A BALIAN–LOW TYPE THEOREM
FOR GABOR RIESZ SEQUENCES OF ARBITRARY DENSITY
ANDREI CARAGEA, DAE GWAN LEE, FRIEDRICH PHILIPP, AND FELIX VOIGTLAENDER

Abstract. We consider Gabor Riesz sequences generated by a lattice \( \Lambda \subset \mathbb{R}^2 \) and a window function \( g \in L^2(\mathbb{R}) \) which is well localized in both time and frequency. When \( g \) belongs to the Feichtinger algebra, we prove that only those time-frequency shifts with parameters from the lattice \( \Lambda \) leave the corresponding Gabor space invariant. This improves on earlier results where only lattices of rational density were considered. A slightly weaker result is proved—again for lattices of general density—under the regularity assumptions of the classical Balian-Low theorem, where both \( g \) and its Fourier transform belong to the Sobolev space \( H^1(\mathbb{R}) \). The proof relies on a combination of methods from time-frequency analysis and the theory of \( C^* \)-algebras, specifically the so-called irrational rotation algebra.

1. Introduction

When working with Gabor frames, the window function \( g \) should have a good time-frequency localization, so that the frame coefficients faithfully reflect the time-frequency behavior of the analyzed function. The Feichtinger algebra \( S_0(\mathbb{R}^d) \) \cite{12,19} is a particularly popular window class. Among other advantages, choosing a window from \( S_0 \) ensures that the canonical dual window also belongs to the Feichtinger algebra \cite{15}, so that for example the membership of a function \( f \in L^2(\mathbb{R}^d) \) in the modulation space \( M^{p,q}(\mathbb{R}^d) \) can be characterized in terms of the decay properties of its frame coefficients. One crucial obstruction, however, is that a Gabor system with window belonging to \( S_0(\mathbb{R}^d) \) can not form an orthonormal basis—in fact not even a Riesz basis—for \( L^2(\mathbb{R}^d) \). We call this phenomenon the \( S_0 \) Balian-Low theorem; it is a consequence of the Amalgam Balian-Low theorem \cite[Theorem 3.2]{2}. The same no-go type result holds for the case where \( g \) belongs to the space \( H^1(\mathbb{R}^d) \) consisting of functions in the \( L^2 \)-Sobolev space \( H^1(\mathbb{R}^d) \) whose Fourier transform also belongs to \( H^1(\mathbb{R}^d) \). This is the classical Balian-Low theorem; see \cite[Theorem 8.4.5]{14} for the case of orthonormal bases, and \cite[Theorem 2.3]{10} for the general case.

Yet, even though a Gabor system with \( g \in S_0(\mathbb{R}^d) \) cannot form a Riesz basis for all of \( L^2(\mathbb{R}^d) \), it might still be a Riesz sequence, that is, a Riesz basis for its closed linear span \( G(g, \Lambda) \), at least if \( G(g, \Lambda) \) is a proper subspace of \( L^2(\mathbb{R}^d) \). In this case, one might wonder about further properties—in addition to being a proper subspace of \( L^2(\mathbb{R}^d) \)—that the Gabor space \( G(g, \Lambda) \) has to have. One important property in time-frequency analysis is the invariance of \( G(g, \Lambda) \) under time-frequency shifts \( T_a M_b \). For lattices \( \Lambda \) of rational density and for dimension \( d = 1 \), it was observed in \cite{4} that if \( (g, \Lambda) \) is a Riesz sequence and if \( g \in S_0(\mathbb{R}) \), then the set of parameters \( (a, b) \in \mathbb{R}^2 \) such that \( G(g, \Lambda) \) is invariant under the time-frequency shift \( T_a M_b \) is exactly equal to \( \Lambda \). A multi-dimensional variant of this was derived in \cite{5}.

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These results generalize the $S_0$ Balian-Low theorem to subspaces of $L^2(\mathbb{R})$. Indeed, to derive the $S_0$ Balian-Low theorem from the above result, note that if $G(g, \Lambda) = L^2(\mathbb{R})$ then $G(g, \Lambda)$ is invariant under all time-frequency shifts, even under those with $(a, b) \notin \Lambda$; hence, $g$ cannot belong to $S_0$. A corresponding generalization of the classical Balian-Low theorem was proved in [6]; a quantitative version can be found in [7].

We emphasize that in all articles [4, 5, 6, 7] it is assumed that the generating lattice $\Lambda$ has rational density. This restriction is needed in order to utilize the Zak transform which is used extensively in [4, 5, 6, 7]. It is thus natural to ask whether the results in [4] and [6] still hold for lattices with irrational density.

In a sense, this question has analogies with the research concerning the regularity of the canonical dual window of a Gabor frame. In 1997 it was shown (see [13, Theorem 3.4]) that if $g \in S_0(\mathbb{R}^d)$ generates a Gabor frame for $L^2(\mathbb{R}^d)$ over a lattice of rational density, then the canonical dual window also belongs to $S_0(\mathbb{R}^d)$. It was conjectured in the same article that this property continues to hold for general lattices. Six years later, this conjecture was confirmed by Gröchenig and Leinert [15] by using $C^*$-algebra methods.

Here, we likewise extend the result in [4] to arbitrary lattices:

**Theorem 1.1.** If $g \in S_0(\mathbb{R})$ and $\Lambda \subset \mathbb{R}^2$ is a lattice such that the Gabor system $(g, \Lambda)$ is a Riesz basis for its closed linear span $G(g, \Lambda)$, then the time-frequency shifts $T_a M_b$ that leave $G(g, \Lambda)$ invariant satisfy $(a, b) \in \Lambda$.

As indicated above, the Zak transform is a powerful tool for analyzing Gabor systems generated by lattices with rational density; yet, it is not of much use in the case of irrational density lattices. Consequently, the methods used in the present paper differ substantially from those in [4, 5, 6, 7]; instead of applying the Zak transform and thus dealing with functions on $\mathbb{R}^2$, we work directly with the given objects and exploit the rich theory of time-frequency analysis. Along the way, we obtain several new statements related to time-frequency shift invariance that are interesting in their own right.

The proof of Theorem 1.1 consists of several steps. First, for $g \in S_0(\mathbb{R})$ and only assuming that $(g, \Lambda)$ is a frame sequence—that is, a frame for its closed linear span—we prove the following dichotomy:

**Either $(g, \Lambda)$ spans all of $L^2(\mathbb{R})$, or the set of $(a, b) \in \mathbb{R}^2$ for which $T_a M_b$ leaves $G(g, \Lambda)$ invariant is a lattice containing $\Lambda$ as a sublattice;**

see Theorem 4.2. This result significantly reduces the range of parameters $(a, b)$ that we need to consider. Next, we give a characterization for the invariance of $G(g, \Lambda)$ under a time-frequency shift $T_a M_b$ with $(a, b) \notin \Lambda$ in terms of the adjoint system of $(g, \Lambda)$; see Theorem 4.2. This characterization holds for general $g \in L^2(\mathbb{R})$, not only for $g \in S_0(\mathbb{R})$. Combining this characterization with a deep existing result about traces of projections in the so-called irrational rotation algebra (see [24, 25]), we arrive at the conclusion of Theorem 1.1.

With Theorem 1.1 established for $g$ in the Feichtinger algebra, it is natural to ask whether the same statement holds in the setting of the classical Balian-Low theorem, that is, when $g$ has finite uncertainty product $(\int x^2 |g(x)|^2 \, dx) \cdot (\int |\omega|^2 |\hat{g}(\omega)|^2 \, d\omega) < \infty$, a condition which we simply write as $g \in H^1$. Unfortunately, we were not able to prove a full-fledged version of Theorem 1.1 for $g \in H^1$; the best we could do is to show that the dichotomy (D) described above for $g \in S_0$ still holds for $g \in H^1$.

The outline of the paper is as follows: After recalling the necessary background on Janssen’s representation, time-frequency shift invariance, symplectic operators, and the
two spaces \( S_0(\mathbb{R}) \) and \( H^1 \) in Section 2, the paper proper starts in Section 3 where we prove the dichotomy \([D]\) described above, for \( g \in S_0(\mathbb{R}) + H^1 \). Next, in Section 4 we show that one can reduce to the case of a separable lattice \( \Lambda = \alpha \mathbb{Z} \times \beta \mathbb{Z} \), with an additional time-frequency shift of the form \( T_{\alpha/\nu} \) for some \( \nu \in \mathbb{N}_{\geq 2} \). For this setting, we then derive a characterization in terms of the adjoint Gabor system. Throughout Section 4, the generating function \( g \) is only assumed to be in \( L^2(\mathbb{R}) \). The paper culminates in Section 5, where we prove Theorem 1.1. Finally, Appendix A contains a short treatise on the irrational rotation algebra and a corresponding result that is crucial for our proof of Theorem 1.1.

2. Preliminaries

For \( a, b \in \mathbb{R} \) and \( f \in L^2(\mathbb{R}) \) we define the operators of translation by \( a \) and modulation by \( b \) as

\[
T_a f(x) := f(x-a) \quad \text{and} \quad M_b f(x) := e^{2\pi i bx} f(x),
\]

respectively. Both \( T_a \) and \( M_b \) are unitary operators on \( L^2(\mathbb{R}) \) and hence so is the time-frequency shift

\[
\pi(a,b) := T_a M_b = e^{-2\pi i ab} M_b T_a.
\]

A lattice \( \Lambda \subset \mathbb{R}^2 \) is any set of the form \( \Lambda = A \mathbb{Z}^2 \) with an invertible matrix \( A \in \mathbb{R}^{2 \times 2} \). The density of \( \Lambda \) is defined by \( d(\Lambda) = \lvert \det A \rvert^{-1} \). Note that \( A \mathbb{Z}^2 = \mathbb{Z}^2 \) if and only if \( A \in \mathbb{Z}^{2 \times 2} \) and \( \det A = \pm 1 \). This will be used heavily in the proof of Proposition 3.1 below.

A lattice \( \Lambda \) is called separable if \( A \) can be chosen to be diagonal, i.e., \( \Lambda = \alpha \mathbb{Z} \times \beta \mathbb{Z} \) with \( \alpha, \beta > 0 \). The next lemma shows that every lattice can be transformed into a separable one by means of a symplectic matrix; this will be used frequently.

**Lemma 2.1.** Let \( A \in \mathbb{R}^{2 \times 2} \) be a non-singular matrix. Then there exists \( C \in \mathbb{R}^{2 \times 2} \) with \( \det C = 1 \) such that \( CA \) is diagonal, i.e., \( CA \mathbb{Z}^2 \) is separable.

**Proof.** Write \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and note \( \Delta := ad - bc \neq 0 \). If \( a \neq 0 \), choose \( C = \begin{pmatrix} 1 + bc/\Delta & ab/\Delta \\ -c/a & 1 \end{pmatrix} \).

Then a simple calculation yields \( \det C = 1 \) and \( CA = \text{diag}(a, \Delta/a) \). In the case \( a = 0 \) we have \( b \neq 0 \neq c \) as \( A \) is non-singular. Then \( C := \begin{pmatrix} -d/b & 1 \\ a & 0 \end{pmatrix} \) satisfies \( \det C = 1 \) and \( CA = \text{diag}(c, -b) \). \( \square \)

For a subset \( M \subset L^2(\mathbb{R}) \), we denote its closure by \( \overline{M} \). Then, for \( g \in L^2(\mathbb{R}) \) and a lattice \( \Lambda \subset \mathbb{R}^2 \) we set

\[
(g, \Lambda) := \{ \pi(\lambda)g : \lambda \in \Lambda \} \quad \text{and} \quad \mathcal{G}(g, \Lambda) := \overline{\text{span} (g, \Lambda)} \subset L^2(\mathbb{R}).
\]

For the Fourier transform, we use the normalization \( \mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} \, dx \) for \( f \in L^1(\mathbb{R}) \). It is well-known that \( \mathcal{F} \) extends to a unitary map \( \mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \).

2.1. Bessel vectors and Janssen’s representation

Let \( \Lambda = \alpha \mathbb{Z} \times \beta \mathbb{Z} \) be a separable lattice with \( \alpha, \beta > 0 \). The adjoint lattice of \( \Lambda \) is defined as \( \Lambda^o = \frac{1}{\beta} \mathbb{Z} \times \frac{1}{\alpha} \mathbb{Z} \). We say that \( g \in L^2(\mathbb{R}) \) is a Bessel vector for \( \Lambda \) if the system \((g, \Lambda)\) is a Bessel system in \( L^2(\mathbb{R}) \), meaning that the analysis operator \( C_{\Lambda,g} \) corresponding to \((g, \Lambda)\) is bounded as an operator from \( L^2(\mathbb{R}) \) to \( l^2(\mathbb{Z}^2) \). It is defined by

\[
C_{\Lambda,g} f = ((f, T_{m\alpha} M_{n\beta} g))_{m,n \in \mathbb{Z}}, \quad f \in L^2(\mathbb{R}).
\]
We denote the set of Bessel vectors for $\Lambda$ by $B_{\Lambda}$. This is a linear subspace of $L^2(\mathbb{R})$ which is dense because it contains the Schwartz space $\mathcal{S}(\mathbb{R})$; see [14 Corollary 6.2.3]. It is well known that $B_{\Lambda} = B_{\Lambda^0}$ (see [20 Theorem 2.2(a)]) and that

$$\sum_{m,n\in \mathbb{Z}} \langle f, T_{ma} M_{n\beta}g \rangle \langle T_{ma} M_{n\beta}h, u \rangle = \frac{1}{\alpha\beta} \sum_{k,l\in \mathbb{Z}} \langle h, T_k M_{l\alpha}g \rangle \langle T_k M_{l\alpha}f, u \rangle$$  \hspace{1cm} (2.1)

whenever at least three of $f, g, h, u \in L^2(\mathbb{R})$ are Bessel vectors for $\Lambda$; this follows from [20 Proposition 2.4]. Formula (2.1) yields a useful representation (the so-called Janssen representation) of the cross frame operator $S_{\Lambda,g,h} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ associated to Bessel vectors $g, h \in B_{\Lambda}$. This operator is defined by

$$S_{\Lambda,g,h} f := \sum_{m,n\in \mathbb{Z}} \langle f, T_{ma} M_{n\beta}g \rangle \cdot T_{ma} M_{n\beta} h, \quad f \in L^2(\mathbb{R}).$$  \hspace{1cm} (2.2)

Equation (2.1) implies that

$$S_{\Lambda,g,h} f = \frac{1}{\alpha\beta} \sum_{k,l\in \mathbb{Z}} \langle h, T_k M_{l\alpha}g \rangle \cdot T_k M_{l\alpha} f \quad \text{if} \quad f, g, h \in B_{\Lambda}.$$  \hspace{1cm} (2.3)

The series in Equations (2.2) and (2.3) both converge unconditionally in $L^2(\mathbb{R})$.

### 2.2. Time-frequency shift invariance

For a closed linear subspace $\mathcal{G} \subset L^2(\mathbb{R})$, we denote by $\mathcal{I}(\mathcal{G})$ the set of all pairs $(a,b) \in \mathbb{R}^2$ such that $\mathcal{G}$ is invariant under the time-frequency shift $\pi(a,b)$; that is,

$$\mathcal{I}(\mathcal{G}) := \{ z \in \mathbb{R}^2 : \pi(z)\mathcal{G} \subset \mathcal{G} \}.$$  

If $\mathcal{G} = \mathcal{G}(g, \Lambda)$ for some $g \in L^2(\mathbb{R})$ and a lattice $\Lambda \subset \mathbb{R}^2$, then clearly $\Lambda \subset \mathcal{I}(\mathcal{G})$. Any time-frequency shift $\pi(z)$ with $z \in \mathcal{I}(\mathcal{G}) \setminus \Lambda$ will be called an additional time-frequency shift for $\mathcal{G}(g, \Lambda)$. For Gabor spaces $\mathcal{G} = \mathcal{G}(g, \Lambda)$, the set $\mathcal{I}(\mathcal{G})$ has some additional structure:

**Lemma 2.2** ([3 Proposition A.1]). Let $g \in L^2(\mathbb{R})$, let $\Lambda \subset \mathbb{R}^2$ be a lattice, and define $\mathcal{G} := \mathcal{G}(g, \Lambda)$. If $z \in \mathbb{R}^2$, then $z \in \mathcal{I}(\mathcal{G})$ if and only if $\pi(z)g \in \mathcal{G}$. Moreover, $\mathcal{I}(\mathcal{G})$ is a closed additive subgroup of $\mathbb{R}^2$.

Lemma 2.2 shows that $z \in \mathcal{I}(\mathcal{G})$ implies $-z \in \mathcal{I}(\mathcal{G})$, i.e., $\pi(z)\mathcal{G} \subset \mathcal{G}$ and $\pi(z)^{-1}\mathcal{G} \subset \mathcal{G}$. Hence, we have $\pi(z)\mathcal{G} = \mathcal{G}$ whenever $z \in \mathcal{I}(\mathcal{G})$.

The next lemma characterizes the case when $\mathcal{G}$ is invariant under all time-frequency shifts.

**Lemma 2.3.** For a closed linear subspace $\mathcal{G} \subset L^2(\mathbb{R})$, $\mathcal{G} \neq \{0\}$, we have $\mathcal{I}(\mathcal{G}) = \mathbb{R}^2$ if and only if $\mathcal{G} = L^2(\mathbb{R})$.

**Proof.** Clearly, if $\mathcal{G} = L^2(\mathbb{R})$, then $\mathcal{I}(\mathcal{G}) = \mathbb{R}^2$. Conversely, assume that $\mathcal{I}(\mathcal{G}) = \mathbb{R}^2$ and let $f \in \mathcal{G}^\perp$ and $g \in \mathcal{G} \setminus \{0\}$. Then $\langle f, \pi(z)g \rangle = 0$ for all $z \in \mathbb{R}^2$, so that the short-time Fourier transform $V_g f$ of $f$ with window $g$ satisfies $V_g f \equiv 0$. By [14 Corollary 3.2.2] and since $g \neq 0$, this implies $f = 0$. We have thus shown $\mathcal{G}^\perp = \{0\}$, whence $\mathcal{G} = L^2(\mathbb{R})$, since $\mathcal{G}$ is a closed subspace of $L^2(\mathbb{R})$.  \hspace{1cm} \blacksquare
2.3. Symplectic operators

It is often useful to reduce a statement involving a non-separable lattice to one that involves a separable lattice, since separable lattices are usually easier to handle. For this reduction, we will use so-called symplectic operators (see [14, Section 9.4]). Since we are working in dimension $d = 1$, a matrix $B \in \mathbb{R}^{2 \times 2}$ is symplectic if and only if $\det B = 1$; see [14, Lemma 9.4.1]. For any such matrix $B$, it is shown in [14, Equation (9.39)] that there exists a unitary operator $U_B : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that

$$U_B \rho(z) = \rho(Bz)U_B, \quad z \in \mathbb{R}^2,$$

where (as in [14, Page 185 and Equation (9.25)])

$$\rho(a,b) := e^{\pi i ab} \cdot \pi(a,b).$$

In the sequel, we fix for each $B \in \mathbb{R}^{2 \times 2}$ with $\det B = 1$ one choice of the operator $U_B$, and for functions $g \in L^2(\mathbb{R})$, closed subspaces $\mathcal{G} \subset L^2(\mathbb{R})$, and sets $\Lambda \subset \mathbb{R}^2$ we write

$$g_B := U_B g, \quad \mathcal{G}_B := U_B \mathcal{G}, \quad \text{and} \quad \Lambda_B := B\Lambda.$$

As shown in [14, Page 197], given $B,C \in \mathbb{R}^{2 \times 2}$ with $\det B = \det C = 1$, we have $U_BU_C = \theta_{B,C}U_{BC}$ for some $\theta_{B,C} \in \mathbb{C}$ with $|\theta_{B,C}| = 1$.

Note that (2.4) implies

$$\pi(z)g \in \mathcal{G} \iff \pi(Bz)g_B \in \mathcal{G}_B, \quad z \in \mathbb{R}^2. \quad (2.5)$$

Therefore, $(g, \Lambda)$ is a frame (Riesz basis, resp.) for its closed linear span $\mathcal{G}$ if and only if $(g_B, \Lambda_B)$ is a frame (Riesz basis, resp.) for its closed linear span $\mathcal{G}_B$. Thanks to Lemma 2.2, the equivalence (2.5) also implies that

$$\mathcal{J}(\mathcal{G}_B) = B \mathcal{J}(\mathcal{G}). \quad (2.6)$$

2.4. The Feichtinger algebra

We denote by $S_0(\mathbb{R})$ the Feichtinger algebra, which is the space of functions $f \in L^2(\mathbb{R})$ such that $(f, \pi(\cdot)\varphi) \in L^1(\mathbb{R}^2)$ for some (and hence every; see [14, Proposition 12.1.2]) Schwartz function $\varphi \neq 0$.

Recall that $S_0(\mathbb{R})$ is invariant under each operator $U_B$ (cf. [14, Proposition 12.1.3]), so that $g \in S_0(\mathbb{R})$ always implies $g_B \in S_0(\mathbb{R})$ for $B \in \mathbb{R}^{2 \times 2}$ with $\det B = 1$. Also, each $g \in S_0(\mathbb{R})$ is a Bessel vector for any (separable) lattice (see e.g. [14, Propositions 6.2.2 and 12.1.4]). Since for $g,h \in S_0(\mathbb{R})$ and any $\alpha, \beta > 0$ the sequence $(\langle h, T_{\alpha\beta}M_{\alpha\beta}g \rangle)_{m,n \in \mathbb{Z}}$ belongs to $\ell^1(\mathbb{Z}^2)$ (see [14, Corollary 12.1.12]), it follows from (2.3) and from the density of $B_\Lambda$ in $L^2(\mathbb{R})$ that

$$S_{\Lambda,g,h} = \frac{1}{\alpha\beta} \sum_{k,l \in \mathbb{Z}} \langle h, T_{\alpha}M_{\alpha}g \rangle \cdot T_{\beta}M_{\beta}g \quad \text{with} \quad \Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}, \quad (2.7)$$

where the series converges absolutely in operator norm.

2.5. The space $\mathbb{H}^1$

Let $H^1(\mathbb{R})$ denote the space of all functions $f$ in $L^2(\mathbb{R})$ for which the weak derivative $f'$ exists and belongs to $L^2(\mathbb{R})$. In other words, $H^1(\mathbb{R}) = W^{1,2}(\mathbb{R})$ is an $L^2$-Sobolev-space. It is well known (see [21, Theorem 7.16]) that each $f \in H^1(\mathbb{R})$ has a representative that is absolutely continuous on $\mathbb{R}$ and whose classical derivative exists and coincides with the weak derivative $f'$ almost everywhere.
By \( \mathbb{H}^1 \) we denote the space of all functions \( f \in H^1(\mathbb{R}) \) whose Fourier transform \( \hat{f} \) also belongs to \( H^1(\mathbb{R}) \). Equivalently, a function \( f \in L^2(\mathbb{R}) \) is in \( \mathbb{H}^1 \) if and only if \( f', Xf \in L^2(\mathbb{R}) \), where \( Xf \) represents the function \( \mathbb{R} \to \mathbb{C}, x \mapsto xf(x) \). The space \( \mathbb{H}^1 \) also coincides with the modulation space \( M^2_m(\mathbb{R}) \) with the weight \( m(x, \omega) = 1+\sqrt{x^2 + \omega^2} \); see [16, Corollary 2.3].

As shown in [16, Proof of Theorem 1.4], the space \( \mathbb{H}^1 \) is invariant under symplectic operators, meaning that \( U_B g \in \mathbb{H}^1 \) if \( g \in \mathbb{H}^1 \) and \( B \in \mathbb{R}^{2 \times 2} \) with \( \det B = 1 \).

### 3. Time-frequency shift invariance: A closer look

In this section, we first establish a certain trichotomy concerning the set of invariant time-frequency shifts. We then show that one of the three cases of the trichotomy is excluded if the generator function \( g \) is “sufficiently nice”.

The next theorem establishes the trichotomy: the invariance set \( \mathcal{I}(G) \) either fills the whole space \( \mathbb{R}^2 \), or it consists of equispaced lines that are aligned with the lattice, or it is a refinement of \( \Lambda \) (and in particular a lattice itself). Note that this holds regardless of the regularity of the generator \( g \) or the (frame) properties of the Gabor system \( (g, \Lambda) \).

**Proposition 3.1.** Let \( H \) be a closed additive subgroup of \( \mathbb{R}^2 \) and suppose that \( H \supset \Lambda \) for a non-degenerate lattice \( \Lambda \subset \mathbb{R}^2 \). Then there exist \( \lambda_1, \lambda_2 \in \Lambda \) satisfying \( \Lambda = \mathbb{Z} \cdot \lambda_1 + \mathbb{Z} \cdot \lambda_2 \) and \( m, n \in \mathbb{N}_{\geq 1} \) such that exactly one of the following conditions holds:

1. \( H = \mathbb{R}^2 \).
2. \( H = \mathbb{R} \cdot \lambda_1 + \mathbb{Z} \cdot \frac{\lambda_2}{m} \).
3. \( H = \mathbb{Z} \cdot \frac{\lambda_1}{n} + \mathbb{Z} \cdot \frac{\lambda_2}{n} \).

In particular, if \( \Lambda \subset \mathbb{R}^2 \) is a lattice and \( g \in L^2(\mathbb{R}) \), then one of the above cases holds for \( H = \mathcal{I}(G(g, \Lambda)) \).

**Proof.** By [17, Theorem 9.11], there are \( \alpha, \beta \in \mathbb{N}_0 \) and linearly independent vectors \( x_1, \ldots, x_\alpha, y_1, \ldots, y_\beta \in \mathbb{R}^2 \) (hence, \( \alpha + \beta \leq 2 \)) such that

\[
H = \mathbb{R} x_1 + \cdots + \mathbb{R} x_\alpha + \mathbb{Z} y_1 + \cdots + \mathbb{Z} y_\beta.
\]

Since \( H \) contains the non-degenerate lattice \( \Lambda \) (and thus two linearly independent vectors), we must have \( \alpha + \beta = 2 \). Hence, there are three cases:

(i) \( (\alpha, \beta) = (2, 0) \) and hence \( H = \mathbb{R}^2 \),
(ii) \( (\alpha, \beta) = (1, 1) \), so that \( H = \mathbb{R} v + \mathbb{Z} w \) with linearly independent \( v, w \in \mathbb{R}^2 \),
(iii) \( (\alpha, \beta) = (0, 2) \), so that \( H \) is a (non-degenerate) lattice.

Clearly, in Case (i), Condition (1) of the statement of the theorem holds. Let us discuss the case (ii): \( H = \mathbb{R} \cdot v + \mathbb{Z} \cdot w \). Let \( \Lambda = \mathbb{Z} \cdot \mu + \mathbb{Z} \cdot \lambda \) be an arbitrary representation of \( \Lambda \). Since \( \Lambda \subset H \), there exist \( m, n \in \mathbb{Z} \) and \( s, t \in \mathbb{R} \) such that

\[
[\mu, \lambda] = [sv + mw, tv + nw] = [v, w] \begin{bmatrix} s & t \\ m & n \end{bmatrix}.
\]

Note that \( \mu, \lambda \) are linearly independent, and hence \( sn - tm \neq 0 \), so that \( d = (sn - tm)^{-1} \) is well-defined. Furthermore, we see

\[
[v, w] = d \cdot [\mu, \lambda] \begin{bmatrix} n & -t \\ -m & s \end{bmatrix}.
\]
which shows that \( v = d(n\mu - m\lambda) \) and thus \( R \cdot v = R \cdot \lambda_1 \) with some \( \lambda_1 \in \Lambda \). By rescaling \( \lambda_1 \), we can ensure that \( \frac{1}{k} \lambda_1 \notin \Lambda \) for each \( k \in \mathbb{Z} \setminus \{-1, 0, 1\} \). Note because of \( R \cdot v = R \cdot \lambda_1 \) that \( H = R \cdot \lambda_1 + Z \cdot w \).

Now, there exists \( \lambda_2 \in \Lambda \) such that \( \Lambda = Z \cdot \lambda_1 + Z \cdot \lambda_2 \). Indeed, writing \( \Lambda = AZ^2 \) with \( A = [a_1, a_2] \in \mathbb{R}^{2 \times 2} \) invertible, there exist \( i, j \in \mathbb{Z} \) such that \( \lambda_1 = ia_1 + ja_2 \). The numbers \( i, j \) are necessarily coprime, since \( \frac{1}{k} \lambda_1 \notin \Lambda \) for \( k \in \mathbb{Z} \setminus \{-1, 0, 1\} \). Hence, by Bézout’s lemma there exist \( k, \ell \in \mathbb{Z} \) such that \( ik - jk = 1 \). Set \( \lambda_2 = ka_1 + \ell a_2 \). Then \( [\lambda_1, \lambda_2]Z^2 = A \left[ \begin{array}{c} i \\ j \end{array} \right] Z^2 = AZ^2 = \Lambda \). Since \( \lambda_2 \in \Lambda \subset H \), there exist \( \sigma \in \mathbb{R} \) and \( \nu \in \mathbb{Z} \) such that \( \lambda_2 = \sigma \lambda_1 + \nu w \). Then \( \nu \neq 0 \) and so \( H = R \cdot \lambda_1 + Z \cdot \left( \frac{\lambda_2}{\nu} - \frac{\sigma}{\nu} \lambda_1 \right) = R \cdot \lambda_1 + Z \cdot \frac{\lambda_2}{\nu} \). Hence, Condition (2) of the statement of the theorem holds.

Assume now that Case (iii) holds: \( H \) is a lattice, i.e., \( H = Z \cdot v + Z \cdot w \) with linearly independent vectors \( v, w \in \mathbb{R}^2 \). Write \( \Lambda = Z \cdot \mu + Z \cdot \lambda \) with linearly independent \( \mu, \lambda \in \mathbb{R}^2 \). Then, because of \( \Lambda \subset H \), there exist \( a, b, c, d \in \mathbb{Z} \) such that

\[
[\mu, \lambda] = [av + cw, bv + dw] = [v, w] \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] =: [v, w] \cdot A,
\]

with \( A \in \mathbb{Z}^{2 \times 2} \). Let \( A = MJD^{-1} \) be the Smith canonical form of \( A \) (see, for instance [22, Theorem 26.2] or [18, Theorem 3.8]), where \( M, N, D \in \mathbb{Z}^{2 \times 2} \) with det \( M = \det N = 1 \) and \( D \) is a diagonal matrix. Note that \( A \) (and hence \( D \)) is invertible; this follows from (3.1) since \( \mu \) and \( \lambda \) are linearly independent. Moreover, note that \( [\mu, \lambda] N D^{-1} = [v, w] M \).

Define \( \lambda_1, \lambda_2 \in \Lambda \) via \( [\lambda_1, \lambda_2] := [\mu, \lambda] N \). Then

\[
\Lambda = [\mu, \lambda] Z^2 = [\mu, \lambda] N Z^2 = [\lambda_1, \lambda_2] Z^2 = Z \cdot \lambda_1 + Z \cdot \lambda_2.
\]

Further, writing \( D = \text{diag}(m, n) \) with \( m, n \in \mathbb{Z} \setminus \{0\} \), we see

\[
H = [v, w] Z^2 = [v, w] M Z^2 = [\mu, \lambda] N D^{-1} Z^2 = [\lambda_1, \lambda_2] D^{-1} Z^2 = Z \cdot \frac{\lambda_1}{|m|} + Z \cdot \frac{\lambda_2}{|n|}.
\]

This completes the proof of the theorem, since the conditions (1)–(3) are clearly mutually exclusive.

The example below shows that Case (2) in Proposition 3.1 can occur for \( H = \mathcal{J}(G(g, \Lambda)) \) for every lattice \( \Lambda \) with density smaller than one—even if \((g, \Lambda)\) is a Riesz sequence.

**Example 3.2.** Due to Equation (2.6) and Lemma 2.1 it suffices to construct an example for a separable lattice \( \Lambda = \alpha \mathbb{Z} \times \beta \mathbb{Z} \) with \( \alpha, \beta > 0 \), \( \alpha \beta > 1 \). For \( m \in \mathbb{Z} \), define \( E_m := m\alpha + [0, \frac{1}{\beta}] \), and let \( g := \sqrt{\beta} \cdot \mathbf{1}_{E_0} \). Then

\[
M_{n\beta} T_{m\alpha} g(x) = \sqrt{\beta} \cdot e^{2\pi i m \beta x} \cdot \mathbf{1}_{[0, \frac{1}{\beta}]}(x - m\alpha) = \sqrt{\beta} \cdot e^{2\pi i m \beta x} \cdot \mathbf{1}_{E_m}(x).
\]

Hence, for any \( m \in \mathbb{Z} \) the system \((T_{m\alpha} M_{n\beta} g)_{n \in \mathbb{Z}}\) is an orthonormal basis for the subspace \( L^2(E_m) \) of \( L^2(\mathbb{R}) \). Note that, since \( \frac{1}{\beta} < \alpha \), we have \([0, \frac{1}{\beta}] \subseteq [0, \alpha] \). The system \((g, \Lambda)\) is thus an orthonormal basis for \( \mathcal{G} := L^2(E) \subset L^2(\mathbb{R}) \), where \( E = \bigcup_{m \in \mathbb{Z}} E_m \). Note that \( \mathbb{R} \setminus E \) has positive (even infinite) measure, so \( \mathcal{G} \subseteq L^2(\mathbb{R}) \). Moreover, for any \( \omega \in \mathbb{R} \) we have \( M_{n\beta} g = \sqrt{\beta} \cdot e^{2\pi i n \omega} \cdot \mathbf{1}_{E_0} \in L^2(E_0) \subset \mathcal{G} \). Therefore, Lemma 2.2 shows \([0] \times \mathbb{R} \subset \mathcal{J}(\mathcal{G})\), which can only occur in Case (2) of Proposition 3.1 since we would have \( \mathcal{G} = L^2(\mathbb{R}) \) in Case (1), see Lemma 2.3.

Note that the function \( g \) in Example 3.2 is not well localized in frequency. In the remainder of this section, we show that Case (2) in Proposition 3.1 cannot occur if \((g, \Lambda)\) is a frame sequence with a sufficiently nice window \( g \). In this case, the trichotomy
from Proposition 3.1 becomes a dichotomy. By $g$ being “sufficiently nice” we mean that $g \in \mathcal{W}(C, \ell^2)$, where

$$\mathcal{W}(C, \ell^2) := \{f \in L^2(\mathbb{R}) : U_B f \in W(C, \ell^2) \text{ for all } B \in \mathbb{R}^{2 \times 2} \text{ with } \det B = 1\}.$$ 

Here, $W(C, \ell^2)$ is the so-called Wiener Amalgam space consisting of all continuous functions $f : \mathbb{R} \to \mathbb{C}$ satisfying

$$\|f\|_{W(C, \ell^2)} := \left(\sum_{k \in \mathbb{Z}} \sup_{x \in [k-1,k+1]} |f(x)|^2 \right)^{1/2} < \infty.$$ 

Recall from Section 2.3 that if $g \in \mathcal{W}(C, \ell^2)$ and if $B, C \in \mathbb{R}^{2 \times 2}$ satisfy $\det B = \det C = 1$, then there is $\theta_{B,C} \in \mathbb{C}$ satisfying $U_C U_B g = \theta_{B,C} U_{CB} g \in W(C, \ell^2)$. This shows that $U_{Bg} \in \mathcal{W}(C, \ell^2)$ whenever $g \in \mathcal{W}(C, \ell^2)$ and $\det B = 1$.

Before we prove the announced theorem let us show that the function classes considered in this paper (namely, $S_0(\mathbb{R})$ and $H^1$) are contained in $\mathcal{W}(C, \ell^2)$.

**Lemma 3.3.** We have $S_0(\mathbb{R}) \subset \mathcal{W}(C, \ell^2)$ and $H^1 \subset \mathcal{W}(C, \ell^2)$.

**Proof.** If $g \in S_0(\mathbb{R})$, then [14, Proposition 12.1.3] shows that $U_{Bg} \in S_0(\mathbb{R})$ for each $B \in \mathbb{R}^{2 \times 2}$ with $\det B = 1$. Similarly, if $g \in H^1$, then [6, Proof of Theorem 1.4] shows that $U_{Bg} \in H^1$ for each $B \in \mathbb{R}^{2 \times 2}$ with $\det B = 1$. Therefore, it suffices to show that $S_0(\mathbb{R}) \subset W(C, \ell^2)$ and $H^1 \subset W(C, \ell^2)$.

First, if $f \in S_0(\mathbb{R})$, then [14, Proposition 12.1.4] shows that $\hat{f} \in L^1(\mathbb{R})$. By Fourier inversion, this implies that $f$ has a continuous representative. Furthermore, by [14, Proposition 12.1.4] we have $f \in W(L^\infty, \ell^1)$. Combined with the embedding $\ell^1(\mathbb{Z}) \hookrightarrow \ell^2(\mathbb{Z})$, this easily implies $f \in W(L^\infty, \ell^2)$ and thus $f \in W(C, \ell^2)$.

Next, if $f \in H^1 \subset H^1 = W^{1,2}(\mathbb{R})$, then [21, Theorem 7.16] shows (after changing $f$ on a null-set) that $f$ is absolutely continuous, and hence continuous, and satisfies $f(x) - f(y) = \int_y^x f'(t) \, dt$ for all $y < x$, where $f' \in L^2(\mathbb{R})$ is the weak derivative of $f$. Now, note that if $n \in \mathbb{Z}$ and $x, y \in [n-1, n+1]$, then

$$|f(x)| \leq |f(y)| + \int_{\max\{x,y\}}^{\min\{x,y\}} |f'(t)| \, dt \leq |f(y)| + \int_{n-1}^{n+1} |f'(t)| \, dt$$

$$\leq |f(y)| + \sqrt{2} \left(\int_{n-1}^{n+1} |f'(t)|^2 \, dt \right)^{1/2},$$

and hence $|f(x)|^2 \leq 2|f(y)|^2 + 4 \int_{n-1}^{n+1} |f'(t)|^2 \, dt$. Integrating this over $y \in [n-1, n+1]$ gives

$$2|f(x)|^2 \leq 2 \int_{n-1}^{n+1} |f(y)|^2 \, dy + 8 \int_{n-1}^{n+1} |f'(t)|^2 \, dt,$$ 

for all $x \in [n-1, n+1]$, which finally implies

$$\|f\|^2_{L^2} \leq \sum_{n \in \mathbb{Z}} \left(\int_{n-1}^{n+1} |f(y)|^2 \, dy + 4 \int_{n-1}^{n+1} |f'(t)|^2 \, dt \right) \leq \|f\|^2_{L^2} + \|f'\|^2_{L^2} < \infty,$$

and hence $f \in W(C, \ell^2)$. □

Our next result shows that Case (2) in Proposition 3.1 cannot occur if $(g, \Lambda)$ is a frame sequence with generator $g \in \mathcal{W}(C, \ell^2) \setminus \{0\}$.
Theorem 3.4. Let \( g \in \mathcal{W}(C, \ell^2) \setminus \{0\} \) and let \( \Lambda \subset \mathbb{R}^2 \) be a lattice such that \((g, \Lambda)\) is a frame for \( \mathcal{G} = \mathcal{G}(g, \Lambda) \). Then either \( \mathcal{I}(\mathcal{G}) = \mathbb{R}^2 \) or there exist \( \lambda_1, \lambda_2 \in \Lambda \) and \( m, n \in \mathbb{N}_{\geq 1} \) such that
\[
\Lambda = \mathbb{Z} \cdot \lambda_1 + \mathbb{Z} \cdot \lambda_2 \quad \text{and} \quad \mathcal{I}(\mathcal{G}) = \mathbb{Z} \cdot \frac{1}{m} + \mathbb{Z} \cdot \frac{1}{n}.
\] (3.2)

Proof. Let us assume \( \mathcal{I}(\mathcal{G}) \subsetneq \mathbb{R}^2 \). Writing \( H = \mathcal{I}(\mathcal{G}) \), the two possibilities in (3.2) represent the cases (1) and (3) from the trichotomy in Proposition 3.1. It is thus enough to show that Case (2) from that theorem cannot occur. Therefore, we assume towards a contradiction that Case (2) holds, i.e., there are \( \lambda_1, \lambda_2 \in \Lambda \) and \( n \in \mathbb{N}_{\geq 1} \) such that \( \Lambda = \mathbb{Z} \cdot \lambda_1 + \mathbb{Z} \cdot \lambda_2 \) and \( \mathcal{I}(\mathcal{G}) = \mathbb{Z} \cdot \frac{1}{n} + \mathbb{R} \cdot \lambda_2 \).

Step 1. We first derive a contradiction for the case \( \lambda_1 = (\alpha, 0)^T \) and \( \lambda_2 = (0, \beta)^T \) with some \( \alpha, \beta > 0 \). Then \( \Lambda = \alpha \mathbb{Z} \times \beta \mathbb{Z} \), and \( \{0\} \times \mathbb{R} \subset \mathcal{I}(\mathcal{G}) \). For \( f \in \mathcal{G} \) we thus have \( M_\omega f \in \mathcal{G} \) for all \( \omega \in \mathbb{R} \). By [27, Theorem 9.17] (applied to the translation invariant space \( \mathcal{F}^{-1} \mathcal{G} \), with \( \mathcal{F} \) denoting the Fourier transform), there exists a Borel measurable set \( E \subset \mathbb{R} \) such that \( \mathcal{G} = \mathcal{L}^2(E) \), where we consider \( \mathcal{L}^2(E) \) as a closed subspace of \( \mathcal{L}^2(\mathbb{R}) \), in the sense that \( \mathcal{L}^2(E) = \{ f \in \mathcal{L}^2(\mathbb{R}) : f = 0 \text{ a.e. on } \mathbb{R} \setminus E \} \).

Our goal is to show that \( E = \mathbb{R} \), up to null-sets. This will imply \( \mathcal{G} = \mathcal{L}^2(E) = \mathcal{L}^2(\mathbb{R}) \) and hence \( \mathcal{I}(\mathcal{G}) = \mathbb{R}^2 \), providing the desired contradiction. Towards proving \( E = \mathbb{R} \), let us consider for given \( f \in \mathcal{L}^2(\mathbb{R}) \) the continuous function \( \Gamma_f : \mathbb{R} \to \mathbb{R} \) defined by
\[
\Gamma_f(\omega) := \langle SM_\omega f, M_\omega f \rangle, \quad \omega \in \mathbb{R},
\]
where \( S : \mathcal{L}^2(\mathbb{R}) \to \mathcal{G} \) denotes the frame operator of \((g, \Lambda)\). By [14, Proposition 7.1.1], the operator \( S \) has the Walnut representation
\[
\langle Sf, h \rangle = \beta^{-1} \sum_{n \in \mathbb{Z}} \langle G_n \cdot T_{\frac{\omega}{\beta}} f, h \rangle \quad \forall f, h \in L^\infty(\mathbb{R}) \text{ with compact support},
\]
where only finitely many terms of the sum do not vanish, and where
\[
G_n(x) := \sum_{m \in \mathbb{Z}} g(x - ma) \cdot \overline{g(x - \frac{n}{\beta} - ma)}, \quad x \in \mathbb{R}, n \in \mathbb{Z}.
\]
The fact that \( g \in \mathcal{W}(C, \ell^2) \) easily implies that the series defining \( G_n \) converges locally uniformly, and that the \( G_n \) are continuous functions. Since \( G_n \) is also \( \alpha \)-periodic, this means that each \( G_n \) is bounded.

Now, since multiplication with \( G_n \) commutes with the modulation \( M_\omega \), using the identity \( T_{\frac{\omega}{\beta}} M_\omega = e^{-2\pi i \frac{\omega}{\beta}} M_\omega T_{\frac{\omega}{\beta}} \), we get
\[
\Gamma_f(\omega) = \beta^{-1} \sum_{n \in \mathbb{Z}} e^{-2\pi i \frac{\omega}{\beta} n} \langle G_n \cdot T_{\frac{\omega}{\beta}} f, f \rangle \quad \forall f \in L^\infty(\mathbb{R}) \text{ with compact support},
\] (3.3)
where there are only finitely many \( n \in \mathbb{Z} \) (depending only on \( f \), but not on the choice of \( \omega \)) for which \( \langle G_n \cdot T_{\frac{\omega}{\beta}} f, f \rangle \neq 0 \).

As \((g, \Lambda)\) is a frame for \( \mathcal{G} \) and \( M_\omega f \in \mathcal{G} \) for all \( \omega \in \mathbb{R} \) and \( f \in \mathcal{G} \), there exists \( A > 0 \) such that \( \Gamma_f(\omega) = \langle SM_\omega f, M_\omega f \rangle \geq A \| f \|_{L^2}^2 \) for all \( f \in \mathcal{G} \). Let us write \( L^\infty_c(\mathbb{R}) \) for the set of all compactly supported \( f \in L^\infty(\mathbb{R}) \) which satisfy \( f = 0 \) on \( \mathbb{R} \setminus E \), and note that \( L^\infty_c(\mathbb{E}) \subset L^2(E) = \mathcal{G} \). For \( f \in L^\infty_c(\mathbb{E}) \), integrate the estimate \( \Gamma_f(\omega) \geq A \| f \|_{L^2}^2 \) over \([0, \beta] \) and apply Equation (3.3) to see
\[
\beta A \| f \|_{L^2}^2 \leq \beta^{-1} \sum_{n \in \mathbb{Z}} \langle G_n \cdot T_{\frac{\omega}{\beta}} f, f \rangle \int_0^\beta e^{-2\pi i \frac{\omega}{\beta} n} d\omega = \langle G_0 f, f \rangle = \langle hf, f \rangle,
\]
where \( h := G_0 = \sum_{m \in \mathbb{Z}} |T_{ma}g|^2 \). We have thus shown
\[
\int_E (h(x) - \beta A) \cdot |f(x)|^2 \, dx \geq 0 \quad \forall f \in L^\infty(E).
\]
Using standard arguments, this implies that \( h(x) \geq \beta A \) for almost all \( x \in E \).

Since \( T_{ma}g \in \mathcal{G} = L^2(E) \) and thus \( T_{ma}g(x) = 0 \) for almost all \( x \in \mathbb{R} \setminus E \) and arbitrary \( m \in \mathbb{Z} \), it follows that \( h(x) = 0 \) for almost all \( x \in \mathbb{R} \setminus E \). Recall from above that \( h(x) \geq \beta A \) for almost all \( x \in E \); thus, \( h(x) \in \{0\} \cup [\beta A, \infty) \) almost everywhere. Also recall from above that \( h = G_0 \) is continuous. Hence, the open set \( \{0\} \cup [\beta A, \infty) \) has measure zero and is thus empty; that is, \( h(x) \in \{0\} \cup [\beta A, \infty) \) for all \( x \in \mathbb{R} \). By the intermediate value theorem, this implies that \( h(x) \geq \beta A \) for all \( x \in \mathbb{R} \) (since \( h \geq |g|^2 \) and \( g \neq 0 \)) and thus, indeed, \( E = \mathbb{R} \) (up to null-sets), since \( h(x) = 0 \) a.e. on \( \mathbb{R} \setminus E \).

**Step 2.** Let \( \Lambda \) be a general lattice. Recall that \( \Lambda = \mathbb{Z} \lambda_1 + \mathbb{Z} \lambda_2 \) and \( \mathcal{J}(\mathcal{G}) = \mathbb{Z}_{\alpha, \beta} - \mathbb{R} \lambda_2 \). By Lemma 2.1 there exists \( B \in \mathbb{R}^{2 \times 2} \) with det \( B = 1 \) such that \( B[\lambda_1, \lambda_2] = \text{diag}(\alpha, \beta) \) for certain \( \alpha, \beta \in \mathbb{R} \setminus \{0\} \). We thus obtain \( \Lambda_B = B\Lambda = B[\lambda_1, \lambda_2] \mathbb{Z}^2 = |\alpha| \mathbb{Z} \times |\beta| \mathbb{Z} \) and
\[
\mathcal{J}(\mathcal{G}_B) = B\mathcal{J}(\mathcal{G}) = B[\lambda_1, \lambda_2] \text{diag}(\alpha, \beta) \mathbb{Z} \times \mathbb{R} = |\alpha| \mathbb{Z} \times |\beta| \mathbb{Z} \times \mathbb{R};
\]
see (2.6). In particular, \( \{0\} \times \mathbb{R} \subset \mathcal{J}(\mathcal{G}_B) \). Hence, since \( g_B = U_Bg \in \mathcal{W}(C, \ell^2) \) and \( (g_B, \Lambda_B) \) is a frame for \( \mathcal{G}_B = U_B \mathcal{G} \subset \ell^2(\mathbb{R} \setminus \{0\}) \) (cf. Subsection 2.3), we are in the situation of Step 1 of which we proved to be impossible. \( \square \)

By combining Theorem 3.4 and Lemma 3.3, we obtain the following corollary.

**Corollary 3.5.** Let \( g \in S_0(\mathbb{R}) \setminus \{0\} \) or \( g \in \mathbb{H}_1 \setminus \{0\} \) and let \( \Lambda \subset \mathbb{R}^2 \) be a lattice such that \( (g, \Lambda) \) is a Riesz basis for \( \mathcal{G} := \mathcal{G}(g, \Lambda) \). Then \( \mathcal{J}(\mathcal{G}) \) is a refinement of \( \Lambda \) as in (3.2).

**Proof.** By the Balian-Low theorem [10] Theorem 2.3] and the Amalgam Balian-Low theorem [2 Theorem 3.2], it is not possible that \( \mathcal{G} = L^2(\mathbb{R}) \). Therefore, Lemma 2.3 implies \( \mathcal{J}(\mathcal{G}) \neq \mathbb{R}^2 \). The rest follows from Lemma 3.3 and Theorem 3.4. \( \square \)

### 4. Time-frequency shift invariance: Duality

Let us consider a Gabor Riesz sequence \( (g, \Lambda) \) with \( g \in \mathcal{W}(C, \ell^2) \) as in the previous section, and assume that \( \mathcal{G} := \mathcal{G}(g, \Lambda) \subset L^2(\mathbb{R}) \), but that there exists an additional time-frequency shift, meaning \( \mathcal{J}(\mathcal{G}) \neq \Lambda \). In view of Theorem 1.1 it is our goal to show that this is impossible, at least if \( g \in S_0 \). To make the situation more accessible, we first reduce to the case where \( \Lambda = \alpha \mathbb{Z} \times \beta \mathbb{Z} \) is separable, and where the additional time-frequency shift is of the form \( (\frac{\nu m}{n}, 0)^T \) for some \( \nu \in \mathbb{N}_{\geq 2} \), meaning that \( T_{\nu/\nu} g \in \mathcal{G} \). After that, we provide a characterization of this simplified condition in terms of the adjoint Gabor system. It is this characterization that we will use to prove our main result, Theorem 1.1 in the next section.

**Lemma 4.1.** Let \( g \in \mathcal{W}(C, \ell^2) \setminus \{0\} \) and let \( \Lambda \subset \mathbb{R}^2 \) be a lattice such that \( (g, \Lambda) \) is a frame for \( \mathcal{G} := \mathcal{G}(g, \Lambda) \). If \( \mathcal{G} \neq L^2(\mathbb{R}) \) and \( \mathcal{J}(\mathcal{G}) \neq \Lambda \), there exist a matrix \( B \in \mathbb{R}^{2 \times 2} \) with det \( B = 1 \) and \( \alpha, \beta > 0 \) such that \( \Lambda_B = \alpha \mathbb{Z} \times \beta \mathbb{Z} \) and \( (\frac{\nu m}{n}, 0)^T \in \mathcal{J}(\mathcal{G}_B) \) for some \( \nu \in \mathbb{N}, \nu \geq 2 \) (i.e., \( T_{\nu/\nu} g_B \in \mathcal{G}_B \)).

**Proof.** Due to Theorem 3.4 and Lemma 2.3 we have \( \Lambda = [\lambda_1, \lambda_2] \mathbb{Z}^2 \) and \( \mathcal{J}(\mathcal{G}) = [\frac{\lambda_1}{m}, \frac{\lambda_2}{n}] \mathbb{Z}^2 \) for suitable vectors \( \lambda_1, \lambda_2 \in \mathbb{R}^2 \) and \( m, n \in \mathbb{N} \setminus \{0\} \). We may safely assume that \( m \neq 1 \).
Indeed, since \( \mathcal{J}(G) \neq \Lambda \), we have \((m,n) \neq (1,1)\). If \( m = 1 \), then with \( J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) also \( \Lambda = [\lambda_1, \lambda_2]JZ^2 = [-\lambda_2, \lambda_1]Z^2 \) and \( \mathcal{J}(G) = \begin{bmatrix} -\lambda_2 \\ \lambda_1 \end{bmatrix}JZ^2 \).

Now, by Lemma 2.1 there exists a matrix \( B \in \mathbb{R}^{2 \times 2} \) with \( \det B = 1 \) such that \( B[\lambda_1, \lambda_2] = \text{diag}(\alpha, \beta) \), where \( \alpha, \beta \in \mathbb{R} \setminus \{0\} \). Hence, \( \Lambda_B = B\Lambda = |\alpha|Z \times |\beta|Z \) and

\[
\mathcal{J}(G_B) = B\mathcal{J}(G) = B[\lambda_1, \lambda_2] \text{diag}(\frac{1}{m}, \frac{1}{n})Z^2 = \frac{|\alpha|}{m}Z \times \frac{|\beta|}{n}Z;
\]

see \((2.6)\). In particular, \((\frac{|\alpha|}{m}, 0)^T \in \mathcal{J}(G_B) \) and \( m \geq 2 \). \( \square \)

In what follows, fix \( g \in L^2(\mathbb{R}) \), \( \alpha, \beta > 0 \), \( \Lambda = \alpha Z \times \beta Z \), and \( \nu \in \mathbb{N}_{\geq 2} \), and assume that \((g, \Lambda)\) is a frame for \( G = G(g, \Lambda) \). The adjoint system \( \mathcal{F} := \{T_{k/\beta}M_{\ell/\alpha}g : k, \ell \in \mathbb{Z}\} \) is then a frame for its closed linear span \( K \) by \cite{24} Theorem 2.2 (c). Note that \( K = L^2(\mathbb{R}) \) if and only if \((g, \Lambda)\) is a Riesz sequence (cf. \cite{24} Thm. 2.2 (e)) or \cite{14} Theorem 7.4.3).

It is a natural question to ask what the existence of an additional time-frequency shift of the form \( T_{\frac{s}{\alpha}} g \in G \) means for the adjoint system \( \mathcal{F} \). To describe this, we set

\[
\mathcal{F}_s := \{T_{k/\beta}M_{\ell/\alpha}M_{s/\alpha}g : k, \ell \in \mathbb{Z}\}, \quad s = 0, \ldots, \nu - 1.
\]

Again by \cite{24} Theorem 2.2 (c), \( \mathcal{F}_0 \) is a frame sequence if and only if the system \((g, \frac{\alpha}{\nu}Z \times \beta Z)\) is a frame sequence. In this case, each \( \mathcal{F}_s \) is a frame sequence because \( M_{s/\alpha} \mathcal{F}_0 \) is, and multiplying the vectors of a frame sequence by unimodular constants results in a frame sequence. We set \( \mathcal{L}_s := \text{span} \mathcal{F}_s \) for \( s = 0, \ldots, \nu - 1 \). Note that

\[
K = \mathcal{L}_0 + \cdots + \mathcal{L}_{\nu-1}.
\]

Indeed, the inclusion \( \subset \) is trivial. Conversely, since \( \mathcal{F} \) is a frame sequence, each \( f \in K \) satisfies \( f = \sum_{k, \ell \in \mathbb{Z}} c_{k, \ell}T_{k/\beta}M_{\ell/\alpha}g \) with a suitable sequence \( c = (c_{k, \ell})_{k, \ell \in \mathbb{Z}} \in \ell^2(\mathbb{Z}^2) \). Since \( \mathcal{F} \) is a Bessel sequence, \( f_s := \sum_{k, \ell \in \mathbb{Z}} c_{k, \ell}T_{k/\beta}M_{\ell/\alpha}M_{s/\alpha}g \in \mathcal{L}_s \) is well-defined for \( s = 0, \ldots, \nu - 1 \), and \( f = f_0 + \cdots + f_{\nu-1} \in \mathcal{L}_0 + \cdots + \mathcal{L}_{\nu-1} \). Finally, it is clear that \( \mathcal{L}_s = M_{s/\alpha} \mathcal{L}_0 \).

In the sequel, the symbol \( \oplus \) denotes the direct (not necessarily orthogonal) sum of subspaces, whereas \( \oplus \) is used to denote an orthogonal sum. The next theorem characterizes the existence of an additional time-frequency shift for \( G \) in terms of properties of the adjoint system \( \mathcal{F} \).

**Theorem 4.2.** Let \( g \in L^2(\mathbb{R}) \) and \( \alpha, \beta > 0 \), and assume that \((g, \alpha \mathbb{Z} \times \beta \mathbb{Z})\) is a frame sequence with canonical dual window \( \gamma \) in \( G \), where \( G = G(g, \alpha \mathbb{Z} \times \beta \mathbb{Z}) \). Let \( \nu \in \mathbb{N}_{\geq 2} \), and define the systems \( \mathcal{F}_s \) and the spaces \( \mathcal{K}, \mathcal{L}_s \) as above, and set \( S_{s, \gamma} := S_{\frac{\alpha}{\nu}Z \times \frac{\beta}{\nu}Z, \gamma} \), with notation as in Equation \((2.2)\). Then the following statements are equivalent:

**(i)** \( T_{\frac{s}{\alpha}} g \in G \).

**(ii)** \( (\alpha \beta)^{-1}S_{s, \gamma}M_{\frac{s}{\alpha}}g = \delta_{s, 0} \cdot g \) for \( s = 0, \ldots, \nu - 1 \).

**(iii)** \( \mathcal{K} = \mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_{\nu-1} \).

**(iv)** \( \langle T_{\frac{k}{\beta}}M_{\frac{\ell}{\alpha}}g, \gamma \rangle = 0 \) for all \( k \in \mathbb{Z} \) and all \( \ell \in \mathbb{Z} \setminus \nu \mathbb{Z} \).

If one of (i)–(iv) holds, then for each \( s = 0, \ldots, \nu - 1 \) the system \( \mathcal{F}_s \) is a frame for \( \mathcal{L}_s \) and the operator \( P_s := (\alpha \beta)^{-1}M_{s/\alpha}S_{\gamma, \gamma}M_{s/\alpha} \) is the (possibly non-orthogonal) projection onto \( \mathcal{L}_s \) with respect to the decomposition \( L^2(\mathbb{R}) = (\mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_{\nu-1}) \oplus \mathcal{K}^\perp \).
Proof. First, note by Ron-Shen duality (see [20] Theorem 2.2(c))) that $\mathcal{F}$ is a frame sequence. We will frequently use the following fact (see [20] Theorem 2.3]):

$$(\alpha\beta)^{-1}\gamma$$ is the canonical dual window of $\mathcal{F} = \{T_k M_{\ell} g : k, \ell \in \mathbb{Z}\}; \quad (4.1)$$

in particular, $\gamma \in \text{span} \mathcal{F} = \mathcal{K}$.

For the rest of the proof we set $P := (\alpha\beta)^{-1} S_{\gamma,g} = (\alpha\beta)^{-1} S_{\frac{1}{\sqrt{\nu}} \mathbb{Z} \times \frac{1}{\alpha} \mathbb{Z}, \gamma,g}$. It is well known (see for instance [14] Equation (5.25))] that

$$PT_k M_{\ell} \frac{Z}{\alpha} = T_k M_{\ell} P \frac{Z}{\alpha} \quad \text{for all } k, \ell \in \mathbb{Z}. \quad (4.2)$$

Moreover, Equation (2.3) applied to the lattice $\frac{1}{\sqrt{\nu}} \mathbb{Z} \times \frac{1}{\alpha} \mathbb{Z}$ shows for $f \in B_{\frac{1}{\sqrt{\nu}} \mathbb{Z} \times \frac{1}{\alpha} \mathbb{Z}}$ that

$$Pf = \sum_{m,n \in \mathbb{Z}} c_{m,n} \cdot T_{\frac{m,n}{\alpha}} M_{\nu} f \quad \text{with} \quad c_{m,n} = \frac{1}{\nu} \langle g, T_{\frac{m,n}{\alpha}} M_{\nu} \rangle. \quad (4.3)$$

Let us denote the orthogonal projection onto the subspace $\mathcal{K} = \text{span} \mathcal{F}$ by $P_{\mathcal{K}}$. Note that Equation (4.1) implies $S_{\frac{1}{\sqrt{\nu}} \mathbb{Z} \times \frac{1}{\alpha} \mathbb{Z}, (\alpha\beta)^{-1}, \gamma,g} |_{\mathcal{K} \perp} \equiv 0$, so that $P_{\mathcal{K}} = S_{\frac{1}{\sqrt{\nu}} \mathbb{Z} \times \frac{1}{\alpha} \mathbb{Z}, (\alpha\beta)^{-1}, \gamma,g}$. Similarly, the orthogonal projection $P_{\mathcal{G}}$ onto $\mathcal{G}$ satisfies $P_{\mathcal{G}} = S_{\frac{1}{\sqrt{\nu}} \mathbb{Z} \times \frac{1}{\alpha} \mathbb{Z}, \gamma,g}$. Next, using (4.3) and the elementary identity $\sum_{s=0}^{\nu-1} e^{2\pi i \frac{m\nu}{\nu}} = \nu \cdot 1_{\nu \mathbb{Z}}(m)$, we obtain

$$M_{-\frac{1}{\alpha}} P M_{-\frac{1}{\alpha}} f = \sum_{s=0}^{\nu-1} \sum_{m,n \in \mathbb{Z}} c_{m,n} \cdot M_{-\frac{1}{\alpha}} T_{\frac{m,n}{\alpha}} M_{\nu} M_{-\frac{1}{\alpha}} f = \sum_{m,n \in \mathbb{Z}} \sum_{s=0}^{\nu-1} c_{m,n} \cdot T_{\frac{m,n}{\alpha}} M_{\nu} M_{-\frac{1}{\alpha}} f = \sum_{m,n \in \mathbb{Z}} \sum_{s=0}^{\nu-1} \langle g, T_{\frac{m,n}{\alpha}} M_{\nu} M_{-\frac{1}{\alpha}} \rangle \cdot T_{\frac{m,n}{\alpha}} M_{\nu} M_{-\frac{1}{\alpha}} f \quad \text{(Equation 2.3)}$$

for all $f \in B_{\frac{1}{\sqrt{\nu}} \mathbb{Z} \times \frac{1}{\alpha} \mathbb{Z}}$ and hence for all $f \in L^2(\mathbb{R})$ by density. Here, we used that $f \in B_{\frac{1}{\sqrt{\nu}} \mathbb{Z} \times \frac{1}{\alpha} \mathbb{Z}}$ implies $M_{-\frac{1}{\alpha}} f \in B_{\frac{1}{\sqrt{\nu}} \mathbb{Z} \times \frac{1}{\alpha} \mathbb{Z}} \subset B_{\frac{1}{\sqrt{\nu}} \mathbb{Z} \times \frac{1}{\alpha} \mathbb{Z}}$. Next, for $s = 0, \ldots, \nu - 1$, we see by another application of Equation (4.3) that

$$M_{-\frac{1}{\alpha}} P M_{-\frac{1}{\alpha}} g = \sum_{m,n \in \mathbb{Z}} \langle g, T_{\frac{m,n}{\alpha}} M_{\nu} M_{-\frac{1}{\alpha}} \rangle \cdot M_{-\frac{1}{\alpha}} T_{\frac{m,n}{\alpha}} M_{\nu} M_{-\frac{1}{\alpha}} g = \sum_{m,n \in \mathbb{Z}} \sum_{r=0}^{\nu-1} \langle g, T_{\frac{m-n\alpha}{\nu}} M_{\nu} M_{-\frac{1}{\alpha}} \rangle \cdot M_{-\frac{1}{\alpha}} T_{\frac{m-n\alpha}{\nu}} M_{\nu} M_{-\frac{1}{\alpha}} g$$

$$= \nu \sum_{m,n \in \mathbb{Z}} \sum_{r=0}^{\nu-1} \langle g, T_{\frac{m-n\alpha}{\nu}} M_{\nu} M_{-\frac{1}{\alpha}} \rangle \cdot M_{-\frac{1}{\alpha}} T_{\frac{m-n\alpha}{\nu}} M_{\nu} M_{-\frac{1}{\alpha}} g$$

$$= \nu \sum_{r=0}^{\nu-1} e^{2\pi i \frac{r\nu}{\nu}} \cdot T_{-\frac{r\nu}{\nu}} \sum_{m,n \in \mathbb{Z}} \langle g, T_{\frac{m\nu}{\nu}} M_{\nu} M_{-\frac{1}{\alpha}} \rangle \cdot T_{\frac{m\nu}{\nu}} M_{\nu} M_{-\frac{1}{\alpha}} g$$

$$= \nu \sum_{r=0}^{\nu-1} e^{2\pi i \frac{r\nu}{\nu}} \cdot T_{-\frac{r\nu}{\nu}} P_{\mathcal{G}} T_{\frac{r\nu}{\nu}} g,$$

where $P_{\mathcal{G}}$ is the orthogonal projection onto $\mathcal{G}$. Equation (4.3) shows that the vectors

$$v = \left( M_{-\frac{1}{\alpha}} P M_{-\frac{1}{\alpha}} \right)_{s=0}^{\nu-1} \quad \text{and} \quad u = \left( T_{-\frac{r\nu}{\nu}} P_{\mathcal{G}} T_{\frac{r\nu}{\nu}} \right)_{r=0}^{\nu-1}$$

...
in \((L^2(\mathbb{R}))^\nu\) satisfy \(F_\omega u = \sqrt{\nu} \cdot v\), where \(F_\omega\) is the DFT-matrix \(F_\omega = \nu^{-1/2} (\omega^{sr})_{s,r=0}^{\nu-1}\) with \(\omega = e^{2\pi i/\nu}\).

With this, we now prove the equivalence of the statements (i)–(iv).

(i) \iff (ii): If \(T_\omega g \in \mathcal{G}\), then Lemma 2.2 shows that \(T_\omega g \in \mathcal{G}\) for all \(r \in \mathbb{Z}\), so that \(T_\omega g \in \mathcal{G}\) for all \(r \in \mathbb{Z}\). Since \(\frac{1}{\nu} \sum_{s=0}^{\nu-1} e^{2\pi i s\omega} = \delta_{s,0}\) for \(s \in \{0, \ldots, \nu - 1\}\), Property (ii) then follows from (4.5). Conversely, if (ii) holds, then \(v = (g_0, 0, \ldots, 0)\), which implies that \(u = \sqrt{\nu} \cdot F_\omega v = (g_0, g_1, \ldots, g_{\nu-1})\). In particular, \(T_{-\alpha/\nu} g = g\), i.e., \(T_{\alpha/\nu} g \in \mathcal{G}\).

(ii) \implies (iii): Since \(P g = g\), it is a consequence of (1.2) that \(P|_{\mathcal{L}_0} = I|_{\mathcal{L}_0}\). Furthermore, for \(s \in \{1, \ldots, \nu - 1\}\) and \(k, \ell \in \mathbb{Z}\), Equation (1.2) implies
\[
PT_{\frac{\alpha}{\nu}} M_{\ell,\ell} M_\alpha g = T_{\frac{k}{\nu}} M_{\ell,\ell} M_\alpha g = 0,
\]
which shows \(P|_{\mathcal{L}_0} = 0\). By using these observations and by noting \(\mathcal{L}_r = M_{r/\alpha} \mathcal{L}_0\), we see for \(r, s \in \{0, \ldots, \nu - 1\}\) that \(P|_{\mathcal{L}_r} = M_{r/\alpha} P M_{-r/\alpha}|_{\mathcal{L}_r} = I|_{\mathcal{L}_r}\) and furthermore \(P|_{\mathcal{L}_s} = M_{s/\alpha} P M_{-s/\alpha}|_{\mathcal{L}_s} = 0\) for \(s \neq r\). Hence, the sum \(K = \mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_{\nu-1}\) is direct, and \(P|_{K} = M_{s/\alpha} P M_{-s/\alpha}|_{K}\) is the projection onto \(\mathcal{L}_s\) with respect to this decomposition. Finally, since \(\gamma \in K\) and since \(K\) is invariant under \(T_{k/\beta} M_{\ell/\alpha}\), it follows by definition of \(P_s = (\alpha\beta)^{-1} M_{s/\alpha} S_{\frac{1}{\alpha\beta} k,\ell} M_{s/\alpha} S_{\frac{1}{\alpha\beta} k,\ell} M_{s/\alpha}\) that \(P_s|_{K^\perp} = 0\). Therefore, \(P_s\) is the projection onto \(\mathcal{L}_s\) with respect to the decomposition \(L^2(\mathbb{R}) = (\mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_{\nu-1}) \oplus K^\perp\).

Finally, we show that \(\mathcal{F}_0\) is a frame for \(\mathcal{L}_s\), where it clearly suffices to show this for \(s = 0\). Since \(\mathcal{F}_0\) is a Bessel sequence, [8, Corollary 5.5.2] shows that we only need to prove that the synthesis operator
\[
D : \ell^2(\mathbb{Z}^2) \to L^2(\mathbb{R}), \quad (c_{k,\ell})_{k,\ell \in \mathbb{Z}} \mapsto \sum_{k,\ell \in \mathbb{Z}} c_{k,\ell} T_{\frac{k}{\nu}} M_{\ell,\ell} g
\]
has closed range \(\text{ran } D = \mathcal{L}_0\). By definition of \(\mathcal{L}_0 = \overline{\text{span } \mathcal{F}_0}\), we see \(\text{ran } D \subset \mathcal{L}_0\). Conversely, if \(f \in \mathcal{L}_0\), then \(f = P f = (\alpha\beta)^{-1} S_{\gamma,g} f = (\alpha\beta)^{-1} D c \in \text{ran } D\) for the sequence \(c = (c_{k,\ell})_{k,\ell \in \mathbb{Z}} \in \ell^2(\mathbb{Z}^2)\) given by \(c_{k,\ell} = \langle f, T_{k/\beta} M_{\ell/\alpha} \gamma \rangle\).

(iii) \implies (ii): Since \(P = (\alpha\beta)^{-1} S_{\gamma,g}\), we see by definition of \(S_{\gamma,g}\) that \(P \mathcal{L} \subset \mathcal{L}_0\). Hence, \(P g \in \mathcal{L}_0\). On the other hand, again as a consequence of \(\text{ran } P \subset \mathcal{L}_0\), we see that \(M_{s/\alpha} P M_{-s/\alpha} g \in \mathcal{L}_s\), so that Equation (4.4) implies
\[
\mathcal{L}_0 \ni P g - g = P g - P K g = -\sum_{s=1}^{\nu-1} M_{s/\alpha} P M_{-s/\alpha} g \in \mathcal{L}_1 + \cdots + \mathcal{L}_{\nu-1}, \quad (4.6)
\]
and thus \(P g = g\) since the sum \(\mathcal{L}_0 + \cdots + \mathcal{L}_{\nu-1}\) is direct. Similarly, for any \(s \in \{1, \ldots, \nu - 1\}\) we get because of \(\text{ran } P \subset \mathcal{L}_0\) that \(M_{s/\alpha} P M_{-s/\alpha} g \in \mathcal{L}_s\); but this implies as in Equation (4.6) that
\[
\mathcal{L}_s \ni P g - g = P g - \sum_{r \neq s} M_{s/\alpha} P M_{-s/\alpha} g \in \mathcal{L}_0 + \text{span } \{ \mathcal{L}_r : r \neq s \} = \text{span } \{ \mathcal{L}_r : r \neq s \}.
\]
Again, since \(\mathcal{L}_0 + \cdots + \mathcal{L}_{\nu-1}\) is a direct sum, this implies \(P M_{s/\alpha} g = 0\) for \(s = 1, \ldots, \nu - 1\). Since \(P\) commutes with \(M_{s/\alpha}\) (see (4.2)), we have \(P M_{(\nu-1)/\alpha} g = 0\) and therefore \(P M_{s/\alpha} g = 0\) for \(s = 1, \ldots, \nu - 1\).

(i) \implies (iv): Note that \((\gamma, \alpha\mathbb{Z} \times \beta\mathbb{Z})\) is a frame sequence and that \(\mathcal{G} = \mathcal{G}(\gamma, \alpha\mathbb{Z} \times \beta\mathbb{Z})\). Further, Lemma 2.2 shows that \(T_{\alpha/\nu} g \in \mathcal{G}\) if and only if \(\mathcal{G}\) is invariant under \(T_{\alpha/\nu}\), if and
only if $T_{α/β}γ ∈ G$. Let us consider the setting above with $g$ and $γ$ interchanged: Define

$$F^*_s := \{ T_{k/α} M_{αγ}, k, ℓ ∈ Z \}, \quad s = 0, \ldots, ν − 1.$$  

Then, by using the implication “(i)⇒(iii)” in this setting, we get $K^*_α = L^*_α \oplus \cdots \oplus L^*_ν$, where $L^*_s := \text{span} F^*_s$ and $K^* = \text{span} F^*$ with $F^* := \{ T_{k/β} M_{ℓ/αγ}, k, ℓ ∈ Z \}$. Note that $K = K^*$ by Equation (4.1).

We have $S_{γ, γ} = S^∗_{γ, γ}$. Hence, $M_{s/α} P^∗ M_{−s/α}$ is the projection onto $L^*_s$ with respect to the decomposition $L^2(ℝ) = (L^*_0 \oplus \cdots \oplus L^*_ν) \oplus K$. In particular, using the general formula $(\text{ker } T)^⊥ = \text{ran } T^*$ for a bounded operator $T : H → H$ (see [9, Remarks after Theorem II.2.19]) and the elementary identity $(A + B)^⊥ = A^⊥ ∩ B^⊥$ for subspaces $A, B ⊂ H$, we get

$$L^*_0 = \text{ran } T^* = (\text{ker } P)^⊥ = (L_1 \oplus \cdots \oplus L_ν)^⊥ \cap K.$$  

For $k, ℓ ∈ Z$ and $s ∈ \{1, \ldots, ν − 1\}$, we have

$$PM_{αγ}^s g = \frac{1}{αβ} ∑_{k, ℓ ∈ Z} \langle M_{αγ}^s (T_{ℓ/α} M_{αγ}), T_{k/α} M_{αγ}^s g \rangle T_{k/α} M_{αγ}^s g = 0.$$  

Thanks to Equation (4.1), this implies $g = P_ξ g = Pg$. Overall, we have thus shown $PM_{s/α} g = δ_{s, 0} g$ for all $s ∈ \{0, \ldots, ν − 1\}$. □

Note that with $P := (αβ)^{-1} S_{γ, γ}$, the condition $P f = f$ for $f ∈ L_0$ means that $(αβ)^{-1} γ$ is a dual window for the frame sequence $F_0 = (g, 1/β × 1/α Z)$. However, it is possible that $γ \notin L_0 = \text{span } F_0$.

5. Proof of the main theorem

In this section, we prove our main result, Theorem 1.1 which we state here once more for the convenience of the reader.

Theorem 1.1. If $g ∈ S_0(ℝ)$ and $Λ ⊂ ℝ^2$ is a lattice such that the Gabor system $(g, Λ)$ is a Riesz basis for its closed linear span $G(g, Λ)$, then the time-frequency shifts $T_{aM}^b$ that leave $G(g, Λ)$ invariant satisfy $(a, b) ∈ Λ$.

Proof. The claim is true if $Λ$ has rational density; see [11, Theorem 1]. Thus, assume that $Λ$ has irrational density $d(Λ) ∈ ℝ \setminus Q$. Write $Λ = A \mathbb{Z}^2$ with an invertible matrix $A ∈ ℝ^{2×2}$.

Due to the Amalgam Balian-Low theorem [21, Theorem 3.2], it is not possible that $G := G(g, Λ) = L^2(ℝ)$. Hence, $G \neq L^2(ℝ)$. Suppose towards a contradiction that $G(G) ⊃ Λ$. According to Lemma 1.1 there exist $B ∈ ℝ^{2×2}$ with det $B = 1$ and $α, β > 0$ such that $Λ_B = α \mathbb{Z} × β \mathbb{Z}$ and $T_{1/α} g_B ∈ G(B)$ for some $ν ∈ ℕ_{≥2}$. Set $h := g_B$ and $G_h := G(h, α \mathbb{Z} × β \mathbb{Z})$. Then $h ∈ S_0(ℝ)$ by [14, Proposition 12.1.3] and $T_{1/α} h ∈ G_h$. Furthermore, note that with $(g, Λ)$, also $(h, α \mathbb{Z} × β \mathbb{Z}) = (g_B, Λ_B)$ is a Riesz sequence (cf. Subsection 4.3).

Let $γ$ be the canonical dual window for $(h, α \mathbb{Z} × β \mathbb{Z})$. By Ron-Shen duality (see [14, Theorem 7.4.3]), the adjoint system $(h, 1/β × 1/α Z)$ is a frame for $L^2(ℝ)$. Let $γ^∗$ denote
the canonical dual window of \((h, \frac{1}{\beta}Z \times \frac{1}{\alpha}Z)\), and note by Wexler-Raz orthogonality (see [14] Theorem 7.3.1) that \(\langle h, \gamma^\delta \rangle = (\alpha \beta)^{-1}\). Next, note that [20] Theorem 2.3 shows \(\gamma^\delta = (\alpha \beta)^{-1} \gamma\) and hence \(\langle h, \gamma \rangle = \alpha \beta \cdot \langle h, \gamma^\delta \rangle = 1\), which will be used below.

Since \(T_{\alpha/\nu}h \in \mathcal{G}^\nu\), Theorem 4.2 implies that \(P_0 := (\alpha \beta)^{-1} S_{\frac{1}{\nu}Z \times \frac{1}{\nu}Z, \gamma, h}\) is an idempotent (i.e., \(P_0^2 = P_0\)). We now wish to apply Theorem A.1 to derive a contradiction. To this end, first note that \(\alpha \beta = (d(\Lambda_B))^{-1} = |\det BA| = |\det A| = (d(\Lambda))^{-1} \in \mathbb{R} \setminus \mathbb{Q}\). Next, set \(U := M_\beta\) and \(V := T_{\frac{2}{\nu}}\). A direct calculation shows that

\[
UV = e^{2\pi i \theta} VU, \quad \text{where} \quad \theta := \frac{\alpha \beta}{\nu} \in \mathbb{R} \setminus \mathbb{Q}.
\]

Note that also \(\gamma \in S_0(\mathbb{R})\); see [1] Theorem 7. Hence, we may use Equation (2.7) and obtain

\[
P_0 = \frac{1}{\nu} \sum_{m,n \in \mathbb{Z}} \langle h, T_{\frac{m}{\nu}} M_{n\gamma} \rangle T_{\frac{m}{\nu}} M_{n\gamma} = \sum_{m,n \in \mathbb{Z}} \frac{1}{\nu} \langle h, V^m U^n \gamma \rangle V^m U^n, \quad \text{(5.1)}
\]

with coefficient sequence \(a = (a_{m,n})_{m,n \in \mathbb{Z}} := \left(\frac{1}{\nu} \langle h, V^m U^n \gamma \rangle\right)_{m,n \in \mathbb{Z} \in \ell^1(\mathbb{Z}^2)\}. Therefore, Theorem A.1 shows that \(\frac{1}{\nu} = a_{0,0} \in \mathbb{Z} + \theta \mathbb{Z}\), say \(\frac{1}{\nu} = m + n \theta\) for some \(m, n \in \mathbb{Z}\). We must have \(n \neq 0\), since otherwise \(\frac{1}{\nu} = m \in \mathbb{Z}\), in contradiction to \(\nu \geq 2\). Thus, \(\theta = \frac{1}{m} - \frac{n}{\nu} \in \mathbb{Q}\), which is the desired contradiction, since \(\theta = \frac{\alpha \beta}{\nu}\) is irrational. \(\square\)

**Remark 5.1.** On a first look, it might appear as if the proof of Theorem 1.1 would also apply in case of \(g \in \mathbb{H}^1\): First, the classical Balian-Low theorem implies that \(\mathcal{G} := \mathcal{G}(g, \Lambda) \subset L^2(\mathbb{R})\), so that Lemma 4.1 allows the reduction to a Gabor Riesz sequence \((h, \Lambda)\) with \(h \in \mathbb{H}^1\), a separable lattice \(\Lambda = \alpha \mathbb{Z} \times \beta \mathbb{Z}\), and an additional time-frequency shift of the form \(T_{\alpha/\nu} h \in \mathcal{G}\). One can then apply Theorem A.2 to see that that \(L^2(\mathbb{R}) = \mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_{\nu-1}\). In the \(S_0\)-case, we then employed Janssen’s representation (5.1) for the projection \(P_0 = \langle \alpha \beta \rangle^{-1} S_{\frac{1}{\nu}Z \times \frac{1}{\nu}Z, \gamma, h}\), which then led to success in the proof of Theorem 1.1 thanks to existing results concerning the structure of the irrational rotation algebra. However, in the case \(h \in \mathbb{H}^1\) the series in (5.1) might not converge in operator norm, so that one does not know whether \(P_0\) belongs to the irrational rotation algebra. Thus, the proof breaks down at this point.

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A. Appendix

In this section, we make use of a deep result concerning the structure of the irrational rotation algebra $A_\theta$ (see [11, 24, 25]) to prove the following auxiliary statement, which is a crucial ingredient for the proof of our main result, Theorem 1.1. As usual, we denote the set of all bounded linear operators from a Hilbert space $\mathcal{H}$ into itself by $B(\mathcal{H})$.

**Theorem A.1.** Let $\mathcal{H} \neq \{0\}$ be a Hilbert space and let $U, V \in B(\mathcal{H})$ be unitary and such that $UV = e^{2\pi i\theta}VU$, for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$. If $a = (a_{k,\ell})_{k,\ell \in \mathbb{Z}} \in l^2(\mathbb{Z}^2)$ is such that the operator $P_a := \sum_{k,\ell \in \mathbb{Z}} a_{k,\ell}V^kU^\ell$ satisfies $P_a^2 = P_a$, then $a_{0,0} \in \mathbb{Z} + \theta \mathbb{Z}$.

The proof will make use of some parts of the theory of $C^*$-algebras, which we recall here for the convenience of the reader, based on [23]. Readers familiar with $C^*$-algebras will probably want to skip this part—except possibly Lemma A.2.

A $C^*$-algebra is a (complex) Banach algebra $(A, \| \cdot \|)$, additionally equipped with a map $A \rightarrow A, x \mapsto x^*$ (called the involution on $A$), satisfying the following properties:

- $(x+y)^* = x^* + y^*$, $(\lambda x)^* = \overline{\lambda} x^*$, and $(xy)^* = y^* x^*$ for all $x, y \in A$ and $\lambda \in \mathbb{C}$;
- $\|x^*\| = \|x\|$ and $\|x^*x\| = \|x\|^2$ for all $x \in A$.

An element $p \in A$ is called an idempotent if $p^2 = p$. An idempotent $p$ is called a projection if additionally $p = p^*$ holds. A $C^*$-algebra $A$ is called unital if it contains a (necessarily unique) element $1 \in A$ satisfying $1 \neq 0$ and $1x = x$ for all $x \in A$. In a unital $C^*$-algebra $A$, an element $x \in A$ is called unitary if $xx^* = 1 = x^*x$. If $A$ is a unital $C^*$-algebra and $a \in A$, then $\sigma(a^*a) \subset [0,\infty)$; see [23, Theorem 2.2.4]. Here, $\sigma(b) = \{ \lambda \in \mathbb{C} : b - \lambda 1 \text{ not invertible in } A \}$.

**Lemma A.2.** Any idempotent $e$ in a unital $C^*$-algebra $A$ is similar to a projection $p \in A$. That is, there exist a projection $p \in A$ and an invertible element $a \in A$ such that $e = a^{-1}pa$.

**Proof.** We set $b := e^* - e$ and $z := 1 + b^*b$. Note that $z$ is invertible since $\sigma(b^*b) \subset [0,\infty)$. We have

$$ez = e + (e - ee^*)(e^* - e) = ee^*e = e + (e - e^*)(e^*e - e) = ze.$$ 

Consequently, $ez^{-1} = z^{-1}e$ and, as $z = z^*$, also $e^*z^{-1} = z^{-1}e^*$. Now, define the element $p := ez^{-1}e^*$. We have $p^* = p$. Furthermore, since we just saw that $z^{-1}$ commutes with $e$ and $e^*$ and that $ee^*e = ze$, we also see that $p^2 = z^{-2}(ee^*e)e^* = z^{-1}ee^*p$. Hence, $p$ is a projection. We further observe that $ep = p$ and $pe = ez^{-1}e^*e = z^{-1}ee^*e = z^{-1}ze = e$. Set $a := 1 - p + e$. Then we see because of

$$(1 \mp p \mp e)(1 \mp p \mp e) = 1 \mp p \mp e \mp p - e \mp e \mp e - p - e = 1$$

that $a$ is invertible with $a^{-1} = 1 + p - e$. Hence, from $ae = e - pe + e = e$ we obtain

$$aee^{-1} = e(1 + p - e) = e + ep - e = ep = p,$$

which proves the lemma. 

A closed subspace $B$ of a $C^*$-algebra $A$ is called a $C^*$-subalgebra of $A$ if it is closed under both multiplication and involution. It is clear that $B$ is then itself a $C^*$-algebra. As usual, given a subset $S \subset A$, there is a smallest (with respect to inclusion) $C^*$-subalgebra of $A$ containing $S$. We call it the $C^*$-algebra generated by $S$, and denote it by $C^*(S)$. 


A map $\varphi : A \to B$ between two $C^*$-algebras $A$ and $B$ is called a *-homomorphism if it is linear and satisfies $\varphi(xy) = \varphi(x)\varphi(y)$ as well as $\varphi(x^*) = [\varphi(x)]^*$ for all $x, y \in A$. A bijective *-homomorphism is called a *-isomorphism. Any *-homomorphism $\varphi : A \to B$ necessarily satisfies $\|\varphi(x)\|_B \leq \|x\|_A$ for all $x \in A$, and is hence continuous; see [23, Theorem 2.1.7].

**Proof of Theorem A.1.** We will make use of the so-called irrational rotation algebra $A_\theta$, as introduced for instance in [11, Chapter VI]. The actual definition of this algebra is not relevant for us; we will only need to know that it satisfies the following properties:

- $A_\theta$ is a unital $C^*$-algebra;
- The algebra $A_\theta$ is universal among all unital $C^*$-algebras generated by unitary elements $U, V$ satisfying $UV = e^{2\pi i\theta}VU$. Thus, defining $A := C^*(U, V)$ as a $C^*$-subalgebra of $B(\mathcal{H})$ with $U, V$ as in the statement of Theorem A.1, there is a *-isomorphism $\varphi : A \to A_\theta$; this follows from [11, Theorem VI.1.4].
- As shown in [11, Corollary VI.1.2 and Proposition VI.1.3], there is a unique (unital) trace $\tau : A_\theta \to \mathbb{C}$. By definition of a trace, this means in particular that $\tau$ is linear and continuous, satisfying $\tau(1) = 1$ and $\tau(xy) = \tau(yx)$ for all $x, y \in A_\theta$.
- For any projection $p \in A_\theta$, we have $\tau(p) \in \mathbb{Z} + \theta\mathbb{Z}$; see [25, Theorem 1.2]. We remark that this result was originally proven in [24].

Let us define $\tau^2 := \tau \circ \varphi$, and note that $\tau^2 : A \to \mathbb{C}$ is continuous. It is easy to see that $\tau^2$ is linear with $\tau^2(id_U) = 1$ and $\tau^2(AB) = \tau^2(BA)$ for all $A, B \in A$; this is called the cyclicity of the trace. Next, from the relation $UV = e^{2\pi i\theta}VU$, we immediately get for $k, \ell \in \mathbb{Z}$ that

$$V^kU^\ell = e^{-2\pi i\theta}V^{k-1}U^\ell V = e^{-2\pi i\theta}UV^kU^\ell.$$  

Thus, noting that $V^kU^\ell \in A$, we obtain

$$\tau^2(V^kU^\ell) = e^{-2\pi i\theta}\tau^2(V^kU^\ell) = e^{-2\pi i\theta}\tau^2(V^kU^\ell)$$

by cyclicity. As $\theta$ is irrational, this implies $\tau^2(V^kU^\ell) = \delta_{k,\ell}0_{k,0}$. Next, since we have $\|V^kU^\ell\| = 1$ for all $k, \ell \in \mathbb{Z}$ and since $a \in \ell^1(\mathbb{Z}^2)$, we see that $P_a = \sum_{k,\ell \in \mathbb{Z}} a_{k,\ell}V^kU^\ell \in A$, with unconditional convergence of the defining series. Hence,

$$\tau^2(P_a) = \sum_{k,\ell \in \mathbb{Z}} a_{k,\ell}0_{k,\ell}.$$

Since $P_a = P_a$ and since $\varphi : A \to A_\theta$ is a *-homomorphism, we see that $e := \varphi(P_a) \in A_\theta$ is an idempotent. By Lemma A.2 there exist $b, p \in A_\theta$ such that $b$ is invertible, $p$ is a projection, and $e = b^{-1}pb$. Thanks to the cyclicity of the trace, we thus see that $a_{0,0} = \tau^2(P_a) = \tau(e) = \tau(b^{-1}pb) = \tau(p) \in \mathbb{Z} + \theta\mathbb{Z}$, as claimed.

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**Author Affiliations**

**A. Caragea:** KU EICHSTÄTT-INGOLSTD, Mathematisch-Geographische Fakultät, Ostenstrasse 26, Kollegiengebäude I Bau B, 85072 Eichstätt, Germany  
Email address: andrei.caragea@gmail.com

**D.G. Lee:** KU EICHSTÄTT-INGOLSTD, Mathematisch-Geographische Fakultät, Ostenstrasse 26, Kollegiengebäude I Bau B, 85072 Eichstätt, Germany  
Email address: daegwans@gmail.com

**F. Philipp:** Technische Universität Ilmenau, Institute for Mathematics, Weimarer Strasse 25, D-98693 Ilmenau, Germany  
Email address: friedrich.philipp@tu-ilmenau.de

**F. Voigtländer:** KU EICHSTÄTT-INGOLSTD, Lehrstuhl Reliable Machine Learning, Ostenstrasse 26, 85072 Eichstätt, Germany  
Email address: felix.voigtlander@ku.de