PROPERTIES OF GENERALIZED BERWALD CONNECTIONS

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ABSTRACT. Recently the present authors introduced a general class of Finsler connections which leads to a smart representation of connection theory in Finsler geometry and yields to a classification of Finsler connections into the three classes. Here the properties of one of these classes namely the Berwald-type connections which contains Berwald and Chern(Rund) connections as a special case is studied. It is proved among the other that the hv-curvature of these connections vanishes if and only if the Finsler space is a Berwald one. Some applications of this connection is discussed.

1. Introduction

Always there were a hope to find a solution for some of the unsolved problems by developing a connection theory, thus it is useful to introduce new connections in Finsler geometry. As it is mentioned in [12], the study of hv-curvature of Finsler connections is of urgent necessity for the Finsler geometry as well as for theoretical physics. Similarly as another application of Finsler connections in physics one can mention an example in Relativistic field theory. In this theory different connections

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have been defined in Finsler geometry where the connections, torsions, or curvatures can be related to fields which might be identified as electromagnetic or Yang-Mills fields. In this relation, one can refer to [2], [5] and [11].

Historically, in Riemannian geometry, the connection of choice was that constructed by Levi-Civita, which has two remarkable attributes; metric-compatibility and torsion-freeness. In 1926, L. Berwald [5] introduced a connection and two curvature tensors. The Berwald connection is torsion-free, but is not necessarily metric-compatible. It was Berwald who first successfully extended the notion of Riemann curvature to Finsler spaces. He also introduced a notion of non-Riemannian quantity called Berwald curvature. From this point of view, Berwald is the founder of differential geometry of Finsler spaces [14]. Next, Cartan in 1934 has found locally the coefficients of a metric-compatible and h-torsion free connection, called later, Cartan connection. The global construction of this connection is given in a remarkable work of Akbar-Zadeh in 1967 [1]. Other progress came in 1948, when the Chern (Rund) connection was defined. In 1943 Chern studied the equivalence problem for Finsler spaces using the Cartan exterior differentiation method [8]. Chern came back to his connection in 1993, in a joint paper with Bao [3] and shows its usefulness in treating global problems in Finsler geometry. The Berwald and Chern connections also fail slightly but expectedly, to be metric-compatible. The Chern connection has a simpler form, while the Berwald connection affects a leaner hh-curvature for spaces of constant flag curvature. Indeed the Berwald connection is particularly convenient when dealing with Finsler spaces of constant flag curvature. It is most directly related to the nonlinear connection coefficients and most amenable to the study of the geometry of paths. These connections (Berwald and Chern) coincide when the underlying Finsler structure is of Landsberg type. They further reduce to a linear connection on $M$, when the Finsler structure is of Berwald type [9], [10].

Recently the present authors have defined a general class of Finsler connections which leads to a general representation of some Finsler connections in Finsler geometry and yields to a classification of Finsler connections into the three classes, namely, Berwald-type, Cartan-type and Shen-type connections [6]. In the present work we study the properties of the former connection which contains Berwald and Chern(Rund) connections as special cases and is the most general connection of this kind. We prove in continuation of Berwald’s and Chern’s works that the
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hv-curvature of the Berwald-type connection characterizes the Berwald structure.

A distinguished property of the introduced connection is its adaptive form for different applications. In fact one can use a suitable special case of this connection to find a geometric interpretation for solutions of certain differential equations formed by Cartan tensor and its derivatives. For example in section 4 we prove: Let \((M, F)\) be a complete Finsler manifold with bounded Landsberg tensor. Then \(F\) is a Landsberg metric if and only if its hv-curvature \(P_{jkl}\) vanishes.

2. Preliminaries

Let \(M\) be a n-dimensional \(C^\infty\) manifold. For a point \(x \in M\), denoted by \(T_x M\) the tangent space of \(M\) at \(x\). The tangent bundle of \(M\) is the union of tangent spaces. \(TM := \bigcup_{x \in M} T_x M\). We will denote elements in \(TM\) by \((x, y)\) if \(y \in T_x M\). Let \(TM_0 = TM \setminus \{0\}\). The natural projection \(\pi : TM \to M\) is given by \(\pi(x, y) := x\).

Throughout this paper, we use Einstein summation convention for expressions with indices.\(^1\)

A Finsler structure on a manifold \(M\) is a function \(F : TM \to [0, \infty)\) with the following properties:

(i) \(F\) is \(C^\infty\) on \(TM_0\).
(ii) \(F\) is positively 1-homogeneous on the fibers of tangent bundle \(TM\):

\[
\forall \lambda > 0 \quad F(x, \lambda y) = \lambda F(x, y).
\]

(iii) The Hessian of \(F^2\) with elements \(g_{ij}(x, y) := \frac{1}{2}[F^2(x, y)]_{y_i y_j}\) is positively defined on \(TM_0\).

Then the pair \((M, F)\) is called a Finsler manifold. \(F\) is Riemannian if \(g_{ij}(x, y)\) are independent of \(y \neq 0\).

Nonlinear connection.

Let us consider the tangent bundle \((TM, \pi, M)\) of the manifold \(M\). The tangent bundle of the manifold \(TM\) is \((TTM, \pi_*, TM)\), where \(\pi_*\) is the tangent mapping of the projection \(\pi\). A tangent vector field on \(TM\) can

\(^1\)That is where ever an index is appeared twice as a subscript as well as a superscript, then that term is assumed to be summed over all values of that index.
be represented in the local natural frame \( \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i} \right) \) on \( TM \) by
\[
\tilde{X} = X^i(x, y)\frac{\partial}{\partial x^i} + Y^i(x, y)\frac{\partial}{\partial y^i}.
\]
It can be written in the form \( \tilde{X} = (x, y, X^i, Y^i) \) or, shorter, \( \tilde{X} = (x, y, X, Y) \). The mapping \( \pi_* : TT M \to TM \) has the local form
\[
\pi_*(x, y, X, Y) = (x, y).
\]
Put \( VTM := \ker \pi_* = \text{span}\{ \frac{\partial}{\partial y^i} \}_{i=1}^n \). \( VTM \) is an \( n \)-dimensional sub-bundle of \( T(TM_0) \), whose fiber \( V_uTM \) at \( v \) is just the tangent space \( T_v(T_xM) \subset T_v(TM_0) \). \( VTM \) is called the **vertical tangent bundle** of \( TM_0 \).

We can write the vertical subbundle as \( (VTM, \pi_{VTM}, TM) \). Its fibres are the linear vertical spaces \( V_uTM, u \in TM \). The points of submanifold \( VTM \) are of the form \( (x, y, 0, Y) \). Hence, the fibers \( V_uTM \) of the vertical bundle are isomorphic to the real vector space \( \mathbb{R}^n \).

Let us consider the pullback tangent bundle \( \pi^*TM \) defined as follows (\( \mathbb{R}^n \)):
\[
\pi^*TM = \{(u, v) \in TM \times TM | \pi(u) = \pi(v)\}.
\]
Take a local coordinate system \( \{x^i\} \) in \( M \). The local natural frame \( \{\frac{\partial}{\partial x^i}\} \) for \( T_xM \) determines a local natural frame \( \partial_i \) for \( \pi^*TM, \partial_i|_y := (y, \frac{\partial}{\partial x^i}|_x) \), \( y \in T_xM \). This gives rise to a linear isomorphism between \( \pi^*TM|_y \) and \( T_yM \) for every \( y \in T_xM \). There is a canonical section \( \ell \) of \( \pi^*TM \) defined by \( \ell = \ell^i\partial_i \), where \( \ell^i = y^i/F(x, y) \).

The fibers of \( \pi^*TM \), i.e., \( \pi_u^*TM \) are isomorphic to \( T_{\pi(u)}M \). One can define the following morphism of vector bundle \( \rho : TT M \to \pi^*TM \), \( \rho(X_u) = (u, \pi_*(\tilde{X}_u)) \). It follows that
\[
\ker \rho = \ker \pi_* = VTM.
\]
By means of these consideration one can see without any difficulties that the following sequence is exact
\[
0 \to VTM \xrightarrow{i} TT M \xrightarrow{\rho} \pi^*TM \to 0,
\]
where \( i \) is natural inclusion map.

A **nonlinear connection** on the manifold \( TM \) is a left splitting of the exact sequence (1.1). Therefore, a nonlinear connection on \( TM \) is a vector bundle morphism \( C : TT M \to VTM \), with the property \( C \circ i =
\]
1. The kernel of the morphism $C$ is a vector bundle of the tangent bundle $(TTM, \pi_*, TM)$, denoted by $(HTM, \pi_{HTM}, TM)$ and called the horizontal subbundle. Its fibres $H_uTM$ determine a distribution $u \in TM \to H_uTM \subset T_uTM$, supplementary to the vertical distribution $u \in TM \to V_uTM \subset T_uTM$. Therefore, a nonlinear connection $N$ induces the following Whitney sum:

\[(2.2) \quad TTM = HTM \oplus VTM.\]

Let we put

\[(2.3) \quad \frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j},\]

where the above $N^j_i$ are the components of $N$ and are known in the trade as the nonlinear connection coefficients on $TM_0$.

Restriction of the morphism $\rho : TTM \to \pi^*TM$ to the $HTM$ is an isomorphism of vector bundles, for which we have

\[(2.4) \quad \rho(\frac{\partial}{\partial x^i}) = \partial_i, \quad \rho(\frac{\partial}{\partial y^j}) = 0.\]

Let $\nabla$ be a linear connection on $\pi^*TM$, $\nabla : \chi(TM_0) \times \pi^*TM \to \pi^*TM$ such that $\nabla : (\hat{X}, Y) \to \nabla \hat{X}Y$. A Finsler connection is a pair of a linear connection $\nabla$, and a nonlinear connection $N$.

Given a Finsler metric $F$ on $M$, $F(y) = F(y^i \frac{\partial}{\partial x^i}|_x)$ is a function of $(y^i) \in \mathbb{R}^n$ at each point $x \in M$. Finsler metric $F$ defines a fundamental tensor $g : \pi^*TM \otimes \pi^*TM \to [0, \infty)$ by the formula $g(\partial_i|_v, \partial_j|_v) = g_{ij}(x, y)$, where $v = y^i \frac{\partial}{\partial x^i}|_x$ and $g_{ij}$ are defined in the definition of Finsler structure. Then $(\pi^*TM, g)$ becomes a Riemannian vector bundle over $TM_0$. Let

\[A_{ijk}(x, y) = \frac{1}{2} F(x, y) [F^2(x, y)] y^i y^j y^k.\]

Clearly, $A_{ijk}$ is symmetric with respect to $i, j, k$. The Cartan tensor $A : \pi^*TM \otimes \pi^*TM \otimes \pi^*TM \to \mathbb{R}$ is defined by $A(\partial_i|_v, \partial_j|_v, \partial_k|_v) = A_{ijk}(x, y)$. In some literatures $C_{ijk} = A_{ijk}/F$ is called Cartan tensor. Riemannian manifolds are characterized by $A \equiv 0$. $F$ is positively homogenous of degree 1 on $M$ then by the Euler’s theorem we see that $y^i F_{y^i} = F$ and then $y^i F_{y^i y^j} = 0$ using this the canonical section $\ell$ satisfies:

\[g(\ell, \ell) = 1 \quad , \quad A(X, Y, \ell) = 0,\]
where the second equation is equivalent to $A(X, Y, \frac{\partial}{\partial x^i}) = \ell_i A(X, Y, \frac{\partial}{\partial x^i}) = 0$. Let $\tilde{\ell}$ denote the unique vector field in $HTM$ such that $\rho(\tilde{\ell}) = \ell$. We call $\tilde{\ell}$ a geodesic or spray field on $TM_0$.

Let $\nabla$ be the Berwald (or Chern) connection, by means of $\nabla$, the tensor $\dot{A}$ is defined by $\dot{A}$:

$$
\dot{A}(X, Y, Z) := \tilde{\ell} A(X, Y, Z) - A(\nabla_{\tilde{\ell}} X, Y, Z) - A(X, \nabla_{\tilde{\ell}} Y, Z) - A(X, Y, \nabla_{\tilde{\ell}} Z).
$$

Putting $A_{ijk} = \dot{A}_{ijk}$, $\ddot{A}_{ijk} = \dddot{A}_{ijk}, \forall m \in \mathbb{N}$ we define $^{m+1}A$ as follow

$$
^{m+1}A(X, Y, Z) := \tilde{\ell} A(X, Y, Z) - m A(\nabla_{\tilde{\ell}} X, Y, Z) - m A(X, \nabla_{\tilde{\ell}} Y, Z) - m A(X, Y, \nabla_{\tilde{\ell}} Z).
$$

Obviously, $\forall m \in \mathbb{N}$, the tensors $A_{ijk}$ are symmetric with respect to three indices. Moreover, using $\nabla_{\tilde{\ell}} \tilde{\ell} = 0$ we have $^{m}A(X, Y, \ell) = 0, \forall m \in \mathbb{N}$. $A$ and $\dot{A}$ are basic tensors in Finsler geometry. In the Riemannian case, both of them vanish. Therefore by the above definition we know that in the Riemannian case $\forall m \in \mathbb{N}$, $^{m}A = 0$.

A Finsler metric $F(x, y)$ on a manifold $M$ is called Berwald metric if in any standard local coordinate system $(x^i, y^i)$ in $TM_0$, the Christoffel symbols $\Gamma^k_{ij} = \Gamma^k_{ij}(x)$ are functions of $x \in M$ alone. In this case $G^i(x, y) = \frac{1}{2} \Gamma^i_{jk}(x)y^j y^k$ are quadratic in $y = y^i \frac{\partial}{\partial x^i}$ and $F(x, y)$ is called a Landsberg metric if $L^i_{jk}(x, y) = 0$, that is

$$
L^i_{jk}(x, y) = \frac{\partial^2 G^i}{\partial y^j \partial y^k}(x, y) - \Gamma^i_{jk}(x, y).
$$

Clearly Minkowski and Riemannian metrics are trivial Berwald metrics. If $F(x, y)$ is a Berwald metric, it is a Landsberg metric. But the converse might not be true, although no counter-example has been found yet \[14\]. A fundamental theorem in Finsler geometry says that a Finsler metric $F$ is a Berwald metric if and only if the Cartan tensor is covariantly constant along all horizontal directions on the slit tangent bundle $TM_0$ (see \[15\] and \[4\] for proof). Thus in the Berwald case, $^{m}A_{ijk}$ vanish $\forall m \in \mathbb{N}$.

**Flag curvature.** A flag curvature is a geometrical invariant that generalizes the sectional curvature of Riemannian geometry. Let $x \in M$, $0 \neq y \in T_x M$ and $V := V^i \frac{\partial}{\partial x^i}$. Flag curvature is obtained by carrying
out the following computation at the point \((x, y) \in T M_0\), and viewing \(y, V\) as section of \(\pi^* TM\):

\[
K(y, V) := \frac{V^i (y^j R_{jikl} y^l)v^k}{g(y, y)g(V, V) - [g(y, V)]^2},
\]

where \(g\) is a Riemannian metric on \(\pi^* TM\). If \(K\) is independent of the transverse edge \(V\), we say that our Finsler space has scalar flag curvature. Denote this scalar by \(\lambda = \lambda(x, y)\). When \(\lambda(x, y)\) has no dependence on either \(x\) or \(y\), then Finsler manifold is said to be of constant flag curvature.

3. Berwald-type connection on \(\pi^* TM\)

In this section we introduce a new family of Finsler connections which are torsion-free and almost compatible with the Finsler metric. In the sequel we will refer to this connection by “Berwald-type connection”.

**Definition 3.1.** Let \((M, F)\) be a Finsler n-manifold. Let \(g\) and \(A\) denote the fundamental and the Cartan tensors in \(\pi^* TM\), respectively. Let \(D\) be a Finsler connection on \(M\).

(i) \(D\) is torsion-free, if \(\forall X, Y \in \chi X(TM_0),\)

\[
\Sigma_D(\dot{X}, \dot{Y}) := D_{\dot{X}}\rho(\dot{Y}) - D_{\dot{Y}}\rho(\dot{X}) - \rho([\dot{X}, \dot{Y}]) = 0.
\]

(ii) \(D\) is almost compatible with the Finsler structure in the following sense: if for all \(X, Y \in \pi^* TM\) and \(\dot{Z} \in T_v(TM_0),\)

\[
(D_{\dot{Z}}g)(X, Y) := \dot{Z}g(X, Y) - g(D_{\dot{Z}}X, Y) - g(X, D_{\dot{Z}}Y),
\]

or equivalently

\[
(D_{\dot{Z}}g)(X, Y) = -2k_1 A(\rho(\dot{Z}), X, Y) - \cdots - 2k_m A(\rho(\dot{Z}), X, Y)
\]

\[
+ 2F^{-1} A(\mu(\dot{Z}), X, Y),
\]

where \(\rho(\dot{Z}) := (v, \pi_*(\dot{Z}))\), \(\mu(\dot{Z}) := D_{\dot{Z}}F\ell, m \in \mathbb{N}\) and \(k_i \in \mathbb{R}\).

The bundle map \(\mu : T(TM_0) \to \pi^* TM\) defined in above definition satisfies \(\mu(\frac{\partial}{\partial y^i}) = \partial_i\). To prove this, take \(\ell = \ell^i \frac{\partial}{\partial x^i}\), where \(\ell = \ell \partial_i\). Now \(\rho(\ell) = \ell\), so from (2.1)

\[
\mu(\frac{\partial}{\partial y^i}) = D_{\frac{\partial}{\partial y^i}} F\ell = \rho([\frac{\partial}{\partial y^i}, y^k \frac{\partial}{\partial x^k}]) = \partial_i.
\]
Theorem 3.1. Let \((M, F)\) be a Finsler n-manifold. Then there is a unique linear torsion-free connection \(D\) in \(\pi^*T M\), which is almost compatible with the Finsler structure in the sense of (2.2).

Proof. In a standard local coordinate system \((x^i, y^j)\) in \(TM_0\), we write

\[ D\frac{\partial}{\partial x^i} \partial_j = \Gamma^k_{ij} \partial_k, \quad D\frac{\partial}{\partial y^j} \partial_i = F^k_{ij} \partial_k. \]

By replacing \(\hat{X}, \hat{Y}\) in (2.1) with the basis of \(T_v(TM_0)\) i.e. \(\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}\) we get

\[ \Gamma^k_{ij} = \Gamma^k_{ji}, \]

\[ F^k_{ij} = 0, \]

and by replacing \(X, Y\) (resp. \(\hat{Z}\)) in (2.2) with the basis of \(\pi^*TM\) i.e. \(\{\partial_i\}\) (resp. with the basis of \(T_v(TM_0)\)) we get

\[ \frac{\partial}{\partial x^k}(g_{ij}) = \Gamma^l_{ki} g_{lj} + \Gamma^l_{kj} g_{li} - 2k_i \dot{A}_{ijk} - \cdots - 2k_m \dot{A}^{(m)}_{ijk} + 2\gamma_{kl} \left\{ A_{ijm} \Gamma^m_{lb} - A_{jlm} \Gamma^m_{ib} - A_{lim} \Gamma^m_{jb} \right\} \ell_b, \]

\[ \frac{\partial}{\partial y^k}(g_{ij}) = F^l_{kj} g_{li} + F^l_{ik} g_{jl} - 2k_i \dot{A}_{ijk} - \cdots - k_m \dot{A}^{(m)}_{ijk} \} F^l_{mk} \ell_m + 2F^{-1} A_{ijk}, \]

where \(g_{ij}, A_{ijk}\) and \(A^{(m)}_{ijk}, \forall m \in \mathbb{N}\) are all functions of \((x, y)\). We shall compute \(\Gamma^k_{ij}\) by "Christoffel’s trick" from (2.4) and (2.6). Then making a permutation to \(i, j, k\) in (2.6), and using (2.4), we obtain

\[ \Gamma^k_{ij} = \gamma^k_{ij} + k_i \dot{A}^k_{ij} + \cdots + k_m \dot{A}^{(m)}_{ij} \]

\[ + g^{kl} \left\{ A_{ijm} \Gamma^m_{lb} - A_{jlm} \Gamma^m_{ib} - A_{lim} \Gamma^m_{jb} \right\} \ell_b, \]

where we have put

\[ \gamma^k_{ij} = \frac{1}{2} g^{kl} \left\{ \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^l} \right\}, \]

and \(A^k_{ij} = g^{kl} A_{ijl}\). Multiplying (2.8) by \(\ell^l\), we obtain

\[ \Gamma^k_{ib} \ell^b = \gamma^k_{ib} \ell^b - A^k_{im} \Gamma^m_{lb} \ell^l. \]

Multiplying (2.9) by \(\ell^b\), yields

\[ \Gamma^k_{ab} \ell^a \ell^b = \gamma^k_{ab} \ell^a \ell^b. \]
Substituting (2.10) into (2.9), we obtain
\[ \Gamma_{ib}^{k \ell} = \gamma_{ib}^{k \ell} - A_{im}^{k \ell} \gamma_{ab}^{m \ell} \gamma_{ib}^{a \ell}. \]

Substituting (2.11) in (2.8), we obtain
\[ \Gamma_{ij}^{k} = \gamma_{ij}^{k} + k_{i} A_{ij}^{k} + \cdots + k_{m} A_{ij}^{m} + g^{kl} \left\{ A_{ijm} \gamma_{ib}^{m} - A_{jim} \gamma_{ib}^{m} - A_{lim} \gamma_{ib}^{m} \right\} \gamma_{ab}^{l \ell} \gamma_{ib}^{a \ell} \]
\[ \quad + \left\{ A_{jm} A_{is}^{m} + A_{im} A_{js}^{m} - A_{sm} A_{ij}^{m} \right\} \gamma_{ab}^{s \ell} \gamma_{ib}^{a \ell}. \]

Then using (1.3), (2.12) become
\[ \Gamma_{jk}^{i} = \frac{g^{is}}{2} \left\{ \frac{\delta g_{sj}}{\delta x^{k}} - \frac{\delta g_{jk}}{\delta x^{s}} \right\} + k_{i} \dot{A}_{jk}^{i} + \cdots + k_{m} \dot{A}_{jk}^{m} \gamma_{ab}^{l \ell} \gamma_{ib}^{a \ell} \gamma_{ab}^{l \ell} \gamma_{ib}^{a \ell}. \]

This proves the uniqueness of \( D \). The set \( \{ \Gamma_{ij}^{k}, F_{ij}^{k} = 0 \} \) where \( \{ \Gamma_{ij}^{k} \} \) are given by (2.13), define a linear connection \( D \) on \( \pi^{*}TM \) satisfying (2.1) and (2.2).

**Definition 3.2.** Let \( (M, F) \) be a Finsler manifold. A Finsler connection is called of Berwald-type (resp. of Cartan-type or Shen-type) if and only if vanishing of its \( h^2 \)-curvature, reduces the Finsler structure to the Berwaldian (resp. Landsbergian or Riemannian) one.

From this viewpoint one can compare some of the non-Riemannian Finsler connections according to the compatibility of the tensors \( S \) and \( T \).

| Connection         | Compatible tensors | Metric compatibility | Torsion   |
|--------------------|--------------------|----------------------|-----------|
| 1. Berwald         | \( A + \dot{A} \) | 0                    | almost compatible | free       |
| 2. Chern- Rund     | \( \dot{A} \)     | 0                    | almost compatible | free       |
| 3. Berwald-type    | \( A + \kappa_{i} \dot{A} + \cdots + \kappa_{m} \ddot{A} \) | 0                    | almost compatible | free       |
| 4. Cartan          | \( A \)           | \( A \)              | metric compatible | not free   |
| 5. Hashiguchi      | \( A + \dot{A} \) | \( A \)              | almost compatible | not free   |
| 6. Cartan-type     | \( A + \kappa_{i} \dot{A} + \cdots + \kappa_{m} \ddot{A} \) | \( A \)              | depends on \( \kappa_{i} \) | not free   |
| 7. Shen            | 0                 | 0                    | almost compatible | free       |
| 8. Shen-type       | \( \kappa_{i} \dot{A} + \cdots + \kappa_{m} \ddot{A} \) | 0                    | almost compatible | free       |
| 9. General-type    | \( \kappa_{i} A + \kappa_{i} \dot{A} + \cdots + \kappa_{m} \ddot{A} \) | \( rA \)              | depends on \( \kappa_{i} \) and \( r \) | depends on \( r \) |
In this table $A, \dot{A}, \ddot{A}, \cdots, \overline{m}$ are Cartan tensor and their covariant derivatives, $\kappa_i$ and $r$ are arbitrary real constants. The connections 1, 2, and 3 belong to the Berwald-type category. The connections 4, 5, and 6 are Cartan-type connections. The connections 7 and 8 belong to the Shen-type Category. The connection 9 contains all other connections.

**Remark 3.1.** The Berwald and Chern connections are special cases of Berwald-type connection in the following way:

Putting $k_1 = \cdots = k_m = 0$ yields the Chern connection.

Putting $k_2 = \cdots = k_m = 0$ and $k_1 = 1$ yields the Berwald connection.

The bundle map $\mu : T(TM_0) \to \pi^*TM$ defined in Definition 1 can be expressed in the following form:

$$\mu \left( \frac{\partial}{\partial x^i} \right) = N^k_i \partial_k, \quad \mu \left( \frac{\partial}{\partial y^i} \right) = \partial_i,$$

where $N^k_i = F^{ij}_{kl} \Gamma^i_{jkl} = F \{ \gamma^i_{jkl} - A_{ik}^j \gamma^i_{kjl} \}$. Using the nonlinear connection coefficients, for Berwald-type connection we have

$$\Gamma^i_{jk} = \gamma^i_{jk} + k_1 \dot{A}^i_{jk} + \cdots + k_m \overline{m}^i_{jk} - g^{il} \{ C_{ijls} N^s_k - C_{jiks} N^s_l + C_{jklr} N^r_s \}.$$
Let \( \{e_i\}_{i=1}^n \) be a local orthonormal (with respect to \( g \)) frame field for the vector bundle \( \pi^*TM \) such that \( g(e_i, e_n) = 0, i = 1, \ldots, n - 1 \) and \( e_n := \ell \). Put \( \ell_i := g_{ij} \ell^j = F_y^{ij} \). Let \( \{\omega^i\}_{i=1}^n \) be its dual co-frame field.

The \( \omega^i \)'s are local sections of the dual bundle \( \pi^*TM \). One readily finds that \( \omega^i : \frac{\partial F}{\partial y^i} dx^i = \omega \), which is the Hilbert form. It is obvious that \( \omega(\ell) = 1 \).

Put \( \rho = \omega^i \otimes e_i \), \( De_i = \omega_i^j \otimes e_j \), \( \Omega e_i = 2\Omega_i^j \otimes e_j \).

\( \{\Omega_i^j\} \) and \( \{\omega_i^j\} \) are called the curvature forms and connection forms of \( D \) with respect to \( \{e_i\} \). We have \( \mu := DF\ell = F\{\omega^i + d(\log F)\delta_n^i \} \otimes e_i \).

Put \( \omega^{n+i} := \omega_n^i + d(\log F)\delta_n^i \). It is easy to see that \( \{\omega^i, \omega^{n+i}\}_{i=1}^n \) is a local basis for \( T^*(TM_0) \).

According to Theorem 1 there exits a connection 1-forms \( \{\omega_j^i\} \) which satisfy the following torsion-freeness and almost compatibility as follows.

\[
\frac{d\omega^i}{\partial x^j} = \omega^i \wedge \omega_j^i, \quad (4.2)
\]

\[
\frac{dg_{ij}}{\partial x^k} = g_{ik}\omega^k_j + g_{jk}\omega^k_i - 2\{k_1 \dot{A}_{ijk} + \cdots + k_m \dot{A}_{ijk}\} \omega^k + 2A_{ijk}\omega^{n+k}, \quad (4.3)
\]

In fact using the local orthonormal frame field \( \{e_i\}_{i=1}^n \) for the vector bundle \( \pi^*TM \) and its dual co-frame field \( \{\omega^i\}_{i=1}^n \), (2.1) and (2.2) respectively, after a straightforward calculation analogous to the proof of Theorem 1, become (3.2) and (3.3).

Let us we put

\[
\frac{dg_{ij} - g_{ij}k^k}{\partial x^k} = g_{ik}\omega^k_j + g_{jk}\omega^k_i - g_{ij+k}\omega + g_{ij,k}\omega^{n+k}, \quad (4.4)
\]

where \( g_{ij,k} \) and \( g_{ijjk} \) are respectively the vertical and horizontal covariant derivative of \( g_{ij} \). This gives

\[
g_{ijjk} = -2\{k_1 \dot{A}_{ijk} + \cdots + k_m \dot{A}_{ijk}\}, \quad (4.5)
\]

and

\[
g_{ij,k} = 2A_{ijk}, \quad (4.6)
\]

Moreover the torsion freeness is equivalent to following

\[
\omega^i_j = \Gamma^i_{jk} dx_k, \quad (4.7)
\]

Clearly (3.1) is equivalent to

\[
d\omega^i_j = -\omega^k_i \wedge \omega^j_k = \Omega^i_j, \quad (4.8)
\]
Since the $\Omega^j_i$ are 2-forms on the manifold $TM_0$, they can be generally expanded as

$$\Omega^j_i = \frac{1}{2} R^j_{ikl} \omega^k \wedge \omega^l + P^j_{ikl} \omega^k \wedge \omega^{n+l} + \frac{1}{2} Q^j_{ikl} \omega^{n+k} \wedge \omega^{n+l}, \quad (4.9)$$

Let $\{\tilde{e}_i, \dot{e}_i\}_{i=1}^n$ be the local basis for $T(TM_0)$, which is dual to $\{\omega^i, \omega^{n+i}\}_{i=1}^n$, i.e., $\tilde{e}_i \in HTM, \dot{e}_i \in VTM$ such that $\rho(\tilde{e}_i) = e_i, \mu(\dot{e}_i) = F e_i$. The objects $R, P$ and $Q$ are respectively the hh-, hv- and vv-curvature tensors of the connection $D$ and with $R(\tilde{e}_k, \tilde{e}_l)e_i = R^j_{ikl}e_j, \quad P(\tilde{e}_k, \dot{e}_l)e_i = P^j_{ikl}e_j, \quad Q(\dot{e}_k, \dot{e}_l)e_i = Q^j_{ikl}e_j$. From (3.9) we see that

$$R^j_{ikl} = - R^j_{lki}, \quad Q^j_{ikl} = - Q^j_{lki}. \quad (4.10)$$

If $D$ is a torsion-free, then $Q = 0$. Differentiating (3.2), then we have the first Bianchi identity

$$\omega^i \wedge \Omega^j_i = 0, \quad (4.11)$$

which implies the first Bianchi identity for $R$:

$$R^j_{ikl} + R^j_{lki} + R^j_{lik} = 0, \quad (4.12)$$

and

$$P^j_{ikl} = P^j_{kil}. \quad (4.13)$$

The exterior differentiation of (3.8) gives rise to the Second Bianchi identity:

$$d\Omega^j_i - \omega^i \wedge \Omega^j_k + \omega^j_k \wedge \Omega^i_k = 0. \quad (4.14)$$

We decompose the covariant derivatives of the Cartan tensor on $TM$

$$dA_{ijk} - A_{ijkl} \omega^i - A_{ilk} \omega^j - A_{ijl} \omega^k = A_{ijk l} \omega^l + A_{ijk} \omega^{n+l}, \quad (4.15)$$

and in the similar way $\forall m \in \mathbb{N}$, for $\tilde{A}_{ijk}$ we have:

$$d\tilde{A}_{ijk} - \tilde{A}_{ijkl} \omega^i - \tilde{A}_{ilk} \omega^j - \tilde{A}_{ijl} \omega^k = \tilde{A}_{ijk l} \omega^l + \tilde{A}_{ijk} \omega^{n+l}. \quad (4.16)$$

Clearly from (3.15) and (3.16), we find that for each $l$ and $\forall m \in \mathbb{N}$

$$A_{ijkl}, \tilde{A}_{ijkl}, \tilde{A}_{ijk l} \quad \text{and} \quad \tilde{A}_{ijk l},$$

are symmetric in $i, j, k$. Put $\tilde{A}_{ijk} = \tilde{A}(e_i, e_j, e_k)$ and $\tilde{A}^i_{ij} = g^{kl} \tilde{A}_{ijk}, \quad \forall m \in \mathbb{N}$. By definition of $A$ and $\tilde{A}$ one has,

$$A_{ijk l} = \tilde{A}_{ijk}, \quad (4.18)$$
where we use the notation $\tilde{A}_{ijk|n} = \tilde{A}_{ijk|s}^s$ for all $m \in \mathbb{N}$ and

$$m\tilde{A}_{ijk|n} = m\tilde{A}_{ijk}.$$  

(4.19)

It follows from (3.15)

$$A_{njk|l} = 0, \quad A_{njk,l} = -A_{jkl},$$

(4.20)

and from (3.16) we have

$$\forall m \in \mathbb{N}, \quad m\tilde{A}_{njk|l} = 0, \quad m\tilde{A}_{njk,l} = -m\tilde{A}_{jkl}.$$  

(4.21)

In this relation the following results are well known:

**Theorem A.** ([7], [10]) Let $(M, F)$ be a Finsler manifold. Then for the Cartan connection (or Hashiguchi connection), hv-curvature $P_{ijkl} = 0$ if and only if $F$ is a Landsberg metric.

**Theorem B.** ([4]) Let $(M, F)$ be a Finsler manifold. Then for the Chern connection (or Berwald connection), hv-curvature $P_{ijkl} = 0$ if and only if $F$ is a Berwald metric.

**Theorem C.** ([13]) Let $(M, F)$ be a Finsler manifold. Then for the Shen connection, hv-curvature $P_{ijkl} = 0$ if and only if $F$ is Riemannian.

Analogously we have the following result.

**Theorem 4.1.** Let $(M, F)$ be a Finsler manifold. Then for the Berwald-type connection, hv-curvature $P_{ijkl} = 0$ if and only if $F$ is a Berwald metric.

**Proof.** Let $(M, F)$ be a Finsler manifold. Differentiating (3.3) and using (3.2), (3.3), (3.8), (3.15), (3.16), (3.17), (3.18), (3.19), (3.20) and (3.21) leads to

\[ g_{kj} \Omega^k_i + g_{ik} \Omega^k_j = -2A_{ijk \Omega^k_i - 2A_{ijk|l}^|l} + 2A_{ijk|l}^|l(\omega^{n+k} \wedge \omega^{n+l}) + k_1(\tilde{A}_{ijk|l}^|l + \tilde{A}_{ijk|l}^|l) \wedge \omega^{n+k} + \cdots \\
+ k_m(\tilde{A}_{ijk|l}^|l + \tilde{A}_{ijk|l}^|l) \wedge \omega^{n+k}. \]

(4.22)

By using (3.9) and (3.22) we gives the following

\[ R_{ijkl} + R_{jikl} = 2k_1 \{ \tilde{A}_{ijkl} - \tilde{A}_{ijkl} \} + \cdots + 2k_m \{ \tilde{A}_{ijkl} - \tilde{A}_{ijkl} \} - 2A_{ijkl} R_{ijkl} \]

(4.23)

\[ P_{ijkl} + P_{jikl} = -2(k_1 \tilde{A}_{ijkl} + \cdots + k_m \tilde{A}_{ijkl}) - 2A_{ijkl} \tilde{A}_{ijkl} - 2A_{ijkl} P_{ijkl}. \]

(4.24)
\[(4.25)\]
\[A_{ijk,l} = A_{ijl,k}.\]
Permuting \(i, j, k\) in (3.24) yields
\[(4.26)\]
\[P_{ijkl} = -\{k_1 \dot{A}_{ijk,l} + \cdots + k_m \dot{A}_{ijk,l}\} - (A_{ijl[k} + A_{jkl|i} - A_{kil|j})
+ A_{kis} P_{n,jl} - A_{jks} P_{n,il} - A_{ijl} P_{n,kl},\]
and
\[(4.27)\]
\[P_{njkl} = \{k_1 \dot{A}_{jkl} + \cdots + k_m \dot{A}_{jkl}\} - \dot{A}_{jkl},\]
that is because \(P_{njnl} = 0\). Now if \(F\) is Berwald metric from (3.26) and (3.27) we conclude \(P = 0\).
Conversely let \(P = 0\). It follows from (3.27),
\[(4.28)\]
\[k_1 \dot{A}_{jkl} + \cdots + k_m \dot{A}_{jkl} = \dot{A}_{jkl},\]
By means of (3.26) we have
\[k_1 \dot{A}_{ijk,l} + \cdots + k_m \dot{A}_{ijk,l} = A_{kil|j} - A_{ijl|k} - A_{jkl|i}.\]
Permuting \(i, j, k\) in the above identity yields
\[k_1 \dot{A}_{ijk,l} + \cdots + k_m \dot{A}_{ijk,l} = A_{jkl|i} - A_{kil|j} - A_{ijl|k},\]
then
\[A_{ijl|k} = A_{jkl|i}.\]
Letting \(k = n\) in the above relation, we can conclude
\[(4.29)\]
\[\dot{A}_{ijk} = 0.\]
It is obvious that
\[(4.30)\]
\[\forall m \in \mathbb{N}, \quad \dot{A}_{ijk} = 0.\]
Therefore from (3.24), (3.26), (3.27) and (3.30) we conclude that \(A_{ijkl} = 0\), thus \(F\) is Berwald metric. \(\square\)

5. Some Applications

5.1. Preliminaries on geodesics and completeness. In this section we explore the notion of geodesics to introduce the concept of completeness for Finsler manifolds. Let \(c : [a, b] \rightarrow M\) be a unit speed \(C^\infty\) curve in \((M, F)\). The canonical lift of \(c\) to \(TM_0\) is defined by
\[\dot{c} := \frac{dc}{dt} \in TM_0.\]
It is easy to see that $\rho(\frac{d\hat{c}}{dt}) = \ell_{\hat{c}}$, where $c$ is called a geodesic if its canonical lift $\hat{c}$ satisfies

$$\frac{d\hat{c}}{dt} = \ell_{\hat{c}},$$

where $\ell$ is the geodesic field on $TM_0$ defined for $\ell \in HTM$ by $\rho(\ell) = \ell$.

Let $I_xM = \{v \in T_xM, F(v) = 1\}$ and $IM = \bigcup_{p \in M} I_xM$. Where $I_xM$ is called the indicatrix, and it is a compact set. We can show that the projection of integral curve $\varphi(t)$ of $\ell$ with $\varphi(0) \in IM$ is a unit speed geodesic $c$ whose canonical lift is $\hat{c}(t) = \varphi(t)$.

A Finsler manifold $(M, F)$ is said to be backward geodesically complete (or forward geodesically complete) if every geodesic $c(t)$, $a \leq t < b$ ($a < t \leq b$), parameterized to have constant Finslerian speed, can be extended to a geodesic defined on $a \leq t < \infty$ ($-\infty < t \leq b$). A Finsler manifold $(M, F)$ is said to be complete if it is both forward and backward geodesically complete.

Let $c$ be a unit speed geodesic in $M$. A section $X = X(t)$ of $\pi^*TM$ along $\hat{c}$ is said to be parallel if $D\frac{d\hat{c}}{dt}X = 0$. For $v \in TM_0$, define $\|A\|_v = \sup A(X, Y, Z)$ and $\|\hat{A}\|_v = \sup \hat{A}(X, Y, Z)$, where the supremum is taken over all unit vectors of $\pi^*TM$. Put $\|A\|_v = \sup_{v \in IM}\|A\|_{v}$ and $\|\hat{A}\|_v = \sup_{v \in IM}\|\hat{A}\|_{v}$.

### 5.2. Application of Berwald-type connections

In this subsection we are going to use two especial cases of Berwald-type connections introduced in section 2. A useful property of this connection is, its adaptive form for applying to the different applications. In fact one can use a suitable special case of this connection to find a geometric interpretation for solutions of some differential equations formed by Cartan tensor and its derivatives in Finsler spaces. For example we prove the following theorem.

**Theorem 5.1.** Let $(M, F)$ be a complete Finsler manifold with bounded Landsberg tensor. Then $F$ is a Landsberg metric if and only if $P_{jkl} = 0$.

**Proof.** To prove this theorem we introduce a connection for which we have put $k_1 = k_3 = \cdots = k_m = 0$ and $k_2 \neq 0$ in (3.27). Let $F$ be a Landsberg metric, then from (3.27) we find that $P_{jkl} = 0$. Conversely if
$P_{jkl} = 0$ then, we have following differential equation:
\begin{equation}
(5.1) \quad k_m A^{(m)} + \cdots + k_2 A^{(2)} + (k_1 - 1) \ddot{A} = 0.
\end{equation}
If $k_1 = k_3 = \cdots = k_m = 0$ and $k_2 \neq 0$ then we find an special Berwald-type connection for which we have
\begin{equation}
(5.2) \quad k_2 \ddot{A} - \dot{A} = 0.
\end{equation}
On the other hand
\begin{equation}
(5.3) \quad \frac{d\dot{A}}{dt} = \ddot{A}.
\end{equation}
We have $\dot{A} = e^{k_2 t} \ddot{A}(0)$. Using $\|\dot{A}\| < \infty$, and letting $t \to +\infty$, then $\ddot{A}(0) = \dot{A}(X, Y, Z) = 0$, or $\dot{A} = 0$ i.e., $F$ is a Landsberg metric. $\Box$

By mean of the Theorem 3, every compact Finsler manifold is a Landsberg space if and only if $P_{jkl}$ vanishes. Next we consider a special Berwald-type connection and give another proof for the following well-known result.

**Corollary 5.1.** Let $(M, F)$ be a complete Finsler manifold with negative constant flag curvature and bounded Cartan tensor. Then $F$ is Riemannian.

**Proof.** Let $(M, F)$ be a complete Finsler manifold with constant flag curvature $\lambda$. If $\lambda \neq 0$ we put in (3.27) $k_2 = k_4 = \cdots = k_m = 0$, $k_1 = 2$ and $k_3 = \frac{1}{\lambda} \neq 0$. We obtain a connection for which the hv-curvature $P$ become
\begin{equation}
(5.4) \quad P_{ijkl} =:\{-2A_{ijkl} + \frac{1}{\lambda} A_{ijkl} \} - (A_{ijl}^k + A_{jkl}^i - A_{kil}^j)
\end{equation}
and
\begin{equation}
(5.5) \quad P_{njkl} = \frac{1}{\lambda} A + \dot{A}.
\end{equation}
As $M$ has constant flag curvature we have
\begin{equation}
(5.6) \quad \ddot{A} + \lambda A = 0.
\end{equation}
From which we have $P_{njkl} = \frac{1}{\lambda} \dddot{A} + \ddot{A} = 0$. By solving this differential equation we find
\begin{equation}
(5.7) \quad A(t) = c_1 + c_2 e^{\sqrt{-\lambda}t} + c_3 e^{-\sqrt{-\lambda}t}.
\end{equation}
By the assumption that the Cartan tensor is bounded, and letting \( t \to \infty \) and \( t \to -\infty \), we see that \( c_2 = c_3 = 0 \). Then \( A = c_1 \) therefore \( \dot{A} = 0 \) and \( F \) is a Landsberg metric. From (4.6), it is easy to see that \( A = 0 \). □

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