SU(2) and SU(3) chiral perturbation theory analyses on baryon masses in 2+1 flavor lattice QCD

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We investigate the quark mass dependence of baryon masses in 2+1 flavor lattice QCD using SU(3) heavy baryon chiral perturbation theory up to one-loop order. The baryon mass data used for the analyses are obtained for the degenerate up-down quark mass of 3 MeV to 24 MeV and two choices of the strange quark mass around the physical value. We find that the SU(3) chiral expansion fails to describe both the octet and the decuplet baryon data if phenomenological values are employed for the meson-baryon couplings. The SU(2) case is also examined for the nucleon. We observe that higher order terms are controlled only around the physical point. We also evaluate finite size effects using SU(3) heavy baryon chiral perturbation theory, finding small values of order 1% even at the physical point.

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I. INTRODUCTION

The aim of the PACS-CS project is full QCD calculations on the physical point avoiding any contamination due to chiral extrapolations. At the first stage of this project\textsuperscript{1} we have succeeded in reducing the up-down quark mass from 67 MeV, the minimum value reached by the previous CP-PACS/JLQCD work\textsuperscript{2}, to 3 MeV, corresponding to the decrease of pion mass from 702 MeV to 156 MeV. This work allowed us to make detailed chiral analyses on the pseudoscalar meson sector with the use of chiral perturbation theory (ChPT). An important finding is that the strange quark mass is not small enough to be treated by the SU(3) ChPT up to the next-to-leading order (NLO). For the octet and decuplet baryon masses we simply employed a linear chiral extrapolation to the physical point assuming isospin symmetry and analyticity of the strange quark contribution around its physical mass.

In this article we reinvestigate the quark mass dependence of the octet and decuplet baryon masses employing the SU(3) heavy baryon chiral perturbation theory (HBChPT) up to NLO\textsuperscript{3, 4, 5}. The results are compared with those of the linear chiral extrapolation obtained in Ref.\textsuperscript{1}. We also examine the convergence property of HBChPT fits to the lattice results. For the nucleon mass we examine the SU(2) HBChPT with an analytic expansion of the low energy constants (LECs) in terms of the strange quark mass around its physical value. The SU(2) BChPT analyses in 2 flavor lattice QCD were previously made by other collaborations and reported in Refs.\textsuperscript{6, 7}. We also discuss the magnitude of finite size effects based on the NLO SU(3) HBChPT.

This paper is organized as follows. In Sec. II we briefly review the results obtained in Ref.\textsuperscript{1} to make the paper self-contained. In Sec. III we apply SU(3) HBChPT analyses to the octet baryon masses. We present the fit results for the LECs and discuss the convergence behavior up to NLO. The same analysis is repeated for the decuplet baryon masses. Section IV describes the fit results of the nucleon mass with the SU(2) HBChPT. In Sec. V we discuss the magnitude of finite size effects for the baryon masses based on the SU(3) HBChPT. Our conclusions are summarized in Sec. VI.

II. SU(2) LINEAR CHIRAL FIT FOR BARYON MASSES

We give a quick review of the chiral analyses employed in Ref.\textsuperscript{1}. The extrapolation to the physical point is performed with the following fit formula:

\[ m = \alpha + \beta m_{\text{ud}}^{\text{AWI}} + \gamma m_{s}^{\text{AWI}}, \]

(1)

where \( m_{\text{ud}}^{\text{AWI}} \) denotes the Axial Ward Identity quark mass for the up-down quark and \( m_{s}^{\text{AWI}} \) for the strange quark. Since we choose \( s_{\alpha} \) around the physical strange quark mass, the above formula is essentially an SU(2) chiral expansion with the strange quark contribution analytically expanded around its physical value. Actually we can rewrite the formula (1) as

\[ m = \alpha' + \beta m_{\text{ud}}^{\text{AWI}} + \gamma (m_{s}^{\text{AWI}} - m_{s,\text{ph}}^{\text{AWI}}), \]

(2)
where $\alpha' = \alpha + \gamma m_{s,\text{ph}}^{\text{AWI}}$ with $m_{s,\text{ph}}^{\text{AWI}}$ the physical value of the strange quark mass. This is a linear expansion of the baryon mass around $(m_{u,\text{AWI}}, m_{s,\text{AWI}}) = (0, m_{s,\text{ph}}^{\text{AWI}})$.

The simulation points and the measured hadron masses are given in Table I and Table III in Ref. [1], respectively. The fit range is the lightest four points at $\kappa_{ud} \geq 0.13754$, where the pion mass varies from 156 MeV to 410 MeV and $m_{u,\text{AWI}}$ from 3 MeV to 24 MeV in the $\overline{\text{MS}}$ scheme. In Figs. 1 and 2 we present the fit results for the octet and the decuplet baryon masses, respectively. Star symbols denote the extrapolated values at the physical point whose numerical values are listed in Tables I and II. The physical point together with the lattice cutoff is determined with $m_{u}, m_{K}, m_{\Omega}$ inputs by applying the NLO SU(2) ChPT fit to the $\pi$ meson sector [1]. The data are reasonably described by the linear function Eq. (1). The values for $\alpha, \beta, \gamma$ and $\chi^2/dof$ are summarized in Tables III and IV. In order to investigate the convergence behavior the contribution of each term in Eq. (2) is drawn in Fig. 3 for the octet baryon masses, where $m_{u,\text{AWI}}$ is fixed at the measured value for $(\kappa_{ud}, \kappa_{s}) = (0.13754, 0.13640)$.

\[ \mathcal{L} = \left( B_{\text{iv}} \cdot DB \right) + 2\alpha_{M} \left( \overline{B}BM_{+} \right) + 2\beta_{M} \left( \overline{B}M_{-} \right) + 2\gamma M_{+} \left( \overline{B}B \right) \text{tr} (M_{+}) \]
\[ - \left( \overline{T}^{\mu} \left\{ iv \cdot D - \Delta \right\} T_{\mu} \right) + 2\gamma_{M} \left( \overline{T}^{\mu} \overline{M}_{+} T_{\mu} \right) - 2\gamma M \left( \overline{T}^{\mu} T_{\mu} \right) \text{tr} (M_{+}) \]
\[ + 2\alpha \left( B\overline{S}^{\mu}A_{\mu} \right) + 2\beta \left( \overline{B}S^{\mu}A_{\mu} \right) + 2\gamma \left( \overline{B}T^{\mu}A_{\mu} \right) \]
\[ + \frac{\sqrt{3}}{2} \mathcal{C} \left[ \left( \overline{T}^{\nu}A_{\nu} \right) + \left( \overline{B}A_{\nu}T^{\nu} \right) \right], \] 

where $B$ and $T$ represent the velocity-dependent octet and decuplet baryon fields with the four-velocity $v_{\mu}$, respectively. This lagrangian contains 9 LECs: $\alpha_{M}, \beta_{M}, \sigma_{M}, \gamma, \sigma_{3}, \alpha, \beta, \gamma, C$. Hereafter we use the axial couplings $F, D$ instead of $\alpha, \beta$. They are related with

\[ \alpha = \frac{2}{3} D + 2F, \] 
\[ \beta = -\frac{5}{3} D + F. \] 

The decuplet-octet mass difference denoted by $\Delta = m_{T} - m_{B}$ has a comparable magnitude to inter-multiplet mass splittings of both the octet and the decuplet. We are allowed to treat it as a perturbation. The octet pseudoscalar mesons couple derivatively to the baryon fields through the vector combination $V_{\mu}$, the axial vector one $A_{\mu}$ and the chiral covariant derivative $D_{\mu}$. The light quark masses $m_{u}, m_{d}, m_{s}$ are contained in $M_{+}$. We refer to Ref. [5] for the explicit expression of $V_{\mu}, A_{\mu}, D_{\mu}, M_{+}$.

In this section the physical $m_{u,\text{AWI}}$ and $m_{s}$ are determined by the SU(3) ChPT analyses in the pseudoscalar meson sector including $m_{u}$ and $m_{K}$; $m_{\Omega}$ is an additional input for the lattice cutoff [1].

We observe that the $O(m_{s} - m_{\text{ph}})$ terms are small, and that $O(m_{ud})$ contributions are less than 20% of that of $\alpha'$ for $m_{u,\text{AWI}} \leq 0.01$. This suggests that higher order terms in $m_{ud}$ and $m_{s} - m_{\text{ph}}$ would be small. Similar trends are found for the decuplet baryon masses in Fig. 4.

### III. SU(3) HBCHPT ANALYSES ON BARYON MASSES

#### A. Lagrangian to leading order

We use the continuum HBChPT in this article, leaving the development of the Wilson HBChPT, which incorporates the chiral symmetry breaking effects of the Wilson-type quark action, for future work. We follow the notation described in Ref. [3]. The lagrangian of HBChPT is written in terms of velocity-dependent baryon fields with a perturbative derivative expansion. To fix the notation we present the leading order terms:

\[ m_{B_{i}} = m_{B} - m_{B_{i}}^{(1)} - m_{B_{i}}^{(3/2)} - \cdots, \] 

where $m_{B}$ is the octet baryon mass in the chiral limit under the SU(3) flavor symmetry, and $m_{B_{i}}^{(n)}$ is the $O(m_{q}^{n})$ contribution to the $i$-th octet baryon. The LO corrections are given by

\[ m_{X_{i}}^{(1)} = (2\alpha_{M} + 2\beta_{M} + 4\sigma_{M})m_{ud} + 2\sigma_{M}m_{s}, \]
\[ m_{A_{i}}^{(1)} = (\alpha_{M} + 2\beta_{M} + 4\sigma_{M})m_{ud} + (\alpha_{M} + 2\sigma_{M})m_{s}, \]
\[ m_{X_{i}}^{(1)} = \left( \frac{5}{3} \alpha_{M} + \frac{2}{3} \beta_{M} + 4\sigma_{M} \right) m_{ud} \]
\[ + \left( \frac{4}{3} \alpha_{M} + \frac{4}{3} \beta_{M} + 2\sigma_{M} \right) m_{s}, \]
\[ m_{X_{i}}^{(1)} = \left( \frac{1}{3} \alpha_{M} + \frac{4}{3} \beta_{M} + 4\sigma_{M} \right) m_{ud} \]
\[ + \left( \frac{5}{3} \alpha_{M} + \frac{2}{3} \beta_{M} + 2\sigma_{M} \right) m_{s}. \]
with \( m_{ud} \) the averaged up-down quark mass, \( m_s \) the strange quark mass, and \( \alpha_M, \beta_M, \sigma_M \) are LECs in the lagrangian \([2]\).

\[
m_{B_i}^{(3/2)} = \frac{2}{(4\pi f_0)^2} \sum_{\phi=\pi,K,\eta} \left[ A_{\phi}^B \mathcal{F}(m_\phi, 0, \mu) + C^2 B_{\phi}^B \mathcal{F}(m_\phi, \Delta, \mu) \right],
\]

where \( \Delta \) is the octet-decuplet baryon mass difference in the SU(3) chiral limit. The pion decay constant \( f_0 \) is also

\[
\mathcal{F}(m_\phi, \Delta, \mu) = (m_\phi^2 - \Delta^2) \left[ \sqrt{\Delta^2 - m_\phi^2 + i\epsilon} \ln \left( \frac{\Delta - \sqrt{\Delta^2 - m_\phi^2 + i\epsilon}}{\Delta + \sqrt{\Delta^2 - m_\phi^2 + i\epsilon}} \right) - \Delta \ln \left( \frac{m_\phi^2}{\mu^2} \right) - \frac{1}{2} \Delta m_\phi^2 \ln \left( \frac{m_\phi^2}{\mu^2} \right) \right]
\]

with \( \mathcal{F}(m_\phi, 0, \mu) = \pi m_\phi^3 \), and \( \mu \) the renormalization scale. This formula assumes \( \Delta \geq m_\phi \). For \( \Delta < m_\phi \) we apply the analytic continuation:

\[
\sqrt{\Delta^2 - m_\phi^2 + i\epsilon} \ln \left( \frac{\Delta - \sqrt{\Delta^2 - m_\phi^2 + i\epsilon}}{\Delta + \sqrt{\Delta^2 - m_\phi^2 + i\epsilon}} \right) \rightarrow \arccos \left( \frac{\Delta}{m_\phi} \right).
\]

We revisit the function \( \mathcal{F} \) later in Sec.\( \text{V} \) to discuss finite size effects. The contributions from the octet-octet- and the decuplet-octet-axial couplings are factored out by \( A_{\phi}^B \) and \( C^2 B_{\phi}^B \), respectively. We summarize their values in Table\( \text{V} \). The LECs are phenomenologically estimated as\([6, 7, 8]\).

\[
D = 0.80, \quad F = 0.47, \quad C = 1.5 \quad (14)
\]

We first present the fit results up to leading order employing the formula \( m_{B_i} = m_B - m_{B_i}^{(1)} \). The values for \( m_B, \alpha_M, \beta_M, \sigma_M \) are given in Table\( \text{V} \) together with \( \chi^2/\text{dof} \). We also present the extrapolated values at the physical point in Table\( \text{VI} \) in comparison with the results of the SU(2) linear fit Eq.\( \text{(1)} \). The two set of results are consistent within 2\sigma error. Figure\( \text{I} \) shows that the data are reasonably described by the formula. The convergence behavior in Fig.\( \text{I} \) however, is disappointing because of the sizable contribution of the \( O(m_s) \) term. This point is in a sharp contrast to the SU(2) linear expansion of Sec.\( \text{II} \) where the contributions of the \( O(m_{ud}, m_s - m_{s,ph}) \) terms are well controlled in the range of \( m_{ud} \leq 0.01 \).

In the NLO fit the number of LECs increases up to seven. We give the fit results in Fig.\( \text{I} \) and Tables\( \text{VI} \) and \( \text{II} \). Although the fit works in a reasonable manner, a critical observation is that the results for \( D, F, C \) are essentially consistent with zero showing a significant deviation from the phenomenological estimates in Eq.\( \text{(14)} \). To examine the contributions of the NLO terms, we make a fit with \( D, F, C \) fixed at the phenomenological estimates. Figure\( \text{I} \) and Table\( \text{II} \) show that this fit assumption is strongly disfavored because of a prohibitively large value of \( \chi^2/\text{dof} \). The reason is found in Fig.\( \text{I} \). If \( D, F, C \) are fixed at the phenomenological estimates, the magnitude of the NLO contribution is 2–5 times larger than that of the data even at \( m_{ud} = 0 \). We also remark that the same situation holds if either of \( D, F, C \) is fixed at the phenomenological estimate.

### C. Decuplet baryons

Let us turn to the decuplet baryons. The chiral expansion of the \( i \)-th decuplet baryon mass is written as

\[
m_{T_i} = m_T + m_{T_i}^{(1)} + m_{T_i}^{(3/2)} + \cdots \quad (15)
\]

with \( m_T \) the decuplet baryon mass in the SU(3) chiral limit. The LO and NLO corrections are given by

\[
\begin{align*}
m_{T}^{(1)} &= \frac{2}{3} \gamma_M (3m_{ud}) - 2\sigma_M (2m_{ud} + m_s), \\
m_{T}^{(2)} &= \frac{2}{3} \gamma_M (2m_{ud} + m_s) - 2\sigma_M (2m_{ud} + m_s), \\
m_{T}^{(3)} &= \frac{2}{3} \gamma_M (m_{ud} + 2m_s) - 2\sigma_M (2m_{ud} + m_s), \\
m_{T}^{(4)} &= \frac{2}{3} \gamma_M (3m_s) - 2\sigma_M (2m_{ud} + m_s)
\end{align*}
\]
$$m_{T_i}^{(3/2)} = -\frac{1}{(4\pi f_0)^2} \sum_{\phi = \pi, K, \eta} \left[ \frac{10}{9} \mathcal{H}^2 A^T_{\phi} \mathcal{F}(m_{\phi}, 0, \mu) + C^2 B^T_{\phi} \mathcal{F}(m_{\phi}, -\Delta, \mu) \right]$$

where $\gamma_M, \bar{M}, \mathcal{H}$ are additional LECs, and $\mathcal{H}^2 A^T_{\phi}$ and $C^2 B^T_{\phi}$ denote the contributions coming from the decuplet-decuplet- and the decuplet-octet-axial couplings, respectively, whose complete list is given in Table VII. Note that the function $\mathcal{F}(m_{\phi}, -\Delta, \mu)$ yields an imaginary part if $\Delta > m_{\phi}$. Our analyses are made by taking account of the real part only. In the region of $\Delta > m_{\phi}$ it might be problematic to apply the continuum chiral expansion for infinite spatial volume to lattice data. We leave the proper treatment to future work.

The LO fit results with the formula $m_{T_i} = m_T + m_{T_i}^{(1)}$ are presented in Figs. 10 and 11 and Tables VII and II. We find a reasonable value for the physical strange quark mass: $m_s(m_{ud} = 0.01)$, employing the experimental axial coupling $g_A = 1.267$, and the value of $f$ already determined from the SU(2) ChPT fit for $M, f_{\pi}, f_K$ in Ref. 1. The results are given in Fig. 13 and Table IX. The value of $\chi^2$/dof is sufficiently small. However, we should remark two points. Firstly, Fig. 13 shows that the fit results fail to predict the quark mass dependence of the data beyond $m_{ud}^{AWI} = 0.01$. The extrapolated value at the physical point, for which we obtain $m_N = 0.382(25)$, also undershoots the experimental one sizably. Secondly, the LO and NLO contributions quickly increase as the quark mass increases so that the good convergence region is restricted near the chiral limit.

IV. SU(2) HBCHPT ANALYSES ON NUCLEON MASS

In the framework of SU(2) HBChPT the nucleon mass up to $O(m_{ud}^{(3/2)})$ is given by

$$m_N = m_0 - 4c_1m_\pi^2 - \frac{3g_A^2}{16\pi f^2}m_\pi^3,$$ \hfill (21)

where $m_0$ and $f$ are defined in the SU(2) chiral limit, $c_1$ is a low energy constant and $g_A$ denotes the axial vector coupling of the nucleon. We further expand $m_0$ around the physical strange quark mass:

$$m_0 = \bar{m}_0 + \bar{m}_0'(m_s - m_{s, ph}).$$ \hfill (22)

Since $|m_s - m_{s, ph}|$ is comparable to $m_{ud}$ in our simulations, the analytic expansion of $c_1, g_A, f$ in terms of $m_s - m_{s, ph}$ yields higher order corrections beyond $O(m_{ud}^{(3/2)})$.

We apply the formula (21) to the four data points with $m_{ud} \lesssim 0.01$, employing the experimental axial coupling $g_A = 1.267$, and the value of $f$ already determined from the SU(2) ChPT fit for $M, f_{\pi}, f_K$ in Ref. 1. The results are given in Fig. 15 and Table IX. The value of $\chi^2$/dof is sufficiently small. However, we should remark two points. Firstly, Fig. 15 shows that the fit results fail to predict the quark mass dependence of the data beyond $m_{ud}^{AWI} = 0.01$. The extrapolated value at the physical point, for which we obtain $m_N = 0.382(25)$, also undershoots the experimental one sizably. Secondly, the LO and NLO contributions quickly increase as the quark mass increases so that the good convergence region is restricted near the chiral limit.

V. FINITE SIZE EFFECTS

The one-loop correction for the baryon mass is evaluated by the following function,

$$\mathcal{F}^{(\infty)}(m_{\phi}, \Delta) = -8\pi^2 \int_0^\infty \frac{d^4\vec{p}}{(2\pi)^3} \frac{\vec{p}^2}{(ip - \Delta)(p_0^2 + \vec{p}^2 + m_{\phi}^2)},$$ \hfill (23)

where the integral is defined in the Euclidean space-time. This leads to Eq. (12) after the dimensional regularization and the renormalization in the $\overline{MS}$ scheme with the scale $\mu$. In a finite spacial volume of linear size $L$, the
integral over the spatial components of the loop momentum $\vec{p}$ is replaced by a sum over discrete momenta $\vec{p} = (2\pi/L)\vec{n}$, where we assume that the time direction is infinite. We define the finite size correction for $F$ as

\[ F(L)(m_\phi, \Delta) = -8\pi^2 \int \frac{dp_4}{2\pi} \sum_{\vec{p}} \frac{1}{L^3} \frac{\vec{p}}{(ip_4 - \Delta)(p^2 + \vec{p}^2 + m_\phi^2)}, \]  

(24)

\[ \delta_L F(m_\phi, \Delta) \equiv F(L)(m_\phi, \Delta) - F(\infty)(m_\phi, \Delta) = 4\pi^2 \int_0^\infty d\lambda \left[ \delta_L(p^2 + \beta_\Delta^2)^{-\frac{3}{2}} - \beta_\Delta^2 \delta_L(p^2 + \beta_\Delta^2)^{-\frac{3}{2}} \right], \]  

(25)

where $\beta_\Delta^2 = \lambda^2 + 2\lambda\Delta + m_\phi^2$ and

\[ \delta_L(p^2 + m^2)^{-r} \equiv \frac{1}{L^3} \sum_{\vec{p}} (\vec{p}^2 + m^2)^{-r} - \int \frac{d^3p}{(2\pi)^3} (\vec{p}^2 + m^2)^{-r}. \]  

(26)

With the use of the master formula[16, 17]

\[ \delta_L(p^2 + m^2)^{-r} = \frac{1}{(4\pi)^{\frac{3}{2}} \Gamma(r)} \sum_{\vec{n} \neq 0} \left( \frac{L|\vec{n}|}{2m} \right)^{r-\frac{3}{2}} K_{r-\frac{3}{2}}(mL|\vec{n}|), \]  

(27)

where $K_n(z)$ is a modified Bessel function of the second kind, one finds that the finite size corrections to the baryon masses at next-to-leading order are

\[ \delta_L m_{B_i} = -\frac{2}{(4\pi f_0)^2} \sum_{\phi=\pi, K, \eta} \left[ A^{B_i}_\phi \delta_L F(m_\phi, 0) + B^{B_i}_\phi \delta_L F(m_\phi, \Delta) \right], \]  

(29)

\[ \delta_L m_{T_i} = -\frac{1}{(4\pi f_0)^2} \sum_{\phi=\pi, K, \eta} \left[ \frac{10}{9} \mathcal{H}^T_\phi \delta_L F(m_\phi, 0) + C^2B^{T_i}_\phi \delta_L F(m_\phi, -\Delta) \right], \]  

(30)

where $\delta_L m = m(L) - m(\infty)$ and $\delta_L F(m_\phi, \Delta)$ is given by
\[
\delta_L F(m_\phi, \Delta) = 2 \int_0^\infty d\lambda \beta_\Delta \sum_{\pi \neq 0} \left[ \frac{1}{L/|\pi| L} K_1(L/\beta_\Delta |\pi|) - \beta_\Delta K_0(L/\beta_\Delta |\pi|) \right],
\]
(31)

\[
\delta_L F(m_\phi, 0) = -\pi m_\phi^2 \sum_{\pi \neq 0} \frac{1}{L/|\pi|} e^{-m_\phi L/|\pi|}.
\]
(32)

In order to evaluate the magnitude of the finite size effects, let us consider the asymptotic form of \(\delta_L F\) [13].

\[
\delta_L F(m_\phi, \Delta) = -6\sqrt{2\pi} m_\phi^{5/2} \frac{1}{L^{3/2}} e^{-m_\phi L} + \ldots,
\]
(33)

\[
\delta_L F(m_\phi, 0) = -6\pi m_\phi^2 \frac{1}{L} e^{-m_\phi L}.
\]
(34)

The finite size effect for the baryon mass \(\delta_L m = m(L) - m(\infty)\) is expressed as

\[
\delta_L m = A \left\{ \frac{3}{8\pi} \frac{m_\pi^3}{f_0^2 (m_\pi L)} e^{-m_\pi L} \right\} + B \left\{ \frac{6\sqrt{2\pi}}{16\pi^2} \frac{m_\pi^4}{f_0^4 \Delta} \frac{1}{(m_\pi L)^{3/2}} e^{-m_\pi L} \right\} \equiv AE_1 + BE_2,
\]
(35)

where we neglect the sub-leading contributions from \(m_K\) and \(m_\eta\). An intriguing point is that \(E_1\) and \(E_2\) diminish as the pion mass decreases if the product of \(m_\pi L\) is kept fixed. The values of \(E_1\) and \(E_2\) with \(aL = 32\) at the physical point can be evaluated as

\[
aE_1 = 6.61 \times 10^{-4},
\]
(36)

\[
aE_2 = 1.43 \times 10^{-4},
\]
(37)

where we employ the following results in Ref. [1]:

\[
m_{ud}^{\text{ph}} B_0 = 0.00859(11) \, \text{[GeV]}^2,
\]
(38)

\[
a f_0 = 0.0546(39),
\]
(39)

\[
a^{-1} = 2.176(31) \, \text{[GeV]},
\]
(40)

We also use \(\Delta = m_T - m_B = 0.16\) obtained by the LO SU(3) HBChPT fit for the baryon masses. Table [X] summarizes the coefficients \(A, B\) and the relative finite size correction normalized by the baryon mass extrapolated at the physical point with the formula [1]. To evaluate the numerical value of \(A\) and \(B\) we use the phenomenological estimates \(D = 0.80\), \(F = 0.47\) and \(C = 1.5\) and assume \(H = 1\) which is comparable to our fit results in Table [VIII]. We find that the HBChPT predicts fairly small finite size corrections even at the physical point:

The magnitude is less than 1% for all the channels. We additionally remark that the \(\Omega\) baryon mass is free from the leading contribution of the finite size effects. This is another fascinating feature to choose \(m_\Omega\) as one of the physical inputs.

VI. CONCLUSION

We have investigated the chiral behavior of the octet and the decuplet baryon masses based on the SU(3) HBChPT. At LO we find reasonable fit results both for the octet and the decuplet baryon masses, though rather large strange quark contributions are observed. This point is contrary to the SU(2) linear chiral expansion where the LO strange quark contribution is well controlled around the physical \(m_s\). Inclusion of the NLO contributions makes the situation worse: The fit results are incompatible with the phenomenological estimates for the low energy constants \(D, F, C\) both for the octet and the decuplet cases. We have also applied the NLO SU(2) HBChPT to the nucleon mass. The quark mass dependence is reasonably described below \(m_{ud}^{\text{AW}} < 0.01\). The good convergence property, however, is observed only
near the chiral limit. The finite size effects predicted by the SU(3) HBChPT turn out to be fairly small: at most less than 1%. However, we should be aware of the possibility that the value \( m_{\pi} l \approx 2 \) at the physical point for the \( aL = 32 \) lattice may be beyond the applicability range of Eq. (35). Comparisons with lattice data on larger lattices should settle the issue here.

In order to avoid the difficulties associated with the chiral extrapolation of the baryonic quantities we are now carrying out a simulation directly on the physical point. This simulation is being made on a larger lattice size so that a direct study of finite size effects is possible.

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TABLE I: Octet baryon masses at the physical point. NLO results are obtained without (case 1) and with (case 2) fixing $D, F, \xi$ at the phenomenological estimate.

|      | linear | LO     | case 1 | NLO    | case 2 |
|------|--------|--------|--------|--------|--------|
| $N$  | 0.438(20) | 0.4532(75) | 0.447(15) | 0.322(32) |
| $\Lambda$ | 0.502(10) | 0.5199(72) | 0.517(11) | 0.387(82) |
| $\Sigma$ | 0.531(11) | 0.5439(77) | 0.546(12) | 0.53(16) |
| $\Xi$  | 0.5992(75) | 0.5987(80) | 0.600(14) | 0.62(18) |

TABLE II: Decuplet baryon masses at the physical point. NLO results are obtained without (case 1) and with (case 2) fixing $\xi$ at the phenomenological estimate.

|      | linear | LO     | case 1 | NLO    | case 2 |
|------|--------|--------|--------|--------|--------|
| $\Delta$ | 0.586(19) | 0.612(9) | 0.604(19) | 0.545(20) |
| $\Sigma^*$ | 0.657(15) | 0.666(9) | 0.663(14) | 0.604(17) |
| $\Xi^*$ | 0.718(12) | 0.720(9) | 0.721(14) | 0.694(17) |
| $\Omega$ | 0.769(11) | 0.774(11) | 0.777(17) | 0.815(21) |

TABLE III: Fit results with the linear formula Eq. (1) for the octet baryon masses.

|      | $N$       | $\Lambda$ | $\Sigma$ | $\Xi$        |
|------|-----------|-----------|----------|--------------|
| $\alpha$ | 0.371(50) | 0.375(28) | 0.381(33) | 0.408(21) |
| $\beta$  | 11.6(2.4) | 9.6(1.0)  | 8.0(1.0)  | 4.24(46)   |
| $\gamma$ | 1.8(1.3)  | 3.90(76)  | 4.7(9)    | 6.24(58)   |
| $\chi^2$/dof | 0.63(2.5) | 1.3(2.3)  | 0.6(1.4)  | 0.09(59)   |

TABLE IV: Same as Table III for the decuplet baryon masses.

|      | $\Delta$ | $\Sigma^*$ | $\Xi^*$ | $\Omega$ |
|------|----------|------------|---------|---------|
| $\alpha$ | 0.527(60) | 0.538(48)  | 0.548(40) | 0.552(33) |
| $\beta$  | 10.7(2.1) | 6.3(1.5)   | 3.4(82)   | 1.80(57) |
| $\gamma$ | 1.6(1.6)  | 3.8(1.3)   | 5.5(1.1)  | 7.2(9)   |
| $\chi^2$/dof | 0.4(1.4)  | 0.2(1.3)   | 0.00(21)  | 0.6(1.7) |

TABLE V: Coefficients for the octet baryons $A_{B_i}^{B_i}$ and $B_{B_i}^{B_i}$.

| $\phi$ | $\pi$ | $A_{B_i}^{B_i}$ | $\eta$ | $\pi$ | $B_{B_i}^{B_i}$ | $\eta$ |
|--------|------|----------------|-------|------|----------------|-------|
| $N$    | $\frac{3}{2}(D + F)^2$ | $\frac{1}{3}(5D^2 - 6DF + 9F^2)$ | $\frac{1}{6}(D - 3F)^2$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 |
| $\Lambda$ | $2D^2$ | $\frac{4}{3}(D^2 + 9F^2)$ | $\frac{4}{3}D^2$ | 1 | $\frac{1}{3}$ | 0 |
| $\Sigma$ | $\frac{2}{3}(D^2 + 6F^2)$ | $2(D^2 + F^2)$ | $\frac{2}{3}D^2$ | $\frac{2}{3}$ | $\frac{2}{3}$ | 0 |
| $\Xi$  | $\frac{1}{2}(D - F)^2$ | $\frac{1}{3}(5D^2 + 6DF + F^2)$ | $\frac{1}{6}(D + 3F)^2$ | $\frac{1}{3}$ | 1 | $\frac{1}{3}$ |
TABLE VI: Fit results with the SU(3) HBChPT for the octet baryon masses. NLO results are obtained with (case 2) and without (case 1) fixing $D$, $F$, $C$ at the phenomenological estimate.

|             | LO       | NLO       | phenomen. |
|-------------|----------|-----------|-----------|
|             | case 1   | case 2    |           |
| $m_B$       | 0.410(14)| 0.391(39) | −0.15(9)  |
| $\alpha_M$  | −2.262(62)| −2.62(62) | −15.3(2.0)|
| $\beta_M$   | −1.740(58)| −2.6(1.5) | −21.3(3.0)|
| $\sigma_M$  | −0.53(12) | −0.71(34) | −9.6(1.4) |
| $D$         | 0.000(16)×10^{−8}| 0.80 fixed| 0.80      |
| $F$         | 0.000(9)×10^{−8}| 0.47 fixed| 0.47      |
| $C$         | 0.36(30)  | 1.5 fixed  | 1.5       |
| $\chi^2$/dof| 1.10(63) | 1.39(77)  | 153(82)   |

TABLE VII: Coefficients for the decuplet baryons $A_{i,\phi}^T$ and $B_{i,\phi}^T$.

| $\phi$ | $\pi$ | $K$ | $\eta$ | $\pi$ | $K$ | $\eta$ | $\pi$ | $K$ | $\eta$ | $\pi$ | $K$ | $\eta$ |
|--------|-------|-----|--------|-------|-----|--------|-------|-----|--------|-------|-----|--------|
| $\Delta$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $\Sigma^*$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $\Xi^*$    | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $\Omega$  | 0      | $\frac{2}{3}$ | $\frac{2}{3}$ | 0         | $\frac{2}{3}$ | $\frac{2}{3}$ | 0         | $\frac{2}{3}$ | $\frac{2}{3}$ | 0         | $\frac{2}{3}$ | $\frac{2}{3}$ |

TABLE VIII: Fit results with the SU(3) HBChPT for the decuplet baryon masses. NLO results are obtained with (case 2) and without (case 1) fixing $D$, $F$, $C$ at the phenomenological estimate.

|             | LO       | NLO       | phenomen. |
|-------------|----------|-----------|-----------|
|             | case 1   | case 2    |           |
| $m_T$       | 0.570(16)| 0.550(43) | 0.359(43) |
| $\gamma_M$  | 2.745(80)| 3.4(1.1)  | 3.88(25)  |
| $\sigma_M$  | −0.56(15)| −0.96(75) | −2.88(40) |
| $C$         | 0.000(21)×10^{−8}| 1.5 fixed| 1.5      |
| $\mathcal{H}$ | 0.50(49) | 0.000(4)×10^{−8} | 21.5(9.5) |
| $\chi^2$/dof| 0.46(48) | 0.50(60)  | 21.5(9.5) |

TABLE IX: Fit results with SU(2) HBChPT up to NLO for the nucleon mass.

|             | NLO       |
|-------------|-----------|
| $\bar{m}_0$ | 0.258(63) |
| $m_0^2$     | 2.8(1.4)  |
| $c_1$       | −3.14(68) |
| $\chi^2$/dof| 0.2(9)    |
TABLE X: Relative finite size correction normalized by the baryon mass extrapolated at the physical point with the formula Eq. (1). LECs are chosen to be $D = 0.80$, $F = 0.47$, $C = 1.5$ and $\mathcal{H} = 1$.

|       | $A$                              | $B$     | $R = \delta \Delta m / m (%)$ |
|-------|----------------------------------|---------|--------------------------------|
| $m_N$ | $3(D + F)^2$                     | $\frac{4}{3}C^2$ | 0.89                           |
| $m_A$ | $4D^2$                           | $2C^2$  | 0.45                           |
| $m_\Sigma$ | $\frac{5}{4}(D^2 + 6F^2)$       | $\frac{15}{8}C^2$ | 0.34                           |
| $m_\Xi$ | $3(D - F)^2$                  | $\frac{7}{4}C^2$  | 0.07                           |
| $m_\Delta$ | $\frac{1}{3}H^2$               | $\frac{5}{8}C^2$  | 0.13                           |
| $m_\Sigma^*$ | $\frac{35}{96}H^2$            | $\frac{5}{8}C^2$  | 0.08                           |
| $m_\Xi^*$  | $\frac{1}{12}H^2$              | $\frac{7}{8}C^2$  | 0.03                           |
| $m_\Omega$ | 0                                | 0        | 0                              |
FIG. 1: Fit results with the linear formula Eq. (1) for the octet baryon masses. Experimental values are given in lattice units with $a^{-1} = 2.176$ GeV in Ref. [1].
FIG. 2: Fit results with the linear formula Eq. (1) for the decuplet baryon masses. Experimental values are given in lattice units with $a^{-1} = 2.176$ GeV in Ref. [1].
FIG. 3: Convergence behavior for the octet baryon masses with the linear formula Eq. (1). $m_a$ is fixed at the measured value at $(\kappa_{ud}, \kappa_s) = (0.13754, 0.13640)$. 
FIG. 4: Convergence behavior for the decuplet baryon masses with the linear formula Eq. (1). $m_a$ is fixed at the measured value at $(\kappa_{ud}, \kappa_s) = (0.13754, 0.13640)$. 
FIG. 5: Fit results with the SU(3) HBChPT up to LO for the octet baryon masses. Experimental values are given in lattice units with $a^{-1} = 2.176$ GeV in Ref. [1].
FIG. 6: Convergence behavior for the octet masses with the SU(3) HBChPT up to LO. $m_\pi$ is fixed with the measured value at $(\kappa_{ud}, \kappa_s) = (0.13754, 0.13640)$. 
FIG. 7: Fit results with the SU(3) HBChPT up to NLO for the octet baryon masses. Experimental values are given in lattice units with $a^{-1} = 2.176$ GeV in Ref. [1].
FIG. 8: Fit results with the SU(3) HBChPT up to NLO for the octet baryon masses. $D, F, C$ are fixed at the phenomenological estimates. Experimental values are given in lattice units with $a^{-1} = 2.176$ GeV in Ref. [1].
FIG. 9: Convergence behavior for the octet masses with the SU(3) HBChPT up to NLO. $D, F, C$ are fixed at the phenomenological estimates. $m_s$ is fixed with the measured value at $(\kappa_{ud}, \kappa_s) = (0.13754, 0.13640)$. 
FIG. 10: Fit results with the SU(3) HBChPT up to LO for the decuplet baryon masses. Experimental values are given in lattice units with $a^{-1} = 2.176$ GeV in Ref. [1].
FIG. 11: Convergence behavior for the decuplet masses with the SU(3) HBChPT up to LO. $m_s$ is fixed with the measured value at $(\kappa_{ud}, \kappa_s) = (0.13754, 0.13640)$. 
FIG. 12: Fit results with the SU(3) HBChPT up to NLO for the decuplet baryon masses. Experimental values are given in lattice units with $a^{-1} = 2.176$ GeV in Ref. [1].
FIG. 13: Fit results with the SU(3) HBChPT up to NLO for the decuplet baryon masses. $C$ is fixed at the phenomenological estimates. Experimental values are given in lattice units with $\alpha^{-1} = 2.176$ GeV in Ref. [1].
FIG. 14: Convergence behavior for the decuplet masses with the SU(3) HBChPT up to NLO. C is fixed at the phenomenological estimates. $m_s$ is fixed with the measured value at $(\kappa_{ud}, \kappa_s) = (0.13754, 0.13640)$. 
FIG. 15: Fit results with the SU(2) HBChPT up to NLO for the nucleon mass (left) and comparison of LO and NLO contributions (right). Experimental values are given in lattice units with $a^{-1} = 2.176$ GeV in Ref. [1].