Nonrelativistic and Relativistic Continuum Mechanics

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Abstract

There is described a spacetime formulation of both nonrelativistic and relativistic elasticity. Specific attention is devoted to the causal structure of the theories and the availability of local existence theorems for the initial-value problem. Much of the presented material is based on joint work of B.G.Schmidt and the author (in Class.Quantum Grav. 20 (2003), 889-904).

1 Introduction

The ancient field theory of continuum mechanics, created by the mathematicians J.Bernoulli, Euler and Cauchy, has grown into a subject of great importance both for its mathematical interest and its applications in material science. In its relativistic guise this theory has not been developed very far yet. Important references are [16], [2], [3], [11], [17] and [10]. The book [15] is also an excellent source.

We start, in Section 2, by describing the nonrelativistic theory in a framework akin to that used in relativity, namely that of Galilean space-times. In the following Section 3 we describe the concept of hyperbolicity appropriate for the resulting system of 2nd-order partial differential equations. We then describe a way of rewriting such a system in symmetric first-order form. The condition of symmetric hyperbolicity for this latter system, although true in many cases of physical interest, is however a more stringent requirement than that of hyperbolicity for the original second-order system. In Section 4 we write down a class of states, the so-called "natural states", which satisfy the assumptions of (both the first- and second-order) hyperbolic theory outlined in Section 3 so that there is available, for initial states sufficiently close to natural ones, a local existence theorem for the Cauchy problem.\footnote{There are in fact theorems even on global existence (see [14]).}

Finally, in Section 5 we de-
scribe the necessary changes as one goes from the nonrelativistic theory (or rather: "Galilean relativity") to Einsteinian relativity.

2 Nonrelativistic Theory

We start with a Galilean spacetime \( M \) (see e.g. [1]). This is furnished by \( \mathbb{R}^4 \), endowed with a symmetric, degenerate contravariant metric \( h^{\mu \nu} \) of signature \((0+++)\) and a choice of covector field \( \tau^{\mu} \) satisfying

\[ h^{\mu \nu} \tau^{\nu} = 0 \]  

We also assume we are given a flat connection \( \nabla_{\mu} \) which annihilates both \( h^{\mu \nu} \) and \( \tau^{\mu} \). The matter flow is described by a vector field \( v^{\mu} \) normalized by \( v^{\mu} \tau^{\mu} = 1 \), i.e. of the form

\[ v^{\mu} = \partial_{\mu} + v^{i} \partial_{i}, \]  

where \( v^{i} = v^{i}(t, x^{i}) \) and \( (x^{\mu}) = (t, x^{i}) \) are flat coordinates in which \( h^{\mu \nu} \partial_{\mu} \partial_{\nu} = \delta^{ij} \partial_{i} \partial_{j} \) and \( \tau^{\mu} dx^{\mu} = dt \). The given flat connection \( \nabla \) is not the only one annihilating \((h^{\mu \nu}, \tau^{\nu})\). One can use this freedom to describe the effect of gravity by using the connection \( \bar{\nabla}_{\mu} \omega_{\nu} = \nabla_{\mu} \omega_{\nu} - C^{\lambda}_{\mu \nu} \omega_{\lambda} \) with \( C^{\lambda}_{\mu \nu} = \tau_{\rho} \tau_{\lambda} h^{\lambda \sigma} \nabla_{\sigma} U \), \( U \) being the gravitational potential. One easily checks that the connection \( \bar{\nabla} \) again annihilates \((h^{\mu \nu}, \tau^{\nu})\). Having described the kinematical arena, we now turn to the specific class of physical models we consider. We start by writing down a "stress-mass" tensor. This tensor is not such a natural object in the nonrelativistic theory as the stress-energy tensor is in relativity, for two reasons: the lack of a non-degenerate spacetime metric and the fact that the Lagrangian of the nonrelativistic theory breaks the Galilean invariance. We will nonetheless use the concept of stress-mass here, since it greatly facilitates the task of moving back and forth between the Galilean and the Einsteinian theory.

The mass-stress tensor has the form

\[ T^{\mu \nu} = \rho v^{\mu} v^{\nu} + t^{\mu \nu}, \]  

where the Cauchy stress tensor \( t^{\mu \nu} \) is purely spatial in the sense that \( t^{\mu \nu} \tau_{\nu} = 0 \). Furthermore we have that \( \rho = nm_{0} \), where \( n > 0 \) is the particle number density and \( m_{0} \) the mass per particle. The continuity equation in our language takes the following form. Take \( \varepsilon \), the volume form on \( M \) defined in the adapted coordinates as \( \varepsilon_{ijkl} = \epsilon_{ijkl} \), and consider the three-form \( N \) given by \( N_{\mu \nu \lambda} = n \varepsilon_{\mu \nu \lambda} v^{\rho} \). Then conservation of mass is given simply by \( dN = 0 \). In the standard coordinates, \( N \) is given by

\[ N = n \varepsilon_{ijk}(dx^{i} - v^{i} dt)(dx^{j} - v^{j} dt)(dx^{k} - v^{k} dt) \]  

and \( dN = 0 \) is of course equivalent to

\[ \partial_{\mu}(nv^{\mu}) = \partial_{t} n + \partial_{i}(nv^{i}) = 0. \]  

The matter field equations are given by

\[ \nabla_{\mu} T^{\mu \nu} = \partial_{\nu} T^{\mu \nu} = 0 \]
in the absence of gravity, otherwise the derivative $\nabla_\mu$ has to be used in Eq. (6). We immediately see that the equation $\tau_\mu \nabla_\nu T^{\mu\nu} = 0$ is already implied by the continuity law Eq. (5). In order to turn Eq. (6) into proper field equations we have to specify the dependent variables. These are furnished by maps $f$ sending points of spacetime $M$ into a manifold $B$ called body or material manifold. This material manifold should be viewed as an abstract set of labels which parametrize the particles making up the continuum. Thus $f$ is the “back-to-labels-map”. If we choose coordinates $(X^A)$ with $A = 1, 2, 3$ on $B$, we can write $X^A = f^A(t, x^i)$. The relationship between the map $f^A$ and the vector field $v^\mu$ is given by

$$v^\mu \partial_\mu f^A = \partial_t f^A + v^i \partial_i f^A = 0 \quad (7)$$

Suppose that $B$ is endowed with a volume form $\Omega_{ABC}$ and set

$$f^* \Omega = N, \quad (8)$$

where $f^*$ denotes pull back under the map $f$. Eq. (8) defines $n$ in terms of $f^A$, namely there holds $n = \text{det}(\partial_i f^A)$. We assume the map $f^A$ to be such that $f(t,.)$ is a diffeomorphism onto its image in $B$ for all $t$ and oriented so that $n$ is positive. This implies that the map $f$ is of maximal rank. Hence, given $f$, Eq. (7) has a unique solution $v^i$. Here, and in what follows, the spacetime field $v^\mu$ is always viewed as a function of $(f^A, \partial_\mu f^B)$. We now introduce the concept of “strain” by means of quantities $H_{AB}$ defined by

$$H_{AB} = h^{\mu\nu} (\partial_\mu f^A) (\partial_\nu f^B) \quad (9)$$

Clearly $H_{AB}$ is positive definite. Consequently there exists the inverse $H_{AB}$ defined by

$$H_{AB} H_{BC} = \delta^A_C \quad (10)$$

We now assume that the Cauchy stress tensor in Eq. (3) is of the form

$$t^{\mu\nu} = n \tau_{AB} (\partial_\mu f^A) (\partial_\nu f^B) h^{\rho\sigma} h^{\sigma\nu} \quad (11)$$

with

$$\tau_{AB} = 2 \frac{\partial e}{\partial H_{AB}} \quad (12)$$

for some function $e = e(f^A(x), H_{BC}(x))$, called stored-energy function in the elastic literature. As an example take the case where $e$ just depends on $n$. Note this makes sense since $n$ can be written in terms of $H_{AB}$, namely

$$6n^2 = H^{AA'} H^{BB'} H^{CC'} \Omega_{ABC} \Omega_{A'B'C'} \Omega_{A'B'C'} \quad (13)$$

When $e$ depends only on $n$ one finds that

$$t^{\mu\nu} = ph^{\mu\nu} \quad (14)$$

where $p$ is defined by

$$p = n \frac{\partial e}{\partial n} \quad (15)$$

The field equations here are the Euler equations for a perfect fluid. For completeness we outline the proof of Eq. (14). We first claim that

$$\frac{\partial n}{\partial H_{AB}} = \frac{n}{2} H_{AB} \quad (16)$$
The trick how to obtain Eq. (16) is to first compute $\frac{\partial n}{\partial H_{AB}} H^{BC}$, using Eq. (13). One quickly finds that

$$\frac{\partial n}{\partial H_{AB}} H^{BC} = \frac{n}{2} \delta^C_A$$  \hspace{1cm} (17)

We next define a quantity $F^\mu_A$ by

$$F^\mu_A = (\partial_\nu f^B) h^\mu\nu H_{AB}$$  \hspace{1cm} (18)

One checks that

$$(\partial_\mu f^A) F^\mu_B = \delta^A_B$$  \hspace{1cm} (19)

Using $F^\mu_A \tau_\mu = 0$ and $(\partial_\mu f^A) v^\mu = 0$, it follows that

$$F^\mu_A (\partial_\nu f^A) = \delta^\mu_\nu - v^\mu \tau_\nu$$  \hspace{1cm} (20)

Using Eq.'s (16,20) in Eq.(11), we immediately obtain Eq.(14) together with (15).

Let us return to the case of a general elastic solid. The field equations are of the following form

$$\frac{\partial T^\mu\lambda}{\partial (\partial_\nu f^A)} \partial_\lambda \partial_\nu f^A = \text{lower-order derivatives of } f^A,$$  \hspace{1cm} (21)

which, by the remark following Eq.(2), are equivalent to

$$M^\mu_{AB} \partial_\mu \partial_\nu f^B = G_A$$  \hspace{1cm} (22)

where $M^\mu_{AB}$, defined as

$$M^\mu_{AB} = (\partial_\lambda f^C) H_{CA} \frac{\partial T^\mu\lambda}{\partial (\partial_\nu f^B)}.$$

(23)

and $G_A$ are functions of $(f^A, \partial_\mu f^B)$. Remarkably the quantities $M^\mu_{AB}$ turn out to satisfy the symmetry

$$M^\mu_{AB} = M^\mu_{BA}$$  \hspace{1cm} (24)

The symmetry Eq.(24) is no accident. It is due to the fact that the field equations are the Euler-Lagrange equations of a variational principle. Explicitly we find

$$M^\mu_{AB} = -\rho v^\mu v^\nu H_{AB} + n[\tau_{AB} H_{CD} + 2\tau_{CI}(A H_{B)D} + 2 \frac{\partial \tau_{AC}}{\partial H_{BD}} \partial^\mu f^C \partial^\nu f^D],$$  \hspace{1cm} (25)

where we have defined $\partial^\mu f^A = h^\mu\nu \partial_\nu f^A$. The associated Lagrangian, which in particular satisfies

$$-M^\mu_{AB} = \frac{\partial^2 \mathcal{L}}{\partial (\partial_\mu f^B) \partial (\partial_\nu f^A)},$$  \hspace{1cm} (26)

is given by

$$\mathcal{L} = n\left(\frac{1}{2} m_0 v^i \delta_{ij} - \varepsilon\right).$$
Note that the quantities $n, e, v^i$ should be all regarded as functions of $(f^A, \partial_\mu f^B)$.

This ends our description of nonrelativistic elasticity in its spatial (or rather: "spacetime") form. We remark that all standard treatments in the literature (see e.g. [8] or [13]) prefer the material form based on the map $F^i(t, X^A)$ defined by

$$f^A(t, F^i(t, X^B)) = X^A.$$  

(27)

From the relativistic point of view the spacetime form is preferable, since, in a relativistic spacetime, there does not exist the standard $t = \text{const}$-foliation available in Galilean spacetime.

3 Some Hyperbolic Theory

An equation of the form of (22) will be called hyperbolic if $M^{\mu\nu}_{AB}$ satisfies the symmetry (24) and the following holds: firstly there should exist a subcharacteristic covector, i.e. a covector $\xi^\mu$ so that

$$M^{\mu\nu}_{AB} \xi^\mu \xi^\nu$$

is negative definite

(28)

Secondly there should exist a timelike vector, i.e. a vector $X^\mu$ so that

$$M^{\mu\nu}_{AB} \eta^\mu m^A \eta^\nu m^B$$

is positive definite

(29)

for all nonzero $\eta^\mu$ with $X^\mu \eta^\mu = 0$. Let us pause for a moment to explain by means of an example simpler than elasticity how the change of sign between Eq. (28) and Eq. (29) arises. The example is the equation for wave maps, where $M^{\mu\nu}_{AB} = g^{\mu\nu} G_{AB}$ with $g^{\mu\nu}$ the spacetime metric and $G_{AB}$ the Riemannian metric on the target space. Now the notion of timelike has its standard Lorentzian meaning, and subcharacteristic covectors are timelike covectors. The sign change between (28) and (29) is then simply due to the fact that (co-)vectors orthogonal to a timelike (co-)vector are spacelike.

A vector $X^\mu$ is called causal if $X^\mu \xi^\mu \neq 0$ for all subcharacteristic covectors $\xi^\mu$. Clearly all timelike vectors are causal. In general there will be a gap between timelike and noncausal vectors due for example to the existence of different characteristic (sound) cones. A covector $k_\mu$ is called characteristic if the symbol of the PD operator on the l.h. side of Eq. (22), namely the quadratic form $M_{AB}(k)$ given by

$$M_{AB}(k) = M^{\mu\nu}_{AB} k_\mu k_\nu$$

(30)

is degenerate. A vector $X^\mu$ is called (bi-)characteristic if it is of the form

$$X^\mu = M^{\mu\nu}_{AB} k_\nu m^A m^B$$

for a characteristic covector $k_\mu$ and $m^A$ such that $M^{\mu\nu}_{AB} k_\nu m^B = 0$. This characteristic vector is tangent to the characteristic sheet to which $k$ belongs where this sheet is a regular surface.
The above definitions are essentially taken from the book [5].\(^2\) The notion of characteristic vector from [5] as above is new, the classical one breaking down when the set of characteristic covectors (also called "normal cone" or "slowness cone" in the literature) has singularities due to intersections between different sheets (there are in general three such sheets which are given by the zero-level set of the eigenvalues of \(M_{AB}\)). Even in the absence of singularities of the normal cone the set of characteristic vectors (also called "ray cone" or "wave cone" in the literature) will in general have cusps (see e.g. Chapter VI of [4]).

A yet more general notion of hyperbolicity, due to Kreiss, of which the above is a special case, is that of strong hyperbolicity (see [12]).

A key point regarding these definitions is the availability of an existence theorem independently of the form of the lower-order terms in Eq.\((22)\).\(^3\) Namely, suppose there is a hypersurface \(S\) of spacetime, together with initial data for \(f^A\) and \(\partial_\mu f^A\) on \(S\) so that, for these data, the surface \(S\) has everywhere subcharacteristic conormal. Then choose a vector field \(X^\mu\) which is timelike on \(S\), whence transversal to \(S\). Use this vector field to Lie-drag \(S\) into the future. Since the properties of being subcharacteristic and timelike are "open" conditions, we thus obtain a spacetime neighbourhood \(N \subset M\) of \(S\), which is foliated by surfaces which are subcharacteristic for all maps \(f\) close to the initial one and where \(X^\mu\) is timelike with respect to such configurations. Now take coordinates \((y^0,y^i)\) so that the leaves of the foliations are given by \(y^0 = \text{const}\) and the vector field \(X^\mu\) is given by \(X^\mu \partial_\mu = \partial_0\). In these coordinates the equation \((22)\) takes the form

\[
[M_{AB}^0 \partial_0^2 + (M_{AB}^0 + M_{AB}^i \partial_i + M_{AB}^j \partial_j)]f^B = \text{lower-order derivatives of } f^A.
\]

Furthermore there should hold

\[
M_{AB}^0 l^Am^B < 0, \quad M_{AB}^i l^i l^Am^B > 0
\]

with \(l^i, m^A\) both nonzero and all values of \((f^A, \partial_\mu f^B)\) close to those corresponding to the initial data. If the neighbourhood \(N\) is of the form \(\{y^0 \in [0, T]\} \times \{y \in \mathbb{R}^3\}\) and the initial data satisfy some decay properties for large \(|y|\) one can now appeal to a basic theorem in [10] to infer existence of a unique solution for sufficiently small \(T\). (We do not spell out the precise differentiability requirements.) The asymptotic conditions imposed in the above theorem are of course not always appropriate, and one would in any case like a local statement amounting to uniqueness in the "domain of dependence" of initial data in open subsets of \(S\). Such a theorem is proved in [5], the appropriate notion of domain of dependence being based on causal curves (in the sense of causal vectors as described above). Consequently the nonlocal nature of the uniqueness part of the theorem in [9] is in fact irrelevant.

In [1] the authors chose to cast the equations of elasticity theory into that of a first-order symmetric hyperbolic system, which goes as follows:

\(^2\)We have only added the word "timelike" for vectors having the property in Eq.\((29)\) and "subcharacteristic" for covectors satisfying Eq.\((28)\).

\(^3\)In [1] a result to that extent is stated without proof.
Define 5-index quantities $W_{\mu \nu AB}^{(\lambda)}$ by

$$W_{\mu \nu AB}^{(\lambda)}(\lambda) := X^\mu M^{\lambda \nu AB} - 2X^{[\lambda \mu]}M^{\nu BA},$$

(34)

where $X$ is a timelike vector. We now replace (22) by the following first-order system:

$$W_{\mu \nu AB}^{(\lambda)}(\lambda)(f, F) \partial_\lambda F^B_\nu = X^\mu (f, F) G_A(f, F)$$

(35)

$$-X^\lambda (f, F) \partial_\lambda f^A = -X^\lambda (f, F) F^A_\lambda$$

(36)

together with the constraint $\partial_\mu f^A = F^A_\mu$. Since

$$W_{\mu \nu AB}^{(\lambda)}(\lambda) = W_{\nu \mu BA}^{(\lambda)},$$

(37)

the system (35, 36) is symmetric. One then finds that (35, 36) is equivalent to the original second-order system (22).

4 Natural States

We now further specify the ”equation of state” given by the stored-energy function, as follows: We assume $B$ to be endowed with a flat Riemannian metric $G_{AB}$ and that the volume form $\Omega$ is compatible with $G_{AB}$. The stored-energy function $e$ is then assumed to be of the form $e = e(H_{AB})$, i.e. $W_{\mu \nu AB}^{(\lambda)}(\lambda)$ is negative definite in the variables $m^\lambda$. Suppose $\xi$ is subcharacteristic for the second-order system and $X^\mu \xi_\mu > 0$: is it then subcharacteristic also for the system (35, 36)?

The answer in general is ”no” as we will see in the next section.

4 In (1) we used a special property of elasticity to prove this equivalence. It is not hard to see that this equivalence works generally for the system (22) when $X^\mu$ is timelike.
i.e. that $\vec{v}^\mu$ is a rigid motion. The absence of linear terms in Eq. (38) furthermore implies that natural states are stressfree, i.e. have $\tau_{AB} = 0$. Using also that $\hat{n} = 1$ we find that

$$M_{\mu\nu}^{AB} = -m_0 \vec{v}^\mu \vec{v}^\nu \delta_{AB} + E_{ACBD} R^C_{\mu} R^D_{\nu}$$

with $R^A_{\mu} = R^A_{\mu B} \delta^B_{\nu}$ and $R^A_{\nu}$ a (in general time dependent) rotation matrix. The covector $\tau_{\mu}$, from Eq. (42), is clearly subcharacteristic. Furthermore the vector $\vec{v}^\mu$ is timelike iff the conditions (39) are valid. Therefore the second-order equations are hyperbolic at natural states.

We now turn to hyperbolicity of the first-order system. Taking $X^\mu = \vec{v}^\mu$ and using Eqs. (42, 34), we see that

$$\hat{W}_{\mu\nu}^{AB}(\lambda) \tau^\lambda = -m_0 \vec{v}^\mu \vec{v}^\nu \delta_{AB} - E_{ACBD} R^C_{\mu} R^D_{\nu}$$

Thus $\tau_{\mu}$ is subcharacteristic for natural states iff

$$E_{ABCD} m^{AB} m_{CD} > 0$$

for all nonzero elements $m^{AB} = m^{(AB)}$, which is clearly a stronger requirement than (39).

The only case which is easy to analyze fully is the isotropic case \(^5\) where

$$E_{ABCD} = \lambda \delta_{AB} \delta_{CD} + 2\mu \delta_{C(A} \delta_{B)D}$$

The constants $\lambda, \mu$ in Eq. (45) are the standard Lamé constants. They should not be confused with indices $(\mu, \nu)$. The ”rank-one convexity” condition Eq. (44) is equivalent to

$$m_0 c_1^2 = \mu > 0, \quad m_0 c_2^2 = 2\mu + \lambda > 0$$

The eigenvalues of $\frac{1}{m_0} M_{AB}(k)$ relative to $\delta_{AB}$, which are real of course since $M_{AB}$ is symmetric, are given by

$$\lambda_1(k) = c_1^2 \frac{1}{g^{\mu\nu}} k_\mu k_\nu$$

and

$$\lambda_2(k) = \lambda_3(k) = c_2^2 \frac{1}{g^{\mu\nu}} k_\mu k_\nu,$$

where

$$g^{\mu\nu} = h^{\mu\nu} - \frac{1}{c_1^2} \vec{v}^\mu \vec{v}^\nu, \quad \frac{1}{g^{\mu\nu}} = h^{\mu\nu} - \frac{1}{c_2^2} \vec{v}^\mu \vec{v}^\nu.$$  

(A Lorentzian metric of the above form is nowadays called ”acoustic metric” or ”Unruh metric”.) Thus the normal cone consists of two sheets. The one corresponding to $\lambda_1$ is associated with a longitudinal (”pressure”) mode propagating at speed $c_1$, the second one corresponding to two transversal (”shear”) modes of speed $c_2$. If $c_2 < c_1$, the first sheet lies inside the second sheet. Subcharacteristic covectors, for which both $\lambda_1$ and $\lambda_2$ are negative, lie inside the inner cone. One such covector is $\tau_{\mu}$. In fact, $\tau_{\mu}$ lies on the central ray inside the two cones in the following sense: the vector $v^\mu$ defines a family of parallel hyperplanes in the

\(^5\)In \cite{1} we mistakenly interchanged the constants $\lambda$ and $\mu$ in equation (4.16) of that paper.
cotangent space. These intersect the normal cones in 2-surfaces which, in the metric $h^{\mu\nu}$, are standard spheres centered at the point where the ray of $\tau_\mu$ intersects this hyperplane. The ray cones dual to the above normal ones are given by the equations

\[ g^{\mu\nu} X^\mu X^\nu = (h_{\mu\nu} - c_1^2 \tau_\mu \tau_\nu) X^\mu X^\nu, \]

\[ g^{\mu\nu} X^\mu X^\nu = (h_{\mu\nu} - c_2^2 \tau_\mu \tau_\nu) X^\mu X^\nu, \]

(50)

where $h_{\mu\nu}$ is the unique tensor satisfying

\[ h^{\mu\nu} h_{\nu\rho} = \delta^\mu_\rho - v^\mu _\tau \rho, \]

\[ h^{\mu\nu} v_{\nu\rho} = 0 \]

(51)

Note that $g_{\mu\nu}$ (resp. $g^{\mu\nu}$) are the inverses of $g^{\mu\nu}$ (resp. $g_{\mu\nu}$). Clearly, the cone of shear waves is now the one lying inside. Timelike vectors $X$, for which all covectors $k_\nu$ with $X^\mu k_\mu = 0$ have $\lambda_1$ and $\lambda_2$ both positive, lie inside this inner ray cone. One such timelike vector is $v^\mu$, in fact, it lies on the central ray of the two ray cones in a fashion exactly dual to that explained for $\tau_\mu$. Causal vectors may be "faster" in that they lie inside or on the outer ray cone.

We now turn, finally in this section, to the question of hyperbolicity of the first-order theory at natural isotropic states. The condition (44) is valid if and only if

\[ c_1^2 > \frac{4}{3} c_2^2 > 0, \]

(52)

which is physically entirely reasonable, since elastic materials typically have $c_1/c_2$ approximately 1,7. But one can do better than that. One first notices that both the symmetries of $M_{AB}^{\mu\nu}$ and the equation (22) remain untouched if the quantities $M_{AB}^{\mu\nu}$ are replaced by $\tilde{M}_{AB}^{\mu\nu}$ given by

\[ \tilde{M}_{AB}^{\mu\nu} = M_{AB}^{\mu\nu} + \Lambda_{AB}^{\mu\nu}, \]

(53)

where $\Lambda_{AB}^{\mu\nu} = \Lambda^{[\mu\nu]}_{AB} = \Lambda^{\mu\nu}_{AB}$. (In fact this replacement can be viewed as coming from adding a total divergence to the underlying Lagrangian (see \[4\])). While it is easily seen that second-order hyperbolicity is unaffected by this replacement, this is not the case for the associated first-order system. In the case at hand we can, by adding to $E_{ACBD}$ a term of the form

\[ (4c_2^2 - 2\epsilon)\delta_{A[C}\delta_{D]B}, \]

(54)

arrange for Eq. (22) to hold if we take $\epsilon$ in the range $0 < \epsilon < \min(2c_2^2, -\frac{c_1^2}{c_2^2})$, which of course can always be satisfied when $c_1$ and $c_2$ are both non-zero.

Let us point out that the standard equations of linearized elasticity at a natural state are obtained by simply freezing the coefficients in Eq. (22) and setting the right-hand side equal to zero. The Cauchy problem for these equations is studied e.g. in [6]. In the isotropic case one finds that the solution at $(t,x^i)$ does not depend on data at $t = 0$ inside $|x| < c_2 t$, i.e. inside the past inner ("shear") ray cone.

5 Relativistic Theory

Having available the spacetime form of the nonrelativistic theory it is easy to write down its relativistic version, so we will be brief, merely pointing
out the necessary changes. We start out with a relativistic spacetime $(M, g_{\mu\nu})$ with $g_{\mu\nu}$ a Lorentz metric. The configurations are again maps from spacetime to $B$, the latter endowed with a Riemannian metric $G_{AB}$ and compatible volume form $\Omega_{ABC}$. The maps $f$ should be of maximal rank and such that the inverse image under $f$ of each point of $B$ in the image of $f$ is a timelike curve in $M$. Thus there are timelike vectors $u^\mu$, unique up to scale, so that

$$ (\partial_\mu f^A)u^\mu = 0 \tag{55} $$

We denote henceforth by $u^\mu$ the unique solution vector of Eq. (55) which is future-pointing and normalized by $g_{\mu\nu}u^\mu u^\nu = -1$. The quantities $H_{AB}$ are defined as

$$ H_{AB} = (\partial_\mu f^A)(\partial_\nu f^B)g^{\mu\nu}. \tag{56} $$

and are again positive definite. The particle number density $n$ is defined by

$$ 6n^2 = \Omega_{A'B'C'}\Omega_{ABC}H_{AA'}^{\mu}H_{BB'}^{\nu}H_{CC'}^{\rho}, \quad n > 0 \tag{57} $$

The Lagrangian of the theory (we set the speed of light equal to one) is taken to be

$$ L = -\rho = -n(m_0 + e), \tag{58} $$

with $e$ a function of $(f^A, H_{BC})$. Varying $L$ with respect to $g^{\mu\nu}$ gives the stress energy tensor

$$ T^{\mu\nu} = \rho u^\mu u^\nu + 2n\tau_{AB}(\partial^\mu f^A)(\partial^\nu f^B), \tag{59} $$

where $\tau_{AB} = 2\frac{\partial e}{\partial H_{AB}}$, as before and $\partial^\mu f^A = g^{\mu\nu}\partial_\nu f^A$. The field equations are again of the form Eq. (22) with

$$ M^{\mu\nu}_{AB} = -\mu_{AB}u^\mu u^\nu + U_{ACBD}F^{C\mu}F^{D\nu}, \tag{60} $$

where

$$ \mu_{AB} = \rho H_{AB} + n\tau_{AB} \tag{61} $$

and

$$ U_{ACBD} = n(\tau_{AB}H_{CD} + \tau_{AC}H_{BD} + \tau_{BD}H_{AC} + 2\frac{\partial\tau_{BD}}{\partial H_{AC}}) + 2\rho H_{A[C}H_{D]B}. \tag{62} $$

The last term in Eq. (62) does not contribute to the equations of motion. The second term on the right in Eq. (61) is clearly a relativistic contribution.

There are slight complications in curved spacetime regarding the notion of a natural state: in order for a natural state to exist, the spacetime metric would have to allow Born rigid motions and the material metric $G_{AB}$ would have to be isometric to the metric on the quotient of $M$ by the action of this motion. We avoid this difficulty here by conforming ourselves to special relativity and taking the natural configuration to be at rest in some inertial system. Thus we assume $(M, g_{\mu\nu})$ to be Minkowski space.

We also suppose the "natural" motion to be of the form $\ddot{\nu}^\mu \partial_\mu = \dot{\theta}_i$ in coordinates $(t, x^i)$ in which $g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \delta_{ij}dx^i dx^j$. We also assume $G_{AB} = \delta_{AB}$ as before. Using (44), a natural map corresponds to a time independent rotation in an inertial system which, in the isotropic case,
can be taken to be the identity without loss. One then merely replaces, in the expressions (47,48) for the different cones, the covector $\tau_\mu$ by the covector $\vec{u}_\mu = -g_{\mu\nu} \vec{u}^\nu$ and the symmetric tensor $h^{\mu\nu}$ by $\vec{h}^{\mu\nu}$ given by
\[ \vec{h}^{\mu\nu} = g^{\mu\nu} - \vec{u}_\mu \vec{u}_\nu. \] (63)
Furthermore one replaces the tensor $\hat{h}^{\mu\nu}$ in (51) by $\vec{h}^{\mu\nu}$ given by
\[ \vec{h}^{\mu\nu} = g^{\mu\nu} + \vec{u}_\mu \vec{u}_\nu. \] (64)
Thus, writing $\vec{c}$ for either $c_1$ or $c_2$, the associated normal cone is now given by the Lorentz metric
\[ \vec{g}^{\mu\nu} = g^{\mu\nu} + (1 - \frac{1}{c^2}) \hat{u}_\mu \hat{u}_\nu \] (65)
and the ray cone by the inverse metric, namely
\[ \vec{g}_{\mu\nu} = g_{\mu\nu} + (1 - \frac{1}{c^2}) \hat{u}_\mu \hat{u}_\nu. \] (66)
One easily infers from these relations that in the present coordinates the special relativistic equations, linearized at a natural state, are exactly identical with the nonrelativistic ones, when the latter are written in coordinates where $v^\mu \partial_\mu = \partial_t$ with $v^\mu$ an inertial motion. Of course the geometrical objects in both theories are different - and thus the behaviour of the two linearized theories under change of coordinates is also different.

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