IRREDUCIBLE $SU(2,\mathbb{C})$-METABELIAN REPRESENTATIONS OF BRANCHED TWIST SPINS

MIZUKI FUKUDA

Abstract. An $(m, n)$-branched twist spin is a fibered 2-knot in $S^4$ which is determined by a 1-knot $K$ and coprime integers $m$ and $n$. For a 1-knot, Lin proved that the number of irreducible $SU(2,\mathbb{C})$-metabelian representations of the knot group of a 1-knot up to conjugation is determined by the knot determinant of the 1-knot. In this paper, we prove that the number of irreducible $SU(2,\mathbb{C})$-metabelian representations of the knot group of an $(m, n)$-branched twist spin up to conjugation is determined by the determinant of a 1-knot in the orbit space by comparing a presentation of the knot group of the branched twist spin with the Lin’s presentation of the knot group of the 1-knot.

1. Introduction

A 2-knot is a smoothly embedded 2-sphere in $S^4$. A 2-knot is said to be fibered if its complement admits a fibration structure over the circle with some natural structure in a tubular neighborhood of the 2-knot. Although it is very difficult to see how the 2-knot is embedded in $S^4$, the idea of admitting a fibration helps us to construct many examples of 2-knots, such as spun knots, twist spun knots, rolling, deformed spun knots, branched twist spins, fibered homotopy-ribbon knots, etc [1, 2, 5, 7, 9, 16, 17]. A branched twist spin is a 2-knot which admits an $S^1$-action in its exterior. The terminology “branched twist spin” appears in the book of Hillman [5]. It is known by Pao and Plotnick that a fibered 2-knot is a branched twist spin if and only if its monodromy is periodic [14]. Therefore, this class has special importance among other known classes of fibered 2-knots. Note that spun knots and twist spun knots are included in the class of branched twist spins.

We give here a short introduction of branched twist spins based on the classification of locally smooth $S^1$-actions on the 4-sphere. Montgomery and Yang showed that effective locally smooth $S^1$-actions are classified into four types [10] and Fintushel and Pao showed that there is a bijection between orbit data and weak equivalence classes of $S^1$-actions on $S^4$ [8, 13]. Suppose that $S^1$ acts locally smoothly and effectively on $S^4$ and the orbit space is $S^3$. Then there are at most two types of exceptional orbits called $\mathbb{Z}_m$-type and $\mathbb{Z}_n$-type, where $m, n$ are coprime positive integers. Let $E_m$ (resp. $E_n$) be the set of exceptional orbits of $\mathbb{Z}_m$-type (resp. $\mathbb{Z}_n$-type) and $F$ be the fixed point set. The image of the orbit map of $E_n$, denoted by $E_n^*$, is an open arc in the orbit space $S^3$, and that of $F$, denoted by $F^*$, is the two points in $S^3$ which are the end points of $E_n^*$. It is known that $E_m^* \cup E_n^* \cup F^*$ constitutes a 1-knot $K$ in $S^3$ and $E_n \cup F$ is diffeomorphic to the 2-sphere. The $(m, n)$-branched twist spin of $K$ is defined as $E_n \cup F$. Note that the $(m, 1)$-branched

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twist spin is the \( m \)-twist spun knot and the \((0, 1)\)-branched twist spin is the spun knot. If \( K \) is a torus knot or a hyperbolic knot then its \((m, n)\)-branched twist spins with \( m > n \) and \( m \geq 3 \) are non-trivial. This follows from the fact that \( K^{m,n} \) is not reflexive known by Hillman and Plotnick [6].

An oriented \( k \)-knot \( K \) is said to be equivalent to another oriented \( k \)-knot \( K' \), denoted by \( K \sim K' \), if there exists a smooth isotopy \( H_t : S^{k+2} \to S^{k+2} \) such that \( H_0 = id \) and \( H_1(K) = K' \) as oriented \( k \)-knots. In [4], the author studied the elementary ideal of the fundamental group of the complement of a branched twist spin and gave a criterion to detect if two branched twist spins \( K_{m1,n1} \) and \( K_{m2,n2} \) are inequivalent. Similar to the results in [8, 12], the number of such representations is given by the knot determinant as follows:

**Theorem 1.1.** The number of irreducible \( SU(2, \mathbb{C}) \)-metabelian representations of \( \pi_1(S^4 \setminus \text{int}N(K^{m,n})) \) is

\[
\frac{|\Delta_K(-1)| - 1}{2} \quad (m : \text{even}) \\
0 \quad (m : \text{odd}),
\]

where \( N(K^{m,n}) \) is a compact tubular neighborhood of \( K^{m,n} \) in \( S^4 \).

As an immediate corollary, we obtain the same criterion as in [4].

**Corollary 1.2 (F. [4]).** Branched twist spins \( K_{m1,n1} \) and \( K_{m2,n2} \) are inequivalent if one of the following holds:

1. \( m_1 \) and \( m_2 \) are even and \( |\Delta_{K_1}(-1)| \neq |\Delta_{K_2}(-1)| \),
2. \( m_1 \) is even, \( m_2 \) is odd and \( |\Delta_{K_1}(-1)| \neq 1 \).

This paper is organized as follows: In Section 1, we define an \((m, n)\)-branched twist spin \( K^{m,n} \) as an oriented 2-knot and introduce Plotnick’s presentation of \( \pi_1(S^4 \setminus \text{int}N(K^{m,n})) \). In Section 2, we state the Lin’s presentation of a 1-knot and the Nagasato-Yamaguchi’s presentation of the \( m \)-fold cyclic branched cover of \( S^3 \) along \( K \). In Section 3, we observe irreducible \( SU(2, \mathbb{C}) \)-metabelian representations of \( \pi_1(S^4 \setminus \text{int}N(K^{m,n})) \) and prove Theorem 1.1.

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### 2. TWO PRESENTATIONS OF BRANCHED TWIST SPINS

#### 2.1. The \((m, n)\)-branched twist spin

Suppose that \( S^4 \) has an effective locally smooth \( S^1 \)-action. Let \( E_m \) be the set of exceptional orbits of \( \mathbb{Z}_m \)-type, where \( m \) is a positive integer, and \( F \) be the fixed point set. Set \( E_m^* \) and \( F^* \) to be the image of \( E_m \) and \( F \) by the orbit map, respectively. Montgomery and Yang showed that effective locally smooth
$S^1$-actions are classified into the following four types: (1) $\{D^3\}$, (2) $\{S^3\}$, (3) $\{S^4, m\}$, (4) $\{(S^3, K), m, n\}$, which are called orbit data \cite{H}. The 3-ball and the 3-sphere in these notations represent the orbit spaces. In case (4), the union $E_m^* \cup E_n^* \cup F^*$ constitutes a 1-knot $K$ in the orbit space $S^3$ and the union $E_n \cup F$ is diffeomorphic to the 2-sphere. This 2-sphere is embedded in $S^4$, and is called the $(m, n)$-branched twist spin of $K$, denoted by $K^{m,n}$. In case (3), for an arc $A^*$ in $S^3$ whose end points are $F^*$, the preimage of $A^*$ is denoted by $A$. Then the union $A \cup F$ is diffeomorphic to the 2-sphere, and is called a twist spun knot. We may regard an $m$-twist spun knot as $K^{m,1}$, where $K$ is $A^* \cup E_n^* \cup F^*$.

We recall the definition of $(m, n)$-branched twist spins for $(m, n) \in \mathbb{Z} \times \mathbb{N}$ in \cite{H}. First, we remark that the definition in \cite{H} depends on the choice of the orientation of $K$. Actually, in the definition we fixed a preferred meridian-longitude $(\theta, \phi)$ of $S^3 \setminus \text{int}N(K)$, where $N(K)$ is a compact tubular neighborhood of $K$, and replacing $(\theta, \phi)$ by $(-\theta, -\phi)$ may change the equivalence class of $K^{m,n}$.

We give the definition of $K^{m,n}$. Let $K$ be a 1-knot in $S^3$ and $(m, n)$ be a pair of integers in $(\mathbb{Z} \setminus \{0\}) \times \mathbb{N}$ such that $|m|$ and $n$ are coprime. We decompose the orbit space $S^3$ into five pieces as follows:

$$S^3 = (S^3 \setminus \text{int}N(K)) \cup (E_m^{cs} \times D^2) \cup (E_n^{cs} \times D^2) \cup (D_1^{3s} \cup D_2^{3s}),$$

where $D_1^{3s} \cup D_2^{3s}$ is a compact neighborhood of $F^*$ and $E_m^{cs}$ and $E_n^{cs}$ are the connected components of $K \setminus \text{int}(D_1^{3s} \cup D_2^{3s})$ such that $E_m^{cs} \subset E_m^{cs}$ and $E_n^{cs} \subset E_n^{cs}$, see Figure 1. Considering the preimage of the orbit map, we decompose $S^4$ as follows:

$$(2.1) \quad S^4 = ((S^3 \setminus \text{int}N(K)) \times S^1) \cup (E_m \times D^2) \cup (E_n \times D^2) \times (D_1 \cup D_2).$$

Let $p$ denote the orbit map. Choosing a point $z_m^{*} \in E_m^{cs}$, let $D_2^{2*}$ be a 2-disk in $S^3$ centered at $z_m^{*} \in E_m^{cs}$ and transversal to $E_m^{cs}$. The preimage $p^{-1}(D_2^{2*})$ is a solid torus $V_m$ whose core is the exceptional orbit of $\mathbb{Z}_m$-type.

![Figure 1. Decomposition of $S^3$](image)

Now we discuss the orientations of $V_m$ and $E_m^{cs}$. Let $K$ be an oriented 1-knot in $S^3$. First, fix the orientation of $S^4$ and those of orbits such that they coincide with the direction of the $S^1$-action. These orientations determine the orientation of $V_m \times E_m^{cs}$. Let $(\theta, \phi)$ be the preferred meridian-longitude pair of $K$ such that the orientation of the longitude $\phi$ coincides the orientation of $K$. From the decomposition (2.1), we can see that $\phi$ is regarded as a coordinate of the second factor of $V_m \times E_m^{cs}$. We assign the orientation of $V_m$ so that the orientation of $V_m \times E_m^{cs}$ coincides with the given orientation of $S^4$. Finally, we choose the meridian and longitude pair $(\Theta, H)$ of $V_m \cong D^2 \times S^1$ such that $H$ becomes
the meridian of \( V_n \) in the decomposition \( V_m \cup V_n = p^{-1}(\partial D^3_i) \) and the orbits of the \( S^1 \)-action are in the direction \( \varepsilon n \Theta + |m|H \) with \( n > 0 \), where \( \varepsilon = 1 \) if \( m \geq 0 \) and \( \varepsilon = -1 \) if \( m < 0 \).

**Definition 2.1** (Branched twist spin). Let \( K \) be an oriented knot in \( S^3 \). For each pair \((m, n) \in \mathbb{Z} \times \mathbb{N}\) with \( m \neq 0 \) such that \( |m| \) and \( n \) are coprime, let \( K^{m,n} \) denote the 2-knot \( E_n \cup F \). If \((m, n) = (0, 1)\) then define \( K^{0,1} \) to be the spun knot of \( K \). The 2-knot \( K^{m,n} \) is called an \((m, n)\)-branched twist spin of \( K \).

Note that the branched twist spin \( K^{m,1} \) constructed from \( \{(S^3, K), m, 1\} \) is an \( m \)-twist spun knot of \( K \).

**Remark 2.2.** Let \(-K\) be an oriented knot obtained from \( K \) by reversing the orientation of \( K \). From the construction of \( K^{m,n} \), we see that \( K^{m,n} \) is equivalent to \(-(-K)^{-m,n}\).

Let \( K \) be a \( k \)-knot in \( S^{k+2} \). The fundamental group of the knot complement \( S^{k+2} \setminus \text{int}N(K) \) is called the knot group of \( K \), where \( N(K) \) is a compact tubular neighborhood of \( K \).

**Lemma 2.3** ([13]). Let \( K \) be an oriented 1-knot and \( K^{m,n} \) be the \((m, n)\)-branched twist spin of \( K \) with \((m, n) \in \mathbb{Z} \times \mathbb{N}\), where \( |m| \) and \( n \) are coprime. Let \( \langle y_1, \ldots, y_s \mid r_1, \ldots, r_l \rangle \) be a presentation of the knot group of \( K \) such that \( y_i \) is a meridian. Then the knot group of \( K^{m,n} \) has the presentation

\[
\pi_1(S^4 \setminus \text{int} N(K^{m,n})) \cong \langle y_1, \ldots, y_s, h \mid r_1, \ldots, r_l, y_i h y_i^{-1} h^{-1}, y_i^{m} h^{\beta} \rangle,
\]

where \( \beta \) is an integer such that \( n \beta \equiv \varepsilon \pmod{m} \) if \( m \) is non-zero and \( \beta = 1 \) if \( m = 0 \).

Recall that \( \varepsilon = 1 \) if \( m \geq 0 \) and \( \varepsilon = -1 \) if \( m < 0 \).

Note that \( \pi_1(S^4 \setminus \text{int} N(K^{m,n})) \) is isomorphic to \( \pi_1(S^4 \setminus \text{int} N((-K)^{-m,n})) \) by Remark 2.2.

**2.2. Plotnick’s presentation.** Assume that \( m \neq 0 \). We ignore the orientation of \( K^{m,n} \) since we are interested in the fundamental group of its complement. By Remark 2.2 changing the orientation of \( K \) and the sign of \( m \) if necessary, we can assume that \( m \) is positive. Pao constructed the knot complement of \( K^{m,n} \) as follows [13]: Let \( M_K \) be the \( m \)-fold cyclic branched cover of \( S^3 \) along \( K \) and \( \tau : M_K \to M_K \) be the diffeomorphism associated with the canonical deck transformation of \( M_K \). Let \( M_K \times_{\tau^n} S^1 \) be the manifold obtained from \( M_K \times I \) by identifying \( M_K \times \{0\} \) with \( M_K \times \{1\} \) by \((z, 1) \mapsto (\tau^n z, 0)\), where \( \tau^n \) means the \( n \)-th power of composite of \( \tau \). Note that \( M_K \times_{\tau^n} S^1 \) has the natural \( S^1 \)-action \( \varphi_s(y, t) = (y, t+s) \), where \( (y, t) \) denotes the image of \((y, t) \in M_K \times I \) by the identification. Let \( x \) be a branch point of \( M_K \). Then the orbit of \( (x, 0) \) is a circle in \( M_K \times_{\tau^n} S^1 \). There is a neighborhood of the orbit which is invariant by the \( S^1 \)-action, denoted by \( T \). It is known in [13] that the knot complement of \( K^{m,n} \) is diffeomorphic to \( (M_K \times_{\tau^n} S^1) \setminus \text{int}T \), which is also diffeomorphic to \( \text{punc}(M_K) \times_{\tau^n} S^1 \), where \( \text{punc}(M_K) = M_K \setminus \partial D^3 \) with \( D^3 \) being a 3-ball in \( M_K \). Note that \( K^{m,n} \) is regarded as the branch set of the \( n \)-fold cyclic branched cover of \( S^4 \) along the \( m \)-twist spun knot of \( K \).

The following lemma is shown by Plotnick in [15].

**Lemma 2.4** (Plotnick [15]). Let \( K^{m,n} \) be a branched twist spin of \( K \). Then the following holds:

\[
\pi_1(S^4 \setminus \text{int} N(K^{m,n})) \cong \pi_1(\text{punc}(M_K)) * \langle \eta \rangle / \langle \eta(\tau^n z) \eta^{-1} = z \rangle \quad \text{for all} \ z \in \pi_1(M_K),
\]
where \( \eta \) is a meridian of \( K^{m,n} \).

2.3. Lin’s presentation. Let \( K \) be a 1-knot in \( S^3 \). A Seifert surface \( S \) of \( K \) is called free if \( S^3 = N(S) \cup (S^3 \setminus \text{int}N(S)) \) gives a Heegaard splitting of \( S^3 \). It is known that any 1-knot has a free Seifert surface. A presentation of \( \pi_1(S^3 \setminus \text{int}N(K)) \) is obtained from the Heegaard splitting associated to a free Seifert surface as follows: Let \( S \) be a free Seifert surface of \( K \) of genus \( g \) and \( W \) be a spine of \( S \). Then \( H_1 = S \times [-1, 1] \) and \( H_2 = S^3 \setminus \text{int}H_1 \) is a Heegaard splitting of \( S^3 \). Let \( K' \) be a simple closed curve obtained from \( K \) by pushing it into \( H_1 \) slightly. Choose a base point \( * \) in \( W \subset S \times \{0\} \) such that \( * \) does not on \( K \) and \( K' \). Since \( H_1 \) and \( H_2 \) are handlebodies with genus \( 2g \), we may choose generators \( a_1, \ldots, a_{2g} \) of \( \pi_1(H_1) \) and generators \( x_1, \ldots, x_{2g} \) of \( \pi_1(H_2) \). Let \( \alpha_i^+ \), \( \alpha_i^- \), \( \alpha_i \) denote the loops \( a_1 \times \{1\}, \ldots, a_{2g} \times \{1\} \) and \( a_1 \times \{-1\}, \ldots, a_{2g} \times \{-1\} \). Each \( \alpha_i^+ \) (resp. \( \alpha_i^- \)) is written in a word of \( x_1, \ldots, x_{2g} \) by the homeomorphism from \( \partial H_2 \) to \( \partial H_1 \). The words of \( \alpha_i^+ \) (resp. \( \alpha_i^- \)) are denoted by \( \alpha_i \) (resp. \( \beta_i \)) for \( i = 1, \ldots, 2g \). There is a unique arc \( c \), up to isotopy, such that \( (* \times [-1, 1]) \cup c \) is a meridian of \( K' \). The homotopy class of this loop is denoted by \( \mu \). From van Kampen theorem, the following theorem holds:

Lemma 2.5 (Lin [8]). Let \( K \) be a 1-knot in \( S^3 \) and \( S \) be a free Seifert surface of \( K \). Let \( \pi_1(S^3 \setminus \text{int}N(K)) \) has the following presentation:

\[
\pi_1(S^3 \setminus \text{int}N(K)) \cong \langle x_1, \ldots, x_{2g}, \mu \mid \mu \alpha_i \mu^{-1} = \beta_i \rangle,
\]

where \( g \) is the genus of \( S \), and \( \alpha_i, \beta_i \) are the words in \( x_1, \ldots, x_{2g} \) determined above.

Let \( \langle x_1, \ldots, x_{2g}, \mu \mid \mu \alpha_i \mu^{-1} = \beta_i \rangle \) be a Lin’s presentation of \( \pi_1(S^3 \setminus \text{int}N(K)) \). Denote the sum of indices of \( x_j \) in \( \alpha_i \) by \( v_{ij} \) and that in \( \beta_i \) by \( u_{ij} \). Then the \( 2g \times 2g \) matrix \( V = (v_{ij}) \) is defined. The matrix \( V \) is called a Seifert matrix and \( \det(V + \imath V) \) is called the knot determinant of \( K \), which equals to \( \Delta_K(-1) \). Note that all generators \( x_1, \ldots, x_{2g} \) are commutators of \( \pi_1(S^3 \setminus \text{int}N(K)) \). Let \( \rho_0 : \pi_1(S^3 \setminus \text{int}N(K)) \to SU(2, \mathbb{C}) \) be an irreducible \( SU(2, \mathbb{C}) \)-metabelian representation. Since all \( x_1, \ldots, x_{2g} \) are commutators of \( \pi_1(S^3 \setminus \text{int}N(K)) \), we can assume that

\[
\rho_0(x_i) = \begin{pmatrix} \alpha_i & 0 \\ 0 & \overline{\alpha_i} \end{pmatrix}, \quad \rho_0(\mu) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

up to conjugation (cf. [11]). Since \( \alpha_i \) and \( \beta_i \) are written in words \( x_1, \ldots, x_{2g} \), each \( \rho_0(\alpha_i) \) and \( \rho_0(\beta_i) \) is a diagonal matrix. From (2.4), Lin checked directly the number of irreducible \( SU(2, \mathbb{C}) \)-metabelian representations of \( \pi_1(S^3 \setminus \text{int}N(K)) \).

Theorem 2.6 (Lin [8]). The number of irreducible \( SU(2, \mathbb{C}) \)-metabelian representations of \( \pi_1(S^3 \setminus \text{int}N(K)) \) is

\[
\frac{|\Delta_K(-1)| - 1}{2}
\]

2.4. Nagasato-Yamaguchi’s presentation. Let \( M_K \) be the \( m \)-fold cyclic branched cover of \( S^3 \) along \( K \) and \( \tau \) be the canonical deck transformation on \( M_K \). The fundamental region of \( M_K \) contains a free Seifert surface of \( K \). Nagasato and Yamaguchi gave a presentation of \( \pi_1(M_K) \) from the Lin’s presentation of \( \pi_1(S^3 \setminus \text{int}N(K)) \).
Theorem 2.7 (Nagasato,Yamaguchi [12]). Let \( \langle x_1, \ldots, x_{2g}, \mu \mid \mu \alpha_i \mu^{-1} = \beta_i \rangle \) be a Lin’s presentation of a 1-knot \( K \). Then \( \pi_1(M_K) \) has the following presentation:

\[
\pi_1(M_K) \cong \langle \tau^0 \bar{x}_1, \ldots, \tau^0 \bar{x}_{2g}, \ldots, \tau^{m-1} \bar{x}_1, \ldots, \tau^{m-1} \bar{x}_{2g} \mid \bar{\alpha}_i^{(j)} = \bar{\beta}_i^{(j-1)} \rangle,
\]

where \( \bar{x}_i \) is the lift of \( x_i \) to \( M_K \), and \( \bar{\alpha}_i^{(j)}, \bar{\beta}_i^{(j)} \) are the words obtained from \( \alpha_i, \beta_i \) by replacing \( x_1, \ldots, x_{2g} \) with \( \tau^j \bar{x}_1, \ldots, \tau^j \bar{x}_{2g} \) for \( i = 1, \ldots, 2g \) and \( j \equiv 0, \ldots, m-1 \) (mod \( m \)).

We can rewrite the presentation in Lemma 2.4 by applying a Nagasato-Yamaguchi’s presentation to \( punc(M_K) \) as follows:

\[
\langle \tau^0 \bar{x}_1, \ldots, \tau^0 \bar{x}_{2g}, \ldots, \tau^{m-1} \bar{x}_1, \ldots, \tau^{m-1} \bar{x}_{2g}, \eta \mid \bar{\alpha}_i^{(j)} = \bar{\beta}_i^{(j-1)}, \eta \tau^{j+n} \bar{x}_i \eta^{-1} = \tau^j \bar{x}_i \rangle.
\]

(2.5)

3. Proof of Theorem 1.1

We first introduce a property of irreducible \( SU(2, \mathbb{C}) \)-metabelian representations of \( \pi_1(S^4 \setminus \text{int}(N(K^{m,n})) \) from (2.5).

Lemma 3.1. Let \( \rho \) be an irreducible \( SU(2, \mathbb{C}) \)-metabelian representation of \( \pi_1(S^4 \setminus \text{int}(N(K^{m,n})) \). Then, up to conjugation, \( \rho \) is of the form

\[
\rho(\tau^j \bar{x}_i) = \begin{pmatrix}
\lambda_i^{(j)} & 0 \\
0 & \lambda_i^{(j)}
\end{pmatrix},
\rho(\eta) = \begin{pmatrix} 0 & -1 \\
1 & 0 \end{pmatrix},
\]

where \( i = 1, \ldots, 2g, j \equiv 0, \ldots, m-1 \) (mod \( m \)), and \( \lambda_i^{(j)} \neq \lambda_i^{(j)} \) for some \( i, j \).

Proof. Since \( \rho \) is a metabelian representation, \( \rho([\pi_1(S^4 \setminus \text{int}(N(K^{m,n}))), \pi_1(S^4 \setminus \text{int}(N(K^{m,n})))]) \) is an abelian group. Up to conjugation of \( \rho \), we can assume that \( \rho(x) \) is a diagonal matrix for any \( x \in [\pi_1(S^4 \setminus \text{int}(N(K^{m,n}))), \pi_1(S^4 \setminus \text{int}(N(K^{m,n})))]. \) Since the generators \( \tau^j \bar{x}_i \) are on the Seifert surface of \( K^{m,n} \), all \( \tau^j \bar{x}_i \) are commutators in \( \pi_1(S^4 \setminus \text{int}(N(K^{m,n}))) \). Then \( \rho(\tau^j \bar{x}_i) \) are of the forms

\[
\rho(\tau^j \bar{x}_i) = \begin{pmatrix}
\lambda_i^{(j)} & 0 \\
0 & \lambda_i^{(j)}
\end{pmatrix},
(\lambda_i^{(j)} \in \mathbb{C}, |\lambda_i^{(j)}|^2 = 1),
\]

see [11]. The matrix \( \rho(\eta) \) is determined by the relations \( \eta \tau^{j+n} \bar{x}_i \eta^{-1} = \tau^j \bar{x}_i \) as follows. Set \( \rho(\eta) = \begin{pmatrix} a & b \\
c & d \end{pmatrix} \in SU(2, \mathbb{C}) \). Then \( \rho(\eta \tau^{j+n} \bar{x}_i) \) and \( \rho(\tau^j \bar{x}_i \eta) \) are given as

\[
\rho(\eta \tau^{j+n} \bar{x}_i) = \begin{pmatrix} a & b \\
c & d \end{pmatrix} \begin{pmatrix}
\lambda_i^{(j+n)} & 0 \\
0 & \lambda_i^{(j+n)}
\end{pmatrix} = \begin{pmatrix} a\lambda_i^{(j+n)} & b\lambda_i^{(j+n)} \\
c\lambda_i^{(j+n)} & d\lambda_i^{(j+n)} \end{pmatrix},
\]

\[
\rho(\tau^j \bar{x}_i \eta) = \begin{pmatrix}
\lambda_i^{(j)} & 0 \\
0 & \lambda_i^{(j)}
\end{pmatrix} \begin{pmatrix} a & b \\
c & d \end{pmatrix} = \begin{pmatrix} a\lambda_i^{(j)} & b\lambda_i^{(j)} \\
c\lambda_i^{(j)} & d\lambda_i^{(j)} \end{pmatrix}.
\]

These two matrices must be the same. Assume that \( \lambda_i^{(j+n)} = \lambda_i^{(j)} \) for all \( i, j \). Since \( m \) and \( n \) are coprime, \( \lambda_i^{(j)} = \lambda_i^{(0)} \) for any \( i, j \). If \( \lambda_i^{(0)} = \lambda_i^{(0)} \) for any \( i, j \), then \( \rho(\tau^j \bar{x}_i) = \begin{pmatrix} \pm 1 & 0 \\
0 & \pm 1 \end{pmatrix} \) for
any \(i, j\). Then \(\rho\) is not irreducible. If \(\lambda_i^{(0)} \neq \bar{\lambda}_i^{(0)}\) for some \(i\), then \(\rho(\eta)\) is a diagonal matrix and \(\rho(\pi_1(S^4 \setminus \text{int}N(K^{m,n})))\) becomes an abelian group. It also contradicts the irreducibility of \(\rho\). Therefore \(\lambda_i^{(j + n)} \neq \lambda_i^{(j)}\) for some \(i, j\). In this case, \(a = d = 0\) and \(\rho(\eta) = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\).

Set \(B = \begin{pmatrix} b^2 & 0 \\ 0 & b^\top \end{pmatrix}\). Since \(B\rho(\eta)B^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) and \(B\rho(\tau^j\bar{x}_i)B^{-1} = \rho(\tau^j\bar{x}_i)\), we have \(\rho(\eta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) up to conjugation. \(\square\)

Let \(S_0\) be a free Seifert surface of \(K\) contained in a fundamental region of \(M_K\), where \(M_K\) is a fiber of \(K^{m,n}\). Let \(S_1, \ldots, S_{m-1}\) be copies of \(S_0\) by the deck transformations. We want to know relation between \(\lambda_i^{(j)}\) and \(\lambda_i^{(j+1)}\) for \(j = 1, \ldots, m-1\) (mod \(m\)). The relation \(\eta\tau^j\eta^{-1} = \tau^j\bar{x}_i\) means that the conjugation by \(\eta\) brings \(\tau^j\eta\bar{x}_i\) on \(S_{j+n}\) to \(\tau^j\bar{x}_i\) on \(S_j\). Let \(q\) be an integer such that \(nq \equiv 1\) (mod \(m\)) and take conjugation of \(\tau^j\bar{x}_i\) by \(\eta^q\). Then we obtain the relation

\[
\tau^j\bar{x}_i = \eta^{\tau^j\eta^{-1}} = \eta^{\tau^j+nq}\bar{x}_i\eta^{-q} = \eta^{\tau^j+1}\bar{x}_i\eta^{-q},
\]

which brings \(\tau^j\bar{x}_i\) on \(S_{j+1}\) to \(\tau^j\bar{x}_i\) on \(S_j\), where we used \(nq \equiv 1\) (mod \(m\)).

Let \(\rho\) be an irreducible \(SU(2, \mathbb{C})\)-metabelian representation of \(\pi_1(S^4 \setminus \text{int}N(K^{m,n}))\) in Lemma 3.1. From the relation (3.1) and Lemma 3.1

\[
\begin{pmatrix} \lambda_i^{(j)} & 0 \\ 0 & \lambda_i^{(j+1)} \end{pmatrix} = \begin{cases} \begin{pmatrix} \lambda_i^{(j)} & 0 \\ 0 & \bar{\lambda}_i^{(j)} \end{pmatrix} & (q : \text{even}) \\ \begin{pmatrix} \bar{\lambda}_i^{(j)} & 0 \\ 0 & \lambda_i^{(j)} \end{pmatrix} & (q : \text{odd}). \end{cases}
\]

Suppose that \(m\) is even. Then \(q\) is odd since \(m\) and \(q\) are coprime. We define the representation \(\overline{\rho}\) by

\[
\overline{\rho}(x) = \rho(\eta^q x \eta^{-q}) = \rho(\eta)\rho(x)\rho(\eta^{-1})
\]

for all \(x \in \pi_1(S^4 \setminus \text{int}N(K^{m,n}))\). Note that \(\rho(\eta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) by Lemma 3.1. In particular,

\[
\overline{\rho}(x) = \rho(x).
\]

By (3.2), \(\rho(\tau^j\bar{x}_i) = \overline{\rho}(\tau^j\bar{x}_i)\) for all \(i, j\). Since \(\tilde{\alpha}_i^{(j)}\) and \(\tilde{\beta}_i^{(j)}\) are words written in \(\tau^j\bar{x}_1, \ldots, \tau^j\bar{x}_{2g}\), we have

\[
\rho(\tilde{\alpha}_i^{(j+1)}) = \overline{\rho}(\tilde{\alpha}_i^{(j)}), \quad \rho(\tilde{\beta}_i^{(j+1)}) = \overline{\rho}(\tilde{\beta}_i^{(j)}).
\]

On the other hand,

\[
\tilde{\beta}_i^{(j)} = \tilde{\alpha}_i^{(j+1)} = \eta^{-q}\tilde{\alpha}_i^{(j)}\eta^q
\]

holds, where the relation (3.1) is applied to the second equality. Since \(q\) is odd, \(\rho(\eta^q x \eta^{-q}) = \rho(\eta^q)\rho(x)\rho(\eta^{-q}) = \rho(\eta)\rho(x)\rho(\eta^{-1}) = \overline{\rho}(x)\). Hence, by (3.2), we have

\[
\rho(\tilde{\beta}_i^{(j)}) = \overline{\rho}(\tilde{\alpha}_i^{(j)}).
\]

From (3.3) and (3.5), one can see that the relations of representations of the first relations \(\tilde{\alpha}_i^{(j)} = \tilde{\beta}_i^{(j-1)}\) in (2.5) are equivalent to \(\rho(\tilde{\beta}_i^{(0)}) = \rho(\eta\tilde{\alpha}_i^{(0)}\eta^{-1})\).
The second relations \( \eta \tau^{j+n} \tilde{x}_i \eta^{-1} = \tau^j \tilde{x}_i \) in (2.3) are equivalent to \( \eta \tau^{j+1} \tilde{x}_i \eta^{-q} = \tau^j \tilde{x}_i \) for all \( j \) as checked in (3.1). Therefore \( \rho(\eta \tau^{j+n} \tilde{x}_i \eta^{-1}) = \rho(\tau^j \tilde{x}_i) \) are equivalent to \( \rho(\eta \tau^{j+1} \tilde{x}_i \eta^{-1}) = \rho(\tau^j \tilde{x}_i) \). Hence the number of irreducible \( SU(2, \mathbb{C}) \)-representations of the presentation (2.5) is equal to that of representations of the group presented by (3.7) \( \langle \tau^0 \tilde{x}_1, \ldots, \tau^0 \tilde{x}_{2g}, \ldots, \tau^{m-1} \tilde{x}_1, \ldots, \tau^{m-1} \tilde{x}_{2g}, \eta \mid \eta \tilde{a}_i(0) \eta^{-1} = \tilde{b}_i(0), \eta \tau^{j+1} \tilde{x}_i \eta^{-1} = \tau^j \tilde{x}_i \rangle \).

Now, we reduce the generators \( \tau^j \tilde{x}_i, \ldots, \tau^j \tilde{x}_{2g}, \ldots, \tau^{m-1} \tilde{x}_1, \ldots, \tau^{m-1} \tilde{x}_{2g} \) and the relations \( \eta \tau^{j+1} \tilde{x}_i \eta^{-1} = \tau^j \tilde{x}_i \) from the above presentation to simplify counting the number of irreducible \( SU(2, \mathbb{C}) \)-metabelian representations of \( \pi_1(S^4 \setminus \text{int}N(K^{m,n})) \).

**Lemma 3.2.** Let \( m \) be an even integer. Then the number of irreducible \( SU(2, \mathbb{C}) \)-metabelian representations of \( \pi_1(S^4 \setminus \text{int}N(K^{m,n})) \) coincides that of the group \( G \) presented by

(3.7) \( \langle \tau^0 \tilde{x}_1, \ldots, \tau^0 \tilde{x}_{2g}, \eta \mid \eta \tilde{a}_i(0) \eta^{-1} = \tilde{b}_i(0) \rangle \).

**Proof.** A representation of (3.6) is a representation of (3.7). So, we prove the converse. The representation of \( \tau^j \tilde{x}_i \) for \( j \equiv 1, \ldots, m-1 \) (mod \( m \)) is determined by the equality \( \rho(\tau^{j+1} \tilde{x}_i) = \overline{\rho}(\tau^j \tilde{x}_i) \) obtained from (3.2). Hence, it is enough to prove that any irreducible \( SU(2, \mathbb{C}) \)-metabelian representation \( \rho \) of (3.7) has the property \( \rho(\eta \tau^0 \tilde{x}_i \eta^{-1}) = \overline{\rho}(\tau^0 \tilde{x}_i) \).

Since the presentation in (3.7) is exactly of the same form as the Lin’s presentation (2.3), all \( \rho(\tau^0 \tilde{x}_i) \) and \( \rho(\eta) \) are of the forms

\[
\rho(\tau^0 \tilde{x}_i) = \begin{pmatrix} \lambda_i(0) & 0 \\ 0 & \lambda_i(0) \end{pmatrix}, \quad \rho(\eta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]

for \( i = 1, \ldots, 2g \) up to conjugation, see [11]. Then, by the definition of \( \overline{\rho} \),

\[
\overline{\rho}(\tau^0 \tilde{x}_i) = \begin{pmatrix} \overline{\lambda}_i(0) & 0 \\ 0 & \overline{\lambda}_i(0) \end{pmatrix}, \quad \overline{\rho}(\eta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

hold. Therefore \( \rho(\eta \tau^0 \tilde{x}_i \eta^{-1}) = \begin{pmatrix} \overline{\lambda}_i(0) & 0 \\ 0 & \overline{\lambda}_i(0) \end{pmatrix}, \) and this is \( \overline{\rho}(\tau^0 \tilde{x}_i) \).

**Proof of** Theorem 1.1 We decompose the proof into two cases: (1) \( m \) is even or (2) \( m \) is odd. In case (1), by Lemma 3.2, we only need to count the number of irreducible \( SU(2, \mathbb{C}) \)-metabelian representations of \( G \) in the Lemma. Since the presentation in (3.7) is exactly of the same form as the Lin’s presentation (2.3), each \( \lambda_1(0), \ldots, \lambda_{2g}(0) \) must satisfy the following equations as explained in [8]:

(3.8) \( \lambda_1(0) \omega_i(0) \cdots \lambda_{2g}(0) \omega_{i+g} = 1 \) for \( i = 1, \ldots, 2g, \)

where the matrix \( \omega = V + {^t}V \) is defined in Section 2.3. With some linear algebra, one can see that the number of non-trivial solutions of (3.8) is \(|\det(V + {^t}V)| = 1 = |\Delta_K(-1)| = 1. \)

If \( \{\gamma_i\}_{0 \leq i \leq 2g} \) is a solution of (3.8), then \( \{\tau^i \}_{0 \leq i \leq 2g} \) is also, which is given by the conjugation of \( \rho \). Therefore the number of irreducible \( SU(2, \mathbb{C}) \)-metabelian representations of \( \pi_1(S^4 \setminus \text{int}N(K^{m,n})) \) is \( \frac{|\Delta_K(-1)| - 1}{2} \).
In case (2), if $q$ is even, then $\lambda_i^{(j)} = \lambda_i^{(0)}$ for all $j$ by (3.2). Then the relation $\eta \tau^{j+n} \tilde{x}_i \eta^{-1} = \tau^j \tilde{x}_i$ in (2.5) gives $\lambda_i^{(j)} = \lambda_i^{(0)}$ for all $i,j$. In this case, there is no irreducible $SU(2, \mathbb{C})$-metabelian representation of $\pi_1(S^4 \setminus int(N(K^{m,n}))$ by Lemma 3.1. Suppose that $q$ is odd. From the relations (3.1), we have $\tau^j \tilde{x}_i = \eta^q \tau^{j+1} \tilde{x}_i \eta^{-q} = \eta^{mq} \tau^{j+m} \tilde{x}_i \eta^{-mq} = \eta^{mq} \tau^j \tilde{x}_i \eta^{-mq}$. Then we have $\lambda_i^{(j)} = \lambda_i^{(j)}$ for all $i,j$ since $mq$ is odd, and hence there is no irreducible $SU(2, \mathbb{C})$-metabelian representation of $\pi_1(S^4 \setminus int(N(K^{m,n}))$ by Lemma 3.1. Thus the assertion holds.

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