Turnaround physics beyond spherical symmetry

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Abstract.
The concept of turnaround radius in an accelerating universe is generalized to arbitrarily large deviations from spherical symmetry, as needed by astronomy. As a check, previous results for small deviations from spherical symmetry are recovered.

1. Introduction: spherical symmetry
The turnaround radius in an accelerated universe has been studied in spherical symmetry as a possible way to test dark energy or even the theory of gravity [1, 2, 3] in the present accelerated era of the universe, but the concept of TR is older. In 2015, it was reported that the upper bound set by General Relativity (GR) on the turnaround radius is significantly exceeded in the galaxy group NGC 5353/4 [4, 5], but the claim was later retracted [6] because the error introduced by the non-sphericity of the system had been underestimated. What is more, one should expect a distribution of values of the turnaround radius for different astronomical systems: an excess in this quantity would be significant from the statistical point of view rather than for individual systems [7]. Currently, the observational search for the turnaround radius focusses on galaxy groups with web-like structures in their neighbouring zones, and six more groups exceeding the GR prediction for the turnaround radius have been reported [7].

Let us begin by reviewing the concept of turnaround radius in spherical symmetry. Consider an accelerated Friedmann-Lemaître-Robertson-Walker (FLRW) universe with a spherical inhomogeneity: massive test particles with zero radial initial velocity cannot collapse if they are outside a critical radius $R_c$ (turnaround radius), but they only expand. A dust shell of proper radius $R < R_c$ initially at rest collapses under its self-gravity, while one with $R > R_c$ disperses and never falls back. $R_c$ is the upper limit to the radius of spherical bound structures in an accelerated universe. On the turnaround sphere, the attraction of the local overdensity balances exactly the cosmic acceleration and a test particle experiences zero acceleration.

In a perturbed, spatially flat, FLRW universe dominated by dark energy with equation of state $P_{\text{DE}} = w\rho_{\text{DE}}$, and with line element

$$ds^2 = a^2(\eta) \left[ - (1 + 2\phi) d\eta^2 + (1 - 2\phi) (dr^2 + r^2 d\Omega^2) \right],$$

(1)
timelike radial geodesics obey

$$\ddot{R} = \frac{\ddot{a}}{a} R - \frac{GM(r)}{R^2},$$

(2)
where $R(t, r) \equiv a(t) r$ is the physical radius and

$$\mathcal{M}(r) = 4\pi \int_0^r dR R^2 \rho_{\text{total}}$$

(with $\rho_{\text{total}}$ sourcing the Newtonian potential $\phi$), which yields the turnaround radius

$$R_c = \left( \frac{3\mathcal{M}}{4(3w + 1)\pi\rho_{\text{DE}}} \right)^{1/3}$$

(other expressions apply in modified gravity [8, 9, 10, 11]). Therefore, one can test gravity, or the dark energy equation of state parameter $w$ in the $\Lambda$CDM model.

2. Turnaround surface without spherical symmetry

Now allow for arbitrarily large deviations from spherical symmetry, but assume small perturbations of the FLRW metric. The key idea consists of identifying the turnaround surface with an equipotential surface of the local metric perturbation potential with the special property that, if a test particle initially sits on this surface at rest with respect to it, it remains on it. There is a unique critical surface $S^*$ on which the two opposing forces of local attraction and cosmic expansion balance. On any other closed surface $S$ nearby, particles will collapse (if $S$ lies inside $S^*$) or disperse (if $S$ lies outside $S^*$).

The spacetime metric in the conformal Newtonian gauge is

$$ds^2 = a^2(\eta) \left\{ -(1 + 2\Phi) d\eta^2 + (1 - 2\Phi) \left[ dr^2 + r^2 d\Omega^2(\theta, \phi) \right] \right\},$$

where $\eta$ is the conformal time and $\Phi(x^i)$ is the Newtonian perturbation (since we consider structures much smaller than the Hubble radius $H^{-1}$, the time-dependence of $\Phi$ can be safely neglected). Test particles lying on $S^*$ follow timelike geodesics with proper time $\tau$ and four-tangents $u^\mu = dx^\mu/d\tau$ that satisfy

$$\frac{du^0}{d\tau} = \frac{1}{a(2\Phi + 1)} \left\{ a \eta \left[ 2\left( \Phi u^2 \right)^2 \sin^2 \theta + \Phi u^2 u^2 + \left( u^2 \right)^2 - \left( u^0 \right)^2 \right] \right\},$$

$$\frac{du^1}{d\tau} = \frac{1}{a(2\Phi - 1)} \left\{ 2u^1 u^0 a \eta (1 - 2\Phi) + a \left[ \left( u^2 \right)^2 + \left( u^3 \right)^2 \sin^2 \theta \right] \right\},$$

$$\frac{du^2}{d\tau} = \frac{1}{r^2 a(2\Phi - 1)} \left\{ 2r^2 u^2 u^0 a \eta (1 - 2\Phi) + a \left[ -2r^2 u^2 u^2 \Phi_{,\varphi} \right. \right. \right.$$}

$$\left. \left. + \left( -2\left( u^2 \right)^2 + r^2 \left( u^3 \right)^2 \sin^2 \theta + \left( u^1 \right)^2 + (u^0)^2 \right) \Phi_{,\theta} \right. \right.$$}

$$\left. + 2r \left( \left( u^3 \right)^2 \sin^2 \theta + (u^0)^2 \right) \Phi - r \left( u^3 \right)^2 \sin^2 \theta - r \left( u^2 \right)^2 \right\},$$

$$\frac{du^3}{d\tau} = \frac{1}{r^2 a(2\Phi - 1)} \left\{ 2r^2 u^2 u^0 a \eta (1 - 2\Phi) + a \left[ -2r^2 u^2 u^2 \Phi_{,\varphi} \right. \right. \right.$$}

$$\left. \left. + \left( -2\left( u^2 \right)^2 + r^2 \left( u^3 \right)^2 \sin^2 \theta + \left( u^1 \right)^2 + (u^0)^2 \right) \Phi_{,\theta} \right. \right.$$}

$$\left. + 2r \left( \left( u^3 \right)^2 \sin^2 \theta + (u^0)^2 \right) \Phi - r \left( u^3 \right)^2 \sin^2 \theta - r \left( u^2 \right)^2 \right\},$$

1 Thus, the emphasis shifts from the size of the turnaround surface to the surface itself.
\[
\frac{du^3}{d\tau} = \frac{1}{r^2 u(2\Phi - 1)} \left\{ 2r^2 u^3 u^0 a_{\gamma\eta}(1 - 2\Phi) + a \left[ (r^2 u^3)^2 + (u^1)^2 + (u^0)^2 \right] \csc^2 \theta - r^2 (u^3)^2 \right\} \Phi \dot{\varphi} \\
+ 2ru^3 \left[ -ru^1 \Phi_r - ru^2 \Phi_\theta + ru^2 \cot \theta + u^1 \right] \\
- 4ru^3 \left( ru^2 \cot \theta + u^1 \right) \Phi \] . 
\tag{10}
\]

The four-tangent is
\[
u^\mu = v_{(0)}^\mu + \delta u^\mu = \left( u_{(0)}^0 + \delta u^0, \delta u \right) = \left( \frac{1}{a}, \delta u^0, \delta u \right)
\tag{11}
\]
and the normalization \(u^\mu u_\mu = -1\) yields \(\delta u^0 = -\Phi/a\) to first order. Assuming \(O(\delta u^1) = O(\delta u^2) = O(\delta u^3)\), we have \(O(u^1) = O(\delta u^1) = O(\Phi)\) \((i = 1, 2, 3)\) and
\[
\frac{d(\delta u^i)}{d\tau} + \frac{2a_{\gamma\eta}}{a^2} \delta u^i + g^{ij} \partial_j \Phi = 0 .
\tag{12}
\]

In spherical symmetry, the turnaround sphere is an equipotential surface of \(\Phi\) but this is not sufficient to identify it. We require the extra property that, if test particles initially lay on this surface and have zero initial velocity with respect to it, they remain on it as the latter evolves.\(^2\) This is still insufficient to identify the turnaround surface because many timelike geodesics cross it: we further restrict to timelike geodesics that initially have zero velocity with respect to this surface. Since they satisfy the second order geodesic equation, assigning their initial position (on \(\Sigma_{t_0}\)) and initial velocity (at rest on \(\Sigma_{t_0}\)) specifies them completely. These massive test particles stay on \(\Sigma_{t_0}\) initially and at all later times. Finally we require that, on this surface, the Newtonian attraction balances the cosmic expansion.

Formally, the turnaround surface \(\Sigma_t\) at (comoving) time \(t\) is a two-dimensional, closed, simply connected surface that is an equipotential surface of the perturbation \(\Phi\) such that:

i) The time evolution of the surface is such that the three-dimensional components of the tangent to the timelike geodesics crossing \(\Sigma_t\) are locally proportional to the gradient \(\nabla \Phi\) (and therefore perpendicular to \(\Sigma_t\) in the three-dimensional sense):
\[
u^i|_{\Sigma_t} = \sigma(t) g^{ij} \partial_j \Phi|_{\Sigma_t} .
\tag{13}
\]

ii) A dust particle initially comoving with the surface remains on it, \(i.e.,\) if
\[
u^i|_{\Sigma_{t_0}} = \sigma(t_0) g^{ij} \partial_j \Phi|_{\Sigma_{t_0}} ,
\tag{14}
\]
then at \(t > t_0\) its 3-velocity satisfies
\[
u^i|_{\Sigma_t} = \sigma(t) g^{ij} \partial_j \Phi|_{\Sigma_t} .
\tag{15}
\]

iii) In an unperturbed FLRW universe, the (purely radial) acceleration of a massive test particle is \(\ddot{r} = \ddot{a}r/a;\) on the initial surface \(\Sigma_{t_0}\) (assumed to be convex), the acceleration of a massive test particle normal to this surface vanishes because the local attraction \(-\nabla \Phi\)

\(^2\) These dust particles, and this surface, are not comoving with the FLRW background: they are slowed down by the attraction of the mass inside the turnaround surface.
balances exactly the force per unit mass due to the cosmic expansion at that point \( \frac{\ddot{a}(t_0)}{a(t_0)} \mathbf{x}_\perp \) (on \( \Sigma_{t_0} \) there is no sideways acceleration due to the local perturbation because \( \Sigma_{t_0} \) is an equipotential surface of \( \Phi \)). Or: if \( \mathbf{x} \) is a point on \( \Sigma_{t_0} \) and

\[
\mathbf{n} = \left. \frac{\nabla \Phi}{|\nabla \Phi|} \right|_{\Sigma_{t_0}}
\]

is the normal to \( \Sigma_{t_0} \), we impose

\[-\nabla \Phi = \frac{\ddot{a}(t_0)}{a(t_0)} \mathbf{x}_\perp \text{ on } \Sigma_{t_0},\]

with \( \mathbf{x}_\perp \equiv (\mathbf{x} \cdot \mathbf{n}) \mathbf{n} \), which implies

\[-|\nabla \Phi|^2 = \frac{\ddot{a}(t_0)}{a(t_0)} \mathbf{x} \cdot \nabla \Phi \text{ on } \Sigma_{t_0}.\]

Now the general result (12) reads [12]

\[
\frac{d(\delta u_i)}{dt} + 2H\delta u_i + g^{ij} \partial_j \Phi = 0;
\]

since \( u_i^0(0) = d\eta/d\tau = d\eta/dt = 1/a \) to order \( \mathcal{O}(\Phi^0) \), Eq. (12) leads to [12]

\[
du_i|_{\Sigma_t} = \frac{a^2(t_0)}{a^2(t)} \delta u_i|_{\Sigma_{t_0}} - \frac{1}{a^2(t)} \int_{t_0}^{t} h^{ij}(x^a(t')) \partial_j \Phi(x^a(t')) dt'. \tag{18}\]

Astronomical observations of the turnaround radius only span a small redshift interval near the time when the light was emitted, hence we linearize \( a(t) \) and, to first order in \( t - t_0 \),

\[
\delta u_i|_{\Sigma_t} = \left[ (1 - 2H(t_0)\epsilon) \sigma(t_0) - \frac{\epsilon}{a^2(t_0)} \right] h^{ij} \partial_j \Phi|_{\Sigma_{t_0}} + \mathcal{O}(H^2(t_0)\epsilon^2) \tag{19}\]

\[
= \left[ \sigma(t_0) - \left( 2\pi(t_0)H(t_0) + \frac{1}{a^2(t_0)} \right) \epsilon \right] h^{ij} \partial_j \Phi|_{\Sigma_{t_0}} + \mathcal{O}(H^2(t_0)\epsilon^2). \tag{20}\]

This equation reproduces known formulas [13] in the spherical case and for small deviations from sphericity [14], and is generalized to scalar-tensor gravity in [12].

3. Outlooks

Turnaround physics constitutes a potential cosmological test of gravity and/or a probe of dark energy, but it is crucial to move beyond the spherical approximation [6]. We have provided a characterization of turnaround surface for arbitrary deviations from spherical symmetry [12]. When applied to small deviations from sphericity, this definition reproduces previous results obtained with a completely different method (the splitting of the Hawking quasilocal energy into local and cosmological parts [13]). Our characterization needs a numerical implementation to enable quick use in astronomy.

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