On the Regularity of Weak Solutions to the Magneto Hydrodynamics System near the curved part of the boundary

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Abstract

We prove a sufficient conditions of local regularity of suitable weak solutions to the MHD system for the point from $C^3$-smooth part of the boundary. Our conditions are the generalizing of the Caffarelli-Kohn-Nirenberg theorem for Navier-Stokes equations.

1 Introduction

Assume $\Omega \subset \mathbb{R}^3$ is a $C^3$–smooth bounded domain and $Q_T = \Omega \times (0,T)$. In this paper we investigate the boundary regularity of solutions to the principal system of magnetohydrodynamics (the MHD equations):

\begin{align}
\partial_t v + (v \cdot \nabla)v - \Delta v + \nabla p &= \text{rot} \, H \times H & \text{in } Q_T, \\
\text{div } v &= 0.
\end{align}

\begin{align}
\partial_t H + \text{rot} \, \text{rot} \, H &= \text{rot} (v \times H) & \text{in } Q_T, \\
\text{div } H &= 0.
\end{align}

Here unknowns are the velocity field $v : Q_T \to \mathbb{R}^3$, pressure $p : Q_T \to \mathbb{R}$, and the magnetic field $H : Q_T \to \mathbb{R}^3$. We impose on $v$ and $H$ the boundary conditions:

\begin{align}
v|_{\partial \Omega \times (0,T)} &= 0, & H_v|_{\partial \Omega \times (0,T)} &= 0, & (\text{rot } H)_v|_{\partial \Omega \times (0,T)} &= 0.
\end{align}

Here by $\nu$ we denote the outer normal to $\partial \Omega$ and $H_v = H \cdot \nu$, $(\text{rot } H)_v = \text{rot } H - \nu (\text{rot } H \cdot \nu)$. These conditions correspond to the case of liquid flowing in the area bounded by ideal conductor.

Definition 1.1. Assume $\Gamma \subset \partial \Omega$. The functions $(v, H, p)$ are called a boundary suitable weak solution to the system (1.1), (1.2) near $\Gamma_T \equiv \Gamma \times (0, T)$ if
1) \( v \in L_{2,\infty}(Q_T) \cap W^{1,0}_2(Q_T) \cap W^{2,1}_2(Q_T), \)
\( H \in L_{2,\infty}(Q_T) \cap W^{1,0}_2(Q_T), \)
2) \( p \in L^{\frac{1}{2}}_2(Q_T) \cap W^{1,0}_2(Q_T), \)
3) \( \operatorname{div} v = 0, \ \operatorname{div} H = 0 \ a.e. \ in \ Q_T, \)
4) \( v|_{\partial \Omega} = 0, \ H|_{\partial \Omega} = 0 \ in \ the \ sense \ of \ traces, \)
5) for any \( w \in L_2(\Omega) \) the functions
\[
 t \mapsto \int_{\Omega} v(x,t) \cdot w(x) \, dx \quad \text{and} \quad t \mapsto \int_{\Omega} H(x,t) \cdot w(x) \, dx
\]
are continuous,
6) \( (v,H) \) satisfy the following integral identities: for any \( t \in [0,T] \)
\[
 \int_{\partial \Omega} \left( -v \cdot \partial_t \zeta + (\nabla v \cdot \nu + H \otimes H) \cdot \nabla \eta \right) \, dx dt = 0,
\]
for all \( \eta \in W^{1,1}_{2,2}(Q_t) \) such that \( \eta|_{\partial \Omega \times (0,t)} = 0, \)
\[
 \int_{\Omega} H(x,t) \cdot \psi(x,t) \, dx \bigg|_{x,t} = 0,
\]
for all \( \psi \in W^{1,1}_{2,2}(Q_t) \) such that \( \psi|_{\partial \Omega \times (0,t)} = 0. \)
7) For every \( z_0 = (x_0,t_0) \in \Gamma_T \) such that \( \Omega_R(x_0) \times (t_0 - R^2, t_0) \subset Q_T \)
where \( \Omega(x_0,R) \equiv \Omega \cap B(x_0,R) \) and for any \( \zeta \in C^0(\overline{B_R(x_0)} \times (t_0 - R^2, t_0)) \) such that \( \frac{\partial \zeta}{\partial s}|_{\partial \Omega} = 0 \) the following "local energy inequality" near \( \Gamma_T \) holds:
\[
 \sup_{t \in (t_0 - R^2, t_0)} \int_{\Omega_R(x_0)} \zeta \left( |v|^2 + |H|^2 \right) \, dx + \sup_{t \in (t_0 - R^2, t_0)} \int_{\Omega_R(x_0)} \zeta \left( |\nabla v|^2 + |\operatorname{rot} H|^2 \right) \, dx dt \leq \int_{t_0 - R^2}^{t_0} \int_{\Omega_R(x_0)} \left( |v|^2 + |H|^2 \right) (\partial_t \zeta + \Delta \zeta) \, dx dt + \int_{t_0 - R^2}^{t_0} \int_{\Omega_R(x_0)} \left( |v|^2 + 2p \right) v \cdot \nabla \zeta \, dx dt + \int_{t_0 - R^2}^{t_0} \int_{\Omega_R(x_0)} (H \otimes H) : \nabla^2 \zeta \, dx dt + \int_{t_0 - R^2}^{t_0} \int_{\Omega_R(x_0)} (v \times H)(\nabla \zeta \times H) \, dx dt \tag{1.4}
\]
Here \( L_{s,t}(Q_T) \) is the anisotropic Lebesgue space equipped with the norm
\[
 \| f \|_{L_{s,t}(Q_T)} := \left( \int_0^T \left( \int_{\Omega} |f(x,t)|^s \, dx \right)^{1/s} \, dt \right)^{1/t},
\]
and we use the following notation for the functional spaces:

\[ W^{1,0}_{s,t}(Q_T) \equiv L_t(0,T;W^1_s(\Omega)) = \{ u \in L_{s,t}(Q_T) : \nabla u \in L_{s,t}(Q_T) \}, \]

\[ W^{2,1}_{s,t}(Q_T) = \{ u \in W^{1,0}_{s,t}(Q_T) : \nabla^2 u, \partial_t u \in L_{s,t}(Q_T) \}, \]

\[ \tilde{W}^1_s(\Omega) = \{ u \in W^1_s(\Omega) : u|_{\partial \Omega} = 0 \}, \]

and the following notation for the norms:

\[ \|u\|_{W^{1,0}_{s,t}(Q_T)} = \|u\|_{L_{s,t}(Q_T)} + \|\nabla u\|_{L_{s,t}(Q_T)}, \]

\[ \|u\|_{W^{2,1}_{s,t}(Q_T)} = \|u\|_{W^{1,0}_{s,t}(Q_T)} + \|\nabla^2 u\|_{L_{s,t}(Q_T)} + \|\partial_t u\|_{L_{s,t}(Q_T)}, \]

**Theorem 1.1.** For any sufficiently smooth divergent-free \( v_0, H_0 \) satisfying (1.3) there exists at least one boundary suitable weak solution near \( \partial \Omega \times (0,T) \) which satisfies the initial conditions:

\[ \|v(\cdot,t) - v_0(\cdot)\|_{L^2(0)} \to 0, \quad \|H(\cdot,t) - H_0(\cdot)\|_{L^2(0)} \to 0 \quad \text{as} \quad t \to +0, \]

and additionally satisfies the global energy inequality

\[ \|v\|_{L^\infty(Q_T)} + \|H\|_{L^\infty(Q_T)} + \|\nabla v\|_{L^2(Q_T)} + \|\nabla H\|_{L^2(Q_T)} \leq \|v_0\|_{L^2(\Omega)} + \|H_0\|_{L^2(\Omega)} \]

The global existence of weak solutions to the MHD equations (1.1) — (1.3) was established originally in [6]. The proof of Theorem 1.1 can be found in [18].

**Notations** In this paper we will use the following notations

\[ B(x_0,R) = \{ x \in \mathbb{R}^3 : |x - x_0| < R \}, \]

\[ B^{(2)}(x_0,R) = \{ x' \in \mathbb{R}^3 : |x' - x_0| < R \}, \]

\[ B^{(2)}(R) = B^{(2)}(0,R), \quad B^{(2)} = B^{(2)}(1) \]

\[ B^+_3(R) = \{ x \in B(0,R) : x_3 > 0 \}, \quad Q^+_3(R) = B^+_3(R) \times (-R^2,0) \]

**2 Main Results**

**Main condition on \( \partial \Omega \).** There exist positive numbers \( \mu \) and \( R_0 \) depending only on \( \partial \Omega \) such that for each point \( x_0 \in \partial \Omega \) we can choose a Cartesian coordinate system \( \{y_i\}_{i=1}^3 \) associated to the origin \( x_0 \), and some function \( \varphi_{x_0} \in C^3(B^{(2)}(R_0)) \) such that

\[ \Omega(x_0,R_0) \equiv \Omega \cap B(x_0,R_0) = \{ y \in B(R_0) : y_3 > \varphi_{x_0}(y_1,y_2) \}, \]

and

\[ \varphi_{x_0}(0) = 0, \quad \nabla \varphi_{x_0}(0) = 0, \quad \|\varphi_{x_0}\|_{W^{2,1}_{s,t}} \leq \mu. \quad (2.1) \]

The main results of the present paper are the following theorems on boundary regularity of suitable weak solutions of MHD system.
Theorem 2.1. Assume that $\partial \Omega$ satisfies Main Condition. Then for any $K > 0$ there exists $\varepsilon_0(K) > 0$ with the following property. Assume $(v, H, p)$ is a boundary suitable weak solution in $Q_T$ and $z_0 = (x_0, t_0) \in \partial \Omega \times (0, T)$. If
\[
\limsup_{r \to 0} \left( \frac{1}{r} \int_{t_0 - r^2}^{t_0} |\nabla H|^2 \, dx \, dt \right)^{1/2} < K
\]
and
\[
\limsup_{r \to 0} \left( \frac{1}{r} \int_{t_0 - r^2}^{t_0} |\nabla v|^2 \, dx \, dt \right)^{1/2} < \varepsilon_0,
\]
then the functions $v$ and $H$ are Hölder continuous in some neighborhood of $z_0$.

Theorem 2.2. Assume that $\partial \Omega$ satisfies Main Condition and $(v, H, p)$ is a boundary suitable weak solution in $Q_T$. Then there exists a closed set $\Sigma \subset \partial \Omega \times (0, T]$ such that for any $z_0 \in (\Gamma \setminus \Sigma) \times (0, T]$ the functions $(v, H)$ are Hölder continuous in some neighborhood of $z_0$.

Our Theorem 2.1 presents for the MHD equations a result which is a boundary analogue of the famous Caffarelli–Kohn–Nirenberg (CKN) theorem for the Navier-Stokes system, see [1], see also [8]. The boundary regularity of solutions to the Navier-Stokes equations was originally investigated by G. Seregin in [10] and [11] in the case of a plane part of the boundary and by G. Seregin, T. Shilkin, and V. Solonnikov in [14] in the case of a curved boundary.

The internal partial regularity of solutions to the MHD system was originally proved by C. He and Z. Xin in [4], see also [16], [17]. The local regularity near the plan part of the boundary for MHD equations was investigated in [18] (see also [19]). In the case of boundary regularity due to boundary conditions on the derivatives of $H$ there will be some problems if try to directly generalize the approach used for Navier-Stokes equations. To solve this problem we will consider the equation (1.2) as a linear heat equation on $H$ and this gives us some additional estimates. This idea was originally proposed in [4] to obtain regularity theorems with just boundedness conditions instead of smallness on magnetic component the internal case. In the present paper corresponding estimates are proved in the sections 6 and 7.

Note that using the methods of our paper one can prove various $\varepsilon$–regularity conditions involving various scale–invariant functionals (such it was done for the plane part of the boundary in [19], see also [9]). In the present paper we concentrate on the condition of the theorem 2.1 as this condition provides the optimal estimate of the Hausdorff measure of the singular set $\Sigma$ in Theorem 2.2. Note that we also have (2.4) in the internal case, so combining these two estimates we will obtain (2.4) for the set of singular points in any bounded domain $\Omega$ with $C^3$ smooth boundary.
Our paper is organized as follows: in Section 3 using symmetries of (1.1)-(1.2) we present more convenient statement of Theorem 2.1. In Section 4 we describe coercive estimates for solutions of the Stokes equations near the boundary. Section 5 contains the proof of the Decay Lemma and the sketch of the proof of Theorem 3.2. Sections 6 and 7 is concerned with the estimate of some Morrey functional for weak solutions to the heat equation near the boundary. These estimates together with the estimates of the scale invariant energy functionals obtained in Section 8 turn to be crucial for the prove of theorem 2.1 presented in section 9.

3 Symmetry group of MHD system and new statement of the main results

The solutions of MHD system (1.1), (1.2) have the same set of symmetries as the Navier-Stokes equations i.e. they are invariant under translations, rotations and scaling

\[ v_R(y, s) = R v(R y, R^2 s), \]
\[ H_R(y, s) = R H(R y, R^2 s), \]
\[ p_R(y, s) = R^2 p(R y, R^2 s). \]

So we can consider that in the statement of theorem 2.1 \( z_0 = 0 \) and the boundary of the domain is described by function \( \varphi \) satisfying (2.1). Also it will be convenient to consider the function \( \varphi \) as a part of the problem data and deal with the "local version" of suitable weak solution in parabolic cylinder

\[ Q^+(R) \equiv (-R^2, 0) \times B^+(R), \]

where

\[ B^+(R) \equiv \{ x \in B(R) : x_3 > \varphi(x_1, x_2) \}. \]

**Definition 3.1.** Let \( R > 0 \) and \( \varphi \in C^2(B^+(R)) \) satisfies (2.1). The functions \( (v, H, p, \varphi) \) are called a boundary suitable weak solution to the system (1.1), (1.2) in \( Q^+(R) \) if there is a domain \( \Omega \) such that \( \Gamma = \{ x_3 = \varphi(x_1, x_2) \} \subseteq \partial \Omega \) and functions \( v, \ p, \ H \) can be extended up to suitable weak solution near \( \Gamma \).

Then theorem 2.1 can be formulated by following way

**Theorem 3.1.** For any \( K > 0 \) there exists \( \varepsilon_0(K) > 0 \) with the following property. Assume \( (v, H, p, R) \) is a boundary suitable weak solution in \( Q^+(R) \) for some \( R > 0 \). If

\[ \limsup_{r \to 0} \left( \frac{1}{r} \int_{Q^+(r)} |\nabla H|^2 \, dx \, dt \right)^{1/2} < K \]

(3.2)

and

\[ \limsup_{r \to 0} \left( \frac{1}{r} \int_{Q^+(r)} |\nabla v|^2 \, dx \, dt \right)^{1/2} < \varepsilon_0, \]

(3.3)

then there exists \( \rho_* > 0 \) such that the functions \( v \) and \( H \) are Hölder continuous on the closure of \( Q^+(\rho_*) \).
To prove this theorem we will generalize the approach introduced in [18] in the case of the plane part of the boundary. The first step is the following theorem

**Theorem 3.2.** There exists an absolute constant \( \varepsilon_* > 0 \) with the following property. Assume \((v, H, p, \varphi)\) is a boundary suitable weak solution in \(Q^+(R)\). If there exists \(0 < r_0 < R\) such

\[
\frac{1}{r_0^2} \int_{Q^+(r_0)} \left( |v|^3 + |H|^3 + |p|^3 \right) dxdt < \varepsilon_*
\]

and

\[\|\varphi\|_{C^2(B(2)(r_0))} < \varepsilon_*, \tag{3.4}\]

then the functions \(v\) and \(H\) are Hölder continuous on the closure of \(Q^+(r_0^2)\).

Note, that (3.4) is just the condition on the smallness of \(r_0\). Indeed if \((v, p, H, \varphi)\) are the suitable weak solution in \(Q^+(R)\), then if we apply the scaling transformations (3.1), then \((v_R, p_R, H_R, \varphi_R)\) where

\[\varphi_R = \frac{1}{R} \varphi\]

will be the solution in \(Q^+(1)\) and from Taylor formula we have

\[\|\varphi\|_{C^2(B(2))} \leq R\|\varphi\|_{C^2}.
\]

### 4 Estimates for perturbed Stokes system

In this section we describe coercive estimates for linearisation of the (1.1). We start from the Stokes problem in \(Q^+\) with some \(\varphi \in C^3(B(2))\)

\[
\begin{align*}
\partial_t u - \Delta u + \nabla p &= f, \\
\nabla \cdot u &= 0, \\
u |_{x_3 = \varphi(x_1, x_2)} &= 0, \\
in Q^+.
\end{align*}
\tag{4.1}
\]

and define new coordinates \(\{y_i\}^3_{i=1}\) connected with the original ones via formula

\[x = e(y) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 + \varphi(y_1, y_2) \end{pmatrix}. \tag{4.2}\]

Denote by \(\mathcal{L}\) the Jacobi matrix of the map \(x = e(y)\) i.e.

\[\mathcal{L} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \varphi_1 & \varphi_2 & 1 \end{pmatrix}, \tag{4.3}\]

and by \(\tilde{\nabla}_\varphi\) and \(\tilde{\Delta}_\varphi\) the following differential operators with variable coefficients:

\[
\begin{align*}
(\tilde{\nabla}_\varphi p)_i &= p_k e_{k,i}, \\
(\tilde{\Delta}_\varphi v)_i &= v_{i,j} e_{j,k} e_{k,i} + v_{i,j} e_{j,k,k},
\end{align*}
\tag{4.4}
\]

\[
\tilde{\nabla}_\varphi \cdot v = v_{i,k} e_{k,i}
\]
Also we have the relation
\[ \text{rot}_x H = \mathcal{L} \text{rot}_y (\mathcal{L}^T \tilde{H}), \]  
(4.5)

here we have used that \( \det \mathcal{L} = 1. \)

Then the system (4.1) transforms into so called perturbed Stokes system
\[
\begin{align*}
\partial_t u - \Delta \varphi u + \nabla \varphi p &= f \\
\nabla \varphi \cdot u &= 0 \\
u|_{y=0} &= 0.
\end{align*}
\]  
(4.6)

Without loss of generality we can consider (4.6) in \( Q_0^+ \).

We recall that the function \( \varphi \) satisfy the following relations
\[ \varphi(0) = 0, \quad \nabla \varphi(0) = 0, \quad \| \varphi \|_{W^{2,\infty}} < \mu, \]  
(4.7)

and we can think, that the constant \( \mu \) is sufficiently small.

For our further arguments we will need two lemmas about the solutions of this problem. We start from consideration of initial-boundary problem for perturbed Stokes system (4.6) with homogenous data:
\[ u|_{\partial B^+} = 0, \quad u|_{t=-1} = 0 \]  
(4.8)

**Lemma 4.1.** Assume that \( s, l \in (1, \infty) \). There is \( \mu_* \in (0, \frac{1}{100}) \) if \( \varphi \) satisfies (4.7) for some \( \mu < \mu_* \), then there is a pair of functions \((u, p)\) such that
\[ u \in W^{2,1}_{s,l}(Q^+), \quad \nabla p \in L_{s,l}(Q^+), \]  
(v, p) satisfy (4.6) with initial-boundary conditions (4.8) and the following estimate holds
\[ \| v \|_{W^{2,1}_{s,l}(Q^+)} + \| \nabla p \|_{L_{s,l}(Q^+)} \leq C_* \| f \|_{L_{s,l}(Q^+)} \]  
(4.9)

where \( C_* \) is an absolute constant depending only on the dimension.

The second lemma is the coercive estimate for the solution of (4.6).

**Lemma 4.2.** Assume that \( s, m, l \in (1, \infty) \), \( m \geq s \). Assume that \( \varphi \in C^3(B^{(2)}_{l}) \) satisfies (4.7) for some \( \mu < \mu_* \). Then for every functions \( u \in W^{2,1}_{s,l}(Q^+), \nabla p \in L_{s,l}(Q^+) \) and \( f \in L_{m,l}(Q^+) \) satisfying (4.6) we have
\[ u \in W^{2,1}_{m,l}(Q^+), \quad \nabla p \in L_{m,l}(Q^+ \leftarrow \frac{1}{2}) \]  
and the following estimate holds
\[ \| v \|_{W^{2,1}_{m,l}(Q^+ \leftarrow \frac{1}{2})} + \| \nabla p \|_{L_{m,l}(Q^+ \leftarrow \frac{1}{2})} \leq C \left( \| f \|_{L_{s,l}(Q^+ \leftarrow \frac{1}{2})} + \| \nabla v \|_{L_{s,l}(Q^+ \leftarrow \frac{1}{2})} + \| p - p_0 \|_{L_{s,l}(Q^+ \leftarrow \frac{1}{2})} \right) \]  
(4.10)

for some absolute constant \( C \) and arbitrary function \( p_0 = p_0(t), \) \( p \in L_{l}(-1, 0) \).
5 The main criterion of $\varepsilon$-regularity

In this section we will prove the theorem 3.2. Let $(v, p, H, \varphi)$ are the suitable weak solution in $Q^+$, then we will use the following notations

$$Y_\tau(v) = \left( \frac{1}{|Q(\tau)|} \int_{Q^+(\tau)} |v|^3 \, dx \, dt \right)^{1/3},$$

$$\tilde{Y}_\tau(H) = \left( \frac{1}{|Q(\tau)|} \int_{Q^+(\tau)} |H - b_R(H)|^3 \, dx \, dt \right)^{1/3},$$

$$\hat{Y}_\tau(p) = \tau \left( \frac{1}{|Q(\tau)|} \int_{Q^+(\tau)} |p - [p]_{B^+(\tau)}|^{3/2} \, dx \, dt \right)^{2/3},$$

$$Y_\tau(v, p, H) = Y_\tau(v) + \tilde{Y}_\tau(H) + \hat{Y}_\tau(p),$$

$$b_\tau(H) = \mathcal{L} \begin{pmatrix} (h_1)_\tau \\ (h_2)_\tau \\ 0 \end{pmatrix},$$

here $h = \mathcal{L}(H \circ e)$, where $e$ and $\mathcal{L}$ defined by (4.2) and (4.3).

Note, that $\tilde{Y}_\tau(H)$ is equivalent to norm $H$ in Morrey space with taking into account boundary conditions. It can be easily seen if we replace $H$ by $\mathcal{L}h$ and use, that matrix $\mathcal{L}$ is close to the identity one.

We start our considerations from the modification of local energy inequality.

**Lemma 5.1.** Assume $(v, H, p, \varphi)$ is a boundary suitable weak solution satisfied the MHD equations in $Q^+(R)$. Let $\zeta \in C_0^\infty(B \times (-R^2, 0))$ be a cut-off function such that $\frac{\partial \zeta}{\partial \nu}
|x_3=\varphi(x_1,x_2)| = 0$. Assume $b \in \mathbb{R}^3$ is an arbitrary constant vector of the form $b = (b_1, b_2, 0)$. Then the following
inequality holds

\[
\int_{B^+(R)} \zeta \left( |v|^2 + |\bar{H}|^2 \right) \, dx
\]
\[
+ 2 \int_{B^+(R) \times (t_0, t)} \zeta \left( |\nabla v| + |\text{rot} \, \bar{H}|^2 \right) \, dz
\]
\[
\leq \int_{B^+(R) \times (t_0, t)} (\partial_t \zeta + \Delta \zeta) \left( |v|^2 + |\bar{H}|^2 \right) \, dz
\]
\[
+ \int_{B^+(R) \times (t_0, t)} \left( |v|^2 + 2\bar{\rho} \right) v \cdot \nabla \zeta \, dz
\]
\[
- 2 \int_{B^+(R) \times (t_0, t)} (H \otimes \bar{H}) : \nabla \zeta^2 \, dz +
\]
\[
+ 2 \int_{B^+(R) \times (t_0, t)} (v \times H)(\nabla \zeta \times \bar{H}) \, dz
\]
\[
- \int_{B^+(R) \times (t_0, t)} \left[ |\text{rot} \, \mathcal{L} b|^2 \zeta + \nabla \zeta \cdot \nabla (|\mathcal{L} b|^2) - (v \times H) \cdot \text{rot}(\mathcal{L} b) \zeta \right] \, dz,
\]

(5.1)

where $\bar{H} = H - \mathcal{L} b$.

**Proof.** We use (1.4) and transform the remaining terms. Via integration by parts formula

\[
\int_{B^+(R) \times (t_0, t)} (\partial_t \zeta + \Delta \zeta) |\mathcal{L} b|^2 \, dz
\]
\[
= \int_{B^+(R)} \zeta |\mathcal{L} b|^2 \, dx - \int_{B^+(R) \times (t_0, t)} \nabla \zeta \cdot \nabla (|\mathcal{L} b|^2) \, dz
\]

(5.2)

Also we have

\[
\int_{B^+(R) \times (t_0, t)} \partial_t \zeta \mathcal{H} \cdot b \, dz = \int_{B^+(R)} \zeta \mathcal{H} \cdot b \, dx +
\]
\[
+ \int_{B^+(R) \times (t_0, t)} (- \partial_t \mathcal{H} \cdot b) \, dz.
\]

(5.3)

Now we consider two terms: one from left hand side another from right
hand side of (5.1)
\[
\int_{B^+ (\Omega \times (t_0, t))} \Delta \zeta \cdot \mathcal{L} b \, dz - \int_{B^+ (\Omega \times (t_0, t))} \text{rot} H \cdot \text{rot}(\mathcal{L} b) \, dz =
\]
\[
= - \int_{B^+ (\Omega \times (t_0, t))} \left[ \text{rot} H \cdot (\nabla \zeta \times (\mathcal{L} b)) + \text{rot} H \cdot \text{rot}(\mathcal{L} b) \zeta \right] \, dz
\]
\[
- H \otimes (\mathcal{L} b) : \nabla^2 \zeta \, dz
\]
\[
= - \int_{B^+ (\Omega \times (t_0, t))} H \otimes (\mathcal{L} b) : \nabla^2 \zeta \, dz - \int_{B^+ (\Omega \times (t_0, t))} \text{rot} \text{rot} H \cdot (\mathcal{L} b \zeta) \, dz.
\]

The last terms of (5.3) and (5.4) can be modified using equation (1.2)
\[
\int_{B^+ (\Omega \times (t_0, t))} (\partial_t H \cdot \mathcal{L} b + \text{rot} \text{rot} H \cdot (\mathcal{L} b \zeta)) \, dz =
\]
\[
= \int_{B^+ (\Omega \times (t_0, t))} \text{rot}(v \times H) \cdot (\mathcal{L} b \zeta) \, dz =
\]
\[
= \int_{B^+ (\Omega \times (t_0, t))} (v \times H) \cdot \text{rot}(\mathcal{L} b \zeta) -
\]
\[
- \int_{B^+ (\Omega \times (t_0, t))} (v \times H) \cdot (\nabla \zeta \times (\mathcal{L} b)) \, dz.
\]

Combining (5.2)-(5.5) we obtain the statement of lemma.

**Lemma 5.2.** There exists an absolute constant \( \varepsilon_* > 0 \) such that for any \( M > 0 \) there exists \( \mathcal{C}_* = \mathcal{C}_*(M) \) with the following properties. For any boundary suitable weak solution \((v, H, p, \varphi)\) of the MHD system (1.1), (1.2) near the boundary in \( Q^+ (1) \) the following implication holds: if
\[
Y_1(v, H, p) + \|\varphi\|_{C^2} < \varepsilon_0,
\]
and
\[
b_R (H) \leq M
\]
then
\[
Y_\tau (v, H, p) \leq \mathcal{C}_* \tau^{1/3} Y_1(v, H, p)
\]

**Proof.** Arguing by contradiction we assume there exists a sequence of numbers \( \varepsilon_m \to 0 \), and a sequence of boundary suitable weak solutions \((v^m, H^m, p^m, \varphi^m)\) such that
\[
Y_1(v^m, H^m, p^m) + \|\varphi^m\|_{C^2} < \varepsilon_m \to 0,
\]
and
\[
Y_\tau (v^m, H^m, p^m) \geq \mathcal{C}_* \tau^{1/3} \delta_m
\]
From the equations (5.8), (5.9) we also obtain the estimate
\[ \varphi^m \text{ obtained from } (1.2) \text{ multiplied by the test function } \psi. \]

Let us introduce functions
\[ u^m(y, s) = \frac{1}{\delta_m} u^m \circ e_m, \]
\[ q^m(y, s) = \frac{1}{\delta_m} \left( p^m(x, t) - [p^m][A_+](t) \right) \circ e_m, \]
\[ h^m(y, s) = \frac{1}{\delta_m} \left( H^m(x, t) - b_1(H^m) \right) \circ e_m, \]
here \( e_m \) denotes the map (4.2) corresponding to \( \varphi^m \). Then
\[ Y_1(u^m, h^m, q^m) = 1, \quad Y_r(u^m, h^m, q^m) \geq C r^{1/3} \quad (5.7) \]
and \( (u^m, h^m, q^m) \) satisfy the following equations in \( D'(Q^+) \)
\[ \begin{aligned}
\partial_t u^m + \delta_m (u^m \cdot \nabla \psi^m)(u^m - \tilde{\Delta}_e \psi^m u^m + \tilde{\nabla} \psi^m q^m) &= \mathcal{L}_m \mathcal{R} \mathcal{L}_m^{-1} h^m + b_1(H^m), \\
\mathcal{L}_m^{-2} \cdot \nabla u^m &= 0,
\end{aligned} \quad (5.8) \]
\[ \begin{aligned}
\partial_t h^m - \tilde{\Delta}_e \psi^m h^m &= \mathcal{L}_m \mathcal{R} \mathcal{L}_m^{-1} \left( u^m \times (\delta_m h^m + (H^m)) \right), \\
\mathcal{L}_m^{-2} \cdot \nabla h^m &= 0.
\end{aligned} \quad (5.9) \]
The conditions (5.7) imply in particular the boundedness
\[ \sup_m \left( \| u^m \|_{L^2(Q^+)} + \| h^m \|_{L^2(Q^+)} + \| q^m \|_{L^2(Q^+)} \right) < +\infty \quad (5.10) \]
From the local energy inequality near the boundary and the relation obtained from (1.2) multiplied by the test function \( \psi = \zeta(H^m) \) we obtain
\[ \begin{aligned}
\| u^m \|_{L^2_0(Q^+_0(t))} + \| h^m \|_{L^2_0(Q^+_0(t))} + \| q^m \|_{L^2_0(Q^+_0(t))} + \| u^m \|_{W^{-1,0}_2(Q^+_0(t))} + \| h^m \|_{W^{-1,0}_2(Q^+_0(t))} + \| q^m \|_{W^{-1,0}_2(Q^+_0(t))} \leq C. \end{aligned} \quad (5.11) \]
From the equations (5.8), (5.9) we also obtain the estimate
\[ \begin{aligned}
\| \partial_t u^m \|_{L^2_0(-1, 0, T^{-1}(B^+_0))} + \| \partial_t h^m \|_{L^2_0(-1, 0, T^{-1}(B^+_0))} \leq C.
\end{aligned} \]
Hence we can extract subsequences
\[ \begin{aligned}
u^m &\to u \quad \text{in } L^2_3(Q^+_0), \\
h^m &\to h \quad \text{in } L^2_3(Q^+_0), \\
q^m &\to q \quad \text{in } L^2_2(Q^+_0), \\
u^m &\to u \quad \text{in } W^{1,0}_2(Q^+_0(t)), \\
h^m &\to h \quad \text{in } W^{1,0}_2(Q^+_0(t)), \\
u^m &\to u \quad \text{in } L^3(Q^+_0(t)), \\
h^m &\to h \quad \text{in } L^3(Q^+_0(t)), \\
\varphi^m &\to 0 \quad \text{in } C^2(B^{2^*)}, \\
b_1(H^m) &\to a \quad \text{in } C^2,
\end{aligned} \quad (5.12, 5.13, 5.14) \]
here \( a \in \mathbb{R}^3 \) is the constant vector.
Passing to the limit in the equations (5.8), (5.9) we obtain
\[ \partial_t u - \Delta u + \nabla q = \text{rot } h \times a \quad \text{in } Q_0^+, \]
\[ \text{div } u = 0 \quad \text{in } Q_0^+, \quad u\big|_{y_3 = 0} = 0, \]
\[ \partial_t h - \Delta h = \text{rot}(u \times a) \quad \text{in } Q_0^+, \quad \text{div } h = 0 \quad \text{in } Q_0^+, \quad h_3|_{y_3 = 0} = 0, \quad \frac{\partial h}{\partial y_3}|_{y_3 = 0} = 0. \]

For the solution to the linear problem (5.15) — (5.16) by a standard way (see [18] Theorem 4.1) we obtain
\[ Y_{\tau}(u) + \hat{Y}_{\tau}(h) \leq C(M) \tau^{1/3} Y_1(u, h, q) \quad (5.17) \]
Moreover from the second relation in (5.7) we have
\[ \liminf_{m \to \infty} Y_{\tau}(u^m, p^m, h^m) \geq c \tau^{1/3}. \quad (5.18) \]
On the other hand we will show that
\[ \limsup_{m \to \infty} Y_{\tau}(u^m, p^m, h^m) \leq c_* \tau^{1/3} \quad (5.19) \]
for some constant $c_*$. Taking in (5.18) a constant $c > c_*$ we obtain a contradiction. This contradiction will prove the theorem.

From (5.14) we conclude
\[ \lim_{m \to +\infty} Y_{\tau}(u^m) = Y_{\tau}(u), \quad \lim_{m \to +\infty} \hat{Y}_{\tau}(h^m) = \hat{Y}_{\tau}(h) \]
and hence
\[ \limsup_{m \to \infty} Y_{\tau}(u^m, h^m, q^m) \leq Y_{\tau}(u) + \hat{Y}_{\tau}(h) + \limsup_{m \to \infty} \hat{Y}_{\tau}(q^m). \quad (5.20) \]

Then to prove (5.19) it is sufficient to show that
\[ \limsup_{m \to \infty} \hat{Y}_{\tau}(q^m) \leq c(M) \tau^{1/4}. \quad (5.21) \]
For this purpose we decompose $(u^m, q^m)$ and $(u, q)$ as
\[ u^m = u_1^m + u_2^m, \quad q^m = q_1^m + q_2^m, \]
\[ u = u_1 + u_2, \quad q = q_1 + q_2, \]
where $(u_1^m, q_1^m) \in W^{2,1}_0(Q_0^+) \times W^{1,0}_0(Q_0^+)$ are determined as a solutions of the following initial boundary-value problems in $Q_0^+$:
\[ \partial_t u_1^m - \Delta_{\varphi_n} u_1^m + \nabla_{\varphi_n} q_1^m = f^m \quad \text{in } Q_0^+, \quad \nabla_{\varphi_n} \cdot u_1^m = 0 \quad \text{in } Q_0^+, \]
\[ u_1^m|_{t=0} = 0, \quad u_1^m|_{y_3 = 0} = 0, \]
\[ u_2^m|_{t=-1} = 0, \quad u_2^m|_{y_3 = 0} = 0. \]
where \( f^m \) is defined by the expression \( \mathcal{L}_m \text{rot} \mathcal{L}_m^{T} h^m \times (\delta_m h^m + b_1 (H^m)) - \delta_m (u^m \cdot \nabla \varphi^m) u^m \) on the set \( Q^+_0 \) and extended by zero onto the whole \( Q^+_0 \). Similarly, \((u_1, q_1)\) are determined by the relations

\[
\begin{align*}
\partial_t u_1 - \Delta u_1 + \nabla q_1 &= f \quad \text{in} \quad Q^+_0, \\
\text{div} u_1 &= 0 \quad \text{in} \quad Q^+_0, \\
|u_1|_{t=1} &= 0, \quad |u_1|_{q_3=0} = 0,
\end{align*}
\]

(5.22)

where \( f \) determined by the expression \( \text{rot} h \times a \) on the set \( Q^+_0 \) and extended by zero onto the whole \( Q^+_0 \).

As functions \( u^m_1 - u_1, q^m_1 - q_1 \) are the solution of the first initial boundary-value problem in \( Q^+_0 \) with the right-hand side \( f^m - f \) and zero initial and boundary conditions from lemma 4.1, we obtain the estimate

\[
\|u^m_1\|_{W^{2,1}_m(Q^+_0)} + \|\nabla q^m_1\|_{L^m(Q^+_0)} \leq C\|f^m\|_{L^m(Q^+_0)}
\]

\[
\|u^m_1 - u_1\|_{W^{2,1}_m(Q^+_0)} + \|\nabla q^m_1 - \nabla q_1\|_{L^m(Q^+_0)} \leq C\|f^m - f\|_{L^m(Q^+_0)}
\]

(5.23)

Note that

\[
\|f^m\|_{L^m(Q^+_0)} \leq C(M)
\]

\[
\|f^m - f\|_{L^m(Q^+_0)} \rightarrow 0, \quad \text{as} \quad m \rightarrow \infty.
\]

(5.24)

So, taking into account the imbedding \( W^{1,0}_m(Q^+_0) \hookrightarrow L^m(Q^+_0) \) we can conclude that

\[
q^m_1 \rightarrow q_1 \quad \text{in} \quad L^m(Q^+_0)
\]

and hence for any \( \tau \in (0, \frac{a}{r^m}) \)

\[
\lim_{m \rightarrow \infty} Y_T(q^m_1) = Y_T(q_1).
\]

On the other hand, \((u_1, q_1)\) is a solution of the linear Stokes problem in \( Q^+_0 \). Hence from lemma 4.2 we conclude

\[
Y_T(q_1) \leq C(M) \tau^{1/3} Y_{\frac{a}{mr}}(q_1)
\]

We need to estimate \( Y_{\frac{a}{mr}}(q_1) \). From imbedding theorem \( L^m(B^+_0(\frac{a}{r})) \hookrightarrow W^{1,0}_m(B^+_0(\frac{a}{r})) \) we conclude

\[
Y_{\frac{a}{mr}}(q_1) \leq C \|\nabla q_1\|_{L^m(B^+_0(\frac{a}{r}))}
\]

For the solution \((u_1, q_1)\) of the initial-boundary value problem (5.22) we have the estimate

\[
\|u_1\|_{W^{2,1}_m(Q^+_0)} + \|\nabla q_1\|_{L^m(Q^+_0)} \leq C(M) \|\nabla h\|_{L^m(Q^+_0)}
\]
Using Hölder inequality \( \| \nabla h \|_{L^2(\Omega^+_{0}(\tau))} \leq C \| \nabla h \|_{L^2(\Omega^+_{0}(\tau))} \) and taking into account the weak convergence (5.13) from which we conclude

\[
\| \nabla h \|_{L^2(\Omega^+_{0}(\tau))} \leq \liminf_{m \to \infty} \| \nabla h^m \|_{L^2(\Omega^+_{0}(\tau))},
\]

and using (5.11) we obtain

\[
Y_M(q_{1}) \leq C(M).
\]

Now we consider functions \((u^m_2, q^m_2)\) determined by relations

\[
u^m_2 := u^m - u^m_1, \quad q^m_2 := q^m - q^m_1. \tag{5.25}
\]

These functions satisfy the homogeneous Stokes problems in \(Q^+_0(\frac{\tau}{M})\):

\[
\begin{align*}
&\partial_t u^m_2 - \Delta u^m_2 + \nabla q^m_2 = 0 \quad \text{in} \quad Q^+_0(\frac{\tau}{M}), \\
&\text{div} u^m_2 = 0 \quad \text{in} \quad Q^+_0(\frac{\tau}{M}), \\
&u^m_2|_{\tau=0} = 0, \\
&\partial_t u_2 - \Delta u_2 + \nabla q_2 = 0 \quad \text{in} \quad Q^+_0(\frac{\tau}{M}), \\
&\text{div} u_2 = 0 \quad \text{in} \quad Q^+_0(\frac{\tau}{M}), \\
&u_2|_{\tau=0} = 0.
\end{align*}
\]

Then

\[
\|u^m_2\|_{L^{2}(Q^+_0(\frac{\tau}{M}))} + \|\nabla q^m_2\|_{L^2(\Omega^+_0(\tau))} \leq C \left(\|u^m_2\|_{L^3(Q^+_0(\frac{\tau}{M}))} + \|q^m_2\|_{L^2(Q^+_0(\frac{\tau}{M}))}\right)
\]

Note that due to (5.25), (5.10) and the first inequalities in (5.23), (5.24) we have the estimate

\[
\begin{align*}
\|u^m_2\|_{L^3(Q^+_0(\frac{\tau}{M}))} + \|q^m_2\|_{L^2(Q^+_0(\frac{\tau}{M}))} &\leq \\
\leq \|u^m\|_{L^3(Q^+_0(\frac{\tau}{M}))} + \|q^m\|_{L^2(Q^+_0(\frac{\tau}{M}))} + \|u^m_1\|_{L^3(Q^+_0(\frac{\tau}{M}))} + \|q^m_1\|_{L^2(Q^+_0(\frac{\tau}{M}))} &\leq \\
&\leq C(M)
\end{align*}
\]

On the other hand, from the Hölder inequality and lemma 4.2 we obtain for any \(\tau \in (0, \frac{\tau}{M})\)

\[
\hat{Y}(q_{2}^m) = \tau \left( \frac{1}{|Q^+_0(\tau)|} \int_{Q^+_0(\tau)} \left| q_{2}^m - [q_{2}^m]_{Q^+_0(\tau)} \right|^2 dx dt \right)^{\frac{1}{2}} \leq
\]

\[
C \tau^2 \left( \frac{1}{|Q^+_0(\tau)|} \int_{Q^+_0(\tau)} \left| \nabla q_{2}^m \right|^2 dx dt \right)^{\frac{1}{2}} \leq
\]

\[
C \tau^2 \| \nabla q_{2}^m \|_{L^2(\Omega^+_0(\frac{\tau}{M}))} \leq C(M) \tau^{\frac{7}{8}}
\]

Summarizing all previous estimates we arrive at

\[
\limsup_{m \to \infty} \hat{Y}(q^m) \leq \lim_{m \to \infty} \hat{Y}(t^m_{1}) + \limsup_{m \to \infty} \hat{Y}(q^m_{2}) \leq C(M) \tau^{\frac{7}{8}}
\]

which gives us a contradiction with (5.18).

Iterating (5.6) and using scaling argument it is easy to obtain the following lemma (see [2] and [14] for details).
Lemma 5.3. There exists an absolute constant $\varepsilon_{**} > 0$ such that for any $M > 0$ and $\beta \in (0, 1/3)$ there exists $\tau \in (0, 1/2)$ with the following properties. For any boundary suitable weak solution $(v, H, p, \varphi)$ of the MHD system (1.1), (1.2) near the boundary in $Q^+ \cup \Pi$ the following implication holds: if

$$Y_1(v, H, p) + \|\varphi\|_{C^2} \leq \varepsilon_{**},$$

and

$$b_R(H) \leq M$$

then

$$Y_{\gamma k}(v, H, p) \leq \tau^{\gamma k} Y_1(v, H, p) \quad (5.26)$$

Theorem 3.1 follows from this lemma in the standard way by scaling arguments, and combination of boundary estimates with the internal estimates obtained in [16]. See details in [10], [11], [14], [13].

6 Estimates of solutions of the heat equation with homogeneous boundary data

In this section we will obtain some estimates for $L_2$-norms of solutions of homogeneous initial and boundary problem for the heat equation in half-ball. Namely, we consider the following problem

$$\begin{align*}
\partial_t h - \Delta h &= f & \text{in } Q_0^+(R), \\
h_3|_{x_3=0} &= 0, & \frac{\partial h_i}{\partial x_3}|_{x_3=0} &= 0 & \text{in } Q^+(R), \quad i = 1, 2, \\
h|_{t=-R^2} &= 0,
\end{align*} \quad (6.1)$$

Here $h : Q_0^+(R) \to \mathbb{R}^3$ is an unknown function.

The main result of this section is the following theorem

Theorem 6.1. Let $f \in L^{2,1}_{\gamma} (Q_0^+(R))$, and $h$ is the solution of (6.1). Then the following estimate holds

$$\|h\|_{L_2(Q_0^+(R))} \leq cR^2 \|f\|_{L^{2,1}_{\gamma} (Q_0^+(R))}. \quad (6.2)$$

We note, that conditions for $h$ on a plain part of a boundary allow us to extend this function into whole $B$ by the following way: components $h_1$ and $h_2$ will be extended as even functions and component $h_3$ as odd function. The right hand side can be extended by the same manner. We also put $f \equiv 0$ in $\mathbb{R}^3 \setminus B^+(R)$. So it is sufficient to prove the theorem for the solution of the following Cauchy problem for the heat equation.

$$\begin{align*}
\partial_t h - \Delta h &= f & \text{in } \Pi_R, \\
h|_{t=-R^2} &= 0,
\end{align*} \quad (6.3)$$

Here $\Pi_R = \mathbb{R}^3 \times [-R^2, 0)$.
To prove this theorem we will need the Young inequality for convolutions (see. [3], [15]). Namely, let
\[ g(x) = \int_{\mathbb{R}^n} K(x - y)f(y) \, dy, \]
then for arbitrary \( 1 \leq p \leq q \leq \infty \), the following estimate holds
\[ \|g\|_q \leq \|K\|_l \|f\|_p, \quad \text{here} \quad 1 - \frac{1}{p} + \frac{1}{q} = \frac{1}{l}. \quad (6.4) \]
In particular we will use an inequality
\[ \|g\|_2 \leq \|K\|_6 \|f\|_3 \quad (6.5) \]

**Lemma 6.1.** Let \( f \in L_1(-R^2, 0) \), for some \( R > 0 \)

\[ g_t(t) = \int_{-R^2}^t f(\tau) \, d\tau - \int_{-R^2}^t \frac{f(\tau)}{(t - \tau)^{\frac{3}{2}}} \, d\tau. \quad (6.6) \]
then
\[ \|g\|_{L_2(-R^2, 0)} \leq cR^\frac{5}{2} \|f\|_{L_1(-R^2, 0)} \quad (6.7) \]

**Proof.** In the case \( R = 1 \) inequality (6.7) is the corollary of (6.4) with \( p = 1 \) and \( q = l = 2 \). In general case we make scaling transformations. Namely, let consider the functions
\[ f^*(s) = f(R^2s), \quad g^*(s) = g(R^2s), \quad \hat{g}(s) = \int_{-R^2}^s \frac{f^*(\sigma) \, d\sigma}{(s - \sigma)^{\frac{3}{2}}} \]
Changing variables under the integral, we have
\[ \|f^*\|_{L_1(-1, 0)} = R^{-2} \|f\|_{L_1(-R^2, 0)}, \quad \|g^*\|_{L_2(-R^2, 0)} = R \|g^*\|_{L_2(-1, 0)}. \]
And then we get
\[ \|g\|_{L_2(-R^2, 0)} \leq R \|g^*\|_{L_2(-1, 0)} = R^\frac{5}{2} \|f\|_{L_1(-R^2, 0)} \leq cR^\frac{5}{2} \|f^*\|_{L_1(-1, 0)} = cR^\frac{5}{2} \|f\|_{L_1(-R^2, 0)}. \]

**Proof of theorem 6.1.** The solution of (6.3) can be found as follows
\[ h(t, x) = \int_{-R^2}^t \int_{\mathbb{R}^3} e^{-\frac{|x - y|^2}{4(t - \tau)}} f(\tau, y) \, dy \, d\tau. \]
We fix \( t \), look on excretion under the integral by time as function with values in Banach space \( L_2(\mathbb{R}^3) \) and use inequality to the norm of its integral
\[ \|h(t, \cdot)\|_{L_2(\mathbb{R}^3)} \leq \int_{-R^2}^t \int_{\mathbb{R}^3} \left\| e^{-\frac{|x - y|^2}{4(t - \tau)}} f(\tau, y) \right\| \, dy \, d\tau. \quad (6.8) \]
By direct computations we find
\[
\left( \int_{\mathbb{R}^3} e^{-\frac{z^2}{4(t-\tau)}} \, dz \right)^{\frac{5}{6}} = c(t-\tau)^{\frac{5}{4}}.
\] (6.9)

Then from (6.6), (6.8) and (6.9) we obtain
\[
\|h(t, \cdot)\|_{L^2_2(\mathbb{R}^3)} \leq c t \int_0^t \|f(\tau, \cdot)\|_{L^2_2(\mathbb{R}^3)} \cdot (t-\tau)^{\frac{5}{4}} \, d\tau.
\]

Next form (6.7)
\[
\|h\|_{L^2_2(\pi \mathbb{R})} \leq cR^{\frac{1}{2}} \|f\|_{L^2_2(\mathbb{R})}.
\]

### 7 Estimates for the magnetic component

In this section we will get estimates for magnetic component of the suitable weak solution in \(Q^+(R)\).

We recall, that \(H\) satisfy the following integral identity
\[
\int_{Q^+(R)} \left( -H \cdot \partial_t \psi + \text{rot} \, H \cdot \text{rot} \psi - \left( \mathcal{L} \text{rot} \mathcal{L}^T \tilde{H}, \mathcal{L} \text{rot} \mathcal{L}^T \tilde{\psi} \right) - \left( \mathcal{L} \times \tilde{H}, \mathcal{L} \text{rot} \mathcal{L}^T \tilde{\psi} \right) \right) \, dx \, dt = 0,
\] (7.1)

for all \(\psi \in C^\infty(Q^+(R))\) such that \(\psi_\nu|_{\partial \Omega \times (-R^2, 0)} = 0\) and \(\psi(-R^2, x) = \psi(0, x) = 0\). Also we have \(H_\nu|_{\mathcal{R} \times (-R^2, 0)} = 0\) and \(\text{div} \, H = 0\). Note, that without loss of generality we can assume, that (7.1) holds only for test functions with \(\text{div} \, \psi = 0\).

**Theorem 7.1.** Assume that (7.1) holds for some function \(v \in W^2_2\). Then there exist absolute positive constants \(\varepsilon_1, \alpha, c\) such that for any \(\varepsilon \in (0, \varepsilon_1)\) and any \(K > 0\) if
\[
\sup_{r \in (0, 1)} E(r) < \varepsilon, \quad \|\varphi\|_{C^2} < \varepsilon \quad \text{and} \quad \sup_{r \in (0, 1)} E_\alpha(r) < K
\] (7.2)
then for any \(0 < r < R \leq 1\)
\[
F_2(r) \leq c \left( \frac{r}{R} \right)^2 F_2(R) + c\varepsilon(F_2(R) + K + 1).
\] (7.3)

**Proof.** We proceed to the coordinates (4.2). Then (7.1) transforms to
\[
\int_{-R^2}^{0} \left[ -(\tilde{H}, \partial_t \tilde{\psi}) + (\mathcal{L} \text{rot} \mathcal{L}^T \tilde{H}, \mathcal{L} \text{rot} \mathcal{L}^T \tilde{\psi}) - (\tilde{\psi} \times \tilde{H}, \mathcal{L} \text{rot} \mathcal{L}^T \tilde{\psi}) \right] \, dt = 0,
\]

here \((\cdot, \cdot)\) is \(L^2\) inner product.
Next we introduce new functions $h = L^{-1} \hat{H}$ and $\eta = L^{-1} \hat{\psi}$. Then the last identity can be written as follows

$$\int_{-R^2}^0 \left[ -\left(h, \partial_t \eta \right) + \left( A \text{rot} \, h, \text{rot} \, \eta \right) - \left( \hat{v} \times \hat{H}, \mathcal{L} \text{rot} \, \eta \right) \right] \, dt = 0 \quad (7.4)$$

Here $A = L^T \mathcal{L}$.

Note, that $\nabla_y \cdot h = \nabla_y \cdot L^{-1} \hat{H} = L^{-T} \nabla_y \cdot \hat{H} = \nabla_x \cdot H \circ \epsilon = 0$. Also $H \cdot \nu \circ \epsilon = L h \cdot \tilde{\nu} = h \cdot L^T \tilde{\nu} = -h_3$. Similar identities holds for function $\eta$.

As the result we can consider function $h$ as the generalized solution of parabolic system which corresponds to holding the identity (7.4) for arbitrary function $\eta \in C^\infty$, such that

$$\text{div} \, \eta = 0 \quad \text{and} \quad \eta_3 |_{y_3 = 0} = 0. \quad (7.5)$$

Then to estimate $L_2$-norm of $H$ it will sufficient to obtain inequality for $\|h\|_2$. To do this we decompose it into three parts

$$h = h^{(1)} + h^{(2)} + h^{(3)}. \quad (7.6)$$

Here $h^{(1)}$ is the solution of the following initial-boundary problem

$$\partial_t h^{(1)} - \Delta h^{(1)} = A \text{rot} \left( L^T (\hat{v} \times \hat{H}) \right) \quad (7.7)$$

Boundary conditions for function $h^{(1)}$ imply that the following identity

$$\int_{-R^2}^0 \left[ -\left(h^{(1)}, \partial_t \eta \right) + \left( \text{rot} \, h^{(1)}, \text{rot} \, \eta \right) - \left( \hat{v} \times \hat{H}, \mathcal{L} \text{rot} \, \eta \right) \right] \, dt = 0 \quad (7.9)$$

for every function $\eta$ satisfying (7.5). As (7.8) and (7.9) are stored at the replacing $h^{(1)}$ to the solenoidal component of its Weil decomposition without loss of generality we can assume, that $\text{div} \, h^{(1)} = 0$.

The second component of (7.6) is the solution of the following problem

$$\int_{-R^2}^0 \left[ -\left(h^{(2)}, \partial_t \eta \right) + \left( \text{rot} \, h^{(2)}, \text{rot} \, \eta \right) \right] \, dt = 0 \quad (7.10)$$
for arbitrary function $\eta$ satisfying (7.5) with the initial and boundary conditions

$$h_3^{(2)}|_{y_3=0} = 0, \quad h_3^{(2)}|_{t=-R^2} = 0.$$  

To obtain the estimate for $h^{(2)}$ we consider the dual problem

$$\partial_t \eta + \Delta \eta = -h^{(2)}$$

$$\eta|_{y_3=0} = 0, \quad \frac{\partial \eta}{\partial y_3}|_{y_3=0} = 0, \quad \frac{\partial h_3}{\partial y_3}|_{y_3=0} = 0, \quad \eta|_{t=0} = 0. \quad (7.11)$$

For the right hand side of (7.10) we have the following identity

$$(\text{rot} \ h, \text{rot} \ \eta) - (A \text{rot} \ A h, \text{rot} \ A \eta) =$$

$$(\text{rot} \ h, \text{rot}(I - \mathcal{A}) \eta) + (\text{rot}(I - \mathcal{A}) h, \text{rot} \ A \eta) + ((I - \mathcal{A}) \text{rot} \ A h, \text{rot} \ A \eta). \quad (7.12)$$

Substituting into (7.10) the solution of (7.11) we have

$$\|h^{(2)}\|_2 = \int_{-R^2}^{0} [(h, \partial_t (\mathcal{A} - I)) + (\text{rot} \ h, \text{rot} \ \eta) - (A \text{rot} \ A h, \text{rot} \ A \eta)] \, dt$$

Note that the matrix $\mathcal{A}$ is close the identity, so from (7.12) and coercive estimates for (7.10) we obtain

$$F_2(R, h^{(2)}) \leq c \|\varphi\|_{C^2} (F_2(R) + E_\ast(R)). \quad (7.13)$$

The third component of (7.6) satisfy to the homogenous boundary problem for the heat equation

$$\partial_t h^{(3)} - \Delta h^{(3)} = 0$$

$$h_3^{(3)}|_{y_3=0} = 0, \quad \frac{\partial h_3^{(3)}}{\partial y_3}|_{y_3=0} = 0, \quad \frac{\partial h_3^{(3)}}{\partial y_3}|_{y_3=0} = 0. \quad (7.14)$$

Extending $h^{(3)}$ into whole cylinder and using mean value theorem we have

$$F_2(r, h^{(3)}) \leq c \left(\frac{r}{R}\right)^2 F_2(R, h^{(3)})$$

$$\leq c \left(\frac{r}{R}\right)^2 \left(F_2(R, h) + F_2(R, h^{(1)} + h^{(2)})\right). \quad (7.15)$$

Combining (7.8), (7.13) and (7.15) we obtain the statement of the theorem.

8 Estimates of Energy Functionals

Now we define few more functionals. Note that all these functionals are invariant with respect to the natural scaling of the MHD system. For $r \leq 1$, $q \in [1, \frac{3}{2}]$, $s \in [1, \frac{3}{2}]$ and $(v, p, H, \varphi)$ suitable weak solution to
the MHD system in $Q^+(R)$ $0 < r < R < 1$ we introduce the following quantities:

$$A(r) \equiv \left( \frac{1}{2} \sup_{t \in (-r^2,0)} \int_{B^+(r)} |v|^2 \, dy \right)^{1/2},$$

$$A_*(r) \equiv \left( \frac{1}{2} \sup_{t \in (-r^2,0)} \int_{B^+(r)} |H|^2 \, dy \right)^{1/2},$$

$$C_Q(r) \equiv \left( \frac{1}{r^{s-q}} \int_{Q^+(r)} |v|^q \, dydt \right)^{1/q},$$

$$D(r) \equiv \left( \frac{1}{r^{s-q}} \int_{Q^+(r)} \left| p - [p]_{B^+(r)} \right|^{3/2} \, dydt \right)^{2/3},$$

$$D_*(r) = R^{3-s} \left( \int_{-r^2}^r \left( \int_{B^+(r)} \left| \nabla p \right|^s \, dy \right)^{\frac{1}{s-1}} \, dt \right)^{2/3},$$

$$C(r) = C_3(r), \quad F(r) = F_3(r), \quad D_*(r) = D_{36}(r).$$

First we formulate the set of results following from the general theory of functions:

**Lemma 8.1.** Let $R > 0$, $\varphi \in C^2(B^{(2)}(R))$, $v$, $H \in W^{1,0}_2(Q^+(R))$ and $p \in W^{1,0}_2(Q^+(R))$ are arbitrary functions. Assume $v|_{x_3=\varphi(x_1,x_2)} = 0$. Then for any $0 < r < R$ the following inequalities hold:

$$C(r) \leq A^2(r)E^2(r), \quad F(r) \leq A^2(r)[E^2(r) + F_2^2(r)] \quad (8.1)$$

$$D(r) \leq cD_1(r), \quad D_1(r) \leq cD_*(r), \quad \forall s > 1. \quad (8.2)$$

The proof of this lemma follows from interpolation inequalities and imbedding theorems. Proof of the similar inequalities for the Navier-Stokes system can be found in [5].

**Lemma 8.2.** Assume $(v,p,H,\varphi)$ is a boundary suitable weak solution to the MHD equations in $Q^+$. Then for any $r \in (0,1)$ the following inequality holds

$$A(r/2) + A_*(r/2) + E(r/2) + E_*(r/2) \leq c \left( C_2(r) + F_2(r) + C^2(r)D^2(r) + C^2(r) \right) +$$

$$+ c \left( C^2(r)A^2(r)E^2(r) + F^2(r)A^2(r)E^2(r) \right) \quad (8.3)$$

**Proof.** Estimate (8.3) follows from (1.4) in a standard way. We just explain the specific estimates of the terms

$$I_1 := \int_{Q^+(r)} |H|^2(v \cdot \nabla \zeta) \, dxdt \quad \text{and} \quad I_2 := \int_{Q^+(r)} (v \cdot H)(H \cdot \nabla \zeta) \, dxdt.$$
Applying the inequality \( \| f - [f]_{(B^+(r))} \|_{L^2(B^+(r))} \leq c \| \nabla f \|_{L^1(B^+(r))} \), we arrive at

\[
|I_1| \leq \frac{c}{r} \int_{-r^2}^{0} \left\| \nabla |H|^2 \right\|_{L^1(B^+(r))} \| v \|_{L^3(B^+(r))} \, dt \leq \\
\leq \frac{c}{r} \int_{-r^2}^{0} \| H \|_{L^2(B^+(r))} \| \nabla H \|_{L^2(B^+(r))} \| v \|_{L^3(B^+(r))} \, dt \leq \\
\leq \frac{c}{r^{3/2}} \| H \|_{L^2(Q^+(r))} \| \nabla H \|_{L^2(Q^+(r))} \| v \|_{L^3(Q^+(r))} \leq c_{\ast} (r) E_{\ast}(r) C(r)
\]

For \( I_2 \) we obtain relations

\[
I_2 = \int_{Q^+(r)} \left( \langle v \cdot H \rangle - [v \cdot H]_{B^+(r)} \right) \left( H \cdot \nabla \zeta \right) \, dx dt
\]

Hence

\[
|I_2| \leq \frac{c}{r} \int_{-r^2}^{0} \left\| \nabla \cdot [v \cdot H]_{B^+(r)} \right\|_{L^2(B^+(r))} \| H \|_{L^2(B^+(r))} \, dt \leq \\
\leq \frac{c}{r} \| H \|_{L^2(Q^+(r))} \int_{-r^2}^{0} \left\| \nabla (v \cdot H) \right\|_{L^2(B^+(r))} \, dt \leq \frac{c}{r} \| H \|_{L^2(Q^+(r))} \times \\
\times \int_{-r^2}^{0} \left( \| \nabla v \|_{L^2(B^+(r))} \| H \|_{L^2(B^+(r))} + \| \nabla H \|_{L^2(B^+(r))} \| v \|_{L^3(B^+(r))} \right) \, dt \leq \\
\leq \frac{c}{r^{3/2}} \| H \|_{L^2(Q^+(r))} \left( \| \nabla v \|_{L^2(Q^+(r))} \| H \|_{L^2(Q^+(r))} + \| \nabla H \|_{L^2(Q^+(r))} \| v \|_{L^3(Q^+(r))} \right)
\]

So, we obtain

\[
|I_2| \leq c_{\ast} (r) \left( E(r) F(r) + E_{\ast}(r) C(r) \right)
\]

Lemma 8.3. Assume \((v, p, H, \varphi)\) is a boundary suitable weak solution to the MHD equations in \(Q^+\) and \(\| \varphi \|_{C^4(\overline{B}(2))} < \mu < \frac{\mu_{\ast}}{2} \) where \(\mu_{\ast}\) is the constant defined in lemma 4.1. Then for any \(r \in (0, 1)\) and \(\theta \in (0, \frac{1}{2})\) the following inequality holds

\[
D_{\ast}(\theta r) \leq c_{\ast} \theta^\mu \left( D_{\ast}(r) + E(r) \right) + \\
+ c(\theta) \left( A^\mu(r) E(r) + A^\nu_{\ast}(r) F_{\ast}(r) E_{\ast}(r) \right)
\]

(8.4)

Proof. To obtain (8.4) we apply the method developed in [10], [12], see also [14]. Let \( c(y) \) is the map defined by (4.2). We fix \( r \in (0, 1) \) and \( \theta \in (0, \frac{1}{2}) \) and without loss of generality we can assume, that \( e^{-1}(Q^+(\theta r)) \subset Q^0_{\theta}(2\theta r) \subset Q^0_{\theta}(r/2) \subset e^{-1}(Q^+(r)) \). Then we decompose \( v \) and \( p \) as

\[
v = \tilde{v} + \hat{v}, \quad p = \tilde{p} + \hat{p},
\]

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where \((\tilde{v}, \tilde{p})\) is a solution of the perturbed Stokes initial boundary value problem in a half-space

\[
\begin{aligned}
\partial_t \tilde{v} - \tilde{\Delta}_\nu \tilde{v} + \nabla \cdot \tilde{v} \tilde{p} &= \mathcal{L} \cdot (\mathcal{L}^{-1} \mathcal{H} \times \mathcal{H} - (\tilde{v} \cdot \nabla)\tilde{v}), & \text{in } Q^0_{\nu}(\frac{R}{2}), \\
\nabla \cdot \tilde{v} &= 0, & \text{in } Q^0_{\nu}(\frac{R}{2}), \\
\tilde{v}|_{t=0} &= 0, & \tilde{v}|_{y_3=0} &= 0,
\end{aligned}
\]

and \((\tilde{v}, \tilde{p})\) is a solution of the homogeneous perturbed Stokes system in \(Q^0_{\nu}(\frac{R}{2})\):

\[
\begin{aligned}
\partial_t \tilde{v} - \tilde{\Delta}_\nu \tilde{v} + \nabla \cdot \tilde{v} \tilde{p} &= 0, & \text{in } Q^0_{\nu}(\frac{R}{2}), \\
\nabla \cdot \tilde{v} &= 0, & \text{in } Q^0_{\nu}(\frac{R}{2}), \\
\tilde{v}|_{y_3=0} &= 0.
\end{aligned}
\]

For \(\nabla \tilde{p}\) and \(\nabla \tilde{p}\) from lemmas 4.1 and 4.2 we have the following estimates.

\[
\|\nabla \tilde{p}\|_{L^{\frac{36}{35}}(Q^0_{\nu}(\frac{R}{3}))} + \frac{1}{r} \|\nabla \tilde{v}\|_{L^{\frac{36}{35}}(Q^0_{\nu}(\frac{R}{3}))} \leq \frac{c}{r^\frac{3}{4}} \left( \|H \times \nabla v\|_{L^{\frac{36}{35}}(Q^0_{\nu}(\frac{R}{3}))} \right),
\]

\[
\|\nabla \tilde{p}\|_{L^{\frac{36}{35}}(Q^0_{\nu}(\frac{R}{3}))} \leq c \frac{1}{r} \|\nabla \tilde{v}\|_{L^{\frac{36}{35}}(Q^0_{\nu}(\frac{R}{3}))} + \|\nabla \tilde{p}\|_{L^{\frac{36}{35}}(Q^0_{\nu}(\frac{R}{3}))}.
\]

From the Hölder inequality we obtain

\[
\|H \times \nabla \tilde{v}\|_{L^{\frac{36}{35}}(Q^0_{\nu}(\frac{R}{3}))} \leq c r^\frac{3}{8} \frac{1}{r^\frac{3}{4}} \left( \|H\|_{L^8(Q^0(\frac{R}{3}))} \right) \|\nabla \tilde{v}\|_{L^2(Q^0(\frac{R}{3}))} \|\tilde{v}\|_{L^2(Q^0(\frac{R}{3}))}
\]

\[
\|(v \cdot \nabla)\tilde{v}\|_{L^{\frac{36}{35}}(Q^0_{\nu}(\frac{R}{3}))} \leq c r^\frac{3}{8} \frac{1}{r^\frac{3}{4}} \left( \|v\|_{L^2(Q^0(\frac{R}{3}))} \right) \|\nabla \tilde{v}\|_{L^2(Q^0(\frac{R}{3}))} \|\tilde{v}\|_{L^2(Q^0(\frac{R}{3}))}
\]

Representing \(\tilde{v} = v - \hat{v}, \tilde{p} = p - \hat{p}\) and gathering all above estimates for \(\hat{p}\) and \(\hat{v}\) we obtain

\[
D_*(\theta r) \leq c \theta^\frac{3}{8} \left( D_*(r) + E(r) + A_0^2(r) E^\frac{3}{4}_*(r) + A_0^2(r) E^\frac{5}{4}_*(r) \right) + c(\theta) \left( A_0^2(r) E^\frac{3}{4}_*(r) + A_0^2(r) E^\frac{5}{4}_*(r) \right)
\]

\[
\square
\]

9 CKN condition and Partial Regularity of Solutions

In this section we present the proofs of Theorems 3.1 and 2.2. We start from proof of the modified version of (7.3).

Lemma 9.1. For any \(K > 0\) there exists a constants \(c(K) > 0\) and \(\varepsilon_2 > 0\) such that for any \(\varepsilon \in (0, \varepsilon_2]\) and any boundary suitable weak solution \((v, H, p, \varphi)\) of the MHD system in \(Q^+\) if

\[
\sup_{r \in (0,1)} E(r) \leq \varepsilon, \quad \|\varphi\|_{C^2(B(2r))} \leq \varepsilon, \quad \sup_{r \in (0,1)} E_*(r) \leq K, \quad (9.1)
\]
then for some $\alpha > 0$ and any $0 < r < R \leq 1$
\[
F_2(r) \leq c \left( \frac{r}{R} \right)^\alpha F_2(R) + c(K)\varepsilon_2. \tag{9.2}
\]

**Proof.** We will use standard iteration technic. Let $R > 0$ and $\theta \in (0, 1/2)$. We fix $\varepsilon_1$ from theorem 7.1. Then from (7.3) we have
\[
F_2(\theta R) \leq c_1(\theta^2 + \varepsilon_1)F_2(R) + c\varepsilon_1(K + 1). \tag{9.3}
\]
Next we choose $\theta$ and $\varepsilon_2 < \varepsilon_1$ such that
\[
c_1\theta^2 \leq \frac{1}{4}, \quad c_1\varepsilon_2 \leq \frac{1}{4}
\]
Then from (9.3) we obtain
\[
F_2(\theta R) \leq \frac{1}{2} F_2(R) + c\varepsilon_2(K + 1).
\]
Next we will iterate the last inequality
\[
F_2(\theta^k R) \leq \frac{1}{2} F_2(\theta^{k-1} R) + c\varepsilon_2(K + 1) \leq
\[
\leq \frac{1}{4} F_2(\theta^{k-2} R) + \left( 1 + \frac{1}{2} \right) c\varepsilon_2(K + 1) \leq
\]
\[
\leq \frac{1}{2^k} F_2(R) + c\varepsilon_2(K + 1).
\]
Finally we put $\alpha = \log \frac{1}{2}$ and chose $k > 0$ such that $\theta^{k+1} R \leq r \leq \theta^k R$. Then from (9.4) we obtain
\[
F_2(r) \leq c F_2(\theta^k R) \leq c \frac{1}{2^k} F_2(R) + c(K)\varepsilon_2 \leq
\]
\[
\leq c^{\theta^{-\alpha}} F_2(R) + c(K)\varepsilon_2 \leq
\]
\[
\leq c \left( \frac{r}{R} \right)^\alpha F_2(R) + c(K)\varepsilon_2.
\]

**Lemma 9.2.** Denote by $E(r)$ the following functional
\[
E(r) = A(r) + A_*(r) + D_*(r),
\]
and let $\varepsilon_2 > 0$ be the absolute constant defined in lemma 9.1. For any $K > 0$ there exists a constant $c(K) > 0$ such that for any $\varepsilon \in (0, \varepsilon_2]$ and any boundary suitable weak solution $(v, H, p, \varphi)$ of the MHD system in $Q^+$ if
\[
\sup_{r \in (0,1)} E(r) \leq \varepsilon, \quad \|\varphi\|_{C^2(\beta/2)} < \varepsilon, \quad \sup_{r \in (0,1)} E_*(r) \leq K, \tag{9.5}
\]
and
\[
F_2(1) \leq M, \tag{9.6}
\]
then for any $0 < r < R \leq 1$
\[
E(r) \leq c \left( \frac{r}{R} \right)^\beta E(R) + c(K)(1 + R^\alpha M). \tag{9.7}
\]
where $\beta > 0$ is some absolute constant.)
Fixing

Taking into account (9.8) and (9.5) we obtain

Applying the Young inequality we get

and the Young inequality we obtain

Proof. Without loss of generality we can assume $K \geq 1$. Then from (9.2) we obtain

$$F_2(R) \leq c\alpha M + c\varepsilon_2 C(K).$$

From this inequality and (8.1) we obtain

$$C(R) \leq c \mathcal{E}_1^\frac{1}{2} + c \mathcal{E}_1^\frac{1}{2}, \quad F(R) \leq c \mathcal{E}_1^\frac{1}{2} \left(C(K) + R \mathcal{M}\right) \tag{9.8}$$

Assume $r \in (0,1)$ and $\theta \in (0,\frac{1}{2})$. From (8.3) with the help of (8.2) and the Young inequality we obtain

$$\mathcal{E}(\theta R) \leq c \left(F_2(2\theta R) + D_\alpha(2\theta R)\right) +$$

$$+ c(\theta) \left(C_2(R) + C(R) + C_\beta(R) + C_\alpha(R) E_\beta(R) + F_\beta(R) A_\beta(R) E_\beta(R)\right) \tag{9.9}$$

Taking into account (9.8) and (9.5) we obtain

$$\mathcal{E}(\theta R) \leq c \left(F_2(2\theta R) + D_\alpha(2\theta R)\right) +$$

$$+ c(\theta) \left(\mathcal{E}_1^\frac{1}{2} + \mathcal{E}_1^\frac{1}{2} + \varepsilon_2^\frac{1}{2} \mathcal{E}_1^\frac{1}{2} \theta R^\frac{2}{2} + \mathcal{E}_1^\frac{1}{2} (R) \theta R^\frac{2}{2} +$$

$$+ (C(K) + R \mathcal{M}) \mathcal{E}_1^\frac{1}{2} (R) \theta R^\frac{2}{2}\right)$$

Applying the Young inequality $ab \leq c a^p + c b^p$ we obtain

$$\mathcal{E}(\theta R) \leq \frac{1}{4} \mathcal{E}(R) + c \left(F_2(2\theta R) + D_\alpha(2\theta R)\right) + c(\theta) c(K) + c(\theta) R \alpha M.$$

From (9.2) and (8.4) we obtain

$$F_2(2\theta R) + D_\alpha(2\theta R) \leq c\theta \alpha \left(F_2(R) + D_\alpha(R)\right) + C(K,\theta) \varepsilon_2 +$$

$$+ c(\theta) \left(A_\beta(R) E_\beta(R) + A_\beta(R) F_\beta(R) E_\beta(R)\right)$$

Taking into account (9.8) and the obvious inequality $F_2(R) \leq A_\alpha(R)$ we arrive at

$$F_2(2\theta R) + D_\alpha(2\theta R) \leq c\theta \alpha \mathcal{E}(R) + c(K,\theta) +$$

$$+ c(\theta) \left(\mathcal{E}_1^\frac{1}{2} + \varepsilon_2^\frac{1}{2} + \mathcal{E}_1^\frac{1}{2} (R) (C(K) + R \mathcal{M}) K\right)$$

Applying the Young inequality we get

$$F_2(2\theta R) + D_\alpha(2\theta R) \leq \left(\frac{1}{4} + c\theta \alpha \right) \mathcal{E}(R) + C(\theta, K) (1 + R \alpha M)$$

Gathering the estimates we obtain

$$\mathcal{E}(\theta R) \leq \left(\frac{1}{4} + c\theta \alpha \right) \mathcal{E}(R) + C(\theta, K) (1 + R \alpha M).$$

Fixing $\theta \in (0,\frac{1}{2})$ so that

$$c\theta \alpha \leq \frac{1}{4}$$

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Hence
\[ E(\theta R) \leq \frac{1}{2} E(R) + C(K)(1 + R^\alpha M). \]

Next with the help of technic used in the proof of lemma 9.1 we obtain (9.7).

\[ \square \]

**Lemma 9.3.** Assume all conditions of Theorem 9.2 hold and fix \( R_0 \in (0, 1) \) so that
\[ R_0^\beta M \leq 1. \] (9.10)

Then for any \( 0 < r < R \leq R_0 \) the following estimates hold:
\[ A(r) + A^*_r(r) \leq c (\frac{r}{R})^{\gamma} \left( A(R) + A^*_R(R) \right) + \varepsilon \left( \frac{1}{2} D(R) + G(K, \varepsilon) \right) \] (9.11)
\[ D(r) \leq c \left( \frac{r}{R} \right)^{\gamma} D(R) + c(K) \left( A^*_R(R) + A^*_R(R) \right) + G(K, \varepsilon) \] (9.12)
where \( \gamma > 0 \) is some absolute constant and \( G \) is a continuous function possessing the following property:
\[ \text{for any fixed } K > 0 \text{ } G(K, \varepsilon) \to 0 \text{ as } \varepsilon \to 0. \] (9.13)

**Proof.** From (8.1) taking into account (9.10) we obtain
\[ C(r) \leq A^*_r(r), \quad F(r) \leq A^*_r(r)C(K) \] (9.14)

Take arbitrary \( r \in (0, R_0) \) and \( \theta \in (0, \frac{1}{2}) \). Denote by \( E_*(R) \) the following functional
\[ E_*(R) = A(R) + A^*_R(R). \]

Then from (8.3) similar to (9.9) using (9.14) we derive
\[ E_*(\theta R) \leq F_2(\theta R) + C^* f_2(\theta R) D^*_F(\theta R) + c(\theta) \left( E^*_F(R) + E^*_F(R)K + E^*_F(R)K \right) \]

Applying the Young inequality and using (8.2) we obtain
\[ E_*(\theta R) \leq \frac{1}{8} E_*(R) + c(\theta)G(K, \varepsilon) + F_2(\theta R) + C^*_F(\theta R) D^*_F(\theta R) \] (9.15)

From (9.2) we conclude
\[ F(\theta R) \leq c \theta^\alpha E_*(R) + G(K, \varepsilon). \] (9.16)

From (8.4) for \( R \leq R_0 \) with the help of (9.14) and the Young inequality we obtain
\[ D_*(\theta R) \leq c \theta^\alpha D_*(R) + C(\theta, K)E^*_F(R) + c(\theta)G(K, \varepsilon) \] (9.17)

Hence from (9.14) we obtain
\[ C^* \left( \frac{1}{2} D^*_F(\theta R) \leq c(\theta)E^*_F(R) + \frac{1}{2} D^*_F(R) \right) + C(\theta, K)E^*_F(R) + c(\theta)G(K, \varepsilon) \]
Applying the Young inequality we arrive at
\[ C^{\frac{1}{2}}(2\theta R)D^{\frac{1}{2}}(2\theta R) \leq \frac{1}{8}E_\varepsilon(R) + \frac{1}{2}\varepsilon^\frac{1}{2}D_\varepsilon(R) + c(\theta)G(K,\varepsilon) \quad (9.18) \]

Gathering estimates (9.15) — (9.18) we obtain the inequality
\[ E_\varepsilon(\theta R) \leq \left( \frac{1}{4} + c\theta^\alpha \right) E_\varepsilon(R) + \frac{1}{2}\varepsilon^\frac{1}{2}D_\varepsilon(R) + c(\theta)G(K,\varepsilon) \]

Choosing \( \theta \in (0, \frac{1}{4}) \) so that
\[ \frac{1}{4} + c\theta^\alpha = \frac{1}{2} \]
we obtain
\[ E_\varepsilon(\theta R) \leq \frac{1}{2} E_\varepsilon(R) + \frac{1}{2}\varepsilon^\frac{1}{2}D_\varepsilon(R) + c(\theta)G(K,\varepsilon) \]

Iterating this inequality we obtain (9.11).

Choosing in (9.17) \( \theta \in (0, \frac{1}{2}) \) so that
\[ c\theta^\beta = \frac{1}{2} \]
and iteration the inequality we derive (9.12).

\[ \square \]

**Theorem 9.1.** For any \( K > 0 \) there exists a constant \( \varepsilon_0(K) > 0 \) such that if the condition (9.5) holds with \( \varepsilon \leq \varepsilon_0 \), then there exists \( \rho_* \in (0, 1) \) such that
\[ \left( C(\rho_*) + F(\rho_*) + D(\rho_*) \right) < \varepsilon_{*}^\frac{1}{2}, \]
where the constant \( \varepsilon_* > 0 \) is defined in Theorem 3.2.

**Proof.** From (9.7) we obtain
\[ \limsup_{r \to 0} D_\varepsilon(r) \leq c(K). \]
From (9.11) we derive
\[ \limsup_{r \to 0} \left( A(r) + A_\varepsilon(r) \right) \leq \varepsilon^{\frac{1}{2}} \limsup_{\rho \to 0} D(\rho) + G(K, \varepsilon) \leq \varepsilon^{\frac{1}{2}} c(K) + G(K, \varepsilon). \]
From (9.12) we obtain
\[ \limsup_{r \to 0} D_\varepsilon(r) \leq c(K) \limsup_{\rho \to 0} \left( A^{\frac{1}{2}} + A^{\frac{1}{2}}_\varepsilon(\rho) \right) + G(K, \varepsilon) \leq c(K) \left( \varepsilon^{\frac{1}{2}} c(K) + G(K, \varepsilon) \right)^{\frac{1}{2}} + G(K, \varepsilon). \]
From (8.1) we conclude
\[ \limsup_{r \to 0} \left( C(r) + F(r) \right) \leq (\varepsilon^{\frac{1}{2}} + c(K)) \limsup_{r \to 0} \left( A(r) + A_\varepsilon(r) \right) \leq (\varepsilon^{\frac{1}{2}} + c(K)) \left( \varepsilon^{\frac{1}{2}} c(K) + G(K, \varepsilon) \right)^{\frac{1}{2}}. \]
Taking into account (9.13) for any $K > 0$ we can find $\varepsilon_0(K) > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$
\[
c(K) \left( \varepsilon^{\frac{1}{2}} c(K) + G(K, \varepsilon) \right)^{\frac{1}{2}} + G(K, \varepsilon) < \frac{\varepsilon^{\frac{1}{2}}}{2}
\]
and
\[
(\varepsilon^{\frac{1}{2}} + K^{\frac{1}{2}}) \left( \varepsilon^{\frac{1}{2}} c(K) + G(K, \varepsilon) \right)^{\frac{1}{2}} < \frac{\varepsilon^{\frac{1}{2}}}{2}.
\]
Hence for $\varepsilon \in (0, \varepsilon_0)$
\[
\limsup_{r \to 0} \left( C(r) + F(r) + D_\star(r) \right) < \varepsilon^{\frac{1}{2}}.
\]
\[\square\]

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