Cooperative Online Learning

Tommaso R. Cesari  
ANITI, Artificial and Natural Intelligence Toulouse Institute, Toulouse, France  
TSE, Toulouse School of Economics, Toulouse, France

Riccardo Della Vecchia  
Artificial Intelligence Lab, Institute for Data Science & Analytics, Bocconi University, Milano, Italy

Abstract

In this preliminary (and unpolished) version of the paper, we study an asynchronous online learning setting with a network of agents. At each time step, some of the agents are activated, requested to make a prediction, and pay the corresponding loss. Some feedback is then revealed to these agents and is later propagated through the network. We consider the case of full, bandit, and semi-bandit feedback. In particular, we construct a reduction to delayed single-agent learning that applies to both the full and the bandit feedback case and allows to obtain regret guarantees for both settings. We complement these results with a near-matching lower bound.

Keywords: online mirror descent, follow-the-regularized-leader, regret minimization, multiagent learning, online learning with delays

1. Introduction

Distributed online learning settings with communication constraints arise naturally in several applications. For example, large-scale learning systems are often geographically distributed, and in domains such as finance or online advertising, each agent must serve high volumes of prediction requests. Suppose agents keep updating their local models in an online fashion. In that case, bandwidth and computational constraints may preclude a central processor from having access to all the observations from all sessions and synchronizing all local models at the same time. Furthermore, interacting learning agents can also experience delays due to communication. Concretely, consider a network of geographically distributed ad servers using real-time bidding to sell their inventory. Each server sequentially learns how to set the auction parameters (e.g., reserve price) to maximize the network’s overall revenue, and shares feedback information with other servers to speed up learning. However, the rate at which data is exchanged through the communication network is slower than the typical rate at which ads are served. This causes each learner to acquire feedback information from other servers with a delay that depends on the network’s structure.

Motivated by these real-life applications, we introduce and analyze an online learning setting in which a network of agents solves a common online convex optimization problem, in the full and partial feedback setting, by sharing feedback with their network neighbours. We also study the impact of delay on the global performance of these agents, which do not have to be synchronized. At each time step, only some of them are requested to make a prediction and pay the corresponding loss: we call these agents ”active” and the set of active agents at time $t$ is denoted with $\mathcal{A}_t$. If $v \in \mathcal{A}_t$, it predicts with $x_t(v) \in X$ and the network incurs the loss $l_t(x_t(v))$. Besides observing their own feedback, each agent obtains some
information previously broadcast by other agents with a delay equal to the shortest-path distance between the agents. Namely, at time $t$ an agent learns what the active agents at shortest-path distance $s$ did at time $t - s$ for each $s = 1, ..., d$, where $d$ is a delay parameter. The goal is to minimize the total network regret after $T$ time steps

$$R_T = \sum_{t=1}^{T} \sum_{v \in A_t} \ell_t(x_t(v)) - \inf_{x \in X} \sum_{t=1}^{T} \sum_{v \in A_t} \ell_t(x).$$

(1)

In words, this is the difference between the cumulative loss of the “active” agents and the loss that they would have incurred had they consistently made the best prediction in hindsight. The lack of global synchronization implies that agents who are not requested to make a prediction can get some ”free feedback” with a certain delay. Since in online convex optimization the sequence of loss functions is fully arbitrary, it is not clear whether this free feedback can improve the system’s performance. Following some previous work in (Cesa-Bianchi et al., 2019a) and (Cesa-Bianchi et al., 2019b), we will show to which extent such improvements are possible, and we will see that we can characterize the improvement of the cooperative learning of the system in terms of the $d$-th independence number $\alpha_d$ of $G$ is the cardinality of the biggest subset of agents, no two of which have shortest-path distance $d$ or less.

We study the problem under two types of feedback, under the full-information feedback we study the family of algorithm of Online Mirror Descent (OMD), but, since this doesn’t directly give the interesting case of Hedge we resort to a specific analysis for it that makes use of the update of Follow The Regularized Leader (FTRL). The other important case is partial information feedback. We study the case of a network of agents that cooperate to solve the same nonstochastic bandit problem, and we extend the analysis also to the case of semi-bandits on $m$-sets.

On the network, the communication between agents can happen with delays that depend on the topology of the graph itself. For this reason, the first step in our analysis is to obtain the regret for algorithms that play with generic delays in different settings. This part of the analysis, presented in Section 3, relies heavily on previous work by Joulani et al. (2016) and Zimmert and Seldin (2019) with some minor adaptation to recover the particular case of Hedge (with adaptive learning rates) from FTRL and the case of semi-bandits on $m$-sets. In Section 4 we present the main novelty of our paper, which is an algorithm and an analysis that lets one transform a general algorithm that plays with delays into an algorithm on the communication network and retains a neat study for the total regret. We differentiate between two types of activation. In the first one, in Section 4.1, we treat the case of single agent activation whose analysis is more straightforward than the multiple agent activation of Section 4.2. We anticipate that the difference is that in the single agent activation setting, we obtain results for both full and partial information feedback. In contrast, multiple agent activation is studied just in the full information setting with the techniques developed up to there. To fill this gap, in Section 5 we propose an ad-hoc analysis for multiple agents activation and semi-bandits which generalizes the work of Cesa-Bianchi et al. (2019b).
2. Related work

The study of cooperative nonstochastic online learning on networks was pioneered by Awerbuch and Kleinberg (2008), who investigated a bandit setting in which the communication graph is a clique, agents belong to clusters characterized by the same loss, and some agents may be non-cooperative. In our multi-agent setting, the end goal is to control the total network regret (1). This objective was already studied by Cesa-Bianchi et al. (2019a) in the full-information case. A similar line of work was pursued by Cesa-Bianchi et al. (2019b), where the authors consider networks of learning agents that cooperate to solve the same nonstochastic bandit problem. In their setting, all agents are simultaneously active at all time steps, and the feedback propagates throughout the network with a maximum delay of $d$ time steps, where $d$ is a parameter of the proposed algorithm. The authors introduce a cooperative version of Exp3 that they call Exp3-COOP with regret of order $\sqrt{(d + 1 + K\alpha_d/N)(T \log K)}$ where $K$ is the number of arms in the nonstochastic bandit problem, $N$ is the total number of agents in the network, and $\alpha_d$ is the independence number of the $d$-th power of the communication network. The case $d = 1$ corresponds to information that arrives with one round of delay and communication limited to first neighbours. In this setting Exp3-COOP has regret of order $\sqrt{(1 + K\alpha_1/N)(T \log K)}$. Cesa-Bianchi et al. (2019a) present a full information scenario where agents play instances of OMD and exchange information just with first neighbours. In a stochastic activation setting, at each time step $t$ each agent $v \in G$ is independently active with probability $q_v$, where $q_v$ is a fixed and unknown number in $[0, 1]$. Under this assumption, they show that when each agent runs OMD, the network regret is $O(\sqrt{\alpha_1 T})$, where $\alpha_1 \leq N$ is the independence number of the communication graph. The bound smoothly interpolates the two extreme cases of no communication ($\alpha_1 = N$) and full communication ($\alpha_1 = 1$). They also find a matching lower bound for their algorithm. More recently, Della Vecchia and Cesari (2020) considered the case of asynchronous online combinatorial semi-bandits on a network of communicating agents and stochastic activation of agents. They introduce Coop-FTPL, the first algorithm that is computationally efficient in this cooperative setting.

Our work can be seen as an extension of these settings along with two directions. On one side, we are interested in the case in which information is broadcast through the network up to a particular delay $d$ and is successively dropped. On the other hand, we take two types of stochastic activations for the agents: a setting in which a single agent is activated per time-step, and another one, where multiple agents are activated together. From an algorithmic point of view, we extend the analysis of Cesa-Bianchi et al. (2019b) to the case of semi-bandits on $m$-sets where the study follows a very general proof strategy in the single agent activation, while follows more closely (Cesa-Bianchi et al., 2019b) for the multiple agents one. We point out that if the network consists of a single node, our cooperative setting always collapses into a single-agent setting. In particular, for combinatorial bandits, when the number of arms is $k$ and $m = 1$, this becomes the well-known adversarial multiarmed bandit problem (see (Auer et al., 2002)). Hence, ours is a proper generalization of all the settings mentioned above. The main result of our work is stated in Theorem 6 for a general algorithm (for experts or bandits) in a delayed setting, with regret guarantee of the following
where \(a, b_1, b_2, c_2\) are constants. Our theorem states that such an algorithm has a correspondent cooperative counterpart that, in the case of single agent activation, satisfies
\[
R_T^{\text{coop}} \leq a \alpha d + \sqrt{b_1 \alpha d T + b_2 \alpha d T} + \sqrt{c_1 \alpha d T + c_2 d T}.
\]

The regret bound of the cooperative algorithm with multiple agents activation for the full-information setting is instead
\[
\mathbb{E}[R_T^{\text{coop}}] \leq a \frac{\alpha d + Q}{1 - e^{-1}} + Q \sqrt{\frac{b_1}{1 - e^{-1}} \left( \frac{\alpha d}{Q} + 1 \right) T + Q \sqrt{b_2 d T}} + Q \sqrt{\frac{c_1}{1 - e^{-1}} \left( \frac{\alpha d}{Q} + 1 \right) T + c_2 d T},
\]
where \(Q = \sum_{v \in V} q(v)\). These results are very general and explain how to pass from a delayed setting to a cooperative one, also showing how the two are deeply related. From this general formulation, it is then possible to recover the bounds for specific algorithms through a general and much more straightforward analysis. Through a unique framework, it is possible to treat the broadcasting of information through the network and the stochastic activation of agents in an elegant way.

3. Single agent with delay

In recent years, learning with delays has received a significant amount of attention in both stochastic and nonstochastic settings, under full and partial information feedback assumptions. In many practical applications, the learner does not have instant access to the feedback. For example, the time between clicking on a link and buying a product could be minutes, days, weeks, or longer. Similarly, the response to a drug does not come immediately. In most cases, the learner does not have the choice to wait before making the next decision because the arrival of new buyers and patients is beyond their control.

To the best of our knowledge, Weinberger and Ordentlich (2002) were the first to study online learning with delays in the full-information setting. Following their work, several extensions and variations have emerged in both stochastic and nonstochastic bandits, (Joulani et al., 2016; Cesa-Bianchi et al., 2019b; Pike-Burke et al., 2018; Desautels et al., 2014).

In this section, we consider delayed online optimization under both full-information and partial feedback.

3.1 Online linear optimization with delay

The online linear optimization protocol with delays is parameterized by a nonempty, convex, and closed decision set \(\mathcal{X} \subseteq \mathbb{R}^k\) and by an unknown sequence of pairs \((\ell_t, d_t)\) for \(t = 1, 2, \ldots\),
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where each $\ell_t(\cdot) = \langle \ell_t, \cdot \rangle$ is a linear loss function (we abuse language by writing $\ell_t$ to denote both a real-valued linear function and its gradient) and each $0 \leq d_t \leq d$ is an integer delay with $d$ a known constant.

For all $t = 1, 2, \ldots$
1. the learner makes a prediction $x_t \in \mathcal{X}$
2. the learner suffers loss $\ell_t(x_t)$ and receives as feedback the set of pairs
   $$f_t = \{(s, \ell_s) : s \in \{1, \ldots, t\}, s + d_s = t\}.$$ 

We write $D_T$ to denote the total delay $d_1 + \cdots + d_T$ up to horizon $T$. The learner’s goal is to minimize the regret, defined for any time horizon $T$ by

$$R_T = \sup_{x \in \mathcal{X}} R_T(x) \quad \text{where} \quad R_T(x) = \sum_{t=1}^{T} \left( \ell_t(x_t) - \ell_t(x) \right).$$

Without loss of generality, we assume that when the regret is measured at time $T$, then all outstanding pairs are $(t, \ell_t)$ are received at time $T$, implying that $t + d_t \leq T$ for all $t < T$. Note that when $d_t = 0$ for all $t$, this setting reduces to standard online linear optimization with full feedback. Although regret bounds can be proven even without assuming a constant bound on the delay, in our setting the delay is naturally limited by the diameter of the communication network.

Our basic building block for the study of the multi-agent setting is the well-known OMD algorithm (Online Mirror Descent, see Algorithm 1). Given a convex, closed, and set $\mathcal{X}' \subseteq \mathbb{R}^k$ with nonempty interior, OMD is parameterized by a regularizer $F: \mathcal{X}' \to \mathbb{R}$ that is 1-strongly convex with respect to a norm $\|\cdot\|$ on $\mathcal{X}'$ and continuously differentiable on $\text{int}(\mathcal{X}')$. The decision set is $\mathcal{X} \subseteq \text{int}(\mathcal{X}')$, and the update (line 4) is computed in terms of the Bregman divergence $\mathcal{B}_F: \mathcal{X}' \times \text{int}(\mathcal{X}') \to \mathbb{R}$ with respect to $F$, defined by

$$\mathcal{B}_F(x, y) = F(x) - F(y) - \langle \nabla F(y), x - y \rangle \quad \forall (x, y) \in \mathcal{X}' \times \text{int}(\mathcal{X}').$$

The Bregman diameter $R$ of $\mathcal{X} \subseteq \text{int}(\mathcal{X}')$ is $\sup_{x, x' \in \mathcal{X}} \mathcal{B}_F(x, x') \leq 2R^2$. Note that $R$ is not necessarily finite.

**Algorithm 1** Online Mirror Descent (OMD) for linear losses

**input:** a nonincreasing sequence of strictly positive learning rates $\eta_1, \eta_2, \ldots$, and an initial prediction $x_1 \in \mathcal{X}$

1. for $t = 1, 2, \ldots$ do
2. predict $x_t$
3. incur loss $\ell_t(x_t) = \langle \ell_t, x \rangle$
4. let $x_{t+1} \leftarrow \arg\min_{x \in \mathcal{X}} \left\{ \langle \ell_t, x \rangle + \frac{1}{\eta_t} \mathcal{B}_F(x, x_t) \right\}$
5. end for

In order to make OMD robust to delayed feedbacks, we use the wrapper SOLID (Single-Instance Online Learning wIth Delays, Algorithm 2) introduced by Joulani et al. (2016).
SOLID runs on any base algorithm for non-delayed online convex optimization and forces the base algorithm to make an update whenever some new feedback pair \((s, \ell_s)\) is received. If multiple pairs are received simultaneously, they are all processed in the same time step, from the oldest to the newest.

**Algorithm 2 SOLID (Single-Instance Online Learning wIth Delays)**

**input:** an algorithm BASE for the non-delayed setting

**initialization:** let \(x_1\) be the first prediction of BASE

1: for \(t = 1, 2, \ldots \) do
2: predict \(x_t\) and incur loss \(\ell_t(x_t)\)
3: receive the feedback set \(f_t\)
4: if \(f_t \neq \emptyset\) then
5: let \(x_{t+1} \leftarrow x_t\)
6: else
7: for each \((s, \ell_s) \in f_t\) in increasing order of \(s\), do
8: update BASE with \(\ell_s\).
9: end for
10: \(x_{t+1} \leftarrow\) the next prediction of BASE
11: end if
12: end for

The next result is an upper bound on the regret of SOLID run with OMD as base algorithm. The theorem, proven in Appendix A.1 (Theorem 1), is a direct adaptation of a result by Joulani et al. (2016).

**Theorem 1 (Joulani et al. (2016))** The regret of SOLID, run using OMD with a 1-strongly convex regularizer with respect to \(\|\cdot\|\) as base algorithm and with an appropriate choice of learning rates, satisfies

\[
R_T \leq 2LR\sqrt{2T + 4D_T} + LR\sqrt{2d(2d - 1)}.
\]

The learning rates depend only on \(d\), the Bregman radius \(R\), and the Lipschitz constant \(L = \max_{t \leq T} \ell_t, \|\cdot\|_s\), where \(\|\cdot\|_s\) is the dual norm of \(\|\cdot\|\).

In view of extending the analysis to the partial information setting, we now focus on the special case where the decision set \(X\) corresponds to the \(k\)-dimensional simplex of probabilities \(\Delta\). In this case, the standard approach is to use OMD with negative entropy regularization,

\[
F(x) = \sum_{i=1}^{k} x_i \ln x_i. \quad (2)
\]

However, when using dynamic learning rates, the analysis of OMD with negative entropy becomes problematic. To circumvent this issue, we replace OMD with its variant FTRL (Follow The Regularized Leader, Algorithm 3) and keep the negative entropy as regularizer. As it is not clear whether the analysis of Joulani et al. (2016) can work for FTRL with dynamic learning rates, we instead adapt the techniques of Zimmert and Seldin (2019) to derive in Theorem 2 an upper bound on the regret of FTRL with delays. The proof is
deferred to Appendix A.2 (Theorem 2). Before stating the theorem, we define the number of outstanding observations at the beginning of round \( t \) as

\[
\mathcal{d}_t = \sum_{s=1}^{t-1} \mathbb{I}\{s + d_s \geq t\}.
\] (3)

The quantity \( \mathcal{d}_t \) counts how many observations for the previous actions we are missing at the beginning of round \( t \). Unlike the delays \( d_t \), \( \mathcal{d}_t \) is an observable quantity. As such, \( \mathcal{d}_t \) can be used in the dynamic learning rates. Note also that, because we assume \( t + d_t \leq T \) for all \( t < T \), \( \mathcal{D}_T = \mathcal{D}_T \), where \( \mathcal{D}_T = \mathcal{d}_1 + \cdots + \mathcal{d}_T \).

**Algorithm 3** Follow The Regularized Leader (FTRL) with linear losses and delays

**input:** a sequence of regularizers \( F_1, F_2, \ldots \)

**initialization:** let \( L^{\text{obs}} = 0 \)

1. for \( t = 1, 2, \ldots \) do
2. \quad let \( x_t \leftarrow \arg \min_{x \in \Delta} \{ \langle x, L_t^{\text{obs}} \rangle + F_t(x) \} \)
3. \quad predict \( x_t \) and incur loss \( \ell_t(x_t) \)
4. \quad receive the feedback set \( f_t \)
5. \quad if \( f_t \neq \emptyset \) then
6. \quad \quad for each \( (s, \ell_s) \in f_t \) do
7. \quad \quad \quad update \( L^{\text{obs}} \leftarrow L^{\text{obs}} + \ell_s \)
8. \quad \quad end for
9. \quad end if
10. end for

**Theorem 2** The regret of Algorithm 3, run with decreasing learning rates \( (\eta_t)_{t \in \mathbb{N}} \) and regularizer sequence \( F_t = \eta_t^{-1}F \), where \( F \) is the negative entropy regularizer (2), satisfies

\[
R_T \leq \frac{\ln k}{\eta_T} + \frac{1}{2} \sum_{t=1}^{T} \eta_t (1 + 2d_t) .
\]

Moreover, if learning rates are chosen as \( \eta_t = \sqrt{\frac{\ln k}{t+2D_T}} \) for all \( t \), then \( R_T \leq 2\sqrt{(\ln k)(T + D_T)} \).

3.2 Partial information feedback with delay

In this section we extend our analysis to the case of partial and delayed feedback in the bandit and semi-bandit settings. In the bandit setting, at each round \( t \) the learner chooses \( x_t \in \Delta \) and then draws \( I_t \sim x_t \) from the set \( \{1, \ldots, k\} \), whose elements we call actions. Similarly to the full information case, the loss at time \( t \) is \( \ell_t(x_t) = \langle \ell_t, x_t \rangle \). However, the feedback set received at the beginning of round \( t \) is now

\[
f_t = \{ (I_s, \ell_t(I_s)) : s \in \{1, \ldots, t\}, s + d_s = t \}
\]

where we use \( \ell_t(i) \) to indicate the \( i \)-th component of the loss vector \( \ell_t \) (i.e., the \( i \)-th component of the gradient of the linear loss function).
Zimmert and Seldin (2019) analyzed FTRL for bandits with delay (Algorithm 4). We re-formulate their main result (Zimmert and Seldin, 2019, Theorem 1) as Theorem 3 below here. The loss estimators \( \hat{\ell}_s(i) \) used by FTRL are the standard importance-weighted estimates,

\[
\hat{\ell}_s(i) = \ell_s(i) x_s(i) I\{I_s = i\} \quad i = 1, \ldots, k
\]

where \( x_s(i) \) is the algorithm’s probability of selecting action \( i \) at round \( s \). The regularizer is of the form \( F_t = F_{t,1} + F_{t,2} \), where

\[
F_{t,1}(x) = -2\sqrt{t} \sum_{i=1}^{k} x_i^{1/2} \quad \text{and} \quad F_{t,2}(x) = \eta_t^{-1} \sum_{i=1}^{k} x_i \log(x_i).
\]

\( F_{t,1} \) is the \( \frac{1}{2} \)-Tsallis entropy with a dynamic (but delay-insensitive) learning rate \( t^{-1/2} \), \( F_{t,2} \) is the negative entropy with a dynamic learning rate \( \eta_t \) that will be chosen depending on the observed delay sequence.

**Algorithm 4** FTRL for bandits with delay

**input:** a sequence of regularizers \( F_1, F_2, \ldots \)

**initialization:** let \( \hat{L}_{\text{obs}} = 0 \)

1: for \( t = 1, 2, \ldots \) do
2: let \( x_t = \arg \min_{x \in \Delta} \langle x, \hat{L}_{\text{obs}} \rangle + F_t(x) \)
3: sample \( I_t \sim x_t \)
4: receive the feedback set \( f_t \)
5: if \( f_t \neq \emptyset \) then
6: for each \( (s, \ell_s(I_s)) \in f_t \) do
7: compute \( \hat{\ell}_s \)
8: update \( \hat{L}_{\text{obs}} \leftarrow \hat{L}_{\text{obs}} + \hat{\ell}_s \)
9: end for
10: end if
11: end for

**Theorem 3 (Zimmert and Seldin (2019))** The regret of Algorithm 4 run with the regularizer (5) and nonincreasing learning rates \( (\eta_t)_{t \in \mathbb{N}} \) satisfies

\[
R_T \leq 4\sqrt{kT} + \eta_T^{-1} \ln k + \sum_{t=1}^{T} \eta_t d_t.
\]

Furthermore, if learning rates are chosen as \( \eta_t = \sqrt{\frac{\ln k}{2T^2}} \), then \( R_T \leq 4\sqrt{kT} + \sqrt{(8 \ln k)D_T} \).

In the combinatorial extension of the multi-armed bandit problem, the set \( \mathcal{A} \) of available actions is a subset of the \( k \)-dimensional Boolean hypercube \( \{0, 1\}^k \). Randomized learning algorithms use the convex hull \( \text{co}(\mathcal{A}) \) as decision set. At each time step \( t \), given \( x_t \in \text{co}(\mathcal{A}) \), the algorithm draws an action \( V_t \in \mathcal{A} \) from a distribution \( P_t \) on \( \mathcal{A} \) such that

\[
\sum_{a \in \mathcal{A}} a P_t(a) = x_t.
\]
We work in the semi-bandit feedback model with delays, in which the feedback set takes the form

\[ f_t = \{(s, V_s(1)\ell_s(1), \ldots, V_s(k)\ell_s(k)) : s \in \{1, \ldots, t\}, s + d_s = t\} \]

We focus on an important special case is when \( \mathcal{A} \) is the \( m \)-set (the set of all vectors with exactly \( m \) ones) of size \( \binom{k}{m} \). In this setting, we run FTRL (see Algorithm 5) replacing the loss estimators (4) with their for semi-bandit variants

\[ \hat{\ell}_s(i) = \frac{\ell_s(i) V_s(i)}{x(i)} \quad i = 1, \ldots, k. \]

The quantities \( \mathfrak{d}_t \) and \( \mathfrak{D}_t \) are defined as before.

**Algorithm 5** FTRL for semi-bandits on m-sets with delays

**input:** a sequence of regularizers \( F_1, F_2, \ldots \)

**initialization:** let \( \hat{L}^{\text{obs}} = 0 \)

1: for \( t = 1, 2, \ldots \) do
2: let \( x_t = \arg\min_{x} \left\{ \langle x, \hat{L}^{\text{obs}} \rangle + F_t(x) \right\} \)
3: Compute \( P_t \) such that \( \sum_{a \in \mathcal{A}} a P_t(a) = x_t \)
4: sample \( V_t \sim P_t \)
5: receive the feedback set \( f_t \)
6: if \( f_t \neq \emptyset \) then
7: for each \( (s, V_s(1)\ell_s(1), \ldots, V_s(k)\ell_s(k)) \in f_t \) do
8: compute \( \hat{\ell}_s \)
9: update \( \hat{L}^{\text{obs}} \leftarrow \hat{L}^{\text{obs}} + \hat{\ell}_s \)
10: end for
11: end if
12: end for

As before, we use the hybrid regularizer (5) and run Algorithm 5 with this choice. In Appendix B.2 (Theorem 4) we prove the following bound.

**Theorem 4** Algorithm 5 with nonincreasing learning rates \((\eta_t)_{t=1,\ldots,n}\) and regularizers (5) satisfies

\[ R_T \leq 3\sqrt{kmT} + \frac{m}{\eta_T} \ln \frac{k}{m} + k \sum_{t=1}^{T} \eta_t \mathfrak{d}_t \]

Furthermore, for \( \eta_t = \sqrt{\frac{m}{2D_t}}(1 + \ln \frac{k}{m}) \) then

\[ R_T \leq 3\sqrt{kmT} + 2\sqrt{\left(2km \ln \frac{k}{m}\right) D_T}. \]
4. From delayed single-agent to cooperative multi-agent

Let \( G = (\mathcal{V}, \mathcal{E}) \) be an undirected graph. We say that \( G \) is a communication network for a set \( \mathcal{V} \) of agents. For any agent \( v \in \mathcal{V} \) and any nonnegative integer \( d \), let

\[
\mathcal{N}_d(v) \equiv \{ w \in \mathcal{V} : \delta(v, w) \leq d \}
\]

where \( \delta_G(v, w) \) is the shortest-path distance on the graph. Note that \( v \in \mathcal{N}_d(v) \) and \( \mathcal{N}_1(v) \) is the union of \( \{v\} \) with the set of all \( w \in \mathcal{V} \) such that \((v, w) \in \mathcal{E}\). The \( d \)-th independence number \( \alpha_d \) of \( G \) is the cardinality of the biggest subset of agents such that their pairwise shortest-path distances are strictly bigger than \( d \). Hence \( \alpha_1 \) is the standard independence number of \( G \).

We study the following cooperative extension of online linear optimization parameterized by a nonempty, convex, and closed decision set \( \mathcal{X} \subseteq \mathbb{R}^k \), by a nonnegative integer \( d \), and by an unknown sequence of pairs \((\ell_t, \mathcal{A}_t)\) for \( t = 1, 2, \ldots \), where each \( \ell_t(\cdot) = \langle \ell_t, \cdot \rangle \) is a linear loss function and each \( \mathcal{A}_t \subseteq \mathcal{V} \) is a subset of active agents.

At each time step \( t = 1, 2, \ldots \)

1. Each active agent \( v \in \mathcal{A}_t \) predicts with \( x_t(v) \in \mathcal{X} \), incurs loss \( \ell_t(x_t(v)) \), and sends to all neighbours in \( \mathcal{N}_1(v) \) a message \( m_t(v) \), containing the received feedback \( f_t(v) \) with possibly additional information;
2. Each agent \( v \in \mathcal{A} \) receives all messages \( m_{t-s}(v') \) such that \( v' \in \mathcal{A}_{t-s} \) and \( \delta_G(v, v') = s \leq d \) (hence an agent \( v \) active at time \( t \) immediately receives its own message \( m_t(v) \)).

The goal is to minimize the network regret as a function of the number \( T \) of time steps:

\[
R_T = \sum_{t=1}^{T} \sum_{v \in \mathcal{A}_t} \ell_t(x_t(v)) - \inf_{x \in \mathcal{X}} \sum_{t=1}^{T} \sum_{v \in \mathcal{A}_t} \ell_t(x). \tag{6}
\]

Note that only the losses of active agents contribute to the network regret. We will study the problem under different feedback types and under two types of activation mechanisms.

**Feedback type.** We consider the bandit, semi-bandit, and full-info feedback types that we introduced in Section 3. In the bandit case, the feedback \( f_t(v) \) received by \( v \) at time \( t \) is the loss \( \ell_t(x_t(v)) \) of the prediction \( x_t(v) \). In the semi-bandit case, \( \mathcal{X} = \text{co}(\mathcal{A}) \), where \( \mathcal{A} = \{a \in \{0, 1\}^k : \sum_{i=1}^{k} a_i = m\} \) for some \( m \in \mathbb{N} \), the agent \( v \) predicts with \( x_t(v) \in \mathcal{A} \), losses \( \ell_t \) are linear, and the feedback \( f_t(v) \) is the vector \( \langle \ell_t, x_t(v), \ldots, \ell_t, x_{t,k}(v) \rangle \) where, with a slight abuse of notation, we write \( \ell_t(x) = \langle \ell_t, x \rangle \) (i.e., we think \( \ell_t \) as a vector in \( \mathbb{R}^k \)). In the full-information case, the feedback \( f_t(v) \) is the whole loss function \( \ell_t \).

**Activations.** We consider the two distinct settings in which a single agent is activated at each time step or multiple agents are. In the single-activation setting, we assume that there exists a distribution \( q \) on \( \mathcal{V} \) and, for each time step \( t \), the set \( \mathcal{A}_t \) contains only one agent that is drawn i.i.d. from \( q \). In the multiple-activation setting, we assume that there exists an activation probability \( q(v) \in [0, 1] \) for each agent \( v \in \mathcal{V} \) and, at each time step \( t \), each agent \( v \in \mathcal{V} \) is activated i.i.d. with probability \( q(v) \). In the case of full-information we are able to
treat multiple activations, while the single activation setting is the one adopted for learning with partial feedback. This difference is due to technical reasons related to the difficulty of building loss estimators for partial feedback when the pieces of information on the loss at a specific time can arrive at possibly different later rounds. Despite these challenges, we present in Section 5 a different analysis for dealing with multiple agent activations and semi-bandit feedback.

Now we show how the regret guarantees of an algorithm for a delayed single-agent $v$ (Theorems 1, 2, 3, 4) translates to cooperative multi-agent setting (for different choices of feedback type and activations). To this end, we define the random variable $D_t(v)$ by

$$D_t(v) = \sum_{s=t+1}^{t+\delta_t(v)} \mathbb{I}\{\exists v' \in \mathcal{A}_s \cap \mathcal{N}_d(v)\},$$

where $\delta_t(v) = \min_{v' \in \mathcal{A}_t} \delta_G(v', v)$,

with the understanding that $D_t(v) = 0$ if $\delta_t(v) = 0$. At a high-level, the random variable $D_t(v)$ represents the delay of loss $\ell_t$ from the perspective of $v$.

**Algorithm 6** Delay Into Cooperation (DIC)

**input:** maximum delay $d$, single-agent non-delayed algorithm BASE

for each time step $t = 1, 2, \ldots$

1. for each active agent $v \in \mathcal{A}_t$
   
   (a) $v$ outputs the prediction $x_t(v) \in \mathcal{X}$ generated by SOLID(BASE)
   
   (b) $v$ receives some feedback $f_t(v)$ and sends the message $m_t(v) = \langle t, v, f_t(v) \rangle$

2. for each agent $v \in \mathcal{V}$
   
   (a) $v$ receives from its neighbours all past messages $m_{t-s}(w)$ and $m'_{t-s}(w)$ (see last item) such that $w \in \mathcal{A}_{t-s}$ and $\delta_G(v, w) = s \in \{1, \ldots, d\}$;
   
   (b) $v$ drops the messages that are older than $t - d$ and forwards the remaining ones
   
   (c) $v$ makes a step of the SOLID(BASE) algorithm for each newly received message
   
   (d) depending on the setting, $v$ sends to its neighbors a message $m'_t(v) = \langle t, v, i_t(v) \rangle$

   containing some local information $i_t(v)$

We define the total delay of loss $\ell_t$ from the perspective of $v$ by

$$\sum_{t=1}^{T} D_t(v) \mathbb{I}\{\exists v' \in \mathcal{A}_t \cap \mathcal{N}_d(v)\}.$$

The following lemma controls the total (expected) delay of the losses delay of from the perspective of each node $v$.

**Lemma 5** For all agents $v \in \mathcal{V}$, for both single and multiple agents activation, we have

$$\mathbb{E}\left[\sum_{t=1}^{T} D_t(v) \mathbb{I}\{\exists v' \in \mathcal{A}_t \cap \mathcal{N}_d(v)\}\right] \leq T d Q_d(v)^2,$$

where $Q_d(v) = \mathbb{P}\{\exists v' \in \mathcal{A}_t \cap \mathcal{N}_d(v)\}$.  

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Proof For all agents $v \in V$, we have

\[
\mathbb{E} \left[ \sum_{t=1}^{T} D_t \mathbb{I}\{ \exists v' \in A_t \cap N_d(v) \} \right] \\
= \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{E}_t \left[ D_t \mathbb{I}\{ \exists v' \in A_t \cap N_d(v) \} \right] \right] \\
= \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{E}_t \left[ D_t \mathbb{I}\{ \exists v' \in A_t \cap N_d(v) \} | \delta_t(v) \right] \right] \\
= \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{E}_t \left[ D_t | \delta_t(v) \right] \mathbb{I}\{ \exists v' \in A_t \cap N_d(v) \} | \delta_t(v) \right] \\
= \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{E}_t \left[ D_t | \delta_t(v) \right] \mathbb{I}\{ \delta_t(v) \leq d \} \right] \\
= \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{E}_t \left[ D_t | \delta_t(v) = b \right] \mathbb{P}_t [\delta_t(v) = b] \mathbb{I}\{ \delta_t(v) \leq d \} \right] \\
= \mathbb{E} \left[ \sum_{t=1}^{T} \left( D_t | \delta_t(v) = 0 \right) q(v) \right] \\
+ \mathbb{E} \left[ \sum_{t=1}^{T} \sum_{b=0}^{d} \mathbb{E}_t \left[ D_t | \delta_t(v) = b \right] (Q_b(v) - Q_{b-1}(v)) (1 - Q_{b-1}(v)) \right] \\
\leq \mathbb{E} \left[ \sum_{t=1}^{T} \left( D_t | \delta_t(v) = 0 \right) q(v) \right] \\
+ \mathbb{E} \left[ \sum_{t=1}^{T} \sum_{b=1}^{d} \mathbb{E}_t \left[ D_t | \delta_t(v) = b \right] (Q_b(v) - Q_{b-1}(v)) \right] \\
= \sum_{t=1}^{T} \sum_{b=1}^{d} b Q_d(v)(Q_b(v) - Q_{b-1}(v)) \\
\leq \sum_{t=1}^{T} d Q_d(v) \sum_{b=1}^{d} (Q_b(v) - Q_{b-1}(v)) \\
\leq \sum_{t=1}^{T} d Q_d(v)^2 \\
= T d Q_d(v)^2,
\]

where we used the fact that $\mathbb{P}[\delta_t(v) = b] = (Q_b(v) - Q_{b-1}(v)) (1 - Q_{b-1}(v))$. \qed
Theorem 6 Fix any algorithm (for experts or bandits) for the delayed setting having regret guarantees

\[ R^\text{delay}_T \leq a + \sqrt{b_1 T} + \sum_{t=1}^{T} d_t + \sqrt{c_1 T + c_2 \sum_{t=1}^{T} d_t}, \]

where the quantities \(a, b_1, b_2, c_2\) are positive and possibly depend in arbitrary ways on all the relevant parameters of the problem and \(d_1, \ldots, d_T\) are the delays and they are upper bounded by \(d\). Then, the regret of the correspondent cooperative algorithm with single agent activation satisfies

\[ R^\text{coop}_T \leq a \alpha d + \sqrt{b_1 \alpha d T} + \sqrt{b_2 d T} + \sqrt{c_1 \alpha d T} + c_2 d T, \]

where \(\alpha d\) is the \(d\)-th independence number of graph \(G\). The regret bound of the cooperative algorithm with multiple agents activation for the full-information setting satisfies instead

\[ \mathbb{E}[R^\text{coop}_T] \leq a \alpha d + \frac{Q}{1 - e^{-1}} + Q \sqrt{\frac{b_1}{1 - e^{-1}} \left( \frac{\alpha d}{Q} + 1 \right) T} + Q \sqrt{b_2 d T} \]

\[ + Q \sqrt{\frac{c_1}{1 - e^{-1}} \left( \frac{\alpha d}{Q} + 1 \right) T} + c_2 d T, \]

where \(Q = \sum_{v \in V} q(v)\).

Proof Fix any \(x \in X\) and let \(r_t(v) = \ell_t(x_t(v)) - \ell_t(x)\). The regret of agent \(v\) on a communication network is

\[ \sum_{t=1}^{T} r_t(v) \mathbb{I}\{ \exists v' \in A_t \cap N_d(v) \} \]

\[ \leq a + \sqrt{b_1 \sum_{t=1}^{T} \mathbb{I}\{ \exists v' \in A_t \cap N_d(v) \}} + \sqrt{b_2 \sum_{t=1}^{T} D_t(v) \mathbb{I}\{ \exists v' \in A_t \cap N_d(v) \}} \]

\[ + \sqrt{c_1 \sum_{t=1}^{T} \mathbb{I}\{ \exists v' \in A_t \cap N_d(v) \}} + c_2 \sum_{t=1}^{T} D_t(v) \mathbb{I}\{ \exists v' \in A_t \cap N_d(v) \}. \]

We now take expectations to both sides with respect to the activations of the nodes. The expectation of the left-hand side is

\[ \mathbb{E} \left[ \sum_{t=1}^{T} r_t(v) \mathbb{I}\{ \exists v' \in A_t \cap N_d(v) \} \right] = \mathbb{E} \left[ \sum_{t=1}^{T} r_t(v) Q_d(v) \right]. \]

By Jensen’s inequality and Lemma 5, the expectation of the right-hand side can be upper bounded by

\[ a + \sqrt{b_1 Q_d(v) T} + \sqrt{b_2 d Q_d(v)^2 T} + \sqrt{c_1 Q_d(v) T} + c_2 d Q_d(v)^2 T, \]
Putting everything together and dividing both sides by $Q_d(v)$ yields

$$
\mathbb{E} \left[ \sum_{t=1}^{T} r_t(v) \right] \leq \frac{a}{Q_d(v)} + \sqrt{b_1 \frac{1}{Q_d(v)} T + \sqrt{b_2 d T} + \sqrt{c_1 \frac{1}{Q_d(v)} T + c_2 d T}},
$$

Hence, by Jensen’s inequality, we get

$$
\mathbb{E}[R_T^\text{coop}] = \mathbb{E} \left[ \sum_{t=1}^{T} \sum_{v \in \mathcal{V}} r_t(v) \mathbb{I} \{ v \in \mathcal{A}_t \} \right] = \sum_{v \in \mathcal{V}} \sum_{t=1}^{T} q(v) \mathbb{E}[r_t(v)] = \sum_{v \in \mathcal{V}} q(v) \sum_{t=1}^{T} \mathbb{E}[r_t(v)]
$$

$$
\leq \sum_{v \in \mathcal{V}} q(v) \left( \frac{a}{Q_d(v)} + \sqrt{b_1 \frac{1}{Q_d(v)} T + \sqrt{b_2 d T} + \sqrt{c_1 \frac{1}{Q_d(v)} T + c_2 d T}} \right). \tag{7}
$$

The probability $Q_d(v) = \mathbb{P} [\exists v' \in \mathcal{A}_d \cap \mathcal{N}_d(v)]$ has the following two expressions depending on the type of activation

$$
Q_d(v) = \begin{cases} 
\sum_{v' \in \mathcal{N}_d(v)} q(v') & \text{for single agent activation,} \\
1 - \prod_{v' \in \mathcal{N}_d(v)} (1 - q(v')) & \text{for multiple agents activation.}
\end{cases}
$$

Therefore continuing from Eq. (7) we have

$$
\sum_{v \in \mathcal{V}} q(v) \left( \frac{a}{Q_d(v)} + \sqrt{b_1 \frac{1}{Q_d(v)} T + \sqrt{b_2 d T} + \sqrt{c_1 \frac{1}{Q_d(v)} T + c_2 d T}} \right)
$$

$$
= a \sum_{v \in \mathcal{V}} \frac{q(v)}{Q_d(v)} + Q \sum_{v \in \mathcal{V}} \sqrt{\frac{b_1 \frac{1}{Q_d(v)} T + Q \sqrt{b_2 d T} + Q \sum_{v \in \mathcal{V}} \frac{q(v)}{Q} \sqrt{c_1 \frac{1}{Q_d(v)} T + c_2 d T}}}
$$

$$
\leq a \sum_{v \in \mathcal{V}} \frac{q(v)}{Q_d(v)} + Q \sqrt{b_1 \frac{1}{Q} \sum_{v \in \mathcal{V}} \frac{q(v)}{Q_d(v)} T + Q \sqrt{b_2 d T} + Q \sqrt{c_1 \frac{1}{Q} \sum_{v \in \mathcal{V}} \frac{q(v)}{Q_d(v)} T + c_2 d T}},
$$

where we recall that $Q = 1$ for single agent activation.

The analysis of the quantity $\sum_{v \in \mathcal{V}} q(v)/Q_d(v)$ differs at this point for the two types of activations. For single agent activation, using Lemma 2 in Cesa-Bianchi et al. (2019a), we get

$$
\mathbb{E}[R_T^\text{coop}] \leq a \alpha_d + \sqrt{b_1 \alpha_d T + \sqrt{b_2 d T}} + \sqrt{c_1 \alpha_d T + c_2 d T}.
$$

In the case of multiple agents activation we use Lemma 16 (setting $p(i, v)$ equal to $q(v)$) and this leads to the following bound

$$
\mathbb{E}[R_T^\text{coop}] \leq a \frac{\alpha_d + \sqrt{Q}}{1 - e^{-1}} + Q \sqrt{\frac{b_1}{1 - e^{-1}} \left( \frac{\alpha_d}{Q} + 1 \right) T + Q \sqrt{b_2 d T}}
$$

$$
+ Q \sqrt{\frac{c_1}{1 - e^{-1}} \left( \frac{\alpha_d}{Q} + 1 \right) T + c_2 d T}.
$$

\[\blacksquare\]
4.1 Cooperative learning with single agent activation

At this point we have all the tools to show how the regret guarantees of the algorithms in Theorems 1, 2, 3, 4, translate when algorithms are played in a cooperative multi-agent setting. In this section anyway we treat the case of single agent activation which is the simplest and is available for all the algorithms that we have presented so far. We postpone to the next section a description of the technical difficulties encountered when trying to do the same thing in a partial information setting. Going in the same order of Section 3 we have the following corollaries.

**Corollary 7** The regret of DIC, when it is run with maximum delay $d$ in a single agent activation setting and the BASE algorithm is OMD, has regret bound that satisfies

$$R_T^{coop} \leq 2LR\sqrt{2T(\alpha_d + 2d)} + LR\sqrt{2d(d + 1)}.$$ 

**Proof** Exploiting the result of Theorem 1 we have the following regret

$$R_T \leq 2LR\sqrt{2\sum_{t=1}^{T}(1 + 2d_t)} + LR\sqrt{2d(d + 1)},$$

and from Theorem 6 we obtain the following regret for the communication network

$$R_T^{coop} \leq 2LR\sqrt{2T(\alpha_d + 2d)} + LR\sqrt{2d(d + 1)}.$$ 

Under some mild conditions OMD and FTRL are the same family of algorithms (see Orabona (2019)). Therefore, we are interested to give a regret bound for a special member of these families which is Hedge. The regret of Hedge on a communication network follows in the corollary below.

**Corollary 8** The regret of Hedge, when it is run with maximum delay $d$ in a single agent activation setting is

$$R_T^{coop} \leq 2\sqrt{\log(k)T(\alpha_d + d)}.$$ 

**Proof** Exploiting the result of Theorem 2, we have the following regret

$$R_T \leq 2\sqrt{\log(k)\left(T + \sum_{t=1}^{T}d_t\right)},$$

and from Theorem 6 we obtain the following regret for the communication network

$$R_T^{coop} \leq 2\sqrt{\log(k)T(\alpha_d + d)}.$$ 

For the case of bandit feedback on the simplex we have the following corollary.
Corollary 9  The regret of FTRL for bandits, when it is run with maximum delay \( d \) in a single agent activation setting is

\[
R_T^{\text{coop}} \leq 4\sqrt{\alpha_d kT} + \sqrt{8Td \log k}.
\]

Proof  Exploiting the result of Theorem 3 we have the following regret

\[
R_T \leq 4\sqrt{kT} + \sqrt{8T \sum_{t=1}^{T} d_t \log k}.
\]

and from Theorem 6 we obtain the following regret for the communication network

\[
R_T^{\text{coop}} \leq 4\sqrt{\alpha_d kT} + \sqrt{8Td \log k}.
\]

The cooperative regret of the optimal algorithm for learning with semi-bandit feedback is given in the following corollary.

Corollary 10  The regret of FTRL for semi-bandits on \( m \)-sets with the regularizer in Eq. (5), when it is run with maximum delay \( d \) in a single agent activation setting is

\[
R_T^{\text{coop}} \leq 3\sqrt{\alpha_d T km} + 2\sqrt{2km \log(k/m) dT}.
\]

Proof  Exploiting the result of Theorem 4 we have the following regret

\[
R_T \leq 3\sqrt{T km} + 2\sqrt{2km \log(k/m) \left( \sum_{t=1}^{T} d_t \right)}.
\]

and from Theorem 6 we obtain the following regret for the communication network

\[
R_T^{\text{coop}} \leq 3\sqrt{\alpha_d T km} + 2\sqrt{2km \log(k/m) dT}.
\]

We note that with the entropic regularizer, we obtain the following regret bound instead (which is not optimal):

\[
R_T^{\text{coop}} \leq 2\sqrt{2km(1 + \log(k/m))T(\alpha_d + d)}.
\]

4.2 Cooperative learning with multiple agents activation

The case of partial information is more complicated to treat for multiple agents activation than the full-info. This stems from the fact that is less trivial to construct the estimators for the losses. In fact, given a node \( v \) and its neighbourhood \( \mathcal{N}_d(v) \) let us assume is possible to take two different nodes \( v', v'' \in \mathcal{N}_d(v) \) such that \( \delta_\mathcal{G}(v, v') \neq \delta_\mathcal{G}(v, v'') \). Furthermore, assuming \( v', v'' \in \mathcal{S}_t \) they received feedbacks \( f_t(v') \) and \( f_t(v'') \) and both of these feedbacks contain some, a priori different, information on the loss \( \ell_t \). Since \( v' \) and \( v'' \) have different
distances from $v$ the two feedbacks will get to node $v$ with two different delays and this doesn’t allow for a direct applications of the techniques proposed in Zimmert and Seldin (2019). For this reason the corollaries of this section just refer to the full-information case and are the following two.

**Corollary 11** The regret of DIC, when it is run with maximum delay $d$ in a multiple agents activation setting and the BASE algorithm is OMD, has regret bound that satisfies

$$R_{T}^{\text{coop}} \leq 2LRQ \sqrt{\frac{2}{1-e^{-1}} \left( \frac{\alpha_d}{Q} + 1 \right) T + 4dT + LR \frac{\alpha_d + Q}{1-e^{-1}} \sqrt{2d(d+1)}}.$$  

**Proof** Exploiting the result of Theorem 1 we have the following regret

$$R_{T} \leq 2LR \sqrt{2 \sum_{t=1}^{T} (1 + 2d_t) + LR \sqrt{2d(d+1)},}$$

and from Theorem 6 we obtain the following regret for the communication network

$$R_{T}^{\text{coop}} \leq 2LRQ \sqrt{\frac{2}{1-e^{-1}} \left( \frac{\alpha_d}{Q} + 1 \right) T + 4dT + LR \frac{\alpha_d + Q}{1-e^{-1}} \sqrt{2d(d+1)}}.$$

**Corollary 12** The regret of Hedge run in a cooperative multiple agents activation setting, with maximum delay $d$ is

$$R_{T}^{\text{coop}} \leq 2Q \sqrt{\log(k) T \left( \frac{1}{1-e^{-1}} \left( \frac{\alpha_d}{Q} + 1 \right) + d \right)}.$$  

**Proof** Exploiting the result of Theorem 2 we have the following regret

$$R_{T} \leq 2 \sqrt{\log(k) \left( T + \sum_{t=1}^{T} d_t \right)},$$

and from Theorem 6 we obtain the following regret for the communication network

$$R_{T}^{\text{coop}} \leq 2Q \sqrt{\log(k) T \left( \frac{1}{1-e^{-1}} \left( \frac{\alpha_d}{Q} + 1 \right) + d \right)}.$$
Algorithm 7 BaditCoopMsets

**Parameters:** learning rate $\eta > 0$

**Initialization:** each agent $v \in V$ sets weights $w_1(i, v) = 1/k$ and $\overline{w}_1(i, v) = m/k$ for all $i \in \{1, \ldots, k\}$

**For:** $t = 1, 2, \ldots$

1. for each active agent $v \in S_t$
   
   (a) $v$ computes a probability distribution $P_t(v) = (P_t(a, v))_{a \in A}$ on $A$ such that
   $$\overline{w}_t(i, v) = \sum_{a \in A} a P_t(a, v), \quad \forall i \in \{1, \ldots, k\}$$

   (a2) $v$ outputs the prediction $x_t(v) \in A$ drawn according to $P_t(v)$
   
   (b) $v$ receives the feedback $f_t(v) = (x_{t,1}, x_{t,2}, \ldots, x_{t,k})$ and sends the message
   $$m_t(v) = \langle t, v, f_t(v) \rangle$$

2. for each agent $v \in V$

   (a) $v$ receives from its neighbours all past messages $m_{t-s}(w)$ and $m'_{t-s}(w)$ (see last item) such that $w \in A_{t-s}$ and $\delta_G(v, w) = s \in \{1, \ldots, d\}$
   
   (b) $v$ drops the messages that are older than $t - d$ and forwards the remaining ones
   
   (c1) $v$ performs the update
   $$w_{t+1}(i, v) = \overline{w}_t(i, v) \exp(-\eta \hat{\ell}_t(i, v)), \quad \forall i \in \{1, \ldots, k\} \quad (8)$$
   
   where
   $$\hat{\ell}_t(i, v) = \begin{cases} \ell_{t-d(i)} - B_{d,t-d}(i, v) & \text{if } t > d \\ 0 & \text{otherwise} \end{cases}$$

   with
   $$B_{d,t-d}(i, v) = \mathbb{I}\{\exists v' \in N_d(v) \cap S_{t-d} : x_{t-d}(i, v') = 1\}$$
   
   and
   $$\overline{B}_{d,t-d}(i, v) = 1 - \prod_{v' \in N_d(v)} \left(1 - \overline{w}_{t-d}(i, v') q(v')\right)$$

   (c2) each agent $v \in V$ computes $\overline{w}_t(v) = (\overline{w}_t(1, v), \ldots, \overline{w}_t(k, v))$ as
   $$\overline{w}_{t+1}(i, v) = m w_{t+1}(i, v) \overline{w}_{t+1}(v), \quad \forall i \in \{1, \ldots, k\} \quad \text{where} \quad W_{t+1}(v) = \sum_{j=1}^{\infty} w_{t+1}(j, v)$$

   (d1) each agent $v \in A_t$ sends to its neighbors the message $m'_t(v) = \langle t, v, i_t(v) \rangle$, where
   $$i_t(v) = (\overline{w}_t(v), x_t(v))$$

   (d2) if $t > 1$ each agent $v \in V \setminus A_t$ such that $\overline{x}_t(v) \neq \overline{x}_{t-1}(v)$ sends to its neighbors the message $m'_t(v) = \langle t, v, i_t(v) \rangle$, where $i_t(v) = (\overline{w}_t(v))$
5. Cooperative multiple agents activation setting for semibandits

As we anticipated in Section 4 the case of partial information is more complicated to treat for multiple agents activation than the full-info. For this reason we propose here Algorithm 7 that uses the loss estimator in Eq. (8). Its analysis is similar in spirit to the analysis contained in Cesa-Bianchi et al. (2019b), the main difference here is that we extend the analysis therein to the case of semi-bandit feedback on \( m \)-sets and stochastic activation of the agents on the communication network. The following lemma provides a deterministic bound for a single agent \( v \), its analysis mimics very closely the analysis that is done for EXP3 and its proof can be found in Appendix C, Lemma 13.

**Lemma 13** If agent \( v \in V \) runs the Algorithm 7 with learning rate \( \eta > 0 \), the following deterministic bound holds for all \( i \in \{1, \ldots, k\} \):

\[
\sum_{t=1}^{T} \sum_{i=1}^{k} \frac{\tilde{x}_t(i, v)}{m} \tilde{\ell}_t(i, v) - \sum_{t=1}^{T} \frac{\tilde{\ell}_t(i^*, v)}{m} \leq \frac{\ln k}{\eta} + \frac{\eta}{2} \sum_{t=1}^{T} \sum_{i=1}^{k} \frac{\tilde{x}_t(i, v)}{m} \tilde{\ell}_t(i, v)^2 .
\]

Next we have a lemma to bound the additive drift of the algorithm in a semibandit setting and it is the analogous of Lemma 1 in Cesa-Bianchi et al. (2019b) for bandits. We postpone its proof to Appendix C, Lemma 14.

**Lemma 14** If agent \( v \in V \) runs BaditCoopMsets with learning rate \( \eta > 0 \), the following deterministic bounds for the drift probabilities hold for all \( i \in \{1, \ldots, k\} \):

\[
-\eta \frac{\tilde{x}_t(i, v)}{m} \tilde{\ell}_t(i, v) \leq \frac{\tilde{x}_{t+1}(i, v)}{m} - \frac{\tilde{x}_t(i, v)}{m} \leq \eta \frac{\tilde{x}_{t+1}(i, v)}{m} - \frac{\tilde{x}_t(i, v)}{m} \sum_{j=1}^{k} \frac{\tilde{x}_t(j, v)}{m} \tilde{\ell}_t(j, v) .
\]

A lemma to bound the drift in a multiplicative way follows. It is the analogous of Lemma 2 in Cesa-Bianchi et al. (2019b) for bandits. Its proof is in Appendix C, Lemma 15.

**Lemma 15** If agent \( v \in V \) runs BaditCoopMsets with learning rate \( \eta \in \left( 0, \frac{m}{k e^{(d+1)}} \right) \), the following deterministic bound holds for all \( i \in \{1, \ldots, k\} \):

\[
\tilde{x}_{t+1}(i, v) \leq \left( 1 + \frac{1}{d} \right) \tilde{x}_t(i, v) .
\]

Finally, a last lemma is used in Theorem 17 to link the regret for the network, that is obtained summing over the agents, to the independence number of the corresponding graph, which is characteristic of the network topology (see Cesa-Bianchi et al. (2019b) for a proof).

**Lemma 16** Let \( G = (V, E) \) be an undirected graph with independence number \( \alpha_1 \). For each \( v \in V \), let \( N_1(v) \) be the neighborhood of node \( v \) (including \( v \) itself), and \( p_t(v) = (p(1, v), \ldots, p(k, v)) \) has positive entries. Then, for all \( i \in [k] \),

\[
\sum_{v \in V} \frac{p(i, v)}{q(i, v)} \leq \frac{1}{1 - e^{-1}} \left( \alpha_1 + \sum_{v \in V} p(i, v) \right) \text{ where } q(i, v) = 1 - \prod_{v' \in N_1(v)} (1 - p(i, v')) .
\]
Theorem 17 If BaditCoopMsets (Algorithm 7) is run with \( \eta > 0 \), its regret satisfies
\[
R_T \leq 2dQm + \frac{m \ln k}{\eta}Q + 4\eta T Q \left( \frac{k}{Q} \alpha_d + md \right),
\]
where \( Q = \sum_{v \in V} q(v) \). Choosing, in particular, \( \eta = Q \sqrt{(m \ln k)/(4TQ((k/Q) \alpha_d + md))} \), yields
\[
R_T \leq 2dQm + 2Q \sqrt{mT \ln(k) \left( \frac{k}{Q} \alpha_d + md \right)}.
\]

Proof The standard analysis of the exponentially-weighted algorithm in Lemma 13 with importance-sampling estimates gives for each agent \( v \) and each \( i^* \in \{1, \ldots, k\} \), the deterministic bound
\[
\sum_{t=1}^{T} \sum_{i=1}^{k} \frac{\bar{x}_t(i, v)}{m} \hat{e}_t(i, v) - \sum_{t=1}^{T} \hat{e}_t(i^*, v) \leq \frac{\ln k}{\eta} + \frac{\eta}{2} \sum_{t=1}^{T} \sum_{i=1}^{k} \frac{\bar{x}_t(i, v)}{m} \hat{e}_t(i, v)^2.
\]
Iterative applications of the first inequality in Lemma 14 gives, for \( t > d \),
\[
\frac{\bar{x}_t(i, v)}{m} \geq \frac{\bar{x}_{t-d}(i, v)}{m} - \eta \sum_{s=1}^{k} \frac{\bar{x}_{t-s}(i, v)}{m} \hat{e}_{t-s}(i, v),
\]
so that, setting for brevity \( A_t(i, v) = \sum_{s=1}^{k} \frac{\bar{x}_{t-s}(i, v)}{m} \hat{e}_{t-s}(i, v) \) we have
\[
\sum_{t=1}^{T} \sum_{i=1}^{k} \frac{\bar{x}_t(i, v)}{m} \hat{e}_t(i, v) \geq \sum_{t=d+1}^{T} \sum_{i=1}^{k} \frac{\bar{x}_t(i, v)}{m} \hat{e}_t(i, v)
\]
\[
\geq \sum_{t=d+1}^{T} \sum_{i=1}^{k} \frac{\bar{x}_{t-d}(i, v)}{m} \hat{e}_{t-d}(i, v) - \eta \sum_{t=d+1}^{T} \sum_{i=1}^{k} A_t(i, v) \hat{e}_t(i, v).
\]
Hence
\[
\mathbb{E} \left[ \sum_{t=1}^{T} \sum_{i=1}^{k} \frac{\bar{x}_t(i, v)}{m} \hat{e}_t(i, v) \right]
\]
\[
\geq \mathbb{E} \left[ \sum_{t=d+1}^{T} \sum_{i=1}^{k} \frac{\bar{x}_{t-d}(i, v)}{m} \hat{e}_{t-d}(i, v) \right] - \eta \mathbb{E} \left[ \sum_{t=d+1}^{T} \sum_{i=1}^{k} A_t(i, v) \hat{e}_t(i, v) \right]
\]
\[
= \mathbb{E} \left[ \sum_{t=d+1}^{T} \sum_{i=1}^{k} \frac{\bar{x}_t(i, v)}{m} \hat{e}_{t-d}(i, v) \right] - \eta \mathbb{E} \left[ \sum_{t=d+1}^{T} \sum_{i=1}^{k} A_t(i, v) \hat{e}_t(i, v) \right]
\]
\[
\geq 2d - \eta T d
\]
where the last step follows by

\[
E \left[ \sum_{i=1}^{k} A_t(i, v) \ell_{t-d}(i) \right] \leq E \left[ \sum_{i=1}^{k} A_t(i, v) \right] = E \left[ \sum_{i=1}^{k} \sum_{s=1}^{k} \frac{\overline{x}_{i,s}(i, v)}{m} \ell_{t-s}(i, v) \right] \\
= E \left[ \sum_{i=1}^{k} \sum_{s=1}^{k} \frac{\overline{x}_{i,s}(i, v)}{m} \ell_{t-s}(i) \right] \leq E \left[ \sum_{i=1}^{k} \sum_{s=1}^{k} \frac{\overline{x}_{i,s}(i, v)}{m} \right] = d.
\]

Similarly, for the second sum in (9), we have

\[
E \left[ \sum_{t=d+1}^{T} \ell_t(i^*, v) \right] = \sum_{t=d+1}^{T} \ell_{t-d}(i^*) \leq \sum_{t=1}^{T} \ell_t(i^*).
\]

Finally for the third sum in (9), an iterative application of Lemma 15 and the inequality \((1 + \frac{1}{d})^k \leq e\) yields, for \(t > d\),

\[
\frac{\overline{x}_t(i, v)}{m} \leq \left( 1 + \frac{1}{d} \right)^k \frac{\overline{x}_{t-d}(i, v)}{m} \leq e \frac{\overline{x}_{t-d}(i, v)}{m},
\]

so that we can finally write

\[
E \left[ \sum_{t=d+1}^{T} \sum_{i=1}^{k} \frac{\overline{x}_t(i, v)}{m} \ell_t(i, v)^2 \right] = \frac{1}{m} E \left[ \sum_{t=d+1}^{T} \sum_{i=1}^{k} \overline{x}_{t-d}(i, v) \ell_t(i, v)^2 \right] \\
\leq \frac{1}{m} E \left[ \sum_{t=d+1}^{T} \sum_{i=1}^{k} \frac{\overline{x}_t(i, v)}{B_{d,t-d}(i, v)} \right] \\
\leq \frac{e}{m} E \left[ \sum_{t=d+1}^{T} \sum_{i=1}^{k} \frac{\overline{x}_{t-d}(i, v)}{B_{d,t-d}(i, v)} \right].
\]

Therefore, putting everything together and multiplying by \(m\), we have, for any \(i^* \in \{1, \ldots, k\}\),

\[
E \left[ \sum_{t=1}^{T} \sum_{i=1}^{k} \ell_t(i) \overline{x}_t(i, v) \right] - m \sum_{t=1}^{T} \ell_t(i^*) \leq 2dm + \eta dm T + \frac{m \ln k}{\eta} + \eta e E \left[ \sum_{t=d+1}^{T} \sum_{i=1}^{k} \frac{\overline{x}_{t-d}(i, v)}{B_{d,t-d}(i, v)} \right].
\]

(10)

We will use this estimate to upper bound the regret. Let

\[
a^* \in \arg \min_{a \in A} \sum_{t=1}^{T} \sum_{v \in V} \sum_{j=1}^{k} \ell_t(j) a(j) \mathbb{I}\{v \in S_t\}
\]

\[
i^* \in \arg \min_{j \in \{1, \ldots, k\}} \sum_{t=1}^{T} m \ell_t(j) q(v)
\]
Then

\[
R_T = E \left[ \sum_{t=1}^{T} \sum_{v \in V} \sum_{i=1}^{k} \ell_t(i) x_t(i, v) \mathbb{1}\{v \in S_t\} \right] - \min_{a \in A} E \left[ \sum_{t=1}^{T} \sum_{v \in V} \sum_{j=1}^{k} \ell_t(j) a(j) \mathbb{1}\{v \in S_t\} \right]
\]

\[
\leq E \left[ \sum_{t=1}^{T} \sum_{v \in V} \sum_{i=1}^{k} \ell_t(i) \mathbb{1}_t(i, v) q(v) \right] - \sum_{t=1}^{T} \sum_{v \in V} \left( \sum_{j=1}^{k} \ell_t(j) a^*(j) \right) q(v)
\]

\[
\leq \sum_{v \in V} \left( E \left[ \sum_{t=1}^{T} \sum_{i=1}^{k} \ell_t(i) \mathbb{1}_t(i, v) \right] - m \sum_{t=1}^{T} \ell_t(i^*) \right) q(v)
\]

\[
\leq \sum_{v \in V} \left( 2dm + \eta dmT + \frac{m \ln k}{\eta} + \frac{\eta e}{2} E \left[ \sum_{t=d+1}^{T} \sum_{i=1}^{k} \frac{\mathbb{1}_t(i, v)}{B_{d,t-d}(i, v)} \right] q(v) \right)
\]

\[
= 2dmQ + \eta dmTQ + \frac{m \ln k}{\eta} Q + \frac{\eta e}{2} E \left[ \sum_{t=d+1}^{T} \sum_{i=1}^{k} \sum_{v \in V} \frac{\mathbb{1}_t(i, v)}{B_{d,t-d}(i, v)} q(v) \right]
\]

For the last term, applying Lemma 16 to the \(d\)-th power of the graph \(G\), we get the following upper bound,

\[
E \left[ \sum_{t=d+1}^{T} \sum_{i=1}^{k} \sum_{v \in V} \mathbb{1}_t(i, v) q(v) \right] \leq \frac{1}{(1 - e^{-1})} E \left[ \sum_{t=d+1}^{T} \sum_{i=1}^{k} \alpha_d + \sum_{v \in V} \mathbb{1}_t(i, v) q(v) \right]
\]

\[
\leq \frac{1}{(1 - e^{-1})} T(k \alpha_d + mQ)
\]

\[
= \frac{1}{(1 - e^{-1})} TQ \left( \frac{k}{Q} \alpha_d + m \right).
\]

Putting all together

\[
R_T \leq 2dmQ + \eta dmTQ + \frac{m \ln k}{\eta} Q + \frac{\eta e TQ}{2 (1 - e^{-1})} \left( \frac{k}{Q} \alpha_d + m \right).
\]
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Appendix A. Full-information setting with delays

We recall some notation that we will use extensively in the analysis of OMD. We denote the topological interior of $\mathcal{X}'$ by $\text{int}(\mathcal{X}')$ and define the Bregman divergence $\mathcal{B}_F: \mathcal{X}' \times \text{int}(\mathcal{X}') \to \mathbb{R}$ with respect to a differentiable function $F: \text{int}(\mathcal{X}') \to \mathbb{R}$ as

$$\mathcal{B}_F(x, y) = F(x) - F(y) - \langle \nabla F(y), x - y \rangle \quad \forall (x, y) \in \mathcal{X}' \times \text{int}(\mathcal{X}')$$ (11)

We recall that for all $z \in \mathbb{R}^k$, the dual norm of $z$ (or, equivalently, of the linear functional $\langle z, \cdot \rangle$) is given by

$$\|z\|_* = \max \{ \langle z, x \rangle : x \in \mathbb{R}^k, \|x\| \leq 1 \}$$ (12)

We also recall that a differentiable function $f$ is called $\sigma$-strongly convex with respect to a norm $\|\cdot\|$, if

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\sigma}{2} \|x - y\|^2$$

for all $x, y$ and for some $\sigma > 0$. Note that the previous expression is equivalent to

$$\mathcal{B}_F(x, y) \geq \frac{\sigma}{2} \|x - y\|^2$$ (13)

by definition of Bregman divergence.

Finally, let $F$ be a convex function and $\mathcal{X} = \text{dom}(F)$ and $C = \text{int}(\mathcal{X})$. Then $F$ is Legendre if

1. $C$ is nonempty;
2. $F$ is differentiable and strictly convex on $C$
3. $\lim_{n \to \infty} \|\nabla F(x_n)\|_2 = \infty$ for any sequence $(x_n)_n$ with $x_n \in C$ for all $n$ and $\lim_{n \to \infty} x_n = x$ and some $x \in \partial C$

A.1 Analysis of Online Mirror Descent with delays

Let us introduce the delayed environment following the work of Joulani et al. (2016). For all $s$, we denote by $\rho(s)$ the time step in which the $s$-th feedback pair, that BASE receives from SOLID (line 8 of Algorithm 2), was generated. Next, for all $s$, we denote by $\tilde{\tau}_s$ the number of feedbacks that BASE receives between the time $\rho(s)$ in which SOLID makes the prediction $x_{\rho(s)}$ (i.e., when the learner incurs loss $\ell_{\rho(s)}$) and that in which $\ell_{\rho(s)}$ is received by BASE (i.e., when the learner receives the loss $\ell_{\rho(s)}$ at round $\rho(s) + d_{\rho(s)}$). For all $s$, we set $\tilde{\ell}_s = \ell_{\rho(s)}$. In words, $\tilde{\ell}_1, \tilde{\ell}_2, \ldots$ is the sequence of losses in the order received by BASE. In the same spirit, for all $s$ we denote the prediction made by BASE after receiving $\tilde{\ell}_s$ by $\tilde{x}_{s+1}$. Note that $\tilde{x}_{s+1} = x_{\rho(s)}$. Furthermore, without loss of generality we will assume that for any $1 \leq t \leq T$, $t + d_t \leq T$, i.e., all feedbacks are received by the end of round $T$. This does not restrict generality because the feedbacks that arrive in round $T$ are not used to make any predictions and hence do not influence the regret of SOLID. Note that under this assumption $\sum_{s=1}^T \tilde{\tau}_s = \sum_{t=1}^T d_t$ (both count over time the total number of outstanding feedbacks), and $(\rho(s))_{1 \leq s \leq T}$ is a permutation of the integers $\{1, \ldots, T\}$.

We recall here an important identity that was proven in Joulani et al. (2016).
Let BASE be any deterministic algorithm for the non-delayed setting. For every $x \in X$ and all time horizons $T$, the regret of SOLID with input BASE satisfies

$$R_T(x) = \tilde{R}_T(x) + \sum_{s=1}^{T} \tilde{D}_{s, \tilde{r}_s}$$

where

$$\tilde{R}_T(x) = \sum_{s=1}^{T} \tilde{\ell}_s(\tilde{x}_s) - \sum_{s=1}^{T} \tilde{\ell}_s(x)$$

is the regret of BASE relative to $x$ for the sequence of losses $\tilde{\ell}_1, \ldots, \tilde{\ell}_T$ and

$$\tilde{D}_{s, \tilde{r}_s} = \tilde{\ell}_s(\tilde{x}_s - \tilde{r}_s) - \tilde{\ell}_s(\tilde{x}_s) = \ell_{\rho(s)}(x_{\rho(s)}) - \tilde{\ell}_s(\tilde{x}_s)$$

is the prediction drift of BASE while feedback $\tilde{\ell}_s$ is outstanding.

In the following we will use Online Mirror Descent (OMD) as BASE and we study the regret of OMD in a delayed environment bounding separately the two contributions coming from the non-delayed regret of BASE and its prediction drift. Let us start with the following lemma that bounds the stability for OMD.

**Lemma 19** If OMD is run with inputs $X, F, (\eta_t)_{t \in \mathbb{N}}, x_1$ and $F$ is $1$-strongly convex with respect to a norm $\|\cdot\|$, then, for all $t = 1, 2, \ldots$

$$\|x_t - x_{t+1}\| \leq \eta_t \|g_t\|_*$$

**Proof** Fix any $t$. Without loss of generality, assume that $\|x_t - x_{t+1}\| > 0$. Recall that by definition of subgradient, a point $x^*$ is the minimum of a convex function $f$ if and only if $\langle \nabla f(x^*), x - x^* \rangle \geq 0$ for all $x$, where $\nabla f(x^*)$ is any subgradient of $f$ at $x^*$. Since for all $t$, $x_{t+1} = \min_{x \in X} \{ f_t(x) \}$, where $f_t(x) = \langle g_t, x \rangle + \frac{1}{\eta_t} \mathcal{B}_F(x, x_t)$ is convex, we have that

$$\eta_t \langle \nabla f_t(x_{t+1}), x_t - x_{t+1} \rangle + \eta_t \|g_t\|_* \|x_t - x_{t+1}\| \geq \langle \nabla F(x_t), x_t - x_{t+1} \rangle$$

or, equivalently,

$$\langle \eta_t g_t, x_t - x_{t+1} \rangle + \eta_t \|g_t\|_* \|x_t - x_{t+1}\| \geq \langle \nabla F(x_t) - \nabla F(x_{t+1}), x_t - x_{t+1} \rangle$$

(15)

By the 1-strong convexity of $F$, we get

$$\|x_t - x_{t+1}\|^2 = \frac{1}{2} \|x_t - x_{t+1}\|^2 + \frac{1}{2} \|x_{t+1} - x_t\|^2$$

$$\leq \mathcal{B}_F(x_t, x_{t+1}) + \mathcal{B}_F(x_{t+1}, x_t) = \langle \nabla F(x_t) - \nabla F(x_{t+1}), x_t - x_{t+1} \rangle$$

Further upper bounding with Eq. (15) and by definition of dual norm, we have

$$\|x_t - x_{t+1}\|^2 \leq \langle \eta_t g_t, x_t - x_{t+1} \rangle \leq \eta_t \|g_t\|_* \|x_t - x_{t+1}\|$$

The result follows by dividing both the left and the right hand side by $\|x_t - x_{t+1}\|$. □

The following known result states an upper bound for the regret of OMD (run in a non-delayed setting).
Theorem 20: The regret of OMD (Algorithm 1) for the sequence of losses \( \tilde{\ell}_1, \ldots, \tilde{\ell}_T \) with \( \{\tilde{\eta}_s\}_{s=1}^{T} \) as the set of learning rates, \( u \in \mathcal{X} \), \( F \) a 1-strongly convex regularizer w.r.t. norm \( \|\cdot\| \) is:

\[
\tilde{R}_T^{BASE}(u) \leq \frac{\max_{1 \leq s \leq T} \mathcal{B}_F(u, \tilde{x}_s)}{\tilde{\eta}_T} + \frac{1}{2} \sum_{s=1}^{T} \tilde{\eta}_s \|\tilde{g}_s\|_*^2.
\]

If we assume that \( \max_{1 \leq s \leq T} \mathcal{B}_F(u, \tilde{x}_s) \leq 2R^2 \) for a positive constant \( R > 0 \), then the regret is

\[
\tilde{R}_T^{BASE}(u) \leq \frac{2R^2}{\tilde{\eta}_T} + \frac{1}{2} \sum_{s=1}^{T} \tilde{\eta}_s \|\tilde{g}_s\|_*^2.
\]

From Lemma 19 the stability of OMD is bounded by \( \|\tilde{x}_j - \tilde{x}_{j+1}\| \leq \tilde{\eta}_j \|\tilde{g}_j\|_* \), and the drift term for linear losses is

\[
\tilde{D}_{s,\tilde{r}_s} = \tilde{\ell}_s(\tilde{x}_s - \tilde{r}_s) - \tilde{\ell}_s(\tilde{x}_s) = \sum_{j=s-\tilde{r}_s}^{s-1} \tilde{\ell}_s(\tilde{x}_j) - \tilde{\ell}_s(\tilde{x}_{j+1})
\]

\[
\leq \sum_{j=s-\tilde{r}_s}^{s-1} \langle \nabla \tilde{\ell}_s(\tilde{x}_j), \tilde{x}_j - \tilde{x}_{j+1} \rangle
\]

\[
\leq \sum_{j=s-\tilde{r}_s}^{s-1} \|\nabla \tilde{\ell}_s(\tilde{x}_j)\|_* \|\tilde{x}_j - \tilde{x}_{j+1}\|
\]

and now if losses are linear, from \( \tilde{\ell}_s(\tilde{x}_j) = \langle \tilde{\ell}_s, \tilde{x}_j \rangle \) we get \( \nabla \tilde{\ell}_s(\tilde{x}_j) = \tilde{\ell}_s \), and the drift term becomes

\[
\tilde{D}_{s,\tilde{r}_s} \leq \sum_{j=s-\tilde{r}_s}^{s-1} \|\tilde{\ell}_s\|_* \|\tilde{x}_j - \tilde{x}_{j+1}\| \leq \sum_{j=s-\tilde{r}_s}^{s-1} \tilde{\eta}_j \|\tilde{\ell}_s\|_* \|\tilde{\ell}_j\|_*.
\]

We need now a technical lemma to prove a regret bound for OMD in the delayed feedback environment.

Lemma 21: For all \( j, s \in \{1, \ldots, T\} \), let

\[
\tilde{G}_j^{fwd} = 1 + 2 \sum_{s=j+1}^{T} \mathbb{I}\{s - \tilde{r}_s \leq j\} \quad \text{and} \quad \tilde{G}_s^{bck} = 1 + 2\tilde{r}_s,
\]

with the understanding that \( \tilde{G}_T^{fwd} = 1 \). For all \( t \in \{1, \ldots, T\} \), let \( \tilde{G}_t^{fwd} = \sum_{s=1}^{t} \tilde{G}_s^{fwd} \), \( \tilde{G}_{1:t}^{bck} = \sum_{s=1}^{t} \tilde{G}_s^{bck} \) and \( d \equiv \max_{s=1,\ldots,T}\{d_s\} \). Then, for all \( t \in \{1, \ldots, T\} \),

\[
\tilde{G}_{1:t}^{bck} \leq \tilde{G}_{1:t}^{fwd} \leq \tilde{G}_{1:t}^{bck} + d(2d - 1)
\]

and \( \tilde{G}_{1:T}^{bck} = \tilde{G}_{1:T}^{fwd} \).
Proof From the definitions, for all \( t \in \{1, \ldots, T\} \),

\[
\hat{G}_{bck}^{1:t} = \sum_{s=1}^{t} \hat{G}_{bck}^s = \sum_{s=1}^{t} (1 + 2 \tilde{\tau}_s) = \sum_{s=1}^{t} 1 + 2 \sum_{s=1}^{t} \sum_{j=s-\tilde{\tau}_s}^{s-1} 1
\]

\[
\leq \sum_{j=1}^{t} 1 + 2 \sum_{j=1}^{t} \sum_{s=j+1}^{T} \mathbb{I}\{s - \tilde{\tau}_s \leq j\}
\]

\[
= \sum_{j=1}^{t} 1 + 2 \sum_{j=1}^{t} \sum_{s=j+1}^{T} \mathbb{I}\{s - \tilde{\tau}_s \leq j\}
\]

\[
- 2 \sum_{j=1}^{t} \sum_{s=j+1}^{t} \mathbb{I}\{s - \tilde{\tau}_s \leq j\}
\]

\[
= \sum_{j=1}^{t} \hat{G}_{fwd}^j - 2 \sum_{j=1}^{t} \sum_{s=j+1}^{T} \mathbb{I}\{s - \tilde{\tau}_s \leq j\}
\]

\[
\leq \hat{G}_{fwd}^{1:t}
\]

furthermore for \( t = T \) we have \(-2\sum_{j=1}^{t} \sum_{s=t+1}^{T} \mathbb{I}\{s - \tilde{\tau}_s \leq j\} = 0\) and therefore we have \(\hat{G}_{bck}^{1:T} = \hat{G}_{fwd}^{1:T}\). We want to lower bound the negative term now to conclude the proof. Let us define \(\tau^* = \max_{s=1, \ldots, T}\{\tilde{\tau}_s\}\). We notice that for \(s > t\) and \(j \leq s - \tau^*\) the indicator function \(\mathbb{I}\{s - \tilde{\tau}_s \leq j\}\) is equal to zero. Also note that \(\mathbb{I}\{s - \tilde{\tau}_s \leq j\} = 0\) for \(s > j + \tau^*\). Hence

\[
\sum_{j=1}^{t} \sum_{s=t+1}^{T} \mathbb{I}\{s - \tilde{\tau}_s \leq j\} = \sum_{j=t+1}^{t+\tau^*} \sum_{s=t+1}^{j} \mathbb{I}\{s - \tilde{\tau}_s \leq j\}
\]

\[
\leq \sum_{j=t+1}^{t+\tau^*} (j + \tau^* - t)
\]

\[
= \sum_{i=1}^{\tau^*} i = \frac{1}{2} \tau^* (\tau^* + 1).
\]

If the maximum delay is \(d = \max_{s=1, \ldots, T}\{d_s\}\), from the definition of \(\tilde{\tau}_s\) we have that \(\tau^* \leq 2d - 1\). We conclude that

\[
\hat{G}_{fwd}^{1:t} \leq \hat{G}_{bck}^{1:t} + \tau^* (\tau^* + 1) \leq \hat{G}_{bck}^{1:t} + d(2d - 1). \]

Lemma 22 (see McMahan and Streeter (2014), Lemma 9) For any sequence of real numbers \(x_1, x_2, \ldots, x_n\) such that \(x_{1:t} = \sum_{s=1}^{t} x_s > 0\) for all \(t = 1, 2, \ldots, n\), we have

\[
\sum_{t=1}^{n} \frac{x_t}{\sqrt{x_{1:t}}} \leq 2\sqrt{x_{1:n}}.
\]
We have the following theorem for the regret of OMD in the delayed environment.

**Theorem 1** Suppose losses are linear and we run SOLID in a delayed-environment. Let $\tilde{\eta}_j$ denote the learning rates that BASE uses in its simulated non-delayed run inside SOLID environment. If $\alpha = \sqrt{\frac{2R}{L}}$ and $\tilde{\eta}_j = \alpha / \sqrt{\hat{G}_{1:j} + d(2d - 1)}$ then the regret of SOLID with OMD can be bounded as

$$R_T \leq 2LR \sqrt{\sum_{t=1}^T (1 + 2d_t) + LR \sqrt{d(2d - 1)}}.$$

**Proof** The total regret is bounded in the following way, where from $L$-Lipschitzness of the losses we have $\|\tilde{\ell}_j\|_* \leq L$ for each $j = 1, \ldots, T$:

$$R_T \leq \frac{2R^2}{\eta_T} + \frac{1}{2} \sum_{s=1}^T \tilde{\eta}_s \|\tilde{\ell}_s\|_*^2 + \sum_{s=1}^T \sum_{j=s-\tau_s}^{s-1} \tilde{\eta}_j \|\tilde{\ell}_j\|_* \tilde{\ell}_j \|_*$$

$$\leq \frac{2R^2}{\eta_T} + \frac{L^2}{2} \left( \sum_{s=1}^T \tilde{\eta}_s + 2 \sum_{s=1}^T \sum_{j=s-\tau_s}^{s-1} \tilde{\eta}_j \right)$$

$$= \frac{2R^2}{\eta_T} + \frac{L^2}{2} \left( \sum_{j=1}^T \tilde{\eta}_j + 2 \sum_{j=1}^T \tilde{\eta}_j \sum_{s=j+1}^T \|s - \tau_s \leq j\| \right)$$

$$= \frac{2R^2}{\eta_T} + \frac{L^2}{2} \left( \sum_{j=1}^T \tilde{\eta}_j \left( 1 + 2 \sum_{s=j+1}^T \|s - \tau_s \leq j\| \right) \right)$$

$$= \frac{2R^2}{\eta_T} + \frac{L^2}{2} \left( \sum_{j=1}^T \tilde{\eta}_j \tilde{G}_{1:j}^{fwd} \right).$$

Let us define

$$\tilde{\eta}_j = \frac{\alpha}{\sqrt{\hat{G}_{1:j} + d(2d - 1)}} = \frac{\alpha}{\sqrt{\sum_{i=1}^j (1 + 2\tau_i) + d(2d - 1)}} \quad (16)$$

Then

$$\frac{2R^2}{\eta_T} = 2R^2 \sqrt{\sum_{s=1}^T (1 + 2\tau_s) + d(2d - 1)} \frac{\alpha}{\alpha}$$

$$\leq 2R^2 \sqrt{\sum_{s=1}^T (1 + 2\tau_s) + \sqrt{d(2d - 1)}} \frac{\alpha}{\alpha}$$

$$= 2R^2 \sqrt{\sum_{t=1}^T (1 + 2d_t) + \sqrt{d(2d - 1)}} \frac{\alpha}{\alpha},$$
where the last equality follows thanks to the identity \( \sum_{s=1}^{T} \tilde{\tau}_s = \sum_{t=1}^{T} d_t \) and we conclude that
\[
\begin{align*}
\sum_{j=1}^{T} \tilde{\eta}_j \tilde{G}^{fwd}_j &= \alpha \sum_{j=1}^{T} \sqrt{\tilde{G}^{bck}_{1:j} + d(2d - 1)} \\
&\leq 2\alpha \sqrt{\tilde{G}^{fwd}_{1:T}} = 2\alpha \sqrt{\tilde{G}^{bck}_{1:T}} = 2\alpha \sum_{t=1}^{T} (1 + 2d_t),
\end{align*}
\]
where in the second inequality we use Lemma 22. Choosing \( \alpha = \sqrt{2R_L} \) the upper bound on the regret becomes
\[
R_T \leq 2LR \sqrt{2 \sum_{t=1}^{T} (1 + 2d_t)} + LR \sqrt{2d(2d - 1)}.
\]

**A.2 Analysis of Hedge with delays**

In order to simplify the reasoning of Zimmert and Seldin (2019) to adapt it to the study of Hedge with delays and in a second moment to its cooperative version on the communication network we remind the following definitions taken exactly like in Zimmert and Seldin (2019).

For a convex function \( F \) we use \( F^* \) to denote its convex conjugate and \( F^* \) the constrained convex conjugate. They are defined as
\[
F^*_t(y) = \max_{x \in \mathbb{R}^k} \langle x, y \rangle - F(x),
\]
\[
F^*_{-1}(y) = \max_{x \in \mathcal{X}} \langle x, y \rangle - F(x),
\]
where here in Appendix A.2 we consider the case of \( \mathcal{X} = \Delta^{k-1} \) and the negative entropy regularizer
\[
F_t(x) = \eta_t^{-1} \sum_{i=1}^{k} x_i \log(x_i).
\]

**A.2.1 Standard properties of FTRL analysis**

We introduce properties of FTRL that will be useful in the following section.

**Claim 23** \( f_t''(x) : \mathbb{R}_+ \to \mathbb{R}_+ \) are monotonically decreasing functions and \( f_t' : \mathbb{R} \to \mathbb{R}_+ \) are convex and monotonically increasing.

**Proof** By definition \( f_t''(x) = \eta_t^{-1} x^{-1} \), which concludes the first statement. Since \( f_t \) are Legendre functions, we have \( f_t''(f_t'(x)) = x \) and, taking derivatives on both sides and
applying the chain rule, we get the identity \( f''_k(y) f''_i(x) = 1 \). We set \( y = f'_i(x) \), with inverse \( f'_i(y) = x \). Therefore, substituting in the previous identity and inverting thanks to monotonicity, we obtain

\[
f''_k(y) = f''_i(f'_i(y))^{-1} > 0. \tag{18}
\]

Therefore the function is monotonically increasing. Since both \( f''_k(x) \), as well as \( f''_i(y) \) are increasing, the composition is as well and \( f''_k \geq 0 \) and this implies convexity of \( f''_k \). ■

**Claim 24** For any convex \( F, L \in \mathbb{R}^k \) and \( c \in \mathbb{R} \):

\[
\overline{F}^*(L + c \overline{1}) = \overline{F}^*(L) + c.
\]

**Proof** By definition \( \overline{F}^*(L + c \overline{1}) = \max_{x \in \Delta^{k-1}} \langle x, L + c \overline{1} \rangle - F(x) = \max_{x \in \Delta^{k-1}} \langle x, L \rangle - F(x) + c = \overline{F}^*(L) + c. \)

**Claim 25** For any \( x_t \) there exists \( \lambda \in \mathbb{R} \) such that:

\[
x_t = \nabla F_t^*(-\widehat{L}_t^{\text{obs}}) = \nabla F_t^*(-\widehat{L}_t^{\text{obs}} + \lambda \overline{1}) = \nabla F_t^*(\nabla F_t(x_t)).
\]

**Proof** By the KKT conditions, there exists \( \lambda \in \mathbb{R} \) such that

\[
x_t = \arg \max_{x \in \Delta^{k-1}} \left\{ \langle x, -\widehat{L}_t^{\text{obs}} \rangle + F_t(x) \right\} = \arg \max_{x \in \text{dom}(F_t)} \left\{ \langle x, -\widehat{L}_t^{\text{obs}} \rangle + F_t(x) + \lambda \left( \sum_{i=1}^k x_i - 1 \right) \right\}
\]

satisfies \( \nabla F_t(x_t) = -\widehat{L}_t^{\text{obs}} + \lambda \overline{1} \). The rest follows from the standard result of \( \nabla F = (\nabla F^*)^{-1} \) for Legendre \( F \). ■

**Claim 26** For any Legendre function \( F \) and \( L \in \mathbb{R}^k \) it holds that

\[
\overline{F}^*(L) \leq F^*(L),
\]

with equality iff there exists \( x \in \Delta^{k-1} \) such that \( L = \nabla F(x) \).

**Proof** The first statement follows from the definition since for any \( A \subseteq B \): \( \max_{x \in A} f(x) \leq \max_{x \in B} f(x) \). The second part follows because \( L = \nabla F(x) \) for some \( x \in \Delta^{k-1} \) holds if and only if \( \nabla F^*(L) = x \) which is equivalent to \( \arg \max_{x' \in \mathbb{R}^k} \langle x', L \rangle - F(x') = \nabla F^*(L) = x \in \Delta^{k-1} \). Therefore, if the unrestricted maximum \( x \) is on the simplex then the maximum restricted to the simplex will be also at the same point \( x \). This statement is equivalent to \( \overline{F}^*(L) = F^*(L) \) from the properties of Legendre functions. ■

**Claim 27** For any \( x \in \Delta^{k-1}, L \in [0, \infty)^k \) and \( i \in [k] \):

\[
\nabla F_t^*(\nabla F_t(x) - L)_i \geq \nabla F_t^*(\nabla F_t(x) - L)_i.
\]
Proof As in the proof of Claim 25, there exists \( \lambda \in \mathbb{R} \) : \( \nabla F_t^*(\nabla F_i(x) - L) = \nabla F_t^*(\nabla F_i(x) - L + \lambda \vec{I}) \). The statement is equivalent to \( \lambda \) being non-negative, since \( f^* \) are monotonically increasing. If \( \lambda < 0 \), then observing that \( \nabla F_t^*(y) \in \Delta^{k-1} \) for all \( y \) we have

\[
1 = \sum_{i=1}^{k} \left( \nabla F_t^*(\nabla F_i(x) - L) \right)_i = \sum_{i=1}^{k} \left( \nabla F_t^*(\nabla F_i(x) - L + \lambda \vec{I}) \right)_i = k \sum_{i=1}^{k} f_t^{*'}(f_t'(x_i) - L + \lambda) < \sum_{i=1}^{k} f_t^{*'}(f_t'(x_i)) = \sum_{i=1}^{k} x_i = 1 ,
\]

which is a contradiction and completes the proof.

Claim 28 For any Legendre function \( f \) with monotonically decreasing second derivative, \( x \in \text{dom}(f) \) and \( \ell \in [0, \infty) \) such that \( f'(x) - \ell \in \text{dom}(f^*) \):

\[
B_{f^*}(f'(x) - \ell, f'(x)) \leq \frac{\ell^2}{2f''(x)} .
\]

Proof Based on Taylor’s theorem, there exists an \( \tilde{x} \in [f^*'(f'(x) - \ell), x] \), such that \( B_{f^*}(f'(x) - \ell, f'(x)) = \frac{\ell^2}{2f''(\tilde{x})} \). \( \tilde{x} \) is smaller than \( x \), since \( f^{*'} \) is monotonically increasing. Finally using the fact that the second derivative is decreasing allows us to bound \( f''(\tilde{x})^{-1} \leq f''(x)^{-1} \).

A.2.2 Proof of Theorem 2

Let us define the cumulative loss vector \( L_t \) as follows, for every arm \( i \) we have

\[
L_{t,i} = \sum_{s=1}^{i-1} \ell_{s,i};
\]

The arm with the best cumulative loss in hindsight is \( i^* = \arg \min_{i \in [k]} \sum_{t=1}^{T} \ell_{t,i} \).

Lemma 29 For any \( t \) it holds

\[
F_t^*(-L_t^{obs} - \ell_t) - F_t^*(-L_t^{obs}) + \langle x_t, \ell_t \rangle \leq \frac{\eta_t}{2} .
\]
Proof We have

\[
\begin{align*}
\sum_{i=1}^k B_{f_t^*}(f_t'(x_{t,i}) - \ell_{t,i}, f_t'(x_{t,i})) \\
\leq \frac{1}{2} \sum_{i=1}^k \ell_{t,i}^2 f_t''(x_{t,i})^{-1}
\end{align*}
\]

where the first equality follows using Claim 25, the second equality is from Claim 24, the first inequality is from both parts of Claim 26, the second inequality follows from Claim 28 and finally the last equality is from the expression of \( f_t''(x) = \frac{1}{x} \) that holds for the negative entropy regularizer.

Lemma 30 For any non-increasing learning rate \( \eta_t \), it holds that

\[
\sum_{t=1}^T \left( F_t^*(-L_t) - F_t^*(-L_{t+1}) - \langle e_t^*, \ell_t \rangle \right) \leq \frac{\log(k)}{\eta T}.
\]

Proof Let \( \tilde{x}_t = \arg \max_{x \in \Delta_{k-1}} \{ \langle x, -L_t \rangle - F_t(x) \} \), then

\[
\begin{align*}
F_t^*(-L_t) &= \langle \tilde{x}_t, -L_t \rangle - F_t(\tilde{x}_t).
\end{align*}
\]

Furthermore, since \( F_t^*(-L_t) = \max_{x \in \Delta_{k-1}} \{ \langle x, -L_t \rangle - F_t(x) \} \), we have

\[
\begin{align*}
- \bar{F}_{t-1}(-L_t) &\leq \langle \tilde{x}_t, L_t \rangle + F_{t-1}(\tilde{x}_t) \\
- \bar{F}_T(-L_{T+1}) &\leq \langle e_t^*, L_{T+1} \rangle + F_T(e_t^*) = \sum_{t=1}^T \langle e_t^*, \ell_t \rangle.
\end{align*}
\]
Plugging these inequalities into the LHS leads to

\[
\sum_{t=1}^{T} \left( \bar{F}_t^* (-L_t) - \bar{F}_t (-L_{t+1}) - \langle e_{t*}, \ell_t \rangle \right)
\]

\[
= \sum_{t=1}^{T} \bar{F}_t^* (-L_t) - \sum_{t=2}^{T} \bar{F}_{t-1}^* (-L_t) - \sum_{t=1}^{T} \langle e_{t*}, \ell_t \rangle
\]

\[
\leq \sum_{t=1}^{T} \bar{F}_t^* (-L_t) - \sum_{t=2}^{T} \bar{F}_{t-1}^* (-L_t)
\]

\[
\leq \sum_{t=1}^{T} \langle \tilde{x}_t, -L_t \rangle - \sum_{t=1}^{T} F_t(\tilde{x}_t) + \sum_{t=2}^{T} F_{t-1}(\tilde{x}_t)
\]

\[
= -F_1(\tilde{x}_1) + \sum_{t=2}^{T} F_{t-1}(\tilde{x}_t) - F_t(\tilde{x}_t)
\]

\[
\leq \sum_{t=1}^{T} F_{t-1}(\tilde{x}_t) - F_t(\tilde{x}_t)
\]

\[
= \sum_{t=1}^{T} \left( \frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) (-F(\tilde{x}_t))
\]

\[
\leq \sum_{t=1}^{T} \left( \frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) \max_{x \in \Delta_{k-1}} (-F(x))
\]

\[
= \frac{1}{\eta_T} \max_{x \in \Delta_{k-1}} \{-F(x)\},
\]

where in the first inequality we used Eq. (20), in the second inequality we used Eq. (19), and with a slight abuse of notation we defined \( \eta_0^{-1} = 0 \) in the third inequality. We are left with computing the maximum, and using Jensen’s inequality

\[
\max_{x \in \Delta_{k-1}} \{-F(x)\} = \max_{x \in \Delta_{k-1}} \left\{ \frac{1}{\eta} \sum_{i=1}^{k} x_i \log \left( \frac{1}{x_i} \right) \right\}
\]

\[
\leq \log(k).
\]

Lemma 31 For any \( t \) it holds that

\[
\bar{F}_t^* (-L_t^{obs}) - \bar{F}_t^* (-L_t^{obs} - \ell_t) - \bar{F}_t (-L_t) + \bar{F}_t^* (-L_{t+1}) \leq \eta_t \delta_t.
\]
\textbf{Proof} We define $L_{t}^{\text{miss}} = L_{t} - L_{t}^{\text{obs}}$. Then we have

\[ -\mathcal{F}_{t}^{\prime}(-L_{t}) + \mathcal{F}_{t}^{\prime}(-L_{t+1}) = -\int_{0}^{1} \left\langle \ell_{t}, \nabla \mathcal{F}_{t}^{\prime}(-L_{t} - x\ell_{t}) \right\rangle \, dx \]

\[ = -\int_{0}^{1} \left\langle \ell_{t}, \nabla \mathcal{F}_{t}^{\prime}(-L_{t}^{\text{obs}} - L_{t}^{\text{miss}} - x\ell_{t}) \right\rangle \, dx \]

where the first equality uses the fundamental theorem of calculus. In the same way we have

\[ \mathcal{F}_{t}^{\prime}(-L_{t}^{\text{obs}}) - \mathcal{F}_{t}^{\prime}(-L_{t}^{\text{obs}} - \ell_{t}) = \int_{0}^{1} \left\langle \ell_{t}, \nabla \mathcal{F}_{t}^{\prime}(-L_{t}^{\text{obs}} - x\ell_{t}) \right\rangle \, dx \]

Now, putting the previous equations together and defining $\tilde{z}(x) = \nabla \mathcal{F}_{t}^{\prime}(-L_{t}^{\text{obs}} - x\ell_{t})$ we have the following

\[ \mathcal{F}_{t}^{\prime}(-L_{t}^{\text{obs}}) - \mathcal{F}_{t}^{\prime}(-L_{t}^{\text{obs}} - \ell_{t}) - \mathcal{F}_{t}^{\prime}(-L_{t}) + \mathcal{F}_{t}^{\prime}(-L_{t+1}) \]

\[ = \int_{0}^{1} \left\langle \ell_{t}, \nabla \mathcal{F}_{t}^{\prime}(-L_{t}^{\text{obs}} - x\ell_{t}) \right\rangle \, dx - \int_{0}^{1} \left\langle \ell_{t}, \nabla \mathcal{F}_{t}^{\prime}(-L_{t}^{\text{obs}} - L_{t}^{\text{miss}} - x\ell_{t}) \right\rangle \, dx \]

\[ = \int_{0}^{1} \left\langle \ell_{t}, \tilde{z}(x) - \nabla \mathcal{F}_{t}^{\prime}\left(\nabla \mathcal{F}_{t}(\tilde{z}(x)) - L_{t}^{\text{miss}}\right) \right\rangle \, dx \]

\[ \leq \int_{0}^{1} \left\langle \ell_{t}, \tilde{z}(x) - \nabla \mathcal{F}_{t}^{\prime}\left(\nabla \mathcal{F}_{t}(\tilde{z}(x)) - L_{t}^{\text{miss}}\right) \right\rangle \, dx \]

\[ = \sum_{i=1}^{k} \int_{0}^{1} \ell_{t,i} \left( \tilde{z}(x) - \nabla \mathcal{F}_{t}^{\prime}\left(\nabla \mathcal{F}_{t}(\tilde{z}(x)) - L_{t}^{\text{miss}}\right) \right) \, dx \]

\[ = \sum_{i=1}^{k} \int_{0}^{1} \ell_{t,i} \left( \tilde{z}(x) - f^{\prime\prime}_{t}(f^{\prime}_{t}(\tilde{z}(x)) - L_{t,i}^{\text{miss}}) \right) \, dx \]

\[ \leq \sum_{i=1}^{k} \int_{0}^{1} \ell_{t,i} f^{\prime\prime\prime}_{t}(f^{\prime}_{t}(\tilde{z}(x))) L_{t,i}^{\text{miss}} \, dx \]

\[ = \sum_{i=1}^{k} \int_{0}^{1} \ell_{t,i} \left( f^{\prime\prime}_{t}(\tilde{z}(x)) \right)^{-1} L_{t,i}^{\text{miss}} \, dx \]

\[ = \eta_{t} \int_{0}^{1} \left( \sum_{i=1}^{k} \ell_{t,i} L_{t,i}^{\text{miss}} \tilde{z}_{i}(x) \right) \, dx \]

where the second equality uses the definition of $\tilde{z}(x)$ and Claim 25, the first inequality applies Claim 27, the second inequality follows because $f^{\prime\prime}_{t}$ is convex, so $-f^{\prime\prime}_{t}(f^{\prime}_{t}(\tilde{z}_{i}) - \ell) \leq -\tilde{z}_{i} + f^{\prime\prime\prime}_{t}(f^{\prime}_{t}(\tilde{z}_{i}))\ell$, the second to last equality follows by Eq. (18), and the last follows
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because \( f''_t(x) = \frac{1}{t^2} \). From the previous series of inequalities we have

\[
\begin{align*}
F^*_t(-L^\text{obs}_t) - F^*_t(-L^\text{obs}_t - \ell_t) - F^*_t(-L_t) + F^*_t(-L_{t+1}) \\
\leq \eta_t \int_0^1 \left( \sum_{i=1}^k \ell_{t,i} L^{\text{miss}}_{t,i} \tilde{z}_i(x) \right) \, dx \\
= \eta_t \int_0^1 \left( \sum_{i=1}^k \ell_{t,i} \left( \sum_{s: s < t} I\{s + d_s \geq t\} \tilde{z}_i(x) \right) \right) \, dx \\
\leq \eta_t \int_0^1 \left( \sum_{i=1}^k \left( \sum_{s: s < t} I\{s + d_s \geq t\} \right) \tilde{z}_i(x) \right) \, dx \\
= \eta_t \sum_{s: s < t} I\{s + d_s \geq t\} \sum_{i=1}^k \tilde{z}_i(x) \, dx \\
= \eta_t \sum_{s: s < t} I\{s + d_s \geq t\} \sum_{i=1}^k \tilde{z}_i(x) \, dx \\
= \eta_t \sum_{s: s < t} I\{s + d_s \geq t\} \\
= \eta_t \mathfrak{d}_t,
\end{align*}
\]

where the first equality follows by expanding the definition of \( L^{\text{miss}}_{t,i} = \sum_{s: s < t} I\{s + d_s \geq t\} \ell_{s,i} \), the second inequality bounds losses with one, finally the second to last equality follows by the fact that by definition \( \tilde{z}(x) \in \Delta^{k-1} \).

\[ \square \]

**Theorem 2** Algorithm 3 with decreasing learning rates \((\eta_t)_{t=1,\ldots,n}\) satisfies

\[
R_T \leq \frac{\log(k)}{\eta_T} + \frac{1}{2} \sum_{t=1}^T \eta_t + \sum_{t=1}^T \eta_t \mathfrak{d}_t.
\]

Furthermore if one chooses \( \eta_t = \sqrt{\frac{\log k}{\sum_{s=1}^t 1 + 2d_s}} \) then

\[
R_T \leq 2 \sqrt{\log(k) \left( T + \sum_{t=1}^T \mathfrak{d}_t \right)}.
\]

**Proof** We have the following decomposition of \( R_T \)

\[
R_T = \sum_{t=1}^T \langle x_t - e_i, \ell_t \rangle = \sum_{t=1}^T \langle x_t, \ell_t \rangle - \langle e_i, \ell_t \rangle \\
= \sum_{t=1}^T \left( F^*_t(-L_t) - F^*_t(-L_{t+1}) - \langle e_i, \ell_t \rangle \right) \\
+ \sum_{t=1}^T \left( F^*_t(-L^\text{obs}_t - \ell_t) - F^*_t(-L^\text{obs}_t) + \langle x_t, L_t \rangle \right)
\]

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Appendix B. Analysis of partial information settings

B.1 Standard properties of FTRL analysis for semi-bandits

In this section we present an adaptation of the reasoning of Zimmert and Seldin (2019) to the study of semi-bandits with delays.

We use the following hybrid regularizer $F_t = F_{t,1} + F_{t,2}$, where each of the two parts of the regularizer has its own learning rate.

\[
F_t(x) = \sum_{i=1}^{k} f_t(x_i) = \sum_{i=1}^{k} 2\sqrt{t}x_i^{1/2} + \eta_t^{-1} \sum_{i=1}^{k} x_i \log(x_i).
\] (21)

The first part of the regularizer $F_{t,1}(x) = \sqrt{t} F_1(x)$ is the $\frac{1}{2}$ Tsallis entropy $F_1(x) = -2 \sum_{i=1}^{k} \sqrt{x_i}$ with learning rate $\frac{1}{\sqrt{t}}$, which is non-adaptive to the problem. The second part of the regularizer $F_{t,2}(x) = \eta_t^{-1} F_2(x)$ is the negative entropy $F_2(x) = \sum_{i=1}^{k} x_i \log(x_i)$ with adaptive learning rate $\eta_t$. We define this regularizer on the domain $\text{co}(A) = \{x \in [0, 1]^k : \sum_{i=1}^{k} x_i = m\}$ which corresponds to the probability simplex just in the case of $m = 1$.

**Claim 32** $f_t''(x) : \mathbb{R}_+ \to \mathbb{R}_+$ are monotonically decreasing functions and $f_t' : \mathbb{R} \to \mathbb{R}_+$ are convex and monotonically increasing. An analogous result holds for the function $f_{t,2}$.

**Proof** By definition $f_t''(x) = \sqrt{t} x^{-3/2} + \eta_t^{-1} x^{-1}$, which concludes the first statement. Since $f_t$ are Legendre functions, we have $f_t''(y) = f_t''(f_t'(y))^{-1} > 0$. Therefore the function is monotonically increasing. Since both $f_t''(x)$, as well as $f_t'(y)$ are increasing, the composition is as well and $f_t''' > 0$. The result for $f_{t,2}$ follows immediately from its definition in the same way as for $f_t$.

**Claim 33** For any convex $F$, $L \in \mathbb{R}^k$ and $c \in \mathbb{R}$:

\[
F'(L + c \bar{1}) = F'(L) + mc.
\]

**Proof** By definition $F'(L + c \bar{1}) = \max_{x \in \text{co}(A)} \langle x, L + c \bar{1} \rangle - F(x) = \max_{x \in \text{co}(A)} \langle x, L \rangle - F(x) + mc = F'(L) + mc$. 

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Claim 34 For any $x_t$ there exists $\lambda \in \mathbb{R}$ such that:

$$x_t = \nabla F_t^*(-\hat{L}^{obs}_t) = \nabla F_t^*(\nabla F_t(x_t) + \lambda \hat{1}) = \nabla F_t^*(\nabla F_t(x_t)).$$

An analogous result also holds if we use as a regularizer $F_{t,2}$ in place of the hybrid $F_t$.

Proof By the KKT conditions, there exists $\lambda \in \mathbb{R}$ such that $x_t = \arg\max_{x \in \text{co}(A)} \langle x, -\hat{L}^{obs}_t \rangle + F_t(x)$ satisfies $\nabla F_t(x_t) = -\hat{L}^{obs}_t + \lambda \hat{1}$. The rest follows from the standard result of $\nabla F = (\nabla F^*)^{-1}$ for Legendre $F$.

Claim 35 For any Legendre function $F$ and $L \in \mathbb{R}^k$ it holds that

$$\mathcal{F}^*(L) \leq F(L),$$

with equality iff there exists $x \in \text{co}(A)$ such that $L = \nabla F(x)$.

Proof The first statement follows from the definition since for any $A \subseteq B$: $\max_{x \in A} f(x) \leq \max_{x \in B} f(x)$. The second part follows because equality means that $\arg\max_{x} \langle x, L \rangle - F(x) = \nabla F^*(L) \in \text{co}(A)$, which is equivalent to the statement.

Claim 36 For any $x \in \text{co}(A)$, $L \in [0, \infty)^k$ and $i \in [k]$:

$$\nabla \mathcal{F}^*_t(\nabla F_t(x) - L)_i \geq \nabla F_t^*(\nabla F_t(x) - L)_i.$$

Proof By Claim 34, there exists $\lambda \in \mathbb{R}: \nabla \mathcal{F}^*_t(\nabla F_t(x) - L) = \nabla F_t^*(\nabla F_t(x) - L + \lambda \hat{1})$. The statement is equivalent to $\lambda$ being non-negative, since $f_t^*$ are monotonically increasing. If $\lambda < 0$, then

$$m = \sum_{i=1}^{k} (\nabla \mathcal{F}^*_t(\nabla F_t(x) - L))_i = \sum_{i=1}^{k} (\nabla F_t^*(\nabla F_t(x) - L + \lambda \hat{1}))_i = \sum_{i=1}^{k} f_t^*(f_t'(x_i) - L_i + \lambda) = \sum_{i=1}^{k} f_t''(f_t'(x_i)) = \sum_{i=1}^{k} x_i = m,$$

which is a contradiction and completes the proof.

Claim 37 For any Legendre function $f$ with monotonically decreasing second derivative, $x \in \text{dom}(f)$ and $\ell \in \mathbb{R}$ such that $f'(x) - \ell \in \text{dom}(f^*)$:

$$\mathcal{B}_{f^*}(f'(x) - \ell, f'(x)) \leq \frac{\ell^2}{2f''(x)}.$$
Proof For any Legendre function \( f \) with monotonically decreasing second derivative, \( x \in \text{dom}(f) \) and \( \ell \in [0, \infty) \) such that \( f'(x) - \ell \in \text{dom}(f^*) \):
\[
B_{f^*}(f'(x) - \ell, f'(x)) \leq \frac{\ell^2}{2f''(x)}.
\]

Claim 38 For each \( j \neq i \) and \( c > 0 \) holds
\[
\nabla F^*(-L)_i \geq \nabla F^*(-L + ce_j)_i
\]
Proof Let \( x = \nabla F^*(-L + ce_j) \), this definition is equivalent to \( \nabla F(x) - ce_j = -L \) then
\[
\nabla F^*(-L)_i = \nabla F^*(\nabla F(x) - ce_j)_i \geq \nabla F^*(\nabla F(x) - ce_j)_i
\]
\[
= f^*(f'(x_i) - c(e_j)_i) = f^*(f'(x_i)) \leq x_i
\]
\[
= \nabla F^*(-L + ce_j)_i
\]

B.2 Proof of Theorem 4
Lemma 39 For any \( t \) it holds
\[
\sum_{t=1}^{T} \mathbb{E} \left[ \sum_{t=1}^{T} \left[ F^*_t(-\hat{L}^{obs}_t - \hat{\ell}_t) - F^*_t(-\hat{L}^{obs}_t) + \langle x_t, \hat{\ell}_t \rangle \right] \right] \leq \sqrt{Tkm}.
\]
Proof We have
\[
\sum_{t=1}^{T} \mathbb{E} \left[ \sum_{t=1}^{T} \left[ F^*_t(-\hat{L}^{obs}_t - \hat{\ell}_t) - F^*_t(-\hat{L}^{obs}_t) + \langle x_t, \hat{\ell}_t \rangle \right] \right] \leq \sqrt{Tkm}.
\]
\[
= \sum_{i=1}^{k} B_{f^*_t}(f'_t(x_{t,i}) - \hat{\ell}_{t,i}, f'_t(x_{t,i}))
\]
\[
= \sum_{i=1}^{k} B_{f^*_t}(f'_t(x_{t,i}) - \frac{A_{t,i} \ell_{t,i}}{x_{t,i}}, f'_t(x_{t,i}))
\]
\[
\leq \frac{1}{2} \sum_{i=1}^{k} A_{t,i} \frac{\ell_{t,i}^2}{x_{t,i}^2} f''(x_{t,i})^{-1}
\]

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\[ \leq \frac{1}{2} \sum_{i=1}^{k} A_{t,i} \ell_{t,i}^{2} j_{t,1}^{-1}(x_{t,i}) \]

\[ = \sum_{i=1}^{k} A_{t,i} \ell_{t,i}^{2} x_{t,i}^{\frac{3}{2}} \sqrt{t} \]

\[ = \sum_{i=1}^{k} A_{t,i} \ell_{t,i}^{2} x_{t,i}^{-\frac{1}{2}} \sqrt{t}, \]

where the first equality follows using lemma 34, the second equality follows by Lemma 33, the first inequality is from Lemma 35 and finally the second inequality follows from Lemma 37. In expectation we get

\[ E[F_{t}^{*}(-\hat{L}_{t} - \hat{\ell}_{t}) - F_{t}^{*}(-\hat{L}_{t+1}) + \left\langle x_{t}, \hat{\ell}_{t} \right\rangle] \]

\[ \leq \frac{1}{2} \sum_{i=1}^{k} E[A_{t,i}] \ell_{t,i}^{2} x_{t,i}^{\frac{1}{2}} \sqrt{t} = \frac{1}{2} \sum_{i=1}^{k} \frac{x_{t,i}^{\frac{1}{2}} \ell_{t,i}^{2}}{\sqrt{t}} \leq \frac{1}{2} \sum_{i=1}^{k} \frac{x_{t,i}^{\frac{1}{2}}}{\sqrt{t}}. \]

Summing over \( t \) and using Cauchy-Schwarz gives

\[ \frac{1}{2} \sum_{t=1}^{T} \sum_{i=1}^{k} \frac{x_{t,i}^{\frac{1}{2}}}{\sqrt{t}} \leq \sqrt{Tkm}. \]

\[ \square \]

**Lemma 40**  For any non-increasing learning rate \( \eta_{t} \), it holds that for every \( a \in A \)

\[ \sum_{t=1}^{T} \left( F_{t}^{*}(-\hat{L}_{t}) - F_{t}^{*}(-\hat{L}_{t+1}) - \left\langle a, \hat{\ell}_{t} \right\rangle \right) \leq 2\sqrt{Tkm} + m \log \left( \frac{k}{m} \right) \eta_{T}. \]

**Proof**  Let \( \tilde{x}_{t} = \arg \max_{x \in \co(A)} \left\{ \left\langle x, -\hat{L}_{t} \right\rangle - F_{t}(x) \right\} \), then

\[ F_{t}^{*}(-\hat{L}_{t}) = \left\langle \tilde{x}_{t}, \hat{L}_{t} \right\rangle - F_{t}(\tilde{x}_{t}). \]

Furthermore, since \( F_{t}^{*}(-\hat{L}_{t}) = \max_{x \in \co(A)} \left\{ \left\langle x, -\hat{L}_{t} \right\rangle - F(x) \right\} \), we have

\[ -F_{t-1}^{*}(-\hat{L}_{t}) \leq \left\langle \tilde{x}_{t}, \hat{L}_{t} \right\rangle + F_{t-1}(\tilde{x}_{t}) \]

\[ -F_{t}^{*}(-\hat{L}_{t+1}) \leq \left\langle a, \hat{L}_{t+1} \right\rangle + F_{t}(a) = \sum_{t=1}^{T} \left\langle a, \hat{\ell}_{t} \right\rangle. \]
Plugging these inequalities into the LHS leads to
\[
\sum_{t=1}^{T} \left( F'_t(-\tilde{L}_t) - F'_t(-\tilde{L}_{t+1}) - \langle a, \hat{\ell}_t \rangle \right)
= \sum_{t=1}^{T} F'_t(-\tilde{L}_t) - \sum_{t=2}^{T} F'_{t-1}(-\tilde{L}_t) - F'_T(-\tilde{L}_{T+1}) - \langle a, \hat{\ell}_t \rangle
\leq \sum_{t=1}^{T} F'_t(-\tilde{L}_t) - \sum_{t=2}^{T} F'_{t-1}(-\tilde{L}_t)
= \sum_{t=1}^{T} \langle \hat{x}_t, -\tilde{L}_t \rangle - \sum_{t=1}^{T} F_t(\tilde{x}_t) + \sum_{t=2}^{T} \langle \hat{x}_t, \tilde{L}_t \rangle + \sum_{t=2}^{T} F_{t-1}(\tilde{x}_t)
\leq \sum_{t=1}^{n} F_{t-1}(\tilde{x}_t) - F_t(\tilde{x}_t)
= \sum_{t=1}^{n} F_{1,t-1}(\tilde{x}_t) + F_{2,t-1}(\tilde{x}_t) - F_{1,t}(\tilde{x}_t) - F_{2,t}(\tilde{x}_t)
\leq \sum_{t=1}^{n} \left( \sqrt{t} - \sqrt{t-1} \right) \max_{x \in \co(A)} (-F_1(x)) + \sum_{t=1}^{n} \left( \frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) \max_{x \in \co(A)} (-F_2(x))
= \sum_{t=1}^{n} \left( \sqrt{t} - \sqrt{t-1} \right) \max_{x \in \co(A)} (-F_1(x)) \leq \sqrt{T} \max_{x \in \co(A)} (-F_1(x)) \leq 2\sqrt{km} \;
\]

We are left with computing the maximum and using Hölder we get:
\[
\max_{x \in \co(A)} \{-F_1(x)\} = \max_{x \in \co(A)} \left\{ 2 \sum_{i=1}^{k} x_i^{\frac{1}{2}} \right\}
\leq 2\sqrt{km} ,
\]
where the inequality is due to Cauchy-Schwarz; also
\[
\max_{x \in \co(A)} \{-F_2(x)\} = \max_{x \in \co(A)} \left\{ \sum_{i=1}^{k} x_i \log \left( \frac{1}{x_i} \right) \right\}
\leq m \log \left( \frac{k}{m} \right) ,
\]
where we used Jensen’s inequality. Follows that
\[
\sum_{t=1}^{T} \left( F'_t(-\tilde{L}_t) - F'_t(-\tilde{L}_{t+1}) - \langle a, \hat{\ell}_t \rangle \right) \leq 2\sqrt{Tkm} + \frac{m \log \left( \frac{k}{m} \right)}{\eta T} .
\]
Lemma 41 For any $t$ it holds that
\[
F_t^*(-\hat{L}_t^{\text{obs}}) - F_t^*(-\hat{L}_t^{\text{obs}} - \hat{t}_t) - \tilde{F}_t^*(\hat{L}_t) + \tilde{F}_t^*(\hat{L}_t+1) \leq \eta_t k\delta_t.
\]

Proof We define $\tilde{L}_t^{\text{miss}} = \hat{L}_t - \hat{L}_t^{\text{obs}}$. Then we have
\[
-\tilde{F}_t^*(\hat{L}_t) + \tilde{F}_t^*(\hat{L}_t+1)
\]
\[
= -\int_0^1 \left( \hat{t}_t, \nabla \tilde{F}_t^*(-\hat{L}_t - x\hat{t}_t) \right) dx
\]
\[
= -\int_0^1 \left( \hat{t}_t, \nabla \tilde{F}_t^*(-\hat{L}_t^{\text{obs}} - \hat{L}_t^{\text{miss}} - x\hat{t}_t) \right) dx
\]
\[
= -\int_0^1 \sum_{i=1}^k \frac{A_{t,i}\hat{t}_t^i}{x_{t,i}} \nabla \tilde{F}_t^*(-\hat{L}_t^{\text{obs}} - \hat{L}_t^{\text{miss}} - x\hat{t}_t) dx
\]
\[
\leq -\sum_{i: A_{t,i}=1} \int_0^1 \frac{\hat{t}_t^i}{x_{t,i}} \nabla \tilde{F}_t^*(-\hat{L}_t^{\text{obs}} - \hat{L}_t^{\text{miss}}) + \sum_{j: A_{t,j}=0} \hat{L}_t^{\text{miss}} \hat{e}_j - x\hat{t}_t) dx
\]
\[
= -\sum_{i: A_{t,i}=1} \int_0^1 \frac{\hat{t}_t^i}{x_{t,i}} \nabla \tilde{F}_t^*(-\hat{L}_t^{\text{obs}} - \sum_{j=1}^k A_{t,j}\hat{L}_t^{\text{miss}} \hat{e}_j - x\hat{t}_t) dx
\]
\[
= -\int_0^1 \left( \hat{t}_t, \nabla \tilde{F}_t^*(-\hat{L}_t^{\text{obs}} - \sum_{j=1}^k A_{t,j}\hat{L}_t^{\text{miss}} \hat{e}_j - x\hat{t}_t) \right) dx,
\]
where the first equality uses the fundamental theorem of calculus and the inequality follows from Claim 38. Now let us define $\tilde{z}(x) = \nabla F_t^*(-\hat{L}_t^{\text{obs}} - x\hat{t}_t)$. We have the following
\[
\tilde{F}_t^*(\hat{L}_t^{\text{obs}}) - \tilde{F}_t^*(\hat{L}_t^{\text{obs}} - \hat{t}_t) - \tilde{F}_t^*(\hat{L}_t) + \tilde{F}_t^*(\hat{L}_t+1)
\]
\[
\leq \int_0^1 \left( \hat{t}_t, \nabla \tilde{F}_t^*(-\hat{L}_t^{\text{obs}} - x\hat{t}_t) \right) dx - \int_0^1 \left( \hat{t}_t, \nabla \tilde{F}_t^*(-\hat{L}_t^{\text{obs}} - \sum_{j=1}^k A_{t,j}\hat{L}_t^{\text{miss}} \hat{e}_j - x\hat{t}_t) \right) dx
\]
\[
= \int_0^1 \left( \hat{t}_t, \tilde{z}(x) - \nabla F_t^*(\tilde{z}(x)) - \sum_{j=1}^k A_{t,j}\hat{L}_t^{\text{miss}} \hat{e}_j \right) dx
\]
\[
\leq \int_0^1 \left( \hat{t}_t, \tilde{z}(x) - \nabla F_t^*(\tilde{z}(x)) - \sum_{j=1}^k A_{t,j}\hat{L}_t^{\text{miss}} \hat{e}_j \right) dx
\]
\[
= \sum_{i=1}^k \int_0^1 \hat{t}_t^i \left( \tilde{z}(x) - \nabla F_t^*(\tilde{z}(x)) - \sum_{j=1}^k A_{t,j}\hat{L}_t^{\text{miss}} \hat{e}_j \right) \right) dx
\]
\[
= \sum_{i=1}^k \int_0^1 \hat{t}_t^i (\tilde{z}_i(x) - f_t^i(\tilde{z}_i(x)) - \hat{L}_t^{\text{miss}}) dx
\]

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\[
\leq \sum_{i=1}^{k} \int_{0}^{1} \frac{\hat{\ell}_{t,i}}{f_t''(\hat{z}_i(x))} \hat{L}_{t,i}^{\text{miss}} \, dx \\
= \sum_{i=1}^{k} \int_{0}^{1} \frac{\hat{\ell}_{t,i}}{f_t''(\hat{z}_i(x))} \hat{L}_{t,i}^{\text{miss}} \, dx \\
\leq \sum_{i=1}^{k} \int_{0}^{1} \frac{\hat{\ell}_{t,i}}{f_t''(\hat{z}_i(x))} \hat{L}_{t,i}^{\text{miss}} \, dx \\
= \frac{\eta t}{k} \int_{0}^{1} \left( \sum_{i=1}^{k} \hat{\ell}_{t,i} \hat{L}_{t,i}^{\text{miss}} \hat{z}_i(x) \right) \, dx 
\]

where the first inequality uses the fundamental theorem of calculus together with the inequality above, the first equality substitutes \( \hat{z}(x) = \nabla \hat{F}_t^* \left( -\hat{L}_{t}^{\text{obs}} \right) \) and applies Claim 34. The second inequality applies Claim 36 and the third uses the fact that \( f''(t) \) is convex, so \( -f''(f'(\hat{z}_A) - \ell) \leq -\hat{z}_A + f''(f'(\hat{z}_A)) \). Taking the expected value we have

\[
\mathbb{E} \left[ \hat{F}_t'(\hat{L}_{t}^{\text{obs}}) - \hat{F}_t'(\hat{L}_{t}^{\text{obs}} - \hat{\ell}_t) \right] \leq \frac{\eta t}{k} \int_{0}^{1} \left( \sum_{i=1}^{k} \hat{\ell}_{t,i} \hat{L}_{t,i}^{\text{miss}} \hat{z}_i(x) \right) \, dx \\
= \frac{\eta t}{k} \int_{0}^{1} \left( \sum_{i=1}^{k} \hat{L}_{t,i}^{\text{miss}} \hat{z}_i(x) \right) \, dx \\
= \frac{\eta t}{k} \int_{0}^{1} \left( \sum_{i=1}^{k} \hat{L}_{t,i}^{\text{miss}} \hat{\ell}_{t,i} \right) \\
= \frac{\eta t}{k} \int_{0}^{1} \left( \sum_{i=1}^{k} \hat{L}_{t,i}^{\text{miss}} \hat{\ell}_{t,i} \right) \\
\leq \frac{\eta t}{k} \sum_{i=1}^{k} \hat{\ell}_{t,i} \\
= \frac{\eta t}{k} \sum_{i=1}^{k} \hat{\ell}_{t,i} \\
\leq \frac{\eta t}{k} \sum_{i=1}^{k} \sum_{s:s<t} \mathbb{I} \{s + d_s \geq t\} \hat{\ell}_{s,i} \\
= \frac{\eta t}{k} \sum_{i=1}^{k} \sum_{s:s<t} \mathbb{I} \{s + d_s \geq t\} \hat{\ell}_{s,i} \\
= \frac{\eta t}{k} \sum_{i=1}^{k} \sum_{s:s<t} \mathbb{I} \{s + d_s \geq t\} \ell_{s,i} \\
\leq \frac{\eta t}{k} \sum_{s:s<t} \mathbb{I} \{s + d_s \geq t\} \\
\leq \eta t \mathbf{d}_t .
\]
Theorem 4 Algorithm 5 with proper learning rates \((\eta_t)_{t=1,...,n}\) satisfies
\[
R_T \leq \frac{m \log\left(\frac{k}{m}\right)}{\eta_T} + 3\sqrt{Tkm} + k \sum_{t=1}^{T} \eta_t \delta_t
\]
Furthermore if one chooses \(\eta_t = \sqrt{\frac{m(1+\log(1/k))}{2k\sum_{s=1}^{T} \delta_s}}\) then
\[
R_T \leq 3\sqrt{Tkm} + 2\sqrt{2km\log\left(\frac{k}{m}\right)\left(\sum_{t=1}^{T} \delta_t\right)}.
\]

Proof We have the following decomposition of \(R_T\)
\[
R_T = \mathbb{E}\left[\sum_{t=1}^{T} \left(\bar{F}_t^*\left(-\hat{L}_t\right) - \bar{F}_t^*\left(-\hat{L}_{t+1}\right) - \langle a, \hat{\ell}_t \rangle\right)\right]
\]
\[
+ \sum_{t=1}^{T} \left(\bar{F}_t^*\left(-\hat{L}_{t}^{obs}\right) - \bar{F}_t^*\left(-\hat{L}_{t}^{obs}\right) + \langle x_t, \hat{\ell}_t \rangle\right)
\]
\[
+ \sum_{t=1}^{T} \left(\bar{F}_t^*\left(-\hat{L}_{t}^{obs}\right) - \bar{F}_t^*\left(-\hat{L}_{t}^{obs}\right) - \bar{F}_t^*\left(-\hat{L}_t\right) + \bar{F}_t^*\left(-\hat{L}_{t+1}\right)\right)
\]
\[
= 2\sqrt{Tkm} + \frac{m \log\left(\frac{k}{m}\right)}{\eta_T} + \sqrt{Tkm} + \sum_{t=1}^{T} \eta_t k \delta_t
\]
Note that, for all \( t, i, v \), we have \( \frac{w_t(i,v)}{W_t^l(v)} = \frac{w_t'(i,v)}{W_t'(v)} \), where \( W_t'(v) = \sum_{i=1}^{k} w_t'(i,v) \). Then,

\[
\frac{W_{t+1}^l(v)}{W_t^l(v)} = \sum_{i=1}^{k} \frac{w_{t+1}^l(i,v)}{W_t^l(v)} = \frac{w_{t+1}^l(i,v)}{W_t^l(v)} e^{-\eta \hat{\ell}_t(i,v)} = \sum_{i=1}^{k} \frac{w_t(i,v)}{W_t^l(v)} e^{-\eta \hat{\ell}_t(i,v)} = \sum_{i=1}^{k} \frac{\pi_t(i,v)}{m} e^{-\eta \hat{\ell}_t(i,v)} \leq \sum_{i=1}^{k} \frac{\pi_t(i,v)}{m} \left(1 - \eta \hat{\ell}_t(i,v) + \frac{1}{2} \eta^2 \left(\hat{\ell}_t(i,v)\right)^2\right) = 1 - \eta \sum_{i=1}^{k} \frac{\pi_t(i,v)}{m} \hat{\ell}_t(i,v) + \frac{\eta^2}{2} \sum_{i=1}^{k} \frac{\pi_t(i,v)}{m} \left(\hat{\ell}_t(i,v)\right)^2.
\]

Taking the logarithm and using the inequality \( \ln(1 + x) \leq x \) for all \( x > -1 \), and summing over \( t = 1, \ldots, T \) yields

\[
\ln \frac{W_{T+1}^l(v)}{W_1^l(v)} \leq -\eta \sum_{t=1}^{T} \sum_{i=1}^{k} \frac{\pi_t(i,v)}{m} \hat{\ell}_t(i,v) + \frac{\eta^2}{2} \sum_{t=1}^{T} \sum_{i=1}^{k} \frac{\pi_t(i,v)}{m} \left(\hat{\ell}_t(i,v)\right)^2.
\]

Moreover, for any fixed comparison action \( i^* \), we also have

\[
\ln \frac{W_{T+1}^l(v)}{W_1^l(v)} \geq \ln \frac{w_T'(i^*,v)}{W_1^l(v)} = -\eta \sum_{t=1}^{T} \hat{\ell}_t(i^*,v) - \ln k
\]

Putting this together and rearranging gives

\[
\sum_{t=1}^{T} \sum_{i=1}^{k} \frac{\pi_t(i,v)}{m} \hat{\ell}_t(i,v) - \sum_{t=1}^{T} \hat{\ell}_t(i^*,v) \leq \frac{\ln k}{\eta} + \frac{\eta^2}{2} \sum_{t=1}^{T} \sum_{i=1}^{k} \frac{\pi_t(i,v)}{m} \left(\hat{\ell}_t(i,v)\right)^2.
\]

\[\blacksquare\]

**Lemma 14** If agent \( v \in \mathcal{V} \) runs BaditCoopMsSets with learning rate \( \eta > 0 \), the following deterministic bounds for the drift probabilities hold for all \( i \in \{1, \ldots, k\} \):

\[-\eta \frac{\pi_t(i,v)}{m} \hat{\ell}_t(i,v) \leq \frac{\pi_{t+1}(i,v)}{m} - \frac{\pi_t(i,v)}{m} \leq \eta \frac{\pi_{t+1}(i,v)}{m} \sum_{j=1}^{k} \frac{\pi_t(j,v)}{m} \hat{\ell}_t(j,v).\]
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**Proof** Directly from the definition of the update \( w_{t+1}(i, v) \leq \mathcal{F}_t(i, v) \) for all \( i \in \{1, \ldots, k\} \), so that \( W_{t+1}(v) \leq m \) which in turn implies

\[
  w_{t+1}(i, v) \leq \frac{w_{t+1}(i, v)}{W_{t+1}(v)/m} = \frac{\mathcal{F}_{t+1}(i, v)}{m}.
\]

Therefore,

\[
  \frac{x_{t+1}(i, v)}{m} - \frac{x_t(i, v)}{m} \geq \frac{w_{t+1}(i, v)}{m} - \frac{x_t(i, v)}{m} = \frac{x_t(i, v)}{m} \left( 1 - e^{-\eta \hat{\ell}_t(i, v)} \right) \geq -\frac{x_t(i, v)}{m} \hat{\ell}_t(i, v),
\]

the last inequality using \( 1 - e^{-x} \leq x \) for \( x \geq 0 \). Similarly,

\[
  \frac{x_{t+1}(i, v)}{m} - \frac{x_t(i, v)}{m} \leq \frac{w_{t+1}(i, v)}{m} - \frac{x_t(i, v)}{m} = \frac{x_t(i, v)}{m} \left( 1 - \frac{w_{t+1}(i, v)}{m} \right) = \frac{x_t(i, v)}{m} \sum_{j=1}^k \left( \frac{x_t(j, v)}{m} - \frac{w_{t+1}(j, v)}{m} \right) = \frac{x_t(i, v)}{m} \sum_{j=1}^k \left( \frac{x_t(j, v)}{m} \hat{\ell}_t(j, v) \right) \leq \eta \sum_{j=1}^k \left( \frac{x_t(j, v)}{m} \hat{\ell}_t(j, v) \right).
\]

\[\blacksquare\]

**Lemma 15** If agent \( v \in \mathcal{V} \) runs BaditCoopMsets with learning rate \( \eta \in \left( 0, \frac{m}{k \epsilon(d+1)} \right) \), the following deterministic bound holds for all \( i \in \{1, \ldots, k\} \):

\[
  \mathcal{F}_{t+1}(i, v) \leq \left( 1 + \frac{1}{d} \right) \mathcal{F}_t(i, v).
\]

**Proof** We proceed by induction over \( t \). For all \( t \leq d \), \( \hat{\ell}_t(\cdot) = 0 \). Hence \( \mathcal{F}_t(\cdot) = \frac{m}{\mathcal{K}} \) and this lemma trivially holds. For \( t > d \) we can write

\[
  \sum_{i=1}^k \frac{x_t(i, v)}{m} \hat{\ell}_t(i, v) = \sum_{i=1}^k \frac{x_t(i, v)}{m} \frac{\ell_{t-d}(i)}{B_{d,t-d}(i, v)} B_{d,t-d}(i, v) \leq \sum_{i=1}^k \frac{x_t(i, v)}{m B_{d,t-d}(i, v)}
\]

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\[
\leq \sum_{i=1}^{k} \left(1 + \frac{1}{d}\right)^k \frac{\pi_{t-d}(i, v)}{mB_{d,t-d}(i, v)} \\
\leq \left(1 + \frac{1}{d}\right)^k \frac{K}{m} \\
\leq \frac{K}{m}e
\]

where the second inequality follows by the inductive hypothesis. Hence, using Lemma 14 we have

\[
\frac{\pi_{t+1}(i, v)}{m} \left(1 - \frac{Ke}{m}\right) \leq \frac{\pi_t(i, v)}{m} \left(1 - \frac{\eta}{m} \sum_{j=1}^{k} \frac{\pi_t(j, v)}{m} \ell_t(j, v)\right) \leq \frac{\pi_t(i, v)}{m}
\]

Appendix D. Table of notation

| Symbol | Description |
|--------|-------------|
| \(k\)  | dimension of \(\mathbb{R}^k\) and number of arms |
| \(d_t\) | loss \(\ell_t\) arrives at time \(t + d_t\) |
| \(d\) | is the maximum delay \(d = \max_{t=1,...,T} d_t\) |
| \(B_F\) | Bregman divergence with respect to \(F\) |
| \(d_t\) | number of outstanding observations at the beginning of round \(t\): \(d_t = \sum_{s=1}^{t-1} I\{s + d_s \geq t\}\) |
| \(S_t\) | number of received observations up to the beginning of time \(t\): \(S_t = \sum_{s=1}^{t-1} I\{s + d_s < t\} = t - 1 - d_t\) |
| \(D_t\) | \(D_t = \sum_{s=1}^{t} d_t\) |
| \(T\) | time horizon |
| \(\mathcal{G}\) | a graph \(\mathcal{G} = (V, \mathcal{E})\) |
| \(V\) | the set of vertices (a.k.a. agents) of a graph \(\mathcal{G} = (V, \mathcal{E})\) |
| \(\mathcal{E}\) | the set of edges of a graph \(\mathcal{G} = (V, \mathcal{E})\) |
| \(\mathcal{A}_t\) | the set of active agents at round \(t\), thus \(\mathcal{A}_t \subset V\) |
| \(q(v)\) | is the probability of activation of an agent \(v \in V\) |
| \(Q_{d+1}(v)\) | is the probability \(P[\exists v' \in \mathcal{A}_t \cap \mathcal{N}_{d+1}(v)]\) |
| \(\delta_{\mathcal{G}}(v, v')\) | is the graph distance between two agents \(v, v' \in V\) |
| \(\mathcal{N}_j(v)\) | the \(j\)-neighborhood of \(v\), i.e., the set of all agents \(v'\) with \(\delta_{\mathcal{G}}(v, v') \leq j\) |
| \(\text{diam}(\mathcal{G})\) | the diameter of \(\mathcal{G}\), i.e., \(\max_{v, v' \in V} \delta_{\mathcal{G}}(v, v')\) |