On testing mean proportionality of multivariate normal variables

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Abstract: This short note considers the problem of testing the null hypothesis that the mean values of two multivariate normal variables are proportional. We show that the usual likelihood ratio $\chi^2$-test is valid non-asymptotically. Our proof relies on expressing the test statistic as the minimum eigenvalue of a Wishart variable and using a representation of its distribution using Legendre polynomials.

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1. Introduction

Suppose $X$ and $Y$ are independent $p$-dimensional random vectors, $X \sim N(\mu_1, I_p)$ and $Y \sim N(\mu_2, I_p)$. This paper is concerned with testing the hypothesis that their mean values are proportional, $H_0 : \mu_1 \propto \mu_2$. That is, we are interested in testing the hypothesis that there exists a scalar $\eta$ such that $\mu_2 = \beta \mu_1$. This problem arises naturally in many applications, such as instrumental variables regression [10] and genetic colocalization analysis [8]. In fact, our investigation is motivated by the last application. In genetic colocalization analysis, the measurements $X$ and $Y$ are the regression coefficients of two phenotypes on the same genotypes in a genomic region. In practice, they are usually obtained from different genome-wide association studies. Under the assumption that the two phenotypes share a single causal genetic variant in this region, the two sets of regression coefficients should have proportional means [7].

The mean proportionality testing problem is closely related to Fieller’s theorem and errors-in-variables regression. Assuming that $H_0$ is indeed true, Fieller [2] proposes to construct a confidence interval for $\beta$ by using the pivot

$$R(\beta) = (Y - \beta X)^T(Y - \beta X)/(1 + \beta^2) \sim \chi^2_p.$$  \hspace{1cm} (1)

Because the only unknown quantity in (1) is $\beta$, one can obtain an exact confidence interval for $\beta$ by using suitable quantiles of $\chi^2_p$. However, our interest lies in testing the existence of such $\beta$ instead of estimating $\beta$ when it is assumed to exist. Thus, the problem being considered here is a special case of goodness-of-fit testing for errors-in-variables regression.
Although we have assumed that both $X$ and $Y$ have identity covariance, the same formulation applies to the more general setting where $X \sim N(\mu_1, \Sigma)$ and $Y \sim N(\mu_2, \Sigma)$, where the $p \times p$ matrix $V$ is known. In such case, we can simply consider the transformed variables $\Sigma^{-1/2}X$ and $\Sigma^{-1/2}Y$, whose mean values, $\Sigma^{-1/2}\mu_1$ and $\Sigma^{-1/2}\mu_2$, are still proportional under $H_0$. The invariance of $H_0$ under scaling also means that we can allow the covariance matrix of $Y$ to be $c\Sigma$, where $c$ can be an unknown scalar.

To test the existence of $\beta$, Plagnol et al. [7] propose to compare the minimum value of $R(\beta)$ over $\beta$ with quantiles of $\chi^2_{p-1}$. More specifically, let $\hat{\beta} = \arg\min_\beta R(\beta)$. Plagnol et al. [7] propose to reject $H_0$ at level $(1 - \alpha)$ if $R(\hat{\beta}) > \chi^2_{p-1}(1-\alpha)$, where $\chi^2_{p-1}(1-\alpha)$ is the $(1-\alpha)$ quantile of $\chi^2_{p-1}$ for $0 < \alpha < 1$. This test is also described in Wallace [8] and implemented in a popular R package called coloc.

To justify the aforementioned $\chi^2$-test, Plagnol et al. [7] and Wallace [8] cite asymptotic theory but do not provide a formal argument. It is straightforward to verify that $R(\beta)$ is twice the negative profile log-likelihood of $\beta$ under $H_0$ (up to an additive constant). Moreover, $R(\hat{\beta})$ is exactly twice the negative log likelihood ratio statistic for testing $H_0 : \mu_1 \propto \mu_2$ versus $H_1 : \mu_1$ and $\mu_2$ are unrestricted. Because $R(\beta) \sim \chi^2_p$ and one degree of freedom is spent on estimating $\beta$, intuitively one may expect that $R(\hat{\beta})$ converges in distribution to $\chi^2_{p-1}$. However, this does not immediately follow from Wilk’s theorem or the classical asymptotic theory for likelihood ratio tests, because the dimension of the parameter space is changing. The null model can be parameterized by the $(p+1)$ dimensional vector $(\mu^T_1, \beta)$, while the full model is parameterized by the $2p$ dimensional vector $(\mu^T_1, \mu^T_2)$. Thus both the null and full model spaces have growing dimensions. In fact, the very statement $R(\hat{\beta}) \rightarrow \chi^2_{p-1}$ in distribution as $p \rightarrow \infty$ is not rigorous, because the distributional limit is changing with $p$.

Another potential concern is that the standard likelihood theory may not apply if $\mu^T_1 \mu_1$ does not grow as fast as $p$ when $p \rightarrow \infty$ [10].

Notice that $R(\beta)$ is stochastically dominated by $\chi^2_p$, meaning that its $(1-\alpha)$ quantile is smaller than the $\chi^2_p(1-\alpha)$ for all $0 < \alpha < 1$. This result is trivial because $R(\hat{\beta}) \leq R(\beta)$ by definition and $R(\beta) \sim \chi^2_p$. In the rest of this article, we will show that the distribution of $R(\hat{\beta})$ is also stochastically dominated by $\chi^2_{p-1}$ for all $p \geq 2$.

2. Exact distribution of $R(\hat{\beta})$

Our result relies on classical distributional results on the eigenvalues of a Wishart random variable and is non-asymptotic (does not require $p \rightarrow \infty$). Let $c(\beta) = (\beta, -1)^T / \sqrt{1 + \beta^2}$; notice that $c(\beta)$ has $\ell_2$-norm equal to 1. Let $S = (XY)^T (XY)$. 


Observe that
\[
R(\hat{\beta}) = \inf_\beta R(\beta) = \inf_\beta (Y - \beta X)^T (Y - \beta X) / (1 + \beta^2)
\]
\[
= \inf_\beta c(\beta)^T (XY)^T (XY)c(\beta)
\]
\[
= \lambda_2(S),
\]
where \(\lambda_1(S) \geq \lambda_2(S) \geq 0\) are the two eigenvalues of \(S\). Because \(S\) follows a Wishart distribution (non-central if \(\mu_1 \neq 0\)), this allows us to use classical distributional results on the eigenvalues of a Wishart random variable [3].

The eigenvalue distribution is much simpler when the Wishart distribution is central. When the scale matrix is identity, Muirhead [6, Corollary 3.2.19] has derived the joint density function of the eigenvalues. When \(S\) is \(2 \times 2\), this is given by

\[
f_c(\lambda_1, \lambda_2) \propto e^{-(\lambda_1 + \lambda_2)/2(\lambda_1 \lambda_2)^{(p-3)/2} / (\lambda_1 - \lambda_2)}, \quad \lambda_1 > \lambda_2 > 0.
\]

The normalizing constant can be found in Muirhead [6].

The non-central case is more complicated. James [3, equation 68] has given the joint density function of the eigenvalues of a non-central Wishart random variable. For the case we are considering (\(S\) is \(2 \times 2\) and scale matrix is identity), the joint density is given by, for \(\lambda_1 > \lambda_2 > 0\),

\[
f(\lambda_1, \lambda_2) \propto \binom{p}{2} \binom{(1 + \beta^2) \mu_1^T \mu_1 / 4}{0} \binom{\lambda_1}{0} \frac{\lambda_1}{\lambda_2} e^{-(\lambda_1 + \lambda_2)/2(\lambda_1 \lambda_2)^{(p-3)/2} / (\lambda_1 - \lambda_2)},
\]

where \(\binom{p}{2}\) is the generalized hypergeometric function of two matrix arguments defined in, for instance, James [3]. The normalizing constant for \(f(\lambda_1, \lambda_2)\) can also be found there. By definition, \(f(\lambda_1, \lambda_2)\) reduces to \(f_c(\lambda_1, \lambda_2)\) when \(\mu_1 = 0\).

By using equation (1.13) in Muirhead [5], we can write \(\binom{p}{2}\) as a series

\[
\binom{p}{2} \binom{(1 + \beta^2) \mu_1^T \mu_1 / 4}{0} \binom{\lambda_1}{0} \frac{\lambda_1}{\lambda_2} e^{-(\lambda_1 + \lambda_2)/2(\lambda_1 \lambda_2)^{(p-3)/2} / (\lambda_1 - \lambda_2)},
\]

where \(P_j\) is the Legendre polynomial of degree \(j\) and \((p/2)_j\) is the rising factorial \((p/2)_j = (p/2)(p/2 + 1) \cdots (p/2 + j - 1)\). The Legendre polynomials can be found using the initial polynomials \(P_0(x) = 0, P_1(x) = x\), and Bonnet’s recursion formula

\[
(j + 1)P_{j+1}(x) = (2j + 1)xP_j(x) - jP_{j-1}(x), \quad j = 1, 2, \ldots
\]
The derivative of the Legendre polynomials can be computed by

$$\frac{d}{dx} P_j(x) = \frac{jxP_j(x) - jP_{j-1}(x)}{x^2 - 1}, \quad j = 1, 2, \ldots .$$

(5)

These formulae will be quite useful in our proof below.

There has been a lot of investigations on the distribution of the largest eigenvalue of a sample covariance matrix, as it is closely related to selecting the number of principal components in a principal component analysis [see e.g., 4]. There has been relatively less interest in studying the distribution of the smallest eigenvalue. Edelman [1] has derived a recursion for the distribution of the smallest eigenvalue of a central Wishart variable without resorting to zonal polynomials and hypergeometric functions of matrix arguments. Although that formula can be used to efficiently compute the distribution numerically, we could not use it to prove that $\lambda_2(S)$ is stochastically dominated by $\chi^2_{p-1}$. Instead, in the proof below we will directly obtain the distribution of $\lambda_2(S)$ by integrating $f(\lambda_1, \lambda_2)$ over $\lambda_1$.

3. Proof of stochastic dominance

Consider two cumulative distribution functions $F(x)$ and $G(x)$. Suppose they are defined on the same support (in our case, $(0, \infty)$) and have density functions $f(x)$ and $g(x)$, respectively. The distribution $F$ is said to have (first-order) stochastic dominance over $G$ if $F(x) \leq G(x)$ for all $x$. To show stochastic dominance, it is sufficient to establish monotone likelihood ratio property, that is, $f(x)/g(x)$ is increasing in $x$ [9, Proposition 4.3].

We first consider the central case of our problem, as the distribution of $\lambda_2(S)$ is much simpler. In this case, the marginal density function of $\lambda_2$ is given by

$$f_c(\lambda_2) \propto \int_{\lambda_2}^{\infty} e^{-(\lambda_1+\lambda_2)/2} (\lambda_1\lambda_2)^{(p-3)/2} (\lambda_1 - \lambda_2) d\lambda_1$$

$$\propto e^{-\lambda_2/2} \lambda_2^{(p-3)/2} \left[ \frac{\lambda_2}{2} \Gamma \left( \frac{p+1}{2}, \frac{\lambda_2}{2} \right) - \frac{\lambda_2}{2} \Gamma \left( \frac{p-1}{2}, \frac{\lambda_2}{2} \right) \right],$$

where $\Gamma$ is the incomplete gamma function $\Gamma(s, x) = \int_x^{\infty} t^{s-1} e^{-t} dt$. To show that $\lambda_2(S)$ is stochastically dominated by $\chi^2_{p-1}$, whose density function is proportional to $e^{-\lambda_2/2}\lambda_2^{(p-3)/2}$, it suffices to show that the likelihood ratio (up to a multiplicative constant)

$$g_c(\lambda_2) = \Gamma \left( \frac{p+1}{2}, \frac{\lambda_2}{2} \right) - \frac{\lambda_2}{2} \Gamma \left( \frac{p-1}{2}, \frac{\lambda_2}{2} \right)$$

is decreasing in $\lambda_2$. This can be verified by checking that the derivative of $g_c(\lambda_2)$ is equal to $-\Gamma((p-1)/2, \lambda_2/2) < 0$, using $(\partial/\partial x)\Gamma(s, x) = -x^{s-1} e^{-x}$.

For the non-central case, the marginal density of $\lambda_2$

$$f(\lambda_2) = \int_{\lambda_2}^{\infty} f(\lambda_1, \lambda_2) d\lambda_1$$
is much more complicated. A natural idea is to show that the distribution in the non-central case is dominated by the central case. However, this is not true (see next section).

Motivated by the proof for the central case, it suffices to show that the likelihood ratio of $\lambda_2(S)$ to $\chi^2_{q-1}$ is decreasing, that is,

$$\frac{\partial}{\partial \lambda_2} \left( \frac{f(\lambda_2)}{e^{-\lambda_2/2}\chi^2_{(p-3)/2}} \right) \leq 0.$$  

By using (2), (3), and Leibniz’s rule for differentiation, this is equivalent to showing

$$\int_{\lambda_2}^{\infty} e^{-\lambda_1/2}\chi^2_{(p-3)/2} \cdot \frac{\partial}{\partial \lambda_2} \sum_{j=0}^{\infty} (\lambda_1 - \lambda_2) \frac{(1 + \beta^2) \mu_1^j \mu_2^{j/2}}{4^j (p/2)_j j!} P_j \left( \frac{\lambda_1 + \lambda_2}{2\sqrt{\lambda_1 \lambda_2}} \right) d\lambda_1 \leq 0.$$ 

Thus, it suffices to show that for all $\lambda_1 > \lambda_2$,

$$\frac{\partial}{\partial \lambda_2} (\lambda_1 - \lambda_2) \chi^2_{1/2} P_j \left( \frac{\lambda_1 + \lambda_2}{2\sqrt{\lambda_1 \lambda_2}} \right) \leq 0, \quad j = 0, 1, \ldots.$$ 

This trivially holds for $j = 0$. Let $x = (\lambda_1 + \lambda_2)/(2\sqrt{\lambda_1 \lambda_2})$, so it suffices to show the following inequality for $j \geq 1$:

$$\{(\lambda_1/\lambda_2)(j/2) - (j/2 + 1)\} P_j(x) + (\lambda_1 - \lambda_2) \frac{d}{dx} P_j(x) \frac{\partial x}{\partial \lambda_2} \leq 0.$$ 

By using $\partial x/\partial \lambda_2 = (\lambda_2 - \lambda_1)/(4\sqrt{\lambda_1 \lambda_2})$, this is equivalent to

$$\{(\lambda_1(j/2) - \lambda_2(j/2 + 1)) P_j(x) \leq \frac{(\lambda_1 - \lambda_2)^2}{4\sqrt{\lambda_1 \lambda_2}} \frac{d}{dx} P_j(x).$$ 

By using (5) and $x - \sqrt{x^2 - 1} = \sqrt{\lambda_2/\lambda_1}$, this can be shown to be equivalent to

$$\frac{P_{j-1}(x)}{P_j(x)} \leq \frac{j + 1}{j} \left( x - \sqrt{x^2 - 1} \right) \quad \text{for all } x > 1. \quad (6)$$ 

In summary, we have reduced the proof of stochastic dominance to showing the above inequality for Legendre polynomials.

We prove the inequality (6) using induction. It is easy to check that (6) holds for $j = 1$. Next we assume (6) holds for $j$. By Bonnet’s recursion (4) and
plugging in (6), we get
\[
\frac{P_{j+1}(x)}{P_j(x)} = \frac{2j+1}{j+1} x - \frac{j}{j+1} \frac{P_{j-1}(x)}{P_j(x)} \\
\geq \frac{2j+1}{j+1} x - \frac{j}{j+1} (x - \sqrt{x^2 - 1}) \\
= \frac{j}{j+1} (x + \sqrt{x^2 - 1}) + \frac{1}{j+1} \sqrt{x^2 - 1}
\]
By using \((x + \sqrt{x^2 - 1})(x - \sqrt{x^2 - 1}) = 1\), this can be written as
\[
\frac{P_{j+1}(x)}{P_j(x)} - \frac{j+1}{j+2} \frac{1}{x - \sqrt{x^2 - 1}} \geq \frac{(j+1)\sqrt{x^2 - 1} - x}{(j+1)(j+2)}.
\]
The right hand side is non-negative if and only if \(x \geq (j+1)/\sqrt{j(j+2)}\). Therefore, we cannot rely on using the induction hypothesis to prove
\[
\frac{P_j(x)}{P_{j+1}(x)} \leq \frac{j+2}{j+1} (x - \sqrt{x^2 - 1}) \quad (7)
\]
for \(1 < x < (j+1)/\sqrt{j(j+2)}\). Instead, we prove this inequality directly for such \(x\) by noticing that \(P_{j-1}(x) \leq P_j(x)\) for all \(x \geq 1\) and \(j = 1, 2, \ldots\) (this can be shown by induction). Thus
\[
\frac{P_{j+1}(x)}{P_j(x)} = \frac{2j+1}{j+1} x - \frac{j}{j+1} \frac{P_{j-1}(x)}{P_j(x)} \geq \frac{2j+1}{j+1} x - \frac{j}{j+1} x - \frac{j}{j+1} x = x,
\]
Notice that for \(x < (j+1)/\sqrt{j(j+2)}\), we have
\[
\left(\frac{j+1}{j+2}\right) (x + \sqrt{x^2 - 1}) = \left(\frac{j+1}{j+2}\right) \left(1 + \sqrt{1 - \frac{1}{x^2}}\right) < 1.
\]
It is then straightforward to verify (7).

4. Numerical illustration

Figure 1 shows the simulated distribution distribution of the test statistic \(R(\hat{\beta})\) in various settings. Apart from the dotted curve which corresponds to \(\chi^2_{p-1}\), each curve is obtained from 100,000 simulations. Notice that the distribution only depends on \(\beta\) and \(\mu_1\) through \(\kappa = (1 + \beta^2)\mu_1^T \mu_1\), which can be shown using equation 68 in James [3]. We will call \(\kappa\) the noncentrality parameter. From this figure, it appears that the distribution function of \(R(\hat{\beta})\) becomes smaller and approaches \(\chi^2_{p-1}\) as \(\kappa\) increases. However, for a fixed \(\kappa\), \(\chi^2_{p-1}\) becomes a worse approximation as \(p\) increases. Table 1 lists the simulated size (type I error) of the \(\chi^2\)-test in various settings. The test becomes more conservative as the noncentrality parameter \(\kappa\) decreases and the dimension \(p\) increases.
Fig 1. Simulated distribution function of the test statistic $R(\hat{\beta})$. The different curves correspond to different values of the noncentrality parameter $\kappa = (1 + \beta^2)\mu_1^T \mu_1$; the rightmost dotted curve in each panel is the distribution function of $\chi^2_{p-1}$.

Table 1
Simulated size (type I error) of the $\chi^2$-test for different dimension $p$, significance level $\alpha$, and noncentrality parameter $\kappa = (1 + \beta^2)\mu_1^T \mu_1$.

|       | $\kappa = 0$ | $\kappa = 5$ | $\kappa = 20$ | $\kappa = 0$ | $\kappa = 5$ | $\kappa = 20$ | $\kappa = 0$ | $\kappa = 5$ | $\kappa = 20$ |
|-------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| $p = 2$ | 0% 0.2% 0.7% | 0.3% 2.2% 4.3% | 1.3% 5.8% 9% | 0% 0.2% 0.7% | 0.3% 2.2% 4.3% | 1.3% 5.8% 9% |
| $p = 5$ | 0% 0.1% 0.4% | 0.1% 0.8% 3.2% | 0.7% 2.5% 7.2% | 0% 0.1% 0.4% | 0.1% 0.8% 3.2% | 0.7% 2.5% 7.2% |
| $p = 10$ | 0% 0.1% 0.4% | 0.1% 0.8% 3.2% | 0.7% 2.5% 7.2% | 0% 0.1% 0.4% | 0.1% 0.8% 3.2% | 0.7% 2.5% 7.2% |
| $p = 20$ | 0% 0% 0.3% | 0.1% 0.5% 2.5% | 0.5% 1.7% 6% | 0% 0% 0.3% | 0.1% 0.5% 2.5% | 0.5% 1.7% 6% |
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