SVD analysis in application to full waveform inversion of multicomponent seismic data

Ilya Silvestrov and Vladimir Tcheverda
Institute of Petroleum Geology and Geophysics SB RAS, Novosibirsk, 630090, Russia
E-mail: SilvestrovIV@ipgg.nsc.ru

Abstract. An inverse problem of recovery the Earth’s interior by multi-shot/multi-offset multicomponent seismic data is considered in this work. This problem may be considered as a nonlinear operational equation, and local derivative-based techniques are commonly used for its solution. Such method is known in seismic processing as ”full-waveform inversion”. The major properties of the inversion process are governed by a Frechet derivative of the forward map. We show and study these properties by means of singular value decomposition(SVD) truncation. This decomposition depends strongly on the acquisition system and on the parameterization of the problem. We show, that it is very important to study the inverse problem in each particular case, otherwise unreliable results may be obtained. Surface and cross-well acquisition systems are considered in this work. Appropriate parameterizations for them are determined, and typical behavior of the inverse problem solution is studied.

1. Introduction
An inverse problem of recovery the Earth’s interior by multi-shot/multi-offset multicomponent seismic data is considered in this work. The data are treated as an image of some nonlinear operator (forward map) which is implicitly introduced by initial-boundary value problem for elastic wave equations. In order to recover elastic parameters, this nonlinear operator equation should be resolved. The usual way to do this numerically is implementation of some local derivative-based minimization technique like conjugate gradient or Gauss-Newton methods and various their modifications. In seismic processing this method is known as ”full-waveform inversion”.

This approach was proposed in early 1980’s [1] and was intensively studied and developed by many researches. The method seemed to be very promising at first, because of its apparent ability to recover elastic parameters under a few preprocessing of data. However its application to realistic synthetic and field data exposed its major drawbacks. The most significant of them were troubles in reconstruction of the smooth velocity constituent, the so-called macrovelocity or migration model. Certain difficulties were found in connection with coupling of elastic parameters[2] and different sensitivity of the method to their perturbations[3].

Our belief is that all peculiarities of the inversion procedure are controlled mainly by the Frechet derivative of the forward map, because this derivative determines the gradient and Hessian of the L2 data misfit functional. Singular Value Decomposition of this derivative is a powerful tool for analyzing its key features. In current work this analysis is applied to 2D elastic full-waveform inversion. We show that all of the mentioned above drawbacks of the method are explained and predicted by the structure of right singular vectors. The notion of stable
subspace in model space is introduced and discussed and its geometry is analyzed. The results of synthetic data processing for surface and for cross-well acquisition systems are presented and discussed.

2. Statement of the problem

The seismic inverse problem, which we are targeted to solve, formally may be written as a following equation:

$$B\vec{m} = \vec{u}_{obs},$$

where $\vec{u}_{obs}$ defines recorded seismic data (seismograms), $\vec{m}$ defines unknown elastic parameters of the subsurface, and $B$ is a nonlinear operator, which is defined implicitly by some boundary-value problem for elastic wave equations. A common way to solve such kinds of problems is an application of Newton method to this nonlinear equation. This leads us to the following well-known iterative process:

$$L[\vec{m}_{k}](\vec{m}_{k+1} - \vec{m}_{k}) = \vec{u}_{obs} - B(\vec{m}_{k}),$$

where operator $L$ is a formal Frechet derivative of the considered operator $B$ and $k$ defines a step number of the iterative process. Certainly, the question of differentiability of this operator is a non-trivial in each particular case, however it is out of scope of this work. On each step of the iteration process we are required to solve the following linear equation:

$$L\delta\vec{m} = \delta\vec{u},$$

where $\delta\vec{u}$ is a small perturbation in the recorded data, and $\delta\vec{m}$ is a small unknown update to the elastic parameters of the subsurface. Two-dimensional isotropic elastic medium is defined by three parameters (e.g. Lame parameters and density) and the recorded data in this case may have two components (e.g. horizontal and vertical displacements). Therefore equation (1) may be written in the form:

$$
\begin{pmatrix}
L_{11} & L_{12} & L_{13} \\
L_{21} & L_{22} & L_{23}
\end{pmatrix}
\begin{pmatrix}
\delta m_1 \\
\delta m_2 \\
\delta m_3
\end{pmatrix}
=
\begin{pmatrix}
\delta u_1 \\
\delta u_2
\end{pmatrix},
$$

where $L_{ij}$ defines partial derivative with respect to corresponding parameter. A common way to obtain these derivatives is to apply a single-scattering (Born’s) approximation, according to which we obtain the following integral equations of first kind:

$$
\int_{\Omega} K_{ij}(\vec{x}_s, \vec{x}_r, \vec{x}, \omega)\delta m_j(\vec{x})d\vec{x} = \delta u_j(\vec{x}_s, \vec{x}_r, \omega),
$$

where the seismic data are considered as functions of frequency $\omega$ after applying Fourier transform. $\vec{x}_s$, $\vec{x}_r$ stand for sources and receivers position correspondingly, and $\Omega$ defines the subdomain in the subsurface, where the elastic parameters are targeted to be inverted. The kernels $K_{ij}$ of these integral operators are defined by Green’s function of the considered problem.

From the above relations we see, that on each step of Newton method we have to solve system of integral equations of first kind, which is a classical ill-posed problem, and it requires regularization for its solution. In this work we will focus on a truncated SVD regularization method, particularly on its generalization for compact operators in Hilbert spaces that were developed in works of Cheverda and Kostin[4, 5]. In this technique the so-called r-solution is constructed as a projection of a true solution onto a ”stable subspace”, which is defined as a linear span of singular vectors, which correspond to largest singular values. The number of the
involved singular vectors is controlled by the acceptable condition number and therefore by the noise level in the data and by the error due to numerical approximating of the operator. Using this technique in the rest part of the paper we will construct the regularized solutions for two problems of seismic prospecting, and will analyze the stable subspaces for both of them. We will study only the first step of the non-linear iterative process. However, this step usually brings main contribution into solution, and its consideration allows us to determine main features of the problem.

One of the most important step in solution of the inverse problem is a proper choice of the unknown parameters. For example, in case of isotropic elastic medium we may consider following sets of unknown elastic properties: pressure($V_p$) and shear($V_s$) waves velocities and density($\rho$); or Lame parameters ($\lambda, \mu$) and density; or pressure($IP$) and shear($IS$) impedances and density. These parameters are connected with each other by following relations:

$$V_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad V_s = \sqrt{\frac{\mu}{\rho}}, \quad IP = \rho V_p, \quad IS = \rho V_s.$$  

First of all we note, that the parameters inside these sets have different physical dimensions, and therefore a compulsory nondimensionalization is required. For example, instead of $V_p, V_s, \rho$ one may use following dimensionless parameters for inversion: $\frac{\delta V_p}{V_p^0}, \frac{\delta V_s}{V_s^0}, \frac{\delta \rho}{\rho^0}$, where $V_p^0, V_s^0, \rho^0$ define elastic properties in some reference medium. However, even after such normalization not every set of parameter is suitable for inversion[3]. As will be shown in next section it requires very careful study in each particular case.

3. Numerical results of SVD analysis

![Figure 1](image1.png)  
**Figure 1.** A model and acquisition systems used for numerical experiments. Sources are marked by (∇) character, and receivers by (△). Red color stands for surface acquisition, and blue color for cross-well experiment.  

![Figure 2](image2.png)  
**Figure 2.** Singular values of linearized forward modeling operator in case of surface acquisition for different sets of parameters: velocities(blue line), impedances(red line), Lame parameters(green line).
Let us consider a model illustrated in Figure 1. We assume that the structure inside the area marked by black square is unknown, and our aim is to determine elastic parameters there. We will consider two cases. The first one is inversion of surface data, in which sources and receivers of seismic waves are placed uniformly at the Earth’s surface. The second one is so-called cross-well data inversion, in which we assume that there are two wells at the vertical edges of the model, and the sources are placed into one well, while receivers are placed into another. This acquisition system is marked by blue rectangles.

The most time and numerically consuming part of the utilized technique is a construction of numerical approximation of operator DB and its singular value decomposition. To reduce this cost in current work we use rather simple approach, which however allows us to catch main features of the inverse problems. We assume that the reference medium, with respect to which we are going to obtain perturbations of elastic parameters, is homogeneous. It allows us to use explicit well-known form of Green’s function for such medium. It is also assumed that source generates only pressure wave. The next simplifications are made by assuming only far field approximation to the Green’s function, and by ignoring the free surface. The first of them is common for seismic prospecting because only far field effects usually contain the desired information about the subsurface, and the second one is usually achieved by additional preprocessing of the recorded data and suppressing the surface waves. All these simplifications allow us to use explicit formulas for constructing the matrix approximation of operator DB.

We consider following parameters of homogeneous background medium: \( V_p = 3000 \text{m/s}, V_s = 1500 \text{m/s}, \rho = 1000 \text{kg/m}^3 \). To construct the matrix approximation, elementary step-like finite-dimensional bases were introduced into the model and into the data spaces. In the model space a grid with cell size of 10m in both directions is used, and in the data space the frequency range from 0.1Hz to 60Hz is uniformly sampled by interval of 0.5Hz. A source wavelet is Ricker and the central frequency is 30Hz.

3.1. Surface data inversion
The singular values of the constructed matrices for different parameterizations are illustrated in Figure 2 in logarithmic scale. The problem is overdetermined and the total number of the shown values is equal to number of the unknown parameters. We see that the singular values rapidly tend to zero, and it is explained by compactness of the considered integral operator. The graphic inflection near 2500 singular value corresponds to loss of machine accuracy, because a normal matrix form is used for calculating the SVD. Therefore, the rest singular values are incorrect.

We observe, that the singular values are very similar for different parameterizations. It means that from point of view of conditioning of the problem, all considered parameters sets are equivalent for inversion. The next step is to understand what part of the model may be inverted, when different parameterizations are used. Let us consider at first the model, in which there is inhomogeneity with respect to only one parameter Lame \( \lambda \). The another Lame parameter \( \mu \) and density are homogeneous. However, when we are solving the inverse problem we do not usually have such information, and we would like to invert for all parameters simultaneously. To predict what kind of the model may be inverted, we truncate the SVD of the corresponded matrix and construct the projection of the true model onto largest singular vectors. We consider a condition number of 10 to be acceptable and use 800 largest singular vectors. Figure 3 illustrates the true perturbation in the target area and its constructed projection that can be treated as a result of inversion. First of all we see, that we do not invert any interior of the layers, but we invert only positions of interfaces in the model. Also we see, that initial perturbation with respect to \( \lambda \) leaks into density and slightly into \( \mu \). Thus using such result of inversion we may come to a wrong conclusion, that the model has interfaces with respect to all parameters. Certainly, this is a numerical artifact and it should be eliminated by using for example some another
3.2. Cross-well data inversion

In this example we will use the same model as in previous one, but another positions of sources and receivers will be considered, as shown in Figure 1. The singular values for matrix approximation of operator $DB$ in case of the cross-well acquisition system are shown.
Figure 5. Inversion of elastic impedances and density in case of surface acquisition. The projections of single parameter perturbation onto 800 largest right singular vectors are shown for: a) pressure impedance; b) shear impedance; c) density.

in Figure 6, compared with values for surface acquisition. We see that singular values for these two experiments are different. As in previous example we again fix condition number to be 10, and it corresponds to 1000 largest right singular vectors that is approximately equal to the number of singular vectors used for surface acquisition.

Previously we obtained, that impedances are the best parameters for inversion using surface data. Now we repeat for cross-well data the numerical experiment shown in Figure 5(b), where single P impedance perturbations were recovered. Using the same technique we construct the projection of such perturbation onto largest singular vectors and obtain the result shown in Figure 7. We see, that while the true model is the same, the projection in case of cross-well acquisition is totally different. We observe a strong low density anomaly in the result, that means a strong coupling between pressure impedance and density. Therefore this parameterization can not be used for inversion in case of cross-well acquisition. Figure 8 illustrates, that velocities are the most appropriate parameters in this case. We note that P wave velocity is recovered almost ideally. The inversion process is able in that case to detect not only the interfaces, but interior of the layers. No coupling between different parameters is observed. The fail to obtain the shear wave velocity is explained because of sources type, which generate only P wave, and apparently that pressure waves propagation is not sensitive to shear waves velocity perturbations. Therefore, for shear velocity we may obtain only interfaces, where waves conversion occurs. The density is also almost not recovered, however some week interfaces are observed.
Figure 6. Singular values of linearized forward modeling operator in case of cross-well acquisition (blue line) and their comparison with values for surface acquisition (red line): a) singular values overview; b) zoom view of largest singular values.

Figure 7. Inversion of pressure impedance perturbation in case of cross-well acquisition. The projection onto 1000 largest right singular vectors is shown.

4. Conclusion
In this work we use singular value decomposition to study the properties of multiparameter seismic inverse problem. As was expected, a Frechet derivative of the non-linear forward modeling operator determines major properties of the inverse problem, and truncating its SVD allows us to analyze them in detail. We show that these properties strongly depend on the considered arrangement of sources and receivers, and they should be studied in each particular case. In this paper we were focused on a correct parameterization of the problem that is one of the important step for successful solution. As was shown, a wrong choice of parameters may lead to totally incorrect results of inversion. It was proved, that pressure and shear impedances are the best choice for inversion using surface acquisition system. However density can not be inverted in that case. Such acquisition allows us to determine only interfaces of reflectors. In case of cross-well experiment waves velocities are the most appropriate parameters for inversion. We obtain, that pressure wave velocity may be inverted almost ideally, and this result is very stable. The results were obtained by analyzing projections of the true solution onto largest right singular vectors. However, we do not analyze here these vectors themselves, that also may provide us additional useful information about the inverse problem. We hope that it will be done in future works. We would like also to note, that though only the SVD truncation regularization technique is considered in this work, the obtained results make sense for others regularization methods such as Tikhonov regularization, or regularization by iterative procedures, because there is a strong relation between all of them.
Figure 8. Inversion of elastic waves velocities and density in case of cross-well acquisition. The projections of single parameter perturbation onto 1000 largest right singular vectors are shown for: a) pressure velocity; b) shear velocity; c) density.

Acknowledgments
This research was done in cooperation with Schlumberger Moscow Research and was partially supported by Russian Foundation of Basic Research, projects 10-05-00233, 11-05-00238, 11-05-00947.

References
[1] Tarantola A 1984 Inversion of seismic reflection data in the acoustic approximation Geophysics 49 1259–1266
[2] Assous F and Collino F 1990 A numerical method for the explanation of sensitivity: the case of the identification of the 2D stratified elastic medium Inverse Prob. 6 487–514
[3] Tarantola A 1986 A strategy for nonlinear elastic inversion of seismic reflection data Geophysics 51 1893–1903
[4] Cheverda V A and Kostin V I 1995 r-pseudoinverses for compact operators in hilbert spaces: existence and stability J. of Inverse and Ill-posed Prob. 3 131–148
[5] Cheverda V A, Clement F, Khaidukov V G and Kostin V I 1998 Linearized inversion of data of multi-offset data for vertically inhomogeneous background J. of Inverse and Ill-posed Prob. 6 453–484