Determination of the parameter of the differential equation of fractional order with the Caputo derivative in Hilbert space

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Abstract. We consider an abstract differential equation of the first order with unbounded linear operator $D^\alpha u(t) = Au(t) + f(t)$ in a Hilbert space $H$, where $D^\alpha$ is the Caputo fractional derivative. For this equation the Cauchy problem is studied with initial data $u(0) = u_0$ at $t = 0$. We assume that the operator $A$ is self-adjoint and non-positive. The inverse problem to determine the nonhomogeneous member is considered under the assumption that this term has the following representation $f(t) = \varphi(t)p$, where the scalar function $\varphi(t)$ is given and the element $p$ is an unknown element of the space $H$. This problem belongs to the class of inverse problems. Inverse problems for abstract differential equations was discussed initially with this inhomogeneous structure member but for equations with an operator generating a $C_0$-semigroup and usual derivative. An additional condition $u(T) = u_1$ is specified for the determination of the unknown $p$, where $u_1$ is a given element of the space $H$. Thus we get the two-point problem which is only beginning to be studied for the case of fractional derivatives. The question of existence and uniqueness of the classical solution of the inverse problem is studied. Sufficient conditions of correct solvability of the inverse problem are obtained. Explicit formula to determine the unknown element in the differential equation is given.

1. Introduction
Consider the following Cauchy problem

$$
\begin{cases}
D^\alpha u(t) = Au(t) + f(t), & 0 \leq t \leq T, \\
u(0) = u_0,
\end{cases}
$$

where $A$ is self-adjoint and non-positive operator in a Hilbert space $H$, $f(t) \in C([0, T]; H)$, the element $u_0 \in H$, $0 < \alpha < 1$ and $D^\alpha$ is the Caputo fractional derivative

$$(D^\alpha u)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left( \int_0^t \frac{u(s)}{(t-s)^\alpha} ds - u(0) \right),$$

($\Gamma(x)$ is the Gamma function).

Well-posedness of the problem (1) is connected with well-posedness of homogeneous problem
\[ \begin{cases} D^\alpha u(t) = Au(t), & 0 \leq t \leq T, \\ u(0) = u_0. \end{cases} \tag{2} \]

We will consider a strong solution for problems (1) and (2). It means that

(i) \( u \in C([0, T], D(A)) \cup C([0, T], H) \),

(ii) \( \int_0^t \frac{u(s)}{(t-s)^\alpha} ds - u(0) \in C^1([0, T], H) \).

Consider the following function of Wright type

\[ \Phi_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n! \Gamma(-\alpha n + 1 - \alpha)} \]

and two functions

\[ \varphi_{t,\alpha}(s) = t^{-\alpha} \Phi_\alpha(st^{-\alpha}), \]
\[ \psi_{t,\alpha}(s) = \alpha st^{-\alpha-1} \Phi_\alpha(st^{-\alpha}). \]

These functions satisfy the following relations

\[ \int_0^\infty \varphi_{t,\alpha}(s)e^{\lambda s} ds = E_{\alpha,1}(t^\alpha \lambda), \]
\[ \int_0^\infty \psi_{t,\alpha}(s)e^{\lambda s} ds = t^{\alpha-1} E_{\alpha,\alpha}(t^\alpha \lambda), \]

where

\[ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{ \Gamma(\alpha n + \beta)} \]

Mittag-Leffler function.

The self-adjoint and non-positive operator \( A \) is a generator of some \( C_0 \)-semigroup \( V(t) \). In this case the strong solution of problem (2) exists, it is unique for every \( u_0 \in D(A) \) and one has the form

\[ u(t) = S_\alpha(t)u_0, \]

and the strong solution of problem (1) exists, it is unique for every \( u_0 \in D(A), f \in C^\beta([0, T]), H \) \((0 < \beta \leq 1)\) and one has the form

\[ u(t) = S_\alpha(t)u_0 + \int_0^t P_\alpha(t-s)f(s)ds, \]

where two strongly continuous operator functions \( S_\alpha(t) \) and \( P_\alpha(t) \) are related to this semigroup \( V(t) \) as

\[ S_\alpha(t) = \int_0^{+\infty} \varphi_{t,\alpha}(s)V(s)ds \tag{3} \]
and
\[ P_\alpha(t) = \int_0^{+\infty} \psi_{t,\alpha}(s)V(s)ds. \]  
\[ (4) \]

It is known that for the bounded operator \( A \) the equality two relation
\[ S_\alpha(t) = E_{\alpha,1}(t^\alpha A), \quad P_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha A), \]
are true.

Applying operator calculus of self-adjoint operators, it is easy to prove that these equations are also valid for any self-adjoint and non-positive operator. In fact, let \( E_{\lambda} \) is a spectral resolution of the operator \( A \), then

\[ S_\alpha(t) = \int_0^{+\infty} \varphi_{t,\alpha}(s) \left( \int_{-\infty}^{0} e^{s_\lambda}dE_{\lambda} \right) ds = \int_{-\infty}^{0} \left( \int_0^{+\infty} \varphi_{t,\alpha}(s)e^{s_\lambda}ds \right) dE_{\lambda} = \]
\[ = E_{\alpha,1}(t^\alpha \lambda)dE_{\lambda} = E_{\alpha,1}(t^\alpha A), \]

\[ P_\alpha(t) = \int_0^{+\infty} \psi_{t,\alpha}(s) \left( \int_{-\infty}^{0} e^{s_\lambda}dE_{\lambda} \right) ds = \int_{-\infty}^{0} \left( \int_0^{+\infty} \psi_{t,\alpha}(s)e^{s_\lambda}ds \right) dE_{\lambda} = \]
\[ = 0 \int_{-\infty}^{0} t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha \lambda)dE_{\lambda} = t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha A). \]

Therefore, the formula for solving the direct problem (1) can be written as follows
\[ u(t) = E_{\alpha,1}(t^\alpha A)u_0 + \int_0^{t} (t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^\alpha A)f(s)ds. \]  
\[ (5) \]

2. Inverse problem
Consider the problem of determining the function \( u(t) \) with values of \( H \) and the element \( p \in H \) from the system of equations
\[ \begin{cases} D^\alpha u(t) = Au(t) + \varphi(t)p, & 0 \leq t \leq T, \\ u(0) = u_0, \quad u(T) = u_1. \end{cases} \]  
\[ (6) \]

If we choose in (5) \( f(t) = \varphi(t)p \) and then put \( t = T \), we get the following equation for the unknown element \( p \)
\[ Bp = h, \]
where
\[ B = \int_0^{T} (T-s)^{\alpha-1}E_{\alpha,\alpha}((T-s)^\alpha A)\varphi(s)ds, \]
\[ h = u_1 - E_{\alpha,1}(T^\alpha A)u_0. \]
We introduce the concept of the characteristic function of the inverse problem (6). This function is determined by the equality

\[ F(z) = \int_0^T (T - s)^{\alpha - 1} E_{\alpha,\alpha}(z(T - s)^\alpha) \varphi(s) ds. \]  

(7)

Let us transform the expression that represents the operator \( B \). For this we use the spectral decomposition of the operator \( A \)

\[ A = \int_{-\infty}^0 \lambda dE_\lambda \]

According to the operator calculus of self-adjoint operators, we have

\[ E_{\alpha,\alpha}((T - s)^\alpha A) = \int_{-\infty}^0 E_{\alpha,\alpha}(\lambda(T - s)^\alpha) dE_\lambda. \]

Therefore

\[ B = \int_0^T \left( \int_{-\infty}^0 (T - s)^{\alpha - 1} E_{\alpha,\alpha}(\lambda(T - s)^\alpha) \varphi(s) dE_\lambda \right) ds. \]

By Fubini’s theorem we have

\[ B = \int_{-\infty}^0 \left( \int_0^T (T - s)^{\alpha - 1} E_{\alpha,\alpha}(\lambda(T - s)^\alpha) \varphi(s) ds \right) dE_\lambda = F(A). \]

Therefore, the equation for the unknown element \( p \) takes the form

\[ F(A)p = h. \]  

(8)

Equation (8) was studied in [1]. Direct application of the results of this work leads to the following statement.

**Theorem 1.** Let us assume that \( H \) is a Hilbert space, operator \( A \) is self-adjoint and non-positive, \( 0 < \alpha < 1 \), function \( \varphi \in C^1[0, T] \), it is not identically equal to zero, the elements \( u_0, u_1 \in H \). The solution of problem (6) is unique if and only if point spectrum of the operator \( A \) does not contain zeroes of the characteristic function (7). The solution of problem (6) exists if and only if

\[ \int_{-\infty}^0 |F(\lambda)|^{-2} d(E_\lambda h, h) < \infty, \]

where \( E_\lambda \) is a spectral resolution of the operator \( A \) and the element

\[ h = u_1 - E_{\alpha,1}(T^\alpha A)u_0. \]

We note that the same inverse problem for the Riemann-Liouville fractional derivative was considered in [2]. In this paper, the inverse problem is also reduced to the equation (8). Moreover, the characteristic functions of the inverse problems for the Riemann-Liouville and
Caputo derivatives are the same. The only difference is in the formulas for the element $h$. For the equation with the Riemann-Liouville derivative, the element $h$ is determined by the equality
\[ h = u_1 - T^{\alpha-1} E_{\alpha,\alpha}(T^\alpha A)u_0. \]

This circumstance allows to transfer the results of article [2] to the inverse problem (6) with the derivative of Caputo. In particular, we obtain the following statement.

**Theorem 2.** Let us assume that $H$ is a Hilbert space, operator $A$ is self-adjoint and non-positive, $0 < \alpha < 1$, function $\varphi \in C^1[0,T]$, everywhere either $\varphi \geq 0$ or $\varphi \leq 0$ and $\varphi(T) \neq 0$, $u_0, u_1 \in D(A)$. Then solution of problem (6) exists and it is unique.

Similarly, we can obtain explicit formulas for the solution of the inverse problem for the model case.

**Corollary 1.** Let us assume that $H$ is a Hilbert space, operator $A$ is self-adjoint and non-positive, $0 < \alpha < 1$, $\beta \geq 0$, $u_0, u_1 \in D(A)$.

exists and it is unique and is given by formulas
\[ p = \frac{1}{\Gamma(\beta + 1)} T^{\alpha+\beta} E_{\alpha,\alpha+\beta+1}(T^\alpha A) (u_1 - E_{\alpha,1}(T^\alpha A)u_0), \]
\[ u(t) = E_{\alpha,1}(t^\alpha A)u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha A)s^\beta pds. \]

**Corollary 2.** Let us assume that $H$ is a Hilbert space, operator $A$ is self-adjoint and non-positive, $0 < \alpha < 1$, $u_0, u_1 \in D(A)$.

exists and it is unique and is given by formulas
\[ p = \frac{1}{T^\alpha} E_{\alpha,\alpha+1}(T^\alpha A) (u_1 - E_{\alpha,1}(T^\alpha A)u_0), \]
\[ u(t) = E_{\alpha,1}(t^\alpha A)u_0 + t^\alpha E_{\alpha,\alpha+1}(t^\alpha A)p. \]

**References**

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[2] Orlovsky D.G. 2015 Parameter determination in a Differential Equation of Fractional Order with Riemann-Liouville Fractional Derivative in a Hilbert Space. *Journal of Siberian Federal University. Mathematics and Physics*. 8 no 1, 55