On the Concavity of Auxiliary Function in Classical-Quantum Channels

Hao-Chung Cheng  
National Taiwan University, Taiwan (R.O.C.) &  
University of Technology Sydney, Australia  
Email: F99942118@ntu.edu.tw  
Min-Hsiu Hsieh  
University of Technology Sydney, Australia  
Email: Min-Hsiu.Hsieh@uts.edu.au

Abstract

The auxiliary function of a classical channel appears in two fundamental quantities that upper and lower bound the error probability, respectively. A crucial property of the auxiliary function is its concavity, which leads to several important results in finite block length analysis. In this paper, we prove that the auxiliary function of a classical-quantum channel also enjoys the same concave property, extending an earlier partial result to its full generality. The key component in our proof is a beautiful result of geometric means of operators.

I. INTRODUCTION

Denote by \( \mathcal{P}(\mathcal{X}) \) the set of probability distributions on a finite set \( \mathcal{X} = \{1, 2, \ldots, |\mathcal{X}|\} \). For any fixed \( P \in \mathcal{P}(\mathcal{X}) \) and \( s \geq 0 \), the auxiliary function \( E_0(s, P) \) of a classical communication channel \( Q(y|x) \) with the output set \( \mathcal{Y} = \{1, 2, \ldots, |\mathcal{Y}|\} \) is defined as

\[
E_0(s, P) \triangleq -\log \left[ \sum_{y \in \mathcal{Y}} \left( \sum_{x \in \mathcal{X}} P(x) Q(y|x)^{\frac{1}{1+s}} \right)^{1+s} \right].
\]

This function appears in two fundamental quantities in classical information theory: for any \( R \geq 0 \),

\[
E_r(R) \triangleq \max_{0 \leq s \leq 1} \left\{ \max_{P \in \mathcal{P}(\mathcal{X})} E_0(s, P) - sR \right\},
\]

and

\[
E_{sp}(R) \triangleq \sup_{s \geq 0} \left\{ \max_{P \in \mathcal{P}(\mathcal{X})} E_0(s, P) - sR \right\},
\]

where \( E_r(R) \) is called the random coding exponent and \( E_{sp}(R) \) is called the sphere-packing exponent of the classical channel \( Q \). These two quantities are critical since, for any block length \( n \) and any rate \( R \geq 0 \), the error probability \( P_e(n, R) \), minimized over all possible coding strategies, satisfies \[1\]

\[
2^{-nE_{sp}(R)} \leq P_e(n, R) \leq 2^{-nE_r(R)}.
\]

Consequently, properties of the auxiliary function \( E_0(s, P) \) reveal important functional behaviour of the two exponents, and lead to a deeper understanding of the error probability of a given classical channel.
It is well-known (and easy to show) [1]: \( \forall s \geq 0, \)
\[
\begin{align*}
E_0(s, P) & \geq 0; \\
\frac{\partial E_0(s, P)}{\partial s} & > 0; \\
\frac{\partial^2 E_0(s, P)}{\partial s^2} & \leq 0.
\end{align*}
\] (5) (6) (7)

It turns out that \( E_0(s, P) \) is concave in \( s \geq 0 \). In addition to other important contributions in finite block length analysis, this fact also provides an alternative proof to Shannon’s noiseless channel coding theorem [2].

In recent years, much attention has been paid to understanding the error probability of a quantum channel. In this scenario, it suffices to consider a classical-quantum channel, which is a mapping \( W : x \in \mathcal{X} \mapsto W_x \in \mathcal{S}(\mathcal{H}) \) from the finite set \( \mathcal{X} \) to \( \mathcal{S}(\mathcal{H}) \), i.e., the set of density operators (positive semi-definite operators with unit trace) on a fixed Hilbert space \( \mathcal{H} \). Given a classical-quantum channel \( W \) and a distribution \( P \) on the input \( \mathcal{X} \), we can similarly define the auxiliary function \( E_0(s, P) \)\(^1\) [3, 4]: \( \forall s \geq 0, \)
\[
E_0(s, P) \triangleq -\log \mathrm{Tr} \left[ \left( \sum_{x \in \mathcal{X}} P(x) \cdot W_x^{1+s} \right)^{1+s} \right].
\] (8)

This quantity is a quantum generalization of Eq. (1), and recovers Eq. (1) when all \( \{W_x\}_{x \in \mathcal{X}} \) commute.

The auxiliary function \( E_0(s, P) \) in Eq. (8) also appears in the random coding exponent \( E_r(R) \) and the sphere-packing exponent \( E_{sp}(R) \) of a classical-quantum channel \( W \), which can be similarly defined as that in Eqs. (2) and (3), respectively. However, the relations between these two exponents and the error probability of the underlying classical-quantum channel \( W \) are much harder to obtain. The random coding exponent \( E_r(R) \) is shown to be an upper bound to the error probability of a classical-quantum channel \( W \) when every \( W_x \) is pure (i.e. the density operator \( W_x \) is a rank-one matrix) in Ref. [3], and it is conjectured to hold for general quantum states. Furthermore, the sphere-packing bound that lower bounds the error probability of \( W \) was recently proved in Ref. [5]\(^2\). These results are highly nontrivial due to the non-commutative nature of the density operators involved in their definitions. Many important questions in quantum information theory are still left open. Notably, it is still unknown whether the auxiliary function \( E_0(s, P) \) in Eq. (8) is concave for all \( s \geq 0 \). This might be one reason that the error probability of any finite block length \( n \) is less understood in the quantum regime. Note that \( E_0(s, P) \) has been shown to be concave in \( 0 \leq s \leq 1 \) in Ref. [6]. Its proof relies on an ad-hoc operator inequality in order to show that the second-order derivative of \( E_0(s, P) \) is non-positive for \( s \in [0, 1] \). However, this method seems impossible to work for all \( s \geq 0 \).

In this paper, we are able to prove that \( E_0(s, P) \) of a classical-quantum channel \( W \) is concave for all \( s \geq 0 \). Our proof culminates the latest development of operator algebra; in particular, the beautiful theory of a general geometric mean of operators [7]. Our proof can be viewed as a direct generalization of its classical proof in Ref. [1, Theorem 5.6.3].

The paper is organized as follows. Sec II presents the main technical tool, i.e., the “\( s \)-weighted geometric mean”. The main result is presented in Sec III, and our conclusion is given in Sec IV.

II. TECHNICAL TOOLS

Denote by \( \mathbb{M}_d^+ \) and \( \mathbb{M}_d^{++} \) the set of positive semi-definite matrices and positive definite matrices, respectively. For two \( d \times d \) Hermitian matrices \( A \) and \( B \), we denote by \( A \succeq B \) if \( A - B \in \mathbb{M}_d^+ \). Let

\(^1\)Here, we slightly abuse the notation since it should be clear from the context the underlying channel is quantum or classical.

\(^2\)However, this bound only works in the asymptotic regime \( n \to \infty \), unlike the classical case in Eq. (4) that holds for any \( n \in \mathbb{N} \) and \( R \geq 0 \).
Lemma 5 (See, e.g. [15, Section 2.2]). Let \( f \) be a convex function on real lines. Then \( A \preceq B \) implies
\[
\text{Tr}[f(A)] \leq \text{Tr}[f(B)].
\]
Lemma 6 (Matrix Hölder’s Inequality [12, Corollary IV.2.6]). Let $A, B \in M^+_d$. Then
\[
\text{Tr}[AB] \leq \left( \text{Tr} \left[ A^{\frac{l}{l}} \right] \right)^\theta \left( \text{Tr} \left[ B^{\frac{1}{1-l}} \right] \right)^{1-\theta}
\] (18)
for all $0 \leq \theta \leq 1$.

III. MAIN RESULT

We first recall a few notations. Let $\mathcal{X} = \{1, 2, \ldots, |\mathcal{X}|\}$ be a finite alphabet. Denote by $\mathcal{P}(\mathcal{X})$ the set of probability distributions on $\mathcal{X}$. Fix a (separable) Hilbert space $\mathcal{H}$. The set of density operators (i.e. positive semi-definite operators with unit trace) on $\mathcal{H}$ is defined as $S(\mathcal{H})$. Denote the set of all classical-quantum (c-q) channels $W$ from $\mathcal{X}$ to $S(\mathcal{H})$ by $\mathcal{W}(\mathcal{X})$.

Theorem 7. Given a classical-quantum channel $W \in \mathcal{W}(\mathcal{X})$ and a distribution $P \in \mathcal{P}(\mathcal{X})$, the auxiliary function $E_0(s, P)$ is concave in $s \geq 0$.

Proof. We first present the proof that only works when all $\{W_x\}_{x \in \mathcal{X}}$ are full rank. The proof can then be relaxed to include the non-invertible case.

Let $X$ be a random variable with distribution $P$, and denote by $\mathbb{E}$ the expectation with respect to $P$.

Then it suffices to prove the convexity of the map:
\[
f : t \mapsto \log \text{Tr} \left[ \left( \mathbb{E} W_X^t \right)^t \right]
\] (19)
for all $t \geq 1$.

Before starting the proof, we first prepare the following lemma that is crucial in our derivations.

Lemma 8. Let $A, B \in M^+_d$. Then, for every $t \geq 1$ and $0 \leq \lambda \leq 1$, we have
\[
\text{Tr} \left[ (A^\lambda B)^t \right] \leq \text{Tr} \left[ A^{(1-\lambda)t} B^{\lambda t} \right].
\] (20)

Proof. From Lemma 2, we have
\[
\text{Tr} [A^\lambda B] \leq \text{Tr} \left[ A^{\frac{1}{2}} B^{\frac{1}{2}} A^\lambda \right] \leq \text{Tr} \left[ \left( A^{\frac{(1-\lambda)}{2}} B^{\lambda t} A^{\frac{(1-\lambda)}{2}} \right)^t \right],
\] (21)
where the last inequality follows from Lemma 3. Next, applying Lemma 4 on the above inequality yields
\[
\text{Tr} \left[ (A^\lambda B)^t \right] \leq \text{Tr} \left[ \left( A^{\frac{(1-\lambda)}{2}} B^{\lambda t} A^{\frac{(1-\lambda)}{2}} \right)^t \right],
\] (22)
which completes the proof.

We now begin the proof of Theorem 7. These steps follow closely with those in Ref. [1, Theorem 5.6.3]. Let $l, r$, and $\theta$ be arbitrary numbers $1 \leq l \leq r$, $0 \leq \theta \leq 1$, and define
\[
t = \theta l + (1-\theta)r.
\] (24)

Let $t \equiv 1 + s \geq 1$. Then we prove the convexity of the map $f$, i.e.
\[
f(t) \leq \theta f(l) + (1-\theta)f(r).
\] (25)

Define the number $\lambda$ by
\[
\lambda = \frac{l \theta}{t}, \quad 1 - \lambda = \frac{r(1-\theta)}{t}.
\] (26)
Then it follows that
\[
\frac{1}{t} = \frac{\theta}{t} + \frac{1 - \theta}{t} = \frac{\lambda}{t} + \frac{1 - \lambda}{r}.
\] (27)

The convexity of the geometric means (see item (f) in Proposition 1) implies that
\[
E[W^{1/t}] = E[W^{\lambda/l}W^{(1-\lambda)/r}]
\] (28)
\[
= E[W^{1/l}]#_{1-\lambda}W^{1/r}
\] (29)
\[
\leq E[W^{1/l}]#_{1-\lambda}E[W^{1/r}].
\] (30)

Now let \(A \equiv E[W^{1/l}]\) and \(B \equiv E[W^{1/r}].\) Since \(x \mapsto x^t\) for \(t \geq 1\) is a convex function, Lemma 5 leads to
\[
\text{Tr} \left[ (E[W^{1/l}])^t \right] \leq \text{Tr} \left[ (A#_{1-\lambda}B)^t \right]
\] (31)
\[
\leq \text{Tr} \left[ A^{\lambda}B^{(1-\lambda)} \right]
\] (32)
\[
= \text{Tr} \left[ A^\theta B^{(1-\theta)} \right],
\] (33)
where Eq. (32) follows from Eq. (8). Finally, applying matrix Hölder’s inequality, Lemma 6, on the right-hand side of Eq. (33), we have
\[
\text{Tr} \left[ (E[W^{1/l}])^t \right] \leq (\text{Tr} [A^t])^\theta (\text{Tr} [B^t])^{1-\theta}
\] (34)
\[
= \left( \text{Tr} \left( E[W^{1/l}] \right) \right)^\theta \left( \text{Tr} \left( E[W^{1/r}] \right) \right)^{1-\theta}.
\] (35)

Taking logarithm on the above inequality arrives at \(f(t) \leq \theta f(l) + (1 - \theta)f(r).\) This completes the proof for the special case of invertible channel outputs.

The above proof assumes that every realization of the density operator \(W_x\) is positive definite. Hence, each density operator \(W_x^{\lambda/l}W_x^{(1-\lambda)/r}\) can be expressed as a geometric mean \(W_x^{1/l}#_sW_x^{1/r}.\) However, if \(W_x\) is not invertible for some \(x \in X,\) then consider a sequence of positive definite operators \(W_{x,\epsilon} \triangleq W_x + \epsilon I\) that approximate \(W_x,\) i.e., \(\lim_{\epsilon \to 0} W_{x,\epsilon} = W_x.\) The geometric mean of \(W_x^{1/l}\) and \(W_x^{1/r}\) is defined as
\[
W_x^{1/l}#_sW_x^{1/r} \triangleq \lim_{\epsilon \to 0} W_{x,\epsilon}^{1/l}#_sW_{x,\epsilon}^{1/r},
\] (36)
by the continuity of the geometric means (see item (h) in Proposition 1). Note that the concavity of the geometric means, and Lemmas 2 and 8 still hold if we use the definition in Eq. (36). We can thus obtain a complete proof.

\[\square\]

IV. CONCLUSION

In this paper, we proved an open question that was originally raised in [4]. A partial result to this question was obtained in [6]; however, we can extend the concavity of the auxiliary function \(E_0(s, P)\) for all \(s \geq 0.\) Consequently, the definition of auxiliary function (8) of a classical-quantum channel exactly recovers its classical counterpart [1], a quantity that plays a crucial role in classical information theory. We hope that this concave property will also allow us to better characterize the error probability of a classical-quantum channel in the finite regime.

ACKNOWLEDGEMENTS

MH is supported by an ARC Future Fellowship under Grant FT140100574.
REFERENCES

[1] R. Gallager, *Information Theory and Reliable Communication*. Wiley, 1968, ISBN: 978-0-471-29048-3.

[2] C. E. Shannon, “A mathematical theory of communication,” *The Bell System Technical Journal*, vol. 27, pp. 379–423, 623656–, 1948.

[3] M. V. Burnashev and A. S. Holevo, “On the reliability function for a quantum communication channel,” *Problems of information transmission*, vol. 34, no. 2, pp. 97–107, 1998. arXiv:quant-ph/9703013.

[4] A. Holevo, “Reliability function of general classical-quantum channel,” *IEEE Transaction on Information Theory*, vol. 46, no. 6, pp. 2256–2261, 2000. DOI: 10.1109/18.868501.

[5] M. Dalai, “Lower bounds on the probability of error for classical and classical-quantum channels,” *IEEE Transactions on Information Theory*, vol. 59, no. 12, pp. 8027–8056, 2013. DOI: 10.1109/tit.2013.2283794. arXiv:1208.1924 [cs.IT].

[6] J. I. Fujii, R. Nakamoto, and K. Yanagi, “Concavity of the auxiliary function appearing in quantum reliability function,” *IEEE Transaction on Information Theory*, vol. 52, no. 7, pp. 3310–3313, 2006. DOI: 10.1109/tit.2006.876248.

[7] F. Kubo and T. Ando, “Means of positive linear operators,” *Mathematische Annalen*, vol. 246, no. 3, pp. 205–224, 1980. DOI: 10.1007/bf01371042.

[8] G. Corach, H. Porta, and L. Recht, “Convexity of the geodesic distance on spaces of positive operators,” *Illinois Journal of Mathematics*, vol. 38, pp. 87–94, 1994.

[9] J. Lawson and Y. Lim, “Metric convexity of symmetric cones,” *Osaka Journal of Mathematics*, vol. 4, no. 4, pp. 795–816, 2007. [Online]. Available: http://projecteuclid.org/euclid.ojm/1199719405.

[10] R. Bhatia, *Positive Definite Matrices*. Princeton University Press, 2009. DOI: 10.1515/9781400827787.

[11] F. Hiai, “Log-majorizations and norm inequalities for exponential operators,” *Banach Center Publications*, vol. 38, no. 1, pp. 119–181, 1997.

[12] R. Bhatia, *Matrix Analysis*. Springer New York, 1997, ISBN: 978-0387948461.

[13] E. Hamza and A. Joye, “Spectral transition for random quantum walks on trees,” *Communications in Mathematical Physics*, vol. 326, no. 2, pp. 415–439, 2014. DOI: 10.1007/s00220-014-1882-7.

[14] J. S. Matharu and J. S. Aujla, “Some inequalities for unitarily invariant norm,” *Linear Algebra and its Applications*, vol. 436, pp. 1623–1631, 2012. DOI: 10.1016/j.laa.2010.08.013.

[15] E. Carlen, “Trace inequalities and quantum entropy: an introductory course,” in *Contemporary Mathematics*, vol. 529, American Mathematical Society, 2010, pp. 73–140. DOI: 10.1090/conm/529/10428.