ON CERTAIN PROPERTIES OF THE COHEN-RAMANUJAN SUM

PATRICK KÜHN AND NICOLAS ROBLES

Abstract. Explicit formulae of the Cohen-Ramanujan sum are derived and their prime-number-theorem analogues are proved.

1. Introduction

For \(\sigma, t \in \mathbb{R}\), we denote by \(s = \sigma + it\) a general complex number. In [9] Ramanujan introduced the following trigonometrical sum.

Definition 1.1. The Ramanujan sum is defined by

\[
c_q(n) = \sum_{(h,q) = 1} e^{2\pi i nh/q},
\]

where \(q\) and \(n\) are in \(\mathbb{N}\) and the summation is over a reduced residue system \((\text{mod } q)\).

Cohen [3] generalized this arithmetical function in the following way.

Definition 1.2. Let \(\beta \in \mathbb{N}^+\). The Cohen-Ramanujan sum is defined by

\[
c^{(\beta)}_q(n) = \sum_{(h,q^\beta)_{\beta=1}^k} e^{2\pi i nh/q^\beta},
\]

where \(h\) ranges over the the non-negative integers less than \(q^\beta\) such that \(h\) and \(q^\beta\) have no common \(\beta\)-th power divisors other than 1.

It follows immediately that when \(\beta = 1\), (1.2) becomes the Ramanujan sum (1.1). Among the most important properties of \(c^{(\beta)}_q(n)\) we mention that it is a multiplicative function of \(q\), i.e.

\[
c^{(\beta)}_{pq}(n) = c^{(\beta)}_p(n) c^{(\beta)}_q(n), \quad (p,q) = 1.
\]

Note that \(c^{(\beta)}_q(n)\) reduces to the Mőbius function for \(n = 1\), i.e. \(c^{(\beta)}_q(1) = \mu(q)\) for all \(\beta \in \mathbb{N}^+\).

By an explicit formula we mean, as is customary in the literature of the Riemann zeta-function, a formula relating a truncated sum of an arithmetical function on one side, and a sum over the zeros, \(\rho\) and \(-2k\) with \(k = 1, 2, \ldots\), of the Riemann zeta-function on the other side. In this case, the arithmetical function of interest is \(c^{(\beta)}_q(n)\) and the sum is on the interval \(1 \leq q \leq x\).

The purpose of this article is to derive explicit formulae involving \(c^{(\beta)}_q(n)\) in terms of the non-trivial zeros \(\rho\) and establish arithmetic theorems.

Originally, explicit formulae involving the Merten’s function \(M(x) = \sum_{n \leq x} \mu(n)\) were proved by Titchmarsh under the assumption of the Riemann hypothesis, see §14.16 of [10]. Specifically, his result was conditioned on the following: for each \(\varepsilon > 0\), \(T \in \mathbb{N}\), there is a \(t \in (T, T + 1)\) such that

\[
\frac{1}{\zeta(s)} = O(t^\varepsilon),
\]

for \(\frac{1}{2} \leq \sigma \leq 2\), if RH is true. Thus, under RH, for each \(\varepsilon > 0\) we can find a sequence \(T_n\) such that

\[
n \leq T_n \leq n + 1, \quad n = 1, 2, 3, \ldots
\]

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and

\[(1.4)\quad |\zeta(\sigma + iT_n)|^{-1} < T_n^\epsilon.\]

In order to obtain unconditional results we use an idea put forward by Bartz [2]. The key is to use the following result of Montgomery, see [7] and Theorem 9.4 of [5].

**Lemma 1.1.** For any given \(\varepsilon > 0\) there exists a \(T_0 = T_0(\varepsilon)\) such that for \(T \geq T_0\) the following holds: between \(T\) and \(2T\) there exists a value of \(t\) for which

\[|\zeta(\sigma \pm iT)|^{-1} < c_1 t^\varepsilon \quad \text{for} \quad -1 \leq \sigma \leq 2,
\]

with an absolute constant \(c_1 > 0\).

That is, for each \(\varepsilon > 0\), there is a sequence \(T_\nu\), where

\[(1.5)\quad 2^{\nu - 1} T_0(\varepsilon) \leq T_\nu \leq 2^\nu T_0(\varepsilon), \quad \nu = 1, 2, 3, \ldots
\]

such that

\[(1.6)\quad |\zeta(\sigma \pm iT_\nu)|^{-1} < c_1 T_\nu^\varepsilon \quad \text{for} \quad -1 \leq \sigma \leq 2.
\]

Finally, towards the end we will need the following bracketing condition: \(T_m (m \leq T_m \leq m + 1)\) are chosen so that

\[(1.7)\quad |\zeta(\sigma + iT_m)|^{-1} < T_m^c_1
\]

for \(-1 \leq \sigma \leq 2\) and \(c_1\) is an absolute constant. The existence of such a sequence of \(T_m\) is guaranteed by Theorem 9.7 of [10].

We will use either bracketing \((1.5)-(1.6)\) or \((1.7)\) depending on the necessity. These choices will lead to different bracketings of the sum over the zeros in the various explicit formulae appearing in the theorems of this note.

Section 2 will be devoted to the proof of the unconditional explicit formula and its graphic illustration by means of plots comparing \(\sum_{q \leq x} c_q^{(\beta)}(n)\) against its analytic counterpart. Moreover, we will prove equivalent conditions for the RH in terms of the Cohen-Ramanujan sum. These conditions generalize existing ones due to Littlewood, see [6].

Since \(c_q^{(\beta)}(n)\) generalises the Möbius function, in Section 3 we will construct a generalized von Mangoldt function \(\Lambda_m^{(\beta)}(q)\) and prove its explicit formula. This will allow us to prove the associated “prime number theorem” of \(\Lambda_m^{(\beta)}(q)\) with an error term due to the standard zero-free region of \(\zeta(s)\). We will also establish this prime number theorem under the assumption of RH.

Finally, Section 4 contains a generalization of a function constructed by Bartz involving \(\mu(n)\) and \(\rho\), see [1] and [2].

## 2. The Explicit Formula

We begin with some auxiliary results.

**Definition 2.1.** Let \(z \in \mathbb{C}\). The **generalized divisor function** \(D_z^{(\beta)}(n)\) is the sum of the \(q^{th}\) powers of those divisors of \(n\) which are \(\beta^{th}\) powers of integers, i.e.

\[D_z^{(\beta)}(n) = \sum_{d^{\beta}|n} d^{\beta z}.
\]

The first immediate result is as follows.

**Lemma 2.1.** The generalized divisor function \(D_z^{(\beta)}(n)\) satisfies the following bound for \(z \in \mathbb{C}\), \(n \in \mathbb{N}\)

\[|D_z^{(\beta)}(n)| \leq D_{\Re(z)}^{(\beta)}(n) \leq n^{\beta \max(0, \Re(z)) + 1}.
\]

In [3] the following properties of \(c_q^{(\beta)}(n)\) are derived.
Lemma 2.2 (Cohen, 1949). For $\beta$ and $n$ integers one has
\[
e^{(\beta)}_q(n) = \sum_{d|q} \mu\left(\frac{q}{d}\right) d^\beta
\]
where $\mu$ denotes the Möbius function.

Lemma 2.3 (Cohen, 1949). For $\text{Re}(s) > 1$ and $\beta \in \mathbb{N}^+$ one has
\[
\sum_{q=1}^{\infty} \frac{e^{(\beta)}_q(n)}{q^s} = \frac{\mathcal{D}^{(\beta)}_{1-s/\beta}(n)}{\zeta(s)}.
\]

Definition 2.2. For $x \geq 1$, we define
\[
\mathcal{C}^{(\beta)}(n, x) = \sum_{q \leq x} e^{(\beta)}_q(n).
\]

To account for discontinuities, we set
\begin{equation}
\mathcal{C}^{(\beta)}_0(n, x) = \begin{cases} 
\mathcal{C}^{(\beta)}(n, x), & \text{if } x \notin \mathbb{N}, \\
\mathcal{C}^{(\beta)}(n, x) - \frac{1}{x} \mathcal{C}^{(\beta)}_0(n), & \text{if } x \in \mathbb{N}.
\end{cases}
\end{equation}

Figure 2.1. Plots of $\mathcal{C}^{(\beta)}_0(n, x)$ with $n = 24$ (left) and $n = 48$ (right) for $\beta = 1$ (blue), $\beta = 2$ (red), $\beta = 3$ (yellow), $\beta = 4$ (green) for $1 \leq x \leq 47$ and $1 \leq x \leq 90$.

The explicit formula for $\mathcal{C}^{(\beta)}(n, x)$ is then as follows.

Theorem 2.1. Let $\rho$ and $\rho_m$ denote non-trivial zeros of $\zeta(s)$ of multiplicity 1 and $m \geq 2$ respectively. Fix integers $\beta$, $n$. There is an $\varepsilon > 0$ and a $T_0 = T_0(\varepsilon)$ such that (1.3) and (1.4) hold for a sequence $T_n$ and
\[
\mathcal{C}^{(\beta)}_0(n, x) = -2\mathcal{D}^{(\beta)}_1(n) + \sum_{|\gamma|<T_n} \frac{\mathcal{D}^{(\beta)}_1-\rho/\beta(n)}{\zeta'(\rho)} x^\rho + K_{T_n}(x) - \sum_{k=1}^{\infty} \frac{(-1)^k (2\pi/x)^{2k}}{(2k)! k \zeta(2k+1)} \mathcal{D}^{(\beta)}_{1+2k/\beta}(n) + E_{T_n}(x),
\]
where the error term satisfies
\[
E_{T_n}(x) = O\left(\frac{x \log x}{T_n^{1-\varepsilon}}\right),
\]
and where for the zeros of multiplicity $m \geq 2$ we have
\[
K_{T_n}(x) = \sum_{m \geq 2} \sum_{|\gamma|<T_n} \kappa(\rho_m, x), \quad \kappa(\rho_m, x) = \frac{1}{(m-1)!} \lim_{s \to \rho_m} \frac{d^{m-1}}{ds^{m-1}} \left( (s - \rho_m)^m \mathcal{D}^{(\beta)}_{1-s/\beta}(n) x^s / s \right).
\]
Moreover, in the limit $\nu \to \infty$ we have
\[
\mathcal{C}^{(\beta)}_0(n, x) = -2\mathcal{D}^{(\beta)}_1(n) + \lim_{\nu \to \infty} \sum_{|\gamma|<T_n} \frac{\mathcal{D}^{(\beta)}_1-\rho/\beta(n)}{\zeta'(\rho)} x^\rho + \lim_{\nu \to \infty} K_{T_n}(x) - \sum_{k=1}^{\infty} \frac{(-1)^k (2\pi/x)^{2k}}{(2k)! k \zeta(2k+1)} \mathcal{D}^{(\beta)}_{1+2k/\beta}(n).
Proof. From Lemma 2.2 one has the following bound

\[ |c_q^{(\beta)}(n)| \leq \sum_{d|q} \frac{d\beta}{d^{|m|n}} = D_1^{(\beta)}(n). \]

Suppose \( x \) is a fixed non-integer. We set \( a_q = c_q^{(\beta)}(n) \) and we use the lemma in §3.12 of [10] to see that we can take \( \varphi(q) = D_1^{(\beta)}(n) \). We note that for \( \sigma > 1 \) we have

\[ \sum_{q=1}^{\infty} \left| \frac{c_q^{(\beta)}(n)}{q^\sigma} \right| = \sum_{q=1}^{\infty} \frac{1}{q^\sigma} = D_1^{(\beta)}(n) \zeta(\sigma) = O \left( \frac{1}{\sigma - 1} \right) \]

so that \( \alpha = 1 \). Moreover, if in that lemma we put \( s = 0 \), \( c = 1 + 1/\log x \) and \( w \) replaced by \( s \), then we obtain

\[ C_0^{(\beta)}(n, x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{D_1^{(\beta)}(n)}{\zeta(s)} \frac{x^s}{s} \, ds + E_{1,T}(x), \]

where \( E_{1,T}(x) \) is an error term that will be evaluated later. If \( x \) is an integer, then \( \frac{1}{2^\beta x}(n) \) is to be subtracted from the left-hand side. Let us now consider the positively oriented path \( C \) made up of the line segments \( [c-iT, c+iT, -2N-1+iT, -2N-1-iT] \) where \( T \) is not the ordinate of a non-trivial zero.

![Figure 2.2. The path of integration C.](image)

Then, by residue calculus we have

\[ \frac{1}{2\pi i} \int_C \frac{D_1^{(\beta)}(n)}{\zeta(s)} \frac{x^s}{s} \, ds = R_0 + R_\rho(T) + K(x, T) + R_{-2k}(N), \]

where the each term is given by the residues inside \( C \)

\[ R_0 = \text{res}_{s=0} \frac{D_1^{(\beta)}(n)}{\zeta(s)} \frac{x^s}{s} = -2D_1^{(\beta)}(n), \]

and for \( k = 1, 2, 3, \ldots \) we have

\[ R_{-2k}(N) = \sum_{k=1}^{N} \text{res}_{s=-2k} \frac{D_1^{(\beta)}(n)}{\zeta(s)} \frac{x^s}{s} = \sum_{k=1}^{N} \frac{D_1^{(\beta)}(n)}{\zeta'(-2k)} x^{-2k} = \sum_{k=1}^{N} \frac{(-1)^{k-1}(2\pi/x)^{2k}}{(2k)!k\zeta(2k+1)} D_1^{(\beta)}(n). \]
For the non-trivial zeros we must distinguish two cases. For the simple zeros $\rho$ we have

$$R_\rho(T) = \sum_{|\gamma|<T} \text{res}_{s=\rho} \frac{\mathcal{D}_{1-s/\beta}^{(\beta)}(n)x^s}{\zeta(s)} s = \sum_{|\gamma|<T} \frac{\mathcal{D}_{1-\rho/\beta}^{(\beta)}(n)x^\rho}{\zeta'(\rho)/\rho},$$

and by the formula for the residues of order $m$ we see that $K(x, T)$ is of the form indicated in the statement of the theorem. We now bound the vertical integral on the far left

$$\int_{-2N-1-iT}^{-2N-1+iT} \frac{\mathcal{D}_{1-s/\beta}^{(\beta)}(n)x^s}{\zeta(s)} s ds = \int_{2N+1-iT}^{2N+1+iT} \frac{\mathcal{D}_{1-(1-s)/\beta}^{(\beta)}(n)x^{1-s}}{\zeta(1-s)} 1 - s ds$$

$$= \int_{2N+1-iT}^{2N+1+iT} \frac{\mathcal{D}_{1-(1-s)/\beta}^{(\beta)}(n)x^{2s-1}\pi^s}{\zeta(s)} 1 ds$$

$$= O\left(\frac{1}{T}\right)^{2N+2} e^{(2N+3)\log(n)+2N+2-(2N+\frac{3}{2})\log(2N+2)dt},$$

since by the use of Lemma 2.31 we have $\mathcal{D}_{1-(1-s)/\beta}^{(\beta)}(n) = O(\mathcal{D}_{1-(1-2N)/\beta}^{(\beta)}(n)) = O(n^{2N+3}).$ This tends to zero as $N \to \infty$, for a fixed $T$ and a fixed $n$. Hence we are left with

$$c_0^{(\beta)}(n, x) = -2\mathcal{D}_1^{(\beta)}(n) + \sum_{|\gamma|<T} \frac{\mathcal{D}_{1-\rho/\beta}^{(\beta)}(n)x^\rho}{\zeta'(\rho)/\rho} + K(x, T)$$

$$\int_{-\infty+iT}^{\infty+iT} \frac{\mathcal{D}_{1-s/\beta}^{(\beta)}(n)x^s}{\zeta(s)} s ds = \int_{-\infty+iT}^{\infty+iT} \frac{\mathcal{D}_{1-(1-s)/\beta}^{(\beta)}(n)x^{1-s}}{\zeta(1-s)} 1 - s ds$$

$$= \int_{-\infty+iT}^{\infty+iT} \frac{\mathcal{D}_{1-(1-s)/\beta}^{(\beta)}(n)x^{2s-1}\pi^s}{\zeta(s)} 1 ds$$

$$= O\left(\frac{1}{T}\right)^{2N+2} e^{(2N+3)\log(n)+2N+2-(2N+\frac{3}{2})\log(2N+2)dt},$$

where the last two terms are to be bounded. For the second integral, we split the range of integration in $(-\infty + iT, -1 + iT) \cup (-1 + iT, c + iT)$ and we write

$$\int_{-\infty+ iT}^{c+ iT} \frac{\mathcal{D}_{1-s/\beta}^{(\beta)}(n)x^s}{\zeta(s)} s ds = \int_{c+ iT}^{c+ iT} \frac{\mathcal{D}_{1-(1-s)/\beta}^{(\beta)}(n)x^{1-s}}{\zeta(1-s)} 1 - s ds$$

$$= \int_{c+ iT}^{c+ iT} \frac{\mathcal{D}_{1-(1-s)/\beta}^{(\beta)}(n)x^{2s-1}\pi^s}{\zeta(s)} 1 ds$$

$$= O\left(\frac{1}{T}\right)^{2N+2} e^{(2N+3)\log(n)+2N+2-(2N+\frac{3}{2})\log(2N+2)dt},$$

We can now choose for each $\varepsilon > 0$, $T = T_\nu$ satisfying (1.5) and (1.6) such that

$$\frac{1}{\zeta(s)} = O(t^\varepsilon), \quad \frac{1}{2} \leq \sigma \leq 2, \quad t = T_\nu.$$

Thus the other part of the integral is

$$\int_{-1+iT_\nu}^{c+iT_\nu} \frac{\mathcal{D}_{1-s/\beta}^{(\beta)}(n)x^s}{\zeta(s)} s ds = O\left(\int_{-1}^{\min(\beta, c)} T_\nu^{-\varepsilon} e^{(\beta+\varepsilon+1)\log n+\sigma-\frac{1}{2}\varepsilon\log\sigma} ds + \int_{\min(\beta, c)}^{c} T_\nu^{-\varepsilon} x^\varepsilon ds\right) = O(xT_\nu^{-\varepsilon}).$$

The integral over $(2 - iT_\nu, -\infty - iT_\nu)$ is dealt with similarly. It remains to bound $E_{1,T_\nu}(x)$, i.e. the three error terms on the right-hand side of (3.12.1) in [10]. We have $\varphi(q) = \mathcal{D}_1^{(\beta)}(n), s = 0,
$c = 1 + \frac{1}{\log x}$ and $\alpha = 1$. Plugging these yields

$$E_{T_\nu}(x) = E_{2,T_\nu}(x) - E_{1,T_\nu}(x)$$

$$= O(xT_\nu^{-1}) + O\left(\frac{x \log x}{T_\nu}\right) + O\left(\frac{x \mathcal{D}_1^{(\beta)}(n) \log x}{T_\nu}\right) = O\left(\frac{x \log x}{T_\nu^{1-\epsilon}}\right),$$

and this completes the proof.

Note that if we assume, for simplicity, that all non-trivial zeros are simple then term $K_{T_\nu}(x)$ disappears. It now follows from this theorem that

$$\sum \frac{\mathcal{D}_{1-\rho/\beta}^{(\beta)}(n)}{|\rho\zeta'(\rho)|}$$

is divergent. To see this, note that if it were convergent then

$$\sum \frac{\mathcal{D}_{1-\rho/\beta}^{(\beta)}(n) x^\rho}{\rho \zeta'(\rho)}$$

would be uniformly convergent over any finite interval, and this would mean that $\mathcal{C}^{(\beta)}(n,x)$ is continuous, which it is not. Theorem 3.1 can be illustrated by plotting the explicit formula as follows.

**Figure 2.3.** In blue: $\mathcal{C}^{(1)}_{0}(12,x)$, in red: Theorem 3.1 with 5 and 25 pairs of zeros and $5 \leq x \leq 100$.

Increasing the value of $\beta$ does not affect the match. For $\beta = 2$:

**Figure 2.4.** In blue: $\mathcal{C}^{(2)}_{0}(24,x)$, in red: Theorem 3.1 with 5 and 25 pairs of zeros and $1 \leq x \leq 100$.

For $\beta = 3$ we have the same effect.
We can unconditionally extend Cohen’s result to the line $\text{Re}(s) = 1$.

**Theorem 2.2.** For fixed $\beta$ and $n$, we have

\[
\frac{\mathcal{D}_{1-s/\beta}(n)}{\zeta(s)} = \sum_{q=1}^{\infty} \frac{c_q^{(\beta)}(n)}{q^s}
\]

at all points on the line $\text{Re}(s) = 1$.

**Proof.** Once again we shall use the lemma in §3.12 of [10]. Take $a_q = c_q^{(\beta)}(n)$, $\alpha = 1$ and let $x$ be half an odd integer. Let $s = 1 + it$, then

\[
\sum_{q < x} c_q^{(\beta)}(n) q^s = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\mathcal{D}_{1-(s+w)/\beta}(n)}{\zeta(s+w)} \frac{x^{w}}{w} dw + O\left(\frac{x^c}{T} \right) + O\left(\frac{1}{T} \mathcal{D}_{1}^{(\beta)}(n) \log x\right)
\]

\[
\frac{\mathcal{D}_{1-s/\beta}(n)}{\zeta(s)} = \sum_{q=1}^{\infty} \frac{c_q^{(\beta)}(n)}{q^s}
\]

where $c > 0$ and $\delta$ is small enough that $\zeta(s+w)$ has no zeros for

\[
\text{Re}(w) > -\delta, \quad |\text{Im}(s+w)| = |t + \text{Im}(w)| \leq |t| + T.
\]

It is known from §3.6 of [10] that $\zeta(s)$ has no zeros in the region $\sigma > 1 - A \log^{-9} t$, where $A$ is a positive constant. Thus, we can take $\delta = A \log^{-9} T$. The contribution from the vertical integral is given by

\[
\int_{-\delta-iT}^{-\delta+iT} \frac{\mathcal{D}_{1-s/\beta}(n)}{\zeta(s+w)} \frac{x^{w}}{w} dw = O\left(x^{-\delta} n^\delta \log^7 T \int_{-T}^{T} \frac{dv}{\sqrt{T^2 + v^2}}\right) = O\left(x^{-\delta} n^\delta \log^7 T \int_{-T/\delta}^{T/\delta} \frac{dv}{\sqrt{1 + v^2}}\right)
\]

\[
= O(x^{-\delta} n^\delta \log^{8} T).
\]

For the top horizontal integral we get

\[
\int_{-\delta+iT}^{c-iT} \frac{\mathcal{D}_{1-s/\beta}(n)}{\zeta(s+w)} \frac{x^{w}}{w} dw = O\left(\frac{\log^7 T}{T} \left(\int_{-\delta}^{\min(c,\beta-1)} x^{\nu} d\nu + \int_{\min(c,\beta-1)}^{c} x^{\nu} d\nu\right)\right)
\]

\[
= O\left(\frac{\log^7 T}{T} x^{\nu} \left(\int_{-\delta}^{\min(c,\beta-1)} n^{\beta-u} d\nu + c\right)\right) = O\left(\frac{\log^7 T}{T} x^{\nu} n^{\delta}\right),
\]

provided $x > 1$. For the bottom horizontal integral we proceed the same way. Consequently, we have the following

\[
\sum_{q < x} c_q^{(\beta)}(n) q^s - \frac{\mathcal{D}_{1-s/\beta}(n)}{\zeta(s)} = O\left(\frac{x^c}{T} \right) + O\left(\frac{1}{T} \mathcal{D}_{1}^{(\beta)}(n) \log x\right)
\]
$$+ O(x^{-\delta}n^\delta \log T) + O\left(\frac{\log^7 T}{T} x^n \delta\right).$$

Now, we choose $c = 1/\log x$ so that $x^c = e$; and take $T = \exp\{\log x\}^{1/10}$ so that $\log T = (\log x)^{1/10}$, $\delta = A(\log x)^{-9/10}$ and $x^\delta = T^A$. Then it is seen that the right-hand side tends to zero as $x \to \infty$ and the result follows. \(\square\)

As a consequence one obtains a result which is, necessarily, of the same depth as a generalized prime number theorem.

**Corollary 2.4.** One has that

$$\sum_{q=1}^{\infty} \frac{c_q^{(\beta)}}{q^{\beta}} = \frac{D^{(\beta)}_0(n)}{\zeta(\beta)}, \quad \sum_{q=1}^{\infty} \frac{c_q^{(\beta)}}{q} = 0$$

for all $\beta, n \in \mathbb{N}^+$. In particular $\sum_{q=1}^{\infty} c_q(n)/q = 0$ and $\sum_{q=1}^{\infty} \mu(q)/q = 0$.

**Proof.** This follows by making the change $s \to \beta s$ so that

$$\sum_{q=1}^{\infty} \frac{c_q^{(\beta)}}{q^{\beta s}} = \frac{D^{(\beta)}_{1-s}(n)}{\zeta(\beta s)}$$

for $\beta \geq 1$ and, by Theorem 3.2, $\Re(s) \geq 1$. If $s = 1$ then the first equation follows. If in (2.2) we set $s = 1$ then the second equation follows. Setting $s = \beta = 1$ yields the third equation. Finally, putting $n = 1$ in the third equation yields the fourth equation. \(\square\)

The plots of Corollary 2.4 are illustrated below.

**Figure 2.6.** Plot of $\sum_{q=1}^{x} c_q^{(1)}(24)/q$ and of $\sum_{q=1}^{x} c_q^{(2)}(24)/q$ for $1 \leq x \leq 1000$.

**Figure 2.7.** Plot of $\sum_{q=1}^{x} c_q^{(2)}(24)/q^2 - \frac{D^{(2)}_0(24)}{\zeta(2)}$ and of $\sum_{q=1}^{x} c_q^{(3)}(24)/q^3 - \frac{D^{(3)}_0(24)}{\zeta(3)}$ for $1 \leq x \leq 1000$.

It is possible to further extend the validity of (2.3) deeper into the critical strip, however, this is done at the cost of the Riemann hypothesis.
Theorem 2.3. Let $\beta, n \in \mathbb{N}^+$. The Riemann hypothesis is true if and only if

$$\sum_{q=1}^{\infty} \frac{c_q^{(\beta)}(n)}{q^s}$$

is convergent and its sum is $\mathcal{D}_{1-s/\beta}(n)/\zeta(s)$, for every $s$ with $\sigma > \frac{1}{2}$.

This is a generalization of a theorem proved by Littlewood (see [6] and §14.25 of [10]) for the special case where $n = 1$.

Proof of Theorem 2.3. Once again, in the lemma of §3.12 of [10], take $a_q = c_q^{(\beta)}(n)$, $f(s) = \mathcal{D}_{1-s/\beta}(n)/\zeta(s)$, $c = 2$, $x$ half an odd integer. Then

$$\sum_{q<x} \frac{c_q^{(\beta)}(n)}{q^s} = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{\mathcal{D}_{1-s/\beta}(n)}{\zeta(s+w)} \frac{x^w}{w} dw + O \left( \frac{x^2}{T} \right)$$

$$= \frac{1}{2\pi i} \left( \int_{2-iT}^{-\sigma+\delta-iT} + \int_{2+iT}^{-\sigma-\delta+iT} \right) \frac{\mathcal{D}_{1-s/\beta}(n)}{\zeta(s+w)} \frac{x^w}{w} dw$$

$$+ \frac{\mathcal{D}_{1-s/\beta}(n)}{\zeta(s)} + O \left( \frac{x^2}{T} \right),$$

where $0 < \delta < \sigma - \frac{1}{2}$. If we assume the Riemann hypothesis, then $\zeta(s) = O(T^\varepsilon)$ for $\sigma > \frac{1}{2}$ and $\forall \varepsilon > 0$ so that the first and third integrals are

$$O \left( T^{-1+\varepsilon} \int_{-\sigma+\delta}^{\min(\beta-\sigma,2)} n^{\beta-\sigma+v} x^v dv + \int_{\min(\beta-\sigma,2)}^{2} x^v dv \right) = O(T^{-1+\varepsilon} x^2),$$

provided $x > 1$. The second integral is

$$O \left( x^{2-\sigma+\delta} n^{\beta+\delta+1} \int_{-T}^{T} (1+|t|)^{-1+\varepsilon} dt \right) = O(x^{2-\sigma+\delta+T^\varepsilon}).$$

Thus we have

$$\sum_{q<x} \frac{c_q^{(\beta)}(n)}{q^s} = \frac{\mathcal{D}_{1-s/\beta}(n)}{\zeta(s)} + O(x^{2-\sigma+\delta+T^\varepsilon}) + O(x^2 T^{\varepsilon-1}).$$

Taking $T = x^3$ the $O$-terms tend to zero as $x \to \infty$, and the result follows. Conversely, if (2.4) is convergent for $\sigma > \frac{1}{2}$, then it is uniformly convergent for $\sigma > \sigma_0 > \frac{1}{2}$, and so in this region represents an analytic function, which is $\mathcal{D}_{1-s/\beta}(n)/\zeta(s)$ for $\sigma > 1$ and so throughout the region. This means that the Riemann hypothesis is true and the proof is now complete. □

Theorem 2.4. A necessary and sufficient condition for the Riemann hypothesis is

$$\mathcal{C}^{(\beta)}(n, x) = O_{n, \beta}(x^{1+\varepsilon})$$

Proof. Once again, in the lemma of §3.12 of [10], take $a_q = c_q^{(\beta)}(n)$, $f(w) = \mathcal{D}_{1-w/\beta}(n)/\zeta(w)$, $c = 2$, $s = 0$, $\delta > 0$ and $x$ half an odd integer. Then

$$\mathcal{C}^{(\beta)}(n, x) = \sum_{q<x} c_q^{(\beta)}(n) = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{\mathcal{D}_{1-w/\beta}(n)}{\zeta(w)} \frac{x^w}{w} dw + O \left( \frac{x^2 \mathcal{D}_{1-w/\beta}(n)}{T} \right)$$

$$= \frac{1}{2\pi i} \left( \int_{2-iT}^{\frac{1}{2}+\delta-iT} + \int_{\frac{1}{2}+\delta+iT}^{2+iT} \right) \frac{\mathcal{D}_{1-w/\beta}(n)}{\zeta(w)} \frac{x^w}{w} dw + O \left( \frac{x^2}{T} \right)$$

$$+ \frac{\mathcal{D}_{1-w/\beta}(n)}{\zeta(w)} + O \left( \frac{x^2}{T} \right).$$
for each \( \varepsilon > 0 \).

We recall that the von Mangolt function \( \Lambda(k) \) is defined as

\[
\Lambda(k) := \sum_{m \mid k} \mu(m) \log \frac{k}{m}.
\]

Since \( c_q^{(\beta)}(n) \) is a generalization of the Möbius function, a natural question to ask is: how can a new \( \Lambda(n) \) be constructed to incorporate the arithmetic information encoded in the variable \( q \) and the parameter \( \beta \)? To this end we set the following.

**Definition 3.1.** The \( \beta, k, m \in \mathbb{N}^+ \) the COHEN-VON MANGOLDT function is defined as

\[
\Lambda_{k,m}^{(\beta)}(n) = \sum_{d \mid n} c_q^{(\beta)}(m) \log^k \delta.
\]

We note the special case

\[
\Lambda_{1,1}^{(\beta)}(n) = \sum_{d \mid n} \mu(d) \log \delta = \Lambda(n).
\]

for all \( \beta \in \mathbb{N}^+ \). The Dirichlet series are given are given by the following result.

**Lemma 3.1.** For \( \text{Re}(s) > 1 \) and \( \beta, k \in \mathbb{N}^+ \) one has

\[
\sum_{n=1}^{\infty} \frac{\Lambda_{k,m}^{(\beta)}(n)}{n^s} = (-1)^k \mathcal{O}_1^{\beta(-1)}(m) \frac{\zeta(k)(s)}{\zeta'(s)},
\]

where \( \zeta(k)(s) \) is the \( k \)th derivative of the Riemann zeta-function.

**Proof.** By Lemma 2.3, and

\[
\sum_{n=1}^{\infty} \frac{\log^k n}{n^s} = (-1)^k \zeta(k)(s)
\]

for \( \text{Re}(s) > 1 \), the result follows by Dirichlet convolution. \( \square \)

We will, for the sake of simplicity, work with \( k = 1 \) and set \( \Lambda_{1,m}^{(\beta)}(n) = \Lambda_{m}^{(\beta)}(n) \). The generalization for \( k \geq 1 \) is straightforward but requires dealing with results involving (computable) polynomials of degree \( k - 1 \), see for instance §12.4 of [4]. Using once again Perron’s formula we obtain the explicit formula for \( \Lambda_{m}^{(\beta)}(n) \). First, from Lemma 3.1 we deduce that

\[
\Lambda_{m}^{(\beta)}(n) = O(n^{\varepsilon})
\]

for each \( \varepsilon > 0 \), otherwise the sum would not be absolutely convergent for \( \text{Re}(s) > 1 \).
**Definition 3.2.** The Cohen-Chebyshev function is defined by

\[ \psi_{m}^{(\beta)}(x) = \sum_{n \leq x} \Lambda_{m}^{(\beta)}(n), \]

for \( \beta, m \in \mathbb{N}^+ \).

Let us set, as usual,

\[ \psi_{0,m}^{(\beta)}(x) = (\psi_{m}^{(\beta)}(x^+) + \psi_{m}^{(\beta)}(x^-)) / 2. \]

The explicit formula for the Cohen-Chebyshev function is given by the following result.

**Theorem 3.1.** Let \( c > 1, \beta \in \mathbb{N}_+, x \geq m, T \geq 2 \) and let \( \langle x \rangle \) denote the distance from \( x \) to the nearest prime power other than \( x \) itself. Then

\[ \psi_{0,m}^{(\beta)}(x) = \mathcal{D}_{1-1/\beta}(m) x - \sum_{|\gamma| \leq T} \mathcal{D}_{1-\beta/\beta}(m) x^{\beta/\rho} - \mathcal{D}_{1}(m) \log(2\pi) - \sum_{k=1}^{\infty} \mathcal{D}_{1+2k/\beta}(m) x^{2k} / 2k + R(x,T), \]

where

\[ R(x,T) = O\left(x^\varepsilon \min\left(1, \frac{x}{T\langle x \rangle}\right) + \frac{x^{1+\varepsilon} \log x}{T} + \frac{\varepsilon^2 T}{x}\right), \]

for all \( \varepsilon > 0 \).

**Proof.** It is known, see for instance Lemma 12.2 of [8], that for each real number \( T \geq 2 \) there is a \( T_1, T \leq T_1 \leq T + 1 \), such that

\[ \frac{\zeta'}{\zeta}(\sigma + iT_1) \ll (\log T)^2 \]

uniformly for \(-1 \leq \sigma \leq 2\). By using Perron’s inversion formula with \( \sigma_0 = 1 + 1/\log x \) we obtain

\[ \psi_{0,m}^{(\beta)}(x) = -\frac{1}{2\pi i} \int_{\sigma_0 - iT_1}^{\sigma_0 + iT_1} \mathcal{D}_{1-\beta}(m) \frac{\zeta'}{\zeta}(s) x^s / s ds + R_1, \]

where

\[ R_1 = O\left(\sum_{x/2 < n < 2x \atop n \neq x} \Lambda_{m}^{(\beta)}(n) \min\left(1, \frac{x}{T|x - n|}\right) + \frac{x}{T} \sum_{n=1}^{\infty} \frac{\Lambda_{m}^{(\beta)}(n)}{n^{\sigma_0}}\right). \]

The second sum is

\[ -\mathcal{D}_{1-\sigma_0/\beta}(m) \frac{\zeta'}{\zeta}(\sigma_0) \gg \mathcal{D}_{1-\sigma_0/\beta}(m) = \mathcal{D}_{1-(1+1/\log x)/\beta}(m) \log x. \]

The term involving the generalized divisor function can be bounded in the following way:

\[ \mathcal{D}_{1-(1+1/\log x)/\beta}(m) \leq m^{\beta+1/\log x} \]

if \( \frac{1}{\log x} \leq \beta - 1 \), and \( \leq m \) otherwise. In both cases, this is bounded in \( x \). The first sum requires some more work. The terms for which \( x + 1 \leq n < 2x \) contribute an amount which is

\[ O\left(\sum_{x+1 \leq n < 2x} x^{1+\varepsilon}/T(n-x)\right) = O\left(\frac{x^{1+\varepsilon} \log x}{T}\right). \]

The terms for which \( x/2 < n \leq x - 1 \) are dealt with in a similar way. The remaining terms for which \( x - 1 < n < x + 1 \) contribute an amount which is

\[ O\left(x^\varepsilon \min\left(1, \frac{x}{T\langle x \rangle}\right)\right), \]

therefore, the final bound for \( R_1 \) is

\[ R_1 = O\left(x^\varepsilon \min\left(1, \frac{x}{T\langle x \rangle}\right) + \frac{x^{1+\varepsilon} \log x}{T}\right). \]
We denote by $N$ an odd positive integer and by $\mathcal{D}$ the contour consisting of line segments connecting $\sigma_0 - iT_1, -N - iT_1, -N + iT_1, \sigma_0 + iT_1$. An application of Cauchy’s residue theorem yields
\[
\psi_0^{(\beta)}(x) = M_0 + M_1 + M_0 + M_{-2k} + R_1 + R_2
\]
where the $M$’s are the residues at $s = 0, s = 1$, the non-trivial zeros $\rho$ and at the trivial zeros $-2k$ for $k = 1, 2, 3, \cdots$ and where
\[
R_2 = -\frac{1}{2\pi i} \int_{\mathcal{D}} \frac{\zeta'(s)}{\zeta} \frac{d^s}{ds} ds.
\]
The residues are easily evaluated. For the constant term we have
\[
M_0 = \res_{s=0} \mathcal{D}^{(\beta)}_{1-s/\beta}(m) \frac{\zeta'}{\zeta}(s) \frac{x^s}{s} = \mathcal{D}^{(\beta)}_1(m) \frac{\zeta'}{\zeta}(0) = \mathcal{D}^{(\beta)}_1(m) \log(2\pi),
\]
and for the leading term
\[
M_1 = \res_{s=1} \mathcal{D}^{(\beta)}_{1-s/\beta}(m) \frac{\zeta'}{\zeta}(s) \frac{x^s}{s} = \mathcal{D}^{(\beta)}_{1-1/\beta}(m) x.
\]
The fluctuating term coming from the non-trivial zeros yields
\[
M_{-2k} = \sum_{k=1}^{\infty} \res_{s=-2k} \mathcal{D}^{(\beta)}_{1-s/\beta}(m) \frac{\zeta'}{\zeta}(s) \frac{x^s}{s} = \sum_{k=1}^{\infty} \mathcal{D}^{(\beta)}_{1+2k/\beta}(m) \frac{x^{-2k}}{-2k}.
\]
Since $|\sigma \pm iT_1| \geq T$, we see, by our choice of $T_1$, that
\[
\int_{-1 \pm iT_1}^{\sigma_0 \pm iT_1} \mathcal{D}^{(\beta)}_{1-s/\beta}(m) \frac{\zeta'}{\zeta}(s) \frac{x^s}{s} ds = O \left( \log^2 \frac{2T}{T} \left( \int_{-1}^{\min(\beta, \sigma_0)} \left( \frac{x}{m} \right)^{\sigma} d\sigma + \int_{\min(\beta, \sigma_0)}^{\sigma_0} x^\rho d\sigma \right) \right)
\]
\[
= O \left( \frac{x \log^2 T}{T \log x} \right) = O \left( \frac{x \log^2 T}{T} \right).
\]
Next, we invoke the following result, see Lemma 12.4 of [8]: if $A$ denotes the set of points $s \in \mathbb{C}$ such that $\sigma \leq -1$ and $|s + 2k| \geq 1/4$ for every positive integer $k$, then
\[
\frac{\zeta'}{\zeta}(s) = O(\log(|s| + 1))
\]
uniformly for $s \in A$. This, combined with the fact that
\[
\frac{\log |\sigma \pm iT_1|}{|\sigma \pm iT_1|} = O \left( \frac{\log T}{T} \right),
\]
gives us
\[
\int_{-N \pm iT_1}^{1 \pm iT_1} \mathcal{D}^{(\beta)}_{1-s/\beta}(m) \frac{\zeta'}{\zeta}(s) \frac{x^s}{s} ds = O \left( \frac{\log T}{T} \int_{-N}^{-1} \left( \frac{x}{m} \right)^{\sigma} d\sigma \right) = O \left( \frac{\log T}{xT \log x} \right) = O \left( \frac{\log T}{T} \right).
\]
Thus this bounds the horizontal integrals. Finally, for the left vertical integral, we have that $|-N + iT| \geq N$ and by the above result regarding the bound of the logarithmic derivative we also see that
\[
\int_{-N \pm iT_1}^{N \pm iT_1} \mathcal{D}^{(\beta)}_{1-s/\beta}(m) \frac{\zeta'}{\zeta}(s) \frac{x^s}{s} ds = O \left( \frac{\log NT}{N} \frac{x}{m} \mathcal{D}^{(\beta)}_{1+N/\beta}(m) \right) = O \left( \frac{T \log NT}{N} \right).
\]
This last term goes to 0 as $N \to \infty$ since $x \geq m$. This proves the theorem. □
Finally, we can bound the sum
\[ \sum_{k=1}^{\infty} \mathcal{D}^{(\beta)}_{1+2k/\beta}(m) \frac{x^{-2k}}{2k} \leq m^{\beta+1} \sum_{k=1}^{\infty} m^{2k} \frac{x^{-2k}}{2k} = \frac{1}{2} m^{\beta+1} \log \left( 1 - \left( \frac{x}{m} \right)^{-2} \right) = o(1). \]

Pictorially, one has the usual match in the explicit formula.

![Figure 3.1](image.png)

**Figure 3.1.** In blue: \( \psi_{0,12}^{(2)} \), in red: Theorem 3.1 with 5 and 25 pairs respectively of zeros for \( 1 \leq x \leq 50 \).

We now follow Chapter 18 of [4]. Let us denote by \( \rho = \beta^* + i\gamma \) a non-trivial zero. With this in mind, we aim at proving the prime number theorem for \( \psi_m^{(\beta)}(x) \). For this, we will use the result that if \( |\gamma| < T \), where \( T \) is large, then \( \beta^* < 1 - c_1/\log T \), where \( c_1 \) is a positive absolute constant. This immediately yields
\[ |x^\rho| = x^{\beta^*} < xe^{-c_1 \log x/\log T}. \]

Moreover, \( |\rho| \geq \gamma \), for \( \gamma > 0 \). We recall that the number of zeros \( N(t) \) up to height \( t \) is
\[ N(t) = \frac{t}{2\pi} \log \frac{t}{2\pi} - \frac{t}{2\pi} + O(\log t) = O(t \log t). \]

We need to estimate the following sum
\[ \sum_{0<\gamma<T} \frac{\mathcal{D}_{1-\gamma/\beta}^{(\beta)}(m)}{\gamma}. \]

This is
\[ O \left( \int_0^T \frac{\mathcal{D}_{1-t/\beta}^{(\beta)}(m)}{t} N(t)dt \right) = O \left( m^{\frac{\beta}{2}} \int_0^T \frac{\log t}{t} dt + m^{\beta+1} \int_T^\infty \frac{\log t}{tm^t} dt \right) = O(\log^2 T). \]

Therefore,
\[ \sum_{|\gamma|<T} \left| \mathcal{D}_{1-\gamma/\beta}^{(\beta)}(m) \frac{x^\rho}{\rho} \right| = O(x(\log T)^2 e^{-c_1 \log x/\log T}). \]

Without loss of generality we take \( x \) to be an integer in which case the error term of the explicit formula of Theorem 3.1 becomes
\[ R(x,T) = O \left( \frac{x^{1+\varepsilon} \log x}{T} + \frac{x \log^2 T}{T} \right). \]

Finally, we can bound the sum
\[ \sum_{k=1}^{\infty} \mathcal{D}^{(\beta)}_{1+2k/\beta}(m) \frac{x^{-2k}}{2k} \leq m^{\beta+1} \sum_{k=1}^{\infty} m^{2k} \frac{x^{-2k}}{2k} = \frac{1}{2} m^{\beta+1} \log \left( 1 - \left( \frac{x}{m} \right)^{-2} \right) = o(1). \]

Thus, we have the following
\[ |\psi_m^{(\beta)}(x) - \mathcal{D}_{1-1/\beta}^{(\beta)}(m)x| = O \left( \frac{x^{1+\varepsilon} \log x}{T} + \frac{x \log^2 T}{T} + x(\log T)^2 e^{-c_1 \log x/\log T} \right), \]

for large \( x \). Let us now take \( T \) as a function of \( x \) by setting \( (\log T)^2 = \log x \) so that
\[ |\psi_m^{(\beta)}(x) - \mathcal{D}_{1-1/\beta}^{(\beta)}(m)x| = O(x^{1+\varepsilon} \log x e^{-(\log x)^{1/2}} + x(\log x)e^{-c_1(\log x)^{1/2}}) = O(x^{1+\varepsilon} e^{-c_2(\log x)^{1/2}}), \]
for all $\varepsilon > 0$ provided that $c_0$ is a suitable constant that is less than both 1 and $c_1$. This proves the “prime number theorem” associated to $\Lambda_m^{(\beta)}(n)$.

Next, if we assume the Riemann hypothesis, then $|x^\rho| = x^{1/2}$ and the other estimate regarding $\sum \mathcal{D}^{(\beta)}_{1-\gamma/\beta}(m)\gamma^{-1}$ stays the same. Thus, the explicit formula yields

$$|\psi_m^{(\beta)}(x) - \mathcal{D}^{(\beta)}_{1-1/\beta}(m)x| = O\left(x^{1/2}\log^2 T + \frac{x^{1+\varepsilon}\log x}{T} + \frac{x\log^2 T}{T}\right)$$

provided that $x$ is an integer. Taking $T = x^{1/2}$ leads us to the optimal error term

$$\psi_m^{(\beta)}(x) = \mathcal{D}^{(\beta)}_{1-1/\beta}(m)x + O(x^{1/2}\log^2 x + x^{1/2+\varepsilon}\log x) = \mathcal{D}^{(\beta)}_{1-1/\beta}(m)x + O(x^{1/2+\varepsilon})$$

for each $\varepsilon > 0$.

4. A FUNCTIONAL EQUATION FOR A FUNCTION INVOLVING THE RESIDUES

Let us set $\mathbb{H} = \{x + iy, x \in \mathbb{R}, y > 0\}$.

**Definition 4.1.** Suppose that $z \in \mathbb{H}$, define the **Bartz-Cohen** function by

$$\varpi_n^{(\beta)}(z) = \lim_{m \to \infty} \sum_{0 < \text{Im} \rho < T_m} \frac{\mathcal{D}^{(\beta)}_{1-s/\beta}(n)}{\zeta'(\rho)} e^{\rho z}. \quad (4.1)$$

**Remark 4.1.** This definition does not depend on a bracketing condition. However, the following theorems will, thus we will adapt the bracketing condition given by (1.7).

The goal of this section is to describe the analytic character of $\varpi_n^{(\beta)}(z)$. Specifically, we will construct its analytic continuation to a meromorphic function of $z$ on the whole complex plane and prove that it satisfies a functional equation. This functional equation takes into account values of $\varpi_n^{(\beta)}(z)$ at $z$ and at $\bar{z}$; therefore one may deduce the behavior of $\varpi_n^{(\beta)}(z)$ for $\text{Im}(z) < 0$.

Finally, we will study the singularities and residues of $\varpi_n^{(\beta)}(z)$. To this end, we now look at the contour integral

$$\Upsilon^{(\beta)}(n, z) = \oint_{\Omega} \frac{\mathcal{D}^{(\beta)}_{1-s/\beta}(n)}{\zeta'(s)} e^{sz} ds$$

taken around the path $\Omega = [-1/2, 3/2, 3/2 + iT_n, -1/2 + iT_n]$. For the upper horizontal integral we have

$$\left| \int_{-1/2 + iT_n}^{3/2} \frac{\mathcal{D}^{(\beta)}_{1-s/\beta}(n)}{\zeta(s)} e^{sz} ds \right| \leq \min(\beta, 3/2) \int_{-1/2}^{3/2} \left| \frac{\mathcal{D}^{(\beta)}_{1-s/\beta}(n)e^{sz}}{\zeta'(\sigma + iT_n)} \right| d\sigma + \int_{-1/2}^{3/2} \left| \frac{\mathcal{D}^{(\beta)}_{1-s/\beta}(n)e^{sz}}{\zeta'(\sigma + iT_n)} \right| d\sigma$$

$$= O\left(T_m^{\beta - 1} e^{-T_m y} \int_{-1/2}^{3/2} n^{-\sigma} e^{\sigma x} d\sigma + n e^{-T_m y} \int_{-1/2}^{3/2} e^{\sigma x} d\sigma \right) \to 0$$

as $m \to \infty$. An application of Cauchy’s residue theorem yields

$$\varpi_n^{(\beta)}(z) = \lim_{m \to \infty} \sum_{0 < \text{Im} \rho < T_m} \frac{1}{(k\rho - 1)!} \frac{d^{k\rho - 1}}{ds^{k\rho - 1}} \left[ (s - \rho)^{k\rho} \frac{\mathcal{D}^{(\beta)}_{1-s/\beta}(n)}{\zeta'(\rho)} e^{sz} \right]_{s = \rho} \quad (4.2)$$
with $k_\rho$ denoting the order of multiplicity of the non-trivial zero $\rho$ of the Riemann zeta-function.

We denote by $\varpi_{1,n}^{(\beta)}(z)$ and by $\varpi_{2,n}^{(\beta)}(z)$ the first and second integrals on the left hand-side of (4.2) respectively. If we operate under assumption that there are no multiple zeros, then the above can be simplified to (4.1). This is done for the sake of clarity, since straightforward modifications are needed to relax this assumption.

**Theorem 4.1.** The function $\varpi_{n}^{(\beta)}(z)$ is holomorphic on the upper half-plane $\mathbb{H}$ and for $z \in \mathbb{H}$ we have

$$2\pi i \varpi_{n}^{(\beta)}(z) = \varpi_{1,n}^{(\beta)}(z) + \varpi_{2,n}^{(\beta)}(z) - e^{3z/2} \sum_{q=1}^{\infty} \frac{c_{q}^{(\beta)}(n)}{q^{3/2}(z - \log q)}$$

where the last term on the right hand-side is a meromorphic function on the whole complex plane with poles at $z = \log q$ whenever $c_{q}^{(\beta)}(n)$ is not equal to zero. Moreover, $\varpi_{1,n}^{(\beta)}(z)$ is analytic on $\mathbb{H}$ and $\varpi_{2,n}^{(\beta)}(z)$ is regular on $\mathbb{C}$.

**Proof.** If $z \in \mathbb{H}$ then by (4.2) one has

$$2\pi i \varpi_{n}^{(\beta)}(z) = \varpi_{1,n}^{(\beta)}(z) + \varpi_{2,n}^{(\beta)}(z) + \varpi_{3,n}^{(\beta)}(z),$$

where the last term is given by the vertical integral on the right of the $\Omega$ contour

$$\varpi_{3,n}^{(\beta)}(z) = \int_{3/2}^{3/2+i\infty} \frac{\mathcal{D}_{1-s/\beta}(n)}{\zeta(s)} e^{sz} ds.$$

By the use of the Dirichlet series of $c_{q}^{(\beta)}(n)$ given in Lemma 2.3 and since we are in the region of absolute convergence we see that

$$\varpi_{3,n}^{(\beta)}(z) = \sum_{q=1}^{\infty} c_{q}^{(\beta)}(n) \int_{3/2}^{3/2+i\infty} e^{sz-s\log q} ds = -e^{3z/2} \sum_{q=1}^{\infty} \frac{c_{q}^{(\beta)}(n)}{q^{3/2}(z - \log q)}.$$

By standard bounds of Stirling and the functional equation of the Riemann zeta-function we have that

$$\frac{1}{\zeta(-\frac{1}{2} + it)} = O(1),$$

as $|t| \to \infty$. Therefore, we see that

$$|\varpi_{1,n}^{(\beta)}(z)| = \left| \int_{-1/2}^{-1/2+i\infty} \frac{\mathcal{D}_{1-s/\beta}(n)}{\zeta(s)} e^{sz} ds \right| = O \left( n^{\beta+3/2} e^{-z/2} \int_{0}^{\infty} e^{-ty} dt \right) = O \left( \frac{n^{\beta+3/2} e^{-x/2}}{y} \right)$$

and $\varpi_{1,n}^{(\beta)}(z)$ is absolutely convergent for $y = \text{Im}(z) > 0$. We know that $\varpi_{n}^{(\beta)}(z)$ is analytic for $y > 0$ and the next step is to show that it can be meromorphically continued for $y > -\pi$. To this end, we go back to the integral

$$\varpi_{1,n}^{(\beta)}(z) = -\int_{-1/2}^{-1/2+i\infty} \frac{\mathcal{D}_{1-s/\beta}(n)}{\zeta(1-s)} e^{sz} ds$$

with $y > 0$. The functional equation of $\zeta(s)$ yields

$$\varpi_{1,n}^{(\beta)}(z) = -\int_{-1/2}^{-1/2+i\infty} \frac{\mathcal{D}_{1-s/\beta}(n)}{\zeta(1-s)} \Gamma(s) e^{s(z-\log 2\pi-i\pi/2)} ds$$

$$\varpi_{1,n}^{(\beta)}(z) = -\int_{-1/2}^{-1/2+i\infty} \frac{\mathcal{D}_{1-s/\beta}(n)}{\zeta(1-s)} \Gamma(s) e^{s(z-\log 2\pi+i\pi/2)} ds$$

(4.3)
Since one has by standard bounds that
\[ \frac{\Gamma\left(-\frac{1}{2}+it\right)}{\zeta\left(\frac{3}{2}-it\right)} = O(e^{-\pi t/2}) \]
it then follows that
\[ \Phi^{(\beta)}_{11,n}(z) = O\left(n^{3+\beta/2} e^{-\pi t/2} e^{z-ty+\pi t/2} \right) = O\left(e^{-x/2 n^{3+\beta/2}} \right), \]
and hence \( \Phi^{(\beta)}_{11,n}(z) \) is regular for \( y > 0 \). Similarly,
\[ \Phi^{(\beta)}_{12,n}(z) = O\left(n^{3+\beta/2} e^{-\pi t/2} \sum_{0}^{\infty} e^{-(\pi+y)t} dt \right) = O\left(n^{3+\beta/2} e^{-x/2} \right), \]
so that \( \Phi^{(\beta)}_{12,n}(z) \) is regular for \( y > -\pi \). Let us further split \( \Phi^{(\beta)}_{11,n}(z) \)
\[ \Phi^{(\beta)}_{11,n}(z) = \left( \sum_{-1/2+i\infty}^{-1/2-i\infty} \right) e^{s(z-\log 2\pi-\pi i/2)} \mathcal{D}^{(\beta)}_{1-s/\beta}(n) \frac{\Gamma(s)}{\zeta(1-s)} ds = I^{(\beta)}_{1,n}(z) + I^{(\beta)}_{2,n}(z). \]

By the same technique as above, it follows that the integral \( I^{(\beta)}_{2,n}(z) \) is convergent for \( y \leq \pi \).
Moreover, since \( \Phi^{(\beta)}_{11,n}(z) \) is regular for \( y > 0 \), then it must be that \( I^{(\beta)}_{1,n}(z) \) is convergent for \( 0 < y < \pi \). Let
\[ f(n,q,s,z) = \mathcal{D}^{(\beta)}_{1-s/\beta}(n) e^{s(z-\log 2\pi-\pi i/2+\log q) \Gamma(s)}. \]

By the theorem of residues we see that
\[ \sum_{-1/2+i\infty}^{-1/2-i\infty} f(n,q,s,z) ds = \sum_{1-i\infty}^{1+i\infty} f(n,q,s,z) ds + 2\pi i \text{ Res} f(n,q,s,z) \]
\[ = \sum_{1-i\infty}^{1+i\infty} f(n,q,s,z) ds + 2\pi i \mathcal{D}^{(\beta)}_{1}(n). \]

This last integral is equal to
\[ \sum_{1-i\infty}^{1+i\infty} f(n,q,s,z) ds = 2\pi i \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( e^{-z} \frac{2\pi i}{q} \right)^k \mathcal{D}^{(\beta)}_{1+k/\beta}(n), \]
where the last sum is absolutely convergent. To prove this, first, note that
\[ \text{Re}(e^{-(z-\log 2\pi-\pi i/2+\log q)}) = (e^{-x/2} \pi/q) \sin y > 0 \]
for \( 0 < y < \pi \). Next, consider the path of integration with vertices \((1 \pm iT)\) and \((-N \pm iT)\),
where \( N \) is an odd positive integer. Once again, by Cauchy’s theorem
\[ \left( \sum_{1+iT}^{1+iT} - \sum_{-N+iT}^{-N+iT} \right) \left( e^{z} \frac{2\pi i}{q} \right)^{-s} \mathcal{D}^{(\beta)}_{1-s/\beta}(n) \Gamma(s) ds \]
\[ = 2\pi i \sum_{k=0}^{N-1} \frac{(-1)^k}{k!} \left( e^{-z} \frac{2\pi i}{q} \right)^k \mathcal{D}^{(\beta)}_{1+k/\beta}(n). \]
The third integral on the far left of the path can be bounded in the following way
\[ \int_{-N-iT}^{-N+iT} \left( e^{-z} \frac{2\pi i}{q} \right)^{-s} \mathcal{D}^{(\beta)}_{1-s/\beta}(n) \Gamma(s) ds = O\left( \int_{-T}^{T} e^{-x} \frac{2\pi i}{q} e^{(y-\frac{\pi}{2})} n^{\beta+N+1} e^{-\frac{\pi}{2} |t|} dt \right) \]
\[ = O\left( e^{-x} \frac{2\pi i}{q} \sum_{-T}^{T} e^{(y-\frac{\pi}{2})} e^{-\frac{\pi}{2} |t|} dt \right) \]
It is now easy to see that all of the three parts tend to 0 as \( N \) thus the result follows. Thus, putting together (4.5) with (4.3) and (4.4) gives us

\[
T > \frac{N(x - \log \frac{2\pi}{qN})}{\min(y, \pi - y)},
\]

We now bound the horizontal parts. For the top one

\[
\int_{-N-iT}^{1+iT} \left( e^{-\frac{2\pi i}{q}} \right)^{-s} D_{1-s/\beta}(n) \Gamma(s) ds = O \left( \int_{-N}^{1} \left( e^{-\frac{2\pi}{qN}} \right) \sigma T e^{T(y-\pi)} \right.
\]

\[
= O \left( T^2 e^{T(y-\pi)} \int_{-N}^{1} \left( e^{-\frac{2\pi}{qN}} \right) \sigma T e^{T y} \right.
\]

\[
= O \left( T^2 e^{T(y-\pi)} \left( e^{-\frac{2\pi}{qN}} \right)^{-N} \right) = O(T^2 e^{N\frac{2\pi}{qN} y}).
\]

and analogously for the bottom one

\[
\int_{-N-iT}^{-1+iT} \left( e^{-\frac{2\pi i}{q}} \right)^{-s} D_{1-s/\beta}(n) \Gamma(s) ds = O \left( \int_{-N}^{1} \left( e^{-\frac{2\pi}{qN}} \right) \sigma T e^{T(y-\pi)} \right.
\]

\[
= O \left( T^2 e^{-T y} \left( e^{-\frac{2\pi}{qN}} \right)^{-N} \right) = O(T^2 e^{-N \log \frac{2\pi}{qN} y}).
\]

Let now \( T = T(N) \) such that

\[
T > \frac{N(x - \log \frac{2\pi}{qN})}{\min(y, \pi - y)}.
\]

It is now easy to see that all of the three parts tend to 0 as \( N \to \infty \) through odd integers, and thus the result follows. Thus, putting together (4.5) with (4.3) and (4.4) gives us

\[
I_{1,n}(z) = -\int_{-1/2-i\infty}^{-1/2+i\infty} e^{s(z-\log 2\pi - i\pi/2)} \frac{D_{1-s/\beta}(n) \Gamma(s)}{\zeta(1-s)} ds
\]

\[
= -\sum_{q=1}^{\infty} \frac{\mu(q)}{q} \int_{-1/2-i\infty}^{-1/2+i\infty} e^{s(z-\log 2\pi - i\pi/2)} q^s D_{1-s/\beta}(n) \Gamma(s) ds
\]

\[
= -\sum_{q=1}^{\infty} \frac{\mu(q)}{q} \left( 2\pi i \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( e^{-\frac{2\pi i}{q}} \right)^k D_{1-k/\beta}(n) - 2\pi i D_{1}(n) \right)
\]

(4.6)

since \( \sum_{q=1}^{\infty} \mu(q)/q = 0 \). Moreover,

\[
|(2\pi i)^{-1} I_{1,n}(z)| = \left| \sum_{q=1}^{\infty} \frac{\mu(q)}{q} \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( e^{-\frac{2\pi i}{q}} \right)^k D_{1-k/\beta}(n) - D_{1}(n) \right) \right|
\]

\[
\leq n^{\beta+1} \sum_{q=1}^{\infty} \frac{1}{q} \left( \sum_{k=1}^{\infty} \frac{1}{k!} \left( e^{-\frac{2\pi i}{q}} \right)^k \right) \left( e^{-\frac{2\pi i}{q}} \right)^n
\]

\[
= n^{\beta+1} \sum_{q=1}^{\infty} \frac{1}{q} \left( \exp \left( e^{-\frac{2\pi i}{q}} \right) - 1 \right)
\]
and the series on the right hand-side of (4.6) is absolutely convergent for all \( y < \pi \). Thus, this proves the analytic continuation of \( \varpi^{(\beta)}_{1,n}(z) \) to \( y > -\pi \). For \( |y| < \pi \) one has

\[
\varpi^{(\beta)}_{1,n}(z) = I^{(\beta)}_{1,n}(z) + I^{(\beta)}_{2,n}(z) + \varpi^{(\beta)}_{12,n}(z)
\]

\[
= -2\pi i \sum_{q=1}^{\infty} \frac{\mu(q)}{q} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( e^{-\frac{2\pi i}{q}} \right)^k \mathcal{D}^{(\beta)}_{1+k/\beta}(n)
\]

\[
+ -1/2 \int_{-1/2-i\infty}^{-1/2+i\infty} e^{s(z-\log 2\pi-\pi/2)} \mathcal{D}^{(\beta)}_{1-s/\beta}(n) \frac{\Gamma(s)}{\zeta(1-s)} \, ds
\]

\[
- -1/2 \int_{-1/2-i\infty}^{-1/2+i\infty} e^{s(z-\log 2\pi+\pi/2)} \mathcal{D}^{(\beta)}_{1-s/\beta}(n) \frac{\Gamma(s)}{\zeta(1-s)} \, ds
\]

where the first term is holomorphic for all \( y \), the second one for \( y < \pi \) and the third for \( y > -\pi \). Hence, this last equation shows the continuation of \( \varpi^{(\beta)}_{n}(z) \) to the region \( y > -\pi \). To complete the proof of the theorem, one then considers the function

\[
\hat{\varpi}^{(\beta)}_{n}(z) = \lim_{m \to \infty} \sum_{\rho \in \mathbb{C}} \frac{\mathcal{D}^{(\beta)}_{1-\rho/\beta}(n)}{\zeta'(\rho)} e^{\rho z},
\]

where the zeros are in the lower part of the critical strip and \( z \) now belongs to the lower half-plane \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) < 0 \} \). It then follows by repeating the above argument that

\[
\hat{\varpi}^{(\beta)}_{1,n}(z) = \hat{\varpi}^{(\beta)}_{11,n}(z) + \hat{\varpi}^{(\beta)}_{12,n}(z),
\]

where

\[
\hat{\varpi}^{(\beta)}_{11,n}(z) = - \int_{-1/2-i\infty}^{-1/2+i\infty} e^{s(z-\log 2\pi-\pi/2)} \mathcal{D}^{(\beta)}_{1-s/\beta}(n) \frac{\Gamma(s)}{\zeta(1-s)} \, ds
\]

is absolutely convergent for \( y < \pi \) and

\[
\hat{\varpi}^{(\beta)}_{12,n}(z) = - \int_{-1/2-i\infty}^{-1/2+i\infty} e^{s(z-\log 2\pi+\pi/2)} \mathcal{D}^{(\beta)}_{1-s/\beta}(n) \frac{\Gamma(s)}{\zeta(1-s)} \, ds
\]

is absolutely convergent for \( y < 0 \). Splitting up the first integral just as before and using a similar analysis to the one we have just carried out, but using the fact that \( \zeta(s) = \zeta(s) \) and choosing \( T_m (m \leq T_m \leq m + 1) \) such that

\[
\left| \frac{1}{\zeta(\sigma - iT_m)} \right| < T_m^{e^1}, \quad -1 \leq \sigma \leq 2,
\]

yields that

\[
\hat{\varpi}^{(\beta)}_{1,n}(z) = -2\pi i \sum_{q=1}^{\infty} \frac{\mu(q)}{q} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( e^{-\frac{2\pi i}{q}} \right)^k \mathcal{D}^{(\beta)}_{1+k/\beta}(n)
\]

\[
- -1/2 \int_{-1/2-i\infty}^{-1/2+i\infty} e^{s(z-\log 2\pi-\pi/2)} \mathcal{D}^{(\beta)}_{1-s/\beta}(n) \frac{\Gamma(s)}{\zeta(1-s)} \, ds
\]
Consequently, \( \hat{\varphi}_n^{(\beta)}(z) \) admits an analytic continuation from \( y < 0 \) to the half-plane \( y < \pi \).

**Theorem 4.2.** The function \( \varphi_n^{(\beta)}(z) \) can be continued analytically to a meromorphic function on \( \mathbb{C} \) which satisfies the functional equation

\[
\varphi_n^{(\beta)}(z) + \varphi_n^{(\beta)}(\bar{z}) = A_n^{(\beta)}(z) = -\sum_{q=1}^{\infty} \frac{\mu(q)}{q} \sum_{k=0}^{\infty} \frac{1}{k!} \left\{ \left( e^{-z} \frac{2\pi i}{q} \right)^k + \left( -e^{-z} \frac{2\pi i}{q} \right)^k \right\} \mathcal{D}_{1+k/\beta}(n),
\]

where the function \( A_n^{(\beta)}(z) \) is entire and satisfies

\[
A_n^{(\beta)}(z) = 2 \sum_{k=1}^{\infty} \frac{(-1)^k (2\pi)^{2k} e^{-2k\pi i z} \mathcal{D}_{1+k/\beta}(n)}{(2k)!} \zeta(1+2k).
\]

**Proof.** Adding up our two results

\[
\varphi_{1,n}^{(\beta)}(z) + \hat{\varphi}_{1,n}^{(\beta)}(z) = -2\pi i \sum_{q=1}^{\infty} \frac{\mu(q)}{q} \sum_{k=0}^{\infty} \frac{1}{k!} \left\{ \left( e^{-z} \frac{2\pi i}{q} \right)^k + \left( -e^{-z} \frac{2\pi i}{q} \right)^k \right\} \mathcal{D}_{1+k/\beta}(n).
\]

The other terms do not contribute since

\[
\varphi_{2,n}^{(\beta)}(z) + \hat{\varphi}_{2,n}^{(\beta)}(z) = \left( \int_{-1/2}^{3/2} \left( \int_{-1/2}^{1/2} \right) e^{\sum_{k=1}^{\infty} \frac{D_{1-s/\beta}(n)}{s^{1+2k}}} ds \right) = 0,
\]

and by the Theorem 4.1 we have

\[
\varphi_{3,n}^{(\beta)}(z) + \hat{\varphi}_{3,n}^{(\beta)}(z) = 0.
\]

Consequently, we have

\[
\varphi_n^{(\beta)}(z) + \hat{\varphi}_n^{(\beta)}(z) = -\sum_{q=1}^{\infty} \frac{\mu(q)}{q} \sum_{k=0}^{\infty} \frac{1}{k!} \left\{ \left( e^{-z} \frac{2\pi i}{q} \right)^k + \left( -e^{-z} \frac{2\pi i}{q} \right)^k \right\} \mathcal{D}_{1+k/\beta}(n)
\]

for \( |y| < \pi \). Thus, once again, by the previous theorem for all \( y < \pi \)

\[
\varphi_n^{(\beta)}(z) = -\hat{\varphi}_n^{(\beta)}(z) = -\sum_{q=1}^{\infty} \frac{\mu(q)}{q} \sum_{k=0}^{\infty} \frac{1}{k!} \left\{ \left( e^{-z} \frac{2\pi i}{q} \right)^k + \left( -e^{-z} \frac{2\pi i}{q} \right)^k \right\} \mathcal{D}_{1+k/\beta}(n)
\]

by analytic continuation, and for \( y > -\pi \)

\[
\hat{\varphi}_n^{(\beta)}(z) = -\varphi_n^{(\beta)}(z) = -\sum_{q=1}^{\infty} \frac{\mu(q)}{q} \sum_{k=0}^{\infty} \frac{1}{k!} \left\{ \left( e^{-z} \frac{2\pi i}{q} \right)^k + \left( -e^{-z} \frac{2\pi i}{q} \right)^k \right\} \mathcal{D}_{1+k/\beta}(n).
\]

This shows that \( \varphi_n^{(\beta)}(z) \) and \( \hat{\varphi}_n^{(\beta)}(z) \) can be analytically continued over \( \mathbb{C} \) as a meromorphic function and that for all \( z \) and consequently \( \mathbb{D}_{1-s/\beta} \) holds for all \( z \). To prove the functional equation, we look at the zeros. If \( \rho \) is a non-trivial zero of \( \zeta(s) \) then so is \( \bar{\rho} \). For \( z \in \mathbb{H} \) one has

\[
\varphi_n^{(\beta)}(z) = \lim_{m \to \infty} \sum_{\rho \in \mathbb{H} \text{ with } 0<\text{Im}\rho<T_m} \frac{\mathcal{D}_{1-\rho/\beta}(n)}{\zeta'(\rho)} e^{\rho z}.
\]

By using \( \mathcal{D}_{1-\rho/\beta}(n) = \mathcal{D}_{1-\bar{\rho}/\beta}(n) \) and since \( \zeta(\bar{s}) = \overline{\zeta(s)} \) we get

\[
\varphi_n^{(\beta)}(z) = \sum_{\rho \in \mathbb{H} \text{ with } 0<\text{Im}\rho<T_m} \frac{\mathcal{D}_{1-\rho/\beta}(n)}{\zeta'(\rho)} e^{\rho z} = \sum_{\rho \in \mathbb{H} \text{ with } 0<\text{Im}\rho<T_m} \frac{\mathcal{D}_{1-\bar{\rho}/\beta}(n)}{\zeta'(\bar{\rho})} e^{\bar{\rho} z}.
\]
as well as

\[ \zeta^{(\beta)}(\rho) = e^{\rho z} = \overline{\zeta^{(\beta)}(\bar{z})}. \]

Invoking (4.8) with \( z \in \mathbb{H} \) we see that

\[ \zeta_n^{(\beta)}(z) = \zeta_n^{(\beta)}(\bar{z}) - \sum_{q=1}^{\infty} \frac{\mu(q)}{q} \sum_{k=0}^{\infty} \frac{1}{k!} \left\{ \left( e^{-\frac{2\pi i}{q}} \right)^k + \left( e^{-\frac{2\pi i}{q}} \right)^k \right\} D_{1+\mu/\beta}(n) \]

and by complex conjugation for \( z \in \mathbb{H} \), and by analytic continuation for \( z \) with \( y = \text{Im}(z) = 0 \). This proves the functional equation (4.7). Another expression can be found which depends on the values of the Riemann zeta-function at odd integers

\[ A_n^{(\beta)}(z) = -\sum_{q=1}^{\infty} \frac{\mu(q)}{q} \sum_{k=0}^{\infty} \frac{1}{k!} \left( e^{-\frac{2\pi i}{q}} \right)^k \left( e^{-\frac{2\pi i}{q}} \right)^k D_{1+\mu/\beta}(n) - 2 D_1^{(\beta)}(n) \]

which converges absolutely. Thus \( A_n^{(\beta)}(z) \) defines an entire function.

**Theorem 4.3.** The only singularities of \( \zeta_n^{(\beta)}(z) \) are simple poles at the points \( z = \log q \) on the real axis, where \( q \) is an integer such that \( c_q^{(\beta)}(n) \neq 0 \), with residue

\[ \text{res}_{z=\log q} \zeta_n^{(\beta)}(z) = -\frac{1}{2\pi i} c_q^{(\beta)}(n). \]

**Proof.** This now follows from Theorem 4.2. \( \square \)

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**Institut für Mathematik, Universität Zürich**
**Winterthurerstrasse 190, CH-8057 Zürich, Switzerland**

gmail: patrick.kuehn@math.uzh.ch
email: nicolas.robes@math.uzh.ch