ABSTRACT

Deep Residual Networks present a premium in performance in comparison to conventional networks of the same depth and are trainable at extreme depths. It has recently been shown that Residual Networks behave like ensembles of relatively shallow networks. We show that these ensembles are dynamic: while initially the virtual ensemble is mostly at depths lower than half the network’s depth, as training progresses, it becomes deeper and deeper. The main mechanism that controls the dynamic ensemble behavior is the scaling introduced, e.g., by the Batch Normalization technique. We explain this behavior and demonstrate the driving force behind it. As a main tool in our analysis, we employ generalized spin glass models, which we also use in order to study the number of critical points in the optimization of Residual Networks.

1 INTRODUCTION

Residual Networks (He et al., 2015) (ResNets) are neural networks with skip connections. These networks, which are a specific case of Highway Networks (Srivastava et al., 2015), present state of the art results in the most competitive computer vision tasks including image classification and object detection.

The success of residual networks was attributed to the ability to train very deep networks when employing skip connections (He et al., 2016). A complementary view is presented by Veit et al. (2016), who attribute it to the power of ensembles and present an unravelled view of ResNets that depicts ResNets as an ensemble of networks that share weights, with a binomial depth distribution around half depth. They also present experimental evidence that short paths of lengths shorter than half-depth dominate the ResNet gradient during training.

The analysis presented here shows that ResNets are ensembles with a dynamic depth behavior. When starting the training process, the ensemble is dominated by shallow networks, with depths lower than half-depth. As training progresses, the effective depth of the ensemble increases. This increase in depth allows the ResNet to increase its effective capacity as the network becomes more and more accurate.

Our analysis reveals the mechanism for this dynamic behavior and explains the driving force behind it. This mechanism remarkably takes place within the parameters of Batch Normalization (Ioffe & Szegedy, 2015), which is mostly considered as a normalization and a fine-grained whitening mechanism that addresses the problem of internal covariate shift and allows for faster learning rates.

We show that the scaling introduced by batch normalization determines the depth distribution in the virtual ensemble of the ResNet. These scales dynamically grow as training progresses, shifting the effective ensemble distribution to bigger depths.

The main tool we employ in our analysis is spin glass models. Choromanska et al. (2015) have created a link between conventional networks and such models, which leads to a comprehensive study of the critical points of neural networks based on the spin glass analysis of Auffinger et al. (2013). In our work, we generalize these results and link ResNets to generalized spin glass models. These models allow us to analyze the dynamic behavior presented above. Finally, we apply the results of Auffinger & Arous (2013) in order to study the loss surface of ResNets.
We briefly summarize Choromanska et al. (2015), which connects the loss function of multilayer networks with the hamiltonian of the p spherical spin glass model, and state their main contributions and results. The notations of our paper are summarized in Appendix A and slightly differ from those in Choromanska et al. (2015).

A simple feed forward fully connected network $N$, with $p$ layers and a single output unit is considered. Let $n_i$ be the number of units in layer $i$, such that $n_0$ is the dimension of the input, and $n_p = 1$. It is further assumed that the ReLU activation functions denoted by $R()$ are used. The output $Y$ of the network given an input vector $x \in \mathbb{R}^d$ can be expressed as

$$Y = \sum_{i=1}^{d} \sum_{j=1}^{\gamma} x_{ij} A_{ij} \prod_{k=1}^{p} w_{ij}^{(k)},$$ 

where the first summation is over the network inputs $x_1 \ldots x_d$, and the second is over all paths from input to output. There are $\gamma = \prod_{i=1}^{p} n_i$ such paths and $\forall i, \ x_{i1} = x_{i2} = \ldots x_{i\gamma}$. The variable $A_{ij} \in \{0,1\}$ denotes whether the path is active, i.e., whether all of the ReLU units along this path are producing positive activations, and the product $\prod_{k=1}^{p} w_{ij}^{(k)}$ represents the specific weight configuration $w_{ij1}^{(k)} \ldots w_{ijp}^{(k)}$, multiplying $x_i$ given path $j$. It is assumed throughout the paper that the input variables are sampled i.i.d from a normal Gaussian distribution.

**Definition 1.** The mass of the network $N$ is defined as $\psi = \prod_{i=0}^{p} n_i$.

$A_{ij}$ are modeled as independent Bernoulli random variables with a success probability $\rho$, i.e., each path is equally likely to be active. Therefore,

$$\mathbb{E}_A[Y] = \sum_{i=1}^{d} \sum_{j=1}^{\gamma} x_{ij} \rho \prod_{k=1}^{p} w_{ij}^{(k)}.$$ 

The task of binary classification using the network $N$ with parameters $w$ is considered, using either the hinge loss $\mathcal{L}_N^h$ or the absolute loss $\mathcal{L}_N^a$:

$$\mathcal{L}_N^h(w) = \mathbb{E}_A[max(0, 1 - Y x Y)], \quad \mathcal{L}_N^a(w) = \mathbb{E}_A[|Y x - Y|]$$

where $Y$ is a random variable corresponding to the true label of sample $x$. In order to equate either loss with the hamiltonian of the p-spherical spin glass model, a few key approximations are made:

**A1** Variable independence - The inputs $x_{ij}$ are modeled as independent normal Gaussian random variables.

**A2** Redundancy in network parameterization - It is assumed that the set of all the network weights $w_1, w_2 \ldots w_N$ contains only $\Lambda$ unique weights such that $\Lambda < N$.

**A3** Uniformity - It is assumed that all unique weights are close to being evenly distributed on the graph of connections defining the network $N$. Practically, this means that we assume every node is adjacent to an edge with any one of the $\Lambda$ unique weights.

**A4** Spherical constraint - The following is assumed:

$$\frac{1}{\Lambda} \sum_{i=1}^{\Lambda} w_i^2 = C^2$$

for some constant $C > 0$.

These assumptions are made for the sake of analysis and do not hold. For example, A1 does not hold since each input $x_i$ is associated with many different paths and $x_{i1} = x_{i2} = \ldots x_{i\gamma}$. See Choromanska et al. (2015) for further justification of these approximations.

Under A1–A4, the loss takes the form of a centered Gaussian process on the sphere $S^{\Lambda-1}(\sqrt{\Lambda})$. Specifically, it is shown to resemble the hamiltonian of the a spherical $p$-spin glass model given by:

$$\mathcal{H}_{p,\Lambda}(\tilde{w}) = \frac{1}{\Lambda} \sum_{i_1 \ldots i_p} x_{i_1 \ldots i_p} \tilde{w}_{i_1} \tilde{w}_{i_2} \ldots \tilde{w}_{i_p}$$
with spherical constraint

\[ \frac{1}{A} \sum_{i=1}^{A} w_i^2 = 1 \]  

(6)

where \( x_{i_1, \ldots, i_p} \) are independent normal Gaussian variables.

In [Auffinger et al. (2013)], the asymptotic complexity of spherical \( p \) spin glass model is analyzed based on random matrix theory. In [Choromanska et al. (2015)] these results are used in order to shed light on the optimization process of neural networks. For example, the asymptotic complexity of spherical spin glasses reveals a layered structure of low-index critical points near the global optimum. These findings are then given as a possible explanation to several central phenomena found in neural networks optimization, such as similar performance of large nets, and the improbability of getting stuck in a “bad” local minima.

As part of our work, we follow a similar path. First, a link is formed between residual networks and the general multi interaction spherical spin glass model. Then, using [Auffinger & Arous (2013)], we obtain insights on residual networks. The other part of our work studies the dynamic behavior of neural networks using the same spin glass models.

### 3 Residual Nets and General Spin Glass Models

We begin by establishing a connection between the loss function of deep residual networks and the hamiltonian of the general spherical spin glass model. We consider a simple feed forward fully connected network \( \mathcal{N} \), with ReLU activation functions and residual connections. For simplicity of notations without the loss of generality, we assume \( n_1 = \ldots = n_p = n \), \( n_0 = d \) as before. In our ResNet model, there exist \( p - 1 \) identity connections skipping a single layer each, starting from the first hidden layer. The output of layer \( l > 1 \) is given by:

\[ \mathcal{N}_l(x) = \mathcal{R}(W^\top \mathcal{N}_{l-1}(x)) + \mathcal{N}_{l-1}(x) \]  

(7)

where \( W_l \) denotes the weight matrix connecting layer \( l - 1 \) with layer \( l \). Notice that the first hidden layer has no parallel skip connection, and so \( \mathcal{N}_1(x) = \mathcal{R}(W^\top x) \). Without loss of generality, the scalar output of the network is the sum of the outputs of the output layer \( p \) and is expressed as

\[ Y = \sum_{r=1}^{p} \sum_{i=1}^{d} \gamma_r x_{ij}^{(r)} A_{ij}^{(r)} \prod_{k=1}^{r} w_{ij}^{(r)(k)} \]  

(8)

where \( A^{(r)} \in \{0, 1\} \) denotes whether path \( j \) of length \( r \) is open, and \( \forall j, j', r, r' \ x_{ij}^{r} = x_{ij'}^{r'} \). The residual connections in \( \mathcal{N} \) imply that the output \( Y \) is now the sum of products of different lengths, indexed by \( r \). Each path of length \( r \) includes \( r - 1 \) non-skip connections (those involving the first term in Eq. 7 and not the second, identity term) out of layers \( l = 2..p \). Therefore, \( \gamma_r = \binom{p-1}{r-1} n^r \).

We define the following measure on the network:

**Definition 2.** The mass of a depth \( r \) subnetwork in \( \mathcal{N} \) is defined as \( \psi_r = d \gamma_r \).

The properties of redundancy in network parameters and their uniform distribution, as described in Sec. 2, allow us to re-index Eq. 8.

**Lemma 1.** Assuming assumptions A2 – A3 hold, and \( \frac{\lambda}{A} \in \mathbb{Z} \), then the output can be expressed after reindexing as:

\[ Y = \sum_{r=1}^{p} \sum_{i_1, i_2, \ldots, i_r}^{A} \sum_{j=1}^{\frac{\lambda}{A}} x_{i_1, i_2, \ldots, i_r}^{(j)} A_{i_1, i_2, \ldots, i_r}^{(j)} \prod_{k=1}^{r} w_{i_k} \]  

(9)

All proofs can be found in Appendix B.

Making the modeling assumption that the ReLU gates are independent Bernoulli random variables with probability \( \rho \), we obtain that for every path of length \( r \), \( \mathbb{E} A_{i_1, i_2, \ldots, i_r}^{(j)} = \rho^r \) and

\[ \mathbb{E}_A[Y] = \sum_{r=1}^{p} \sum_{i_1, i_2, \ldots, i_r}^{A} \sum_{j=1}^{\frac{\lambda}{A}} x_{i_1, i_2, \ldots, i_r}^{(j)} \rho^r \prod_{k=1}^{r} w_{i_k} \]  

(10)

3
In order to connect ResNets to generalized spherical spin glass models, we denote the variables:

$$\xi_{i_1,i_2,...,i_r} = \sum_{j=1}^{p} x_{i_1,i_2,...,i_r}^j, \quad \hat{x}_{i_1,i_2,...,i_r} = \frac{\xi_{i_1,i_2,...,i_r}}{\mathbb{E}_x[\xi_{i_1,i_2,...,i_r}^2]^{\frac{1}{2}}}$$ (11)

Note that since the input variables $x_1,...,x_d$ are sampled from a centered Gaussian distribution (dependent or not), then the set of variables $\hat{x}_{i_1,i_2,...,i_r}$ are dependent normal Gaussian variables.

**Lemma 2.** Assuming A2 - A3 hold, and $n \in \mathbb{N}$ then $\forall r_{i_1,...,i_r}$ the following holds:

$$\frac{1}{d} (\psi_r)^2 \leq \frac{\mathbb{E}[\xi_{i_1,i_2,...,i_r}^2]}{\psi_r} \leq (\frac{\psi_r}{A})^2.$$ (12)

We approximate the expected output $\mathbb{E}_A(Y)$ with $\tilde{Y}$ by assuming the minimal value in [12] holds such that $\forall r_{i_1,...,i_r} \mathbb{E}[\xi_{i_1,i_2,...,i_r}^2] = \frac{1}{\psi_r}$. The following expression for $\tilde{Y}$ is then obtained:

$$\tilde{Y} = \frac{1}{C} \sum_{r=1}^{p} \frac{\rho_C}{\sqrt{d}} \psi_r \frac{\tilde{x}_{i_1,i_2,...,i_r}}{\Lambda} \prod_{k=1}^{r} w_{i_k}.$$ (13)

The independence assumption A1 was not assumed yet, and [13] holds regardless. Assuming A4 and denoting the scaled weights $\tilde{w}_{i} = \frac{1}{C} w_i$, we can link the distribution of $\tilde{Y}$ to the distribution on $\hat{x}$:

$$\tilde{Y} = \frac{1}{C} \sum_{r=1}^{p} \frac{\rho_C}{\sqrt{d}} \psi_r \frac{\tilde{x}_{i_1,i_2,...,i_r}}{\Lambda} \prod_{k=1}^{r} \tilde{w}_{i_k}$$

$$= z \sum_{r=2}^{p} \frac{\epsilon_r}{\Lambda^{r-1}} \sum_{i_1,i_2,...,i_r=1}^{\Lambda} \hat{x}_{i_1,i_2,...,i_r} \prod_{k=1}^{r} \tilde{w}_{i_k}$$ (14)

where $\epsilon_r = \frac{1}{z} (\frac{\psi_{r-1}}{\psi_r}) (\frac{\rho_C}{\sqrt{d}})^{r-1}$ and $z$ is a normalization factor such that $\sum_{r=1}^{p} \epsilon_r^2 = 1$.

The following lemma gives a generalized expression for the binary and hinge losses of the network.

**Lemma 3 [Choromanska et al., 2015].** Assuming assumptions A2 - A4 hold, then both the losses $\mathcal{L}_N^\rho(x)$ and $\mathcal{L}_N^\lambda(x)$ can be generalized to a distribution of the form:

$$C_1 + C_2 \sum_{r=1}^{p} \frac{\epsilon_r}{\Lambda^{r-1}} \sum_{i_1,i_2,...,i_r=1}^{\Lambda} \hat{x}_{i_1,i_2,...,i_r} \prod_{k=1}^{r} \tilde{w}_{i_k}$$ (15)

where $C_1, C_2$ are positive constants that do not affect the optimization process, and will be omitted in the following sections.

The model in Eq. [15] has the form of a spin glass model, except for the dependency between the variables $\hat{x}_{i_1,i_2,...,i_r}$. We later use an assumption similar to A1 of independence between these variables in order to link the two binary classification losses and the general spherical spin glass model. However, for the results in this section, this is not necessary.

We denote the important quantities:

$$\beta = \frac{\rho n C}{\sqrt{d}}, \quad \epsilon_r = \frac{1}{z} \left( \frac{p-1}{r-1} \right) \beta^r$$ (16)

The series $(\epsilon_r)_{r=1}^{p}$ determines the weight of interactions of a specific length in the loss surface. Notice that for constant depth $p$ and large enough $\beta$, $\arg \max_{r} (\epsilon_r) = p$. Therefore, for wide networks, where $n$ and, therefore, $\beta$ are large, interactions of order $p$ dominate the loss surface, and the effect of the residual connections diminishes. Conversely, for constant $\beta$ and a large enough $p$ (deep networks), we have that $\arg \max_{r} (\epsilon_r) < p$, and can expect interactions of order $r < p$ to dominate the loss. The asymptotic behavior of $\epsilon$ is captured by the following lemma:

**Theorem 1.** Assuming $\frac{\beta}{1+\beta} p \in \mathbb{N}$, we have that:

$$\lim_{p \to \infty} \frac{1}{p} \arg \max_{r} (\epsilon_r) = 1 + \frac{\beta}{\beta + 1}$$ (17)
As the next theorem shows, the epsilons are concentrated in a narrow band near the maximal value.

**Theorem 2.** For any $\alpha_1 < \frac{\beta}{1+\beta} < \alpha_2$, and assuming $\alpha_1 p, \alpha_2 p, \frac{\beta}{1+\beta} p \in \mathbb{N}$, it holds that:

$$\lim_{p \to \infty} \sum_{r=\alpha_1 p}^{\alpha_2 p} \epsilon_r^2 = 1$$

(18)

Thm. 2 implies that for deep residual networks, the contribution of weight products of order far away from the maximum $\frac{\beta}{1+\beta} p$ is negligible. The loss is, therefore, similar in complexity to that of an ensemble of potentially shallow conventional nets. In a common weight initialization scheme for neural networks, $C = \frac{1}{\sqrt{n}}$ (Orr & Müller, 2003; Glorot & Bengio, 2010). With this initialization and $\Lambda = n$, $\beta = \rho$ and the maximal weight is obtained at less than half the network’s depth $\lim_{p \to \infty} \arg \max_r (\epsilon_r) < \frac{p}{2}$. Therefore, at the initialization, the loss function is primarily influenced by interactions of considerably lower order than the depth $p$, which facilitates easier optimization.

4 DYNAMIC BEHAVIOR OF RESIDUAL NETS

The expression for the output of a residual net in Eq. (14) provides valuable insights into the machinery at work when optimizing such models. Thm. 1 and 2 imply that the loss surface resembles that of an ensemble of shallow conventional nets (although not a real ensemble due to obvious dependencies), with various depths concentrated in a narrow band. As noticed in Veit et al. (2016), viewing ResNets as ensembles of relatively shallow networks helps in explaining some of the apparent advantages of these models, particularly the apparent ease of optimization of extremely deep models, since deep paths barely affect the overall loss of the network. However, this alone does not explain the increase in accuracy of deep residual nets over actual ensembles of standard networks. In order to explain the improved performance of ResNets, we make the following claims:

1. The mixture vector $\epsilon$ determines the distribution of the depths of the networks within the ensemble, and is controlled by the scaling parameter $C$.
2. During training, $C$ changes and causes a shift of focus from a shallow ensemble to deeper and deeper ensembles, which leads to an additional capacity.
3. In networks that employ batch normalization, $C$ is directly embodied as the scale parameter $\lambda$. The starting condition of $\lambda = 1$ offers a good starting condition that involves extremely shallow nets.

The next lemma validates item 1 from this list of claims. It shows that we can shift the effective depth to any value by simply controlling $C$.

**Lemma 4.** For any integer $1 \leq k \leq p$ there exists a global scaling parameter $C$ such that $\arg \max_r (\epsilon_r(\beta)) = k$.

A simple global scaling of the weights is, therefore, enough to change the loss surface, from an ensemble of shallow conventional nets, to an ensemble of deep nets. This is illustrated in Fig. 1(a-c) for various values of $\beta$.

In order to gain additional insight into this dynamic mechanism, we investigate the derivative of the loss with respect to the scale parameter $C$. By noticing that $\frac{\partial}{\partial C} = r \frac{\partial}{\partial \epsilon_r}$, and using Eq. (15) we obtain:

$$\frac{\partial L_N(x, w)}{\partial C} = \sum_{r=1}^{p} \epsilon_r \sum_{i_1, i_2, \ldots, i_r=1}^{\Lambda} \bar{x}_{i_1,i_2,\ldots,i_r} \prod_{k=1}^{r} \bar{w}_{ik}$$

(19)

Notice that the addition of a multiplier $r$ indicates that the derivative is increasingly influenced by deeper networks.

4.1 BATCH NORMALIZATION

Batch normalization has shown to be a crucial factor in the successful training of deep residual networks. As we will show, batch normalization layers offer an easy starting condition for the
network, such that the gradients from early in the training process will originate from extremely shallow paths.

We consider a simple batch normalization procedure, which ignores the additive terms, has the output of each ReLU unit in layer $l$ normalized by a factor $\sigma_l$ and then is multiplied by some parameter $\lambda_l$. The output of layer $l > 1$ is therefore:

$$N_l(x) = \frac{\lambda_l}{\sigma_l} R(W_l^T N_{l-1}(x)) + N_{l-1}(x) \quad (20)$$

where $\sigma_l$ is the mean of the estimated standard deviations of various elements in the vector $R(W_l^T N_{l-1}(x))$. Furthermore, a typical initialization of batch normalization parameters is to set $\lambda_l = \sigma_l = 1$ for every unit in the same layer. The recursive relation $E[N_l(x)^2] = 1 + E[N_l(x)^2]$ holds for any unit $j$ in layer $l$. In this case, providing that units in the same layer have equal variance $\sigma_l$, the recursive relation $E[N_l(x)^2] = 1 + E[N_l(x)^2]$ holds for any unit $j$ in layer $l$. This, in turn, implies that the output of the ReLU units should have increasing variance $\sigma_l^2$ as a function of depth. Multiplying the weight parameters in deep layers with an increasingly small scaling factor $\frac{\lambda_l}{\sigma_l}$, effectively reduces the influence of deeper paths, so that extremely short paths will dominate the early stages of optimization. We next analyze how the weight scaling, as introduced by batch normalization, provides a driving force for the effective ensemble to become deeper as training progresses.

### 4.2 The Driving Force Behind the Scale Increase

In the following analysis, we examine the mechanics of a simple example, which can be extrapolated to more general architectures.

We consider a simple network of depth $p$, with a single residual connection skipping $p - m$ layers. We further assume that batch normalization is applied at the output of each ReLU unit as described in Eq. $20$. We denote by $\bar{l}_1, \ldots, \bar{l}_m$, the indices of layers that are not skipped by the residual connection, and $\bar{\lambda}_m = \prod_{i=1}^{m} \frac{\lambda_{\bar{l}_i}}{\sigma_{\bar{l}_i}}, \bar{\lambda}_p = \prod_{i=1}^{p} \frac{\lambda_{\bar{l}_i}}{\sigma_{\bar{l}_i}}$. Since every path of length $m$ is multiplied by $\bar{\lambda}_m$, and every...
path of length $p$ is multiplied by $\hat{\lambda}_p$, the expression for the loss can be written:

$$\mathcal{L}_N(x, w) = \frac{\epsilon_m}{\Lambda^2} \hat{\lambda}_m \sum_{i_1, i_2 \ldots i_m=1} \tilde{x}_{i_1, i_2 \ldots i_m} \prod_{k=1}^m \tilde{w}_{i_k} + \frac{\epsilon_p}{\Lambda^2} \hat{\lambda}_p \sum_{i_1, i_2 \ldots i_p=1} \tilde{x}_{i_1, i_2 \ldots i_p} \prod_{k=1}^p \tilde{w}_{i_k}$$

$$= L_m(x, w) + L_p(x, w) \quad (21)$$

We denote by $\nabla_w$ the derivative operator with respect to the parameters $w$, and the gradient $g = \nabla_w \mathcal{L}_N(x, w) = g_m + g_p$ evaluated at point $w$.

**Theorem 3.** Considering the loss in $21$ and assuming $\frac{\partial \mathcal{L}_N(x, w)}{\partial \lambda_l} = 0$, then for a small learning rate $0 < \mu < 1$ the following hold:

1. For any $\lambda_l, l \in l_1 \ldots l_m$, we have:

$$\left| \lambda_l - \mu \frac{\partial \mathcal{L}_N(x, w - \mu g)}{\partial \lambda_l} \right| > |\lambda_l| \quad (22)$$

2. Assuming $\|g_p\|_2 > \|g_m\|_2$, for any $\lambda_l, l \notin l_1 \ldots l_m$ we have:

$$\left| \lambda_l - \mu \frac{\partial \mathcal{L}_N(x, w - \mu g)}{\partial \lambda_l} \right| > |\lambda_l| \quad (23)$$

Thm. 3 suggests that $|\lambda_l|$ will increase for layers $l$ that do not have skip-connections. Conversely, if layer $l$ has a parallel skip connection, then $|\lambda_l|$ will increase when the gradient from deeper paths $g_p$ becomes dominant as shallow paths reach a local minima. Notice that an increase in $|\hat{\lambda}_l|, l \notin l_1 \ldots l_m$ results in an increase in $|\hat{\lambda}_p|$, while $|\hat{\lambda}_m|$ remains unchanged, therefore shifting the balance into deeper ensembles.

This steady increase of $|\lambda_l|$, as predicted in our theoretical analysis, is also backed in experimental results, as depicted in Fig. 1(d). Note that the first layer, which cannot be skipped, behaves differently than the other layers.

It is worth noting that the mechanism for this dynamic property of residual networks can also be observed without the use of batch normalization, as a steady increase in the $L2$ norm of the weights, as shown in Fig. 1(e). In order to model this, consider the residual network as discussed above, without batch normalization layers. Recalling, $\|w\|_2 = C \sqrt{\sum_i w_i^2}$, the loss of this network is expressed as:

$$\mathcal{L}_N(x, w) = \frac{\epsilon_m}{\Lambda^2} \sum_{i_1, i_2 \ldots i_m=1} \tilde{x}_{i_1, i_2 \ldots i_m} \prod_{k=1}^m \tilde{w}_{i_k} + \frac{\epsilon_p}{\Lambda^2} \sum_{i_1, i_2 \ldots i_p=1} \tilde{x}_{i_1, i_2 \ldots i_p} \prod_{k=1}^p \tilde{w}_{i_k}$$

$$= L_m(x, w) + L_p(x, w) \quad (24)$$

**Theorem 4.** Considering the loss in $24$ and assuming $\frac{\partial \mathcal{L}_N(x, w)}{\partial \lambda_l} = 0$, then for a small learning rate $0 < \mu < 1$ the following hold:

$$\frac{\partial \mathcal{L}_N(x, w - \mu g)}{\partial C} \approx - \frac{1}{C} (m \|g_m\|_2^2 + p \|g_p\|_2^2 + (m + p)g_p^\top g_m) \quad (25)$$

Thm. 4 indicates that when deeper gradients become dominant (for example, near local minimas of the shallow network), the scaling of the weights $C$ will increase. This expansion will, in turn, emphasize the contribution of deeper paths, and increase the overall capacity of the residual network.

5 THE LOSS SURFACE OF ENSEMBLES

We now present the results of Auffinger & Arous (2013) regarding the asymptotic complexity in the case of $\lim_{n \to \infty}$ of the multi-spherical spin glass model given by:

$$\mathcal{H}_{e, \Lambda} = - \sum_{r=2}^{\infty} \frac{\epsilon_r}{\Lambda^2} \sum_{i_1 \ldots i_r=1} J_{i_1 \ldots i_r} \tilde{w}_{i_1} \ldots \tilde{w}_{i_r} \quad (26)$$
where $J_{i_1...i_r}$ are independent centered standard Gaussian variables, and $\epsilon = (\epsilon_r)_{r \geq 2}$ are positive real numbers such that $\sum_{r=2}^{\infty} \epsilon_r 2^r < \infty$. A configuration $w$ of the spin spherical spin-glass model is a vector in $R^\Lambda$ satisfying the spherical constraint:

$$\frac{1}{\Lambda} \sum_{i=1}^{\Lambda} w_i^2 = 1, \quad \sum_{r=2}^{\infty} \epsilon_r^2 = 1$$

(27)

Note that the variance of the process is independent of $\epsilon$:

$$E[H_{\epsilon,\Lambda}^2] = \sum_{r=2}^{\infty} \Lambda^{1-r} \epsilon_r^2 (\sum_{i=1}^{\Lambda} w_i^2)^r = \Lambda \sum_{r=1}^{\infty} \epsilon_r^2 = \Lambda$$

(28)

**Definition 3.** We define the following:

$$v' = \sum_{r=2}^{\infty} \epsilon_r^2 r, \quad v'' = \sum_{r=2}^{\infty} \epsilon_r^2 (r-1), \quad \alpha^2 = v'' + v' - v'^2$$

(29)

Note that for the single interaction spherical spin model $\alpha^2 = 0$. The index of a critical point of $H_{\epsilon,\Lambda}$ is defined as the number of negative eigenvalues in the hessian $\nabla^2 H_{\epsilon,\Lambda}$ evaluated at the critical point $w$.

**Definition 4.** For any $0 \leq k < \Lambda$ and $u \in \mathcal{R}$, we denote the random number $Crt_{\lambda, k}(u, \epsilon)$ as the number of critical points of the Hamiltonian in the set $BX = \{ AX | X \in (-\infty, u) \}$ with index $k$. That is:

$$Crt_{\lambda, k}(u, \epsilon) = \sum_{w: \nabla H_{\epsilon,\Lambda} = 0} \mathbb{I} \{ H_{\epsilon,\Lambda} \in \Lambda u \} \mathbb{I} \{ i(\nabla^2 H_{\epsilon,\Lambda}) = k \}$$

(30)

Furthermore, define $\theta_k(u, \epsilon) = \lim_{\Lambda \rightarrow \infty} \frac{1}{\Lambda} \log E[Crt_{\lambda, k}(u\epsilon)]$. Corollary 1.1 of Auffinger & Arous (2013) states that for any $k > 0$:

$$\theta_k(\mathbb{R}, \epsilon) = \frac{1}{2} \log (v'' - v') - \frac{v'''}{v'' + v'}$$

(31)

Eq. [31] provides the asymptotic mean total number of critical points with non-diverging index $k$. It is presumed that the SGD algorithm will easily avoid critical points with a high index that have many descent directions, and maneuver towards low index critical points. We, therefore, investigate how the mean total number of low index critical points vary as the ensemble distribution embodied in $(\epsilon_r)_{r \geq 2}$ changes its shape by a steady increase in $\beta$.

Fig. [f] shows that as the ensemble progresses towards deeper networks, the mean amount of low index critical points increases, which might cause the SGD optimizer to get stuck in local minima. This is, however, resolved by the fact that by the time the ensemble becomes deep enough, the loss function has already reached a point of low energy as shallower ensembles were more dominant earlier in the training. In the following theorem, we assume a finite ensemble such that $\sum_{r=p+1}^{\infty} \epsilon_r 2^r \approx 0$.

**Theorem 5.** For any $k \in \mathbb{N}$, $p > 1$, we denote the solution to the following constrained optimization problems:

$$\epsilon^* = \arg \max_{\epsilon} \theta_k(\mathbb{R}, \epsilon) \quad s.t. \quad \sum_{r=2}^{p} \epsilon_r^2 = 1$$

(32)

It holds that:

$$\epsilon_r^* = \begin{cases} 1, & r = p \\ 0, & otherwise \end{cases}$$

(33)

Theorem [5] implies that any heterogeneous mixture of spin glasses contains fewer critical points of a finite index, than a mixture in which only $p$ interactions are considered. Therefore, for any distribution of $\epsilon$ that is attainable during the training of a ResNet of depth $p$, the number of critical points is lower than the number of critical points for a conventional network of depth $p$. 


6 CONCLUSION

Ensembles are a powerful model for ResNets, which unravels some of the key questions that have surrounded ResNets since their introduction. Here, we show that ResNets display a dynamic ensemble behavior, which explains the ease of training such networks even at very large depths, while still maintaining the advantage of depth. As far as we know, the dynamic behavior of the effective capacity is unlike anything documented in the deep learning literature. Surprisingly, the dynamic mechanism typically takes place within the outer multiplicative factor of the batch normalization module.

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A SUMMARY OF NOTATIONS

Table 1 presents the various symbols used throughout this work and their meaning.

B PROOFS

Proof of Lemma 7 There are a total of $ψ_r$ paths of length $r$ from input to output, and a total of $Λ^r$ unique $r$ length configurations of weights. The uniformity assumption then implies that each configuration of weights is repeated $ψ_r Λ^r$ times. By summing over the unique configurations, and re indeximg the input we arrive at Eq. 9.

Proof of Lemma 2 From 11 we have that for each $ξ_{i_1,i_2,\ldots,i_r}$ there exists a sequence $β = (β_i)_{i=1}^P ∈ \mathbb{N}$ such that $\sum_{i=1}^P β_i = \frac{ψ_r}{Λ^r}$, and $ξ_{i_1,i_2,\ldots,i_r} = \sum_{i=1}^P β_i x_i$. We, therefore, have that $E[ξ^2_{i_1,i_2,\ldots,i_r}] = \|β\|_2^2$. Note that the minimum value of $E[ξ^2_{i_1,i_2,\ldots,i_r}]$ is a solution to the following:

$$
\min(E[ξ^2_{i_1,i_2,\ldots,i_r}]) = \min_β(\|β\|_2) \quad s.t \quad \|β\|_1 = \frac{ψ_r}{Λ^r}, \quad (β_i)_{i=1}^P ∈ \mathbb{N}, \quad (34)
$$
Table 1: Notations

| SYMBOL | DESCRIPTION |
|--------|-------------|
| $x$    | Input vector $\in \mathbb{R}^d$, sampled from a normal distribution |
| $d$    | The dimensionality of the input $x$ |
| $N_i(x)$ | The output of layer $i$ of network $N$ given input $x$ |
| $Y$    | The final output of the network $N$ |
| $L$    | The true label of input $x$ |
| $L_N$  | Loss function of network $N$ |
| $H^L_N$ | Hinge loss |
| $\alpha$ | Absolute loss |
| $p$    | The depth of network $N$ |
| $w$    | Weights of the network $w \in \mathbb{R}^\Lambda$ |
| $c$    | A positive scale factor such that $\|w\|_2 = \sqrt{\lambda_c}$ |
| $\hat{w}$ | Scaled weights such that $\hat{w} = \frac{1}{\lambda_c}w$ |
| $n$    | The number of unique weights in the network |
| $N$    | The total number of weights in the network $N$ |
| $W_l$  | The weight matrix connecting layer $l-1$ to layer $l$ in $N$. |
| $H_{p,\lambda}$ | The hamiltonian of the $p$ interaction spherical spin glass model. |
| $H_{e,\lambda}$ | The hamiltonian of the general spherical spin glass model. |
| $\gamma$ | Total number of paths from input to output in network $N$ |
| $\gamma_r$ | Total number of paths from input to output in network $N$ of length $r$ |
| $\rho$ | The mean of the estimated standard deviation various elements in $\mathcal{R}(W_l^T N_{l-1}(x))$. |

which achieves its minimal value at $\forall i, \beta_i = \frac{1}{d} \frac{\psi_r}{\psi_r}$. Similarly, the maximum value is achieved at $\beta_i = \frac{1}{\lambda_c} \delta_i$ for some index $i$. 

Proof of Thm. 1.

We use the stirling approximation, which states $\lim_{p \to \infty} \frac{1}{p} log \left( \binom{p}{\alpha} \right) = H(\alpha)$, where $H(\alpha) = -\alpha \log(\alpha) - (1 - \alpha) \log(1 - \alpha)$. Ignoring the constants which do not depend on $\alpha$, we have:

$$\lim_{p \to \infty} \frac{1}{p} log \left( \binom{p}{\alpha} \right) \beta^{\alpha p} = H(\alpha) + \alpha \log(\beta)$$

which achieves its maximum value at $\alpha = \alpha^*$. 

Proof of Thm. 2.

For brevity, we provide a sketch of the proof. It is enough to show that $\lim_{p \to \infty} \sum_{r=1}^{\alpha_1} \epsilon_r^2 = 0$ for $\beta < 1$. Ignoring the constants in the binomial terms, we have:

$$\lim_{p \to \infty} \sum_{r=1}^{\alpha_1} \epsilon_r^2 = \lim_{p \to \infty} \frac{\sum_{i=1}^{\alpha_1} (\binom{p}{i})^2 \beta^{2r}}{z^2} \leq \lim_{p \to \infty} \frac{\alpha_1 p (\binom{p}{\alpha_1})^2 \beta^{2\alpha_1 p}}{z^2}$$

(36)

Where $z^2 = \sum_{r=1}^{p} (\binom{p}{r})^2 \beta^{2r}$, which can be expressed using the Legendre polynomial of order $p$:

$$z^2 = (1 - \beta^2)^p \frac{p}{1 - \beta^2}$$

(37)
In order to compute the limit of Eq.\[36\] we use the asymptotic of the Legendre polynomial of order \(p\) for \(x > 1\), \(P_p(x) \sim \frac{1}{\sqrt{2\pi p}} \left(\frac{x+\sqrt{x^2-1}}{2}\right)^p\). For the term in the nominator of Eq.\[36\] we use the Stirling approximation for factorials \(p! \sim \sqrt{2\pi p}(\frac{p}{e})^p\). Substituting both approximations in Eq.\[36\] and taking the limit completes the proof. 

**Proof of Lemma 2** For simplicity, we ignore the constants in the binomial coefficient, and therefore \(\lambda_l\) multiplies every path in the network. Therefore, \(\frac{\partial L_N(x, w)}{\partial \lambda_l} = \frac{1}{\lambda_l}(L_m(x, w) + L_p(x, w))\). Using taylor series expansion:

\[
\frac{\partial L_N(x, w) - \mu g_w}{\partial \lambda_l} = \frac{\partial L_N(x, w)}{\partial \lambda_l} - \mu \nabla_w \frac{\partial L_N(x, w)}{\partial \lambda_l} g
\]  

Substituting \(\nabla_w \frac{\partial L_N(x, w)}{\partial \lambda_l} = \frac{1}{\lambda_l}(g_m + g_p)\) in \(38\) we have:

\[
\frac{\partial L_N(x, w) - \mu g_w}{\partial \lambda_l} \approx 0 - \frac{1}{\lambda_l}(g_m + g_p)\top (g_m + g_p) = -\mu \frac{1}{\lambda_l} \|g_m + g_p\|_2^2 < 0
\]  

And hence:

\[
\lambda_l - \mu \frac{\partial L_N(x, w) - \mu g_w}{\partial \lambda_l} = \lambda_l + \mu^2 \frac{1}{\lambda_l} \|g_m + g_p\|_2^2
\]

Finally:

\[
|\lambda_l(1 + \mu^2 \frac{1}{\lambda_l} \|g_m + g_p\|_2^2)| = |\lambda_l|(1 + \mu^2 \frac{1}{\lambda_l} \|g_m + g_p\|_2^2) \geq |\lambda_l|
\]  

2. Since paths of length \(m\) skip layer \(l\), we have that \(\nabla_w \frac{\partial L_N(x, w)}{\partial \lambda_l} = \frac{1}{\lambda_l} g_p\). Therefore:

\[
\frac{\partial L_N(x, w) - \mu g}{\partial \lambda_l} \approx 0 - \frac{1}{\lambda_l}(g_m + g_p)\top g_p = -\mu \frac{1}{\lambda_l} (g_m\top g_p + \|g_p\|_2^2)
\]

The condition \(\|g_p\|_2 > \|g_m\|_2\) implies that \(g_m\top g_p + \|g_p\|_2^2 > 0\), completing the proof. 

**Proof of Thm 2** Notice that \(\frac{\partial L_N(x, w)}{\partial c} = \frac{\partial L_N(x, w)}{\partial w} \frac{\partial w}{\partial \|w\|_2} \sqrt{\lambda} = g\top \hat{w} = 0\), and hence the gradient is orthogonal to the weights. Using \(19\) we have that \(\frac{\partial L_N(x, w)}{\partial c} = \frac{1}{C}(mL_m(x, w) + pL_p(x, w))\). Using taylor series expansion we have:

\[
\frac{\partial L_N(x, w)}{\partial c} \approx \frac{\partial L_N(x, w)}{\partial c} - \mu \nabla_w \frac{\partial L_N(x, w)}{\partial c} g
\]  

For the last term we have:

\[
\nabla_w \frac{\partial L_N(x, w)}{\partial c} g = (mL_m(x, w) + pL_p(x, w))\nabla_w \frac{\sqrt{\lambda}}{\|w\|_2} g + \frac{1}{C}(m(g_m + g_p)\top g)
\]

\[
= (mL_m(x, w) + pL_p(x, w))\frac{g}{C \|w\|_2} + \frac{1}{C}(m(g_m + g_p)\top g) = \frac{1}{C}(m(g_m + g_p)\top g)
\]  

where the last step stems from the fact that \(w\top g = 0\). Substituting \(\nabla_w \frac{\partial L_N(x, w)}{\partial c} = \frac{1}{C}(m(g_m + g_p)\top g)\) in \(43\) we have:

\[
\frac{\partial L_N(x, w) - \mu g_w}{\partial c} \approx 0 - \mu \frac{1}{C}(m(g_m + g_p)\top (g_m + g_p))
\]

\[
= -\mu \frac{1}{C}(m\|g_m\|_2^2 + p\|g_p\|_2^2 + (m + p)g_p\top g_m)
\]  

\[\square\]
**Proof of Thm 5.** Inserting Eq. 29 into Eq. 31 we have that:

\[
\theta_k(\mathbb{R}, \epsilon) = \frac{1}{2} \log \left( \frac{\sum_{r=2}^{p} \epsilon_{i}^2 (r - 1)}{\sum_{r=2}^{p} \epsilon_{i}^2 (r - 2)} - \frac{\sum_{r=2}^{p} \epsilon_{i}^2 (r - 2)}{\sum_{r=2}^{p} \epsilon_{i}^2 (r - 2)} \right)
\]

(46)

We denote the matrices \( V' \) and \( V'' \) such that \( V_{ij}' = r \delta_{ij} \) and \( V_{ij}'' = r(r - 1) \delta_{ij} \). We then have:

\[
\theta_k(\mathbb{R}, \epsilon) = \frac{1}{2} \log \left( \frac{\epsilon^\top V'' \epsilon}{\epsilon^\top V' \epsilon} \right) - \frac{\epsilon^\top (V'' - V') \epsilon}{\epsilon^\top (V'' - V') \epsilon}
\]

(47)

\[
\max_{\epsilon} \theta_k(\mathbb{R}, \epsilon) \leq \max_{\epsilon} \left( \frac{1}{2} \log \left( \frac{\epsilon^\top V'' \epsilon}{\epsilon^\top V' \epsilon} \right) \right) - \min_{\epsilon} \left( \frac{\epsilon^\top (V'' - V') \epsilon}{\epsilon^\top (V'' - V') \epsilon} \right)
\]

\[
= \frac{1}{2} \log \left( \max_i (V''_{ii} V'_{ii}^{-1}) \right) - \min_i \left( (V''_{ii} - V'_{ii}) (V''_{ii} + V'_{ii})^{-1} \right)
\]

\[
= \frac{1}{2} \log (p - 1) - (1 - \frac{2}{p}) = \theta_k(\mathbb{R}, \epsilon^*)
\]

(48)