The weighted tunable clustering in local-world networks with incremental behaviors *

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Abstract: Since some realistic networks are influenced not only by increment behavior but also by tunable clustering mechanism with new nodes to be added to networks, it is interesting to characterize the model for those actual networks. In this paper, a weighted local-world model, which incorporates increment behavior and tunable clustering mechanism, is proposed and its properties are investigated, such as degree distribution and clustering coefficient. Numerical simulations are fit to the model characters and also display good right skewed scale-free properties. Furthermore, the correlation of vertices in our model is studied which shows the assortative property. Epidemic spreading process by weighted transmission rate on the model shows that the tunable clustering behavior has a great impact on the epidemic dynamic.

Keywords: Weighted network, increment behavior, tunable cluster, epidemic spreading.

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1 Introduction

Since Barabási and Albert introduced the classical BA model [1], lots of empirical measurements have been used to discover some properties on real-world complex systems. However, networks as well known are far from general scale-free structure and people find in many real networks, for example, Internet, power grids, WWW and so on [2]-[6]. Nodes connect each other closer within a group than to other groups, such a group is defined to be a local-world which plays an important role in micro dynamics of scale-free networks including both boolean and weighted complex networks [7].

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In some complex systems, when a new individual enters the system, it takes a great deal of searching in the global preferential attachments. The preferential attachment of local economy regions exists in World Trade Web [8]. In Internet on the router level, a host only has information of others who are in a same local domain [9, 10]. Motivated by the above phenomenon, Li and Chen [8] presented a Local World model in which a new node makes local preferential attachment in a local-world. Zhang et al. [11] introduced an evolving scale-free network model with a continuously adjustable clustering coefficient to modify the small cluster of Li’s. Chen et al. [12] proposed a multi-local-world model to mimic the Internet. Wu and Liu [13] presented a high clustering coefficient in a local-world model. On the other hand, there are many models of growing networks with community structures, for example, Noh et al. [14] proposed a growing network model with group structures basing on creation or joining mechanism. Further, Xuan et al. [15] gave a hierarchical and modular network by adding a growing rule with the preferential attachment rule and the Motter’s modeling procedure. Pollner et al. introduced a model with the dynamics of overlapping communities. But these papers basically consider boolean growing networking with local small-worlds, while the weighted networks are conformity with the real-world whose links between nodes display heterogeneity in the capacity and intensity [17]-[19]. Figure 1 shows the local micro dynamics in networks.

![Figure 1](image.png)

Figure 1: It is an increment behavior mechanism. That is, if a new node $n$ connects to both the existing node $i$ and it’s neighbor $j$, then the link weight between $i$ and $j$ increases $\delta$.

In previous studies, when a new node $v$ is added to the network and builds links with an existing node $v_i$ and it’s neighbor $v_j$, the weight of link between $v_i$ and $v_j$ does not change. However, $v_i$ and $v_j$ is in one local-world with high probability, they will have more common knowledge and interest if they received same information from a node, and then the probability that they communicate with each other will increase, that is reflected by the increasing of the like weight between them. We call this phenomenon *increment behavior*, which had been noticed by Dorogovstev et al. [20]. Increment behavior exists universally in real-life networks. For example, in wide airport networks(WAN) [21, 22], the amount of passengers between two airports will increase when they both build airlines with a common airport, in SCN [25, 26], two cooperative scientists will strengthen their cooperation to a great extent if they are collaborated with a common scientist in different research fields. Dorogovstev et al. [20] proposed a growing weighted network model with a power-law weight distribution based with the preferential attachment rule and ”increment behavior”. But the communities of this growing network model was not discussed and studied in this paper.

In this paper, in order to character the real-life networks more precisely, we present a weighted local-world evolving network model with increment behavior and tunable clustering mechanism which can capture both the dynamic of local preferential attachment(LPA) [8] and triad formation(TF) [27]. The three main mechanisms of a weighted growing networks with local small-world
are the preferential attachment of communities, strength preferential attachment and increment
behavior. It is shown that the degree of increment behavior of this network model has a great
impact on topology property and social behaviors of networks. The geometric characteristics
of the model both analytically and numerically are discussed, which shows that the analytical
expressions are agreement with the numerical simulations well. Moreover, we analyze the effect
of tunable clustering behaviors and increment behaviors on the weighted local-world network.
Finally, by way of the weighted transmission rate, we find that the tunable clustering behavior
has a great impact on the spreading dynamic networks.

2 The model construction

We consider not only attachment mechanisms between local-worlds but also inter locals prefer-
ential attachment mechanism. First, we define three kinds of evolving attachment rules.

Local-world preferential attachment \((LPA)\): a new node \(v\) is joining an existing local-
world \(\Omega_i\), the probability of choosing \(\Omega_i\), \(\prod(\Omega_i)\), depends on it’s size \(|\Omega_i| = N_i\), that is

\[
\prod(\Omega_i) = \frac{N_i}{\sum_j N_j}.
\]

Strength preferential attachment \((SPA)\): If a node \(v\) had joined in a local world,
without loss of generality, \(\Omega_i\), then \(v\) connected to a node \(v_j\) which strength is \(s_j\) in \(\Omega_i\) with probability \(\prod(s_j)\),

\[
\prod(s_j) = \frac{s_j}{\sum_l s_l}.
\]

For the sake of convenience, local-world preferential attachment \((LPA)\) and strength prefer-
ential attachment \((SPA)\) are called by a joint name preferential attachment, and a link is called
PA link if it obtained by the preferential attachment.

Increment behavior mechanism: If a new node \(v\) joins the two nodes \(v_i\) and \(v_j\) of a
link \(v_iv_j\) in a network, then the weight of the link \(v_iv_j\) increases \(\delta\). The network growing way
according to this mechanism is illustrated in Fig. 1.

In some real-life networks, clustering structure sometime can be quantified by large clustering
coefficients, that means there are many triangles in networks, this property will be shown in
the following construction for this model which induced by the tunable clustering mechanism.
Besides PA links, we also introduce triad-formation\((TF)\) links which means if a new node \(v\) joins
node \(v_i\) at last time, then a neighbor \(v_j\) of node \(v_i\) is selected to connect \(v\) with probability

\[
\prod(w_{ij}) = \frac{w_{ij}}{s_i},
\]

where \(w_{ij}\) is the weight of link \(v_iv_j\). It is easy to see that an increment behavior appears on link
\(v_iv_j\) with emergence of TF links.

Next, we produce our model:

Process of generation: The weighted tunable cluster local-world with increment behavior
network model.
Initializing an undirect weighted network with \( c_0 \) communities \((c_0 > 1)\), each community is an \( n_0 \) complete connected network, and there is a link for each pair communities, that is, there are \( c_0(c_0 - 1)/2 \) inter-local links to make \( c_0 \) local-worlds connected, the weight for each link \( v_i v_j \) is set \( w_{ij} = w_0 = 1 \).

**BEGIN:**

**step 1,** with probability \( q \), a new local-world \( \Omega \) with \( n_0 \) completely connected nodes is added and the weight for each link is \( w_0 = 1 \). Choose a node in \( \Omega \) randomly to connect \( m \) existing nodes in other local-worlds according to preferential attachment, it comes into \( m PA \) links.

**Step 2,** with probability \( 1 - q \), a new node \( v \) is added with \( m \) links, and the generated mechanism is preferential attachment.

**Step 3,** with probability \( p \), a \( TF \) link is introduced by equation (3). Both \( PA \) link and \( TF \) link induced the increment behavior. **END**

For simply, we set the initial weight of \( PA \) links and \( TF \) links with \( w_0 = 1 \) and increment amount with \( \delta \) for each old link. In the generating process, after \( t \) steps, there are \( c_0 + qt \) local-worlds and \( c_0 n_0 + (qn_0 + (1 - q)) t \) nodes and \( c_0(n_0(n_0 - 1)/2 + c_0(c_0 - 1)/2 + (q(n_0(n_0 - 1)/2 + m(1 + p)) + (1 - q)m(1 + p + p\delta))t \). And when \( p = q = 0 \), the model is consistent with weighted BA model, which sets the weight of edges to 1 in BA model. So, at this time, all nodes \( v_i \) in this model has \( k_i = s_i \). In the following section, we analysis the topology properties of the weighted tunable cluster local-world network model.

### 3 Topology properties of the Model

#### 3.1 Distributions of local world on size, degree and link weight

By mean-field theory [28], we can obtain the distribution of local-world size, strength, degree, and link weight in the weighted tunable cluster local-world network model. The mean strength \( s \) is \( \langle s \rangle = \frac{qn_0(n_0 - 1) + 2m(1 + p + p\delta)}{2} \), when \( p, q \) and \( \delta \) are fixed and \( t \) large enough, therefor, \( \langle s \rangle \) and \( \langle w \rangle = \langle s \rangle/2 \) are considered as constants sometime.

First, considering the size distribution of local-worlds. Assume the size of local-world is continuous, hence, \( LPA \) is also interpreted as continuous rate of change on local-world \( \Omega_i \),

\[
\frac{\partial N_i}{\partial t} = \frac{N_i}{\sum_j N_j}.
\]

Then the total number of nodes in one local-world at time \( t \) is

\[
N = \sum_j N_j = c_0 n_0 + (qn_0 + (1 - q)) t \approx (qn_0 + (1 - q)) t.
\]

By initial condition \( N(t = t_i) = n_0 \), we get

\[
N_i = n_0 \left( \frac{t}{t_i} \right)^{(1/1-q+qn_0)}.
\]

For convenience, we simplify probability density of \( t_i \) by

\[
p(t_i) = \frac{1}{c_0 + qt}.
\]
Then the size distribution of local-world $\Omega_i$ is

$$P(N_i(t) \geq n) = P(t_i \leq \frac{n_0^{1-q+qn_0}}{n^{1-q+qn_0}t}) = \frac{1}{c_0 + qt} \left( \frac{n_0}{n} \right)^{1+q(n_0-1)}. \quad (8)$$

Equation (8) shows that the size distribution of $\Omega_i$ obeys power-law rule, $P(N) \sim N^{-\gamma}$ and $\gamma = 1 + q(n_0 - 1) \geq 1$. This property is in conformity with many real-world networks [2, 3, 18, 19, 20, 21], which are scale-free local-world with scaling exponent $\gamma \in [1, 2]$.

Next, we analyze the strength distribution, the degree distribution, and the weight distribution.

If a new node $v$ is selected by SPA and added to a local-world, the strength $s_i$ of an old node $v_i$ will be affected by $v$ in following three cases, those microscopic variety mechanisms for $s_i$ are shown in Figure 2.

![Figure 2: The microscopic variety mechanisms for $s_i$. The dashed edges are TF links. (a) $v$ joins $v_i$ by LPA, a neighbor $v_j$ of $v_i$ is chosen to build a TF link. In this case, $s_i = s_i + 1 + \delta$; (b) $v$ connects to $v_j$ which is a neighbor of $v_i$ by SPA, and $v_i v$ is a TF link. In this case, $s_i = s_i + 1 + \delta$. (c) $v$ joins $v_i$ by SPA but none of $v_i$’s neighbor is selected to build TF link with $v$. In this case, $s_i = s_i + 1$.](image)

By Figure 2 it is easy to see that the strength change rate of $v_i$ is

$$\frac{\partial s_i}{\partial t} = m \frac{N_i}{\sum N_j} \sum_{v_j \in \text{local}} s_j \left( \sum_{v_l \in \Omega_i} \frac{w_{il}}{s_l} \frac{p(1 + \delta)}{s_i} \right) + \sum_{v_l \in \Omega_i} \frac{w_{il}}{s_l} \frac{p(1 + \delta)}{s_i} + \sum_{j \in \Omega_i} \frac{w_{ij}}{s_j} \frac{(1 - p)}{s_i} \right)$$

$$= m \frac{N_i}{\sum N_j} \sum_{v_j \in \text{local}} s_j \left( 1 + p + 2p\delta \right), \quad (9)$$

and after $t$ steps, the total strength of each local-world in model, on average, is

$$\sum_{v_j \in \text{local}} s_j = \langle s \rangle N_i. \quad (10)$$

From Figure 2 the degree of $v_i$ is affected in cases (a) and (b). However, increment behavior does not have any impact in case (c). Therefor, degree change rate of $v_i$ is

$$\frac{\partial k_i}{\partial t} = m \frac{N_i}{\sum N_j} \left( \frac{s_i}{\sum_{v_l \in \text{local}} s_l} + \sum_{v_j \in \Omega_i} \frac{s_j}{\sum_{v_l \in \text{local}} s_l} \frac{w_{ij}}{s_j} \right)$$

$$= m \frac{N_i}{\sum N_j} \sum_{v_j \in \text{local}} s_j \left( 1 + p \right). \quad (11)$$
When $t$ is large enough and after $t$ steps, the amount of nodes in any local-world is

$$\sum_j N_j = c_0 n_0 + (q_0 + (1-q)) t \approx (q_0 + (1-q)) t.$$  \tag{12}$$

Hence, we obtain

$$\frac{\partial s_i}{\partial t} = \frac{m(1 + p + 2p\delta)}{(s)(q_0 + 1 - q)} t = A s_i$$, \tag{13}$$

where $A = m(1 + p + 2p\delta)/(s)(q_0 + 1 - q))$. And

$$\frac{\partial k_i}{\partial t} = \frac{m(1 + p)}{(s)(q_0 + 1 - q)} t = C s_i$$, \tag{14}$$

where $C = m(1 + p)/(s)(q_0 + 1 - q))$. With the initial time $t_i$ of $v_i$, $s_i(t_i) = k_i(t_i) = m(1 + p)$ and equations (13) (14), we have

$$s_i(t) = m(1 + p)\left(\frac{t}{t_i}\right)^A,$$ \tag{15}$$

$$k_i(t) = \frac{s_i(t) + (A/C - 1)m(1 + p)}{A/C}.$$ \tag{16}$$

Therefore, by mean-field theory, the strength distribution is

$$p(s) = \frac{\partial P(s_i(t) < s)}{\partial s} = \frac{1}{c_0 n_0 + (q_0 + (1-q)) t} \frac{m(1 + p)^{1/A}}{A} s^{-(1+1/A)},$$ \tag{17}$$

since the time density is $p(t = t_i) = \frac{1}{c_0 n_0 + (q_0 + (1-q)) t_i}$. By equation (10), clearly, it is a linear relationship between strength and degree of $v_i$. Therefore, the degree distributions have the same distribution with strength, hence $P(k) \sim k^{-\gamma}$ and $\gamma = 1 + 1/A$.

When a new node $v$ connects two ends of a link $v_i v_j$, the link weight $w_{ij}$ will change by (a) and (b) in Figure 2 that is,

$$\frac{\partial w_{ij}}{\partial t} = m \sum_{j} N_j \left( \frac{s_i}{\sum_{v_i \in \text{local}} s_i} s_i + \sum_{v_j \in \text{local}} \frac{s_j}{\sum_{v_j \in \text{local}} s_j} s_j \right)$$

$$= \frac{2mp\delta}{(s)(q_0 + 1 - q)} w_{ij} = D \frac{w_{ij}}{t},$$ \tag{18}$$

where $D = (2mp\delta)/(s)(q_0 + 1 - q))$. Therefore, $w_{ij} = \left(\frac{t}{t_i}\right)^D$ with initial time $t_i$ of $v_i$ and $w_{ij}(t_i)=1$. So the density of the link weight distribution is

$$p(w) = \frac{\partial P(w_{ij}(t) < w)}{\partial w} = \frac{1}{c_0 n_0 + (q_0 + (1-q)) D} w^{-(1+1/D)}.$$ \tag{19}$$

The left graph in Fig 3 displays the strength distribution on three values of $\delta$ and the right graph exhibits the degree distribution and the weight distribution in a special case with theoretical curves. All those simulations agree well with Eqs. (17) and (19) obtained from theory. It is observed that there is some deviation from power-law behavior alone with small values of $s$ and $k$, which are originated by fluctuation on the number of new links built-up by systems (30, 31). In many real-life networks, such as the World Wide Web [5, 6], the actor collaboration network [32] and scientific collaboration networks [25, 26], truly exhibit those phenomena of deviation from power-law in small $s$ and $k$ values.
3.2 Generalization of clustering on the weighted networks

Large clustering coefficient is a typical property of social acquaintance networks, where two individuals with a common friend are likely to know each other. The clustering coefficient of a node indicates the close level among its neighborhood. The clustering coefficient $c_i$ of a node $v_i$ is the ratio of the existed link size $e_i$ and all possible links size among its $n_i$ neighbors, that is, $c_i = 2e_i/(n_i(n_i-1))$. The clustering coefficient of a network is the average for all individual $c_i$. In order to understand the structure organization of weighted networks much better, the weighted clustering coefficient $c_{iw}$ is introduced by Barrat et al. [33].

$$c_{iw} = \frac{1}{s_i(n_i-1)} \sum_{j,h} \frac{(w_{ij} + w_{ih})}{2} a_{jh}, \quad (20)$$

where $s_i$ is the strength of $v_i$, $w_{ij}$ is the weight of link $v_i v_j$ and $a_{jh}$ is the element of adjacent matrix in which $a_{jh} = 1$ if $v_j v_h$ is a link, $a_{jh} = 0$ otherwise. In here, the weight for each link is positive. If weights of some links are zero, then the topological structures of the weighted networks will be different. Therefore, the weighted clustering coefficient without an edge is not the same as if the weight of the link is put to zero. The clustering coefficient of weighted networks depends on both its topological structure and its weight of each link. However, Onnela et al. in [34] and Holme in [35] provided a different definition of the clustering coefficient of weighted networks, which is only dependent on its weight, while independent of its topological structure. For the sake of stressing known topological structure of networks, we adapt the definition of the clustering coefficient of weighted networks in [33].

The weighted clustering coefficient is only related to the weights of links adjacent to $v_i$ and does not reflect the relations among the neighbors. So, we extend $c_{iw}$ to $\tilde{c}_{iw}$ which took all links weight among $v_i$’s neighbors and $v_i$ into account.

$$\tilde{c}_{iw} = \frac{1}{s_i(k_i-1)} \sum_{j,h} \frac{(w_{ij} + w_{ih} + w_{jh})}{3} a_{jh} = \frac{\langle w_{ijh} \rangle}{s_i(k_i-1)} c_i,$$

where $\langle w_{ijh} \rangle$ is the average weight of all the new triangles $v_i v_j v_h$ with $v_i$ newly added to networks. In all the new triangles $v_i v_j v_h$, the edge weight $w_{ij} = w_{ih} = 1$ since $v_i v_j$ and $v_i v_h$ are new edges,
and \(w_{jh} = w_{jh} + \delta\) because \(v_jv_h\) is an old edge with the increment behavior arising when the new node \(v_i\) is joined \(v_j\) and \(v_h\). So \(\langle w_{jh} \rangle\) can be approximated by \(\frac{1}{3} (2 + (\frac{1}{3})^D + \delta)\) with \(w_{ij} = (\frac{1}{3})^D\). Therefore,

\[
e_i^w = \frac{2 + (\frac{1}{3})^D + \delta}{s_i(k_i - 1)} c_i.
\]

In equation (21), the main parameter is links size \(e_i\), in Fig.4 illustrates microscopic change mechanisms of \(e_i\), where \(v\) is a new node selected by LPA and added to a local-world, there are six microscopic variety mechanisms showing in Fig.4. We can compute the variety rate of \(e_i\) in Figures 4 and 2. Therefore,

![Figure 4: The dashed links show the change microscopic mechanisms of \(e_i\). (a) \(v_i\) connects to \(v\) by \(PA\) step, and potentially following by a \(TF\) step which successes \(vv_j\) be a link. (b) By a \(PA\) step, \(v\) attaches to \(v_j\) which is one of neighbor of \(v_i\), and by one of \(TF\) step \(v\) conversely links to \(v_i\). (c) \(v_i\) connects to \(v\) by a \(PA\) step, \(v_j\), a neighbor of \(v_i\), in another \(PA\) step is selected to connect \(v\). (d) By a \(TF\) step, \(v_i\) connects to \(v\), and by another \(TF\) step, \(v\) joins \(v_j\), a neighbor of \(v_i\). (e) \(v\) connects to \(v_i\) by a \(PA\) step, and following by a potential \(TF\) step such that \(v\) connects to \(v_j\), a neighbor of \(v_i\). (f) \(v_i\) connects to \(v\) by a \(TF\) step, and in a following potential \(PA\) step, \(v\) links to \(v_j\).

\[
\frac{\partial e_i}{\partial t} = 2pm \sum N_i \sum_{v_j \in local} s_j \sum_{v_k \in \Gamma_i} \frac{s_i}{s_j} w_{ik} s_k p + \sum N_i \sum_{v_k \in \Gamma_i} \sum_{v_j \in local} s_j s_k w_{ik} s_k p
\]

\[
+ \sum s_i \sum_{v_j \in \Gamma_i} \sum_{v_k \in local} s_j s_k w_{ik} s_k p (m - 1) N_i \sum_{v_j \in local} s_j s_k w_{ik} s_k p
\]

\[
+ \sum s_i \sum_{v_k \in \Gamma_i} \sum_{v_j \in local} s_j s_k w_{ik} s_k p N_i \sum_{v_j \in local} s_j s_k w_{ik} s_k p
\]

\[
+ \sum s_i \sum_{v_j \in \Gamma_i} \sum_{v_k \in local} s_j s_k w_{ik} s_k p (m - 1) \sum_{v_j \in local} s_j s_k w_{ik} s_k p
\]

\[
= 2pm \sum N_i \sum_{v_j \in local} s_j + m(m - 1)(1 + p)^2 (\sum N_i)^2 \sum s_i \sum_{v_k \in \Gamma_i} \sum_{v_j \in local} s_j s_k w_{ik} s_k p
\]

\[
= \Delta_1 + \Delta_2,
\]

where \(\Gamma_i\) is the neighborhood of \(v_i\).
As at the initial time \( t = t_i \) of \( v_i \), \( s_i(t = t_i) = m(1 + p) \), and by equations (5), (10) and (15), we calculate \( e_i \) by integrating both sides in (22).

The first part in the right hand of (22) is integrated as

\[
\Delta_1 = \int_{t_i}^{t_N} \frac{2mp}{\langle s \rangle (qn_0 + 1 - q)} \frac{s_i(t)}{t} \, dt = \frac{2mp}{A\langle s \rangle (qn_0 + 1 - q)} (s_i(t_N) - s_i(t_i)).
\] (23)

In order to obtain the second term \( \Delta_2 \) in equation (22), first, we consider the weighted tunable cluster local-world model in section 2. The degree of node \( v_i \) is given by (16) at time \( t \), and with equation (15), and neglecting the constant part, then \( k_i(t) \) is

\[
k_i(t) \approx \frac{C}{A}s_i(t) = \frac{C}{A}m(1 + p)(\frac{t}{t_i})^A.
\] (24)

Clearly the density of time is

\[
p_i(t) = \frac{1}{c_0n_0 + (qn_0 + 1 - q)t} = \frac{1}{N}
\] (25)

by equation (5). Then the degree of \( v_i \) can also be expressed by

\[
k_i(N) = C_An_0m(1 + p)(\frac{N}{t_i})^A.
\]

Second, we consider the neighborhood information of nodes in our network model, the average degree of nodes in \( \Gamma_i \) is

\[
k_{nn}^i = \frac{1}{k_i} \sum_{j \in \Gamma_i} k_j.
\] (26)

Let \( p_c(k' | k) \) be conditional probability such that a link adjacent a node with degree \( k \) to a node with degree \( k' \), then equation (26) is

\[
\langle k_{nn} \rangle = \sum_{k'} k'p_c(k' | k) = \sum_{k'} \frac{(k')^2}{\langle k \rangle}p(k') = \frac{\langle k^2 \rangle}{\langle k \rangle},
\] (27)

where \( p_c(k' | k) = \frac{k'p(k')}{\langle k \rangle} \) in uncorrelated network. Therefore, by (24), (25) and (27), we obtain

\[
\langle k^2 \rangle = \sum_{i=1}^{N} k_i^2(t)p_i(t) = \left( \frac{Cm(1 + p)}{A} \right)^2 \frac{1 - N^{2A-1}}{-2A + 1}.
\] (28)

Hence, the degree sum of all nodes in \( \Gamma_i \) is followed by equations (28) and (27)

\[
\sum_{j \in \Gamma_i} k_j(t) = k_i\langle k_{nn} \rangle = \left( \frac{C}{A} \right)^2 \frac{(m(1 + p))^2}{\langle k \rangle} \frac{N^{2A-1} - 1}{-2A + 1} k_i(t).
\] (29)

Consequently, the strength sum of all nodes in neighborhood \( \Gamma_i \) of \( v_i \) is

\[
\sum_{j \in \Gamma_i} s_j(t) = U N^{2A-1} - 1 \frac{1}{-2A + 1} s_i(t),
\] (30)

since \( k_i(t) \) and \( s_i(t) \) are linear related, where \( U = \left( \frac{C}{A} \right)^2 \frac{(m(1 + p))^2}{\langle k \rangle} \) and \(-2A + 1 \neq 0\).
Case 1. \(-2A + 1 < 0\). That is, \(A > 1/2\). Then \(\frac{N^{-2A+1}-1}{-2A+1} \approx \frac{1}{2A-1}\) with \(N \to \infty\), therefore, equation (30) is approximately \(\sum_{j \in N_i} s_j(t) \approx \frac{U}{2A-1} s_i(t)\).

In this case, the second term of (22) is

\[
\Delta_2 = \int_{t_i}^{t_N} (m)(m-1)(1+p)^2 \frac{U}{2A-1} s_i(t)^2 \, \text{d}t
\]

\[
= \frac{Um(m-1)(1+p)^2}{(s)^2(qn_0 + 1 - q)^2(2A-1)} \int_{t_i}^{t_N} s_i(t)^2 \, \text{d}t
\]

\[
= E_1 \int_{t_i}^{t_N} \frac{(m+1)(p)^2 t^2 A^{-2}}{2A-1} \, \text{d}t
\]

\[
= E_1 \frac{(m+1)(p)^2}{t^2 A^{-2}} \frac{t^2 A}{2A-1} |_{t_i}^{t_N}
\]

where \(E_1 = \frac{Um(m-1)(1+p)^2}{(s)^2(qn_0 + 1 - q)^2(2A-1)}\). Hence, by equations (23) and (31), we can get \(e_i\) from equation (22).

\[
e_i = \Delta_1 + \Delta_2
\]

\[
= \frac{2mp}{A(s)(qn_0 + 1 - q)} (s_i(t_N) - s_i(t_i)) + \frac{E_1(m+1)(p)^2}{t^2 A} \frac{t^2 A}{2A-1} |_{t_i}^{t_N}
\]

\[
= \Delta_0 + \frac{2mp}{A(s)(qn_0 + 1 - q)} s_i(t_N) + \frac{E_1 s_i(t_N)^2}{2A-1} |_{t_i}^{t_N}
\]

\[
\approx \frac{2mp}{A(s)(qn_0 + 1 - q)} s_i(t_N) + \frac{E_1 s_i(t_N)^2}{2A-1} |_{t_i}^{t_N}
\]

where \(\Delta_0 = \frac{2mp}{A(s)(qn_0 + 1 - q)} s_i(t_i) - \frac{E_1 s_i(t_i)^2}{2A-1} |_{t_i}^{t_N}\) is a constant, since \(t_i\) is the initial time and \(\Delta_0\) can be neglected. By the above analysis, the weighted clustering coefficient of \(v_i\) with large strength \(s_i(t_N)\) is obtained by the following,

\[
\bar{c}_i^{w} = \frac{\langle w_{ijh} \rangle}{s_i(t_N)(k_i(t_N) - 1)} e_i
\]

\[
\approx \frac{\langle w_{ijh} \rangle}{s_i(t_N)(k_i(t_N) - 1)} \left( \frac{2mp}{A(s)(qn_0 + 1 - q)} s_i(t_N) + \frac{E_1 s_i(t_N)^2}{2A-1} |_{t_i}^{t_N} \right)
\]

\[
\approx \frac{2mp \langle w_{ijh} \rangle}{C(s)(qn_0 + 1 - q) s_i(t_N)} \frac{1}{2A-1} + \frac{E_1 \langle w_{ijh} \rangle}{C(t_N)} A \frac{1}{2A-1} |_{t_i}^{t_N}
\]

Case 2. \(-2A + 1 > 0\). Then \(\frac{N^{-2A+1}-1}{-2A+1} \approx \frac{(qn_0 + 1 - q)t^{-2A+1}}{-2A+1}\) and \(\sum_{j \in \Gamma_i} s_j(t) \approx U((qn_0 + 1 - q)t^{-2A+1}) |_{2A-1}^{-2A+1} s_i(t)\).

Hence

\[
\Delta_2 = \int_{t_i}^{t_N} m(m-1)(1+p)^2 U(qn_0 + 1 - q)^{-2A+1} s_i(t)^2 \, \text{d}t
\]

\[
= E_2 \int_{t_i}^{t_N} \frac{(m+1)(p)^2 t^{-1}}{t^2 A} \, \text{d}t
\]

\[
= E_2 \frac{(m+1)(p)^2}{t^2 A} |_{t_i}^{t_N} (\log t_N - \log t_i)
\]
where $E_2 = \frac{m(m-1)(1+p)^2U \left(qn_0 + 1 - q\right) - 2A + 1}{(s)^2(qn_0 + 1 - q)^2}$. Ignoring the constant, the weighted clustering coefficient of $v_i$ with large strength $s_i(t_N)$ can be given by

$$
\tilde{c}_i^w = \frac{\langle w_{ijh} \rangle}{s_i(t_N)(k_i(t_N) - 1)}(\Delta_1 + \Delta_2) \approx \frac{2mp}{s_i(t_N)(k_i(t_N) - 1)}\frac{s_i(t_N)}{A(s)(qn_0 + 1 - q)} + E_2s_i^2(t_N)\frac{\ln t_N}{t_N^A}
$$

(35)

In our model, the parameter $p$ effects the system by allowing the formation of triads, the value $\delta$ also affects $\tilde{c}_i^w$ without changing its power-law slope. From Fig. 5, we can find the weighted clustering coefficient of each node can be adjusted continuously and growing monotonically with an increasing $\delta$. In the expression of $\tilde{c}_i^w$ of Cases 1 and 2, the first term can be viewed as the triad formation induced clustering, and it shows the $s^{-1}$ behavior that has been observed in several real-life systems [21, 22].

$\tilde{c}_i^w = \frac{e_i}{k_i(k_i - 1)/2} \approx \frac{4p}{1 + p} \frac{1}{k_i} + \left(1 + p\right)^2 \frac{m - 1}{4(1 + p)} \left(\ln N\right)^2.$

(36)

In [36], the first term can be ascribed to the triad formation which induced by clustering and displayed $k^{-1}$ behavior. We set $p$ any value in interval $(0, 1)$, then $\tilde{c}_i^w$ can be adjusted continuously and it will grow monotonically increasing with $p$. In this case, the model may be regarded as a growing network with uniform attachment, which is similar to the triad formation model proposed by Szabó et al. [36]. Therefore Eq. (36) is similar to the expression of the clustering coefficient in [36].

3.3 Comparison with SCN Networks

In order to verify the effectiveness of our in weighted tunable cluster local-world network model, some real data of typical social networks are applied to our model in this section. In scientific
co-authorship network (SCN) of MathSciNet, nodes are defined as scientists and two scientists (nodes) are connected if they have coauthored at least one paper. Furthermore, we define the link’s weight between each pair of authors in an article with $n$ authors by $1/(N - 1)$. Specifically, our model can be applied to boolean networks if $\delta = 0$. We initialize our model with $c_0 = 3, n_0 = 3, p = 0.05, q = 0.1, m = 4$, when $T \geq 29981$, a boolean network with the same number nodes as SCN in 2004 [37] is obtained. As listed in Table 1, with the same number of nodes $N$ and very similar number of edges $E$, our model produces the same degree scaling exponent $\gamma$ with boolean SCN networks and produces very similar average clustering coefficient $c$ and local worlds number $L$ and modularity $Q$.

**Table 1. Topological properties of our model and boolean SCN(2004)**

| $N$   | $E$   | $\gamma$ | $c$  | $L$  | $Q$  |
|-------|-------|-----------|------|------|------|
| SCN(2004) | 30561 | 125959     | 3    | 1069 | 0.668|
| Model  | 30561 | 115387     | 3    | 1172 | 0.682|
| Relative error | 0 | 0.08 | 0 | 0.04 | 0.09 | 0.02 |

By cumulative distributions, the following figures shown that our model owns very close degree distribution, strength distribution and weight distribution compared to SCN(2004) in Fig.6, the curves in SCN(2004) are almost completely overlapped. In this way, the network generated by our model matches to SCN2004 well.

Figure 6: Comparisons of our model and SCN(2004) by cumulative distributions, (a) degree distribution $P(k)$, (b) strength distribution $P(s)$, (c) weight distribution $P(w)$.

On the other hand, our model is also compared with weighted SCN(2008). We initialized our model with $c_0 = 3, n_0 = 3, p = 004, q = 0.09, m = 3$, when $T \geq 9244$, a weighted network with the same number of nodes as SCN(2008) is obtained. Similarly, our model is succeeded in capturing the structure properties of strength scaling exponent $\gamma$, average clustering coefficient $c$ and local world number $L$ and modularity $Q$ with the same number of nodes $N$ and very similar number of edges $E$, the results are shown in Table 2.

**Table 2. Topological properties of our model and weighted SCN(2008)**

| $N$   | $E$   | $\gamma$ | $c$  | $n$  | $Q$  |
|-------|-------|-----------|------|------|------|
| SCN(2008) | 10136 | 31174     | 3.1  | 254  | 0.679|
| Model  | 10136 | 30269     | 3.1  | 231  | 0.682|
| Relative error | 0 | 0.03 | 0 | 0.015 | 0.09 | 0.004 |

Similarly, our model fits well the scale-invariant properties of degree distribution, strength distribution and weight distribution of the SCN(2008), as shown in Fig[7].
Figure 7: Comparisons of our model and SCN(2008), (a) degree distribution $P(k)$, (b) strength distribution $P(s)$, (c) weight distribution $P(w)$.

4 The correlation of vertices in model

In the following section, we analyze the effect of degree correlation between two nodes. In real-world weighted networks, a high degree vertex connects to low degree vertices with small weight, while connects to high degree vertices with large weight. For instance, in WAN, a busy airport $v_i$ has a lot of direct flights to another dense airport $v_j$, while has a few of flight to a spare airport. In order to describe those phenomena, Barrat et al.\[33\] proposed the weighted degree-dependent average nearest-neighbor with degree $k$, that is,

$$k_{nn}^{w}(k) = \frac{1}{|\sum v_i|} \sum_{d_i = k} \frac{1}{s_i} \sum_{j \in \Gamma_i} w_{ij} k_j.$$ (37)

In Fig. 8, $k_{nn}^{w}(k)$ exhibits increasing power-law behavior for $\delta \geq 0$ which indicated that the networks are assortative. It is worth noting that a link introduced at an early time, which connects two old vertices having similar degrees together, tends to be high-weighted as $\delta \geq 0$. This leads to the observed weighted assortative behavior.

Figure 8: The weighted degree-dependent average nearest-neighbor with degree $k$, $k_{nn}^{w}(k)$, is a function of degree $k$ for different $\delta$ in a log-log plot with $m = 3$, $p = 0.1$ and $q = 0.01$. Each data point is obtained over 20 independent average realizations with network size $N = 10^4$.

Although we can find a network is assortative or disassortative by $k_{nn}^{w}(k)$ increasing or decreasing with $k$, a quantity directly describing the weighted assortatively is needed, it is introduced by Barrat et al.\[33\].
\[ r^w = \frac{H^{-1} \sum_{\phi} (\varpi_{\phi} \prod_{i \in F(\phi)} k_i) - \left[ \frac{H^{-1}}{2} \sum_{\phi} (\varpi_{\phi} \prod_{i \in F(\phi)} k_i^2) \right]^2}{\frac{H^{-1}}{2} \sum_{\phi} (\varpi_{\phi} \prod_{i \in F(\phi)} k_i^2) - \left[ \frac{H^{-1}}{2} \sum_{\phi} (\varpi_{\phi} \prod_{i \in F(\phi)} k_i) \right]^2}. \]  

(38)

where \( \varpi_i \) is the weight of the \( \phi \)th link, \( F(\phi) \) is the set of the two vertices connected by the \( \phi \)th link and \( H \) is the total weight of all links in the network. \( r^w \) lies between -1 and 1. Moreover, \( r^w \) is positive for weighted assortative networks, is negative for otherwise.

Figure 9: The weighted assortativity coefficient \( r^w \) as a function of the node degree \( \delta \) for different \( p \) with \( m = 3, q = 0.01 \). Each data point is obtained by averaging over 20 independent realizations with network size \( N = 10^4 \) fixed.

In Fig. 9 it is shown that \( r^w \) is positive and increases with the growth of \( \delta \) and for all \( q \). Thus, the networks are assortativity for all \( \delta \), which is in accord with the definition of \( k_{\text{nn}}^w(k) \). Therefore, the probability \( p \) of adding new TF links and the value of \( \delta \) significantly determine the degree-degree mixing patterns of evolving weighted networks in our model. This phenomenon can be easily understood since the local-world inherits the property of the hierarchy structure in networks.

5 Epidemic spreading in our model with weighted transmission rate

Epidemic spreading theory on complex networks has a board practical background. To study the dynamics of infectious diseases spreading on the weighted tunable cluster local-world with increment behavior network model making the transmission rate accord with the realistic cases much more, we take into account the effects of the weights of edges and the strengths of nodes which are of great importance measures in the weighted networks [18]-[21]. For epidemic spreading, the weight can indicate the extent of frequency of the contacting of two nodes in scale-free networks, the larger the weight is, the more intensively the two nodes communicate, the more possible a susceptible individual will be infected through the edge, where the transmission rate is larger.

We make use of the spreading related weight \( w_{kk'} \) between two nodes with degree \( k \) and \( k' \) represented as a function of their degrees [18]-[24], \( w_{kk'} = w_0(kk')^\beta \), where the basic parameter \( w_0 \) and the exponent \( \beta \) depend on the particular complex networks (e.g., in the E.coli metabolic network \( \beta = 0.5 \); in the US airport network (USAN) \( \beta = 0.9 \) [38]; in the scientist collaboration networks (SCN) \( \beta = 0 \) [18]). Noteworthily, the spreading related weight \( w_{kk'} \) belongs to an edge
which connect a node of degree $k$ and a node of degree $k'$, the spreading related weights of links connected to all nodes of degree $k$ is $s_k = k \sum_{k'} P(k'/k) w_{kk'}$, and $s_k$ is also the strength of a node of degree $k$. In the following, we focus on uncorrelated (also called non-assortative mixing) networks where the conditional probability satisfies $P(k'/k) = k'P(k'/\langle k \rangle)$ \cite{39}. Thus, one can obtain $s_k = w_0(k^{1+\beta}) k^{1+\beta}/\langle k \rangle$.

Here, the total transmission rate of each $k$-degree node is $\lambda k$, since the transmission is $\lambda$ for each link adjacent to this node. A transmission rate on the edge from the $k$-degree node to $k'$-degree node, say $\lambda_{kk'}$, is defined as follows \cite{33},

$$\lambda_{kk'} = \lambda k \frac{w_{kk'}}{s_k}.$$ \hspace{1cm} (39)

In \cite{39}, the more proportion of $s_k$ that $w_{kk'}$ of an edge holds, the more possible the disease will transmit through this edge. $\lambda_{kk'} = \lambda k \beta (k) \langle k^{1+\beta} \rangle$ in uncorrelated networks.

In the following, we investigate the modified $SI$ model \cite{39} in which the weighted transmission rate and nonlinear infectivity are introduced. The results we obtain might deliver some useful information for the epidemic outbreak. And for a better analysis, we firstly describe the general differential equations for $SI$ model based on the mean field theory, as follows:

$$\partial I_k(t)/\partial t = k(1 - I_k(t)) \sum_{k'} P(k'/k) I_{k'}(t) \lambda_{kk'},$$ \hspace{1cm} (40)

where $I_k$ and $\lambda_{kk'}$ denote the $k$-degree nodes' infectivity and the transmission rate from $k$-degree nodes to $k'$-degree nodes respectively. Neglecting the terms of $O(k^2)$ in the expansion of (40), the simplified result is

$$\partial I_k(t)/\partial t = \frac{\lambda k^{1+\beta} \langle k \rangle}{\langle k^{1+\beta} \rangle} \sum_{k'} k' P(k') I_{k'}(t) = \frac{\lambda k^{1+\beta} \langle k \rangle}{\langle k^{1+\beta} \rangle} \theta_k(t).$$ \hspace{1cm} (41)

In uncorrelated heterogeneous networks, $\theta_k(t)$ is independent of the degree of vertex, then $\theta_k(t) = \theta(t) = \sum_k \frac{k P(k)}{\langle k \rangle} I_k(t)$ and every infected neighbor may be the initial seeds (infected at $t = 0$) or be infected at $t > 0$. Therefore,

$$I_k(t) = I_0 + I_0 \frac{\lambda k^{1+\beta} \langle k \rangle}{\langle k^{1+\beta} \rangle} (e^{t/\tau} - 1),$$ \hspace{1cm} (42)

where $\tau = \frac{\langle k^{1+\beta} \rangle}{\lambda k^{2+\beta}}$, and the total infection density is

$$I(t) = \sum_k P(k) I_k(t) = I_0 + I_0 \frac{\lambda \langle k^{1+\beta} \rangle \langle k \rangle}{\langle k^{1+\beta} \rangle} (e^{t/\tau} - 1),$$ \hspace{1cm} (43)

since $\partial \theta(t)/\partial t = \sum_k \frac{k P(k)}{\langle k \rangle} \frac{\partial I_k(t)}{\partial t} = \frac{\lambda \langle k^{2+\beta} \rangle}{\langle k^{1+\beta} \rangle} \theta(t)$ and the uniform initial condition $I_k(t = 0) = I_0$.

In the weighted tunable cluster local-world with increment behavior network model, the probability density of $k$-degree node is $p(k) = ak^{-1-1/A}$ and this model is also an uncorrelated networks, $\tau$ is obtained in the following

$$\tau = \frac{\langle k^{1+\beta} \rangle}{\lambda \langle k^{2+\beta} \rangle} = \frac{k_{\max}}{\lambda (1 + \beta - \frac{1}{A})} \frac{\langle k^{\beta+1} \rangle}{\langle k^{\beta+2} \rangle} = 2 + \beta - \frac{1}{A} \frac{k_{\max}^{\beta+1 - \frac{1}{A}} - 1}{k_{\max}^{\beta+2 - \frac{1}{A}} - 1}.$$ \hspace{1cm} (44)
Generally, when $1 + \beta - \frac{1}{A} > 0$, namely $2 + \beta > \gamma = 1 + \frac{1}{A}$. If $k_{\text{max}}$ big enough, one can get

$$\tau \approx \frac{2 + \beta - \frac{1}{A}}{\lambda(1 + \beta - \frac{1}{A})} \cdot \frac{1}{k_{\text{max}}}$$  \hspace{1cm} (45)$$

The parameter $A$ can be treated as a function of $p$, $q$ and $\delta$, according to (43), then $\tau$ and $I(t)$ increase with the degree of tunable clustering mechanism and inversely effect $\beta$. In order to illustrate this property, some numerical simulations are shown in the left graph of Fig.10 $\beta$ accelerates the dynamical evolution when $\beta$ is less and the inner graph shown the time scale $\tau$ agrees well with the theoretical results. The dynamical evolution of infection density under various value $p$ is explained by the middle picture of Fig.10. The larger value of $p$, that is, the larger probability of $TF$ links occurs, the fast the dynamical evolution reaches steady state. In the right of Fig.10 plots the effect with different $\delta$ on the dynamic spreading behavior. However, $\delta$ doesn’t have obvious effect on the dynamic of $I(t)$. A reasonable explanation is that the increment behaviors do not change the value of $k$ nor $p(k)$, and therefor do not effect the weighted transmission rate $\lambda_{kk'}$. The current analysis and simulations provide an intuitive description of the spreading phenomena in complex networks, and it helps us to further understand some real-world propagation mechanisms and spreading behaviors.

6 Conclusion

Tunable clustering mechanism and the increment behavior are two important dynamic mechanisms in real world networks. In order to interpret the real network accurately, we propose a weighted tunable cluster local-world evolving model with increment behavior. The growth dynamics include some new local-world and new nodes added by local preferential attachment. A lot of interesting properties of the generated networks display good right-skewed distribution characters, which have been discovered in most realistic networks such as the distributions of strength, degree and link weight display of a power law. We also performed numerical simulations and verified the experimental results which are agreement with theoretical analysis very well. The effects of tunable clustering mechanism and increment behavior on correlations of
vertices of weighted networks are studied also. The accurate values of the weighted clustering coefficient, weighted average nearest-neighbor degree and weighted assortatively coefficient are obtained. All of these results exhibit the assortative behaviors of our model and the dynamic mechanisms properties. Finally, it is discovered that the tunable clustering behavior has a great impact on the spreading dynamic. However, due to the particularity of the epidemic spreading model, the increment behavior doesn’t have obvious effect on the spreading dynamic in our model.

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