Spectral dimension and Bohr’s formula for Schrödinger operators on unbounded fractal spaces

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Abstract

We establish an asymptotic formula for the eigenvalue counting function of the Schrödinger operator $-\Delta + V$ for some unbounded potentials $V$ on several types of unbounded fractal spaces. We give sufficient conditions for Bohr’s formula to hold on metric measure spaces which admit a cellular decomposition, and then verify these conditions for fractafolds and fractal fields based on nested fractals. In particular, we partially answer a question of Fan, Khandker, and Strichartz regarding the spectral asymptotics of the harmonic oscillator potential on the infinite blow-up of a Sierpinski gasket.

Keywords: fractal, Bohr’s formula, Schrödinger operator

(Some figures may appear in colour only in the online journal)

1. Introduction

In this paper we present an asymptotic formula for the eigenvalue counting function of the Schrödinger operator $-\Delta + V$ for unbounded potentials $V$ on several types of unbounded fractal spaces. Such an asymptotic formula is often attributed to Niels Bohr in the Euclidean setting. We identify a set of sufficient conditions for Bohr’s formula to hold on locally self-similar metric measure spaces which admit a cellular decomposition, and then verify these conditions for fractafolds [53, 58] and fractal fields [23] based on nested fractals. In particular,
we are able to partially answer a question of Fan, Khandker, and Strichartz [18] regarding the spectral asymptotics of the harmonic oscillator potential on the infinite blow-up of a Sierpinski gasket (abbreviated SG).

All these results have similarities in the classical theory of 1D Sturm–Liouville operators (see [46]). The deep analogy between nested fractals (the typical representative being SG) and the real line \( \mathbb{R}^1_+ = [0, \infty) \) is related to the fact that all of them are finitely ramified. (A set is said to be finitely ramified if it can be divided into several disconnected subsets upon removing a finite number of points from the set. For \( \mathbb{R}^1_+ \), it suffices to remove one point; for SG, two points.)

Let us recall several known results from the spectral theory of the 1D Schrödinger operator

\[ H \psi = -\psi'' + V(x)\psi, \quad x \geq 0 \quad (1.1) \]

with boundary condition at \( x = 0 \) of either Dirichlet type, \( \psi(0) = 0 \), or Neumann type, \( \psi'(0) = 0 \).

I. Assume that \( V(x) \to +\infty \) as \( |x| \to +\infty \). Then, by the result of H Weyl, the spectrum of \( H \) in \( L^2([0, \infty), dx) \) is discrete and, under some technical conditions,

\[ N(\lambda, V) := \# \{ \lambda_i(H) \leq \lambda \} \approx \frac{1}{\pi} \int_0^\infty \sqrt{\lambda - V(x)} \, dx. \quad (1.2) \]

This is known as N Bohr’s formula, see [26, 32, 34].

II. Assume that \( V(x) \) is compactly supported, or (weaker assumption) vanishing fast enough (see below). Put \( V(x) = V_f(x) - V_c(x) \), where \( V_f = \max(0, V) \) and \( V_c = \max(0, -V) \), and

\[ N_c(V) := \# \{ \lambda_i \leq 0 \} \leq N_c(-V_c(\cdot)). \quad (1.3) \]

The estimate of \( N_c(V) \) as a result can be reduced to the negative potentials (potential wells).

We use the notation \( N_c(V) \) assuming here that \( V(x) = -V_c(x) \leq 0 \). The following estimates of \( N_c(V) \) are popular in applications (see [46]):

(a) (Bargmann)

\[ N_c(V) \leq 1 + \int_0^\infty x V(x) \, dx. \quad (1.4) \]

(b) (Calogero) If \( V(x) \) decreases with \( |x| \) as \( |x| \to \infty \), then

\[ N_c(V) \leq c_0 \int_0^\infty \sqrt{V(x)} \, dx. \quad (1.5) \]

The Calogero estimate has the correct scaling in the following sense.

(c) Consider the operator

\[ H_\sigma \psi = -\psi'' + \sigma V_0(x) \psi, \quad x \geq 0 \quad (\text{plus boundary condition}). \quad (1.6) \]

Then as \( \sigma \to \infty \),

\[ N_c(\sigma V_0) \sim c_1 \sigma^{1/2} \int_0^\infty \sqrt{V(x)} \, dx. \quad (1.7) \]

This is the so-called quasiclassical asymptotics. It is an important problem to find such an estimate for \( N_c(V) \), which has true scaling in \( \mathbb{R}^d, d \geq 2 \), i.e., for any \( \sigma \),

\[ N_c(\sigma V_0) \leq \sigma^{d/2} \Phi(V_0). \quad (\text{Cwicłl–Lieb–Rosenblum}) \quad (1.8) \]
For $d \geq 3$ this is the CLR estimate

$$N_\nu(V) \leq c_d \int_{\mathbb{R}^d} |V(x)|^{d/2} \, dx. \quad (1.9)$$

For $d = 2$ the recent results by Grigor’yan and Nadirashvili [21] and Shargorodsky [51] give the desirable (though not simple) estimate. The paper [21] contains the justification of the physical conjecture by Madau and Wu on $N_\nu(V)$ for 2D operators. The case $d = 1$ was studied in the relatively recent papers by Naimark, Rozenblum, Solomyak et al (see [35, 50] and references therein).

In this paper we address the item I. above in detail. Items II(a), II(b), II(c) will be the subject of future work.

Our main objective is to consider, instead of the Euclidean space, a fractafold, which according to Strichartz [53] is defined as ‘a space that is locally modeled on a specified fractal, the fractal equivalent of a manifold.’ The first instance of a fractafold is the infinite SG (figure 1). As shown by Barlow and Perkins [8], the heat kernel $p_t(x, y)$ on the infinite SG satisfies a sub-Gaussian estimate with respect to the Euclidean metric $d(\cdot, \cdot)$:

$$p_t(x, y) \asymp c_1 t^{-d/2} \exp \left\{ -c_2 \left( \frac{d(x, y)^2}{t} \right)^{1/(d_x-1)} \right\},$$

where $d_x = 2 \log 3/\log 5$ and $d_w = \log 5/\log 2 > 2$. Here $\asymp$ means that there are upper and lower estimates, but the constants $c_1$, $c_2$ in them may be different. We would like to note, however, that the heat kernel is not immediately relevant for spectral analysis, partially because its form is complicated, but mostly because the domain of the Laplacian is not an algebra under multiplication [11]. Other typical examples of fractafolds that we consider, see [58] and section 6 for details, are shown in figures 2, 3, and 4. For background concerning spectral analysis on fractafolds, see [2, 4, 15, 18, 27–29, 41–43, 45, 47–49, 54, 59]. Existence
of gaps in the spectrum is investigated in [9, 16, 24, 55]. Wave equation on fractals is discussed in [3, 13, 14, 39, 57]. Physics applications, and spectral zeta functions in particular, are given in [1, 10, 17, 25, 52, 60].

2. Main results

2.1. Spectral asymptotics of $-\Delta + V$

In all the examples to follow, $K$ is a compact set in $\mathbb{R}^d$ endowed with a Borel probability measure $\mu$ and a ‘well-defined boundary’ $\partial K$ which has $\mu$-measure zero. We shall assume that there exists a well-defined self-adjoint Laplacian operator $-\Delta$ (respectively $-\Delta'$) on $L^2(K, \mu)$ satisfying the Dirichlet (respectively Neumann) condition on $\partial K$. Note that $\partial K$ might not coincide with the boundary of $K$ in the topological sense. We assume (as is well known in examples) that both $-\Delta$ and $-\Delta'$ have compact resolvents, and hence have pure point spectra. It then makes sense to introduce the eigenvalue counting function

$$N^b(K, \mu, \lambda) := \# \left\{ \lambda_i(-\Delta^b) \leq \lambda \right\}, \quad b \in \{ \wedge, \vee \}. \quad (2.1)$$
Assumption 2.1. There exists a positive constant $d_s$ such that
\[
0 < \lim_{\lambda \to \infty} \lambda^{-d_s/2} N^b(K, \mu, \lambda) \leq \lim_{\lambda \to \infty} \lambda^{-d_s/2} N^b(K, \mu, \lambda) < \infty,
\] (2.2)
where $b \in \{\land, \lor\}$.

A stronger condition than assumption 2.1 is

Assumption 2.2 (Weyl asymptotics of the bare Laplacian). There exist a positive constant $d_s$ and a right-continuous with left limits (càdlàg), $T$-periodic function $G : \mathbb{R} \to \mathbb{R}_+$ satisfying

\begin{align*}
\text{(G1)} & \quad 0 < \inf G \leq \sup G < \infty. \\
\text{(G2)} & \quad G \text{ is independent of the boundary condition } b \in \{\land, \lor\} \text{ such that as } \lambda \to \infty,
\end{align*}
\[
N^b(K, \mu, \lambda) = \lambda^{d_s/2} \left[ G \left( \frac{1}{2} \log \lambda \right) + R^b(\lambda) \right],
\] (2.3)
where $R^b(\lambda)$ denotes the remainder term of order $o(1)$.

Remark 2.3. The parameter $d_s$ is often identified with the spectral dimension of the bare Laplacian $-\Delta$ on $L^2(K, \mu)$. If $K$ is a domain in $\mathbb{R}^d$ with a nice boundary, and $\mu$ is the Lebesgue measure, then $d_s = d$ and $G$ is an explicit constant $(2\pi)^{-d} \mu(B) / \mu(K)$, where $B$ is the unit ball in $\mathbb{R}^d$. However, there are classes of fractals $K$ for which (2.3) holds with $G$ being possibly nonconstant.

In many examples, the leading-order term in $R^b(\lambda)$ gives information about the boundary of the domain. For an Euclidean domain in $\mathbb{R}^d$ with nice boundary, the leading-order term of $R^b(\lambda)$ scales with $\lambda^{-1/2}$, and the sign of this term is negative (respectively positive) if $b = \land$ (respectively if $b = \lor$) \cite{12, 20, 30, 33}.

We now consider an unbounded space $K_\infty$ which admits a cellular decomposition into copies of $K$. Formally, let $K_\infty := \bigcup_\alpha K_\alpha$, where

- Each $K_\alpha$ is isometric to $K$ via the map $\varphi_\alpha : K \to K_\alpha$.
- We identify $\partial K_\alpha := \varphi_\alpha(\partial K)$ to be the boundary of $K_\alpha$, and $K_\alpha^\circ := K_\alpha \setminus \partial K_\alpha$ the interior of $K_\alpha$.
- (Cells adjoin only on the boundary.) For all $\alpha \neq \alpha'$, $\left( K_\alpha \cap K_{\alpha'} \right) = \left( \partial K_\alpha \cap \partial K_{\alpha'} \right)$.

Let $\mu_\alpha := \mu \circ \varphi_\alpha^{-1}$ be the push-forward measure of $\mu$ onto $K_\alpha$. For any $\alpha \neq \alpha'$, it is direct to define the ‘glued’ measure $\mu_{\alpha, \alpha'}$ on $K_\alpha \cup K_{\alpha'}$ in the natural way:
\[
\forall B \in B\left( K_\alpha \cup K_{\alpha'} \right), \quad \mu_{\alpha, \alpha'}(B) = \mu_{\alpha}(B \cap K_\alpha) + \mu_{\alpha'}(B \cap K_{\alpha'}).
\] (2.4)
By extension we define the measure $\mu_\infty$ on $K_\infty$.

Proposition 2.4 (Decoupling of $L^2$). For all $\alpha \neq \alpha'$ we have $K_\alpha \cap K_{\alpha'} = \emptyset$ and $L^2(K_\alpha^\circ \cup K_{\alpha'}^\circ, \mu_{\alpha, \alpha'}) = L^2(K_\alpha^\circ, \mu_\alpha) \oplus L^2(K_{\alpha'}^\circ, \mu_{\alpha'})$.

Proposition 2.4 allows one to decouple the Laplacian on the glued measure space into the direct sum of the Laplacians on the individual components (see \cite[proposition 13.15.3]{46}):
\[ \Delta_{K_0 \cup K_0'}^b := \Delta_{K_0}^b \oplus \Delta_{K_0'}^b, \]  

from which it follows that

\[ N^b(K_0 \cup K_0', \mu_{K_0} \cup \mu_{K_0'}, \lambda) = N^b(K_0, \mu_{K_0}, \lambda) + N^b(K_0', \mu_{K_0'}, \lambda). \]  

By extension we have that

\[ N^b(K_0, \mu_{K_0}, \lambda) = \sum_{\alpha} N^b(K_0, \mu_{\alpha}, \lambda). \]  

For future purposes we also put a metric \( d : K_\infty \times K_\infty \rightarrow [0, \infty) \), and fix an origin \( 0 \in K_\infty \). In proving our main results, the metric \( d \) does not play a major role. However for practical applications, such as determining the spectral dimension of the Schrödinger operator, one needs to understand the interplay between the metric \( d \) and the measure \( \mu_{K_\infty} \); see remark 2.9 and section 6.

Let the potential \( V \) be a nonnegative, locally bounded measurable function on \( K_\infty \). (In general, \( V \) can be a real-valued, locally bounded measurable function which is bounded below. By adding a suitable constant to \( V \) one retrieves the case of a nonnegative potential.)

**Assumption 2.5.** There exists a self-adjoint Laplacian \( -\Delta \) on \( L^2(K_\infty, \mu_{K_\infty}) \) (equivalently, a local regular Dirichlet form \( \mathcal{E}, \mathcal{F} \) on \( L^2(K_\infty, \mu_{K_\infty}) \)), and that the potential \( V(x) \rightarrow +\infty \) as \( d(0, x) \rightarrow +\infty \).

**Proposition 2.6.** Under assumption 2.5, the Schrödinger operator \( (-\Delta + V) \), regarded as a sum of quadratic forms, is self-adjoint on \( L^2(K_\infty, \mu_{K_\infty}) \), and has pure point spectrum.

**Proof.** This uses the min-max principle as stated in [46, theorem 13.2], and then follows the proof of [46, theorem 13.16].

By virtue of proposition 2.6, we can define the eigenvalue counting function for \((-\Delta + V)\) on \( K_\infty \):

\[ N(K_\infty, \mu_{K_\infty}, V, \lambda) := \# \left( \lambda \left( -\Delta + V \right) \leq \lambda \right). \]  

We are interested in the asymptotics of \( N(K_\infty, \mu_{K_\infty}, V, \lambda) \) as \( \lambda \rightarrow \infty \). In order to state the precise results, we will impose some mild conditions on the potential \( V \).

Given a potential \( V \) on \( K_\infty \), let \( V^\wedge \) (respectively \( V^\vee \)) be the function which is piecewise constant on each cell \( K_\alpha \), and takes value \( \sup_{x \in K_\alpha} V(x) \) (respectively \( \inf_{x \in K_\alpha} V(x) \)) on \( K_\alpha \). We introduce the associated distribution functions

\[ F^\wedge(V, \lambda) := \mu_{K_\infty} \left( \left\{ x \in K_\infty : V^\wedge(x) \leq \lambda \right\} \right), \]

\[ F^\vee(V, \lambda) := \mu_{K_\infty} \left( \left\{ x \in K_\infty : V^\vee(x) \leq \lambda \right\} \right). \]

Note that \( F^\wedge(V, \lambda) \leq F^\vee(V, \lambda) \).

**Assumption 2.7.** There exists a constant \( C > 0 \) such that \( F^\vee(V, 2\lambda) \leq CF^\wedge(V, \lambda) \) for all sufficiently large \( \lambda \).
Note that this assumption implies that both $F^v(V, \cdot)$ and $F^\wedge(V, \cdot)$ have the doubling property: there exist $C^v, C^\wedge > 0$ such that
\[ F^v(V, 2\lambda) \leq C^v F^v(V, \lambda) \quad \text{and} \quad F^\wedge(V, 2\lambda) \leq C^\wedge F^\wedge(V, \lambda) \] (2.11)
for all sufficiently large $\lambda$.

**Assumption 2.8.** The potential $V$ on $K_\infty$ satisfies
\[ \frac{F^v(V, \lambda)}{F^v(V, \lambda)} = 1 + o(1) \quad \text{as} \quad \lambda \to \infty. \] (2.12)

**Remark 2.9.** To understand assumption 2.7 or 2.8, it helps to keep the following example in mind. Let $(K_\infty, \mu_\infty, d)$ be a metric measure space which admits a cellular decomposition into copies of the compact metric measure space $(K, \mu, d)$. Let $\text{diam}_d(K)$ be the diameter of $K$ in the $d$-metric. Further suppose that $\mu_\infty$ is Ahlfors-regular: there exist positive constants $c_1, c_2$, and $\alpha$ such that
\[ c_1 r^\alpha \leq \mu_\infty(B_d(x, r)) \leq c_2 r^\alpha \] (2.13)
for all $x \in K_\infty$ and sufficiently large $r > 0$. As for the potential $V$, assume that there exist $\beta > 1$ and $\gamma \in (0, 1]$ such that
\[ c_3 d(0, x)^\beta \leq V(x) \leq c_4 d(0, x)^\beta, \] (2.14)
\[ \frac{|V(x) - V(y)|}{d(x, y)\gamma} \leq c_5 [\max(d(0, x), d(0, y))]^{\beta-\gamma} \] (2.15)
for all $x, y \in K_\infty$. By a direct calculation one can verify that (2.14) implies
\[ c_6 x^{\beta/\gamma} \leq F^b(V, \lambda) \leq c_7 x^{\beta/\gamma}, \] (2.16)
which satisfies assumption 2.7. Meanwhile, (2.15) implies
\[ V^\wedge(x) - V^\wedge(y) \leq c_8 [\text{diam}_d(K)]^\beta d(0, x)^\beta - \gamma. \] (2.17)
Thus (2.14) and (2.15) together imply assumption 2.8.

Our main results are the following.

**Theorem 2.10 (Existence of spectral dimension).** Under assumptions 2.1, 2.5, and 2.7, we have that
\[ 0 < \lim_{\lambda \to \infty} \frac{N(K_\infty, \mu_\infty, V, \lambda)}{\lambda^{d/2} F(V, \lambda)} \leq \lim_{\lambda \to \infty} \frac{N(K_\infty, \mu_\infty, V, \lambda)}{\lambda^{d/2} F(V, \lambda)} < \infty, \] (2.18)
where $F(V, \lambda) := \mu_\infty(\{x \in K_\infty : V(x) \leq \lambda\})$. In particular, if $F(V, \lambda) = \Theta(\lambda^\beta)$ as $\lambda \to \infty$, then $d_\text{s}(V) = d_\text{i} + 2\beta$ is the effective spectral dimension of the Schrödinger operator $(-\Delta + V)$.

**Theorem 2.11 (Bohr’s formula).** Under assumptions 2.2, 2.5, and 2.8,
\[ \lim_{\lambda \to \infty} \frac{N(K_\infty, \mu_\infty, V, \lambda)}{g(V, \lambda)} = 1, \] (2.19)
where
\[ g(V, \lambda) := \int_{K_\infty} \left[(\lambda - V(x))_+\right]^{d/2} G\left(\frac{1}{2} \log(\lambda - V(x))_+\right) \mu_\infty(dx), \] (2.20)
and \((f)_+ = \max\{f, 0\}\).

In what follows we shall refer to \(g\) as ‘Bohr’s asymptotic function’.

The proof of theorem 2.11, discussed in section 3, utilizes Dirichlet–Neumann bracketing on the eigenvalue counting function and on Bohr’s asymptotic function. This is a relatively standard technique which is explained in the mathematical physics literature; see e.g. [46, section 13]. The novelty of our approach is to restate the sufficient condition on the potential \(V\) in terms of its distribution function, which allows us to extend the classical Bohr’s formula to a wider class of settings, such as on unbounded fractal spaces.

2.2. Laplace transform version

There are also analogs of theorems 2.10 and 2.11 for the Laplace–Stieltjes transform of the eigenvalue counting function
\[ \mathcal{L}(K_\infty, \mu_\infty, V, t) := \text{Tr}_{K_\infty}\{e^{-t(-\Delta + V)}\} = \int_0^\infty e^{-\lambda t} N(K_\infty, \mu_\infty, V, d\lambda). \] (2.21)

When \(V = 0\) this is the trace of the heat semigroup associated with the bare Laplacian \(-\Delta\). More generally, it can be regarded as the trace of the Feynman–Kac semigroup associated to the Markov process driven by \(-\Delta\) subject to killing with rate \(V(x)\) at \(x \in K_\infty\).

The reason for stating the analog versions is because for certain compact metric measure spaces, it is not known whether an explicit Weyl asymptotic formula for the bare Laplacian (assumption 2.2) exists. However it may be the case that an asymptotic formula for the heat kernel trace (in some literature it is also called the partition function)
\[ \mathcal{L}(K, \mu, t) := \text{Tr}\{e^{t\Delta}\} = \int_K p_t(x, x) \mu(dx) \] (2.22)
exists in the \(t \downarrow 0\) limit. Here \(p_t(x, y)\) \((t > 0, x, y \in K)\) is the heat kernel associated to the Markov semigroup \(e^{t\Delta}\) generated by the self-adjoint Laplacian \(-\Delta\) on \(L^2(K, \mu)\). To be more precise, we denote by \(\mathcal{L}^b(K, \mu, t)\) the heat kernel trace of the Laplacian \(-\Delta^b\) on \(L^2(K, \mu)\) with boundary condition \(b \in \{\land, \lor\}\). Then
\[ \mathcal{L}^b(K, \mu, t) = \int_0^\infty e^{-\lambda t} N^b(K, \mu, d\lambda) = \int_K p^b_t(x, x) \mu(dx), \] (2.23)
where \(N^b(K, \mu, \lambda)\) is as in (2.1), and \(p^b_t(x, y)\) is the heat kernel associated with the infinitesimal generator \(-\Delta^b\).

**Assumption 2.12** (Existence of the spectral dimension for the bare Laplacian). There exists a positive constant \(d_s\) such that
\[ 0 < \lim_{t \downarrow 0} t^{d_s/2} \mathcal{L}^b(K, \mu, t) \leq \lim_{t \downarrow 0} t^{d_s/2} \mathcal{L}^b(K, \mu, t) < \infty \] (2.24)
for \(b \in \{\land, \lor\}\).
Theorem 2.13. Under assumptions 2.5, 2.7, and 2.12, we have that
\[ 0 < \lim_{t \to 0} \frac{\mathcal{L}(K_{\infty}, \mu_{\infty}, V, t)}{t^{-d/2} F(V, t)} \leq \lim_{t \to 0} \frac{\mathcal{L}(K_{\infty}, \mu_{\infty}, V, t)}{t^{-d/2} F(V, t)} < \infty, \] (2.25)
where
\[ F(V, t) = \int_{K_{\infty}} e^{-tV(x)} \mu_{\infty}(dx). \] (2.26)
In particular, if \( F(V, \lambda) := \mu_{\infty}(\{ x \in K_{\infty} : V(x) \leq \lambda \}) = \Theta(\lambda^3) \) as \( \lambda \to \infty \), then \( d_e(V) = d_\infty + 2\beta \) is the spectral dimension for the Schrödinger operator \( -\Delta + V \).

Assumption 2.14 (Weak Weyl asymptotics for the bare Laplacian). There exists a positive constant \( d_\infty \) and a continuous function \( H : \mathbb{R}_+ \to \mathbb{R}_+ \), independent of the boundary condition \( b \in \{ \land, \lor, \} \) and with \( 0 < \inf H \leq \sup H < \infty \), such that as \( t \downarrow 0 \),
\[ \mathcal{L}^b(K, \mu, t) = t^{-d/2} \left[ H(t) + \rho^b(t) \right], \] (2.27)
where \( \rho^b(t) \) denotes the remainder term of order \( o(1) \).

Theorem 2.15 (Laplace transform version of Bohr’s formula). Under assumptions 2.5, 2.14, and 2.8, we have that
\[ \lim_{t \to 0} \frac{\mathcal{L}(K_{\infty}, \mu_{\infty}, V, t)}{t^{-d/2} F(V, t)} = 1, \] (2.28)
Note that (2.28) can also be interpreted as the asymptotic factorization of the trace of the Feynman–Kac semigroup:
\[ \lim_{t \to 0} \frac{\text{Tr}_{K_{\infty}} \left\{ e^{-t(-\Delta + V)} \right\}}{\text{Tr}_{K_{\infty}} \left\{ e^{\alpha \Delta} \right\} \cdot \text{Tr}_{K_{\infty}} \left\{ e^{-tV} \right\}} = 1. \] (2.29)

Remark 2.16. We make a few comments concerning the connections between assumption 2.1/2.2 and assumption 2.12/2.14.

(i) Assumption 2.1 is equivalent to assumption 2.12.
(ii) Assumption 2.2 implies assumption 2.14. However, the reverse implication is possibly not true, since the classical technique of Tauberian theorems may not be applicable in this context.
(iii) In order to prove Bohr’s formula (theorem 2.11), we impose in assumption 2.2 that the function \( G \) be a periodic function. This is natural in light of the fractal examples we are interested in. However, to prove the Laplace transform version of Bohr’s formula (theorem 2.15), one does not need to assume the log-periodicity in assumption 2.14. This leads to the question of whether one could relax the periodicity of \( G \) and still be able to prove the original Bohr’s formula in greater generality (we do not address this question in the present work).
2.3. Application of the main results

To illustrate how our main results can be used, we now describe the ‘harmonic oscillator’ problem on the SG which was investigated in [18]. For discussions of more general unbounded potentials on other fractal-like spaces, see section 6.

Example 2.17  Harmonic oscillator on the infinite blow-up of the SG. Let K be the SG. To construct SG, we first set the three vertices \( \{p_1, p_2, p_3\} \) of an equilateral triangle in \( \mathbb{R}^2 \), and then introduce the contraction maps \( \Psi_j : \mathbb{R}^2 \to \mathbb{R}^2 \), \( \Psi_j(x) = \frac{1}{2}(x - p_j) + p_j \), \( j = 1, 2, 3 \). Then SG is the unique fixed point \( K \) under the iterated function system consisting of the \( \Psi_j \): \( K = \bigcup_{j=1}^3 \Psi_j(K) \). Let \( w = w_1w_2\cdots w_m \) be a word of length \( |w| = m \) where each letter \( w_j \in \{1, 2, 3\} \), and define the map \( \Psi_w = \Psi_{w_1} \circ \cdots \circ \Psi_{w_m} \).

We endow SG with the uniform self-similar measure \( \nu \) with \( \nu(\Psi_w K) = 3^{-|w|} \). The theory of Kigami [38] allows us to define the standard Laplacian on \( L^2(SG, \nu) \) with either Dirichlet or Neumann condition on the boundary \( \partial(SG) = \{p_1, p_2, p_3\} \). Moreover, Kigami and Lapidus [31] proved that the eigenvalue counting function for the standard Laplacian satisfies

\[
N^b(\Sigma, \nu, \lambda) = \lambda^{d/2} \left[ G\left(\frac{1}{2} \log \lambda + o(1)\right) \right] (b \in \{\land, \lor\}),
\]

where \( d = 2 \log 3 / \log 5 \), and \( G \) is a càdlàg periodic function with period \( \frac{1}{2} \log 5 \) and contains discontinuities. Thus assumption 2.2 is satisfied.

Next, for each infinite word \( w = w_1w_2\cdots \) which is not eventually constant, define

\[
SG^w_\infty := \bigcup_{m=0}^{\infty} \left( \Psi_w^{-1} \cdots \Psi_{w_m}^{-1}(SG) \right)
\]

to be the infinite blow-up of SG associated with the word \( w \). This is an unbounded fractal space where the neighborhood of any point \( x \in K_\infty \) is homeomorphic to SG, and thus is a fractal analog of a manifold, called a fractafold by Strichartz [54]. Properties of the Laplacian on \( SG^w_\infty \) are discussed in [53]. Here we point out that by construction, \( SG^w_\infty \) admits a cellular decomposition into copies of SG which intersect on the boundary only. Thus the measure \( \nu \) on SG can be readily extended to the measure \( \nu_\infty \) on \( SG^w_\infty \).

In [18] Fan, Khanderi, and Strichartz studied the spectral problem of a harmonic oscillator potential \( V \) on a class of infinite blow-ups of SG. They defined \( V \) to be a solution to \( -\Delta V = -1 \) on \( SG^w_\infty \) which grows unboundedly as \( d(0, x) \to \infty \) and attains a minimum at some vertex \( x_0 \in K_\infty \). (The first condition is a suitable replacement of \( V(x) = \frac{1}{2} |x|^2 \), which is available only in the Euclidean setting.) Note that this implies that \( V(x) \) grows at infinity at rate comparable to a positive power of \( R(x_0, x) \), where \( R(\cdot, \cdot) \) is the effective resistance metric on \( SG^w_\infty \). This verifies assumption 2.7. However we cannot verify assumption 2.8 for general words \( w \). Paper [18] also contains information about spectral dimension, which depends on the blow-ups of SG. Through a mix of computations and numerical simulations, the authors of [18] were able to find properties of the low-lying eigenfunctions, as well as the asymptotic growth rate of the eigenvalue counting function of \( -\Delta + V \) [18, theorem 8–1 and equation (8.18a)]:

\[
c\lambda^d \leq N(\Sigma^w_\infty, \nu_\infty, V, \lambda) \leq C\lambda^d,
\]

Among the open questions posed in [18, problem 8–3 and conjecture 8–4] is finding the asymptotic ‘Weyl ratio’ \( \lambda^{d(V)/2} N(\Sigma^w_\infty, \nu_\infty, V, \lambda) \) of the eigenvalue counting function. Here we can provide an indirect answer. Given that assumptions 2.2, 2.5, and 2.8 are satisfied, Bohr’s formula (theorem 2.11) says that as \( \lambda \to \infty \),
This in some sense answers the Weyl ratio question, in spite of the non-explicit nature of the integral on the right-hand side.

The rest of this paper is organized as follows. In section 3 we describe the tools needed to establish Bohr’s formula in the setting of an unbounded space which admits a cellular decomposition according to the setup in section 2.1. In section 4 we show how to restate the general sufficient condition for Bohr’s formula in terms of distribution functions of \( V^- \) and \( V^+ \), and also give a ‘weak’ version of Bohr’s formula. We can show how the addition of an unbounded potential leads to the absence of gaps in the spectrum of \(-\Delta + V\). This is of independent interest since the spectrum of the bare Laplacian on certain fractals (e.g. the SG) has gaps. In section 5 we establish the Laplace transform version of Bohr’s formula. Finally, in section 6, we discuss applications of our main results to various unbounded potentials on several types of unbounded fractal spaces.

3. The general Bohr’s formula

In this section and the next section, assumptions 2.2 and 2.5 are in force.

3.1. Bohr’s asymptotic functions

Let \(-\Delta^\circ\) (respectively \(-\Delta^\prime\)) be the Laplacian on \(L^2(K_\infty, \mu_\infty)\) with Dirichlet (respectively Neumann) conditions on the gluing boundary \(\bigcup_{i_a} \partial K_{a_i}\). For each potential \(V\), let \(V^-\) (respectively \(V^\prime\)) be the piecewise constant function which takes value \(\sup_{x \in K_a} V(x)\) (respectively \(\inf_{x \in K_a} V(x)\)) on \(K_a\). Thanks to proposition 2.6, one can introduce the eigenvalue counting functions

\[
N(K_\infty, \mu_\infty, V, \lambda) := \# \{ \lambda_i (-\Delta + V) \leq \lambda \},
\]

\[
N^\circ(K_\infty, \mu_\infty, V, \lambda) := \# \{ \lambda_i (-\Delta^\circ + V^-) \leq \lambda \},
\]

\[
N^\prime(K_\infty, \mu_\infty, V, \lambda) := \# \{ \lambda_i (-\Delta^\prime + V^\prime) \leq \lambda \}.
\]

Note that since \((-\Delta^\prime + V^\prime) \leq (-\Delta + V) \leq (-\Delta^\circ + V^-)\) in the sense of quadratic forms,

\[
N^\circ(K_\infty, \mu_\infty, V, \lambda) \leq N(K_\infty, \mu_\infty, V, \lambda) \leq N^\prime(K_\infty, \mu_\infty, V, \lambda).
\]

We shall show that under some mild additional conditions on \(V\), \(N(K_\infty, \mu_\infty, V, \lambda)\) is asymptotically comparable to the ‘Bohr’s asymptotic function’

\[
g(V, \lambda) := \int_{K_{\infty}} \left[ (\lambda - V(x))_+ \right]^{d/2} G \left( \frac{1}{2} \log(\lambda - V(x)_+) \right) d\mu_\infty(x),
\]

where \((f)_+ := \max\{f, 0\}\), and \(G\) is as appeared in assumption 2.2. In order to estimate this rate of convergence, we introduce the functions
\[ g^b(V, \lambda) := \int_{K_{\infty}} \left[ (\lambda - V^b(x))_+ \right]^{d/2} G \left( \frac{1}{2} \log (\lambda - V^b(x)_+) \right) d\mu_{\infty}(x) \]  

(3.6)

and

\[ \mathcal{R}^b(V, \lambda) := \int_{K_{\infty}} \left[ (\lambda - V^b(x))_+ \right]^{d/2} R^b \left( (\lambda - V^b(x)_+) \right) d\mu_{\infty}(x) \]  

(3.7)

for \( b \in \{\land, \lor\} \), where \( \mathcal{R}^b \) is the remainder term which appeared in assumption 2.2. Observe that since \( V^b(x) \) is constant on cells, the right-hand side expressions in (3.6) and (3.7) are really discrete sums:

\[ g^b(V, \lambda) = \sum_{\{\alpha: V^b\mid_{K_\alpha} < \lambda\}} \left[ \lambda - V^b\mid_{K_\alpha} \right]^{d/2} G \left( \frac{1}{2} \log (\lambda - V^b\mid_{K_\alpha}) \right) . \]  

(3.8)

\[ \mathcal{R}^b(V, \lambda) = \sum_{\{\alpha: V^b\mid_{K_\alpha} < \lambda\}} \left[ \lambda - V^b\mid_{K_\alpha} \right]^{d/2} R^b \left( \lambda - V^b\mid_{K_\alpha} \right) . \]  

(3.9)

Moreover, by proposition 2.4, \( K_{\infty} \) decouples into the various \( K_\alpha \) according to the Dirichlet or Neumann boundary condition, so

\[ N^b(K_{\infty}, \mu_{\infty}, V, \lambda) = \sum_{\{\alpha: V^b\mid_{K_\alpha} < \lambda\}} N^b(K_\alpha, \mu_\alpha, \lambda - V^b\mid_{K_\alpha}) . \]  

(3.10)

Pulling (2.3), (3.8), (3.9), and (3.10) together we obtain

\[ N^b(K_{\infty}, \mu_{\infty}, V, \lambda) = g^b(V, \lambda) + \mathcal{R}^b(V, \lambda) . \]  

(3.11)

### 3.2. Monotonicity of Bohr’s asymptotic functions

A key monotonicity result we need is

**Proposition 3.1.** Fix a potential \( V \). Then each of the functions \( \lambda \mapsto g(V, \lambda) \), \( \lambda \mapsto g^\land(V, \lambda) \), and \( \lambda \mapsto g^\lor(V, \lambda) \) is monotone nondecreasing for all \( \lambda > 0 \). Moreover, \( g^\land(V, \lambda) \leq g(V, \lambda) \leq g^\lor(V, \lambda) \).

This follows from the monotonicity of the integrand of the \( g \) function.

**Proposition 3.2.** The function \( W(\lambda) = \lambda^{d/2} G \left( \frac{1}{2} \log \lambda \right) \) is nondecreasing.

**Remark 3.3.** This proposition implies, in particular, that \( G \) has a càdlàg version. Although the result is very simple, we could not find it in the literature.

**Proof of proposition 3.2.** If \( W \) was not nondecreasing, then there would existed \( \lambda_2 > \lambda_1 > 0 \) such that \( W(\lambda_2) = W(\lambda_1) = -\delta < 0 \), which contradicts (2.3) and the monotonicity of \( N^b(K, \mu, \lambda) \). □
Proof of proposition 3.1. Fix a potential $V$. For each $\lambda > 0$ and $x \in K_\infty$, put

$$W(\lambda, V, x) = \left(\lambda - V(x)\right)^{d/2} \left(\frac{1}{2} \log\left(\lambda - V(x)\right)\right)$$

and

$$W^b(\lambda, V, x) = \left(\lambda - V^b(x)\right)^{d/2} \left(\frac{1}{2} \log\left(\lambda - V^b(x)\right)\right).$$

Observe that $W(\lambda, V, x) = W((\lambda - V(x))_+) = W^b(\lambda, V, x) = W((\lambda - V^b(x))_+).$ Using proposition 3.2 we deduce the following two consequences. First, $\lambda \mapsto W(\lambda, V, x)$ is nonnegative and monotone nondecreasing for each $x$. And since $g(V, \lambda)$ is the weighted integral of $W(\lambda, V, x)$ over $x$, it follows that $\lambda \mapsto g(V, \lambda)$ is also monotone nondecreasing.

The monotonicity of $\lambda \mapsto g^b(V, \lambda)$ is proved in exactly the same way. Second, the monotonicity of $W(\lambda, V, x)$ implies that $W^b(\lambda, V, x) \leq W(\lambda, V, x) \leq W^\vee(\lambda, V, x)$ for each $x$, and upon integration over $x$ we get $g^b(V, \lambda) \leq g(V, \lambda) \leq g^\vee(V, \lambda)$.

3.3. Bohr's asymptotics via Dirichlet–Neumann bracketing

We have all the necessary pieces to state the error of approximating $N(K_\infty, \mu_\infty, V, \lambda)$ by $g(V, \lambda)$.

Theorem 3.4 (Error estimate in Bohr's approximation). Under assumptions 2.2 and 2.5, we have

$$\left|\frac{N(K_\infty, \mu_\infty, V, \lambda)}{g(V, \lambda)} - 1\right| \leq \max_{b \in \{\land, \lor\}} \left|\frac{\tilde{g}^b(V, \lambda) - 1 + \mathcal{R}^b(V, \lambda)}{\tilde{g}^b(V, \lambda)}\right|,$$

where $\tilde{b} = \land$ (respectively $\tilde{\bar{b}} = \lor$) if $b = \lor$ (respectively if $b = \land$).

Proof. From (3.4) we have

$$N^\land(K_\infty, \mu_\infty, V, \lambda) \leq N(K_\infty, \mu_\infty, V, \lambda) \leq N^\lor(K_\infty, \mu_\infty, V, \lambda).$$

Meanwhile by proposition 3.1,

$$g^\land(V, \lambda) \leq g(V, \lambda) \leq g^\lor(V, \lambda).$$

Therefore

$$\frac{N^\land(K_\infty, \mu_\infty, V, \lambda)}{g^\land(V, \lambda)} \leq \frac{N(K_\infty, \mu_\infty, V, \lambda)}{g(V, \lambda)} \leq \frac{N^\lor(K_\infty, \mu_\infty, V, \lambda)}{g^\lor(V, \lambda)}.$$

Subtract 1 from every term in the inequality (3.17), and then use (3.11) to replace $N^b(K_\infty, \mu_\infty, V, \lambda)$ with $\tilde{g}^b(V, \lambda) + \mathcal{R}^b(V, \lambda)$. Finally, we can estimate the absolute value of the middle term of the inequality by the maximum of the absolute value on either side of the inequality.
Having established the main error estimate, theorem 3.4, we can now give an abstract condition on $V$ for which Bohr’s formula holds.

**Assumption 3.5.** The potential $V$ on $K_\infty$ satisfies
\[
\frac{g^+(V, \lambda)}{g^-(V, \lambda)} = 1 + o(1) \text{ as } \lambda \to \infty.
\] (3.18)

**Theorem 3.6 (Strong Bohr’s formula).** Under assumptions 2.2, 2.5, and 3.5, we have
\[
\lim_{\lambda \to \infty} \frac{N(K_\infty, \mu_\infty, V, \lambda)}{g(V, \lambda)} = 1.
\] (3.19)

**Proof of theorem 3.6.** Observe that assumptions 2.2 and 3.5 together imply that the error term stated in theorem 3.4 is $o(1)$.

---

4. Connection between Bohr’s formula and the distribution function of the potential

Assumption 3.5 can be too abstract for applications dealing with fractal spaces. We now explain how this assumption can be restated in terms of distribution functions of $V$:

- $F(V, \lambda) := \mu_\infty\left( \{ x \in K_\infty : V(x) \leq \lambda \} \right)$ and
- $F^b(V, \lambda) := \mu_\infty\left( \{ x \in K_\infty : V^b(x) \leq \lambda \} \right)$.

(4.1)

**Lemma 4.1.** We have that
\[
g(V, \lambda) = \int_0^W F(V, \lambda - W^{-1}(t))dt \quad \text{and}
\]
\[
\times \quad g^b(V, \lambda) = \int_0^W F^b(V, \lambda - W^{-1}(t))dt,
\] (4.2)

where
\[
W^{-1}(t) = \inf \{ \lambda \geq 0 : W(\lambda) \geq t \}
\] (4.3)
is the generalized inverse of $W(\lambda) = \lambda^{d/2}G_\text{c}^{-1}(\log \lambda)$.

**Proof.** We start with a fundamental identity in measure theory. For any nonnegative function $f$ on a $\sigma$-finite measure space $(X, m)$, Fubini’s theorem tells us that
\[
\int_X f(x) m(dx) = \int_0^\infty m(\{ x \in X : f(x) \geq t \}) dt.
\] (4.4)

Applying this identity to $g(V, \lambda)$ we find
\[
g(V, \lambda) = \int_{K_\infty} W\left( (\lambda - V(x))^+ \right) d\mu_\infty(x)
\]
\[
= \int_0^\infty \mu_\infty\left( \{ x \in K_\infty : W\left( (\lambda - V(x))^+ \right) \geq t \} \right) dt.
\] (4.5)
Since $W$ is monotone nondecreasing (proposition 3.2), it has a well-defined generalized inverse $W^{-1}$, which satisfies
\[ W(\lambda) \geq t \iff \lambda \geq W^{-1}(t). \] (4.6)

So the right-hand term in (4.5) can be further rewritten as
\[ \int_0^{\infty} \mu(x) \left\{ x \in \mathcal{K}_{\infty} : (\lambda - V(x))_+ \geq W^{-1}(t) \right\} dt. \] (4.7)

Now by assumption $V$ is a nonnegative potential, so $W^{-1}(t) \leq (\lambda - V(x))_+ \leq \lambda$, or equivalently, $t \leq W(\lambda)$. This places an upper bound on the integral, and we get
\[ g(V, \lambda) = \int_{0}^{W(\lambda)} \mu(x) \left\{ x \in \mathcal{K}_{\infty} : V(x) \leq \lambda - W^{-1}(t) \right\} dt \]
\[ = \int_{0}^{W(\lambda)} F(V, \lambda - W^{-1}(t)) dt. \] (4.8)

The proof for $g^b(V, \lambda)$ is identical. \qed

Observe that for $\lambda \leq \lambda'$,
\[ g^\vee(V, \lambda) - g^\wedge(V, \lambda') = \int_{0}^{W(\lambda)} \left[F^\vee(V, \lambda - W^{-1}(t)) - F^\wedge(V, \lambda' - W^{-1}(t))\right] dt \]
\[ - \int_{W(\lambda)}^{W(\lambda')} F^\wedge(V, \lambda' - W^{-1}(t)) dt, \] (4.9)

and
\[ g^\vee(V, \lambda') - g^\wedge(V, \lambda) = \int_{0}^{W(\lambda)} \left[F^\vee(V, \lambda' - W^{-1}(t)) - F^\wedge(V, \lambda - W^{-1}(t))\right] dt \]
\[ + \int_{W(\lambda)}^{W(\lambda')} F^\vee(V, \lambda' - W^{-1}(t)) dt. \] (4.10)

These identities suggest that if the difference of the distribution functions $F^\vee(V, \lambda) - F^\wedge(V, \lambda)$ can be controlled, then one can control the difference $g^\vee(V, \lambda) - g^\wedge(V, \lambda)$. Indeed we have

**Proposition 4.2.** assumption 2.8 implies assumption 3.5. Therefore, the strong Bohr’s formula (theorem 3.6) holds under assumptions 2.2, 2.5, and 2.8.

**Proof.** Let $h(V, \lambda) = \frac{F^\vee(V, \lambda)}{F^\wedge(V, \lambda)} - 1 \geq 0$. Then
\[ 0 \leq g^\vee(V, \lambda) - g^\wedge(V, \lambda) \]
\[ = \int_{0}^{W(\lambda)} \left[1 + h(V, \lambda - W^{-1}(t)) - 1\right] F^\wedge(V, \lambda - W^{-1}(t)) dt \] (4.11)

\[ = \int_{0}^{W(\lambda)} h(V, \lambda - W^{-1}(t)) F^\wedge(V, \lambda - W^{-1}(t)) dt \] (4.12)
\[
\leq \left( \sup_{0 \leq r \leq W(\lambda)} h(V, \lambda - W^{-1}(t)) \right) \int_0^{W(\lambda)} F^\wedge(V, \lambda - W^{-1}(t)) dt
\] (4.13)

\[
= \left( \sup_{0 \leq r \leq \lambda} h(V, s) \right) g^\wedge(V, \lambda).
\] (4.14)

Assumption 2.8 implies that \( h(V, s) = o(1) \) as \( \lambda \to \infty \), so we obtain assumption 3.5.

\section{4.1. A weak version of Bohr’s formula}

Motivated by [18, 42, 43], we also give a weak version of Bohr’s formula as follows.

\textbf{Theorem 4.3 (Weak Bohr’s formula).} Let \( \lambda^* > \lambda \) with \( \lambda^* - \lambda = o(\lambda) \) and

\[
\frac{F^\lambda(V, \lambda)}{F^\lambda(V, \lambda^*)} = 1 + o(1)
\] and
\[
\frac{F^\wedge(V, \lambda)}{F^\wedge(V, \lambda^*)} = 1 + o(1).
\] (4.15)

Then, with assumptions 2.2 and 2.5, we have

\[
\lim_{\lambda \to \infty} \frac{N(K_{\lambda}, \mu_{\lambda}, V, \lambda)}{g(V, \lambda^*)} = 1.
\] (4.16)

The statement of theorem 4.3 is reminiscent of the situation when one compares two non-decreasing distribution jump functions with closely spaced jumps. When the jumps asymptotically coincide, then the difference of corresponding measures tends to zero in the sense of weak convergence.

\textbf{Proof.} By mimicking the proof of theorem 3.4 we get

\[
\left| \frac{N(K_{\lambda}, \mu_{\lambda}, V, \lambda)}{g(V, \lambda^*)} - 1 \right| \leq \max_{b \in \{\land, \lor\}} \left| \frac{g^\lambda(V, \lambda)}{g^\lambda(V, \lambda^*)} - 1 + \frac{\mathcal{R}^\lambda(V, \lambda)}{g^\lambda(V, \lambda^*)} \right|.
\] (4.17)

Since \( \lambda^* - \lambda = o(\lambda) \) as \( \lambda \to \infty \), the ratio \( \mathcal{R}^\lambda(V, \lambda)/g^\lambda(V, \lambda^*) \) can be made to be \( o(1) \). So the key estimate is to show that \( g^\lambda(V, \lambda)/g^\lambda(V, \lambda^*) = 1 + o(1) \) for both \( b = \land \) and \( b = \lor \). (This is to contrast with the case \( V = \lambda \) as shown in proposition 4.2, where a one-sided bound suffices because \( g^\wedge(V, \lambda) = g^\wedge(V, \lambda) \geq 0 \).

From (4.9) we find

\[
\left| g^\wedge(V, \lambda) - g^\wedge(V, \lambda^*) \right| \leq W(\lambda) \left( \sup_{0 \leq s \leq \lambda} \left[ F^\wedge(V, s) - F^\wedge(V, s + \lambda^* - \lambda) \right] \right).
\] (4.18)

\[
+ \left( W(\lambda^*) - W(\lambda) \right) \left( \sup_{0 \leq s \leq \lambda^* - \lambda} F^\wedge(V, s) \right).
\] (4.19)

According to the first condition in (4.15), \( \sup_{0 \leq s \leq \lambda} \left[ F^\wedge(V, s) - F^\wedge(V, s + \lambda^* - \lambda) \right] = o(F^\wedge(V, \lambda)) \) and \( \sup_{0 \leq s \leq \lambda^* - \lambda} F^\wedge(V, s) = o(F^\wedge(V, \lambda^*)) \). This implies that the absolute value
on the RHS of (3.14) is $\alpha(1)$ for $b = \wedge$. Similarly, the second condition in (4.15) implies that the absolute value on the RHS of (3.14) is $\alpha(1)$ for $b = \vee$ also.

5. Laplace transform (heat kernel trace) version of Bohr’s formula

In this section we impose assumption 2.5 and either one of assumptions 2.12 and 2.14, and prove theorems 2.13 and 2.15. Let us introduce the traces

\[ \mathcal{L}(K_x, \mu_x, V, t) := \text{Tr}_{K_x} \left\{ e^{-t(-\Delta + V)} \right\}, \]

\[ \mathcal{L}^\wedge(K_x, \mu_x, V, t) := \text{Tr}_{K_x} \left\{ e^{-t(-\Delta + V)} \right\}, \]

\[ \mathcal{L}^\vee(K_x, \mu_x, V, t) := \text{Tr}_{K_x} \left\{ e^{-t(-\Delta + V)} \right\}. \]

Observe that $\mathcal{L}^\wedge(K_x, \mu_x, V, t) \leq \mathcal{L}(K_x, \mu_x, V, t) \leq \mathcal{L}^\wedge(K_x, \mu_x, V, t)$. Since $L^2(K_x, \mu_x) = \mathcal{G}_0 L^2(K_x, \mu_x)$, it follows that

\[ \mathcal{L}^b(K_x, \mu_x, V, t) = \sum_{\alpha} \mathcal{L}^b(K_{\alpha}, \mu_{\alpha}, V, t), \quad b \in \{ \wedge, \vee \}, \]

where

\[ \mathcal{L}^b(K_{\alpha}, \mu_{\alpha}, V, t) = \text{Tr}_{K_{\alpha}} \left\{ e^{-t(-\Delta + V b)} \right\} = \mathcal{L}^b(K_{\alpha}, \mu_{\alpha}, t) \cdot \exp \left(-iVb \big|_{K_{\alpha}} \right). \]

Let

\[ \mathcal{F}(V, t) := \int_{K_x} e^{-iV(x)} \mu_x(dx). \]

Similarly define

\[ \mathcal{F}^b(V, t) := \int_{K_x} e^{-iVb(x)} \mu_x(dx) \]

for $b \in \{ \wedge, \vee \}$. Observe that $\mathcal{F}^\wedge(V, t) \leq \mathcal{F}(V, t) \leq \mathcal{F}^\vee(V, t)$, and that assumption 2.5 ensures that $\mathcal{F}(V, t)$ and $\mathcal{F}^b(V, t)$ are finite for $t > 0$.

**Proof of theorem 2.13.** Let us first note that

\[ \frac{\mathcal{L}^\wedge(K_x, \mu_x, V, t)}{t^{-d/2} \mathcal{F}^\wedge(V, t)} \leq \frac{\mathcal{L}(K_x, \mu_x, V, t)}{t^{-d/2} \mathcal{F}(V, t)} \leq \frac{\mathcal{L}^\vee(K_x, \mu_x, V, t)}{t^{-d/2} \mathcal{F}^\wedge(V, t)}. \]

By (5.4),

\[ \mathcal{L}^b(K_x, \mu_x, V, t) = \sum_{\alpha} \mathcal{L}^b(K_{\alpha}, \mu_{\alpha}, V, t), \]

\[ = \sum_{\alpha} \mathcal{L}^b(K_{\alpha}, \mu_{\alpha}, t) \cdot \exp \left(-iVb \big|_{K_{\alpha}} \right), \]

\[ = \sum_{\alpha} \mathcal{L}^b(K_{\alpha}, \mu_{\alpha}, t) \cdot \int_{K_{\alpha}} e^{-iVb(x)} \mu_{\alpha}(dx). \]

Under assumption 2.12, there exist positive constants $C_1$ and $C_2$ such that for all sufficiently small $t$,
Meanwhile, by Fubini’s theorem and by the nonnegativity of $V$, we have

$$\mathcal{F}^b(V, t) = \int_0^\infty \mu_\infty \left( \left\{ x \in K_\infty : e^{-b V(x)} \geq s \right\} \right) ds$$

$$= \int_{-\infty}^\infty \mu_\infty \left( \left\{ x \in K_\infty : e^{-b V(x)} \geq e^{-\lambda} \right\} \right) e^{-t \lambda} d\lambda$$

$$= \int_0^\infty \mu_\infty \left( \left\{ x \in K_\infty : V^b(x) \leq \lambda \right\} \right) e^{-t \lambda} d\lambda$$

$$= \int_0^\infty F^b(V, \lambda) e^{-t \lambda} d\lambda. \quad \text{(5.17)}$$

Hence under assumption 2.7, there exists $\lambda_0 > 0$ such that

$$\mathcal{F}^\vee(V, t) = \int_0^\infty F^\vee(V, \lambda) e^{-t \lambda} d\lambda$$

$$= \int_0^{\lambda_0} F^\vee(V, \lambda) e^{-t \lambda} d\lambda + \int_{\lambda_0}^{\infty} F^\vee(V, \lambda) e^{-t \lambda} d\lambda$$

$$\leq F^\vee(V, \lambda_0) \int_0^{\lambda_0} e^{-t \lambda} d\lambda + C \int_{\lambda_0}^{\infty} F^\vee(V, \lambda) \cdot 2e^{-2t \lambda} d\lambda$$

$$= F^\vee(V, \lambda_0) (1 - e^{-t \lambda_0}) + C \int_{\lambda_0/2}^{\infty} F^\vee(V, \lambda_0) \cdot 2e^{-2t \lambda} d\lambda$$

$$\leq F^\vee(V, \lambda_0) (1 - e^{-t \lambda_0}) + C \mathcal{F}^\wedge(V, 2t). \quad \text{(5.22)}$$

Therefore

$$\mathcal{F}^\vee(V, t) \leq \frac{\mathcal{F}^\vee(V, t)}{\mathcal{F}^\wedge(V, t)} \leq C + F^\vee(V, \lambda_0) \frac{1 - e^{-t \lambda_0}}{\mathcal{F}^\wedge(V, 2t)}. \quad \text{(5.23)}$$

Since $\lim_{t \to 0} (1 - e^{-t \lambda_0}) = 0$ and $t \mapsto \mathcal{F}^\wedge(V, 2t)$ is monotone decreasing, it follows that

$$\lim_{t \to 0} \frac{\mathcal{F}^\vee(V, t)}{\mathcal{F}^\wedge(V, t)} \leq C + F^\vee(V, \lambda_0) \lim_{t \to 0} \frac{1 - e^{-t \lambda_0}}{\mathcal{F}^\wedge(V, 2t)} = C. \quad \text{(5.24)}$$

Putting everything together we find

$$\lim_{t \to 0} \frac{\mathcal{L}(K_\infty, \mu_\infty, V, t)}{t^{-d/2} \mathcal{F}(V, t)} \leq \left( \lim_{t \to 0} \frac{t^{d/2} \mathcal{L}^\vee(K_\infty, \mu_\infty, V, t)}{\mathcal{F}^\vee(V, t)} \right) \left( \lim_{t \to 0} \frac{\mathcal{F}^\vee(V, t)}{\mathcal{F}^\wedge(V, t)} \right) \left( \lim_{t \to 0} \frac{\mathcal{F}^\wedge(V, t)}{\mathcal{F}^\vee(V, t)} \right). \quad \text{(5.25)}$$

$$\lim_{t \to 0} \frac{\mathcal{L}(K_\infty, \mu_\infty, V, t)}{t^{-d/2} \mathcal{F}(V, t)} \geq \left( \lim_{t \to 0} \frac{t^{d/2} \mathcal{L}^\wedge(K_\infty, \mu_\infty, V, t)}{\mathcal{F}^\wedge(V, t)} \right) \left( \lim_{t \to 0} \frac{\mathcal{F}^\vee(V, t)}{\mathcal{F}^\wedge(V, t)} \right) \left( \lim_{t \to 0} \frac{\mathcal{F}^\wedge(V, t)}{\mathcal{F}^\vee(V, t)} \right). \quad \text{(5.26)}$$
Thus

\[ C_2 C^{-1} \leq \lim_{t \to 0} \frac{\mathcal{L}(K_{\infty}, \mu_{\infty}, V, t)}{t^{-d/2} \mathcal{F}(V, t)} \leq \lim_{t \to 0} \frac{\mathcal{L}(K_{\infty}, \mu_{\infty}, V, t)}{t^{-d/2} \mathcal{F}(V, t)} \leq C_1. \] (5.27)

Finally, regarding the spectral dimension of \(-\Delta + V\), we note that \(F(V, \lambda) = \Theta(\lambda^2)_{\lambda \to \infty}\) is equivalent to \(\mathcal{F}(V, t) = \Theta(t^{-\beta})_{t \to 0}\), an easy consequence of Laplace transform. Thus according to (5.27), \(\mathcal{L}(K_{\infty}, \mu_{\infty}, V, t) \asymp t^{-(d+2\beta)/2}\) as \(t \downarrow 0\).

**Proof of theorem 2.15.** Combining (5.11) with assumption 2.14 we obtain

\[ \mathcal{L}^b(K_{\infty}, \mu_{\infty}, V, t) = t^{-d/2} \left[ H(t) + \rho^b(t) \right] \mathcal{F}^b(V, t) \] (5.28)

which, together with (5.8) and after some manipulation, implies

\[
\left| \frac{\mathcal{L}(K_{\infty}, \mu_{\infty}, V, t)}{t^{-d/2} H(t) \mathcal{F}(V, t)} - 1 \right| \leq \max_{b \in [\lambda, \sqrt{\lambda}]} \left| \left(1 + \frac{\rho^b(t)}{H(t)} \right) \frac{\mathcal{F}^b(V, t)}{\mathcal{F}^b(V, t)} - 1 \right|.
\] (5.29)

Next, by assumption 2.8 and (5.18),

\[ \mathcal{F}^\lambda(V, t) = \mathcal{F}^\lambda(V, t) + \int_0^\infty \left( F^\lambda(V, \lambda) - F^\lambda(V, \lambda) \right) t e^{-t\lambda} d\lambda \] (5.30)

\[ = \mathcal{F}^\lambda(V, t) + o(\mathcal{F}^\lambda(V, t))_{t \to 0} \] (5.31)

because for any \(\delta > 0\) there is \(\lambda_\delta > 0\) such that \(F^\lambda(V, \lambda) - F^\lambda(V, \lambda) < \delta F^\lambda(V, \lambda)\) when \(\lambda > \lambda_\delta\). Thus \(\frac{\mathcal{F}^\lambda(V, t)}{\mathcal{F}^\lambda(V, t)} = 1 + o(1)\) as \(t \downarrow 0\). Hence (5.29) implies (2.28).

**6. Examples**

In this section we provide several instances on both classical and fractal settings whereby the existence of the spectral dimension of \(-\Delta + V\) can be proved, and moreover, Bohr’s formula holds.

**6.1. Euclidean spaces**

One would be remiss not to mention the most classical setting, which is the Schrödinger operator \(-\Delta + V\) on \(\mathbb{R}^d\), where \(\Delta = \sum_{i=1}^d (\partial^2 / \partial x_i^2)\) and \(V\) is an unbounded potential. See e.g. [46, section 13.15]. The key idea is to partition \(\mathbb{R}^d\) (the unbounded space \(K_{\infty}\)) into cubes of side 1 (the cells \(K_n\)). Then by applying the machinery outlined in the previous section, one arrives at the following well-known result: if \(V(x) = \Theta(|x|^2)\) as \(|x| \to \infty\), then Bohr’s formula holds, and the spectral dimension of this Schrödinger operator is \(d(1 + 2/\beta)\).

In dimension 1 Bohr’s formula can be established for logarithmically diverging potentials. The proof method involves solving a Sturm–Liouville ODE, which appears rather particular to one-dimensional settings, and may be difficult to generalize to higher dimensions. We refer the reader to [26, 35] for more details.
6.2. Infinite fractafolds based on nested fractals

Nested fractals are introduced first by Lindstrøm [40]. The typical examples to keep in mind are the SG(n), where n denotes the length scale of the subdivision. There are also higher-dimensional analogs of SG.

On nested fractals, and more generally post-critically finite (pcf) fractals, one can define a notion of the Laplacian (or a Brownian motion). See e.g. [3, section 2–4], [38, ch 2–3], [56, ch 1–2] for the relevant definitions and results. We will need just one result on the spectral asymptotics of the Laplacian on pcf fractals with regular harmonic structure.

Proposition 6.1 ([31, theorem 2.4], [38, theorem 4.1.5]). Let K be a pcf fractal, and μ be a self-similar measure on K with weight (μ_i)_{i=1}^{∞}. Assume that μ_i < 1 for all i ∈ {1, 2, ..., N}.

Let d_i be the unique number d which satisfies \( \sum_{i=1}^{N} \gamma_i^d = 1 \), where \( \gamma_i = \sqrt{\mu_i} \). Let \( N^b(K, \mu, \lambda) \) (respectively \( N^s(K, \mu, \lambda) \)) be the eigenvalue counting function for the Laplacian on \( L^2(K, \mu) \) with Dirichlet (respectively Neumann) boundary condition. Then for \( b \in \{ \land, \lor \} \),

\[
0 < \lim_{\lambda \to \infty} \lambda^{-d/2}N^b(K, \mu, \lambda) \leq \lim_{\lambda \to \infty} \lambda^{-d/2}N^b(K, \mu, \lambda) < \infty. \tag{6.1}
\]

Moreover:

(a) Non-lattice case: if \( \sum_{i=1}^{N} \mathbb{Z}\log \gamma_i \) is a dense subgroup of \( \mathbb{R} \), then the limit exists, and is independent of the boundary conditions.

(b) Lattice case: if \( \sum_{i=1}^{N} \mathbb{Z}\log \gamma_i \) is a discrete subgroup of \( \mathbb{R} \), let \( T > 0 \) be its generator. Then as \( \lambda \to \infty \),

\[
N^b(K, \mu, \lambda) = \left[ G\left( \log \frac{\lambda}{2} \right) + o(1) \right] \lambda^{d/2}, \tag{6.2}
\]

where G is a right-continuous, T-periodic function with \( 0 < \inf G \leq \sup G < \infty \), and is independent of the boundary conditions.

We remark that the proof of proposition 6.1 relies upon Feller’s renewal theorem [19].

Our goal is to state Bohr’s formula for the Schrödinger operator on a class of unbounded fractal spaces. As mentioned in section 1, one such candidate is a fractafold based on a nested fractal. We shall consider two types:

(i) The infinite blow-ups of a nested fractal in \( \mathbb{R}^d, d \geq 2 \), (see figure 1).

(ii) Infinite periodic fractafolds \( K_\infty \) based on the planar SG \( K = SG(n) \), equipped with a metric R. (In practice, R is taken to be the resistance metric, but the results to follow do not depend explicitly on the specifics of R.) The examples we will consider can be constructed by first defining an infinite ‘cell graph’ \( \Gamma \), and then replacing each vertex of \( \Gamma \) by a copy of \( K \), and gluing the \( K_n \) in a consistent way. With this construction the metric R on K extends to a metric R on \( K_\infty \) in the obvious way. For instance, one can construct the ladder periodic fractafold (figure 2) and the hexagonal periodic fractafold (figure 3).

To establish Bohr’s formula, we will need information about the measure growth of balls in \( K_\infty \). For the infinite blow-ups of a nested fractal, it is direct to verify that for all \( x \in K_\infty \) and \( r > 0 \),
where \( d_{f,R} \) is the Hausdorff dimension of \( K \) with respect to the metric \( R \) on \( K \).

For the periodic fractafolds a slightly different analysis is needed. Let \( d_{\Gamma} \) be the graph metric of the cell graph \( \Gamma \), and \( B_{\Gamma}(z, r) = \{ y \in \Gamma : d_{\Gamma}(z, y) \leq r \} \) be the ball of radius \( r \) centered at \( z \) in \( \Gamma \). Since \( K_\infty \) is constructed by replacing each vertex of \( \Gamma \) by a copy of \( K \), we can estimate the volume growth of balls in \( K_\infty \) using the cardinality of balls in \( \Gamma \).

**Proposition 6.2.** Let \( D(K) \) := \( \text{diam}_R(K) \). For all \( x \in K_\infty \) and all \( r > 2D(K) \),

\[
\mu_\infty \left( B_{\Gamma}(\psi(x), r - 2D(K)) \right) \leq \mu_\infty \left( B_{\Gamma}(x, r) \right) \leq \mu_\infty \left( B_{\Gamma}(\psi(x), r + 2D(K)) \right),
\]

where \( \psi(x) \) is the vertex in \( \Gamma \) which is replaced by the cell \( K_\alpha \supset x \) in the periodic fractafold construction.

**Proof.** Let \( \eta(r) = r/D(K) > 2 \). Then \( B_{\Gamma}(x, r) = B_{\Gamma}(x, \eta(r)D(K)) \) and

\[
B_{\Gamma}(y, ([\eta(r)] - 1)D(K)) \subseteq B_{\Gamma}(x, \eta(r)D(K)) \subseteq B_{\Gamma}(y, ([\eta(r)] + 1)D(K))
\]

for any \( y \) which lies in the same cell \( K_\alpha \) as \( x \). Here \([\alpha] \) (respectively \([\alpha] \)) denotes the largest integer less than or equal to \( \alpha \) (respectively the smallest integer greater than or equal to \( \alpha \)). It is then direct to show that there exist \( y \) such that \( B_{\Gamma}(y, ([\eta(r)] + 1)D(K)) \) is covered by the union of all cells \( K_\alpha \) which are at most distance \(([\eta(r)] + 1) \) from \( y \) in the \( \Gamma \) metric. Since each cell has \( \mu \)-measure 1, the \( \mu \)-measure of the cover is equal to the cardinality of \( B_{\Gamma}(\psi(x), [\eta(r)] + 1) \). The upper bound in (6.4) follows by overestimating \([\eta(r)] + 1 \) by \( \eta(r) + 2 \). The proof of the lower bound is similar.

We can now state the main result of this subsection.

**Proposition 6.3.** On the infinite blow-up of a nested fractal (respectively the ladder periodic fractafold based on \( \text{SG}(n) \), the hexagonal periodic fractafold based on \( \text{SG}(n) \)), Bohr’s formula holds for potential of the form \( V(x) \sim R(0, x)^{\beta} \) for any \( \beta > 0 \). In particular, the spectral dimension of \( -\Delta + V \) is \( d_s(V) = d_s + 2(d_h/\beta) \), where \( d_h \) equals the Hausdorff dimension of the nested fractal with respect to the metric \( R \) (respectively 1, 2).

**Proof.** Since each \( K_\alpha \) which makes up the cellular decomposition of \( K_\infty \) is isometric to the same nested fractal \( K \), by proposition 6.1 we have that assumption 2.2 holds.

Because the cells \( K_\alpha \) intersect at boundary points in a natural way, the Dirichlet form \( \mathcal{E}(\mathcal{F}) \) corresponding to the Laplacian \(-\Delta\) on \( L^2(K_\infty, \mu_\infty) \) can be built up as a sum of the constituent Dirichlet forms on \( L^2(K_\alpha, \mu_\alpha) \). Hence one can show self-adjointness of \(-\Delta\) in the sense of quadratic forms. And since the potential \( V(x) \) grows unboundedly as \( d(0, x) \rightarrow +\infty \), assumption 2.5 then implies that \( (-\Delta + V) \) has pure point spectrum.

For condition (i), one can confirm that there exist constants \( c \) and \( C \) such that for all \( x \in K_\infty \) and all sufficiently large \( r > 0 \),

\[
cr^{d_h} \leq \mu_\infty (B_{R}(x, r)) \leq Cr^{d_h}.
\]
For the infinite blow-up (6.6) follows from (6.3) with \( d_h = d_{h,R} \). As for the periodic fractafolds, note that the corresponding cell graphs \( \Gamma \) satisfy
\[
|B_\Gamma(z, r)| \asymp r^{d_{h,R}} \quad \text{for all } z \in \Gamma \text{ and } r > 0,
\] (6.7)
where \( d_{h,R} \) equals 1 (respectively 2) in the case of the ladder fractafold (respectively the hexagonal fractafold). Combining this with proposition 6.2 we get (6.6) with \( d_h = d_{h,R} \). In all cases, we the find
\[
F(\lambda) = \mu_{\infty} \left( \{ x : V(x) < \lambda \} \right) \approx \mu_{\infty} \left( B_\Gamma(0, \lambda^{1/\beta}) \right) \approx \lambda^{d_{h,R}/\beta},
\] (6.8)
and the same asymptotics holds for \( F^N(\lambda) \) and \( F^U(\lambda) \). Finally, to see that condition (ii) holds, we use the inequality
\[
\left[ V^h(x) - V^v(x) \right] \leq [R(0, x) + 1]^{\beta} - [R(0, x) - 1]^{\beta} \leq C_\beta [R(0, x) + 1]^{\beta-1},
\] (6.9)
where \( C_\beta \) is an explicit constant depending on \( \beta \) only. Observe that the RHS is uniformly bounded from above by a constant multiple of \( \lambda^{1-\beta-} \) for all \( x \) in the set \( \{ x : V(x) \leq \lambda \} \). □

### 6.3. Infinite fractal fields based on nested fractals

There is another notion of an unbounded space based on compact fractals, which are known as fractal fields. The name originated from Hambly and Kumagai [23], who were interested in studying fractal penetrating Brownian motions. Fractal fields differ from the fractafolds of the previous subsection in that we do not require neighborhoods of (junction) points in \( K_\infty \) to be homeomorphic to \( K \).

First consider the triangular lattice finely ramified Sierpinski fractal field introduced in [58, section 6], see figure 4. Notice that this fractal field admits a cellular decomposition whereby cells adjoin at boundary vertices of \( SG(n) \). As a result, the proof strategy from the previous proposition 6.3 applies in this setting.

**Proposition 6.4.** On the triangular lattice finely ramified fractal field based on \( SG(n) \), Bohr’s formula holds for potential of the form \( V(x) \sim R(0, x)^\beta \) for any \( \beta > 0 \). In particular, the spectral dimension of \( -\Delta + V \) is \( d_s(V) = d_t + (4/\beta) \).

Next we consider the double-ladder fractal field based on \( SG(2) \), see figure 5. An important difference here is that pairs of \( SG(2) \) cells may adjoin either at a point or along a boundary segment, which makes this space infinitely ramified. In order to analyze this example using our methods, one needs to understand the eigenvalue problem for the Laplacian on \( SG(2) \) with boundary condition \( \{ \text{top vertex and the bottom edge} \} \), where \( \partial \Omega \) consists of the top vertex and the bottom edge of \( SG(2) \). This was investigated by Qiu [44], whose result we quote below.

**Proposition 6.5 ([44, theorem 3.10]).** Let \( N_0^b(\lambda) \) be the eigenvalue counting function for the Laplacian on \( SG(2) \) with boundary condition \( b \in \{ \wedge, \vee \} \) on the top vertex and the bottom edge of \( SG(2) \). Then there exists a càdlàg \( \log 5 \)-periodic function \( G : \mathbb{R} \to \mathbb{R} \), with \( 0 < \inf G < \sup G < \infty \) and independent of \( b \), such that
\[
N_0^b(\lambda) = G(\log \lambda) \lambda^{3/\log 5} + O \left( \lambda^{3/\log 5} \log \lambda \right)
\] (6.10)
as \( \lambda \to \infty \).

Using this result we can show the validity of Bohr’s formula in this setting.
Proposition 6.6. On the double-ladder fractal field based on \( \text{SG}(2) \), Bohr’s formula holds for potential of the form \( V(x) \sim R(0, x)^\beta \) for any \( \beta > 0 \).

Proof. The one nontrivial assumption to check is assumption 2.2, which is furnished by proposition 6.5. The other two assumptions, 2.5 and 2.8, are verified easily. The result then follows from theorem 2.11.

There are some obvious extensions of the double-ladder fractal field example, which we leave to the reader. An interesting open problem is to study the applicability of Bohr’s formula to the original fractal field (or gasket tiling) in [23], shown in figure 6. We note that heat kernel estimates are established on this fractal field [23, theorem 1.1]. However, to the best of the authors’ knowledge, there is no corresponding Weyl asymptotic (or heat kernel trace asymptotic) estimate which is sharp to an \( o(1) \) remainder. In particular, the fact that the SG cells adjoin along edges rather than at points makes the analysis more delicate.

6.4. Infinite Sierpinski carpets

Let \( F \subset \mathbb{R}^d \) (\( d > 2 \)) be a generalized Sierpinski carpet in the sense of [6, 7], and let \( F_n \) be its \( n \)th-level approximation. Following [6], we call \( \tilde{F} = \bigcup_{n \in \mathbb{N}_0} \ell^n F_n \) the pre-carpet, and \( F_\infty = \bigcup_{n \in \mathbb{N}_0} \ell^n F \) the infinite carpet. The difference between the two is that \( \tilde{F} \) is tiled by unit squares and has nonzero Lebesgue measure, whereas \( F_\infty \) is tiled by copies of the same Sierpinski carpet \( F \) and has zero Lebesgue measure. In both cases, we adopt the Euclidean metric \( \| \cdot \| \) and regard \( (K_\infty, \mu_\infty, \| \cdot \|) \) as the metric measure space, which has volume growth

\[
c_1 r^{d_f} \leq \mu_\infty(B(x, r)) \leq c_2 r^{d_f} \quad (x \in K_\infty, r > 0),
\]

where \( d_f = (\log m/\log \ell) \) is the Hausdorff dimension of the carpet \( F \) with respect to the Euclidean metric.
Proposition 6.7. Bohr’s formula holds on the pre-carpet $\tilde{F}$ with potential $V(x) \sim |x|^\beta$ for any $\beta > 0$. In particular, the spectral dimension of $(-\Delta + V)$ on $\tilde{F}$ is $d + 2(d_f/\beta)$, where $d$ is the dimension of the ambient space $\mathbb{R}^d$ in which $\tilde{F}$ lies.

The case of the infinite carpet is more nuanced. Hambly [22] and Kajino [36] proved that the heat kernel trace of the bare Laplacian on $F$ satisfies assumption 2.14, with $H$ a continuous periodic function of $\log t$ (though it is NOT known whether $H$ is non-constant). Kajino [37] further showed the asymptotics of the heat kernel trace to all orders of the boundary terms. Note that their results imply that the eigenvalue counting function satisfies the asymptotics $c_1 \lambda^{d_f/2} \leq N^{b}(F, \mu, \lambda) \leq c_2 \lambda^{d_f/2}$, but do NOT necessarily imply the sharper estimate, assumption 2.2. As mentioned earlier, this is because the classical techniques of Tauberian theorems cannot be applied here.

Proposition 6.8. The Laplace transform version of Bohr’s formula holds on the infinite carpet $F_\infty$ with potential $V(x) \sim |x|^\beta$ for any $\beta > 0$. In particular, the spectral dimension of $(-\Delta + V)$ on $F_\infty$ is $d_s + 2(d_f/\beta)$, where $d_s$ is the spectral dimension of the bare Laplacian on $F$.

Proof. By [22, theorem 1.1] and [36], assumption 2.14 is satisfied on the constituent Sierpinski carpet $F$. In fact, [37, theorem 4.10] provides a sharper result of the form

$$\mathcal{L}^b(F, \mu, t) = t^{-d_f/2}H(-\log t) + \sum_{k=1}^{d} t^{-d_f/d_k}G_k^b(-\log t) + O\left(\exp\left(-c t^{-\frac{1}{d_w-1}}\right)\right)$$

as $t \downarrow 0$, where $H$ and the $G_k$ are continuous periodic functions, $d_k$ is the Minkowski dimension of $F \cap \{x = (x_1, \cdots, x_d) \in \mathbb{R}^d : x_1 = \cdots = x_d = 0\}$, and $d_s$ and $d_w$ are respectively the spectral dimension and the walk dimension of $F$. 
We turn our attention next to the potential term $F^b(V, t)$. It is direct to verify that for any $\beta > 0$,

$$
\int_{K_\infty} e^{-t|x|^\beta} d\mu_\infty(x) \leq \int_0^\infty e^{-\lambda} \frac{d\mu_\infty(\{x : |x|^{\beta} < \lambda\})}{d\lambda} d\lambda \quad (6.13)
$$

$$
= \int_0^\infty t e^{-\lambda} \mu_\infty(\mathcal{B}(0, \lambda^{1/\beta})) d\lambda \quad (6.14)
$$

$$
\leq c_2 t \int_0^\infty e^{-\lambda} \lambda^{d_f/\beta} d\lambda \leq C_2 (d_f, \beta) t^{d_f/\beta}, \quad (6.15)
$$

and similarly

$$
\int_{K_\infty} e^{-t|x|^\beta} d\mu_\infty(x) \geq C_1 (d_f, \beta) t^{d_f/\beta}. \quad (6.16)
$$

Using the inequality $e^s \geq 1 + s$ for $s \in \mathbb{R}$, we find

$$
|e^{-t|x-y|^\beta} - e^{-t|x-z|^\beta}| \leq \max\left(e^{-t|x-y|^\beta}, e^{-t|x-z|^\beta}\right) \cdot t \left(|x - y|^\beta - |x - z|^\beta\right)
$$

$$
\leq C_3 \cdot t \cdot \max\left(e^{-t|x-y|^\beta}, e^{-t|x-z|^\beta}\right) \cdot |y - z|.
$$

It follows that as $t \downarrow 0$,

$$
F^*(V, t) - F^*(V, t) \leq C \cdot O(t F(V, t)) = o(F(V, t)), \quad (6.17)
$$

leading to the error estimate

$$
\left| \frac{\mathcal{L}(F_{K_\infty} \mu_\infty, V, t)}{t^{-d_f/\beta} H(-\log t) F(V, t)} - 1 \right| = O\left(t^{(d_f - d_i)/d_a}\right) \quad (6.18)
$$

as $t \downarrow 0$. The Laplace transform version of Bohr’s formula then follows.

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