UNIQUENESS FOR THE BREZIS-NIRENBERG TYPE PROBLEMS ON SPHERES AND HEMISPHERES

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Abstract. In this work, we study some nonlinear partial differential equations on spheres and hemispheres, with zero Neumann boundary data, which is a Brezis-Nirenberg type problem, and find conditions such that equations admit only constant solutions. Moreover, we study that uniqueness problem for some nonlinear partial differential systems.

1. Introduction and main results

Let \((M^n, g)\), \(n \geq 3\), be a compact Riemannian manifold (possibly with non-empty boundary). We consider the following problem \((P)\)
\[
\begin{cases}
-L_g u = f(u) & \text{in } M \\
u > 0 & \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M
\end{cases}
\]
where \(L_g\) is a second order partial differential operator on \(M^n\) with respect to the metric \(g\), and \(\frac{\partial u}{\partial \nu}\) is the normal derivative of \(u\) with respect to the unit exterior normal vector field \(\nu\) of the boundary \(\partial M\), and \(f : (0, \infty) \to \mathbb{R}\) is a smooth function. If the boundary of \(M\) is empty, we do not assume \(\frac{\partial u}{\partial \nu} = 0\) on \(\partial M\) in the problem \((P)\). Our main interest here is to find conditions on \(f\) and on the geometry or topology of \(M\) which imply that the problem \((P)\) admits only constant solutions. A particular case of the problem \((P)\) is the following one:
\[
\begin{cases}
-\Delta_g u + \lambda u - F(u)u^{\frac{n+2}{n-2}} = 0 & \text{in } M \\
u > 0 & \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M
\end{cases}
\]
where \(\Delta_g\) is the Laplace-Beltrami operator on \(M^n\) with respect to the metric \(g\), \(\lambda\) is a real smooth function on \(M\) and \(F : (0, \infty) \to \mathbb{R}\) is a real smooth function. Note that when \(\lambda > 0\) is a constant and \(F(u)u^{\frac{n+2}{n-2}} = u^p\), \(p > 1\), then \(u = \lambda^{\frac{2}{p-1}}\) is a solution of the problem \((Q)\). In the case where \(F\) is a constant and \(\lambda = \frac{(n-2)}{4(n-1)} R_g\), where \(R_g\) denote the scalar curvature of the Riemannian manifold \((M, g)\), the problem \((Q)\) is just the Yamabe problem in the conformal geometry for the closed case or, if \(\partial M\) is not empty, for the case of minimal boundary. See Escobar’s work [23]. If \(M^n = \mathbb{S}^n\) is the standard unit \(n\)-sphere and \(g\) is the standard metric, there are infinitely many solutions for the Yamabe problem with respect to the

\begin{flushleft}
\textit{Date:} June 25, 2019.
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\textit{2010 Mathematics Subject Classification.} 35J60, 35C21.
\end{flushleft}

\begin{flushleft}
\textit{Key words and phrases.} Unit sphere; Uniqueness; Fractional Laplacian; Elliptic system; Positive solution.
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The authors were supported by FAPEMIG and CNPq grants.
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metric $g$ since the conformal group of the standard unit $n$-sphere is also infinite. If $(M^n, g)$ is an Einstein manifold which is conformally distinct from the standard $n$-sphere, it was showed by Obata that the Yamabe problem has unique solution. However, R. Schoen has proved that there are at least 3 solutions for the Yamabe problem on $S^1 \times S^{n-1}$ with respect to the standard product metric. In a more specific situation, that problem $(Q)$ were studied by Lin, Ni and Takagi for the case when $f(u)u^{\frac{n+2}{n-2}} = w^p$, $p > 1$, $\lambda > 0$ is a constant function and $M$ is a bounded convex domain with smooth boundary in the Euclidean space $\mathbb{R}^n$ (see [19], [50] and the references therein). When $p$ is a subcritical exponent, that is, $p < (n+2)/(n-2)$, Lin, Ni and Takagi [19] showed that problem has a unique solution if $\lambda$ is sufficiently small. Such kind of uniqueness results about radially symmetric solution of $(P)$ were also obtained by Lin and Ni in [2] when $\Omega$ is an annulus and $p > 1$ or when $\Omega$ is a ball and $p > (n+2)/(n-2)$. It was then conjectured by Lin and Ni in [2] that, for any $p > 1$, there exists a $\lambda > 0$ such that the problem $(P)$ has only a constant solution if $0 < \lambda < \lambda$. However, that conjecture of Lin-Ni is not true in general, especially when $p$ is the critical Sobolev exponent, that is, $p = (n+2)/(n-2)$. In this case, when $\Omega$ is a unit ball and $n = 4, 5, 6$, it was shown by Adimurthi and Yadava [1] that the problem $(P)$ has at least two radial solutions if $\lambda > 0$ and is close to 0 (see also [3]). However, if $\Omega$ is not a ball or $n$ is different from 4, 5 or 6, that conjecture is open.

In [11], H. Brezis and Y.Y. Li studied the problem $(P)$ for the case of the standard unit sphere $(S^n, g)$, where $g$ is the standard metric, and, using results due to Giga, Ni and Nirenberg [24], they showed that the problem $(P)$ admits only constant solutions provided that $L_g = \Delta_g$ and $f$ is such that the function $h(t) = t^{-\frac{n+2}{4}}(f(t) + n(n-2)t/4)$ is a decreasing function on $(0, +\infty)$. Hence, considering the particular problem $(Q)$ on the standard sphere, they showed that if $0 < \lambda < \frac{n(n-2)}{4}$ and $F$ is a decreasing function on $(0, +\infty)$, then the only positive solution to $(Q)$ is the constant one.

Motivated by the results in [11], and by the technique applied in that work, we study first the following nonlinear elliptic equations and systems:

$$\begin{align*}
-\Delta_g u &= f(u), \quad u > 0 \quad \text{in } S^n_+,
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial S^n_+,
\end{align*}$$

and

$$\begin{align*}
-\Delta_g u_1 &= f_1(u_1, u_2) \quad \text{in } S^n_+,
-\Delta_g u_2 &= f_2(u_1, u_2) \quad \text{in } S^n_+,
u_1, \quad u_2 > 0 \quad \text{in } S^n_+,
\frac{\partial u_1}{\partial \nu} &= \frac{\partial u_2}{\partial \nu} = 0 \quad \text{on } \partial S^n_+,
\end{align*}$$

where $g$ is the standard metric on the hemisphere $S^n_+$, $n \geq 3$, $\frac{\partial}{\partial \nu}$ is the derivate with respect to the outward normal vector field $\nu$, and $f : (0, +\infty) \rightarrow \mathbb{R}$, $f_1$, $f_2 : (0, +\infty) \times (0, +\infty) \rightarrow \mathbb{R}$ are continuous functions.

Our first results are the following.

**Theorem 1.1.** Assume that

$$h_1(t) := t^{\frac{n+2}{4}} \left( f(t) + \frac{n(n-2)}{4} t \right)$$

is decreasing in $(0, +\infty)$. \hspace{1cm} (1.3)
Then the problem (1.1) admits only constant solutions.

A typical example of that case is the following:
\[ f(t) = t^p - \lambda t, \quad p > 0, \ \lambda > 0. \]
So that (1.1) becomes
\[
\begin{align*}
-\Delta u &= u^p - \lambda u, \quad u > 0 \quad \text{in } S^n_+,
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial S^n_+.
\end{align*}
\]

(1.4)

Corollary 1.2. Assume that \( p \leq (n + 2)/(n - 2) \) and \( \lambda \leq n(n - 2)/4 \), and at least one of these inequalities is strict. Then the only solution of (1.4) is the constant \( u \equiv \lambda^{1/(p-1)} \).

We define for \( i = 1, 2 \):
\[ h_{i1}(t_1, t_2) := t_i^{-\frac{n+2}{n}} \left( f_i(t_1, t_2) + \frac{n(n - 2)}{4} t_i \right), \quad t_1 > 0, \ t_2 > 0. \]

Similarly to the Theorem 1.1 for system, we have:

Theorem 1.3. Assume that for \( i, j = 1, 2 \):
\[ h_{i1}(t_1, t_2) \text{ is nondecreasing in } t_j > 0, \quad \text{with } i \neq j, \]
\[ h_{i1}(t_1, t_2) t_i^{-\frac{n+2}{n}} \text{ is nondecreasing in } t_i > 0, \]
\[ h_{i1}(a_1 t, a_2 t) \text{ is } \begin{cases} \text{decreasing in } t > 0 & \text{for any } a_i > 0. \end{cases} \]

Then the problem (1.2) admits only constant solutions.

An example for system is the case
\[ f_1(t_1, t_2) = t_1^{\alpha_1} t_2^{\alpha_2} - \lambda_1 t_1, \quad f_2(t_1, t_2) = t_1^{\beta_1} t_2^{\beta_2} - \lambda_2 t_2, \]
where \( \alpha_i > 0, \beta_i > 0, \lambda_i > 0, \ i = 1, 2 \). So that (1.2) becomes
\[
\begin{align*}
-\Delta u_1 &= u_1^{\alpha_1} u_2^{\alpha_2} - \lambda_1 u_1 \quad \text{in } S^n_+, \\
-\Delta u_2 &= u_1^{\beta_1} u_2^{\beta_2} - \lambda_2 u_2 \quad \text{in } S^n_+, \\
u_1, u_2 &> 0 \quad \text{in } S^n_+, \\
\frac{\partial u_1}{\partial \nu} &= \frac{\partial u_2}{\partial \nu} = 0 \quad \text{on } \partial S^n_+.
\end{align*} 
\]

(1.6)

Corollary 1.4. Assume that \( \alpha_1 + \alpha_2 \leq (n + 2)/(n - 2) \), \( \beta_1 + \beta_2 \leq (n + 2)/(n - 2) \), \( \lambda_1 \leq n(n - 2)/4 \) and \( \lambda_2 \leq n(n - 2)/4 \), and at least two of these inequalities are strict. Denote \( \gamma = \alpha_2 \beta_1 - (\alpha_1 - 1)(\beta_2 - 1) \).

- If \( \gamma \neq 0 \), then the only solution of (1.6) are \( u_1 \equiv \lambda_1^{\frac{1-\beta_2}{\alpha_2}} \lambda_2^{\frac{1-\alpha_2}{\beta_2}} \) and \( u_2 \equiv \lambda_1^{\frac{1-\alpha_2}{\beta_2}} \lambda_2^{\frac{1-\alpha_2}{\beta_2}} \);
- If \( \gamma = 0 \), \( \lambda_1^{\beta_2 - 1} = \lambda_2^{\alpha_2} \) and \( \lambda_1^{\alpha_2} = \lambda_2^{\beta_2 - 1} \), then the problem (1.6) has infinite constant solutions;
- If \( \gamma = 0 \), and \( \lambda_1^{\beta_2 - 1} \neq \lambda_2^{\alpha_2} \) or \( \lambda_1^{\alpha_2} \neq \lambda_2^{\beta_2 - 1} \), then the problem (1.6) has no solutions.

In [24], the authors used the method of moving planes and some forms of maximum principle to obtain symmetry of the solutions of elliptical problems. However, we used the method of moving planes and some techniques based on inequalities of integrals. These techniques are used in works concerning Liouville type theorems for
elliptic equation and system with general nonlinearity (see e.g. [5] [19] [29] [30] [46] [47] and references therein), but these techniques were originally based on the ideas of S. Terracini [44]. On the other hand, this method was also widely used in integral equation and system that are closely related to the fractional differential equation and system, see e.g., [16] [17] [18] [48] and references therein.

We consider now the problem \((P)\) where the operator \(L_g\) is replaced by a fourth order partial differential operator. Let \((M^n, g)\) be a smooth compact Riemannian n-manifold, \(n \geq 5\). The Paneitz operator [41] is defined by

\[
P^g_2 u = \Delta^2_g u - \text{div}_g(a_n R_g g + b_n \text{Ric}_g)du + \frac{n-4}{2} Q_g u,
\]

where \(R_g\) denotes the scalar curvature of \((M^n, g)\), \(\text{Ric}_g\) denotes the Ricci curvature of \((M^n, g)\), \(a_n\) and \(b_n\) are constants dependent of \(n\), and \(Q_g\) is called \(Q\)-curvature. See [10] [21] for details about the properties of \(P^g_2\). On the unit sphere \((S^n, g)\), \(n \geq 5\), the operator \(P^g_2\) has the expression

\[
P^g_2 u = \Delta^2_g u - c_n \Delta_g u + d_n u,
\]

where \(c_n = (n^2 - 2n - 4)/2\) and \(d_n = (n - 4)n(n^2 - 4)/16\).

We study the following equations:

\[
\begin{align*}
\Delta^2_g u - c_n \Delta_g u &= f(u) \quad \text{in } S^n, \\
u &> 0 \quad \text{in } S^n;
\end{align*}
\]

and

\[
\begin{align*}
\Delta^2_g u_1 - c_n \Delta_g u_1 &= f_1(u_1, u_2) \quad \text{in } S^n, \\
\Delta^2_g u_2 - c_n \Delta_g u_2 &= f_2(u_1, u_2) \quad \text{in } S^n, \\
u_1, u_2 &> 0 \quad \text{in } S^n,
\end{align*}
\]

where \(g\) is the standard metric in \(S^n\), \(n > 4\), and \(f : (0, +\infty) \to \mathbb{R}\), \(f_1, f_2 : (0, +\infty) \times (0, +\infty) \to \mathbb{R}\) are continuous functions.

We use the same arguments used in the proof of Theorems [11] and [13] to show the following results.

**Theorem 1.5.** Assume that

\[
\begin{align*}
h_2(t) := t^{-\frac{n+4}{4}}(f(t) + d_n t) \text{ is decreasing non-negative in } (0, +\infty) \text{ and } \\
h_2(t)t^{\frac{n+4}{4}} \text{ is nondecreasing in } (0, +\infty).
\end{align*}
\]

Then the problem \((L, f)\) admits only constant solutions.

An other typical example is the case

\[
\Delta^2_g u - c_n \Delta_g u = u^p - \lambda u, \quad u > 0 \quad \text{in } S^n.
\]

Then we have the following result.

**Corollary 1.6.** Assume that \(p \leq (n + 4)/(n - 4)\) and \(0 < d_n\), and at least one of these inequalities is strict. Then the only solution of \((L, f)\) is the constant \(u \equiv \lambda^{1/(p-1)}\).

Now, we define for \(i = 1, 2\):

\[
h_{i2}(t_1, t_2) := t_i^{-\frac{n+4}{4}}(f_i(t_1, t_2) + d_n t_i), \quad t_1 > 0, \quad t_2 > 0.
\]

Similarly to Theorem [13], for system, we have:
**Theorem 1.7.** Assume that for \(i, j = 1, 2\): \(h_{i,2}\) are non-negative,
\[ h_{i,2}(t_1, t_2) \text{ is nondecreasing in } t_j > 0, \text{ with } i \neq j, \]
\[ h_{i,2}(t_1, t_2)t_i^{n-2} \text{ is nondecreasing in } t_i > 0, \]
\[ h_{i,2}(a_1, a_2 t) \text{ is decreasing in } t > 0 \text{ for any } a_i > 0. \]

Then the problem (1.8) admits only constant solutions.

An example for the system is the case
\[
\begin{align*}
\Delta^2_\gamma u_1 - c_1 \Delta u_1 &= u_1^{a_1} u_2^{a_2} - \lambda_1 u_1 &\text{in } \mathbb{S}^n, \\
\Delta^2_\gamma u_2 - c_2 \Delta u_2 &= u_1^{b_1} u_2^{b_2} - \lambda_2 u_2 &\text{in } \mathbb{S}^n
\end{align*}
\]  

where \(\alpha_i > 0\), \(\beta_i > 0\) and \(\lambda_i > 0\) for \(i = 1, 2\). Then we have the following result.

**Corollary 1.8.** Assume that \(\alpha_1 + \alpha_2 \leq (n + 4)/(n - 4)\), \(\beta_1 + \beta_2 \leq (n + 4)/(n - 4)\), \(\lambda_1 \leq d_n\) and \(\lambda_2 \leq d_n\), and at least two of these inequalities are strict. Denote \(\gamma = \alpha_2 \beta_1 - (\alpha_1 - 1)(\beta_2 - 1)\). Then

- if \(\gamma \neq 0\), then the only solution of (1.12) are \(u_1 \equiv \lambda_1^{1-\beta_2} \lambda_2^{\alpha_2} \) and \(u_2 \equiv \lambda_2^{1-\beta_1} \lambda_1^{\alpha_1} \);
- if \(\gamma = 0\), then the problem (1.12) has infinite constant solutions provided that \(\lambda_1^{\beta_2 - 1} = \lambda_2^{\alpha_2}\) and \(\lambda_1^{\beta_1} = \lambda_2^{\alpha_1 - 1}\);
- if \(\gamma = 0\), then the problem (1.12) has not solutions provided that \(\lambda_1^{\beta_2 - 1} \neq \lambda_2^{\alpha_2}\) or \(\lambda_1^{\beta_1} \neq \lambda_2^{\alpha_1 - 1}\).

In [31], Graham, Jenne, Mason and Sparling constructed a sequence of conformally covariant elliptic operators \(P^s_k\), on Riemannian manifolds \((M, g)\) for all positive integers \(k\) if \(n\) is odd, and for \(k \in \{1, \ldots, n/2\}\) if \(n\) is even. Moreover, \(P^s_1\) is the well known conformal Laplacian \(L_g := -c(n)\Delta_g + R_g\), where \(c(n) = 4(n-1)/(n-2)\), \(n \geq 3\), and \(P^\infty_1\) is the Paneitz operator. The problem of prescribing scalar curvature and Paneitz curvature on \(\mathbb{S}^n\) was studied extensively in last years, see e.g., [0] [1] [13] [32] [33] and [1] [21] [35].

Making use of a generalized Dirichlet to Neumann map, Graham and Zworski [32] introduced a meromorphic family of conformally invariant operators on the conformal infinity of asymptotically hyperbolic manifolds (see Mazzeo and Melrose [39]). Recently, Chang and Gonzlez [13] reconciled the way of Graham and Zworski to define conformally invariant operators \(P^\infty_s\) of non-integer order \(s \in (0, n/2)\) and the localization method of Caffarelli and Silvestre [13] for fractional Laplacian \((-\Delta)^s\) on the Euclidean space \(\mathbb{R}^n\). These lead naturally to a fractional order curvature \(R^g_s = P^\infty_s(1)\), which called \(s\)-curvature. There are several works on these conformally invariant equations of fractional order and prescribing \(s\)-curvature problems (fractional Yamabe problem and fractional Nirenberg problem), see, e.g., [27] [28] [34] [35] [36] and references therein.

In \((\mathbb{S}^n, g)\), \(n > 2\) with \(g\) is a standard metric, the operator \(P^g_s\) has the formula (see, e.g., [9])
\[
P^g_s = \frac{\Gamma(B + \frac{1}{2} + s)}{\Gamma(B + \frac{1}{2})} \cdot \frac{\Gamma(B + \frac{1}{2})}{\Gamma(B + \frac{1}{2} + s)} \cdot B = \sqrt{-\Delta_g + \left(\frac{n - 1}{2}\right)^2},
\]
Then the problem (1.16) admits only constant solutions.

Similarly to Theorem 1.9 for system, we have:

Then the problem (1.15) admits only constant solutions.

where $\Gamma$ is the Gamma function. When $s \in (0, 1)$, Pavlov and Sanko \[12\] showed that

$$P^q_s(u)(q) = C_{n,s} \int_{S^n} \frac{u(q) - u(z)}{|q - z|^{n+2s}} \, d\text{vol}_g(z) + P^q_s(1)u(q), \; u \in C^2(S^n), \; q \in S^n,$$

where $C_{n,s} = \frac{2^{2s} \pi^{n/2}}{\Gamma(1-s)} |.|$ is the Euclidean distance in $\mathbb{R}^{n+1}$ and $\int_{S^n}$ is understood as $\lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon}.$

We denote

$$D^2_s u(q) := C_{n,s} \int_{S^n} \frac{u(q) - u(z)}{|q - z|^{n+2s}} \, dz, \; u \in C^2(S^n) \quad \text{and} \quad d_{n,s} := \frac{\Gamma\left(\frac{n}{2} + s\right)}{\Gamma\left(\frac{n}{2} - s\right)} = P^q_s(1).$$

In the section 4, we study the following equations

$$\begin{cases}
D^q_s u = f(u) & \text{in } S^n, \\
u > 0 & \text{in } S^n;
\end{cases}
$$

and

$$\begin{cases}
D^q_s u_1 = f_1(u_1, u_2) & \text{in } S^n, \\
D^q_s u_2 = f_2(u_1, u_2) & \text{in } S^n, \\
u_1, u_2 > 0 & \text{in } S^n,
\end{cases}
$$

where $g$ is the standard metric in $S^n$, $n > 4$, and $f : (0, +\infty) \to \mathbb{R}$, $f_1$, $f_2 : (0, +\infty) \times (0, +\infty) \to \mathbb{R}$ are continuous functions. Then, we have the following.

**Theorem 1.9.** Let $s \in (0, 1)$. Assume that

$$h_s(t) := t^{-\frac{n+2s}{n-2s}} (f(t) + d_{n,s} t) \quad \text{is decreasing in } (0, +\infty).$$

Then the problem (1.15) admits only constant solutions.

An example of the equation (1.15) is the case

$$D^q_s u = u^p - \lambda u, \; u > 0 \text{ in } S^n,$$

where $p > 0$ and $\lambda > 0$. Then we have the following result.

**Corollary 1.10.** Assume that $p \leq (n + 2s)/(n - 2s)$ and $\lambda \leq d_{n,s}$, and at least one of these inequalities is strict. Then the only solution of (1.18) is the constant $u \equiv \lambda^{1/(p-1)}$.

We define for $i = 1, 2$:

$$h_{is}(t_1, t_2) := t_i^{-\frac{n+2s}{n-2s}} (f_i(t_1, t_2) + d_{n,s} t_i), \; t_1 > 0, \; t_2 > 0.$$

Similarly to Theorem 1.9 for system, we have:

**Theorem 1.11.** Let $s \in (0, 1)$. Assume that for $i, j = 1, 2$:

$$\begin{cases}
h_{is}(t_1, t_2) \text{ is nondecreasing in } t_j > 0, \; \text{with } i \neq j, \\
h_{is}(t_1, t_2), t_i^{-\frac{n+2s}{n-2s}} \text{ is nondecreasing in } t_i > 0, \\
h_{is}(a_1 t, a_2 t) \text{ is decreasing in } t > 0 \text{ for any } a_i > 0.
\end{cases}
$$

Then the problem (1.16) admits only constant solutions.
An example of the system (1.16) is the case

\[
\begin{cases}
    D^2 u_1 = u_1^{\alpha_1} u_2^{\alpha_2} - \lambda_1 u_1 & \text{in } S^n, \\
    D^2 u_2 = u_1^{\beta_1} u_2^{\beta_2} - \lambda_2 u_2 & \text{in } S^n,
\end{cases}
\]

(1.20)

where \( \alpha_i > 0, \beta_i > 0 \) and \( \lambda_i > 0 \) for \( i = 1, 2 \). Then we have the following result.

**Corollary 1.12.** Assume that \( \alpha_1 + \alpha_2 \leq (n+2s)/(n-2s) \), \( \beta_1 + \beta_2 \leq (n+2s)/(n-2s) \), \( \lambda_1 \leq d_{n,s} \) and \( \lambda_2 \leq d_{n,s} \), and at least two of these inequalities are strict. Denote \( \gamma = \alpha_2 \beta_1 - (\alpha_1 - 1)(\beta_2 - 1) \). Then

- if \( \gamma \neq 0 \), then the only solution of (1.20) are \( u_1 = \lambda_1^{1-\beta_2} \lambda_2^{\alpha_2} \), and \( u_2 = \lambda_1^{\beta_1} \lambda_2^{1-\alpha_1} \); 
- if \( \gamma = 0 \), then the problem (1.20) has infinite constant solutions provided that \( \lambda_1^{\beta_2 - 1} = \lambda_2^{\alpha_2} \) and \( \lambda_1^{\beta_1} = \lambda_2^{\alpha_1 - 1} \); 
- if \( \gamma = 0 \), then the problem (1.20) has not solutions provided that \( \lambda_1^{\beta_2 - 1} \neq \lambda_2^{\alpha_2} \) or \( \lambda_1^{\beta_1} \neq \lambda_2^{\alpha_1 - 1} \).

\[\text{2. PROOF OF THEOREMS 1.1 AND 1.3}\]

Let \( p \) be an arbitrary point on \( \partial S^n_+ \), which we will rename the north pole \( N \). Let \( \pi^{-1} : S^n_+ \setminus \{N\} \to \mathbb{R}^n_+ \) be the stereographic projection

\[\pi(y) = \left( \frac{2y}{1 + |y|^2}, \frac{|y|^2 - 1}{|y|^2 + 1} \right), \quad y \in \mathbb{R}^n.\]

For each \( 0 < s < n/2 \), let

\[\xi_s(y) = \left( \frac{2}{1 + |y|^2} \right)^{\frac{n-2s}{2}}, \quad y \in \mathbb{R}^n.\]

Considering the new unknown \( v \) defined on \( \mathbb{R}^n_+ \) by

\[v(y) = \xi_1(y) u(\pi(y)),\]

we have

\[v \in L^{\frac{2n}{n-2s}}(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+).\]

(2.2)

From (1.11) and standard computations gives

\[\begin{cases}
    - \Delta v = h_1 \left( \frac{v}{\xi_1(y)} \right) v^{\frac{n+2}{n+2}}, & v > 0 \quad \text{in } \mathbb{R}^n_+,
    \\
    \frac{\partial v}{\partial y^n} = 0 & \text{on } \partial \mathbb{R}^n_+.
\end{cases}\]

(2.3)

In order to show Theorem 1.1, we used the moving plane method to prove symmetry with respect to the axis \( y^n \) of solutions of problem (2.3). The following results are based on [5] [7].

**Lemma 2.1.** Let \( u \) be a solution of (1.1). Under the assumptions of Theorem 1.1, \( v = \xi_1(u \circ \pi) \) is symmetric with respect to the axis \( y^n \).

**Proof.** Given \( t \in \mathbb{R} \) we set

\[Q_t = \{ y \in \mathbb{R}^n_+; \ y^1 < t \}; \quad U_t = \{ y \in \mathbb{R}^n_+; \ y^1 = t \},\]
where \( y_t := \mathcal{I}_t(y) := \langle 2t - y^1, y^2 \rangle \) is the image of point \( y = (y^1, y^2) \) under of reflection through the hyperplane \( \mathcal{U}_t \). We define the reflected function by \( v^t(y) := v(y_t) \). The proof is carried out in three steps. In the first step we show that

\[
\Lambda := \inf \{ t > 0; v \geq v^t \text{ in } Q_\mu, \forall \mu \geq t \}
\]

is well-defined, i.e. \( \Lambda < +\infty \). The second step consists in proving that if \( \Lambda > 0 \) then \( v = v^\Lambda \) in \( Q_\Lambda \). The third step we conclude the symmetry of \( v \).

**Step 1.** \( \Lambda < +\infty \).

Assume that there is \( t > 0 \) such that \( v^t(y) \geq v(y) \) for some \( y \in Q_t \). Since \( |y| < |y_t| \), we have, \( \frac{v(y)}{\xi_t(y)} < \frac{v(y_t)}{\xi_t(y_t)} \) and

\[
-\Delta(v^t - v) = h_1 \left( \frac{v^t}{\xi_t} \right) (v^t)^{\frac{n+2}{n-2}} - h_1 \left( \frac{v}{\xi_t} \right) v^{\frac{n+2}{n-2}} \leq h_1 \left( \frac{v^t}{\xi_t} \right) (v^t)^{\frac{n+2}{n-2}} - h_1 \left( \frac{v}{\xi_t} \right) v^{\frac{n+2}{n-2}} = h_1 \left( \frac{v^t}{\xi_t} \right) ((v^t)^{\frac{n+2}{n-2}} - v^{\frac{n+2}{n-2}}) \leq \frac{n+2}{n-2} \max \left\{ h_1 \left( \frac{v^t}{\xi_t^t} \right), 0 \right\} (v^t)^{\frac{1}{n-2}} (v^t - v) = C(v^t)^{\frac{1}{n-2}} (v^t - v), \tag{2.4}
\]

where the last inequality is consequence of \( h_1 \left( \frac{v}{\xi_t} \right) \in L^\infty(\mathbb{R}^n_+) \) and \( C \) is a constant.

Since \( v(y) \to 0 \) as \( |y| \to \infty \), then for \( \varepsilon > 0 \), we can take \( (v^t - v - \varepsilon)^+ \) as test function with compact support in \( Q_t \). Then, from (2.4) we obtain

\[
\int_{Q_t} |\nabla (v^t - v - \varepsilon)^+|^2 dy \leq C \int_{Q_t} (v^t)^{\frac{1}{n-2}} (v^t - v)(v^t - v - \varepsilon)^+ dy.
\]

Using (2.2), we obtain that the right hand of the inequality above is limited by the integral of a function that independent of \( \varepsilon \). In fact, if \( (v^t(y) - v(y) - \varepsilon)^+ > 0 \) for some \( y \in Q_t \), then \( v^t(y) > v(y) \) and

\[
(v^t)^{\frac{1}{n-2}} (v^t - v)(v^t - v - \varepsilon)^+ \leq 4(v^t)^{\frac{2}{n-2}} \in L^1(\mathbb{R}^n_+).
\]

Using Fatou’s lemma, Holder and Sobolev inequalities, and Dominate Convergence, we have

\[
\int_{Q_t} [(v^t - v)^+]^{\frac{2n}{n-2}} dy \leq \liminf_{\varepsilon \to 0} \int_{Q_t} [[(v^t - v - \varepsilon)^+]^{\frac{2n}{n-2}} dy
\]

\[
\leq \liminf_{\varepsilon \to 0} C \left( \int_{Q_t} |\nabla (v^t - v - \varepsilon)^+|^2 dy \right)^{\frac{n}{n-2}} \leq C \liminf_{\varepsilon \to 0} \left( \int_{Q_t} (v^t)^{\frac{1}{n-2}} (v^t - v)(v^t - v - \varepsilon)^+ dy \right)^{\frac{n}{n-2}} = C \left( \int_{Q_t} (v^t)^{\frac{1}{n-2}} [(v^t - v)^+]^2 dy \right)^{\frac{n}{n-2}} \leq C \left( \int_{Q_t} (v^t)^{\frac{2n}{n-2}} dy \right)^{\frac{2}{n}} \left( \int_{Q_t} [(v^t - v)^+]^{\frac{2n}{n-2}} dy \right)^{\frac{n-2}{n}}
\]
contradicting the definition of $\Lambda$.

Hence, for $t > t_1$, such that

$$\phi(t) = C \left( \int_{Q_t} (v' - v)^+ \frac{n}{n-2} dy \right), \quad (2.5)$$

where $\phi(t) = C \left( \int_{Q_t} (v' - v)^+ \frac{n}{n-2} dy \right)$. Since $v^{\frac{n}{n-2}} \in L^1(\mathbb{R}^n_+)$, then $\lim_{t \to +\infty} \phi(t) = 0$.

Thus, choosing $t_1 > 0$ large sufficiently such that $\phi(t_1) < 1$, we have from (2.5)

$$\int_{Q_t} [(v' - v)^+]^{\frac{n}{n-2}} dy = 0, \quad \text{for all } t > t_1.$$

This implies $(v' - v)^+ \equiv 0$, for $t > t_1$. Therefore $\Lambda$ is well defined, i.e. $\Lambda < +\infty$.

**Step 2.** If $\Lambda > 0$ then $v \equiv v_\Lambda$ in $Q_\Lambda$.

By definition of $\Lambda$ and continuity of the solution, we get $v \geq v_\Lambda$ and $\xi_1 > \xi_1^\Lambda$ in $Q_\Lambda$.

Suppose there is a point $y_0 \in Q_\Lambda$ such that $v(y_0) = v_\Lambda(y_0)$. Then, there is $r > 0$ sufficiently small such that

$$\frac{v}{\xi_1} < \frac{v_\Lambda}{\xi_1^\Lambda} \text{ in } B(y_0, r).$$

Hence, for $y \in B(y_0, r)$,

$$-\Delta (v(y) - v_\Lambda(y)) = h_1 \left( \frac{v}{\xi_1} \right) \frac{v^{\frac{n+2}{n-2}}}{v^{\frac{n+2}{n-2}}(y) - h_1 \left( \frac{v_\Lambda}{\xi_1^\Lambda} \right) (v_\Lambda) \frac{n+2}{n-2}$$

$$\geq h_1 \left( \frac{v_\Lambda}{\xi_1^\Lambda} \right) (v^{\frac{n+2}{n-2}}(y) - (v_\Lambda) \frac{n+2}{n-2})$$

$$\geq -C(v(y) - v_\Lambda(y)), \quad (2.6)$$

where $C$ is a non-negative constant. The last inequality is consequence of $v$, $v_\Lambda$, $h_1(v^\infty) \in L^\infty(\mathbb{R}^n)$. From Maximum Principle, we obtain $v \equiv v_\Lambda$ in $B(y_0, r)$. As the set $\{y \in Q_\Lambda; v(y) = v_\Lambda(y)\}$ is open and closed in $Q_\Lambda$, then $v \equiv v_\Lambda$ in $Q_\Lambda$.

Now, assume that $v > v_\Lambda$ in $Q_\Lambda$. We can choose a compact $K \subset Q_\Lambda$ and a number $\delta > 0$ such that $\forall t \in (\Lambda - \delta, \Lambda)$ we have $K \subset Q_t$ and

$$\phi(t) = C \left( \int_{Q_t \setminus K} (v')^{\frac{2n}{n-2}} dy \right) \frac{2n}{n-2} < \frac{1}{2}.$$ \quad (2.7)

Moreover, there exists $0 < \delta_1 < \delta$, such that

$$v > v', \text{ in } K \forall t \in (\Lambda - \delta_1, \Lambda). \quad (2.8)$$

Using (2.7) and proceeding as in Step 1, since the integrals are over $Q_t \setminus K$, we see that $(v' - v)^+ \equiv 0$ in $Q_\Lambda \setminus K$. By (2.8) we get $v > v'$ in $Q_t$ for all $t \in (\Lambda - \delta_1, \Lambda)$, contradicting the definition of $\Lambda$.

**Step 3. Conclusion.**

If $\Lambda > 0$, then

$$h_1 \left( \frac{v}{\xi_1} \right) = -\frac{\Delta v}{v^{\frac{n+2}{n-2}}} = -\frac{\Delta v_\Lambda}{v_\Lambda^{\frac{n+2}{n-2}}} = h_1 \left( \frac{v_\Lambda}{\xi_1^\Lambda} \right) = h_1 \left( \frac{v}{\xi_1} \right) < h_1 \left( \frac{v}{\xi_1} \right) \text{ in } Q_\Lambda.$$

This is a contradiction. Thus $\Lambda = 0$. By continuity of $v$, we have $v(y) \leq v(y)$ for all $y \in Q_0$. We can also perform the moving plane procedure from the left and find a corresponding $\Lambda'$. An analogue to Step 1 and Step 2 we can assume $\Lambda' = 0$. Then
we get \( v^0(y) \geq v(y) \) for \( y \in Q_0 \). This fact and the opposite inequality imply that
\( v \) is symmetric with respect to \( U_0 \). Therefore, if \( \Lambda = \Lambda' = 0 \) for all directions that are vertical to the \( y^n \) direction, then \( v \) is symmetric with respect to the axis \( y^n \).

\( \square \)

**Lemma 2.2.** Let \( u \) be a solution of (1.1). Under the assumptions of Theorem 1.1, we have that for each \( r \in [0, \pi/2] \), \( u \) is constant in
\[
A_r = \{ q \in \mathbb{S}_+^n : r = \inf\{|q - p|_{\mathbb{S}^n} : p \in \partial \mathbb{S}_+^n\}\},
\]
where \(|.|_{\mathbb{S}^n}\) is the distance in \( \mathbb{S}^n \).

**Proof.** Let \( q_1, q_2 \in A_r, r > 0 \). Then there is \( p \in \partial \mathbb{S}_+^n \) such that \( |q_1 - p|_{\mathbb{S}^n} = |q_2 - p|_{\mathbb{S}^n} \). Let \( \pi^{-1} : \mathbb{S}^n_+ \setminus \{p\} \to \mathbb{R}_+^n \) be the stereographic projection. Then \( \pi(q_1) \) and \( \pi(q_2) \) are symmetrical points with respect to the axis \( y^n \). We define \( v = \xi_1(u \circ \pi) \) in \( \mathbb{R}^n_+ \).

From Lemma 2.1 we obtain that \( v \) is symmetric with respect to the axis \( y^n \) and by definition of \( v \) and by symmetry of \( \xi_1 \), we have \( u(q_1) = u(q_2) \).

Therefore \( u \) is constant in \( A_r \) for each \( r \in (0, \pi/2] \). By continuity of \( u \), we have \( u \) is constant in \( A_0 \).

\( \square \)

**Proof. Theorem 1.1**

Let \( u \) be the solution of problem (1.1). We take an arbitrary point \( p \in \partial \mathbb{S}_+^n \), and let \( \pi^{-1} : \mathbb{S}^n_+ \setminus \{p\} \to \mathbb{R}_+^n \) be the stereographic projection. We consider the equation (2.3), where \( v = \xi_1(u \circ \pi) \) in \( \mathbb{R}^n_+ \).

We define
\[
v^*(y) = \begin{cases} v(y', y^n), & \text{if } y^n \geq 0, \\
v(y', -y^n), & \text{if } y^n < 0, \end{cases}
\]
where \( y = (y', y^n) \in \mathbb{R}^n_+ \). Then \( v^* \in C^1(\mathbb{R}^n) \) is a weak solution of problem
\[
-\Delta v^* = h \left( \frac{v^*}{\xi_1} \right) v^{\frac{n+2}{2}}, \quad v^* > 0 \text{ in } \mathbb{R}^n.
\]

We can apply Lemma 2.1 for \( v^* \) in whole space \( \mathbb{R}^n \) to get the same conclusions. This fact is due to the fact that we can apply the Maximum Principle [29, Proposition 3.7] for \( v^* \) in (2.6). Then we have that \( v^* \) is radially symmetrical. This implies
\[
v^*(y) = v(y) = C, \quad \forall y \in \mathbb{R}^n_+, |y| = 1, \tag{2.10}
\]
where \( C \) is a constant.

On the other hand, the set \( \pi(\{y \in \mathbb{R}^n_+ : |y| = 1\}) \) intersects perpendicularly to \( A_r \) for any \( r \in (0, \pi/2] \). Therefore, from (2.10) we have that \( u \) is constant in \( \pi(\{y \in \mathbb{R}^n_+ : |y| = 1\}) \) and from Lemma 2.2 we have that \( u \) is constant in \( \mathbb{S}_+^n \).

\( \square \)

Now, we consider the functions \( v_1, v_2 \) defined on \( \mathbb{R}^n_+ \) by
\[
v_1(y) = \xi_1(y)u_1(\pi(y)), \quad v_2(y) = \xi_1(y)u_2(\pi(y)),
\]
where \( \xi_1 \) is defined in (2.1) and \( (u_1, u_2) \) is the solution of (1.2). Then we have that
\[
v_1, v_2 \in L^{\frac{2n}{n+2}}(\mathbb{R}^n_+) \cap L^{\infty}(\mathbb{R}^n_+). \tag{2.11}
\]
From (1.2) and standard computations gives

\[
\begin{align*}
-\Delta v_1 &= h_{11} \left( \frac{v_1}{\xi_1(y)} \right) \left( \frac{v_2}{\xi_1(y)} \right)^{n+2} v_1^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}_+^n, \\
-\Delta v_2 &= h_{21} \left( \frac{v_1}{\xi_1(y)} \right) \left( \frac{v_2}{\xi_1(y)} \right)^{n+2} v_2^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}_+^n, \\
v_1, v_2 &> 0 \quad \text{in } \mathbb{R}_+^n, \\
\frac{\partial v_1}{\partial y^n} = \frac{\partial v_2}{\partial y^n} = 0 \quad \text{on } \partial \mathbb{R}_+^n.
\end{align*}
\]

(2.12)

To show the Theorem 1.3, we will use the same arguments that were used in the proof of the Theorem 1.1. The following results are based on [3, 17, 29].

**Lemma 2.3.** Let \((u_1, u_2)\) be a solution of (1.2). Then \(v_1 = \xi_1(u_1 \circ \pi)\) and \(v_2 = \xi_1(u_2 \circ \pi)\) are symmetric with respect to the axis \(y^n\).

**Proof.** Given \(t \in \mathbb{R}\) we set

\[
Q_t = \{ y \in \mathbb{R}_+^n; y^n < t \}; \quad U_t = \{ y \in \mathbb{R}_+^n; y^n = t \},
\]

where \(y_t := I_t(y) := (2t - y^1, y^n)\) is the image of point \(y = (y^1, y^n)\) under reflection through the hyperplane \(U_t\). We define the reflected function by \(v_t^1(y) := v_1(y_t), v_t^2 := v_2(y_t)\). The proof is carried out in three steps. In the first step we show that

\[
\Lambda := \inf \{ t > 0; v_1 \geq v_t^1, v_2 \geq v_t^2 \text{ in } Q_t, \forall \mu \geq t \}
\]
is well-defined, i.e. \(\Lambda < +\infty\). The second step consists in proving that if \(\Lambda > 0\) then \(v_1 \equiv v_1^\Lambda\) or \(v_2 \equiv v_2^\Lambda\) in \(Q_{\Lambda}\). The third step we conclude the symmetries of \(v_1\) and \(v_2\).

**Step 1.** \(\Lambda < +\infty\).

Assume that there is \(t > 0\) such that \(v_t^1(y) \geq v_1(y)\) for some \(y \in Q_t\). Since \(|y| < |y_t|\), we have, \(\frac{v_t^1}{\xi_1} < \frac{v_1^\Lambda}{\xi_1}\).

If \(v_2 > v_2^\Lambda\), then from (1.5) we have

\[
-\Delta (v_1^t - v_1) = h_{11} \left( \frac{v_t^1}{\xi_1} \right) \left( \frac{v_1^t}{\xi_1} \right)^{n+2} - h_{11} \left( \frac{v_1}{\xi_1} \right) \left( \frac{v_2}{\xi_1} \right)^{n+2}
\]

\[
\leq h_{11} \left( \frac{v_t^1}{\xi_1} \right) \left( \frac{v_1^t}{\xi_1} \right)^{n+2} - h_{11} \left( \frac{v_1}{\xi_1} \right) \left( \frac{v_2}{\xi_1} \right)^{n+2}
\]

\[
= h_{11} \left( \frac{v_t^1}{\xi_1} \right) \left( \frac{v_1^t}{\xi_1} \right)^{n+2} - h_{11} \left( \frac{v_1}{\xi_1} \right) \left( \frac{v_2}{\xi_1} \right)^{n+2} - h_{11} \left( \frac{v_t^1}{\xi_1} \right) \left( \frac{v_1^t}{\xi_1} \right)^{n+2} - h_{11} \left( \frac{v_1}{\xi_1} \right) \left( \frac{v_2}{\xi_1} \right)^{n+2}
\]

\[
\leq h_{11} \left( \frac{v_t^1}{\xi_1} \right) \left( \frac{v_1^t}{\xi_1} \right)^{n+2} - h_{11} \left( \frac{v_1}{\xi_1} \right) \left( \frac{v_2}{\xi_1} \right)^{n+2}
\]

\[
= h_{11} \left( \frac{v_t^1}{\xi_1} \right) \left( \frac{v_1^t}{\xi_1} \right)^{n+2} - h_{11} \left( \frac{v_1}{\xi_1} \right) \left( \frac{v_2}{\xi_1} \right)^{n+2} - h_{11} \left( \frac{v_t^1}{\xi_1} \right) \left( \frac{v_1^t}{\xi_1} \right)^{n+2} - h_{11} \left( \frac{v_1}{\xi_1} \right) \left( \frac{v_2}{\xi_1} \right)^{n+2}
\]

\[
\leq C(v_1^t)^{\frac{n+2}{n-2}} \max \left\{ \frac{v_t^1}{\xi_1}, \frac{v_1^t}{\xi_1} \right\} , 0 \right\} (v_1^t - v_1)
\]

\[
\leq C(v_1^t)^{\frac{n+2}{n-2}} (v_1^t - v_1),
\]

(2.13)
where $C$ is a non-negative constant.

If $v_2 > v_1$, then \( \frac{v_1 v_2}{\xi_1} > \frac{v_1 v_2}{\xi_1} \) and

\[
-\Delta (v_1 - v_1) = h_{11} \left( \frac{v_1}{\xi_1}, \frac{v_2}{\xi_1} \right) (v_1^{1/2}) \frac{\nabla v_1}{\xi_1} - h_{11} \left( \frac{v_1}{\xi_1}, \frac{v_2}{\xi_1} \right) v_1^{1/2}
\]

\[
= h_{11} \left( \frac{v_1}{\xi_1}, \frac{v_2}{\xi_1} \right) (v_1^{1/2}) \frac{\nabla v_1}{\xi_1} - h_{11} \left( \frac{v_1 v_2}{\xi_1 v_2}, \frac{v_2 v_1}{\xi_1 v_1} \right) v_1^{1/2}
\]

\[
\leq h_{11} \left( \frac{v_1}{\xi_1}, \frac{v_2}{\xi_1} \right) (v_1^{1/2}) \frac{\nabla v_1}{\xi_1} - h_{11} \left( \frac{v_1 v_2}{\xi_1 v_2}, \frac{v_2 v_1}{\xi_1 v_1} \right) (v_1^{1/2}) v_1^{1/2}
\]

\[
\leq h_{11} \left( \frac{v_1}{\xi_1}, \frac{v_2}{\xi_1} \right) (v_1^{1/2}) \frac{\nabla v_1}{\xi_1} - h_{11} \left( \frac{v_1 v_2}{\xi_1 v_2}, \frac{v_2 v_1}{\xi_1 v_1} \right) (v_1^{1/2}) v_1^{1/2}
\]

\[
= (v_2^{1/2} - v_1^{1/2}) h_{11} \left( \frac{v_1}{\xi_1}, \frac{v_2}{\xi_1} \right) \left( (v_1^{1/2}) \frac{\nabla v_1}{\xi_1} - (v_1 v_2^{1/2}) (v_1^{1/2}) v_1^{1/2} \right)
\]

\[
\leq C(v_1^{1/2}) \max \left\{ 
\right. \left( \frac{v_1}{\xi_1}, \frac{v_2}{\xi_1} \right), 0 \left. \right\} [ (v_1^{1/2} - v_1^{1/2}) + (v_1 v_2^{1/2}) (v_1^{1/2}) (v_1^{1/2}) v_1^{1/2}]
\]

\[
\leq C(v_1^{1/2}) \pi \alpha_2 [ (v_1^{1/2} - v_1^{1/2}) + (v_1 v_2^{1/2}) (v_1^{1/2}) (v_1^{1/2}) v_1^{1/2}]
\]

\[
\leq C(v_1^{1/2}) \pi \alpha_2 [ (v_1^{1/2} - v_1^{1/2}) + (v_1 v_2^{1/2}) (v_1^{1/2}) (v_1^{1/2}) v_1^{1/2}]
\]

(2.14)

where the last inequality is consequence of \((u_1 \circ \pi)^t((u_2 \circ \pi)^t)^{-1} \in L^\infty(\mathbb{R}^n)\) and $C$ is a non-negative constant. Since $v_1(y) \to 0$ as $|y| \to \infty$, then for $\epsilon > 0$, we can take $(v_1^{1/2} - v_1^{1/2})^+$ as test function with compact support in $Q_t$ for (2.13) and (2.14).

Then we obtain

\[
\int_{Q_t} \left| \nabla (v_1^{1/2} - v_1^{1/2})^+ \right|^2 \, dy \leq C \int_{Q_t} \left( v_1^{1/2} \right)^{\alpha_2} [ (v_1^{1/2} - v_1^{1/2}) + (v_1 v_2^{1/2}) ] (v_1^{1/2} - v_1^{1/2})^+ \, dy
\]

By (2.11), we obtain that the right hand of the inequality above is limited by the integral of a function independent of $\epsilon$. In fact, if $(v_1^{1/2} - v_1^{1/2})^+ > 0$ and $(v_2^{1/2} - v_2^{1/2})^+ > 0$ for some $y \in Q_t$, then $v_1(y) > v_1(y)$, $v_2(y) > v_2(y)$ and

\[
(v_1^{1/2} \pi \alpha_2) [ (v_1^{1/2} - v_1^{1/2}) + (v_1 v_2^{1/2}) ] (v_1^{1/2} - v_1^{1/2})^+ \leq 4(v_1^{1/2} \pi \alpha_2)(v_1^{1/2} + v_2^{1/2}) \in L^1(\mathbb{R}^n_+).
\]

Using Fatou’s lemma, Holder and Sobolev inequalities, and Dominate Convergence, we have

\[
\left( \int_{Q_t} [ (v_1^{1/2} - v_1^{1/2})^+ \right)^{\alpha_2} \, dy \right)^{\alpha_2}
\]

\[
\leq \liminf_{\epsilon \to 0} \left( \int_{Q_t} [ (v_1^{1/2} - v_1^{1/2})^+ \right)^{\alpha_2} \, dy \right)^{\alpha_2}
\]

\[
\leq \liminf_{\epsilon \to 0} C \int_{Q_t} \left| \nabla (v_1^{1/2} - v_1^{1/2})^+ \right|^2 \, dy
\]

\[
\leq C \liminf_{\epsilon \to 0} \int_{Q_t} \left( v_1^{1/2} \right)^{\alpha_2} [ (v_1^{1/2} - v_1^{1/2}) + (v_1 v_2^{1/2}) ] (v_1^{1/2} - v_1^{1/2})^+ \, dy
\]
\[
= C \int_{Q_t} (v_1^t)^\frac{n}{n-2} [(v_1^t - v_1) + (v_2^t - v_2)^+ (v_1^t - v_1)^+] dy
\]
\[
\leq C \int_{Q_t} (v_1^t)^\frac{4}{n} \{(v_1^t - v_1)^2 + (v_2^t - v_2)^+ (v_1^t - v_1)^+\} dy
\]
\[
\leq C\psi_1(t) \left[ \left( \int_{Q_t} [(v_1^t - v_1)^+]^{\frac{2n}{n-2}} dy \right)^{\frac{n-2}{2n}} + \left( \int_{Q_t} [(v_2^t - v_2)^+]^{\frac{2n}{n-2}} dy \right)^{\frac{n-2}{2n}} \right],
\]
where
\[
\psi_1(t)^\frac{2}{n} = \int_{Q_t} (v_1^t)^{\frac{2n}{n-2}} dy.
\]
This implies that
\[
(1 - C\psi_1(t)) \left( \int_{Q_t} [(v_1^t - v_1)^+]^{\frac{2n}{n-2}} dy \right)^{\frac{n-2}{2n}} \leq C\psi_1(t) \left( \int_{Q_t} [(v_2^t - v_2)^+]^{\frac{2n}{n-2}} dy \right)^{\frac{n-2}{2n}}. \tag{2.15}
\]
Similarly, we have
\[
(1 - C\psi_2(t)) \left( \int_{Q_t} [(v_2^t - v_2)^+]^{\frac{2n}{n-2}} dy \right)^{\frac{n-2}{2n}} \leq C\psi_2(t) \left( \int_{Q_t} [(v_1^t - v_1)^+]^{\frac{2n}{n-2}} dy \right)^{\frac{n-2}{2n}}. \tag{2.16}
\]
where
\[
\psi_2(t)^\frac{2}{n} = \int_{Q_t} (v_2^t)^{\frac{2n}{n-2}} dy.
\]
Since \(v_1, v_2 \in L^{\frac{2n}{n-2}}(\mathbb{R}^n)\), we obtain
\[
\lim_{{t \to \infty}} \psi_1(t) = \lim_{{t \to \infty}} \psi_2(t) = 0.
\]
Then, we can choose a \(t_1 \in \mathbb{R}\), such that
\[
C\psi_1(t) < \frac{1}{2} \quad \text{and} \quad C\psi_2(t) < \frac{1}{2}, \quad \text{for all} \quad t > t_1, \tag{2.17}
\]
and from (2.15) - (2.17), we have
\[
\int_{Q_t} [(v_1^t - v_1)^+]^{\frac{2n}{n-2}} dy = 0, \quad \int_{Q_t} [(v_2^t - v_2)^+]^{\frac{2n}{n-2}} dy = 0 \quad \text{for all} \quad t > t_1.
\]
Therefore, \((v_1^t - v_1)^+ \equiv 0\) and \((v_2^t - v_2)^+ \equiv 0\) in \(Q_t\) for all \(t > t_1\); and \(\Lambda\) is well defined, i.e. \(\Lambda < +\infty\).

**Step 2.** If \(\Lambda > 0\) then \(v_1 \equiv v_1^\Lambda\) or \(v_2 \equiv v_2^\Lambda\) in \(Q_{\Lambda}\).

By definition of \(\Lambda\) and continuity of solutions, we get \(v_1 \geq v_1^\Lambda\) and \(v_2 \geq v_2^\Lambda\) in \(Q_{\Lambda}\). Then
\[
\begin{cases}
-\Delta(v_1 - v_1^\Lambda) = h_{11} \left( \frac{v_1}{v_1^\Lambda} \right) v_1^{\frac{n}{n-2} - 1} \left( v_2^\Lambda \right)^{\frac{n}{n-2}} \geq 0 \quad \text{in} \quad Q_{\Lambda},
-\Delta(v_2 - v_2^\Lambda) = h_{22} \left( \frac{v_2}{v_2^\Lambda} \right) v_2^{\frac{n}{n-2} - 1} \left( v_1^\Lambda \right)^{\frac{n}{n-2}} \geq 0 \quad \text{in} \quad Q_{\Lambda},
 v_1 - v_1^\Lambda \geq 0, v_2 - v_2^\Lambda \geq 0
\end{cases}
\]
From Maximum Principle, we obtain either \( v_i \equiv v_i^A \) in \( Q_\Lambda \) for some \( i = 1, 2 \) or \( v_i > v_i^A \) in \( Q_\Lambda \) for all \( i = 1, 2 \). Suppose \( v_1 > v_1^A \) and \( v_2 > v_2^A \) in \( Q_\Lambda \). We can choose a compact \( K \subset Q_\Lambda \) and a number \( \delta > 0 \) such that \( \forall t \in (\Lambda - \delta, \Lambda) \) we have \( K \subset Q_t \) and

\[
C\psi_i(t) = \left( \int_{Q_t\setminus K} (v_i^A)^{2-} dy \right) < \frac{1}{4} \quad \text{for} \quad i = 1, 2. \tag{2.18}
\]

On the other hand, there exists \( 0 < \delta_1 < \delta \), such that

\[
v_i > v_i^1 \quad \text{and} \quad v_2 > v_2^1 \quad \text{in} \quad K, \quad \forall t \in (\Lambda - \delta_1, \Lambda). \tag{2.19}
\]

Using (2.19) and proceeding as in Step 1, since the integrals are over \( Q_t \setminus K \), we see that \( (v_i^1 - v_1^1)^+ \equiv 0 \) in \( Q_t \setminus K \) for \( i = 1, 2 \). By (2.19) we get \( v_1 > v_1^1 \) and \( v_2 > v_2^1 \) in \( Q_t \) for all \( t \in (\Lambda - \delta_1, \Lambda) \), contradicting the definition of \( \Lambda \).

**Step 3. Conclusion.**

Suppose \( \Lambda > 0 \). From Step 2 we can assume \( v_1 = v_1^A \). Then

\[
h_{11} \left( \frac{v_1}{\xi_1}, \frac{v_2}{\xi_1} \right) = -\frac{\Delta v_1}{v_1^{2-}} = -\frac{\Delta v^A_1}{(v^A_1)^{2-}} = h_{11} \left( \frac{v^A_1}{\xi_1}, \frac{v^A_1}{\xi_1} \right)
\]

\[
\leq h_{11} \left( \frac{v_1}{\xi_1}, \frac{v_2}{\xi_1} \right) < h_{11} \left( \frac{v_1^A}{\xi_1}, \frac{v_1^A}{\xi_1} \right).
\]

This is a contradiction. Thus \( \Lambda = 0 \). By continuity of \( v_1 \) and \( v_2 \), we have \( v_1^0(y) \leq v_1(y) \) and \( v_2^0(y) \leq v_2(y) \) for all \( y \in Q_0 \). We can also perform the moving plane procedure from the left and find a corresponding \( \Lambda' \). An analogue to Step 1 and Step 2 we can show that \( \Lambda' = 0 \). Then we get \( v_1^0(y) \geq v_1(y) \) and \( v_2^0(y) \geq v_1(y) \) for \( y \in Q_0 \). This fact and the above inequality imply that \( v_1 \) and \( v_2 \) are symmetric with respect to \( U_0 \). Therefore, if \( \Lambda = \Lambda' = 0 \) for all directions that are vertical to the \( y^n \) direction, then \( v_1 \) and \( v_2 \) are symmetric with respect to the axis \( y^n \).

**Lemma 2.4.** Let \( (u_1, u_2) \) be a solution of (1.2). Then for each \( r \in [0, \pi/2] \), we have that \( u_1 = u_1(r) \) and \( u_2 = u_2(r) \) in \( A_r \), where \( A_r \) is defined by (2.9).

**Proof.** The arguments for the proof are the same as the Lemma 2.2.

**Proof. Theorem 1.3**

Let \( (u_1, u_2) \) be the solution of problem (1.2). We take a \( p \in \partial \mathcal{S}^n_+ \) and let \( \pi^{-1}: \mathcal{S}^n_+ \{p\} \to \mathbb{R}^n_+ \) be the stereographic projection. We consider the problem (2.12), where \( v_1 = \xi_1 (u_1 \circ \pi) \) and \( v_2 = \xi_1 (u_2 \circ \pi) \) in \( \mathbb{R}^n_+ \). We denote \( y = (y', y^n) \in \mathbb{R}^n_+ \).

We define

\[
v_1^*(y) = \begin{cases} v_1(y', y^n) & \text{if} \quad y^n \geq 0, \\ v_1(y', -y^n) & \text{if} \quad y^n < 0, \end{cases}
\]

\[
v_2^*(y) = \begin{cases} v_2(y', y^n) & \text{if} \quad y^n \geq 0, \\ v_2(y', -y^n) & \text{if} \quad y^n < 0. \end{cases}
\]

Then, \( (v_1^*, v_2^*) \in C^1(\mathbb{R}^n) \times C^1(\mathbb{R}^n) \) is a weak solution of problem

\[
\begin{align*}
-\Delta v_1^* &= h_{11} \left( \frac{v_1^*}{\xi_1}, \frac{v_2^*}{\xi_1} \right) \quad &\text{in} \quad \mathbb{R}^n, \\
-\Delta v_2^* &= h_{21} \left( \frac{v_1^*}{\xi_1}, \frac{v_2^*}{\xi_1} \right) \quad &\text{in} \quad \mathbb{R}^n, \\
v_1^* > 0, v_2^* > 0 \quad &\text{in} \quad \mathbb{R}^n.
\end{align*}
\]
We can apply Lemma 2.3 for \((v_1^*, v_2^*)\) in whole space \(\mathbb{R}^n\) to get the same conclusions. Then we have that \(v_1^*\) and \(v_2^*\) are radially symmetrical. This implies

\[
v_1^*(y) = v_1(y) = C_1 \text{ and } v_2^*(y) = v_2(y) = C_2, \quad \forall y \in \mathbb{R}_+^n, |y| = 1, \tag{2.20}
\]

where \(C_1\) and \(C_2\) are constant.

On the other hand \(\pi(\{y \in \mathbb{R}_+^n; |y| = 1\})\) intersects perpendicularly to \(A_r\) for all \(r \in (0, \pi/2)\). Therefore, from (2.20) \(u_1\) and \(u_2\) are constant in \(\pi(\{y \in \mathbb{R}_+^n; |y| = 1\})\) and from Lemma 2.4 we have that \(u_1\) and \(u_2\) are constant in \(S^n_r\).

\[\square\]

3. Proof of Theorems 1.5 and 1.7

Let \(p\) be an arbitrary point on \(S^n\), which we will rename the north pole \(N\). Let \(\pi^{-1}: S^n \setminus \{N\} \to \mathbb{R}^n\) be the stereographic projection.

Let \(u\) be a solution of (1.7). We define

\[
v(y) = \xi_2(y) u(\pi(y)), \quad y \in \mathbb{R}^n,
\]

where \(\xi_2\) is defined by (2.1). Then we have

\[
\begin{align*}
|y|^{-2} v &\in L^2(\mathbb{R}^n \setminus B_r) \cap L^\infty(\mathbb{R}^n \setminus B_r), \\
|y|^{-2} v \Delta v &\in L^1(\mathbb{R}^n \setminus B_r) \cap L^\infty(\mathbb{R}^n \setminus B_r),
\end{align*}
\]

where \(B_r\) is any ball with center zero and radius \(r > 0\). By standard computations we have the following equation

\[
\Delta^2 v = h_2 \left( \frac{v}{\xi_2} \right) v^{\frac{n+4}{n}}, \quad v > 0 \text{ in } \mathbb{R}^n, \tag{3.2}
\]

where

\[
h_2(t) = t^{-\frac{n+4}{n}} (f(t) + d_n) t, \quad t > 0
\]

and \(d_n = n(n-4)(n^2-4)/16\).

We denote \(w_1 = v\) and \(w_2 = -\Delta w_1\). Then we have

\[
\begin{align*}
-\Delta w_1 &= w_2 & \text{in } \mathbb{R}^n, \\
-\Delta w_2 &= h_2 \left( \frac{w_1}{\xi_2} \right) w_1^{\frac{n+4}{n}} & \text{in } \mathbb{R}^n. \tag{3.3}
\end{align*}
\]

In order to show Theorem 1.5 we used the moving plane method to prove radial symmetry of solution of problem (3.2). The following results are based on [4, 30].

**Lemma 3.1.** \(w_2 = -\Delta w_1\) is non-negative in \(\mathbb{R}^n\).

**Proof.** Suppose that there exists \(y_0 \in \mathbb{R}^n\) such that \(w_2(y_0) < 0\). Without loss of generality, we assume that \(y_0 = 0\). We introduce the spherical average of a function

\[
\overline{w}(r) = \frac{1}{|S_r|} \int_{S_r} w \, d\sigma,
\]

where \(|S_r|\) is the measure of the sphere of radius \(r\). By definition of \(w_1\), we have \(\overline{w_1} \in L^\infty(\mathbb{R}^n)\) and

\[
\begin{align*}
-\Delta \overline{w_1} &= \overline{w_2}, \\
-\Delta \overline{w_2} &= h_2 \left( \frac{w_1}{\xi_2} \right) \overline{w_1}^{\frac{n+4}{n}}.
\end{align*}
\]
Since $\nabla w_2(0) = w_2(0) < 0$ and $-\Delta w_2 = h_2(\frac{w_2}{\xi_2})\frac{w_2}{\xi_2} = 0$, from Maximum Principle, we obtain for all $r > 0$,
\[
\begin{cases}
\nabla w_2(r) \leq w_2(0) < 0,
\quad -\Delta w_1(r) = w_2(r) \leq w_2(0),
\end{cases}
\]
or
\[-\frac{1}{r} \frac{d}{dr}(r \frac{d}{dr} w_1) \leq w_2(0). \tag{3.4}\]
Integrating in (3.4), we have
\[w_1(r) \geq w_1(0) - \frac{w_2(0)}{4} r^2, \text{ for all } r > 0.
\]
Since $w_2(0) < 0$, we have $w_1(r) \to +\infty$ as $r \to +\infty$. This leads to a contradiction. \hfill \Box

**Proof.** Theorem 1.5

Let $u$ be the solution of problem 1.7. We take an arbitrary point $p \in \partial \mathbb{S}^n_+$, and let $\pi^{-1} : \mathbb{S}^n \setminus \{p\} \to \mathbb{R}^n$ be the stereographic projection. We define $v = \xi_2(u \circ \pi)$ in $\mathbb{R}^n$. Given $t \in \mathbb{R}$ we set
\[Q_t = \{ y \in \mathbb{R}^n; \ y^1 < t \}; \ U_t = \{ y \in \mathbb{R}^n; \ y^1 = t \},\]
where $y_t := I_t(y) := (2t-y^1, y')$ is the image of point $y = (y^1, y')$ under of reflection through the hyperplane $U_t$. We define the reflected function by $v^t(y) := v(y_t)$. We denote $w_1 := v$ and $w_2 := -\Delta w_1$. The proof is carried out in three steps. In the first step we show that
\[\Lambda := \inf\{ t > 0; \ w_1 \geq w_1^\mu, \ w_2 \geq w_2^\mu \text{ in } Q_t, \forall \mu \geq t \}
\]
is well-defined, i.e. $\Lambda < +\infty$. The second step consists in proving that if $\Lambda > 0$ then $w_1 = w_1^\Lambda$ or $w_2 = w_2^\Lambda$ in $Q_\Lambda$. In the third step we conclude the proof.

**Step 1.** $\Lambda < +\infty$.

For $\varepsilon > 0$ and $t > 0$, we denote $W_{i,\varepsilon}^t = w_i^t - w_i - \varepsilon$ and $W_i^t = w_i^t - w_i$ for $i = 1, 2$. Then
\[
\int_{Q_t} |\nabla (W_{1,\varepsilon}^t + |y_t|^{-1})|^2 dy
\]
\[= \int_{Q_t} |\nabla (W_{1,\varepsilon}^t + |y_t|^{-2})|^2 |y_t|^{-2} dy + \int_{Q_t} 2|y_t|^{-1} \nabla (W_{1,\varepsilon}^t + |y_t|^{-1}) \cdot \nabla (|y_t|^{-1}) dy
\]
\[+ \int_{Q_t} (W_{1,\varepsilon}^t + |y_t|^{-1})^2 (\nabla (|y_t|^{-1}))^2 dy
\]
\[= \int_{Q_t} \nabla (W_{1,\varepsilon}^t) \cdot \nabla (W_{1,\varepsilon}^t + |y_t|^{-2}) dy + \int_{Q_t} (W_{1,\varepsilon}^t + |y_t|^{-1})^2 (\nabla (|y_t|^{-1}))^2 dy
\]
\[= \int_{Q_t} \nabla (W_{1,\varepsilon}^t) \cdot \nabla (W_{1,\varepsilon}^t + |y_t|^{-2}) dy + \int_{Q_t} (W_{1,\varepsilon}^t + |y_t|^{-1})^2 (\nabla (|y_t|^{-1}))^2 dy
\]
\[= \int_{Q_t} \nabla W_{1,\varepsilon}^t \cdot \nabla (W_{1,\varepsilon}^t + |y_t|^{-2}) dy + \int_{Q_t} (W_{1,\varepsilon}^t + |y_t|^{-1})^2 (\nabla (|y_t|^{-1}))^2 dy. \tag{3.5} \]
Since $w_1(y) \to 0$ as $|y| \to \infty$, then for $\varepsilon > 0$, we can take $(W_{1,\varepsilon}^t + |y_t|^{-2}$ as test function with compac support in $Q_t$ for the problem (3.3). Then, from (3.5) we
By (3.1) we can see that if \((W_{1,e}^t)^+|y_t|^{-1}\) then
\[
\int_{Q_t} |\nabla ((W_{1,e}^t)^+ |y_t|^{-1})|^2 \, dy
\]
\[
= \int_{Q_t} (w_2^t - w_2)^+ (W_{1,e}^t)^+ |y_t|^{-2} \, dy + \int_{Q_t} [(W_{1,e}^t)^+]^2 (\nabla \{ |y_t|^{-1}\})^2 \, dy
\]
\[
\leq \int_{Q_t} (W_{2}^t)^+ (W_{1,e}^t)^+ |y_t|^{-2} \int_{Q_t} [(W_{1,e}^t)^+]^2 |y_t|^{-4} \, dy
\]
\[
= I_\varepsilon + II_\varepsilon. \quad (3.6)
\]
By (3.1) we can see that if \((W_{1,e}^t)^+(y) > 0\) and \((W_{2}^t)^+(y) \geq 0\) for some \(y \in Q_t\), then \(w_1^t(y) > w_1(y)\), \(w_2^t(y) \geq w_2(y)\) and
\[
(W_{2}^t)^+ (W_{1,e}^t)^+ |y_t|^{-2} \leq 4w_1^t w_2^t |y_t|^{-2} \in L^1(\mathbb{R}^n)
\]
\[
[(W_{1,e}^t)^+]^2 |y_t|^{-4} \leq 4(w_1^t)^2 |y_t|^{-4} \in L^1(\mathbb{R}^n).
\]
Thus, by Fatou’s lemma, Sobolev’s inequality and Dominate Convergence we get
\[
\left( \int_{Q_t} [(W_{1}^t)^+ |y_t|^{-1}]^{\frac{2n}{n-2}} \, dy \right)^{\frac{n-2}{2}} \leq \liminf_{\varepsilon \to 0} \left( \int_{Q_t} [(W_{1,e}^t)^+ |y_t|^{-1}]^{\frac{2n}{n-2}} \, dy \right)^{\frac{n-2}{2}}
\]
\[
\leq C \liminf_{\varepsilon \to 0} \int_{Q_t} |\nabla ((W_{1}^t)^+ |y_t|^{-1})|^2 \, dy
\]
\[
\leq \liminf_{\varepsilon \to 0} (I_\varepsilon + II_\varepsilon) < +\infty. \quad (3.7)
\]
From Holder’s inequality we obtain
\[
I_\varepsilon \leq \left( \int_{Q_t} [(W_{2}^t)^+]^2 \, dy \right)^{\frac{1}{2}} \left( \int_{Q_t} \left[ \frac{(W_{1,e}^t)^+ |y_t|}{|y_t|^2} \right]^2 \, dy \right)^{\frac{1}{2}}
\]
\[
\leq \left( \frac{n-2}{2} \right)^2 \left( \int_{Q_t} [(W_{2}^t)^+]^2 \, dy \right)^{\frac{1}{2}} \left( \int_{Q_t} |\nabla ((W_{1,e}^t)^+ |y_t|^{-1})|^2 \, dy \right)^{\frac{1}{2}}, \quad (3.8)
\]
where we have used Hardy’s inequality,
\[
\left( \frac{n-2}{2} \right)^2 \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} \, dx \leq \int_{\mathbb{R}^n} |\nabla u|^2 \, dx, \quad u \in H^1(\mathbb{R}^n).
\]
Moreover,
\[
II_\varepsilon = \int_{Q_t} \left[ \frac{(W_{1,e}^t)^+ |y_t|}{|y_t|^2} \right]^2 \, dy \leq \left( \frac{n-2}{2} \right)^2 \int_{Q_t} |\nabla ((W_{1,e}^t)^+ |y_t|^{-1})|^2 \, dy. \quad (3.9)
\]
By (3.10), (3.8) and (3.9), we have
\[
\left( 1 - \left( \frac{n-2}{2} \right)^2 \right)^2 \int_{Q_t} |\nabla ((W_{1,e}^t)^+ |y_t|^{-1})|^2 \, dy \leq \int_{Q_t} [(W_{2}^t)^+]^2 \, dy \quad (3.10)
\]
On the other hand, for \( t > 0 \), we get \( \xi_2 > \xi_1^t \) in \( Q_t \). By conditions on \( h_2 \) we have: if \( w_1^t > w_1 \),
\[
- \Delta W_2^t = h_2 \left( \frac{w_1^t}{\xi_2^t} \right) (w_1^t)^{\frac{n+4}{n-4}} - h_2 \left( \frac{w_1}{\xi_2} \right) w_1^{\frac{n+4}{n-4}} \\
\leq h_2 \left( \frac{w_1^t}{\xi_2^t} \right) ((w_1^t)^{\frac{n+4}{n-4}} - (w_1)^{\frac{n+4}{n-4}}) \\
\leq \frac{n+4}{n-4} (w_1^t)^{\frac{n-4}{n+4}} h_2 \left( \frac{w_1^t}{\xi_2^t} \right) (w_1^t - w_1);
\]
if \( w_1^t < w_1 \),
\[
- \Delta W_2^t \leq h_2 \left( \frac{w_1^t}{\xi_2^t} \right) (w_1^t)^{\frac{n+4}{n-4}} - h_2 \left( \frac{w_1}{\xi_2} \right) w_1^{\frac{n+4}{n-4}} \\
\leq 0.
\]
Thus,
\[
- \Delta W_2^t \leq C (w_1^t)^{\frac{n+4}{n-4}} (w_1^t - w_1)^+ , \tag{3.11}
\]
where the last inequality is consequence of \( h_2 \left( \frac{w_1}{\xi_2} \right) \in L^\infty(\mathbb{R}^n) \), and \( C \) is a positive constant. Since \( w_2(y) \to 0 \) as \( |y| \to \infty \), then for \( \varepsilon > 0 \), we can take \( (W_2^{t,\varepsilon})^+ |y|^2 \) as test function with compact support in \( Q_t \) for (3.3). Thus we get
\[
\int_{Q_t} |\nabla \{(W_2^{t,\varepsilon})^+ |y|\}|^2 dy = \int_{Q_t} \nabla W_2^{t,\varepsilon} \nabla \{(W_2^{t,\varepsilon})^+ |y|\} dy + \int_{Q_t} [(W_2^{t,\varepsilon})^+]^2 dy \\
\leq C \int_{Q_t} (w_1^t)^{\frac{n+4}{n-4}} (W_1^t)^+ |y|^2 dy + \int_{Q_t} [(W_2^{t,\varepsilon})^+]^2 dy \\
= III_\varepsilon + IV_\varepsilon. \tag{3.12}
\]
From Holder, Sobolev and Hardy inequalities, \cite{[bib]}, we have
\[
III_\varepsilon \\
\leq C \left( \int_{Q_t} [(w_1^t)^{\frac{n+4}{n-4}} |y|^2]^{\frac{2n}{n+4}} dy \right)^{\frac{1}{n}} \left( \int_{Q_t} \left[ \frac{(W_1^t)^+}{|y|} \right]^{\frac{2n}{n-2}} dy \right) \left( \int_{Q_t} \left[ \frac{(W_2^{t,\varepsilon})^+}{|y|} \right]^{\frac{2n}{n-2}} dy \right) \left( \int_{Q_t} \nabla \{(W_2^{t,\varepsilon})^+ |y|\}^2 dy \right)^{\frac{1}{2}}, \tag{3.13}
\]
where
\[
\varphi(t) = C \left( \int_{Q_t} (w_1^t)^{\frac{4n}{n+4}} |y|^n dy \right)^{\frac{n}{4}} \quad \text{and} \quad \lim_{t \to 0} \varphi(t) = 0, \tag{3.14}
\]
because \( (w_1^t)^{\frac{4n}{n+4}} |y|^n \in L^1(\mathbb{R}^n) \); and from Hardy’s inequality,
\[
IV_\varepsilon \leq \left( \frac{2}{n-2} \right)^2 \int_{Q_t} |\nabla \{(W_2^{t,\varepsilon})^+ |y|\}|^2 dy. \tag{3.15}
\]
Then, by (3.12), (3.13) and (3.15), we get
\[
\begin{align*}
\int_{Q_t} |\nabla \{(W_{2,\varepsilon}^t)^+|y_t|\}|^2 dy &
\leq \varphi(t) \left( \int_{Q_t} \left[ (W_{1}^t)^+|y_t|^{n-1} \right]^{\frac{2n}{n-2}} dy \right)^{\frac{n-2}{n}} \\
& \quad + \left( \frac{2}{n-2} \right)^2 \int_{Q_t} |\nabla \{(W_{2,\varepsilon}^t)^+|y_t|\}|^2 dy,
\end{align*}
\]

hence, for all \(\varepsilon > 0\),
\[
\begin{align*}
\left( 1 - \left( \frac{2}{n-2} \right)^2 \right)^2 \left( \int_{Q_t} |\nabla \{(W_{2,\varepsilon}^t)^+|y_t|\}|^2 dy \right)
& \leq \varphi(t)^2 \left( \int_{Q_t} \left[ (W_{1}^t)^+|y_t|^{n-1} \right]^{\frac{2n}{n-2}} dy \right)^{\frac{n-2}{n}}.
\end{align*}
\]

From Fatou’s lemma, Hardy’s inequality and (3.16), we obtain
\[
\int_{Q_t} [(W_{2}^t)^+]^2 dy \leq \liminf_{\varepsilon \to 0} \int_{Q_t} [(W_{2,\varepsilon}^t)^+]^2 dy
\]
\[
\leq \left( \frac{2}{n-2} \right)^2 \liminf_{\varepsilon \to 0} \int_{Q_t} |\nabla \{(W_{1,\varepsilon}^t)^+|y_t|\}|^2 dy
\]
\[
\leq C \varphi(t)^2 \left( \int_{Q_t} \left[ (W_{1}^t)^+|y_t|^{n-1} \right]^{\frac{2n}{n-2}} dy \right)^{\frac{n-2}{n}},
\]
where \(C\) is a positive constant depending of \(n\). From Sobolev’s inequality, (3.10), (3.17) and taking \(\varepsilon \to 0\), we have
\[
\left( \int_{Q_t} [(W_{1}^t)^+|y_t|^{n-1}]^{\frac{2n}{n-2}} dy \right)^{\frac{n-2}{n}} \leq C \varphi(t)^2 \left( \int_{Q_t} \left[ (W_{1}^t)^+|y_t|^{n-1} \right]^{\frac{2n}{n-2}} dy \right)^{\frac{n-2}{n}}.
\]

Thus, choosing \(t_1\) sufficiently large such that \(\varphi(t)^2 < \frac{1}{C}\) for all \(t > t_1\), we have
\[
\int_{Q_t} [(W_{1}^t)^+|y_t|^{n-1}]^{\frac{2n}{n-2}} dy \equiv 0, \quad \text{in } Q_t \text{ for all } t > t_1.
\]

Then, \((W_{1}^t)^+ \equiv 0\) in \(Q_t\) for all \(t > t_1\), and from (3.17), gets \((W_{2}^t)^+ \equiv 0\) in \(Q_t\) for all \(t > t_1\).

Therefore, \(\Lambda\) is well-defined, i.e. \(\lambda < +\infty\).

**Step 2.** If \(\Lambda > 0\), then \(w_1 \equiv w_1^A\) or \(w_2 \equiv w_2^A\) in \(Q_\Lambda\).

By definition of \(\Lambda\) and continuity of solutions, we get \(w_1 \geq w_1^A\) and \(w_2 \geq w_2^A\) in \(Q_\Lambda\), and from (3.9) and (3.3) we have

\[
\begin{align*}
-\Delta (w_1 - w_1^A) &= w_2 - w_2^A \geq 0 \quad \text{in } Q_\Lambda, \\
-\Delta (w_2 - w_2^A) &= h_2 \left( \frac{w_1}{w_2} \right) \frac{W_{1,\varepsilon}^{\frac{2n}{n-2}}}{\varepsilon^2} - h_2 \left( \frac{w_1^A}{w_2^A} \right) \left( w_1^A \right)^{\frac{2n}{n-2}} \geq 0 \quad \text{in } Q_\Lambda, \\
w_1 - w_1^A \geq 0, \quad w_2 - w_2^A \geq 0
\end{align*}
\]

Then, from Maximum Principle we have either \(w_i \equiv w_i^A\) in \(Q_\Lambda\) for some \(i = 1, 2\) or \(w_i > w_i^A\) in \(Q_\Lambda\) for all \(i = 1, 2\). Suppose \(w_1 > w_1^A\) and \(w_2 > w_2^A\) in \(Q_\Lambda\). We can
choose a compact $K \subset Q_\Lambda$ and a number $\delta > 0$ such that $\forall t \in (\Lambda - \delta, \Lambda)$ we have $K \subset Q_t$ and

$$C \varphi(t)^2 = C \left( \int_{Q_\Lambda \setminus K} (w_1')^{\frac{n}{n-4}} |y_t|^2 dy \right)^{\frac{n}{n-4}} < \frac{1}{2}. \tag{3.18}$$

On the other hand, there exists $0 < \delta_1 < \delta$, such that

$$w_1 > w_1', \ w_2 > w_2' \text{ in } K \ \forall t \in (\Lambda - \delta_1, \Lambda). \tag{3.19}$$

Using (3.18) and proceeding as in Step 1, considering the integrals are over $Q_t \setminus K$, we see that $(w_1' - w_1)^+ \equiv 0$ in $Q_t \setminus K$. By (3.19) we get $w_1 > w_1'$ in $Q_t$ for all $t \in (\Lambda - \delta_1, \Lambda)$, contradicting the definition of $\Lambda$.

**Step 3. Conclusion**

Suppose $\Lambda > 0$. From Step 2 we can assume $w_2 \equiv w_2^\Lambda$ in $Q_\Lambda$. Then

$$h_2 \left( \frac{w_1}{\xi_2} \right) = -\frac{\Delta w_2}{w_1'} = -\frac{\Delta w_2^\Lambda}{w_1'} = h_2 \left( \frac{w_1^\Lambda}{\xi_2} \right) \left( \frac{w_1^\Lambda}{w_1} \right)^{\frac{n+4}{n-4}} \leq h_2 \left( \frac{w_1}{\xi_2} \right) < h_2 \left( \frac{w_1}{\xi_2} \right).$$

This is a contradiction.

Therefore, $\Lambda = 0$ for all directions. This implies that $w_1$ is radially symmetrical in $\mathbb{R}^n$. By definition of $w_1$, we obtain $u$ is constant on every $(n-1)$-sphere whose elements $q \in S^n$ satisfy $|q - N| = \text{constant}$. Since $p \in S^n$ is arbitrary on $S^n$, $u$ is constant.

\[ \square \]

Now, we will show the Theorem (1.7). Let $(u_1, u_2)$ be a solution of (1.8). We define the functions

$$v_1(y) = \xi_2(y) u_1(\pi(y)), \ v_2(y) = \xi_2(y) u_2(\pi(y)), \ y \in \mathbb{R}^n,$$

where $\xi_2$ is defined in (2.1). Then we have that

$$|y|^{-2} v_1, \ |y|^{-2} v_2 \in L^2(\mathbb{R}^n \setminus B_r) \cap L^\infty(\mathbb{R}^n \setminus B_r),$$

$$|y|^{-2} \Delta v_1, \ |y|^{-2} \Delta v_2 \in L^1(\mathbb{R}^n \setminus B_r) \cap L^\infty(\mathbb{R}^n \setminus B_r), \tag{3.20}$$

where $B_r$ is any ball with center zero and radius $r > 0$. By standard computations we have the following system

$$\begin{cases} 
\Delta^2 v_1 = h_{12} \left( \frac{v_1}{\xi_2(y)}, \frac{v_2}{\xi_2(y)} \right)^{\frac{n+4}{n+2}}, \ v_1 > 0 \text{ in } \mathbb{R}^n, \\
\Delta^2 v_2 = h_{22} \left( \frac{v_1}{\xi_2(y)}, \frac{v_2}{\xi_2(y)} \right)^{\frac{n+4}{n+2}}, \ v_2 > 0 \text{ in } \mathbb{R}^n, 
\end{cases} \tag{3.21}$$

where

$$h_{ij}(t_1, t_2) = t_i^{-\frac{n+4}{n+2}} (f_i(t_1, t_2) + d_n t_i), \ t_1 > 0, \ t_2 > 0 \text{ for } i = 1, 2,$$

and $d_n = n(n-4)(n^2-4)/16.$
Denote \( w_{11} = v_1, w_{12} = -\Delta w_{11}, w_{21} = v_2 \) and \( w_{22} = -\Delta w_{21} \). Then we have\(^{(3.22)}\)
\[
\begin{aligned}
-\Delta w_{11} &= w_{12}, & \text{in } & \mathbb{R}^n \\
-\Delta w_{12} &= h_{12} \left( \frac{w_{11}}{\xi_2(y)}, \frac{w_{12}}{\xi_2(y)} \right) w_{11}^{\mu+1} & \text{in } & \mathbb{R}^n \\
-\Delta w_{21} &= w_{22}, & \text{in } & \mathbb{R}^n \\
-\Delta w_{22} &= h_{22} \left( \frac{w_{11}}{\xi_2(y)}, \frac{w_{12}}{\xi_2(y)} \right) w_{21}^{\mu+1} & \text{in } & \mathbb{R}^n.
\end{aligned}
\]

In order to show Theorem 1.7 we used the moving plane method to prove radial symmetry of solution of problem \((3.21)\). The following results are based on \([5, 30]\).

**Lemma 3.2.** For \( i = 1, 2 \), we have \(-\Delta w_{11}\) are non-negative in \( \mathbb{R}^n \).

**Proof.** The arguments for the proof are the same as the Lemma 3.1. \(\square\)

**Proof. Theorem 1.7**

Let \((u_1, u_2)\) be the solution of problem \((1.8)\). We take a \( p \in \partial \mathbb{S}_t^\mu \) and let \( \pi^{-1} : \mathbb{S}^n \setminus \{p\} \rightarrow \mathbb{R}^n \) be the stereographic projection. We define \( v_1 = \xi_2(u_1 \circ \pi) \) and \( v_2 = \xi_2(u_2 \circ \pi) \) in \( \mathbb{R}^n \). Given \( t \in \mathbb{R} \) we set
\[
Q_t = \{ y \in \mathbb{R}^n; y^1 < t \}; U_t = \{ y \in \mathbb{R}^n; y^1 = t \},
\]
where \( y_t := I_t(y) := (2t - y^1, y') \) is the image of point \( y = (y^1, y') \) under of reflection through the hyperplane \( U_t \). We define the reflected function by \( v_t^i(y) := v_i(y_t) \), \( i = 1, 2 \). We denote \( w_{11} := v_1 \) and \( w_{12} := -\Delta w_{11}, i = 1, 2 \). The proof is carried out in three steps. In the first step we show that
\[
\Lambda := \inf \{ t > 0; w_{ij} \geq w_{ij}^t, \text{ in } Q_t, \forall \mu \geq t, i, j = 1, 2, \}.
\]
is well-defined, i.e. \( \Lambda < +\infty \). The second step consists in proving that if \( \Lambda > 0 \) then \( w_{ij} \equiv w_{ij}^\Lambda \) in \( Q_{\Lambda} \), for some \( i, j, 1, 2 \). In the third step we conclude the proof.

**Step 1.** \( \Lambda < +\infty \).

For \( \varepsilon > 0 \) and \( t > 0 \), we denote \( W_{ij}^t = w_{ij}^t - w_{ij} - \varepsilon \) and \( W_{ij}^t = w_{ij}^t - w_{ij} \) for \( i, j = 1, 2 \). Then we get
\[
\int_{Q_t} |\nabla \{(W_{11}^t)^+|y_t|^{-1}\}|^2 dy = \int_{Q_t} \nabla W_{11}^t \cdot \nabla \{(W_{11}^t)^+|y_t|^{-2}\} dy \\
+ \int_{Q_t} [(W_{11}^t)^+]^2 |\nabla \{|y_t|^{-1}\}|^2 dy, \quad i = 1, 2. \quad (3.23)
\]

Since \( w_{11}(y) \rightarrow 0 \) as \( |y| \rightarrow \infty \), then for \( \varepsilon > 0 \), we can take \((W_{11}^t)^+|y_t|^{-2}\) as test function with compact support in \( Q_t \) for the problem \((3.22)\). Then, from \((3.23)\), we obtain
\[
\int_{Q_t} |\nabla \{(W_{11}^t)^+|y_t|^{-1}\}|^2 dy \\
= \int_{Q_t} (w_{12} - w_{12})(W_{11}^t)^+|y_t|^{-2} dy + \int_{Q_t} [(W_{11}^t)^+]^2 |\nabla \{|y_t|^{-1}\}|^2 dy \\
\leq \int_{Q_t} (W_{12}^t)^+(W_{11}^t)^+|y_t|^{-2} dy + \int_{Q_t} [(W_{11}^t)^+]^2 |y_t|^{-4} dy \\
= I_{1,\varepsilon} + II_{1,\varepsilon}, \quad i = 1, 2. \quad (3.24)
\]
By (3.20) we can see that if $(W_{21}^i)^+(y) > 0$ and $(W_{22}^i)^+(y) > 0$ for $i = 1, 2$, then $w_{11}^i(y) > w_{11}(y), w_{12}^i(y) > w_{12}(y)$ and

$$(W_{11}^i)^+(W_{11}^c)^+|y|^2 \leq 4w_{11}^i w_{12}^i |y|^2 \in L^1(\mathbb{R}^n)$$

$$(W_{11}^i)^+|y|^4 \leq 4(w_{11}^i)^2|y|^4 \in L^1(\mathbb{R}^n).$$

Thus, by Fatou’s lemma, Sobolev’s inequality and Dominance Convergence we get

$$\left(\int_{Q_i} [(W_{11}^i)^+|y|^{-1}]^{\frac{2n}{n-2}} dy \right)^{\frac{n-2}{n}} \leq \liminf_{\varepsilon \to 0} \left(\int_{Q_i} [(W_{11}^{i,\varepsilon})^+|y|^{-1}]^{\frac{2n}{n-2}} dy \right)^{\frac{n-2}{n}}$$

$$\leq C \liminf_{\varepsilon \to 0} \int_{Q_i} |\nabla((W_{11}^{i,\varepsilon})^+|y|^{-1})|^2 dy$$

$$\leq C \liminf_{\varepsilon \to 0}(I_{i,\varepsilon} + II_{i,\varepsilon}) < +\infty \quad \text{for} \quad i = 1, 2.$$  

(3.25)

From H"older’s inequality we obtain

$$I_{i,\varepsilon} \leq \left(\int_{Q_i} [(W_{12}^i)^+|^y|^2 dy \right)^{\frac{1}{2}} \left(\int_{Q_i} \left[\frac{(W_{11}^{i,\varepsilon})^+}{|y|^2} \right]^2 dy \right)^{\frac{1}{2}}$$

$$\leq \left(\frac{2}{n-2} \right) \left(\int_{Q_i} [(W_{12}^i)^+|^y|^2 dy \right)^{\frac{1}{2}} \left(\int_{Q_i} |\nabla((W_{11}^{i,\varepsilon})^+|y|^{-1})|^2 dy \right)^{\frac{1}{2}} \quad \text{for} \quad i = 1, 2.$$  

(3.26)

where we have used Hardy’s inequality. Moreover,

$$II_{i,\varepsilon} = \int_{Q_i} \left[\frac{(W_{11}^i)^+}{|y|^2} \right]^2 dy \leq \left(\frac{2}{n-2} \right)^{2} \left(\int_{Q_i} |\nabla((W_{11}^{i,\varepsilon})^+|y|^{-1})|^2 dy \right)^{\frac{1}{2}} \quad \text{for} \quad i = 1, 2.$$  

(3.27)

By (3.24), (3.26) and (3.27), we have

$$\left(1 - \left(\frac{2}{n-2} \right)^2 \right)^{2} \int_{Q_i} |\nabla((W_{11}^{i,\varepsilon})^+|y|^{-1})|^2 dy \leq \int_{Q_i} [(W_{12}^i)^+|^y|^2 dx \quad \text{for} \quad i = 1, 2.$$  

(3.28)

On the other hand, for $t > 0$, we get $\xi_2 > \xi_2^t$ in $Q_t$. Since $h_{12}(\frac{w_{11}^t}{\xi_2^t}, \frac{w_{21}^t}{\xi_2^t}) \in L^\infty(\mathbb{R}^n)$ and by condition 1.11, we have: if $w_{11} > w_{11}^t$ and $w_{21} > w_{21}^t$, then

$$-\Delta W_{12}^t = h_{12} \left(\frac{w_{11}^t}{\xi_2^t}, \frac{w_{21}^t}{\xi_2^t} \right) \left(w_{11}^t\right)^{\frac{n+4}{n-4}} - h_{12} \left(\frac{w_{11}^t}{\xi_2^t}, \frac{w_{21}^t}{\xi_2^t} \right) \left(w_{11}^t\right)^{\frac{n+4}{n-4}}$$

$$\leq h_{12} \left(\frac{w_{11}^t}{\xi_2^t}, \frac{w_{21}^t}{\xi_2^t} \right) \left(w_{11}^t\right)^{\frac{n+4}{n-4}} - h_{12} \left(\frac{w_{11}^t}{\xi_2^t}, \frac{w_{21}^t}{\xi_2^t} \right) \left(w_{11}^t\right)^{\frac{n+4}{n-4}}$$

$$\leq h_{12} \left(\frac{w_{11}^t}{\xi_2^t}, \frac{w_{21}^t}{\xi_2^t} \right) \left(w_{11}^t\right)^{\frac{n+4}{n-4}} - h_{12} \left(\frac{w_{11}^t}{\xi_2^t}, \frac{w_{21}^t}{\xi_2^t} \right) \left(w_{11}^t\right)^{\frac{n+4}{n-4}}$$

$$\leq 0;$$  

(3.29)

if $w_{11} < w_{11}^t$ and $w_{21} > w_{21}^t$, then $\frac{w_{11}}{\xi_2} < \frac{w_{11}^t}{\xi_2}$ and

$$-\Delta W_{12}^t = h_{12} \left(\frac{w_{11}^t}{\xi_2^t}, \frac{w_{21}^t}{\xi_2^t} \right) \left(w_{11}^t\right)^{\frac{n+4}{n-4}} - h_{12} \left(\frac{w_{11}^t}{\xi_2^t}, \frac{w_{21}^t}{\xi_2^t} \right) \left(w_{11}^t\right)^{\frac{n+4}{n-4}}$$
\begin{align}
\leq h_{12} \left( \frac{w_{11}^t}{\xi_2}, \frac{w_{21}^t}{\xi_2} \right) \left( w_{11}^t \right)^{\frac{n+4}{4}} - h_{12} \left( \frac{w_{11}^t w_{21}}{\xi_2}, \xi_2 \right) \left( w_{21}^t \right)^{\frac{n+4}{4}} \\
\leq h_{12} \left( \frac{w_{11}^t}{\xi_2}, \frac{w_{21}^t}{\xi_2} \right) \left( w_{11}^t \right)^{\frac{n+4}{4}} - h_{12} \left( \frac{w_{11}^t}{\xi_2}, \frac{w_{21}}{\xi_2} \right) \left( w_{21}^t \right)^{\frac{n+4}{4}} \\
\leq h_{12} \left( \frac{w_{11}^t}{\xi_2}, \frac{w_{21}^t}{\xi_2} \right) \left( w_{11}^t \right)^{\frac{n+4}{4}} - h_{12} \left( \frac{w_{11}^t}{\xi_2}, \frac{w_{21}}{\xi_2} \right) \left( w_{21}^t \right)^{\frac{n+4}{4}} \\
\leq C(w_{11}^t)^{\frac{n}{n+4}} (w_{11}^t - w_{21}^t) \tag{3.30}
\end{align}

where $C$ is a non-negative constant; if $w_{11} > w_{11}^t$ and $w_{21} < w_{21}^t$, then $\frac{w_{11}}{\xi_2} < \frac{w_{21}^t}{\xi_2}$ and

\begin{align}
-\Delta W_{12}^t &= h_{12} \left( \frac{w_{11}^t}{\xi_2}, \frac{w_{21}^t}{\xi_2} \right) \left( w_{11}^t \right)^{\frac{n+4}{4}} - h_{12} \left( \frac{w_{11}^t w_{21}}{\xi_2}, \xi_2 \right) \left( w_{21}^t \right)^{\frac{n+4}{4}} \\
&\leq h_{12} \left( \frac{w_{11}^t}{\xi_2}, \frac{w_{21}^t}{\xi_2} \right) \left( w_{11}^t \right)^{\frac{n+4}{4}} - h_{12} \left( \frac{w_{11}^t w_{21}}{\xi_2}, \xi_2 \right) \left( w_{21}^t \right)^{\frac{n+4}{4}} \\
&\leq h_{12} \left( \frac{w_{11}^t}{\xi_2}, \frac{w_{21}^t}{\xi_2} \right) \left( w_{11}^t \right)^{\frac{n+4}{4}} - h_{12} \left( \frac{w_{11}^t}{\xi_2}, \frac{w_{21}}{\xi_2} \right) \left( w_{21}^t \right)^{\frac{n+4}{4}} \\
&\leq h_{12} \left( \frac{w_{11}^t}{\xi_2}, \frac{w_{21}^t}{\xi_2} \right) \left( w_{11}^t \right)^{\frac{n+4}{4}} - h_{12} \left( \frac{w_{11}^t}{\xi_2}, \frac{w_{21}}{\xi_2} \right) \left( w_{21}^t \right)^{\frac{n+4}{4}} \\
&\leq \left( w_{21}^t \right)^{\frac{n+4}{4}} h_{12} \left( \frac{w_{11}^t}{\xi_2}, \frac{w_{21}^t}{\xi_2} \right) \left( w_{21}^t \right)^{\frac{n+4}{4}} - \left( w_{21}^t \right)^{\frac{n+4}{4}} \\
&\leq C(w_{11}^t)^{\frac{n}{n+4}} (w_{11}^t - w_{21}^t) \tag{3.31}
\end{align}

where the last inequality is consequence of $w_{11}^t (w_{21}^t)^{-1} \in L^{\infty}(\mathbb{R}^n)$ and $C$ is a non-negative constant; if $w_{11} < w_{11}^t$ and $w_{21} < w_{21}^t$, then $\frac{w_{11}^t w_{21}}{\xi_2} < \frac{w_{11}^t w_{21}}{\xi_2}$ and

\begin{align}
-\Delta W_{12}^t &= h_{12} \left( \frac{w_{11}^t}{\xi_2}, \frac{w_{21}^t}{\xi_2} \right) \left( w_{11}^t \right)^{\frac{n+4}{4}} - h_{12} \left( \frac{w_{11}^t w_{21}}{\xi_2}, \xi_2 \right) \left( w_{21}^t \right)^{\frac{n+4}{4}} \\
&\leq h_{12} \left( \frac{w_{11}^t}{\xi_2}, \frac{w_{21}^t}{\xi_2} \right) \left( w_{11}^t \right)^{\frac{n+4}{4}} - h_{12} \left( \frac{w_{11}^t w_{21}}{\xi_2}, \xi_2 \right) \left( w_{21}^t \right)^{\frac{n+4}{4}} \\
&\leq h_{12} \left( \frac{w_{11}^t}{\xi_2}, \frac{w_{21}^t}{\xi_2} \right) \left( w_{11}^t \right)^{\frac{n+4}{4}} - h_{12} \left( \frac{w_{11}^t}{\xi_2}, \frac{w_{21}}{\xi_2} \right) \left( w_{21}^t \right)^{\frac{n+4}{4}} \\
&\leq h_{12} \left( \frac{w_{11}^t}{\xi_2}, \frac{w_{21}^t}{\xi_2} \right) \left( w_{11}^t \right)^{\frac{n+4}{4}} - h_{12} \left( \frac{w_{11}^t}{\xi_2}, \frac{w_{21}}{\xi_2} \right) \left( w_{21}^t \right)^{\frac{n+4}{4}} \\
&\leq \left( w_{21}^t \right)^{\frac{n+4}{4}} h_{12} \left( \frac{w_{11}^t}{\xi_2}, \frac{w_{21}^t}{\xi_2} \right) \left( w_{21}^t \right)^{\frac{n+4}{4}} - \left( w_{21}^t \right)^{\frac{n+4}{4}} \\
&\leq C(w_{11}^t)^{\frac{n}{n+4}} (w_{11}^t - w_{21}^t) \tag{3.32}
\end{align}

where $C$ is a non-negative constant. Thus, from \eqref{3.29}-\eqref{3.32}, gets

\begin{align}
-\Delta W_{12}^t \leq C(w_{11}^t)^{\frac{n}{n+4}} [(w_{11}^t - w_{11})^+ + (w_{21}^t - w_{21})^+], \tag{3.33}
\end{align}
and similarly,

\[- \Delta W_{22}^t \leq C(w_{21}^t)^{\frac{2}{n-2}}[(w_{11}^t - w_{11})^+ + (w_{21}^t - w_{21})^+], \tag{3.34}\]

where \(C\) is a positive constant. Since \(w_{i2}(y) \to 0\) as \(|y| \to \infty\), then we can take \((W_{i2}^t)^+ |y_i|^2\) as test function in (3.22). By (3.33) and (3.34), we have

\[
\int_{Q_t} |\nabla \{(W_{i2}^t)^+ |y_i|\}|^2 dy = \int_{Q_t} \nabla W_{i2}^t \nabla \{(W_{i2}^t)^+ |y_i|^2\} dy + \int_{Q_t} [(W_{i2}^t)^+]^2 dy
\]

\[
\leq C \int_{Q_t} w_{i1}^t \nabla \{(W_{11}^t)^+ + (W_{21}^t)^+\}[(W_{i2}^t)^+ |y_i|^2 dy
\]

\[
+ \int_{Q_t} [(W_{i2}^t)^+]^2 dy
\]

\[
= C \int_{Q_t} w_{i1}^t (W_{11}^t)^+ (W_{i2}^t)^+ |y_i|^2 dy
\]

\[
+ C \int_{Q_t} w_{i1}^t (W_{21}^t)^+ (W_{i2}^t)^+ |y_i|^2 dy + \int_{Q_t} [(W_{i2}^t)^+]^2 dy
\]

\[
= III_{i,t} + IV_{i,t} + V_{i,t} \quad \text{for} \quad i = 1, 2. \tag{3.35}\]

From Holder, Sobolev and Hardy inequalities, (3.25), we have

\[
III_{i,t}
\]

\[
\leq C \left( \int_{Q_t} \left( \frac{(w_{21}^t)^{\frac{2}{n-2}}}{|y_i|} \right)^{\frac{2}{n-2}} d\right)^{\frac{n}{2}} \left( \int_{Q_t} \left( \frac{(W_{11}^t)^+ |y_i|^2}{|y_i|} \right)^{\frac{2}{n-2}} d\right)^{\frac{1}{2}} \left( \int_{Q_t} \left( \frac{(W_{i2}^t)^+ |y_i|^2}{|y_i|} \right)^{2} d\right)^{\frac{1}{2}}
\]

\[
\leq \varphi_i(t) \left( \int_{Q_t} [(W_{11}^t)^+ |y_i|^{-1}]^{\frac{2}{n-2}} d\right)^{\frac{n-2}{2(n-2)}} \left( \int_{Q_t} \nabla \{(W_{i2}^t)^+ |y_i|\}^2 dy \right)^{\frac{1}{2}}, \tag{3.36}\]

where \(2^* := \frac{2n}{n-2}\),

\[
\varphi_i(t) = C \left( \int_{Q_t} (w_{21}^t)^{\frac{4}{n-2}} |y_i|^n d\right)^{\frac{n}{2}} \text{ and } \lim_{t \to 0} \varphi_i(t) = 0, \tag{3.37}\]

because \((w_{21}^t)^{\frac{4}{n-2}} |y_i|^n \in L^1(\mathbb{R}^n)\) for \(i = 1, 2;\)

\[
IV_{i,t} \leq \varphi_i(t) \left( \int_{Q_t} [(W_{21}^t)^+ |y_i|^{-1}]^{\frac{2}{n-2}} d\right)^{\frac{n-2}{2(n-2)}} \left( \int_{Q_t} \nabla \{(W_{i2}^t)^+ |y_i|\}^2 dy \right)^{\frac{1}{2}}
\]

\[
\tag{3.38}\]

and, from Hardy’s inequality,

\[
V_{i,t} \leq \left( \frac{2}{n-2} \right)^2 \int_{Q_t} |\nabla \{(W_{i2}^t)^+ |y_i|\}|^2 dy.
\]

\[
\tag{3.39}\]
Then, by (3.35), (3.36), (3.38) and (3.39), we get
\[
\int_{Q_t} |\nabla ((W_{i,2,2})^t + |y_t|)|^2 \, dy
\]
\[
\leq \varphi_i(t) \left( \int_{Q_t} [(W_{11})^t + |y_t|]^{\frac{2n}{n-2}} \, dy \right)^{\frac{n-2}{n}} \left( \int_{Q_t} |\nabla ((W_{i,2,2})^t + |y_t|)|^2 \, dy \right)^{\frac{1}{2}}
\]
\[
+ \varphi_i(t) \left( \int_{Q_t} [(W_{21})^t + |y_t|]^{\frac{2n}{n-2}} \, dy \right)^{\frac{n-2}{n}} \left( \int_{Q_t} |\nabla ((W_{i,2,2})^t + |y_t|)|^2 \, dy \right)^{\frac{1}{2}}
\]
\[
+ \left( \frac{2}{n-2} \right)^2 \int_{Q_t} |\nabla ((W_{i,2,2})^t + |y_t|)|^2 \, dy, \quad \text{for } i = 1, 2.
\]
Hence, for all \( \varepsilon > 0 \), \( i = 1, 2 \), we obtain
\[
\left( 1 - \left( \frac{2}{n-2} \right)^2 \right) \left( \int_{Q_t} |\nabla ((W_{i,2,2})^t + |y_t|)|^2 \, dy \right)^{\frac{1}{2}}
\]
\[
\leq \varphi_i(t) \left[ \left( \int_{Q_t} [(W_{11})^t + |y_t|]^{\frac{2n}{n-2}} \, dy \right)^{\frac{n-2}{n}} + \left( \int_{Q_t} [(W_{21})^t + |y_t|]^{\frac{2n}{n-2}} \, dy \right)^{\frac{n-2}{n}} \right].
\]
(3.40)

From Fatou’s lemma, Hardy’s inequality and (3.40), we obtain
\[
\int_{Q_t} [(W_{i,2}^t)^2] \, dy \leq \liminf_{\varepsilon \to 0} \int_{Q_t} [(W_{i,2,2}^t)^2] \, dy
\]
\[
\leq \left( \frac{2}{n-2} \right)^2 \liminf_{\varepsilon \to 0} \int_{Q_t} |\nabla ((W_{i,2,2})^t + |y_t|)|^2 \, dy
\]
\[
\leq C \varphi_i(t)^2 \left[ \left( \int_{Q_t} [(W_{11})^t + |y_t|]^{\frac{2n}{n-2}} \, dy \right)^{\frac{n-2}{n}} + \left( \int_{Q_t} [(W_{21})^t + |y_t|]^{\frac{2n}{n-2}} \, dy \right)^{\frac{n-2}{n}} \right]
\]
(3.41)

for \( i = 1, 2 \), where \( C \) is a positive constant depending of \( n \). From Fatou’s lemma, Sobolev’s inequality, (3.28), (3.41) and taking \( \varepsilon \to 0 \), we have, for \( i = 1, 2 \),
\[
\left( \int_{Q_t} [(W_{11}^t)^t + |y_t|]^{\frac{2n}{n-2}} \, dy \right)^{\frac{n-2}{n}} \leq \liminf_{\varepsilon \to 0} \left( \int_{Q_t} [(W_{11,2,2}^t)^t + |y_t|]^{\frac{2n}{n-2}} \, dy \right)^{\frac{n-2}{n}}
\]
\[
\leq C \liminf_{\varepsilon \to 0} \int_{Q_t} |\nabla ((W_{i,2,2}^t)^t) + |y_t|)|^2 \, dy
\]
\[
\leq C \varphi_i(t)^2 \left[ \left( \int_{Q_t} [(W_{11}^t)^t + |y_t|]^{\frac{2n}{n-2}} \, dy \right)^{\frac{n-2}{n}} + \left( \int_{Q_t} [(W_{21}^t)^t + |y_t|]^{\frac{2n}{n-2}} \, dy \right)^{\frac{n-2}{n}} \right].
\]

So,
\[
\left( \int_{Q_t} [(W_{11}^t)^t + |y_t|]^{\frac{2n}{n-2}} \, dy \right)^{\frac{n-2}{n}} \leq \varphi(t)^2 \left[ \left( \int_{Q_t} [(W_{11}^t)^t + |y_t|]^{\frac{2n}{n-2}} \, dy \right)^{\frac{n-2}{n}} + \left( \int_{Q_t} [(W_{21}^t)^t + |y_t|]^{\frac{2n}{n-2}} \, dy \right)^{\frac{n-2}{n}} \right],
\]
where \( \varphi(t)^2 = C(\varphi_1(t)^2 + \varphi_2(t)^2) \). We can choose \( t_1 \) sufficiently large such that \( \varphi(t)^2 < 1 \) for all \( t > t_1 \). Then
\[
\int_{Q_t} [(W_{i1}^t)^+ |y|^{-1}]^{\frac{n}{n-2}} dy \equiv 0, \text{ in } Q_t \text{ for all } t > t_1, \ i = 1, 2.
\]
Thus, \((W_{i1}^t)^+ \equiv 0 \) in \( Q_t \) for all \( t > t_1 \), and from (3.41), gets \((W_{i2}^t)^+ \equiv 0 \) in \( Q_t \) for all \( t > t_1, \ i = 1, 2 \).

Therefore, \( \Lambda \) is well-defined, i.e. \( \lambda < +\infty \).

**Step 2.** If \( \Lambda > 0 \) then \( w_{ij} \equiv w_{ij}^\Lambda \) in \( Q_{\Lambda} \) for some \( i, j = 1, 2 \).

By definition of \( \Lambda \) and continuity of solutions, we get \( w_{ij} \geq w_{ij}^\Lambda, \ i, j = 1, 2 \) in \( Q_{\Lambda} \), and from (1.11) and (3.22) we have for \( i, j = 1, 2 \):
\[
\begin{align*}
-\Delta(w_{1i} - w_{1i}^\Lambda) & = w_{12} - w_{12}^\Lambda \geq 0 \quad \text{in } Q_{\Lambda}, \\
-\Delta(w_{2j} - w_{2j}^\Lambda) & = h_{12} \left( \frac{w_{11}^\Lambda}{\xi_2^2} - \frac{w_{11}}{\xi_2^2} \right) w_{11}^{\frac{n-4}{2}} - h_{12} \left( \frac{w_{11}^\Lambda}{\xi_2^2} - \frac{w_{11}}{\xi_2^2} \right) (w_{11}^\Lambda)^{\frac{4-n}{2}} \geq 0 \quad \text{in } Q_{\Lambda}, \\
w_{ij} - w_{ij}^\Lambda & \geq 0
\end{align*}
\]

Then, from Maximum Principle we have either \( w_{ij} \equiv w_{ij}^\Lambda \) in \( Q_{\Lambda} \) for some \( i, j = 1, 2 \), or \( w_{ij} > w_{ij}^\Lambda \) in \( Q_{\Lambda} \) for all \( i, j = 1, 2 \). Suppose \( w_{ij} > w_{ij}^\Lambda \) in \( Q_{\Lambda} \). We can choose a compact \( K \subset Q_{\Lambda} \) and a number \( \delta > 0 \) such that \( \forall t \in (\Lambda - \delta, \Lambda) \) we have \( K \subset Q_t \) and
\[
C \varphi_i(t)^2 = C \left( \int_{Q_t \setminus K} (w_{i1}^t)^{\frac{n}{n-2}} |y|^{2} dy \right)^{\frac{2}{n}} < \frac{1}{2}. \quad (3.42)
\]

On the other hand, there exists \( 0 < \delta_1 < \delta \), such that
\[
w_{ij} > w_{ij}^\Lambda \text{ in } K \forall t \in (\Lambda - \delta_1, \Lambda), \ \forall i, j = 1, 2. \quad (3.43)
\]

Using (3.42) and proceeding as in Step 1, considering the integrals are over \( Q_t \setminus K \), we see that \((w_{ij} - w_{ij}^\Lambda)^+ \equiv 0 \) in \( Q_t \setminus K \) for all \( i, j = 1, 2 \). By (3.43) we get \( w_{ij} > w_{ij}^\Lambda \) in \( Q_t \) for all \( t \in (\Lambda - \delta_1, \Lambda) \), contradicting the definition of \( \Lambda \).

**Step 3. Conclusion**

Suppose \( \Lambda > 0 \). From Step 2 we can assume \( w_{12} \equiv w_{12}^\Lambda \) in \( Q_{\Lambda} \). Then
\[
h_{12} \left( \frac{w_{11}}{\xi_2^2}, \frac{w_{21}}{\xi_2^2} \right) = -\Delta w_{12}^{\frac{n}{n-2}} = -\Delta w_{12}^{\frac{n}{n-2}} = h_{12} \left( \frac{w_{11}^\Lambda}{\xi_2^2}, \frac{w_{21}^\Lambda}{\xi_2^2} \right) \left( \frac{w_{11}^\Lambda}{w_{11}} \right)^{\frac{4-n}{2}} \leq h_{12} \left( \frac{w_{11}}{\xi_2^2}, \frac{w_{21}}{\xi_2^2} \right) < h_{12} \left( \frac{w_{11}}{\xi_2^2}, \frac{w_{21}}{\xi_2^2} \right).
\]

This is a contradiction.

Therefore, \( \Lambda = 0 \) for all directions. This implies that \( w_{ij} \) is radially symmetrical in \( \mathbb{R}^n \) for \( i, j = 1, 2 \). By definition of \( v_1 = w_{11} \) and \( v_2 = w_{21} \), we obtain \( u_1 \) and \( u_2 \) are constant on every \((n-1)\)-sphere whose elements \( q \in \mathbb{S}^n \) satisfy \(|q - N| = \text{constant}\). Since \( p \in \mathbb{S}^n \) is arbitrary on \( \mathbb{S}^n \), \( u_1 \) and \( u_2 \) are constant.

\( \square \)
4. Proof of Theorem 1.9 and 1.11

Let $p$ be an arbitrary point on $\mathbb{S}^n$, which we will rename the north pole $N$. Let $\pi^{-1}: \mathbb{S}^n \setminus \{N\} \rightarrow \mathbb{R}^n$ be the stereographic projection.

The operator $P^2$ can be seen more concretely on $\mathbb{R}^n$ using stereographic projection (see for more details [9, 11]). For $u \in C^2(\mathbb{S}^n)$, we have

$$P^2_s(u)(\pi(y)) = \xi_s(y)^{\frac{n+2\tilde{s}}{n-2}}(-\Delta)^s(\xi_s(y)(u \circ \pi)(y)), \quad y \in \mathbb{R}^n,$$

where $\xi_s$ is defined in (2.1) and $(-\Delta)^s$ is the fractional Laplacian operator (see, e.g., page 2 of [22]).

Let $s \in (0, 1)$. Let $u$ be a solution of (1.15). We define

$$v(y) = \xi_s(y)u(\pi(y)), \quad y \in \mathbb{R}^n.$$ 

Then we have

$$v \in L^{2^{\frac{n}{n+2}}} (\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).$$

By (4.1), (1.13) and (1.14) gets the following equation

$$(-\Delta)^sv = h_s\left(\frac{v}{\xi_s}\right)^{\frac{n+2\tilde{s}}{n-2}}, \quad v > 0 \text{ in } \mathbb{R}^n,$$

where

$$h_s(t) = t^{-\frac{n+2\tilde{s}}{n-2}}(f(t) + d_{n,s}t), \quad t > 0 \quad \text{and} \quad d_{n,s} = \frac{\Gamma(n/2 + s)}{\Gamma(n/2 - s)}.$$

Given $t \in \mathbb{R}$ we set

$$Q_t = \{y \in \mathbb{R}^n; \quad y_1 < t\}; \quad U_t = \{y \in \mathbb{R}^n; \quad y_1 = t\},$$

where $y_1 := I_t(y) := (2t - y_1, y')$ is the image of point $y = (y_1, y')$ under of reflection through the hyperplane $U_t$. We define the reflected function by $v^t(y) := v(y_t)$. We define the following functions for $t \geq 0$ and $\varepsilon > 0$:

$$w^t_\varepsilon(y) = \begin{cases} (v^t(y) - v(y) - \varepsilon)^+, & y \in Q_t, \\
(v^t(y) - v(y) + \varepsilon)^-, & y \in Q^c_t, \end{cases} \quad \text{and} \quad w^t_\varepsilon(y) = \begin{cases} (v^t(y) - v(y))^+, & y \in Q_t, \\
(v^t(y) - v(y))^-, & y \in Q^c_t, \end{cases}$$

where $(v^t - v)^+ = \max\{v^t - v, 0\}$ and $(v^t - v)^- = \min\{v^t - v, 0\}$. Following the arguments of [5, 22], we have the following.

**Lemma 4.1.** Under the assumptions of Theorem 1.9, there exists a constant $C > 0$ such that, for $t > 0$, gets

$$\left(\int_{Q_t} |w^t_\varepsilon|^{\frac{n+2\tilde{s}}{n-2}} dy\right)^{\frac{n-2}{n+2\tilde{s}}} \leq C \int_{Q_t} (-\Delta)^s(v^t - v)(v^t - v - \varepsilon)^+ dy. \quad (4.5)$$

**Proof.** Given $t > 0$, we have

$$w^t_\varepsilon(y) = \max\{v^t(y) - v(y) - \varepsilon, 0\} = -\min\{v^t(y_t) - v(y_t) + \varepsilon, 0\} = -w^t_\varepsilon(y_t), \quad \text{for } y \in Q_t,$$

and, similarly, $w^t_\varepsilon(y) = -w^t_\varepsilon(y_t)$ for $y \in Q^c_t$. So

$$w^t_\varepsilon(y) = -w^t_\varepsilon(y_t) \quad \text{for all } y \in \mathbb{R}^n. \quad (4.6)$$

This implies

$$\int_{\mathbb{R}^n} |w^t_\varepsilon|^{\frac{n+2\tilde{s}}{n-2}} dy = \int_{Q_t} |w^t_\varepsilon|^{\frac{n+2\tilde{s}}{n-2}} dy + \int_{Q^c_t} |w^t_\varepsilon|^{\frac{n+2\tilde{s}}{n-2}} dy = 2 \int_{Q_t} |w^t_\varepsilon|^{\frac{n+2\tilde{s}}{n-2}} dy. \quad (4.7)$$
Moreover, we see that for any \( y \in Q_t \cap \text{supp}(w^v_t) \) we obtain \( w^v_t = v^t(y) - v(y) - \varepsilon \), and

\[
(-\Delta)^s w^v_t(y) - (-\Delta)^s (v^t - v - \varepsilon)(y) \\
= \int_{\mathbb{R}^n} \frac{w^v_t(y) - w^v_t(z)}{|y - z|^{n+2s}} dz - \int_{\mathbb{R}^n} \frac{(v^t - v - \varepsilon)(y) - (v^t - v - \varepsilon)(z)}{|y - z|^{n+2s}} dz \\
= \int_{\mathbb{R}^n} \frac{(v^t - v - \varepsilon)(y) - (v^t - v - \varepsilon)(z)}{|y - z|^{n+2s}} dz \\
= \int_{Q_t \cap \text{supp}(w^v_t)^c} \left( \frac{(v^t - v - \varepsilon)(y)}{|y - z|^{n+2s}} - \frac{(v^t - v - \varepsilon)(z)}{|y - z|^{n+2s}} \right) dz \\
\leq 0,
\]

where the last two integrals are finite, \( v^t - v - \varepsilon \leq 0 \) in \( Q_t \cap \text{supp}(w^v_t)^c \) and \( |y - z| < |y - z_t| \) for \( y, z \in Q_t \). Using the same arguments as in (2.4) and (2.5), we have

\[
(-\Delta)^s (v^t - v)(y) \leq C v^t(y)^{\frac{4s}{n-2s}} (v^t - v)(y) \quad \text{for} \quad v^t(y) \geq v(y), \quad y \in Q_t. \tag{4.9}
\]

From (4.8), (4.10) and (4.9), we get

\[
\int_{Q_t} (-\Delta)^s w^v_t w^v_t dy \leq \int_{Q_t} (-\Delta)^s (v^t - v - \varepsilon)(v^t - v - \varepsilon)^+ dy \\
= \int_{Q_t} (-\Delta)^s (v^t - v)(v^t - v - \varepsilon)^+ dy \tag{4.10}
\]

\[
\leq C \int_{Q_t} (v^t)^{\frac{4s}{n-2s}} (v^t - v)(v^t - v - \varepsilon)^+ dy \\
\leq 4C \int_{Q_t} (v^t)^{\frac{4s}{n-2s}} (v^t)^2 dy < \infty,
\]

where the last inequality is a consequence of (4.2). From here the following integrals are finite and, by (4.10), we obtain that

\[
\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} w^v_t|^2 dy = \int_{Q_t} |(-\Delta)^{\frac{s}{2}} w^v_t|^2 dy + \int_{Q_t} |(-\Delta)^{\frac{s}{2}} w^v_t|^2 dy = 2 \int_{Q_t} |(-\Delta)^{\frac{s}{2}} w^v_t|^2 dy. \tag{4.11}
\]

Using Sobolev’s inequality, (4.7), (4.10) and (4.11) we obtain

\[
\left( \int_{Q_t} |w^v_t|^2 \frac{4s}{n-2s} dy \right)^{\frac{n-2s}{4s}} = \left( \frac{1}{2} \int_{\mathbb{R}^n} |w^v_t|^2 \frac{4s}{n-2s} dy \right)^{\frac{n-2s}{4s}} \leq C \int_{Q_t} |(-\Delta)^{\frac{s}{2}} w^v_t|^2 dy
\]
where $U$ through the hyperplane $R$ we show that $\Lambda$ is well-defined, i.e. $\Lambda < \infty$.

This completes the proof of Lemma.

\begin{proof}

Proof. Theorem \ref{t1.9}

Let $u$ be the solution of problem \ref{1.15}. We take an arbitrary point $p \in \partial S^*_n$, and let $\pi^{-1} : \overline{S}^n \setminus \{p\} \to \mathbb{R}^n$ be the stereographic projection. We define $v = \xi_s (u \circ \pi)$ in $\mathbb{R}^n$. Given $t \in \mathbb{R}$ we set

\[ Q_t = \{ y \in \mathbb{R}^n ; \ y^1 < t \}; \ U_t = \{ y \in \mathbb{R}^n ; \ y^1 = t \}, \]

where $y_t := I_t(y) := (2t - y^1, y^1)$ is the image of point $y = (y^1, y')$ under of reflection through the hyperplane $U_t$. The proof is carried out in three steps. In the first step we show that

\[ \Lambda := \inf \{ t > 0; \ v \geq v^\mu, \ \text{in} \ Q_\mu, \forall \mu \geq t \}. \]

is well-defined, i.e. $\Lambda < +\infty$. The second step consists in proving that if $\Lambda = 0$. In the third step we conclude the proof.

\textbf{Step 1. $\Lambda < +\infty$.}

For $\varepsilon > 0$ and $t > 0$, we consider the functions $w^\varepsilon_z$ and $w^t$ defined by (4.4). Using Fatou's lemma, Lemma 4.1, (4.9), Holder and Sobolev inequalities, and Dominate Convergence, we find that

\begin{align*}
\left( \int_{Q_t} |w^t|^{\frac{2n}{n-2\varepsilon}} \, dy \right)^{\frac{n-2\varepsilon}{2}} &\leq \liminf_{\varepsilon \to 0} \left( \int_{Q_t} |w^\varepsilon_z|^{\frac{2n}{n-2\varepsilon}} \, dy \right)^{\frac{n-2\varepsilon}{2}} \\
&\leq C \liminf_{\varepsilon \to 0} \int_{Q_t} (\Delta)^s (v^t - v)(v^t - v - \varepsilon)^+ \, dy \\
&\leq C \liminf_{\varepsilon \to 0} \int_{Q_t} (v^t)^{\frac{2n}{n-2\varepsilon}} (v^t - v)(v^t - v - \varepsilon)^+ \, dy \\
&\leq C \int_{Q_t} (v^t)^{\frac{2n}{n-2\varepsilon}} [(v^t - v)^+]^2 \, dy \\
&\leq C \left( \int_{Q_t} (v^t)^{\frac{2n}{n-2\varepsilon}} \, dy \right)^{\frac{2n}{2n}} \left( \int_{Q_t} [(v^t - v)^+]^{\frac{2n}{n-2\varepsilon}} \, dy \right)^{\frac{n-2\varepsilon}{2n}} \\
&\leq \phi(t) \left( \int_{Q_t} |w^t|^{\frac{2n}{n-2\varepsilon}} \, dy \right)^{\frac{n-2\varepsilon}{2n}}, \tag{4.12}
\end{align*}

where $\phi(t) = C(\int_{Q_t} (v^t)^{\frac{2n}{n-2\varepsilon}} \, dy)^{\frac{2n}{2n}}$. Since $v^{\frac{2n}{n-2\varepsilon}} \in L^1(\mathbb{R}^n)$, then $\lim_{t \to +\infty} \phi(t) = 0$. Thus, choosing $t_1 > 0$ large sufficiently such that $\varphi(t_1) < 1$, we have from (4.12)

\[ \int_{Q_t} |w^t|^{\frac{2n}{n-2\varepsilon}} \, dy = 0, \quad \text{for all} \ t > t_1. \]
This implies \((v^t - v)^+ \equiv 0\) in \(Q_t\) for \(t > t_1\). Therefore \(\Lambda\) is well defined, i.e. \(\Lambda < +\infty\).

**Step 2.** \(\Lambda = 0\).

Assume \(\Lambda > 0\). By definition of \(\Lambda\) and continuity of the solution, we get \(v \geq v^\Lambda\) and \(\xi_s > \xi_s^\Lambda\) in \(Q_\Lambda\).

Suppose there is a point \(y_0 \in Q_\Lambda\) such that \(v(y_0) = v^\Lambda(y_0)\). Using the fact of \(h\) is decreasing, we have

\[
(-\Delta)^s v(y_0) - (-\Delta)^s v^\Lambda(y_0) = h_s \left( \frac{v(y_0)}{\xi_s(y_0)} \right) \frac{v^\Lambda(y_0)}{\xi_s^\Lambda(y_0)} v(y_0) \frac{\alpha_s^+}{\alpha_s^+ - 2} - h_s \left( \frac{v^\Lambda(y_0)}{\xi_s^\Lambda(y_0)} \right) v^\Lambda(y_0) \frac{\alpha_s^+}{\alpha_s^+ - 2}
\]

\[= \left[ h_s \left( \frac{v(y_0)}{\xi_s(y_0)} \right) \left( \frac{v^\Lambda(y_0)}{\xi_s^\Lambda(y_0)} \right) v(y_0) \right] \frac{\alpha_s^+}{\alpha_s^+ - 2} > 0. \tag{4.13} \]

On the other hand,

\[
(-\Delta)^s v(y_0) - (-\Delta)^s v^\Lambda(y_0)
\]

\[= - \int_{\mathbb{R}^n} \frac{v(z) - v(z_\Lambda)}{|y_0 - z|^{n+2s}} \, dz
\]

\[= - \int_{Q_\Lambda} \frac{v(z) - v(z_\Lambda)}{|y_0 - z|^{n+2s}} \, dz - \int_{Q_\Lambda^c} \frac{v(z) - v(z_\Lambda)}{|y_0 - z|^{n+2s}} \, dz
\]

\[= - \int_{Q_\Lambda} (v(z) - v(z_\Lambda)) \left( \frac{1}{|y_0 - z|^{n+2s}} - \frac{1}{|y_0 - z_\Lambda|^{n+2s}} \right) \, dz \leq 0,
\]

which contradicts (4.13). As a sequence, \(v > v^\Lambda\) in \(Q_\Lambda\).

We can choose a compact \(K \subset Q_\Lambda\) and a number \(\delta > 0\) such that \(\forall t \in (\Lambda - \delta, \Lambda)\) we have \(K \subset Q_t\) and

\[
\phi(t) = C \left( \int_{Q_t \setminus K} (v^t)^{\frac{2s}{n+2s}} \, dy \right) < \frac{1}{2}. \tag{4.14}
\]

On the other hand, there exists \(0 < \delta_1 < \delta\), such that

\(v > v^t\), in \(K \forall t \in (\Lambda - \delta_1, \Lambda)\). \tag{4.15}

Using (4.14), (4.15) and proceeding as in Step 1, in (4.12), since the integrals are over \(Q_t \setminus K\), we see that \((v^t - v)^+ \equiv 0\) in \(Q_t \setminus K\). By (4.15) we get \(v > v^t\) in \(Q_t\) for all \(t \in (\Lambda - \delta_1, \Lambda)\), contradicting the definition of \(\Lambda\).

**Step 3. Conclusion.**

By Step 2 we have \(\Lambda = 0\) for all directions. This implies that \(v\) is radially symmetrical in \(\mathbb{R}^n\). By definition of \(v\), we obtain \(u\) is constant on every \((n - 1)\)-sphere whose elements \(q \in \mathbb{S}^n\) satisfy \(|q - N| = \text{constant}\). Since \(p \in \mathbb{S}^n\) is arbitrary on \(\mathbb{S}^n\), \(u\) is constant.

\[\square\]

Now, we will show the Theorem [1.11] Let \((u_1, u_2)\) be a solution of (1.10). We define

\[v_1(y) = \xi_s(y)u_1(\pi(y)), \quad v_2(y) = \xi_s(y)u_2(\pi(y)),\]

where \(\xi_s\) is defined in (2.1). Then we have that

\[v_1, \ v_2 \in L^{\frac{2s}{n+2s}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n). \tag{4.16}\]
Proof. By (4.1), (1.13) and (1.14) gets the following equation
\[
\begin{cases}
(-\Delta)^s v_1 = h_{1s} \left( \frac{v_1}{\xi_s(y)}, \frac{v_2}{\xi_s(y)} \right) v_1^{-\frac{n+2s}{2}}, & v_1 > 0 \text{ in } \mathbb{R}^n, \\
(-\Delta)^s v_2 = h_{2s} \left( \frac{v_1}{\xi_s(y)}, \frac{v_2}{\xi_s(y)} \right) v_2^{-\frac{n+2s}{2}}, & v_2 > 0 \text{ in } \mathbb{R}^n,
\end{cases}
\tag{4.17}
\]
where
\[h_{is}(t) = t^{-\frac{n+2s}{2}}(f(t) + d_{n,s}t), \quad t > 0, \quad i = 1, 2, \text{ and } d_{n,s} = \frac{\Gamma(\frac{n}{2} + s)}{\Gamma(\frac{n}{2} - s)}.
\]
We define the following functions for \( t \geq 0, i = 1, 2: \)
\[
w_{i,\varepsilon}^t(y) = \begin{cases}
(v_i^t(y) - v_i(y) - \varepsilon)^+, \quad y \in Q_t, \\
(v_i^t(y) - v_i(y) + \varepsilon)^-, \quad y \in Q_t^c.
\end{cases}
\tag{4.18}
\]
We have the following lemma.

**Lemma 4.2.** Under the assumptions of Theorem [3,4] there exists a constant \( C > 0 \) such that, for \( t > 0 \) and \( \varepsilon > 0 \), gets
\[
\left( \int_{Q_t} |w_{i,\varepsilon}^t|^\frac{2s}{n-2s} dy \right)^\frac{n-2s}{2s} \leq C \int_{Q_t} (-\Delta)^s(v_i^t - v_i)(v_i^t - v_i - \varepsilon)^+ dy, \quad i = 1, 2. \tag{4.19}
\]

**Proof.** Given \( t > 0, \varepsilon > 0 \) and proceeding as (4.16), (4.17) and (4.18) we have, for \( i = 1, 2, \)
\[
w_{i,\varepsilon}(y) = -w_{i,\varepsilon}(y_t) \quad \text{for all } y \in \mathbb{R}^n, \quad \text{and} \quad \int_{\mathbb{R}^n} |w_{i,\varepsilon}^t|^\frac{2s}{n-2s} dy = 2 \int_{Q_t} |w_{i,\varepsilon}^t|^\frac{2s}{n-2s} dy,
\tag{4.20}
\]
and
\[
(-\Delta)^s w_{i,\varepsilon}^t(y) \leq (-\Delta)^s(v_i^t - v_i - \varepsilon)(y), \quad y \in Q_t \cap \text{supp}(w_{i,\varepsilon}). \tag{4.21}
\]
Using the same arguments as in (2.13) and (2.14), we have
\[
(-\Delta)^s(v_i^t - v_i)(y) \leq C v_i^t(y) \left( \int \frac{v_i^t}{v_2} dy \right)^\frac{n-2s}{2s} \left[ (v_i^t - v_1) + (v_2^t - v_2 - \varepsilon) \right] (v_i^t - v_i)(y), \tag{4.22}
\]
for \( v_i^t(y) \geq v_i(y) \), \( y \in Q_t \), \( i = 1, 2 \). From (4.21), (4.17) and (4.22), we get
\[
\int_{Q_t} (-\Delta)^s v_i^t w_{i,\varepsilon}^t dy \leq \int_{Q_t} (-\Delta)^s (v_i^t - v_i)(v_i^t - v_i - \varepsilon)^+ dy \tag{4.23}
\]
\[
\leq C \int_{Q_t} \left( v_i^t \left[ (v_i^t - v_1)^+ + (v_2^t - v_2)^+ \right] (v_i^t - v_i - \varepsilon)^+ dy.
\]
By (4.16), we obtain that the right hand of the inequality above is limited by the integral of a function independent of \( \varepsilon \). In fact, if \( (v_i^t(y) - v_i(y) - \varepsilon)^+ > 0 \) and \( (v_2^t(y) - v_2(y) + \varepsilon)^+ > 0 \) for some \( y \in Q_t \), then \( v_i^t(y) > v_i(y) \), \( v_2^t(y) > v_2(y) \) and
\[
(v_i^t)^{\frac{n-2s}{2s}}(v_i^t - v_1)^+ + (v_2^t - v_2)^+ (v_i^t - v_1 - \varepsilon)^+ \leq 4(v_i^t)^{\frac{n+2s}{2s}} (v_i^t + v_2^t) \in L^1(\mathbb{R}^n), \quad i = 1, 2.
\]
From here the following integrals are finite and, by (4.20), we obtain that
\[
\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} w_{i,\varepsilon}^t|^2 dy = 2 \int_{Q_t} |(-\Delta)^{\frac{s}{2}} w_{i,\varepsilon}^t|^2 dy, \quad i = 1, 2. \tag{4.24}
\]
Using Sobolev’s inequality, (1.20), (1.23) and (1.24) we obtain
\[
\left( \int_{Q_t} |w_{t,c}^i|^{\frac{2n}{n-2\sigma}} dy \right)^{\frac{n-2\sigma}{n}} = \left( \frac{1}{2} \int_{\mathbb{R}^n} |w_{t,c}^i|^{\frac{2n}{n-2\sigma}} dy \right)^{\frac{n-2\sigma}{n}} \leq C \int_{Q_t} |(-\Delta)^{\frac{\sigma}{2}} w_{t,c}^i|^2 dy \\
= \frac{C}{2} \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\sigma}{2}} w_{t,c}^i|^2 dy = \frac{C}{2} \int_{\mathbb{R}^n} (-\Delta)^{\frac{\sigma}{2}} w_{t,c}^i w_{t,c}^i dy \\
= C \int_{Q_t} (-\Delta)^{\frac{\sigma}{2}} w_{t,c}^i w_{t,c}^i dy \\
\leq C \int_{Q_t} (-\Delta)^{\frac{\sigma}{2}} (v_t^i - v_1)(v_t^i - v_1 - \varepsilon)^+ dy, \quad \text{for } i = 1, 2.
\]

This completes the proof of Lemma.

\[\square\]

**Proof. Theorem 1.11**

Let \((u_1, u_2)\) be the solution of problem (1.10). We take a \(p \in \partial B^n\) and let \(\pi^{-1} : \mathbb{R}^n \setminus \{p\} \rightarrow \mathbb{R}^n\) be the stereographic projection. We define \(v_1 = \xi_s(u_1 \circ \pi)\) and \(v_2 = \xi_s(u_2 \circ \pi)\) in \(\mathbb{R}^n\). Given \(t \in \mathbb{R}\) we set

\[Q_t = \{y \in \mathbb{R}^n; \ y^1 < t\}; \ U_t = \{y \in \mathbb{R}^n; \ y^1 = t\},\]

where \(y_t := I_t(y) := (2t - y^1, y')\) is the image of point \(y = (y^1, y')\) under of reflection through the hyperplane \(U_t\). The proof is carried out in three steps. In the first step we show that

\[\Lambda := \inf\{t > 0; \ v_i \geq v_\mu, \ \text{in } Q_\mu, \forall \mu \geq t, \ i = 1, 2\}.\]

is well-defined, i.e. \(\Lambda < +\infty\). The second step consists in proving that \(\Lambda = 0\). In the third step we conclude the proof.

**Step 1.** \(\Lambda < +\infty\).

For \(\varepsilon > 0\) and \(t > 0\) we consider the functions \(w_{t,c}^i\) and \(w_t^i\) defined by (1.18). Using Fatou’s lemma, Lemma 1.12, 1.22, Holder and Sobolev inequalities, and Dominate Convergence, we have that

\[
\left( \int_{Q_t} |w_t^i|^\frac{2n}{n-2\sigma} dy \right)^{\frac{n-2\sigma}{n}} \leq \liminf_{\varepsilon \to 0} \left( \int_{Q_t} |w_{t,c}^i|^{\frac{2n}{n-2\sigma}} dy \right)^{\frac{n-2\sigma}{n}} \\
\leq C \liminf_{\varepsilon \to 0} \int_{Q_t} (-\Delta)^{\frac{\sigma}{2}} (v_t^i - v_1)(v_t^i - v_1 - \varepsilon)^+ dy \\
\leq C \liminf_{\varepsilon \to 0} \int_{Q_t} (v_t^i)^{\frac{4\sigma}{n-2\sigma}} (v_t^i - v_1)(v_t^i - v_1 - \varepsilon)^+ dy \\
+ C \liminf_{\varepsilon \to 0} \int_{Q_t} (v_t^i)^{\frac{4\sigma}{n-2\sigma}} (v_t^i - v_2)(v_t^i - v_1 - \varepsilon)^+ dy \\
\leq C \int_{Q_t} (v_t^i)^{\frac{4\sigma}{n-2\sigma}} [(v_t^i - v_1)^{+}]^2 dy \\
+ C \int_{Q_t} (v_t^i)^{\frac{4\sigma}{n-2\sigma}} (v_t^i - v_2)^+ (v_t^i - v_1)^+ dy \\
\leq C \left( \int_{Q_t} (v_t^i)^{\frac{2n}{n-2\sigma}} dy \right)^{\frac{2n}{n}} \left( \int_{Q_t} [(v_t^i - 1)^+]^{\frac{2n}{n-2\sigma}} dy \right)^{\frac{n-2\sigma}{n}}
\]
On the other hand, choosing \( t_1 > 0 \) large sufficiently such that \( \phi_i(t_1) < 1/4 \) for \( i = 1, 2 \), we have from (4.25) and (4.26)

\[
\int_{Q_t} |w_i^1|^{\frac{2n}{n-2s}} dy = 0, \quad \text{for all } t > t_1, \ i = 1, 2.
\]

This implies \((v_i^1 - v_i)^+ \equiv 0\) in \( Q_t \) for \( t > t_1 \) and \( i = 1, 2 \). Therefore \( \Lambda \) is well defined, i.e. \( \Lambda < +\infty \).

**Step 2.** \( \Lambda = 0 \).

Assume \( \Lambda > 0 \). By definition of \( \Lambda \) and continuity of the solution, we get \( v_i \geq v_i^\Lambda \) in \( Q_\Lambda \) for \( i = 1, 2 \).

Suppose there is a point \( y_0 \in Q_\Lambda \) such that \( v_1(y_0) = v_2^\Lambda(y_0) \). Using the conditions of \( h \), we have

\[
(-\Delta)^sv_1(y_0) - (-\Delta)^sv_2^\Lambda(y_0)
= h_{1s} \left( \frac{v_1(y_0)}{\xi_s(y_0)} - \frac{v_2(y_0)}{\xi_s(y_0)} \right) v_1(y_0)^{\frac{2s}{n}} - h_{1s} \left( \frac{v_1^\Lambda(y_0)}{\xi_s^\Lambda(y_0)} - \frac{v_2^\Lambda(y_0)}{\xi_s^\Lambda(y_0)} \right) v_1^\Lambda(y_0)^{\frac{2s}{n}}
\]

\[
\geq h_{1s} \left( \frac{v_1(y_0)}{\xi_s(y_0)} - \frac{v_2(y_0)}{\xi_s(y_0)} \right) v_1(y_0)^{\frac{2s}{n}} > 0.
\]

On the other hand,

\[
(-\Delta)^sv_1(y_0) - (-\Delta)^sv_2^\Lambda(y_0)
= -\int_{Q_\Lambda} (v_1(z) - v_2(z)) \left( \frac{1}{y_0 - z}^{n+2s} - \frac{1}{y_0 - z_{\Lambda}}^{n+2s} \right) dz \leq 0,
\]

which contradicts (4.27). As a sequence, \( v_1 \geq v_2^\Lambda \) in \( Q_\Lambda \).

Similarly, we have that \( v_2 > v_2^\Lambda \) in \( Q_\Lambda \). We can choose a compact \( K \subset Q_\Lambda \) and a number \( \delta > 0 \) such that \( \forall t \in (\Lambda - \delta, \Lambda) \) we have \( K \subset Q_t \) and

\[
\phi_i(t) = C \left( \int_{Q_t \setminus K} (v_i^t)^{\frac{2n}{n-2s}} dy \right)^{\frac{2n}{2n}} < \frac{1}{2}, \ i = 1, 2.
\]
On the other hand, there exists $0 < \delta_1 < \delta$, such that
\[ v_1 > v_1^t \quad \text{and} \quad v_2 > v_2^t, \quad \text{in} \ K \forall t \in (\Lambda - \delta_1, \Lambda). \] (4.29)

Using (4.28), (4.29) and proceeding as in Step 1, in (4.25), since the integrals are over $Q_t \setminus K$, we see that $(v^t - v)^+ \equiv 0$ in $Q_t \setminus K$. By (4.29) we get $v_i > v_i^t$ in $Q_i$ for all $t \in (\Lambda - \delta_1, \Lambda)$ and $i = 1, 2$, contradicting the definition of $\Lambda$.

**Step 3. Conclusion.**

By Step 2 we have that $\Lambda = 0$ for all directions. This implies that $v$ is radially symmetrical in $\mathbb{R}^n$. By definition of $v$, we obtain $u$ is constant on every $(n - 1)$-sphere whose elements $q \in S^n$ satisfy $|q - N| = \text{constant}$. Since $p \in S^n$ is arbitrary on $S^n$, $u$ is constant. \(\square\)

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