RATIONAL CURVES ON QUOTIENTS OF ABELIAN VARIETIES BY FINITE GROUPS

BO-HAE IM AND MICHAEL LARSEN

Abstract. In [3], it is proved that the quotient of an abelian variety $A$ by a finite order automorphism $g$ is uniruled if and only if some power of $g$ satisfies a numerical condition $0 < \text{age}(g^k) < 1$. In this paper, we show that $\text{age}(g^k) = 1$ is enough to guarantee that $A/\langle g \rangle$ has at least one rational curve.

1. Introduction

Let $G$ be a finite group of automorphisms of an abelian variety $A/\mathbb{C}$. It is a classical result [4, II. §1] that $A$ itself cannot contain a rational curve. For $|G| > 1$, there may or may not be rational curves on $A/G$. For general abelian varieties, $\text{Aut}(A) = \pm 1$, and Pirola proved [7] that for $A$ sufficiently general and of dimension at least three, $A/\pm 1$ has no rational curves. At the other extreme, regarding $A = E^n$ as the set of $n + 1$-tuples of points on the elliptic curve $E$ which sum to 0, $A/S_{n+1}$ can be interpreted as the set of effective divisors linearly equivalent to $(n+1)[0]$ and, as such, is just $\mathbb{P}^n$. More generally, Looijenga has shown [6] that the quotient of $E^n$ by the Weyl group of a root system of rank $n$ is a weighted projective space.

Rational curves on $A/G$ over a field $K$ are potentially a source of rational points over $G$-extensions of $K$. For instance, the method [5] for finding pairs $a, b \in \mathbb{Q}^\times$ such that the quadratic twists $E_a, E_b,$ and $E_{ab}$ all have positive rank amounts to finding a rational curve on $E^3/(\mathbb{Z}/2\mathbb{Z})^2$. Likewise, the theorem of Looijenga cited above gives for each elliptic curve $E$ over a number field $K$ and for each Weyl group $W$, a source of $W$-extensions $L_i$ of $K$ such that the representation of $W$ on each $E(L_i) \otimes \mathbb{Q}$ contains the reflection representation. On the other hand, the result of Pirola cited above dims the hope of using geometric methods to show that every abelian variety over a number field $K$ gains rank over infinitely many quadratic extensions of $K$. Thus, it is desirable from the viewpoint of arithmetic to understand when $A/G$ can be expected to have a rational curve over a given field $K$.

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and to begin with, one would like to know when $A/G$ has a rational curve over $\mathbb{C}$.

Any automorphism $g$ of an abelian variety $A$ defines an invertible linear transformation (also denoted $g$) on $\text{Lie}(A)$. If $g$ is of finite order, there exists a unique sequence of rationals $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n < 1$ such that the eigenvalues of $g$ are $e(x_1), \ldots, e(x_n)$, where $e(x) := e^{2\pi i x}$. We say $g$ is of type $(x_1, \ldots, x_n)$. Following Kollár and Larsen [3], we write $\text{age}(g) = x_1 + \cdots + x_n$. For instance, $\text{age}(g) = 1/2$ for every reflection $g$. The main result of [3] asserts that $A/G$ is uniruled if and only if $0 < \text{age}(g) < 1$ for some $g \in G$.

In this paper, we prove that to find a single rational curve in $A/G$, it suffices that $\text{age}(g) \leq 1$. Since we need only consider the case $\text{age}(g) = 1$, we first classify all types of weight 1. This requires a combinatorial analysis, which we carried out using a computer algebra system to minimize the risk of an oversight. There are thirty-five cases (see Table 2 below), and our strategy for finding rational curves depends on case analysis. Abelian surfaces play a special role, since here we can use known results on K3 surfaces. The other key idea is to find a non-singular projective curve $X$ on which $G$ acts with quotient $\mathbb{P}^1$ and a $G$-equivariant map from $X$ to $A$, or, equivalently, a $G$-homomorphism from the Jacobian variety of $X$ to $A$.

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2. Classifying types

If $A = V/\Lambda$, then the Hodge decomposition $\Lambda \otimes \mathbb{C} \cong V \oplus \bar{V}$ respects the action of $\text{Aut}(A)$. Therefore, if $g$ is of finite order with eigenvalues $e(x_1), \ldots, e(x_n)$, then the multiset

$$(*) \quad \{e(x_1), \ldots, e(x_n), e(-x_1), \ldots, e(-x_n)\}$$

is $\text{Aut}(\mathbb{C})$-stable. By a type, we mean a multiset $\{x_1, \ldots, x_n\}$ with $x_i \in [0, 1)$ such that the multiset $(*)$ is $\text{Aut}(\mathbb{C})$-stable. Equivalently, a type can be identified with a finitely supported function $f : \mathbb{Q}/\mathbb{Z} \to \mathbb{N}$ such that $f(x) + f(-x)$ depends only on the order of $x$ in $\mathbb{Q}/\mathbb{Z}$. By the weight of $\{x_1, \ldots, x_n\}$, we mean the sum $x_1 + \cdots + x_n$, so that $\text{age}(g)$ is the weight of the type of $g$.

A type is reduced if 0 does not appear, and the reduced type of a given type is obtained by discarding all copies of 0. The sum of types is the union in the sense of multisets; at the level of associated functions on $\mathbb{Q}/\mathbb{Z}$ it is the usual sum. A type which is not the sum of non-zero types is primitive. All the elements of a primitive type appear with multiplicity one, and they all have the same denominator. Every type can be realized (not necessarily uniquely) as a sum of primitive types; if the weight of the type is 1, each of the primitive types has weight $\leq 1$, so our first task is to classify primitive types with weight $\leq 1$. 

A primitive type $X$ of denominator $n \geq 2$ consists of fractions $a_i/n$ where $0 < a_i < n$, and $(a_i, n) = 1$. Moreover, if $n \geq 3$, for each positive integer $a < n$ prime to $n$, exactly one of $a/n$ and $1- a/n$ belongs to $X$. If $a < b < n/2$ and $1- a/n \in X$, then the weight of $X$ exceeds 1 since either $b/n$ or $1- b/n$ belongs to $X$. Thus, if $0 < a < n/2$ and $1- a/n \in X$, then $a$ must be the largest integer in $(0, n/2)$ prime to $n$.

**Lemma 1.** If $\phi(n) > 24$, then

$$\sum_{x \in S_n} \min(x, n-x) > 2n,$$

where $S_n$ is the set of positive integers $< n$ and prime to $n$. Moreover, the largest integer $n$ such that $\phi(n) \leq 24$ is 90.

**Proof.** Note that

$$
\min(x, n-x) > \frac{x(n-x)}{n} = \frac{n^2 - x^2 - (n-x)^2}{2n}.
$$

In order to prove the first statement, we want to prove if $\phi(n) > 24$, then

$$
\sum_{x \in S_n} \left( n^2 - x^2 - (n-x)^2 \right) > 4n^2,
$$

or equivalently,

$$
\phi(n)n^2 - 2 \sum_{x \in S_n} x^2 - 4n^2 > 0.
$$

By Möbius inversion, one can prove that

$$
\sum_{x \in S_n} x^2 = \frac{\phi(n)n^2}{3} + (-1)^d_n \frac{\phi(f(n))n}{6},
$$

where $f(n)$ denotes the largest squarefree divisor of $n$ and $d_n$ is the number of distinct prime divisors of $n$. Thus, if $\phi(n) > 24$, then $\phi(n) > 24 \geq 12\frac{n}{n-1}$, so $(n-1)\phi(n) - 12n > 0$ and since $\phi(f(n)) \leq \phi(n)$,

$$
\phi(n)n^2 - 2 \sum_{x \in S_n} x^2 - 4n^2 \geq \left( \frac{\phi(n)}{3} - 4 \right)n^2 - \frac{\phi(n)n}{3}
= \frac{n((n-1)\phi(n) - 12n)}{3} > 0,
$$

which is the desired inequality.

For the second statement, if $\phi(n) \leq 24$ and $p$ is a prime factor of $n$, then $\phi(p) = p - 1 \leq \phi(n) \leq 24$. Hence $p \leq 23$. Writing

$$
n = 2^{n_2}3^{n_3}5^{n_5}7^{n_7}11^{n_{11}}13^{n_{13}}17^{n_{17}}19^{n_{19}}23^{n_{23}},
$$

we have

$$
0 \leq n_2 \leq 5, 0 \leq n_3 \leq 3, 0 \leq n_5 \leq 2,
$$

and $0 \leq n_i \leq 1$ for $7 \leq i \leq 23$. Case analysis now shows $n \leq 90$. 

□
Proposition 2. There are 28 primitive types with weight $\leq 1$:

| # | n | primitive types | weight |
|---|---|----------------|--------|
| 1 | 2 | 1/2 | 1/2 |
| 2 | 3 | 1/3 | 1/3 |
| 3 | 3 | 2/3 | 2/3 |
| 4 | 4 | 1/4 | 1/4 |
| 5 | 4 | 3/4 | 3/4 |
| 6 | 5 | 1/5, 2/5 | 3/5 |
| 7 | 5 | 1/5, 3/5 | 4/5 |
| 8 | 6 | 1/6 | 1/6 |
| 9 | 6 | 5/6 | 5/6 |
| 10 | 7 | 1/7, 2/7, 3/7 | 6/7 |
| 11 | 7 | 1/7, 2/7, 4/7 | 1 |
| 12 | 8 | 1/8, 3/8 | 1/2 |
| 13 | 8 | 1/8, 5/8 | 3/4 |
| 14 | 9 | 1/9, 2/9, 4/9 | 7/9 |
| 15 | 9 | 1/9, 2/9, 5/9 | 8/9 |
| 16 | 10 | 1/10, 3/10 | 2/5 |
| 17 | 10 | 1/10, 7/10 | 4/5 |
| 18 | 12 | 1/12, 5/12 | 1/2 |
| 19 | 12 | 1/12, 7/12 | 2/3 |
| 20 | 14 | 1/14, 3/14, 5/14 | 9/14 |
| 21 | 14 | 1/14, 3/14, 9/14 | 13/14 |
| 22 | 15 | 1/15, 2/15, 4/15, 7/15 | 14/15 |
| 23 | 15 | 1/15, 2/15, 4/15, 8/15 | 1 |
| 24 | 16 | 1/16, 3/16, 5/16, 7/16 | 1 |
| 25 | 18 | 1/18, 5/18, 7/18 | 13/18 |
| 26 | 18 | 1/18, 5/18, 11/18 | 17/18 |
| 27 | 20 | 1/20, 3/20, 7/20, 9/20 | 1 |
| 28 | 24 | 1/24, 5/24, 7/24, 11/24 | 1 |

Table 1

Proof. For $n \geq 3$, the weight of a primitive type of denominator $n$ is at least

$$\sum_{\{x \in S_n | x < n/2\}} \frac{x}{n} \geq \frac{1}{2n} \sum_{x \in S_n} \min(x, n - x).$$

By Lemma 1, it suffices to carry out an exhaustive search up to $n = 90$. □

Lemma 3. There are 35 types with age 1 of automorphisms given in Table 2 below.
### Table 2

| #  | n  | types          | notes     |
|----|----|----------------|-----------|
| 1  | 2  | 1/2, 1/2       | Prop. 6   |
| 2  | 3  | 1/3, 1/3, 1/3  | Th. 12    |
| 3  | 4  | 1/3, 2/3       | Prop. 7   |
| 4  | 6  | 1/4, 1/4, 1/4, 1/4, 1/4 | Th. 12 |
| 5  | 6  | 1/4, 1/4, 2/4  | $g^2 \rightarrow \#1$ |
| 6  | 6  | 1/4, 3/4       | $g^2 \rightarrow \#1$ |
| 7  | 6  | 1/6, 1/6, 1/6, 1/6, 1/6, 1/6 | Th. 12 |
| 8  | 6  | 1/6, 1/6, 1/6, 1/6, 2/6 | Th. 12 |
| 9  | 6  | 1/6, 1/6, 1/6, 1/6, 3/6 | $g^2 \rightarrow \#2$ |
| 10 | 6  | 1/6, 1/6, 1/6, 4/6 | $g^3 \rightarrow \#1$ |
| 11 | 6  | 1/6, 5/6       | $g^3 \rightarrow \#1$ |
| 12 | 6  | 1/6, 2/6, 3/6  | $g^3 \rightarrow \#1$ |
| 13 | 6  | 1/6, 1/6, 2/6, 2/6 | $g^3 \rightarrow \#1$ |
| 14 | 7  | 1/7, 2/7, 4/7  | Cor. 10   |
| 15 | 8  | 1/8, 2/8, 5/8  | $g^4 \rightarrow \#1$ |
| 16 | 8  | 1/8, 3/8, 4/8  | $g^4 \rightarrow \#1$ |
| 17 | 8  | 1/8, 1/8, 3/8, 3/8 | Th. 12 |
| 18 | 8  | 1/8, 2/8, 2/8, 3/8 | $g^4 \rightarrow \#1$ |
| 19 | 10 | 1/10, 2/10, 3/10, 4/10 | $g^5 \rightarrow \#1$ |
| 20 | 12 | 4/12, 2/12, 1/12, 5/12 | $g^6 \rightarrow \#1$ |
| 21 | 12 | 4/12, 3/12, 3/12, 2/12 | $g^6 \rightarrow \#1$ |
| 22 | 12 | 6/12, 1/12, 5/12 | $g^6 \rightarrow \#1$ |
| 23 | 12 | 3/12, 3/12, 1/12, 5/12 | $g^4 \rightarrow \#3$ |
| 24 | 12 | 2/12, 2/12, 1/12, 5/12 | $g^6 \rightarrow \#1$ |
| 25 | 12 | 1/12, 1/12, 5/12, 5/12 | Th. 12 |
| 26 | 12 | 4/12, 1/12, 7/12 | $g^6 \rightarrow \#1$ |
| 27 | 12 | 2/12, 2/12, 1/12, 7/12 | $g^6 \rightarrow \#1$ |
| 28 | 12 | 3/12, 3/12, 2/12, 2/12, 2/12 | $g^6 \rightarrow \#1$ |
| 29 | 15 | 1/15, 2/15, 4/15, 8/15 | Cor. 11   |
| 30 | 16 | 1/16, 3/16, 5/16, 7/16 | Cor. 9    |
| 31 | 20 | 1/20, 3/20, 7/20, 9/20 | Cor. 9    |
| 32 | 24 | 1/24, 5/24, 7/24, 11/24 | Cor. 9    |
| 33 | 24 | 8/24, 4/24, 3/24, 9/24 | $g^{12} \rightarrow \#1$ |
| 34 | 24 | 3/24, 9/24, 2/24, 10/24 | $g^{12} \rightarrow \#1$ |
| 35 | 24 | 4/24, 4/24, 4/24, 3/24, 9/24 | $g^{12} \rightarrow \#1$ |

**Proof.** Let $[a_i]$ be a formal variable representing the $i$th primitive type in Table 1, and let $w_i$ denote the weight of the $i$th type. A monomial $\prod a_i^{m_i}$ stands for a sum of primitive types in which the $i$th type appears $m_i$ times. The g.c.d. of the denominators of the $w_i$ is 5040. Let $y = x^{1/5040}$, so $(1-[a_i]x^{w_i})^{-1}$ is a power series in $y$ for every $i$. By MAPLE 13, the coefficient
of \( y^{5040} \) in the product
\[
\prod_{i=1}^{28} (1 - [a_i]x^{w_i})^{-1} = \prod_{i=1}^{28} (1 - [a_i]y^{w_i \cdot 5040})^{-1},
\]
is \( a_1^2 + a_2^3 + a_2 a_3 + a_4 a_5 + a_8 a_9 + a_4 a_13 + a_6 a_{16} + a_4^2 a_1 + a_{28} + a_{27} + a_{23} + a_{24} + a_{11} + a_{19} a_2 + a_{19} a_8^2 + a_{18} a_1 + a_{18} a_4^2 + a_{18} a_8^3 + a_{18} a_{12} + a_{18}^2 + a_{12} a_1 + a_{12} a_4^2 + a_{12} a_8^3 + a_{12}^2 + a_8^2 a_2^2 + a_8^2 a_3 + a_8^3 a_1 + a_8^3 a_4^2 + a_8^4 a_2 + a_8^6 + a_4^4 + a_{18} a_8 a_2 + a_{12} a_8 a_2 + a_8 a_1 a_2 + a_8 a_2 a_4^2.

Each monomial in this sum corresponds to an entry in Table 2.

\[\square\]

3. Rational curves in \( A/\langle g \rangle \)

In this section we explain how to find rational curves on \( A/\langle g \rangle \) in each case in Table 2.

**Lemma 4.** If \( A/\langle g^n \rangle \) has a rational curve for some positive integer \( n \), then \( A/\langle g \rangle \) has a rational curve.

**Proof.** The morphism \( A/\langle g^n \rangle \to A/\langle g \rangle \) is finite, so the image of a rational curve is again a rational curve. \[\square\]

**Proposition 5.** Let \( A \) be an abelian variety and \( g \) an automorphism of finite order. Suppose that for every abelian variety \( B \) and finite-order automorphism \( h \in \text{Aut}(B) \) whose type is the reduced type of \( g \), \( B/\langle h \rangle \) has a rational curve. Then \( A/\langle g \rangle \) has a rational curve.

**Proof.** Let \( B \) denote the image of \( 1 - g \) acting on \( A \). Then \( B \) is an abelian subvariety of \( A \), and \( g \) restricts to an automorphism \( h \) of \( B \) whose type is the reduced type of \( g \). As \( B/\langle h \rangle \subset A/\langle g \rangle \) has a rational curve, the same is true of \( A/\langle g \rangle \). \[\square\]

The following proposition is well known.

**Proposition 6.** If \( A \) is an abelian surface, then \( A/\pm 1 \) has a rational curve.

**Proof.** Resolving the 16 singularities of \( A/\pm 1 \), we obtain a K3 surface with Picard number \( \geq 16 \geq 5 \). By work of Bogomolov and Tschinkel [1], any such surface is either elliptic or has infinite automorphism group and in either case has infinitely many rational curves, all but finitely many of which lie on \( A/\pm 1 \).

Note that Proposition [6] covers not only case \#1 in Table 2 but twenty other cases as well, namely those (indicated in the “notes” column) for which the reduced type of some power of \( g \) is \((1/2, 1/2)\).

**Proposition 7.** Let \( A \cong V/\Lambda \) be an abelian surface with an automorphism \( g \) of type \((1/3, 2/3)\). Then \( A/\langle g \rangle \) contains a rational curve.
Proof. Let $G = \langle g \rangle$ and $X = A/G$. Regarding $1 - g$ as an isogeny of $A$, the number of fixed points of $g$ is
\[
\deg(1 - g) = \# \ker(1 - g) = \det(1 - g|A) = 3^2 = 9.
\]
These are singularities of type $A_2$, since under $(x, y) \mapsto (\omega x, \omega^2 y)$ where $\omega$ is a cube root of unity, the invariants are generated by $X = x^3, Y = y^3, Z = xy$, and so
\[
\mathbb{C}[[x, y]]^G = \mathbb{C}[[X, Y, Z]]/(XY - Z^3).
\]
This is isomorphic to $\mathbb{C}[[x, y, z]]/(x^2 + y^2 + z^3)$, which has a Du Val singularity of type $A_2$ (see [8, Ch.4, 4.2]).

Consider the minimal resolution $f : Y \to X$, for which the 9 exceptional divisors $Y_i$ each consists of two projective lines $D_{i,1}$ and $D_{i,2}$ intersecting at one point. The canonical divisor of $Y$ is $K_Y = f^*K_X = 0$. Hence $Y$ is a K3-surface of Picard number $\geq 18$, and again by [1], we deduce that $X$ has infinitely many rational curves. □

**Theorem 8.** Let $B$ be an abelian variety and $h \in \text{Aut}(B)$ an automorphism of finite order such that $h$ and $h^{-1}$ have disjoint types. Let $A$ be an abelian variety with a finite order automorphism $g$ whose type is contained in that of $h$. If there is a rational curve on $B/\langle h \rangle$ then there is a rational curve on $A/\langle g \rangle$.

Proof. It suffices to prove that there exists a surjective homomorphism $p : B \to A$ such that the diagram
\[
\begin{array}{ccc}
B & \xrightarrow{p} & A \\
\downarrow h & & \downarrow g \\
B & \xrightarrow{p} & A
\end{array}
\]
commutes. Writing $B = \text{Lie}(B)/\Lambda_B$ and $A = \text{Lie}(A)/\Lambda_A$, the goal is to find a surjective $\mathbb{C}[t]$-linear map $\phi : \text{Lie}(B) \to \text{Lie}(A)$ (where $t$ acts as $g$ on $\text{Lie}(A)$ and as $h$ on $\text{Lie}(B)$) such that $\phi(\Lambda_B) \subseteq \Lambda_A$. If $\psi$ is a surjective $\mathbb{C}[t]$-linear map $\text{Lie}(B) \to \text{Lie}(A)$ such that $\psi(\Lambda_B \otimes \mathbb{Q}) = \Lambda_A \otimes \mathbb{Q}$, then we can define $\phi := n\psi$ for $n$ a sufficiently divisible positive integer. It suffices, therefore, to find $\psi$ with the desired properties.

As the type of $A$ is a subset of the type of $B$, there exists a surjective $\mathbb{Q}[t]$-linear map $T : \Lambda_B \otimes \mathbb{Q} \to \Lambda_A \otimes \mathbb{Q}$. Extending scalars to $\mathbb{C}$, $T \otimes 1$ maps $\Lambda_B \otimes \mathbb{C} = \text{Lie}(B) \oplus \text{Lie}(B)$ to $\Lambda_A \otimes \mathbb{C} = \text{Lie}(A) \oplus \text{Lie}(A)$. The type of $g$ acting on $\text{Lie}(A)$ is the same as the type of $g^{-1}$ acting on $\text{Lie}(A)$ and therefore disjoint from the type of $g$ acting on $\text{Lie}(A)$, and the same is true for the type of $h$ acting on $\text{Lie}(B)$. As $\text{Lie}(B)$ and $\text{Lie}(A)$ are direct sums of certain $t$-eigenspaces of $\Lambda_B \otimes \mathbb{C}$ and $\Lambda_A \otimes \mathbb{C}$ respectively and as the spectrum of $t$ acting on $\text{Lie}(A)$ is the intersection of the spectra of $t$ acting on $\text{Lie}(B)$ and on $\Lambda_A \otimes \mathbb{C}$, it follows that $T \otimes 1$ maps $\text{Lie}(B)$ to $\text{Lie}(A)$. The restriction of $\psi$ to $\text{Lie}(B)$ is therefore the desired map. □
Corollary 9. Let $n \geq 3$ be a positive integer, and let $m = \lceil n/2 \rceil - 1$. If $A$ is an abelian variety and $g \in \text{Aut}(A)$ is an automorphism of order $n$ whose type is contained in $\{1/n, 2/n, \ldots, m/n\}$, then $A/\langle g \rangle$ has a rational curve.

Proof. Let $X$ denote the non-singular projective hyperelliptic curve of genus $m$ which contains the affine curve $y^2 = x^n - 1$. The order-$n$ automorphism $h(x, y) = (e(1/n)x, y)$ extends to an automorphism of $X$ and therefore defines an automorphism of $B := \text{Jac}(X)$. The Lie algebra $\text{Lie}(B)$ can be identified with the space $H^0(X, \Omega_X)$ of holomorphic differential forms on $X$, which has a basis

$$\left\{ \frac{dx}{y}, \frac{x dx}{y}, \ldots, \frac{x^{m-1} dx}{y} \right\}.$$ 

Therefore, the type of $h$ acting on $B$ is $\{1/n, 2/n, \ldots, m/n\}$, which is disjoint from the type of $h^{-1}$. On the other hand, $B/\langle h \rangle$ contains the rational curve $X/\langle h \rangle$. Thus, Theorem 8 applies. \qed

Corollary 10. If $A$ is an abelian 3-fold and $g$ is an automorphism of $A$ of type $(1/7, 2/7, 4/7)$, then $A/\langle g \rangle$ has a rational curve.

Proof. Let $X$ denote the Klein quartic:

$$X : x^3 y + y^3 z + z^3 x = 0$$

and $B$ the Jacobian of $X$. The self-map $h(x, y, z) = (\zeta_7 x, \zeta_4^2 y, \zeta_7^2 z)$ of $X$ belongs to the automorphism group $\text{PSL}_2(\mathbb{F}_7)$ of $X$ which acts non-trivially on the Jacobian variety $B$ and therefore on $\text{Lie}(B) = H^0(X, \Omega_X)$. Conjugating $h$ by the cyclic permutations of $(x, y, z)$, we see that $h$ is conjugate to $h^2$ and $h^4$ in $\text{Aut}(X)$, and therefore the type of $h$ is invariant under multiplication by 2 (mod 1). It is therefore $(1/7, 2/7, 4/7)$ or $(3/7, 5/7, 6/7)$, and replacing $h$ by $h^{-1}$ if necessary, we may assume that it is the former. \qed

We remark that $B/\langle h \rangle$ in the proof of Corollary 10 has appeared in the literature; it is known to have a Calabi-Yau resolution \cite[Example 6.3]{IML1}.

Corollary 11. If $A$ is an abelian variety and $g$ an automorphism such that the type of $A$ is contained in $(1/15, 2/15, 3/15, 4/15, 8/15, 9/15)$, then $A/\langle g \rangle$ has a rational curve.

Proof. Let $X$ be the non-singular projective curve which has a (singular, affine) model $X' : y^{15} = x^2(x - 1)$. This is singular only at $(0, 0)$, and the inverse image of this singularity under the normalization map $X \setminus \{P_\infty\} \to X'$ is a single point $P_0 \in X$. The automorphism $h : (x, y) \mapsto (x, e(1/15)y)$ of the affine curve induces an automorphism of $X$ of order 15. As $15y^{14} dy = (3x^2 - 2x)dx$, any differential form $\frac{x^m y^n dy}{3x^2 - 2x}$, $m, n \geq 0$, is holomorphic except possibly at $P_0$ and $P_\infty$. One checks that

$$\frac{dy}{3x - 2}, \frac{y dy}{3x - 2}, \frac{y^2 dy}{3x - 2}, \frac{y^3 dy}{3x^2 - 2x}, \frac{y^7 dy}{3x^2 - 2x}, \frac{y^8 dy}{3x^2 - 2x}.$$
are all holomorphic, and their eigenvalues under $h$ are $e(1/15)$, $e(2/15)$, $e(3/15)$, $e(4/15)$, $e(8/15)$, $e(9/15)$ respectively. Applying the Riemann-Hurwitz theorem to the map $X \to \mathbb{P}^1$ given by $y$, we see that $X$ is of genus 6, and therefore, that these differential forms form a basis of $\text{Lie}(\text{Jac}(B)) = H^0(X, \Omega_X)$. \hfill $\Box$

**Theorem 12.** If $A$ is an abelian variety and $g$ is an automorphism of finite order such that $g$ and $g^{-1}$ have disjoint types and the type of $g$ is a sum of primitive types at least one of which has weight less than 1, then $A/\langle g \rangle$ has a rational curve.

**Proof.** For every primitive type, there exists an abelian variety $B_i$ with complex multiplication and an automorphism $h_i$ of $B_i$ with the given type. Indeed, the primitive types of denominator $n$ are in natural correspondence with CM-types on $\mathbb{Q}(\zeta_n) = \mathbb{Q}(e(1/n))$. Any CM-type $\Phi$ on $\mathbb{Q}(\zeta_n)$, $n \geq 3$ defines an embedding $\mathbb{Q}(\zeta_n) \to \mathbb{C}^{\phi(n)/2}$. The image of $\mathbb{Z}[\zeta_n]$ defines a lattice $\Lambda \subset \mathbb{C}^{\phi(n)/2}$, and the quotient $\mathbb{C}^{\phi(n)/2}/\Lambda$ is a complex torus with a natural action of $\mathbb{Z}/n\mathbb{Z}$ of the type associated with $\Phi$. The quotient $\mathbb{C}^{\phi(n)/2}/\Lambda$ admits a polarization [9], II 6 Theorem 4], so there exists a pair $(B_i, h_i)$ as claimed. If $\text{age}(h_1) < 1$, then by [3], $B_1/\langle g_1 \rangle$ has a rational curve. If $A = A_1 \times \cdots \times A_m$, and $g = (g_1, \ldots, g_m)$ is a finite order automorphism of $A$ which stabilizes each factor, then $A_1/\langle g_1 \rangle \subset A/\langle g \rangle$, so $A/\langle g \rangle$ has a rational curve. The theorem now follows from Theorem [3]. \hfill $\Box$

To summarize, we have the following theorem:

**Theorem 13.** Let $A$ be an abelian variety with a nontrivial automorphism $g$ of finite order such that $\text{age}(g) \leq 1$. Then $A/\langle g \rangle$ contains a rational curve.

**Corollary 14.** Let $A$ be an abelian variety of dimension $n$ with a nontrivial automorphism $g$ of finite order. If $\dim(\ker(1 - g)) \geq n - 2$ (i.e. the codimension of the fixed subspace of $A$ under $g$ is less than or equal to 2), then the quotient $A/\langle g \rangle$ contains a uniruled hypersurface.

**Proof.** Since $\dim(\ker(1 - g)) \geq n - 2$, $B := \text{im}(1 - g)$ is an abelian variety of dimension $n - \dim(\ker(1 - g)) \leq 2$. Let $h$ denote the restriction of $g$ to $B$. As $\text{age}(h) + \text{age}(h^{-1}) \geq 2$, we may assume without loss of generality that $\text{age}(h) \leq 1$, so $B/\langle h \rangle$ has a rational curve $Z$ by Theorem [3]. Let $C$ denote the identity component of $\ker(1 - g)$, which is an abelian subvariety of dimension $\dim \ker(1 - g)$ on which $g$ acts trivially. The addition morphism $B \times C \to A$ is an isogeny and respects the action of $\langle g \rangle$. We therefore obtain a finite morphism from

$$Z \times C \subset (B/\langle g \rangle) \times C \cong (B \times C)/\langle g \rangle$$

to $A/\langle g \rangle$. The image of an $n - 1$-dimensional ruled variety under a finite morphism is a uniruled hypersurface. \hfill $\Box$
Corollary 15. Let \( E \) be an elliptic curve. If \( W \) is a Weyl group of simple roots of rank \( n \geq 3 \) acting on \( E^n \) and \( W^+ \) is an index 2-subgroup of \( W \), then the quotient \( E^n/W^+ \) contains a rational curve.

Proof. \( W \) is generated by reflections \( s_j \) of simple roots. Since \( W^+ \) is an index 2-subgroup of \( W \), there exist two reflections \( s_1 \) and \( s_2 \) such that \( s_1s_2 \in W^+ \). Then for each \( i \), \( \ker(1 - s_i) \) has codimension 1 and their intersection has codimension \( \leq 2 \). Since \( \ker(1 - s_1s_2) \) contains the intersection of \( \ker(1 - s_1) \) and \( \ker(1 - s_2) \), this follows from Corollary \([4] \). \( \square \)

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