Quantum Symmetries in 2D Massive Field Theories.
We review various aspects of (infinite) quantum group symmetries in 2D massive quantum field theories. We discuss how these symmetries can be used to exactly solve the integrable models. A possible way for generalizing to three dimensions is shortly described.
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INTRODUCTION

Symmetries of the S-matrices of 4D quantum field theories are subject to the severe constraints of the Coleman-Mandula theorem [1]. In general, these possible symmetries do not allow for non-perturbative solutions of the theories. In two dimensions, this theorem breaks down and there is room for richer symmetries. The aim of the lecture is to describe new symmetries of 2D massive QFT, known as quantum group symmetries. This paper is mainly a review of published papers completed by few remarks and comments.

There are at least two motivations for studying quantum symmetries in 2D quantum field theories:

(i) The study of the possible symmetries of 2D QFT. The quantum group symmetries we will analyse in these lectures are characterized by the fact that, unlike standard Lie algebra symmetries, they do not act additively, and by the fact that they are generated by non-local currents having in general non-integer Lorentz spins. They thus provide non-Abelian extension of the Lorentz group.

(ii) An algebraic formulation of 2D QFT and exact solutions. Our (desesperate ?) goal is to formulate an algebraic approach to 2D QFT, based on their symmetries (local and non-local), which could offer a way to solve the integrable two-dimensional quantum field theories from symmetry data, in analogy with the approach used in conformal field theories [2].

It is of some interest to compare the approach which as been used recently in CFT and in 2D integrable models. a) Rational conformal field theories are invariant under chiral vertex operator algebras, which could be local, e.g. the Virasoro or affine algebras, or non-local, e.g. the parafermionic algebras. The conformal field theories are reformulated as representation theories of the chiral algebras: Hilbert spaces of the CFT’s are direct sums of representations of the chiral algebras; conformal primary fields are intertwiners for the chiral algebras, etc... The chiral algebras are non-abelian and this is, for a large part, at the origin of the exact solvability of the CFT’s. Being completely integrable, the CFT’s also possess infinitely many local integrals of motion in involution. However, almost none of these integrals of motion are actually used to solve the conformal models. (It is even difficult to express them in terms the generators of the chiral algebras.) b) In contrast, a way of studying integrable models [3] consists in extracting local integrals of motion which are in involution. These integrals of motion thus form an abelian algebra. There existence ensures the integrability of the theory and the factorization of the S-matrix. In general,
they do not provide enough informations for solving the theory, e.g. for determining the S-matrix.

Thus, almost none of the techniques used in one of these fields is used in the other. However, besides these local integrals of motion, the 2D integrable models and the conformal models also possess non-local integrals of motion. These non-local conserved charges are the generators of non-abelian algebras known as the quantum symmetry algebras of the models. It is hoped that these new symmetry algebras will allow us to define a framework which could be apply simultaneously to the conformal field theories and to the massive integrable models.

To characterize the massive quantum field theories uniquely by their symmetry algebras requires:

(I) that the asymptotic particles form multiplets of the symmetry algebras and that the invariance for the S-matrix determine it uniquely. As we will see in the course of this lecture, because quantum group symmetries in 2D do not commute with the Lorentz group, they provide algebraic relations on the S-matrices.

(II) that all the fields of the QFT can be gathered into field multiplets transforming covariantly under the symmetry algebras and that the intertwining properties for the field multiplets determine them uniquely (in analogy with the minimal assumption in rational conformal field theories). This amounts to demand that the Ward identities have unique solutions. As we will describe, the components of the field multiplets, the descendents and the highest vector fields, are related through the Ward identities, once more in complete analogy with conformal field theories.

The symmetry algebras having these requirements could be called complete symmetry algebras. For a given model there could be more than one complete symmetry algebra. The problem of solving the integrable massive models reduce to the problem of finding a complete symmetry algebra. As we already said, the local integrals of motion which are involution do not form (in general) a complete symmetry algebra. The quantum symmetry we are going to describe provide in general more informations than these abelian integrals of motion. To our knowldege, it is not known if they form or not a complete symmetry algebra. However, it is tempting to conjecture that the algebra generated by the local integrals of motion together with the generators of the quantum symmetry form a complete symmetry algebra.
An example: \( \hat{sl}_q(2) \) -symmetry in the Sine-Gordon models. The quantum sine-Gordon theory is described by the Euclidean action

\[
S = \frac{1}{4\pi} \int d^2x \left[ \partial_x \Phi \partial_x \Phi + \lambda : \cos \left( \hat{\beta} \Phi \right) : \right].
\]  

(1)

The parameter \( \hat{\beta} \) is a coupling constant; it is related to the conventionally normalized coupling by \( \hat{\beta} = \beta / \sqrt{4\pi} \). For \( \hat{\beta} \leq \sqrt{2} \) the action can be renormalized by normal-ordering the \( \cos \left( \hat{\beta} \Phi \right) \) interaction and absorbing the infinities into \( \lambda \); the coupling constant \( \hat{\beta} \) is thereby unrenormalized [4]. The sine-Gordon theory has a well known topological current:

\[
J^\mu(x, t) = \hat{\beta} \frac{2}{\pi} \epsilon_{\mu\nu} \partial_\nu \Phi(x, t) \text{ where } \epsilon_{\mu\nu} = -\epsilon_{\nu\mu}.
\]

The topological charge is:

\[
T = \hat{\beta} \frac{2}{\pi} \int_{-\infty}^{+\infty} dx \partial_x \Phi = \frac{\hat{\beta}}{2\pi} \left( \Phi(x = \infty) - \Phi(x = -\infty) \right).
\]  

(2)

The topological solitons that correspond to single particles in the quantum theory are described classically by field configurations with \( T = \pm 1 \). These solitons are kinks that connect two neighboring vacua in the \( \cos(\hat{\beta} \Phi) \) potential.

The sine-Gordon model possesses infinitely many local integrals of motion with odd Lorentz spins, we denote them by \( J_n \). Besides those, the sine-Gordon model also admits four non-local conserved currents [5]:

\[
\partial_\mu J^\pm_\mu(x, t) = \partial_\mu \overline{J}^\pm_\mu(x, t) = 0.
\]  

(3)

The Lorentz spin \( s \) of the currents \( J^\pm_\mu \) \( (\overline{J}^\pm_\mu) \) are \( s = \frac{2}{\beta^2} \left( -\frac{2}{\beta^2} \right) \). From these conserved currents we define four conserved charges, \( Q_\pm \) and \( \overline{Q}_\pm \), respectively associated to the currents \( J^\pm_\mu(x, t) \) and \( \overline{J}^\pm_\mu(x, t) \). The Lorentz spins of the conserved charges are:

\[
\text{spin} (Q_\pm) = -\text{spin} (\overline{Q}_\pm) = \frac{2 - \hat{\beta}^2}{\beta^2} = \frac{8\pi - \beta^2}{\beta^2}.
\]  

(4)

The conserved currents whose exact expressions are given in ref. [3] are Mandelstam like vertex operators [3] and are thus non-local.

The algebra of the non-local charges is:

\[
Q_\pm \overline{Q}_\pm - q^2 \overline{Q}_\pm Q_\pm = 0
\]  

(5a)

\[
Q_\pm \overline{Q}_\mp - q^{-2} \overline{Q}_\mp Q_\pm = a \left( 1 - q^{\pm 2} \right)
\]  

(5b)

\[
\left[ T , Q_\pm \right] = \pm 2 Q_\pm
\]  

(5c)

\[
\left[ T , \overline{Q}_\pm \right] = \pm 2 \overline{Q}_\pm
\]  

(5d)
where \( q = \exp(-2\pi i/\beta^2) \). and \( a \) some constant. The algebra \( \mathfrak{sl}(2) \) is a known infinite dimensional algebra, namely the q-deformation of the \( sl(2) \) affine Kac-Moody algebra, denoted \( \mathfrak{sl}_q(2) \), with zero center \( \mathfrak{sl}_q(2) \). Only the Serre relations for \( \mathfrak{sl}_q(2) \) are missing in \( \mathfrak{sl}(2) \).

The non-local charges \( \mathfrak{sl}_q(2) \) provide relevant information; for example, the S-matrix of the Sine-Gordon solitons \( \mathfrak{sl}_q(2) \) can be deduced from this \( \mathfrak{sl}_q(2) \) symmetry plus its unitary and crossing symmetry property. However, they probably do not form a complete symmetry algebra because the local integrals of motion do not seem to be generated by them. To prove the conjecture that the local conserved charges \( J_n \) and these non-local charges generate a complete symmetry algebra for the sine-Gordon models will be very illuminating.

1. QUANTUM SYMMETRIES IN 2D LATTICE FIELD THEORY.

We consider vertex models, i.e. models of two-dimensional statistical mechanics in which the discrete spin variables live on the midpoints of the links of a square lattice, and the Boltzmann weights are associated to the vertices of the lattice \( \mathfrak{sl}_q(2) \). The Boltzmann weight of a given vertex depends on the spin variables \( \sigma_1, \ldots, \sigma_4 \) at the four sites surrounding the vertex, and is denoted by \( R_{\sigma_1 \sigma_2}^{\sigma_3 \sigma_4} \). It is useful to view \( R \) as an operator \( V \otimes V \rightarrow V \otimes V \), where \( V \) is the vector space spanned by a set of basis vectors \( e_\sigma \) labeled by the possible values of the spin variable:

\[
Re_{\sigma_1} \otimes e_{\sigma_2} = R_{\sigma_1 \sigma_2}^{\sigma_3 \sigma_4} e_{\sigma_3} \otimes e_{\sigma_4}.
\]  

(1.1)

Partition function and correlation functions are defined as follows. Consider the system in a finite square box of size \( N \times N \). Then the square lattice \( \Lambda = \mathbb{Z}^2 / N^2 \mathbb{Z}^2 \) contains points with integer coordinates \( (x, t) \) called space and time. The spin variables live on the lattice \( \Lambda' \) of points of the form \( (x + \frac{1}{2}, t) \) and \( (x, t + \frac{1}{2}) \) with \( x, t \) integers modulo \( N \). For each \( i \in \Lambda' \) introduce a copy \( V_i \) of the space \( V \). For \( (x, t) \in \Lambda \) define \( R(x, t) \) to be the matrix \( R \) mapping \( V_{(x-\frac{1}{2}, t)} \otimes V_{(x,t-\frac{1}{2})} \) to \( V_{(x+\frac{1}{2}, t)} \otimes V_{(x,t+\frac{1}{2})} \). The matrix of Boltzmann weights,

\[
B = \bigotimes_{(x,t) \in \Lambda} R(x, t),
\]  

(1.2)

is an operator from \( \bigotimes_{i \in \Lambda'} V_i \) to \( \bigotimes_{j \in \Lambda'} V_j \).

The partition function is \( Z_N = \text{tr} B \).
For any operator \( \mathcal{O} \in \text{End} V \) define the insertion \( \mathcal{O}(j) \) of \( \mathcal{O} \) at the point \( j \in \Lambda' \) to be the operator \( 1 \otimes \cdots \otimes 1 \otimes \mathcal{O} \otimes 1 \otimes \cdots \otimes 1 \) acting on \( V_j \) in the tensor product \( \otimes_{i \in \Lambda'} V_i \). The correlation functions of operator insertions are defined as

\[
\langle \mathcal{O}_1(j_1) \cdots \mathcal{O}_n(j_n) \rangle_N = \frac{1}{Z_N} \text{tr} \left( \prod_{k=1}^{n} \mathcal{O}(j_k) B \right).
\] (1.3)

Classical examples are \( \mathcal{O}_e \sigma = \sigma e \sigma \) and \( \mathcal{O}_e \sigma = \delta_{\sigma \overline{\sigma}} e \sigma \) for some \( \overline{\sigma} \). Their correlation functions are the usual spin correlation functions and the joint probabilities that the spin \( \sigma_{j_k} \) assume given values.

An alternative formalism is the transfer matrix formulation. In this formalism, one assigns to each \( x = 1, \ldots, N \) a copy \( V_x \) of \( V \). The transfer matrix \( T \) is:

\[
T = \text{tr}_0 \left( R_{0N} \cdots R_{02} R_{01} \right)
\] (1.4)

with \( R_{nm} \) the matrix \( R \) acting on \( V_n \) and \( V_m \) in \( V_0 \otimes V_1 \otimes \cdots \otimes V_N \), and the trace is over \( V_0 \).

For \( \mathcal{O} \in \text{End} V \) and \( x \in \{1, \ldots, N\} \) define \( \mathcal{O}(x) \) as \( \mathcal{O} \) acting on \( V_x \) in \( V_1 \otimes \cdots \otimes V_N \) and \( \mathcal{O}(x - \frac{1}{2}) \) as \( \mathcal{O}(x - \frac{1}{2}) = T^{-t} \text{tr}_0 \left( R_{0N} \cdots R_{0x} \mathcal{O}_0 R_{0,x-1} \cdots R_{01} \right) \). Heisenberg fields \( \mathcal{O}(j), j \in \Lambda' \) are defined as:

\[
\begin{align*}
\mathcal{O}(x - \frac{1}{2}, t) &= T^{-t} \mathcal{O}(x - \frac{1}{2}) T^t \\
\mathcal{O}(x, t - \frac{1}{2}) &= T^{-t} \mathcal{O}(x) T^t
\end{align*}
\] (1.5)

The partition function in the transfer matrix formalism is \( Z = \text{tr} T^N \), and if the time coordinates of \( j_1, \ldots, j_n \) are ordered (with smaller times on the right of larger times) the correlation function \( \langle \mathcal{O}_1(j_1) \cdots \mathcal{O}_n(j_n) \rangle_N \) defined above coincides with

\[
\langle \mathcal{O}_1(j_1) \cdots \mathcal{O}_n(j_n) \rangle_N = \frac{1}{Z_N} \text{tr} \left( \mathcal{O}_1(j_1) \cdots \mathcal{O}_n(j_n) T^N \right)
\] (1.6)

We have defined point-like operator insertions in two different formalisms. It is sometimes useful to define operator insertions associated to some finite set of neighboring points in \( \Lambda' \) as linear combination of products of point-like insertions at the points of the set. This is the lattice analogue of the operator product expansion of field theory.

1a) Quantum symmetries and conserved currents.
(i) **Lie algebra symmetry.** Suppose that the Boltzmann weights are invariant under some Lie algebra \( \mathcal{G} \). This means that \( V \) carries a representation of \( \mathcal{G} \) and for each generator \( T_a \) of \( \mathcal{G} \) in that representation we have

\[
R(T_a \otimes 1 + 1 \otimes T_a) = (T_a \otimes 1 + 1 \otimes T_a)R
\]

i.e. \( R \) is an intertwiner. It is useful to represent this equation graphically. If \( T_a \) is represented by a little cross, we have

\[
\begin{array}{c}
- - - + - - - = - - - + - - - \\
\end{array}
\]

Introduce now a local current \( J^\mu(x; t; X) \), linear in \( X \in \mathcal{G} \), for each vertex \((x, t)\) of the lattice. The components \( J^t(x, t; X) \), \( J^x(x, t; X) \) are defined by the insertion of the matrix \( X = \sum_a X_a T_a \) at the site \((x, t - \frac{1}{2})\) or the site \((x - \frac{1}{2}, t)\), respectively. Graphically,

\[
\begin{aligned}
J^t(x, t; X) &= - - (x, t) \\
J^x(x, t; X) &= - - (x, t)
\end{aligned}
\]

Then, in any correlation function (with no insertion of other fields at the sites surrounding \((x, t)\)), (1.7) reads

\[
J^t(x, t + 1; X) - J^t(x, t; X) + J^x(x + 1, t; X) - J^x(x, t; X) = 0
\]

which is the lattice version of the continuity equation \( \partial_\mu J^\mu = 0 \). As in the continuum, this equation implies the conservation of the charge \( Q(X) = \sum_x J^t(x, t; X) \).

As it is obvious from the pictures (1.9), the operators \( J^\mu(x; X) \) are local operators: they satisfy equal-time commutation relations:

\[
J^\mu(x; X) J^\nu(y; Y) = J^\nu(y; Y) J^\mu(x; X) ; \quad \forall x \neq y
\]

for all \( X, Y \in \mathcal{G} \).
(ii) Quantum invariance. We now generalize \[ 11 \] the preceding construction to an invariance under a Hopf algebra. Recall that a Hopf algebra $A$ is an algebra with unit $1$ and associative product $m: A \otimes A \to A$, equipped with a coproduct $\Delta: A \to A \otimes A$, a counit $\epsilon: A \to \mathbb{C}$, and an antipode $S: A \to A$ so that: (i) $\Delta$, $\epsilon$ are algebra homomorphisms, $S$ is an algebra antihomomorphism; (ii) $(1 \otimes \Delta) \Delta(X) = (\Delta \otimes 1) \Delta(X)$; (iii) $(1 \otimes \epsilon) \Delta(X) = (\epsilon \otimes 1) \Delta(X) = X$; (iv) $m(1 \otimes S) \Delta(X) = m(S \otimes 1) \Delta(X) = \epsilon(X)1$, for all $X \in A$. The Hopf algebra $A$ we will consider are those generated by elements $T_a$, $\Theta^b_a$, $\hat{\Theta}^b_a$ with, among the relations, $\Theta^c_a \hat{\Theta}^b_a = \hat{\Theta}^c_a \Theta^b_a = \delta^b_a$. We also assume that the comultiplication in $A$ is defined by:

\[
\Delta(T_a) = T_a \otimes 1 + \Theta^b_a \otimes T_b \\
\Delta(\Theta^b_a) = \Theta^c_a \otimes \Theta^b_c \\
\Delta(\hat{\Theta}^b_a) = \hat{\Theta}^c_a \otimes \hat{\Theta}^a_c
\]

The definition of the counit and the antipode in $A$ are found from the Hopf algebra axioms. The motivation for introducing these algebras will be given later. Lie superalgebras provide an example of such algebras.

The correct generalization of the invariance eq. (1.7) is $R \, \Delta(X) = \sigma \circ \Delta(X) \, R$, with $\sigma X \otimes Y = Y \otimes X$. Explicitly,

\[
R \,(T_a \otimes 1 + \Theta^b_a \otimes T_b) = (1 \otimes T_a + T_b \otimes \Theta^b_a) \, R \tag{1.13a}
\\
R \, \Theta^c_a \otimes \Theta^b_c = \Theta^c_b \otimes \Theta^c_a \, R \tag{1.13b}
\]

These equations have a graphical interpretation. The generators $T_a$, $\Theta^b_a$, $\hat{\Theta}^b_a$ are conveniently represented in terms of crosses and oriented wavy lines:

$T_a = a \times$ ; $\Theta^b_a = a^b$ ; $\hat{\Theta}^b_a = b^a$

The graphical representation of (1.13a) is then

\[
- - - - + - - - = - - - - + - - - -
\]

with the convention that where pieces of wavy lines join an implicit contraction of indices is understood. The currents $J^\mu(x, t; X)$, $X = \sum_a X_a T_a$, are then constructed as for
parafermionic currents, namely with a disorder line (the wavy line) attached:

\[
J^t_a(x,t) = J^t(x,t; T_a) = -\ldots - \ldots - \ldots - \ldots - \ldots - \ldots - (x,t) \\
\phantom{J^t_a(x,t)} = -\ldots - \ldots - \ldots - \ldots - \ldots - \ldots - (x,t) \\
\phantom{J^r_a(x,t)} = -\ldots - \ldots - \ldots - \ldots - \ldots - \ldots - (x,t)
\]

The disorder line ends at some specified point on the boundary of the lattice. The identity (1.13) implies that the disorder line may be deformed (away from insertions of observables) without changing the value of correlation functions. It behaves as the holonomy of a flat connection, just as for ordinary disorder fields [12]. Equation (1.13a) implies the continuity equation (1.10) for non-local currents.

In the operator formalism, the time component of the current is an operator (in the Schrödinger picture) acting on the finite volume Hilbert space \( V \otimes \ldots \otimes V \) (\( N \) factors) as

\[
J^t_a(x) = J^t(x; T_a) = \Theta_a^1 \Theta^{a_2} \ldots \Theta^{a_{x-1}} \otimes T_{a_{x-2}} \otimes T_{a_{x-1}} \otimes 1 \otimes \ldots \otimes 1.
\]

The space component has a more cumbersome operator representation which we omit.

(iii) The braiding relations. By construction the currents (1.14) are non-local. They satisfy braided equal-time commutation relations. These braiding relations arise due to the topological obstructions that one encounters when trying to move the wavy string attached to the currents through a point on which a field is located. In order to write simple closed formula for the braiding relations we now assume that we have completed the set of generators, \( T_a, \Theta^b_a, \hat{\Theta}^b_a \) such that they close under the adjoint action. This implies that there exists a c-number matrix \( R^{bd}_{ac} \) such that:

\[
\Theta_n^a T_c \hat{\Theta}^b_n = R^{bd}_{ac} T_d \\
R^{ab}_{nm} \Theta^a_n \Theta^m_c = \Theta^b_m \Theta^a_n R^{nm}_{cd}
\]

Then, a simple computation shows that:

\[
J^\mu_a(x) J^\nu_b(y) = R^{cd}_{ab} J^\nu_d(y) J^\mu_c(x), \quad \text{for } x > y
\]
(iii) Global symmetry algebra. The algebra $A$ acts on the Hilbert space $V^\otimes N$ by the coproduct $\Delta_N$, defined recursively by $\Delta_2 = \Delta$, $\Delta_{n+1} = \Delta_n(1 \otimes \Delta)$. For generators, we have the formulae

$$
\Delta_N(\Theta^b_a) = \Theta^{a_1}_a \otimes \Theta^{a_2}_{a_1} \otimes \cdots \otimes \Theta^{a_{N-1}}_{a_N},
$$

$$
\Delta_N(\hat{\Theta}^b_a) = \hat{\Theta}^{a_1}_a \otimes \hat{\Theta}^{a_2}_{a_1} \otimes \cdots \otimes \hat{\Theta}^{a_{N-1}}_{a_N},
$$

$$
\Delta_N(T_a) = \sum_{x=1}^{N} \Theta^{a_1}_{a_1} \otimes \Theta^{a_2}_{a_2} \otimes \cdots \otimes \Theta^{a_{x-1}}_{a_{x-2}} \otimes T_{a_{x-1}} \otimes 1 \otimes \cdots \otimes 1.
$$

(1.18)

Comparing with (1.15), we see that $\Delta_N(T_a)$ is the charge corresponding to the current $J^\mu_a(x)$:

$$
\Delta_N(T_a) = \sum_{x=1}^{N} J^i_a(x).
$$

(1.19)

The global charges $\Delta_N(\Theta^b_a)$ can be interpreted as topological charges.

The global charges $\Delta_N(T_a)$ and $\Delta_N(\Theta^b_a)$, satisfy the same algebra as the original generators $T_a$, $\Theta^b_a$ (because the comultiplication $\Delta$ is an homomorphism from $A$ to $A \otimes A$). If we assume, as we did in the previous section, that the generators $T_a$, $\Theta^b_a$ are closed under the adjoint action, then there also exist c-numbers $f^a_{bc}$ such that :

$$
T_a T_b - R^{cd}_{ab} T_d T_c = f^c_{ab} T_c.
$$

(1.20)

These are generalized braided Lie commutation relations.

Remark: Because the currents are non-local, the local conservation laws for the currents do not systematically imply those of the global charges. The conservation laws for the charges can be broken by boundary terms which depends on the sector on which the charges are acting.

1b) Fields multiplets.

The symmetry algebra $A$ acts also on the field operators. In the operator formalism, the fields are operators $O \in \text{End}(V^\otimes N)$. Any element $X \in A$ of an Hopf algebra $A$ acts an operator $O$ by

$$
Q_X O = \sum_i X_i O S(X^i)
$$

(1.21)

with $\Delta(X) = \sum_i X_i \otimes X^i$ and $S$ the antipode in $A$. In our case, for the generators $T_a$ and $\Theta^b_a$, this becomes:

$$
Q_a O = \Delta_N(T_a) O - (Q^b_a O) \Delta_N(T_b)
$$

$$
Q^b_a O = \Delta_N(\Theta^c_a) O \Delta_N(\hat{\Theta}^b_c)
$$

(1.22)
The multiplets of fields are collections of fields transforming in a representation of the symmetry algebra $A$. More precisely, let $V_A$ be a representation space for $A$. A field multiplet at $x$ is an operator $\Phi^A(x; v)$ acting on $V^\otimes n$ depending linearly on a vector $v$ in $V_A$, with transformation property

$$Q_X \Phi^A(x; v) = \Phi^A(x; Xv). \tag{1.23}$$

It is clear that, in general, the fields $\Phi^A(x; v)$ are necessarily non-local. However, in a given multiplet there can be local fields.

Field multiplets can be constructed from the following data: Representation spaces $W$, $W'$ and operators $\phi$ and $\Omega$, $\phi : V_A \otimes W \to W'$ and $\Omega : V_A \otimes V \to V_A \otimes V$, with (twisted) intertwining properties

$$\phi \Delta(X) = X \phi$$

$$\Omega \Delta(X) = \sigma \circ \Delta(X) \Omega, \tag{1.24}$$

for all $X \in A$. The spaces $W$, $W'$ are in the simplest case equal to $V$, or may be tensor products $V^\otimes n$, $V^\otimes n'$. In terms of a basis $\{e_i\}$ of $V_A$, with $\phi(e_i \otimes w) = \phi_i w$, $\Omega(e_i \otimes v) = e_j \otimes \Omega_i^j v$, $X e_i = e_j X_i^j$, the field multiplets are operators acting on $V^\otimes N$ defined by,

$$\Phi^A(x; e_i) \equiv \Phi^A_i(x) = \Omega_{i_1}^{i_i} \otimes \Omega_{i_1}^{i_2} \otimes \cdots \otimes \Omega_{i_{x-2}}^{i_{x-1}} \otimes \phi_{i_{x-1}} \otimes 1 \otimes \cdots \otimes 1, \tag{1.25}$$

with $\phi_{i_x}$ acting on the tensor product of the $x$th to the $(x-1+n)$th factor *. Graphically this is represented as

$$\Phi^A_i(x) \equiv \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array}$$

where the circle represents an insertion of $\phi$ and each crossing of the double line with a single line represents an $\Omega$.

The transformation property (1.23) follows from (1.24) and Hopf algebra properties. The action of the algebra of fields is by definition (1.22):

$$Q^b_a \Phi^A(x; v) = \Theta^a_{a_1} \otimes \Theta^a_{a_2} \otimes \cdots \otimes \Theta^c_{a_{N-1}} \Phi^A(x; v) \Theta^b_{b_1} \otimes \Theta^b_{b_2} \otimes \cdots \otimes \Theta^b_{b_N},$$

$$Q^a_a \Phi^A(x; v) = \sum_{y=1}^N \left( J^a_y \Phi^A(x; v) - Q^b_{a} \Phi^A(x; v) J^b_y \right). \tag{1.26}$$

* If $n \neq n'$ the insertion of a field produces a deformation of the lattice. It is understood that we consider correlation functions where the total $n$ is equal to the total $n'$ so that at infinity the lattice is the regular square lattice.
Because the terms with \( y > x + n - 1 \) cancel in the sum, the action of the charges \( Q_a \) can written as a contour integral on the lattice. Indicating graphically the summation by an integration contour on the dual lattice, we have:

\[
Q_a \Phi^A_i(x) \equiv \frac{a}{i} = \begin{array}{cccccc}
\quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\end{array}
\]

(1.27)

The integration contour is surrounding the fields. The intertwining properties (1.24) of the microscopic data \( \phi \) and \( \Omega \) imply that the field multiplets defined in (1.25) transform covariantly:

\[
Q^b_a \Phi^A_i(x) = \Theta^{bj}_{ai} \Phi^A_j(x)
\]

\[
Q_a \Phi^A_i(x) = T^j_{ai} \Phi^A_j(x)
\]

(1.28)

**Remark:** Because the field multiplets are non-local they satisfy equal-time braiding relations. But the braiding relations between the field multiplets are constrained by the quantum invariance. Let \( V_\Lambda \) and \( V_{\Lambda'} \) be two representation spaces of \( A \) with basis \( e_i \in V_\Lambda \) and \( e'_\alpha \in V_{\Lambda'} \). Let \( \Phi_i(x) \equiv \Phi(x; e_i) \) and \( \Phi'_\alpha(y) \equiv \Phi'(y; e'_\alpha) \) be two field multiplets. Denote by \( \mathcal{R} : V_{\Lambda'} \otimes V_\Lambda \rightarrow V_{\Lambda'} \otimes V_\Lambda \), with \( \mathcal{R}(e'_\alpha \otimes e_i) = \mathcal{R}_{\alpha i}^{\beta j} e'_\beta \otimes e_j \), the braiding matrix of these field multiplets:

\[
\Phi_i(x)\Phi'_\alpha(y) = \mathcal{R}_{\alpha i}^{\beta j} \Phi'_\beta(y)\Phi_j(x) \quad \text{for } x > y
\]

(1.29)

By quantum invariance,

\[
\mathcal{R} \Delta(X) = \sigma \circ \Delta(X) \mathcal{R} \quad , \quad \forall X \in A
\]

(1.30)

Thus, the braiding matrices intertwines the quantum algebra. Examples will be given in the following sections.

1c) **Examples.**

(i) **Yangian invariance.** The Yangians are deformations of loop algebras which have been introduced by Drinfel’d [7]. They are related to rational solutions of the quantum Yang-Baxter equation.

Let us first recall what are the Yangians. Let \( \mathcal{G} \) be a simple Lie algebra with structure constants \( f_{abc} \) in an orthonormalized basis. The Yangian, denoted \( \mathcal{Y}(\mathcal{G}) \), is the associative
algebra with unity generated by the elements \( t_a \) and \( T_a, \quad a = 1, \ldots, \text{dim} \mathcal{G}, \) satisfying the relations:

\[
\begin{align*}
[t_a, t_b] &= f_{abc} t_c \\
[t_a, T_b] &= f_{abc} T_c \\
[T_a, \{T_b, t_c\}] - [t_a, \{T_b, T_c\}] &= A_{lmn}^{abc} \{t_l, t_m, t_n\}
\end{align*}
\]  

(1.31)

with \( A_{abc}^{def} = \frac{1}{24} f_{adk} f_{bel} f_{cfm} f_{klm} \) and \( \{x_1, x_2, x_3\} = \sum_{i \neq j \neq k} x_i x_j x_k \). In particular, the elements \( t_a \) generate the Lie algebra \( \mathcal{G} \) and the elements \( T_a \) are \( \mathcal{G} \)-intertwiners taking values in the adjoint representation of \( \mathcal{G} \). (for \( \mathcal{G} = SU(2) \) one must add another Serre-like relation.) The Yangians \( \mathcal{Y}(\mathcal{G}) \) are Hopf algebras with comultiplication \( \Delta \), counit \( \epsilon \) and antipode \( S \) defined by:

\[
\begin{align*}
\Delta t_a &= t_a \otimes 1 + 1 \otimes t_a ; \\
\epsilon(t_a) &= 0 ; \quad S(t_a) = -t_a \\
\Delta T_a &= T_a \otimes 1 + 1 \otimes T_a - \frac{1}{2} f_{abc} t_b \otimes t_c ; \\
\epsilon(T_a) &= 0 ; \quad S(T_a) = -T_a - \frac{C_{\text{Ad}}}{4} t_a
\end{align*}
\]  

(1.32c)

with \( C_{\text{Ad}} \) the Casimir in the adjoint representation of \( \mathcal{G} \): \( f_{abc} f_{bcd} = C_{\text{Ad}} \delta_{ad} \).

The Yangian invariant \( R \)-matrices are those which satisfy the intertwining relation \( R \Delta(Y) = \sigma \circ \Delta(Y) \ R \) for all \( Y \in \mathcal{Y}(\mathcal{G}) \). We suppose that the vertex models we are considering in this section are defined from \( \mathcal{Y}(\mathcal{G}) \)-invariant Boltzmann weights. The non-local conserved currents we will describe in this section are the lattice analogues of those hidden in 2D massive current algebras \([13] [14]\), see section 4.

From the defining relations of \( \mathcal{Y}(\mathcal{G}) \), it is obvious that the Yangians \( \mathcal{Y}(\mathcal{G}) \) possess the properties we need in order to apply our formalism. We therefore can define Yangian currents. For simplicity, we just define the currents associated to the generators \( t_a \) and \( T_a \); we denote them \( J_a^\mu(x, t) \) and \( J_a^\mu(x, t) \), respectively. By applying the general construction, we deduce that the conserved currents \( J_a^\mu(x, t) \) are local (as it should be); they are defined by local insertions of the matrices \( t_a \) representing the Lie algebra \( \mathcal{G} \). The conserved currents \( J_a^\mu(x, t) \) are non-local; in the operator formalism we have:

\[
J_a^\mu(x, t) = T_a^\mu(x, t) + \frac{1}{2} f_{abc} J_b^\mu(x, t) \phi_c(x, t)
\]  

(1.33)
with
\[ \phi_c(x, t) = \sum_{y < x} J^c(y, t) \]  
(1.34)

In eq. (1.33) and (1.34), the notations \( J^\mu_a(x, t) \) and \( \mathcal{T}^\mu_a(x, t) \) refer to insertions of the matrices \( t_a \) or \( \mathcal{T}_a \) on the link oriented in the direction \( \mu \) and ending at the point \( (x, t) \). Notice the similarity between the expression of these lattice non-local currents and of their continuous partners \[13\] \[14\] and section 4b.

The braiding relations for the currents \( J^\mu_a(x, t) \) and \( \mathcal{T}^\mu_a(y, t) \) follows from our previous discussion:
\[
J^\mu_a(x, t) J^\nu_b(y, t) = J^{\nu b}_{y, t} J^{\mu a}_{x, t} ; \forall x \neq y \\
\mathcal{T}^\mu_a(x, t) J^\nu_b(y, t) = J^{\nu b}_{y, t} \mathcal{T}^{\mu a}_{x, t} ; \text{ for } x < y \\
\mathcal{T}^\mu_a(x, t) \mathcal{T}^\nu_b(y, t) = J^{\nu b}_{y, t} \mathcal{T}^{\mu a}_{x, t} \right) J^\mu_m(x, t) ; \text{ for } x > y
\]  
(1.35)

(ii) Quantum universal enveloping algebras. For any (affine) Kac-Moody algebra \( \mathcal{G} \) with Cartan matrix \( a_{ij}, 0 \leq i, j \leq r \) and any complex number \( q \neq 0 \), Drinfel’d [7] and Jimbo [8] define an universal quantum enveloping algebra (QUEA) \( U_q(\mathcal{G}) \). Let \( d_i \) be positive integers such that the matrix \( d_i a_{ij} \) is symmetric, and let \( q_i = q^{d_i} \). The algebra \( A = U_q(\mathcal{G}) \) has generators \( E^+_i, E^-_i, K^2_i, K^{-2}_i \), \( 0 \leq i \leq r \), and relations
\[
K^2_i K^{\pm 2}_j = K^{\pm 2}_j K^2_i, \\
K^2_i K^{-2}_j = K^{-2}_j K^2_i = 1, \\
K^2_i E^\pm_j = q_i^{\pm a_{ij}} E^\pm_j K^2_i, \\
E^+_i E^-_j - q_i^{a_{ij}} E^-_j E^+_i = \delta_{ij} (K^4_i - 1),
\]  
(1.36)

plus Chevalley-Serre relations to be written below. The Hopf algebra structure is defined by the coproduct
\[
\Delta(K^\pm_i) = K^\pm_i \otimes K^\pm_i, \\
\Delta(E^\pm_i) = E^\pm_i \otimes 1 + K^2_i \otimes E^\pm_i,
\]  
(1.37)

counit \( \epsilon(E^\pm_i) = 0, \epsilon(K^\pm_i) = 1 \), and antipode \( S(E^\pm_i) = -K^{-2}_i E^\pm_i, S(K^\pm_i) = K^{\mp 2}_i. \) The adjoint representation is then defined as usual, eq. (1.21), and the Chevalley-Serre relations are
\[
\text{Ad}_{E^\pm_i} E^\pm_j = 0.
\]  
(1.38)
We see that this is a very simple example of the Hopf algebras described in the introduction: the generators $T_a$ are $E_i^\pm$ and $\Theta_a^b$ is diagonal with entries $K_i^2$.

Statistical models with QUEA symmetry are defined by trigonometric solutions of the Yang-Baxter equation, the simplest case being the six-vertex model \[10\].

The simple currents are defined by insertions of $E_i^\pm$ with disorder lines given by insertions of $K_i^2$. In the operator formalism, the time components of the currents are

$$J_i^{t\pm}(x) = K_i^2 \otimes \cdots \otimes K_i^2 \otimes E_i^\pm \otimes 1 \otimes \cdots \otimes 1.$$

The corresponding charges are the generators $\Delta_N(E_i^\pm)$ acting on the whole space $V \otimes N$. The braiding relations are, for $x > y$:

$$J_i^{\mu\pm}(x)J_j^{\nu\pm}(y) = q_i^{\pm a_{ij}} J_j^{\nu\pm}(y)J_i^{\mu\pm}(x),$$

$$J_i^{\mu\pm}(x)J_j^{\nu\mp}(y) = q_i^{\mp a_{ij}} J_j^{\nu\mp}(y)J_i^{\mu\pm}(x).$$

These relations are the same as the braiding relations of chiral vertex operators of a free massless field $\phi$ taking value in the Cartan subalgebra of $G$ with canonical momentum $\pi$. This suggests the continuum limit identification

$$J_i^{t\pm} \sim \exp \left( i\beta \alpha_j \left( \pm \phi(x) + \int_{-\infty}^x \pi(y)dy \right) \right),$$

with $q = e^{i\beta^2}$, $\alpha_j$ the simple roots, with inner product $\alpha_i \alpha_j = d_i a_{ij}$. The space component in the continuum limit is $J_j^{x\pm} = \mp i J_j^{t\pm}$.

2. CLASSICAL ORIGIN OF QUANTUM SYMMETRIES: DRESSING TRANSFORMATIONS.

The dressing transformations form the (hidden) symmetry groups of solitons equations. Dressing transformations were first introduced by V. Zakharov and A. Shabat [15] and further developed by the Kyoto group in their Tau-function approach to soliton equations [16]. Their Poisson structure was disantangled by M. Semenov-Tian-Shansky [17]. The author’s understanding of these transformations emerged from a joint work with O. Babelon [18].

2a) What are the dressing transformations?
(i) Equations of motion and Lax connexions. Suppose that the equations of motion of a set of fields $\phi$ are described by a set of non-linear differential equations. Suppose moreover that these equations admit a Lax representation. This means that there exists a field dependent connexion, called the Lax connexion, $D_\mu$,

$$
D_\mu = \partial_\mu - A_\mu[\phi],
$$
such that the equations of motion are equivalent to the zero curvature condition for $D_\mu$,

$$
\left[ D_\mu , D_\nu \right] = 0 \quad (2.1)
$$
The Lax connexion takes value in some Lie algebra $G$ with Lie group $G$.

Notice that, thanks to the zero-curvature condition, the Lax connexion is a pure gauge; i.e. there exists a $G$-valued function $\Psi(x, t)$ such that:

$$
\left( \partial_\mu - A_\mu \right) \Psi = 0 \quad \text{or} \quad A_\mu = \left( \partial_\mu \Psi \right) \Psi^{-1} \quad (2.2)
$$
The function $\Psi(x, t)$ is defined up to a right multiplication by a space-time independent group element. This freedom is fixed by imposing a normalization condition on $\Psi$; e.g. $\Psi(x_0) = 1$ for some point $x_0$.

(ii) Construction of the dressing transformations. The dressing transformations are non-local gauge transformations acting on the Lax connexion $A_\mu \rightarrow A_\mu^g$ and leaving its form invariant. They thus induce a transformation of the field variables $\phi \rightarrow \phi^g$ mapping a solution of the equations of motion into another.

They are constructed as follows. First let us study the set of gauge transformations mapping the Lax connexion $A_\mu$ on a given connexion $A_\mu^g$. Suppose that there exist two $G$ valued functions, $\Theta^g_+$ and $\Theta^g_-$, such that:

$$
A_\mu^g = \left( \partial_\mu \Theta^g_\pm \right) \Theta^g_\pm^{-1} + \Theta^g_\pm A_\mu \Theta^g_\pm^{-1} \quad (2.3)
$$
Since $A_\mu$ is a pure gauge, $A_\mu = \left( \partial_\mu \Psi \right) \Psi^{-1}$, $A_\mu^g$ is also a pure gauge, $A_\mu^g = \partial_\mu \left( \Theta^g_\pm \Psi \right) \left( \Theta^g_\pm \Psi \right)^{-1}$. This implies that $\left( \Theta^g_+ \Psi \right)$ and $\left( \Theta^g_- \Psi \right)$ differ by a right multiplication by a space-time independent group element which we denote by $g$. Equivalently:

$$
\Theta^g_-^{-1} \Theta^g_+ = \Psi \ g \ \Psi^{-1} \quad (2.4)
$$
The main idea underlaying the dressing transformations is to consider eq. (2.4) as a factorization problem; i.e. we look for two subgroups $B_\pm \subset G$ such that any element $h \in G$
admits a unique decomposition \( h = h_\pm^{-1} h_+ \) with \( h_\pm \in B_\pm \). The requirement that \( \Theta_\pm \) belongs to \( B_\pm \) then specify them uniquely from eq. (2.4). The subgroups \( B_\pm \) are found by demanding that the transformations (2.3) preserve the form of the Lax connexion.

The factorization problem in \( G \),

\[
g = g_\pm^{-1} g_+ \quad \text{with} \quad g_\pm \in B_\pm
\]

is a called an algebraic Riemann Hilbert problem (by analogy with the classical Riemann Hilbert problem). For the dressing transformations to be well-defined this decomposition as to be unique.

The gauge transformation (2.3) induces a transformation of the group valued function \( \Psi : \Psi \to \Psi^g \). Decompose the group element \( g \in G \) as \( g = g_\pm^{-1} g_+ \) with \( g_\pm \in B_\pm \), in the way specified by the algebraic factorization problem discussed above, then,

\[
\Psi^g = (\Psi g \Psi^{-1})_+ \Psi g_+^{-1} = (\Psi g \Psi^{-1})_- \Psi g_-^{-1}
\]

The transformation (2.6) is well defined on the phase space because it preserves the normalization condition \( \Psi(x_0) = 1 \).

(iii) The composition law for the dressing transformations. It is not the composition law in \( G \) \([7] [19]\). Let \( g, h \in G \) with decomposition, \( g = g_\pm^{-1} g_+ \) and \( h = h_\pm^{-1} h_+ \), their composition law in the dressing group is:

\[
(h_+, h_-) \cdot (g_+, g_-) = (h_+ g_+, h_- g_-)
\]

In particular, the plus and minus components commute. We denote by \( G_R \) the new group equipped with this multiplication law. The group law (2.7) can be derived by using the dressing of \( \Psi \). First, we dress \( \Psi \to \Psi^g \) by the \( g = g_\pm^{-1} g_+ \) according to eq. (2.6). Then, we dress \( \Psi^g \to (\Psi^g)^h \) by \( h = h_\pm^{-1} h_+ \):

\[
(\Psi^g)^h = \Theta_+^h \Psi^g h_\pm^{-1} \quad \text{with} \quad \Theta_\pm^h = (\Psi^g h \Psi^{-1}_g)_\pm
\]

Using the definition (2.6) of \( \Psi^g \), the factorization of \( \Psi^g h \Psi^{-1} \) can be written as follows:

\[
\Theta_-^h \Theta_+^g = \Psi^g h \Psi^g \Psi^{-1} = \Theta_-^g \Psi(h_- g_-)^{-1}(h_+ g_+) \Psi^{-1} \Theta_+^g
\]

This implies that \( (\Theta_-^h \Theta_+^g)^{-1}(\Theta_-^g \Theta_+^h) = \Psi(h_- g_-)^{-1}(h_+ g_+) \Psi^{-1} \), or equivalently:

\[
\left( \Psi(h_- g_-)^{-1}(h_+ g_+) \Psi^{-1} \right)_\pm = (\Psi^g h \Psi^{-1}_g)_\pm (\Psi g \Psi^{-1})_\pm = \Theta_\pm^h \Theta_\pm^g
\]
This proves eq. (2.7).

For infinitesimal transformations, \( g \simeq 1 + X, \ X \in \mathcal{G} \), and \( g_\pm \simeq 1 + X_\pm \) with \( X = X_+ - X_- \), the dressing transformations are:

\[
\delta_X \Psi = Y_\pm \Psi - \Psi X_\pm \quad \text{with} \quad Y_\pm = (\Psi X \Psi^{-1})_\pm.
\]  

(2.9)

The composition law is, for \( X, Z \in \mathcal{G} \):

\[
\left[ X, Z \right]_R = \left[ X_+, Z_+ \right] - \left[ X_-, Z_- \right]
\]  

(2.10)

This defines a new Lie algebra \( \mathcal{G}_R \) which is the Lie algebra of \( G_R \).

2b) Few of their properties.

(i) They are non-local. This is obvious from their definition as \( \Psi \) is non-local. The dressing transformations can be used to construction solutions of the soliton equations having non-trivial topological numbers from solutions with trivial topological numbers:

\[
\phi(x) \rightarrow \phi^g(x) \quad ; \quad \forall g \in G_R;
\]  

(2.11)

In particular, by dressing local conserved currents, the dressing transformations provide a way to construct non-local conserved currents :

\[
J_\mu(x,t) \rightarrow J^g_\mu(x,t) \quad ; \quad \forall g \in G_R.
\]  

(2.12)

In the quantum theories, these non-local currents are turned into the generators of the quantum group symmetries.

(ii) They induce Lie Poisson actions. The dressing transformations induce an action of the group \( G_R \) on the space of solutions of the classical equations of motion, i.e. on the phase space. This action is (in general) compatible with the Poisson structure; more precisely, it is a Lie Poisson action. It means that the Poisson brackets transform covariantly if the group \( G_R \) is equipped with a non-trivial Poisson structure. This Poisson structure, which, by construction, is compatible with the multiplication in \( G_R \), turns the group \( G_R \) into a Lie Poisson group.

Let us be a more precise. Denote by \( \mathcal{P} \) the phase space and by \( \{ , \}_\mathcal{P} \) the Poisson bracket on it. The dressing transformations define an action of \( G_R \) on the function over the phase space: \( f^g(x) = f(g^{-1} \cdot x) \) for \( f \in \text{Funct}(\mathcal{P}) \) and \( x \in \mathcal{P} \). Suppose that the group \( G_R \) of is equipped with a Poisson bracket which we denote by \( \{ , \}_{G_R} \). The statement that
the dressing transformation are Lie Poisson action is equivalent to the covariance of the Poisson brackets:

\[ \{ f_1, f_2 \}_P = \{ f_1^g, f_2^g \}_{P \times G_R} \quad \forall f_1, f_2 \in \text{Funct}(P) ; \forall g \in G_R \]  (2.13)

The Poisson bracket on \( P \times G_R \) is the product Poisson structure.

(iii) A standard example. As is well known, a Lie group \( G \) can be equipped with the following Poisson bracket (Sklyanin’s Poisson bracket) [20]:

\[ \left\{ \Psi(x) \otimes, \Psi(x) \right\} = \left[ r^\epsilon, \Psi(x) \otimes \Psi(x) \right] \] (2.14)

with matrices \( r^\epsilon, \epsilon = \pm \), solutions of the classical Yang-Baxter equation. By \( G \)-invariance, in eq. (2.14) we can choose any of two solutions \( r^+ \) or \( r^- \) of the classical Yang-Baxter equation provided that their difference is the tensor Casimir \( C = r^+ - r^- \). A direct computation [14] [21] shows that the Sklyanin’s Poisson brackets are covariant under the transformation (2.6), \( \Psi \rightarrow \Psi^g \), only if there are non-trivial Poisson brackets among the \( g \)'s but vanishing Poisson brackets between the \( g \)'s and the fields \( \Psi \). The Poisson brackets in \( G_R \) are:

\[
\begin{align*}
\left\{ g_+ \otimes, g_+ \right\} &= \left[ r^{\pm}, g_+ \otimes g_+ \right] \quad (2.15a) \\
\left\{ g_- \otimes, g_- \right\} &= \left[ r^{\pm}, g_- \otimes g_- \right] \quad (2.15b) \\
\left\{ g_- \otimes, g_+ \right\} &= \left[ r^-, g_- \otimes g_+ \right] \quad (2.15c) \\
\left\{ g_+ \otimes, g_- \right\} &= \left[ r^+, g_+ \otimes g_- \right] \quad (2.15d)
\end{align*}
\]

For \( g = g_-^{-1} g_+ \), the Poisson brackets are the Semenov-Tian-Shansky brackets:

\[
\left\{ g \otimes, g \right\} = (g \otimes 1)r^+(1 \otimes g) + (1 \otimes g)r^-(g \otimes 1) - (g \otimes g)r^+ - r^\mp (g \otimes g) \] (2.16)

It is easy to check that the multiplication in \( G_R \) (not in \( G \)) is a Poisson mapping for the Poisson structure defined in eq. (2.16), or in eq. (2.15). Therefore \( G_R \) is a Poisson Lie group and the actions (2.6) are Lie Poisson actions.

3. AN EXAMPLE OF DRESSING TRANSFORMATIONS: CURRENT ALGEBRAS.

We illustrate the general construction explained in the previous section on the example of the classical current algebras. We essentially follow [22]. Dressing transformations in the Toda theories were treated in [18].
3a) The equations of motion.

The field variables are one-forms, denoted by $J_\mu(x)$, valued in a semi-simple Lie algebra $G$: $J_\mu(x) = \sum_a J^a_\mu(x) t^a$ where $t^a$, $a = 1, \ldots, \dim G$, form a basis of $G$. By definition the equations of motion impose to $J_\mu(x)$ to be a curl-free conserved current:

$$\partial_\mu J^a_\mu(x) = 0$$

The equations of motion (3.1) admit a Lax representation: they are equivalent to the zero curvature condition, $[D_\mu(\lambda), D_\nu(\lambda)] = 0$, for the connexion $D_\mu(\lambda)$,

$$D_\mu(\lambda) = \partial_\mu + \frac{\lambda^2}{\lambda^2 - 1} J_\mu(x) + \frac{\lambda}{\lambda^2 - 1} \epsilon_{\mu\nu} J_\nu(x)$$

(3.2)

The Lax connexion is an element of the loop algebra $\hat{G} = G \otimes C[\lambda, \lambda^{-1}]$.

Remark 1: The linear problem $(\partial_\mu - A_\mu(x)) \Psi(x) = 0$ associated to the Lax representation (3.2) is equivalent to the following $(\epsilon_{\mu\nu} \epsilon^{\nu\sigma} = \delta^\sigma_\mu)$:

$$\left( \partial_\mu - \lambda \epsilon_{\mu\nu} \partial_\nu - \lambda \epsilon_{\mu\nu} J_\nu \right) \Psi(x) = 0$$

(3.3)

Remark 2: In the light-cone components of the Lax connexion are $(\epsilon^\pm_\mu = \pm 1)$:

$$A_\pm = -\frac{\lambda}{\lambda \pm 1} J_\pm$$

(3.4)

The Lax connexion is therefore completely characterized by the following two conditions: i) $A_\pm$ have a simple pole at $\lambda = \pm 1$ (we then set $\text{Res}_{\lambda=\pm 1} A_\pm = \mp J_\pm$) and ii) $A_\pm(\lambda = 0) = 0$. Therefore, for a gauge transformation to be a symmetry it only has to preserve these two conditions.

Remark 3: The gauge condition $A_\pm(\lambda = 0) = 0$ implies that $\Psi(\lambda = 0)$ is space-time independent. In the following we fix the gauge on $\Psi$ by setting $\Psi(\lambda = 0) = 1$. Moreover we have:

$$J_\nu(x) = \partial_\lambda \left( \epsilon_{\nu\mu} A_\mu(x) \right)_{\lambda=0} = \partial_\lambda \left( \epsilon_{\nu\mu} (\partial_\mu \Psi) \Psi^{-1} \right)_{\lambda=0}.$$

(3.5)

3b) Dressing transformations and the Riemann Hilbert problem.

1 We suppose the $t^a$ orthonormalized. We use the convention: $[t^a, t^b] = f^{abc} t^c$ where $f^{abc}$ denote the structure constants of $G$. 

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First, because the Lax connexion takes value in the loop algebra, we have to define the factorization problem (2.5) in the loop group. It can be formulated as follows: Let $\Gamma$ be a contour around the origin $\lambda = 0$. Denote by $\Gamma_-$ ($\Gamma_+$) the exterior (interior) domain of $\Gamma$. We choose $\Gamma$ such that the points $\lambda = \pm 1$ belong to $\Gamma_-$. The factorization problem consists in factorizing any regular element of the loop group, $G(\lambda)$, $\lambda \in \Gamma$, into the product of two $\lambda$-dependent group elements $G_\pm(\lambda)$ respectively analytic on $\Gamma_\pm$:

$$G(\lambda) = G_-^{-1}(\lambda) \, G_+(\lambda) \quad ; \quad \lambda \in \Gamma$$

(3.6)

This is the Riemann-Hilbert factorization problem. It is known that it admits a unique solution up to a left multiplication, $G_\pm \to M G_\pm$, by a $\lambda$-independent group element $M$.

The definition of the Riemann-Hilbert factorization is cooked up such that the transformations we will now define are symmetries of the equations of motion of the classical current algebras.

To dress the Lax connexion (3.2), we follow the general procedure:

(i) We pick up an element $G(\lambda)$ of the loop group. We fix the gauge in the Riemann-Hilbert factorization by imposing $G_+(\lambda = 0) = 1$.

(ii) We define $\Theta^G(\lambda) = \Psi \, G(\lambda) \Psi^{-1}$ and factorize it according to the Riemann-Hilbert problem:

$$\Theta^G(\lambda) = \Psi \, G(\lambda) \Psi^{-1} = \Theta^G(\lambda) \, \Theta^G_+(\lambda).$$

(3.7)

We impose the gauge condition $\Theta^G_+(\lambda = 0) = 1$. The solution to eq. (3.7) is then unique.

(iii) We define the dressed Lax connexion by:

$$A^G_\mu = (\partial_\mu \Theta^G_\pm) \, \Theta^G_\pm^{-1} + \Theta^G_\pm \, A_\mu \, \Theta^G_\pm^{-1}.$$  

(3.8)

Because we can implement the dressing either using $\Theta_+$ or using $\Theta_-$, it easy to check that the dressed connexion $A^G_\mu$ possesses the same poles with the same orders than the original connexion $A_\mu$. The gauge conditions we choose for the Riemann-Hilbert factorization also ensure that $A^G_\mu$ satisfy the same gauge condition as $A_\mu$. Therefore, the dressing transformation $A_\mu \to A^G_\mu$, preserving the structure of the Lax connexion, induce a symmetry of the equations of motion. The dressed currents $J^G_\mu(x)$ are defined via the eq. (3.5) with $A^G_\mu$ instead of $A_\mu$:

$$J^G_\mu(x) = J_\mu(x) + \epsilon_{\mu\nu} \partial_\nu \left( \partial_\lambda \Theta^G_+ \right)_{\lambda=0}$$

(3.9)
(iv) For infinitesimal transformations, \( G(\lambda) = 1 + X(\lambda) + \cdots \), where \( X(\lambda) \in \hat{G} \), \( X(\lambda) = X_+(\lambda) - X_-(\lambda) \) with \( X_\pm(\lambda) \) analytic in \( \Gamma_\pm \), the dressings are:

\[
\begin{align*}
\delta_X A_\mu &= \partial_\mu Y_\pm + \left[ Y_\pm, A_\mu \right] \\
\delta_X \Psi &= Y_\pm \Psi - \Psi X_\pm
\end{align*}
\]

with \( Y(\lambda) = (\Psi X(\lambda) \Psi^{-1})(\lambda) = Y_+(\lambda) - Y_-(\lambda) \). In particular for the current:

\[
\delta_X J_\mu = \epsilon_{\mu\nu} \partial_\nu \left( \partial_\lambda Y_+ \right)_{\lambda=0}
\]

3c) Non-local conserved currents.

The problem consists now in solving the Riemann-Hilbert factorization, eq. (3.6). The main point is that we will find differential equations which solve this problem recursively. In the following we restrict ourselves to the dressing of the current \( J_\mu(x) \).

(i) Projection on \( \hat{G}_+ \). Recall that by definition, eq. (3.6), \( \hat{G}_+ \) is the algebra of \( G \)-vector fields regular at the origin \( \lambda = 0 \). Therefore, if \( Y(\lambda) \) is an element of the loop algebra \( \hat{G} \), its projection \( Y_+(\lambda) \) on \( \hat{G}_+ \) is:

\[
Y_+(\lambda) = \oint_{\Gamma} \frac{dz}{2i\pi} \frac{Y(z)}{z - \lambda}; \quad \lambda \in \Gamma_+
\]

(ii) The dressing transformations act on \( J_\mu \) by non-local gauge transformations. The dressing of \( J_\mu \) is defined in eq. (3.9), or its infinitesimal form (3.11). To compute it we use the explicit expression of \( Y(\lambda) \) and the projector (3.12):

\[
\partial_\mu \left( \partial_\lambda Y_+ \right)_{\lambda=0} = \oint \frac{dz}{2i\pi z^2} \left[ (\partial_\nu \Psi(z)) \Psi^{-1}(z) , \Psi(z) X(z) \Psi^{-1}(z) \right] = \epsilon_{\mu\nu} \left( \partial_\nu Z_+ + [J_\nu, Z_+] \right)
\]

with:

\[
Z_+ = \oint \frac{dz}{2i\pi z} \left( \Psi(z) X(z) \Psi^{-1}(z) \right)
\]

To derive the last equation we used the linear problem in the form (3.3). The variation of \( J_\mu \) is therefore:

\[
\delta_X J_\mu = \partial_\mu Z_+ + [J_\mu, Z_+]
\]
Recursion relation for $Z_{+}$. The last step consists in solving for $Z_{+}$ by recursion. Let $X \in \hat{G}$ be $X_{n}(\lambda) = v\lambda^{-n}$ with $n = 0, 1, \cdots$ and $v \in G$. Denote by $Z^{n}$ the corresponding solution to eq. (3.14):

$$Z^{n} = \oint \frac{dz}{2i\pi z} (\Psi(z)v\Psi^{-1}(z))z^{-n}. \quad (3.16)$$

Then using once more the differential equation (3.3), we have:

$$\partial_{\mu}Z^{n+1} = \epsilon_{\mu\nu} \left( \partial_{\nu}Z^{n} + [J_{\nu}, Z^{n}] \right). \quad (3.17)$$

As advertised, this solves recursively the Riemann-Hilbert problem. The dressed currents, $\delta_{v}^{n}J_{\mu}$, are recursively defined by eqs. (3.13) and (3.17). This recursive construction is equivalent to the construction of ref. [23]. The conservation law for the dressed currents $\delta_{v}^{n}J_{\mu}$ can be checked directly.

The two first conserved currents. The first ones are the local currents $J_{\mu}^{0}(x)$ since, for $X = v \in G$,

$$\delta_{v}^{0}J_{\mu}(x) = \left[ J_{\mu}(x), v \right]. \quad (3.18)$$

For $X = v\lambda^{-1}, v \in G$, we have $\partial_{\mu}Z^{1} = \epsilon_{\mu\nu} [J_{\nu}, v]$, or equivalently,

$$Z^{1}(x) = \left[ \Phi(x), v \right] \quad \text{with} \quad \Phi(x) = \int_{C_{x}} \ast J \quad (3.19)$$

where $C_{x}$ is a curve ending at the point $x$. The dressing of $J_{\mu}$ is:

$$\delta_{v}^{1}J_{\mu}(x) = \epsilon_{\mu\nu} \left[ J_{\nu}(x), v \right] + \left[ J_{\mu}(x), \left[ \Phi(x), v \right] \right]. \quad (3.20)$$

In particular, projecting on the adjoint representation, we find the following non-local conserved currents:

$$f^{abc}_{\partial_{\mu}^{1}J_{\mu}^{b}(x) \propto} J_{\mu}^{c}(x) \propto J_{\mu}^{(1)a}(x)$$

$$J_{\mu}^{(1)a}(x) = \epsilon_{\mu\nu}J_{\mu}^{a}(x) + \frac{1}{2}f^{abc}_{J_{\mu}^{b}(x) \Phi^{c}(x)} \quad (3.21)$$

4. QUANTIFICATION: YANGIANS IN MASSIVE CURRENT ALGEBRAS.

We use the example of the massive current algebras in order to describe how non-local conserved currents can be defined in a non-perturbative way and to illustrate few of their
properties, (e.g. how they act on the states, on the fields, etc...). But the approach is more
general, see e.g. ref. [3].

The currents $J^{(1)}_{\mu}(x)$ are the currents we want to quantize. There are different ways
to specify the quantum theory, e.g. by defining it on the lattice, or as perturbation of its
U.V. fixed point, etc... Here we use an alternative approach: we look for the conditions
that we have to impose on the operator algebra in order to be able to define the quantum
non-local conserved currents. Therefore, we are interested in a quantum models satisfying
the following hypothesis:

(a) There exist quantum local conserved currents, $J^a_\mu(x)$, taken values in the Lie algebra $G$:
\[ \partial_\mu J^a_\mu(x) = 0. \] (4.1)

Furthermore, because the currents $J^a_\mu$ have to be one-forms, we impose that they have
scaling dimensions one.

(b) The currents $J^a_\mu(x)$ satisfy the quantum version of the equations of motion (3.1); i.e.
the quantum currents are curl-free:
\[ \partial_\mu J^a_\nu(x) - \partial_\nu J^a_\mu(x) + f^{abc} : J^b_\mu(x) J^c_\nu(x) : = 0 \] (4.2)

where the double dots denote an appropriate regularization of $f^{abc} J^b_\mu(x) J^c_\nu(x)$, e.g. by a
point splitting. This hypothesis imposes constraints on the operator product expansion
(OPE) of the currents.

(c) The only fields taking values in the adjoint representation of $G$ and having scaling
dimensions zero, one or two are either $J^a_\mu$ or $\partial_\nu J^a_\mu$. This fixes the OPE $f^{abc} J^b_\mu(x) J^c_\nu(0)$ up
to the order $O(|x|^{1-0})$:
\[ f^{abc} J^b_\mu(x) J^c_\nu(0) = C^\rho_{\mu\nu}(x) J^a_\rho(0) + D^{\sigma\rho}_{\mu\nu}(x) \left( \partial_\sigma J^a_\rho(0) \right) + O(|x|^{1-0}) \] (4.3)

The quantum currents $J^a_\mu(x)$ satisfying these three hypothesis generate what could be
called a massive current algebra.

4a) OPE’s in massive quantum current algebras.

We now show that these hypothesis ensure that the currents satisfy the commutation
relations of a Kac-Moody algebra but also that they satisfy the following OPE’s [3]:

---

We used the following space-time conventions $x^\nu \equiv (x^0 = t, x^1 = x); \ x^\pm = x \pm t$; and
\[ ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - dx^2. \]
\[ J^b_\pm (x)J^c_\pm (0) = - \frac{k \delta^{ab}}{8i\pi} \frac{1}{(x^\pm)^2} - \frac{f^{abc} J^e_\pm (0)}{2i\pi} x^\pm + \mathcal{O}(|x|^{-0}) \quad (4.4a) \]

\[ \frac{1}{2} f^{abc} \left( J^b_+ (x)J^c_+ (0) - J^b_- (x)J^c_- (0) \right) \]

\[ = \frac{C_{Adj}}{8i\pi} \log (M^2 x^+ x^-) \left( \partial_+ J^a_+ (0) - \partial_- J^a_- (0) \right) + \mathcal{O}(|x|^{1-0}) \quad (4.4b) \]

\[ C_{Adj} \text{ is the Casimir of } G \text{ in the adjoint representation and } M \text{ is the mass scale. The product } J^a_\pm (x)J^c_\pm (0) \text{ is logarithmically divergent.} \]

We solve for the OPE (4.3) using our hypothesis. The proof goes in few steps:

(i) First, locality, PT-invariance and Lorentz covariance determine the general tensor form of \( C_{\mu\nu}(x) \) and \( D_{\mu\nu}^\sigma(x) \). Notice also that the conservation law for \( J^a_\mu \) allows us to choose \( D_{\mu\nu}^\sigma(x) \) to be traceless: \( \eta_{\sigma\rho} D_{\mu\nu}^\rho(x) = 0 \). Therefore, under the conditions (a) to (c), the generators \( J^a_\mu(x) \) of a massive current algebra satisfy the following OPE's [13]:

\[ f^{abc} J^b_\mu(x)J^c_\nu(0) = \]

\[ \left( C_1 x^2 \eta_{\mu\nu} x^\rho + C_2 x^2 (x_\mu \delta^\rho_\nu + x_\nu \delta^\rho_\mu) + C_3 x_\mu x_\nu x^\rho \right) \left( J^a_\rho (0) + \frac{1}{2} x^\sigma \partial_\sigma J^a_\rho (0) \right) \]

\[ + \left( D_1 x^\rho (x_\mu \delta^\sigma_\nu - x_\nu \delta^\sigma_\mu) + D_2 x^\sigma (x_\mu \delta^\rho_\nu - x_\nu \delta^\rho_\mu) \right) \left( \partial_\sigma J^a_\rho (0) \right) + \mathcal{O}(|x|^{1-0}) \quad (4.5) \]

The coefficients \( C_i \) and \( D_i \) only depend on \( x^2 \). Furthermore, the conservation law for the currents implies the following differential equations for the functions \( C_i \) and \( D_i \) [13]:

\[ x^2 \frac{d}{dx^2} C_2 = - \frac{1}{2} (C_1 + 5C_2) \quad (4.6a) \]

\[ x^2 \frac{d}{dx^2} (C_1 + C_2 + C_3) = - (C_1 + C_2 + 2C_3) \quad (4.6b) \]

and

\[ x^2 \frac{d}{dx^2} D_1 = - D_1 - \frac{x^2}{4} C_1 \quad (4.7a) \]

\[ x^2 \frac{d}{dx^2} D_2 = - D_2 - \frac{x^2}{4} C_2 \quad (4.7b) \]

\[ x^2 \frac{d}{dx^2} (D_1 + D_2) = \frac{x^2}{4} C_3 \quad (4.7c) \]
(ii) The differential equations (4.6) and (4.7) do not specify uniquely the unknown coefficients $C_i(x^2)$ and $D_i(x^2)$. But we can use the hypothesis on the scaling dimension of the currents to fix the leading behaviour of the functions $C_i(x^2)$:

$$C_i(x^2) = \frac{\alpha_i}{(x^2)^2} + \mathcal{O}(|x|^{-3-0}) \quad ; \quad i = 1, 2, 3,$$

with $\alpha_i$ some constants. We assume that there is no leading logarithmic corrections.

Solving the differential equations (4.6) and (4.7), we find:

$$C_1(x^2) = -\frac{\alpha}{(x^2)^2} + \mathcal{O}(|x|^{-3-0}) \quad (4.9a)$$

$$C_2(x^2) = \frac{\alpha}{(x^2)^2} + \mathcal{O}(|x|^{-3-0}) \quad (4.9b)$$

$$C_3(x^2) = -\frac{\gamma \alpha}{(x^2)^2} + \mathcal{O}(|x|^{-3-0}) \quad (4.9c)$$

$$D_k(x^2) = -\frac{\alpha_k}{4x^2} \log (-M_k^2 x^2) + \mathcal{O}(|x|^{-1-0}) \quad ; \quad k = 1, 2 \quad (4.9d)$$

The constants $M_k$ are related to the mass scale and $\gamma = 2 \log(M_2/M_1)$. The constant $\alpha$ depends on the normalization of the currents: we fixe the normalization such that $\alpha = -\frac{C_{adj}}{2i\pi}$.

(iii) We finally impose the zero curvature condition. From eqs. (4.5) and (4.9), we have:

$$\epsilon_{\mu\nu} \left[ f^{abc} J^b_\mu(x) J^c_\nu(0) + Z(-x^2) \left( \partial_\mu J^a_\nu(0) - \partial_\nu J^a_\mu(0) \right) \right] = -\frac{\alpha \gamma}{4x^2} \frac{x^\mu}{2} \left( x^\rho \epsilon^{\mu\alpha} + x^\alpha \epsilon^{\mu\rho} \right) \left( \partial_\sigma J^a_\rho(0) + \partial_\rho J^a_\sigma(0) \right) \quad (4.10)$$

with $Z(-x^2) = \frac{\alpha}{4} \log(-M_1 M_2 x^2)$.

The curl-free equation (4.2) is then an immediate consequence of (4.10) if $\gamma$ vanishes. The normal order in (4.2) is defined in such a way to cancel the logarithmic divergence in $f^{abc} J^b_\mu J^c_\nu$. Therefore, the curl-free equation fixes the two mass scale to be equal

$$\gamma = 2 \log(M_2/M_1) = 0$$

The same conclusion could have been reached by imposing the chiral splitting of the leading terms of the OPE (4.3). The approach based on the chiral splitting assumption was described in ref. [14].

3 The case $\gamma \neq 0$ is also quite interesting; it probably corresponds to the 2D $O(n)$ models. In particular, in this case the leading terms of the OPE of the currents do not satisfy the chiral splitting. In other words the chiral components of the currents, $J_\pm^a$ and $J_\mp^a$, are mixed in the leading terms of the OPE.

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(iv) **Commutation relations of the currents.** Finally, other current OPE’s can be deduced using the same techniques. In particular we have:

\[
J^a_\mu(x)J^c_\nu(0) = -\frac{k\delta^{ab}}{2i\pi x^2} \left( x_\mu x_\nu - \frac{1}{2}x^2 \eta_{\mu\nu} \right) - f^{abc} \frac{1}{2i\pi} \left( x_\mu \delta^\rho_\nu + x_\nu \delta^\rho_\mu - x^2 \eta_{\mu\nu} x^\rho \right) J^a_\rho(0) + \mathcal{O}(|x|^{-0}).
\]

These OPE’s reduce to eq. (4.4a). The products of the quantum operators are defined by:

\[
J^a_\mu(x, t)J^b_\nu(y, t) = \lim_{\epsilon \to 0^+} J^a_\mu(x, t + i\epsilon)J^b_\nu(y, t).
\]

Therefore, using \(\lim_{\epsilon \to 0^+} \frac{i\epsilon}{\pi^2 + \epsilon^2} = i\pi \delta(x)\), the OPE’s (4.11) implies:

\[
\begin{align*}
[J^a_t(x), J^b_t(0)] &= f^{abc}J^c_t(0)\delta(x) - \frac{k}{2}\delta^{ab}\delta'(x) \\
[J^a_\tau(x), J^b_\tau(0)] &= f^{abc}J^c_\tau(0)\delta(x) \\
[J^a_x(x), J^b_x(0)] &= f^{abc}J^c_x(0)\delta(x)
\end{align*}
\]

They are the commutation relations of a current algebra: the light cone component \(J^a_\pm\) satisfy the commutation relations of the affine Kac-Moody algebra \(G^{(1)}\).

**Remark 1:** Two hidden consequences of the definition of the massive current algebras we choose are: i) their ultra-violet limits are WZW models with \(G^{(1)} \otimes G^{(1)}\) symmetry; and ii) they describe perturbations of affine Kac-Moody algebras by the perturbing fields \(\Phi_{\text{pert}}(x) = \sum_a J^a_\mu(x)J^a_\mu(x)\).

**Remark 2:** The massive current algebras are characterized by the level \(K\) of the affine Kac-Moody algebras. However the OPE’s (4.4) and the curl-free equation (4.2) are model independent in the sense that they do not depend on the level.

**Remark 3:** Because the WZW models are the U.V. fixed point of the massive chiral algebras, they are also \(Y(G)\) invariant. Actually, The WZW models are \(Y(G) \otimes Y(G)\) invariant (at least classically [24]). They are also \(U_q(G) \times U_q(G)\) invariant. The perturbing field \(J^a_\mu J^a_\mu\) breaks these symmetries down to the diagonal \(Y(G)\) symmetry times a fractional supersymmetry. It could be interesting to solve the WZW models from their non-local symmetries. This will provide a test of the idea we are trying to develop for the massive integrable models.

4b) **The quantum non-local conserved currents.**
(i) Their definition. Having proved that the quantum conserved currents satisfy the quantum form (4.2) of the equations of motion (3.1), it is now easy to defined the quantum conserved currents \( J^{(1)}(x, t) \). We define them by a point splitting regularization (\( \delta > 0 \)):

\[
J^{(1)}_{\mu}(x, t) = \lim_{\delta \to 0^+} J^{(1)}_{\mu}(x, t|\delta)
\]

\[
J^{(1)}_{\mu}(x, t|\delta) = Z(\delta) \epsilon_{\mu\nu} J^0_{\nu}(x, t) + \frac{1}{2} f^{abc} J^b_{\mu}(x, t) \phi^c(x - \delta, t)
\]

where \( \phi^c(x, t) \), which satisfies \( d\phi^c = *J^c \), is defined by: \( \phi^c(x, t) = \int_{C_x} *J^c \) The contour of integration \( C_x \) is a curve from \( -\infty \) to \( x \).

The renormalization constant \( Z(\delta) \) is fixed by requiring that \( J^{(1)}_{\mu}(x, t) \) are finite and conserved. First it is easily seen from eq. (4.14) that \( J^{(1)}_{\mu}(x, t) \) is finite whenever \( Z(\delta) = \frac{\alpha}{2} \log(\delta) + \text{constant} \). The constant is fixed by demanding the conservation law for \( J^{(1)}_{\mu} \). (The other subleading terms in \( Z(\delta) \) are meaningless.) Using eq. (2.4) we deduce,

\[
\partial_{\mu} J^{(1)}_{\mu}(x, t|\delta) = \frac{1}{2} \epsilon_{\mu\nu} \left[ Z(\delta) \left( \partial_{\mu} J^a_{\nu} - \partial_{\nu} J^a_{\mu} \right) (x, t) + f^{abc} J^b_{\mu}(x, t) J^c_{\nu}(x - \delta, t) \right]
\]

(4.15)

Therefore, from eq. (4.4b) or (4.10), we learn that \( \partial_{\mu} J^{(1)}_{\mu}(x, t|\delta) \) vanishes when \( \delta \to 0 \) if \( Z(\delta) = \frac{\alpha}{2} \log(M\delta) + O(\delta^{1-0}) \).

(ii) Non-locality: the braiding relations. The non-local character of the currents \( J^{(1)}(x, t) \) is encoded in their braiding relations, the equal time commutation relations. The latter are described as follows: Let \( \Phi(y, t) \) be a quantum field local with respect to the currents \( J^a_{\mu}(x, t) \). Then it satisfies the following equal-time braiding relations [14]:

\[
J^{(1)}_{\mu}(x, t) \Phi(y, t) = \Phi(y, t) J^{(1)}_{\mu}(x, t) \quad ; \quad \text{for } x < y
\]

(4.16a)

\[
J^{(1)}_{\mu}(x, t) \Phi(y, t) = \Phi(y, t) J^{(1)}_{\mu}(x, t) - \frac{1}{2} f^{abc} Q^b_0(\Phi(y, t)) J^c_{\mu}(x, t) \quad ; \quad \text{for } x > y
\]

(4.16b)

where \( Q^b_0 \) are the global charges associated with the local conserved current \( J^b_{\mu} \). They are the same as on the lattice, eq. (1.35).

The proof of the braiding relations (4.16 ) is the same as the proof of the braiding relations for disorder fields. It only relies on the way to deform the contour \( C_x \) entering in the definition of the currents \( J^{(1)}(x, t) \) The relative positions of the contours \( C_x \) depend if \( J^{(1)}(x, t) \) acts first or second: if \( J^{(1)}(x, t) \) acts first (second) the contour is slightly under (above) the equal-time slice \( t = \text{cst} \), we denote them \( C^-_x (C^+_x) \). (Remember that product of operators are defined by time ordering.) The relation (1.16a) follows because, in this case,
there is no topological obstruction for moving the contour from the configuration $C^+_x$ to the configuration $C^-_x$. In the case of the relation (4.16), there is an obstruction for moving the contour $C^+_x$ onto the contour $C^-_x$. This implies non-trivial braiding relations. All the non-locality of the currents $J^{(1)\alpha}_\mu(x,t)$ is concentrated in the fields $\phi^c(x,t)$, eq. (2.4). For $x > y$ the exchange relation between $\phi^c(x,t)$ and $\Phi(y,t)$ is:

$$
\phi^c(x,t)\Phi(y,t) = \int_{z \in C^+_x} *J^c(z)\Phi(y,t) = \int_{z \in \gamma(y)} *J^c(z)\Phi(y,t) + \int_{z \in C^-_x} *J^c(z)\Phi(y,t)
$$

(4.17)

The contour $\gamma(y)$ is a small contour surrounding the point $y$. Plugging back eq. (4.17) into the definition of the non-local current $J^{(1)\alpha}_\mu$ proves the braiding relations (4.16).

4c) The non-local conserved charges and their algebra.

Given conserved currents the associated charges are defined by integrating their dual forms along some curves. The charges depend weakly on the contours of integration because the dual forms are closed. The global conserved charges acting on the states of the physical Hilbert space are defined by choosing the domain of integration to be an equal-time slice. Namely for a current $\mathcal{J}_\mu(x,t)$:

$$
Q = \int_{t=\text{cst}} dx \mathcal{J}_t(x,t)
$$

(4.18)

We denote by $Q^a_0$ and $Q^a_1$ the global charges associated to the currents $J^a_\mu(x)$ and $J^{(1)\alpha}_\mu(x)$.

The (non-local) conserved charges generate a non-abelian extension of the two-dimensional Lorentz algebra. In two dimensions the Poincaré algebra which is generated by the momentum operators $P_\mu$ and the Lorentz boosts $L$ is abelian. The momentum operators $P_\mu$ are the global charges associated with the conserved stress-tensor $T_{\mu\nu}(x)$:

$$
\partial_\nu T_{\mu\nu}(x) = 0
$$

The Lorentz boost $L$ is the global charge associated with the conserved boost current:

$$
L_\mu(x) = \frac{1}{2} \epsilon^{\rho\sigma} \left( x_\rho T_{\mu\sigma}(x) - x_\sigma T_{\mu\rho}(x) \right)
$$

(4.19)

The (non-local) charges satisfy the following algebraic relations:

$$
\begin{align*}
\left[ Q^a_0 , Q^b_0 \right] &= f^{abc}Q^c_0 ; & \left[ Q^a_0 , Q^b_1 \right] &= f^{abc}Q^c_1 \\
\left[ L , Q^a_0 \right] &= 0 ; & \left[ L , Q^a_1 \right] &= -\frac{C_{\text{Adj}}}{4i\pi} Q^a_0
\end{align*}
$$

(4.20)
The relations (4.20) are part of the defining relations of the semi-direct product of the Yangians \( Y(G) \) by the Poincaré algebra. Only the Serre relations are missing. (They are more difficult to prove because they involve commutation relations between the non-local charges.) Moreover, as we will soon show, the comultiplications are those in \( Y(G) \).

The three first relations are easily proved. The last relation is more interesting and can be proved in geometrical way. It consists in imposing a Lorentz boost \( \mathcal{R}_{2\pi} \) of angle \( (2\pi) \) to the non-local currents \( J^{(1)\mu}_a(x,t) \). It is a rotation of \( (2\pi) \) in the Euclidian plane. Because the currents \( J^{(1)\mu}_a(x,t) \) are non-local this transformation does not act trivially on them: the string \( C_x \) winds around the point \( x \). By decomposing this winding contour into the sum of a contour from \( -\infty \) to \( x \) plus a small contour surrounding \( x \) we obtain:

\[
\mathcal{R}_{2\pi} J^{(1)\mu}_a(x,t) \mathcal{R}^{-1}_{2\pi} = J^{(1)\mu}_a(x,t) - \frac{1}{2} f^{abc} Q^c_0 \left( J^b_\mu(x,t) \right) \tag{4.21}
\]

Integrating the time-component of eq. (4.21) over an equal-time slice gives

\[
\mathcal{R}_{2\pi} Q^a_1 \mathcal{R}^{-1}_{2\pi} = Q^a_1 - \frac{1}{2} C_{Adj} Q^a_0 \tag{4.22}
\]

in agreement with the relations (4.20) because \( \mathcal{R}_{2\pi} = \exp(i2\pi L) \).

4d) Action on the asymptotic states and the S-matrices. Non-perturbative results on the S-matrices can be deduced by looking at the action of the quantum charges on the asymptotic states. The constraints on the S-matrices we obtain arise by requiring that they commute with the non-local charges. These commutation relations imply algebraic equations which are nothing but the exchange relations for the quantum symmetry algebra (the Yangians \( Y(G) \) in the case of massive current algebras). In general, these algebraic equations implies non-trivial constraints on the S-matrices which are sometimes enough to determine them.

Example: the \( SO(N) \) Gross-Neveu models. The \( SO(N) \) Gross-Neveu models are equivalent to the \( SO(N) \) massive current algebras at level \( K = 1 \). In the \( SO(N) \) Gross-Neveu models the fundamental asymptotic particles are Majorana fermions taking values in the vector representation of \( SO(N) \). In the \( SO(N) \) Gross-Neveu models, the \( Y(SO(N)) \) charges acting on the asymptotic fermions are given by:

\[
Q^{kl}_0 = T^{kl} \tag{4.23a}
\]

\[
Q^{kl}_1 = -\frac{\theta}{i\pi} \left( \frac{N-2}{\mu} \right) (T^{kl}) \tag{4.23b}
\]

\[
\Delta Q^{kl}_1 = Q^{kl}_1 \otimes 1 + 1 \otimes Q^{kl}_1 - \sum_n (T^{kn} \otimes T^{nl} - T^{ln} \otimes T^{nk}) \tag{4.23c}
\]

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where the $T^{kl}$'s form the vector representation $\Box$ of $SO(N)$: 
$$(T^{kl})^{mn} = \delta^{km}\delta^{ln} - \delta^{lm}\delta^{kn}.$$ 

The charges $Q_0^a$ and $Q_1^0$ defined in eq. (4.23) satisfy the algebra (4.20); on-shell the boost operator $L$ acts as $\frac{\partial}{\partial \theta}$. They define an irreducible representation of the $SO(N)$-Yangians in the vector representation of $SO(N)$. Eq. (4.23) is the comultiplication in $Y((SO(N))$.

Denote by $S(\theta_12), \theta_12 = \theta_1 - \theta_2$, the S-matrix of the two-fermion scattering. $S(\theta)$ acts from $\Box \otimes \Box$ into itself. As an $SO(N)$ representation the tensor product $\Box \otimes \Box$ decomposes into $\Box + \Box + \Box$. We denote by $P_-, P_+$ and $P_0$ the respective projectors. By $SO(N)$-invariance, $S(\theta)$ decomposes on these projectors:

$$S(\theta) = \sigma_+(\theta)P_+ + \sigma_-(\theta)P_- + \sigma_0(\theta)P_0$$ (4.24)

where $\sigma_n(\theta)$ are scattering amplitudes. The non-local charges $Q_{1}^{kl}$ are conserved and therefore they commute with the S-matrix. For the two-fermion scattering, the $Y(SO(N))$ exchange relations imply the following algebraic relations between the scattering amplitudes:

$$\frac{\sigma_-(\theta)}{\sigma_+(\theta)} = \frac{\theta(N-2) + i2\pi}{\theta(N-2) - i2\pi}; \quad \frac{\sigma_0(\theta)}{\sigma_+(\theta)} = \frac{\theta + i\pi}{\theta - i\pi}$$ (4.25)

Eq. (4.25) determine $S(\theta)$ up to an overall function which could be fixed by closing the bootstrap program [9].

4e) Action on the fields and the field multiplets.

(i) The definition of the action. The definitions of charges acting on the states and on the fields differ by the choice of the contour along which the conserved current is integrated. The charges acting on a field $\Phi(y)$ located at a point $y$ are defined by choosing the contour of integration $\gamma(y)$ from $-\infty$ to $-\infty$ but surrounding the point $y$:

$$Q_0^a(\Phi(y)) = \int_{z\in\gamma(y)} dz_{\mu}\epsilon^{\nu\mu} J^{(k)a}_\nu(z)\Phi(y)$$ (4.26)

Compare with the lattice definition (1.27).

For the currents $J^a_\mu(x)$ and the charges $Q_0^a$ deforming the contour $\gamma(y)$ proves that

$$Q_0^a(\Phi(y)) = Q_0^a(\Phi(y)) - \Phi(y) Q_0^a$$ (4.27)

When the currents and the field $\Phi(y)$ are not respectively local the situation is more subtle. The contour $\gamma(y)$ can no more be closed and the action of the charges on the field is no more a pure commutator. For the non-local conserved currents $J^{(1)}(x)$ the relation
between the global charges (4.18) acting on the states and the charges (4.26) acting on
the fields is the following:

\[ Q^a_1 \Phi(y) = Q^a_1 \Phi(y) - \Phi(y) Q^a_1 + \frac{1}{2} f^{abc} Q^b_0 \Phi(y) Q^c_0 \]  

(4.28)

The proof of eq. (4.28) consists in decomposing the contour of integration \( \gamma(y) \) into the
difference of two contours \( \gamma_+ \) and \( \gamma_- \) which are respectively above and under the point \( y \),
and in using the braiding relation (4.16) when the current \( J^b(x) \) is on \( \gamma_- \).

(ii) The comultiplications. We now derive the comultiplication from the braiding re-
lations. The comultiplications just encode how the charges act on a product of fields,
say \( \Phi_1(y_1)\Phi_2(y_2) \cdots \). We denote them by \( \Delta \). In the case of the charges \( Q^a_0 \) and for fields
\( \Phi_n(y_n) \) which are local with respect to the currents \( J^a_\mu(x) \) all the contours can be deformed
without troubles and we have:

\[ Q^a_0 \left( \Phi_1(y_1) \Phi_2(y_2) \right) = Q^a_0 \left( \Phi_1(y_1) \right) \Phi_2(y_2) + \Phi_1(y_1) Q^a_0 \left( \Phi_2(y_2) \right) \]  

\[ \Delta Q^a_0 = Q^a_0 \otimes 1 + 1 \otimes Q^a_0 \]  

(4.29)

It is the standard Lie algebra comultiplication as it should be.

In the case of the non-local charges \( Q^a_1 \) the standard comultiplication is deformed due
to the non-trivial braiding relations between the non-local currents and the fields. Let
\( \Phi_n(y_n) \) be quantum fields local with respect to the currents \( J^a_\mu(x) \). Then we have the
following comultiplication for the non-local conserved charges \( Q^a_1 \):

\[ Q^a_1 \left( \Phi_1(y_1) \Phi_2(y_2) \right) = Q^a_1 \left( \Phi_1(y_1) \right) \Phi_2(y_2) + \Phi_1(y_1) Q^a_1 \left( \Phi_2(y_2) \right) \]  

\[ - \frac{1}{2} f^{abc} Q^b_0 \left( \Phi_1(y_1) \right) Q^c_0 \left( \Phi_2(y_2) \right) \]  

(4.30a)

\[ \Delta Q^a_1 = Q^a_1 \otimes 1 + 1 \otimes Q^a_1 - \frac{1}{2} f^{abc} Q^b_0 \otimes Q^c_0 \]  

(4.30b)

Eqs. (4.29) and (4.30) are the comultiplication in \( Y(G) \). Equation (4.30a) can be proved by
decomposing the contour \( \gamma_{12} \) used in defining the action of \( Q^a_1 \) on the product \( \Phi_1(y_1)\Phi_2(y_2) \).
The contour \( \gamma_{12} \) is surrounding the two points \( y_1 \) and \( y_2 \). It decomposes into the sum of
two contours \( \gamma_1 \) and \( \gamma_2 \) surrounding \( y_1 \) and \( y_2 \) respectively. But on the contour \( \gamma_2 \) we have
to use the braiding relations (4.16) in order to pass the string \( C_z \) through the point \( y_1 \).
Eq. (4.30 ) can also be proved starting from the graded commutators (1.28).

(iii) The field multiplets. To any (local) field \( \Phi(x,t) \) is associated a multiplet which is
constructed by acting on the field with as many charges as possible:

\[ Q^{A_1} \cdots Q^{A_p} \Phi(x,t) \]  

(4.31)
with, in the case \(Y(\mathcal{G})\) symmetry, \(Q^A = Q^a_0, Q^a_1\) or any element of the algebra generated by them. By construction, the fields (4.31) form a field multiplet in the sense of eq. (1.22). In general the field multiplets are infinite dimensional.

The main property of the field multiplets resides, (assuming the knowledge of the action on the asymptotic states), in the fact that if the field \(\Phi(x,t)\) is known, then all its descendents, \(Q^{A_1} \cdots Q^{A_P} \Phi(x,t)\) are also known. In other words, the descendents are completely determined by the data of the fields \(\Phi(x,t)\) and of the values of the charges on the asymptotic states.

This property follows from the Ward identities expressing the quantum invariance:

\[
\Delta^{(M)}(Q^A) \langle \Phi_1(x_1) \cdots \Phi_M(x_M) \rangle = 0 \tag{4.32}
\]

where \(\Delta^{(M)}\) the \(M\)th comultiplication with \(\Delta Q^A = Q^A \otimes 1 + \Theta^A_B \otimes Q^B\). Here we have assumed that the vacuum is quantum group invariant: \(Q^A|0\rangle = 0, \langle 0|Q^A = 0\). The identity (4.32) can be formulated on the form factors. The form factors are the matrix elements of the fields between asymptotic states. By crossing symmetry, only matrix elements between the vacuum and the asymptotic particles are relevant. Let us denote by \(Z^\alpha(\theta)\) the asymptotic particles with rapidity \(\theta\); they form a representation \(W\) of the quantum symmetry algebra. The form factors of the fields \(\Phi_i(x, t)\) are defined by:

\[
\mathcal{F}^\alpha_{\alpha_1, \cdots, \alpha_M}(\theta_1, \cdots, \theta_M) = \langle 0|\Phi_i^A(0)|Z^{\alpha_1}(\theta_1) \cdots Z^{\alpha_M}(\theta_M)\rangle \tag{4.33}
\]

On the form factors, the Ward identities (4.32) become:

\[
\langle 0| \left( Q^A \Phi_i(x) \right) |Z^{\alpha_1}(\theta_1), \cdots, Z^{\alpha_M}(\theta_M)\rangle = -\langle 0| \left( \Theta^A_B \Phi_i(x) \right) \left( \Delta^{(M)} Q^B |Z^{\alpha_1}(\theta_1) \cdots Z^{\alpha_M}(\theta_M)\rangle \right) \tag{4.34}
\]

with \(s\) the antipode. Eqs. (4.34) give the form factors of the field \(Q^A(\Phi_i(x, t))\) in terms of those of the fields \(\Theta^A_B \Phi_i(x, t)\) and of the action of the charges \(Q^A\) on the asymptotic particles \(Z^\alpha(\theta)\).

**Example: action on the stress-tensor.** In massive current algebras, the stress-tensor and the current are in the same \(Y(\mathcal{G})\) - multiplets. We have:

\[
Q_1^a(T_{\mu\nu}) \propto \epsilon_{\mu\rho} \partial_\rho J_\nu^a + \epsilon_{\nu\rho} \partial_\rho J_\mu^a. \tag{4.35}
\]
This relation was proved in [25] using form factor technique, it can also be deduced from the hypothesis we made for defining the massive current algebras.

**Remark 1:** The Ward identity (4.34) can be written for any element in the enveloping algebra. Choosing a particular element associated to the square of the antipode leading to the so-called deformed KZ equations for the form factors [26].

**Remark 2:** Assuming, as in conformal field theory, that the (complete) symmetry algebra possess free field vertex representations, the form factors will also admit free field representations. The Zamolodchikov creation operators $Z^{\lambda}(x)$ as well as the field operators $\Phi^{\Lambda}(x)$ will be represented as quantum vertex operators in analogy with the vertex operator representations of the quantum affine algebras. This is suggested by the explicit formula for the form factors found by Smirnov [27]. Their generic forms are as follows:

$$
\mathcal{F}(\theta_1, \cdots, \theta_M) = \int \prod_k d\mu(\alpha_k) \, P(\alpha_1, \cdots, \alpha_k|\theta_1, \cdots, \theta_M) \\
\times \prod_{k<l} G_1(\alpha_k - \alpha_l) \prod_{i<j} G_2(\theta_i - \theta_j) \prod_{k,j} G_3(\alpha_k - \theta_j)
$$

with $d\mu(\alpha)$ some integration measure, the functions $G_n$ are some models dependent kernels and $P(\alpha_k|\theta_i)$ are polynomials. Formula (4.36) suggest the following vertex operator representations:

$$
\mathcal{F}(\theta_1, \cdots, \theta_M) = \int \prod_k d\mu(\alpha_k) \langle \mathcal{O} \prod_k V(\alpha_k) \prod_i Z(\theta_i) \rangle
$$

where the $V(\alpha_k)$ 's are “screening” operators, the $Z(\theta)$ 's are vertex operator representations of the Zamolodchikov operators and $\mathcal{O}$ an operator representing the fields. The kernel between these operators can be deduced from the formula (4.36).

**Remark 3:** The braiding relations between the quantum field multiplets are determined by the quantum symmetries: the braiding matrices intertwine the quantum symmetries, cf e.g. eqs. (1.29) and (1.30). Moreover the braiding relations are scale invariant; i.e. they are renormalization group invariant. This is obvious from their definitions but this also follows from the topological origin of the braiding relations. The braiding relations just reflect the monodromy of the field multiplet correlation functions. Therefore, the renormalization group induces isomonodromy deformations [28]. The connection between isomonodromy deformations and quantum group symmetries could provide another starting point for determining the correlation functions in massive two dimensional quantum field theories.
5. CONCLUSIONS.

Few open problems: Quantum symmetries have been used with some success to study integrable perturbations of conformal field theories. Some examples are [29]: the $\Phi_{(1,3)}$ and the $\Phi_{(1,2)}$, $\Phi_{(2,1)}$ perturbations of the minimal conformal models, the $G_K \otimes G_L / G_{K+L}$ cosets models, the $Z_N$ parafermionic models, and the fractional supersymmetric models, etc... Most of the results obtained in these papers concern the S-matrices of these massive models. The main open problem is the derivation of the off-shell properties of the models, (the correlation functions and the form factors), from their quantum symmetries. As we mentioned in the introduction, this requires checking if the quantum symmetries form a complete symmetry algebra or not. Other few technical problems have been formulated in the previous sections, most of them as remarks. In particular, the connection between isomonodromy deformations and quantum group symmetries could open a new way of solving for the correlation functions.

3D generalizations? Let us discuss how these constructions could possibly be generalized to three dimensions. In two dimensions, quantum group symmetries require non-local currents: the non-locality, which is reflected in the equal-time commutation relations, imply the non-trivial comultiplications. The 2D non-local currents are fields localized on points but with a “string” attached to them. The currents are generically express as products of disorder fields (the “wavy string” in the lattice description) by spin fields (which are local fields). This is analogue to the definition of the 2D parafermions.

In any dimensions, to have more general symmetry than supersymmetry we need fields with non-trivial equal-time commutation relations. In three dimensions, this requires to consider fields localized on curves (with a sheet attached to them). Once again, examples are provided by disorder and parafermionic fields. The latters can be described as follows: consider a group G invariant spin lattice model in three dimensions. The disorder fields $\mu_g(C)$, $g \in G$, are defined by splitting all the spin variables $\sigma$ which leave on a surface $\Sigma_C$ bounded by $C$: $\sigma \rightarrow g\sigma$. By G-invariance, $\mu_g(C)$ depend weakly on $\Sigma_C$. The 3D parafermions $\Psi_g(C; x)$ are defined as product of disorder fields $\mu_g(C)$ by spin fields $\sigma(x)$:

$$\Psi_g(C; x) = \mu_g(C) \sigma(x) ; \quad x \in C. \quad (5.1)$$

They satisfy non-trivial commutation relations analogous to the two-dimensional case. Anions are particular examples of this construction, with the curve C a small two-dimensional
cone extending to the spacial infinity \cite{30}. Generalizing eq. (5.1) by considering product of spin fields all along the curve \( C \) gives the parafermionic string which has been considered in the 3D Ising model \cite{31}.

Thus, if quantum group symmetry exists in three dimensions, it is a theory of quantum fields localized on curves, i.e. a theory of quantum loops. A (formal) example is given by Polyakov’s string representation of gauge theories in three dimensions \cite{32}. Let \( W(C) \) be the Wilson loops:

\[
W(C) = P \exp \left( \oint_C A \right)
\]

(5.2)

where \( A \) is the Yang-Mills connection. Define the functional current \( \mathcal{J}_\mu(C; x) \) by:

\[
\mathcal{J}_\mu(C; x) = W(C)^{-1} \frac{\delta}{\delta x_\mu} W(C)
\]

(5.3)

This functional current is conserved and curl-free:

\[
\frac{\delta}{\delta x_\mu} \mathcal{J}_\mu(C; x) = 0
\]

\[
\frac{\delta}{\delta x_\mu} \mathcal{J}_\nu(C; x') - \frac{\delta}{\delta x_\nu} \mathcal{J}_\mu(C; x) + \left[ \mathcal{J}_\mu(C; x), \mathcal{J}_\nu(C; x') \right] = 0
\]

(5.4)

\[
t_\mu \mathcal{J}_\mu(C; x) = 0
\]

with \( t_\mu \) the vector tangent to the curve \( C \) at the point \( x \). Formally the following non-local current, \( \mathcal{J}_\mu^{(1)}(C; x) \), localized on the curve \( C \) is functionally conserved:

\[
\mathcal{J}_\mu^{(1)}(C; x) = \epsilon_{\mu\nu\rho} t_\nu \mathcal{J}_\rho(C; x) + \frac{1}{2} \left[ \mathcal{J}_\mu(C; x), \mathcal{P}(C; x) \right]
\]

(5.5)

with \( \delta \mathcal{P}(C; x)/\delta x_\mu = \epsilon_{\mu\nu\rho} t_\nu \mathcal{J}_\rho(C; x) \). The analogy with the 2D current algebras is appealing, cf. eq. (3.21). It seems to indicate the possibility of having generalized non-local symmetry in 3D gauge theories. Unfortunately, the equations of motion (5.4) are not very rigorous; only the discretized lattice version has been proved and, up to our knowledge, no concrete result on the quantum continuous case has never been proved. However, to construct generalized quantum group symmetry in three dimensions remains a very attractive challenge.

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