MARTINGALE HARDY-AMALGAM SPACES: ATOMIC DECOMPOSITIONS, MARTINGALE EMBEDDINGS AND DUALITY

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Abstract. In this paper, we introduce the notion of martingale Hardy-amalgam spaces: \( H_{p,q}^S \), \( H_{p,q}^x \), \( H_{p,q}^* \), \( Q_{p,q} \) and \( P_{p,q} \). We present two atomic decompositions for these spaces, and martingale embeddings. The dual space of \( H_{p,q}^* \) for \( 0 < p \leq q \leq 1 \) is shown to be a Campanato-type space.

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1. Introduction

We owe the martingale theory to J. L. Doob from his seminal work [8]. The theory was later developed by D. L. Burkholder, A. M. Garsia, R. Cairoli, B. J. Davis and their collaborators (see [4, 5, 6, 7, 11, 19, 21] and the references therein). Martingales are particularly interesting because of their connection and applications in Fourier analysis, complex analysis and classical Hardy spaces (see for example [3, 4, 8, 9, 17, 21] and the references therein).

The classical martingale Hardy spaces \( H_p \) are defined as the spaces of martingales whose maximal function, quadratic variation or conditional quadratic variation belongs to the usual Lebesgue spaces \( L_p \) with a probability measure. The atomic decompositions, martingale embeddings and dual spaces of these spaces and related spaces are discussed by F. Weisz in [21]. This type of studies has been
considered by several authors for some generalizations of the classical Lebesgue spaces as Lorentz spaces, Orlicz spaces, Orlicz-Musielak spaces... (see [14, 16, 18, 20, 22, 23, 24] and the references therein).

In this paper, inspired by the recent introduction of Hardy-amalgam spaces in the classical Harmonic analysis ([1] [2] [26]), we replace Lebesgue spaces in the definition of the classical martingale Hardy spaces by the Wiener amalgam spaces, introducing then the notion of martingale Hardy-amalgam spaces. As Wiener amalgam spaces generalize Lebesgue spaces, martingale Hardy-amalgam spaces then generalize the martingale Hardy spaces presented in [21]. We provide atomic decompositions, martingale inequalities and we characterize the dual spaces of these martingale Hardy-amalgam spaces and their associated spaces of predictive martingales and martingales with predictive quadratic variation.

2. Preliminaries: Menagerie of spaces

We introduce here some function spaces in relation with our concern in this paper.

2.1. Wiener Amalgam Spaces. Let $\Omega$ be an arbitrary non-empty set and let $\{\Omega_j\}_{j \in \mathbb{Z}}$ be a sequence of nonempty subsets of $\Omega$ such that $\Omega_j \cap \Omega_i = \emptyset$ for $j \neq i$, and

$$\bigcup_{j \in \mathbb{Z}} \Omega_j = \Omega.$$ 

For $0 < p, q \leq \infty$, the classical amalgam of $L_p$ and $l_q$, denoted $L_{p,q}$, on $\Omega$ consists of functions which are locally in $L_p$ and have $l_q$ behaviour (c.f [10]), in the sense that the $L_p$-norms over the subsets $\Omega_j \subset \Omega$ form an $l_q$-sequence i.e. for $p, q \in (0, \infty)$,

$$L_{p,q} = \{ f : \|f\|_{p,q} := \|f\|_{L_{p,q}(\Omega)} < \infty \}$$

where

$$\|f\|_{p,q} := \|f\|_{L_{p,q}(\Omega)} := \left[ \sum_{j \in \mathbb{Z}} \left( \int_{\Omega} |f|^p 1_{\Omega_j} \, dP \right)^{\frac{q}{p}} \right]^{\frac{1}{q}},$$

for $0 < q < \infty$, and for $q = \infty$,

$$L_{p,\infty} = \left\{ f : \|f\|_{p,\infty} := \|f\|_{L_{p,\infty}(\Omega)} := \sup_{j \in \mathbb{Z}} \left( \int_{\Omega} |f|^p 1_{\Omega_j} \, dP \right)^{\frac{1}{p}} < \infty \right\}.$$ 

We observe the following:
• Endowed with the (quasi)-norm $\| \cdot \|_{p,q}$, the amalgam space $L_{p,q}$ is a complete space, and a Banach space for $1 \leq p, q \leq \infty$.

• $\| f \|_{p,p} = \| f \|_p$ for $f \in L_p(\Omega)$.

• $\| f \|_{p,q} \leq \| f \|_p$ if $p \leq q$ and $f \in L_p(\Omega)$.

• $\| f \|_p \leq \| f \|_{p,q}$ if $q \leq p$ and $f \in L_{p,q}(\Omega)$.

Amalgam function spaces have been essentially considered in the case $\Omega = \mathbb{R}^d$, $d \in \mathbb{N}$, and in the case $d = 1$, the subsets $\Omega_j$ are just the intervals $[j, j+1)$, $j \in \mathbb{Z}$. It is also known that different appropriate choices of the sequence of sets $(\Omega_j)_{j \in \mathbb{Z}}$ provide the same spaces (see for example [1, 13]). For more on amalgam spaces, we refer the reader to [10, 15].

2.2. Martingale Hardy Spaces via Amalgams. In the remaining of this text, all the spaces are defined with respect to the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\mathcal{F}_n)_{n \geq 0} := (\mathcal{F}_n)_{n \in \mathbb{Z}^+}$ be a non-decreasing sequence of $\sigma$-algebra with respect to the complete ordering on $\mathbb{Z}^+ = \{0, 1, 2 \cdots \}$ such that

$$\sigma \left( \bigcup_{n \in \mathbb{N}} \mathcal{F}_n \right) = \mathcal{F}.$$ 

For $n \in \mathbb{Z}^+$, the expectation operator and the conditional expectation operator relatively to $\mathcal{F}_n$ are denoted by $\mathbb{E}$ and $\mathbb{E}_n$ respectively. We denote by $\mathcal{M}$ the set of all martingales $f = (f_n)_{n \geq 0}$ relatively to the filtration $(\mathcal{F}_n)_{n \geq 0}$ such that $f_0 = 0$. We recall that for $f \in \mathcal{M}$, its martingale difference is denoted $d_n f = f_n - f_{n-1}$, $n \geq 0$ with the convention that $d_0 f = 0$.

We recall that the stochastic basis, $(\mathcal{F}_n)_{n \geq 0}$ is said to be regular if there exists $R > 0$ such that $f_n \leq R f_{n-1}$ for all non-negative martingale $(f_n)_{n \geq 0}$.

A martingale $f = (f_n)_{n \geq 0}$ is said to be $L_p$ bounded $(0 < p \leq \infty)$ if $f_n \in L_p$ for all $n \in \mathbb{Z}^+$ and we define

$$\| f \|_p := \sup_{n \in \mathbb{N}} \| f_n \|_p < \infty.$$ 

We recall that

$$\| f \|_p = (\mathbb{E}(|f|^p))^{\frac{1}{p}} = \left( \int_{\Omega} |f|^p d\mathbb{P} \right)^{\frac{1}{p}}.$$ 

We denote by $\mathcal{T}$ the set of all stopping times on $\Omega$. For $\nu \in \mathcal{T}$, and $(f_n)_{n \geq 0}$ an integrable sequence, we recall that the associated stopped sequence $f^\nu = (f^\nu_n)_{n \geq 0}$ is defined by

$$f^\nu_n = f_{n \wedge \nu}, \quad n \in \mathbb{Z}^+.$$
For a martingale \( f = (f_n)_{n \geq 0} \), the quadratic variation, \( S(f) \), and the conditional quadratic variation, \( s(f) \), of \( f \) are defined by

\[
S(f) = \left( \sum_{n \in \mathbb{N}} |d_n f|^2 \right)^{\frac{1}{2}} \quad \text{and} \quad s(f) = \left( \sum_{n \in \mathbb{N}} \mathbb{E}_{n-1} |d_n f|^2 \right)^{\frac{1}{2}}
\]

respectively. We shall agree on the notation

\[
S_n(f) = \left( \sum_{i=1}^{n} |d_i f|^2 \right)^{\frac{1}{2}} \quad \text{and} \quad s_n(f) = \left( \sum_{n=1}^{n} \mathbb{E}_{i-1} |d_i f|^2 \right)^{\frac{1}{2}}.
\]

The maximal function \( f^* \) or \( M(f) \) of the martingale \( f \) is defined by

\[
M(f) = f^* := \sup_{n \in \mathbb{N}} |f_n|.
\]

We now introduce the martingale Hardy-amalgam spaces: \( H^S_{p,q} \), \( H^s_{p,q} \), \( H^*_{p,q} \), \( Q_{p,q} \) and \( P_{p,q} \). Let \( 0 < p < \infty \) and \( 0 < q \leq \infty \). The first three spaces are defined as follows.

i. \( H^S_{p,q}(\Omega) = \{ f \in \mathcal{M} : \| f \|_{H^S_{p,q}(\Omega)} := \| S(f) \|_{p,q} < \infty \} \).

ii. \( H^s_{p,q}(\Omega) = \{ f \in \mathcal{M} : \| f \|_{H^s_{p,q}(\Omega)} := \| s(f) \|_{p,q} < \infty \} \).

iii. \( H^*_{p,q}(\Omega) = \{ f : \| f \|_{H^*_{p,q}(\Omega)} := \| f^* \|_{p,q} < \infty \} \).

Let \( \Gamma \) be the set of all sequences \( \beta = (\beta_n)_{n \geq 0} \) of adapted (i.e. \( \beta_n \) is \( F_n \)-measurable for any \( n \in \mathbb{Z}_+ \)) non-decreasing, non-negative functions and define

\[
\beta_\infty := \lim_{n \to \infty} \beta_n.
\]

iv. The space \( Q_{p,q}(\Omega) \) consists of all martingales \( f \) for which there is a sequence of functions \( \beta = (\beta_n)_{n \geq 0} \in \Gamma \) such that \( S_n(f) \leq \beta_{n-1} \) and \( \beta_\infty \in L_{p,q}(\Omega) \). We endow \( Q_{p,q}(\Omega) \) with

\[
\| f \|_{Q_{p,q}(\Omega)} := \inf_{\beta \in \Gamma} \| \beta_\infty \|_{p,q}.
\]

v. The space \( P_{p,q}(\Omega) \) consists of all martingales \( f \) for which there is a sequence of functions \( \beta = (\beta_n)_{n \geq 0} \in \Gamma \) such that \( |f_n| \leq \beta_{n-1} \) and \( \beta_\infty \in L_{p,q}(\Omega) \). We endow \( P_{p,q}(\Omega) \) with

\[
\| f \|_{P_{p,q}(\Omega)} := \inf_{\beta \in \Gamma} \| \beta_\infty \|_{p,q}.
\]

A martingale \( f \in P_{p,q}(\Omega) \) is called predictive martingale and a martingale \( f \in Q_{p,q}(\Omega) \) is a martingale with predictive quadratic variation.
In the sequel, when there is no ambiguity, the spaces $H^{S}_{p/q}(\Omega)$, $H^{s}_{p/q}(\Omega)$, $H^{*}_{p/q}(\Omega)$, $Q_{p,q}(\Omega)$ and $\mathcal{P}_{p,q}(\Omega)$ will be just denoted $H^{S}_{p,q}$, $H^{s}_{p,q}$, $H^{*}_{p,q}$, $Q_{p,q}$ and $\mathcal{P}_{p,q}$ respectively. The same will be done for the associated (quasi)-norms.

Remark 2.3. • Observe that when $0 < p = q < \infty$, the above spaces are just the spaces $H^{S}_{p}$, $H^{s}_{p}$, $H^{*}_{p}$, $Q_{p}$ and $P_{p}$ defined and studied in [21].

• Hardy-amalgam spaces of classical functions $H_{p,q}$ of $\mathbb{R}^{d}$ ($d \geq 1$) were introduced recently by V. P. Ablé and J. Feuto in [1] where they provided an atomic decomposition for these spaces for $0 < p, q \leq 1$. In [2], they also characterized the corresponding dual spaces, for $0 < p \leq q \leq 1$.

A generalization of their definition and their work was pretty recently obtained in [26].

• Our definitions here are inspired from the work [1] and the usual definition of martingale Hardy spaces.

3. Presentation of the results

We start by defining the notion of atoms.

Definition 3.1. Let $0 < p < \infty$, and $\max(p, 1) < r \leq \infty$. A measurable function $a$ is a $(p, r)^{s}$-atom (resp. $(p, r)^{S}$-atom, $(p, r)^{*}$-atom) if there exists a stopping time $\nu \in \mathcal{T}$ such that

\begin{align*}
(\text{a1}) & \quad a_{n} := \mathbb{E}_{n}a = 0 \text{ if } \nu \geq n; \\
(\text{a2}) & \quad \|s(a)\|_{r,r} := \|s(a)\|_{r} \text{ (resp. } \|S(f)\|_{r}, \|a^{*}\|_{r}) \leq \mathbb{P}(B_{\nu})^{\frac{1}{r} - \frac{1}{p}}.
\end{align*}

where $B_{\nu} = \{\nu \neq \infty\}$.

We also have the following other definition of an atom.

Definition 3.2. Let $0 < p < \infty$, $0 < q \leq \infty$ and $\max(p, 1) < r \leq \infty$. A measurable function $a$ is a $(p, q, r)^{s}$-atom (resp. $(p, q, r)^{S}$-atom, $(p, q, r)^{*}$-atom) if there exists a stopping time $\nu \in \mathcal{T}$ such that condition (a1) in Definition 3.1 is satisfied and

\begin{align*}
(\text{a3}) & \quad \|s(a)\|_{r} \text{ (resp. } \|S(a)\|_{r}, \|a^{*}\|_{r}) \leq \mathbb{P}(B_{\nu})^{\frac{1}{r}}\|1_{B_{\nu}}\|_{p,q}^{-1}.
\end{align*}

We denote by $\mathcal{A}(p, q, r)^{s}$ (resp. $\mathcal{A}(p, q, r)^{S}$, $\mathcal{A}(p, q, r)^{*}$) the set of all sequences of triplets $(\lambda_{k}, a^{k}, \nu^{k})$, where $\lambda_{k}$ are nonnegative numbers, $a^{k}$ are $(p, r)^{s}$-atoms (resp. $(p, r)^{S}$-atoms, $(p, r)^{*}$-atoms) and $\nu^{k} \in \mathcal{T}$ satisfying conditions (a1) and (a2) in Definition 3.1 and such that for any $0 < \eta \leq 1$,

$$
\sum_{k} \left( \frac{\lambda_{k}}{\mathbb{P}(B_{\nu^{k}})^{\frac{1}{r}}} \right)^{\eta} 1_{B_{\nu^{k}}} \in L^{p, q}_{\eta}. 
$$
We denote by $B(p, q, r)^s$ (resp. $B(p, q, r)^S, B(p, q, r)^*$) the set of all sequences of triplets $(\lambda_k, a^k, \nu^k)$, where $\lambda_k$ are nonnegative numbers, $a^k$ are $(p, q, r)^s$-atoms (resp. $(p, q, r)^S$-atoms, $(p, q, r)^*$-atoms) and $\nu^k \in T$ satisfying conditions $(a1)$ and $(a3)$ in Definition 3.2 and such that for any $0 < \eta \leq 1$,

$$\sum_k \left( \frac{\lambda_k}{\|1_{B_{\nu^k}}\|_{p, q}} \right)^{\eta} 1_{B_{\nu^k}} \in L_{\frac{p}{\eta}, \frac{q}{\eta}}.$$

We observe that $\mathcal{A}(p, q, r)^s \subseteq B(p, q, r)^s$ if $p \leq q$ and $B(p, q, r)^s \subseteq \mathcal{A}(p, q, r)^s$ if $q \leq p$. The same relation holds between the other sets of triplets.

Our first atomic decomposition of the spaces $H^S_{p,q}$, $H^s_{p,q}$, $H^*_{p,q}$ is as follows.

**Theorem 3.3.** Let $0 < p < \infty$, $0 < q \leq \infty$ and let $\max(p, 1) < r \leq \infty$. If the martingale $f \in \mathcal{M}$ is in $H^s_{p,q}$ (resp. $H^S_{p,q}$, $H^*_{p,q}$), then there exists a sequence of triplets $(\lambda_k, a^k, \nu^k) \in \mathcal{A}(p, q, r)^s$ (resp. $\mathcal{A}(p, q, r)^S, \mathcal{A}(p, q, r)^*$) such that for all $n \in \mathbb{N}$,

$$\sum_{k \in \mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n$$

and for any $0 < \eta \leq 1$,

$$\left\| \sum_{k \geq 0} \left( \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} 1_{B_{\nu^k}} \right\|_{\frac{p}{\eta}, \frac{q}{\eta}} \leq C \|f\|_{H^s_{p,q}} \left( \text{resp. } \|f\|_{H^S_{p,q}}, \|f\|_{H^*_{p,q}} \right).$$

Moreover,

$$\sum_{k=l}^{m} \lambda_k a^k \rightarrow f$$

in $H^s_{p,q}$ (resp. $H^S_{p,q}$, $H^*_{p,q}$) as $m \rightarrow \infty$, $l \rightarrow -\infty$.

Conversely if $f \in \mathcal{M}$ has a decomposition as in (2), then for any $0 < \eta \leq 1$,

$$\|f\|_{H^s_{p,q}} \left( \text{resp. } \|f\|_{H^S_{p,q}}, \|f\|_{H^*_{p,q}} \right) \leq C \left\| \sum_{k \in \mathbb{Z}} \left( \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} 1_{B_{\nu^k}} \right\|_{\frac{p}{\eta}, \frac{q}{\eta}}.$$

Using our second definition of atoms, we also obtain the following atomic decomposition.

**Theorem 3.4.** Let $0 < p < \infty$, $0 < q \leq \infty$ and let $\max(p, 1) < r \leq \infty$. If the martingale $f \in \mathcal{M}$ is in $H^s_{p,q}$ (resp. $H^S_{p,q}$, $H^*_{p,q}$), then there exists a sequence of triplets $(\lambda_k, a^k, \nu^k) \in B(p, q, r)^s$ (resp. $B(p, q, r)^S, B(p, q, r)^*$) such that for all $n \in \mathbb{N}$,

$$\sum_{k \in \mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n$$
and for any \(0 < \eta \leq 1\),

\[
\left\| \sum_{k \geq 0} \left( \frac{\lambda_k}{\| 1_{B_{\nu k}} \|_{p,q}} \right)^{\eta} 1_{B_{\nu k}} \right\|^{\frac{1}{\eta}} \leq C \| f \|_{H_p^s, \nu} \left( \text{resp. } \| f \|_{H_p^s, \nu}, \| f \|_{H_p^s, \nu} \right).
\]

Moreover,

\[
\sum_{k=l}^{m} \lambda_k a_k \rightharpoonup f
\]

in \(H_p^s (\text{resp. } H_p^s, H_p^s)\) as \(m \to \infty, l \to -\infty\).

Conversely if \(f \in \mathcal{M}\) has a decomposition as in (4), then for any \(0 < \eta \leq 1\),

\[
\| f \|_{H_p^s, \nu} \left( \text{resp. } \| f \|_{H_p^s, \nu}, \| f \|_{H_p^s, \nu} \right) \leq C \sum_{k \in \mathbb{Z}} \left( \frac{\lambda_k}{\| 1_{B_{\nu k}} \|_{p,q}} \right)^{\eta} 1_{B_{\nu k}}^{\eta}.
\]

For the last two spaces, we obtain the two following atomic decompositions.

**Theorem 3.5.** Let \(0 < p < \infty\) and \(0 < q \leq \infty\). If the martingale \(f \in \mathcal{M}\) is in \(Q_{p,q}\) (resp. \(P_{p,q}\)), then there exists a sequence sequence of triplets \((\lambda_k, a^k, \nu^k) \in \mathcal{A}(p, q, \infty)^S\) (resp. \(\mathcal{A}(p, q, \infty)^*\)) such that for any \(n \in \mathbb{N}\),

\[
\sum_{k \in \mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n
\]

and for any \(0 < \eta \leq 1\),

\[
\left\| \sum_{k \in \mathbb{Z}} \left( \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} 1_{B_{\nu^k}} \right\|^{\frac{1}{\eta}} \leq C \| f \|_{Q_{p,q}, \nu} \left( \text{resp. } \| f \|_{P_{p,q}, \nu} \right).
\]

Moreover,

\[
\sum_{k=l}^{m} \lambda_k a^k \rightharpoonup f
\]

in \(Q_{p,q}\) (resp. \(P_{p,q}\)) as \(m \to \infty, l \to -\infty\).

Conversely, if \(f \in \mathcal{M}\) has a decomposition as in (6), then for any \(0 < \eta \leq 1\),

\[
\| f \|_{Q_{p,q}, \nu} \left( \text{resp. } \| f \|_{P_{p,q}, \nu} \right) \leq C \left\| \sum_{k \in \mathbb{Z}} \left( \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} 1_{B_{\nu^k}} \right\|^{\frac{1}{\eta}}.
\]

**Theorem 3.6.** Let \(0 < p < \infty\) and \(0 < q \leq \infty\). If the martingale \(f \in \mathcal{M}\) is in \(Q_{p,q}\) (resp. \(P_{p,q}\)), then there exists a sequence sequence of triplets \((\lambda_k, a^k, \nu^k) \in \mathcal{B}(p, q, \infty)^S\) (resp. \(\mathcal{B}(p, q, \infty)^*\)) such
that for any \( n \in \mathbb{N} \),

\[
\sum_{k \in \mathbb{Z}} \lambda_k E_n a^k = f_n
\]

and for any \( 0 < \eta \leq 1 \),

\[
\left\| \sum_{k \in \mathbb{Z}} \left( \frac{\lambda_k}{\|1_{B_{\nu k}}\|_{p,q}} \right)^{\eta} \right\|_{L^\eta_{\frac{p}{\eta}, \frac{q}{\eta}}} \leq C \|f\|_{Q_{p,q}} \text{ (resp. } \|f\|_{P_{p,q}} \).
\]

Moreover,

\[
\sum_{k=l}^{m} \lambda_k a^k \rightarrow f
\]

in \( Q_{p,q} \) (resp. \( P_{p,q} \)) as \( m \to \infty \), \( l \to -\infty \).

Conversely if \( f \in \mathcal{M} \) has a decomposition as in (8), then for any \( 0 < \eta \leq 1 \),

\[
\|f\|_{Q_{p,q}} \text{ (resp. } \|f\|_{P_{p,q}}) \leq C \left\| \sum_{k \in \mathbb{Z}} \left( \frac{\lambda_k}{\|1_{B_{\nu k}}\|_{p,q}} \right)^{\eta} \right\|_{L^\eta_{\frac{p}{\eta}, \frac{q}{\eta}}}.
\]

We now present the martingale inequalities that relate the five sets. For this we further assume that the sequence \( \{\Omega_j\}_{j \in \mathbb{Z}} \) in the definition of \( L_{p,q}(\Omega) \) is such that \( \Omega_j \in \mathcal{F}_n \) for any \( j \in \mathbb{Z} \) and any \( n \geq 1 \). We then have the following.

**Theorem 3.7.** Let \( 0 < q \leq \infty \). Then

(i) \( \|f\|_{H^s_{p,q}} \leq C \|f\|_{H^s_{p,q}} \), \( \|f\|_{H^s_{p,q}} \leq C \|f\|_{H^s_{p,q}} \) \( (0 < p \leq 2) \),

(ii) \( \|f\|_{H^s_{p,q}} \leq C \|f\|_{H^s_{p,q}} \), \( \|f\|_{H^s_{p,q}} \leq C \|f\|_{H^s_{p,q}} \) \( (2 \leq p < \infty) \),

(iii) \( \|f\|_{H^s_{p,q}} \leq C \|f\|_{P_{p,q}} \), \( \|f\|_{H^s_{p,q}} \leq C \|f\|_{P_{p,q}} \) \( (0 < p \leq \infty) \),

(iv) \( \|f\|_{H^s_{p,q}} \leq C \|f\|_{Q_{p,q}} \), \( \|f\|_{H^s_{p,q}} \leq C \|f\|_{Q_{p,q}} \) \( (0 < p < \infty) \),

(v) \( \|f\|_{H^s_{p,q}} \leq C \|f\|_{P_{p,q}} \), \( \|f\|_{H^s_{p,q}} \leq C \|f\|_{Q_{p,q}} \) \( (0 < p < \infty) \).

Moreover, if \( (\mathcal{F}_n)_{n \geq 0} \) is regular, then \( H^s_{p,q}, H^s_{p,q}, H^s_{p,q}, Q_{p,q} \) and \( P_{p,q} \) are all equivalent.

**Remark 3.8.** The hypothesis \( \Omega_j \in \mathcal{F}_n \) for any \( j \in \mathbb{Z} \) and any \( n \geq 1 \) comes from the definition of martingale Hardy-amalgam spaces and the fact that we want to exploit the martingale inequalities for the usual martingale hardy spaces. Hence for \( f \in \mathcal{M} \), we need \( f1_{\Omega_j} \) to also be an element of \( \mathcal{M} \) for any \( j \in \mathbb{Z} \) and have the flexibility to take the corresponding characteristic (indicator) function out of any conditional expectation for a better use of the definition of amalgam spaces. Obviously, there is no need for \( \Omega_j \in \mathcal{F}_0 \) since we are assuming that \( f_0 = 0 \) for \( f \in \mathcal{M} \).
A simple case where the above situation happens, is as follows: for $\Omega = \mathbb{R}$, we assume that the probability measure $\mathbb{P}$ is continuous with respect to the Lebesgue measure on $\mathbb{R}$. Recall that a dyadic interval is any interval of the form $I_{k,n} = [k2^{-n}, (k+1)2^{-n})$, with $n, k \in \mathbb{Z}$. Denote by $D_n$ the set
\[ D_n := \{I_{k,n} : k \in \mathbb{Z}\}. \]

For $n \geq 0$, we take as $\mathcal{F}_n$ the $\sigma$-algebra generated by $D_n$. It is then enough to put $\Omega_j = [j, j+1)$.

Denote by $L^0_{2}$ the set of all $f \in L^2$ such that $E_0 f = 0$. For $f \in L^0_{2}$, put $f_n = E_n f$. We recall that $(f_n)_{n \geq 0}$ is in $\mathcal{M}$ and $L^2$-bounded. Moreover, $(f_n)_{n \geq 0}$ converges to $f$ in $L^2$ (see [19]).

Define the function $\phi : \mathcal{F} \longrightarrow (0, \infty)$ by
\[ \phi(A) = \frac{\|1_A\|_{p,q}}{\mathbb{P}(A)} \]
for all $A \in \mathcal{F}$, $\mathbb{P}(A) \neq 0$. We then define the Campanato space $L_{2,\phi}$ as
\[ L_{2,\phi} := \left\{ f \in L^0_2 : \|f\|_{L_{2,\phi}} := \sup_{\nu \in \mathcal{T}} \frac{1}{\phi(B_\nu)} \left( \frac{1}{\mathbb{P}(B_\nu)} \int_{B_\nu} |f - f_\nu|^2 d\mathbb{P} \right)^{\frac{1}{2}} < \infty \right\}. \]

Our characterization of the dual space of $H^{s}_{p,q}$ spaces for $0 < p \leq q \leq 1$ is as follows.

**Theorem 3.9.** Let $0 < p \leq q \leq 1$. For $\kappa \in (H^{s}_{p,q})^*$, the dual space of $H^{s}_{p,q}$, there exists $g \in L_{2,\phi}$ such that
\[ \kappa(f) = \mathbb{E}[fg] \quad \text{for all} \quad f \in H^{s}_{p,q}, \]
and
\[ \|g\|_{L_{2,\phi}} \leq c\|\kappa\|. \]

Conversely, let $g \in L_{2,\phi}$. Then the mapping
\[ \kappa_g(f) = \mathbb{E}[fg] = \int_{\Omega} fg \, d\mathbb{P}, \quad \forall f \in L^2(\Omega) \]
can be extended to a continuously linear functional on $H^{s}_{p,q}$ such that
\[ \|\kappa\| \leq c\|g\|_{L_{2,\phi}}. \]

For next result, we assume again that the sequence $\{\Omega_j\}_{j \in \mathbb{Z}}$ in the definition of $L_{p,q}(\Omega)$ is such that $\Omega_j \in \mathcal{F}_n$ for any $j \in \mathbb{Z}$ and $n \geq 1$. 
Theorem 3.10. If either $1 < q \leq p \leq 2$ or $2 \leq p \leq q < \infty$, then the dual space of $H_{p,q}^s$ identifies with $H_{p',q'}^{s'}$ where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.

4. PROOF OF ATOMIC DECOMPOSITIONS

We prove here the atomic decompositions of the spaces $H_{p,q}^s$, $H_{p,q}^s$, $H_{p,q}^s$, $Q_{p,q}$ and $P_{p,q}$.

Proof of Theorem 3.3. We only present here the case of $H_{p,q}^s$. The proof of the atomic decomposition of the other spaces is obtained mutatis mutandis.

Let $f$ be in $H_{p,q}^s$ and define the stopping time as

$$\nu^k := \inf\{n \in \mathbb{N} : s_{n+1}(f) > 2^k\}. \quad (10)$$

It is clear that $(\nu^k)_{k \in \mathbb{Z}}$ is nonnegative and nondecreasing. Take $\lambda_k = 2^{k+1}\mathbb{P}(\nu^k \neq \infty)^{\frac{1}{2}}$, and

$$a^k = \frac{f_{\nu^{k+1}} - f_{\nu^k}}{\lambda_k}, \quad \text{if } \lambda_k \neq 0 \quad \text{and} \quad a^0 = 0, \quad \text{if } \lambda_k = 0. \quad (11)$$

Recalling that

$$d_n f_{\nu^k} = f_{n}^{\nu^k} - f_{n-1}^{\nu^k} = \sum_{m=0}^{n} 1_{\{\nu^k \geq m\}} d_{m} f - \sum_{m=0}^{n-1} 1_{\{\nu^k \geq m\}} d_{m} f = 1_{\{\nu^k \geq n\}} d_{n} f,$$

we obtain that

$$s(f_{\nu^k}) = \left( \sum_{n \in \mathbb{N}} E_{n-1} |d_{n} f_{\nu^k}|^2 \right)^\frac{1}{2} = \left( \sum_{n \in \mathbb{N}} E_{n-1} 1_{\{\nu^k \geq n\}} |d_{n} f|^2 \right)^\frac{1}{2} = \left( \sum_{n=0}^{\nu^k} E_{n-1} |d_{n} f|^2 \right)^\frac{1}{2} = s_{\nu^k}(f).$$

Thus by the definition of our stopping time, $s(f_{\nu^k}) = s_{\nu^k}(f) \leq 2^k$. Moreover,

$$\sum_{k \in \mathbb{Z}} (f_{\nu^{k+1}} - f_{\nu^k}) = f_n \text{ a.e.}$$

It follows that $(f_n^{\nu^k})_{n \geq 0}$ is an $L_2$-bounded martingale and so is $(a_n^{\nu^k})_{n \geq 0}$. Consequently, the limit

$$\lim_{n \to \infty} a_n^{\nu^k}$$

exists a.e. in $L_2$. Hence (2) is satisfied.
Let us check that \( a^k \) is an \((p, r)^s\)-atom. We start by noting that on the set \( \{ n \leq \nu^k \} \) we have

\[
a^k_n = \frac{f_{\nu^k+1} - f_{\nu^k}}{\lambda_k} = \frac{f_n - f_n}{\lambda_k} = 0
\]

by definition of stopped martingales and thus \( E_n a^k = 0 \) when \( \nu^k \geq n \). Hence \( a^k \) satisfies condition (a1) in Definition 3.1.

Also we note that equation (12) also implies that

\[
1_{\{ \nu^k = \infty \}} [s(a^k)]^2 \leq \sum_{n \in \mathbb{N}} 1_{\{ \nu^k \geq n \}} E_{n-1} |d_n a^k|^2 = \sum_{n \in \mathbb{N}} 1_{\{ \nu^k \geq n \}} E_{n-1} |1_{\{ \nu^k \geq n \}} d_n a^k|^2 = 0.
\]

That is the support of \( s(a^k) \) is contained in \( B_{\nu^k} = \{ \nu^k \neq \infty \} \).

Observing that

\[
d_n a^k = \frac{d_n (f_{\nu^k+1} - f_{\nu^k})}{\lambda_k} = \frac{(d_n f) 1_{\{ \nu^k < n \leq \nu^k+1 \}}}{\lambda_k},
\]

we obtain the following

\[
[s(a^k)]^2 = \sum_{n \in \mathbb{N}} E_{n-1} |d_n a^k|^2 \leq \left( \frac{s_{\nu^k+1}(f)}{\lambda_k} \right)^2 \leq \left( \frac{2^{k+1}}{\lambda_k^2} \right)^2 = \left( \mathbb{P}(B_{\nu^k})^{-\frac{1}{p'}} \right)^2.
\]

Therefore \( s(a^k) \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p'}} \) and as \( s(a^k) = 0 \) outside \( B_{\nu^k} \), we easily obtain that

\[
\| s(a^k) \|_r \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{r'} + \frac{1}{p'}}.
\]

Thus condition (a2) in the definition of an \((p, r)^s\)-atom also holds.

We next check that

\[
\sum_k \lambda_k a^k \rightarrow f \text{ in } H^s_{p,q}.
\]

As

\[
\lambda_k a^k = f_{\nu^k+1} - f_{\nu^k},
\]

we obtain that

\[
\sum_{k=l}^m \lambda_k a^k = \sum_{k=l}^m (f_{\nu^k+1} - f_{\nu^k}) = f_{\nu^{m+1}} - f_{\nu^l}.
\]

Hence we have that

\[
f - \sum_{k=l}^m \lambda_k a^k = (f - f_{\nu^{m+1}}) + f_{\nu^l}.
\]

Now by definition,

\[
\| f - f_{\nu^{m+1}} \|_{H^s_{p,q}} = \| s(f - f_{\nu^{m+1}}) \|_{p,q}.
\]
Thus for $\Omega_j$ as in the definition of amalgam spaces,

$$
\|s(f - f^{\nu_{m+1}})\|_{p,q} = \left[ \sum_{j \in \mathbb{Z}} \left( \int_{\Omega_j} s^p(f - f^{\nu_{m+1}}) \mathbf{1}_{\Omega_j} \, d\mathbb{P} \right)^{\frac{q}{q'}} \right]^{\frac{1}{q'}}
$$

As $s^p(f - f^{\nu_{m+1}}) \leq s^p(f)$ and

$$
\int_{\Omega_j} s^p(f) \, d\mathbb{P} < \infty,
$$

it follows from the Dominated Convergence Theorem that $\|s(f - f^{\nu_{m+1}})\|_{p,q} \rightarrow 0$ as $m \rightarrow \infty$.

Hence as $\sum_{j \in \mathbb{Z}} \|s^q(f)\|_{p,\Omega_j} = \|f\|_{H^{s,p}_q} < \infty$, applying the Dominated Convergence Theorem for the sequence space $\ell_q$, we conclude that

$$
\|f - f^{\nu_{m+1}}\|_{H^{s,p}_q} = \|s(f - f^{\nu_{m+1}})\|_{p,q} \rightarrow 0
$$
as $m \rightarrow \infty$.

Also since $s(f^{\nu_k}) \leq 2^k$, we have that

$$
\|f^{\nu^l}\|_{H^{s,p}_q} = \|s(f^{\nu^l})\|_{p,q} \leq 2^l.
$$

Hence $\|f^{\nu^l}\|_{H^{s,p}_q} \rightarrow 0$ as $l \rightarrow -\infty$. Thus (13) implies that

$$
\left\| f - \sum_{k=l}^{m} \lambda_k a^k \right\|_{H^{s,p}_q} = \|f - f^{\nu_{m+1}} + f^{\nu^l}\|_{H^{s,p}_q}
$$

$$
\leq \|f - f^{\nu_{m+1}}\|_{H^{s,p}_q} + \|f^{\nu^l}\|_{H^{s,p}_q} \rightarrow 0
$$
as $m \rightarrow \infty$ and $l \rightarrow -\infty$. Hence

$$
\sum_{k=l}^{m} \lambda_k a^k \rightarrow f
$$
in $H^{s}_p$ as $m \rightarrow \infty$, $l \rightarrow -\infty$.

Let us now establish (13). Let $\Omega_j \subset \Omega$ be as in the definition of amalgam space. Then by definition,

$$
\left\| \sum_{k \in \mathbb{Z}} \left( \frac{\lambda_k}{\mathbb{P}(B_{\nu_k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu_k}} \right\|_{\frac{p}{2},q,\frac{q}{2}} = \left[ \sum_{j \in \mathbb{Z}} \left( \int_{\Omega} \left( \sum_{k \in \mathbb{Z}} \left( \frac{\lambda_k}{\mathbb{P}(B_{\nu_k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu_k}} \right) \frac{\mathbf{1}_{\Omega_j}}{\mathbb{P}} \right)^{\frac{q}{q'}} \right]^{\frac{1}{q'}}.
$$
Considering the inner sum, we see that
\[
\sum_{k \in \mathbb{Z}} \left( \frac{\lambda_k}{\mathbb{P}(B_{\nu_k})^{\frac{1}{p}}} \right)^\eta 1_{B_{\nu_k}} = \sum_{k \in \mathbb{Z}} \left( \frac{2^{k+1} \mathbb{P}(\nu_k \neq \infty)^{\frac{1}{p}}}{\mathbb{P}(B_{\nu_k})^{\frac{1}{p}}} \right)^\eta 1_{B_{\nu_k}} = \sum_{k \in \mathbb{Z}} (2^{k+1})^\eta 1_{B_{\nu_k}}.
\]
We shall borrow an idea from [23, pp 21-22]. Let \( G_k = B_{\nu_k} \setminus B_{\nu_{k+1}} \) where \( B_{\nu_k} = \{\nu_k \neq \infty\} \). Then \( G_k \) are disjoint such that \( B_{\nu_k} = \bigcup_{r=k}^{\infty} G_r \) and

(14)
\[
1_{B_{\nu_k}} = \sum_{r=k}^{\infty} 1_{G_r}.
\]

Hence
\[
\sum_{k \in \mathbb{Z}} (2^{k+1})^\eta 1_{B_{\nu_k}} = \sum_{k \in \mathbb{Z}} \sum_{r=k}^{\infty} 1_{G_r} = \sum_{r \in \mathbb{Z}} \sum_{k \leq r} 2^{(k+1)\eta} 1_{G_r} \leq \frac{2^\eta}{2^\eta - 1} \sum_{k \in \mathbb{Z}} 2^{(k+1)\eta} 1_{G_k}.
\]

Thus
\[
\sum_{k \in \mathbb{Z}} (2^{k+1})^\eta 1_{B_{\nu_k}} \leq \frac{4^\eta}{2^\eta - 1} \left( \sum_{k \in \mathbb{Z}} s(f) 1_{G_k} \right)^\eta = \frac{(4^\eta s(f)^\eta}{2^\eta - 1} \sum_{k \in \mathbb{Z}} 1_{G_k}.
\]

It follows that
\[
\left\| \sum_{k \in \mathbb{Z}} \left( \frac{\lambda_k}{\mathbb{P}(B_{\nu_k})^{\frac{1}{p}}} \right)^\eta 1_{B_{\nu_k}} \right\|_{L^{\frac{p}{1+\eta}, \frac{q}{1+\eta}}} = \left\| \sum_{j \in \mathbb{Z}} \left( \int_{\Omega} \left( \sum_{k \in \mathbb{Z}} \left( \frac{\lambda_k}{\mathbb{P}(B_{\nu_k})^{\frac{1}{p}}} \right)^\eta 1_{B_{\nu_k}} \right)^\frac{1}{\eta} \Omega_{\nu_k} \right)^\frac{1}{\eta} \right\|_{L^{\frac{p}{1+\eta}, \frac{q}{1+\eta}}} \leq \left\| \sum_{j \in \mathbb{Z}} \left( \int_{\Omega} \left( \frac{4^\eta s(f)^p}{(2^\eta - 1)^\eta} \sum_{k \in \mathbb{Z}} 1_{G_k} \Omega_{\nu_k} \right)^\frac{1}{p} \right)^\frac{1}{\eta} \right\|_{L^{\frac{p}{1+\eta}, \frac{q}{1+\eta}}} \leq \left( \frac{4^\eta}{2^\eta - 1} \right) \left\| s(f) \right\|_{p,q} = \left( \frac{4^\eta}{2^\eta - 1} \right)^{\frac{1}{\eta}} \| f \|_{H^{p,q}}.
\]
That is
\[
\left\| \sum_{k \in \mathbb{Z}} \left( \frac{\lambda_j}{\mathbb{P}(B_{\nu_k})^{\frac{1}{p}}} \right)^\eta 1_{B_{\nu_k}} \right\|_{L^{\frac{p}{1+\eta}, \frac{q}{1+\eta}}} \leq \left( \frac{4^\eta}{2^\eta - 1} \right)^{\frac{1}{\eta}} \| s(f) \|_{p,q} = \left( \frac{4^\eta}{2^\eta - 1} \right)^{\frac{1}{\eta}} \| f \|_{H^{p,q}}.
\]

The first part of the theorem is then established.
Conversely, let the martingale \( f \) have a representation as in (2). Then as \( s(a^k) \leq \mathbb{P}(\nu^k \neq \infty)^{\frac{1}{p}} \) with support in \( B_{\nu^k} \), we obtain that

\[
\| f \|_{H^s_{p,q}} = \| s(f) \|_{p,q} \leq \left\| \sum_{k \in \mathbb{Z}} \lambda_k s(a^k) \right\|_{p,q}
\]

\[
\leq \left\| \sum_{k \in \mathbb{Z}} \lambda_k \mathbb{P}(B_{\nu^k})^{-\frac{1}{p}}1_{B_{\nu^k}} \right\|_{p,q}
\]

\[
= \left\| \sum_{k \in \mathbb{Z}} \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}1_{B_{\nu^k}} \right\|_{p,q}.
\]

Let us quickly check that

\[
\left\| \sum_{k \in \mathbb{Z}} \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}1_{B_{\nu^k}} \right\|_{p,q} \leq \left\| \sum_{k \in \mathbb{Z}} \left( \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^\eta 1_{B_{\nu^k}} \right\|_{\frac{1}{\eta}, \frac{q}{\eta}}
\]

for \( 0 < \eta < 1 \). Indeed by definition,

\[
\left\| \sum_{k \in \mathbb{Z}} \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}1_{B_{\nu^k}} \right\|_{p,q} = \left\[ \sum_{j \in \mathbb{Z}} \left( \int \Omega \left( \sum_{k \in \mathbb{Z}} \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}1_{B_{\nu^k}} \right)^p 1_{\Omega_j} \mathbb{d}\mathbb{P} \right)^{\frac{\eta}{q}} \right\]^{\frac{1}{\eta}}
\]

\[
= \left\[ \sum_{j \in \mathbb{Z}} \left( \int \Omega \left( \sum_{k \in \mathbb{Z}} \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}1_{B_{\nu^k}} \right)^{\eta} 1_{\Omega_j} \mathbb{d}\mathbb{P} \right)^{\frac{q}{\eta}} \right\]^{\frac{1}{\eta}}
\]

\[
= \left\| \sum_{k \in \mathbb{Z}} \left( \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^\eta 1_{B_{\nu^k}} \right\|_{\frac{1}{\eta}, \frac{q}{\eta}}.
\]

Hence

\[
\| f \|_{H^s_{p,q}} \leq \left\| \sum_{k \in \mathbb{Z}} \left( \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^\eta 1_{B_{\nu^k}} \right\|_{\frac{1}{\eta}, \frac{q}{\eta}},
\]

establishing the converse. The theorem is proved.

\( \square \)

**Proof of Theorem 3.4.** The proof of Theorem 3.4 follows similarly. The main changes are as follows.

Let \( f \) be in \( H^s_{p,q} \) and define the stopping time

\[
\nu^k := \inf \{ n \in \mathbb{N} : s_{n+1}(f) > 2^k \}.
\]

(15)
The sequence here is taken as $\lambda_k = 2^{k+1} \| 1_{B_{\nu_k}} \|_{p,q}$, and again

$$a^k = \frac{f^{\nu_k+1} - f^{\nu_k}}{\lambda_k}, \text{ if } \lambda_k \neq 0 \text{ and } a^k = 0, \text{ if } \lambda_k = 0.$$  

(16)

As above one obtains that

$$s(a^k) \leq \| 1_{B_{\nu_k}} \|_{p,q}^{-1}$$

and $s(a^k) = 0$ on \{ $\nu_k = \infty$ \}. It follows that

$$\| s(a^k) \|_r \leq \| 1_{B_{\nu_k}} \|_{p,q}^{-1} \mathbb{P}(B_{\nu_k})^{\frac{1}{r}}.$$  

The remaining of the proof follows as above.

Proof of Theorem 3.3. Let $f \in Q_{p,q}$ (resp. $f \in P_{p,q}$) Then there exists an adapted non-decreasing, non-negative sequence $(\beta_n)_{n \in \mathbb{N}}$ such that

$$S_n(f) \leq \beta_{n-1} \text{ (resp. } |f_n| \leq \beta_{n-1})$$

and

$$\| \beta_\infty \|_{p,q} \leq 2 \| f \|_{Q_{p,q}} \text{ (resp. } \| f \|_{P_{p,q}}).$$

As stopping time, we take

$$\nu^k := \inf\{ n \in \mathbb{N} : \beta_n > 2^k \}$$

(17)

and define $\lambda_k = 2^{k+2} \mathbb{P}(\nu_k \neq \infty)^{\frac{1}{2}}$, and

$$a^k = \frac{f^{\nu_k+1} - f^{\nu_k}}{\lambda_k} \text{ if } \lambda_k \neq 0 \text{, and } a^k = 0 \text{ otherwise.}$$

(18)

As in Theorem 3.3, we obtain that $a^k$ satisfies condition (a1) in the definition of $(p, \infty)^S$-atom (resp. $(p, \infty)^\ast$-atom). Also, $a^k_n = 0$ on \{ $\nu_k = \infty$ \} for all $n \geq 0$, and the support of $S(a^k)$ (resp. $(a^k)^\ast$) is contained in $B_{\nu_k}$.

We have that

$$S(f^{\nu_k}) = S_{\nu^k}(f) \leq \beta_{\nu^k-1} \leq 2^k \text{ (resp. } (f^{\nu_k})^\ast \leq \beta_{\nu^k-1} \leq 2^k).$$
We also obtain that
\[
[S(a^k)]^2 = \sum_{n \geq 0} |d_n a^{\nu^k}|^2 \leq \sum_{n \geq 0} \left| \frac{(d_n f) 1_{\nu^k < n \leq \nu^k + 1}}{\lambda_k} \right|^2
\]
\[
\leq \left( \frac{S_{\nu^k + 1}(f)}{\lambda_k} \right)^2 \leq \left( \frac{2^{k + 1}}{\lambda_k} \right)^2 \leq P(B_{\nu^k})^{-\frac{2}{p}}.
\]
Also
\[
\left( \text{resp. } (a^k)^s \leq \frac{(f_{\nu^k + 1})^s + (f_{\nu^k})^s}{\lambda_k} \leq \frac{2^{k + 2}}{\lambda_k} = P(B_{\nu^k})^{-\frac{1}{p}} \right).
\]
Thus \(\|S(a^k)\|_\infty \leq P(\nu^k \neq \infty)^{-\frac{1}{p}}\) (resp. \(\|(a^k)^s\|_\infty \leq P(\nu^k \neq \infty)^{-\frac{1}{p}}\)). Hence condition (a2) in the definition of an \((p, \infty)\)\(^s\)-atom (resp. \((p, \infty)^s\)-atom) is satisfied.

We next prove that \(\sum_{k=1}^m \lambda_k a^k\) converges to \(f\) in \(Q_{p,q}\) (resp. \(P_{p,q}\)) as \(l \to -\infty\) and \(m \to \infty\). As usual, define
\[
\zeta_{n-1}^j = 1_{\{\nu^k \leq n-1\}} \|S(a^k)\|_\infty \quad \text{and} \quad (\zeta_{n-1})^2 = \sum_{k=m+1}^{\infty} \lambda_k^2 (\zeta_{n-1}^k)^2
\]
\[
\left( \text{resp. } \zeta_{n-1}^j = 1_{\{\nu^k \leq n-1\}} \|(a^k)^s\|_\infty \right. \quad \text{and} \quad \zeta_{n-1}^j = \sum_{k=m+1}^{\infty} \lambda_k (\zeta_{n-1}^k). \]
Then we have (see [21, p. 17])
\[
S_n(f - f_{\nu^m + 1}) \leq \left( \sum_{k=m+1}^{\infty} \lambda_k^2 (\zeta_{n-1}^k)^2 \right)^{\frac{1}{2}} = \zeta_{n-1} \left( \text{resp. } |f_n - f_{\nu^m}| \leq \zeta_{n-1}. \right)
\]
Putting \(T(a^k) = S(a^k), (a^k)^s\), we obtain
\[
\zeta_{n-1} \leq \sum_{k=m+1}^{\infty} \lambda_k \zeta_{n-1}^k \leq \sum_{k=m+1}^{\infty} \lambda_k \|T(a^k)\|_\infty 1_{\{\nu^k \leq n-1\}} \leq \sum_{k=m+1}^{\infty} \frac{\lambda_k}{P(B_{\nu^k})^\frac{1}{p}} 1_{B_{\nu^k}}.
\]
It follows that
\[
S(f - f_{\nu^m + 1}) \left( \text{resp. } (f - f_{\nu^m + 1})^s \right) \leq \lim_{n \to \infty} \zeta_n \leq \sum_{k=m+1}^{\infty} \frac{\lambda_k}{P(B_{\nu^k})^\frac{1}{p}} 1_{B_{\nu^k}} = \sum_{k=m+1}^{\infty} 2^{k+2} 1_{B_{\nu^k}}.
\]
Hence
\[
\|f - f_{\nu^m + 1}\|_{Q_{p,q}}^q \left( \text{resp. } \|f - f_{\nu^m + 1}\|_{P_{p,q}}^q \right) \leq \|\zeta_n\|_{p,q}^q \leq \sum_{j \in \mathbb{Z}} \left\| \left( \sum_{k=m+1}^{\infty} 2^{k+2} 1_{B_{\nu^k}} \right) 1_{\Omega_j} \right\|^p_q.
\]
Proceeding as in the proof of Theorem 3.3, we obtain that
\[
\left( \sum_{k=m+1}^{\infty} 2^{k+2} 1_{B_{\nu^k}} \right) 1_{\Omega_j} \leq C \beta_\infty 1_{\Omega_j},
\]
Hence as $\|\beta_\infty 1_{\Omega_j}\|_p < \infty$, it follows from the the Dominated Convergence Theorem that

$$\left\| \left( \sum_{k=m+1}^{\infty} 2^{k+2} 1_{B_k} \right) 1_{\Omega_j} \right\|_p \to 0 \text{ as } m \to \infty.$$  

As

$$\left\| \left( \sum_{k=m+1}^{\infty} 2^{k+2} 1_{B_k} \right) 1_{\Omega_j} \right\|_p \leq C \|\beta_\infty 1_{\Omega_j}\|_p$$

and as

$$\sum_{j \in \mathbb{Z}} \|\beta_\infty 1_{\Omega_j}\|_p^q = \|\beta_\infty\|_{p,q}^q < \infty,$$

an application of the Dominated Convergence Theorem for sequence spaces leads to

$$\sum_{j \in \mathbb{Z}} \left\| \left( \sum_{k=m+1}^{\infty} 2^{k+2} 1_{B_k} \right) 1_{\Omega_j} \right\|_p^q \to 0 \text{ as } m \to \infty.$$  

Thus $\|f - f^\nu m+1\|_{Q_{p,q}}$ (resp. $\|f - f^\nu m+1\|_{P_{p,q}}$) $\to 0$ as $m \to \infty$.

Similarly, we obtain that $\|f^\nu\|_{Q_{p,q}}$ (resp. $\|f^\nu\|_{P_{p,q}}$) $\to 0$ as $l \to -\infty$. Therefore

$$\|f - \sum_{k=l}^{m} \lambda_k a^k\|_{Q_{p,q}} \text{ (resp. } \|f - \sum_{k=l}^{m} \lambda_k a^k\|_{P_{p,q}}) \to 0$$

as $m \to \infty$ and $l \to -\infty$. Hence

$$\sum_{k=l}^{m} \lambda_k a^k \to f$$

in $Q_{p,q}$ (resp. $P_{p,q}$) as $m \to \infty$, $l \to -\infty$ and thus for all $n \in \mathbb{N}$,

$$\sum_{k \in \mathbb{Z}} \lambda_k E_n a^k = f_n.$$  

Now, as in Theorem 3.3, we obtain

$$\left\| \sum_{k \in \mathbb{Z}} \left( \frac{\lambda_k}{\mathbb{P}(B_k)^{\frac{1}{p}} \mathbb{P}(B_k)^{\frac{1}{q}}} \right)^{\eta} 1_{B_k} \right\|_{\frac{1}{\eta}} \leq \left\| \sum_{k \in \mathbb{Z}} (2^{k+2})^{\eta} 1_{B_k} \right\|_{\frac{1}{\eta}} \leq C \|\beta_\infty\|_{p,q} \leq 2C\|f\|_{Q_{p,q}} \text{ (resp. } \|f\|_{P_{p,q}}).$$

Conversely, assume that $f \in \mathcal{M}$ has the decomposition (6). Define $\beta_n$ by

$$\beta_n := \sum_{k \in \mathbb{Z}} \lambda_k \|S(a^k)\|_{\infty} 1_{\{\nu^k \leq n\}} \text{ (resp. } \beta_n := \sum_{k \in \mathbb{Z}} \lambda_k \|(a^k)^*\|_{\infty} 1_{\{\nu^k \leq n\}}).$$
Then \((\beta_n)_{n \geq 0}\) is a nondecreasing nonnegative adapted sequence also, for \(n \geq 0\),

\[
S_n(f) \leq \beta_{n-1} \quad \text{(resp. } |f_n| \leq \beta_{n-1} \text{)}.
\]

As \(\|S(a^k)\|_\infty \) (resp. \(\|(a^k)^*\|_\infty\)) \(\leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}}\), it follows that

\[
\|\beta_\infty\|_{p,q} \leq \left\| \sum_{k \in \mathbb{Z}} \frac{\lambda_k}{\mathbb{P}(B_{\nu,k})^{\frac{1}{p}}} 1_{B_{\nu,k}} \right\|_{p,q} \leq \left\| \sum_{k \in \mathbb{Z}} \left( \frac{\lambda_k}{\mathbb{P}(B_{\nu,k})^{\frac{1}{p}}} \right)^\eta 1_{B_{\nu,k}} \right\|_{\frac{1}{\eta}, \frac{\eta}{p}}.
\]

Thus

\[
\|f\|_{Q_{p,q}} \quad \text{(resp. } \|f\|_{P_{p,q}} \text{)} \leq \|\beta_\infty\|_{p,q} \leq \left\| \sum_{k \in \mathbb{Z}} \left( \frac{\lambda_k}{\mathbb{P}(B_{\nu,k})^{\frac{1}{p}}} \right)^\eta 1_{B_{\nu,k}} \right\|_{\frac{1}{\eta}, \frac{\eta}{p}}.
\]

The proof is complete. \(\square\)

The proof of Theorem 3.6 follows similarly. We leave it to the interested reader.

## 5. Proof of martingale inequalities

In this section we shall discuss the various inclusions of the martingale Hardy-amalgam spaces \(H^S_{p,q} \), \(H^*_{p,q} \), \(H^{*}_{p,q} \), \(Q_{p,q}\) and \(P_{p,q}\). Weisz [21] has discussed the classical cases including the Doob’s maximal inequality for \(p > 1\) and the concavity and the convexity theorems. We recall that here, we assume that the disjoint cover \((\Omega_j)_{j \in \mathbb{Z}}\) is such that \(\Omega_j \in \mathcal{F}_n\) for all \(j \in \mathbb{Z}\) and all \(n \geq 1\).

We refer to [21] Theorem 2.11 for the following.

**Proposition 5.1.** For any \(f \in \mathcal{M}\), the following hold.

(i) \(\|f\|_{H^p_{\nu}} \leq C_p \|f\|_{H^p_{\nu}}\), \(\|f\|_{H^p_{\nu}} \leq C_p \|f\|_{H^p_{\nu}}\) \((0 < p \leq 2)\)

(ii) \(\|f\|_{H^p_{\nu}} \leq C_p \|f\|_{H^p_{\nu}}\), \(\|f\|_{H^p_{\nu}} \leq C_p \|f\|_{H^p_{\nu}}\) \((2 \leq p < \infty)\)

(iii) \(\|f\|_{H^p_{\nu}} \leq C_p \|f\|_{P_{p,q}}\), \(\|f\|_{H^p_{\nu}} \leq C_p \|f\|_{P_{p,q}}\) \((0 < p < \infty)\)

(iv) \(\|f\|_{H^p_{\nu}} \leq C_p \|f\|_{Q_{p,q}}\), \(\|f\|_{H^p_{\nu}} \leq C_p \|f\|_{Q_{p,q}}\) \((0 < p < \infty)\)

(v) \(\|f\|_{H^p_{\nu}} \leq C_p \|f\|_{P_{p,q}}\), \(\|f\|_{H^p_{\nu}} \leq C_p \|f\|_{Q_{p,q}}\) \((0 < p < \infty)\).

Moreover, if the \((\mathcal{F})_{n \geq 0}\) is regular, the above five spaces are equivalent.

We observe the following.

**Lemma 5.2.** Assume that \(A \in \mathcal{F}_n\) for all \(n \geq 1\). Then if \(f \in \mathcal{M}\), then \(f 1_A = (f_n 1_A)_{n \geq 0}\) is also a martingale in \(\mathcal{M}\). Moreover, if \(T\) is any of the operators \(s\), \(S\) and \(M\) (the maximal operator), then

\[
T(f 1_A) = T(f) 1_A.
\]
Combining the above lemma with [21, Lemma 2.20], we obtain the following.

**Lemma 5.3.** Assume that $A \in F_n$ for all $n \geq 1$. Then for any martingale $f \in \mathcal{M}$ and $0 < p < \infty$, we have

$$
\mathbb{E} \left[ \sup_n \mathbb{E}_{n-1} (|f_n|^p 1_A) \right] \leq 2 \mathbb{E} \left( (f^*)^p 1_A \right)
$$

and

$$
\mathbb{E} \left[ \sup_n \mathbb{E}_{n-1} (|S_n(f)|^p 1_A) \right] \leq 2 \mathbb{E} \left( S(f)^p 1_A \right).
$$

**Proof of Theorem 3.7.** Let $T$ be any of the operators $s, S$ and $M$, and $H^T_p(\Omega), H^T_{p,q}(\Omega)$ the corresponding martingale spaces, then by the hypothesis on the sequence $(\Omega_j)_{j \in \mathbb{Z}}$ and Lemma 5.2, we have that for any martingale $f \in \mathcal{M}$ and any $j \in \mathbb{Z},$

$$
\int_{\Omega} T(f)^p 1_{\Omega_j} d\mathbb{P} = \int_{\Omega} T(f 1_{\Omega_j})^p d\mathbb{P} = \|f 1_{\Omega_j}\|_{H^T_p}^p,
$$

and consequently,

$$
\|T(f)\|_{H^T_{p,q}}^q = \sum_j \|T(f) 1_{\Omega_j}\|_{L^p}^q = \sum_j \|f 1_{\Omega_j}\|_{H^T_p}^q.
$$

The two first assertions of the theorem then follow from (19) and Proposition 5.1.

To obtain the other assertions, following (19) and Proposition 5.1, we only need to prove that

$$
\sum_j \|f 1_{\Omega_j}\|_{Q_p}^q \leq C \|f\|_{Q_{p,q}}^q
$$

and

$$
\sum_j \|f 1_{\Omega_j}\|_{P_p}^q \leq C \|f\|_{P_{p,q}}^q.
$$

We only prove (20) as (21) follows similarly.

Let $(\beta_n)_{n \geq 0}$ be an arbitrary nonnegative nondecreasing adapted sequence such that

$$
S_n(f) \leq \beta_{n-1}, \text{ and } \|\beta_\infty\|_{p,q} < \infty.
$$

We have that the sequence $(\gamma_n)_{n \geq 0} = (\beta_n 1_{\Omega_j})_{n \geq 0}$ is also nonnegative nondecreasing and adapted, and

$$
S_n(f 1_{\Omega_j}) = S_n(f 1_{\Omega_j}) \leq \beta_{n-1} 1_{\Omega_j} = \gamma_{n-1}^j, \text{ and } \|\gamma_\infty^j\|_p = \|\beta_\infty 1_{\Omega_j}\|_p \leq \|\beta_\infty\|_{p,q} < \infty.
$$
It follows that
\[ \sum_j \| f \mathbf{1}_{\Omega_j} \|_{Q_p}^q \leq \sum_j \| \gamma_j \|_{p}^q = \sum_j \| \beta_\infty \mathbf{1}_{\Omega_j} \|_{p}^q = \| \beta_\infty \|_{p,q}^q. \]

As the sequence \((\beta_n)_{n \geq 0}\) was chosen arbitrarily, we conclude that
\[ \sum_j \| f \mathbf{1}_{\Omega_j} \|_{Q_p}^q \leq \inf_{\beta \in \Gamma} \| \beta_\infty \|_{p,q}^q = \| f \|_{Q_p,q}^q. \]

Let us now assume that the \((\mathcal{F}_n)_{n \geq 0}\) is regular. To prove the equivalence between the five spaces, we only need to prove that

\[ \| f \|_{Q_{p,q}} \leq C \| f \|_{H^S_{p,q}} \] \hspace{1cm} (22)

and

\[ \| f \|_{P_{p,q}} \leq C \| f \|_{H^*_p} \] \hspace{1cm} (23)

We only prove the (22) since the proof of (23) use similar arguments.

Let \( f = (f_n)_{n \geq 0} \) be a martingale in \( H^S_{p,q}(\Omega) \). Then using the definition of the regularity, one obtain that

\[ S_n(f) \leq \left[ C_p \left( S^p_{n-1}(f) + \mathbb{E}_{n-1}(S^p_n(f)) \right) \right]^{\frac{1}{p}} \] \hspace{1cm} (24)

(see [21, p. 39]). Define the sequence \( \beta = (\beta_n)_{n \geq 0} \) by

\[ \beta_n = \left[ C_p \left( S^p_n(f) + \mathbb{E}_n(S^p_{n+1}(f)) \right) \right]^{\frac{1}{p}}. \]

Then \( \beta \in \Gamma \) and by (24),

\[ S_n(f) \leq \beta_{n-1}. \]

Also, we have that

\[ \beta_\infty = \sup_n \beta_n = \left[ C_p \left( S^p(f) + \sup_n \mathbb{E}_n(S^p_{n+1}(f)) \right) \right]^{\frac{1}{p}}. \]

Then using Lemma 5.2 and Lemma 5.3 we obtain for any \( j \in \mathbb{Z} \),

\[ \| \beta_\infty \mathbf{1}_{\Omega_j} \|_{p} \leq 3C_p \| S^p(f) \mathbf{1}_{\Omega_j} \|_{p}. \]

Hence

\[ \| f \|_{Q_{p,q}} \leq \| \beta_\infty \|_{p,q} \lesssim \| S(f) \|_{p,q} = \| f \|_{H^S_{p,q}}. \]
The proof is complete.

6. Proof of duality results

We start this section by introducing the following result which is essentially [2, Proposition 2.1]. We provide a proof for a continuous reading.

Proposition 6.1. Let $0 < p < 1$ and $0 < q \leq 1$. For all finite sequence $\{f_n\}_{n=-m}^m$ of elements in $L_{p,q}(\Omega)$, we have

$$
\sum_{n=-m}^m \|f_n\|_{p,q} \leq \left\| \sum_{n=-m}^m |f_n| \right\|_{p,q}.
$$

Proof. Let $0 < p < 1$, $0 < q \leq 1$ and let $\{f_n\}_{n=0}^m$ be a finite sequence of elements of $L_{p,q}(\Omega)$. For $q = 1$, using the reverse Minkowski’s inequality in $L_p$ (see [12, p. 11-12]), we obtain

$$
\sum_{n=-m}^m \|f_n\|_{p,1} = \sum_{j \in \mathbb{Z}} \left\| \sum_{n=-m}^m |f_n 1_{\Omega_j}| \right\|_{p,1} \leq \left\| \sum_{n=-m}^m |f_n| \right\|_{p,1}.
$$

Now assume that $0 < q < 1$ and set

$$
x_n := \{\|f_n 1_{\Omega_j}\|_p\}_{j \in \mathbb{Z}} \quad \forall n = -m, \ldots, m.
$$

Applying the reverse Minkowski’s inequality in $\ell^q$ and $L_p$, we obtain

$$
\sum_{n=-m}^m \|f_n\|_{p,q} = \sum_{n=-m}^m \|x_n\|_{\ell^q} \leq \left\| \left\{ \sum_{n=-m}^m \|f_n 1_{\Omega_j}\|_p \right\}_{j \in \mathbb{Z}} \right\|_{\ell^q}
\leq \left\| \left\{ \sum_{n=-m}^m \|f_n 1_{\Omega_j}\|_p \right\}_{j \in \mathbb{Z}} \right\|_{\ell^q} = \left\| \sum_{n=-m}^m |f_n| \right\|_{p,q}.
$$

We next prove our first duality result.

Proof of Theorem 3.9. Let us start by defining some spaces. For $\nu$ a stopping time, we define

$$
L_2^\nu(\Omega) := \{f \in L_2(\Omega) : \mathbb{E}_{n}(f) = 0, \text{ for } \nu \geq n, n \in \mathbb{N}\}
$$

and

$$
L_2^\nu(B_\nu) := \{f \in L_2^\nu(\Omega) : \text{supp}(f) \subseteq B_\nu\}.
$$
We endow $L^p_2(B_\nu)$ with
\[ \|f\|_{L^p_2(B_\nu)} := \left( \int_{B_\nu} |f|^2 d\mathbb{P} \right)^{1 \over 2} < \infty. \]

We will first prove that any continuous linear functional on $H^{s\nu}_{p,q}(\Omega)$ is also continuous on $L^p_2(B_\nu)$.

Let $f \in L^p_2(B_\nu) \setminus \{0\}$ and consider
\[ a(\omega) := C\|f\|^{-1}_{L^p_2(B_\nu)} |P(B_\nu)^{1 \over p \nu} f(\omega), \quad \omega \in \Omega. \]

Then for an appropriate choice of the constant (for a choice of the constant, use [21, Proposition 2.6 and Theorem 2.11]), $a$ is an $(p, 2)^s$-atom associated to $\nu \in T$. Observing with Theorem 3.3 that
\[ \|a\|_{H^{p,q}_s(\Omega)} \lesssim \|1_{B_\nu}\|_{p,q} P(B_\nu)^{-1 \over p}, \]
and recalling that $p \leq q$, we obtain
\[ \|f\|_{H^{p,q}_s(\Omega)} = C^{-1} \|f\|_{L^p_2(B_\nu)} P(B_\nu)^{1 \over p \nu} \|a\|_{H^{p,q}_s(\Omega)} \lesssim \|f\|_{L^p_2(B_\nu)} P(B_\nu)^{1 \over p \nu} \|1_{B_\nu}\|_{p,q} P(B_\nu)^{-1 \over p} \lesssim \|f\|_{L^p_2(B_\nu)} P(B_\nu)^{1 \over p \nu} \|a\|_{H^{p,q}_s(\Omega)}. \]

It follows that for any continuous linear functional $\kappa$ on $H^{p,q}_s(\Omega)$ with operator norm $\|\kappa\|$,\n\[ |\kappa(f)| \leq \|\kappa\| \|f\|_{H^{p,q}_s(\Omega)} \lesssim \|\kappa\| \|P(B_\nu)^{1 \over p \nu} \|f\|_{L^p_2(B_\nu)}. \]

Hence $\kappa$ is continuous on $L^p_2(B_\nu)$ with operator norm\n\[ \|\kappa\|_{(L^p_2(B_\nu))^\ast} := \sup_{\|f\|_{L^p_2(B_\nu)} \leq 1} \|\kappa(f)\| \lesssim \|P(B_\nu)^{1 \over p \nu} \| \|\kappa\|. \]

As $L^p_2(B_\nu)$ is a subspace of $L^2(B_\nu) = L^2(B_\nu, d\mathbb{P})$, it follows from the above observation and the Hahn-Banach Theorem that any continuous linear functional $\kappa$ on $H^{p,q}_s(\Omega)$ can be extended to a continuous linear functional $\kappa_\nu$ on $L^2(B_\nu)$. As $L^2(B_\nu)$ is auto-dual, it follows that there exists $g \in L^2(B_\nu)$ such that
\[ \kappa_\nu(f) = \int_{B_\nu} fg d\mathbb{P}, \quad \forall f \in L^2(B_\nu). \]

Consequently,
\[ \kappa(f) = \kappa_\nu(f) = \int_{B_\nu} fg d\mathbb{P}, \quad \forall f \in L^p_2(B_\nu). \]
Next, as $L_2(\Omega)$ is a dense in $H^s_{p,q}(\Omega)$ (this follows from the fact that $p \leq q < 2$ and Theorem 33), we have that any element $\kappa$ of the dual space of $H^s_{p,q}(\Omega)$ can be represented by

$$\kappa(f) = \int_\Omega f g \, dP, \quad \forall f \in L_2(\Omega). \tag{25}$$

We are going to prove that the function $g$ in (25) is in $L^2(\phi).$

Let $\nu \in \mathcal{T}$ and let $f \in L^2_{\nu}(B_\nu)$ with $\|f\|_{L^2(B_\nu)} \leq 1$. Define

$$a(\omega) = C\mathbb{P}(B_\nu)^{\frac{1}{2} - \frac{1}{p}} \frac{(f - f^\nu)1_{B_\nu}(\omega)}{\|f - f^\nu\|_{L^2(\Omega)}}, \quad \omega \in \Omega.$$ 

Then for an appropriate choice of the constant, $a$ is an $(p, 2)^*$-atom associated to the stopping time $\nu$ and $a \in L_2(\Omega)$. Hence

$$\kappa(a) = \int_\Omega a g \, dP = \int_{B_\nu} a g \, dP.$$ 

Thus

$$\left| \int_{B_\nu} a(g - g^\nu) \, dP \right| = \left| \int_{B_\nu} a g \, dP \right| = |\kappa(a)| \leq \|\kappa\| \|a\|_{H^s_{p,q}(\Omega)} \lesssim \|\kappa\| \|1_{B_\nu}\|_{p,q} \mathbb{P}(B_\nu)^{-\frac{1}{p}}.$$ 

Hence

$$\left| \int_{B_\nu} f(g - g^\nu) \, dP \right| \leq \left| \int_{B_\nu} (f - f^\nu)(g - g^\nu) \, dP \right| \lesssim C^{-1}\|f - f^\nu\|_{L^2(\Omega)} \mathbb{P}(B_\nu)^{\frac{1}{2} - \frac{1}{p}} \|\kappa\| \|1_{B_\nu}\|_{p,q} \mathbb{P}(B_\nu)^{-\frac{1}{p}} \lesssim \mathbb{P}(B_\nu)^{-\frac{1}{2}} \|1_{B_\nu}\|_{p,q} \|\kappa\|.$$ 

Thus

$$\left( \int_{B_\nu} |g - g^\nu|^2 \, dP \right)^{\frac{1}{2}} : = \sup_{\substack{f \in L^2_{\nu}(B_\nu) \\|f\|_{L^2(B_\nu)} \leq 1}} \left| \int_{B_\nu} f(g - g^\nu) \, dP \right| \lesssim \mathbb{P}(B_\nu)^{-\frac{1}{2}} \|1_{B_\nu}\|_{p,q} \|\kappa\|.$$ 

This gives us

$$\frac{1}{\phi(B_\nu)} \left( \frac{1}{\mathbb{P}(B_\nu)} \int_{B_\nu} |g - g^\nu|^2 \, dP \right)^{\frac{1}{2}} \lesssim \|\kappa\|, \quad \forall \nu \in \mathcal{T}.$$
Hence \( g \in L_{2,\phi}(\Omega) \), and the proof of the first part of the theorem is complete.

Conversely, let \( g \in L_{2,\phi}(\Omega) \). Let \( f \in H_{p,q}^s(\Omega) \). We know that for the stopping times

\[
\nu^k := \inf\{n \in \mathbb{N} : s_{n+1}(f) > 2^k\}, \quad k \in \mathbb{Z},
\]

\[
\tag{26}
\left\| \sum_{k \in \mathbb{Z}} \left( \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} 1_{B_{\nu^k}} \right\|_{L^{\frac{1}{\eta},\frac{1}{\eta}}} \leq C\|f\|_{H_{p,q}^s}.
\]

and moreover,

\[
\sum_{k=1}^{m} \lambda_k a^k \longrightarrow f
\]

in \( H_{p,q}^s \) as \( m \to \infty, \, l \to -\infty \), where \( (\lambda_k, a^k, \nu_k) \in \mathcal{A}(p, q, 2)^s \). Also since \( a^k \) is \( L_2 \)-bounded, for \( f \in H_{p,q}^s(\Omega) \),

\[
\kappa_g(f) = \mathbb{E}[fg] = \sum_{k \geq 0} \mathbb{E}[a^k g]
\]

is well defined and linear. Using this, Schwartz’s inequality, and the fact that \( \|s(a^k)\|_2 \leq \mathbb{P}(B_{\nu^k})^{\frac{1}{2} - \frac{1}{p}} \), we obtain

\[
|\kappa_g(f)| \leq \sum_{k \in \mathbb{Z}} \lambda_k \left| \int_{\Omega} a^k (g - g^{\nu^k}) d\mathbb{P} \right| \leq \sum_{k \in \mathbb{Z}} \lambda_k \|a^k\|_2 \left( \int_{B_{\nu^k}} |g - g^{\nu^k}|^2 d\mathbb{P} \right)^{\frac{1}{2}}
\]

\[
\leq \sum_{k \in \mathbb{Z}} \lambda_k \|s(a^k)\|_2 \left( \int_{B_{\nu^k}} |g - g^{\nu^k}|^2 d\mathbb{P} \right)^{\frac{1}{2}}
\]

\[
= \sum_{k \in \mathbb{Z}} \lambda_k \left\| 1_{B_{\nu^k}} \right\|_{p,q} \frac{1}{\mathbb{P}(B_{\nu^k})^{\frac{1}{2}}} \left( \frac{1}{\mathbb{P}(B_{\nu^k})} \int_{B_{\nu^k}} |g - g^{\nu^k}|^2 d\mathbb{P} \right)^{\frac{1}{2}}.
\]

Hence using Proposition \ref{prop:6.1}, we deduce that

\[
|\kappa_g(f)| \leq \|g\|_{L_{2,\phi}} \sum_{k \in \mathbb{Z}} \left\| \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} 1_{B_{\nu^k}} \right\|_{p,q} \leq \|g\|_{2,\phi} \left\| \sum_{k \geq 0} \left( \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} 1_{B_{\nu^k}} \right\|_{\frac{1}{p},\frac{1}{p}} \lesssim \|f\|_{H_{p,q}^s} \|g\|_{2,\phi}.
\]

Thus \( \kappa_g(f) = \mathbb{E}[fg] \) extends continuously on \( H_{p,q}^s(\Omega) \) and the proof is complete. \( \square \)

We finish this section with the proof of Theorem \ref{thm:3.10} For this, we will first prove the following.

**Lemma 6.2.** Let \( 2 \leq p \leq q < \infty \). Then the space \( H_{p,q}^s \) is uniformly convex.

**Proof.** We recall that a Banach space \( \mathcal{H} \) is uniformly convex if for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( x, y \in \mathcal{H} \) with \( \|x\|_{\mathcal{H}} \leq 1, \|y\|_{\mathcal{H}} \leq 1 \) and \( \|x - y\|_{\mathcal{H}} \geq \epsilon \), then \( \|x + y\|_{\mathcal{H}} \leq 2(1 - \delta) \).
We recall that for $1 \leq r < \infty$ and for $a, b > 0$,

$$(a + b)^r \leq 2^{r-1}(a^r + b^r) \quad \text{and} \quad a^r + b^r \leq (a + b)^r.$$ 

Let $\epsilon > 0$, and assume that $f, g \in H^s_{p,q}$ with $\|f\|_{H^s_{p,q}} \leq 1$, $\|g\|_{H^s_{p,q}} \leq 1$ and $\|f - g\|_{H^s_{p,q}} \geq \epsilon$. We start by observing that

$$s^2(f + g) + s^2(f - g) = 2(s^2(f) + s^2(g)).$$

We then obtain

$$(s^2(f + g))^{\frac{p}{q}} + (s^2(f - g))^{\frac{p}{q}} \leq (s^2(f + g) + s^2(f - g))^{\frac{p}{q}} \leq 2^{p-1}[s^p(f) + s^p(g)].$$

Hence for any $j \in \mathbb{Z}$,

$$\|s(f + g)1_{\Omega_j}\|_p^p + \|s(f + g)1_{\Omega_j}\|_p^p \leq 2^{p-1} (\|s(f)1_{\Omega_j}\|_p^p + \|s(g)1_{\Omega_j}\|_p^p).$$

Raising both members of the last inequality to the power $\frac{p}{q} \geq 1$, we obtain

$$\|s(f + g)1_{\Omega_j}\|_p^q + \|s(f + g)1_{\Omega_j}\|_p^q \leq \left(\|s(f + g)1_{\Omega_j}\|_p^p + \|s(f + g)1_{\Omega_j}\|_p^p\right)^{\frac{q}{p}} \leq 2^{\frac{q}{p}(p-1)} (\|s(f)1_{\Omega_j}\|_p^p + \|s(g)1_{\Omega_j}\|_p^p)^{\frac{q}{p}} \leq 2^{\frac{q}{p}(p-1)2^{p-1}} (\|s(f)1_{\Omega_j}\|_p^p + \|s(g)1_{\Omega_j}\|_p^p).$$

Hence taking the sum over $j \in \mathbb{Z}$, we obtain

$$\|s(f + g)\|_{p,q}^q + \|s(f - g)\|_{p,q}^q \leq 2^{q-1} (\|s(f)\|_{p,q}^q + \|s(g)\|_{p,q}^q)$$

and so

$$\|s(f + g)\|_{p,q}^q \leq 2^{q-1} (\|s(f)\|_{p,q}^q + \|s(g)\|_{p,q}^q) - \|s(f - g)\|_{p,q}^q$$

$$\leq 2^q - \epsilon^q.$$ 

Thus $\|s(f + g)\|_{p,q} \leq 2(1 - \delta)$ with $\delta = 1 - \left(1 - \frac{\epsilon^q}{2^q}\right)^{\frac{1}{q}}$. The proof is complete.

From the above lemma and Milman’s theorem (see [25, p. 127]), we deduce the following.

**Corollary 6.3.** Let $2 \leq p \leq q < \infty$. Then the space $H^s_{p,q}$ is reflexive.

It follows from the above corollary that to prove Theorem 3.10, we only need to prove the following results.
Theorem 6.4. Suppose that 1 < q ≤ p ≤ 2. Assume that for any j ∈ \( \mathbb{Z} \), \( \Omega_j \in \mathcal{F}_n \) for all \( n \geq 1 \). Then the dual of \( H^s_{p,q} \) identifies with \( H^s_{p',q'} \) where \( \frac{1}{p'} + \frac{1}{q'} = \frac{1}{q} + \frac{1}{q} = 1 \).

Proof. Let \( g \in H^s_{p',q'} \) and
\[
\kappa_g(f) := \mathbb{E}\left( \sum_{n=0}^{\infty} d_n f d_n g \right) \quad (f \in H^s_{p,q}).
\]

Hence by Schwarz’s inequality, we have that
\[
|\kappa_g(f)| \leq \int_{\Omega} \sum_{n=0}^{\infty} \mathbb{E}_{n-1} |d_n f||d_n g|dP
= \sum_{j \in \mathbb{Z}} \int_{\Omega_j} \sum_{n=0}^{\infty} \mathbb{E}_{n-1} |d_n f||d_n g|dP
\leq \sum_{j \in \mathbb{Z}} \int_{\Omega_j} \left( \sum_{n=0}^{\infty} \mathbb{E}_{n-1} |d_n f|^2 \right)^{\frac{1}{q'}} \left( \sum_{n=0}^{\infty} \mathbb{E}_{n-1} |d_n g|^2 \right)^{\frac{1}{q'}} dP
\leq \sum_{j \in \mathbb{Z}} \int_{\Omega_j} s(f)s(g)dP.
\]

Applying the Hölder’s inequality to the right hand of the last inequality, we obtain
\[
|\kappa_g(f)| \leq \sum_{j \in \mathbb{Z}} \|s(f)1_{\Omega_j}\|_p \|s(g)1_{\Omega_j}\|_{p'}
\leq \left( \sum_{j \in \mathbb{Z}} \|s(f)1_{\Omega_j}\|_p^q \right)^{\frac{1}{q}} \left( \sum_{j \in \mathbb{Z}} \|s(g)1_{\Omega_j}\|_{p'}^q \right)^{\frac{1}{q'}}
= \|f\|_{H^s_{p,q}} \|g\|_{H^s_{p',q'}}.
\]

Thus
\[
|\kappa_g(f)| \leq C\|g\|_{H^s_{p',q'}}.
\]

Conversely, let \( \kappa \) be a continuous linear functional on \( H^s_{p,q} \). Then as \( H^s_{p,q} \) embeds continuously into \( H^s_{p} \) (since \( q < p \)), we have by the Hahn-Banach theorem that \( \kappa \) can be extended to a continuous linear functional \( \tilde{\kappa} \) on \( H^s_{p} \) having the same operator norm as \( \kappa \). It follows from [21, Theorem 2.26] that there exists some \( g \in H^s_{p} \) such that
\[
\tilde{\kappa}(f) = \mathbb{E}(fg) \quad (\forall f \in H^s_{p}).
\]
In particular

\[(27)\quad \kappa(f) = \bar{\kappa}(f) = \mathbb{E}(fg) \quad (\forall f \in H^*_{p,q}).\]

Let us prove that

\[(28)\quad \|g\|_{H^*_{p',q'}} \lesssim \sup_{f \in H^*_{p'}, \|f\|_{H^*_{p,q}} \leq 1} |\kappa(f)| < \infty.\]

Obviously, this holds if \(\|g\|_{H^*_{p',q'}} = 0\). Hence we assume that \(\|g\|_{H^*_{p',q'}} \neq 0\).

We recall that by assumption, \(\Omega_j \in \mathcal{F}_n\) for all \(j \in \mathbb{Z}\) and \(n \geq 1\). Set

\[(29)\quad \mu_n = \sum_{j \in \mathbb{Z}} \frac{s^{p'-2}_n(g)1_{\Omega_j}}{\|s(g)\|^{q'-2}_{p',q'} \|s(g)1_{\Omega_j}\|_{p'-q'}}.\]

Since the \(\Omega_j\)'s are pairwise disjoint, we have that

\[
\mu_n^2 = \sum_{j \in \mathbb{Z}} \frac{s_n^{2p'-4}(g)1_{\Omega_j}}{\|s(g)\|^{2q'-2}_{p',q'} \|s(g)1_{\Omega_j}\|_{p'}^{2(p'-q')}}.
\]

From the definition of \(s(\cdot)\), we have that \(\mu_n\) is \(\mathcal{F}_{n-1}\)-measurable. We define \(h\) as the martingale transform of \(g\) by \(\mu_n\). That is

\[(30)\quad d_nh = \mu_n d_ng.\]

We then obtain

\[
\sum_{n=0}^{\infty} \mathbb{E}_{n-1} |d_nh|^2 = \sum_{n=0}^{\infty} \mu_n^2 \mathbb{E}_{n-1} |d_ng|^2
\]

or equivalently

\[
s^2(h) = \sum_{n=0}^{\infty} \sum_{j \in \mathbb{Z}} \frac{s_n^{2p'-4}(g)1_{\Omega_j}}{\|s(g)\|^{2q'-2}_{p',q'} \|s(g)1_{\Omega_j}\|_{p'}^{2(p'-q')}} \mathbb{E}_{n-1} |d_ng|^2.
\]

Therefore

\[
s^2(h) = \sum_{j \in \mathbb{Z}} \frac{1_{\Omega_j}}{\|s(g)\|^{2q'-2}_{p',q'} \|s(g)1_{\Omega_j}\|_{p'}^{2(p'-q')}} \sum_{n=0}^{\infty} s_n^{2p'-4}(g) \mathbb{E}_{n-1} |d_ng|^2
\]

\[
= \sum_{j \in \mathbb{Z}} \frac{1_{\Omega_j}}{\|s(g)\|^{2q'-2}_{p',q'} \|s(g)1_{\Omega_j}\|_{p'}^{2(p'-q')}} \sum_{n=0}^{\infty} s_n^{2p'-4}(g)(s_n^2(g) - s_{n-1}^2(g))
\]

\[
= \frac{1}{\|s(g)\|^{2q'-2}_{p',q'}} \sum_{j \in \mathbb{Z}} \frac{1_{\Omega_j}}{\|s(g)1_{\Omega_j}\|_{p'}^{2(p'-q')}} \sum_{n=0}^{\infty} [s_n^{2p'-2}(g) - s_n^{2p'-4}(g)s_{n-1}^2(g)].
\]
It follows that
\[ s^2(h) \leq \frac{1}{\|s(g)\|^{2p-2}_{p',q'}} \sum_{j \in \mathbb{Z}} \frac{1_{\Omega_j}}{\|s(g)\|^{2p'}_{p'} \|s(g)\|^{2p'}_{p'}} \sum_{n=0}^{\infty} \left[ s_n^{2p-2}(g) - s_{n-1}^{2p-2}(g) \right] \]
\[ = \frac{1}{\|s(g)\|^{2p-2}_{p',q'}} \sum_{j \in \mathbb{Z}} s_n^{2p-2}(g) 1_{\Omega_j}. \]

Thus, by disjointedness of the \( \Omega_j \)'s,

\[ (31) \quad s(h) \leq \frac{s^{p-1}(g)}{\|s(g)\|^{p-1}_{p',q'}} \sum_{j \in \mathbb{Z}} \frac{1_{\Omega_j}}{\|s(g)\|^{p-1}_{p'} \|s(g)\|^{p-1}_{p'}}. \]

We also have that for any \( k \in \mathbb{Z} \),

\[ s(h) 1_{\Omega_k} \leq \sum_{j \in \mathbb{Z}} \frac{s^{p-1}(g)}{\|s(g)\|^{p-1}_{p',q'} \|s(g)\|^{p-1}_{p'} 1_{\Omega_k}} \frac{1_{\Omega_j}}{\|s(g)\|^{p-1}_{p'} \|s(g)\|^{p-1}_{p'}} = \frac{s^{p-1}(g)}{\|s(g)\|^{p-1}_{p',q'} \|s(g)\|^{p-1}_{p'} 1_{\Omega_k}}. \]

Therefore

\[ \|s(h) 1_{\Omega_k}\|_p \leq \frac{\|s^{p-1}(g) 1_{\Omega_k}\|_p}{\|s(g)\|^{q-1}_{p',q} \|s(g)\|^{q-1}_{p'} \|s(g)\|^{q-1}_{p'} 1_{\Omega_k}} = \frac{\|s(g)\|^{p-1}_{p'} \|s(g)\|^{p-1}_{p'}}{\|s(g)\|^{p-1}_{p',q'} \|s(g)\|^{p-1}_{p',q'}} = \frac{\|s(g)\|^{p-1}_{p',q'}}{\|s(g)\|^{p-1}_{p',q'}}. \]

Hence

\[ \sum_{k \in \mathbb{Z}} \|s(h) 1_{\Omega_k}\|_p^q \leq \sum_{k \in \mathbb{Z}} \frac{\|s(g)\|^{q(q-1)}_{p',q'}}{\|s(g)\|^{q(q-1)}_{p',q'} \|s(g)\|^{q(q-1)}_{p',q'}} = \sum_{k \in \mathbb{Z}} \frac{\|s(g)\|^{q}_{p',q} \|s(g)\|^{q}_{p',q} \|s(g)\|^{q}_{p',q}}{\|s(g)\|^{q}_{p',q} \|s(g)\|^{q}_{p',q}} = 1. \]

That is

\[ \|h\|_{H_{p,q}} \leq 1. \]

We now test \( (28) \) with the martingale \( h \) above. First proceeding as in \( (21) \), p.37] (this is why we need \( p \) to be smaller than 2), we obtain

\[ |\kappa(h)| = E \left( \sum_{n=0}^{\infty} d_n h d_n g \right) = E \left( \sum_{n=0}^{\infty} \mu_n |d_n g|^2 \right) \]
\[ = \frac{1}{\|s(g)\|^{q-1}_{p',q'}} E \left( \sum_{n=0}^{\infty} \sum_{j \in \mathbb{Z}} \frac{s_n^{p-2}(g) 1_{\Omega_j}}{\|s(g)\|^{p-2}_{p'} \|s(g)\|^{p-2}_{p'}} \sum_{n=0}^{E_{n-1}} |d_n g|^2 \right) \]
\[ = \frac{1}{\|s(g)\|^{q-1}_{p',q'}} E \left( \sum_{n=0}^{\infty} \sum_{j \in \mathbb{Z}} \frac{s_n^{p-2}(g) 1_{\Omega_j}}{\|s(g)\|^{p-2}_{p'} \|s(g)\|^{p-2}_{p'}} (s_n^2(g) - s_{n-1}^2(g)) \right) \]
\[ \geq \frac{2}{p'} \frac{1}{\|s(g)\|^{q-1}_{p',q'}} \sum_{j \in \mathbb{Z}} \frac{1}{\|s(g)\|^{p-1}_{p'} \|s(g)\|^{p-1}_{p'}} E \left( 1_{\Omega_j} \sum_{n=0}^{\infty} \frac{s_n^2(g) - s_{n-1}^2(g)}{p'} \right). \]
It follows that
\[
|\kappa(h)| \geq \frac{2}{p'} \frac{1}{\|s(g)\|_{p',q}^{q-1}} \sum_{j \in \mathbb{Z}} \frac{1}{\|s(g)1_{\Omega_j}\|_{p',q}^{p'-q}} \mathbb{E} \left( \mathbb{1}_{\Omega_j} s^{p'}(g) \right) 
\]
\[
= \frac{2}{p'} \frac{1}{\|s(g)\|_{p',q}^{q-1}} \sum_{j \in \mathbb{Z}} \frac{1}{\|s(g)1_{\Omega_j}\|_{p',q}^{p'-q}} \int_{\Omega} \mathbb{1}_{\Omega_j} s^{p'}(g) d\mathbb{P} 
\]
\[
= \frac{2}{p'} \frac{1}{\|s(g)\|_{p',q}^{q-1}} \sum_{j \in \mathbb{Z}} \frac{1}{\|s(g)1_{\Omega_j}\|_{p',q}^{p'-q}} \|s(g)1_{\Omega_j}\|_{p'}^{p'} 
\]
\[
= \frac{2}{p'} \frac{1}{\|s(g)\|_{p',q}^{q-1}} \sum_{j \in \mathbb{Z}} \|s(g)1_{\Omega_j}\|_{p',q}^{q'} 
\]
\[
= \frac{2}{p'} \frac{\|s(g)\|_{p',q}^{q'}}{\|s(g)\|_{p',q}^{q'-1}} = \frac{2}{p'} \|s(g)\|_{p',q'}.
\]

The proof is complete. \(\square\)

7. Concluding comments

We note that to obtain the martingale inequalities, we further supposed that the disjoint cover \((\Omega_j)_{j \in \mathbb{Z}}\) of the set \(\Omega\) was such that for each \(j \in \mathbb{Z}\), \(\Omega_j \in \mathcal{F}_n\) for any \(n \geq 1\). We do not know if these inequalities still hold without this assumption. The question also holds for the characterization of the dual spaces of \(H_{p,q}^s(\Omega)\) for \(1 < p, q < \infty\). Nevertheless, under this hypothesis, we observe that using Lemma 5.2 and [21, Theorem 2.12], one obtains the following extension of Burkholder-Davis-Gundy’s inequality.

**Proposition 7.1.** The spaces \(H_{p,q}^s\) and \(H_{p,q}^*\) are equivalent for \(1 \leq p, q \leq \infty\), namely,

\[
c_p \|f\|_{H_{p,q}^s} \leq \|f\|_{H_{p,q}^*} \leq C_p \|f\|_{H_{p,q}^s} \quad (1 \leq p, q < \infty)
\]

and

\[
c_p \|f\|_{H_{p,\infty}^s} \leq \|f\|_{H_{p,\infty}^*} \leq C_p \|f\|_{H_{p,\infty}^s} \quad (1 \leq p < \infty).
\]

We note that our methods do not allow us to obtain a characterization of the dual space of \(H_{p,q}^s(\Omega)\) for \(1 \leq p < q \leq 2\) and \(2 \leq q < p < \infty\).

Also it would be interesting to know if the spaces studied in this paper can be generalized by considering for example martingale analogues of the Orlicz-Slice Hardy spaces introduced in [26] that are a generalization of Hardy-amalgam spaces of [11]. These questions will be considered in a subsequent work.
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