Non-extremal Martingale with Brownian Filtration

Sakrani Samia

Abstract: Let \((B_t)_{t \geq 0}\) be the filtration of a Brownian motion \((B_t)_{t \geq 0}\) on \((\Omega, \mathcal{F}, P)\). An example is given of a non-extremal martingale which generates the filtration \((\mathcal{F}_t)_{t \geq 0}\). We also discuss a property of pure martingales, we show here that it is a property of a filtration rather than a martingale.

Key Words: Extremal martingale, Brownian filtration, Pure martingale, Pure filtration.

Contents

1 Introduction 1
2 Preliminaries 2
3 Example of non-extremal martingale with Brownian filtration 2
4 Examples of extremal non-pure martingales with Brownian filtrations 4
5 A martingale class that satisfy property (*) 5
6 Appendix 6

1. Introduction

Among the series of questions asked at the end of the chap.V of [12]) (or also in [13] and [15]) is the following question: a filtration being given on a probability space, how to recognize if it is generated by a Brownian motion or not? This question is especially of interest for a weakly Brownian filtration (there exists an \(\mathcal{F}\)-Brownian motion which has the predictable representation property (PRP) with respect to \(\mathcal{F}\), see [11] for application of this important property). In all generality, there are weakly Brownian filtrations which are not Brownian, as it is shown in [6], paper that was followed by other examples of non-Brownian filtrations given in [4], [7], [14]. These works are important progress that raises many new questions, including how to establish the non-Brownian character of a weakly Brownian filtration?

In all the works above, it is the notion of non-cosiness (introduced by Tsirel’son in [14] and that we will not discuss in this paper) of these filtrations which serves as a criterion to show that they are non-Brownian, see [4], [10] for different types of cosiness: I-cosiness, D-cosiness and T-cosiness. One might think that a filtration generated by a non-pure extremal martingale or non-extremal martingale can not be Brownian. In fact we show in Section 3 that this is not true. The non-Brownian character of a weakly Brownian filtration is much more delicate. Section 4 shows that Brownian filtration can be generated by non-pure extremal martingale. In section 5, we discuss the following property denoted by (*) in [1]: If \(M\) is a continuous martingale and \(\mathcal{F} = \mathcal{F}^M\), for every, \(\mathcal{F}\)-stopping time \(T\) finite a.s such that \(\mathbb{P}(M_T = 0) = 0\), then

\[ \mathcal{F}^+_T = \mathcal{F}^-_T \vee \sigma(M_T < 0), \]

where \(G_T = \sup\{s \leq T, M_s = 0\}, T \in [0, \infty[.\) Authors of [1] have shown that property (*) is satisfied by any pure martingale. It is understood here that (*) is a property of a filtration rather than a martingale.

2010 Mathematics Subject Classification: 60G44, 60J65.
Submitted November 25, 2018. Published May 25, 2019
2. Preliminaries

We will only consider completed probability spaces and right continuous filtrations. We denote $\int HdX$ the stochastic integral of $H$ with respect to $X$ and $\mathcal{F}^X$ the natural filtration of $X$. An $\mathcal{F}$–continuous local martingale $X$ has the PRP (the predictable representation property) if for every $\mathcal{F}$–continuous local martingale $M$ there exists an $\mathcal{F}$–predictable process $H$ such that

$$M = M_0 + \int HdX,$$

where $X$ is called $\mathcal{F}$–extremal if $\mathcal{F}_0$ is trivial and $X$ has the $\mathcal{F}$–PRP. If $\mathcal{F}^X = \mathcal{F}$ then $X$ is called extremal martingale. (this terminology is justified by the fact that the law of an extremal martingale is an extremal point in the convex set of all probability measures on $W = C(\mathbb{R}^+; \mathbb{R})$, which make the coordinate process a local martingale). A continuous local martingale $X$ with $(X)_\infty = \infty$ is pure if $\mathcal{F}^X = \mathcal{F}^B_\infty$ where $B$ is the Brownian motion of Dubins-Schwartz (DDS) associated with $X$, which is equivalent to say that for all $t$, $(X)_t$ is $\mathcal{F}^B_\infty$–measurable.

Every pure martingale is extremal but the opposite is not true. Yor has given in [15] an example of an extremal martingale which is not pure; we will prove here that its natural filtration is Brownian.

**Definition 2.1.** A filtration $\mathcal{F}$ is said to be immersed in a filtration $\mathcal{G}$ (defined on the same probability space) if any $\mathcal{F}$-martingale is $\mathcal{G}$-martingale.

3. Example of non-extremal martingale with Brownian filtration

We have the following characterization of extremal martingales with respect to Brownian filtration:

**Lemma 3.1.** If $B$ is a Brownian motion, $\mathcal{B}$ its natural filtration and $M$ is a $\mathcal{B}$– martingale, then $M$ is $\mathcal{B}$–extremal if and only if $d(M)$ is equivalent to $\lambda$ a.s, where $\lambda$ is the Lebesgue measure on $\mathbb{R}^+$.

**Proof.** $M$ is a $\mathcal{B}$–martingale, so there exists a $\mathcal{B}$–predictable process $H$ such that:

$$M = M_0 + \int HdB$$

$$H^2 = \frac{d(M)}{d\lambda}$$

If $M$ is $\mathcal{B}$–extremal, then there exists a $\mathcal{B}$–predictable process $K$ such that $B = \int KdM$ and $d\lambda = K^2d\langle M \rangle$, that is $d\langle M \rangle$ is equivalent to $\lambda$. If now, $d\langle M \rangle$ is equivalent to $\lambda$, it is enough to represent $B$ as a stochastic integral with respect to $M$. We have $H \neq 0$, $\lambda \otimes dP$ a.s so $B = \int \frac{H}{H^2}dM$. $\square$

Lane [9], gave partial answers to the following question [12]: If $B$ is a Brownian motion, $f$ is borel function and $M$ is the local martingale $\int f(B) dB$, under what conditions the filtration $\mathcal{F}^M$ is Brownian? An important example is when $f \geq 0$ and $\mu(\{f = 0\}) > 0$ but the set $\{f = 0\}$ does not contain any interval ($\mu$ is the Lebesgue measure on $\mathbb{R}$). This case was studied by knight [8] with $F = \{f = 0\}$ is a subset of $[0, 1]$, defined by the Cantor method: removing $\left[\frac{1}{3}, \frac{2}{3}\right]$ then $\left[\frac{1}{9}, \frac{7}{9}\right]$ and $\left[\frac{11}{32}, \frac{31}{32}\right]$ and so on. We define the set $F_n$ by means of its complementary $F^c_n$,

$$F^c_1 = \left[\frac{3}{8}, \frac{5}{8}\right], F^c_2 = F^c_1 \cup \left[\frac{5}{32}, \frac{7}{32}\right] \cup \left[\frac{19}{32}, \frac{21}{32}\right],$$

$$F^c_n = F^c_{n-1} \cup \bigcup_{k=1}^{2^{n-1}} A^k_n, n \geq 2,$$

where $A^k_n = [a^k_n, b^k_n]$ are disjoint intervals of length $\frac{1}{4^n}$. Finally

$$F^c = \bigcup_n F^c_n = 2^{\ell_n} \bigcup_{n \geq 1} A^k_n,$$

with $\ell_n = \sum_{k=0}^{n-1} 2^k = 2^n - 1$. Hence $\mu(F^c) = \lim_{n \to \infty} \mu(F^c_n) = \sum_{n=1}^{\infty} 2^{n-1} \frac{1}{4^n} = \frac{1}{2}$. 

Theorem 3.2. Let $B$ be a Brownian motion, $\mathcal{B}$ its natural filtration and $M$ the martingale defined by

$$M = c' \int 1_{\{B < 0\}} dB + c'' \int 1_{\{B > 1\}} dB + \sum_{n \geq 1} \sum_{k=n}^{\ell_n} c_n^k \int 1_{A_n^k}(B) dB,$$

where the numbers $(c_n^k)$, $n \geq 1$, $k \in \{1, ..., \ell_n\}$, $c'$ and $c''$ are strictly positive and all different. The martingale $M$ is not extremal and we have $\mathcal{F}^M = \mathcal{B}$.

Remark 3.3. In order not to burden the proof of Theorem 1, at the end of this paper (in the appendix) we have gathered some non-detailed points.

Proof. The processes $B^-$ and $(B - 1)^+$ are $\mathcal{F}^M$-adapted (Point 1), it remains to show that $B_t 1_{\{0 < B_t < 1\}}$ is $\mathcal{F}^M$-adapted. We consider the martingales

$$M_n^k = \int 1_{A_n^k}(B) dB$$

($M_n^k$) are also $\mathcal{F}^M$-adapted (Point 1). The stopping times $\{(S_n^k)^r, (T_n^k)^r\} r \geq 1$ of the successive entries and exits of $B$ in the set $A_n^k$ are $\mathcal{F}_n^M$-measurable because these are the moments where $\Delta C_n^k > 0$, with $C_n^k$ the inverse of $< M_n^k >$.

Fix $n \in \mathbb{N}^*$, $k \in \{1, ..., \ell_n\}$ and for every $r \in \mathbb{N}^*$

$$S^r := (S_n^k)^r, \quad T^r := (T_n^k)^r, \quad A_n^k = ]a, b[ \quad , \quad N := M_n^k \quad \text{and} \quad \alpha := c_n^k.$$ (Attention! $a, b, N$ and $\alpha$ depend on $k$ and $n$).

Let us show that the sequence $(B_{S^r}, B_{T^r}) r \geq 1$ is $\mathcal{F}^M_t$-measurable. We have, $N_t = 0$ until $S^1$ and $B_{S^1} = a$. If $t \in [S^1, T^1)$, then

$$N_t = \int_{S^1}^t dB_s = B_t - a.$$ So, we know $B_{T^1}$ and for every $r \geq 1$ and $t \in [S^r, T^r]$ we have

$$M_t - M_{S^r} = \alpha (N_t - N_{S^r}) = \alpha (B_t - B_{S^r}) \quad (1)$$

Therefore

$$M_t - M_{S^r} = \alpha (B_{T^r} - B_{S^r})$$

Then, if we know $M$ and $B_{T^r}$, we can know $B_{S^r}$ (and the inverse is true).

If $M_{T^r} - M_{S^r} > 0$ then $B_{T^r} = b$ and $B_{S^r} = a$. If $M_{T^r} - M_{S^r} < 0$ then $B_{T^r} = a$ and $B_{S^r} = b$.

If $M_{T^r} - M_{S^r} = 0$ so $B_{T^r} = b$ (and then $B_{T^r} = B_{S^r}$). Remark that

$$B_{T^r} = B_{S^r + 1} \quad (2)$$

Indeed, if $B$ is above $]a, b[$ after $T^r$, then $B_{T^r} = b = B_{S^r + 1}$, and if $B$ is below $]a, b[$ after $T^r$, then $B_{T^r} = a = B_{S^r + 1}$.

Suppose we know $M$ until time $t$, since we know $B_{T^1}$, then, from (2), we can know $B_{S^2}$ and $B_{T^2}$ and so on, we can know the sequence $(B_{T^2}, B_{S^2})$ for $T^r, S^r \leq t$.

To finish the proof, let $t_0 \leq t$, the set $\{B_{t_0} \in F^c\}$ is $\mathcal{F}^M_{t_0}$-measurable (Point 2). If $B_{t_0} \in F^c$, then there exists $n$ and $k$ such that $B_{t_0} \in A_n^k$ and so, there exists $r$ such that $t_0 \in ]S^r, T^r[$. We have

$$B_{t_0} = B_{t_0} - B_{S^r} + B_{S^r}.$$ and equality (1) gives

$$B_{t_0} = \frac{1}{\alpha} (M_{t_0} - M_{S^r}) + B_{S^r}.$$ Since $F^c$ is dense in $[0, 1]$ (Point 3), we have

$$B_t 1_{\{0 < B_t < 1\}} = \limsup_{s \to t} B_s 1_{\{B_s \in F^c\}} \text{ and } \mathcal{F}^M = \mathcal{B}.$$ It remains to establish that $M$ is non-extremal. This follows easily from Lemma 1, since $\lambda(F) > 0$. □
4. Examples of extremal non-pure martingales with Brownian filtrations

We will now show that the filtration of the extremal non-pure martingale given in [15] is Brownian.

**Theorem 4.1.** Brownian filtration is generated by a non-pure extremal martingale.

**Proof.** Let \( B \) be a Brownian motion and \( \mathcal{F} \) its natural filtration. We start by considering the stochastic equation
\[
    dX_t = \varphi(X_t)dB_t, \quad X_0 = 0,
\]
where \( \varphi(x) = \frac{1}{\sqrt{2\pi t}}x + \frac{1}{\sqrt{2t+1}} \).

We easily check that:
\[
    |\varphi(x) - \varphi(x')|^2 \leq c \left| \frac{1}{\varphi(x)} - \frac{1}{\varphi(x')} \right|^2 \\
    \leq c \left| \frac{x}{1+|x|} - \frac{x'}{1+|x'|} \right|
\]
and
\[
    \frac{1}{\sqrt{3}} \leq \varphi(x) \leq 1, \forall x, x' \in \mathbb{R}.
\]

The function \( \frac{x}{\sqrt{1+|x|}} \) is strictly increasing, we apply theorem 3.5(iii), chap.IX of [12] and we get \( \mathcal{F}^X = \mathcal{F} \).

We have, \( \langle X \rangle = \int \varphi^2(X_t)dt \), since \( \varphi^2 \) is continuous and strictly decreasing
\[
    \mathcal{F}^{\langle X \rangle} = \mathcal{F}^X.
\]

We define the martingale
\[
    M_t = \tilde{\gamma}(X)_t,
\]
where \( \tilde{\gamma}_t = \int_0^t \text{sgn} \gamma_s d\gamma_s \) and \( \gamma \) is the DDS Brownian motion associated to \( X \). We have \( \langle X \rangle = \langle M \rangle \) then
\[
    \mathcal{F}^{\langle M \rangle} = \mathcal{F}^M = \mathcal{F}.
\]

It remains to show that \( M \) is extremal but non-pure. Since \( \varphi \) is strictly positive, \( d\langle M \rangle \) is equivalent to Lebesgue measure and \( \mathcal{F}^M \) is a Brownian filtration, therefore, using Lemma 1, we deduce that \( M \) is extremal. \( M \) is non-pure because
\[
    \mathcal{F}^{\langle \tilde{\gamma} \rangle}_\infty \subsetneq \mathcal{F}^{\langle \mathcal{M} \rangle}_\infty = \mathcal{F}^M._\infty.
\]

Here is an other example of non-pure extremal martingale with Brownian filtration :

**Theorem 4.2.** Let \( B \) be a Brownian motion. There exists a strictly positive predictable process \( H \) such that \( N_t = \int_0^t H(B_u, u \leq s)dB_s \) is non-pure extremal martingale.

**Proof.** Let \( (T_t) \) be absolutely continuous and strictly increasing time change of Theorem 4.1 of [7]. Then \( M_t := (B_{T_t}) \) generates non-Brownian filtration. We have \( M_t = \int_0^t f(M_u, u \leq s)d\gamma_s \) (see Proposition 3.8, Chap V of [12]), for \( \gamma \) a Brownian motion and \( f \) predictable process which can be choose strictly positive. Since \( M \) is pure by construction (so \( \mathcal{F}^M = \mathcal{F}^B \)), \( B_t = \int_0^t g(B_u, u \leq s)dB_{C_s} \), where \( g \) is \( \mathcal{F}^B \)-predictable process and \( C \) the inverse of \( T \), so
\[
    \gamma_{C_t} = \int_0^t H_s dB_s,
\]
with \( H = \frac{1}{\sqrt{3}} \). Since the filtration of \( M \) is non Brownian, \( \mathcal{F}^M \neq \mathcal{F}^\gamma \) and the martingale \( N = \gamma_C \) is not pure. But \( \mathcal{F}^N = \mathcal{F}^B \) and \( H \) is strictly positive, then \( N \) is extremal by Lemma 1.

**Remark 4.3.** Theorem 3 responds affirmatively to the following question asked at the end of Chap V of [12]: is there a strictly positive predictable process \( H \) such that the martingale \( N_t = \int_0^t H_s dB_s \) is not pure?
5. A martingale class that satisfy property (∗)

In [1], authors discussed a property (∗) verified by all pure martingales and gave some examples of non-pure extremal martingales and non-extremal martingales that nevertheless satisfy property (∗). In [2], we better understand this property that we reset here: Let \( M \) be a continuous martingale and \( \mathcal{F} = \mathcal{F}^{M} \), for every, \( \mathcal{F} \)-stopping time \( T \) finite a.s such that \( \mathcal{P}(M_{T} = 0) = 0 \), we have

\[
\mathcal{F}_{G_{T}}^{+} = \mathcal{F}_{G_{T}}^{-} \vee \sigma(M_{T} < 0),
\]

where \( G_{T} = \sup\{s \leq T, M_{s} = 0\}, T \in [0, \infty[. \) The example given in [1] of non-pure extremal martingale satisfying property (∗) is in fact the example of Yor [15]. We have shown that its filtration is Brownian and therefore, it is obvious that this martingale satisfies (∗) using Barlow’s property proven in [2]. In the same way, our non-extremal martingale of Theorem 1, satisfies (∗).

In general, the following proposition can be stated:

**Proposition 5.1.** Let \( \mathcal{F} \) be a filtration such that all \( \mathcal{F} \)-martingales are continuous and \( SpMult|\mathcal{F} \leq 2 \) (see the definition below), then all martingales generating \( \mathcal{F} \) satisfy property (∗).

Before proving the proposition, we recall the following definition:

**Definition 5.2.** Let \((\Omega, A, \mathcal{P})\) be probability space and \( \mathcal{F} \) a sub-field of \( \mathcal{A} \). Let \( \Omega \) be the set of all finite measurable partitions of \((\Omega, A)\), for \( Q \in \Omega \), \( |Q| \) is the cardinal of \( Q \). The conditional multiplicity of \( A \) with respect to \( \mathcal{F} \) is the random variable with values in \( \mathbb{N}^* \cup \{\infty\} \)

\[
Mult[A \mid \mathcal{F}] = \text{ess} \sup_{Q, \in \mathcal{Q}} |Q| \mathbf{1}_{S_B(Q)}
\]

where \( S_B(Q.) = \{\forall A \in Q, \mathcal{P}(A \mid \mathcal{F}) > 0\} \). The splitting multiplicity of a filtration \( \mathcal{F}, SpMult[\mathcal{F}] \) is the smallest integer \( n \) such that: \( Mult[\mathcal{F}_{L^n} \mid \mathcal{F}_L] \leq n \), for any honest time \( L \) of \( \mathcal{F} \).

**Proof.** Using proposition 1 of [1], it is enough to show (∗) for \( T = t \).

Let \( A = \{M_t > 0\} \), we have \( \mathcal{E}[M_t \mid \mathcal{F}_{G_t}] = 0 \) a.s, because \( M_{G_t} = 0 \) a.s (by Theorem XX-35 of [5]). Then a.s

\[
\mathcal{E}[M_t \mathbf{1}_A \mid \mathcal{F}_{G_t}] = -\mathcal{E}[M_t \mathbf{1}_{A^c} \mid \mathcal{F}_{G_t}]. \tag{3}
\]

We define the sets \( C_1 = \{\mathcal{P}(A \mid \mathcal{F}_{G_t}) = 0\} \) and \( C_2 = \{\mathcal{P}(A^c \mid \mathcal{F}_{G_t}) = 0\} \) which are in \( \mathcal{F}_{G_t} \). We have \( \mathcal{P}(A \cap C_1) = 0 \) and \( \mathcal{P}(A^c \cap C_2) = 0 \).

And for every \( n \in \mathbb{N} \):

\[
\mathcal{E}[\mathbf{1}_{C_1} M_t \mathbf{1}_{\{0 < M_t < n\}} \mid \mathcal{F}_{G_t}] \leq n \mathcal{P}(A \cap C_1 \mid \mathcal{F}_{G_t}) = 0,
\]

then

\[
\mathbf{1}_{C_1} \mathcal{E}[M_t \mathbf{1}_A \mid \mathcal{F}_{G_t}] = 0
\]

and from (3), we have

\[
\mathbf{1}_{C_1} \mathcal{E}[M_t \mathbf{1}_{A^c} \mid \mathcal{F}_{G_t}] = 0.
\]

So, \( \mathcal{E}[M_t \mathbf{1}_{A \cap A^c}] = 0 \) and \( C_1 \subset \{M_t = 0\} \).

Similarly, we have \( C_2 \subset \{M_t = 0\} \) Applying hypothesis \( \mathcal{P}(M_t = 0) \) is null, we get \( \mathcal{P}(C_1 \cup C_2) = 0 \) So

\[
\mathcal{F}_{G_t}^{+} = \mathcal{F}_{G_t} \vee \sigma(M_t > 0),
\]

according to proposition 3 of [2] (see also Lemma 4.3 ,Chap . I of [3]).

Here is an example of a filtration with \( SpMult \leq 2 \).

**Definition 5.3.** A filtration generated by a pure martingale is called pure filtration.
Proposition 5.4. Let $\mathcal{F}$ be a filtration, $C = (C_t)$ time change for $\mathcal{F}$ and $\hat{\mathcal{F}} = (\mathcal{F}_{C_t})$. We have:

(a) $\text{SpMult}(\mathcal{F}) \leq \text{SpMult}(\hat{\mathcal{F}})$. If moreover $C$ is strictly increasing, we have: $\text{SpMult}(\mathcal{F}) = \text{SpMult}(\hat{\mathcal{F}})$. In particular, if $\mathcal{F}$ is pure (non-trivial), then $\text{SpMult}(\mathcal{F}) = 2$.

(b) Let $\mathcal{F}$ be the natural filtration of a continuous martingale $M$ and $C$ the inverse of $\langle M \rangle$. We suppose that $\langle M \rangle$ is strictly increasing and $\langle M \rangle_\infty = \infty$. If $\hat{\mathcal{F}}$ is Brownian, then $M$ is extremal and $\mathcal{F}$ is pure.

Proof. (a) Suppose $\text{SpMult}(\mathcal{F}) = n \in \mathbb{N}^*$.

Let $M$ be $\mathcal{F}$-spider martingale of multiplicity $n + 1$, bounded and $M_0 = 0$. Then $M_C = \mathbb{E}[M_\infty | \hat{\mathcal{F}}]$ is $\hat{\mathcal{F}}$-spider martingale of multiplicity $n + 1$ vanishing at the origin, Proposition 13 of [2] gives $M_\infty = 0$ a.s and $\text{SpMult}(\mathcal{F}) \leq n$. If $C$ is strictly increasing and if $\tau$ is its inverse, then by Lemma 5.9 of [13], we have

$$\hat{\mathcal{F}}_{\tau} = \mathcal{F}_{C_{\tau}} = \mathcal{F}.$$ 

If $\mathcal{F}$ is pure, then there exists a time change which we also note $C$, such that $\mathcal{F}_C$ is Brownian, then $\text{SpMult}(\hat{\mathcal{F}}) = 2$ and $\text{SpMult}(\mathcal{F}) \leq 2$.

(b) Let $W$ be a Brownian motion that generates $\mathcal{F}$ and $X$ the martingale $W_{\langle M \rangle}$ (by construction, $X$ is pure).

Let us show that $M$ is extremal: let $B$ be the DDS Brownian motion of $M$, $B$ is $\hat{\mathcal{F}}$-Brownian motion that has $\mathcal{F} - \text{PRP}$ (because $\mathcal{F}$ is Brownian), as $\mathcal{F}_{C_0}$ is trivial, $\mathcal{F}_0$ is too, and $M$ is extremal. Notice now that

$$\mathcal{F}^X_\infty = \mathcal{F}^W_\infty = \hat{\mathcal{F}}_\infty = \mathcal{F}_\infty.$$ (4)

and

$$M_t = \int_0^t \varepsilon_{\langle M \rangle} dX_s,$$

with $\varepsilon_t = \frac{d(B, W)_\tau}{dt}$. Hence $X$ is $\mathcal{F}$-extremal (and since it is extremal), Proposition 7.1 of [13], gives us that $\mathcal{F}^X$ is immersed in $\mathcal{F}$. So we have $\mathcal{F} = \mathcal{F}^X$ using (4). \qed

The next question naturally arises: The reciprocal of proposition 1 is it true? i.e if all the martingales that generate a filtration $\mathcal{F}$ satisfy the property $(\star)$, do we have $\text{SpMult}(\mathcal{F}) = 2$?

For now, we do not have a general answer to this question. In any case, let us note that the following example given in [1] section 6, does not give a negative answer, let

$$M_t = \int_0^t \frac{X_s dY_s - Y_s dX_s}{(X_s^2 + Y_s^2)^\alpha},$$

where $(X_t + iY_t)$ is a planar Brownian motion starting from $z \in \mathbb{C}^*$ and $\alpha \in \left[ -\infty, \frac{1}{2} \right]$. Let $\mathcal{F}$ be the filtration of $M$, $C$ the inverse of $\langle M \rangle$ and $\hat{\mathcal{F}} = (\mathcal{F}_{C_t})_{t \geq 0}$, $\hat{\mathcal{F}}$ is Brownian, so $\mathcal{F}$ is pure and according to proposition 1, $M$ satisfy property $(\star)$.

6. Appendix

Point 1. We have

$$\int 1_{\{B < 0\}} dB = \frac{1}{c'} \int 1_{\{B < 0\}} dM$$

and

$$\int 1_{\{B > 1\}} dB = \frac{1}{c'} \int 1_{\{B > 1\}} dM.$$ 

Hence, by applying Skorokhod’s Lemma (Lemma 2.1, Chap.VI of [12]) it is sufficient to see that the sets $\{B_t < 0\}$ and $\{B_t > 1\}$ are $\mathcal{F}_t^{M}$-measurable:

$$\{B_t < 0\} = \left\{ \frac{d(M)}{dt}(t) = c' \right\} \text{ and } \{B_t > 1\} = \left\{ \frac{d(M)}{dt}(t) = c'' \right\},$$
and similarly for martingales \((M_n^k), n \geq 1, k \in \{1, \ldots, \ell_n\}\).

**Point 2.** According to Point 1, the martingale \(\int 1_{F^c}(B)dB = \sum_n \sum_k M_n^k\) is \(\mathcal{F}^M\)-adapted, so that’s its quadratic variation.

**Point 3.** We will only show that \(0 \in F^c\), more precisely \(\inf F^c = 0\). Let \(x_n = \inf F_n^c\). We have

\[
x_n = \frac{x_{n-1}}{2} - \frac{1}{2 \times 4^n}, n \geq 2
\]

and \(x_1 = \frac{3}{2}\).

Hence

\[
x_n = \frac{x_1}{2^{n-1}} - \sum_{k=2}^{n} \frac{1}{2^{n+1-k} \times 4^k}.
\]

But

\[
\sum_{k=2}^{n} \frac{1}{2^{-k} \times 4^k} = \frac{1}{2^n \times 4} (1 - \left(\frac{1}{2}\right)^{n-1}),
\]

and then

\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{1}{2^{n+1}} (1 - \frac{1}{2^n}) = 0.
\]

**References**

1. J. Azéma, C. Rainer, and M. Yor. Une propriété des martingales pures. Séminaire de probabilités de Strasbourg, 30:243–254, 1996.

2. M. T. Barlow, M. Émery, F. B. Knight, S. Song, and M. Yor. Autour d’un théorème de tsirelson sur des filtrations Browniennes et non Browniennes. In Séminaire de Probabilités XXXII, pages 264–305. Springer, 1998.

3. S. Beghdadi Sakrani. Martingales continues, Filtrations faiblement Browniennes et Mesures signées. PhD thesis, Paris 6, 2000.

4. S. Beghdadi-Sakrani and M. Emery. On certain probabilities equivalent to coin-tossing, d’après schachermayer. In Séminaire de Probabilités XXXIII, pages 240–256. Springer, 1999.

5. C. Dellacherie. Probabilités et potentiel: Tome 5, Processus de Markov (fin): Compléments de calcul stochastique, volume 5. Hermann, 2008.

6. L. Dubins, J. Feldman, M. Smorodinsky, B. Tsirelson, et al. Decreasing sequences of sigma-fields and a measure change for Brownian motion. The Annals of Probability, 24(2):882–904, 1996.

7. M. Émery and W. Schachermayer. Brownian filtrations are not stable under equivalent time-changes. Séminaire de probabilités de Strasbourg, 33:267–276, 1999.

8. F. B. Knight. On invertibility of martingale time changes. In Seminar on Stochastic Processes, 1987, pages 193–221. Springer, 1988.

9. D. A. Lane. On the fields of some Brownian martingales. The Annals of Probability, pages 499–508, 1978.

10. S. Laurent. On standardness and i-cosiness. In Séminaire de Probabilités XLIII, pages 127–186. Springer, 2011.

11. L. PETROVIĆ and D. VALJAREVIĆ. Statistical causality and martingale representation property with application to stochastic differential equations. Bulletin of the Australian Mathematical Society, 90(2):327–338, 2014.

12. D. Revuz and M. Yor. Continuous martingales and Brownian motion, volume 293. Springer Science & Business Media, 2013.

13. D. Stroock and M. Yor. On extremal solutions of martingale problems. In Annales scientifiques de l’École Normale Supérieure, volume 13, pages 95–164, 1980.

14. B. Tsirelson. Triple points: from non-Brownian filtrations to harmonic measures. Geometric and Functional Analysis, 7(6):1096–1142, 1997.

15. M. Yor. Sur l’étude des martingales continues extrémales. Stochastics: An International Journal of Probability and Stochastic Processes, 2(1-4):191–196, 1979.