Macdonald denominators for affine root systems, orthogonal theta functions, and elliptic determinantal point processes

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Abstract

Rosengren and Schlosser introduced notions of $R_N$-theta functions for the seven types of irreducible reduced affine root systems, $R_N = A_{N-1}, B_N, B'_N, C_N, C'_N, BC_N, D_N, N \in \mathbb{N}$, and gave the Macdonald denominator formulas. We prove that, if the variables of the $R_N$-theta functions are properly scaled with $N$, they construct seven sets of biorthogonal functions, each of which has a continuous parameter $t \in (0, t^*]$ with $0 < t^* < \infty$. Following the standard method in random matrix theory, we introduce seven types of one-parameter ($t \in (0, t^*)$) families of determinantal point processes in one dimension, in which the correlation kernels are expressed by the biorthogonal theta functions. We demonstrate that they are elliptic extensions of the classical determinantal point processes whose correlation kernels are expressed by trigonometric and rational functions. In the scaling limits associated with $N \to \infty$, we obtain four types of elliptic determinantal point processes with an infinite number of points and parameter $t \in (0, t^*)$. We give new expressions for the Macdonald denominators using the Karlin–McGregor–Lindström–Gessel–Viennot determinants for noncolliding Brownian paths, and show the realization of the associated elliptic determinantal point processes as noncolliding Brownian brides with a time duration $t^*$, which are specified by the pinned configurations at time $t = 0$ and $t = t^*$.

1 Introduction

Random $N$-point process, $N \in \mathbb{N} \equiv \{1, 2, \ldots \}$, on a space $S \in \mathbb{R}^d$ is a statistical ensemble of nonnegative integer-valued Radon measures

$$\Xi() = \sum_{j=1}^{N} \delta_{X_j}(),$$

where $\delta_y(), y \in S$ denotes the delta measure such that $\delta_y(\{x\}) = 1$ if $x = y$ and $\delta_y(\{x\}) = 0$ otherwise, provided that the distribution of points $\{X_j\}_{j=1}^{N}$ on $S$ is governed by a probability measure $P$. We assume that $P$ has density $p$ with respect to the Lebesgue measure $dx = \prod_{j=1}^{N} dx_j$, i.e., $P(X \in dx) = p(x)dx$, $x \in S^N$. For the point process $(\Xi, P)$, $n$-point correlation function of a set $\{x_1, \ldots, x_n\} \in S^n$, $1 \leq n \leq N$, is defined by

$$\rho(\{x_1, \ldots, x_n\}) = \frac{1}{(N-n)!} \int_{S^{N-n}} \prod_{j=n+1}^{N} dx_j p(x_1, \ldots, x_n, x_{n+1}, \ldots, x_N).$$

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Then, for any set of observables $\chi_\ell, \ell = 1, 2, \ldots, N$, we have the following useful formulas for expectations,

$$
E \left[ \int_{S^n} \prod_{\ell=1}^n \chi_\ell(x_\ell) dz(x_\ell) \right] = \int_{S^n} \prod_{\ell=1}^n \left\{ dz_\ell \chi_\ell(x_\ell) \right\} \rho(x_1, \ldots, x_n), \quad n = 1, 2, \ldots, N.
$$

If any correlation function is expressed by a determinant in the form

$$
\rho(x_1, \ldots, x_n) = \det_{1 \leq j, k \leq n} [K(x_j, x_k)]
$$

with a two-point continuous function $K(x, y), x, y \in S$, then the point process is said to be determinantal and $K$ is called the correlation kernel [27, 25, 26, 1, 13].

A typical example of determinantal point process is the eigenvalue distribution on $S = \mathbb{R}$ of Hermitian random matrices in the Gaussian unitary ensemble (GUE) studied in random matrix theory [21, 6, 1]. The probability measure is given as

$$
P_{\text{GUE}}(X \in dx) = p_{\text{GUE}}(x)dx = \frac{1}{C_{\text{GUE}}} \prod_{\ell=1}^N e^{-x_\ell^2} \prod_{1 \leq j < k \leq N} (x_k - x_j)^2 dx,
$$

which is normalized as $(1/N!) \int_{\mathbb{R}^N} P_{\text{GUE}}(x)dx = 1$ with $C_{\text{GUE}} = 2^{-N(N-1)/2} \prod_{n=1}^{N-1} n! \pi^{N/2}$. It is not obvious that one can perform integrations (1.1) for (1.3) and obtained results are generally expressed by determinants as (1.2). The verification is, however, not difficult, if we have the following preliminaries [21, 6, 1, 13].

[P1] The factor $\prod_{1 \leq j < k \leq N} (x_k - x_j)$ in (1.3) obeys the Weyl denominator formula for the classical root system $A_{N-1}$,

$$
\det_{1 \leq j, k \leq N} [x_k^{j-1}] = \prod_{1 \leq j < k \leq N} (x_k - x_j) \qquad (1.4)
$$

[P2] By a basic property of determinant, without change of value, we can replace the entries $x_k^{j-1}$ in LHS of (1.4) by any monic polynomials of $x_k$ with order $j - 1$. Here we choose them as the monic Hermitian polynomials $2^{-(j-1)} H_{j-1}(x) \equiv 2^{-(j-1)} e^{x^2} (-d/dx)^{j-1} e^{-x^2}$, and obtain the following equality including the square roots of Gaussian weights in (1.3),

$$
\prod_{\ell=1}^N e^{-x_\ell^2/2} \det_{1 \leq j, k \leq N} [x_k^{j-1}] = \det_{1 \leq j, k \leq N} \left[ 2^{-(j-1)} e^{-x_\ell^2/2} H_{j-1}(x_k) \right].
$$

The reason of this choice is that they satisfy the orthogonal relation,

$$
\int_{\mathbb{R}} \left\{ 2^{-(j-1)} e^{-x^2/2} H_{j-1}(x) \right\} \left\{ 2^{-(k-1)} e^{-x^2/2} H_{k-1}(x) \right\} dx = h_j \delta_{jk}, \quad j, k \in \mathbb{N},
$$

where $h_j = 2^{-(j-1)}(j - 1)! \pi^{1/2}$.

Then integrals (1.1) are given by determinants (1.2) with the correlation kernel,

$$
K_{\text{GUE}}(x, y) = \sum_{n=1}^N \frac{1}{h_n} \left\{ 2^{-(n-1)} e^{-x^2/4} H_{n-1}(x) \right\} \left\{ 2^{-(n-1)} e^{-y^2/4} H_{n-1}(y) \right\}, \quad x, y \in \mathbb{R} \quad (1.5)
$$

See Appendix C for proof in a general setting.

In [24], Rosengren and Schlosser extended the Weyl denominator formulas for classical root systems to the Macdonald denominator formulas for seven types of irreducible reduced affine root systems, $R_N = A_{N-1}, B_N, C_N, D_N, E_6, E_7, E_8$. They expressed the result using the theta functions and stated that they are elliptic extensions of the classical results. In the present paper, we use their result as an elliptic
extension of the preliminary [P1]. We report in this paper an elliptical extension of the preliminary [P2], and then construct seven types of determinantal point processes in the elliptic level, \((E^{GN}, P^{R\alpha}, t \in (0, t_\ast))\), \(R_N = A_{N-1}, B_N, B_N', C_N, C_N', BC_N, D_N, N \in \mathbb{N}\), in the sense that their correlation kernels are expressed by the orthogonal theta functions and, if we take appropriate limits of parameters, they are reduced to the classical ones expressed by trigonometric and rational functions.

Once the \(N\)-point systems have been proved to be determinantal, by taking proper scaling limit associated with \(N \to \infty\) limit of the correlation kernels, we can define the determinantal point processes with an infinite number of points. Remark that any \(N \to \infty\) limit of the probability measure \(P^{GUE_N}\) is meaningless, since as shown by (1.3) it is absolutely continuous to the Lebesgue measure of \(N\) dimensions, \(dx = \prod_{j=1}^N dx_j\), and \(N \to \infty\) limit of \(dx\) cannot be mathematically defined. Taking the scaling limit called the bulk scaling limit, we obtain the following kernel from (1.5) [21, 6, 1, 13],

\[
K^\sin(x, y) = \frac{\sin\{\pi \rho(x-y)\}}{\pi (x-y)}, \quad x, y \in \mathbb{R}. \tag{1.6}
\]

This is called the sine kernel and it governs the determinantal point process on \(\mathbb{R}\) with an infinite number of points, which is spatially homogeneous on \(\mathbb{R}\) with constant density of points \(\rho > 0\).

Our elliptic determinantal point processes have two positive parameters \(t_\ast\) and \(r\). We demonstrate that in the limit \(t_\ast \to \infty\), our seven types of determinantal point processes in the elliptic level are reduced to the four types of determinantal point processes in the trigonometric level, in which the correlation kernels are expressed by sine functions. If we take the further limit \(r \to \infty\), they are reduced to the three types of sine kernels, one of which is identified with (1.6). The bulk scaling limit is realized in our systems by taking the double limit \(N \to \infty, r \to \infty\) with ratio \(N/r\) fixed. We construct four types of determinantal point processes in the elliptic level with an infinite number of particles. The reductions of them in the limit \(t_\ast \to \infty\) to the classical infinite determinantal point processes are also shown.

The determinantal point process of GUE, \((E^{GUE_N}, P^{GUE_N})\), is related with an interacting particle system consisting of \(N\) Brownian motions on \(\mathbb{R}, N \in \mathbb{N}\). The transition probability density of the one-dimensional standard Brownian motion (BM) from a point \(v\) at time \(s\) to a point \(x\) at time \(t\) is given by \(p^{BM}(s, v; t, x) = e^{-(v-x)^2/(2(t-s))}/\sqrt{2\pi(t-s)}, 0 \leq s < t, v, x \in \mathbb{R}\). As a function of \(x\), this is nothing but the probability density function of the Gaussian distribution with mean \(v\) and variance \(t-s\). The square of products of differences, \(\prod_{1 \leq j < k \leq N} (x_k-x_j)^2\), in (1.3) shows that the points \(\{x_j\}_{j=1}^N\) on \(\mathbb{R}\) are distributed exclusively. The corresponding stochastic process is then realized as a system of Brownian motions conditioned never to collide with each other [13]. Consider the Weyl chamber, \(W_N = \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N\}\).

For \(v, x \in W_N\), the total probability mass of \(N\)-tuple of Brownian paths, in which the \(j\)-th path starts from \(v_j\) at time \(s\) and arrives at \(x_j\) at time \(t > s\), \(j = 1, 2, \ldots, N\), is given by a determinant \(\det[p(s, v; t, x)]\). Here \(p(s, v; t, x)\) is the \(N \times N\) matrix whose \((j, k)\)-entry is given by \(p^{BM}(s, v_j; t, x_k), j, k \in \{1, 2, \ldots, N\}\). This is known as the Karlin–McGregor–Lindström–Gessel–Viennot (KMLGV) formula [10, 20, 9]. Here we consider the situation such that \(N\) BMs start from a given configuration \(v \in W_N\) at time \(0\), execute noncolliding process, and then return to the configuration \(v\) at time \(t > 0\). Such a process is called the \(N\)-particle system of noncolliding Brownian bridges from \(v\) to \(v\) in time duration \(t_\ast\) (see, for instance, Part I, IV.4.22 of [3] for the original Brownian bridge of a single path). The probability density at time \(t\) of this \(N\)-particle process is then given by (see Section V.C of [16])

\[
p_t^{v \to v}(x; t_\ast) = \frac{\det[p^{BM}(0, v; t, x)] \det[p^{BM}(t, x; t_\ast, v)]}{\det[p^{BM}(0, v; t_\ast, v)]}, \quad x \in W_N, \quad t \in (0, t_\ast).
\]

We can prove that the limit \(v \to 0\) \((0, \ldots, 0) \in \mathbb{R}^N\) exists (see, for instance, Section 3.3 in [13]), and we obtain

\[
p_t^{0 \to 0}(x; t_\ast) = \frac{1}{C(N, t, t_\ast)} \prod_{1 \leq j < k \leq N} (x_k-x_j)^2, \quad x \in W_N, \quad t \in (0, t_\ast), \tag{1.7}
\]

with a normalization factor \(C(N, t, t_\ast)\) which does not depend on \(x\). If we put \(t_\ast = 2\) and \(t = t_\ast/2 = 1\), (1.7) coincides with \(p^{GUE_N}(x)\) in (1.3). In other words, the \(N\)-particle system of noncolliding Brownian bridges from \(0\) to \(0\) with time duration \(t_\ast\) realizes a one-parameter extension of determinantal point process of GUE.
Each type of elliptic determinantal point processes studied in this paper makes a family with one continuous parameter \( t \in (0,t_*) \) (in addition to a discrete parameter \( N \in \mathbb{N} \)). We can show that, \( (\mathcal{X}^{A_{N-1}}, \mathbf{P}_t^{A_{N-1}}, t \in (0,t_*) ) \) is realized as an \( N \)-particle system of noncolliding Brownian bridges on a circle with radius \( r \), for \( R_N = B_N, B_N', C_N, C_N', BC_N, (\mathcal{X}^{R_N}, \mathbf{P}_t^{R_N}, t \in (0,t_*)) \) are realized as \( N \)-particle systems of noncolliding Brownian bridges in an interval \([0, \pi r]\) with absorbing boundary conditions at both edges, and \( (\mathcal{X}^{D_N}, \mathbf{P}_t^{D_N}, t \in (0,t_*)) \) is realized as noncolliding \( N \)-Brownian bridges in \([0, \pi r]\) with reflecting boundary conditions at both edges. These Brownian bridges are specified by the pinned configurations \( \bar{v}_t^{R_N} \) at the initial time \( t = 0 \) and at the final time \( t = t_* \).

The paper is organized as follows. In Section 2 we first list up the Macdonald denominators \( W_t^{R_N}(\xi; \tau) \) for the seven types of irreducible reduced affine root systems, \( R_N = A_{N-1}, B_N, B_N', C_N, C_N', BC_N, D_N \), and give explicit expressions of theta functions used by Rosengren and Schlosser for the Macdonald denominator formulas [24]. In this paper we use the classical expressions of Jacobi’s theta functions (as shown in Appendix A) in order to clarify the conditions that the functions are real-valued or complex-valued, and to show dependence on the parameters \( t_* \) and \( r \) explicitly. We prove that, if we put the two variables \( (\xi, \tau) \) in these theta functions as functions of \((x,t)\) properly depending on the value of \( N \), then the obtained sets of functions \( \{M_t^{R_N}(x,t)\}_{j=1}^N \) constructing seven families of biorthogonal systems with respect to the integral over \( x \) which have a continuous parameter \( t \in (0,t_*) \) (Lemma 2.1). In Section 3 we introduce seven types of point processes \( (\mathcal{X}^{R_N}, \mathbf{P}_t^{R_N}, t \in (0,t_*)) \), associated with the seven sets of biorthogonal theta functions after giving the nonnegative conditions (Lemma 3.1) and the normalization conditions (Lemma 3.2) for \( \mathbf{P}_t^{R_N} \). As a byproduct of the latter, the Selberg-type integral formulas including the Jacobi theta functions are derived as shown in Appendix B. Then we prove that they are all determinantal with parameter \( t \in (0,t_*) \) (Theorem 3.3). The proof of the theorem with derivation of correlation kernels is given by the standard method in random matrix theory as explained in Appendix C. We discuss the temporally homogeneous limit \( t_* \to \infty \) in Section 3.2 and the scaling limit associated with \( N \to \infty \) limit in Section 3.3 (Theorem 3.6) by analyzing the correlation kernels expressed by \( \{M_t^{R_N}(x,t)\}_{j=1}^N \). Reductions of the determinantal point processes from the present elliptic level to the trigonometric and rational function levels are shown by studying some limit transitions. In Section 4.1 we give new expressions for the Macdonald determinants by the KMLGV determinants of noncolliding BMs (Proposition 4.2). Then in Section 4.2 we show the realizations of \( (\mathcal{X}^{R_N}, \mathbf{P}_t^{R_N}, t \in (0,t_*)) \) as \( N \)-particle systems of Brownian bridges with time duration \( t_* \) (Theorem 4.4). Concluding remarks are given in Section 5.

2 Orthogonal Theta Functions

2.1 Macdonald denominator formulas of Rosengren and Schlosser

Assume that \( N \in \mathbb{N} \equiv \{1,2,\ldots\} \). As extensions of the Weyl denominators for classical root systems, Rosengren and Schlosser studied the Macdonald denominators for the seven types of irreducible reduced affine root systems, \( W_t^{R_N}(\xi), \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{C}^N, R_N = A_{N-1}, B_N, B_N', C_N, C_N', BC_N, D_N, N \in \mathbb{N} \) [24]. Up to trivial factors they are written using the Jacobi theta functions as follows. (Notations and formulas
of the Jacobi theta functions used in this paper are shown in Appendix A.)

\[
\begin{align*}
W^{A_{N-1}}(\xi; \tau) &= \prod_{1 \leq j < k \leq N} \vartheta_1(\xi_k - \xi_j; \tau), \\
W^{B_N}(\xi; \tau) &= \prod_{\ell=1}^N \vartheta_1(\xi_\ell; \tau) \prod_{1 \leq j < k \leq N} \left\{ \vartheta_1(\xi_k - \xi_j; \tau)\vartheta_1(\xi_k + \xi_j; \tau) \right\}, \\
W^{B_N}(\xi; \tau) &= \prod_{\ell=1}^N \vartheta_1(2\xi_\ell; 2\tau) \prod_{1 \leq j < k \leq N} \left\{ \vartheta_1(\xi_k - \xi_j; \tau)\vartheta_1(\xi_k + \xi_j; \tau) \right\}, \\
W^{C_N}(\xi; \tau) &= \prod_{\ell=1}^N \vartheta_1(\xi_\ell + \frac{\tau}{2}; \tau) \prod_{1 \leq j < k \leq N} \left\{ \vartheta_1(\xi_k - \xi_j; \tau)\vartheta_1(\xi_k + \xi_j; \tau) \right\}, \\
W^{C_N}(\xi; \tau) &= \prod_{\ell=1}^N \left\{ \vartheta_1(\xi_\ell; \tau)\vartheta_1(2\xi_\ell; 2\tau) \right\} \prod_{1 \leq j < k \leq N} \left\{ \vartheta_1(\xi_k - \xi_j; \tau)\vartheta_1(\xi_k + \xi_j; \tau) \right\}, \\
W^{D_N}(\xi; \tau) &= \prod_{1 \leq j < k \leq N} \left\{ \vartheta_1(\xi_k - \xi_j; \tau)\vartheta_1(\xi_k + \xi_j; \tau) \right\},
\end{align*}
\]  

(2.1)

where \( \tau \in \mathbb{C}, \Im \tau > 0 \). They introduced the notions of \( A_{N-1} \)-theta function of norm \( \alpha \) and \( R_N \)-theta function for \( R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N \). Then they proved that, if \( f_j^{A_{N-1}}, j = 1, 2, \ldots, N \) are \( A_{N-1} \)-theta function of norm \( \alpha \), then

\[
\det_{1 \leq j < k \leq N} \left[ f_j^{A_{N-1}}(\xi_k; \tau) \right] = C^{A_{N-1}}(\tau)\vartheta_1 \left( \sum_{\ell=1}^N \xi_\ell + \tilde{\alpha} \right) W^{A_{N-1}}(\xi; \tau) \tag{2.2}
\]

with \( \alpha = e^{2\pi i\tilde{\alpha}} \), and if \( f_j^{R_N}, j = 1, 2, \ldots, N \), are \( R_N \)-theta functions,

\[
\det_{1 \leq j < k \leq N} \left[ f_j^{R_N}(\xi_k; \tau) \right] = C^{R_N}(\tau)W^{R_N}(\xi; \tau), \quad R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N, \tag{2.3}
\]

where \( C^{R_N}(\tau) \) depend on \( \tau \) and \( N \) but not on \( \xi \). The factors \( C^{R_N}(\tau) \) are explicitly determined in Proposition 6.1 in [24] and the equalities (2.2) and (2.3) are called the Macdonald denominator formulas. See also [18, 28].

### 2.2 Biorthogonality

Assume that \( 0 < r < \infty \). Let \( i = \sqrt{-1} \), and put

\[
\xi(x) = \xi(x; r) = \frac{x}{2\pi r}, \quad \tau(t) = \tau(t; r) = \frac{it}{2\pi r^2},
\]

and

\[
N^{R_N} = \begin{cases} 
N, & R_N = A_{N-1}, \\
2N - 1, & R_N = B_N, \\
2N, & R_N = B_N^\vee, C_N^\vee, \\
2(N + 1), & R_N = C_N, \\
2N + 1, & R_N = BC_N, \\
2(N - 1), & R_N = D_N.
\end{cases}
\]  

(2.5)
In the present paper, we consider the following seven sets of functions of \((x, t) \in \mathbb{R} \times [0, \infty), \{M_j^{R_N}(x, t)\}_{j=1}^N\), which are defined using the \(A_{N-1}\)-theta function of norm \(\alpha = e^{2\pi i N \tau}\) with
\[
\tilde{\alpha}_N = \begin{cases} 
N\tau(t)/2, & \text{if } N \text{ is even,} \\
(1 + N\tau(t))/2, & \text{if } N \text{ is odd,}
\end{cases} \tag{2.6}
\]
and the \(R_N\)-theta functions, \(R_N = B_{N'}, C_{N'}, C_{N'}', BC_N, D_N\), of Rosengren and Schlosser as
\[
M_j^{R_N}(x, t) = f_j^{R_N}(\xi(x); N^{R_N}\tau(t)), \quad j = 1, 2, \ldots, N. \tag{2.7}
\]
Note that the choice of norm (2.6) for the \(A_{N-1}\)-theta function is different from the previous papers [12, 14, 15].

The explicit expressions of these functions are given by follows,
\[
M_j^{A_{N-1}}(x, t) = M_j^{A_{N-1}}(x, t; r)
\]
\[
e^{2\pi i J^{A_{N-1}}(j)(x)} \varphi_1\left(N^{A_{N-1}}\{J^{A_{N-1}}(j)\tau(t) + \xi(x); (N^{A_{N-1}})^2\tau(t)\right), \tag{2.8}
\]
\[
M_j^{R_N}(x, t) = M_j^{R_N}(x, t; r)
\]
\[
e^{2\pi i J^{R_N}(j)(x)} \varphi_1\left(N^{R_N}\{J^{R_N}(j)\tau(t) + \xi(x); (N^{R_N})^2\tau(t)\right), \tag{2.9}
\]
\[
M_j^{R_N}(x, t) = M_j^{R_N}(x, t; r)
\]
\[
e^{2\pi i J^{R_N}(j)(x)} \varphi_1\left(N^{R_N}\{J^{R_N}(j)\tau(t) - \xi(x); (N^{R_N})^2\tau(t)\right), \tag{2.10}
\]
\[
M_j^{D_N}(x, t) = M_j^{D_N}(x, t; r)
\]
\[
e^{2\pi i J^{D_N}(j)(x)} \varphi_1\left(N^{D_N}\{J^{D_N}(j)\tau(t) - \xi(x); (N^{D_N})^2\tau(t)\right), \tag{2.11}
\]
where
\[
J^{R_N}(j) = \begin{cases} 
\frac{j}{2}, & R_N = A_{N-1}, C_{N'}, \\
-j, & R_N = B_{N'}, D_N, \\
j, & R_N = C_{N}, BC_N. \tag{2.12}
\end{cases}
\]
The complex conjugates of these functions are given as
\[
\overline{M_j^{A_{N-1}}(x, t)} = M_j^{A_{N-1}}(-x, t),
\]
\[
\overline{M_j^{R_N}(x, t)} = M_j^{R_N}(x, t) \in \mathbb{R}, \quad \text{for } R_N = B_{N'}, D_N,
\]
\[
\overline{M_j^{R_N}(x, t)} = -M_j^{R_N}(x, t) \in i\mathbb{R}, \quad \text{for } R_N = C_{N}, BC_N. \tag{2.13}
\]

In RHS of (2.7) the setting of the second variable be \(N^{R_N}\tau(t)\) with (2.5) instead of \(\tau(t)\) is essential for establishing the following biorthogonal relations.

**Lemma 2.1** Assume \(0 < \tau_*, < \infty\). For any \(t \in (0, t_*), if j, k \in \{1, 2, \ldots, N\}, then
\[
\int_0^{2\pi} \overline{M_j^{A_{N-1}}(x, t_*) - t)} M_k^{A_{N-1}}(x, t) dx = m_j^{A_{N-1}}(t_*) \delta_{jk}, \tag{2.14}
\]
\[
\int_0^{2\pi} \overline{M_j^{R_N}(x, t_*)} M_k^{R_N}(x, t) dx = m_j^{R_N}(t_*) \delta_{jk}, \quad \text{for } R_N = B_{N'}, C_{N}, C_{N'}, BC_N, D_N. \tag{2.15}
\]
where

\[ m_j^{R_N}(t_s) = 2\pi r \vartheta_2 \left( N^{R_N} J^R_N(j) \tau(t_s); (N^{R_N})^2 \tau(t_s) \right), \quad \text{for } R_N = A_{N-1}, C_N, C_N^*, BC_N, \quad (2.16) \]

\[ m_j^{R_N}(t_s) = \begin{cases} 4\pi r \vartheta_2 \left( 0; (N^{R_N})^2 \tau(t_s) \right), & j = 1, \\ 2\pi r \vartheta_2 \left( N^{R_N} J^R_N(j) \tau(t_s); (N^{R_N})^2 \tau(t_s) \right), & j \in \{2, 3, \ldots, N\}, \end{cases} \quad \text{for } R_N = B_N, B_N^*, \quad (2.17) \]

\[ m_j^{D_N}(t_s) = \begin{cases} 4\pi r \vartheta_2 \left( 0; (N^{D_N})^2 \tau(t_s) \right), & j = 1, \\ 2\pi r \vartheta_2 \left( N^{D_N} J^D_N(j) \tau(t_s); (N^{D_N})^2 \tau(t_s) \right), & j \in \{2, 3, \ldots, N-1\}, \end{cases} \quad (2.18) \]

**Proof.** (i) First we prove (2.14) for the type \( A_{N-1} \). By (2.5), (2.12), and (2.13),

\[
I_{jk}^{A_{N-1}} = \int_0^{2\pi} M_j^{A_{N-1}}(x, t_s - t) M_k^{A_{N-1}}(x, t) dx \\
= \int_0^{2\pi} dx e^{2\pi i (k-j)\xi} \vartheta_2 \left( N \{(j-1/2)\tau(t_s - t) - \xi(x)\}; N^2 \tau(t_s - t) \right) \\
\times \vartheta_2 \left( N \{(k-1/2)\tau(t) + \xi(x)\}; N^2 \tau(t) \right) \\
= \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} e^{(n-1/2)2N^2\tau(t_s - t) \pi i + (2m-1)(n-1/2)\tau(t_s - t) \pi i} \\
\times e^{(m-1/2)2N^2\tau(t) \pi i + (2m-1)(n-1/2)\tau(t) \pi i} \int_0^{2\pi} e^{2\pi i ((k-j) - N(n-m)) \xi(x)} dx,
\]

where we have used the definition of \( \vartheta_2 \) given by (A.1). By (2.4),

\[
\int_0^{2\pi} e^{2\pi i (k-j) - N(n-m)} \xi(x) dx = 2\pi r \int_0^1 e^{2\pi i (k-j) - N(n-m)} \xi d\xi.
\]

Here we use the equality

\[
\int_0^1 e^{2\pi i \theta \xi} d\xi = 1(\theta = 0),
\]

where \( 1(\omega) \) is the indicator function of condition \( \omega \); \( 1(\omega) = 1 \) if \( \omega \) is satisfied, and \( 1(\omega) = 0 \) otherwise. The integral \( I_{jk}^{A_{N-1}} \) is nonzero, if and only if

\[
(k-j) - N(n-m) = 0.
\]

Since \( j,k \in \{1, 2, \ldots, N\} \) and \( n, m \in \mathbb{Z} \), we see that \(-(N-1) \leq k-j \leq N-1 \), and \( N(n-m) \in \mathbb{NZ} \). Hence (2.19) is satisfied if and only if, \( j = k \) and \( n = m \). Therefore, we can conclude \( I_{jk}^{A_{N-1}} = 0 \), if \( j \neq k \), and

\[
I_{jj}^{A_{N-1}} = 2\pi r \vartheta_2 \left( N \{(j-1/2)\tau(t_s) + \tau(t)\}; N^2 \tau(t_s) \right) \\
= 2\pi r \vartheta_2 \left( N \{(j-1/2)\tau(t_s); N^2 \tau(t_s) \} \right).
\]

Then for \( R_N = A_{N-1} \) the proof of (2.14) with (2.16) is complete.

(ii) Next we prove (2.15) for \( R_N = B_N \) and \( B_N^* \). By (2.13), LHS of (2.15) is given by

\[
I_{jk}^{R_N} = \int_0^{2\pi} M_j^{R_N}(x, t_s - t) M_k^{R_N}(x, t) dx \\
= I_{jk,++}^{R_N} - I_{jk,+-}^{R_N} - I_{jk,-+}^{R_N} + I_{jk,--}^{R_N}.
\]
where
\[
I_{jk,\pm}^{R_N} = \int_0^{\pi r} dx e^{2\pi i ((j-1)\pm(k-1))\xi(x)} \vartheta_1 \left( N^R_N \{(j-1)\tau(t_+ - t) \pm \xi(x)\}; (N^R_N)^2 \tau(t_+ - t) \right) \\
\times \vartheta_1 \left( N^R_N \{(k-1)\tau(t) \pm \xi(x)\}; (N^R_N)^2 \tau(t) \right).
\]
By changing the sign of integral variables appropriately, \(x \to -x\), we obtain
\[
I_{jk,+}^{R_N} = I_{jk,-}^{R_N} + I_{jk,-}^{R_N}
\]
\[
= \int_{-\pi r}^{\pi r} dx e^{2\pi i ((j+k-2))\xi(x)} \vartheta_1 \left( N^R_N \{(j-1)\tau(t_+ - t) + \xi(x)\}; (N^R_N)^2 \tau(t_+ - t) \right) \\
\times \vartheta_1 \left( N^R_N \{(k-1)\tau(t) + \xi(x)\}; (N^R_N)^2 \tau(t) \right),
\]
\[
I_{jk,-}^{R_N} = -\sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} (-1)^{n+m} e^{(n-1/2)^2(N^R_N)^2 \tau(t_+ - t) + (2m-1)N^R_N (k-1)\tau(t_+ - t) + (2n-1)N^R_N (j-1)\tau(t_+ - t) + n\pi i} \\
\times e^{(m-1/2)^2(N^R_N)^2 \tau(t_+ - t) + (2m-1)N^R_N (k-1)\tau(t_+ - t) + (2n-1)N^R_N (j-1)\tau(t_+ - t) + m\pi i} \\
\times e^{2\pi i ((j-k)+N^R_N (n+m))\xi(x)} dx.
\]
Here we note that
\[
\int_{-\pi r}^{\pi r} e^{2\pi i ((j+k-2)+N^R_N (n+m-1))\xi(x)} dx = 2\pi r \int_{-1/2}^{1/2} e^{2\pi i ((j+k-2)+N^R_N (n+m-1))\xi} d\xi
\]
\[
= 2\pi r 1 \left( (j+k-2) + N^R_N (n + m - 1) = 0 \right),
\]
\[
\int_{-\pi r}^{\pi r} e^{2\pi i ((j-k)+N^R_N (n-m))\xi(x)} dx = 2\pi r \int_{-1/2}^{1/2} e^{2\pi i ((j-k)+N^R_N (n-m))\xi} d\xi
\]
\[
= 2\pi r 1 \left( (j-k) + N^R_N (n - m) = 0 \right).
\]
Since \(j, k \in \{1, 2, \ldots, N\}\), we see that
\[
0 \leq j + k - 2 \leq 2N - 2 < N^R_N = \begin{cases} 2N - 1, & R_N = B_N, \\
2N, & R_N = B_N', \end{cases}
\]
and \(0 \leq j - k \leq N - 1 < N^R_N\). The condition \((j+k-2) + N^R_N (n+m-1) = 0\) is satisfied if and only if
\[
j + k - 2 = 0, \quad n + m - 1 = 0 \iff j = k = 1, \quad m = -n + 1,
\]
and the condition \((j-k) + N^R_N (n-m) = 0\) is satisfied if and only if \(j = k, n = m\). Hence
\[
I_{jk,+}^{R_N} = -2\pi r \delta_{j1} \delta_{k1} \sum_{n \in \mathbb{Z}} (-1)^{n(n-1/2)^2(N^R_N)^2 \tau(t_+ - t) + n\pi i} e^{(n-1/2)^2(N^R_N)^2 \tau(t_+ - t) + n\pi i} \\
\times e^{2\pi i (j_1)\vartheta_2 \{0; (N^R_N)^2 \tau(t_+)\}}.
\]
and \( I_{jk}^{R_N} = -2\pi \delta_{jk} \vartheta_2(N^{R_N} (j - 1) \tau(t_*) + (N^{R_N})^2 \tau(t_*)) \). Therefore, we obtain

\[
I_{jk}^R = I_{jk,+}^R - I_{jk,-}^R = 2\pi \delta_{jk} \left\{ \delta_{1j} \vartheta_2 \left( 0; (N^{R_N})^2 \tau(t_*) \right) + \vartheta_2 \left( N^{R_N} (j - 1) \tau(t_*); (N^{R_N})^2 \tau(t_*) \right) \right\}.
\]

This proves (2.15) for \( R_N = B_N \) and \( B_N' \) with (2.17).

(iii) Now we prove (2.15) for \( R_N = C_N \) and \( BC_N \). By (2.13), LHS of (2.15) is given by

\[
I_{jk}^R = - \int_0^{\pi} M_t^{R_N}(x,t_* - t)M_k^{R_N}(x,t)dx = - I_{jk,+}^R + I_{jk,-}^R,
\]

where

\[
I_{jk,+}^R = \int_{-\pi}^{\pi} dx e^{2\pi i (j+k) \xi(x)} \vartheta_2 \left( N^{R_N} \{ j \tau(t_* - t) + \xi(x) \}; (N^{R_N})^2 \tau(t_* - t) \right) \times \vartheta_2 \left( N^{R_N} \{ k \tau(t) + \xi(x) \}; (N^{R_N})^2 \tau(t) \right),
\]

\[
I_{jk,-}^R = \int_{-\pi}^{\pi} dx e^{2\pi i (j-k) \xi(x)} \vartheta_2 \left( N^{R_N} \{ j \tau(t_* - t) + \xi(x) \}; (N^{R_N})^2 \tau(t_* - t) \right) \times \vartheta_2 \left( N^{R_N} \{ k \tau(t) - \xi(x) \}; (N^{R_N})^2 \tau(t) \right).
\]

By the definition of \( \vartheta_2 \) given by (A.1), we have

\[
I_{jk,+}^R = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} e^{(n-1/2)^2(N^{R_N})^2 \tau(t_* - t) \pi i + (2n-1)N^{R_N} j \tau(t_* - t) \pi i} \times e^{(m-1/2)^2(N^{R_N})^2 \tau(t) \pi i + (2m-1)N^{R_N} k \tau(t) \pi i} \int_{-\pi}^{\pi} e^{2\pi i (j+k + N^{R_N} (n+m-1)) \xi(x)} dx,
\]

\[
I_{jk,-}^R = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} e^{(n-1/2)^2(N^{R_N})^2 \tau(t_* - t) \pi i + (2n-1)N^{R_N} j \tau(t_* - t) \pi i} \times e^{(m-1/2)^2(N^{R_N})^2 \tau(t) \pi i + (2m-1)N^{R_N} k \tau(t) \pi i} \int_{-\pi}^{\pi} e^{2\pi i (j-k + N^{R_N} (n-m)) \xi(x)} dx.
\]

Here we note that

\[
\int_{-\pi}^{\pi} e^{2\pi i (j+k + N^{R_N} (n+m-1)) \xi(x)} dx = 2\pi \tau \left( j + k + N^{R_N} (n + m - 1) = 0 \right),
\]

\[
\int_{-\pi}^{\pi} e^{2\pi i (j-k + N^{R_N} (n-m)) \xi(x)} dx = 2\pi \tau \left( j - k + N^{R_N} (n - m) = 0 \right).
\]

Since \( j, k \in \{1, 2, \ldots, N\} \), we see that

\[
2 \leq j + k \leq 2N < N^{R_N} = \begin{cases} 2(N + 1), & R_N = C_N, \\ 2N + 1, & R_N = BC_N, \end{cases}
\]

and \( 0 \leq j - k \leq N - 1 < N^{R_N} \). The condition \( j + k + N^{R_N} (n + m - 1) = 0 \) is satisfied, and thus \( I_{jk,+}^R \equiv 0 \). The condition \( j - k + N^{R_N} (n - m) = 0 \) is satisfied if and only if \( j = k, n = m \). Hence

\[
I_{jk}^R = I_{jk,-}^R = 2\pi \delta_{jk} \sum_{n \in \mathbb{Z}} e^{(n-1/2)^2(N^{R_N})^2 \tau(t) \pi i + (2n-1)N^{R_N} j \tau(t) \pi i} = 2\pi \delta_{jk} \vartheta_2 \left( N^{R_N} j \tau(t_*); (N^{R_N})^2 \tau(t_*) \right).
\]
This proves (2.15) for $R_N = C_N$ and $BC_N$ with (2.16).

(iv) We prove (2.15) for $R_N = C_N^\nu$. By (2.13), LHS of (2.15) is given by

$$I_{jk}^{C_N^\nu} = - \int_{0}^{\pi} M_{j}^{C_N^\nu}(x, t_\ast - t)M_{k}^{C_N^\nu}(x, t)dx = -I_{jk,+}^{C_N^\nu} + I_{jk,-}^{C_N^\nu},$$

where

$$I_{jk,+}^{C_N^\nu} = \int_{-\pi}^{\pi} dx e^{2\pi i(j+k-1)\xi(x)} \vartheta_2(N^{C_N^\nu}(j-1/2)\tau(t_\ast - t) + \xi(x)); (N^{C_N^\nu})^2\tau(t_\ast - t)$$

$$\times \vartheta_2(N^{C_N^\nu}(k-1/2)\tau(t) + \xi(x)); (N^{C_N^\nu})^2\tau(t),$$

$$I_{jk,-}^{C_N^\nu} = \int_{-\pi}^{\pi} dx e^{2\pi i(j-k)\xi(x)} \vartheta_2(N^{C_N^\nu}(j-1/2)\tau(t_\ast - t) + \xi(x)); (N^{C_N^\nu})^2\tau(t_\ast - t)$$

$$\times \vartheta_2(N^{C_N^\nu}(k-1/2)\tau(t) - \xi(x)); (N^{C_N^\nu})^2\tau(t)).$$

We follow the similar argument to the case (iii). Here the key inequalities are $1 \leq j + k - 1 \leq 2N - 1 < C_N^\nu = 2N$ and $0 \leq j - k \leq N - 1 < C_N^\nu$, $j, k \in \{1, 2, \ldots, N\}$. Then we can conclude $I_{jk,+}^{C_N^\nu} \equiv 0$ and

$$I_{jk}^{C_N^\nu} = I_{jk,-}^{C_N^\nu} = 2\pi r \delta_{jk} \vartheta_2(N^{C_N^\nu}(j-1/2)\tau(t_\ast)); (N^{C_N^\nu})^2\tau(t_\ast)).$$

This proves (2.15) for $R_N = C_N^\nu$ with (2.16).

(v) Finally we prove (2.15) for $R_N = D_N$. By (2.13), LHS of (2.15) is given by

$$I_{jk}^{D_N} = \int_{0}^{\pi} M_{j}^{D_N}(x, t_\ast - t)M_{k}^{D_N}(x, t)dx = I_{jk,+}^{D_N} + I_{jk,-}^{D_N},$$

where

$$I_{jk,+}^{D_N} = \int_{-\pi}^{\pi} dx e^{2\pi i(j+k-2)\xi(x)} \vartheta_2(N^{D_N}(j-1)\tau(t_\ast - t) + \xi(x)); (N^{D_N})^2\tau(t_\ast - t)$$

$$\times \vartheta_2(N^{D_N}(k-1)\tau(t) + \xi(x)); (N^{D_N})^2\tau(t),$$

$$I_{jk,-}^{D_N} = \int_{-\pi}^{\pi} dx e^{2\pi i(j-k)\xi(x)} \vartheta_2(N^{D_N}(j-1)\tau(t_\ast - t) + \xi(x)); (N^{D_N})^2\tau(t_\ast - t)$$

$$\times \vartheta_2(N^{D_N}(k-1)\tau(t) - \xi(x)); (N^{D_N})^2\tau(t)).$$

By the definition of $\vartheta_2$ given by (A.1), we have

$$I_{jk,+}^{D_N} = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} e^{i(n-1/2)!N^{D_N}(j-1)\tau(t_\ast - t)\tau(t)} \times e^{i(n-1)!N^{D_N}(j-1)\tau(t_\ast - t)\tau(t)}$$

$$\times e^{i(m-1)!N^{D_N}(2m-1)\tau(t)\tau(t)} \int_{-\pi}^{\pi} e^{2\pi i(j+k-2)N^{D_N}(n+m-1)\xi(x)} dx,$$

$$I_{jk,-}^{D_N} = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} e^{i(n-1/2)!N^{D_N}(j-1)\tau(t_\ast - t)\tau(t)} \times e^{i(n-1)!N^{D_N}(j-1)\tau(t_\ast - t)\tau(t)}$$

$$\times e^{i(m-1)!N^{D_N}(2m-1)\tau(t)\tau(t)} \int_{-\pi}^{\pi} e^{2\pi i(j-k)N^{D_N}(n-m)\xi(x)} dx.$$

Here we note that

$$\int_{-\pi}^{\pi} e^{2\pi i(j+k-2)N^{D_N}(n+m-1)\xi(x)} dx = 2\pi r 1((j + k - 2) + N^{D_N}(n + m - 1) = 0),$$

$$\int_{-\pi}^{\pi} e^{2\pi i(j-k)N^{D_N}(n-m)\xi(x)} dx = 2\pi r 1((j - k) + N^{D_N}(n - m) = 0).$$
Since \( j, k \in \{1, 2, \ldots, N\} \), we see that
\[
0 \leq j + k - 2 \leq 2(N - 1) = N^{D_N},
\]
and \( 0 \leq j - k \leq N - 1 < N^{D_N} \). The condition \((j + k - 2) + N^{D_N}(n + m - 1) = 0\) is satisfied, if
\[
j + k - 2 = 0, \quad n + m - 1 = 0 \iff j = k = 1, \quad m = -n + 1,
\]
or if
\[
j + k - 2 = 2(N - 1), \quad n + m - 1 = -1 \iff j = k = N, \quad m = -n.
\]
And the condition \((j - k) + N^{D_N}(n - m) = 0\) is satisfied if and only if \(j = k, n = m\). Hence we see that
\[
I_{j}^{D_N} = \delta_{j1}\delta_{k1}J_1 + \delta_{jN}\delta_{kN}J_2,
\]
where \(J_1 = 2\pi r\theta(0; (N^{D_N})^2 \tau(t_*))\), and
\[
J_2 = 2\pi r \sum_{n \in \mathbb{Z}} e^{(n - 1/2)^2(N^{D_N})^2 \tau(t_* - t) + (2n - 1)N^{D_N} \tau(t_* - t)}
\]
\[
\times e^{(-n - 1/2)^2(N^{D_N})^2 \tau(t) + (2n - 1)N^{D_N} \tau(t)}.
\]
By the fact \(N^{D_N} = 2(N - 1)\), it is easy to verify that
\[
(n - 1/2)^2(N^{D_N})^2 \tau(t_* - t) + (2n - 1)N^{D_N} \tau(t) + (-2n - 1)N^{D_N} \tau(t)
\]
\[
= (n - 1/2)^2(N^{D_N})^2 \tau(t_*) + (2n - 1)N^{D_N} \tau(t_*),
\]
and thus \(J_2 = 2\pi r\theta(0; (N^{D_N})^2 \tau(t_*)); (N^{D_N})^2 \tau(t_*)\). We also obtain
\[
I_{jk}^{D_N} = 2\pi r \delta_{jk}\theta(0; (N^{D_N})^2 \tau(t_*)); (N^{D_N})^2 \tau(t_*))
\]
Therefore, we can conclude
\[
I_{jk}^{D_N} = I_{jk,+}^{D_N} + I_{jk,-}^{D_N}
\]
\[
= 2\pi r \delta_{jk} \left\{ \delta_{j1}\theta(0; (N^{D_N})^2 \tau(t_*)) + \theta(0, (N^{D_N})^2 \tau(t_*)) \right\}
\]
\[
+ \delta_{jN}\theta(0; (N^{D_N})^2 \tau(t_*)); (N^{D_N})^2 \tau(t_*)) \right\}.
\]
This proves (2.15) for \(R_N = D_N\) with (2.18). The proof is complete.

**Remark 1** When \(t = t_*/2\), the functions \(\{M_{2}^{R_N}(x, t_*/2)\}_{j=1}^{N}\) form orthogonal sets with respect to the inner product \((f|g) = \int_{0}^{L} f(x)g(x)dx\) with \(L = 2\pi r\) for \(R_N = A_{N-1}\) and \(L = \pi r\) for \(R_N = B_N, B_N^c, C_N, C_N^c, BC_N, D_N\). For the case \(R_N = A_{N-1}\), this fact was announced on page 217 in [6].

### 3 Determinantal Point Processes

#### 3.1 Main results

As functions of \(\tau\), we define
\[
q(\tau) = e^{\pi i \tau}, \quad q_0(\tau) = \prod_{n=1}^{\infty} (1 - q(\tau)^{2n}).
\]

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In our setting (2.7) with (2.4), the Macdonald denominator formula (2.2) and (2.3) of Rosengren and Schlosser (Proposition 6.1 in [24]) are written as follows. (For type $A_{N-1}$, we set the norm as $\alpha = e^{2\pi i \sigma_N}$ with (2.6), which is different from the choice in Lemma 2.4 in [15]).

$$\det_{1 \leq j,k \leq N} \left[ M_j^{A_{N-1}}(x_k, t) \right] = \begin{cases} i^{-N(N+1)/2} a^{A_{N-1}}(t) \partial_\theta \left( \sum_{j=1}^{N} \xi(x_j); N^{A_{N-1}-1}(t) \right) W^{A_{N-1}}(\xi(x); N^{A_{N-1}-1}(t)), & \text{if } N \text{ is even,} \\ i^{-(N-1)(N-2)/2} a^{A_{N-1}}(t) \partial_\theta \left( \sum_{j=1}^{N} \xi(x_j); N^{A_{N-1}-1}(t) \right) W^{A_{N-1}}(\xi(x); N^{A_{N-1}-1}(t)), & \text{if } N \text{ is odd,} \end{cases}$$

where

$$a^{A_{N-1}}(t) = q(N^{A_{N-1}-1}(t))^{-N/2} q_0(N^{A_{N-1}-1}(t))^{-1/(N-1)},$$

$$a^{B_N}(t) = 2q(N^{B_N}(t))^{-N/2} q_0(N^{B_N}(t))^{-1/(N-1)},$$

$$a^{B_N^\vee}(t) = 2q(N^{B_N^\vee}(t))^{-N/2} q_0(N^{B_N^\vee}(t))^{-1/(N-1)},$$

$$a^{C_N}(t) = q(N^{C_N}(t))^{-N/2} q_0(N^{C_N}(t))^{-1/(N-1)},$$

$$a^{C_N^\vee}(t) = q(N^{C_N^\vee}(t))^{-N/2} q_0(N^{C_N^\vee}(t))^{-1/(N-1)},$$

$$a^{D_N}(t) = 4q(N^{D_N}(t))^{-N/2} q_0(N^{D_N}(t))^{-1/(N-1)},$$

and $\xi(x) = \xi(x_1), \xi(x_2), \ldots, \xi(x_N))$. Note that

$$q(N^{R_N}(t)) = e^{-N^{R_N}(t)/2t^2} > 0,$$

$$q_0(N^{R_N}(t)) = \prod_{n=1}^{\infty} (1 - e^{-nN^{R_N}(t)/2t^2}) \geq 0, \quad \text{if } 0 \leq t < \infty.$$

Consider the following Weyl alcoves,

$$W_N^{[0,2\pi]} \equiv \{ x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : 0 \leq x_1 < x_2 < \cdots < x_N < 2\pi \},$$

$$W_N^{[0,\pi]} \equiv \{ x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : 0 \leq x_1 < x_2 < \cdots < x_N < \pi \}.$$

By (A.4), $\partial_\theta \left( \sum_{j=1}^{N} \xi(x_j); N^{A_{N-1}-1}(t) \right) \geq 0$ for $s = 0, 3$, if $t \geq 0$, and the definitions of Macdonald denominators (2.1) imply that

$$W^{A_{N-1}}(\xi(x); N^{A_{N-1}-1}(t)) \geq 0, \quad \text{if } x \in W_N^{[0,2\pi]}, \ t \geq 0,$$

$$W^{R_N}(\xi(x); N^{R_N}(t)) \geq 0, \quad \text{if } x \in W_N^{[0,\pi]}, \ t \geq 0, \text{ for } R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N.$$

Now we introduce

$$q^R_{t}(x) = \det_{1 \leq j,k \leq N} \left[ M_j^{R_N}(x_k, t) - t \right] \det_{1 \leq m,n \leq N} \left[ M_j^{R_N}(x_m, t) \right], \quad t \in (0, t_*).$$

By the basic properties of the Jacobi theta functions (A.2)-(A.4), the product form of (3.3) guarantees the follows.
Lemma 3.1 If \( t \in (0,t_*) \), \( q_{1t}^{R_N}(x) \geq 0 \), \( x \in \mathbb{R}^N \), for \( R_N = A_{N-1}, B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N \).

Moreover, we can verify the following.

Lemma 3.2 For \( t \in (0,t_*) \),
\[
\int_{W_N^{[0,2\pi r]}} q_{1t}^{A_{N-1}}(x) dx = \prod_{n=1}^{N} m_n^{A_{N-1}}(t_*), \tag{3.4}
\]
\[
\int_{W_N^{[0,\pi r]}} q_{1t}^{R_N}(x) dx = \prod_{n=1}^{N} m_n^{R_N}(t_*), \quad R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N. \tag{3.5}
\]

Proof Let \( S^{A_{N-1}} = W_N^{[0,2\pi r]} \), \( L^{A_{N-1}} = 2\pi r \), and \( S^{R_N} = W_N^{[0,\pi r]} \), \( L^{R_N} = \pi r \) for \( R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N \). By the Heine identity (C.4) in Appendix C,
\[
\int_{S^{R_N}} q_{1t}^{R_N}(x) dx = \det_{1 \leq j, k \leq N} \left[ \int_{0}^{L^{R_N}} M_{jk}^{R_N}(x, t_* - t) M_{k}^{R_N}(x, t) dx \right].
\]

By the biorthogonality given by Lemma 2.1, this is equal to \( \det_{1 \leq j, k \leq N} [m_j^{R_N}(t_*) \delta_{jk}] \), and hence (3.4) and (3.5) are proved. \( \blacksquare \)

Remark 2 Combining this lemma with the Macdonald denominator formulas (3.1) with (3.2), we readily obtain the Selberg-type integral formulas (see, for instance, Chapter 14 of [6]) for products of Macdonald denominators. See Appendix B. They seem to be much simpler than the formulas known as elliptic Selberg integrals (see Section 4.4 of [7], Exercise 4.14 in [6], and references therein).

Then the seven types of one-parameter \( (t \in (0,t_*)) \) families of probability measures \( \mathbf{P}_{t}^{R_N} \) are defined as
\[
\mathbf{P}_{t}^{R_N}(X \in dx) = \frac{\prod_{n=1}^{N} m_n(t_*)}{q_{1t}^{R_N}(x)} dx, \quad \text{for } R_N = A_{N-1} - 1,
\]
\[
\mathbf{P}_{t}^{R_N}(X \in dx) = \frac{\prod_{n=1}^{N} m_n(t_*)}{q_{1t}^{R_N}(x)} dx, \quad \text{for } R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N, \tag{3.6}
\]
which are normalized as
\[
\int_{W_N^{[0,2\pi r]}} \mathbf{P}_{t}^{A_{N-1}}(x) dx = 1,
\]
\[
\int_{W_N^{[0,\pi r]}} \mathbf{P}_{t}^{R_N}(x) dx = 1, \quad R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N. \tag{3.7}
\]

Under these probability measures \( \mathbf{P}_{t}^{R_N} \) with one parameter \( t \in (0,t_*) \), we consider seven types of point processes,
\[
\Xi^{A_{N-1}}(\cdot) = \sum_{j=1}^{N} \delta_{X_j^{A_{N-1}}}(\cdot) \quad \text{on } S = [0,2\pi r),
\]
and
\[
\Xi^{R_N}(\cdot) = \sum_{j=1}^{N} \delta_{X_j^{R_N}}(\cdot) \quad \text{on } S = [0,\pi r], \quad \text{for } R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N.
\]

Given the determinantal expressions (3.3) and (3.6) for the probability measures associated with the biorthogonal functions (2.8)-(2.11), we can readily prove the following fact by the standard method in random matrix theory [21, 6, 1, 13]. We give a sketch of proof for a general statement in Appendix C for convenience of the reader.
Theorem 3.3 The seven types of one-parameter families of point processes, \((\Xi_{RN}, P_{RN}^t, t \in (0, t_*))\), \(R_N = A_{N-1}, B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N\), are determinantal with the correlation kernels,

\[
K_{RN}^t(x, y; t, r) = \sum_{n=1}^{N} \frac{1}{m_n^{RN}(t)} M_n^{RN}(x, t)M_n^{RN}(y, t - t), \quad t \in (0, t_*),
\]

\(x, y \in [0, 2\pi r]\), for \(R_N = A_{N-1}\),

\(x, y \in [0, \pi r]\), for \(R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N\). \hfill (3.8)

3.2 Temporally homogeneous limit at \(t = t_*/2\)

We consider the determinantal point processes at \(t = t_*/2\). The correlation kernels (3.8) become

\[
K_{t_*/2}^{RN}(x, y; t_*, r) = \sum_{n=1}^{N} \frac{1}{m_n^{RN}(t_*)} M_n^{RN}(x, t_*/2)M_n^{RN}(y, t_*/2), \quad x, y \in [0, 2\pi r]
\]

\(x, y \in [0, 2\pi r]\) for \(R_N = A_{N-1}\), and \(x, y \in [0, \pi r]\) for \(R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N\).

By the asymptotics of the Jacobi theta function (A.5), the temporally homogeneous limit \(t_* \to \infty\) of (3.9) are obtained as follows.

(i) For \(R_N = A_{N-1}\),

\[
K_{A_{N-1}}^{A_{N-1}}(x, y; r) = \lim_{t_* \to \infty} K_{t_*/2}^{A_{N-1}}(x, y; t_*, r)
= \frac{1}{2\pi r} \sum_{n=1}^{N} e^{2\pi i (n-1)(\xi(x) - \xi(y))} = \frac{1}{2\pi r} \sin\{N(x - y)/2r\} \quad x, y \in [0, 2\pi r]. \hfill (3.10)
\]

(ii) For \(R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N\),

\[
K_{RN}^{RN}(x, y; r) = \lim_{t_* \to \infty} K_{t_*/2}^{RN}(x, y; t_*, r)
= \frac{2}{\pi r} \sum_{n=1}^{N} \sin\{N^{RN} - 2J^{RN}(n)\xi(x)\} \sin\{N^{RN} - 2J^{RN}(n)\xi(y)\}
= \begin{cases} 
\frac{1}{2\pi r} \left[ \sin\{(N^{RN} + 1)(x - y)/2r\/ \sin\{(x - y)/2r\} - \sin\{(N^{RN} + 1)(x + y)/2r\}/ \sin\{(x + y)/2r\} \right], & \text{if } R_N = B_N, B_N^\vee, \\
\frac{1}{2\pi r} \left[ \sin\{(N^{RN} - 1)(x - y)/2r\}/ \sin\{(x - y)/2r\} - \sin\{(N^{RN} - 1)(x + y)/2r\}/ \sin\{(x + y)/2r\} \right], & \text{if } R_N = C_N, BC_N, \\
\frac{1}{2\pi r} \left[ \sin\{N^{RN}(x - y)/2r\}/ \sin\{(x - y)/2r\} - \sin\{N^{RN}(x + y)/2r\}/ \sin\{(x + y)/2r\} \right], & \text{if } R_N = C_N^\vee, 
\end{cases}
\]

\(x, y \in [0, \pi r]\).

(iii) For \(R_N = D_N\),

\[
K_{D_N}^{D_N}(x, y; r) = \lim_{t_* \to \infty} K_{t_*/2}^{D_N}(x, y; t_*, r)
= \frac{2}{\pi r} \sum_{n=1}^{N} \cos\{2\pi (N - n)\xi(x)\} \cos\{2\pi (N - n)\xi(y)\}
= \frac{1}{2\pi r} \left[ \sin\{(2N - 1)(x - y)/2r\}/ \sin\{(x - y)/2r\} + \sin\{(2N - 1)(x + y)/2r\}/ \sin\{(x + y)/2r\} \right], \quad x, y \in [0, \pi r]. \hfill (3.11)
\]
Lemma 3.5

For \( K^{B_N}(x, y; r) = K^{BC_N}(x, y; r) = K^{C_N}(x, y; r) \)

\[
= \frac{1}{2\pi r} \left[ \frac{\sin \{N(x - y)/r\}}{\sin \{(x - y)/2 \}} - \frac{\sin \{N(x + y)/r\}}{\sin \{(x + y)/2 \}} \right], \quad x, y \in [0, \pi r],
\]

(3.12)

\[ K^{C_N}(x, y; r) = K^{B_N}(x, y; r) \]

\[
= \frac{1}{2\pi r} \left[ \frac{\sin \{(2N + 1)(x - y)/2 \}}{\sin \{(x - y)/2 \}} - \frac{\sin \{(2N + 1)(x + y)/2 \}}{\sin \{(x + y)/2 \}} \right], \quad x, y \in [0, \pi r].
\]

(3.13)

Corollary 3.4

Put \( t = t_*/2 \) in Theorem 3.3. In the limit \( t_* \to \infty \), the seven types of determinantal point processes \( (\mathbb{Z}^{R_N}, \mathbf{D}^{R_N}_{t_*}) \) are degenerated into the four types of determinantal point processes specified by the correlation kernels \( K^{N-1}(x, y; r) \), \( K^{B_N}(x, y; r) \), \( K^{C_N}(x, y; r) \), and \( K^{D_N}(x, y; r) \) as shown by (3.10), (3.12), (3.13), and (3.11), respectively.

3.3 Infinite determinantal point processes

We fix the density of points as

\[
\rho = \begin{cases} 
\frac{N}{2\pi r}, & R_N = A_{N-1}, \\
\frac{N}{2\pi r}, & R_N = B_N, B'_N, C_N, C'_N, BC_N, D_N,
\end{cases}
\]

(3.14)

and take double limit \( N \to \infty \), \( r \to \infty \). Then we obtain the following limits of correlation kernels.

Lemma 3.5

For \( t \in (0, t_*) \), the following scaling limits are obtained for correlation kernels.

(i) For \( R_N = A_{N-1} \),

\[
K^A_t(x, y; t_*, \rho) \equiv \lim_{N \to \infty, r \to \infty, \rho \to 0} K^{A_{N-1}}_t(x, y; t_*, r) = \int_0^\rho d\lambda e^{2\pi i(x-y)\lambda} \frac{\vartheta_2(\lambda x + 2\pi it\rho \lambda; 2\pi it \rho^2) \vartheta_2(\lambda y - 2\pi i(t_* - t)\rho \lambda; 2\pi i(t_* - t)^2 \rho^2)}{\vartheta_2(2\pi it \rho \lambda; 2\pi it^2 \rho^2)},
\]

(3.15)

\( x, y \in \mathbb{R} \).

(ii) For \( R_N = B_N, B'_N \),

\[
K^B_t(x, y; t_*, \rho) \equiv \lim_{N \to \infty, r \to \infty, \rho \to 0} K^{B_N}_t(x, y; t_*, r) = \frac{1}{2} \left[ \int_{-\rho}^\rho d\lambda e^{\pi i(x-y)\lambda} \frac{\vartheta_1(\lambda x + \pi it \rho \lambda; 2\pi it \rho^2) \vartheta_1(\lambda y - \pi i(t_* - t)\rho \lambda; 2\pi i(t_* - t)^2 \rho^2)}{\vartheta_2(2\pi it \rho \lambda; 2\pi it^2 \rho^2)} \right. \\
- \left. \int_{-\rho}^0 d\lambda e^{\pi i(x+y)\lambda} \frac{\vartheta_1(\lambda x + \pi it \rho \lambda; 2\pi it \rho^2) \vartheta_1(-\lambda y - \pi i(t_* - t)\rho \lambda; 2\pi i(t_* - t)^2 \rho^2)}{\vartheta_2(2\pi it \rho \lambda; 2\pi it^2 \rho^2)} \right],
\]

(3.16)
$(x, y) \in [0, \infty)$.
(iii) For $R_N = C_N, C_N^\vee, BC_N$, 
\[
K_i^C(x, y; t_*, \rho) = \lim_{N \to \infty, \rho \to \infty} K_i^{R_N}(x, y; t_*, \rho)
\]
\[
= \frac{1}{2} \left[ \int_{-\rho}^{\rho} d\lambda e^{\pi i(x-y)\lambda} \frac{\vartheta_2(\rho x + \pi it\rho \lambda; 2\pi it\rho^2) \vartheta_2(\rho y - \pi i(t_* - t)\rho \lambda; 2\pi i(t_* - t)^2 \rho^2)}{\vartheta_2(\pi it_* \rho \lambda; 2\pi it_* \rho^2)} - \int_{-\rho}^{\rho} d\lambda e^{\pi i(x+y)\lambda} \frac{\vartheta_2(\rho x + \pi it\rho \lambda; 2\pi it\rho^2) \vartheta_2(-\rho y - \pi i(t_* - t)\rho \lambda; 2\pi i(t_* - t)^2 \rho^2)}{\vartheta_2(\pi it_* \rho \lambda; 2\pi it_* \rho^2)} \right],
\]
(3.17)

$x, y \in [0, \infty)$.
(iv) For $R_N = D_N$,
\[
K_i^D(x, y; t_*, \rho) = \lim_{N \to \infty, \rho \to \infty} K_i^{D_N}(x, y; t_*, \rho)
\]
\[
= \frac{1}{2} \left[ \int_{-\rho}^{\rho} d\lambda e^{\pi i(x-y)\lambda} \frac{\vartheta_2(\rho x + \pi it\rho \lambda; 2\pi it\rho^2) \vartheta_2(\rho y - \pi i(t_* - t)\rho \lambda; 2\pi i(t_* - t)^2 \rho^2)}{\vartheta_2(\pi it_* \rho \lambda; 2\pi it_* \rho^2)} + \int_{-\rho}^{\rho} d\lambda e^{\pi i(x+y)\lambda} \frac{\vartheta_2(\rho x + \pi it\rho \lambda; 2\pi it\rho^2) \vartheta_2(-\rho y - \pi i(t_* - t)\rho \lambda; 2\pi i(t_* - t)^2 \rho^2)}{\vartheta_2(\pi it_* \rho \lambda; 2\pi it_* \rho^2)} \right],
\]
(3.18)

$x, y \in [0, \infty)$.

Proof Here we give proof for (iii). Other cases are similarly proved. The explicit expressions for $K_i^{R_N}(x, y; t_*, \rho)$ for $R_N = C_N, C_N^\vee, BC_N$ are given by
\[
K_i^{R_N}(x, y; t_*, \rho) = \frac{1}{2\pi r} \sum_{n=1}^{N} \frac{1}{\vartheta_2(N^{R_N} J^{R_N}(n) \tau(t_*)); (N^{R_N})^2 \tau(t_*)} \times \left\{ e^{2\pi i J^{R_N}(n) \xi(x)} \vartheta_2(N^{R_N} J^{R_N}(n) \tau(t) + \xi(x)); (N^{R_N})^2 \tau(t) \right\}
- e^{-2\pi i J^{R_N}(n) \xi(x)} \vartheta_2(N^{R_N} J^{R_N}(n) \tau(t) - \xi(x)); (N^{R_N})^2 \tau(t)) \right\}
\times e^{2\pi i J^{R_N}(n) \xi(y)} \vartheta_2(N^{R_N} J^{R_N}(n) \tau(t_* - t) + \xi(y)); (N^{R_N})^2 \tau(t_* - t))
- e^{-2\pi i J^{R_N}(n) \xi(y)} \vartheta_2(N^{R_N} J^{R_N}(n) \tau(t_* - t) - \xi(y)); (N^{R_N})^2 \tau(t_* - t)) \right\},
\]
(3.19)

where $N^{C_N} = 2(N + 1)$, $N^{C_N^\vee} = 2N$, $N^{BC_N} = 2N + 1$, and $J^{C_N}(n) = J^{BC_N} = n$, $J^{C_N^\vee}(n) = n - 1/2$. By (2.4) and (3.14), we see that
\[
\frac{1}{2\pi r} = \frac{\rho}{2N}, \quad N^{R_N} J^{R_N}(n) \tau(t) = \frac{N^{R_N} 2\pi it\rho^2 J^{R_N}(n)}{2N}, \quad (N^{R_N})^2 \tau(t) = \left( \frac{N^{R_N}}{2N} \right)^2 2\pi it\rho^2,
\]
\[
2\pi i J^{R_N}(n) \xi(x) = \pi \rho x J^{R_N}(n) \rho x, \quad N^{R_N} \xi(x) = \frac{N^{R_N}}{2N} \rho x.
\]

Since $N^{R_N}/2N \to 1$ as $N \to \infty$, (3.19) given by summation converges uniformly on any compact subset of $[0, \pi r]^2 \ni (x, y)$ to the following integral with an integral variable $u \sim J^{R_N}(n)/N$,
\[
- \frac{1}{2} \int_{0}^{1} du \frac{1}{\vartheta_2(\pi it_* \rho^2 u; 2\pi it_* \rho^2)} \left\{ e^{\pi i \rho u} \vartheta_2(\pi it_* \rho^2 u - \pi ; 2\pi it_* \rho^2) - e^{-\pi i \rho u} \vartheta_2(\pi it_* \rho^2 u - \rho x; 2\pi it_* \rho^2) \right\}
\times e^{\pi i \rho u} \vartheta_2(\pi i(t_* - t)^2 \rho^2 u + \pi y ; 2\pi i(t_* - t) \rho^2) - e^{-\pi i \rho u} \vartheta_2(\pi i(t_* - t)^2 \rho^2 u - \pi y ; 2\pi i(t_* - t) \rho^2) \right\}.
\]
(3.20)
We change the integral variable $u \rightarrow \lambda$ by $\lambda = \rho u$. If we use the symmetry of Jacobi’s theta functions, (A.2), we can verify that (3.20) is rewritten as (3.17).

The uniform convergence of correlation kernels implies the convergence of all correlation kernels. Then we conclude the following.

**Theorem 3.6** In the scaling limit $N \rightarrow \infty$, $r \rightarrow \infty$ with constant density of points (3.14), the seven types of one-parameter families of determinantal point processes, $(\Xi^{RN}, \mathbf{P}_{t}^{RN}, t \in (0, t_{*}))$, $R_{N} = A_{N-1}$, $B_{N}$, $C_{N}$, $C_{N}^{\nu}$, $BC_{N}$, $D_{N}$, converge in the sense of distribution to the four types of infinite dimensional point processes as follows,

$$
(\Xi^{AN-1}, \mathbf{P}_{t}^{AN-1}, t \in (0, t_{*})) \rightarrow (\Xi^{A}, \mathbf{P}^{A}, t \in (0, t_{*})) \quad \text{as } N \rightarrow \infty, r \rightarrow \infty \text{ with } \frac{N}{2\pi r} = \rho,
$$

$$
(\Xi^{BN}, \mathbf{P}_{t}^{BN}, t \in (0, t_{*})) \rightarrow (\Xi^{B}, \mathbf{P}^{B}, t \in (0, t_{*})) \quad \text{as } N \rightarrow \infty, r \rightarrow \infty \text{ with } \frac{N}{\pi r} = \rho,
$$

$$
(\Xi^{CN}, \mathbf{P}_{t}^{CN}, t \in (0, t_{*})) \rightarrow (\Xi^{C}, \mathbf{P}^{C}, t \in (0, t_{*})) \quad \text{as } N \rightarrow \infty, r \rightarrow \infty \text{ with } \frac{N}{\pi r} = \rho,
$$

$$
(\Xi^{BCN}, \mathbf{P}_{t}^{BCN}, t \in (0, t_{*})) \rightarrow (\Xi^{D}, \mathbf{P}^{D}, t \in (0, t_{*})) \quad \text{as } N \rightarrow \infty, r \rightarrow \infty \text{ with } \frac{N}{\pi r} = \rho,
$$

where $(\Xi^{A}, \mathbf{P}^{A}, t \in (0, t_{*})), (\Xi^{B}, \mathbf{P}^{B}, t \in (0, t_{*})), (\Xi^{C}, \mathbf{P}^{C}, t \in (0, t_{*})), \text{ and } (\Xi^{D}, \mathbf{P}^{D}, t \in (0, t_{*}))$ are infinite determinantal point processes associated with the correlation kernels $K_{t}^{A}, K_{t}^{B}, K_{t}^{C}$, and $K_{t}^{D}$, $t \in (0, t_{*})$, which are given by (3.15), (3.16), (3.17), and (3.18), respectively.

Put $t = t_{*}/2$ in (3.15)-(3.18). By (A.5), we see that

$$
\lim_{t_{*} \to \infty} \frac{\partial \lambda(\rho x + \pi it_{*}\rho \lambda/2; \pi it_{*}\rho \lambda/2; \pi it_{*}\rho \lambda)}{\partial \lambda(\pi i t_{*}\rho \lambda/2; \pi it_{*}\rho \lambda)} = \begin{cases} 
\frac{e^{-\pi \rho(x-y)} - \rho}{\pi (x-y)}, & \text{if } \lambda > 0, \\
\frac{e^{\pi \rho(x-y)}}{\pi (x-y)}, & \text{if } \lambda < 0,
\end{cases}
$$

for $s = 1, 2$. Then we obtain the following three types of limits,

$$
K^{A}(x, y; \rho) = \lim_{t_{*} \to \infty} K_{t_{*}/2}^{A}(x, y; t_{*}; \rho) = e^{-\pi \rho(x-y)} \int_{0}^{\rho} e^{2\pi i(x-y)\lambda} d\lambda
$$

$$
= \frac{\sin \{\pi \rho(x-y)\}}{\pi (x-y)}, \quad x, y \in \mathbb{R}, \quad \text{(3.21)}
$$

$$
K^{C}(x, y; \rho) = \lim_{t_{*} \to \infty} K_{t_{*}/2}^{C}(x, y; t_{*}; \rho) = e^{\pi \rho(x-y)} \int_{0}^{\rho} e^{2\pi i(x-y)\lambda} d\lambda
$$

$$
= \frac{\sin \{\pi \rho(x-y)\} - \sin \{\pi \rho(x+y)\}}{\pi (x-y)} - \frac{\sin \{\pi \rho(x+y)\}}{\pi (x+y)}, \quad \text{for } R = B, C, \quad x, y \in [0, \infty), \quad \text{(3.22)}
$$

$$
K^{D}(x, y; \rho) = \lim_{t_{*} \to \infty} K_{t_{*}/2}^{D}(x, y; t_{*}; \rho) = e^{\pi \rho(x-y)} \int_{0}^{\rho} e^{2\pi i(x-y)\lambda} d\lambda
$$

$$
= \frac{\sin \{\pi \rho(x-y)\} + \sin \{\pi \rho(x+y)\}}{\pi (x+y)}, \quad x, y \in [0, \infty). \quad \text{(3.23)}
$$

**Remark 4** The kernel (3.21) is known as the sine kernel with density $\rho$, which governs the bulk scaling limit of the determinantal point process in GUE, as explained in Section 1. The statistical ensemble of nonnegative square roots of eigenvalues of $M^{2}M$, in which $\{M\}$ are $(N + \nu) \times N$ rectangular complex matrices and the real and imaginary parts of their entries are independently and normally distributed, is called the chiral GUE with parameter $\nu$. In the scaling limit associated with $N \rightarrow \infty$ called hard-edge scaling limit, the correlation kernel of this determinantal point process is given by

$$
K^{\nu}_{\text{chGUE}}(x, y) = \frac{2\sqrt{\nu}}{x^{2} - y^{2}} \{J_{\nu}(2x)yJ_{\nu}'(2y) - J_{\nu}(2y)xJ_{\nu}'(2x)\}, \quad x, y \in [0, \infty),
$$

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where \( J_\nu(z) \) is the Bessel function and \( J'_\nu(z) = dJ_\nu(z)/dz \) (see [6, 16, 17] and references therein). Since
\[
J_{1/2}(z) = \sqrt{2/(\pi z)} \sin z \quad \text{and} \quad J_{-1/2}(z) = \sqrt{2/(\pi z)} \cos z,
\]
we can see that
\[
K_{\pm 1/2}^{\text{chGUE}}(x, y) = \frac{\sin\{2(x - y)\}}{\pi(x - y)} + \frac{\sin\{2(x + y)\}}{\pi(x + y)}.
\]
The kernels (3.22) and (3.23) are the scale changes of \( K_{\pm 1/2}^{\text{chGUE}} \).

**Remark 5** If we take the scaling limit \( N \to \infty, r \to \infty \) with constant density (3.14) in the four types of correlation kernels in the trigonometric level, (3.10)–(3.13), the three types of sine kernels (3.21)–(3.23) are obtained.

## 4 Realization as Systems of Noncolliding Brownian Bridges

### 4.1 New expressions of Macdonald denominators by KMLGV determinants

Consider the one-dimensional standard BM, \( B(t), t \in [0, \infty) \) governed by the Wiener measure denoted by \( P \). The transition probability density of BM, starting from \( x \) at time \( s \) and arriving at \( y \) at time \( t \), \( x, y \in \mathbb{R}, 0 \leq s < t < \infty \), is denoted as \( p_{\text{BM}}(s, x; t, y) \) and defined by
\[
P(B(t) \in dy | B(s) = x) = p_{\text{BM}}(s, x; t, y)dy,
\]
with
\[
p_{\text{BM}}(s, x; t, y) = p_{\text{BM}}(s, y; t, x) = \frac{1}{\sqrt{2\pi(t - s)}} e^{-(x-y)^2/(2(t-s))}.
\]
By the Markov property of BM, the Chapman-Kolmogorov equation holds,
\[
\int_{\mathbb{R}} p_{\text{BM}}(s, x; t, y)p_{\text{BM}}(t, y; u, z)dy = p_{\text{BM}}(s, x; u, z), \quad 0 \leq s < t < u < \infty, \quad x, z \in \mathbb{R}. \tag{4.1}
\]
For \( 0 \leq s < t < \infty \), define
\[
p_{\text{circ}}(s, x; t, y) = p_{\text{circ}}(s, x; t, y; r)
\]
\[
= \begin{cases} 
\sum_{w \in \mathbb{Z}} (-1)^w p_{\text{BM}}(s, x; t, y + 2\pi rw), & \text{if } N \text{ is even,} \\
\sum_{w \in \mathbb{Z}} p_{\text{BM}}(s, x; t, y + 2\pi rw), & \text{if } N \text{ is odd,}
\end{cases}
\]
\[
= \begin{cases} 
p_{\text{BM}}(s, x; t, y)\vartheta_0(i(x - y)r/(t - s); -1/\tau(t - s)), & \text{if } N \text{ is even,} \\
p_{\text{BM}}(s, x; t, y)\vartheta_3(i(x - y)r/(t - s); -1/\tau(t - s)), & \text{if } N \text{ is odd,}
\end{cases}
\]
\[
= \begin{cases} 
\frac{1}{2\pi r} \vartheta_0(\xi(x - y); \tau(t - s)), & \text{if } N \text{ is even,} \\
\frac{1}{2\pi r} \vartheta_3(\xi(x - y); \tau(t - s)), & \text{if } N \text{ is odd,}
\end{cases} \tag{4.2}
\]
x, y \in [0, 2\pi r], where \( \xi(x) \) and \( \tau(t) \) are defined by (2.4) and in the last equalities Jacobi’s imaginary trans-
The function $p_{\text{abs}}(x; t, y)$ was used, and
\[
p^{\text{abs}}_j(s; x; t, y) = \sum_{k \in \mathbb{Z}} \left\{ p_{\text{BM}}(s; x; t, y + 2\pi kr) - p_{\text{BM}}(s; x; t, y + 2\pi kr) \right\},
\]
for $x, y \in [0, \pi r]$. By (4.1), we can readily confirm that
\[
\int_0^{2\pi r} p^{\text{circ}}(s; x; t; y)dy = p^{\text{circ}}(s; x; t; y), \quad x, z \in [0, 2\pi r),
\]
\[
\int_0^{2\pi r} p^{\text{ref}}(s; x; t; y)dy = p^{\text{ref}}(s; x; u, z), \quad x, z \in [0, \pi r], \quad z = \text{abs, ref}. \quad (4.5)
\]
The function $p_{\text{abs}}$ (resp. $p_{\text{ref}}$) can be interpreted as the transition probability density of BM in an interval $[0, \pi r]$ with the absorbing (resp. reflecting) boundary conditions both at $x = 0$ and $x = \pi r$. These facts are proved by the reflection principle of BM (see, for instance, Appendices 1.5 and 1.6 in [3]).

We introduce time dependent $N \times N$ matrices, $p_j^{\text{ref}}(s; x; t; y)$, $0 \leq t < \infty$, with entries
\[
(p_j^{\text{ref}}(s; x; t; y))_{jk} = p_j^{\text{ref}}(s; x_j; t; y_k), \quad j, k = 1, 2, \ldots, N,
\]
for $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $y = (y_1, \ldots, y_N) \in \mathbb{R}^N$. We see
\[
p_j^{\text{ref}}(0; x; t; y) = p_j^{\text{ref}}(u - t, y; u, x), \quad \sharp = \text{circ, abs, ref}, \quad (4.6)
\]
for any $0 < t < \infty$, $x, y \in \mathbb{R}^N$. By the Heine identity (C.4) in Appendix C, the Chapman-Kolmogorov equations (4.5) can be extended to the following determinantal versions,
\[
\int_{W_N^{[0,2\pi r]}} dy \det[p^{\text{circ}}(s; x; t; y)] \det[p^{\text{circ}}(s; u; z)] = \det[p^{\text{circ}}(s; x; u, z)], \quad x, z \in W_N^{[0,2\pi r]},
\]
\[
\int_{W_N^{[0,\pi r]}} dy \det[p^{\text{circ}}(s; x; t; y)] \det[p^{\text{ref}}(s; u; z)] = \det[p^{\text{ref}}(s; x; u, z)], \quad x, z \in W_N^{[0,\pi r]},
\]
for $\sharp = \text{abs, ref}. \quad (4.7)$

The determinant $\det[p^{\text{circ}}(s; x; t; y)]$ with $x, y \in W_N^{[0,2\pi r]}$, $0 \leq s < t$, is the KMLGV determinant giving the total probability mass of $N$-tuples of noncolliding Brownian paths on a circle with radius $r > 0$, starting from the unlabeled configuration $x$ at time $s$ and arriving at the unlabeled configuration $y$ at time $t > s$ [4, 8, 19]. The determinant $\det[p^{\text{abs}}(s; x; t; y)]$ (resp. $\det[p^{\text{ref}}(s; x; t; y)]$) can be regarded as the KMLGV determinant for the noncolliding BMs in the interval $[0, \pi r]$ with the absorbing (resp. reflecting) boundary conditions at both edges.

We consider the following eight types of configurations of $N$ points, $v^N = v^N(r)$ with the elements,
\[
v^N_j = v^N_j(r) = \frac{2\pi r}{N}(j - 1),
\]
\[
v^N_j = v^N_j(r) = \frac{2\pi r}{N^{R_N}}(j - 1/2), \quad \text{for } R_N = B_N, B_N^c,
\]
\[
v^N_j = v^N_j(r) = \frac{2\pi r}{N^{R_N}}j, \quad \text{for } R_N = C_N, C_N^c, BC_N,
\]
\[
v^N_j = v^N_j(r) = \frac{\pi r}{N^{D_N} - 1}(j - 1), \quad j = 1, 2, \ldots, N. \quad (4.8)
\]
The configurations \( u^{RN} \) make equidistant series of points in \([0, 2\pi r]\) for \( R_N = A_{N-1} \) and in \([0, \pi r]\) for others. We also consider \( N \times N \) matrices whose entries are given by the biorthogonal theta functions studied in Section 2.2,

\[
M^{RN}(x, t) = \left( M^{RN}_{jk}(x_k, t) \right)_{1 \leq j, k \leq N}.
\]

Then the following relations hold between matrices.

**Lemma 4.1** Consider the \( N \times N \) matrices \( r^{RN}(t) \) with the following entries; for \( j = 1, \ldots, N, \)

\[
(r^{A_{N-1}}(t))_{jk} = \frac{2\pi r}{N} e^{-\pi i (j-1/2)^2 r} \cos \left( \frac{\pi (N-j)(k-1)}{N} \right), \quad k = 1, \ldots, N,
\]

\[
(r^{R_{N}}(t))_{jk} = \frac{4\pi r}{iN_{RN}} e^{-\pi i (jR_{N}(t))^2 \tau(t)} \sin \left( N^2 R_{N} - 2j R_{N} \right) \frac{v_{R_{N}}}{2r}, \quad k = 1, \ldots, N, \quad \text{for } R_{N} = B_N', C_N, B'C_N,
\]

\[
(r^{D_{N}}(t))_{jk} = \begin{cases} 
\frac{\pi r}{N-1} e^{-\pi i (j-1)^2 \tau(t)}, & k = 1, \\
\frac{2\pi r}{N-1} e^{-\pi i (j-1)^2 \tau(t) \cos} \left[ \frac{\pi (N-j)(k-1)}{N-1} \right], & k = 2, \ldots, N-1, \\
\frac{\pi r}{N-1} e^{-\pi i (j-1)^2 \tau(t) \cos} \left[ \frac{\pi (N-j)}{N} \right], & k = N.
\end{cases}
\]

Then for \( t \in [0, \infty) \),

\[
r^{A_{N-1}}(t) p^{circ}_{N} (0, u^{A_{N-1}}; t, x) = M^{A_{N-1}}(t, x), \quad x \in \mathbb{W}^{[0, 2\pi r]}_N, \tag{4.10}
\]

\[
r^{R_{N}}(t) p^{abs}_{N} (0, u^{R_{N}}; t, x) = M^{R_{N}}(t, x), \quad x \in \mathbb{W}^{[0, \pi r]}_N, \quad R_{N} = B_N, B_N', C_N, C_N', B'C_N, \tag{4.11}
\]

\[
r^{D_{N}}(t) p^{circ}_{N} (0, u^{D_{N}}; t, x) = M^{D_{N}}(t, x), \quad x \in \mathbb{W}^{[0, \pi r]}_N. \tag{4.12}
\]

**Proof** First we prove (4.10) when \( N \) is even. By the definitions of \( r^{A_{N-1}}(t) \) and \( p^{circ}_{N} (0, u^{A_{N-1}}; t, x) \) with (4.8), the \((j, k)\)-entry of LHS of (4.10) is given by

\[
L^{A_{N-1}}_{jk} = \frac{2\pi r}{N} e^{-\pi i (j-1/2)^2 \tau(t)}
\]

\[
\times \sum_{\ell=1}^{N} e^{2\pi i (j-1/2-N/2)(t-1)/N} \frac{1}{2\pi r} \theta_2 \left( \frac{\ell - 1}{N} - \frac{x_k}{2\pi r}, \tau(t) \right)
\]

\[
= \frac{1}{N} e^{-\pi i (j-1/2)^2 \tau(t)} \sum_{n \in \mathbb{Z}} \frac{e^{(n-1/3)^2 \tau(t) \pi - (2n-1)(x_k/2\pi r) \pi}}{1 \leq j, k \leq N,}
\]

where we have used the definition of \( \theta_2 \) given by (A.1). We note \((j-1/2-N/2) + (n-1/2) \in \mathbb{Z}\) for \( N \) even and use the equality

\[
\sum_{\ell=1}^{N} e^{2\pi i (\ell-1) \theta/N} = N \sum_{m \in \mathbb{Z}} 1(\theta + Nm = 0), \quad \theta \in \mathbb{Z}, \quad N \in \mathbb{N}. \tag{4.13}
\]
Then we obtain
\[ L_{jk}^{A_{N-1}} = e^{-\pi i(j-1/2)^2 \tau(t)} \sum_{m \in \mathbb{Z}} e^{i(mN + (j-1/2-N/2))^2 \tau(t) \pi i + 2(mN + (j-1/2-N/2))(x_k/(2\pi \tau)) \pi i}. \]

It is easy to confirm the equality
\[
-\pi i(j-1/2)^2 \tau(t) + \{mN + (j-1/2-N/2)\}^2 \tau(t) \pi i + 2\{mN + (j-1/2-N/2)\} \frac{x_k}{2\pi \tau} \pi i = (m-1/2)^2 N^2 \tau(t) \pi i + (2m-1)N \left\{(j-1/2) \tau(t) + \frac{x_k}{2\pi \tau}\right\} \pi i + 2\pi i(j-1/2) \frac{x_k}{2\pi \tau} \pi i.
\]

Hence we have the equality \( L_{jk} = M_{jk}^{A_{N-1}}(x_k, t) \). We can similarly prove (4.10) for odd \( N \).

Next we explain how to prove (4.11) for \( R_N = C_N \). The \((j, k)\)-entry of LHS is
\[
L_{jk}^{C_N} = \frac{i e^{-\pi j^2 \tau(t)}}{N+1} \sum_{\ell=1}^{N} \left[ \frac{\pi \{j - (N+1)\} \ell}{N+1} \right] \vartheta_3 \left( \frac{\ell}{2(N+1)} - \frac{x_k}{2\pi \tau} \tau(t) \right) \]
\[
- \sum_{\ell=1}^{N} \left[ \frac{\pi \{j - (N+1)\} \ell}{N+1} \right] \vartheta_3 \left( \frac{\ell}{2(N+1)} + \frac{x_k}{2\pi \tau} \tau(t) \right), \quad 1 \leq j, k \leq N.
\]

By the fact that \( \sin[\pi \{j - (N+1)\} \ell/(N+1)] = 0 \) when \( \ell = 0 \) and \( \ell = N+1 \), and by the parity \( \sin(-\pi \nu) = -\sin(\pi \nu) \), \( \vartheta_3(-\nu; \tau) = \vartheta_3(\nu; \tau) \), this entry is equal to
\[
L_{jk}^{C_N} = \frac{i e^{-\pi j^2 \tau(t)}}{N+1} \sum_{\ell=1}^{N} \left[ \frac{\pi \{j - (N+1)\} \ell}{N+1} \right] \vartheta_3 \left( \frac{\ell}{2(N+1)} - \frac{x_k}{2\pi \tau} \tau(t) \right) \]
\[
- \sum_{\ell=1}^{N} \left[ \frac{\pi \{j - (N+1)\} \ell}{N+1} \right] \vartheta_3 \left( \frac{\ell}{2(N+1)} + \frac{x_k}{2\pi \tau} \tau(t) \right), \quad 1 \leq j, k \leq N.
\]

Note that, in the last expression, the number of terms of summation is equal to \( 2N + 2 = N^{C_N} \). Then we use the definition (A.1) of the Jacobi theta function \( \vartheta_3 \), and rewrite the above as
\[
L_{jk}^{C_N} = \frac{e^{-\pi j^2 \tau(t)}}{2(N+1)} \sum_{n \in \mathbb{Z}} e^{(n^2 \tau(t) - 2n x_k/(2\pi \tau)) \pi i} \times \sum_{\ell=-N}^{N+1} \left\{ e^{2\pi i(j-(N+1)+\ell)/(2(N+1))} - e^{2\pi i(j-(N+1)-\ell)/(2(N+1))} \right\}, \quad 1 \leq j, k \leq N.
\]

By the equality (4.13) with the replacement \( N \to N^{C_N} = 2(N+1) \), we can verify \( L_{jk} = M_{jk}^{C_N}(x_k, t) \). For other types of \( R_N \) than \( A_{N-1} \) and \( C_N \), we can show that the \((j, k)\)-entries of LHS of (4.11) and (4.12) are
Proposition 4.2. Written with the Macdonald denominator formulas of Rosengren and Schlosser [24], which are written as (3.1) and (4.12). The proof is complete.

If we take the determinants of both sides of the equalities (4.10)–(4.12), we obtain the equalities (4.14) also for \( R_N = B_N, B_N', C_N', BC_N \) and (4.12). The proof is complete.

If we take the determinants of both sides of the equalities (4.10)–(4.12), we obtain the equalities

\[
\det[p^{R_N}(t)] \det[p^s(0, v^{R_N}; t, x)] = \det[M^{R_N}(t, x)],
\]

where \( s = \text{circ} \) for \( R_N = A_{N-1}, s = \text{abs} \) for \( R_N = B_N, B_N', C_N, C_N', BC_N \), and \( s = \text{ref} \) for \( R_N = D_N \). Combine them with the Macdonald denominator formulas of Rosengren and Schlosser [24], which are written as (3.1) in the present paper, we obtain new determinantal expressions for the Macdonald denominators.

**Proposition 4.2.** For the irreducible reduced affine root systems, \( R_N = A_{N-1}, B_N, B_N', C_N, C_N', BC_N, D_N \), the Macdonald denominators \( W^{R_N} \) defined by (2.1) are proportional to the KMLGV determinants for noncolliding Brownian paths starting from the configurations \( v^{R_N} \) given by (4.8) as follows. Let \( s(N) = 0 \) if \( N \) is even, and \( s(N) = 3 \) if \( N \) is odd, then

\[
\vartheta_{s(N)} \left( \sum_{j=1}^{N} \xi(x_j); N \tau(t) \right) W^{A_{N-1}}(\xi(x); N \tau(t)) = b^{A_{N-1}}(t) \det[p^{\text{circ}}(0, v^{A_{N-1}}; t, x)],
\]

\[
W^{R_N}(\xi(x); N \tau(t)) = \begin{cases} 
\left( b^{R_N}(t) \det[p^{\text{abs}}(0, v^{R_N}; t, x)] \right), & \text{for } R_N = B_N, B_N', C_N, C_N', BC_N, \\
\left( b^{D_N}(t) \det[p^{\text{ref}}(0, v^{D_N}; t, x)] \right), & \text{for } R_N = D_N,
\end{cases}
\]
with the coefficients
\[ b^{R_N}(t) = \begin{cases} 
  i^{N(N+1)/2} \frac{\det[r^{A_{N-1}}(t)]}{a^{A_{N-1}}(t)} & \text{for } R_N = A_{N-1}, \text{ if } N \text{ is even}, \\
  i^{(N-1)(N-2)/2} \frac{\det[r^{A_{N-1}}(t)]}{a^{A_{N-1}}(t)} & \text{for } R_N = A_{N-1}, \text{ if } N \text{ is odd}, \\
  \frac{\det[r^{R_N}(t)]}{a^{A_{N-1}}(t)} & \text{for } R_N = B_N, B_N^\sim, D_N, \\
  i^N \frac{\det[r^{R_N}(t)]}{a^{A_{N-1}}(t)} & \text{for } R_N = C_N, C_N^\sim, BC_N, 
\end{cases} \]

where the factors \(a^{R_N}(t)\) and the entries of the matrices \(r^{R_N}(t)\) are given by (3.2) and (4.9), respectively.

Remark 6 Forrester proved the equality (4.15) for the type \(A_{N-1}\) independently of the Macdonald denominator formulas given by Rosengren and Schlosser [24]. For \(N\) even (resp. \(N\) odd), (4.15) is a special case with \(\alpha = 1/2 + 1/N\) (resp. \(\alpha = 1/N\)) of Eq. (5.111) (resp. Eq. (5.110)) in Proposition 5.6.3 in [6]. The matrix relation (4.10) was also used to prove (4.14) for \(R_N = A_{N-1}\) in pages 216-217 in [6]. Moreover, explicit evaluation of \(\det[r^{A_{N-1}}(t)]\) was found there. If we use this result, we obtain
\[ b^{A_{N-1}}(t) = \left(\frac{2\pi i}{N}\right)^N \eta(N\tau(t))^{(N-1)(N-2)/2} q_0(N\tau(t))^{(N-1)(N-2)/2} \]
\[ = \left(\frac{2\pi i}{N}\right)^N \eta(N\tau(t))^{(N-1)(N-2)/2}, \]

where \(\eta(\tau)\) is the Dedekind modular function (see, for instance, Sec.23.15 in [23]),
\[ \eta(\tau) = q(\tau)^{1/12} q_0(\tau) = e^{\tau \pi i/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n}). \]

See also [5] and references therein. Lemma 4.1 and Proposition 4.2 are extensions of Forrester’s results to other six types of matrices and their determinants. Here we identify LHS of the equations (4.15) and (4.16) as the Macdonald denominators and the determinants in RHS of them as the KMLGV determinants of noncolliding Brownian paths.

4.2 Noncolliding Brownian bridges

The following is derived by Lemma 4.1.

Proposition 4.3 The probability densities given by (3.6) for the determinantal point processes, \((\Xi^{R_N}, P_i^{R_N}, t \in (0, t_\star))\), have the following expressions,
\[ P_i^{A_{N-1}}(x) = \frac{\det[p^{\text{circ}}(0, x^{A_{N-1}}; t, x)] \det[p^{\text{circ}}(t, x; t_\star, x^{A_{N-1}})]}{\det[p^{\text{circ}}(0, x^{A_{N-1}}; t_\star, x^{A_{N-1}})]}, \quad x \in \mathbb{W}^{(0, 2\pi i)^N}, \]
\[ P_i^{R_N}(x) = \frac{\det[p^{\text{circ}}(0, x^{R_N}; t, x)] \det[p^{\text{circ}}(t, x; t_\star, x^{R_N})]}{\det[p^{\text{circ}}(0, x^{R_N}; t_\star, x^{R_N})]}, \quad x \in \mathbb{W}^{(0, \pi i)^N}, \]
\[ P_i^{D_N}(x) = \frac{\det[p^{\text{circ}}(0, x^{D_N}; t, x)] \det[p^{\text{circ}}(t, x; t_\star, x^{D_N})]}{\det[p^{\text{circ}}(0, x^{D_N}; t_\star, x^{D_N})]}, \quad x \in \mathbb{W}^{(0, \pi i)^N}. \]

Proof In the equalities (4.14), if we replace \(t\) by \(t_\star - t\) and consider the complex conjugate of the obtained equalities, then by (4.6), we have
\[ \det[r^{R_N}(t_\star - t)] \det[p^2(t, x; t_\star, x^{R_N})] = \det[r^{R_N}(t_\star - t, x)]. \]
Hence (3.6) with (3.3) gives

$$p_t^{R_N}(x) = e^{R_N} \det[p_t^0(0, v_t; t, x)] \det[p_t^1(t, x; t_*, v_t^{R_N})]$$

with constants $e^{R_N}$ which do not depend on $x$. Since $p_t^{R_N}(x)$ is normalized as (3.7), the Chapman-Kolmogorov equations (4.7) determine the constants as $e^{R_N} = 1/\det[p_t^0(0, v_t; t_*, v_t^{R_N})]$. The proof is complete.  \[ \Box \]

From the expressions (4.17) in Proposition 4.3, we can conclude the following. (See, for instance, Part I, IV.4.22 of [3] and Section V.C of [16] for Brownian bridges.)

**Theorem 4.4** (i) The one-parameter family of determinantal point process, $(\Xi^{A_{N-1}}_N, P_t^{A_{N-1}}, t \in (0, t_*))$, is realized as the particle configuration at time $t \in (0, t_*)$ of the noncolliding Brownian bridges on a circle with radius $r$, starting from and returning to the configuration $v_t^{A_{N-1}} = (2\pi r(j - 1)/N)_{j=1}^N$.

(ii) For $R_N = B_N, B_N^\ast, C_N, C_N^\ast, BC_N$, each one-parameter family of determinantal point process, $(\Xi^{R_N}_N, P_t^{R_N}, t \in (0, t_*))$, is realized as the particle configuration at time $t \in (0, t_*)$ of the noncolliding Brownian bridges starting from and returning to the configuration $v_t^{R_N}$ given by (4.8) in the interval $[0, \pi r]$ with the absorbing boundary conditions at both edges.

(iii) The one-parameter family of determinantal point process, $(\Xi^{D_N}_N, P_t^{D_N}, t \in (0, t_*))$, is realized as the particle configuration at time $t \in (0, t_*)$ of the noncolliding bridges starting from and returning to the configuration $v_t^{D_N} = (\pi r(j - 1)/(N - 1))_{j=1}^N$ in the interval $[0, \pi r]$ with the reflecting boundary conditions at both edges.

**5 Concluding Remarks**

In the present paper we have constructed seven types of one-parameter families of determinantal point processes, $(\Xi^{R_N}_N, P_t^{R_N}, t \in (0, t_*))$, $R_N = A_{N-1}, B_N, B_N^\ast, C_N, C_N^\ast, BC_N, D_N$. These point processes can be interpreted as configurations at time $t \in (0, t_*)$ of the noncolliding Brownian bridges starting from and returning to the equidistant configurations $v_t^{R_N}$ given by (4.8). In this picture, the variety of elliptic determinantal processes is due to various choices of configurations pinned at the initial time $t = 0$ and at the final time $t = t_*$. If we regard these Brownian bridges on a circle with radius $r$, $P^1(r)$, or in an interval $[0, \pi r]$ with time duration $t_*$ as the statistical ensembles of noncolliding paths on the spatio-temporal cylinder $P^1(r) \times (0, t_*)$ or on the spatio-temporal plane $[0, \pi r] \times (0, t_*)$, $v_t^{R_N}$ gives a boundary condition to the paths. The degeneracy of types in the scaling limit $N \to \infty$, $r \to \infty$ with constant density $\rho$ of paths shown by Theorem 3.6 is caused by vanishing of the boundary effect in this bulk limit.

In previous papers [12, 14, 15], the processes associated with the affine root systems $A_{N-1}, B_N, C_N$, and $D_N$ were characterized as solutions of some systems of stochastic differential equations (SDEs). Characterization of the present determinantal point processes $(\Xi^{R_N}_N, P_t^{R_N}, t \in (0, t_*))$ in terms of SDEs should be further studied. The noncolliding Brownian bridges discussed in Section 4 are determinantal [2, 17, 13] and the spatio-temporal correlation kernels should be determined.

As mentioned in Section 1 and in Remark 4 in Section 3.3, the present determinantal point processes are elliptic extensions of the eigenvalue ensembles of Hermitian random matrices in GUE and chiral GUE. The trigonometric reductions discussed in Section 3.2 are related with the eigenvalue distributions of random matrices in circular ensembles [21, 6]. It is an interesting future problem to find the statistical ensembles of random matrices in the elliptic level whose eigenvalues realize the present seven types of elliptic determinantal point processes.

In [5] Forrester studied the quantum $N$-particle systems in two dimensions with doubly periodic boundary conditions, in which the $N$-body potentials and wave functions are described using the Jacobi theta functions. He constructed the doubly periodic probability measures on a complex plane and discussed solvability and universality of the obtained two-dimensional systems. From the view point of the present study, his systems are of type $A_{N-1}$ and they are truly elliptic. Generalization of his study to the quantum systems associated with other six types of irreducible reduced affine root systems will be an important future problem.
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A The Jacobi Theta Functions

Let 
\[ z = e^{\tau \pi i}, \quad q = e^{\tau \pi i}, \]
where \( v, \tau \in \mathbb{C} \) and \( \Im \tau > 0 \). The Jacobi theta functions are defined as follows [29, 23],

\[ \vartheta_0(v; \tau) = -ie^{\pi i(v+\tau/4)} \vartheta_1 \left( v + \frac{\tau}{2}; \tau \right) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} z^{2n} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{\tau \pi i n^2} \cos(2n\pi v), \]

\[ \vartheta_1(v; \tau) = i \sum_{n \in \mathbb{Z}} (-1)^n q^{(n-1/2)^2} z^{2n-1} = 2 \sum_{n=1}^{\infty} (-1)^{n-1} e^{\tau \pi i (n-1/2)^2} \sin((2n-1)\pi v), \]

\[ \vartheta_2(v; \tau) = \vartheta_1 \left( v + \frac{1}{2}, \tau \right) = \sum_{n \in \mathbb{Z}} q^{n/2} z^{2n-1} = 2 \sum_{n=1}^{\infty} e^{\tau \pi i (n-1/2)^2} \cos((2n-1)\pi v), \]

\[ \vartheta_3(v; \tau) = e^{\pi i(v+\tau/4)} \vartheta_1 \left( v + \frac{1 + \tau}{2}; \tau \right) = \sum_{n \in \mathbb{Z}} q^n z^{2n} = 1 + 2 \sum_{n=1}^{\infty} e^{\tau \pi i n^2} \cos(2n\pi v). \quad (A.1) \]

(Note that the present functions \( \vartheta_\mu(v; \tau), \mu = 1, 2, 3 \) are denoted by \( \vartheta_\mu(v, q) \), and \( \vartheta_0(v; \tau) \) by \( \vartheta_4(v, q) \) in [29].) For \( \Im \tau > 0 \), \( \vartheta_\mu(v; \tau), \mu = 0, 1, 2, 3 \) are holomorphic for \( |v| < \infty \) and satisfy the partial differential equation

\[ \frac{\partial \vartheta_\mu(v; \tau)}{\partial \tau} = \frac{1}{4\pi i} \frac{\partial^2 \vartheta_\mu(v; \tau)}{\partial v^2}. \]

The parity with respect to \( v \) is given by

\[ \vartheta_1(-v; \tau) = -\vartheta_1(v; \tau), \quad \vartheta_\mu(-v; \tau) = \vartheta_\mu(v; \tau), \quad \mu = 0, 2, 3, \quad (A.2) \]

and they have the quasi-periodicity; for instance, \( \vartheta_1 \) satisfies

\[ \vartheta_1(v + 1; \tau) = -\vartheta_1(v; \tau), \quad \vartheta_1(v + \tau; \tau) = -e^{-\pi i(2v+\tau)} \vartheta_1(v; \tau). \quad (A.3) \]

By the definition (A.1), when \( \Im \tau > 0 \),

\[ \vartheta_1(0; \tau) = \vartheta_1(1; \tau) = 0, \quad \vartheta_1(x; \tau) > 0, \quad x \in (0, 1), \]

\[ \vartheta_2(-1/2; \tau) = \vartheta_2(1/2; \tau) = 0, \quad \vartheta_2(x; \tau) > 0, \quad x \in (-1/2, 1/2), \]

\[ \vartheta_3(x; \tau) > 0, \quad x \in \mathbb{R}. \quad (A.4) \]

We see the asymptotics

\[ \vartheta_0(v; \tau) \sim 1, \quad \vartheta_1(v; \tau) \sim 2e^{\pi i/4} \sin(\pi v), \quad \vartheta_2(v; \tau) \sim 2e^{\pi i/4} \cos(\pi v), \quad \vartheta_3(v; \tau) \sim 1, \]

in \( \Im \tau \to +\infty \) (i.e., \( q \to e^{\tau \pi i} \to 0 \)). \quad (A.5)

The following functional equalities are known as Jacobi’s imaginary transformations [29, 23],

\[ \vartheta_0(v; \tau) = e^{\pi i/4} - 1/2 e^{-\pi i v^2/\tau} \vartheta_2 \left( \frac{v}{\tau}; - \frac{1}{\tau} \right), \]

\[ \vartheta_1(v; \tau) = e^{3\pi i/4} - 1/2 e^{-\pi i v^2/\tau} \vartheta_1 \left( \frac{v}{\tau}; - \frac{1}{\tau} \right), \]

\[ \vartheta_3(v; \tau) = e^{\pi i/4} - 1/2 e^{-\pi i v^2/\tau} \vartheta_3 \left( \frac{v}{\tau}; - \frac{1}{\tau} \right). \quad (A.6) \]
B Selberg-type Integral Formulas Including the Jacobi Theta Functions

Apply the Macdonald denominator formulas (3.1) with (3.2) into (3.3). Then Lemma 3.2 gives the following Selberg-type integral formulas including the Jacobi theta functions. Let \( s(N) = 0 \) if \( N \) is even, and \( s(N) = 3 \) if \( N \) is odd. For \( 0 < t_s < \infty, t \in (0, t_s) \),

\[
\int_{[0,2\pi]^N} d\vartheta_s(N) \left( \sum_{j=1}^{N} \xi(x_j); N^{A_{N-1}^{-1}} \tau(t_s - t) \right) \vartheta_s(N) \left( \sum_{j=1}^{N} \xi(x_j); N^{A_{N-1}^{-1}} \tau(t) \right)
\times W^{A_{N-1}}(\xi(x); N^{A_{N-1}^{-1}} \tau(t_s - t)) W^{A_{N-1}}(\xi(x); N^{A_{N-1}^{-1}} \tau(t)) = \frac{N! \prod_{n=1}^{N} m_{A_{N-1}^{-1}}(t_s)}{a^{A_{N-1}^{-1}}(t_s - t)a^{A_{N-1}^{-1}}(t)},
\]

\[
\int_{[0,\pi]^N} dx \left\{ \vartheta_s(N) \left( \sum_{j=1}^{N} \xi(x_j); N^{A_{N-1}^{-1}} \tau(t_s/2) \right) W^{A_{N-1}}(\xi(x); N^{A_{N-1}^{-1}} \tau(t_s/2)) \right\}^2 = \frac{N! \prod_{n=1}^{N} m_{R_N}(t_s)}{(a^{R_N}(t_s/2))^2},
\]

for \( R_N = B_N, B_N^*, C_N, C_N^*, BC_N, D_N \).

In particular, if we set \( t = t_s/2 \), we have the following,

\[
\int_{[0,2\pi]^N} d\vartheta_s(N) \left( \sum_{j=1}^{N} \xi(x_j); N^{A_{N-1}^{-1}} \tau(t_s) \right) W^{A_{N-1}}(\xi(x); N^{A_{N-1}^{-1}} \tau(t_s)) = \frac{N! \prod_{n=1}^{N} m_{A_{N-1}^{-1}}(t_s)}{(a^{A_{N-1}^{-1}}(t_s))^2},
\]

\[
\int_{[0,\pi]^N} dx \left\{ W^{R_N}(\xi(x); N^{R_N} \tau(t_s/2)) \right\}^2 = \frac{N! \prod_{n=1}^{N} m_{R_N}(t_s)}{(a^{R_N}(t_s))^2},
\]

for \( R_N = B_N, B_N^*, C_N, C_N^*, BC_N, D_N \).

C Determinantal Point Processes and Correlation Kernels

Let \( N \in \mathbb{N}, S \subset \mathbb{R}^d \). Assume that the probability measure of point process, \( \Xi(\cdot) = \sum_{j=1}^{N} \delta x_j(\cdot) \), is given by

\[
P(X \in dx) = p(x)dx = \frac{1}{C(N)} \det_{1 \leq i,j \leq N} [f_j(x_k)] \det_{1 \leq \ell,m \leq N} [g_{\ell}(x_m)], \quad x \in S_N,
\]

with the biorthogonal relations

\[
\int_S f_j(x)g_k(x)dx = h_{j,k}, \quad j,k \in \{1,2,\ldots,N\},
\]

where \( h_j > 0, j = 1,2,\ldots,N \). Let \( C_0(S) \) be a collection of all continuous real functions with a compact support in \( S \). For \( \psi \in C_0(S), \theta \in \mathbb{R} \), the characteristic function of \( (\Xi, P) \) is defined as

\[
\Psi[\psi; \theta] = \mathbb{E} \left[ e^{\theta \sum_{j=1}^{N} \psi(x_j)} \right] = \frac{1}{N^N} \int_{S^N} d\vartheta e^{\theta \sum_{j=1}^{N} \psi(x_j)} p(x),
\]

which can be regarded as the Laplace transform of \( p \). Put \( \chi(x) = 1 - e^{\theta \varphi(x)} \). By performing binomial expansion, we obtain

\[
\Psi[\psi; \theta] = \frac{1}{N^N} \int_{S^N} d\vartheta \prod_{j=1}^{N} (1 - \chi(x_j)) p(x)
= 1 + \sum_{n=1}^{N} \frac{(-1)^n}{n!} \int_{S^n} \prod_{\ell=1}^{n} \{dx \chi(x_\ell)\} \rho(\{x_1, \ldots, x_n\}),
\]

(C.3)
where \( \rho(\{x_1, \ldots, x_n\}) \) is given by (1.1). This implies that if we regard \( \Psi[\psi; \theta] \) as a functional of \( \chi \), it gives the generating function of correlation functions [17]. Insert (C.1) into (C.3) and use the Heine identity

\[
\frac{1}{N!} \int_{S^n} dx \det_{1 \leq j, k \leq N} [\phi_j(x_k)] \det_{1 \leq \ell, m \leq N} [\varphi_{\ell}(x_m)] = \det_{1 \leq j, k \leq N} \left[ \int_S dx \phi_j(x) \varphi_k(x) \right]
\]  

(C.4)

for square integrable functions \( \phi_j, \varphi_j, j \in \{1, 2, \ldots, N\} \). Then we have

\[
\Psi[\psi; \theta] = \frac{1}{C(N)} \det_{1 \leq j, k \leq N} \left[ \int_S dx f_j(x)(1 - \chi(x))g_k(x) \right]
\]

\[
= \frac{\det_{1 \leq j, k \leq N} \left[ \int_S dx f_j(x)(1 - \chi(x))g_k(x) \right]}{\det_{1 \leq j, k \leq N} \left[ \int_S dx f_j(x)g_k(x) \right]}
\]

\[
= \det_{1 \leq j, k \leq N} \left[ \int_S dx f_j(x)g_k(x) - \int_S dx f_j(x)g_k(x) \right]
\]

where we used the normalization condition \( \Psi[\psi; 0] = 1 \) at the second equality. We introduce the \( N \times N \) matrices \( A \) and \( A[\chi] \) with the entries

\[
(A)_{jk} = \int_S dx f_j(x)g_k(x), \quad (A[\chi])_{jk} = \int_S dx f_j(x)\chi(x)g_k(x), \quad 1 \leq j, k \leq N.
\]

Since (C.2) is assumed, \( A \) is a regular matrix, and the above is written as

\[
\Psi[\psi; \theta] = \det_{1 \leq j, k \leq N} \left[ \delta_{jk} - (A^{-1}A[\chi])_{jk} \right],
\]

where

\[
(A^{-1}A[\chi])_{jk} = \int_S dx B_j(x)g_k(x) \quad \text{with} \quad B_j(x) = \sum_{\ell=1}^N (A^{-1})_{j\ell}f_{\ell}(x)\chi(x).
\]

Now we apply the Fredholm expansion formula. Then we can verify that [17]

\[
\Psi[\psi; \theta] = 1 + \sum_{n=1}^N (-1)^n \frac{1}{n!} \int_{S^n} \prod_{\ell=1}^n dx_{\ell} \det_{1 \leq j, k \leq N} \left[ \sum_{m=1}^N g_m(x_j)B_m(x_k) \right].
\]

By the orthogonality (C.2), \((A^{-1})_{j\ell} = (1/h_j)\delta_{j\ell}\) and hence \( B_m(x) = f_m(x)\chi(x)/h_m \). We define

\[
K(x, y) = \sum_{m=1}^N g_m(x)f_m(y)/h_m, \quad x, y \in S.
\]  

(C.5)

Then we arrive at the expression

\[
\Psi[\psi; \theta] = 1 + \sum_{n=1}^N (-1)^n \frac{1}{n!} \int_{S^n} \prod_{\ell=1}^n \{dx_{\ell}\chi(x_{\ell})\} \det_{1 \leq j, k \leq N} [K(x_j, x_k)].
\]  

(C.6)

For any \( \chi \in C_c(S) \), we have proved that (C.3) is equal to (C.6). Hence we can conclude that

\[
\rho(\{x_1, \ldots, x_n\}) = \det_{1 \leq j, k \leq N} [K(x_j, x_k)], \quad n = 1, 2, \ldots, N.
\]

In summary, if (C.1) and (C.2) are satisfied, then the point process \((\Xi, \mathbb{P})\) is determinantal and the correlation kernel is given by (C.5). We note that (C.6) defines the Fredholm determinant associated with the integral kernel \( K(x, y)\chi(y) \) and it is written as \( \text{Det}_{x, y \in S} \left[ \delta(x - y) - K(x, y)\chi(y) \right] \) (see, for instance, [13]).
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