Positivity conjectures for Kazhdan-Lusztig theory on twisted involutions: the finite case

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Abstract

Let \((W, S)\) be any Coxeter system and let \(w \mapsto w^*\) be an involution of \(W\) which preserves the set of simple generators \(S\). Lusztig and Vogan have shown that the corresponding set of twisted involutions (i.e., elements \(w \in W\) with \(w^{-1} = w^*\)) naturally generates a module of the Hecke algebra of \((W, S)\) with two distinguished bases. The transition matrix between these bases defines a family of polynomials \(P_{y,w}^\sigma\) which one can view as a “twisted” analogue of the much-studied family of Kazhdan-Lusztig polynomials of \((W, S)\). The polynomials \(P_{y,w}^\sigma\) can have negative coefficients, but display several conjectural positivity properties of interest, which parallel positivity properties of the Kazhdan-Lusztig polynomials. This paper reports on some calculations which verify four such positivity conjectures in several finite cases of interest, in particular for the non-crystallographic Coxeter systems of types \(H_3\) and \(H_4\).

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1 Introduction

1.1 Overview

Let \((W, S)\) be any Coxeter system, and write \(H_q^2\) for the associated Hecke algebra with parameter \(q^2\): this is the usual Hecke algebra (namely, a certain \(\mathbb{Z}[q^{\pm 1/2}]\)-algebra with a basis \((T_w)_{w \in W}\) indexed by \(W\)), but with \(q\) replaced by \(q^2\) in its defining relations. (A precise definition appears in Section 1.3.) Next, fix an automorphism \(*: W \to W\) with order one or two which preserves the set of simple generators \(S\). Write \(I_*\) for the corresponding set of twisted involutions (i.e., elements \(w \in W\) with \(w^{-1} = w^*\), and let \(M_q^2\) be the free \(\mathbb{Z}[q^{\pm 1/2}]\)-module which this set generates.

Lusztig and Vogan [11, 12] have shown that \(M_q^2\) naturally carries a nontrivial \(H_q^2\)-module structure, which gives rise to a distinguished basis of \(M_q^2\) sharing many formal properties with the much-studied Kazhdan-Lusztig basis of \(H_q^2\). In particular, the transition matrix from the standard basis of \(M_q^2\) to its distinguished basis defines a family of “twisted Kazhdan-Lusztig polynomials” \((P_{\sigma y,w}^\sigma)_{y,w \in I_*} \subset \mathbb{Z}[q]\), formally similar to the Kazhdan-Lusztig polynomials \((P_{y,w})_{y,w \in W} \subset \mathbb{Z}[q]\) attached to \((W, S)\). The details of these constructions are given in Sections 1.2 and 1.3.

Several remarkable properties of the Kazhdan-Lusztig basis of \(H_q^2\) appear to have “twisted” analogues for the module \(M_q^2\). For example, one of the most famous aspects of the original Kazhdan-Lusztig polynomials \((P_{y,w})_{y,w \in W}\), only recently proved in complete generality by Elias and Williamson [5], is that their coefficients are always nonnegative. The twisted Kazhdan-Lusztig polynomials \((P_{\sigma y,w}^\sigma)_{y,w \in I_*}\) can have negative coefficients, but Lusztig and Vogan [11] have shown by geometric arguments that the modified polynomials \(\frac{1}{2}(P_{y,w} - P_{\sigma y,w}^\sigma)\) for \(y, w \in I_*\) have nonnegative coefficients whenever \((W, S)\) is crystallographic. In fact, for any choice of \((W, S)\) and \(*\), the polynomials \(\frac{1}{2}(P_{y,w} - P_{\sigma y,w}^\sigma)\) belong to \(\mathbb{Z}[q]\) by [11, Theorem 9.10], and Lusztig [11, Conjecture 9.12] has conjectured that their coefficients are always nonnegative.

Section 1.4 presents three additional conjectural positivity properties related to the “Kazhdan-Lusztig basis” of the \(H_q^2\)-module \(M_q^2\). We prove that these positivity properties hold for arbitrary Coxeter systems if they hold for irreducible Coxeter systems, provided that analogous positivity conjectures related to the ordinary Kazhdan-Lusztig polynomials hold. In addition, we report on some calculations performed using extensions [14] to du Cloux’s program Coxeter [4], which verify our four positivity properties in several finite cases (in particular, for the non-crystallographic Coxeter systems of types \(H_3\) and \(H_4\)). A more detailed summary of our results appears in Section 1.5 at the end of this introduction, after some minimal preliminaries in Sections 1.2, 1.3, and 1.4.

1.2 Kazhdan-Lusztig theory

Throughout we write \(\mathbb{Z}\) for the integers and \(\mathbb{N} = \{0, 1, 2, \ldots\}\) for the nonnegative integers, and we adopt the following conventions:

- Let \((W, S)\) be a Coxeter system with length function \(\ell: W \to \mathbb{N}\).
- Let \(\leq\) denote the Bruhat order on \(W\).
- Let \(\mathcal{A} = \mathbb{Z}[v, v^{-1}]\) be the ring of Laurent polynomials over \(\mathbb{Z}\) in an indeterminate \(v\).
- Let \(q = v^2\). In the sequel, we will refer to \(v\) in place of the parameter \(q^{1/2}\) in Section 1.3.
For background on Coxeter systems and the Bruhat order, see for example [11, 7, 10].

Here we briefly recall the definition of the Kazhdan-Lusztig polynomials attached to \((W,S)\). Let \(\mathcal{H}_q\) denote the free \(\mathcal{A}\)-module with basis \(\{t_w : w \in W\}\). This module has a unique \(\mathcal{A}\)-algebra structure with respect to which the multiplication rule

\[
t_s t_w = \begin{cases} 
    t_{sw} & \text{if } sw > w \\
    qt_{sw} + (q-1)t_w & \text{if } sw < w
\end{cases}
\]

holds for each \(s \in S\) and \(w \in W\). We remark that the element \(t_w \in \mathcal{H}_q\) is more often denoted in the literature by the symbol \(T_w\), but here we reserve the latter notation for the Hecke algebra \(\mathcal{H}_{q^2}\), to be introduced in the next section.

We refer to the algebra \(\mathcal{H}_q\) as the Hecke algebra of \((W,S)\) with parameter \(q\). Standard references for this much-studied object include, for example, [11, 7, 9, 10]. The Hecke algebra possesses a unique ring involution \(\ast : \mathcal{H}_q \to \mathcal{H}_q\) with \(v^n = v^{-n}\) and \(t_w = (t_{w^{-1}})^{-1}\) all \(n \in \mathbb{Z}\) and \(w \in W\), referred to as the bar operator, and this gives rise to the following theorem-definition from Kazhdan and Lusztig’s seminal paper [9].

**Theorem-Definition 1.1** (Kazhdan and Lusztig [9]). For each \(w \in W\) there is a unique family of polynomials \((P_{y,w})_{y \in W} \subset \mathbb{Z}[q]\) with the following three properties:

1. The element \(c_w \overset{\text{def}}{=} v^{-\ell(w)} \cdot \sum_{y \in W} P_{y,w} \cdot t_y \in \mathcal{H}_q\) has \(\overline{v}_w = c_w\).
2. \(P_{y,w} = \delta_{y,w}\) if \(y \not< w\) in the Bruhat order.
3. \(P_{y,w}\) has degree at most \(\frac{1}{2}(\ell(w) - \ell(y) - 1)\) as a polynomial in \(q\) whenever \(y < w\).

**Remark.** Here and in the sequel, the Kronecker delta \(\delta_{y,w}\) has the usual meaning of \(\delta_{y,w} = 1\) if \(y = w\) and \(\delta_{y,w} = 0\) otherwise.

The polynomials \((P_{y,w})_{y,w \in W}\) are the Kazhdan-Lusztig polynomials of the Coxeter system \((W,S)\). Property (b) implies that the elements \((c_w)_{w \in W}\) form an \(\mathcal{A}\)-basis for \(\mathcal{H}_q\), which one calls the Kazhdan-Lusztig basis. We note the following well-known multiplication formula for use later.

**Theorem 1.2** (Kazhdan and Lusztig [9]). Let \(w \in W\) and \(s \in S\). Then \(c_s = v^{-1}(t_s + 1)\) and

\[
c_sc_w = \begin{cases} 
    (v + v^{-1})c_w & \text{if } sw < w \\
    c_{sw} + \sum_{z \in W; sz < z < w} \mu(z, w)c_z & \text{if } sw > w
\end{cases}
\]

where \(\mu(z, w)\) denotes the coefficient of \(v^{\ell(w) - \ell(z) - 1}\) in the polynomial \(P_{z,w} \in \mathbb{Z}[v^2]\).

### 1.3 Twisted Kazhdan-Lusztig theory

Following [11, 12], we now introduce a slightly different Hecke algebra \(\mathcal{H}_{q^2}\), possessing an analogous pair of \(W\)-indexed \(\mathcal{A}\)-bases which we will denote using capital letters by \((T_w)_{w \in W}\) and \((C_w)_{w \in W}\). Explicitly, let \(\mathcal{H}_{q^2}\) denote the free \(\mathcal{A}\)-module with basis \(\{T_w : w \in W\}\). This module has a unique \(\mathcal{A}\)-algebra structure with respect to the slightly altered multiplication rule

\[
T_s T_w = \begin{cases} 
    T_{sw} & \text{if } sw > w \\
    q^2T_{sw} + (q^2 - 1)T_w & \text{if } sw < w
\end{cases}
\]
holds for each $s \in S$ and $w \in W$. We refer to $H_{q^2}$ with this structure as the Hecke algebra of $(W, S)$ with parameter $q^2$. This algebra likewise possesses a unique ring involution $\overline{\cdot} : H_{q^2} \to H_{q^2}$ with $\overline{v^m} = v^{-n}$ and $\overline{T_w} = (T_{w^{-1}})^{-1}$ for all $n \in \mathbb{Z}$ and $w \in W$, which fixes each of the elements

$$C_w \overset{\text{def}}{=} q^{-\ell(w)} \cdot \sum_{y \in W} P_{y,w}(q^2) \cdot T_y \in H_{q^2} \quad \text{for } w \in W.$$ 

The elements $(C_w)_{w \in W}$ form an $A$-basis of $H_{q^2}$ which one refers to as its Kazhdan-Lusztig basis.

The construction which is the main topic of this work is now given as follows. Choose an automorphism $w \mapsto w^*$ of $W$ with order $\leq 2$ such that $s^* \in S$ for each $s \in S$, and write $I_*$ for the corresponding set of twisted involutions

$$I_* = \{ w \in W : w^* = w^{-1} \}.$$ 

Lusztig and Vogan’s paper [12] first established the following trio of Theorem-Definitions in the case that $W$ is a Weyl group or affine Weyl group and $*$ is trivial; Lusztig’s paper [11] then extended these results to arbitrary Coxeter systems.

**Notation.** Given $s \in S$ and $w \in I_*$, let $s \prec w$ denote the unique element in the intersection of $\{sw, sws^*\}$ with $I_* \setminus \{w\}$. Explicitly, $s \prec w = sw = ws^*$ if $w = sws^*$ and $s \prec w = sws^*$ otherwise.

**Theorem-Definition 1.3** (Lusztig and Vogan [12]; Lusztig [11]). Let $M_{q^2}$ be the free $A$-module with basis $\{a_w : w \in I_*\}$. Then $M_{q^2}$ has a unique $H_{q^2}$-module structure with respect to which the following multiplication rule holds for each $s \in S$ and $w \in I_*:

$$T_s a_w = \begin{cases} 
  a_{s \prec w} & \text{if } s \prec w = sws^* > w \\
  (q+1)a_{s \prec w} + qa_w & \text{if } s \prec w = sw > w \\
  (q^2 - q)a_{s \prec w} + (q^2 - q - 1)a_w & \text{if } s \prec w = sw < w \\
  q^2a_{s \prec w} + (q^2 - 1)a_w & \text{if } s \prec w = sws^* < w.
\end{cases}$$

**Theorem-Definition 1.4** (Lusztig and Vogan [12]; Lusztig [11]). There is a unique $\mathbb{Z}$-linear involution $\overline{\cdot} : M_{q^2} \to M_{q^2}$ such that $\overline{a_1} = a_1$ and $\overline{h \cdot m} = \overline{h} \cdot \overline{m}$ for all $h \in H_{q^2}$ and $m \in M_{q^2}$.

Lusztig [11] has shown moreover that the bar operator just defined acts on the standard basis of $M_{q^2}$ by the formula $\overline{a_w} = (-1)^{\ell(w)} \cdot (T_{w^{-1}})^{-1} \cdot a_{w^{-1}}$ for $w \in I_*$. 

**Theorem-Definition 1.5** (Lusztig and Vogan [12]; Lusztig [11]). For each $w \in I_*$ there is a unique family of polynomials $(P_{y,w}^\sigma)_{y \in I_*} \subset \mathbb{Z}[q]$ with the following three properties:

(a) The element $A_w \overset{\text{def}}{=} v^{-\ell(w)} \cdot \sum_{y \in I_*} P_{y,w}^\sigma \cdot a_y \in M_{q^2}$ has $\overline{A_w} = A_w$.

(b) $P_{y,w}^\sigma = \delta_{y,w}$ if $y \preceq w$ in the Bruhat order.

(c) $P_{y,w}^\sigma$ has degree at most $\frac{1}{2}(\ell(w) - \ell(y) - 1)$ as a polynomial in $q$ whenever $y < w$.

The elements $(A_w)_{w \in I_*}$ form an $A$-basis for the module $M_{q^2}$, which we sometimes refer to this as the “twisted Kazhdan-Lusztig basis.” Likewise, we call the polynomials $P_{y,w}^\sigma$ the twisted Kazhdan-Lusztig polynomials of the triple $(W, S, *)$. 

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1.4 Positivity properties

The primary results of this paper concern four conjectural positivity properties of the twisted Kazhdan-Lusztig polynomials $P_{y,w}^\sigma$. These are patterned on the following (partially conjectural) properties of the Kazhdan-Lusztig polynomials $P_{y,w}$ of an arbitrary Coxeter system $(W,S)$.

**Property A.** The polynomials $P_{y,w}$ have nonnegative coefficients for all $y, w \in W$.

**Property B.** The polynomials $P_{y,w}$ are decreasing for fixed $w$, in the sense that the difference $P_{y,w} - P_{z,w}$ has nonnegative coefficients whenever $y, z, w \in W$ and $y \leq z$.

Let $(h_{x,y,z})_{x,y,z \in W}$ denote the structure constants of $\mathcal{H}_q$ in the Kazhdan-Lusztig basis, i.e., the Laurent polynomials in $A$ satisfying $c_xc_y = \sum_{z \in W} h_{x,y,z}c_z$ for $x, y, z \in W$.

**Property C.** The Laurent polynomials $h_{x,y,z}$ have nonnegative coefficients for all $x, y, z \in W$.

**Property D.** For any $x, y, z \in W$, if the Laurent polynomial $h_{x,y,z}$ has degree $d$ in $v$ (where we consider 0 to have degree 0), then $v^d h_{x,y,z}$ is a unimodal polynomial in $q = v^2$.

**Remark.** Recall that a polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in \mathbb{Z}[x]$, where each $a_i \in \mathbb{Z}$, is unimodal if $0 \leq a_0 \leq \cdots \leq a_{i-1} \leq a_i \geq a_{i+1} \geq \cdots \geq a_n \geq 0$ for some index $i$. The content of Property D really only concerns unimodality, for it always holds that $v^d h_{x,y,z}$ is a polynomial in $q$ if $d$ is the degree of $h_{x,y,z}$ as a Laurent polynomial in $v$; see Corollary 3.5.

Naturally accompanying the preceding properties is this conjecture.

**Conjecture 1.6.** Properties A, B, C, and D hold for all Coxeter systems $(W,S)$.

Elias and Williamson’s recent proof of Soergel’s conjecture [5] shows that at least Properties A and C hold for all Coxeter systems. Property D is known to hold for all finite Coxeter systems by work of Irving [8] and du Cloux [3]. Property D has received the least attention in the literature. Computations of du Cloux [3] at least show that this property holds for dihedral Coxeter systems and in types $H_3$ and $H_4$. Using du Cloux’s program Coxeter [4] we have in turn checked (appealing to Proposition 3.5 below) that Property D holds for all finite Coxeter systems whose irreducible factors have rank at most five. (Recall that the rank of $(W,S)$ is the size of $S$.)

The appropriate “twisted” analogues of the preceding properties are not the obvious ones suggested by the formal parallels between Theorem-Definitions 1.1 and 1.5. Instead we proceed as follows. Define $P_{y,w}^+, P_{y,w}^- \in \mathbb{Q}[q]$ by

$$P_{y,w}^\pm = \frac{1}{2} \left( P_{y,w} \pm P_{y,w}^\sigma \right)$$

for each $y, w \in \mathcal{I}_*$. While the polynomials $P_{y,w}^\sigma$ may have negative coefficients, Lusztig proves that the polynomials $P_{y,w}^\pm$ actually have integer coefficients [11, Theorem 9.10]. Consider the following conjectural properties related to these polynomials:

**Property A’.** Both $P_{y,w}^+$ and $P_{y,w}^-$ have nonnegative coefficients for all $y, w \in \mathcal{I}_*$.

**Property B’.** The polynomials $P_{y,w}^\pm$ are decreasing for fixed $w$, in the sense that the differences $P_{y,w}^+ - P_{z,w}^+$ and $P_{y,w}^- - P_{z,w}^-$ have nonnegative coefficients whenever $y, z, w \in \mathcal{I}_*$ and $y \leq z$. 

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To give analogues of Properties \(C\) and \(D\) for each \(x \in W\) and \(y \in I_x\) define \((h_{x,y;z}^\sigma)_{z \in I_x} \subset A\) and \((h_{x,y;y}^\sigma)_{z \in I_x} \subset A\) as the Laurent polynomials satisfying

\[
c_x c_y c_{x-1} = \sum_{z \in I_x} h_{x,y;z}^\sigma c_z \quad \text{and} \quad C_x A_y = \sum_{z \in I_x} h_{x,y;z}^\sigma A_z.
\]

(1.1)

Note that \(c_x, c_y, c_z \in \mathcal{H}_q\) while \(C_x \in \mathcal{H}_q^2\) and \(A_y \in \mathcal{M}_q^2\). Now, define \(h_{x,y;z}^+\) and \(h_{x,y;z}^-\) for each \(x \in W\) and \(y, z \in I_x\).

Though not clear \textit{a priori}, these Laurent polynomials too always have integer coefficients \[13\] Proposition 2.11.

\textbf{Property C'}. Both \(h_{x,y;z}^+\) and \(h_{x,y;z}^-\) have nonnegative coefficients for all \(x \in W\) and \(y, z \in I_x\).

\textbf{Property D'}. For any \(x \in W\) and \(y, z \in I_x\), if the Laurent polynomials \(h_{x,y;z}^+\) and \(h_{x,y;z}^-\) have degrees \(d_+\) and \(d_-\) in \(v\), then \(v^{d_+} h_{x,y;z}^+\) and \(v^{d_-} h_{x,y;z}^-\) are unimodal polynomials in \(q = v^2\).

The main purpose of this paper is to provide evidence to support the following conjecture:

\textbf{Conjecture 1.7}. Properties A', B', C', and D' hold for all triples \((W, S, \ast)\) where \((W, S)\) is a Coxeter system and \(\ast \in \text{Aut}(W)\) is an \(S\)-preserving involution.

Lusztig and Vogan have shown that Property \(A'\) holds when \(W\) is a Weyl group or affine Weyl group; see [12 §3.2 and §7]. In these cases, [12 Section 5] also mentions without proof that Property \(C'\) holds (when \(\ast\) is trivial). The companion work [13] establishes Properties \(A'\) and \(B'\) in the case that \((W, S)\) is a universal Coxeter system. We did not consider Property \(D'\) in [13], but one can likely adapt the arguments in [13] to also prove this fourth property in the universal case.

### 1.5 Outline of main results

Section 2 reviews several formulas concerning the module \(\mathcal{M}_q^2\) from Lusztig’s paper [11]. Notably, Section 2.2 gives a recurrence for the polynomials \(P_{y,w}^\sigma\). In Section 2.1 we discuss how this recurrence leads to an algorithm capable of verifying our positivity properties in finite cases.

In Section 3 we show that the eight properties in Section 1.4 hold for all Coxeter systems if they hold for all irreducible Coxeter systems. (For the definition of irreducibility, see 7 Section 6.1.) Table 1 enumerates all triples \((W, S, \ast)\) where \((W, S)\) is an irreducible finite Coxeter system and \(\ast \in \text{Aut}(W)\) is an \(S\)-preserving involution. In more detail, let \(X\) be one of the letters A, B, C, or D, so that our positivity properties may each be referred to as either Property X or Property X'. We adopt the following convention: whenever we say that Property X' holds for a Coxeter system \((W, S)\), we mean that the Property X' holds for \((W, S)\) with respect to all choices of \(S\)-preserving involution \(\ast \in \text{Aut}(W)\). Propositions 3.8 and 3.9 together imply the following.

\textbf{Theorem 1.8}. Let \(X \in \{A, B, C, D\}\) and let \((W, S)\) be a Coxeter system. If Properties X and X' hold for all irreducible factors of \((W, S)\), then Properties X and X' hold for \((W, S)\).
In Section 4 we prove that Properties A′ and B′ hold for all Coxeter systems of rank two. Our further results are computational in nature. We have obtained these from extensions [13] we have written to the final version du Cloux’s C++ program Coxeter [4]. Our extended version of Coxeter allows a user to compute the polynomials \( P^\sigma_{y,w}, h^\sigma_{x,y,z}, h^\sigma_{x,y,z}, \) and \( h^\sigma_{x,y,z} \) for a given finite Coxeter system with involution.

Using the extended program, we have been able to check directly that Properties A′ and B′ hold for each of the triples \((W, S, \ast)\) listed in Table 2 and that Properties C′ and D′ hold for the triples listed in Table 3. Of the cases considered, type \( H_4 \) is by far the most computationally intensive, requiring for the calculation of the polynomials \( (h^\pm_{x,y,z})_{x \in W; y,z \in I} \) around 10 days’ computing time (on a 2.26 GHz MacBook Pro with 4 GB of main memory) and around 93 GB of memory to store all uncompressed output files. Even in this type, verifying Properties A′ and B′ only takes a few minutes, however. Combining this discussion with the results of [12] and with Theorem 1.8, we arrive at the following statement.

**Theorem 1.9.** Let \((W, S)\) be a Coxeter system with an \( S \)-preserving involution \( \ast \in \text{Aut}(W) \). Properties A′, B′, C′, D′ then hold in at least the following cases:

(a) Property A′ holds whenever \((W, S)\) is finite.

(b) Property B′ holds if all irreducible factors of \((W, S)\) are finite with rank at most 6.

(c) Properties C′, D′ hold if all irreducible factors of \((W, S)\) are finite with rank 1, 3, 4, or 5.

Our calculations actually show a little more than this statement indicates. Specifically, Property B′ also holds if the irreducible factors of \((W, S)\) include Coxeter systems of types \( A_7 \) or \( A_8 \). Properties C′ and D′ hold if the irreducible factors of \((W, S)\) include Coxeter systems of types \( A_6 \) or \( I_2(m) \) for \( m \leq 100 \). It is of course expected that Properties C′ and D′ hold for all Coxeter systems of rank two, and this can probably be shown by some technical but elementary calculations in the dihedral case. We do not attempt to carry these out in the present work, however.

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## 2 Computations for arbitrary Coxeter systems

Here we review some general information about the distinguished bases of \( \mathcal{H}_q \) and \( \mathcal{H}_{q^2} \) and \( \mathcal{M}_{q^2} \).

### 2.1 Lusztig’s recurrence for the twisted polynomials

Let \((W, S)\) denote a Coxeter system and \( \ast \in \text{Aut}(W) \) any \( S \)-preserving involution of \( W \). To describe how the standard basis \((T_w)_{w \in W}\) of \( \mathcal{H}_{q^2} \) acts on the distinguished basis \((A_w)_{w \in I_*}\) of \( \mathcal{M}_{q^2} \), we introduce the following notation.

**Notation.** Recall that \( q = v^2 \) and \( P^\sigma_{y,w} \in Z[q] \). Given \( y, w \in I_* \), let \( \mu^\sigma(y, w) \) and \( \nu^\sigma(y, w) \) respectively denote the coefficients of \( v^{\ell(w) - \ell(y) - 1} \) and \( v^{\ell(w) - \ell(y) - 2} \) in \( P^\sigma_{y,w} \). In turn, for each \( s \in S \)
define $\mu^\sigma(y, w; s)$ as the integer given by
\[
\mu^\sigma(y, w; s) \overset{\text{def}}{=} \nu^\sigma(y, w) + \delta_{y, y^\ast} \mu^\sigma(sy, w) - \delta_{sw, ws^\ast} \mu^\sigma(y, sw) - \sum_{x \in I; x < x} \mu^\sigma(y, x) \mu^\sigma(x, w).
\]

As usual, the Kronecker delta here means $\delta_{a,b} = 1$ if $a = b$ and $\delta_{a,b} = 0$ otherwise. Note that $\mu^\sigma(y, w)$ (respectively, $\nu^\sigma(y, w)$) is nonzero only if $y \leq w$ and $\ell(w) - \ell(y)$ is odd (respectively, even). Define $m^\sigma(y \rightarrow w) \in A$ for $y, w \in I_\ast$ and $s \in S$ as the Laurent polynomial
\[
m^\sigma(y \rightarrow w) = \begin{cases} 
\mu^\sigma(y, w)(v + v^{-1}) & \text{if } \ell(w) - \ell(y) \text{ is odd} \\
\mu^\sigma(y, w; s) & \text{if } \ell(w) - \ell(y) \text{ is even.}
\end{cases}
\]

Finally, let $\text{Des}_L(w) = \{ s \in S : \ell(sw) < \ell(w) \}$ and $\text{Des}_R(w) = \{ s \in S : \ell(ws) < \ell(w) \}$.

Lusztig proves the following as [11, Theorem 6.3].

**Theorem 2.1** (Lusztig [11]). Let $w \in I_\ast$ and $s \in S$. Then $C_s = q^{-1}(T_s + 1)$ and
\[
C_s A_w = \begin{cases}
(q + q^{-1}) A_w & \text{if } s \in \text{Des}_L(w) \\
(v + v^{-1}) A_w + \sum_{y \in I_\ast; s y < s w} m^\sigma(y \rightarrow w) A_y & \text{if } s \notin \text{Des}_L(w) \text{ and } sw = ws^\ast \\
A_{sws^\ast} + \sum_{y \in I_\ast; s y < s ws^\ast} m^\sigma(y \rightarrow w) A_y & \text{if } s \notin \text{Des}_L(w) \text{ and } sw \neq ws^\ast.
\end{cases}
\]

Comparing coefficients of $a_y$ on both sides of the preceding equation yields our next result, which also appears as [13, Corollary 2.7].

**Corollary 2.2.** Let $y, w \in I_\ast$ with $y \leq w$ and $s \in \text{Des}_L(w)$.

(a) $P^\sigma_{y, w} = P^\sigma_{s y, y, w}$.

(b) If $s \in \text{Des}_L(y)$ and we let $w' = s \prec w$ and $c = \delta_{sw, ws^\ast}$ and $d = \delta_{y, y^\ast}$, then
\[
(q + 1) c P^\sigma_{y, w} = (q + 1)^d P^\sigma_{s y, y, w'} + q(q - d)P^\sigma_{y, w'} - \sum_{x \in I; x \prec z, y \leq z < w} v^{\ell(w) - \ell(z) + c} \cdot m^\sigma(z \rightarrow w') \cdot P^\sigma_{y, z}.
\]

The preceding theorem and corollary give recursive formulas which can be used to compute the polynomials $P^\sigma_{y, w}$ and $h^\sigma_{x, y, z}$, though there is some subtlety in how to go about this. The following algorithms carrying out such calculations are implemented in our extensions [14] to du Cloux’s program Coxeter [4].

**Algorithm for computing the polynomials $P^\sigma_{y, w}$.** If $y \neq w$ then $P^\sigma_{y, w} = \delta_{y, w}$. If $\ell(w) > 1$ then it is usually possible to compute $P^\sigma_{y, w}$ inductively by applying Corollary 2.2 in a straightforward way, since this result usually gives a formula for $P^\sigma_{y, w}$ in terms of polynomials $P^\sigma_{y', w'}$ with either $y' \leq w' < w$ or $y < y' < w' = w$, which one may assume to be already known. However, terms on the right hand side of the recurrence (2.2) can sometimes depend on terms on the left. There is actually only one such term: the summand indexed by $z = y$ when $sw = ws^\ast$ and $\ell(w) - \ell(y)$ is odd. In this case, Corollary 2.2(b) assumes the form
\[
(q + 1) P^\sigma_{y, w} = f + v^{\ell(w) - \ell(y) + 1} \mu^\sigma(y, w) + \text{other terms}.
\]
where \( f \in \mathbb{Z}[q] \) is the polynomial
\[
f = (q + 1)^d P_{s:y,w}^\sigma + q(q - d)P_{y,w}^\sigma - \sum_{x \in \mathbb{N}_+: z < z} v^{\ell(w) - \ell(z) + c} \cdot m^\sigma(z \to w') \cdot P_{y,w}^\sigma
\]
where our notation is as in Corollary 2.2. The definition of \( f \) depends on quantities which one can assume to be already known, and once \( f \) is computed, it is straightforward to extract \( P_{y,w}^\sigma \) from the equation (2.3).

Algorithm for computing the structure constants \( h_{x,y,z}^\sigma \). By definition \( h_{1,y,z}^\sigma = \delta_{y,z} \), and if \( s \in S \) then \( h_{s,y,z}^\sigma \) is determined by Theorem 2.1. When \( x \in W \) has length greater than one so that there exists \( s \in S \) with \( sx < s \), then (2.4) affords the recurrence
\[
h_{x,y,z}^\sigma := \sum_{z' \in \mathbb{N}_+} h_{s:z',z}^\sigma h_{s,x,z'}^\sigma - \sum_{z' \in \mathbb{N}_+: sx < x} \mu(x', sx) h_{x',y,z}^\sigma
\]
which expresses \( h_{x,y,z}^\sigma \) in terms of quantities which may be assumed to have been already computed. Recall here that \( \mu(x', sx) \) denotes the coefficient of \( v^{\ell(x') - \ell(sx) - 2} \) in \( P_{x',sx} \).

Du Cloux’s papers [2, 3] describe efficient algorithms (implemented in Coxeter [4]) for computing the Kazhdan-Lusztig polynomials \( P_{y,w} \) and the structure constants \( h_{x,y,z} \). Once the arrays \( (P_{y,w})_{y,w \in \mathbb{N}_+} \) and \( (P_{y,w}^\sigma)_{y,w \in \mathbb{N}_+} \) have been computed for a finite Coxeter system, it is straightforward to check Properties [A] and [B]. To similarly check Properties [C] and [D], one must first compute the arrays \( (h_{x,y,z})_{x,y,z \in \mathbb{N}_+} \) and \( (h_{x,y,z}^\sigma)_{x,y,z \in \mathbb{N}_+} \), and then calculate the Laurent polynomials \( h_{x,y,z} \) via the identity
\[
\tilde{h}_{x,y,z} = \sum_{z' \in \mathbb{N}_+} h_{x,y,z} h_{z',(x^*)^{-1},z} \quad \text{for } x \in W \text{ and } y, z \in \mathbb{N}_+.
\]
Implementing this formula presents its own challenges in cases when the array \( (h_{x,y,z})_{x,y,z \in \mathbb{N}_+} \) is very large; however, given \( \tilde{h}_{x,y,z} \) and \( h_{x,y,z}^\sigma \) it is again straightforward to check our remaining positivity properties for a particular finite Coxeter system with involution. All of these checks are implemented in [4].

2.2 A special case of the twisted involution module

If \( (W', S') \) and \( (W'', S'') \) are Coxeter systems, then we define the direct product
\[
(W, S) = (W', S') \times (W'', S'')
\]
to be the Coxeter system with \( W = W' \times W'' \) and \( S = \{(s, 1) : s \in S'\} \cup \{(1, s) : s \in S''\} \). The following proposition shows that the Kazhdan-Lusztig polynomials \( P_{y,w} \) can occur as instances of the twisted polynomials \( P_{y,w}^\sigma \) for certain choices of \( (W, S) \) and \( \ast \).

Proposition 2.3. Suppose that \( (W, S) = (W', S') \times (W', S') \) for some Coxeter system \( (W', S') \), and that \( \ast \in \text{Aut}(W) \) acts by \( (x, y)^\ast = (y, x) \) for \( y, w \in W' \). Let \( \mathcal{H}_{q^2} \) denote the Hecke algebra of \( (W', S') \) with parameter \( q^2 \), and define
\[
\iota : \mathcal{H}_{q^2} \to \mathcal{H}_{q^2} \quad \text{and} \quad \iota^\ast : \mathcal{M}_{q^2} \to \mathcal{M}_{q^2}
\]
as the unique \( \mathcal{A} \)-linear maps with \( \iota(T_w) = T_{(w, 1)} \) and \( \iota^\ast(T_w) = a_{(w, w^{-1})} \) for \( w \in W' \).
(a) The map \( \iota \) is injective and the map \( \iota^\sigma \) is bijective, and for all \( T, T' \in \mathcal{H}_q \) we have

\[
\iota(TT') = \iota(T)\iota(T') \quad \text{and} \quad \iota^\sigma(TT') = \iota(T)\iota^\sigma(T') \quad \text{and} \quad \iota^\sigma(T) = \iota^\sigma(T).
\]

(b) For all \( y, w \in W' \), we have \( P_{\sigma(y,y^{-1}), (w,w^{-1})}^\sigma = P_{y,w}(q^2) \).

**Proof.** The map \( w \mapsto (w, w^{-1}) \) clearly defines a poset isomorphism \( (W', \leq) \xrightarrow{\sim} (I_s, \leq) \), as is noted in [6, Example 3.2] and also [15, Example 10.1]. It follows that \( \iota^\sigma \) is a bijection. On the other hand, it is easy to check that the map \( \iota \) is an injective \( A \)-algebra homomorphism. Now, since \( s \varphi w = s w s^* \) for all \( s \in S \) and \( w \in W \), it follows from Theorem-Definition 1.3 that \( \iota^\sigma(T_sT_w) = \iota(T_s)\iota^\sigma(T_w) = T_{s(w,w^{-1})}a_{w,w^{-1}} \) for all \( s \in S' \) and \( w \in W' \). Since \( \iota \) is a homomorphism and \( \iota^\sigma \) is \( A \)-linear, we conclude that \( \iota^\sigma(TT') = \iota(T)\iota^\sigma(T') \) for all \( T, T' \in \mathcal{H}_q' \). One can similarly check that \( \iota^\sigma(T_w) = \iota^\sigma(T_w) \) for \( w \in W' \), which establishes the last assertion in part (a) by \( A \)-linearity.

To prove part (b), note that \( \ell((w, w^{-1})) = 2\ell(w) \) for \( w \in W' \), where on the left \( \ell \) is interpreted as the length function of \( (W, S) \) and on the right as the length function of \( (W', S') \). In light of this and Theorem-Definition 1.3, it follows from the uniqueness specified in Theorem-Definition 1.3 that \( P_{\sigma(y,y^{-1}), (w,w^{-1})}^\sigma = P_{y,w}(q^2) \) for \( y, w \in W' \).

When \((W, S, \ast)\) is as in Proposition 2.3, one can express the polynomials \( P_{\pm}(w, y, z) \) and \( h_{\pm}(w, y, z) \) entirely in terms of polynomials attached to \((W', S')\). To prove these formulas, we require two lemmas. Our first lemma applies to an arbitrary Coxeter system \((W, S)\) with an \( S \)-preserving involution \( \ast \in \text{Aut}(W) \).

**Lemma 2.4.** If \( x \in W \) and \( y, z \in I_s \) then \( h_{x,y,z}^\sigma = h_{x^*,y^*,z^*}^\sigma \).

**Proof.** Let \( \varphi : \mathcal{H}_q \rightarrow \mathcal{H}_q \) and \( \varphi^\sigma : M_q \rightarrow M_q \) be the unique \( A \)-linear maps with \( \varphi(T_w) = T_w^* \) and \( \varphi^\sigma(a_w) = a_w^* \). It is clear that \( \varphi(T_s)\varphi^\sigma(a_w) = \varphi^\sigma(T_s a_w) \) for all \( s \in S \) and \( w \in I_s \). Because \( \ast \) is an automorphism of \( W \) preserving \( S \), the map \( \varphi \) is an automorphism of the \( A \)-algebra \( \mathcal{H}_q \), and from this and \( A \)-linearity it follows that \( \varphi(T)\varphi^\sigma(a) = \varphi^\sigma(T a) \) for all \( T \in \mathcal{H}_q \) and \( a \in M_q \). It is a straightforward exercise to show that

\[
\varphi^\sigma(A_y) = A_y^* \quad \text{for all} \quad y \in I_s \quad \text{and} \quad \varphi(C_x) = C_x^* \quad \text{for all} \quad x \in W
\]

and consequently \( \varphi^\sigma \left( \sum_{z \in W} h_{x,y,z}^\sigma A_z \right) = \varphi^\sigma(C_x A_y) = \varphi(C_x)\varphi^\sigma(A_y) = C_x^* A_y^* \). The left hand side of the preceding equation is equal to \( \sum_{z \in W} h_{x^*,y^*,z^*}^\sigma A_z^* \) while the right hand side is equal to \( \sum_{z \in W} h_{x^*,y^*,z^*}^\sigma A_z^* \), so since \( (A_z)_{z \in I_s} \) is a basis for \( M_q \), we conclude that \( h_{x,y,z}^\sigma = h_{x^*,y^*,z^*}^\sigma \).

Our second lemma applies only to the special situation of Proposition 2.3.

**Lemma 2.5.** Suppose that \((W, S), (W', S')\), and \( \ast \) are defined as in Proposition 2.3.

(a) For all \( y, w, w' \in W' \), we have \( P_{(y,y'), (w,w')} = P_{y,w} P_{y',w'} \), and it holds that

\[
c(y,w) = c(y,1)c(1,w) = c(1,w)c(y,1) \quad \text{and} \quad C(y,w) = C(y,1)C(1,w) = C(1,w)C(y,1)
\]

in the respective Hecke algebras \( \mathcal{H}_q \) and \( \mathcal{H}_q' \) attached to \((W, S)\).

(b) For all \( x, y, z \in W' \), we have \( h_{(x,1), (y,1)(z,1)} = h_{(1,y^{-1})(1,x^{-1})(1,z^{-1})} = h_{x,y,z} \).
(c) For all \( x, y, z \in W' \), we have \( h^\sigma_{(x,1), (y,y^{-1}), (z,z^{-1})} = h^\sigma_{(1,y^{-1}), (x,x^{-1}), (z,z^{-1})} = h_{x,y,z}(v^2) \).

**Proof.** Parts (a) and (b) follow as a straightforward exercise from Theorem-Definition 1.1. To prove part (c), recall the definitions of the maps \( \iota \) and \( \iota^\sigma \) above, and note from part (a) and Proposition 2.3(c) that for \( w \in W' \) we have \( \iota(C_w) = C_{(w,1)} \) and \( \iota^\sigma(C_w) = A_{(w,w^{-1})} \). Therefore for \( x, y, z \in W' \) it holds that

\[
\sum_{z \in W'} h^\sigma_{(x,1), (y,y^{-1}), (z,z^{-1})} A_{(z,z^{-1})} = C_{(x,1)} A_{(y,y^{-1})} = \iota^\sigma(C_x C_y) = \sum_{z \in W'} h_{x,y,z}(v^2) A_{(z,z^{-1})}.
\]

Note that on the right \( h_{x,y,z} \) is evaluated at \( v^2 \) rather than at \( v \) because we are computing the product \( C_x C_y \) in a Hecke algebra with parameter \( q^2 \) rather than \( q \). Thus \( h^\sigma_{(x,1), (y,y^{-1}), (z,z^{-1})} = h_{x,y,z}(v^2) \). Combining this fact with the well-known identity \( h_{x,y,z} = h_{y^{-1}, x^{-1}, z^{-1}} \) (see [3, Section 2.1]) and Lemma 2.4 gives the second equality in part (c). 

We may now state the main result of this section.

**Proposition 2.6.** Suppose that \((W,S), (W',S')\), and \(*\) are defined as in Proposition 2.3

(a) The set of polynomials \( \{ P^\pm_{y,w} : y, w \in I_* \} \) is equal to

\[
\left\{ \frac{1}{2} \left( P_{y,w}(q^2) \pm P_{y,w}(q^2) \right) : y, w \in W' \right\}.
\]

(b) The set of polynomials \( \{ P^\pm_{y,w} - P^\pm_{z,w} : y, z, w \in I_* \}, y \leq z \} \) is equal to

\[
\left\{ \frac{1}{2} \left( P_{y,w}(q^2) - P_{z,w}(q^2) \right) \pm \frac{1}{2} \left( P_{y,w}(q^2) - P_{z,w}(q^2) \right) : y, z, w \in W', y \leq z \right\}.
\]

(c) The set of polynomials \( \{ h^\pm_{x,y,z} : x \in W, y, z \in I_* \} \) is equal to

\[
\left\{ \frac{1}{2} \left( f_{w,x,y,z}(v^2) \pm f_{w,x,y,z}(v^2) \right) : w, x, y, z \in W' \right\}
\]

where we define \( f_{w,x,y,z} = \sum_{g \in W'} h_{w,x,g} h_{y,g,z} \in A \) for \( w, x, y, z \in W' \).

**Remark.** Observe that the polynomials \( f_{w,x,y,z} \) defined in part (c) of this result are the structure constants satisfying \( c_{w,c_x,c_y} = \sum_{z \in W'} f_{w,x,y,z} \) for \( w, x, y \in W' \).

**Proof.** Parts (a) and (b) follow by comparing the definition of the polynomials \( P^\pm_{y,w} \) for \( y, w \in I_* \) with Proposition 2.3(c) and Lemma 2.5(a), while noting the well-known identity \( P_{y,w} = P_{y^{-1}, w^{-1}} \) \([\text{?}]\). To prove part (c) it suffices to show that for \( w, x, y, z \in W' \) we have

\[
\tilde{h}_{(w,y^{-1}), (x,x^{-1}), (z,z^{-1})} = f_{w,x,y,z}(v^2) \quad \text{and} \quad \tilde{h}^\sigma_{(w,y^{-1}), (x,x^{-1}), (z,z^{-1})} = f_{w,x,y,z}(v^2).
\]

To check the left identity, we compute from Lemma 2.5(a) that

\[
c_{(w,y^{-1})} c_{(x,x^{-1})} c_{(y,w)} = \left( c_{(w,1)} c_{(x,1)} c_{(y,1)} \right) \cdot \left( c_{(1,y^{-1})} c_{(1,x^{-1})} c_{(1,w^{-1})} \right).
\]
Applying Lemma 2.5(b) to the products \((c_{(w,1)}c_{(x,1)}) \cdot c_{(y,1)}\) and \(c_{(1,y^{-1})} \cdot (c_{(1,x^{-1})}c_{(1,w^{-1})})\), noting our parenthesizations, shows that

\[
c_{(w,1)}c_{(x,1)}c_{(y,1)} = \sum_{z \in W'} f_{w,x,y;z}c_{(z,1)} \quad \text{and} \quad c_{(1,y^{-1})}c_{(1,x^{-1})}c_{(1,w^{-1})} = \sum_{z \in W'} f_{w,x,y;z}c_{(1,z^{-1})}.
\]

By Lemma 2.5(a) we thus have \(c_{(w,y^{-1})}c_{(x,x^{-1})}c_{(1,1)} = \sum_{z,z' \in W'} f_{w,x,y;z}f_{w,x,y;z'} \cdot c_{(z',z^{-1})}\), from which the first identity in (2.6) follows. To check the second equality in (2.6), we note from Lemma 2.5(a) that \(C_{(w,y^{-1})}A_{(x,x^{-1})} = C_{(1,y^{-1})}C_{(w,1)}A_{(x,x^{-1})}\), which implies by Lemma 2.5(c) that

\[
h_{(w,y^{-1}), (x,x^{-1}),(z,z^{-1})} = \sum_{g \in W'} h_{(w,1),(x,x^{-1}),(g,g^{-1})}h_{(1,y^{-1}),(g,g^{-1}),(z,z^{-1})} = f_{x,w,y;z}(v^2)
\]

as desired. Therefore both identities in (2.6) hold so part (c) holds.

\[\square\]

3 Reduction to the irreducible case

We devote this section to the proof of Theorem 1.8 from the introduction. Our proof depends on a few preliminary facts, which occupy the next three subsections.

3.1 Facts about unimodal polynomials

Recall the definition of unimodality from Section 1.4. In particular, note that if a polynomial \(f \in \mathbb{Z}[x]\) is unimodal then automatically \(f \in \mathbb{N}[x]\). Let \(f \in \mathcal{A}\) be a Laurent polynomial with degree \(d\) in \(v\). We say that \(f\) is balanced if \(v^df\) is polynomial in \(q = v^2\) such that

\[v^df = a_0 + a_1q + a_2q^2 + \cdots + a_dq^d\]

for some integers \(a_i \in \mathbb{Z}\) with \(a_i = a_{d-i}\) for all \(0 \leq i \leq d\). We say that \(f\) is balanced unimodal if \(f\) is balanced and additionally \(v^df\) is a unimodal polynomial in \(q\). Note that 0 is balanced unimodal since we consider the zero polynomial to have degree 0.

**Lemma 3.1.** Suppose \(f, g \in \mathcal{A}\) are nonzero and balanced unimodal. Then the product \(fg\) is balanced unimodal, while the sum \(f + g\) is balanced unimodal if and only if the degrees of \(f\) and \(g\) as polynomials in \(v\) are either both even or both odd.

**Proof.** Suppose \(f\) and \(g\) have degrees \(d\) and \(d'\) as Laurent polynomials in \(v\). Then \(v^df\) and \(v^dg\) are “symmetric unimodal” in the sense of [16] and it follows that \(fg\) is balanced unimodal by [16, Observation 2]. The remainder of the lemma follows by inspection. \(\square\)

We also require the following technical lemma.

**Lemma 3.2.** Let \(f \in \mathcal{A}\) and define \(f^\pm = \frac{1}{2}(f(v)^2 \pm f(v^2))\).

(a) Both \(f^+\) and \(f^-\) belong to \(\mathcal{A}\).

(b) If \(f\) has nonnegative coefficients then \(f^+\) and \(f^-\) have nonnegative coefficients.

(c) If \(f\) is balanced unimodal then \(f^+\) and \(f^-\) are balanced unimodal.
Proof. Parts (a) and (b) follow by computing the coefficients of \( f^\pm \) in terms of those of \( f \). To prove part (c), suppose \( f \in A \setminus \{0\} \) is balanced unimodal. We may assume \( f \) is nonzero with degree \( d \) as a polynomial in \( v \). We need only show that \( 2f^+ \) and \( 2f^- \) are balanced unimodal. To this end, note that \( f\) is a linear combination with nonnegative integer coefficients of polynomials of the form 
\[
h_i \overset{\text{def}}{=} v^{-d}(q^i + \cdots + q^{d-i})
\]
for integers \( 0 \leq i \leq d/2 \). In particular, we may write 
\[
f = \sum_{0 \leq i \leq d/2} a_i h_i \quad \text{for some nonnegative integers } a_i \in \mathbb{N} \text{ with } a_0 \neq 0,
\]
and we then have 
\[
2f^\pm = \sum_{0 \leq i \leq d/2} a_i (h_i^2 \pm h_i(v^2)) + \sum_{0 \leq i \leq d/2} (a_i^2 - a_i)h_i^2 + \sum_{0 < i < j \leq d/2} (2a_i a_j)h_i h_j.
\] (3.1)

To show that \(2f^\pm\) is balanced unimodal, it is enough by Lemma 3.1 to check that the terms 
\(h_i^2 \pm h_i(v^2)\) and \(h_i h_j\) occurring in the three sums in (3.1) are balanced unimodal polynomials 
whose degrees in \( v \) are all even. This is a simple exercise, which we leave to the reader. \( \square \)

3.2 Facts about the structure constants

In both propositions in this section, we let \( u = v + v^{-1} \) and we let \((W, S)\) denote an arbitrary 
Coxeter system with an \( S \)-preserving involution \( * \).

Proposition 3.3. Suppose \( x, y, z \in W \).

(a) If \( \ell(x) + \ell(y) + \ell(z) \) is odd then \( h_{x,y,z} \in u\mathbb{Z}[u^2] \).

(b) If \( \ell(x) + \ell(y) + \ell(z) \) is even then \( h_{x,y,z} \in \mathbb{Z}[u^2] \).

Proof. Since \( h_{1,y,z} = \delta_{y,z} \), the proposition holds when \( \ell(x) = 0 \), and since \( \mu(z, y) \) is nonzero only if \( \ell(y) = \ell(z) \) is odd, Theorem 1.2 shows that the proposition also holds when \( \ell(x) = 1 \). Assume \( \ell(x) \geq 2 \) and that the proposition holds if we replace \( x \) by any element of shorter length. Choose \( s \in \text{Des}_L(x) \). By Theorem 1.2 we have 
\[
c_x = c_x c_{sx} - \sum_{x' \in W; sx' < sx} \mu(x', sx)c_{x'},
\]
and so
\[
h_{x,y,z} = \sum_{x' \in W} h_{sx,y,z} h_{s,x';z} - \sum_{x' \in W; sx' < sx} \mu(x', sx)h_{x',y;z}.
\] (3.2)

Since \( \mu(x', sx) \) is nonzero only if \( \ell(x) - \ell(x') \) is even, it follows by our inductive hypothesis that 
\[
\sum_{x' \in W; sx' < sx} \mu(x', sx)h_{x,y,z} \text{ belongs to } u\mathbb{Z}[u^2] \text{ or } \mathbb{Z}[u^2]
\]
if \( \ell(x) + \ell(y) + \ell(z) \) is odd or even respectively. On the other hand, for all \( z' \in W \) the parities of 
\[
\ell(sx) + \ell(y) + \ell(z') \quad \text{and} \quad \ell(s) + \ell(z') + \ell(z)
\]
are distinct or equal according to whether \( \ell(x) + \ell(y) + \ell(z) \) is odd or even respectively. Therefore it 
follows likewise by hypothesis that 
\[
\sum_{x' \in W} h_{sx,y,z} h_{s,x';z} \text{ belongs to } u\mathbb{Z}[u^2] \text{ or } \mathbb{Z}[u^2]
\]
if \( \ell(x) + \ell(y) + \ell(z) \) is odd or even respectively. The proposition thus holds for all \( x \) by (3.2) and induction. \( \square \)

Proposition 3.4. Suppose \( x \in W \) and \( y, z \in I_* \).

(a) If \( \ell(y) + \ell(z) \) is even then \( \tilde{h}_{x,y;z} \) and \( h^\sigma_{x,y;z} \) and \( h^\pm_{x,y;z} \) all belong to \( \mathbb{Z}[u^2] \).

(b) If \( \ell(y) + \ell(z) \) is odd then \( \tilde{h}_{x,y;z} \) and \( h^\sigma_{x,y;z} \) and \( h^\pm_{x,y;z} \) all belong to \( u\mathbb{Z}[u^2] \).
Proof. Since \( \ell((x^*)^{-1}) = \ell(x) \), the parities of \( \ell(x) + \ell(y) + \ell(z) \) and \( \ell(z) + \ell((x^*)^{-1}) + \ell(z) \) are either always equal or always distinct for \( z' \in W \), according to whether \( \ell(y) + \ell(z) \) is even or odd respectively. Since \( \tilde{h}_{x,y,z} = \sum_{z' \in W} h_{x,y,z} h_{z',(x^*)^{-1};z} \), it follows from Proposition 3.3 that \( \tilde{h}_{x,y,z} \) belongs to \( \mathbb{Z}[u^2] \) if \( \ell(y) + \ell(z) \) is even and to \( u\mathbb{Z}[u^2] \) otherwise.

We next establish the claim that \( h_{x,y,z}^r \) belongs to \( \mathbb{Z}[u^2] \) or \( u\mathbb{Z}[u^2] \) according to whether \( \ell(y) + \ell(z) \) is even or odd. The proof of this fact is similar to that of Proposition 3.3. Since \( h_{x,y,z}^r = \delta_{y,z} \) our claim holds if \( \ell(x) = 0 \). Since \( m^q(z \xrightarrow{a} y) \) belongs to \( \mathbb{Z}[u^2] \) if \( \ell(y) + \ell(z) \) is even and to \( u\mathbb{Z}[u^2] \) otherwise (see (2.4)), Theorem 2.4 shows that our claim also holds when \( \ell(x) \leq 1 \). Finally, when \( \ell(x) \geq 2 \) and \( s \in \text{Des}_L(x) \), our claim follows by induction using (2.4) exactly as in the proof of Proposition 3.3.

Combining the preceding paragraphs demonstrates that the polynomials \( h_{x,y,z}^r \), which automatically belong to \( \mathcal{A} = \mathbb{Z}[v,v^{-1}] \) by [13] Proposition 2.11, also belong to \( \mathbb{Q}[u^2] \) or \( u\mathbb{Q}[u^2] \) according to whether \( \ell(y) + \ell(z) \) is even or odd. It is straightforward to check that \( \mathcal{A} \cap \mathbb{Q}[u^2] \subset \mathbb{Z}[u^2] \) and \( \mathcal{A} \cap u\mathbb{Q}[u^2] \subset u\mathbb{Z}[u^2] \), which establishes the proposition in full.

All elements of \( \mathbb{Z}[u^2] \) have the form \( a_0 + a_2(v^2 + v^{-2}) + \cdots + a_d(v^d + v^{-d}) \) while all elements of \( u\mathbb{Z}[u^2] \) have the form \( a_1(v + v^{-1}) + a_3(v^3 + v^{-3}) + \cdots + a_d(v^d + v^{-d}) \) for some integers \( a_i \in \mathbb{Z} \). From this observation and the preceding propositions derives the following corollary.

Corollary 3.5. The Laurent polynomials \( h_{x,y,z}, \tilde{h}_{x,y,z}, h_{x,y,z}^r, h_{x,y,z}^\pm \in \mathcal{A} \) are always balanced.

3.3 Reductions

Propositions 3.8 and 3.9 in this section together imply Theorem 1.8 in the introduction. Before proceeding to these results we require two additional lemmas.

Lemma 3.6. Suppose that \((W, S), (W', S')\), and * are defined as in Proposition 2.6 and let \( L \) be one of the letters A, B, C, or D. If Property X holds for \((W', S')\), then Property X' holds for the triple \((W, S, *)\).

Proof. We know that Property A holds for \((W', S')\), and it follows that Property A' holds for \((W, S, *)\) from Proposition 2.6(a) and Lemma 3.2(b). Suppose Property B holds for \((W, S, *)\). Fix \( y, z, w \in W' \) with \( y \leq z \) and let \( f = P_{y,w} \) and \( g = P_{z,w} \). Then \( f - g \in \mathbb{N}[q] \) and also \( f, g \in \mathbb{N}[q] \), since Property B implies Property A and so

\[
(f(q)^2 - g(q)^2) + (f(q^2) - g(q^2)) = (f - g)^2 \pm (f(q^2) - g(q^2)) + 2(f - g) \in \mathbb{N}[q].
\]

Property B' therefore holds for \((W, S, *)\) by Proposition 2.6(b).

For the remainder of the proof, the constant terms \( w, x, y, z \in W' \) and write \( f = f_{w, x, y, z} \) as in Proposition 2.6(c). Then Properties C' and D' are respectively equivalent to the assertions that the polynomials \( f^\pm \) always have nonnegative coefficients and always are balanced unimodal. Since Property C' always holds for \((W', S')\) we have \( f \in \mathbb{N}[v, v^{-1}] \) so \( f^\pm \in \mathbb{N}[v, v^{-1}] \) by Lemma 3.2(b).

Suppose Property D holds for \((W', S')\). The structure constants \( h_{x, y, z} \) are then always balanced unimodal, and so by Lemma 3.1(a) the product \( h_{w, x, y} h_{g, y, z} \) for each \( g \in W' \) is likewise balanced unimodal. Let \( u = v + v^{-1} \). For all \( w \in W' \), it holds by Proposition 3.3 that \( h_{w, x, y} h_{g, y, z} \) belongs...
to \(u\mathbb{Z}[u^2]\) or \(\mathbb{Z}[u^2]\) according to whether \(\ell(w) + \ell(x)\) and \(\ell(y) + \ell(z)\) have distinct or equal parities. Thus the degrees of the products \(h_{w, x, g}^* h_{y, g, z}^*\) for \(g \in W'\) all have the same parity, so \(f\), being equal to sum of such products, is balanced unimodal by Lemma 3.2(b). By Lemma 3.2(c) it follows that the polynomials \(f^\pm\) are therefore balanced unimodal, so Property D holds for \((W, S, \ast)\).

In the next statement and for the duration of this section, we fix an arbitrary Coxeter system \((W, S)\) with an \(S\)-preserving involution \(* \in \text{Aut}(W)\), and we let \(S' \subset S\) and \(S'' = S \setminus S'\) be (possibly empty) sets of simple generators such that

(i) \(S'\) and \(S''\) are each preserved by \(*\);

(ii) Every \(s' \in S'\) commutes with every \(s'' \in S''\).

We write \(W' = \langle S' \rangle\) and \(W'' = \langle S'' \rangle\) for the subgroups generated by \(S'\) and \(S''\), and let \(I_0 = W' \cap I_*\) and \(I_0'' = W'' \cap I_*\).

**Lemma 3.7.** For each \(w \in W\) there are unique elements in \(W'\) and \(W''\), which we denote \(w'\) and \(w''\) respectively, such that \(w = w'w''\). This decomposition has the following properties:

(a) If \(w \in W\) then \(w \in I_*\) if and only if \(w' \in I_*\) and \(w'' \in I_0''\).

(b) For all \(w, x, y, z \in W\) we have \(P_{y, w}^* = P_{y, w'}^{*} P_{y''}^{*} w' w''\) and \(h_{x, y, z} = h_{x', y', z'} h_{x'', y'', z''}\).

(c) For all \(x \in W\) and \(y, y, z \in I_*\) we have

\[
P_{y, w}^* = P_{y', w'}^{*} P_{y''}^{*} w' w''\quad\text{and}\quad h_{x, y, z}^* = h_{x', y', z'}^* h_{x'', y'', z''}.
\]

**Remark.** In part (b), we identify \(P_{y, w'}^{*}\) and \(P_{y''}^{*}\) with Kazhdan-Lusztig polynomials of the Coxeter systems \((W', S')\) and \((W'', S'')\). Similar identifications apply to the structure constants \(h_{x', y', z'}^*\) and \(h_{x'', y'', z''}\). In part (c), likewise, we identify \(P_{y', w'}^{*}\), \(h_{x', y', z'}^*\) and \(P_{y''}^{*}\), \(h_{x'', y'', z''}\) with polynomials attached to the triples \((W', S', \ast)\) and \((W'', S'', \ast)\). Note that this makes sense since \(*\) restricts to an involution of \(W'\) and of \(W''\) which preserves \(S'\) and \(S''\).

**Proof.** The first assertion and part (a) follow from basic group theory and properties of the Bruhat order (see [11 Exercise 2.3]). Since in the Hecke algebras \(\mathcal{H}_q\) and \(\mathcal{H}_q^*\) we have \(t_w = t_{w'} t_{w''}\) and \(T_w = T_{w'} T_{w''}\) for all \(w \in W\), parts (b) and (c) follow as consequences of the uniqueness specified in Theorem-Definitions [11 and 15].

The following result is presumably well-known to experts, but we could not locate a reference in the literature.

**Proposition 3.8.** Suppose Property \(\mathbf{B}\) (respectively, \(\mathbf{D}\)) holds for all irreducible factors of a Coxeter system \((W, S)\). Then Property \(\mathbf{B}\) (respectively, \(\mathbf{D}\)) holds for \((W, S)\).

Of course, the corresponding statement for Properties \(\mathbf{A}\) and \(\mathbf{C}\) holds vacuously since these properties hold for all Coxeter systems by [5].

**Proof.** Let \(X\) stand for one of the letters B or D, and assume Property X holds for all irreducible factors of \((W, S)\). We may assume without loss of generality that the rank of \((W, S)\) is finite, since any finite set of elements of \(W\) belong to a Coxeter subgroup of \(W\) generated by a finite subset of \(S\), and so we can view the polynomials \(P_{y, w}^*\) and \(h_{x, y, z}\) as attached to a finite rank Coxeter system.
We now proceed by induction on the finite rank of \((W, S)\). If \((W, S)\) is irreducible then the proposition holds automatically. If \((W, S)\) is not irreducible, then \(S'\) and \(S''\) can both be chosen (taking \(*\) to be trivial) to be proper subsets of \(S\). In this case we may assume by induction that Property \(X\) holds for the Coxeter systems \((W', S')\) and \((W'', S'')\), since these both have rank strictly less than that of \((W, S)\). If \(X = D\) then it follows by Proposition 3.8 that Property \(X\) holds for \((W, S)\). If \(X = B\) then Property \(X\) holds for \((W, S)\) since in the notation of Lemma 3.7 we have

\[
P_{y,w} - P_{z,w} = \frac{1}{2} (P_{y',w' - P_{z',w'}})(P_{y'',w''} - P_{z'',w''}) + \frac{1}{2} (P_{y',w'} - P_{z',w'})(P_{y'',w''} + P_{z'',w''})
\]

for all \(y, z, w \in W\) with \(y \leq z\), and by induction all parenthesized terms on the right hand side of this identity belong to \(\mathbb{N}[y]\).

In our second proposition, recall that when we say that “Property \(X'\) holds for \((W, S)\)” we mean that the property in question holds with respect to the Coxeter system \((W, S)\) for all choices of \(S\)-preserving involution \(* \in \text{Aut}(W)\).

**Proposition 3.9.** Let \(X\) stand for one of the letters \(A, B, C,\) or \(D\), and let \((W, S)\) be a Coxeter system with an \(S\)-preserving involution \(* \in \text{Aut}(W)\). If Properties \(X\) and \(X'\) hold for all irreducible factors of \((W, S)\), then Property \(X'\) holds for the triple \((W, S, *)\).

**Proof.** As in the proof of Proposition 3.8, we proceed by induction on the rank of \((W, S)\), which we may assume to be finite, supposing Properties \(X\) and \(X'\) hold with respect to any choice of involution for all irreducible factors of our Coxeter system.

If \(S'\) and \(S''\) cannot both be chosen to be proper subsets of \(S\), then either \((W, S)\) is irreducible, or there are disjoint subsets \(J', J'' \subset S\) with \(S = J' \cup J''\) such that \(\{s^* : s \in J'\} = J''\) and such that the Coxeter systems \((W_{J'}, J')\) and \((W_{J''}, J'')\) are both irreducible, where \(W_{J'} = \langle J' \rangle\) and \(W_{J''} = \langle J'' \rangle\). In the first case Property \(X'\) holds for the triple \((W, S, *)\) by hypothesis. In the second case, \(W_{J'} \cong W_{J''}\) and we may identify the triple \((W, S, *)\) with a Coxeter system with involution of the form in Proposition 2.3. In this situation, it follows by Proposition 3.8 that Property \(X\) holds for the Coxeter system \((W_{J'}, J')\), and so it follows in turn by Lemma 3.6 that Property \(X'\) holds for \((W, S, *)\).

On the other hand suppose \(S'\) and \(S''\) can both be chosen to be proper subsets of \(S\). Let \(x \in W \) and \(y, z, w \in \mathbb{I}_x\) and observe that in the notation of Lemma 3.7 the following identities hold:

- \(P_{y,w} = P_{y',w'} + P_{y'',w''} + P_{y',w'} P_{y'',w''}\)
- \(P_{y,w} - P_{z,w} = \frac{1}{2} (P_{y',w'} + P_{z',w'}) (P_{y'',w''} - P_{z'',w''}) + \frac{1}{2} (P_{y',w'} - P_{z',w'}) (P_{y'',w''} + P_{z'',w''})\)
- \(h_{x,y;z} = h_{x',y';z} h_{x'',y'';z''} + h_{x',y';z} h_{x'',y'';z''}\)

As we may assume by induction Property \(X'\) holds for \((W', S')\) and \((W'', S'')\), these identities (together Lemma 3.1 and with Corollary 3.5) imply that Property \(X'\) holds for \((W, S, *)\).

**4 Computations for finite dihedral Coxeter systems**

Fix a positive integer \(m \in \{3, 4, 5, \ldots\} \) and suppose \((W, S)\) is the finite Coxeter system of type \(I_2(m)\). (We require \(m \geq 3\) so that \((W, S)\) is irreducible.) We take \(S = \{s, t\}\) to be a set with two elements, and define

\[
W = \langle s, t : s^2 = t^2 = (st)^m = 1 \rangle
\]
as the dihedral group of order $2m$. It is well-known that $P_{y,w} = 1$ for all $y, w \in W$ with $y \leq w$ (see \textbf{[3]} \S4.2), and we prove here the analogous result that in the finite dihedral case, for any choice of $S$-preserving involution $* \in \text{Aut}(W)$ one has likewise $P_{y,w}^* = 1$ for all $y, w \in I_*$ with $y \leq w$. The same statement holds in the infinite dihedral case by \textbf{[13]} Proposition 3.8], and so we are able to deduce here that Properties $\mathcal{A}$ and $\mathcal{B}$ hold for all Coxeter systems of rank two.

**Remark.** Du Cloux has derived explicit formulas in the dihedral for the structure constants $h_{x,y,z}$; see \textbf{[3]} Propositions 4.4 and 4.6]. We imagine that similar formulas can be derived and used to show that Properties $\mathcal{C}$ and $\mathcal{D}$ for dihedral Coxeter systems, but the calculations necessary for this appear significantly more involved, and we do not undertake them here.

To denote the elements of the dihedral group $W$, we define for positive integers $i$

$$[s,i] = \underbrace{sststs\cdots}_{i \text{ factors}} \quad \text{and} \quad [t,i] = \underbrace{tstst\cdots}_{i \text{ factors}}.$$  

There exist exactly two $S$-preserving involution $*$ of $W$: either $*$ is the identity automorphism or $*$ is the automorphism interchanging $s$ and $t$. If $*$ is trivial, then $I_*$ consists of the identity, the longest element, and all elements of $W$ of odd length, i.e.,

$$1, \quad [s,1], [s,3], [s,5], \ldots \quad [t,1], [t,3], [t,5], \ldots \quad \text{and} \quad [s,m] = [t,m].$$

In the nontrivial case $I_*$ consists of the longest element and all elements of even length, i.e.,

$$1, \quad [s,2], [s,4], [s,6], \ldots \quad [t,2], [t,4], [t,6], \ldots \quad \text{and} \quad [s,m] = [t,m].$$

Fix an arbitrary choice of $S$-preserving involution $* \in \text{Aut}(W)$ and write $w_0 = [s,m] = [t,m]$ for the longest element in $W$. Every $w \in W$ has a unique reduced expression except $w_0$, which has exactly two reduced expressions given by $ststs\cdots$ and $tstst\cdots$ (each with $m$ factors). The Bruhat order on $W$ has the simple description that $y < w$ if and only if $\ell(y) < \ell(w)$.

We note two lemmas before stating our main result.

**Lemma 4.1.** Suppose $r \in S$ and $w \in I_*$ with $rw = wr^*$. Then $m$ is odd or $*$ is trivial, such that:

(a) If $m$ is odd and $*$ is trivial then $w \in \{1,r\}$.

(b) If $m$ is odd and $*$ is nontrivial then $w \in \{w_0, rw_0\}$.

(c) If $m$ is even and $*$ is trivial and $w \in \{w_0, rw_0\}$.

**Proof.** Since $rw = wr^*$ if and only if $rw' = w'r^*$ where $w' = r \prec w$, we may assume $rw > w$. If $\ell(w) = 0$ then $sw = ws^*$ if and only if $s = s^*$. If $0 < \ell(w) < m - 1$ then it follows from the previous lemma that $rw \neq wr^*$. It remains only to consider the case when $\ell(w) = m - 1$ (since when $\ell(w) = m$ it cannot hold that $rw > w$). In this situation $rw = wr^*$ if and only if $w_0 = rw = r(rw)r^* = rw_0r^*$. One checks that this holds precisely when $m = \ell(w_0)$ is odd and $*$ is nontrivial or $m$ is even and $*$ is trivial. \hfill $\square$

**Lemma 4.2.** Suppose $y, w \in I_*$ and $\ell(w) - \ell(y) = 1$. Then $m$ is odd or $*$ is trivial, such that:

(a) If $m$ is odd and $*$ is trivial then $y = 1$ and $w \in S$. 

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(b) If \( m \) is odd and \( * \) is nontrivial then \( y \in \{sw_0, tw_0\} \) and \( w = w_0 \).

(c) If \( m \) is even and \( * \) is trivial then \( y \in \{sw_0, tw_0\} \) and \( w = w_0 \), or \( y = 1 \) and \( w \in S \).

**Proof.** The claims here follow by inspecting the lists of elements in \( I_* \), given before Lemma 4.1, noting that the elements \([s, i]\) and \([t, i]\) have length \( i \) when \( i \leq m \).

We now have the main result of this section. Despite the simplicity of this statement, we know of no easier proof than the following somewhat lengthy inductive argument using Corollary 2.2.

**Theorem 4.3.** Suppose \((W, S)\) is of dihedral type \( I_2(m) \), with \( 3 \leq m < \infty \). Let \( * \in \text{Aut}(W) \) be either \( S \)-preserving involution. Then \( P_{y,w}^\sigma = 1 \) for all \( y, w \in I_* \) with \( y \leq w \).

**Proof.** Let \( y, w \in I_* \) such that \( y \leq w \). If \( w = 1 \) then \( y \leq w \) implies \( y = w \) so \( P_{y,w}^\sigma = 1 \) as desired. If \( \ell(w) \in \{1, 2\} \), then \( w = r \times 1 \) for some \( r \in S \), in which case \( y \leq w \) if and only if \( y \in \{1, w\} \), whence \( P_{y,w}^\sigma = P_{r \times y, w}^\sigma = 1 \) by the first part of Corollary 2.2.

For the remainder of this proof we assume that \( \ell(w) \geq 3 \). We may assume that \( y \leq w \) since \( P_{w,w}^\sigma = 1 \), and may take an inductive hypothesis that \( P_{y', w'}^\sigma = 1 \) when \( w' < w \) or when \( w = w' \) and \( y' > y \). Let \( r \in \text{Des}_L(w) \) and set \( w' = r \times w \). If \( r \notin \text{Des}_L(y) \) then \( P_{y,w}^\sigma = P_{r \times y, w}^\sigma = 1 \) by hypothesis, so assume \( r \in \text{Des}_L(y) \). This implies that \( y \neq 1 \), and that \( r \times y \leq w' \).

Suppose \( y \leq w' \). Then \( \ell(y) = \ell(w') \), so the only element \( z \in I_* \), with \( y \leq z < w \) is \( z = y \), and the second part of Corollary 2.2 becomes

\[
(q + 1)^2 P_{y,w}^\sigma = (q + 1)^d - v_\ell(w) \times (y, w) \times \mu^\sigma(y, w) = 0
\]

where \( c = \delta_{r \times w, w} \) and \( d = \delta_{r \times y, r} \). To express \( m^\sigma(y, w) \) more simply, we note that since \( \ell(y) = \ell(w') \), we have

\[
u^\sigma(y, w') = \mu^\sigma(y, x) \mu^\sigma(x, w') = 0 \quad \text{for all } x \in I_*,
\]

and also

\[
\delta_{r \times y, r} \mu^\sigma(y, w') = \delta_{r \times y, r} \quad \text{and} \quad \delta_{r \times w, w} \mu^\sigma(y, r \times w') = \delta_{r \times w, w} \mu^\sigma(y, w).
\]

Thus, by the definition 2.1, our previous equation becomes

\[
(q + 1)^c P_{y,w}^\sigma = (q + 1)^d - q(d - c \times \mu^\sigma(y, w)).
\]

If \( c = 0 \) then this reduces to the formula \( P_{y,w}^\sigma = (q + 1)^d - dq \) which is equal to 1 for all \( d \in \{0, 1\} \).

If \( c = 1 \) then \( \ell(y) = \ell(w') = \ell(w) - 1 \) so \( \mu^\sigma(y, w) \) is the constant coefficient of \( P_{y,w}^\sigma \) and therefore equal to 1. In this case we must have \( d = 0 \) since (using Lemma 4.1) the only element \( x \in I_* \), with \( r x = x r \) and \( \ell(x) = \ell(w) - 1 \) is \( w' \) which by assumption is distinct from \( y \). Thus if \( c = 1 \) then \( d = 0 \) and our equation becomes \((q + 1)P_{y,w}^\sigma = q + 1 \) so \( P_{y,w}^\sigma = 1 \) again as desired.

From now on we assume \( y \leq w' \leq w \). Since \( r \in \text{Des}_L(y) \setminus \text{Des}_L(w') \), we must actually have \( y < w' \). Further, since \( y \neq 1 \) and \( w' \neq w_0 \), it follows from Lemma 4.2 that \( \ell(w') - \ell(y) \geq 2 \).

Continuing, by the second part of Corollary 2.2 and our inductive hypothesis, we have

\[
(q + 1)^c P_{y,w}^\sigma = q^2 + 1 - \sum_{z \in I_* : rz < z < w \leq y \leq z < w} v_\ell(z) \times (z, w)
\]

where \( c = \delta_{r \times w, w} \). (There are no \( d \)'s here because \((q + 1)^d + q(q - d) = q^2 + 1 \) for all \( d \in \{0, 1\} \).) We wish to replace the right hand side of this equation with a more elementary expression. To this end, suppose \( z \in I_* \) such that \( rz < z \) and \( y \leq z < w \). We make the following observations:
Thus, noting that \( \ell \) two additional observations:

(b) By definition and inductive hypothesis, \( \nu^\sigma(z, w') = \begin{cases} 1 & \text{if } \ell(w') - \ell(z) = 2 \\ 0 & \text{otherwise.} \end{cases} \)

(c) \( \delta_{rz, z'} \mu^\sigma(rz, w') = 0 \). This follows as \( \mu^\sigma(rz, w') = 0 \) unless \( \ell(w') - \ell(rz) = 1 \), which by Lemma 4.2 occurs only if \( rz = 1 \) and \( w' \in S \) (since \( w' \neq w_0 \)). By assumption, however, we have \( \ell(w') \geq \ell(y) + 2 \geq 3 \).

(d) \( \delta_{w', w'} \mu^\sigma(w, w') = c \cdot \mu^\sigma(z, w) \) by definition.

(e) \( \mu^\sigma(z, x) \mu^\sigma(x, w') = 0 \) for all \( x \in I_\ast \) with \( r \in \text{Des}_L(x) \). This follows as the product can only be nonzero if \( z < x < w' \), in which case by hypothesis the product is 1 if and only if \( \ell(x) = \ell(z) + 1 = \ell(w') - 1 \) and is 0 otherwise. If \( \ell(x) = \ell(z) + 1 \), however, then \( x \neq 1 \), so \( \ell(x) \neq \ell(w') - 1 \) as \( w' \neq w_0 \), by Lemma 4.2

In consequence of (a), we deduce that \( \mu^\sigma(z \rightarrow w') = 0 \) if \( \ell(w') - \ell(z) \) is odd, and in consequence of (b)-(e), we deduce that if \( \ell(w') - \ell(z) \) is even then

\[
\mu^\sigma(z \rightarrow w') = \nu^\sigma(z, w') - c \cdot \mu^\sigma(z, w).
\]

Thus, noting that \( \ell(w) + c = \ell(w') + 2 \), we have

\[
(q + 1)^c \mathcal{P}_{\mu, \nu}^\sigma = q^2 + 1 - \left( \sum_z \nu^{\ell(w') - \ell(z) + 2} \cdot \nu^\sigma(z, w') \right) + \left( \sum_z \nu^{\ell(w') - \ell(z) + 2} \cdot c \cdot \mu^\sigma(z, w) \right)
\]

where both sums are over \( z \in I_\ast \) with \( rz < z \) and \( y \leq z < w \) and \( \ell(w') - \ell(z) \) even. Recall that \( \ell(w') - \ell(y) \geq 2 \) and that \( \ell(y) \geq 1 \). From this and the description of the elements of \( I_\ast \), we note two additional observations:

- There exists exactly one element \( z \in I_\ast \) with \( y \leq z < w \) and \( rz < z \) and \( \ell(w') - \ell(z) \) even and \( \nu^\sigma(z, w') \neq 0 \). This is the element \( z = r' \times w' \) where \( r' \in \text{Des}_L(w') \subset S \) is the generator distinct from \( r \in S \), for which \( \ell(w') - \ell(z) = 2 \) and \( \nu^\sigma(z, w') = 1 \) by claim (b) above. It follows that the first parenthesized sum in (4.1) is equal to \( q^2 \).

- If \( c = 1 \) then by Lemma 4.1 we must have \( w = w_0 \), since \( \ell(w) \geq 3 \) and \( r \in \text{Des}_L(w) \). In this case there exists exactly one element \( z \in I_\ast \) with \( y < z < w \) (note that we exclude the case \( y = z \)) and \( rz < z \) and \( \ell(w') - \ell(z) \) even and \( \mu^\sigma(z, w) \neq 0 \). Namely, this element \( z \) is given by the unique twisted involution of length \( m - 1 \) distinct from \( w' = rw \). This element has \( \ell(w') - \ell(z) = 0 \) and \( \mu^\sigma(z, w) = 1 \), by inductive hypothesis. It follows that the second parenthesized sum in (4.1) is equal to

\[
c \cdot q + c \cdot \nu^{\ell(w') - \ell(y) + 1} \cdot \mu^\sigma(y, w).
\]

The second term here corresponds to the summand indexed by \( z = y \). Such a summand occurs if and only if \( \ell(w') - \ell(y) \) is even, but our expression accounts for this circumstance because if \( \ell(w') - \ell(y) \) is odd and \( c \neq 0 \) then nevertheless \( \mu^\sigma(y, w) = 0 \), as \( \ell(w) - \ell(y) \) would then not be odd.
Substituting these facts into (4.1) gives

\[(q + 1)^c P_{y,w}^\sigma = 1 + c \cdot q + c \cdot v^{\ell(w) - \ell(y)} + 1 \cdot \mu^\sigma(y, w). \]

(4.2)

If \(c = 0\) then it follows immediately that \(P_{y,w}^\sigma = 1\). Suppose \(c = 1\). If \(\ell(w) - \ell(y)\) is even then \(\mu^\sigma(y, w) = 0\) so the preceding equation becomes \((q + 1)P_{y,w}^\sigma = q + 1\) and we get likewise \(P_{y,w}^\sigma = 1\). Assume therefore that \(\ell(w) - \ell(y)\) is odd. Define

\[\mu_n = \mu^\sigma(y, w) \quad \text{and} \quad n = \frac{\ell(w) - \ell(y) - 1}{2}\]

so that by definition \(P_{y,w}^\sigma = \mu_n q^n + \mu_{n-1} q^{n-1} + \cdots + \mu_0\) for some integers \(\mu_0, \ldots, \mu_{n-1}\). In this notation, our equation (4.2) becomes

\[(q + 1)(\mu_n q^n + \mu_{n-1} q^{n-1} + \cdots + \mu_0) = 1 + q + q^{n+1} \mu_n.\]

As the left hand side is equal to \(\mu_n q^{n+1} + \sum_{i=1}^{n} (\mu_i + \mu_{i-1}) q^n + \mu_0\), equating coefficients of \(q^i\) gives \(\mu_0 = 1\) and \(\mu_0 + \mu_1 = 1\) and \(\mu_i + \mu_{i-1} = 0\) for \(i = 2, 3, \ldots, n\). The only solution to this system of equations is to set \(\mu_0 = 1\) and \(\mu_1 = \mu_2 = \cdots = \mu_n = 0\); hence even in this final case we get \(P_{y,w}^\sigma = 1\) as desired.

It follows that when \((W, S)\) is a finite dihedral Coxeter system, the polynomials \(P_{y,w}^\sigma\) are all zero for \(y, w \in I^*\), while the polynomials \(P_{y,w}^+\) are 0 or 1 according to whether \(y \not\leq w\) or \(y \leq w\). We thus are left with the following corollary.

**Corollary 4.4.** Properties \(A'\) and \(B'\) hold for all Coxeter systems of rank two.

**Proof.** This follows from Theorem 4.3 and the preceding discussion (which covers the finite irreducible dihedral case), \([13, \text{Theorem 3.13}]\) (which covers the infinite dihedral case), and Theorem 1.8 (which covers type \(A_1 \times A_1\)).
Table 1: Irreducible finite Coxeter systems with involution; see Section 3

| Name     | Coxeter diagram for \((W, S)\) | Involution \(\ast \in \text{Aut}(W)\)          |
|----------|--------------------------------|------------------------------------------------|
| \(A_n\)  \((n \geq 1)\) | \(s_1 \cdots s_n\)               | Identity                                      |
| \(2A_n\) \((n \geq 2)\) | \(s_1 \cdots s_n\)               | \(\text{Diagram } s_i \mapsto s_{n+1-i}\) |
| \(BC_n\) \((n \geq 3)\) | \(s_1 \cdots s_n\)               | Identity                                      |
| \(D_n\)  \((n \geq 4)\) | \(s_1 \cdots s_n\)               | Identity                                      |
| \(2D_n\) \((n \geq 4)\) | \(s_1 \cdots s_n\)               | \(\text{Diagram } \begin{cases} s_1 \leftrightarrow s_2 \\ s_i \mapsto s_i \ (i \geq 3) \end{cases}\) |
| \(E_6\)  | \(s_1 \cdots s_6\)               | Identity                                      |
| \(2E_6\) | \(s_1 \cdots s_6\)               | \(\text{Diagram } \begin{cases} s_1 \leftrightarrow s_3 \\ s_3 \leftrightarrow s_5 \\ s_i \mapsto s_i \ (i = 2, 4) \end{cases}\) |
| \(E_7\)  | \(s_1 \cdots s_7\)               | Identity                                      |
| \(E_8\)  | \(s_1 \cdots s_8\)               | Identity                                      |
| \(F_4\)  | \(s_1 \cdots s_4\)               | Identity                                      |
| \(2F_4\) | \(s_1 \cdots s_4\)               | \(\text{Diagram } s_i \mapsto s_{5-i}\)     |
| \(H_3\)  | \(s_1 \cdots s_3\)               | Identity                                      |
| \(H_4\)  | \(s_1 \cdots s_4\)               | Identity                                      |
| \(I_2(m)\) \((m \geq 4)\) | \(s_1 \cdots s_m\)           | Identity                                      |
| \(2I_2(m)\) \((m \geq 4)\) | \(s_1 \cdots s_m\)           | \(\text{Diagram } s_i \mapsto s_{3-i}\)     |

All Coxeter diagrams are labeled to coincide with the indexing conventions in Coxeter [4]. The types \(BC_2, 2BC_2, G_2, 2G_2\) are omitted since they coincide with types \(I_2(m), 2I_2(m)\) for \(m = 4, 6\).
Table 2: Maximum nonzero coefficients in KL-type polynomials; see Section 1.4

| Type     | $P_{y,w}$ ($y, w \in I_*$) | $P_{y,w}^{\sigma}$ | $-P_{y,w}^{\sigma}$ | $P_{y,w}^+$ | $P_{y,w}$ |
|----------|----------------------------|---------------------|----------------------|-------------|-----------|
| $A_1$, $A_2$, $A_3$ | 1 1 -1 1 (all polynomials zero) | 2 2 -1 2 (all polynomials zero) | 4 4 -1 4 | 15 7 -1 11 | 4 4 |
| $A_4$    | 1 1 -1 1 | 1 1 -1 1 | 4 4 -1 4 | 15 7 -1 11 | 4 4 |
| $A_5$    | 4 4 -1 4 | 4 4 -1 4 | 15 7 -1 11 | 4 4 |
| $A_6$    | 73 25 -1 49 | 73 25 -1 49 | 4 4 |
| $A_7$    | 362 82 -1 222 | 362 82 -1 222 | 4 4 |
| $A_8$    | 1 1 1 1 | 1 1 1 1 | 4 4 |
| $A_4$    | 1 1 1 1 | 1 1 1 1 | 4 4 |
| $A_5$    | 4 2 1 3 | 4 2 1 3 | 1 1 |
| $A_6$    | 15 3 2 8 | 15 3 2 8 | 4 4 |
| $A_7$    | 73 5 3 38 | 73 5 3 38 | 4 4 |
| $A_8$    | 460 12 6 232 | 460 12 6 232 | 4 4 |
| $BC_3$   | 1 1 1 1 | 1 1 1 1 | 4 4 |
| $BC_4$   | 5 3 1 4 | 5 3 1 4 | 4 4 |
| $BC_5$   | 35 10 3 21 | 35 10 3 21 | 4 4 |
| $BC_6$   | 454 48 8 246 | 454 48 8 246 | 4 4 |
| $D_4$    | 4 3 2 3 | 4 3 2 3 | 4 4 |
| $D_5$    | 17 8 3 11 | 17 8 3 11 | 4 4 |
| $D_6$    | 217 25 12 121 | 217 25 12 121 | 4 4 |
| $E_6$    | 581 54 10 293 | 581 54 10 293 | 4 4 |
| $E_7$    | 748 18 3 374 | 748 18 3 374 | 4 4 |
| $F_4$    | 12 8 2 9 | 12 8 2 9 | 4 4 |
| $G_2$    | 12 2 1 6 | 12 2 1 6 | 4 4 |
| $H_3$    | 3 1 1 2 | 3 1 1 2 | 4 4 |
| $H_4$    | 5,116 213 9 2,651 | 5,116 213 9 2,651 | 4 4 |

We obtained this data by running our extended version of Coxeter [14] for the triples ($W, S, \ast$) of the types listed. Our computations verify Properties A" and B" in each of these types.
Table 3: Maximum nonzero coefficients in KL-type structure constants; see Section 1.4

| Type   | $h_{x,y,z}$ ($x \in W; y, z \in I$) | $h_{x,y,z}^0$ | $-h_{x,y,z}^0$ | $h_{x,y,z}^+ - h_{x,y,z}^-$ | $h_{x,y,z}^-$ |
|--------|----------------------------------|---------------|----------------|-------------------------------|--------------|
| $A_1$  | 2                                | 1             | -1             | 1                             | 1            |
| $A_2$  | 10                               | 2             | -1             | 5                             | 5            |
| $A_3$  | 132                              | 10            | -1             | 66                            | 66           |
| $A_4$  | 3,748                            | 61            | -1             | 1,892                         | 1,856        |
| $A_5$  | 922,740                          | 912           | -1             | 461,826                       | 460,914      |
| $A_6$  | 179,487,027                      | 20,367        | -1             | 89,753,697                    | 89,733,330   |
| $A_2^2$| 10                               | 2             | 1              | 5                             | 5            |
| $A_3^2$| 132                              | 7             | 3              | 66                            | 66           |
| $A_4^2$| 4,698                            | 36            | 10             | 2,358                         | 2,340        |
| $A_5^2$| 922,740                          | 506           | 162            | 461,404                       | 461,336      |
| $A_6^2$| 186,996,750                      | 4,080         | 1,994          | 93,499,109                    | 93,497,641   |
| $BC_2$ | 14                               | 2             | 1              | 8                             | 6            |
| $BC_3$ | 905                              | 28            | 8              | 451                           | 454          |
| $BC_4$ | 397,846                          | 767           | 156            | 199,042                       | 198,804      |
| $BC_5$ | 1,319,190,596                    | 42,248        | 9,924          | 659,608,306                   | 659,582,290  |
| $G_2$  | 14                               | 2             | 1              | 8                             | 6            |
| $G_2^2$| 14                               | 2             | 1              | 8                             | 6            |
| $D_4$  | 42,384                           | 246           | 85             | 21,226                        | 21,225       |
| $D_5$  | 89,307,651                       | 11,123        | 3,319          | 44,652,166                    | 44,655,485   |
| $D_4^2$| 42,384                           | 116           | 30             | 21,225                        | 21,159       |
| $D_5^2$| 89,307,651                       | 4,748         | 1,538          | 44,655,112                    | 44,652,539   |
| $F_4$  | 108,380,588                      | 8,995         | 2,007          | 54,192,072                    | 54,188,516   |
| $F_4^2$| 108,380,588                      | 2,600         | 86             | 54,191,394                    | 54,188,994   |
| $G_2$  | 22                               | 2             | 2              | 12                            | 10           |
| $G_2^2$| 22                               | 2             | 2              | 12                            | 10           |
| $H_3$  | 15,676                           | 106           | 49             | 7,870                         | 7,806        |
| $H_4$  | 59,133,414,193,112,056           | 467,325,554   | 60,353,800    | 29,566,707,126,594,414        | 29,566,707,066,517,642 |

We obtained this data by running our extended version of Coxeter [14] for the triples $(W, S, \ast)$ of the types listed. Our computations verify Properties $C'$ and $D'$ in each of these types.

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