Positive Lyapunov exponent for random perturbations of predominantly expanding multimodal circle maps

Alex Blumenthal*  Yun Yang†

May 24, 2018

Abstract

We study the effects of IID random perturbations of amplitude $\epsilon > 0$ on the asymptotic dynamics of one-parameter families $\{f_a : S^1 \to S^1, a \in [0,1]\}$ of smooth multimodal maps which “predominantly expanding”, i.e., $|f'_a| \gg 1$ away from small neighborhoods of the critical set $\{f'_a = 0\}$. We obtain, for any $\epsilon > 0$, a checkable, finite-time criterion on the parameter $a$ for random perturbations of the map $f_a$ to exhibit (i) a unique stationary measure, and (ii) a positive Lyapunov exponent comparable to $\int_{S^1} \log |f'_a| \, dx$. This stands in contrast with the situation for the deterministic dynamics of $f_a$, the chaotic regimes of which are determined by typically uncheckable, infinite-time conditions. Moreover, our finite-time criterion depends on only $k \sim \log(\epsilon^{-1})$ iterates of the deterministic dynamics of $f_a$, which grows quite slowly as $\epsilon \to 0$.

1 Introduction and statement of results

A fundamental goal in dynamical systems is to determine the asymptotic behavior of various dynamical systems. Away from the uniformly expanding, Anosov and Axiom A settings, maps can have “mixed” dynamical behavior, e.g., hyperbolicity on some parts of phase space and contractive behavior on others. On the collection of maps with this ‘mixed’ behavior, various dynamical regimes (e.g., asymptotically stable orbits with large basins of attraction versus more ‘chaotic’ asymptotic behavior) can be intermingled, in the space of maps, in an extremely convoluted way.

These issues are already present in the deceptively simple example of the one-parameter family of quadratic maps $f_a : [0,1] \to [0,1], f_a(x) := ax(1-x)$ for $a \in [0,4]$. Let us agree to say that for a parameter $a \in [0,4]$, the map $f_a$ is regular if phase space $[0,1]$ is covered Lebesgue almost-surely by the basins of periodic sinks, while $f_a$ is chaotic if it possesses a unique a.c.i.m. with a positive Lyapunov exponent. For the family $\{f_a\}$, it is known (e.g., [21] and many others) that the parameter space $[0,4]$ is Lebesgue-almost surely partitioned into two sets, $A \cup B$, with the following properties:

- For all $a \in A$, the map $f_a$ is regular, and for all $a \in B$, the map $f_a$ is chaotic.
- The set $A$ is open and dense in $[0,4]$, while $B$ has positive Lebesgue measure. In particular, every $a \in B$ is the limit point of a sequence $\{a_n\} \subset A$.

In particular, the chaotic property is extremely structurally unstable with respect to the parameter $a$: any $a \in B$ is the limit point of a sequence $\{a_n\} \subset A$.

Aside from ‘exceptional’ cases (e.g., $a = 4$), it is typically impossible to rigorously determine, even with the help of a computer, the dynamical regime corresponding to a given parameter $a \in [0,4]$, as this determination would require infinite-precision knowledge of infinite-length trajectories. For the quadratic family and other families of 1D maps with mixed expansion and contraction, the core issue is the difficulty in ruling out the formation of sinks of high period: even if, for a given $a$, sinks of period $\leq N$ are ruled out
for some extremely large $N$, one cannot rule out the existence of a sink of period $N + 1$ or greater. Indeed, the trajectory of a sink of large period may ‘look’ chaotic before the full period has elapsed.

Although fewer results are known for higher-dimensional models, one anticipates a similar degree of convoluted intermingling of dynamical regimes: see, e.g., the class of examples now known as Newhouse phenomena \[23\]. A somewhat more complete account of coexistence phenomena is available for the famous Chirikov standard map family \[11\], a one-parameter family $\{F_L, L > 0\}$ of volume-preserving maps on the torus $\mathbb{T}^2$ exhibiting simultaneously both strong hyperbolicity and elliptic-type behavior on phase space. As the parameter $L$ increases, so too does the proportion of phase space on which $F_L$ is hyperbolic, as well as the “strength” of this hyperbolicity. However, even for large $L$, a small amount of elliptic-type behavior is intermingled with hyperbolic behavior in the parameter space. Indeed, for a residual set of large $L$, it is known that elliptic islands for $F_L$ are approximately $L^{-1}$-dense in $\mathbb{T}^2$ (Duarte 1994 \[14\]; see also \[12\]), while the set of points with a positive Lyapunov exponent has Hausdorff dimension 2 and is approximately $L^{-1/3}$-dense in $\mathbb{T}^2$ (Gorodetski 2012 \[15\]). To the authors’ knowledge, it is still not known whether $F_L$ has positive metric entropy (equivalently, a positive Lyapunov exponent on a positive-volume set) for any fixed value of $L$.

Random perturbations

The real world is inherently noisy, and so it is natural to consider IID random perturbations of otherwise deterministic dynamics and seek to understand the corresponding asymptotic behavior. For concreteness, let us consider a smooth, deterministic map $f : S^1 \to S^1$ and seek to understand the corresponding asymptotic behavior. For concreteness, let us consider a smooth, deterministic map $f : S^1 \to S^1$ and seek to understand the corresponding asymptotic behavior. For concreteness, let us consider a smooth, deterministic map $f : S^1 \to S^1$ and seek to understand the corresponding asymptotic behavior. For concreteness, let us consider a smooth, deterministic map $f : S^1 \to S^1$ and seek to understand the corresponding asymptotic behavior. For concreteness, let us consider a smooth, deterministic map $f : S^1 \to S^1$ and seek to understand the corresponding asymptotic behavior.

Parametrizing $S^1 \cong [0, 1)$ and doing arithmetic “modulo 1”, at time $n$ we perturb $f$ to the map $f_{\omega_n-1}(x) = f(x + \omega_n - 1)$, where $\omega_0, \omega_1, \cdots$ are IID random variables uniformly distributed in $[-\epsilon, \epsilon]$. Here, the noise amplitude $\epsilon > 0$ is a fixed parameter. We will consider the asymptotic dynamics of compositions of the form

$$f^n_{\omega} = f_{\omega_n-1} \circ \cdots \circ f_{\omega_0}$$

given a sample $\omega = (\omega_0, \omega_1, \cdots)$.

When $\epsilon \gtrsim 1$, random trajectories $X_n = f^n_{\omega}(X_0)$, $n \geq 1$ are essentially IID themselves; in this situation it is a straightforward exercise to check (i) uniqueness of the stationary measure for the process $(X_n)$ on $S^1$ and (ii) that the Lyapunov exponent $\lambda = \lim_{n \to \infty} \frac{1}{n} \log \| (f^n_{\omega})'(x) \|$ exists and is constant for every $x \in S^1$ and a.e. sample $\omega$. What is more subtle is the situation when $\epsilon \ll 1$, in which case the composition $f^n_{\omega}$ may develop one or more random sinks; here, for our purposes, a random sink is a stationary measure for $(X_n)$ with a negative Lyapunov exponent.

Random sinks can develop if, for instance, the map $f$ itself has a periodic sink $z \in S^1$. Indeed, it is not hard to check that the sink $z$ persists in the form of a random sink for all $\epsilon > 0$ sufficiently small (see, e.g., Section 3.1 of this paper for a worked example). On the other hand, one anticipates that sinks of $f$ of high period $N$ can be “destroyed” in the presence of a small but sufficient amount of noise, i.e., when $\epsilon \geq \epsilon_N$, where $\epsilon_N \to 0$ as $N \to \infty$. As described previously, these high-period sinks are precisely those responsible for the convoluted intermingling of dynamical regimes in one-parameter families of unimodal or multimodal maps.

In an alternative perspective: given a fixed noise amplitude $\epsilon > 0$, the only sinks of $f$ which could possibly persist as random sinks for $(f^n_{\omega})$ are those of period $\leq k_\epsilon := \max\{N : \epsilon < \epsilon_N\}$. A crucial point here is that, for a given map $f$, it is virtually always possible to check for sinks of period less than some given value. For these reasons, one anticipates that for a reasonably large class of $f$ as above and a given noise amplitude $\epsilon > 0$, it should be possible to determine the asymptotic chaotic regime of the corresponding random composition $f^n_{\omega}$ based on checkable criteria involving only finitely many iterates of the map $f$.

The present paper is a step in this direction for a model of one-parameter families of multimodal circle maps $f = f_\alpha$ exhibiting strong expansion $\| (f^n_\alpha)' \| \gg 1$ away from a small neighborhood of the critical set $\{f^n_\alpha = 0\}$. We obtain a checkable sufficient criterion on the parameter $\alpha$, involving only finitely many iterates of the map $f_\alpha$ (in particular, precluding sinks of low period, as above), for deducing asymptotic chaotic behavior for the random composition $f^n_{\omega}$ when the noise parameter $\epsilon$ is not too small. An appealing feature of these results is that, given $\epsilon > 0$, the criterion involves only approximately $\log(\epsilon^{-1})$ iterates, which grows quite slowly as $\epsilon \to 0$. 

2
1.1 Statement of results

The model

Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the unit circle, parametrized by the interval $[0,1)$. We assume throughout that $\psi : S^1 \to \mathbb{R}$ is a $C^2$ function for which the following conditions hold:

(H1) the critical set $C'_\psi = \{ \hat{x} \in S^1 : \psi(\hat{x}) = 0 \}$ has finite cardinality, and

(H2) we have $\{\psi'' = 0\} \cap C'_\psi = \emptyset$.

We consider maps of the form

$$f = f_{L,a} := L\psi + a \pmod{1},$$

for $L > 0, a \in [0,1)$, where $\pmod{1} : \mathbb{R} \to S^1 \cong \mathbb{R}/\mathbb{Z}$ is the natural projection. Observe that for $L \gg 1$, the map $f$ is strongly expanding away from $C'_\psi$.

When $\epsilon > 0$ is specified, we write $\Omega = \Omega^\epsilon = \{[-\epsilon, \epsilon]\}_{i \geq 0}$ for the sample space for our perturbations. Elements $\omega \in \Omega$ are written $\omega = (\omega_0, \omega_1, \omega_2, \cdots)$ where $\omega_i \in [-\epsilon, \epsilon], i \geq 0$. With $\nu^\epsilon$ denoting the uniform distribution on $[-\epsilon, \epsilon]$, we define $P = P^\epsilon = (\nu^\epsilon)^{\otimes \mathbb{Z}_{\geq 0}}$ on $\Omega$. We write $\mathcal{F}$ for the product $\sigma$-algebra on $\Omega$ and for $n \geq 0$ we write $\mathcal{F}_n = \sigma(\omega_0, \omega_1, \cdots, \omega_n) \subset \mathcal{F}$.

When $f = f_{L,a}$ is specified, we consider random maps of the form $f_\omega : S^1 \to S^1, f_\omega(x) := f(x + \omega)$, where it is understood implicitly that the argument for $f$ is taken $(\mod{1})$. Given a sample $\omega \in \Omega$, we have a corresponding random composition

$$f^n_\omega := f_{\omega_{n-1}}, \cdots, f_{\omega_1}, f_{\omega_0}$$

for $n \geq 0$.

Alternatively, we can view the random maps $f^n_\omega$ as giving rise to a Markov chain $(X_n)_n$ on $S^1$ defined, for fixed initial $X_0 \in S^1, X_{n+1} := f_{\omega_n}(X_n)$. The corresponding Markov transition kernel $P(\cdot, \cdot)$ is defined for $x \in S^1$ and Borel $B \subset \mathbb{S}$ by

$$P(x, B) := P(X_1 \in B | X_0 = x) = \nu^\epsilon(\{\omega \in [-\epsilon, \epsilon] : f_\omega(x) \in B\}).$$

We say that a Borel measure $\mu$ on $S^1$ is stationary if

$$\mu(B) = \int_{S^1} P(x, B) \, d\mu(x)$$

for all Borel $B \subset S^1$.

Results

Our results concern the following checkable, finite-time criterion $(H3)_{c,k}$ on the dynamics of $f$. For now, $c > 0$ and $k \in \mathbb{N}$ are arbitrary.

$$(H3)_{c,k} \quad \text{For } \hat{x} \in C'_\psi, \text{ we have } d(f^l(\hat{x}), C'_\psi) \geq c \quad \text{for all } 1 \leq l \leq k. \quad (1)$$

We now state our results.

**Theorem A.** Let $\beta, c \in (0,1)$. Let $L > 0$ be sufficiently large, depending on these constants, and assume $f = f_{L,a}$ satisfies $(H3)_{c,k}$ for some arbitrary $k \in \mathbb{N}$. Finally, assume $\epsilon \geq L^{-(2k+1)(1-\beta)}$. Then, the random composition $f^n_\omega$ admits a unique (hence ergodic) stationary measure $\mu$ supported on all of $S^1$.

**Theorem B.** Let $\beta, c \in (0,1)$. Let $L > 0$ be sufficiently large, depending on these constants, and assume $f = f_{L,a}$ satisfies $(H3)_{c,k}$ for some arbitrary $k \in \mathbb{N}$. Finally, assume $\epsilon \geq L^{-(2k+1)(1-\beta)+\alpha}$ where $\alpha \geq 0$ is arbitrary. Then, the Lyapunov exponent

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \log \| (f^n_\omega)'(x) \|$$

exists and

$$\lambda > 0.$$
exists and is constant over \( x \in S^1 \) and \( \mathbb{P} \)-almost every \( \omega \in \Omega \), and satisfies the estimate

\[
\lambda \geq \lambda_0 \log L,
\]

where \( \lambda_0 = \lambda_0(\alpha, k) := \min\{\frac{\alpha}{k+1}, \frac{1}{10}\} \).

Theorems A and B are essentially sharp, in the sense that \((H3)_{c,k}\) is compatible with the formation of sinks of period \( k+1 \), while such sinks persist under random perturbations of order \( \epsilon \lesssim L^{-(2k+1)} \). See Proposition 2 in Section 3.1 for more information.

A satisfying feature of our results is that, for fixed sufficiently large \( L \) and any given \( \epsilon > 0 \), to deduce a large positive exponent for \( f = f_{L,a} \) requires validating condition \((H3)_{c,k}\) with \( k = k_{\epsilon} \approx \log(\epsilon^{-1}) \). The value of \( k_{\epsilon} \) grows only logarithmically with \( \epsilon^{-1} \), which means that even for quite small \( \epsilon > 0 \), Theorems A and B are already valid when \((H3)_{c,k}\) is verified for a relatively small value of \( k \).

**Prior work**

There is a substantial and growing literature on random dynamical systems in low dimensions: we recall below some of the literature on random dynamical systems closest to the present paper, i.e., dealing with random maps having strong expansion mixed with some contraction in phase space.

Lian and Stenlund \([20]\) consider random perturbations of predominantly expanding (expanding on most of phase space with a small exceptional set) multimodal maps, more-or-less equivalent to the model in the present paper. They prove that for large enough noise amplitudes, the random system has a unique ergodic stationary measure and a positive Lyapunov exponent. They develop a similar condition with smaller noise amplitude assuming a ‘one time-step’ condition on the dynamics, essentially equivalent to \((H3)_{c,1}\) in our paper. Because we deal with higher-iterate dynamical assumptions, the perturbations we may consider are substantially smaller than those in \([20]\).

Stenlund and Sulku \([25]\) obtain exponential loss of memory for IID compositions \( T^n = T_1 \circ \cdots \circ T_1 \) of random circle maps which are “expanding on average”: contractive behavior \((\inf |T'| \approx 0)\) can appear with positive probability, but the random variable \( \inf |T'| \) satisfies a moment condition. The random maps we consider in the present paper always have critical points, and so do not satisfy the conditions of \([25]\).

In a joint work between the first author, Xue and Young \([8, 9]\), random perturbations of a model of “predominantly hyperbolic” two-dimensional maps are considered. The paper \([8]\) considers a volume-preserving model encompassing the Chirikov standard map, and \([9]\) considers a dissipative (volume-compressing) model of maps having qualitative similarities to the Henon maps. Chaotic properties of the deterministic dynamics in each case are anticipated to hold on large subsets of parameter space, but rigorous verification is largely beyond the scope of current studies. What \([8, 9]\) show is that sufficiently large random perturbations have the effect of “unlocking” the hyperbolicity of these systems (positive Lyapunov exponent proportional to the Lebesgue average \( \int \log \|dF_x\| \, dx \), estimate of decay of correlations). A different but related analysis is carried out in the paper of Ledrappier, Simó, Shub and Wilkinson \([19]\), which considers IID perturbations applied to a twist map on the sphere.

Additionally, \([8, 9]\) allow smaller random perturbations on assuming a checkable condition involving the first several iterates of the deterministic map, consistent with the finite-time checkable criterion given in the present paper.

To reiterate, the papers \([20, 25, 19, 8, 9]\) are emphasized because they deal with random perturbations of maps for which very little is assumed: in these studies, the randomness itself is leveraged in a crucial way to ‘shake loose’ hyperbolicity. Other works examine random compositions of maps with ‘good’ asymptotic behavior: by way of example, we mention works on smooth \([24, 3]\) and piecewise \([10]\) expanding maps, maps with a neutral fixed point \([1]\), and work on quadratic \([24, 7, 2, 17]\) and Henon-like maps \([6]\) satisfying (uncheckable) infinite-time conditions. In contrast with the present work, we do not assume strong chaoticity of our unperturbed maps.

The study of deterministic one-dimensional maps with critical points (unimodal or multimodal) has a long history, a small part of which we recall here. Naturally we inherit and use some of the ideas developed in this literature. Indeed, our criterion \((H3)_{c,k}\) is a checkable, finite-time version of various criteria on postcritical orbits of unimodal and multimodal maps as used by, e.g., Misiurewicz \([22]\), Jakobson \([16]\), Collet-Eckmann \([12]\) and Benedicks and Carleson \([3, 5]\). We note as well the more expository account by Wang and Young \([20]\), which we found remarkably helpful in preparing this work.
Organization of the paper.

In Section 2, we derive elementary properties of our model used throughout the paper, especially the notion of bound period defined in Section 2.2. In Section 3.1, we discuss the possible formation of sinks of period \( k + 1 \) under the condition \((H3)_{k+1}\), verifying the relative sharpness of Theorems A, B; ergodicity as Theorem A is then proved in Section 3.2. The material in Section 3 depends on Section 2 but is otherwise logically isolated from the rest of the manuscript. The proof of Theorem B occupies the remainder of the paper, Sections 4–6.

Notation

- Throughout, we parametrize \( S^1 \) by the half-open interval \([0, 1) \cong \mathbb{R}/\mathbb{Z}\). For \( s \in \mathbb{R} \), we write \( s \pmod{1} \) for the projection of \( s \) to \([0, 1) \cong \mathbb{R}/\mathbb{Z}\) modulo 1.
- The lift \( \tilde{f} : S^1 \to \mathbb{R} \) of \( f \) by \( \tilde{f}(x) = L\psi(x) + a \) (i.e., without projecting (mod 1) to \( S^1 \)). We regard \( \tilde{f} \) as a map \( \mathbb{R} \to \mathbb{R} \) by extending the domain periodically to all of \( \mathbb{R} \). We write \( \tilde{f}_\omega(x) = \tilde{f}(x + \omega) \). We define the corresponding Markov process \((\tilde{X}_n)_{n \geq 0}\) on \( \mathbb{R} \) by setting \( \tilde{X}_{n+1} = \tilde{f}_{\omega_n}(\tilde{X}_n) \).
- We write \( d(\cdot, \cdot) \) for the metric induced on \( S^1 \) via the identification with \( \mathbb{R}/\mathbb{Z} \cong [0, 1) \). Note that in our parameterization, we have the identity \( d(x, y) = \min\{|x - y|, |x - y + 1|\} \).
- For a set \( A \subseteq S^1 \) and a set \( A \subseteq S^1 \), we define the minimal distance \( d(x, A) = \inf_{a \in A} d(x, a) \). For sets \( A, B \subseteq S^1 \), we define \( d(A, B) = \inf_{a \in A} d(a, B) = \inf_{a \in A, b \in B} d(a, b) \).
- Given a set \( A \subseteq S^1 \) or \( \mathbb{R} \) and \( z \in S^1 \) or \( \mathbb{R} \), we write \( A - z = \{a - z : a \in A\} \) for the set \( A \) shifted by \( z \).
- Given a partition \( \zeta \) of \( S^1 \) (resp. \( \mathbb{R} \)) and a set \( A \subseteq S^1 \) (resp. \( A \subseteq \mathbb{R} \)), we write \( \zeta|_A \) for the partition on \( A \) consisting of atoms of the form \( C \cap A, C \in \zeta, C \cap A \neq \emptyset \).
- When it is clear from context, we write \( \mathbb{E} \) for the expectation with respect to \( \mathbb{P} \).

2 Preliminaries: predominant expansion and bound periods

Bound periods: a heuristic

Consider the dynamics of a smooth unimodal or multimodal map \( f : S^1 \to S^1 \). In the pursuit of finding maps \( f \) accumulating a positive Lyapunov exponent, the main obstruction is the formation of sinks, and so a natural assumption to make is that the postcritical orbits \( f^n\hat{x}, \hat{x} \in \{f' = 0\}, n \geq 1 \) remain enough far away from \( \{|f'| \leq 1\} \) so that \( |(f^n)'(f\hat{x})| > e^{\alpha n} \) for some \( \alpha > 0 \).

If, for some \( x \in S^1 \), the orbit \((f^n x)_n\) reaches a small neighborhood of some \( \hat{x} \in \{f' = 0\} \) at time \( t \), then the subsequent iterates \( f^{i+p}x \) will closely shadow \( f^i\hat{x} \) for \( i \leq p = p(d(f^i x, \hat{x})) \). The time interval \([t + 1, t + p]\) is referred to as the bound period for \( x \) at time \( t \). As we assumed expansion along the postcritical orbit \((f^i \hat{x})_{i \geq 1}\), one anticipates that the derivative growth \((f^p)'(f^{i+p}x) \) accumulated along the bound period will balance out the derivative ‘damage’ due to \( f'(f^i x) \) (possibly \( \ll 1 \) when \( f^i \hat{x} \) are quite close), so that, for instance, \((f^{i+p})'(f^{i+p}x) \sim e^{i(p+1)\alpha'} \) holds for some \( \alpha' < \alpha \).

This is a rough summary of a mechanism by which 1D maps with critical points (unimodal and multimodal) can accumulate a positive Lyapunov exponent for typical trajectories. For an exposition of this method, see [20].

Our aim in Section 2 is to apply a variation of this idea to our model: the condition \((H3)_{k+1}\) involves the first \( k \) iterates of postcritical trajectories, and so bound periods of length up to \( k \) are available to recover derivative growth. In Section 2.1 we carry out some essential preliminaries used in the rest of the paper, and in Section 2.2 we will discuss bound periods for our random compositions.
2.1 Preliminaries

2.1.1 The basic setup

We fix, below and throughout the paper, a function $\psi : S^1 \rightarrow \mathbb{R}$ satisfying (H1) and (H2), as well as parameters $c \in (0, 1), \beta \in (0, \frac{1}{100})$ (restricting to $\beta$ in this range incurs no loss of generality). Moreover, we implicitly fix the parameter $L > 0$, and are allowed to take it sufficiently large depending on $c, \beta$ and the function $\psi$.

On rescaling the function $\psi$ in relation to the parameter $L$, we will assume going forward that the following condition holds in addition to (H1) – (H2).

(H4) We have $\|\psi^\prime\|_{C^0}, \|\psi^{\prime\prime}\|_{C^0} \leq \frac{1}{10}$.

Separately (i.e., independently of $L$), $k \in \mathbb{N}$ is fixed, and a parameter $a \in [0, 1)$ is fixed for which $(H3)_{c, k}$ holds for the mapping $f = f_{L, a} := L\psi + a \pmod 1$. Finally, we fix a parameter $\epsilon > 0$, on which constraints (depending on all the previous parameters) will be made as we go along.

2.1.2 Partition of phase space

The conditions (H1) – (H2) imply that there is a constant $K_1 = K_1(\psi) > 0$ with the property that for any $x \in S^1$,

$$|\psi^\prime(x)| \geq K_1 d(x, C^\prime \psi).$$

(2)

We use (2) repeatedly, often without mention. For $\eta < 0$, we define

$$B(\eta) = \{x \in S^1 : d(x, C^\prime \psi) \leq K_1^{-1} L^\eta\}.$$  

(3)

It is clear that for $x \notin B(\eta)$, we have $|f^\prime(x)| \geq L^{n+1}$, while $B(\eta)$ is the union of $#C^\prime \psi$-intervals of length $\sim L^n$ each.

Define the partition $S^1 = G \cup I \cup B$, where

$$G = S^1 \setminus B(-\beta), \quad I = B(-\beta) \setminus B(-\frac{1}{2} - \beta), \quad B = B(-\frac{1}{2} - \beta).$$

We have, then, that

$$|f^\prime|_{\partial G} \geq L^{1-\beta}, \quad \text{and} \quad |f^\prime|_{\partial I} \geq L^{1-\beta}.$$ 

Similar estimates apply to $f^\prime_\omega$ on the shifted sets $G_\omega := G - \omega, I_\omega := I - \omega$ for $\omega \in [-\epsilon, \epsilon]$.

Observe that $|f^\prime|_B$ can be arbitrarily small. To address this, we subdivide $B = \cup_{k=1}^k B^l$ in the following way: set

$$B^k = B(-\frac{k}{2} - \beta),$$

and for $1 \leq l < k$,

$$B^l = B(-\frac{l}{2} - \beta) \setminus B(-\frac{l+1}{2} - \beta).$$

Notice that the definition above is consistent with the identification $I = B^0$. We also use the notation $B^l_\omega := B^l - \omega$ for $\omega \in [-\epsilon, \epsilon]$. Using (2), one checks that

$$|f^\prime_{\omega}|_{B^l_\omega} \geq L^{1-\omega - \beta} \quad \text{for} \quad 1 \leq l < k,$$

while on $B^k$ we have no lower bound on $|f^\prime_{\omega}|$.

The partitions $S^1 = G \cup I \cup B = G \cup I \cup B_1 \cup \cdots \cup B^k$ are used repeatedly throughout the paper. We will abuse notation and regard these as partitions of $\mathbb{R}$ as well, extended by periodicity via the parametrization $S^1 \cong [0, 1) \cong \mathbb{R}/\mathbb{Z}$. 

6
2.2 Bound periods

The following lemma confirms that a random orbit \((f^n_\omega x)\), initiated at \(x \in B^l, 1 \leq l \leq k\), will closely shadow a postcritical orbit \((f^i \hat{x})\) for \(l\) steps, i.e., will have a bound period of length \(l\).

In Lemma 1 below we do not assume \((H3)_{c,k}\).

**Lemma 1.** Let \(L\) be sufficiently large, and let \(k \in \mathbb{N}\) be arbitrary. Assume that

\[
\epsilon < L^{-\max(7-1,4)} - \beta. \tag{4}
\]

Then, we have the following. Let \(1 \leq l \leq k\) and fix an arbitrary sample \(\omega \in \Omega\). Let \(J_0\) be any connected component of \(B(-\frac{L^2}{1-\beta})\) and let \(\hat{x} = C^c_\omega \cap J\) be the (unique) critical point contained in \(J_0\).

Then, for all \(1 \leq i \leq l\) we have that

\[
f^{i}_{\omega}(J_0) \subset N_{L^{-\beta/2}}(f^i \hat{x}).
\]

The reason for the upper bound \((4)\) is that if the perturbation amplitude \(\epsilon\) is too large, then \(f^{i}_{\omega}|_{B^\epsilon_{\omega_0}}\) may diverge from \(f^i \hat{x}\) for some \(i < k\), thereby spoiling the corresponding bound periods.

From Lemma 1 and noting \(B^l \subset B(-\frac{L^2}{1-\beta})\), it is straightforward to check that if \(L\) is sufficiently large and \(f = f_a\) satisfies \((H3)_{c,k}\), then \(f^i \hat{x}\) is well inside \(\mathcal{G}\) for \(1 \leq i \leq k\). It follows that for any \(1 \leq l \leq k\) and \(x \in B^l_{\omega_0}\), we have \(f^l_\omega(x) \in \mathcal{G}\) for all \(1 \leq i \leq l\), and the derivative estimate

\[
|\langle f^l_\omega(x) \rangle | \geq L^{(1-\beta)}.
\]

Moreover, if \(1 \leq l < k\) then we have \(|\langle f^l_\omega \rangle | \geq L^{(1-\beta)}\), hence

\[
|\langle f^{l+1}_\omega \rangle | \geq L^{(1+1)(1-\beta)}.
\]

For the purposes of the preceding paragraph, it suffices to take \(L\) large enough so that \(L^\beta \gg 2/(cK_1)\); note in particular that \(L\) does not depend on \(k\).

**Proof of Lemma 1** In the following proof, the lift \(\tilde{f} : S^1 \to \mathbb{R}\) of \(f\) is defined by \(\tilde{f}(x) = L\psi(x) + a\), i.e., leaving out the “\((\text{mod } 1)\)” in the definition of \(f\). We extend the domain of \(\tilde{f}\) to all of \(\mathbb{R}\) by periodicity.

Without loss, we regard \(J_0\) as an interval in \(\mathbb{R}\). Let \(\hat{x} \in C^c_\omega \cap J_0\) be the (unique) critical point in \(J_0\). Define \(I_0 = N_\epsilon(J_0)\) and inductively set \(I_{i+1} = \tilde{f}(I_i), I_{i+1} = N_\epsilon(I_{i+1})\). Since \(f^i \hat{x} \in J_i\) for all \(i\), it suffices to show \(\text{Len}(J_i) \leq L^{-\beta/2}\) for all \(1 \leq i \leq l\).

To start, decompose \(I_0 = I_0^* \cup I_0^-\) where \(I_0^- = [\hat{x} - \epsilon - K^{-1}L^{-\frac{\beta \epsilon}{1-\beta}}, \hat{x}], I_0^* = [\hat{x} - \epsilon + K^{-1}L^{-\frac{\beta \epsilon}{1-\beta}}]\). Noting that the images \(\tilde{f}(I_0^-), \tilde{f}(I_0^*)\) share the left (resp. right) endpoint \(\tilde{f}(\hat{x})\) if \(f''(\hat{x}) > 0\) (resp. \(f''(\hat{x}) < 0\)), we have the estimate

\[
\text{Len}(I_1) \leq \max\{\tilde{f}(I_0^*), \tilde{f}(I_0^-)\} \leq \frac{1}{2}L\|\psi''\|_{C^0} \cdot (\epsilon + K^{-1}L^{-\frac{\beta \epsilon}{1-\beta}})^2
\]

\[
\leq L \max\{\epsilon, \text{Len}(J_0)\}^2
\]

using \((H4)\) in the last step. For each \(i > 1\), we estimate

\[
\text{Len}(I_i) = \text{Len}(\tilde{f}(I_{i-1})) \leq L\|\psi''\|_{C^0} \text{Len}(I_{i-1}) \leq L \max\{\epsilon, \text{Len}(J_{i-1})\}.
\]

by estimating \(\text{Len}(I_{i-1}) \leq 2\epsilon + \text{Len}(J_{i-1}) \leq 3 \max\{\epsilon, \text{Len}(J_{i-1})\}\) and using \((H4)\). Bootstrapping, we conclude

\[
\text{Len}(J_i) \leq L^{-i-1} \max\{\epsilon, \text{Len}(J_1)\} \leq \max\{L^{-i-1} \epsilon, L^i \epsilon^2, L^i \text{Len}(J_0)^2\}.
\]

The first two terms are \(< L^{-\beta}\) by \((3)\) for all \(i \leq k\). For \(i \leq l\), the third term is \(\leq L^i \cdot 4K^{-2}L^{-\beta/2} \leq L^{-\beta/2}\).

This completes the proof. \(\Box\)
3 Ergodicity

In Section 3.1, we prove Proposition 2 which confirms the sharpness of Theorems 1 and 3 in the following sense. To start, condition (H3)_{c,k} for the map \( f = f_a \) is compatible with the formation of a sink of period \( k + 1 \). For all \( \epsilon > 0 \) sufficiently small, such sinks persist as random sinks for the random compositions \( (f^n) \), i.e., stationary measures for the Markov chain \( (X_n) \), admitting a negative Lyapunov exponent. In Proposition 2 we make this quantitative by exhibiting a scenario when \( f = f_a \) (i) satisfies (H3)_{c,k}; (ii) admits a sink of period \( k + 1 \); and (iii) the random composition \( (f^n) \) admits a random sink for all \( \epsilon \gtrsim L^{-2(k+1)} \). This upper bound for \( \epsilon \) approximately matches the upper bound in Theorems 1 and 3 confirming the view that these results are sharp.

Having established this, in Section 3.2 we proceed with the proof of Theorem 1. We note that in terms of logical dependence, Section 3 depends on Section 2 and is otherwise independent of the remainder of the paper, Sections 4 – 6.

3.1 Sinks

Let us take on the assumptions made for the map \( f = f_{L,a} \) as in Section 2.1.1, except that for Proposition 2 we need not assume (H3)_{c,k} holds. Observe, however, that the hypothesis of Proposition 2, i.e., the existence of a sink of period \( k + 1 \) for \( f = f_{L,a} \), is entirely compatible with (H3)_{c,k}. 

**Proposition 2.** For all \( L \) sufficiently large, depending only on \( \psi \), we have the following. Let \( k \in \mathbb{N} \) be arbitrary, and assume \( f = f_{L,a} \) has the property that \( f^{k+1} \hat{x} = \hat{x} \) for some \( \hat{x} \in \mathbb{C}_\psi' \). Then, for any \( \epsilon \leq \frac{1}{49} L^{-2(k+1)} \), we have that the random composition \( f^n \) admits a stationary measure \( \mu \) for which

(a) the support of \( \mu \text{ Supp}(\mu) \) is contained in a \( \frac{1}{2} L^{-(k+1)} \)-neighborhood of the orbit \( \hat{x}, f\hat{x}, \cdots, f^k \hat{x} \) (in particular, \( \text{Supp} \mu \subseteq S^1 \)); and

(b) \( \lambda(\mu) < 0 \).

**Proof.** We will show that there is a neighborhood \( U \) of \( \hat{x} \) such that for a.e. sample \( \omega \in \Omega \),

(i) \( f^{k+1}(U) \subset U \); and

(ii) \(|(f^{k+1})'(x)| < \frac{1}{2} \) for all \( x \in U \).

By standard arguments, (i) – (ii) imply the existence of a stationary measure \( \mu \) with Lyapunov exponent \( \lambda(\mu) \leq -\frac{\log 2}{k+1} < 0 \) supported in \( \{f^{i_\omega} x : x \in U, \omega \in \Omega, 0 \leq i \leq k \} \). At the end, we will estimate the size of this support.

Let \( \gamma \in (0, 1) \) be a constant, to be taken sufficiently small below, and throughout assume that \( \epsilon \leq \gamma L^{-2(k+1)} \). Let \( U \) to be the closed neighborhood of \( \hat{x} \) of radius \( r_U = \sqrt{\gamma} L^{-(k+1)} \). We estimate

\[
\sup_{z \in U} |(f^{i_\omega})'(z)| \leq \|f'\|_{C^0}^{-1} \cdot (\epsilon + \sqrt{\gamma} L^{-(k+1)}) \cdot \|f''\|_{C^0} \leq L^i \cdot 2 \sqrt{\gamma} L^{-(k+1)} \leq 2 \sqrt{\gamma} L^{-(k+1)},
\]

having used the elementary bound \( |f_\omega(z)| \leq |z + \omega - \hat{x}| \cdot |f''|_{C^0} \leq L|z + \omega - \hat{x}| \) for \( z \) near \( \hat{x} \). In particular, at \( i = k + 1 \) we have that

\[
|(f^{k+1})'(z)| \leq 2 \sqrt{\gamma},
\]

hence \( U \) maps to an interval \( f^{k+1}(U) \) of length \( |f^{k+1}(U)| \leq 2 \sqrt{\gamma} \cdot |U| = 4 \sqrt{\gamma} \cdot r_U \).

Let us now estimate \( d(\hat{x}, f^{k+1}(\hat{x})) \). For simplicity, we pass to the lifts \( \tilde{f}, \tilde{f}_\omega \): write \( \tilde{x}^i = f^i \hat{x}, \hat{x}^i = f^i \hat{x} \) for \( 0 \leq i \leq k + 1 \). To start,

\[
|\hat{x}^{i+1} - \hat{x}^i| = |\tilde{f}(\hat{x}) - \tilde{f}(\hat{x} + \omega_0)| \leq \epsilon \cdot \sup_{d(z, \hat{x}) \leq \epsilon} |f'(z)| \leq \epsilon^2 L.
\]

Next, for \( i > 0 \),

\[
|\hat{x}^{i+1} - \hat{x}^i| = |\tilde{f}(\hat{x}^i) - \tilde{f}(\hat{x}^{i+1} + \omega_i)| \leq L(\epsilon + |\hat{x}^{i+1} - \hat{x}^i|).
\]
Collecting, we obtain
\[
d(\hat{x}, f_\omega^{k+1}(\hat{x})) \leq |\hat{x} - \hat{x}_\omega| \leq (L + L^2 + \cdots + L^k) \epsilon + L^{k+1} \epsilon^2
\]
\[
\leq 2L^k \epsilon + L^{k+1} \epsilon^2 \leq 3\gamma L^{-(k+1)},
\]
here having assumed \(L > 2\). We deduce
\[
d(\hat{x}, f_\omega^{k+1}(\hat{x})) \leq 3\sqrt{\gamma} \cdot r_U.
\]
It is easy to check that the same bound \(d(\hat{x}^i, f_\omega^i(\hat{x})) \leq 3\sqrt{\gamma} \cdot r_U\) holds for any \(0 \leq i \leq k\) as well.

To conclude: for (i) it suffices (see (5)) to take \(\gamma \leq 1/16\). For (ii) we estimate as follows for \(z \in U\):
\[
d(f_\omega^{k+1}(z), \hat{x}) \leq d(\hat{x}, f_\omega^{k+1}(\hat{x})) + |f_\omega^{k+1}(U)| \leq 7\sqrt{\gamma} \cdot r_U. \tag{6}
\]
We conclude that \(f_\omega^{k+1}(U) \subset U\) almost surely as long as \(\gamma \leq 1/49\).

Finally, to estimate the support of \(\mu\) it suffices to repeat the estimate (6) with \(f_\omega(z), z \in U\) replacing \(f_\omega^{k+1}(z)\). We conclude that \(\mu\) is supported in the \(7\sqrt{\gamma} \cdot r_U\)-neighborhood of the periodic sink \(\{f^i \hat{x}\}_{0 \leq i \leq k}\).

### 3.2 Ergodicity

As already seen in the proofs of Lemma 1 and Proposition 2 the noise amplitude \(\epsilon\) is amplified by the strong expansion \(L \gg 1\) exhibited by \(f = f_{L,a}\). Each of these results depended on the noise being \emph{small enough} to control this amplification. Quite to the contrary, in Section 3.2 we will take advantage of this amplification to show that our process \((X_n)\) explores all of phase space \(S^1\) with some positive probability. The amplification of noise by expansion is a core motif in this paper, one which we will return to in Sections 5 – 6.

Before proceeding to the proof of Theorem A let us establish the setting and a brief reduction. Throughout, we assume the setup for \(f = f_{L,a}\) in Section 2.1.1 including (H3)\(_{c,k}\).

**Reductions.** We first argue that without loss of generality, in the hypotheses of Theorem A we may assume that \(\epsilon, k\) are such that the upper bound in (3) is satisfied, so that Lemma 1 applies. To justify this, consider the following alternative cases: (a) \(L^{-(k-1)} \leq \epsilon < L^{-1}\); (b) \(L^{-1} \leq \epsilon < L^{-1/2}\); and (c) \(\epsilon \geq L^{-1/2}\). For (a), let \(k' \in \mathbb{N}\) be such that \(L^{-k'} \leq \epsilon < L^{-(k'-1)}\). Clearly \(k' < k\), hence (H3)\(_{c,k}\) implies (H3)\(_{c,k'}\), while \(\epsilon \geq L^{-k'} \geq L^{-(2k'+1)(1-\beta)+\beta}\). So, it makes no difference to replace \(k\) with \(k'\) and proceed as before. In case (b), we can replace \(k\) with 1 and proceed as before. Finally, Theorem A in case (c) is a simple exercise left to the reader—see also Theorem 1 in [20], where ergodicity as in Theorem A is proved for \(\epsilon \gtrsim L^{-1}\) for a very similar model of multimodal circle maps.

In addition, on shrinking the parameter \(\beta\) we will assume the slightly stronger hypothesis
\[
\epsilon \geq L^{-(2k+1)(1-\beta)+\beta}
\]
on the noise parameter \(\epsilon\). In relation to Theorem A this incurs no loss of generality.

**Notation.** Given an initial \(X_0 \in S^1\), we write \(X_n = f^n_{\omega}(X_0)\) for the Markov chain evaluated at the sample \(\omega \in \Omega\) (notation as in Section 1.1). We write \(P_{X_0}\) for the law of \(X_n\) conditioned on the value of \(X_0 \in S^1\). Moreover, for \(n, m \geq 0\), random variables \(Z_1, Z_2, \cdots, Z_m : \Omega \rightarrow \mathbb{R}\), and \(X_0 \in S^1\), we write
\[
P^n(X_0, \{\{Z_j, 1 \leq j \leq m\}) = P_{X_0}(X_n \in |\sigma(Z_1, \cdots, Z_m))
\]
for the law of \(X_n\) conditioned on \(\sigma(Z_1, Z_2, \cdots, Z_m)\).

With the setup and reduction established, we now turn to the proof of Theorem A. We break this up into two parts, Propositions 3 and 4 below.

**Proposition 3.** There exist \(N \in \mathbb{N}, c > 0\) with the property that for any sample \(\omega\) and any \(X_0 \in B^k_{\omega_0}\) we have that \(P^N(X_0, \{\{\omega_i, 0 \leq i \leq N, i \neq 1\}) \geq c\text{Leb}(\cdot)\).
What this means is that random trajectories initiated in $B^k$ reach all of $S^1$ with some positive probability. Note that in Proposition \ref{proposition:random-dynamical-systems}, we randomize only in $\omega_1$. One reason is that since $X_0 \in B^k_{\omega_1}$, we have that $X_1, X_2, \ldots, X_k$ experience a bound period of length $k$, and so $\omega_1$ is the only perturbation which experiences the full $k$ steps of expansion guaranteed by Lemma \ref{lemma:random-dynamical-systems}. Meanwhile, it is technically more convenient to work with one perturbation $\omega_i$ at a time.

By Proposition \ref{proposition:random-dynamical-systems} it suffices to check that almost every trajectory enters $B^k$ after a finite time. Define the stopping time
\[ T := \min\{i \geq 0 : X_i \in B^k_{\omega_i}\}. \]

**Proposition 4.** Assume the hypotheses of Theorem \ref{theorem:random-dynamical-systems}. Then, there exists $N \in \mathbb{N}$ such that for any $X_0 \in S^1$, we have $\mathbb{P}_{X_0}(T \leq N) > 0$.

**Proof of Theorem \ref{theorem:random-dynamical-systems} assuming Propositions \ref{proposition:random-dynamical-systems} \ref{proposition:random-dynamical-systems}**. Observe that ergodic measures $\mu$ (1) exist by a standard tightness argument, and (2) automatically inherit absolute continuity w.r.t. Lebesgue on $S^1$ from the same property for our random perturbations $\omega_i, i \geq 0$. So, to conclude uniqueness it suffices to check that for all $X_0 \in S^1$, $P^M(X_0, \cdot)$ is supported on all of $S^1$ (i.e., assigns positive mass to all open intervals) for some $M = M(X_0) \in \mathbb{N}$. For more details, see, e.g., the characterization of ergodicity for stationary measures of random dynamical systems in Lemma 2.4 on pg. 19 of \cite{MR2005}.

To complete the proof, fix $X_0 \in S^1$ and let $n \leq N$ be such that $\mathbb{P}_{X_0}(T = n) > 0$. Then, for any interval $J \subset S^1$ with nonempty interior,
\[
P^{n+N}(X_0, J) = \mathbb{E}\left(P^n(X_n, J|\{\omega_i\}_{0 \leq i \leq n+N, i \neq n+1})\right) \\
\geq \mathbb{E}\left(\chi_{T=n} \cdot P^n(X_n, J|\{\omega_i\}_{0 \leq i \leq n+N, i \neq n+1})\right) \\
\geq \mathbb{E}\left(\chi_{T=n} \cdot c \text{Leb}(I)\right) = c \cdot \mathbb{P}_{X_0}(T = n) \cdot \text{Leb}(I) > 0.
\]

Here, $\mathbb{E}_{X_0}$ refers to the expectation conditioned on the value of $X_0$.

This completes the proof. It remains to check Propositions \ref{proposition:random-dynamical-systems} \ref{proposition:random-dynamical-systems}.

---

*In the remainder of Section 3, we prove Propositions \ref{proposition:random-dynamical-systems} \ref{proposition:random-dynamical-systems} in that order. With the above setup assumed, we hereafter fix $\epsilon \in [L^{-2(k+1)(1-\beta)+\beta}, L^{-\max\{k-1, 0\}}].$

### 3.2.1 Constructions and a Preliminary Lemma

Define $R$ to be the partition of $S^1$ into the connected components of the sets $G, I = B^0, B^1, \ldots, B^k$. For $\omega \in [-\epsilon, \epsilon]$, let $R_\omega$ denote the partition into atoms of the form $\alpha - \omega, \alpha \in R$. Extending by periodicity, we regard $R, R_\omega$ as partitions on $\mathbb{R}$ as well. Given an interval $J \subset \mathbb{R}$, let us write $R|_J = \{\alpha \cap J : \alpha \in R\}$. For $\omega \in [-\epsilon, \epsilon]$, the partition $R_\omega|_J$ of $J$ is defined analogously.

**Lemma 5.** Assume $\tilde{J} \subset \mathbb{R}$ is an interval with $|\tilde{J}| < L^{-\beta}$. Let $J$ be the longest atom of $R|_J$. Then, $|J| \geq \kappa|\tilde{J}|$, where $\kappa = \min\{\frac{l}{5}, K_{1}^{-1}\}$.

**Proof of Lemma 5.** Some notation for this proof: given $\hat{x} \in \{\hat{j} = 0\} \subset \mathbb{R}$ and $0 \leq l \leq k$, define $B^{l,+}(\hat{x})$ to be the connected component of $B^l$ to the immediate right of $\hat{x}$, and $B^{l,-}(\hat{x})$ to be the connected component to the immediate left. Let us write $B(\hat{x})$ for the component of $B$ containing $\hat{x}$.

If $R|_J$ has only one or two atoms of positive length, then $|J| \geq \frac{1}{2}|\tilde{J}|$ holds trivially. Hereafter we assume $R|_J$ consists of three or more atoms of positive length. In particular, $\tilde{J}$ contains a connected component of $B^l$ for some $0 \leq l \leq k$, since $|\tilde{J}| < L^{-\beta}$ was assumed. Let $\hat{x} \in \{\hat{j} = 0\}$ be the nearest critical point to $\tilde{J}$.

Define
\[ l_1 = \min\{0 \leq l \leq k : \tilde{J} \text{ contains a component of } B^l\}. \]

There are two cases: (i) $J \subset B^{l_1}$, in which case $J = B^{l_1, \pm}(\hat{x})$ for some choice of $\pm$, or (ii) $J \cap B^{l_1} = \emptyset$. 

For case (i), assume first that $l_1 = 0$. WLOG we assume $J = \mathcal{B}^{0,+}(\hat{x})$. Note that $\tilde{J} \cap \mathcal{G}$ consists of at most two components, hence $|\tilde{J} \cap \mathcal{G}| \leq 2|J|$, while $\tilde{J} \cap \mathcal{B}$ has one component, hence $|\tilde{J} \cap \mathcal{B}| \leq 2K_1^{-1}L^{-\frac{1}{2}+\beta} \leq 2L^{-\frac{1}{2}}|J|$. Finally, $\tilde{J} \cap \mathcal{I}$ has at most two components, and so $|\tilde{J} \cap \mathcal{I}| \leq 2|J|$. In total,

$$|\tilde{J}| \leq |\tilde{J} \cap \mathcal{G}| + |\tilde{J} \cap \mathcal{I}| + |\tilde{J} \cap \mathcal{B}| \leq (4 + 2L^{-\frac{1}{2}})|J| \leq 5|J|.$$  

Assuming now that $l_1 > 0$, WLOG we have $J = \mathcal{B}^{l_1,+}(\hat{x})$. Moreover, $\tilde{J} \subset \bigcup_{i=1}^k \mathcal{B}^{l_i}$; otherwise, $\tilde{J}$ would contain an intact component of $\mathcal{B}^{l_1-1}$, a contradiction. As before, $\tilde{J} \cap \mathcal{B}^{l_1-1}$ has at most two components, each of length $\leq |J|$, while $\tilde{J} \cap \bigcup_{i=1}^k \mathcal{B}^{l_i}$ has at most one component of length

$$\leq 2K_1^{-1}L^{-\frac{1}{2}+\beta} \leq 2L^{-\frac{1}{2}}|J| \ll |J|,$$

unless $l_1 = k$, in which case we can ignore this contribution. As before, we conclude $|\tilde{J}| \leq 3|J|$. 

For case (ii), if $l_1 = 0$, then $J \subset \mathcal{G}$. Note $\tilde{J}$ does contains some atom $\mathcal{B}^{0,\pm}(\hat{x})$, hence $|\tilde{J}| \geq K_1^{-1}L^{-\beta} > K_1^{-1}|J|$, having assumed in Lemma 5 that $|\tilde{J}| < L^{-\beta}$.

If $l_1 > 0$, then likewise it is not hard to show that $J \subset \mathcal{B}^{l_1-1}$. As before, $\tilde{J}$ contains some $\mathcal{B}^{l_1,\pm}(\hat{x})$ and so $|\tilde{J}| \geq K_1^{-1}L^{-\beta} \geq \beta$ holds. One now repeats the same arguments as for case (i), $l_1 > 0$. 

3.2.2 Proof of Proposition 4

To prove Proposition 4 we introduce the random interval process $(J_t)_{t \geq 0}$ of subintervals of $\mathbb{R}$, defined as follows. Fix $X_0 \in S^1$. To start, $J_0 := X_0 + [-\varepsilon, \varepsilon]$, regarded as an interval in $\mathbb{R}$. We set $J_1 := \hat{f}(J_0)$ and define $J_1$ to be the longest atom of $\mathcal{R}_{\omega_0}|_{J_1}$; if more than one atom has maximal length, then select $J_1$ to be the rightmost one. Inductively, given $J_0, \cdots, J_t$, define $J_{t+1} := \hat{f}_\omega(J_t)$ and $J_{t+1}$ to be the longest atom of $\mathcal{R}_{\omega_{t+1}}|_{J_{t+1}}$, with the same rule if there is a tie for longest atom.

We terminate the process $(J_t)$, at the stopping time $\tau := \min\{\sigma_1, \sigma_2\}$, where

$$\sigma_1 := \min\{i : |\tilde{J}_i| > L^{-\beta}\}, \quad \sigma_2 := \min\{i : J_i \subset B^k_{\omega_i}\}.$$  

**Lemma 6.** There exists $\tilde{N} = \tilde{N}(k, \beta) \in \mathbb{N}$ for which $\mathbb{P}_{X_0}(\sigma \leq \tilde{N} - 1) > 0$ holds.

**Proof of Proposition 4 assuming Lemma 6** Observe that for each $i \geq 0$,

$$\tilde{J}_i \subset \hat{f}^{\sigma_i - 1} \circ \hat{f}(\mathcal{N}_0(X_0)),$$

hence the projection $\tilde{J}_i$ (mod 1) of $\tilde{J}_i$ to $S^1$ is a subset of the support of the measure $\mathbb{P}_{X_0}(X_i \in \cdot | \{\omega_i\}_{i \neq 0})$.

On the event $\sigma = \sigma_1 = m$ for some $m \geq 0$, it is not hard to see that $|\hat{f}_{\omega_m}(\tilde{J}_m)| \gg 1$ (see Section 2), hence on the event $\{\sigma = \sigma_1\}$ we have $T \leq \sigma_1 + 1$. Meanwhile, $T \leq \sigma_2$ holds unconditionally (note $X_m \in B^k_{\omega_m}$ iff $X_m \in B^k_{\omega_{m+1}}$), hence

$$T \leq \sigma + 1$$

holds almost surely.

To complete the proof of Proposition 4 it remains to prove Lemma 6. 

**Proof of Lemma 6.** We will show that conditioned on $\{\sigma_2 > \tilde{N}\}$, we have $\sigma_1 \leq \tilde{N}$.

Define $t_1 = \min\{t : J_t \subset B_{\omega_0}\}$ and let $p_1 \in \{1, \cdots, k - 1\}$ be such that $J_{t_1} \subset B^p_{\omega_1}$. Inductively, for $j > 1$ set

$$t_j = \min\{t > t_{j-1} : J_t \subset B_{\omega_j}\}$$

and let $p_j$ be such that $J_{t_j} \subset B^p_{\omega_j}$. We let $q \geq 0$ be such that $t_q \leq \tilde{N} < t_{q+1}$ (note $q = 0$ is allowed).

At time $t_j$, the interval process $J_{t_j}$ is said to initiate a bound period of length $p_j$; that is, $J_{t_j+1}, \cdots, J_{t_{j+p_j}}$, shadow some postcritical orbit in the sense of Lemma 1. In particular, $t_j + p_j + 1 \leq t_{j+1}$ for all $j$. For $t_j + p_j + 1 \leq t_{j+1}$, we say that the interval $J_{t_j}$ is free.

When $t$ is free, expansion on $\mathcal{G} \cup \mathcal{I}$ (see Section 2) and Lemma 5 imply

$$|J_{t+1}| \geq \kappa |J_{t+1}| \geq \kappa L^\frac{1}{2} - \beta |J_t|,$$

(7)
while along bound periods (having conditioned on \( \{ \sigma_2 > \bar{N} \} \), it follows that \( p_j < k \) for all \( j \leq q \) we have

\[
J_{i_j + p_j, i_j + 1} \geq \kappa |J_{i_j + p_j, i_j + 1}| \geq \kappa L^{(\frac{1}{2} - \beta)(p_j + 1)} |J_{i_j}|
\]

(8)
since, by Lemma \( \square \) we have \( \tilde{J}_{i_j + p_j, i_j + 1} = \tilde{f}_{p_j, i_j}_{\tilde{a}} \) (i.e., no cutting can occur during a bound period). We obtain that when \( J_t \) is free, we have

\[
|\tilde{J}_t| \geq \left( \kappa L^{\frac{1}{2} - \beta} \right)^t \cdot 2\epsilon \geq L^{(\frac{1}{2} - \beta) \cdot \epsilon}.
\]

when \( L \) is sufficiently large. Since, for any \( t \), the interval \( J_t \) is free for at least one \( t' \in \{ t, \cdots, t + k \} \), and \( \epsilon \geq L^{-(2k+1)(1-\beta)+\beta} \) was assumed, it follows that \( \sigma_1 \leq \bar{N} \), where \( \bar{N} = \bar{N}(k, \beta) \) depends on \( k, \beta \) alone. \( \square \)

### 3.2.3 Proof of Proposition \( \square \)

Assume \( X_0 \in B_{\omega_0}^k \). We form what is essentially the same interval process as before, starting now with the interval

\( J_1 := X_1 + [-\epsilon, \epsilon] \),

again regarded as a subset of \( \mathbb{R} \), and taking \( \tilde{J}_2 := \tilde{f}(J_1) \), and \( J_2 \in \mathcal{R}|_{\tilde{J}_2} \) the longest atom. The intervals \( J_3, J_4, \cdots \) are defined the same as before.

As in the proof of Lemma \( \square \) no cutting occurs during the initial bound period of length \( k \), hence \( \tilde{J}_{k+1} = \tilde{f}^{k+1} \circ \tilde{f}(\mathcal{N}_c(X_1)) \). By Lemma \( \square \) and Lemma \( \square \), this implies

\[
|J_{k+1}| \geq \kappa |\tilde{J}_{k+1}| \geq L^{-(k+1)(1-\beta)+\beta/2},
\]

perhaps taking \( L \) sufficiently large (independently of \( k \)).

With \( t_1 = 0, p_1 = k \) and \( t_j, p_j, j \geq 2 \) defined as in the proof of Lemma \( \square \) note that if \( p_j < k \) then \( \square \) holds, while if \( t \) is free we have that \( \square \) holds. It remains to check that some interval growth occurs when \( p_j = k \); we do so below.

#### Lemma 7. Assume \( L \) is sufficiently large, depending on \( \beta \). Let \( J \subset B^k_{\omega_0} \) be an interval for which \( |J| \geq L^{-(k+1)(1-\beta)+\gamma} \) for some constant \( \gamma > \beta/2 \). Then, \( |\tilde{f}^{k+1}(J)| \geq L^{-(k+1)(1-\beta)+\frac{3}{2}\gamma} \).

**Proof.** It suffices to estimate the length of \( \tilde{f}_{\omega_0}(J) \). For this, let us subdivide \( J = J^+ \cup J^- \), where \( J^+ \) is to the right of the critical point and \( J^- \) to the left. WLOG let \( J^+ \) be the longer of the two intervals, so \( |J^+| \geq \frac{1}{2} |J| \) holds.

Writing \( J^+ = [\hat{x} - \omega_0, \hat{x} - \omega_0 + b^+] \), \( b^+ > 0 \) (noting \( b^+ \geq \frac{1}{2} |J| \)), we have

\[
(*) = |\tilde{f}_{\omega_0}(J^+)| = \int_{\hat{x} + b^+}^{\hat{x} + b^+} |\tilde{f}'(x)| dx \geq K_1 \int_0^{b^+} x dx = \frac{1}{2} K_1 (b^+)^2 \geq \frac{1}{8} K_1 |J|^{1/2}
\]

Plugging in the lower bound for \( |J| \) gives \( (*) \geq \frac{1}{8} K_1 L^{-2(k+1)(1-\beta)+2\gamma} \geq L^{-2(k+1)(1-\beta)+\frac{3}{2}\gamma} \). From here, using Lemma \( \square \) we estimate

\[
|\tilde{f}^{k+1}(J)| \geq |\tilde{f}_{\omega_0}^{k+1}(J)| \geq L^{-(k+1)(1-\beta)+\frac{3}{2}\gamma}.
\]

**Proposition \( \square \)** now follows from a similar argument to that for Lemma \( \square \) where \( N = N(k, \beta) \in \mathbb{N} \) and the constant \( c > 0 \) depends on \( N \) as well as \( L \). Details are left to the reader.

### 4 Itineraries and distortion

For the remainder of the paper we turn our attention to the proof of Theorem \( \square \). In essence, this proof will be an elaboration of the idea, used heavily in Section 3.2, that the predominant expansion of \( f = f_{L,a} \) has the effect of amplifying the noise \( \epsilon \). On the other hand, in Section 3.2 and the proof of ergodicity as in Theorem \( \square \) we were able to avoid exerting any precise control on the densities of the conditional laws
$P^n(X_0, \cdot \{[\omega, i \neq 0]\})$. For our purposes in Section 6, however, we will need some control on these densities, which amounts to controlling distortion of the random compositions $f^n_\omega$.

Our objective in Section 4, then, is to establish some control on the distortion of $f^n_\omega$. As is typical of systems exhibiting nonuniform expansion, distortion of $f^n_\omega$ for some $n \geq 1$ can only be controlled along sufficiently small intervals $J \subset S^1$ (see, e.g., [26]). Establishing just how small these intervals need to be is a crucial component of our argument.

In Section 4.1, we formulate *itineraries* for the random dynamics of $f^n_\omega$, a form of symbolic dynamics for the trajectories of $f^n_\omega$ with the property (checked in Section 4.2) that the distortion of $f^n_\omega$ can be controlled along subintervals with the same itinerary (symbolic sequence) out to time $n - 1$.

The preceding paragraphs apply equally well to deterministic as well as random compositions of interval maps—indeed, the assignment of itineraries to control distortion is an old idea (see the references in [26] for more information). Something to keep in mind, however, is that since the condition (H3)$_{c,k}$ only guarantees bound periods up to length $k$, we lose control of the dynamics of $f^n_\omega$ upon the first visit to the ‘worst possible’ neighborhood $B^k$ of $\{f' = 0\}$. Thus the itinerary subdivision procedure and and resulting distortion estimates we obtain below are only valid up until this first visit to $B^k$. This issue will be addressed in Section 5.

### 4.1 Itineraries

Throughout, in addition to the preparations in Section 2.1.1, we assume the parameter $\epsilon$ satisfies the upper bound (4), so that Lemma (4) holds. No lower bound on $\epsilon$ is assumed.

(A) Partition construction.

To start, we define the partition $\mathcal{P}$ of $S^1$ as follows. Recall the notation $\mathcal{B}^0 = \mathcal{I}$.

- $\mathcal{P}|_\mathcal{G}$ is the partition of $\mathcal{G}$ into connected components.
- To define $\mathcal{P}|_{\mathcal{B}^l}$, $0 \leq l < k$, start by cutting $\mathcal{B}^l$ into connected components. For each such component $J$, $\mathcal{P}|_J$ is defined as any partition of $J$ into intervals of length

$$
\epsilon \in [(l + 1)^{-2} L^{\frac{-143}{2}} - \beta, 2(l + 1)^{-2} L^{\frac{-143}{2}} - \beta].
$$

- $\mathcal{P}|_{\mathcal{B}^k}$ is the partition of $\mathcal{B}^k$ into connected components.

We write $\mathcal{P}_\omega$ for the partition of $S^1$ with atoms of the form $C - \omega, C \in \mathcal{P}$. Abusing notation somewhat, we regard $\mathcal{P}, \mathcal{P}_\omega$ as partitions of $\mathbb{R}$, extended by periodicity.

**Definition 8.** For a bounded, connected interval $I \subset S^1$ (or $\subset \mathbb{R}$) which is not a singleton, we define the partition $\mathcal{P}_\omega(I)$ of $I$ as follows. To start, form $\mathcal{P}_\omega|_I = \{J \cap I : J \in \mathcal{P}_\omega, J \cap I \neq \emptyset\}$, and write $J_1, J_2, \cdots, J_N$ for the non-singleton atoms of this partition in increasing order from left to right (note that $N = 1$ is possible).

- If $N = 1, 2$ or 3, then set $\mathcal{P}_\omega(I) := \{I\}$.
- If $N \geq 4$, then set $\mathcal{P}_\omega(I) = \{J_1 \cup J_2, J_3, J_4, \cdots, J_{N-2}, J_{N-1} \cup J_N\}$.

We define the *bound period* $p(I)$ of an interval $I$ as follows. First, $p : S^1 \to \{0, \cdots, k\}$ (or $\mathbb{R} \to \{0, \cdots, k\}$) is defined by setting $p|_{\mathcal{B}^l} := p$ for all $1 \leq p \leq k$, and $p_{|\mathcal{I} \cup \mathcal{G}} = 0$. Next, for an interval $I \subset S^1$ or $\mathbb{R}$, we define

$$p(I) = \max_{x \in I} p(x).$$

For $\omega \in [-\epsilon, \epsilon]$, we define $p_\omega(\cdot) = p(\cdot - \omega)$.

**Remark 9.** For an atom $C \in \mathcal{P}$ or $\mathcal{P}_\omega$, write $C^+$ for the union of $C$ with its two adjacent atoms. Observe that for any interval $I$, we have that each atom $J \in \mathcal{P}_\omega(I)$ is contained in $C^+$ for some $C \in \mathcal{P}_\omega(I)$. By this line of reasoning, for any $J \in \mathcal{P}_\omega(I)$ with $p = p(J) \in \{1, \cdots, k - 1\}$, we have the estimate

$$|J| \leq 6p^{-2} L^{\frac{-143}{2}} - \beta.$$
Similarly, if \( J \in \mathcal{P}_\omega(I) \), \( J \cap \mathcal{B}^k_\omega \neq \emptyset \) (i.e. \( p(J) = k \)) then \(|J| \leq 3 \max\{1, K_1^{-1}\} L^{-\frac{k}{2} - \beta} \).

For a lower bound: if in the above setting we have that there are at least two distinct atoms in \( \mathcal{P}_\omega(I) \), then any atom \( J \in \mathcal{P}_\omega(I) \) with \( p = p_\omega(J) > 0 \) must contain an atom \( C \in \mathcal{P}_\omega|_{G_\omega} \). Thus
\[
|J| \geq (p + 1)^{-2} L^{-\frac{p+1}{2} - \beta}.
\]

**Remark 10.** Fix a sample \( \omega \in \Omega \) and let \( J \) be a connected interval contained in \( C^+ \) for some \( C \in \mathcal{P}_\omega \). If \( p := p_\omega(J) > 0 \), then
\[
\tilde{f}_\omega^i(J) \subset \mathcal{G} \quad \text{for all } 1 \leq i \leq p,
\]
even though \( J \) is not necessarily a subset of \( \mathcal{B}^k_\omega \). This is because \( \mathcal{P}|_{B^{p-1}} \)-atoms are small enough so that \( J \subset B(-\frac{p+1}{2} \beta) \) must hold, Lemma\[i\]
implies that \( \tilde{f}_\omega^i(B(-\frac{p+1}{2} \beta)) \subset \mathcal{G} \) for all \( 1 \leq i \leq p \) and all samples \( \omega \). Note, in particular, that \( \tilde{f}_\omega^i(J) \) meets at most one component of \( \mathcal{G} \) for each \( 1 \leq i \leq p \), hence \( \mathcal{P}_\omega(\tilde{f}_\omega^i(J)) = \{ \tilde{f}_\omega^i(J) \} \).

**(B) Time-\( n \) itineraries for an interval \( I \subset S^1 \).**

Let \( I \subset S^1 \) be an interval (which we regard as a subset of \( \mathbb{R} \)) and fix a sample \( \omega \in \Omega \). For each time \( i \geq 1 \), we define a partition \( \mathcal{Q}_i = \mathcal{Q}_i(I; (\omega_0, \ldots, \omega_i)) \) of \( I \), the atoms of which correspond to points in \( I \) with the same itinerary for the map \( \tilde{f}_\omega^i \).

The definition is inductive. To start, we define \( \mathcal{Q}_0 = \mathcal{P}_\omega(I) \). Assuming \( \mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_i \) have been constructed, for each \( C_i \in \mathcal{Q}_i \), we define \( \mathcal{Q}_{i+1} \geq \mathcal{Q}_i \) as follows:
\[
\mathcal{Q}_{i+1} \cap C_i = (\tilde{f}_\omega^i)^{-1}\left\{ \mathcal{P}_\omega_{i+1}(\tilde{f}_\omega^i(C_i)) \right\}.
\]

In what follows, we will only attempt to keep track of itineraries until a first “near visit” to the set \( \mathcal{B}^k \). Precisely, we define a ‘terminating’ stopping time \( \tau = \tau[I] : I \times \Omega \rightarrow \mathbb{Z}_{\geq 0} \cup \{ \infty \} \) as follows:
\[
\tau(x, \omega) = \min\{i \geq 0 : \tilde{f}_\omega^i(C_i(x)) \cap \mathcal{B}^k_\omega \neq \emptyset\}.
\]

Here, \( C_i(x) \) denotes the \( \mathcal{Q}_i \)-atom containing \( x \). Notice that \( \tau \) is adapted to \( (\mathcal{Q}_i)_i \), i.e., \( \{ \tau > i \} \) is a union of \( \mathcal{Q}_i \)-atoms for each \( i \geq 0 \). In particular, \( \{ \tau > i \} \) depends only on \( \omega_0, \ldots, \omega_i \).

**(C) Bound and free periods of an itinerary**

Fix \( n \geq 1 \) and \( C_n \in \mathcal{Q}_n \) such that \( \tau|_{C_n} \geq n \). For each \( i < n \), let \( C_i \in \mathcal{Q}_i \) denote the atom containing \( C_n \). For \( 1 \leq i \leq n \), we write \( I_i = \tilde{f}_\omega^i(C_i) \).

Define
\[
t_1 = \min\{n\} \cup \{i \geq 0 : I_i \cap \mathcal{B}_{\omega_i} \neq \emptyset\}, \quad \text{and}
\]
\[
t_j = \min\{n\} \cup \{i > t_{j-1} : I_i \cap \mathcal{B}_{\omega_i} \neq \emptyset\} \quad \text{for } j \geq 2,
\]
and let \( q \geq 0 \) be the index for which \( t_{q+1} = n \). For \( 1 \leq j \leq q \), define
\[
p_j = p_{\omega_j}(I_j).
\]

At time \( t_j, 1 \leq j \leq q \), the itinerary \( C_n \) initiates a bound period of length \( p_j \) (Remark\[i\]): in particular, \( t_j \) is a period for all \( 1 \leq j < q \). We say that \( C_n \) is **bound at time** \( t \) if \( t \in [t_j + 1, t_j + p_j] \) for some \( 1 \leq j < q \) and that \( C_n \) is **free at time** \( t \) if it is not bound at time \( t \).

By Remark\[i\] and the fact that \( \tau|_{C_n} \geq n \), we have the following.

**Lemma 11.** Let \( 1 \leq i \leq n \) and assume \( C_n \in \mathcal{Q}_n \) is such that \( \tau|_{C_n} \geq n \).

(a) If \( C_n \) is **free at time** \( i \), then
\[
|\tilde{f}_\omega^i|_{C_n} \geq L^{(\frac{i}{2} - \beta)}.
\]

(b) If \( C_n \) is **bound at time** \( i \), i.e., \( i \in [t_j + 1, t_j + p_j] \) for some \( 1 \leq j \leq q \), then
\[
|\tilde{f}_\omega^i|_{C_n} \geq L^{t_j \left( \frac{i}{2} - (1-\beta)(i-(t_j+1)) - \frac{p_j-1}{2} \beta \right)}.
\]

In this case, \( C_{t_j} = C_{t_j+1} = \cdots = C_{t_j+p_j} = C_i \) and \( C_n \) is **free at time** \( t_j + p_j + 1 \). Note that \( C_{t_j+p_j+1} \subset C_i \) is possible.

\[1\] Here, for two partitions \( \zeta, \xi \), we write \( \zeta \leq \xi \) if each atom of \( \zeta \) is a union of \( \xi \)-atoms.
4.2 Distortion estimates

Let \( I \subset S^1 \) be a connected interval, \( \omega \in \Omega \) a sample. Assume that the partitions \((Q_i)_{i \geq 0}, Q_i = Q_i(I; (\omega_0, \ldots, \omega_i))\) and the stopping time \( \tau = \tau[I] \) have been constructed as in Section 4.1. Here we prove a time-\( n \) distortion estimate for trajectories with the same time-\( n \) itineraries, i.e., belonging to the same \( \mathcal{Q}_n \)-atom.

Our approach to distortion estimates is inspired from the treatment in [26], which in turn is a version of estimates first appearing in [45].

**Proposition 12.** For all \( L \) sufficiently large, the following holds. Let \( n \geq 1 \). Assume \( C_n \in \mathcal{Q}_n \) is free at time \( n \) and \( \tau|_{C_n} \geq n \). Let \( x, x' \in C_n \). Then,

\[
\left( \frac{\tilde{f}_n^x \omega}(x) - \tilde{f}_n^x \omega(x') \right) \leq e^{K_2 L^{-\frac{4}{3} + 4\|\psi''\|_{C^0} L^2} \|\psi\|_{C^0} L^{-1 - 2\eta}|f_n^x x - f_n^x x'|}. 
\]

We start with a preliminary Lemma.

**Lemma 13.** Let \( L \) be sufficiently large, and let \( \eta \in [-\frac{3}{4}, 0] \). Let \( y, y' \in S^1, i \geq 1 \), and define \( J \) to be the interval between \( y, y' \). If \( \tilde{f}_n^x(J) \subset B(\eta)^c \) for all \( 0 \leq j < i \), then

\[
\left| \log \frac{\tilde{f}_n^x(y)}{\tilde{f}_n^x(y')} \right| \leq 2\|\psi''\|_{C^0} L^{-1 - 2\eta}|\tilde{f}_n^x(y) - \tilde{f}_n^x(y')| .
\]

**Proof.** Define \( y_j = \tilde{f}_n^x y, y_j' = \tilde{f}_n^x y' \). We estimate

\[
(*) \quad \left| \log \frac{\tilde{f}_n^x(y_j)}{\tilde{f}_n^x(y_j')} \right| \leq \sum_{j=0}^{i-1} \left| \log \frac{\tilde{f}_n^x(y_j)}{\tilde{f}_n^x(y_{j+1})} \right| \leq \sum_{j=0}^{i-1} \frac{L\|\psi''\|_{C^0} L^{1+\eta} |y_j - y_j'|}{L^{1+\eta}} = \|\psi''\|_{C^0} L^{-\eta} \sum_{j=0}^{i-1} |y_j - y_j'| .
\]

We bound \( |y_j - y_j'| \leq L^{-1 + \eta(i-j)}|y_i - y_i'| \), hence

\[
(*) \leq \|\psi''\|_{C^0} L^{-\eta} \left( \sum_{j=0}^{i-1} L^{-1 + \eta(i-j)} \right) |y_i - y_i'| \leq 2\|\psi''\|_{C^0} L^{-1 - 2\eta} |y_i - y_i'|. 
\]

In view of \([26]\), observe that the above estimates can be written in the following alternative form: writing \( J_j \) for the interval between \( y_j, y_j' \), we have that

\[
\sum_{j=0}^{i-1} \frac{|J_j|}{d(J_j, C^0 \omega - \omega_j)} \leq 2\|\psi''\|_{C^0} L^{-1 - 2\eta} |J_i| .
\]

**Proof of Proposition 12.** Below, we write \( C \) to refer to a generic positive constant; the value of \( C \) may change from line to line, but always depends only on the function \( \psi \).

With \( n \geq 1 \) and \( C_n \in \mathcal{Q}_n \) fixed and free at time \( n \), we adopt the notation of Section 4.1 (C). Write \( x_i = \tilde{f}_n^x(x), x_i' = \tilde{f}_n^x(x') \). By hypothesis, \( x, x' \) belong to the same \( \mathcal{Q}_i \) atom \( C_i \) for all \( 0 \leq i \leq n \).

We decompose

\[
\left| \log \frac{\tilde{f}_n^x \omega(x)}{\tilde{f}_n^x \omega(x')} \right| \leq \sum_{i=0}^{n-1} \left| \log \frac{\tilde{f}_n^x \omega(x_i)}{\tilde{f}_n^x \omega(x_i')} \right|.
\]

Using \([22]\), each summand is bounded by

\[
\left| \log \frac{\tilde{f}_n^x \omega(x_i)}{\tilde{f}_n^x \omega(x_i')} \right| \leq C \frac{|J_i|}{d(J_i, C^0 \omega - \omega_i)} ,
\]

where \( J_i \) is the interval from \( x_i + \omega_i \) to \( x_i' + \omega_i \).

\[\text{\(\square\)}\]
With \( t_j, p_j \) as in \([9], [10]\), we decompose the time interval from 0 to \( n \) into the succession of free and bound periods experienced by the atom \( C_n \in Q_n \) containing \( x, x' \):

\[
0 \leq t_1 < t_1 + p_1 < t_2 < t_2 + p_2 < \cdots < t_q < t_q + p_q < t_{q+1} := n.
\]

We assume going forward that \( q \geq 1 \), i.e., \( C_n \) experiences at least one bound period. If \( q = 0 \), then Proposition [12] follows easily from Lemma [13] applied to \( \eta = -\frac{1}{2} - \beta \); details are left to the reader.

We now decompose \( \sum_{i=0}^{n-1} \) as follows:

\[
\sum_{i=0}^{n-1} \frac{|J_i|}{d(J_i, C'_\psi - \omega_i)} = \sum_{i=0}^{t_1-1} q_i + \sum_{j=t_1}^{t_j+p_j} \left( \sum_{i=t_j}^{t_j+p_j} \sum_{i=t_j+p_j+1}^{t_j+1} \right) =: D_0 + \sum_{j=1}^{q} (D_j + D_j')
\]

Above, a summand of the form \( \sum_{m=1}^{m-1}, m \in \mathbb{N} \) is regarded as empty and the corresponding summation is defined to be 0 (as may happen for some of the \( D_j' \) terms). The \( D_j, D_j' \) are estimated separately below.

Before proceeding, observe that \( |J_{t_j+p_j+1}| \geq L^{(p_j+1)(\frac{1}{2} - \beta)}|J_{t_j}| \) and \( |J_{t+1}| \geq L^{(\frac{1}{2} - \beta)}|J_t| \) for all \( t \) such that \( C_t, C_{t+1} \) are free. In particular,

\[
|J_{t_j+1}| \geq L^{(t_j+1-t_j)(\frac{1}{2} - \beta)}|J_{t_j}|
\]

for all \( 1 \leq i \leq q \).

**Bounding \( \sum_{j=1}^{q} D_j \):** Let \( 1 \leq j \leq q \).

**Claim 14.**

\[
\sum_{i=t_j+1}^{t_j+p_j} \frac{|J_i|}{\text{dist}(J_i, C'_\psi - \omega_i)} \leq CL^{2\beta} \frac{|J_{t_j}|}{d(J_{t_j}, C'_\psi - \omega_{t_j})}
\]

Assuming the Claim, we now bound \( \sum_{j=1}^{q} D_j \). For \( 1 \leq p < \kappa \), let \( K_p = \{ 1 \leq j \leq q : p_j = p \} \). Let \( j^*_p = \max K_p \), and observe that \( |J_{t_j}| \leq |J_{t_j^*}| \cdot L^{-((t_j-t_j^*)(\frac{1}{2} - \beta))} \) for all \( j \in K_p \) by \([11]\). Thus

\[
\sum_{j \in K_p} D_j \leq CL^{2\beta} \sum_{j \in K_p} \frac{|J_{t_j}|}{\text{dist}(J_{t_j}, C'_\psi - \omega_{t_j})} \leq \frac{CL^{2\beta}}{1 - L^{-(\frac{1}{2} - \beta)}} \cdot \frac{|J_{t_j^*}|}{\frac{1}{2}K_1^{-1}L^{-\frac{s_p}{2} - \beta}} \leq CL^{2\beta} \frac{|J_{t_j^*}|}{L^{-\frac{s_p}{2} - \beta}}
\]

Here we are using that \( \text{dist}(J_{t_j}, C'_\psi - \omega_{t_j}) \geq \frac{1}{2}K_1^{-1}L^{-\frac{s_p}{2} - \beta} \) for all \( j \in K_p \). By Remark \([9]\) we have \( |J_{t_j^*}| \leq 6p-2L^{-\frac{s_p}{2} - \beta} \). So,

\[
\sum_{j \in K_p} D_j \leq CL^{2\beta} \frac{6pL^{-\frac{s_p}{2} - \beta}}{L^{-\frac{s_p}{2} - \beta}} \leq Cp^{-2}L^{-\frac{1}{2} + 2\beta}
\]

hence

\[
\sum_{j=1}^{q} D_j = \sum_{j=1}^{K-1} \sum_{p=1}^{j} D_j \leq \sum_{p=1}^{K-1} Cp^{-2}L^{-\frac{1}{2} + 2\beta} \leq CL^{-\frac{1}{2} + 2\beta}.
\]

**Proof of Claim.** Assume \( I_{t_j} \) meets the component of \( B_{\omega_{t_j}} \) near \( x_{t_j} \in C_{\psi} - \omega_{t_j} \); write \( \hat{x}_i = \hat{f}^{t_i-t_j}(\hat{x}_{t_j}) \) for \( i > t_j \). Assume, without loss, that

\[
|x'_{t_j} - \hat{x}_{t_j}| \leq |x_{t_j} - \hat{x}_{t_j}| ;
\]

in the alternative case, exchange the roles of \( x_i, x'_i \) in what follows.

For \( t_j < i \leq t_j + p_j \), we have

\[
\frac{|J_i|}{\text{dist}(J_i, C'_\psi - \omega_i)} = \frac{|x_i - x'_i|}{|x_i - \hat{x}_i|} \cdot \frac{|x_i - \hat{x}_i|}{\text{dist}(J_i, C'_\psi - \omega_i)}
\]
By Lemmas 11 and 13, we have that the first right-hand factor is

$$\leq 2 \frac{|x_{t,j+1} - x_{t,j+1}'|}{|x_{t,j+1} - x_{t,j+1}'|}$$

The numerator of 13 coincides with $|f''_{\omega_j}(\zeta)| \cdot |x_{t,j} - x_{t,j}'| \cdot (\zeta - \hat{x}_{t,j}| \leq L \|\psi''\|_{C^0} |\zeta - \hat{x}_{t,j}| |x_{t,j} - x_{t,j}'| \cdot |x_{t,j} - x_{t,j}'|$, for some $\zeta \in J_{t,j}$. Moreover, $|f''_{\omega_j}(\zeta)| = |f''_{\omega_j}(\zeta')| \cdot |\zeta - \hat{x}_{t,j}| \leq L \|\psi''\|_{C^0} |\zeta - \hat{x}_{t,j}| |x_{t,j} - x_{t,j}'|$, for some $\zeta'$ between $\zeta$ and $\hat{x}_{t,j}$. By (12) we have $|\zeta - \hat{x}_{t,j}| \leq |x_{t,j} - x_{t,j}'|$, and so conclude that the numerator of 13 is $\leq L \|\psi''\|_{C^0} |x_{t,j} - x_{t,j}'| (x_{t,j} - \hat{x}_{t,j})^2$ for some $\zeta''$ between $x_{t,j}$ and $\hat{x}$.

For $L$ sufficiently large and all $\varepsilon$ satisfying (1), we have that $\min_{x \in N(0)} = \frac{1}{2} \min_{x \in N(0)} (\hat{x}) : \hat{x} \in C'_{\varepsilon}$ from (H1), (H2). We have therefore that the denominator of 13 is $\geq \frac{1}{2} c_1 L |x_{t,j} - \hat{x}_{t,j}|^2$.

Collecting,

$$\frac{|J|}{\text{dist}(J, C'_\varepsilon - \omega_i)} \leq C \frac{|J|}{\text{dist}(J, C'_\varepsilon - \omega_i)} \frac{|J|}{\text{dist}(J, C'_\varepsilon - \omega_i)},$$

since $|x_{t,j} - \hat{x}_{t,j}|^{-1} \leq d(J, C'_\varepsilon - \omega_i)^{-1}$ by assumption, and so

$$\sum_{i=t,j}^{t_1+j} \frac{|J|}{d(J, C'_\varepsilon - \omega_i)} \leq C \frac{|J|}{\text{dist}(J, C'_\varepsilon - \omega_i)} \left( \sum_{i=t,j}^{t_1+j} \frac{|J|}{\text{dist}(J, C'_\varepsilon - \omega_i)} \right) \frac{|J|}{\text{dist}(J, C'_\varepsilon - \omega_i)}.$$

By Lemma 13 applied to $\eta = -\beta$, the parenthetical sum is bounded $\leq CL^{1+2\beta} |x_{t,j+p_1} - x_{t,j+p_1}|$. Since $|x_{t,j+p_1} - x_{t,j+p_1}| \leq L^{-\beta/2} \ll 1$ (see the proof of Lemma 1), we bound $|x_{t,j+p_1} - x_{t,j+p_1}| \leq CL$, hence the parenthetical sum is $\leq CL^{2\beta}$. This completes the proof.

Bounding $\sum_{j=0}^{q} D'_j$. For each $1 \leq j < q$, we have from Lemma 13 applied to $\eta = -\frac{1}{2} - \beta$ that

$$D'_j \leq L^{1-2(-\frac{1}{2} - \beta)} |J_{t,j+1}| = CL^{2\beta} |J_{t,j+1}|.$$

Similarly, we estimate $D'_0 \leq CL^{2\beta} |J_0|$. Since $|J_0| \leq L^{-\beta/2} \ll 1$ for all $1 \leq j \leq q$ by (11), we conclude $\sum_{j=0}^{q} D'_j \leq CL^{2\beta} |J_0|$. The proof of Proposition 12 is now complete.

5 Selective averaging process

We aim to get more refined control on the conditional laws $P^n(X_0, \cdot |\{\omega_i, i \neq 0\}, n \geq 0$. Towards this end, the itinerary subdivision procedure in Section 4 applied to $J = X_0 + [-\varepsilon, \varepsilon]$ can be used to control the density of $P^n(X_0, \cdot |\{\omega_i, i \neq 0\}, X_0 + \omega_0 \in C_n)$ for some $C_n \in \mathcal{Q}_n$, i.e., conditioning on $X_0 + \omega_0$ belonging to a single subdivision $C_n$ of $\mathcal{Q}_n$. This is only valid, however, up until the first ‘near visit’ to $B^k$, the closest neighborhood to the critical set $\{f'' = 0\}$. Afterwards, the material in Section 4 is no longer valid and we lose control over distortion, hence over the conditioned law $P^n(X_0, \cdot |\{\omega_i, i \neq 0\})$.

A rough idea of how to proceed is as follows: visits to $B^k$ ‘spoil’ the random parameter $\omega_0$, and so if $X_m$ comes too close to $B^k$ for some $m \geq 0$, we will ‘freeze’ $\omega_0$ (essentially, treat as deterministic) and ‘smear’ (average) the record of perturbation $\omega_{m+1}$, i.e., for $n \geq m$, work with the conditional law $P^n(X_0, \cdot |\{\omega_i, i \neq m + 1\})$.

Let us make all this more precise. Fix $X_0 \in S^1$ and define the Markov chain $(\tilde{X}_n)$ on $\mathbb{R}$ by $\tilde{X}_n = \tilde{f}^n_{m}(X_0) = \tilde{f}^n_{m-1} (\tilde{X}_{n-1})$. We will obtain in this section an increasing filtration $(\mathcal{H}_n)_{n \geq 0}$, $\mathcal{H}_n \subset \mathcal{F}_n := \sigma(\omega_0, \omega_1, \cdots, \omega_n)$ (depending also on $X_0$), designed so that the conditional measures

$$\nu_n(\cdot) := \mathbb{P}(\tilde{X}_n \in \cdot |\mathcal{H}_n)$$

have the following desirable properties:

(i) the measures $\nu_n$ are absolutely continuous;

(ii) $\rho_n := \frac{d\nu_n}{d\text{Leb}}$ is more-or-less constant on the interval of support $I_n := \text{supp} \nu_n$; and
(iii) the intervals $I_n = \text{supp}(\nu_n)$ are, for large $n$, rather long with high probability.

In this section, we focus on the construction of $\mathcal{H}_n, I_n, \nu_n$ as above; property (ii) will fall out as a natural consequence of our construction and the distortion estimate in Proposition 12.

The plan is as follows: first, in Section 5.1 we will describe an algorithm constructing the supporting intervals $I_n$ as above, in a way completely parallel to the itinerary construction given in Section 4.1. From this construction, it will be clear when ‘smearing’ in a new $\omega$ is necessary: this decision is made according to a sequence $\tau_1 < \tau_2 < \cdots$ of stopping times roughly related to the first arrival to the neighborhood $B^k$ (closely related to the stopping time $\tau$ as in Section 4.1). In Section 5.2 we will construct the filtration $(\mathcal{H}_n)$ and then describe the resulting conditional measures $\nu_n$ in Section 5.3.

In addition to the preparations in Section 2.1.1 we assume the parameter $\epsilon$ satisfies (1), so that Lemma 7 holds. No lower bound on $\epsilon$ is assumed.

5.1 The supporting intervals $I_n$

We define here an interval-valued stochastic process $(I_n)_{n \geq 1}$ for which $I_n \subset \mathbb{R}$ is $\mathcal{F}_n$-measurable for all $n$.

Embed $X_0 =: \tilde{X}_0 \in \mathbb{R}$ via the identification $S^1 \cong [0, 1)$. Throughout, the dependence of the $I_n$ on the sample $\omega = (\omega_i)_{i \geq 0} \in \Omega$ is implicit (keeping in mind that $I_n$ depends on $\omega_i, 0 \leq i \leq n$).

**Base cases:** Set $I_0 = \tilde{X}_0 + [-\epsilon, \epsilon]$. To determine $I_1$, there are two cases:

- If $I_0 \cap B^k_{\omega_0} \neq \emptyset$, then define $I_1 = \tilde{X}_1 + [-\epsilon, \epsilon]$.
- Otherwise, form $\mathcal{P}_{\omega_1}(\hat{f}(I_0))$ and let $I_1$ be the atom containing $\tilde{X}_1$.

Note that since $\epsilon > 0$ is assumed to satisfy (4), we have automatically that $\mathcal{P}(I_0)$ consists of a single atom.

**Inductive step:** Assume the intervals $I_0, I_1, \cdots, I_n$ have been constructed, with $n \geq 1$.

(a) If $I_n \cap B^k_{\omega_n} = \emptyset, I_{n-1} \cap B^k_{\omega_{n-1}} = \emptyset$, then form $\mathcal{P}_{\omega_{n+1}}(\hat{f}_{\omega_n}(I_n))$ and define $I_{n+1}$ to be the atom containing $\tilde{X}_{n+1}$.

(b) If $I_n \cap B^k_{\omega_n} \neq \emptyset$, then define $I_{n+1} = \tilde{X}_{n+1} + [-\epsilon, \epsilon]$. Form $\mathcal{P}_{\omega_{n+1}}(\hat{f}(I_{n+1}))$ and let $I_{n+2}$ be the atom containing $\tilde{X}_{n+2}$.

From Lemma 1 and Remark 10 it is simple to check that cases (a) – (b) are exhaustive and mutually exclusive. Note in case (b) that $I_{n+1} \subset G_{\omega_{n+1}}$ holds (Lemma 4 and 11).

**Definition 15.** We define a sequence of $(\mathcal{F}_n)$-adapted stopping times $0 =: \tau_0 < \tau_1 < \tau_2 < \cdots$ as follows: for $i > 0$, set

$$\tau_i = \min\{m > \tau_{i-1} : I_m \cap B^k_{\omega_m} \neq \emptyset\}.$$

Observe that case (b) above is observed iff $n = \tau_i$ for some $i$.

As formulated below, between ‘near visits’ to $B^k$ (i.e., the times $\tau_1, \tau_2, \cdots$), the procedure defining the $(I_n)$ process is completely parallel to the itinerary construction in Section 4.1. The proof is straightforward and left to the reader.

**Lemma 16.** Fix $i \geq 0$ and $0 \leq m < n$. 

\footnote{For our purposes, an interval is a bounded, connected subset of $\mathbb{R}$, with either open or closed endpoints. Since we care only about $P$-typical trajectories, we need not specify what to do with endpoints.}
Let us first describe more transparently what the conditional measures are. To start, for \( \omega \in \mathcal{O} \), the conditional measures \( \nu \) consist of atoms of the form

\[
\{ \omega_0 \} \times \{ \omega_1 \} \times \cdots \times \{ \omega_m \} \times J \times \{ \omega_{m+2} \} \times \cdots \times \{ \omega_n \},
\]

where \( J \) ranges over the atoms of \( \mathcal{Q}_{n-m-1}(I_{m+1};(0,\omega_{m+2},\cdots,\omega_n)) \) (i.e., we have \( \tau_i > n \in \mathcal{F}_n \) for all \( i, n \)). Define as well the events \( S_{i,m,n} = \{ \tau_i = n-1 \} \), and observe that the collection

\[
\mathcal{P}_n = \{ S_{i,m,n} : i \geq 1 \} \cup \{ S_{i,m,n} : i \geq 1, 0 \leq m < n \}
\]

is a partition of \( [-\epsilon, \epsilon]^{n+1} \). We define \( A_n \geq \mathcal{P}_n \) on each \( \mathcal{P}_n \)-atom separately.

- For each set of the form \( S_{i,m,n} \in \mathcal{P}_n, i \geq 0, 0 \leq m < n \), we define \( A_n \mid S_{i,m,n} \) to consist of atoms of the form

\[
\{ \omega_0 \} \times \{ \omega_1 \} \times \cdots \times \{ \omega_m \} \times J \times \{ \omega_{m+2} \} \times \cdots \times \{ \omega_n \},
\]

as \( J \) ranges over the atoms of \( \mathcal{Q}_{n-m-1}(I_{m+1};(0,\omega_{m+2},\cdots,\omega_n)) \). Here we identify \( [-\epsilon, \epsilon] \) with \( I_{m+1} = \hat{X}_{m+1} + [-\epsilon, \epsilon] \) in the obvious way.

- On each set \( S_{i,n} \in \mathcal{P}_n, i \geq 1 \), we define \( A_n \mid S_{i,n} \) to consist of atoms of the form

\[
\{ \omega_0 \} \times \{ \omega_1 \} \times \cdots \times \{ \omega_{n-1} \} \times [-\epsilon, \epsilon].
\]

With \( A_n \) completely described, the construction of \( \mathcal{H}_n := \sigma(A_n) \) is complete. It is not hard to check that \( \mathcal{H}_n \) is a filtration, i.e., \( \mathcal{H}_n \supset \mathcal{H}_{n-1} \): to do this, one verifies that the partition sequence \( A_n \) is increasing by inspecting each \( \mathbb{B}_n \)-atom separately.

The following is a straightforward consequence of Lemma 16.

**Lemma 17.** For each \( n \geq 1 \), the random interval \( I_n \) is \( \mathcal{H}_n \)-measurable. Moreover, the measure \( \nu_n(\cdot) = \mathbb{P}(\hat{X}_n \in \cdot \mid \mathcal{H}_n) \) satisfies \( \text{supp}(\nu_n) = I_n \).

### 5.3 The conditional measures \( \nu_n \)

Let us first describe more transparently what the conditional measures \( \nu_n(\cdot) = \mathbb{P}(\hat{X}_n \in \cdot \mid \mathcal{H}_n) \) actually are. To start, for \( \omega \in S_{i,m,n}, i \geq 0, n \geq 1 \), we have that \( \nu_n = \delta_{\hat{X}_n} \ast \nu^{\epsilon} \) is the uniform distribution on \( I_n = \hat{X}_n + [-\epsilon, \epsilon] \). The following characterizes \( \nu_n \) on the event \( S_{i,m,n}, i \geq 0, 0 \leq m < n \).

**Lemma 18.** Let \( i \geq 0, 0 \leq m < n \) and condition on the event \( S_{i,m,n} = \{ \tau_i = m, \tau_{i+1} \geq n \} \). Define \( \hat{F}_{m,n} : [-\epsilon, \epsilon] \to \mathbb{R} \) to be the map sending \( \omega \to \hat{X}_n = \hat{f}_{\omega_{m-1}} \circ \cdots \circ \hat{f}_{\omega_{m+2}} \circ \hat{f}_{\omega}(\hat{X}_{m+1}) \).

Let \( J \in \mathcal{Q}_{n-m-1}(\hat{X}_{m+1};(0,\omega_{m+2},\cdots,\omega_n)) \) (regarded as a partition of \( [-\epsilon, \epsilon] \)) be the atom containing \( \omega_{m+1} \). Then, \( F_{m,n} : J \to I_n \) is a diffeomorphism, and

\[
\nu_n = \frac{1}{\nu^{\epsilon}(J)}(\hat{F}_{m,n})_{*}(\nu^{\epsilon}|_J). \tag{14}
\]
The proof is a case-by-case verification of the above formula and is left to the reader.

Recall that \( J \subset [-\epsilon, \epsilon] \) appearing in (13) has the property that points in \( \tilde{X}_{m+1} + J \) have the same itinerary under \( \tilde{f}_{g_m + \alpha} \). In that notation, we have that the density \( \rho_n = \frac{d\nu_n}{d\tilde{\nu}_n} \) at a point \( x \in I_n \) is, up to a constant scalar, given by

\[
(\tilde{F}_{m,n})'(\omega) = (f_{g_n + \alpha} \circ \tilde{f})'(\tilde{X}_{m+1} + \omega)
\]

where \( \omega \in [-\epsilon, \epsilon] \) is such that \( x = \tilde{F}_{m,n}(\omega) \). In view of Proposition 19 and Lemma 10 then, we obtain a distortion estimate for the density \( \rho_n = \frac{d\nu_n}{d\tilde{\nu}_n} \).

**Corollary 19.** Let \( n \geq 1 \) be such that \( I_n \) is free. Then, for all \( x, x' \in I_n \), we have the estimate

\[
\frac{\rho_n(x)}{\rho_n(x')} \leq \exp\left( K_2L^{-1/2} + 4\|\psi''\|C\beta|x - x'| \right).
\]

(15)

6 Lyapunov exponents

Finally, we come to the estimation of Lyapunov exponents in Theorem 13. Throughout, we assume the setup of Section 2.1.1 and that \( \epsilon \geq L^{-2(k+1)(1-\beta)+\alpha} \) for some \( \alpha \geq 0 \). By Theorem A it follows that there is a unique ergodic stationary measure \( \mu \) supported on \( S^1 \).

By (a version of) the Birkhoff ergodic theorem (see Corollary 2.2 on pg. 24 of 18), we have that

\[
\lambda = \lim_{n \to \infty} \frac{1}{n} \log |(f_\omega^n)'(x)|
\]

exists and is constant over \( \mathbb{P}\text{-a.e. } \omega \in \Omega \) and \( \mu\text{-a.e. } x \in S^1 \). Since, however, \( \mu \) is absolutely continuous and supported on all of \( S^1 \), we can promote this limit to every \( x \in S^1 \) and \( \mathbb{P}\text{-a.e. } \omega \in \Omega \); details are left to the reader.

It remains to estimate \( \lambda \) from below, for which we use the following.

**Lemma 20.** In the above setting, we have that

\[
\lambda \geq \inf_{x \in S^1} \liminf_{n \to \infty} \frac{1}{n} \mathbb{E}(\log |(f_\omega^n)'(x)|)
\]

for all \( x \in S^1 \).

**Proof.** The limit

\[
\lambda = \lim_{n \to \infty} \frac{1}{n} \int_{S^1} \mathbb{E}(\log |(f_\omega^n)'(x)|) \, d\mu(x)
\]

follows from the \( L^1 \)-Mean Ergodic Theorem applied to the skew product \( \tau : S^1 \times \Omega \to S^1 \times \Omega \) defined by setting \( \tau(x, \omega) = (f_\omega x, \theta\omega) \), on noting that \( \mu \) is a stationary ergodic measure iff \( \mu \otimes \mathbb{P} \) is an ergodic invariant measure for \( \tau \) (Theorem 2.1 on pg. 20 in 18).

As is not hard to check, for all \( x \in S^1 \) we have \(-d(L, \epsilon) \leq \mathbb{E}(\log |f'_\omega(x)|) \leq \log L \) where \( d(L, \epsilon) > 0 \) is a constant depending only on \( \epsilon, L \). These bounds pass to the averages \( g_n := \frac{1}{n} \mathbb{E}(\log |(f_\omega^n)'(x)|) \). Applying Fatou’s Lemma to the nonnegative sequence \( g_n + d(L, \epsilon) \), we conclude

\[
\inf_{x \in S^1} \liminf_{n \to \infty} \frac{1}{n} \mathbb{E}(\log |(f_\omega^n)'(x)|) \leq \int_{S^1} \liminf_{n \to \infty} g_n \, d\mu \leq \lim_{n \to \infty} \int_{S^1} g_n \, d\mu = \lambda.
\]

The remaining work is to estimate \( \liminf_n \frac{1}{n} \mathbb{E}(\log |(f_\omega^n)'(x)|) \) for arbitrary \( x \in S^1 \).

**Proposition 21.** For all \( x \in S^1 \), we have

\[
\liminf_{n \to \infty} \frac{1}{n} \mathbb{E}(\log |(f_\omega^n)'(x)|) \geq \lambda_0 \log L,
\]

where \( \lambda_0 = \min\{ \frac{\alpha}{k+1}, \frac{1}{10} \} \).
The proof of Proposition 21 occupies the remainder of Section 6.

Reductions. We make here some slight modifications to the upper and lower bounds on \( \epsilon \) and the parameter \( \beta \). To start, on shrinking the parameter \( \beta \), we assume
\[
\epsilon \geq L^{-k_{\beta}^{1-\beta}(k+1)+\alpha}.
\]
Second, we can assume without loss that \( \epsilon < L^{-\min(k-1,\frac{1}{2})} \) as in the hypothesis \([1]\) for Lemma \([1]\). If not, then we can reduce to this case by a similar line of reasoning as to the reductions in Section 3.2 in the proof of Theorem \([A]\) to which we refer for details.

Finally, a minor technical point: we will assume that \( k, \beta \) satisfy the relation
\[
\left(\frac{3}{10} - \frac{5}{2}\beta - \beta^2\right)k \geq 2\beta(1 + \beta).
\]
(16)

For \( k \geq 6 \), (16) is automatic for all \( \beta \in (0, 1/10) \), and \([10]\) while holds for all \( k \in \mathbb{N} \) when \( \beta \in (0, 1/100) \). This entails no loss of generality.

With \( \beta \) fixed once and for all, we let \( L \) be sufficiently large, in terms of \( \beta \), and take on the assumptions of Section \([2.1.1]\). The parameter \( \epsilon \) is as above, and for our choice of \( k \in \mathbb{N} \) we assume \([10]\) holds. Finally, the constructions of Section 5 (namely, the filtration \( H_n \)) are applied to the arbitrary initial condition \( x = X_0 \in S^1 \).

6.1 Decomposing the sum

Fix \( n \geq 1 \). Define \( T_i = \log |\hat{f}_{\omega_i}^n(X_i)| \), \( X_i := f_{\omega_i}^n(x) \). With \( \tau_0 \equiv 0 < \tau_1 < \tau_2 < \cdots \) as in Section 5, define the random index \( J \in \mathbb{Z}_{\geq 0} \) to satisfy
\[
\tau_j < n \leq \tau_{j+1}.
\]
Note that \( \tau_1 \geq n \) implies \( J = 0 \) since \( \tau_0 = 0 \).

We decompose
\[
(*) := \log |f_{\omega}^n(x)| = \sum_{i=0}^{n-1} T_i = \sum_{i=0}^{\min\{\tau_1,n\}-1} T_i + \sum_{j=1}^{\infty} \chi_{J \geq j}(T_{\tau_j} + \sum_{i=\tau_{j+1}}^{\min\{\tau_{j+1},n\}-1} T_i)
\]
and will bound \( \mathbb{E}(\ast) \) from below; here, for an event \( A \) we write \( \chi_A \) for the indicator function of \( A \). The main obstacles are the terms \( T_{\tau_j}, 1 \leq j \leq J \), which we bound from below using conditional expectations w.r.t. the filtration \((\mathcal{H}_n)_n\).

Proposition 22. Let \( j \geq 2 \) and condition on the event \( \tau_j = m \). Then,
\[
\mathbb{E}(T_m | \mathcal{H}_m) \geq -\gamma \log L,
\]
(17)
where \( \gamma := \max\{(1+\beta)((\frac{1}{2}+\beta)k + 2\beta), k(1-\beta) - \alpha\} \).

Proposition 22 is proved in Section 6.2.

We apply Proposition 22 by replacing the terms \( \chi_{J \geq j} T_{\tau_j}, j \geq 2 \) under \( \mathbb{E} \) with the conditional expectations
\[
(*)_j := \mathbb{E}(\chi_{J \geq j} T_{\tau_j} | \mathcal{H}_\tau_j) = \sum_{m=1}^{n-1} \mathbb{E}(\chi_{\tau_j=m} T_m | \mathcal{H}_m) = \sum_{m=1}^{n-1} \chi_{\tau_j=m} \cdot \mathbb{E}(T_m | \mathcal{H}_m).
\]
Here, we use that \( \{J \geq j\} = \cup_{m=1}^{n-1} \{\tau_j = m\} \) for all \( j \geq 1 \). By Proposition 22 for \( j \geq 2 \) we have
\[
(*)_j \geq -\gamma \log L \cdot \chi_{J \geq j}.
\]

For the \( j = 1 \) term, we use the following crude estimate:

\footnote{For a filtration \((\mathcal{G}_n)\) and an adapted stopping time \( \eta \), we write \( \mathcal{G}_\eta \) for the stopped \( \sigma \)-algebra consisting of the set of measurable sets \( A \) for which \( A \cap \{\eta \leq m\} \in \mathcal{G}_m \) for all \( m \).}
Lemma 23. We have

\[(*)_1 := \mathbb{E}(\chi_{J \geq 1} T_{\tau_1} | \mathcal{H}_{\tau_1}) \geq -2(2k+1) \log L =: -\gamma_1 \log L.\]

We prove Lemma 23 in Section 6.2.

Applying these estimates, we have

\[
\begin{align*}
\mathbb{E}(\ast) & \geq \mathbb{E} \left[ \sum_{i=0}^{\min(\tau_1,n)-1} T_i + (*)_1 + \chi_{J \geq 1} \sum_{i=\tau_1+1}^{\min(\tau_2,n)-1} T_i + \sum_{j=2}^{\infty} \left( (*)_j + \chi_{J \geq j} \sum_{i=\tau_j+1}^{\min(\tau_{j+1},n)-1} T_i \right) \right] \\
& \geq \mathbb{E} \left[ \sum_{i=0}^{\min(\tau_1,n)-1} T_i + \chi_{J \geq 1} \left( -\gamma_1 \log L + \sum_{i=\tau_1+1}^{\min(\tau_2,n)-1} T_i \right) + \sum_{j=2}^{\infty} \chi_{J \geq j} \left( -\gamma \log L + \sum_{i=\tau_j+1}^{\min(\tau_{j+1},n)-1} T_i \right) \right] \\
& =: \mathbb{E}[I + II + III].
\end{align*}
\]

To complete the estimate, we decompose according to the events \{J = K\}, \(K = 0, 1, 2, \ldots\).

(A) Estimate of \(\mathbb{E}(\chi_{J=0}(I + II + III))\).

We have \(II = III = 0\) and

\[
\mathbb{E}[\chi_{J=0} \cdot I] = \mathbb{E}\left[ \chi_{J=0} \sum_{i=0}^{n-1} T_i \right]
\]

Conditioned on \(J = 0\), we have \(\tau_1 \geq n\) and so Lemma 11 may be applied (see also Lemma 16). We obtain a lower bound using the worst possible case that \(p_{\nu_{n-1}}(I_{n-1}) = k - 1\), i.e., \(I_{n-1}\) initiates a bound period of length \(k - 1\) at time \(n - 1\) (corresponding to \(t_j = n - 1, p_j = k - 1\) in the notation of Lemma 11(b)). So,

\[
\sum_{i=0}^{n-1} T_i = \log \| (f_2^n)'(x_0) \| \geq L^{(n-1)\left(\frac{1}{2} - \beta\right) - \frac{k-1}{2}} - \beta.
\]

We conclude

\[
\mathbb{E}[\chi_{J=0} \cdot I] \geq \left( (n - 1) \left( \frac{1}{2} - \beta \right) \log L - \left( \frac{k - 1}{2} + \beta \right) \log L \right) \cdot \mathbb{P}(J = 0).
\]

(B) Estimate of \(\mathbb{E}(\chi_{J=1}(I + II + III))\).

Here we have \(III = 0\) and

\[
\mathbb{E}[\chi_{J=1} \cdot (I + II)] = \mathbb{E}\left[ \chi_{J=1} \left( \sum_{i=0}^{\tau_1-1} T_i - \gamma_1 \log L + \sum_{i=\tau_1+1}^{n-1} T_i \right) \right]
\]

By Lemma 11(a) we have \(\sum_{i=0}^{\tau_1-1} T_i \geq \tau_1 \cdot (\frac{1}{2} - \beta) \log L\). The second summation \(\sum_{i=\tau_1+1}^{n-1} T_i\) is estimated as in paragraph (A): we have

\[
\mathbb{E}\left[ \chi_{J=1} \sum_{i=\tau_1+1}^{n-1} T_i \right] \geq \left( (n - 2 - \tau_1) \left( \frac{1}{2} - \beta \right) \log L - \left( \frac{k - 1}{2} + \beta \right) \log L \right) \cdot \mathbb{P}(J = 1),
\]

and so collecting, we get

\[
\mathbb{E}[\chi_{J=1} \cdot (I + II)] \geq \left( (n - 2) \left( \frac{1}{2} - \beta \right) \log L - \gamma_1 \log L + \left( 1 - \frac{k}{2} - \beta \right) \log L \right) \cdot \mathbb{P}(J = 1).
\]

(C) Estimate of \(\mathbb{E}(\chi_{J=K}(I + II + III))\) for \(K > 1\).
We bound $\mathbb{E}[\chi_{J=K} \cdot (I + II)]$ as in paragraph (A), obtaining

$$\mathbb{E}[\chi_{J=K} \cdot (I + II)] \geq \mathbb{E}\left[ \chi_{J=K} \left( (\tau_2 - 1) \left( \frac{1}{2} - \beta \right) \log L - \gamma_1 \log L \right) \right].$$

Conditioned on $\{J = K\}$ for $K > 1$, the $III$ term has the form

$$III = \sum_{j=2}^{K-1} \left( -\gamma \log L + \sum_{i=\tau_j+1}^{\tau_{j+1}-1} T_i \right) + \left( -\gamma \log L + \sum_{i=\tau_K+1}^{n-1} T_i \right).$$

For each summand $IV_j, j \geq 2$, observe that $\tilde{X}_i \in \mathcal{G}$ for each $i = \tau_j + 1, \ldots, \tau_j + k$, hence $\sum_{i=\tau_j+1}^{\tau_{j+1}-1} T_i \geq k(1 - \beta) \log L$. If $\tau_j + k + 1 \leq \tau_{j+1} - 1$, then the summands $\tau_j + k + 1 \leq i \leq \tau_{j+1} - 1$ are estimated as in Lemma 11(a). In total,

$$\mathbb{E}[\chi_{J=K} \cdot IV_j] \geq \mathbb{E}\left[ \chi_{J=K} \left( (k(1 - \beta) - \gamma) \cdot \log L + (\tau_{j+1} - 1 - \tau_j - k) \cdot \left( \frac{1}{2} - \beta \right) \log L \right) \right].$$

Observe that

$$k(1 - \beta) - \gamma = \min\{\alpha, \left( \frac{1}{2} - \frac{5}{2} \beta - \beta^2 \right) k - 2\beta(1 + \beta)\} \geq \min\{\alpha(k + 1), \frac{1}{5} k\}$$

holds from (13). Dividing the latter by $k + 1$ yields an estimate for the average growth rate $\lambda_0$ as follows:

$$\frac{k(1 - \beta) - \gamma}{k + 1} \geq \min\{\alpha, \frac{1}{10}\} =: \lambda_0 = \lambda_0(\alpha, k),$$

hence

$$\mathbb{E}[\chi_{J=K} \cdot IV_j] \geq \mathbb{E}[\chi_{J=K} (\tau_{j+1} - \tau_j) \cdot \lambda_0 \log L].$$

This telescopes, and so

$$\mathbb{E}[\chi_{J=K} \sum_{j=2}^{K-1} IV_j] \geq \mathbb{E}[\chi_{J=K} (\tau_K - \tau_2) \cdot \lambda_0 \log L]$$

Using Lemma 11(b) we bound $IV_K$ from below by

$$IV_K = -\gamma \log L + \sum_{j=\tau_K+1}^{n-1} T_i \geq -\gamma \log L + (n - \tau_K - 2) \left( \frac{1}{2} - \beta \right) \log L - \left( \frac{k-1}{2} + \beta \right) \log L$$

hence

$$\mathbb{E}[\chi_{J=K} \cdot III] \geq \mathbb{E}\left[ \chi_{J=K} \left( (n - 2 - \tau_2) \cdot \lambda_0 \log L - \gamma \log L - \left( \frac{k-1}{2} + \beta \right) \log L \right) \right]$$

and in total,

$$\mathbb{E}[\chi_{J=K} (I + II + III)] \geq \left( (n - 3) \lambda_0 \log L - (\gamma + \gamma_1) \log L - \left( \frac{k-1}{2} + \beta \right) \log L \right) \cdot \mathbb{P}(J = K).$$

**Putting it together.**

The lower bounds obtained for $K > 1$ as in paragraph (C) are the worst of the three cases examined already, hence

$$\mathbb{E}^* = \mathbb{E}[I + II + III] = \sum_{K=0}^{\infty} \mathbb{E}[\chi_{J=K} (I + II + III)] \geq (n - 3) \lambda_0 \log L - (\gamma + \gamma_1) \log L - \left( \frac{k-1}{2} + \beta \right) \log L.$$

On dividing by $n$ and taking $n \to \infty$, we conclude that $\lim_{n \to \infty} \frac{1}{n} \mathbb{E}^* \left( |f_\omega(x)| \right) \geq \lambda_0 \log L$, as desired.
6.2 Proofs of Proposition 22 and Lemma 23

Below, $C > 0$ refers to a constant depending only on $\psi$, and may change in value from line to line.

We start with the following preliminary estimate.

**Lemma 24.** Let $I \subset \mathcal{B}$ be any connected interval. Then,

$$\int_I \log |f'(z)| \, dz \geq |I| \cdot \log(L^{1-\beta}|I|).$$

This is a simple consequence of (20) and follows on taking $L$ sufficiently large, depending only on $\beta$ and $\psi$; details are left to the reader.

**Proof of Proposition 22.** Unconditionally, for any $m \geq 0$ the conditional expectation $\mathbb{E}(T_m | \mathcal{H}_m)$ is given by

$$\mathbb{E} = \int_{I_m} \log |f'_{\omega_m}(z)| \, d\nu_m(z).$$

by Lemma 17.

Conditioning on $\{\tau_j = m\}$, recall (Remark 9) that $|J_m| \leq CL^{-\frac{1}{2}-\beta}$ since $I_m$ is an atom of $\mathcal{P}_{\omega_m}(\tilde{f}_{\omega_{m-1}}(I_{m-1}))$. Our distortion control on $\rho_m = \frac{d\nu_m}{d\nu_0}$ as in Corollary 19 along $I_m$ implies $\log \frac{\rho_m(z)}{\rho_0(z)} \leq K_2 L^{-1/2} + 2K_1 L^{-\frac{1}{2}+\beta} \leq CL^{-1/2+\beta}$ for $z, z' \in I_m$, hence

$$(**) \geq (1 + CL^{-1/2+\beta}) \frac{1}{|I_m|} \left( \int_{I_m} \log |f'_{\omega_m+1}(z)| \, dz \right).$$

From Lemma 24 applied to $I = I_m$, we conclude

$$(**) \geq (1 + CL^{-1/2+\beta}) \log(L^{1-\beta}|I_m|) \geq (1 + \beta) \log(L^{1-\beta}|I_m|). \quad \text{(19)}$$

We now bound $|I_m|$ from below.

**Lemma 25.** On the event $\{\tau_j = m\}, j \geq 2, m \geq 1$, we have the estimate

$$|I_m| \geq \min\{L^{-1-(\frac{k}{4}+\beta)k-\beta}, L^{k(1-\beta)}\}.$$

Assuming this and plugging in $\epsilon \geq L^{-1+\frac{\beta}{2}}k-(1-\beta)(k+1)+\alpha$, we conclude

$$(***) \geq (1 + \beta) \log \min\{L^{-\left(\frac{k}{4}+\beta\right)k-2\beta}, L^{(k+1)(1-\beta)}\}$$

$$\geq \min\{(1 + \beta)(-2\beta - \frac{1}{2} + \beta), (1 + \beta)(\alpha - k \frac{1 - \beta}{1 + \beta})\}$$

$$\geq -\max\{(1 + \beta)(\frac{1}{2} + \beta)k + 2\beta), k(1 - \beta) - \alpha \} \log L =: -\gamma \log L.$$

To finish the proof of Proposition 22, it remains to prove Lemma 25.

**Proof of Lemma 25.** We distinguish two cases:

(a) $I_i = f_{\omega_{i-1}}(I_{i-1})$ for each $\tau_{j-1} + 2 \leq i \leq m = \tau_j$

(b) $I_i \not\subset f_{\omega_{i-1}}(I_{i-1})$ for some $\tau_{j-1} + 2 \leq i \leq m = \tau_j$.

In case (a), we easily have $|I_{\tau_{j-1}+k+1}| \geq L^{k(1-\beta)}\epsilon$, and since no additional cuts are made, we estimate

$$\begin{align*}
|I_m| &= |\tilde{f}_{\omega_{m-1}} \circ \cdots \circ \tilde{f}_{\omega_{\tau_{j-1}+k+1}}(I_{\tau_{j-1}+k+1})| \\
&\geq L^{-\left((\tau_{j-1}+k+1)\left(\frac{k}{4}+\beta\right)k+2\beta\right)}|I_{\tau_{j-1}+k+1}| \geq L^{k(1-\beta)}\epsilon.
\end{align*}$$

24
In case (b), set \(i^* = \max\{i \leq \tau_j : I_i \subseteq \hat{f}_{\omega_{i-1}}(I_{i-1})\}\) (note \(i^* = m\) is possible), and note that if \(i^* < m\) then
\[
I_m = \hat{f}_{\omega_{m-1}} \circ \cdots \circ \hat{f}_{\omega_1}(I_{i^*}).
\]
To bound \(|I_{i^*}|\) we split further to the cases (i) \(p_{\omega_{i^*}}(I_{i^*}) = 0\), (ii) \(p_{\omega_{i^*}}(I_{i^*}) \in \{1, \ldots, k - 1\}\) and (iii) \(p_{\omega_{i^*}}(I_{i^*}) = k\). Note that in all cases, \(P_{\omega_{i^*}}(f_{\omega_{i-1}}(I_{i-1}))\) contains at least two elements, hence \(I_{i^*}\) contains at least one atom of \(P_{\omega_{i^*}}\) (Remark \([1]\)).

In case (b)(i), \(I_{i^*} \subset I_i\), \(\omega_{i^*} \in \mathcal{G}_{\omega_{i^*}}\). Either \(I_{i^*}\) contains an atom of \(\mathcal{G}_{\omega_{i^*}}\), in which case \(|I_{i^*}|\) is bounded from below by \(\frac{1}{2} \min\{d(\hat{x}, \hat{x}'): \hat{x}, \hat{x}' \in C'_{\omega}, \hat{x} \neq \hat{x}'\}\), or \(I_{i^*}\) contains an atom of \(P_{\omega_{i^*}}|_{I_{i^*}}\), hence \(|I_{i^*}| \geq L^{-\frac{2}{4} - \beta}\) (the latter bound being the worse of the two). Since \(I_m = I_{\tau_j}\) is free, we conclude \(|I_m| \geq |I_{i^*}| \geq L^{-\frac{2}{4} - \beta}\) from Lemma \([11]\).

In case (b)(ii), we have automatically that \(I_{i^*}\) is free and initiates a bound period of length \(p^* = p_{\omega_{i^*}}(I_{i^*})\). Since \(0 < p^* < k - 1\) by assumption, we cannot have \(i^* = \tau_j = m\) (since then \(p^* = k\)) and so conclude \(i^* < \tau_j\) in this case—indeed, we have \(i^* + p^* + 1 \leq m = \tau_j\), since \(I_{\tau_j}\) is free. From Remark \([8]\) we have
\[
|I_{i^*}| \geq (p^* + 1)^{-2} L^{-\frac{p^* + 2}{2} - \beta} \geq L^{-\frac{p^* + 2}{2} - \beta(p^* + 1)},
\]
on taking \(L\) large enough so \(\beta > 2/\log L\). Moreover, since \(I_m = I_{\tau_j}\) is free, we have
\[
|I_m| \geq |I_{i^*} + p^* + 1| = |f_{\omega_{i^*}}^{{p^* + 1}}(I_{i^*})| \geq L^{(p^* + 1)(\frac{3}{4} - \beta)}|I_{i^*}| \geq L^{(p^* + 1)(\frac{3}{4} - \beta)} L^{-\frac{p^* + 2}{2} - \beta(p^* + 1)} = L^{-1 - \frac{1}{4} - \beta(p^* + 1)}
\]
The worst possible case is \(p^* = k - 1\), and so we conclude \(|I_m| \geq L^{-1 - \frac{1}{4}k - \beta}\) in case (ii).

In case (b)(iii), we have necessarily that \(i^* = m = \tau_j\). In the worst case, \(I_m\) contains an atom of \(P_{\omega_m}|_{B_{\omega_m}^{1-1}}\), and so
\[
|I_m| \geq k^{-2} L^{-\frac{k-2}{2} - \beta} \geq L^{-1 - \frac{1}{4} + \beta(k-\beta)}.
\]

**Proof of Lemma 26.** Arguing in parallel to the proof of Proposition \([22]\) (see \([19]\)) we have, on the event \(\{\tau_1 = m\}\), the estimate
\[
E(T_m|H_m) \geq (1 + \beta) \log(L^{1 - \beta}|I_m|)
\]
As before, we estimate \(|I_m|\) from below.

**Lemma 26.** On the event \(\{\tau_1 = m\}\), we have the estimate \(|I_m| \geq \min\{L^{-1 - \frac{1}{4} + \beta(k-\beta)}, \epsilon\}\).

Assuming this, we easily obtain
\[
E(T_m|H_m) \geq (1 + \beta) \log(L^{1 - \beta} \min\{L^{-1 - \frac{1}{4} + \beta(k-\beta)}, \epsilon\}) \geq -2(2k + 1) \log L,
\]
as claimed. It remains to prove Lemma \([26]\).

**Proof of Lemma 26.** Condition on \(\tau_1 = m\). The proof is very much parallel to that of Lemma \([25]\). Case (b) can be repeated verbatim, and yields the identical estimate \(|I_m| \geq L^{-1 - \frac{1}{4} + \beta(k-\beta)}\).

The only difference is in case (a). Here, we observe that \(I_m\) must be free, and so (Lemma \([11]\) a)) we have
\[
|I_m| \geq L^m(\frac{1}{2} - \beta) \cdot 2 \epsilon \geq \epsilon.
\]
This completes the proof of Lemma \([26]\).

**References**

[1] Romain Aimino, Huyi Hu, Matthew Nicol, Andrei Török, and Sandro Vaienti. Polynomial loss of memory for maps of the interval with a neutral fixed point. *Discrete & Continuous Dynamical Systems-A*, 35(3):793–806, 2015.

[2] Viviane Baladi, Michael Benedicks, and Véronique Maume-Deschamps. Decay of random correlation functions for unimodal maps. *Reports on mathematical physics*, 46(1-2):15–26, 2000.
[3] Viviane Baladi and L-S Young. On the spectra of randomly perturbed expanding maps. *Communications in Mathematical Physics*, 156(2):355–385, 1993.

[4] Michael Benedicks and Lennart Carleson. On iterations of $1-ax^2$ on $(-1,1)$. *Annals of Mathematics*, pages 1–25, 1985.

[5] Michael Benedicks and Lennart Carleson. The dynamics of the Hénon map. *Annals of Mathematics*, 133(1):73–169, 1991.

[6] Michael Benedicks and Marcelo Viana. Random perturbations and statistical properties of Hénon-like maps. In *Annales de l’Institut Henri Poincare (C) Non Linear Analysis*, volume 23, pages 713–752. Elsevier, 2006.

[7] Michael Benedicks and Lai-Sang Young. Absolutely continuous invariant measures and random perturbations for certain one-dimensional maps. *Ergodic Theory and Dynamical Systems*, 12(1):13–37, 1992.

[8] Alex Blumenthal, Jinxin Xue, and Lai-Sang Young. Lyapunov exponents for random perturbations of some area-preserving maps including the standard map. *Annals of Mathematics*, 185(1):285–310, 2017.

[9] Alex Blumenthal, Jinxin Xue, and Lai-Sang Young. Lyapunov exponents and correlation decay for random perturbations of some prototypical 2D maps. *Communications in Mathematical Physics*, 359(1):347–373, 2018.

[10] Jérôme Buzzi. No or infinitely many ACIP for piecewise expanding $C^r$ maps in higher dimensions. *Communications in Mathematical Physics*, 222(3):495–501, 2001.

[11] Boris V Chirikov. A universal instability of many-dimensional oscillator systems. *Physics reports*, 52(5):263–379, 1979.

[12] Pierre Collet and J-P Eckmann. Positive Liapunov exponents and absolute continuity for maps of the interval. *Ergodic Theory and Dynamical Systems*, 3(1):13–46, 1983.

[13] P Duarte. Elliptic isles in families of area-preserving maps. *Ergodic Theory and Dynamical Systems*, 28(6):1781–1813, 2008.

[14] Pedro Duarte. Plenty of elliptic islands for the standard family of area preserving maps. In *Annales de l’Institut Henri Poincare (C) Non Linear Analysis*, volume 11, pages 359–409. Elsevier, 1994.

[15] Anton Gorodetski. On stochastic sea of the standard map. *Communications in Mathematical Physics*, 309(1):155–192, 2012.

[16] Michael V Jakobson. Absolutely continuous invariant measures for one-parameter families of one-dimensional maps. *Communications in Mathematical Physics*, 81(1):39–88, 1981.

[17] Anatole Katok and Yuri Kifer. Random perturbations of transformations of an interval. *Journal D’Analyse Mathematique*, 47(1):193–237, 1986.

[18] Yuri Kifer. *Ergodic theory of random transformations*, volume 10. Springer Science & Business Media, 2012.

[19] François Ledrappier, Michael Shub, Carles Simó, and Amie Wilkinson. Random versus deterministic exponents in a rich family of diffeomorphisms. *Journal of statistical physics*, 113(1-2):85–149, 2003.

[20] Zeng Lian and Mikko Stenlund. Positive Lyapunov exponent by a random perturbation. *Dynamical Systems*, 27(2):239–252, 2012.

[21] Mikhail Lyubich. Almost every real quadratic map is either regular or stochastic. *Annals of Mathematics*, pages 1–78, 2002.
[22] Michael Misiurewicz. Absolutely continuous measures for certain maps of an interval. *Publications Mathématiques de l’Institut des Hautes Études Scientifiques*, 53(1):17–51, 1981.

[23] Sheldon E Newhouse. The abundance of wild hyperbolic sets and non-smooth stable sets for diffeomorphisms. *Publications Mathématiques de l’Institut des Hautes Études Scientifiques*, 50(1):101–151, 1979.

[24] William Ott, Mikko Stenlund, and Lai Sang Young. Memory loss for time-dependent dynamical systems. *Mathematical Research Letters*, 16(3):463–475, 2009.

[25] Mikko Stenlund and Henri Sulku. A coupling approach to random circle maps expanding on the average. *Stochastics and Dynamics*, 14(04):1450008, 2014.

[26] Qiudong Wang and Lai-Sang Young. Nonuniformly expanding 1D maps. *Communications in mathematical physics*, 264(1):255–282, 2006.

[27] L-S Young. Decay of correlations for certain quadratic maps. *Communications in mathematical physics*, 146(1):123–138, 1992.