A relative-geometric treatment of ruled surfaces

Georg Stamou, Stylianos Stamatakis, Ioannis Delivos

Aristotle University of Thessaloniki
Department of Mathematics
GR-54124 Thessaloniki, Greece
e-mail: stamoug@math.auth.gr

Abstract
We consider relative normalizations of ruled surfaces with non-vanishing Gaussian curvature $K$ in the Euclidean space $\mathbb{R}^3$, which are characterized by the support functions $(\alpha)q = |K|^{\alpha}$ for $\alpha \in \mathbb{R}$. All ruled surfaces for which the relative normals, the Pick invariant or the Tchebychev vector field have some specific properties are determined. We conclude the paper by the study of the affine normal image of a non-conoidal ruled surface.

MSC 2010: 53A25, 53A15, 53A40

Keywords: Ruled surfaces, relative normalizations, affine normal image

1 Introduction
Relatively normalized hypersurfaces with non-vanishing Gaussian curvature $K$ in the Euclidean space $\mathbb{R}^{n+1}$, whose relative normalizations are characterized by the support functions $(\alpha)q = |K|^{\alpha}$ for $\alpha \in \mathbb{R}$ (see Section 2), have been studied in the last two decades by a number of authors and many results have been derived. The one-parameter family of relative normalizations $(\alpha)\bar{y}$, which is determined by the support functions $(\alpha)q$ and was introduced by F. Manhart [5], deserves special interest, because in this family are contained among other relative normalizations the Euclidean normalization (for $\alpha = 0$) as well as the equiaffine normalization (for $\alpha = 1/(n+2)$). A class of surfaces in $\mathbb{R}^3$, relatively normalized by $(\alpha)\bar{y}$, which has been investigated in the past, is the one of ruled surfaces (see e.g. [3], [4], [10]). These surfaces are further discussed in the first part of the present work. The affine normal image of a non-conoidal ruled surface is studied in the second part.
2 Relative normalizations of surfaces

In the Euclidean space $\mathbb{R}^3$ let $\Phi : \bar{x} = \bar{x}(u^1, u^2) : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be an injective $C^r$-immersion with Gaussian curvature $K \neq 0$ $\forall (u^1, u^2) \in U$. A $C^s$-mapping $\bar{y} : U \rightarrow \mathbb{R}^3$ ($r > s \geq 1$) is called a $C^s$-relative normalization if

$$\bar{y}(\alpha) \notin T_P\Phi, \quad \frac{\partial \bar{y}}{\partial u^i}(\alpha) \in T_P\Phi \quad (i = 1, 2), \quad P = \bar{x}(\alpha),$$

(2.1)

at every point $P \in \Phi$, where $T_P\Phi$ is the tangent vector space of $\Phi$ in $P$.

The covector $\bar{X}$ of the tangent plane is defined by

$$\langle \bar{X}, \frac{\partial \bar{x}}{\partial u^i} \rangle = 0 \quad (i = 1, 2) \quad \text{and} \quad \langle \bar{X}, \bar{y} \rangle = 1,$$

(2.2)

where $\langle , \rangle$ denotes the standard scalar product in $\mathbb{R}^3$. Using $\bar{X}$, the relative metric $G$ is introduced by

$$G_{ij} = \langle \bar{X}, \frac{\partial^2 \bar{x}}{\partial u^i \partial u^j} \rangle.$$

(2.3)

From now on, we will use the tensor $G_{ij}$ for raising and lowering the indices in the sense of classical tensor notation. Let $\xi : U \rightarrow \mathbb{R}^3$ be the Euclidean normalization of $\Phi$. The support function of the relative normalization $\bar{y}$ is defined by

$$q := \langle \xi, \bar{y} \rangle : U \rightarrow \mathbb{R}, \quad q \in C^s(U).$$

(2.4)

By virtue of (2.1), the support function $q$ is not vanishing in $U$; moreover because of (2.2) it is

$$\bar{X} = q^{-1} \xi.$$

(2.5)

From (2.3) and (2.5) we obtain

$$G_{ij} = q^{-1} h_{ij},$$

(2.6)

where $h_{ij}$ are the components of the second fundamental form of $\Phi$. We mention that given a support function $q$ the relative normalization $\bar{y}$ is uniquely determined and possesses the following parametrization (see [5, p. 197])

$$\bar{y} = -h^{(ij)} \frac{\partial q}{\partial u^i} \frac{\partial \bar{x}}{\partial u^j} + q\xi,$$

(2.7)

where $h^{(ij)}$ are the components of the inverse tensor of $h_{ij}$.

Let $^G\nabla f$ denote the covariant derivative with respect to $G$ of a differentiable function $f$, which is defined on $\Phi$. The symmetric Darboux tensor is defined by

$$A_{ijkl} := \langle \bar{X}, ^G\nabla_i^G\nabla_k \frac{\partial \bar{x}}{\partial u^j} \rangle,$$

(2.8)

For notations and definitions the reader is referred to [8] and [9].
the Tchebychev vector by

\[ T_i = \frac{1}{2} A_{ijk} G^j k = \frac{1}{2} A^j_j, \]  

(2.9)

and the Pick invariant by

\[ J := \frac{1}{2} A_{jkl} A^{jkl}. \]  

(2.10)

3 Relatively normalized ruled surfaces by \((\alpha) \bar{y}\)

3.1. Let \(\Phi \subset \mathbb{R}^3\) be a skew ruled \(C^2\)-surface. We denote by \(\bar{s}(u), u \in I \ (I \subset \mathbb{R}\) open interval) the position vector of the line of striction of \(\Phi\) and by \(\bar{e}(u)\) the unit vector pointing along the rulings. Moreover we can choose the parameter \(u\) to be the arc length along the spherical curve \(\bar{e}(u)\). Then a parametrization of the ruled surface \(\Phi\) over the region \(U := I \times \mathbb{R}\) of the \((u, v)\)-plane is

\[ \bar{x}(u, v) = \bar{s}(u) + v \bar{e}(u) \quad \text{with} \quad |\bar{e}| = |\bar{e}'| = 1, \quad \langle \bar{s}'(u), \bar{e}(u) \rangle = 0 \text{ in } I, \]  

(3.1)

where the prime denotes differentiation with respect to \(u\). The moving frame of \(\Phi\), consisting of the vector \(\bar{e}(u)\), the central normal vector \(\bar{n} := \bar{e}'\) and the central tangent vector \(\bar{z} := \bar{e} \times \bar{n}\), moves along the line of striction and satisfies the relations [2, p. 62f]

\[ \bar{e}' = \bar{n}, \quad \bar{n}' = -\bar{e} + \kappa \bar{z}, \quad \bar{z}' = -\kappa \bar{n}, \]  

(3.2)

where \(\kappa = (\bar{e}, \bar{e}', \bar{e}'')\) denotes the conical curvature of \(\Phi\). Consider the parameter of distribution \(\delta = (\bar{s}', \bar{e}, \bar{e}')\) and the striction \(\sigma := \angle(\bar{e}, \bar{s}') \ (-\frac{\pi}{2} < \sigma \leq \frac{\pi}{2}, \ \text{sign} \sigma = \text{sign} \delta)\). Then the tangent vector \(\bar{s}'\) of the line of striction has the expression

\[ \bar{s}' = \delta (\lambda \bar{e} + \bar{z}) \quad \text{with} \quad \lambda := \cot \sigma. \]  

(3.3)

When the invariants \(\kappa(u), \delta(u)\) and \(\lambda(u)\) (fundamental invariants) are given, then there exists up to rigid motions of the space \(\mathbb{R}^3\) a unique ruled surface \(\Phi\), whose fundamental invariants are the given. The components \(g_{ij}\) and \(h_{ij}\) of the first and the second fundamental tensors in the coordinates \(u^1 := u, u^2 := v\) are the following

\[ (g_{ij}) = \begin{pmatrix} v^2 + \delta^2 (\lambda^2 + 1) & \delta \lambda \\ \delta \lambda & 1 \end{pmatrix}, \]  

(3.4)

\[ (h_{ij}) = \frac{1}{w} \begin{pmatrix} -[\kappa v^2 + \delta' v + \delta^2 (\kappa - \lambda)] & \delta \\ \delta & 0 \end{pmatrix}, \]  

(3.5)

where \(w := \sqrt{v^2 + \delta^2}\). For the Gaussian curvature \(K\) of \(\Phi\) the following relation holds

\[ K = -\frac{\delta^2}{w^4}. \]  

(3.6)
We consider now the relative normalizations \( (\alpha)\tilde{y} : U \rightarrow \mathbb{R}^3 \), which are introduced by F. Manhart [5] and, on account of (2.7), are defined by the support functions
\[
(\alpha)q := |K|^\alpha, \quad \alpha \in \mathbb{R}.
\]
We denote by \( (\alpha)J \) the Pick invariant. In order to compute it we firstly have from (2.8) the relation [5, p. 196]
\[
A_{jkl} = \frac{1}{q} \left( \xi, \frac{\partial^2 \tilde{x}}{\partial u^j \partial u^k} \right) - \frac{1}{2} \left( \frac{\partial G_{jk}}{\partial u^l} + \frac{\partial G_{kl}}{\partial u^j} + \frac{\partial G_{ij}}{\partial u^k} \right), \quad (3.8)
\]
Using (3.8), we find for the ruled surface (3.1), which is relatively normalized by \( (\alpha)\tilde{y} \),
\[
A_{112} = \frac{(4\alpha - 1) |\delta|^{2-2\alpha}}{2w^{3+4\alpha}} \left[ \kappa v^3 + 2\delta' v^2 + \delta^2 (\kappa - \lambda) v - \delta^2 \delta' \right], \quad (3.9)
\]
\[
A_{221} = \frac{\epsilon (1 - 4\alpha) |\delta|^{2\alpha - 1}}{w^{3+4\alpha}} v, \quad A_{222} = 0, \quad (3.10)
\]
\[
A_{111} = \frac{\epsilon |\delta|^{2\alpha - 1}}{2w^{3+4\alpha}} \left\{ (\delta \kappa' - 6\alpha \delta' \kappa) v^4 + [-2\delta^2 (1 + \kappa \lambda) + \delta \delta'' - 6\alpha \delta^{-2}] v^3 + \delta^2 \delta (2\kappa' + \lambda') - \delta' \kappa + 2 (3\alpha - 1) \delta' \lambda] v^2 + \delta^2 [ -2\delta^2 (1 + \kappa \lambda) + \delta \delta'' + 3 (2\alpha - 1) \delta^{-2}] v + \delta^4 \right\}, \quad (3.11)
\]
where \( \epsilon = \text{sign} \delta \). Furthermore, since the tensor \( A_{jkl} \) is symmetric, we have
\[
A_{112} = A_{211} = A_{121}, \quad A_{221} = A_{212} = A_{122}. \quad (3.12)
\]
The components \( A^{ijkl} \) can be found from \( A^{ijkl} = G^{ij}G^{km}G^{lr}A_{imr} \). Inserting \( A_{jkl} \) and \( A^{ijkl} \) in (2.10), it turns out that
\[
(\alpha)J = \frac{3 (4\alpha - 1)^2 |\delta|^{2\alpha - 2}}{2w^{4\alpha + 3}} \left[ \kappa v^4 + \delta^2 (\kappa - \lambda) v^2 + \delta^2 \delta' v \right]. \quad (3.13)
\]
Let \( (\alpha)\tilde{T} \) be the corresponding Tchebychev vector of \( (\alpha)\tilde{y} \). Applying similar computations as above we can find \( (\alpha)\tilde{T} \) as follows: In view of \( T^i = G^{ij}T_j \) and using (2.9) we obtain
\[
T^1 = \frac{\epsilon (1 - 4\alpha) |\delta|^{2\alpha - 1}}{w^{4\alpha + 1}} v, \quad (3.14)
\]
\[
T^2 = \frac{(1 - 4\alpha) |\delta|^{2\alpha - 2}}{2w^{4\alpha + 1}} \left[ 2\kappa v^3 + \delta' v^2 + 2\delta^2 (\kappa - \lambda) v + \delta^2 \delta' \right]. \quad (3.15)
\]
Then, substituting (3.14) and (3.15) into
\[
(\alpha)\tilde{T} = T^1 \frac{\partial \tilde{x}}{\partial u} + T^2 \frac{\partial \tilde{x}}{\partial v}, \quad (3.16)
\]
we obtain
\[
(\alpha)\tau = \frac{(1-4\alpha)|\delta|^{2\alpha-2}}{2w^{4\alpha+1}} \left[ (2\kappa v^3 + \delta v^2 + 2\delta^2 \kappa v + \delta^2 \delta) \bar{e} + 2\delta^2 v \bar{n} + 2\delta^2 v \bar{z} \right]. \tag{3.17}
\]

In the following paragraphs we will discuss questions on ruled surfaces relatively normalized by \((\alpha)\bar{y}\), which are related with the relative normals, the Pick invariant and the Tchebychev vector field.

3.2. In this paragraph we treat ruled surfaces, whose relative normal vectors \((\alpha)\bar{y}\) are parallel to a fixed plane \(E\). The vectors \((\alpha)\bar{y}\) are given by the relation (see \[10\] p. 212)
\[
(\alpha)\bar{y} = A_1 \bar{e} + A_2 \bar{n} + A_3 \bar{z}, \tag{3.18}
\]
where
\[
A_1 = \frac{2\alpha (2\kappa v + \delta') |\delta|^{2\alpha-2}}{w^{4\alpha+1}}, \tag{3.19}
\]
\[
A_2 = \frac{\varepsilon \left(4\alpha v^2 + \delta^2\right) |\delta|^{2\alpha-1}}{w^{4\alpha+1}}, \tag{3.20}
\]
\[
A_3 = \frac{(4\alpha - 1) |\delta|^{2\alpha}}{w^{4\alpha+1}} v. \tag{3.21}
\]
Let \(\bar{c} \neq \bar{0}\) be a constant normal vector of the plane \(E\). Because of (3.19)-(3.21) the assumption \(\langle (\alpha)\bar{y}, \bar{c}\rangle = 0\) leads to the relation
\[
4\alpha \kappa \langle \bar{e}, \bar{c}\rangle v^3 + 2\alpha (\delta' \langle \bar{e}, \bar{c}\rangle + 2\delta \langle \bar{n}, \bar{c}\rangle) v^2
+ \delta^2 [4\alpha \kappa \langle \bar{e}, \bar{c}\rangle + (4\alpha - 1) \langle \bar{z}, \bar{c}\rangle] v + 2\alpha \delta^2 \delta' \langle \bar{e}, \bar{c}\rangle + \delta^3 \langle \bar{n}, \bar{c}\rangle = 0. \tag{3.22}
\]

The polynomial on the left hand side of (3.22) is of degree three in \(v\) and vanishes for all \(u \in I\) and infinite values for \(v \in \mathbb{R}\). Comparing its coefficients with those of the zero polynomial we obtain
\[
\alpha \kappa \langle \bar{e}, \bar{c}\rangle = 0, \tag{3.23}
\]
\[
\alpha (\delta' \langle \bar{e}, \bar{c}\rangle + 2\delta \langle \bar{n}, \bar{c}\rangle) = 0, \tag{3.24}
\]
\[
4\alpha \kappa \langle \bar{e}, \bar{c}\rangle + (4\alpha - 1) \langle \bar{z}, \bar{c}\rangle = 0, \tag{3.25}
\]
\[
2\alpha \delta^2 \delta' \langle \bar{e}, \bar{c}\rangle + \delta^3 \langle \bar{n}, \bar{c}\rangle = 0. \tag{3.26}
\]
If \(\alpha \neq 1/4\), then from (3.23)-(3.25) we get \(\langle \bar{n}, \bar{c}\rangle = \langle \bar{z}, \bar{c}\rangle = 0\), i.e. \(\bar{e}/\bar{c}\), which is impossible. If \(\alpha = 1/4\), then according to [6] \(\Phi\) is a conoidal surface (\(\kappa = 0\)). The same result follows also from (3.23)-(3.26). The above discussion gives rise to the following

**Proposition 1** Let \(\Phi\) be a ruled \(C^3\)-surface \(\Phi\), free of torsal rulings, which is relatively normalized by \((\alpha)\bar{y}\) (\(\alpha \in \mathbb{R}\)). If the relative normals of \(\Phi\) are parallel to a fixed plane, then \(\alpha = 1/4\) and \(\Phi\) is a conoidal surface.
3.3. In this paragraph we classify the ruled surfaces \( \Phi \subset \mathbb{R}^3 \), whose Tchebychev vectors \( ^{(\alpha)}\hat{T} \) \((\alpha \neq 1/4)\) are tangent to some geometrically distinguished families of curves of \( \Phi \).

A first result in this direction is obtained immediately from (3.17): \textit{The vectors \( ^{(\alpha)}\hat{T} \) are tangent to the orthogonal trajectories of the rulings, if and only if \( \kappa = \delta' = 0 \), i.e. if and only if \( \Phi \) is a conoidal surface with constant parameter of distribution.}

We consider a directrix \( \Gamma \) of \( \Phi \) defined by \( v = v(u) \). In view of (3.17) we find that the vectors \( ^{(\alpha)}\hat{T} \) along \( \Gamma \) are

- tangent to \( \Gamma \) if and only if
  \[ 2\kappa v^3 + \delta' v^2 + 2\delta [\delta (\kappa - \lambda) - v'] v + \delta^2 \delta' = 0, \]
  and \[(\delta \lambda + v') (2\kappa v + \delta') + 2\delta v = 0. \]

Furthermore we consider the following curves of \( \Phi \):

- a) the curved asymptotic lines,
- b) the curves of constant striction distance (\( u \)-curves) and
- c) the \( K \)-curves, i.e. the curves along which the Gaussian curvature is constant \[7\].

The corresponding differential equations of these families of curves are

\[ \kappa v^2 + \delta' v + \delta^2 (\kappa - \lambda) - 2\delta v' = 0, \] \[(\delta \lambda + v') (2\kappa v + \delta') + 2\delta v = 0. \]

respectively. On account of (3.27), we find that the vectors \( ^{(\alpha)}\hat{T} \) \((\alpha \neq 1/4)\) are tangent to one of these families of curves if and only if

\[ \kappa v^3 + \delta^2 (\kappa - \lambda) v + \delta^2 \delta' = 0, \]
\[ 2\kappa v^3 + \delta' v^2 + 2\delta^2 (\kappa - \lambda) v + \delta^2 \delta' = 0, \]
\[ \kappa v^3 + \delta^2 (\kappa - \lambda) v + \delta^2 \delta' = 0, \]

respectively. Since each condition is satisfied for all \((u, v) \in U\), we obtain \( \kappa = \lambda = \delta' = 0 \). Therefore \( \Phi \) lies on a right helicoid. The same result arises, when we suppose that the Pick invariant \( ^{(\alpha)}J \) \((\alpha \neq 1/4)\) vanishes identically, as it follows immediately by means of (3.13). The above constitute the proof of

**Proposition 2** \textit{Let \( \Phi \subset \mathbb{R}^3 \) be a ruled \( C^3 \)-surface, free of torsal rulings, which is relatively normalized by \( ^{(\alpha)}\tilde{y} \) \((\alpha \in \mathbb{R}/\{1/4\})\). The following properties are equivalent:}

(i) The Tchebychev vectors \( ^{(\alpha)}\hat{T} \) are tangent to the curved asymptotic lines or to the curves of constant striction distance or to the \( K \)-curves.
(ii) The Pick invariant vanishes identically.
(iii) \( \Phi \) lies on a right helicoid.
Continuing this line of work, we require now that the vectors \((\alpha)\bar{T}\) are tangent to the orthogonal trajectories of the \(u\)-curves or to the \(K\)-curves. On account of (3.30) and (3.31) and by virtue of (3.28), we obtain in the first case the condition
\[
2 (1 + \kappa \lambda) v + \delta \lambda = 0 \quad (3.35)
\]
and in the second case the condition
\[
2 \delta' \kappa v^3 + \left[ \delta'^2 + 4 \delta^2 (1 + \kappa \lambda) \right] v^2 - 2 \delta^2 \delta' (\kappa - \lambda) v - \delta^2 \delta'^2 = 0, \quad (3.36)
\]
which are satisfied for every \((u, v) \in U\). Thus, we have
\[
\delta' = 1 + \kappa \lambda = 0. \quad (3.37)
\]
Next, we assume that the vectors \((\alpha)\bar{T}\) are tangent to one family of (Euclidean) lines of curvature. We substitute the derivative \(v'(u)\) from (3.27) into the differential equation of the lines of curvature
\[
g_{12} h_{11} - g_{11} h_{12} + (g_{22} h_{11} - g_{11} h_{22}) v' + (g_{22} h_{12} - g_{12} h_{22}) v^2 = 0 \quad (3.38)
\]
and taking into account (3.4) and (3.5), we obtain once more (3.36). Hence we get again the conditions (3.37), which show that \(\Phi\) is a ruled surface with constant parameter of distribution and whose line of striction is a line of curvature. This characterizes the Edlinger surfaces (see [1, p. 31]) which, by definition, are ruled surfaces whose osculating quadrics are rotational hyperboloids (see [1, p. 36]). Thus the following proposition is valid:

**Proposition 3** Let \(\Phi \subset \mathbb{R}^3\) be a ruled \(C^3\)-surface, free of torsal rulings, which is relatively normalized by \((\alpha)\bar{y}\) \((\alpha \in \mathbb{R}/\{1/4\})\). Assume that the Tchebychev vectors \((\alpha)\bar{T}\) are tangent to the orthogonal trajectories of the curves of constant striction distance or to the \(K\)-curves or tangent to one family of (Euclidean) lines of curvature. Then \(\Phi\) is an Edlinger surface.

**Remark 4** (a) On a right helicoid the three families of curves, which are mentioned in Proposition 2(i), coincide.
(b) On an Edlinger surface the curves of constant striction distance and the \(K\)-curves coincide and they are lines of curvature.

3.4. In this paragraph we will study ruled surfaces, whose divergence or rotation (curl), with respect to the metric \(g_{ij} du' dv'\), of the vector field \((\alpha)\bar{T}\) \((\alpha \neq 1/4)\) vanishes identically.

The divergence of \((\alpha)\bar{T}(u, v)\) is given by the relation [11 p. 121]
\[
\text{div} \left( (\alpha)\bar{T}(u, v) \right) = \frac{1}{w} \left( \frac{\partial (wT^1)}{\partial u} + \frac{\partial (wT^2)}{\partial v} \right). \quad (3.39)
\]
After a short calculation we obtain
\[
\text{div} \left( (\alpha) \bar{T}(u, v) \right) = \frac{(1 - 4\alpha) |\delta|^{2\alpha - 2}}{w^{4\alpha + 3}} \left\{ (3 - 4\alpha) \kappa v^4 + \delta^2 [4(1 - \alpha) \kappa + (4\alpha - 1) \lambda] v^2 - 4\alpha \delta^2 \delta' v + \delta^4 (\kappa - \lambda) \right\}. 
\]

The identical vanishing of \( \text{div} \left( (\alpha) \bar{T}(u, v) \right) \) implies the following conditions:
\[
(3 - 4\alpha) \kappa = 0, 
\]
\[
4(1 - \alpha) \kappa + (4\alpha - 1) \lambda = 0, 
\]
\[
\alpha \delta' = 0, 
\]
\[
\kappa - \lambda = 0. 
\]

Elementary treatment of the above system yields: a) if \( \alpha = 0 \), then \( \kappa = \lambda = 0 \), i.e. \( \Phi \) is a right conoid, b) if \( \alpha \in \mathbb{R} \setminus \{0, 1/4\} \), then \( \kappa = \lambda = \delta' = 0 \), which means that \( \Phi \) lies on a right helicoid.

We compute now the rotation of \( (\alpha) \bar{T}(u, v) \). According to [11, p. 125] holds
\[
\text{rot} \left( (\alpha) \bar{T}(u, v) \right) = \frac{1}{w} \left[ \frac{\partial}{\partial u} \left( T^1 g_{12} + T^2 g_{22} \right) - \frac{\partial}{\partial v} \left( T^1 g_{11} + T^2 g_{12} \right) \right]. 
\]

On account of (3.4), (3.14) and (3.15) we then find
\[
\text{rot} \left( (\alpha) \bar{T}(u, v) \right) = \frac{\varepsilon (1 - 4\alpha) |\delta|^{2\alpha - 3}}{2w^{4\alpha + 4}} \left\{ [4(\alpha - 1) \delta' \kappa + 2\delta \kappa'] v^5 + \delta^2 [4\delta \kappa' - 6\delta' \kappa + (4\alpha - 1) \delta' \lambda] v^3 + \delta^2 [-3\delta'^2 + 2\delta \delta' + 2\delta^2 (4\alpha - 3) (1 + \kappa \lambda)] v^2 + \delta^4 [-2(2\alpha + 1) \delta' \kappa + 2\delta \kappa' + (4\alpha - 1) \delta' \lambda] v + \delta^4 [- (2\alpha + 1) \delta'^2 + \delta \delta' - 2\delta^2 (1 + \kappa \lambda)] \right\},
\]
which vanishes identically if and only if
\[
2(\alpha - 1) \delta' \kappa + \delta \kappa' = 0, 
\]
\[
2(\alpha - 1) \delta'^2 + \delta \delta' + 4\delta^2 (2\alpha - 1) (1 + \kappa \lambda) = 0, 
\]
\[
4\delta \kappa' - 6\delta' \kappa + (4\alpha - 1) \delta' \lambda = 0, 
\]
\[
-3\delta'^2 + 2\delta \delta' + 2\delta^2 (4\alpha - 3) (1 + \kappa \lambda) = 0, 
\]
\[
-2(2\alpha + 1) \delta' \kappa + 2\delta \kappa' + (4\alpha - 1) \delta' \lambda = 0, 
\]
\[
-(2\alpha + 1) \delta'^2 + \delta \delta' - 2\delta^2 (1 + \kappa \lambda) = 0. 
\]

From (3.47) and (3.49) we obtain
\[
\delta' (2\kappa - \lambda) = 0. 
\]
and from (3.50) and (3.52)
\[ \delta \delta' - 2 \alpha \delta' = 0. \tag{3.54} \]

From (3.52) and (3.54), and if \( \lambda = 2 \kappa \), we deduce that
\[ \delta^2 + 2 \delta' + 4 \delta^2 \kappa^2 = 0, \]
a contradiction, since \( \delta \neq 0 \). From (3.47) and for \( \delta' = 0 \) we then obtain \( \kappa' = 0 \) and from (3.52) \( 1 + \kappa \lambda = 0 \). Thus, \( \Phi \) is an Edlinger surface, whose osculating hyperboloids are congruent \[1, \text{p.} 36\]. The above discussion can be summarized in

**Proposition 5** Let \( \Phi \subset \mathbb{R}^3 \) be a ruled \( C^3 \)-surface, free of torsal rulings, which is relatively normalized by \( ^{(\alpha)} \vec{y} \) (\( \alpha \in \mathbb{R}/ \{1/4\} \)). For the Tchebychev vector field \( ^{(\alpha)} \vec{T} \) the following properties are valid:

(i) \( \text{div}^{(0)} \vec{T} \equiv 0 \), when \( \Phi \) is a right conoid.
(ii) \( \text{div}^{(\alpha)} \vec{T} \equiv 0 \) (\( \alpha \in \mathbb{R}/ \{0, 1/4\} \)), when \( \Phi \) lies on a right helicoid.
(iii) \( \text{rot}^{(\alpha)} \vec{T} \equiv 0 \) (\( \alpha \in \mathbb{R}/ \{1/4\} \)), when \( \Phi \) is an Edlinger surface with congruent osculating hyperboloids.

### 4 The affine normal image of a ruled surface

We consider a ruled surface \( \Phi \) with the parametrization (3.1). A parametrization of the affine normal image \( \Phi^* \) of \( \Phi \) is obtained setting in (3.18) \( \alpha = 1/4 \):

\[ \vec{x}^* := \left( \frac{1}{4} \right) \vec{y} = \varepsilon |\delta|^{-1/2} \vec{n} + \frac{(2 \kappa \nu + \delta')}{2} |\delta|^{-3/2} \vec{e}, \quad \varepsilon = \text{sign} \delta. \tag{4.1} \]

From the above parametrization we have: Two ruled surfaces \( \Phi, \tilde{\Phi} \), parametrized by (3.2), with parallel rulings and the same parameter of distribution have the same affine normal image.

Hereafter, we consider only non-conoidal \( (\kappa \neq 0) \) ruled surfaces. In this case \( \Phi^* \) is a ruled surface, whose rulings are parallel to the corresponding rulings of \( \Phi \). Its directrix
\[ \Gamma : \vec{r}^* = \varepsilon |\delta|^{-1/2} \vec{n} \tag{4.2} \]
is a geometrically distinguished curve of \( \Phi^* \): it is the sectional curve of the director-cone of the surface of the central normals of \( \Phi \) with \( \Phi^* \). We can easily find the line of striction of \( \Phi^* \) to be

\[ s^* = \frac{\delta' |\delta|^{-3/2}}{2} \vec{e} + \varepsilon |\delta|^{-1/2} \vec{n}. \tag{4.3} \]

We observe that setting in (4.1) \( \nu = 0 \), we find the above vector. So we have: The lines of striction of \( \Phi \) and \( \Phi^* \) correspond to each other. Furthermore we conclude: The line of striction of \( \Phi^* \) coincides with the curve \( \Gamma \) if and only if the parameter of distribution of \( \Phi \) is constant.
After a short computation we find as fundamental invariants \((\kappa^*, \delta^*, \lambda^*)\) of \(\Phi^*\):

\[
\kappa^* = \kappa, \quad (4.4)
\]

\[
\delta^* = \varepsilon \kappa |\delta|^{-1/2}, \quad (4.5)
\]

\[
\lambda^* = \frac{2\delta\delta^* - 3\delta^2 - 4\delta^2}{4\delta^2\kappa}. \quad (4.6)
\]

From these relations we obtain:

a) \(\Phi^*\) is an orthoid ruled surface \((\lambda^* = 0)\) if and only if \(\delta = 1/(c_1 \sin u + c_2 \cos u)^2\), where \(c_1, c_2 = \text{const.}\)

b) The line of striction of \(\Phi^*\) is a line of curvature \((1 + \kappa^* \lambda^* = 0)\) if and only if \(\delta = \text{const.} \neq 0\) or \(\delta = c_1/u^2\), where \(c_1 = \text{const.} \neq 0\).

The ruled surfaces \(\Phi\) and \(\Phi^*\) are congruent if and only if \(\kappa = \kappa^*, \delta = \delta^*, \lambda = \lambda^*\) or, because of \((4.4)-(4.6),

\[
\kappa = |\delta|^{3/2}, \quad \lambda = \frac{2\delta\delta^* - 3\delta^2 - 4\delta^2}{4|\delta|^{7/2}}. \quad (4.7)
\]

Thus we deduce

**Proposition 6** Let \(\Phi \subset \mathbb{R}^3\) be a non-conoidal ruled \(C^3\)-surface, free of torsal rulings, which is parametrized by \((3.1)\). Then \(\Phi\) is congruent with its affine normal image \(\Phi^*\) if and only if the fundamental invariants of \(\Phi\) are associated with \((4.7)\).

By virtue of \((3.3)\), we can easily confirm that the vectors \(\bar{s}'(u)\) and \(\bar{r}'(u)\) are parallel or orthogonal if and only if \(\delta' = 1 + \kappa \lambda = 0\) or \(\kappa = \lambda\) respectively. The surface \(\Phi\) is an Edlinger surface in the first case. In the second case the line of striction of \(\Phi\) is an asymptotic line. We formulate these results in the following

**Proposition 7** Let \(\Phi \subset \mathbb{R}^3\) be a non-conoidal ruled \(C^3\)-surface, free of torsal rulings, and \(\Phi^*\) its affine normal image. If the tangents of the line of striction of \(\Phi\) and the directrix \(\Gamma\) of \(\Phi^*\) in the corresponding points are parallel (resp. orthogonal), then \(\Phi\) is an Edlinger surface (resp. the line of striction of \(\Phi\) is an asymptotic line).

Let \(\Phi^*\) be an Edlinger surface, i.e. \(\delta^* = 1 + \kappa^* \lambda^* = 0\). On account of \((4.4)-(4.6)\) we obtain \(\kappa = \text{const.}, \delta = \text{const.}\) or

\[
\kappa = \frac{c_0}{u}, \quad \delta = \frac{c_1}{u^2}, \quad c_0, c_1 = \text{const.}, \quad c_0 c_1 \neq 0. \quad (4.8)
\]

The condition \(\kappa = \text{const.}\) means that \(\Phi\) is of constant slope. So we can state

**Proposition 8** Let \(\Phi \subset \mathbb{R}^3\) be a non-conoidal ruled \(C^3\)-surface, free of torsal rulings, which is parametrized by \((3.1)\). If the affine normal image \(\Phi^*\) of \(\Phi\) is an Edlinger surface, then \(\Phi\) is a ruled surface of constant slope and constant parameter of distribution or the relations \((4.8)\) are valid.
If both ruled surfaces $\Phi$ and $\Phi^*$ are Edlinger surfaces, then their fundamental invariants are constant. Consequently we have

**Proposition 9** Assume that the affine normal image of an Edlinger $C^3$-surface is an Edlinger surface too. Then the osculating hyperboloids of each of them are congruent.

We conclude this work by studying the rulings-preserving mapping $f : \Phi \rightarrow \Phi^*$ defined by considering the parametrizations (3.1), (4.1) and making the association $\bar{x}(u, v) \rightarrow \bar{x}^*(u, v)$. The components $g^*_{ij}$ of the first fundamental tensor of $\Phi^*$ in the coordinates $u^1 := u$, $u^2 := v$ are the following

$$g^*_{11} = |\delta|^{-5} \left[ \frac{2\delta\delta^* - 3\delta^2 - 4\delta^2}{4} + \frac{(2\delta\kappa' - 3\delta\kappa)v}{2} \right]^2 + \kappa^2 v^2 |\delta|^{-3} + \kappa^2 |\delta|^{-1}, \quad (4.9)$$

$$g^*_{12} = g^*_{21} = \varepsilon \kappa |\delta|^{-4} \left[ \frac{2\delta\delta^* - 3\delta^2 - 4\delta^2}{4} + \frac{(2\delta\kappa' - 3\delta\kappa)v}{2} \right], \quad (4.10)$$

$$g^*_{22} = \kappa^2 |\delta|^{-3}. \quad (4.11)$$

Let $f$ be an area-preserving mapping. Then $\det(g_{ij}) = \det(g^*_{ij})$, which, on account of (3.4) and (4.9)-(4.11), is equivalent to

$$\kappa^4 |\delta|^{-6} v^2 + \kappa^4 |\delta|^{-4} = v^2 + \delta^2. \quad (4.12)$$

Hence

$$\kappa = \varepsilon_0 |\delta|^{3/2}, \quad \varepsilon_0 = \pm 1. \quad (4.13)$$

Let the mapping $f$ be conformal. Then

$$\frac{g^*_{11}}{g_{11}} = \frac{g^*_{12}}{g_{12}} = \frac{g^*_{22}}{g_{22}}. \quad (4.14)$$

Inserting (4.10) in (4.14) and taking into account (3.4), we find

$$\kappa = c |\delta|^{3/2}, \quad \lambda = \frac{2\delta\delta^* - 3\delta^2 - 4\delta^2}{4c|\delta|^{7/2}}, \quad c = \text{const.} \neq 0. \quad (4.15)$$

Especially, the conformal mapping $f$ is an isometry (of Minding) if and only if $c = \varepsilon_0$. So we have the following result:

**Proposition 10** The above mentioned mapping $f : \Phi \rightarrow \Phi^*$ is an area-preserving mapping (resp. conformal), if the fundamental invariants of $\Phi$ are associated with (4.13) (resp. (4.15)). In particular $f$ is an isometry if and only if $c = \varepsilon_0$.

**Remark 11** By an isometry $f : \Phi \rightarrow \Phi^*$ the fundamental invariants $(\kappa^*, \delta^*, \lambda^*)$ of $\Phi^*$ are $(\kappa, \varepsilon_0 \delta, \lambda)$. This means that the ruled surfaces $\Phi$ and $\Phi^*$ are congruent or opposite congruent (in German “gegensinnig kongruent”) if $\varepsilon_0 = +1$ or $\varepsilon_0 = -1$ respectively [1, p. 23].

**Acknowledgement**

The authors would like to express their thanks to the referee for his useful remarks.

11
References

[1] Hoschek J.: Liniengeometrie. Bibliographisches Institut, Zürich 1971. Zbl 0227.53007

[2] Kruppa E.: Analytische und konstruktive Differentialgeometrie. Springer-Verlag, Wien 1957. Zbl 0077.15401

[3] Manhart F.: Uneigentliche Relativsphären, die Regelflächen oder Rückungsflächen sind. Geometry, Proc. Congr., Thessaloniki/Greece 1987, 106-113 (1988). Zbl 0641.53004

[4] Manhart F.: Eigentliche Relativsphären, die Regelflächen oder Rückungsflächen sind. Anz. Österreich. Akad. Wiss. Math.-Natur. Kl. 125 (1988), 37-40.

[5] Manhart F.: Relativgeometrische Kennzeichnungen Euklidischer Hypersphären. Geom. Dedicata 29 (1989), 193-207. Zbl 0733.53006

[6] Opozda B.; Sasaki T.: Surfaces whose affine normal images are curves. Kyushu J. Math. 49 (1995), 1-10. Zbl 0837.53012

[7] H. Sachs: Einige Kennzeichnungen der Edlinger-Flächen. Monatsh. Math. 77 (1973), 241-250. Zbl 0259.53005

[8] Schirokow P.A.; Schirokow A.P.: Affine Differentialgeometrie, B. G. Teubner Verlagsgesellschaft, Leipzig 1962. Zbl 0106.14703

[9] Schneider R.: Zur affinen Differentialgeometrie im Grossen I. Math. Z. 101 (1967), 375-406. Zbl 0156.20101

[10] Stamou G.; Magkos A.: Regelflächen relativgeometrisch behandelt. Beitr. Algebra Geom. 45 (2004), 209-215. Zbl 1060.53011

[11] Strubecker K.: Differentialgeometrie II. Sammlung Göschen, Walter de Gruyter & Co, Berlin 1969. Zbl 0169.23501