Structural Properties of Optimal Test Channels for Distributed Source Coding with Decoder Side Information for Multivariate Gaussian Sources with Square-Error Fidelity

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Abstract

This paper focuses on the structural properties of test channels, of Wyner’s [1] operational information rate distortion function (RDF), \( R(\Delta_X) \), of a tuple of multivariate correlated, jointly independent and identically distributed Gaussian random variables (RVs), \( \{X_t, Y_t\}_{t=1}^{\infty} \), \( X_t: \Omega \rightarrow \mathbb{R}^{n_x} \), \( Y_t: \Omega \rightarrow \mathbb{R}^{n_y} \), with average mean-square error at the decoder, \( \frac{1}{n} \sum_{t=1}^{n} ||X_t - \hat{X}_t||^2 \leq \Delta_X \), when \( \{Y_t\}_{t=1}^{\infty} \) is the side information available to the decoder only. We construct optimal test channel realizations, which achieve the informational RDF, \( R(\Delta_X) \equiv \inf_{M(\Delta_X)} I(X; Z | Y) \), where \( M(\Delta_X) \) is the set of auxiliary RVs \( Z \) such that, \( P_{Z|X,Y} = P_{Z|X} \), \( \hat{X} = f(Y, Z) \), and \( \mathbb{E}[||X - \hat{X}||^2] \leq \Delta_X \). We show the fundamental structural properties: (1) Optimal test channel realizations that achieve the RDF, \( R(\Delta_X) \), satisfy conditional independence,

\[
P_{X|\hat{X},Y,Z} = P_{X|\hat{X},Y} = P_{X|\bar{X}}, \quad \mathbb{E}\{X|\hat{X},Y,Z\} = \mathbb{E}\{X|\bar{X}\} = \hat{X} \tag{1}
\]

and (2) similarly for the conditional RDF, \( R_{X|Y}(\Delta_X) \equiv \inf_{M(\Delta_X)} \mathbb{E}[||X - \hat{X}||^2] \leq \Delta_X \) \( I(X; \hat{X}|Y) \), when \( \{Y_t\}_{t=1}^{\infty} \) is available to both the encoder and decoder, and the equality \( R(\Delta_X) = R_{X|Y}(\Delta_X) \).

This paper also shows that the optimal test channel realization of the RDF of the distributed remote source coding problem, [2, Theorem 4] and [3, Theorem 3A] (a noisy version of Wyner’s \( R(\Delta_X) \)), when specialized to Wyner’s RDF \( R(\Delta_X) \), do not generate Wyner’s value of this RDF of scalar RVs.

I. INTRODUCTION, PROBLEM STATEMENT AND MAIN RESULTS

A. Wyner [1] and Wyner and Ziv [4] Lossy Compression Problem and Generalizations

Wyner and Ziv [4] derived an operational information definition for the lossy compression problem of Fig. 1 with respect to a single-letter fidelity of reconstruction, when the joint sequence of random variables (RVs) \( \{(X_t, Y_t) : t = 1, 2, \ldots\} \) takes values in sets of finite cardinality, \( \{\mathcal{X}, \mathcal{Y}\} \), and it is generated independently.
Fig. 1: The Wyner and Ziv [4] block diagram of lossy compression. If switch A is closed then the side information is available at both the encoder and the decoder; if switch A is open the side information is only available at the decoder.

according to the joint probability distribution function \( P_{X,Y} \). Wyner [1] generalized [4] to RVs \( \{(X_t,Y_t) : t = 1,2,\ldots\} \) that takes values in abstract alphabet spaces \( \{X,Y\} \), and hence include continuous-valued RVs.

(A) Switch A Closed. When the side information \( \{Y_t : t = 1,2,\ldots\} \) is available, noncausally, at both the encoder and the decoder, Wyner [1] (see also Berger [5]) characterized the infimum of all achievable operational rates (denoted by \( R_1(\Delta_X) \) in [1]), subject to a single-letter fidelity with average distortion less than or equal to \( \Delta_X \in [0,\infty) \), by the single-letter operational information theoretic conditional rate distortion function (RDF):

\[
R_{X|Y}(\Delta_X) \triangleq \inf_{M_0(\Delta_X)} I(X;\hat{X}|Y), \quad \Delta_X \in [0,\infty)
\]

(2)

\[
\text{where } M_0(\Delta_X) \text{ is specified by the set }
\]

(4)

and where \( \hat{X} \) is the reproduction of \( X \), \( I(X;\hat{X}|Y) \) is the conditional mutual information between \( X \) and \( \hat{X} \) conditioned on \( Y \), and \( d_X(\cdot,\cdot) \) is the fidelity criterion between \( x \) and \( \hat{x} \). The infimum in (2) is over all joint distributions \( P_{X,Y,\hat{X}} \) such that the marginal distribution \( P_{X,Y} \) is the joint distribution of the source \( (X,Y) \).

(B) Switch A Open. When the side information \( \{Y_t : t = 1,2,\ldots\} \) is available, noncausally, only at the decoder, Wyner [1] characterized the infimum of all achievable operational rates (denoted by \( R^*(\Delta_X) \) in [1]), subject to a single-letter fidelity with average distortion less than or equal to \( \Delta_X \), by the single-letter operational information theoretic RDF, as a function of an auxiliary RV \( Z : \Omega \rightarrow Z \):

\[
R(\Delta_X) \triangleq \inf_{M(\Delta_X)} \left\{ I(X;Z) - I(Y;Z) \right\}, \quad \Delta_X \in [0,\infty)
\]

(5)

\[
\text{where } M(\Delta_X) \text{ is specified by the set of auxiliary RVs } Z,
\]

(7)

\[
M(\Delta_X) \triangleq \left\{ Z : \Omega \rightarrow Z : P_{Z|X,Y} = P_{Z|X}, \exists \text{ measurable function } f : Y \times Z \rightarrow \hat{X}, \hat{X} = f(Y,Z), \mathbb{E}\{d_X(X,\hat{X})\} \leq \Delta_X \right\}.
\]
Wyner’s realization of the joint measure $P_{X,Y,Z,X}$ induced by the RVs $(X,Y,Z,\hat{X})$, is illustrated in Fig. 2 where $Z$ is the output of the “test channel”, $P_{Z|X}$. 

Special Case of Switch A Open with Causal Side Information. When the side information $\{Y_t : t = 1, 2, \ldots \}$ is causally available, only at the decoder, it follows from [1], that the infimum of all achievable operational rates, denoted by $R^*_{\text{CSI}}(\Delta_X)$, is characterized by a degenerate version of $\overline{R}(\Delta_X)$, given by

$$\overline{R}_{\text{CSI}}(\Delta_X) \triangleq \inf_{M(\Delta_X)} I(X;Z), \Delta_X \in [0, \infty).$$

Throughout [1] the following assumption is imposed.

**Assumption 1.** $I(X;Y) < \infty$ (see [1]).

For scalar-valued RVs $(X,Y,\hat{X},Z)$ with square-error distortion, Wyner [1] constructed the optimal realizations $\hat{X}$ and $\hat{X} = f(X,Z)$ that achieve the characterizations of the RDFs $R_{X|Y}(\Delta_X)$ and $\overline{R}(\Delta_X)$, respectively, and showed $\overline{R}(\Delta_X) = R_{X|Y}(\Delta_X)$.

The main objective of this paper is to generalize Wyner’s [1] optimal realizations $\hat{X}$ and $\hat{X} = f(X,Z)$ that achieve the RDFs $R_{X|Y}(\Delta_X)$ and $\overline{R}(\Delta_X)$, to multivariate-valued RVs $(X, Y, \hat{X}, Z)$, and to show $\overline{R}(\Delta_X) = R_{X|Y}(\Delta_X)$. Our main contribution lies in the derivation of structural properties of the optimal test channels that achieve the RDFs, and their realizations. Further, these structural properties are indispensable in other problems of rate distortion theory. In particular, it is verified (see Remark 5) that the optimal realization of the test channel that achieves the RDF of the remote source coding problem given in [2, Theorem 4 and Abstract] and [3, Theorem 3A], when specialized to scalar RVs, and Wyner’s RDF $\overline{R}(\Delta_X)$, do not generate Wyner’s value of the RDF $\overline{R}(\Delta_X)$ and optimal test channel realization that achieves it, contrary to the current belief, i.e., [2] and [3, Theorem 3A]. The remote sensor problem is introduced by Draper and Wornell in [6].

(C) **Marginal RDF.** If there is no side information $\{Y_t : t = 1, \ldots, \}$, or the side information is independent of the source $\{X_t : t = 1, \ldots, \}$, the RDFs $R_{X|Y}(\Delta_X), \overline{R}(\Delta_X)$ degenerate to the marginal RDF $R_X(\Delta_X)$, defined by

$$R_X(\Delta_X) \triangleq \inf_{P_{X|\Delta_X}} \mathbb{E}\{d(X,\hat{X})\} \leq \Delta_X I(X;\hat{X}), \Delta_X \in [0, \infty).$$

(D) **Gray’s Lower Bounds.** Related to the RDF $R_{X|Y}(\Delta_X)$ is Gray’s [7, Theorem 3.1] lower bound,

$$R_{X|Y}(\Delta_X) \geq R_X(\Delta_X) - I(X;Y).$$

(E) **The Draper and Wornell Distributed Remote Source Coding Problem.** Draper and Wornell [6] generalized the RDF $\overline{R}(\Delta_X)$, when the source to be estimated at the decoder is $S : \Omega \to \mathcal{S}$, and it is not directly observed at the encoder. Rather, the encoder observes a RV $X : \Omega \to \mathcal{X}$ (which is correlated with $S$), while the decoder observes another RV, as side information, $Y : \Omega \to \mathcal{Y}$, which provides information on $(S,X)$. The aim is to reconstruct $S$ at the decoder by $\hat{S} : \Omega \to \hat{\mathcal{S}}$, subject to an average distortion $\mathbb{E}\{d_S(S,\hat{S})\} \leq \Delta_S$.

1The RDF of the remote sensor problem is a generalization Wyner’s RDF $\overline{R}(\Delta_X)$, with the encoder observing a noisy version of the RVs generated by the source.

2Remark 4 implies that, for multivariate-valued Gaussian RVs, the characterization of the RDF of the remote sensor problem and the test channel that achieves it, are currently not known.
by a function \( \hat{S} = f(Y, Z) \).

The RDF for this problem, called distributed remote source coding problem, is defined by \[ R_{PO}(\Delta S) = \inf_{M^{PO}(\Delta S)} I(X; Z|Y) \] (11)

where \( M^{PO}(\Delta S) \) is specified by the set of auxiliary RVs \( Z \),

\[ M^{PO}(\Delta S) \triangleq \left\{ Z : \Omega \rightarrow Z : P_{Z|S,X,Y} = P_{Z|X}, \exists \text{ measurable function } f^{PO} : \mathcal{Y} \times \mathcal{Z} \rightarrow \hat{S}, \right. \]

\[ \hat{S} = f^{PO}(Y, Z), \quad E\{d_S(S, \hat{S})\} \leq \Delta S \} \]. (12)

It should be mentioned that if \( S = X - \text{a.s. (almost surely)} \) then \( R_{PO}(\Delta S) \) degenerates to \( R(\Delta X) \).

For scalar-valued jointly Gaussian RVs \((S, X, Y, Z, \hat{X})\) with square-error distortion, Draper and Wornell [6, eqn(3) and Appendix A] derived the characterization of the RDF \( R_{PO}(\Delta S) \), and constructed the optimal realization \( \hat{S} = f^{PO}(Y, Z) \) that achieves this characterization.

In [2] the authors investigated the RDF \( R_{PO}(\Delta S) \) for the multivariate jointly Gaussian RVs \((S, X, Y, Z, \hat{X})\), with square-error distortion, and derived a characterization for the RDF \( R_{PO}(\Delta S) \) in [2, Theorem 4] and [3, Theorem 3A] (see [3, eqn(26)]). However, as it apparent in Remark [5] when \( S = X - \text{almost surely} \), and hence \( R_{PO}(\Delta S) = R(\Delta X) \), the optimal test channel realizations that are used to derive [2, Theorem 4] and [3, Theorem 3A], when substituted into the RDF \( R(\Delta X) \), i.e., [5], do not produce Wyner’s [1] value of the RDF, test channel realization (and also do not produce the known RDF and test channel realization of memoryless sources). This observation is sufficient to raise concerns regarding the validity of the water-filling solution given in [2, Theorem 4] and [3, Theorem 3A].

**B. Problem Statement and Main Contributions**

In this paper we consider a tuple of jointly independent and identically distributed multivariate Gaussian random variables (RVs) \((X^n, Y^n) = \{(X_t, Y_t) : t = 1, 2, \ldots, n\}\), with respect to the square-error fidelity, as

\[3\text{This implies the optimal test channel that achieves the characterization of the RDF } R_{PO}(\Delta S) \text{ should degenerate to the optimal test channel that achieves the characterization of the RDF } R(\Delta X).\]
defined below.

\[ X_t : \Omega \rightarrow \mathbb{R}^{n_x} = \mathcal{X}, \quad Y_t : \Omega \rightarrow \mathbb{R}^{n_y} = \mathcal{Y}, \quad t = 1, 2, \ldots, n, \]  

\[ X_t \in \mathcal{N}(0, Q_X), \quad Y_t \in \mathcal{N}(0, Q_Y), \]  

\[ Q_{(X_t, Y_t)} = E\left\{ \begin{bmatrix} X_t \\ Y_t \end{bmatrix} \begin{bmatrix} X_t \\ Y_t \end{bmatrix}^T \right\} = \begin{bmatrix} Q_X & Q_{XY} \\ Q_{YX}^T & Q_Y \end{bmatrix}, \]  

\[ P_{X_t, Y_t} = P_{X,Y} \text{ multivariate Gaussian distribution}, \]  

\[ \hat{X}_t : \Omega \rightarrow \mathbb{R}^{n_x} = \mathcal{X}, \quad t = 1, 2, \ldots, n, \]  

\[ D_X(x^n, \hat{x}^n) = \frac{1}{n} \sum_{t=1}^{n} ||x_t - \hat{x}_t||_{2}^{2}, \]  

where \( n_x, n_y \) are arbitrary positive integers, \( I_{n_y} \) is the \( n_y \times n_y \) diagonal matrix, \( X \in \mathcal{N}(0, Q_X) \) means \( X \) is a Gaussian RV, with zero mean and covariance matrix \( Q_X \), and \( || \cdot ||_{2}^{2} \) is the Euclidean distance on \( \mathbb{R}^{n_x} \).

To give additional insight we often consider the following realization of side information:\(^4\)

\[ Y_t = CX_t + DV_t, \]  

\[ V_t \in \mathcal{N}(0, Q_V), \]  

\[ C \in \mathbb{R}^{n_y \times n_x}, \quad D \in \mathbb{R}^{n_y \times n_y}, \quad DD^T \succ 0, \quad Q_V = I_{n_y} \]  

\[ V^n \text{ independent of } X^n. \]  

For the above specification of the source and distortion criterion, we derive the following results.

1) **Theorem 1, Fig. 3** Structural properties of optimal realization of \( \hat{X} \) that achieves the RDFs, \( R_{X|Y}(\Delta_X) \), closed form expression for \( R_{X|Y}(\Delta_X) \).

2) **Theorem 2** Structural properties of optimal realization of \( \hat{X} \) and \( \hat{X} = f(Y, Z) \) that achieve the RDF, \( \overline{R}(\Delta_X) \), and closed form expressions for \( \overline{R}(\Delta_X) \).

3) A proof that \( \overline{R}(\Delta_X) \) and \( R_{X|Y}(\Delta_X) \) coincide, calculation of positive surface such that Gray’s lower bound \(^{10}\) holds with equality, and a proof that the optimal test channel realization for the remote sensor problem, that is used to derive \[^2\] Theorem 4] is incorrect (Remark 5).

In Remark 4 we consider the tuple of scalar-valued, jointly Gaussian RVs \((X, Y)\), with square error distortion function, and verify that our optimal realizations of \( \hat{X} \) and closed form expressions for \( R_{X|Y}(\Delta_X) \) and \( \overline{R}(\Delta_X) \) are identical to Wyner’s \[^{11}\] realizations and RDFs.

We emphasize that past literature often deals with the calculation of RDFs using optimization techniques, without much emphasis on the structural properties of the realizations of the test channels, that achieve the characterizations of the RDFs. Because of this, often the optimization problems appear intractable, while closed form solutions are rare. It will be indefensible to claim that solving an optimization problem of a RDF, without specifying the realization of the optimal test channel that achieves the value of the RDF, fully characterizes the RDF. As demonstrated by Wyner \[^{1}\] for a tuple of scalar jointly Gaussian RVs \((X, Y)\) with square-error distortion criterion, the identity \( \overline{R}(\Delta_X) = R_{X|Y}(\Delta_X) \) holds, because the realizations that achieve these RDFs

\[^4\]The condition \( DD^T \succ 0 \) ensures \( I(X; Y) < \infty \), and hence Assumption 1 is respected.
are explicitly constructed. Although, in the current paper the emphasis is on 1), 2) above, our derivations are
generic and bring new insight into the construction of optimal test channels for other distributed source coding
problems.

C. Additional Literature Review

The formulation of Fig. 1 is generalized to other multiterminal or distributed lossy compression problems, such
as, relay networks, sensor networks etc., under various code formulations and assumptions. Oohama [8] analyzed
the lossy compression problems for a tuple of scalar correlated Gaussian memoryless sources with square error
distortion criterion, and determined the rate-distortion region, in the special case when one source provides partial
side information to the other source. Oohama [9] analyzed the separate lossy compression problem for $L + 1$
scalar correlated Gaussian memoryless sources, when $L$ act as sources partial side information at the decoder
for the reconstruction of the remaining source, and gave a partial answer to the rate distortion region. Oohama
[9] also proved that his problem gives as a special case the additive white Gaussian CEO problem analyzed by
Viswanathan and Berger [10]. In addition, Ekrem and Ulukus [11] and Wang and Chen [12] expanded Oohama
[9] main results, by deriving an outer bound on the rate region of the vector Gaussian multiterminal source. The
vast literature on multiterminal or distributed lossy compression of jointly Gaussian sources with square-error
distortion (mentioned above), is often confined to a tuple of correlated RVs $X : \Omega \rightarrow \mathbb{R}, Y : \Omega \rightarrow \mathbb{R}$. The
above literature treats the optimization problems of RDFs without much emphasis on the structural properties
of the optimal test channels that achieve the characterizations of the RDFs.

D. Main Theorems of the Paper

The characterizations of the RDFs $R_{X|Y}(\Delta_X)$ and $\overline{R}(\Delta_X)$ are encapsulated in Theorem 1 and Theorem 2
stated below. These theorems include, structural properties of optimal test channels or realizations of $\hat{X}$, that
induce joint distributions, which achieve the RDFs, and closed form expressions of the RDFs based on a
water-filling. The realization of the optimal test channel of $R_{X|Y}(\Delta_X)$ is shown in Fig. 3.

First, we introduce some notation. We denote the covariance of $X$ and $Y$ by
\[ Q_{X,Y} \triangleq \text{cov}(X,Y). \] (23)

We denote the covariance of $X$ conditioned on $Y$ by,
\[ Q_{X|Y} \triangleq \text{cov}(X,X|Y) \]
\[ = \mathbb{E}\left\{ \left( X - \mathbb{E}(X|Y) \right) \left( X - \mathbb{E}(X|Y) \right)^T \right\} \text{ if } (X,Y) \text{ is jointly Gaussian.} \] (24)

where the second equality is due to a property of jointly Gaussian RVs.

The first theorem gives the optimal test channel that achieves the characterization of the RDF $R_{X|Y}(\Delta_X)$,
and its water-filling representation.

**Theorem 1. Characterization and water-filling solution of $R_{X|Y}(\Delta_X)$**

Consider the RDF $R_{X|Y}(\Delta_X)$ defined by (2), for the multivariate Gaussian source with mean-square error
distortion defined by (13)-(22).

Then the following hold.
(a) The optimal realization $\hat{X}$ that achieves $R_{X|Y}(\Delta_X)$ is represented by

$$
\hat{X} = HX + (I_{n_x} - H)Q_{X,Y}Q_Y^{-1}Y + W
$$

(25)

where

$$
H \triangleq I_{n_x} - \Sigma_\Delta Q_{X,Y}^{-1} = I_{n_x} - Q_{X,Y}^{-1}\Sigma_\Delta = H^T \succeq 0,
$$

(27)

$$
W = H\Psi, \quad \Psi \in N(0, Q_\Psi), \quad Q_\Psi = \Sigma_\Delta H^{-1} = H^{-1}\Sigma_\Delta.
$$

(28)

$$
Q_W \triangleq H\Sigma_\Delta = \Sigma_\Delta - \Sigma_\Delta Q_{X,Y}^{-1}\Sigma_\Delta = \Sigma_\Delta H \succeq 0,
$$

(29)

$$
\Sigma_\Delta \triangleq E\left\{ (X - \hat{X}) (X - \hat{X})^T \right\},
$$

(30)

$$
Q_{X|Y} = Q_{X} - Q_{X,Y}Q_Y^{-1}Q_{X,Y}, \quad Q_{X,Y} = Q_X C^T, \quad Q_Y = C Q_X C^T + DD^T.
$$

(31)

Moreover, the following structural properties hold:

1. The optimal test channel satisfies

   $$(i) \quad P_{X|\hat{X},Y} = P_{X|\hat{X}},$$

   (32)

   $$(ii) \quad E\{ X|\hat{X},Y \} = E\{ X|\hat{X} \} = \hat{X} \quad \Rightarrow \quad E\{ X|Y\} = E\{ \hat{X}|Y\}.$$

   (33)

2. The matrices

   \{\Sigma_\Delta, Q_{X|Y}, H, Q_W\} have spectral

   decompositions w.r.t the same unitary matrix $UU^T = I_{n_x}, U^TU = I_{n_x}.$

   (34)

(b) The RDF $R_{X|Y}(\Delta_X)$ is given by the water-filling solution:

$$
R_{X|Y}(\Delta_X) = \frac{1}{2} \log \max \left\{ 1, \det(Q_{X|Y}\Sigma_\Delta^{-1}) \right\} = \frac{1}{2} \sum_{i=1}^{n_x} \log \frac{\lambda_i}{\delta_i}
$$

(35)

where

$$
E\{ \|X - \hat{X}\|_{R^{n_x}} \} = \text{tr} (\Sigma_\Delta) = \sum_{i=1}^{n_x} \delta_i = \Delta_X,
$$

\[ \delta_i = \begin{cases} \mu, & \text{if } \mu < \lambda_i \\ \lambda_i, & \text{if } \mu \geq \lambda_i \end{cases} \]

(36)

and where $\mu \in [0, \infty)$ is a Lagrange multiplier (obtained from the Kuch-Tucker conditions), and

$$
Q_{X|Y} = U \Lambda U^T, \quad \Lambda = \text{diag} \{ \lambda_1, \ldots, \lambda_{n_x} \}, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_x}
$$

(37)

$$
\Sigma_\Delta = U \Delta U^T, \quad \Delta = \text{diag} \{ \delta_1, \ldots, \delta_{n_x} \}, \quad \delta_1 \geq \delta_2 \geq \cdots \geq \delta_{n_x}.
$$

(38)

(c) The optimal $\hat{X}$ of part (a) that achieves $R_{X|Y}(\Delta_X)$ is realized by the parallel channel scheme depicted in Fig. [5]

(d) If $X$ and $Y$ are independent or $Y$ is replaced by a RV that generates the trivial information, i.e., the $\sigma-$algebra of $Y$, is $\sigma\{ Y \} = \{ \Omega, \emptyset \}$ (or $C = 0$ in (19)), then (a)-(c) hold with $Q_{X|Y} = Q_X, Q_{X,Y} = 0$, and $R_{X|Y}(\Delta_X) = R_X(\Delta_X)$, i.e. becomes the marginal RDF of $X$.

The proof of Theorem [1] is given Section [III] and it is based on the derivation of the structural properties. Some of the implications are briefly described below.
Fig. 3: $R_{X|Y}(\Delta_X)$: A realization of optimal reproduction $\hat{X}$ over parallel additive Gaussian noise channels, where $h_i \triangleq 1 - \frac{\delta_i}{\lambda_i} \geq 0$, $i = 1, \ldots, n_x$ are the diagonal elements of the spectral decomposition of the matrix $H = U \text{diag}(h_1, \ldots, h_{n_x}) U^T$, and $W_i \in N(0, h_i \delta_i)$, $i = 1, \ldots, n_x$.

**Conclusion 1.** The construction and the structural properties of the optimal test channel $P_{X|Y, \hat{X}}$ that achieves the water-filling characterization of the RDF $R_{X|Y}(\Delta_X)$ of Theorem 1 are not documented elsewhere in the literature.

(i) Structural property (32) strengthens Gray’s [7, Theorem 3.1] (see proof of (10)), inequality,

$$I(X; \hat{X}|Y) \geq I(X; \hat{X}) - I(X; Y).$$

(39)

to the equality

$$I(X; \hat{X}|Y) = I(X; \hat{X}) - I(X; Y) \in [0, \infty) \quad \text{if} \quad P_{X|\hat{X}, Y} = P_{X|\hat{X}}$$

(40)

$$= \frac{1}{2} \log \left\{ \det(Q_{X|Y} \Sigma_\Delta^{-1}) \right\}, \quad Q_{X|Y} - \Sigma_\Delta \succeq 0, \quad E\{||X - \hat{X}||_{F^n_x}^2\} = \text{tr}(\Sigma_\Delta) \leq \Delta_X$$

(41)

Structural property (33) means the subtraction of equal quantities $E\{X|Y\}$ and $E\{\hat{X}|Y\}$ at the encoder and decoder, respectively, without affecting the information measure, see Fig. 3.

Theorem 1(b), (c), are obtained, with the aid of part (a), and Hadamard’s inequality that shows $Q_{X|Y}$ and $\Sigma_\Delta$ have the same eigenvectors.

Structural property (32) implies that Gray’s [7, Theorem 3.1] lower bound (10) holds with equality for a strictly
positive surface $\Delta_X \leq \mathcal{D}_C(X|Y) \subseteq [0, \infty)$, i.e.,

$$R_{X|Y}(\Delta_X) = R_X(\Delta_X) - I(X;Y), \quad \Delta_X \leq \mathcal{D}_C(X|Y) \triangleq \{ \Delta_X \in [0, \infty) : \Delta_X \leq n_x \lambda_{n_x} \}. \quad (42)$$

That is, the set $\mathcal{D}_C(X|Y)$ excludes values of $\Delta_X \in [0, \infty)$ for which water-filling is active in (35), (36).

(ii) Structural property (2), i.e., the matrices $\{ \Sigma_\Delta, Q_{X|Y}, H, Q_W \}$ are nonnegative symmetric, and have a spectral decomposition with respect to the same unitary matrix $UU^\dagger = I_{n_x} \quad (13)$, implies that the test channel is equivalently represented by parallel additive Gaussian noise channels (subject to pre-processing and post-processing at the encoder and decoder).

(iii) Remark 4 shows that the realization of optimal $\hat{X}$ of Fig. 3 that achieves the RDF of Theorem 1 degenerates to Wyner’s [1] optimal realization that achieves the RDF $R_{X|Y}(\Delta_X)$, for the tuple of scalar-valued, jointly Gaussian RVs $(X, Y)$, with square error distortion function.

The second theorem gives the optimal test channel that achieves the characterization of the RDF $\overline{R}(\Delta_X)$, and further states that, there is no loss of compression rate if side information is only available at the decoder. That is, although in general, $\overline{R}(\Delta_X) \geq R_{X|Y}(\Delta_X)$, an optimal reproduction $\hat{X} = f(Y,Z)$ of $X$, where $f(\cdot,\cdot)$ is linear, is constructed such that the inequality holds with equality.

**Theorem 2.** Characterization and water-filling solution of $\overline{R}(\Delta_X)$

Consider the RDF $\overline{R}(\Delta_X)$ defined by (6), for the multivariate Gaussian source with mean-square error distortion, defined by (13)-(22).

Then the following hold.

(a) The characterization of the RDF $\overline{R}(\Delta_X)$, satisfies

$$\overline{R}(\Delta_X) \geq R_{X|Y}(\Delta_X) \quad (43)$$

where $R_{X|Y}(\Delta_X)$ is given in Theorem 1(b).

(b) The optimal realization $\hat{X} = f(Y,Z)$ that achieves the lower bound in (43), i.e., $\overline{R}(\Delta_X) = R_{X|Y}(\Delta_X)$, is represented by

$$\hat{X} = f(Y,Z)$$

$$= I - H \right) Q_{X,Y} Q_{Y,1} Y + Z, \quad (44)$$

$$Z = H \left( X + H^{-1} W \right), \quad (45)$$

$$(H,Q_W) \quad \text{given by (27)-(31), and (34) holds.} \quad (46)$$

Moreover, the following structural properties hold:

(1) The optimal test channel satisfies

(i) $P_{X|\hat{X},Y,Z} = P_{X|\hat{X},Y} = P_{X|\hat{X}}, \quad (48)$

(ii) $E\{X|\hat{X},Y,Z\} = E\{X|\hat{X}\} = \hat{X} \implies E\{X|Y\} = E\{\hat{X}|Y\}. \quad (49)$

(2) Structural property (2) of Theorem 2(a) holds.

\(^5\) See Gray [7] for definition.
The proof of Theorem 2 is given in Section III and it is based on the derivation of the structural properties and Theorem 1. Some implications are discussed below.

**Conclusion 2.** The optimal reproduction \( \hat{X} = f(X,Z) \) or test channel distribution \( P_{X|\hat{X},Y,Z} \) that achieve the RDF \( R(\Delta_X) \) of Theorem 2 are not reported in the literature.

(i) From the structural property (1) of Theorem 2, i.e., \( \frac{\Delta X}{\sigma_X} \) then follows the lower bound \( R(\Delta_X) \geq R_{X|Y}(\Delta_X) \) is achieved by the realization \( \hat{X} = f(Y,Z) \) of Theorem 2(b), i.e., for a given \( Y = y \), then \( \hat{X} \) uniquely defines \( Y \).

(ii) If \( X \) is independent of \( Y \) or \( Y \) generates a trivial information, then the RDFs \( R(\Delta_X) = R_{X|Y}(\Delta_X) \) degenerate to the classical RDF of the source \( X \), i.e., \( R_X(\Delta_X) \), as expected. This is easily verified from (44), i.e., \( Q_{X,Y} = 0 \) which implies \( \hat{X} = Y \).

For scalar-valued RVs, \( X : \Omega \to \mathbb{R}, Y : \Omega \to \mathbb{R}, X \in N(0, \sigma^2_X) \), and \( X \) independent of \( Y \), then the optimal realization reduces to
\[
\hat{X} = Z = \left(1 - \frac{\Delta_X}{\sigma_X^2}\right)X + \sqrt{\left(1 - \frac{\Delta_X}{\sigma_X^2}\right)}\Delta_XW, \quad W \in N(0,1), \quad \sigma^2_X \leq \Delta_X, \quad (50)
\]
\[
Q_{\hat{X}} = Q_Z = \sigma^2_{\hat{X}} = \sigma^2_X - \Delta_X. \quad (51)
\]
as expected.

(iii) Remark 4 shows that the realization of optimal \( \hat{X} = f(Y,Z) \) that achieves the RDF \( R(\Delta_X) \) of Theorem 2 degenerates to Wyner’s [1] realization that achieves the RDF \( R(\Delta_X) \), of the tuple of scalar-valued, jointly Gaussian RVs \( (X,Y) \), with square error distortion function.

(iv) Remark 5 shows that, when specialized to Wyner’s RDF \( R(\Delta_X) \), the optimal test channel realizations that achieve the RDFs of the distributed remote source coding problems in [2, Theorem 4], does not degenerate to Wyner’s optimal test channel realization, \( (\hat{X},Z) \), that achieves the RDF \( R(\Delta_X) \), contrary to what is expected [2, Abstract].

The next corollary follows from the above two theorems.

**Corollary 1.** Characterization of \( R^{CSI}_{\Delta_X} \)

Consider the RDF \( R^{CSI}_{\Delta_X} \) defined by (8), for the multivariate Gaussian source with mean-square error distortion, defined by (27)–(22).

The optimal test channel of the RDF \( R^{CSI}_{\Delta_X} \) is induced by the realization \( Z = H\left(X + H^{-1}W\right) \), where \( (H,Q_W) \) are given by (27)–(31) of Theorem 2 and
\[
R^{CSI}_{\Delta_X} = \inf_{Q(\Delta_X)} \left\{ H(X) - H(X|Z) \right\}
= \inf_{Q(\Delta_X)} \frac{1}{2} \log \left\{ \det(Q_X Q^{-1}_X|Z) \right\}
= \inf_{Q(\Delta_X)} \frac{1}{2} \log \left\{ \det(Q_X Q^{-1}_X|Z) \right\}
\]
(52)

where \( (H,Q_W) \) are given by (27)–(31), and
\[
Q(\Delta_X) \triangleq \left\{ \Sigma_X \succeq 0 : \text{tr}(\Sigma_X) \leq \Delta_X, \quad HQ_X H^T + Q_W \preceq \Sigma_X \right\}
\]
(53)
\[
Q_{X|Z} = Q_X - HQ_X H^T H Q_X H^T + Q_W \preceq Q_X
\]
(54)
\[
Q_{X|Z} = Q_X - HQ_X H^T H Q_X H^T + Q_W \preceq Q_X
\]
(55)

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The rest of the paper is organized as follows. In Section II, we review Wyner’s operational definition of lossy compression and state a fundamental theorem on mean-square estimation that we use throughout the paper. In Section III we prove the structural properties and the two main theorems.

II. PRELIMINARIES

In this section we review the Wyner source coding problems with fidelity of Fig. 1.

We begin with the notation, which follows closely [1].

A. Notation

Let \( Z = \{ \ldots, -1, 0, 1, \ldots \} \) the set of all integers, \( \mathbb{N} = \{ 0, 1, 2, \ldots \} \) the set of natural integers, \( Z_+ = \{ 1, 2, \ldots \} \). For \( n \in Z_+ \) denote the following finite subset of the above defined set, \( Z_n = \{ 1, 2, \ldots, n \} \).

Denote the real numbers by \( \mathbb{R} \) and the set of positive and of strictly positive real numbers, respectively, by \( \mathbb{R}_+ = [0, \infty) \) and \( \mathbb{R}_{++} = (0, \infty) \). For any matrix \( A \in \mathbb{R}^{p \times m} \), \( (p, m) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \), we denote its transpose by \( A^\top \), and for \( m = p \), we denote its trace by \( \text{tr}(A) \), and by \( \text{diag}(A) \), the matrix with diagonal entries \( A_{ii}, \ i = 1, \ldots, p \), and zero elsewhere. The identity matrix with dimensions \( p \times p \) is designated as \( I_p \).

Denote an arbitrary set or space by \( U \), and the product space formed by \( n \) copies of it by \( U^n \triangleq \times_{i=1}^n U \). Denote a probability space by \((\Omega, F, \mathbb{P})\). For a sub-sigma-field \( G \subseteq F \), and \( A \in F \), denote by \( \mathbb{P}(A|G) \) the conditional probability of \( A \) given \( G \), i.e., \( \mathbb{P}(A|G) = \mathbb{P}(A|G)(\omega), \omega \in \Omega \) is a measurable function on \( \Omega \).

On the above probability space, consider two real valued random variables (RV) \( X : \Omega \to \mathcal{X}, Y : \Omega \to \mathcal{X} \), where \( (\mathcal{X}, \mathcal{B}(\mathcal{X})), (\mathcal{Y}, \mathcal{B}(\mathcal{Y})) \) are arbitrary measurable spaces. The measure (or joint distribution if \( \mathcal{X}, \mathcal{Y} \) are Euclidean spaces) induced by \( (X, Y) \) on \( \mathcal{X} \times \mathcal{Y} \) is denoted by \( \mathbb{P}_{X,Y} \) or \( \mathbb{P}(dx, dy) \) and their marginals on \( \mathcal{X} \) and \( \mathcal{Y} \) by \( \mathbb{P}_{X} \) and \( \mathbb{P}_{Y} \), respectively. The conditional measure of RV \( X \) conditioned on \( Y \) is denoted by \( \mathbb{P}_{X|Y} \) or \( \mathbb{P}(dx|y) \), when \( Y = y \) is fixed.

On the above probability space, consider three real valued RVs \( X : \Omega \to \mathcal{X}, Y : \Omega \to \mathcal{X}, Z : \Omega \to \mathcal{X} \). We say that RVs \( (Y, Z) \) are conditionally independent given RV \( X \) if \( \mathbb{P}_{Y,Z|X} = \mathbb{P}_{Y|X}\mathbb{P}_{Z|X} - \text{a.s} \) (almost surely) or equivalently \( \mathbb{P}_{Z|X,Y} = \mathbb{P}_{Z|X} - \text{a.s} \); the specification a.s is often omitted. We often denote the above conditional independence by the Markov chain (MC) \( Y \leftrightarrow X \leftrightarrow Z \).

Finally, for RVs \( X, Y \) etc. \( H(X) \) denotes differential entropy of \( X \), \( H(X|Y) \) conditional differential entropy of \( X \) given \( Y \), \( I(X;Y) \) the mutual information between \( X \) and \( Y \), as defined in standard books on information theory. [14]. [15].

The notation \( X \in \mathcal{N}(0, Q_X) \) means \( X \) is a Gaussian distributed RV with zero mean and covariance \( Q_X \geq 0 \), where \( Q_X \geq 0 \) (resp. \( Q_X > 0 \)) means \( Q_X \) is positive semidefinite (resp. positive definite).

B. Wyner’s Coding Theorems with Side Information at the Decoder

For the sake of completeness, we introduce certain results from Wyner’s paper [1], that we use in this paper. On a probability space \((\Omega, F, \mathbb{P})\), consider a tuple of jointly independent and identically distributed RVs \((X^n, Y^n) = \{ (Y_t, Y_t) : t = 1, 2, \ldots, n \} \),

\[
X_t : \Omega \to \mathcal{Y}, \quad Y_t : \Omega \to \mathcal{Y}, \quad t = 1, 2, \ldots, n
\] (56)
with induced distribution $P_{X_t,Y_t} = P_{X,Y}, \forall t$. Consider also the measurable function $d_X : \mathcal{X} \times \hat{\mathcal{X}} \rightarrow [0, \infty)$, for a measurable space $\hat{\mathcal{X}}$. Let

$$I_M \triangleq \{0, 1, \ldots, M - 1\}, \ M \in \mathbb{Z}_M.$$  \hfill (57)

be a finite set.

A code $(n, M, D_X)$, when switch $A$ is open in Fig. 1 is defined by two measurable functions, the encoder $F_E$ and the decoder $F_D$, with average distortion, as follows.

$$F_E : \mathcal{X}^n \longrightarrow I_M, \quad F_D : I_M \times \mathcal{Y}^n \longrightarrow \hat{\mathcal{X}}^n,$$

$$\frac{1}{n} \mathbb{E} \left\{ \sum_{t=1}^{n} d_X(X_t, \hat{X}_t) \right\} = D_X$$

where $\hat{\mathcal{X}}^n$ is again a sequence of RVs, $\hat{\mathcal{X}}^n = F_D(Y^n, F_E(X^n)) \in \hat{\mathcal{X}}^n$. A non-negative rate distortion pair $(R, \Delta X)$ is said to be achievable if for every $\epsilon > 0$, and $n$ sufficiently large there exists a code $(n, M, D_X)$ such that

$$M \leq 2^n (R + \epsilon), \quad D_X \leq \Delta X + \epsilon$$

Let $\mathcal{R}$ denote the set of all achievable pairs $(R, \Delta_X)$, and define, for $\Delta_X \geq 0$, the infimum of all achievable rates by

$$R^*(\Delta_X) = \inf_{(R, \Delta_X) \in \mathcal{R} \setminus \{0\}} R$$  \hfill (58)

If for some $\Delta_X$ there is no $R < \infty$ such that $(R, \Delta_X) \in \mathcal{R}$, then set $R^*(\Delta_X) = \infty$. For arbitrary abstract spaces Wyner [1] characterized the infimum of all achievable rates $R^*(\Delta_X)$, by the single-letter RDF, $\overline{R}(\Delta_X)$, given by (56), (57), in terms of an auxiliary RV $Z : \Omega \rightarrow \hat{Z}$. Wyner’s realization of the joint measure $P_{X,Y,Z,\hat{X}}$, induced by the RVs $(X, Y, Z, \hat{X})$, is illustrated in Fig. 2, where $Z$ is the output of the “test channel”, $P_{Z|X}$.

Wyner proved the following coding theorems.

**Theorem 3.** [1]

Suppose Assumption [1] holds.

(a) Converse Theorem. For any $\Delta_X \geq 0$, $R^*(\Delta_X) \geq \overline{R}(\Delta_X)$.

(b) Direct Theorem. If the conditions stated in [1] pages 64-65, (i), (ii) hold, then $R^*(\Delta_X) \leq \overline{R}(\Delta_X)$, $0 \leq \Delta_X < \infty$.

When switch $A$ is closed in Fig. 1 and the tuple of jointly independent and identically distributed RVs $(X^n, Y^n)$, is defined as in Section II-B, Wyner [1] generalized Berger’s [5] characterization of all achievable pairs $(R, \Delta_X)$, from finite alphabet spaces to abstract alphabet spaces.

A code $(n, M, D_X)$, when switch $A$ is closed in Fig. 1 is defined as in Section II-B, with the encoder $F_E$, replaced by

$$F_E : \mathcal{X}^n \times \mathcal{Y}^n \longrightarrow I_M.$$  \hfill (59)

Let $\mathcal{R}_1$ denote the set of all achievable pairs $(R, \Delta_X)$, again as defined in Section II-B. For $\Delta_X \geq 0$, define the infimum of all achievable rates by

$$\overline{R}_1(\Delta_X) = \inf_{(R, \Delta_X) \in \mathcal{R}_1} R$$  \hfill (60)
Wyner [1] characterized the infimum of all achievable rates $\overline{R}_1(\Delta_X)$, by the single-letter RDF $R_{X|Y}(\Delta_X)$ given by [2, 4]. The coding Theorems are given by Theorem 3 with $R^*(\Delta_X)$ and $\overline{R}(\Delta_X)$ replaced by $\overline{R}_1(\Delta_X)$ and $R_{X|Y}(\Delta_X)$, respectively. That is, $\overline{R}_1(\Delta_X) = R_{X|Y}(\Delta_X)$ (using Wyner’s notation [1 Appendix A]) These coding theorems generalized earlier work of Berger [5] for finite alphabet spaces.

Wyner also derived a fundamental lower bound on $R^*(\Delta_X)$ in terms of $\overline{R}_1(\Delta_X)$, as stated in the next remark.

Remark 1. Wyner [1] Remarks, page 65]

(A) For $Z \in \mathcal{M}(\Delta_X)$, $\hat{X} = f(Y; Z)$, and thus $P_{Z|X,Y} = P_{Z|X}$, then by a property of conditional mutual information and the data processing inequality:

$$I(X; Z|Y) = I(X; Z, f(Y, Z)|Y) \geq I(X; \hat{X}|Y) \geq R_{X|Y}(\Delta_X)$$  \hspace{1cm} (61)

where the last equality is defined since $\hat{X} \in \mathcal{M}_0(\Delta_X)$ (see [1] Remarks, page 65]). Moreover,

$$R^*(\Delta_X) \geq R_{X|Y}(\Delta_X).$$  \hspace{1cm} (62)

(B) Inequality (62) holds with equality, i.e., $R^*(\Delta_X) = R_{X|Y}(\Delta_X)$ if the $\hat{X} \in \mathcal{M}_0(\Delta_X)$, which achieves $I(X; \hat{X}|Y) = R_{X|Y}(\Delta_X)$ can be generated as in Fig. 2 with $I(X; Z|Y) = I(X; \hat{X}|Y)$. This occurs if and only if $I(X; Z|\hat{X}, Y) = 0$, and follows from the identity and lower bound

$$I(X; Z|Y) = I(X; Z, \hat{X}|Y) = I(X; Z|Y, \hat{X}) + I(X; \hat{X}|Y)$$ \hspace{1cm} (63)

$$\geq I(X; \hat{X}|Y)$$ \hspace{1cm} (64)

where the inequality holds with equality if and only if $I(X; Z|\hat{X}, Y) = 0$.

C. Mean-Square Estimation of Conditionally Gaussian RVs

Below, state a well-known property of conditionally Gaussian RVs, which we use in our derivations.

Proposition 1. Conditionally Gaussian RVs

Consider a pair of multivariate RVs $X = (X_1, \ldots, X_{n_x})^T : \Omega \to \mathbb{R}^{n_x}$ and $Y = (Y_1, \ldots, Y_{n_y})^T : \Omega \to \mathbb{R}^{n_y}$, $(n_x, n_y) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, defined on some probability distribution $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-$\sigma$-algebra. Assume the conditional distribution of $(X, Y)$ conditioned on $\mathcal{G}$, i.e., $P(dx, dy|\mathcal{G})$ is $\mathbb{P}$-a.s. (almost surely) Gaussian, with conditional means

$$\mu_{X|\mathcal{G}} \triangleq \mathbb{E}(X|\mathcal{G}), \quad \mu_{Y|\mathcal{G}} \triangleq \mathbb{E}(Y|\mathcal{G})$$ \hspace{1cm} (65)

and conditional covariances

$$Q_{X|\mathcal{G}} \triangleq \text{cov}(X, X|\mathcal{G}), \quad Q_{Y|\mathcal{G}} \triangleq \text{cov}(Y, Y|\mathcal{G}),$$ \hspace{1cm} (66)

$$Q_{X,Y|\mathcal{G}} \triangleq \text{cov}(X, Y|\mathcal{G}).$$ \hspace{1cm} (67)
Then, the vectors of conditional expectations \( \mu_{X|Y,G} \triangleq \mathbf{E}(X|Y,G) \) and matrices of conditional covariances \( Q_{X|Y,G} \triangleq \text{cov}(X,X|Y,G) \) are given, \( \mathbb{P} \)-a.s., by the following expressions\(^6\):

\[
\mu_{X|Y,G} = \mu_{X|G} + Q_{X,Y|G}^{-1}(Y - \mu_{Y|G}),
\]

\[
Q_{X|Y,G} \triangleq Q_{X|G} - Q_{X,Y|G}Q_{Y|G}^{-1}Q_{X,Y|G}^T.
\]

If \( G \) is the trivial information, i.e., \( G = \{ \Omega, \emptyset \} \), then \( G \) is removed from the above expressions.

Note that \( G = \{ \Omega, \emptyset \} \) then \( (68), (69) \) reduce to the well-known conditional mean and conditional covariance of \( X \) conditioned on \( Y \).

III. PROOFS OF THEOREM 1 AND THEOREM 2

In this section we derive the statements of Theorem 1 and Theorem 2. The proofs are based on several intermediate results, some of which also hold for general abstract alphabet spaces.

A. Side Information at Encoder and Decoder

We start our analysis with the following achievable lower bound on the conditional mutual information \( I(X;\hat{X}|Y) \), which appears in the definition of \( R_{X|Y}(\Delta_X) \), given by \( \text{[2]} \), that strengthen Gray’s lower bound \( \text{[10]} \), given in \( \text{[7]} \) Theorem 3.1.

**Lemma 1.** Achievable lower bound on conditional mutual information

Let \( (X,Y,\hat{X}) \) be a triple of arbitrary RVs on the abstract spaces \( X \times Y \times \hat{X} \), with distribution \( P_{X,Y,\hat{X}} \) and joint marginal the fixed distribution \( P_{X,Y} \) of \( (X,Y) \).

Then the following hold.

(a) The inequality holds:

\[
I(X;\hat{X}|Y) \geq I(X;\hat{X}) - I(X;Y)
\]

Moreover, if

\[
P_{X|\hat{X},Y} = P_{X|\hat{X}} - \text{a.s.}
\]

or equivalently \( Y \leftrightarrow \hat{X} \leftrightarrow X \) is a MC then the equality holds,

\[
I(X;\hat{X}|Y) = I(X;\hat{X}) - I(X;Y)
\]

(b) If \( Y \leftrightarrow \hat{X} \leftrightarrow X \) is a Markov chain then the equality holds

\[
R_{X|Y}(\Delta_X) = R_X(\Delta_X) - I(X;Y), \quad \Delta_X \leq \mathcal{D}_C(X|Y)
\]

for a strictly positive set \( \mathcal{D}_C(X|Y) \).

**Proof.** See Appendix V-A

The next theorem is used to derive the characterization of \( R_{X|Y}(\Delta_X) \).

\(^6\)If the inverse \( Q_{Y|G}^{-1} \) does not exists then it is replaced by the pseudo inverse \( Q_{Y|G}^\dagger \).
Theorem 4. Achievable lower bound on conditional mutual information and mean-square error estimation

(a) Let \((X, Y, \hat{X})\) be a triple of arbitrary RVs on the abstract spaces \(X \times Y \times \hat{X}\), with distribution \(P_{X,Y,\hat{X}}\) and joint marginal the fixed distribution \(P_{X,Y}\) of \((X, Y)\).

Define the conditional mean of \(X\) conditioned on \((\hat{X}, Y)\) by
\[
\bar{X}_{\text{cm}} \triangleq \mathbb{E}(X | \hat{X}, Y).
\]

(74)

Then the inequality holds:
\[
I(X; \hat{X}|Y) \geq I(X; \bar{X}_{\text{cm}}|Y)
\]
(75)

Moreover,
\[
\text{if } \bar{X}_{\text{cm}} = \hat{X} - \text{a.s} \text{ then } I(X; \hat{X}|Y) = I(X; \bar{X}_{\text{cm}}|Y). \tag{76}
\]

(b) In part (a) let \((X, Y, \hat{X})\) be a triple of arbitrary RVs on \(X \times Y \times \hat{X} = \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_x}\), \((n_x, n_y) \in \mathbb{Z}_+ \times \mathbb{Z}_+\).

For all measurable functions \((y, \hat{x}) \rightarrow g(y, \hat{x}) \in \mathbb{R}^{n_x}\) the mean-square error satisfies
\[
\mathbb{E}\left\{ \|X - g(Y, \hat{X})\|_{\mathbb{R}^{n_x}}^2 \right\} \geq \mathbb{E}\left\{ \|X - \mathbb{E}(X|Y, \hat{X})\|_{\mathbb{R}^{n_x}}^2 \right\}, \quad \forall g(\cdot).
\]

(77)

Proof. (a) By properties of conditional mutual information \([15]\) then
\[
I(X; \hat{X}|Y) \overset{(1)}{=} I(X; \hat{X}, \bar{X}_{\text{cm}}|Y) \tag{78}
\]

\[
\overset{(2)}{=} I(X; \hat{X}|\bar{X}_{\text{cm}}, Y) + I(X; \bar{X}_{\text{cm}}|Y) \tag{79}
\]

\[
\overset{(3)}{=} I(X; \bar{X}_{\text{cm}}|Y) \tag{80}
\]

where (1) is due to \(\bar{X}_{\text{cm}}\) is a function of \((Y, \hat{X})\), and a well-known property of the mutual information \([15]\), (2) is due to the chain rule of mutual information \([15]\), and (3) is due to \(I(X; \hat{X}|\bar{X}_{\text{cm}}, Y) \geq 0\). If \(\hat{X} = \bar{X}_{\text{cm}}\) - a.s, then \(I(X; \hat{X}|\bar{X}_{\text{cm}}, Y) = 0\), and hence the inequality (75) becomes an equality.

(b) The inequality (77) is well-known, due to the orthogonal projection theorem. \(\square\)

For jointly Gaussian RVs \((X, Y, \hat{X})\), in the next theorem we identify simple sufficient conditions for the lower bound of Theorem 4 to be achievable.

Theorem 5. Sufficient conditions for the lower bounds of Theorem 4 to be achievable

Consider the statement of Theorem 4 for a triple of jointly Gaussian RVs \((X, Y, \hat{X})\) on \(\mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_x}\), \((n_x, n_y) \in \mathbb{Z}_+ \times \mathbb{Z}_+\), i.e., \(P_{X,Y,\hat{X}} = P_{X,Y,\hat{X}}^G\) and joint marginal the fixed Gaussian distribution \(P_{X,Y} = P_{X,Y}^G\) of \((X, Y)\).

Suppose Conditions 1 and 2 hold:

Condition 1. \(\mathbb{E}(X | Y) = \mathbb{E}(\hat{X} | Y)\) \(\tag{81}\)

Condition 2. \(\text{cov}(X, \hat{X}|Y)\text{cov}(\hat{X}, \hat{X}|Y)^{-1} = I_{n_x}\) \(\tag{82}\)

Then
\[
\bar{X}_{\text{cm}} = \hat{X} - \text{a.s} \tag{83}\]
and the inequality (75) holds with equality, i.e., \(I(X; \hat{X}|Y) = I(X; \hat{X}^{cm}|Y)\).

**Proof.** By use of Proposition 1 [68], and letting \(Y = \hat{X}\) and \(G\) the information generated by \(Y\), then

\[
\hat{X}^{cm} \overset{a}{=} E\left(X | \hat{X}, Y\right) = E\left(X | Y\right) + \text{cov}(X, \hat{X}|Y)\text{cov}(\hat{X}, \hat{X}|Y)^{-1}\left(\hat{X} - E\left(\hat{X} | Y\right)\right) \tag{86}\]

\[
\overset{(a)}{=} \hat{X} - \text{a.s.} \tag{87}
\]

where \((a)\) is due to Conditions 1 and 2.

Now, we turn our attention to the optimization problem \(R_{X|Y}(\Delta_X)\) defined by (2), for the multivariate Gaussian source with mean-square error distortion defined by (13)-(22). In the next lemma we derive a preliminary parametrization of the optimal reproduction distribution \(P_{\hat{X}|X,Y}\) of the RDF \(R_{X|Y}(\Delta_X)\).

**Lemma 2. Preliminary parametrization of optimal reproduction distribution of \(R_{X|Y}(\Delta_X)\)**

Consider the RDF \(R_{X|Y}(\Delta_X)\) defined by (2) for the multivariate Gaussian source, i.e., \(P_{X,Y} = P_{\hat{X},Y}^G\), with mean-square error distortion defined by (13)-(22).

(a) For every joint distribution \(P_{X,Y,\hat{X}}\) there exists a jointly Gaussian distribution denoted by \(P_{\hat{X},Y}^G\), with marginal the fixed distribution \(P_{\hat{X},Y}^G\), which minimizes \(I(X; \hat{X}|Y)\) and satisfies the average distortion constraint, i.e., with \(d_X(x, \hat{x}) = ||x - \hat{x}||^2\).\(\Delta_X\)

(b) The conditional reproduction distribution \(P_{\hat{X}|X,Y} = P_{\hat{X}|X,Y}^G\) is induced by the parametric realization of \(\hat{X}\) (in terms of \(H, G, Q_W\)),

\[
\hat{X} = HX + GY + W, \quad H \in \mathbb{R}^{n_x \times n_x}, \quad G \in \mathbb{R}^{n_x \times n_y}, \quad W \in N(0, Q_W), \quad Q_W \succeq 0, \quad W \text{ independent of } (X,V). \tag{91}
\]

and \(\hat{X}\) is a Gaussian RV.

(c) \(R_{X|Y}(\Delta_X)\) is characterized by the optimization problem.

\[
R_{X|Y}(\Delta_X) \overset{a}{=} \inf_{\mathcal{M}_o^G(\Delta_X)} I(X; \hat{X}|Y), \quad \Delta_X \in [0, \infty) \tag{92}
\]

where \(\mathcal{M}_o^G(\Delta_X)\) is specified by the set

\[
\mathcal{M}_o^G(\Delta_X) \overset{a}{=} \{\hat{X} : \Omega \rightarrow \hat{X} : \overset{[88] - [91]}{\text{hold, and }} E\{||X - \hat{X}||^2_{\mathbb{R}^{n_x}}\} \leq \Delta_X\}. \tag{93}
\]

(d) If there exists \((H, G, Q_W)^*\) such that \(\hat{X}^{cm} = \hat{X} - \text{a.s}\), then a further lower bound on \(R_{X|Y}(\Delta_X)\) is achieved in the subset \(\mathcal{M}_o^{G,*}(\Delta_X) \subseteq \mathcal{M}_o^G(\Delta_X)\) defined by

\[
\mathcal{M}_o^{G,*}(\Delta_X) \overset{a}{=} \{\hat{X} : \Omega \rightarrow \hat{X} : \overset{[88] - [91]}{\text{hold, } \hat{X} = \hat{X}^{cm} - \text{a.s, } E\{||X - \hat{X}||^2_{\mathbb{R}^{n_x}}\} \leq \Delta_X\} \tag{94}
\]
and the corresponding characterization of the RDF is

\[ R_{X|Y}(\Delta_X) \triangleq \inf_{Q_{\Delta}} I(X; \hat{X}|Y), \quad \Delta_X \in [0, \infty) \]  

(95)

Proof. \((a)\) This is omitted since it is similar to the classical unconditional RDF \(R_X(\Delta_X)\) of a Gaussian message \(X \in N(0, Q_X)\). \((b)\) By \((a)\) the conditional distribution \(P_{X|X,Y}^{G}\) is such that, its conditional mean is linear in \((X,Y)\), its conditional covariance is nonrandom, i.e., constant, and for fixed \((X,Y) = (x, y)\), \(P_{X|X,Y}^{G}\) is Gaussian. Such a distribution is induced by the parametric realization \((88)\)\-(91). \((c)\) Follows from parts \((a)\) and \((b)\). \((d)\) Follows from Theorem 5 and \((77)\), by letting \(g(y, \hat{x}) = \hat{x}\). □

In the next theorem we identify the optimal triple \((H, G, Q_W)\) such that \(X_{\text{opt}} = \hat{X} - a.s\), and thus establish its existence. We also characterize the RDF by \(R_{X|Y}(\Delta_X) \triangleq \inf_{Q_{\Delta}} I(X; \hat{X}|Y)\), and construct a realization \(\hat{X}\) that achieves it.

**Theorem 6. Characterization of RDF \(R_{X|Y}(\Delta_X)\)**

Consider the RDF \(R_{X|Y}(\Delta_X)\), defined by \((2)\) for the multivariate Gaussian source with mean-square error distortion, defined by \((12)\)\-\((22)\).

The characterization of the RDF \(R_{X|Y}(\Delta_X)\) is

\[ R_{X|Y}(\Delta_X) \triangleq \inf_{Q(\Delta)} I(X; \hat{X}|Y) \]

\[ = \inf_{Q(\Delta)} \frac{1}{2} \log \left\{ \det(Q_X \Sigma_{\Delta}^{-1}) \right\} \]

(96)\-(97)

where

\[ Q(\Delta) \triangleq \left\{ \Sigma_{\Delta} : \text{tr}(\Sigma_{\Delta}) \leq \Delta_X \right\}, \]

\[ \Sigma_{\Delta} \triangleq \mathbb{E}\left\{ (X - \hat{X})(X - \hat{X})^T \right\}, \]

\[ Q_{X|Y} = Q_X - Q_{X,Y} Q_Y^{-1} Q_{Y,Y}', \quad Q_{X|Y} = \Sigma_{\Delta} \geq 0, \]

\[ Q_{X,Y} = Q_X C', \quad Q_Y = C Q_X C' + D D' \]

(98)\-(101)

and the optimal reproduction \(\hat{X}\) which achieves \(R_{X|Y}(\Delta_X)\) is

\[ \hat{X} = HX + \left(I_{n_x} - H\right)Q_{X,Y} Q_Y^{-1} Y + W \]

\[ H \triangleq I_{n_x} - \Sigma_{\Delta} Q_{X,Y}^{-1} \Sigma_{\Delta}, \quad G \triangleq \left(I_{n_x} - H\right)Q_{X,Y} Q_Y^{-1}, \]

\[ Q_W \triangleq \Sigma_{\Delta} H^T = \Sigma_{\Delta} - \Sigma_{\Delta} Q_{X,Y}^{-1} \Sigma_{\Delta} = H \Sigma_{\Delta} \geq 0. \]

(102)\-(104)

Moreover, the realization \((102)\) satisfies, almost surely,

\[ P_{X|\hat{X},Y} = P_{X|\hat{X}}, \]

\[ \mathbb{E}(X|\hat{X},Y) = \hat{X}, \]

\[ \mathbb{E}(X|Y) = \mathbb{E}(\hat{X}|Y) = Q_{X,Y} Q_Y^{-1} Y, \]

\[ \text{cov}(X, \hat{X}|Y) = \text{cov}(\hat{X}, \hat{X}|Y). \]

(105)\-(108)
Proof. See Appendix V-B.

Remark 2. Structural properties of realization of Theorem 6

For the characterization of the RDF $R_{X|Y}(\Delta_X)$ of Theorem 6 for the tuple of multivariate jointly Gaussian RVs $(X,Y)$, we can proceed one step further to show that the optimal $\hat{X}$ defined by (102)-(104) in terms of the matrices $\{\Sigma_{\Delta}, Q_{X|Y}, H, Q_W\}$, is such that

\begin{enumerate}
  \item $H = H^T \succeq 0$, \hspace{1cm} (109)
  \item $\{\Sigma_{\Delta}, \Sigma_{X|Y}, H, Q_W\}$ have spectral decompositions w.r.t the same unitary matrix $UU^T = I_{n_x}$. \hspace{1cm} (110)
\end{enumerate}

We show this in Corollary 3.

To prove the structural property of Remark 2 we use the next corollary, which is a degenerate case of [16, Lemma 2] (i.e., the structural properties of test channel of Gorbunov and Pinsker [17] nonanticipatory RDF of Markov sources).

Corollary 2. Structural properties of realization of optimal $\hat{X}$ of characterization of $R_{X|Y}(\Delta_X)$

Consider the characterization of the RDF $R_{X|Y}(\Delta_X)$ of Theorem 6.

Suppose $Q_{X|Y} \geq 0$ and $\Sigma_{\Delta} \succeq 0$ commute, that is,

$$Q_{X|Y} \Sigma_{\Delta} = \Sigma_{\Delta} Q_{X|Y}. \hspace{1cm} (111)$$

Then

\begin{enumerate}
  \item $H = I_{n_x} - \Sigma_{\Delta} Q_{X|Y}^{-1} = H^T$, $Q_W = \Sigma_{\Delta} H^T = \Sigma_{\Delta} H = H \Sigma_{\Delta} = Q_W^T \hspace{1cm} (112)$
  \item $\{\Sigma_{\Delta}, Q_{X|Y}, H, Q_W\}$ have spectral decompositions w.r.t the same unitary matrix $UU^T = I_{n_x}$, $U^T U = I_{n_x}$. \hspace{1cm} (113)
\end{enumerate}

that is, the following hold.

$$Q_{X|Y} = U \text{ diag } \{\lambda_1, \ldots, \lambda_{n_x}\} U^T, \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n_x}, \hspace{1cm} (114)$$

$$\Sigma_{\Delta} = U \text{ diag } \{\delta_1, \ldots, \delta_{n_x}\} U^T, \quad \delta_1 \geq \delta_2 \geq \ldots \geq \delta_{n_x}, \hspace{1cm} (115)$$

$$H = U \text{ diag } \{1 - \frac{\delta_1}{\lambda_1}, \ldots, 1 - \frac{\delta_{n_x}}{\lambda_{n_x}}\} U^T, \hspace{1cm} (116)$$

$$Q_W = U \text{ diag } \{(1 - \frac{\delta_1}{\lambda_1}) \delta_1, \ldots, (1 - \frac{\delta_{n_x}}{\lambda_{n_x}}) \delta_{n_x}\} U^T. \hspace{1cm} (117)$$

Proof. See Appendix V-C.

In the next corollary we re-express the realization of $\hat{X}$ which characterizes the RDF of Theorem 6 using a translation of $X$ and $\hat{X}$, by subtracting their conditional means with respect to $Y$, making use of (107). Then we apply corollary 2 to establish that the optimal matrices of the RDF $R_{X|Y}(\Delta_X)$ of Theorem 6 are such that $\{\Sigma_{\Delta}, Q_{X|Y}, H, Q_W\}$ have a spectral decomposition w.r.t the same unitary matrix $UU^T = I_{n_x}$. 

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**Corollary 3.** Equivalent characterization of $R_{X|Y}(\Delta_X)$

Consider the characterization of the RDF $R_{X|Y}(\Delta_X)$ of Theorem 6. Define the translated RVs

$$X \triangleq X - \mathbf{E}\{X|Y\} = X - Q_{X,Y}Q_Y^{-1}Y, \quad \hat{X} \triangleq \hat{X} - \mathbf{E}\{\hat{X}|Y\} = \hat{X} - Q_{X,Y}Q_Y^{-1}Y$$

(118)

where the equalities are due to (107). Let

$$Q_{X,Y} = U \text{ diag}\{\lambda_1, \ldots, \lambda_{n_x}\} U^T, \quad UU^T = I_{n_x}, \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n_x},$$

(119)

$$\hat{X} \triangleq U^T X, \quad \hat{\hat{X}} \triangleq U^T \hat{X}.$$  

(120)

Then

$$\hat{X} = HX + W,$$

(121)

$$I(X; \hat{X}|Y) = I(X; \hat{X}) = I(U^T X; U^T \hat{X}),$$

(122)

$$\mathbf{E}\|X - \hat{X}\|^2_{R_{n_x}} = \mathbf{E}\|X - \hat{X}\|^2_{R_{n_x}} = \mathbf{E}\|U^T X - U^T \hat{X}\|^2_{R_{n_x}} = \text{tr} \left( \Sigma_\Delta \right).$$

(123)

where $(H, Q_W)$ are given by (103) and (104).

Further, the characterization of the RDF $R_{X|Y}(\Delta_X)$ (97) satisfies the following equalities and inequality:

$$R_{X|Y}(\Delta_X) \triangleq \inf_{Q(\Delta_X)} I(X; \hat{X}|Y) = \inf_{Q(\Delta_X)} \frac{1}{2} \log \max \left\{ 1, \det(Q_X) \Sigma^{-1}_\Delta \right\}$$

(124)

$$= \inf_{Q(\Delta_X)} \mathbf{E}\|X - \hat{X}\|^2_{R_{n_x}} \leq \Delta_X$$

(125)

$$= \inf_{Q(\Delta_X)} \mathbf{E}\|U^T X - U^T \hat{X}\|^2_{R_{n_x}} \leq \Delta_X$$

(126)

$$\geq \inf_{Q(\Delta_X)} \mathbf{E}\|U^T X - U^T \hat{X}\|^2_{R_{n_x}} \leq \Delta_X \sum_{t=1}^{n_x} I(X_t|\hat{X}_t)$$

(127)

Moreover, the inequality (127) is achieved if $Q_{X,Y} \succeq 0$ and $\Sigma_\Delta \succeq 0$ commute, that is, if (111) holds, and

$$R_{X|Y}(\Delta_X) = \inf_{\sum_{i=1}^{n_x} \delta_i \leq \Delta_X} \frac{1}{2} \sum_{i=1}^{n_x} \log \max \left\{ 1, \frac{\lambda_i}{\delta_i} \right\}$$

(128)

where

$$\text{diag}\left\{ \mathbf{E}\left(U^T X - U^T \hat{X}\right) \left(U^T X - U^T \hat{X}\right)^T \right\} = \text{diag}\{\delta_1, \delta_2, \ldots, \delta_{n_x}\}.$$  

(129)

**Proof.** By Theorem 6 then

$$\hat{X} = HX + GY + W$$

(130)

$$= HX + \left( I - H \right) Q_{X,Y} Q_Y^{-1} Y + W$$

(131)

$$= H \left( X - Q_{X,Y} Q_Y^{-1} Y \right) + Q_{X,Y} Q_Y^{-1} Y + W$$

(132)

$$\Rightarrow \hat{X} = Q_{X,Y} Q_Y^{-1} Y = H \left( X - Q_{X,Y} Q_Y^{-1} Y \right) + W$$

(133)

$$\Rightarrow \hat{X} = HX + W.$$  

(134)
The last equation establishes \((121)\). By properties of conditional mutual information and the properties of optimal realization \(\hat{X}\) then the following equalities hold.

\[
I(X; \hat{X}|Y) = I(X - Q_{X,Y}Q_Y^{-1}Y; \hat{X} - Q_{X,Y}Q_Y^{-1}Y|Y) = I(X; \hat{X}|Y), \quad \text{by} \ (118), \ i.e., \ (107)
\]

\[
= H(\hat{X}|Y) - H(\hat{X}|Y, X)
\]

\[
= H(\hat{X}) - H(\hat{X}|Y, X), \quad \text{by \ indep. \ of \ X \ and \ Y}
\]

\[
= H(\hat{X}) - H(\hat{X}|X), \quad \text{by \ indep. \ of \ W \ and \ Y \ for \ fixed \ X}
\]

\[
= I(\hat{X}; \hat{X})
\]

\[
= I(U^TX; U^T\hat{X})
\]

\[
= I(\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_{n_x}; \hat{X}_1, \hat{X}_2, \ldots, \hat{X}_{n_x}) \geq \sum_{t=1}^{n_x} I(\hat{X}_t; \hat{X}_t), \quad \text{by \ mutual \ independence \ of \ } \hat{X}_t, t = 1, 2, \ldots, n_x
\]

Moreover, inequality \((143)\) holds with equality if \((\hat{X}_t; \hat{X}_t), t = 1, 2, \ldots, n_x\) are jointly independent.

The average distortion function is then given by

\[
E\|X - \hat{X}\|_{R_{n_x}}^2 = E\|X - \hat{X} - Q_{X,Y}Q_Y^{-1}Y + Q_{X,Y}Q_Y^{-1}Y\|_{R_{n_x}}^2
\]

\[
= E\|X - \hat{X}\|_{R_{n_x}}^2, \quad \text{by} \ (118), \ i.e., \ (107)
\]

\[
= E\|U^TX - U^T\hat{X}\|_{R_{n_x}}^2 = tr(\Sigma_{\Delta}), \quad \text{by} \ UU^T = I_{n_x}
\]

By Corollary 2 if \((111)\) holds, that is, \(Q_{X|Y} \succeq 0\) and \(\Sigma_{\Delta} \succeq 0\) satisfy \(Q_{X|Y} \Sigma_{\Delta} = \Sigma_{\Delta} Q_{X|Y}\) (i.e., commute), then \((112)-(117)\) hold, and by \((128)\), then

\[
\hat{X} \overset{\Delta}{=} U^TX = U^THX + U^TW = U^T\hat{X} = U^THUU^TX + U^TW
\]

\[
= U^THU\hat{X} + U^TW, \quad U^THU \text{ is diagonal and } U^TW \text{ has indep. components}
\]

Hence, if \((111)\) holds then the lower bound in \((143)\) holds with equality, because \((\hat{X}_t; \hat{X}_t), t = 1, 2, \ldots, n_x\) are jointly independent. Moreover, if \((111)\) holds then from, say, \((124)\), the expressions \((128), (129)\) are obtained.

The above equations establish all claims.

**Proposition 2.** Theorem 7 is correct.

**Proof.** By invoking Corollary 3 Theorem 5 and the convexity of \(R_{X|Y}(\Delta_X)\) given by \((128)\), then we arrive at the statements of Theorem 7 which completely characterize the RDF \(R_{X|Y}(\Delta_X)\), and constructs a realization of the optimal \(\hat{X}\) that achieves it.

Next, we discuss the degenerate case, when the statements of Theorem 6 and Theorem 7 reduce to the RDF \(R_X(\Delta_X)\) of a Gaussian RV \(X\) with square-error distortion function. We illustrate that, the identified structural property of the realization matrices \(\{\Sigma_{\Delta}, Q_{X|Y}, H, Q_W\}\) leads to the well-known water-filling solution.
Remark 3. Degenerate case of Theorem 6

Consider the characterization of the RDF \( R_{X|Y}(\Delta X) \) of Theorem 6 and assume \( X \) and \( Y \) are independent or \( Y \) generates the trivial information, i.e., the \( \sigma \)-algebra of \( Y \), is \( \sigma\{Y\} = \{\Omega, \emptyset\} \) or \( C = 0 \) in (19)-(22).

(a) By the definitions of \( Q_{X,Y}, Q_{X|Y} \)

\[
Q_{X,Y} = 0, \quad Q_{X|Y} = Q_X.
\]

Substituting (149) into the expressions of Theorem 6, then the RDF \( R_{X|Y}(\Delta X) \) reduces to

\[
R_{X|Y}(\Delta X) = R_X(\Delta X) \triangleq \inf_{Q(\Delta X)} I(X; \hat{X})
\]

\[
= \inf_{Q^m(\Delta X)} \frac{1}{2} \log \left\{ \det(Q_X \Sigma^{-1}) \right\}
\]

where

\[
Q^m(\Delta X) \triangleq \left\{ \Sigma_\Delta : \text{tr}(\Sigma_\Delta) \leq \Delta_X \right\}
\]

and the optimal reproduction \( \hat{X} \) reduces to

\[
\hat{X} = (I_{n_x} - \Sigma_\Delta Q^{-1}_{X}) X + W, \quad Q_X \succeq \Sigma_\Delta.
\]

(153)

\[
Q_W = (I_{n_x} - \Sigma_\Delta Q^{-1}_{X}) \Sigma_\Delta \succeq 0.
\]

(154)

Thus, \( R_X(\Delta X) \) is the well-known RDF of a multivariate memoryless Gaussian RV \( X \) with square-error distortion.

(b) For the RDF \( R_X(\Delta X) \) of part (a), it is known [18] that \( \Sigma_\Delta \) and \( Q_X \) have a spectral decomposition with respect to the same unitary matrix, that is,

\[
Q_X = U \Lambda_X U^T, \quad \Sigma_\Delta = U \Delta U^T, \quad UU^T = I
\]

(155)

\[
\Lambda_X = \text{diag}\{\lambda_{X,1}, \ldots, \lambda_{X,n_x}\}, \quad \Delta = \text{diag}\{\delta_1, \ldots, \delta_{n_x}\}
\]

(156)

where the entries of \( (\Lambda_X, \Delta) \) are in decreasing order.

Define

\[
\chi^p \triangleq U^T X, \quad \hat{\chi}^p \triangleq U^T \hat{X}, \quad W^p \triangleq U^T W.
\]

(157)

Then a parallel channel realization of the optimal reproduction \( \hat{\chi}^p \) is obtained given by,

\[
\hat{\chi}^p = HX^p + W^p,
\]

(158)

\[
H = I_{n_x} - \Delta \Lambda_X^{-1} = \text{diag}\{1 - \frac{\delta_1}{\lambda_{X,1}}, \ldots, 1 - \frac{\delta_{n_x}}{\lambda_{X,n_x}}\},
\]

(159)

\[
Q_{W^p} = H \Delta = \text{diag}\{1 - \frac{\delta_1}{\lambda_{X,1}}\delta_1, \ldots, 1 - \frac{\delta_{n_x}}{\lambda_{X,n_x}}\delta_{n_x}\}.
\]

(160)

The RDF \( R_X(\Delta X) \) is then computed from the reverse water-filling equations, as follows.

\[
R_X(\Delta X) = \frac{1}{2} \sum_{i=1}^{n_x} \log \frac{\lambda_{X,i}}{\delta_i}
\]

(161)

where

\[
\sum_{i=1}^{n_x} \delta_i = \Delta_X, \quad \delta_i = \begin{cases} \mu, & \text{if } \mu < \lambda_{X,i} \\ \sigma_i, & \text{if } \mu \geq \lambda_{X,i} \end{cases}
\]

(162)

and where \( \mu \in [0, \infty) \) is a Lagrange multiplier (obtained from the Kuch-Tucker conditions).
B. Side Information only at Decoder

In general, when the side information is available only at the decoder the achievable operational rate $R^*(\Delta_X)$ is greater than the achievable operational rate $\overline{R}_1(\Delta_X)$, when the side information is available to the encoder and the decoder [1]. By Remark [1] $\overline{R}(\Delta_X) \geq R_{X|Y}(\Delta_X)$, and equality holds if $I(X;Z|\hat{X},Y) = 0$.

In view of the characterization of $R_{X|Y}(\Delta_X)$ and the realization of the optimal reproduction $\hat{X}$ of Theorem [1] which is presented in Fig. [3] we observe that we can re-write (25) as follows.

$$
\hat{X} = \left( I_{n_x} - \Sigma_\Delta Q_{X|Y}^{-1} \right) X + \Sigma_\Delta Q_{X|Y}^{-1} Q_{XY} Q_Y^{-1} Y + W, \quad (163)
$$

$$
= \Sigma_\Delta Q_{X|Y}^{-1} Q_{XY} Q_Y^{-1} Y + Z \quad (164)
$$

$$
= f(Y, Z) \quad (165)
$$

$$
Z = \left( I_{n_x} - \Sigma_\Delta Q_{X|Y}^{-1} \right) \left( X + \left( I_{n_x} - \Sigma_\Delta Q_{X|Y}^{-1} \right)^{-1} W \right), \quad (166)
$$

$$
H = I_{n_x} - \Sigma_\Delta Q_{X|Y}^{-1}, \quad Q_W = H \Sigma_\Delta, \quad \text{defined by (27)-(34)}. \quad (167)
$$

$$
P_{Y|X} = P_{Z|X}, \quad (\hat{X}, Y) \quad \text{uniquely defined} \ Z, \quad \text{which implies} \ I(X;Z|\hat{X},Y) = 0. \quad (168)
$$

The realization $\hat{X} = f(Y, Z)$ is shown in Fig. [3]

**Proposition 3. Theorem [2] is correct.**

**Proof.** From the above realization of $\hat{X} = f(Y, Z)$, we have the following. (a) By Wyner, see Remark [1] then the inequalities (61) and (62) hold, and equalities holds if $I(X;Z|\hat{X},Y) = 0$. That is, for any $\hat{X} = f(Y, Z)$, by properties of conditional mutual information then

$$
I(X;Z|Y) \overset{(a)}{=} I(X;Z,\hat{X}|Y) \quad (169)
$$

$$
\overset{(b)}{=} I(X;Z|\hat{X},Y) + I(X;\hat{X}|Y) \quad (170)
$$

$$
\overset{(c)}{=} I(X;\hat{X}|Y) \quad (171)
$$

where (a) is due to $\hat{X} = f(Y, Z)$, (b) is due to the chain rule of mutual information, and (c) is due to $I(X;Z|\hat{X},Y) \geq 0$. Hence, (43) is obtained (as as in Wyner [1] for a tuple of scalar jointly Gaussian RVs).

(b) Equality holds in (171) if there exists an $\hat{X} = f(Y, Z)$ such that $I(X;Z|\hat{X},Y) = 0$, and the average distortion is satisfied. Taking $\hat{X} = f(Y, Z) = (I_{n_x} - H)Q_{XY} Q_Y^{-1} Y + Z$, where $Z = g(X, W)$ is specified by (163)-(167), then $I(X;Z|\hat{X},Y) = 0$ and the average distortion is satisfied. Since the realization (163)-(167) is identical to the realization (41)-(47), then part (b) is also shown. (c) This follows directly from the optimal realization.

**Remark 4. Relation to Wyner’s [1] optimal test channel realizations**

Now, we verify that our optimal realizations of $\hat{X}$ and closed form expressions for $R_{X|Y}(\Delta_X)$ and $\overline{R}(\Delta_X)$
(a) RDF $R_{X|Y}(\Delta X)$: Wyner’s $[1]$ optimal realization of $\hat{X}$ for RDF $R_{X|Y}(\Delta X)$ of (172)-(175).

(b) RDF $\overline{R}(\Delta X)$: Wyner’s $[1]$ optimal realization $\hat{X} = f(X, Z)$ for RDF $\overline{R}(\Delta X)$ of (172)-(175).

Fig. 4: Wyner’s realizations of optimal reproductions for RDFs $R_{X|Y}(\Delta X)$ and $\overline{R}(\Delta X)$

are identical to Wyner’s $[1]$ realizations and RDFs (see Fig. 4), for the tuple of scalar-valued, jointly Gaussian RVs $(X, Y)$, with square error distortion function,

$$d_X(x, \hat{x}) = (x - \hat{x})^2,$$

$$Q_X = \sigma_X^2, \quad Q_{X,Y} = \alpha^2 \sigma_X^2, \quad Q_Y = \alpha^2 \sigma_Y^2,$$

$$H = 1 - \Delta X Q_X^{-1} X_{X|Y}^1 = \frac{c}{\alpha} \sigma_Y^2, \quad H Q_{X,Y} Q_Y^{-1} = \frac{c}{\alpha}, \quad \sigma_{\overline{R}}^2 = \sigma_{\overline{R}}^2,$$

Moreover, by Theorem $[1]$ (b) the optimal reproduction $\hat{X} \in M_0(d)$ and $R_{X|Y}(d)$ are,

$$\hat{X} = a(X - \frac{c}{a} Y) + \frac{c}{a} Y + a\Psi, \quad \sigma_{\overline{R}}^2 = \Delta X > 0,$$

This shows our realization of Fig. 3 degenerates to Wyner’s $[1]$ realization of Fig. 4a.

(b) RDF $\overline{R}(\Delta X)$. Now, we show that our realization of optimal $\hat{X} = f(Y, Z)$ that achieves the RDF $\overline{R}(\Delta X)$ of Theorem $[1]$ degenerates to Wyner’s $[1]$ realization that achieves the RDF $\overline{R}(\Delta X)$, of the tuple of scalar-valued,
jointly Gaussian RVs \((X, Y)\), with square error distortion function given by (172)-(175). This is verified below. By Theorem 2.(b) applied to (172)-(175), and using the calculations (176)-(179), then

\[
\hat{X} = f(Y, Z) = \frac{c}{\alpha}(1 - a)Y + Z \quad \text{by (179), (182),}
\]

(181)

\[
Z = a(X + \Psi), \quad (a, \Psi) \text{ defined in (177), (178)}
\]

(182)

\[
R(\Delta_X) = R_{X|Y}(\Delta_X) = \text{by evaluating } I(X; Z) - I(Y; Z), \text{ i.e., using (5) and (182).}
\]

(183)

This shows our value of \(R(\Delta_X)\) and optimal realization \(\hat{X} = f(Y, Z)\), reproduce Wyner’s optimal realization and value of \(R(\Delta_X)\) given in [1] (i.e., Fig. 4b).

Remark 5. On the optimal test channel realization of distributed source coding problem [2] and [3]

We show that contrary to the claim in [2, Abstract] and [3, Theorem 3A], the optimal test channels used in the derivations of the RDF for the distributed remote source coding problem are incorrect, and do not produce Wyner’s value of the RDF \(R(\Delta_X)\), and the optimal test channel that achieves it (i.e. the solution presented in Remark 4).

(a) Tian and Chen [2] considered the following formulation\(^7\) of (11), (12):

\[
R^{PO,1}_{X} (\Delta_S) = \inf_{Z; P_{Z|X,Y,S} = P_{Z|X}, \hat{S} = E\{S|Z,Y\}, \mathbb{E}\{||S - \hat{S}||^2\} \leq \Delta_S} I(X; Z|Y).
\]

(184)

For multivariate correlated jointly Gaussian RVs \((S, X, Y, Z, \hat{S})\), with square-error distortion function \(d_S(s, \hat{s}) = ||s - \hat{s}||^2\), the RDF \(R^{PO,1}_{X} (\Delta_S)\) is given in [2, Theorem 4].

Clearly,

(i) if \(S = X - a.s\) (almost surely) then the RDF \(R^{PO,1}_{X} (\Delta_S)\) degenerates to Wyner’s RDF \(R(\Delta_X)\), and

(ii) if \(S = X - a.s\) and the RV \(X\) is independent of the RV \(Y\) or \(Y\) generates a trivial information, then the RDF \(R^{PO,1}_{X} (\Delta_S)\) degenerates to the classical RDF of the source \(X\), i.e., \(R_X(\Delta_X)\), as verified from (44)-(47), i.e., \(Q_{X,Y} = 0\) which implies \(\hat{X} = Z\).

We examine (i), i.e., under the restriction \(S = X - a.s.,\) by recalling the optimal realization of RVs \((Z, \hat{X})\) used in the derivation of [2, Theorem 4].

The derivation of [2, Theorem 4], uses the following RVs (see [2, eqn(4)]) adopted to our notation:

\[
X = K_{xy}Y + N_1,
\]

(185)

\[
S = K_{sx}X + K_{sy}Y + N_2
\]

(186)

\[
S = K_{sx}K_{xy}Y + N_1 + K_{sy}Y + N_2,
\]

(187)

\[
= \left(K_{sx}K_{xy} + K_{sy}\right)Y + K_{sx}N_1 + N_2
\]

(188)

where \(N_1\) and \(N_2\) are independent Gaussian RVs with zero mean, \(N_1\) is independent \(Y\) and \(N_2\) is independent of \((X, Y)\).

\(^7\)In the notation of [2] the RVs \((S, X, Y, Z, \hat{S})\) are represented by \((X, Y, Z, W, \tilde{X})\).
The condition \( X = S - a.s. \) implies,

\[
K_{sx}K_{xy} + K_{sy} = K_{xy}, \quad K_{sx} = I, \quad N_2 = 0 - a.s. \tag{189}
\]

\[
\implies K_{xy} + K_{sy} = K_{xy} \implies K_{sy} = 0. \tag{190}
\]

The optimal realization of the auxiliary random variable \( Z \) used to achieve the RDF in the derivation of [2] Theorem 4 [see [2] 3 lines above eqn(32)] using our notation is

\[
Z = UK_{sx}X + N_3
\]

\[
= UX + N_3, \quad \text{by (189)} \tag{191}
\]

where \( U \) is a unitary matrix and \( N_3 \in N(0, Q_{N_3}) \), i.e., Gaussian, such that \( Q_{N_3} \) is a diagonal covariance matrix, with diagonal elements given by

\[
\sigma_{3,i}^2 = \frac{\min(\lambda_i, \delta_i)}{\lambda_i - \min(\lambda_i, \delta_i)} \tag{193}
\]

For scalar-valued RVs \([192], [193]\), reduce to

\[
Z = X + N_3, \quad N_3 \in N\left(0, \frac{\Delta X}{\sigma_{X|Y}^2 - \Delta X}\right) \tag{194}
\]

It is easy to verify, by letting \((X, Y)\) as in \([174], [175]\), that the realization of the auxiliary RV \( Z \) given by \([194]\) is different from Wyner’s auxiliary RV \( Z \) given by \([182]\), and gives a value of \( I(X; Z) - I(Y; Z) \), which also different from Wyner’s value of the RDF \( \mathcal{R}(\Delta X) \), i.e., \([180]\). In particular, if \( \sigma_{X|Y}^2 = \Delta X \) then it should be \( Z = 0 - a.s \) almost surely (as verified from the realization of \( Z \) given by \([182]\), which reduces to \( Z = 0 - a.s \) almost surely, if \( Q_{X|Y} = \Delta X \), i.e., the value of parameter \( H \) is \( H = a = 0 \)), but instead the variance of \( Z \) takes the value \(+\infty\).

We also examine (ii) (above), i.e., setting \( S = X - a.s \), and taking \( X \) to be independent of \( Y \) or \( Y \) generates a trivial information. Clearly, RDFs \( \mathcal{R}(\Delta S) \) degenerates to the classical RDF of the source \( X \), i.e., \( \mathcal{R}(\Delta X) \), as it is verified from \([44]-[47]\), i.e., \( Q_{X,Y} = 0 \) which implies \( \tilde{X} = Z \). For scalar-valued RVs the optimal reproduction \( \tilde{X} = Z \) degenerates to \([50], [51]\). On the other hand, \([194]\) does not reduce to \([50], [51]\), and moreover the variance of \( Z \) defined by \([194]\) is \( \sigma_Z^2 = \sigma_X^2 + \frac{\Delta X}{\sigma_{X|Y}^2 - \Delta X} \), and this is fundamentally different from the variance \( Q_{\tilde{X}} = \sigma_{\tilde{X}}^2 = \sigma_Z^2 - \Delta X \) of \([51]\).

(b) Similarly to part (a) above, if we repeat the above steps, under the condition \( S = X - a.s \), the RDF of the remote sensor problem analyzed in \([3]\) reduces to Wyner’s RDF \( \mathcal{R}(\Delta X) \). Moreover, optimal realization of the auxiliary RV \( Z \), which is used to achieve the RDF in the derivation of \([3] \) Theorem 3A [see \([3]\) eqn(26)] using our notation) reduces to

\[
Z = UX + \nu \tag{195}
\]

where \( U \) is a unitary matrix and \( \nu \in N(0, Q_{\nu}) \) is a zero mean Gaussian vector with independent components, with variances across the diagonal of \( Q_{\nu} \) given by

\[
\sigma_{\nu_i}^2 = \frac{\lambda_i \min(\lambda_i, \delta_i)}{\lambda_i - \min(\lambda_i, \delta_i)}. \tag{196}
\]
For scalar-valued RVs \((195), (196)\), reduce to
\[
Z = X + \nu, \quad \nu \in N\left(0, \frac{\sigma^2_{X|Y}}{\sigma^2_X - \Delta_X} \Delta_X\right)
\]  
(197)

Clearly, by letting \((X,Y)\) as in \((174), (175)\), the auxiliary RV \(Z\) given by \((197)\) is different from Wyner’s auxiliary RV \(Z\) given by \((182)\), and does not produce Wyner’s value \(R(\Delta_X) = I(X; Z) - I(Y; Z)\) given by \((183)\). In particular, if \(\sigma^2_{X|Y} = \Delta_X\) then it should be \(Z = 0\)–almost surely (as verified from the realization of \(Z\) given by \((182)\), which reduces to \(Z = 0\)–almost surely, if \(Q_{X|Y} = \Delta_X\), i.e., the value of parameter \(H\) is \(H = a = 0\)), but instead the variance of \(Z\) takes the value \(+\infty\).

It is also noted that the variance of the auxiliary RV \(Z\) of \((2)\) given by \((194)\) is different from the variance of the auxiliary RV of \((3)\) given by \((197)\), although both are designed to achieve the same value of \(I(X; Z) - I(Y; Z)\).

**Remark 6.** On the realization of test channels

(a) It should be mentioned that unless a realization of \(\hat{X}\) is identified that achieves the RDFs \(R_{X|Y}(\Delta_X)\) and \(\overline{R}(\Delta_X)\), such that the joint distribution \(P_{X,Y,\hat{X}}\) has marginal the fixed source distribution \(P_{X,Y}\), then the characterization of the RDFs is incomplete.

(b) Corollary \(7\) follows from the two main theorems, and its complete solution is generated similarly to the two main theorems.

**IV. Conclusion**

We derived structural properties of optimal test channels realizations that achieve the characterizations of RDFs for a tuple of multivariate jointly independent and identically distributed Gaussian random variables with mean-square error fidelity, when side information is available to the decoder and not to the encoder, and when side information is available to both. We derived achievable lower bounds on conditional mutual information, and applied properties of mean-square error estimation to identify structural properties of optimal test channels that achieve these bounds. We also applied the structural properties of optimal test channels to construct realizations of optimal reproductions.

**V. Appendix**

A. Proof of Lemma \(7\)

(a) By the chain rule of mutual information then
\[
I(X; \hat{X}, Y) = I(X; Y | \hat{X}) + I(X; \hat{X})
\]
(198)
\[
= I(X; \hat{X} | Y) + I(X; Y)
\]
(199)

Since \(I(X; Y | \hat{X}) \geq 0\) then from above it follows
\[
I(X; \hat{X}) \leq I(X; \hat{X} | Y) + I(X; Y)
\]
(200)
\[
I(X; \hat{X} | Y) \geq I(X; \hat{X}) - I(X; Y)
\]
(201)
The above shows (70). To show equality, we note the following,
\[
I(X; \hat{X}|Y) = E\left[\log \frac{P_{X|Y}}{\hat{P}_{X|Y}}\right] = E\left[\log \frac{P_{X|Y}}{P_X} \cdot \frac{P_X}{\hat{P}_X|Y}\right] = E\left[\log \frac{P_{X|Y}}{P_X} - \log \frac{P_X}{\hat{P}_X|Y}\right], \text{ if } P_{X|Y,Y} = P_X|\hat{X}.
\]

This completes the statement of equality of (70), i.e., it establishes equality (72). (b) Consider a test channel \(P_{X|\hat{X},Y}\) such that \(E[\|X - \hat{X}\|^2] \leq \Delta_X\), i.e., \(\hat{X} \in M(\Delta_X)\), and such that \(P_{X|\hat{X},Y} = P_{X|\hat{X}}\), for \(\Delta_X \leq D_C(X|Y) \subseteq [0, \infty)\). By (72) taking the infimum of both sides over \(\hat{X} \in M(\Delta_X)\) such that \(P_{X|\hat{X},Y} = P_{X|\hat{X}}\) then (73) is obtained, for a nontrivial surface \(\Delta_X \leq D_C(X|Y)\), which exists due to continuity and convexity of \(R_X(\Delta_X)\) for \(\Delta_X \in (0, \infty)\). This completes the proof.

B. Proof of Theorem 6

We identify the triple \((H, G, Q_{W})\) that satisfied Conditions 1 and 2 of Theorem 5 which then implies \(\hat{X} = X^{mse}\), from which the claimed statements follow. Consider the realization given by (88).

**Condition 1**, i.e., (81). The left hand side part of (81) is given by (this follows from mean-square estimation theory, or an application of (68) with \(\mathcal{G} = \{\Omega, \emptyset\}\))
\[
E(X|Y) = E(X) + \text{cov}(X, Y)Y^{-1}\left(Y - E(Y)\right) \tag{202}
\]
\[
= \text{cov}(X, Y)Y^{-1}Y = Q_{X,Y}Y^{-1}Y = Q_X C^* Q_Y^{-1}Y \quad \text{by model (19)-(22)} \tag{203}
\]

Similarly, the right hand side of (81) is given by
\[
E(\hat{X}|Y) = E(\hat{X}) + \text{cov}(\hat{X}, Y)Y^{-1}\left(Y - E(Y)\right) \tag{205}
\]
\[
= \text{cov}(\hat{X}, Y)Y^{-1}Y = \left(HQ_{X,Y} + GQ_Y\right)Q_Y^{-1}Y \tag{206}
\]
\[
= \left(HQ_X C^* + GQ_Y\right)Q_Y^{-1}Y \quad \text{by (19)-(22)} \tag{207}
\]

Equating (203) and (206) or (207) then
\[
E(X|Y) = E(\hat{X}|Y) \tag{208}
\]

\[
\implies Q_{X,Y}Q_Y^{-1}Y = \left(HQ_{X,Y} + GQ_Y\right)Q_Y^{-1}Y \quad \text{by (206)} \tag{209}
\]

\[
\implies Q_X C^* Q_Y^{-1}Y = \left(HQ_X C^* + GQ_Y\right)Q_Y^{-1}Y \quad \text{by (19)-(22), (207)} \tag{210}
\]

\[
\implies G = \left(I - H\right)Q_X C^* Q_Y^{-1} \tag{211}
\]

\[
\implies G = \left(I - H\right)Q_X Y Q_Y^{-1} \tag{212}
\]
Hence, $G$ is obtained, and the reproduction is represented by

$$\hat{X} = HX + (I - H)Q_{X,Y}Q_{Y}^{-1}Y + W,$$

(213)

$$\text{cov}(\hat{X}, Y) = Q_{X,Y}, \quad \mathbf{E}(\hat{X} \mid Y) = Q_{X,Y}Q_{Y}^{-1}Y = \mathbf{E}(X \mid Y),$$

(214)

$$\hat{X} - \mathbf{E}(\hat{X} \mid Y) = HX - HQ_{X,Y}Q_{Y}^{-1}Y + W.$$  

(215)

**Condition 2, i.e., (82).** To apply (82) the following calculations are needed.

$$Q_{X|Y} \triangleq \text{cov}(X, X|Y)$$

(216)

$$= \mathbf{E}\left\{ \left( X - \mathbf{E}(X \mid Y) \right) \left( X - \mathbf{E}(X \mid Y) \right)^\top \right\}$$

$$= Q_X - Q_{X,Y}Q_{Y}^{-1}Q_{X,Y}$$

(217)

$$= Q_X - Q_X C^\top Q_{Y}^{-1}CQ_X \quad \text{by (19)-(22)}$$

(218)

$$\text{cov}(X, \hat{X} \mid Y) \triangleq \mathbf{E}\left\{ \left( X - \mathbf{E}(X \mid Y) \right) \left( \hat{X} - \mathbf{E}(\hat{X} \mid Y) \right)^\top \right\}$$

(219)

$$= \mathbf{E}\left\{ \left( X - \mathbf{E}(X \mid Y) \right) \left( \hat{X} - \mathbf{E}(\hat{X} \mid Y) \right)^\top \right\} \quad \text{by (214)}$$

$$= \mathbf{E}\left\{ \left( X - \mathbf{E}(X \mid Y) \right) \left( \hat{X}^\top \right)^\top \right\} \quad \text{by orthogonality}$$

(220)

$$= Q_X H^\top - Q_{X,Y}Q_{Y}^{-1}Q_{Y,X}H^\top \quad \text{by (213), (214)}$$

(221)

$$= Q_X H^\top - Q_X C^\top Q_{Y}^{-1}CQ_X H^\top \quad \text{by (19)-(22)}$$

(222)

$$= \left( Q_X - Q_X C^\top Q_{Y}^{-1}CQ_X \right) H^\top$$

$$= Q_{X|Y} H^\top. \quad \text{(223)}$$

$$\text{cov}(\hat{X}, \hat{X} \mid Y) \triangleq \mathbf{E}\left\{ \left( \hat{X} - \mathbf{E}(\hat{X} \mid Y) \right) \left( \hat{X} - \mathbf{E}(\hat{X} \mid Y) \right)^\top \right\}$$

(224)

$$= HQ_X H^\top + Q_W - HQ_{X,Y}Q_{Y}^{-1}Q_{Y,X}H^\top \quad \text{by (215)}$$

$$= HQ_X H^\top + Q_W - HQ_{X,Y}C^\top Q_{Y}^{-1}CQ_X H^\top \quad \text{by (19)-(22)}$$

(225)

$$= H \left( Q_X - Q_X C^\top Q_{Y}^{-1}CQ_X \right) H^\top + Q_W$$

$$= HQ_{X|Y} H^\top + Q_W. \quad \text{(226)}$$

By Condition 2 and (223) and (226) then

$$\text{cov}(X, \hat{X} \mid Y) \text{cov}(\hat{X}, \hat{X} \mid Y)^{-1} = I_{n_x}$$

(227)

$$\implies Q_{X|Y} H^\top \left( HQ_{X|Y} H^\top + Q_W \right)^{-1} = I_{n_x}$$

$$\implies Q_W = Q_{X|Y} H^\top - H\Sigma_{X|Y} H^\top$$

(228)

$$\implies Q_W = \left( I_{n_x} - H \right) Q_{X|Y} H^\top. \quad \text{(229)}$$
Now, we determine \( H \) as follows.

\[
\Sigma_\Delta \triangleq \text{cov}(X, X|Y \hat{X}) \\
= \text{cov}(X, X|Y) - \text{cov}(X, \hat{X}|Y)\text{cov}(\hat{X}, \hat{X}|Y)^{-1}\text{cov}(X, \hat{X}|Y)^\top, \quad \text{by prop. } \[68\] (230)
\]

\[
= \text{cov}(X, X|Y) - \text{cov}(X, \hat{X}|Y)^\top, \quad \text{by } (82) (231)
\]

\[
= Q_{X|Y} - HQ_{X|Y}, \quad \text{by } (223) (232)
\]

\[
\Rightarrow HQ_{X|Y} = Q_{X|Y} - \Sigma_\Delta (233)
\]

\[
\Rightarrow H = I - \Sigma_\Delta Q_{X|Y}^{-1} (234)
\]

Hence, \( H \) is obtained. Moreover, \( Q_W \) is obtained by substituting \( (235) \) into \( (229) \). From the above specification of parameters \((H, G, Q_W)\) then the realization \( (102)-(104) \) follows. From the realization \( (102)-(104) \) it then follows the property \( P_{X|\hat{X},Y} = P_{X|\hat{X}} - as \). Moreover, \( (96)-(100) \) are obtained from the realization.

### C. Proof of Corollary 2

(a) This part is a special case of a related statement in \([16]\). However, we include it for completeness. By linear algebra \([13]\), given two matrices \( A \in S^{k \times k}_+, B \in S^{k \times k}_+ \), the following statements are equivalent: (1) \( AB \) is normal, (2) \( AB \succeq 0 \), where \( AB \) normal means \( (AB)(AB)^\top = (AB)^\top(AB) \). Note that \( AB \) is normal if and only if \( AB = BA \), i.e., commute. Let \( A = U_A D_A U_A^\top, B = U_B D_B U_B^\top, U_A U_A^\top = I_k, U_B U_B^\top = I_k \), i.e., there exists a spectral representation of \( A, B \) in terms of unitary matrices \( U_A, U_B \) and diagonal matrices \( D_A, D_B \). Then, \( AB \succeq 0 \) if and only if the matrices \( A \) and \( B \) commute, i.e., \( AB = BA \), and \( A \) and \( B \) commute if and only if \( U_A = U_B \).

Suppose \( (11) \) holds. Letting \( A = Q_{X|Y}, B = \Sigma_\Delta \), then \( A = U_A D_A U_A^\top, B = U_B D_B U_B^\top, U_A U_A^\top = I_{n_x}, U_B U_B^\top = I_{n_x}, U_A = U_B \). Since \( Q_{X|Y}^{-1} = A^{-1} = U_A D_A^{-1} U_A^\top, \) then \( \Sigma_\Delta Q_{X|Y}^{-1} = Q_{X|Y}^{-1} \Sigma_\Delta \), i.e., they commute. Hence,

\[
H = (I_{n_x} - (\Sigma_\Delta Q_{X|Y}^{-1})^\top) = I_{n_x} - (Q_{X|Y}^{-1})^\top \Sigma_\Delta = I_{n_x} - Q_{X|Y}^{-1} \Sigma_\Delta
\]

\[
= I_{n_x} - \Sigma_\Delta Q_{X|Y}^{-1} = H \quad \text{since } Q_{X|Y} \text{ and } \Sigma_\Delta \text{ commute.} (236)
\]

By the definition of \( Q_W \) given by \( (104) \) we have

\[
Q_W = \Sigma_\Delta H^\top = Q_W^\top = H \Sigma_\Delta. (237)
\]

Substituting \( (236) \) into \( (237) \), then

\[
Q_W = \Sigma_\Delta H. (238)
\]

Hence, \( \{\Sigma_\Delta, \Sigma_{X|Y}, H, Q_W\} \) are all elements of \( S^{p \times p}_+ \) having a spectral decomposition wrt the same unitary matrix \( UU^\top = I_{n_x} \).

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