EXPONENTIAL CARMICHAEL FUNCTION
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Abstract. Consider exponential Carmichael function $\lambda^{(e)}$ such that $\lambda^{(e)}$ is multiplicative and $\lambda^{(e)}(p^a) = \lambda(a)$, where $\lambda$ is usual Carmichael function. We discuss the value of $\sum \lambda^{(e)}(n)$, where $n$ runs over certain subsets of $[1, x]$, and provide bounds on the error term, using analytic methods and especially estimates of $\int_T^1 |\zeta(\sigma + it)|^m dt$.

1. Introduction

Consider an operator $E$ over arithmetic functions such that for every $f$ the function $Ef$ is multiplicative and

$$(Ef)(p^a) = f(a), \quad p \text{ is prime}.$$ 

For various functions $f$ (such as the divisor function, the sum-of-divisor function, Möbius function, the totient function and so on) the behaviour of $Ef$ was studied by many authors, starting from Subbarao [12]. The bibliography can be found in [10].

The notation for $Ef$, established by previous authors, is $f^{(e)}$.

Carmichael function $\lambda$ is an arithmetic function such that

$$\lambda(p^a) = \begin{cases} 
\phi(p^a), & p > 2 \text{ or } a = 1, 2, \\
\phi(p^a)/2, & p = 2 \text{ and } a > 2,
\end{cases}$$

and if $n = p_1^{a_1} \cdots p_m^{a_m}$ is a canonical representation, then

$$\lambda(n) = \text{lcm}(\lambda(p_1^{a_1}), \ldots, \lambda(p_m^{a_m})).$$

This function was introduced at the beginning of the XX century in [1], but intense studies started only in 1990-th, e. g. [2]. Carmichael function finds applications in cryptography, e. g. [3].

Consider also the family of multiplicative functions

$$\delta_r(p^a) = \begin{cases} 
0, & a < r, \\
1, & a \geq r,
\end{cases} \quad r \text{ is integer}.$$ 

Function $\delta_2$ is a characteristic function of the set of square-full numbers, $\delta_3$ — of cube-full numbers and so on. Of course, $\delta_1 \equiv 1$.

Denote $\lambda_r^{(e)}$ for the product of $\delta_r$ and $\lambda^{(e)}$:

$$\lambda_r^{(e)}(n) = \delta_r(n)\lambda^{(e)}(n).$$

The aim of our paper is to study asymptotic properties of $\lambda^{(e)} \equiv \lambda_1^{(e)}, \lambda_2^{(e)}, \lambda_3^{(e)}$ and $\lambda_4^{(e)}$.

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2. Notations

Letter $p$ with or without indexes denotes a prime number.

We write $f \ast g$ for Dirichlet convolution

$$(f \ast g)(n) = \sum_{d|n} f(d)g(n/d).$$

Denote

$$\tau(a_1, \ldots, a_k; n) := \sum_{d_1 \cdots d_k = n} a_1 \cdots a_k.$$  

In asymptotic relations we use $\sim$, $\asymp$, Landau symbols $O$ and $o$, Vinogradov symbols $\ll$ and $\gg$ in their usual meanings. All asymptotic relations are given as an argument (usually $x$) tends to the infinity.

Everywhere $\varepsilon > 0$ is an arbitrarily small number (not always the same even in one equation).

As usual $\zeta(s)$ is Riemann zeta-function. Real and imaginary components of the complex $s$ are denoted as $\sigma := \Re s$ and $t := \Im s$, so $s = \sigma + it$.

For a fixed $\sigma \in [1/2, 1]$ define

$$m(\sigma) := \sup \left\{m \mid \left| \int_1^T |\zeta(\sigma + it)|^m \, dt \ll T \varepsilon \right. \right\}.$$  

and

$$\mu(\sigma) := \limsup_{t \to \infty} \frac{\log|\zeta(\sigma + it)|}{\log t}.$$  

Below $H_{2005} = (32/205 + \varepsilon, 269/410 + \varepsilon)$ stands for Huxley’s exponent pair from [5].

3. Preliminary Lemmas

**Lemma 1.** Let $F: \mathbb{Z} \to \mathbb{C}$ be a multiplicative function such that $F(p^a) = f(a)$, where $f(n) \ll n^\beta$ for some $\beta > 0$. Then

$$\limsup_{n \to \infty} \frac{\log F(n)}{\log n} = \sup_{n \geq 1} \frac{\log f(n)}{n}.$$  

*Proof.* See [13].

**Lemma 2.** Let $f(t) \geq 0$. If

$$\int_1^T f(t) \, dt \ll g(T),$$

where $g(T) = T^\alpha \log^\beta T$, $\alpha \geq 1$, then

$$I(T) := \int_1^T \frac{f(t)}{t} \, dt \ll \begin{cases} \log^{\beta+1} T & \text{if } \alpha = 1, \\ T^{\alpha-1} \log^\beta T & \text{if } \alpha > 1. \end{cases}$$

*Proof.* Let us divide the interval of integration into parts:

$$I(T) \ll \sum_{k=0}^{\log_2 T} \int_{T/2^{k+1}}^{T/2^k} \frac{f(t)}{t} \, dt \ll \sum_{k=0}^{\log_2 T} \frac{1}{T/2^{k+1}} \int_{1}^{T/2^k} f(t) \, dt \ll \sum_{k=0}^{\log_2 T} \frac{g(T/2^k)}{T/2^{k+1}}.$$  

Now the lemma’s statement follows from elementary estimates.

**Lemma 3.** For $\sigma \geq 1/2$ and for any exponent pair $(k, l)$ such that $l-k \geq \sigma$ we have

$$\mu(\sigma) \ll \frac{k + l - \sigma}{2} + \varepsilon.$$  

*Proof.* See [6, (7.57)].
A well-known application of Lemma 3 is
\[ \mu(1/2) \leq 32/205, \]
following from the choice \( (k, l) = H_{2005}. \) Another (maybe new) application is
\[ \mu(3/5) \leq 1409/12170, \]
following from
\[ (k, l) = \left( \frac{269}{2434} \cdot \frac{1755}{2434} \right) = ABABH_{2005}, \]
where \( A \) and \( B \) stands for usual \( A \)- and \( B \)-processes [7] Ch. 2.

**Lemma 4.** Let \( \eta > 0 \) be arbitrarily small. Then for growing \(|t| \geq 3\)
\[ \zeta(s) \ll \begin{cases} |t|^{1/2-(1-2\mu(1/2))\sigma}, & \sigma \in [0, 1/2], \\ |t|^{2\mu(1/2)(1-\sigma)}, & \sigma \in [1/2, 1 - \eta], \\ |t|^{2\mu(1/2)(1-\sigma)} \log^{2/3} |t|, & \sigma \in [1 - \eta, 1], \\ \log^{2/3} |t|, & \sigma \geq 1. \end{cases} \]
More exact estimates for \( \sigma \in [1/2, 1 - \eta] \) are also available, e. g.
\[ \mu(\sigma) \ll \begin{cases} 10(\mu(3/5) - \mu(1/2))\sigma + (6\mu(1/2) - 5\mu(3/5)), & \sigma \in [1/2, 3/5], \\ 5\mu(3/5)(1 - \sigma)/2, & \sigma \in [3/5, 1 - \eta], \end{cases} \]

**Proof.** Estimates follow from Phragmén–Lindelöf principle, exact and approximate functional equations for \( \zeta(s) \) and convexity properties. See [14, Ch. 5] and [6, Ch. 7.5] for details.

**Lemma 5.** For any integer \( r \)
\[ \max_{n \leq x} \lambda_r^{(e)}(n) \ll x^e. \]

**Proof.** Surely \( \lambda_r^{(e)}(n) \ll \lambda^{(e)}(n) \). By Lemma 5 we have
\[ \limsup_{n \to \infty} \frac{\log \lambda^{(e)}(n) \log \log n}{\log n} = \sup_m \frac{\log \lambda(m)}{m} = \frac{\log 4}{5} =: c, \]
because \( \lambda(m) \ll m - 1 \). It implies
\[ \max_{n \leq x} \lambda^{(e)}(n) \ll x^{e/\log \log n} \ll x^e. \]

**Lemma 6.** Let \( L_r(s) \) be the Dirichlet series for \( \lambda_r^{(e)}:\)
\[ L_r(s) := \sum_{n=1}^{\infty} \lambda_r^{(e)}(n)n^{-s}. \]
Then for \( r = 1, 2, 3, 4 \) we have \( L_r(s) = Z_r(s)G_r(s) \), where
\[ Z_1(s) = \zeta(s)\zeta(3s)\zeta^2(5s), \]
\[ Z_2(s) = \zeta(2s)\zeta^2(3s)\zeta(4s)\zeta^2(5s), \]
\[ Z_3(s) = \zeta^2(3s)\zeta^2(4s)\zeta^4(5s), \]
\[ Z_4(s) = \zeta^2(4s)\zeta^4(5s)\zeta^2(6s)\zeta^6(7s), \]
**Dirichlet series** \( G_1(s), \) \( G_2(s), \) \( G_3(s) \) converge absolutely for \( \sigma > 1/6 \) and \( G_4(s) \) converges absolutely for \( \sigma > 1/8. \)
Proof. Follows from the identities
\[
1 + \sum_{a \geq 1} \lambda^e(p^n)x^a = 1 + x + x^2 + 2x^3 + 2x^4 + 4x^5 + 2x^6 + 6x^7 + O(x^8)
\]
\[
= \frac{1 + O(x^8)}{(1-x)(1-x^2)(1-x^3)(1-x^5)^2},
\]
\[
1 + \sum_{a \geq 2} \lambda^e(p^n)x^a = \frac{1 + O(x^8)}{(1-x^2)(1-x^4)^2(1-x^7)^2},
\]
\[
1 + \sum_{a \geq 3} \lambda^e(p^n)x^a = \frac{1 + O(x^8)}{(1-x^4)^2(1-x^5)^2(1-x^7)^3},
\]
\[
1 + \sum_{a \geq 4} \lambda^e(p^n)x^a = \frac{1 + O(x^8)}{(1-x^4)^2(1-x^5)^2(1-x^7)^5}.
\]

\[\square\]

Lemma 7. Let \( \Delta(x) \) be the error term in the well-known asymptotic formula for \( \sum_{n \leq x} \tau(a_1, a_2, a_3, a_4; n) \), let \( A_1 = a_1 + a_2 + a_3 + a_4 \) and let \((k,l)\) be any exponent pair. Suppose that the following conditions are satisfied:

1. \((k+l+2)a_4 < (k+l)a_1 + A_4\),
2. \(2(k+l+1)a_1 \leq (2k+1)(a_2 + a_3)\).
3. \(l \leq k a_2 \) and \((k+l+1)a_1 \geq k(a_2 + a_3)\)
   or
4. \(l \geq k a_2 \) and \((l-k)(2k+1)a_3 \leq (2l-2l-1)(k+l+1)a_1 + (2k(k-l+1)+1)a_2\).

Proof. This is \([8]\) Th. 3] with \( p = 4 \).

\[\square\]

Lemma 8.

\[
m(\sigma) \geq \begin{cases} 
4/(3-4\sigma), & 1/2 \leq \sigma \leq 5/8, \\
10/(5-6\sigma), & 5/8 \leq \sigma \leq 35/54, \\
19/(6-6\sigma), & 35/54 \leq \sigma \leq 41/60, \\
2112/(859-948\sigma), & 41/60 \leq \sigma \leq 3/4, \\
12408/(4537-4890\sigma), & 3/4 \leq \sigma \leq 5/6, \\
4324/(1031-1044\sigma), & 5/6 \leq \sigma \leq 7/8, \\
98/(31-32\sigma), & 7/8 \leq \sigma \leq 0.91591..., \\
(24\sigma-9)/(4\sigma-1)(1-\sigma), & 0.91591... \leq \sigma \leq 1-\varepsilon.
\end{cases}
\]

Proof. See \([8]\) Th. 8.4].

\[\square\]

4. Main results

Theorem 1.

\[
\sum_{n \leq x} \lambda^e(n) = c_{11}x + c_{13}x^{1/3} + (c'_{15} \log x + c_{15})x^{1/5} + O(x^{1+15/6073+\varepsilon}),
\]

where \(c_{11}, c_{13}, c_{15}\) and \(c'_{15}\) are computable constants.

Proof. Lemma \([8]\) and equation \(\square\) implies that \(\lambda^e(n) = \tau(1,3,5,5;\cdot) \ast g_1\), where

\[
\sum_{n \leq x} g_1(n) \ll x^{1/6+\varepsilon}. \quad \text{Due to} \quad \square
\]

\[
\sum_{n \leq x} \tau(1,3,5,5;n) = x\zeta(3)\zeta^2(5) \sum_{s=1} \zeta(s) + 3x^{1/3}\zeta(1/3)\zeta^2(5/3) \sum_{s=1} \zeta(3s) + 5x^{1/5}\zeta(1/5)\zeta(3/5) \sum_{s=1} \zeta^2(5s) + R(x).
\]


To estimate $R(x)$ we use Lemma 7 with $a_1 = 1$, $a_2 = 3$, $a_3 = a_4 = 5$. Exponent pair $(k, l) = H_{2005}$ satisfies conditions 1, 2 and 3.2 and thus

$$R(x) \ll x^{(k+l+2)/(k+l+14)} = x^{1153/6073+\varepsilon}, \quad 1/6 < 1153/6073 < 1/5.$$  

Now the convolution argument completes the proof. □

Exponential totient function $\phi^{(c)}$ has similar to $\lambda^{(c)}$ Dirichlet series:

$$\sum_{n=1}^{\infty} \phi^{(c)}(n) = \zeta(s)\zeta(3s)\zeta^2(5s)H(s),$$

where $H(s)$ converges absolutely for $\sigma > 1/6$. Theorem 4 can be extended to this case without any changes, so

$$\sum_{n\leq x} \phi^{(c)}(n) = c_{11}x + c_{13}x^{1/3} + (c'_{15} \log x + c_{15})x^{1/5} + O(x^{1153/6073+\varepsilon}).$$

This improves the result of Pétermann [11], who obtained $\sum_{n\leq x} \phi^{(c)}(n) = c_{11}x + c_{13}x^{1/3} + O(x^{1/5} \log x)$.

**Theorem 2.**

$$\sum_{n\leq x} \lambda^{(c)}_2(n) = c_{22}x^{1/2} + \left(c'_{23} \log x + c_{23}\right)x^{1/3} + c_{24}x^{1/4} + O(x^{1153/5586+\varepsilon}),$$

where $c_{22}$, $c_{23}$, $c'_{23}$ and $c_{24}$ are computable constants.

**Proof.** Similar to Theorem 4 with following changes: now by (6)

$$\lambda^{(c)}_2 = \tau(2, 3, 3, 4; \cdot) * g_2,$$

where $\sum_{n\leq x} g_2(n) \ll x^{1/6+\varepsilon}$. But

$$\sum_{n\leq x} \tau(2, 3, 3, 4; n) = 2x^{1/2}\zeta(2)\zeta(3/2) \text{ res}_{s=1/2} \zeta(2s) +$$

$$+ 3x^{1/3}\zeta(2/3)\zeta(4/3) \text{ res}_{s=1/3} \zeta(3s) + 4x^{1/4}\zeta(1/2)\zeta(3/4) \text{ res}_{s=1/4} \zeta(4s) + R(s).$$

Again by Lemma 7 with $a_1 = 2$, $a_2 = a_3 = 3$, $a_4 = 4$, $(k, l) = H_{2005}$ we get

$$R(x) \ll x^{(k+l+2)/(k+l+12)} = x^{1153/5586+\varepsilon}, \quad 1/5 < 1153/5586 < 1/4.$$  

□

**Theorem 3.**

(9) \hspace{1cm} \sum_{n\leq x} \lambda^{(c)}_3(n) = (c'_{33} \log x + c_{33})x^{1/3} + (c'_{34} \log x + c_{34})x^{1/4} +

$$+ P_{35}(\log x)x^{1/5} + O(x^{1/6+\varepsilon}),$$

where $c_{33}$, $c'_{33}$, $c_{34}$ and $c'_{34}$ are computable constants, $P_{35}$ is a polynomial of degree 3 with computable coefficients.

**Proof.** Lemma 6 and equation (7) implies that $\lambda^{(c)}_3 = z_3 * g_3$, where $z_3$ is defined implicitly by

$$\sum_{n=1}^{\infty} z_3(n)n^{-s} = Z_3(s) = \zeta(3s)\zeta^2(4s)\zeta^4(5s),$$

and $g_3$ is a multiplicative function such that $\sum_{n\leq x} g_3(n) \ll x^{1/6+\varepsilon}$.

The main term at the right side of (9) equals to

$$M_3(x) := \left(\text{res}_{s=1/3} + \text{res}_{s=1/4} + \text{res}_{s=1/5}\right) \left(\zeta(3s)\zeta^2(4s)\zeta^4(5s)x^s s^{-1}\right).$$
To obtain the desirable error term it is enough to prove that
\[ \sum_{n \leq x} z_2(n) = M_3(x) + O(x^{1/6 + \varepsilon}). \]

By Perron formula for \( e := 1/3 + 1/\log x \) we have
\[ \sum_{n \leq x} z_3(n) = \frac{1}{2\pi i} \int_{c - iT}^{c + iT} Z_3(s)x^s s^{-1} ds + O(x^{1/6 + \varepsilon} T^{-1}). \]

Substituting \( T = x \) and moving the contour of the integration till \([1/6 - ix, 1/6 + ix]\) we get
\[ \sum_{n \leq x} f_3(n) = M_3(x) + O(I_0 + I_- + I_+ + x^\varepsilon), \]

where
\[ I_0 := \int_{1/6 - ix}^{1/6 + ix} Z_3(s)x^s s^{-1} ds, \quad I_{\pm} := \int_{1/6 \pm ix}^{c \pm ix} Z_3(s)x^s s^{-1} ds. \]

Firstly,
\[ I_+ \ll x^{-1} \int_1^{e} Z_3(\sigma + ix)x^\sigma d\sigma. \]

Let \( \alpha(\sigma) \) be a function such that \( Z_3(\sigma + ix) \ll x^{\alpha(\sigma) + \varepsilon} \). By (5) we have
\[ \alpha(\sigma) \leq \begin{cases} (16 - 68\sigma)\mu(1/2) < 4/5, & \sigma \in [1/6, 1/5), \\ (8 - 28\sigma)\mu(1/2) < 3/4, & \sigma \in [1/5, 1/4), \\ (4 - 12\sigma)\mu(1/2) < 2/3, & \sigma \in [1/4, 1/3), \\ 0, & \sigma \in [1/3, \varepsilon]. \end{cases} \]

This means that \( I_+ \ll x^\varepsilon \). Plainly, the same estimate holds for \( I_- \).

Secondly, it remains to prove that \( I_0 \ll x^{1/6 + \varepsilon} \). Here
\[ I_0 \ll x^{1/6} \int_1^{e} Z_3(1/6 + it)t^{-1} dt \]

and taking into account Lemma 2 it is enough to show \( \int_1^{e} Z_3(1/6 + it) dt \ll x^{1+\varepsilon} \).

Applying Cauchy inequality twice we obtain
\[
\begin{aligned}
\int_1^{e} Z_3(1/6 + it) dt &\ll \left( \int_1^{e} |\zeta^4(1/2 + it)| dt \right)^{1/2} \\
&\times \left( \int_1^{e} |\zeta^4(2/3 + it)| dt \right)^{1/4} \left( \int_1^{e} |\zeta^{16}(5/6 + it)| dt \right)^{1/4} \\
&\ll x^{(1+\varepsilon)/2} x^{(1+\varepsilon)/4} x^{(1+\varepsilon)/4} \ll x^{1+\varepsilon}
\end{aligned}
\]

since by Lemma 3 \( m(1/2) \geq 4, m(2/3) \geq 8 \) and \( m(5/6) \geq 16 \). \( \square \)

**Theorem 4.**
\[ \sum_{n \leq x} \lambda_4^{(n)}(n) = (c_{44}' \log x + c_{44})x^{1/4} + P_{45}(\log x)x^{1/5} + (c_{46}' \log x + c_{46})x^{1/6} + P_{47}(\log x)x^{1/7} + O(x^{C_4 + \varepsilon}), \]

where \( c_{44}, c_{44}', c_{46} \) and \( c_{46}' \) are computable constants, \( P_{45} \) and \( P_{47} \) are computable polynomials, \( \deg P_{45} = 3, \deg P_{47} = 5 \).

\[ C_4 = \frac{7863059 - \sqrt{13780693090921}}{85962240} \approx 0.134656 \ldots, \quad 1/8 < C_4 < 1/7. \]
Proof. We shall follow the outline of Theorem 3. Let us prove that for \( c := 1/4 + 1/\log x \) we can estimate

\[
I_+ := \int_{C_4 + i\epsilon}^{c + i\epsilon} Z_4(s)x^s s^{-1} ds \ll x^{C_4 + \epsilon}
\]

and

\[
I_0 := \int_{C_4 - i\epsilon}^{c + i\epsilon} Z_4(s)x^s s^{-1} ds \ll x^{C_4 + \epsilon}.
\]

We start with \( I_+ \ll x^{-1} \int_{C_4}^c Z_4(s + ix)x^\sigma d\sigma \). Now let \( \alpha(\sigma) \) be a function such that \( Z_4(\sigma + ix) \ll x^{\alpha(\sigma) + \epsilon} \). By (3) and (5) we have

\[
\alpha(\sigma) \leq \begin{cases} 
(16 - 80\sigma)\mu(1/2) < 5/6, & \sigma \in [1/7, 1/6), \\
(12 - 56\sigma)\mu(1/2) < 4/5, & \sigma \in [1/6, 1/5), \\
(4 - 16\sigma)\mu(1/2) < 3/4, & \sigma \in [1/5, 1/4), \\
0, & \sigma \in [1/4, 1]. 
\end{cases}
\]

So \( \int_{1/7}^{c/4} Z_4(\sigma + ix)x^{\sigma - 1} d\sigma \ll x^{\epsilon} \) and the only case that requires further investigations is \( \sigma \in [C_4, 1/7] \). Instead of (3) we apply (4) together with (1) and (2) to obtain

\[
\alpha(\sigma) \leq \frac{1045018}{249485} - \frac{2459357}{99794} \sigma, \quad \sigma \in [1/8, 1/7],
\]

which implies \( \int_{C_4}^{1/7} x^{\alpha(\sigma) + \sigma - 1} d\sigma \ll x^{C_4 + \epsilon} \) as soon as

\[
C_4 \geq 1591066/12296785 = 0.129388 \ldots
\]

Our choice of \( C_4 \) in (10) is certainly the case.

Let us move on \( I_0 \) and prove that \( \int_{1}^{x} Z_4(C_4 + i\epsilon) d\epsilon \ll x^{1+\epsilon} \). For \( q_1, q_2, q_3, q_4 \) such that

\[
(11) \quad 1/q_1 + 1/q_2 + 1/q_3 + 1/q_4 = 1 \quad \text{and} \quad q_1, q_2, q_3, q_4 \geq 1
\]

by Hölder inequality we have

\[
\int_{1}^{x} Z_4(C_4 + i\epsilon) d\epsilon \ll \left( \int_{1}^{x} |\zeta^{q_1}(4s + i\epsilon)| d\epsilon \right)^{1/q_1} \left( \int_{1}^{x} |\zeta^{q_2}(5s + i\epsilon)| d\epsilon \right)^{1/q_2} \times \\
\times \left( \int_{1}^{x} |\zeta^{q_3}(6s + i\epsilon)| d\epsilon \right)^{1/q_3} \left( \int_{1}^{x} |\zeta^{q_4}(7s + i\epsilon)| d\epsilon \right)^{1/q_4}.
\]

Choose

\[
q_1 = m(4C_4)/2, \quad q_2 = m(5C_4)/4, \quad q_3 = m(6C_4)/2, \quad q_4 = m(7C_4)/6
\]

One can make sure by substituting the value of \( C_4 \) from (10) into Lemma 8 that such choice of \( q_k \) satisfies (11). Thus we obtain

\[
\int_{1}^{x} Z_4(C_4 + i\epsilon) d\epsilon \ll x^{(1+\epsilon)/q_1 x^{(1+\epsilon)/q_2} x^{(1+\epsilon)/q_3} x^{(1+\epsilon)/q_4}} \ll x^{1+\epsilon},
\]

which finishes the proof. \( \square \)

5. Decrease of \( C_4 \)

In this section we obtain lower value of \( C_4 \) by improving lower bounds of \( m(\sigma) \) from Lemma 8.

Estimates below depend on values of

\[
(13) \quad \inf_{(k,i) \in \mathbb{N}^2} \frac{ak + bl + c}{dk + el + f},
\]
where \((k, l)\) runs over the set of exponent pairs and satisfies certain linear inequalities. A method to estimate \([13]\) without linear constrains was given by Graham \([4]\).

In the recent paper \([9]\) we have presented an effective algorithm to deal with \((13)\) under a nonempty set of linear constrains.

Let \(c\) be an arbitrary function such that \(c(\sigma) \geq \mu(\sigma)\). Define \(\theta\) by an implicit equation

\[2c(\theta(\sigma)) + 1 + \theta(\sigma) - 2(1 + c(\theta(\sigma)))\sigma = 0.\]

Finally, define

\[f(\sigma) = 2\frac{1 + c(\theta(\sigma))}{c(\theta(\sigma))}.\]

Due to Lemma \([3]\) one can take \(c(\sigma) = \inf_{-k \geq \sigma} (k + l - \sigma)/2\), where \((k, l)\) runs over the set of exponent pairs. However even rougher choice of \(c\) leads to satisfiable values of \(f\) such as in \([6]\) (8.71)].

**Lemma 9.** Let \(\sigma \geq 5/8\). Compute

\[
\alpha_1 = \frac{4 - 4\sigma}{1 + 2\sigma}, \quad \beta_1 = -\frac{12}{1 + 2\sigma}, \quad m_1 = \frac{1 - \alpha_1}{\mu(\sigma)} - \beta_1, \\
\alpha_2(k, l) = \frac{4(1 - \sigma)(k + l)}{(2 + 4\sigma)(2 + 4\sigma)(1 - 2k) - 2l}, \quad \beta_2(k, l) = -\frac{4(1 + 2k + 2l)}{(2 + 4\sigma)(2 + 4\sigma)(1 - 2k) - 2l}, \\
m_2(k, l) = \frac{1 - \alpha_2(k, l)}{\mu(\sigma)} - \beta_2(k, l), \quad m_2 = \inf_{\alpha_2(k, l) \leq 1} m_2(k, l),
\]

where \((k, l)\) runs over the set of exponent pairs. Then

\[m(\sigma) \geq \min(m_1, m_2, 2f(\sigma)).\]

**Note that for** \(\sigma \geq 2/3\) the condition \(\alpha_2(k, l) \leq 1\) is always satisfied.

**Proof.** Follows from \([6]\) (8.97)] and from \(T^\alpha V^\beta \ll TV^{\beta + (\alpha - 1)/\mu(\sigma)}\) for \(\alpha < 1\) and \(V \ll T^{\mu(\sigma)}\).

Substituting pointwise estimates of \(m(\sigma)\) from Lemma \(9\) instead of segmentwise from Lemma \(8\) into \([12]\) we obtain following result.

**Theorem 5.** The statement of Theorem \(4\) remains valid for

\[C_4 = 0.133437785 \ldots\]

6. Conclusion

We have obtained nontrivial error terms in asymptotic estimates of \(\sum_{n \leq x} \lambda_r^{(\sigma)}(n)\) for \(r = 1, 2, 3, 4\). Cases of \(r = 1\) and \(r = 2\) depend on the method of exponent pairs.

Cases of \(r = 3\) and \(r = 4\) depend on lower bounds of \(m(\sigma)\). Note that case of \(r = 4\) may be improved under Riemann hypothesis up to \(C_4 = 1/8\), because Riemann hypothesis implies \(\mu(\sigma) = 0\) and \(m(\sigma) = \infty\) for \(\sigma \in [1/2, 1]\).

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