MANIFOLDS NOT CONTAINING GOMPF NUCLEI

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Abstract. In this note we show that there are 4-manifolds not containing Gompf nucleus \( N_2 \); in this way we answer Problem 4.98 of Kirby’s problem list (see \([K]\)) in the negative.

1. Introduction

This note is devoted to answer a question in Kirby’s problem list \([K]\) asking whether every simply connected smooth 4-manifold with \( b^+ \geq 3 \) contains a Gompf nucleus \( N_2 \) (Problem 4.98 in \([K]\)). We prove that by doing logarithmic transformations on three linearly independent tori in the \( K3 \)-surface we get a 4-manifold not containing \( N_2 \). To make our statements more precise we need a few definitions.

The hypersurface \( X = \{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3 \mid \sum_{i=0}^{3} z_i^4 = 0 \} \subset \mathbb{C}P^3 \) is a simply connected, smooth 4-manifold with \( c_1(X) = 0 \), hence it is a \( K3 \)-surface. It is known that all simply connected complex surfaces with vanishing first Chern class are diffeomorphic, consequently from the differential topological point of view \( X \) is the \( K3 \)-surface. The complex surface \( X \) admits a holomorphic fibration \( \pi: X \rightarrow \mathbb{C}P^1 \) such that the generic fiber is a smooth elliptic curve — a 2-dimensional torus. Such fibrations are called elliptic fibrations. It can be assumed that \( \pi \) has a singular fiber homeomorphic to the 2-dimensional sphere \( S^2 \) — such fibers are called cusp fibers. The fibration also has a section \( \sigma: \mathbb{C}P^1 \rightarrow X \), the image of which is a sphere \( S \subset X \) with square \( [S]^2 = -2 \). The Gompf nucleus \( N_2 \) is by definition the tubular neighborhood of the union of a cusp fiber and a section in \( X \). The manifold \( N_2 \) admits a handle decomposition with one 0-handle and two 2-handles, where these two 2-handles are attached to \( S^3 = \partial(0 - \text{handle}) \) according to the Kirby diagram shown in Figure 1.

It can be shown that the \( K3 \)-surface \( X \) contains three disjoint copies of \( N_2 \). If a 4-manifold \( M^4 \) contains a 2-dimensional torus \( T \) with square 0, then we can perform a logarithmic transformation on \( T \): deleting the tubular neighborhood of \( T \) (which is diffeomorphic to \( D^2 \times T^2 \)) and regluing it via a diffeomorphism \( \varphi: \partial(M \setminus D^2 \times T^2) \rightarrow \partial(D^2 \times T^2) \) we get a new manifold \( M_\varphi \). It turns out that if \( T \) is the fiber in a Gompf nucleus \( N_2 \subset M \), then the diffeomorphism type of \( M_\varphi \) will depend only on one nonnegative number \( p \) associated to \( \varphi \). This number is called the multiplicity of the logarithmic transformation. For more about elliptic surfaces, nuclei and logarithmic transformations see \([G]\), \([FS]\) and \([GS]\).
Now perform logarithmic transformations of multiplicity 2 on the three tori contained by the three disjoint nuclei in the \( K3 \)-surface \( X \). The resulting manifold will be denoted by \( X_{2,2,2} \).

**Theorem 1.1.** The 4-manifold \( X_{2,2,2} \) does not contain a Gompf nucleus \( N_2 \).

**Remark 1.2.** One of the most interesting questions in 4-manifold theory is whether all 4-manifolds are of simple type or not. (For the definition of simple type see Section 2.) It is known that if \( M^4 \) contains a homologically essential torus with square 0, then \( M \) is of simple type. There is no example of a 4-manifold with \( b^+ \geq 3 \) and not containing a torus with square 0. One can ask whether every 4-manifold contains a cusp neighborhood (a tubular neighborhood of a cusp fiber) — or even a Gompf nucleus \( N_2 \). The above theorem shows an example of a manifold which contains no \( N_2 \); \( X_{2,2,2} \) is still of simple type, however.

One can modify the question by trying to find more general nuclei \( N_n \) in 4-manifolds. The manifold \( N_n \) is described by the Kirby diagram shown in Figure 2 — it can be defined alternatively as the tubular neighborhood of the union of a cusp fiber and a section in a simply connected elliptic surface (admitting a section) with Euler characteristic \( 12n \) (see [3]).

**Theorem 1.3.** For every \( n \) there is a manifold \( Y_n \) such that \( Y_n \) does not contain \( N_n \).
A related question would be to find a 4-manifold $M$ with the property that $M$ does not contain $N_2(p, q)$; here $N_2(p, q)$ denotes the 4-manifold with boundary we get by performing two logarithmic transformations (of multiplicity $p$ and $q$) along the fiber of the Gompf nucleus $N_2$. Using a recent construction of Fintushel and Stern [FS3] such $M$ can be found, see [SSZ].

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2. Basic classes

In studying differential topological properties of smooth 4-manifolds, Seiberg-Witten basic classes turn out to be very powerful tools. For the definition of these objects see [A], [M] or [GS], here we restrict ourselves to a very short outline.

Assume that $M^4$ is a smooth, oriented, simply connected, closed 4-manifold with $b_+^2(M) \geq 3$. A cohomology class $K \in H^2(M; \mathbb{Z})$ with $K \equiv w_2(M)$ (mod 2) uniquely determines a spin$^c$ structure on $M$, and for such a structure a certain pair of partial differential equations (the so-called Seiberg-Witten equations) can be described — involving a choice of metric on $M$, a coupled Dirac operator and a perturbation 2-form. By a delicate ”counting argument” of the solutions of these equations a number $SW_M(K)$ can be associated to the cohomology class $K$. It turns out that this number (up to sign) is a smooth invariant of the manifold $M$, more precisely

**Theorem 2.1.** If $f: M' \rightarrow M$ is an orientation preserving diffeomorphism then $SW_M(K) = \pm SW_{M'}(f^*K)$. Moreover, for a fixed 4-manifold $M$ there are only finitely many classes $K$ with $SW_M(K) \neq 0$, and $SW_M(-K) = \pm SW_M(K)$. \hfill \Box

**Definition 2.2.** The cohomology class $K \in H^2(M; \mathbb{Z})$ is called a Seiberg-Witten basic class if $SW_M(K) \neq 0$. The 4-manifold $M$ is of simple type if $SW_M(K) \neq 0$ implies that $K^2 = 3\sigma(M) + \chi(M)$; here $\sigma(M)$ and $\chi(M)$ stand for the signature and Euler characteristic of $M$.

The most important relation between the smooth topology of a 4-manifold $M$ and its basic classes $\{K_i \mid i = 1, \ldots, n\}$ is shown by the generalized adjunction formula:

**Theorem 2.3.** (Kronheimer-Mrowka) If $\Sigma^2 \subset M^4$ is a smooth, connected 2-dimensional surface of genus $g(\Sigma)$, $[\Sigma] \neq 0$ and $[\Sigma]^2 \geq 0$, then for every basic class $K$ we have

$$2g(\Sigma) - 2 \geq |\Sigma|^2 + |K([\Sigma])|.$$ \hfill \Box

3. Proofs of Theorems 1.1 and 1.3

Assume that $T_1, T_2$ and $T_3$ are three tori in the $K3$-surface lying in three disjoint Gompf nuclei. Perform logarithmic transformations of multiplicity 2 on each $T_i$. The basic classes of the resulting manifold $X_{2,2,2}$ are determined by Fintushel and Stern [FS3].

**Proposition 3.1.** The basic classes of $X_{2,2,2}$ are the Poincaré duals of the homology classes $\pm[T_1] \pm [T_2] \pm [T_3]$. \hfill \Box
Proof of Theorem 1.1: Assume that $N_2 \subset X_{2,2,2}$. The homology class of the fiber and the section in $N_2$ will be denoted by $f$ and $s$ respectively; note that $f$ and $g = f + s$ have square 0 and can be represented by tori. Consequently (by the generalized adjunction formula) a basic class $K$ of $X_{2,2,2}$ evaluates trivially on $f$ and $g$. Now $f \cdot (\frac{T_3}{2} + \frac{T_2}{2} + \frac{T_1}{2}) = 0$ and $f \cdot (-\frac{T_3}{2} + \frac{T_2}{2} + \frac{T_1}{2}) = 0$ ($\{i,j,k\} = \{1,2,3\}$) implies that $f \cdot \frac{T_1}{2} = 0$, similarly $g \cdot \frac{T_1}{2} = 0$ for $i = 1,2,3$. Since the complement of the three disjoint nuclei in $X$ have negative definite intersection form, the above equalities imply that $f = \sum_{i=1}^{3} \alpha_i[T_i]$ and similarly $g = \sum_{i=1}^{3} \beta_j[T_j]$. These two latter equations, however, give a contradiction, since $f \cdot g = 1$ but $(\sum_{i=1}^{3} \alpha_i[T_i]) \cdot (\sum_{i=1}^{3} \beta_j[T_j]) = 0$. Consequently $N_2$ does not embed in $X_{2,2,2}$.

Proof of Theorem 1.3: Perform logarithmic transformations of multiplicity 2 on $T_1, T_2$ and $T_3$ as above; the resulting 4-manifold $X_{2n,2n,2n}$ will be denoted by $Y_n$. The basic classes of $Y_n$ are determined in [FS2]: these are the Poincaré duals of the homology classes $\alpha_1(\frac{T_1}{2n}) + \beta(\frac{T_2}{2n}) + \gamma(\frac{T_3}{2n})$ where $\alpha, \beta, \gamma \in \{\pm(2j-1) \mid j = 1, \ldots, n\}$. Assume now that $N_n \subset Y_n$. As before, $f$ and $s$ will denote the homology classes of the fiber and the section in $N_n$ respectively. Note that $f^2 = 0$ and $s^2 = -n$. The class $f$ can be represented by a torus, while $g = s + nf$ can be represented by a surface of genus $n$; moreover $g^2 = n$. The same argument as in the proof of Theorem 1.1 gives that $f = \sum_{i=1}^{3} \alpha_i[T_i]$. To show that every basic class of $Y_n$ evaluates trivially on $g$ involves a little trick: take the basic classes $\pm[\frac{T_1}{2}] \pm[\frac{T_2}{2}] \pm[\frac{T_3}{2}]$ and evaluate them on $g$. If one of them, say $K$, evaluates nontrivially on $g$, then for $L = (2n-1)K$ (which is also a basic class) we have $|L(g)| \geq 2n - 1$, but then $L$ and $g$ would violate the generalized adjunction formula. Consequently $(\pm[\frac{T_1}{2}] \pm[\frac{T_2}{2}] \pm[\frac{T_3}{2}]) \cdot g = 0$ implying that $g \cdot [T_i] = 0$ ($i = 1,2,3$). These latter equations show that $g = \sum_{j=1}^{3} \beta_j[T_j]$, and we have the same contradiction as before.

Remark 3.2. Note that the proof given above also shows that no $N_k$ with $k \leq n$ embeds in $Y_n = X_{2n,2n,2n}$.

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