ON A CLASS OF MIXED
CHOQUARD-SCHRÖDINGER-POISSON SYSTEMS

MARIUS GHERGU * AND GURPREET SINGH

School of Mathematics and Statistics
University College Dublin
Belfield, Dublin 4, Ireland

To Professor Vicentiu Radulescu
on the occasion of his 60th anniversary with deep esteem and consideration

ABSTRACT. We study the system
\begin{equation}
\begin{aligned}
-\Delta u + u &= (I_\alpha * |u|^p)|u|^{p-2}u + K(x)\phi |u|^{q-2}u \quad \text{in } \mathbb{R}^N, \\
-\Delta \phi &= K(x)|u|^q \quad \text{in } \mathbb{R}^N,
\end{aligned}
\end{equation}
where \( N \geq 3, \alpha \in (0, N) \), \( p, q > 1 \) and \( K \geq 0 \). Using a Pohozaev type identity we first derive conditions in terms of \( p, q, N, \alpha \) and \( K \) for which no solutions exist. Next, we discuss the existence of a ground state solution by using a variational approach.

1. Introduction. In this paper, we study the following system
\begin{equation}
\begin{aligned}
-\Delta u + u &= (I_\alpha * |u|^p)|u|^{p-2}u + K(x)\phi |u|^{q-2}u \quad \text{in } \mathbb{R}^N, \\
-\Delta \phi &= K(x)|u|^q \quad \text{in } \mathbb{R}^N,
\end{aligned}
\end{equation}
where \( p, q > 1 \) are real numbers and \( K \geq 0 \) satisfies some more properties as we shall precise below. Here \( I_\alpha : \mathbb{R}^N \rightarrow \mathbb{R} \) is the Riesz potential of order \( \alpha \in (0, N) \), \( N \geq 3 \), given by
\begin{equation}
I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}}, \quad \text{with} \quad A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2}) \pi^{N/2} 2^\alpha}.
\end{equation}
When \( K \equiv 0 \), system (1) reduces to the single equation
\begin{equation}
-\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N
\end{equation}
which bears the name Choquard or Choquard-Pekar equation.

For \( N = 3, p = \alpha = 2 \), equation (3) was introduced in 1954 by S.I. Pekar [24] as a model in quantum theory of a Polaron at rest (see also [12]). In 1976, P. Choquard used (3) in a certain approximation to Hartree-Fock theory of one component plasma (see [16]). In 1996, equation (3) appears in a different context, being employed by R. Penrose [25] as a model of self-gravitating matter (see, e.g., [14, 22]) and it is known in this context as the Schrödinger-Newton equation.

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* Corresponding author: Marius Ghergu.
If $u$ solves (3), then the function $\psi$ defined by $\psi(t,x) = e^{it}u(x)$ is a solitary wave of the focusing time dependent Hartree-Fock equation

$$i\psi_t + \Delta \psi + (I_\alpha * |\psi|^p)|\psi|^{p-2}\psi = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^N.$$ 

The Choquard equation (3) has been investigated for a few decades by variational methods starting with the pioneering works of E.H. Lieb [16] and P.-L. Lions [17, 18]. More recently, new and improved techniques have been devised to deal with various forms of (3) (see, e.g., [1, 23, 28] and the references therein). In [23] existence, regularity, positivity, asymptotic behavior and radial symmetry of solutions to (1) is discussed for optimal range of parameters. We also mention here the works [10, 11] where the fractional version of (3) is considered. For a nonvariational approach to Choquard equation the reader may consult [13, 19].

Back to (1), we should point out that since for all $\phi \in C^\infty_0(\mathbb{R}^N)$, $I_\alpha \phi \rightarrow \phi$ as $\alpha \rightarrow 0$, the system

$$\begin{cases}
-\Delta u + u = |u|^{2p-2}u + K(x)\phi|u|^{q-2}u & \text{in } \mathbb{R}^N, \\
-\Delta \phi = K(x)|u|^q & \text{in } \mathbb{R}^N,
\end{cases}$$

(4)

may be seen as a formal limit of (1) when $\alpha \rightarrow 0$. The nonlocal nonlinear Schrödinger equation

$$i\psi_t + \Delta \psi + V_{\text{ext}}(x)\psi + (I_2 * |\psi|^2)\psi - |\psi|^{p-2}\psi = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3$$

is used as an approximation to Hartree-Fock model of a quantum many-body system of electrons under the presence of an external potential $V_{\text{ext}}$ (see [15]). In such a setting, (4) and its stationary counterpart bear the name of Schrödinger-Poisson-Slater [5], Schrödinger-Poisson-X$_\alpha$ [2, 20], or Maxwell-Schrödinger-Poisson [3, 7] equations. The convolution term in (4) represents the Coulombic repulsion between the electrons. The local term $|u|^{2p-2}u$ was introduced by Slater [27] as a local approximation of the exchange potential in the Hartree-Fock model [5, 20].

**Notations.** Throughout in this paper we use the following notations.

- $H^1(\mathbb{R}^N)$ denotes the standard Sobolev space endowed with the usual norm

$$\|u\|^2 = \int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2)dx.$$ 

We shall denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^1(\mathbb{R}^N)$ and its dual $H^{-1}(\mathbb{R}^N)$.

- $D^{1,2}(\mathbb{R}^N)$ is the Hilbert space

$$D^{1,2}(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N)\}$$

endowed with the standard norm

$$\|u\|_{D^{1,2}}^2 = \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$ 

and the associated scalar product

$$\langle u, v \rangle_{D^{1,2}} = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v.$$ 

- $L^s(\mathbb{R}^N)$ is the usual Lebesgue space in $\mathbb{R}^N$ of order $s \in [1, \infty]$ whose norm will be denoted by $\| \cdot \|_s$. 

2. Main results. Our first result provides sufficient conditions for the nonexistence of solutions to (1).

**Theorem 2.1.** Assume $K \in C^1(\mathbb{R}^N)$, $K \geq 0$. If one of the following hold

(i) $x \cdot \nabla K(x) + \gamma K(x) \geq 0$ in $\mathbb{R}^N$ for some $\gamma \in (-\infty, \frac{N+2}{2})$ and

$$N < \min \left\{ \frac{N + \alpha}{p}, \frac{N + 2 - 2\gamma}{q} \right\};$$

(ii) $x \cdot \nabla K(x) + \gamma K(x) \leq 0$ in $\mathbb{R}^N$ for some $\gamma \in \mathbb{R}$ and

$$N > \max \left\{ \frac{2 + \frac{N + \alpha}{p} + \frac{N + 2 - 2\gamma}{q}}{2} \right\};$$

then, the only solution $(u, \phi)$ of (1) that satisfies

$$u \in H^1(\mathbb{R}^N) \cap L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N), \quad \phi \in H^1(\mathbb{R}^N)$$

and

$$K(x)|u|^q \in L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N), \quad |\nabla u| \in H^1_{loc}(\mathbb{R}^N) \cap L^{\frac{2N}{N+\alpha}}_{loc}(\mathbb{R}^N)$$

is $u \equiv \phi \equiv 0$.

By taking $K \equiv 0$, for suitable choice of $\gamma$ in (5) and (6) we obtain that if $p \geq \frac{N + \alpha}{N - 2}$ or $p \leq \frac{N + \alpha}{N}$ then the only solution of (3) which satisfies $u \in H^1(\mathbb{R}^N) \cap L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ and $|\nabla u| \in H^1_{loc}(\mathbb{R}^N) \cap L^{\frac{2N}{N+\alpha}}_{loc}(\mathbb{R}^N)$ is the trivial one. We thus recover the result in [23, Theorem 2].

By taking $\gamma = 0$ in Theorem 2.1 we obtain:

**Corollary 1.** Let $K \equiv \text{const} > 0$. If one of the following conditions hold

$$p \geq \frac{N + \alpha}{N - 2} \quad \text{and} \quad \frac{N + \alpha}{p} \leq \frac{N + 2}{q};$$

or

$$p \leq \frac{N + \alpha}{N} \quad \text{and} \quad \frac{N + \alpha}{p} > \frac{N + 2}{q},$$

then the only solution $(u, \phi)$ of (1) satisfying (7)-(8) is the trivial one.

**Corollary 2.** Let $K(x) = (1 + |x|^2)^{-\gamma/2}$. If $\gamma \in (0, \frac{N+2}{2})$ and (5) holds or $\gamma \leq 0$ and (6) holds then the only solution $(u, \phi)$ of (1) satisfying (7)-(8) is the trivial one.

Let us now discuss the existence of a solution to (1). Crucial to our approach will be the Hardy-Littlewood-Sobolev inequality

$$\int_{\mathbb{R}^N} |I_\alpha * u|^\frac{N \cdot N}{N-\alpha} \leq C \left( \int_{\mathbb{R}^N} |u|^s \right)^\frac{N}{N-\alpha}$$

for any $u \in L^s(\mathbb{R}^N), s \in \left(1, \frac{N}{s}\right)$

which also implies

$$\left| \int_{\mathbb{R}^N} (I_\alpha * u)v \right| \leq C \|u\|_s \|v\|_t \quad \text{for any } u \in L^s(\mathbb{R}^N), v \in L^t(\mathbb{R}^N), \frac{1}{s} + \frac{1}{t} = 1 + \frac{\alpha}{N}. \quad (10)$$

It is more convenient to reduce our system (1) to a single equation. More exactly, for any $u \in H^1(\mathbb{R}^N)$ define

$$T_u : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}, \quad T_u(v) = \int_{\mathbb{R}^N} K(x)|u|^q v dx.$$
If \( K \in L^r(\mathbb{R}^N) \), with
\[
\frac{1}{r} + \frac{q + 1}{2^*} = 1 \quad \text{and} \quad 1 < q < \frac{N + 2}{N - 2},
\] (11)
then, by Hölder and Sobolev inequality one gets that \( T_u \) is linear and continuous. By Lax-Milgram theorem, there exists a unique \( \phi_u \in D^{1,2}(\mathbb{R}^N) \) such that
\[
T_u(v) = (\phi_u, v)_{D^{1,2}} \quad \text{for all } v \in D^{1,2}(\mathbb{R}^N).
\] (12)
As a result, \( \phi_u \) solves
\[
-\Delta \phi_u = K(x)|u|^q \quad \text{in } \mathbb{R}^N,
\]
and
\[
\phi_u(x) = A_2 \int_{\mathbb{R}^N} \frac{K(y)|u|^p(y)}{|x - y|^{N-2}} \, dy \quad \text{where } A_2 \text{ corresponds to } (2).
\]
Hence
\[
\phi_u = I_2 * (K|u|^q).
\] (13)
More properties of \( \phi_u \) are given in Lemma 3.1 below. We should finally note that with \( \phi_u \) given by (13), system (1) reduces implicitly to the single equation
\[
-\Delta u + u = (I_2 * |u|^p)|u|^{p-2}u + K(x)\phi_u|u|^{q-2}u \quad \text{in } \mathbb{R}^N.
\] (14)

Let us remark that (14) has a variational structure. If \( \frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2} \) and \( q, r \) satisfy (11) then then functional
\[
\mathcal{J}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) - \frac{1}{2p} \int_{\mathbb{R}^N} (I_2 * |u|^p)|u|^{p} - \frac{1}{2q} \int_{\mathbb{R}^N} K(x)\phi_u|u|^q
\]
is well defined for all \( u \in H^1(\mathbb{R}^N) \) and any critical point \( u \) of \( \mathcal{J} \) is a weak solution to (14).

Our existence result is the following.

**Theorem 2.2.** Assume \( 1 < q < \frac{N+2}{N-2}, \frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}, q < p \) and \( K \in L^r(\mathbb{R}^N) \), with \( r \) given by (11). Then, problem (1) has a solution \( (u, \phi) \in H^1(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N) \). Moreover, \( u \) is a ground state of (14).

In order to deal with the lack of compactness of \( H^1(\mathbb{R}^N) \) into the Lebesgue spaces \( L^s(\mathbb{R}^N), 2 \leq s \leq 2^* \), we rely on a careful analysis of the Palais-Smale (in short \( (PS) \)) sequences for \( \mathcal{J} \) restricted to its Nehari manifold. Roughly speaking, we have that any \( (PS) \) either converges strongly to its weak limit or differs from it by a finite number of sequences, which are nothing but translated solutions of (3), centered at points whose distances from the origin and whose interdistances go to infinity (see Proposition 3). Then, a further evaluation of the energy levels of \( \mathcal{J} \) allows us to locate some ranges for which the compactness is still preserved. Such an approach was successfully applied for the Schrödinger-Poisson system (4) in [8, 9] and recently adapted to the study of the non-autonomous fractional Choquard equation in [10]. Unlike the approach in [10] where a direct energy estimation is possible due to the presence of suitable non-autonomous terms, we shall rely essentially on several nonlocal Brezis-Lieb type results as we describe in Section 3.2.

The remaining part of the paper is organised as follows. Section 3 contains some preliminary results which we will use in the study of existence of a ground state to (1). Sections 4 and 5 contain the proofs of our main results.
3. Preliminary results.

3.1. Some properties of $\phi_u$.

**Lemma 3.1.** We have

(i) $\phi_u \geq 0$ for any $u \in H^1(\mathbb{R}^N)$;

(ii) $\phi_{tu} = t^q \phi_u$ for any $t > 0$;

(iii) if $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, then $\phi_{u_n} \to \phi_u$ strongly in $D^{1,2}(\mathbb{R}^N)$.

**Proof.** (i) and (ii) follow from the definition of $\phi_u$.

(iii) For a proof of this part in dimension $N = 3$ the reader may consult [8, Proposition 2.2(a)]. Here we provide a different argument.

Let us note first that from the definition of $\phi_u$ in (12) we deduce

$$\|\phi_u\|_{D^{1,2}} = \|T_u\|_{L(D^{1,2})}.$$ 

For any $v \in D^{1,2}(\mathbb{R}^N)$ we have

$$|T_{u_n}(v) - T_u(v)| \leq \int_{\mathbb{R}^N} K(x) |u_n|^q - |u|^q |v|$$

$$\leq \|v\|_{D^{1,2}} \left( \int_{\mathbb{R}^N} K(x) \frac{2N}{2N+2} |u_n|^q - |u|^q \right)^{\frac{N+2}{2N}}.$$ 

Using the continuous embedding of $H^1(\mathbb{R}^N)$ into $L^s(\mathbb{R}^N)$, $2 \leq s \leq 2^*$ and Lemma 3.3 below, it follows that

$$\left| |u_n|^q - |u|^q \right|^{\frac{2N}{2N+2}} \to 0 \quad \text{weakly in } L^\frac{N+2}{N-2}(\mathbb{R}^N).$$

Thus, since $K \frac{2N}{2N+2} \in L^\frac{N+2}{N-2}(\mathbb{R}^N)$ we deduce

$$\|\phi_{u_n} - \phi_u\|_{D^{1,2}} = \|T_{u_n} - T_u\|_{L(D^{1,2})}$$

$$\leq \left( \int_{\mathbb{R}^N} K(x) \frac{2N}{2N+2} |u_n|^q - |u|^q \right)^{\frac{N+2}{2N}} \to 0.$$ 

\[\square\]

3.2. Some nonlocal versions of Brezis-Lieb lemma. In this part we collect some useful results in dealing with the existence of a ground state solution to (3).

We first recall the concentration-compactness lemma of P.-L. Lions formulated in an inequality setting.

**Lemma 3.2.** ([18, Lemma 1.1], [23, Lemma 2.3]) Let $s \in [2, 2^*]$. There exists a constant $C > 0$ such that for any $u \in H^1(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} |u|^s \leq C\|u\| \left( \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u|^s \right)^{1-\frac{s}{2}}.$$

**Lemma 3.3.** ([4, Proposition 4.7.12]) Let $s \in (1, \infty)$. Assume $(w_n)$ is a bounded sequence in $L^s(\mathbb{R}^N)$ that converges to $w$ almost everywhere. Then $w_n \rightharpoonup w$ weakly in $L^s(\mathbb{R}^N)$.

Using a similar proof to that in the original Brezis-Lieb lemma [6, Theorem 2] (see also [29, Proposition 4.7.30]) we have
4.2. Proof of Theorem 2.1 completed.

Let \( s \in (1, \infty) \). Assume \((w_n)\) is a bounded sequence in \( L^s(\mathbb{R}^N) \) that converges to \( w \) almost everywhere. Then, for every \( q \in [1, s] \) we have

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \left| |w_n|^q - |w_n - w|^q - |w|^q \right|^\frac{2}{q} = 0,
\]
and

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \left| |w_n|^{q-1}w_n - |w_n - w|^{q-1}(w_n - w) - |w|^{q-1}w \right|^\frac{2}{q} = 0.
\]

A first nonlocal version of Bezis-Lieb lemma in the literature appeared in \[23\] (see also \[21\]) and reads as follows.

\[ \text{Lemma 3.4. (Local Brezis-Lieb lemma) Let } s \in (1, \infty). \text{ Assume } (w_n) \text{ is a bounded sequence in } L^s(\mathbb{R}^N) \text{ that converges to } w \text{ almost everywhere. Then, for every } q \in [1, s] \text{ we have } \]

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \left| |w_n|^q - |w_n - w|^q - |w|^q \right|^\frac{2}{q} = 0,
\]
and

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \left| |w_n|^{q-1}w_n - |w_n - w|^{q-1}(w_n - w) - |w|^{q-1}w \right|^\frac{2}{q} = 0.
\]

Below we state another nonlocal version of Brezis-Lieb lemma.

\[ \text{Lemma 3.6. (Nonlocal Brezis-Lieb lemma, [23, Lemma 2.4]) Let } \alpha \in (0, N) \text{ and } p \in [1, \frac{2N}{N+\alpha}). \text{ Assume } (u_n) \text{ is a bounded sequence in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N) \text{ that converges almost everywhere to some } u : \mathbb{R}^N \to \mathbb{R}. \text{ Then, for any } h \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N) \text{ we have } \]

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} (I_{\alpha} \ast |u_n|^p)|u_n|^{p-2}u_nh = \int_{\mathbb{R}^N} (I_{\alpha} \ast |u|^p)|u|^{p-2}uh.
\]

4. Proof of Theorem 2.1.

4.1. A Pohozaev identity. The main tool in proving Theorem 2.1 is the following Pohozaev type identity.

\[ \text{Proposition 1. Let } (u, \phi) \text{ be a solution of } (1) \text{ that satisfies } (7)-(8). \text{ Then } \]

\[
\int_{\mathbb{R}^N} \left( \frac{N-2}{2} |\nabla u|^2 + \frac{N}{2} |u|^2 \right)
= \frac{N + \alpha}{2p} \int_{\mathbb{R}^N} (I_{\alpha} \ast |u|^p)|u|^p + \frac{N + 2}{2q} \int_{\mathbb{R}^N} K(x)|\phi|^q + \frac{1}{q} \int_{\mathbb{R}^N} |\phi|^q \cdot \nabla K(x).
\]

4.2. Proof of Theorem 2.1 completed. Let \((u, \phi)\) be a solution of (1) which satisfies (7)-(8). It is enough to show that \( u \equiv 0 \) as the second equation of (1) together with \( \phi \in H^1(\mathbb{R}^N) \) will imply \( \phi \equiv 0 \). Suppose by contradiction that the solution \((u, \phi)\) satisfies \( u \not\equiv 0 \).

For convenience, let us denote

\[
A(u) = \int_{\mathbb{R}^N} K(x)|\phi|^q, \quad B(u) = \int_{\mathbb{R}^N} |\phi|^q \cdot \nabla K(x), \quad C(u) = \int_{\mathbb{R}^N} (I_{\alpha} \ast |u|^p)|u|^p.
\]

From Proposition 1 we have

\[
\frac{N - 2}{2} \|\nabla u\|^2 + \frac{N}{2} \|u\|^2 = \frac{N + \alpha}{2p} C(u) + \frac{N + 2}{2q} A(u) + \frac{1}{q} B(u).
\]
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Since $u$ is a solution of (14) we also have

$$\|u\|^2 = A(u) + C(u). \quad (17)$$

(i) Assume $x \cdot \nabla K(x) + \gamma K(x) \geq 0$ in $\mathbb{R}^N$ for some $\gamma \in (-\infty, \frac{N+2}{2})$ and that (5) holds. Then

$$B(u) \geq -\gamma A(u)$$

so that from (16) and (17) we obtain

$$\frac{N}{2} \|u\|^2 \geq \frac{N + \alpha}{2p} C(u) + \frac{N + 2 - 2\gamma}{2q} A(u)$$

that is,

$$\left(N - \frac{N + \alpha}{p}\right) C(u) \geq \left(\frac{N + 2 - 2\gamma}{q} - N\right) A(u)$$

But this last inequality is impossible since $C(u) \geq 0$, $A(u) \geq 0$ and $p, q, N, \alpha, \gamma$ satisfy (5).

(ii) Assume $x \cdot \nabla K(x) + \gamma K(x) \leq 0$ in $\mathbb{R}^N$ for some $\gamma \in \mathbb{R}$ and that (6) holds. It follows that

$$B(u) \leq -\gamma A(u)$$

so that (16) together with (17) yield

$$\frac{N - 2}{2} \|u\|^2 \leq \frac{N + \alpha}{2p} + \frac{N + 2 - 2\gamma}{2q} A(u)$$

that is,

$$\left(N - 2 - \frac{N + \alpha}{p}\right) C(u) \leq \left(\frac{N + 2 - 2\gamma}{q} - N + 2\right) A(u).$$

Note that the above inequality is impossible since $C(u) \geq 0$, $A(u) \geq 0$ and $p, q, N, \alpha, \gamma$ satisfy (6). This concludes our proof.

5. Proof of Theorem 2.2.

5.1. The Nehari manifold associated with (14). Define the Nehari manifold associated with $\mathcal{J}$ as

$$\mathcal{N} = \{ u \in H^1(\mathbb{R}^N) \setminus \{0\} : \langle \mathcal{J}'(u), u \rangle = 0 \} \quad (18)$$

and let

$$m_{\mathcal{J}} = \inf_{u \in \mathcal{N}} \mathcal{J}(u).$$

Remark that for $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ and $t > 0$ we have

$$\langle \mathcal{J}'(tu), tu \rangle = t^2 \|u\|^2 - t^{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p - t^{2q} \int_{\mathbb{R}^N} K(x) \phi_u |u|^{q-2} u.$$

Since $p > q > 1$, the equation $\langle \mathcal{J}'(tu), tu \rangle = 0$ has a unique positive solution $t = t(u)$ and the corresponding element $t(u)u \in \mathcal{N}$ is called the projection of $u$ on $\mathcal{N}$. The main properties of the Nehari manifold which we use in this paper are stated below.

Proposition 2. (i) $\mathcal{J}|_{\mathcal{N}}$ is bounded from below by a positive constant;
(ii) If $u$ is a critical point of $\mathcal{J}$ in $\mathcal{N}$ then $u$ is a free critical point of $\mathcal{J}$;
(iii) If $(u_n)$ is a $\langle PS \rangle$ sequence for $\mathcal{J}|_{\mathcal{N}}$ then $(u_n)$ is a $\langle PS \rangle$ sequence for $\mathcal{J}$. 

Proof. (i) Using the Hardy-Littlewood-Sobolev inequality (10) together with the continuous embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{\frac{2Np}{N-p}}(\mathbb{R}^N)$, for any $u \in \mathcal{N}$ we have

$$0 = \langle J'(u), u \rangle = \|u\|^2 - \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p - \int_{\mathbb{R}^N} K(x)\phi_u |u|^q \geq \|u\|^2 - C(\|u\|^{2p} + \|u\|^{2q}).$$

Hence, there exists $C_0 > 0$ such that

$$\|u\| \geq C_0 > 0 \quad \text{for all } u \in \mathcal{N}. \quad (19)$$

Using this fact we have

$$J(u) = \left(\frac{1}{2} - \frac{1}{2q}\right)\|u\|^2 + \left(\frac{1}{2} - \frac{1}{2p}\right) \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \geq \left(\frac{1}{2} - \frac{1}{2q}\right)C_0^2 > 0.$$

(ii) For $u \in H^1(\mathbb{R}^N)$ let $G(u) = \langle J'(u), u \rangle$. If $u \in \mathcal{N}$, by (19) we obtain

$$\langle G'(u), u \rangle = 2\|u\|^2 - 2p \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p - 2q \int_{\mathbb{R}^N} K(x)\phi_u |u|^q = 2(1-q)\|u\|^2 - 2(p-q) \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p$$

$$\leq -2(q-1)\|u\|^2 < -2(q-1)C_0.$$

Assume now that $u \in \mathcal{N}$ is a critical point of $J$ in $\mathcal{N}$. By the Lagrange multiplier theorem, there exists $\lambda \in \mathbb{R}$ such that $J'(u) = \lambda G'(u)$. In particular $\langle J'(u), u \rangle = \lambda \langle G'(u), u \rangle$. Since $\langle G'(u), u \rangle < 0$, it follows that $\lambda = 0$ so $J'(u) = 0$.

(iii) Let $(u_n) \subset \mathcal{N}$ be a $(PS)$ sequence for $J|_{\mathcal{N}}$. Since

$$J(u_n) \geq \left(\frac{1}{2} - \frac{1}{2q}\right)\|u_n\|^2,$$

it follows that $(u_n)$ is bounded in $H^1(\mathbb{R}^N)$. Let us show that $J'(u_n) \to 0$. Since

$$o(1) = J'|_{\mathcal{N}}(u_n) = J'(u_n) - \lambda_n G'(u_n),$$

for some $\lambda_n \in \mathbb{R}$, it follows that

$$\lambda_n (G'(u_n), u_n) = \langle J'(u_n), u_n \rangle + o(1) = o(1).$$

By (20) we deduce $\lambda_n \to 0$ which further yields $J'(u_n) \to 0$. \hfill \Box

5.2. A compactness result. Let

$$E : H^1(\mathbb{R}^N) \to \mathbb{R}, \quad E(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p,$$

be the energy functional corresponding to (3). Also, consider its Nehari manifold

$$\mathcal{N}_E = \{ u \in H^1(\mathbb{R}^N) \setminus \{0\} : \langle E'(u), u \rangle = 0 \}$$

and let

$$m_E = \inf_{u \in \mathcal{N}_E} E(u).$$
Proposition 3. Let \( (u_n) \subset \mathcal{N} \) be a \((PS)\) sequence of \( J|_{\mathcal{N}} \), that is,
(a) \( (\mathcal{J}(u_n)) \) is bounded;
(b) \( J'|_{\mathcal{N}} (u_n) \to 0 \) strongly in \( H^{-1}(\mathbb{R}^N) \).

Then, there exists a solution \( u \in H^1(\mathbb{R}^N) \) of (14) such that replacing \( (u_n) \) with a subsequence the following alternative holds
(1) either \( u_n \to u \) strongly in \( H^1(\mathbb{R}^N) \);
or
(2) \( u_n \to u \) weakly (but not strongly) in \( H^1(\mathbb{R}^N) \) and there exists a positive integer \( k \geq 1, k \) functions \( u_1, u_2, \ldots, u_k \in H^1(\mathbb{R}^N) \) which are nontrivial weak solutions to
(3) and \( k \) sequence of points \( (y_{n,1}), (y_{n,2}), \ldots, (y_{n,k}) \subset \mathbb{R}^N \) such that:

(i) \( |y_{n,j}| \to \infty \) and \( |y_{n,j} - y_{n,i}| \to \infty \) if \( i \neq j, n \to \infty \);

(ii) \( u_n - \sum_{j=1}^{k} u_j (\cdot + y_{n,j}) \to u \) in \( H^1(\mathbb{R}^N) \);

(iii) \( \mathcal{J}(u_n) \to \mathcal{J}(u) + \sum_{j=1}^{k} \mathcal{E}(u_j) \);

Proof. Because \( (u_n) \) is bounded in \( H^1(\mathbb{R}^N) \), there exists \( u \in H^1(\mathbb{R}^N) \) such that, up to a subsequence, we have
\[
\begin{aligned}
\begin{cases}
  u_n \rightharpoonup u \quad \text{weakly in } H^1(\mathbb{R}^N), \\
  u_n \to u \quad \text{weakly in } L^s(\mathbb{R}^N), \ 2 \leq s \leq 2^*, \\
  u_n \to u \quad \text{a.e. in } \mathbb{R}^N.
\end{cases}
\end{aligned}
\]

We also need the following result:

Lemma 5.1. We have

(i) \( \int_{\mathbb{R}^N} K(x) \phi_{u_n} |u_n|^q = \int_{\mathbb{R}^N} K(x) \phi_u |u|^q + o(1) \);

(ii) \( \int_{\mathbb{R}^N} K(x) \phi_{u_n} |u_n|^{q-2} u_n h = \int_{\mathbb{R}^N} K(x) \phi_u |u|^{q-2} u h + o(1), \text{ for all } h \in H^1(\mathbb{R}^N) \).

Proof. We shall prove only (ii) as the (i) part is similar.

Note first that
\[
\left| \int_{\mathbb{R}^N} K(x) \phi_{u_n} |u_n|^{q-2} u_n h - \int_{\mathbb{R}^N} K(x) \phi_u |u|^{q-2} u h \right| \leq \int_{\mathbb{R}^N} |K(x)||\phi_{u_n} - \phi_u||u|^{q-1} |h|
\]

and we get
\[
= \|K\|_{r} \|\phi_{u_n} - \phi_u\|_{2^*} \|u_n\|_{2^*} \|h\|_{2^*} = o(1).
\]

Using Lemma 3.1(iii) and Hölder’s inequality we find
\[
\int_{\mathbb{R}^N} |K(x)||\phi_{u_n} - \phi_u||u_n|^{q-1} |h| \leq \|K\|_{r} \|\phi_{u_n} - \phi_u\|_{2^*} \|u_n\|_{2^*} \|h\|_{2^*}.
\]

By Lemma 3.3 we have \( |u_n|^{q-2} u_n \rightharpoonup |u|^{q-2} u \) weakly in \( L^{\frac{2^*}{q-2}}(\mathbb{R}^N) \).
Since $K(x)\phi_u h \in L^{2^*_{N-\gamma-\gamma'}}(\mathbb{R}^N)$ it follows that
\[
\int_{\mathbb{R}^N} K(x)\phi_u h(|u_n|^q-2u_n - |u|^{q-2}u) = o(1). \tag{24}
\]

Now, the proof follows by combining (22)-(24). \qedhere

We now return to the proof of Proposition 3. By (21), Lemma 3.6 and Lemma 5.1(ii) it follows that $J'(u) = 0$ so $u \in H^1(\mathbb{R}^N)$ is a solution of (14).

If $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$ then the first alternative in the statement of Proposition 3 holds and we are done. Assume in the following that $(u_n)$ does not converge strongly in $H^1(\mathbb{R}^N)$ to $u$ and define $z_{n,1} = u_n - u$. Then $(z_{n,1})$ converges weakly and not strongly to zero in $H^1(\mathbb{R}^N)$ and
\[
\|u_n\|^2 = \|u\|^2 + \|z_{n,1}\|^2 + o(1). \tag{25}
\]

By Lemma 3.5 we have
\[
\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p + \int_{\mathbb{R}^N} (I_\alpha * |z_{n,1}|^p)|z_{n,1}|^p + o(1). \tag{26}
\]

Using (25), (26) and Lemma 5.1(i) we deduce
\[
\mathcal{J}(u_n) = \mathcal{J}(u) + \mathcal{E}(z_{n,1}) + o(1). \tag{27}
\]

For any $h \in H^1(\mathbb{R}^N)$, by Lemma 3.6 and Lemma 5.1(ii) we have
\[
\langle \mathcal{E}'(z_{n,1}), h \rangle = o(1). \tag{28}
\]

Next, by Lemma 3.5 and Lemma 5.1(i) we have
\[
0 = \langle \mathcal{J}'(u_n), u_n \rangle = \langle \mathcal{J}'(u), u \rangle + \langle \mathcal{E}'(z_{n,1}), z_{n,1} \rangle + o(1)
= \langle \mathcal{E}'(z_{n,1}), z_{n,1} \rangle + o(1),
\]

which yields
\[
\langle \mathcal{E}'(z_{n,1}), z_{n,1} \rangle = o(1). \tag{29}
\]

Let
\[
\delta := \limsup_{n \to \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |z_{n,1}|^{\frac{2N}{N-\gamma+\alpha}} \right) \geq 0.
\]

We claim that $\delta > 0$. Indeed, if $\delta = 0$, by Lemma 3.2 we deduce $z_{n,1} \to 0$ strongly in $L^{\frac{2N}{N-\gamma+\alpha}}(\mathbb{R}^N)$. Then, by Hardy-Littlewood-Sobolev inequality (10) we find
\[
\int_{\mathbb{R}^N} (I_\alpha * |z_{n,1}|^p)|z_{n,1}|^p = o(1).
\]

This fact combined with (29) yields $z_{n,1} \to 0$ strongly in $H^1(\mathbb{R}^N)$ in contradiction to our assumption.

Hence, $\delta > 0$ so that we may find $y_{n,1} \in \mathbb{R}^N$ with
\[
\int_{B_1(y_{n,1})} |z_{n,1}|^{\frac{2N}{N-\gamma+\alpha}} > \frac{\delta}{2}. \tag{30}
\]
Considering the sequence \( (z_{n,1}(\cdot + y_{n,1})) \), there exists \( u_1 \in H^1(\mathbb{R}^N) \) such that, up to a subsequence, we have

\[
\begin{align*}
z_{n,1}(\cdot + y_{n,1}) & \rightharpoonup u_1 \quad \text{weakly in } H^1(\mathbb{R}^N), \\
z_{n,1}(\cdot + y_{n,1}) & \rightarrow u_1 \quad \text{strongly in } L^{\frac{2Np}{N+p}}(\mathbb{R}^N), \\
z_{n,1}(\cdot + y_{n,1}) & \rightarrow u_1 \quad \text{a.e. in } \mathbb{R}^N.
\end{align*}
\]

Passing to the limit in (30) we find

\[
\int_{B_1(0)} |u_1|^{\frac{2Np}{N+p}} \geq \frac{\delta}{2},
\]

so \( u_1 \neq 0 \). Also, since \( (z_{n,1}) \) converges weakly to zero in \( H^1(\mathbb{R}^N) \) it follows that \( (y_{n,1}) \) is unbounded. Passing to a subsequence we may assume \( |y_{n,1}| \rightarrow \infty \). From (29) we also obtain \( \mathcal{E}'(u_1) = 0 \), so \( u_1 \) is a nontrivial solution of (3).

Set next

\[
z_{n,2}(x) = z_{n,1}(x) - u_1(x - y_{n,1}).
\]

As above we have

\[
\|z_{n,1}\|^2 = \|u_1\|^2 + \|z_{n,2}\|^2 + o(1).
\]

and by Lemma 3.5 we derive

\[
\int_{\mathbb{R}^N} (I_\alpha * |z_{n,1}|^p) |z_{n,1}|^p = \int_{\mathbb{R}^N} (I_\alpha * |u_1|^p) |u_1|^p + \int_{\mathbb{R}^N} (I_\alpha * |z_{n,2}|^p) |z_{n,2}|^p + o(1).
\]

Thus,

\[
\mathcal{E}(z_{n,1}) = \mathcal{E}(u_1) + \mathcal{E}(z_{n,2}) + o(1)
\]

so, by (27) one has

\[
\mathcal{J}(u_n) = \mathcal{J}(u) + \mathcal{E}(u_1) + \mathcal{E}(z_{n,2}) + o(1).
\]

Using the above techniques one can also derive

\[
\langle \mathcal{E}'(z_{n,2}), h \rangle = o(1) \quad \text{for any } h \in H^1(\mathbb{R}^N)
\]

and

\[
\langle \mathcal{E}'(z_{n,2}), z_{n,2} \rangle = o(1).
\]

If \( (z_{n,2}) \) converges strongly to zero, the proof finishes (and take \( k = 1 \) in the statement of Proposition 3). Assuming that \( z_{n,2} \rightarrow 0 \) weakly and not strongly in \( H^1(\mathbb{R}^N) \), we iterate the process. In \( k \) number of steps we find a set of sequences \( (y_{n,j}) \subset \mathbb{R}^N, 1 \leq j \leq k \) with

\[
|y_{n,j}| \rightarrow \infty \quad \text{and} \quad |y_{n,i} - y_{n,j}| \rightarrow \infty \quad \text{as } i \neq j, n \rightarrow \infty
\]

and \( k \) nontrivial solutions \( u_1, u_2, \ldots, u_k \in H^1(\mathbb{R}^N) \) of (3) such that, denoting

\[
z_{n,j}(x) := z_{n,j-1}(x) - u_{j-1}(x - y_{n,j-1}), \quad 2 \leq j \leq k,
\]

we have

\[
z_{n,j}(x + y_{n,j}) \rightarrow u_j \quad \text{weakly in } H^1(\mathbb{R}^N)
\]

and

\[
\mathcal{J}(u_n) = \mathcal{J}(u) + \sum_{j=1}^{k} \mathcal{E}(u_j) + \mathcal{E}(z_{n,k}) + o(1).
\]

Since \( \mathcal{E}(u_j) \geq m_\mathcal{E} \) and \( (\mathcal{J}(u_n)) \) is bounded, the process can be iterated only a finite number of times. This concludes our proof. \( \square \)
Corollary 3. Let \( c \in (0, m_\varepsilon) \). Then, any \((PS)_c\) sequence of \( J \) is relatively compact.

Proof. Let \((u_n)\) be a \((PS)_c\) sequence of \( J \). Since \( \mathcal{E}(u_j) \geq m_\varepsilon \) in Proposition 3, it follows that up to a subsequence \( u_n \to u \) strongly in \( H^1(\mathbb{R}^N) \) and \( u \) is a solution of (14).

5.3. Proof of Theorem 2.2 completed. In view of Proposition 3 and Corollary 3, it is enough to show that there exists \( M > 0 \) such that \( K \in L^r(\mathbb{R}^N) \) imply

\[
m_J < m_\varepsilon.
\]

Let \( w \in H^1(\mathbb{R}^N) \) be a ground state of (3); by [23, Theorem 1] we know that such a ground state exists. Let \( tw \) be the projection of \( w \) on \( N \), that is, \( t = t(w) > 0 \) is the unique real number such that \( tw \in N \) (with \( N \) defined in (18)). Denote

\[
A(w) = \int_{\mathbb{R}^N} K(x) \phi_w |w|^q, \quad B(w) = \int_{\mathbb{R}^N} (I_\alpha * |w|^p) |w|^p.
\]

Since \( w \in N_\varepsilon \) and \( tw \in N \) we have

\[
||w||^2 = B(w) \tag{31}
\]

and

\[
t^2||w||^2 = t^{2q}A(w) + t^{2p}B(w).
\]

Hence \( t < 1 \). We claim that

\[
J(tw) < m_\varepsilon = \mathcal{E}(w).
\]

Indeed, this can be written as

\[
\frac{1}{2}t^2||w||^2 - \frac{1}{2q}t^{2q}A(w) - \frac{1}{2p}t^{2p}B(w) < \frac{1}{2}||w||^2 - \frac{1}{2p}B(w).
\]

Using (31), this amounts to

\[
A(w) \leq \frac{q}{p} \frac{(1-t^{2p})(p-1)}{t^{2q}(q-1)} ||w||^2. \tag{32}
\]

On the other hand

\[
A(w) \leq ||K||_r \phi_w ||_2 \cdot ||w||_2^q.
\]

Hence, if \( ||K||_r \) is small, (32) holds which implies \( m_J \leq J(tw) < m_\varepsilon \) and proves our result.

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E-mail address: marius.ghergu@ucd.ie
E-mail address: gurpreet.singh@ucdconnect.ie