Abstract

We exhibit canonical middle-inverse Choice maps within categorical (Free-Variable) Theory of Primitive Recursion as well as in Theory of partial PR maps over Theory of Primitive Recursion with predicate abstraction. Using these choice-maps, defined by µ-recursion, we address the consistency problem for a minimal Quantified extension Q of latter two theories: We prove, that Q’s ∃-defined µ-operator coincides on PR predicates with that inherited from theory of partial PR maps. We strengthen Theory Q by axiomatically forcing the lexicographical order on its ωω to become a well-order: “finite descent”. Resulting theory admits non-infinit PR-iterative descent schema (π) which constitutes Cartesian PR Theory πR introduced in RCF 2.

A suitable Cartesian subSystem of Q + wo(ωω) above, extension of πR “inside” Theory Q + wo(ωω), is shown to admit code self-evaluation: extension of formally partial code evaluation of πR. Appropriate diagonal argument then shows inconsistency of this subSystem and (hence) of its extensions Q + wo(ωω) and ZF.

1 Introduction

We begin with Proof of a local, middle-inverse form ACCmi of—Countable—Choice. This for fundamental Free-Variables (categorical) Theory PR, as well as for Theory P̃RA = PRA of partial maps over Theory
\[ \text{PR}_A = \text{PR} + \text{(abstr) of Primitive Recursion with predicate abstraction} \]
\[ (\chi : A \to 2) \mapsto \text{Object } \{A | \chi\}. \]
Equational (!) Axiom \( \text{ACC}_{\text{mi}} \) is preserved by theory strengthening, and by theory extension—the latter with respect to \( \text{PR}_A \)-defined maps.

[\( \text{AC} \) cannot hold for Theory \( \text{PR}_A \) itself consistently]

What we can prove is “even” middle inverse form \( \text{ACC}_{\text{mi}} \) of \( \text{AC} \), for “classically” quantified Arithmetical Theory \( Q = \text{PR}_A + \exists \forall \), having (possibility of) “discrete” map-definition, via left-total, right-unique binary predicates \( \varphi = \varphi(a, b) : A \times B \to 2 \), a possibility for map-definition typical for set theory(s).

For Ordinal \( \mathbb{N}[\omega] \subseteq \mathbb{N}^* = \omega^\omega \) we recall schema \( (\pi) = (\pi_{n[\omega]} \text{(of finite descent for Complexity Controlled Iteration with complexity values in } \mathbb{N}[\omega], \text{and definition of strengthening } \pi\text{R} = \text{PR}_A + (\pi) \text{ of } \text{PR}_A \text{ : within } \pi\text{R} \text{ the defined-arguments enumerations of its CCI's are forced to become epi, “onto”: These CCI's on-terminate within Theory } \pi\text{R}, \text{ in particular so does formally partial—iterative—code evaluation of Theory } \pi\text{R}, \text{ cf. part RCF} 2. \]

“Critical” Theory, namely Theory \( Q^{\text{wo}} : \hat{\text{PR}}_A \subseteq \text{PR}_A \), enriched by existential Quantification giving “total” predicates \( \exists_n \varphi(a, n) : A \to 2 \) from “total”—\( \text{PR}_A \)—predicates \( \varphi : A \times \mathbb{N} \to 2 \), makes the (canonical) middle-inverse partial maps, middle-inverse to defined-arguments enumerations of \( \pi\text{R}'s \text{CCI's, into “total” maps, maps within } Q^{\text{wo}}. \) Adding these \( Q^{\text{wo}}\)-maps as “total” maps to Theory \( \pi\text{R}, \text{ i.e. forcing by } Q^{\text{wo}}\)-consistent axiom the enriched Theory—a priori only PR monoidal—to become Cartesian, allows for resulting (Cartesian PR) Theory \( \pi\hat{\text{R}} \text{ code self-evaluation } \hat{\varepsilon}(u, a) : \pi\hat{\text{R}} \times \mathbb{X} \to \mathbb{X} \text{ (within } \pi\text{R}). \]

From this then results—by appropriate diagonal argument—inconsistency of \( \pi\hat{\text{R}} \) as well as of its extensions \( Q^{\text{wo}}, \text{ PA + wo}(\omega^\omega) \) and ZF.

## 2 Middle-Inverse Choice Maps in Theories \( \text{PR} \) and \( \hat{\text{PR}}_A \)

**Definition:** For a given map (term) \( f = f(a) : A \to B \) of a (categorical) theory \( T \), a \( T \)-map \( f' : B \to A \) is called a Choice map for \( f \) in the middle-inverse sense, if

\[
\begin{align*}
f &= f \circ f' \circ f : A \to B \overset{f'}{\to} A \\
&= f \circ f' \circ f : A \overset{f'}{\to} B \overset{f}{\to} A
\end{align*}
\]

If the given \( T \) map \( f : A \to B \) is a \( T \)-epi, then obviously \( f' : B \to A \) is a \( T \)-section for \( f \).

**Definition:** A (categorical) Theory \( T \) with terminal Object \( 1 \)—or at least a half-terminal Object \( 1 : A \to 1 \)—is said to admit (middle-inverse) Choice, or to satisfy Axiom \( \text{AC}_{\text{mi}} \) if each \( T \) map \( f : A \to B \) coming with a point \( a_0 : 1 \to A \), admits a middle inverse map \( f' : B \to A \) in the sense above.
Remarks:

- If $T$ satisfies $AC_{mi}$, then each “pointed” $T$-epi is a retraction: $T$ satisfies the (local) Axiom of Choice $AC$. And—dually—each pointed $T$-mono then is a section.

- In set theories, requirement of pointed Domains seems to be redundant, since non-empty sets have points, by extensionality axiom. But are these points available for “construction” below, without (set-theoretical) Axiom $ACC$ of Countable Choice? In our case yes: by the “set-theoretical” $\mu$-operator, available e.g. in $PA$ : “(Classical) $PA = PRA + \exists$”

Countable Choice Theorem for PR and $P\hat{R}_A$:

(i) Fundamental theory $PR$ of Primitive Recursion—Objects: finite (binary bracketed) powers of $N$, not yet formal extensions (abstractions) $\{ A | \chi : A \to 2 \}$ —, admits, within itself, middle-inverse Choice maps $f^{' : B \to A}$ for all of its maps $f : A \to B$.

In particular, all epis of this fundamental theory turn out to be retractions: $PR$ satisfies $AC$ (here $ACC$).

[All Objects $A$ of $PR$ are pointed, by—componentwise defined—zero $0 : 1 \to A$. We just need any point. $PR$ is not a pointed category, since maps are not required to map “canonical” points into canonical ones.]

(ii) Theory $P\hat{R}_A$, of partial PR maps over basic Theory $P\hat{R}_A = PR +$ (abstr) of Primitive Recursion with predicate abstraction, again admits axiom $ACC$ of (Countable) Choice, in the form of middle-inverse partial PR maps to arbitrary partial PR maps.

(iii) Middle-inverse form $AC_{mi}$ of $AC$ is clearly inherited by strengthenings of a theory, because of its purely equational character: To each map is associated a map in the converse direction, with “characteristic” middle-inverse equation—maintained.

(iv) Problem: Does Theory $P\hat{R}_A$ “itself” admit $AC$?

Middle-inverse $f^{' : B \to A}$ to a $P\hat{R}_A$ map $f : A \to B$ is in general not $P\hat{R}_A$. Use of $AC$ in its epis-have-sections form cannot be inherited by $P\hat{R}_A$ from $P\hat{R}_A$ since $P\hat{R}_A$ epis are a priori not $P\hat{R}_A$ epis: a “direct” proof would need $P\hat{R}_A$ p.b.’s to pull back epis into epis, and this is excluded in general, by an argument discussed in part RCF 2.

Proof of assertion (i) by recursive case distinction on the structure of $f : A \to B$ in $PR, A, B$ fundamental, i.e. of form of a (binary bracketed) finite power of object $N$:

- Case of map-constants: All of these come with retractions or with sections, in particular since each of the fundamental (!) Objects $A$ comes with a (componentwise defined) zero $0_A : 1 \to A$. 

3
• Composition \( f = h \circ g : A \to B \to C : f' = \text{def} \, g' \circ h' : C \to B \to A \).

• Cylindrification \( f = (C \times g) : C \times A \to C \times B : f' = \text{def} \, (C \times g') : C \times B \to C \times A \).

• Iteration \( f = g^\delta = g^\delta(a, n) : A \times \mathbb{N} \to A : (\text{id}_A, 0!_A) : A \to A \times \mathbb{N} \) is a section to \( f \).

Proof of assertion (ii): middle-inverse Choice for Theory \( \text{PR}_A \):

For \( f = \langle (d_f, \hat{f}) : D_f \to A \times B \rangle : A \to B \) within \( \text{PR}_A \), we could choose middle-inverse just (graph-) opposite to \( f \), namely

\[
\hat{f}^- = \text{by def} \, \langle (\hat{f}, d_f) : D_f \to B \times A \rangle : B \to A.
\]

But wanted proof of middle-inverse property

\[
f \circ \hat{g} \circ f = f \circ \hat{f}^- \circ f \cong f : A \to B \to A \to B
\]

is more conceptual—and simpler—if we use definition of partial maps inside \( \text{PR}_A \) via \( \mu \)-recursion, cf. RCF 1:

We define our middle-inverse candidate \( g = g(b) : B \to A \) in \( \mu \text{R} \cong \text{PR}_A \) as follows, (essentially) via a (partial) \( \mu \text{R} \)-map

\[
\mu_g = \mu_g(b) = \text{def} \, \mu \{ \hat{a} \in D_f \mid \hat{f}(\hat{a}) \equiv_B b \} : B \to D_f,
\]

this with respect to canonical, CANTOR ordering of Object \( D_f = \{ D \mid \zeta \} \) inherited from \( \mathbb{N} \) via \( D \) fundamental.

Partial map \( g : B \to A \) is then choosen as

\[
g = \text{def} \, d_f \hat{\circ} \mu_g : B \to D_f \to A, \text{ with } b \in B \text{ free :}
\]

\[
g(b) = \text{def} \, d_f(\mu \{ \hat{a} \in D_f \mid \hat{f}(\hat{a}) \equiv_B b \}) : B \to A.
\]

This \( g : B \to A \) is a middle-inverse to \( f : A \to B \), since—preliminary result:

\[
f \circ \hat{g} \circ f \circ d_f = f \circ \hat{g} \circ f \circ d_f(\hat{a})
\]

\[
\cong f \circ \hat{g} \circ \hat{f}(\hat{a})
\]

\[
= f \circ d_f \circ \hat{f}(\hat{a}) \equiv \mu \{ \hat{a}' \in D_f \mid \hat{f}(\hat{a}') \equiv_B \hat{f}(\hat{a}) \}
\]

\[
= \hat{f} \circ (\min \{ \hat{a}' \leq D_f \, \hat{a} \mid \hat{f}(\hat{a}') \equiv_B \hat{f}(\hat{a}) \})
\]

\[
= \hat{f}(\hat{a}) \equiv f \circ d_f(\hat{a}) = f \circ d_f : D_f \to A \to B.
\]

In order to get rid of the leading \( d_f : D_f \to A \) on both sides of the (resulting) \( \text{PR}_A \) equation above, we use the commuting \( \text{PR}_A \) Basic Partial Map diagram of Structure Theorem for \( \text{PR}_A \) out of RCF 1:
In fact, with both “structural” $\mathcal{PR}_A$ equations of the diagram, we get from our $\mathcal{PR}_A$ equation:

\[
\begin{align*}
&f \circ g \circ f \equiv f \circ g \circ \hat{f} \circ d_f \circ \hat{d_f} \equiv f \circ \hat{g} \circ \hat{f} \circ d_f \circ \hat{d_f} \\
&\equiv f \circ \hat{d_f} \circ \hat{d_f} \quad \text{by equation above} \\
&\equiv \hat{f} \circ \hat{d_f} \equiv f : A \rightarrow B \rightarrow A \rightarrow B \quad \text{q.e.d.}
\end{align*}
\]

3 Choice within Classically Quantified Arithmetics

**Define** Theory $Q = \mathcal{PR}_A + \forall \exists !$ as as Cartesian (!) PR extension of $\mathcal{PR}_A$ by Quantification—considered to give $\exists_b \varphi(a, b) : A \rightarrow 2$ and $\forall_b \varphi(a, b) : A \rightarrow 2$ as total maps, this (intuitive) totality formally expressed by—axiomatically maintained—Cartesianness, and by possibility—axiomatically forced either—of map-definition via (formal) unique existence of values to given arguments.

Formally, we define “minimal classical” (categorical) Theory $Q$ by the following additional schemata over $\mathcal{PR}_A$:

- “Quantified” law of excluded middle:

\[
(\text{no-mid}) \quad \varphi = \varphi(a, b) : A \times B \rightarrow 2 \text{ in } Q \\
Q \vdash [\forall_b \varphi(a, b) \lor \exists_b \neg \varphi(a, b)] = \text{true}_A(a) : A \rightarrow 2
\]

- “Discrete” Map definition by unique existence:

\[
(\forall \exists !) \quad \varphi = \varphi(a, b) : A \times B \rightarrow 2 \text{ functional from } A \text{ to } B, \text{ i.e.} \\
Q \vdash (\forall a \in A) (\exists! b \in B) \varphi(a, b) \\
[\text{Unique existence is formalised as usual by a} \\
\text{Free-Variables implication between maps.}] \\
Q \vdash [f_{\varphi(a) \equiv_B b}] = \varphi(a, b) : A \times B \rightarrow 2
\]

Forgoing schema—including its uniqueness clause—then gives, for all $Q$-maps $f, g : A \rightarrow B$:

- argumentwise functionality:

\[
Q \vdash \forall a \exists b \ [f(a) \equiv b], \text{ in FV form, for } a \in A : \\
Q \vdash \exists b [f(a) \equiv b] : A \rightarrow 2, \ a \in A \text{ free}
\]

- argumentwise definition of map-equality:

\[
Q \vdash f = g : A \rightarrow B \quad \text{iff} \\
Q \vdash [f(a) \equiv_B g(a)] : A \rightarrow 2, \ a \in A \text{ free} \quad (\text{Equality Definability}) \\
\text{iff} \quad Q \vdash \forall a [f(a) \equiv_B g(a)] : 1 \rightarrow 2
\]
What we want to show for Theory Q is a (map-theoretical) local version of the Axiom of Choice, AC, necessarily here just—pointed—Countable Choice ACC.

**µ-Inherit Lemma:** Q (and hence Q^wo) inherit PR_A’s µ-operator: for PR_A-predicate \( \varphi = \varphi(a, n) : A \times N \rightarrow 2 \),

\[
Q \vdash \mu^{PR_A} \varphi = \mu^{PR_A} \{n | \varphi(a, n)\} : A \rightarrow N
\]

\[
\cong \mu^Q \{n | \varphi(a, n)\} = \text{by def } \begin{cases} 
\min\{n' | \varphi(a, n')\} \text{ if } \exists n \varphi(a, n) \\
\text{undefined if } \forall n \neg \varphi(a, n)
\end{cases}
\]

: \( A \rightarrow N \)

**Proof:** Asserted partial-map equality \( Q \vdash \mu^Q \varphi \cong \mu^{PR_A} \varphi : A \rightarrow N \) for PR_A-predicates \( \varphi = \varphi(a, n) : A \times N \rightarrow 2 \) is due to the fact that the two µ-recursive (partial) maps are compared—in both directions—by suitable Q-total maps with respect to their graphs, as follows:

Consider—within \( Q \)—defining PR_A-diagram for \( \mu^{PR_A} \varphi(a) : A \rightarrow N \), namely

\[
\begin{array}{c}
\{ (a, n) \in A \times N | \varphi(a, n) \} \\
\downarrow \quad \downarrow \\
A \\
\downarrow \mu^{PR_A} \varphi \\
N
\end{array}
\]

Partial µ-recursive map \( \mu^Q \varphi = \mu^Q \varphi(a) \rightarrow N \) defines—“over” Q—an equal partial map by

\[
\begin{array}{c}
\{ a \in A | \exists n \varphi(a, n) \} \\
\downarrow \downarrow \\
A \\
\downarrow \mu^Q \varphi \\
N
\end{array}
\]

Partial-map-equality \( \mu^{PR_A} \varphi = \mu^Q \varphi \) in \( Q \) (“over Q”) established by Q-maps

\[
i := \ell \circ \subseteq : \{ A \times N | \varphi \} \rightarrow \{ a \in A | \exists n \varphi(a, n) \}
\]

\[
j := (a, \mu^Q \varphi(a)) : \{ a \in A | \exists n \varphi(a, n) \} \rightarrow \{ A \times N | \varphi \}
\]

Both \( i, j \) total Q-maps, since

\[
\mu^Q \varphi(a) = \text{by def } \min\{n' | \varphi(a, n')\} : \{ a \in A | \exists n \varphi(a, n) \} \rightarrow N
\]

is Q-total q.e.d.

This Lemma gives
Middle-Inverse Countable-Choice Theorem for Theory \(Q\):

- Since \(Q\) extends \(PR_A\) and \(PR_A^\#\), it inherits middle-inverse-property from \(PR_A^\#\). In particular for a pointed \(PR_A\)-map \(f = f(a) : A \rightarrow B\) (point \(a_0 : 1 \rightarrow A\) given), \(Q\) inherits earlier partial \(\mu_{PR_A}\) map
  
  \[ f' = f^{-1}(b) =_{by\,def} \text{count}_A(\mu_{PR_A}\{n | f(\text{count}_A(n)) \equiv_B b\}) \]

  \[ \equiv \text{count}_A(\mu_Q\{n | f(\text{count}_A(n)) \equiv_B b\}) : B \rightarrow \mathbb{N} \rightarrow A \]

  as (partial) middle inverse.

  [\text{count}_A(n) : N \rightarrow A\,\text{retractive count, available via point}\,a_0]

- for a (pointed) \(PR_A\)-map \(f : A \rightarrow B\), the \(Q\)-map

\[
 f' = f'(b) : B \rightarrow A \overset{\text{def}}{=} \begin{cases} f^{-1}(b) & \text{if } f^{-1}(b) \text{ defined, i.e.} \\ a_0 & \text{otherwise,} \\ \text{i.e. } & \text{if } \forall n f(\text{count}_A(n)) \neq_B b & \end{cases}
\]

is definitionally complemented into a (“non-constructive”) total \(Q\)-map, \(f' : B \rightarrow A\), middle-inverse to given \(f : A \rightarrow B\) in the sense of schema \(ACC_{mi}\).

Comment: We will not rely on latter middle-inverse \(Q\)-choice map \(f' : B \rightarrow A\)—which involves “\(\forall\)” and the Quantified law of excluded middle.

\(\forall\)-Elimination: For our argument below we may drop formal universal Quantor “\(\forall\)”\, Quantified law of excluded middle, and replace schema (\(\forall\exists!\)) above by schema of map definition by unique value-existence

\[
 \varphi = \varphi(a, b) : A \times B \rightarrow 2 \\
 \text{FV}/\exists! \text{ functional from } A \text{ to } B, \text{ i.e.}
\]

(FV/\exists!)

\[
 Q \vdash (\exists b \in B) \varphi(a, b) : A \rightarrow 2, \ a \in A \text{ free}
\]

\[
 f_{\varphi} = f_{\varphi}(a) : A \rightarrow B, \\
 \text{this map } f_{\varphi} \text{ in } Q \text{ (“again”) characterised by}
\]

\[
 Q \vdash [f_{\varphi}(a) \equiv_B b] = \varphi(a, b) : A \times B \rightarrow 2, \\
 a \in A, \ b \in B \text{ free}
\]

as well as (canonical) map definition via “multivalued” predicate,

\[
 \varphi = \varphi(a, b) : A \times B \rightarrow 2 \text{ in } PR_A,
\]

(FV/\exists)

\[
 Q \vdash (\exists b \in B) \varphi(a, b) : A \rightarrow 2, \ a \in A \text{ free}
\]

\[
 Q \vdash f_{\varphi} = f_{\varphi}(a) =_{\text{def}} \mu_Q\{b \in B \mid \varphi(a, b)\} : A \rightarrow B \text{ (“total”)} \\
 = \mu_{PR_A}\{b \in B \mid \varphi(a, b)\} : A \rightarrow B \text{ in } PR_A < Q \text{ (subSystem)}
\]

This \(Q\)-map \(f_{\varphi}\) is characterised within \(Q\) by

\[
 Q \vdash [f(a) \equiv_B b] = \varphi(a, b) : A \times B \rightarrow 2, \\
 a \in A, \ b \in B \text{ free, and value-minimality:}
\]

\[
 Q \vdash [\varphi(a, b) \Rightarrow f(a) \leq_B b] : A \times B \rightarrow 2
\]
in the order (canonically) inherited by \( B \) (pointed) from that of \( N \), via retraction \( \text{count}_B = \text{count}_B(n) : N \to B \).

So the critical properties of \( Q \) are those of its \textit{existential} Quantification. \textit{First:} this quantification yields \textit{total predicates}, in the formal sense that it leads never out of \textit{Cartesianness}, and \textit{second:} it allows—by sheer (established) \textit{formal existence} of “values”—\textit{definition} of maps via (even “infinite”) argument/value tables.

4 Complexity Controlled Iteration Recalled

Complexity Controlled Iteration—CCI\(_O\)—is \textit{Iteration} of a \textit{predecessor} (endo) step, decreasing \textit{Complexity} of argument—Complexity measured in (a given) \textit{Ordinal} \( O \)—as long as complexity zero is not “yet” reached. \textit{Result} then is the argument \textit{reached}, with complexity zero. We choose here (axis case) \( O := N[\omega] \subset \omega^0 \), the set of \textit{polynomial coefficient} strings (no trailing zeros).

It is highly plausible, and a Theorem in \( PA \)—at least in \( PA + \text{well-order of } \omega^0 \)—that such CCI’s \textit{terminate}, on each initial argument given. So our first step in direction of \textit{Terminating Recursiveness}—strengthening \( PR_A \)—can (and will) be formalisation first of the concept CCI of \textit{Complexity Controlled Iteration} (“over” \( O := N[\omega] \)) and—second—introduction of \textit{axiom} schema for conceiving \textit{weakest} Theory \( \pi R \) (strengthening \( PR_A \) and) admitting \textit{termination} of all these CCI’s.

We attempt to formalise wanted Theory within the \textit{partial-map} framework of theory \( \hat{PR}_A \sqsupseteq PR_A \), which is a \textit{definitional, conservative} extension of Theory \( PR_A \). It contains (Cartesian) \( PR_A \) embedded as a \textit{monoidal PR subCategory}.

\textbf{Definition:} For “\textit{Ordinal}” \( N[\omega] \), schema (CCI) below (quote from part RCF 2) is to \textit{define} a \textit{Complexity Controlled Iteration}—CCI—with \textit{complexity values} in \( N[\omega] \), as a \textit{(formally) partial map} \( \text{wh}[c > 0 \mid p] : A \to A \), definition based on suitable data \( c \) (complexity) and \( p \) (predecessor step) as follows, within any theory \( S \) strengthening \( PR_A \):

\[
\begin{align*}
  c &: A \to O \text{ in } S, \text{ complexity,} \\
  p &: A \to A \text{ S-endo, \ predecessor step,} \\
  S \vdash c(a) > 0_O \implies p(c(a)) < c(a) : A \to 2 \quad \text{(Desc)} \\
  \text{strict descent above complexity zero,} \\
  S \vdash c(a) \equiv 0_O \implies p(a) =_A a : A \to 2 \quad \text{(Stat)} \\
  \text{stationarity at complexity zero}
\end{align*}
\]

(CCI)

\[
\text{wh}[c > 0 \mid p] : A \to A \text{ in } \hat{S} \text{ (partial map):}
\]

\[
\text{wh}[c > 0 \mid p] : A \to A \text{ realises the CCI (as a while loop). As a partial map it is given \textit{defined arguments enumeration}}
\]

\[
\begin{align*}
  d_{\text{wh}[c > 0 \mid p]}(a, n) &= \text{by def } a : \\
  D_{\text{wh}[c > 0 \mid p]} &= \{(a, n) \in A \times N \mid c p^n(a) \equiv 0_O \} \subseteq A \times N \xrightarrow{i} A,
\end{align*}
\]

8
and (calculation) rule
\[
\hat{\text{wh}}[c > 0 | p] = \hat{\text{wh}}[c > 0 | p](a, n) = \text{def} \ p^n(a) : D_{\text{wh}}[c > 0 | p] \to A.
\]

**Comment:** Essential “ingredient” for above iteration \(\hat{\text{wh}}[c > 0 | p]\) is its (formally) partial termination-index
\[
\mu[c > 0 | p] = \mu[c > 0 | p](a) : A \to \mathbb{N} \text{ given as}
\]
\[
\mu[c > 0 | p](a) = \text{def} \ \mu\{n \mid c p^n(a) = 0\} : A \to \mathbb{N},
\]
and as such characterised—as partial map: within \(\hat{S}\)—by
\[
\hat{S} \vdash c p^n(a, \mu[c > 0 | p](a)) = 0, \text{ and}
\]
\[
\mu[c > 0 | p] : A \to \mathbb{N} \text{ (argumentwise) minimal in this regard.}
\]

Partial map
\[
(id_A, \mu[c > 0 | p]) = (A \times \mu[c > 0 | p]) \circ \Delta : A \to A^2 \to A \times \mathbb{N}
\]
is just the—pointwise minimised (“canonical”)—opposite partial map
\[
d^- = d^-_{\text{wh}}[c > 0 | p] : A \to D_{\text{wh}}[c > 0 | p],
\]
opposite to \(d = d_{\text{wh}}[c > 0 | p]\). As opposite, this \(d^-\) has partial section property \(d \circ d^- \subseteq \text{id}_A\) within \(\hat{S}\), maximally.

5 Cartesian Code Self-Evaluation “inside” \(Q^{wo}\)

We question here—on consistency—Theory \(Q^{wo} = \text{def} \ Q + wo(\omega^\omega) = \text{by def} \ PR_A + \forall \exists! + wo(\omega^\omega)\) of classically Quantified Arithmetic with well-ordered \(\omega^\omega\), subsystem of set theory \(ZF\).

We attempt to exhibit a Cartesian subsystem \(\pi \hat{R}\) of \(Q^{wo}\) which admits “total” (i.e. Cartesian) self-evaluation \(\hat{\varepsilon} = \hat{\varepsilon}(u, a) : \pi \hat{R} \times X \to X\). As a consequence, \(Q^{wo}\) will turn out to be inconsistent.

We first form the monoidal closure \(\pi \hat{R} + (d^-_{\text{wh}}[c > 0 | p])\), within \(\hat{\pi} \hat{R}\), of \(\pi \hat{R}\) under all (formally partial) \(\mu\)-recursive maps opposite, (canonically) middle-inverse, to the (PR) defined-arguments enumerations
\[
d^-_{\text{wh}}[c > 0 | p] : D_{\text{wh}}[c > 0 | p] = \{(a, n) \mid c p^n = 0 \in \mathbb{N}[\omega]\} \xrightarrow{a} A
\]
of all CCI’s given by (PR) complexity \(c\) and \(\mathbb{N}[\omega]\)-descending (PR) step \(p : A \to A\).

Interpreted in frame \(Q^{wo}\), these \(d^-_{\text{wh}}[c > 0 | p]\) become total, since
\[
Q^{wo} \vdash \exists n \left[ c p^n(a) = 0 \right] : A \to 2, \text{ (} a \in A \text{ free), hence}
\]
\[
Q^{wo} \vdash d^-_{\text{wh}}[c > 0 | p](a) = \text{by def} \ (a, \mu\{n \mid c p^n(a) = 0\})
\]
\[
= \text{by def} \ (a, \mu^{PR_A}\{n \mid c p^n(a) = 0\})
\]
\[
= (a, \mu^{Q^{wo}}\{n \mid c p^n(a) = 0\}) : (\mu\text{-inherit})
\]
\[
A \to D_{\text{wh}}[c > 0 | p] \text{ total :}
\]
\[
A \to D_{\text{wh}}[c > 0 | p] \subseteq A \times \mathbb{N}
\]
and they are, again within $Q^{\text{wo}}$, sections to their

$$d_{\text{wh}[c>0|p]} : D_{\text{wh}[c>0|p]} \rightarrow A,$$

(not only partial sections, cf. the above).

Categorically, this totality means that the Godement equations hold, “even” when these additional $\overline{\text{PR}}_A$-maps $\overline{d}$ are involved.

So the following Theory $\pi \overline{\text{R}}$, strengthening of monoidal theory $\pi \text{R} + (\overline{d}_{\text{wh}[c>0|p]}) \subseteq \pi \overline{\text{R}}$ by the following two axioms (“schemata”) is—as a subsystem of $Q^{\text{wo}}$—consistent relative to $Q^{\text{wo}}$:

\[(\text{Gode})\quad f : C \rightarrow A, \quad g : C \rightarrow B \implies \pi \overline{\text{R}} + (\overline{d}_{\text{wh}[c>0|p]})\]

For “induced” $(f, g) = \text{by def} (f \times g) \circ \Delta_C:

\pi \overline{\text{R}} \vdash \ell \circ (f, g) = f : C \rightarrow A \times B \xrightarrow{\ell} A \quad \text{and} \quad \pi \overline{\text{R}} \vdash r \circ (f, g) = g : C \rightarrow A \times B \xrightarrow{r} B

[We could add as an axiom section property $d_{\text{wh}} \circ \overline{d}_{\text{wh}[c>0|p]} = \text{id}_A$, given within $Q^{\text{wo}}$ as well, but forcing Cartesianness—standing for totality of all of $\pi \overline{\text{R}}$ maps—will be sufficient for our argument.]

Since evaluation $\varepsilon = \varepsilon(u, a) : \text{PR}_A \times X \rightarrow X$ is defined as a CCI within $\overline{\text{PR}}_A$,

$$d_{\varepsilon} = \overline{d}_{\varepsilon}(u, a) = \text{by def} \left( (u, a), \mu \{ n \mid c_{\text{PR}} e^n(u, a) \neq 0 \} \right) : \pi \overline{\text{R}} \times X \rightarrow D_{\varepsilon} \subseteq (\text{PR}_A \times X) \times \mathbb{N}

is in fact a $\overline{\text{PR}}$-map (considered total as such).

Is it possible to extend this $\overline{\text{PR}}_A$-evaluation $\varepsilon(u, a) : \pi \overline{\text{R}} \times X \rightarrow X$—defined as a $\overline{\text{PR}}_A$-map—into a code self-evaluation

$$\hat{\varepsilon} = \hat{\varepsilon}(u, a) : \pi \overline{\text{R}} \times X \rightarrow X?$$

For this end, let us treat the additional maps

$$d^- = \overline{d}_{\text{wh}[c>0|p]} : A \rightarrow D = D_{\text{wh}[c>0|p]} \subseteq A \times \mathbb{N}$$

as “basic” with respect to the evaluating CCI to be constructed:

$$\hat{c}(u, a) = c_{\hat{\varepsilon}}(u, a) : \pi \overline{\text{R}} \times X \xrightarrow{u} \pi \overline{\text{R}} \rightarrow \mathbb{N}[\omega]

\text{is PR defined from } c_{\text{PR}} : \pi \overline{\text{R}} \times X \rightarrow \pi \overline{\text{R}} \rightarrow \mathbb{N}[\omega]

\text{by adding the clause}

\text{wh = wh}[c > 0|p] \text{ CCI } \implies \hat{\varepsilon}(\overline{d}_{\text{wh}}^-) \neq 1.

Extended evaluation step

$$\hat{\varepsilon} = \hat{\varepsilon}(u, a) : \pi \overline{\text{R}} \times X \rightarrow \pi \overline{\text{R}} \times X$$

then is PR defined from $e : \pi \overline{\text{R}} \times \pi \overline{\text{R}} \rightarrow \pi \overline{\text{R}} \times \pi \overline{\text{R}}$

by addition of (Objectivity) clause

$\text{wh = wh}[c > 0|p] \text{ CCI } \implies \hat{\varepsilon}(\overline{d}_{\text{wh}}^-, a) = \text{def} \left( \overline{\text{id}}^-, \overline{d}_{\text{wh}}(a) \right)$

$\left[ \in \pi \overline{\text{R}} \times (A \times \mathbb{N}) \subseteq \pi \overline{\text{R}} \times X \right]$
(Self-) evaluation $\hat{\varepsilon} = \hat{\varepsilon}(u,a) : \pi\hat{R} \times X \to X$ then is defined—within Cartesian theory $\pi\hat{R}$ itself—by CCI (!)

$\hat{\varepsilon} = \hat{\varepsilon}(u,a) =_{\text{def}} r \circ \text{wh}[\dot{\varepsilon}] (u,a) : \pi\hat{R} \times X \to \pi\hat{R} \times X \xrightarrow{\tau} X$

with $\hat{\Pi}_A$ middle-inverse

$d^{-}_{\dot{\varepsilon}}(u,a) = d^{-}_{\text{wh}[\dot{\varepsilon}>0]}(u,a) : \pi\hat{R} \times X \to D_{\dot{\varepsilon}} = \{(u,a), n) | \dot{\varepsilon} \dot{\varepsilon}^{n}(u,a) \neq 0\}$

a $\hat{\Pi}_A$ (partial) map, more: a $\hat{\Pi}$ map. By Structure Theorem for $\hat{\Pi}_A$, we have for any (partial) $\hat{\Pi}_A$-map $f = ((d_f, \hat{f}) : D_f \to A \times B) : A \to B :$

$\hat{\Pi}_A \vdash f \equiv \hat{f} \circ d_{\dot{\varepsilon}} : A \to D_f \to B.$

So for CCI's:

$\hat{\Pi} \vdash \text{wh}[c > 0 \mid p] = \text{wh} \circ d^{-}_{\text{wh}} : A \to D_{\text{wh}} \to A$

is represented as a $\pi\hat{R}$-map $\text{wh} :$

$\hat{\Pi} \vdash \text{wh}[c > 0 \mid p] \equiv \text{wh} =_{\text{def}} \text{wh} \circ d^{-}_{\text{wh}} : A \to D_{\text{wh}} \to A.$

As a special case then

$\hat{\Pi} \vdash \hat{\varepsilon} = \hat{\varepsilon}(u,a) = \hat{\varepsilon} \circ d^{-}_{\dot{\varepsilon}} (u,a) : \pi\hat{R} \times X \to D_{\dot{\varepsilon}} \xrightarrow{\dot{\varepsilon}} X$

is a (is represented as) map in $\pi\hat{R}$ itself. It constitutes a (code) self-evaluation for Theory $\pi\hat{R}$ since—only further property needed—it is Objective as an evaluation, will say

$$\text{(Obj)} \quad f = f(a) : A \to B \quad \text{a } \pi\hat{R} \text{-map} \quad \hat{\Pi} \vdash \hat{\varepsilon}(\text{r} f \text{\flat}, a) = f(a) : A \to B$$

**Proof:** Objectivity $\pi\Pi_A \vdash \varepsilon(\text{r} f \text{\flat}, a) = f(a) : A \to B$ of “fundamental” evaluation $\varepsilon : \Pi_A \times X \to X$ has been shown in RCF 2, by external $\Pi$ on depth $f$ which relies on Peano Induction (on the iteration counter $n$) in case $(\text{r} f \text{\flat}, a) \in \Pi_A \times X$ of form of an iterated:

$(\text{r} f \text{\flat}, a) = (\text{r} g^{\delta \text{\flat}}, (b; n))$.

[Free-Variables Peano Induction is available in $\Pi_A$ and strengthenings]

Same $\Pi$ argument works in present case of self-evaluation

$\hat{\varepsilon}(u,a) = r \circ \text{wh}[\dot{\varepsilon} > 0 \mid \dot{\varepsilon}] : \pi\hat{R} \times X \to \pi\hat{R} \times X \xrightarrow{\dot{\varepsilon}} X$.

The reason is that the evaluation clause for the additional maps is given as an Objective instance:

$$\hat{\varepsilon}(\text{r} d^{-}_{\text{wh}[c > 0] \mid p} \text{\flat}, a) =_{\text{by def}} (\text{r} id \text{\flat}, d^{-}_{\text{wh}} (a)),$$

$$\hat{\Pi} \vdash \hat{\varepsilon}(\text{r} d^{-}_{\text{wh}[c > 0] \mid p} \text{\flat}, a) = d^{-}_{\text{wh}} (a) :$$

$X \supset A \to D_{\text{wh}} \subset A \times N \subset X$
So Objectivity is preserved by extension of evaluation $\varepsilon$ to $\varepsilon : \pi \mathcal{R} \times \mathbb{X} \to \mathbb{X}$ q.e.d.

But (Objective) code self-evaluation of any Cartesian PR Theory $T$ renders $T$ inconsistent, as we will show in detail—final section—by the “appropriate” diagonal argument.

Since self-evaluating theory $Q_w$ is an extension of inconsistent $\pi \mathcal{R}$, $Q_w$ itself turns out to be inconsistent. So in particular Peano Arithmetic $PA + wo(\omega)$ with the lexicographical order on $\omega = \mathbb{N}^* \supset \mathbb{N}[\omega]$ a well-order, as well as set theory $ZF$ are shown to be inconsistent.

6 Liar via Code Self-Evaluation

Any Code Self-Evaluation family

$$\varepsilon = \varepsilon_{A,B} (u, a) : [A, B]_T \times A \to B$$

of a Cartesian PR Theory $T$ within Theory $T$ itself, which is Objective as (self-) evaluation—see above—establishes a contradiction within $T$, by the ("usual") diagonal argument below: formalisation of “Antinomie Richard”.

Remains to develop that diagonal argument “against” (consistent) code self-evaluation for Theory $T$ in general—skip, if you are used to such diagonal argument –, same argument as in RCF 3: Map-Code Interpretation via Closure.

In presence of such (Objective) self-evaluation family $\varepsilon$ define (anti) diagonal $d : \mathbb{N} \to 2$ within general Cartesian Arithmetical theory $T$:

$$d = \text{def} \ (\neg \circ \varepsilon_{N,2} \circ (\#, \text{id}_N) : \mathbb{N} \to [N, 2]_T \times \mathbb{N} \to 2 \to 2,$$

with $\# = \#(n) : \mathbb{N} \supset [N, 2]_T$ the —isomorphic—PR count of all (internal) predicate codes, of Theory $T$.

As expected in such diagonal argument, we substitute — within Theory $T$—the counting index $q = \text{def} \ #^{-1}(\neg d) = (\neg \circ \#^{-1} \circ \neg d) : 1 \to [N, 2]_T \supset \mathbb{N}$

of $d$’s code, into $T$-map $d : \mathbb{N} \to 2$ itself, and get a “liar” map $\text{liar} : 1 \to 2$, namely

$$T \vdash \text{liar} = \text{def} \ d \circ q : 1 \to \mathbb{N} \to 2$$

$$= \text{by def} \ d \circ \#^{-1} \circ \neg d$$

$$= \text{by def} \ (\neg \circ \varepsilon_{N,2} \circ (\#, \text{id}_N) \circ \#^{-1} \circ \neg d)$$

$$= \neg \circ \varepsilon_{N,2} \circ (\neg d, \#^{-1} \circ \neg d) \quad (T \ Cartesian)$$

$$= \text{by def} \ \neg \circ \varepsilon_{N,2} (\neg d, q)$$

$$= \neg \circ d(q) \quad (Objectivity of \varepsilon)$$

$$\neg \circ d \circ q = \text{by def} \ \neg \text{liar} : 1 \to 2 \to 2.$$
a contradiction, whence

**Conclusion** (again): Code-self-evaluating Theory $\pi \hat{R}$ is **inconsistent** and so are all of its **extensions**, in particular “minimal” Quantified Arithmetical Theory $Q^{\omega_0} = Q + wo(\omega^\omega)$ with $\forall \exists!$ **definition** of maps out of (binary, PR) predicates as well as its **extensions** such as $PA + wo(\omega^\omega)$, and extension $ZF$ of the latter theory.

[Without well-order of “one of the first” countable “Ordinals”, namely of $\omega^\omega$, (countable) well-order which is expressible within the language of first-order set-theory $1ZF$ and already within that of $PA$, the theory of (countable) Ordinals would be rather poor]

7 Discussion

- Our **inconsistency** argument applies to **Peano-Arithmetic**, if this theory is presented as predicate calculus (“full quantification”) for description of Algebra $\&$ Order on $\mathbb{N}$, plus induction schema $P5$, i.e. if $PA$ is conceived as $PR\forall\exists! : PR$ $infinity$ plus “full” (classical) predicate calculus, with “set theoretical” possibility of **map-definition**, see above.

  But following Lawvere—and Goodstein—Algebra $\&$ Order can be expressed by equations, in particular by use of **truncated subtraction** $[m - n] : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, which yields order and equality predicates on $\mathbb{N} \times \mathbb{N}$, as well as (constructive) “existence” of $b$ such that $a + b = c$, namely $b := c - a$ for “given” $a,c \in \mathbb{N}$ satisfying $a \leq c$, see the WIKIPEDIA-article on Peano Arithmetic, and the fact that Peano induction $P5$ can be expressed equationally, within (categorical) Free-Variables Calculus.

  [In $PR_A$, induction axiom $P5$ is a consequence of **uniqueness** of maps defined by the full schema of Primitive Recursion.]

  But in Free-Variables setting—my guess—the PR schema, in form of (special one-fold successor case of) **iteration schema** of Eilenberg $\&$ Elgot, plus Freyd’s **uniqueness schema** for the initialised iterated, are **needed** for (unique) **definition** of the more complex PR maps such as exponentiation, faculty etc. which are classically obtained from addition, multiplication, and order by use of (formal) **existence**.

- On the “constructive” side, Free-Variables categorical Primitive Recursion Theory $PR_A$ above, strengthens into theorie(s) $\pi_0R$ ($O$ an Ordinal extending $\mathbb{N}[\omega]$) of on-terminating (not: “retractively” terminating) **Complexity-Controlled Iterations** with complexity measured in $O$: These theories “just” exclude **infinite descending chains** in “their” Ordinal $O$, and seem therefore to be almost as consistent as **basic** Theory $PR_A$ (conservative extension of fundamental Theory $PR$.) Theories $\pi_0R$ are—“on the other hand”—strong enough to derive their own (Free-Variable) **Consistency formulae**, see part RCF 2 mentioned above.

- **Question:** Does our inconsistency argument equally apply to **Arithmetical first order Elementary Theory of Topoi** $1ET\mathbb{N}$ (Topoi with
NNO) in place of Theory Q? As far as I can see, our argument could possibly be adapted to this case. Theory 1ETTN has two truth-Objects, one arithmetical, \(2 =_{\text{by def}} \{N \mid < 2\} = 1 \oplus 1\), inherited from its sub-System PR\(_A\), as well as its genuine, intuitionistic subobject classifier \(\top\) for its “specific” logic, in particular “receiving” (intuitionistic) Quantifier \(\exists\).

1ETTN admits schema

\[
\varphi = \varphi(a, n) : A \times N \to 2
\]

arithmetical predicate, in PR\(_A\)

\[
\exists_r \varphi = \exists n \varphi(a, n) : A \to \Omega \text{ a (“total”) 1ETTN-map}
\]

—“\(\exists\) fits (already) in Cartesian frame”—

+ universal properties characterising map \(\exists_r \varphi : A \to \Omega\) within 1ETT.

My guess is that 1ETTN further admits schema of \(\exists\)-dominated \(\mu\)-recursion

\[
\varphi(a, n) : A \times N : A \times N \to 2 \text{ in PR\(_A\)}
\]

\[
\mu \varphi(a, n) : A \to \Omega
\]

Formally partial PR\(_A\)-map \(\mu_r \varphi = \mu\{n \mid \varphi(a, n)\} : A \to N\)

“total”, i.e. represented by a 1ETTN map \(\mu_r \varphi : A \to N\)

[ and 1ETTN \vdash \varphi(a, \mu_r \varphi(a)) = \text{true}_A(a) : A \to 2,
\mu_r \varphi : A \to 2 \text{ (argumentwise) minimal in this regard}]

“Latter instance of (overall) defined \(\mu_n \varphi(a, n) : A \to 2\), fits into (given) Cartesian frame of 1ETTN.”

We saw above that we do not need formal universal Quantor “\(\forall\)”, and in particular not Booleanness of Quantification—we could drop schema of Excluded Middle.

So, if 1ETTN should admit latter schema \((\mu)\), of \(\exists\)-dominated totality of \(\mu_r \varphi : A \to N\), then our inconsistency argument would apply to first order arithmetical Theory 1ETT + wo(\(\omega^\omega\)) of Elementary Topoi, with lexicographical Order on \(\omega^\omega\) a well-order.

If so, then the final question is: Do real-life Topoi, i.e. interesting Topoi of sheaves, have an NNO?
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