n - 2^k$ is written as
\[
(n - 2^k) = m_{k-1} \cdots m_2 m_1 m_0
\]
where $m_j \in \{0, 1\}$ for $j = 0, 1, \cdots k - 1$. Define
\[
W_L(n) := \delta_{m_0}2^\left(\sum_{i=1}^{\lfloor \frac{k-1}{2} \rfloor} 2^{i-1+m_1+m_2+1}(d_{2i+m_2+1})\right)
\]
and
\[
W_R(n) := \left(\prod_{i=1}^{\lfloor \frac{k-1}{2} \rfloor} 2^{2i-1+m_1+m_2+1}(d_{2i-1+m_2+1})\right)^2
\]
where $\delta_{m_0} = \varepsilon$ if $m_0 = 0$ and 1 if $m_0 = 1$. Let $W(n) := W_L(n)W_R(n)$.

Lemma 6. For every integer $n$ satisfying $2^k \leq n < 2^{k+1}$ for some $k \geq 0$, we have

1. if $k$ is even, then $W_L(n) \triangleright \sigma^k(d_{k+1})$ and $W_R(n) \triangleleft \sigma^{k+1}(d_{k})$,
2. if $k$ is odd, then $W_L(n) \triangleright \sigma^{k+1}(d_{k+2})$ and $W_R(n) \triangleleft \sigma^k(d_{k-1})$.

Proof. For $k = 0, 1, 2$, the result can be verified directly from the definition of $W_L$ and $W_R$. Suppose the result hold for all $m \leq k$. Now we prove it for $m = k + 1$. Let $2^{k+1} \leq n < 2^{k+2}$ with $(n - 2^{k+1}) = m_k m_{k-1} \cdots m_1 m_0$. Set $n^* = 2^{k+1} + m_k m_{k-1} \cdots m_1 m_0$.

When $k + 1$ is odd, write $k = 2\ell$. Then $W_L(n) = W_R(n^*) \triangleleft \sigma^{k+1}(d_k)$ and
\[
W_L(n) = \delta_{m_0}2^\left(\prod_{i=1}^{\ell} 2^{2i+m_2+1}(d_{2i+m_2+1})\right)
\]
\[
= \delta_{m_0}2^\left(\prod_{i=1}^{\ell-1} 2^{2i+m_2+1}(d_{2i+m_2+1})\right)\sigma^{2\ell+m_2+1}(d_{2\ell+m_2+1})
\]
\[
= W_L(n^*)\sigma^{k+m_1}(d_{k+m_1}).
\]
When \( m_k = 0 \), by the induction hypothesis,
\[
W_L(n')\sigma^{k+m_k}(d_{k+m_k}) \succ \sigma^k(d_{k+1})\sigma^k(d_k) = \sigma^k(d_{k+2})\sigma^k(d_k) = \sigma^{k+1}(d_{k+2}) \succ \sigma^{(k+1)+1}(d_{(k+1)+2}).
\]

When \( m_k = 1 \), by the induction hypothesis,
\[
W_L(n')\sigma^{k+m_k}(d_{k+m_k}) \succ \sigma^k(d_{k+1})\sigma^{k+1}(d_{k+1}) \succ \sigma^{k+1}(d_{k+3})\sigma^{k+1}(d_{k+1}) = \sigma^{(k+1)+1}(d_{(k+1)+2}).
\]

So, \( W_L(n) \succ \sigma^{(k+1)+1}(d_{(k+1)+2}) \).

When \( k+1 \) is even, write \( k = 2\ell + 1 \). Then \( W_L(n) = W_L(n') \succ \sigma^{k+1}(d_{k+2}) \) and
\[
W_R(n) = \sigma^{k+m_k}(d_{k+m_k})W_R(n')
\]
\[
\begin{cases}
\sigma^k(d_k)\sigma^k(d_{k-1}) = \sigma^{k+1}(d_{k+1}), & \text{if } m_k = 0, \\
\sigma^{k+1}(d_{k+1})\sigma^k(d_{k-1}), & \text{if } m_k = 1,
\end{cases}
\]
\[
\succ \sigma^{k+2}(d_{k+1}).
\]

This completes the induction. \( \square \)

**Proposition 2.** For all \( n \geq 1 \), \( W(n) \in \mathcal{F}_1 \) and \( \tau_1(W(n))^R \in \mathcal{F}_1 \). Moreover,
\[
\text{DS}(W(n)) = n + \lfloor \log_2 n \rfloor + 1 \quad \text{and} \quad \text{DS}(\tau_1(W(n))^R) = n - \lfloor \log_2 n \rfloor - 1.
\]

**Proof.** For all \( n \) with \( 2^k \leq n < 2^{k+1} \), by Lemma 3,
\[
W(n) = W_L(n)W_R(n) \prec \sigma^k(d_{k+2}d_kd_{k-2}) \prec \sigma^{k+1}(d_{k+2}d_{k-2}).
\]

Since \( (d_i)_{i \geq 1} \) is periodic, \( d_{k+2}d_{k-2} \in \{21, 10, 02\} \subset \mathcal{F}_1(2) \). Hence \( W(n) \in \mathcal{F}_1 \). By Lemma 3, we know \( \tau_1(W(n))^R \in \mathcal{F}_1 \).

According to the definition of \( W_L \) and \( W_R \),
\[
|W(n)|_2 - |W(n)|_0 = 2 + \sum_{i=1}^{k-1}(|\sigma^{i+m_i}(d_{i+m_i})|_2 - |\sigma^i(d_{i+m_i})|_0)
\]
\[
= 2 + k - 1 \quad \text{(by Lemma 3)}
\]
\[
= \lfloor \log_2(n) \rfloor + 1.
\]

Then the results follow from (3.1) and the definition of \( \tau_1 \). \( \square \)

### 3.3. Accessible values of digit sums

We shall prove the following intermediate value property of digit sums of all the factors of length \( n \).

**Proposition 3.** For all \( n \geq 1 \) and all integer \( k \) satisfying \( n - \lfloor \log_2 n \rfloor - 1 < k < n + \lfloor \log_2 n \rfloor + 1 \), there exists \( u \in \mathcal{F}_k(n) \) such that \( \text{DS}(u) = k \).

Before proving Proposition 3, we first study the behavior of digit sums during the shift (to the right). Denote by \( \mathcal{I}(u) \) the set of all the indexes (or positions) of occurrences of \( u \), i.e., for every \( i \in \mathcal{I}(u) \), \( t_it_{i+1}\cdots t_{i+n-1} = u \). Since \( \mathbf{t} \) is uniformly recurrent, \( \mathcal{I}(u) \) is an infinite set for all \( u \in \mathcal{F}_1 \). For every \( i \in \mathcal{I}(u) \), set
\[
\rho_i(u) = \min\{j > i : g_n(j) > \text{DS}(u)\},
\]
where \( g_n(j) := \text{DS}(t_{j+1}t_{j+2}\cdots t_{j+n-1}) \). Set \( \min\emptyset = -\infty \).
Lemma 7. Let \( u \in \mathcal{F}_t(n) \). If DS(u) \( \neq \) DS(W(n)), then \( r_i(u) \) is finite and
\[
g_n(r_i(u)) - DS(u) = 1 \text{ or } 2.
\]
Moreover, if \( g_n(r_i(u)) - DS(u) = 2 \), then \( g_n(r_i(u) - 1) = DS(u) \).

Proof. By Proposition 1 and 2, if DS(u) \( \neq \) DS(W(n)), then DS(u) < DS(W(n)). For any occurrence of \( u \), say \( t_1t_2\cdots t_{i+n-1} = u \), since \( t \) is uniformly recurrent, there exists \( j > i \) such that \( t_jt_{j+1}\cdots t_{j+n-1} = W(n) \). Thus, \( r_i(u) < j \).

Now suppose \( r_i(u) \) is finite. Write \( k := r_i(u) \). Then, \( g_n(k - 1) \leq DS(u) \). Since
\[
g_n(k) - g_n(k - 1) = t_{k+n-1} - t_{k-1} \in \{0, \pm 1, \pm 2\}
\]
and \( g_n(k) \geq DS(u) \geq g_n(k - 1) \), we know that \( g_n(k) - DS(u) = 1 \) or 2. Moreover, if \( g_n(k - 1) < DS(u) \), then \( 0 < g_n(k) - DS(u) < g_n(k) - g_n(k - 1) \leq 2 \) which implies \( g_n(k) - DS(u) = 1 \). \( \square \)

The key to prove Proposition 3 is to figure out how many times we need to do the shift in order to increase the digit sum of a given factor by 1. The following two lemmas deal with the problem. The first one is a technical lemma. Let \( u, v \in \mathcal{F}_t(3) \). Write \( \sigma^0(u) = u_0u_1\cdots u_{191} \) and \( \sigma^0(v) = v_0v_1\cdots v_{191} \). For \( 64 \leq i, j < 128 \) satisfying \( u_i = 0 \) and \( v_j = 2 \), \( 0 < m < 192 - \max(i, j) \) and \( 0 < p \leq \min(i, j) \), set
\[
R(u, v, i, j, m) = \sum_{\ell=0}^{m}(u_{i+\ell} - u_{i+\ell}),
\]
\[
L(u, v, i, j, -p) = \sum_{\ell=1}^{p}(u_{i-\ell} - v_{j-\ell}).
\]

Lemma 8. For all \( u, v \in \mathcal{F}_t(3) \) and \( 64 \leq i, j < 128 \) satisfying \( u_i = 0 \) and \( v_j = 2 \), \( R(u, v, i, j, m) = 1 \) for some \( 0 < m < 192 - \max(i, j) \) or \( L(u, v, i, j, -p) = 1 \) for some \( 0 < p \leq \min(i, j) \).

Proof. Since the choices of variables of both \( L \) and \( R \) are finite, the result can be verified exhaustively. (This can be easily checked by a computer. We give the pseudocode for the corresponding procedures in Appendix A.) \( \square \)

Lemma 9. Let \( n > 128 \). For every \( u \in \mathcal{F}_t(n) \) with DS(u) \( \neq \) DS(W(n)), there exists \( z \in \mathcal{F}_t(n) \) satisfying DS(z) – DS(u) = 1.

Proof. Let \( i \in I(u) \) with \( i \geq 2^8 \). Set \( j = r_i(u) - 1 \). By Lemma 4 if \( g_n(r_i(u)) - DS(u) = 1 \), then we are done. If \( g_n(r_i(u)) - DS(u) = 2 \), then \( g_n(j) = DS(u) \) which also implies \( t_j = 0 \) and \( t_{j+n} = 2 \). Write \( w = t_jt_{j+1}\cdots t_{j+n-1} \).

The word \( w \) has the following decomposition:
\[
w = (t_{j}t_{j+1}\cdots t_{j+r-1})\sigma^0(v)(t_{j+n-r}\cdots t_{j+n-1}) < \sigma^0(xvy)
\]
where \( v \in \mathcal{F}_t, t_jt_{j+1}\cdots t_{j+r-1}\sigma^0(x) \) and \( t_{j+n-r}\cdots t_{j+n-1} < \sigma^0(y) \) for some \( x, y \in \{0, 1, 2\} \). Note that \( \ell, r \leq 64 \). Further, we have
\[
w < \sigma^0(bvxyd)
\]
where \( v \in \mathcal{F}_t, bx, yd \in \mathcal{F}_t(2) \) and \( bvxyd \in \mathcal{F}_t \). Let \( j = j \) (mod 64) and \( j' = j + n - 1 \) (mod 64). Let \( a, c \in \{0, 1, 2\} \) with \( a < v \) and \( c > v \). Then,
\[
g_n(j + m) - DS(w) = R(bxa, cyd, j + 64, j' + 64, m),
\]
\[
g_n(j - p) - DS(w) = L(bxa, cyd, j + 64, j' + 64, p).
\]
By Lemma 5 one of the following is true:
there exists a factor $v \in \{w \}.$

The property of digit sums. It is easy to know that the intermediate value property holds for all integers $w \in \mathbb{Z}$ such that for every integer $n \geq N$ and every integer $i$ satisfying

$$\min_{u \in \mathcal{F}_u(n)} DS(u) \leq i \leq \max_{u \in \mathcal{F}_u(n)} DS(u),$$

there exists a factor $v \in \mathcal{F}_u(n)$ such that $DS(v) = i.$

According to Proposition 3, the Thue-Morse like sequence $t$ has the intermediate value property of digit sums. It is easy to know that the intermediate value property holds for all $w \in \{0, 1\}^\mathbb{N}$. However, this property does not always hold. The following is a counter example.

Let the alphabet $A = \{a, b, c\}$ and the morphism $\sigma_3 : a \mapsto abc, b \mapsto bca, c \mapsto cab.$ Consider the automatic sequence $w = \sigma_3^\infty(a)$ generated by $\sigma_3$. It follows from [10] that its abelian complexity $\{\rho_u(n)\}_{n \geq 3} = 766766766 \cdots$.

Recall that $\Psi_w(n)$ is the set of the Parikh vectors of all the factors of length $n$ of $w$. Following from the proof of [10, Proposition 4.1], it is not hard to check the following proposition.

**Proposition 4.** Let $w = \sigma_3^\infty(a)$ with $\sigma_3 : a \mapsto abc, b \mapsto bca, c \mapsto cab$ and set $I = (1, 1, 1)$. For every integer $n \geq 3$ where $n = 3m + r$ for some $r = 0, 1, 2$, we have

- if $r = 0$, then
  $$\Psi_w(n) = mI + \{(1, 0, -1), (0, 0, 0), (1, -1, 0), (0, 1, -1), (-1, 1, 0), (-1, 0, 1), (0, -1, 1)\}.$$

- if $r = 1$, then
  $$\Psi_w(n) = mI + \{(1, 1, -1), (1, -1, 1), (0, 1, 0), (1, 0, 0), (0, 0, 1), (-1, 1, 1)\}.$$

- if $r = 2$, then
  $$\Psi_w(n) = mI + \{(2, 0, 0), (0, 2, 0), (0, 0, 2), (0, 1, 1), (1, 1, 0), (1, 0, 1)\}.$$

**Example 1.** Given a coding $\tau : a \mapsto x, b \mapsto y, c \mapsto z$ where $x < y < z \in \mathbb{Z},$ consider the infinite word $w' = \tau(w)$ with $w$ is defined in Proposition 3. We will discuss the sum set of all the factors of length $n$ in $w'$ by the value of $r$.

- If $r = 0$, then we have
  $$\{DS(u) : u \in \mathcal{F}_w(n)\} = \{< (x, y, z), \psi(n) > : \psi(n) \in \Psi_w(n)\} = x + y + z + \{x - z, y - z, x - y, 0, y - x, z - y, z - x\}$$
If $r = 1$, then we have
\[
\{DS(u) : u \in F_w(n)\} = \{< (x, y, z), \psi(n) >: \psi(n) \in \Psi_w(n)\}
\]
\[
= x + y + z + \{x + y - z, y + x - z, x + z - y, x, y, z\}
\]

If $r = 2$, then we have
\[
\{DS(u) : u \in F_w(n)\} = \{< (x, y, z), \psi(n) >: \psi(n) \in \Psi_w(n)\}
\]
\[
= x + y + z + \{x + y, y + x, x + z, 2x, 2y, 2z\}
\]

Hence the intermediate value property of $w'$ holds if and only if $z = y + 1 = x + 2$. It follows that there are many infinite words such as $\tau(w)$ with $\tau : a \mapsto 0, b \mapsto 1, c \mapsto 3$ in which the intermediate value property fails.

As a consequence, we have the following interesting questions.

**Question 1.** Given a finite set $A \subset \mathbb{Z}$, under what condition an infinite sequence $w \in A^\mathbb{N}$ will have the intermediate value property?

In addition, it follows from Proposition 3 and Lemma 1 that for the Thue-Morse like sequence $t$, we have for every $n \geq 1$,
\[
\max_{a, b \in \{0, 1, 2\}} \max_{u \in F_t(n)} \{|u|_a - |u|_b\} = 2\lceil \log_2 n \rceil + 3.
\]

On the basis of the above equation, for an infinite word $w$ on a finite alphabet $A$, we can define a measure for the evenness degree of the factors of length $n$ in $w$
:\[
E_w(n) := \max_{a, b \in A} \max_{u \in F_w(n)} \{|u|_a - |u|_b\}.
\]

Note that the evenness function $E_w(n)$ is different with the balance function $B_t(n)$ which is first introduced in [1]. For every primitive morphic word $w$, Adamczewski [1] showed that the asymptotic behaviour of the balance function $B_w(n)$ is in part ruled by the spectrum of the incidence matrix associated with the substitution. Here we have the following question.

**Question 2.** How about the asymptotic behaviour of the evenness function $E_w(n)$ for the primitive morphic word $w$?

**Acknowledgement**

This work was supported by NSFC (No. 11701202), the Fundamental Research Funds for the Central Universities (Nos. 2017MS110, 2662015QD016) and the Characteristic innovation project of colleges and universities in Guangdong (No. 2016KTSCX007).
Appendix A: Pseudocode for Lemma 8.

Algorithm 1 For every input $u, v, i, j$ which is present in Lemma 8, the outputs of two following procedures can not be both false.

1: procedure RightShiftTimes
2: Input: $u, v, i, j$ with $64 \leq i, j < 128$ satisfying $u_i = 0$ and $v_j = 2$
3: Output: $m$ or false
4: lword $\leftarrow \sigma^6(u)$
5: rword $\leftarrow \sigma^6(v)$
6: $m \leftarrow 0$
7: $s \leftarrow 0$
8: while $m \leq 192 - \max(i, j)$ do
9: \hspace{1em} $s \leftarrow s + rword(j + m) - lword(i + m)$
10: \hspace{1em} if $s = 1$ then
11: \hspace{2em} return $m$
12: \hspace{1em} $m \leftarrow m + 1$
13: return false

14: procedure LeftShiftTimes
15: Input: $u, v, i, j$ with $64 \leq i, j < 128$ satisfying $u_i = 0$ and $v_j = 2$
16: Output: $-p$ or false
17: lword $\leftarrow \sigma^6(u)$
18: rword $\leftarrow \sigma^6(v)$
19: $p \leftarrow 1$
20: $s \leftarrow 0$
21: while $p \leq \min(i, j)$ do
22: \hspace{1em} $s \leftarrow s + lword(i - p) - rword(j - p)$
23: \hspace{1em} if $s = 1$ then
24: \hspace{2em} return $-p$
25: \hspace{1em} $p \leftarrow p + 1$
26: return false

Appendix B: Pseudocode for Proposition 3 for $1 \leq n \leq 128$.

Since $t$ is uniformly recurrent, for every positive integer $n$, there exits an integer $R(n) > n$ such that for every $u \in F_t(n)$, we have $u = t_0 \cdots t_{R(n)}$. At the mean time, using the analogue of the proof of Proposition 5.1.9, we can show the subword complexity function $\rho_t(n)$: $\rho_t(1) = 3$, $\rho_t(2) = 9$, and for $n \geq 3$,

$$
\begin{align*}
\rho_t(2n) &= \rho_t(n) + \rho_t(n + 1), \\
\rho_t(2n + 1) &= 2\rho_t(n + 1).
\end{align*}
$$

Hence it is possible to find the index $R(n)$ for every $1 \leq n \leq 128$ with the help of a computer.
Algorithm 2 For every input \( n, k \), the output of the following procedure always be true.

1: procedure HAVEDESIREDDIGITSUM
2: Input: \( n, k \) with \( 1 \leq n \leq 128 \) and \( n - \lfloor \log_2 n \rfloor - 1 < k < n + \lfloor \log_2 n \rfloor + 1 \)
3: Output: true or false
4: \( i \leftarrow 0 \)
5: while \( i \leq R(n) - n + 1 \) do
6: \( ds \leftarrow 0 \)
7: \( j \leftarrow i \)
8: while \( j \leq i + n - 1 \) do
9: \( ds \leftarrow ds + t_j \)
10: \( j \leftarrow j + 1 \)
11: if \( ds = k \) then
12: return true
13: \( i \leftarrow i + 1 \)
14: return false

References

[1] B. Adamczewski, Balances for fixed points of primitive substitutions, Theoret. Comput. Sci. 307(1) (2003) 47-75.

[2] H. Ardal, T. Brown, V. Jungić and J. Sahasrabudhe, On abelian and additive complexity in infinite words, INTEGERS - Elect. J. Combin. Number Theory 12 (2012), #A21.

[3] J. P. Allouche and J. Shallit, Automatic Sequences, Theory, Applications, Generalizations, Cambridge, 2003.

[4] L. Balková, K. Břinda and O. Turek. Abelian complexity of infinite words associated with quadratic Parry numbers, Theoret. Comput. Sci. 412 (45) (2011) 6252-6260.

[5] J. Cassaigne, G. Fici, M. Sciortino and L. Q. Zamboni, Cyclic complexity of words, in International Symposium on Mathematical Foundations of Computer Science (pp. 159-170). Springer Berlin Heidelberg (2014, August).

[6] E. M. Coven and G. A. Hedlund, Sequences with minimal block growth, Math. Systems Theory 7 (1973) 138-153.

[7] N. P. Fogg, Substitutions in Dynamics, Arithmetics and Combinatorics (Lecture Notes in Mathematics, 1794) . Eds. Berth V., Ferenczi S., Mauduit C. and Siegel A.. Springer, Berlin, 2002

[8] X.-T. Lü, J. Chen, Z.-X. Wen and W. Wu, On the abelian complexity of the Rudin-Shapiro sequence. J. Math. Anal. Appl. 451 (2017) 822838.

[9] X.-T. Lü, J. Chen and W. Wu, On the \( k \)-abelian complexity of the Cantor sequence. J. Combin. Theory Ser. A 155 (2014) 287302.

[10] Kabor I, Kientga B. Abelian Complexity of Thue-Morse Word over a Ternary Alphabet. InInternational Conference on Combinatorics on Words (pp. 132-143). Springer, Cham (2017 Sep).
[11] J. Karhumaki, A. Saarela and L. Q. Zamboni, *On a generalization of Abelian equivalence and complexity of infinite words*, J. Combin. Theory Ser. A 120(8) (2013) 2189-2206.

[12] B. Madill and N. Rampersad. *The abelian complexity of the paperfolding word*, Discrete Math. 313 (7) (2013) 831-838.

[13] G. Pirillo and S. Varricchio. *On uniformly repetitive semigroups*, Semigroup Forum 49 (1994) 125129.

[14] M. Rigo and P. Salimov, *Another generalization of abelian equivalence: Binomial complexity of infinite words*, Theoret. Comput. Sci. 601 (2015) 47-57.

[15] G. Richomme, K. Saari and L. Q. Zamboni, *Balance and Abelian complexity of the Tribonacci word*, Adv. in Appl. Math. 45(2) (2010) 212-231.

[16] G. Richomme, K. Saari and L. Zamboni, *Abelian complexity of minimal subshifts*, J. Lond. Math. Soc. 83 (1) (2011) 79-95.

[17] N. J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences*, [http://oeis.org](http://oeis.org) Sequence A071858.