THE DIRICHLET PROBLEM FOR A CLASS OF HESSIAN QUOTIENT EQUATIONS IN LORENTZ-MINKOWSKI SPACE $\mathbb{R}^{n+1}_1$

YAO GAO, YANLING GAO, JING MAO*

ABSTRACT. In this paper, under suitable settings, we can obtain the existence and uniqueness of solutions to a class of Hessian quotient equations with Dirichlet boundary condition in Lorentz-Minkowski space $\mathbb{R}^{n+1}_1$, which can be seen as a prescribed curvature problem and a continuous work of [14].

Keywords: Spacelike hypersurfaces, Lorentz-Minkowski space, Hessian quotient equations, curvature estimates, Dirichlet boundary condition.

MSC 2020: 35J60, 35J65, 53C50.

1. Introduction

Throughout this paper, let $\mathbb{R}^{n+1}_1$ be the $(n+1)$-dimensional $(n \geq 2)$ Lorentz-Minkowski space with the following Lorentzian metric
\[ \langle \cdot, \cdot \rangle_L = dx_1^2 + dx_2^2 + \cdots + dx_n^2 - dx_{n+1}^2. \]

In fact, $\mathbb{R}^{n+1}_1$ is an $(n+1)$-dimensional Lorentz manifold with index 1. Denote by $\mathcal{H}^n(1) = \{(x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1}_1 | x_1^2 + x_2^2 + \cdots + x_n^2 - x_{n+1}^2 = -1 \text{ and } x_{n+1} > 0\}$, which is exactly the hyperbolic plane of center $(0,0,\ldots,0)$ (i.e., the origin of $\mathbb{R}^{n+1}_1$) and radius 1 in $\mathbb{R}^{n+1}_1$.

Assume that
\[ (1.1) \quad \mathcal{G} := \{(x,u(x)) | x \in M^n \subset \mathcal{H}^n(1)\} \]
is a spacelike graphic hypersurfaces defined on some bounded piece $M^n \subset \mathcal{H}^n(1)$, with the boundary $\partial M^n$, of the hyperbolic plane $\mathcal{H}^n(1)$, where $\sup_{M^n} \frac{Du}{u} \leq \rho < 1$. Let $x$ be a point on $\mathcal{H}^n(1)$ which is described by local coordinates $\xi^1, \ldots, \xi^n$, that is, $x = x(\xi^1, \ldots, \xi^n)$. By the abuse of notations, let $\partial_i$ be the corresponding coordinate vector fields on $\mathcal{H}^n(1)$ and $
abla$ the Levi-Civita connection of $\mathcal{G}$ w.r.t. the metric $g_{\mathcal{H}^n(1)} = \sigma_{ij} = g_{\mathcal{H}^n(1)}(\partial_i, \partial_j)$ be the induced Riemannian metric on $\mathcal{H}^n(1)$. Of course, $\{\sigma_{ij}\}_{i,j=1,2,\ldots,n}$ is also the metric on $M^n \subset \mathcal{H}^n(1)$. Denote by $u_i := D_i u$, $u_{ij} := D_j D_i u$, and $u_{ijk} := D_k D_j D_i u$ the covariant derivatives of $u$ w.r.t. the metric $g_{\mathcal{H}^n(1)}$, where $D$ is the covariant connection on $\mathcal{H}^n(1)$. Let $\nabla$ be the Levi-Civita connection of $\mathcal{G}$ w.r.t. the metric $g := u^2 g_{\mathcal{H}^n(1)} - dr^2$ induced from the Lorentzian metric $\langle \cdot, \cdot \rangle_L$ of $\mathbb{R}^{n+1}_1$. Clearly, the tangent vectors of $\mathcal{G}$ are given by
\[ X_i = (1,Du) = \partial_i + u_i \partial_r, \quad i = 1, 2, \ldots, n. \]

The induced metric $g$ on $\mathcal{G}$ has the form
\[ g_{ij} = \langle X_i, X_j \rangle_L = u^2 \sigma_{ij} - u_i u_j, \]

* Corresponding author.

1 Clearly, for accuracy, here $D_i u$ should be $D_{\partial_i} u$. In the sequel, without confusion and if needed, we prefer to simplify covariant derivatives like this. In this setting, $u_{ij} := D_j D_i u$, $u_{ijk} := D_k D_j D_i u$ mean $u_{ij} = D_{\partial_j} D_{\partial_i} u$ and $u_{ijk} = D_{\partial_k} D_{\partial_j} D_{\partial_i} u$, respectively.
and its inverse is given by

\[ g^{ij} = \frac{1}{u^2} \left( \sigma^{ij} + \frac{u^i u^j}{u^2} \right). \]

Then the future-directed timelike unit normal of \( G \) is given by

\[ \nu = \frac{1}{v} \left( \partial_r + \frac{1}{u^2} u^i \partial_i \right), \]

where \( u^j := \sigma^{ij} u_i \) and \( v := \sqrt{1 - u^{-2} |Du|^2} \) with \( Du \) the gradient of \( u \). Of course, in this paper we use the Einstein summation convention – repeated superscripts and subscripts should be made summation from 1 to \( n \). The second fundamental form of \( G \) is

\[ h_{ij} = -\langle \nabla X_i X_j, \nu \rangle_L = \frac{1}{v} \left( u_{ij} + u \sigma_{ij} - \frac{2}{u} u_i u_j \right), \]

with \( \nabla \) the covariant connection in \( \mathbb{R}^{n+1} \). Denote by \( \lambda_1, \lambda_2, \ldots, \lambda_n \) the principal curvatures of \( G \), which are actually the eigenvalues of the matrix \( (h_{ij})_{n \times n} \) w.r.t. the metric \( g \). The so-called \( k \)-th Weingarten curvature at \( X = (x, u(x)) \in G \) is defined as

\[ \sigma_k(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}. \]

We also need the following conception:

**Definition 1.1.** For \( 1 \leq k \leq n \), let \( \Gamma_k \) be a cone in \( \mathbb{R}^n \) determined by

\[ \Gamma_k = \{ \lambda \in \mathbb{R}^n | \sigma_j(\lambda) > 0, \ j = 1, 2, \ldots, k \}. \]

A smooth spacelike graphic hypersurface \( G \subset \mathbb{R}^{n+1}_1 \) is called \( k \)-admissible if at every point \( X \in G \), \((\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Gamma_k \).

**Remark 1.1.** We kindly refer readers to [12] Remark 1.1 for a brief introduction to the \( \sigma_k \) operator on \( G \subset \mathbb{R}^{n+1}_1 \) and our previous works (see, e.g., [7, 8, 9, 10, 11]) related to this operator.

In this paper, we investigate the existence and uniqueness of solutions for a class of nonlinear partial differential equations (PDEs for short) given as follows

\[ \begin{cases} \sigma_k = \psi(x, u, \vartheta), & x \in M^n \subset \mathcal{H}^n(1) \subset \mathbb{R}^{n+1}_1, \ 2 \leq k \leq n, \ 0 \leq l \leq k - 2, \\ u = \varphi, & x \in \partial M^n, \end{cases} \tag{1.4} \]

where \( \psi \), depending on \( X := (x, u), \ \vartheta := -\langle X, \nu \rangle_L \), and \( \varphi \) are functions defined on \( M^n \).

**Remark 1.2.** (1) Obviously, when \( l = 0 \), the LHS of the first equation in \( (1.4) \) degenerates into \( \sigma_k(\lambda_1, \lambda_2, \ldots, \lambda_n) \) exactly – the \( k \)-th Weingarten curvature of \( G \), and then the system \( (1.4) \) becomes the prescribed curvature problem (PCP for short) studied in [12] with \( k = 2, 3, \ldots, n \). In this sense, we also say that \( (1.4) \) is a PCP and should be a continuity of our previous work [12].

(2) In fact, our experience in [12] tells that if \( M^n \) in the PCP \( (1.4) \) was replaced by bounded domain \( \Omega \subset \mathbb{R}^n \subset \mathbb{R}^{n+1}_1 \), then similar existence and uniqueness conclusion (as in Theorem 1.4 below) for the new PCP could be expected provided suitable assumptions made to the positive function \( \psi \) and the Dirichlet boundary condition (DBC for short) \( \varphi \). We prefer to leave this problem as an exercise for readers since the possible a priori estimates can be carried out by using the ones (we have shown here) for reference.

(3) The PCPs (with or without boundary condition) in Euclidean space or even more general

\[ \text{case } k = 1 \] has also been discussed in [12].
Riemannian manifolds were extensively studied – see, e.g., [4, 5, 17, 21] and the references therein for details. Affected by the study of Geometry of Submanifolds, it is natural to consider PCPs in the pseudo-Riemannian context. In fact, many other important results on PCPs in pseudo-Riemannian manifolds have been obtained. For instance, in the Lorentz-Minkowski space or general Lorentz manifolds, Bartnik [1], Bartnik-Simon [2], Gerhardt [13, 14] solved the Dirichlet problem for the prescribed mean curvature equation, Delanoë [6], Guan [16] considered the prescribed Gauss-Kronecker curvature equation with DBC, while Bayard [3], Gerhardt [15], Urbas [23] worked for the prescribed scalar curvature equation. From this brief introduction, one can see that when considering the PCPs (with or without boundary condition) in either Riemannian manifolds or pseudo-Riemannian manifolds, equations involving the \( \sigma_k(\lambda(A)) \) were considered a lot – this is natural, since in general \( \sigma_k(\lambda(A)) \) corresponds to the \( k \)-th Weingarten curvature of submanifolds in some geometric space considered. Our research experience in curvature flows inspires us that maybe it is possible to improve our previous existence and uniqueness result [12, Theorem 1.4] to more general setting – replacing \( \sigma_k(\lambda(A)) \) by the \( (k,l) \)-Hessian quotient \( \frac{\sigma_k(\lambda(A))}{\sigma_l(\lambda(A))} \) of \( \lambda(A) \), with \( 2 \leq k \leq n, 0 \leq l \leq k-2 \).

However, in the Riemannian context, Guan-Ren-Wang [18] showed that \( C^2 \)-estimates fail for the curvature equation of the form
\[
\frac{\sigma_k(\lambda(A))}{\sigma_l(\lambda(A))} = f(X, \nu(X)), \quad \forall X \in N^n \subset \mathbb{R}^{n+1},
\]
where \( \nu(X) \) therein denotes the outward unit normal vector of closed hypersurface \( N^n \) (in the Euclidean \( (n + 1) \)-space \( \mathbb{R}^{n+1} \)) at \( X \). This fact brings us negative attitude to consider the PCP (1.4), and one might worries about that maybe same story happens in the pseudo-Riemannian context. But luckily, we almost overcome this difficulty and successfully obtain the existence and uniqueness of solutions to the PCP (1.4) for some special \( k \) and \( l \) – see Theorem 1.4 below for details.

For the PCP (1.4), first, we can get the following curvature maximum estimate:

**Theorem 1.2.** Suppose that \( u \in C^4(M^n) \cap C^2(M^n) \) is a spacelike, \( k \)-admissible solution of the PCP (1.4), \( 0 < \psi \in C^\infty(M^n) \) and that \( \psi^{\frac{1}{k-l}}(X, \vartheta) \) is convex in \( \vartheta \) and satisfies
\[
\frac{\partial \psi^{\frac{1}{k-l}}(X, \vartheta)}{\partial \vartheta} \cdot \vartheta \geq \psi^{\frac{1}{k-l}}(X, \vartheta) \quad \text{for fixed } X \in \mathcal{G}.
\]
Then the second fundamental form \( A \) of \( \mathcal{G} \) satisfies
\[
\sup_{M^n} ||A|| \leq C \left( 1 + \sup_{\partial M^n} ||A|| \right),
\]
where \( C \) depends only on \( n, ||\varphi||_{C^1(M^n)}, ||\psi||_{C^2(M^n \times \left[ \inf_{\partial M^n} u, \sup_{\partial M^n} u \right] \times \mathbb{R})} \).

**Remark 1.3.** It is not hard to find some \( \psi \) satisfying assumptions in Theorem 1.2. For instance, (i) \( \psi(x, u, \vartheta) = \vartheta^p h(x, u) \) for \( p \geq k - l \); (ii) \( \psi(x, u, \vartheta) = e^{\vartheta^p h(x, u)} \) for \( p \geq k - l \).

The following interior curvature estimate can also be obtained:

\[ \sigma_k(\lambda(\cdot)) \text{ denotes the } k \text{-th elementary symmetric function of eigenvalues of a given tensor – the second fundamental form } A. \]
Theorem 1.3. Suppose that $u \in C^4(M^n) \cap C^2(\bar{M}^n)$ is a spacelike, $k$-admissible solution of the PCP \[(1.4)\], $0 < \psi \in C^\infty(\bar{M}^n)$ and that $\psi^\perp(X, \vartheta)$ is convex in $\vartheta$ and satisfies
\[(1.7)\]
\[
\frac{\partial\psi^\perp(X, \vartheta)}{\partial \vartheta} \cdot \vartheta > \psi^\perp(X, \vartheta) \quad \text{for fixed } X \in G.
\]
Furthermore, suppose that $M^n \subset \mathcal{H}^n(1)$ is $C^2$ and uniformly convex, and that $\varphi$ is spacelike and affine. If $u \in C^4(M^n)$ is a spacelike, $k$-admissible solution of the PCP \[(1.4)\], then
\[
\sup_{\bar{M}^n} |A| \leq C\left(\bar{M}^n\right)
\]
for any $\bar{M}^n \subset \subset M^n$, where $C\left(\bar{M}^n\right)$ depends only on $n, \zeta, M^n, \text{dist}(\bar{M}^n, \partial M^n), \|\varphi\|_{C^1(\bar{M}^n)}$, and $\|\psi\|_{C^2\left(\bar{M}^n \times \left[\inf_{\partial \bar{M}^n} u, \sup_{\partial \bar{M}^n} u\right] \times \mathbb{R}\right)}$.

Combining the above curvature estimates and the boundary $C^2$-estimates proven in the sequel, we can get the following result:

Theorem 1.4. For $k = 2, l = 0$, suppose that $M^n$ is a smooth bounded domain of $\mathcal{H}^n(1)$ and is strictly convex, while $\psi$ is a smooth positive function and $\psi^\perp$ is convex in $\vartheta$ satisfying
\[
\frac{\partial\psi^\perp(X, \vartheta)}{\partial \vartheta} \cdot \vartheta \geq \psi^\perp(X, \vartheta) \quad \text{for fixed } X \in G.
\]
Then for any spacelike, affine function $\varphi$, there exist a uniquely smooth spacelike, $2$-admissible graphic hypersurface $G$ (define over $M^n$) with the prescribed $(2,0)$-Hessian quotient $\frac{\sigma_2(L(A))}{\sigma_0(L(A))} = \sigma_2(\lambda(A)) = \psi$ and Dirichlet boundary data $\varphi$.

Remark 1.4. (1) Although nearly the whole part of the a priori estimates works for $2 \leq k \leq n, 0 \leq l \leq k - 2$, the estimate for the double normal second derivatives on the boundary only works for $k = 2$, which leads to the situation that so far, we can get the existence and uniqueness of solutions to the PCP \[(1.4)\] only for $k = 2, l = 0$. However, we do hope that this restriction can be overcome in the future, i.e., the estimate for the double normal second derivatives on the boundary can also be obtained for $3 \leq k \leq n$. If so, that would be a surprising breakthrough.

(2) Clearly, when $k = 2, l = 0$, the existence and uniqueness conclusion of Theorem 1.4 is the same with our previous result \[12\, Theorem 1.4\]. But, the reason why we still write down this paper is that the a priori estimates here (especially the part of the $C^2$ boundary estimates) is much complicated than the one in \[12\]. In fact, because of this reason, we did not show the details of the $C^2$ boundary estimates for the solutions to the PCP considered in \[12\].

(3) In our previous work \[12\] and this paper, we insist on numbering (by subscripts) nearly all constants appearing in the process of doing a priori estimates, and we believe that this way can reveal the relations among constants clearly to readers.

The paper is organized as follows. Some useful formulae for spacelike graphic hypersurfaces defined over $M^n \subset \mathcal{H}^n(1)$ will be introduced in Section 2. These formulae have been proven carefully in \[12\]. Section 3 devotes to the gradient estimate. Curvature maximum principle will be shown in Section 4. Interior $C^2$-estimates will be proven in Section 5, and boundary $C^2$-estimates will be proven in Section 6, which, together with the method of continuity, lead to the existence and uniqueness result for the PCP \[(1.4)\], i.e. Theorem 1.4.
2. Some elementary formulas

For the spacelike graphic hypersurface $G \subset \mathbb{R}^{n+1}$ given by (1.1) and $X = (x, u(x)) \in G$, set $X_{ij} := \partial_i \partial_j X - \Gamma_{ij}^k X_k$ with $\Gamma_{ij}^k$ the Christoffel symbols of the metric on $G$. Then it is easy to know

$$h_{ij} = -\langle X_{ij}, \nu \rangle_L,$$

and have the following identities

(2.1) (Gauss formula) $X_{ij} = h_{ij} \nu$,

(2.2) (Weingarten formula) $\nu,_{i} = h_{ij} X_{j}$.

By [7, Section 2], we have

(2.3) $R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}$,

(2.4) $\nabla_k h_{ij} = \nabla_j h_{ik}$, (i.e., $h_{ij,k} = h_{ik,j}$)

and

(2.5) $\Delta h_{ij} = H_{ij} - H h_{ik} h^k_{j} + h_{ij} |A|^2$.

As usual, here the comma “,” in subscript of a given tensor means doing covariant derivatives. Besides, we make an agreement that, for simplicity, in the sequel the comma “,” in subscripts will be omitted unless necessary. BTW, formulae (2.1)-(2.5) have also been mentioned in our previous works [8, 9, 10, 12].

For any equation

(2.6) $F(A) = f(\lambda_1, \lambda_2, \ldots, \lambda_n),$

where $A$ is the second fundamental form of the spacelike graphic hypersurface $G \subset \mathbb{R}^{n+1}$. We can prove the following two conclusions:

**Proposition 2.1.** Let $G$, defined by (1.7), be a smooth $k$-admissible spacelike graphic hypersurface in $\mathbb{R}^{n+1}$, $0 \leq l \leq k - 2$, $2 \leq k \leq n$. Then the operator

$$\left( \frac{\sigma_k}{\sigma_l} [u] \right)^{1/4},$$

is elliptic.

**Proof.** To prove the ellipticity of the operator $\sigma_k/\sigma_l$, it is equivalent to prove

$$\frac{\partial}{\partial \lambda_i} \left( \frac{\sigma_k}{\sigma_l} [u] \right) > 0 \quad \text{for all } i = 1, 2, \ldots, n,$$

where $\lambda_i$’s are the principal curvatures of $G$.

By direct calculation, we have

$$\frac{\partial}{\partial \lambda_i} \left( \frac{\sigma_k}{\sigma_l} [u] \right) = \frac{\sigma_{k-1}(\lambda_i)\sigma_l - \sigma_k\sigma_{l-1}(\lambda_i)}{\sigma_l^2},$$

$$= \frac{\sigma_{k-1}(\lambda_i)\sigma_l(\lambda_i) - \sigma_k(\lambda_i)\sigma_{l-1}(\lambda_i)}{\sigma_l^2}.$$

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Here, for accuracy, $\frac{\partial}{\partial \lambda_i} [u]$ should be $\frac{\partial}{\partial \lambda_i} (\lambda(A(u)))$. We write as $\frac{\partial}{\partial \lambda_i} [u]$ for the purpose of simplifying and emphasizing that $A$ and its eigenvalues depend on the graphic function $u$. Based on this fact, if necessary, sometimes we also write $\frac{\partial}{\partial \lambda_i} (\lambda(A(u)))$ (or $\frac{\partial}{\partial \lambda_i}$) as $\frac{\partial}{\partial \lambda_i} [u]$ to emphasize this connection. This simplification will be used similarly in the sequel.
If \( \sigma_k(\lambda|i) < 0 \), this proposition follows. If \( \sigma_k(\lambda|i) > 0 \), then one has \( \sigma_m(\lambda|i) > 0 \), \( m = 1, 2, \cdots, k-1 \). So, using the generalized Newton-Maclaurin inequality (see, e.g., [20, 22]), we have

\[
\frac{\sigma_k(\lambda|i)/C_n^k}{\sigma_l(\lambda|i)/C_n^l} \leq \frac{\sigma_{k-1}(\lambda|i)/C_n^{k-1}}{\sigma_{l-1}(\lambda|i)/C_n^{l-1}},
\]

which implies

\[
\frac{\sigma_k(\lambda|i)}{\sigma_l(\lambda|i)} < \frac{\sigma_k(\lambda|i)}{\sigma_l(\lambda|i)} \cdot \frac{k}{l} \cdot \frac{n-l+1}{n-k+1} \leq \frac{\sigma_{k-1}(\lambda|i)}{\sigma_{l-1}(\lambda|i)}.
\]

Then we have

\[
\sigma_{k-1}(\lambda|i)\sigma_l(\lambda|i) - \sigma_k(\lambda|i)\sigma_{l-1}(\lambda|i) > 0,
\]

and the ellipticity of the operator \( \sigma_k/\sigma_l \) follows directly.

By [12] Lemmas 2.2 and 2.3, we have:

**Lemma 2.2.** For the function \( F \) defined by (2.6) and the quantity \( \vartheta \) given in the PCP (1.4), one has

\[
F^{ij}\nabla_i\nabla_j\nu = \nu F^{ij}h_j^n h_{im} + F^{ij}\nabla_i h_j^m X_m,
\]

\[
\Delta \vartheta = \sigma_1 + \nabla^i \sigma_1 (X, X)_L + |A|^2 \vartheta,
\]

and

\[
F^{ij}\nabla_i\nabla_j \sigma_1 = -F^{ij,pq}h^{k}_{ij} \nabla_k h_{pq} + F^{ij}h_j^m h_{im} \sigma_1 - F^{ij}h_j |A|^2 + \Delta f,
\]

\[
F^{ij}\nabla_i\nabla_j h_{mn} = -F^{ij,pq}h^{k}_{ij} \nabla_m h_{pq} + F^{ij}h_j^l h_{il} h_{mn} - F^{ij}h_j^l h_{ln} h_{ij} + \nabla _m \nabla_n f,
\]

where \( F_{ij} := \partial F/\partial h_{ij}, \quad F^{ij,pq} := \partial^2 F/\partial h_{ij} \partial h_{pq} \).

3. \( C^1 \) estimate

3.1. Boundary estimate. In \( \mathbb{R}^{n+1}_1 \), for any \( k \in \{1, \cdots, n\} \), we assume the existence and uniqueness of solutions to the prescribed \( k \)-th Weingarten curvature problem with DBC. Let \( s^+ \) be the solution of the following Dirichlet problem

\[
\begin{cases}
\sigma_2[s] = C_n^2 \left( \frac{C_n^l}{C_n^k} \psi(x, s, \vartheta) \right)^2 & x \in M^n \subset \mathcal{H}^n(1) \subset \mathbb{R}^{n+1}_1, \\
S = \varphi & x \in \partial M^n.
\end{cases}
\]

From the generalized Newton-Maclaurin inequality, we have

\[
\sigma_2[s^+] \leq \sigma_2[u].
\]

By the comparison principle, we have \( u \leq s^+ \) in \( M^n \), and thus \( \partial u/\partial \nu \geq \partial s^+/\partial \nu \). In order to get a lower barrier, let \( s^- \) be the solution of the following Dirichlet problem

\[
\begin{cases}
\sigma_{k-1}[s] = \psi(x, s, \vartheta) \frac{k-1}{k} \cdot C_n^l \cdot C_n^{k-1}(n, k, l) & x \in M^n \subset \mathcal{H}^n(1) \subset \mathbb{R}^{n+1}_1, \\
S = \varphi & x \in \partial M^n,
\end{cases}
\]

with \( C(n, k, l) := (C_n^l)^{-k-1} (C_n^k)^{-k-1} (C_n^{k-1})^{k-1} \). Also from the generalized Newton-Maclaurin inequality, we have

\[
\sigma_{k-1}[s^-] \geq \sigma_{k-1}[u].
\]

So \( s^- \leq u \) in \( M^n \), and thus \( \partial u/\partial \nu \leq \partial s^-/\partial \nu \).
Remark 3.1. The existence and uniqueness of solutions to the prescribed 2-th Weingarten curvature problem with DBC have been shown in [12], we can only take \( k = 2, l = 0 \) here. So, for the first equations, one has

\[
\begin{cases}
\sigma_2[s] = \psi(x, s, \vartheta) & x \in M^n \subset \mathcal{H}^n(1) \subset \mathbb{R}^{n+1}_1, \\
s = \varphi & x \in \partial M^n,
\end{cases}
\]

while, for the second equations, one has

\[
\begin{cases}
\sigma_1[s] = \psi^{\frac{1}{2}}(x, s, \vartheta) \cdot n \left( \frac{n(n-1)}{2} \right)^{\frac{s}{2}} & x \in M^n \subset \mathcal{H}^n(1) \subset \mathbb{R}^{n+1}_1, \\
s = \varphi & x \in \partial M^n.
\end{cases}
\]

3.2. Maximum principle. The upper bound on \( Du \) amounts to an upper bound on \( W := \frac{1}{v} = \frac{1}{\sqrt{1-|D\pi|^2}} \), where \( \pi := \ln u \). Therefore, it would follow from the boundary estimate once one can prove that \( We^{S\pi} \) cannot attain an interior maximum for \( S \) sufficiently large under control.

Proposition 3.1. Let \( u \) be the admissible solution of the PCP \( (1.1) \). Then

\[
\sup_{M^n} W \leq \left( \sup_{\partial M^n} W \right) e^{S_2 \left( 2 \sup_{\partial M^n} |\varphi| + \text{diam}(M^n) \right)},
\]

where as usual \( \text{diam}(M^n) \) stands for the diameter of the bounded domain \( M^n \subset \mathcal{H}^n(1) \).

Proof. By contradiction, suppose that \( \sup_{M^n} We^{S\pi} \) is achieved at an interior point \( x_0 \in M^n \). At \( x_0 \), we choose a nice basis for the convenience of computations, that is, let \( \{e_1, e_2, \cdots, e_n\} \) be an orthonormal basis of \( T_{x_0}M^n \) (i.e., the tangent space at \( x_0 \) diffeomorphic to \( \mathbb{R}^n \)) such that \( D\pi(x_0) = |D\pi(x_0)|e_1 \), and moreover, the matrix \( \left( (D^2\pi(x_0))_{ij} \right)_{(n-1) \times (n-1)}, 2 \leq i, j \leq n \), is orthogonal under the basis \( \{e_2, \cdots, e_n\} \). Since \( |\pi_1| \leq |D\pi| \) on \( M^n \) and \( \pi_1(x_0) = |D\pi(x_0)| \). The function

\[
\ln \left( \frac{1}{\sqrt{1-\pi_1^2}} \right) + S\pi = -\frac{1}{2} \ln (1 - \pi_1^2) + S\pi
\]

has a maximum at \( x_0 \) as well. Hence, at \( x_0 \), for any \( i \in \{1, \cdots, n\} \), one has

\[
\frac{\pi_{1i}\pi_1}{1 - \pi_1^2} + S\pi_i = 0.
\]

So the matrix of the curvature operator is diagonal, with diagonal entries \( \frac{\pi_{1i}}{uv} (1 + \frac{\pi_{1i} + 2(\pi_{1i})^2}{uv^3}) \). Moreover, still at \( x_0 \), one has \( \pi_{111}\pi_1 \leq -\pi_{11}^2 - \frac{2(\pi_{11})^2}{1 - \pi_1^2} - S\pi_{11}(1 - \pi_1^2) \), and for \( i \geq 2 \), \( \pi_{1ii}\pi_1 \leq -(1 - \pi_1^2) S\pi_{ii} \). Then we have

\[
\sum_{i=1}^{n} \left( \frac{\sigma_k}{\sigma_1} \right)_{\lambda_i, 1} \cdot \lambda_{i, 1} = \sum_{i=1}^{n} \left( \frac{\sigma_k}{\sigma_1} \right)_{\lambda_i} \cdot h_{i, 1} = \psi_1.
\]

Since \( h_{1} = \frac{1}{uv} \left( 1 + (\sigma^{jk} + \frac{\pi_{1jk}}{uv^3}) \pi_{1k} \right) \), we have

\[
h_{1, 1} = \frac{\pi_{111}}{uv} + \frac{\pi_{111}}{uv^3}, \quad h_{i, 1} = \frac{\pi_{1i1}}{uv} - \frac{\pi_{1i} \pi_{1i}(S + 1)}{uv} - \frac{\pi_1 (S + 1)}{uv} \quad \text{for } i \geq 2.
\]
The differentiated equation, multiplied by $\pi_1$, becomes:

\[
\left(\frac{\sigma_k}{\sigma_l}\right)_{\lambda_1} \left( \frac{\pi_1(3S^2 - 1)}{uv} + \frac{\pi_{111}}{uv^3} \right) + \sum_{i \geq 2} \frac{\sigma_k}{\sigma_l}_{\lambda_i} \left( \frac{\pi_{i11}}{uv} - \frac{\pi_1 \pi_{ii} (S + 1)}{uv} - \frac{\pi_1 (S + 1)}{uv} \right) = \pi_1 \psi_1.
\]

From the maximum conditions, we have

\[
\frac{\pi_1^2 (3S^2 - 1)}{uv} + \frac{\pi_{111} \pi_1}{uv^3} \leq \frac{\pi_1^2 (S^2 - 1)}{uv},
\]

and, since $\pi_{1ii} = \pi_{ii1} - \pi_1$, we have

\[
\frac{\pi_{ii1} \pi_1}{uv} - \frac{\pi_1^2 \pi_{ii} (S + 1)}{uv} \leq -\frac{1}{u} \pi_1^2 \pi_{ii} (S + 1).
\]

Then we can infer

\[
\left(\frac{\sigma_k}{\sigma_l}\right)_{\lambda_1} \cdot \frac{\pi_1^2 (S^2 - 1)}{uv} - \sum_{i \geq 2} \left(\frac{\sigma_k}{\sigma_l}\right)_{\lambda_i} \left( \frac{1}{u} v S \pi_{ii} + \frac{\pi_1^2 S}{uv} + \frac{\pi_1^2 \pi_{ii} (S + 1)}{uv} \right) \geq \pi_1 \psi_1,
\]

so we have

\[
-(k - l)v^2 \frac{\sigma_k}{\sigma_l} - (k - l)(S + 1) \pi_1^2 \frac{\sigma_k}{\sigma_l}
\]

\[
+ \frac{v}{u} \mathcal{M} S + \frac{1}{uv} \mathcal{M} \pi_1^2 - \left(\frac{\sigma_k}{\sigma_l}\right)_{\lambda_1} \frac{1}{uv}(v^2 S^2 + \pi_1^2) \geq \pi_1 \psi_1,
\]

where $\mathcal{M} := \sum_{i=1}^n (\frac{\sigma_k}{\sigma_l})_{\lambda_i}$, and then, since $(\frac{\sigma_k}{\sigma_l})_{\lambda_i} > 0$, there exists a positive constant $m > 1$ such that $(\frac{\sigma_k}{\sigma_l})_{\lambda_1} = \frac{1}{m} \mathcal{M}$. Then, we have

\[
(k - l)v S \leq \mathcal{M} \left( \frac{v}{u} S + \frac{\pi_1^2}{uv} - \frac{1}{m} \frac{v}{u} S (S - 1) \right) - \pi_1 \psi_1.
\]

We want

\[
\frac{v}{u} S + \frac{\pi_1^2}{uv} - \frac{1}{m} \frac{v}{u} S (S - 1) \leq 0,
\]

which is equivalent to

\[
v^2 S + \pi_1^2 \leq \frac{1}{m} v^2 S (S - 1).
\]

Since $\pi_1^2 \leq \rho^2 < 1$, choosing $S = S_1$ large enough such that

\[
\rho^2 \leq 1 - \frac{m(S + 1)}{S(S - 1)},
\]

So we have

\[
(k - l)v S \leq \sup_{\mathcal{M}} |D\psi|.
\]

Then choosing $S_2 > \max \left\{ \frac{\sup_{\mathcal{M}} |D\psi|}{(k - l) \min_S S_1}, S_1 \right\}$, we reach a contradiction. \square
4. Curvature maximum principle

We write (1.4) in the form

\begin{equation}
F(A) = \left(\frac{\sigma_k}{\sigma_l}\right)^{\frac{1}{k-l}} (A) = \psi \left(\frac{1}{k-l}(X, \vartheta) = f(X, \vartheta) \right) \text{ for any } X \in \mathcal{G}.
\end{equation}

Proof of Theorem 1.2. Consider the function

\[ W(A) = \sigma_1(A), \]

which attains its maximum value at some \(X_0 = (x_0, u(x_0)) \in \mathcal{G}\). If \(x_0 \in \partial M^n\), then our claim (1.6) follows directly. Now, we try to prove this claim in the case that \(x_0 \notin \partial M^n\). Choose the frame fields \(e_1, e_2, \cdots, e_n, \nu\) at \(X_0\) such that \(e_1, e_2, \cdots, e_n \in T_{X_0} \mathcal{G}\) at \(X_0\) and \((h_{ij})_{n \times n}\) is diagonal at \(X_0\) with eigenvalues \(h_{11} \geq h_{22} \geq \cdots \geq h_{nn}\). Here, as usual, \(T_{X_0} \mathcal{G}\) denotes the tangent space of the graphic hypersurface \(\mathcal{G}\) at \(X_0\). For each \(i = 1, \ldots, n\), we have

\[ \nabla_i \sigma_1 = 0 \] at \(X_0\).

Therefore, at \(X_0\), it follows that

\begin{equation}
0 \geq F^{ij} \nabla_i \nabla_j \sigma_1
\end{equation}

\[ = -F^{ij,pq} \nabla_i h_{ij} \nabla_p h_{pq} + F^{ij} h_{im} h_{mj} \sigma_1 - F^{ij} h_{ij} |A|^2 + \Delta f. \]

Since \(f\) is convex in \(\vartheta\), together with Lemma 2.2 we have

\begin{equation}
\Delta f = \frac{\partial^2 f}{\partial X^\alpha \partial X^\beta} \nabla_i X^\alpha \nabla_i X^\beta + 2 \frac{\partial^2 f}{\partial X^\alpha \partial \vartheta} \nabla_i X^\alpha \nabla_i \vartheta
\end{equation}

\[ + \frac{\partial f}{\partial \vartheta^2} |\nabla \vartheta|^2 + \frac{\partial f}{\partial X^\alpha} \Delta X^\alpha + \frac{\partial f}{\partial \vartheta} \Delta \vartheta
\]

\[ \geq \frac{\partial f}{\partial \vartheta} \Delta \vartheta + \frac{\partial^2 f}{\partial \vartheta^2} |\nabla \vartheta|^2 - c_1 \sigma_1 - c_2
\]

\[ \geq \frac{\partial f}{\partial \vartheta} |A|^2 - c_1 \sigma_1 - c_2, \]

where positive constants \(c_1, c_2\) depend on \(\|\varphi\| \in C^1 \left(\overline{M^n}\right), \|\psi\| \in C^2 \left(\overline{M^n} \times \left[\inf u, \sup u\right] \times R\right)\), and \(X^\alpha := \langle X, \partial_\alpha \rangle_L, \alpha = 1, 2, \ldots, n + 1\). Obviously, \(\partial_1, \partial_2, \cdots, \partial_n\) are the corresponding coordinate vector fields on \(\mathcal{H}^n(1)\), \(\partial_{n+1} := \partial_r\). Putting (4.3) into (4.2) yields

\begin{equation}
0 \geq F^{ij} \nabla_i \nabla_j \sigma_1
\end{equation}

\[ \geq -F^{ij,pq} \nabla_i h_{ij} \nabla_p h_{pq} + F^{ij} h_{im} h_{mj} \sigma_1
\]

\[ + \left(\frac{\partial f}{\partial \vartheta} \cdot \vartheta - f\right) |A|^2 - c_1 \sigma_1 - c_2
\]

\[ \geq F^{ij} h_{im} h_{mj} \sigma_1 - c_1 \sigma_1 - c_2, \]
where we have used (1.5) and the concavity of $F$. On the other hand, using the properties of Garding cone and the Cauchy inequality, we have

$$F^{ij} h_{im} h_{mj} = \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i} \left[ \frac{\sigma_k}{\sigma_l} \right]^\frac{1}{k-l} \lambda_i^2,$$

(4.5)

$$\geq n \sum_{i=1}^{n} \lambda_i^2 \geq \frac{1}{m} \sum_{i=1}^{n} \lambda_i^2 \geq \frac{1}{m'} \sum_{i=1}^{n} \lambda_i^2 \sigma_1^2,$$

where $\left[ \frac{\sigma_k}{\sigma_l} \right]^\frac{1}{k-l} \lambda_{\min} := \min \left\{ \frac{\partial}{\partial \lambda_i} \left[ \frac{\sigma_k}{\sigma_l} \right]^\frac{1}{k-l} \right\}, i = 1, 2, \ldots, n$. Since $\left[ \frac{\sigma_k}{\sigma_l} \right]^\frac{1}{k-l} \lambda_i > 0$, there exists a positive constant $m'$ such that $\left[ \frac{\sigma_k}{\sigma_l} \right]^\frac{1}{k-l} \lambda_{\min} = \frac{1}{m} \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i} \left[ \frac{\sigma_k}{\sigma_l} \right]^\frac{1}{k-l}$. Taking (4.5) into (4.4), it is easy to know that $\sigma_1$ is bounded. Then the conclusion of Theorem 1.2, i.e. (1.6), follows naturally.

5. Curvature estimates

Let

$$\mathcal{P}(\lambda) := F(A) = \left( \frac{\sigma_k}{\sigma_l} \right)^\frac{1}{k-l} (A) = f(X, \vartheta) \quad \text{for any } X \in \mathcal{G}.$$ 

Set

$$\frac{\sigma_k}{\sigma_l} (\lambda_1, \lambda_2, \ldots, \lambda_n) = \mathcal{P}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

(5.1)

$$\text{tr} F^{ij} = \sum_{i=1}^{n} P_i, \quad P_i = \frac{\partial}{\partial \lambda_i} \mathcal{P}.$$ 

(5.2)

First, we list a useful lemma (see, e.g., [23]).

Lemma 5.1. For any symmetric matrix $\eta = (\eta_{ij})$, we have

$$F^{ij,pq} \eta_{ij} \eta_{pq} = \sum_{i=1}^{n} \frac{\partial^2 \mathcal{P}}{\partial \lambda_i \partial \lambda_j} \eta_{ii} \eta_{jj} + \sum_{i\neq j} \frac{\mathcal{P}_i - \mathcal{P}_j}{\lambda_i - \lambda_j} \eta_{ij}^2.$$ 

(5.3)

The second term on RHS of (5.3) is nonpositive if $\mathcal{P}$ is concave, and it is interpreted as the limit if $\lambda_i = \lambda_j$.

Proof of Theorem 1.3. Using a similar argument to that in the proof of [12, Theorem 1.3].

Let $\eta = \varphi - u$, as observed at the beginning of the proof of [12, Theorem 1.3], one knows that $\eta > 0$ in $M^n$. We now consider the function

$$G = \eta^\alpha e^{\Phi(\vartheta)} h_{ij} \tau_i \tau_j,$$

achieving its maximum value at some $X_0 \in \mathcal{G}$, where $\alpha \geq 1$, $\Phi$ is a function determined later and satisfies $\Phi' := \frac{\partial \Phi}{\partial \vartheta} \geq 0$. Without loss of generality, one may choose the frame field $e_1 = \tau, e_2, \ldots, e_n, \nu$ such that $e_1, e_2, \ldots, e_n \in T_{X_0} \mathcal{G}, \nabla_{e_i} e_j = 0$ at $X_0$ for all $i = 1, 2, \ldots, n$, and
\( (h_{ij})_{n \times n} \) is diagonal at \( X_0 \) with eigenvalues \( h_{11} \geq h_{22} \geq \cdots \geq h_{nn} \). At \( X_0 \), for each \( i, j = 1, 2, \cdots, n \), one has

\[
\alpha \frac{\nabla_i \eta}{\eta} + \psi \nabla_i \vartheta + \left( \frac{\nabla_i \eta}{\eta} \right) = 0,
\]

\[
\alpha \left( \frac{\nabla_i \nabla_j \eta}{\eta} - \frac{\nabla_i \eta \nabla_j \eta}{\eta^2} \right) + \psi' \nabla_i \vartheta \nabla_j \vartheta + \frac{\nabla_i \nabla_j h_{11}}{h_{11}} - \frac{\nabla_i \nabla_j h_{j}}{h_{11}^2} \leq 0.
\]

Therefore, by Lemma 2.2, we have

\[
0 \geq \alpha F^{ij} \left( \frac{\nabla_i \nabla_j \eta}{\eta} - \frac{\nabla_i \eta \nabla_j \eta}{\eta^2} \right) + \psi'' F^{ij} \nabla_i \vartheta \nabla_j \vartheta + \psi' F^{ij} \nabla_i \eta \nabla_j \eta + F^{ij} h_{im} h_{jm} + \frac{\nabla_1 \nabla_1 f}{h_{11}} - \frac{1}{h_{11}} F^{ij} \eta \eta \nabla_1 h_{ij} \nabla_1 h_{pq} - F^{ij} \nabla_i \nabla_j h_{11} - \frac{h_{11}^2}{h_{11}^2}.
\]

We also find that

\[
F^{ij} \nabla_i \nabla_j \vartheta = \partial F^{ij} h_{im} h_{jm} + f + \nabla_i f \langle X, X_i \rangle_L.
\]

Consequently,

\[
0 \geq \alpha F^{ij} \left( \frac{\nabla_i \nabla_j \eta}{\eta} - \frac{\nabla_i \eta \nabla_j \eta}{\eta^2} \right) + \psi'' F^{ij} \nabla_i \vartheta \nabla_j \vartheta + \psi' F^{ij} \nabla_i \eta \nabla_j \eta + F^{ij} h_{im} h_{jm} + \frac{\nabla_1 \nabla_1 f}{h_{11}} - \frac{1}{h_{11}} F^{ij} \eta \eta \nabla_1 h_{ij} \nabla_1 h_{pq} - F^{ij} \nabla_i \nabla_j h_{11} - \frac{h_{11}^2}{h_{11}^2}.
\]

Since \( f \) is convex in \( \vartheta \), we have

\[
\nabla_1 \nabla_1 f = \frac{\partial^2 f}{\partial X^\alpha \partial X^\beta} \nabla_1 X^\alpha \nabla_1 X^\beta + 2 \frac{\partial^2 f}{\partial X^\alpha \partial X^\beta} \nabla_1 X^\alpha \nabla_1 \vartheta + \frac{\partial^2 f}{\partial \vartheta^2} |\nabla_1 \vartheta|^2
\]

\[
\geq \frac{\partial f}{\partial \vartheta} \nabla_1 \nabla_1 \vartheta - c_3 h_{11} - c_4
\]

\[
= \frac{\partial f}{\partial \vartheta} (\vartheta h_{11} + \vartheta h_{11} (X, X_i) L) - c_3 h_{11} - c_4,
\]

where \( c_3, c_4 \) are positive constant depending on \( ||\varphi||_{C^2(M^n)}, ||\psi||_{C^2(M^n \times \inf_0 u, sup_0 u \times \mathbb{R})} \). Inserting this into (5.5) yields

\[
0 \geq \alpha F^{ij} \left( \frac{\nabla_i \nabla_j \eta}{\eta} - \frac{\nabla_i \eta \nabla_j \eta}{\eta^2} \right) + \psi'' F^{ij} \nabla_i \vartheta \nabla_j \vartheta + \psi' F^{ij} \nabla_i \eta \nabla_j \eta + F^{ij} h_{im} h_{jm} + \frac{\nabla_1 \nabla_1 f}{h_{11}} - \frac{1}{h_{11}} F^{ij} \eta \eta \nabla_1 h_{ij} \nabla_1 h_{pq} - F^{ij} \nabla_i \nabla_j h_{11} - c_3,
\]

where we have assumed that \( h_{11} \) is sufficiently large. Otherwise, the assertion of Theorem 1.3 holds.
Next, we assume that \( \varphi \) has been extended to be constant in the \( \partial r \) direction. Therefore,

\[
\nabla_i \nabla_j \eta = \sum_{\alpha, \beta = 1}^{n} \frac{\partial^2 \varphi}{\partial X^\alpha \partial X^\beta} \nabla_i X^\alpha \nabla_j X^\beta + \sum_{\alpha = 1}^{n} \frac{\partial \varphi}{\partial X^\alpha} \nabla_i \nabla_j X^\alpha - u_{ij}
\]

\[
\geq \sum_{\alpha = 1}^{n} \frac{\partial \varphi}{\partial X^\alpha} \nu^\alpha h_{ij} - c_5 h_{ij} v,
\]

where \( c_5 > 0 \) depends on \( \| \varphi \|_{C^1(M^n)} \) and we have again used Gaussian formula and the assumption that \( \varphi \) is affine. Consequently,

\[
F^{ij} \nabla_i \nabla_j \eta \geq \left( \sum_{\alpha = 1}^{n} \frac{\partial \varphi}{\partial X^\alpha} - c_5 v \right) F^{ij} h_{ij} \geq -c_6,
\]

where positive constant \( c_6 \) depends on \( c_5, \| \psi \|_{C^0(M^n \times [\inf_{\partial M^n} u, \sup_{\partial M^n} u] \times \mathbb{R})} \) and \( \| \varphi \|_{C^1(M^n)} \). Combining (5.6) and (5.7), at \( X_0 \), we have

\[
0 \geq -\frac{c_6 \alpha}{\eta} - \alpha F^{ij} \frac{\nabla_i \eta \nabla_j \eta}{\eta^2} + \Psi'' F^{ij} \nabla_i \vartheta \nabla_j \vartheta + \Psi' \nabla_i f(X, X_l) L
\]

\[
+ (\frac{\partial f}{\partial \vartheta} \cdot \vartheta - f) h_{11} + (\Psi' \vartheta + 1) F^{ij} h_{im} h_{jm} + \frac{\partial f}{\partial \vartheta} \nabla_{ij} h_{11} (X, X_l) L
\]

\[
- \frac{1}{h_{11}} F^{ij, pq} \nabla_1 h_{ij} \nabla_1 h_{pq} - F^{ij} \nabla_i h_{11} \nabla_j h_{11} - c_3.
\]

We now estimate the remaining terms in (5.8), and divide the argument into two cases.

**Case 1.** Assume that there exists a positive constant \( \zeta \) to be determined such that

\[
h_{nn} \leq -\zeta h_{11}.
\]

Using the critical point condition (5.4), we have

\[
F^{ij} \frac{\nabla_i h_{11} \nabla_j h_{11}}{h_{11}^2} = F^{ij} \left( \alpha \frac{\nabla_i \eta}{\eta} + \Psi' \nabla_i \vartheta \right) \left( \alpha \frac{\nabla_j \eta}{\eta} + \Psi' \nabla_j \vartheta \right)
\]

\[
\leq (1 + \varepsilon^{-1}) \alpha^2 F^{ij} \frac{\nabla_i \eta \nabla_j \eta}{\eta^2} + (1 + \varepsilon)(\Psi')^2 F^{ij} \nabla_i \vartheta \nabla_j \vartheta
\]

for any \( \varepsilon > 0 \). Since \( |\nabla \eta| \leq c_7(M^n) \), so

\[
F^{ij} \frac{\nabla_i \eta \nabla_j \eta}{\eta^2} \leq c_8 \frac{tr F^{ij}}{\eta^2},
\]

where \( c_8 \) depends on \( c_7 \). Therefore, at \( X_0 \), we have

\[
0 \geq -\frac{c_6 \alpha}{\eta} - c_9 \left[ \alpha + (1 + \varepsilon^{-1}) \alpha \right] \frac{tr F^{ij}}{\eta^2} + \left[ \Psi'' - (1 + \varepsilon)(\Psi')^2 \right] F^{ij} \nabla_i \vartheta \nabla_j \vartheta
\]

\[
+ (\frac{\partial f}{\partial \vartheta} \cdot \vartheta - f) h_{11} + (\Psi' \vartheta + 1) F^{ij} h_{im} h_{jm} - c_3
\]

\[
+ \frac{\partial f}{\partial \vartheta} \frac{\nabla_{ij} h_{11} (X, X_l) L}{h_{11}} + \Psi' \nabla_i f(X, X_l) L,
\]
where \(c_9 := \max\{1, c_8\}\) and the concavity of \(F(A)\) has been used. On the other hand, from (\ref{5.3}), the last two terms of the RHS of (\ref{5.10}) are bounded from below

\[
\frac{\partial f}{\partial \vartheta} \nabla_{l11}(X, X_l) + \Psi' \nabla_l f(X, X_l) \\
= (\Psi' \nabla_l f - \frac{\partial f}{\partial \vartheta} \nabla_l \eta) - \frac{\partial f}{\partial \vartheta} \Psi' \nabla_l \vartheta)(X, X_l)
\]

where \(c_{10}\) is a positive constant depending on \(c_7\), \(\|\varphi\|_{C^1(M^n)}\), \(\|\psi\|_{C^1(M^n × \left[\inf u, \sup u\right] × R)}\), and \(c_{11}\) depends on \(\|\varphi\|_{C^1(M^n)}\), \(\|\psi\|_{C^2(M^n × \left[\inf u, \sup u\right] × R)}\). Therefore

\[
0 \geq - \frac{c_{12}c}{\eta} - c_9 \left[\alpha + (1 + \varepsilon^{-1})\alpha^2\right] tr F_{ij} + \left[\Psi'' - (1 + \varepsilon)(\Psi')^2\right] F_{ij} \nabla_i \vartheta \nabla_j \vartheta
\]

where constant \(c_{12} > 0\) depends on \(c_6\), \(c_{10}\), and constant \(c_{13} > 0\) depends on \(c_3\) and \(c_{11}\). By the Weingarten formula, it follows that

\[
F_{ij} \nabla_i \vartheta \nabla_j \vartheta = F_{ij} h_{il} h_{jk} \langle X, X_l \rangle \langle X, X_k \rangle \leq c_{14} F_{ij} h_{il} h_{jk},
\]

where \(c_{14}\) is a positive constant depending on \(\|\varphi\|_{C^1(M^n)}\), and then we can take a function \(\Psi\) satisfying

\[
\Psi'' - (1 + \varepsilon)(\Psi')^2 \leq 0.
\]

Since \(M^n\) is bounded and \(C^2\), there exists a positive constant \(a = a(\rho) > \sup u\) such that

\[
-a \leq \vartheta < - \sup u\]as \(M^n\).

Let us take

\[
\Psi(\vartheta) = - \log(2a + \vartheta),
\]

so we have (\ref{5.12}) and

\[
\Psi' \vartheta + 1 + c_{14} \left[\Psi'' - (1 + \varepsilon)(\Psi')^2\right] \geq \frac{1}{2} \quad \text{for } \varepsilon \leq \frac{2a^2}{c_{14}}.
\]

From (\ref{5.11}), together with

\[
F_{ij} h_{im} h_{jm} = F_{ij} h_{il} h_{jk} \geq \frac{c^2}{n} h_{11}^2 tr F_{ij},
\]

which follows from the assumption (\ref{5.9}) and the fact \(F_{mn} \geq \frac{1}{n} tr F_{ij}\), at \(X_0\), we have that

\[
0 \geq - \frac{c_{12} \alpha}{\eta} - c_9 \left[\alpha + (1 + \varepsilon^{-1})\alpha^2\right] \frac{tr F_{ij}}{\eta^2}
\]

\[
+ \left(\frac{\partial f}{\partial \vartheta}\right) \vartheta - f h_{11} + \frac{\zeta^2}{2n} h_{11}^2 tr F_{ij} - c_{13},
\]

which implies an upper bound

\[
\eta h_{11} \leq \frac{c_{15}}{\zeta} \quad \text{at } X_0.
\]
Since
\[ \text{tr} F^{ij} = \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i} \left[ \left( \frac{\sigma_k}{\sigma_l} \right)^{\frac{1}{n-1}} \right] \geq \left( \frac{C_n}{C_l} \right)^{\frac{1}{n-1}} > 0, \]
where \( c_{15} \) is a positive constant depending on \( c_9, c_{12}, c_{13}, \alpha, M^n, \| \varphi \|_{C^0(M^m)} \).

**Case 2.** We now assume that
\[ (5.13) \quad h_{nn} \geq -\zeta h_{11}. \]
Since \( h_{11} \geq h_{22} \geq \cdots \geq h_{nn} \), we have
\[ h_{ii} \geq -\zeta h_{11} \quad \text{for all} \quad i = 1, \cdots, n. \]

For a positive constant, assume to be 4, we divide \( \{1, \cdots, n\} \) into two parts as follows
\[ I = \{ i : \mathcal{P} \leq 4P^{11}\}, \quad J = \{ j : \mathcal{P}^{ij} > 4P^{11}\}, \]
where \( \mathcal{P}^{ii} := \frac{\partial P}{\partial h_{ii}} = \mathcal{P}_i \) is evaluated at \( \lambda(X_0) \). Then for each \( i \in I \), by (5.4), we have
\[ \alpha \mathcal{P}_i \frac{|\nabla_i h_{11}|^2}{h_{11}^2} = \mathcal{P}_i \left( \frac{\alpha |\nabla_i \eta|^2}{\eta^2} + \Psi' \nabla_i \vartheta \right)^2 \leq (1 + \varepsilon^{-1})\alpha^2 \mathcal{P}_i \frac{|\nabla_i \eta|^2}{\eta^2} + (1 + \varepsilon) (\Psi')^2 \mathcal{P}_i |\nabla_i \vartheta|^2 \]
for any \( \varepsilon > 0 \). For each \( j \in J \), we have
\[ \alpha \mathcal{P}_j \frac{|\nabla_j \eta|^2}{\eta^2} = \alpha^{-1} \mathcal{P}_j \left( \frac{\nabla_j h_{11}}{h_{11}} + \Psi' \nabla_j \vartheta \right)^2 \leq \frac{1 + \varepsilon}{\alpha} (\Psi')^2 \mathcal{P}_j |\nabla_j \vartheta|^2 + \frac{1 + \varepsilon^{-1}}{\alpha} \mathcal{P}_j \frac{|\nabla_j h_{11}|^2}{h_{11}^2} \]
for any \( \varepsilon > 0 \). Consequently,
\[
\alpha \sum_{i=1}^{n} \mathcal{P}_i \frac{|\nabla_i \eta|^2}{\eta^2} + \sum_{i=1}^{n} \mathcal{P}_i \frac{|\nabla_i h_{11}|^2}{h_{11}^2} \\
\leq \left[ \alpha + (1 + \varepsilon^{-1})\alpha^2 \right] \sum_{i=1}^{n} \mathcal{P}_i \frac{|\nabla_i \eta|^2}{\eta^2} + \left[ 1 + \varepsilon \right] (\Psi')^2 \sum_{i=1}^{n} \mathcal{P}_i |\nabla_i \vartheta|^2 \\
\quad + \frac{1 + \varepsilon}{\alpha} (\Psi')^2 \sum_{j=1}^{n} \mathcal{P}_j |\nabla_j \vartheta|^2 + \left[ 1 + (1 + \varepsilon^{-1})\alpha^{-1} \right] \sum_{j=1}^{n} \mathcal{P}_j \frac{|\nabla_j h_{11}|^2}{h_{11}^2} \\
\leq 4n \left[ \alpha + (1 + \varepsilon^{-1})\alpha^2 \right] \mathcal{P}_1 \frac{|\nabla_1 \eta|^2}{\eta^2} + \left[ 1 + \varepsilon \right] (1 + \alpha^{-1})(\Psi')^2 \sum_{i=1}^{n} \mathcal{P}_i |\nabla_i \vartheta|^2 \\
\quad + \left[ 1 + (1 + \varepsilon^{-1})\alpha^{-1} \right] \sum_{j=1}^{n} \mathcal{P}_j \frac{|\nabla_j h_{11}|^2}{h_{11}^2}. \]
Using this estimate and (5.8), the follows inequality
\[
0 \geq -\frac{c_6 \alpha}{\eta} - 4n \left[ \alpha + (1 + \epsilon^{-1}) \alpha^2 \right] P_1 \frac{||\nabla \eta||^2}{\eta^2} + \left[ \Psi'' - (1 + \epsilon)(1 + \alpha^{-1})(\Psi')^2 \right] P_1 ||\nabla \eta||^2 \\
+ \Psi' \nabla_i f \langle X, X_i \rangle_L + \left( \frac{\partial f}{\partial \vartheta} \cdot \vartheta - f \right) h_{11} + (\Psi' \vartheta + 1) F^{ij} h_{im} h_{jm} + \frac{\partial f}{\partial \vartheta} \nabla_i h_{11} \langle X, X_i \rangle_L \\
- \frac{1}{h_{11}} F^{ij,pq} \nabla_i \nabla_j \nabla_k \nabla_l \nabla_m \nabla_n \nabla_p - \left[ 1 + (1 + \epsilon^{-1}) \alpha^{-1} \right] \sum_{j \in J} P_j \frac{||\nabla_j h_{11}||^2}{h_{11}^2} - c_{13}
\]
holds at \( X_0 \). Then as Case 1, we have that for an appropriate selection of \( \Psi \),
\[
0 \geq -\frac{c_{12} \alpha}{\eta} - c_{16}(\alpha + \alpha^2) \frac{P_1}{\eta^2} + \frac{1}{2n} P_1 h_{11}^2 + \left( \frac{\partial f}{\partial \vartheta} \cdot \vartheta - f \right) h_{11} - c_{13}
\]
(5.14)
\[
- \frac{1}{h_{11}} F^{ij,pq} \nabla_i \nabla_j \nabla_k \nabla_l \nabla_m \nabla_n \nabla_p - \left[ 1 + c_{17} \alpha^{-1} \right] \sum_{j \in J} P_j \frac{||\nabla_j h_{11}||^2}{h_{11}^2},
\]
where \( c_{16} > 0 \) depends on \( n, \epsilon^{-1}, \) and \( c_{17} = (1 + \epsilon^{-1}) \).

We claim that
\[
- \frac{1}{h_{11}} F^{ij,pq} \nabla_i \nabla_j \nabla_k \nabla_l \nabla_m \nabla_n \nabla_p - \left[ 1 + c_{17} \alpha^{-1} \right] \sum_{j \in J} P_j \frac{||\nabla_j h_{11}||^2}{h_{11}^2} \geq 0.
\]
(5.15)
If (5.15) holds, then from (5.14) we have
\[
\left( \frac{\partial f}{\partial \vartheta} \cdot \vartheta - f \right) h_{11} + \frac{1}{2n} P_1 h_{11}^2 \leq c_{18}(1 + \frac{1}{\eta} + \frac{P_1}{\eta^2}),
\]
from which we again get a bound for \( \eta h_{11} \) at \( X_0 \) due to condition (1.7), where \( c_{18} > 0 \) depends on \( c_{12}, c_{13}, c_{16}, c_{17} \) and \( \alpha \). The proof of this claim can be found at the end of the proof of [12, Theorem 1.3], so we omit it here. Then, the proof of Theorem 1.3 is finished. \( \square \)

6. \( C^2 \) boundary estimates

Throughout this section, we just assume that \( M^n \) is strictly convex. By N. M. Ivochkina, M. Lin, N. S. Trudinger [19, 20] and Pierre Bayard [3], we have the follow inequality: \( \exists B_0 = B_0(n, k, l) \) such that in \( \Gamma_k, \forall i \in \{1, \cdots, k\} \),
\[
\left( \frac{\sigma_k}{\sigma_l} \right) \cdot \lambda_i \leq \left( \frac{\sigma_k}{\sigma_l} \right) + B_0 \sum_{i \neq j} \left( \frac{\sigma_k}{\sigma_l} \right) \cdot \lambda_j.
\]
(6.1)
Let \( x_0 \) be a boundary-point, and \( \{e_1, \cdots, e_n\} \) be an adapted basis such that at \( x_0 \), \( \sup_{M^n} |Du| = \sup_{M^n} |D\pi| \leq \rho < 1 \).

**Lemma 6.1.** Let \( g : \overline{M^n} \cap \overline{B}_r(x_0) \times B(0, 1) \to \mathbb{R}, (x, p) \mapsto g(x, p) \) be a function of class \( C^2 \), concave with respect to \( p \), where \( B_r(x_0) := \{ x \in \mathbb{R}^{n+1} | |x - x_0| \leq r \}, B(0, 1) := \{ x \in \mathbb{R}^{n+1} | |x| \leq 1 \} \), and \( \mathcal{W} = g(., Du) - \frac{g}{2} \sum_{s=1}^{n-1} (u_s - u_s(x_0))^2 \). If \( D_{r, \rho} \) denotes the compact \( \overline{M^n} \cap \overline{B}_r(x_0) \times \overline{B}(0, \rho) \), for \( B = B(n, k, l, \rho, \mathcal{B}_0, \|g\|_{1, D_{r, \rho}}) \) sufficiently large, \( \mathcal{W} \) satisfies on
\[ M^n \cap B_r(x_0) \text{ the inequality:} \]
\[
\sum_{i,j} \frac{\partial}{\partial q_{ij}} \left( \frac{\sigma_k}{\sigma_l}(u) \right) W_{ij} \leq B_1 \left( 1 + |D^2 W| + \sum_{i,j} \frac{\partial}{\partial q_{ij}} \left( \frac{\sigma_k}{\sigma_l}(u) \right) W_i W_j + \frac{\sigma_k - 1}{\sigma_l}(u) \right),
\]

where \( B_1 = B_1(n, k, l, M^n, \psi, \rho, B_0, \|g\|_{2,D_n}, \|g\|_2, \rho, B_0) \).

**Proof.** Let us denote by \( \tilde{\tau}_\alpha, \alpha = 1, \ldots, n \), vectors of \( H^n(1) \) induced by the map \( x \mapsto X := (x, u(x)) \), an orthonormal basis of principle vectors of \( TXG \), and \( (\tilde{\eta}^\alpha) \) such that, \( \forall s \in \{1, \ldots, n\} \), \( e_s = \sum_{\alpha=1}^n \tilde{\eta}^\alpha \tilde{\tau}_\alpha \). Define \( (\tilde{\tau}^\alpha) \) such that, \( \forall \alpha \in \{1, \ldots, n\} \), \( \tilde{\tau}_\alpha = \sum_{s=1}^n \tilde{\tau}_s e_s \). We thus have from the definition \( (\tilde{\eta}^\alpha) = (\tilde{\tau}^\alpha)^{-1} \).

We will use the Greek letters and the Latin letters for derivatives in the basis \( \{\tilde{\tau}_\alpha, \alpha = 1, \ldots, n\} \) and \( \{e_s, s = 1, \ldots, n\} \), respectively. For instance, \( u_{\alpha\beta} \) and \( u_{s\alpha} \) will denote respectively \( D^2 u(\tilde{\tau}_\alpha, \tilde{\tau}_\beta) \) and \( D^2 u(e_s, \tilde{\tau}_\alpha) \). In view of the choice of the \( \tilde{\tau}_\alpha \), the quantities \( \frac{1}{u}(u_{\alpha\alpha} + u - \frac{2}{u} u_{\alpha}^2), \alpha = 1, \ldots, n \), are the principal curvatures of the graph \( G \) of \( u \). The inequality in Lemma 6.1 may then be written as

\[
\sum_{\alpha=1}^n \left( \frac{\sigma_k}{\sigma_l} \right) W_{\alpha\alpha} \leq B_1 \left( 1 + |D^2 W| + \sum_{\alpha=1}^n \left( \frac{\sigma_k}{\sigma_l} \right) W_{\alpha\alpha}^2 + \frac{\sigma_k - 1}{\sigma_l} \right).
\]

For the first and second derivatives of \( W \), we have \( \forall \alpha \in \{1, \ldots, n\} \),

\[
W_\alpha = g_\alpha + \sum_{l=1}^n g_{pl} u_{l\alpha} - B \sum_{s=1}^{n-1} u_{s\alpha} (u_s - u_s(x_0)),
\]

and

\[
W_{\alpha\alpha} = g_{\alpha\alpha} + 2 \sum_{l=1}^n g_{\alpha pl} u_{l\alpha} + \sum_{s,t=1}^n g_{ppl} u_{l\alpha} u_{s\alpha}
\]

\[+ \sum_{l=1}^n g_{pl} u_{l\alpha} - B \left( u_{\alpha\alpha}(u_s - u_s(x_0)) + u_{\alpha\alpha}^2 \right). \]

The following formula can represent the third derivatives of \( u \)

\[
\left( \frac{\sigma_k}{\sigma_l}(u) \right)_{,i} = \left( \frac{\sigma_k}{\sigma_l}(u) \right)_{,\lambda_{\alpha,i}} = \left( \frac{\sigma_k}{\sigma_l}(u) \right)_{,\lambda_{\alpha,i}} \cdot \lambda_{\alpha,i}
\]

\[= \left( \frac{\sigma_k}{\sigma_l}(u) \right)_{,\lambda_{\alpha,i}} \cdot \left[ \frac{u_{\alpha i}}{u} \left( \frac{u_{\alpha i}}{u^2} - \frac{u_{\alpha i} u_i}{u^2} \right) (u_{\alpha\alpha} + u - \frac{2}{u} u_{\alpha i}^2) + (u_{\alpha i} + u_i - \frac{4 u_{\alpha i} u_i}{u} + \frac{2 u_{\alpha i}^2}{u^2}) \right]
\]

\[= \psi_i. \]
Since \( u_{imm} = u_{mmi} - u_i \), by the use of \( u_{sa} = \tilde{\eta}_s^a u_{aa} \), we get:

\[
\sum_{\alpha=1}^{n} \frac{\partial}{\partial \lambda_\alpha} \left( \frac{\sigma_k}{\sigma_l} \right) \mathbb{W}_{aa}^n
= \sum_{t=1}^{n} \mathfrak{g}_{pt} \psi_t - B \sum_{s=1}^{n-1} \psi_s (u_s - u_s(x_0))
+ \sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_\alpha} \left\{ \mathfrak{g}_{aa} + 2 \sum_{t=1}^{n} \tilde{\eta}_t^a \mathfrak{g}_{aap} u_{aa} + \sum_{s=1}^{n} \mathfrak{g}_{ps} \tilde{\eta}_s^a \tilde{\eta}_t^a u_{aa}^2 
- B u_{aa}^2 \sum_{s=1}^{n-1} |\tilde{\eta}_s^a|^2 + \left( 4\pi_\alpha - \frac{\pi_\alpha \lambda_\alpha}{u} \right) \mathbb{W}_\alpha + \left( \frac{\pi_\alpha \lambda_\alpha}{u} - 4\pi_\alpha \right) \mathfrak{g}_a 
- \sum_{t=1}^{n} \mathfrak{g}_{pt} \left[ 2u_t (1 + \pi_\alpha^2 - \pi_\alpha^2 \pi_t \lambda_\alpha) \right] \right\},
\]

where \( \pi = \log u \). It’s easy to estimate

\[
\sum_{t=1}^{n} \mathfrak{g}_{pt} \psi_t - B \sum_{s=1}^{n-1} \psi_s (u_s - u_s(x_0)) \leq b_1 (1 + |D\mathbb{W}|),
\]

where positive constant \( b_1 \) depends on \( \psi \). Moreover, the term

\[
\sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_\alpha} \left\{ \mathfrak{g}_{aa} + 2 \sum_{t=1}^{n} \tilde{\eta}_t^a \mathfrak{g}_{aap} u_{aa} + \left( \frac{\pi_\alpha \lambda_\alpha}{u} - 4\pi_\alpha \right) \mathfrak{g}_a - \sum_{t=1}^{n} \mathfrak{g}_{pt} \left[ 2u_t (1 + \pi_\alpha^2 - \pi_\alpha^2 \pi_t \lambda_\alpha) \right] \right\}
\]

is easily estimated by \( b_2 \left( \frac{\sigma_k}{\sigma_l} + \sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_\alpha} |\lambda_\alpha| \right) \), positive constant \( b_2 \) depends on \( \|\mathfrak{g}\|_{2,D_r,p} \), \( \rho, M^n, k, l \). Recalling that \( \mathfrak{g} \) is concave w.r.t. \( p \), for all \( \alpha \in \{1, \ldots, n\} \), \( \sum_{s,t=1}^{n} \mathfrak{g}_{ps} \tilde{\eta}_s^a \tilde{\eta}_t^a \leq 0 \), and then we thus finally get the estimate of \( \sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_\alpha} \mathbb{W}_\alpha^n \):

\[
\sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_\alpha} \mathbb{W}_\alpha^n
\leq b_3 \left( 1 + |D\mathbb{W}| + \frac{\sigma_k-1}{\sigma_l} + \sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_\alpha} |\lambda_\alpha| \right)
+ \sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_\alpha} \left( \right. - B u_{aa}^2 \sum_{s=1}^{n-1} |\tilde{\eta}_s^a|^2
+ \left. \left( 4\pi_\alpha - \frac{\pi_\alpha \lambda_\alpha}{u} \right) \mathbb{W}_\alpha + B \sum_{s=1}^{n-1} \left[ 2u_s (1 + \pi_\alpha^2 - \pi_\alpha^2 \pi_s \lambda_\alpha) \right] (u_s - u_s(x_0)) \right)
\leq b_3 \left( 1 + |D\mathbb{W}| + \frac{\sigma_k-1}{\sigma_l} + \sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_\alpha} |\lambda_\alpha| \right)
+ \sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_\alpha} \left( -b_4 B \lambda_\alpha^2 \sum_{s=1}^{n-1} |\tilde{\eta}_s^a|^2 + b_5 |\mathbb{W}_\alpha| + b_6 |\lambda_\alpha| |\mathbb{W}_\alpha| \right),
\]

where the positive constant \( b_3 \) depends on \( b_1, b_2, \) positive constants \( b_4, b_5, b_6 \) depend on \( \rho, M^n \).

By \( \mathfrak{B} \) Lemma 4.3, we denote \( \delta_\varepsilon := \delta_\varepsilon(\varepsilon, \rho, n) \), where \( \varepsilon \in (0, 1) \). Let us consider two types of
points: either \( \forall \alpha \in \{1, \cdots, n\}, \sum_{s=1}^{n-1} |\tilde{\eta}_s^\alpha|^2 \geq \delta \), or \( \exists \alpha \in \{1, \cdots, n\} \) (for example \( \alpha = 1 \)) such that \( \sum_{s=1}^{n-1} |\tilde{\eta}_s^\alpha|^2 < \delta \).

For the points of the first type, one has

\[
\sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \lambda_\alpha \left( -b_4 B \lambda_\alpha^2 \sum_{s=1}^{n-1} |\tilde{\eta}_s^\alpha|^2 + b_5 |\mathcal{W}_\alpha| + b_6 |\lambda_\alpha||\mathcal{W}_\alpha| \right)
\leq \sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \lambda_\alpha \left( b_7 (1 + \frac{b_8}{\delta}) |\mathcal{W}_\alpha|^2 + b_9 \delta_{\lambda_\alpha}^2 - b_3 B \delta_{\lambda_\alpha}^2 \right),
\]

where the positive constant \( b_7 \) depends on \( b_5, b_6 \), the positive constant \( b_8 \) depends on \( b_6, b_7, b_8 \), and the positive constant \( b_9 \) depends on \( b_6, b_7, b_8 \). Since the term \( b_3 \sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) |\lambda_\alpha| \) in (6.3) can be estimated by

\[
b_3 \sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) |\lambda_\alpha| \leq b_{10} \left( \frac{\sigma_{k-1}}{\sigma_l} \right) + \frac{\delta}{2} \sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \lambda_\alpha^2,
\]

where the positive constant \( b_{10} \) depends on \( b_3 \), taking \( B \) large enough, we can get a bound on \( \sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) |\lambda_\alpha| \) of the expected form.

For the points of the second type, let us consider two cases:

**First case.** \( \lambda_1 \leq 0 \). The inequality (6.1) then becomes

\[
\left( \frac{\sigma_k}{\sigma_l} \right) \lambda_1^2 \leq B_0 \sum_{\alpha=2}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \lambda_\alpha^2.
\]

From [3], Lemma 4.3, for \( \alpha \geq 2, \exists \delta_{\lambda_\alpha} := \delta_{\lambda_\alpha}(n, \rho, \varepsilon) \), s.t. \( \sum_{s=1}^{n-1} |\tilde{\eta}_s^\alpha|^2 \geq \delta_{\lambda_\alpha} \). Hence

\[
\sum_{\alpha=2}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \lambda_\alpha^2 \leq \frac{1}{\delta_{\lambda_\alpha}} \left( \sum_{\alpha=2}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \lambda_\alpha \cdot \lambda_\alpha^2 \sum_{s=1}^{n-1} |\tilde{\eta}_s^\alpha|^2 \right),
\]

and by using inequality (6.4), it follows that

\[
\sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \lambda_\alpha^2 \leq \frac{B_0 + 1}{\delta_{\lambda_\alpha}} \left( \sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \lambda_\alpha \cdot \lambda_\alpha^2 \sum_{s=1}^{n-1} |\tilde{\eta}_s^\alpha|^2 \right).
\]

Thus, we can obtain

\[
\sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \left( -b_4 B \lambda_\alpha^2 \sum_{s=1}^{n-1} |\tilde{\eta}_s^\alpha|^2 + b_5 |\mathcal{W}_\alpha| + b_6 |\lambda_\alpha||\mathcal{W}_\alpha| \right)
\leq \sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \left( b_5 |\mathcal{W}_\alpha| + b_6 |\lambda_\alpha||\mathcal{W}_\alpha| - \frac{B}{b_{11}} \lambda_\alpha^2 \right)
\leq \sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \left( b_7 (1 + \frac{b_8}{\delta}) |\mathcal{W}_\alpha|^2 + b_9 \delta_{\lambda_\alpha}^2 - \frac{B}{b_{11}} \lambda_\alpha^2 \right),
\]

with \( b_{11} = \frac{B_0 + 1}{\delta_{\lambda_\alpha}} \), and

\[
b_3 \sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \lambda_\alpha \leq b_{12} \left( \frac{\sigma_{k-1}}{\sigma_l} \right) + \frac{1}{2b_{13}} \sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \lambda_\alpha^2,
\]
where positive constants $b_{12}$, $b_{13}$ depend on $b_3$. Now one can give the expected bound on $\sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_\alpha} \mathbb{W}_{\alpha\alpha}$ with $\mathcal{B}$ large enough.

**Second case.** $\lambda_1 > 0$. Then one has

\[
\sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_\alpha} (b_5 |\mathbb{W}_\alpha| + b_6 |\lambda_\alpha| |\mathbb{W}_\alpha|)
\]

\[
= \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_1} (b_5 |\mathbb{W}_1| + b_6 |\lambda_1| |\mathbb{W}_1|) + \sum_{\alpha=2}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_\alpha} (b_5 |\mathbb{W}_\alpha| + b_6 |\lambda_\alpha| |\mathbb{W}_\alpha|)
\]

\[
\leq b_{14} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_1} \cdot |\lambda_1| \mathbb{W}_1 + b_{15} \sum_{\alpha=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_\alpha} \mathbb{W}_\alpha^2 + \sum_{\alpha=2}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_\alpha} \cdot |\lambda_\alpha|^2,
\]

where the positive constants $b_{14}$, $b_{15}$ depend on $b_5$, $b_6$. Let us bound $Q = \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_1} |\mathbb{W}_1|$. $Q = \mathbb{W}_1 \left( \psi - \bar{\Gamma}(k, l, 1) \right)$ with $\psi = \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_\alpha} \cdot |\lambda_\alpha| + \bar{\Gamma}(k, l, \alpha)$, where $\alpha = 1, \cdots, n$, $\bar{\Gamma}(k, l, \alpha) = \frac{\sigma_{k\alpha}(\lambda|\alpha|) - \sigma_{k\alpha}(\lambda|\alpha|)}{\sigma_{\alpha\lambda}} + \frac{\sigma_{k\alpha}(\lambda|\alpha|)}{\sigma_{\alpha\lambda}}$, and $\left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_1}$ is positive. We must again consider two cases:

If $\bar{\Gamma}(k, l, 1) \geq 0$,

\[
|Q| \leq |\mathbb{W}_1| \left( \psi - \bar{\Gamma}(k, l, 1) \right) \leq |\mathbb{W}_1| \psi \leq b_{16} |D\mathbb{W}|
\]

where the positive constant $b_{16}$ depends on $\psi$.

If $\bar{\Gamma}(k, l, 1) < 0$,

\[
(6.5) \quad |Q| \leq -|\mathbb{W}_1| \bar{\Gamma}(k, l, 1) + |\mathbb{W}_1| \psi \leq -|\mathbb{W}_1| \bar{\Gamma}(k, l, 1) + b_{16} |D\mathbb{W}|
\]

Since

\[
\mathbb{W}_1 = g_1 + \sum_{l=1}^{n} g_{p_l} u_{11} \tilde{n}_k^1 - \mathcal{B} \sum_{s=1}^{n} u_{11} \tilde{n}_s^1 (u_s - u_s(x_0))
\]

If $|\sum_{l=1}^{n} g_{p_l} \tilde{n}_k^1 - \mathcal{B} \sum_{s=1}^{n} \tilde{n}_s^1 (u_s - u_s(x_0))| < \tilde{\beta}^1$ and $|g_1| < b_{17}$, then

\[
|\mathbb{W}_1| \leq \tilde{\beta} \lambda_1 + b_{17}
\]

Besides, $\forall s \in \{1, \cdots, n - 1\}$, $|\tilde{n}_s^1| \leq (\delta_\epsilon)^{1/2}$. Choosing $\epsilon = \epsilon(n, \rho, \mathcal{B})$ sufficiently small such that

\[
(6.6) \quad \mathcal{B} (\delta_\epsilon)^{1/2} \leq 1,
\]

and then, with such a choice of $\epsilon$ we get: $\forall s \in \{1, \cdots, n - 1\}$, $|\mathcal{B} \tilde{n}_s^1| \leq 1$. Let us then take $\tilde{\beta}$ as

\[
(6.7) \quad \tilde{\beta} = \sup_{Dr, \rho} \left( \sum_{l=1}^{n} |g_{p_l}| + 2(n - 1) \right).
\]

Hence,

\[
-|\mathbb{W}_1| \bar{\Gamma}(k, l, 1) \leq -\bar{\Gamma}(k, l, 1) \tilde{\beta} \lambda_1 - b_{17} \bar{\Gamma}(k, l, 1)
\]

\[
\leq -\bar{\Gamma}(k, l, 1) \tilde{\beta} \lambda_1 + b_{17} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_1} \cdot |\lambda_1 - \psi|.
\]
Let us find a bound on \(-\bar{\Gamma}(k, l, 1) \cdot \lambda_1\):

\[
\sum_{\alpha=2}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \cdot \lambda_\alpha^2 = \sum_{\alpha=2}^{n} \left( \psi - \bar{\Gamma}(k, l, \alpha) \right) \lambda_\alpha \\
= \psi \sigma_1(\lambda|1) - \sum_{\alpha=2}^{n} \bar{\Gamma}(k, l, \alpha) \lambda_\alpha \\
= \psi \sigma_1(\lambda|1) - \sum_{\alpha=1}^{n} \bar{\Gamma}(k, l, \alpha) \lambda_\alpha + \bar{\Gamma}(k, l, 1) \lambda_1 \\
= \psi \sigma_1(\lambda|1) + (k - l) \frac{\sigma_k}{\sigma_l} - \frac{\sigma_k}{\sigma_l} \sigma_1 + \bar{\Gamma}(k, l, 1) \lambda_1.
\]

So, we have

\[
-\bar{\Gamma}(k, l, 1) \lambda_1 = \psi \sigma_1(\lambda|1) + (k - l) \frac{\sigma_k}{\sigma_l} - \frac{\sigma_k}{\sigma_l} \sigma_1 - \sum_{\alpha=2}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \lambda_\alpha^2 \\
\leq -\psi \lambda_1 + (k - l) \frac{\sigma_k}{\sigma_l} + \sum_{\alpha=2}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \lambda_\alpha^2 \\
\leq (k - l) \frac{\sigma_k}{\sigma_l} + \sum_{\alpha=2}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \lambda_\alpha^2.
\]

(6.9)

Last, we estimate \(\left( \frac{\sigma_k}{\sigma_l} \right) \lambda_1 \) as follows:

\[
\left( \frac{\sigma_k}{\sigma_l} \right) \lambda_1 = (k - l) \frac{\sigma_k}{\sigma_l} - \sum_{\alpha=2}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \lambda_\alpha \\
\leq (k - l) \frac{\sigma_k}{\sigma_l} + \frac{1}{4}\sum_{\alpha=2}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \lambda_\alpha + \gamma \sum_{\alpha=2}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \lambda_\alpha^2 \\
\leq (k - l) \frac{\sigma_k}{\sigma_l} + \frac{1}{4} \frac{(n-k+1)\sigma_l - (n-l+1)\sigma_k}{\sigma_l} \frac{\sigma_l}{\sigma_l} \\
\leq (k - l) \frac{\sigma_k}{\sigma_l} + \frac{n-k+1}{4\gamma} \frac{\sigma_k}{\sigma_l} + \frac{\gamma}{\sigma_l} \sum_{\alpha=2}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \lambda_\alpha^2 \\
\leq b_{18} \left( 1 + \frac{\sigma_{k-1}}{\sigma_l} \right) + \frac{n-k+1}{4\gamma} \frac{\sigma_k}{\sigma_l} + \frac{\gamma}{\sigma_l} \sum_{\alpha=2}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \lambda_\alpha^2
\]

(6.10)

where the positive constant \(b_{18}\) depends on \(\psi, k, l, \gamma, \) and \(\gamma\) is to be specified. With \(\gamma\) sufficiently small, inequalities (6.5), (6.8), (6.9) and (6.10) then give

\[
|Q| \leq b_{19} \left( 1 + |D \bar{W}| + \frac{\sigma_{k-1}}{\sigma_l} \right) + b_{20} \sum_{\alpha=2}^{n} \left( \frac{\sigma_k}{\sigma_l} \right) \lambda_\alpha^2,
\]

where the positive constant \(b_{19}\) depends on \(\bar{\beta}, k, l, \psi, b_{16}, b_{17}, b_{18}\), the positive constant \(b_{20}\) depends on \(\bar{\beta}, b_{17}\). Recalling the inequality
\[
\sum_{a=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_a} \cdot \mathcal{W}_{\alpha \alpha} \leq b_3 \left( 1 + |D\mathcal{W}| + \frac{\sigma_{k-1}}{\sigma_l} + \sum_{a=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_a} \right) - b_4 B \delta \varepsilon \sum_{a=2}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_a} \cdot \lambda_\alpha^2
\]

\[
+ \sum_{a=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_a} (b_5 |\mathcal{W}| + b_6 |\lambda_a||\mathcal{W}|),
\]

with estimates (6.10).

\[
b_3 \sum_{a=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_a} \cdot |\lambda_a| \leq b_{21} \left( 1 + \frac{\sigma_{k-1}}{\sigma_l} \right) + b_{22} \sum_{a=2}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_a} \cdot \lambda_\alpha^2
\]

and

\[
\sum_{a=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_a} (b_5 |\mathcal{W}| + b_6 |\lambda_a||\mathcal{W}|)
\]

\[
\leq b_{23} \left( 1 + |D\mathcal{W}| + \frac{\sigma_{k-1}}{\sigma_l} \right) + b_{24} \sum_{a=2}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_a} \cdot \lambda_\alpha^2 + b_{15} \sum_{a=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_a} \mathcal{W}_{\alpha \alpha}^2,
\]

where the positive constant \(b_{21}\) depends on \(b_3, b_{18}\), the positive constant \(b_{22}\) depends on \(b_3, \gamma\), the positive constant \(b_{23}\) depends on \(b_{14}, b_{19}\), and the positive constant \(b_{24}\) depends on \(b_{20}\). Let \(B = B(n, k, l, \rho, \|g\|_{1,D_{r,\rho}}, B_0)\) be large enough, independent of \(\varepsilon\), and compatible with (6.6). Then we have

\[
\sum_{a=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_a} \cdot \mathcal{W}_{\alpha \alpha}
\]

\[
\leq b_{25} \left( 1 + |D\mathcal{W}| + \frac{\sigma_{k-1}}{\sigma_l} + \sum_{a=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_a} \mathcal{W}_{\alpha \alpha}^2 \right)
\]

\[
+ b_{26} \sum_{a=2}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_a} \cdot \lambda_\alpha^2 - b_4 B \delta \varepsilon \sum_{a=2}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_a} \cdot \lambda_\alpha^2,
\]

where the positive constant \(b_{25}\) depends on \(b_3, b_{15}, b_{21}, b_{23}\), and the positive constant \(b_{26}\) depends on \(b_{22}, b_{24}\). Now we can give the expected bound on \(\sum_{a=1}^{n} \left( \frac{\sigma_k}{\sigma_l} \right)_{\lambda_a} \mathcal{W}_{\alpha \alpha}\) with \(B\) large enough. This completes the proof of Lemma 6.1.

Setting \(\tilde{\mathcal{W}} = \exp \left( -B_1 g(x_0, Du(x_0)) \right) - \exp \left( -B_1 \mathcal{W} \right) - b|x - x_0|^2\), from Lemma 6.1 and an appropriate \(b\) as in [19], we can get the following crucial inequality.

**Lemma 6.2.** For \(b = b(n, B_1, \|g\|_{0,D_{r,\rho}}, B)\) sufficiently large, \(\tilde{\mathcal{W}}\) satisfies

\[
\sum_{i,j} \frac{\partial}{\partial q_{ij}} \left( \frac{\sigma_k}{\sigma_l} (u) \right) \mathcal{W}_{ij} \leq B_2 \left( 1 + |D\mathcal{W}| \right)
\]

on \(M^n \cap B_r(x_0)\), where \(B_2 = B_2(B_1, b, r, \|g\|_{0,D_{r,\rho}}, B)\).
Lemma 6.3. Let \( \tilde{\psi} \in C^2(\partial M^n \cap \bar{B}_r(x_0)) \) and \( a_0 \in \mathbb{R} \). The function \( \tilde{v} = -a_0|x - x_0|^2 - \bar{h}(d) + \tilde{\psi}(x') \), with \( \bar{h}(d) = \mathcal{B}_3(1 - e^{-B_d}) \), satisfies: for any positive function \( p, Dp \in \mathbb{R}^n \), s.t. \( \frac{|Dp|}{p} \leq \rho < 1, \forall i \in \{1, \cdots, k\}, F_i(Dp, D^2\tilde{v}) > 0 \), and

\[
(6.11) \quad \sum_{i,j} \frac{\partial}{\partial q_{ij}} \left( \frac{F_k}{F_l} \right)(u) \tilde{v}_{ij} \geq B_2(1 + |D\tilde{v}|) \quad \text{on} \quad M^n \cap B_r(x_0)
\]

for suitable parameters \( \mathcal{B}_3, \mathcal{B}_4 \), depending only on \( n, k, l, M^n, \psi, a_0, \|\tilde{\psi}\|_{2, \partial M^n \cap \bar{B}_r(x_0)} \) and \( B_2 \).

Here \( d \) denotes the distance function to the boundary of \( M^n, x = (x', x_n) \) in a given adapted basis at the boundary-point \( x_0 \), and \( F_k(Dp, q) = F_k(g^{ij}(Dp)q), g^{ij}(Dp) = \frac{1}{p^2} \left( \sigma^{ij} + \frac{p^2|p|}{p^2 - |Dp|^2} \right), \)

\( F_k(q) \) denotes the \( k^{th} \) symmetric function of the eigenvalues of \( q \). Particularly, \( F_k(u) = \sigma_k(u) \).

To prove Lemma 6.3 we will need the following fact.

Lemma 6.4. If \( q \) is a symmetric non-negative matrix, then for any positive function \( p, Dp \in \mathbb{R}^n \), s.t. \( \frac{|Dp|}{p} \leq \rho < 1, 2 \leq k \leq n, 0 \leq l \leq k - 2 \),

\[
\frac{F_k}{F_l}(Dp, q) \geq (1 - \rho^2) \left( \frac{1}{p^2} \right)^{k-l} \frac{F_k}{F_l}(q).
\]

Proof. Set \( 2 \leq k \leq n, 0 \leq l \leq k - 2, \forall (Dp, q) \in B(0, 1) \times S_n(\mathbb{R}), \) where \( S_n(\mathbb{R}) \) denotes \( n \times n \) order real symmetric matrix,

\[
(6.12) \quad F_k(Dp, q) = F_k(g^{ij}(Dp)q).
\]

The expression \( (6.12) \), independent of the orthonormal basis of \( \mathbb{R}^n \), is chosen to express \( p \) and \( q \). Take an orthonormal basis \( \{e_1, \cdots, e_n\} \) with \( e_1 \) directed along \( Dp \). Then

\[
go^{ij}(Dp)q = \begin{pmatrix}
\frac{1}{p^2 - |Dp|^2} & 1/p^2 \\
1/p^2 & \cdots \\
& \cdots \\
& 1/p^2
\end{pmatrix} q
= \begin{pmatrix}
\frac{1}{p^2 - |Dp|^2}q_{11} \\
1/p^2 \cdot q_{21} & \frac{1}{p^2 - |Dp|^2}q_{12} \\
& \cdots \\
& \cdots \\
& 1/p^2 \cdot q_{n1} & \frac{1}{p^2 - |Dp|^2}q_{2n} \\
& & \cdots \\
& & 1/p^2 \cdot q_{n2} & \cdots & 1/p^2 \cdot q_{nn}
\end{pmatrix}.
\]

Thus

\[
F_k(g^{ij}(Dp)q) = \frac{1}{p^2 - |Dp|^2} \times \left( \frac{1}{p^2} \right)^{k-1} \sum_{1<i_1<\cdots<i_k \leq n} q_{11}q_{i_1i_2} \cdots q_{i_1i_k} \\
+ \left( \frac{1}{p^2} \right)^{k} \sum_{I=(i_1, \cdots, i_k)} \quad 1<i_1<\cdots<i_k \leq n
\]

\[
F_l(g^{ij}(Dp)q) = \frac{1}{p^2 - |Dp|^2} \times \left( \frac{1}{p^2} \right)^{l-1} \sum_{1<i_1<\cdots<i_l \leq n} q_{11}q_{i_1i_2} \cdots q_{i_1i_l} \\
+ \left( \frac{1}{p^2} \right)^{l} \sum_{J=(i_1, \cdots, i_l)} \quad 1<i_1<\cdots<i_l \leq n
\]
Using $\frac{|Dp|}{p} \leq \rho < 1$, since the determinants in the first sum are non-negative when $q$ is non-negative, Lemma 6.3 follows.

**Proof of Lemma 6.3.** We first show that for any positive function $p$, $Dp \in \mathbb{R}^n$, s.t. $\frac{|Dp|}{p} \leq \rho < 1$, $0 \leq l \leq k - 2$, $2 \leq k \leq n$,

\begin{equation}
\frac{\mathcal{F}_k}{\mathcal{F}_l}(Dp, D^2\tilde{\nu}) \geq \mathcal{B}_5(1 + |D\tilde{\nu}|)^{k-l},
\end{equation}

where $\mathcal{B}_5$ is as large as desired, and if $\mathcal{B}_3$, $\mathcal{B}_4$ are suitable parameters. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of $\mathbb{H}^n(1)$, $q \in S_n(\mathbb{R})$. Denote by $Dp'$ the component of $Dp$ on $\{e_1, \cdots, e_{n-1}\}$, by $q' \in S_{n-1}(\mathbb{R})$ the restriction of $q$ on $\{e_1, \cdots, e_{n-1}\}$, and by $q_{(n,n)}$ the $n \times n$ matrix deduced from $q$ by setting its $(n,n)$ coefficient to 0. It’s easy to show that

\begin{equation}
\mathcal{F}_k(Dp, q) = \frac{1}{p^2} \left( \frac{p^2 - |Dp'|^2}{p^2 - |Dp|^2} q_{nn} \mathcal{F}_{k-1}(Dp', q') + \mathcal{O}\left(|q_{(n,n)}|^k\right) \right),
\end{equation}

\begin{equation}
\mathcal{F}_l(Dp, q) = \frac{1}{p^2} \left( \frac{p^2 - |Dp'|^2}{p^2 - |Dp|^2} q_{nn} \mathcal{F}_{l-1}(Dp', q') + \mathcal{O}\left(|q_{(n,n)}|^l\right) \right),
\end{equation}

where $O\left(|q_{(n,n)}|^k\right)$, $O\left(|q_{(n,n)}|^l\right)$ denote quantities estimated by $C|q_{(n,n)}|^k$, $C|q_{(n,n)}|^l$ with $C$ depending on $Dp, p$. So, we have

\begin{equation}
\frac{\mathcal{F}_k}{\mathcal{F}_l}(Dp, q) = \frac{\mathcal{F}_{k-1}(Dp', q') + \mathcal{O}(1)}{\mathcal{F}_{l-1}(Dp', q') + \mathcal{O}(1)}.
\end{equation}

Let $x \in M^n \cap B_r(x_0)$, and $\{e_1, \cdots, e_n\}$ be an adapted basis at the boundary-point $y$ minimizing the distance between $x$ and $\partial M^n$. We get

\begin{equation}
\frac{\mathcal{F}_k}{\mathcal{F}_l}(Dp, D^2\tilde{\nu}) = \mathcal{F}_{k-1}(Dp', D^2\tilde{\nu}') + \mathcal{O}(1),
\end{equation}

where

\begin{equation}
D^2\tilde{\nu} = \tilde{h}' \text{diag} \left( -2a_0 \frac{\kappa_1}{h'} + \frac{\kappa_1}{1 - \kappa_1 d}, \cdots, -2a_0 \frac{\kappa_{n-1}}{h'} + \frac{\kappa_{n-1}}{1 - \kappa_{n-1} d}, -2a_0 + \frac{\tilde{h}''}{h'} \right) + \left( \frac{D^2\tilde{\nu}}{dx_i dx_j} \right)_{1 \leq i,j \leq n-1}.
\end{equation}

Let us bound $\frac{\mathcal{F}_{k-1}}{\mathcal{F}_{l-1}}(Dp', D^2\tilde{\nu}')$ from below: from the definition of $\mathcal{F}_{k-1}$, $\mathcal{F}_{l-1}$, one knows that $\left( \frac{1}{\tilde{h}'} \right)^{k-l} \mathcal{F}_{k-1}(Dp', D^2\tilde{\nu}')$ tends to

\begin{equation}
\frac{\mathcal{F}_{k-1}}{\mathcal{F}_{l-1}} \left( g^{ij}(Dp') \text{diag}(\kappa_1, \cdots, \kappa_{n-1}) \right),
\end{equation}

where $d$ tends to 0, $\tilde{h}'$ tends to $+\infty$, $\kappa_1, \cdots, \kappa_{n-1}$ denote the principal curvatures of $\partial M^n$ associated with $\{e_1, \cdots, e_{n-1}\}$. By Lemma 6.4, (6.18) can be estimated from below. Taking $r$ small and $\tilde{h}'$ large, we thus obtain the estimate

\begin{equation}
\frac{\mathcal{F}_{k-1}}{\mathcal{F}_{l-1}}(Dp', D^2\tilde{\nu}') \geq \delta \left( \frac{\tilde{h}'}{h'} \right)^{k-l},
\end{equation}

where $\delta = \delta(n, a_0, d, \kappa_1, \cdots, \kappa_{n-1})$.

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where $\delta$ is a positive constant under control. This allows the estimate of $\frac{F_k}{F_l}(Dp, D^2\tilde{v})$: from (6.17), we have
\[ \frac{F_k}{F_l}(Dp, D^2\tilde{v}) \geq \delta \left( \frac{h}{h'} \right)^{k-l}. \]

Since $\tilde{h}^l \geq a_0 (1 + |D\tilde{v}|)$ if $\tilde{h}'$ is chosen sufficiently large ($a_0$ positive constant), we can get (6.11) by taking $B_3$ and $B_4$ sufficiently large such that $\tilde{h}'$ are large. In particular, we thus have: $\forall x \in M^n \cap B_r(x_0)$, $D^2\tilde{v}(x) \in \Gamma_k(Du(x))$. We may then use the concavity of $\left( \frac{F_k}{F_l} \right)^{\frac{1}{k-l}}$ with respect to $q$ to bound $\sum_{i,j} \frac{\partial}{\partial q_{ij}} \left( \frac{F_k}{F_l}(u) \right) \tilde{v}_{ij}$ from below:
\[ \frac{1}{k-l} \sum_{i,j} \frac{\partial}{\partial q_{ij}} \left( \frac{F_k}{F_l}(u) \right) \tilde{v}_{ij} \geq \left( \frac{F_k}{F_l} \right)^{\frac{1}{k-l}}(u) \left( \frac{F_k}{F_l} \right)^{\frac{1}{k-l}}(Du, D^2\tilde{v}), \]
where $\left( \frac{F_k}{F_l} \right)^{\frac{1}{k-l}}(u)$ is itself estimated from below. In view of (6.13), $\left( \frac{F_k}{F_l} \right)^{\frac{1}{k-l}}(Du, D^2\tilde{v})$ is large than $\mathcal{B}_5 (1 + |D\tilde{v}|)$ if $B_3$ and $B_4$ are suitable parameters, where $\mathcal{B}_5$ is as large as desired, i.e. Lemma 6.3 holds for sufficiently large $\mathcal{B}_3, \mathcal{B}_4$ under control.

\[ \square \]

**Estimate of mixed second derivatives.** Let $\{e_1, \cdots, e_n\}$ still denote an adapted basis at the boundary-point $x_0$ and let $t \in \{1, \cdots, n-1\}$. Our purpose is to estimate $u_{tn}(x_0)$. For any $x \in \overline{M^n \cap B_r(x_0)}$, let $\xi = e_t + \tilde{\rho}(x')e_n$, where $\tilde{\rho}$ denotes the function locally defined on $T_{x_0} \partial M^n$ whose graph is $\partial M^n$. Set
\[ g(x, \tilde{p}) = \langle p, \xi(x) \rangle = p_t + \tilde{\rho}(x')p_n. \]

Using Lemma 6.1 and Lemma 6.2, there exists the $\mathcal{B} = \mathcal{B}(n, k, l, \rho, B_0, \|g\|_{1, D_{r, \rho}}, \mathcal{B}_1 = \mathcal{B}_1(n, k, l, M^n, \psi, \rho, B_0, \|g\|_{2, D_{r, \rho}})$ and $b = b(n, B_1, \|g\|_{0, D_{r, \rho}}, \mathcal{B})$ sufficiently large such that the function
\[ \tilde{W} = \exp \left( -\mathcal{B}_1 u_{\xi}(x_0) \right) - \exp \left( -\mathcal{B}_1 u_{\xi} + \frac{\mathcal{B}_1}{2} \sum_{s=1}^{n-1} (u_s - u_s(x_0))^2 \right) - b|x - x_0|^2 \]
satisfies
\[ \sum_{i,j} \frac{\partial}{\partial q_{ij}} \left( \frac{F_k}{F_l}(u) \right) \tilde{W}_{ij} \leq \mathcal{B}_2 \left( 1 + |D\tilde{W}| \right), \]
where $\mathcal{B}_2 = \mathcal{B}_2(B_1, b, r, \|g\|_{0, D_{r, \rho}}, \mathcal{B})$. Define
\[ \tilde{v}(x') = \exp \left( -\mathcal{B}_1 \varphi_{\xi}(x_0) \right) - \exp \left( -\mathcal{B}_1 \varphi_{\xi}(x') \right) \cdot \exp \left( \mathcal{B}_1 \sum_{s=1}^{n-1} \left( \varphi_s(x') - \varphi_s(x_0) \right)^2 + 2 \mathcal{B}_1 \mathcal{B} \left( |\varphi_n(x')|^2 + 1 \right) |D\tilde{\rho}(x')|^2 \right) \]
and $\tilde{v} = -a_0|x - x_0|^2 - \tilde{h}(d) + \tilde{v}(x')$, where $\tilde{h}(d) = \mathcal{B}_3 \left( 1 - e^{-\mathcal{B}_1 d} \right)$. For suitable constants $a_0, \mathcal{B}_3, \mathcal{B}_4$, from Lemma 6.3 we have $\tilde{v} \leq \tilde{W}$ on $\partial (M^n \cap B_r(x_0))$ and $\sum_{i,j} \frac{\partial}{\partial q_{ij}} \left( \frac{F_k}{F_l}(u) \right) \tilde{v}_{ij} \geq \mathcal{B}_2 \left( 1 + |D\tilde{v}| \right)$ on $\overline{M^n \cap B_r(x_0)}$. By the comparison principle in 3, Lemma 4.5, we have $\tilde{v} \leq \tilde{W}$ on $\overline{M^n \cap B_r(x_0)}$, and then since $\tilde{v}(x_0) = \tilde{W}(x_0)$, we get $\tilde{v}_n(x_0) \leq \tilde{W}_n(x_0)$, i.e., $\tilde{v}_n(x_0) \leq B_1 u_{tn}(x_0) \exp \left( -\mathcal{B}_1 \varphi_{\xi}(x_0) \right)$. In other words,
\[ u_{tn}(x_0) \geq \mathcal{B}_6, \]
Remark 6.1. In [3, pp. 27-28], so we omit it.

derivatives on the boundary. to that in [12, Section 5], the existence and uniqueness of the PCP 1.4 with

\[ \frac{F_k}{F_i}(u) = A_{k,l} \cdot h_{mn} + B_{k,l} = \psi, \]

where \( B_{k,l} \) depends only on \( u, Du, \) the tangential and mixed second derivatives of \( u, \) which are already estimated.

Lemma 6.5. \( A_{k,l} \) is given by

\[
A_{k,l} = \frac{1}{u^2} \frac{|\partial \varphi|^2}{u^2 v^2} \cdot \frac{F_k - F_l}{2} \frac{F_k - F_l - 1}{2} (\partial \varphi, \partial^2 \varphi + u, \partial \gamma) \cdot \frac{F_k}{F_l} - \frac{F_l - 1}{2} (\partial \varphi, \partial^2 \varphi + u, \partial \gamma),
\]

where \( \partial \) denotes the tangential gradient and \( \gamma \) the future-directed unit normal to \( \partial M^n, \)

\[
F_{k-1,l-1} := \frac{F_{k-1,l-1}}{F_l}.
\]

We continue to proceed as follows: estimate \( h_{mn} \) via the previous method at a point \( y \) of \( \partial M^n \) where \( F_{k-1,l-1} (\partial \varphi, \partial^2 \varphi + u, \partial \gamma) \) is minimum. It implies a lower bound on \( F_{k-1,l-1} (\partial \varphi, \partial^2 \varphi + u, \partial \gamma) \) because

\[
F_{k-1,l-1} (\partial \varphi, \partial^2 \varphi + u, \partial \gamma)(y) \geq \frac{F_{k-1,l-1}}{F_l} \frac{F_k}{F_l}(u)(y) \geq B_0 \frac{F_k}{F_l}(u)(y).
\]

For the last inequality see [19], and the positive constant \( B_0 \) depends on \( n, k, l, \rho, M^n. \) From the
definition of \( y, \) the function \( F_{k-1,l-1}(\partial \varphi, \partial^2 \varphi + u, \partial \gamma) \) admits itself a lower bound on \( \partial M^n, \) and so does \( A_{k,l}, \) which yields an estimate on the second normal derivatives at every boundary-point.

Let us take \( g(x,p) = F_{k-1,l-1} \left( \partial \varphi(x'), \partial^2 \varphi(x') + \langle p, \gamma(x') \rangle \partial \gamma(x') \right) \), where \( x = (x', x_n) \) in the basis \( \{e_1, \cdots, e_n\}. \) A priori \( g \) is concave with respect to \( p \) only for \( k = 2, l = 0, \) then \( g(x,p) = F_1 \left( \partial \varphi(x'), \partial^2 \varphi(x') + \langle p, \gamma(x') \rangle \partial \gamma(x') \right). \) The rest is almost the same as the argument in [3, pp. 27-28], so we omit it.

Remark 6.1. Nearly the whole part of our proof in this section is valid for any \( k = 2, \cdots, n, \) \( 0 \leq l \leq k - 2. \) However, the auxiliary function \( g(x,p) \) in Lemma 6.1 is concave with respect to \( p, \) and then we need to use the constraint \( k = 2, l = 0 \) to estimate the double normal second derivatives on the boundary.

By the method of continuity and the a priori estimates obtained here, using a similar argument to that in [12, Section 5], the existence and uniqueness of the PCP [4.4] with \( k = 2, l = 0 \) can be proven. This completes the proof of Theorem [4.4]
Acknowledgments

This work is partially supported by the NSF of China (Grant Nos. 11801496 and 11926352), the Fok Ying-Tung Education Foundation (China) and Hubei Key Laboratory of Applied Mathematics (Hubei University).

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Faculty of Mathematics and Statistics, Key Laboratory of Applied Mathematics of Hubei Province, Hubei University, Wuhan 430062, China

Email address: Echo-gaoya@outlook.com, 1766019437@qq.com, jiner1200163.com