The Edge State Network Model and the Global Phase Diagram

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The effects of randomness are investigated in the fractional quantum Hall systems. Based on the Chern-Simons Ginzburg-Landau theory and considering relevant quasi-particle tunneling, the edge state network model for the hierarchical state is introduced and the plateau-plateau transition and the liquid-insulator transition are discussed. This model has duality which corresponds to the relation of the quantum Hall liquid phase and the Hall insulating phase and reveals a mechanism of plateau transition in the weak coupling regime.

1. Introduction

The integer and fractional quantum Hall effect appears in two-dimensional electron systems in a strong magnetic field. For the quantum Hall effect weak randomness is needed. A large number of experimental and theoretical studies have been made on the effects of randomness. Especially, the quantum phase transition from a quantum Hall liquid to a different quantum Hall liquid or the Hall insulating phase which occurs as the randomness strength or the magnetic field increase are meaningful. In one experiment, Shahar et. al. observed a reflection symmetry in the $I-V_{xx}$ characteristics near the transition point of a $\nu = 1/3$ fractional quantum Hall liquid to a Hall insulator. They insisted that the observed symmetry corresponds to the charge-flux duality. Near the critical (i.e. self-dual) point the phase separation should occur, Simshoni et al. and Pryadko et al. introduced the edge state network model and discussed the non-linear $I-V_{xx}$ characteristics. The edge state network model was originally introduced by Chalker and Coddington, to study the universal properties of the localization length in the strong magnetic field in the framework of the non-interacting electron systems. Chalker-Coddington model is constructed by quantum tunneling of the electron near the Fermi levels which are located along the equal potential lines.

Meanwhile in the study of the liquid-liquid transition, the topological arguments and the numerical studies have been successful in integer quantum Hall systems. Using the flux attached composite fermion mean field theory Kivelson, Lee, and Zhang(KLZ) mapped the fractional quantum Hall state to integer quantum Hall state and they proposed the global phase diagram. There are also experimental discussions on the validity of the global phase diagrams. Their results seem to be slightly different from KLZ’s diagram.

In this paper we introduce a general fractional filling version of the network model. Our model shows dual symmetry which parallels the charge-flux duality. In weak coupling regime this model describes the hierarchical quantum Hall liquid phase, whereas the strong coupling regime corresponds to the Hall insulator. Analyzing the model the mechanisms of plateau transitions are revealed. We discuss the global phase diagram finally.

2. The edge state network model

The Chern-Simons gauge field theory is a long wave length and low energy effective theory of the fractional quantum Hall state. In this theory one regards the electrons as the flux attached bosons. In the case of filling factor $\nu = 1/q$ ($q$ is an odd integer) the Lagrangian is written as

$$
\mathcal{L}[a_\mu, \varphi] = -\frac{\nu}{4\pi} e^{\nu\mu\rho} a_\mu a_\rho
+ \nu \partial_\mu [e(a_0 + A_0)] \varphi
+ \frac{1}{2m} [\nabla - e(a + A)] \varphi^2
+ \mu |\varphi|^2 + \lambda |\varphi|^4
$$

(1)

where $\varphi$ is the field of the composite bosons, $a_\mu$ is the Chern-Simons gauge field, $A$ is a vector potential for the external magnetic field and $A_0 = V_{imp}$ is the impurity potential. The grand state of the fractional quantum Hall systems is characterized as Bose condensed state of the composites. In the dual description, the vortices condensed state describes the Hall insulator phase. Blok and Wen extended it to the hierarchical quantum Hall states. In the case of $\nu = N/(2mN \pm 1)$, the dual form Lagrangian is given as

$$
\mathcal{L}[b_{I\mu}, j^\mu_{vortex}] = \frac{1}{4\pi} K_{IJ} e^{\mu\nu\rho} b_{I\mu} \partial_\nu b_{J\rho}
- \frac{1}{2g_I} f_{I\mu\nu} f_{J}^{\mu
u} + b_{I\mu} j^\mu_{vortex}
$$

(2)

where $K$ is the K-matrix, $j^\mu_{vortex}$ is the current of the vortices, and dual gauge field $b_\mu$ is defined as $J^\mu = \frac{1}{2\pi} e^{\mu\nu\rho} \partial_\nu b_\rho$. $J_\mu$ is the electric current vector of the composite bosons. For simplicity we consider the case $\nu = N/(2mN + 1 \equiv \nu_0(N,m)$ in this paper. The extension to $\nu = N/(2mN - 1)$ is straightforward.

The vortices are quasi-particles in the quantum Hall liquids. Increasing the magnetic field, the vortices are introduced and are trapped by the impurity potential. Assuming the impurity potential varies slowly, many vortices gather around the maximum points of the potential. We can regard their regions as void of the quantum Hall liquids. Namely there are two kind of regions in the
systems, one is filled and the other is empty with the quantum Hall liquids. The quantum Hall liquids have the gapless edge modes on the boundaries of their systems. If the impurity potential exists, the edge modes exist not only at the sample edges but also on the boundaries between full and empty regions in the bulk of the sample. In the case the number of the vortices is small, since the area of the empty space is also small, these inner edge modes hardly contribute to the transport and the transport coefficients are determined by the edge modes of the sample boundary which is connected with the source and the drain electrodes. Increasing the number of vortices, however, inter-boundaries edge tunneling occur frequently and they affect the transports.

Since the vortex excitations are gapfull, we neglect their dynamics. Instead we consider the gapless edge modes and their tunneling. To see the dynamics of (8) in the strong coupling regime, we derive the effective action which includes only the non-linear degrees of the freedoms in eq.(5) and (6).

\[ Z = \int D\phi \left( 1 - S_0[\phi] - S_1[\phi_I - \phi_J] \right) \]
\[ = \int D\phi \int D\theta \prod_{ij,\alpha} \delta(\theta_I(x_\alpha) - \phi_I(x_\alpha) + \phi_J(x_\alpha)) \exp(-S_0[\phi_I] - S_1[\theta(x_\alpha)]) \]
\[ = \int D\phi \int D\theta \int D\lambda \exp(-S_0[\phi] - S_1[\theta]) \]
\[ - \sum_{ij,\alpha} \int d\tau \lambda_I(x_\alpha) (\theta_I(x_\alpha) - \phi_I(x_\alpha) - \phi_J(x_\alpha)) \]
\[ = \int D\theta \exp(-S_{eff}[\theta]) \]

where

\[ S_{eff}[\theta] = \sum_\omega \frac{|\omega|}{4\pi} M_{\alpha\beta} \theta_I(x_\alpha, -\omega) K_{IJ} \theta_J(x_\beta, \omega) \]
\[ - \sum_I \int d\tau \ u_I(x_\alpha) \cos(\theta_I(x_\alpha, \tau)) \]

(8)

\[ \theta_I(x_\alpha) = \phi_I(x_\alpha) - \phi_J(x_\alpha) \]. The matrix \( M_{\alpha\beta} \) depends on the geometry of the distribution of the empty and the filled region. Pryadko and Chaltikian, calculated this matrix in very simple cases at \( \nu = 1/q \). If all the tunneling points are widely separated from each other, we can regard \( M_{\alpha\beta} \) as Kronecker’s delta \( \delta_{\alpha\beta} \). According to Caldeira and Leggett, the diagonal terms of (8) are friction terms of quantum mechanical particles. In the periodic potential, Bloch states are formed and \( \theta \) can take arbitrary values. However the frictions make these particles practically localized, but slightly tunnel between the minima of the periodic potentials. In the strong coupling limit, \( \theta_I \)’s are fixed to \( 2\pi n \) where \( n \) is an integer. Because the edge currents are given as the time differential of the \( \phi \), all quasi-particles tunnel at the tunneling points.

3. Dual transformation

To study the dynamics of (8) in the strong coupling regime, we introduce the dilute instanton gas approximation. We consider the number of instantons \( n \) and their configurations as the trace of the partition function:
\[ Z = \text{Tr} \ e^{-\beta H} \]
\[ = \sum_{\alpha, I} \sum_{n=1}^{\infty} \sum_{\{\epsilon_i^n\}} \tilde{u}_I^n(x_\beta) \int_0^\beta d\tau_1^I \int_0^{\tau_1^I} d\tau_{n-1}^I \]
\[ \cdots \int_0^{\tau_2^I} d\tau_1^I \ e^{-S_{\text{ins}}} \] (9)

where \( \tilde{u}_I^n(x_\beta) \sim e^{-S_{\text{ins}}} \) is the fugacity of the instantons, \( S_{\text{ins}} \) is the action of the single instanton.

\[ S_D = \sum_\omega \frac{|\omega|}{4\pi} K_{IJ} M_{\alpha\beta} \frac{1}{\omega^2} \sum_{ij} \epsilon_i^{I\alpha} \epsilon_j^{I\beta} \]
\[ \times h_{IJ}(-\omega) h_{\alpha\beta}(\omega) e^{i\omega(\tau_1^{i\alpha} - \tau_j^{i\beta})} \]
\[ \simeq \sum_{ij} \frac{\pi}{\beta} K_{IJ} \sum_\omega \left[ \frac{1}{|\omega|} e^{i\omega(\tau_1^{i\alpha} - \tau_j^{i\beta})} \right] \epsilon_i^{I\alpha} M_{\alpha\beta} \epsilon_j^{I\beta} \] (10)

\( h_{IJ}(\tau) \) is the differential of the instanton solution which has the step around \( \tau = 0 \), \( h_{IJ}(\omega) \) is its Fourier transform and \( \epsilon_i^{I\alpha} \) is the charge of the instantons.

Using Stratonovich-Hubbard transformation we have

\[ e^{-S_{\text{D}}} = \int \mathcal{D}q_{I\alpha}(\tau) \exp\left( -\frac{1}{4\pi} K_{IJ} \sum_\omega |\omega| q_{I\alpha}(-\omega) M_{\alpha\beta} q_{J\beta}(\omega) \right) \]
\[ + \sum_\omega \frac{1}{\sqrt{\beta}} \sum_i \epsilon_i^{I\alpha} e^{-i\omega \tau_i^{I\alpha}} q_{I\alpha}(\omega). \] (11)

Substituting (11) for (9), we get the dual representation of (8).

\[ Z = \int \mathcal{D}q_{I\alpha} \ e^{-\tilde{S}} \] (12)

\[ \tilde{S} = \sum_\omega \frac{|\omega|}{4\pi} M_{\alpha\beta}^{-1} q_{I\alpha}(-\omega) K_{IJ} q_{J\beta}(\omega) \]
\[ + \sum_{I\alpha} \int d\tau \tilde{u}_I(x_\alpha) \cos(q_{I\alpha}) \]
\[ = \sum_\omega \frac{|\omega|}{4\pi} M_{\alpha\beta}^{-1} \tilde{\theta}_I(x_\alpha, -\omega) K_{IJ} \tilde{\theta}_J(x_\beta, \omega) \]
\[ + \sum_{I\alpha} \int d\tau \tilde{u}_I(x_\alpha) \cos(K_{IJ} \tilde{\theta}_J(x_\alpha, \tau)) \] (13)

where \( \tilde{\theta}_I = K_{IJ}^{-1} q_{IJ} \) is the dual field of \( \theta_I \). The first term of (13) is equivalent to the first term of (8) except the difference of \( M_{\alpha\beta} \) and \( M_{\alpha\beta}^{-1} \). Contrarily, the second term of (8) describes the tunneling of the quasi-particles, the second term of (13) describes the electron tunneling.

Eq.(13) describes the quantum droplets junction systems with weak electron tunneling. This accords with our intuitive picture of the Hall insulator. Actually the changing \( M \) into \( M^{-1} \) shows the changing of the geometry between the quasi-particle tunneling picture of QH liquid phase and the electron tunneling picture of Hall insulator phase.

4. Plateau transition

Now let us consider the effects of randomness. We read the randomness strength as the number of the minima and maxima of the impurity potential. In the case the value of the magnetic field is just \( B = B_0(N, m) \) and the randomness strength is weak enough, everywhere of the sample is filled by the quantum Hall liquids and there are no vortices and no edge modes in the bulk of the sample. If the impurity potential is dominant rather than the electron-electron interaction, the quantum Hall liquids are broken. Next we consider the case the magnetic field is slightly increased. The vortices are introduced to the sample and pinned by the potential. Then many empty regions exist and many edge modes are formed. Assuming the randomness is weak enough we can expect to apply the weak coupling edge state network model. We consider what happen as the randomness is increased. As increase of the number of the minima and maxima, the number of the empty regions and the tunneling points are also increased. So the quasi-particles tunneling becomes more frequent. Therefore in the framework of the edge state network model, increasing the randomness strength is regarded effectively as increasing of the tunneling amplitudes \( u(x_n) \). Increasing the magnetic field instead of the randomness strength, the area of the empty space is increased and the tunneling amplitudes are also increased. As we stated above, the values of the tunneling amplitudes \( u_I \) depend on the channel \( I \) which corresponds to the Landau levels of the composite fermions in the Jain’s theory. On every tunneling points, the amplitudes of the highest Landau level \( u_N \) are largest, \( u_{N-1} \) is the next, and so on. Therefore increasing \( \{u_I\} \), the \( I = N \) edge modes are pinned first, and the next is \( I = N - 1 \). To see the transport coefficients, let us consider a two terminal Hall bar. For simplify we consider the two terminal conductance. The discussion of the two terminal conductance is replaced as problem of the percolation as following.

In the weak tunneling limit, the transport properties are determined by the charged edge modes theory of sample boundary \( \phi_c = \sum_{i=1}^N \phi_i / N \) which is connected with source and drain electrodes and has the Luttinger liquid parameter given as \( K_c = \nu_0(N, m) / N = 1 / (2Nm + 1) \). Using linear response theory, we have the two-terminal conductance \( G = K_c N e^2 / h = \frac{N}{2(Nm+1)} e^2 / h \). Increasing the tunneling amplitudes, the modes \( I = N \) is pinned and edge quasi-particles on the boundary of the sample are carried into the bulk and reach the other side of the boundary of the sample. Finally, they return to the source electrode, i.e. the \( I = N \) channel do not con-
tribute to the transport. In this case, the charged mode is redefined as \( \phi_{r,\pm} = \sum_{l=1}^{N-1} \phi_l/(N-1) \) and the Luttinger parameter is \( K_c = \nu_0(N-1, m)/(N-1) \). So the two terminal conductance changes from \( G = \frac{N}{2N_m + 1} e^2/h \) to \( G = K_c(N-1)e^2/h = \frac{N-1}{2(N-1)m + 1} e^2/h \). Thus the plateaus transition is described as the percolation problem.

In the above arguments, we assume that the impurity potential \( V_{imp} \) varies slowly compared with the magnetic length \( l_B \). Increasing randomness, the ratio \( |\nabla V_{imp}/V_{imp}| \) is increased. In the case of \( |\nabla V_{imp}/V_{imp}| \sim l_B^{-1} \), we can expect that the tunneling amplitudes \( u_I \) scarcely depend on the channel index \( I \). We can understand this in composite fermion picture as following. The each channels of composite fermions near the Fermi level are very close. Then if the value of the magnetic field is changed, the quantum Hall liquid phase at \( \nu = N/(2Nm+1) \) may translate to the Hall insulator phase.

5. Global phase diagram

Now we discuss the global phase diagram. We first consider the regime that the randomness is weak. If the value of the magnetic field is just \( B = B_0(N, m) \equiv 2\pi \rho_c/\nu_0(N, m) \), the system is filled by quantum Hall liquid and doesn’t matter how randomness changes. If the value of the magnetic field is shifted as \( \Delta B = B - B_0(N, m) > 0 \), we can introduce the edge state network model as we said above. If \( \Delta B \) is small, the quasi-particle tunneling is rare. Since increasing \( \Delta B \) is regarded as increase of the tunneling amplitudes \( u_I \) in the network model, the quantum Hall liquids are translated to the other quantum Hall liquid according the selection rules as \( \Delta B \) increase. This statement is consistent with changing of the filling factor in Jain’s hierarchical theory. This is the behavior of the global phase diagram when randomness is fixed and the magnetic field is changed. Next we consider changing the randomness with fixed magnetic field. In the case of \( \Delta B = 0 \), the quantum Hall state is stable against weak randomness as a perturbation. Thus increasing the randomness, nothing goes on. If \( V_{imp} \) is more dominant than the \( e^2/4\pi l_B \) which is the characteristic energy of general quantum Hall liquids, the condensation of the composite boson is broken and the system becomes the Anderson insulator. Then the plateaus transition does not occur as increasing randomness in \( B = B_0(N, m) \). Because the percolation is not allowed, this result is also applicable to the case of \( B \lesssim B_0(N, m) \). In the other case i.e. \( B \) is larger than \( B_0(N, m) \) but smaller than \( B_0(N-1, m) \), as randomness increases from the weak randomness regime, quasi-particle tunneling becomes frequent and the quantum Hall liquid-quantum Hall liquid transition occurs. Because we set the value of the magnetic field smaller than \( B_0(N-1, m) \) and the filling factor becomes \( \nu_0(N-1, m) \) effectively, the additional transition to the quantum Hall liquid at \( \nu_0(N-2, m) \) doesn’t occur.

These results are consistent with an interesting experiment by Reznikov et al. They observed a fractional charge \( e/5 \) in the shot noise experiment at \( \nu = 2/5 \) restricted quantum Hall liquid. The two-terminal conductance shows a transition from a plateau at \( G = (2/5)e^2/h \) to another plateau at \( G = (1/3)e^2/h \) as the constriction is increase. On the second plateau they observed a current carrying particle with charge \( e/3 \). Motivated by the experiment K.Imura and one of the authors(K.N) calculated the shot noise at \( G = (2/5)e^2/h \) regime and \( G = (1/3)e^2/h \) regime in the framework of the chiral Tomonaga-Luttinger liquid theory in the single point contact systems. The edge state network model is regarded as a generalization of above analysis.

6. Conclusion

In summary, we introduced a general filling version of the edge state network model which has the dual description and discussed the transition between different quantum Hall liquid states. The results are following.

i) If \( B \) is almost \( B_0(N, m) \) or smaller than \( B_0(N, m) \), the plateau transition from \( \nu = \nu_0(N, m) \) quantum Hall liquid phase doesn’t occur as randomness increases.

ii) If \( B \) is enough larger than \( B_0(N, m) \), \( \nu = \nu_0(N, m) \) quantum Hall liquid shows a plateau transition to \( \nu = \nu_0(N-1, m) \) quantum Hall liquid phase. However additional transition doesn’t occur. The condition for the plateau transition have been discussed in the section 4.

So far we have discussed the case \( \nu = N/(2mN+1) \). To construct the effective theory for the \( \nu = N/(2mN-1) \) quantum Hall state, one should change the sign of the charge of the partons in the theory of the parton construction. Eventually we should regard decrease of the magnetic field as increase of that in above theory. These statements lead to a broad form of the global phase diagram shown as Fig.1 which seems like Kravechenko’s one. If randomness is weak, the plateau to plateau transitions occur as the magnetic field is increased, or decreased for \( \nu = N/(2mN+1) \) and \( \nu = N/(2mN-1) \), respectively. Contrary, in the regime where randomness is strong to some degree, the transition from a plateau is not to the other plateau but to the insulator. Consequently, the striking point which differs from KLZ’s diagram is that the hierarchical regimes border upon the Hall insulating regime. We expect that if one study the transport properties near the transition point of the hierarchical quantum Hall liquid states to the Hall insulating states, charge-flux duality symmetry is observed as the experiments by Shahar et al.
FIG. 1. The proposed global phase diagram based on the network model. In the weak randomness regime the plateau-plateau transitions occur. Contrary, in the strong randomness regime, the quantum Hall liquid-insulator transitions occur.