A QUANTUM ANOMALY FOR RIGID PARTICLES

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Abstract

Canonical quantisation of rigid particles is considered paying special attention to the restriction on phase space due to causal propagation. A mixed Lorentz-gravitational anomaly is found in the commutator of Lorentz boosts with world-line reparametrisations. The subspace of gauge invariant physical states is therefore not invariant under Lorentz transformations. The analysis applies for an arbitrary extrinsic curvature dependence with the exception of only one case to be studied separately. Consequences for rigid strings are also discussed.

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1. Introduction

Following Polyakov’s proposal\cite{1,2} of adding to the ordinary Nambu-Goto action of string theory a scale invariant term quadratic in the world-sheet extrinsic curvature – as a means of smoothening out the world-sheet structure at short distances, and possibly also obtaining a low-energy effective theory for QCD –, there has been a revival of interest in theories of relativistic particles whose action includes a dependence on the world-line extrinsic curvature and torsion, so called rigid particles\cite{3-18}. These theories not only provide a simpler setting in which to study some of the difficulties presented by rigid strings due to their nonlinear higher derivative actions, especially at the quantum level, but it has also been suggested that rigid particles could define a unifying formalism for quantum particles of arbitrary spin using spacetime coordinates only (even half-integer spins have been claimed\cite{6,10}, though erroneously\cite{6,19}, to be possible). However, the understanding of quantum rigid particles presently available in the literature is not really satisfactory, being rather confusing and even self-contradictory at times.

In view of this situation, the present letter reports on results of a detailed analysis of the canonical quantisation of rigid particles whose action includes an arbitrary dependence on the world-line extrinsic curvature but is independent of its extrinsic torsion. Leaving details of the analysis to a separate publication\cite{20}, the main emphasis here is on the existence of a Lorentz anomaly for physical states. Namely, at the quantum level, while both the local world-line gauge algebra and the spacetime Poincaré algebra are each anomaly free, world-line reparametrisations do not commute with Lorentz boosts. Consequently, quantum physical states do not transform covariantly under the Lorentz group. Even though the system only has a finite number of degrees of freedom, the associated anomaly arises due to operator ordering and the need to perform a certain map from the original restricted phase space to an unrestricted one. The initial restriction originates from the requirement that classical world-line trajectories are time-like and causal, namely that the velocity of the particle is always less than the speed of light so that its trajectories always lie inside the light-cone (in spite of this, classical solutions always include tachyonic but nevertheless causal ones). The present anomaly is quite analogous to the mixed triangular anomaly in four dimensions for two gravitons and one U(1) gauge boson\cite{21}, Poincaré invariance playing the rôle of an internal symmetry for rigid particles.

The discussion outlined in this letter applies whatever the dependence on the extrinsic curvature, with one exception however referred to as “the degenerate case”. In contradistinction with the “generic case”, the degenerate case possesses\cite{6,20} additional first- and second-class constraints\cite{22} and actually requires an independent analysis using more advanced techniques. Having not been completed yet, the discussion of the degenerate case is not included here. Nevertheless, no fundamental difference with the generic case is to be expected. The same type of anomaly as the one described above for the generic case would presumably appear in the quantisation of degenerate rigid particles.
2. Classical Rigid Particles

Consider the following action for rigid particles

\[ S[x^\mu, q^\mu, \lambda^\mu] = -\mu c \int_{\tau_i}^{\tau_f} d\tau \left[ \sqrt{-q^2} F(\kappa^2 K^2) + \lambda_\mu (q^\mu - \dot{x}^\mu) \right] . \]  

(2.1)

These particles, whose trajectories are described by functions \( x^\mu(\tau) \) of the world-line parameter \( \tau \), propagate in a \( D \)-dimensional Minkowski spacetime with metric \( \eta_{\mu\nu} \) of signature mostly plus signs (as usual, dots above quantities refer to \( \tau \) derivatives and \( \mu, \nu = 0, 1, \cdots, D-1 \)). The system is characterized by two fundamental positive constants \( \mu \) and \( \kappa \) with dimensions of mass and length respectively, while \( c \) denotes the speed of light. The coordinates \( x^\mu \) have dimension of length and \( \tau \) is taken to be dimensionless.

The overall sign in (2.1) is a matter of convention\(^{[22]} \), chosen such that for \( F(\kappa^2 K^2) \) constant and positive, solutions of positive energy propagate forward in time thus describing particles as opposed to antiparticles. The dimensionless variables \( \lambda^\mu(\tau) \) are Lagrangian multipliers for the constraints \( q^\mu(\tau) = \dot{x}^\mu(\tau) \) identifying the degrees of freedom \( q^\mu(\tau) \) with the particle velocity. Finally, \( F(\kappa^2 K^2) \) is a given dimensionless non constant function specifying the dependence of the action on the world-line extrinsic curvature

\[ K^\mu = \frac{(q\dot{q})q^\mu - q^2 \dot{q}^\mu}{(q^2)^2} , \quad K^2 = \frac{q^2 \dot{q}^2 - (q\dot{q})^2}{(q^2)^3} . \]  

(2.2)

When using \( q^\mu = \dot{x}^\mu \), we have indeed the correspondence

\[ K^\mu = \frac{d n^\mu}{d s} = \gamma^{-1/2} \frac{d}{d\tau} \left[ \gamma^{-1/2} \dot{x}^\mu \right] , \quad n^\mu = \frac{d x^\mu}{d s} = \gamma^{-1/2} \dot{x}^\mu , \]  

(2.3)

where \( n^\mu \) is the normalised world-line tangent vector, \( \gamma = -\dot{x}^2 \) the induced world-line metric and \( ds \) the proper-time line element \( ds = \gamma^{1/2} d\tau \).

In defining the action, it is understood that only time-like trajectories are to be considered, corresponding to the restriction \((-q^2 > 0)\) and implying a causal propagation inside the local light-cone (the unphysical case of supraluminal rigid particles has also been considered in the literature\(^{[6]} \)). Consequently, we have

\[ n^2 = -1 , \quad K^2 > 0 , \quad nK = 0 , \]  

(2.4)

showing that any function \( F(x) \) which is well defined for positive arguments is acceptable in the definition of (2.1). Nevertheless, classical tachyonic solutions of constant extrinsic curvature always exist\(^{[6,12,13,20]} \) for arbitrary choices of \( F(\kappa^2 K^2) \).

The degenerate case mentioned in the introduction corresponds to the function

\[ F(x) = \alpha_0 \sqrt{x} + \beta_0 . \]  

(2.5)

However, solutions then exist only\(^{[6,12,20]} \) when \( \beta_0 \neq 0 \) and \( \alpha_0/\beta_0 > 0 \), in which case all solutions are\(^{[12]} \) of constant extrinsic curvature and include tachyonic ones for curvatures
\[ \sqrt{\kappa^2 K^2} > \beta_0 / \alpha_0. \] Any other choice for \( F(\kappa^2 K^2) \) different from (2.5) defines the generic case to which the present discussion applies. Only degenerate rigid particles require a separate treatment not included here.

By construction, (2.1) is invariant under world-line reparametrisations – including orientation reversing ones – with the coordinates \( x^\mu \) transforming as scalars, the velocities \( q^\mu \) as vectors and the Lagrange multipliers \( \lambda^\mu \) as pseudo-scalars. In fact, due to this local symmetry and the presence of Lagrange multipliers, (2.1) defines a system whose Hamiltonian formulation includes constraints on phase space\(^{[22]} \). For generic rigid particles, one ends up with the following description\(^{[6,20]} \). Phase space degrees of freedom are the coordinates \( x^\mu \) and \( q^\mu \) and their respective conjugate momenta

\[ P_\mu = \frac{\partial L}{\partial \dot{x}^\mu}, \quad Q_\mu = \frac{\partial L}{\partial \dot{q}^\mu}. \] (2.6)

Here, \( L \) is the Lagrangian defining the action \( S = \int d\tau L \) in (2.1). The associated symplectic structure is given by the canonical Poisson brackets

\[ \{ x^\mu, P^\nu \} = \eta^{\mu \nu}, \quad \{ q^\mu, Q^\nu \} = \eta^{\mu \nu}. \] (2.7)

The sector related to the Lagrange multipliers decouples when solving for the second-class constraints \((\chi^\mu_1 = \chi^\mu_2 = \pi^\mu = \partial L / \partial \dot{\lambda}_\mu)\) and \((\chi^\mu_2 = P^\mu - \mu c \lambda^\mu)\) using Dirac brackets\(^{[22]} \) (this is also true in the degenerate case). Consequently, these degrees of freedom do not appear in the Hamiltonian description. Actually, (2.1) being a spacetime Poincaré invariant action, \( P^\mu \) also defines the total energy-momentum of the particle, while its total angular-momentum is given by

\[ M^{\mu \nu} = L^{\mu \nu} + S^{\mu \nu}, \] (2.8)

with the orbital and internal spin contributions, respectively

\[ L^{\mu \nu} = P^\mu x^\nu - P^\nu x^\mu, \quad S^{\mu \nu} = Q^\mu q^\nu - Q^\nu q^\mu. \] (2.9)

With the brackets (2.7), \( P^\mu \) and \( M^{\mu \nu} \) of course obey the Poincaré algebra. Both \( Q^\mu \) and \( S^{\mu \nu} \) are a measure of the world-line extrinsic curvature, since (2.1) implies

\[ Q^\mu = -2\mu c K F(\kappa^2 K^2) K^\mu / \sqrt{-q^2} \] \( \Phi(q^2 Q^2) \), \quad S^{\mu \nu} = 2\mu c K F(\kappa^2 K^2) \kappa(n^\mu K^\nu - n^\nu K^\mu). \] (2.10)

These extrinsic curvature contributions suggest the possibility that rigid particles could actually provide a unifying scheme for quantum particles of different spins. In fact, assuming for a moment that this were indeed the case, only integer spins would be obtained. Indeed, the Heisenberg algebra of commutators \([q^\mu, Q^\nu] = i\hbar \eta^{\mu \nu}\) associated to (2.7) only supports single valued wave-function representations thereby always leading to integer spin representations. However, the quantum anomaly described later on shows that even such a possibility is unfortunately not tenable.

The above phase space \( \{ x^\mu, P^\mu, q^\mu, Q^\mu \} \) is subject to the two first-class constraints

\[ \phi_1 = qQ, \quad \phi_2 = qP + \mu c \sqrt{-q^2} \Phi(q^2 Q^2), \] (2.11)
with the Poisson bracket
\[
\{ \phi_1, \phi_2 \} = -\phi_2 .
\] (2.12)

Here, \( \Phi(q^2Q^2) \) is given by
\[
\Phi(q^2Q^2) = F(x_0) - 2x_0F'(x_0) ,
\] (2.13)
with \( x_0 \) a solution to the equation
\[
x_0F''(x_0) = \frac{-q^2Q^2}{(2\mu\epsilon\kappa)^2} > 0 .
\] (2.14)

Therefore, in order to define the Hamiltonian description of generic rigid particles, the function \( F(x) \) must also be such that given any \( y > 0 \) there always exists a unique \( x > 0 \) for which \( xf''(x) = y \). This condition puts some restriction on the class of acceptable functions \( F(x) \) in (2.1), which is assumed to be met in our analysis. However, one may also take the point of view that the Hamiltonian formulation is not necessarily directly related to the Lagrangian one in (2.1), in which case only \( \Phi(q^2Q^2) \) needs to be given and may be assumed to be any arbitrary non constant function (\( \Phi(q^2Q^2) \) constant corresponds to the degenerate rigid particle). Finally, the total Hamiltonian for the system is simply
\[
H = \lambda_1\phi_1 + \lambda_2\phi_2 ,
\] (2.15)
with \( \lambda_1 \) and \( \lambda_2 \) being Lagrange multipliers for the two first-class constraints.

As is always[22] the case with first-class constraints, \( \phi_1 \) and \( \phi_2 \) are generators of local (Hamiltonian) gauge symmetries. The transformations associated to \( \phi_1 \) – a constraint equivalent to the property \( nK = 0 \) in (2.4) – are simply[20] local rescalings of the variables \( q^\mu, Q^\mu \) and \( \lambda_2 \) and a local shift in \( \lambda_1 \). On the other hand, local world-line reparametrisations preserving the world-line orientation are generated[20] by the constraint \( \phi_2 \) (to which a specific contribution from the constraint \( \phi_1 \) must also added). In fact, the combination \( \lambda_3 = \lambda_1 + \lambda_2/\lambda_2 \) defines[20] an intrinsic world-line metric through \( \lambda_3 = \dot{e}/e \) with \( e(\tau) \) being the world-line einbein. Using the freedom offered by these gauge symmetries, it is always possible to choose for the Lagrange multipliers
\[
\lambda_1 = 0 , \quad \lambda_2 = 1 .
\] (2.16)
Indeed, modular space[22] for this system reduces[20] to a single point. The configuration (2.16) is gauge equivalent to this single point and actually defines[20] an admissible[22], i.e. a complete and global gauge fixing of the system.

3. The Unrestricted Phase Space Map

Naively, canonical quantisation of rigid particles would proceed from their Hamiltonian formulation above. Heisenberg commutation relations for the fundamental degrees of freedom would simply follow from the Poisson brackets (2.7) and in the associated
representation space of quantum states – necessarily equivalent to a wave-function representation –, physical states would be identified as being those states annihilated by two quantum operators in direct correspondence with the first-class constraints \( \phi_1 \) and \( \phi_2 \) for some consistent choice of normal ordering of composite operators. However, this approach – the one so far always adopted for rigid particles\(^{[5,6,18,19]} \) – overlooks one important feature concerning the degrees of freedom \( q^\mu \), namely the fact that this sector of phase space is restricted by the requirement \( (-q^2 > 0) \) or in other words that \( q^\mu \) must lie inside the light-cone. Hence, in the same way that the canonical quantisation of the nonrelativistic particle moving freely on the positive real axis needs some specification\(^{[23]} \), we must first find a (canonical) transformation for the restricted degrees of freedom \( q^\mu \) and \( Q^\mu \) such that the new set of variables is unrestricted and preferably, is also equipped with a canonical symplectic structure. Then, in terms of the transformed degrees of freedom, the above quantisation program may be applied.

Such a set of transformed degrees of freedom indeed exists for rigid particles. It is defined by the following relations

\[
y^0 = \eta \sqrt{-q^2}, \quad R^0 = \eta \frac{[q^0 Q^0 + \vec{q} \cdot \vec{Q}]}{\sqrt{-q^2}}, \quad \tag{3.1a}
\]

\[
y^i = \eta \frac{q^i}{\sqrt{-q^2}}, \quad R^i = \eta \sqrt{-q^2} [Q^i - \frac{q^i}{q^0} Q^0], \quad \tag{3.1b}
\]

where \( \eta \) is the sign of \( q^0 \) and \( (i = 1, 2, \ldots, D - 1) \) are space indices. The inverse relations are

\[
q^0 = y^0 \sqrt{1 + \vec{y}^2}, \quad Q^0 = \sqrt{1 + \vec{y}^2} \left[ -R^0 + \frac{\vec{y} \cdot \vec{R}}{y^0} \right], \quad \tag{3.2a}
\]

\[
q^i = y^0 y^i, \quad Q^i = \frac{R^i}{y^0} + y^i \left[ -R^0 + \frac{\vec{y} \cdot \vec{R}}{y^0} \right]. \quad \tag{3.2b}
\]

In geometrical terms, \( y^0 \) measures the invariant length of the vector \( q^\mu \) with a sign related to whether \( q^\mu \) lies in the forward or in the backward light-cone, while the remaining variables \( y^i \) are in fact the parameters of the Lorentz boost in the direction \( \vec{q} \) mapping the vector \( q^\mu = (q^0, \vec{q}) \) into the vector \( (y^0, \vec{y}) \). The remaining variables \( R^0 \) and \( R^i \) are then obtained as degrees of freedom conjugate to \( y^0 \) and \( y^i \) respectively. Namely, the Poisson brackets (2.7) and the following canonical brackets

\[
\{y^0, R^0\} = 1, \quad \{y^i, R^j\} = \delta^{ij}, \quad \tag{3.3}
\]

are mapped into one another under the transformations (3.1) and (3.2).

Clearly, the canonically conjugate degrees of freedom \((y^0, R^0, y^i, R^i)\) are no longer restricted as are the original ones \((q^\mu, Q^\mu)\), thereby achieving the required properties. However, the price to pay is a loss of manifest Lorentz covariance. Spacetime translations generated by \( P^\mu \) and space rotations generated by \( M^{ij} = L^{ij} + S^{ij} \) are still manifest
symmetries in the transformed representation of the system, but this is no longer the case for Lorentz boost generators \( M^{0i} = L^{0i} + S^{0i} \). Indeed, while the expressions (2.9) for \( L^{\mu\nu} \) are not affected by the redefinitions (3.2), those for the spin tensor become

\[
S^{0i} = -R^i \sqrt{1 + \vec{y}^2} , \quad S^{ij} = R^i y^j - R^j y^i .
\] (3.4)

Nevertheless, it is a straightforward calculation to check that with the brackets (3.3), the full Poincaré algebra is still obtained for the generators \( P^\mu \) and \( M^{\mu\nu} \) expressed in terms of the transformed variables (3.1), thereby establishing the consistency of this alternative Hamiltonian description of rigid particles (the redefinitions (3.1) are of course also applicable in the degenerate case). In the generic case, the first-class constraints (2.11) and associated Hamiltonian (2.15) are then given by

\[
\phi_1 = y^0 R^0 ,
\] (3.5)

and

\[
\phi_2 = y^0 [\vec{y} \vec{R} - P^0 \sqrt{1 + \vec{y}^2}] + \eta \mu c y^0 \Phi \left( (y^0 R^0)^2 - (\vec{y} \vec{R})^2 - \vec{R}^2 \right) ,
\] (3.6)

with

\[
q^2 Q^2 = (y^0 R^0)^2 - (\vec{y} \vec{R})^2 - \vec{R}^2 .
\] (3.7)

From these expressions and the brackets (3.3), the gauge algebra (2.12) is obviously also recovered.

4. The Mixed Lorentz-Gravitational Anomaly

The quantisation of rigid particles is thus specified by the Heisenberg commutation relations

\[
[x^\mu, P^\nu] = i\hbar \eta^{\mu\nu} , \quad [y^0, R^0] = i\hbar , \quad [y^i, R^j] = i\hbar \delta^{ij} ,
\] (4.1)

and an abstract representation space of this algebra equipped with an inner product for which these operators are all hermitian and self-adjoint. Representations of this algebra are unitarily equivalent to wave-function ones either in position or in momentum space for each pair of conjugate degrees of freedom. This determines the space of quantum states for such systems, each of these states being therefore of positive norm.

Turning to the ordering problem, let us first consider the situation for the Poincaré generators. Clearly, \( P^\mu \) does not require an ordering prescription. For \( L^{\mu\nu} \) and \( S^{\mu\nu} \) we choose

\[
L^{\mu\nu} = P^\mu x^\nu - P^\nu x^\mu ,
\] (4.2)

and

\[
S^{0i} = -\frac{1}{2} \left[ R^i \sqrt{1 + \vec{y}^2} + \sqrt{1 + \vec{y}^2} \vec{R}^i \right] , \quad S^{ij} = R^i y^j - R^j y^i ,
\] (4.3)

in order that these operators be hermitian and self-adjoint. Obviously, \( L^{\mu\nu} \) and \( P^\mu \) generate the Poincaré algebra. On the other hand, while it is clear that \( S^{ij} \) generates the algebra
of rotations in space, it is not difficult to check that with the choice of ordering in (4.3) the operators $S_{ij}$ and $S_{0i}$ in fact obey the whole Lorentz algebra. Thus, the total angular-momentum ($M^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$) and energy-momentum $P^\mu$ operators generate the whole Poincaré algebra, thereby establishing that this algebra is anomaly free and that in spite of the loss of manifest spacetime covariance, quantum states of rigid particles indeed span a linear representation space for spacetime translations and Lorentz transformations. As already pointed out above, this space can only support integer spin representations of the Lorentz group.

Let us now turn to the ordering problem for the first-class constraints $\phi_1$ and $\phi_2$, a necessary prerequisite in order to define physical states of quantum rigid particles. Again, in order to have hermitian and self-adjoint operators, we must choose for the quantum constraints

$$\phi_1 = \frac{1}{2} [ y^0 R^0 + R^0 y^0 ] ,$$  \hspace{1cm} (4.4a)

and

$$\phi_2 = y^0 [\bar{y}.\bar{R} - P^0 \sqrt{1 + \bar{y}^2}] + \frac{1}{2} \eta \mu c \left[ y^0 \Phi : (y^0 R^0)^2 : - ; (\bar{y}.\bar{R})^2 : - R^2 \right] +$$

$$+ \Phi : (y^0 R^0)^2 : - ; (\bar{y}.\bar{R})^2 : - R^2 : y^0 ,$$ \hspace{1cm} (4.4b)

where $: (y^0 R^0)^2 :$ and $: (\bar{y}.\bar{R})^2 :$ stand for normal ordered expressions of the corresponding operators to be specified presently. By considering all possible orderings for the products in these operators, one concludes that the most general choices possible all reduce to expressions of the following form

$$: (y^0 R^0)^2 : = R^0 y^0 y^0 R^0 + i\hbar A_1 y^0 R^0 + \hbar^2 A_2 ,$$  \hspace{1cm} (4.5a)

$$: (\bar{y}.\bar{R})^2 : = R^i y^i y^i R^i + i\hbar B_1 y^i R^i + \hbar^2 B_2 ,$$  \hspace{1cm} (4.5b)

where $A_1, A_2, B_1$ and $B_2$ are undetermined free complex coefficients. Requiring that these operators be also hermitian and self-adjoint only leads to the restrictions

$$A_1^* = -A_1 , \quad A_2^* = A_2 - A_1 ,$$ \hspace{1cm} (4.6a)

$$B_1^* = -B_1 , \quad B_2^* = B_2 - (D - 1)B_1 .$$ \hspace{1cm} (4.6b)

With these definitions, it is now possible to determine the commutation relations for the quantum gauge algebra. One easily finds

$$[ \phi_1 , \phi_2 ] = -i\hbar \phi_2 .$$ \hspace{1cm} (4.7)

Comparison with the classical bracket (2.12) shows that the gauge algebra is indeed anomaly free. Therefore, both local world-line reparametrisations and the local rescalings generated by $\phi_1$ are symmetries of quantised generic rigid particles. From that point of view, it is thus meaningful to define their quantum physical states $|\psi >$ as being the solutions to the conditions

$$\phi_1 |\psi > = 0 , \quad \phi_2 |\psi > = 0 ,$$ \hspace{1cm} (4.8)
thereby ensuring invariance of these states under all local gauge symmetries including world-line reparametrisations. However, this definition must also be consistent with the other symmetries of the system. Namely, the generators of gauge symmetries must commute with those of spacetime Poincaré transformations. Otherwise, \textit{physical} states solving (4.8) cannot define linear representations of the Poincaré group. In other words, a state physical in a given reference frame would no longer be physical in some other frame! Nor would it be possible to define consistently the mass or spin of \textit{physical} states!

Clearly, this type of problem does not arise for the gauge generator \( \phi_1 \) since
\[
[ L^{\mu \nu}, \phi_1 ] = 0 , \quad [ S^{\mu \nu}, \phi_1 ] = 0 , \quad [ M^{\mu \nu}, \phi_1 ] = 0 , \quad [ P^\mu, \phi_1 ] = 0 .
\] (4.9)

Moreover, we also have for the generator of world-line reparametrisations
\[
[ P^\mu, \phi_2 ] = 0 .
\] (4.10)

Therefore, at least the energy-momentum hence also the mass of quantum physical states are well defined observables for generic rigid particles. To analyse the situation for the remaining commutators \([ M^{\mu \nu}, \phi_2 ]\), it is useful to decompose \( \phi_2 \) in (4.4b) as \( \phi_2 = \chi_1 + \chi_2 \) with
\[
\chi_1 = y^0 [\bar{y} \cdot \vec{R} - P^0 \sqrt{1 + \bar{y}^2}] .
\] (4.11)

A simple calculation then finds that
\[
[ L^{0i}, \chi_1 ] = i \hbar (P^0 y^0 y^i - P^i y^0 \sqrt{1 + \bar{y}^2}) = -[ S^{0i}, \chi_1 ] ,
\] (4.12a)
\[
[ L^{ij}, \chi_1 ] = i \hbar (P^i y^0 y^j - P^j y^0 y^i) = -[ S^{ij}, \chi_1 ] ,
\] (4.12b)
leading to
\[
[ M^{\mu \nu}, \chi_1 ] = 0 ,
\] (4.13)
and
\[
[ M^{\mu \nu}, \phi_2 ] = [ M^{\mu \nu}, \chi_2 ] .
\] (4.14)

Moreover, since \( L^{\mu \nu} \) clearly also commutes with \( \chi_2 \), only the commutators of \( S^{\mu \nu} \) with \( \chi_2 \) are left to be computed. In fact, since both \( y^0 \) and \( R^0 \) commute with \( S^{\mu \nu} \), the crucial commutators to be determined are those of \( S^{\mu \nu} \) with \( : (y^0 R^0)^2 : - : (\bar{y} \cdot \vec{R})^2 : - \vec{R}^2 : \). Using the normal ordered expressions (4.5), a direct calculation shows that
\[
[ S^{ij}, : (y^0 R^0)^2 : - : (\bar{y} \cdot \vec{R})^2 : - \vec{R}^2 ] = 0 ,
\] (4.15)
so that finally
\[
[ M^{ij}, \phi_2 ] = 0 .
\] (4.16)

This result is indeed to be expected owing to the manifest rotation covariance of the quantisation procedure. On the other hand, the commutator with Lorentz boost generators gives
\[
[ S^{0i}, : (y^0 R^0)^2 : - : (\bar{y} \cdot \vec{R})^2 : - \vec{R}^2 ] = \frac{1}{2} i \hbar^3 \frac{y^i}{(1 + \bar{y}^2)^{3/2}} + \\
+ \frac{1}{2} \hbar^2 B_1 \left[ R^i \frac{1}{\sqrt{1 + \bar{y}^2}} + \frac{1}{\sqrt{1 + y^2}} R^i \right].
\] (4.17)
Hence, we certainly have for any choice of $F(\kappa^2 K^2)$ in the generic case

$$[ M^{0i}, \phi_2 ] \neq 0 .$$  

(4.18)

This is the mixed Lorentz-gravitational anomaly of the title. Generally, this anomaly is of order $\hbar^2$ unless an ordering for $\phi_2$ corresponding to $B_1 = 0$ in (4.5b) happens to be chosen, in which case the anomaly is of order $\hbar^3$. Therefore, given any ordering for the generator of local world-line reparametrisations, physical states (4.8) do not transform covariantly under Lorentz boosts! The subspace of physical states (4.8) is not closed under the action of Lorentz generators, even though these generators act covariantly on the entire space of states.

5. Conclusions

This letter has described how by properly accounting for the restriction of causal propagation inside the light-cone, rigid particles cannot be quantised in a way which is compatible with the requirements of local gauge invariance under world-line reparametrisations and of spacetime Poincaré covariance both at the same time. The origin of the problem lies with a mixed Lorentz-gravitational anomaly in the commutator of Lorentz boosts with the generator of world-line reparametrisations. Consequently, even though the Poincaré and local gauge algebras are both anomaly free on the complete space of states, Lorentz boosts map outside of the subspace of physical states defined to be all states invariant under gauge transformations – which includes world-line reparametrisations. In fact, the only Poincaré invariant quantum observable which is well defined for physical states is their mass; the concept of spin has no meaning for these states. We must therefore conclude that rigid particles are not consistent quantum models for particle physics. To be precise, the present analysis applies for any possible dependence on the world-line extrinsic curvature with only one exception, corresponding to a degenerate situation for which classical solutions are all of constant extrinsic curvature. This degenerate case requires a separate treatment still to be completed. Nevertheless, the same type of anomaly as the one above would presumably be obtained in that case as well. Most probably, the same conclusion would extend further to theories which also include a dependence on the extrinsic torsion and other such invariants of higher order still.

One may also argue that the quantum anomaly for rigid particles is the strongest indication yet as to the probable inconsistency of quantised rigid strings. It is widely believed\cite{24} that the higher derivative couplings of rigid string theories lead to quantum physical states either of negative norm or of energy unbounded below. In fact, a semi-classical analysis of Polyakov’s rigid strings has revealed\cite{25} instabilities of the latter type. However, rigid strings possess collapsed configurations corresponding to rigid particles. Since quantised rigid particles are not consistent, quantised rigid strings cannot be consistent either. Note that quantum inconsistency of rigid particles is not related either to negative-norm physical states nor to energy unbounded below but actually follows from a quantum anomaly. Strictly speaking, if this type of reasoning is justified, the conclusion applies so far only to those rigid strings whose collapsed configurations are not degenerate
rigid particles. Specifically, consider the dimensional reduction\textsuperscript{[26,8]} of a \((D+1)\)-dimensional rigid string whose action depends on the world-sheet extrinsic curvature through some function \(G(x)\),

\[
S[\phi^M] = -\frac{\mu_c}{\kappa} \int d\tau d\sigma \sqrt{-g} G(\kappa^2 \Delta \phi^M \Delta \phi_M) .
\] (5.1)

Here, \(\phi^M(M = 0, 1, \cdots, D)\) are the string coordinates, \(g_{\alpha\beta} = \eta_{MN} \partial_\alpha \phi^M \partial_\beta \phi^N\) is the induced world-sheet metric (\(\eta_{MN}\) is the Minkowski metric in \((D + 1)\) dimensions), \(\Delta\) is the Laplacian

\[
\Delta = \frac{1}{\sqrt{-g}} \partial_\alpha \sqrt{-g} g^{\alpha\beta} \partial_\beta ,
\] (5.2)

and as usual \(\xi^{\alpha=0} = \tau\) and \(\xi^{\alpha=1} = \sigma\) with \(\alpha, \beta = 0, 1\). When identifying\textsuperscript{[26,8]} one of the space coordinates \(\phi^M\) with \(\sigma\) and assuming that the remaining string coordinates \(x^\mu\) are independent of \(\sigma\), (5.2) reduces to (2.1) with

\[
F(x) = G(x) \int d\sigma
\] (5.3)

(the integral is over the finite range of \(\sigma\)). Thus in particular, Polyakov’s rigid strings\textsuperscript{[1,8]} correspond to the choice

\[
F(x) = \alpha_0 x + \beta_0 .
\] (5.4)

Since this function does not define the degenerate case (2.5), we must conclude from the analysis of this paper that Polyakov’s rigid strings cannot be consistent fundamental quantum theories. Of course, this does not necessarily exclude their possible relevance as effective theories for some \textit{semi-classical} approximation to other more fundamental theories.

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REFERENCES

[1] A. M. Polyakov, Nucl. Phys. B\textbf{268} (1986) 406.
[2] H. Kleinert, Phys. Lett. B\textbf{174} (1986) 335.
[3] R. D. Pisarski, Phys. Rev. D\textbf{34} (1986) 670.
[4] C. Battle, J. Gomis and N. Roman-Roy, J. Phys. A\textbf{21} (1988) 2693.
[5] V. V. Nesterenko, J. Phys. A\textbf{22} (1989) 1673; Theor. Math. Phys. 86 (1991) 168; Mod. Phys. Lett. A\textbf{6} (1991) 719.
[6] M. S. Plyushchay, Mod. Phys. Lett. A3 (1988) 1299; ibid A4 (1989) 837; ibid A4 (1989) 2747; Int. J. Mod. Phys. A4 (1989) 3851; Phys. Lett. B235 (1990) 47; ibid B236 (1990) 291; ibid B243 (1990) 383; ibid B248 (1990) 107; ibid B248 (1990) 299; ibid B253 (1991) 50; ibid B262 (1991) 71; ibid B280 (1992) 232; Nucl. Phys. B362 (1991) 54.
[7] A. Dhar, Phys. Lett. B214 (1988) 75.
[8] J. Grundberg, J. Isberg, U. Lindström and H. Nordström, Phys. Lett. B231 (1989) 61.
[9] J. P. Gauntlett, K. Itoh and P. K. Townsend, Phys. Lett. B238 (1990) 65.
[10] M. Pavšič, Phys. Lett. B205 (1988) 231; ibid B221 (1989) 264.
[11] M. Huq, P. I. Obiakor and S. Singh, Int. J. Mod. Phys. A5 (1990) 4301.
[12] H. Arodz, A. Sitarz and P. Wegrzyn, Acta Phys. Polonica B20 (1989) 921.
[13] T. Dereli, D. H. Harley, M. Önder and R. W. Tucker, Phys. Lett. B252 (1990) 601.
[14] D. Zoller, Phys. Rev. Lett. 65 (1990) 2236.
[15] G. Fiorentini, M. Gasperini and G. Scapetta, Mod. Phys. Lett. A6 (1991) 2033.
[16] A. M. Polyakov, Mod. Phys. Lett. A3 (1988) 325.
[17] S. Iso, C. Itoi and H. Mukaida, Phys. Lett. B236 (1990) 287; Nucl. Phys. B346 (1990) 293.
[18] Yu. A. Kuznetsov and M. S. Plyushchay, “The Model of the Relativistic Particle with Curvature and Torsion”, Protvino preprint IHEP 91-162 (October 1991), and references therein.
[19] J. Isberg, U. Lindström and H. Nordström, Mod. Phys. Lett. A5 (1990) 2491.
[20] J. Govaerts, in preparation.
[21] L. Alvarez-Gaumé and E. Witten, Nucl. Phys. B234 (1983) 269.
[22] For a recent review, see
J. Govaerts, Hamiltonian Quantisation and Constrained Dynamics, Lecture Notes in Mathematical and Theoretical Physics 4, (Leuven University Press, Leuven, 1991).
[23] C. J. Isham, in Relativity, Groups and Topology II, Les Houches 1983, eds. B. S. DeWitt and R. Stora (North Holland, Amsterdam, 1984), p. 1162.
[24] See for example
J. Polchinski and Z. Yang, “High Temperature Partition Function of the Rigid String”,
Texas/Rochester preprint UTTG-08-92, UR-1254, ER-40685-706.
[25] E. Braaten and C. K. Zachos, Phys. Rev. D34 (1987) 1512.
[26] M. J. Duff, P. S. Howe, T. Inami and K. S. Stelle, Phys. Lett. B191 (1987) 70.