Null controllability of a nonlinear age and two-sex population dynamics structured model

Amidou Traore * Okana S. Sougué † Yacouba Simporé ‡ Oumar Traore§

Abstract

This paper is devoted to study the null controllability properties of a nonlinear age and two-sex population dynamics structured model without spatial structure. Here, the nonlinearity and the coupling are at birth level. In this work we consider two cases of null controllability problem. The first problem is related to the extinction of male and female subpopulation density. The second case concerns the null controllability of male or female subpopulation individuals. In both cases, if $\alpha$ is the maximal age expectancy, a time interval of duration $\alpha$ after the extinction of males or females, one must get the total extinction of the population.

Our method uses first an observability inequality related to the adjoint of an auxiliary system, a null controllability of the linear auxiliary system and after the Kakutani’s fixed point theorem.

Keywords : two-sex population dynamics model, Null controllability, method of characteristics, Observability inequality, Kakutani fixed point.

1 Introduction and main results

In this paper, we study the null controllability of an infinite dimensional nonlinear coupled system describing the dynamics of two-sex structured population without spatial position. Let $(m, f)$ be the solution of the following system :

$$\begin{cases}
m_t + m_a + \mu_m m = \chi_m v_m & \text{in } Q, \\
f_t + f_a + \mu_f f = \chi_f v_f & \text{in } Q, \\
m(a, 0) = m_0(a) & f(a, 0) = f_0(a) & \text{in } Q_A, \\
m(0, t) = (1 - \gamma) \int_0^\alpha \beta(a, M) f(a, t) da & \text{in } Q_T, \\
f(0, t) = \gamma \int_0^\alpha \beta(a, M) f(a, t) da & \text{in } Q_T, \\
M = \int_0^\alpha \lambda(a) m(a, t) da & \text{in } Q_T,
\end{cases}$$

(1.1)

where $T$ is a positive number, $Q = (0, A) \times (0, T)$, $\Theta = (0, a_2) \times (0, T)$, $\Xi = (a_1, a_2) \times (0, T)$ and $\Xi' = (b_1, b_2) \times (0, T)$. Here $a_1 < a_2 \leq A$, $b_1 < b_2 \leq A$, $Q_A = (0, A)$ and $Q_T = (0, T)$.

We denote the density of males and females of age $a$ at time $t$ respectively by $m(a, t)$ and $f(a, t)$. Moreover, $\mu_m$ and $\mu_f$ denote respectively the natural mortality rate of males and females. The control functions are $v_m$ and $v_f$ and depend on $a$ and $t$. In addition $\chi_m$ and $\chi_f$ are the characteristic functions of the support of the control $v_m$ and $v_f$ respectively.

We have denoted by $\beta$ the positive function describing the fertility rate that depends on $a$ and also on

$$M = \int_0^\alpha \lambda(a) m(a, t) da,$$

where $\lambda$ is the fertility function of the male individuals. Thus the densities of newborn male and female individuals at time $t$ are given respectively by $m(0, t) = (1 - \gamma)N(t)$ and $f(0, t) = \gamma N(t)$ where

$$N(t) = \int_0^\alpha \beta(a, M) f(a, t) da.$$
We assume that the fertility rate $\beta$, $\lambda$ and the mortality rate $\mu_f$, $\mu_m$ satisfy the demographic properties:

$$(H_1) \begin{cases} 
\mu_m(a) \geq 0, \quad \mu_f(a) \geq 0 \text{ a.e } a \in (0, A) \\
\mu_m \in L^1_{loc}(0, A), \quad \mu_f \in L^1_{loc}(0, A) \\
\int_0^A \mu_m(a) da = +\infty, \quad \int_0^A \mu_f(a) da = +\infty
\end{cases}$$

$$(H_2) \begin{cases} 
\beta(a, p) \in C([0, A] \times \mathbb{R}) \\
\beta(a, p) \geq 0 \text{ for every } (a, p) \in [0, A] \times \mathbb{R}.
\end{cases}$$

We further assume that the birth function $\beta$ and the fertility function $\lambda$ verify the following hypothesis:

$$(H_3) \begin{cases} 
\text{there exists } b \in (0, A) \text{ such that } \beta(a, p) = 0, \forall (a, p) \in (0, b) \times \mathbb{R}, \\
\text{there exists } ||\beta||_{\infty} > 0 \text{ such that } 0 \leq \beta \leq ||\beta||_{\infty}, \forall (a, p) \in (0, b) \times \mathbb{R}, \\
\beta(a, 0) = 0, \forall a \in (0, A).
\end{cases}$$

$$(H_4) \begin{cases} 
\lambda \in C^1([0, A]) \\
\lambda(a) \geq 0 \text{ for every } a \in [0, A], \\
\lambda \mu_m \in L^1(0, A).
\end{cases}$$

The assumption $\beta(a, 0) = 0$ for $a \in (0, A)$ means that, the birth rate is zero if there are no fertile male individuals. We can now state the main results. If $(a_1, a_2) \subset (b_1, b_2)$, we have the following theorem:

**Theorem 1.1.** Let us assume that the assumptions $(H_1)-(H_4)$ hold true. If $a_1 < b$, for every time $T > a_1 + A - a_2$ and for every $(m_0, f_0) \in (L^2(\Omega))^2$, there exists $(v_m, v_f) \in L^2(\Omega) \times L^2(\Omega)$ such that the associated solution $(m, f)$ of system (1.1) verifies:

$$(1.2) \quad m(a, T) = f(a, T) = 0 \quad a.e \ a \in (0, A).$$

**Theorem 1.2.** Let us assume that the assumptions $(H_1)-(H_4)$ hold true. We have:

1. let $v_f = 0$. For any $p > 0$, for every time $T > A - a_2$ and for every $(m_0, f_0) \in (L^2(\Omega))^2$, there exists a control $v_m \in L^2(\Theta)$ such that the associated solution $(m, f)$ of system (1.1) verifies:

$$(1.3) \quad m(a, T) = 0 \quad a.e \ a \in (p, A)$$

where $\Theta = (0, a_2) \times (0, T)$.

2. let $v_m = 0$. For every time $T > a_1 + A - a_2$ and for every $(m_0, f_0) \in (L^2(\Omega))^2$, there exists a control $v_f \in L^2(\Omega)$ such that the associated solution $(m, f)$ of system (1.1) verifies:

$$(1.4) \quad f(a, T) = 0 \quad a.e \ a \in (0, A).$$

**Remark 1.1.** The first condition of $(H_3)$ is not necessary for the Theorem 1.2-(1).

In practice this study takes place in the fight against malaria. Malaria is a serious disease (in 2017 there were 219 million cases in the world [10]) and our work takes its importance in the strategy to fight against it.

In fact Malaria is a vector-borne disease transmitted by an infective female anophelous mosquito. A malaria control strategy in Brazil or Burkina Faso consists of releasing genetically modified male mosquitoes (precisely sterile males) in the nature. This can reduce the reproduction of mosquito since females mate only once in their life cycle.

In the theoretical framework, very few authors have studied control problems of two-sex structured population dynamics model.

The control problems of coupled systems of population dynamics models take an intense interest and are widely investigated in many papers. Among them we have [1], [7], [11] and the references therein. In fact, in [1] the authors studied a coupled reaction-diffusion equations describing interaction between a prey population and a predator one. The goal of the above work is to look for a suitable control supported on a small spatial subdomain which guarantees the stabilization of the predator population to zero. In [11], the objective was different. More precisely, the authors consider an age-dependent prey predator system and they prove the existence and uniqueness of an optimal control (called also "optimal effort") which gives the
maximal harvest via the study of the optimal harvesting problem associated to their coupled model. In [6] He and Ainseba study the null controllability of a butterfly population by acting on eggs, larvas and female moths in a small age interval. 

In [7], the authors analyze the growth of a two-sex population with a fixed age-specific sex ratio without diffusion. The model is intended to give an insight into the dynamics of a population where the mating process takes place at random choice and the proportion between females and males is not influenced by environmental or social factors, but only depends on a differential mortality or on a possible transition from one sex to the other (e.g., in sequential hermaphrodite species). Simeone and Traore study in [9] the null controllability of a nonlinear age, space and two-sex structured population dynamics model. They first study an approximate null controllability result for an auxiliary cascade system and prove the null controllability of the nonlinear system by means of Schauder’s fixed point theorem.

Unlike the model treated in [9], we consider a nonlinear cascade system with two different fertility rates and without space variable. The fertility rate of the male $\lambda$ and the fertility rate $\beta$ of the female depend on the total population of the fertile males.

We use the technique of [9] and [8] combining final-state observability estimates with the use of characteristics to establish the observability inequalities necessary for the null controllability property of the auxiliary systems. Roughly, in our method we first study the null controllability result for an auxiliary cascade system. Afterwards, we prove the null controllability result for the system (1.1) by means of Kakutani’s fixed point theorem.

The remainder of this paper is as follows: in Section 2 we study the existence and uniqueness of a positive solution for the model. Section 3 is devoted to the proof of the Theorem 1.1 and the Theorem 1.2.

## 2 Well posedness result

In this section, we study the existence of positive solution of the model. For this, we assume that the so-called demographic conditions $(H_1)$, $(H_2)$, $(H_3)$ and $(H_4)$ are verified. Moreover, here, we suppose that

$$(H_b) \begin{cases} 
\beta(a, p) = \beta_1(a)\beta_2(p) \text{ for all } (a, p) \in (0, A) \times \mathbb{R}, \\
\text{there exists } C > 0 \text{ such that } |\beta_2(p) - \beta_2(q)| \leq C|p - q| \text{ for all } p, q \in \mathbb{R}, \\
\beta_1\mu_f \in L^1(0, A) 
\end{cases}$$

holds true.

Thus, we have the following result.

**Theorem 2.1.**

Assume that $(H_1) - (H_5)$ hold. For every $(m_0, f_0) \in (L^2(0, A))^2$ and $(v_m, v_f) \in (L^2(Q))^2$, the system (1.1) admits a unique solution $(m, f) \in (L^2((0, A) \times (0, T)))^2$ and we have the following estimations:

$$
\begin{align*}
\|m\|_{L^2((0, A) \times (0, T))} &\leq K\left(\|f_0\|_{L^2(0, T)} + \|m_0\|_{L^2(0, T)} + \|v_m\|_{L^2(Q)} + \|v_f\|_{L^2(Q)}\right), \\
\|f\|_{L^2((0, A) \times (0, T))} &\leq C\left(\|m_0\|_{L^2(0, T)} + \|v_f\|_{L^2(Q)}\right)
\end{align*}
$$

where $K$ and $C$ are positive constants.

Moreover, suppose that $m_0, f_0 \geq 0$ a.e $(0, A)$ and $v_m, v_f \geq 0$ a.e $Q$; then $(m, f)$ is also positive.

**Proof of Theorem 2.1**

Let $p$ be fixed in $L^2(0, T)$, $h$ and $h'$ be fixed in $L^2(Q)$ and consider the following system

$$
\begin{cases}
\begin{align*}
m_t + m + \mu_m m &= h \quad \text{in } Q, \\
f_t + f + \mu_f f &= h' \quad \text{in } Q, \\
m(a, 0) &= m_0(a) \quad f(a, 0) = f_0(a) \quad \text{in } Q_A, \\
m(0, t) &= (1 - \gamma) \int_0^A \beta \left(a, \int_0^A \lambda(a)p(a, t)\right) f(a, t)da \quad \text{in } Q_T, \\
f(0, t) &= \gamma \int_0^A \beta \left(a, \int_0^A \lambda(a)p(a, t)\right) f(a, t)da \quad \text{in } Q_T,
\end{align*}
\end{cases}
$$

(2.2)
For every \( f_0 \in L^2(0, A) \) and \( h' \in L^2(Q) \), the following system

\[
\begin{align*}
  f_t + f_a + \mu f_f &= h' & \text{in } Q, \\
  f(a, 0) &= f_0(a) & \text{in } Q_A, \\
  f(0, t) &= \gamma \int_0^A \beta \left( a, \int_0^A \lambda(a)p(a, t) \right) f(a, t)da & \text{in } Q_T,
\end{align*}
\]  

(2.3)

admits a unique positive solution in \( L^2(Q) \), see [2] and one has

\[
\|f\|^2_{L^2(Q)} \leq C \left( \|f_0\|^2_{L^2(0, A)} + \|h'\|^2_{L^2(Q)} \right),
\]

(2.4)

where \( C \) is a positive constant and independent of \( p \) because \( \beta \in L^\infty((0, T) \times (0, A)) \).

Now, \( f \) and \( h' \) are being known, the system

\[
\begin{align*}
  m_t + m_a + \mu m m &= h & \text{in } Q, \\
  m(a, 0) &= m_0(a) & \text{in } Q_A, \\
  m(0, t) &= (1 - \gamma) \int_0^A \beta \left( a, \int_0^A \lambda(a)p(a, t) \right) f(a, t)da & \text{in } Q_T,
\end{align*}
\]

(2.5)

admits a unique positive system in \( L^2(Q) \) and we have the following estimation

\[
\|m\|^2_{L^2(Q)} \leq K \left( \|f_0\|^2_{L^2(0, A)} + \|m_0\|^2_{L^2(0, A)} + \|h\|^2_{L^2(Q)} + \|h'\|^2_{L^2(Q)} \right),
\]

where \( K \) is a positive constant and independent of \( p \) because \( \beta \in L^\infty((0, T) \times (0, A)) \).

Let define \( \Phi : L^2_+(Q) \rightarrow L^2_+(Q) \), \( \Phi p = m(p) \) where \( m(p) \) is the unique solution of the system (2.5).

For any \( p, q \in L^2_+(Q) \), we set

\[
B_1(a, t) = \int_0^A \lambda(a)p(t, a)da \quad \text{and} \quad B_2(a, t) = \int_0^A \lambda(a)q(t, a) \quad \text{a.e. } \ t \in (0, A) \times (0, T),
\]

and \( w = (m(p) - m(q))e^{-\gamma_0 t} \) where \( \gamma_0 \) is a positive parameter that will be chosen later; \( w \) is solution of

\[
\begin{align*}
  w_t + w_a + (\gamma_0 + \mu m)w &= 0 & \text{in } Q, \\
  w(a, 0) &= 0 & \text{in } Q_A, \\
  w(0, t) &= (1 - \gamma)e^{-\gamma_0 t} \times \\
  & \quad \int_0^A [\beta_2(B_1) - \beta_2(B_2)] \beta_1(a)f(p) + (f(p) - f(q))\beta_2(B_2)\beta_1(a)da & \text{in } Q_T,
\end{align*}
\]

(2.6)

Multiplying (2.6) by \( w \) and integrating over \( (0, A) \times (0, t) \), and using Young’s inequality we get

\[
\frac{1}{2} \|w(t)\|^2_{L^2(0, A)} + \int_0^t \int_0^A (\gamma_0 + \mu m)w^2(s, a)da ds \leq \int_0^t \left( \int_0^A |\beta_2(B_1) - \beta_2(B_2)| \beta_1(a)f(p)da \right)^2 ds \\
+ \int_0^t \left( \int_0^A (f(p) - f(q))\beta_2(B_2)\beta_1(a)da \right)^2 ds \\
\leq C^2 \|\lambda\|^2_\infty \int_0^t \left( \int_0^A p(s, a)da - \int_0^A q(s, a)da \right)^2 \left( \int_0^A \beta_1(a)f(p(s))da \right)^2 ds \\
+ \int_0^t \left( \int_0^A (f(p) - f(q))\beta_2(B_2)\beta_1(a)da \right)^2 ds \\
\leq C^2 A \|\lambda\|^2_\infty \int_0^t \int_0^A |p(s) - q(s)|^2 \left( \int_0^A \beta_1(a)f(p(s))da \right)^2 ds \\
+ \|\beta_1\|^2_\infty \|\beta_2\|^2_\infty A \int_0^t \int_0^A |f(p) - f(q)|^2 ds.
\]
Hence for every $\gamma_0 > 0$, there is a constant $C = \max \left\{2C^2 A \parallel \lambda \parallel_{2, \infty}^2; 2 \parallel \beta_1 \parallel_{2, \infty}^2 \parallel \beta_2 \parallel_{2, \infty}^2 A \right\}$ such that

\[
(2.7) \quad \|w(t)\|_{L^2(0,A)}^2 \leq C \left( \int_0^t \int_0^A |p(s) - q(s)|^2 \left( \int_0^A \beta_1(a) f(p(s)) da \right)^2 ds + \int_0^t \int_0^A |f(p(s)) - f(q(s))|^2 ds \right)
\]

Now set $F = (f(p) - f(q))e^{-\delta t}$ where $\delta$ is a positive parameter that will be chosen later. Then, $F$ solves the following auxiliary system

\[
(2.8) \quad \begin{cases}
F_t + F_a + (\delta + \mu_f) F = 0 & \text{in } Q, \\
F(a,0) = 0 & \text{in } Q_A,
\end{cases}
\]

Similarly as above, we have

\[
\delta \int_0^t \int_0^A F(a,s)^2 ds \leq C \left( \int_0^t \int_0^A |p(s) - q(s)|^2 \left( \int_0^A \beta_1(a) f(p(s)) da \right)^2 ds + \int_0^t \int_0^A |F(a,s)|^2 ds \right).
\]

Hence, there is a positive constant $C'$ such that

\[
(2.9) \quad \int_0^t \int_0^A F(a,s)^2 ds \leq C' \int_0^t \int_0^A |p(s) - q(s)|^2 \left( \int_0^A \beta_1(a) f(p(s)) da \right)^2 ds.
\]

Setting $Y(t) = \int_0^A \beta_1(a) f(p) da$ a.e in $(0, T)$, $Y$ solves the following system

\[
(2.10) \quad \begin{cases}
Y_t = \int_0^A \beta_1(a) h'(a,t) da + \int_0^A \beta_1'(a) f(a,t) da - \int_0^A \mu_f(a) \beta_1(a) f(a,t) da & \text{in } (0, T), \\
Y(0) = \int_0^A \beta_1(a) f_0(a) da
\end{cases}
\]

Multiplying (2.10) by $Y$, integrating over $(0, t)$ and using Young’s inequality we get

\[
Y^2(t) \leq Y^2(0) + \int_0^t Y^2(s) ds + \int_0^t \left( \int_0^A \beta_1(a) h'(a,s) da + \int_0^A \beta_1'(a) f(a,s) da + \int_0^A \mu_f(a) \beta_1(a) f(a,s) da \right)^2 ds
\]

\[
\leq Y^2(0) + \int_0^t Y^2(s) ds + 3 \int_0^t \left( \int_0^A \beta_1(a) h'(a,s) da \right)^2 ds + 3 \int_0^t \left( \int_0^A \beta_1'(a) f(a,s) da \right)^2 ds
\]

\[
+ 3 \int_0^t \left( \int_0^A \beta_1(a) \mu_f(a) f(a,s) da \right)^2 ds.
\]

So,

\[
(2.11) \quad Y^2(t) \leq \left( \int_0^A \beta_1(a) f_0(a) da \right)^2 + \int_0^T Y^2(t) dt + 3 \int_0^T \left( \int_0^A \beta_1(a) h'(a,t) da \right)^2 dt + 3 \int_0^T \left( \int_0^A \beta_1'(a) f(a,t) da \right)^2 dt
\]

\[
+ 3 \int_0^T \left( \int_0^A \beta_1(a) \mu_f(a) f(a,t) da \right)^2 dt.
\]

Let us set $\tilde{f} = e^{-\lambda_0 t} f$. Then, from (2.3) $\tilde{f}$ satisfies the following system

\[
(2.12) \quad \begin{cases}
\tilde{f}_t + \tilde{f}_a + (\lambda_0 + \mu_f) \tilde{f} = e^{-\lambda_0 t} h' & \text{in } Q, \\
\tilde{f}(a,0) = f_0(a) & \text{in } Q_A,
\end{cases}
\]

\[
\tilde{f}(0,t) = \gamma \int_0^A \beta \left( a, \int_0^A \lambda(a) p(a,t) \right) \tilde{f}(a,t) da & \text{in } Q_T.
\]
Multiplying the first equation of (2.12) by \( \tilde{f} \), integrating on \( Q \) and using the inequality of Young we get
\[
\int_0^T \int_0^A (\lambda_0 + \mu_f(a)) \tilde{f}^2(a,t)dadt \leq \frac{1}{2} ||f_0||^2_{L^2(0,A)} + \frac{1}{2} ||h'||^2_{L^2(Q)} + \frac{1}{2} \int_0^T \tilde{f}^2(0,t)dt.
\]
Using the inequality of Cauchy Schwartz and choosing \( \lambda_0 = \frac{3}{2} + ||\beta||^2_{\infty} \), we obtain
\[
\int_0^T \int_0^A \mu_f(a) \tilde{f}^2(a,t)dadt \leq \frac{1}{2} \left( ||f_0||^2_{L^2(0,A)} + ||h'||^2_{L^2(Q)} \right).
\]
So,
\[
(2.13) \quad \int_0^T \int_0^A \mu_f(a) \tilde{f}^2(a,t)dadt \leq \frac{1}{2} \left( ||f_0||^2_{L^2(0,A)} + ||h'||^2_{L^2(Q)} \right).
\]
Using (2.11), (2.13) and against Young’s inequality we have
\[
Y^2(t) \leq ||\beta_1||^2_{\infty} A ||f_0||^2_{L^2(0,A)} + ||\beta_1||^2_{\infty} A ||f||^2_{L^2(Q)} + 3 ||\beta_1||^2_{\infty} A ||h'||^2_{L^2(Q)}
\]
\[
+ 3 ||\beta||^2_{\infty} A ||f||^2_{L^2(Q)} + 3C||\beta_1|| ||\beta_1||_{\infty} ||\beta_1||_{\infty} ||f||^2_{L^2(0,A)}
\]
\[
+ 3C||\beta||_{\infty} ||\beta_1||_{\infty} ||f||^2_{L^2(0,A)} ||h'||^2_{L^2(Q)}.
\]
From (2.4), we have just proved the existence of a positive constant \( C \) such that ,
\[
(2.14) \quad Y^2(t) \leq C \left( ||f_0||^2_{L^2(0,A)} + ||h'||^2_{L^2(Q)} \right)
\]
The estimate (2.14) means also that \( Y \in L^\infty(0,T) \).
Combining (2.7), (2.9) and (2.14), we get the following estimate
\[
(2.15) \quad ||(\Phi p - \Phi q)(t)||^2_{L^2(0,A)} \leq \sigma \int_0^t \| p(s) - q(s) \|^2_{L^2(0,A)} ds,
\]
where \( \sigma \) is a positive constant.
Let us define the metric \( d \) on \( L^2_+(Q) \) by setting
\[
d(h_1, h_2) = \left( \int_0^T ||(h_1 - h_2)(t)||^2_{L^2(0,A)} \exp\{-2\sigma t\} dt \right)^{\frac{1}{2}}, \quad \text{for } h_1, h_2 \in L^2_+(Q).
\]
We have
\[
d(\Phi p, \Phi q)^2 = \int_0^T ||(\Phi p - \Phi q)(t)||^2_{L^2(0,A)} \exp\{-2\sigma t\} dt \leq \sigma \int_0^T \exp\{-2\sigma t\} \int_0^t || p - q(s) ||^2_{L^2(0,A)} ds dt
\]
Using the Fubini theorem, we conclude that
\[
d(\Phi p, \Phi q)^2 = \int_0^T ||(\Phi p - \Phi q)(t)||^2_{L^2(0,A)} \exp\{-2\sigma t\} dt \leq \int_0^T || p - q(s) ||^2_{L^2(0,A)} \times \int_0^T \sigma e^{-2\sigma t} ds dt \leq \frac{1}{2} d(p,q)^2.
\]
Then, \( \Phi \) is a contraction on the complete metric space \( L^2_+(Q) \) into itself. Using Banach’s fixed point theorem, we conclude the existence of a unique fixed point \( m \). Moreover, \( m \) is nonnegative. Hence, the unique couple \((m, f)\) is the unique solution to our problem (1.1).

\[ \square \]

### 3 Null controllability results

For the sequel, the hypothesis \((H_5)\) is not necessary.

#### 3.1 Null controllability of an auxiliary coupled system

This section is devoted to the study of an auxiliary system obtained from the system (1.1).
Let $p$ be a $L^2(Q_T)$ function, we define the auxiliary system given by:

$$
\begin{cases}
    m_t + m_n + \mu_m m = \chi \varepsilon v_m & \text{in } Q, \\
    f_t + f_n + \mu f = \chi \varepsilon v_f & \text{in } Q, \\
    m(0,0) = m_0(a) \quad f(0,0) = f_0(a) & \text{in } Q_A, \\
    m(0,t) = (1-\gamma) \int_0^A \beta(a,p)f(a,t)da & \text{in } QT, \\
    f(0,t) = \gamma \int_0^A \beta(a,p)f(a,t)da & \text{in } QT.
\end{cases}
$$

(3.1)

Let $p$ be fixed in $L^2(0,T)$, for $(m_0, f_0) \in (L^2(Q_A))^2$ and $(v_m, v_f) \in L^2(\Xi) \times L^2(\Xi')$ the system (3.1) admits a unique solution $(m, f) \in (L^2(Q))^2$, see Section 2. The system above is null approximately controllable. Indeed we have the following result:

**Theorem 3.1.** Let us assume that assumptions $(H_1)$ -- $(H_2)$ hold. For every time $T > a_1 + A - a_2$, for every $\kappa, \theta > 0$ and for every $(m_0, f_0) \in (L^2(Q_A))^2$, there exists a control $(v_m, v_f)$ such that the solutions $m$ and $f$ of the system (3.1) verify

$$\|m(., T)\|_{L^1(0,A)} \leq \kappa \quad \text{and} \quad \|f(., T)\|_{L^2(0,A)} \leq \theta.$$  

The adjoint system of (3.1) is given by:

$$
\begin{cases}
    -n_t - n_a + \mu_m n = 0 & \text{in } Q, \\
    -l_t - l_a + \mu f = (1-\gamma)\beta(a,p)n(0,t) + \gamma\beta(a,p)l(0,t) & \text{in } Q, \\
    n(a,T) = n_T(a), \quad l(a,T) = l_T(a) & \text{in } Q_A, \\
    n(A,t) = 0, \quad l(A,t) = 0 & \text{in } QT.
\end{cases}
$$

(3.3)

The main idea in this part is to establish an observability inequality of (3.3) that will allow us to prove the approximate null controllability of (3.1).

The basic idea for establishing this inequality is the estimation of non-local terms. For that, let us start by formulating a representation of the solution of the adjoint system by the method of characteristic semi-group. For every $(n_T, l_T) \in (L^2(Q_A))^2$, under the assumptions $(H_1)$ and $(H_2)$, the coupled system (3.3) admits a unique solution $(n, l)$. Moreover integrating along the characteristic lines, the solution $(n, l)$ of (3.3) is as follows:

$$
n(t) = \begin{cases}
    \frac{\pi_1(a + T - t)}{\pi_1(a)} n_T(a + T - t) & \text{if } T - t \leq A - a, \\
    0 & \text{if } A - a < T - t
\end{cases}
$$

(3.4)

and

$$
l(t) = \begin{cases}
    \frac{\pi_2(a + T - t)}{\pi_2(a)} l_T(a + T - t) + \int_t^T \frac{\pi_2(a + s - t)}{\pi_2(a)} \beta(a + s - t, p(s))(1-\gamma)n(0,s) + \gamma l(0,s))ds & \text{if } T - t \leq A - a, \\
    \int_t^{A-a} \frac{\pi_2(a + s - t)}{\pi_2(a)} \beta(a + s - t, p(s))(1-\gamma)n(0,s) + \gamma l(0,s))ds & \text{if } A - a < T - t,
\end{cases}
$$

(3.5)

where $\pi_1(a) = e^{-\int_a^T \mu_m(r)dr}$ and $\pi_2(a) = e^{-\int_a^T \mu_f(r)dr}$. Suppose that the assumptions $(H_1)$, $(H_2)$, $(H_3)$ and $(H_4)$ are fulfilled, then we have the following result.

**Theorem 3.2.** Under the assumptions of Theorem 1.1, if $(a_1, a_2) \subset (b_1, b_2)$, there exists a constant $C_T > 0$ independent of $p$ such that the couple $(n, l)$ solution of (3.3) verifies the following inequality:

$$
\int_0^A n^2(a,0)da + \int_0^A l^2(a,0)da \leq C_T \left( \int\int_{\Xi} n^2(a,t)da dt + \int\int_{\Xi'} l^2(a,t)da dt \right).
$$

(3.6)

For the proof of Theorem 3.2, we state the following estimations of the non-local terms.
Proposition 3.1. Under the assumptions of Theorem 1.1, there exists $C > 0$ such that

$$\int_0^{T-\eta} n^2(0, t) dt \leq C \int_0^T \int_{a_1}^{a_2} n^2(a, t) dadt,$$

where $a_1 < \eta < T$.

In particular, for every $g > 0$, if $a_1 = 0$ and $\eta T(a) = 0$ a.e $a \in (0, g)$; there is $C_{g, T} > 0$ such that:

$$\int_0^T n^2(0, t) dt \leq C \int_0^T \int_{a_1}^{a_2} n^2(a, t) dadt.$$

Moreover, if (??) hold, we have the inequality

$$\int_0^{T-\eta} \tilde{n}^2(0, t) dt \leq C \int \tilde{n}^2(a, t) dadt,$$

for every $\eta$ such that $b_1 < b$ and $b_1 < \eta < T$.

Proof of Proposition 3.1

The state $n$ of (3.3) verifies

$$\begin{cases}
-\frac{\partial n}{\partial t} - \frac{\partial n}{\partial a} + \mu_m n = 0 & \text{in } (0, a_2) \times (0, T), \\
n(a, T) = n_T & \text{in } (0, a_2).
\end{cases}$$

We denote by $\tilde{n}(a, t) = n(a, t) e^{-\int_0^a \mu_m(s) ds}$. Then, $\tilde{n}$ satisfies

$$\frac{\partial \tilde{n}}{\partial t} + \frac{\partial \tilde{n}}{\partial a} = 0 \quad \text{in } (a_1, a_2) \times (0, T).$$

Proving the inequality

$$\int_0^{T-\eta} \tilde{n}^2(0, t) dt \leq C \int_0^T \int_{a_1}^{a_2} \tilde{n}^2(a, t) dadt$$

lead to get inequality (3.7).

Indeed, we have

$$\int_0^{T-\eta} n^2(0, t) dt = \int_0^{T-\eta} \tilde{n}^2(0, t) dt \leq C \int_0^T \int_{a_1}^{a_2} \tilde{n}^2(a, t) dadt = C \int_0^T \int_{a_1}^{a_2} e^{-2 \int_0^a \mu_m(s) ds} n^2(a, t) dadt$$

$$\leq C \int_0^T \int_{a_1}^{a_2} n^2(a, t) dadt.$$

We consider the following characteristics trajectory $\gamma(\lambda) = (T - \lambda, T + t - \lambda)$. If $\lambda = T$, the backward characteristic starting from $(0, t)$. If $T < a_1$, the trajectory $\gamma(\lambda)$ never reaches the observation region $(a_1, a_2)$. So we choose $T > a_1$.

Without loss of generality, let us assume that $\eta < a_2 < T$.

**step 1: Estimation of $n(0, t), t \in (0, T - \eta)$**

- **Estimation for** $t \in (0, T - a_2)$.

We denote by:

$$w(\lambda) = \tilde{n}(T - \lambda, T + t - \lambda), \lambda \in (T - a_2, T).$$

Then, $\frac{\partial w}{\partial \lambda} = 0$ for all $\lambda \in (T - a_2, T)$. In particular, $w(\lambda)$ is constant for all $\lambda \in (T - a_2, T)$. Since $\eta < a_2$, we have

$$w(t) = \frac{1}{a_2 - \eta} \int_{T-a_2}^{T-\eta} w(\lambda) d\lambda.$$

Therefore,

$$w(T) = \tilde{n}(0, t) = \frac{1}{a_2 - \eta} \int_{T-a_2}^{T-\eta} \tilde{n}(T - \lambda, T + t - \lambda) d\lambda = \frac{1}{a_2 - \eta} \int_{\eta}^{a_2} \tilde{n}(s, t + s) ds.$$

Using the fact that

$$\left( \int_{T-a_2}^{T-\eta} w(\lambda) d\lambda \right)^2 \leq (a_2 - \eta) \int_{T-a_2}^{T-\eta} w^2(\lambda) d\lambda$$
and integrating with respect to \( t \) over \((0, T - a_2)\) we get
\[
\int_0^{T-a_2} \tilde{n}^2(0, t)dt \leq C \int_{a_1}^{a_2} \int_0^{T-a_2} \tilde{n}^2(s, t + s)dt ds = C \int_{a_1}^{a_2} \int_s^{T-a_2+s} \tilde{n}^2(s, u)du ds.
\]
Finally
\[
(3.13) \quad \int_0^{T-a_2} \tilde{n}^2(0, t)dt \leq C \int_0^{T} \int_{a_1}^{a_2} \tilde{n}^2(a, t)da dt.
\]

- **Estimation for \( t \in (T - a_2, T - \eta) \)**

We define:
\[
w(\lambda) = \tilde{n}(T - \lambda, T + t - \lambda), \; \lambda \in (T - a_1, T).
\]
Then, \( \frac{\partial w}{\partial \lambda} = 0 \) for all \( \lambda \in (T - a_2, T) \). In particular, \( w(\lambda) \) is constant for all \( \lambda \in (T - a_2, T) \). Thus we have
\[
w(t) = \frac{1}{\eta - a_1} \int_{t-\eta}^{t-a_1} w(\lambda)d\lambda = \frac{1}{\eta - a_1} \int_{a_1}^{\eta} \tilde{n}(s, t + s)ds.
\]
Integrating with respect to \( t \) over \((T - a_2, T - \eta)\), we get
\[
\int_{T-a_2}^{T-\eta} \tilde{n}^2(0, t)dt \leq C(\eta, a_1) \int_{a_1}^{\eta} \int_{T-a_2}^{T-\eta} \tilde{n}(s, t + s)dt ds.
\]
Finally
\[
(3.14) \quad \int_{T-a_2}^{T-\eta} \tilde{n}^2(0, t)dt \leq C(\eta, a_1) \int_0^{a_2} \tilde{n}(a, t)da dt.
\]
Combining (3.13) and (3.14), we obtain
\[
\int_0^{T-\eta} \tilde{n}^2(0, t)dt \leq C(\eta, a_1, a_2) \int_0^{a_2} \tilde{n}(a, t)da dt,
\]
leading to (3.7).
Remark that \( \lim_{\eta \to a_1^+} \frac{1}{\eta - a_1} = +\infty \) then, \( \lim_{\eta \to a_1^+} C(\eta, a_1, a_2) = +\infty \).
Suppose now that \( a_1 = 0 \). From the above, we have for all \( \rho > 0 \), the existence of a constant depending on \( \rho \) such that:
\[
\int_0^{T-\rho} \tilde{n}^2(0, t)dt \leq C(\rho, a_2) \int_0^{a_2} \tilde{n}^2(a, t)da dt.
\]
Moreover, if \( n_T(a) = 0 \) in \((0, \rho)\), we have \( w(t) = 0 \) in \((T - \rho, T) \). Then \( w \equiv 0 \). So \( n(0, t) = 0 \) in \((T - \rho, T) \).
Finally, if \( n_T(a) = 0 \) in \((0, \rho)\), we have the following inequality:
\[
\int_0^{T} \tilde{n}^2(0, t)dt \leq C(\rho, a_2) \int_0^{a_2} \tilde{n}^2(a, t)da dt.
\]

**step 2: Estimation of \( l(0, t), \; t \in (0, T - \eta) \)**

Considering \( \nu = \min\{b, b_2\} \), the state \( l \) of the adjoint system can be rewritten as:
\[
\begin{align*}
- \frac{\partial l}{\partial t} - \frac{\partial l}{\partial a} + \mu l &= 0 \quad \text{in} \; (0, \nu) \times (0, T), \\
l(a, T) &= l_T \quad \text{in} \; (0, \nu).
\end{align*}
\]
We denote by \( \tilde{l}(a, t) = l(a, t)e^{-\int_0^t \mu \ell(s)ds} \). Then, \( \tilde{l} \) satisfies
\[
\frac{\partial \tilde{l}}{\partial t} + \frac{\partial \tilde{l}}{\partial a} = 0 \quad \text{in} \; (b_1, \nu) \times (0, T).
\]

- **Estimation for \( t \in (0, T - \nu) \).**

Defining \( w \) as:
\[
w(\lambda) = \tilde{l}(T - \lambda, T + t - \lambda), \; \lambda \in (T - \nu, T - b_1),
\]
then, \( w(\lambda) \) is constant for all \( \lambda \in (T - \nu, T - b_1) \) and we have \( w(t) = \frac{1}{\nu - \eta} \int_{t - \nu}^{t - \eta} w(\lambda) d\lambda \). Likewise the step 1, we obtain

\[
\int_0^{T - \nu} \tilde{I}^2(0, t) dt \leq C(\eta, \nu) \int_{\mathbb{R}} \bar{I}^2(a, t) d\lambda dt
\]

• Estimation for \( t \in (T - \nu, T - \eta) \).

We denote by:

\[
w(\lambda) = \tilde{I}(T - \lambda, T + t - \lambda), \quad \lambda \in (T - \eta, T - b_1).
\]

As above \( w(\lambda) \) is constant for all \( \lambda \in (T - \eta, T - b_1) \) and we set \( w(t) = \frac{1}{\eta - b_1} \int_{t - \eta}^{t - b_1} w(\lambda) d\lambda \). Then,

\[
\int_0^{T - \eta} \tilde{I}^2(0, t) dt \leq C(\eta, b_1) \int_{\mathbb{R}} \bar{I}^2(a, t) d\lambda dt
\]

Combining (3.15) and (3.16), the inequality (3.9) follows.

\[
\text{Proposition 3.2.} \quad \text{Let us assume the assumptions } (H_1) - (H_3). \quad \text{For every } T > \sup \{a_1, A - a_2\} \text{ there exists } C_T > 0 \text{ such that the solution } (n, l) \text{ of the system (3.1) verifies the following inequality:}
\]

\[
\int_0^A n^2(a, 0) da \leq C_T \int_{\mathbb{R}} n^2(a, t) d\lambda dt.
\]

Note that for every \( T > \sup \{a_1, A - a_2\} \), there exists \( a_0 \in (a_1, a_2) \) such that \( n(a, 0) = 0 \) for all \( a \in (a_0, A) \). This is a consequence of the following lemma.

\[
\text{Lemma 3.1.} \quad \text{Let us suppose that } T > \sup \{a_1, A - a_2\}. \quad \text{Then there exists } a_0 \in (a_1, a_2) \text{ such that } T > A - a_0 > A - a \text{ for all } a \in (a_0, A). \quad \text{Therefore, } n(a, 0) = 0 \text{ for all } a \in (a_0, A).
\]

Proof of Lemma 3.1

Suppose that \( T > A - a_2 \), then there exists \( \kappa > 0 \) (we choose \( \kappa \) such that \( \kappa < a_2 - a_1 \)) \( T > A - a_2 + \kappa \). So \( T > A - (a_2 - \kappa) \) and we denote \( a_0 = a_2 - \kappa \). Then, \( T > A - a_0 > A - a \) for all \( a \in (a_0, A) \). Finally, from (3.4) for all \( (a, t) \) such that \( T - t > A - a \), we get \( n(a, 0) = 0 \) for all \( a \in (a_0, A) \).

Proof of Proposition 3.2

From the Lemma 3.1, we have to prove the following inequality:

\[
\int_0^{a_0} n^2(a, 0) da \leq C_T \int_0^T \int_{a_1}^{a_2} n^2(a, t) d\lambda dt.
\]

We set \( \check{n}(a, t) = e^{-\int_0^a \mu_m(s) ds} n(a, t) \). Then, from the first equation of (3.3), \( \check{n} \) satisfies

\[
\frac{\partial \check{n}}{\partial t} + \frac{\partial \check{n}}{\partial a} = 0 \quad \text{in } (0, A) \times (0, T).
\]

Inequality (3.11) is a consequence of the following estimation:

\[
\int_0^{a_0} \check{n}^2(a, 0) da \leq C \int_0^T \int_{a_1}^{a_2} \check{n}^2(a, t) d\lambda dt.
\]

Indeed, we have

\[
\int_0^{a_0} n^2(a, 0) da = \int_0^{a_0} e^{\int_0^a \mu_m(s) ds} \check{n}^2(a, 0) da \leq e^{2||\mu_m||_{L^1(0, a_0)}} \int_0^{a_0} \check{n}^2(a, 0) da \leq C e^{4||\mu_m||_{L^1(0, a_0)}} \int_0^T \int_{a_1}^{a_2} n^2(a, t) d\lambda dt.
\]

We consider in this proof the characteristics \( \gamma(\lambda) = (a + \lambda, \lambda) \). For \( \lambda = 0 \) the characteristics starting from \( (a, 0) \). We have two cases.

**Case 1:** \( T < a_2 \).

Two situations can arise:

- \( b_0 = a_2 - T < a_1 < a_0 \), in this situation we split the interval \( (0, a_0) \) as:

\[
(0, a_0) = (0, b_0) \cup (b_0, a_1) \cup (a_1, a_0).
\]
\[ a_1 < b_0 < a_0, \text{ in this situation we split the interval } (0, a_0) \text{ as:} \]

\[(0, a_0) = (0, a_1) \cup (a_1, a_0).\]

**Case 2: \( T \geq a_2. \)**

In this case, we split the interval \((0, a_0)\) as:

\[(0, a_0) = (0, a_1) \cup (a_1, a_0).\]

We make the proof for the second case. For the proof in the first case, see [9].

**Upper bound on \((0, a_1):\)**

For \( a \in (0, a_1), \) we set \( w(\lambda) = \tilde{n}(T + a - \lambda, T - \lambda), \) \( \lambda \in (a_1, T). \) We prove easily that \( w \) is a constant, see the proof of Proposition 3.1. Then, we set

\[ w(t) = \frac{1}{a_2 - a_1} \int_{t-a_2}^{t-a_1} w(\lambda) d\lambda. \]

Integrating with respect to \( a \) over \((0, a_1)\) we get

\[ \int_0^{a_1} \tilde{n}^2(a, 0) da \leq C \int_0^{a_1} \int_{a_1}^{a_2} \tilde{n}^2(a + \alpha, \alpha) d\alpha da. \]

Finally, we obtain

\[ \int_0^{a_1} \tilde{n}^2(a, 0) da \leq C \int_0^{a_1} \int_{a_1}^{a_2} \tilde{n}^2(a, t) dadt. \]

**Upper bound on \((a_1, a_0):\)**

For \( a \in (a_1, a_0), \) we set \( w(\lambda) = \tilde{n}(T + a - \lambda, T - \lambda), \) \( \lambda \in (a_0, T) \) and

\[ w(t) = \frac{1}{a_2 - a_0} \int_{t-a_0}^{t-a_2} w(\lambda) d\lambda. \]

Making as a above and integrating with respect to \( a \) over \((a_1, a_0)\) it follows that

\[ \int_{a_1}^{a_0} \tilde{n}^2(a, 0) da \leq C \int_{a_1}^{a_0} \int_{a_1}^{a_2} \tilde{n}^2(a, t) dadt. \]

The inequalities (3.20) and (3.21) give the desired result.

\[ \square \]

We also need the following estimate for the proof of the Theorem 3.2.

**Proposition 3.3.** Let us assume the assumptions \((H_1) - (H_2), \) let \( b_1 < a_0 < b \) and \( T > b_1. \)

Then, there exists \( C_T > 0 \) such that the solution \( l \) of (3.3) verifies

\[ \int_0^{a_0} l^2(a, 0) da \leq C_T \int_{\Xi} l^2(a, t) dadt. \]

**Proof of Proposition 3.3**

We suppose that \( \beta = 0 \) in \((0, b), \) the function \( l \) verifies:

\[
\begin{cases}
\frac{\partial l}{\partial t} + \frac{\partial l}{\partial a} + \mu l = 0 & \text{in } (0, b) \times (0, T), \\
l(a, T) = l_T & \text{in } (0, b).
\end{cases}
\]

Proceeding as in the proof of Proposition 3.2, we get the desired result.

\[ \square \]

For the proof of Theorem 3.2, we start by the following lemma.

**Lemma 3.2.** Suppose that \( T > A - a_2 + a_1 \) and \( a_1 < b. \)

Then, there exists \( a_0 \in (a_1, b) \) and \( \kappa > 0 \) such that

\[ T > T - (a_1 + \kappa) > A - a_0 > A - a \quad \text{for all } a \in (a_0, A). \]

Therefore

\[ l(a, 0) = \int_0^{A-a} \pi_2(a + s) \beta(a + s, p(s))(l(0, s) + n(0, s)) ds \quad \text{for all } a \in (a_0, A). \]
Proof of Lemma 3.2
Notice that the solution \( l(t) \) of (3.3) is given by
\[
l(t) = \begin{cases}
\pi_2(a + T - t) & 
\frac{\pi_2(a + T - t)}{\pi_2(a)} \left( 1 - (1 - \gamma)n(0, s) + \gamma l(0, s) \right) ds \quad \text{if } T - t \leq A - a,

\int_t^{1 + A - a} \pi_2(a + s - t) \left( 1 - (1 - \gamma)n(0, s) + \gamma l(0, s) \right) ds & \quad \text{if } A - a < T - t.
\end{cases}
\]

Without loss of generality, we suppose that \( a_2 = b \).
Suppose that \( T > a_1 + A - a_2 \Leftrightarrow T - a_1 > A - a_2 \). Then there exists \( \kappa > 0 \) (we choose \( \kappa \) such that \( 2\kappa < a_2 - a_1 \)) such that \( T - (a_1 + \kappa) > A - (a_2 - \kappa) \). We denote by \( a_0 = a_2 - \kappa \) and as \( A - a_0 > A - a \) for all \( a \in (a_0, A) \), then \( T > T - (a_1 + \kappa) > A - a_0 > A - a \) for all \( a \in (a_0, A) \).
Moreover, for \( (a, t) \) such that \( T - t > A - a \), we have
\[
l(a, t) = \int_t^{1 + A - a} \pi_2(a + s - t) \beta(a + s - t, p(s)) \left( 1 - (1 - \gamma)n(0, s) + \gamma l(0, s) \right) ds.
\]
For \( t = 0 \) and \( a \in (a_0, A) \), \( T - 0 > A - a_0 > A - a \), one has
\[
l(a, 0) = \int_0^{A - a} \pi_2(a + s) \frac{\pi_2(a)}{\pi_2(a)} \beta(a + s, p(s)) \left( 1 - (1 - \gamma)n(0, s) + \gamma l(0, s) \right) ds.
\]
Remark that as \( (a_1, a_2) \subset (b_1, b_2) \), if \( a_0 \in (a_1, a_2) \) then \( a_0 \in (b_1, b_2) \).

Now, we can prove the Theorem 3.2.
Proof of Theorem 3.2
Let \( a_0 \) as in the Lemma 3.2, we have:
\[
\int_0^A l^2(a, 0) da = \int_0^{a_0} l^2(a, 0) da + \int_{a_0}^A l^2(a, 0) da.
\]
Using the results of Lemma 3.2, the assumption of \( \beta \) and the regularity of \( \frac{\pi_2(a + s)}{\pi_2(a)} \), we can prove the existence of a constant \( K_T \) independent of \( p \) (\( \beta \in L^\infty((0, T) \times (0, A)) \) such that:
\[
\int_0^A l^2(a, 0) da \leq K_T \left( \int_0^{T - (a_1 + \kappa)} n^2(0, t) dt + \int_0^{T - (a_1 + \kappa)} l^2(0, t) dt \right).
\]
Moreover, we have \( b_1 \leq a_1 \leq a_1 + \kappa \). Using Proposition 3.1, it follows that
\[
\int_0^{T - (a_1 + \kappa)} n^2(0, t) dt \leq C_T \int_\Xi n^2(a, t) dadt \quad \text{and} \quad \int_0^{T - (a_1 + \kappa)} l^2(0, t) dt \leq C_T \int_\Xi l^2(a, t) dadt.
\]
Finally, adding the above to the results of Proposition 3.2 and Proposition 3.3, we get:
\[
\int_0^A n^2(a, 0) da + \int_0^A l^2(a, 0) da \leq C_T \left( \int_\Xi n^2(a, t) dadt + \int_\Xi l^2(a, t) dadt \right).
\]
\(\square\)

For \( \epsilon > 0 \) and \( \theta > 0 \), we consider the functional \( J_{\epsilon, \theta} \) defined by:
\[
J_{\epsilon, \theta}(v_m, v_f) = \frac{1}{2} \int_\Xi v_m^2 dadt + \frac{1}{2} \int_\Xi v_f^2 dadt + \frac{1}{2\epsilon} \int_0^A m^2(a, T) da + \frac{1}{2\epsilon} \int_0^A f^2(a, T) da,
\]
where \( (m, f) \) is the solution of the following system
\[
\begin{align*}
\begin{cases}
m_t + m_a + \mu m & = \chi v_m \quad \text{in } Q,
m_t + f_a + \mu f & = \chi v_f \quad \text{in } Q,
m(0, a) = m_0(a), & \quad f(0, a) = f_0(a) \quad \text{in } Q_A,\\
m(0, t) = (1 - \gamma) \int_0^A \beta(a, p) f(a, t) da, & \quad f(0, t) = \gamma \int_0^A \beta(a, p) f(a, t) da \quad \text{in } Q_T.
\end{cases}
\end{align*}
\]
Lemma 3.3. The functional $J_{\varepsilon, \theta}$ is continuous, strictly convex and coercive. Consequently, $J_{\varepsilon, \theta}$ reaches its minimum at a point $(v_{m, \varepsilon}, v_{f, \theta}) \in L^2(\Xi) \times L^2(\Xi^*)$. Setting $(m_{\varepsilon}, f_{\theta})$ the associated solution of (3.24) and $(n_{\varepsilon}, l_{\theta})$ the solution of (3.3) with

$$n_{\varepsilon}(a, T) = -\frac{1}{\varepsilon}m_{\varepsilon}(a, T) \quad \text{and} \quad l_{\theta}(a, T) = -\frac{1}{\theta}f_{\theta}(a, T),$$

we have

$$\chi_{\Xi} v_{m, \varepsilon} = \chi_{\Xi} n_{\varepsilon} \quad \text{and} \quad \chi_{\Xi} v_{f, \theta} = \chi_{\Xi} l_{\theta}.$$  

Moreover, there exist $C_1, C_2, C_3, C_4 > 0, 1 \leq i \leq 4$, independent of $\varepsilon$ and $\theta$ such that

$$\int_{\Xi} n_{\varepsilon}^2(a, t)dadt \leq C_1 \left( \int_{\Xi} m_{\varepsilon}^2(a)da + \int_{\Xi} f_{\theta}^2(a)da \right),$$

$$\int_{\Xi} m_{\varepsilon}^2(a, T)da \leq \varepsilon C_2 \left( \int_{\Xi} m_{\varepsilon}^2(a)da + \int_{\Xi} f_{\theta}^2(a)da \right),$$

$$\int_{\Xi} l_{\theta}^2(a, t)dadt \leq C_3 \left( \int_{\Xi} m_{\varepsilon}^2(a)da + \int_{\Xi} f_{\theta}^2(a)da \right),$$

$$\int_{\Xi} f_{\theta}^2(a, T)da \leq \theta C_4 \left( \int_{\Xi} m_{\varepsilon}^2(a)da + \int_{\Xi} f_{\theta}^2(a)da \right).$$

Proof of Lemma 3.3

It is easy to check that $J_{\varepsilon, \theta}$ is coercive, continuous and strictly convex. Then, it admits a unique minimiser $(v_{\varepsilon}, v_{f})$. The maximum principle gives

$$\chi_{\Xi} v_{m, \varepsilon} = \chi_{\Xi} n_{\varepsilon} \quad \text{and} \quad \chi_{\Xi} v_{f, \theta} = \chi_{\Xi} l_{\theta}$$

where the couple $(n_{\varepsilon}, l_{\theta})$ is the solution of the system:

$$\begin{cases}
-\partial_t n_{\varepsilon} - \partial_\theta n_{\varepsilon} + \mu m_{\varepsilon} = 0 \quad &\text{in } Q, \\
-\partial_l l_{\theta} - \partial_\theta l_{\theta} + \mu f \theta = (1 - \gamma)\beta(a, p)n_{\varepsilon}(0, t) + \gamma \beta(a, p)l_{\theta}(0, t) \quad &\text{in } Q, \\
n_{\varepsilon}(a, T) = -\frac{1}{\varepsilon}m_{\varepsilon}(a, T), \quad l_{\theta}(a, T) = -\frac{1}{\theta}f_{\theta}(a, T) \quad &\text{in } QA, \\
n_{\varepsilon}(A, t) = 0, \quad l_{\theta}(A, t) = 0 \quad &\text{in } QT.
\end{cases}$$

(3.26)

Multiplying the first and the second equation of (3.26) by respectively $m_{\varepsilon}$ and $f_{\theta}$, integrating with respect to $Q$ and using (3.25) we get

$$\int_{\Xi} n_{\varepsilon}^2(a, t)dadt + \frac{1}{\varepsilon} \int_{\Xi} m_{\varepsilon}^2(a, T)da = - \int_{\Xi} m_{\varepsilon}(a)n_{\varepsilon}(a, 0)da - (1 - \gamma) \int_{\Xi} \beta(a, p)f_{\theta}(a, t)n_{\varepsilon}(0, t)dadt$$

and

$$\int_{\Xi} l_{\theta}^2(a, t)dadt + \frac{1}{\theta} \int_{\Xi} f_{\theta}^2(a, T)da = - \int_{\Xi} f_{\theta}(a)l_{\theta}(a, 0)da - (1 - \gamma) \int_{\Xi} \beta(a, p)f_{\theta}(a, t)n_{\varepsilon}(0, t)dadt.$$

Combining (3.27) and (3.28), we obtain

$$\int_{\Xi} n_{\varepsilon}^2(a, t)dadt + \frac{1}{\varepsilon} \int_{\Xi} m_{\varepsilon}^2(a, T)da + \int_{\Xi} l_{\theta}^2(a, t)dadt + \frac{1}{\theta} \int_{\Xi} f_{\theta}^2(a, T)da = - \int_{\Xi} m_{\varepsilon}(a)n_{\varepsilon}(a, 0)da$$

$$- \int_{\Xi} f_{\theta}(a)l_{\theta}(a, 0)da.$$  

Using the inequality of Young, we have for any $\delta > 0$,

$$\int_{\Xi} n_{\varepsilon}^2(a, t)dadt + \frac{1}{\varepsilon} \int_{\Xi} m_{\varepsilon}^2(a, T)da + \int_{\Xi} l_{\theta}^2(a, t)dadt + \frac{1}{\theta} \int_{\Xi} f_{\theta}^2(a, T)da \leq \frac{\delta}{2} \int_{\Xi} m_{\varepsilon}^2(a)da$$

$$+ \frac{1}{2\delta} \int_{\Xi} n_{\varepsilon}^2(a, 0)da + \frac{\delta}{2} \int_{\Xi} l_{\theta}^2(a, 0)da + \frac{1}{2\delta} \int_{\Xi} f_{\theta}^2(a, 0)da.$$  

Using the observability inequality (3.6) and choosing $\delta = C T$ in the previous inequality, it follows that

$$\frac{1}{2} \int_{\Xi} n_{\varepsilon}^2(a, t)dadt + \frac{1}{\varepsilon} \int_{\Xi} m_{\varepsilon}^2(a, T)da + \frac{1}{2} \int_{\Xi} l_{\theta}^2(a, t)dadt + \frac{1}{\theta} \int_{\Xi} f_{\theta}^2(a, T)da \leq \frac{C_T}{2} \left( \int_{\Xi} m_{\varepsilon}^2(a)da + \int_{\Xi} f_{\theta}^2(a)da \right).$$
This gives the desired result necessary to the proof of the main one.

Now, we consider the system

\[
\begin{align*}
\frac{\partial}{\partial t}m_\epsilon(p) + \frac{\partial}{\partial x}m_\epsilon(p) + \mu_m m_\epsilon(p) &= \chi_{\leq n_\epsilon} \quad \text{in } Q, \\
\frac{\partial}{\partial t}f_\theta(p) + \frac{\partial}{\partial x}f_\theta(p) + \mu_m f_\theta(p) &= \chi_{\leq l_\theta} \quad \text{in } Q,
\end{align*}
\]

(3.29)

\[
m_\epsilon(p)(a,0) = m_0(a), \quad f_\theta(p)(a,0) = f_0(a) \quad \text{in } Q_A,
\]

\[
m_\epsilon(p)(0,t) = (1 - \gamma) \int_0^T \beta(a,p)f_\theta(p)(t)da, \quad f_\theta(p)(0,t) = \gamma \int_0^T \beta(a,p)f_\theta(p)(t)da \quad \text{in } QT,
\]

where \((n_\epsilon, l_\theta)\) is the solution of (3.26) that minimizes the functional \(J_{\epsilon, \theta}\). We have the following result:

**Lemma 3.4.** Under the assumptions of the Theorem 1.1, the solutions \(m_\epsilon\) and \(f_\theta\) verify the following inequalities:

\[
\int_0^A m_\epsilon^2(a,T)da + \int_0^T \int_0^A (1 + \mu_m)m_\epsilon^2(a,t)dt\,da \leq C \left( \int_0^A m_0^2(a)da + \int_0^A f_0^2(a)da \right)
\]

and

\[
\int_0^A f_\theta^2(a,T)da + \int_0^T \int_0^A (1 + \mu_f)f_\theta^2(a,t)dt\,da \leq C \left( \int_0^A m_0^2(a)da + \int_0^A f_0^2(a)da \right).
\]

**Proof of Lemma 3.4**

We denote by

\[(y_\epsilon, z_\theta) = (e^{-\lambda_\epsilon t}m_\epsilon, e^{-\lambda_\theta t}f_\theta).\]

The functions \(y_\epsilon\) and \(z_\theta\) verify

\[
\frac{\partial}{\partial t}y_\epsilon + \frac{\partial}{\partial x}y_\epsilon + (\lambda_0 + \mu_m)y_\epsilon = \chi_{\leq e^{-\lambda_\epsilon t}m_\epsilon}
\]

and

\[
\frac{\partial}{\partial t}z_\theta + \frac{\partial}{\partial x}z_\theta + (\lambda_0 + \mu_f)z_\theta = \chi_{\leq e^{-\lambda_\theta t}l_\theta}.
\]

Multiplying the equality (3.32) and the equality (3.33) by respectively \(y_\epsilon\) and \(z_\theta\) and integrating with respect to \(Q\), we get

\[
\frac{1}{2} \int_0^A y_\epsilon^2(a,T)da + \frac{1}{2} \int_0^T y_\epsilon^2(A, t)dt + \int_0^T \int_0^A (\lambda_0 + \mu_m(a))y_\epsilon^2(a,t)dt\,da = \frac{1}{2} \int_0^A y_0^2(a)da
\]

\[(1 - \gamma)^2 \int_0^T \left( \int_0^A \beta(a,p)z_\theta da \right)^2 dt + \gamma^2 \int_0^T \left( \int_0^A \beta(a,p)z_\theta da \right)^2 dt + \int_0^T \int_0^A \chi_{\leq e^{-\lambda_\theta t}l_\theta}z_\theta \,dz \,dt.
\]

Using the Young inequality, Cauchy Schwartz inequality and the fact that \(\beta\) is \(L^\infty\), we prove that:

\[
(1 - \gamma)^2 \int_0^T \left( \int_0^A \beta(a,p)z_\theta da \right)^2 dt + \int_0^T \int_0^A \chi_{\leq e^{-\lambda_\theta t}l_\theta}z_\theta \,dz \,dt \leq ||\beta||_\infty^2 ||z_\theta||_L^2(Q) + \frac{1}{2} ||y_\epsilon||_L^2(Q) + \frac{1}{2} ||n_\epsilon||_L^2(\Xi)
\]

and

\[
\gamma^2 \int_0^T \left( \int_0^A \beta(a,p)z_\theta da \right)^2 dt + \int_0^T \int_0^A \chi_{\leq e^{-\lambda_\theta t}l_\theta}z_\theta \,dz \,dt \leq ||\beta||_\infty^2 ||z_\theta||_L^2(Q) + \frac{1}{2} ||z_\theta||_L^2(Q) + \frac{1}{2} ||l_\theta||_L^2(\Xi).
\]

Therefore, choosing \(\lambda_0 > (||\beta||_\infty^2 + 3/2)\), we get:

\[
\frac{1}{2} \int_0^A z_\theta^2(a,T)da + \int_0^T \int_0^A (1 + \mu_f(a))z_\theta^2(a,t)dt\,da \leq \frac{1}{2} \left( ||f_0||_{Q,A}^2 + ||l_\theta||_{L^2(\Xi)}^2 \right).
\]
Finally, applying the result of Lemma 3.3 to the above inequality, it follows that

\[
\frac{1}{2} \int_0^A \dot{z}_0^2(a, T)da + \int_0^T \int_0^A (1 + \mu_f(a)) \dot{z}_0^2(a, t) da dt \leq C \left( \int_0^A f_0^2(a)da + \int_0^A m_0^2(a)da \right)
\]

and then the inequality (3.31) holds.

Likewise, we have

\[
\frac{1}{2} \int_0^A y_0^2(a, T)da + \int_0^T \int_0^A (1 + \mu_m)y_0^2(a, t) da dt \leq \frac{1}{2} \|m_0\|_{Q,A}^2 + \|\beta\|_{L^2(Q)}^2 + \frac{1}{2} \|n_e\|_{L^2(\Xi)}^2
\]

Using the above inequality, Lemma 3.3 and the inequality (3.36) we obtain

\[
\int_0^A y_0^2(a, T)da + \int_0^T \int_0^A (1 + \mu_m)y_0^2(a, t) da dt \leq C \left( \int_0^A f_0^2(a)da + \int_0^A m_0^2(a)da \right)
\]

and then, we get the desired result. \(\square\)

Finally, from the Lemma 3.3 and the Lemma 3.4, if \((e, \theta) \to (0, 0)\) we get:

\[
(\chi n_e, \chi \Xi l_\theta) \to (\chi v_m, \chi \Xi v_f)\) and \((m_e, f_\theta) \to (m, f)\),
\]

with \((m, f)\) solution of the problem (3.1) and

\[
m(., T) = f(., T) = 0 \quad a.e \quad a \in (0, A).
\]

We have now the necessary ingredients for the proof of Theorem 1.1.

### 3.2 Proof of Theorem 1.1

In this section, we established the existence of a fixed point for the preceding auxiliary problem. Indeed, we consider that \((H_5)\) hold and we suppose to simplify that \(\lambda(0) = \lambda(A) = 0\). For each \(p \in L^2(0, T)\), let us denote by \(\Lambda(p) \subset L^2(0, T)\) the set of all \(\int_0^A \lambda(a)m(p)da\), where the couple \((m(p), f(p))\) is the solution of the following system:

\[
\begin{cases}
\partial_t m(p) + \partial_a m(p) + \mu_m m(p) = \chi n_e \quad \text{in } Q, \\
\partial_t f(p) + \partial_a f(p) + \mu_m f(p) = \chi \Xi l \quad \text{in } Q, \\
m(p)(a, 0) = m_0(a), \quad f(p)(a, 0) = f_0(a) \quad \text{in } QA, \\
m(p)(0, t) = (1 - \gamma) \int_0^A \beta(a, p) f(p)(a, t) da, \quad f(p)(0, t) = \gamma \int_0^A \beta(a, p) f(p)(a, t) da \quad \text{in } QT,
\end{cases}
\]

and \((m(p), l(p))\) the corresponding solution of the minimizer of \(J_{e, \theta}\) with \(m(p)(a, T) = f(p)(a, T) = 0\) for almost every \(a \in (0, A)\).

We have the following result.

**Proposition 3.4.** Under the assumptions of the Theorem 1.1, for any \(p \in L^2(Q_T)\) the solution of problem (3.37) satisfies

\[
|Y(t)| + \left\| \frac{d}{dt} Y \right\|_{L^2(0, T)} \leq C \left( \|m_0\|_{L^2(Q_A)} + \|f_0\|_{L^2(Q_A)} \right),
\]

where \(Y(t) = \int_0^A \lambda(a)m(p)da\) and the constant \(C\) is independent of \(p, m_0\) and \(f_0\).

**Proof of Proposition 3.4**

Let \(Y(t) = \int_0^A \lambda(a)m(p)da\). It is easy to prove that \(Y\) is solution of system

\[
\begin{cases}
\partial_t Y + \int_0^A \mu_m(a)\lambda(a)m(p)da = R(t) \quad \text{in } QT, \\
Y(0) = \int_0^A \lambda(a)m_0(a)da,
\end{cases}
\]

where

\[
R(t) = \int_0^A \lambda'(a)m(p)da + (1 - \gamma)\lambda(0) \int_0^A \beta(a, p)f(p)da + \int_0^A \lambda(a)n(p)da.
\]
Using the Lemma 3.4 and the assumptions on $\beta$ and $\lambda$, we infer that there exists $K > 0$ such that
\begin{equation}
||R||_{L^2(Q_T)} \leq K \left(||m_0||_{L^2(Q_A)} + ||f_0||_{L^2(Q_A)}\right).
\end{equation}
By using (3.38), the Young inequality and integrating on $Q_T$, we obtain
\begin{equation}
\int_0^T |\partial_t Y|^2 dt \leq 2 \int_0^T |R(t)|^2 dt + 2 \int_0^T \left(\int_0^A \mu_m(a)\lambda(a)m_r(p)da\right)^2 dt.
\end{equation}
Moreover, the Cauchy Schwartz inequality leads to
\begin{equation}
\int_0^T \left(\int_0^A \mu_m(a)\lambda(a)m_r(p)da\right)^2 dt \leq \int_0^A \mu_m(a)\lambda(a)da \int_0^T \int_0^A \mu_m(a)\lambda(a)m_r^2(p)dadt.
\end{equation}
The inequality (3.30) and the fact that $\lambda \in C([0, A])$ give
\begin{equation}
\int_0^T \int_0^A \mu_m(a)\lambda(a)m_r^2(p)dadt \leq K_1 \left(||m_0||_{L^2(Q_A)}^2 + ||f_0||_{L^2(Q_A)}^2\right),
\end{equation}
where $K_1 > 0$ is independent of $p$, $\epsilon$ and $\theta$. Moreover as $\lambda m \in L^1(0, A)$, and using (3.39), it follows that
\begin{equation}
\left\|\frac{d}{dt} Y\right\|_{L^2(0, T)} \leq C \left(||m_0||_{L^2(Q_A)} + ||f_0||_{L^2(Q_A)}\right).
\end{equation}
Now, let $\tilde{Y} = e^{-\lambda_0 t} Y$. Then, $\tilde{Y}$ satisfies
\begin{equation}
\begin{cases}
\partial_t \tilde{Y} + \lambda_0 \tilde{Y} + e^{-\lambda_0 t} \int_0^A \mu_m(a)\lambda(a)m_r(p)da = e^{-\lambda_0 t} R(t) \quad \text{in } Q_T, \\
\tilde{Y}(0) = \int_0^A \lambda(a)m_0(a)da.
\end{cases}
\end{equation}
Multiplying the first equation of (3.41) by $\tilde{Y}$, integrating on $(0, t)$ and using successively Cauchy Schwartz and Young inequalities, we deduce that
\begin{equation}
||\tilde{Y}(t)||^2 + \lambda_0 \int_0^t \tilde{Y}^2 dt \leq ||\tilde{Y}(0)||^2 + \int_0^t \tilde{Y}^2 dt + \int_0^T \left(\int_0^A \mu_m(a)\lambda(a)m_r(p)da\right)^2 dt + ||R||_{L^2(Q_T)}.
\end{equation}
Using the above calculations and choosing $\lambda_0 > 2$, we get
\begin{equation}
||\tilde{Y}(t)||^2 \leq K_2 \left(||m_0||_{L^2(Q_A)}^2 + ||f_0||_{L^2(Q_A)}^2\right).
\end{equation}
The desired result comes from (3.40) and (3.42). \hfill \Box
It is obvious that $\Lambda(p)$ is convex, and let
\begin{equation}
W(0, T) = \left\{Y \in L^\infty(0, T), ||Y||_{L^\infty(0, T)} \leq N; \left\|\frac{dY}{dt}\right\|_{L^2(0, T)} \leq N\right\},
\end{equation}
with $N = C \left(||m_0||_{L^2(Q_A)} + ||f_0||_{L^2(Q_A)}\right)$. We have $W(0, T) \subset W^{1,1}(0, T)$. Moreover the injection of $W^{1,1}(0, T)$ into $L^2(0, T)$ is compact, see [3] Page 129. So $W(0, T)$ is relatively compact in $L^2(0, T)$. From Proposition 3.4 we have $\Lambda(W(0, T)) \subset W(0, T)$, and we see that $\Lambda(W(0, T))$ is a relatively compact subset of $L^2(0, T)$. Let us now prove that $\Lambda$ is upper-semicontinuous. This is equivalent to prove that for any closed subset $K$ of $L^2(0, T)$, $\Lambda^{-1}(K)$ is closed in $L^2(0, T)$. Let $(p_k) \in \Lambda^{-1}(K)$ such that $p_k$ converges towards $p$ in $L^2(0, T)$. Then, $p_k$ is bounded and for all $k$ there exists $P_k \in K$ such that $P_k \in \Lambda(p_k)$. Therefore, from the definition of $\Lambda$, there exists $(m_k, f_k) \in (L^2((0, T) \times (0, A)))^2$ associated to $(n_k, l_k) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ solution of (3.37) such that $P_k = \int_0^A \lambda(a)m_k(p_k)da$ and satisfying the inequalities of the Lemma 3.3 and Lemma 3.4. Consequently $(m_k, f_k)$ (and $(n_k, l_k)$) are bounded respectively in $(L^2((0, T) \times (0, A)))^2$ and $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. Thus, there exists a subsequences still denoted by $(m_k, f_k)$ and $(n_k, l_k)$ that converge weakly to $(m, f)$ in $(L^2((0, T) \times (0, A)))^2$ and $(n, l) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ respectively. Using hypothesis (H3), it follows that $\int_0^A \lambda(a)m_k(p_k)da$ converges strongly to $\int_0^A \lambda(a)m(p)da$ in $L^2(0, T)$. Now, by standard device we see that $(m, f)$ associated to $(n, l)$ are solution of (3.37) and satisfy the inequalities of the Lemma 3.3 and Lemma 3.4. This implies that $P \in \Lambda(p)$. On the other hand, thanks to the Proposition 3.4, one can extract a subsequence also denoted by $P_k$ that converges strongly towards the function $P$ in $L^2(0, T)$. Since $K$ is closed we deduce that $P \in K$. Finally, we deduce that $p \in \Lambda^{-1}(P)$. Applying the Kakutani fixed point theorem [4] in the space $L^2(0, T)$ to the mapping $\Lambda$, we infer that there is at least one $Y \in W(0, T)$ such that $Y \in \Lambda(Y)$. This completes the null controllability proof of the model (1.1).
3.3 Proof of Theorem 1.2

3.3.1 Proof of Theorem 1.2-(1)

In this section, we always consider the following system:

\[
\begin{align*}
    m_t + m_a + \mu_m m &= \chi\omega v_m \quad \text{in } Q, \\
    f_t + f_a + \mu_f f &= 0 \quad \text{in } Q, \\
    m(a,0) &= m_0, \quad f(a,0) = f_0 \quad \text{in } Q_A, \\
    m(0,t) &= (1 - \gamma) \int_0^A \beta(a,p)d\alpha, \quad f(0,t) = \gamma \int_0^A \beta(a,p)d\alpha \quad \text{in } QT,
\end{align*}
\]

for every \( p \in L^2(Q_T) \). Under the assumptions of Theorem 1.2, the controllability problem that is to find \( v_m \in L^2(\Omega) \) such that \( (m,f) \) solution of the system (3.43) verifies

\[
m(.,T) = 0 \quad a \in (g,A)
\]

is equivalent to the following observability inequality.

**Proposition 3.5.** Let us assume the assumptions \((H_1)-(H_3)\), for every \( T > A-a_2 \) and for any \( g > 0 \), if \( h_T(a) = 0 \) a.e in \((0, g)\), there exists \( C_{\Theta,T} > 0 \) such that the following inequality

\[
\int_0^A h^2(a,0)da + \int_0^T g^2(a,0)da \leq C_{\Theta,T} \int_\Theta h^2(a,t)dadt
\]

holds, where \((h,g)\) is the solution of

\[
\begin{align*}
    -h_t - h_a + \mu_m h &= 0 \quad \text{in } Q, \\
    -g_t - g_a + \mu_f g &= (1 - \gamma)\beta(a,p)g(0,t) + \gamma\beta(a,p)h(0,t) \quad \text{in } Q, \\
    h(a,T) &= h_T, \quad g(a,T) = 0 \quad \text{in } Q_A, \\
    h(A,t) &= 0, \quad g(A,t) = 0 \quad \text{in } Q_T.
\end{align*}
\]

For the proof of the Proposition 3.5, we state the following estimate.

**Proposition 3.6.** Under the assumptions \((H_1)\) and \((H_2)\), there exists a constant \( C > 0 \) such that the solution \((h,g)\) of the system (3.45) verifies

\[
\int_0^A g^2(a,0)da + \int_0^T (1 + \mu_f)g^2(a,t)dadt \leq C \int_0^T h^2(0,t)dt.
\]

Moreover, we deduce for \( h_T = 0 \) a.e in \((0, g)\) that there exists a constant \( C_{\Theta,T} > 0 \) such that

\[
\int_0^A g^2(a,0)da \leq C_{\Theta,T} \int_0^T h^2(0,t)dt \leq C_{\Theta,T} \int_0^T \int_0^2 h^2(a,t)dadt.
\]

**Proof of Proposition 3.6**

Setting \( y = e^{\lambda_0 t}g \), the function \( y \) verifies

\[
-\partial_t y - \partial_a y + (\lambda_0 + \mu_f) y = \gamma\beta(a,p)e^{\lambda_0 t}h(0,t) + (1 - \gamma)\beta(a,p)y(0,t).
\]

Multiplying the equality (3.48) by \( y \) and integrating on \( Q \), we obtain:

\[
\frac{1}{2} \int_0^A g^2(a,0)da + \frac{1}{2} \int_0^T g^2(0,t)dt + \int_0^T \int_0^A (\lambda_0 + \mu_f)g^2(a,t)dadt = \int_0^T \int_0^A \left( \gamma\beta(a,p)e^{\lambda_0 t}h(0,t) + (1 - \gamma)\beta(a,p)y(0,t) \right)ydadt.
\]

Using Young inequality and the condition on \( \beta \), we get

\[
\int_0^T \int_0^A \left( \gamma\beta(a,p)e^{\lambda_0 t}h(0,t) + (1 - \gamma)\beta(a,p)y(0,t) \right)ydadt \leq \frac{e^{2\lambda_0 T}||\beta||_{\infty}^2||h(0,.)||_{L^2(Q_T)}^2}{25} + \frac{\delta}{2}||y||_{L^2(\Omega)}^2 + \frac{||\beta||_{\infty}^2||g(0,.)||_{L^2(Q_T)}^2}{25} + \frac{\delta}{2}||y||_{L^2(\Omega)}^2.
\]
Choosing $\delta = ||\beta||_{\infty}$, we obtain
\[
\frac{1}{2} \int_0^A y^2(a,0)da + \int_0^T \int_0^A (\lambda_0 + \mu_f) y^2(a,t)dadt \leq \frac{e^{2\lambda_0 T}}{2}||h(0,.)||_{L^2(Q_T)}^2 + ||\beta||_{\infty}^2 \|y\|_{L^2(Q)}^2.
\]
Finally, choosing $\lambda_0 > ||\beta||_{\infty}^2 + 1$ it follows that
\[
\frac{1}{2} \int_0^A y^2(a,0)da + \int_0^T \int_0^A (1 + \mu_f) y^2(a,t)dadt \leq \frac{e^{2\lambda_0 T}}{2} \int_0^A h^2(0,t)dt.
\]
So,
\[
\int_0^A y^2(a,0)da \leq e^{2\lambda_0 T} \int_0^A h^2(0,t)dt.
\]
Finally, we get
\[
\int_0^A g^2(a,0)da \leq e^{2\lambda_0 T} \int_0^A h^2(0,t)dt.
\]
Combining the above inequality and the inequality (3.50) of the Proposition 3.1, for $h_T = 0$ a.e in $(0, T)$ we get
\[
(3.50)
\int_0^A g^2(a,0)da \leq C_{\theta,T} \int_0^T \int_0^2 h^2(a,t)dadt.
\]
□

Proof of Proposition 3.5
We use the results of Proposition 3.2 and Proposition 3.6. Indeed, combining (3.17) and (3.50) the desired result is obtained. □

Now, let $\epsilon > 0$ and $\rho > 0$. We consider the functional $J_\epsilon$ defined by
\[
(3.51)
J_\epsilon(v_m) = \frac{1}{2} \int_0^T \int_{a_1}^{a_2} v_m^2(a,t)dadt + \frac{1}{2\epsilon} \int_0^A m^2(a,T)da,
\]
where $(m, f)$ is the solution of the following system
\[
(3.52)
\begin{align*}
m_t + ma + \mu mm &= \chi_\Omega v_m &\text{in Q,} \\
f_t + fa + \mu f &= 0 &\text{in Q,} \\
m(a, 0) &= m_0(a) & f(a, 0) &= f_0(a) &\text{in QA}, \\
m(0, t) &= (1 - \gamma) \int_0^A \beta(a,p)f(a,t)da &\text{in QT}, \\
f(0, t) &= \gamma \int_0^A \beta(a,p)f(a,t)da &\text{in QT}.
\end{align*}
\]
We have the following lemma.

Lemma 3.5. The functional $J_\epsilon$ is continuous, strictly convex and coercive. Consequently, $J_\epsilon$ reaches its minimum at one has $v_{m,\epsilon} = \chi_\Omega h_{\epsilon}$ and there exists a positive constants $C_1$, $C_2$ independent of $\epsilon$ such that
\[
\int_0^T \int_{a_1}^{a_2} h_{\epsilon}^2(a,t)dadt \leq C_1 \left( \int_0^A m_0^2(a)da + \int_0^A f_0^2(a)da \right)
\]
and
\[
\int_0^A m_{\epsilon}^2(a,T)da \leq C_2 \left( \int_0^A m_0^2(a)da + \int_0^A f_0^2(a)da \right).
\]

Proof of Lemma 3.5
The proof is similar to that of Lemma 3.3. □

By making $\epsilon$ tending towards zero, we thus obtain that $\chi_\Omega h_{\epsilon} \rightarrow \chi_\Omega v_m$ and $(m, f) \rightarrow (m, f)$, where $(m, f)$ is the solution of the system (3.52) that verifies
\[
m(., T) = 0 \text{ a.e in } (\rho, A).
\]
Finally, a similar function $A$ is defined and a similar procedure is followed to get the null controllability for the nonlinear problem.
3.3.2 Proof of Theorem 1.2-(2)

Let \( p \in L^2(Q_T) \), under the assumptions of Theorem 1.2, the following controllability problem find \( v_f \in L^2(\Theta) \) such that the solution of the system

\[
\begin{align*}
    f_t + f_a + \mu f &= \chi Z^f v_f & \text{in } Q, \\
    f(a,0) &= f_0(a) & \text{in } Q_A, \\
    f(0,t) &= \gamma \int_0^A \beta(a,p) f(a,t) \, da & \text{in } Q_T
\end{align*}
\]

(3.53)

verifies

\[ f(., T) = 0 \text{ a.e in } (0, A). \]

is equivalent to the following observability inequality.

**Proposition 3.7.** Let us assume true the assumptions \((H_1) - (H_3) - (H_4)\). For any \( T > a_1 + A - a_2 \) there exists \( C_T > 0 \) such that

\[
\int_0^A g^2(a,0) \, da \leq C_T \int_{\Xi} g^2(a,t) \, dadt,
\]

where \( g \) is solution of the system

\[
\begin{align*}
    -g_t - g_a + \mu f &= \gamma \beta(a,p) g(0,t) & \text{in } Q, \\
    g(a, T) &= g_T & \text{in } Q_A, \\
    g(A, t) &= 0 & \text{in } Q_T.
\end{align*}
\]

(3.55)

Proof of Proposition 3.7

Using the inequality (3.9) of Proposition 3.1, the result of Proposition 3.3 and the representation of the solution of the system (3.55), we get the desired result. \( \square \)

To conclude, a similar function \( \Lambda \) is defined and a similar procedure is followed to get the null controllability for the nonlinear system. We omit all details because the extension is straightforward.

**Conclusion**

In this paper, we have proved the null controllability of a nonlinear age and two-sex structured population dynamics model. We considered two controllability issues.

The first problem is related to the total extinction, which means that, we have estimated a time \( T \) to bring the male and female subpopulation density to zero.

The second deals with the null controllability of the density of male or female individuals. In this case, the control is made to act either on the males or on the females. In the case where control acts on males, we show that the density of male individuals can be reduced to zero over part of the age range. In the event that the control acts on the females, the density of female individuals can be reduced to zero over the entire age range.

In the desire to consolidate even more the result obtained in this article, we will think in our future work of the case of controllability with positivity constraint and also to establish algorithms allowing to calculate the control from its characterization. These outstanding questions are therefore in this order:

- **Controllability with positivity constraints:** We will be interested in controllability with positivity constraint.

- **Numerical implementation:** For a given fertility rate \( \beta \) and \( \lambda \), the mortality rate \( \mu_m \) and \( \mu_f \), the initial conditions \( m_0, f_0 \) and a positive parameters \( \epsilon, \theta > 0 \), how to determine a numerical algorithm allowing to determine the \((\epsilon, \theta)-\)approximate null control functions \( v_m \) and \( v_f \)?

**References**

[1] B. Ainseba, S. Anita, *Internal stabilizability for a reaction-diffusion problem modelling a predator-prey system*, Nonlinear analysis, 61 (2005), 491 – 501

[2] S. Anita, *Analysis and Control of Age-Dependent Population Dynamics*, MATHEMATICAL MODELLING: Theory and Applications
[3] H. Brezis, Analyse fonctionnelle : Théorie et Applications, Masson, Paris, 1983
[4] K. Deimling, Nonlinear Functional Analysis, Springer Verlag, Berlin, 1985
[5] A. V Fursikov, O. Y Imanuvilov, Controllability of evolution equations, volume 34 of Lecture Notes Series. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996
[6] Y. He, B. Ainseba, Exact null controllability of the Lobesia botrana model with diffusion, J. Math. Anal. Appl. 409(2014) 530 – 543
[7] M. Iannelli, J. Pipol, Two-sex age structured dynamics in a fixed sex-ratio population
[8] D. Maity, On the Null Controllability of the Lotka-Mckendrick System, Mathematical Control and Related Fields, AIMS, 2019
[9] Y. Simporé, O. Traoré, Null controllability of a nonlinear age, space and two-sex population dynamics structured model, AMS subject classifications. 93B03, 93B05, 92D25
[10] WHO, World malaria report, 2018 (WHO) ISBN:9789241565653
[11] C. Zhao, M. Wang, P. Zhao Optimal control of harvesting for age-dependent predator-prey system, Mathematical and Computer Modelling, 42 (2005), 573 – 584