The Statistical Analysis of Gaussian and Poisson Signals Near Physical Boundaries

Mark Mandelkern and Jonas Schultz
Department of Physics and Astronomy
University of California, Irvine, California 92697

Abstract

We propose a construction of frequentist confidence intervals that is effective near unphysical regions and unifies the treatment of two-sided and upper limit intervals. It is rigorous, has coverage, is computationally simple and avoids the pathologies that affect the Likelihood Ratio and related constructions. Away from non-physical regions, the results are exactly the usual central two-sided intervals. The construction is based on including the physical constraint in the derivation of the estimator, leading to an estimator with values that are confined to the physical domain.

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I. INTRODUCTION

Obtaining confidence intervals near physical boundaries is a long-standing problem. Experiments designed to detect a non-zero neutrino mass by observing neutrino oscillation or to detect a small resonance signal in the presence of background are examples in which a negative result may be obtained for a quantity that is intrinsically positive. The difficulty arises when the estimate for the Gaussian or Poisson mean, as obtained from the data, is near or beyond the physical boundary, in which case the standard (classical) result of Neyman’s construction is an unphysical or null interval as illustrated in Figs. 1 and 2.

For the Gaussian case, Fig. 1, one obtains central confidence intervals for the mean $\mu$ constrained to be non-negative, using the sample mean $\bar{x}$ as the estimator for $\mu$. $\bar{x}$ sufficiently negative leads to the null interval. Despite the fact that the construction has coverage $\alpha$, which means that, for any given true mean, the confidence interval includes that value with probability $\alpha$, the null interval cannot contain a true non-zero mean. It is necessarily one of the measured intervals that, with probability $1 - \alpha$, fail to contain the true mean. Even the non-null intervals obtained by this method for some negative values of the estimator are unphysically small in that, for most possible (true) means, the confidence interval does not contain the true mean.

The other difficult case, illustrated in Fig. 2, is that of Poisson distributed data with unknown signal mean $\mu \geq 0$, in the presence of a background with known mean $b$; $n$ is the result of a single observation. For $n < b$ the interval for $\mu$ is unphysically small. For sufficiently small $n$ the interval is null. The implausibility of the resulting intervals is well illustrated by the example shown. For a background-free ($b = 0$) experiment that measures zero events ($n = 0$), the 90% upper limit for $\mu$ is 2.62, for the explicit construction exhibited in Fig. 2 [1]. For an experiment with known mean background $b = 3.0$ that measures $0(1)$ events, the upper limit for $\mu$ is $0(1.7)$. Thus the poorer experiment has the potential to yield a much smaller (but not believable) upper limit.
When the estimator takes on a value near or beyond the physical limit, we have information greater than that available when no boundary is present since we know \textit{a priori} that the true value is not beyond the boundary. For the Gaussian case, where the confidence intervals are of fixed length for measurements away from the boundary, we expect smaller confidence intervals for measurements near or beyond the boundary. The classical construction gives this feature. We also know that an estimate for the parameter beyond the physical limit is relatively improbable. The flaw in the standard classical method is that increasingly improbable estimates lead to increasingly small and ultimately null confidence intervals. One cannot accurately estimate a parameter by making an extremely improbable observation. The best result for the determination of a parameter should follow from the most probable measurement and, arguably, the smallest confidence interval should be obtained for that observation, i.e. $\bar{x} = \mu$ for the Gaussian case and $n = b + \mu$ for the Poisson case.

II. PREVIOUSLY SUGGESTED METHODS FOR OBTAINING IMPROVED CONFIDENCE INTERVALS

A number of suggestions have been made for estimating believable confidence intervals for bounded parameters. In the Review of Particle Properties \cite{2,3}, the Particle Data Group suggests several options for revising the intervals described above to make them conservative, leading to overcoverage for small true values, and also discusses the use of “Bayesian upper limit(s), which must necessarily contain subjective feelings about the possible values of the parameter”.

Recently, several authors have suggested the use of different \textit{selection principles} for the construction of intervals. In the Neyman construction, the confidence belt depends both on the properties of the estimator and a selection principle. The Neyman construction can be simply described by means of a plot containing values of the estimator on the abscissa and values of the parameter on the ordinate. According to some prescription, i.e. the
selection principle, one selects, for any given value of the parameter, a horizontal interval corresponding to a designated probability (the \textit{coverage}) as determined by the sampling distribution of the estimator. The region mapped out in this way for all values of the parameter constitutes the \textit{confidence belt}. After an experiment is performed, yielding a specific value for the estimator, the corresponding confidence interval for the parameter, with the designated coverage, is the vertical interval contained in the confidence belt at that value of the estimator. The most commonly used selection principles (for coverage \(\alpha\)) are \textit{central} (probability \(\alpha\) within the belt and equal probabilities on either side) and \textit{one sided} (0 lower limit and thus probability \(\alpha\) to the left of \(\bar{x}_{\text{upper}}\)). One has the freedom to depart from the usual selection principles by, for example, invoking a selection which makes the confidence belt as narrow as possible \[4\].

In recently suggested modifications, Ref. \[5\] addresses both the Gaussian and Poisson cases while Ref. \[6\] deals only with the Poisson case. These approaches employ \textit{ordering principles} for the selection, i.e. rules which order the outcome probabilities before aggregating to give total probability \(\alpha\) for each value of the parameter. In particular, the ordering is based on the Likelihood Ratio Construction \[7\] (and a variant), where the physical constraint on the parameter space is used in the computation. These constructions produce finite confidence intervals for all values of the classical estimator and also achieve the admirable unifying feature that one need not decide beforehand whether to set a confidence interval or a confidence bound. However, the intervals obtained are small for \textit{improbable} values of the estimators and share with the classical central construction the difficulty that, for a quite improbable value, the confidence interval approaches the null interval. Thus, for the Gaussian case, a very negative measured value yields a very small confidence interval with lower limit zero. Table X of Ref. \[5\] gives the confidence interval for the (non-negative) Gaussian mean \(\mu\) for measured value \(x_0\). For measured value \(-3.0\) (unit variance assumed), the 68.27\% confidence interval is \([0.00, 0.04]\). Despite the fact that this construction has 68.27\% coverage, the confidence interval derived from this measurement does not contain the true value \textit{for most possible true values} of the Gaussian mean (excepting those in \([0.00, 0.04]\).
that can lead to the measurement. The resulting confidence interval is unphysically small. It does not imply, in the words of the authors, a high “degree of belief” that the true value is within the interval. Our construction, which is described below, yields [0, 1.0].

For the Poisson example cited above, of an experiment with known mean background $b$ of 3.0 and a single observation yielding $n = 0$, the 90% interval for the signal $\mu$ given by Ref.s [5] and [6] are [0, 1.08] and [0, 1.86] respectively, smaller than the interval given for $n = 0, b = 0$ of [0, 2.44]. Ref. [6] emphasizes that the reason for obtaining small upper limits for $n < b$ is not increased sensitivity to the signal but just that fewer background events than expected are observed, and views it as “an undesirable feature from the physical point of view” for the upper limit to decrease as $b$ increases. Our construction, described below, yields [0, 2.62] for the $b = 0$ case and [0, 4.69] for the $b = 3.0$ case, thus a larger rather than smaller interval for 0 events measured when background is present. Of the constructions discussed here, ours is the only one where the upper limit increases rather than decreases as $b$ increases for fixed $n$.

In recognition of the problem of unphysical intervals, Ref. [5] introduces the concept of “sensitivity” to handle cases in which the measurement is less than the estimated background and the confidence interval is suspect. This, however, requires quoting a second value, a characteristic of the experiment itself, in addition to the interval quoted. No substitute interval is offered.

The authors of Ref. [8] construct confidence intervals for the Poisson case. They point out that the observation $n = 0$ implies that zero signal is seen, thus the estimate for $\mu$ (zero) is independent of $b$. They argue, therefore, that the confidence interval for $\mu$ for n=0 must be independent of $b$. Extending the argument, they note that for any observation $n$, one has observed a signal $n$ from the Poisson pdf $p(n; \mu + b)$ and at most a background $n$. Thus they formulate a method of obtaining confidence intervals based on the conditional probability to observe $n$ given a background $\leq n$ and obtain the desired result for $n = 0$ and approximately the classical confidence intervals for $n > b$. While they identify their method as an ordering principle, it is not one in the same sense as Ref.s [5] and [6].
confidence belt is not constructed from the sampling distribution of an estimator and hence
does not have coverage in the usual sense. The method gives intervals that are intuitively
more satisfying as measures of confidence. However, because the method does not provide
coverage, one cannot precisely state the probability that the interval encloses the true value.

Although the intervals determined by the method of \([8]\) do not have coverage, they can
be easily modified so that they do, by restructuring the confidence belt, retaining the lower
limit and adjusting the upper limit so that all horizontal intervals contain probability \(\alpha\).
If one thus modifies the construction, the procedure represents another selection principle
applied to the Poisson pdf for the sample mean. For \(n = 0\) independent of \(b\), this method
gives a 90\% upper limit of 2.42.

### III. FREQUENTIST VS BAYESIAN CONFIDENCE INTERVALS

The methods of Refs. \([5]\) and \([6]\) are frequentist, as they are constructed from the sampling
distribution of an estimator, in this case the sample mean, and have coverage by construction.
However any estimator may be chosen for the Neyman construction. The method used to
choose the estimator is arbitrary. The estimator may be a guess, or arrived at by the
usual techniques of moments or the Maximum Likelihood Method. Although it is in general
desirable for an estimator to be sufficient and unbiased \([12]\), it need not have these properties,
so long as it possesses other desirable features, e.g. gives an appropriate point estimate of
the parameter of interest and leads to confidence intervals that are restrictive and believable
from a physical point of view. Coverage is guaranteed by construction.

Bayesian confidence intervals are constructed from the Bayesian posterior density, which
is interpreted as the probability density for the unknown parameter. A selection principle is
again needed to specify the parameter interval containing the designated probability. The
Bayesian procedure for confidence intervals does not guarantee coverage because it is not
obtained from the probability density of a statistic or random variable and can be criticized
for the subjectiveness inherent in establishing the required Bayesian prior. For a discussion
of Bayesian methods, the reader is referred to Ref. [9]. Our interest is in a frequentist method, as described in the following section.

**IV. INTERVALS BASED ON AN ESTIMATOR DERIVED FROM A LIKELIHOOD FUNCTION THAT CONTAINS THE PHYSICAL CONSTRAINTS**

The authors cited above have focused on modifying the selection principle to make the confidence intervals more believable. However the reason that their constructions lead to unphysically small confidence intervals near the boundary of a physical region is that the method used to obtain the estimator does not take into account the physical constraint on the parameter of interest and the resulting estimator is thus the same as if there were no boundary. Even though that estimator is efficient, it is appropriate for a problem other than the one under consideration.

We propose a frequentist method and use the Maximum Likelihood Method to derive the estimator employed. Among methods for determining estimators, the Maximum Likelihood Method is preferred in that if a consistent estimator exists, the method will produce it [10–12]. The Likelihood Function chosen explicitly contains the physical constraint and leads to an estimator with values within the physical domain that is appropriate for the problem. The confidence intervals obtained consequently from the sampling distribution of the estimator have coverage by construction, are more physical and support a higher degree of belief that the parameter of interest lies within the interval.

This method follows classical estimation theory; the only new element is that the Likelihood Function explicitly excludes non-physical true values. The determination of the estimator, its sampling distribution and the confidence intervals follow directly without further assumptions. We emphasize that the procedure we are following is not Bayesian and that the exclusion of non-physical true values is not equivalent to a uniform Bayesian prior for the physical region any more than the usual unconstrained Likelihood Function is viewed as
containing a uniform Bayesian prior for the entire domain.

A. Gaussian variates

We assume that \( x \) is normally distributed with non-negative mean \( \mu \) and variance \( \sigma^2 \).

\[
f(x|\mu) = \frac{1}{\sqrt{2\pi\sigma}} exp \left[ -\frac{(x - \mu)^2}{2\sigma^2} \right]
\]  

(1)

The likelihood function, when there are \( N \) measurements \( x_1, x_2, ... x_n \), is:

\[
L(\mu) = \prod_{i=1}^{N} f(x_i|\mu) \theta(\mu)
\]  

(2)

\[
w(\mu) = \ln L(\mu) = \sum_{i=1}^{N} \left( -\frac{(x_i - \mu)^2}{2\sigma^2} \right) - N\ln(\sqrt{2\pi\sigma}) + \ln\theta(\mu)
\]  

(3)

where \( \theta(\mu) \) is a step function; \( \theta(\mu) = 0 \) for \( \mu < 0 \), \( \theta(\mu) = 1 \) for \( \mu \geq 0 \). The estimator for \( \mu \), which we denote by \( \mu^* \), is the function of the measurements, \( \mu(x_i) \), that maximizes \( w \).

Since \( w = -\infty \) for \( \mu < 0 \), \( \mu^* \) must be \( \geq 0 \). We set

\[
\frac{dw}{d\mu} = \sum_{i=1}^{N} \frac{x_i - \mu}{\sigma^2} = 0
\]  

(4)

For the sample mean \( \bar{x} \equiv \frac{1}{N} \sum_{i=1}^{N} x_i \geq 0 \), \( \mu^* = \bar{x} \). For \( \bar{x} < 0 \), \( \frac{dw}{d\mu} < 0 \) for all \( \mu \geq 0 \), so the maximum of \( w \) is at \( \mu^* = 0 \). \( \bar{x} \) has a normal distribution with mean \( \mu \) and variance \( \sigma^2_N = \sigma^2/N \). The probability density function for \( \mu^* \) is normal with the usual normalization for \( \mu^* > 0 \) and a delta function at \( \mu^* = 0 \) normalized to the remaining probability

\[
P_0(\mu) \equiv \frac{1}{\sqrt{2\pi\sigma_N}} \int_{-\infty}^{0} \exp \left[ -\frac{(x - \mu)^2}{2\sigma^2_N} \right] dx = \frac{1 - erf(\mu/\sqrt{2\sigma_N})}{2}
\]  

(5)

Thus the probability density function for \( \mu^* \) is given by:

\[
P(\mu^*|\mu) = \frac{1}{\sqrt{2\pi\sigma_N}} \exp \left[ -\frac{(\mu^* - \mu)^2}{2\sigma^2_N} \right] + \delta(\mu^*)P_0
\]  

(6)

The mean and variance of \( \mu^* \) are given by:
\[ E(\mu^*) = \frac{\sigma_N}{\sqrt{2\pi}} \exp\left[-\frac{\mu^2}{2\sigma_N^2}\right] + \mu(1 - P_0) \]  
\[ V(\mu^*) = (\sigma_N^2 + \mu^2 P_0)(1 - P_0) - \frac{\mu \sigma_N(1 - 2P_0)}{\sqrt{2\pi}} \exp\left[-\frac{\mu^2}{2\sigma_N^2}\right] - \frac{\sigma_N^2}{2\pi} \exp\left[-\frac{\mu^2}{\sigma_N^2}\right] \]

\( E(\mu^*) \) approaches \( \mu \) and \( V(\mu^*) \) approaches \( \sigma_N^2 \) for \( N \) large. For finite \( N \), \( E(\mu^*) \) does not equal \( \mu \), so \( \mu^* \) is a consistent but not unbiased estimator for \( \mu \). It is, however, asymptotically unbiased. From Estimation Theory \cite{10, 12} we know that If the Likelihood Equation has a solution \( \mu^* \) which is a consistent estimator of \( \mu \), then \( \mu^* \) is asymptotically normally distributed with a mean of \( \mu \) and a variance of \( [-NE(d^2ln(f(x)|\mu))/d\mu^2]^{-1} \). \( V(\mu^*) \) equals 0.34\( \sigma_N^2 \) at \( \mu = 0 \), monotonically increasing to \( \sigma_N^2 \) at large \( \mu \). For finite \( N \), \( V(\mu^*) \) is smaller than \( \sigma_N^2 \).

Nevertheless, \( V(\mu^*) \) satisfies the usual Cramer-Rao inequality \cite{12}

\[ V(\mu^*) \geq \frac{(dE(\mu^*)/d\mu)^2}{I_X} \]

where \( I_X \) is the Fisher Information, the usual measure of the information contained in the measurements. One finds \( dE(\mu^*)/d\mu = 1 - P_0 \) and \( I_X = \frac{1}{\sigma_N^2} \) and by explicit calculation one can show that

\[ V(\mu^*) \geq (1 - P_0)^2 \sigma_N^2 \longrightarrow \sigma_N^2 \quad N \to \infty \]

We note that \( \mu^* \) does not satisfy the criteria for sufficiency. However for the purpose of supplying a point or interval estimate for this special case where there is a boundary, it contains all of the necessary information. (For \( \bar{x} < 0 \), the best estimate of \( \mu \) is zero.) We demonstrate the construction of the 68.27% central confidence belt, in units of \( \sigma_N = \sigma/\sqrt{N} \), in Fig. 3. We invoke the Neyman construction and select, for any given value of \( \mu \), the “central” interval of \( \mu^* \) that contains 68.27% of the \( \mu^* \) sampling distribution. For \( \mu = 0 \), 50% of the \( \mu^* \) probability distribution is associated with \( \mu^* = 0 \). The remaining 18.27% of the
68.27% belt is contained in the $\mu^*$ interval between 0 and what we call $\delta_\mu$. A straightforward calculation gives $erf(\delta_\mu/\sqrt{2}) = 2 \times 0.1827$, or $\delta_\mu = 0.475$.

As $\mu$ increases from 0 to 1, the upper endpoint of the 68.27% interval rises linearly with unit slope. For $\mu > 1$, the central 68.27% interval in $\mu^*$ is $\mu - 1 \leq \mu^* \leq \mu + 1$. It is the requirement of exactly 68.27% coverage, and the fact that the finite probability associated with $\mu^* = 0$ must be taken into account, that introduces a discontinuity in the central interval at $\mu = 1$.

Once the confidence belt is constructed, as in Fig. 3, it follows from the Neyman method that confidence intervals of $\mu$ with corresponding coverage can be read off as vertical intervals of the belt for any measured $\bar{x}$. We need only keep in mind that all $\bar{x} < 0$ correspond to $\mu^* = 0$.

In our formulation, the necessary “lift up” [2] of the estimate from an unphysical to a physical value and/or the raising of an upper bound to a non-null value comes naturally from the estimator derived from the Likelihood Function. In other approaches, [2,5] the “lift-up” is obtained somewhat arbitrarily by ad hoc procedures or by specifying an ordering principle. The latter methods do not solve the problem that, in the words of Ref. [2], “in some (rare) cases it is necessary to quote an interval known to be wrong.”

**B. Poisson variates with background**

We consider $n$ to be a single Poisson distributed variate with non-negative signal mean $\mu$ and known mean background $b$. Let $p(n|m) = m^n e^{-m}/n!$ denote the Poisson probability for obtaining the measurement $n$ when the mean is $m$. Then

$$f(n|\mu) = p(n|\mu + b)$$

$$L(\mu) = f(n|\mu)\theta(\mu)$$

$$w(\mu) = lnL(\mu) = nln(\mu + b) - (\mu + b) - ln(n!) + ln\theta(\mu)$$
\(\mu^*\) is the function of \(n\) that maximizes \(L\) and is thus the estimator for \(\mu\). For \(n > b\), \(\mu^* = n - b\). For \(n \leq b\), \(\mu^* = 0\). Thus the estimator for \(\mu\) is non-negative. The probability of \(\mu^*\) for a given \(\mu\) is \(P(\mu^*|\mu, b) = p(\mu^* + b|\mu + b)\) for \(\mu^* > 0\) and a value at \(\mu^* = 0\) given by \(\sum_{n \leq b} p(n|\mu + b)\). Rather than work with the estimator \(\mu^*\), it is more convenient to define an integer estimator, \(n^*\), such that \(n^* = 0\) for \(n \leq b\) and \(n^* = n - b^-\) for \(n \geq b\), where \(b^-\) is the largest integer less than or equal to \(b\). Thus \(n^* = \mu^* + (b - b^-)\).

We demonstrate the construction of the 90% confidence belt by means of an example, shown in Fig. 4, where the known mean background \(b\) is equal to 2.8. \(b\) is chosen non-integer to illustrate this slightly more complicated case. We also show the confidence belt consisting of central intervals \([n_1(\mu_0), n_2(\mu_0)]\) containing at least 90% of the probability for unknown Poisson mean \(\mu_0\) in the absence of any known background (dotted) and the 90% one-sided belt consisting of intervals \([0, n_{os}(\mu_0)]\)(dashed). Our 90% confidence belt is defined only for \(\mu \geq 0\) and \(\mu^* \geq 0\). We define a coordinate system \((n^*, \mu)\) by placing the ordinate \(\mu = 0\) at \(\mu_0 = b\) and choosing the integer abscissa value \(n^* = 0\) to coincide with \(n = b^-\).

Let \(\mu'_0\) be the largest value of \(\mu_0\) such that \([n_1(\mu_0), n_2(\mu_0)]\) contains \(b^-\). (In the example given, \(\mu'_0 = 6.2\), corresponding to a value of \(\mu = 6.2 - 2.8 = 3.4\), and \(n_{os}(\mu'_0) = 9\).) For \(0 \leq \mu \leq \mu'_0 - b\) (i.e. \(b \leq \mu_0 \leq \mu'_0\)), the 90% horizontal interval is \([b^-, n_{os}(\mu_0)]\). For \(\mu > \mu'_0 - b\) (i.e. \(\mu_0 > \mu'_0\)), the 90% horizontal interval is \([n_1(\mu_0), n_2(\mu_0)]\). The resulting confidence belt is shown in solid lines. The set of joined horizontal and vertical line segments is simple and continuous and no compensatory remedies are required. To obtain the 90% confidence intervals for \(\mu\), given a measurement \(n\), we need simply find the appropriate vertical interval from the plot. By the Neyman construction, it has \(\geq 90\%\) coverage.

Let \([c_1(m), c_2(m)]\) denote the usual (i.e. in the absence of known background) Poisson 90% confidence interval for the mean, \(\mu_0\), for \(m\) observed events (the dotted horizontal lines in Fig. 4) . Also, let \(c_{os}(m)\) denote the usual 90% one-sided lower limit for \(m\) observed events (the dashed horizontal lines in Fig. 4). Then for \(n \leq b\), \(n^* = 0\) and we obtain the upper limit for \(\mu\) of \(c_2(b^-) - b\). For \(b < n \leq n_{os}(b)\) we obtain the upper limit \(c_2(n) - b\);
for \( n_{os}(b) < n \leq n_{os}(\mu'_0) \) we obtain the interval \([c_{os}(n) - b, c_2(n) - b]\); for \( n = n_{os}(\mu'_0) + 1 \) we obtain the interval \([\mu'_0 - b, c_2(n) - b]\) and for \( n > n_{os}(\mu'_0) + 1 \) we obtain the interval \([c_1(n) - b, c_2(n) - b]\). We note that any Poisson interval with known background can be obtained from a single figure or table.

It is straightforward to generalize to the case of \( N \) independent measurements. For measured mean \( \bar{n} \geq b \), \( \mu^* = \bar{n} - b \). For \( \bar{n} < b \), \( \mu^* = 0 \). The probability for \( \mu^* \) is Poisson for \( \mu^* > 0 \) plus a value at \( \mu^* = 0 \) normalized to the remaining probability.

\[
P(\mu^*|\mu, b)_{\mu^*>0} = p(N(\mu^* + b)|N(\mu + b))
\]

\[
P(0|\mu, b) = \sum_{m \leq Nb} p(m|N(\mu + b))
\]

In this case we can find the confidence interval for \( \mu^* \) by relabeling the axes in Fig. 4 as follows: \( n \to N\bar{n}, \mu_0 \to N\mu_0, n^* \to Nn^*, \mu \to N\mu \), and the origin of the inner coordinate system is \((Nb^-, Nb)\).

V. MASS SquARED OF THE ELECTRON NEUTRINO

As an example we obtain the 68.27% confidence interval for the mass squared of the electron neutrino, disregarding the possibility that the source of negative measurements is physics (fitting to the wrong function) rather than statistical variation. Using the measurement quoting the smallest error, that of Ref. [14] giving \(-22 \pm 4.8 \text{ eV}^2\), and assuming Gaussian probability we obtain the interval \([0, 4.8]\). The classical Neyman interval is null and the interval offered by Ref. [14] is \([0, 0.02]\) [15].

VI. CONCLUSION
We have demonstrated a rigorous method for obtaining frequentist confidence intervals that incorporates the physical constraints of the problem into the Likelihood Function, thus yielding an estimator that is suitable to the presence of physical boundaries. Using a central ordering principle, we obtain either upper limits or central intervals with a smooth transition. The intervals are physical in that they support a high degree of belief that the true value is within the interval, avoiding the pathologies of null or unphysically small intervals and the consequent possibility of obtaining a better result (smaller confidence interval) for a worse experiment. The construction is not equivalent to the Likelihood Ratio Construction which does not give satisfactory intervals near unphysical regions.

VII. ACKNOWLEDGMENTS

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[1] We note that, depending upon the particular choice of construction, the 90% upper limit obtained for the case \( b = 0, n = 0 \) can vary over a small range; e.g. the limit is 2.30 for a one-sided upper limit construction, 2.44 for the methods of Ref.s [4] and [5] and 2.62 for the construction presented here.

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as close as possible to 5%, on the left gives slightly less symmetrical intervals. For the latter choice the 90% Poisson upper limit for \( n = 0 \) is \( \mu_0 = 3.0 \) compared to \( \mu_0 = 2.62 \) for our choice. For \( \mu_0 < 2.62 \), according to this prescription, one cannot construct an interval containing probability > 90% that does not include \( n = 0 \) and we adopt 90% one-sided intervals.

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FIG. 1. Confidence belt, in the usual construction, giving 68.27% central confidence intervals for the unknown mean of a Gaussian with variance $\sigma^2$, in units of $\sigma_N = \sigma / \sqrt{N}$, where $\bar{x}$ is the sample mean of $N$ measurements.
FIG. 2. The classical construction of the 90% central confidence belt (solid) for unknown non-negative Poisson signal $\mu$ in the presence of a Poisson background with known mean $b$ taken to be 3.0, where $n$ is the result of a single observation. Here $\mu_0 = \mu + b$ is the parameter representing the mean of signal plus background. For $n = 0$ the confidence interval is null.
FIG. 3. Confidence belt, in our construction, giving 68.27% central confidence intervals for the unknown mean of a Gaussian with variance $\sigma^2$, in units of $\sigma_N = \sigma/\sqrt{N}$, where $\bar{x}$ is the sample mean of $N$ measurements. For $\bar{x} \leq 0$, $\mu^* = 0$ and the interval is $[0, 1]$. For $0 < \bar{x} \leq \delta\mu$ the interval is $[0, \bar{x} + 1]$. For $\delta\mu < \bar{x} \leq 1 + \delta\mu$ it is $[\bar{x} - \delta\mu, \bar{x} + 1]$. For $1 + \delta\mu < \bar{x} \leq 2$ the interval is $[1, \bar{x} + 1]$ and for $\bar{x} > 2$ we obtain the usual central interval $[\bar{x} - 1, \bar{x} + 1]$. $\delta\mu = 0.475$. 

\[ \delta\mu = 0.475. \]

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FIG. 4. The 90% central confidence belt (solid) for unknown non-negative Poisson signal $\mu$ in the presence of a Poisson background with known mean $b$ taken to be 2.8, where $n$ is the result of a single observation. We show the confidence belt consisting of central intervals $[n_1(\mu_0), n_2(\mu_0)]$ containing at least 90% of the probability for unknown Poisson mean $\mu_0$ in the absence of background (dotted) and the 90% one-sided belt consisting of intervals $[0, n_{os}(\mu_0)]$ (dashed). For $\mu_0 < 2.62$, only one-sided intervals can be constructed. For $b = 2.8$, $b^- = 2$, $\mu_0' = 6.2$, $n_{os}(b) = 5$, and $n_{os}(\mu_0') = 9$ (see text for definitions). For $n \leq b$, the confidence interval for $\mu$ is $[0, c_2(b^-) - b = 3.4]$ and the examples given are for $n \leq 2$. For $b < n \leq 5$, the interval is $[0, c_2(n) - b]$ where the example given is for $n = 4$ and the interval is $[0, 5.8]$. For $5 < n \leq 9$, the interval is $[c_{os}(n) - b, c_2(n) - b]$ where the example given is for $n=7$ and the interval is $[1.1, 9.7]$; for $n = 10$, the interval is $[\mu_0' - b, c_2(n) - b]$; and for $n > 10$, the interval is $[c_1(n) - b, c_2(n) - b]$. Here $[c_1(m), c_2(m)]$ is the Poisson central 90% confidence interval and $c_{os}(m)$ is the one-sided Poisson 90% lower limit, both for a single observation giving $m$ in the absence of any known background.