Giant Faraday rotation induced by the Berry phase in bilayer graphene under strong terahertz fields

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Abstract
High-order terahertz (THz) sideband generation in semiconductors is a phenomenon with physics similar to that of high-order harmonic generation but in a regime of much lower frequency. Our previous paper [1] found that the electron–hole pair excited by a weak optical laser can accumulate a Berry phase along a cyclic trajectory under the driving of a strong elliptically polarized THz field. Furthermore, the Berry phase appears as the Faraday rotation angle of the emission signal under short-pulse excitation in monolayer MoS2. In this paper, the theory of the Berry phase in THz extreme nonlinear optics is applied to biased bilayer graphene with Bernal stacking, which has similar Bloch band features and optical properties to monolayer MoS2, such as the time-reversal related valleys and the valley contrasting optical selection rule. However, the biased bilayer graphene has much larger Berry curvature than monolayer MoS2, which leads to a large Berry phase of the quantum trajectory and in turn a giant Faraday rotation of the optical emission (∼1 rad for a THz field with frequency 1 THz and strength 8 kV cm−1). This surprisingly big angle shows that the Faraday rotation can be induced more efficiently by the Berry curvature in momentum space than by the magnetic field in real space. It provides opportunities to use bilayer graphene and THz lasers for ultrafast electro-optical devices.
Keywords: Faraday rotation, Berry phase, bilayer graphene, THz nonlinear optics

In semiconductors irradiated by a strong terahertz (THz) laser of frequency $\omega$, the electron–hole pairs excited by a weak optical laser of frequency $\Omega$ are driven into large amplitude oscillations. They subsequently acquire kinetic energies much greater than the THz photon energy during the oscillations, leading to optical emissions at frequencies $\Omega \pm N\omega$, where $N$ is an integer [2, 3]. This high-order THz sideband generation (HSG) in semiconductors is a generalization of the high-order harmonic generation (HHG) [4–7] to the THz frequency regime. The core physics of HHG and HSG is captured by the quantum trajectory theory [2, 6]. When the electrons (or electron–hole pairs) are driven by the strong laser field, their oscillation amplitudes are much larger than the wavepacket diffusion range. Therefore, the quantum evolution of the electrons (or electron–hole pairs) is well described by quantum trajectories that satisfy the stationary phase condition (i.e. the least action condition in classical mechanics) in the formalism of Dirac–Feynman path integrals [2, 6].

A fundamental difference between HHG and HSG is that the electron–hole pair in semiconductors can have nontrivial Bloch states [1]. As a result, when the electron (or hole) is driven by the THz field in the conduction (or valence) band, not only does the quasi-momentum $k$ change [8], but also the Bloch wavefunction evolves with $k$. Since the THz frequency is much smaller than the band gaps of semiconductors, the evolution is adiabatic (i.e. no interband tunneling is induced by the THz field). This adiabatic evolution along the path in $k$-space can accumulate a geometrical phase, and in particular a Berry phase along a closed loop [9]. A rich structure of the Bloch states in condensed matter systems (such as in topological insulators [10, 11]) thus leads to a variety of phase effects in extreme nonlinear optics. An additional, important difference between the physics of HHG and that of HSG is that the electron–hole pairs are elementary optical excitations in solids. Therefore, the quantum trajectories in semiconductors, unlike those in atoms, can be triggered by lasers on demand at designed frequencies [12] or times (relative to the THz field oscillation). This excitation at will provides a great deal of controllability and flexibility for studying the quantum trajectories in extreme nonlinear optics.

In [1], we studied the Berry phases of quantum trajectories in monolayer MoS$_2$ with a band gap in the visible wavelength regime. We found that the optical emission delayed by integer multiples of the THz period after the pulse excitation acquired a Faraday rotation, and the rotation angle was exactly the Berry phase of the quantum trajectory ($\sim 0.01$ rad for a THz field with frequency $1 \text{ THz}$ and strength $8 \text{ kV cm}^{-1}$). In this paper, however, we consider Bernal stacked bilayer graphene with an interlayer gate bias [13–16], which has similar Bloch band features and optical properties to monolayer MoS$_2$ but with a smaller band gap, tunable up to $\sim 250 \text{ meV}$. Its conduction and valence band edges are located at the corners of the 2D hexagonal Brillouin zone, and the optical interband transitions at the two time-reversal (TR) related valleys have nearly perfect but opposite polarization selection rules [17, 18]. In bilayer graphene without interlayer bias, the conduction (or valence) state acquires a pseudo-spin winding number $\pm 2$ along any closed path around the Dirac points [14, 19], and the Berry curvature is completely concentrated on the singular monopole (with topological charge 2) at the Dirac point. In biased bilayer graphene (BBG), the electrons (or holes) within each valley still have a strong chirality such that they undergo a large pseudo-spin rotation along a closed
path around the Dirac point. This rotation gives the Berry phase of the path, which is much larger than in many other materials such as monolayer MoS$_2$. In BBG, the Berry curvature is concentrated in a small region in the vicinity of the Dirac points, and near the band edge is about 100 times larger than that in monolayer MoS$_2$ (see figure 1). Thus the Berry phase of the quantum trajectory in BBG is also about 100 times larger than that in monolayer MoS$_2$, and in turn leads to a giant Faraday rotation. This property makes BBG an excellent candidate for use in both the experimental investigation of Berry phases of quantum trajectories and applications in ultrafast electro-optical devices without magnetic fields. Besides this, a gate bias can be applied to tune the band gap between zero and mid-infrared energies [15, 16], which offers extra controllability.

In this paper, we apply the Berry phase dependent theory of optical response in semiconductors under strong THz fields to the BBG. The quantum trajectory theory shows that the Faraday rotation angle of the optical emission from BBG delayed by multiples of the THz period is exactly equal to the Berry phase of the stationary quantum trajectory. Since BBG has a very large Berry curvature near the band edge, the stationary trajectory has a large Berry phase, which in turn leads to a giant Faraday rotation ($\sim 1$ rad for a THz field with frequency 1 THz and strength 8 kV cm$^{-1}$). This giant Faraday rotation is verified by numerical simulations. Our study shows that the Faraday rotation can be induced more efficiently by the Berry curvature in momentum space than by the magnetic field in real space, as in the traditional case. This hence provides a promising possibility for using BBG and THz lasers in ultrafast electro-optical devices without magnetic fields.

Figure 1. Energy bands, optical selection rules and Berry curvatures near the Dirac points of biased bilayer graphene; (a) shows the energy spectrum at the $K$ valleys, where $k$ is expanded around the respective Dirac points and $k_z = 0$; (b) shows the degrees of circular polarization for the interband dipole moment (defined in equation (14)), where we see the valley contrasting optical selection rules near the two Dirac points (see also [18]); (c) shows the combined Berry curvature of the bottom conduction band and the top valence band (see also [20]). The solid (dashed) line gives the values in valley $K_-$ ($K_+ \pm K_0$). The interlayer bias is chosen as $2\Delta = 0.3$ eV.
We consider a semiconductor under an elliptically polarized THz field
\[ \mathbf{F}(t) = F \left( \cos \theta \cos (\omega t), \sin \theta \sin (\omega t), 0 \right) , \] (1)
with \( \omega \) much smaller than the energy gap of the material (so the THz field does not induce interband tunneling). The Hamiltonian in the Bloch state representation \( H(k) \) evolves adiabatically in the \( k \)-space \( H(k) \to H(\tilde{k}(t)) \) under the driving of this field, with
\[ \tilde{k}(t) = k + eA(t) = \left( k_x - k_0 \cos \theta \sin (\omega t), k_y + k_0 \sin \theta \cos (\omega t), k_z \right) , \] (2)
where \( k_0 = eF/\omega \) and \( A(t) \) is the electromagnetism vector potential with \( \mathbf{F} = -\partial A/\partial t \). Then we study the interaction of this system with a weak optical laser that creates electron–hole pairs at the band edge, where the interaction Hamiltonian is \( \hat{H}_i = -\hat{\mathbf{P}} \cdot \mathbf{E}_i e^{-i\gamma t} + \text{h.c.} \). The interband polarization operator in the Bloch state representation is
\[ \hat{\mathbf{P}} = \int d^3k \hat{\mathbf{P}}_{\mu, k} \hat{h}_{\nu, -k} \mathbf{d}_{\mu, k} \] (3)
where \( \hat{\mathbf{P}} \) and \( \hat{\mathbf{h}} \) are electron and hole operators, and the interband dipole moment \( \mathbf{d}_{\mu, k} \) is given by [21, 22]
\[ \mathbf{d}_{\mu, k} = -e \left\langle +, \mu, k \left| \mathbf{V}_k \right| -, \nu, k \right\rangle = \frac{e \left\langle +, \mu, k \left| \mathbf{V}_k H(k) \right| -, \nu, k \right\rangle}{E_k^+ - E_k^-} . \] (4)

Here, + and – denote the conduction and valence bands, respectively, \( \mu \) and \( \nu \) are the pseudo-spin or valley indices introduced to label degenerate bands, and \( \left| \pm, \mu, k \right\rangle \) are the Bloch states of the conduction (valence) bands with eigenenergies \( E_k^\pm \). We assume that the initial state is the vacuum state \( \left| G \right\rangle \) with empty conduction bands and filled valence bands. After excitation by an optical laser, the electron (hole) is driven into adiabatic evolution in the conduction (valence) band by the THz field, and thus obtains a geometric phase \( \int \mathcal{T} e^{i \int \mathcal{A}_{\mu
u}^z \cdot d \mathbf{k}} \) in addition to the dynamical phase \( -\int E_k^z d\tau \), where \( \mathcal{A}_{\mu
u}^{\pm} = i \left\langle \pm, \mu, \tilde{k} \left| \mathbf{V}_k \right| \pm, \nu, \tilde{k} \right\rangle \) are the Berry connections with \( \left| \pm, \mu, \tilde{k} \right\rangle \) being the instantaneous eigenstates of \( H(\tilde{k}(t)) \) with instantaneous eigenenergies \( E_k^z \). In general, the Berry connection can be non-Abelian (i.e. \( \mathcal{A}_{\mu
u}^{\pm} = i \left\langle \pm, \mu, \tilde{k} \left| \mathbf{V}_k \right| \pm, \nu, \tilde{k} \right\rangle \neq 0 \) for \( \mu \neq \nu \)) and the geometric phase factor becomes a unitary matrix \( \mathcal{T} e^{i \int \mathcal{A}_{\mu
u}^z \cdot d \mathbf{k}(t)} \), where \( \mathcal{T} \) is the time-ordering operator. The electron and hole recombine at a later time, leading to optical emission modified by the geometric phase.

The linear optical response of semiconductors driven by THz fields has been derived in [1]. If the THz field does not mix different pseudo-spin or valley states of the degenerate energy bands, the Berry connections are Abelian and the optical response is simplified to
\[ \mathbf{P}(t) = \sum_{\mu} \int_{-\infty}^{t} dt' \int d\mathbf{k} \mathbf{d}_{\mu, k(t)} \mathbf{d}_{\mu, k(t')} \cdot \mathbf{E}_i(t') e^{-i \int_{t(t')}^{t} \mathcal{A}_{\mu
u}(\mathbf{k}) \cdot d\mathbf{k}(t') - i \gamma (t - t')} , \] (5)
where \( \epsilon_k = E_k^+ - E_k^- \) is the energy of the electron–hole pair and \( A_k = A_k^+ - A_k^- \) is the combined Berry connection of the electron–hole pair. This general formula can be applied to any band material, including both Schrödinger-type materials (such as GaAs) and non-Schrödinger materials (such as monolayer MoS\(_2\) and BBG). In formula (5), we have included a damping term \( e^{-\gamma t} \) of the electron–hole pair, which takes into account the recombination of the electron–hole pair, phonon scattering, and so on. In the following discussions, for simplicity, we assume that the electron–hole pair damping is absorbed into the energy \( \epsilon_k \). For a system with TR symmetry, we have the relations

\[
\phi_B^{(n)}(k) = \phi_B^{(n)}(k) = -\phi_B^{(n)}(k) = \int_0^{\tau_B} A_k \cdot d\vec{k}.
\]

where the pseudo-spin state \( \uparrow \) denotes one state of the Kramers pair and the other is denoted as \( \downarrow \). We apply a short optical laser pulse to the system at time \( t_0 \), with the duration much shorter than the THz period \( T = 2\pi/\omega \), such that the pulse can be approximated by a \( \delta \)-pulse \( E_t(t) \approx \delta(t) \). Then the response at \( t = nT \) is simplified as

\[
P(t_n) = i \int \mathbf{d}k e^{-i \int_0^{\tau} \epsilon_{k(t)} d\tau + i\phi_B^{(n)}(k)} d_{k(0)}^* d_{k(0)} \cdot \mathbf{E} + i \int \mathbf{d}k e^{-i \int_0^{\tau} \epsilon_{k(t)} d\tau - i\phi_B^{(n)}(k)} d_{-k(0)} d_{-k(0)}^* \cdot \mathbf{E},
\]

where

\[
\phi_B^{(n)}(k) = \phi_B^{(n)}(k) = -\phi_B^{(n)}(k) = \int_0^{\tau_B} A_k \cdot d\vec{k}.
\]

Note that \( \phi_B^{(n)} \) is the gauge invariant Berry phase since the electron (hole) undergoes a cyclic evolution in the conduction (valence) band at \( t_0 = nT \), and the two TR related states have exactly the opposite Berry phases. Thus \( P(t_n) \) is given by the interference between two responses with the same dynamical phase but opposite Berry phases, which, as shown below, results in a Faraday rotation of the optical emission.

The main consequence of equation (7) can be understood using the stationary phase formalism (or the quantum trajectory theory) [1, 2, 6]. In the path integral, the electron–hole pair moves along all possible trajectories when driven by the THz field, with the phase given by the action \( S(k) = \int_0^{\tau_B} \epsilon_{k(t)}(t) d\tau \). The Berry phase is dropped from this action that determines the electron–hole motion, since it is generally much smaller than the dynamical phase (see figure 2). As the THz field is strong, the motion amplitude of the electron–hole pair is much larger than the wavepacket diffusion range and the response is dominated by the stationary phase points of the action, i.e. trajectories that satisfy the least action condition

\[
\nabla_k S(k) = \int_0^{\tau_B} v_k d\tau = 0.
\]

Here, \( v_k = \nabla_k \epsilon_k \) is the semiclassical velocity of the electron–hole pair, and hence the stationary phase condition (9) means that the electron under the acceleration by the THz field returns to the hole after \( nT \) for recombination. Then the transient response is mainly modified by the Berry phase accumulated along the stationary trajectory:

\[
P(t_n) = e^{i\phi_B^{(n)}(k)} i \int \mathbf{d}k e^{-i \int_0^{\tau} \epsilon_{k(t)} d\tau} d_{k(0)}^* d_{k(0)} \cdot \mathbf{E} + e^{-i\phi_B^{(n)}(k)} i \int \mathbf{d}k e^{-i \int_0^{\tau} \epsilon_{k(t)} d\tau} d_{-k(0)} d_{-k(0)}^* \cdot \mathbf{E},
\]
where $d_{k(i)} = d_{k(i)}$ near the band edge has been used and $k_s$ is the stationary phase point. When multiple solutions exist, the response is given by the interference of all possible stationary trajectories. The optical selection rule near the Dirac points in BBG is such that the interband dipole moment $d_{v,k} \approx d \epsilon_{v,k}$, i.e. the optical transition at valley $⇑$ ($⇓$) is coupled to the $\sigma^+$ ($\sigma^-$) polarized light (see [18] or figure 1(b)). Thus the linear susceptibilities for the $\sigma^\pm$-polarized lights are

$$
\chi_{\pm}(\epsilon) = e^{i\phi_{\pm}(\epsilon)} \int dk e^{-i\epsilon_k d_{v,k}dk} d_{v,k}(\epsilon).
$$

We see that if the excitation light is linearly polarized in the $x$–$y$ plane, the Faraday rotation of the optical emission at $t_n$ is exactly given by the Berry phase $\phi_{F,R}(t_n) = \phi_{B}(k)$. This Faraday rotation induced by the Berry phase can be intuitively understood as following [1]. The linearly polarized optical laser is a superposition of two opposite circular polarizations, which cause respective transitions of the two Kramers states. Thus the electron–hole pair created by the optical pulse is in a superposition of the Kramers states $⇑ + ⇓$. After the cyclic evolution under the THz field, the $⇑$ and $⇓$ states obtain the same dynamical phase $\phi_D$ and opposite Berry phases $\pm \phi_B$. Thus the final state of the electron–hole pair is $e^{i\phi_D} (e^{i\phi_B} |⇑⟩ + e^{-i\phi_B} |⇓⟩)$, which gives emission with linear polarization rotated by an angle $\phi_B$.

In bilayer graphene with Bernal (A–B) stacking, the band structure is well described by the tight-binding Hamiltonian with an intralayer nearest-neighbor hopping $t \approx 0.4$ eV and an interlayer bias $2\Delta$ [13–15].

**Figure 2.** The dynamical phase $\phi_D^{(1)}(k)$ and Berry phase $\phi_B^{(1)}(k)$ near the $K_s$ valley ($k_s = 0$) over one THz period of evolution. Here we follow the notation of equations (6) and (8). The interlayer bias is $\Delta = 0.3$ eV, the frequency of the THz field is $\omega = 4$ meV and the THz field is circularly polarized (i.e. $\theta = \pi/4$). In (a), the THz field strength $F = 2, 4$ and $8$ kV cm$^{-1}$ (i.e. $k_0 = 0.0039, 0.0078$ and $0.0157$ π/a), and the dynamical phase has two kinds of stationary phase points (indicated by the arrows) at the Dirac point and the band edge. In (b), the THz field strength $F = 12, 16$ and $20$ kV cm$^{-1}$ (i.e. $k_0 = 0.0235, 0.0313$ and $0.0392$ π/a), and only the Dirac point is a stationary phase point (indicated by the arrow).
\[
H(K) = \begin{pmatrix}
\Delta & f(K) & 0 & 0 \\
f^*(K) & \Delta & t_\perp & 0 \\
0 & t_\perp & -\Delta & f(K) \\
0 & 0 & f^*(K) & -\Delta
\end{pmatrix},
\]

(12)

Here, \( f(K) = -t(e^{iKb_1} + e^{iKb_2} + e^{iKb_3}) \), where \( b_{1,2} = \left( \frac{a}{\sqrt{3}}, \pm \frac{a}{2} \right) \) and \( b_3 = \left( -\frac{a}{\sqrt{3}}, 0 \right) \) with \( a \approx 2.46 \text{ Å} \) being the lattice constant. Expanding the momentum near the two Dirac points \( K_\pm = (0, \pm \frac{\pi}{\sqrt{3}}) \), we have \( f(K_\pm + k) = iv_j (k_y \mp ik_x) \), where \( v_j = \frac{\gamma}{\sqrt{3}} at \ (\approx 10^6 \text{ m s}^{-1}) \). The states at \( K_\pm \) valleys are related by TR operation and the corresponding Hamiltonians are given by

\[
H_{\theta/\bar{\theta}}(k) = \begin{pmatrix}
\Delta & \mp i v_j k_x & 0 & 0 \\
\mp iv_j k_x & \Delta & t_\perp & 0 \\
0 & t_\perp & -\Delta & \mp iv_j k_x \\
0 & 0 & \mp iv_j k_x & -\Delta
\end{pmatrix},
\]

(13)

where \( k_x = k_x \pm i k_y \) and the \( K_\pm \) valley is denoted by the pseudo-spin notation \( \uparrow/\downarrow \). In this paper, it is assumed that the valley coherence time in bilayer graphene is longer than the THz period, such that the intervalley scattering can be neglected. The calculated band structure (as shown in figure 1(a)), reproduces the well-known Mexican hat structure.

The bilayer graphene has two conduction bands (positive energy bands) and two valence bands (negative energy bands). We assume that the Fermi level is kept in the energy gap (e.g. by doping the devices chemically) while tuning the gap value [15]. Hence the system is initially in the vacuum state \( |G\rangle \) with empty conduction bands and filled valence bands. Since the optical laser is near resonant with the energy gap between the bottom conduction band and the top valence band, only the optical transitions between these two bands are considered, with \( |\pm, \mu, k\rangle (\mu = \uparrow, \downarrow) \) now denoting the corresponding Bloch states. Furthermore, it was shown in [18] that there is a nearly perfect optical selection rule for the interband transitions between the two bands near the Dirac points, where the valley \( K_\pm \) favors the \( \sigma_\pm \) polarized transitions, respectively. This can be seen from figure 1(b), which plots the degree of circular polarization

\[
[\eta(k)]_\mu = \left( \frac{|d^+_{\mu,k}|^2 - |d^-_{\mu,k}|^2}{\left| d^+_{\mu,k} \right|^2 + \left| d^-_{\mu,k} \right|^2} \right),
\]

(14)

with \( d^\pm_{\mu,k} = d_{\mu,x} \cdot (e_1 \pm i e_3)/\sqrt{2} \). Figure 1(c) shows the combined Berry curvature of the bottom conduction band and the top valence band, which is defined as

\[
\left( \Omega^\pm_{\mu\mu} \right)_{\mu\mu} = \left( \Omega^+_{\mu\mu} \right)_{\mu\mu} - \left( \Omega^-_{\mu\mu} \right)_{\mu\mu},
\]

(15)
where \[9, 23\]

\[
\left( \Omega_{ij}^{\mu} \right)_{\mu \mu} = \partial_{\mu} \left( \mathcal{A}_{ij}^{\mu} \right)_{\mu \mu} - \partial_{\mu} \left( \mathcal{A}_{ij}^{\mu} \right)_{\mu \mu} = i \sum_{n \neq j} \left\langle j, \mu, k | \partial_{\mu} H_{\mu}(k) | n, \mu, k \right\rangle \left\langle n, \mu, k | \partial_{\mu} H_{\mu}(k) | j, \mu, k \right\rangle \left( E_{k}^{j} - E_{k}^{n} \right)^{-2} - (\chi \leftrightarrow y),
\]

with \( j = \pm \) and \( |n, \mu, k\rangle \) denoting the four eigenstates of \( H_{\mu}(k) \) with eigenenergy \( E_{k}^{n} \). The states at the two valleys have opposite Berry curvatures, which lead to opposite Berry phases. In BBG with an energy gap opened by the bias, the Berry curvature is concentrated in a small region in the vicinity of the Dirac points (see figure 1), which leads to a Berry curvature that is much larger than that in monolayer MoS\(_2\) \((\approx 22.3 (\pi / a)^{-2})\) [1, 17]. Since the Berry phase equals the Berry curvature flux enclosed by the path in the \( k \)-space, the electron–hole pair in bilayer graphene acquires a much larger Berry phase during the cyclic evolution, which leads to a giant Faraday rotation of the optical emission.

Then we study the stationary trajectories under the driving of the THz field, i.e. the stationary phase points of the dynamical phase \( \phi_{D}^{(n)}(k) = \int_{0}^{\kappa} \hat{\epsilon}_{k(\tau)} \, d\tau \). Figure 2 plots the distribution of the dynamical phase \( \phi_{D}^{(1)}(k) \) over one THz period and the corresponding Berry phase \( \phi_{B}^{(1)}(k) \) at the \( K \) valley for different field strengths, \( F \). The dynamical phase is indeed much larger than the Berry phase. This justifies the approximation in equation (9) for determining the quantum trajectories. When the THz field is weak (i.e. \( k_{0} \), the amplitude of the quantum trajectory in \( k \)-space, is small as compared with the radius of the ‘Mexican hat’ ring \((\approx 0.0233 \pi / a)\) around the Dirac point), we can expand \( \hat{\epsilon}_{k(\tau)} \) around \( k \) and get

\[
\phi_{D}^{(1)}(k) \approx \epsilon_{i} T + \frac{1}{2} \frac{\partial^{2} \hat{\epsilon}_{k}}{\partial k_{0}^{2}} k_{0}^{2} T.
\]

Hence the stationary phase points are exactly the extreme points of the energy band, i.e. the Dirac points and the ring-shaped band edges (figure 2(a)). Note that the trajectory around the Dirac point has a Berry phase close to that of the trajectory around the band edge point (figure 2(a)). Thus the Faraday rotation is well approximated by the Berry phase of the stationary trajectory around the Dirac point. When the THz field is strong enough (i.e. \( k_{0} \) is comparable to the Mexican hat ring radius \((\approx 0.0233 \pi / a)\)), only the Dirac points are stationary phase points (figure 2(b)). In this case, the Faraday rotation is also given by the Berry phase of the stationary trajectory centered at the Dirac point.

To verify the prediction of the quantum trajectory method (i.e. the Faraday rotation equals the Berry phase of the stationary trajectory), we calculate the Faraday rotation directly through numerical integration of equation (5) for bilayer graphene and compare it with the Berry phase \( \phi_{B}^{(n)} \) of the quantum trajectories. In the numerical calculation, the frequency of the THz field is \( \omega = 4 \) meV and the optical pulse has the Gaussian form \( E_{0} \exp \left( -i\Omega t - t^{2} / \delta t^{2} \right) \), with \( \Omega = 2\Delta \) and \( \omega \delta t = 0.2 \) \((\ll 2\pi)\) (the optical laser spectrum width is \( \sim 2/\delta t = 40 \) meV, much smaller than the valence/conduction band splitting).
For the sake of simplicity, we have not included the Coulomb interaction between electrons and holes [24]. The free electron–hole pair approximation is justified since the Berry phase is mainly determined by the quantum trajectories of the electron–hole pair in the scattering states. After quantum tunneling from the bound exciton states to the scattering states and before recombination, the quantum trajectories of the electron–hole pair are well described by the motions of free particles driven by the strong THz fields, since the effect of the Coulomb attraction on the scattering states is much weaker. Under a circularly polarized THz field with strength $F = 8 \text{ kV/cm}$ (i.e. $k_y = 0.0157\pi/a$). In (a), the THz field strength $F = 8 \text{ kV/cm}$ (i.e. $k_y = 0.0157\pi/a$). In (b), the THz field is circularly polarized (i.e. $\theta = \pi/4$). We choose the damping rate $\gamma = 4 \text{ meV}$. The other parameters are the same as in figure 2.

For the sake of simplicity, we have not included the Coulomb interaction between electrons and holes [24]. The free electron–hole pair approximation is justified since the Berry phase is mainly determined by the quantum trajectories of the electron–hole pair in the scattering states. After quantum tunneling from the bound exciton states to the scattering states and before recombination, the quantum trajectories of the electron–hole pair are well described by the motions of free particles driven by the strong THz fields, since the effect of the Coulomb attraction on the scattering states is much weaker. Under a circularly polarized THz field with strength $F = 8 \text{ kV/cm}$, the real-space amplitude of the trajectory is about 200 nm, which is much larger than the exciton radius ($\sim 10–40 \text{ nm}$) [24]. This confirms the assumption that the relative motion of the electron–hole pair in the scattering state is largely free of the Coulomb attraction. In fact, the free particle trajectory theory has been successfully applied to explain the experiments on HSG in quantum wells [3], and also to describe HHG [4–7] in atoms where the binding between electrons and atom cores is even stronger. We expect that the Coulomb interaction in the scattering states, though much weaker than in the bound states, may slightly modify the quantum trajectories and hence the Berry phase. It is certainly of interest to study the effects of Coulomb interaction for quantitative comparison with future experiments. The free particle model, notwithstanding its simplicity, captures the essential physics and can produce reasonably accurate results.

The results calculated for various THz field ellipticities $\theta$ and THz field strengths $F$ are shown in figure 3, which confirms that the Faraday rotation is well approximated by the Berry phase of the trajectory around the Dirac point. The Faraday rotation angle is about 100 times larger than in monolayer MoS$_2$ [1], which is consistent with the Berry curvature difference between the two materials. The Berry phase is given by the Berry curvature flux through the
area element $\pi k_0^2 \sin \theta \cos \theta$ enclosed by the closed path in $k$-space (equation (2)), which explains the approximate sinusoidal and parabolic dependence of the rotation angle on $\theta$ and $F$, respectively. Also, as expected from the quantum trajectory analysis, the Berry phases around the band edge points deviate appreciably from the numerical results when the THz field is strong enough (see figure 3(b)), which means that the band edge points fail to be the stationary phase points.

We perform the calculations with different damping rates ($\gamma \sim 1-6$ meV) of the electron–hole pair. All of the calculations give the same Faraday rotation as that shown in figure 3. In fact the electron–hole pair damping, which is similar to that for the dynamical phase, has the same value at the two valleys $K_e$. Thus while the damping can reduce the overall intensity of the optical signal, it does not affect the Faraday rotation, even when the decay time is shorter than the THz period. However, the valley coherence time needs to be longer than the period of the THz field. Although the valley coherence time in bilayer graphene has not been determined, to the best of our knowledge, the experiments for an analogous material (monolayer WSe$_2$) show that the valley coherence time can be even longer than the recombination time of the electron–hole pairs [25], which indicates that the observation of the predicted giant Faraday rotation is promising. On the other hand, the Faraday rotation can also be employed to study the intervalley coherence.

In summary, the Berry phase dependent theory of THz extreme nonlinear optics developed in [1] is applied to the BBG. The Faraday rotation angle of emission light delayed by integer multiples of the THz period is the Berry phase of the quantum trajectory. As the bilayer graphene has large Berry curvatures near the Dirac points, the Berry phase of the stationary trajectory is large, which leads to a surprisingly big Faraday rotation. This prediction of the giant Faraday rotation is confirmed by numerical simulations. Our study suggests that the Faraday rotation can be induced more efficiently by the Berry curvature flux enclosed by the electron–hole trajectory in momentum space than by the magnetic flux in real space, as in the traditional case. Thus, it provides an opportunity for using BBG and elliptically polarized THz lasers in ultrafast electro-optical devices.

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