Heterogeneous Weibull count distribution

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Abstract. In this paper, we present a generalized model for count data based upon an assumed Weibull interarrival times. Weibull interarrival times could handle overdispersed data with shape parameter $0 < c < 1$, and underdispersed data with $c > 1$, and is reduced as exponential when $c = 1$. Weibull count model will be obtained by Taylor expansion and convolution method. By mixing, we obtain heterogeneous Weibull count model. Finally, we fit this model and several other models to non-equidispersed data and show that this model fit the data better than Poisson.

1. Introduction
Count data is a numerical data which contains non-negative integers. Count data is usually the outcomes of an underlying count process in continuous time [1]. Count process is stochastic process \{\(N(t)\mid t > 0\)\} in which \(N(t)\)'s are non-decreasing and represents the total number of events occurring by time \(t\), and thus the values are non-negative integers [5].

One of the distributions which often used to fit count data is Poisson count model. It is quite simple because its process satisfies the characteristics of count process and it is a distribution with a discrete random variable. Poisson count model has exponential interarrival times. However, Poisson count model is only valid if the data satisfy equidispersion assumption (the variance of the data equals its mean). Applying Poisson count model to the significantly non-equidispersed data could lead to misspecification of the distribution of the data [1]. In that case, the data could either be overdispersed (the variance is more than its mean) or underdispersed (the variance is less than its mean). Many models that can handle overdispersion have been developed. Statisticians have also tried to provide ways to manage underdispersion problem, but not many offer the conceptual elegance and usefulness of the Poisson-exponential connection [4].

Winkelmann has developed a count model to handle underdispersion based on gamma interarrival times which is a generalization of exponential, but there are limitations of the model. One of those is the hazard function of the gamma interarrival times is not a closed-form [6]. Weibull, which is also a generalization of exponential, on the other hand, has a closed-form hazard function. Another characteristic of Weibull distribution is the shape parameter \((c)\) could vary. Weibull interarrival times could handle overdispersed data when \(0 < c < 1\), and underdispersed data when \(c > 1\), and is reduced as exponential when \(c = 1\) [4]. Therefore, Weibull interarrival times is considered in developing count model that can handle underdispersion. The count model will be named later for its interarrival times, Weibull count model.

Weibull count model is obtained by Taylor expansion and convolution method. Assuming that the rate across the population of units follow gamma distribution, Weibull count model is extended to
heterogeneous Weibull count by mixing. Heterogeneous Weibull count distribution reduces to negative binomial distribution when \( c = 1 \). However, it is difficult to show this mathematically. Therefore, we arrange and run the R script which provides the evidence. Finally, we will fit this model and several other models to both underdispersed and overdispersed data and show that this model fit the data better than Poisson and even turns out to be the best among those models in fitting the underdispersed data.

2. Weibull count model

Before deriving the Weibull count model, we first discuss about the relationship between a count model and its underlying interarrival times. Let \( Y_n \) denote the time needed for the \( n \)th event to occur. Let \( N(t) \) denote the number of events that occur up to time \( t \). We could state that the time needed until the \( n \)th event occurs is less than or equal to \( t \) if and only if the number of events occuring up to time \( t \) is more than or equal to \( n \). Then we have the formula

\[
Y_n \leq t \iff N(t) \geq n
\]  

(1)

Based on (1), we will derive Weibull count model, that is a probability function that \( n \) events would occur in time interval \((0, t]\) and will be denoted by \( C_n(t) \). Based on (1), we have

\[
C_n(t) = P(N(t) = n) = P(N(t) \geq n) - P(N(t) \geq n + 1) = P(Y_n \leq t) - P(Y_{n+1} \leq t) = F_n(t) - F_{n+1}(t)
\]  

(2)

Let \( T_1, T_2, \ldots, T_n \) denote the i.i.d Weibull interarrival times, then \( F_n(t) \) is derived as the cumulative distribution of \( Y_n = T_1 + T_2 + \ldots + T_n \) using convolution by solving this integral [2]

\[
F_n(t) = \int_0^t F(t-s)f(s)ds
\]  

(3)

However, not like exponential, (3) does not have a proper solution for Weibull distribution [1]. This is why the Taylor expansion is needed for an approach in solving the integral in (3).

Weibull distribution has probability density function (PDF) \( f(t) = \lambda c t^{c-1} e^{-\lambda t^c} (c, \lambda \in \mathbb{R}^+) \) and cumulative distribution function (CDF) \( F(t) = 1 - e^{-\lambda t^c} (c, \lambda \in \mathbb{R}^+) \) which respectively are reduced to pdf and cdf of exponential when \( c = 1 \). The Weibull cdf summation form in (4) is obtained by first expanding the \( e^{-\lambda t^c} \).

\[
F(t) = 1 - e^{-\lambda t^c} = 1 - \left( 1 - \frac{\lambda t^c}{1!} + \frac{(\lambda t^c)^2}{2!} - \frac{(\lambda t^c)^3}{3!} + \ldots \right) = \frac{\lambda t^c}{1!} - \frac{(\lambda t^c)^2}{2!} + \frac{(\lambda t^c)^3}{3!} - \ldots = \sum_{h=1}^{\infty} (-1)^{h+1}(\lambda t^c)^h \frac{1}{\Gamma(h+1)},
\]  

(4)

for \( t > 0, \ c, \lambda > 0 \). Because the cdf is a continuous function of \( t \), it will then be differentiated with respect to \( t \) to obtain the summation form of Weibull pdf shown in (5).

\[
f(t) = \frac{dF(t)}{dt} = \frac{d}{dt} \left( \sum_{h=1}^{\infty} (-1)^{h+1}(\lambda t^c)^h \frac{1}{\Gamma(h+1)} \right) = \sum_{h=1}^{\infty} (-1)^{h+1} c \lambda^h t^{c-1} \frac{h}{\Gamma(h+1)},
\]  

(5)

for \( t > 0, \ c, \lambda > 0 \).

Using (2) and initializing \( F_0(t) = 1 \) and \( F_1(t) = F(t) \), it is obtained

\[
C_n(t) = F_n(t) - F_{n+1}(t) = e^{-\lambda t^c} = \sum_{h=0}^{\infty} (-1)^h (\lambda t^c)^h \frac{1}{\Gamma(h+1)}.
\]  

(6)

Meanwhile, for \( n > 0 \),

\[
C_n(t) = F_n(t) - F_{n+1}(t) = \int_0^t F_n(t-s) f(s)ds - \int_0^t F_n(t-s) f(s)ds = \int_0^t C_{n+1}(t-s) f(s)ds.
\]  

(7)
So, \( C_t(t) \) is obtained as follows.

\[
C_t(t) = \int_0^t C_0(t-s)f(s)\,ds
\]

\[
= \int_0^t \left( \sum_{h=0}^{\infty} \left( -1 \right) \frac{\lambda^h (t-s)^h}{\Gamma(h+1)} \left( \sum_{k=1}^{\infty} \left( -1 \right)^{k+1} \frac{ck\lambda^k}{\Gamma(k+1)} \frac{(t-s)^{ck-1}}{\Gamma(ck)} \right) \right)\,ds
\]

\[
= \int_0^t \sum_{h=0}^{\infty} \sum_{k=1}^{\infty} \left( -1 \right)^{k+1} \frac{\lambda^h \lambda^k}{\Gamma(h+1)\Gamma(k+1)} \int_0^s c^k(t-s)^{ck-1}\,ds
\]

\[
= \sum_{h=0}^{\infty} \sum_{k=1}^{\infty} \left( -1 \right)^{k+1} \frac{\lambda^h \lambda^k}{\Gamma(h+1)\Gamma(k+1)} \int_0^s c^k \left( \frac{t}{s} \right)^{ck} \left( 1-\frac{s}{t} \right)^{ck-1} \,ds.
\]

To solve the integral part in (8), consider the pdf of Beta distribution with three parameters \((ck, ch+1, t)\),

\[
\left( \frac{s}{t} \right)^{ck} \left( 1-\frac{s}{t} \right)^{ch} \frac{1}{s \Gamma(ch+1)\Gamma(ck+1)} \quad \text{for} \quad 0 < s < t.
\]

If we integrate it from 0 to \( t \) the result would be 1. Thus,

\[
\int_0^t \left( \frac{s}{t} \right)^{ck} \left( 1-\frac{s}{t} \right)^{ch} \frac{1}{s} \,ds = \frac{\Gamma(ch+1)\Gamma(ck+1)}{(ch+ck+1)}.
\]

So, \( C_t(t) \) becomes

\[
C_t(t) = \sum_{h=0}^{\infty} \sum_{k=1}^{\infty} \left( -1 \right)^{k+1} \frac{\lambda^h \lambda^k}{\Gamma(h+1)\Gamma(k+1)} \int_0^s c^k \left( \frac{t}{s} \right)^{ck} \left( 1-\frac{s}{t} \right)^{ck-1} \,ds
\]

By changing the variables \( m = h \) we obtain

\[
C_t(t) = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \left( -1 \right)^m \left( -1 \right)^{k+1} \frac{\lambda^m (\lambda)^k}{\Gamma(m+1)\Gamma(k+1)} \int_0^s c^k \left( \frac{t}{s} \right)^{ck} \left( 1-\frac{s}{t} \right)^{ck-1} \,ds
\]

and by changing the variables \( l = m+k \) we obtain

\[
C_t(t) = \sum_{l=1}^{\infty} \sum_{m=0}^{\infty} \left( -1 \right)^m \left( -1 \right)^{l+1} \frac{(\lambda)^m (\lambda)^l}{\Gamma(m+1)\Gamma(l-m+1)} \int_0^s c^k \left( \frac{t}{s} \right)^{ck} \left( 1-\frac{s}{t} \right)^{ck-1} \,ds
\]

or it can be stated as

\[
C_t(t) = \sum_{l=1}^{\infty} \frac{\left( -1 \right)^{l+1} (\lambda)^l (t)^{cl} \alpha_l}{\Gamma(cl+1)}.
\]

(9)
where $\alpha'_{l} = \sum_{m=0}^{l+1} \Gamma(cm+1)\Gamma(cl-cm+1) / \Gamma(m+1)\Gamma(l-m+1)$.

After obtaining $C_{l}(t)$, in a similar way we can obtain $C_{2}(t)$ by the recursive formula stated in (7).

\[
C_{2}(t) = \int_{0}^{t} C_{1}(t-s) f(s) ds
\]

\[
= \int_{0}^{t} \left( \sum_{i=0}^{t-1} (-1)^{t-i} (\lambda (t-s)^{t})^{i} \alpha_{i}^{2} \right) \times \left( \sum_{k=0}^{t-1} (-1)^{k+1} ckxk^{t-1} \right) ds
\]

\[
= \int_{0}^{t} \sum_{l=0}^{t-1} \sum_{k=0}^{t-1} (-1)^{l} (-1)^{k+1} \lambda^{l} (t-s)^{cl} ckxk^{s-1} \frac{\Gamma(cl+1)\Gamma(k+1)}{\Gamma(cl+1)\Gamma(k+1)} ds
\]

By changing the variables $q = l$ and $p = q + k$ we obtain

\[
C_{2}(t) = \sum_{p=2}^{\infty} \left[ \sum_{q=p-1}^{\infty} (-1)^{p-1} (\lambda^{q-p+1}) \frac{(t)^{p-q+1} \Gamma(cp+1)\Gamma(cp-cq+1)}{\Gamma(p+1)} \times \left( \sum_{q=p}^{\infty} \left( \sum_{m=0}^{q-1} \Gamma(cm+1)\Gamma(cm+1)\Gamma(p-q+1) \right) \right) \right]
\]

or it can be stated as

\[
C_{2}(t) = \sum_{p=2}^{\infty} \frac{(-1)^{p-1} (\lambda)^{p} (t)^{p} \alpha_{p}^{2}}{\Gamma(cp+1)}
\]

where $\alpha_{p}^{2} = \left( \sum_{q=p}^{\infty} \left( \sum_{m=0}^{q-1} \frac{\Gamma(cm+1)\Gamma(cm+1)}{\Gamma(m+1)\Gamma(q-m+1)} \right) \right) \frac{\Gamma(p+1)}{\Gamma(p+1)}$.

Before obtaining the general form of $C_{n}(t)$, we will first obtain the general form of $\alpha_{n}^{n}$. Note that if we denote $\alpha_{l}^{0} = \Gamma(cl+1) / \Gamma(l+1)$, then

\[
\alpha_{l}^{0} = \sum_{m=0}^{l} \frac{\Gamma(cl+1)\Gamma(cl-cm+1)}{\Gamma(m+1)\Gamma(l-m+1)} \frac{\Gamma(cm+1)\Gamma(l-cm+1)}{\Gamma(l-m+1)} = \sum_{m=0}^{l} \alpha_{m}^{0} \Gamma(cl-cm+1) / \Gamma(l-m+1).
\]
\[
\alpha_i^n = \left\{ \sum_{m=n}^{l-1} \frac{\Gamma(cl+1)\Gamma(cm-cm+1)}{\Gamma(l+1)\Gamma(l-m+1)} \right\} = \sum_{m=n}^{l-1} \frac{\alpha_i^n}{\Gamma(l-m+1)}.
\]

Thus, the general recursive form of \( \alpha_i^n \) is

\[
\alpha_i^n = \begin{cases} 
\frac{\Gamma(cl+1)}{\Gamma(l+1)}, & \text{for } n = 0, \\
\sum_{m=n}^{l-1} \frac{\alpha_i^n}{\Gamma(l-m+1)}, & \text{for } n > 0 
\end{cases}.
\]

(9) and (10) lead us to an idea that the general form for \( C_n(t) \) is

\[
C_n(t) = \sum_{i=0}^{\infty} \frac{(-1)^{i+1} (t)^i}{\Gamma(cl+1)} \alpha_i^n
\]

which is confirmed by mathematical induction.

\[
C_{n+1}(t) = \int_0^t C_n(t-s) f(s)ds
\]

\[
= \int_0^t \left( \sum_{h=0}^{\infty} \frac{(-1)^{h+1} (\lambda(t-s))^h}{\Gamma(ch+1)} \alpha_h^n \right) \times \left\{ \sum_{k=1}^{\infty} \frac{(-1)^{k+1} ck\lambda^k s^{k-1}}{\Gamma(k+1)} \right\} ds
\]

\[
= \sum_{h=0}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{h+1} (-1)^{k+1} (\lambda)^h x \alpha_h^n}{\Gamma(ch+1)\Gamma(k+1)} \int_0^t c(t-s)^x s^{k-1} ds
\]

\[
= \sum_{h=0}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{h+1} (-1)^{k+1} (\lambda)^h x \alpha_h^n}{\Gamma(ch+1)\Gamma(k+1)} \times \left( \frac{(t)^x}{\Gamma(ch+1)\Gamma(c+1)} + \frac{(ch+1)^x}{\Gamma(ch+c+1)} \right)
\]

\[
= \sum_{h=0}^{\infty} \frac{(-1)^{h+1} (\lambda t)^x}{\Gamma(ch+1)\Gamma(c+1)} \times \left( \sum_{m=n}^{l-1} \frac{\alpha_i^n}{\Gamma(l-m+1)} \right)
\]

\[
C_{n+1}(t) = \sum_{i=n}^{\infty} \frac{(-1)^{i+1} (\lambda t)^x \alpha_i^{n+1}}{\Gamma(cl+1)}.
\]

\( C_{n+1}(t) \) is in the same form as our assumption of general \( C_n(t) \). Therefore, we have one of the result of this paper, that is Weibull count model:

\[
P(N(t) = n) = C_n(t) = \sum_{h=0}^{\infty} \frac{(-1)^{h+1} (\lambda t)^x}{\Gamma(ch+1)\Gamma(c+1)}, \quad n = 0, 1, 2, \ldots
\]

where \( \alpha_h^n = \left\{ \frac{\Gamma(ch+1)}{\Gamma(h+1)}, \quad \text{for } n = 0, \quad h = 0, 1, 2, \ldots \right\} \)

The expectation of this model is given by

\[
E(N(t)) = \sum_{n=1}^{\infty} nC_n(t) = \sum_{n=1}^{\infty} \sum_{h=0}^{\infty} \frac{n(-1)^{h+1} (\lambda t)^x}{\Gamma(ch+1)\Gamma(h+1)}
\]

and its variance is given by

\[
Var(N(t)) = E([N(t)]^2) - (E[N(t)])^2 = \sum_{n=1}^{\infty} \sum_{h=0}^{\infty} \frac{n^2(-1)^{h+1} (\lambda t)^x}{\Gamma(ch+1)\Gamma(h+1)} - \left( \sum_{n=1}^{\infty} \sum_{h=0}^{\infty} \frac{n(-1)^{h+1} (\lambda t)^x}{\Gamma(ch+1)\Gamma(h+1)} \right)^2.
\]
and the moment generating function is stated as follows

\[
M_{N(t)}(u) = E\left(e^{N(t)u}\right) = \sum_{n=0}^{\infty} \sum_{h=0}^{\infty} \frac{(-1)^{h+n} (\lambda t)^h \alpha_n^h}{\Gamma(ch+1)} = \sum_{n=0}^{\infty} \sum_{h=0}^{\infty} \frac{e^{nu} (-1)^{h+n} (\lambda t)^h \alpha_n^h}{\Gamma(ch+1)}.
\]

(13)

So, to obtain the \(i^{th}\) moment of the model we differentiate (13) \(i\) times with respect to \(u\)

\[
\frac{d^i}{du^i} M_{N(t)}(u) = \frac{d^i}{du^i} \sum_{n=0}^{\infty} \sum_{h=0}^{\infty} \frac{e^{nu} (-1)^{h+n} (\lambda t)^h \alpha_n^h}{\Gamma(ch+1)} = \sum_{n=0}^{\infty} \sum_{h=0}^{\infty} \frac{n^i e^{nu} (-1)^{h+n} (\lambda t)^h \alpha_n^h}{\Gamma(ch+1)}
\]

and then set \(u\) equal to zero

\[
M_{N(t)}^{(i)}(u) = \sum_{n=0}^{\infty} \sum_{h=0}^{\infty} \frac{n^i (-1)^{h+n} (\lambda t)^h \alpha_n^h}{\Gamma(ch+1)}.
\]

### 3. Heterogeneous Weibull count model

In Weibull count model, the value of lambda \((\lambda)\) or the rate of event is assumed to be constant for every unit, but in reality it is difficult to fulfill such assumption. Let \(\lambda_i\) be the value of \(\Lambda_i\) which denotes the rate of event of unit \(i\). In this section we also discuss the heterogeneous Weibull count model as an expansion of Weibull count where the “heterogeneous” means that the rate varies across the population of units and it is assumed to follow gamma distribution with pdf

\[
g(\lambda_i \mid r,a) = a'(\lambda_i)^{r-1}e^{-\lambda_i a}/\Gamma(r).
\]

Let the probability that the number of events from unit \(i\) up until time \(t\) is \(n\), conditional on its rate \(\lambda_i\) follows Weibull count distribution

\[
P(N(t)=n \mid \Lambda_i = \lambda_i) = \sum_{h=0}^{\infty} \frac{(-1)^{h+n} (\lambda_i t)^h \alpha_n^h}{\Gamma(ch+1)}, \ n = 0, 1, 2, \ldots,
\]

where \(\alpha_n^h = \frac{\Gamma(ch+1)}{\Gamma(h+1)}\), for \(n = 0, h = 0, 1, 2, \ldots\)

\[
\sum_{m=0}^{n-1} \alpha_n^m \frac{\Gamma(ch-cm+1)}{\Gamma(h-m+1)}, \text{ for } n = 1, 2, \ldots, h = n+1, n+2, n+3, \ldots
\]

Then, the unconditional probability that the number of events from unit \(i\) up until time \(t\) is \(n\) can be obtained by mixing.

\[
P(N(t)=n) = \int_0^\infty P\{N(t)=n \mid \Lambda_i = \lambda_i\} \cdot g(\lambda_i \mid r,a) d\lambda_i
\]

\[
= \int_0^\infty \sum_{h=0}^{\infty} \frac{(-1)^{h+n} (\lambda_i t)^h \alpha_n^h}{\Gamma(ch+1)} \times \frac{a'(\lambda_i)^{r-1}e^{-\lambda_i a}}{\Gamma(r)} d\lambda_i
\]

\[
= \sum_{h=0}^{\infty} \frac{(-1)^{h+n} (\lambda_i t)^h \alpha_n^h}{\Gamma(ch+1)} \int_0^\infty \frac{a'(\lambda_i)^{r-1}e^{-\lambda_i a}}{\Gamma(r)} d\lambda_i
\]

\[
= \sum_{h=0}^{\infty} \frac{(-1)^{h+n} (\lambda_i t)^h \alpha_n^h}{\Gamma(ch+1)} \int_0^\infty \frac{\lambda_i^h a^h a a'(\lambda_i)^{r-1}e^{-\lambda_i a}}{\Gamma(r)} d\lambda_i
\]
Note that the integral part above is a gamma function \( \int_0^\infty (a\lambda_i)^{hy-1} e^{-a\lambda_i} d\lambda_i = \Gamma(r+h) \). Thus, (14) becomes

\[
P(N(t) = n) = \sum_{h=0}^\infty \frac{(-1)^{h+n}}{\Gamma(c(h+1))} \frac{\alpha^h}{\Gamma(r) a^h} \Gamma(r+h) 
\]

with

\[
\alpha^h = \begin{cases} 
\Gamma(c(h+1)) & \text{for } n = 0, \\
\sum_{a=0}^{\infty} \frac{\Gamma(c(h+1))}{\Gamma(l-1)} & \text{for } n > 0.
\end{cases}
\]

The expectation of this model is given by

\[
E(N(t)) = \sum_{n=1}^\infty n C_n(t) = \sum_{n=1}^\infty \sum_{h=0}^\infty \frac{n(-1)^{h+n}}{\Gamma(c(h+1))} \frac{\alpha^h}{\Gamma(r) a^h} \Gamma(r+h),
\]

and its variance is given by

\[
\text{Var}(N(t)) = \left( E[N(t)]^2 \right) - \left( E[N(t)] \right)^2
\]

\[
= \sum_{n=1}^\infty \sum_{h=0}^\infty \frac{n^2(-1)^{h+n} (t^r)^h a^n}{\Gamma(c(h+1))} \frac{\alpha^h}{\Gamma(r) a^h} \Gamma(r+h) \left( \sum_{n=1}^\infty \sum_{h=0}^\infty \frac{n(-1)^{h+n}(t^r)^h a^n}{\Gamma(c(h+1))} \frac{\alpha^h}{\Gamma(r) a^h} \Gamma(r+h) \right)^2.
\]

and the moment generating function is stated as follows

\[
M_{N(t)}(u) = E(e^{uN(t)}) = \sum_{n=1}^\infty \sum_{h=0}^\infty \frac{n(-1)^{h+n}(t^r)^h a^n}{\Gamma(c(h+1))} \frac{\alpha^h}{\Gamma(r) a^h} \Gamma(r+h).
\]

So to obtain the \( l \)th moment of the model we differentiate (16) \( l \) times with respect to \( u \)

\[
\frac{d^l}{du^l} M_{N(t)}(u) = \sum_{n=1}^\infty \sum_{h=0}^\infty \frac{n^l(-1)^{h+n}(t^r)^h a^n}{\Gamma(c(h+1))} \frac{\alpha^h}{\Gamma(r) a^h} \Gamma(r+h)
\]

and then set \( u \) equal to zero

\[
M^{(l)}_{N(t)}(u) = \sum_{n=1}^\infty \sum_{h=0}^\infty \frac{n^l(-1)^{h+n}(t^r)^h a^n}{\Gamma(c(h+1))} \frac{\alpha^h}{\Gamma(r) a^h} \Gamma(r+h).
\]

Heterogeneous Weibull count distribution is reduced to negative binomial distribution when \( c = 1 \). It is difficult to show this mathematically. However, by defining function (15) in R software with
script shown below, we have run some \( r \) and \( a \) values while \( c=1 \). The results do not differ significantly from the outcomes of negative binomial pdf with the same \( r \) and \( a \) values.

\[
\begin{align*}
\text{alpha} &\leftarrow \text{function}(c,h,n) \{
\text{sum} \leftarrow 0
\text{if} \ (n == 0) \{
\text{sum} \leftarrow \text{gamma}(c*h+1)/\text{gamma}(h+1)
\} \text{ else if} \ (n > 0) \{
\text{for} \ (m \ \text{in} \ (n-1):(h-1)) \{
\text{sum} \leftarrow \text{sum} + \text{alpha}(c,m,(n-1)) \* \text{gamma}(c*h-c*m+1)/\text{gamma}(h-m+1))
\}
\text{return}(\text{sum})
\}
\text{dis} &\leftarrow \text{function}(c,n,a,r,h) \{
\text{u} \leftarrow ((-1)^(h+n)) \* \text{alpha}(c,h,n) \* \text{gamma}(r+h)
\text{d} \leftarrow \text{gamma}(c*h+1) \* \text{gamma}(r) \* a^h
\text{return}(\text{u}/\text{d})
\}
\text{summ} &\leftarrow \text{function}(c,n,a,r,hmax) \{
\text{sum} \leftarrow 0
\text{for} \ (h \ \text{in} \ n:hmax) \{
\text{sum} \leftarrow \text{sum}+\text{dis}(c,n,a,r,h)
\}
\text{return}(\text{sum})
\}
\text{pdf} &\leftarrow \text{function}(c,n,a,r) \{
\text{tr} \leftarrow 0.0000001
\text{max} \leftarrow 500
\text{hmax} \leftarrow n
\text{cont} \leftarrow \text{TRUE}
\text{while}(\text{cont} \&\& \text{hmax}<500) \{
\text{diff} \leftarrow \text{dis}(c,n,a,r,hmax)
\text{hmax} \leftarrow \text{hmax}+1
\text{if}(\text{abs}(\text{diff})<\text{tr}) \{
\text{cont} \leftarrow \text{FALSE}
\}
\}
\text{return}(\text{summ}(c,n,a,r,hmax))
\}
\]

4. Testing and results
In this section, we fit the underdispersed data below with several distribution which are heterogeneous Weibull count model, negative binomial model, and Poisson model in order to compare the suitability of those models.

| Table 1. Underdispersed data. |
|-------------------------------|
| Observation | 0 | 1 | 2 | 3 | 4 | 5 | 6+ |
| Frequency   | 66 | 242 | 152 | 33 | 5 | 1 | 1 |
The mean of the data is 1.352 which is higher than the variance 0.7495952 so it is clear that the data is underdispersed. The table of parameters and the Schwarz Bayesian Criterion score of the models is shown below.

**Table 2.** Several model parameters of the data in table 1.

| Parameters     | Heterogeneous Weibull count | Negative Binomial | Poisson |
|----------------|-----------------------------|-------------------|---------|
| c              | 2                           | -                 | -       |
| r              | 5                           | 592.44635290      | -       |
| a              | 2                           | 438.1998939       | -       |
| \( \lambda \)  | -                           | -                 | 1.352   |
| Negative log likelihood | 618.4196 | 665.3779 | 663.872 |
| \( \frac{p}{2} \ln(500) \) | 9.321912148 | 6.214608098 | 3.107304049 |
| Schwarz Bayesian Criterion | 627.7415121 | 671.5925081 | 666.979304 |

The Schwarz Bayesian Criterion (SBC) score is negative log likelihood + \( \frac{p}{2} \ln(500) \) where \( p \) is the number of parameters of the model and 500 is the number of observations. The lowest SBC score in the table 2 is the SBC score of heterogeneous Weibull count. Thus, it could be concluded that the best model among the three for the underdispersed data in table 1 is heterogeneous Weibull count model [3].

Afterwards, we also fit an overdispersed data with heterogeneous Weibull count model, negative binomial model, and Poisson model.

**Table 3.** Overdispersed data.

| Observation | 0 | 1 | 2 | 3 | 4 | 5 | 6+ | 7+ |
|-------------|---|---|---|---|---|---|----|----|
| Frequency   | 225 | 132 | 59 | 27 | 15 | 9 | 2  | 1  |

The mean of the data is 0.91 which is lower than the variance 1.569038 so it is clear that the data is overdispersed. The table of parameters and the Schwarz Bayesian Criterion score of the models is shown below.

**Table 4.** Several model parameters of the data in table 3.

| Parameters | Heterogeneous Weibull count | Negative Binomial | Poisson |
|------------|-----------------------------|-------------------|---------|
| c          | 0.75                        | -                 | -       |
| r          | 5                           | 1.18083921        | -       |
| a          | 6                           | 1.297625933       | -       |
| \( \lambda \) | -                           | -                 | 0.91    |
| Negative log likelihood | 664.3644 | 660.4317 | 699.6264 |
| \( \frac{p}{2} \ln(500) \) | 9.321912148 | 6.214608098 | 3.107304049 |
| Schwarz Bayesian Criterion | 673.6863121 | **666.6463081** | 702.733704 |

The lowest SBC score in the table 4 is the SBC score of negative binomial. Thus, it concludes that negative binomial is better than heterogeneous Weibull count model in modeling overdispersed data in table 1 [3]. However, heterogeneous Weibull count is still way better than Poisson in this case [3].
5. Conclusion
In this paper, we have derived Weibull count distribution, which is based on Weibull interarrival
times, by first obtaining the summation form of probability density function and cumulative
distribution function of Weibull distribution. Afterwards, by assuming that the rate of event varies
across the population of units (heterogeneous) and follows gamma distribution, we then derived
the heterogeneous Weibull count distribution by mixing Weibull count distribution with the gamma
distribution. We have also arranged and run the R script which gives us evidence that heterogeneous
Weibull count distribution reduces to negative binomial distribution when \( c = 1 \). Finally, we indeed
cannot generalize the result from only the data in table 1 and table 3, but we have shown the purpose
of heterogeneous Weibull count model.

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