**Abstract.** Let $\mathbb{Z}^2$ be the two-dimensional integer lattice. For an integer $k \geq 1$, a non-zero lattice point is $k$-free if the greatest common divisor of its coordinates is a $k$-free number. We consider the proportions of $k$-free and twin $k$-free lattice points on a path of an $\alpha$-random walker in $\mathbb{Z}^2$. Using the second-moment method and tools from analytic number theory, we prove that these two proportions are $1/\zeta(2k)$ and $\prod_p (1 - 2p^{-2k})$, respectively, where $\zeta$ is the Riemann zeta function and the infinite product takes over all primes.

**1. Introduction**

For an integer $k \geq 1$, a positive integer is said to be $k$-free if it is not divisible by any $k$-th power of primes. In the two-dimensional integer lattice $\mathbb{Z}^2$, we say a non-zero lattice point $(m, n)$ is $k$-free if $\gcd(m, n)$ is $k$-free, where $\gcd$ is the greatest common divisor function. Particularly, a non-zero lattice point $(m, n) \in \mathbb{Z}^2$ is 1-free if and only if $\gcd(m, n) = 1$, which is equivalent to $(m, n)$ is a visible lattice point (from the origin). In 2013, Pleasants and Huck [8] showed that the density of $k$-free lattice points in $\mathbb{Z}^2$ is $1/\zeta(2k)$, where $\zeta$ is the Riemann zeta function. We also refer to [6] for dynamical properties of $k$-free lattice points and [1, 2, 5, 8] for related topics.

In 2015, Cilleruelo, Fernández, and Fernández [3] first considered visible (i.e. 1-free) lattice points in $\mathbb{Z}^2$ from the viewpoint of random walk. For $0 < \alpha < 1$, an $\alpha$-random walk starting at the origin of $\mathbb{Z}^2$ is defined by

$$P_{i+1} = P_i + \begin{cases} (1, 0) & \text{with probability } \alpha, \\ (0, 1) & \text{with probability } 1 - \alpha, \end{cases}$$

for $i = 0, 1, 2, \cdots$, where $P_i = (x_i, y_i)$ is the coordinate of the $\alpha$-random walker at the $i$-th step and $P_0 = (0, 0)$. This means, a random walker moving up and right in the integer lattice from the origin, with probabilities $\alpha$ and $1 - \alpha$, respectively. Cilleruelo, Fernández, and Fernández showed that the proportion of $r$ consecutively visible lattice points on a path of an $\alpha$-random walker is almost surely $b_r(\alpha) \prod_{p \geq r} (1 - rp^{-2})$, where $b_r(\alpha)$ is a polynomial in $\alpha$ with rational coefficients that can be explicitly computed. Particularly, they obtained $b_r(\alpha) = 1$ for $r = 1, 2$. It follows that the asymptotic proportions of visible ($r = 1$) and twin visible ($r = 2$) lattice points on a path of an $\alpha$-random walker are independent of the probability $\alpha$. But for $r \geq 3$, this phenomenon does not hold.

In this paper, we generalize the above results of [3] for $r = 1, 2$ to the case of $k$-free lattice points in $\mathbb{Z}^2$. It is also reasonable to consider the cases of $r \geq 3$, but here we will not go further in this direction.
For an $\alpha$-random walk given by (1.1), consider a sequence of random variables $\{X_i\}_{i \in \mathbb{N}}$ with

$$X_i = \begin{cases} 1, & P_i \text{ is } k\text{-free}, \\ 0, & \text{otherwise}. \end{cases}$$

For $n \geq 1$, define

$$S_n = S(n, k, \alpha) := \frac{X_1 + X_2 + \cdots + X_n}{n},$$

then $S_n$ indicates the proportion of $k$-free lattice points in the first $n$ steps of an $\alpha$-random walk.

**Theorem 1.1.** Under the above notations, for any $\alpha \in (0, 1)$ and integer $k \geq 1$, we have

$$\lim_{n \to +\infty} S_n = \frac{1}{\zeta(2k)}$$

almost surely, where $\zeta$ is the Riemann zeta function.

We remark that the limit proportion in Theorem 1.1 is independent on the probability $\alpha$. Particularly, if $k = 1$, then Theorem 1.1 gives Theorem A of [3]. We also remark that the limit proportion in Theorem 1.1 is the same as the density of $k$-free lattice points in $\mathbb{Z}^2$.

For a path of an $\alpha$-random walk given by (1.1), if two consecutive lattice points $P_i, P_{i+1}$ are both $k$-free, then we say $P_i, P_{i+1}$ are twin $k$-free lattice points. For $n \geq 1$, define

$$T_n = T(n, k, \alpha) := \frac{X_1X_2 + X_2X_3 + \cdots + X_nX_{n+1}}{n},$$

then $T_n$ indicates the proportion of twin $k$-free lattice points in the first $n + 1$ steps of an $\alpha$-random walk.

**Theorem 1.2.** Under the above notations, for any $0 < \alpha < 1$ and integer $k \geq 1$, we have

$$\lim_{n \to +\infty} T_n = \prod_p \left(1 - \frac{2}{p^{2k}}\right)$$

almost surely, where the infinite product takes over all primes.

Again, the limit proportion in Theorem 1.2 is independent on the probability $\alpha$.

1.1. **Notations.** As usual, for real functions $f$ and $g$, we use the expressions $f = O(g)$ and $f \ll g$ to mean $|f| \leq Cg$ for some constant $C > 0$. When this constant $C$ depends on some parameter $\varepsilon$, we write $f \ll_{\varepsilon} g$ and $f = O_{\varepsilon}(g)$. We use $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ to denote the sets of all real numbers, integers and positive integers, respectively. Moreover, we use $\mathbb{P}$, $\mathbb{E}$ and $\mathbb{V}$ to denote taking probability, expectation and variance, respectively.

2. **Preliminaries**

2.1. **Tools from probability theory.** We need the following two results from probability theory. Lemma 2.1 is the second-moment method (see Lemma 2.5, [3]) and Lemma 2.2 is the local central limit theorem (see Theorem 3.5.2, [4]).

**Lemma 2.1.** For a sequence of uniformly bounded random variables $(W_i)_{i \geq 1}$, let $S_n = (W_1 + \cdots + W_n)/n$. If the expectation $\mathbb{E}(S_n)$ and the variance $\mathbb{V}(S_n) \equiv S_n$ satisfy

$$\lim_{n \to \infty} \mathbb{E}(S_n) = \mu$$

and

$$\lim_{n \to \infty} \mathbb{V}(S_n) = \sigma^2,$$
and
\[ \forall (S_n) \ll \delta n^{-\delta} \]
for some \( \delta > 0 \) and any \( n \geq 1 \), then we have
\[ \lim_{n \to \infty} S_n = \mu \]
a almost surely.

**Lemma 2.2.** Let \( \alpha \in (0, 1) \) be fixed. For any integer \( n \geq 1 \), we have
\[ \max_{0 \leq l \leq n} \binom{n}{l} \alpha^l(1-\alpha)^{n-l} = O_\alpha \left( \frac{1}{\sqrt{n}} \right), \]
where \( \binom{n}{l} \), \( 0 \leq l \leq n \) are binomial coefficients.

### 2.2. Divisor functions.

For \( l \geq 2 \), let
\[ \tau_l(n) := \sum_{n = d_1d_2 \ldots d_l} 1 \]
be the \( l \)-fold divisor function. By (1.81) of [7], we have
\[ \tau_l(n) = O_{\epsilon, \delta}(n^{l \epsilon}) \]
for any \( \epsilon > 0 \). By the above upper bound and partial summation, we have the following estimates involving the 3-fold divisor function, which will be used in the proof of our theorems.

**Lemma 2.3.** For any integer \( n \geq 1 \) and any \( \epsilon > 0 \), we have
\[ \sum_{1 \leq i < j \leq n} \frac{\tau_3(j)\tau_3(i)}{\sqrt{i}} = O_{\delta}(n^{3/2+\epsilon}) \quad \text{and} \quad \sum_{1 \leq i < j \leq n} \frac{\tau_3(j)}{\sqrt{j-i}} = O_{\epsilon}(n^{3/2+\epsilon}). \]

**Lemma 2.4.** For any integer \( n \geq 1 \) and any \( \epsilon > 0 \), we have
\[ \sum_{1 \leq i \leq n} \sum_{i+1 < j \leq n} \frac{\tau_3(i)\tau_3(i+1)\tau_3(j)\tau_3(j+1)}{\sqrt{i}} = O_{\epsilon}(n^{3/2+\epsilon}) \]
and
\[ \sum_{1 \leq i \leq n} \sum_{i+1 < j \leq n} \frac{\tau_3(j)\tau_3(j+1)}{\sqrt{j-i-1}} = O_{\epsilon}(n^{3/2+\epsilon}). \]

The factor \( n^\epsilon \) in Lemmas 2.3 and 2.4 can be replaced by a power of \( \log n \), but this is not necessary here.

We also use the unitary divisor function \( \tau^*(n) \), which is defined by
\[ \tau^*(n) := \sum_{n = n_1n_2 \text{ gcd}(n_1, n_2) = 1} 1. \]
Note that \( \tau^*(n) \) is multiplicative and
\[ \tau^*(p^m) = 2 \]
for any prime power \( p^m \). Obviously, for \( n \geq 1 \) we have
\[ |\tau^*(n)| \leq \tau_2(n) = O_{\epsilon}(n^\epsilon). \]
2.3. Estimates for sums of binomial probabilities. To prove our theorems, we also need two results for sums of binomial probabilities. The following result is Lemma 2.1 of [3].

**Lemma 2.5.** For any $\alpha \in (0, 1)$, there is a constant $B_\alpha > 0$ such that for integers $n \geq 1$, $1 \leq d \leq n$ and $r \in \{0, 1, \ldots, d - 1\}$, there holds

$$\left| \sum_{l \equiv r \mod d} \binom{n}{l} \alpha^l (1 - \alpha)^{n-l} - \frac{1}{d} \right| \leq B_\alpha \sqrt{n}.$$

Using Lemma 2.5, we derive the following result.

**Lemma 2.6.** For any $\alpha \in (0, 1)$ and integer $l \geq 1$, let $u_j, 1 \leq j \leq l$ be pairwise coprime positive integers. If positive integers $d_j$ satisfy $d_j \mid u_j$ for any $1 \leq j \leq l$, then for $n \geq 1$ and any integers $a_1, \ldots, a_j$, we have

$$\sum_{\substack{0 \leq s \leq n \quad \text{gcd}(s+a_j, u_j)=d_j, \\ 1 \leq j \leq l}} \binom{n}{s} \alpha^s (1 - \alpha)^{n-s} = \frac{1}{d_1 \cdots d_k} \sum_{r_j|(u_j/d_j), 1 \leq j \leq l} \mu(r_1) \cdots \mu(r_l) \frac{\prod \tau_2(u_j/d_j)}{r_1 \cdots r_l} + O\left( \frac{1}{\sqrt{n}} \prod_{1 \leq j \leq l} \tau_2(u_j/d_j) \right),$$

where the implied $O$-constant depends only on $\alpha$ and $l$.

**Proof.** For simplicity, denote the left hand side of the above equation by

$$S := \sum_{\substack{0 \leq s \leq n \quad \text{gcd}(s+a_j, u_j)=d_j, 1 \leq j \leq l}} C_\alpha(n, s),$$

where

$$C_\alpha(n, s) := \binom{n}{s} \alpha^s (1 - \alpha)^{n-s}.$$

Then we have

$$S = \sum_{\substack{0 \leq s \leq n \quad \text{gcd}(s+a_j, u_j)=d_j, 1 \leq j \leq l}} C_\alpha(n, s).$$

Applying the formula

$$\sum_{r \mid n} \mu(r) = \begin{cases} 1, & n = 1, \\ 0, & \text{otherwise}, \end{cases}$$

where $\mu$ is the Möbius function, we obtain

$$S = \sum_{\substack{0 \leq s \leq n \quad \text{gcd}(s+a_j, u_j)=d_j, 1 \leq j \leq l}} C_\alpha(n, s) \prod_{1 \leq j \leq l} \left( \sum_{r_j \mid \text{gcd}(s+a_j)/d_j, u_j/d_j} \mu(r_j) \right).$$

Changing the order of summations, we obtain

$$S = \sum_{r_j|(u_j/d_j), 1 \leq j \leq l} \mu(r_1) \cdots \mu(r_l) \sum_{\substack{0 \leq s \leq n \quad \text{gcd}(s+a_j, r_j/d_j), 1 \leq j \leq l}} C_\alpha(n, s).$$
Note that \( u_1, \ldots, u_l \) are pairwise coprime and \( r_j d_j \mid u_j \) for \( 1 \leq j \leq l \). Then we have \( r_j d_1, \ldots, r_l d_l \) are pairwise coprime. By the Chinese remainder theorem, there exists an integer

\[
a \in \left\{ 0, \ldots, \prod_{1 \leq j \leq l} r_j d_j - 1 \right\}
\]

such that

\[
S = \sum_{r_j \mid \prod_{1 \leq j \leq l} u_j} \mu(r_1) \cdots \mu(r_l) \sum_{0 \leq s \leq n \mod \prod_{1 \leq j \leq l} r_j d_j} C_\alpha(n, s).
\]

Breaking the sum \( S \) into two sums according to \( \prod_{1 \leq j \leq l} r_j d_j \leq n \) or not, we write

\[
(2.7) \quad S = S_1 + S_2,
\]

where

\[
S_1 = \sum_{r_j \mid \prod_{1 \leq j \leq l} u_j, \prod_{1 \leq j \leq l} r_j d_j \leq n} \mu(r_1) \cdots \mu(r_l) \sum_{0 \leq s \leq n \mod \prod_{1 \leq j \leq l} r_j d_j} C_\alpha(n, s)
\]

and

\[
S_2 = \sum_{r_j \mid \prod_{1 \leq j \leq l} u_j, \prod_{1 \leq j \leq l} r_j d_j > n} \mu(r_1) \cdots \mu(r_l) \sum_{0 \leq s \leq n \mod \prod_{1 \leq j \leq l} r_j d_j} C_\alpha(n, s).
\]

For \( S_1 \), by Lemma 2.5, we obtain

\[
S_1 = \frac{1}{d_1 \cdots d_l} \sum_{r_j \mid \prod_{1 \leq j \leq l} u_j, \prod_{1 \leq j \leq l} r_j d_j \leq n} \frac{\mu(r_1) \cdots \mu(r_l)}{r_1 \cdots r_l} + O_{\alpha, l} \left( \frac{1}{\sqrt{n}} \prod_{1 \leq j \leq l} \tau_2(u_j/d_j) \right).
\]

The first term in the above equation is equal to

\[
\frac{1}{d_1 \cdots d_l} \sum_{r_j \mid \prod_{1 \leq j \leq l} u_j, \prod_{1 \leq j \leq l} r_j d_j \leq n} \frac{\mu(r_1) \cdots \mu(r_l)}{r_1 \cdots r_l} + O_{\alpha, l} \left( \frac{1}{n} \prod_{1 \leq j \leq l} \tau_2(u_j/d_j) \right),
\]

since

\[
\sum_{r_j \mid \prod_{1 \leq j \leq l} u_j, \prod_{1 \leq j \leq l} r_j d_j > n} \frac{\mu(r_1) \cdots \mu(r_l)}{r_1 \cdots r_l} \ll \frac{d_1 \cdots d_l}{n} \prod_{1 \leq j \leq l} \tau_2(u_j/d_j).
\]

It follows that

\[
(2.8) \quad S_1 = \frac{1}{d_1 \cdots d_l} \sum_{r_j \mid \prod_{1 \leq j \leq l} u_j, \prod_{1 \leq j \leq l} r_j d_j \leq n} \frac{\mu(r_1) \cdots \mu(r_l)}{r_1 \cdots r_l} + O_{\alpha, l} \left( \frac{1}{\sqrt{n}} \prod_{1 \leq j \leq l} \tau_2(u_j/d_j) \right).
\]

For \( S_2 \), by Lemma 2.2, we obtain

\[
(2.9) \quad S_2 = O_{\alpha, l} \left( \frac{1}{\sqrt{n}} \prod_{1 \leq j \leq l} \tau_2(u_j/d_j) \right).
\]

Now our desired result follows from inserting (2.8) and (2.9) into (2.7). \( \square \)
2.4. Two arithmetic functions. For $k \geq 1$, define

\begin{equation}
 g_k(n) := \sum_{\substack{d \mid n \\text{is } k-\text{free}}} \mu(r),
\end{equation}

where $\mu$ is the Möbius function. Note that $g_k(n)$ is multiplicative and

\[ g_k(p^m) = \begin{cases} -1, & m = k, \\ 0, & \text{otherwise}, \end{cases} \]

for any prime power $p^m$. It follows that

\begin{equation}
 \sum_{n=1}^{\infty} \frac{g_k(n)}{n^2} = \prod_p \left(1 - \frac{1}{p^{2k}}\right) = \frac{1}{\zeta(2k)}.
\end{equation}

Here and in the following, the symbol $\prod_p$ always means taking product over all primes. Similarly, since $\tau^*(n)$ is also multiplicative, we have

\begin{equation}
 \sum_{n=1}^{\infty} \frac{g_k(n)\tau^*(n)}{n^2} = \prod_p \left(1 - \frac{2}{p^{2k}}\right),
\end{equation}

where we have used (2.4). Obviously, for $n \geq 1$ we have

\begin{equation}
 |g_k(n)| \leq \tau_2(n) = O_\varepsilon(n^\varepsilon).
\end{equation}

For $k \geq 1$, define

\begin{equation}
 f_k(n) = \sum_{\substack{d \mid n \\text{is } k-\text{free}}} \frac{\mu(r)}{rd}.
\end{equation}

Obviously, for $n \geq 1$ we have

\begin{equation}
 |f_k(n)| \leq \tau_3(n) = O_\varepsilon(n^\varepsilon).
\end{equation}

**Lemma 2.7.** For any $\varepsilon > 0$, we have

\[ \sum_{1 \leq n \leq N} f_k(n) = \frac{N}{\zeta(2k)} + O_\varepsilon \left( N^\varepsilon \right) \]

for $N \geq 1$.

**Proof.** Let $rd = w$ in (2.14). Then we have

\begin{equation}
 f_k(n) = \sum_{w \mid n} \frac{1}{w} \sum_{\substack{d \mid w \\text{is } k-\text{free}}} \mu(r) = \sum_{w \mid n} \frac{g_k(w)}{w}.
\end{equation}

It follows that

\[ \sum_{1 \leq n \leq N} f_k(n) = \sum_{1 \leq w \leq N} \frac{g_k(w)}{w} \sum_{w \mid n} \mu(r) = \sum_{w \leq N} \frac{g_k(w)}{w} \sum_{1 \leq n \leq N} \frac{1}{n \equiv 0 \pmod{w}}, \]

where we have changed the order of summations. It follows that

\[ \sum_{1 \leq n \leq N} f_k(n) = \sum_{w \leq N} \frac{g_k(w)}{w} \left( \frac{N}{w} + O(1) \right), \]
which gives
\[ \sum_{1 \leq n \leq N} f_k(n) = N \sum_{w \leq N} \frac{g_k(w)}{w^2} + O\left( \sum_{w \leq N} \frac{|g_k(w)|}{w} \right). \]
Using bound (2.13) to estimate the O-term, we obtain
\[ \sum_{1 \leq n \leq N} f_k(n) = N \sum_{w \leq N} \frac{g_k(w)}{w^2} + O_\varepsilon(N^\varepsilon). \]
Extending the range of the sum over \( w \), we obtain
\[ \sum_{1 \leq n \leq N} f_k(n) = N \sum_{w=1}^{\infty} \frac{g_k(w)}{w^2} + O\left( N \sum_{w > N} \frac{|g_k(w)|}{w^2} \right) + O_\varepsilon(N^\varepsilon). \]
Using bound (2.13) again to estimate the first O-term in the above, we obtain
\[ \sum_{1 \leq n \leq N} f_k(n) = N \sum_{w=1}^{\infty} \frac{g_k(w)}{w^2} + O_\varepsilon(N^\varepsilon). \]
This together with (2.11) gives our desired result. \( \square \)

**Lemma 2.8.** For any \( \varepsilon > 0 \), we have
\[ \sum_{1 \leq n \leq N} f_k(n)f_k(n+1) = N \prod_p \left( 1 - \frac{2}{p^{2k}} \right) + O_\varepsilon(N^\varepsilon) \]
for \( N \geq 1 \).

**Proof.** It follows from (2.16) that
\[ \sum_{1 \leq n \leq N} f_k(n)f_k(n+1) = \sum_{1 \leq n \leq N} \sum_{n_1} \frac{g_k(w_1)}{w_1} \sum_{n_2(n+1)} \frac{g_k(w_2)}{w_2}. \]
Changing the order of summations, we obtain
\[ \sum_{1 \leq n \leq N} f_k(n)f_k(n+1) = \sum_{w_1 \leq N, w_2 \leq N+1} \frac{g_k(w_1)g_k(w_2)}{w_1 w_2} \sum_{1 \leq n \leq N \atop n \equiv 0 \mod w_1, n \equiv -1 \mod w_2} 1. \]
It follows from the Chinese remainder theorem that
\[ \sum_{1 \leq n \leq N} f_k(n)f_k(n+1) = \sum_{w_1 \leq N, w_2 \leq N+1 \atop (w_1, w_2) = 1} \frac{g_k(w_1)g_k(w_2)}{w_1 w_2} \left( \frac{N}{w_1 w_2} + O(1) \right), \]
which gives
\[ \sum_{1 \leq n \leq N} f_k(n)f_k(n+1) = N \sum_{w_1 \leq N, w_2 \leq N+1 \atop (w_1, w_2) = 1} \frac{g_k(w_1)g_k(w_2)}{(w_1 w_2)^2} + O\left( \sum_{w_1 \leq N} \frac{|g_k(w_1)|}{w_1} \sum_{w_2 \leq N+1} \frac{|g_k(w_2)|}{w_2} \right). \]
Using bound (2.13) to estimate the O-term, we obtain
\[ \sum_{1 \leq n \leq N} f_k(n)f_k(n+1) = N \sum_{w_1 \leq N, w_2 \leq N+1 \atop (w_1, w_2) = 1} \frac{g_k(w_1)g_k(w_2)}{(w_1 w_2)^2} + O_\varepsilon(N^\varepsilon). \]
Since $g_k(n)$ is multiplicative, we have
\[
\sum_{1 \leq i \leq n} f_k(n)f_k(n+1) = N \sum_{w_1 \leq N, w_2 \leq N+1} \frac{g_k(w_1 w_2)}{(w_1 w_2)^2} + O_\varepsilon(N^\varepsilon)
\]
Letting $w = w_1 w_2$, we have
\[
\sum_{1 \leq i \leq n} f_k(n)f_k(n+1) = N \sum_{w \leq N(N+1)} \frac{g_k(w)\tau^*(w;N)}{w^2} + O_\varepsilon(N^\varepsilon),
\]
where
\[
\tau^*(w;N) = \sum_{w = w_1 w_2 \atop w_1 \leq N, w_2 \leq N+1} 1.
\]
Note that for $w \leq N$, we have $\tau^*(w;N) = \tau^*(n)$, where $\tau^*(n)$ is given by (2.3). Then we have
\[
\sum_{1 \leq i \leq n} f_k(n)f_k(n+1) = N \sum_{w \leq N} \frac{g_k(w)\tau^*(w)}{w^2} + N \sum_{w > N} \frac{g_k(w)\tau^*(w;N)}{w^2} + O_\varepsilon(N^\varepsilon)
= N \sum_{w \leq N} \frac{g_k(w)\tau^*(w)}{w^2} + O_\varepsilon(N^\varepsilon),
\]
where we have used bounds $\tau^*(w;N) \leq \tau^*(w)$, (2.5), (2.13) and
\[
\sum_{w > N} \frac{g_k(w)\tau^*(w;N)}{w^2} \ll \sum_{w > N} \frac{\tau^2(w)}{w^2} \ll N^{-1+\varepsilon}
\]
for any $\varepsilon > 0$. Extending the range of the sum over $w$, we obtain
\[
\sum_{1 \leq i \leq n \leq N} f_k(n)f_k(n+1) = N \sum_{w=1}^\infty \frac{g_k(w)\tau^*(w)}{w^2} + O\left(N \sum_{w > N} \frac{g_k(w)\tau^*(w)}{w^2}\right) + O_\varepsilon(N^\varepsilon).
\]
With the help of bounds (2.5) and (2.13) again, we estimate the first $O$-term in the above and obtain
\[
\sum_{1 \leq i \leq n \leq N} f_k(n)f_k(n+1) = N \sum_{w=1}^\infty \frac{g_k(w)\tau^*(w)}{w^2} + O_\varepsilon(N^\varepsilon).
\]
This together with (2.12) gives our desired result. \qed

3. Proof of Theorem 1.1

According to Lemma 2.1, to prove Theorem 1.1, it is sufficient to compute the expectation and estimate the variance of $S_n$, respectively. We first prove the following result.

Lemma 3.1. Let $0 < \alpha < 1$ is fixed. For any integers $k \geq 1$ and $a, b, n$ with $b, n \in \mathbb{N}$, we have
\[
\sum_{0 \leq m \leq n \atop \gcd(m+a, b) \text{ is } k\text{-free}} C_\alpha(n, m) = f_k(b) + O_\alpha\left(\frac{\tau_3(b)}{\sqrt{n}}\right),
\]
where $C_\alpha(n, m)$ is given by (2.6).
Proof. For simplicity, denote the left hand side of the above equation by $R$. Then we have
\[ R := \sum_{d|b \; d \text{ is } k\text{-free}} \sum_{0 \leq m \leq n, \gcd(m+a,b) = d} C_\alpha(n, m). \]
Applying Lemma 2.6 to the sum over $m$, we obtain
\[ R = \sum_{d|b \; d \text{ is } k\text{-free}} \left( \sum_{r|d} \frac{\mu(r)}{rd} + O_\alpha \left( \frac{\tau_2(b/d)}{\sqrt{n}} \right) \right) \]
The contribution of the $O$-term to $R$ is \[ \ll_\alpha \frac{1}{\sqrt{n}} \sum_{d|b} \tau_2(b/d) = \frac{\tau_3(b)}{\sqrt{n}}. \]
Hence, we have
\[ R = f_k(b) + O_\alpha \left( \frac{\tau_3(b)}{\sqrt{n}} \right), \]
where $f_k(b)$ is given by (2.14). This completes our proof. □

3.1. The expectation of $S_n$. In this subsection, we compute the expectation $\mathbb{E}(S_n)$ and prove the following result.

Proposition 3.2. For integer $k \geq 1$ and $\alpha \in (0, 1)$, we have
\[ \mathbb{E}(S_n) = \frac{1}{\zeta(2k)} + O_{k, \alpha}(n^{-1/2+\varepsilon}) \]
for any $\varepsilon > 0$ and $n \geq 1$.

To prove Proposition 3.2, we first write
\[ \mathbb{E}(S_n) = \frac{1}{n} \sum_{1 \leq i \leq n} \mathbb{E}(X_i). \]
For each $1 \leq i \leq n$, by the definition of $X_i$, we have
\[ \mathbb{E}(X_i) = \mathbb{P}(P_i \text{ is } k\text{-free}), \]
where $P_i = (x_i, y_i)$ is the coordinate of the $\alpha$-random walker at the $i$-th step. Observe that $x_i + y_i = i$, thus we can write $P_i = (l, i - l)$ for some $l = 0, 1, \ldots, i$. The probability that $P_i = (l, i - l)$ is
\[ \mathbb{P}(P_i = (l, i - l)) = C_\alpha(i, l), \]
where $C_\alpha(i, l)$ is given by (2.6). By Lemma 3.1 and $\gcd(l, i - l) = \gcd(l, i)$, we have
\[ \mathbb{E}(X_i) = \sum_{0 \leq l \leq i, \gcd(l, i) \text{ is } k\text{-free}} C_\alpha(i, l) = f_k(i) + O_\alpha \left( \frac{\tau_3(i)}{\sqrt{i}} \right), \]
where $f_k$ is given by (2.14). Inserting (3.18) to (3.17), we obtain
\[ \mathbb{E}(S_n) = \frac{1}{n} \sum_{1 \leq i \leq n} f_k(i) + O_\alpha \left( \frac{1}{n} \sum_{1 \leq i \leq n} \frac{\tau_3(i)}{\sqrt{i}} \right). \]
Using bound (2.2) to estimate the $O$-term, we obtain

$$
\mathbb{E}(\overline{S}_n) = \frac{1}{n} \sum_{1 \leq i \leq n} f_k(i) + O(n^{-\frac{1}{2} + \varepsilon})
$$

for any $\varepsilon > 0$. This together with Lemma 2.7 yields Proposition 3.2.

3.2. Estimating the variance of $\overline{S}_n$. In this subsection, we estimate the variance of $\overline{S}_n$ and prove the following result.

Proposition 3.3. For integer $k \geq 1$ and $\alpha \in (0, 1)$, we have

$$
\mathbb{V}ar(\overline{S}_n) = O_{k, \alpha}(n^{-1/2 + \varepsilon})
$$

for any $\varepsilon > 0$ and $n \geq 1$.

To prove Proposition 3.3, we first write

$$
\mathbb{V}ar(\overline{S}_n) = \mathbb{E}(\overline{S}_n^2) - \mathbb{E}^2(\overline{S}_n).
$$

Further, by the definition of $\overline{S}_n$, we have

$$
\mathbb{V}ar(\overline{S}_n) = \frac{1}{n^2} \sum_{1 \leq i \leq n} \mathbb{E}(X_i^2) + \frac{2}{n^2} \sum_{1 \leq i < j \leq n} \mathbb{E}(X_i X_j) - \frac{1}{n^2} \mathbb{E}^2\left(\sum_{1 \leq i \leq n} X_i\right).
$$

For the first term on the right hand side of (3.20), since $\mathbb{E}(X_i^2) = \mathbb{E}(X_i)$ for each $1 \leq i \leq n$ by the definition of $X_i$. Then we have

$$
\sum_{1 \leq i \leq n} \mathbb{E}(X_i^2) = \sum_{1 \leq i \leq n} \mathbb{E}(X_i) = O(n),
$$

where we have used $\mathbb{E}(X_i) = \mathbb{P}(X_i) = O(1)$ for any $1 \leq i \leq n$.

For the third term on the right hand side of (3.20), by the definition of $\overline{S}_n$ and (3.19), we obtain

$$
\mathbb{E}^2\left(\sum_{1 \leq i \leq n} X_i\right) = \left(\sum_{1 \leq i \leq n} f_k(i)\right)^2 + O_\varepsilon(n^{3/2 + \varepsilon}),
$$

for any $\varepsilon > 0$, where we have used Lemma 2.7 to obtain the $O$-term in the above.

Now we deal with the second term on the right hand side of (3.20). For $1 \leq i < j \leq n$, let $P_i$ and $P_j$ be the coordinates of the $i$-th and $j$-th steps of a path of the $\alpha$-random walker, respectively. Here, we remark that $P_j$ depends on $P_i$. By the definition of $X_i$, we have

$$
\mathbb{E}(X_i X_j) = \mathbb{P}(P_i, P_j \text{ are both } k-\text{free}).
$$

Note that $P_i = (l, i - l)$ for some $0 \leq l \leq i$, then we have $P_j = (l + m, j - l - m)$ for some $0 \leq m \leq j - i$. The probability that $P_i$ and $P_j$ are both $k$-free is

$$
\sum_{0 \leq l \leq i} \sum_{\gcd(l, i - l) \text{ is } k-\text{free}} \sum_{0 \leq m \leq j - i} \gcd(l + m, j - l - m) \text{ is } k-\text{free} \mathbb{P}(P_i = (l, i - l), P_j = (l + m, j - l - m)).
$$

Note that

$$
\mathbb{P}(P_i = (l, i - l), P_j = (l + m, j - l - m)) = C_\alpha(i, l)C_\alpha(j - i, m).
$$
Since $\gcd(l, i - 1) = \gcd(l, i)$ and $\gcd(l + m, j - l - m) = \gcd(l + m, j)$, then we have

\begin{equation}
\mathbb{E}(X_i X_j) = \sum_{0 \leq l \leq i \atop \gcd(l, i) \text{ is } k\text{-free}} C_\alpha(i, l) T_k(l, i, j, \alpha),
\end{equation}

where

\begin{equation*}
T_k(l, i, j, \alpha) = \sum_{0 \leq m \leq j - i \atop \gcd(l + m, j) \text{ is } k\text{-free}} C_\alpha(j - i, m).
\end{equation*}

For $T_k(l, i, j, \alpha)$, applying Lemma 3.1 to the sum over $m$, we obtain

\begin{equation}
T_k(l, i, j, \alpha) = f_k(j) + O_\alpha\left(\frac{\tau_3(j)}{\sqrt{j - i}}\right),
\end{equation}

where $f_k(j)$ is given by (2.14). Inserting (3.24) to (3.23), we obtain

\begin{equation*}
\mathbb{E}(X_i X_j) = \sum_{0 \leq l \leq i \atop \gcd(l, i) \text{ is } k\text{-free}} C_\alpha(i, l) \left(f_k(j) + O_\alpha\left(\frac{\tau_3(j)}{\sqrt{j - i}}\right)\right).
\end{equation*}

By the binomial theorem, the contribution of the $O$-term to $\mathbb{E}(X_i X_j)$ is $O(\tau_3(j)/\sqrt{j - i})$. Hence, we have

\begin{equation*}
\mathbb{E}(X_i X_j) = f_k(j) \sum_{0 \leq l \leq i \atop \gcd(l, i) \text{ is } k\text{-free}} C_\alpha(i, l) + O_\alpha\left(\frac{\tau_3(j)}{\sqrt{j - i}}\right).
\end{equation*}

Using Lemma 3.1 again, we obtain

\begin{equation*}
\mathbb{E}(X_i X_j) = f_k(j) \left(f_k(i) + O_\alpha\left(\frac{\tau_3(i)}{\sqrt{i}}\right)\right) + O_\alpha\left(\frac{\tau_3(j)}{\sqrt{j - i}}\right)
= f_k(i) f_k(j) + O_\alpha\left(\frac{\tau_3(j) \tau_3(i)}{\sqrt{i}}\right) + O_\alpha\left(\frac{\tau_3(j)}{\sqrt{j - i}}\right),
\end{equation*}

where we have used bound (2.15). Summing over $1 \leq i < j \leq n$ and using Lemma 2.3 to estimate the contribution of the above $O$-terms, we obtain

$$\sum_{1 \leq i < j \leq n} \mathbb{E}(X_i X_j) = \sum_{1 \leq i < j \leq n} f_k(i) f_k(j) + O_{\alpha, \varepsilon}(n^{3/2 + \varepsilon})$$

for any $\varepsilon > 0$. Note that

$$2 \sum_{1 \leq i < j \leq n} f_k(i) f_k(j) = \left(\sum_{1 \leq i \leq n} f_k(i)\right)^2 - \sum_{1 \leq i \leq n} f_k^2(i) = \left(\sum_{1 \leq i \leq n} f_k(i)\right)^2 + O_\varepsilon(n^{1 + \varepsilon}),$$

where we have used

$$\sum_{1 \leq i \leq n} f_k^2(i) \ll \sum_{1 \leq i \leq n} \tau_3^2(i) \ll \varepsilon n^{1 + \varepsilon}.$$

Then we have

\begin{equation}
\sum_{1 \leq i < j \leq n} \mathbb{E}(X_i X_j) = \frac{1}{2} \left(\sum_{1 \leq i \leq n} f_k(i)\right)^2 + O_{\alpha, \varepsilon}(n^{3/2 + \varepsilon}).
\end{equation}

Now Proposition 3.3 follows from combining (3.21), (3.25) and (3.22) with (3.20).
4. Proof of Theorem 1.2

The frame of the proof of Theorem 1.2 is similar to the proof of Theorem 1.1. According to Lemma 2.1, it is also sufficient to compute the expectation and estimate the variance of \( T_n \), respectively. We first prove the following counterpart of Lemma 3.1.

Lemma 4.1. For \( \alpha \in (0, 1) \) and any integers \( a_1, a_2, b_1, b_2, n, k \) with \( b_1, b_2, n, k \in \mathbb{N} \) and \( \gcd(b_1, b_2) = 1 \), we have

\[
\sum_{0 \leq m \leq n \atop \gcd(m+a_1,b_1) \text{ is } k\text{-free}} C_\alpha(n, m) = f_k(b_1)f_k(b_2) + O_\alpha\left(\frac{\tau_3(b_1)\tau_3(b_2)}{\sqrt{n}}\right),
\]

where where \( C_\alpha(n, m) \) is given by (2.6) and \( f_k \) is given by (2.14).

Proof. For simplicity, denote the left hand side of the above equation by \( R \). Then we have

\[
R := \sum_{d_1 | b_1, d_2 | b_2 \atop d_1, d_2 \text{ are } k\text{-free}} \sum_{0 \leq m \leq n \atop \gcd(a_1+m,b_1)=d_1 \atop \gcd(a_2+m,b_2)=d_2} C_\alpha(n, m).
\]

By Lemma 2.6, we obtain

\[
R = \sum_{d_1 | b_1, d_2 | b_2 \atop d_1, d_2 \text{ are } k\text{-free}} \left( \frac{1}{d_1 d_2} \sum_{r_1 | (b_1/d_1)} \mu(r_1) \frac{\mu(r_2)}{r_2} \right) + O_\alpha\left(\frac{\tau_3(b_1)\tau_3(b_2)}{\sqrt{n}}\right).
\]

The contribution of the \( O \)-term to \( R \) is

\[
\ll_{\alpha} \frac{1}{\sqrt{n}} \left( \sum_{d_1 | b_1} \tau_2(b_1/d_1) \right) \left( \sum_{d_2 | b_2} \tau_2(b_2/d_2) \right) = \frac{\tau_3(b_1)\tau_3(b_2)}{\sqrt{n}}.
\]

Then we have

\[
R = \left( \sum_{r_1 d_1 | b_1 \atop d_1 \text{ is } k\text{-free}} \frac{\mu(r_1)}{r_1 d_1} \right) \left( \sum_{r_2 d_2 | b_2 \atop d_2 \text{ is } k\text{-free}} \frac{\mu(r_2)}{r_2 d_2} \right) + O_\alpha\left(\frac{\tau_3(b_1)\tau_3(b_2)}{\sqrt{n}}\right),
\]

which gives our desired result. \( \square \)

4.1. The expectation of \( T_n \). In this subsection, we compute the expectation \( \mathbb{E}(T_n) \) and prove the following result.

Proposition 4.2. For \( \alpha \in (0, 1) \) and positive integers \( k, n \), we have

\[
\mathbb{E}(T_n) = \prod_p \left( 1 - \frac{2}{p^{2k}} \right) + O_{k, \alpha}(n^{-1/2+\varepsilon})
\]

for any \( \varepsilon > 0 \).

By the definition of \( T_n \) and the linearity of the expectation, we have

\[
\mathbb{E}(T_n) = \frac{1}{n} \sum_{1 \leq i \leq n} \mathbb{E}(X_i X_{i+1}).
\]

For each \( 1 \leq i \leq n \), by the definition of \( X_i \), we have

\[
\mathbb{E}(X_i X_{i+1}) = \mathbb{P}(P_i, P_{i+1} \text{ are both } k\text{-free}),
\]

where \( \mathbb{P} \) denotes the probability.
where $P_i$, $P_{i+1}$ are the $i$-th and $(i+1)$-th steps of the $\alpha$-random walker, respectively. It is the same as the expectation of $S_n$. We denote $P_i = (l, i - l)$ for $0 \leq l \leq i$. Thus we have $P_{i+1} = (l + m, i + 1 - l - m)$ for $m \in \{0, 1\}$. The probability that $P_i$, $P_{i+1}$ both are $k$-free is of the form

$$\sum_{0 \leq l \leq i} C_\alpha(i, l) \sum_{0 \leq m \leq 1} C_\alpha(1, m).$$

Since $\gcd(l, i - l) = \gcd(l, i)$ and $\gcd(l + m, i + 1 - l - m) = \gcd(l + m, i + 1)$, by (4.27), we have

$$\mathbb{E}(X_iX_{i+1}) = \sum_{0 \leq l \leq i} C_\alpha(i, l) \sum_{0 \leq m \leq 1} C_\alpha(1, m),$$

which gives

$$\mathbb{E}(X_iX_{i+1}) = (1 - \alpha)M_1 + \alpha M_2,$$

where

$$M_1 = \sum_{0 \leq l \leq i} \sum_{\gcd(l, i) \text{ is } k\text{-free}} C_\alpha(i, l) \quad \text{and} \quad M_2 = \sum_{0 \leq l \leq i} \sum_{\gcd(l + l, i + 1) \text{ is } k\text{-free}} C_\alpha(i, l).$$

We apply Lemma 4.1 to $M_1$ and obtain

$$M_1 = f_k(i)f_k(i + 1) + O_\alpha \left( \frac{\tau_3(i)\tau_3(i + 1)}{\sqrt{i}} \right).$$

Similarly, by Lemma 4.1 again, we also have

$$M_2 = f_k(i)f_k(i + 1) + O_\alpha \left( \frac{\tau_3(i)\tau_3(i + 1)}{\sqrt{i}} \right).$$

Inserting (4.30) and (4.31) to (4.29) and with help of bound (2.2), we obtain

$$\mathbb{E}(X_iX_{i+1}) = f_k(i)f_k(i + 1) + O_{\alpha, \epsilon}(i^{-1/2 + \epsilon})$$

for any $\epsilon > 0$. We then have

$$\sum_{1 \leq i \leq n} \mathbb{E}(X_iX_{i+1}) = \sum_{1 \leq i \leq n} f_k(i)f_k(i + 1) + O_{\alpha, \epsilon}(n^{1/2 + \epsilon}).$$

Inserting this to (4.26) and applying Lemma 2.8 to the sum over $i$, we obtain our desired result.
4.2. **Estimating the variance of** $T_n$. In this subsection, we estimate the variance of $T_n$ and prove the following result.

**Proposition 4.3.** For any $\alpha \in (0, 1)$, $\varepsilon > 0$ and integer $k \geq 1$, we have

$$\text{Var}(T_n) = O_{k, \alpha}(n^{-1/2+\varepsilon}).$$

By the definition of $T_n$, we have

$$\text{Var}(T_n) = \frac{1}{n^2} \left( \sum_{1 \leq i, j \leq n} \mathbb{E}(X_i X_{i+1} X_j X_{j+1}) - \mathbb{E}^2(\sum_{1 \leq i \leq n} X_i X_{i+1}) \right).$$

It follows that

$$\text{Var}(T_n) = \frac{1}{n^2} \sum_{1 \leq i \leq n} \mathbb{E}(X_i^2 X_{i+1}^2) + \frac{2}{n^2} \sum_{1 \leq i < j \leq n} \mathbb{E}(X_i X_{i+1} X_j X_{j+1}) - \frac{1}{n^2} \mathbb{E}^2(\sum_{1 \leq i \leq n} X_i X_{i+1}).$$

For the first term on the right hand side of (4.34), by the definition of $X_i$, we have

$$\sum_{1 \leq i \leq n} \mathbb{E}(X_i^2 X_{i+1}^2) = \sum_{1 \leq i \leq n} \mathbb{E}(X_i X_{i+1}) = O(n),$$

where we have used $\mathbb{E}(X_i X_{i+1}) = \mathbb{P}(P_i, P_{i+1} \text{ are both } k - \text{free}) = O(1)$ for any $1 \leq i \leq n$.

For the third term on the right hand side of (4.34), by (4.33), we have

$$\mathbb{E}^2(\sum_{1 \leq i \leq n} X_i X_{i+1}) = \left( \sum_{1 \leq i \leq n} f_k(i) f_k(i + 1) \right)^2 + O_{\alpha, \varepsilon}(n^{3/2+\varepsilon})$$

for any $\varepsilon > 0$, where we have used Lemma 2.8 to obtain the $O$-term in the above.

Now we deal with the second term on the right hand side of (4.34). Note that

$$\sum_{1 \leq i < j \leq n} \mathbb{E}(X_i X_{i+1} X_j X_{j+1}) = \sum_{1 \leq i \leq n} \sum_{i+1 < j \leq n} \mathbb{E}(X_i X_{i+1} X_j X_{j+1}) + O(n),$$

since $\mathbb{E}(X_i X_{i+1} X_{i+2}) = \mathbb{P}(P_i, P_{i+1}, P_{i+2} \text{ are } k - \text{free}) = O(1)$. For $i + 1 < j$, we have

$$\mathbb{E}(X_i X_{i+1} X_j X_{j+1}) = \sum_{0 \leq i \leq l \leq i} C_2(i, l) \sum_{0 \leq m_1 \leq i} C(1, m_1) \sum_{0 \leq m_2 \leq j-i-1} C(j-i-1, m_2) \sum_{0 \leq m_3 \leq l} C(1, m_3).$$

Writing the inner sum over $m_3$ explicitly, we have

$$\mathbb{E}(X_i X_{i+1} X_j X_{j+1}) = \sum_{0 \leq i \leq l \leq i} C_2(i, l) \sum_{0 \leq m_1 \leq i} C_2(1, m_1) \sum_{0 \leq m_2 \leq j-i-1} C_2(j-i-1, m_2) \sum_{0 \leq m_3 \leq l} C(1, m_3) \left((1 - \alpha)M_3 + \alpha M_4\right),$$

where

$$M_3 = \sum_{0 \leq m_2 \leq j-i-1} \sum_{\gcd(l+m_1+m_2,j) \text{ is } k-\text{free}} C_2(j-i-1, m_2).$$
and

\[ M_4 = \sum_{0 \leq m_2 \leq j - 1 \atop \gcd(l + m_1 + m_2, j) \text{ is } k\text{-free}} C_\alpha(j - i - 1, m_2). \]

Applying Lemma 4.1 to the sum \( M_3 \) over \( m_2 \), we obtain

\[ M_3 = f_k(j)f_k(j + 1) + O_\alpha \left( \frac{\tau_3(j)\tau_3(j + 1)}{\sqrt{j - i - 1}} \right). \tag{4.39} \]

Similarly, applying Lemma 4.1 again to the sum \( M_4 \) over \( m_2 \), we also obtain

\[ M_4 = f_k(j)f_k(j + 1) + O_\alpha \left( \frac{\tau_3(j)\tau_3(j + 1)}{\sqrt{j - i - 1}} \right). \tag{4.40} \]

Inserting (4.39), (4.40) to (4.38), we obtain

\[ \mathbb{E}(X_iX_{i+1}X_jX_{j+1}) = \sum_{0 \leq l \leq i \atop \gcd(l, j) \text{ is } k\text{-free}} C_\alpha(i, l) \sum_{0 \leq m_1 \leq 1 \atop \gcd(l + m_1, i + 1) \text{ is } k\text{-free}} C_\alpha(1, m_1)
\begin{aligned}
&\left( f_k(j)f_k(j + 1) + O_\alpha \left( \frac{\tau_3(j)\tau_3(j + 1)}{\sqrt{j - i - 1}} \right) \right).
\end{aligned} \]

By the binomial theorem and bound (2.2), we have

\[ \mathbb{E}(X_iX_{i+1}X_jX_{j+1}) = f_k(j)f_k(j + 1)M_5 + O_\alpha \left( \frac{\tau_3(j)\tau_3(j + 1)}{\sqrt{j - i - 1}} \right), \]

where

\[ M_5 = \sum_{0 \leq l \leq i \atop \gcd(l, j) \text{ is } k\text{-free}} C_\alpha(i, l) \sum_{0 \leq m_1 \leq 1 \atop \gcd(l + m_1, i + 1) \text{ is } k\text{-free}} C_\alpha(1, m_1). \]

Note that \( M_5 \) is the same as \( E(X_iX_{i+1}) \) given by (4.28). Then by (4.32) we have

\[ M_5 = f_k(i)f_k(i + 1) + O_\alpha \left( \frac{\tau_3(i)\tau_3(i + 1)}{\sqrt{i}} \right). \]

It follows that

\[ \mathbb{E}(X_iX_{i+1}X_jX_{j+1}) = f_k(i)f_k(i + 1)f_k(j)f_k(j + 1) + O_\alpha \left( \frac{\tau_3(i)\tau_3(i + 1)\tau_3(j)\tau_3(j + 1)}{\sqrt{i}} \right) \]
\[ + O_\alpha \left( \frac{\tau_3(j)\tau_3(j + 1)}{\sqrt{j - i - 1}} \right), \]

where we have used

\[ f_k(j)f_k(j + 1) \ll \tau_3(j)\tau_3(j + 1). \]

Summing over \( 1 \leq i \leq n \) and \( i + 1 < j \leq n \), we obtain

\begin{align*}
\sum_{1 \leq i \leq n} \sum_{i + 1 < j \leq n} \mathbb{E}(X_iX_{i+1}X_jX_{j+1}) &= V_k(n) + O_\alpha \sum_{1 \leq i \leq n} \sum_{1 \leq i + 1 < j \leq n} \frac{\tau_3(i)\tau_3(i + 1)\tau_3(j)\tau_3(j + 1)}{\sqrt{i}} \\
&\quad + O_\alpha \left( \sum_{1 \leq i \leq n} \sum_{1 \leq i + 1 < j \leq n} \frac{\tau_3(j)\tau_3(j + 1)}{\sqrt{j - i - 1}} \right),
\end{align*}
where
\[ V_k(n) = \sum_{1 \leq i \leq n} \sum_{i+1 < j \leq n} f_k(i)f_k(i+1)f_k(j)f_k(j+1). \]

By Lemma 2.4, the contribution of \( O \)-terms is \( O(\varepsilon n^{3/2+\varepsilon}) \) for any \( \varepsilon > 0 \). Then we have
\[ \sum_{1 \leq i \leq n} \sum_{i+1 < j \leq n} \mathbb{E}(X_iX_{i+1}X_jX_{j+1}) = V_k(n) + O(\varepsilon n^{3/2+\varepsilon}). \]

For \( V_k(n) \), replacing \( i+1 < j \leq n \) by \( i < j \leq n \) up to an error term \( \ll \varepsilon n^{1+\varepsilon} \), we have
\[ 2V_k(n) = 2 \sum_{1 \leq i < j \leq n} f_k(i)f_k(i+1)f_k(j)f_k(j+1) + O(\varepsilon n^{1+\varepsilon}), \]

which gives
\[
2V_k(n) = \left( \sum_{1 \leq i \leq n} f_k(i)f_k(i+1) \right)^2 - \sum_{1 \leq i \leq n} \left( f_k(i)f_k(i+1) \right)^2 + O(\varepsilon n^{1+\varepsilon})
\]
\[ = \left( \sum_{1 \leq i \leq n} f_k(i)f_k(i+1) \right)^2 + O(\varepsilon n^{1+\varepsilon}). \]

Hence, we have
\[
\sum_{1 \leq i \leq n} \sum_{i+1 < j \leq n} \mathbb{E}(X_iX_{i+1}X_jX_{j+1}) = \frac{1}{2} \left( \sum_{1 \leq i \leq n} f_k(i)f_k(i+1) \right)^2 + O(\varepsilon n^{3/2+\varepsilon}).
\]

Inserting (4.35), (4.36) and (4.41) to (4.34), we obtain
\[ \mathbb{V}(T_n) = O(\varepsilon n^{-1/2+\varepsilon}) \]
for any \( \varepsilon > 0 \). This completes our proof.

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