Parametrized post-Newtonian equations of motion of $N$ mass monopoles with the SEP violation

Sergei A. Klioner

Lohrmann-Observatorium, Technische Universität Dresden, 01062 Dresden, Germany

(Dated: January 25, 2018)

Post-Newtonian equations of motion of a system of $N$ mass monopoles in the framework of the PPN formalism with two parameters $\beta$ and $\gamma$ are derived for the case when the Strong Equivalence Principle can be violated. The derivation is based on the previously published general framework. The multipole moments of each body are defined as a PPN-generalization of the corresponding Blanchet-Damour multiple moments in a relevant local reference system defined for that body. The classical ten integrals of the derived equations of motion are given. A special version of the equations of motion, for which seven of these integrals are exact, is discussed. The derived equations of motion can be used to test the Strong Equivalence Principle in various solar system experiments.

I. INTRODUCTION

Recent advances in observational technique and future accurate stellar catalogues will soon drastically improve the accuracy of routine observations of asteroids and other minor bodies of the solar system. Within a decade from now the accuracy of such observations are expected to achieve an accuracy level of several milliarcseconds. Even more accurate observations will be available from the ESA second-generation astrometric mission Gaia (see e.g., [7] and [18]) that is expected to achieve an accuracy between 0.2 and 3 milliarcseconds depending on the apparent brightness of the asteroid. This gives an improvement by a factor of 100–500 compared to the typical accuracy of classical Earth-bound positional observations of asteroids (about 1″). Moreover, Gaia is expected to provide routinely observations of about 500000 asteroids that will allow one to boost our knowledge of the short-term dynamics of the Solar system.

One of the interesting applications of the drastically improved accuracy is the use of asteroid motion to test various aspects of relativity. Already 4 years after the discovery of asteroid Icarus, Gylvarry [8] has suggested to use its motion to test general-relativistic perihelion precession. This idea has been used several times [12, 21–24, 26] and led to an independent determination of the relativistic perihelion precession with a precision of currently 4%. Although the perihelion precession of Icarus (10.05″ per century) is significantly smaller than that for Mercury (∼43″ per century) it has been recognized already by Dicke [6] that asteroids with their large inclinations and their range of semi-major axes allow one to distinguish between the general-relativistic perihelion precession and possible effects due to the solar oblateness (quadrupole) while it is well known that such a distinction is virtually impossible if only the motion of Mercury is considered. Although in recent years the analysis of motion of the whole system of inner planets did allow to determine separately the solar quadrupole moment and the relativistic precession [20] it remains unclear how reliable these estimates are.

One additional aspect of the solar system dynamics is related to the possible violation of
the Strong Equivalence Principle (SEP). Solar system dynamics was used in the very first quasi-empirical demonstration of the so-called Nordtvedt effect which is directly related to possible violations of the SEP [14]. This phenomenological approach was used in [15, 16, 19] for some tests of the SEP using asteroid motion. A more rigorous discussion of this phenomenon in the motion of asteroids is still to be done. In this paper we discuss the equations of motion of N mass monopoles in the Parametrized Post-Newtonian (PPN) framework with parameters $\beta$ and $\gamma$. After a short discussion of the Newtonian framework in Section II we derive and discuss the rigorous post-Newtonian equations of motion in Section III.

II. THE EQUATIONS OF MOTION IN THE NEWTONIAN FRAMEWORK

We start with purely empirical Newtonian considerations. We consider Newtonian N-body problem and assume that for each body $A$ we have two different masses: the inertial mass $M_{A}^{\text{iner}}$ appearing in the Newtonian second law and the gravitational mass $M_{A}$ appearing in the formula for the attractive force. Then the equations of motion read

$$\ddot{x}_{A}^{i} = -f_{A} \sum_{B \neq A} \mu_{B} \frac{r_{AB}^{i}}{r_{AB}^{3}} ,$$

where $x_{A}^{i}$ is the position of body $A$, $r_{AB}^{i} = x_{A}^{i} - x_{B}^{i}$, $\mu_{A} = GM_{A}$ is the mass parameter of body $A$, and $f_{A} = \frac{M_{A}}{M_{A}^{\text{iner}}}$ is the ratio of gravitational and inertial masses of body $A$. We see that the motion of body $A$ in an inertial reference system depends on gravitational mass parameters $\mu_{B}$ of other bodies and the mass ratio $f_{A}$ of body $A$. The mass ratios $f_{B}$ of other bodies play no role here.

We assume in this Section that $f_{A} = \text{const}$ for each body. The Lagrange function of these equations obviously read

$$L = \frac{1}{2} \sum_{A} f_{A}^{-1} \mu_{A} \dot{x}_{A}^{2} - \frac{1}{2} \sum_{A} \sum_{B \neq A} \frac{\mu_{A} \mu_{B}}{r_{AB}} .$$

From this Lagrange function it is clear that among ten classical integrals only the integral of energy involves both gravitational and inertial masses. The other nine integrals involve only inertial masses. For the dynamical modeling of the N-body problem the mass center integral plays an important role since it defines the origin of a convenient inertial reference system to be used. The condition that the mass center of the system coincides with the origin of the reference system involves only inertial masses $M_{A}^{\text{iner}} = f_{A}^{-1} M_{A}$ and read:

$$\sum_{A} f_{A}^{-1} \mu_{A} x_{A}^{i} = 0 ,$$
$$\sum_{A} f_{A}^{-1} \mu_{A} \dot{x}_{A}^{i} = 0 .$$

This condition with $f_{A} = 1$ for all bodies is usually applied to the solar system to define the barycentric coordinates. Either these conditions must be satisfied by positions and velocities of all bodies at some initial moment of time or Eqs. (1)–(3) can be used to eliminate one body (e.g., the Sun) from (1).
Eq. (1) shows that if the SEP is assumed implying \( f_A = 1 \) for all bodies, the equations of motion of the body under consideration are the same as in the Newtonian theory. In the framework of the PPN formalism, one can assume that \( f_A = 1 \) for laboratory test bodies and minor planets (see Section III) [27]. In this case the only effect of the violation of the SEP in the barycentric equations of motion of a minor body is the change of the definition of the center of mass as given by (3).

Larger effects from the possible violation of the SEP should be expected in the motion of the Sun and major planets (bodies for which \( f_A - 1 \) may be expected to be maximal). The changes in the positions of these bodies, in turn, influence also the motion of minor planets and thus, appear indirectly in (1). In order to make these effects explicit one should consider equations of relative motion (e.g., heliocentric).

### III. RIGOROUS DERIVATION OF THE EIH EQUATIONS WITH THE SEP VIOXATION

Our goal now is to derive the post-Newtonian equations of motion for \( N \) mass monopoles with a possible SEP violation rigorously and without using any empirical arguments. Klioner & Soffel [9] formulated a general theory of local reference systems in the framework of the PPN formalism with parameters \( \beta \) and \( \gamma \). That theory is a generalization of the Brumberg-Kopeikin and Damour-Soffel-Xu formalisms [1–5, 10, 11] for the case of the special subset of the PPN formalism [25]. The theory in [9] covers a number of aspects including the definition of body’s multipole moments in its own local reference system, the post-Newtonian tidal forces, various equations of motions, etc. It is straightforward to use the formalism from [9] to derive the relevant equations of motion for the case under study.

#### A. The equations of motions of \( N \) mass monopoles

The barycentric equations of motion of a system of \( N \) bodies characterized by their mass monopoles and spin dipoles were derived in Section IX.G of [9]. The derivation [9] is an immediate generalization of the derivation of the Einstein-Infeld-Hoffmann (EIH) equations given by Damour et al. [3] for general relativity. Below we denote the equation numbers of [9] as “KS(xx)”. The equations of motion KS(9.69) are derived rigorously using the PPN definition of Blanchet-Damour-like multipole moments and the assumptions on the multipole structure of gravitational field of each body given by KS(9.48)–KS(9.52) and the assumption that the spin of each body vanishes [28]:

\[
\mathcal{S}^a = \mathcal{O}(c^{-2}).
\]

The notations in this paper follow those of [9] (see e.g. Section II of that work). These assumptions represent a PPN generalization of the gravitational field of a mass monopole. The effect of the violation of the SEP (or the Nordtvedt effect) for assumptions KS(9.48)–KS(9.52) and (5) is given by the second term \(-R^a_i Q^m_a\) on the right-hand side of KS(9.69), where \( Q^a_m \) is given by KS(9.58) and KS(9.40). Let us further simplify the equation for the “Nordtvedt acceleration” \(-R^a_i Q^m_a\). Substituting KS(9.40) into KS(9.58) and multiplying the
result with \(-R^a_i\) one gets

\[
-R^a_i \, Q_a^m = \frac{1}{c^2} \eta \frac{\Omega_E}{M} a_E^i \\
- \frac{1}{c^2} \eta \frac{N}{M} \overline{Q}_{ij} a_E^j + \frac{1}{6c^2} (1 - \gamma) \frac{N}{M} \ddot{a}_E^i \\
+ \frac{1}{6c^2} (1 - \gamma) \frac{N}{M} \dot{a}_E^i + O(c^{-4}),
\]

(6)

where \(\overline{Q}_{ij} = R^a_i R^b_j Q_{ab}\) is the tidal quadrupole of the external gravitational field given by KS(7.3) and projected onto the local spatial axes. First, note that the simplified Nordtvedt acceleration does not depend on \(\dot{N}\) while from KS(9.58) and KS(9.40) one may assume such a dependence. Second, since \(N = \int_V \Sigma X^2 \, d^3X\) (as given by KS(8.13)) one has

\[
N = 2 \mathcal{P}, \\
\mathcal{P} = \int_V \Sigma X^a \, d^3X.
\]

(7)

Note that one needs these relations only in Newtonian approximation. Therefore, one can formulate one more assumption for the structure of the gravitational field of the bodies that should hold together with assumptions KS(9.48)–KS(9.52) and (5):

\[
\mathcal{P} = O(c^{-2}).
\]

(8)

With this assumption the last term in (6) vanishes. Third, \(N\) is related to Newtonian moment of inertia of the body and \(M = \int_V \Sigma \, d^3X + O(c^{-2})\) is the body’s mass in Newtonian limit. One can, therefore, always write \(N = k M L^2\), where \(L\) is the radius of a sphere encompassing the body and \(k\) is a numerical coefficient, \(k \sim 1\) for small bodies and significantly smaller for giant planets and the Sun. Using numerical characteristics of the solar system bodies one can demonstrate that both terms in (6) proportional to \(\frac{N}{M}\) are smaller than \(10^{-16}\) of the Newtonian barycentric acceleration of the body (factors \(\eta\) and \(1 - \gamma\) are assumed to be of order 1 in this estimate). This is significantly smaller than the first term in (6), which is between \(10^{-11}\) (for the Moon) and \(10^{-8}\) (for the Sun) of the Newtonian acceleration (again \(\eta\) is ignored in this estimate).

In this way, we give a rigorous derivation of the EIH equations with the single effect from
the violation of the SEP – the effect related to \( f_A \) in the equations of motion below:

\[
\ddot{x}_A = -f_A \sum_{B \neq A} \mu_B \frac{r^2_{AB}}{r_{AB}^3} \left[ (2 \gamma + 2 \beta + 1) \frac{\mu_A}{r_{AB}} + (2 \beta - 1) \sum_{C \neq A,B} \frac{\mu_C}{r_{BC}} + 2(\gamma + \beta) \sum_{C \neq A} \frac{\mu_C}{r_{AC}} \right]
\]

\[
+ \frac{1}{c^2} \sum_{B \neq A} \mu_B \frac{r^2_{AB}}{r_{AB}^3} \left\{ \frac{3 (r^3_{AB})^2}{2 r_{AB}^2} - \frac{1}{2} \sum_{C \neq A,B} \mu_C \frac{r^2_{AB} r^j_{BC}}{r_{BC}} \right\}
\]

\[
- (1 + \gamma) \dot{x}_B^j \dot{x}_B^j - \gamma \dot{x}_A^i \dot{x}_A^i + 2(1 + \gamma) \dot{x}_A^i \dot{x}_B^i \}
\]

\[
+ \frac{1}{c^2} \sum_{B \neq A} \mu_B \frac{r^2_{AB}}{r_{AB}^3} \left\{ 2(1 + \gamma) \dot{x}_A^i - (2 \gamma + 1) \dot{x}_B^j \right\} (\dot{x}_A^i - \dot{x}_B^j)
\]

\[
- \frac{1}{c^2} \left( 2 \gamma + \frac{3}{2} \right) \sum_{B \neq A} \frac{\mu_B}{r_{AB}} \sum_{C \neq A,B} \mu_C \frac{r^2_{BC}}{r_{BC}} + \mathcal{O}(c^{-4}),
\]

\[
f_A = 1 + \frac{1}{c^2} \frac{\Omega_A}{\mathcal{M}_A} + \mathcal{O}(c^{-4}),
\]

\[
\mu_A = G \mathcal{M}_A.
\]

These equations are valid in the PPN formalism with parameters \( \gamma \) and \( \beta \) for a system of \( N \) bodies, the gravitational fields of which satisfy assumptions KS(9.48)–KS(9.52), (9), and (10). The second and third terms on the right-hand side of (10) are neglected here because of their numerical smallness for applications in solar system.

These equations of motion are in a nice agreement with empirical considerations given in Section I

Note that the acceleration \( \ddot{x}_A \) of body \( A \) as given by (9) depend on the mass of the body itself \( \mu_A \). Those “self-terms” are explicitly shown in (11) as proportional to \( \mu_A \) in the first term in the curly braces. If the motion of a minor body is considered, its gravitational influence of the motion of the massive bodies can be neglected and this term in (9) can be omitted.

Eq. (11) gives the ratio between the gravitational and inertial masses of body \( A \) in the framework of PPN formalism, \( \eta \) being the Nordtvedt parameter (\( \eta = 0 \) if the SEP is satisfied). In the PPN formalism with two parameters \( \gamma \) and \( \beta \) considered by Klioner & Soffel [9] one has \( \eta = 4 \gamma - \beta - 3 \). A more general situation was discussed e.g. by Will [25].

B. The Lagrange function for the \( N \)-body problem

The equations of motion (9) are equivalent to the following Lagrange function

\[
L = \frac{1}{2} \sum_A f_A^{-1} \mu_A \dot{x}_A^2 \left[ 1 + \frac{1}{4c^2} \dot{x}_A^2 \right] + \frac{1}{2} \sum_A \sum_{B \neq A} \mu_A \mu_B \frac{\mu_B}{r_{AB}} \left[ 1 + \frac{2 \gamma + 1}{c^2} \dot{x}_A^2 \right]
\]

\[
- \frac{4 \gamma + 3}{2c^2} \dot{x}_A \cdot \dot{x}_B - \frac{1}{2c^2} \frac{\dot{x}_A \cdot r_{AB}}{r_{AB}} \dot{x}_B \cdot \frac{r_{AB}}{r_{AB}} - \frac{2 \beta - 1}{c^2} \sum_{C \neq A} \frac{\mu_C}{r_{AC}} r_{AC}^2 \right].
\]
Eq. (9) can be derived from (12) up to the terms \( O(c^{-4}) \) using
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i_A} - \frac{\partial L}{\partial x^i_A} = 0.
\]

C. Integrals of motion

Using
\[
\sum_A \frac{\partial L}{\partial \dot{x}^i_A} = P^i = \text{const},
\]
\[
\sum_A \frac{\partial L}{\partial \dot{x}^i_A} \dot{x}^j_A - L = h = \text{const},
\]
\[
\sum_A \varepsilon_{ijk} x^j_A \frac{\partial L}{\partial \dot{x}^k_A} = c^i = \text{const}
\]
it is easy to demonstrate that the equations of motion have the following ten classical integrals defined here with the post-Newtonian accuracy. Here we add one more assumption: the gravitation binding energy of each body is constant
\[
\Omega_A = -\frac{1}{2} G \int_T \int_T \frac{\Sigma(T, X) \Sigma(T, X')}{|X - X'|} d^3X d^3X' = \text{const}.
\]
This means that for each body \( f_A = \text{const} \). The six integrals of the center of mass read
\[
\sum_A f_A^{-1} \mu_A \dot{x}_A \left(1 + \frac{1}{2c^2} \left( \dot{x}_A^2 - f_A \frac{\mu_B}{r_{AB}} \right) \right)
- \frac{1}{2c^2} \sum_A \sum_{B \neq A} \frac{\mu_A \mu_B}{r_{AB}^3} \left( r_{AB} \cdot \dot{x}_A \right) r_{AB} = P = \text{const},
\]
\[
\sum_A f_A^{-1} \mu_A x_A \left(1 + \frac{1}{2c^2} \left( \dot{x}_A^2 - f_A \frac{\mu_B}{r_{AB}} \right) \right) + O(c^{-4}) = P t + Q, \quad Q = \text{const}.
\]
The integral of energy reads
\[
\frac{1}{2} \sum_A f_A^{-1} \mu_A \ddot{x}_A \left(1 + \frac{3}{4c^2} \dot{x}_A^2 \right) - \frac{1}{2} \sum_A \sum_{B \neq A} \frac{\mu_A \mu_B}{r_{AB}} \left(1 - \frac{2\gamma + 1}{c^2} \dot{x}_A^2 + \frac{4\gamma + 3}{2c^2} \dot{x}_A \cdot \dot{x}_B \right)
+ \frac{1}{2c^2} \frac{\dot{x}_A \cdot r_{AB} \dot{x}_B \cdot r_{AB}}{r_{AB}^2} - \frac{2\beta - 1}{c^2} \sum_{C \neq A} \frac{\mu_C}{r_{AC}} = h = \text{const}.
\]
Finally, the integral of angular momentum reads
\[
\sum_A f_A^{-1} \mu_A x_A \times \ddot{x}_A \left(1 + \frac{1}{2c^2} \dot{x}_A^2 + \frac{2\gamma + 1}{c^2} f_A \sum_{B \neq A} \frac{\mu_B}{r_{AB}} \right)
- \frac{1}{2c^2} \sum_A \sum_{B \neq A} \frac{\mu_A \mu_B}{r_{AB}} \left(4\gamma + 3\right) x_A \times \ddot{x}_B - \frac{x_A \times \ddot{x}_B \cdot r_{AB}}{r_{AB}^2} = c = \text{const}.
\]
Note that when all terms in (12)–(21) that are explicitly proportional to $c^{-2}$ are omitted (while keeping $f_A$ as a symbol), one gets the exact Newtonian Lagrange function and the corresponding integrals of the Newtonian equations with a violation of the SEP discussed above. In these equations $\mu_A = GM_A$ corresponds to the gravitational masses of the bodies, while $f^{-1}_A \mu_A = GM_A^{\text{iter}}$ corresponds to the inertial masses ($f_A = M_A/M_A^{\text{iter}}$).

D. Exact equations of motion from the Lagrange function

The Lagrange function (12) gives the equations of motion which agree with the EIH-like equations (9) only approximately up to the terms $O(c^{-4})$. Here again, when computing the equations of motion from (12) and (13), the acceleration $\ddot{x}_A$ in the post-Newtonian terms was replaced using the Newtonian equations of motion. This means again that if one considers (9) as exact and integrates these equations numerically, the integrals (18), (20), and (21) are not exactly constants because of the terms $O(c^{-4})$, which are neglected in the analytical calculations. The equations of motion that exactly agree with the Lagrange function (12) read

$$\ddot{x}_A^i = F_A^{-1} \left( H_A^i - \frac{1}{c^2} f_A^{-1} \dot{x}_A^i \dot{x}_A^j \dot{x}_A^j + \frac{4\gamma + 3}{2c^2} \sum_{B \neq A} \frac{\mu_B \dot{x}_B^i}{r_{AB}^3} + \frac{1}{2c^2} \sum_{B \neq A} \mu_B \frac{r_{AB}^i}{r_{AB}^3} \dot{x}_B^j r_{AB}^j \right),$$

$$(22)$$

$$H_A^i = - \sum_{B \neq A} \mu_B \frac{r_{AB}^i}{r_{AB}^3} \left( 2(\beta - 1) \frac{\mu_A}{r_{AB}} + (2\beta - 1) \sum_{C \neq A,B} \frac{\mu_C}{r_{BC}} + (2\beta - 1) \sum_{C \neq A} \frac{\mu_C}{r_{AC}} \right)$$

$$+ \frac{1}{c^2} \sum_{B \neq A} \mu_B \frac{r_{AB}^i}{r_{AB}^3} \left( (2\gamma + 1)(\dot{x}_A^i - \dot{x}_B^i)(\dot{x}_A^j - \dot{x}_B^j) - \dot{x}_A^i \dot{x}_A^j + 2(1 + \gamma) \dot{x}_A^i \dot{x}_B^j \right),$$

$$(23)$$

$$F_A = f_A^{-1} \left( 1 + \frac{1}{2c^2} \dot{x}_A^2 \right) + \frac{1}{c^2} (2\gamma + 1) \sum_{B \neq A} \frac{\mu_B}{r_{AB}^3}. \hspace{1cm}$$

$$(24)$$

If the accelerations in the post-Newtonian terms in (22) are replaced using the Newtonian approximation these equations are equivalent to (9). Because of the last three terms in (22) Eqs. (22)–(24) are implicit with respect to the accelerations $\ddot{x}_A$ and can be solved e.g. by iterations using the Newtonian equations for $\ddot{x}_A$ or the approximation $\ddot{x}_A = F_A^{-1} H_A$ as initial approximation.

Again all the terms in $\ddot{x}_A$ explicitly proportional to $\mu_A$ are explicitly shown in (23) as the first term in the curly braces. This term should be omitted if the motion of a minor body is considered so that its gravitational influence on the motion of other bodies is neglected.
E. Character of the integrals of motion

Integrals (18), (20), and (21) are exact integrals of the equations of motion with the Lagrange function (12). Therefore, these integrals remain exactly constant if the equations of motion (22)–(24) are used. Integral (19) is derived by integrating (18) and using Newtonian equations of motion in the post-Newtonian terms to replace the accelerations. It means that when computing (19) numerically along solutions of the equations of motion with the Lagrange function (12) the quantity is not exactly linear function of time as specified in (19), but also has some small non-linear deviations (basically, \( Q = \text{const} + O(c^{-4}) \) and those \( c^{-4} \) terms depend on time in a non-linear way).

It remains unclear if one can find a form of (19) that is satisfied exactly with either form of the equations of motion discussed above.

IV. CONCLUSIONS

The derived equations of motions and their integrals can be used for tests of the Strong Equivalence Principle in the future solar system experiments. In particular, the equations can be useful for the tests using high-accuracy asteroid observations from the ESA space mission Gaia as well as the SEP tests planned as a part of the BepiColombo mission [13].

Acknowledgments

The author thanks Léo Bernus for finding and correcting an error in Eq. (23).

[1] Brumberg, V.A., Kopejkin, S.M. 1989 in Reference Frames, ed. J. Kovalevsky, I.I. Mueller, B.Kolaczek, Kluwer, Dordrecht, 115
[2] Brumberg, V.A., Kopejkin, S.M. 1989 Nuovo Cimento, 103B, 63
[3] Damour, T., Soffel, M., Xu, C. 1991 Phys. Rev. D 43, 3273
[4] Damour, T., Soffel, M., Xu, C. 1992 Phys. Rev. D 45, 1017
[5] Damour, T., Soffel, M., Xu, C. 1993 Phys. Rev. D 47, 3124
[6] Dicke, R.H. 1965, Astron.J., 70, 395
[7] ESA, 2000, ESA-SCI(2000)4, Noordwijk: European Space Agency
[8] Gylvarry, J.J. 1953, Phys. Rev., 89, 1046
[9] Kliooner, S.A., & Soffel, M.H. 2000, Phys. Rev. D, 62, ID 024019
[10] Kliooner, S.A., & Soffel, M.H. 1999, Phys. Rev. D 48, 1451
[11] Kopejkin, S.M. 1988 Celestial Mechanics, 44, 87
[12] Kopejkin, S.M. 1989 Celestial Mechanics, 44, 87
[13] Lieske, J.H., Null, G.W., 1969, Astron.J., 74, 297
[14] Milani, A., Vokrouhlický, D., Villani, D., Bonanno, C., Rossi, A., 2002 Phys. Rev. D, 66, 082001
[15] Nordtvedt, K., Jr. 1968, Phys. Rev., 169, 1014
[16] Orellana, R.B., Vucetich, H. 1988, A&A, 200, 248
[17] Overduin, J.M. 2000, Phys. Rev. D, 62, 102001
[25] Let us note that in some other alternative theories of gravity $f_A - 1$ can be relatively large even for minor solar system bodies (see, e.g. Overduin [17]). This would mean that a test of the SEP is possible directly using Eq. (1).

[28] Section IX.G of Klioner & Soffel [9] contains also the equations of motion for the bodies with non-vanishing spin dipole $S^a$. However, this case will not be considered here.