Mathematical model of a single-acting pneumatic cylinder actuator

A A Nikitin, E A Sorokin, T N Nikitina, D A Sokolov and I V Andreychikov

Siberian Federal University, 660074, 26a, Academician Kirensky street, Krasnoyarsk, Russia

E-mail: allynev@mail.ru

Abstract. Pneumatic actuators are widely used in industry as the most reliable automation tools. The efficiency of automated systems depends on the speed of pneumatic actuators. Therefore, the development of methods studying the dynamic processes of pneumatic actuators is an urgent task. This paper concerns a mathematical model of a single-acting pneumatic cylinder actuator, which is convenient for computer calculations using universal mathematical packages. As an example of the application of a mathematical model, paper shows the influence of the piston diameter on transient processes in a pneumatic actuator.

1. Introduction

Increasing the efficiency of pneumatic actuators means increasing their responsiveness. Since pneumatic actuators are part of technological equipment, the time of their actuation is included in the total time of the operating cycle of technological equipment, affecting its efficiency [1, 2].

The problem of increasing the productivity of pneumatic actuators is directly related to the development of methods of dynamic research of fabrication system, with the creation of methods of dynamic analysis of pneumatic mechanisms [1].

The most important directions of development of pneumatic actuators include research of the dynamics of pneumatic actuators.

Papers [1-4] are devoted to the dynamics of pneumatic systems. More complete list of papers on the dynamics of a pneumatic actuators can be found in [1].

In [1], the dynamics of a single-acting pneumatic actuator is described by a system of two equations

\[ \frac{d^2 y}{dt^2} = \frac{S}{m_p}(p - p_a) - \mathcal{F}_p, \]  
\[ \frac{dp}{dt} = \frac{k\mu_s\kappa p_m \sqrt{RT}}{s y} \varphi(\sigma) - \frac{k p}{y} \frac{dy}{dt}, \]

where \( m_p \) - reduced to the piston mass of the mechanism units, set in motion by the rod; \( y \) - piston movement; \( S \) - area of the piston from the side of the working cavity of the pneumatic cylinder, \( S = \pi D_p^2 / 4 \); \( D_p \) - piston diameter; \( p \) - pressure in the working cavity of the pneumatic cylinder; \( p_a \) - atmosphere pressure; \( \mathcal{F}_p \) - the value of the drag force reduced to the piston; \( k \) - is the adiabatic index; \( \mu_s \) - flow rate coefficient of the hole through which air enters the working cavity of the pneumatic cylinder; \( \varphi(\sigma) \) - area of the hole through which air enters the working cavity of the pneumatic cylinder, \( \varphi(\sigma) = \frac{\pi d_c^2}{4} \); \( d_c \) - diameter of the hole through which air enters the working cavity of the pneumatic cylinder;
$p_m$ - pressure of the gas in the pipeline; $R$ - gas constant of the gas; $T_M$ - absolute temperature of the gas coming from the pipeline; $K$ - a parameter determined by the formula $K = \sqrt{2k/(k-1)}$; $\varphi (\sigma)$ - flow rate function, $\varphi (\sigma) = \sqrt{\sigma^{(k-1)/k} - \sigma^{(k+1)/k}}$; $\sigma$ - pressure ratio, $\sigma = p/p_m$.

In equation (2), the values of the flow rate function $\varphi(\sigma)$ are
\[
\varphi(\sigma) = \sqrt{\sigma^{2/k} - \sigma^{(k+1)/k}} \text{ at } \sigma_* < \sigma < 1
\]

or
\[
\varphi(\sigma) = \varphi(\sigma_*) \text{ at } 0 < \sigma < \sigma_*
\]

where $\sigma_*$ - critical pressure ratio, $\sigma_* = [2k/(k+1)]^{k/(k-1)}$.

Two formulas were obtained in paper [1] to determine the time of filling the piston cavity of a single-acting pneumatic drive,
\[
t_f = \frac{y_0S(\sigma_d - \sigma_a)}{k\mu S_c K \sqrt{RT_M} \varphi(\sigma_*)} \text{ at } 0 < \sigma_a < \sigma < \sigma_d \leq \sigma_*
\]

and
\[
t_f = \frac{2y_0S\left(\sqrt{1-\sigma_*^{(k-1)/k}} - \sqrt{1-\sigma_d^{(k-1)/k}}\right)}{(k-1)\mu S_c K \sqrt{RT_M}} \text{ at } \sigma_* < \sigma < \sigma < \sigma_d < 1,
\]

where $\sigma_a$ - pressure ratio, $\sigma_a = p_a/p_m$; $\sigma_d$ - pressure ratio, $\sigma_d = p_d/p_m$; $p_d$ - pressure value in the piston cavity of the pneumatic cylinder, when exceeded, the piston begins to move during the working stroke of the piston.

If $0 < \sigma_a < \sigma_*$, and $\sigma_* < \sigma_d$, then the filling time is determined by the sum
\[
t_f = t_{f1} + t_{f2},
\]

where
\[
t_{f1} = \frac{y_0S(\sigma_* - \sigma_a)}{k\mu S_c K \sqrt{RT_M} \varphi(\sigma_*)} \text{ when } 0 < \sigma_a < \sigma \leq \sigma_*
\]
\[
t_{f2} = \frac{2y_0S\left(\sqrt{1-\sigma_*^{(k-1)/k}} - \sqrt{1-\sigma_d^{(k-1)/k}}\right)}{(k-1)\mu S_c K \sqrt{RT_M}} \text{ when } \sigma_* < \sigma < \sigma_d < 1.
\]

2. Mathematical model of a single-acting pneumatic actuator. Forward stroke

Let us carry out a mathematical description of a single-acting pneumatic actuator, the diagram of which is shown in figure 1. In a mathematical description, the processes occurring in a pneumatic actuator is considered as quasi-stationary, air is considered a perfect gas and heat exchange with the environment will be neglected. The compressed air pressure in the line is taken as a constant value.

![Diagram of single-acting pneumatic actuator](image_url)
The equation of piston motion at a constant reduced mass \( m_p \) can be written in the form
\[
m_p \frac{d\nu_p}{dt} = S(p - p_a) - F_p, \tag{10}\]
where \( \nu_p \) - piston speed.

The union between displacement and piston speed is determined by the formula
\[
\nu_p = \frac{d\nu_p}{dt}. \tag{11}\]

To determine the dependences \( y(t) \), \( \nu_p(t) \) and \( p(t) \), the system of equations (10) and (11) must be supplemented with one more equation. We obtain this equation from the energy equation for gas of variable mass in a pneumatic cylinder without heat exchange [3]
\[
c_p T_M G_M dt = c_v d(\rho VT) + pdV, \tag{12}\]
where \( c_p \) and \( c_v \) - specific heat capacities at constant pressure and at constant volume, respectively; \( T_M \) - absolute temperature of the gas coming from the main; \( G_M \) - mass velocity of the gas entering the pneumatic cylinder from the main; \( d\nu \) - infinitely small time interval; \( \rho \) - density of the gas in the pneumatic cylinder; \( V \) - volume of gas in the pneumatic cylinder; \( T \) - absolute temperature of the gas in the pneumatic cylinder; \( p \) - pressure of the gas in the pneumatic cylinder; \( dV \) - change in the volume of gas in the pneumatic cylinder.

The density of a perfect gas can be found using the equation of state
\[
\rho = \frac{p}{RT}, \tag{13}\]
where \( V \) - volume of gas in the pneumatic cylinder; \( R \) - gas constant of the gas.

After substituting the expression for the density from formula (13), we bring equation (12) taking into account the formula
\[
k = \frac{c_p}{c_v} (k \text{ - adiabatic index})\]
to the form
\[
kRT_M G_M dt = Vdp + pd(V) + \frac{R}{c_v} pdV. \tag{14}\]

The specific heat capacities \( c_p \) and \( c_v \) are related by the Mayer equation
\[
c_p - c_v = R. \tag{15}\]

We express the volume of the working cavity through the area of the piston \( S \) and its displacement \( y \)
\[
V = yS. \tag{16}\]

Replacing in equation (14) the volume \( V \) of the gas by the expression from the formula (16) and the gas constant \( R \) in the right-hand side of equation (14) by the expression from the relationship (15), taking into account the formula (11), after simple transformations, the equation (14) is reduced to the form [1, 4]
\[
kRT_M G_M = yS \frac{dp}{dt} + kS \nu_p. \tag{17}\]

Considering the process of filling the working cavity of the pneumatic cylinder with gas as quasi-stationary, the mass flow rate of gas \( G_M \) coming from the main into the working cavity of the pneumatic cylinder can be determined for each moment of time \( t \) by the formula obtained for the steady motion of perfect gas [3]
\[
G_M = \mu_c S_c \sqrt[1/k]{\frac{p_m}{\sqrt{RT_M}}} \left( \frac{p_c}{p_m} \right)^{1/k} \frac{2k}{R} \frac{1}{k-1} \sqrt{1 - \left( \frac{p_c}{p_m} \left( \frac{k-1}{k} \right) \right)^{k-1}}, \tag{18}\]
where \( \mu_c \) - flow rate coefficient of the hole through which air enters the working cavity of the pneumatic cylinder; \( S_c \) - area of the hole through which air enters the working cavity of the pneumatic cylinder,
\( S_c = \pi d_c^2 / 4; \)  
\( d_c \) – diameter of the hole through which air enters the working cavity of the pneumatic cylinder;  
\( p_m \) - pressure of the gas in the pipeline;  
\( p_{c1} \) - pressure in the gas stream flowing into the working cavity of the pneumatic cylinder with pressure \( p \), is determined by the ratio

\[
p_{c1} = \begin{cases} 
  p, & \text{if } p > p_{k1} \\
  p_{k1}, & \text{if } p \leq p_{k1}.
\end{cases}
\]

(19)

where \( p_{k1} \) – critical pressure at which the gas velocity is equal to the local speed of sound, is determined by the formula

\[
p_{k1} = p_m \left( \frac{2}{k+1} \right)^{k/(k-1)}.
\]

(20)

Relations (19) for the gas pressure in the jet flowing into the working cavity of the pneumatic cylinder can be written using logical operators (for example, in the computer mathematical system Mathcad) in the form

\[
p_{c1}(p) = p(p > p_{k1}) + p_{k1}(p \leq p_{k1}).
\]

(21)

If the condition in the bracket \( p > p_{k1} \) is satisfied, this bracket and the first term on the right side of formula (21) takes the value 1, but when the condition in the bracket \( p \leq p_{k1} \) is not satisfied and it and the second term on the right side of the formula (21) take the value 0. And vice versa, if the condition in the parenthesis \( p > p_{k1} \) is not met, then this parenthesis and the first term on the right side of formula (21) take the value 0, while the condition in the parenthesis \( p \leq p_{k1} \) is fulfilled and, therefore, it and the second term on the right-hand side of formula (21) take the value 1.

Taking into account relations (21), formula for determining the mass flow rate of gas can be written in the form

\[
G_M(p) = \mu_c S_c \frac{p_m}{\sqrt{RT_m}} \left( \frac{p_{c1}(p)}{p_m} \right)^{1/k} \sqrt{\frac{2k}{k-1}} \left[ 1 - \left( \frac{p_{c1}(p)}{p_m} \right)^{(k-1)/k} \right].
\]

(22)

The system of three equations (10), (11) and (17), taking into account formulas (21) and (22), is a mathematical model of a one-way pneumatic drive for direct travel, we will bring it to the Cauchy form [5-8]

\[
\frac{dy}{dt} = v_p;
\]

(23)

\[
\frac{dv_p}{dt} = \frac{1}{m_p} \left[ S(p - p_a) - F_p \right];
\]

(24)

\[
\frac{dp}{dt} = \frac{k}{y} \left\{ \frac{RT_m G_M(p)}{S} - p v_p \right\}.
\]

(25)

To solve the mathematical model, the system of equations (23) - (25) should be supplemented with the initial conditions (the values of the sought variables at the initial moment of time):

\[
y(t_0) = y_0;
\]

(26)

\[
v_p(t_0) = v_{p,0};
\]

(27)

\[
p(t_0) = p_d.
\]

(28)

The air pressure in the working cavity of pneumatic cylinder at the beginning of movement can be found from equation (24) under the condition of a stationary piston \((v_p = 0 and \frac{dv_p}{dt} = 0)\)

\[
p_a = p_a + \frac{F_p}{S}.
\]

(29)
To determine the time during which the pressure in the working cavity with a stationary piston (figure 1, piston 3 occupies the extreme left position) increases from the initial value \( p = p_a \) to the value \( p = p_d \), we use equation (17), which at \( y = y_0 \) and \( u_p = 0 \), taking into account formula (22), takes the form

\[
kRT_M G_M (p) = y_0 S \frac{dp}{dt},
\]

where \( y_0 \) – value of the y coordinate at the initial position of the piston.

Separating the variables, we reduce equation (30) to the following form:

\[
dt = \frac{y_0 S}{kRT_M G_M (p)} dp.
\]

By integrating equation (31), we find the filling time \( t_f \) of the working cavity of the pneumatic cylinder

\[
t_f = \int_0^{Pd} \frac{y_0 S}{kRT_M G_M (p)} dp.
\]

After substitution of the expression for the mass flow rate of gas from formula (22) into formula (32), we obtain the final formula for determining the filling time of the working cavity of the pneumatic cylinder with a stationary piston

\[
t_f = \int_0^{Pd} \left( \frac{y_0 S}{kRT_M G_M (p)} \right) dp.
\]

The air pressure in the working cavity of pneumatic cylinder at the beginning of the movement is determined by the formula (29), and the speed of piston at the beginning of the movement is equal to zero, in this case initial conditions (26) - (28) will take the following form:

\[
\begin{align*}
y(t_0) &= y_0; \quad \text{(34)} \\
u_p(t_0) &= 0; \quad \text{(35)} \\
p(t_0) &= p_a + \frac{F_p}{S}. \quad \text{(36)}
\end{align*}
\]

The formula for determining final time \( t_e \) with a forward stroke, during which the pressure in the working cavity with a stationary piston (figure 1, piston 3 occupies the extreme right position) increases from the value \( p = p_{d,k} \) (at the end of the working stroke) to the value \( p = p_m \) (mainline), it is easy to obtain using equation (17), which with \( y = y_0 + L_x \) and \( u_p = 0 \), taking into account formula (22), will take the form

\[
kRT_M G_M (p) = (y_0 + L_x) S \frac{dp}{dt},
\]

where \( L_x \) – piston stroke.

Having performed the calculations deriving formula (33) for determining the filling time \( t_f \), taking into account equation (37), we obtain a formula for determining the final time \( t_e \)

\[
t_e = \int_{Pd}^{Pm} \left( \frac{(y_0 + L_x) S}{kRT_M G_M (p)} \right) dp.
\]

The pressure \( p_{d,k} \) in the working cavity at the end of working stroke is found by solving the system of equations (23) - (25).
3. Calculation results
A program has been developed for calculation in the computer mathematical system Mathcad using one of the built-in functions for solving the system of ordinary differential equations of the mathematical model of a drive with a single-acting pneumatic actuator.

Figures 3-5 show calculation results of pneumatic lift with a single-acting pneumatic cylinder for two piston diameters at the same load (14700 N) and line pressure (6 bar) in the load lifting mode.

Figure 3 shows the graphs of the dependences of the pressure in the piston cavity of pneumatic cylinder on time for two values of the piston diameter. From the graphs (figure 3) it follows that the dependences of the pressure in the piston cavity of pneumatic cylinders with different diameters of pistons on time have an oscillatory character, at the beginning of movement the amplitude of oscillations has a maximum value, then the oscillations quickly damp. The maximum value of the amplitude of pressure fluctuations in a pneumatic cylinder with a large piston is 44% of the steady-state value, and in a pneumatic cylinder with a smaller piston diameter it is 54%. In a pneumatic cylinder with a smaller piston diameter, vibrations are damped faster than in a pneumatic cylinder with a large piston.

![Figure 2](image2.png)

**Figure 2.** Dependence of pressure in the piston cavity on time for two values of the piston diameter: 320 mm; 250 mm.

Figure 4 shows graphs of the piston displacement versus time for two values of the piston diameter. It follows that the time dependences of the displacement of pistons with different diameters have a monotonic character close to linear from the graphs (figure 4). Graphs (figure 4) show that the stroke time of the piston with a large diameter is longer than with a smaller piston diameter.

Figure 5 shows the graphs of the piston speed versus time for two values of the piston diameter. It follows from the graphs (figure 5) that the dependences of the piston speed on time have an oscillatory character, at the beginning of the movement the amplitude of the oscillations has a maximum value, then the oscillations die out. The maximum value of the amplitude of oscillations of the speed of the piston with a large diameter is 49% of the steady value, and with a smaller piston diameter is 16%. In a pneumatic cylinder with a smaller piston diameter, speed fluctuations damp out faster than in a pneumatic cylinder with a large diameter.

![Figure 3](image3.png)

**Figure 3.** Piston displacement versus time for two values of the piston diameter: 320 mm; 250 mm.
4. Conclusions

The calculation results show that an increase in the piston diameter leads to a decrease in the amplitude of oscillations and the steady-state value of pressure in the piston cavity of the pneumatic cylinder, as well as to an increase in the amplitude of oscillations of the piston speed and the time of the piston stroke.

The obtained mathematical model of a single-acting pneumatic cylinder actuator makes it possible to simplify the study of the dynamics of pneumatic actuators using universal computer mathematical systems. The mathematical model of a pneumatic actuator with a single-acting pneumatic cylinder allows to study the effect of changing the parameters of the pneumatic actuator on the dynamics at the design stage, as well as to select the optimal values of the drive parameters.

References

[1] Herts E V 1985 Dynamics of Pneumatic Systems of Machines (Moscow: Machine building)
[2] Metlyuk N F and Avtushko V P 1980 Dynamics of Fluid and Pneumatic Actuators of Vehicles (Moscow: Machine building)
[3] Popov D N 2002 Mechanics of Fluid and Pneumatic Actuators (Moscow: Bauman Moscow State Technical University Press)
[4] Levitsky N I 1990 The Theory of Mechanisms and Machines (Moscow: Science)
[5] Mandrakov E A and Nikitin A A 2014 Dynamics of Hydraulic Systems: monograph (Moscow: INFRA-M)
[6] Gorbeshko M V 1997 Development of Mathematical Models for the Hydraulic Machinery of Systems Controlling the Moving Components of Water Development Works Hydrotechnical construction 31(12) 745-50
[7] Mobley R K 1999 Fluid Power Dynamics (Oxford: Butterworth Heinemann)
[8] Kaverzina A S, Lunev A S, Afanasov V I, Pilyugaev A I, Andreychikov I V and Zakovryazhin M V 2020 Climatic Effect on Characteristics of a Hydraulic Drive of a Self-propelled Vehicle IOP Conference series: Materials science and engineering 734
[9] Lunev A S, Sokolov D A, Litvinchuk M V, Dunaeva S P and Andreychikov I V 2020 Mathematical Characterization of Fluid Compressibility Impact on Fluid Drive's Output Parameters IOP Conference Series: Materials Science and Engineering 1047