r-STABLE SPACELIKE HYPERSURFACES IN CONFORMALLY STATIONARY SPACETIMES

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Abstract. In this paper we study the r-stability of closed spacelike hypersurfaces with constant r-th mean curvature in conformally stationary spacetimes of constant sectional curvature. In this setting, we obtain a characterization of r-stability through the analysis of the first eigenvalue of an operator naturally attached to the r-th mean curvature. As an application, we treat the case in which the spacetime is the de Sitter space.

1. Introduction

The notion of stability concerning hypersurfaces of constant mean curvature of Riemannian ambient spaces was first studied by Barbosa and do Carmo in [4], and Barbosa, do Carmo and Eschenburg in [5], where they proved that spheres are the only stable critical points of the area functional for volume-preserving variations.

In the Lorentz context, in 1993 Barbosa and Oliker [7] obtained an analogous result, proving that constant mean curvature spacelike hypersurfaces in Lorentz manifolds are also critical points of the area functional for variations that keep the volume constant. They also computed the second variation formula and showed, for the de Sitter space $S^{n+1}_1$, that spheres maximize the area functional for volume-preserving variations.

More recently, Liu and Junlei [15] have characterized the r-stable closed spacelike hypersurfaces with constant scalar curvature in the de Sitter space.

The natural generalization of mean and scalar curvatures for an $n-$dimensional hypersurface is the r-th mean curvatures $H_r$, for $r = 1, \cdots, n$. In fact, $H_1$ is just the mean curvature and $H_2$ defines a geometric quantity which is related to the scalar curvature.

In [10], some of the authors have studied the problem of strong stability (that is, stability with respect to not necessarily volume-preserving variations) for spacelike hypersurfaces with constant r-th mean curvature in a Generalized Robertson-Walker (GRW) spacetime, giving a characterization of r-maximal and spacelike slices.

Here, motivated by these works, we consider closed spacelike hypersurfaces with constant r-th mean curvature in a wide class of Lorentz manifolds, the so-called conformally stationary spacetimes, in order to obtain a relation between r-stability and the spectrum of a certain elliptic operator naturally attached to the r-th mean curvature.

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curvature of the hypersurfaces. Our approach is based on the use of the Newton transformations $P_r$ and their associated second order differential operators $L_r$ (cf. Section 2). More precisely, we prove the following result.

**Theorem 1.1.** Let $\overline{M}^{n+1}_c$ be a conformally stationary Lorentz manifold with constant curvature $c$. Suppose that $\overline{M}^{n+1}_c$ has a closed conformal vector field $V$ and a Killing vector field $W$. Let $x : M^n \to \overline{M}^{n+1}_c$ be a closed spacelike hypersurface, with constant, positive $(r+1)$-th mean curvature $H_{r+1}$ such that

$$\lambda = c(n - r)\binom{n}{r}H_r - nH_1\binom{n}{r+1}H_{r+2} - (r + 2)\binom{n}{r+2}H_{r+2}$$

is constant. Assume also that $\text{Div}_\overline{M}V$ does not vanish on $M^n$. Then $x$ is $r$-stable if and only if $\lambda$ is the first eigenvalue of $L_r$ on $M^n$.

As an application of the previous result, we obtain the following corollary in the de Sitter space.

**Corollary 1.2.** Let $x : M^n \to S^{n+1}_1$ be a closed spacelike hypersurface, contained in the chronological future (or past) of an equator of $S^{n+1}_1$, with positive constant $(r+1)$-th mean curvature such that

$$\lambda = (n - r)\binom{n}{r}H_r - nH_1\binom{n}{r+1}H_{r+2} - (r + 2)\binom{n}{r+2}H_{r+2}$$

is constant. Then $x$ is $r$-stable if and only if $\lambda$ is the first eigenvalue of $L_r$ on $M^n$.

2. Preliminaries

Let $\overline{M}^{n+1}$ denote a time-oriented Lorentz manifold with Lorentz metric $\mathcal{F} = \langle \ , \ \rangle$, volume element $d\overline{M}$ and semi-Riemannian connection $\nabla$. In this context, we consider spacelike hypersurfaces $x : M^n \to \overline{M}^{n+1}$, namely, isometric immersions from a connected, $n$-dimensional orientable Riemannian manifold $M^n$ into $\overline{M}$. We let $\nabla$ denote the Levi-Civita connection of $M^n$.

If $\overline{M}$ is time-orientable and $x : M^n \to \overline{M}^{n+1}$ is a spacelike hypersurface, then $M^n$ is orientable (cf. [13]) and one can choose a globally defined unit normal vector field $N$ on $M^n$ having the same time-orientation of $\overline{M}$. Such an $N$ is named future-pointing Gauss map of $M^n$. In this setting, let $A$ denote the shape operator of $M$ with respect to $N$, so that at each $p \in M^n$, $A$ restricts to a self-adjoint linear map $A_p : T_pM \to T_pM$.

For $1 \leq r \leq n$, let $S_r(p)$ denote the $r$-th elementary symmetric function on the eigenvalues of $A_p$; this way one gets $n$ smooth functions $S_r : M^n \to \mathbb{R}$, such that

$$\det(tI - A) = \sum_{k=0}^{n}(-1)^kS_k t^{n-k},$$

where $S_0 = 1$ by definition. If $p \in M^n$ and $\{e_k\}$ is a basis of $T_pM$ formed by eigenvectors of $A_p$, with corresponding eigenvalues $\{\lambda_k\}$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \ldots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[X_1, \ldots, X_n]$ is the $r$-th elementary symmetric polynomial on the indeterminates $X_1, \ldots, X_n$. 

For $1 \leq r \leq n$, one defines the $r$-th mean curvature $H_r$ of $x$ by
\[
{n \choose r} H_r = (-1)^r S_r = \sigma_r(-\lambda_1, \ldots, -\lambda_n).
\]

Also, for $0 \leq r \leq n$, the $r$-th Newton transformation $P_r$ on $M^n$ is defined by setting $P_0 = I$ (the identity operator) and, for $1 \leq r \leq n$, via the recurrence relation
\[
P_r = (-1)^r S_r I + A P_{r-1}.
\]

A trivial induction shows that
\[
P_r = (-1)^r(S_r I - S_{r-1}A + S_{r-2}A^2 - \cdots + (-1)^r A^r),
\]
so that Cayley-Hamilton theorem gives $P_n = 0$. Moreover, since $P_r$ is a polynomial in $A$ for every $r$, it is also self-adjoint and commutes with $A$. Therefore, all bases of $T_p M$ diagonalizing $A$ at $p \in M^n$ also diagonalize all of the $P_r$ at $p$. Let $\{e_k\}$ be such a basis. Denoting by $A_i$ the restriction of $A$ to $\langle e_i \rangle \perp \subset T_p \Sigma$, it is easy to see that
\[
det(tI - A_i) = \sum_{k=0}^{n-1} (-1)^k S_k(A_i) t^{n-1-k},
\]
where
\[
S_k(A_i) = \sum_{1 \leq j_1 < \cdots < j_k \leq n} \lambda_{j_1} \cdots \lambda_{j_k}.
\]

With the above notations, it is also immediate to check that $P_re_i = (-1)^rS_r(A_i)e_i$, and hence (cf. Lemma 2.1 of [3])
\[
\begin{align*}
\text{tr}(P_r) &= (-1)^r(n - r)S_r = b_r H_r; \\
\text{tr}(AP_r) &= (-1)^r(r + 1)S_{r+1} = -b_r H_{r+1}; \\
\text{tr}(A^2P_r) &= (-1)^r(S_1S_{r+1} - (r + 2)S_{r+2}),
\end{align*}
\]
where $b_r = (n - r) {n \choose r}$.

Associated to each Newton transformation $P_r$ one has the second order linear differential operator $L_r : \mathcal{D}(M) \to \mathcal{D}(M)$, given by
\[
L_r(f) = \text{tr}(P_r \text{Hess } f).
\]

For instance, when $r = 0$, $L_r$ is simply the Laplacian operator.

According to [3], if $\overline{M}^{n+1}$ is of constant sectional curvature, then $P_r$ is divergence-free and, consequently,
\[
L_r(f) = \text{div}(P_r \nabla f).
\]

If $x$ is as above, a variation of it is a smooth mapping
\[
X : M^n \times (-\epsilon, \epsilon) \to \overline{M}^{n+1}
\]
satisfying the following conditions:

1. For $t \in (-\epsilon, \epsilon)$, the map $X_t : M^n \to \overline{M}^{n+1}$ given by $X_t(p) = X(t, p)$ is a spacelike immersion such that $X_0 = x$.
2. $X_t|_{\partial M} = x|_{\partial M}$, for all $t \in (-\epsilon, \epsilon)$.
In all that follows, we let $dM_t$ denote the volume element of the metric induced on $M$ by $X_t$ and $N_t$ the unit normal vector field along $X_t$.

The variational field associated to the variation $X$ is the vector field $\left.\frac{\partial X}{\partial t}\right|_{t=0}$. Letting $f = -\langle \frac{\partial X}{\partial t}, N_t \rangle$, we get

$$\left.\frac{\partial X}{\partial t}\right|_{t=0} = f N_t + \left(\frac{\partial X}{\partial t}\right)^\top,$$

where $\top$ stands for tangential components.

The balance of volume of the variation $X$ is the function $V : (-\epsilon, \epsilon) \to \mathbb{R}$ given by

$$V(t) = \int_{M \times [0,t]} X^*(dM),$$

and we say $X$ is volume-preserving if $V$ is constant.

From now on, we will consider only closed spacelike hypersurface $x : M^n \to \mathbb{M}^{n+1}$. The following lemma is enough known and can be found in (cf. \cite{14}).

**Lemma 2.1.** Let $\mathbb{M}^{n+1}$ be a time-oriented Lorentz manifold and $x : M^n \to \mathbb{M}^{n+1}$ a closed spacelike hypersurface. If $X : M^n \times (-\epsilon, \epsilon) \to \mathbb{M}^{n+1}$ is a variation of $x$, then

$$\frac{dV}{dt} = \int_M f dM_t.$$

In particular, $X$ is volume-preserving if and only if $\int_M f dM_t = 0$ for all $t$.

We remark that Lemma 2.2 of \cite{5} remains valid in the Lorentz context, i.e., if $f_0 : M \to \mathbb{R}$ is a smooth function such that $\int_M f_0 dM = 0$, then there exists a volume-preserving variation of $M$ whose variational field is $f_0 N$.

In order to extend \cite{6} to the Lorentz setting, we let the $r$-area functional $A_r : (-\epsilon, \epsilon) \to \mathbb{R}$ associated to the variation $X$ be given by

$$A_r(t) = \int_M F_r(S_1, S_2, \ldots, S_r) dM_t,$$

where $S_r = S_r(t)$ and $F_r$ is recursively defined by setting $F_0 = 1$, $F_1 = -S_1$ and, for $2 \leq r \leq n-1$,

$$F_r = (-1)^r S_r - \frac{c(n - r + 1)}{r - 1} F_{r-2}.$$

We notice that if $r = 0$, the functional $A_0$ is the classical area functional.

The next step is the Lorentz analogue of Proposition 4.1 of \cite{6}. From Lemma 2.2 in \cite{10} we obtain the following result.

**Lemma 2.2.** Let $x : M^n \to \mathbb{M}^{n+1}_c$ be a closed spacelike hypersurface of the time-oriented Lorentz manifold $\mathbb{M}^{n+1}_c$ with constant curvature $c$, and let $X : M^n \times (-\epsilon, \epsilon) \to \mathbb{M}^{n+1}_c$ be a variation of $x$. Then,

$$\left.\frac{\partial S_{r+1}}{\partial t}\right|_{t=0} = (-1)^{r+1} \left[ L_r f + c tr(P_r) f - tr(A^2 P_r) f \right] + \left(\left.\frac{\partial X}{\partial t}\right|_{t=0}\right)^\top \nabla S_{r+1}.$$

The previous lemma allows us to compute the first variation of the $r$-area functional.
2.5 Remark

The time-oriented Lorentz manifold is the mean of the \( (r + 1) \)-th mean curvature and denote by \( (r + 1) \)-mean curvature. We say that \( X \) is \( r \)-stable if and only if \( \mathcal{A}_r(0) \leq 0 \), for all volume-preserving variation of \( x \).

As an immediate consequence of (2.5) we get

\[
\mathcal{J}_r'(t) = \int_M [b_r H_{r+1} + c_r] f dM_t,
\]

where \( b_r = (r + 1) \binom{n}{r+1} \). Therefore, if we choose \( \lambda = c_r + b_r \mathcal{H}_{r+1}(0) \), where

\[
\mathcal{H}_{r+1}(0) = \frac{1}{\mathcal{A}_0(0)} \int_M H_{r+1}(0) dM
\]

is the mean of the \( (r + 1) \)-th curvature \( H_{r+1}(0) \) of \( M \), we arrive at

\[
\mathcal{J}_r'(t) = b_r \int_M [H_{r+1} - \mathcal{H}_{r+1}(0)] f dM_t.
\]

Hence, a standard argument (cf. [10]) shows that \( M \) is a critical point of \( \mathcal{J}_r \) for all variations of \( x \) if and only if \( M \) has constant \( (r + 1) \)-th mean curvature.

We wish to study spacelike immersions \( x : M^n \to \mathcal{M}^{n+1} \) that maximize \( \mathcal{A}_r \) for all volume-preserving variations \( X \) of \( x \). The above discussion shows that \( M \) must have constant \( (r + 1) \)-th mean curvature and, for such an \( M \), one is naturally lead to compute the second variation of \( \mathcal{A}_r \). This motivates the following

Definition 2.4. Let \( \mathcal{M}^{n+1}_c \) be a time-oriented Lorentz manifold of constant curvature \( c \), and \( x : M^n \to \mathcal{M}^{n+1}_c \) be a closed spacelike hypersurface having constant \( (r + 1) \)-th mean curvature. We say that \( x \) is \( r \)-stable if \( \mathcal{A}_r''(0) \leq 0 \), for all volume-preserving variation of \( x \).

Remark 2.5. Let \( x : M^n \to \mathcal{M}^{n+1}_c \) be a closed spacelike hypersurface with constant \( (r + 1) \)-th mean curvature and denote by \( \mathcal{G} \) the set of differential functions \( f : M^n \to \mathbb{R} \) with \( \int_M f dM_t = 0 \). Just as [15] we can establish the following criterion for stability: \( x \) is \( r \)-stable if and only if \( \mathcal{J}_r''(0) \leq 0 \), for all \( f \in \mathcal{G} \).

The sought formula for the second variation of \( \mathcal{J}_r \) is another straightforward consequence of Proposition 2.3.

Proposition 2.6. Let \( x : M^n \to \mathcal{M}^{n+1}_c \) be a closed spacelike hypersurface of the time-oriented Lorentz manifold \( \mathcal{M}^{n+1}_c \), having constant \( (r + 1) \)-mean curvature \( H_{r+1} \). If \( X : M^n \times (-\epsilon, \epsilon) \to \mathcal{M}^{n+1}_c \) is a variation of \( x \), then \( \mathcal{J}_r''(0) \) is given by

\[
\mathcal{J}_r''(0)(f) = (r + 1) \int_M \left[ L_r(f) + \{\text{tr}(P_r) - \text{tr}(A^2 P_r)\}f \right] f dM.
\]
3. A Characterization of r-Stable Spacelike Hypersurfaces

As in the previous section, let $\mathcal{M}^{n+1}$ be a Lorentz manifold. A vector field $V$ on $\mathcal{M}^{n+1}$ is said to be conformal if

$$\mathcal{L}_V\langle\ ,\ \rangle = 2\psi\langle\ ,\ \rangle$$

for some function $\psi \in C^\infty(\mathcal{M})$, where $\mathcal{L}$ stands for the Lie derivative of the Lorentz metric of $\mathcal{M}$. The function $\psi$ is called the conformal factor of $V$.

Since $\mathcal{L}_V(X) = [V, X]$ for all $X \in \mathcal{X}(\mathcal{M})$, it follows from the tensorial character of $\mathcal{L}_V$ that $V \in \mathcal{X}(\mathcal{M})$ is conformal if and only if

$$\langle \nabla_X V, Y \rangle + \langle X, \nabla_Y V \rangle = 2\psi\langle X, Y \rangle,$$

for all $X, Y \in \mathcal{X}(\mathcal{M})$. In particular, $V$ is a Killing vector field relatively to $\overline{g}$ if and only if $\psi \equiv 0$. Observe that the function $\psi$ can be characterized as

$$\psi = \frac{1}{n+1} \text{Div}_\mathcal{M} V.$$

An interesting particular case of a conformal vector field $V$ is that in which $\nabla_X V = \psi X$ for all $X \in \mathcal{X}(\mathcal{M})$; in this case we say that $V$ is closed, an allusion to the fact that its dual 1-form is closed.

Any Lorentz manifold $\mathcal{M}^{n+1}$, possessing a globally defined, timelike conformal vector field is said to be a conformally stationary spacetime.

In what follows we need a formula first derived in [3]. As stated below, it is the Lorentz version of the one stated and proved in [8].

**Lemma 3.1.** Let $\mathcal{M}^{n+1}_c$ be a conformally stationary Lorentz manifold having constant curvature $c$ and conformal vector field $V$. Let also $x : M^n \to \mathcal{M}^{n+1}_c$ be a spacelike hypersurface of $\mathcal{M}^{n+1}_c$ and $N$ be a future-pointing Gauss map on $M^n$. If $\eta = \langle V, N \rangle$, then

$$L_r(\eta) = \{\text{tr}(A^2P_r) - c\text{tr}(P_r)\}\eta - b_rH_rN(\psi) + b_rH_{r+1}\psi + b_r\frac{1}{r+1}(V, \nabla H_{r+1}),$$

where $\psi : \mathcal{M}^{n+1}_c \to \mathbb{R}$ is the conformal factor of $V$, $H_j$ is the $j$-th mean curvature of $M^n$ and $\nabla H_j$ stands for the gradient of $H_j$ on $M^n$.

In particular, we obtain the following

**Corollary 3.2.** Let $\mathcal{M}^{n+1}_c$ be a conformally stationary Lorentz manifold having constant curvature $c$ and Killing vector field $W$. Let also $x : M^n \to \mathcal{M}^{n+1}_c$ be a spacelike hypersurface having constant $(r + 1)$-th mean curvature $H_{r+1}$, $N$ be a future-pointing Gauss map on $M^n$ and $\eta = \langle W, N \rangle$, then

$$L_r(\eta) + \{c\text{tr}(P_r) - \text{tr}(A^2P_r)\}\eta = 0.$$

In particular, if $x : M^n \to \mathcal{M}^{n+1}_c$ is a closed spacelike hypersurface with constant $(r + 1)$-th mean curvature such that $\lambda = c\text{tr}(P_r) - \text{tr}(A^2P_r)$ is constant, then $\lambda$ is an eigenvalue of the operator $L_r$ in $M^n$ with eigenfunction $\eta$.

**Remark 3.3.** Assuming that the conformal vector field $V$ is closed and such that $\text{Div}_\mathcal{M} V$ does not vanish on $M^n$, then there exists an elliptic point in $M^n$ (cf. Corollary 5.5 of [2]). Moreover, if $M^n$ has an elliptic point and $H_{r+1} > 0$ on $M$, for
2 \leq r \leq n - 1\), then \(L_r\) is elliptic (cf. Lemma 3.3 of [3]). In the case \(r = 1\), the hypothesis \(H_2 > 0\) guarantees the ellipticity of \(L_1\) without the additional assumption on the existence of an elliptic point (cf. Lemma 3.2 of [3]).

We can now state and prove our main result.

**Theorem 3.4.** Let \(\overline{M}^{n+1}_c\) be a conformally stationary Lorentz manifold with constant curvature \(c\). Suppose that \(\overline{M}^{n+1}_c\) has a closed conformal vector field \(V\) and a Killing vector field \(W\). Let \(x : M^n \to \overline{M}^{n+1}_c\) be a closed spacelike hypersurface, with positive constant \((r+1)\)-th mean curvature \(H_{r+1}\) such that
\[
\lambda = c(n-r)\left(\frac{n}{r}\right)H_r - nH_1\left(\frac{n}{r+1}\right)H_{r+1} - (r+2)\left(\frac{n}{r+2}\right)H_{r+2}
\]
is constant. Assume also that \(\text{Div}\overline{M} V\) does not vanish on \(M^n\). Then \(x\) is \(r\)-stable if and only if \(\lambda\) is the first eigenvalue of \(L_r\) on \(M^n\).

**Proof.** From Remark 3.3 the operator \(L_r\) is elliptic. On the other hand, by using the formulas (2.2), it is easy to show that \(\lambda = \text{tr}(P_r) - \text{tr}(A^2P_r)\). Therefore, since \(\lambda\) is constant and \(W\) is a Killing field on \(\overline{M}^{n+1}_c\), Corollary 3.2 guarantees that \(\lambda\) is in the spectrum of \(L_r\).

Let \(\lambda_1\) be the first eigenvalue of \(L_r\) on \(M^n\). If \(\lambda = \lambda_1\), then the variational characterization of \(\lambda_1\) gives
\[
\lambda = \min_{f \in \mathcal{G} \setminus \{0\}} \frac{-\int_M fL_r(f)\,dM}{\int_M f^2\,dM}.
\]
It follows that, for any \(f \in \mathcal{G}\),
\[
J''_r(0)(f) = (r+1)\int_M \{fL_r(f) + \lambda f^2\}\,dM
\]
\[
\leq (r+1)(-\lambda + \lambda)\int_M f^2\,dM = 0,
\]
and \(x\) is \(r\)-stable.

Now suppose that \(x\) is \(r\)-stable, so that \(J''_r(0)(f) \leq 0\) for all \(f \in \mathcal{G}\). Let \(f\) be an eigenfunction associated to the first eigenvalue \(\lambda_1\) of \(L_r\). As was already observed, there exists a volume-preserving variation of \(M\) whose variational field is \(fN\). Consequently, by (2.4) we get
\[
0 \geq J''_r(0)(f) = (r+1)(-\lambda_1 + \lambda)\int_M f^2\,dM
\]
and therefore \(\lambda_1 = \lambda\), since that \(\lambda_1 \leq \lambda\). \(\square\)

4. Applications to GRW spacetimes

A particular class of conformally stationary spacetimes is that of generalized Robertson-Walker spacetimes, or GRW for short (cf. [2]), namely, warped products \(\overline{M}^{n+1}_c = -I \times \phi F^n\), where \(I \subseteq \mathbb{R}\) is an interval with the metric \(-ds^2\), \(F^n\) is an \(n\)-dimensional Riemannian manifold and \(\phi : I \to \mathbb{R}\) is positive and smooth. For such a space, let \(\pi_I : \overline{M}^{n+1}_c \to I\) denote the canonical projection onto \(I\). Then the vector field
\[
V = (\phi \circ \pi_I) \frac{\partial}{\partial s}
\]
Corollary 4.1. Let \( x : M^n \to -I \times_\phi F^n \) be a closed spacelike hypersurface with constant \((r+1)\)-th mean curvature \(H_{r+1} > 0\). Suppose also that \(-I \times_\phi F^n\) is of constant curvature \(c\), has a Killing vector field and \(\phi'\) does not vanish on \(M^n\). If

\[
\lambda = c(n-r) \left( \frac{n}{r} - nH \right) H_{r-1} - 2n \left( \frac{n}{r+1} - (r+2) \right) H_{r+2}
\]

is constant, then \(x\) is \(r\)-stable if and only if \(\lambda\) is the first eigenvalue of \(L_r\) on \(M^n\).

A particular example of GRW spacetime is de Sitter space. More precisely, let \(\mathbb{L}^{n+2}\) denote the \((n+2)\)-dimensional Lorentz-Minkowski space \((n \geq 2)\), that is, the real vector space \(\mathbb{R}^{n+2}\), endowed with the Lorentz metric

\[
\langle v, w \rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2} w_{n+2},
\]

for all \(v, w \in \mathbb{R}^{n+2}\). We define the \((n+1)\)-dimensional de Sitter space \(S_1^{n+1}\) as the following hyperquadric of \(\mathbb{L}^{n+2}\)

\[
S_1^{n+1} = \{ p \in \mathbb{L}^{n+2} : \langle p, p \rangle = 1 \}.
\]

From the above definition it is easy to show that the metric induced from \(\langle \, , \rangle\) turns \(S_1^{n+1}\) into a Lorentz manifold with constant sectional curvature 1.

Choose a unit timelike vector \(a \in \mathbb{L}^{n+2}\), then \(V(p) = a - \langle p, a \rangle p, p \in S_1^{n+1}\) is a conformal and closed timelike vector field. It foliates the de Sitter space by means of umbilical round spheres \(M_\tau = \{ p \in S_1^{n+1} : \langle p, a \rangle = \tau \}, \tau \in \mathbb{R}\). The level set given by \(\{ p \in S_1^{n+1} : \langle p, a \rangle = 0 \}\) defines a round sphere of radius one which is a totally geodesic hypersurface in \(S_1^{n+1}\). We will refer to that sphere as the equator of \(S_1^{n+1}\) determined by \(a\). This equator divides the de Sitter space into two connected components, the chronological future which is given by

\[
\{ p \in S_1^{n+1} : \langle a, p \rangle < 0 \},
\]

and the chronological past, given by

\[
\{ p \in S_1^{n+1} : \langle a, p \rangle > 0 \}.
\]

In the context of warped products, the de Sitter space can be thought of as the following GRW

\[
S_1^{n+1} = -\mathbb{R} \times_{\cosh} S^n,
\]
where $S^n$ means Riemannian unit sphere. We observe that there is a lot of possible choices for the unit timelike vector $a \in L^{n+2}$ and, hence, a lot of ways to describe $S_1^{n+1}$ as such a GRW (cf. [12], Section 4). We notice that in this model, the equator of $S_1^{n+1}$ is the slice $\{0\} \times S^n$ and, consequently, $\phi'(s) = \sinh s$ vanishes only on this slice. Finally, the vector field

$$V = \phi'(s) \frac{\partial}{\partial s} = (\sinh s) \frac{\partial}{\partial s}$$

is conformal, timelike and closed in $S_1^{n+1}$.

In order to rewrite Theorem 3.4 for the case of closed spacelike hypersurfaces immersed in de Sitter space, we recall some facts.

(a) Killing vector fields in de Sitter space $S_1^{n+1}$ can be constructed by fixing two vectors $u$ and $v$ in the Lorentz-Minkowski space $L^{n+2}$ and a non-zero constant $k \in \mathbb{R}$, and considering the vector field $W = k\{\langle u, \cdot \rangle v - \langle v, \cdot \rangle u\}$. Geometrically, $W(x)$ determines an orthogonal direction to the position vector $x$ on the subspace spanned by $u$ and $v$ (cf. Example 1 of [11]).

(b) Let $x : M^n \to S_1^{n+1}$ be a closed spacelike hypersurface with positive constant $(r + 1)$-th mean curvature. Assuming that $M^n$ is contained in the chronological future (or past) of the equator of $S_1^{n+1}$ then $\text{Div}_M V$ does not vanish on $M^n$. Also, there exists an elliptic point in $M^n$ (cf. Theorem 7 of [1]) and, if $H_{r+1} > 0$ on $M$ for some $2 \leq r \leq n-1$, then, for all $1 \leq j \leq r$, the operator $L_j$ is elliptic (cf. Lemma 3.3 of [3]). In the case of $L_1$, it is sufficient to require that $R < c$ (cf. Lemma 3.2 of [3]).

We can now state the following corollary of Theorem 3.4.

**Corollary 4.2.** Let $x : M^n \to S_1^{n+1}$ be a closed spacelike hypersurface, contained in the chronological future (or past) of an equator of $S_1^{n+1}$, with positive constant $(r + 1)$-th mean curvature such that

$$\lambda = (n-r)\left(\binom{n}{r}\right)H_r - nH_1\left(\binom{n}{r+1}\right)H_{r+1} - (r+2)\left(\binom{n}{r+2}\right)H_{r+2}$$

is constant. Then $x$ is $r$-stable if and only if $\lambda$ is the first eigenvalue of $L_r$ on $M^n$.

**Remark 4.3.** We remark that the round spheres of $S_1^{n+1}$ are $r$-stable (cf. [9], Proposition 2).

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