Abstract. We give a precise and concise formulation of the orientifold construction in Type II superstring theory. Our results include anomaly cancellation on the worldsheet and a $K$-theoretic computation of the background Ramond-Ramond charge.

Since its inception [Sa, PS, Ho, DLP, BiS] the orientifold construction of Type II superstring theory has proven useful both for the formal development of string theory and for its potential applications to phenomenology. For reviews see [P, D, AnS, BH1, J, BH2, BHHW]. Our work began many years ago when the first author showed the second the formula [DJM, CR, MSS, S, SS] for the Ramond-Ramond charge induced by an orientifold and inquired about its $K$-theoretic significance, especially in view of the unusual cousin of the Hirzebruch $L$-genus contained therein.

In the process of our investigations we were led to more foundational questions about orientifolds and the Type II superstring. In this letter we describe some mathematical foundations we have developed to resolve these questions. In particular, we give careful definitions of both the worldsheet and spacetime fields of an orientifold, including precise Dirac quantization conditions for the $B$-field and Ramond-Ramond (RR) field. The (Neveu-Schwarz)$^2$ $\text{NSNS}$ fields which appear in Definition 2 involve subtle topological structures which, for very different reasons, exactly fit what is needed in both the (short distance) worldsheet and (long distance) spacetime theories. The worldsheet fields are enumerated in Definition 4. A surprising challenge here is to define the integral of the $B$-field over the worldsheet. The lack of a proper orientation leads to a novel prescription, one feature of which is that the $B$-field amplitude has a (classical) anomaly in the sense that it takes values in a complex line not canonically isomorphic to the complex numbers. It then cancels against the more standard (quantum) anomaly from the spinor field on the worldsheet, and the cancellation uses the “twisted” spin structure in spacetime. The spacetime RR field is self-dual, so part of its formulation (Definition 6) involves a certain quadratic function. The general theory of self-dual fields defines an RR charge due to the orientifold background, which we compute with the prime 2 inverted, in which case it localizes to the orientifold fixed point set. Tensoring with the reals we recover the formula (8) which began this project.

Our definitions are new, even in the case of the Type I superstring, and offer some refinements of the standard (non-orientifold) Type II superstring. Taken together the orientifold data is an impressively tight structure, and leads to the most intricate matching we know between topological...
features in a short distance theory and its long distance approximation. We illustrate the tight flow of ideas in Figure 1: the basic hypotheses are in the three turquoise ovals on top, the consequences for the worldsheet theory in the green diamonds, and the consequences for the spacetime theory in the yellow rectangles.

In recent years new abelian objects have entered differential geometry. Most familiar are “gerbes with connection” in various incarnations. Often these objects have cohomological significance and can be studied as part of generalized differential cohomology and twisted versions thereof. The abelian gauge fields in this paper are examples of such objects. Conversely, these geometrical constructs are precisely what we need to formulate orientifolds. The foundations of equivariant generalized differential cohomology have yet to be fully developed; see [SV, O, BSH] for recent accounts of equivariant differential complex K-theory. On the other hand, our results about worldsheet anomalies and the RR background charge are purely topological and do not require these missing foundations.

We are writing longer accounts (e.g. [DEM]) which will on the one hand include detailed definitions, statements, and proofs; and on the other explain the results in a manner more accessible to physicists, including the relationship to previous approaches to orientifolds and several physical
consequences. One of the virtues of Definition 2 is that it tames the zoo of orientifolds into a small set of data; in subsequent papers we will unpack this definition to recreate the zoo. Also, our formulation of the B-field allows us to state new consistency conditions for D-branes in orientifolds, and suggests the existence of new NS-branes with torsion charges. We have not investigated anomalies in the long distance spacetime theory, but have provided a framework in which to investigate them.

Previous work on K-theoretic interpretation of RR charge in orientifolds includes [W2, G, Hor, BGH, OS, dBHKMMS, AH, GL, BGS, BS]. The twisted $K$-theory used here unifies all the various forms of $K$-theory in these papers.

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**NSNS superstring backgrounds**

We begin with some general concepts.

A smooth orbifold of dimension $n$ is a space locally modeled on the quotient of $\mathbb{R}^n$ by the linear action of a finite group, and it has a smooth structure defined similarly to that of a smooth manifold; see [ALR] for a recent exposition. (Smooth manifolds are smooth orbifolds.) An important example to keep in mind is the case of a global quotient, where a discrete group $\Gamma$ acts on a smooth manifold $Y$ with finite stabilizers; the quotient smooth orbifold is denoted $Y/\Gamma$. Orbifolds are presented as groupoids, which may or may not be global quotients. A smooth orbifold can also be termed a ‘smooth real Deligne-Mumford stack’, and this leads to a more invariant description, but as that $s$-word conjures up demons, we avoid it. A smooth orbifold has a geometric realization, which in the case of a global quotient $Y/\Gamma$ is known as the Borel construction. We define the ordinary cohomology of an orbifold to be that of the geometric realization.

We make use of generalized cohomology theories [A], such as periodic $KO$-theory and its connective cover $ko$, and also employ twisted versions. Twistings are geometric objects whose equivalence class can be located in a cohomology theory, at least if the cohomology theory being twisted is suitably multiplicative. The cohomological degree is a particular type of twisting. For example, on an orbifold $X$ the twistings of $K$-theory of interest here [FHT] are classified by the set

$$H^0(X;\mathbb{Z}/2\mathbb{Z}) \times H^1(X;\mathbb{Z}/2\mathbb{Z}) \times H^3(X;\mathbb{Z}).$$

Let $R$ be the Postnikov section $ko(0\cdots4)$ of $ko$; it has homotopy groups concentrated in the indicated degrees. The twistings of $K$-theory we use are classified by the abelian group $R^{-1}(X)$, which as a set is isomorphic to $[1]$. It is crucial for us that $R$ is a multiplicative cohomology theory, more precisely an $E^\infty$ ring: we can multiply and integrate. Twistings and orientations of $ko$ (or of the more familiar periodic version $KO$) induce twistings and orientations of $R$.

Abelian gauge fields and their associated currents are geometric objects which live in differential cohomology theories [F1, F2, FMS2]. The differential theory associated to a cohomology theory $h$ is denoted $h$. A systematic development of some foundations is given in [HIS]. A differential cohomology group is the set of equivalence classes in a groupoid. For example, objects in degree

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2To obtain local objects, we need to consider higher groupoids, or spaces.
What is new in our definition of the B physics, one speaks of “the orientifold of the spacetime Y of the KO twistings of X posititing a trivialization of the usual orbifold construction in superstring theory. It also includes non-orientifold theories by $\beta$-field, even in non-orientifold theories, is the inclusion $\beta$ is a twisting $\beta$ of $KR(X_w)$ whose equivalence class lies in $R^{w-1}(X)$, which is the twisting of $R^{-1}(X)$ induced by the orientifold double cover. As a set $R^{w-1}(X)$ is isomorphic to the product of cohomology groups (1), but with the last factor replaced by $H^2(X;\mathbb{Z}_w)$ for $\mathbb{Z}_w$ the local coefficient system associated to $X_w \to X$. The “curvature” of $\tilde{\beta}$ is a closed twisted $3$-form with integral periods, the $3$-form field strength of the $B$-field. It lifts to an ordinary $3$-form on $X_w$ which is odd under the orientifold involution on $X_w$. What is new in our definition of the $B$-field, even in non-orientifold theories, is the inclusion of the $H^0$ and $H^1$ components in (I). The twisted spin structure in (iv) is an isomorphism of twistings of $KO(X)$: $\Re(\beta)$ is a lift of $\tilde{\beta} + \beta$ to a $KO$-twisting and $\tau^{KO}(V)$ denotes the $KO$-twisting determined by the real vector bundle $V$. In more down-to-earth terms, an ordinary spin structure on an $n$-dimensional Riemannian manifold (or orbifold) is a reduction of the principal $O_n$-bundle of orthonormal frames to a $\text{Spin}_n$-bundle. Note that $\text{Spin}_n$ is a double cover of an index two subgroup of $O_n$. The twisted spin structure in (iv) is a similar reduction, but the double cover of the index two subgroup depends on the topological object $\beta$ underlying the $B$-field. A twisted spin structure is a discrete field in the long distance supergravity theory; it has no differential form field strength.

The $B$-field is classified topologically by $R^{-1}(X)$ as in (I). The $H^0$ component $t$ distinguishes between the usual Type B ($t = 0$) and Type A ($t = 1$). Its inclusion in the $B$-field incorporates the signs in the sum over spin structures of the worldsheet Type II theory [SW, AgMV] as part of the $B$-field amplitude [I], as we explain in [DFM]. Note that since $R^{-1}(X)$ is a group there is a distinguished zero, thus singling out the IIB superstring as “more fundamental.” The $H^1$ component $a$ is a further twisting of a Type II theory. We remark that the existence of a twisted spin structure

Definition 2. An NSNS superstring background consists of:

(i) a 10-dimensional smooth orbifold $X$ together with Riemannian metric and real-valued scalar (dilaton) field;
(ii) a double cover $\pi: X_w \to X$;
(iii) a differential twisting $\tilde{\beta}$, the $B$-field;
(iv) and a twisted spin structure $\kappa: \Re(\beta) \to \tau^{KO}(TX - 10)$.

We call $\pi: X_w \to X$ the orientifold double cover. The symbol ‘w’ is used to denote the orientifold double cover as well as its equivalence class in $H^1(X;\mathbb{Z}/2\mathbb{Z})$. In the case of a global quotient $X = Y/\Gamma$, the double cover may be described by an index two subgroup $\Gamma_0 \subset \Gamma$: then $X_w = Y/\Gamma_0$. In physics, one speaks of “the orientifold of the spacetime $Y$ by the action of $\Gamma$.” Our definition includes the usual orbifold construction in superstring theory. It also includes non-orientifold theories by positing a trivialization of $X_w \to X$. Underlying the $B$-field $\beta$ is a twisting $\beta$ of $KR(X_w)$ whose equivalence class lies in $R^{w-1}(X)$, which is the twisting of $R^{-1}(X)$ induced by the orientifold double cover. As a set $R^{w-1}(X)$ is isomorphic to the product of cohomology groups (1), but with the last factor replaced by $H^2(X;\mathbb{Z}_w)$ for $\mathbb{Z}_w$ the local coefficient system associated to $X_w \to X$. The “curvature” of $\tilde{\beta}$ is a closed twisted $3$-form with integral periods, the $3$-form field strength of the $B$-field. It lifts to an ordinary $3$-form on $X_w$ which is odd under the orientifold involution on $X_w$. What is new in our definition of the $B$-field, even in non-orientifold theories, is the inclusion of the $H^0$ and $H^1$ components in (I). The twisted spin structure in (iv) is an isomorphism of twistings of $KO(X)$: $\Re(\beta)$ is a lift of $\tilde{\beta} + \beta$ to a $KO$-twisting and $\tau^{KO}(V)$ denotes the $KO$-twisting determined by the real vector bundle $V$. In more down-to-earth terms, an ordinary spin structure on an $n$-dimensional Riemannian manifold (or orbifold) is a reduction of the principal $O_n$-bundle of orthonormal frames to a $\text{Spin}_n$-bundle. Note that $\text{Spin}_n$ is a double cover of an index two subgroup of $O_n$. The twisted spin structure in (iv) is a similar reduction, but the double cover of the index two subgroup depends on the topological object $\beta$ underlying the $B$-field. A twisted spin structure is a discrete field in the long distance supergravity theory; it has no differential form field strength.

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implies the following relations between the Stiefel-Whitney classes of spacetime $X$ and the $B$-field:

\begin{align}
    w_1(X) &= tw \\
    w_2(X) &= tw^2 + aw.
\end{align}

The Type I superstring on a spin 10-manifold $Y$ is a special case of Definition 2. Then $X_w = Y$ with trivial involution and quotient $X$ is a “non-effective” orbifold, presented as the groupoid $Y \times \text{pt} // (\mathbb{Z}/2\mathbb{Z})$. The $B$-field reduces to a discrete field in $\text{ob} H^2(Y; \mathbb{Z}/2\mathbb{Z})$—in particular, both $t$ and $a$ vanish—and the twisted spin structure is an ordinary spin structure on $Y$. The RR fields may be modeled as principal $\text{Spin}_{32}$-bundles “without vector structure” \cite{LMST, W3}. As another special case, when $X_w = \mathbb{E}^{10}$ is flat Euclidean space and $X$ the quotient by a reflection in 9 -- $p$ directions, then the resulting constraint among $t$, $a$, and $p$ reproduces the standard list of consistent orientifold projections \cite{P}.

**Worldsheet theory**

We now define a worldsheet in a given NSNS background (in the NSR formalism). Curiously, while low genus surfaces have been extensively investigated in the physics literature, a formulation valid for general worldsheets does not seem to be available, even for the Type I superstring.

**Definition 4.** Fix an NSNS superstring background as in Definition 2. Then a worldsheet consists of

(i) a compact smooth 2-manifold $\Sigma$ (possibly with boundary) with Riemannian structure;

(ii) a spin structure $\alpha$ on the orientation double cover $\hat{\Sigma} \to \Sigma$ whose underlying orientation is that of $\hat{\Sigma}$;

(iii) a smooth map $\phi: \Sigma \to X$;

(iv) an isomorphism $\phi^* w \to \hat{w}$, or equivalently a lift of $\phi$ to an equivariant map $\hat{\Sigma} \to X_w$;

(v) a positive chirality spinor field $\psi$ on $\hat{\Sigma}$ with coefficients in $\hat{\pi}^* \phi^* (TX)$;

(vi) and a negative chirality spinor field $\chi$ on $\hat{\Sigma}$ with coefficients in $T^* \hat{\Sigma}$ (the gravitino).

The orientation double cover $\hat{\Sigma}$ carries a canonical orientation, used in (ii), whereas no orientation is assumed on $\Sigma$, which indeed may be nonorientable. We use ‘$\hat{w}$’ to denote the orientation double cover of $\Sigma$. The spin structure $\alpha$, which is a discrete field on $\Sigma$, is locally on $\Sigma$ a choice of two spin structures with opposite orientations. The isomorphism in (iv) is also a discrete field on $\Sigma$.

In case $X = Y // \Gamma$ is a global quotient with double cover $X_w = Y // \Gamma_0$, a map (iii) is given by a principal $\Gamma$-bundle $P \to \Sigma$ and a $\Gamma$-equivariant map $P \to Y$. Furthermore, $P$ is oriented, elements of $\Gamma_0$ preserve the orientation, and elements of $\Gamma \setminus \Gamma_0$ reverse it. The spinor fields in (v) and (vi) use the spin structure $\alpha$; the chirality refers to the canonical orientation of $\hat{\Sigma}$.

Assume the boundary of $\Sigma$ is empty. The exponentiated Euclidean action, after integrating out $\psi$ and $\chi$, has two factors on which we focus:

\begin{equation}
    \exp \left( 2\pi i \int_\Sigma \hat{\zeta} \cdot \phi^* \hat{\beta} \right) \cdot \text{pfaff} D_{\Sigma, \alpha} (\hat{\pi}^* \phi^* (TX) - T\Sigma),
\end{equation}
the $B$-field amplitude and a pfaffian. In the first factor the pullback $\phi^* \tilde{\beta}$ is an object in $\tilde{R}^{\hat{w}-1}(\Sigma)$, because of (iv) in Definition 4. Sadly, $\Sigma$ is not endowed with an $R$-orientation and so we have introduced a new object $\zeta$ in twisted differential $R$-theory in order to define the integral. We do not give here a precise definition, but remark that one ingredient is a pushforward of $\alpha$ to $\Sigma$, which measures the obstruction to refining $\alpha$ to a pin structure. Of utmost importance is that the first factor in (5) is not a number but rather an element in a certain hermitian "$B$-line" $L_B$: the $B$-field amplitude is anomalous. Let $S$ be a parameter space of non-fermionic worldsheet fields (Definition 4(i)–(iv)). In string theory one integrates over worldsheets, and therefore the exponentiated effective action after integrating out the fermions should be a measure on $S$. The measure turns out to be a product of a measure which is manifestly well-defined on $S$ and $\tilde{\psi}$, whence the latter should be a function on $S$. The pfaffian of the Dirac operator on $\Sigma$ has its usual anomaly: it takes values in a hermitian "pfaffian line" $L_\psi$. In a parametrized family both $L_B$ and $L_\psi$ are flat line bundles over $S$: these are global anomalies. The crucial result is that the tensor product $L_B \otimes L_\psi \to S$ has a natural trivialization, and so defines $\tilde{\psi}$ as a function on $S$. This trivialization uses the twisted spin structure on spacetime $X$ (see Definition 2(iv)); it is the manner in which that twisted spin structure enters the lagrangian worldsheet theory. This anomaly cancellation—more precisely, this “setting of the quantum integrand” $\tilde{\psi}$—is the most subtle example of its kind that we’ve seen. It applies as well in the non-orientifold case: then the $B$-field amplitude is a function, as the worldsheet carries a spin structure, and the trivialization of the pfaffian line bundle $L_\psi \to S$ depends on the spin structure on spacetime.

We do not attempt to couple the worldsheet to the RR field or fermions on spacetime.

The RR field on spacetime

The RR field is self-dual, so we begin with some general remarks about self-dual fields. First, the cohomology theory used to quantize the self-dual charges and fluxes must itself be Pontrjagin self-dual [PMS1, Appendix B]. Furthermore, part of the definition of a self-dual field is a quadratic refinement of the pairing between electric and magnetic currents: this refinement is a topological datum. This quadratic function has a well-defined center of symmetry $\mu$. We interpret $-\mu$ as the self-dual charge induced by the background. If $\mu$ is nonzero and $X$ is compact, then there must be additional charged objects—D-branes—in the theory whose total charge $j_{\text{ext}}$ equals $\mu$. These charges have differential refinements—the currents—and the self-dual gauge field is a (nonflat) isomorphism $\tilde{\mu} \to \tilde{j}_{\text{ext}}$. See [W1, F1, F2, HS, PMS1, BM] for background on self-dual fields.

Fix an NSNS superstring background. The RR charges and fluxes are quantized by a twisted form of periodic $K$-theory on $X$: the $KR$ theory on $X_w$ with its involution. As $X_w$ is an orbifold, not a manifold, we must specify what we mean by its $KR$-theory. Here we do not use the geometric realization or Borel construction, but rather use a geometric model which generalizes the equivariant

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3 An $R$-orientation is the same as an $\tilde{R}$-orientation. Because of the $\hat{w}$ twisting in the integrand we do not quite need an $R$-orientation, which is a spin structure on $\Sigma$, but rather a pin structure. Still, we do not have one.

4 In this letter we ignore all subtleties associated with the measure for supermoduli. We believe these are unrelated to the questions we address here.

5 The lift of $\mu$ to a current may involve an additional topological choice.

6 Objects in $KR(X_w)$ carry a lift of the involution on $X_w$, so descend to $X$ and should be regarded as living on $X$. But there is no standard notation for $KR(X_w)$ as a twisted $K$-group on $X$. 
vector bundles of Atiyah-Segal equivariant $K$-theory. (The analogous model for complex $K$-theory appears in [FH1 §3].) The quadratic function is a single topological choice which induces quadratic functions on families of manifolds of dimension $\leq 12$. Its simplest manifestation is integer-valued and occurs on a 12-dimensional orbifold $M$ which has a double cover $M_w$, a $B$-field $\hat{\beta}$, and a twisted spin structure $\kappa$.  

**Definition 6.** Fix an NSNS superstring background as in Definition 2. Then 

(i) an RR current is an object in $KR^\beta(X_w)$; 

(ii) the required quadratic function on a 12-manifold $M$ is the composition 

$$KR^\beta(M_w) \to KO_{\mathbb{Z}/2\mathbb{Z}}^{\beta}(M_w) \cong KO_{\mathbb{Z}/2\mathbb{Z}}^{KO(TM-4)}(M_w) \to KO_{\mathbb{Z}/2\mathbb{Z}}^{4}(pt) \to \mathbb{Z}$$  

$$j \quad \mapsto \quad \kappa_j j \quad \mapsto \quad \int_{M_w} \kappa_j j \quad \mapsto \quad \epsilon\text{-component}$$  

Notice that the $B$-field $\hat{\beta}$ is used in the definition of the RR current to twist differential $KR$-theory. The twisted spin structure enters into the definition of the quadratic function at the second stage, and we have used Bott periodicity to adjust the degree. At the last stage of the quadratic function we identify the quaternionic representation group $KO_{\mathbb{Z}/2\mathbb{Z}}^{4}(pt)$ of $\mathbb{Z}/2\mathbb{Z}$ with $\mathbb{Z} \oplus \mathbb{Z}_{\epsilon}$, where $\epsilon$ is the sign representation. Our notation is schematic; details will appear in a subsequent paper. The quadratic function (7) generalizes that for non-orientifold Type II [W4, FH1, DMW] and for Type I [MW, F1].

One of our main results is the computation of the center $\mu$ of the quadratic function (7)—which equals minus the RR charge of the orientifold background—but only after inverting 2. Assume $X_w$ is a 10-manifold with involution $\sigma$. Let $i: F \to X_w$ be the fixed point set of the involution and $\nu \to F$ the normal bundle. We apply the localization theorem in $\mathbb{Z}/2\mathbb{Z}$-equivariant $KO$-theory [AS] to compute the image of $\mu$ in a certain localization of $KR^\beta(X)$. The formula generalizes that in [FH2] for the special case of the Type I superstring. There a $KO$-theory analog of the Wu class appears, and in the general story it appears in a twisted form. Let $r = \text{codim}_{X_w}(F)$. The RR charge in (possibly twisted) rational cohomology, obtained as a normalized Chern character, is 

$$-\sqrt{A(X)} \text{ch}(\mu) = \pm 2^{5-r} i_\mu \left( \sqrt{\frac{L'(F)}{L'(\nu)}} \right), \quad L'(V) = \prod \frac{x/4u}{\tanh x/4u},$$  

where the $L'$-genus of a real vector bundle $V$ is expressed as usual in terms of formal degree two classes, $\tilde{A}$ is the A-hat genus, and $u$ is the Bott generator of $K^2(pt)$. The $L'$-genus is reminiscent of Hirzebruch’s $L$-genus, but the factors of 4 in the $L'$-genus are not seen in ordinary index theory. The sign in (8) (as well as omitted powers of the Bott elements in $K$ and $KR$) depends on the twisting $\beta$. We remark that under the usual definition of $O^{\pm}$-planes the sign in (8) is $\pm$. We will give a precise formula for the sign, as well as a proof of (8) and its $K$-theory progenitor, in a subsequent paper.

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7The twisted spin structure on $M$ is an isomorphism $\kappa: \mathbb{R}(\beta) \to \tau KO(TM - 12)$.

8The normalization by $\sqrt{\tilde{A}(X)}$ is discussed in [GHM, CY, MM, F1].
Physical remarks

We conclude with some points where our work illuminates the physics.

(a) In the physics literature on global quotients one finds different species of “orientifold planes”.

Our work gives an intrinsic definition as well as global constraints on the distribution of orientifold planes. (Some examples of these constraints for toroidal orientifolds were investigated in [BHKMMS, BGS, GH].)

(b) Our formula (8) for the RR charge induced by the orientifold is compatible with [MSS, SS] but not with formulæ in [MS, HJ].

(c) In the spacetime supergravity action, there are couplings between the spacetime fermions and the RR fields. Our definition of the twisted spin structure (for the fermions) and the twisting of differential $KR$-theory (for the RR fields) are such as to allow those couplings to be globally consistently defined in an orientifold background.

(d) The case $w = 0$ corresponds to Type II string theory. In this case a twisted spin structure determines a spin structure on $X$. Moreover, the data $(t, a)$, which correspond to a graded real line bundle on $X$, can be used to produce a modified spin structure. One physical interpretation of this pair of spin structures is that they are the spin structures “seen” by the two gravitinos. (When $t = 0$ the two spin structures have the same underlying orientation, and when $t = 1$ they have opposite orientation.) Thus, introducing twistings with nonzero $a$ incorporates and generalizes Scherk-Schwarz compactifications [SSc1, SSc2, KM, H].

(e) For orbifolds which are global quotients $X/\Gamma$ there is a subgroup of discrete $B$-fields classified by $H^3(\Gamma; \mathbb{Z})$; these are known as “discrete torsion”. There has been some controversy and confusion in the literature concerning the generalization to orientifolds. In our formulation the answer is clear: discrete torsion is classified by $R^{w-1}(\Gamma)$. Whereas $H^3(\Gamma; \mathbb{Z})$ classifies central extensions of $\Gamma$ by the circle group $\mathbb{T}$, the group $R^{w-1}(\Gamma)$ classifies non-central $\mathbb{Z}/2\mathbb{Z}$-graded extensions of $\Gamma$ by $\mathbb{T}$, the action of $\Gamma$ on $\mathbb{T}$ being determined by the orientifold. (See also [BS].)

(f) In the case that spacetime is a global quotient $Y/\Gamma$ there is a model for $KR^0(X_w)$ which makes contact with the tachyon field picture of $K$-theoretic charges [W2] and is a slight generalization of both standard equivariant $K$-theory and $KR$-theory. The Chan-Paton bundle of the unstable brane filling spacetime—an object in $KR^0(X_w)$—is a $\mathbb{Z}/2\mathbb{Z}$-graded complex vector bundle with the $\Gamma$ action on $Y$ lifted: elements of $\Gamma_0$ act $\mathbb{C}$-linearly and elements of $\Gamma - \Gamma_0$ act $\mathbb{C}$-antilinearly. The tachyon field is an odd endomorphism graded commuting with this $\Gamma$ action.

(g) Suppose $W \subset X$ is the worldvolume of a D-brane. The open string field configurations on the D-brane include an object in differential $KR$-theory on $W$ with twisting $\tau_W$. The induced RR current is computed by pushing forward under the inclusion $i: W \hookrightarrow X$, from which we deduce an important constraint relating the twisting class on $W$, the topology of $W$, and the $B$-field: namely, there must exist an isomorphism $\tau_W + \tau^{KR}(\nu) \cong i^* \beta$ in $R^{p-w-1}(W)$, where $\nu \to W$ is the normal bundle. In the non-orientifold case this is the spacetime derivation of the anomaly derived in [FW] from the open string worldsheet. In case $W$ is spin and coincides with an orientifold plane, we easily recover the standard rules for when a D-brane supports an orthogonal or symplectic gauge theory. There are many more possibilities in general. The
need to view D-brane sources in orientifolds in a $K$-theoretic way in order to avoid paradoxes is central to the work of [CDE].

(h) Our work raises the possibility that there exist new solitonic objects in superstring theory. The magnetic NS current is an object in $\tilde{R}^{w+0}(X)$ while the electric NS current is an object in $\tilde{R}^{w+8}(X)$. If we consider NS-charged branes with constant charge density on their worldvolume, then the $R$-cohomology classification of NS currents recovers the fundamental string and the solitonic 5-brane. It also predicts the existence of an electrically charged particle with $\mathbb{Z}/2\mathbb{Z}$ torsion charge, as well as magnetically charged $\mathbb{Z}/2\mathbb{Z}$-torsion 7- and 8-branes. The 8-brane seems to be especially curious, being a domain wall between the type IIA and IIB theories.

(i) It would be interesting to reconcile our Dirac charge quantization condition for the $B$-field—which is different from that of the bosonic string—with that of the pure spinor formalism [B].

(j) We hope that our formulation of orientifold theory can help clarify some aspects of and prove useful to investigations in orientifold compactifications, especially in the applications to model building and the “landscape.” In particular, our work suggests the existence of topological constraints on orientifold compactifications which have not been accounted for in the existing literature on the landscape.

References

[A] J. F. Adams, Stable Homotopy and Generalised Homology, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, 1974.

[AgMV] Luis Álvarez-Gaumé, Gregory W. Moore, and Cumrun Vafa, Theta Functions, Modular Invariance, and Strings, Commun. Math. Phys. 106 (1986) 1–40.

[AH] Michael Atiyah and Michael Hopkins, A Variant of K-Theory: $K_\pm$, London Math. Soc. Lecture Notes, vol. 308, pp. 5–17, Cambridge Univ. Press, Cambridge, 2004, arXiv:math/0302128.

[ALR] Alejandro Adem, Johann Leida, and Yongbin Ruan, Orbifolds and stringy topology, Cambridge Tracts in Mathematics, vol. 171, Cambridge University Press, Cambridge, 2007.

[AnS] Carlo Angelantonj and Augusto Sagnotti, Open Strings, Phys. Rept. 371 (2002) 1–150, arXiv:hep-th/0204089.

[AS] Michael F. Atiyah and Graeme B. Segal, The index of elliptic operators. II, Ann. of Math. (2) 87 (1968), 531–545.

[B] Nathan Berkovits, ICTP Lectures on Covariant Quantization of the Superstring, arXiv:hep-th/0209059.

[BGH] Oren Bergman, Eric G. Gimon, and Petr Horava, Brane Transfer Operations and T-Duality of Non-BPS States, JHEP 04 (1999), 010, arXiv:hep-th/9902160.

[BGS] Oren Bergman, Eric G. Gimon, and Shigeki Sugimoto, Orientifolds, RR Torsion, and K-Theory, JHEP 05 (2001), 047, arXiv:hep-th/0103183.

[BH1] Ilka Brunner and Kentaro Hori, Notes on Orientifolds of Rational Conformal Field Theories, JHEP 07 (2004), 023, arXiv:hep-th/0208141.

[BH2] ______, Orientifolds and Mirror Symmetry, JHEP 11 (2004), 005, arXiv:hep-th/0303135.

[BHWW] Ilka Brunner, Kentaro Hori, Kazuo Hosomichi, and Johannes Walcher, Orientifolds of Gepner Models, JHEP 02 (2007), 001, arXiv:hep-th/0401137.

[BS] Massimo Bianchi and Augusto Sagnotti, On the Systematics of Open String Theories, Phys. Lett. B247 (1990) 517–524.

[BM] Dmitriy M. Belov and Gregory W. Moore, Type II Actions from 11-Dimensional Chern-Simons Theories, arXiv:hep-th/0611020.

[BS] Volker Braun and Bogdan Stefanski Jr., Orientifolds and K-Theory, arXiv:hep-th/0206158.

[BSh] U. Bunke and T. Schick, On the Topology of T-Duality, Rev. Math. Phys. 17 (2005) 77–112.
Jose F. Morales, Claudio A. Scrucca, and Marco Serone, Anomalous Couplings for D-Branes and O-Planes, Nucl. Phys. B552 (1999), 291–315, [arXiv:hep-th/9812071]

Gregory W. Moore and Edward Witten, Self-Duality, Ramond-Ramond Fields, and K-Theory, JHEP 05 (2000), 032, [arXiv:hep-th/9912279]

Michael L. Ortiz, Differential Equivariant K-Theory, [arXiv:0905.0476 [math.AT]]

Kasper Olsen and Richard J. Szabo, Constructing D-Branes From K-Theory, Adv. Theor. Math. Phys. 3 (1999), 889–1025, [arXiv:hep-th/9907140]

Joseph Polchinski, String Theory, Volumes I,II Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1998.

Gianfranco Pradisi and Augusto Sagnotti, Open String Orbifolds, Phys. Lett. B216 (1989), 59.

Augusto Sagnotti, Open Strings and their Symmetry Groups, [arXiv:hep-th/0208020]

Claudio A. Scrucca and Marco Serone, Anomaly Inflow and RR Anomalous Couplings, [arXiv:hep-th/9911223]

Joel Scherk and John H. Schwarz, Spontaneous Breaking of Supersymmetry Through Dimensional Reduction, Phys. Lett. B82 (1979), 60.

Richard J. Szabo and Alessandro Valentino, Ramond-Ramond Fields, Fractional Branes and Orbifold Differential K-Theory, [arXiv:0710.2773 [hep-th]].

Nathan Seiberg and Edward Witten, Spin Structures in String Theory, Nucl. Phys. B276 (1986), 272.

Edward Witten, Five-Branes Effective Action in M-Theory, J. Geom. Phys. 22 (1997), 103–133, [arXiv:hep-th/9610234]

W2 D-Branes and K-Theory, JHEP 12 (1998), 019, [arXiv:hep-th/9810188]

W3 Toroidal compactification without vector structure, JHEP 02 (1998), 006, [arXiv:hep-th/9712028]

W4 Duality relations among topological effects in string theory, J. High Energy Phys. (2000) no. 5, Paper 31, 31.