FOURIER TRANSFORMS ON THE BASIC AFFINE SPACE
OF A QUASI-SPLIT GROUP

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Abstract. We extend the Gelfand and Graev construction of generalized Fourier transforms on basic affine space from split groups to quasi-split groups over a local non-archimedean field $F$.

1. Introduction

1.0.1. Notation.
- Let $F$ be a local non-archimedean field with the norm $|·| = |·|_F$, the ring of integers $\mathcal{O}$ and a fixed uniformizer $\varpi$ such that $|\varpi| = q^{-1}$, where $q$ is the cardinality of the residue field.
- We fix a non-trivial additive character $\psi$ throughout the paper. The self-dual Haar measure $dx$ on $F$ with respect to $\psi$ defines the Haar measure $d^\times x = \frac{dx}{|x|}$ on $F^\times$.
- For a quadratic extension $K$ of $F$ we denote by $\chi_K$ the quadratic character of $F^\times$, associated to $K$ by class field theory. We also denote by $\chi_0$ the trivial character of $F^\times$.
- For a space $Y$ over $F$ we denote by $\mathcal{S}^\infty(Y)$ (resp. $\mathcal{S}_c(Y)$) the space of locally constant (resp. locally constant of compact support) functions on $Y$.
- Throughout this paper we use boldface characters for group schemes over $F$, such as $H$, and plain text characters for their group of $F$-points, such as $H$.
- Let $G$ be a simply-connected quasi-split group defined over $F$ with a maximal $F$-split torus $T'$ and the maximal torus $T = Z_G(T')$. We fix a Borel subgroup $B$ of $G$ containing $T$ so that $B = T \cdot U$. We write $U^{\text{opp}}$ for the unipotent radical of the opposite Borel subgroup.
- The Weyl group $W = N_G(T')/T$ acts on $T$ by conjugation and we write $t^w$ for $w^{-1}tw$ for all $t \in T, w \in W$.
- The quotient $X = U \backslash G$ is called the basic affine space of $G$. For any $g \in G$ we write $[g]$ for the element $Ug$ in $X$. The space $X$ admits unique, up to a scalar, $G$-invariant measure $\omega_X$. The precise choice of $\omega_X$ is not important for general $G$, but will be fixed for groups of rank 1.

1.1. Fourier transforms on the basic affine space of a quasi-split group. We define a unitary representation $\theta$ of the group $G \times T$ on $L^2(X, \omega_X)$
by:
\[
\theta(g, t)f([h]) = \delta_B^{1/2}(t)f([t^{-1}hg]),
\]
where \(\delta_B\) is the modular character.

For split groups Gelfand and Graev in [GG73], see also [KL88], [Kaz95], extended the action \(\theta\) of \(G \times T\) to a representation of \(G \times (T \rtimes W)\), so that every element \(w\) of \(W\) acts on \(L^2(X, \omega_X)\) by an operator \(\Phi_w\), called a generalized Fourier transform. Our paper has two goals:

- To extend the construction by Gelfand and Graev to quasi-split groups.
- To show that the Whittaker map intertwines the action of \(W\) on a dense subspace \(S_0(X)\) in \(L^2(X)\) with the natural action of \(W\) on the space of Whittaker vectors. We show (see Theorem 1.2) that this property characterizes uniquely the operators \(\Phi_w\).

1.1.1. Whittaker map. Fix a non-degenerate character \(\Psi\) of \(U^{\text{op}}\). The map
\[
W_\Psi : S_c(X) \to S_c(T), \quad W_\Psi(f)(t) = \int_{U^{\text{op}}} \theta(t)f([u])\Psi^{-1}(u)du,
\]
defines an isomorphism \(S_c(X)_{U^{\text{op}}, \Psi} \simeq S_c(T)\).

We define an action of \(W\) on \(S_c(T)\). For split groups set
\[
w \cdot \varphi(t) = \varphi(t^w).
\]
For quasi-split groups see Definition 5.6.

We define (see 6.1) a \(G \times T\) submodule \(S_0(X)\) that is dense in \(L^2(X)\) and put
\[
S_0(T) = W_\Psi(S_0(X)) \simeq S_0(X)_{U^{\text{op}}, \Psi}.
\]

There is a natural map \(\kappa_\Psi : \text{End}_G(S_0(X)) \to \text{End}_C(S_0(X)_{U^{\text{op}}, \Psi}) = \text{End}_C(S_0(T))\) such that for every \(B \in \text{End}_G(S_0(X))\) the following diagram is commutative.

\[
\begin{array}{ccc}
S_0(X) & \xrightarrow{B} & S_0(X) \\
\downarrow{W_\Psi} & & \downarrow{W_\Psi} \\
S_0(T) & \xrightarrow{\kappa_\Psi(B)} & S_0(T)
\end{array}
\]

We prove in Proposition 6.2 that the map \(\kappa_\Psi\) is injective.

1.1.2. Main Theorem. With this notation we formulate our main result.

**Theorem 1.2.** There exists unique family of unitary operators \(\Phi_w, w \in W\), on \(L^2(X, \omega_X)\), preserving the space \(S_0(X)\) and satisfying:

\[
\begin{cases}
\Phi_w \circ \theta(g, t^w) = \theta(g, t) \circ \Phi_w \quad \forall w \in W, t \in T, g \in G \\
\Phi_w \circ \Phi_w = \Phi_{w_1w_2} \quad \forall w_1, w_2 \in W \\
\kappa_\Psi(\Phi_w)(\varphi) = w \cdot \varphi \quad \forall w \in W, \varphi \in S_0(T)
\end{cases}
\]

(1.3)
Let us sketch the proof.

(1) First consider a quasi-split, almost simple, simply-connected group $G_1$ of rank one. The group $G_1$ is isomorphic to either $\text{Res}_L SL_2$ or $\text{Res}_L SU_3$ for a finite extension $L$ of $F$. Without loss of generality we can assume that $L = F$. In both cases the Weyl group $W = \{e, s\}$ consists of two elements. We shall define the generalized Fourier operator $\Phi_s$, separately for these two cases.

- In the case $G_1 = SL_2$ the set $X$ can be identified with $V - 0$ for a symplectic two dimensional plane $V$. In this case $\Phi_s \in \text{Aut}(L^2(X)) = \text{Aut}(L^2(V))$ is defined to be the classical Fourier transform with respect to the symplectic form on $V$. Theorem 1.2 in this case is proven in Section 3.

- In the case $G_1 = SU_3$, the set $X$ can be identified with the set of non-zero isotropic vectors in a 6 dimensional quadratic space. The treatment of this case is the crux of the paper. In [GK22] we have defined a unitary operator $\Phi \in L^2(X)$ of order 2, commuting with $G_1$ and anti-commuting with $T'$, and provided an explicit formula for the restriction of $\Phi$ to the space $S_c(X)$. We put $\Phi_s = \Phi$ and prove Theorem 1.2 in this case in Section 4.

(2) For a general quasi-split group $G$ and any simple reflection $s$ we, using the results for groups of rank 1, define a unitary involution $\Phi_s \in \text{Aut}(L^2(X))$, satisfying

$$
\begin{cases}
\Phi_s \circ \theta(g, t^s) = \theta(g, t) \circ \Phi_s & \forall t \in T, g \in G \\
\kappa_\varphi(\Phi_s)(\varphi) = s \cdot \varphi & \forall \varphi \in S_0(T)
\end{cases}
$$

(3) For arbitrary $w \in W$ with a presentation $w = s_1 \cdot s_2 \cdot \ldots \cdot s_n$ as a product of simple reflections we define $\Phi_w = \Phi_{s_1} \circ \Phi_{s_2} \ldots \circ \Phi_{s_n}$. Hence the operators $\Phi_w$ are unitary and possess the desired equivariance properties. It remains to prove that $\Phi_w$ does not depend on the presentation. For every $\varphi \in S_0(T)$ one has $\kappa_\varphi(\Phi_w)(\varphi) = w \cdot \varphi$ and so $\kappa_\varphi(\Phi_w)$ does not depend on the presentation of $w$. Since $\kappa_\varphi$ is injective, the operator $\Phi_w$ does not depend on the presentation of $w$ as well. In particular, $\Phi_{w_1} \circ \Phi_{w_2} = \Phi_{w_1 w_2}$ for $w_1, w_2 \in W$ and the operators $\{\Phi_w, w \in W\}$ satisfy 1.3.

**Remark 1.4.** We expect that a similar strategy can be applied to prove Theorem 1.2 for $F = \mathbb{R}$.

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2. ON THE SPACE \( S_0(X) \)

In [BK99] the authors have defined for split groups the spaces

\[
S(X) = \sum_{w \in W} \Phi_w(S_c(X)), \quad S^0(X) = \cap_{w \in W} \Phi_w(S_c(X)).
\]

In particular

\[
S^0(X) \subset S_c(X) \subset S(X) \subset L^2(X, \omega_X)
\]

and the spaces \( S^0(X), S(X) \) are preserved by the family of operators \( \Phi_w, w \in W \).

The space \( S(X) \), called Schwartz space, is potentially important for construction of integral representations of \( L \)-functions.

The description of the Schwartz space \( S(X) \) explicitly is a deep problem. For example for \( G = SL_2 \) one has \( S(X) = S_c(V) \) and for \( G = SU_3 \) the space \( S(X) \) can be identified with the space of smooth vectors in the unitary minimal representation of a group \( SO(8) \) containing \( SU_3 \) inside its Levi subgroup \( GL_1 \times SO(6) \), see [GK22].

The space \( S_0(X) \) in this paper is contained in \( S^0(X) \). Let us highlight its useful properties:

- It is explicitly given as an intersection of kernels of certain partial Mellin transforms.
- The Fourier transforms corresponding to simple reflections preserve this space and can be written as integral operators with explicitly given continuous kernels.
- The family \( \{\Phi_w\} \) is unique for given \( S_0(X) \).

On the other hand this space is not canonical and can easily be replaced by other subspaces in \( S_c(X) \), dense in \( L^2(X, \omega_X) \) and preserved by \( \Phi_w \), for example by \( S^0(X) \).

The space \( S_0(X) \) will be defined separately for the groups of rank one, and, based on this, for general group.

The density of \( S_0(X) \) in \( L^2(X, \omega_X) \) is the consequence of Proposition 2.1 below.

Consider a finite set

\[
\mathbb{B} = \{(L_i, a_i, \chi_i), \quad 1 \leq i \leq k\},
\]

where \( L_i \) is a finite extension of \( F \), \( a_i : L_i^\times \hookrightarrow T \) is an embedding and \( \chi_i \) is a character of \( L_i^\times \). For each \( (L_i, a_i, \chi_i) \) consider a partial Mellin transform \( P_i : S_c(X) \to S^\infty(X) \) defined by

\[
P_i(f) = \int_{L_i^\times} \theta(a_i(y)) f \chi_i(y) d^\times y.
\]

Define \( S_{\mathbb{B}}(X) = \cap_{i=1}^k \text{Ker}(P_i) \). It is a \( G \times T \) invariant subspace of \( S_c(X) \).

The following proposition will be repeatedly used in the paper.

**Proposition 2.1.** The space \( S_{\mathbb{B}}(X) \) is dense in \( L^2(X, \omega_X) \).
proof. Let us prove this first for the case all the characters $\chi_i$ are not unitary. Precisely assume that all $\chi_i$ satisfy $|\chi_i| = |\cdot|^b_i$ with real $b_i \neq 0$ for all $i$. Let $b = \min(|b_i|) > 0$.

To show that the space $S_B(X)$ is dense, assume existence of a non-zero function $f \in S_B(X)^+ \subset L^2(X,\omega_X)$. Since $S_c(X)$ is dense in $L^2(X)$ there exists a function $g \in S_c(X)$ such that $\langle f, g \rangle \neq 0$.

Denote by $\varpi_i$ an uniformizer of $L_i$. For any $n \in \mathbb{N}$ define operators $E_n^i, E_n$ on $S_c(X)$ by $E_n = \Pi_{i=1}^k E_{n_i}$, where

$$E_{n_i}^i = \begin{cases} \text{Id} - \theta(a_i(\varpi_i)^n)\chi_i(\varpi_i^n) & b_i > 0 \\ \text{Id} - \theta(a_i(\varpi_i)^{-n})\chi_i(\varpi_i^{-n}) & b_i < 0 \end{cases}.$$ 

Clearly, $E_n(g) \in S_B(X)$. Set $g_n = g - E_n(g)$. Note that $|\chi_i(\varpi_i)|$ (resp. $|\chi_i(\varpi_i^{-1})|$) is bounded by $q^{-b}$ for $b_i > 0$ (resp. $b_i < 0$) for any $i$. Moreover the action $\theta(a_i(\varpi_i))$ is unitary. This implies $\|g_n\| \leq q^{-nb}(2^k - 1)\|g\|$. Hence

$$0 \neq |\langle g, f \rangle| = |\langle g_n, f \rangle| \leq q^{-nb}(2^k - 1)\|f\| \|f\| \to 0$$

as $n \to \infty$, which is a contradiction.

Now let us treat the general set of characters $\mathbb{B}$. For any compact subset $\mathcal{K}$ in $X$ let $S_{\mathbb{B}}(X;\mathcal{K})$ be the space of functions in $S_{\mathbb{B}}(X)$ supported on $\mathcal{K}$. Obviously, $S_{\mathbb{B}}(X) = \bigcup_{\mathcal{K}} S_{\mathbb{B}}(X;\mathcal{K})$.

Since the action of $T$ on $X$ is free, for any character $\chi$ of $T$ there exists a smooth function $h$ on $X$, such that

$$h([t^{-1}g]) = \chi(t)h([g]), \quad h([g]) \neq 0, \quad \forall [g] \in X, t \in T.$$ 

Multiplication on $h$ defines a $T$-equivariant isomorphism between $S_{\mathbb{B}}(X)$ and $S_{\mathbb{B}'}(X)$, where $\mathbb{B}' = \{\{L_i, a_i, \chi_i \circ a_i\}\}$, which is also homeomorphism between $S_{\mathbb{B}}(X;\mathcal{K})$ and $S_{\mathbb{B}'}(X;\mathcal{K})$ for all compact $\mathcal{K} \subset X$. Hence $S_{\mathbb{B}}(X)$ is dense if and only if $S_{\mathbb{B}}(X)$ is dense in $L^2(X,\omega_X)$. By choosing appropriate $\chi$ we can ensure that $\mathbb{B}'$ does not contain unitary characters. We are done.

\[ \blacksquare \]

3. $G_1 = SL_2$

Let $(V, \langle \cdot, \cdot \rangle_V)$ be a two dimensional symplectic space with the standard basis $e_1, e_2$ such that $\langle e_1, e_2 \rangle_V = 1$.

The group $G_1$ acts on $V$ on the right, preserving the symplectic form. Let $B_1 = T_1 \cdot U_1$ be the Borel group, stabilizing the line $F_{e_2}$. The space $X = U_1 \setminus G_1$ is identified with $[g] \to e_2g$. The $G_1$-invariant measure $\omega_X$ on $X$ is fixed to be the self-dual measure $|dv|$ on $V$ with respect to the additive character $\psi$ and the symplectic form on $V$.

The Fourier transform $\Phi \in \text{Aut}(S_c(V))$ is defined by the formula

$$\Phi(f)(w) = \int_V f(v)\psi((w, v)_V)dv.$$ 

The following properties of $\Phi$ are well-known:
Proposition 3.1.  

1. \( \Phi \) extends to a unitary involution on \( L^2(V, |dv|) = L^2(X, \omega_X) \).
2. \( \theta(t, g) \circ \Phi = \Phi \circ \theta(t^{-1}, g) \) for all \( (t, g) \in T_1 \times G_1 \).

For a function \( f \) on \( X \) the argument will be denoted either as a class \([g]\) or as a vector \((x, y) = xe_1 + ye_2 \in V - 0\).

We define certain typical elements of \( G_1 \):

\[
x(r) = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}, \quad t(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad n_s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

One has \( \alpha(t(a)) = a^2 \) for the unique positive root \( \alpha \) of \( G_1 \) with respect to \( T_1 \).

3.1. The space \( S_0(X) \). Let \( \mathbb{B} \) be the set of two triples

\[
\mathbb{B} = \{(F^\times, t : F^\times \to T_1, \chi_\pm(y) = |y|^{\pm 1})\}.
\]

We define \( S_0(X) \) to be \( S_\mathbb{B}(X) \), see section 2 for the definition. It is obviously a \( G_1 \times T_1 \) representation and is dense in \( L^2(X, \omega_X) \) by Proposition 2.1.

**Proposition 3.2.** The operator \( \Phi \) preserves \( S_0(X) \).

**Proof.** First note, that for any \( f \in S_0(X) \), the function \( \Phi(f) \) belongs to \( S_{\mathbb{B}}(X) \). Indeed, the germ \( [\Phi(f)]_0 \) of \( \Phi(f) \) at zero is constant and equals

\[
[\Phi(f)]_0 = \int \int f(v)dv = \int \int \theta(t(x))f(1, y)|x|d^x(x)dy = \int P(\chi_+)(f)(1, y)dy = 0.
\]

For any character \( \chi \) of \( T_1 \) one has

\[
P(\chi)(\Phi(f))(v) = \int T_1 \theta(t)\Phi(f)(v)\chi(t)dt = \int T_1 \Phi(\theta(t^{-1})f)(v)\chi(t)dt.
\]

Since \( f \) is of compact support, the integral defining \( \Phi(f) \) is taken over a compact set in \( X \), and hence the integral over \( T_1 \) can also be replaced by an integral over a compact set. By interchanging the order of integration we see that if \( f \in \text{Ker} P(\chi^{-1}) \) then \( \Phi(f) \in \text{Ker} P(\chi) \).

Hence for \( f \in S_0(X) \) the function \( \Phi(f) \) belongs to \( S_0(X) \). This proves the Lemma.

\[ \square \]

3.1.1. The Whittaker map. We fix a character \( \Psi \) on \( U_1^{op} \) by \( \Psi(x(r)) = \psi(r) \).

The Whittaker map \( \mathcal{W}_\Psi : S_{\mathbb{C}}(X) \to S_{\mathbb{C}}(T_1) \) is defined by

\[
\mathcal{W}_\Psi(f)(t) = \int_{U_1^{op}} \theta(t)f([u])\Psi^{-1}(u)du.
\]

The map \( \mathcal{W}_\Psi \) defines an isomorphism \( S_0(X)_{U_1^{op}, \Psi} \cong S_0(T_1) \), where \( S_0(T_1) = \mathcal{W}_\Psi(S_0(X)) \), which induces the map

\[
\kappa_\Psi : \text{End}_{G_1}(S_0(X)) \to \text{End}_{\mathbb{C}}(S_0(X)_{U_1^{op}, \Psi}) = \text{End}_{\mathbb{C}}(S_0(T_1)).
\]
Lemma 3.3. $\kappa_\psi$ is injective.

Proof. See the proof of 6.2 for a general quasi-split $G$. \hfill \Box

Definition 3.4. We define an action of $W$ on $S_c(T_1)$ by

$$s \cdot \varphi(t) = \varphi(t^s), \quad \varphi \in S_c(T_1).$$

Proposition 3.5. For any $\varphi \in S_0(T_1)$ one has $\kappa_\psi(\Phi)(\varphi) = s \cdot \varphi$.

Proof. Any function in $S_0(T_1)$ is of the form $W_\Psi(f)$ for $f \in S_0(X)$. It is enough to show that

$$\kappa_\psi(\Phi(f)) = s \cdot f.$$

Indeed, once this is proven one has for $t \in T_1$

$$W_\Psi(\Phi(f))(t) = W_\Psi(\theta(t)\Phi(f))(1) = W_\Psi(\Phi(\theta(t^s)f))(1) = W_\Psi(\theta(t^s)f)(1) = W_\Psi(f)(t^s).$$

There is an injective map with open dense image

$$j : T_1 \times U_1^{op} \rightarrow X, \quad j(t, u) = [t^{-1}u]$$

and the push-forward of the measure $\delta_B(t)dtdu$ on $T_1 \times U_1^{op}$ equals $dv$.

$$\Phi([g]) = \int \int_{U_1^{op} T_1} f([t^{-1}u])\psi([g, [t^{-1}u]]_V)\delta_B(t)dtdu.$$

Hence

$$W_\Psi(\Phi(f))(1) = \int_{U_1^{op}} \Phi(f)([u])\Psi(u)^{-1}du =$$

$$\int_{U_1^{op}} \int_{U_1^{op} T_1} f([t^{-1}u'])\psi([u, [t^{-1}u']]_V)\Psi(u^{-1})\delta_B(t)du' du =$$

$$\int_{U_1^{op} T_1} \left( \int_{U_1^{op}} f([t^{-1}u'])\Psi(u^{-1})du' \right) \psi([1, [t^{-1}u]]_V)\Psi(u)\delta_B(t)dt du =$$

$$\int_{U_1^{op} T_1} W_\Psi(f)(t)\psi([1, [t^{-1}u]]_V)\Psi(u)\delta_B^{1/2}(t)dt du.$$

Put $t = t(b)$ and $u = x(r)$ and notice that $[1, [t^{-1}u]]_V = -br$. Then

$$W_\Psi(\Phi(f))(1) = \int_F \left( \int_F W_\Psi(f)(t(b))\psi(-br)db \right) \psi(r)dr =$$

$$\int \mathcal{F}_\psi(W_\Psi(f))(-r)\psi(r)dr = W_\Psi(f)(1),$$
where \( \mathcal{W}_\psi(f) \) is considered as a function on \( \mathcal{S}_c(F^\times) \) via \( b \mapsto \mathcal{W}_\psi(f)(t(b)) \) and \( \mathcal{F}_\psi : \mathcal{S}_c(F) \to \mathcal{S}_c(F) \) denotes the one-dimensional Fourier transform with respect to \( \psi \) and the self-dual measure \( dx \) on \( F \).

\[ \square \]

Theorem 3.8. There exists a unique unitary operator \( \Phi_s \in \text{Aut}(L^2(X,\omega_X)) \), that preserves the space \( \mathcal{S}_0(X) \) and satisfies

\[
\begin{align*}
\theta(g,t) \circ \Phi_s &= \Phi_s \circ \theta(g,t^*) \quad g \in G_1, t \in T_1 \\
\Phi_s \circ \Phi_s &= \text{Id} \\
\kappa_s(\Phi_s)(\varphi) &= s \cdot \varphi \quad \varphi \in \mathcal{S}_0(T_1)
\end{align*}
\]

(3.9)

Proof. The injectivity of \( \kappa_s \) implies the uniqueness of the operator \( \Phi_s \), hence it is enough to construct such an operator.

We define \( \Phi_s \) to be \( \Phi \). The properties follow from Propositions 3.1, 3.2, 3.5.

\[ \square \]

4. \( G_1 = SU_3 \)

4.1. The structure and compatibility of measures.

4.1.1. The field. Let \( K \) be a quadratic field extension over \( F \) with the Galois involution \( x \mapsto \bar{x} \), the norm \( Nm \) and the trace \( Tr \). We write \( | \cdot |_K \) for the absolute value on \( K \), such that \( |x|_K = |Nm(x)|_F \). We fix an element \( \tau \in \mathcal{O}_F \) such that \( \mathcal{O}_K = \mathcal{O}_F + \sqrt{\tau}\mathcal{O}_F \).

The space \( K \) admits a quadratic form \( x \mapsto Nm(x) \) and the associated bilinear form on \( K \) is \( (x,y) \mapsto Tr(x\bar{y}) \).

We fix on \( K \) a self dual measure \( dx \) with respect to \( \psi \) and \( Nm \). The Fourier transform on \( K \) is denoted by \( \mathcal{F}_\psi,K \), to distinguish it from the Fourier transform \( \mathcal{F}_\psi \) with respect to \( \psi \) and the self-dual measure on \( F \).

4.1.2. The unitary group. Let \( (\mathbb{W}, h) \) be the following Hermitian space

\[ \mathbb{W} = K^3, \quad h(v_1,v_2) = x_1\bar{z}_2 + y_1\bar{y}_2 + z_1\bar{x}_2, \quad v_i = (x_i,y_i,z_i). \]

The group \( G_1 = SU(\mathbb{W}, h) \) is the group of automorphisms of \( \mathbb{W} \), acting on the right, preserving the Hermitian form \( h \) and having determinant 1. Its elements are \( 3 \times 3 \) matrices over \( K \).

We denote by \( B_1 = T_1 \cdot U_1 \) the Borel subgroup of \( G_1 \), preserving the line \( K(0,0,1) \) in \( \mathbb{W} \). The unipotent radical \( U_1 \) is the stabilizer of the vector \( (0,0,1) \). The space \( X = U_1 \backslash G_1 \) is naturally identified with the set \( \mathbb{W}^0 \) of \( h \)-isotropic non-zero vectors in the space \( \mathbb{W} \). We write \( T' \) for the maximal split torus of \( T_1 \).

4.1.3. The measures. The space \( \mathbb{W} \) with \( \dim_F(\mathbb{W}) = 6 \) admits the \( F \)-bilinear form \( \langle v_1,v_2 \rangle = Tr(h(v_1,v_2)) \) and the corresponding quadratic form \( q \) is given by

\[ q(v) = \langle v,v \rangle / 2 = Tr(x\bar{z}) + Nm(y), \quad v = (x,y,z). \]
We fix the self-dual measure $dw$ on $\mathbb{W}$ with respect to $\psi$ and $q$. It gives rise to a measure on the cone $\mathbb{W}^0$ and hence to a measure on $X$ which we denote by $\omega_X$.

We fix bijections

$$x : K \times \sqrt{\tau}F \to U_1^{op}, \quad t : K^\times \to T_1$$

by

$$x(r, y) = \begin{pmatrix} 1 & 0 & 0 \\ -\bar{r} & 1 & 0 \\ -\text{Nm}(r)/2 & y & 1 \end{pmatrix}, \quad t(a) = \begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1}\bar{a} & 0 \\ 0 & 0 & \bar{a}^{-1} \end{pmatrix} \quad a \in K^\times.$$

We also fix a representative $n_s$ of the Weyl element $s$ by

$$n_s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The Haar measures on $K \times \sqrt{\tau}F$ and $K^\times$ define the measures on $U_1^{op}$ and $T_1$ respectively.

By Bruhat decomposition for $G_1$, there is an embedding $j : T_1 \times U_1^{op} \to X$ with dense image, defined by $j(t, u) = [t^{-1}u]$.

It is straightforward to check that for any $f \in \mathcal{S}_c(X)$ one has

$$\int_X f(v)\omega_X(v) = \int_K \int_{\sqrt{\tau}F} \int_{K^\times} f([t(b)^{-1}x(r, y)])|\text{Nm}(b)|^2 d^x b dy dr.$$

The root system with respect to the torus $T'$ is

$$R(G_1, T') = \{ \pm \alpha, \pm 2\alpha \}, \quad \alpha(t(a)) = a, \quad \forall a \in F^\times.$$

The operator $\Phi_s$ for the group $G_1$ is defined using the normalized Radon transform on the cone $X$. Below we recall the definition and the relevant properties. We refer to [GK22] for proofs.

4.2. Mellin transform. Let $\chi$ be a character of $O^\times$, extended to $F^\times$ by setting $\chi(\varpi) = 1$. We write $\chi_s$ for the character $\chi \cdot |^s$ of $F^\times$. The character $\chi_s$ is lifted to the character of $T' \simeq F^\times$ via isomorphism $t(x) \mapsto x$.

Define the Mellin transform $P(\chi, s) : \mathcal{S}_c(X) \to \mathcal{S}^\infty(X)$ along $T'$ by

$$P(\chi, s) = \int_{T'} \theta(t)f_{\chi_s}(t)dt.$$

The image $\mathcal{S}(\chi, s)$ consists of functions $f \in \mathcal{S}^\infty(X)$ satisfying $\theta(t)f = \chi_s^{-1}(t)f$.

The Mellin transform can be also computed on functions on $\mathcal{S}^\infty(X)$, not necessarily of compact support, provided the integral converges.

The following statement is obvious and will be used later.

Lemma 4.1. Let $G : \mathcal{S}_c(X) \to \mathcal{S}(\chi, s)$, such that $G \circ \theta(t) = \theta(t^{-1}) \circ G$ for all $t \in T'$. Then Ker$G$ contains Ker$P(\chi^{-1}, -s)$. 

4.3. The Radon transform. Recall that $X$ can be identified with the space $\mathbb{W}^0$ of non-zero isotropic vectors in $\mathbb{W}$. In this section elements in $X$ will be denoted by $u, v, w \ldots$, isotropic vectors in $\mathbb{W}$.

For any vector $w \in \mathbb{W}^0 = X$ consider an algebraic map 

$$p_w : \mathbb{W}^0 \to F, \quad p_w(v) = \langle v, w \rangle.$$

The measure $\omega_X$ defined above and the measure $dx$ on $F$ give rise to well-defined measure $\omega_{w,a}$ on the fiber $p_w^{-1}(a) = \{ v \in \mathbb{W}^0, \langle v, w \rangle = a \}$ for any $a \in F$.

For any $a \in F$ we define Radon transform $R(a) : \mathcal{S}_c(X) \to \mathcal{S}^\infty(X)$ by

$$R(a)(f)(w) = \int_{p_w^{-1}(a)} f(v)\omega_{w,a}(v).$$

The function $a \mapsto R(a)(f)(w)$ is continuous, of bounded support. The normalized Radon transform on $\mathcal{S}_c(X)$ is defined by

$$\hat{R}(f)(w) = \int_F R(a)(f)(w)\psi(a)da.$$

In addition set

$$R_1(f)(w) = \int_{F^\times} \theta(t(x))f(w)\chi_{K}(x)d^X x.$$

Below we list the properties of the operators $R(a)$ and $\hat{R}$, all proven in [GK22], section 3. The quadratic space $(V_K, q_K)$ in loc. cit. is isomorphic to the quadratic space $(\mathbb{W}, q)$ and the results proven in loc.cit. hold in our setting.

For all $f \in \mathcal{S}_c(X)$, $w \in X$ one has

1. $R(xa)(f)(xw) = |x|^{-1}R(a)(f)(w)$ for all $x \in F^\times$. This implies

$$\hat{R}(f)(xw) = \int_F R(a)(f)(w)\psi(ax)da.$$

2. $R(a) \circ \theta(g,t) = \theta(g,t^{-1}) \circ R(a)$ for $g \in G_1, t \in T'$ and the same is true for $\hat{R}$.

3. There exists a constant $c_{\psi,q}$ such that for $|a|$ small enough one has

$$R(a)(f)(w) = R(0)(f)(w) + c_{\psi,q}\chi_{K}(a)|a|R_1(f)(w).$$

4. The function $x \mapsto \theta(t(x))\hat{R}(f)(w)$ is bounded for $x \in F$.

5. $\hat{R}(f)$ extends to a locally constant function on $\mathbb{W}^0 \cup \{0\}$ whose value at 0 is $\int_X f(v)\omega_X(v)$.

Lemma 4.2. (1) If $f \in \text{Ker} P(\chi_K, z)$ then $\hat{R}(f) \in \text{Ker} P(\chi_K, -z)$ for $Re(z) > 0$. 


(2) Let \( f \in \text{Ker} P(\chi_K, -1) \cap \text{Ker} P(\chi_K, 0) \). For \( w \in X \) the function \( a \mapsto \mathcal{R}(a)(f)(w) \) is of compact support on \( F^\times \).

**Proof.**

(1) The transform \( P(\chi_K, -z)(\hat{\mathcal{R}}(f))(w) \) is well-defined for \( \text{Re}(z) > 0 \) by the property (4). The Lemma 4.1 yields the result.

(2) By properties (1), (2), the map \( \mathcal{R}(0) \) has image in \( S(\chi_0, 1) \) and satisfies the condition of Lemma 4.1. Similarly, the map \( \mathcal{R}(1) \) has image in \( S(\chi_K, 0) \) and satisfies the condition. Hence for \( f \in \text{Ker} P(\chi_K, -1) \cap \text{Ker} P(\chi_K, 0) \) one has \( \mathcal{R}(0)(f) = \mathcal{R}(1)(f) = 0 \). By the property (3), the function \( \mathcal{R}(a)(f)(w) \) vanishes for small \( |a| \) and hence is of compact support on \( F^\times \).

Let us fix terminology for convergence of integrals of locally constant functions, not necessary of compact support, on \( F^\times \). For \( f \in S^\infty(F^\times) \) we say that \( \int_{|x| \leq 1} f(x)d^\times x \) converges absolutely if \( \int_{|x| \leq 1} |f(x)|d^\times x \) converges, it converges if \( \lim_{n \to \infty} \int_{|x| \geq 2^{-n}} f(x)d^\times x \) exists, it stabilizes if the sequence \( \int_{|x| \geq 2^{-n}} f(x)d^\times x \) becomes constant for \( n > N \).

Similarly we say that the integral \( \int_{|x| > 1} f(x)d^\times x \) converges absolutely, (resp. converges or stabilizes) the integral if \( \int_{|x| < 1} f(x^{-1})d^\times x \) converges absolutely, (resp. converges or stabilizes).

Given an integral \( I = \int_{F^\times} f(x)d^\times x \) we say that it stabilizes at zero and converges absolutely at infinity if \( \int_{|x| \leq 1} f(x)d^\times x \) stabilizes and \( \int_{|x| > 1} f(x)d^\times x \) converges absolutely.

For example, for any unitary character \( \chi \) and \( \text{Re}(s) > 0 \) the integral \( \int_{F^\times} \psi(x)|\chi(x)|x^s d^\times x \) stabilizes at infinity and converges absolutely at zero.

**4.4. The operators \( \Phi \).** We define an operator \( \Phi : S_c(X) \to S^\infty(X) \) by

\[
\Phi(f) = \int_{F^\times} \theta(t(x))\hat{\mathcal{R}}(f)(\psi(x^{-1})\chi_K(x)|x|^{-1}d^\times x.
\]

By properties (4), (5) the integral converges absolutely. By property (2) it satisfies the equivariance property for \( G \times T' \).

In [GK22], using the minimal representation for the group \( O(8) \) we proved

**Theorem 4.3.** The operator \( \Phi \)

(1) has its image in the space of functions of bounded support on \( X \).

(2) extends to a unitary involution on \( L^2(X, \omega_X) \),

(3) satisfies \( \theta(g, t) \circ \Phi = \Phi \circ \theta(g, t^*) \) for all \( g \in G_1, t \in T' \).
The operator $\Phi$ is our candidate for Fourier transform. To prove Theorem 1.2 for $G_1$ it remains

- to show that $\Phi$ enjoys the equivariance property with respect to $T_1$,
- to define a space $\mathcal{S}_0(X) \subset \mathcal{S}_c(X)$, preserved by $\Phi$ and dense in $L^2(X, \omega_X)$ and
- to compute $\kappa_{\psi}(\Phi)$ on the space $\mathcal{S}_0(T_1) = \mathcal{W}_{\psi}(\mathcal{S}_0(X))$.

4.5. The space $\mathcal{S}_0(X)$. Define the space $\mathcal{S}_0(X) = \mathcal{S}_{\mathbb{B}}(X)$, where $\mathbb{B}$ is the following finite set of characters of $T$:

$\mathbb{B} = \{\chi_K, \chi_s | \pm 1, |\pm 2\}$.

Proposition 4.4. The operator $\Phi$ preserves $\mathcal{S}_0(X)$.

Proof. We start by showing that for $f \in \mathcal{S}_0(X)$ one has $\Phi(f) \in \mathcal{S}_c(X)$. Since $\Phi(f)$ has bounded support, it is enough to show that the germ $[\Phi(f)]_0$ at zero vanishes.

The operator $\Phi$ can be naturally decomposed as a sum $\Phi = \Phi_1 + \Phi_2$ where

$$\Phi_1(f)(w) = \gamma(\chi_K, \psi) \int_{F^\times} \theta(t(x)) \hat{\mathcal{R}}(f)(w) \chi_K(-x) |x|^{-1} dx,$$

and

$$\Phi_2(f)(w) = \gamma(\chi_K, \psi) \int_{F^\times} \theta(t(x)) \hat{\mathcal{R}}(f)(w) (\psi(x^{-1}) - 1) \chi_K(-x) |x|^{-1} dx.$$

For $f \in \mathcal{S}_0(X)$ one has $\Phi_1(f) = 0$ by Proposition 3.2. Let us show that the germ $[\Phi_2(f)]_0$ is zero for $f \in \mathcal{S}_0(X)$. The function

$$g(x) = \gamma(\chi_K, \psi) (\psi(x^{-1}) - 1) \chi_K(-x) |x|$$

has a bounded support, denote it by $\mathcal{B}$. For $|w|$ small enough, the function $x \mapsto \hat{\mathcal{R}}(f)(xw)$ is constant for $x \in \mathcal{B}$. Hence for $|w|$ small one has

$$\Phi_2(f)(w) = \hat{\mathcal{R}}(f)(w) \int_{F^\times} g(x) dx.$$

By property (5), $\hat{\mathcal{R}}(f)(w) = \int_X f(v) \omega_X(v)$ for $|w|$ small enough. The map $f \mapsto \int_X f(v) \omega_X(v)$ has its image in $\mathcal{S}(\chi_0, |\cdot|^{-2})$. Hence by Lemma 4.1 if $f \in \operatorname{Ker} P(\chi_0, |\cdot|^{-2})$ then $\hat{\mathcal{R}}(f)(w) = 0$ for small $w$ and so $[\Phi_2(f)]_0 = 0$. Hence $\Phi(f)$ is of compact support.

Since $\Phi \circ \theta(t) = \theta(t^{-1}) \circ \Phi$ for $t \in T'$ the properties $f \in \operatorname{Ker} P(\chi, s)$, and $\Phi(f) \in \mathcal{S}_c(X)$ imply $\Phi(f) \in \operatorname{Ker} P(x^{-1}, -s)$, as in Proposition 3.2. This yields the result. \hfill \Box

Proposition 4.5. For $f \in \mathcal{S}_0(X)$ one has

$$\Phi(f)(w) = \int_X f(v) L(\langle v, w \rangle) \omega_X(v),$$
where for $a \in F^\times$

$$L(a) = \gamma(\chi_K, \psi) \int_{F^\times} \psi(ax + x^{-1})\chi_K(-x)|x|d^\times x.$$  

Proof. For $a \in F^\times$, the integral defining $L$ stabilizes both at zero at and infinity. In particular, there exists a compact set $K_1$ in $F^\times$ such that

$$L(a) = \gamma(\chi_K, \psi) \int_{K_1} \psi(ax + x^{-1})\chi_K(-x)|x|d^\times x.$$

For $f \in S_0(X)$, the function $a \mapsto R(a)(f)(w)$ is of compact support on $F^\times$ by Lemma 4.2, part (2). We can assume that the support is contained in $K_1$.

By the Fubini theorem

$$\int_X f(v)L(\langle v, w \rangle)\omega_X(v) = \int_F R(a)(f)(w)\mathcal{L}(a)da = \gamma(\chi_K, \psi) \int_{K_1} \int_{K_1} R(a)(f)(w)\psi(ax)\psi(x^{-1})\chi_K(-x)|x|d^\times x da.$$

We can change the order of integration over compact sets. This gives

$$\gamma(\chi_K, \psi) \int_{K_1} \left( \int_{K_1} R(a)(f)(w)\psi(ax)da \right) \psi(x^{-1})\chi_K(-x)|x|d^\times x da = \gamma(\chi_K, \psi) \int_{F^\times} \theta(t(x))\hat{R}(f)(w)\psi(x^{-1})\chi_K(-x)|x|^{-1}d^\times x da = \Phi(f)(w),$$

as required. \qed

**Proposition 4.7.** One has $\Phi \circ \theta(t^s)(f) = \theta(t) \circ \Phi(f)$ for all $f \in S_0(X)$ and $t \in T_1$.

Proof. This is a straightforward computation and is very similar to the proof of the equivariance property of the classical Fourier transform.

$$\theta(t)\Phi(f)(w) = \delta_B^{1/2}(t) \int_X f(v)L(\langle v, tw \rangle)\omega_X(v).$$

One has $\langle v, tw \rangle = \langle (t^s)^{-1}v, w \rangle$ for all $t \in T_1$. Applying the change of variables $v \mapsto (t^s)^{-1}v$ and taking the measure into account, we get that the integral equals

$$\delta_B^{1/2}(t^s) \int_X f(t_sv)L(\langle v, w \rangle)\omega_X(v) = \Phi(\theta(t^s)f)(w)$$

as required. \qed
4.6. The Whittaker map. It remains to compute $\kappa_\varPsi(\Phi)$.

We fix a character $\Psi$ of $U_1^{\text{op}}$ such that $\Psi(x(r,r')) = \psi(\text{Tr}(r))$. The Whittaker map $\mathcal{W}_\Psi : S_c(X) \to S_c(T_1)$ is defined as in introduction.

**Proposition 4.8.** Let $f \in S_0(X)$.

1. $\mathcal{W}_\Psi(\Phi(f))(1) = \mathcal{W}_\Psi(f)(t(-1))$,  
2. $\mathcal{W}_\Psi(\Phi(f))(t) = \mathcal{W}_\Psi(f)(t(-1)t^*)$.

The proof occupies the rest of this subsection. We start with the following technical Lemmas, whose proofs are postponed to the end of this subsection.

**Lemma 4.9.** For any $x \in F$ and $g \in S_c(K)$ one has
\[
\int \int_{\sqrt{T} F K} g(b)\psi(-x \text{Tr}(b \cdot y)) db dy = |x|^{-1} \int_F g(b) db.
\]

According to Weil, [Wei64] there exists a constant $\gamma(\chi_K, \psi)$, which is a fourth root of unity, satisfying
\[
\int_K \mathcal{F}_{\varPsi,K}(f)(x)\psi(\text{Nm}(x)) dx = \gamma(\chi_K, \psi) \int_K f(x)\psi(-\text{Nm}(x)) dx.
\]

For all $t \in F^\times$ denote by $\psi_t$ the additive character $\psi_t(x) = \psi(tx)$. One has
- $\gamma(\chi_K, \psi_t) = \chi_K(t)\gamma(\chi_K, \psi)$,
- $\gamma(\chi_K, \psi) = 1$ if $K$ is split.

**Lemma 4.11.** For any $g \in S_c(F)$ one has
\[
\int_{K F^\times} \int_F g(x)\psi\left(-\frac{\text{Nm}(r)}{x}\right) \chi_K(x) d^x x \psi(\text{Tr}(r)) dr = \gamma(\chi_K, \psi) \chi_K(-1) \mathcal{F}_\varPsi(g)(1).
\]

**Proof of Proposition 4.8.** It is easy to see that the first part implies the second. Indeed, assuming part (1), for any $t \in T_1$ and $f \in S_0(X)$ one has
\[
\mathcal{W}_\Psi(\Phi)(f)(t) = \mathcal{W}_\Psi(\theta(t)\Phi(f))(1) = \mathcal{W}_\Psi(\Phi(\theta(t^*)f))(1) = \\
\mathcal{W}_\Psi(\theta(t^*)f)(t(-1)) = \mathcal{W}_\Psi(f)(t(-1)t^*).
\]

Using Bruhat decomposition for $G_1$ this equals.
\[
\mathcal{W}_\Psi(\Phi(f))(1) = \int \int \int f([t^{-1}u_1],[u_2]) \mathcal{L}([[t^{-1}u_1],[u_2]]) \Psi(u_2)^{-1} \delta_B(t) du_1 dt du_2 = \\
\int \int \int \psi(t) f([u_2]) \Psi(u_2)^{-1} du_2 \mathcal{L}([[t^{-1}u_1],[1]]) \Psi(u_1) \delta_B^{1/2}(t) dt du_1 = \\
\int \int \mathcal{W}_\Psi(f)(t) \mathcal{L}([[t^{-1}u_1],[1]]) \Psi(u_1) \delta_B^{1/2}(t) dt du_1.
We put\[ t = t(b), b \in K^\times, \quad u_1 = x(r, y), r \in K, y \in \sqrt{\tau}F. \]

To ease notation we write \( f \in S_c(F^\times) \) for the function \( b \mapsto W_\psi(f)(t(b)). \)

One has
\[
\langle [t^{-1}(b) x(r, y)], [1] \rangle = - \text{Tr}(b) \text{Nm}(r)/2 - \text{Tr}(b y).
\]

Hence the above equals
\[
\int_K \int_{\sqrt{\tau}F K^\times} \tilde{f}(b) L(- \text{Tr}(b) \frac{\text{Nm}(r)}{2} - \text{Tr}(b \sqrt{\tau} y)) \psi(\text{Tr}(r)) \text{Nm}(b) |d^x b d r t y|.
\]

Writing explicitly the expression for \( L \) from 4.6 and rearranging the change of integrals this equals
\[
(4.13) \quad \gamma(\chi_K, \psi) \chi_K(-1) \cdot \int \int_{F^\times}
\int \int_{\sqrt{\tau}F K^\times} \left( \tilde{f}(b) \psi(- \text{Tr}(b x \text{Nm}(r)/2)) \right) \psi(- \text{Tr}(b xy)) db dy
\chi_K(x) \psi(x^{-1}) |x| d^x x \psi(\text{Tr}(r)) dr.
\]

We apply Lemma 4.9 to the middle line, i.e. for
\[
g(b) = \tilde{f}(b) \psi(- \text{Tr}(b x \text{Nm}(r)/2)).
\]

Notice that for \( b \in F \) one has \( \text{Tr}(b x \text{Nm}(r)/2) = b x \text{Nm}(r) \). Hence the middle line equals
\[
|x|^{-1} \int_F \tilde{f}(b) \psi(- b x \text{Nm}(r)) db
\]

The integral becomes \( \gamma(\chi_K, \psi) \chi_K(-1) \) times
\[
\int_{r \in K} \int_{b \in F} \int_{x \in F^\times} \psi(- b x \text{Nm}(r)) \chi_K(x) \psi(x^{-1}) d^x x db \psi(\text{Tr}(r)) dr
\]

After the change of variables \( b x \mapsto x^{-1} \) this becomes \( \gamma(\chi_K, \psi) \chi_K(-1) \) times
\[
\int_{r \in K} \int_{x \in F^\times} \left( \int_{b \in F} \tilde{f}(b) \chi_K(b) \psi(b x) db \right) \psi(- \frac{\text{Nm}(r)}{x}) \chi_K(x) d^x x \psi(\text{Tr}(r)) dr =
\]
\[
\gamma(\chi_K, \psi) \chi_K(-1) \int_{r \in K} \int_{x \in F^\times} F_\psi(\tilde{f}(\chi_K))(x) \psi(- \frac{\text{Nm}(r)}{x}) \chi_K(x) d^x x \psi(\text{Tr}(r)) dr.
\]
Applying Lemma 4.11 to $g = \tilde{f}\chi_K$ and the properties of $\gamma(\chi_K, \psi)$ we obtain that $W_\psi(\Phi(f))(1)$ equals

$$(\gamma(\chi_K, \psi)\chi_K(-1))^2 F_\psi(F_\psi(\tilde{f}\chi_K))(1) = \tilde{f}(-1) = W_\psi(f)(t(-1)),$$

as required.

It remains to prove Lemmas.

**Proof of Lemma 4.9.** We fix the isomorphism of vector spaces

$$K \simeq F \oplus F, \quad b_1 + \sqrt{T}b_2 \mapsto (b_1, b_2)$$

which induces the isomorphism $S_c(K) \simeq S_c(F) \otimes S_c(F)$. The self-dual measure on $K$ with respect to $(\psi, \text{Nm})$ is transported under this isomorphism to $|2|^{1/2} db_1 db_2$.

It is enough to prove Lemma for $g = g_1 \otimes g_2$, where $g_1, g_2 \in S_c(F)$, so that $g(b_1 + \sqrt{T}b_2) = g_1(b_1)g_2(b_2)$.

Let us write $y = \sqrt{T}y'$ for $y' \in F$ and $dy = |\tau|^{1/2}dy'$. Then for $b = b_1 + \sqrt{T}b_2$ one has $\text{Tr}(bx\sqrt{T}y') = 2\tau b_2 xy'$.

$$\int_F \int_K g(b)\psi(\text{Tr}(bx\sqrt{T}y))dbdy =$$

$$\int_F g(b_1, b_2)\psi(2\tau b_2 xy)|2\tau| db_1 db_2 dy' = |2\tau| \int_F g_1(b_1)db_1 \int_F F_\psi(g_2)(2\tau xy)dy =$$

$$g_2(0)|x|^{-1} \int_F g_1(b_1)db_1 = |x|^{-1} \int_F g(b)db.$$

**Proof of Lemma 4.11.** Let $c \in F^\times \setminus \text{Nm}(K^\times)$, so that $F^\times = \text{Nm}(K^\times) \cup c\text{Nm}(K^\times)$. The measures $d^x y$ on $K^\times$ and $d^x x$ on $\text{Nm}(K^\times) \subset F^\times$ define Haar measures on the fibers of $\text{Nm}: K^\times \rightarrow F^\times$. All the fibers are compact and have the same measure $C$. By the Fubini theorem for any function $h \in L^1(F^\times)$ one has

$$(4.14) \quad \int_{F^\times} h(x)d^x x = C^{-1} \int_{K^\times} h(\text{Nm}(y)) + h(c\text{Nm}(y))d^x y.$$

Applying this integral over $F^\times$ in the LHS of (4.12) we obtain

$$C^{-1} \int_{K^\times} \int_K g(\text{Nm}(y))\psi(-\text{Nm}(r/y)) - g(c\text{Nm}(y))\psi(-c^{-1}\text{Nm}(r/y))d^x x\psi(\text{Tr}(r))dr.$$
After the change of variables $r \mapsto r \bar{y}$ this equals

$$C^{-1} \times \int_K \left( \int_K g(Nm(y)) \psi(Tr(r \bar{y})) \, dy \right) \psi(-Nm(r)) \, dr -$$

$$\int_K \left( \int_K g(c Nm(y)) \psi(Tr(r \bar{y})) \, dy \right) \psi(-c^{-1} N(r)) \, dr =$$

$$\int_K \mathcal{F}_{K, \psi}(g \circ Nm)(r) \psi(-Nm(r)) \, dr - \int_K \mathcal{F}_{K, \psi}(g_c \circ Nm)(r) \int_K \psi(-c^{-1} Nm(r)) \, dr.$$

where $g_c(x) = g(cx)$ for any $x$. This equals by 4.10

$$C^{-1} \times \int_K \chi_K(-1) \gamma(\chi_K, \psi) \int_K (g \circ Nm) \psi(Nm(r)) \, dr + |c| \int_K (g_c \circ Nm)(r) \psi(c Nm(r)) \, dr.$$

Applying equation 4.14 again this equals

$$\chi_K(-1) \gamma(\chi_K, \psi) \int g(x) \psi(x) \, dx = \gamma(\chi_K, \psi) \chi_K(-1) \mathcal{F}_{\psi}(g)(1).$$

The restriction $W_{\Psi} : \mathcal{S}_0(X) \to \mathcal{S}_c(T)$, whose image we denote by $\mathcal{S}_0(T)$, gives rise to the homomorphism $\kappa_{\Psi} : \text{End}_{\mathbb{C}}(\mathcal{S}_0(X)) \to \text{End}_{\mathbb{C}}(\mathcal{S}_0(T))$.

**Lemma 4.15.** $\kappa_{\Psi}$ is injective.

**Proof.** See the proof of 6.2 for the general case. □

Let us define the action of $W$ on $\mathcal{S}_0(T_1)$.

**Definition 4.16.** The action of $W$ on $\mathcal{S}_0(T_1)$ is defined by

$$s \cdot \varphi(t) = \varphi(t(-1)t^s)$$

**Theorem 4.17.** There exists a unique unitary involution $\Phi_s \in \text{Aut}(L^2(X, \omega_X))$ that preserves the space $\mathcal{S}_0(X)$ and satisfies

$$\begin{cases} 
\theta(g, t) \circ \Phi_s = \Phi_s \circ \theta(g, t^s) & g \in G_1, t \in T_1 \\
\kappa_{\Psi}(\Phi_s)(\varphi) = s \cdot \varphi & \varphi \in \mathcal{S}_0(T_1) 
\end{cases}$$

(4.18)

**Proof.** The injectivity of $\kappa_{\Psi}$ implies uniqueness of such operator, and hence it is enough to construct such $\Phi_s$. We put $\Phi_s = \Phi$. The properties follow from Theorem 4.3, Propositions 4.7, 4.4 and 4.8 part (2). □

5. **QUASI-SPLIT GROUPS**

We recall below the structure of reductive quasi-split groups. Our main reference is [BTS81].
5.1. Relative and absolute root systems. Let $G$ be a reductive, connected, simply-connected quasi-split group over $F$ with a maximal split torus $T'$. We denote by $\text{Lie}(G)$ the Lie algebra of $G$ and by $\text{Ad}$ the adjoint action of $G$ on $\text{Lie}(G)$. Let $T$ be the centralizer of $T'$ and $N$ be the normalizer of $T'$, both defined over $F$.

The root datum of $G$ with respect to $T'$ is a quadruple $(X^*(T'), R, X_*(T'), R^\vee)$, where the set of roots $R \subset X^*(T')$ consists of the weights that appear in the representation $\text{Ad} : T' \to \text{Aut}(\text{Lie}(G))$.

The root system $R$ is not necessarily reduced. For any root $\alpha \in R$, its root ray is defined as $1 \otimes R \cap \mathbb{R}_{\geq 0} \otimes \alpha$, in $\mathbb{R} \otimes X^*(T')$. Each root ray contains one or two elements. We denote by $R$ the set of root rays.

The choice of a Borel subgroup $B$, containing $T$ and defined over $F$, determines the decomposition $R = R^+ \cup R^-$ into the set of positive and negative roots and the subset $\Delta \subset R^+$ of simple roots. We call a root ray positive (resp. negative, resp. simple) if it contains a positive (resp. negative, resp. simple) root.

The groups $G$ and $T$ are split over the separable closure $F_s$ of $F$. There exists a minimal extension $F \subset E \subset F_s$ over which $T$ and hence $G$ splits. Then $E/F$ is Galois. We denote this split $E$-group by $\hat{G}$. It has a root datum $(X^*(T), \hat{R}, X_*(T), \hat{R}^\vee)$. Note that all root rays in $X^*(T) \otimes \mathbb{R}_{\geq 0}$ are singletons.

The Borel subgroup $\hat{B}$ containing $B$ of $\hat{G}$, determines the set $\hat{R}^+$ of positive roots and the set $\hat{\Delta}$ of simple roots. The Galois group $\Gamma = \text{Gal}(E/F)$ acts on $X^*(T), \hat{R}, \hat{R}^+$ and $\hat{\Delta}$.

There is a bijection $\beta \leftrightarrow \hat{\beta}$ between the set $R$ of roots and the set of $\Gamma$ orbits of $\hat{R}$. The restriction of every root in $\hat{R}$ to $T'$ equals to $\beta$.

Definition 5.1. Let $\alpha \in \hat{R}$. The field $L_\alpha = E^{\Gamma_\alpha}$ is called the field of definition of $\alpha$, where $\Gamma_\alpha \in \Gamma$ is the stabilizer of $\alpha$.

Proposition 5.2.  

(1) For any $\gamma \in \Gamma$ and $\alpha \in \hat{R}$ one has $L_{\gamma(\alpha)} = \gamma(L_\alpha)$.

(2) For $\alpha \in \hat{R}$, if $\alpha|_{T'}$ is a divisible root in $R$, then there exist roots $\alpha_1, \alpha_2 \in \hat{R}$ such that

$$\alpha_1|_{T'} = \alpha_2|_{T'} = \alpha/2|_{T'}, \quad \alpha = \alpha_1 + \alpha_2.$$ 

In addition $L_{\alpha_1} = L_{\alpha_2}$ is a quadratic extension of $L_\alpha$.

5.2. The Chevalley-Steinberg pinning. For any $a \in R$ there exists a maximal connected subgroup $U_a$ of $G$, defined over $F$, such that the weights that appear in the representation $\text{Ad} : T' \to \text{Aut}(\text{Lie}(U_a))$ belong to $a$. The group $U_a$ is called the root subgroup corresponding to $a \in R$.

For any simple root ray $\alpha$ in $R$, let $G_\alpha$ be the group generated by $U_a$ and $U_{-a}$. Since the group $G$ is simply-connected, the group $G_\alpha$ is a simply connected group of rank 1 over $F$. We denote by $T_a$ and $T'_a$ the maximal torus and the maximal split torus of $G_\alpha$ respectively. The group $G_\alpha$ in $G$ is $G_\alpha$ considered as a group over $E$. 
The following proposition describes $G_a$ and $\tilde{G}_{a}$.

**Proposition 5.3.** Let $a$ be a root ray. There are two possible cases

- $a = \{\alpha\}$. In this case the group $G_a$ is isomorphic over $E$ to a product of copies of the group $SL_2$, indexed by $\tilde{R}_{\alpha}$.

  There exists an isomorphism $\phi_a : SL_2(L_{\alpha}) \rightarrow G_a$ such that
  $$\phi_a(x(r)) \in U_{-a}, \quad \phi_a(t^i x(r)) \in U_a, \quad \phi_a(n_s) \in N$$

  For any $w \in\tilde{W}$ we define $\tilde{W}_a$ the element of $\tilde{W}$.

- $a = \{\alpha, 2\alpha\}$. In this case the group $G_a$ is isomorphic to a product of copies of $SL_3$ indexed by the set $I$ of subsets $\{\alpha_1, \alpha_2\} \subset \tilde{R}_{\alpha}$, such that $\alpha_1 + \alpha_2 \in \tilde{R}$. The field $L_{\alpha_1} = L_{\alpha_2}$ is a quadratic extension of $L_{\alpha_1 + \alpha_2}$ with a non-trivial automorphism $x \mapsto \bar{x}$. Let $SU_3$ be the group of automorphisms on the Hermitian space $L_3^{1}$ preserving the form $h(x,y,z) = \text{Tr}(\bar{x}z) + Nm(\bar{y}y)$ and having determinant 1. It is a quasi-split group of rank 1 over $L_{\alpha_1 + \alpha_2}$.

  There exists an isomorphism $\phi_a : SU_3(L_{\alpha_1 + \alpha_2}) \rightarrow G_a$ such that
  $$\phi_a(x(r, r')) \in U_{-a}, \quad \phi_a(t^i x(r, r')) \in U_a, \quad \phi_a(n_s) \in N.$$}

From now on we fix a family of isomorphisms $\phi_a, a \in R$ such that $\phi_a$ define a Steinberg-Chevalley pinning of the group $G$. See [BT84], page 78.

**5.3. The Weyl group.** The Weyl group $W$ is isomorphic to $N/T$. For any $a \in R$ the image of the element $n_{sa} = \phi_a(n_s)$ in $W$ is denoted by $s_a$. These elements, called simple reflections, generate $W$.

The roots in the same $W$ orbit have the same field of definition.

For any $w \in W$ we denote by $l(w)$ the length of a reduced presentation of $w$ as a product of simple reflections.

For any $w \in W$ we define $R(w) = R^+ \cap w^{-1}(R^-)$. Then $l(w) = |R(w)|$.

We denote by $w_0$ the longest element of $W$, and by $n_0$ its representative in $N$.

**5.4. The action of $W$ on $S_c(T)$.

**Definition 5.4.** Define for any $w \in W$ the element and

$$t_w = \prod_{a \in R(w)} t_a \in T,$$

where $t_a = \phi_a(t(-1))$ for $a = \{\alpha, 2\alpha\}$ and $t_a = 1$ otherwise.

**Lemma 5.5.**

$$t_{w_2} \cdot (w_2^{-1} t_{w_1} w_2) = t_{w_1} w_2$$

**Proof.** The set $R(w_1 w_2)$ can be written as a disjoint union

$$R(w_1 w_2) = \left(R(w_2) \setminus w_2^{-1} R(w_1)\right) \cup \left(w_2^{-1} R(w_1) \setminus R(w_2)\right).$$

Indeed,

$$R(w_2) \setminus w_2^{-1} R(w_1) = \{\alpha > 0, w_2 \alpha < 0, w_1 w_2 \alpha < 0\},$$

$$w_2^{-1} R(w_1) \setminus R(w_2) = \{\alpha > 0, w_2 \alpha > 0, w_1 w_2 \alpha < 0\}.$$
and the union is $R(w_1w_2)$. Besides $R(w) = -wR(w^{-1})$.

Writing by definition

$$t_{w_2} = \prod_{R(w_2) \setminus -w_2^{-1}R(w_1)} t_a \cdot \prod_{R(w_2) \setminus -w_2^{-1}R(w_1)} t_a$$

and

$$t_{w_1} = \prod_{R(w_1) \setminus -w_2R(w_2)} t_a \cdot \prod_{R(w_1) \setminus -w_2R(w_2)} t_a$$

we conclude that $t_{w_2}w_2^{-1}t_{w_1}w_2 = t_{w_1w_2}$. □

**Proposition 5.6.** The map $W \times \mathcal{S}_c(T) \to \mathcal{S}_c(T)$ defined by

(5.7) \quad $w \cdot \varphi(t) = \varphi(t_w \cdot w^{-1}tw)$

is an action of $W$ on $\mathcal{S}_c(T)$.

**Proof.** For $w_1, w_2 \in W$ one has

$$w_1 \cdot (w_2 \cdot \varphi)(t) = (w_2 \cdot \varphi)(w_1^{-1}tw_1) = \varphi((w_2 \cdot w_2^{-1}w_1w_2) \cdot (w_1w_2)^{-1}t(w_1w_2)),$$

which by Lemma 5.5 equals $w_1w_2 \cdot \varphi(t)$. □

For groups of rank 1 this action was defined in 3.4 and 4.16

6. Generalized Fourier transforms

In this section we generalize Theorems 3.8 and 4.17 that concern the quasi-split groups of $F$-rank one to a general quasi-split group $G$. We keep the notation of Section 5.

For any root ray $a$ of the group $G$ we fix the isomorphisms $\phi_a : G_1 \to G_a$, where $G_1$ is a quasi-split group of rank 1.

To formulate the main result we introduce the spaces $\mathcal{S}_0(X)$, $\mathcal{S}_0(T)$ and the homomorphism $\kappa_\varphi : \text{End}_C(\mathcal{S}_0(X)) \to \text{End}_C(\mathcal{S}_0(T))$.

6.0.1. The space $\mathcal{S}_0(X)$. We define for each positive root ray $a$ a set of triples $\mathbb{B}_a$ as in section 2 as follows.

1. Assume that $a = \{\alpha\}$ and $L_\alpha$ be the field of definition of $\alpha$. Then

$$\mathbb{B}_a = \{(L_\alpha, a_i(x) = \phi_a(t(x)), \chi_\pm(x) = \frac{|x|_L^{\pm 1}}{L_\alpha})\}$$

2. Assume that $a = \{\alpha, 2\alpha\}$ and $L_\alpha \supset L_{2\alpha}$ are the fields of definition of $\alpha$ and $2\alpha$. The set $\mathbb{B}_a$ consists of the triples $(L_i, a_i, \chi_i)$ where

$L_i = L_{2\alpha}, \quad a_i(x) = \phi(a(t(x)), \chi_i \in \{\chi_{L_\alpha}, \chi_{L_\alpha} : \frac{1}{|x|_L^{\pm 1}}, \frac{1}{|x|_L^{\pm 2}}\}.$

**Definition 6.1.** Define $\mathcal{S}_0(X) = \mathcal{S}_B(X)$, where $B = \cup_a \mathbb{B}_a$ and the union is taken over all positive root rays.

In particular $\mathcal{S}_0(X) = \cap_a \mathcal{S}_{B_a}(X)$. The Weyl group acts naturally on the set $\mathbb{B}_a$ by $w(L_\alpha, \phi_a \circ t, \chi_i) = (L_\alpha, \phi_{w(\alpha)} \circ t, \chi_i)$ where $\alpha \in a$. Note that under this action $w(\mathbb{B}_a) = \mathbb{B}_{wa}$.

For groups of rank one, the definition of the space $\mathcal{S}_0(X)$ coincides with the definition given in 3.14 and 4.5.
6.0.2. Whittaker map and the map $\kappa_\Psi$. We define a distinguished non-degenerate character $\Psi : U^\text{op} \to \mathbb{C}$ that is compatible with the fixed family of isomorphisms $\{\phi_a\}$ from section [5].

For a quasi-split group $G_1$ of $F$-rank 1 with Borel subgroup $T \cdot U_1$, we define a complex character $\Psi_1$ of $U_1^{\text{op}}$ by

- $\Psi_1(x(r)) = \psi(\text{Tr}_{L/F}(r))$ if $G_1 = \text{Res}_L SL_2$
- $\Psi_1(x(r,s)) = \psi(\text{Tr}_{K/F}(r))$ if $G_1 = \text{Res}_L SU_3$, corresponding to a quadratic field extension $K/L$.

Let $\Psi$ be the unique character of $U^\text{op}$ such that for every simple root ray $a$ the restriction $\Psi$ to $U_a$ equals $\Psi_a^1 = \Psi_1 \circ \phi_a^{-1}$.

For this $\Psi$ the Whittaker map $\mathcal{W}_\Psi : \mathcal{S}_c(X) \to \mathcal{S}_c(T)$, defined as in the introduction,

$$\mathcal{W}_\Psi(f)(t) = \int_{U^\text{op}} \theta(t) f([u]) \Psi^{-1}(u) du$$

gives rise to an isomorphism $\mathcal{S}_0(X)_{U^\text{op}, \Psi} \simeq \mathcal{S}_0(T)$, where $\mathcal{S}_0(T) = \mathcal{W}_\Psi(\mathcal{S}_0(X))$. This isomorphism induces the map

$$\kappa_\Psi : \text{End}_G(\mathcal{S}_0(X)) \to \text{End}_C(\mathcal{S}_0(X)_{U^\text{op}, \Psi}) = \text{End}_C(\mathcal{S}_0(T))$$

Lemma 6.2. The map $\kappa_\Psi$ is injective.

Proof. Let us show that $\text{Ker} \mathcal{W}_\Psi$ does not contain non-zero $G$-modules. Indeed, assume that $V \subset \text{Ker} \mathcal{W}_\Psi \subset \mathcal{S}_0(X)$ is a non-zero $G$-module. For any character $\chi$ of $T$ the space of coinvariants $\mathcal{S}_0(X)_{T, \chi^{-1}}$ is naturally isomorphic to the normalized principal series representation $\text{Ind}_B^G(\chi)$. The functor of coinvariants induces a map $V_{T, \chi^{-1}} \to \text{Ind}_B^G(\chi)$. For every character $\chi$ in a Zariski-open set one has:

- for some $f \in V$ the Mellin transform $P_\chi(f) = \int_T \theta(t) f \cdot \chi(t) dt \neq 0$,
- the representation $\text{Ind}_B^G(\chi)$ is irreducible.

We pick such $\chi$. Since $f$ does not belong to the kernel of $P_\chi$, so the map $V_{T, \chi^{-1}} \to \text{Ind}_B^G(\chi)$ is non-zero, thus surjective. The functor of coinvariants with respect to $(U^\text{op}, \Psi)$ is exact and hence there is a surjection

$$(V_{T, \chi^{-1}})_{U^\text{op}, \Psi} \to \text{Ind}_B^G(\chi)_{U^\text{op}, \Psi}.$$ 

Since $V \subset \text{Ker} \mathcal{W}_\Psi$, one has $0 = V_{U^\text{op}, \Psi} = (V_{U^\text{op}, \Psi})_{T, \chi^{-1}}$, while $\text{Ind}_B^G(\chi)_{U^\text{op}, \Psi} \neq 0$. This is a contradiction.

Let $\mathcal{B} \in \text{End}_G(\mathcal{S}_0(X))$ such that $\kappa_\Psi(\mathcal{B}) = 0$. Then $\mathcal{W}_\Psi \circ \mathcal{B} = 0$, and $\text{Im}(\mathcal{B})$ is a $G$-module, contained in $\text{Ker} \mathcal{W}_\Psi$ and hence is zero. So $\mathcal{B} = 0$ and $\kappa_\Psi$ is injective.

We have defined all the notation, mentioned in Theorem [1,2]. It states:
There exists a unique family of unitary operators $\Phi_w \in \text{Aut}(L^2(X))$, $w \in W$ that preserves the space $S_0(X)$ and satisfies

\[
\begin{align*}
\theta(g, t) \circ \Phi_w &= \Phi_w \circ \theta(g, t^w) \quad g \in G, t \in T \\
\kappa_\Psi(\Phi_w)(\varphi) &= w \cdot \varphi \quad \varphi \in S_0(T) \\
\Phi_{w_1} \circ \Phi_{w_2} &= \Phi_{w_1w_2} \quad w_1, w_2 \in W
\end{align*}
\]

We begin with the construction of the operators $\Phi_s$ for simple reflections, based on the results for the groups of rank one.

6.0.3. The definition of $\Phi_{s_a}$. The space $L^2(X)$ is the unitary completion $L^2 \cdot \text{ind}^G_1$ of the space $S_0(X) = \text{ind}^G_1$.

For a simple root ray $a$ of $G$ consider a parabolic subgroup $P_a = M_a \cdot U^a$, with the derived group $P'_a = M'_a U^a$, where $M'_a = G_a$ is a semisimple group of rank 1. We denote by $B_a = T_a \cdot U_a$ the Borel subgroup of $G_a$ and put $X_a = U_a \backslash G_a$.

Consider the isomorphism, implied by the transitivity of induction,

$$
\iota_a : L^2(X) \rightarrow L^2 \cdot \text{ind}^G_1 L^2(X_a).
$$

defined by $\iota_a(f)(g)([m]) = f([mg])$.

The isometry $\Phi_s$ on $L^2(X_a)$, defined in sections III and IV gives rise to an isometry on $L^2(X)$ by functoriality of induction. We continue to denote this isometry by $\Phi_{s_a}$.

**Definition 6.4.** The operator $\Phi_{s_a} \in \text{Aut}_G(L^2(X))$ is defined by

$$
\iota_a(\Phi_{s_a}(f))(g) = \Phi_s(\iota_a(f))(g), \quad f \in L^2(X), g \in G.
$$

**Proposition 6.5.** For any simple root ray $a$ the operator $\Phi_{s_a} \in \text{Aut}(L^2(X))$ is a unitary involution satisfying $\theta(g, t) \circ \Phi_{s_a} = \Phi_{s_a} \circ \theta(g, t^a)$.

**Proof.** The only non-trivial statement is the equivariance of $T$ which is enough to prove for $f \in S_0(X)$.

Consider an embedding with dense image

$$
j : T_a \times U_a \hookrightarrow X_a, \quad (t, u) \mapsto t^{-1}n_{s_a}u.
$$

For $f \in S_0(X)$ the Fourier transform is given by

$$
\hat{\Phi}_{s_a}(f)([g]) = \int_{T_a U_a} f(t^{-1}n_{s_a}ug)\mathcal{L}([t^{-1}n_{s_a}u], [1])X_a)\delta_B(t_1)dt_1du.
$$

Here $\mathcal{L} = \psi$ for $a = \{\alpha\}$ and is defined by (4.6) for $a = \{\alpha, 2\alpha\}$.

Assume that $a = \{\alpha\}$. Using (4.7) for $f \in S_c(X)$ one has

$$
\theta(t_1)\Phi_{s_a}(f)([g]) = \delta_B^{1/2}(t_1)\Phi_{s_a}(f)([t_1^{-1}g]) =
$$

$$
= \delta_B^{1/2}(t_1) \int_{T_a U_a} f([t^{-1}n_{s_a}ut_1^{-1}g])\mathcal{L}([1], [t^{-1}n_{s_a}])\delta_B(t)dtdu =
$$

$$
\delta_B^{1/2}(t_1)\delta_{B_a}^{M'_a}(t_1)^{-1} \int_{T_a U_a} f((t_1)^{s_a}s_a^{-1}t^{-1}n_{s_a}ug)\psi([1], [t^{-1}n_{s_a}])\delta_B(t)dtdu =
$$
\[
\Phi_{s_a}(\theta(t_a^s)f)([g]).
\]

We have used the fact that the inner product is \(G_a\) invariant and that
\[
\delta_B^{1/2}(t_1)\delta_{B_a}^{M_a}(t_1)^{-1} = \delta_B^{1/2}(t_1^s).
\]

\[\square\]

**Proposition 6.6.** For any simple root ray \(a\) the operator \(\Phi_{s_a}\) preserves \(S_0(X)\).

**Proof.** We have defined for any root ray \(a\) the set of triples \(B_a\) such that \(S_0(X) \subset S_{B_a}(X) \subset S_c(X)\). In fact \(S_{B_a}(X) = \text{ind}_{P_a}^{G} S_0(X_a)\) which is preserved by \(\Phi_{s_a}\) by Definition 6.1 and by Propositions 3.8, 4.17. In particular, \(\Phi_{s_a}(S_0(X)) \subset S_c(X)\).

For \(f \in S_0(X)\) let us show that \(\Phi_{s_a}(f) \in S_0(X)\). For any triple \((\alpha, \phi_a \circ t, \chi) \in B_a\) denote the Mellin transform by \(P(\chi, \alpha)\).

\[P(\chi, s_a(\alpha))\Phi_{s_a}(f) = 0 \text{ by the equivariance property of } \Phi_{s_a}.
\]

\[\square\]

6.0.4. **The operator \(\kappa_{\Psi}(\Phi_{s_a})\).** In this subsection we compute \(\kappa_{\Psi}(\Phi_{s_a})\) for the character \(\Psi\) defined in 6.0.2.

**Proposition 6.7.** For any \(\varphi \in S_0(T)\) one has

\[\kappa_{\Psi}(\Phi_{s_a})(\varphi) = s_a \cdot \varphi.
\]

**Proof.** We shall show first the statement for \(t = 1\), i.e.

\[\kappa_{\Psi}(\Phi_{s_a})(\varphi)(1) = \theta(t_a)\varphi(1).
\]

Let \(a\) be a positive root ray.

\[\mathcal{W}_{\Psi}(\Phi_{s_a}(f))(1) = \int_{U^{\text{opp}}} \Phi_{s_a}(f)([u])\Psi^{-1}(u)du.
\]

We use decomposition \(U^{\text{opp}} = U^{-a}U_{-a}\) where \(U^{-a}\) is the product of all root subgroups corresponding to the negative root rays, except \(-a\). One has

\[\mathcal{W}_{\Psi}(\Phi_{s_a}(f))(1) = \int_{U^{-a}} \left( \int_{U_{-a}} \Phi_{s_a}(f)([u_1 u_2])\Psi^{-1}(u_1)du_1 \right) \cdot \Psi^{-1}(u_2)du_2.
\]

The character \(\Psi\) restricted to \(U_{-a}\) equals \(\Psi_1^a\) by the definition of \(\Psi\). The inner integral equals

\[\int_{U_{-a}} \Phi_{s_a}(t_a(f)(u_2))([u_1])\Psi_1^a(u_1^{-1})du_1 = \mathcal{W}_{\Psi_1^a}(\Phi_{s_a}(t_a(f)(u_2)))(1).
\]

By Theorems 3.8 and 4.17 it equals to \(\mathcal{W}_{\Psi_1^a}(\theta(t_a)_{-a}(f)(u_2))(1)\).
Thus \( W_\Psi(\Phi_{s_a}(f))(1) \) equals

\[
\int_{U^{-a}} \left( \int_{U^{-a}} (\theta(t_a)\tau_{a}(f))(u_2)([u_3])^{-1}(u_3)du_3 \right) \cdot \Psi^{-1}(u_2)du_2 = W_\Psi(\theta(t_a)f)(1).
\]

For an arbitrary \( t \in T \) one has

\[
\kappa_\Psi(\Phi_{s_a})(\varphi)(t) = \kappa_\Psi(\theta(t)\Phi_{s_a})(\varphi)(1) = \\
\kappa_\Psi(\Phi_{s_a})(\theta(t^{s_a})\varphi)(1) = \theta(t^{s_a})\varphi(1) = \\
\varphi(t^{s_a}) = s_a \cdot \varphi(t)
\]

as required. \( \square \)

Now we are ready to prove Theorem 1.2.

**Proof.** The injectivity of \( \kappa_\Psi \) implies the uniqueness of the family \( \Phi_w, w \in W \).

To prove Theorem it is enough to construct the operators \( \Phi_w \). For any \( w \in W \) there is a presentation \( w = s_{a_1} \cdot \ldots \cdot s_{a_n} \) as a product of simple reflections. We define the operator \( \Phi_w \in \text{Aut}(L^2(X)) \)

\[
\Phi_w(f) = \Phi_{s_{a_1}} \circ \ldots \circ \Phi_{s_{a_n}}.
\]

The operator \( \Phi_w \) is unitary, preserves \( S_0(X) \) and satisfies \( \theta(g, t) \circ \Phi_w = \Phi_w \circ \theta(g, t^w) \) for \( g \in G, t \in T \).

Clearly, \( \kappa_\Psi \) is a homomorphism of algebras. In particular,

\[
kappa_\Psi(\Phi_w)(\varphi) = \kappa_\Psi(\Phi_{s_{a_1}}) \circ \ldots \circ \kappa_\Psi(\Phi_{s_{a_n}})(\varphi) = s_{a_1} \cdot \ldots \cdot s_{a_n} \cdot \varphi = w \cdot \varphi,
\]

and hence \( \kappa_\Psi(\Phi_w) \) does not depend on the presentation of \( w \). Since \( \kappa_\Psi \) is injective, the operator \( \Phi_w \) does not depend on the presentation of \( w \). The property \( \Phi_{w_1w_2} = \Phi_{w_1} \circ \Phi_{w_2} \) is obvious from the definition. \( \square \)

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