CONSTANT MEAN CURVATURE TORI AS STATIONARY SOLUTIONS TO THE DAVEY–STEWARDSON EQUATION

CHRISTOPH BOHLE

Abstract. A well known result of Da Rios and Levi–Civita says that a closed planar curve is elastic if and only if it is stationary under the localized induction (or smoke ring) equation, where stationary means that the evolution under the localized induction equation is by rigid motions. We prove an analogous result for surfaces: an immersion of a torus into the conformal 3–sphere has constant mean curvature with respect to a space form subgeometry if and only if it is stationary under the Davey–Stewartson flow.

1. Introduction

The Davey–Stewartson (DS) hierarchy [6] is a 2+1–dimensional generalization of the non–linear Schrödinger (NLS) hierarchy. The time evolutions in both cases describe integrable deformations of Dirac operator potentials in 1 or 2 dimensions, respectively. It is well know since the work of Hasimoto [7] that the NLS–hierarchy has a geometric realization as an evolution of space curves describing the motion of infinitely thin vortex filaments, the localized induction equation

\[ \dot{\gamma} = J\gamma'' = \gamma' \times \gamma'' \]

introduced by Levi–Civita’s student Da Rios [5] (see also [10]), where \( J \) denotes the 90°– degree rotation in the normal bundle and \( ' \) the derivative with respect to arc length: if a curve \( \gamma \) in \( \mathbb{R}^3 \) moves under the localized induction equation its complex curvature \( \psi = \kappa e^{i\tau} \) evolves under the non–linear Schrödinger equation. While the localized induction equation describes the extrinsic evolution of curves in space, the non–linear Schrödinger equation describes the corresponding evolution of the differential invariants curvature \( \kappa \) and torsion \( \tau \). A curve is stationary under this smoke ring evolution if it evolves by rigid motions and reparametrization only or, which is equivalent, if its complex curvature is a traveling wave solution of the NLS–equation.

An analogous geometric version of the Davey–Stewartson flow as an evolution of surfaces in 4–space was introduced by Konopelchenko [8, 9] who realized that the Weierstrass representation, a correspondence between immersions into Euclidean space and solutions to Dirac equations, allows to deform conformal immersions of surfaces by deforming the corresponding Dirac potentials. While Konopelchenko’s discussion is rather local, Taimanov [12, 13] pointed out that the evolution equations can in fact be globally defined for immersions of tori with trivial normal bundle and preserves their conformal type. A more geometric definition of the Davey–Stewartson flow that avoids the use of Dirac operators and underlines the analogy to the smoke ring evolution of space curves was given by

Date: October 7, 2008.

1991 Mathematics Subject Classification. 35Q55,53C42,53A30.
Author supported by DFG SPP 1154 ”Global Differential Geometry".

Date: October 7, 2008.

1991 Mathematics Subject Classification. 35Q55,53C42,53A30.
Author supported by DFG SPP 1154 ”Global Differential Geometry".
Burstall, Pedit, and Pinkall [4]: the normal part of the Davey–Stewartson evolution of an immersion \( f: T^2 \to S^4 \) into the conformal 4–sphere is
\[
\dot{f}^\perp = J \Pi^0(X, X),
\]
where \( J \) denotes 90° rotation in the normal bundle of \( f \), \( \Pi^0 \) is the trace free second fundamental form with respect to some compatible space form subgeometry, and \( X \) is some holomorphic vector field. In case \( f \) has topologically trivial normal bundle there is an essentially unique way to complement \( \dot{f}^\perp \) to a conformal deformation \( \dot{f} \) by adding a tangential deformation. It should be noted that, in contrast to the case of curves, the resulting conformal deformation \( \dot{f} \) of a torus \( f: T^2 \to S^4 \) with trivial normal bundle is not given explicitly, because the tangential deformation is obtained by solving a \( \bar{\partial} \)-problem. Unlike the smoke ring evolution of curves in \( \mathbb{R}^3 \), the Davey–Stewartson flow depends on the additional choice of a holomorphic vector field or, equivalently, the choice of conformal coordinates on the universal covering of \( T^2 \); scaling these coordinates by some complex factor changes the Davey–Stewartson flow. An immersion \( f: T^2 \to S^4 \) is called stationary under the Davey–Stewartson flow if, for every choice of conformal coordinates, the evolution is by Möbius transformations and reparametrization only.

We prove the following analogue to Da Rios’s and Levi–Civita’s result that the closed planar curves stationary under the localized induction equation are plane elastica:

**Theorem.** A torus in the conformal 3–sphere is stationary under the Davey–Stewartson flow if and only if it is constrained Willmore and strongly isothermic.

Here constrained Willmore means that \( f: T^2 \to S^3 \) is a critical point of the Willmore functional under infinitesimal conformal variations and strongly isothermic means that \( f \) is a critical point of the projection to Teichmüller space, cf. [2]. A theorem of Richter [11, 4] implies that a torus in \( S^3 \) is strongly isothermic and constrained Willmore if and only if it has constant mean curvature with respect to some space form subgeometry.

The main theorem of the paper thus provides a Möbius geometric characterization of constant mean curvature tori in space forms as stationary solutions to the Davey–Stewartson equation. In the appendix we prove a local Möbius geometric characterization of constant mean curvature surfaces in space forms which is due to K. Voss and uses Bryant’s quartic differential [3].

2. **Invariants of Immersions into the Conformal n–Sphere**

We briefly review the Burstall–Pedit–Pinkall–approach [4] to define invariants of immersions into the conformal n–sphere. For details see Sections 3.1 to 3.3 of [4].

As a model for the **conformal n–sphere** \( S^n \) we use the projectivized lightcone in Minkowski space \( \mathbb{R}^{n+1,1} \) of dimension \( n + 2 \), that is, in \( \mathbb{R}^{n+2} \) equipped with the metric
\[
\langle v, w \rangle = -v_0w_0 + v_1w_1 + \ldots + v_{n+1}w_{n+1}.
\]

Given an immersion \( f: M \to S^n \) of a manifold \( M \) into the projective lightcone \( S^n \), the pull back \( g = \langle d\psi, d\psi \rangle \) of the Minkowski metric with respect to a homogeneous lift \( \psi \) of \( f \) is a Riemannian metric on \( M \) whose conformal class is independent of \( \psi \) because \( \bar{g} = \lambda^2 g \) for \( \bar{\psi} = \lambda \psi \). The name conformal n–sphere is justified by the fact that the standard metric of the round n–sphere in Euclidean space is obtained from the embedding \( x \in \{ x \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1 \} \to (1, x) \in S^n \) into the lightcone. The identity component
\(O_o(n + 1, 1)\) of the isometry group of \(\mathbb{R}^{n+1,1}\) acts on the projective lightcone \(S^n\) as the group of orientation preserving Möbius transformations.

Let \(M\) now be a Riemann surface and \(f: M \to S^n\) a conformal immersion which means that the conformal structure induced by \(f\) coincides with the Riemann surface structure of \(M\). The mean curvature sphere congruence of \(f\) is the sphere congruence spanned by

\[ V = \text{Span}\{\psi, \psi_z, \psi_{zz}, \psi_{zzz}\}, \]

where \(\psi\) is an arbitrary homogeneous lift of \(f\) and \(z\) are local conformal coordinates on \(M\). One easily checks that \(V\) is a well defined Minkowski space bundle of rank 4 and therefore indeed describes a congruence of 2–sphere in \(S^n\), namely the unique congruence of 2–spheres tangent to \(f\) for which the mean curvature at every point of contact coincides with that of \(f\) (where the mean curvature is taken with respect to an arbitrary compatible space form subgeometry). The bundle \(V^\perp\) is called the Möbius normal bundle and carries a positive definite metric and a metric connection \(D\), the normal connection, defined by orthogonal projection of the ordinary derivative on \(\mathbb{R}^{n+1,1}\).

We assume now that \(M\) is equipped with a fixed nowhere vanishing conformal vector field \(X\) or, equivalently, with a nowhere vanishing complex holomorphic 1–form \(dz\) dual to \(X\). Then there is a unique future pointing homogeneous lift \(\hat{\psi}\) of \(f\) normalized by the property that the metric induced by \(\hat{\psi}\) coincides with \(|dz|^2\). We call this lift the normalized lift of \(f\) with respect to \(X\). While the lightcone model of \(S^n\) and the mean curvature sphere congruence date back to Darboux and Thomsen, the normalized lift as an efficient means to define Möbius invariants of immersion was only introduced quite recently in [4].

There is a unique section \(\hat{\psi}\) of \(V\) that satisfying \(< \hat{\psi}, \hat{\psi} > = 0\), \(< \hat{\psi}, \hat{\psi} > = -1\) and \(< d\hat{\psi}, \hat{\psi} > = 0\), where \(\psi\) is the normalized lift of \(f\) with respect to \(z\). This yields a normalized frame \((\psi, \psi_z, \psi_{zz}, \hat{\psi})\) of \(V^\perp\). Since \(\psi_{zz}\) is orthogonal to \(\psi, \psi_z\) and \(\psi_{zz}\), there is a complex function \(c\) and a section \(\kappa \in \Gamma((V^\perp)^\perp)\) such that

\[(2.1) \quad \psi_{zz} + \frac{\kappa}{2} \psi = \kappa.\]

The quantities \(\kappa\) and \(c\) are called the conformal Hopf differential and the Schwarzian derivative of \(f\). Their dependence on the choice of \(dz\) is as follows: if \(\bar{\kappa}\) and \(\bar{c}\) are the invariants with respect to \(d\bar{z}\), then

\[(2.2) \quad \frac{\bar{\kappa}}{|d\bar{z}|} \frac{d\bar{z}^2}{|d\bar{z}|} = \kappa \frac{dz^2}{|dz|} \quad \text{and} \quad \bar{c}d\bar{z}^2 = (c - S_z(\hat{\psi})))dz^2,\]

where \(S_z(g) = \left(\frac{g''}{g}' - \frac{1}{2} \left(\frac{g''}{g}\right)^2\right)\) denotes the classical Schwarzian derivative of a holomorphic function \(g\) with respect to \(z\).

Let \((\psi, \psi_z, \psi_{zz}, \hat{\psi})\) be the frame of \(V^\perp\) and let \(\xi \in \Gamma(V^\perp)\) be a section of the Möbius normal bundle. A straightforward calculation shows that the frame equations are

\[(2.3) \quad \begin{align*}
\psi_{zz} &= -\frac{\bar{\kappa}}{2} \psi + \kappa \\
\psi_{zzz} &= -|\kappa|^2 \psi + \frac{1}{2} \hat{\psi} \\
\hat{\psi}_z &= -2|\kappa|^2 \psi_z - c\psi_{zz} + 2D_z \kappa \\
\xi_z &= 2 < D_z \kappa, \xi > \psi - 2 < \kappa, \xi > \psi_z + D_z \xi.
\end{align*}\]
The conformal Gauss, Codazzi and Ricci equation are the integrability equations

\begin{align}
\frac{1}{2}c_\bar{z} &= 2|\kappa|^2 + \langle D_\bar{z} \bar{\kappa}, \kappa \rangle - \langle \bar{\kappa}, D_\bar{z} \kappa \rangle \\
\text{Im}(D_\bar{z} D_\bar{\zeta} + \frac{\bar{c}}{2} \kappa) &= 0 \\
R^D_{\bar{z} \bar{z} \bar{z}} \xi &= 2 < \bar{\kappa}, \xi > \kappa - 2 \langle \kappa, \xi \rangle \bar{\kappa}.
\end{align}

For immersions into the 4–sphere $S^4$ the normal bundle is 2–dimensional and has a complex structure $J$ compatible with metric and orientation. The Ricci equation (2.6) then reads

\begin{equation}
R^D_{\bar{z} \bar{z} \bar{z}} = 2 < J\bar{\kappa}, \kappa > J.
\end{equation}

The invariants $\kappa$, $c$ and $D$ describe the immersions $f$ uniquely up to Möbius transformation. Conversely, given $\kappa$, $c$ and $D$ (with $D$ a metric connection on an abstract bundle $\mathcal{V}^\perp$ and $\kappa$ a section of its complexification) that satisfy the conformal Gauss–Codazzi–Ricci equations, there is a unique conformal immersions $f$ with Möbius monodromy belonging to this data, cf. [4, 1].

An immersion $f$ is called isothermic if locally and away from umbilic points it admits conformal curvature line coordinates. One easily checks (using (2.6) of [4]) that $z = x + iy$ are conformal curvature line coordinates for an immersion $f$ if and only if the conformal Hopf differential $\kappa$ of $f$ with respect to $z$ is a real section of the Möbius normal bundle $\mathcal{V}^\perp$. Thus $f$ is isothermic if locally and away from umbilics it admits conformal coordinates $z$ for which $\kappa$ is real.

If $M = T^2$ is a torus it makes sense to consider isothermic immersions $f$ admitting global conformal curvature line coordinates, that is, conformal coordinates $z$ on the universal covering of $T^2 = \mathbb{C}/\Gamma$ with respect to which the conformal Hopf differential $\kappa$ is real. Such immersions are called strongly isothermic.

It is proven in [4], see equations (34) and (54) there, that a conformal immersion of a torus $M = T^2$ is constrained Willmore, i.e., a critical point of the Willmore energy $\mathcal{W} = \int |\kappa|^2$ under infinitesimal conformal variations, if and only if

\begin{equation}
D_\bar{z} D_\bar{z} \kappa + \frac{1}{2} \bar{c} \kappa = \text{Re}(\lambda \kappa)
\end{equation}

for some $\lambda \in \mathbb{C}$, where $z$ are conformal coordinates on the universal covering.

A theorem by J. Richter [11] (proven in (36) to (40) of [4]) states that an immersion $f : T^2 \to S^3$ of a torus into the conformal 3–sphere is simultaneously strongly isothermic and constrained Willmore if and only if it has constant mean curvature with respect to some space form subgeometry.

It should be mentioned that if the underlying compact Riemann surfaces $M$ has higher genus the definition of strongly isothermic immersions $f : M \to S^n$ and the Euler–Lagrange equation for constrained Willmore immersions $f : M \to S^n$ are more involved, because the coordinates $z$ have to be replaced by a holomorphic quadratic differential, see [2].

### 3. Davey–Stewartson Flow

We describe now the Möbius geometric approach [4] to the Davey–Stewartson flow on the space of immersions $f : T^2 \to S^4$ with topologically trivial normal bundle. As a preparation we discuss the effect of arbitrary infinitesimal conformal deformations of immersions $f : T^2 \to S^4$ on the differential invariants of Section [2] (For this we follow Section 4.1 of [4] and 14.3 of [1].)
We fix conformal coordinates $z$ on the universal covering of the torus $T^2$. An infinitesimal deformation of the homogeneous lift of a conformal immersion $f : T^2 \to S^4$ is of the form

\begin{equation}
\dot{\psi} = a\psi + b\psi_z + \overline{b}\psi_{\overline{z}} + \sigma
\end{equation}

with real function $a$, complex function $b$ and $\sigma \in \Gamma(\mathcal{V}^\perp)$. The deformation $\dot{\psi}$ hat no $\psi$–component, because we deform tangential to the lightcone. The infinitesimal condition $<\dot{\psi}_z, \psi_z> = 0$ for preserving the conformal structure is equivalent to

\begin{equation}
b_z = 2 <\sigma, \kappa>
\end{equation}

and the condition $\text{Re} <\dot{\psi}_z, \psi_{\overline{z}}> = 0$ for preserving the normalization is equivalent to

\begin{equation}
a = - \text{Re} b_z.
\end{equation}

The tangential part $b$ of a conformal deformation $\dot{f}$ is thus essentially determined by the normal part $\sigma$ via the $\overline{\partial}$–equation (3.2) which can be solved if and only if $\int <\sigma, \kappa> = 0$.

In codimension 2, unlike in the codimension 1 case, the tangential and normal deformations $b$ and $\sigma$ do not completely determine the deformation of the Gauss–Codazzi–Ricci data because one can apply infinitesimal gauge transformations of the normal bundle without changing $\psi$. Such infinitesimal normal bundle rotation is given by $\chi : T^2 \to \mathbb{R}$ that describes the normal deformation of a section of the normal bundle $\xi \in \Gamma(\mathcal{V}^\perp)$ via

\begin{equation}
(\xi)^\perp = \chi J\xi,
\end{equation}

where $J$ denotes the complex structure on the normal bundle.

The effect on the Gauss–Codazzi–Ricci–data $\kappa$, $c$ and $D$ of an infinitesimal conformal deformation (3.1) together with an infinitesimal normal bundle rotation (3.3) is

\begin{equation}
\dot{\kappa} = D_z \kappa - \frac{1}{2}b\sigma + (\frac{\overline{b}}{2}b_z - \frac{1}{2}\overline{b}\overline{z})\kappa + bD_z\kappa + \overline{b}D_z\kappa - \chi J\kappa
\end{equation}

\begin{equation}
\dot{c} = b_{zzz} + 2c b_z + bc_z + \overline{b}c_{\overline{z}} + 16 <\sigma, \kappa> <\kappa, \kappa> + 6(<D_z\sigma, D_z\kappa> - <D_z\kappa, D_z\sigma>) + 2(<D_z\sigma, D_z\kappa> - <\sigma, D_zD_z\kappa> + 2(<J\sigma, D_z\kappa> + 2 <J\kappa, \overline{b}\kappa + D_z\sigma>) J\xi.
\end{equation}

We describe now the flows up to order two of the Davey–Stewartson hierarchy on the space of immersions $f : T^2 \to S^4$ with trivial normal bundle.

The 0th–order flow is given by $\sigma = 0$, $b = 0$ and $\chi : T^2 \to \mathbb{R}$ and hence consists of a pure normal bundle rotation (the immersion does not actually move, but normal frames are rotated). By (3.4) the resulting deformation of the invariants is

\begin{equation}
\dot{\kappa} = - \chi J\kappa
\end{equation}

\begin{equation}
\dot{c} = 0
\end{equation}

\begin{equation}
\dot{D} = d\chi J.
\end{equation}

The 1st–order flow is obtained by setting $\sigma = 0$, $b \in \mathbb{C}$ and $\chi = 0$. This corresponds to a reparametrization of the conformal immersion (without normal deformation) under which the invariants evolve by

\begin{equation}
\dot{\kappa} = bD_z\kappa + \overline{b}D_z\kappa
\end{equation}

\begin{equation}
\dot{c} = bc_z + \overline{b}c_{\overline{z}}
\end{equation}

\begin{equation}
\dot{D}_z = 2\overline{b} <J\kappa, \kappa> + J.
\end{equation}
The 2nd–order flow, the Davey–Stewartson flow with respect to the coordinates \(z\), is the conformal deformation whose normal part is

\[
\sigma = 2 \text{Re}(J\kappa) = J\kappa + J\bar{\kappa}
\]

and whose tangential part \(b\) satisfies the \(\bar{\partial}\)–equation (3.2)

\[
\bar{\partial}b_z = 2 <J\bar{\kappa}, \kappa>.
\]

Since we are on a torus such \(b\) exists if and only if \(\int_M <J\bar{\kappa}, \kappa> = 0\) which, by the Ricci–equation (2.6), is equivalent to the normal bundle degree of the immersion being zero. For tori \(f: T^2 \to S^4\) with topologically trivial normal bundle the tangential deformation \(b\) making the Davey–Stewartson deformation conformal is thus uniquely defined up to adding a constant. In other words, the 2nd–order flow is then well defined up to 1st–order flow. It should be noted that changing the chosen coordinates \(z\) changes the Davey–Stewartson flow.

### 4. Davey–Stewartson Stationary Tori

An immersed torus \(f: T^2 \to S^4\) is **stationary under the Davey–Stewartson flow** if the Davey–Stewartson flows with respect to all coordinates acts by Möbius transformations and reparametrizations only. Davey–Stewartson stationary tori generalize homogeneous tori, the tori stationary under the 1st–order “reparametrization flow” which are orbits of 2–parameter groups of Möbius transformations and hence Möbius equivalent to tori of revolution whose profile curves are circles.

Due to the non–explicitness of the Davey–Stewartson equation caused by the \(\bar{\partial}\)–equation describing the tangential part of the flow, the investigation of general Davey–Stewartson stationary immersions \(f: T^2 \to S^4\) into the 4–sphere is difficult (see Lemma 4.3 below for the equations characterizing such tori). In the following we focus on the special case of Davey–Stewartson stationary tori that are contained in a 3–sphere \(S^3 \subset S^4\) and hence have flat normal bundle such that the \(\bar{\partial}\)–problem can be solved explicitly.

**Theorem 4.1.** An conformal immersion \(f: T^2 \to S^3 \subset S^4\) into the conformal 4–sphere that takes values in a 3–sphere is stationary under the Davey–Stewartson flow if and only if it is strongly isothermic and constrained Willmore.

Combined with Richter’s theorem mentioned in Section 2 this implies:

**Corollary 4.2.** A conformal immersion of a torus into the conformal 3–sphere is stationary under the Davey–Stewartson flow if and only if it has constant mean curvature with respect to some space form subgeometry.

We derive now equations characterizing immersions \(f: T^2 \to S^4\) with topologically trivial normal bundle that are Davey–Stewartson stationary.

**Lemma 4.3.** A conformal immersion \(f: T^2 \to S^4\) with trivial normal bundle is stationary under the Davey–Stewartson flow if and only if there are \(\alpha, \beta \in \mathbb{C}\) and \(\mu: T^2 \to \mathbb{C}\) such
that the invariants of \( f \) with respect to a conformal chart \( z \) on the covering of \( T^2 \) satisfy

\[
J(D_z D_z \kappa + \frac{c}{2} \kappa) + \frac{3}{2} b_z \kappa + b D_z \kappa = \alpha D_z \kappa + \beta D_z \kappa - \mu J \kappa
\]

(4.1)

\[
J(D_z D_z \kappa + \frac{c}{2} \kappa) - \frac{1}{2} b_z \kappa + \bar{b} D_z \kappa = \beta D_z \kappa + \bar{\alpha} D_z \kappa - \bar{\mu} J \kappa
\]

(4.2)

\[
b_{zzz} + 2cb_z + bc_z + 16 < J \kappa, \bar{\kappa} > < \kappa, \kappa > + 8(\langle D_z J \kappa, \kappa > - \langle D_z J \bar{\kappa}, D_z \kappa >) = \alpha c_z + \beta c_z
\]

(4.3)

\[
b_cz + 6(\langle D_z D_z J \kappa, \kappa > - \langle D_z J \bar{\kappa}, D_z \kappa >) + 2(\langle D_z J \bar{\kappa}, D_z \kappa > - \langle J \bar{\kappa}, D_z D_z \kappa >) = \beta c_z + \alpha c_z
\]

(4.4)

\[
0 = 2 \beta < \bar{\kappa}, J \kappa > + (\mu)z
\]

(4.5)

\[
2 < \bar{\kappa}, J \kappa > \bar{b} + 2 < D_z \bar{\kappa}, \kappa > - 2 < D_z \kappa, \bar{\kappa} > = 2 \bar{\alpha} < \bar{\kappa}, J \kappa > + (\bar{\mu})z,
\]

(4.6)

where \( b : T^2 \to \mathbb{C} \) is a solution to the \( \bar{\partial} \)-equation \( \bar{b}_z = 2 < J \bar{\kappa}, \kappa >.

**Proof.** We fix conformal coordinates \( z \) on the universal covering of the torus. Because being stationary under the Davey–Stewartson flow means that the second order flows with respect to all conformal coordinates on the universal covering depend linearly on the lower order flows, we express now the flows with respect to rotated coordinates \( \tilde{z} = az \) \((a \in \mathbb{C}^*)\) in terms of the invariants with respect to \( z \).

Because real scalings of the coordinates correspond to real scalings of the flows it is sufficient to consider coordinates of the form \( \tilde{z} = ze^{-i\theta} \) with \( \theta \in \mathbb{R} \). From (2.2) we obtain that the invariants with respect to \( \tilde{z} \) are \( \tilde{\kappa} = \kappa e^{2i\theta} \) and \( \tilde{\bar{\kappa}} = \bar{\kappa} e^{2i\theta} \). Furthermore, if \( b \) solves \( \bar{\partial} \)-problem for the tangential deformation with respect to \( z \), then \( \tilde{b} = be^{i\theta} \) solves it with respect to \( \tilde{z} \), because \( \frac{\partial}{\partial \tilde{z}} = \frac{\partial}{\partial z} e^{i\theta} \).

By (3.4) to (3.6), the evolution of the Gauss–Codazzi–Ricci data under the Davey–Stewartson flow with respect to \( \tilde{z} \) with \( \bar{b} = be^{i\theta} \) and \( \chi = 0 \) is

\[
\tilde{\kappa} = e^{2i\theta}(J(D_z D_z \kappa + \frac{c}{2} \kappa) + \frac{3}{2} b_z \kappa + b D_z \kappa) + e^{-2i\theta}(J(D_z D_z \bar{\kappa} + \frac{c}{2} \bar{\kappa}) - \frac{1}{2} b_z \kappa + \bar{b} D_z \kappa)
\]

\[
\tilde{\bar{\kappa}} = e^{2i\theta}(b_{zzz} + 2cb_z + bc_z + 16 < J \kappa, \bar{\kappa} > < \kappa, \kappa > + 8(\langle D_z J \kappa, \kappa > - \langle D_z J \bar{\kappa}, D_z \kappa >) + 2(\langle D_z J \bar{\kappa}, D_z \kappa > - \langle J \bar{\kappa}, D_z D_z \kappa >)
\]

\[
\tilde{\beta}_z = e^{-2i\theta}(2 < \bar{\kappa}, J \kappa > \bar{b} + 2 < D_z \bar{\kappa}, \kappa > - 2 < D_z \kappa, \bar{\kappa} >)J.
\]

Now \( f \) is stationary if there is a complex function \( \alpha(\theta) \) depending on \( \theta \) and a real function \( g(\theta, p) \) depending on \( \theta \) and \( p \in T^2 \) such that, for every \( \theta \), the above deformation equals

\[
\tilde{\kappa} = \alpha(\theta)e^{i\theta} D_z \kappa + \bar{\alpha}(\theta)e^{-i\theta} D_z \kappa - gJ \kappa
\]

\[
\tilde{\bar{\kappa}} = \alpha(\theta)e^{i\theta} c_z + \bar{\alpha}(\theta)e^{-i\theta} \bar{c}_z
\]

\[
\tilde{\beta}_z = 2\bar{\alpha}(\theta)e^{-i\theta} < J \kappa, \bar{\kappa} > J + g(\theta)zJ.
\]

Fourier decomposition of \( g \) and \( \alpha \) immediately yields that the condition for \( f \) to be Davey–Stewartson stationary is (4.1) to (4.6).
Proof (of Theorem 4.1). Because \( f \) takes values in a 3–sphere, its normal bundle is flat which by (2.6) is equivalent to \( < J\bar{\kappa}, \kappa > = 0 \). Hence \( b = 0 \) solves the \( \bar{\partial} \)-equation \( \bar{b}_z = 2 < J\bar{\kappa}, \kappa > \) which drastically simplifies all equations in Lemma 4.3. In particular, since we are on a torus, equation (4.5) implies that \( \mu: T^2 \to \mathbb{C} \) is constant.

Because \( f \) takes values in a 3–sphere there is a constant vector \( n \in \mathbb{R}^5 \) of length 1 that is contained in every fiber of the Möbius normal bundle \( V^\perp \) and satisfies

\[
<\kappa, n> = 0.
\]

(4.7)

In particular \( \kappa \) pointwise is a complex multiple of \( Jn \). We use this in order to decompose equations (4.1) and (4.2) into \( n \)- and \( Jn \)-parts

\[
D_z D\bar{z} \kappa + \frac{c}{2} \kappa = -\mu \kappa \quad \text{(4.1)}
\]

\[
D_z D\bar{z} \bar{\kappa} + \frac{c}{2} \bar{\kappa} = -\bar{\mu} \kappa \quad \text{(4.2)}
\]

and

\[
\alpha D_z \kappa + \beta D\bar{z} \kappa = 0 \quad \text{(4.1')}\]

\[
\beta D_z \kappa + \bar{\alpha} D\bar{z} \kappa = 0 \quad \text{(4.2')}\]

Moreover, the remaining scalar product terms in (4.3) and (4.4) vanish such that

\[
\alpha c_z + \beta \bar{c}_z = 0 \quad \text{(4.3)}
\]

\[
\beta c_z + \bar{\alpha} \bar{c}_z = 0 \quad \text{(4.4)}
\]

The equations (4.1'), (4.2'), (4.3) and (4.4) can only be solved by \( \alpha = \beta = 0 \) (unless \( f \) is “equivariant” with respect to a 1–parameter group of Möbius transformations).

If \( \mu \neq 0 \) the Codazzi equation (2.5) implies that the imaginary part of (4.2') is \( \text{Im}(\bar{\mu} \kappa) = 0 \).

After rotation of the coordinates \( z \) we can thus assume that \( \kappa \) and \( \mu \) are real. This shows that \( f \) is strongly isothermic. On the other hand, taking the real part of (4.2') implies

\[
D_z D\bar{z} \kappa + \frac{c}{2} \kappa = \text{Re}(-\mu \kappa) = -\mu \kappa
\]

which shows that \( f \) is constrained Willmore, see (2.7).

If \( \mu = 0 \), then \( f \) is Willmore. Because \( f \) takes values in the 3–sphere, (4.6) takes the form

\[
< D_z \bar{\kappa}, \kappa > - < D_\bar{z} \kappa, \bar{\kappa} > = 0
\]

which, after writing \( \kappa \) as \( \kappa = aJn \) with \( a: T^2 \to \mathbb{C} \), becomes \( \bar{a}_z a - a_z \bar{a} = 0 \). Away from umbilic points the ratio \( \bar{a}/a \) is constant such that the argument of \( a \) is a locally constant function defined on the complement of the set of umbilic points. We can thus assume, possibly after rotation of \( z \), that \( \kappa \) is real on a component of the complement of the set of umbilic points. Now Willmore surfaces are analytic because their Euler–Lagrange equation is elliptic such that imaginary part of \( \kappa \) has to vanish globally and \( f \) is strongly isothermic.

Reversing the argumentation one immediately shows that a strongly isothermic, constrained Willmore immersion \( f: T^2 \to S^3 \) is Davey–Stewartson stationary: choosing \( z \) such that \( \kappa \) is real, (2.7) together with the Codazzi equation implies \( D_z D_\bar{z} \kappa + \frac{1}{2} \bar{c} \kappa = \lambda \kappa \) with \( \lambda \in \mathbb{R} \). Thus, (4.1) to (4.6) are satisfied with \( \alpha = \beta = b = 0 \) and \( \mu = -\lambda \). \( \square \)
Appendix

We prove a (local) characterization of Willmore surfaces and constant mean curvature surfaces in space forms in terms of Bryant’s quartic differential \[3\]. This characterization was apparently first obtained by K. Voss \[14\] but seems to be nowhere published. Our proof uses the methods of Burstall, Pedit, and Pinkall \[4\] which are described in Section 2.

For surfaces in the conformal 3–sphere, the Möbius normal bundle \(V^\perp\) has a canonical trivialization by the unique section \(Y \in \Gamma(V^\perp)\) of unit length compatible with the orientation. Using this trivialization, the conformal Hopf differential with respect to a chart \(z\) becomes a complex function \(\kappa\) and the Gauss and Codazzi equations (2.4) and (2.5) become

\[
\frac{1}{2}c_z = (|\kappa|^2)_{z} + 2\kappa_z \kappa \quad \text{and} \quad \text{Im}(\kappa_{zz} + \frac{\bar{\kappa}}{\kappa}) = 0.
\]

Equation (2.3) shows that in our setting Bryant’s quartic differential \(Q\) takes the form

\[
Q = 4(\kappa\kappa_{zz} + |\kappa|^2\kappa^2 - \kappa_z \kappa_z)dz^4.
\]

Holomorphicity of \(Q\) is equivalent to

\[
k\kappa_{zz} + (|\kappa|^2)_{z}\kappa^2 + 2|\kappa|^2\kappa\kappa_z - \kappa_{zz} \kappa_z = 0.
\]

The Gauss–equation implies that, away from umbilic points,

\[
\kappa^2 \left( \frac{\kappa_{zz} + \frac{\bar{\kappa}}{\kappa}}{\kappa} \right)_{z} = \kappa\kappa_{zz} + (|\kappa|^2)_{z}\kappa^2 + 2|\kappa|^2\kappa\kappa_z - \kappa_{zz} \kappa_z.
\]

Thus, away from umbilic points, holomorphicity of \(Q\) is equivalent to

\[(*) \quad \kappa_{zz} + \frac{\bar{\kappa}}{2\kappa} = \lambda \kappa \]

for some anti–holomorphic function \(\lambda\). The function \(\lambda\) vanishes identically if and only if the immersion is Willmore. In case \(Q\) is holomorphic and \(\lambda\) does not vanish identically, away from the isolated zeroes of \(\lambda\) we can introduce local coordinates \(\tilde{z}\) for which \(d\tilde{z} = \sqrt{\lambda}\).

By (2.2) we then have \(\tilde{\kappa}|\lambda|^3/2 = \kappa \lambda\) such that \(\tilde{\kappa}\) is real, because \(\text{Im}(\lambda \kappa) = 0\) by \((*)\) and the Codazzi equation. Without loss of generality we can thus assume that \(\kappa\) is real for the original coordinates \(z\). The Codazzi equation then shows that \((*)\) holds with constant \(\lambda\). But this implies that the immersion has constant mean curvature with respect to some space form subgeometry, see the paragraph in \[4\] that starts with equation (39).

This proves:

**Theorem 4.4 (K.Voss).** The quartic differential \(Q\) of an immersion into the conformal 3–sphere is holomorphic if and only if, away from umbilics and isolated points, the immersion is Willmore or has constant mean curvature with respect to some space form subgeometry.

**References**

[1] C. Bohle, *Möbius Invariant Flows of Tori in \(S^4\).* Thesis, Technische Universität Berlin, 2003.

[2] C. Bohle, G. P. Peters, and U. Pinkall, Constrained Willmore surfaces. *Calc. Var. Partial Differential Equations* 32 (2008), 263-277.

[3] R. L. Bryant, A duality theorem for Willmore surfaces. *J. Differential Geom.* 20 (1984), 23–53.

[4] F. Burstall, F. Pedit, and U. Pinkall, Schwarzian derivatives and flows of surfaces. *Contemp. Math.* 308 (2002), 39–61.

[5] L. S. Da Rios, Sul moto d’un liquido indefinito con un filettò vorticoso di forma qualunque. *Rend. Circ. Mat. Palermo* 22 (1906), 117–135.

[6] A. Davey and K. Stewartson, On three–dimensional packets of surface waves. *Proc. Roy. Soc. London A* 338 (1974), 101–110.
[7] H. Hasimoto, A soliton on a vortex filament. J. Fluid Mech. 51 (1972), 477–485.
[8] B. G. Konopelchenko, Induced surfaces and their integrable dynamics. Stud. Appl. Math. 96 (1996), 9–51.
[9] B. G. Konopelchenko, Weierstrass representations for surfaces in 4D spaces and their integrable deformations via DS hierarchy. Ann. Glob. Anal. Geom. 18 (2000), 61–74.
[10] R. L. Ricca, Rediscovery of Da Rios Equations. Nature 352 (1991), 561–562.
[11] J. Richter, Conformal maps of a Riemannian surface onto the space of quaternions, Thesis, Technische Universität Berlin, 1997.
[12] I. A. Taimanov, Modified Novikov–Veselov equation and differential geometry of surfaces. Amer. Math. Soc. Transl. 179 (1997), 133–151.
[13] I. A. Taimanov, Surfaces in the four-space and the Davey-Stewartson equations. J. Geom. Phys. 56 (2006), 1235–1256.
[14] K. Voss, private communication.

Christoph Bohle, Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany

E-mail address: bohle@math.tu-berlin.de