The Jantzen conjecture for singular characters

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ABSTRACT
We show that the Jantzen filtration of a Verma module (possibly singular) coincides with its radical filtration. It implies that the Jantzen Conjecture on Verma modules holds for all infinitesimal characters, while the regular case was settled by Beilinson and Bernstein by geometric methods and reproved by Williamson by an algebraic approach.

1. Introduction
Let \( g \) be a complex semisimple Lie algebra with a fixed Borel subalgebra \( b \) and a Cartan subalgebra \( h \subset b \). Let \( \Phi \) be the root system of \((g, h)\) with the positive system \( \Phi^+ \) and simple system \( \Delta \) corresponding to \( b \). For any \( \lambda \in h^* \), define the Verma module
\[
M(\lambda) := U(g) \otimes_{U(b)} C_{\lambda - \rho},
\]
where \( C_{\lambda - \rho} \) is a one dimensional \( b \)-module of weight \( \lambda - \rho \) and \( \rho \) is the half sum of positive roots. Those modules are standard modules in the BGG category \( O \) [5]. Denote \( \Phi_+^\lambda = \{ \alpha \in \Phi^+ \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}^{>0} \} \). Here \( \langle , \rangle \) is the pairing between \( h^* \) and \( h \) and \( \alpha^\vee \) is the coroot of \( \alpha \) (see [6, VI, 1]).

1.1. Jantzen filtration
Let \( \lambda \in h^* \). Then \( M(\lambda) \) has a filtration by submodules
\[
M(\lambda) = M(\lambda)^0 \supset M(\lambda)^1 \supset M(\lambda)^2 \supset \cdots
\]
with \( M(\lambda)^i = 0 \) for large \( i \), such that:
(1) Every quotient \( M(\lambda)^i/M(\lambda)^{i+1} \) has a nondegenerate contravariant form.
(2) \( M(\lambda)^1 \) is the unique maximal submodule of \( M(\lambda) \).
(3) There is a formal character formula:
\[
\sum_{i>0} \text{ch } M(\lambda)^i = \sum_{\alpha \in \Phi_+^\lambda} \text{ch } M(s_{\alpha} \lambda).
\]
The equation (1.2) is known as the Jantzen sum formula.
This remarkable filtration introduced by Jantzen [14] provides deep information about Verma modules. Since nonzero homomorphisms between Verma modules are always injective, Jantzen raised the following famous question.
1.2. Jantzen Conjecture

Suppose that \( M(\mu) \subset M(\lambda) \) for \( \lambda, \mu \in h^* \). Then
\[
M(\mu) \cap M(\lambda)^i = M(\mu)^{i-r}
\]
for \( i \geq r \), where \( r = |\Phi_\lambda^+| - |\Phi_\mu^+| \).

Understanding the composition factor multiplicities of Verma modules is a core problem in representation theory around 1980s. A precise conjecture was proposed (for regular integral infinitesimal character) by Kazhdan and Lusztig in their 1979 paper [15]. Soon Gabber and Joseph showed that a weak version of the Jantzen Conjecture lead to the validity of the Kazhdan–Lusztig Conjecture [10]. This implies that the Jantzen Conjecture may be stronger than the Kazhdan-Lusztig Conjecture. Almost at the same time, the Kazhdan-Lusztig Conjecture was proved independently by Beilinson-Bernstein [3] and Brylinsky-Kashiwara [7] through geometric techniques. After that, the singular and nonintegral cases were solved by Soergel [18, 19]. The proof of the Jantzen Conjecture, however, requires an extension of the geometric methods. The main idea was formulated in the early 1980s by Beilinson and Bernstein, while a complete proof (for the regular case) was published in 1993 [4]. In this paper, we find a method to translate the Jantzen filtration on a regular Verma module into the Jantzen filtration on a singular one (Proposition 3.1). Thus the full Jantzen Conjecture (Theorem 4.5) follows from [4] and the rigidity of Verma modules.

It is natural to expect an algebraic approach to these purely algebraic problems. In 1990, Soergel reformulated the Kazhdan-Lusztig Conjecture by using some special modules, so called Soergel modules [19]. These modules can be obtained from the Soergel bimodules [8, 20]. Soergel showed that the Kazhdan-Lusztig Conjecture holds if and only if the indecomposable Soergel bimodules are categorification of the Kazhdan-Lusztig basis of the Hecke algebra. In 2007, he also proved that such a theory can be generalized to any Coxeter system [21]. The analogue of the Kazhdan-Lusztig Conjecture in these settings, which is now known as the Soergel Conjecture, was eventually settled by Elias and Williamson in 2014 [9]. Lately, Williamson also got an algebraic proof of the regular Jantzen Conjecture by giving the local hard Lefschetz theorem for Soergel bimodules [24]. Combined with this, the full Jantzen Conjecture can be obtained by algebraic methods since the approach in this paper is purely algebraic.

The Jantzen filtration also plays a significant role in the classification of unitary Harish-Chandra modules. It is known that parabolic Verma modules appear as intermediate modules in the cohomological induction [16], which can be used to construct all the simple Harish-Chandra modules of a real reductive Lie group. The signatures of invariant Hermitian forms on these intermediate modules may be used to calculate the signatures of forms associated with the cohomologically induced modules and thus determine the unitarity.

The application of the Jantzen filtration for computing these signatures was initiated by Vogan [23]. Yee extended this idea in the setting of highest weight modules. In [25], the signed Kazhdan-Lusztig polynomials are defined to give signatures of forms on regular highest weight modules, while such polynomials were shown to be equal to classical Kazhdan-Lusztig polynomials evaluated at \(-q\) and multiplied by a sign in [26]. The proof depends heavily on the validity of regular Jantzen conjecture. It is expected that results in this paper could shed some light on the singular case of related problem, which have attracted more attention since the irreducible unitary Harish-Chandra modules with strongly regular infinitesimal characters are already known [17]. In [1], a general algorithm was built to find all the unitary representations, where usage of regular Jantzen Conjecture is still an essential step. The result of the singular case might be used to calculate unitary representations more efficiently.

The Jantzen filtration can also be defined in various setting such as quantum groups [2] and Weyl modules [13] and superalgebras [22]. It is natural to consider similar problems in those settings.

The paper is organized as follows. In Section 2, we recall the definition of the Jantzen filtration. In Section 3, we solve a problem raised by Jantzen which states that the translation functor sends Jantzen filtrations of regular Verma modules to the filtrations of singular ones. In Section 4, we show that Jantzen filtrations and radical filtrations are the same for Verma modules. This immediately yields the Jantzen Conjecture for singular infinitesimal characters.
2. Preliminaries

The Jantzen filtration for highest weight modules will be defined at the end of this section, after the necessary notations are introduced. Most results in this section can be found in [11] or [10].

2.1. Complex Lie algebras

Recall that $\mathfrak{g}$ is a complex semisimple Lie algebra with a fixed Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{h}$ is a Cartan subalgebra and $\mathfrak{n}$ the nilradical of $\mathfrak{b}$. Define the root lattice $Q := \mathbb{Z}\Phi$, where $\Phi$ is the root system of $(\mathfrak{g}, \mathfrak{h})$ with a simple system $\Delta$ corresponding to $\mathfrak{b}$. Choose a basis $E_\beta (\beta \in \Phi)$, $H_\alpha (\alpha \in \Delta)$ of $\mathfrak{g}$ so that $E_\beta$ is a nonzero vector in the root space corresponding to $\beta$ and $[E_\alpha, E_\beta] = H_\alpha$. There is an anti-involution $\sigma$ on $\mathfrak{g}$ which fixes $\mathfrak{h}$ and interchanges $E_\beta$ with $E_{-\beta}$ for $\beta \in \Phi$. Set $\overline{\mathfrak{g}} = \mathfrak{g}/\Phi$. Then $\mathfrak{g} = \overline{\mathfrak{g}} \oplus \mathfrak{h} \oplus \mathfrak{n}$. Denote by $W$ the Weyl group of $\Phi$. The action of $W$ on $\mathfrak{h}^*$ is given by $s_\alpha \lambda = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ for $\alpha \in \Phi$ and $\lambda \in \mathfrak{h}^*$. A weight $\lambda \in \mathfrak{h}^*$ is called dominant (resp. anti-dominant) if $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}^>0$ (resp. $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}^<0$) for all $\alpha \in \Delta$.

2.2. Extended characters

Let $A = \mathbb{C}[t]$ for an indeterminate $t$. Set $\mathfrak{g}_A := A \otimes \mathbb{C} \mathfrak{g}$ and similarly define $\mathfrak{b}_A, \mathfrak{h}_A, \mathfrak{n}_A, \overline{\mathfrak{n}}_A, ...$ Note that each $\lambda \in \mathfrak{h}_A^*$ determines an $\mathfrak{h}_A$-module $A_\lambda$ by $h \cdot a = \lambda(h)a$ for $h \in \mathfrak{h}_A$ and $a \in A$, which can be viewed as a $\mathfrak{b}_A$-module with trivial $\mathfrak{n}_A$ action. The Verma module (over $A$) is defined by

$$M(\lambda)_A := U(\mathfrak{g}_A) \otimes_U (\mathfrak{b}_A)_{A, \lambda}^\ast, \rho.$$ 

The modules $M(\lambda)_A$ possess a character $\chi_\lambda$, which is an algebra homomorphism from the center $Z(\mathfrak{g}_A)$ of $U(\mathfrak{g}_A)$ to $A$ so that $z \cdot v = \chi_\lambda(z)v$ for all $z \in Z(\mathfrak{g}_A)$ and $v \in M(\lambda)_A$. Moreover, $\chi_\lambda = \chi_\mu$ when $\mu \in \mathbb{W}_A$.

2.3. The category $\mathcal{K}_C$

The above Verma modules can be well studied in some special subcategories of $\mathfrak{g}_A$-modules. If $M$ is an $\mathfrak{h}_A$-module, for $\lambda \in \mathfrak{h}_A^*$, set

$$M_\lambda := \{ v \in M \mid h \cdot v = \lambda(h)v, \text{ for all } h \in \mathfrak{h}_A \}.$$ 

Fix $\mu \in \mathfrak{h}_A^*$. Put $C := \mu + Q \subset \mathfrak{h}_A^*$. Denote by $\mathcal{K}_C$ the full subcategory of $\mathfrak{g}_A$-modules $M$ such that:

1. $M = \sum_{\nu \in C} M_\nu$. 
2. $U(\mathfrak{n}_A)v$ is a finitely generated $A$-module for any $v \in M$.
3. $M$ is finitely generated as a $\mathfrak{g}_A$-module.

Denote by $\lambda \rightarrow \overline{\lambda}$ the canonical map $\mathfrak{h}_A^* \rightarrow \mathfrak{h}_A^*/\theta \mathfrak{h}_A^* \simeq \mathfrak{h}^*$. We call $D \subset C$ a block if $\overline{D} = \{ \overline{v} \mid v \in D \}$ is a nonempty intersection of $\overline{C} = \{ \overline{v} \mid v \in C \}$ with a $W$-orbit in $\mathfrak{h}^*$. In other words, $\overline{D} = \overline{C} \cap W\lambda$ for a $\lambda \in C$. Set

$$J_D := \cap_{\lambda \in D} \text{Ker } \chi_\lambda.$$ 

For $M \in \mathcal{K}_C$, define the submodule

$$M_D := \{ v \in M \mid \forall x \in J_D, \exists n \in \mathbb{Z}^>0 \text{ such that } x^n \cdot v = 0 \}.$$ 

Proposition 2.1. [10, Proposition 1.8.4] Let $M \in \mathcal{K}_C$. Then $M = \bigoplus D M_D$, where the sum is taken over all blocks $D \subset C$. In particular, $M_D$ is nonzero for only finitely many blocks $D$.

Let $\text{pr}_D : M \rightarrow M_D$ be the natural projection on $\mathcal{K}_C$, that is, $\text{pr}_D M = M_D$. Evidently $\text{pr}_D$ is exact.
2.4. Contravariant forms

Note that the anti-involution \( \sigma \) extends naturally to an anti-automorphism on the enveloping algebra \( U(\mathfrak{g}_A) \). Let \( M \) be a \( \mathfrak{g}_A \)-module, we say a symmetric bilinear form \( \langle , \rangle_M \) on \( M \) is contravariant if
\[
(u \cdot v_1, v_2)_M = (v_1, \sigma(u) \cdot v_2)_M
\]
for all \( v_1, v_2 \in M \) and \( u \in U(\mathfrak{g}_A) \).

**Lemma 2.2.** Let \( \langle , \rangle_M \) be a contravariant form on \( M \in \mathcal{K}_C \). Then
\begin{enumerate}
\item \( (M_\lambda, M_\mu)_M = 0 \) for \( \lambda \neq \mu \) in \( \mathfrak{h}_A^* \).
\item \( (M_{D_1}, M_{D_2})_M = 0 \) for blocks \( D_1 \neq D_2 \) in \( C \).
\item For \( i = 1, 2 \), suppose that \( M_i \) is a \( \mathfrak{g}_A \)-module with contravariant form \( \langle , \rangle_{M_i} \). Then \( \langle , \rangle_M = \langle , \rangle_{M_1} \otimes \langle , \rangle_{M_2} \) is a contravariant form on \( M_1 \otimes M_2 \).
\end{enumerate}

**Proof.** The lemma follows as in the case when \( A \) is the field \( \mathbb{C} \) (e.g., [11, Proposition 3.14], keeping in mind of results in [10, 1.8]). \( \square \)

Note that each \( \lambda \in \mathfrak{h}_A^* \) induces a homomorphism \( \lambda : U(\mathfrak{h}_A) \rightarrow A \). Composing \( \lambda \) with the natural projection \( U(\mathfrak{g}_A) \cong U(\mathfrak{h}_A) \otimes U(\mathfrak{n}_A) \rightarrow U(\mathfrak{h}_A) \), we get a linear map \( \varphi_\lambda : U(\mathfrak{g}_A) \rightarrow A \).

If \( M \) is a highest weight \( \mathfrak{g}_A \)-module generated by a highest weight vector \( v^+ \), we can construct contravariant forms \( \langle , \rangle_M \) on \( M \) with
\[
(u \cdot v^+, u' \cdot v^+) = (v^+, \varphi_\lambda(\sigma(u)u')v^+)_M,
\]
where \( u, u' \in U(\mathfrak{g}_A) \). The form is called canonical if \( (v^+, v^+) = 1 \). The following lemma shows that the canonical form is determined by \( v^+ \).

**Lemma 2.3.** Let \( M \) be a highest weight \( \mathfrak{g}_A \)-module. There exists a unique (up to a unit in \( A \)) nonzero contravariant form \( \langle , \rangle_M \) on \( M \). Moreover, any contravariant form on \( M \) can be written as \( a(\), \rangle_M \) for some \( a \in A \).

2.5. Jantzen filtration

Now we take \( C = \mu + \delta t + Q \) with \( \mu \in \mathfrak{h}_A^* \) anti-dominant and \( \delta \in \mathfrak{h}_A^* \) regular.

**Definition 2.4.** Let \( M \in \mathcal{K}_C \). Fix a contravariant form \( \langle , \rangle_M \) on \( M \). Define
\[
M^j = \{ v \in M | (v, M) \in (t^j) \}
\]
for \( j \in \mathbb{N} \). Then \( M^j \) is a filtration of \( \mathfrak{g}_A \)-modules. It determines a filtration of \( \mathfrak{g} \)-module \( \overline{M} := M/tM \) by \( \overline{M}^j = M^j/(tM \cap M^j) \). In particular, if \( M \) is a highest weight module and \( \langle , \rangle_M \) is a canonical form on \( M \), then we call \( \overline{M}^j \) the Jantzen filtration of \( \overline{M} \).

**Remark 2.5.** For \( \lambda \in \mathfrak{h}_A^* \), the Jantzen filtration \( M(\lambda)^j = \overline{M(\lambda + \delta t)^j} \) of the Verma module \( M(\lambda) \) is defined this way. As shown in [24, 8.4], it is possible to get filtrations of \( M(\lambda) \) with non-semisimple layers when the deformation \( \delta \) is not dominant. In this paper, we will always assume that \( \delta = \rho \) as is done in Jantzen’s original setting [14].

3. Jantzen filtration under translation

This section is devoted to proving a result (Proposition 3.1) proposed by Jantzen [14, 5.17]. Let \( C = \mu + \delta t + Q \) with \( \mu \in \mathfrak{h}_A^* \) anti-dominant and \( \delta = \rho \). Suppose that \( \lambda \in \mu + Q \) is a regular anti-dominant weight. Let \( T^\mu_\lambda \) be the usual translation functor (see [11, 7.1]). Define
\[
\Phi_{[\mu]} := \{ \alpha \in \Phi | (\mu, \alpha^\vee) \in \mathbb{Z} \}
and 
\[ W_{[\mu]} := \{ w \in W \mid w\mu - \mu \in Q \}. \]

Then \( \Phi_{[\mu]} \) is a root system with Weyl group \( W_{[\mu]} \) (see [6] or [11]). Let \( \Delta_{[\mu]} \) be the simple system determined by \( \Phi_{[\mu]}^+ := \Phi_{[\mu]} \cap \Phi^+ \). Set \( J = \{ \alpha \in \Delta_{[\mu]} \mid \langle \mu, \alpha^\vee \rangle = 0 \} \). Denote by \( \Phi_J \) the root system generated by \( J \) and by \( W_J \) the Weyl group generated by reflections \( s_\alpha \) with \( \alpha \in J \). Let \( \ell(, ) \) (resp. \( \ell_{[\mu]}(, ) \)) be the length function on \( W \) (resp. \( W_{[\mu]} \)). Obviously \( \ell = \ell_{[\mu]} \) when \( \mu \) is integral. Set 
\[ W^J_{[\mu]} := \{ w \in W_{[\mu]} \mid \ell_{[\mu]}(ws_\alpha) = \ell_{[\mu]}(w) + 1 \text{ for all } \alpha \in J \}. \]

Thus \( W^J_{[\mu]} \) contains all the shortest representatives of the coset \( W_{[\mu]}/W_J \). For any \( v \in \mathfrak{h}^* \), there exists a unique anti-dominant weight \( \mu \) and \( w \in W^J_{[\mu]} \) such that \( v = w\mu \) (see [11, Proposition 3.5]).

**Proposition 3.1.** Fix \( w \in W_{[\mu]} \). There exists \( k \in \mathbb{N} \) such that 
\[ T^\mu_\lambda M(w\lambda)^j \simeq M(w\mu)^{j-k} \]  
for all \( j \in \mathbb{N} \). In particular, if \( w \in W^J_{[\mu]} \), then \( k = 0 \).

Here \( M(w\mu)^{j-k} = M(w\mu) \) when \( j < k \). For \( v \in \mathfrak{h}^* \), denote by \( L(v) \) the unique simple quotient of \( M(v) \). The following result is quite useful.

**Lemma 3.3.** [11, Theorem 7.9] Fix \( w \in W_{[\mu]} \). Then \( T^\mu_\lambda L(w\lambda) = L(w\mu) \) for \( w \in W^J_{[\mu]} \) and is vanishes otherwise.

### 3.1. Extended translation functor

To prove Proposition 3.1, we need to extend the definition of translation functors. Let \( E(\mu - \lambda) \) be the finite dimensional simple \( \mathfrak{g} \)-module with extremal weight \( \mu - \lambda \). Then \( E(\mu - \lambda)_A := A \otimes_C E(\mu - \lambda) \) is a \( \mathfrak{g}_A \)-module of finite rank. Set \( D = W_{[\mu]}\lambda + \delta t \) and \( D' = W_{[\mu]}\mu + \delta t \). Define the (extended) translation functor \( T^\mu_\lambda \) on \( M \in K_C \) :
\[ T^\mu_\lambda M = \text{pr}_D(E(\mu - \lambda)_A \otimes_A \text{pr}_DM). \]  
Obviously \( T^\mu_\lambda M \simeq T^\mu_\lambda M/\delta T^\mu_\lambda M \) as \( \mathfrak{g} \)-modules.

**Proposition 3.5.** For \( w \in W_{[\mu]} \),
\[ T^\mu_\lambda M(w\lambda + \delta t)_A \simeq M(w\mu + \delta t)_A. \]

**Proof.** Take \( M = M(w\lambda + \delta t)_A \) for \( w \in W_{[\mu]} \). Then \( M \in K_C \) and \( \text{pr}_DM = M_D = M \). The module \( E(\mu - \lambda)_A \otimes_A M(w\lambda + \delta t)_A \) has a filtration with quotients which are isomorphic to \( M(w\lambda + \delta t + v')_A \), where \( v' \in \mathfrak{h}^* \) are weights of \( E(\mu - \lambda) \), counting multiplicities. By [14, 2.9] (see also [11, Lemma 7.5]), \( \text{pr}_D(M(w\lambda + \delta t + v')_A) \) does not vanish if and only if \( v' = w\mu - w\lambda \). The lemma then follows from the exactness of \( \text{pr}_D \).

### 3.2. Filtrations on tensor product and primary decomposition

In this subsection, we give two necessary lemmas for proving Proposition 3.1. For simplicity, put \( E := E(\mu - \lambda)_A \) in this subsection.

**Lemma 3.6.** Let \( M \in K_C \) be a highest weight \( \mathfrak{g}_A \)-module. Fix the contravariant form \( (, ) = (, )_E \otimes (, )_M \) on \( E \otimes_A M \), where \( (, )_E \) and \( (, )_M \) are canonical forms on \( E \) and \( M \). Then for \( j \in \mathbb{N} \), we have
\[ (E \otimes_A M)^j = E \otimes_A M^j \]
Recall that the \( E(\mu - \lambda) \) admits a nondegenerate contravariant form over \( \mathbb{C} [11, \text{Theorem 3.15}] \). This form is extended to a canonical form over \( A \), which can be assumed to be \( (\cdot, \cdot)_E \). The inclusion \( E \otimes_A M^j \subset (E \otimes_A M)^j \) follows immediately from the definition. For the other direction, take a basis \( \{e_1, \ldots, e_n\} \) of \( E(\mu - \lambda) \) over \( \mathbb{C} \) and its dual basis \( \{e'_1, \ldots, e'_n\} \) with respect to the nondegenerate contravariant form on \( E(\mu - \lambda) \). Then \( (e_i, e'_j)_E = \delta_{ij} \). Any \( u \in E \otimes_A M \) can be written as \( \sum_{i=1}^n e_i \otimes v_i \) with \( v_i \in M \). Fix an integer \( 1 \leq k \leq n \). If \( u \in (E \otimes_A M)^j \), it follows from
\[
(v, v_M) = \left( \sum_{i=1}^n e_i \otimes v_i, e'_k \otimes v \right) = (u, e'_k \otimes v) \in (t^j)
\]
for all \( v \in M \) that \( v_k \in M^j \). Thus \( u \in E \otimes_A M^j \).

**Lemma 3.7.** Let \( (\cdot)_M \) be a nondegenerate contravariant form on \( M \in \mathcal{K}_C \). If \( M_D \) is a highest weight module, then there exists \( k \in \mathbb{N} \) such that
\[
(M^j)_D = M^j \cap M_D = (M_D)^j - k
\]
for any \( j \in \mathbb{N} \). Here \( (M_D)^j \) is determined by a canonical form on \( M_D \).

**Proof.** With Lemma 2.2, it suffices to prove the last equality. Suppose that \( (\cdot)' \) is the restriction of \( (\cdot)_M \) on \( M_D \). In view of Lemma 2.3, \( (\cdot)' = a(\cdot)_{M_D} \) for some \( a \in A \), where \( (\cdot)_{M_D} \) is a canonical form on \( M_D \). Since \( A \) is a principal domain, we can find \( k \in \mathbb{N} \) such that \( a \in (t^k) \) and \( a \neq (t^{k+1}) \). Therefore \( M^j \cap M_D = (M_D)^j - k \), keeping in mind that \( M^j \cap M_D \) is a filtration of \( M_D \) determined by \( (\cdot)' \).

### 3.3. Proof of Proposition 3.1

Take \( M = M(w \lambda + \delta t)_A \) for \( w \in W_\mu \). Set \( N = E \otimes_A M \) and \( (\cdot)_N = (\cdot)_E \otimes (\cdot)_M \). Proposition 3.5 implies \( N_D = M(w \mu + \delta t)_A \) is a highest weight module. It follows from Lemmas 3.6 and 3.7 that
\[
T^\mu_\lambda M^j = (E \otimes_A M^j)_D = (N^j)_D = N^j \cap N_D = (N_D)^j - k
\]
for a \( k \in \mathbb{N} \). We get \( T^\mu_\lambda M(w \lambda + \delta t)_A \simeq M(w \mu + \delta t)_A^j - k \). Note that \( M(w \lambda + \delta t)_A = M(w \lambda) \) and \( M(w \mu + \delta t)_A = M(w \mu) \). This immediately yields \( T^\mu_\lambda M(w \lambda) \simeq M(w \mu)^j - k \). Now suppose \( w \in W_{[\mu]} \).

If \( k > 0 \), one has \( T^\mu_\lambda M(w \lambda) = M(w \mu) \) and \( T^\mu_\lambda M(w \lambda)^k \simeq M(w \mu) \) by taking \( j = 0, k \). This forces \( T^\mu_\lambda (M(w \lambda)/M(w \lambda)^k) = 0 \) and \( T^\mu_\lambda L(w \lambda) = 0 \). By Lemma 3.3, we arrive at a contradiction \( L(w \mu) = 0 \).

### 4. Jantzen Conjecture of Verma modules

In this section, we will prove the Jantzen Conjecture for singular infinitesimal characters.

We say a filtration of \( M \in \mathcal{O} \) is a Loewy filtration if it has the shortest possible length provided its successive quotients is semisimple. The length of such a filtration is called the Loewy length of \( M \). The radical filtration and socle filtration are two extreme examples of Loewy filtrations. Denote by \( \text{Rad}^i M \) (resp. \( \text{Soc}^i M \)) the \( i \)-th radical (resp. socle) filtration of \( M \) when \( i \geq 0 \). If \( M = M^0 \supseteq M^1 \supseteq \cdots \) is the descending chain of a Loewy filtration, then \( \text{Rad}^i M \subset M^i \subset \text{Soc}^{i-1} M \), where \( s \) is the Loewy length of \( M \). We say \( M \) is rigid if its radical and socle filtrations coincide (thus its Loewy filtration is unique). If \( M \) is rigid, then \( \text{Rad}^i M = \text{Soc}^{i-1} M \) for \( i \in \mathbb{Z} \). Here \( \text{Rad}^0 M = M = \text{Soc}^0 M = 0 \) when \( i \in \mathbb{Z} < 0 \). Put \( \text{Rad}_s M = \text{Rad}^s M/\text{Rad}^{s+1} M \) and \( \text{Soc}_s M = \text{Soc}^s M/\text{Soc}^{s+1} M \).

As in the previous section, the weight \( \mu \) is anti-dominant and \( \lambda \in \mu + Q \) is regular anti-dominant. Recall that \( J = \{ \alpha \in \Delta_{[\mu]} \ | \ \langle \mu, \alpha \rangle = 0 \} \). Any \( w \in W_{[\mu]} \) can be uniquely written as \( w = yx \) with \( y \in W_{[\mu]} \) and \( x \in W_J \).

**Proposition 4.1.** Suppose that \( w = yx \in W_{[\mu]} \) with \( y \in W^j_{[\mu]} \) and \( x \in W_J \). Then
Theorem 4.5.

(1) \( M(w\mu) \) is rigid with Loewy length \( \ell_{[\mu]}(y) + 1 \).

(2) \( T^\mu_\lambda \text{Rad}^j M(w\lambda) = \text{Rad}^{\ell - \ell_{[\mu]}(x)} M(y\mu) \) for any \( j \in \mathbb{N} \).

**Proof.** With \( \lambda \in \mu + \mathbb{Q} \), we can set \( \ell' := \ell_{[\mu]} = \ell_{[\mu]}(\lambda) \). So \( \ell'(w) = \ell'(y) + \ell'(x) \).

(1) Since \( w\mu = y\mu \), we need to show that \( M(y\mu) \) is rigid with Loewy length \( \ell'(y) + 1 \). If \( \mu \) is integral and regular, this is a consequence of the Kazhdan–Lusztig theory [11, Theorem 8.15]. If \( \mu \) is integral and singular, the result follows from [12, Theorem 1.4.2]. If \( \mu \) is not integral, the proof can be reduced to the integral case by Soergel's category equivalence [19, Theorem 11].

(2) Since \( \lambda \) is regular, \( W^\mu_{[\lambda]} = W_{[\lambda]} = W_{[\mu]} \). The Loewy length of \( M(w\lambda) \) (resp. \( M(y\lambda) \)) is \( \ell'(w) + 1 \) (resp. \( \ell'(y) + 1 \)) by (1). Note that \( M(y\lambda) \subset M(w\lambda) \). Thus

\[
M(y\lambda) \subset \text{Soc}^{\ell'(y)+1}M(w\lambda) = \text{Rad}^{\ell'(x)}M(w\lambda) \subset M(w\lambda).
\]

(4.2)

In view of [11, Theorem 7.6], we have \( T^\mu_\lambda M(w\lambda) = M(w\mu) = M(y\mu) = T^\mu_\lambda M(y\lambda) \). Thus \( T^\mu_\lambda (M(w\lambda)/M(y\lambda)) = 0 \) and \( T^\mu_\lambda \text{Rad}^{\ell'(x)} M(w\lambda) = M(w\mu) \). If \( i < \ell'(x) \), then \( \text{Rad}_i M(w\lambda) \) is a subquotient of \( M(w\lambda)/M(y\lambda) \) by (4.2) and \( T^\mu_\lambda \text{Rad}_i M(w\lambda) = 0 \). We claim that

\[
M(w\mu) = T^\mu_\lambda \text{Rad}^{\ell'(x)} M(w\lambda) \supset \cdots \supset T^\mu_\lambda \text{Rad}^{\ell'(x)+1} M(w\lambda) = 0
\]

is a Loewy filtration of \( M(w\mu) \). In fact, the \( j \)-th quotient of this filtration is equal to \( T^\mu_\lambda \text{Rad}^{\ell'(x)+j} M(w\lambda) \), which is semisimple by Lemma 3.3. The length of this filtration is equal to \( \ell'(w) + 1 - \ell'(x) = \ell'(y) + 1 \), which is the Loewy length of \( M(w\mu) \) in view of (1). Hence the rigidity of \( M(w\mu) \) forces \( T^\mu_\lambda \text{Rad}^{\ell'(x)+j} M(w\lambda) = \text{Rad}_j M(w\mu) \) for \( 0 \leq j \leq \ell'(y) + 1 \).

**Remark 4.3.** If \( \mu \) is integral, Proposition 4.1(2) is obtained in [12, Theorem 2.3.2].

**Theorem 4.4.** For \( w \in W^\mu_{[\lambda]} \) and \( j \in \mathbb{N} \), we have

\[
M(w\mu)^j \simeq \text{Rad}^j M(w\mu).
\]

**Proof.** Proposition 3.1 implies \( T^\mu_\lambda M(w\lambda)^j \simeq M(w\mu)^j \). By [4], one has \( M(w\lambda)^j = \text{Rad}^j M(w\lambda) \). The theorem then follows from Proposition 4.1.

**Theorem 4.5.** The Jantzen Conjecture holds for all infinitesimal characters.

**Proof.** It suffices to consider \( M(x\mu) \subset M(w\mu) \) for \( x, w \in W^\mu_{[\mu]} \). With Theorem 4.4, it suffices to show that \( M(x\mu) \cap \text{Rad}^j M(w\mu) = \text{Rad}^{\ell - \ell_{[\mu]}(x)} M(x\mu) \) for \( j \in \mathbb{N} \) and \( r = |\Phi^+_{w\mu}| - |\Phi^+_{x\mu}| \). By Soergel's category equivalence [19], we can assume that \( \mu \) is integral. In this case, \( W_{[\mu]} = W \). Recall that \( \Phi^+_{w\mu} = \{ \alpha > 0 \mid \langle w\mu, \alpha \rangle \in \mathbb{Z}\} \). So

\[
\Phi^+_{w\mu} = \{ \alpha > 0 \mid \langle \mu, w^{-1}\alpha \rangle \in \mathbb{Z}\} = \{ \alpha > 0 \mid w^{-1}\alpha < 0 \} = \ell(w),
\]

where the second equality follows from the fact that \( w^{-1}\alpha > 0 \) for \( w \in W^\mu_{[\mu]} \) and \( \beta \in \Phi^+ \). Thus, \( r = \ell(w) - \ell(x) \). By Proposition 4.1, the Verma module \( M(w\mu) \) (resp. \( M(x\mu) \)) is rigid with Loewy length \( \ell(w) + 1 \) (resp. \( \ell(x) + 1 \)). Therefore,

\[
M(x\mu) \cap \text{Rad}^j M(w\mu) = M(x\mu) \cap \text{Soc}^{\ell(w)+1-j} M(w\mu) = \text{Soc}^{\ell(w)+1-j} M(x\mu) = \text{Rad}^{\ell - (\ell(w) - \ell(x))} M(x\mu).
\]

\[
\square
\]
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