Topological-Antitopological Fusion
and the Large $N CP^N$ Model

M. Bourdeau
Dept. of Theoretical Physics
Oxford University
Oxford, OX1 3NP

and

M. R. Douglas
Dept. of Physics and Astronomy
Rutgers University
Piscataway, NJ 08855

Abstract

We discuss the large $N$ limit of the supersymmetric $CP^N$ models as an illustration of Cecotti and Vafa’s $tt^*$ formalism. In this limit the ‘$tt^*$ equation’ becomes the long wavelength limit of the 2D Toda lattice, an equation first studied in the context of self-dual gravity. We show how simple finite temperature and large $N$ techniques determine the relevant solution, and verify analytically that it solves the $tt^*$ equation, using Legendre transform techniques from self-dual gravity.

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bourdeau@thphys.ox.ac.uk
mrd@physics.rutgers.edu
1 Introduction

Recently there has been a lot of work on $N = 2$ supersymmetric models in two dimensions, in the context of string theory and integrable and massive quantum field theories. Non-renormalization theorems for these models make them a lot easier to study and classify. Much structure is encoded in a finite dimensional closed subalgebra of chiral primary fields. Many models admit a Landau-Ginsburg description and these can be largely understood in terms of their superpotential. More recently, Cecotti et. al. offer a further classification of models by defining a ‘new index’ $\text{Tr} \left( (-1)^F e^{-\beta H} \right)$ which exhibits even more of the structure of the model. The new index depends only on $F$-term perturbations, thus is much simpler than quantities like the free energy, but unlike Witten’s index $\text{Tr} \left( (-1)^F e^{-\beta H} \right)$ and the chiral ring it can encode information about scale and coupling dependence of the model, both at short distances (e.g. dimensions of perturbations near the UV fixed point) and at long distances (e.g. the soliton spectrum).

In principle the new index is exactly calculable in any two-dimensional $N = 2$ theory, whether or not it is integrable. One first considers the inner product on the space of supersymmetric ground states, which geometrically plays the role of a metric. This metric satisfies a differential equation as a function of the couplings, essentially the same as the one found for Zamolodchikov’s metric in the case of $N = 2$ superconformal theory. In many interesting cases it reduces to a familiar equation of mathematical physics. It is a short step from the metric to the new index (whose physical interpretation is perhaps clearer).

In [3], the authors investigate the new index for several integrable models, such as the $N = 2$ sine-Gordon and minimal $N = 2$ theories. They show how to obtain the new index for integrable theories, given the exact $S$-matrix, by means of the thermodynamical Bethe ansatz. This method requires solving a moderately tractable set of coupled non-linear integral equations. In [7, 8], applications to polymer physics (self-avoiding random walks) are carried out. In all of these papers, the authors uncover previously unknown mathematical structure of $N = 2$ theories and equivalences between solutions of integral equations and differential equations. In the simplest case (the $A_2$ deformed minimal model), the differential equation is a special case of Painlevé III (or the sinh-Gordon equation) and the relevant solution was shown numerically to be equal to the TBA result.

In [2], Cecotti and Vafa study supersymmetric $\sigma$ models and obtain a differential equation for the metrics of the SUSY $\mathbb{C}P^1$ and $\mathbb{C}P^2$ models. This equation is also the sinh-Gordon equation but the metric differs from the $A_2$ minimal case by its different boundary conditions. For $\mathbb{C}P^n$ with $n \geq 3$, the equations have not been studied explicitly.

Historically, the $\mathbb{C}P^n$ models have been studied more extensively than the other models one might consider as applications, and they are very interesting theories with some analogies to QCD: they are asymptotically free, and they have instantons and $\theta$ vacua, leading to a ‘$U(1)$ problem’ with a resolution like that of QCD. (The supersymmetric case, which has fermions of zero bare mass, is more similar to QCD with a massless quark.) Moreover, the $\mathbb{C}P^n$ model is very simple to solve in the large $n$ limit: to leading order in $1/n$, $S$-matrix elements are given by summing tree diagrams, while bulk quantities like the free energy are calculable by simply
extremizing an effective action.

This leads to the question of whether the $tt^*$ methods show comparable simplifications in this limit. In this paper we study the large $n$ supersymmetric $\mathbb{C}P^n$ model, and find that the $tt^*$ equation determining the metric is an equation first studied in the context of self-dual gravity, and related (by a Legendre transform) to a symmetry reduction of Plebański’s ‘heavenly’ equation for a self-dual Kähler potential in $D = 4$. Few explicit solutions of this equation are known.

A calculation of the index (and metric) using large $n$ techniques proceeds in two steps. One can write the model in terms of free fields parametrizing $\mathbb{C}^{n+1}$ with auxiliary fields implementing the reduction to $\mathbb{C}P^n$. Integrating out the free fields gives a quantum effective action, and in the large $n$ limit observables (such as the index) are dominated by a saddle point of this effective action. The second step is to minimize the effective action with respect to the auxiliary fields. It will emerge that in our problem, the minimization can be reinterpreted as precisely a combination of known techniques for finding solutions to self-dual gravity from those for simpler equations via Legendre transform. Thus we will prove that the index and metric, computed independently, solve the $tt^*$ equation.

Our original motivation for this work was simply to have a field theory example in which we could make every element of the $tt^*$ formalism completely explicit. Perhaps the most useful consequence is that in this model, extensions to the original ideas, such as understanding the role of the higher couplings, or of changes to the two-dimensional space-time metric, can be studied explicitly. It may also be possible to study more interesting large $n$ models such as Grassmannian target spaces or other models with $n^2$ degrees of freedom.

In section 2 we review the work of Cecotti, Vafa and collaborators and discuss the $tt^*$ equations and new index. In section 3 we review the supersymmetric $\mathbb{C}P^n$ model, and its solution in the large $n$ limit. In section 4 we derive the large $n$ limit of the $tt^*$ equation for the $\mathbb{C}P^n$ model. In section 5 we will derive its solution and discuss the connections with self-dual gravity. Section 6 contains conclusions.

2 The ground-state metric for $N = 2$ theories and the new index

The situation governed by the $tt^*$ equations is the following. We have a $d = 2$, $N = 2$ supersymmetric field theory quantized on a Euclidean manifold, with metric and boundary conditions preserving $N = 2$ global supersymmetry. We assume there are a discrete set of supersymmetric ground states, that at least one dimension is compact, and that a Hamiltonian defined on a compact hypersurface has a gap. Given all this, certain cleverly chosen correlation functions can be reduced to sums over the ground states.

For superconformal theory this is very easy to arrange; we just need to have a compact dimension with Ramond boundary conditions. Since Weyl transformations act so simply the form of the metric is not important. For general theories we expect some constraints. First, to have an unbroken global supersymmetry, there should be a covariantly constant spinor on the surface, which in two dimensions implies the metric is flat. Second, for a correlation function to reduce to a sum over ground states, it must be expressible as a limit in which the distance
between any pair of operators inserted goes to infinity. Thus the space-time must also have a non-compact dimension. It is natural to require this dimension to be infinite in both directions, in which case space-time is a cylinder.

In some sense the formalism is a particular case of the “nonabelian Berry’s phase” we would see if we varied the parameters of any quantum system with a degenerate ground state. The first role of supersymmetry in the discussion is simply to guarantee that there will be a set of exactly degenerate ground states. \( N = 2 \) supersymmetry came in when we identified these with deformations of the couplings of the theory; in general there is no relation but in \( N = 2 \) spectral flow provides this relation.

An \( N = 2 \) theory can be ‘topologically twisted’\cite{10} one component of the supercharge is singled out to play a role like the BRST charge, and the state space projected to its cohomology. The stress-tensor is also modified to be a BRST commutator, so correlation functions are independent of the positions of operators. This truncation keeps only the ground states and makes the identification of these with operators very simple.

The most basic elements particular to a given \( N = 2 \) theory are the chiral and anti-chiral rings. We list the chiral operators \( \phi_i \) satisfying \( [Q^+, \phi] = 0 \), and anti-chiral operators \( \bar{\phi}_i \) satisfying \( [Q^-, \bar{\phi}] = 0 \). The chiral ring is defined in terms of the operator product algebra as

\[
\phi_i \phi_j = \sum_k C_{ij}^k \phi_k + [Q^+, \Lambda].
\]

Since the derivative of any operator (and the stress tensor itself) is a descendant under \( Q^+ \) (and \( Q^- \)), the positions of the operators on the left hand side do not matter. The anti-chiral ring will have structure constants \( \bar{C}^k_{ij} \).
Equally important are the supersymmetric (Ramond) ground states
\[ H|a\rangle = Q^\pm|a\rangle = 0. \tag{2.4} \]
We could make a correspondence between these and chiral fields by choosing a canonical ground state \(|0\rangle\). Then we can identify
\[ \phi_i|0\rangle = |i\rangle + Q^+|\Lambda\rangle. \tag{2.5} \]
Finally we could project on the true ground state by applying an operator like \(\lim_{T \to \infty} \exp -HT\).

We could also do this with anti-chiral fields \(\bar{\phi}_i\), producing states to be called \(|\bar{i}\rangle\). The structure constants \(C^k_{ij}\) then also give the action of the chiral operators on the ground states:
\[ \phi_i|j\rangle = C^k_{ij}|k\rangle + Q^+|\psi\rangle. \tag{2.6} \]

This construction is not completely satisfactory because it is not clear that the correspondence is one to one; furthermore it depended on the choice of \(|0\rangle\). Both problems are dealt with by making a correspondence using spectral flow. In principle, this constructs the state \(|i\rangle\) by doing a path integral on a hemisphere with an insertion of \(\phi_i\). We need spectral flow to put this state in the Ramond sector, and we can think of it as turning on a \(U(1)\) gauge field coupled to the fermion number current, with holonomy \(e^{i\pi}\) on the boundary. We can then take as \(|0\rangle\) the state produced by inserting the identity operator \(\phi_0 \equiv 1\), and non-degeneracy of the two-point function \(\langle \bar{\phi}_i \phi_j \rangle\) will imply that the correspondence is one to one.

Now we take \(|i\rangle\) and \(|j\rangle\) to denote the basis of ground states corresponding to the fields \(\phi_i\) and \(\phi_j\). CPT will relate \(|i\rangle\) to a state \(\langle \bar{i}\rangle\) so the usual Hilbert space metric will be the hermitian
\[ g_{ij} = \langle \bar{j}|i\rangle. \tag{2.7} \]

Another structure present in the theory is the “real structure” \(M\) expressing one basis in terms of the other:
\[ \langle \bar{i}\rangle = \langle j|M_i^j \tag{2.8} \]
CPT implies \(MM^* = 1\).

A combination of these produces the ‘topological’ metric \(\eta\):
\[ g_{ik} = \eta_{ij}M_i^j \tag{2.9} \]
This is the two-point function in the topologically twisted theory and as such it is in many ways a more basic object than \(g\) or \(M\). We will not use it in the following but instead refer the reader to the extensive literature on topological field theory.

Supersymmetry-preserving perturbations of the action are of two types. In general we need to write a commutator with all four supercharges (or integral \(d^4\theta\)) to preserve all supersymmetries. However we can also write
\[ \delta S = \sum_i \int d^2 x \ \delta t_i \{Q_R, [Q_L, \phi_i]\} + \delta \bar{t}_i \{Q_R^+, [Q_L^+, \bar{\phi}_i]\}. \tag{2.10} \]
where \( \phi_i \) and \( \bar{\phi}_i \) are chiral and anti-chiral fields. A perturbation which can only be written in this form is called an \( F \) term; the others are \( D \) terms.

Given a space of theories \( T \in \mathcal{T} \) defined by perturbing around a base theory \( T_0 \) (and thus with coordinates \( t_i \) and \( \bar{t}_i \)), we would like to define the rings, ground states and metric for each theory \( T \). A simple way to do this locally is to use the same operator basis for the chiral ring for each \( T \), but evaluate the o.p.e. in (2.3) and the path integral in the spectral flow construction using the action for theory \( T \). This will give structure constants and ground states depending on the couplings, and in principle from this we could compute the metric \( g \) as a function of the couplings. There is an important subtlety in this computation. One might think that since an operator \( \{ Q_R, [Q_L^-, \phi_i] \} \) annihilates a supersymmetric ground state, inserting (2.10) into (2.7) would give zero. It is true for \( D \) terms, but not for \( F \) terms. One sees the subtlety most simply by considering a mixed second derivative of the metric, which is evidently expressed as

\[
\partial_k \bar{\partial}_l \langle j | i \rangle = g^{jk} \langle j' | i \rangle \int d^2x \int d^2x' \{ Q_R, [Q_L^-, \phi_k(x)] \} \{ Q_R^+, [Q_L^+, \bar{\phi}_l(x')] \} | i \rangle. \tag{2.11}
\]

We do not have a perturbation of the states like (2.10), followed by projection on the true ground states, but rather some mixture of the two. By considering the action of the supercharges in this formula, one sees that it differs by including contact terms where \( x = x' \) (as in (2.11)).

A nice way to disentangle these is to separate the dependence on the couplings of the states \( \langle j | \) and \( | i \rangle \), varying the action for the two path integrals we use to construct the two states. In the spirit of the non-abelian Berry’s phase, define the gauge connection

\[
A_{i k}^{j} = g^{jk} \langle j' | \partial_i | k \rangle \tag{2.12}
\]

and its conjugate. By definition, the metric \( g \) is covariantly constant with respect to the derivatives

\[
D_i = \partial_i - A_i, \quad \bar{D}_i = \bar{\partial}_i - \bar{A}_i. \tag{2.13}
\]

Then, we might expect covariant combinations like the curvature to be especially simple. Writing these out explicitly and manipulating the supercharges gives

\[
[D_i, D_j] = [\bar{D}_i, \bar{D}_j] = 0 \tag{2.14}
\]

and for the mixed case terms involving insertions of the Hamiltonian, which can be written as total \( x^1 \) and \( x'^1 \) derivatives. Considering the boundary terms in the \( x \) integrals, one limit will produce the same contact term, which cancels in the commutator, while for the other limit, with all operators at large distances, the correlator reduces to a sum over ground states, which is evaluated using (2.6). Thus one finds

\[
[D_i, \bar{D}_j] = -\beta [C_i, \bar{C}_j], \tag{2.15}
\]

a differential equation for the metric. By (2.14) one can choose a basis in which \( \bar{A}_i = 0 \), so \( A_i = g^{-1} \partial_i g \), and it becomes

\[
\bar{\partial}_j (g \partial_i g^{-1}) = \beta [C_i, g C_j^g g^{-1}] \tag{2.16}
\]
These are the $tt^*$ equations, which given enough boundary conditions determine the metric $g$. Different models with the same chiral ring can have different metrics; thus the boundary conditions are a crucial part of the story. In the cases considered in detail the dependence on one relevant coupling is studied. Let this define a mass scale $m$; then small $\beta m$ is weak coupling and this limit of the metric can be found using semiclassical techniques. The large $\beta m$ boundary conditions are even simpler and are best explained in terms of the ‘new index’ (see below).

It is perhaps worth noting that quantum field theory was not really used in deriving the $tt^*$ equations (the original derivation of [12] was in the context of $N = 2$ supersymmetric quantum mechanics!) and whatever quantum field theory structure is there is in some sense fed in through the boundary conditions and the chiral ring structure constants. It is a pleasant surprise then to find that quite non-trivial quantum field theoretic information emerges. This point also holds out some hope that these ideas will have value in $D > 2$. We recall as well that no assumption about the integrability of the theory (in the usual senses of having extra conserved charges or a factorized $S$-matrix) was made.

Another observable depending only on $F$ couplings was given in [3]. Although it is simply related to the metric it has a clearer physical interpretation. It is modeled after the index $\text{Tr}(-1)^F e^{-\beta H}$, which is completely independent of finite perturbations of the theory for $N \geq 1$ supersymmetric theories in any dimension.[4] This index has been very useful in providing criteria for supersymmetry breaking.

For an $N = 2$ theory in two dimensions, we have a conserved $U(1)$ charge $F$ (the ‘fermion number’ of (2.1)), and in [3] Cecotti et al. show that the ‘new index’ $\text{Tr} F (-1)^F e^{-\beta H}$ depends only on $F$-term perturbations.[4] This can also be thought of as a path integral on the cylinder, now written in an $x^0$ as time operator formalism.

The new index is actually a matrix since the boundary conditions at spatial infinity can be any vacuum of the theory. Let the left vacuum be $a$ and the right one $b$, and consider the matrix elements

$$Q_{ab} = \frac{i\beta}{L} \text{Tr}_{ab} (-1)^F F e^{-\beta H}. \tag{2.17}$$

In [3] it is shown that the matrix $Q$ is imaginary and hermitian, and that

$$Q_{ab} = i(\beta g \partial_\beta g^{-1} + n)_{ab} \tag{2.18}$$

where $n$ is the coefficient of the chiral anomaly. This expression comes from writing out the path integral calculation and reinterpreting it in the $x^1$ as time operator language. The fermion number becomes chiral charge, and a relation between this and the stress tensor is used to get $\partial/\partial \beta$.

This quantity is particularly suited for extracting the soliton spectrum and other low temperature properties of the model. The simplest case is a model with a mass gap; clearly $Q$ will be exponentially small in $\beta m$ and typically each of the leading terms in an expansion

\[^1\text{And thus is not an index in the mathematical sense. Rather it is related to what the mathematicians call} \text{‘holomorphic torsion.’}\]
in \(\exp - \beta m\) is the contribution of a single massive particle saturating the Bogomolnyi bound
\(m = |\Delta|\).

For our purposes, it has the additional advantage that its definition does not involve the precise normalization of the ground states \(|i\rangle\), which simplifies its computation.

3 The \(\mathbb{C}P^n\) sigma models

Non-linear sigma models define maps from spacetime into a riemannian target manifold \(M\). Supersymmetric \(d = 2\) sigma models exist for any target manifold. If the target manifold and metric is Kähler, the model will be \(N = 2\). (For a review, see [20, 21, 22]).

A manifestly \(N = 2\) invariant superspace Lagrangian is
\[
\mathcal{L} = \frac{1}{2} \int dx \, d^2 \theta \, d^2 \bar{\theta} \, K(\Phi, \Phi^\dagger).
\]
(3.1)

where \(K\) is the Kähler potential and \(\Phi_i\) are complex chiral superfields
\[
\Phi_i = \Phi_i(x, \theta, \bar{\theta}) = \varphi_i(x) + \sqrt{2} \epsilon_{\alpha\beta} \theta^\alpha \psi^\beta_i(x) + \epsilon_{\alpha\beta} \theta^\alpha \bar{\theta}^\beta F_i(x).
\]
(3.2)

Any term in \(K\) which is globally defined on \(M\) is a \(D\) term, and conversely two choices of \(K\) for which the Kähler forms \(J = dz^i \wedge d\bar{z}^j \partial_i \partial_j K\) are in different complex cohomology classes differ by \(F\) terms.

For \(\mathbb{C}P^n\), \(\dim H^{1,1}(M, \mathbb{R}) = 1\) and the Kähler class is specified by a single parameter. We can take
\[
K(\Phi, \Phi^\dagger) = \frac{1}{g^2} \log(1 + \sum_{i=1}^{\infty} \Phi_i^\dagger \Phi_i).
\]
(3.3)

The supersymmetric ground states of an \(N = 2\) sigma model are in one-to-one correspondence with the complex cohomology classes of the target space, and by spectral flow so are the chiral primaries. Using semiclassical techniques to compute the chiral ring, one finds it to be a deformation of the classical cohomology ring: instantons can contribute to correlation functions of the chiral primaries. In simple cases the possible contributions are determined by the chiral anomaly. For \(\mathbb{C}P^n\), the classical cohomology ring is the powers of the Kähler form \(x\) allowed on a \(2n\)-dimensional manifold, up to \(x^n\). The instanton changes the relation \(x^{n+1} = 0\) to
\[
x^{n+1} = e^{-2\pi / g^2}.
\]
(3.4)

One can introduce as well a coupling \(\theta\) to control a topological term, which weighs a configuration of instanton number \(w\) by a factor \(e^{i\theta w}\). This combines with \(1/g^2\) to make a chiral coupling \(t_1 = 2\pi / g^2 + i\theta\) as in section 2.

For many purposes, a more useful definition of the \(\mathbb{C}P^n\) sigma model is provided by a gauged \(N = 2\) model, which constructs \(\mathbb{C}P^n\) as a quotient of \(\mathbb{C}^N\) (let \(N = n+1\):
\[
\mathcal{L} = \int d^4 \theta \left[ \sum_{i=1}^{N} \bar{S}_i e^{-V} S_i + \frac{N}{g^2} V \right].
\]
(3.5)
$S_i$ are $N$ chiral superfields which become the homogeneous coordinates on $\mathbb{C}P^n$. We have introduced a factor of $N$ with the coupling $1/g^2$ which will make the $N \to \infty$ limit well defined. $V$ is a real vector superfield, whose components become the many auxiliary fields of the following component form of the Lagrangian:

$$L = \frac{N}{g^2} \left\{ (D_\mu n^*_i)(D_\mu n_i) + \bar{\psi}^i(i\mathcal{D} + \sigma + i\pi\gamma^5)\psi_i - \right.$$  
$$\left. (\sigma^2 + \pi^2) - \lambda(n^*_i n_i - 1) + \bar{\chi}n^*_i \psi^i + \bar{\psi}^i n_i \chi \right\}. \quad (3.6)$$

The superfields $S$ have complex components $n_i$ and $\psi_i$. (We rescaled them by $\sqrt{N/g}$.) The constraint $n^*_i n_i = 1$ is imposed by the Lagrange multiplier $\lambda$; the phase of $n$ and $\psi$ is gauged by $A_\mu$ (which appears in $D_\mu = \partial_\mu + iA_\mu$). The fields $\sigma$ and $\pi$ implement 4-fermi interactions and by the equations of motion are equal to $\bar{\psi}^i \psi_i$ and $i\bar{\psi}^i \gamma^5 \psi_i$ respectively. The fermionic auxiliary fields $\chi$ constrain the $\psi_i$ to be tangent to $\mathbb{C}P^n$. They will play a secondary role in our considerations. The action has an additional chiral $U(1)$ symmetry, $\delta \psi_i = \gamma^5 \psi_i$, and $\delta(\sigma + i\pi) = -2i(\sigma + i\pi)$, which in the quantum theory will be anomalous.

Since the fields $S_i$ appear quadratically, it is possible to integrate them out exactly, at least in terms of a one-loop determinant:

$$L = -N\text{Tr} \log(-D_\mu D^\mu + \lambda) + N\text{Tr} \log(i\mathcal{D} + \sigma + i\pi\gamma^5)$$
$$+ \frac{N}{g^2}(\sigma^2 + \pi^2 - \lambda) + \text{fermionic.} \quad (3.7)$$

The determinant can be regulated straightforwardly (e.g. by Pauli-Villars) and by supersymmetry the divergences will cancel in the result. As a functional of the auxiliary fields, it can be evaluated quite explicitly for constant fields and then as an expansion in either the amplitude or frequency of the fluctuations around this. In the large $N$ limit this allows us to solve the model: the $N$ in front of the action means that the remaining integration over auxiliary fields can be done by saddle point, and calculations at leading order in $1/N$ can be done by classical techniques. For example, an S-matrix element would be given by a sum of tree diagrams; for a specified number of external particles these are finite in number.

Writing out the low energy effective action makes the physics of the model clear. At zero temperature we have an effective potential

$$V_{\text{eff}} = \frac{1}{g^2}(\sigma^2 + \pi^2) + \frac{1}{4\pi}(\sigma^2 + \pi^2) \log \frac{\sigma^2 + \pi^2}{\mu^2}$$
$$- \frac{1}{g^2}\lambda - \frac{\lambda}{4\pi} \log \frac{\lambda}{\mu^2}. \quad (3.8)$$

It exhibits the dimensional transmutation in this asymptotically free theory as only the combination $m_0 \equiv \mu \exp -2\pi/g^2$ appears. The effective potential has a minimum at non-zero $\sigma^2 + \pi^2 \equiv m_0^2$ and an extremum at $\lambda = m_0^2$. These give masses to the $n$ and $\psi$ particles, which are equal by supersymmetry. There will be kinetic terms induced for the auxiliary fields: to
lowest order, with the anomaly,
\[ S_{\text{eff}} = N \int d^2x \frac{1}{8\pi m_0^2} \left( F_{\mu\nu}^2 + (\partial_{\mu}\sigma)^2 + (\partial_{\mu}\pi)^2 \right) + \frac{i}{2\pi} e^{\mu\nu} F_{\mu\nu} \text{Im} \log(\sigma + i\pi) + V_{\text{eff}} + \ldots \] (3.9)

Supersymmetry tells us to expect a multiplet of particles associated with the \( \sigma, \pi \) and \( \chi \) fields. This is the point at which we see a much-noted analogy with QCD. The vev for \( \sigma + i\pi \) breaks chiral \( U(1) \) spontaneously, suggesting that its phase is a Goldstone boson. However this suggestion cannot be right in two dimensions (it also contradicts the expectations from supersymmetry) and for finite \( N \) one is happy to find that just as in QCD, instanton effects explicitly break this \( U(1) \) to \( Z_N \). This leaves a bit of a puzzle in that in terms of the coupling in (3.6), rescaled as appropriate for the large \( N \) limit, the instanton action is \( \exp -N/g^2 \) and instantons should be invisible in the limit, so what eliminates the Goldstone boson? The answer to this puzzle is that the role of the instanton in the story was to provide a field configuration in which the integrated anomaly, \( \int d^Dx F^{D/2} \) was non-zero despite being integral of a total derivative. In two dimensions we do not need the instantons - a typical gauge potential grows linearly in space and \( \int d^2xF \) will be non-zero without their help. Rather we must treat the anomaly on the same footing with the other induced kinetic terms, and the result is more analogous to the massless Schwinger model: the \( \pi \) boson is massive (as are the other particles in the supermultiplet), and the gauge field is screened (so unlike the bosonic \( CP^n \) model, the \( n \) and \( \psi \) particles are not confined.)

There are still degenerate vacua labeled by the phase of \( \sigma + i\pi \). Although the phase is not quantized, the difference between the phases at \( x^1 = \pm L \) is quantized. This follows from Gauss’ law and the screening of the gauge field. We have
\[ \frac{1}{4\pi m_0^2} \partial_1 F - \frac{i}{2\pi m_0} \partial_1 \pi = \frac{1}{N} J^0 \] (3.10)

where \( J^0 \) is the electric current of the elementary fields \( n \) and \( \psi \). Integrating \( \int dx^1 \) and realizing that because of the screening, \( F \) vanishes at infinity, shows not only that the phase difference is quantized in units of \( 1/N \) but that we should think of the elementary particles as solitons, in the sense that the associated field configuration interpolates between different choices of vacuum.

The conclusion is that we should think of the chiral \( U(1) \) as being explicitly broken to \( Z_N \) just as for finite \( N \), and the further spontaneous breaking of \( Z_N \) is associated with multiple vacua and solitons. This allows us to identify the concepts of section 2 in this language. The ground states are characterized by the phase of \( \sigma + i\pi \), so we can associate the choice of vacua \( a \) and \( b \) in (2.17) with \( \sigma + i\pi \to x^1 \to -L \ m_0 \exp 2\pi ia/N \) (resp. \( b \) and \( +L \)). Clearly it is a function only of \( a - b \). Furthermore, this difference must be \( O(N^0) \), because we have a lower bound \( m_0|a - b| \) on the energy in this sector. Thus a representative choice is \( \pi \to \pm 2\pi m_0 a/N, \sigma = Om_0 + O(1/N^2) \) and we can think of a ground state as labelled by a value of \( \pi \). This is only one possible basis and it is not in fact the basis defined by spectral flow; we will see this shortly.
One can also write a manifestly $N = 2$ supersymmetric effective Lagrangian. A priori this is a functional of $V$, but by gauge invariance the effective action should depend only on the field-strength superfields $X$ and $\bar{X}$ ($V$ is not gauge invariant):

$$X = D_L \bar{D}_R V, \quad \bar{X} = D_R \bar{D}_L V$$

$$X = (\sigma + i\pi) + \bar{\theta}_L \chi_R + \chi_L \theta_R (\lambda - F)|_{x_{ch}}$$

The effective action then is

$$S_{\text{eff}} = \frac{N}{2\pi} \int d^2 x \left\{ \int d^2 \theta W(X) + \int d^2 \bar{\theta} \bar{W}(\bar{X}) + \int d^4 \theta [Z(X, \bar{X}, \Delta, \bar{\Delta})] \right\}$$

where

$$W(X) = \frac{1}{2\pi} X \left\{ \frac{1}{N} \log X^N - 1 + A(\mu) - i\theta \right\}$$

where $A$ is a renormalized coupling. (For more details, see [17, 6])

Now many $N = 2$ supersymmetric theories in two dimensions admit a Landau-Ginsburg description, meaning they can be described by a superspace Lagrangian of the form

$$\mathcal{L} = \int d^4 \theta \sum_i \phi_i \bar{\phi}_i + \int d^2 \theta W(\phi_i) + h.c.$$ 

The superpotential $W$ is an analytic function of the complex superfields. Since there is a non-renormalization theorem for it, one can directly infer that the ground states of the theory are $dW(\phi) = 0$. The chiral ring is the ring of polynomials generated by the $\phi_i$ modulo the relations $dW(\phi_i)/d\phi_i = D \bar{D} \phi_i \sim 0$. (For a review, see [23].)

Since our $CP^n$ effective action now has the form of a Landau-Ginsburg theory, it follows that its chiral ring is the powers of $X$ mod

$$X^N = \exp - A + i\theta \equiv m_0^N.$$ 

This is an alternate derivation of the chiral ring [3,4]. It would be interesting to get the $N \to \infty$ results below in a manifestly supersymmetric way.

The ground states of our previous section are the solutions of $W'(X) = 0$, in other words $X = m_0 \exp 2\pi i \sigma N$. Clearly these are eigenstates under multiplication by the chiral ring, and therefore not the states defined by $|j\rangle = X^j |0\rangle$. By considering the $\mathbb{Z}_N$ symmetry, one sees that these are the conjugate states

$$|j\rangle = m_0^2 N_j \sum_k e^{2\pi i (j - j_0) k/N} |\sigma + i\pi = m_0 \exp 2\pi i k/N)$$

up to a phase $j_0$ and normalizations $N_j$ not determined by this argument. To get these we should make the spectral flow argument explicit, and we will discuss this below.
4 The ground-state metric for CP^n

In reference [2], the tt* equations are given for the supersymmetric CP^n model on a Kähler manifold M. As we have just seen, the chiral ring is generated by a single element x with the relation

\[ x^N = t^N \]  

(4.1)

where \( t = m_0 e^{i\theta} \) is a complex chiral coupling as in section 2.

The action can be written in the following form

\[ S = -\frac{N}{4\pi} \ln t \int d^2 y d^2 \theta \, x + \text{c.c.} \]

(4.2)

where \( x = x(\Phi_i, \bar{\Phi}_i) = DD \ln(1 + \sum \Phi_i \bar{\Phi}_i) \) represents the Kähler class, \( \ln t \) is the Kähler form and \( \Phi_i, \bar{\Phi}_i \) are chiral superfields.

To write down the tt* equations, we need to find the operator corresponding to a perturbation of \( t \), and its action on the chiral ring \((X^n, \ldots, X, 1)\). It is represented by the matrix

\[
C_t = \frac{N}{4\pi t} \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 \\
t^N & 0 & 0 & \ldots & 0 & 0
\end{pmatrix}
\]

The \( \mathbb{Z}_N \) symmetry implies that the metric \( g_{ij} = \langle j|i \rangle \) is diagonal. Thus, defining

\[ q_i = \ln g_{ii} \quad g_{ij} = \langle j|i \rangle \quad q_{i+N} \equiv 2N \log |t| + q_i \]

(4.3)

the tt* equations are

\[ \frac{16\pi^2 t^2}{N^2 \beta^2} \partial_\beta \partial^*_\beta q_i + e^{(q_{i+1} - q_i)} - e^{(q_i - q_{i-1})} = 0. \]

(4.4)

The metric \( g \) is a function only of \( |t|^2 \), because it is a path integral with total chiral charge zero, and chiral charge non-conservation is proportional to instanton number. Thus the equations become o.d.e.’s in terms of \( |t| \). We can write them in terms of the dimensionless parameter \( x = \beta t/2\pi \), but to do this we need to take out the dimensional factors in \( g_{ij} \) coming from the definition (3.17). Thus we redefine

\[ q_j = \ln g_{jj} + 2j \log |t| \quad q_{j+N} \equiv q_j \]

(4.5)

With this straight, we will use \( x \) as our coupling, and call it \( \beta \) in the following. The tt* equation becomes

\[ \frac{4}{N^2} \partial_\beta \partial^*_\beta x + e^{(q_{i+1} - q_i)} - e^{(q_i - q_{i-1})} = 0. \]

(4.6)

This equation is the affine \( \hat{A}_n \) Toda equation. (In [2] it is shown that on general grounds \( q_i + q_{N-i-1} = 0 \), which reduces the equation to the \( C_m(BC_m) \) Toda equation with \( n = 2m(n = 2m + 1) \), but we will not use this.)
A solution should be determined by the boundary conditions near $\beta \sim 0$ and $\beta \sim \infty$. In [1] these are found explicitly for the small $\beta$ limit by a semiclassical calculation of the metric. For the large $\beta$ limit it would suffice to know (on general grounds) that the solution is exponentially small in $\beta$; in fact the precise form of the leading exponential is determined by the soliton spectrum, which is already known for these (integrable) models.

For the cases of $\mathbb{C}P^1$ and $\mathbb{C}P^2$, the $tt^*$ equations become special cases of the Painlevé III equation, for which the connection formula between small and large $\beta$ asymptotics is known.

A reasonable ansatz for the large $N$ limit would be that the metric and index are continuous functions of the variable $s \equiv i/N$. We will verify that this is true for the boundary conditions of [2]; it will also follow from the explicit calculation in section 5. Computing the metric is very similar to computing the free energy with specified boundary conditions at $x^1 \to \pm L$, which would produce $\exp NS_{eff}$ at an appropriate saddle point. With this motivation we redefine

$$q_j = \frac{1}{N} \log g_{j\bar{j}} + 2 \frac{j}{N} \log |t|$$

(4.7)

The $tt^*$ equation becomes:

$$\frac{4}{N} \partial_\beta \partial_{\beta^*} q_i + e^{N(q_{i+1}-q_i)} - e^{N(q_i-q_{i-1})} = 0$$

(4.8)

with $q_{i+N} = q_i$. We see that the $N$ dependence is consistent with $q(\beta, s)$ having a good large $N$ limit, satisfying ($q' = \frac{\partial q}{\partial s}$)

$$4\partial_\beta \partial_{\beta^*} q + \frac{\partial}{\partial s} e^{q'} = 0.$$  

(4.9)

Defining $H = q'$,

$$4\partial_\beta \partial_{\beta^*} H + \frac{\partial^2}{\partial s^2} e^H = 0.$$  

(4.10)

This equation has been studied in several contexts. It was first noted for a connection with 4D self-dual gravity. More recently, it has been studied in the context of the large $n$ limit of $W_n$ algebra. It is also a well known scaling limit of the two-dimensional infinite Toda lattice. A formal solution of the boundary value (Goursát) problem for the equation has been given in [28].

We still need to specify the boundary conditions to select a solution to the equation. One can take the large $N$ limit of Cecotti and Vafa’s boundary conditions at large and small $\beta$; they will follow independently from the results of section 5 so we will just quote them here. The $s$ boundary conditions are $H(\beta, s) = H(\beta, s+1)$.

We can deduce the large $N$ limit of our metric for small $|\beta|$ from the semi-classical result of Cecotti and Vafa. This is essentially the two-point function $\langle \phi_i \bar{\phi}_j \rangle$ reduced to constant field configurations, or $\int d\phi \ x^i \wedge *x^j$. They find

$$g_{r\bar{r}} = \frac{r!}{(n-1-r)!} [|\beta|(-\ln(|\beta|/2) - \gamma)]^{n-1-2r}$$

(4.11)
where $\gamma$ is Euler’s constant, a factor predicted by the connection formula for the $n = 1, 2$ equations, and described in [3] as a one-loop correction to the semiclassical calculation. This becomes, in the large $N$ limit

$$e^H = \frac{s(1 - s)}{|\beta|^2 (- \ln(|\beta|/2) - \gamma)^2}$$

(4.12)

(for $0 < s < 1$ and defined elsewhere by periodicity).

Actually, this is already an exact solution to (4.10) (this does not depend on the value of $\gamma$). At finite $N$ perturbative calculations around the trivial background are one-loop exact; the small $\beta$ boundary condition failed to be a solution because of instanton corrections. In the large $N$ limit these in some sense become trivial. We saw in section 3 that in terms of the rescaled coupling the instanton weight is $\exp(-2\pi N/\beta^2)$ and that we can understand a lot of physics even if we call this zero. Here the instantons are responsible for the boundary condition $H(\beta, s) = H(\beta, s + 1)$. Whether this prevents (4.12) from being a solution depends sensitively on how we treat the region $s = 0$, since (4.12) has a kink there. One prescription which makes sense is to solve not imposing $H(\beta, s) = H(\beta, s + 1)$ but allowing arbitrary $s$ dependence, and if the answer satisfies $H(\beta, 0) = H(\beta, 1)$, accept it. If we use this definition we cannot attribute the corrections to the small $\beta$ limit to instantons. An analogous problem was studied in [15], that of understanding the theta dependence of the large $N$ bosonic $\mathbb{C}P^n$ model. There instantons were also unimportant and the non-perturbatively small soliton action $S \sim \beta \mu \exp(-1/\beta^2)$ controlled the theta dependence.

For large $\beta$, each sector with one soliton of mass $m$ satisfying the Bogomolnyi bound contributes $(f + 1 - f) \exp(-\beta m)$ times a factor depending on its central charge $\Delta$ to the new index, and by (2.18) to $H$. From section 3 we see we have $N$ such solitons with mass $m = m_0$; the central charge is $\pi(+L) - \pi(-L)$ or one can just linearize (4.10) to see the appropriate boundary condition

$$H(s) \sim -\frac{\exp(-2\pi|\beta|)}{\sqrt{2\pi|\beta|}} \cos(2\pi s)$$

(4.13)

which satisfies our equation to first order. This limit is not a solution and one could use (4.10) to generate corrections to $H$ coming from multi-soliton sectors.

One might at first say that in the $tt^*$ formalism, the existence of the solitons is fed in through the large $\beta$ boundary condition. However even without this it was clear that some physics must modify the solution (4.12) – it is singular at $|\beta| = 2 \exp(-\gamma)$. (Classically, without $\gamma$, this would be the ‘zero volume limit of the target space.’) In [1, 2, 3] it was typically found that requiring regularity on solutions of the $tt^*$ equations was a strong constraint, which to some extent predicted physical boundary conditions. In this sense quantum field theoretic information seems to be emerging from the formalism in a rather mysterious way. We do not know enough about general solutions of (4.10) to make a strong statement here, but we are certainly seeing some form of this novel way to predict non-perturbative corrections.

If we write

$$e^H = -R \ s(1 - s) \ e^{\phi(\beta, \bar{\beta})}$$

(4.14)
\[(1.10)\] reduces to the Liouville equation for a 2d metric with constant curvature \(R\). Its solutions are related by Legendre transform (as we will see in section 5) to self-dual Einstein metrics, and with this motivation, this ansatz was considered in \cite{32}. The \(R > 0\) case is related to the Eguchi-Hanson metric, a gravitational instanton \cite{12} An ‘elliptic’ \(R < 0\) solution can be related to a similar (but singular) metric. Our \(R < 0\) solution \((4.12)\) is the ‘parabolic’ case \cite{36}.

\section{Finite temperature results and the new index}

Recall the effective action \((3.7)\). It has been extensively studied at \(T = 0\) \cite{17} and at finite temperature (typically not in the supersymmetric context, but the results can be easily adapted) \cite{37, 15}. In the large \(N\) limit it is \(O(N)\) and we can calculate bulk quantities like the free energy simply by extremizing it with respect to the auxiliary fields. The ‘new index’ is computed similarly, with the main differences being that we take periodic fermion boundary conditions (and have unbroken supersymmetry), we insert the fermion number operator \(F\) (this will be done by differentiating with respect to a coupling at the end), and we fix the boundary conditions at \(x^1 = \pm L\) to go to two possibly different values of \(\sigma + i\pi\). This last condition means that we need to consider non-constant background fields in the functional integral. For general background fields this is quite complicated, but what saves us is that the required variation is small, of \(O(1/N)\), so we only need the leading terms in an expansion in derivatives and amplitude. The derivative terms (to the accuracy we need them) are

\[S_{\text{eff}} = N \int d^2x \frac{1}{8\pi^2} \left( F_{\mu\nu}^2 + (\partial_\mu \sigma)^2 + (\partial_\mu \pi)^2 \right) + \frac{i}{2\pi} e^{i\mu} F_{\mu\nu} \text{Im log}(\sigma + i\pi) + V_{\text{eff}} + \ldots \]

where we no longer assume \(\langle \sigma \rangle = m_0\). The finite temperature effective potential is

\[V_{\text{eff}} = \frac{1}{g^2} (\sigma^2 + \pi^2) - \sum_{k^0} \int \frac{dk^1}{(2\pi)^2} \text{tr ln}[k_{\mu} \gamma^\mu - A_{\mu} \gamma^\mu - (\sigma + i\pi \gamma^5)] \]

\[-\frac{1}{g^2} \lambda + \sum_{k^0} \int \frac{dk^1}{(2\pi)^2} \text{ln}[(k_{\mu} - A_\mu)^2 + \lambda] \]

\[= V_{\text{eff}}^T = 0 + \int_0^\infty \frac{dk}{2\pi} \text{ln} |1 - e^{-\beta k^2 + \sigma^2} e^{iA_0}|^2 \]

\[-\int_0^\infty \frac{dk}{2\pi} \text{ln} |1 - e^{-\beta k^2 + \lambda} e^{iA_0}|^2. \]

This is essentially the standard expression from statistical mechanics of a free field \cite{38} (with chemical potential \(iA_0\)) with one difference: we incorporated the periodic fermion boundary conditions, which led to the sign change in \((5.3)\).

The new index is

\[Q_{ab} = \frac{i\beta}{L} \text{Tr} \left(-1\right)^F F e^{-\beta H}. \]

where \(a\) and \(b\) characterize the vacua at spatial infinity. We can rewrite it as a path integral with an insertion of the fermion number charge \(\int dx^1 J^0_F\). If we introduce a new gauge field \(B_\mu\)
which replaces $A_\mu$ in coupling to the bosons, the fermion number will be given by differentiating $dV_{\text{eff}}/dA_0$ before imposing $B_\mu = A_\mu$. Thus we will split $S_{\text{eff}}$ into two parts, $S_B(\lambda, B)$ from the integral over the $n_i$, and $S_F(\sigma, \pi, A)$ from the $\psi_i$. All the derivative terms of (3.9) are in $S_F$. We know that $H = 0$ on our ground states, so $Q$ will just be the expectation value of $F$.

It may sound a bit strange to be extremizing a Euclidean action which one might have thought should be non-negative. The reason it need not be (and is not) bounded below is that it depends on a Lagrange multiplier, $\lambda$. There can be several extrema, so we first minimize the fermion effective action, then determine $\lambda$ by supersymmetry. $S_B$ enters only in computing $S_{\text{eff}} = 0$, and we will not write this out in the following.

The boundary conditions $ab$ select supersymmetric vacua. As we saw in section 3, these are determined by expectation values $\text{Im} \log(\sigma + i\pi) = 2\pi a/N$ and $2\pi b/N$, so $Q$ will be a function of $p = (a - b)/N$. We must also specify the other fields: they will be independent of $x^0$ and satisfy the equations of motion: in $A_1 = 0$ gauge,

$$\begin{align*}
- \partial_1^2 A_0 - i\partial_1 \pi &= \frac{\delta V_{\text{eff}}}{\delta A_0} \quad (5.6) \\
- \partial_1^2 \pi + i\partial_1 A_0 &= 0. \quad (5.7)
\end{align*}$$

The general solutions are exponentials, but there are special solutions. One which works independently of $V_{\text{eff}}$ is

$$\partial_1 A_0 = \partial_1^2 \pi = 0. \quad (5.8)$$

One can check that this preserves a supersymmetry (up to $O(1/N)$ corrections).

We will substitute this solution directly into the effective action. Thus we take $\pi = p\sigma x^1/L$ and $A_0$ constant, giving

$$
\begin{align*}
S_F &= N \int d^2x \frac{1}{8\pi} \left( \frac{p}{L} \right)^2 - i \frac{1}{2\pi} (p/L) A_0 + V_F(\sigma, A_0) + \ldots \\
&= \beta NL \left( \frac{1}{8\pi} \left( \frac{p}{L} \right)^2 - i \frac{1}{2\pi} (p/L) A_0 + V_F(\sigma, A_0) + \ldots \right) \quad (5.10)
\end{align*}
$$

The anomaly $\int \pi F$ contributes because we integrate by parts and drop a boundary term (more on this below). The expansion in derivatives of $\pi$ becomes exact in the limit $L \to \infty$. Using $dS_F/dA_0 = 0$ at the saddle point, $Q = i\beta/L \ dV_F/dA_0 = -(\beta/2\pi)(p/L)$ at an extremum of $S_F$ in $\sigma$ and $A_0$, given $p$ and $\beta$.

The basis $|a\rangle$ is not the basis of (3.17). There the basis was defined by acting on a vacuum $|0\rangle$ with chiral fields of definite charge, $X^a$. This is the conjugate basis and we have

$$Q(j, \beta) = \int dp \ e^{i(j - j_0)p} Q(p, \beta). \quad (5.11)$$

Now we need $j_0$, which is determined by the spectral flow construction. We have not done this construction in detail but we believe the essential points are as follows. We need to work on a disk, say with radial coordinate $x^1$ and angular coordinate $x^0$, and boundary $x^1 = 1$. The quantum number $j$ is the variable conjugate to the phase of $\sigma + i\pi$, or working near $\pi = 0$, etc.
conjugate to $\pi$. This is not $\partial_1 \pi$, because the anomaly $\pi \partial_1 A_0$ changes the symplectic structure. Instead it is $j = N(\partial_1 \pi - 2i A_0)/2$. The ground state is clearly $j = 0$, but this is in the ‘Neveu-Schwarz’ sector (antiperiodic fermion boundary conditions on the cylinder) and we must turn on a gauge field $A_0 = 1/2$ to turn it into the corresponding supersymmetric ground state. This shifts $j \to j - N/2$ and the relation to section 4 is $j - j_0 = N(s - 1/2)$. We are then instructed to use this state (with norm 1) as a boundary condition on our cylinder, which justifies dropping the boundary term in (5.9).

Since $j - j_0 = N(s - 1/2)$, at fixed $s$ we can also do this integral by saddle point, producing

$$Q(s, \beta) = -\frac{N}{2\pi i} \frac{d}{ds} \frac{\beta}{NL} S_F(\beta, s)|_{\min}$$

(5.12)

with

$$\frac{\beta}{NL} S_F = -\frac{1}{2\pi} (2\pi s - 1) + \beta A_0 + \beta^2 \sigma^2 \frac{1}{4\pi} \left( \ln \frac{\sigma^2}{m_0^2} - 1 \right) - \frac{\beta}{\pi} \int_0^\infty \frac{dk}{2\pi} \ln \left| 1 - e^{-\beta k^2 + 2i\beta A_0} \right|^2.$$  

(5.13)

Let $u \equiv y^2 \equiv \beta^2 \sigma^2$, $A \equiv \beta A_0$, $s' = s - 1/2$ and $\lambda = \log(\beta m_0/2\pi)$, so

$$\frac{2\pi \beta}{NL} S_F = -\lambda u - (2\pi s' + A)^2 + \frac{1}{2} u (\ln \frac{u}{4\pi^2} - 1)$$

$$- \int_0^\infty \frac{dk}{2\pi} \ln \left| 1 - e^{-\sqrt{k^2 + u + A}} \right|^2.$$  

(5.14)

We will now show that the metric related to the index $Q$ by (2.18) indeed solves the ‘heavenly’ equation (4.10). We still need to minimize the effective action with respect to $\sigma$ and $A_0$. Although this cannot be done in closed form, nevertheless the minimization procedure is natural in this context: it amounts to a double Legendre transform of the effective action from $(\sigma, A_0)$ to $(\beta, s)$.

We will start from a rather little known fact: considered as a function of the background fields $\sigma$ and $A_0$, the effective action $S_F$ satisfies a linear p.d.e. (essentially the Laplace equation):

$$\left[ y \frac{\partial}{\partial y} \frac{1}{y} \frac{\partial}{\partial y} + \frac{\partial^2}{\partial A^2} \right] S_F(y, A) = 0.$$  

(5.15)

Clearly the zero temperature part works. At finite $\beta$ one can verify it by expanding the $\ln$ in (5.14) and integrating termwise to get a sum over Bessel functions as in [39], but it is much clearer in terms of the sum over timelike momenta (5.2):

$$F_0 = \sum_{k_0} \int \frac{dk_1}{(2\pi)^2} \ln [(k_0 - A)^2 + k_1^2 + y^2].$$  

(5.16)

\footnote{This is a straightforward calculation. To see it explicitly, turn on the ‘expandedversion’ switch in the tex file, hep-th/9312095.}
Now (5.15) will be true if
\[ v \equiv \frac{1}{y} \frac{\partial F_0}{\partial y} = \sum_{k_0} \int \frac{dk_1}{(2\pi)^2} \frac{1}{(k_0 - A)^2 + k_1^2 + y^2} \] (5.17)
satisfies the three dimensional Laplace equation
\[ \Delta v = \frac{1}{y} \frac{\partial}{\partial y} y \frac{\partial v}{\partial y} + \frac{\partial^2 v}{\partial A^2} = 0. \] (5.18)

Since this is linear we can verify it for each term in the sum, and shift \( A \) in each to absorb \( k_0 \).

Now, as a function of \( x^\mu \equiv (k_1, A_0, y \cos \theta, y \sin \theta) \), the integrand is \( 1/x^2 \) which solves the four dimensional Laplace equation. Thus the operator \( \Delta \) on the integrand is equal to \( -\partial^2/\partial k_1^2 \), which integrates to zero.

The sum over \( k_0 \) in (5.16) of course does not converge, so to use this we would need to subtract the zero temperature part. The result is that the finite temperature part \( f_0 \) (the integral) in (5.14) satisfies (5.15). Let us combine it with some of the zero mode terms in (5.14), defining
\[ f(u, A) = -A^2 + \frac{1}{2} u \ln u - f_0(u, A). \] (5.19)

Now we do a Legendre transform from \( A \) to \( s \) which minimizes the action in \( A \)
\[ K(s', u) = f(u, A) - 4\pi s' A \] (5.20)
and we have
\[ s' = \frac{1}{4\pi} \frac{\partial f}{\partial A} \quad A = -\frac{1}{4\pi} \frac{\partial K}{\partial s'} \quad \frac{\partial K}{\partial u} = \frac{\partial f}{\partial u}. \] (5.21)

Using this we turn (5.15) (applied to \( f \)) into
\[
0 = 4u \frac{\partial}{\partial u} \left[ \frac{\partial f}{\partial u} \right] + \frac{\partial^2 f}{\partial A^2} \\
= 4u d \left( \frac{\partial f}{\partial u} \right) \wedge dA - d \left( \frac{\partial f}{\partial A} \right) \wedge du \\
= u d \left( \frac{\partial K}{\partial u} \right) \wedge d \left( \frac{\partial K}{\partial s'} \right) + 4\pi^2 ds' \wedge du 
\] (5.22)

This last equation is essentially Plebański’s equation
\[
\frac{\partial^2 K}{\partial u^2} \frac{\partial^2 K}{\partial s'^2} \left( \frac{\partial^2 K}{\partial u \partial s'} \right)^2 = \frac{4\pi^2}{u}. \] (5.23)

We now perform a second Legendre transform from \( u \) to \( \lambda \) which will minimize in \( u \)
\[ 4\pi^2 J(s', \lambda) = K(s', u) - (2\pi s')^2 - \lambda u \] (5.24)
with
\[ \lambda = \frac{\partial K}{\partial u} \quad u = -4\pi^2 \frac{\partial J}{\partial \lambda} \] (5.25)
This turns (5.22) into
\[ (-\frac{\partial J}{\partial \lambda}) d\lambda \wedge d (\frac{\partial J}{\partial s'} + 2s') = -ds \wedge d (-\frac{\partial J}{\partial \lambda}) \] (5.26)
\[ \frac{\partial^2 J}{\partial s'^2} + 2 = \frac{\partial}{\partial \lambda} \log \left(-\frac{\partial J}{\partial \lambda}\right) \] (5.27)

Remembering the result for the index (5.12), and using the relation (2.18),
\[ \frac{\partial^2 J}{\partial s'^2} = -\frac{i}{N} \frac{\partial Q}{\partial s'} \]
\[ = \frac{\partial H}{\partial \lambda} \] (5.28)
\[ (5.29) \]

Using this in the l.h.s. of (5.27) and integrating once $d\lambda$ gives
\[ H + 2\lambda = \log \left(-\frac{\partial J}{\partial \lambda}\right). \] (5.30)

There is a constant of integration which is determined by consistency, for example by considering
the large $\beta$ limit of (5.32). It eliminates a term $-u(1/2 + \log 2\pi)$ in (5.14).

Finally differentiate $d/d\lambda$ (5.26) and substitute to get
\[ e^{-2\lambda} \frac{\partial^2}{\partial \lambda^2} H + \frac{\partial^2}{\partial s'^2} e^H = 0. \] (5.31)

Remembering $\lambda = \log x$, this is our equation (4.10) for the $tt^*$ metric. We also find
\[ u = 4\pi^2 e^{H + 2\lambda}. \] (5.32)

Readers familiar with the work of Boyer and Finley\textsuperscript{31} and Gegenberg and Das\textsuperscript{32} will recognize
that we are applying methods developed in the study of solutions of the complex vacuum
Einstein equations with a self-dual metric admitting at least one Killing vector field. They
relate (by Legendre transform) the ‘heavenly’ equation (4.11) to Plebański’s equation [30, 31]
reduced by symmetry with respect to a so-called ‘rotational’ or ‘non-KSD’ Killing vector field,
i.e. a Killing vector whose covariant derivative has a non-zero anti-self-dual part. Our solution
does not depend on the phase of $\beta$ (the theta parameter), which implies that the self-dual metric
will have two rotational Killing vectors. Both papers suggest that such a self-dual metric must
have a ‘translational’ or ‘KSD’ Killing vector, i.e. one whose covariant derivative is self-dual.
All such self-dual metrics can be obtained by Legendre transform of a solution to the Laplace
equation, thus we could expect that our solution to the ‘heavenly’ equation could be obtained
by performing two Legendre transforms on a solution to the Laplace equation. This expectation
was helpful to us, though we note that the determination of the index as the double Legendre
transform of the effective potential is really a consequence of the physical definition of the index
starting from (3.5) and not these more abstract considerations.
Is the correct solution completely determined from $tt^*$ considerations? The elliptic nature of (4.10) means that if the boundary conditions are correct and ‘reasonable’ then we might expect its solution to be unique. Thus we want to compare the limits $\beta \to 0$ and $\infty$ of this solution with the boundary conditions determined by semiclassical considerations in [2] and quoted in section 4. The large $\beta$ limit is easy, by expanding (5.14) in $e^{-\beta \sigma}$ with $\beta \sigma = y = 2\pi \beta + O(e^{-\beta \sigma})$.

For small $\beta$ we want to make contact with (4.12),

$$e^H = \frac{1 - 4s'^2}{4|\beta|^2(-\ln |\beta| - \gamma + \ln 2)^2}.$$  

The inverse of the above Legendre transforms can be performed analytically in this limit and are essentially the ones done in [31, 32].

The results are that in this limit

$$u = \frac{\pi^2}{\lambda + \gamma - \ln 2} \quad \text{and} \quad s' = \frac{A}{2 \sqrt{u + A^2}},$$

and the function defined in (5.19) has limit

$$f = (\log 2 - \gamma)u - 2\pi \sqrt{u + A^2}. \quad (5.34)$$

This is to be compared with the high temperature limit of the effective action (5.2). This is a one-dimensional limit and the leading term is obtained by keeping only the dominant term in the sum over frequencies $k_0$ (and losing explicit periodicity in $A_0$). The subleading terms are harder but fortunately are known.

$$f_0 = 2\pi \sqrt{u + A^2} + \frac{1}{2} u \log u - A^2 + (\gamma - \log 4\pi - \frac{1}{2})u + \ldots \quad (5.35)$$

and after combining with the zero temperature terms, we see that the boundary conditions agree.

There is a technical point which would have to be addressed to actually prove that the solution is uniquely determined by the (asymptotic) semiclassical boundary conditions. This is the effect of the kink at $s = 0$ in (4.12), which is not present in the true solution. One would hope that the equation (4.10) is stable under such a perturbation.

The minimization procedure brings with it the possibility of a phase transition. Since the $\mathbb{C}P^n$ model has a $\mathbb{Z}_N$ symmetry restoration transition at finite temperature, this would seem quite possible. On the other hand the bosonic sector has no transition and thus with periodic boundary conditions the fermionic effective action will not either. However, in a sector with finite soliton number, $A_0 \neq 0$ so it is not a priori obvious that a transition is impossible. If the $\mathbb{Z}_N$ symmetry restoration transition were present at some $\beta = \beta_c$, we would see $u = \beta^2 \langle \sigma \rangle^2 = 0$ there, and for $\beta < \beta_c$ we would expect to see multiple extrema with $u = 0$ and $u < 0$. This is not present in the $\beta \to 0$ limit (5.33) and we conclude that this transition is not present in the new index. Similarly, TBA calculations of the new index for finite $N$ could have shown phase transitions, but did not.
There should be a similar direct computation of $g_{i\bar{j}}$ itself. This would require doing the spectral flow construction to find the correct normalization of the ground states (3.17). The main point here may be the following: given $|0\rangle$, we do not need spectral flow but can apply chiral fields to the Ramond vacua to produce $|i + 1\rangle = X|i\rangle$. The lowest component of $X$ is $\sigma + i\pi$ and since $\langle \sigma \rangle$ depends on the couplings, so do the normalizations. The point is that $\langle \sigma \rangle$ depends on the state $|i\rangle$. If we build up $|i\rangle$ one step at a time we will find

$$
\log g_{i\bar{i}} = \log \prod_{j=1}^{i} \langle \bar{j} | \sigma | j \rangle^2
\rightarrow \int_{s}^{s'} ds' \log u
\sim \int_{s}^{s'} ds' H
$$

by (5.32). But this is exactly the definition of $H$, so the picture is consistent.

6 Concluding Remarks

We showed that the ‘new index’ for the $\mathbb{C}P^n$ model computed with large $N$ methods agreed with that determined by the tt$^*$ formalism. As in previous work, the formalism seems to be a fertile source of pretty mathematical structure. Much more is known about the basic structure we saw here (continuous Toda and Legendre transforms) than we made use of. An analogous computation of the index for finite $N$, or for other integrable $N = 2$ theories, can be done using the TBA, but it seems out of reach at present to relate it analytically to the tt$^*$ equation. It would be very interesting to know if some version of the structure here generalizes to finite $N$.

Equally importantly, we feel the present work is a step towards a physical understanding of the tt$^*$ formalism. We find the formalism attractive, not just for classifying $N = 2$ theories, but as a prototype of an exact result, revealing dynamical information about a full quantum field theory, but not requiring exactly solving the full theory to get, which is something we would very much like to have in higher dimensions. As an illustration, the large $N \mathbb{C}P^n$ model has the great advantage that an effective action can be derived, in terms of which every element of the formalism can be realized classically. We can see in what sense the formalism is a reduction to $D = 1$, and where two-dimensional physics enters. The dynamics visible in the formalism is simple but non-trivial – in this model, particles interact only through the constant modes of the auxiliary fields.

Since we have an exact, non-perturbative result, we can reevaluate the old debate on the importance of instantons in the large $N$ limit. The role of the instanton correction $x^N = \beta^N$ in the chiral ring is to produce the boundary condition $H(s) = H(s + 1)$. This boundary condition is satisfied for the true tt$^*$ solution but causes the semiclassical approximation to the $\beta \to 0$ boundary condition to be non-analytic at $s = 0$. If we ignored this, we could define a formal limit of the model in which (4.1) determines the metric and in which non-perturbative effects are neglected. As in the non-perturbative effects being neglected are solitons, which contribute $O(\exp -\beta \mu e^{-1/g^2})$. The tt$^*$ solution in this limit is singular. It is an intriguing
aspect of the formalism that regularity of the solutions of the non-linear $tt^*$ equations gives
strong restrictions on allowed boundary conditions. Here the question is whether there is a
unique solution with asymptotics described by (4.12) and (4.13). It would be quite interesting
to make such a statement for (4.10). It seems likely that such a statement would depend on the
boundary condition $H(\beta, s) = H(\beta, s + 1)$ and in this formal sense we would say that instanton
effects are important in the model.

It should be possible to make the spectral flow construction explicit as well. It would also
be nice to know more about the self-dual metric $K$ and whether it has a physical interpretation
in our problem.

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References

[1] S. Cecotti and C. Vafa, Nucl. Phys. B367 (1991) 359.
[2] S. Cecotti and C. Vafa, Phys. Rev. Lett. 68 (1992) 903.
[3] S. Cecotti, P. Fendley, K. Intriligator and C. Vafa, Nucl. Phys. B386 (1992) 405.
[4] L. Dixon, V. Kaplunovsky and J. Louis, Nucl. Phys. B329 (1990) 27;
B. de Wit and A. van Proeyen, Nucl. Phys. B245 (1984) 89;
E. Cremmer, C. Kounnas, A. van Proeyen, J.-P. Derendinger, S. Ferrara, B. de Wit and
L. Girardello, Nucl. Phys. B250 (1985) 385;
S. Cecotti, Comm. Math. Phys. 131 (1990) 517;
A. Strominger, Comm. Math. Phys. 133 (1990) 163.
[5] For a general introduction, see the review “1/$N$” by S. Coleman, in “Aspects of Symmetry”,
Erice Lectures, Cambridge University Press, 1985.
[6] S. Cecotti and C. Vafa , HUTP-92, SISSA-203/92/EP, November 92.
[7] P. Fendley and H. Saleur, Nucl. Phys. B388 (1992) 609.
[8] P. Fendley, H. Saleur and Al.B. Zamolodchikov, BUHEP-93-8 and 9, USC/93-003 and 4,
LPM-93-07 and 8, [hep-th/9304050] and [hep-th/9304051].
[9] T. Eguchi and S-K Yang, Mod. Phys. Lett. A5 (1990) 1693.
[10] E. Witten, Nucl. Phys. B340 (1990) 281.
[11] D. Kutasov, Phys. Lett. B220 (1989) 153.
[12] S. Cecotti, Nucl. Phys. B355 (1991) 755.
[13] E. Witten, Nucl. Phys. B149 (1979) 285.
[14] E. Witten, Nucl. Phys. B202 (1982) 253.
[15] I. Affleck, Nucl. Phys. B162 (1980) 461.
[16] A. D’Adda, P. Di Vecchia and M. Lüscher, Nucl. Phys. B152 (1979) 125.
[17] A. D’Adda, A.C. Davis, P. Di Vecchia and P. Salomonson, Nucl. Phys. B222 (1983) 45.
[18] D. Olive and E. Witten, Phys. Lett. B78 (1978) 97.
[19] B. Zumino, Phys. Lett. B87 (1979) 203.
[20] L. Alvarez-Gaumé and P. Ginsparg, Comm. Math. Phys. 102 (1985) 311.
[21] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Phys. Rep. 116 (1984) 105.
[22] A.M. Perelomov, Phys. Rep. 146 (1987) 137.
[23] W. Lerche, C. Vafa and N.P. Warner, Nucl. Phys. B324 (1989) 427.
[24] E. Witten, Phys. Rev. D16 (1977) 2991.
  P. Di Vecchia and S. Ferrara, Nucl. Phys. B130 (1977) 93.
[25] I. Bakas, Phys. Lett. B228 (1989) 57.
[26] A. Bilal and J.L. Gervais, Phys. Lett. B206 (1988) 412.
[27] Q-H. Park, Phys. Lett. B236 (1990) 429.
[28] R.M. Kashaev, M.V. Saveliev, S.A. Savelieva and A.M. Vershik, in “Ideas and Methods in Mathematical Analysis, Stochastics, and Applications”, eds. S. Albeverio, J.E. Fenstad, H. Holden and T. Lindstrom, Vol. I, Cambridge University Press, 1992.
[29] M.V. Saveliev, Comm. Math. Phys. 121 (1989) 283.
[30] J.F. Plebański, J. Math. Phys. 16 (1975) 2396.
[31] C.P. Boyer and J.D. Finley, III, J. Math. Phys. 23(6) (1982) 1126 and references therein.
[32] J.D. Gegenberg and A. Das, Gen. Rel. and Grav. vol. 16, No 9. (1984) 817.
[33] T. Eguchi and A.J. Hanson, Phys. Lett. B74 (1978) 249.
[34] N.J. Hitchin, A. Karlhede, U. Lindström and M. Roček, Comm. Math. Phys. 108 (1987) 535; 108 (1987) 529.
[35] P. Fendley, ‘Exact information in $N = 2$ theories,’ to appear in the proceedings of SUSY ’93.

[36] N. Seiberg, ‘Notes on quantum Liouville theory and quantum gravity,’ in “Random Surfaces and Quantum Gravity,” eds. O. Alvarez, E. Marinari and P. Windey, Plenum 1991.

[37] A.C. Davis and A.M. Matheson, Nucl. Phys. B258 (1985) 373, Phys. Lett. B179 (1986) 135.

[38] J.I. Kapusta, “Finite-Temperature Field Theory,” Cambridge monographs in Mathematical Physics, Cambridge University press.

[39] H.E. Haber and H.A. Weldon, J. Math. Phys. 23(10) (1982) 1852, Phys. Rev. D25 (1982) 502.

[40] K. Intriligator, private communication.

[41] I.M. Krichever, “The tau function of the universal Whitham hierarchy, matrix models and topological field theories”, LPTENS-92-18, May 1992, [hep-th/9205110](http://arxiv.org/abs/hep-th/9205110).

[42] K. Takasaki and T. Takebe, in Proceedings of the RIMS Research Project 91 “Infinite Analysis”, RIMS-814.

[43] A. Jevicki, Phys.Rev.D20 (1979) 3331.

[44] A. Polyakov, “Gauge Fields and Strings,” Harwood 1987.