Local Hölder regularity for nonlocal equations with variable powers

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Abstract
We prove the local boundedness and local Hölder continuity of weak solutions to nonlocal equations with variable orders and exponents under sharp assumptions.

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1 Introduction

We study regularity theory for weak solutions to the following integro-differential equations:

\[ L_K u(x) = \text{P.V.} \int_{\Omega} |u(x) - u(y)|^{p(x,y) - 2}(u(x) - u(y)) K(x, y) \, dy = 0 \quad \text{in} \ \Omega, \quad (1.1) \]

where \( \Omega \subset \mathbb{R}^n \) is open and bounded, and \( K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) with \( K(x, y) = K(y, x) \) is a suitable kernel with variable order \( s(\cdot, \cdot) \) and variable exponent \( p(\cdot, \cdot) \). We note that when \( p(x, y) \equiv p, \ s(x, y) \equiv s \) and \( K(x, y) = \frac{1}{|x-y|^{n+sp}} \), (1.1) is the \((s,p)\)-fractional \(p\)-Laplace equation \((-\Delta)^s_p u = 0\), and moreover if \( p = 2 \), it is the \((s,s)\)-fractional Laplace equation \((-\Delta)^s u = 0\). In particular, we mainly prove the local Hölder continuity of weak solutions to (1.1), under essentially sharp regularity conditions on \( s(\cdot, \cdot), p(\cdot, \cdot) \) and \( K(\cdot, \cdot) \). We will present our results with the definition of weak solution in Sect. 1.1 below.

In the past two decades, there have been tremendous amount of researches on nonlocal equations of the type (1.1) and relevant variational problems. We refer to for instance survey papers [27, 52] and monographs [9, 49] for the history, development and applications of nonlocal problems. Regarding the regularity theory, Caffarelli and Silvestre [15] proved Harnack’s inequality for the fractional Laplace equation by using an extension argument. After this pioneering work, regularity theory for nonlocal equations of the fractional Laplacian...
type has rapidly developed. We refer to for instance \[5, 6, 14, 16, 17, 33, 34, 43, 53, 55\] and related references. In particular, Caffarelli, Chan and Vasseur applied De Giorgi’s approach to nonlocal equations in \[14\]. More generally, for the fractional \(p\)-Laplace equation, Di Castro, Kuusi and Palatucci \[25, 26\] have proved Harnack’s inequality and the Hölder continuity of weak solutions by using De Giorgi’s approach. Especially, they introduced the so called nonlocal tail (see Sect. 2), and obtained regularity estimates involving this. We refer to \[18, 22, 24, 28, 31, 38–42, 44, 47, 50, 51\] for researches for nonlocal equations of the fractional \(p\)-Laplacian type.

Furthermore, in very recent years, there have also been research activities on regularity theory for nonlocal equations with nonstandard order and exponent. Bass and Kassmann \[5, 6\] and Silvestre \[55\] proved the Hölder continuity and Harnack’s inequality for bounded solutions to nonlocal linear equations with kernels whose prototypes are \(|x−y|^{−s(x)2}\) or \(\varphi(|x−y|)^{−1}\). We also refer to related researches \[3, 4, 37\]. In addition, recently De Filippis and Palatucci \[23\] considered nonlocal equations of the double phase type, and proved Hölder continuity for bounded solutions.

The model problem we have in mind is the \(s(\cdot, \cdot)\)-fractional \(p(\cdot, \cdot)\)-Laplace equation:

\[
(-\Delta)^{s(\cdot, \cdot)} u = 0 \quad \text{in } \Omega,
\]

that is, \(K(x, y) = \frac{1}{|x−y|^{n+s(x,y)p(x,y)}}\) in \((1.1)\), where \(s(\cdot, \cdot)\) and \(p(\cdot, \cdot)\) are variable functions satisfying \((1.4)\) and \((1.5)\), respectively. In this paper, we call \(s(\cdot, \cdot)\) the variable order, \(p(\cdot, \cdot)\) the variable exponent, and both \(s(\cdot, \cdot)\) and \(p(\cdot, \cdot)\) the variable powers. Note that the nonlocal equation \((1.2)\) is the Euler-Lagrange equations of the minimizing problem of the energy functional

\[
v \mapsto \int_{\Omega} \frac{1}{p(x, y)} \frac{|v(x) − v(y)|^{p(x,y)}}{|x−y|^{n+s(x,y)p(x,y)}} \, dydx,
\]

where

\[
C_{\Omega} := (\mathbb{R}^n \times \mathbb{R}^n) \setminus ((\mathbb{R}^n \setminus \Omega) \times (\mathbb{R}^n \setminus \Omega)),
\]

and its admissible set is naturally linked to fractional Sobolev space \(W^{s(\cdot, \cdot), p(\cdot, \cdot)}\). In recent years, there have been a lot of papers studying nonlocal equations of the type \((1.2)\) and fractional Sobolev spaces with variable powers, see for instance \[7, 8, 20, 32, 35, 36, 46, 48, 54, 56\]. We can also find their applications in some of the preceding references. We note that in \[20\] the global boundedness of weak solutions to nonhomogeneous \(s(\cdot, \cdot)\)-fractional \(p(\cdot, \cdot)\)-Laplace equations with the boundary condition \(u \equiv 0\) in \(\mathbb{R}^n \setminus \Omega\) is proved by using De Giorgi’s approach.

The local problem corresponding to \((1.2)\) is the \(p(x)\)-Laplace equation

\[
\Delta_{p(\cdot)} u := \text{div} \left( |Du|^{p(\cdot)−2} Du \right) = 0,
\]

that is a typical model of problems with nonstandard growth, and the one of elliptic equations extensively studied over last two decades. We refer to \[1, 2, 11, 12, 21, 29, 30\] for regularity results of problems modeled by the \(p(x)\)-Laplace equation. In particular, we know from \[30\] that the weak solution to the \(p(x)\)-Laplace equation is locally bounded if \(p(\cdot)\) is continuous, and locally Hölder continuous if \(p(\cdot)\) is so called log-Hölder continuous, that is,

\[
\sup_{0<\tau<\frac{1}{2}} \omega_p(r) \ln(1/r) < \infty,
\]
where \( \omega_p \) is the modulus of continuity of \( p(\cdot) \). Note that the above regularity conditions on \( p(\cdot) \) are essentially sharp.

For nonlocal equations with variable powers \( s(\cdot, \cdot) \) and \( p(\cdot, \cdot) \), despite active research on these problems, we couldn’t find any result on local regularity theory, especially local Hölder continuity or boundedness. In fact, the nonlocal nature makes difficult to prove regularity estimates for nonlocal equations with variable powers. More precisely, even if we investigate the local regularity of weak solutions, we have to estimate integrals over the whole space \( \mathbb{R}^n \).

In the variable powers case, the differences between the supremum and the infimum of the variable powers may be large, hence we can not approximate the nonlocal tails to integrals with constant powers. This requires much more delicate analysis that is not used in local problems with variable exponent.

In this paper, we prove the local boundedness and the Hölder continuity for the weak solutions to nonlocal equations modeled by (1.2) in Theorems 1.1 and 1.2, respectively. These are the natural nonlocal counter parts of the regularity results for the \( p(x) \)-Laplace equations mentioned above and, in our best knowledge, the first regularity results for nonlocal equations with not only the variable exponent \( p(\cdot, \cdot) \) but also the variable order \( s(\cdot, \cdot) \) depending on two space variables. We emphasize that weak solutions are not assumed to be bounded in \( \mathbb{R}^n \). Instead, we assume that weak solutions belong to the so called the tail space with variable powers \( T^s(\cdot, \cdot), p(\cdot, \cdot)(\Omega) \), see Sect. 2, and prove the desired regularity results. Very recently, Chaker and Kim [19] obtained the local Hölder continuity weak solutions to nonlocal problems with variable exponent \( p(\cdot, \cdot) \) and constant differentiability order, i.e., \( s(\cdot, \cdot) \equiv s \), where the condition of the variable exponent \( p(\cdot, \cdot) \) is stronger than the one in this paper and a slightly different approach is adapted.

We prove the theorems by using De Giorgi’s approach, in particular, in [26]. Hence we first obtain a Caccioppoli type estimate and a logarithmic estimate. Then using the Caccioppoli type estimate and the De Giorgi iteration, we prove the local boundedness of weak solution. Finally, using the above results, we prove an oscillation estimate in Lemma 6.1. We would emphasize that we obtain the logarithmic estimate (Lemma 4.3) for locally bounded and nonnegative the weak solutions, and the nonlocal tail term in this estimate contains the \( L^{\infty} \)-norm of weak solution. Note that the local boundedness of weak solution is not assumed in the case of the fractional \( p \)-Laplace equations, and seems unavoidable in the case of the fractional nonlocal equations with nonstandard growth. Technically, we do not use any embedding property of fractional Sobolev spaces with variable powers even in the proof of the existence of weak solution. Instead, we use well-known embedding properties for the fractional Sobolev spaces with constant powers. It is possible by decreasing the order of differentiability, see Lemma 2.2. Moreover, it is unclear that the multiplication of a weak solution and a cut-off function can be directly used as a test function in the weak form of (1.1), that is usually the first step of the proofs of Caccioppoli type estimates (see the beginning of Sect. 4). This problem occurs in the variable exponent case. Nevertheless, we obtain a desired Caccioppoli estimate by using an approximation argument.

Now, we state the main results in this paper.

### 1.1 Main results

We start with defining the variable powers \( s(\cdot, \cdot) \) and \( p(\cdot, \cdot) \), and the kernel \( K(\cdot, \cdot) \). Note that a two variable function \( f = f(x, y) \) is said to be symmetric if \( f(x, y) = f(y, x) \) for all \( x \).
and $y$. Let $s, p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be symmetric and satisfy that
\[
0 < s^+ := \inf_{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n} s(x, y) \leq \sup_{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n} s(x, y) =: s^+ < 1, \tag{1.4}
\]
and
\[
1 < p^- := \inf_{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n} p(x, y) \leq \sup_{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n} p(x, y) =: p^+ < \infty, \tag{1.5}
\]
and let $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be measurable and satisfy that
\[
\frac{\Lambda^{-1}}{|x - y|^{n + s(x, y)p(x, y)}} \leq K(x, y) \leq \frac{\Lambda}{|x - y|^{n + s(x, y)p(x, y)}} \quad \text{for a.e. } (x, y) \in \mathbb{R}^n \times \mathbb{R}^n, \tag{1.6}
\]
for some $\Lambda > 1$. Without loss of generality we shall further assume that $K$ is symmetric, see [44, Section 1.5] for comment on the symmetricity assumption on the kernel.

We next state the definition of weak solution to (1.1) with (1.4)–(1.6). For relevant function spaces $W^{s(\cdot), p(\cdot)}(\Omega)$ and $T^{s(\cdot), p(\cdot)}(\Omega)$, we will introduce in the next section. We say that $u \in W^{s(\cdot), p(\cdot)}(\Omega)$ is a weak solution to (1.1) if
\[
\int_{\mathbb{C}_\Omega} |u(x) - u(y)|^{p(x, y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y)) K(x, y) \, dy \, dx = 0 \quad \tag{1.7}
\]
for every $\varphi \in W^{s(\cdot), p(\cdot)}(\Omega)$ with $\varphi = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, where $\mathbb{C}_\Omega$ is defined in (1.3). In addition, we say $u \in W^{s(\cdot), p(\cdot)}(\Omega)$ is a weak subsolution(resp. supersolution) if (1.7) with replacing “$=$” by “$\leq$ (resp. $\geq$)” holds for every $\varphi \in W^{s(\cdot), p(\cdot)}(\Omega)$ with $\varphi \geq 0$ a.e. in $\mathbb{R}^n$ and $\varphi = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$. We will discuss the existence and the uniqueness of the weak solution in Sect. 3.

Now we introduce regularity assumption on $s(\cdot, \cdot)$ and $p(\cdot, \cdot)$. Set
\[
\omega_s(r) := \sup_{B_r \subset \Omega} \sup_{x_1, x_2, y_1, y_2 \in B_r} |s(x_1, y_1) - s(x_2, y_2)|, \quad r \in (0, 1/2).
\]
This means $\omega_s(\cdot)$, or $\omega_p(\cdot)$, is the oscillation of $s(\cdot, \cdot)$, or $p(\cdot, \cdot)$, near the diagonal region $D := \{(x, x) : x \in \Omega\}$. Hence, for instance, $\lim_{r \to 0} \omega_s(r) = 0$ if and only if $p(\cdot, \cdot)$ is continuous on $D$ uniformly. We then consider the following logarithmic continuity condition:
\[
\sup_{0 < r \leq 1/2} (\omega_p(r) + \omega_s(r)) \ln \left( \frac{1}{r} \right) \leq c_{LH} \quad \text{for some } c_{LH} > 0. \tag{1.8}
\]
Note that this assumption means $\omega_s(\cdot)$ and $\omega_p(\cdot)$ are so called log-Hölder continuous on $D$.

The first main result is the local boundedness of the weak solutions to (1.1).

**Theorem 1.1** Let $s, p, K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be symmetric and satisfy (1.4), (1.5) and (1.6), respectively. Suppose that for $p(\cdot, \cdot)$, $\lim_{r \to 0} \omega_p(r) = 0$. If $u \in W^{s(\cdot), p(\cdot)}(\Omega) \cap T^{s(\cdot), p(\cdot)}(\Omega)$ is a weak solution to (1.1), then $u \in L^\infty_{\text{loc}}(\Omega)$.

The second main result is the local Hölder continuity of the weak solutions to (1.1).

**Theorem 1.2** Let $s, p, K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be symmetric and satisfy (1.4), (1.5) and (1.6), respectively. Suppose that $s(\cdot, \cdot)$ and $p(\cdot, \cdot)$ satisfy (1.8). If $u \in W^{s(\cdot), p(\cdot)}(\Omega) \cap T^{s(\cdot), p(\cdot)}(\Omega)$ is a weak solution to (1.1), then $u \in C^{0, \alpha}_{\text{loc}}(\Omega)$ for some $\alpha \in (0, 1)$ depending on $n, s^\pm, p^\pm, \Lambda$ and $c_{LH}$. 
There are remarks for the above theorems.

**Remark 1.3** We consider the weak solutions belonging to \( T^{s(\cdot), p(\cdot)}(\Omega) \). This allows to handle the nonlocal tail given by (2.1) that is an essential factor in the analysis of local regularity for nonlocal equations. In Remark 2.1 below, we introduce some examples of functions in \( T^{s(\cdot), p(\cdot)}(\Omega) \).

**Remark 1.4** We emphasize that, except the basic assumptions (1.4) and (1.5), we do not assume any regularity condition on \( s(\cdot, \cdot) \) and \( p(\cdot, \cdot) \) in \( (\mathbb{R}^n \times \mathbb{R}^n) \setminus D \).

For the local boundedness, Theorem 1.1, we only assume that the variable exponent \( p(\cdot, \cdot) \) is continuous on \( D \) uniformly (this can be replaced by a smallness condition on the oscillation of \( p(\cdot, \cdot) \), see Remark 5.2), hence the variable order \( s(\cdot, \cdot) \) could be any measurable function satisfying (1.4). On the other hand, for the local Hölder continuity, Theorem 1.2, we assume that \( s(\cdot, \cdot) \) and \( p(\cdot, \cdot) \) are log-Hölder continuous on \( D \).

**Remark 1.5** In above results, we could consider more general nonhomogeneous nonlocal equations such as

\[
(-\Delta)^{s(\cdot, \cdot)}_{p(\cdot, \cdot)} u(x) = f(x, u), \quad x \in \Omega,
\]

where \( f \) satisfies a suitable growth condition. The proofs are almost similar. For modification concerned with the term \( f \), we refer to, for instance, [22].

**Remark 1.6** Recently, the author has also obtained Hölder regularity results for fractional nonlocal double phase problems [13] and fractional nonlocal equations with general growth [10].

The remaining part of the paper is organized as follows. In Sect. 2, we introduce notation and basic tools used in this paper. In Sect. 3, we discuss on the existence of weak solution to (1.1). In Sect. 4, we obtain Caccioppoli type and logarithmic estimates. Finally, in Sects. 5 and 6, we prove Theorems 1.1 and 1.2, respectively.

## 2 Preliminaries

### 2.1 Notation and function spaces

For \( q \geq 1 \) and \( t \in (0, 1] \) with \( tq < n, q_t^* := \frac{ntq}{n-tq} \). Let \( \Omega \subset \mathbb{R}^n \) be open and bounded, and let \( B_r(x_0) \) be a standard open ball in \( \mathbb{R}^n \) centered at \( x_0 \in \mathbb{R}^n \) with radius \( r > 0 \). If the center \( x_0 \) is not important we write \( B_r = B_r(x_0) \). For a measurable function \( v \) in \( \Omega \), \( v_\pm := \max\{\pm v, 0\} \), and by \( (v)_\Omega \) is denoted the mean of \( v \) in \( \Omega \), that is, \( (v)_\Omega := \frac{1}{|\Omega|} \int_{\Omega} v \, dx \).

We always assume that \( s, p, K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) are symmetric and satisfy (1.4), (1.5) and (1.6), respectively. For \( E \subset \mathbb{R}^n \), we define

\[
s_E^- := \inf_{(x,y)\in E \times E} s(x,y) \quad \text{and} \quad s_E^+ := \sup_{(x,y)\in E \times E} s(x,y),
\]

and \( p_E^- \) by the same quantities for \( p(\cdot, \cdot) \). In particular, we write \( p_{E,\pm} = p_\pm \) and \( s_{E,\pm} = s_\pm \).

For a measurable function \( v \) in \( \Omega \), set

\[
\mathcal{E}_{s(\cdot, \cdot), p(\cdot, \cdot)}(v; \Omega) := \int_{\Omega} \int_{\Omega} \frac{|v(x) - v(y)|^{p(x,y)}}{|x-y|^{n+s(x,y)p(x,y)}} \, dy \, dx.
\]
For constant $p \in [1, \infty)$, we define fractional Sobolev space $W^{s, \cdot}_v(\Omega) := \{ v \in L^p(\Omega) : \varrho(s, \cdot)_v(p; \Omega) < \infty \}$. Note that if $s(\cdot, \cdot)$ is constant with $s(\cdot, \cdot) \equiv s$, $W^{s, p}(\Omega)$ is the usual fractional Sobolev space.

We next introduce function spaces related to the weak solution to (1.1). We define $\mathbb{W}^{s, \cdot}_v(\Omega)$ by the set of all measurable functions $v : \mathbb{R}^n \to \mathbb{R}$ satisfying that

$$v|_{\Omega} \in L^{p} (\Omega) \quad \text{and} \quad \int_{\mathbb{C}^{\Omega}} \frac{|v(x) - v(y)|^{p(x,y)}}{|x - y|^{n+s(x,y)}p(x,y)} \, dy \, dx < \infty.$$  

Clearly, $\varrho(s, \cdot)_v(p; \cdot)(u; \Omega) < \infty$ if $v \in \mathbb{W}^{s, \cdot}_v(\Omega)$. We next define a tail space with variable powers $s(\cdot, \cdot)$ and $p(\cdot, \cdot)$. For a measurable functions $v : \mathbb{R}^n \to \mathbb{R}$, we denoted by $v \in T^{s(\cdot, \cdot), p(\cdot, \cdot)}(\Omega)$ if

$$[v]_{T^{s(\cdot, \cdot), p(\cdot, \cdot)}(\Omega)} := \text{ess sup}_{x \in \Omega} \int_{\mathbb{R}^n} \frac{|v(y)|^{p(x,y)-1}}{(1 + |y|)^{n+s(x,y)p(x,y)}} \, dy < \infty.$$  

Note that if $v \in T^{s(\cdot, \cdot), p(\cdot, \cdot)}(\Omega)$, the following quantity is always finite whenever $x_0 \in \mathbb{R}^n$, $r > 0$ and $B_r(x_0) \subset \Omega$:

$$T(v; x_0, r, \rho) := \sup_{x \in B_r(x_0)} \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{|v(y)|^{p(x,y)-1}}{|y - x_0|^{n+s(x,y)p(x,y)}} \, dy,$$  

which we call the nonlocal tail with variable powers. Indeed, since $1 + |y| \leq (1 + \frac{|x_0|+1}{r}) |y - x_0|$ for $y \in \mathbb{R}^n \setminus B_r(x_0)$, we have

$$T(u; x_0, r, \rho) \leq \left( 1 + \frac{|x_0|+1}{r} \right)^{n+s^+ p^+} [v]_{T^{s(\cdot, \cdot), p(\cdot, \cdot)}(\Omega)} < \infty.$$  

Note that they are the variable powers versions of the tail space and the nonlocal tail introduced in [25, 26, 39, 52]. For simplicity, if $x_0$ is obvious, we write $T(v; r, \rho) = T(v; x_0, r, \rho)$.

**Remark 2.1** If $v \in L^\infty(\mathbb{R}^n)$ for some $\gamma \geq p^+ - 1$, or if $v \in L^{p^+ - 1}(B_M(0))$ for some $M > 0$ and is bounded in $\mathbb{R}^n \setminus B_M(0)$, then $v \in T^{s(\cdot, \cdot), p(\cdot, \cdot)}(\Omega)$.

In particular, $v \in L^\infty(\mathbb{R}^n) \subset T^{s(\cdot, \cdot), p(\cdot, \cdot)}(\Omega)$ for every $\Omega$, and if $p(\cdot, \cdot)$ is constant with $p(\cdot, \cdot) \equiv p$, by [27, Proposition 2.1 and Theorem 6.5] we see that

$$W^{s(\cdot, \cdot), p}(\mathbb{R}^n) \subset W^{t, p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n) \subset T^{s(\cdot, \cdot), p(\cdot, \cdot)}(\Omega),$$  

where $t \in (0, s^-)$ is an arbitrary number satisfying $tp < n$.

## 2.2 Inequalities

We introduce inequalities used in this paper. The first inequality is a general version of the inequality in [22, Lemma 4.6] in the setting with variable powers. Note that [22, Lemma 4.6] implies the inclusion $W^{s_2, p_2}(\Omega) \subset W^{s_1, p_1}(\Omega)$ with $0 < s_1 < s_2 < 1$ and $p_1 \leq p_2$. Therefore, the following lemma also implies a similar inclusion for fractional Sobolev spaces with variable powers.

**Lemma 2.2** Let $\Omega \subset \mathbb{R}^n$ be bounded, and $p_1, p_2, s_1, s_2 : \Omega \times \Omega \to (0, \infty)$ be symmetric and satisfy that $s_1(x, y) \leq s_2(x, y)$, $p_1(x, y) \leq p_2(x, y)$, $(p_1)^{\Omega} := \inf_{x, y \in \Omega} p(x, y) > 0$.
and \( d_1 \leq s_2(x, y) - s_1(x, y) \leq d_2 \) for all \( x, y \in \Omega \) and for some \( 0 < d_1 \leq d_2 \). Then we have

\[
\varrho_{s_1(\cdot), p_1(\cdot)}(v, \Omega) \leq M \left\{ \varrho_{s_2(\cdot), p_2(\cdot)}(v, \Omega) + \frac{c(n)}{d_1(p_1)_{\Omega}} |\Omega \cap \{ v \neq 0 \}| \right\},
\]

where

\[
M = \begin{cases} \text{diam} (\Omega)^{d_1(p_1)_{\Omega}} & \text{if diam} (\Omega) \leq 1, \\
\text{diam} (\Omega)^{d_2(p_1)_{\Omega}} & \text{if diam} (\Omega) > 1, 
\end{cases}
\]

and the constant \( c(n) > 0 \) depends only on \( n \).

**Proof** The direct computation yields that

\[
\varrho_{s_1(\cdot), p_1(\cdot)}(v, \Omega) = \int_{\Omega} \int_{\Omega} \left( \frac{|v(x) - v(y)|}{|x - y|^{s_2(x, y)}} \right)^{p_1(x, y)} \frac{dy \, dx}{|x - y|^{n-(s_2(x, y)-s_1(x, y))p_1(x, y)}}
\]

\[
\leq \int_{\Omega} \int_{\Omega} \left( \frac{|v(x) - v(y)|}{|x - y|^{s_2(x, y)}} \right)^{p_2(x, y)} \frac{dy \, dx}{|x - y|^{n-(s_2(x, y)-s_1(x, y))p_1(x, y)}}
\]

\[
+ 2 \int_{\Omega \setminus \{ v \neq 0 \}} \int_{\Omega} \frac{dy \, dx}{|x - y|^{n-(s_2(x, y)-s_1(x, y))p_1(x, y)}}
\]

\[
\leq M \varrho_{s_2(\cdot), p_2(\cdot)}(v, B_r) + 2 \int_{\Omega \setminus \{ v \neq 0 \}} \int_{\Omega} \frac{dy \, dx}{|x - y|^{n-(s_2(x, y)-s_1(x, y))p_1(x, y)}}.
\]

Note that if \( \text{diam} (\Omega) \leq 1 \)

\[
\int_{\Omega} \frac{dy}{|x - y|^{n-(s_2(x, y)-s_1(x, y))p_1(x, y)}} \leq c(n) \int_{0}^{\text{diam} (\Omega)} r^{1+d_1(p_1)_{\Omega}} dr = \frac{c(n)}{d_1 p_{\Omega}} M,
\]

and if \( \text{diam} (\Omega) > 1 \)

\[
\int_{\Omega} \frac{dy}{|x - y|^{n-(s_2(x, y)-s_1(x, y))p_2(x, y)}}
\]

\[
\leq c(n) \left( \int_{1}^{\text{diam} (\Omega)} r^{-1+d_2(p_1)_{\Omega}} dr + \int_{0}^{1} r^{-1+d_1(p_1)_{\Omega}} dr \right)
\]

\[
\leq \frac{c(n)}{d_1(p_1)_{\Omega}} M.
\]

\( \square \)

We next introduce a fractional Sobolev-Poincaré type inequality for the fractional Sobolev spaces with constant powers. This is a simple corollary of [27, Theorem 6.7]. However, for the sake of completeness, we report a proof.

**Lemma 2.3** Let \( 0 < s < 1 \), \( 1 \leq p < \infty \), and \( sp < n \). For any \( v \in L^1(B_r) \) with \( \varrho_{s, p}(v; B_r) < \infty \), we have

\[
\left( \int_{B_r} |v - (v)_{B_r}|^{p} dx \right)^{\frac{p}{p_s}} \leq c r^{sp} \int_{B_r} \frac{|v(x) - v(y)|^{p}}{|x - y|^{n+sp}} dy \, dx.
\]
where \( c = c(n, s, p) > 0 \). Moreover, if \( s < t \) and \( p < q \) we also have that
\[
\left( \int_{B_r} |v - (v)_{B_r}|^{p_s} \, dx \right)^{\frac{q}{p_s}} \leq cr^{tq} \int_{B_r} \int_{B_r} \frac{|v(x) - v(y)|^q}{|x - y|^{n+ tq}} \, dy \, dx
\]
for some \( c = c(n, p, q, s, t) > 0 \), if the right hand side is finite.

**Proof** We first observe from Hölder’s inequality that
\[
\int_{B_r} |v - (v)_{B_r}|^p \, dx \leq \frac{1}{|B_r|} \int_{B_r} \int_{B_r} |v(x) - v(y)|^p \, dy \, dx
\]
\[
\leq \frac{(2r)^{n+sp}}{|B_r|} \int_{B_r} \int_{B_r} \frac{|v(x) - v(y)|^p}{|x - y|^{n+sp}} \, dy \, dx
\]
\[
\leq cr^{sp} \int_{B_r} \int_{B_r} \frac{|v(x) - v(y)|^p}{|x - y|^{n+sp}} \, dy \, dx,
\]
which is in fact the fractional version of Poincaré inequality. Hence \( v - (v)_{B_r} \in W^{s, p}(B_r) \).

Then, by the Sobolev inequality for the fractional Sobolev spaces with constant powers, see for instance [27, Theorems 6.7 and 6.10], we have
\[
\left( \int_{B_r} |v - (v)_{B_r}|^{p_s} \, dx \right)^{\frac{p}{p_s}} \leq cr^{sp} \int_{B_r} \int_{B_r} \frac{|v(x) - v(y)|^p}{|x - y|^{n+sp}} \, dy \, dx + c \int_{B_r} |v - (v)_{B_r}|^p \, dx
\]
\[
\leq cr^{sp} \int_{B_r} \int_{B_r} \frac{|v(x) - v(y)|^p}{|x - y|^{n+sp}} \, dy \, dx.
\]

Therefore, we have the first inequality. The second inequality is implied by the first inequality with the following estimate:
\[
\int_{B_r} \int_{B_r} \frac{|v(x) - v(y)|^p}{|x - y|^{n+sp}} \, dy \, dx
\]
\[
\leq \left( \int_{B_r} \int_{B_r} \frac{|v(x) - v(y)|^q}{|x - y|^{n+ tq}} \, dy \, dx \right)^{\frac{p}{q}} \left( \int_{B_r} \int_{B_r} \frac{1}{|x - y|^{n+ \frac{(s-1)q}{p} - \frac{n+ (s-1)q}{p}}} \, dy \, dx \right)^{\frac{q-p}{q}}
\]
\[
\leq c(n, p, q, s, t) \left( r^{(s-1)q} \int_{B_r} \int_{B_r} \frac{|v(x) - v(y)|^q}{|x - y|^{n+ tq}} \, dy \, dx \right)^{\frac{p}{q}}.
\]

\( \square \)

The next inequality can be found in [22, Lemma 4.3].

**Lemma 2.4** Let \( p > 1 \) and \( a \geq b \geq 0 \). For any \( \varepsilon > 0 \)
\[
a^p - b^p \leq \varepsilon a^p + \left( \frac{p - 1}{\varepsilon} \right)^{p-1} (a - b)^p,
\]
hence
\[
a^p - b^p \leq \varepsilon b^p + (\varepsilon + c \varepsilon^{1-p}) (a - b)^p
\]
for some \( c > 0 \) depending only on \( p \).
We end the subsection recalling [45, Chapter 2, Lemma 4.7].

**Lemma 2.5** Let \( \{ y_i \}_{i=0}^{\infty} \) be a sequence of nonnegative numbers and satisfy
\[
y_{i+1} \leq b_1 b_2^{1+\beta} y_i, \quad i = 0, 1, 2, \ldots
\]
for some \( b_1, \beta > 0 \) and \( b_2 > 1 \). If
\[
y_0 \leq b_1^{-\frac{1}{\beta}} b_2^{-\frac{1}{p^*}},
\]
then \( y_i \to 0 \) as \( i \to \infty \).

## 3 Existence of weak solution

We discuss on the existence of weak solution to (1.1). This is naturally linked to the existence of a minimizer of
\[
\mathcal{E}(v; \Omega) := \int_{\Omega} \frac{1}{p(x, y)} |v(x) - v(y)|^{p(x, y)} K(x, y) \, dy \, dx.
\]  
(3.1)

We say that \( u \in W^{s, p}(\cdot, \cdot) (\Omega) \) is a minimizer of the energy functional (3.1) if
\[
\mathcal{E}(u; \Omega) \leq \mathcal{E}(v; \Omega)
\]
for every \( v \in W^{s, p}(\cdot, \cdot) (\Omega) \) with \( u = v \) a.e. in \( \mathbb{R}^n \setminus \Omega \). By a standard argument, see for instance the proof of [26, Theorem 2.3], we notice that \( u \in W^{s, p}(\cdot, \cdot) (\Omega) \) is a weak solution to (1.1) if and only if \( u \) is a minimizer of (3.1). Hence we prove the existence and uniqueness of a minimizer of (3.1). We emphasize that no regularity condition on \( s(\cdot, \cdot) \) and \( p(\cdot, \cdot) \) is assumed. On the other hand, \( \Omega \) is assumed to be Lipschitz, in order to apply the compact embedding theorem for the fractional Sobolev space \( W^{s, p} \).

**Theorem 3.1** Let \( s, p, K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) be symmetric and satisfy (1.4), (1.5) and (1.6), respectively, \( \Omega \) be bounded, open and Lipschitz, and \( g \in W^{s, p}(\cdot, \cdot) (\Omega) \). There exists a unique minimizer \( u \in W^{s, p}(\cdot, \cdot) (\Omega) \) with \( u = g \) a.e. in \( \mathbb{R}^n \setminus \Omega \) of (3.1).

Moreover, suppose that \( p^+ - 1 < \frac{np^*}{n+s-p} \) or \( s-p^- \geq n \). If \( g \in W^{s, p}(\cdot, \cdot) (\Omega) \cap T^{s, p}(\cdot, \cdot) (\Omega) \), then the minimizer \( u \in W^{s, p}(\cdot, \cdot) (\Omega) \) is in \( T^{s, p}(\cdot, \cdot) (\Omega) \).

**Proof** We follow the argument in the proof of [26, Theorem 2.3]. Note that the uniqueness is the direct consequence of the strict convexity of the mapping \( t \mapsto t^{p(x,y)} \) which is uniform in \( (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \), since \( p^+ > 1 \).

We now prove the existence. Let \( \{ u_k \}_{k=1}^{\infty} \subset W^{s, p}(\cdot, \cdot) (\Omega) \) be a minimizing sequence with \( u_k = g \) a.e. in \( \mathbb{R}^n \setminus \Omega \). (Note that by the definition of \( g \) the admissible set of the energy functional, that is \( W^{s, p}(\cdot, \cdot) (\Omega) = \{ v \in W^{s, p}(\cdot, \cdot) (\Omega) : v = g \) a.e. in \( \mathbb{R}^n \setminus \Omega \}, is non-empty.) Then there exists \( M > 0 \) such that
\[
Q_{s, p}(u_k; \Omega) = \int_{\Omega} \frac{|u_k(x) - u_k(y)|^{p(x, y)}}{|x - y|^{n+s(x, y)p(x, y)}} \, dy \, dx \leq M \quad \text{for all} \quad k = 1, 2, \ldots.
\]
Hence, by Lemma 2.2, we have that for any \( t \in (0, s^-) \), \( Q_{t, p^-}(u_k; \Omega) \) is bounded for \( k \). Moreover, since \( u_k - g = 0 \) a.e. in \( \mathbb{R}^n \setminus \Omega \), by using the fractional Sobolev inequality [27,
Theorem 6.5] we have that for some $B_R = B_R(0) \supset \Omega$ with $R \geq 1$ and $t \in (0, \frac{\alpha}{2})$ with 
\[
\frac{np}{n+tp} =: q > 1,
\]
\[
\left( \int_{B_R} |u_k - g|^p \, dx \right)^{\frac{q}{p}} \leq c_{q,t,q} \left( u_k - g; \mathbb{R}^n \right) 
\leq c_{q,t,q} \left( u_k - g; B_R \right) + c \int_{B_R} |u_k(y) - g(y)|^q \left( \int_{\mathbb{R}^n \setminus B_R} \frac{1}{|x-y|^{n+q}} \, dx \right) \, dy 
\leq c_{q,t,q} \left( u_k - g; B_R \right) + c \int_{B_R} |u_k(y) - g(y)|^q \left( \int_{B_R \setminus B_R} \frac{1}{|x-y|^{n+q}} \, dx \right) \, dy 
\leq c \int_{B_R} \int_{B_R} \frac{|u_k(x) - g(x) - (u_k(y) - g(y))|^q}{|x-y|^{n+q}} \, dx \, dy \leq c_{q,t,q} \left( u_k - g; B_{2R} \right).
\]

Note that in the second inequality we used the fact that 
\[
\int_{\mathbb{R}^n \setminus B_R} \frac{1}{|x-y|^{n+q}} \, dx \leq C(n) \int_{B_{2R} \setminus B_R} \frac{1}{|x-y|^{n+q}} \, dx \text{ for all } y \in B_R.
\]

Again using Lemma 2.2, we have that 
\[
\left( \int_{\Omega} |u_k - g|^p \, dx \right)^{\frac{q}{p}} = \left( \int_{B_R} |u_k - g|^p \, dx \right)^{\frac{q}{p}} 
\leq c R^{q-p} \left( q_{t,(\cdot),p,(\cdot)} (u_k - g); B_{2R} \right) + R^n 
\leq c R^{q-p} \left( \int_{\mathbb{R}^n} |u_k(x) - u_k(y)|^{p(x,y)} \, dy \, dx + \int_{\mathbb{R}^n} \frac{|g(x) - g(y)|^{p(x,y)}}{|x-y|^{n+s(x,y)p(x,y)}} \, dy \, dx + R^n \right) 
\leq c R^{q-p} \left( M + \int_{\mathbb{R}^n} \frac{|g(x) - g(y)|^{p(x,y)}}{|x-y|^{n+s(x,y)p(x,y)}} \, dy \, dx + R^n \right) \text{ for all } k = 1, 2, \ldots.
\]

which implies that $\{u_k\}$ is bounded in $L^p(\Omega)$, hence $\{u_k\}$ is bounded in $W^{t,p}(\Omega)$.

By the compact embedding theorem for the fractional Sobolev space $W^{t,p}(\Omega)$ [27, Theorem 7.1], there exist a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ and $u \in L^p(\Omega)$ such that 
\[
u_{k_j} \rightharpoonup u \text{ as } j \to \infty \text{ strongly in } L^p(\Omega)
\]
and 
\[
u_{k_j} \rightarrow u \text{ as } j \to \infty \text{ a.e. in } \Omega.
\]

We extend $u$ by $g$ in $\mathbb{R}^n \setminus \Omega$. Then Fatou’s lemma yields 
\[
\int_{\mathbb{R}^n} \frac{1}{p(x,y)} |u(x) - u(y)|^{p(x,y)} K(x, y) \, dy \, dx 
\leq \liminf_{j \to \infty} \int_{\mathbb{R}^n} \frac{1}{p(x,y)} |u_{k_j}(x) - u_{k_j}(y)|^{p(x,y)} K(x, y) \, dy \, dx,
\]
that means $u \in W^{t,p}(\cdot, \cdot)(\Omega)$, and it is the minimizer.
Next we further assume \( g \in W^{s(\cdot), p(\cdot)}(\Omega) \cap T^{s(\cdot), p(\cdot)}(\Omega) \), and \( p^+ - 1 < \frac{np^-}{n-s} \) or \( s^+ p^- \geq n \). Then one can find \( t \in (0, s^-) \) such that \( t p^- < n \) and \( p^+ - 1 \leq \frac{np^-}{n-tp} \).

Then \( u \in W^{1,p}(\Omega) \) by Lemma 2.2, where \( u \in W^{s(\cdot), p(\cdot)}(\Omega) \) is the unique minimizer. Therefore, by the embedding theorem for the fractional Sobolev space [27, Theorem 6.7] we have that \( u \in L^{\frac{np^-}{n-tp}}(\Omega) \subset L^{p^+ - 1}(\Omega) \). This and the fact that \( u = g \) a.e. in \( \mathbb{R}^n \setminus \Omega \) imply that the minimizer \( u \) is in \( T^{s(\cdot), p(\cdot)}(\Omega) \).

\[ \square \]

4 Caccioppoli and logarithmic estimates

We obtain a Caccioppoli type estimate and a logarithmic estimate for the weak solutions to (1.1). In order to obtain a Caccioppoli type estimate for a weak solution to a local/nonlocal equation, we take a multiplicative function of the weak solution and a cut-off function as a test function in the weak form of the problem. However, in the variable exponent case, that is, \( p(\cdot, \cdot) \) is not constant, it is unclear that the multiplicative function can be taken as a test function. More precisely, even if \( v \in W^{s(\cdot), p(\cdot)}(\Omega) \), we couldn’t prove that \( v \eta \in W^{s(\cdot), p(\cdot)}(\Omega) \), where \( \eta \) is a standard cut-off function in a ball contained in \( \Omega \). (Note that if \( p^- \) and \( p^+ \) are sufficiently close, it is correct.) To overcome this problem, we will use an approximation argument together with the following lemma.

\[ \textbf{Lemma 4.1} \]

Let \( v \in L^{p^+}(B_{2r}) \) satisfy \( \mathcal{Q}_{s(\cdot), p(\cdot)}(v, B_{2r}) < \infty \), and \( \eta \in W^{1,\infty}_0(B_r) \). Then \( \mathcal{Q}_{s(\cdot), p(\cdot)}(v \eta, \mathbb{R}^n) < \infty \). In particular, \( v \eta \in W^{s(\cdot), p(\cdot)}(\Omega) \) whenever \( \Omega \supset B_{2r} \).

\[ \textbf{Proof} \]

We have to show that

\[ \mathcal{Q}_{s(\cdot), p(\cdot)}(v \eta, \mathbb{R}^n) = \mathcal{Q}_{s(\cdot), p(\cdot)}(v \eta, B_{2r}) + 2 \int_{B_{2r}} \int_{\mathbb{R}^n \setminus B_{2r}} \frac{|v(x)\eta(x)||p(x,y)}{|x-y|^{n+s(x,y)p(x,y)}} \, dx \, dy < \infty. \]

We estimate the first term. By the Mean Value Theorem, (1.4) and (1.5),

\[ \mathcal{Q}_{s(\cdot), p(\cdot)}(v \eta, B_{2r}) \]

\[ \leq c \int_{B_{2r}} \int_{B_{2r}} \frac{|(v(x) - v(y))\eta(y)||p(x,y)}{|x-y|^{n+s(x,y)p(x,y)}} \, dx \, dy + c \int_{B_{2r}} \int_{B_{2r}} \frac{|v(x)(\eta(x) - \eta(y))||p(x,y)}{|x-y|^{n+s(x,y)p(x,y)}} \, dx \, dy \]

\[ \leq c(\|\eta\|_{L^{\infty}(B_r)} + 1)\mathcal{Q}_{s(\cdot), p(\cdot)}(v, B_{2r}) + c(||D\eta||_{L^{p^+}(B_{2r})} + 1) \int_{B_{2r}} (v(x)|^{p^+} + 1) \int_{B_{2r}(x)} |x-y|^{-n+(1-s(x,y))p(x,y)} \, dy \, dx \]

\[ \leq c(\|\eta\|_{L^{p^+}(B_r)} + 1)\mathcal{Q}_{s(\cdot), p(\cdot)}(v, B_{2r}) \]

\[ + c \max\{r(1-s^+)p^-, r(1-s^-)^+\}(||D\eta||_{L^{\infty}} + 1) \left( \int_{B_r} |v(x)|^{p^+} \, dx + r^n \right) < \infty. \]

We next estimate the second term. Using the fact that \( \eta \equiv 0 \) in \( \mathbb{R}^n \setminus B_r \), (1.4) and (1.5),

\[ \int_{B_{2r}} \int_{\mathbb{R}^n \setminus B_{2r}} \frac{|v(x)\eta(x)||p(x,y)}{|x-y|^{n+s(x,y)p(x,y)}} \, dx \, dy = (||\eta||_{L^{p^+}} + 1) \int_{B_r} \int_{\mathbb{R}^n \setminus B_{2r}} \frac{|v(x)||^{p^+} + 1}{|x-y|^{n+s(x,y)p(x,y)}} \, dy \, dx \]

\[ \leq (||\eta||_{L^{p^+}(B_r)} + 1) \int_{B_r} (v(x)|^{p^+} + 1) \int_{\mathbb{R}^n \setminus B_{2r}(x)} |x-y|^{-n-(s(x,y))p(x,y)} \, dy \, dx \]

\[ \leq c \max\{r-s^- p^-, r-s^+ p^+\}(||\eta||_{L^{p^+}} + 1) \left( \int_{B_r} |v(x)|^{p^+} \, dx + r^n \right) < \infty. \]

\[ \square \]

Now we obtain a Caccioppoli type inequality.
Lemma 4.2 Let \( u \in \mathbb{W}^{s, p} \cap T^{s, p} \) be a weak solution to (1.1), and let \( B_{2r} = B_{2r}(x_0) \subseteq \Omega \) satisfying

\[
r \leq \frac{1}{2} \quad \text{and} \quad \omega_p(r) \leq \frac{(1 - s^+) p^-}{2s^+}.
\]

Then for any \( 0 < \rho < r \) and any \( k \in \mathbb{R} \), we have

\[
\begin{align*}
Q_{s, p}((u - k)_\pm, B_\rho) + \int_{B_\rho} (u(x) - k)_\pm \left( \int_{\mathbb{R}^n} \frac{(k - u(y))_{s, p}^p}{|x - y|^{n + s_p(x, y) p(x, y)}} \, dy \right) \, dx \\
\leq \frac{1}{(r - \rho)^{p_1}} \int_{B_r} \int_{B_r} \max\{(u(x) - k)_\pm, (u(y) - k)_\pm\}^{p(x, y)} \, dy \, dx \\
+ c \left( \frac{r}{r - \rho} \right)^{n + s_2 p_2} \int_{B_r} (u(x) - k)_{s, p}^p \left[ \int_{\mathbb{R}^n \setminus B_r} \frac{(u(y) - k)_{s, p}^p}{|y - x_0|^{n + s_p(x, y) p(x, y)}} \, dy \right] \, dx \\
\leq c \left( \frac{r}{r - \rho} \right)^{n + s_2 p_2} \int_{\mathbb{R}^n \setminus B_r} \left( (u(x) - k)_{s, p}^p + 1 \right) \, dx \\
+ c \left( \frac{r}{r - \rho} \right)^{n + s_2 p_2} T((u - k)_\pm, x_0, r, r) \int_{B_r} (u - k)_{s, p}^p \, dx,
\end{align*}
\]

where

\[
s_1 := s_{B_r}, \quad s_2 := s_{B_r}^+, \quad p_1 := p_{B_r}^-, \quad p_2 := p_{B_r}^+,
\]

\[
A_{\pm}(k, r) = A_{\pm}(k, r, x_0) := \{ x \in B_r : (u(x) - k)_\pm > 0 \},
\]

and the constant \( c > 0 \) depends only on \( n, s^+, p^+ \) and \( \Lambda \).

**Proof** We prove the desired estimate only for \( '+' \) sign. Then the estimate for \( '-' \) sign directly follows by considering \( -u \) that is also a weak solution to (1.1). Let \( \eta \in C^\infty_c(B_{r + \rho}) \) with \( 0 \leq \eta \leq 1, \eta \equiv 1 \) in \( B_\rho \) and \( |\nabla \eta| \leq 4/(r - \rho) \) in \( \mathbb{R}^n \). Define \( w_{\pm} := (u - k)_\pm \) and \( w_{\pm}^l := \min\{u, l\} - k \), where \( l > k \). Note that \( w_{\pm}^l(x) \leq w_{\pm}(x) \) and \( |w_{\pm}^l(x) - w_{\pm}(y)| \leq |w_+(x) - w_+(y)| \) for every \( x, y \in \mathbb{R}^n \) and every \( l > k \). By Lemma 4.1, \( w_{\pm}^l \eta p_{2} \in \mathbb{W}^{s, p}(\Omega) \) for every \( l > k \) since \( 0 \leq w_{\pm}^l \leq l - k \) and \( Q_{s, p}(w_{\pm}^l, B_{2r}) \leq Q_{s, p}(w_+, B_{2r}) < \infty \). Hence we take \( w_{\pm}^l \eta p_{2} \) as a test function \( \varphi \) in (1.7), and obtain an estimate for \( w_{\pm}^l \) first. Then by taking the limit as \( l \to \infty \), we obtain the desired estimate.

Recalling (1.7) with \( \varphi = w_{\pm}^l \eta p_{2} \),

\[
0 = \int_{B_r} \int_{B_r} |u(x) - u(y)|^{p(x, y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y))K(x, y) \, dy \, dx \\
+ 2 \int_{B_r} \int_{\mathbb{R}^n \setminus B_r} |u(x) - u(y)|^{p(x, y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y))K(x, y) \, dy \, dx \\
=: I_1 + 2I_2.
\]

We first estimate \( I_1 \). We first have

\[
|u(x) - u(y)|^{p(x, y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y)) = 0, \quad \text{for } x, y \in B_r \setminus A^+(k, r).
\]

(4.3)
If \( x \in A^+(k, r) \) and \( y \in B_r \setminus A^+(k, r) \), then \( u(x) > k \geq u(y) \) hence
\[
|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y)) \\
\geq (w_+(x) + w_-(y))^{p(x,y)-1}w_i^{p}(x)\eta(x)^{p_2} \\
\geq \frac{1}{2} |w_+(x) - w_+(y)|^{p(x,y)}\eta(x)^{p_2} + \frac{1}{2} w_-(y)^{p(x,y)-1} w_i^{p}(x)\eta(x)^{p_2}.
\]
(4.4)

Now we suppose \( x, y \in A^+(k, r) \). If \( u(x) > u(y) > k \) and \( \eta(x)^{p_2} \geq \eta(y)^{p_2} \),
\[
|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y)) \\
= |w_+(x) - w_+(y)|^{p(x,y)-1}(w_i^{p}(x)\eta(x)^{p_2} - w_i^{p}(y)\eta(y)^{p_2}) \\
\geq |w_+(x) - w_+(y)|^{p(x,y)}\eta(x)^{p_2}.
\]

If \( u(x) > u(y) > k \) and \( \eta(x)^{p_2} < \eta(y)^{p_2} \),
\[
|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y)) \\
\geq (w_+(x) - w_+(y))^{p(x,y)}\eta(y)^{p_2} - (w_+(x) - w_+(y))^{p(x,y)-1}w_i^{p}(x)\eta(y)^{p_2} - \eta(x)^{p_2}.
\]

Applying Lemma 2.4 with \( p = p_1, a = \eta(y)^{p_2} / p_1 \), \( b = \eta(x)^{p_2} / p_1 \), and \( \varepsilon = \frac{1}{2} \frac{w_+(x) - w_+(y)}{w_+(x) - w_+(y)} \), and using the facts that \( 0 \leq \eta \leq 1 \) and \( |\eta(y) - \eta(x)| \leq \|\nabla \eta\|_{L^\infty(B_r)}|x - y| \leq c \frac{|x - y|}{r - \rho} \),
\[
\eta(y)^{p_2} - \eta(x)^{p_2} \leq \frac{1}{2} \frac{w_+(x) - w_+(y)}{w_+(x)}\eta(y)^{p_2} + \left\{ \frac{(p_1 - 1)w_+(x)}{w_+(x) - w_+(y)} \right\}^{p_1-1} \\
\times \left( \eta(y)^{p_2 / p_1} - \eta(x)^{p_2 / p_1} \right)^{p_1} \\
\leq \frac{1}{2} \frac{w_+(x) - w_+(y)}{w_+(x)}\eta(y)^{p_2} + c \left( \frac{w_+(x)}{w_+(x) - w_+(y)} \right)^{p(x,y)-1} \frac{|x - y|^{p_1}}{(r - \rho)^{p_1}},
\]
hence
\[
|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y)) \\
\geq \frac{1}{2} |w_+(x) - w_+(y)|^{p(x,y)}\eta(y)^{p_2} - c w_+(x)^{p(x,y)} \frac{|x - y|^{p_1}}{(r - \rho)^{p_1}} \\
- \frac{1}{2} \left( |w_+(x) - w_+(y)|^{p(x,y)} - |w_i^{p}(x) - w_i^{p}(y)|^{p(x,y)} \right) \eta(y)^{p_2}.
\]

Therefore, by considering the symmetry of \( p(\cdot, \cdot) \) and \( s(\cdot, \cdot) \), we have that for every \( x, y \in A^+(k, r) \),
\[
|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y)) \\
\geq \frac{1}{2} |w_+(x) - w_+(y)|^{p(x,y)} \min\{\eta(x), \eta(y)\}^{p_2} \\
- c \max\{w_+(x), w_+(y)\}^{p(x,y)} \frac{|x - y|^{p_1}}{(r - \rho)^{p_1}} \\
- \frac{1}{2} \left( |w_+(x) - w_+(y)|^{p(x,y)} - |w_i^{p}(x) - w_i^{p}(y)|^{p(x,y)} \right) \max\{\eta(x), \eta(y)\}^{p_2}.
\]
(4.5)
Combining the results in (4.3), (4.4) and (4.5), we get

\[ I_1 \geq \frac{1}{2} \left( \varrho_s(\cdot, \cdot, \rho) (w_+^l, B_{\rho}) + \int_{B_{\rho}} w_+(x) \left[ \int_{B_r} \frac{w_-(y)^{p(x,y)-1}}{|x-y|^{n+s(x,y)p(x,y)}} \, dy \right] \, dx \right) \\
- c \frac{r}{(r-\rho)^p} \int_{B_r} \int_{B_r} \frac{\max \{ w_+(x), w_+(y) \}^{p(x,y)}}{|x-y|^{n+s(x,y)p(x,y)-p_1}} \, dy \, dx \\
- \frac{1}{2} \left( \varrho_s(\cdot, \cdot, \rho) (w_+, B_{\rho}) - \varrho_s(\cdot, \cdot, \rho) (w_+^l, B_{\rho}) \right). \]

On the other hand,

\[ I_2 \geq \int_{A^+(k, \rho)} w_+^l(x) \left[ \int_{\{u(x) \geq u(y)\} \setminus B_{\rho}} (u(x) - u(y))^{p(x,y)-1} \frac{(u(y) - u(x))^{p(x,y)-1}}{|x-y|^{n+s(x,y)p(x,y)}} \, dy \right] \, dx \\
- \int_{A^+(k, \frac{r+\rho}{2})} w_+(x) \left[ \int_{\{u(x) < u(y)\} \setminus B_{\rho}} (u(y) - u(x))^{p(x,y)-1} \frac{(u(y) - u(x))^{p(x,y)-1}}{|x-y|^{n+s(x,y)p(x,y)}} \, dy \right] \, dx \\
\geq \int_{B_{\rho}} w_+^l(x) \left[ \int_{\mathbb{R}^n \setminus B_{\rho}} \frac{w_-(y)^{p(x,y)-1}}{|x-y|^{n+s(x,y)p(x,y)}} \, dy \right] \, dx \\
- c \left( \frac{r}{r-\rho} \right)^{n+s_2} \int_{B_{\rho}} w_+(x) \left[ \int_{\mathbb{R}^n \setminus B_{\rho}} \frac{w_+(y)^{p(x,y)-1}}{|y-x_0|^{n+s(x,y)p(x,y)}} \, dy \right] \, dx. \]

Here we used the fact that

\[ |x-y| \geq |y-x_0| - |x-x_0| \geq |y-x_0| - \frac{r+\rho}{2} \frac{|y-x_0|}{r} = \frac{r-\rho}{2r} |y-x_0| \]

for every \( x \in B_{\frac{r+\rho}{2}} \) and \( y \in \mathbb{R}^n \setminus B_{\rho} \), where \( x_0 \) is the center of \( B_{\rho} \).

Therefore, from the above estimates for \( I_1 \) and \( I_2 \) with \( I_1 + 2I_2 = 0 \), we have that for every \( l > k \),

\[ \varrho_s(\cdot, \cdot, \rho) (w_+^l, B_{\rho}) + \int_{B_{\rho}} w_+(x) \left[ \int_{\mathbb{R}^n} \frac{w_-(y)^{p(x,y)-1}}{|x-y|^{n+s(x,y)p(x,y)}} \, dy \right] \, dx \]

\[ \leq \frac{c}{(r-\rho)^p} \int_{B_r} \int_{B_r} \frac{\max \{ w_+(x), w_+(y) \}^{p(x,y)}}{|x-y|^{n+s(x,y)p(x,y)-p_1}} \, dy \, dx \\
+ c \left( \frac{r}{r-\rho} \right)^{n+s_2} \int_{B_{\rho}} w_+(x) \left[ \int_{\mathbb{R}^n \setminus B_{\rho}} \frac{w_-(y)^{p(x,y)-1}}{|y-x_0|^{n+s(x,y)p(x,y)}} \, dy \right] \, dx \\
+ c \left( \varrho_s(\cdot, \cdot, \rho) (w_+, B_{\rho}) - \varrho_s(\cdot, \cdot, \rho) (w_+^l, B_{\rho}) \right). \]

Since \( w_+^l(x) \not\to w_+(x) \) and \( |w_+^l(x) - w_+(y)| \not\to |w_+(x) - w_+(y)| \) as \( l \to \infty \) for a.e. \( x, y \in B_{\rho} \), by the Lebesgue monotone convergence theorem, sending \( l \) in the above estimate to \( \infty \), we obtain the first inequality in (4.2).
The second inequality in (4.2) is obtained from the definition the tail and the following estimates: since \( p_1 - s_2p_2 \geq (1 - s^+) p^- - s^+ \omega_p(r) \geq \frac{(1-s^+)p^-}{2} > 0 \) by (4.1),
\[
\frac{1}{(r - \rho)^{p_1}} \int_{B_r} \int_{B_r} \frac{\max\{w_+(x), w_+(y)\}^{p(x,y)}}{|x - y|^{n+s(x,y)p(x,y)-p_1}} \, dy \, dx \\
\leq \frac{r^{p_2}}{r^{p_1}(r - \rho)^{p_2}} \int_{B_r} \int_{B_r} \frac{\max\{w_+(x), w_+(y)\}^{p(x,y)}}{|x - y|^{n+s_2p_2-p_1}} \, dy \, dx \\
\leq \frac{r^{p_2}}{r^{p_1}(r - \rho)^{p_2}} \int_{A^+(k,r)} (w_+(x)^{p_2} + 1) \int_{B_{2\rho}(x)} \frac{1}{|x - y|^{n+s_2p_2-p_1}} \, dy \, dx \\
\leq c \frac{r^{p_2-s_2p_2}}{(r - \rho)^{p_2}} \int_{A^+(k,r)} (w_+(x)^{p_2} + 1) \, dx.
\]

\[\square\]

Next we obtain a logarithmic estimate that will be used crucially in the proof of the Hölder regularity in Sect. 5. Here we consider a locally bounded and nonnegative solution.

**Lemma 4.3** Let \( u \in W^{s(\cdot),p(\cdot)}(\Omega) \cap T^{s(\cdot),p(\cdot)}(\Omega) \) be a weak solution to (1.1) such that \( u \in L^\infty(\Omega') \) for some \( \Omega' \subseteq \Omega \). If \( r \in (0, 1) \) satisfies
\[
\omega_p(r) \leq \frac{\ln 2}{\ln(1 + \| u \|_{L^\infty(\Omega')})},
\]
\( u \geq 0 \) in \( B_r = B_r(x_0) \subset \Omega' \) and \( d > 0 \), then for any \( B_{\rho} = B_{\rho}(x_0) \) with \( 0 < \rho < r/2 \),
\[
\int_{B_{\rho}} \int_{B_{\rho}} \left| \ln \left( \frac{d + u(x)}{d + u(y)} \right) \right|^{p_2} K(x, y) \, dy \, dx \\
\leq c \rho^{n-s_2p_1+p_1-p_2} + c \max\{d, d^{-1}\}^{-p_3} \rho^{n-s_4p_4} + \frac{c \rho^n}{d^{p_2-1}} T(u_+ + \| u \|_{L^\infty(B_{\rho})}, r, 2\rho)
\]
for some \( c = c(n, s^\pm, p^\pm, \Lambda) > 0 \), where
\[
s_1 := s_{B_{2\rho}}, \quad s_2 := s_{B_\rho}, \quad s_3 := s_{B_{2\beta}}, \quad s_4 := s_{B_{\rho}},
\]
\[
p_1 := p_{B_{2\rho}}, \quad p_2 := p_{B_\rho}, \quad p_3 := p_{B_{2\beta}}, \quad p_4 := p_{B_{\rho}}.
\]

**Proof** Let \( \eta \in C_0^\infty(B_{3\rho/2}) \) such that \( 0 \leq \eta \leq 1, \eta \equiv 1 \) in \( B_{\rho} \) and \( |D\eta| \leq c(n)/\rho \). Note that, since the Mean Value Theorem yields \(|(u(x) + d)^{1-p_2} - (u(y) + d)^{1-p_2}| \leq (p_2 - 1) \max\{d, d^{-1}\}^{-p_2} |u(x) - u(y)| \) and \( 0 < (u + d)^{1-p_2} \leq \min\{1, d\}^{1-p_2} \) for every \( x, y \in B_{2\rho} \), by Lemma 4.1 we can take \( (u + d)^{1-p_2}\eta^{p_2} \) as a test function in (1.7). Hence, we have
\[
0 = \int_{B_{3\rho/2}} \int_{B_\rho} |u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y)) \times \left[ \frac{\eta(x)^{p_2}}{(u(x) + d)^{p_2-1}} - \frac{\eta(y)^{p_2}}{(u(y) + d)^{p_2-1}} \right] K(x, y) \, dy \, dx \\
+ 2 \int_{B_{3\rho/2}} \int_{\mathbb{R}^n \setminus B_{\rho}} |u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y)) \eta(x)^{p_2} K(x, y) \, dy \, dx \\
=: I_1 + 2I_2.
\]

We first estimate \( I_1 \). Suppose \( u(x) > u(y) \) for \( x, y \in B_{2\rho} \). By the second inequality in Lemma 2.4 with \( a = \eta(x)^{p_2} p(x,y)^{p(x,y)-2}, b = \eta(y)^{p_2} p(x,y)^{p(x,y)-2}, p = p(x, y) \) and \( \varepsilon = \delta \frac{u(x) - u(y)}{u(x) + d} \in (0, 1), \)
we have that for any \( \delta \in (0, 1) \),
\[
\eta(x)^{p_2} \leq \eta(y)^{p_2} + \delta \frac{u(x) - u(y)}{u(x) + d} \eta(y)^{p_2}
+ c \left( \delta \frac{u(x) - u(y)}{u(x) + d} \right)^{1-p(x,y)} |\eta(x)|^{p_2} + |\eta(y)|^{p_2} |p(x,y)|^p
\]

(if \( \eta(x) \leq \eta(y) \) the inequality is trivial). Using this inequality,
\[
|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y)) \left[ \frac{\eta(x)^{p_2}}{(u(x) + d)^{p_2-1}} - \frac{\eta(y)^{p_2}}{(u(y) + d)^{p_2-1}} \right] K(x, y)
\leq \frac{(u(x) - u(y))^{p(x,y)-1}}{(u(x) + d)^{p_2-1}} \eta(y)^p \left[ 1 + \delta \frac{u(x) - u(y)}{u(x) + d} - \left( \frac{u(x) + d}{u(y) + d} \right)^{p_2-1} \right] K(x, y)
\]
\[
= J
+ c \delta^{1-p(x,y)} (u(x) + d)^{p(x,y)-p_2} |\eta(x)|^{p_2} + |\eta(y)|^{p_2} |p(x,y)| K(x, y).
\]

Now we estimate
\[
J = \frac{(u(x) - u(y))^{p(x,y)}}{(u(x) + d)^{p_2}} \eta(y)^p \left[ \delta + \frac{1-\left(\frac{u(y)+d}{u(x)+d}\right)^{1-p}}{1-\left(\frac{u(y)+d}{u(x)+d}\right)^{1-p}} \right] K(x, y).
\]

Then we consider \( g(t) := \frac{1-t^{1-p}}{1-t} \), where \( p > 1 \) and \( t \in (0, 1) \). Note that with constant \( p > 1 \)
\[
\frac{1-t^{1-p}}{1-t} \leq -(p-1) \quad \forall t \in (0, 1) \quad \text{and} \quad \frac{1-t^{1-p}}{1-t} \leq -\frac{p-1}{1-t} \quad \forall t \in (0, \frac{1}{2}). \quad (4.9)
\]

If \( \frac{u(y)+d}{u(x)+d} \in (0, \frac{1}{2}] \), by the second inequality in (4.9),
\[
1 - \frac{(u(y)+d)^{1-p}}{u(x)+d} \leq -\frac{(p-1)}{2} \left( \frac{u(x)+d}{u(y)+d} \right)^{p_2-1} \leq -\frac{p}{2} ,
\]

hence, using this inequality and (4.6) and choosing \( \delta \leq \frac{p}{4} \),
\[
J \leq -\frac{p-1}{4} \frac{1}{(u(x) - u(y))^{p_2-p(x,y)}} \frac{(u(x) - u(y))^{p_2}}{(u(y) + d)^{p_2}} K(x, y)
\]
\[
\leq -c \ln \left( \frac{u(x) + d}{u(y) + d} \right)^{p_2} \min\{\eta(x), \eta(y)\}^{p_2} K(x, y).
\]

On the other hand, if \( \frac{u(y)+d}{u(x)+d} \in [\frac{1}{2}, 1) \),
\[
\frac{u(x) - u(y)}{u(x) + d} \geq \frac{1}{2} \frac{u(x) - u(y)}{u(y) + d} \geq \frac{1}{2} \ln \left( \frac{1 + u(x) - u(y)}{u(y) + d} \right) = \frac{1}{2} \ln \left( \frac{u(x) + d}{u(y) + d} \right) ,
\]
hence, using this inequality, the first inequality in (4.9) and \( (u(x) - u(y))^{p_2 - p(x, y)} \leq (1 + 2\|u\|_{L^\infty(\Omega')} \omega_p(r) \leq c \) by (4.6), and choosing \( \delta \leq \frac{\rho - 1}{2} \),

\[
J \leq -c \left| \ln \left( \frac{u(x) + d}{u(y) + d} \right) \right|^{p_2} \min\{\eta(x), \eta(y)\}^{p_2} K(x, y).
\]

By the symmetry of \( p(\cdot, \cdot) \) and \( K(\cdot, \cdot) \), we have the same estimate for \( J \) when \( u(y) > u(x) \). Therefore, we have

\[
I_1 \leq -c \int_{B_2} \int_{B_2} \left| \ln \left( \frac{u(x) + d}{u(y) + d} \right) \right|^{p_2} \min\{\eta(x), \eta(y)\}^{p_2} K(x, y) \, dy \, dx
\]

\[
+ c \int_{B_2} \int_{B_2} |\eta(x)|^{p_2} - |\eta(y)|^{p_2} |p(x, y)| K(x, y) \, dy \, dx.
\]

We further estimate the second integral on the right hand side. Since \( |\eta(x)|^{p_2} - |\eta(y)|^{p_2} |p(x, y)| \leq c \rho^{-1} |x - y| \) for every \( x, y \in B_{2\rho} \),

\[
\int_{B_{2\rho}} \int_{B_{2\rho}} |\eta(x)|^{p_2} - |\eta(y)|^{p_2} |p(x, y)| K(x, y) \, dy \, dx
\]

\[
\leq c \rho^{-p_2} \int_{B_{2\rho}} \int_{B_{2\rho}} |x - y|^{-n + (1 - s(x, y))p(x, y)} \, dy \, dx
\]

\[
\leq c \rho^{-p_2} \int_{B_{2\rho}} \int_{B_{4\rho}(x)} |x - y|^{-n + (1 - s_2)p_1} \, dy \, dx
\]

\[
\leq c \rho^{-s_2p_1 + p_1 - p_2}.
\]

We next estimate

\[
I_2 \leq \int_{B_{2\rho}} \int_{\mathbb{R}^n \setminus B_{2\rho}} \frac{(u(x) - u(y))^{p(x, y) - 1}_+ \eta(x)^{p_2}}{(u(x) + d)^{p_2 - 1}} K(x, y) \, dy \, dx.
\]

Since \( u \geq 0 \) in \( B_r \), for \( x \in B_{2\rho} \) and \( y \in B_r \)

\[
\frac{(u(x) - u(y))^{p(x, y) - 1}_+}{(u(x) + d)^{p_2 - 1}} \leq (u(x) + d)^{p(x, y) - p_2}
\]

\[
\leq \begin{cases} 
(1 + \|u\|_{L^\infty(\Omega')} \max\{1, d\})^{p_4 - p_3} \leq c \max\{1, d\}^{p_4 - p_3} & \text{if } p(x, y) > p_2 \\
\max\{1, d^{-1}\}^{p_4 - p_3} & \text{if } p(x, y) \leq p_2
\end{cases}
\]

\[
\leq c \max\{d, d^{-1}\}^{p_4 - p_3}
\]

and for \( x \in B_{2\rho} \) and \( y \in \mathbb{R}^n \setminus B_r \)

\[
\frac{(u(x) - u(y))^{p(x, y) - 1}_+}{(u(x) + d)^{p_2 - 1}} \leq \frac{\|u\|_{L^\infty(B_r)} + (u(y))_-^{p(x, y) - 1}}{d^{p_2 - 1}}.
\]
Therefore,
\[ I_2 \leq c \max \{d, d^{-1}\} p_4^{-4} p_3 \int_{B_{2\rho}} \int_{B_{2\rho} \setminus B_{\rho}} \frac{\eta(x)^{p_2}}{|x - y|^{p_4} p(x, y)} \, dy \, dx + c \int_{B_{\rho}} \int_{\mathbb{R}^n \setminus B_{\rho}} \frac{|u| |(u(y))_{\rho}(x, y)|^{p(x, y)-1}}{|x - y|^{p_4} p(x, y)} \, dy \, dx \]
\[ \leq c \max \{d, d^{-1}\} p_4^{-4} p_3 \int_{B_{3\rho} \setminus B_{\rho}} \int_{B_{2\rho} \setminus B_{\rho}} \frac{1}{|x - y|^{p_4} p(x, y)} \, dy \, dx + \frac{c \rho^n}{d^{p_2-1}} T(u_\pm + \|u\|_{L^\infty(B_r)}, r, 2\rho) \]
\[ \leq c \max \{d, d^{-1}\} p_4^{-4} p_3 \rho^{n-s_4 p_4} + \frac{c \rho^n}{d^{p_2-1}} T(u_\pm + \|u\|_{L^\infty(B_r)}, r, 2\rho). \]

Finally, combining (4.10), (4.11) and the preceding inequality with \( I_1 + 2I_2 = 0 \), we get the desired estimate. \( \square \)

The preceding lemma directly implies the next result.

**Lemma 4.4** Let \( u \in \mathbb{W}^{s(\cdot), p(\cdot)}(\Omega) \cap \mathbb{T}^{s(\cdot), p(\cdot)}(\Omega) \) be a weak solution to (1.1) such that \( u \in L^\infty(\Omega') \) for some \( \Omega' \Subset \Omega \). If \( r \in (0, 1) \) satisfies (4.6) and \( u \geq 0 \) in \( B_r = B_r(x_0) \subset \Omega' \), then for any \( B_\rho = B_\rho(x_0) \) with \( 0 < \rho < r/2 \), we have

\[ \int_{B_\rho} |v - (v)_{B_\rho}|^{p_2} \, dx \leq c \rho^{(s_1 - s_2)n + p_1 - p_2} + c \max \{d, d^{-1}\} p_4^{-4} p_3 \rho^{s_1 p_1 - s_4 p_4} \]

for some \( c = c(n, s^\pm, p^\pm, \Lambda) > 0 \), where

\[ v := \min\{\ln(a + d) - \ln(u + d), \ln b\} \] with \( a, d > 0 \) and \( b > 1 \),

and \( s_i \) and \( p_i \) \((i = 1, 2, 3, 4)\) are given in (4.7) and (4.8).

**Proof** We first notice that

\[ |v(x) - v(y)| \leq |\ln(u(x) + d) - \ln(u(y) + d)| = |\ln \left( \frac{u(x) + d}{u(y) + d} \right)|. \]

From this inequality and Hölder’s inequality, we have

\[ \int_{B_\rho} |v - (v)_{B_\rho}|^{p_2} \, dx \leq \int_{B_\rho} \int_{B_\rho} |v(x) - v(y)|^{p_2} \, dy \, dx \]
\[ \leq c \rho^{s_1 p_1 - n} \int_{B_\rho} \int_{B_\rho} \frac{|v(x) - v(y)|^{p_2}}{|x - y|^{n + s(x, y)p(x, y)}} \, dy \, dx \]
\[ \leq c \rho^{s_1 p_1 - n} \int_{B_\rho} \int_{B_\rho} \left| \ln \left( \frac{u(x) + d}{u(y) + d} \right) \right|^{p_2} K(x, y) \, dy \, dx. \]

Therefore, by Lemma 4.3, we have the conclusion. \( \square \)

**Remark 4.5** The estimate in Lemma 4.2 with the “+”(resp. “−”) sign still holds for subsolutions(resp. supersolutions) \( u \in \mathbb{W}^{s(\cdot), p(\cdot)}(\Omega) \cap \mathbb{T}^{s(\cdot), p(\cdot)}(\Omega) \) to (1.7). On the other hand, the estimates in Lemmas 4.3 and 4.4 with the “+”(resp. “−”) sign still hold for supersolutions \( u \in \mathbb{W}^{s(\cdot), p(\cdot)}(\Omega) \cap \mathbb{T}^{s(\cdot), p(\cdot)}(\Omega) \) to (1.7).
5 Local boundedness

In this section, we prove the local boundedness of the weak solutions to (1.1).

**Proof of Theorem 1.1** Since \( \lim_{r \to 0} \omega(r) = 0 \), one can find \( r > 0 \) satisfying that (4.1) and

\[
\omega_p(r) \leq \min \left\{ \frac{s^-}{4}, \frac{3s^-(p^-)^2}{4n}, \frac{p^-}{4} \right\}. \tag{5.1}
\]

Fix any \( B_{2r} = B_{2r}(x_0) \subseteq \Omega \), and set

\[
s_1 := s_{B_{2r}}, \quad s_2 := s_{B_{2r}}, \quad p_1 := p_{B_{2r}}, \quad p_2 := p_{B_{2r}}.
\]

Note that if \( s_1 p_1 > n \) then \( u \in W^{s_1, p_1}(B_r) \subseteq L_\infty(B_r) \) by Lemma 2.2 and [27, Theorem 8.2], where \( \tilde{s} \in (0, s) \) is arbitrary satisfying \( \tilde{s}p_1 > n \). Therefore, we assume that \( s_1 p_1 \leq n \).

We only prove \( \text{ess sup}_{B_{r/2}} u_+ < \infty \). Then, since \( -u \) is also a weak solution to (1.7), we also obtain \( \text{ess sup}_{B_{r/2}} u_- < \infty \).

Define

\[
\sigma := \max \{ 2s^- - 1, \frac{3}{4} s^- \} \quad (\iff s^- - \sigma = \min \{ 1 - s^- - \frac{1}{4} s^- \}).
\]

Then we immediately see that \( \frac{3}{4} s^- \leq \sigma < s^- \). Moreover, by (5.1), we have \( p_2 - p_1 \leq \frac{3}{4} s^- p_1 \) hence \( p_2 < (p_1)^*_{\frac{3}{4} s^-} \leq (p_1)^*_{\sigma} \).

Note that \( n - \sigma p_1 > n - s^- p_1 \geq 0 \) hence \( (p_1)^*_{\sigma} := \frac{n p_1}{n - \sigma p_1} \) is well defined.

Let \( r/2 \leq r_1 < r_2 \leq r \), and \( w_k := (u - k)_+ \) with \( k \geq 0 \). Then by Hölder’s inequality and Lemma 2.3, we have

\[
\int_{B_{r_1}} w_k^{p_2} \, dx \leq \left( \frac{|A^+(k, r_1)|}{|B_r|} \right)^{1 - \frac{p_2}{p_1}} \left( \int_{B_{r_1}} w_k^{(p_1)^*_{\sigma}} \, dx \right)^{\frac{p_2}{(p_1)^*_{\sigma}}}
\]

\[
\leq c \left( \frac{|A^+(k, r_1)|}{|B_r|} \right)^{1 - \frac{p_2}{(p_1)^*_{\sigma}}} \left[ r_\sigma p_1 \int_{B_{r_1}} \int_{B_{r_1}} \frac{|w_k(x) - w_k(y)|^{p_1}}{|x - y|^{n + \sigma p_1}} \, dy \, dx + \int_{B_{r_1}} w_k^{p_1} \, dx \right]^{\frac{p_2}{p_1}}.
\]

We first estimate the double integral on the right hand side. Applying Lemma 2.2 to \( v = w_k \), \( \Omega = B_{r_1} \), \( s_1(x, y) \equiv \sigma \), \( s_2(x, y) \equiv s(x, y) \), \( p_1(x, y) \equiv p_1 \), \( p_2(x, y) \equiv p(x, y) \), and Lemma 4.2 to \( \rho = r_1 \) and \( r = r_2 \),

\[
Q_{\sigma, p_1}(w_k, B_{r_1}) \leq c Q_{s_1(-), p_1(-)}(w_k, B_{r_1}) + c |A(k, r_1)|
\]

\[
\leq c \frac{r_2^{(1-s_2)p_2}}{(r_2 - r_1)^{p_2}} \left( \int_{B_{r_2}} w_k^{p_2} \, dx + |A(k, r_2)| \right)
\]

\[
+ c \frac{r_2^{n+s_2 p_2}}{(r_2 - r_1)^{n+s_2 p_2}} \left( \int_{B_{r_2}} w_k \, dx \right) T(w_k, r_1, r_1).
\]

Note that when applying Lemma 2.2 we use the facts that \( r_1 \leq 1 \) and \( d_1 \geq s^- - \sigma \). Therefore, from the preceding two estimates and using the facts that \( r_2^{(1-s_2)p_2 + \sigma p_1} \leq 1, w_k \leq u_+ \) and \( r/2 \leq r_1 \leq r \), we have

\[
\int_{B_{r_1}} w_k^{p_2} \, dx \leq c \left( \frac{|A^+(k, r_2)|}{|B_r|} \right)^{1 - \frac{p_2}{(p_1)^*_{\sigma}}} \left[ r_2^{(1-s_2)p_2 + \sigma p_1} \left( \frac{r_2^{(1-s_2)p_2 + \sigma p_1}}{(r_2 - r_1)^{p_2}} \left( \int_{B_{r_2}} w_k^{p_2} \, dx + |A(k, r_2)| \right) \right)
\]

\[
+ \frac{r_2^{n+s_2 p_2 + \sigma p_1}}{(r_2 - r_1)^{n+s_2 p_2}} \left( \int_{B_{r_2}} w_k \, dx \right) T(u_+, \frac{r}{2}, r_1) \right]^{\frac{p_2}{p_1}}.
\]
Let $0 < h < k$. Then we notice that for any $\rho \in [\frac{r}{2}, r]$,

$$
\frac{|A^+(k, \rho)|}{|B_r|} \leq \frac{1}{|B_r|} \int_{A^+(k, \rho)} \frac{w_h^{p_2}}{(k - h)^{p_2}} \, dx \leq \frac{1}{(k - h)^{p_2}} \int_{B_r} w_h^{p_2} \, dx,
$$

$$
\int_{B_r} w_k^{p_2} \, dx \leq \int_{B_r} w_h^{p_2} \, dx \quad \text{and} \quad \int_{B_r} w_k \, dx \leq \frac{1}{(k - h)^{p_2-1}} \int_{B_r} w_h^{p_2} \, dx.
$$

Hence,

$$
\int_{B_r} w_k^{p_2} \, dx \leq \frac{c}{(k - h)^{p_2(1 - \frac{p_1}{p_2} + \sigma_0)}} \left( \frac{r^{(1-\sigma_2)p_2+\sigma_1}}{(r_2 - r_1)^{p_2-1}} \left( \int_{B_{r_2}} w_h^{p_2} \, dx \right)^{1+\sigma_0} \right),
$$

where

$$
\sigma_0 := \frac{p_2}{p_1} - \frac{p_2}{(p_1)^*} = \frac{\sigma p_2}{n}.
$$

Now we define for $i = 0, 1, 2, \ldots$,

$$
k_i := M(1 - 2^{-i}) \quad \text{with} \quad M > 0, \quad \rho_i := (1 + 2^{-i}) \frac{r}{2}, \quad Y_i := \int_{B_{\rho_i}} w_k^{p_2} \, dx.
$$

By (5.2) with $k = k_i+1$, $h = k_i$, $r_1 = \rho_i$ and $r_2 = \rho_i+1$, we have

$$
Y_{i+1} \leq c \left( \frac{\sigma}{M} \right)^{p_2(1 - \frac{p_1}{p_2} + \sigma_0)} \left[ 2^{p_2} r^{\sigma p_1 - s_2/p_2} \left( 1 + \frac{2^{p_2} r}{M^{p_2}} \right) \right] + \frac{2^{(n + s_2/p_2 + p_2 - 1) \sigma p_1 T(u_+, \frac{r}{2}, r)}}{M^{p_2-1}} Y_i^{1+\sigma_0}.
$$

Fix an arbitrary $\delta \in (0, 1]$. We choose $M > 0$ such that

$$
M \geq M_1 := \delta \left( r^{s_2/p_2} T(u_+, \frac{r}{2}, r) \right)^{\frac{1}{p_2-1}} + \delta \frac{2^{p_2-1}}{p_2}.
$$

Then we have

$$
Y_{i+1} \leq c_0 \delta^{-p_2+1} \rho^{\sigma p_1 - s_2/p_2} M^{-p_2(1 - \frac{p_3}{p_1} + \sigma_0)} \alpha Y_i^{1+\sigma_0}
$$

for some $c_0, \alpha > 0$ depending only on $n, s^\pm, p^\pm$ and $\Lambda$. Furthermore, if we choose $M > 0$ such that

$$
M \geq M_2 := c_0 \delta^{-\frac{p_2-1}{\sigma_0}} \frac{r^{\sigma p_1 - s_2 p_2}}{2^{\sigma_0} \left( \int_{B_r} u_{-}^{p_2} \, dx \right)} \left( \frac{p_2}{p_2(1 - \frac{p_3}{p_1} + \sigma_0)} \right)^{\frac{p_2}{p_2(1 - \frac{p_3}{p_1} + \sigma_0)}}
$$

then

$$
Y_0 = \int_{B_{\frac{r}{2}}} u_+^{p_2} \, dx \leq c_0 \delta^{-\frac{p_2-1}{\sigma_0}} \frac{r^{\sigma p_1 - s_2 p_2}}{M \frac{p_2(1 - \frac{p_3}{p_1} + \sigma_0)}{\sigma_0}} \frac{2^{\sigma_0}}{\sigma_0}.
$$

Therefore, by Lemma 2.5 we get $\lim_{i \to \infty} Y_i = 0$ which implies

$$
\text{ess sup} u_+ \leq M.
$$
where $M$ can be chosen as $M = M_1 + M_2$ so that

$$
M \leq c \left[ \delta \frac{p_2-1}{p_0} r^{-\frac{p_1-2s}{p_0}} \left( \int_{B_r} |u|^{p_2} \, dx \right) \right]^{\frac{p_0}{p_2(1-\frac{\sigma}{p_1} + \sigma_0)}} + \delta \left[ r^{s_2/p_2} T(u_+, \frac{r}{2}, r) \right]^{\frac{1}{p_2-1}} + \delta \frac{p_2-1}{p_2}
$$

for some $c = c(n, \delta, s, p, \Lambda) > 0$. \hfill \Box

**Remark 5.1** In the above theorem, we can also obtain the following $L^\infty$-estimate:

$$
\|u\|_{L^\infty(B_{r/2})} \leq c \left[ \delta \frac{p_2-1}{p_0} r^{-\frac{p_1-2s}{p_0}} \left( \int_{B_r} |u|^{p_2} \, dx \right) \right]^{\frac{p_0}{p_2(1-\frac{\sigma}{p_1} + \sigma_0)}} + \delta \left[ r^{s_2/p_2} T(|u|, \frac{r}{2}, r) \right]^{\frac{1}{p_2-1}} + \delta \frac{p_2-1}{p_2},
$$

where $\delta \in (0, 1]$ is arbitrary. (Note that we can also have almost the same estimate in the case $s_1 p_1 > n$ by using almost the same argument in the above proof.) It looks quite complicated. If we assume that $p(\cdot, \cdot)$ is log-Hölder continuous on the diagonal region $\{(x, x) : x \in \Omega\}$, that is, it satisfies (1.8) without $\omega_p(r)$, then for every $B_{2r} \subseteq \Omega$ with $r > 0$ satisfying (4.1), (5.1) and $\omega_p(r) \leq \frac{\ln 2}{\ln(\theta \cdot \cdot \cdot, p_{\cdot \cdot} (1, \Omega) + \delta + 1)}$ we have

$$
\|u\|_{L^\infty(B_{r/2})} \leq c \delta \frac{p_2-1}{p_0} r^{-\frac{p_1-2s}{p_0}} \left( \int_{B_r} |u|^{p_2} \, dx \right) \frac{1}{p_2} + \delta \left[ r^{s_2/p_2} T(|u|, \frac{r}{2}, r) \right]^{\frac{1}{p_2-1}} + \delta \frac{p_2-1}{p_2}.
$$

Moreover, if $s(\cdot, \cdot) \equiv s$ and $p(\cdot, \cdot) \equiv p$ with $sp < n$, then we can choose $\sigma = s$ in the preceding proof and obtain the last estimate without the term $\delta \frac{p_2-1}{p_2}$, that is, we have

$$
\|u\|_{L^\infty(B_{r/2})} \leq c \delta \frac{(p-1)\ln} {sp^2} \left( \int_{B_r} |u|^p \, dx \right) \frac{1}{p} + \delta \left[ r^{s_2/p_2} T(|u|, \frac{r}{2}, r) \right]^{\frac{1}{p-1}}.
$$

This is exactly the same as the $L^\infty$-estimate for the nonlocal equations with constant powers obtained in [26, Theorem 1.1] (Note that the definition of the nonlocal tail in [26] is slightly different from the one in this paper).

**Remark 5.2** Instead of the continuity assumption of $p(\cdot, \cdot)$ on the diagonal region in Theorem 1.1, we can assume that the oscillation of $p(\cdot, \cdot)$, $\omega_p(r)$, is sufficiently small. Precisely, we may assume that $\omega_p(\cdot)$ satisfies (4.1) and (5.1) for some $r < \frac{1}{2}$.

### 6 Hölder continuity

Finally, we prove Theorem 1.2. Therefore, we assume that $s(\cdot, \cdot)$ and $p(\cdot, \cdot)$ satisfy (1.8) in this section.

Fix any $\Omega' \subseteq \Omega$. Note that since we have seen $u \in L^\infty(\Omega)$ in the preceding section, $\|u\|_{L^\infty(\Omega')} < \infty$. Let $\sigma = \sigma(n, \Lambda, s, p, c_{LH}) \in (0, \frac{1}{2})$ be a small positive number that will be determined later in (6.26) and (6.32). Suppose that $r > 0$ satisfies (4.1), (4.6), (5.1),

$$
r \leq \sigma^{s_1/p_1 - 1}, \quad \omega_r(r) \leq \frac{\ln 2}{\ln(1/\sigma)},
$$

$$
\omega_p(r) \leq \min \left\{ \frac{2s^+ p^+ - (1 - s^+) p^+}{2s^+}, \frac{\ln 2}{\ln(\|u\|_{L^\infty(\Omega')} + 1/\sigma + R_{\Omega'})} \right\},
$$

(6.2)
where $R_{\Omega'} := \sup \{ |x| : x \in \Omega' \} < \infty$. Note that the above inequalities yield
\[
\sigma^{-\omega_s(r)} \leq 2 \quad \text{and} \quad (|v|)_{T(s(\cdot, \cdot), p(\cdot, \cdot), \Omega')} + \|u\|_{L^\infty(\Omega')} + R_{\Omega'} + 1/\sigma^{\omega_p(r)} \leq 2. \tag{6.3}
\]

Let $B_r = B_r(x_0) \subset \Omega'$, and define
\[
K_0 := 2\|u\|_{L^\infty(B_r)} + \left[ r^{\omega_0} T(|u| + \|u\|_{L^\infty(B_r)}, x_0, r, r) \right]^{1/r - 1} + 1, \tag{6.4}
\]
where
\[
s_0 := s(x_0, x_0) \quad \text{and} \quad p_0 := p(x_0, x_0).
\]

For $j = 0, 1, 2, \ldots$, set
\[
r_j := \sigma^j \frac{r}{2}, \quad B_j := B_{r_j}(x_0), \quad 2B_j := B_{2r_j}(x_0),
\]
\[
s_{j, 1} := s_{2B_j}^-, \quad s_{j, 2} := s_{2B_j}^+, \quad p_{j, 1} := p_{2B_j}^- \quad \text{and} \quad p_{j, 2} := p_{2B_j}^+.
\]

In particular, we write
\[
p_1 := p_{0, 1} \quad \text{and} \quad p_2 := p_{0, 2}.
\]

Note that by (1.8)
\[
\frac{1}{r} r_j^{-\omega_s(2r)} \leq c \quad \text{and} \quad \frac{1}{r} r_j^{-\omega_p(2r)} \leq c \tag{6.5}
\]
and that by (2.2) and (6.3)
\[
T(|u| + \|u\|_{L^\infty(B_r)}, x_0, r, r) \leq \left( 1 + \frac{R_{\Omega'} + 1}{r} \right)^{n+s^+} p^+ \left[ v + \|u\|_{L^\infty(\Omega')} \right]_{T(s(\cdot, \cdot), p(\cdot, \cdot), \Omega')}
\leq c \left( \frac{R_{\Omega'} + 1}{r} \right)^{n+s^+} p^+ \left[ v \right]_{T(s(\cdot, \cdot), p(\cdot, \cdot), \Omega')} + \|u\|^p_{L^\infty(\Omega')} + 1
\]
and so
\[
K_0^{p_2-p_1} \leq K_0^{\omega_p(r)} \leq c. \tag{6.6}
\]

Now we prove an oscillation decay estimate.

**Lemma 6.1** Under the above setting, we further suppose that
\[
s_1 p_1 < n + \frac{s^- p^-}{4} \tag{6.7}
\]
Then we have
\[
\theta(r_j) := \sup_{B_j} u - \inf_{B_j} u \leq K_j := \sigma^{\alpha_j} K_0 \quad \text{for all} \quad j = 0, 1, 2, \ldots, \tag{6.8}
\]
for some $\alpha = \alpha(n, \Lambda, s^\pm, p^\pm, c_{LH}) > 0$ satisfying that
\[
\alpha \leq \min \left\{ \frac{s^- p^-}{2(p^+ - 1)}, \ln \frac{1}{\sigma}, \ln \left( 1 - \sigma \frac{s^+ p^+}{p^+ - 1} \right), s^- \right\}. \tag{6.9}
\]
Proof Step 1 (Induction). We first observe from (6.5) and (6.6) that
\[ \sup_{x_1,x_2,y_1,y_2 \in B_j} K_j^{p(x_1,y_1) - p(x_2,y_2)} \leq \sigma^{-\alpha_j \omega_p(r_j)} K_0^{\alpha_p(r_j)} \leq r_j^{-\alpha_p(r_j)} K_0^{\alpha_p(r_j)} \leq c, \] (6.10)
from (6.9) that
\[ s^p - \alpha(p^+ - 1) \geq \frac{s^p}{2}, \] (6.11)
\[ \sigma^\alpha \geq \frac{1}{2}, \] (6.12)
\[ \sigma^\alpha - 1 + \sigma \frac{r(x,y)p(x,y)}{p^+ - 1} \alpha \geq \sigma^\alpha - 1 + \sigma \frac{r^+ p^+}{p^+ - 1} \geq 0, \] for all \( x, y \in \mathbb{R}^n \). (6.13)

We prove the lemma by induction. Since
\[ \sup_{B_0} u - \inf_{B_0} u \leq 2\|u\|_{L^\infty(B_0)} \leq K_0, \] (6.8)
is true when \( j = 0 \). Now assume that
\[ \theta(r_i) \leq K_i \] for all \( i = 0, 1, 2, \ldots, j \), (6.14)
and then we will prove that
\[ \theta(r_{j+1}) \leq K_{j+1}. \] (6.15)

Hence, from now on, \( j \in \mathbb{N} \) is fixed. Without loss of generality, we shall assume that
\[ \theta(r_{j+1}) \geq \frac{1}{2} K_{j+1}. \] Then this and (6.12) imply that
\[ \theta(r_j) \geq \theta(r_{j+1}) \geq \frac{1}{2} K_{j+1} = \frac{1}{2} \sigma^\alpha K_j \geq \frac{1}{4} K_j. \] (6.16)

We notice that one of the following two cases must holds:
\[ \frac{|2B_{j+1} \cap \{u \geq \inf_{B_j} u + \theta(r_j)/2\}|}{|2B_{j+1}|} \geq \frac{1}{2}, \]
\[ \frac{|2B_{j+1} \cap \{u \leq \inf_{B_j} u + \theta(r_j)/2\}|}{|2B_{j+1}|} \geq \frac{1}{2}. \]
If the first case is true we define \( u_j := u - \inf_{B_j} u \). On the other hand, if the second case is true we define \( u_j := \sup_{B_j} u - u = \theta(r_j) - (u - \inf_{B_j} u) \). Note that in any case, \( u_j \geq 0 \) in \( B_j \), \( u_j \) is also a weak solution to (1.1),
\[ \frac{|2B_{j+1} \cap \{u_j \geq \theta(r_j)/2\}|}{|2B_{j+1}|} \geq \frac{1}{2}, \] (6.17)
and
\[ \sup_{B_i} u_j \leq \theta(r_i) \leq K_i \] for all \( i = 0, 1, \ldots, j \). (6.18)

Finally, we define
\[ d_j := \varepsilon \theta(r_j) \] with \( \varepsilon := \sigma \frac{r^+ p^+ - 1}{p^+ - 1} \leq \sigma \frac{r^- p^-}{2(p+1)} \leq c \). (6.19)
Note that by (6.14) with \(i = j\), (6.16), (6.10) and (6.3)

\[
\max\{d_j, d_j^{-1}\}^{p_j, 2 - p_j, 1} \leq c \varepsilon^{-(p_j, 2 - p_j, 1)} \max\{K_j, K_j^{-1}\}^{p_j, 2 - p_j, 1} \leq c \sigma^{-\omega_p(r)^{\alpha + p^+ p^+} p_j, 2 - p_j, 1} \leq c.
\]

(6.20)

Step 2 (Tail estimates). We first estimate \(\frac{r^{p_0}}{d_j^{\alpha - 1}} T (|u_j| + \|u_j\|_{L^\infty(B_j)}, r_j, r_j)\). Here, we note that \(\|u_j\|_{L^\infty(B_j)} = \theta (r_j)\). From (6.18), (6.14), (6.19), (6.16) and (6.10) with the the definitions of \(u_j\), the tail \(T\) and \(K_0\) in (6.4), we have that for every \(x \in B_j\),

\[
d_j^{-(p_0 - 1)}\left(\frac{r_j}{d_j}\right)^{p_0 - \alpha} T (|u_j| + \|u_j\|_{L^\infty(B_j)}, r_j, r_j)
\]

\[
= \frac{1}{d_j^{p_0 - 1}} \left[ \sum_{i=1}^{j} \int_{B_{i-1} \setminus B_{i}} \frac{(u_j(y) + \theta (r_j))^{p_j, x, y - 1}}{|y - x|^{p_j, x, y} p(x, y)} dy + \int_{\mathbb{R}^n \setminus B_0} \frac{(u_j(y) + \theta (r_j))^{p_j, x, y - 1}}{|y - x|^{p_j, x, y} p(x, y)} dy \right]
\]

\[
\leq \frac{c}{(\varepsilon K_j)^{p_0 - 1}} \left[ \sum_{i=1}^{j} \int_{B_{i-1} \setminus B_{i}} K_j^{p_0 - 1 + p_j, x, y - 1} \frac{(u_j(y) + \theta (r_j))^{p_j, x, y - 1}}{|y - x|^{p_j, x, y} p(x, y)} dy + \int_{\mathbb{R}^n \setminus B_0} \frac{(u_j + \|u\|_{L^\infty(B_j)})^{p_j, x, y - 1}}{|y - x|^{p_j, x, y} p(x, y)} dy \right]
\]

\[
\leq c \varepsilon^{-\alpha} \left[ \sum_{i=1}^{j} r_i^{\alpha - 1, 2} \sum_{j=1}^{i} \sigma^{(p_0 - 1)(i - j)} + r^{-\alpha} \sum_{i=1}^{j} \sigma^{(p_0 - 1)(i - j)} \right]
\]

\[
\leq c \varepsilon^{-\alpha} \left[ \sum_{i=1}^{j} r_i^{(p_0 - 1)(i - j)} \right] + c \sum_{i=1}^{j} \sigma^{(p_0 - 1)(i - j)}.
\]

In the last inequality we used the inequality

\[
r_i^{\alpha - 1, 2} \max \{d_j, d_j^{-1}\}^{p_j, 2 - p_j, 1} \leq (\varepsilon r^{-1})^{\alpha - 1, 2} \leq (\varepsilon r^{-1})^{\alpha - 1, 2} \leq (\varepsilon r^{-1})^{\alpha - 1, 2}.
\]

which follows from (6.3) and (6.5). Therefore, recalling the definition of \(\varepsilon\) in (6.19) and using (6.11) and the fact that \(\sigma \in (0, \frac{1}{4})\), we get

\[
\frac{r_j^{p_0}}{d_j^{\alpha - 1}} T (|u_j| + \|u_j\|_{L^\infty(B_j)}, r_j, r_j)
\]

\[
\leq c \varepsilon^{-\alpha} \sum_{i=1}^{j} \sigma^{(p_0 - 1)(i - j)} \leq c \sum_{i=1}^{\infty} \sigma^{(p_0 - 1)(i - j)} \leq c.
\]

(6.21)
We next estimate $r_j^{s_0 p_0} T(\theta(r_j), r_j, r_j)$ in a similar way. By (6.14) with $i = j$, (6.4), (6.6), and (6.10), we have that for every $x \in B_j$,

\[
\int_{\mathbb{R}^n \setminus B_j} \frac{\theta(r_j)^{p(x,y) - 1}}{|y - x_0|^{n + s(x,y)p(x,y)}} \, dy
\]

\[= \sum_{i=1}^j \int_{B_{i-1} \setminus B_i} \frac{\theta(r_j)^{p(x,y) - 1}}{|y - x_0|^{n + s(x,y)p(x,y)}} \, dy + \int_{\mathbb{R}^n \setminus B_0} \frac{\theta(r_j)^{p(x,y) - 1}}{|y - x_0|^{n + s(x,y)p(x,y)}} \, dy
\]

\[\leq c \sum_{i=1}^j \int_{B_{i-1} \setminus B_i} K_i^{p(x,y) - 1} \frac{|y - x_0|^{n + s_i - 1, 2 p_i - 1}}{r_j^{1 - s_i - 1, 2 p_i - 1} + c r_j^{- \sigma_0 p_0} K_i^{p_0 - 1}} \leq \sum_{i=1}^j c K_i^{p_0 - 1} r_i^{- \sigma_0 p_0},
\]

hence

\[r_j^{s_0 p_0} T(\theta(r_j), r_j, r_j) \leq (\sigma^i r_0)^{s_0 p_0} \sum_{i=1}^j (\sigma^{i-1} \alpha K_0)^{p_0 - 1} (\sigma^i r_0)^{- s_0 p_0}
\]

\[= \sigma^{- \alpha(p_0 - 1)} (\sigma^i \alpha K_0)^{p_0 - 1} \sum_{i=1}^j \sigma^{(s_0 p_0 - \alpha(p_0 - 1))(j - i)}
\]

\[(6.8), (6.11) \leq \sigma^{- \alpha(p_0 - 1)} K_j^{p_0 - 1} \sum_{i=0}^\infty 4^{- \frac{\alpha - 2}{2} i}
\]

\[(6.16) \leq c \sigma^{- \alpha(p_0 - 1)} \theta(r_j)^{p_0 - 1}.
\]

**Step 3 (Density estimates).** Define

\[v := \min \left\{ \left\lfloor \ln \left( \frac{\theta(r_j)/2 + d_j}{u_j + d_j} \right) \right\rfloor, k \right\},
\]

where $d_j$ is given in (6.19), and a constant $k > 0$ will be chosen later in (6.25). Applying Corollary 4.4 to $a = \theta(r_j)/2$, $b = \exp(k)$, $d = d_j$ and $\rho = 2 r_j + 1$ and $r = r_j$ and setting $\bar{p}_{j+1} := \sup_{x, y \in B_{4 r_j + 1}} p(x, y)$, we have

\[
\int_{\bar{B}_{j+1}} |v - (v)_{2 B_{j+1}}| \bar{p}_{j+1} \, dx
\]

\[\leq c r_j^{- \omega_x(4 r_j + 1) p - \omega_p(4 r_j + 1)} + c \max\{d, d^{-1}\omega_p(r_j)(\sigma r_j)^{- \omega_p(r_j) s - \omega_x(r_j) p -}
\]

\[+ c r_j^{- \omega_x(4 r_j + 1) s - \omega_x(4 r_j + 1) p -} \max\{d, d^{-1}\omega_p(r_j) r_j^{- s_0 p_0} \frac{r_j^{s_0 p_0}}{d_j^{p_0 - 1} T(|u_j| + \|u_j\|_{L^\infty(B_j)}, r_j, 4 r_j + 1)}
\]

\[(6.3), (6.5), (6.20) \leq c + c r_j^{- \omega_x(4 r_j + 1) s - \omega_x(4 r_j + 1) p -} \max\{d, d^{-1}\omega_p(r_j) r_j^{- s_0 p_0} \frac{r_j^{s_0 p_0}}{d_j^{p_0 - 1} T(|u_j| + \|u_j\|_{L^\infty(B_j)}, r_j, 4 r_j + 1)}
\]

\[(6.21) \leq c.
\]

In particular, we have

\[
\int_{\bar{B}_{j+1}} |v - (v)_{2 B_{j+1}}| \, dx \leq c.
\]

(6.24)
By the definition of $v$ in (6.23) and (6.17), we have
\[
k = \frac{1}{|2B_{j+1} \cap \{u_j \geq \theta(r_j)/2\}|} \int_{2B_{j+1} \cap \{v=0\}} [k-v] \, dx \leq 2 \int_{2B_{j+1}} [k-v] \, dx = 2[k-(v)_{2B_{j+1}}].
\]
This and (6.24) imply
\[
\frac{|2B_{j+1} \cap \{v=k\}|}{|2B_{j+1}|} \leq \frac{2}{k|2B_{j+1}|} \int_{2B_{j+1} \cap \{v=k\}} [k-(v)_{2B_{j+1}}] \, dx \leq \frac{2}{k} \int_{2B_{j+1}} |v-(v)_{2B_{j+1}}| \, dx \leq \frac{c}{k}.
\]

Here, we choose
\[
k = \ln \left( \frac{\theta(r_j)/2 + \epsilon \theta(r_j)}{3\epsilon \theta(r_j)} \right) = \ln \left( \frac{\theta(r_j)/2 + d_j}{2\epsilon \theta(r_j) + d_j} \right),
\]
and assume that $\sigma \in (0, \frac{1}{4})$ satisfies
\[
\sigma \leq 6^{-\frac{4(p^-+1)}{4(p+1)}} \quad \left( \iff \ln 6 \leq \frac{s^- p^-}{4(p^+ - 1)} \ln \left( \frac{1}{\sigma} \right) \right).
\]

Then we see that \( \{v = k\} = \{u_j \leq 2\epsilon \theta(r_j)\} = \{u_j \leq 2d\} \) and
\[
k \geq \ln \left( \frac{1}{6\epsilon} \right) \geq \frac{s^- p^-}{2(p^+ - 1)} \ln \left( \frac{1}{\sigma} \right) - \ln 6 \geq \frac{s^- p^-}{4(p^+ - 1)} \ln \left( \frac{1}{\sigma} \right).
\]

Therefore, combining the above results we have
\[
\frac{|2B_{j+1} \cap \{u_j \leq 2d_j\}|}{|2B_{j+1}|} = \frac{|2B_{j+1} \cap \{v = k\}|}{|2B_{j+1}|} \leq \frac{c}{k} \leq \frac{c_0}{\ln \left( \frac{1}{\sigma} \right)}
\]
for some $c_0 > 0$ depending only on $n, \Lambda, s^\pm, p^\pm$, and $c_{LH}$, but independent of $\sigma$.

**Step 4 (Proof of (6.15)).** Finally, we complete the proof by proving (6.15). Set
\[
\tilde{B} = 2B_{j+1}, \quad \tilde{s}_1 = s_{j+1,1}, \quad \tilde{s}_2 = s_{j+1,2}, \quad \tilde{p}_1 = p_{j+1,1}, \quad \tilde{p}_2 = p_{j+1,2}, \quad d = d_j.
\]

For $i = 0, 1, 2, \ldots$, set
\[
\rho_i := (1 + 2^{-i})r_j, \quad B^i := B_{\rho_i},
\]
\[
k_i := (1 + 2^{-i})d, \quad w_i := (k_i - u_j)_{+},
\]
\[
A_i := \frac{|B^i \cap \{u_j < k_i\}|}{|B^i|} = \frac{|B^i \cap \{w_i > 0\}|}{|B^i|}.
\]

Notice that for every $i = 0, 1, 2, \ldots$,
\[
r_j + 1 < \rho_i + 1 \leq \rho_i \leq 2r_j + 1, \quad d \leq k_i + 1 \leq k_i \leq 2d \quad \text{and} \quad 0 \leq w_i \leq k_i \leq 2d.
\]

We will prove that $A_i \to 0$ as $i \to \infty$, by choosing $\sigma$ sufficiently small. Set
\[
t := \frac{s^-}{2}.
\]

Note that
\[
n - t \tilde{p}_1 \geq n - \frac{s_1}{2}p_2 \geq n - \frac{s_1}{2}p_1 - \frac{s^+}{2} \omega_p(r) \geq n - \frac{s_1 p_1}{2} - \frac{s_1 p_1}{4} > 0.
\]
By Lemma 2.3 and Lemma 2.2
\[
A_{i+1}^{(n-i)p_1}(k_i - k_{i+1})^{\tilde{p}_1} = \left( \frac{1}{|B^{i+1}|} \int_{B^{i+1} \cap \{|u_j < k_{i+1}\}} (k_i - k_{i+1})^{\frac{n-ip_1}{n}} \, dx \right)^{(n-i)p_1}
\]
\[
\leq c \left( \int_{B^{i+1}} w_i^{(n-ip_1)/n} \, dx \right)
\]
\[
\leq c \frac{\rho_i^{p_1}}{|B^{i+1}|} \bar{q}_i (w_i, B^{i+1}) + c \int_{B^{i+1}} w_i^{\tilde{p}_1} \, dx
\]
\[
\leq c \frac{r_j^{p_1}}{|B^{i+1}|} \bar{q}_{s(\cdot), p(\cdot)}(w_i, B^{i+1}) + c \left( r_{j+1}^{\tilde{p}_1} + d^{\tilde{p}_1} \right) A_i
\]
\[
\leq c \frac{r_j^{p_1}}{|B^{i+1}|} \bar{q}_{s(\cdot), p(\cdot)}(w_i, B^{i+1}) + cd^{p_0} A_i.
\]

In the last inequality we used (6.20) and the following estimate:
\[
r_{j+1}^{\tilde{p}_1} = \sigma^{s(j+1)} r_{j+1} \leq \sigma^{s^{-} j} \sigma^{s^{-}} r_{j+1} \leq c \sigma^{s^{-} j} \sigma^{s^{-}} r_{j+1}^{-} \sigma^{s^{-} r} \leq c \sigma^{s^{-} j} \sigma^{s^{-} r}^{-} K_0 \leq cd.
\]

We then estimate the double integral on the right hand side. Applying the first inequality in (4.2) to \(u = u_j, k = k_i, \rho = \rho_{i+1} \text{ and } r = \rho_i,\)
\[
\bar{q}_{s(\cdot), p(\cdot)}(w_i, B^{i+1})
\]
\[
\leq c \left( \frac{\rho_i - \rho_{i+1}}{p_2} \right)^{2p_2} \int_{B^i} B_j \max \{w_i(x)^{p(x,y)}, w_i(y)^{p(x,y)}\} \frac{dy \, dx}{|x - y|^{n+s(x,y) + p_2 - p_1}}
\]
\[
+ c \left( \frac{\rho_i - \rho_{i+1}}{p_2} \right)^{2p_2} \left( \int_{B^i} w_i \, dx \right) T(w_i, \rho_i, \rho_i)
\]
\[
\leq c 2^{ip_2} r_{j+1}^{p_2} d^{p_0} \int_{B^i \cap \{|w_i > 0\}} \int_{B_{j+1}(x)} \frac{1}{|x - y|^{n+s(x,y) + p_2 - p_1}} \, dy \, dx
\]
\[
+ c 2^{i(n+s^p)} d|B^i \cap \{|w_i > 0\}| T(w_i, \rho_i, \rho_i)
\]

In the last inequality, we also use the fact that \(\tilde{s}_2 \tilde{p}_2 - \tilde{p}_1 < 0\) by (6.2). Moreover, for every \(x \in B^i \subset B_j,\)
\[
\int_{\mathbb{R}^n \setminus B^i} \frac{|w_i(y)|^{p(x,y) - 1}}{|y - x_0|^{n+s(x,y)p(x,y)}} \, dy
\]
\[
\leq \int_{B_j \setminus B_{j+1}} \frac{|w_i(y)|^{p(x,y) - 1}}{|y - x_0|^{n+s(x,y)p(x,y)}} \, dy + \int_{\mathbb{R}^n \setminus B_j} \frac{|w_i(y)|^{p(x,y) - 1}}{|y - x_0|^{n+s(x,y)p(x,y)}} \, dy
\]
\[
\leq c \left( \sup_{x, y \in B^i} d^{p(x,y) - 1} \right) \int_{B_j \setminus B_{j+1}} \frac{1}{|y - x_0|^{n+s(x,y)p(x,y)}} \, dy + T(\theta(j), r_j, r_j)
\]
\[
\leq cd^{p_0 - 1} r_{j+1}^{-s_j,2p_j,2} + cr_j^{-s_0 p_0} \sigma^{-\alpha(p_0 - 1)} \theta(j)^{p_0 - 1}
\]
\[
\leq cd^{p_0 - 1} (r_{j+1}^{-s_j,2p_j,2} + r_{j+1}^{-s_0 p_0}) \leq cd^{p_0 - 1} r_{j+1}^{-\tilde{s}_1 \tilde{p}_1}.
\]
Therefore we have
\[ Q_s(\cdot, \cdot, p(\cdot, \cdot))(w_i, B^{i+1}) \leq c 2^{(n+p^+)} d^p_0 |B^i \cap \{ w_i > 0 \}| r_+ \tilde{p}_1. \]
Inserting this into (6.29), we obtain
\[ A_i^{(n-\tilde{p}_1)} (k_i - k_{i+1}) \tilde{p}_1 \leq c 2^{(n+p^+)} d^p_0 A_i. \]
Moreover, by (6.20) and (6.28),
\[ A_i^{1-\frac{s-p^-}{2n}} 2^{-i+1} p^- d^p_0 \leq c A_i^{(n-p^+)} [2^{-i-1} d] \tilde{p}_1 = c A_i^{(n-p^+)} (k_i - k_{i+1}) \tilde{p}_1. \]
The preceding two estimates yield
\[ A_{i+1} \leq c_1 2^{\frac{2(n+p^+)}{2n-s^+p^-}} A_i^{1+\frac{s-p^-}{2n-s^+p^-}} \]
for some \( c_1 > 0 \) depending only on \( n, s^\pm, p^\pm, \Lambda \) and \( c_{LH} \), and independent of \( \sigma \). Therefore, by Lemma 2.5 if
\[ A_0 = \frac{|2B_{j+1} \cap \{ u_j \leq 2d \}|}{|2B_{j+1}|} \leq c_1 2^{\frac{2n-s^+p^-}{s^+p^-}} 2^{\frac{2n(p^+)(2n-s^+p^-)}{(s^-p^-)^2}}, \]
then \( A_i \to 0 \) as \( i \to \infty \) which implies
\[ |B_{j+1} \cap \{ u_j < d \}| \leq c \exp \left( -c_0 c_1 2^{\frac{2n-s^+p^-}{s^+p^-}} 2^{\frac{2n(p^+)(2n-s^+p^-)}{(s^-p^-)^2}} \right). \]
Then in view of Lemma 2.2 and [27, Theorem 8.2], we have $u \in W^{1,p_1}(B_R) \subset C^{0,\beta}(B_R)$ with \( \beta = \frac{tp_1-n}{p_1} \geq \frac{s-p^-}{8p^+} \) hence $u \in C^{0,\frac{s-p^-}{8p^+}}(B_R)$.

On the other hand, if $n - s_1 p_1 > -\frac{s-p^-}{4}$.

The preceding Lemma yields $u \in C^{0,\alpha}(B_R)$ for some $\alpha \in (0, 1)$ satisfying (6.9).

Therefore by standard covering argument we have that $u \in C^{0,\alpha}_{loc}(\Omega')$ for some $\alpha \in (0, 1)$ depending on $n, s^\pm, p^\pm, \Lambda$ and $c_{L,H}$. Since $\Omega' \subset \Omega$ is arbitrary, we have the conclusion. \( \square \)

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