Well-posedness and Finite Element Approximations for Elliptic SPDEs with Gaussian Noises

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Abstract. We analyze the well-posedness and optimal error estimates of finite element approximations for Dirichlet boundary problems with white or colored Gaussian noises. The covariance operator of the proposed noise need not to be commutative with Dirichlet Laplacian. Through the convergence analysis for a sequence of approximate solutions of SPDEs with the noise replaced by its spectral projections, we obtain covariance operator dependent sufficient and necessary conditions for the well-posedness of the continuous problem. These approximate equations with projected noises are then used to construct finite element approximations, for which we establish a general framework of rigorous error estimates. Based on this framework and with the help of Weyl’s law, we derive optimal error estimates of finite element approximations for elliptic SPDEs driven by power-law noises including white noises. In particular, we obtain 1.5 order of convergence for one dimensional white noise driven SPDE, which improves the existing first order results, and remove a usual infinitesimal factor for higher dimensional problems.

1. Introduction

In recent years, random disturbance as a form of uncertainty has been increasingly considered as an essential modeling factor in the analysis of complex phenomena. Adding such uncertainty to partial differential equations (PDEs) which model such physical and engineering
phenomena, one derives stochastic PDEs (SPDEs) as improved mathematical modeling tools. SPDEs derived from fluid flows and other engineering fields are often assumed to be driven by white noises which have constant power spectral densities [8], while most random fluctuations in complex systems are correlated acting on different frequencies in which case the noises are called colored noises [10].

Elliptic SPDEs driven by white noises and colored noises have been considered by many authors, see e.g. [1, 5, 6] for white noises, [12, 13] for colored noises determined by Riesz-type kernels, [3, 4] for fractional noises, and [15] for power-law noises. When one studies finite element methods for elliptic SPDEs, Green’s function framework is applied. In this framework, one first converts an SPDE into a regularized equation by discretizing the noise with piecewise constant process [1, 4, 5] or Fourier truncation [6], and then considers the finite element approximations of the regularized equation.

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \(\mathcal{O} \subset \mathbb{R}^d\) be a bounded domain with regular boundary \(\partial \mathcal{O}\). The main objective of this study is to investigate the well-posedness and optimal error estimate of finite element approximations for the semilinear elliptic SPDE

\[
\begin{align*}
-\Delta u(x) &= f(u(x)) + \dot{W}^Q(x), \quad x \in \mathcal{O}, \\
u(x) &= 0, \quad x \in \partial \mathcal{O}.
\end{align*}
\]

Here \(u\) is an \(\mathbb{R}\)-valued random field, \(f : \mathbb{R} \to \mathbb{R}\) is a Lipschitz continuous function, and \(\dot{W}^Q\) is a class of centered Gaussian noises with covariance operator \(Q\) including white noises and colored noises. The dimension \(d\) varies depending on the type of noises.

The existence of the unique solution for Eq. (1.1) driven by the white noise, i.e., \(Q = I\), has been established in [2] by converting the problem into an integral equation. In this paper, we establish a covariance operator dependent sufficient and necessary condition for the well-posedness of Eq. (1.1) in Theorem 2.1 through the convergence analysis for a sequence of approximate solutions of SPDEs with the noise replaced by its spectral projections. To the best of our knowledge, this seems to be the first well-posed result for general Gaussian noises driven elliptic SPDEs. The aforementioned integral equation is also used as a tool to derive the error estimates of the numerical approximations for elliptic SPDEs (see e.g. [1, 4, 5, 6]). Similar Green’s function framework is also used to study stochastic evolution equations, see e.g. [3, 17] for parabolic SPDEs and [11, 16] for hyperbolic SPDEs.
Our main purpose is to establish a general framework to analyze the error estimate of finite element approximations for Eq. (1.1) with general Gaussian noises. It is known that the difficulty in the error analysis of finite element method for a stochastic problem is the lack of regularity of its solution. In the evolutionary case, Thomée’s finite element error analysis theory for SPDEs with rough solutions is available [11, 17]. The situation is different in the elliptic case. As shown in [1], the required regularity conditions are not satisfied for the standard error estimates of finite element method. To overcome this difficulty, the authors in [1, 6] consider Eq. (1.1) with \( \dot{W}^Q \) replaced by its piecewise constant approximations and Fourier truncations, respectively. They both assume that the eigenvectors of the Laplacian also diagonalize the covariance operator of the noise. In our framework, we do not need any commutative assumption.

Another advantage of our approach is the optimal error estimate of finite element approximations for Eq. (1.1) with Gaussian noises, including white noises and power-law noises whose covariance operators are functionals of Laplacian [15] as well as other types of colored noises. However, either the piecewise constant approximations or the Fourier truncations of noises in Eq. (1.1) fails to achieve the sharp convergence order, even though the exact solution has the required regularity. For this reason, we turn to the truncated approximations of the noises via spectral projection. Applying Weyl’s law in elliptic eigenvalue theory [7], we obtain optimal finite element error estimates in arbitrary piecewise smooth domains. In particular, we obtain 1.5 order convergence for one dimensional white noise driven SPDE like Eq. (1.1) which improves the existing first order results, and remove a usual infinitesimal factor for higher dimensional problems.

The paper is organized as follows. We give a covariance operator dependent sufficient and necessary condition to ensure the existence of a unique solution for Eq. (1.1), through spectral projections on the noise, and establish its Sobolev regularity in Section 2. The error estimation of the spectral truncations as well as the regularity of the truncated solution are also derived. In Section 3, we construct finite element approximations to the spectral truncated noises driven SPDEs and derive their convergence rate. Previous results then apply to the case of power-law noises.
We end this section by introducing several frequently used notations. For $r \in \mathbb{N}$, we use $(\mathbb{H}^r, \| \cdot \|_r)$ to denote the usual Sobolev space

$$
\mathbb{H}^r := \left\{ v : \| v \|_{\mathbb{H}^r} := \left( \sum_{|k| \leq r} \| D^k v \|_2^2 \right)^{1/2} < \infty \right\}
$$

When $r = 0$, $\mathbb{H}^0 := \mathbb{H}$ is the space of square integrable functions on $\partial \mathcal{O}$, whose inner product and norm are denoted by $(\cdot, \cdot)$ and $\| \cdot \|$, respectively. We also use $\mathbb{H}^1_0$ (resp., $\mathbb{H}_0$) to denote the subspace of $\mathbb{H}^1$ (resp., $\mathbb{H}$) whose elements vanish on $\partial \mathcal{O}$. For $s \in \mathbb{R}$, we use $(\tilde{\mathbb{H}}^s, \| \cdot \|_s)$ to denote the interpolation space

$$
\tilde{\mathbb{H}}^s := \left\{ v : \| v \|_s := \left( \sum_{k \in \mathbb{N}_+} \lambda_k^s (v, \varphi_k)^2 \right)^{1/2} < \infty \right\},
$$

where $\{(\lambda_k, \varphi_k)\}_{k \in \mathbb{N}_+}$ is an eigensystem of the negative Dirichlet Laplacian. When $s \in \mathbb{N}$, it is known (see e.g. [14], Lemma 3.1) that $\tilde{\mathbb{H}}^s$ coincides with the usual Sobolev space $\mathbb{H}^s$ with additional boundary conditions. We denote by $C$ a genetic positive constants which is independent of the number of spectral truncation terms and the mesh size of finite element triangulations and may differ from one place to another.

## 2. Spectral Approximations and Error Estimates

In this section, we prove the existence of a unique solution of Eq. (1.1) through the spectral projection on the noise, and establish its Sobolev regularity. We also derive the error estimation of the spectral truncations as well as the regularity of the truncated solution.

### 2.1. Formulations

Recall that an $\mathbb{H}$-valued random field $u = \{u(x) : x \in \mathcal{O}\}$ is said to be a solution of Eq. (1.1) if

$$
u = A^{-1} f(u) + A^{-1} \tilde{W}^Q, \quad \text{a.s.} \tag{2.1}$$

Here $A^{-1} = (-\Delta)^{-1}$ is the inverse of negative Dirichlet Laplacian.

For general bounded and open domain with piecewise smooth boundary $\partial \mathcal{O}$, $A$ subjects to the homogenous Dirichlet boundary condition, as a self-adjoint operator, has discrete and nonnegative eigenvalues
\( \{ \lambda_k \}_{k=1}^{\infty} \) in an ascending order with finite multiplicity and corresponding smooth eigenvectors \( \{ \varphi_k \}_{k=1}^{\infty} \), which vanish on \( \partial \Omega \) and form a complete orthonormal basis in \( H_0 \) (see e.g. [7]), i.e.,

\[
A \varphi_k = \lambda_k \varphi_k, \quad k \in \mathbb{N}_+.
\]

(2.2)

Moreover, the asymptoticity of these eigenvalues is characterized by Weyl’s law (see e.g. [7]):

\[
\lambda_k \asymp k^{\frac{d}{2}}, \quad \text{as} \quad k \to \infty,
\]

(2.3)

where the notation \( A \asymp B \) means that there exists a generic positive constant \( C \) such that \( C^{-1}A \leq B \leq CA \). The above Weyl’s law is our main tool in the optimal error estimation of finite element approximations for Eq. (1.1) with power-law noises in Section 3.2.

The centered Gaussian noise \( \tilde{W}^Q \) is uniquely determined by its covariance operator \( Q \). Assume that \( Q \) has \( \{ (\sigma_k, \psi_k) \}_{k=1}^{\infty} \) as its eigensystem, i.e.,

\[
Q \psi_m = \sigma_m \psi_m, \quad m \in \mathbb{N}_+.
\]

(2.4)

where \( \{ \psi_k \}_{k=1}^{\infty} \) form a complete orthonormal basis in \( \mathbb{H} \). Based on Karhunen-Loève Theorem, one has the following expansion for the infinite dimensional noise \( \tilde{W}^Q \):

\[
\tilde{W}^Q(\omega) = \sum_{m=1}^{\infty} Q^{\frac{1}{2}} \psi_m \eta_m(\omega), \quad \omega \in \Omega,
\]

(2.5)

where \( \{ \eta_m \}_{m=1}^{\infty} \) are independent and normal random variables.

To ensure the well-posedness of Eq. (1.1), we make the following assumption on \( f \).

**Assumption 2.1.** \( f \) is Lipschitz continuous, i.e.,

\[
\| f \|_{\text{Lip}} := \sup_{u,v \in \mathbb{R}, u \neq v} \frac{|f(u) - f(v)|}{|u - v|} < \infty.
\]

(2.6)

We further assume that the Lipschitz constant \( \| f \|_{\text{Lip}} \) is assumed to be smaller than the positive constant \( \gamma \) in the Poincaré inequality:

\[
\| \nabla v \|^2 \geq \gamma \| v \|^2, \quad \forall \ v \in \mathbb{H}_0^1.
\]

(2.7)

We remark that the well-posedness of Eq. (1.1) is also valid for general assumptions on \( f \) possibly depending on the spatial variable which is proposed in [4, 5], i.e., there exist two positive constants \( L_1 < \gamma \) and \( L_2 \) such that for any \( x \in \Omega \) and any \( u, v \in \mathbb{R} \),

\[
(f(x, u) - f(x, v), u - v) \geq -L_1 |u - v|^2
\]
and
\[ |f(x,u) - f(x,v)| \leq L_2(1 + |u - v|). \]

Moreover, our arguments for spectral projection approximations and finite element approximations, by making use of the method in [4, 5], are also available under the above assumption on \( f \). In that case, all the convergence rates halve.

We also make the following assumption on the noise \( \dot{W}^Q \).

**Assumption 2.2.** There exists a \( \beta \in [0, 2] \) such that
\[
\|A^{\frac{\beta-2}{2}}\|_{L_0^2} < \infty, \tag{2.8}
\]
where \( L_0^2 := HS(Q^{\frac{1}{2}}(H), H) \) denotes the space of Hilbert-Schmidt operators from \( Q^{\frac{1}{2}}(H) \) to \( H \) and \( \| \cdot \|_{L_0^2} \) denotes the corresponding norm.

**Remark 2.1.** \( Q \) is a trace operator if and only if (2.8) holds for \( \beta = 2 \). Another class of noises satisfying Assumption 2.2 are the power-law noises, where \( Q = A^\rho \) for certain \( \rho \in \mathbb{R} \), or equivalently, \( \psi_k = \varphi_k \) and \( \sigma_k = \lambda_k^\rho \) for all \( k \in \mathbb{N} \). In particular, if \( \rho = 0 \), the power-law noise becomes the white noise [6]. As pointed out in [15], power-law noises abound in nature and have been observed extensively in both time series and spatially varying environmental parameters.

### 2.2. Well-posedness and Regularity

The parameter \( \beta \) appeared in (2.8) determines the regularity of \( \dot{W}^Q \). In fact,
\[
\mathbb{E} \left[ \|\dot{W}^Q\|_{\dot{H}^{\beta-2}}^2 \right] = \mathbb{E} \left[ \left\| \sum_{k=1}^{\infty} A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}} \psi_k \eta_k \right\|^2 \right] 
= \mathbb{E} \left[ \sum_{m=1}^{\infty} \left( \sum_{k=1}^{\infty} \left( A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}} \psi_k, \varphi_m \right) \eta_k \right)^2 \right] 
= \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \left( A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}} \psi_k, \varphi_m \right)^2 = \left\| A^{\frac{\beta-2}{2}} \right\|_{L_0^2}^2 < \infty, \tag{2.9}
\]
which shows that \( \dot{W}^Q \in \dot{H}^{\beta-2} \). To ensure that the stochastic convolution \( A^{-1}\dot{W}^Q \) is well defined, i.e., \( \mathbb{E}[\|A^{-1}\dot{W}^Q\|^2] < \infty \), we assume that (2.9) holds at least with \( \beta = 0 \). This condition also turns out to be sufficient to ensure the existence of the unique mild solution for Eq. (1.1) in the next theorem. We can also derive a solution in \( L^p(\Omega; \dot{H}^\beta) \) for \( \beta \in [0, 2] \) and any \( p \geq 1 \) provided (2.9) holds. We don’t need \( \beta > 2 \), since in that case standard finite element theory can work directly.
To prove this result, we use the spectral projection operator $\mathbb{P}_N : H \to V_N = \text{span}\{\varphi_m\}_{m=1}^N$ defined by $(u, v) = (\mathbb{P}_N u, v)$ for any $u \in H$, $v \in V_N$, to approximate the noise $\dot{W}^Q$, i.e., we consider the following approximate equation

$$Au_N = f(u_N) + \mathbb{P}_N \dot{W}^Q, \quad N \in \mathbb{N}_+,$$

with vanishing boundary values. Its mild solution is the solution of

$$u_N = A^{-1}f(u_N) + A^{-1}\mathbb{P}_N \dot{W}^Q, \quad N \in \mathbb{N}_+.$$  \(\text{(2.10)}\)

The above spectral truncation equation is also used in the construction of finite element approximations in Section 3.1.

**Theorem 2.1.** Let $p \geq 1$ and Assumptions 2.1 and 2.2 hold. Eq. (1.1) possesses a unique solution $u \in L^p(\Omega; H^\beta)$.  

**Proof.** We first prove the existence of an $L^p(\Omega; H)$-valued solution. For each $N \in \mathbb{N}_+$, the existence of a unique solution $u_N \in H^1_0$ a.s. of Eq. (2.10) follows from the classical elliptic PDE theory. For $M < N$, set $E_{M,N} := A^{-1}(\mathbb{P}_N \dot{W}^Q - \mathbb{P}_M \dot{W}^Q)$. Then

$$u_N - u_M = A^{-1}(f(u_N) - f(u_M)) + E_{M,N}.$$  

Multiplying the above equation by $-(f(u_N) - f(u_M))$ and applying the Lipschitz condition (2.6) and the Poincaré inequality (2.7), we deduce

$$- ||f||_{\text{Lip}} ||u_N - u_M||^2 \leq - (u_N - u_M, f(u_N) - f(u_M))$$

$$= - (A^{-1}(f(u_N) - f(u_M)), f(u_N) - f(u_M)) - (E_{M,N}, f(u_N) - f(u_M))$$

$$\leq - \gamma ||A^{-1}(f(u_N) - f(u_M))||^2 + ||E_{M,N}|| \cdot ||f(u_N) - f(u_M)||.$$

Using Young-type inequality

$$||\phi_1 + \phi_2||^2 \geq \epsilon ||\phi_1||^2 - \frac{2 - \epsilon}{1 - \epsilon} ||\phi_2||^2, \quad \forall \epsilon \in (0, 1), \ \phi_1, \phi_2 \in H$$

with $\phi_1 = u_N - u_M, \phi_2 = -E_{M,N}$ and $\epsilon = \frac{||f||_{\text{Lip}} + \gamma}{2\gamma},$ we obtain

$$||A^{-1}(f(u_N) - f(u_M))||^2 = ||(u_N - u_M) - E_{M,N}||^2$$

$$\geq \frac{||f||_{\text{Lip}} + \gamma}{2\gamma} ||u_N - u_M||^2 - \frac{3\gamma - ||f||_{\text{Lip}}}{\gamma} ||E_{M,N}||^2.$$  

The average inequality \( a \cdot b \leq \gamma \| f \|_{\text{Lip}} a^2 + \frac{\| f \|_{\text{Lip}}^2}{\gamma - \| f \|_{\text{Lip}}} b^2 \) with \( a = \| u_N - u_M \| \) and \( b = \| E_{M,N} \| \) yields
\[
\| E_{M,N} \| \cdot \| f(u_N) - f(u_M) \| \leq \frac{\| f \|_{\text{Lip}}^2}{\gamma - \| f \|_{\text{Lip}}} \| E_{M,N} \|^2 + \frac{\gamma}{4 \| f \|_{\text{Lip}}^2} \| u_N - u_M \|^2.
\]
Substituting the above two inequalities into (2.11), we deduce
\[
\| u_N - u_M \|^2 \leq 4(3 \gamma^2 - \| f \|_{\text{Lip}} \gamma + \| f \|_{\text{Lip}}^2) \| E_{M,N} \|^2.
\]

The moments property of Gaussian process and calculations similarly to (2.9) give
\[
\mathbb{E} \left[ \| E_{M,N} \|^p \right] = C_p \left( \mathbb{E} \left[ \| E_{M,N} \|^2 \right] \right)^{\frac{p}{2}} = C_p \sum_{k=M+1}^{\infty} \sum_{m=1}^{\infty} (A^{-1/2} \psi_m, \varphi_k)^2,
\]
which tends to zero as \( n, m \to \infty \) under the condition (2.8) with \( \beta = 0 \). As a consequence, \( \{ u_N \} \) is a Cauchy sequence in \( L^p(\Omega; \mathbb{H}) \) hence converges to an element \( u \in L^p(\Omega; \mathbb{H}) \). The existence then follows from taking the limit in Eq. (2.10).

Next we prove the uniqueness. Let \( u, v \) be two solutions of Eq. (2.1). Similar arguments as (2.12), in the proof of the existence, yield
\[
\| u - v \| \leq 4(3 \gamma^2 - \| f \|_{\text{Lip}} \gamma + \| f \|_{\text{Lip}}^2) \| A^{-1} W^Q - A^{-1} W^Q \|^2 = 0,
\]
from which we conclude that \( u = v \).

Finally we prove that \( u \in L^p(\Omega; \mathbb{H}^\beta) \) for any \( p \geq 1 \). The Young inequality yields
\[
\mathbb{E} \left[ \| u \|_{\beta}^p \right] \leq \mathbb{E} \left[ \| f(u) \|_{\beta-2}^p \right] + \left( \mathbb{E} \left[ \| W^Q \|_{\beta-2}^2 \right] \right)^{\frac{p}{2}}.
\]
Since the \( \mathbb{H}^\beta \)-norm is increasing with respect to \( \beta \in [0, 2] \), (2.6) yields
\[
\mathbb{E} \left[ \| f(u) \|_{\beta-2}^p \right] \leq \mathbb{E} \left[ \| f(u) \|_{\beta-2}^p \right] \leq C(1 + \mathbb{E} \left[ \| u \|^p \right]) < \infty.
\]
Substituting (2.9) into the above two inequalities, we conclude that \( \mathbb{E}[\| u \|_{\beta}^p] < \infty \) and we complete the proof. □

2.3. Error Estimates for Spectral Truncations. To derive the Sobolev regularity of the solution \( u_N \), we need the regularity of the spectral truncated noise \( \mathbb{P}_N W^Q \). Since \( \{ \lambda_m \}_{m=1}^{\infty} \) is increasing, for any
\[ \alpha \geq \beta - 2 \text{ and any } N \in \mathbb{N}_+, \text{ we have} \]
\[
\mathbb{E} \left[ \left\| P_N \dot{W}^Q \right\|^2_\alpha \right] = \mathbb{E} \left[ \left\| \sum_{k=1}^\infty A^\frac{\beta}{2} P_N Q^\frac{1}{2} \psi_k \eta_k \right\|^2 \right] \\
= \mathbb{E} \left[ \sum_{m=1}^\infty \left( \sum_{k=1}^\infty (P_N A^\frac{\beta}{2} Q^\frac{1}{2} \psi_k, \varphi_m) \eta_k \right)^2 \right] \\
= \sum_{m=1}^N \sum_{k=1}^\infty \lambda_m^{2-\beta+\alpha} \left( A^\frac{\beta-2}{2} Q^\frac{1}{2} \psi_k, \varphi_m \right)^2 \leq \lambda_N^{2-\beta+\alpha} \left\| A^\frac{\beta-2}{2} \right\|^2_{L^0_2}.
\] (2.13)

To establish the convergence rate for the spectral approximations, we need the error estimate for \( E_N := A^{-1}(I - P_N) \dot{W}^Q \). For any \( \alpha \in [0, \beta] \) and any \( N \in \mathbb{N}_+ \),
\[
\mathbb{E} \left[ \left\| E_N \right\|^2_\alpha \right] = \sum_{m=1}^\infty \sum_{k=1}^\infty \lambda_m^{\alpha-\beta} \left( A^\frac{\beta-2}{2} Q^\frac{1}{2} \psi_k, \varphi_m \right)^2 \leq \lambda_N^{\alpha-\beta} \left\| A^\frac{\beta-2}{2} \right\|^2_{L^0_2}.
\] (2.14)

The above calculations lead to the following error estimation between the solution \( u_N \) of Eq. (2.10) and the solution \( u \) of Eq. (2.1), as well as the Sobolev regularity of \( u_N \) which is needed in the error estimation of finite element approximations.

**Theorem 2.2.** Let \( p \geq 1 \) and Assumptions 2.1 and 2.2 hold. Let \( u \) and \( u_N, N \in \mathbb{N}_+ \), be the solutions of Eq. (2.1) and Eq. (2.10), respectively. Then \( u_N \in L^p(\Omega; H^2_0) \) and there exists a constant \( C \) such that
\[
\mathbb{E} \left[ \left\| u_N \right\|^p_2 \right] \leq C \lambda_N^{(2-\beta)p} (1 + \left\| A^\frac{\beta-2}{2} \right\|^p_{L^0_2}).
\] (2.15)

Assume furthermore that \( f \) has bounded derivatives up to order \( r-1 \) with \( r \geq 2 \), with its first derivative being bounded by \( \gamma \), then \( u_N \in L^p(\Omega; \overline{H}^{r+1}) \) and
\[
\mathbb{E} \left[ \left\| u_N \right\|^p_{r+1} \right] \leq C \lambda_N^{(r+1-\beta)p} (1 + \left\| A^\frac{\beta-2}{2} \right\|^p_{L^0_2}).
\] (2.16)

Moreover,
\[
(\mathbb{E} \left[ \left\| u - u_N \right\|^p \right])^{\frac{1}{p}} \leq C \lambda_N^{\frac{\beta}{N+1}} \left( 1 + \left\| A^\frac{\beta-2}{2} \right\|^p_{L^0_2} \right).
\] (2.17)

**Proof:** We first prove (2.15) and (2.16). Since \( f \) is Lipschitz continuous, there exists a constant \( C \) such that
\[
\left\| u_N \right\|_2 = \left\| f(u_N) + P_N \dot{W}^Q \right\| \leq C (1 + \left\| u_N \right\| + \left\| P_N \dot{W}^Q \right\|).
\]
Taking inner product with \( u_N \) in Eq. (2.10), using integration by part formula and Poincaré inequality (2.7), we obtain

\[
(\gamma - \|f\|_{\text{Lip}})\|u_N\|^2 - |f(0)| \cdot \|u_N\| \\
\leq (\nabla u_N, \nabla u_N) - (f(u_N), u_N) \\
= (\mathbb{P}_N \dot{W}^Q, u_N) \leq \|\mathbb{P}_N \dot{W}^Q\| \cdot \|u_N\|,
\]
from which we obtain

\[
\|u_N\| \leq \frac{|f(0)| + \|\mathbb{P}_N \dot{W}^Q\|}{\gamma - \|f\|_{\text{Lip}}}, \tag{2.18}
\]

We conclude (2.15) by combining the above equations and (2.13) with \( \alpha = 0 \). By recursion, we obtain (2.16) by (2.13) with \( \alpha = r - 1 \).

To prove (2.17), we subtract Eq. (2.10) from Eq. (2.1) and get

\[
u - u_N = A^{-1}(f(u) - f(u_N)) + E_N.
\]

Similarly to (2.12), we have

\[
\|u - u_N\|^2 \leq 4(3\gamma^2 - \|f\|_{\text{Lip}} \gamma + \|f\|_{\text{Lip}}^2)\|E_N\|^2. \tag{2.19}
\]

Substituting the estimations (2.18) and (2.14) for \( E_N \), we obtain (2.17).

\[\square\]

3. Finite Element Approximations and Applications

In this section, we establish the general abstract framework to construct the finite element approximation of the spectral truncated noise driven Eq. (2.10) and derive its error estimates. Then we apply this general framework to the discretization of Eq. (1.1) driven by power-law noises.

3.1. Finite Element Approximations. Let \( \mathcal{T}_h \) be a quasiuniform family of triangulations of \( \mathcal{O} \) with meshsize \( h \in (0, 1) \). Let \( V_h \) consist of all continuous piecewise polynomials of degree \( r \) such that

\[
\inf_{v \in V_h} \|v - v_h\|_{\mathbb{H}^s} \lesssim h^{k-s}\|v\|_{\mathbb{H}^k}, \quad \forall \ v \in \mathbb{H}^k, \ s \leq k \leq r + 1. \tag{3.1}
\]

The variational formulation of Eq. (2.10) is to find a \( u_N \in \mathbb{H}^1_0 \) such that

\[
(\nabla u_N, \nabla v) = (f(u_N), v) + (\mathbb{P}_N \dot{W}^Q, v), \quad \forall \ v \in \mathbb{H}^1_0. \tag{3.2}
\]

Then the finite element approximation to (3.2) is to find \( u_N^h \in V_h \) such that

\[
(\nabla u_N^h, \nabla v) = (f(u_N^h), v) + (\mathbb{P}_N \dot{W}^Q, v), \quad \forall \ v \in V_h. \tag{3.3}
\]
In order to estimate the error \( u_N - u_N^h \), we need the Rietz projection operator \( R_h : \mathbb{H}^1_0(D) \to V_h \) defined by
\[
(\nabla R_h w, \nabla v) = (\nabla w, \nabla v), \quad \forall \ v \in V_h, \ w \in \mathbb{H}^1_0(D).
\]
It is well-known that (see e.g. [14])
\[
\| w - R_h w \| \lesssim h^{r+1} \| w \|_{H^r+1}, \quad \forall \ w \in \mathbb{H}^r_0 \cap \mathbb{H}^{r+1}.
\]

**Theorem 3.1.** Let \( p \geq 1 \) and Assumptions 2.1 and 2.2 hold. Let \( u_N \) and \( u_N^h \) be the solutions of Eq. (2.10) and Eq. (3.3), respectively. Then there exists a constant \( C \) such that
\[
(\mathbb{E} [\| u_N - u_N^h \|^p])^{\frac{1}{p}} \leq Ch^2 \frac{2}{\lambda_N^2} \left( 1 + \| A \|_{L^2} \| u_N - u_N^h \|^2 \right).
\]
Assume furthermore that \( f \) has bounded derivatives up to order \( r - 1 \) for some \( r \geq 2 \), with its first derivative being bounded by \( \gamma \), then
\[
(\mathbb{E} [\| u_N - u_N^h \|^p])^{\frac{1}{p}} \leq Ch^{r+1} \frac{r+1}{\lambda_N^2} \left( 1 + \| A \|_{L^2} \| u_N - u_N^h \|^2 \right).
\]

Proof: It follows from (3.2), Eq. (3.3) and (3.4) that
\[
(\nabla(R_h u_N - u_N^h), \nabla(R_h u_N - u_N^h)) = (f(u_N) - f(u_N^h), R_h u_N - u_N^h).
\]
The Assumptions (2.6) and the average inequality \( a \cdot b \leq \frac{\gamma - \| f \|_{Lip}^2}{2} a^2 + \frac{\| f \|_{Lip}^2}{2(\gamma - \| f \|_{Lip})} b^2 \) with \( a = \| u_N - u_N^h \| \) and \( b = \| u_N - R_h u_N \| \) yield
\[
\| \nabla(R_h u_N - u_N^h) \|^2 = (f(u_N) - f(u_N^h), R_h u_N - u_N) + (f(u_N) - f(u_N^h), u_N - u_N^h) \leq \gamma + \frac{\| f \|_{Lip}^2}{2(\gamma - \| f \|_{Lip})} \| u_N - u_N^h \|^2 + \frac{\| f \|_{Lip}^2}{2(\gamma - \| f \|_{Lip})} \| R_h u_N - u_N \|^2.
\]
Applying projection theorem, Poincaré inequality (2.7) and the standard estimation (3.5) with \( r = 1 \), we have
\[
\| u_N - u_N^h \| \leq C \| u_N - R_h u_N \| \leq C h^2 \| u_N \|_2,
\]
and thus (3.6) holds. By (3.5) and (2.16) in Theorem 2.2, we obtain (3.7).}

Now we state and prove our main result about the error estimate between the exact solution \( u \) of Eq. (1.1) and its finite element approximate solution \( u_N^h \).

\[
(\nabla R_h w, \nabla v) = (\nabla w, \nabla v), \quad \forall \ v \in V_h, \ w \in \mathbb{H}^1_0(D).
\]
Theorem 3.2. Let $p \geq 1$ Assumptions 2.1 and 2.2 hold. Let $u$ and $u_N^h$ be the solutions of Eq. (1.1) and Eq. (3.3), respectively. Then there exists a constant $C$ such that

$$\mathbb{E} \left[ \|u - u_N^h\|^p \right]^{\frac{1}{p}} \leq C \left( N^{-\frac{d}{2}} + h^2 N^{\frac{2r-2}{2}} \right) \left( 1 + \|A\|_{L^0}^2 \right).$$

(3.11)

In particular, if $f$ has bounded derivatives up to order $r - 1$ for some $r \geq 2$ with its first derivative being less than $\gamma$,

$$\mathbb{E} \left[ \|u - u_N^h\|^p \right]^{\frac{1}{p}} \leq C \left( N^{-\frac{d}{2}} + h^{r+1} N^{\frac{2r+2}{2}} \right) \left( 1 + \|A\|_{L^0}^2 \right).$$

(3.12)

Proof. The estimations (3.11)–(3.12) follows immediately from (2.17) in Theorem 2.2 and (3.6)–(3.7) in Theorem 3.1 as well as Weyl’s law (2.3).

Remark 3.1. When $h = O(N^{-\frac{d}{2}})$, we obtain the optimal convergence rate, independent of the choice of $r$:

$$\mathbb{E} \left[ \|u - u_N^h\|^p \right]^{\frac{1}{p}} \leq C h^\beta \left( 1 + \|A\|_{L^0}^2 \right),$$

which coincides with the regularity established in Theorem 2.1.

We will see in the next subsection, in the power-law noises case, the finite element approximations can be super-convergent, in the sense that the order of convergence removes a usual infinitesimal factor appearing in the regularity of the solution.

3.2. Applications to Power-law Noises. In this subsection we apply the previous results to Eq. (1.1) with power-law noises, i.e., $Q = A^\rho$ for some $\rho \in \mathbb{R}$.

Combing Theorem 2.1, Theorem 2.2 and Theorem 3.2, we obtain the following well-posed and convergent results for power-law noise driven Eq. (1.1).

Theorem 3.3. Let $p \geq 1$ and Assumption 2.1 hold.

(1) There exists a unique mild solution of power-law noise driven Eq. (1.1) if and only if $\rho < 2 - \frac{d}{2}$. Moreover, $u \in \mathbb{H}^{2-\frac{d}{2}, -\rho-\epsilon}$ a.s. for any positive $\epsilon$.

(2) Suppose that $-\frac{d}{2} < \rho < 2 - \frac{d}{2}$. Then

$$\mathbb{E} \left[ \|u - u_N^h\|^p \right]^{\frac{1}{p}} \leq C \left( N^{\frac{2r-2}{2}} + h^{r+1} N^{\frac{2r+2}{2}} \right).$$

(3.14)

If, in addition, $f$ has bounded derivatives up to order $r - 1$ for some $r \geq 2$ with its first derivative being less than $\gamma$, then

$$\mathbb{E} \left[ \|u - u_N^h\|^p \right]^{\frac{1}{p}} \leq C \left( N^{\frac{2r-2}{2}} + h^{r+1} N^{\frac{2r+2}{2}} \right).$$

(3.15)
Proof. It suffices to verify that the conditions of Theorem 2.1, Theorem 2.2 and Theorem 3.2 hold. Set $Q = A^\rho$, we get, by Weyl’s law (2.3),

$$\|A^{\beta-2}\|_{L_2^2}^2 = \|A^{\beta-2+\rho}\|_{HS}^2 = \sum_{k=1}^{\infty} \lambda_k^{\beta-2+\rho} \asymp \sum_{k=1}^{\infty} k^{2(\beta-2+\rho)/d},$$  

The above series converges if and only if \(\beta < 2 - \frac{d}{2} - \rho\), which is the condition (2.8) in Theorem 2.1. Then (1) follows from Theorem 2.1.

Applying Weyl’s law (2.3), we deduce from (2.19) in Theorem 2.2 and (2.14) with \(\alpha = 0\),

$$\mathbb{E}[\|u - u_N\|^p] \leq \mathbb{E}[\|E_N\|^p] = \left(\sum_{k=N+1}^{\infty} \lambda_k^{\rho-2}\right)^{\frac{p}{2}} \asymp N^{(\frac{p\rho}{2} + \frac{1}{2})p}.$$

Analogously, by (3.10) in Theorem 3.1 and (2.13) with \(\alpha = 0\),

$$\mathbb{E}[\|u - u_N^h\|^p] \lesssim h^{2p} \mathbb{E}[\|u_N\|^p] \lesssim h^{2p} \left(\sum_{k=1}^{\infty} \lambda_k^\rho\right)^{\frac{p}{2}} \asymp h^{2p} \left(\sum_{k=1}^{\infty} k^{2p}\right)^{\frac{p}{2}}.$$

Then (3.14) follows immediately from the elementary inequality

$$\sum_{k=1}^{\infty} k^{\delta} \asymp N^{\delta+1}, \quad \delta > -1.$$

The estimation (3.15) follows from similar arguments and (2.13) with \(\alpha = r - 1\). \(\square\)

Remark 3.2. Let \(h = \mathcal{O}(N^{-\frac{1}{d}})\). We have the optimal error estimate

$$\left(\mathbb{E}[\|u - u_N^h\|^p]\right)^{\frac{1}{p}} \leq C h^{2 - \frac{d}{2} - \rho}.$$  

In particular for white noise driven Eq. (1.1), i.e., \(\rho = 0\), we have

$$\left(\mathbb{E}[\|u - u_N^h\|^p]\right)^{\frac{1}{p}} \leq C h^{2 - \frac{d}{2}}.$$

The above estimation shows that the finite element method is superconvergent, which removes a usual infinitesimal factor appearing in both the regularity of the solution and various numerical approximations of two dimensional white noise driven Eq. (1.1) (see e.g. [5, 9]):

$$\left(\mathbb{E}[\|u - u_h\|^2]\right)^{\frac{1}{2}} \leq C h^{1-\epsilon},$$

where \(\epsilon\) is a small and positive number. Moreover, the convergence order for one dimensional white noise driven Eq. (1.1) is 1.5, which improves the existing convergence results in [1, 6, 9] of first order.
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