Morse theory for Lagrange multipliers and adiabatic limits

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Outline

1. Morse homology
2. Lagrange multiplier
3. Adiabatic limits
I. Morse homology
Let $M$ be a smooth manifold, $f \in C^\infty(M)$. $f$ is a Morse function if $df$ is a transverse section of $T^* M$. 
Let $M$ be a smooth manifold, $f \in C^\infty(M)$. $f$ is a **Morse function** if $df$ is a transverse section of $T^* M$.

By **Morse lemma**, near $p \in \text{Crit} f$, there is a coordinate chart $(x_1, \ldots, x_n)$ such that

$$f(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2.$$
Let $M$ be a smooth manifold, $f \in C^\infty(M)$. $f$ is a Morse function if $df$ is a transverse section of $T^*M$.

By Morse lemma, near $p \in \text{Crit} f$, there is a coordinate chart $(x_1, \ldots, x_n)$ such that

$$f(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2.$$ 

Morse index: $\text{ind}(p, f) = k$, the number of negative eigenvalues of the Hessian.
For any metric $g$ on $M$, the metric dual of $df$ is the gradient $\nabla f$ of $f$. The negative gradient flow of $f$ is the ODE

$$x : \mathbb{R} \to M, \ x'(t) + \nabla f(x(t)) = 0.$$
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The integral of the flow defines a 1-parameter diffeomorphisms group $(\phi_t)_{t \in \mathbb{R}}$ of $M$, i.e.

$$\phi_t : M \to M, \quad \frac{d}{dt} \phi_t(x) + \nabla f(\phi_t(x)) = 0.$$
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unstable/stable manifolds of $p \in \text{Crit} f$:

$$W^u(p) = \left\{ x \in M \mid \lim_{t \to -\infty} \phi_t(x) = p \right\}, \ \text{dim} W^u(p) = \text{ind}(p)$$

$$W^s(p) = \left\{ x \in M \mid \lim_{t \to +\infty} \phi_t(x) = p \right\}, \ \text{dim} W^s(p) = n - \text{ind}(p).$$
The pair \((f, g)\) is Morse-Smale if

\[ \forall p_-, p_+ \in \text{Crit}_f, \; W^u(p_-) \cap W^s(p_+) = \emptyset. \]
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\[ \forall p_-, p_+ \in \text{Crit} f, \ W^u(p_-) \pitchfork W^s(p_+). \]

In this case, the moduli space of solutions to the ODE which are asymptotic to \(p_\pm\) at \(\pm\infty\)

\[ \tilde{M}(p_-, p_+) = W^u(p_-) \cap W^s(p_+) \]

\[ = \left\{ x : \mathbb{R} \to M \mid x'(t) + \nabla f(x(t)) = 0, \lim_{t \to \pm\infty} x(t) = p_\pm \right\} \]

is a smooth manifold. It has dimension \(\text{ind}(p_-) - \text{ind}(p_+)\).
The pair \((f, g)\) is **Morse-Smale** if

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In this case, the **moduli space** of solutions to the *ODE* which are asymptotic to \(p\) at \(\pm \infty\)

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\tilde{M}(p_-, p_+) = W^u(p_-) \cap W^s(p_+)
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= \left\{ x : \mathbb{R} \to M \mid x'(t) + \nabla f(x(t)) = 0, \lim_{t \to \pm \infty} x(t) = p_\pm \right\}
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is a smooth manifold. It has dimension \(\text{ind}(p_-) - \text{ind}(p_+)\).

\[
M(p_-, p_+) := \tilde{M}(p_-, p_+)/\mathbb{R}.
\]
(We need Palais-Smale condition in noncompact case):

\[ \text{ind}(p_-) - \text{ind}(p_+) = 1 \implies \# M(p_-, p_+) < \infty. \]
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Morse-Smale-Witten complex:

\[ CM_k(f; \mathbb{Z}_2) = \bigoplus_{p \in \text{Crit}_k f} \mathbb{Z}_2 \langle p \rangle, \quad \delta_{f,g} : CM_k \rightarrow CM_{k-1}. \]

\[ \delta_{f,g}(p) = \sum_{q \in \text{Crit}_{k-1} f} \#\mathcal{M}(p, q) \cdot q. \]
(We need **Palais-Smale condition** in noncompact case):

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**Morse-Smale-Witten complex:**

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\]

**Slogan:** “Boundary operator defined by the (oriented) counting of isolated trajectories.”
A nontrivial fact is that \((\delta_f,g)^2 = 0\). Then \((CM_*(f;\mathbb{Z}_2),\delta_{f,g})\) is a chain complex, and we define the Morse homology associated to the pair \((f,g)\) by

\[
HM_*(f,g;\mathbb{Z}_2) := H(CM_*(f;\mathbb{Z}_2),\delta_{f,g}).
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HM_*(f, g; \mathbb{Z}_2) := H \left( CM_*(f; \mathbb{Z}_2), \delta f, g \right).
\]

**Theorem**

A generic pair \((f, g)\) is Morse-Smale. The Morse homology is (canonically) independent of the choice of \(f, g\). For compact \(M\), \(HM_*(f, g; \mathbb{Z}_2)\) is isomorphic to the homology of \(M\).
II. Morse theory for Lagrange multipliers
$X$: compact manifold
$f, \mu \in C^\infty(X)$
$0$: regular value of $\mu$
$\overline{X} := \mu^{-1}(0)$. 

\[
\text{Lagrange multiplier: } F : X \times \mathbb{R} \to \mathbb{R}
\quad (x, \eta) \mapsto f(x) + \eta \mu(x)
\]
\[
\text{Crit } F = \{ (x, \eta) \mid df(x) + \eta d\mu(x) = 0, \mu(x) = 0 \}
\]
\[
\simeq \text{Crit } (f|_X) =: \{ p_1, p_2, \ldots, p_k \}
\]

In general, we can consider $\mu = (\mu_1, \ldots, \mu_k) : X \to \mathbb{R}^k$ and
\[
F(x, \eta_1, \ldots, \eta_k) = f(x) + \sum \eta_i \mu_i(x).
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$X$: compact manifold

$f, \mu \in C^\infty(X)$

$0$: regular value of $\mu$

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Lagrange multiplier:

$F : X \times \mathbb{R} \to \mathbb{R}$

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$\text{Crit}F = \{(x, \eta) \mid df(x) + \eta d\mu(x) = 0, \; \mu(x) = 0\}$

$\simeq \text{Crit}(f|_{\overline{X}}) =: \{p_1, p_2, \ldots, p_k\}$. 
\(X\): compact manifold
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In general, we can consider \(\mu = (\mu_1, \ldots, \mu_k) : X \to \mathbb{R}^k\) and

\[ F(x, \eta_1, \ldots, \eta_k) = f(x) + \sum \eta_i \mu_i(x). \]
If $p \in \text{Crit} f|_{\mathcal{X}}$, and locally

$$\mu = x_n, \quad f|_{\mathcal{X}} = -x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{n-1}^2.$$
If $p \in \text{Crit}_f|_X$, and locally

$$\mu = x_n, \quad f|_X = -x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{n-1}^2.$$  

Then

$$\nabla^2(f + \eta x_n) = \nabla^2 F = \begin{bmatrix} A_{n-1} & * & 0 \\ * & * & 1 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} A_{n-1} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$
If \( p \in \text{Crit } f|\overline{X} \), and locally

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\end{bmatrix}.
\]

So the two extra direction, \( x_n \) and \( \eta \) give additional one positive and one negative eigenvalues of the Hessian, and

\[
\text{ind } (p, F) = \text{ind } (p, \text{Crit } f|\overline{X}) + 1.
\]
Choose a metric $g_X$ on $X$, and the standard metric $e$ on $\mathbb{R}$, we define a family of metrics on $X \times \mathbb{R}$ by

$$g_\lambda = g_X \oplus \lambda^{-2}e, \quad \lambda > 0.$$
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The **negative gradient flow equation** is

$$\begin{cases} 
    x'(t) + \nabla f(x(t)) + \eta(t)\nabla \mu(x(t)) &= 0, \\
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Denote by $\mathcal{M}_{\lambda} (p_-, p_+)$ the moduli space of solutions connecting $p_-$ and $p_+$ modulo time translation. Elements are called $\lambda$-trajectories.
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We try to define the Morse homology for $(F, g_\lambda)$, denoted by $\text{HM}^\lambda_\ast(F; \mathbb{Z}_2)$, by counting isolated $\lambda$-trajectories.
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Second, for generic \((f, \mu, g_X)\), there is an isolated set \( \Lambda^{sing} \subset \mathbb{R}^+ \) such that for \( \lambda \in \Lambda^{reg} := \mathbb{R}^+ \setminus \Lambda^{sing} \), \((F, g_\lambda)\) is Morse-Smale. So \( HM^{\lambda}_*(F; \mathbb{Z}_2) \) is defined.
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Moreover, for $\lambda_1, \lambda_2 \in \Lambda^\text{reg}$, there is a canonical isomorphism (induced from a chain map)

$$\Phi : HM^\lambda_1(F; \mathbb{Z}_2) \simeq HM^\lambda_2(F; \mathbb{Z}_2).$$
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The dynamics are varying with $\lambda$, so we are counting different objects for different $\lambda$. It yields different chain complexes $CM_\lambda^*(F)$ which have isomorphic homology.
A natural question to ask is

What are the “limits” of $CM^\lambda_*(F)$ as $\lambda \to 0$ and $\lambda \to \infty$?
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More approachable questions:

1. Given a sequence $\lambda_i$ with $\lim \lambda_i = 0$ or $\infty$, and a sequence $\gamma_i \in \mathcal{M}_\lambda(p_-, p_+)$. Is there a good limit (up to choosing a subsequence) of $\gamma_i$?
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A natural question to ask is

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More approachable questions:

1. Given a sequence $\lambda_i$ with $\lim \lambda_i = 0$ or $\infty$, and a sequence $\gamma_i \in M^\lambda(p_-, p_+)$. Is there a good limit (up to choosing a subsequence) of $\gamma_i$?

2. If we can describe limiting trajectories, do they always arise as limits of $\lambda$-trajectories?

3. Does the counting of the limiting trajectories defines a chain complex which has the same homology?
λ → ∞
$\lambda \to \infty$

In general, if $x'(t) + \nabla h(x(t)) = 0$, then the energy of $x$ is

$$\int_{\mathbb{R}} |x'(t)|^2 = h(x(-\infty)) - h(x(+\infty)).$$
\[ \lambda \to \infty \]

In general, if \( x'(t) + \nabla h(x(t)) = 0 \), then the energy of \( x \) is

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\]

In the case of \((F, g_\lambda)\), for any solution \((x, \eta) : \mathbb{R} \to X \times \mathbb{R}\),

\[
|\nabla f(x) + \eta \nabla \mu(x)|_{L^2}^2 + \lambda^2 |\mu(x)|_{L^2}^2 = F(p_-) - F(p_+).
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\]

\[ \lambda \to \infty \] trajectories converge into \( \overline{X} = \mu^{-1}(0) \).

Indeed, the \( X \)-component of the trajectory converges to (broken) trajectories of \((\overline{f}, \overline{g}) := (f|_{\overline{X}}, g|_{\overline{X}})\). So the limiting dynamical system is well-understood.
**Theorem**

If \( \lambda_i \to \infty \), and \( \gamma_i : \mathbb{R} \to X \times \mathbb{R} \) is a sequence of \( \lambda_i \)-trajectories connecting \( p_- \) and \( p_+ \), then there is a subsequence of \( \gamma_i \) which converges to a broken trajectory of \(-\nabla F\) connecting \( p_- \) and \( p_+ \).
Theorem

If $\lambda_i \to \infty$, and $\gamma_i : \mathbb{R} \to X \times \mathbb{R}$ is a sequence of $\lambda_i$-trajectories connecting $p_-$ and $p_+$, then there is a subsequence of $\gamma_i$ which converges to a broken trajectory of $-\nabla f$ connecting $p_-$ and $p_+$.

Conversely, if $\text{ind} p_- - \text{ind} p_+ = 1$, then there exists $\lambda_0 >> 0$ such that for any trajectory $\bar{y}$ of $-\nabla f$ connecting $p_-$ and $p_+$, for any $\lambda > \lambda_0$, there exists a unique $\lambda$-trajectory (up to time translation) connecting $p_-$ and $p_+$ which is “close” to $\bar{y}$. 
**Theorem**

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**Corollary (Folklore)**

For \( \lambda \in \Lambda^\text{reg} \), there is an isomorphism

\[
HM^\lambda_*(F; \mathbb{Z}_2)[1] \cong HM_*(\bar{f}; \mathbb{Z}_2) \cong H_*(\bar{X}; \mathbb{Z}_2)
\]
Now we consider the other limit of $CM_\lambda^\lambda(F)$ as $\lambda \to 0$. The equation is

\[
\begin{cases}
    x'(t) + \nabla f(x(t)) + \eta(t)\nabla \mu(x(t)) &= 0, \\
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Now we consider the other limit of $CM_{\ast}^\lambda(F)$ as $\lambda \to 0$. The equation is

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    x'(t) + \nabla f(x(t)) + \eta(t)\nabla \mu(x(t)) &= 0, \\
    \eta'(t) + \lambda^2 \mu(x(t)) &= 0 
\end{align*}
\]

This is a special case of the fast-slow ODE:

\[
\begin{align*}
    x'(t) &= A(x(t), y(t)), \\
    y'(t) &= \epsilon B(x(t), y(t)).
\end{align*}
\]

So in normal scale, the variable $\eta$ is “freezed”. Then the variable $x$ travels along the flow of $-\nabla(f + \eta \mu)$. 
Set $\lambda = 0$. It gives the equation for the fast flow is

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On the other hand, in large scale (long time), \( \eta \) can still change at places where \( x \) also changes slowly, i.e., the slow manifold

\[
C_X := \{(x, \eta) \mid \nabla f(x) + \eta \nabla \mu(x) = 0\}.
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$$C_X := \{(x, \eta) \mid \nabla f(x) + \eta \nabla \mu(x) = 0\}.$$

So for generic data, $C_X$ is a smooth curve in $X \times \mathbb{R}$. Moreover

$$(x, \eta) \in C_X \implies x \in \text{Crit}(f + \eta \mu), \quad \text{Crit}F = C_X \cap (\overline{X} \times \mathbb{R}).$$
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So for generic data, $C_X$ is a smooth curve in $X \times \mathbb{R}$. Moreover

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Moreover, in generic case $F|_{C_X}$ is a Morse function, whose negative gradient flow is called the slow flow. Crit$F$ is part of the equilibria of the slow flow.
A fast-slow trajectory connecting $p_-$ and $p_+$ is a finite concatenations

$$
\gamma = (\ldots, \gamma_k^F, \gamma_k^S, \gamma_{k+1}^F, \gamma_{k+1}^S, \ldots)
$$

of trajectories of the fast or slow flow. $\mathcal{M}_{FS}(p_-, p_+)$ is their moduli space.
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of trajectories of the fast or slow flow. \( \mathcal{M}_{FS}(p_-, p_+) \) is their moduli space.

**Theorem (Schecter-G.X.)**

Given \( p_\pm \in \text{Crit}F \), a sequence \( \lambda_i \to 0 \) and a sequence of \( \lambda_i \)-trajectories \( \gamma_i \in \mathcal{M}_{\lambda_i}(p_-, p_+) \), there is a subsequence which converges to a fast-slow trajectory connecting \( p_- \) and \( p_+ \).
A fast-slow trajectory connecting $p_-$ and $p_+$ is a finite concatenations

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**Theorem (Schecter-G.X.)**

*Given $p_\pm \in \text{Crit}F$, a sequence $\lambda_i \to 0$ and a sequence of $\lambda_i$-trajectories $\gamma_i \in \mathcal{M}_{\lambda_i}(p_-, p_+)$, there is a subsequence which converges to a fast-slow trajectory connecting $p_-$ and $p_+$.**

To show that all fast-slow trajectories arise as limits of $\lambda$-trajectories, we have to do the “gluing” part. In the study of fast-slow ODE, this story is called geometric singular perturbation theory.
The “gluing part” of the main theorem is

**Theorem (Schecter–G.X., 2012)**

Suppose \((f, \mu, g_X)\) is generic. Then for any if \(\text{ind} p_- - \text{ind} p_+ = 1\), \(\mathcal{M}_{FS}(p_-, p_+)\) consists of isolated objects. Moreover, there exists a \(\lambda_0 > 0\) such that for every \(\lambda \in (0, \lambda_0)\) and for every \(\gamma \in \mathcal{M}_{FS}(p_-, p_+)\), there exists a unique \(\lambda\)-trajectory \(\gamma_{\lambda} \in \mathcal{M}_\lambda(p_-, p_+)\) which is close to \(\gamma\).
The “gluing part” of the main theorem is

**Theorem (Schecter–G.X., 2012)**

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**Corollary**

The counting of isolated fast-slow trajectories defines a chain complex \(CM_{FS}(f, \mu; \mathbb{Z}_2)\), whose homology is isomorphic to \(HM^\lambda_*(F; \mathbb{Z}_2)\). So there is an isomorphism

\[
HM^FS_*(f, \mu; \mathbb{Z}_2) \cong HM^\lambda_{big}_*(F; \mathbb{Z}_2) \cong HM^\lambda_{small}_*(F; \mathbb{Z}_2) \cong H_*(\overline{X}; \mathbb{Z}_2)[-1].
\]
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We see \( \text{Crit}f_\epsilon = \text{Crit}h \). Then we consider the flow of \(-\nabla f_\epsilon\). In a tubular neighborhood of \( S \), the equation is

\[
\begin{align*}
    v'(t) &= -\nabla f(v(t), s(t)), \\
    s'(t) &= -\epsilon \nabla h(v(t), s(t)).
\end{align*}
\]
The fast flow is just the flow of $-\nabla f$. The slow manifold is $S$. The slow flow is the flow of $-\nabla h$ inside $S$. 
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The fast-slow trajectories in this case are the “cascades”, i.e., a concatenation of flow lines of $-\nabla f$ between two points in two different components of $S$, and flow lines of $-\nabla h$ inside $S$. 
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**Theorem (Banyaga-Hurtubise, 2013)**

If $\text{ind}(p_-) - \text{ind}(p_+) = 1$, then for any $\epsilon$ small enough, there is a one-to-one correspondence between “cascade trajectories” and trajectories of $-\nabla f_\epsilon$ connecting $p_-$ and $p_+$. 
Morse homology
Lagrange multipliers
Adiabatic Limits

\[ \lambda \to \infty \]
\[ \lambda \to 0 \]
In our fast-slow complex, there are only finitely many “fast tunnels” which can be a component of a fast-slow trajectory connecting $p_-$ and $p_+$ if they have adjacent indices.

They are the handle-slides, cusp trajectories, and some index 1 trajectories.
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For the family of functions $f + \eta \mu$, there are finitely many $\eta$’s such that $(f + \eta \mu, g_X)$ is not a Morse-Smale pair on $X$. 
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There may be handle-slides, which are fast trajectories connecting $(x_-, \eta), (x_+, \eta) \in C_X$ with $\text{ind}(x_-, f + \eta \mu) = \text{ind}(x_+, f + \eta \mu)$. 
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There also may be birth-deaths, i.e., for some $(x, \eta) \in C_X$ such that $x$ is a degenerate critical point of $f + \eta \mu$. A cusp trajectory is a type of fast trajectory connecting a birth-death with a non-birth-death point on $C_X$. 
If $p_-$ is a local minimum of the slow flow, then the fast-slow trajectory must start with an index 1 fast trajectory from $p_-$; if $p_+$ is a local maximum of the slow flow, then the fast-slow trajectory must end with an index 1 fast trajectory to $p_+$. 

We have then a map of transportation, consists of those "fast tunnels" and the "slow tracks" gives only finitely many possible routes to transport from $p_-$ to $p_+$. We are actually counting those routes.
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A model case: \( f = -\frac{1}{3}x^3, \mu = x + 1, C_X = \{ \eta = x^2 \} \). The origin is a birth-death.

\[
\begin{align*}
z_1 &= x, \\
z_2 &= \eta, \\
\epsilon &= \lambda^2.
\end{align*}
\]
Thank you!