STABILIZATION OF THE WAVE EQUATION WITH EXTERNAL FORCE.

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Abstract. We study the rate of decay of the energy functional of solutions of the wave equation with localized damping and an external force. We prove that the decay rates of the energy functional is determined from a forced differential equation.

1. Introduction

This article is devoted to the study of stabilization for the wave equation with external force on a compact Riemannian manifold with boundary. In the first part of this paper, we consider the following wave equation with linear internal damping and external force

\[
\begin{aligned}
\partial_t^2 u - \Delta u + a(x) \partial_t u &= f(t,x) \quad \mathbb{R}_+ \times M \\
(\partial_t u)(0) &= (u_0, u_1)
\end{aligned}
\]  

Here \( M = (M, q) \) is a compact, connected Riemannian manifold of dimension \( d \), with \( C^\infty \) boundary \( \partial M \), where \( q \) denotes a Riemannian metric of class \( C^\infty \). \( \Delta \) the Laplace–Beltrami operator on \( M \), \( a(x) \) is a non negative function in \( C^\infty(M) \) and \( f \) is a function in \( L^2(\mathbb{R}_+ \times M) \).

We define the energy space

\[ \mathcal{H} = H^1_0(M) \times L^2(M) \]

where

\[ H^1_0(M) = \{ u \in H^1(M) ; u|_{\partial M} = 0 \} \]

which is a Hilbert space. Linear semigroup theory applied to (1.1), provides the existence of a unique solution \( u \) in the class

\[ u \in C^0(\mathbb{R}_+, H^1_0(M)) \cap C^1(\mathbb{R}_+, L^2(M)) \]

With (1.1) we associate the energy functional given by

\[ E_u(t) = \frac{1}{2} \int_M |\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2 \, dx. \]

The energy \( E_u(t) \) is topologically equivalent to the norm on the space \( \mathcal{H} \). Under these assumptions, the energy functional satisfies the following identity

\[ E_u(t) + \int_s^t \int_M a(x)|\partial_t u|^2 \, dx \, d\sigma = E_u(s) + \int_s^t \int_M f \partial_t u \, dx \, d\sigma \]

for every \( t \geq s \geq 0 \).

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The topic of interest is rate of decay of the energy functional. This problem has a very long history. The connection between controllability, observability and stabilization was discovered \[24\] and effectively used in the context of linear PDE systems.

When \( f = 0 \) and the damping term acts on the hole manifold the problem has been studied by many authors \[2\] and reference therein. For the wave equation with localized linear damping term, we mention the works of Rauch- Taylor \[22\] and Bardos et al \[4\] in which microlocal techniques is used. In particular the notion of geometric control. We cite also the works of Lasiecka et al \[16\] and Triggiani- Yao \[26\] in which another approach based on Riemannian geometry is presented.

Particular attention has been paid to the case when \( M \) is a bounded domain and the damping is linearly bounded \[14\][13] and reference therein. Under certain geometric condition, the energy functional decays exponentially. Damping that does not satisfy such linear bound near the origin (e.g. when the damping has polynomial, exponential or logarithmic behavior near the origin) results in a weaker form of the energy that could be expressed by algebraic, logarithmic (or possibly slower) rates \[13\][11][19]. Finally we mention the work of cavalcanti et al \[6\] when \( M \) is a compact manifold with or without boundary.

When \( f \neq 0 \), the literature is less furnished, we specially mention the works of Haraux \[12\] and Zhu \[27\] when the damping is globally distributed.

We should also remark when the support of the dissipation may be arbitrarily small require more regular initial data and result in very slow (logarithmic or slower ) decay rates as shown in \[7\] and reference therein.

We assume that the geodesics of \( \bar{M} \) have no contact of infinite order with \( \partial M \). Let \( \omega \) be an open subset of \( M \) and consider the following assumption:

(G) \( (\omega,T) \) geometrically controls \( M \), i.e. every generalized geodesic of \( M \), travelling with speed 1 and issued at \( t = 0 \), enters the set \( \omega \) in a time \( t < T \).

This condition is called Geometric Control Condition (see e.g. \[4\]) We shall relate the open subset \( \omega \) with the damper \( a \) by

\[
\omega = \{ x \in M : a(x) > 0 \}.
\]

Under the assumption (G) it was proved in \[4\][18], that the energy decays exponentially, moreover if there exits a maximal generalized geodesic of \( M \) that never meets the support of the damper \( a \), then we don’t have the exponential decay of the energy for initial data in the energy space.

It is known that the exponential decay of the energy is equivalent to the following observability inequality:

(A) **Linear Observability inequality:** There exist positive constants \( T \) and \( \alpha = \alpha(T) \), such that for every initial condition \( \varphi = (u_0,u_1) \in \mathcal{H} \) the corresponding solution satisfies

\[
E_v(t) \leq \alpha \int_0^{t+T} \int_\Omega a(x) |\partial_x v|^2 \, dx \, ds.
\] \hspace{1cm} (1.3)

for every \( t \geq 0 \).

In this paper, under the assumption (G), we show that for the non autonomous case the corresponding observability inequality reads as follows:

(B) **Non autonomous linear Observability inequality:** There exist positive constants \( T \) and \( \alpha = \alpha(T) \), such that for every initial condition \( \varphi = (u_0,u_1) \in \mathcal{H} \) the
corresponding solution satisfies

$$E_v(t) \leq \alpha \int_t^{t+T} \int_M a(x) |\partial_t v|^2 + |f(s,x)|^2 \, dx \, ds. \quad (1.4)$$

for every $t \geq 0$.

From the observability inequality above, we infer that the rate of decay of the energy will depends on $\int_M |f(t,x)|^2 \, dx$. Now we state the main result of the first part of the paper:

**Theorem 1.** Let $u(t)$ is the solution to the linear problem (1.1) with initial condition $(u_0, u_1) \in \mathcal{H}$. We assume that $(\omega, T)$ satisfies the assumption $(G)$ and

$$\Gamma(t) = C_{1,T} \int_M |f(t,x)|^2 \, dx \in L^1(\mathbb{R}_+)$$

with $C_{1,T} \geq 1$. Then

$$E_u(t) \leq 4e^T \left( S(t-T) + \int_{t-T}^t \Gamma(s) \, ds \right), \quad t \geq T$$

where $S(t)$ is the solution of the following ordinary differential equation

$$\frac{dS}{dt} + \frac{1}{TC_T} S = \Gamma(t), \quad S(0) = E_u(0). \quad (1.5)$$

where $C_T \geq 1$.

1.1. **Applications for the linear case.** Setting

$$\Gamma(t) = C_{1,T} \int_M |f(t,x)|^2 \, dx$$

with $C_{1,T} \geq 1$.

The ODE (1.5) governing the energy bound reduces to

$$\frac{dS}{dt} + CS = \Gamma(t) \quad (1.6)$$

where constant $C > 0$ does not depend on $E_u(0)$.

(1) If there are constants $M > 0$ and $\theta > 0$, such that

$$\Gamma(t) \leq Me^{-\theta t}$$

We have

$$\int_{t-T}^t e^{-\theta s} ds \leq \frac{1}{\theta} \left[ e^{\theta T} - 1 \right] e^{-\theta t}, \quad t \geq T$$

Multiply (1.6) both sides by $\exp(Ct)$ and integrate from 0 to $t$, we obtain

(a) $C > \theta$

$$E_u(t) \leq c (1 + E_u(0)) e^{-\theta t}, \quad t \geq 0$$

(b) $C = \theta$

$$E_u(t) \leq c (1 + E_u(0)) (1 + t) e^{-\theta t}, \quad t \geq 0$$

(c) $C < \theta$

$$E_u(t) \leq c (1 + E_u(0)) e^{-Ct}, \quad t \geq 0$$
(2) If there are constants $M > 0$ and $\theta > 1$, such that

$$\Gamma (t) \leq M (1 + t)^{-\theta}$$

We have

$$\int_{t-T}^{t} (1 + s)^{-\theta} ds \leq T (1 + t - T)^{-\theta}, \quad t \geq T$$

In order to obtain the rate of decay in this case, we use proposition 1. Then

$$E_u (t) \leq c (1 + t - T)^{-\theta}, \quad t \geq T$$

where $c > 0$ and depends on $E_u (0)$.

**Remark 1.** If we consider the following system

$$\begin{cases}
\partial_{tt} u - \Delta u + a (x) g (\partial_t u) = f (t, x) & \mathbb{R}_+ \times M \\
u = 0 & \mathbb{R}_+ \times \partial M \\
(u (0), \partial_t u (0)) = (u_0, u_1) \in H^1_0 (M) \times L^2 (M)
\end{cases}$$

with $g$ continuous, monotone increasing function, vanishing at the origin and linearly bounded. Then the result of the theorem above remains true.

### 1.2. The nonlinear case.

In the second part of the paper we study the rate of decay of the energy functional of solution of the wave equation with nonlinear damping and external force. More precisely, we consider the following system

$$\begin{cases}
\partial_{tt} u - \Delta u + a (x) g (\partial_t u) = f (t, x) & \mathbb{R}_+ \times M \\
u = 0 & \mathbb{R}_+ \times \partial M \\
(u (0), \partial_t u (0)) = (u_0, u_1) \in H^1_0 (M) \times L^2 (M)
\end{cases} \tag{1.7}$$

$g$ is a continuous, monotone increasing function vanishing at the origin. Moreover we assume that, there exists a positive constant $m$, such that

$$\frac{1}{m} |s|^2 \leq g (s) s \leq m |s|^2, \quad |s| > \eta \tag{1.8}$$

for some $\eta > 0$. $a (x)$ is a non negative function in $C^\infty (M)$ and $f$ is in $L^2 (\mathbb{R}_+ \times M)$.

Nonlinear semigroup theory applied to (1.1), provides the existence of a unique solution $u$ in the class

$$u \in C^0 \left( \mathbb{R}_+, H^1_0 (M) \right) \cap C^1 \left( \mathbb{R}_+, L^2 (M) \right)$$

Under these assumptions on the behavior on the damping, the energy functional satisfies the following identity

$$E_u (t) + \int_{t-s}^{t} \int_M a (x) g (\partial_t u) \partial_t u dx d\sigma = E_u (s) + \int_{s}^{t} \int_M f \partial_t u dx d\sigma \tag{1.9}$$

for every $t > s \geq 0$.

It is well known, for the nonlinear problem without a external force the corresponding observability inequality \[15\]... reads as follows:

**C) Nonlinear Observability Inequality**: There exists a constant $T > 0$ and a concave, continuous, monotone increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+, \: h (0) = 0$ (possibly dependent on $T$) such that the solution $u (t, x)$ to the nonlinear problem (1.7) with initial data $\varphi = (u_0, u_1)$ and $f \equiv 0$ satisfies

$$E_u (t) \leq h \left( \int_{t}^{t+T} \int_{\Omega} a (x) g (\partial_t u) \partial_t u \ dx ds \right), \quad (1.10)$$
for every $t \geq 0$.

The function $h(s)$ in (1.10) depends on the nonlinear map $g(s)$, and ultimately determines the decay rates for the energy $E_u(t)$. The energy decay for the nonlinear problem will be determined from the following ODE

$$S_t + h^{-1}(CS) = 0, \quad S(0) = E_u(0) \quad (1.11)$$

we show that under the assumption (G) we obtain the following observability inequality

**D) Nonlinear Non-autonomous Observability Inequality:** There exists a constant $T > 0$ and a concave, continuous, monotone increasing function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $h(0) = 0$ (possibly dependent on $T$ ) such that the solution $u(t, x)$ to the nonlinear problem (1.7) with initial data $\varphi = (u_0, u_1)$ satisfies

$$E_u(t) \leq h \left( \int_t^{t+T} \int_{\Omega} a(x) g(\partial_t u) \partial_t u \, dx \, ds + \int_t^{t+T} \int_M |f(s, x)|^2 \, dx \, ds \right), \quad (1.12)$$

for every $t \geq 0$.

Before giving the main result of this section, we will define some needed functions. According to [15] there exists a strictly increasing function $h_0$ with $h_0(0) = 0$ such that

$$h_0(g(s)s) \geq \epsilon_0 \left( |s|^2 + |g(s)|^2 \right), \quad |s| \leq \eta$$

for some $\epsilon_0, \eta > 0$. For the construction of such function we refer the interested reader to [15, 8]. With this function, we define

$$h = I + m_a(M_T) h_0 \circ \frac{I}{m_a(M_T)}$$

where

$$m_a = a(x) \, dx \, dt$$

and $M_T = (0, T) \times M$.

We can now proceed to state the main result of the second part of the paper

**Theorem 2.** Let $u(t)$ be the solution to the nonlinear problem (1.7) with initial condition $(u_0, u_1) \in \mathcal{H}$. We assume that $(\omega, T)$ satisfies the assumption (G) and

$$\Gamma(t) = 2 \int_M |f(t, x)|^2 \, dx + \psi^* \left( \|f(t, .)\|_{L^2(M)} \right) \in L^1(\mathbb{R}_+)$$

where $\psi^*$ is the convex conjugate of the function $\psi$, defined by

$$\psi(s) = \begin{cases} \frac{1}{2T} h^{-1} \left( -|s|^2 \frac{C_T}{2T} e^s \right) & s \in \mathbb{R}_+ \\ +\infty & s \in \mathbb{R}_- \end{cases}$$

with $C_T \geq 1$. Then

$$E_u(t) \leq 4e^{T} \left( S(t - T) + \int_{t-T}^{t} \Gamma(s) \, ds \right), \quad t \geq T$$

where $S(t)$ is the solution of the following ordinary differential equation

$$\frac{dS}{dt} + \frac{1}{4T} h^{-1} \left( \frac{1}{K} S \right) = \Gamma(t), \quad S(0) = E_u(0) \quad (1.13)$$

with, $K \geq C_T$. 

1.2.1. Applications for the nonlinear case.

**Proposition 1.** Let \( p \) a differentiable, positive strictly increasing function on \( \mathbb{R}_+ \). We assume that there exists \( m_1 > 0 \) such that, \( p(x) \leq m_1 x \) for every \( x \in [0, \eta] \) for some \( 0 < \eta << 1 \) and the following property

\[
p(Kx) \geq mp(K)p(y)
\]

holds for some \( m > 0 \) and for every \((K, x) \in [1, +\infty[ \times \mathbb{R}_+ \). \( \Gamma \in C^1(\mathbb{R}_+) \). Let \( S \) satisfying the following differential inequality

\[
\frac{dS}{dt} + p(S) \leq \Gamma (t), \ S(0) \geq 0.
\]

(1) \( \Gamma (t) = 0 \) for every \( t \geq 0 \). We assume that \( S(0) > 0 \). Then

\[
S(t) \leq \psi^{-1}(t), \ \text{for every } t \geq 0
\]

where

\[
\psi(x) = \int_x^{S(0)} \frac{ds}{p(s)}, \quad x \in [0, S(0)].
\]

(2) \( \Gamma (t) > 0 \) for every \( t \geq 0 \).

(a) There exist \( c > 0 \) and \( \kappa \geq 1 \) such that

\[
\frac{d}{dt} p^{-1}(\Gamma(t)) + c \Gamma(t) < 0, \ \text{for every } t \geq 0
\]

and

\[
mp(\kappa) - \kappa c - 1 \geq 0
\]

\[
\kappa p^{-1} \circ \Gamma(0) \geq S(0)
\]

Then

\[
S(t) \leq \kappa \psi^{-1}(ct), \ \text{for every } t \geq 0
\]

where

\[
\psi(x) = \int_x^{p^{-1} \circ \Gamma(0)} \frac{ds}{p(s)}, \quad x \in [0, p^{-1} \circ \Gamma(0)].
\]

(b) There exist \( c > 0 \) and \( \kappa \geq 1 \) such that

\[
\frac{d}{dt} p^{-1}(\Gamma(t)) + c \Gamma(t) \geq 0, \ \text{for every } t \geq 0
\]

and

\[
mp(\kappa) - c \kappa - 1 \geq 0
\]

\[
\kappa p^{-1} \circ \Gamma(0) \geq S(0)
\]

Then

\[
S(t) \leq \kappa p^{-1} \circ \Gamma(t), \ \text{for every } t \geq 0
\]

**Proof.**

(1) If \( S(0) = 0 \), since \( S \) is positive and decreasing then \( S(t) = 0 \) for every \( t \geq 0 \). We assume that \( S(0) > 0 \). Let \( \psi \), the function defined by

\[
\psi(x) = \int_x^{S(0)} \frac{ds}{p(s)}.
\]
then $\psi$ is a strictly decreasing function on $(0, S(0))$ and $\lim_{x \to 0} \psi(x) = +\infty$. We have

$$
\frac{d}{dt} \psi \circ S(t) \geq 1
$$

Integrating from 0 to $t$, we obtain

$$
\psi \circ S(t) \geq t, t \geq 0
$$

since $\psi$ is decreasing

$$
S(t) \leq \psi^{-1}(t), t \geq 0
$$

(2)

(a) Let $\psi$, the function defined by

$$
\psi(x) = \int_x^p \frac{p^{-1}(s)}{\Gamma(0)} ds
$$

then $\psi$ is a strictly decreasing function on $]0, p^{-1} \circ \Gamma(0)]$ and $\lim_{x \to 0} \psi(x) = +\infty$. We have

$$
\frac{d}{dt} \psi \circ p^{-1} \circ \Gamma(t) = -\frac{\frac{d}{dt} p^{-1}(\Gamma(t))}{\Gamma(t)}
$$

from (1.15), we infer that

$$
\frac{d}{dt} \psi \circ p^{-1} \circ \Gamma(t) \geq c
$$

Integrating from 0 to $t$, we obtain

$$
\psi \circ p^{-1} \circ \Gamma(t) \geq ct
$$

this gives

$$
\Gamma(t) \leq p \circ \psi^{-1}(ct), \text{ for every } t \geq 0
$$

(1.17)

Setting

$$
y(t) = \kappa \psi^{-1}(ct), t \geq 0.
$$

We have

$$
y'(t) + p(y(t)) = -c\kappa \psi^{-1}(ct) + p(\kappa(\psi^{-1}(ct)))
$$

Using (1.14) and (1.17)

$$
y'(t) + p(y(t)) \geq (mp(\kappa) - c\kappa) p \circ \psi^{-1}(ct)
$$

$$
\geq (mp(\kappa) - c\kappa) \Gamma(t)
$$

(1.16) gives

$$
y'(t) + p(y(t)) \geq \Gamma(t)
$$

$$
y(0) \geq S(0)
$$

the result follows from the following lemma

**Lemma 1.** Let $p_i$ ($i = 1, 2$) a positive strictly increasing function on $\mathbb{R}_+$. Suppose that $S$ and $y$ are absolutely continuous functions and satisfy

$$
\frac{dS}{dt} + p_1(S) \leq \Gamma(t) \text{ on } [0, +\infty[.
$$

(1.18)

and

$$
\frac{dy}{dt} + p_2(y) \geq \Gamma_1(t) \text{ on } [0, +\infty[.
$$

(1.19)
where \( \Gamma; \Gamma_1 \in L^1([0; \infty)); \Gamma_1 \geq \Gamma \geq 0; p_1 \geq p_2 \geq 0. \) In addition; if \( y(0) \geq S(0) \)

Then
\[
y(t) \geq S(t) \quad \text{for } t \geq 0
\]

First we finish the proof of the proposition, then we give the proof of the lemma.

(b) Setting \( y(t) = \kappa p^{-1} \circ \Gamma (t), t \geq 0. \)

We have
\[
y'(t) + p(y(t)) = \kappa (p^{-1} \circ \Gamma)'(t) + p(\kappa p^{-1} \circ \Gamma (t))
\]

Using (1.14) and the fact that
\[
\frac{d}{dt} p^{-1}(\Gamma (t)) + c\Gamma (t) \geq 0
\]
for some \( c > 0, \) we obtain
\[
y'(t) + p(y(t)) \geq (mp(\kappa) - c\kappa) \Gamma (t)
\]

(1.16) gives
\[
y'(t) + p(y(t)) \geq \Gamma (t)
\]

\( y(0) \geq S(0) \)

the result follows from lemma 1.

\[\square\]

The proof of lemma 1 is borrowed from [27].

Proof of lemma 1. Suppose that there exists \( t_0 \) in \([0, +\infty[\), such that
\[
S(t_0) = y(t_0) \quad \text{and } S(t) \geq y(t) \quad \text{on } [t_0, t_0 + \epsilon]
\]

for some \( \epsilon > 0. \) Integrate (1.18) and (1.19) from \( t_0 \) to \( t_0 + \epsilon, \) we obtain
\[
S(t_0 + \epsilon) - S(t_0) + \int_{t_0}^{t_0+\epsilon} p_1(S(t)) dt \leq \int_{t_0}^{t_0+\epsilon} \Gamma (t) dt
\]
and
\[
y(t_0 + \epsilon) - y(t_0) + \int_{t_0}^{t_0+\epsilon} p_2(y(t)) dt \geq \int_{t_0}^{t_0+\epsilon} \Gamma_1 (t) dt
\]
therefore
\[
S(t_0 + \epsilon) + \int_{t_0}^{t_0+\epsilon} p_1(S(t)) dt \leq y(t_0 + \epsilon) + \int_{t_0}^{t_0+\epsilon} p_2(y(t)) dt
\]
which gives
\[
S(t_0 + \epsilon) - y(t_0 + \epsilon) \leq \int_{t_0}^{t_0+\epsilon} p_2(y(t)) - p_1(S(t)) dt
\]
\[
\leq \int_{t_0}^{t_0+\epsilon} p_1(y(t)) - p_1(S(t)) dt \leq 0
\]
which contradict the fact that \( S(t) > y(t) \) on \([t_0, t_0 + \epsilon].\)  

\[\square\]
Setting
\[ \Gamma(t) = 2 \int_M |f(t, x)|^2 \, dx + \psi^* \left( \|f(t, \cdot)\|_{L^2(M)} \right) \]
where \( \psi^* \) is the convex conjugate of the function \( \psi \), defined by
\[ \psi(s) = \begin{cases} \frac{1}{2T} h^{-1} \left( \frac{s^2}{8C_T e^t} \right) & s \in \mathbb{R}_+ \\ +\infty & s \in \mathbb{R}^- \end{cases} \]
and
\[ \psi^*(s) = \sup_{y \in \mathbb{R}} [sy - \varphi(y)] \]

**Superlinear damping:** Assume
\[ g(s) = \begin{cases} s^2 e^{-\frac{s}{2}} & 0 \leq s < 1 \\ -s^2 e^{-\frac{s}{2}} & -1 < s < 0 \end{cases} \]

We choose \( h^0_0(s) = s^{3/2} e^{-\frac{s}{2}}, 0 < s < \eta < 1 \) and
\[ K >> \max \left( E_u(0) + \|\Gamma\|_{L^1(\mathbb{R}_+)} , C_T \right) \]

We have
\[ \psi^* \left( \|f(t, \cdot)\|_{L^2(M)} \right) \leq C \left( \|f(t, \cdot)\|_{L^2(M)} \left| \ln \left( \|f(t, \cdot)\|_{L^2(M)} \right) \right|^{-\frac{1}{2}} + \|f(t, \cdot)\|_{L^2(M)}^2 \right) \]

The ODE (1.5) governing the energy bound reduces to
\[ \frac{dS}{dt} + CS^{3/2} e^{-\frac{1}{s}} \leq \Gamma(t) \]
with \( C > 0 \) depends on \( E_u(0) \). If there are constants \( M > 0 \) and \( \theta > 1 \), such that
\[ \Gamma(t) \leq M (1 + t)^{-\theta} \]
then
\[ E_u(t) \leq \frac{c_0}{\ln (ct + c_1)} , t \geq T \]
with \( c, c_0, c_1 > 0 \). These constants may depend on \( E_u(0) \).

**Sublinear near the origin:** Assume \( g(s) \approx |s|^{1+r_0}, |s| < 1, r_0 \in (0, 1) \). We choose \( h_0(s) = s^{2r_0/(1+r_0)} \) for \( 0 \leq s \leq 1 \) and
\[ K >> \max \left( E_u(0) + \|\Gamma\|_{L^1(\mathbb{R}_+)} , C_T \right) \]

We have
\[ \psi^* \left( \|f(t, \cdot)\|_{L^2(M)} \right) \leq C \left( \|f(t, \cdot)\|_{L^2(M)}^{r_0+1} + \|f(t, \cdot)\|_{L^2(M)}^2 \right) \]

The ODE (1.5) governing the energy bound reduces to
\[ \frac{dS}{dt} + CS^{(1+r_0)/2r_0} \leq \Gamma(t) \]
with \( C > 0 \) depends on \( E_u(0) \).

(1) If there are constants \( M > 0 \) and \( \theta > 1 \), such that
\[ \Gamma(t) \leq M (1 + t)^{-\theta} \]

Then
(a) \( \theta \in \left[1, \frac{1+\tau_0}{1-\tau_0}\right] \).

\[ E_u(t) \leq c \left(1 + t - T\right)^{-\frac{2\tau_0\theta}{1-\tau_0}}, t \geq T \]

where \( c > 0 \).

(b) \( \theta \geq \frac{1+\tau_0}{1-\tau_0} \)

\[ E_u(t) \leq c \left(t - T\right)^{-\frac{2\tau_0}{1-\tau_0}}, t > T \]

with \( c > 0 \) and depends on \( E_u(0) \).

(2) If there are constants \( M > 0 \) and \( \theta > 0 \), such that

\[ \Gamma(t) \leq Me^{-\theta t} \]

Then

\[ E_u(t) \leq c \left(t - T\right)^{-\frac{2\tau_0}{1-\tau_0}}, t > T \]

with \( c > 0 \) and depends on \( E_u(0) \).

2. The linear case: Proof of theorem 1

2.1. Preliminary results.

**Proposition 2.** Let \( u \) be a solution of (1.1) with initial data in the energy space. Then

\[ E_u(t) \leq \left(1 + \frac{1}{\epsilon}\right) e^{\epsilon(t-s)} \left(E_u(s) + \int_s^t \int_M |f(\sigma, x)|^2 \, dx \, d\sigma\right) \]

(2.1)

for every \( \epsilon > 0 \) and for every \( t \geq s \geq 0 \).

**Proof.** Let \( t \geq s \geq 0 \). From the energy identity

\[ E_u(t) \leq E_u(s) + \int_s^t \int_M f \partial_t u \, dx \, d\sigma \]

Using Young’s inequality

\[ E_u(t) \leq E_u(s) + \frac{1}{\epsilon} \int_s^t \int_M |f(\sigma, x)|^2 \, dx \, d\sigma + \epsilon \int_s^t E_u(\sigma) \, d\sigma \]

for every \( \epsilon > 0 \). Now Gronwall’s lemma, gives

\[ E_u(t) \leq e^{\epsilon(t-s)} \left(E_u(s) + \frac{1}{\epsilon} \int_s^t \int_M |f(\sigma, x)|^2 \, dx \, d\sigma\right). \]

The result below is a generalisation of the comparison lemma of Lasiecka and Tataru [15].

**Lemma 2.** Let \( T > 0 \) and

- \( \Gamma \in L_{\text{loc}}^1(\mathbb{R}_+) \) and, non negative. Setting \( \delta(t) = \int_t^{t+T} \Gamma(s) \, ds \).
- \( W(t) \) be a non negative, continuous function for \( t \in \mathbb{R}_+ \). Moreover we assume that there exists a positive, monotone, increasing function \( \alpha \) such that

\[ W(t) \leq \alpha(t-s) \left[ W(s) + \int_s^t \Gamma(\sigma) \, d\sigma\right], \]

for every \( t \geq s \geq 0 \).
Suppose that $\ell$ and $I - \ell : \mathbb{R}_+ \to \mathbb{R}$ are increasing functions with $\ell(0) = 0$ and
\[
W((m + 1)T) + \ell\{W(mT) + \delta(mT)\} \leq W(mT) + \delta(mT) \tag{2.2}
\]
for $m = 0, 1, 2, \ldots$ where $\ell(s)$ does not depend on $m$. Then
\[
W(t) \leq \alpha(T) \left(S(t - T) + 2 \int_{t-T}^{t} \Gamma(s) \, ds\right), \quad \forall t \geq T
\]
where $S(t)$ is a positive solution of the following nonlinear differential equation
\[
\frac{dS}{dt} + \frac{1}{T} \ell(S) = \Gamma(t) ; \quad S(0) = W(0). \tag{2.3}
\]

**Proof.** To prove this result we use induction. Assume that $W(mT) \leq S(mT)$ and prove that $W((m + 1)T) \leq S((m + 1)T)$ where $S(t)$ is the solution of (2.3).

Integrating the equation (2.3) from $mT$ to $(m + 1)T$ yields
\[
S((m + 1)T) = S(mT) - \frac{1}{T} \int_{mT}^{(m+1)T} \ell(S(t)) \, dt + \delta(mT) \tag{2.4}
\]
On the other hand, we have
\[
\frac{d}{dt} \left(S - \int_{0}^{t} \Gamma(s) \, ds\right) = -\frac{1}{T} \ell(S) \leq 0.
\]
therefore, for $t_1 \geq t_2$
\[
S(t_1) \leq S(t_2) + \int_{t_2}^{t_1} \Gamma(s) \, ds
\]
the function $\ell$ is increasing
\[
\ell(S(t)) \leq \ell\left(S(mt) + \int_{mt}^{t} \Gamma(s) \, ds\right), \quad \text{for } mT \leq t \leq (m + 1)T
\]
\[
\leq \ell(S(mt) + \delta(mT))
\]
Using now, (2.4), we obtain
\[
S((m + 1)T) \geq S(mT) + \delta(mT) - \ell(S(mt) + \delta(mT))
\]
Since the function, $I - \ell$ is increasing
\[
S((m + 1)T) \geq W(mT) + \delta(mT) - \ell(W(mt) + \delta(mT))
\]
(2.2), gives
\[
S((m + 1)T) \geq W((m + 1)T).
\]
Setting $t = mT + \tau$, with $0 \leq \tau < T$. Then we obtain
\[
W(t) \leq \alpha(\tau) \left[W(t - \tau) + \int_{t-\tau}^{t} \Gamma(s) \, ds\right]
\]
\[
\leq \alpha(\tau) \left(S(t - \tau) + \int_{t-\tau}^{t} \Gamma(s) \, ds\right)
\]
\[
\leq \alpha(T) \left(S(t - T) + 2 \int_{t-T}^{t} \Gamma(s) \, ds\right), \quad \text{for every } t \geq T.
\]
Proposition 3. We assume that \((\omega, T)\) satisfies the assumption (G). Then there exists \(\hat{C}_T > 0\), such that the following inequality
\[
E_u(t) \leq \hat{C}_T \left[ \int_t^{t+T} \int_M a(x) |\partial_t u|^2 + |f(s,x)|^2 \, dxds \right]
\] (2.5)
holds for every \(t \geq 0\), for every solution \(u\) of \((1.1)\) with initial data in the energy space, a non-negative sequence \((t_n)_n\) and \(f_n\) in \(L^2(\mathbb{R}_+ \times M)\), such that
\[
E_{u_n}(t_n) \geq n \int_{t_n}^{t_n+T} \int_M a(x) |\partial_t u_n|^2 + |f_n(t,x)|^2 \, dxdt,
\] (2.6)
Moreover, \(u_n\) has the following regularity
\[
u_n \in C(\mathbb{R}_+, H^1_0(M)) \cap C^1(\mathbb{R}_+, L^2(M))
\] (2.7)
Setting \(\alpha_n = (E_{u_n}(t_n))^{1/2} > 0\) and \(v_n(t,x) = \frac{u_n(t_n+x)}{\alpha_n}\). Then \(v_n\) satisfies
\[
\begin{cases}
\partial_t^2 v_n - \Delta v_n + a(x) \partial_t v_n = \frac{1}{\alpha_n} f_n(t_n + x, t) & \mathbb{R}_+ \times M \\
v_n = 0 & \mathbb{R}_+ \times \partial M \\
(v_n(0), \partial_t v_n(0)) = \frac{1}{\alpha_n} (u_n(t_n), \partial_t u_n(t_n))
\end{cases}
\] (2.8)
Moreover,
\[E_{v_n}(0) = 1\]
and
\[1 \geq n \int_0^T \int_M a(x) |\partial_t v_n|^2 + \left| \frac{1}{\alpha_n} f_n(t_n + t, x) \right|^2 \, dxdt,
\] (2.9)
From the inequality above, we infer that
\[
\begin{align*}
\int_0^T \int_M a(x) |\partial_t v_n|^2 \, dxdt & \to 0, \\
\int_0^T \int_M \left| \frac{1}{\alpha_n} f_n(t_n + t, x) \right|^2 \, dxdt & \to 0
\end{align*}
\] (2.10)
We have
\[
v_n \in C([0,T], H^1_0(M)) \cap C^1([0,T], L^2(M))
\] (2.11)
Therefore,
\[
E_{v_n}(t) = E_{v_n}(0) - \int_0^T \int_M a(x) |\partial_t v_n|^2 \, dxdt + \int_0^T \int_M \frac{1}{\alpha_n} f_n(t_n + t, x) \partial_t v_n \, dxdt
\] (2.12)
and using \((2.1)\), we infer that
\[
E_{v_n}(t) \leq 2e^T \left( E_{v_n}(0) + \int_0^T \int_M \left| \frac{1}{\alpha_n} f_n(t_n + t, x) \right|^2 \, dxdt \right)
\]
\[
E_{v_n}(t) \leq 2e^T \left( 1 + \frac{1}{n} \right), \text{ for every } t \in [0,T]
\]
This estimate allows one to show that the sequence \((v_n, \partial_t v_n)\) is bounded in \(L^\infty((0, T), \mathcal{H})\) then it admits a subsequence still denoted by \((v_n, \partial_t v_n)\) that converges weakly-* to \((v, \partial_t v)\) in \(L^\infty((0, T), \mathcal{H})\). Passing to the limit in the system satisfied by \(v_n\), we obtain
\[
\begin{aligned}
\partial_t^2 v - \Delta v &= 0 & & [0, T] \times M \\
\partial_t v &= 0 & & [0, T] \times \omega \\
(v(0), \partial_s v(0)) &\in \mathcal{H}
\end{aligned}
\] (2.13)
and the solution \(v\) is in the class
\[C \left( [0, T], H^1_0(M) \right) \cap C^1 \left( [0, T], L^2(M) \right)\]
We deduce as in J. Rauch and M. Taylor \[22\] or C. Bardos, G. Lebeau, J. Rauch \[4\] that the set of such solutions is finite dimensional and admits an eigenvector \(v\) for \(\Delta\). By unique continuation for second order elliptic operator, we get \(\partial_t v = 0\). Multiplying the equation by \(v\) and integrating, we obtain \(v = 0\).

Now we prove that \(v_n \to 0\), in the strong topology of \(H^1_{loc}((0, T), H^1(\omega))\).

We obtain a contradiction by choosing \(\varphi\) such that \(\int_0^T \varphi(t) E_{v_n}(t) \, dt > 0\). □

2.2. Proof of Theorem 1. Let \(u\) be a solution of (1.1) with initial data in the energy space. From the energy identity we have
\[
\int_t^{t+T} \int_M a(x) |\partial_t u|^2 \, dx dt = E_u(t) - E_u(t + T) + \int_t^{t+T} \int_M f(s, x) \partial_t u \, dx ds
\]
therefore, using Young’s inequality
\[ \int_t^{t+T} \int_M a(x) |\partial_t u|^2 \, dx \, dt \leq E_u(t) - E_u(t+T) \]
\[ + \epsilon \int_t^{t+T} \int_M |\partial_t u|^2 \, dx \, ds + \frac{1}{\epsilon} \int_t^{t+T} \int_M |f(s,x)|^2 \, dx \, ds \]
for every \( \epsilon > 0 \). Now using the observability estimate \((2.5)\) and \((2.1)\), we can show that
\[ \int_t^{t+T} \int_M |\partial_t u|^2 \, dx \, ds \leq 2Te^T \hat{C}_T \left[ \int_t^{t+T} \int_M a(x) |\partial_t u|^2 + |f(s,x)|^2 \, dx \, ds \right] \]
(2.15)
Then setting \( \epsilon = \frac{1}{4Te^T \hat{C}_T} \), we infer that
\[ \frac{1}{2} \int_t^{t+T} \int_M a(x) |\partial_t u|^2 \, dx \, dt \leq E_u(t) - E_u(t+T) + \left( 1 + Te^T \hat{C}_T \right) \int_t^{t+T} \int_M |f(s,x)|^2 \, dx \, ds \]
Hence with \( \hat{C}_T = \left( 1 + Te^T \hat{C}_T \right) \), we have
\[ \int_t^{t+T} \int_M a(x) |\partial_t u|^2 \, dx \, ds \leq 2 \left[ E_u(t) - E_u(t+T) + \hat{C}_T \int_t^{t+T} \int_M |f(s,x)|^2 \, dx \, ds \right] \]
Now from the observability estimate \((2.15)\)
\[ E_u(t) \leq 2\hat{C}_T \left[ E_u(t) - E_u(t+T) + \left( \hat{C}_T + 1 \right) \int_t^{t+T} \int_M |f(s,x)|^2 \, dx \, ds \right] \]
with \( \hat{C}_T \geq 1 \). Setting \( C_{1,T} = 2 \left( \hat{C}_T + 1 \right) \). We remark that
\[ 2\hat{C}_T \left[ E_u(t) - E_u(t+T) + C_{1,T} \int_t^{t+T} \int_M |f(s,x)|^2 \, dx \, ds \right] \geq C_{1,T} \int_t^{t+T} \int_M |f(s,x)|^2 \, dx \, ds \]
Then for \( C_T = 4\hat{C}_T \), we have
\[ E_u(t) + C_{1,T} \int_t^{t+T} \int_M |f(s,x)|^2 \, dx \, ds \leq C_T \left[ E_u(t) - E_u(t+T) \right] \]
Therefore
\[ E_u(t+T) + \frac{1}{C_T} \left[ E_u(t) + C_{1,T} \int_t^{t+T} \int_M |f(s,x)|^2 \, dx \, ds \right] \]
\[ \leq E_u(t) + C_{1,T} \int_t^{t+T} \int_M |f(s,x)|^2 \, dx \, ds \]
Setting \( t = mT \), with \( m \in \mathbb{N} \) and using Lemma 2.4, we conclude that
\[ E_u(t) \leq 4e^T \left( S(t-T) + \int_{t-T}^{t} \Gamma(s) \, ds \right), \quad \forall t \geq T \]
where \( S(t) \) is a positive solution of the following nonlinear differential equation
\[ \frac{dS}{dt} + \frac{1}{C_T T} S = \Gamma(t); \quad S(0) = E_u(0) \]  
(2.16)
and
\[ \Gamma (s) = C_{1,T} \int_{M} |f (s,x)|^2 \, dx \]

3. The nonlinear case: Proof of theorem \( 2 \)

This part is devoted to the proof of theorem \( 2 \). First we give the following energy inequality.

**Proposition 4.** Let \( u \) be a solution of (1.7) with initial data in the energy space. Then the following inequality
\[
E_{u} (t) \leq \left( 1 + \frac{1}{\epsilon} \right) e^{\epsilon(t-s)} \left( E_{u} (s) + \int_{s}^{t} \int_{M} |f (\sigma, x)|^2 \, dx \, d\sigma \right)
\]
holds for every \( \epsilon > 0 \) and for every \( t \geq s \geq 0 \).

For the proof of (3.1), we have only to proceed as in the proof of (2.1). Now we give the proof of theorem \( 2 \).

**Proof of Theorem \( 2 \).** Let \( u \) be the solution of (1.7) with initial condition \((u_0, u_1)\) in the energy space \( \mathcal{H} \). Let \( t \geq 0 \) and \( \phi = u (t + \cdot) \) be the solution of
\[
\left\{ \begin{array}{ll}
\partial_s^2 \phi - \Delta \phi + a (x) g (\partial_s \phi) = f (t + s, x) & \mathbb{R}_+ \times \Omega \\
\phi = 0 & \mathbb{R}_+ \times \partial \Omega \\
(\phi (0), \partial_s \phi (0)) = (u (t), \partial_t u (t)) & \end{array} \right.
\]
We argue as in [9]. Define \( z = \phi - v \), where \( v \) is the solution of (1.1) with initial data \((u (t), \partial_t u (t))\) and \( f = f (t + \cdot, \cdot) \). Then \( z \) satisfies the system
\[
\left\{ \begin{array}{ll}
\partial_t^2 z - \Delta z + a (x) g (\partial_t \phi) - a(x) \partial_t v = 0 & \mathbb{R}_+ \times \Omega \\
z = 0 & \mathbb{R}_+ \times \partial \Omega \\
(z (0), \partial_t z (0)) = 0 & \end{array} \right.
\]
Let \( T > 0 \), such that \((\omega, T)\) satisfies the assumption \((G)\). It is clear that \( a (x) (g (\partial_t \phi) - \partial_t v) \in L^2 ((0, T) \times \Omega) \). This observation permits us to apply energy identity, whence
\[
E_z (T) = \int_{0}^{T} \int_{M} a (x) \left( \partial_t v - g (\partial_t \phi) \right) \partial_t z \, dx \, dt
\]
\[
= - \int_{0}^{T} \int_{M} a (x) \left( |\partial_t v|^2 + g (\partial_t \phi) \partial_t \phi \right) \, dx \, dt + \int_{0}^{T} \int_{M} g (\partial_t \phi) \partial_t v + \partial_t v \partial_t \phi \, dm_a
\]
The monotonicity of \( g (g (s) s \geq 0) \) and the estimate above, gives the following estimate:
\[
\int_{0}^{T} \int_{M} a (x) \left( |\partial_t v|^2 + g (\partial_t \phi) \partial_t \phi \right) \, dx \, dt \leq \int_{0}^{T} \int_{M} g (\partial_t \phi) \partial_t v + \partial_t v \partial_t \phi \, dm_a
\]
Now the observability estimate (2.5), gives
\[
E_u (t) = E_v (0) \leq \hat{C}_T \left( \int_{0}^{T} \int_{M} g (\partial_t \phi) \partial_t v + \partial_t v \partial_t \phi \, dm_a + \int_{0}^{T} \int_{M} |f (t + s, x)|^2 \, dx \, ds \right)
\]
for some \( \hat{C}_T \geq 1 \). From the estimate above we infer that
\[
E_u (t) \leq \hat{C}_T \left( \int_{t}^{t+T} \int_{M} g (\partial_t u) \partial_t \tilde{v} + \partial_t \tilde{v} \partial_t u \, dm_a + \int_{t}^{t+T} \int_{M} |f (s, x)|^2 \, dx \, ds \right)
\]
where \( \tilde{v} (s) = v (s - t), \, s \geq t \geq 0. \)
Lemma 3. Setting 
\[ M_{s,t} = [s, t] \times \Omega, \ t \geq s \geq 0 \] and \( M_{0,t} = M_t \)

Let \( t \geq 0 \). For \( i = 0, 1 \) let
\[ M_t^0 = \{(s, x) \in [t, t + T] \times \Omega; \ |\partial_s u(s, x)| < \eta_0\}, \quad M_t^1 = M_{t,t+T} \setminus M_t^0 \]
and define
\[ \Theta(M_{t,t+T}) = \int_{M_t} |\partial_s u(s) \partial_s v(s-t)| \, dm_a, \quad \Psi(M_t^i) = \int_{M_t^i} |g(\partial_s u(s)) \partial_s v(s-t)| \, dm_a, \]

where \( u \) and \( v \) denotes respectively the solution of (1.7) and (1.1) with initial data \((u_0, u_1)\) and \((v_0, v_1) = u(t), \partial_t u(t)\).

(1) The following inequality holds for every \( \epsilon > 0 \),
\[ \Psi(M_t^0) + \Theta(M_t^0) \leq \epsilon E_u(t) + \frac{C}{\epsilon} m_a(M_T) h_0 \left( \frac{1}{m_a(M_T)} \int_{M_{t,t+T}} g(\partial_s u) \partial_s u \, dm_a \right) \]
+ \( \epsilon \int_t^{t+T} \int_M |f(s, x)|^2 \, dx \, ds \) \hspace{1cm} (3.3)

with \( C > 0 \).

(2) Estimate on the damping near infinity. The following inequality
\[ \Psi(M_t^1) + \Theta(M_t^1) \leq \epsilon E_u(t) + C \epsilon^{-1} \left( \int_{M_{t,t+T}} g(\partial_s u) \partial_s u \, dm_a \right) \]
+ \( \epsilon \int_t^{t+T} \int_M |f(s, x)|^2 \, dx \, ds \) \hspace{1cm} (3.5)

holds for every \( \epsilon > 0 \) with \( C > 0 \).

For the proof of the lemma above we have only to proceed as in [9, Lemma 3.1 cases 1 and 2] and to use the energy inequality (2.1).

Now using (3.3) and (3.5), we deduce that
\[ E_u(t) \leq \tilde{C}_T \left( \epsilon E_u(t) + C_{T,h} \left( \int_{M_{t,t+T}} g(\partial_s u) \partial_s u \, dm_a + \int_{M_{t,t+T}} |f(s, x)|^2 \, dx \, ds \right) \right) \]

for every \( \epsilon > 0 \), where the function \( h = I + m_a(M_T) h_0 \circ \frac{1}{m_a(M_T)} \) and \( \tilde{C}_T \geq 1 \). Setting \( \epsilon \) small enough, e.g. \( \epsilon = \frac{1}{2T} \)
\[ E_u(t) \leq C_{T,h} \left( \int_t^{t+T} \int_M g(\partial_s u) \partial_s u \, dm_a + \int_t^{t+T} \int_M |f(s, x)|^2 \, dx \, ds \right) \] \hspace{1cm} (3.7)

for some \( C_T \geq 1 \). This gives
\[ E_u(t) + \int_t^{t+T} \int_M |f(s, x)|^2 \, dx \, ds \leq 2C_{T,h} \left( \int_t^{t+T} \int_M g(\partial_s u) \partial_s u \, dm_a + \int_t^{t+T} \int_M |f(s, x)|^2 \, dx \, ds \right) \] \hspace{1cm} (3.8)

On the other hand, the energy identity gives
\[ \int_t^{t+T} \int_M a(x) g(\partial_t u) \partial_t u \, dx \, ds \leq E_u(t) - E_u(t + T) + \int_t^{t+T} \int_M |f(\sigma, x) \partial_t u| \, dx \, d\sigma \] \hspace{1cm} (3.9)
Let $\psi$, defined by

$$\psi(s) = \begin{cases} \frac{1}{2T}h^{-1} \left( \frac{s^2}{8C_T} \right) & s \in \mathbb{R}_+ \\ +\infty & s \in \mathbb{R}_- \end{cases}$$

It is clear that $\psi$ convex and proper function. Hence, we can apply Young’s inequality [23]

$$\int_t^{t+T} \int_M |f(\sigma,x) \partial_t u| \, dx \, d\sigma \leq \int_t^{t+T} \|f(\sigma,\cdot)\|_{L^2} \|\partial_t u(\sigma,\cdot)\|_{L^2} \, d\sigma$$

$$\leq \int_t^{t+T} \psi^*(\|f(\sigma,\cdot)\|_{L^2}) + \psi(\|\partial_t u(\sigma,\cdot)\|_{L^2}) \, d\sigma$$

where $\psi^*$ is the convex conjugate of the function $\psi$, defined by

$$\psi^*(s) = \sup_{y \in \mathbb{R}}[sy - \varphi(y)]$$

Using the energy inequality (3.1) and the observability estimate (3.8), we infer that

$$\int_t^{t+T} \psi(\|\partial_t u(\sigma,\cdot)\|_{L^2}) \, d\sigma \leq \frac{1}{2} \left( \int_t^{t+T} \int_M g(\partial_s u) \partial_s u dm_a + \int_t^{t+T} \int_M |f(s,x)|^2 \, dx \, ds \right)$$

then (3.9), gives

$$\int_t^{t+T} \int_M a(x) g(\partial_t u) \partial_t u dx \, ds \leq 2 \left( E_u(t) - E_u(t + T) + \int_t^{t+T} \int_M |f(s,x)|^2 \, dx \, ds \right) + \int_t^{t+T} \psi^*(\|f(\sigma,\cdot)\|_{L^2}) \, d\sigma \quad (3.10)$$

The inequality above combined with the observability estimate (3.7) and the fact $h = I + m_m(M_T) h_0 \circ \frac{I}{m_m(M_T)}$ is increasing, gives

$$E_u(t) \leq C_T h \left( 4 \left( E_u(t) - E_u(t + T) + \int_t^{t+T} \int_M |f(s,x)|^2 \, dx \, ds + \int_t^{t+T} \psi^*(\|f(\sigma,\cdot)\|_{L^2}) \, d\sigma \right) \right)$$

Setting

$$\Gamma(s) = 2 \int_M |f(s,x)|^2 \, dx + \psi^*(\|f(s,\cdot)\|_{L^2})$$

Therefore

$$E_u(t) + \int_t^{t+T} \Gamma(s) \, ds \leq K h \left( 4 \left( E_u(t) - E_u(t + T) + \int_t^{t+T} \Gamma(s) \, dx \, ds \right) \right)$$

with $K \geq C_T$. Setting $\theta(t) = \int_t^{t+T} \Gamma(s) \, ds$. Thus

$$E_u(t + T) + \frac{1}{4} h^{-1} \left( \frac{1}{K} (E_u(t) + \theta(t)) \right) \leq E_u(t) + \theta(t)$$

for every $t \geq 0$. Take $t = mt$, $m \in \mathbb{N}$

$$E_u((m+1)T) + \frac{1}{4} h^{-1} \left( \frac{1}{K} (E_u(mT) + \theta(mT)) \right) \leq E_u(mT) + \theta(mT)$$
Setting $W(t) = E_u(t), \ell(s) = \frac{1}{2}b^{-1} \circ \frac{d}{ds}$ and $\Gamma(t) = 2 \int_M |f(s, x)|^2 \, dx + \psi^* (\|f(s, \cdot)\|_{L^2})$. It is clear that the functions $\ell$ and $I - \ell$ are increasing on the positive axis and $\ell(0) = 0$. The function $\Gamma \in L^1_{loc}(\mathbb{R}^+)$ and non-negative on $\mathbb{R}^+$. According to lemma 2

$$E_u(t) \leq 4e^{T} \left( S(t - T) + \int_{t-T}^{t} \Gamma(s) \, ds \right), \quad \forall t \geq T$$

where $S(t)$ is the solution of the following nonlinear differential equation

$$\frac{dS}{dt} + \frac{1}{T} \ell(S) = \Gamma(t) \quad S(0) = W(0). \quad (3.12)$$

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