Enriched Interpretation

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Abstract

The theory introduced, presented and developed in this paper, is concerned with an enriched extension of the theory of Rough Sets pioneered by Zdzislaw Pawlak [5]. The enrichment discussed here is in the sense of valuated categories as developed by F.W. Lawvere [4]. This paper relates Rough Sets to an abstraction of the theory of Fuzzy Sets pioneered by Lotfi Zadeh [7], and provides a natural foundation for soft computation. To paraphrase Lotfi Zadeh, the impetus for the transition from a hard theory to a soft theory derives from the fact that both the generality of a theory and its applicability to real-world problems are substantially enhanced by replacing various hard concepts with their soft counterparts. Here we discuss the corresponding enriched notions for indiscernibility, subsets, upper/lower approximations, and rough sets. Throughout, we indicate linkages with the theory of Formal Concept Analysis pioneered by Rudolf Wille [6]. We pay particular attention to the all-important notion of a linguistic variable — developing its enriched extension, comparing it with the notion of conceptual scale from Formal Concept Analysis, and discussing the pragmatic issues of its creation and use in the interpretation of data. These pragmatic issues are exemplified by the discovery, conceptual analysis, interpretation, and categorization of networked information resources in WAVE, the Web Analysis and Visualization Environment [3] currently being developed for the management and interpretation of the universe of resource information distributed over the World-Wide Web.

1 Indiscernibility

Indiscernibility, a central concept in Rough Sets theory, is traditionally treated as a hard relationship — either two objects are indiscernible or they are not. In order to define and develop a soft theory of Rough Sets, it would seem quite appropriate, if not necessary, to define and develop a soft or graded version of indiscernibility. We do just that in this paper by using ideas from the theory of valuated categories.

An approximation space [5] is traditionally defined as a pair $\mathcal{G} = (G, E)$ consisting of a set of objects or entities $G$ and an equivalence relation $E \subseteq G \times G$ called indiscernibility. Two objects $g_1, g_2 \in G$ are indiscernible when $g_1 E g_2$; that is, when $E(g_1, g_2) = \text{true}$. Equivalently, an approximation space (function version) is a triple $\langle G, \phi, D \rangle$, where $G$ is a set of objects, $D$ is a set (hard and unenriched!) of values, and $G \xrightarrow{\phi} D$ is a (not necessarily surjective) function called a description function. The description function $\phi$ represents a certain amount of knowledge about the objects in $G$. Two objects $g_1, g_2 \in G$ are indiscernible when the procedure $\phi$ cannot distin-
guish between them, \( \phi(g_1) = \phi(g_2) \); or more generally, when \( \text{Eq}_D(\phi(g_1), \phi(g_2)) = \text{true} \) for some sense and relationship \( \text{Eq}_D \) of identification or approximation of values in \( D \). We are particularly interested in the case where \( D = \phi M \cong 2^M \) consists of subsets of a collection of attributes \( M \), and \( \phi \) maps an object of \( G \) to the subset of all attributes that it satisfies.

One way to soften this definition is to observe the fact that often \( D \) has additional enriched structure — either order-theoretic, topological or algebraic structure. To ignore this structure is to weaken the Rough Set analysis of the situation by using only the less refined, harder representation. In this paper we develop a more general, more flexible, and softer approach to Rough Sets, which handles enriched order-theoretic, metric topological, and fuzzy structure. A full categorical formulation would also handle algebraic structure. To enrich (and yes, fuzzify) Rough Set notions, we allow grades of indiscernibleness by assuming that \( D \) has \( V \)-enriched structure on it, where \( V = (V, \leq, \otimes, \Rightarrow, e) \) is a closed preorder (see Appendix A); that is, we assume that \( D \) is an approximation \( V \)-space.

2 Spaces and Maps

While enriched approximation spaces are the appropriate abstraction of indiscernibility and our main concern in this paper, it seems that these approximation spaces are best defined in terms of an asymmetric generalization called simply an enriched space. A pair \( \mathcal{X} = (X, \mu) \) consisting of a set \( X \) and a function \( \mu: X \times X \rightarrow V \) is called a \( V \)-enriched space or \( V \)-space when it satisfies

**transitivity (triangle axiom):**
\[
\mu(x_1, x_2) \otimes \mu(x_2, x_3) \preceq \mu(x_1, x_3), \text{ for all } x_1, x_2, x_3 \in X.
\]

The function \( \mu \), called a metric, represents a measure of agreement or distance between the elements of \( X \). We view the metric \( \mu \) to be a special square matrix \( \mu = [\mu_{x_1,x_2}] \) of \( V \)-values. We can interpret \( \mu \) to be either an enriched preordering, a generalized distance function, a similarity measure or a gradation.

When \( V = 2 \), the Boolean case, a \( V \)-space \( \mathcal{X} \) is precisely a preorder \( \mathcal{X} = (X, \preceq) \) with order characteristic function \( \preceq: X \times X \rightarrow 2 \). When \( V = \mathbb{R} \), the metric topology case, a \( V \)-space \( \mathcal{X} \) is (generalize) metric space \( \mathcal{X} = (X, \delta) \) with distance function \( \delta: X \times X \rightarrow \mathbb{R} \). When \( V = [0,1] \), the fuzzy case, a \( V \)-space \( \mathcal{X} \) is a fuzzy space \( \mathcal{X} = (X, \mu) \) with similarity measure \( \mu: X \times X \rightarrow [0, 1] \). Any \( V \)-space \( \mathcal{X} = (X, \mu) \) has a dual or opposite \( V \)-space \( \mathcal{X}^{\text{op}} = (X, \mu^{\text{op}}) \), where \( \mu^{\text{op}}(x_1,x_2) = \mu(x_2,x_1) \) is the dual or opposite metric. The sum of any two \( V \)-spaces \( \mathcal{X}_0 = (X_0, \mu_0) \) and \( \mathcal{X}_1 = (X_1, \mu_1) \) is the \( V \)-space \( \mathcal{X}_0 \oplus \mathcal{X}_1 = (X_0 + X_1, \mu) \) defined by \( \mu(x_0, x'_0) = \mu_0(x_0, x'_0), \mu(x_0, x_1) = \bot_V, \mu(x_1, x_0) = \bot_V, \) and \( \mu(x_1, x'_1) = \mu_1(x_1, x'_1) \). In general our metrics are asymmetrical: \( \mu(x_1, x_2) \neq \mu(x_2, x_1) \). A \( V \)-enriched approximation space or approximation \( V \)-space is defined to be a symmetrical \( V \)-space. So the metric \( \mu \), called an indiscernibility measure, is a \( V \)-enriched equivalence relation on \( X \) satisfying reflexivity, transitivity and

**reflexivity (zero law):**
\[
e \preceq \mu(x,x), \text{ for all } x \in X;
\]

**symmetry:** \( \mu(x_2,x_1) = \mu(x_1,x_2), \text{ for all } x_1, x_2 \in X. \)

Any \( V \)-space \( \mathcal{X} = (X, \mu) \) can be symmetrized and made into an approximation space, by defining the junction metric \( \mu^{\text{sym}}(x_1, x_2) = \mu(x_1, x_2) \otimes \mu^{\text{op}}(x_1, x_2) \).
Associated with every \( V \)-space \( X = (X, \mu) \) is an underlying preorder \( \Box_V(X) = (X, \preceq) \) where 
\[ x_1 \preceq x_2 \text{ when } e \preceq \mu(x_1, x_2) \text{ and } x_1 \text{ and } x_2 \text{ are unrelated when } e \not\preceq \mu(x_1, x_2). \]
Two elements \( x_1, x_2 \in X \) are said to be indiscernible when \( x_1 \equiv x_2 \), where \( x_1 \equiv x_2 \) means \( x_1 \preceq x_2 \) and \( x_2 \preceq x_1 \).
A \( V \)-space \( X = (X, \mu) \) is strict when the underlying indiscernibility relation is the identity: if \( x_1 \equiv x_2 \) then \( x_1 = x_2 \). The set of “truth values with implication” \( V = (V, \Rightarrow) \) is a strict \( V \)-space.

Note that \( \Box((X^{\text{op}})) = (\Box X)^{\text{op}} = (X, \succeq) \) the opposite order, and that \( \Box V(X) = (\Box V(X))^{\text{sym}} = (X, \equiv) \) the underlying indiscernibility relation. For a strict \( V \)-space the underlying preorder is a partial order. For a (soft) approximation \( V \)-space the underlying preorder is a (hard) equivalence relation. For the space of generalized truth \( V \)-values \( V \)-isometry \( y : X \rightarrow Y \) is called a \( V \)-isometry. Composition of (the opposite of) a \( V \)-map \( f : X \rightarrow Y \) on the right with the Yoneda embedding, is a \( V \)-isometry \( y : X^{\text{op}} \rightarrow V^Y \). Composition of (the opposite of) a \( V \)-map \( f : X \rightarrow Y \) on the right with the Yoneda embedding \( y : X^{\text{op}} \rightarrow V^Y \), resulting in the \( V \)-map \( f \circ y : X^{\text{op}} \rightarrow V^Y \), allows us to generalize the concept of a \( V \)-map. Such a generalized \( V \)-map, equivalent to a \( V \)-map \( X^{\text{op}} \times Y \rightarrow V \), may be regarded to be a \( V \)-enriched relation or \( V \)-relation from \( X \) to \( Y \). It is denoted by \( X \twoheadrightarrow Y \), with \( \tau(x, y) \) an element of \( V \) interpreted as the “truth-value of the \( \tau \)-relatedness of \( x \) to \( y \)” \[4\]. A \( V \)-relation is an \( |X| \times |Y| \)-matrix, whose \((x, y)\)-th entry is \( \tau(x, y) \). In elementary terms, a \( V \)-relation is an \( |X| \times |Y| \)-matrix, which respects the measures on both left and right: \( \mu(x', x) \otimes \tau(x, y) \leq \tau(x', y) \) and \( \tau(x, y) \otimes \nu(y, y') \leq \tau(x, y') \), \( \forall x, x' \in X, y, y' \in Y \).

As mentioned above, every \( V \)-map \( f : X \rightarrow Y \) determines a \( V \)-relation \( f^* \subset X \times Y \) defined by \( f^* = \}

3 Relations

Each element \( x \in X \) of a \( V \)-space \( X = (X, \mu) \) can be represented as the \( V \)-predicate \( y(x) = \mu(x, -) \) over \( X \) where \( y(x)(x') = \mu(x, x') \) for each \( x' \in X \). The function \( y : X \rightarrow V^X \), which is called the Yoneda embedding, is a \( V \)-isometry \( y : X^{\text{op}} \rightarrow V^X \). Composition of (the opposite of) a \( V \)-map \( f : X \rightarrow Y \) on the right with the Yoneda embedding \( y : X^{\text{op}} \rightarrow V^Y \), resulting in the \( V \)-map \( f \circ y : X^{\text{op}} \rightarrow V^Y \), allows us to generalize the concept of a \( V \)-map. Such a generalized \( V \)-map, equivalent to a \( V \)-map \( X^{\text{op}} \times Y \rightarrow V \), may be regarded to be a \( V \)-enriched relation or \( V \)-relation from \( X \) to \( Y \). It is denoted by \( X \twoheadrightarrow Y \), with \( \tau(x, y) \) an element of \( V \) interpreted as the “truth-value of the \( \tau \)-relatedness of \( x \) to \( y \)” \[4\]. A \( V \)-relation is an \( |X| \times |Y| \)-matrix, whose \((x, y)\)-th entry is \( \tau(x, y) \). In elementary terms, a \( V \)-relation is an \( |X| \times |Y| \)-matrix, which respects the measures on both left and right: \( \mu(x', x) \otimes \tau(x, y) \leq \tau(x', y) \) and \( \tau(x, y) \otimes \nu(y, y') \leq \tau(x, y') \), \( \forall x, x' \in X, y, y' \in Y \).

As mentioned above, every \( V \)-map \( f : X \rightarrow Y \) determines a \( V \)-relation \( f^* \subset X \times Y \) defined by \( f^* = \}
\[ f^{\text{op}} \cdot y_x, \text{ or on elements by } f_*(x, y) = \nu(f(x), y). \]

In particular, the Yoneda embedding becomes the relation \( \mathcal{X} \mathrel{\mu} \mathcal{X} \). Dually every \( \mathbf{V} \)-map \( \mathcal{X} \xrightarrow{f} \mathcal{Y} \) also determines a \( \mathbf{V} \)-relation \( \mathcal{Y} \xleftarrow{f^*} \mathcal{X} \) in the opposite direction defined by \( f^* = y_\mathcal{Y} \cdot V^f \), or on elements by \( f^*(y, x) = \nu(y, f(x)) \).

A pair of \( \mathbf{V} \)-relations \( \mathcal{X} \xrightarrow{\sigma} \mathcal{Y} \) and \( \mathcal{Y} \xrightarrow{\tau} \mathcal{Z} \) can be composed, yielding the \( \mathbf{V} \)-relation \( \mathcal{X} \xrightarrow{\sigma \circ \tau} \mathcal{Z} \) defined to be the supremum (iterated disjunction)

\[
(\sigma \circ \tau)(x, z) = \bigvee_{y \in \mathcal{Y}} (\sigma(x, y) \sqcap \rho(y, z))
\]

Relational composition is viewed as matrix multiplication. One can verify that relational composition is associative \( (\rho \circ \sigma) \circ \tau = \rho \circ (\sigma \circ \tau) \), and that metrics (as \( \mathbf{V} \)-relations) are identities \( \mu \circ \tau = \tau = \tau \circ \mu \). So \( \mathbf{V} \)-spaces and \( \mathbf{V} \)-relations form a category \( \text{Rel}_{\mathbf{V}} \). One can also verify that \( (f \cdot g)_* = f_* \circ g_* \) for any two composable \( \mathbf{V} \)-maps \( \mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z} \), and that \( (\text{Id}_{\mathcal{X}})_* = \mu \) the identity \( \mathbf{V} \)-relation at \( \mathcal{X} \). So the Yoneda embedding determines a functor \( (\_)_* : \text{Space}_{\mathbf{V}} \to \text{Rel}_{\mathbf{V}} \) which makes concrete the concept generalization discussed at the beginning of this section.

Relational composition has a right adjoint called residuation. The resiliation of a pair of \( \mathbf{V} \)-relations \( \mathcal{X} \xrightarrow{\sigma} \mathcal{Y} \) and \( \mathcal{X} \xrightarrow{\varphi} \mathcal{Z} \), denoted by the \( \mathbf{V} \)-relation \( \mathcal{Y} \xleftarrow{\sigma \wedge \varphi} \mathcal{Z} \), is defined to be the infimum (iterated conjunction)

\[
(\sigma \wedge \varphi)(y, z) = \bigwedge_{x \in \mathcal{X}} (\sigma(x, y) \Rightarrow \varphi(x, z))
\]

Note that \( (\sigma \circ \tau)^* \wedge \rho = (\tau^* \wedge (\sigma^* \wedge \rho)) \) for any pair of composable \( \mathbf{V} \)-relations \( \mathcal{X} \xrightarrow{\sigma} \mathcal{Y} \) and \( \mathcal{Y} \xrightarrow{\tau} \mathcal{W} \), and that \( \mu \wedge \rho = \rho \) for identity relation \( \mathcal{X} \xrightarrow{\mu} \mathcal{X} \).

## 4 Subsets

Given any two \( \mathbf{V} \)-spaces \( \mathcal{X} = \langle X, \mu \rangle \) and \( \mathcal{Y} = \langle Y, \nu \rangle \) there is a \( \mathbf{V} \)-space \( \mathcal{X} \otimes \mathcal{Y} \), called the tensor product of \( \mathcal{X} \) and \( \mathcal{Y} \), which enriches the Cartesian product set \( X \times Y \) with the metric defined by

\[
(\mu \otimes \nu)((x_1, y_1), (x_2, y_2)) = \mu(x_1, x_2) \otimes \nu(y_1, y_2)
\]

When \( \mathbf{V} \) is Cartesian closed, the tensor product is the (ordinary) Cartesian product. This tensor product construction has a right adjoint exponential construction making \( \text{Space}_{\mathbf{V}} \) into a closed category \( [4] \). Given any two \( \mathbf{V} \)-spaces \( \mathcal{X} \) and \( \mathcal{Y} \) the set of all \( \mathbf{V} \)-maps from \( \mathcal{X} \) to \( \mathcal{Y} \) is a \( \mathbf{V} \)-space \( \mathcal{Y}^X \), called the exponential \( \mathbf{V} \)-space of \( \mathcal{X} \) and \( \mathcal{Y} \), whose pointwise inf metric \( \mu \) is defined by \( \mu(f, g) = \bigwedge_{x \in X} \mu_Y(f(x), g(x)) \). Notice that the metric \( \mu_X \) is not used to define \( \mu \). The metric \( \mu_X \) is only used to restrict admission to the underlying set of \( \mathcal{Y}^X \).

As an important special case, the power \( \mathbf{V} \)-space \( \mathcal{X}^\mathcal{X} \) of all \( \mathbf{V} \)-valued \( \mathbf{V} \)-maps on \( \mathcal{X} \) is a \( \mathbf{V} \)-space with metric

\[
\phi \Rightarrow \psi = \bigwedge_{x \in \mathcal{X}} (\phi(x) \Rightarrow \psi(x))
\]

We interpret an element of \( \mathcal{X}^\mathcal{X} \), a \( \mathbf{V} \)-map \( \phi : \mathcal{X} \to \mathcal{V} \), to be a \( \mathbf{V} \)-enriched subset, which satisfies the internal pointwise metric constraint \( \mu : \mu(x_1, x_2) \leq \phi(x_1) \Rightarrow \phi(x_2) \) for all \( x_1, x_2 \in X \); or equivalently, by the \( \otimes \Rightarrow \) adjointness, \( \phi(x_1) \otimes \mu(x_1, x_2) \leq \phi(x_2) \) for all \( x_1, x_2 \in X \). Such a characteristic function \( \phi : \mathcal{X} \to \mathcal{V} \), which is constrained by the metric on \( \mathcal{X} \), is called a \( \mathbf{V} \)-enriched predicate or \( \mathbf{V} \)-predicate in \( \mathcal{X} \). It can also be called, using Rough Set terminology, a \( \mathbf{V} \)-enriched definable subset or \( \mathbf{V} \)-definable subset in \( \mathcal{X} \). To use a slogan, “predicate (or definable subset) \( \equiv \) metric-constrained character”.
5 Enriched Concept Analysis

Enriched Concept Analysis starts with the primitive notion of an enriched formal context. A (formal \(V\)-context) is a triple \(\langle G, M, \iota \rangle\) consisting of two approximation spaces \(G = \langle G, \gamma \rangle\) and \(M = \langle M, \mu \rangle\) and an incidence \(V\)-relation \(G \subseteq M\) between \(G\) and \(M\). Intuitively, the elements of \(G\) are thought of as entities or objects with (a priori) approximation structure \(\gamma\) on objects, the elements of \(M\) are thought of as properties, characteristics or attributes with approximation structure \(\mu\) on attributes, and \(\iota(g, m) = v\) asserts that “object \(g\) has attribute \(m\) with measure \(v\).”

Enriched Formal Concept Analysis is based upon the understanding that an enriched concept is a unit of thought consisting of two parts: its extension and its intension. Within the restricted scope of a formal context, the extent of a concept is an enriched subset of objects \(\phi \in V^G\) consisting of all objects belonging to the concept, whereas the intent of a concept is a enriched subset of attributes \(\psi \in V^M\) which includes all attributes shared by the objects. A concept of a given context will consist of an extent/intent pair

\[(\phi, \psi).\]

Of central importance in concept construction are two derivation operators which define the notion of “sharing” or “commonality”. For any subset of objects \(\phi \in V^{G^{op}} = V^G\), regarded as a \(V\)-relation \(G \phi \rightarrow 1\), the direct derivation along \(\iota\) is defined to be \(\phi'_\iota = \phi \setminus \iota\), which pointwise is \(\phi'_\iota(m) = \Lambda_{g \in G}(\phi(g) \Rightarrow \iota(g, m))\), the \(V\)-subset of \(M\) which for each attribute \(m \in M\) provides a soft measurement of the degree to which \(m\) is an attribute of all objects in \(\phi\). For any subset of attributes \(\psi \in V^M\) regarded as a \(V\)-relation \(1 \psi \缘 M\), the inverse derivation along \(\iota\) is defined to be \(\psi'_\iota = \iota \setminus \psi\), which pointwise is \(\psi'_\iota(g) = \Lambda_{m \in M}(\psi(m) \Rightarrow \iota(g, m))\), the \(V\)-subset of \(G\) which for each object \(g \in G\) provides a soft measurement of the degree to which \(g\) has all attributes in \(\psi\). These two derivation operators form an enriched adjointness

\[\phi'_\iota \iff \psi = \phi \Rightarrow \psi'_\iota.\]

To demand that a concept \((\phi, \psi)\) be determined softly by its extent and its intent means that this adjointness should be a soft inverse relationship at the extent/intent pair \((\phi, \psi)\): the intent should contain approximately (with measure the truthvalue \(v\)) those attributes shared by all objects in the extent \(v \preceq \Lambda_{m \in M}(\phi'_\iota(m) \iff \psi(m))\), and vice-versa, the extent should contain approximately those objects sharing all attributes in the extent \(v \preceq \Lambda_{g \in G}(\phi(g) \Rightarrow \psi'_\iota(g))\). Together this means that \(v \preceq (\phi'_\iota \iff \psi) \otimes (\phi \Rightarrow \psi'_\iota)\). A hard concept \((\phi, \psi)\) is a concept whose extent and intent determine each other exactly, satisfying the condition

\[e \preceq (\phi'_\iota \iff \psi) \otimes (\phi \Rightarrow \psi'_\iota).\]

The collection of all hard concepts is enriched by a generalization-specialization metric. One concept \((\phi_1, \psi_1)\) is more specialized (and less general) than another concept \((\phi_2, \psi_2)\) with measure \(\phi_1 \Rightarrow \phi_2 = \Lambda_{g \in G}(\phi_1(g) \Rightarrow \phi_2(g))\); or equivalently, \(\psi_2 \Rightarrow \psi_1 = \Lambda_{m \in M}(\psi_2(m) \Rightarrow \psi_1(m))\). Con-
cepts with this generalization-specialization metric form a concept hierarchy for the context.

**Proposition 1** The concept hierarchy is a complete V-space B(G, M, i) called the enriched concept lattice of (G, M, i).

Completeness means that the underlying preorder □(B(G, M, i)) is a complete lattice. The join of a collection of concepts represents the common attributes (common properties, or shared characteristics) of the concepts. The meet of a collection of concepts represents a coordinated sum of the attributes of the concepts. The top of the conceptual hierarchy (the empty meet) represents the universal concept whose extent consists of all objects.

According to Formal Concept Analysis, in the formal context (G, M, I) an implication Y_1 → Y_2 holds between a pair of attribute subsets Y_1, Y_2 ⊆ M when each object from G having all attributes of Y_1 has also all attributes of Y_2. The set of all implications forms a preorder (2-space).

We here define a softer notion of implication in enriched formal contexts. For any pair of attribute predicates ψ_1, ψ_2 ∈ V^M, a witness for the potential intuitive implication ψ_1 → ψ_2 is an object g ∈ G which satisfies the condition “if g has all attributes of ψ_1 then g also has all attributes of ψ_2”. Witnesses help verify potential implications by their collective measurement. We collect together all witnesses for the potential implication ψ_1 → ψ_2, and we measure the “implication witness” by using the metric for V-predicates over approximation space G.

ψ_1 → ψ_2 = ψ_1 i⇒ ψ_2 i

Let Impl((G, M, i)) = (V^M, i) denote the V-space of implications of (G, M, I). We can, of course, limit implication pairs by requiring a certain threshold measure v ≤ ψ_1 → ψ_2. The notion of implication from Formal Concept Analysis is the derived notion of maximal implication, requiring maximal measure e ≤ ψ_1 → ψ_2. These implications are orderings in the underlying preorder □(Impl((G, M, i))).

6 Linguistic Variables

We describe linguistic variables in terms of a use-case scenario. We start with a collection of objects G = (G, γ). We assume that some observations or experimental measurements have been made, resulting in the production of some “raw” data D = (D, δ). Both objects and data have been enriched as approximation spaces for benefit of flexibility by using soft structures. The data is associated with the objects by a map called a description function

G ⊓ D.

We will use linguistic variables in order (1) to interpret this data and (2) to provide a view or facet of it which is meaningful to the user. The creation of linguistic variables is an act of interpretation. Mathematically, the notion of a linguistic value (or constraint) is represented here by the notion of an enriched subset. A linguistic value over data domain D = (D, δ) is an enriched subset in V^D. A linguistic variable [7] (conceptual scale [1] or distributed constraint [2]) over data domain D = (D, δ) is a collection σ = {σ_m ∈ V^D | m ∈ M} of linguistic values over D, indexed by a collection of attribute symbols M. Using functional notation we can write this as the V-map σ: M → V^D, where
we have enriched the attributes to a (approximation) space \( \mathcal{M} = (M, \mu) \). Equivalently, a linguistic variable can be represented either as the map \( \tilde{\sigma} : \mathcal{D} \rightarrow \mathcal{V}^\mathcal{M} \) where \( \tilde{\sigma}(d)(m) = \sigma(m)(d) \) or as the relation
\[
\mathcal{M} \overset{\sigma}{\rightarrow} \mathcal{D}
\]
where \( \sigma(m,d) = \sigma(m)(d) \). The four parts of a linguistic variable can be interpreted as follows.

1. \( \mathcal{D} \) gives its (raw) data scope or range,
2. \( \mathcal{V} \) represents our interpretation bias or style,
3. \( \mathcal{M} \) gives linguistic terms of the linguistic variable which are meaningful to us, with a priori (approximation) measure.
4. \( \sigma \) connects, attaches or assigns (as you will) linguistic values to linguistic terms.

These are listed in order of volatility — of these four, \( \mathcal{D} \) varies slowest (it is given to us), whereas \( \sigma \) is most volatile. A standard example of a linguistic variable is “age”, where
\[
\begin{align*}
\mathcal{D} & = \{0,1, \ldots, 100\} \\
\mathcal{V} & = \text{the Fuzzy closed poset} \\
\mathcal{M} & = \{\text{“young”, “middle-age”, “old”}\} \\
\sigma(\text{“young”})(d) & = \begin{cases} 
1, & 0 \leq d \leq 20 \\
-\frac{1}{20}d + 2, & 20 \leq d \leq 40 \\
0, & 40 \leq d 
\end{cases}
\end{align*}
\]

etc.

There are two ways to combine linguistic variables, through summation and tensoring.

**Constraint Sum:** Given two linguistic variables on the same data domain \( \mathcal{M}_0 \overset{\sigma_0}{\rightarrow} \mathcal{D} \) and \( \mathcal{M}_1 \overset{\sigma_1}{\rightarrow} \mathcal{D} \), the copairing \( \tilde{\sigma} \) is the linguistic variable
\[
\mathcal{M}_0 \oplus \mathcal{M}_1 \overset{[\sigma_0, \sigma_1]}{\rightarrow} \mathcal{D}
\]
on the unconstrained (or constrained) sum space of terms, which sums the term assignments
\[
[\sigma_0, \sigma_1](m_0, d) = \sigma_0(m_0, d) \\
[\sigma_0, \sigma_1](m_1, d) = \sigma_1(m_1, d).
\]

**Vector Concatenation:** Given two linguistic variables (with no apparent relationships) \( \mathcal{M}_0 \overset{\sigma_0}{\rightarrow} \mathcal{D}_0 \) and \( \mathcal{M}_1 \overset{\sigma_1}{\rightarrow} \mathcal{D}_1 \), the tensor product is the linguistic variable
\[
\mathcal{M}_0 \otimes \mathcal{M}_1 \overset{[\sigma_0 \otimes \sigma_1]}{\rightarrow} \mathcal{D}_0 \otimes \mathcal{D}_1
\]
on the tensor product space of terms and data, which products the term assignments
\[
(\sigma_0 \otimes \sigma_1)((m_0, m_1), (d_0, d_1)) = \sigma_0(m_0, d_0) \otimes \sigma_1(m_1, d_1).
\]

We use the linguistic variable to interpret the meaning of the raw data assigned to objects by \( \phi \). This enriched interpretation, called granulation in Fuzzy Sets or conceptual scaling in Formal Concept Analysis, assigns a view or facet to the data \( \phi \).

This facet takes the form of a \( \mathcal{V} \)-relation (an enriched formal context — see below) \( \mathcal{G} \overset{\iota}{\rightarrow} \mathcal{M} \) called the derived context in Formal Concept Analysis. It is defined by relational composition \( \iota = \phi \circ \sigma_{\text{op}} \). In terms of elements this definition is
\[
\iota(g, m) = \tilde{\sigma}(\phi(g))(m) = \sigma(m)(\phi(g)).
\]
The given indiscernibility \( \gamma \) on objects \( G \) is required to be as fine as the induced indiscernibility \( \gamma_{\phi} \) on objects \( G \), defined via logical \( \mathcal{V} \)-equivalence
\[
\gamma_{\phi}(g_1, g_2) = \land_{m \in M} (\sigma(m)(\phi(g_1)) \iff \sigma(m)(\phi(g_2))).
\]
Granulation of the tensor product of several linguistic variables is called apposition in Formal Concept Analysis.
7 Enriched Interpretation of Networked Information Resources

We are currently developing an information management software system for the World-Wide Web called wave, the Web Analysis and Visualization Environment. wave is a third generation World-Wide Web tool used for navigation and discovery over a universe of networked information resources. Interpretation of resource descriptions, via conceptual scaling or faceted analysis, plays a central role in wave. At the present time, the kernel component of the wave system conceptually analyzes, interprets, and categorizes resources, such as Web textual and image documents, in a crisp fashion.

However, using ideas developed in this paper, an excellent approach for the extension to an enriched wave system is quite clear. The following short list of conceptually scalable attributes indicates that notions of approximation are very important for networked information resources: the visible size of textual documents in pages or some other meaningful unit; the concept extent cardinality as a count of equivalent instances of resources; similarity measures between Web documents based upon numbers of common attributes; relative scores for Waisindex keyword search; the cost of resources; the duration of play for audio/video data; the critical review of resources; etc. We intend to develop in the near future an enriched wave system, which will allow the user to define according to his own judgement various enriched interpretations of networked resource information.

A Closed Preorders

A closed preorder \( \langle V, \leq, \otimes, \Rightarrow, e \rangle \) consist of the following data and axioms.

- \( \langle V, \leq, \otimes, \Rightarrow, e \rangle \) is a monoidal preorder, or ordered monoid, with \( \langle V, \leq \rangle \) a preorder and \( \langle V, \otimes, e \rangle \) a monoid, where the binary operation \( \otimes: V \times V \rightarrow V \), called \( V \)-composition, is monotonic: if both \( u \leq u' \) and \( v \leq v' \) then \( (u \otimes v) \leq (u' \otimes v') \).
- \( \otimes \) is symmetric, or commutative; that is, \( a \otimes b = b \otimes a \) for all elements \( a, b \in V \).
- \( V \) satisfies the closure axiom: the monotonic \( V \)-composition function \( (\cdot) \otimes b: V \rightarrow V \) has a specified right adjoint \( b \Rightarrow (\cdot): V \rightarrow V \) for each element \( b \in B \), called \( V \)-implication, or symbolically \( ((\cdot) \otimes b) \hookrightarrow (b \Rightarrow (\cdot)): V \rightarrow V \); that is, \( a \otimes b \leq c \) iff \( a \leq b \Rightarrow c \) for any triple of elements \( a, b, c \in V \).
- We usually also assume that our closed preorders are bicomplete; that is, the supremum \( \bigvee B \) and the infimum \( \bigwedge B \) exist (and are unique up to equivalence \( \equiv \)) for all subsets \( B \subseteq V \).

The following define special closed preorders.

- A closed preorder is normal when the unit is the top element \( e = \top_V \) and \( V \)-implication is directed-continuous: \( b \Rightarrow (\bigvee_{d \in D} d) \equiv \bigvee_{d \in D} (b \Rightarrow d) \) for all directed subsets \( D \subseteq V \). For normal closed preorders \( a \otimes b \leq a \wedge b \) for all elements \( a, b \in V \).
- When the tensor product \( \otimes \) is the binary infimum or meet \( \wedge \) and the unit \( e \) is the top element \( \top_V \), the closed preorder \( V = \langle V, \leq, \wedge, \Rightarrow, \top_V \rangle \) is called a cartesian closed preorder. The context of cartesian closed
preorders is the context of traditional logic. A characteristic property of cartesian closed preorders is idempotency: \( v \otimes v = v \wedge v = v \) for all elements \( v \in V \). In a cartesian closed preorder, and even in an arbitrary closed preorder, we regard \( V \) as being a set of generalized truth values. Cartesian closed preorders are normal.

We list some important closed preorders which can be used in Rough Sets and Soft Computing for enriched interpretation in linguistic variables.

**Boolean truth-values**

\[ 2 = \langle 2 = \{0,1\}, \leq, \wedge, \rightarrow, 1 \rangle \]

where 0 is false, 1 is true, \( \leq \) is the usual order on truth-values, \( \wedge \) is the truth-table for and, and \( \rightarrow \) is the truth-table for implies. Here 2-spaces \( \mathcal{X} = \langle X, d \rangle \) are preorders \( \mathcal{X} = \langle X, \leq \rangle \) where \( x_1 \leq x_2 \) when \( d(x_1, x_2) = 1 \), strict 2-spaces are posets, and 2-morphisms are monotonic functions.

**Subset truth-values**

\[ \varphi(A) = \langle P(A), \subseteq, \cap, \rightarrow, A \rangle \]

for any set \( A \), where \( P(A) \) is the set \( P(A) = \{ B \mid B \subseteq A \} \) of all subsets of \( A \), \( \cap \) is set intersection, and \( \rightarrow \) is set implication: \( B_1 \rightarrow B_2 = \{ a \in A \mid a \in B_1 \text{ implies } a \in B_2 \} = -B_1 \cup B_2 \). \( \varphi(A) \) is essentially the marking space closed preorder \( \varphi(A) \cong 2^A \) defining the most basic markings-as-fuzzy-subsets interpretation for Petri nets.

**Fuzzy truth-values**

\[ [0,1] = \langle [0,1], \leq, \wedge, \rightarrow, 1 \rangle \]

where 0 is false, 1 is true, \( 0 \leq r \leq 1 \) is some grade of truth-value between false and true, \( \leq \) is the usual order on fuzzy truth-values in the interval, \( \wedge \) is the minimum operation representing the interval truth-table for the fuzzy and, and \( \rightarrow \) is operation \( r \rightarrow s \) defining the interval truth-table for the fuzzy implies. The cartesian closed interval \([0,1]\) is coreflective and normal. This defines the correct context for Fuzzy Set theory.

**Real truth-values**

\[ \mathfrak{R} = \langle \mathfrak{R} = [0, \infty], \geq, +, -, 0 \rangle \]

where \( \geq \) is the usual downward ordering on the nonegative real numbers \( \mathfrak{R} \) (regarded as quantitative truth-values), + is sum, and \( - \) defined by \( s - r \) representing the truth-table for the metrical difference. The quantitative closed preorder of reals \( \mathfrak{R} \) is normal.

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1Point your World-Wide Web browser to the URL http://www.ncsa.uiuc.edu/SDG/IT94/Agenda/Papers-received.html