A GRAPHICAL CATEGORIFICATION OF THE TWO-VARIABLE CHEBYSHEV POLYNOMIALS OF THE SECOND KIND

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ABSTRACT. We show that the $A_2$ clasps in the Karoubi envelope of $A_2$ spider satisfy the recursive formula of the two-variable Chebyshev polynomials of the second kind associated with a root system of type $A_2$. The $A_2$ spider is a diagrammatic description of the representation category for $U_q(sl_3)$ and the $A_2$ clasps are projectors. Our categorification also gives a natural definition of a $q$-deformation of the two-variable Chebyshev polynomials. This paper is constructed based only on the linear skein theory and graphical calculus.

1. INTRODUCTION

In this paper, we give a categorification of a two-variable Chebyshev polynomial of the second kind inspired by Queffelec and Wedrich [QW18a, QW18b]. They categorified the Chebyshev polynomials and power-sum symmetric polynomials through diagrammatic categories. In [QW18a], it is shown that the Jones-Wenzl projectors satisfy the recursive formula of the Chebyshev polynomials of the second kind (resp. first kind) in the split Grothendieck group of the Karoubi envelope of the Temperley-Lieb category (resp. an affine version of the Temperley-Lieb category).

From representation theoretical point of view, many mathematician and physicist have studied multi-variable generalizations of the Chebyshev polynomials associated with root systems. For examples, Koornwinder [Koo74a, Koo74b, Koo74c, Koo74d], Hoffman and Withers [HW88] for type $A$, and [NPST10, NPT11] in general. We consider the following problem:

Question 1.1. Give diagrammatic categorifications of multi-variable generalizations of the Chebyshev polynomials.

We treat with a two-variable generalization of the Chebyshev polynomials, called the $A_2$ Chebyshev polynomials, of the second kind appearing in [Koo74a, Koo74b, Koo74c, Koo74d] and give a solution of the above question. The $A_2$ Chebyshev polynomials of the second kind is a family of two-variable polynomials $\{S_{k,l}(x,y) \mid k, l \in \mathbb{Z}_{\geq 0}\}$ with integer coefficients. It is defined by the following recursive formulas [Koo74c, ?]:

\[
S_{(k+1,l)}(x, y) = xS_{(k,l)}(x, y) - S_{(k-1,l+1)}(x, y) - S_{(k,l-1)}(x, y) \quad \text{for } k, l \geq 1,
\]

\[
S_{(k,l+1)}(x, y) = yS_{(k,l)}(x, y) - S_{(k+1,l-1)}(x, y) - S_{(k-1,l)}(x, y) \quad \text{for } k, l \geq 1,
\]

$S_{(k,l)}(x, y) = 0$ if $k < 0$ or $l < 0$, and $S_{(0,0)}(x, y) = 1$.

The categorification of $S_{(k,l)}$ is given by using the $A_2$ spider. The $A_2$ spider defined in [Kup96] gives a diagrammatic description of the invariant space $\text{Inv}(V_{e_1} \otimes V_{e_2} \otimes \cdots \otimes V_{e_n})$ of the fundamental irreducible representations $V_+$ and $V_-$ of $U_q(sl_3)$ and their tensor...

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products. An invariant vector is described as a linear combination of directed uni-trivalent planar graphs with source and sink vertices. We call it $A_2$ web. By using $A_2$ webs, we can consider the $A_2$ version of the Temperley-Lieb category, namely a linear category of intertwining operators between tensor powers of $V_+$’s and $V_-$’s, through $\text{Hom}(A, B) \cong \text{Inv}(A^* \otimes B)$. We call it also the $A_2$ spider in this paper, and denote it by $\text{Sp}_q$. Kuperberg also defined the (internal) $A_2$ clasp $P_{+k-l}^\pm$ in $\text{Hom}(V_+^{\otimes k} \otimes V_-^{\otimes l}, V_+^{\otimes k} \otimes V_-^{\otimes l})$ for $k, l \geq 0$ as a generalization of the Joens-Wenzl idempotent. Ohtsuki and Yamada gave a recursive definition of $P_{+k-l}^\pm$ in [OY97]. Then, $P_{+k-l}^\pm$ give the $A_2$ Chebyshev polynomial in the following sense. Let $P_{(k,l)}$ be an object in the Karoubi envelope $\text{Kar}(\text{Sp}_q)$ of $\text{Sp}_q$ corresponding to the idempotent $P_{+k-l}^\pm$.

**Theorem 1.2.** The isomorphism classes of $\{P_{(k,l)}\}_{k,l \geq 0}$ satisfy the recursive formula of $\{S_{(k,l)}\}_{k,l \geq 0}$ in the split Grothendieck group $K_0(\text{Kar}(\text{Sp}))$ where $\text{Sp}$ is the $A_2$ spider at $q = 1$.

In the above, we consider the $C$-linear category $\text{Sp}_q$ at $q = 1$. However, we will show a recursive formula of $P_{(k,l)}$ for the $\mathbb{C}(q^\frac{1}{2})$-linear category $\text{Sp}_q$ and obtain the proof by specializing $q = 1$. This recursive formula of $P_{(k,l)}$ gives a natural $q$-deformation of the $A_2$ Chebyshev polynomial. Many kinds of $q$-deformations of the Chebyshev polynomials have been studied in various contexts, for example, [Dup10], [Cig12a, Cig12b].

This paper is organized as follows. In Section 2, we recall definitions and properties of the $A_2$ spider and the $A_2$ clasps. They are an $A_2$ version of the Temperley-Lieb category and the Jones-Wenzl projectors. We categorify the $A_2$ Chebyshev polynomials in Section 3. All proofs are given by diagrammatic calculations in the $A_2$ spider.

## 2. The $A_2$ Spider and the $A_2$ Clasp

In this section, we introduce an $A_2$ version of the Temperley-Lieb category $\text{Sp}_q$ based on Kuperberg’s $A_2$ spider [Kup96]. Our definition of $\text{Sp}_q$ is only for proof of Theorem 1.2. Evans and Pugh studied the details about the $A_2$ version of the Temperley-Lieb category and the $A_2$ planar algebra in [EP10, EP11]. We also define the $A_2$ clasps which play a role analogous to the Jones-Wenzl projectors and show some fundamental properties. This section is constructed by the linear skein theory and diagrammatic calculations.

### 2.1. The $A_2$ Spider

An $A_2$ web is a linear combination of graphs in a disk $D = [0, 1] \times [0, 1]$ with marked points on $[0, 1] \times \{0, 1\}$ which represents an intertwining operator of $U_q(\mathfrak{sl}_3)$ in manner of Reshetikhin and Turaev [RT91]. See a Turaev’s book [Tur94] and [BK01] for details. We only give a combinatorial definition of the $A_2$ spider which is a linear category constructed from $A_2$ webs. See [Kup94, Kup96] for details on relation to representation theory of $U_q(\mathfrak{sl}_3)$.

Let us recall the $A_2$ web defined by Kuperberg [Kup96]. For any $n \in \mathbb{Z}_{\geq 0}$, we denote sets of marked points on $[0, 1] \times \{0, 1\}$ by $P_n = \{p_1, p_2, \ldots, p_n\}$ and $Q_n = \{q_1, q_2, \ldots, q_n\}$ where $p_i = (i/(n+1), 0)$ and $q_i = (i/(n+1), 1)$ for $1 \leq i \leq n$. If $n = 0$, then $P_0 = Q_0 = \emptyset$. A sign of $P_n$ is a map $\varepsilon_{P_n} : P_n \to \{+, -\}$. The sign $\varepsilon_{P_n}$ is defined by the sequence $\varepsilon_{P_n(1)}\varepsilon_{P_n(2)} \ldots \varepsilon_{P_n(n)}$ of $+$ and $-$. A sign of $Q_n$ is defined in the same way. We consider $D$ with marked points $P_k$ and $Q_l$ with signs $\varepsilon_{P_k}$ and $\varepsilon_{Q_l}$ where $k, l \in \mathbb{Z}_{\geq 0}$. A bipartite uni-trivalent graph $G$ in $D$ is a directed graph embedded into $D$ such that every vertex is either trivalent or univalent and the vertices are divided into sinks or sources as follows:
An $A_2$ basis web is the boundary-fixing isotopy class of a bipartite trivalent graph $G$ in $D$ such that any internal face of $D \setminus G$ has at least six sides. Let $\epsilon_k$ and $\epsilon_l$ be sequences of $+$ and $-$ with length $k$ and $l$, respectively. We denote the set of $A_2$ basis webs in $D$ with signed marked points $\bar{\epsilon}_k = \epsilon_k$ and $\bar{\epsilon}_l = \epsilon_l$ by $B(\epsilon_k, \epsilon_l)$. The $A_2$ web space $W(\epsilon_k, \epsilon_l)$ is the $\mathbb{C}(q^{\frac{1}{2}})$-vector space spanned by $B(\epsilon_k, \epsilon_l)$. An $A_2$ web is an element in the $A_2$ web space.

For example, $B(+-+-,-+--)$ has the following $A_2$ basis webs:

\begin{center}
\includegraphics[width=0.8\textwidth]{A2_basis_webs.png}
\end{center}

\textbf{Definition 2.1} (The $A_2$ spider). The $A_2$ spider $\text{Sp}_{q}$ is a $\mathbb{C}(q^{\frac{1}{2}})$-linear category defined as follows.

- An object of $\text{Sp}_{q}$ is a finite sequence of signs, that is, a map $\epsilon_n: \mathbb{N} \to \{+,-\}$ where $\mathbb{N} = \{1 < 2 < \cdots < n\}$ is a finite totally ordered set for $n \in \mathbb{Z}_{>0}$ where the map $\epsilon_0$ is the map from $\emptyset = \emptyset$. A morphism $\text{Sp}_{q}(\epsilon_k, \epsilon_l) = \text{Hom}_{\text{Sp}_{q}}(\epsilon_k, \epsilon_l)$ is the $A_2$ web space $W(\epsilon_k, \epsilon_l)$ where $\epsilon_k$ is the opposite sign of $\epsilon_k$. We remark that $\text{Sp}_{q}(\epsilon_0, \epsilon_0)$ is the 1-dimensional vector space spanned by the empty web $D_0$.
- For $A_2$ basis webs $F \in \text{Sp}_{q}(\epsilon_k, \epsilon_l)$ and $G \in \text{Sp}_{q}(\epsilon_l, \epsilon_m)$, the composition $G \circ F = GF \in \text{Sp}_{q}(\epsilon_k, \epsilon_m)$ is defined by gluing top side of the disk of $F$ and bottom side of the disk of $G$. If $GF$ makes 4-, 2-, and 0-gons, we reduce them by
  - $[4] = [\begin{array}{c}
  \includegraphics[width=0.2\textwidth]{4gon.png}
  \end{array}]$,
  - $[2] = [\begin{array}{c}
  \includegraphics[width=0.2\textwidth]{2gon.png}
  \end{array}]$,
  - $[3] = [\begin{array}{c}
  \includegraphics[width=0.2\textwidth]{3gon.png}
  \end{array}]$,

where $[n] = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})/(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$.
- The identity morphism $1_{\epsilon_k}$ in $\text{Sp}_{q}(\epsilon_k, \epsilon_k)$ is the $k$ parallel edges with no trivalent vertex.
- A tensor product of two objects is defined by their concatenation, that is, $\epsilon_k \otimes \epsilon_{k'}$ is a map $k + k' \to \{+,-\}$ such that $\epsilon_k \otimes \epsilon_{k'}(i) = \epsilon_k(i)$ if $0 \leq i \leq k$ and $\epsilon_k \otimes \epsilon_{k'}(i) = \epsilon_{k'}(i-k)$ if $k+1 \leq i \leq k+k'$. The tensor product of two $A_2$ basis webs $F \in \text{Sp}_{q}(\epsilon_k, \epsilon_l)$ and $G \in \text{Sp}_{q}(\epsilon_l, \epsilon_m)$ is defined by gluing right side of the disk of $F$ and left side of the disk of $G$.

We remark that an $A_2$ web $f \in \text{Sp}_{q}(\epsilon_k, \epsilon_l)$ represents an intertwining operator in $\text{Hom}(V_k, V_l)$ where $V_k = V_{\epsilon_k(1)} \otimes V_{\epsilon_k(2)} \otimes \cdots \otimes V_{\epsilon_k(k)}$ and $V_l = V_{\epsilon_l(1)} \otimes V_{\epsilon_l(2)} \otimes \cdots \otimes V_{\epsilon_l(l)}$. As an invariant vector $f$ is an element in $\text{Inv}(V_k \otimes V_l)$ and $V_k^* \cong V_{\epsilon_k(k)} \otimes \cdots \otimes V_{\epsilon_k(2)} \otimes V_{\epsilon_k(1)}$ because $V_k^* \cong V_{\bar{\epsilon}_k}$. See [Kup96] in detail. The signs of marked points of the $A_2$ web space are labeled according to the signs of the invariant space.
2.2. $A_2$ clasps. We give a diagrammatic definition of an $A_2$ version of the Jones-Wenzl projectors, called the $A_2$ clasps, introduced in [Kup96, OY97]. The purpose of this section is to construct the general form of the $A_2$ clasp and to prove its properties by only using the linear skein theory.

In what follows, we omit “$\otimes$” to describe the tensor product of objects in $\text{Sp}_q$ and + (resp. −) means the constant map from 1 to + (resp. −). Thus, $+^k$ (resp. $-^k$) is the constant map from k to + (resp. −) for any positive integer k.

**Definition 2.2** (The $A_2$ clasp in $\text{Sp}_q( +^k, +^k)$).

(1) $\begin{array}{c} b \hline a \end{array} = \begin{array}{c} 1 \end{array} \in \text{Sp}_q( +, +)$

(2) $\begin{array}{c} k \hline 1 \end{array} = \begin{array}{c} k-1 \hline \frac{k-1}{k} \end{array} \in \text{Sp}_q( +^k, +^k)$

The above recursive formula defines the $A_2$ clasp in $\text{Sp}_q( +^k, +^k)$ and we denote it by $P_{+^k}^k$. The $A_2$ clasp in $\text{Sp}_q(-^k, -^k)$ is also defined by the same way and we denote it by $P_{-^k}^k$.

We introduce the following $A_2$ webs:

(2.1)

$$t_{+^+} = \Bigcup t_{+^+} = \Bigcup t_{+^+} = \bigcup t_{+^+} = \bigcup,$$

$$b_{-^+} = \bigcup b_{-^+} = \bigcup d_{++} = \bigcup d_{--} = \bigcup$$

An $A_2$ basis web $B(\epsilon_k, \epsilon_k)$ provides a tiling of $D$. The following Lemma is easily shown by calculating the Euler number of the tiling.

**Lemma 2.3** (Ohtsuki and Yamada [OY97, Lemma 3.3]). For any $A_2$ basis web $D$ in $\text{Sp}_q(\epsilon_k, \epsilon_k)$ other than $1_b$, has $t_{+^+}^{+^+}, t_{-^+}^{-}, b_{-^+}^+, b_{-^+}^-$ in the top side of D and $t_{+^+}^{+^+}, t_{-^+}^{-}, b_{-^+}^+, b_{-^+}^-$ in the bottom side of $D$.

**Proposition 2.4** (Kuperberg [Kup96], Ohtsuki and Yamada [OY97]). Let $k$ be a non-negative integer, then

1. $P_{+^k}^{+^k}(1_b + a \otimes P_{+^t}^t \otimes 1_b) = P_{+^k}^{+^k} = (1_b + a \otimes P_{+^t}^t \otimes 1_b)P_{+^k}^{+^k} for a + b = k$,

2. $P_{+^k}^{+^k}(1_b \otimes t_{+^+}^+ \otimes 1_b) = 0 = (1_b \otimes t_{+^+}^+ \otimes 1_b)P_{+^k}^{+^k} for a + b + 2 = k$.

The above is true for the opposite sign.

**Proposition 2.5** (Uniqueness). If a non-trivial element $T \in \text{Sp}_q(\epsilon^k, \epsilon^k)$ satisfies $T^2 = T$ and Proposition 2.4(2), then $T = P_{+^k}^{+^k}$ for $\epsilon \in \{+, -, \}$.

**Proof.** $P_{+^k}^{+^k}$ can be expanded as a linear sum of $A_2$ basis webs $P_{+^k}^{+^k} = c1_b + x$ by Lemma 2.3, where $c$ is a constant and $x$ have $t_{+^+}^{+^+}$ and $t_{-^+}^{-}$ in the top side and in the bottom side. We can see that $c = 1$ from $(P_{+^k}^{+^k})^2 = P_{+^k}^{+^k}$ and Proposition 2.4(2). In the same way, we know $T = 1_b + x'$. Therefore,

$$P_{+^k}^{+^k}T = (1_b + x)T = \bigcup$$

$$P_{+^k}^{+^k}T = P_{+^k}^{+^k}(1_b + x) = P_{+^k}^{+^k}$$

We can prove it for $\epsilon = -$ in the same way. \qed
For an $A_2$ basis web $w \in B(\epsilon_k, \epsilon_l)$, we define $w^* \in B(\epsilon_l, \epsilon_k)$ as the reflection of $w$ through the horizontal line $[0, 1] \times \{1/2\}$ with the opposite direction. For the coefficient $\mathbb{C}(q^{\pm})$, the star operator acts on $\mathbb{C}$ by complex conjugate and $(q^{\pm})^* = q^{-\pm}$. In this way, we define a linear map $^* : \text{Sp}_q(\epsilon_k, \epsilon_l) \to \text{Sp}_q(\epsilon_l, \epsilon_k)$.

**Remark 2.6.** Proposition 2.5 implies $(P_{+k}^{+k})^* = P_{+k}$ and $(P_{-k}^{-k})^* = P_{-k}$.

We introduce the $A_2$ clasp $P_{+k-i}^{+k-i}$ in $\text{Sp}_q(+k-l, +k-l)$ based on [OY97].

**Definition 2.7** (The $A_2$ clasp in $\text{Sp}_q(+k-l, +k-l)$),

$P_{+k-i}^{+k-i} = \frac{k}{i} \left( \begin{array}{c} l \\ k \end{array} \right) = \sum_{i=0}^{\min(k,l)} (-1)^i \left( \begin{array}{c} k \\ i \end{array} \right) \left( \begin{array}{c} l \\ k+i+1 \end{array} \right) \left( \begin{array}{c} k \\ k-i \end{array} \right).$

$P_{-k+i}^{+k-i}$ is also defined by the same way.

One can prove a similar statement to Proposition 2.4.

**Proposition 2.8** (Kuperberg [Kup96], Ohtsuki and Yamada [OY97]). Let $k$ and $l$ be non-negative integers, then

1. $P_{+k-i}^{+k-i}(1_+^a \otimes P_{+k-i}^{+k-i} \otimes 1_-) = P_{+k-i}^{+k-i} = (1_+^a \otimes P_{+k-i}^{+k-i} \otimes 1_-)P_{+k-i}^{+k-i}$ for $a + s = k$ and $b + t = l$.
2. $P_{+k-i}^{+k-i}(1_+^a \otimes b_+^t \otimes 1_-) = 0 = (1_+^a \otimes d_+^s \otimes 1_-)P_{+k-i}^{+k-i}$ for $a + 1 = k$ and $b + 1 = l$.

The same equalities hold for $P_{-k+i}^{+k-i}$.

One can prove the uniqueness of $P_{+k-i}^{+k-i}$ in a similar way to Proposition 2.5.

**Proposition 2.9** (Uniqueness). A non-trivial idempotent element in $\text{Sp}_q(+k-l, +k-l)$ satisfying Proposition 2.8 (2) is uniquely determined.

**Proof.** In the same way as the proof of Proposition 2.5. \qed

**Remark 2.10.** Proposition 2.9 implies $(P_{+k-i}^{+k-i})^* = P_{+k-i}^{+k-i}$ and $(P_{-k+i}^{+k-i})^* = P_{-k+i}^{+k-i}$.

We give an explicit definition of a general form of the $A_2$ clasp appear in Kuperberg [Kup96] and Kim [Kim07]. This $A_2$ clasp is an $A_2$ web, no longer idempotent, in $\text{Sp}_q(\epsilon, \delta)$ such that $k = \# \epsilon^1(+) = \# \delta^1(+) + l = \# \epsilon^1(-) = \# \delta^1(-)$. We introduce the following $A_2$ basis webs:

(2.2) \[ H_{+}^{+} = \bullet \quad H_{-}^{-} = \bullet \]

Let $k$ and $l$ be non-negative integers. We take an arbitrary object $\epsilon : k + l \to \{ \pm \}$ in $\text{Sp}_q$ satisfying $k = \# \epsilon^1(+) + l = \# \epsilon^1(-)$. Then, we define an $A_2$ basis web $\sigma_{+k-i}^{\epsilon}$ as follows. We consider the disk $D = [0, 1] \times [0, 1]$ with marked points signed by $(+k, l)$ and $\epsilon$. Join the marked points labeled by $+$ in the bottom side with ones of the upper side by straight arcs. In the same way, join the mark points labeled by $-$ by straight arcs. Then, one can obtain the $A_2$ basis web $\sigma_{+k-i}^{\epsilon}$ by replacing all crossing points by $H_{+}^{+}$.
Definition 2.11. Let $k$ and $l$ be non-negative integers. Then, an $A_2$ clasp $P_{+k,-l}^\epsilon$ in $\text{Sp}_q(\epsilon, k-l, \epsilon)$ is defined by

$$P_{+k,-l}^\epsilon = \sigma_{+k,-l}^\epsilon P_{+k,-l}^{+k,-l}.$$ 

Proposition 2.12. Compositions of $P_{+k,-l}^\epsilon$ with $A_2$ basis webs $t$ and $d$ vanish.

Proof. We prove it by induction on the number $h(\sigma)$ of $H_{-\pm}^{-\pm}$ contained in $\sigma = \sigma_{+k,-l}^\epsilon$. If $h(\sigma) = 0$, it is clear since $P_{+k,-l}^\epsilon = P_{+k,-l}^{+k,-l}$ and Proposition 2.8. If $h(\sigma) = 1$, then

$$\sigma = 1_{+k-1} \otimes H_{-\pm}^{-\pm} 1_{-l-1}.$$ 

One can prove by easy calculations. When $h(\sigma) = n + 1$ $(n \geq 1)$, $\sigma$ is described as a composition of $\sigma'$ with $1_n \otimes H_{-\pm}^{-\pm} 1_\beta$ where $\epsilon = \alpha-\beta$ and $\sigma' = \sigma_{+k,-l}^{\alpha-\beta}$ such that $h(\sigma') = n$. Thus, $\sigma$ is

$$\begin{array}{c|c|c}
\sigma' & \sigma'' & \sigma''''
\end{array}$$

in the first case, one can show by easy calculations and the induction hypothesis. In the second case, we only have to prove $\begin{array}{c|c|c}
\sigma' & \sigma'' & \sigma''''
\end{array} \circ P_{+k,-l}^{+k,-l} = 0$. By construction of $\sigma$, the right leg of $H_{-\pm}^{-\pm}$ and the up-pointing arc on one’s right should have a crossing $(H_{-\pm}^{-\pm})$. Then, there exists $\sigma''$ such that $h(\sigma'') = n - 1$ and

$$\begin{array}{c|c|c}
\sigma'' & \sigma''' & \sigma''''
\end{array} \circ P_{+k,-l}^{+k,-l} = 0,$$

by the induction hypothesis for $\sigma'' \circ P_{+k,-l}^{+k,-l}$. For other cases, we can prove in the same way. 

Proposition 2.13. Let us decompose $\epsilon$ into three subsequences $\alpha\beta\gamma$ such that $\beta$ is expressed as the form $+++\cdots+\cdots+\cdots$ or $-\cdots-\cdots-\cdots$. Then,

1. $P_{+k,-l}^\epsilon = (1_\alpha \otimes P_{+k,-l}^{\beta} \otimes 1_\gamma) P_{+k,-l}^\epsilon$.
2. $(P_{+k,-l}^\epsilon)^\star P_{+k,-l}^\epsilon = P_{+k,-l}^{+k,-l}$.

Especially, $tP_{+k,-l}^\epsilon = 0$ and $dP_{+k,-l}^\epsilon = 0$ where $t$ (resp. $d$) is a tensor product of identity morphisms and $t^\pm_+$ or $t^\pm_-$ (resp. $d^\pm_-$ or $d^\pm_+$).

Proof. (1) is easily shown by expanding $P_{+k,-l}^{\beta}$. We can describe $P_{+k,-l}^\epsilon$ as a product $\tau P_{+k,-l}^\epsilon$ where $\tau$ is a tensor product of identity morphisms and only one $H_{-\pm}^{-\pm}$ or $H_{-\pm}^{-\pm}$. Then, $(P_{+k,-l}^\epsilon)^\star P_{+k,-l}^\epsilon = (P_{+k,-l}^\epsilon)^\star \tau^\star P_{+k,-l}^\epsilon$, and one can finish the proof of (2) by applying the defining relation of the $A_2$ web to $\tau^\star \tau$.

Let us define an $A_2$ clasp $P_{+k,-l}^\epsilon$ in $\text{Sp}_q(\epsilon, k-l, \epsilon)$ by $P_{+k,-l}^\epsilon (P_{-k,-l}^\epsilon)^\star$. Then,

Proposition 2.14.

1. $(P_{+k,-l}^\epsilon)^2 = P_{+k,-l}^\epsilon$.
2. $tP_{+k,-l}^\epsilon$, $dP_{+k,-l}^\epsilon$, $P_{+k,-l}^\epsilon b$, and $P_{+k,-l}^\epsilon t'$ vanish.

In the above, $t'$ (resp. $b$) is a tensor product of identity morphisms and $t^+_+$ or $t^-_-$ (resp. $b^-_+$ or $b^+_+.$)
Proof. 

\[(P_\epsilon')^2 = P_{+k-l}(P_{+k-l}^*P_{+k-l}(P_{+k-l}^*)^*
= P_{+k-l}P_{+k-l}^*(P_{+k-l}^*)^*
= P_{+k-l}(P_{+k-l}^*)^* = P_\epsilon'
\]

\[\square\]

**Proposition 2.15** (Uniqueness). If \(T\) in \(Sp_q(\epsilon, \epsilon)\) satisfies the conditions (1) and (2) of Proposition 2.14, then \(T = P_\epsilon\).

**Proof.** In the same way as the proof of Proposition 2.5. \(\square\)

**Remark 2.16.** Proposition 2.15 implies \((P_\epsilon')^* = P_\epsilon\).

For any non-negative integers \(k\) and \(l\), we consider a subset \(Sp_q^{(k,l)} = \{ \epsilon \in Sp_q \mid k = \#\epsilon^{-1}(+), l = \#\epsilon^{-1}(-) \}\) of \(Sp_q\).

**Proposition 2.17.** There exist a set of \(A_2\) webs \(\{ P_\alpha^\beta \in Sp_q(\alpha, \beta) \mid \alpha, \beta \in Sp_q^{(k,l)} \}\) satisfying

1. \(P_\alpha^\beta P_\alpha^\gamma = P_\alpha^\gamma P_\alpha^\beta\) for any \(\alpha, \beta, \gamma \in Sp_q^{(k,l)}\),
2. \(dP_\alpha^\beta, dP_\alpha^\beta b, \) and \(P_\alpha^\beta b\) vanish for any \(\alpha, \beta \in Sp_q^{(k,l)}\).

**Proof.** Let us define \(P_\alpha^\beta = P_{+k-l}(P_{+k-l}^*)^*\) for any \(\alpha, \beta \in Sp_q^{(k,l)}\). Then, it is obvious that \(P_\alpha^\beta\) satisfies (2) because of Proposition 2.13(1). We show the equation of (1):

\[P_\alpha^\beta P_\alpha^\gamma = P_{+k-l}(P_{+k-l}^*)^* P_{+k-l}^*(P_{+k-l}^*)^*\]

\[= P_{+k-l}P_{+k-l}^*(P_{+k-l}^*)^* \quad \text{by Proposition 2.13(2)}
\]

\[= P_{+k-l}(P_{+k-l}^*)^* = P_\alpha^\beta.
\]

\[\square\]

2.3. Braidings in \(Sp_q\). We introduce a braiding \(\{ c_{\delta, \epsilon} : \delta \otimes \epsilon \rightarrow \epsilon \otimes \delta \}\) which is a family of isomorphisms in \(Sp_q\). A definition of the braiding in the diagrammatic category \(Sp_q\) is given by Kuperberg [Kup94] [Kup96]. In detail about a general theory of braidings in monoidal categories, for example, see [Tur91].

Let us define a description of an \(A_2\) web by using a crossing with over/under information introduced in Kuperberg [Kup96]:

\[c_{+,+} = \begin{array}{c}\bigotimes
\end{array} \bigotimes = q^{\frac{1}{2}} - q^{-\frac{1}{2}},
\]

\[c_{-,+} = \begin{array}{c}\bigotimes
\end{array} \bigotimes = q^{\frac{1}{2}} - q^{-\frac{1}{2}}.
\]

We also describe their inverses as

\[c_{+,+}^{-1} = \begin{array}{c}\bigotimes
\end{array} \bigotimes = c_{+,+},
\]

\[c_{-,+}^{-1} = \begin{array}{c}\bigotimes
\end{array} \bigotimes = c_{-,+}.
\]
By the above description, we can consider $A_2$ webs with over/under crossings. These $A_2$ webs satisfies the Reidemeister moves (R1)–(R4) for framed tangled trivalent graphs, that is, we can confirm the following local moves of $A_2$ webs:

\[(\text{R1}) \quad \begin{array}{c}
\includegraphics[width=1cm]{web1.png} \\
\end{array} \quad \begin{array}{c}
\includegraphics[width=1cm]{web2.png} \\
\end{array}\]

\[(\text{R2}) \quad \begin{array}{c}
\includegraphics[width=1cm]{web3.png} \\
\end{array} \quad \begin{array}{c}
\includegraphics[width=1cm]{web4.png} \\
\end{array}\]

\[(\text{R3}) \quad \begin{array}{c}
\includegraphics[width=1cm]{web5.png} \\
\end{array} \quad \begin{array}{c}
\includegraphics[width=1cm]{web6.png} \\
\end{array}\]

\[(\text{R4}) \quad \begin{array}{c}
\includegraphics[width=1cm]{web7.png} \\
\end{array} \quad \begin{array}{c}
\includegraphics[width=1cm]{web8.png} \\
\end{array}\]

For any objects $\delta$ and $\epsilon$ in $\text{Sp}_q$, we define

$$c_{\delta,\epsilon} = \frac{\epsilon}{\delta} \in \text{Sp}_q(\delta \epsilon, \epsilon \delta), \quad c_{\delta,\epsilon}^{-1} = \frac{\epsilon}{\delta} \in \text{Sp}_q(\epsilon \delta, \delta \epsilon),$$

where an edge labeled by $\delta$ (resp. $\epsilon$) mean an embedding of $1_\delta$ (resp. $1_\epsilon$) along it with the same over/under information at every crossings. The invariance under the Reidemeister moves (R1)–(R4) provides the invariance under (R1)–(R3) for any labeled edges. By the same reason, we can slide $A_2$ claps across over/under crossings, namely,

$$c_{\delta,\epsilon}^{-1}(1_\delta \otimes P^\epsilon_{+}) = (P^\epsilon_{+} \otimes 1_\delta)c_{\delta,\epsilon}, \quad c_{\delta,\epsilon}(P^\delta_{-} \otimes 1_\epsilon) = (1_\epsilon \otimes P^\delta_{-})c_{\delta,\epsilon},$$

$$c_{\delta,\epsilon}^{-1}(1_\delta \otimes P^\delta_{-}) = (P^\delta_{-} \otimes 1_\epsilon)c_{\delta,\epsilon}^{-1}, \quad c_{\delta,\epsilon}(P^\epsilon_{+} \otimes 1_\delta) = (1_\delta \otimes P^\epsilon_{+})c_{\delta,\epsilon}^{-1}.$$  

### 3. A Categorification of $S_{(k,l)}(x,y)$

Let us briefly recall the definition of the Karoubi envelope and the split Grothendieck group.

**Definition 3.1** (The Karoubi envelope). The *Karoubi envelope* of a category $\mathcal{A}$, denoted by $\text{Kar}(\mathcal{A})$, is defined as follows:

- Objects in $\text{Kar}(\mathcal{A})$ is pairs $(X, f)$ of objects $X$ in $\mathcal{A}$ and idempotents $f \in \text{Hom}_{\mathcal{A}}(X, X)$.
- Morphisms in $\text{Hom}_{\text{Kar}(\mathcal{A})}((X, f), (Y, g))$ is morphisms $\phi \in \text{Hom}_{\mathcal{A}}(X, Y)$ satisfying $g \circ \phi \circ f = \phi$ for any objects $(X, f)$ and $(Y, g)$.

We remark that the identity in $\text{Hom}_{\text{Kar}(\mathcal{A})}((X, f), (X, f))$ is given by $f$. If $\mathcal{A}$ is monoidal, then the Karoubi envelope $\text{Kar}(\mathcal{A})$ inherits a tensor product with $(X, f) \otimes (Y, g) = (X \otimes Y, f \otimes g)$.

**Definition 3.2** (The split Grothendieck group). The *split Grothendieck group* of an additive category $\mathcal{A}$ with $\oplus$ is the abelian group $K_0(\mathcal{A})$ generated by isomorphism classes $\langle X \rangle$ of objects $X$ in $\mathcal{A}$ modulo the relations $\langle X_1 \oplus X_2 \rangle = \langle X_1 \rangle + \langle X_2 \rangle$. If $\mathcal{A}$ is monoidal, that is $\mathcal{A}$ equipped with a tensor $\otimes$ and the identity object $e$, then $K_0(\mathcal{A})$ inherits a ring structure with the unit $\langle e \rangle$ and the multiplication $\langle X \otimes Y \rangle = \langle X \rangle \cdot \langle Y \rangle$.

In this section, we consider the pairs $\{(\epsilon, P^\epsilon_{+})\}$ in the Karoubi envelope $\text{Kar}(\text{Sp}_q)$ of the $A_2$ spider and its split Grothendieck group. Firstly, it is easy to see that we can take a standard representatives for the isomorphism class of $\{(\epsilon, P^\epsilon_{+})\}$.

**Lemma 3.3.** For any $\epsilon \in \text{Sp}_q^{(k,l)}$, $(\epsilon, P^\epsilon_{+}) \cong (+^k_{-}, P^{k+l}^{k+l}_{+k+l})$ in $\text{Kar}(\text{Sp}_q)$.

**Proof.** $P^{k+l}_{+k+l}$ and $P^{k+l}_{+k+l}$ give an isomorphism between $(\epsilon, P^\epsilon_{+})$ and $(+^k_{-}, P^{k+l}^{k+l}_{+k+l})$. In fact, Proposition 2.17 shows $P^{k+l}_{+k+l}P^{k+l}_{+k+l} = P^{k+l}_{+k+l}$ and $P^{k+l}_{+k+l}P^{k+l}_{+k+l} = P^{k+l}_{+k+l}$.
Thus, $P_{e + k - l}^+ - P_{e + k - l}^-$ are morphisms in $\text{Kar}(\mathcal{SP}_q)$. By the same reason, $P_{e + k - l}^+ P_{e + k - l}^+ = P_e^+ - P_{e + k - l}^-$ and $P_{e + k - l}^- P_{e + k - l}^+ = P_{e + k - l}^-$. □

The multiplication in $K_0(\text{Kar}(\mathcal{SP}_q))$ is commutative because of a property 2.3 of a braiding.

**Lemma 3.4.** For any objects $\epsilon$ and $\delta$ in $\mathcal{SP}_q$, $P_\epsilon^+ \otimes P_\delta^+ \cong P_\delta^+ \otimes P_\epsilon^+$ by $e_{\epsilon, \delta}$.

Let us denote objects $(+ k - l, P_{+ k - l})$ in $\text{Kar}(\mathcal{SP}_q)$ by $P_{(k, l)}$ for $k, l \geq 1$. We denote the pair $(e_0, D_0)$ of the empty sign and the empty web by $P_{(0, 0)}$. The split Grothendieck group $K_0(\text{Kar}(\mathcal{SP}_q))$ is a ring with with the unit $1 = \langle P_{(0, 0)} \rangle$. We denote $\langle (+, 1) \rangle$ and $\langle (-, 1) \rangle$ by $X$ and $Y$ in $K_0(\text{Kar}(\mathcal{SP}_q))$, respectively.

We will show the isomorphism classes $\langle P_{(k, l)} \rangle$ satisfy the recursive formula of the $A_2$ Chebyshev polynomials in $K_0(\text{Kar}(\mathcal{SP}_q))$:

\begin{align*}
(3.1) & \quad \langle P_{(1, 1)} \rangle = XY - 1, \\
(3.2) & \quad \langle P_{(k+1, 0)} \rangle = X\langle P_{(k, 0)} \rangle - \langle P_{(k-1, 1)} \rangle \quad \text{for } k \geq 1, \\
(3.3) & \quad \langle P_{(k+1, l)} \rangle = X\langle P_{(k, l)} \rangle - \langle P_{(k-1, l+1)} \rangle - \langle P_{(k, l-1)} \rangle \quad \text{for } k, l \geq 1, \\
(3.4) & \quad \langle P_{(0, l+1)} \rangle = Y\langle P_{(0, l)} \rangle - \langle P_{(1, l-1)} \rangle \quad \text{for } l \geq 1, \\
(3.5) & \quad \langle P_{(k, l+1)} \rangle = Y\langle P_{(k, l)} \rangle - \langle P_{(k+1, l-1)} \rangle - \langle P_{(k-1, l-1)} \rangle \quad \text{for } k, l \geq 1.
\end{align*}

We only have to prove $(3.1) - (3.14)$ because of $P_{+ k - l}^+ \cong P_{- k + l}^-$. We use the following well-known fact about an additive category, see [ML98, KS06], for example.

**Lemma 3.5.** Let $\mathcal{A}$ be an additive category and $X_1, X_2, Y$ objects in $\mathcal{A}$. $Y \cong X_1 \oplus X_2$ if and only if there exists morphisms $p_1 : Y \to X_1$, $p_2 : Y \to X_2$, $\iota_1 : X_1 \to Y$, and $\iota_2 : X_2 \to Y$ satisfying the following conditions:

\[
\iota_1 \circ p_1 + \iota_2 \circ p_2 = 1_Y, \quad p_1 \circ \iota_1 = 0, \quad p_1 \circ \iota_1 = 1_{X_1}, \quad p_2 \circ \iota_2 = 1_{X_2}.
\]

**Proposition 3.6.** $\langle P_{(1, 1)} \rangle = XY - 1$

**Proof.** By Definition 2.7

\[
P_{+ -}^+ + \frac{1}{3} b^+ d^- = 1_+ \otimes 1_-.
\]

Let us prove $(++, P_{+ -}^+ + \frac{1}{3} b^+ d^-) \cong P_{(1, 1)} \oplus P_{(0, 0)}$. It is only necessary to construct projections $p_1 : (+, P_{+ -}^+ + \frac{1}{3} b^+ d^-) \to P_{(1, 1)}$ and $p_2 : (+, P_{+ -}^+ + \frac{1}{3} b^+ d^-) \to P_{(0, 0)}$, and inclusions $\iota_1 : P_{(1, 1)} \to (+, P_{+ -}^+ + \frac{1}{3} b^+ d^-)$ and $\iota_2 : P_{(0, 0)} \to (+, P_{+ -}^+ + \frac{1}{3} b^+ d^-)$ satisfying the conditions in Lemma 3.5. These projections are given by $p_1 = P_{+ -}^+$ and $p_2 = \frac{1}{3} d^-$ because

\[
P_{+ -}^+(P_{+ -}^+ + \frac{1}{3} b^+ d^-) = (P_{+ -}^+)^2 + \frac{1}{3} (P_{+ -}^+ b^+ d^-) = P_{+ -}^+
\]

by Proposition 2.8 and

\[
\frac{1}{3} d^- (P_{+ -}^+ + \frac{1}{3} b^+ d^-) = \frac{1}{3} d^- P_{+ -}^+ + \frac{1}{3} (d^- b^+ d^-) = \frac{1}{3} d^-
\]
by Proposition 2.8 and $d_+ b^- = \emptyset = [3]$. These inclusions are given by $\iota_1 = P_{++}^+$ and $\iota_2 = b^+$ because

$$(P_{++}^+ + \frac{1}{[3]} b^+ d_{+-}) P_{++}^+ = (P_{++}^+)^2 + \frac{1}{[3]} (b^+) d_{+-} P_{++}^+ = P_{++}^+$$

and

$$(P_{++}^+ + \frac{1}{[3]} b^+ d_{+-}) b^- = P_{++}^+ b^- + \frac{1}{[3]} b^+ d_{+-} b^- = b^+. $$

In fact, it is easily to see that

$$\begin{align*}
\iota_1 \circ p_2 &= \iota_2 \circ p_1 = 0, \\
\iota_1 \circ p_1 &= P_{++}^+, \\
\iota_2 \circ p_2 &= \frac{1}{[3]} b^+ d_{+-}, \\
p_1 \circ \iota_2 &= p_2 \circ \iota_1 = 0, \\
p_1 \circ \iota_1 &= P_{++}^+, \\
p_2 \circ \iota_2 &= D_0.
\end{align*}$$

Thus, we obtain $P_{(1,1)} \oplus P_{(0,0)} \cong (+, P_{++}^+ + \frac{1}{[3]} b^+ d_{+-})$. In $K_0(\text{Kar}(\text{Sp}_q))$, it is interpreted as $\langle P_{(1,1)} \rangle + 1 = \langle (+, P_{++}^+ + \frac{1}{[3]} b^+ d_{+-}) \rangle = XY$.

\begin{proof}
For any positive integer $k$, $\langle P_{(k+1,0)} \rangle = X \langle P_{(k,0)} \rangle - \langle P_{(k-1,1)} \rangle$.
\end{proof}

\begin{proposition}
For any positive integer $k$, $\langle P_{(k+1,0)} \rangle = X \langle P_{(k,0)} \rangle - \langle P_{(k-1,1)} \rangle$.
\end{proposition}

\begin{proof}
By Definition 2.2, we have

$$P_{++}^{k+1} + \frac{[k]}{[k+1]} (P_{++}^k \otimes 1_+)(1_{+k-1} \otimes t_{++} t_{+-}) (P_{++}^k \otimes 1_+) = P_{++}^k \otimes 1_+$$

Let us denote the LHS of (3.6) by $f$. We show that the pair $(+^{k+1}, f)$ is isomorphic to the direct sum $P_{(k+1,0)} \oplus P_{(k-1,1)}$ in $\text{Kar}(\text{Sp}_q)$. We only have to construct projections $p_1: (+^{k+1}, f) \to P_{(k+1,0)}$ and $p_2: (+^{k+1}, f) \to P_{(k-1,1)}$, and inclusions $\iota_1: P_{(k+1,0)} \to (+^{k+1}, f)$ and $\iota_2: P_{(k-1,1)} \to (+^{k+1}, f)$ satisfying (3.5). We confirm that

$$p_1 = P_{++}^{k+1}\quad \text{and} \quad p_2 = \frac{[k]}{[k+1]} P_{++}^{k-1} (1_{+k-1} \otimes t_{+-}) (P_{++}^k \otimes 1_+),$$

provide projections and

$$\iota_1 = P_{++}^{k+1}\quad \text{and} \quad \iota_2 = (P_{++}^k \otimes 1_+) (1_{+k-1} \otimes t_{++}) P_{++}^{k-1}$$

inclusions. It is easy to see that $p_1$ and $\iota_1$ is a morphism between $(+^{k+1}, f)$ and $P_{(k+1,0)}$. We show that $p_2$ is a morphism in $\text{Kar}(\text{Sp}_q)$, that is,

$$P_{++}^{k-1} P_{++}^{k-1} (1_{+k-1} \otimes t_{++}) (P_{++}^k \otimes 1_+) f = P_{++}^{k-1} (1_{+k-1} \otimes t_{++}) (P_{++}^k \otimes 1_+).$$

By Proposition 2.8, the LHS of the above equation is

$$\frac{[k]}{[k+1]} P_{++}^{k-1} (1_{+k-1} \otimes t_{++}) (P_{++}^k \otimes 1_+) (1_{+k-1} \otimes t_{++}) (P_{++}^k \otimes 1_+).$$
We diagrammatically calculate it using Definition 2.2.

\[
\begin{align*}
\frac{[k]}{[k+1]} & \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
k-1
\end{array}
\end{array}
\end{array}
\end{array}\hspace{1cm} = \hspace{1cm} \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
k-1
\end{array}
\end{array}
\end{array}
\end{array} - \frac{[k]}{[k+1]} \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
k-1
\end{array}
\end{array}
\end{array} = \frac{[k]}{[k+1]} \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
k-1
\end{array}
\end{array}
\end{array} - \frac{[k]}{[k+1]} \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
k-1
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
\]

The third equation uses \([2][k] = [k + 1] + [k - 1]\). To prove \(\iota_2\) is a morphism in \(\text{Sp}_q\), we have to compute \(f(P^+_{+k} \otimes 1_+)(1_{+k-1} \otimes t^{++})P^+_{+k-1} P^+_{+k-1} \). By Proposition 2.8, it reduce to

\[
\frac{[k]}{[k+1]} (P^+_{+k} \otimes 1_+)(1_{+k-1} \otimes t^{++} P^+_{+k-1} P^+_{+k-1}).
\]

We can calculate it by turning the diagrams in (3.7) upside down. Moreover, it can be confirmed that \(\iota_2 \circ p_2 = \frac{[k]}{[k+1]} (P^+_{+k} \otimes 1_+)(1_{+k-1} \otimes t^{++} P^+_{+k-1} P^+_{+k-1}) = P^+_{+k} \otimes 1_+\) by Definition 2.7 and Proposition 2.8 and \(p_2 \circ \iota_2 = P^+_{+k-1} \) by inserting \(P^+_{+k-1} \) into the diagrams of (3.7). Therefore, these projections and inclusions satisfy the condition in Lemma 3.5 and give \((+^{k+1}, f) \equiv P_{(k+1,0)} \oplus P_{(k-1,1)}\) in \(\text{Kar}(\text{Sp}_q)\). In terms of \(K_0(\text{Kar}(\text{Sp}_q))\),

\[
\langle P_{(k,0)} \rangle X = \langle (+^{k+1}, f) \rangle = \langle P_{(k+1,0)} \rangle + \langle P_{(k-1,1)} \rangle.
\]

Let us prepare some lemmata to prove (3.13). We introduce an \(A_2\) basis web represented by a rectangle with a diagonal line which consist of \(H_{+}^{++}\) and \(H_{-}^{-}\).

**Definition 3.8.** Let \(m\) and \(n\) be positive integers.

1. \begin{align*}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
k-1
\end{array}
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
k-1
\end{array}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
k-1
\end{array}
\end{array}
\end{array} \in \text{Sp}_q(\cdot^{+n}, -\cdot^{n})
\end{align*}
\end{equation}

2. \begin{align*}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
k-1
\end{array}
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
k-1
\end{array}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
k-1
\end{array}
\end{array}
\end{array} \circ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
k-1
\end{array}
\end{array}
\end{array} \in \text{Sp}_q(\cdot^{+m}, -\cdot^{m+n})
\end{align*}
\end{equation}

**Lemma 3.9** (Kim [Kim07, Proposition 3.1]). For any positive integer \(k\),

\[
\frac{1}{[k]} \sum_{j=0}^{k-1} (-1)^j \frac{[k-j]}{[k]} \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
k-1
\end{array}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
k-1
\end{array}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
k-1
\end{array}
\end{array}
\end{array}
\end{equation}
Lemma 3.10. Let $k$ be a positive integer and $X(k; i) = \frac{k}{k+1-i}$. Then, $X(k; i) = 0$ for $1 < i < k + 1$ and

$X(k; 1) = \frac{(-1)^k}{[k+1]} \left( \frac{k}{k+1-i} \right) = \frac{(-1)^k}{[k+1]} (1_+ \otimes P_{-k-1}^+ \otimes P_{+k-1}^-).$

Proof. $A_2$ webs appearing in the RHS of Lemma 3.9 contains $t^+_+$ if $j \geq 1$ in the top side. Thus,

by applying Lemma 3.9 to $P_{+k}^+$. The $A_2$ web in the second term has $X(k-1; i)$. We obtain proof by induction on $k$. \qed

Proposition 3.11 (Kim [Kim07, Theorem 3.3]). For any positive integers $k$ and $l$,

$P_{+k}^{+1} \frac{k}{k+1-i} = \left( \frac{k}{k+1} \right) - \frac{k}{k-i} \left( \frac{k}{k+1} \right) - \frac{k}{k+1} \frac{[l]}{[k+1]} \frac{[k]}{[l+1]}.$

Proof. We decompose $P_{+k}^{+1}$ into $(1_+ \otimes P_{+k-1}^+)P_{+k+1-1}^+(1_+ \otimes P_{+k-1}^+)$ and expand the middle $A_2$ clasp $P_{+k+1-1}^+$ by using Definition 2.7. We calculate the following $A_2$ web:

$X(k, l; i) = \frac{k+1-l}{k+1-i} = \left( \frac{k}{k+1} \right) - \frac{k}{k-i} \left( \frac{k}{k+1} \right) - \frac{k}{k+1} \frac{[l]}{[k+1]} \frac{[k]}{[l+1]}.$
By Lemma 3.10 $X(k, l; i) = 0$ for $i > 1$ and

$$X(k, l; 1) = -\frac{[k]}{[k+1]} (-1)^{k-1} \frac{[l-1]}{[l]}$$

We can easily calculate $X(k, l; 0)$ from Definition 2.2

$$X(k, l; 0) = (1_+ \otimes P^{k}_{+k-1})(P^{k+1}_{+k+1} \otimes 1_+)(1_+ \otimes P^{l}_{+k-1})$$

$$= 1_+ \otimes P^{k}_{+k-1} - \frac{[k]}{[k+1]} (1_+ \otimes P^{k}_{+k-1})(t_+^+ t_-^+ \otimes 1_+)(1_+ \otimes P^{l}_{+k-1})$$

We complete the proof by substituting the above solutions of $X(k, l; i)$ into

$$P^{k+1}_{+k+1-1} = \sum_{i=0}^{\min(k+1, l)} (-1)^i \frac{[k+1]}{[k+1+i]} \frac{[l]}{[k+1+i]} X(k, l; i) = X(k, l; 0) - \frac{[k+1][l]}{[k+1][k+l+2]} X(k, l; 1).$$

Lemma 3.12.
Proof. Apply Proposition 3.11 to $P_{n_+ k - l}^+$ as follows:

\[
\begin{align*}
&= \left(\frac{2}{k} - \frac{k - 1}{k}\right) - \frac{k - 1}{k} - \frac{l}{k|k + l + 1|} \\
&= \left(\frac{k + 1}{k}\right) - \frac{l}{k|k + l + 1|}.
\end{align*}
\]

Once again, we apply Proposition 3.11 to $P_{n_+ k - l}^-$.

\[
\begin{align*}
&= \left[\frac{l + 1}{l}\right] - \frac{[l - 1]}{[l]} \\
&= \left[\frac{l + 1}{l}\right].
\end{align*}
\]

Lemma 3.13 (Otsuki-Yamada [OY97, Lemma 5.3]).

\[
\begin{align*}
&= \left[\frac{k + 2}{k + 1}\right] \frac{[k + l + 3]}{[k + l + 2]} P_{n_+ k - l}^+.
\end{align*}
\]
Proof. The $A_2$ web in the left-hand side should be expressed as a scalar multiplication of $P_{+k,l}^{+i}$. This constant can be calculated by taking the closure of the $A_2$ webs in both sides. We remark that the value of the closure of $P_{+k,l}^{+i}$ is $\frac{(k+1)(l+1)[k+l+2]}{2}$.

\[ \langle P_{(k+1,l)} \rangle = X \langle P_{(k,l)} \rangle - \langle P_{(k-1,l+1)} \rangle - \langle P_{(k,l-1)} \rangle. \]

Proof. By Proposition 3.11

(3.9)

\[ 1_+ \otimes P_{+k,l}^{+i} - P_{+k+1,l}^{+i} = \frac{[k]}{[k+1]} P_{+k-1,l}^{+i-1} \otimes P_{+k-1,l}^{+i} (1_+ \otimes P_{+k,l}^{+i}), \]

\[ + \frac{[l][k+l+1]}{[l+1][k+l+2]} (d_{+} \otimes P_{+k-2,l}^{+i-1} (1_+ \otimes P_{+k,l}^{+i}). \]

It is easy to see that the left-hand side is an idempotent and $X \langle P_{(k,l)} \rangle - \langle P_{(k+1,l)} \rangle$ in $K_0(K_0(Sp_2))$. We denote the right-hand side of the above $A_2$ web by $g$. We consider morphisms $p_1 : (k+1, l, g) \to P_{(k-1,l+1)}$ and $p_2 : (k+1, l, g) \to P_{(k,l-1)}$ defined by

\[ p_1 = \frac{[k]}{[k+1]} P_{+k-1,l}^{+i-1} \otimes P_{+k-1,l}^{+i} (1_+ \otimes P_{+k,l}^{+i}), \]

\[ p_2 = \frac{[l][k+l+1]}{[l+1][k+l+2]} (d_{+} \otimes P_{+k-2,l}^{+i-1} (1_+ \otimes P_{+k,l}^{+i}). \]

Let us confirm that $p_1$ and $p_2$ are morphisms in $K_0(Sp_2)$. By a similar way to (3.8) and using Lemma 3.12

\[ P_{+k-2,l}^{+i-1} \otimes P_{+k-2,l}^{+i} (1_+ \otimes P_{+k,l}^{+i}), \]

\[ = \frac{[k]}{[k+1]} P_{+k-1,l}^{+i-1} \otimes P_{+k-1,l}^{+i} (1_+ \otimes P_{+k,l}^{+i}), \]

\[ + \frac{[l][k+l+1]}{[l+1][k+l+2]} (d_{+} \otimes P_{+k-2,l}^{+i-1} (1_+ \otimes P_{+k,l}^{+i}). \]
By Lemma 3.12 and Lemma 5.13

\[ P_{+k-l-1}^k (d_+ \otimes P_{+k-l-1}^k) (1_+ \otimes P_{+k-l-1}^k) g \]

\[ = \begin{array}{c}
\begin{array}{c}
\text{[k]} \\
\text{[k + 1]}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{[l]} \\
\text{[k + 1][k + l + 2]}
\end{array}
\end{array} \]

\[ = \left( 1 - \frac{[l + 1]}{[k + 1][k + l + 1]} \right) \begin{array}{c}
\begin{array}{c}
\text{[k]} \\
\text{[k + 1][k + l + 2]}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{[l]} \\
\text{[k][k + l + 1]}
\end{array}
\end{array} \times \begin{array}{c}
\begin{array}{c}
\text{[l + 1][l + k + 2]} \\
\text{[l][l + k + 1]}
\end{array}
\end{array} \]

\[ = \begin{array}{c}
\begin{array}{c}
\text{[k]} \\
\text{[k + 1][k + l + 1]}
\end{array}
\end{array} = (d_+ \otimes P_{+k-l-1}^k) (1_+ \otimes P_{+k-l-1}^k). \]

Let us define \( \iota_1 : P_{(k-1,l+1)} \to (+_k^{k+1-l-1} g) \) and \( \iota_2 : P_{(k,l-1)} \to (+_k^{k+1-l-1} g) \) by \((1_+ \otimes P_{+k-l-1}^k) (t_+^{k-1-l} \otimes P_{+k-l-1}^k) P_{+k-l-1}^{k-1-l+1} \) and \((1_+ \otimes P_{+k-l-1}^k) (b_+^{k-l} \otimes P_{+k-l-1}^k) \), respectively. Because these webs are obtained by turning the webs in \( p_1 \) and \( p_2 \) upside down, we can confirm these maps are morphisms in \( \text{Kar}(\text{Sp}_q) \) by the same calculation in the above. The rest of the proof is to confirm that \( p_1, p_2, \iota_1, \) and \( \iota_2 \) satisfy the condition in Lemma 3.5. It is easy to see that \( p_i \circ \iota_j = 0 \) if \( i \neq j \). \( p_1 \circ \iota_1 = P_{+k-l-1}^{k-1-l+1} \) and \( p_2 \circ \iota_2 = P_{+k-l-1}^{k-1-l+1} \) are
derived from (3.8) and Lemma 3.13 respectively. By Proposition 3.11

\[ \iota_1 \circ p_1 + \iota_2 \circ p_2 = \frac{[k]}{[k + 1]} + \frac{[l][k + l + 1]}{(l + 1)[k + l + 2]} \]

\[ = \frac{[k]}{[k + 1]} - \frac{[l]}{(l + 1)} \left( \frac{[k + l + 1]}{[k + l + 2]} - \frac{[k]}{[k + 1]} \right) + \frac{[l][k + l + 1]}{(l + 1)[k + l + 2]} \]

\[ = \frac{[k]}{[k + 1]} - \frac{[l]}{(l + 1)} \left( \frac{[k + l + 1]}{[k + l + 2]} - \frac{[k]}{[k + 1]} \right) \cdot \]

The last equation uses a formula \( [a][b] = \sum_{i=1}^{a+b} [a + b - (2i - 1)] \) where \( a \) and \( b \) are integers. Thus, the condition in Lemma 3.5 is satisfied and we obtained the isomorphism \((+k+1-l,g) \cong P_{(k-1,l+1)} \oplus P_{(k,l-1)}\) in Kar\((Sp_q)\). In terms of \( K_0(Kar(Sp_q))\), the equation (3.9) is interpreted as

\[ X \langle P_{(k,l)} \rangle - \langle P_{(k+1,l)} \rangle = \langle (+k+1-l,g) \rangle = \langle P_{(k-1,l+1)} \rangle + \langle P_{(k,l-1)} \rangle. \]

\[ \square \]

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