Stable Iterative Methods for Inversion of Magnetotelluric and Transient Electromagnetic Data

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Abstract. The joint inversion of geophysical data can reduce the ambiguity of model parameters. Joint Inversion of Magnetotelluric (MT) and Transient Electromagnetic (TEM) data was performed to get detailed information of subsurface structure. Whereas TEM data is sensitive to describe in the shallow structure, while the deeper structure is related to MT data. We derived the joint inversion scheme from the second order of Marquardt algorithm using singular value decomposition (SVD). The detailed analyses of the model parameters are performed by damping factors, V-matrix, damped error bounds and importances value. The joint models parameters with damping factor and importances value above 0.9 indicate that the associated eigen parameters are well-resolved.

1. Introduction
Modeling and inversion of geophysical data are crucial for the meaningful interpretation of the processed field response. For this purpose, a sequence of steps based on well-defined physio-mathematical formulations is implemented. These steps include the computations of responses for the assumed geoelectrical model referred as modeling in geophysical literature. Solutions to geophysical inverse problems are generally non-unique. It is usual to reduce the non-uniqueness by restricting the complexity of the Earth models. The restricted class of the earth models consists of horizontally layered, isotropic media, with constant resistivity in each layer.

The analysis of the problem applies to the geophysical inverse problem in which the partial derivatives of the data respect to the model parameters can be calculated which are classified as Important, Unimportant, and Irrelevant. The classification is based on the Singular Value Decomposition [1] of the Jacobian matrix, which also allows the visualization of which combination of earth parameters was most effectively resolved and allocated the confidence intervals for the parameters. Many inverse problems in geophysics are ill-posed (unstable) that is small changes in the data can lead to large changes in the solution model. The unstable nature and consequent numerical instability of the inverse problem are largely contained in Unimportant and Irrelevant parameters.

The detailed analyses of the model parameters are performed by damping factors, V-matrix, damped error bounds, and importances. The damped error bounds measure the expected variation in the well-resolved (Important) parameters, in response to small variation in the data. The bounds attributed to the Unimportant parameters are damped, and only measured the variation due to their correlation with the
well-resolved parameters. The greatest expected variations occur for parameters which are near the threshold or dividing line between the Important and Unimportant parameters.

In this paper, the methods are applied to synthetic data sets of Radiomagnetotelluric (RMT) and Central Loop Transient Electromagnetic (TEM). Joint inversion can increase the numbers of Important parameters of a model to include some which the methods cannot resolve separately. Whereas RMT data is very sensitive to changes in the upper few meters, the TEM data is more sensitive to deeper variations. It is essential to note that this effect cannot be achieved merely by increasing the number or accuracy of data values of a single method. Joint inversion is a means of eliminating some defects inherent in the individual methods.

2. The analysis and solution of the inverse problem

2.1. General notation and the Jacobian matrix

The forward problem generates a set of model data for each setting of restricted earth model can generally denoted by

\[ \mathbf{d} = g(\mathbf{x}) \]  

(1)

With \( \mathbf{d} \) is data that can be apparent resistivity values (\( \rho_a \)) or phase (\( \Phi \)) and \( g(\mathbf{x}) \) the forward operator of model parameters (\( \mathbf{x} \)) while in our examples \( d_i \) is the induced voltage at the \( i \)'th period for the TEM data; and the apparent resistivity or phase at the \( i \)'th period for MT data. In 1D case, models can be defined by the subsurface layer resistivity and thickness. Jupp and Vozoff [1] suggest the value of a model in logarithmic value to avoid a negative value in the inversion result.

The inverse problem determines values of \( \mathbf{x} \) such that \( g(\mathbf{x}) \) matches \( \mathbf{d} \) in some sense, which in this paper is the minimum of the familiar Root Mean Squared (RMS) relative error between model and data,

\[ F(x) = \left( \frac{1}{N} \sum_{i=1}^{N} \left( \frac{d_i - g(x_i)}{d_i} \right)^2 \right)^{1/2} \]  

(2)

The iterative method can improve a current model until the error measure is small, and the parameters are stable on reasonable changes in the model. When the partial derivates of the model data on their parameters can be obtained accurately and relatively easily, the Gauss method [1] is an attractive iterative inverse method. The Gauss method iteratively changes the current \( \mathbf{x} \) by an amount \( \delta \mathbf{x} \) calculated by solving the linear least-squares problem. We expand \( g(\mathbf{x}) \) about \( \mathbf{x} \) in a Taylor expansion to derive it

\[ g(\mathbf{x} + \delta \mathbf{x}) = g(\mathbf{x}) + J \delta \mathbf{x} \]  

(3)

where \( J \) is the Jacobian matrix of the vector function \( g(\mathbf{x}) \) which is a by-product of linearizing a non-linear problem. Its elements are the partial derivatives of the calculated data on the model parameters.

\[ J_{ij} = \frac{\partial g_i}{\partial x_j} |_{i=1,M} \]  

(4)

The Jacobian matrix is also called parameter sensitivity matrix that explains its importance in the course inversion. Statistically, sensitivity values show whether the layer parameters are seen individually in the measured data or not. They qualitatively display which parameter or combinations of parameters are resolved by the data. The model parameter is only well-resolved by the measurements if its sensitivity matrix has large entries in the corresponding column. On the other hand, the normalized Jacobian values or relative sensitivities give an idea of each data point’s sensitivity to a change of an individual parameter [1].
We drop the weighting factor \( W \) which are diagonal matrices corresponding to the mean relative error measure, \( W = (1 / d_i) |_{i=1,N} \). When \( f^T W f \) is non-singular, we find that

\[
\delta x = (f^T W f)^{-1} W (d - g(x))
\]

The MT and TEM data recorded from field contain noise and hence, are inconsistent. Therefore, the inverse problem is ill-conditioned, and solution is normally ill-posed (unstable). Hence, regularized inversion techniques are used to obtain the solution of the inverse problem [1]. In this paper, we have used Marquardt-Levenberg scheme for the inversion and also Jupp and Vozoff approach for the joint inversion.

2.2. The singular values of \( J \)

Singular Value Decomposition (SVD) method has been used for the detailed analysis of the inverse problem. Stabilized solution of the ill-posed inverse problem is obtained by truncating the small eigenvalues of the matrix. The system matrix is decomposed into a product of three matrices as

\[
J = USV^T
\]

Where \( U \) is a NxM matrix containing data space eigenvectors of \( J \) in its columns, \( V \) is a MxM matrix which contains the parameter space eigenvectors, and \( S \) is a MxM diagonal matrix with eigenvalues \( (s_i) \) as its diagonal elements. The \( s_i \)'s are the non-negative square root of eigenvalues of \( f^T J f \), which are arranged in decreasing order as

\[
s_1 \geq s_2 \geq s_3 \cdots s_m \geq 0
\]

By retaining only significant eigenvalues, a generalized stable inverse solution can be formulated. The damping factors act more like thresholds at each iteration. They are generally measures the ‘importances’ of the corresponding layer parameters and vary between 0 and 1 to represent the minimal and maximal parameter resolution respectively. The SVD approach [1] implemented in EMUPLUS code has been applied for detailed analysis of RMT and TEM data in the following sections.

2.3. Marquardt-Levenberg inversion scheme

Levenberg-Marquardt Algorithm is an algorithm that applying the minimization of model perturbation to the Gauss-Newton solution. It can be done by minimizing the objective function \( F \)

\[
F = (d - g(x_0 + \delta x)) + \lambda \| \delta x \|^2
\]

The solution now becomes

\[
\delta x = [J^T W J + \lambda I]^{-1} J^T W (d - g(x))
\]

With the parameter \( \lambda \) is showing of the effect of the model perturbation. Small \( \lambda \) value will make the last equation becomes equal to Gauss – Newton solution in function \( F \). The parameter \( \lambda \) is initialized with a large initial value. If the solution after iteration is worse than the previous iteration \( \lambda \) value is increased, otherwise it is reduced. To ensure the calculation stability the last equation is modified to

\[
\delta x = [J^T W J + \lambda \text{diag}(J^T W J)]^{-1} J^T W (d - g(x))
\]

The method effectively controls the instability caused by the existence of zero or very small eigen value of the system matrix.

2.4. Parameter classification and ‘error bounds’

Eigenstructure (singular values) of \( J \) can be used to classify the parameters as Irrelevant, Unimportant, and Important [1].

i) Irrelevant parameters have no influence on the model data. They correspond to layers which are the range of the measurements, or as in the case of thin resistive layers, to parameter combinations that cannot be resolved from the data.
ii) Unimportant parameters have the only small influence on model data. Inversely, large changes in these parameters can occur for only marginal improvement in fit to the actual data. For this reason, they must be either neglected, or altered only marginally during the inversion process.

iii) Important parameters correspond to the well-resolved, and often gross features that are well represented in the data.

2.5. Joint Inversion

To have a good fitting and avoid ambiguity that is inherent to the individual methods, we can use joint inversion. Joint inversion is a method for inverting two or more data sets from different geophysical methods and using the same model. Application of joint inversion was first introduced by Vozoff and Jupp [1] and first applied by Meju [2] for EM sounding.

The Jacobian matrix $J$ determines how the model should be altered to improve the fitting:

$$J_{ij} = w_i \frac{\partial g_i}{\partial x_j}$$

(11)

where the weight $w_i$ and N-layer earth model of resistivity and thickness are given as

$$\begin{cases} \rho_i \geq 0 & i = 1, N+1 \\
 h_i \geq 0 & i = 1, N 
\end{cases}$$

(12)

The elements of the matrix $J$ can be written as

$$J_{ij} = w_i \frac{\partial g_i}{\partial x_j} = \frac{1}{g_i \log \rho_j} \frac{\partial g_i}{\partial \log \rho_j} \text{ or } \frac{1}{g_i \log h_j} \frac{\partial g_i}{\partial \log h_j}$$

(13)

In this scheme, the Jacobian matrix is made scale free. Since the relative error will be proportional to the model, the two kinds of the data will equally influence the correction that improves the current model. Theoretically, each of them can compensate for the weaknesses of the other, and therefore the overall benefit will be a reduction of ambiguity [3].

3. Applications to TEM and RMT synthetic data

3.1. Synthetic model generated

Synthetic data sets were generated for RMT at the frequency range $10^4 - 10^6$ Hz and TEM data were generated for time range $10^{-5} - 1$ seconds. To make synthetic data close to the field conditions, 20 percent random Gaussian noise were added to each data. For each data set, Marquardt inversion was carried out. The three layers model with conductive layer sandwiched between two resistive layers ($\rho_1 > \rho_2 < \rho_3$) was used to generate the synthetic data. The results of single inversion are more dependent on the starting model, so then we choose the best influence of the starting model that have been observed before. Joint inversion effectively combines the capabilities of each method.

3.2. Detail analysis for the inversion

The damping factors, the V-matrix, calculated error bounds and parameters importance can be used to judge the performance of the inversion. By limits of damping factors to judge the resolution of the model parameter suggested, EP1, EP2, EP3 and EP4 are well-resolved; EP5 is unresolved for single RMT inversion and EP5 is shaky for single TEM inversion. However, in joint inversion case EP1, EP2, EP3, EP4, and EP5 are well-resolved. The structure of V-matrix is convenient as significant parameter combinations in resistivity are often products and ratios. The factors $\rho_1, h_1, \rho_1 h_1, h_i/\rho_i$ and $\rho_i+1/\rho_i$ may all appear as transformed parameters and interpret the V-matrix regarding these combinations.
The V-matrix analysis of single RMT inversion shows that EP1 has relatively large entries in the first rows, it consists of the resistivity product of the top layer. EP2 consists of the thickness product of the first layer. Since EP1 and EP2 have damping factors of 1.0, they are well-resolved. EP3 consists of resistivity of the second layer which is well-resolved since EP3 has damping factor more than 0.9. EP4 and EP5 have very low damping factor, consequently, the thickness of second layer and the basement is not resolved. Error bounds and parameters importances for single RMT inversion also indicates each parameter as the V-matrix and damping factors analysis as they are. The first layer parameters are well-resolved in comparison to the second and third layer parameters.

Similar analysis has been performed for the TEM inversion. EP1 is the third layer resistivity and has damping factor 1.0 while it means the basement is well-resolved. EP2 and EP3 show that the thickness and resistivity of the first layer, and are well-resolved since they have damping factors more than 0.9. EP4 consists of the third layer resistivity and the second layer thickness ratio and EP5 consists of the resistivity thickness ratio of the second layer. Since EP4 has damping factor more than 0.9 so the resistivity of the third layer is completely well-resolved that associated with EP1 but EP5 has damping factor between 0.8 and 0.9 so the thickness of the second layer is not resolved as well or shaky, then exactly for the resistivity of the second layer. Hence, the thickness of the second layer cannot be estimated independently. Error bounds and parameters importances for single TEM inversion indicates that the basement is well-resolved in comparison to the first and the second layer parameters.

The V-matrix analysis for joint inversion shows the better resolution. EP1, EP2, EP3, EP4, and EP5 have damping factors 1.0 respectively while it means they are well-resolved at all. EP1 shows the resolution of the second layer parameters, EP2, EP3, and EP4 show the resolution of the first layer parameters. EP5 is the basement resistivity which is resolved as well. Similar results are also indicated in error bounds and importances.

**Table 3.1** Synthetic data and single and joint inversion of RMT and TEM results.

| Parameter | True Model | Starting Model | Single RMT inverted model | Single TEM inverted model | Joint Inversion model |
|-----------|------------|----------------|--------------------------|--------------------------|----------------------|
| $\rho_1$ (Ωm) | 100 | 400 | 142.6958 | 186.129 | 117.8884 |
| $\rho_2$ (Ωm) | 10 | 30 | 43.7311 | 29.0225 | 11.4097 |
| $\rho_3$ (Ωm) | 1000 | 800 | 737.1805 | 1152.911 | 993.5697 |
| $h_1$ (m) | 30 | 20 | 30.0475 | 31.3023 | 30.9257 |
| $h_2$ (m) | 10 | 20 | 31.1785 | 31.5486 | 30.6399 |
Table 3.2 V matrix and damping factor for single RMT inversion.

|        | EP1  | EP2  | EP3  | EP4  | EP5  |
|--------|------|------|------|------|------|
| PAR1   | 0.6  | 0.3  | 0.20 | 0.10 | 0.28 |
| PAR2   | -    | 0.0  | 0.83 | -    | 0.16 |
| PAR3   | -    | 0.0  | 0.00 | -    | 0.88 |
| PAR4   | 0.0  | 0.8  | 0.17 | -    | 0.18 |
| PAR5   | 0.0  | 0.1  | 0.42 | 0.87 | 0.05 |
| Normalized singular value | 1.0 | 0.2 | 6.21 | 2.42 | 1.71 |
| Damping Factor | 1.0 | 1.0 | 0.99 | 0.03 | 0.00 |
| Parameter Combination | $\rho_1$, $h_1$, $1/\rho_2$, $1/h_2$, $\rho_3$ |

Table 3.3 Error bounds and importance for single RMT inversion.

| Layer resistivities – 68 percent confident interval (damped) |
|-------------|-----------------|-----------------|-----------------|
| RO(I)       | BOUND(1)        | BOUND(2)        | IMPORTANCE     |
| 1           | 142.695         | 121.102         | 0.8135         |
| 2           | 43.7311         | 25.1377         | 0.8911         |
| 3           | 737.180         | 637.338         | 0.1176         |

| Layer thickness – 68 percent confident interval (damped) |
|-------------|-----------------|-----------------|-----------------|
| HI          | BOUND(1)        | BOUND(2)        | IMPORTANCE     |
| 1           | 30.0475         | 20.6738         | 0.9275         |
| 2           | 31.1785         | 11.0028         | 0.1366         |

Table 3.4 V matrix and damping factor for single TEM inversion.

|        | EP1  | EP2  | EP3  | EP4  | EP5  |
|--------|------|------|------|------|------|
| PAR1   | 0.42 | -0.042| 0.81 | -0.43 | 0.188|
| PAR2   | 0.16 | -0.450| 0.14 | 0.27 | 0.768|
| PAR3   | 0.66 | -0.248| 0.41 | 0.53 | 0.176|
| PAR4   | 0.37 | 0.657| -0.28 | 0.12 | 0.008|
| PAR5   | 0.43 | 0.167| 0.25 | 0.48 | 0.009|
| Normalized singular value | 1.00 | 0.718| 0.43 | 0.28 | 0.232|
| Damping Factor | 1.00 | 1.000| 0.95 | 0.90 | 0.875|
| Parameter Combination | $\rho_3$, $1/h_1$, $1/\rho_1$, $\rho_3/h_2$, $\rho_2/h_2$ |

Table 3.5 V matrix and damping factor for single TEM inversion.

|        | EP1  | EP2  | EP3  | EP4  | EP5  |
|--------|------|------|------|------|------|
| PAR1   | 0.42 | -0.042| 0.81 | -0.43 | 0.188|
| PAR2   | 0.16 | -0.450| 0.14 | 0.27 | 0.768|
| PAR3   | 0.66 | -0.248| 0.41 | 0.53 | 0.176|
| PAR4   | 0.37 | 0.657| -0.28 | 0.12 | 0.008|
| PAR5   | 0.43 | 0.167| 0.25 | 0.48 | 0.009|
| Normalized singular value | 1.00 | 0.718| 0.43 | 0.28 | 0.232|
| Damping Factor | 1.00 | 1.000| 0.95 | 0.90 | 0.875|
| Parameter Combination | $\rho_3$, $1/h_1$, $1/\rho_1$, $\rho_3/h_2$, $\rho_2/h_2$ |
### Table 3.5: Error bounds and importance for single TEM inversion.

| Layer resistivities – 68 percent confident interval (damped) | RO(I) | BOUND(1) | BOUND(2) | IMPORTANCE |
|-------------------------------------------------------------|-------|----------|----------|------------|
| 1               | 186.129 | 177.867 | 194.7747 | 0.8746     |
| 2               | 29.0225  | 28.2094 | 29.8591  | 0.7885     |
| 3               | 1152.91  | 1118.295| 1188.599 | 0.9511     |

| Layer thickness – 68 percent confident interval (damped) | H(I)   | BOUND(1) | BOUND(2) | IMPORTANCE |
|----------------------------------------------------------|--------|----------|----------|------------|
| 1               | 31.3023 | 28.7802  | 31.9048  | 0.9198     |
| 2               | 26.4486  | 25.3215  | 27.6259  | 0.8712     |

### Table 3.6: V matrix and damping factor for joint inversion.

| EP1 | EP2 | EP3 | EP4 | EP5 |
|-----|-----|-----|-----|-----|
| PAR1| 0.2 | 0.7 | 0.6 | 0.03|
| PAR2| 0.6 | 0.1 | 0.52| 0.69 |
| PAR3| 0.1 | 0.0 | 0.30| 0.39 |
| PAR4| 0.0 | 0.2 | 0.55| 0.55 |
| PAR5| 0.6 | 0.1 | 0.04| 0.90 |

| Normalized singular value | 1.0 | 0.7 | 0.5 | 0.2 | 9.84E |
|----------------------------|-----|-----|-----|-----|------|
| Damping Factor            | 1.0 | 1.0 | 1.0 | 1.0 | 1.00 |

| Parameter Combination | \( \rho_2/\rho_1 \) | \( 1/\rho_1 \) | \( 1/h_1 \) | \( 1/p_1 \) | \( \rho_3 \) |

### Table 3.7: Error bounds and importance for joint inversion.

| Layer resistivities – 68 percent confident interval (damped) | RO(I) | BOUND(1) | BOUND(2) | IMPORTANCE |
|-------------------------------------------------------------|-------|----------|----------|------------|
| 1               | 117.888 | 103.3135 | 134.5194 | 1.0000     |
| 2               | 11.4097  | 3.1497   | 41.3314  | 0.9923     |
| 3               | 993.569  | 548.668  | 1799.231 | 0.9995     |

| Layer thickness – 68 percent confident interval (damped) | H(I)   | BOUND(1) | BOUND(2) | IMPORTANCE |
|----------------------------------------------------------|--------|----------|----------|------------|
| 1               | 30.9257 | 25.7304  | 39.6127  | 0.9998     |
| 2               | 11.6399  | 3.1284   | 43.3084  | 0.9919     |
3.3. Conclusion

From the above example, which has been demonstrated that joint inversion produces the better results in the equivalence problem in comparison with single inversion of RMT and TEM. However, if the data contain no information about the model parameters, the no inversion scheme is capable for delineation of the parameters.

Starting from the synthetic model with $\mu = 0.01$ and with data perturbed to the 20 per cent relative error level, we obtained the final model of Fig. 3.1. The residual error was 5 per cent, which was mainly the original error, but shows the model fits some of the original error at the expense of its truth. Table 3.2 to 3.7 records the singular values, the damping factors, the V matrix, error bounds, and parameters importances for single and joint inversion of RMT and TEM respectively, all of which have been interpreted as described in Section 3.2. This method provides a basis for its extension to the more realistic situation.

References

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