A hypergeometric test interpretation of a common tf–idf variant

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Abstract

Term frequency–inverse document frequency, or tf–idf for short, is a numerical measure that is widely used in information retrieval to quantify the importance of a term of interest in one out of many documents. While tf–idf was originally proposed as a heuristic, much work has been devoted over the years to placing it on a solid theoretical foundation. Following in this tradition, we here advance the first justification for tf–idf that is grounded in statistical hypothesis testing. More precisely, we first show that the hypergeometric test from classical statistics corresponds well with a common tf–idf variant on selected real-data information retrieval tasks. Then we set forth a mathematical argument that suggests the tf–idf variant functions as an approximation to the hypergeometric test (and vice versa). The hypergeometric test interpretation of this common tf–idf variant equips the working statistician with a ready explanation of tf–idf’s long-established effectiveness.

Keywords: document retrieval, document summarization, information retrieval; probabilistic explanation, term weighting
1 Introduction

There are two general questions that information retrieval practitioners are wont to ask about a collection of documents. The first is: What documents are most relevant to a given query consisting of one or more terms? This is the problem of document retrieval. The second is: What terms in a given document best characterize its subject matter? This is the problem of document summarization, which is a springboard into document classification and document clustering.

A document in the widest sense of the word we shall consider is a piece of written, printed, or electronic textual matter. In information retrieval, documents are commonly represented by the multisets of their constituent terms. For most practical purposes it is unnecessary to distinguish between a simplified representation of a document and the textual matter it represents. We shall, therefore, use the word “document” to refer to a piece of textual matter in some places, and to a multiset of terms in others without undue hair splitting over the potential for ambiguity.

Let \( \mathcal{D} = \{d_1, d_2, \ldots, d_N\} \) be a set of \( N \) documents such that each \( d_i \in \mathcal{D} \) is a multiset of terms taken from the \( M \) term vocabulary \( \mathcal{T} = \{t_1, t_2, \ldots, t_M\} \). The specification of a term-document matrix

\[
S = \begin{pmatrix}
d_1 & d_2 & \cdots & d_N \\
t_1 & s_{11} & s_{12} & \cdots & s_{1N} \\
t_2 & s_{21} & s_{22} & \cdots & s_{2N} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
t_M & s_{M1} & s_{M2} & \cdots & s_{MN}
\end{pmatrix}
\]

for a given set of documents is the textbook starting point for any analysis. The matrix \( S \) is an \( M \times N \) matrix each of whose rows corresponds to a term \( t_i \in \mathcal{T} \) and each of whose columns corresponds to a document \( d_j \in \mathcal{D} \). The entries of \( S \) define a scoring function, \( s : \mathcal{T} \times \mathcal{D} \to \mathbb{R}_{\geq 0} \). Each entry \( s_{ij} \) is a nonnegative score reflecting the “relevance” of term \( t_i \) to document \( d_j \). The more highly a term scores in a document, the more representative it is considered to be of the document’s content.

This conceptualization of a set of documents allows us to recast our opening pair of questions in technical terms. The document retrieval problem amounts to using term-
document matrix scores to calculate a ranking for the documents in $\mathbb{D}$ (think the columns of $S$) according to their relevance to a user submitted query $q = \{t_{l_1}, t_{l_2}, \ldots, t_{l_m}\}$ of $1 \leq m \leq M$ terms from $\mathbb{T}$ (think one or more rows of $S$). By contrast, document summarization is achieved by using the top $1 \leq m \leq M$ scoring terms in $\mathbb{T}$ (think one or more rows of $S$) as a proxy for the subject matter of a given document of interest from $\mathbb{D}$ (think of a column of $S$). The choice of a specific value for $m$ is a subject to which we will return at a later stage.

Prerequisite to analysis is the adoption of a concretely defined scoring function. The term frequency–inverse document frequency (tf–idf) scoring function has been a mainstay of information retrieval science for nearly half a century. It scores term $t_i \in \mathbb{T}$ in document $d_j \in \mathbb{D}$ in accordance with the formula $tf$-$idf(t_i, d_j) = -k_{ij} \times \log(K_i/N)$, where the term frequency ($tf$) $k_{ij}$ is the number of times $t_i$ occurs in $d_j$, and $K_i$ the number of documents in $\mathbb{D}$ containing at least one occurrence of the term $t_i$. The name inverse document frequency (idf) is given to the factor $\log(N/K_i) = -\log(K_i/N)$. The particular tf–idf variant that shall concern us in the present work, which we call tp-idf, is given by

$$tp$-$idf(t_i, d_j) = -\frac{k_{ij}}{n_j} \times \log(K_i/N).$$

In the equation, the term $n_j = \sum_{i=1}^{M} k_{ij}$ is the total number of term occurrences in $d_j$ counting multiplicities. The term proportion (tp) $k_{ij}/n_j$ term is the document length normalized term frequency. The intuition behind tf–idf is that a term which is disproportionately concentrated in a few documents tends to score more highly in those documents than do terms occurring frequently in many documents such as articles, prepositions and conjunctions. Hence a document’s highest tf–idf scoring terms can ordinarily be expected to serve as an informative characterization of its subject matter.

Term scoring functions based on tf–idf variants are a fixture of contemporary document retrieval systems used for ranking documents by relevance in response to user submitted textual queries. To take a simple illustration, consider a user who wishes to query a set of documents, $\mathbb{D}$, on the $m$ terms in $q = \{t_{l_1}, t_{l_2}, \ldots, t_{l_m}\}$ a subset of $\mathbb{T}$. One very basic document ranking scheme assigns a score to each document $d_j \in \mathbb{D}$ by summing the tf–idf scores of the query terms: $Score(d_j, q) = \sum_{i=1}^{m} tf$-$idf(t_{l_i}, d_j)$. Documents are ranked in decreasing order of their score. By the same token, tf–idf variants have been used for
document summarization in both document classification (e.g., Joachims (1997); Kim and Gil (2019)) and document clustering (e.g., Balna et al. (2016)) algorithms.

Although tf–idf was once considered to be a heuristic, a series of efforts have been undertaken to explain its empirical success within one or another theoretical framework. We will review the existing crop of theoretical justifications for tf–idf in Section 2. In this paper we use the framework of statistical hypothesis testing to explain why tf–idf works as well as it does. In particular, we show that the reason why tp–idf works is precisely because it approximates the negative logarithm of the hypergeometric (hypothesis) test from classical statistics.

In our present setting, the hypergeometric test calculates the probability of drawing term $t_i$ at least $k_{ij}$ times out of $n_j$ draws from a population of $N = \sum_{j=1}^{N} n_j$ terms out of which term $t_i$ occurs $K_i = \sum_{j=1}^{N} k_{ij}$ times as the hypergeometric distribution tail probability

$$P(k_{ij}, n_j, K_i, N) = \sum_{k=k_{ij}}^{n_j} \frac{\binom{K_i}{k} \binom{N-K_i}{n_j-k}}{\binom{N}{n_j}}.$$  \hfill (2)

According to the test, the term $t_i$ is said to be over-represented in document $d_j$ with respect to the background set of documents $D$, if $P(k_{ij}, n_j, K_i, N)$, which is interpreted as a P-value, falls short of a preselected significance level. Defining the hypergeometric test scoring function as the negative logarithm of Eq. (2), that is,

$$hgt(k_{ij}, n_j, K_i, N) = -\log \left( \sum_{k=k_{ij}}^{n_j} \frac{\binom{K_i}{k} \binom{N-K_i}{n_j-k}}{\binom{N}{n_j}} \right),$$ \hfill (3)

ensures that it takes on the same range of values as does tp–idf; namely, values in the interval $[0, \infty)$. Note that the transformation, being monotonic, will have no effect on resulting term rankings.

In Section 3 we take to comparing the performance of tp–idf with that of the hypergeometric test in a real data setting. To this end, we carried out a host of document retrieval and document summarization tasks on a real-world text collection consisting of roughly 10,000 online English news articles pertaining to the highly publicized criminal case New York v. Strauss-Kahn (see Dermouche et al. (2014)). We found that tp–idf consistently produces results very similar to those of the hypergeometric test, and that the degree of agreement between the two scoring functions cannot be accounted for by the effect of the tp
term alone. The level of agreement between tp–idf and the hypergeometric test is particularly striking in the document summarization case for reasons that will be explained below. We additionally describe in this section what, to our knowledge, is the first appearance in the literature of a multivariate generalization of the hypergeometric test.

In Section 4 we establish a mathematical connection between tp–idf and the hypergeometric test. To do so, we analyze a pair of continuous valued functions that serve as convenient surrogates for tp–idf and the hypergeometric test. The key to defining these functions lies in permitting the term proportion $k_{ij}/n_j$, the document proportion $K_i/N$ and the total term proportion $K_i/N$ to take on values in the open unit interval. From there we show that the tp–idf and the hypergeometric test surrogates have similar functional forms in the region of highest term significance. This, in essence, comprises our argument.

Section 5 concludes the paper with a brief discussion on some potential future research projects in this area.

In summary, there are multiple ways to explain why tf–idf works well in practice. Our primary contribution in this paper is to advance a novel justification for tf–idf that ties its success to its variant tp–idf behaving roughly the same as the hypergeometric test. In this way, we establish the first-ever theoretical basis for tf–idf within the framework of statistical hypothesis testing.

2 Related work

The earliest appeal to tf in the context of computer aided literature searching is attributed to Luhn (1957). Spärck Jones (1972) advanced the idf metric in all but name. To be technically correct it is the equivalent of the formula $\log(N) - \log(K_i) + 1$ that one finds written in her paper. Salton and Yang (1973) were quick to take the next logical step of multiplying the tf quantity by Spärck Jones’s new formula, which they christened idf, and tf–idf was born. Never mind that the idf formula specifies a logarithmically scaled inverse proportion, rather than an inverse frequency. Roelleke (2013) reviews most, if not all, of the many tf–idf variants that have cropped up in the literature.

There is an appreciable body of scholarship on tf–idf theoretical foundations. Robertson (2004) reviews some the more notable attempts to place the idf metric on a sound theoretical
footing. His paper is, moreover, a fair starting-point for learning the layout of the theoretical foundations for tf-idf landscape. However it is Thomas Roelleke who has emerged as the primary authority on the subject. Roelleke (2013) in essence identifies four types of theoretical argument for tf-idf: those based on information theory, those based on probabilistic relevance modelling, those based on statistical language modelling, and those based on divergence from randomness models.

Let us now take stock of the existing theoretical arguments for tf-idf within the framework of Roelleke’s classification scheme.

2.1 Information theory approaches

Aizawa (2003) hit on a connection between mutual information and tf-idf by building on the earlier work of Robertson (1974) and Wong and Yao (1992) to place idf in an information theoretic framework. Robertson (2004) writes that “it is difficult to see it [Aizawa’s analysis] as an explanation of the value of IDF” (p. 508), leading one to presume his difficulty carries over to tf-idf by extension. His criticism almost certainly inspired Roelleke to seek out a logically incontestable explanation for tf-idf within the realm of information theory. We are happy to report that Roelleke was wholly successfully in this effort. For in Roelleke (2013) Section 3.11, he convincingly relates a form of tf-idf to the difference between two Kullback-Leibler divergences.

2.2 Probabilistic relevance modelling approaches

A number of historically influential document retrieval models and their many variants were devised within this Robertson and Jones (1976) conceived modelling framework. The framework, in broad outline, is as follows. The goal is to rank the documents of a given document set, \( D \), in accordance with their “relevance” to a given query, \( q \). The “relevance” of document \( d_j \in D \) to query \( q \) is made precise in the form of a probability model, \( P(R_{d_j,q} | d_j, q) \), over the indicator random variable \( R_{d_j,q} \), which is defined to be 1 when \( d_j \) is relevant to \( q \) and 0 otherwise. By ranking the documents of \( D \) in decreasing order of the odds ratio \( O(R_{d_j,q} | d_j, q) = P(R_{d_j,q} = 1 | d_j, q) / P(R_{d_j,q} = 0 | d_j, q) \), the desired goal is attained.
The Binary Independence Model (BIM) of Yu and Salton (1976) is the progenitor of a simple class of probabilistic relevance models. In a BIM, each document \( d_j \in D \) is represented as the binary vector of its terms \( \vec{d}_j = (x_{1j}, \ldots, x_{Mj})^T \), where \( x_{ij} = 1 \) if the corresponding term \( t_i \in T \) occurs in \( d_j \) and \( x_{ij} = 0 \) otherwise. The query \( q \) is likewise represented as the binary vector \( \vec{q} = (q_1, \ldots, q_M)^T \) such that \( q_i = 1 \) if \( t_i \in q \) and \( q_i = 0 \) otherwise. The odds ratio \( O(R_{d_j,q}|D_j = \vec{d}_j, Q = \vec{q}) \) is calculated through a routine application of Bayes theorem, assuming \( D_j = (X_{1j}, \ldots, X_{Mj})^T \) is an independent binary vector and \( Q = (Q_1, \ldots, Q_M)^T \) is an independent binary vector. Each of Robertson (2004), de Vries and Roelleke (2005), and Manning et al. (2008a) arrive at theoretical justifications for idf starting from one or another elaboration on this core modelling scheme. Only de Vries and Roelleke (2005) go on to draw a connection, albeit a tenuous one, between tf–idf and a common BIM variant.

To truly justify tf–idf within the framework of probabilistic relevance modelling it is necessary to entertain non-binary models. The Poisson Model, as described in Roelleke (2013), is a straightforward count data modelling BIM generalization. This time each document \( d_j \in D \) is represented as the vector of term frequencies \( \vec{d}_j = (k_{1j}, \ldots, k_{Mj})^T \). In addition, the random vector \( D_j = (K_{1j}, \ldots, K_{Mj})^T \) is composed of independent Poisson-distributed random variables. More specifically, \( K_{ij} \sim \text{Pois}(\lambda_i) \) with \( \lambda_i = K_i/N \) for \( i = 1, \ldots, M \). Odds ratio score evaluation differs from that of the BIM special case only in mathematical detail. In an important advance in tf–idf foundations, Roelleke and Wang (2006) show under which conditions tf–idf emerges from the Poisson Model. This marks the first convincing derivation of tf–idf within anything resembling a classical probabilistic framework. Robertson and Walker (1994) used the 2-Poisson Model, in which term counts are modelled as a mixture of two Poisson distributions, as a starting point for developing their celebrated BM25 scoring function. Robertson (2004) expresses BM25 as a generalized tf–idf scoring function. The connection between tf–idf and BM25 is further elucidated in Roelleke and Wang (2008) and Roelleke (2013). Wu et al. (2008) interpret tf–idf as a special case of a probabilistic relevance model of their own contrivance.

Finally, we note that Joachims (1997) presents a very rough analysis of tf–idf within the context of probabilistic relevance modelling-based document classification.
2.3 Statistical language modeling approaches

Hiemstra (1998) and Ponte and Croft (1998) more or less simultaneously pioneered this approach to information retrieval in the late 1990s. Abandoned is the modelling of relevance as a random variable, which had a certain contrived quality about it from the outset, in favor of a more traditional probabilistic approach. To define a statistical language model as a probability distribution over the multi-subsets of $T$ would not be far wrong. And since this definition facilitates our exposition, we shall adopt it without further ado. The conditional probability $P(d_j|q) \propto P(q|d_j)P(d_j)$ provides a general framework for ranking the documents of $D$ in response to a given query, $q = \{t_{l_1}, t_{l_2}, \ldots, t_{l_m}\}$. If term probabilities are independent, as is usually assumed, then the likelihood function $P(q|d_j)$ is the product of the individual query term probabilities $P(t_{l_1}|d_j) \cdots P(t_{l_m}|d_j)$. A naïve assignment for $P(t_{l_r}|d_j)$ is the document proportion $k_{l_r,j}/n_j$. Assigning $P(d_j) = 1/N$ completes a very simple statistical language model up to a normalizing constant. This barely scratches the surface of the state of modern statistical language modelling research. An overview of basic concepts and techniques is found in Manning et al. (2008b).

Hiemstra (2000) advanced the first statistical language modelling-based interpretation of tf–idf. By way of an elementary probabilistic argument, Hiemstra essentially arrives at the formula

$$\log P(d_j|q) = \sum_{r=1}^{m} \log \left( 1 + \frac{1 - \gamma}{\gamma} \times \frac{k_{l_r,j}}{n_j} \times \frac{\sum_{i=1}^{M} K_i}{K_{l_r}} \right) + \log(N/C)$$

for the log probability of document $d_j \in D$ given query $q = \{t_{l_1}, t_{l_2}, \ldots, t_{l_m}\}$ with free parameter $0 < \gamma < 1$ and normalizing constant $C$. He assumes $P(d_j) = 1/N$ and $P(t_{l_r}|d_j) = \gamma k_{l_r,j}/n_j + (1 - \gamma)K_i/\sum_{i=1}^{M} K_i$ in addition to the terms in $q$ being conditionally independent given $d_j$. Specifying $P(t_{l_r}|d_j)$ as a linear combination of $k_{l_r,j}/n_j$ and $K_i/\sum_{i=1}^{M} K_i$ constitutes a frequentist approach to regularization. The probability $P(t_{l_r}|d_j)$ tends to the maximum likelihood estimate $k_{l_r,j}/n_j$ as $\gamma$ approaches unity. The $K_i/\sum_{i=1}^{M} K_i$ term is included to compensate for the term frequency sparseness empirical phenomenon described by Manning et al. (2008c). The free parameter $\gamma$ is to be optimized from training data. Hiemstra directly interprets his formula for $\log P(d_j|q)$ as a tf–idf document scoring function.
Roelleke and Wang (2008) improve upon Hiemstra’s finding by identifying the conditions under which the decidedly more \( \text{tf–idf-like formula} \)

\[
\log P(d_j|q) = \sum_{t_{ir} \in d_j \cap q} k_{ir,j} \log (N/K_{ir}) - \log(N)
\]

emerges from the statistical language model \( P(d_j|q) = \prod_{t_i \in d_j} P(t_i|q)^{k_{ij}} \). The trick is to assign \( P(t_i|q) \) differently depending on whether or not \( t_i \) is a query term. In particular, they assume \( P(t_i|q) = 1 \), if \( t_i \in q \) and \( P(t_i|q) = K_i/N \) otherwise. The uniform prior \( P(d_j|q) = 1/N \) over documents is the only other assumption required to derive their result. And, to put a cherry on top of this cake, Roelleke (2013) derives \( \text{tf–idf as a limiting case of a generalization of Hiemstra’s above described mixture model.} \)

Elkan (2005) teases out a loose connection between \( \text{tf–idf and the Dirichlet-multinomial distribution Fisher kernel.} \) We have seen that Hiemstra (2000) and Roelleke and Wang (2008) assigned the equal probability \( 1/N \) to each document in \( \mathbb{D} \). In statistical language modelling, the multinomial probability \( P(d_j) \propto \prod_{t_i \in d_j} P(t_i)^{k_{ij}} \) is a well-established document probability assignment. But even this comparatively elaborate formulation fails to adequately account for the empirical fact that, as Church and Gale (1995) have shown, any given term tends to bunch up inside particular documents. Madsen et al. (2005) fruitfully modelled this phenomenon, called burstiness, by assigning document probability according to the Dirichlet-multinomial distribution in place of standard multinomial probabilities. Elkan (2005) carried the success further, showing an approximation of the partial derivative of the Dirichlet-multinomial distribution log-likelihood function has components that are similar to the \( \text{tf and idf factors in the tf–idf variant} \) \( \log(k_{ij} + 1) \times \log(N/K_i) \). However, Roelleke (2013) establishes a more direct relationship between \( \text{tf–idf and the statistical language modelling of burstiness.} \)

### 2.4 Divergence from randomness approaches

Havrlant and Kreininovich (2017) derive \( \text{tf–idf (at least approximately)} \) from a simple probabilistic model with statistical language model characteristics. The model assumes the probability of observing \( k_{ij} \) occurrences of term \( t_i \) in document \( d_j \) is binomially distributed as \( P(k_{ij}|\mathbb{D}) = \binom{n_j}{k_{ij}} p^{k_{ij}}(1 - p)^{n_j - k_{ij}} \) with \( p = 1/N \). They show that \( -\log P(k_{ij}|\mathbb{D}) \approx \)
$k_{ij} \log (N/K_i)$ so long as the inequalities $1 \ll k_{ij} \ll n_j \ll N$ hold true. What they evidently did not realize is that Amati and Van Rijsbergen (2002) had already derived virtually the same result in a divergence from randomness modelling context.

Baeza-Yates and Ribeiro-Neto (2011) write that “The idea [underlying the divergence from randomness approach] is to compute term weights by measuring the divergence between a term distribution produced by a random process and the actual term distribution.” (p. 113). Stated precisely, the weight of term $t_i$ in document $d_j$ is defined as $w_{ij} = -\log P(t_i|\mathcal{D}) \times (1 - P(t_i|d_j))$. The $-\log P(t_i|\mathcal{D})$ factor quantifies the amount of information content of term $t_i$ under the assumption it is randomly distributed over the entire set of documents $\mathcal{D}$. The $1 - P(t_i|d_j)$ factor constitutes an ad hoc characterization of the information content of term $t_i$ in document of interest $d_j$. Amati and Van Rijsbergen (2002) obtain a variety of tf–idf variants by defining $P(t_i|\mathcal{D})$ in different ways. Roelleke (2013), in characteristic fashion, sheds further light on the relationship between tf–idf and divergence from randomness modelling.

3 A real data example

We availed ourselves of the New York v. Strauss-Kahn (NYSK) dataset to assess the correspondence between tf–idf and the hypergeometric test on selected document retrieval and document summarization tasks. The NYSK dataset is a collection of 10,421 online English news articles about a highly publicized criminal case relating to allegations that former IMF director Dominique Strauss-Kahn had sexually assaulted a hotel maid. Dermouche et al. (2014) curated the data, and it has been made publicly available for download at the UCI Machine Learning Repository website; credit Dua and Graff (2019). In our comparisons below, we consistently find that tp–idf and the hypergeometric test agree with each other significantly more than either scoring function agrees with a tp scoring function baseline.

3.1 Data preprocessing

We subjected the raw NYSK data to a series of routine preprocessing steps as a prelude to analysis. The procedure we implemented in Python 3.7.4. For each article in the collection,
we first excised and tokenized the body text, then removed any non-ASCII terms, then converted all terms to lowercase, then removed any punctuation marks, then substituted textual representations for any terms representing integers, then removed all stop words, and finally lemmatized any remaining terms. In addition to standard Python libraries, we made use of the natural language processing libraries **nltk** 1.11.0 (see Bird et al. (2009)), **BeautifulSoup4** 1.9.4, **inflect** 2.1.0 and **contractions** 0.0.21. The total number of unique terms after preprocessing is 74,004. Computer code to run our preprocessing procedure on the raw NYSK data is available at [www.github.com/paul-sheridan/hgt-tfidf](www.github.com/paul-sheridan/hgt-tfidf).

### 3.2 Experimental setup

To compare tp–idf with the hypergeometric test, we calculated three NYSK dataset term-document matrix variants: one based on the tp–idf scoring function of Eq. (1), one based on the hypergeometric test scoring function of Eq. (3), and another based on the tp scoring function for use as a baseline measure. We evaluated the agreement between the tp–idf and the hypergeometric test in two different scenarios: document retrieval and document summarization. The questions we provide quantitative answers to, precisely stated, are as follows:

**Document retrieval scenario:** To what extent do the the tp–idf and hypergeometric test agree on which documents are most relevant to a given user submitted query?

**Document summarization scenario:** To what extent do the the tp–idf and hypergeometric test agree on which among a given document’s terms best characterize its subject matter?

We used the *Precision at 10* (P@10) evaluation measure to quantify how well rival scoring function outputs agree on particular information retrieval tasks. P@10 generally measures the number of relevant items in the ten leading positions of an item ranking. In the document retrieval scenario the items correspond to documents. In the document summarization scenario the items correspond to terms. Now suppose $S_1$ and $S_2$ are scoring functions. To calculate the P@10 score between $S_1$ and $S_2$ for a given retrieval task, we take the top ten items according to $S_1$ as the relevant items, and then return the number of relevant items
found in the top ten scoring items according to $S_2$. Distinct ordinal numbers are assigned to equally scoring items in a randomized manner. Note that the P@10 score comes out to be the same when $S_1$ and $S_2$ are swapped, so that the order does not matter. R code to reproduce our results is available at [www.github.com/paul-sheridan/hgt-tfidf](http://www.github.com/paul-sheridan/hgt-tfidf).

### 3.3 Results

#### 3.3.1 The document retrieval scenario

We ran one- and two-term queries on the NYSK documents, of which one will recall there are $N = 10,421$ in number, to compare tp–idf with the hypergeometric test.

Consider first the one-term query case. We queried the NYSK documents on each of the $M = 74,004$ unique terms making up the dataset. For each one-term query $q = \{t_i\}$ ($1 \leq i \leq M$), we ranked the documents in decreasing order of tp–idf, hypergeometric test, and tp score, respectively.

The average P@10 score between tp–idf and the hypergeometric test works out to $7.70 \pm 2.97$ in the case when we confine ourselves to those 15,788 terms occurring in at least ten documents. Contrast with this with the average P@10 score of $3.84 \pm 3.10$ that we obtained by calculating P@10 scores between corresponding pairs of randomized document rankings. This shows that the hypergeometric test and tp–idf are more similar to each other than either is to a baseline measure generated from the P@10 scores of random orderings of documents. It is important to note, however, that the average P@10 score between hypergeometric test and tp equals that of the hypergeometric test and tp–idf. This is because tp and tp–idf give rise to identical document rankings, disregarding the random breaking of ties, on account that the idf factor in tp–idf is constant across all documents for any one-term query. It will come as no surprise, then, when we report an average P@10 score of $9.93 \pm 0.31$ between tp and tp–idf.

Figure 1(A) shows average P@10 scores for terms occurring in at least $C$ documents plotted over a broad range of $C$. In what preceded we examined the special case when $C$ is equal to ten. But a cursory inspection of the plot reveals that the conclusions we drew in the $C = 10$ case hold generally true, namely: 1) the hypergeometric test and tp–idf produce top ten document rankings that are much more similar each other than they are to random
Figure 1: The hypergeometric test produces outcomes similar to those of tp–idf on the NYSK dataset in (A) the document retrieval scenario for one-term queries and (B) the document summarization scenario. Panel (A) shows an average of one-term query P@10 scores (along the vertical axis) plotted as a function of a cutoff value, C, (along the logarithmically scaled horizontal axis). The average P@10 score for a given value of C is calculated by adding up the P@10 scores of precisely those terms occurring in at least C documents and then dividing by the total number of scores. The hypergeometric test/tp–idf average P@10 scores (solid line) conspicuously exceed those calculated between the hypergeometric test and the random document overlap baseline (dotted line). Note, however, that one-term queries cannot be used to discriminate between tp and tp–idf (dashed line). Panel (B) shows histograms of P@10 scores as measured between the hypergeometric test and each of three alternatives in the document summarization scenario. The agreement between the hypergeometric test and tp–idf (blue) is quite strong with an average P@10 score of 8.47. This result cannot be explained by summarizing documents with randomly selected terms (red). Nor can it be explained by the effect of tp alone (green), seeing as the average P@10 score between the hypergeometric test and tp is a mere 4.18.
orderings of documents, but 2) it is impossible to differentiate tp–idf from tp on the basis of one-term queries. The latter conclusion is problematic as our objective is to compare the hypergeometric test with tp–idf, rather than the naïve tp scoring function.

This brings us to the two-term query case. We are immediately confronted with the problem of how to go about constructing queries that discriminate tp–idf from tp as much as possible. To meet this challenge we isolated a small pool of “bursty” terms from which individual query terms are to be selected. A term is said to be bursty, loosely speaking, when its occurrences are concentrated in very few documents. We evaluated the Irvine and Callison-Burch (2017) proposed term burstiness measure

\[ B(t_i) = \frac{1}{K_i} \sum_{j=1}^{N} \frac{k_{ij}}{n_j} \]

on those terms occurring in at least ten documents. The top 106 highest scoring terms we identified as bursty, and separated into two groups based on high/low document proportion, \( K_i/N \). The six bursty terms with highest \( K_i/N \) value we call common (i.e., “strausskahn,” “say,” “new,” “imf,” “comment,” and “lagarde”), and the remaining 100 bursty terms we call rare. Each common bursty term we paired with each rare bursty terms for a total of six different sets of 100 two-term queries.

Table summarizes our findings for the six different two-term query experiments. Two-term query scores are evaluated as the sum of the scores of their constituent terms. Each common bursty term doubles as an experiment name. While the agreement between the hypergeometric test and tp–idf falls short of what was observed in the one-term query experiment, it remains comfortably in excess of the agreement observed between the hypergeometric test and the tp baseline, not to mention the hypergeometric test and the random document overlap baseline. In the “strausskahn” experiment, for instance, we find an average P@10 score of 2.90 ± 3.52 between tp–idf and the hypergeometric test, as compared with 0.29 ± 1.34 between the hypergeometric test and the tp baseline. And unlike with the one-term query experiment, the agreement between the hypergeometric test and tp–idf is not solely be attributable to tp, as the average P@10 score between tp and tp–idf is just 0.62 ± 1.55. Results from the five other two-term query experiments can be similarly interpreted. Empirical evidence in support of the claim that tp–idf agrees with the hypergeometric test beyond what the tp baseline can explain is thus supplied.
Table 1: Scoring function comparison results from six two-term query experiments. The first column lists in decreasing order of document proportion (second column) the six common bursty terms described in the main text. A total of 100 two-term queries were formed for each experiment by pairing a common bursty term with each of the 100 rare bursty terms (see the main text). The rightmost four columns show average P@10 scores between selected scoring functions with standard deviations. The hypergeometric test (HGT) agrees with tp–idf (fifth column) beyond what can be accounted for by the random document overlap baseline (third column). Unlike with the one-term query experiment, the agreement between the hypergeometric test and tp–idf is not explained by the tp scoring function alone (fourth and sixth columns).

| Experiment name | $K_i/N$ | Random | HGT / tp | HGT / tp–idf | tp–idf / tp |
|-----------------|---------|--------|----------|--------------|-------------|
| strausskahn     | 0.92    | 0.01 ± 0.00 | 0.29 ± 1.34 | 2.90 ± 3.52 | 0.62 ± 1.55 |
| say             | 0.85    | 0.01 ± 0.00 | 1.86 ± 1.48 | 3.91 ± 3.65 | 0.82 ± 1.86 |
| new             | 0.83    | 0.01 ± 0.00 | 0.25 ± 1.34 | 2.13 ± 2.99 | 0.80 ± 1.81 |
| imf             | 0.78    | 0.01 ± 0.00 | 0.17 ± 1.13 | 3.38 ± 3.78 | 1.18 ± 2.29 |
| comment         | 0.39    | 0.02 ± 0.00 | 0.20 ± 1.19 | 1.08 ± 2.29 | 4.80 ± 2.65 |
| lagarde         | 0.23    | 0.04 ± 0.00 | 0.40 ± 1.53 | 1.34 ± 2.16 | 4.58 ± 3.32 |

Table 2: Scoring function comparison results from six different two-term query experiments based on a multivariate generalization of the hypergeometric test (MvHGT).

| Exp. name | $K_i/N$ | Random | MvHGT / tp | MvHGT / tp–idf | tp–idf / tp |
|-----------|---------|--------|------------|----------------|-------------|
| strausskahn | 0.92    | 0.01 ± 0.00 | 0.30 ± 1.31 | 2.36 ± 2.78 | 0.62 ± 1.55 |
| say       | 0.85    | 0.01 ± 0.00 | 1.72 ± 1.15 | 3.20 ± 2.90 | 0.82 ± 1.86 |
| new       | 0.83    | 0.01 ± 0.00 | 0.18 ± 0.94 | 2.29 ± 2.60 | 0.80 ± 1.81 |
| imf       | 0.78    | 0.01 ± 0.00 | 0.13 ± 0.85 | 2.20 ± 2.67 | 1.18 ± 2.29 |
| comment   | 0.39    | 0.02 ± 0.00 | 1.70 ± 1.04 | 0.89 ± 1.58 | 4.80 ± 2.65 |
| lagarde   | 0.23    | 0.04 ± 0.00 | 0.27 ± 0.97 | 1.21 ± 1.97 | 4.58 ± 3.32 |
In the two-term query experiments, we defined the hypergeometric test score of a two-term query, \( q = \{t_{l_1}, t_{l_2}\} \) \((t_{l_1}, t_{l_2} \in T)\), on a document, \( d_j \in D \), as the sum of two one-term hypergeometric test scores, \( hgt(k_{l_1j}, n_j, K_i, N) + hgt(k_{l_2j}, n_j, K_i, N) \). Table 2 shows what happens when this convenient formula is replaced by a proper multivariate generalization of the hypergeometric test. One might expect that tp–idf would better agree with the multivariate hypergeometric test, than it does with our crude additive version, but this is not so. However, it is reassuring to find that the interpretation of the experimental results does not change even when the multivariate hypergeometric test is employed.

3.3.2 The document summarization scenario

We summarized each NYSK document according to its top ten highest hypergeometric test scoring terms and compared these results with those of our three now familiar alternative scoring functions. The blue barred histogram of Fig. 1(B) casts into clear relief a striking correspondence between the hypergeometric test and tp–idf. The average P@10 score is 8.47 with a standard deviation of 1.04. It is interesting to observe that this is significantly higher than the average P@10 score of 4.18 ± 1.68 between the hypergeometric test and the tp baseline. The green barred histogram of Fig. 1(B) serves to reinforce this finding. Lastly, the red barred histogram of Fig. 1(B) shows the P@10 score distribution for a simple baseline measure where each document is summarized by ten randomly selected terms. The average P@10 score is 0.58 with a standard deviation of 0.71. To sum up: the hypergeometric test overwhelmingly agrees with tp–idf in the document summarization scenario, and this cannot be explained by the effect of the tp baseline function, let alone by lists of randomly selected terms.

4 A mathematical justification

In this section we provide a mathematical argument that explains the close correlation we observed between the hypergeometric test and tp–idf on the preceding information retrieval tasks.

For notational convenience we shall drop the subscripts on \( k_{ij}, n_j, K_i \) and \( K_i \). Let
\[ p = k/n \text{ and } q = K/N. \] Note that we will treat \( p \) and \( q \) as continuous variables taking on values in the open unit interval. Define the \( \text{tp–idf} \) surrogate \( f(p, q) \) as follows:

\[
\text{tp-idf}(k, n, K, N) \overset{\text{def}}{=} -\frac{k}{n} \log \left( \frac{K}{N} \right),
\]

\[
= -\beta \frac{k}{n} \log \left( \frac{K}{N} \right) - \frac{k}{n},
\]

\[
= -\beta p \log q - p,
\]

\[
\overset{\text{def}}{=} f(p, q).
\]

The second line is obtained by assuming \(-\log(K/N)\) is a linear function of \(-\log(K/N)\) with slope \( \beta > 0 \) and intercept \( \alpha = 1 \). It will be seen from the plot of Fig. 2(A) that this assumption holds approximately true for the NYSK dataset. The ordinary least-squares regression line of best fit has slope \( \hat{\beta} = 2.47 \) and intercept \( \hat{\alpha} = 1.03 \). The associated R-squared value of 0.97 is indicative of a pretty good fit of the linear model to the data. We explored modeling \( K/N \) as \( \gamma K/N \) with constant of proportionality \( \gamma > 0 \), but the plot of Fig. 2(B) suggests a nonlinear relationship. Modeling the logarithms of \( K/N \) and \( K/N \) helps to mollify this effect. In the next-to-last line, we substitute \( p \) for \( k/n \), \( q \) for \( K/N \), and for \( \alpha \) unity. The domain of \( f(p, q) \) is the open unit square.

Next we devise a surrogate for the hypergeometric test. \cite{Chvatal1979} has established the upper bound

\[
P(k, n, K, N) \leq \left( \frac{q}{p} \right)^p \left( \frac{1-q}{1-p} \right)^{1-p} n
\]

(5)

on the hypergeometric test probability of Eq. (2) under the condition that \( q < p \). The constraint \( q < p \) is inconsequential since a term scores highly in a document precisely when \( p \) is large (i.e., the term occurs in the document frequently) and \( q \) small (i.e., the term occurs in the collection infrequently). Now define the hypergeometric test surrogate \( g(p, q) \) as follows:

\[
hgt(k, n, K, N) \overset{\text{def}}{=} -\log P(k, n, K, N),
\]

\[
> -np \log q + np \log p - n(1-p) \log(1-q) + n(1-p) \log(1-p),
\]

\[
\overset{\text{def}}{=} g(p, q).
\]

The first line is just a restatement of the hypergeometric test scoring function of Eq. (3). In going from the first to the second line, we apply Chvátal’s inequality and algebraically
Figure 2: NYSK data derived scatterplot with ordinary least-squares regression line (red) for (A) the negated logarithmically scaled document proportion \(-\log(K/N)\) versus the negated logarithmically scaled total term proportion \(-\log(K/N)\), and (B) the document proportion \(K/N\) versus the total term proportion \(K/N\).

simplify the result. The domain of \(g(p,q)\) is \(\{(p,q)|0 < p < 1, q < p\}\) which is the half of the open unit square lying below the diagonal line \(p = q\).

Two points warrant discussion. First, \(g(p,q)\) sets a lower bound on \(hgt(k,n,K,N)\) so long as \(K/N < k/n\). At present, very little is known about the tightness of Chvátal’s bound. But the tighter the bound, the more secure is our argument. Second, \(g(p,q)\) contains a niggling factor of \(n\) that cannot be rewritten in terms of \(p = k/n\). We deal with this inconvenience by assuming \(n\) is fixed and that \(k\) varies from 0 to \(n\). In other words, we treat \(n\) as a constant.

The contour plots of Figs. 3(A) and 3(B) reveal a definite topological correspondence between the surfaces of \(f(p,q)\) and \(g(p,q)\). Both functions spike around the lower-right corner of the unit square in the region where \(p\) is moderate to large and \(q\) small. Note however that \(p\) will tend to be quite small in practice. To see why, recall that \(p\) serves as a proxy for the fraction of terms in a document equal to a given term. Even a value of \(p\) as
Figure 3: Contour plots for (A) the tp–idf surrogate \( f(p, q) \) and (B) the hypergeometric test surrogate \( g(p, q) \).

low as 0.25 would mean that a single term accounts for a quarter of all terms occurrences in a document. This is clearly unrealistic. By way of comparison, we found \( \max(p) \approx 0.01 \) and \( \max(q) \approx 0.01 \) in the case of the NYSK dataset.

In light of this reality, we are compelled to shift focus to the transformed functions \( f(\lambda p, \lambda q) \) and \( g(\lambda p, \lambda q) \) for scaling factor \( 0 < \lambda \leq 1 \). In practice, we will be interested in values of \( \lambda \) that are close to 0. Figures 4(A) and 4(B) show the contour plots of \( f(\lambda p, \lambda q) \) and \( g(\lambda p, \lambda q) \), respectively, with \( \lambda \) fixed at 0.01. In each case, the overall surface topology is not much changed under the transformation \( p \rightarrow \lambda p \) and \( q \rightarrow \lambda q \). The formulae for \( f(\lambda p, \lambda q) \) and \( g(\lambda p, \lambda q) \) work out to be

\[
f(\lambda p, \lambda q) = \lambda f(p, q) - \lambda \log \lambda \beta p
\]

and

\[
g(\lambda p, \lambda q) \approx n \lambda p \log q + n \lambda p \log p + n \lambda^2 p(q + 1) + n \lambda(q - p),
\]

\[
= \lambda g(p, q) - n \lambda(1 - p) \log(1 - p) + n \lambda(\lambda + 1)pq - \lambda(1 - \lambda)p.
\]

Eq. (7) yields a simple functional relationship between \( f(p, q) \) and \( f(\lambda p, \lambda q) \). Eq. (8) is
Figure 4: Contour plots for (A) the tp–idf surrogate $f(\lambda p, \lambda q)$ with $\lambda = 0.01$ and (B) the hypergeometric test surrogate $g(\lambda p, \lambda q)$ likewise with $\lambda = 0.01$.

obtained by appeal to the first-order Taylor approximation of the logarithms $\log(1 - \lambda p) \approx -\lambda p$, $\log(1 - \lambda q) \approx -\lambda q$ and $\log(1 - q) \approx -q$. But it is evident that the functional relationship between $g(p, q)$ and $g(\lambda p, \lambda q)$ established through Eq. (9) is somewhat more complicated.

We are interested in comparing $f(\lambda p, \lambda q)$ and $g(\lambda p, \lambda q)$ in the neighborhood of the point $(\lambda, 0)$, since both functions tend to infinity as $(\lambda p, \lambda q)$ approaches $(\lambda, 0)$ (i.e., as $p \to 1$ and $q \to 0$). Reformulating $f(\lambda p, \lambda q)$ and $g(\lambda p, \lambda q)$ in polar coordinates facilitates the analysis. Fix $O = (\lambda, 0)$ as the point of origin. Let $\epsilon > 0$ be the distance from $O$ to any other point $P = (\lambda p, \lambda q)$ such that $0 < p < 1$ and $q < p$. Let $0 < \theta < \pi/2$ be the angle between the horizontal axis and the line $OP$. In polar coordinates, the point $P$ is represented by the ordered pair $(\lambda - \epsilon \cos(\theta), \epsilon \sin(\theta))$ where $\lambda p = \lambda - \epsilon \cos(\theta)$ and $\lambda q = \epsilon \sin(\theta)$.

Rewriting $f(\lambda p, \lambda q)$ and $g(\lambda p, \lambda q)$ in polar form gives

$$f(\epsilon, \theta) = -\lambda \beta \log(\epsilon \sin \theta) + \beta \epsilon \cos \theta \log(\epsilon \sin \theta) + \epsilon \cos \theta - \log \lambda$$

(10)
and
\[ g(\epsilon, \theta) \approx -n\lambda \log(\epsilon \sin \theta) + n\epsilon \cos \theta \log(\epsilon \sin \theta) + n\epsilon(1 - \log \lambda \cos \theta - \cos \theta - \lambda) \]
\[ + n\epsilon^2(\lambda^{-1} \cos^2 \theta + \cos \theta \sin \theta) + n(1 - \lambda + \epsilon \cos \theta) \log(1 - \lambda + \epsilon \cos \theta) \]
\[ + n\lambda \log \lambda, \tag{11} \]
respectively. The approximation in Eq. (11) stems from employing the first-order Taylor expansions \( \log(1 - \epsilon \sin \theta) \approx -\epsilon \sin \theta \) and \( \log(1 - \epsilon \lambda^{-1} \cos \theta) \approx -\epsilon \lambda^{-1} \cos \theta \).

Notice that \( f(\epsilon, \theta) \) is dominated by \( \log(1/\epsilon) \) in the limit as \( \epsilon \) approaches 0. The \( \beta\epsilon \cos \theta \log(\epsilon \sin \theta) \) term in Eq. (10) can be ignored as \( \lim_{x \to 0} x \log(x) = 0 \). A cursory inspection of \( g(\epsilon, \theta) \) reveals that it too is dominated by \( \log(1/\epsilon) \) as \( \epsilon \) goes to 0. Thus, the fundamental reason why the hypergeometric test agrees with tp–idf to as great a degree as it does is because they are approximately proportional in the region of the most significant terms. This concludes the argument.

5 Discussion

In this paper we have presented a novel hypergeometric test interpretation of a common tf–idf variant. In so doing we have established the first theoretical foundation for a tf–idf variant within the framework of classical statistical hypothesis testing. The hypergeometric test justification for tf–idf offers a new and intuitive way of understanding why tf–idf has proved to be so effective in practice.

The hypergeometric test is commonly used in bioinformatics research to identify statistically over-represented genes in lists of genetic pathways (Boyle et al. (2004); Huang et al. (2009); Maere et al. (2005); Warde-Farley et al. (2010); Zheng and Wang (2008)). It is curious fact that the tf–idf numerical measure has not yet been brought to bear on this problem. Conversely, the hypergeometric test has received surprisingly little attention from the information retrieval community. Outside of the present work, the performance of tf–idf as compared with that of the hypergeometric test on information retrieval tasks is found only in Önsjö and Sheridan (2020). In Section 3.3.1 we described the implementation of a multivariate generalization of the hypergeometric test, but its evaluation is computationally intensive. One interesting line of future work would be to develop a more clever
implementation of the multivariate hypergeometric test, and compare its performance with that of tf-idf on document retrieval tasks.

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