Abstract—The $\mathcal{H}_2$ guaranteed cost decentralized control problem is investigated in this work. More specifically, on the basis of an appropriate $\mathcal{H}_2$ re-formulation that we put in place, the optimal control problem in the presence of parameter uncertainties is solved in parameter space. It is shown that all the stabilizing decentralized controller gains for the uncertain system are parameterized in a convex set, and then the formulated problem is converted to a conic optimization problem. It facilitates the use of the symmetric Gauss-Seidel (sGS) semi-proximal augmented Lagrangian method (ALM), which attains high computational efficiency. A comprehensive analysis is given on the application of the approach in solving the optimal decentralized control problem; and subsequently, the preserved decentralized structure, robust stability, and robust performance are all suitably guaranteed with the proposed methodology. Furthermore, illustrative examples are presented to demonstrate the effectiveness of the proposed optimization approach.

Index Terms—Convex optimization, optimal control, decentralized control, parameter uncertainties, parameter space, augmented Lagrangian method.

I. INTRODUCTION

Decentralized control has been widely applied to various large-scale systems in view of several important attractive properties [1], such as parallel computation of control variables, ease of communication among controllers, sensors, and actuators, etc. Typical applications of decentralized control in these situations include electrical power network models [2], transportation systems [3], communication networks [4], robotics [5], etc. Further, in general, a large-scale multi-input-multi-output (MIMO) system can essentially be decomposed into a set of physically coupled subsystems, and it then allows the independent design of controllers suitably. As the decentralized control approach has a multi-level control architecture, the control system thus also exhibits some good robustness properties towards various structural perturbations that can occur at the high-level control function, such as the breakdown of information exchanges between subsystems [6]. Stabilization of a MIMO system by decentralized control has been rather thoroughly studied in the literature, with methodologies such as Lyapunov-type methods [7] and algebraic-type methods [8], [9]. On the other hand, although there are already some optimal and robust control techniques that are relatively straightforward and well-established, it is still not easy to apply these relevant optimal and robust control techniques in decentralized control; such as the common-place usage of algebraic Riccati equations (AREs) for Linear Quadratic Regulator (LQR), Linear Quadratic Gaussian (LQG), $\mathcal{H}_2$, and $\mathcal{H}_\infty$ control [10] in the non-decentralized control situation. The key evident reason is that the decentralized controller gain matrix exhibits specific sparsity constraints, i.e. zero blocks in the off-diagonal entries of the composite gain matrix. In view of the sparsity constraints due to this decentralized control architecture, various alternate controller design methods have been presented to cater to this shortcoming. Thus in [11], through a problem re-formulation, sufficient conditions are given such that the standard LQG theory could be applied. Elsewhere in [12], in the LQR and $\mathcal{H}_2$ control problems, it is noted that the gradient of a predefined objective function with respect to the controller gain admits a closed-form solution, which enables the use of the gradient descent method (after the projection of the gradient onto the decentralized constraint hyperplanes). However, this approach only gives the local optimum that is highly dependent on controller initialization, and it does not take the robustness issue into consideration.

To further address these shortcomings, several works on parameterization have been proposed where approaches are developed such that the constrained optimization problem can be solved in extended parameter space. For example in [13], it is shown that the sparsity constraints can be explicitly expressed in the parameter space, thereby invoking the development and utilization of a cutting-plane optimization algorithm. In such a framework, the global optimum is then determined through an iterative framework, and also robustness towards parameter uncertainties is ensured [14], [15]. Nevertheless, a very real practical constraint is that the convergence rate of the cutting-plane optimization algorithm is rather slow. The first reason is that, due to the requirement in the procedure for the generation
of separating hyperplanes, additional equality constraints are added to the linear programming problem during each iteration (thereby adding to the computational burden). The second reason is that the nonlinear constraints are solved through outer linearization, which is not computationally efficient; and these constraints are not always exhaustively exactly satisfied but violated in a small number of instances, even upon completion of all iterations.

Added to all the above observations, it is useful to note that [16] reveals that all stabilizing controllers can be characterized by convex constraints on the Youla-Kucera parameter; and under this framework, synthesis procedures are provided to obtain a feasible controller that admits the optimal performance. Some similar research results and insights are also reported in [17]–[19]. However, these techniques are also computationally expensive, and the required conditions based on quadratic invariance can be rather stringent for many practical cases. Moreover, the transfer-function-level operations involved in these techniques are inherently less stable when processed in numerical computations. Thus, related works have reported that several appropriate mathematical tools can be used to solve the resulting optimization problem, such as the use of Linear Matrix Inequality (LMI), but a reliable solution is not always achievable; and these relevant works include an $H_\infty$ decentralized fuzzy control presented in [20], and a robust decentralized control strategy based on the inclusion principle in [21].

In more recent works, a new optimization technique called the augmented Lagrangian method (ALM) has been presented, which has attracted considerable attention from researchers in the optimal control area [22]. However, for some ill-conditioned optimization problems, a very slow rate of convergence can result from the application of the conventional ALM, and for some large-scale optimization problems, the conventional ALM cannot even be used to solve the given problem successfully. Since the scale of the $H_2$ optimization problem under parameter uncertainties is usually very large, and the condition number of the resulting optimization problem is typically not clear, the conventional ALM is not feasible as a proposed methodology for the specified optimization problem here. To cater to the shortcomings of the conventional ALM, in [23], a semi-proximal term is introduced into the augmented Lagrangian function to ensure that the related sub-problems can be solved efficiently. Additionally, in [24], an approach denoted as the symmetric Gauss-Seidel (sGS) technique is applied to ensure the large-scale problem can be separated into a group of sub-problems. Nevertheless, while promising, all these methodologies essentially provide only rather generic guidelines at this stage, and a more definite strategy and formulation (with accompanying comprehensive analysis) is still lacking on how to extend their usages in real-world problems.

With all of the above descriptions as a back-drop, in this work, we develop and propose a definite strategy and formulation that uses the semi-proximal ALM and the sGS to specifically solve the optimal decentralized control problem with parameter uncertainties. On the basis of a parameter space formulation, the optimization problem is further reformulated as a convex optimization problem, where the sGS semi-proximal ALM is utilized to efficiently obtain the optimal solution. Importantly here in our formulation, the sparsity constraints resulting from the decentralized control structure are satisfied. Additionally, robust stability in the presence of parameter uncertainties is guaranteed, and the system performance is maintained within a prescribed level. This approach thus addresses all the shortcomings as previously highlighted, and also attains a high computational efficiency in solving the optimal decentralized control problem.

The remainder of this paper is organized as follows. In Section II, the problem formulation of the $H_2$ guaranteed cost decentralized control problem is given in its usual commonly-encountered form. In Section III, we present and show how the resulting optimization problem can be suitably re-formulated in the parameter space. Next, since this problem now takes the structure of a convex optimization problem, we then present and develop the specific procedures utilizing the sGS semi-proximal ALM for efficient optimization in this specific parameter space re-formulation of the $H_2$ guaranteed cost decentralized control problem. In Section V, two illustrative examples are given to validate the results. Finally, pertinent conclusions are drawn in Section VI.

## II. Problem Statement

The following notations are used in the remaining text. $\mathbb{R}^{m \times n}$ ($\mathbb{R}^n$) denotes the real matrix with $m$ rows and $n$ columns ($n$ dimensional real column vector). $\mathbb{S}^n$ ($\mathbb{S}_+^n$) denotes the $n$ dimensional (positive semi-definite) real symmetric matrix. The symbol $A > 0$ ($A \succeq 0$) means that the matrix $A$ is positive (semi-)definite, and $A > B$ ($A \succeq B$) means $A - B$ is positive (semi-)definite. $A^T$ ($x^T$) denotes the transpose of the matrix $A$ (vector $x$). $I_n$ represents the identity matrix with a dimension of $n \times n$. The operator $\text{Tr}(A)$ denotes the trace of the square matrix $A$. The operator $\langle A, B \rangle$ denotes the Frobenius inner product i.e. $\langle A, B \rangle = \text{Tr}(A^T B)$ for all $A, B \in \mathbb{R}^{m \times n}$. The norm operator based on the inner product operator is defined by $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in \mathbb{R}^{m \times n}$. $\|H(s)\|_2$ represents the $H_2$-norm of $H(s)$. $\otimes$ denotes the Kronecker product. $\text{eig}(A)$ represents all the eigenvalues of the matrix $A$. $\text{blockdiag}\{A_1, A_2, \cdots, A_n\}$ denotes a block diagonal matrix with diagonal entries $A_1, A_2, \cdots, A_n$. $\text{diag}\{a_1, a_2, \cdots, a_n\}$ denotes a diagonal matrix with diagonal entries $a_1, a_2, \cdots, a_n$.

Consider a linear time-invariant (LTI) system

$$\dot{x}(t) = Ax(t) + B_2u(t) + B_1w(t), \quad (1)$$

$$z(t) = Cx(t) + Du(t), \quad (2)$$

with a static state feedback controller

$$u(t) = -Kx(t), \quad (3)$$

where $x \in \mathbb{R}^n$ is the state vector with $x(0) = x_0$, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^l$ is the exogenous disturbance input, $z \in \mathbb{R}^n$ is the controlled output, $K \in \mathbb{R}^{m \times n_2}$ is the feedback gain matrix. $A, B_1, B_2, C,$ and $D$ are constant real matrices with appropriate dimensions. It is assumed that there is no cross weighting between the state variables and the control variables, i.e. $C^TD = 0$, and the control weighting matrix is...
nonsingular, i.e., $D^TD > 0$. Also, as a usual practice, it is assumed that $(A, B_2)$ is stabilizable and the pair $(A, C)$ has no unobservable modes on the imaginary axis. Remarkably, for a decentralized control problem, $K$ is constrained to be block diagonal.

Here, the objective function is defined as

$$J = \int_0^\infty z(t)^Tz(t)\,dt.$$  

(4)

To optimize (4), it is equivalent to minimize the $H_2$-norm of the transfer function

$$H(s) = (C - DK)(sI_n - A + B_2K)^{-1}B_1,$$  

(5)

from $w$ to $z$, and the objective function (4) can be reformulated as

$$J(K) = \|H(s)\|_2^2 = \text{Tr}(C - DK)W_c(C - DK)^T = \text{Tr}(B_1^TW_cB_1),$$  

(6)

where $W_c$ and $W_o$ are the controllability Gramian and the observability Gramian associated with the closed-loop system.

If all the associated matrices in the system are precisely known (there is no parameter uncertainty), the problem of finding the optimal decentralized $K$ to minimize the objective function (4) is considered as an $H_2$ decentralized control problem. On the other hand, with the existence of parameter uncertainties, the design of a decentralized controller such that the upper bound to the $H_2$-norm is minimized is referred to as an $H_2$ guaranteed cost decentralized control problem.

III. OPTIMIZATION PROBLEM FORMULATION IN PARAMETER SPACE

It is assumed that $A$ and $B_2$ are subjected to parameter uncertainties. Define $p = n + m$, and then the following extended matrices are introduced for an alternative representation of the system:

$$F = \begin{bmatrix} A & -B_2 \\ 0 & 0 \end{bmatrix} \in R^{p\times p}, \quad G = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \in R^{p\times m},$$

$$Q = \begin{bmatrix} B_1B_1^T & 0 \\ 0 & 0 \end{bmatrix} \in S^p, \quad R = \begin{bmatrix} C^TC & 0 \\ 0 & D^TD \end{bmatrix} \in S^p.$$  

(7)

Assumption 1. The parameter uncertainties are structural and convex-bounded.

Followed by Assumption 1, it is assumed that $F$ belongs to a polyhedral domain, which is expressed by a convex combination of the extreme matrices, where $F = \sum_{i=1}^M \xi_i F_i$, $\xi_i \geq 0$, $\sum_{i=1}^M \xi_i = 1$, $F_i = \begin{bmatrix} A_i & -B_{2i} \\ 0 & 0 \end{bmatrix} \in R^{p\times p}$ denotes the extreme vertex of the uncertain domain. Remarkably, the precisely known system is a special case of the above expression, where $M = 1$.

Define the matrix

$$W = \begin{bmatrix} W_1 & W_2 \\ W_2^T & W_3 \end{bmatrix} \in S^p,$$  

(8)

with $W_1 \in S^n_+$, $W_2 \in R^{n\times m}$, $W_3 \in S^m$, and then define $\Theta(W) = F_1W + WF_2^T + Q$, which can also be denoted in the block matrix form, where

$$\Theta(W) = \begin{bmatrix} \Theta_1(W) & \Theta_2(W) \\ \Theta_3(W) & \Theta_4(W) \end{bmatrix} \in S^p,$$  

(9)

with $\Theta_1(W) \in S^n, \Theta_2(W) \in R^{n\times m}, \Theta_3(W) \in S^m$. Theorem 1 presents a set consisting of all the controller gains that preserve the decentralized structure, robust stability, and robust performance, in the presence of parameter uncertainties.

Theorem 1. For the controller with a specific decentralized structure

$$K = \text{blockdiag} \{K_{D,1}, K_{D,2}, \ldots, K_{D,m}\},$$  

(10)

with $K_{D,i} \in R^{1\times D_i}$, $\forall i = 1, \ldots, m$, one can define the set $\mathcal{C}$ of $W \in S^p : W \geq 0, \Theta(W) \leq 0, W_1 = \text{blockdiag}(W_{1i,1}, W_{1i,2}, \ldots, W_{1i,m})$ with $W_{1i,i} \in R^{D_i \times D_i}$, $W_2 = \text{blockdiag}(W_{2i,1}, W_{2i,2}, \ldots, W_{2i,m})$ with $W_{2i,i} \in R^{D_i}$. Then we have

$$K = W_2^TW_1^{-1} = \text{blockdiag}\left\{W_{2i,1}W_{1i,1}^{-1}W_{2i,2}W_{1i,2}^{-1}, \ldots, W_{2i,m}W_{1i,m}^{-1}\right\},$$  

(11)

which holds the decentralized structure in (10).

For Statement (b), $\Theta_1(W) \leq 0$ is equivalent to

$$A_iW_1 - B_2W_2 + W_1A_i^T - W_2B_{2i}^T + B_1B_1^T \leq 0.$$  

(12)

Since $W_1 > 0$, we have

$$(A_i - B_2W_2^{-1}W_1^{-1})W_1 + W_1(A_i - B_2W_2^{-1}W_1^{-1})^T + B_1B_1^T \leq 0.$$  

(13)

From the Lyapunov stability condition, it is straightforward that $K = W_2^{-1}W_1^{-1}$ stabilizes the extreme system and then we have $W_2 = W_1K^T$, and subsequently we can construct $W = \begin{bmatrix} W_1 & W_1K^T \end{bmatrix}$. From Schur complement, $W_3$ is a free variable to choose such that $W \geq 0$ is ensured. $W \geq 0$ is not a necessary condition to stabilize the extreme systems, but it indeed provides a norm bound to the controller gain. It is straightforward that, to ensure the stability over the entire uncertain domain which is convex, it suffices to check the stability at the vertices of the convex polyhedron. Therefore, if the stability holds for all the extreme systems, then the stability for the entire uncertain domain is guaranteed.

For Statement (c), (13) is equivalent to

$$(A_i - B_2iK)W_1 + W_1(A_i - B_2iK)^T + B_1B_1^T \leq 0.$$  

(14)
The controllability Gramian \( W_c > 0 \), and it satisfies
\[
(A_i - B_{2i}K)W_c + W_c(A_i - B_{2i}K)^T + B_1B_1^T = 0.
\]
We claim that \( W_1 \geq W_c \) and it can be proved by contradiction. From (14) and (15), we have
\[
(A_i - B_{2i}K)(W_c - W_1) + (W_c - W_1)(A_i - B_{2i}K)^T \geq 0.
\]
Assume \( W_c > W_1 \), and then from the Lyapunov instability theorem [25], it contradicts the fact that \( K \) stabilizes the closed-loop system. Therefore, we have \( W_1 \geq W_c \). This completes the proof of the claim.

Then, from Schur complement, \( W \geq 0 \) leads to
\[
W_3 \geq W_2W_1^{-1}W_2
= KW_1K^T
\geq KWW_1K^T.
\]
Therefore, it follows that
\[
\langle R, W \rangle = \text{Tr} \left( CW_1C^T + DW_3D^T \right)
\geq \text{Tr} \left( CW_1C^T + DKW_1K^TD \right)
= \text{Tr} \left( (C - DK)W_c(C - DK)^T \right)
= \|H(s)\|_2^2.
\]

This completes the proof of Theorem 1.

**Definition 1.** The system (2) is called robustly decentralized stabilizable if \( \mathcal{C} \neq \emptyset \).

It can be seen that \( \langle R, W \rangle \) provides an upper bound to \( \|H(s)\|_2^2 \). Then, it is aimed to solve the optimization problem \( W = \arg\min \{ \langle R, W \rangle : W \in \mathcal{C} \} \), which yields \( K = W_2W_1^{-1} \in \mathcal{X} \), such that the upper bound to the \( H_2 \)-norm is minimized. Hence, the optimization problem is summarized in the following form:

\[
\begin{aligned}
\text{minimize} & \quad \langle R, W \rangle \\
\text{subject to} & \quad W \geq 0 \\
& \quad \Theta_1(W) \leq 0 \\
& \quad W \in \Phi(s),
\end{aligned}
\]

where \( \Phi(s) \) represents the set of all \( W \) satisfying the sparsity constraints.

It is obvious that the operator \( \Theta_1(W) \) is a bounded linear operator (affine to \( W \)), and the sparsity constraints are linear equality constraints. Since the objective function is also a linear function, and the optimization variable \( W \) is confined in a convex cone, thus it falls into the category of the convex optimization problem, where \( \mathcal{C} \) is a convex set. In particular, the optimization problem is a linear conic programming problem. In this paper, the sGS semi-proximal ALM is introduced to solve the given optimization problem.

To express the optimization problem explicitly, we define a matrix \( V = \begin{bmatrix} I_n & 0_{n\times m} \end{bmatrix} \), and then the optimization problem can be equivalently expressed in the matrix form, where
\[
\begin{aligned}
\text{minimize} & \quad \langle R, W \rangle \\
\text{subject to} & \quad W \in \mathbb{S}^p_+ \\
& \quad -V(F_1W + WF_1^T + Q)V^T \in \mathbb{S}^n_+ \\
& \quad -V(F_2W + WF_2^T + Q)V^T \in \mathbb{S}^n_+ \\
& \quad \vdots \\
& \quad -V(F_MW + WF_M^T + Q)V^T \in \mathbb{S}^n_+ \\
& \quad V_{11}WV_{12} = 0 \\
& \quad V_{21}WV_{22} = 0 \\
& \quad \vdots \\
& \quad V_{N1}WV_{N2} = 0.
\end{aligned}
\]

**Remark 1.** By splitting \( W_1 \) and \( W_2 \) into \( m^2 \) sub-blocks, the zero blocks in \( W_1 \) can be expressed by \( m(m - 1)/2 \) equality constraints (because \( W_1 \) is symmetric), and the zero blocks in \( W_2 \) can be expressed by another \( m(m - 1) \) equality constraints. Therefore, \( N = 3m(m - 1)/2 \). Obviously, adjacent zero blocks can be combined into one single equality constraint, thus \( N = 3m(m - 1)/2 \) is not the minimum number of the equality constraints. For the sake of illustration purposes and without loss of generality, the adjacent zero blocks are not combined in the following analysis.

Then the optimization problem can be denoted in a compact form, which is shown as
\[
\begin{aligned}
\text{minimize} & \quad \langle R, W \rangle \\
\text{subject to} & \quad G(W) \in \mathcal{K},
\end{aligned}
\]
where \( G(W) \) is a linear mapping which is given by
\[
G(W) = \begin{bmatrix}
W \\
-V(F_1W + WF_1^T + Q)V^T \\
-V(F_2W + WF_2^T + Q)V^T \\
\vdots \\
-V(F_MW + WF_M^T + Q)V^T \\
V_{11}WV_{12} \\
V_{21}WV_{22} \\
\vdots \\
V_{N1}WV_{N2}
\end{bmatrix},
\]
and the convex cone \( \mathcal{K} \) can be denoted as
\[
\mathcal{K} = \mathbb{S}^p_+ \times \mathbb{S}^n_+ \times \cdots \times \mathbb{S}^n_+ \\
\times \left\{ v_{i1}v_{i2} \times \cdots \times \{ v_{N1}v_{N2} \} \right\},
\]
where for all \( i = 1, 2, \ldots, N \), the scalars \( v_{i1} \) and \( v_{i2} \) denote the number of rows of the matrix \( V_{i1} \) and the number of the columns of the matrix \( V_{i2} \), respectively. Since the positive semi-definite cone is self-dual, it is straightforward to express
the dual cone $\mathcal{K}^*$ as

$$
\mathcal{K}^* = S^p \times S^n \times \cdots \times S^n \\
\times \mathbb{R}^{v_{11} \times v_{12}} \times \mathbb{R}^{v_{21} \times v_{22}} \times \cdots \times \mathbb{R}^{v_{N} \times v_{N}^2},
$$

(24)

Define the linear space $\mathcal{X}$ in terms of the cone $\mathcal{K}$, which is given by

$$
\mathcal{X} = S^p \times S^n \times \cdots \times S^n \\
\times \mathbb{R}^{v_{11} \times v_{12}} \times \mathbb{R}^{v_{21} \times v_{22}} \times \cdots \times \mathbb{R}^{v_{N} \times v_{N}^2},
$$

(25)

and it is straightforward that the cone $\mathcal{K} \subset \mathcal{X}$ and the dual cone $\mathcal{K}^* \subset \mathcal{X}$.

**Assumption 2.** Strong duality always holds for the optimization problem (21).

Under Assumption 2, the optimal solution to the linear conic optimization problem (21) can be obtained by solving the corresponding dual problem, due to the difficulty to deal with the primal problem directly. In terms of the optimization problem (21), the Lagrangian function is introduced, which is expressed as

$$
\mathcal{L}(W; X) = \langle R, W \rangle - \langle X, \mathcal{G}(W) \rangle,
$$

(26)

where $X = (X_0, X_1, \ldots, X_{M+N}) \in \mathcal{K}^*$ is the Lagrange multiplier. It follows that the Lagrangian dual function $\theta(X)$ is obtained by

$$
\theta(X) = \min_{W \in \mathcal{X}^p} \mathcal{L}(W; X) \\
= \min_{W \in \mathcal{X}^p} \left\{ \langle R, W \rangle - \langle X, \mathcal{G}(W) \rangle \right\}.
$$

(27)

It is shown that the Lagrangian dual function can be denoted in the explicit form, where

$$
\min_{W \in \mathcal{X}^p} \left\{ \langle R, W \rangle - \langle X, \mathcal{G}(W) \rangle \right\} \\
= \langle X_1 + X_2 + \cdots + X_M, VQV^T \rangle + \langle R, W \rangle - \langle X_0, W \rangle \\
+ \langle F_1^T V^TX_1V + V^TX_1VF_1, W \rangle \\
+ \langle F_2^T V^TX_2V + V^TX_2VF_2, W \rangle \\
\vdots \\
+ \langle F_M^T V^TX_MV + V^TX_MVF_M, W \rangle \\
+ \langle \frac{1}{2} V^T_{11} X_{M+1} V_{12} + \frac{1}{2} V_{12}^T X_{M+1} V_{11}, W \rangle \\
+ \langle \frac{1}{2} V^T_{21} X_{M+2} V_{22} + \frac{1}{2} V_{22}^T X_{M+2} V_{21}, W \rangle \\
\vdots \\
+ \langle \frac{1}{2} V^T_{N1} X_{M+N} V_{N2} + \frac{1}{2} V_{N2}^T X_{M+N} V_{N1}, W \rangle
$$

For the sake of simplicity in the remaining text, we define

$$
\mathcal{F}(X_1, X_2, \ldots, X_M) = -\langle X_1 + X_2 + \cdots + X_M, VQV^T \rangle \\
\mathcal{A}(X_0, X_1, \ldots, X_{M+N}) = \\
- X_0 \\
+ F_1^T V^TX_1V + V^TX_1VF_1 \\
+ F_2^T V^TX_2V + V^TX_2VF_2 \\
\vdots \\
+ F_M^T V^TX_MV + V^TX_MVF_M \\
+ \frac{1}{2} V^T_{11} X_{M+1} V_{12} + \frac{1}{2} V_{12}^T X_{M+1} V_{11} \\
+ \frac{1}{2} V^T_{21} X_{M+2} V_{22} + \frac{1}{2} V_{22}^T X_{M+2} V_{21} \\
\vdots \\
+ \frac{1}{2} V^T_{N1} X_{M+N} V_{N2} + \frac{1}{2} V_{N2}^T X_{M+N} V_{N1}.
$$

(29)

For any linear space $\mathcal{Y}$ and any convex set $\mathcal{C} \subset \mathcal{Y}$, define the indicator function $\delta_C(v)$ such that for any $v \in \mathcal{Y}$, it follows that

$$
\delta_C(v) = \begin{cases} 
0 & \text{if } v \in \mathcal{C} \\
\infty & \text{otherwise}. 
\end{cases}
$$

(32)

Finally, the optimization problem can be equivalently converted to the form with only one linear equality constraint, which is given in the following form:

$$
\min_{X \in \mathcal{X}} \mathcal{F}(X_1, X_2, \ldots, X_M) + \delta^p_{\mathcal{C}}(X_0) + \delta^n_{\mathcal{C}}(X_1) + \delta^p_{\mathcal{C}}(X_2) + \cdots + \delta^p_{\mathcal{C}}(X_M) \\
\text{subject to } \mathcal{A}(X) + R = 0.
$$

(33)

**IV. SGS SEMI-PROXIMAL ALM FOR OPTIMAL DECENTRALIZED CONTROL**

In the following text, the sGS semi-proximal ALM is presented to solve the dual problem (28). Notably, in the
remaining text in this section, variable $X$ is considered as the optimization variable of the problem (33) and the variable $W$ is considered as the Lagrange multiplier.

**Step 1: Initialization**

Choose the optimization parameter $\tau \in (0, (1 + \sqrt{3})/2)$. In practice, in case that the rate of convergence is influenced, $\tau$ is usually chosen as 1.618 such that a high rate of convergence can be obtained, and the convergence of the optimization algorithm can be preserved.

As for the optimization parameter $\sigma$, the choice of $\sigma$ can influence the rate of convergence significantly. If $\sigma$ is chosen to be a very small value, the augmented Lagrangian function is similar to the classical Lagrangian function. Then, the optimization direction can be too inaccuracy to be accepted. If $\sigma$ is chosen to be a very large value, the sub-problem in each step is similar to a quadratic optimization problem. The solution to the sub-problem in each step is too conservative to be applied. However, even though the choice of the optimization parameter $\sigma$ is essential, it is not easy to determine the best choice, since it highly depends on the given optimization problem. Besides, the choice of the optimization parameter $\sigma$ does not influence the property of the convergence with the proposed optimization algorithm, though it may destroy the property of convergence in many other types of optimization algorithms.

Then, choose the initial matrices $(X^0; W^0) \in X \times \mathcal{S}^p$. Notably, the initial matrices are not required to satisfy any constraint. One simple choice of the initial matrices is the zero matrix with appropriate dimensions. Notably, although the global optimal point can always be achieved because the given optimization problem is convex, and all the sub-problems in our proposed optimization algorithm is strictly convex, the choice of the initial matrix can also influence the rate of convergence significantly.

After that, choose the parameter of the stopping criterion $\varepsilon > 0$. The stopping criterion is given based on the duality gap between the primal optimization problem and the dual optimization problem. Based on Assumption 2 the optimal point is achieved once the duality gap reaches zero. Since the numerical error always exists during each iteration, the duality gap only approaches zero in finite time. Therefore, once the stopping criterion is satisfied, we can obtain the nearly optimal solution to the given optimization problem.

Finally, set the iteration index $k = 0$.

**Step 2: Update of the optimization variable $X$ by semi-proximal ALM.**

**Step 2.1: Backward sGS sweep**

Define the augmented Lagrangian function as

$$
\mathcal{L}_\sigma(X; W) = F(X_1, X_2, \ldots, X_M) + \delta_{\mathcal{S}^p}(X_0) + \delta_{\mathcal{S}^p}(X_1)
+ \delta_{\mathcal{S}^p}(X_2) + \cdots + \delta_{\mathcal{S}^p}(X_M)
+ \frac{\sigma}{2} \| A(X) + R - \sigma^{-1}W \|^2 - \frac{1}{2\sigma} \| W \|^2,
$$

where $W \in \mathcal{S}^p$ is the Lagrange multiplier for the augmented Lagrangian function, $\sigma$ is a positive parameter. Notice that there are three groups of variables in the optimization problem: the variable $X_0$ is related to the positive semi-definite constraint for the Lagrange multiplier $W$; the variables $X_1, X_2, \ldots, X_M$ are regarding the positive semi-definite constraints of a group of linear functions; the variables $X_{M+1}, X_{M+2}, \ldots, X_{M+N}$ are related to the linear equality sparsity constraints for the Lagrange multiplier $W$. The sub-problems in the backward sGS sweep will be solved in terms of these three groups of variables separately.

We begin with the sub-problem of the third group. For all $i = N, N - 1, \ldots, 1$, the sub-problem with respect to the variable $X_{M+i}$ is given by

$$
\bar{X}^{k+1}_{M+i} = \text{argmin}_{X_{M+i} \in \mathbb{R}^{v_i \times u_i}} \mathcal{L}_\sigma\left(X_0^k, X_1^k, \ldots, X_{M+i-1}^k, X_{M+i}, \bar{X}^{k+1}_{M+i+1}, \bar{X}^{k+1}_{M+i+2}, \ldots, \bar{X}^{k+1}_{M+N}; W^k\right).
$$

where $\bar{X}^{k+1}_{M+i}$ denotes the solution to the sub-problem in the $(k + 1)$th iteration, $X_0^k, X_1^k, \ldots, X_{M+i-1}^k, W^k$ denote the values of the optimization variable and the Lagrange multiplier in the $k$th iteration, respectively, and $X_{M+i}^k, \bar{X}^{k+1}_{M+i+1}, \bar{X}^{k+1}_{M+i+2}, \ldots, \bar{X}^{k+1}_{M+N}$ denote the updated optimization variables in the $(k + 1)$th iteration.

Before the optimality condition is constructed, Lemma 1 and Property 1 are presented, which are used in the sequel.

**Lemma 1.** For a matrix norm function given by

$$
\mathcal{H}_{i1}(X) = \left\| \frac{1}{2} V_{i1}^T XV_{i2}^T + \frac{1}{2} V_{i2}^T V_{i1} + H_{i1} \right\|^2,
$$

where $H_{i1}$ is an arbitrary constant symmetric matrix with appropriate dimensions, one has

$$
\frac{\partial \mathcal{H}_{i1}(X)}{\partial X} = 2V_{i1} \left( \frac{1}{2} V_{i1}^T XV_{i2}^T + \frac{1}{2} V_{i2}^T V_{i1} + H_{i1} \right) V_{i2}.
$$

**Proof of Lemma 1.** The derivative of the matrix norm function in the form of (36) can be obtained by using some properties of derivative of trace operator. The procedures are simple but tedious, thus the proof is omitted.

**Property 1.** For all the matrices regarding the structural constraints of the matrix $W$, one has the following properties:

$$
\begin{align*}
V_{i1} V_{i1}^T &= I_{v_{i1}}, \\
V_{i2} V_{i2} &= I_{v_{i2}}, \\
V_{i1} V_{i2} &= 0_{v_{i1} \times v_{i2}}.
\end{align*}
$$

**Proof of Property 1.** First, it is easy to extract $W_1$ from $W$ by $W_1 = VWV^T$. As presented in Theorem 1, $W_1$ can be split into $m^2$ blocks where all the non-diagonal entries are zero blocks, i.e.

$$
W_1 = \begin{bmatrix}
W_{1D,1} & 0 & \cdots & 0 \\
0 & W_{1D,2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & W_{1D,m}
\end{bmatrix},
$$

(39)
Take the zero block in the first row and second column as an example, this zero block can be expressed by

$$G_1 W_1 H_1 = 0_{D_1 \times D_2}, \tag{40}$$

where $G_1 = \left[ I_{D_1}, 0_{D_1 \times D_2}, 0_{D_1 \times D_3}, \cdots, 0_{D_1 \times D_n} \right]$ and $H_1 = \left[ 0_{D_2 \times D_1}, I_{D_2}, 0_{D_2 \times D_3}, \cdots, 0_{D_2 \times D_n} \right]^T$. Then, it can be verified that $G_1 V^T G_1 = I_{D_1}, \ (V^T H_1)^T V^T H_1 = I_{D_2}$, and $G_1 V^T H_1 = 0_{D_1 \times D_2}$. Also, it is easy to derive similar results for all the other zero blocks, which shows that Property 2 holds for $W_1$.

Then, $W_2$ can be extracted from $W$ by $W_2 = S_1 W S_2$, where $S_1 = [I_n, 0_{n \times m}]$ and $S_2 = [0_{m \times n}, I_m]^T$. In the same manner, $W_2$ can be split into $m^2$ blocks, where all the non-diagonal entries are zero blocks.

The analysis for $W_2$ proceeds along the same way as $W_1$, and it can also be verified that Property 1 holds for $W_2$. This completes the proof of Property 1.

By using Lemma 1 and Property 1, the optimality condition to the sub-problem can be obtained. Since the sub-problem is an unconstrained optimization problem, the optimality condition is given by

$$0 \in \partial X_{M+i} \mathcal{L}_\sigma \left( X_{0}, X_{1}, \cdots, X_{M+i-1}, X_{M+i}; X_{M+i+1}; X_{M+i+2}; \cdots, X_{M+N}; W^k \right)$$

$$= V_1 \left[ A \left( X_{0}, X_{1}, \cdots, X_{M+i-1}, X_{M+i}; X_{M+i+1}; X_{M+i+2}; \cdots, X_{M+N} \right) + R - \sigma^{-1}W^k \right] V_2.$$

and then we have

$$\hat{X}_{M+i}^{k+1} = -2V_1 \left[ A_{M+i} \left( X_{0}, X_{1}, \cdots, X_{M+i-1}, X_{M+i+1}; X_{M+i+2}; \cdots, X_{M+N} \right) + R - \sigma^{-1}W^k \right] V_2. \tag{42}$$

The solution to the sub-problem of the second group is introduced. Before the optimality condition is given, Lemma 2 is introduced, which is used in the sequel.

**Lemma 2.** For a matrix norm function given by

$$\mathcal{H}_{2}(X) = \left\| F_i^T V^T XV + V^T XV F_i + H_{i2} \right\|^2, \tag{43}$$

where $H_{i2}$ is an arbitrary constant symmetric matrix with appropriate dimensions, one has

$$\frac{\partial \mathcal{H}_{2}(X)}{\partial X} = 2 \left[ VF_i \left( F_i^T V^T XV + V^T XV F_i + H_{i2} \right) V^T + V \left( F_i^T V^T XV + V^T XV F_i + H_{i2} \right) F_i^T V^T \right]. \tag{44}$$

**Proof of Lemma 2.** Similar to Lemma 1, the proof is omitted.

In terms of minimizing the augmented Lagrangian function with respect to the variable $X_i$ for all $i = M, M-1, \cdots, 1$, a proximal term must be considered in the sub-problem during the iterations. To include the proximal term without influencing the convergence of the algorithm, we firstly introduce the positive linear operator. For any linear space $\mathcal{Y}$, a linear operator $S : \mathcal{Y} \rightarrow \mathcal{Y}$ is positive, which means that for all $v \in \mathcal{Y}$, it follows that $\langle v, S(v) \rangle \geq 0$.

Now we can derive the optimality condition to the sub-problem by using Lemma 2 and Property 1, and the sub-problem in terms of the variable $X_i$ in the backward sGS sweep is given by

$$\hat{X}_{i+1}^{k+1} = \arg \min_{X_i \in \mathbb{R}^n} \mathcal{L}_\sigma \left( X_{0}, X_{1}, \cdots, X_{i-1}, X_i, \hat{X}_{i+1}^{k+1}; X_{i+2}, \cdots, \hat{X}_{M+N}^{k+1}; W^k \right) + \frac{1}{2} \left\| X_i - \hat{X}_{i+1}^{k+1} \right\|^2_{S_i}, \tag{45}$$

where $S_i$ is a positive linear operator. To solve this sub-problem efficiently, $S_i$ is chosen as

$$S_i(X) = \alpha_s I(X) - \sigma V F_i^T V^T X - \sigma V F_i^T V^T X V^T F_i V^T$$

$$- \sigma V F_i^T V^T X V^T F_i^T V^T - \sigma X V F_i^T V^T,$$ \hspace{1cm} \tag{46}

where $I$ denotes an identity operator, and $\alpha_s$ is chosen such that $S_i(X)$ is a positive linear operator. One positive choice of $\alpha_s$ is the maximum eigenvalue of the vectorization matrix, which is

$$\alpha_s = \sigma \max \left\{ \epsilon \left( V F_i^T V^T \otimes I + I \otimes V F_i^T V^T \right. \right.$$

$$+ \left. \left( V F_i^T V^T \otimes V F_i^T V^T + (V F_i^T V)^T \otimes (V F_i^T V^T) \right) \right\}. \tag{47}$$

Before presenting the optimality condition to the sub-problem, Theorem 2 is introduced to determine the projection operator.

**Theorem 2.** The projection operator $\Pi_{C}(\cdot)$ with respect to the convex cone $C$ can be expressed as

$$\Pi_{C} = (I + \alpha \partial \mathcal{C})^{-1}, \tag{48}$$

where $\partial \mathcal{C}$ denotes the sub-differential operator, and $\alpha \in \mathbb{R}$ can be an arbitrary real number.

**Proof of Theorem 2.** Define a finite dimensional Euclidean space $\mathcal{X}$ equipped with an inner product and its induced norm such that $C \subset \mathcal{X}$. For any $x \in \mathcal{X}$, there exists $z \in \mathcal{X}$ such that $z \in (I + \alpha \partial \mathcal{C})^{-1}(x)$. Then it follows that

$$x \in (I + \alpha \partial \mathcal{C})(z) = z + \alpha \partial \mathcal{C} \tag{49}$$

Note that the projection operator $\Pi_{C}(z)$ can be expressed as

$$\Pi_{C}(z) = \arg \min_{z \in W} \left\{ \mathcal{C}(x) + \frac{1}{2\alpha} \left\| z - x \right\|^2 \right\}. \tag{50}$$

Since the optimization problem in (50) is strictly convex, the sufficient and necessary optimality condition for the optimization problem of the projection operator can be expressed as

$$0 \in \alpha \partial \mathcal{C} \tag{51}$$

which is equivalent to (49). Note that the projection onto a convex cone is unique. Therefore, the mapping from $x$ to $z$ is also unique, which means the operator $(I + \alpha \partial \mathcal{C})^{-1}(\cdot)$ is a point-to-point mapping. This completes the proof of Theorem 2.
Then the optimality condition to the sub-problem of the second group is given by

\[
0 \in \partial_X L_\sigma \left( X^{k_1} , X^k_1 , \cdots , X^{k_n} ; X^{k+1}_i , X^{k+1}_{i+1} \right) = \left( V Q V^T - \sigma F_i F_i^T X_i^k + \sigma F_i^T V X_i^k V F_i T - \sigma V F_i F_i^T X_i^k + \sigma F_i^T V X_i^k V F_i T \right) \in \alpha_i X_i + \partial \delta_{S_+^n} ( X_i ) ,
\]

and then it follows that

\[
\text{LHS}_i \left( X_i^0 , X_i^1 , \cdots , X_i^{k-1} , X_i^k , X_i^{k+1} , X_i^{k+2} , \cdots , X_i^{k+M+N} ; W^k \right) \in \alpha_i X_i + \partial \delta_{S_+^n} ( X_i ) ,
\]

which is equivalent to

\[
\tilde{X}_i^{k+1} = \Pi_{S_+^n} \left[ \alpha_i^{-1} \text{LHS}_i \left( X_i^0 , X_i^1 , \cdots , X_i^{k-1} , X_i^k , X_i^{k+1} , X_i^{k+2} , \cdots , X_i^{k+M+N} ; W^k \right) \right] ,
\]

where \( \Pi_{S_+^n} ( \cdot ) \) denotes the projection operator in terms of the positive semi-definite cone, and the projection results can be obtained by Lemma 3.

**Lemma 3.** Projection onto the positive semi-definite cone can be computed explicitly. Let \( X = \sum_{i=1}^n \lambda_i v_i v_i^T \in S^n \) be the eigenvalue decomposition of the matrix \( X \) with the eigenvalues satisfying \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \), where \( v_i \) denotes the eigenvector corresponding to the \( i \)th eigenvalue. Then the projection onto the positive semi-definite cone of the matrix \( X \) can be expressed by

\[
\Pi_{S_+^n} ( X ) = \sum_{i=1}^n \max \left\{ \lambda_i , 0 \right\} v_i v_i^T .
\]

**Proof of Lemma 3:** By using eigenvalue decomposition, we have \( X = U \Lambda U^T \) where \( U \) is an orthogonal matrix, and \( \Lambda = \text{diag} \{ \lambda_1 , \lambda_2 , \cdots , \lambda_n \} \) with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \). Then by the definition of the projection operator, we have

\[
\Pi_{S_+^n} ( X ) = \sum_{i=1}^n \max \left\{ \lambda_i , 0 \right\} v_i v_i^T .
\]

**Step 2.2: Forward sGS sweep**

By using the solutions to the sub-problems in the backward sGS sweep, we can complete the forward sGS sweep similarly.
For all $i = 1, 2, \cdots, M$, the second group optimization variables in the forward sGS sweep are given by

$$X_{i}^{k+1} = \Pi_{\Sigma_{2}^{n}} \left[ \alpha_{i}^{-1} LHS \left( X_{0}^{k+1}, X_{1}^{k+1}, \cdots, X_{i-1}^{k+1}, X_{i+1}^{k+1}, \cdots, X_{M+N}^{k+1}; W^{k} \right) \right].$$

(63)

For all $i = 1, 2, \cdots, N$, the third group optimization variables in the forward sGS sweep are given by

$$X_{M+i}^{k+1} = -2 V_{i} \left[ A_{M+i} \left( X_{0}^{k+1}, X_{1}^{k+1}, \cdots, X_{M+i-1}^{k+1}, X_{M+i+1}^{k+1}, \cdots, \bar{X}_{M+N}^{k+1} \right) + R - \sigma^{-1} W^{k} \right] V_{i}. \tag{64}$$

**Step 3: Update of the Lagrange multiplier $W$.**

The Lagrange multiplier can be determined by

$$W^{k+1} = W^{k} - \tau \left[ A \left( X_{0}^{k+1}, X_{1}^{k+1}, \cdots, X_{M+N}^{k+1} \right) + R \right]. \tag{65}$$

**Step 4: Stopping criterion.**

Define the Lagrangian function of the problem \([33]\) as

$$\mathcal{L}_{s}(X; W) = \mathcal{F}(X_{1}, X_{2}, \cdots, X_{M}) + \delta_{\Sigma_{2}^{n}}(X_{0}) + \delta_{\Sigma_{2}^{n}}(X_{2}) + \cdots + \delta_{\Sigma_{2}^{n}}(X_{M}) - \langle W, A(X) + R \rangle.$$ \tag{66}

The Karush-Kuhn-Tucker (KKT) conditions of the problem \([33]\) are given by

$$0 \in \partial \mathcal{L}_{s}(X_{0}, X_{1}, \cdots, X_{M+N}; W)$$

$$0 \in \partial \mathcal{L}_{s}(X_{0}, X_{1}, \cdots, X_{M+N}; W)$$

$$\vdots$$

$$0 \in \partial \mathcal{L}_{s}(X_{0}, X_{1}, \cdots, X_{M+N}; W)$$

$$0 = A(X_{0}, X_{1}, \cdots, X_{M+N}) + R,$$ \tag{67}

which are equivalent to

$$0 = X_{0} - \Pi_{\Sigma_{2}^{n}}(X_{0} - W)$$

$$0 = X_{1} - \Pi_{\Sigma_{2}^{n}} \left( VQV^{T} + VF_{1}WV^{T} + VWF_{1}^{T} V^{T} + X_{1} \right)$$

$$0 = X_{2} - \Pi_{\Sigma_{2}^{n}} \left( VQV^{T} + VF_{2}WV^{T} + VWF_{2}^{T} V^{T} + X_{2} \right)$$

$$\vdots$$

$$0 = X_{M} - \Pi_{\Sigma_{2}^{n}} \left( VQV^{T} + VF_{M}WV^{T} + VWF_{M}^{T} V^{T} + X_{M} \right)$$

$$0 = V_{11}WV_{12}$$

$$0 = V_{21}WV_{22}$$

$$\vdots$$

$$0 = V_{N1}WV_{N2}$$

$$0 = A(X_{0}, X_{1}, \cdots, X_{M+N}) + R.$$ \tag{68}

Therefore, the relative residual error in terms of the optimization variable $X_{0}$ is given by

$$\text{err}^{k}_{X_{0}} = \frac{\| X_{0} - \Pi_{\Sigma_{2}^{n}}(X_{0} - W) \|}{1 + \| W \| + \| X_{0} \|}, \tag{69}$$

the relative residual errors in terms of the optimization variables $X_{1}, X_{2}, \cdots, X_{M}$ are given by

$$\text{err}^{k}_{X_{1}} = \frac{\| X_{1} - \Pi_{\Sigma_{2}^{n}} \left( VQV^{T} + VF_{1}WV^{T} + VWF_{1}^{T} V^{T} + X_{1} \right) \|}{1 + \| VQV^{T} \| + \| VF_{1}WV^{T} \| + \| VWF_{1}^{T} V^{T} \| + \| X_{1} \|},$$

$$\text{err}^{k}_{X_{2}} = \frac{\| X_{2} - \Pi_{\Sigma_{2}^{n}} \left( VQV^{T} + VF_{2}WV^{T} + VWF_{2}^{T} V^{T} + X_{2} \right) \|}{1 + \| VQV^{T} \| + \| VF_{2}WV^{T} \| + \| VWF_{2}^{T} V^{T} \| + \| X_{2} \|},$$

$$\vdots$$

$$\text{err}^{k}_{X_{M}} = \frac{\| X_{M} - \Pi_{\Sigma_{2}^{n}} \left( VQV^{T} + VF_{M}WV^{T} + VWF_{M}^{T} V^{T} + X_{M} \right) \|}{1 + \| VQV^{T} \| + \| VF_{M}WV^{T} \| + \| VWF_{M}^{T} V^{T} \| + \| X_{M} \|}, \tag{70}$$

the relative residual errors in terms of the optimization variables $X_{M+1}, X_{M+2}, \cdots, X_{M+N}$ are given by

$$\text{err}^{k}_{X_{M+1}} = \| V_{11}WV_{12} \|,$n

$$\text{err}^{k}_{X_{M+2}} = \| V_{21}WV_{22} \|,$n

$$\vdots$$

$$\text{err}^{k}_{X_{M+N}} = \| V_{N1}WV_{N2} \|, \tag{71}$$

and the relative residual error in terms of the equality constraint is given by

$$\text{err}^{k}_{eq} = \frac{\| A(X_{0}^{k}, X_{1}^{k}, \cdots, X_{M+N}^{k}) + R \|}{1 + \| R \|}. \tag{72}$$

Define

$$\text{err}_{\text{max}} = \max \left\{ \text{err}^{k+1}_{X_{0}}, \cdots, \text{err}^{k+1}_{X_{M+N}}, \text{err}^{k+1}_{eq} \right\}, \tag{73}$$

if $\text{err}_{\text{max}} < \epsilon$, then terminate the optimization process, and $W^{k+1}$ is the optimal solution to the optimization problem \([21]\).

**Step 5: Increase of the iteration index**

Set $k = k + 1$. Go back to Step 2.

With the above analysis, the proposed algorithm is summarized by Algorithm 1.

**Remark 2.** The proposed methodology is not only suitable for uncertain systems, but also applicable to precisely known systems. As an important additional property, all the extreme systems are replaced by a single precise system, thereby the upper bound to the $\mathcal{H}_{2}$-norm is reduced to the optimal $\mathcal{H}_{2}$ cost in case of precisely known systems.
Algorithm 1 sGS semi-proximal ALM for optimal decentralized control

Input: Choose the parameters $\sigma > 0$ and $\tau \in \left(0, \frac{1-\sqrt{2}}{2}\right)$, the parameter $\alpha$ such that the linear operator $\mathcal{S}_i$ is a positive operator, the initial matrices $(X^0, W^0) \in \mathcal{X} \times \mathcal{S}$, and the parameter of the stopping criterion $\epsilon > 0$. Set the iteration index $k = 0$. For $k = 0, 1, 2, \cdots$, perform the $k$th iteration

Output: $K$

1: while true do
2: Determine $\hat{X}^{k+1}_{M+N}, \bar{X}^{k+1}_{M+N-1}, \cdots, \bar{X}^{k+1}_1$ and $X_0^{k+1}$ by (35), (42), and (62).
3: Determine $\bar{X}^{k+1}_1, X_2^{k+1}, \cdots, X_{M+N}^{k+1}$ by (63) and (64).
4: Determine $W^{k+1}$ by (65).
5: Determine $\text{err}^{k+1}_{X_0}, \text{err}^{k+1}_X, \cdots, \text{err}^{k+1}_{X_{M+N}}$, and $\text{err}^{k+1}_{eq}$ by (69), (70), (71), and (72).
6: Determine $\text{err}_{\text{max}}$ by (73).
7: if $\text{err}_{\text{max}} \leq \epsilon$ then
8: $K = (W^{k+1}_2)^{-1} (W^{k+1}_1)^{-1}$
9: break
10: end if
11: end while
12: return $K$

V. ILLUSTRATIVE EXAMPLE

To illustrate the effectiveness of the above results, two examples are presented. First of all, Example 1 presents a decentralized controller design problem with a precise model, where the state matrix and the control input matrix are randomly chosen such that their elements are stochastic variables uniformly distributed over $[0, 1]$. Next, Example 2 is reproduced from [26] with some modifications, which presents a two-link manipulator decentralized controller design problem in the presence of parameter uncertainties.

In both examples, the optimization algorithm is implemented in Python 3.7.5 with Numpy 1.16.4, and executed on a computer with 16G RAM and a 2.2GHz i7-8750H processor (6 cores). The parameters for initialization is given by: $\sigma = 0.1$, $\tau = 0.618$, $X^0 = 0$, $W^0 = 0$, and the stopping criterion is set as $\epsilon = 10^{-12}$.

Example 1. Consider $x = [x_1 \ x_2 \ x_3]^T$ and a linear system

\[
\dot{x} = Ax + B_2 u + B_1 w,
\]

\[
z = C x + D u,
\]

\[
u = -K x,
\]

where

\[
A = \begin{bmatrix}
0.0461 & 0.8269 & 0.7079 \\
0.7716 & 0.5949 & 0.7541 \\
0.2339 & 0.6478 & 0.8203
\end{bmatrix}, \quad B_1 = I_3,
\]

\[
B_2 = \begin{bmatrix}
0.8221 & 0.2301 \\
0.6399 & 0.2384 \\
0.6129 & 0.6109
\end{bmatrix},
\]

and

\[
C = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad D = \begin{bmatrix}
0 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}.
\]

The stopping criterion is reached with 7312 iterations in 9.8480 seconds, and the change of the duality gap is shown in Fig. 1. The optimal $W$ is given by

\[
W = \begin{bmatrix}
0.8477 & -0.3729 & 0.2806 & 0 & 0.3729 & 0.5942 & 0.8840 \\
0 & 0 & 0.1021 & 0 & 1.0121 & 0.8840 & 0.2452 \\
0 & 0 & 0.2174 & 0 & 0.2174 & 0 & 4.6845
\end{bmatrix},
\]

and then the optimal decentralized controller gain $K$ is determined, where

\[
K = \begin{bmatrix}
1.3614 & 2.3422 & 0 \\
0 & 0 & 2.1514
\end{bmatrix}.
\]

It can be seen that the decentralized structure is preserved, and the upper bound to $\|H(s)\|_2$ is $\text{Tr}(RW) = 8.5790$. In the simulation, $w$ is characterized as a vector of the impulse disturbance, and the responses of all the state variables are plotted in Fig. 2. It can be seen that the closed-loop system is stabilized by the controller determined.
Example 2. Consider $x = [x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6]^T$ and a linear system

$$\dot{x} = Ax + B_2 u + B_1 w,$$
$$z = C x + D u,$$
$$u = -K x,$$

where

$$A = \begin{bmatrix}
-1/\tau_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
k_{11} & -k_{11} & c_{1} - k_{12} & 0 & 0 & c_2 \\
0 & 0 & 0 & -1/\tau_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & c_3 & k_{21} & -k_{21} & -k_{22}
\end{bmatrix},$$

$$B_1 = I_6, \quad B_2 = \begin{bmatrix}
1/\tau_1 \\
0 \\
0 \\
0 \\
0 \\
1/\tau_2 \\
0
\end{bmatrix},$$

and

$$C = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad D = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 0 \\
1 & 0
\end{bmatrix},$$

where $k_{11} = k_{21} = 10$, $k_{12} = k_{22} = 2$, $\tau_1 = \tau_2 = 0.1$, $c_1 = 0.2$, $c_2 = c_3 = 0.1$. Suppose that $c_1$, $c_2$, and $c_3$ are uncertain parameters with $c_1 \in [0, 0.4]$, $c_2 \in [0, 0.2]$, and $c_3 \in [0, 0.2]$, which result in $2^3 = 8$ extreme systems.

The stopping criterion is reached with 13690 iterations in 56.8478 seconds, and the change of the duality gap is shown in Fig. 3. The optimal $W$ is obtained, and then the optimal decentralized controller gain $K$ is determined, where

$$K = \begin{bmatrix}
0.1606 & 0.2536 & 0.1735 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.1460 & 0.2682 & 0.1567
\end{bmatrix}. $$

Similarly, the decentralized structure of the controller gain is preserved, and the upper bound to $\|H(s)\|_2^2$ is $\text{Tr}(RW) = 0.7314$. Also, in the simulation, $w$ is characterized as a vector of the impulse disturbance, and an extreme system with all the uncertain parameters at their lower bounds is used. The responses of all the state variables are illustrated in Fig. 4 and it can be seen that the robust stability is well guaranteed for this extreme system. It is worthwhile to mention that the stability is ensured for all the other uncertain systems within the uncertain domain.

VI. CONCLUSION

This paper presents a highly effective optimization algorithm for the optimal decentralized control problem with parameter uncertainties. With parameterization of the stabilizing controller gain in the extended parameter space, the problem is re-formulated as a convex optimization problem, which is solved by the sGS semi-proximal ALM. Furthermore, in the detailed development and methodology that we present here (which goes beyond just “generic” guidelines), both the closed-loop stability and the system performance are guaranteed in the presence of model uncertainties. Illustrative examples demonstrate the applicability and effectiveness of the proposed approach.

REFERENCES

[1] L. Bakule, “Decentralized control: an overview,” Annual Reviews in Control, vol. 32, no. 1, pp. 87–98, 2008.
[2] Y. Guo, D. J. Hill, and Y. Wang, “Nonlinear decentralized control of large-scale power systems,” Automatica, vol. 36, no. 9, pp. 1275–1289, 2000.
[3] A. A. Malikopoulos, C. G. Cassandras, and Y. J. Zhang, “A decentralized energy-optimal control framework for connected automated vehicles at signal-free intersections,” Automatica, vol. 93, pp. 244–256, 2018.
[4] P. Yang, R. A. Freeman, G. J. Gordon, K. M. Lynch, S. S. Srinivasa, and R. Sukthankar, “Decentralized estimation and control of graph connectivity for mobile sensor networks,” Automatica, vol. 46, no. 2, pp. 390–396, 2010.
[5] S. Omidshafiei, A.-A. Agha-Mohammadi, C. Amato, S.-Y. Liu, J. P. How, and J. Vian, “Decentralized control of multi-robot partially observable markov decision processes using belief space macro-actions,” The International Journal of Robotics Research, vol. 36, no. 2, pp. 231–258, 2017.
[6] S. M. Asghari and A. Nayar, “Dynamic teams and decentralized control problems with substitutable actions,” IEEE Transactions on Automatic Control, vol. 62, no. 10, pp. 5302–5309, 2016.
[7] R. Bellman, “Vector Lyapunov functions,” Journal of the Society for Industrial and Applied Mathematics, Series A: Control, vol. 1, no. 1, pp. 32–34, 1962.
\[ W = \begin{bmatrix}
0.0743 & -0.0088 & -0.1943 & 0 & 0 & 0 & -0.0240 & 0 \\
-0.0088 & 0.3301 & -0.5250 & 0 & 0 & 0 & -0.0088 & 0 \\
-0.1943 & -0.5250 & 2.6951 & 0 & 0 & 0 & 0.3031 & 0 \\
0 & 0 & 0 & 0.0647 & -0.0153 & -0.1275 & 0 & -0.0146 \\
0 & 0 & 0 & -0.0153 & 0.3201 & -0.5136 & 0 & 0.0032 \\
0 & 0 & 0 & -0.1275 & -0.5136 & 2.4634 & 0 & 0.2995 \\
-0.0240 & -0.0088 & 0.3031 & 0 & 0 & 0 & 0.0465 & 0 \\
0 & 0 & 0 & -0.0146 & 0.0032 & 0.2295 & 0 & 0.0347
\end{bmatrix}, \]

Jun Ma (S’15-M’18) received the B.Eng. degree with First Class Honours in electrical and electronic engineering from the Nanyang Technological University, Singapore, in 2014, and the Ph.D. degree in electrical and computer engineering from the National University of Singapore, Singapore, in 2018.

From 2018 to 2019, he was a Research Fellow with the Department of Electrical and Computer Engineering, National University of Singapore, Singapore. In 2019, he was a Research Associate with the Department of Electronic and Electrical Engineering, University College London, London, U.K. He is currently a Visiting Scholar with the Department of Mechanical Engineering, University of California, Berkeley, Berkeley, CA, USA. His research interests include control and optimization, precision mechatronics, robotics, and medical technology.

He was a recipient of the Singapore Commonwealth Fellowship in Innovation.

Zilong Cheng (S’19) received the B.Eng. degree in automotive engineering from the Jilin University, China, in 2014. He is a recipient of the NUS Graduate School for Integrative Sciences and Engineering Scholarship and is currently working towards his Ph.D. degree with the NUS Graduate School for Integrative Sciences and Engineering, National University of Singapore, Singapore.

He has been attached to the Mechatronics Group, Singapore Institute of Manufacturing Technology since 2018, with which he is actively involved in control system design and development. His research interests include optimal control, robust control, intelligent control, and precision mechatronics.

Xiaoxue Zhang (S’19) received the B.Eng. and the M.Sc. degrees in automotive engineering from the China Agricultural University, China, in 2015 and 2018, respectively. She is a recipient of the NUS Graduate School for Integrative Sciences and Engineering Scholarship and is currently working towards her Ph.D. degree with the NUS Graduate School for Integrative Sciences and Engineering, National University of Singapore, Singapore.

Her research interests include motion planning, optimal control, robust control, and intelligent control.
Masayoshi Tomizuka (M’86-SM’95-F’97-LF’17) received the B.S. the and M.S. degrees in mechanical engineering from the Keio University, Tokyo, Japan, and the Ph.D. degree in mechanical engineering from the Massachusetts Institute of Technology in February 1974.

In 1974, he joined the faculty of the Department of Mechanical Engineering at the University of California at Berkeley, where he currently holds the Cheryl and John Neerhout, Jr., Distinguished Professorship Chair. His current research interests are optimal and adaptive control, digital control, motion control, and their applications to robotics and vehicles.

He served as Program Director of the Dynamic Systems and Control Program of the Civil and Mechanical Systems Division of NSF (2002-2004). He served as Technical Editor of the ASME Journal of Dynamic Systems, Measurement and Control, J-DSMC (1988-93) and Editor-in-Chief of the IEEE/ASME Transactions on Mechatronics (1997-99). He is a Life Fellow of the ASME and IEEE, and a Fellow of International Federation of Automatic Control (IFAC) and the Society of Manufacturing Engineers. He is the recipient of the ASME/DSCD Outstanding Investigator Award (1996), the Charles Russ Richards Memorial Award (ASME, 1997), the Rufus Oldenburger Medal (ASME, 2002), the John R. Ragazzini Award (2006), and the Richard Bellman Control Heritage Award (2018).

Tong Heng Lee received the B.A. degree with First Class Honours in the Engineering Tripos from the Cambridge University, Cambridge, U.K., in 1980, the M.Eng. degree from the National University of Singapore (NUS), Singapore, in 1985, and the Ph.D. degree from the Yale University, New Haven, CT, USA, in 1987.

He is a Professor in the Department of Electrical and Computer Engineering at the National University of Singapore, and also a Professor in the NUS Graduate School, NUS NGS. He was a Past Vice-President (Research) of NUS. Dr. Lee’s research interests are in the areas of adaptive systems, knowledge-based control, intelligent mechatronics, and computational intelligence.

He currently holds Associate Editor appointments in the IEEE Transactions in Systems, Man and Cybernetics, Control Engineering Practice (an IFAC journal), and the International Journal of Systems Science (Taylor and Francis, London). In addition, he is the Deputy Editor-in-Chief of IFAC Mechatronics journal.