STANDING WAVES FOR QUASILINEAR SCHRÖDINGER EQUATIONS WITH INDEFINITE POTENTIALS

SHIBO LIU AND JIAN ZHOU
School of Mathematical Sciences, Xiamen University, Xiamen 361005, China

Abstract. We consider quasilinear Schrödinger equations in $\mathbb{R}^N$ of the form

$$-\Delta u + V(x)u - u\Delta(u^2) = g(u),$$

where $g(u)$ is 4-superlinear. Unlike all known results in the literature, the Schrödinger operator $-\Delta + V$ is allowed to be indefinite, hence the variational functional does not satisfy the mountain pass geometry. By a local linking argument and Morse theory, we obtain a nontrivial solution for the problem. In case that $g$ is odd, we get an unbounded sequence of solutions.

1. Introduction

In this paper we consider quasilinear stationary Schrödinger equations in $\mathbb{R}^N$ of the form

$$-\Delta u + V(x)u - u\Delta(u^2) = g(u).$$

This kind of equations arise when we are looking for standing waves $\psi(t, x) = e^{-i\omega t}u(x)$ for the time-dependent quasilinear Schrödinger equation

$$i\partial_t \psi = -\Delta \psi + U(x)\psi - \psi\Delta(|\psi|^2) - \bar{g}(|\psi|^2)\psi, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N,$$

which was used for the superfluid film equation in plasma physics by Kurihar [17]. As mentioned by Ruiz-Siciliano [32], it is also introduced in [4, 5, 15] to study a model of self-trapped electrons in quadratic or hexagonal lattices. Note that the potential $V$ in (1.1) is given by

$$V(x) = U(x) - \omega.$$ 

Thus, if the frequency $\omega$ is large, $V$ may be indefinite in sign. For physical reason, it is natural to make the following assumption on the nonlinearity

$$\lim_{t \to 0} \frac{g(t)}{t} = 0.$$ 

To the best of our knowledge, for nonlinearity of the form $g(u) = |u|^{p-2}u$, the first mathematical studies of the equation (1.1) seems to be Poppenberg et. al. [30] for the one dimensional case and Liu-Wang [19] for higher dimensional case. The proofs in these papers are based on constrained minimization argument.

To consider more general nonlinearities, we should seek for a free variational formulation for the problem. Formally, the solutions of the problem (1.1) should be critical points of the following functional

$$J(u) = \frac{1}{2} \int \left(1 + 2u^2\right)|\nabla u|^2 + \frac{1}{2} \int V(x)u^2 - \int G(u), \quad \text{where } G(t) = \int_0^t g(\tau)d\tau.$$ 

Unfortunately, the functional $J$ could not be defined for every $u \in H^1(\mathbb{R}^N)$. Therefore the standard variational methods could not be applied. To overcome this difficulty, Liu et. al. [22] and Colin-Jeanjean [10] introduced a nonlinear transformation $f$ so that if $v \in H^1(\mathbb{R}^N)$ is a critical point of $\Phi : H^1(\mathbb{R}^N) \to \mathbb{R}$,

$$\Phi(v) = \frac{1}{2} \int |\nabla v|^2 + \frac{1}{2} \int V(x)f^2(v) - \int G(f(v)),$$

This work was supported by NSFC (11671331) and NSFFJ (2014J06002).
then \( u = f(v) \) is a solution of (1.1).

Since the publication of [10, 22], many results about (1.1) have been obtained by various authors using this transformation. For example, the case that the potential \( V \) is \( \mathbb{Z}^N \)-periodic (or asymptotically periodic) and \( g \) is 4-superlinear is studied in Silva-Vieira [34], Yang [38], Fang-Szulkin [13] and Zhang et. al. [39]. In [37], Wu studied the case that the nonlinearity is 4-superlinear and the potential has a positive lower bound and satisfies the condition (V) below. In Furtado et. al. [14], the asymptotically linear case (the nonlinearity behaves like \( t \) at the origin and like \( t^3 \) at infinity) is investigated, where the potential \( V \) satisfies the same conditions as in [37]. For equations with concave and convex nonlinearities, one can consult do Ó and Severo [11]. For problems with critical nonlinearities, see Silva-Vieira [33] and Wang-Zou [35]. For the supercritical case, we refer the reader to Moameni [29] and Miyagaki-Moreira [28].

In all these papers, it is required that the potential \( V \) satisfies the positive condition

\[
\alpha := \inf_{\mathbb{R}^N} V > 0. \tag{1.6}
\]

With this condition and suitable conditions on the nonlinearity, one can show that the zero function \( \nu = 0 \) is a local minimizer of the functional \( \Phi \), and \( \Phi \) would then verify the mountain pass geometry and the mountain pass theorem can be applied to produce a solution. However, from (1.3) we can see that, if we want to find standing waves \( \psi(t, x) = e^{-i\omega t} u(x) \) of (1.2) with large \( \omega \), then the potential \( V \) in (1.1) could not satisfy (1.6).

In the literature there are some existence results which allow the potential \( V \) to be negative somewhere. The strategy is to write \( V = V^+ - V^- \) with \( V^\pm = \max\{0, \pm V\} \). Then if \( V^- \) is in some sense small, it can be absorbed and the functional \( \Phi \) still verifies the mountain pass geometry. We refer the reader to Fang-Han [12] and Maia et. al. [26] for this kind of results.

In Zhang et. al. [40], the authors studied the problem (1.1) with sign-changing potential. Their potential satisfies conditions slightly general than our condition (V) below. Because \( V \) is sign-changing, the function

\[
u \mapsto \left( \int (|\nabla u|^2 + V(x)u^2) \right)^{1/2}
\]

is no longer a norm on the function space. This will bring some problems for verifying the boundedness of asymptotically critical sequences. To overcome this difficulty, they chose a constant \( V_0 > 0 \) such that

\[
\tilde{V}(x) = V(x) + V_0 > 0
\]

and consider the equivalent problem

\[-\Delta u + \tilde{V}(x) u - u\Delta(|u|^2) = \tilde{g}(x, u), \quad x \in \mathbb{R}^N,\]

where \( \tilde{g}(x, u) = g(x, u) + V_0 u \). Unfortunately, from

\[\mathcal{G}(x, u) := \frac{1}{4} g(x, u) u - G(x, u) \geq 0\]

in their condition (\( G_2 \)), we could not deduce

\[\mathcal{G}(x, u) := \frac{1}{4} \tilde{g}(x, u) u - \tilde{G}(x, u) = \mathcal{G}(x, u) - \frac{1}{4} V_0 u^2 \geq 0.\]

Therefore unlike what the authors declared at the beginning of [40, §2], this new nonlinearity \( \tilde{g}(x, u) \) does not satisfy their condition (\( G_2 \)) any more. Because \( \mathcal{G}(x, u) \geq 0 \) is crucial to get

\[\text{meas}(\Omega_n(r, +\infty)) \to 0\]

in [40, Page 1768], their result can only be valid for the case that \( V \) is positive.

In conclusion, for the quasilinear Schrödinger equation (1.1), as far as we know, up to now in the literature there is no research devoted to the situation that the zero function \( \nu = 0 \) fails to be a local minimizer of \( \Phi \).
The purpose of this paper is to present the first results in this indefinite situation. Firstly, we present our assumptions on the potential \( V(x) \) and the nonlinearity \( g(u) \).

\((V)\) \( V \in C(\mathbb{R}^N) \) is bounded from below and, \( m(V^{-1}(-\infty, M]] < \infty \) for all \( M > 0 \), where \( m \) is the Lebesgue measure on \( \mathbb{R}^N \).

\((g_0)\) \( g \in C(\mathbb{R}) \) and there exist \( C > 0 \) and \( p \in (4, 2^*) \) such that

\[
|g(t)| \leq C (|t| + |t|^{p-1}),
\]

where the critical Sobolev exponent \( 2^* = 2N/(N - 2) \) for \( N \geq 3 \) and \( 2^* = \infty \) for \( N = 2 \).

\((g_1)\) there exists \( \mu > 4 \) such that for \( t \neq 0 \) there holds

\[
0 < \mu G(t) \leq tg(t).
\]

As we shall see, for problem (1.1) with indefinite potential \( V \), there are several interesting features. Firstly, it is well known that the mountain pass theorem is not applicable in this situation. For semilinear problems, we may try to apply the linking theorem. But for our quasilinear problem, because

\[
Q(v) = \frac{1}{2} \int |\nabla v|^2 + \frac{1}{2} \int V(x)f^2(v),
\]

the principle part of \( \Phi \), is not a quadratic form on \( v \), the linking theorem is also not applicable. The reason is that for applying the linking theorem, one needs to decompose the function space according to certain quadratic form, while in our functional \( \Phi \), there is no natural quadratic form. Even if we decompose the space according to some quadratic forms like \( \mathfrak{B} \) given by

\[
\mathfrak{B}(u) = \frac{1}{2} \int \left( |\nabla u|^2 + V(x)u^2 \right),
\]

the non-quadratic feature of \( Q \) will also prevent us to verify the linking geometry. Our key observation is that, just like our previous study [7] on the Schrödinger-Poisson systems

\[
\begin{cases}
-\Delta u + V(x)u + \phi u = g(u) \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi = u^2 \quad \text{in } \mathbb{R}^3,
\end{cases}
\]

the functional \( \Phi \) has a local linking [18, 21] at the origin with respect to the decomposition \( X = X^- \oplus X^+ \), where \( X \) is our working space that will introduced later, \( X^- \) and \( X^+ \) are the negative and positive spaces of the quadratic form \( \mathfrak{B} \), respectively.

Secondly, for indefinite semilinear Schrödinger equations

\[
-\Delta u + V(x)u = g(u) \quad \text{in } \mathbb{R}^N
\]

and the Schrödinger-Poisson systems (1.8), the principle part of the variational functional is the quadratic form \( \mathfrak{B} \). Therefore, to obtain the boundedness of asymptotically critical sequences \( \{u_n\} \), one usually needs to decompose \( u_n = u^-_n + u^+_n \), where \( u^\pm_n \) is the orthogonal projection of \( u_n \) on \( X^\pm \), see e.g., [16, 24, 25]. In our quasilinear case, again, the non-quadratic feature of \( Q \) makes such decomposition useless for verifying the boundedness of the sequences.

Before state our results, we shall introduce the suitable function space in which we will find critical points of \( \Phi \). Since \( V \) is bounded from below, as in [7, Page 45] we choose \( m > 0 \) such that

\[
\bar{V}(x) = V(x) + m > 1
\]

for all \( x \in \mathbb{R}^N \). On the subspace

\[
X = \left\{ u \in H^1(\mathbb{R}^N) \left| \int V(x)u^2 < \infty \right. \right\}
\]

we equip the inner product

\[
\langle u, v \rangle = \int (\nabla u \cdot \nabla v + \bar{V}(x)uv)
\]
and corresponding norm \(\|u\| = \langle u, u \rangle^{1/2}\). Then \(X\) is a Hilbert space and by Bartsch-Wang [3] we have a compact embedding \(X \hookrightarrow L^s(\mathbb{R}^N)\) for \(s \in [2, 2^\ast)\).

By the compactness of the embedding \(X \hookrightarrow L^2(\mathbb{R}^N)\), applying the spectral theory of self-adjoint compact operators, we see that the eigenvalue problem

\[
-\Delta u + V(x)u = \lambda u, \quad u \in X
\]

possesses a complete sequence of eigenvalues

\[
-\infty < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_i \to +\infty.
\]

Each \(\lambda_i\) has been repeated in the sequence according to its finite multiplicity. We denote by \(\phi_i\) the eigenfunction of \(\lambda_i\) with \(|\phi_i|_2 = 1\), where \(|\cdot|_q\) is the \(L^q\)-norm. Now we can state our main results.

**Theorem 1.1.** Suppose that \((V), (g_0), (g_1)\) and (1.4) hold. If 0 is not an eigenvalue of (1.10), then (1.1) has a nontrivial solution.

**Theorem 1.2.** Suppose that \((V), (g_0)\) and (1.1) hold. If \(g\) is odd, then (1.1) has a sequence of solutions \(\{u_n\}\) such that \(J(u_n) \to +\infty\).

**Remark 1.3.** Note that in Theorem 1.2 we don’t need the local condition (1.4).

The paper is organized as follows. In Section 2 we recall the transformation \(f\) introduced in [10], which converts the problem of solving (1.1) into searching for critical points of the functional \(\Phi\) given in (1.5). Then, as the first step of dealing with \(\Phi\) we show that it satisfies the Cerami condition. We will prove Theorem 1.1 by applying Morse theory. Therefore in Section 3 we recall some necessary concepts and results in Morse theory. Then we verify the local linking property for \(\Phi\) and compute the critical groups of \(\Phi\) at infinity. With these preparation we give the proof of Theorem 1.1. Finally, in Section 4, we apply the symmetric mountain pass theorem of Ambrosetti-Rabinowitz [1] to prove Theorem 1.2.

### 2. Variational Reformulation and Cerami Condition

Following Colin-Jeanjean [10], we make the change of variables by \(u = f(v)\), where \(f\) is an odd function defined by

\[
\hat{f}(t) = \frac{1}{\sqrt{1 + 2f^2(t)}}, \quad f(0) = 0
\]

on \([0, +\infty)\). Then we have the following proposition, whose proof can be found in [9, 10].

**Proposition 2.1.** The function \(f\) satisfies the following properties:

1. \(f \in C^\infty(\mathbb{R})\) is strictly increasing, therefore is invertible.
2. \(|f(t)| \leq t\) and \(|\hat{f}(t)| \leq 1\) for all \(t \in \mathbb{R}\). Moreover,

\[
\hat{f}(0) = \lim_{t \to 0} \frac{\hat{f}(t)}{t} = 1.
\]

3. for all \(t > 0\) we have

\[
\frac{1}{2}f(t) \leq \hat{f}(t)t \leq f(t).
\]

4. for all \(t \in \mathbb{R}\) we have \(f^2(t) \geq f(t)\hat{f}(t)t\) and \(|f(t)| \leq 2^{1/4}\sqrt{\lambda}|t|^{1/2}.

5. there exists a positive constant \(\kappa\) such that

\[
|f(t)| \geq \kappa|t| \quad \text{for } |t| \leq 1, \quad |f(t)| \geq \kappa|t|^{1/2} \quad \text{for } |t| \geq 1.
\]

6. for each \(\lambda > 0\), there exists a positive constant \(C_\lambda\) such that \(f^2(\lambda t) \leq C_\lambda f^2(t)\).

By the growth condition of the nonlinearity \(g\) and the properties of the transformation \(f\), it is easy to verify that the functional

\[
\Phi(v) = J(f(v)) = \frac{1}{2} \int |\nabla v|^2 + \frac{1}{2} \int V(x)f^2(v) - \int G(f(v))
\]
is well defined and of class $C^1$ on the Sobolev space $X$ introduced in Section 1, and if $v$ is a critical point of $\Phi$, then $u = f(v)$ is a solution of our problem (1.1), see [10] for the details.

Hence, to prove our main results, we shall look for critical points of the functional $\Phi$. Firstly, we need to show that the functional $\Phi$ satisfies the Cerami condition. To this end, we set

$$\tilde{g}(t) = g(t) + mt, \quad \tilde{G}(t) = \int_0^t \tilde{g}(\tau)d\tau = G(t) + \frac{m}{2}t^2$$

and rewrite $\Phi$ in the following form

$$\Phi(v) = \frac{1}{2} \int |\nabla v|^2 + \frac{1}{2} \int \tilde{V}(x)f^2(v) - \int \tilde{G}(f(v)),$$

where $\tilde{V}$ is given in (1.9). Note that by $(g_1)$, the new nonlinearity $\tilde{g}$ satisfies

$$\tilde{G}(t) - \frac{1}{\mu} \tilde{g}(t)t \leq \left(\frac{1}{2} - \frac{1}{\mu}\right)mt^2. \quad (2.1)$$

**Lemma 2.2.** Under the assumptions $(V)$, $(g_0)$ and $(g_1)$, the functional $\Phi$ satisfies the Cerami condition.

*Proof.* Let $\{v_n\}$ be a Cerami sequence of $\Phi$, that is,

$$\Phi(v_n) \to c, \quad (1 + ||v_n||)\Phi'(v_n) \to 0$$

for some $c \in \mathbb{R}$. We claim that there exists $C > 0$ such that

$$\rho_n := \left\{ \int \left(|\nabla v_n|^2 + \tilde{V}(x)f^2(v_n)\right) \right\}^{1/2} \leq C. \quad (2.2)$$

If this is not true, we may assume $\rho_n \to +\infty$. Consider the sequence

$$h_n = \frac{f(v_n)}{\rho_n}.$$

Then since $|\dot{f}(t)| \leq 1$, we have

$$||h_n||^2 = \frac{1}{\rho_n^2} \int \left(|\nabla f(v_n)|^2 + \tilde{V}(x)f^2(v_n)\right)$$

$$= \frac{1}{\rho_n^2} \int \left(|\dot{f}(v_n)|^2|\nabla v_n|^2 + \tilde{V}(x)f^2(v_n)\right)$$

$$\leq \frac{1}{\rho_n^2} \int \left(|\nabla v_n|^2 + \tilde{V}(x)f^2(v_n)\right) = 1. \quad (2.3)$$

Consequently, $\{h_n\}$ is bounded in $X$. Up to a subsequence, by the compactness of the embedding $X \hookrightarrow L^2(\mathbb{R}^N)$, we may assume that

$$h_n \to h \text{ in } X, \quad h_n \to h \text{ in } L^2, \quad h_n \to h \text{ a.e. in } \mathbb{R}^N.$$

As in Colin-Jeanjean [10, Page 225], set

$$\phi_n = \sqrt{1 + 2f^2(v_n)}f(v_n).$$

Then using Proposition 2.1 (2) and (4) we have

$$||\nabla \phi_n|| = \left(1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)}\right)||\nabla v_n|| \leq 2||\nabla v_n||,$$

$$||\phi_n||^2 = \int \left(|\nabla \phi_n|^2 + \tilde{V}(x)\phi_n^2\right)$$

$$\leq 4 \int |\nabla v_n|^2 + \int \tilde{V}(x)f^2(v_n) + 2\int \tilde{V}(x)f^4(v_n)$$

$$\to 0$$

as $n \to \infty$. This proves Lemma 2.2.
\[ \leq 4 \int |\nabla v_n|^2 + \int \tilde{V}(x)v_n^2 + 2 \int \tilde{V}(x)(2v_n^2) \leq 5\|v_n\|^2. \]

Therefore, by (2.1), we have
\[
c + o(1) = \Phi(v_n) - \frac{1}{\mu} \Phi'(v_n), \phi_n) \leq \frac{1}{2} \int (|\nabla v_n|^2 + \tilde{V}(x)f^2(v_n)) - \int \tilde{G}(f(v_n)) - \frac{1}{\mu} \left\{ \left( \left( 1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)} \right) |\nabla v_n|^2 + \tilde{V}(x)f^2(v_n) \right) - \int \tilde{G}(f(v_n))f(v_n) \right\}
= \left( \frac{1}{2} - \frac{1}{\mu} \right) \int (|\nabla v_n|^2 + \tilde{V}(x)f^2(v_n)) - \frac{1}{\mu} \int mf^2(v_n). \]

Multiplying both sides by \( \rho_n^{-2} \), we deduce
\[
\left( \frac{1}{2} - \frac{1}{\mu} \right) m|h_n|^2 = \frac{m}{\rho_n^2} \left( \frac{1}{2} - \frac{1}{\mu} \right) \int f^2(v_n) \geq \frac{1}{2} - \frac{2}{\mu}. \]

Since \( h_n \to h \) in \( L^2 \) and \( \mu > 4 \), it follows that \( h \neq 0 \), the zero function. Hence the set\[
\Theta = \{ x \in \mathbb{R}^N | h(x) \neq 0 \}\]
is of positive Lebesgue measure.

By our assumption (g1), it is well known that\[
\frac{\tilde{G}(t)}{t^2} = \frac{1}{t^2} \left( G(t) + \frac{1}{2}mt^2 \right) \to +\infty \]
as \( |t| \to \infty \). For \( x \in \Theta \), we have \( h_n(x) \to h(x) \neq 0 \) and\[
|f(v_n(x))| = \rho_n|h_n(x)| \to \infty, \quad \frac{\tilde{G}(f(v_n(x)))}{f^2(v_n(x))} h_n^2(x) \to +\infty. \]

By the Fatou lemma and noting that \( \tilde{G}(t) \geq 0 \), we deduce\[
\int \frac{\tilde{G}(f(v_n))}{\rho_n^2} \geq \int \frac{\tilde{G}(f(v_n))}{f^2(v_n)} h_n^2 \to +\infty. \quad (2.4) \]

Therefore for large \( n \) we have\[
c - 1 \leq \Phi(v_n) = \frac{1}{2} \int (|\nabla v_n|^2 + \tilde{V}(x)f^2(v_n)) - \int \tilde{G}(f(v_n)) \]
\[
\leq \rho_n^2 \left( \frac{1}{2} - \frac{1}{\mu} \right) \int \frac{\tilde{G}(f(v_n))}{\rho_n^2} \to -\infty. \]

This is impossible. Therefore, our claim (2.2) is true.

Next, we can follow the same argument as Wu [37, Page 2626–2628] (for the special case \( \alpha = 1 \), see Remark 2.3 below) to show that \( \{v_n\} \) is bounded in \( X \), and has a convergent subsequence.

More precisely, using Proposition 2.1 (5) and (6) we can show that there exists a constant \( C > 0 \) which may be depend on the sequence \( \{v_n\} \), such that\[
\rho_n^2 \geq C\|v_n\|^2, \]
this and the boundedness of \( \{ \rho_n \} \) imply that \( \{ v_n \} \) is bounded. By the compactness of the embedding \( X \hookrightarrow L^2(\mathbb{R}^N) \), up to a subsequence we may assume
\[
v_n \to v \quad \text{in} \quad X, \quad v_n \to v \quad \text{in} \quad L^1(\mathbb{R}^N)
\]
for \( s \in [2, 2^*) \). By the growth condition (1.7) and the properties of \( f \) described in Proposition 2.1, we deduce
\[
\int \left[ \tilde{g}(f(v_n)) \hat{f}(v_n) - \tilde{g}(f(v)) \hat{f}(v) \right] (v_n - v) \to 0.
\]
As in the proof of [13, Lemma 3.11] we can prove that there is a constant \( C > 0 \) such that
\[
\int |\nabla (v_n - v)|^2 + \int \tilde{V}(x) \left[ f(v_n) \hat{f}(v_n) - f(v) \hat{f}(v) \right] (v_n - v) \geq C \| v_n - v \|^2.
\]
Consequently,
\[
C \| v_n - v \|^2 \leq \int |\nabla (v_n - v)|^2 + \int \tilde{V}(x) \left[ f(v_n) \hat{f}(v_n) - f(v) \hat{f}(v) \right] (v_n - v)
\]
\[
- \int \left[ \tilde{g}(f(v_n)) \hat{f}(v_n) - \tilde{g}(f(v)) \hat{f}(v) \right] (v_n - v) + o(1)
\]
\[
= (\Phi'(v_n) - \Phi'(v), v_n - v) + o(1).
\]
Since \( \{ v_n \} \) is a bounded Cerami sequence and \( v_n \to v \), the right hand side goes to zero as \( n \to \infty \), and we deduce \( v_n \to v \) in \( X \).

Remark 2.3. In [37], Wu considered slightly general equations of the form
\[
-\Delta u + V(x)u - |u|^{2\alpha - 2}u\Delta(\sqrt{\alpha}u) = g(x,u).
\]
Obviously, if \( \alpha = 1 \) this equation reduces to our (1.1). Of course, he only studied the positive definite case \( \text{inf}_{\mathbb{R}^N} V > 0 \).

3. Proof of Theorem 1.1

To prove Theorem 1.1, we will apply the infinite dimensional Morse theory, see, e.g., Chang [6] and Mawhin-Willem [27, Chapter 8]. We start by recalling some concepts and results.

Let \( X \) be a Banach space, \( \varphi : X \to \mathbb{R} \) be a \( C^1 \)-functional, \( u \) is an isolated critical point of \( \varphi \) and \( \varphi(u) = c \). Then
\[
C_i(\varphi, u) := H_i(\varphi, \varphi \setminus \{0\}), \quad i \in \mathbb{N} = \{0, 1, 2, \ldots, \},
\]
is called the \( i \)-th critical group of \( \varphi \) at \( u \), where \( \varphi_c := \varphi^{-1}(-\infty, c] \) and \( H_i \) stands for the singular homology with coefficients in \( \mathbb{Z} \).

If \( \varphi \) satisfies Cerami condition and the critical values of \( \varphi \) are bounded from below by \( \alpha \), then following Bartsch-Li [2], we define the \( i \)-th critical group of \( \varphi \) at infinity by
\[
C_i(\varphi, \infty) := H_i(X, \varphi_c), \quad i \in \mathbb{N}.
\]
It is well known that the homology on the right side does not depend on the choice of \( \alpha \).

Proposition 3.1 ([2, Proposition 3.6]). If \( \varphi \in C^1(X, \mathbb{R}) \) satisfies the Cerami condition and \( C_i(\varphi, 0) \neq C_i(\varphi, \infty) \) for some \( i \in \mathbb{N} \), then \( \varphi \) has a nonzero critical point.

Proposition 3.2 ([20, Theorem 2.1]). Suppose \( \varphi \in C^1(X, \mathbb{R}) \) has a local linking at 0 with respect to the decomposition \( X = Y \oplus Z \), i.e., for some \( \varepsilon > 0 \),
\[
\varphi(u) \leq 0 \quad \text{for} \quad u \in Y \cap B_\varepsilon,
\]
\[
\varphi(u) > 0 \quad \text{for} \quad u \in (Z \setminus \{0\}) \cap B_\varepsilon,
\]
where \( B_\varepsilon = \{ u \in X \mid \| u \| \leq \varepsilon \} \). If \( \ell = \dim Y < \infty \), then \( C_\ell(\varphi, 0) \neq 0 \).
Now we are ready to prove Theorem 1.1. Set $\lambda_0 = -\infty$, since 0 is not an eigenvalue of (1.10), we may assume that for some $\ell \geq 0$ we have $0 \in (\lambda_\ell, \lambda_{\ell+1})$. For $\ell \geq 1$ we set
\[
X^- = \text{span} \{ \phi_1, \ldots, \phi_{\ell} \}, \quad X^+ = (X^-)^\perp.
\]
If $\ell = 0$, we set $X^- = \{0\}$ and $X^+ = X$. Then $X^-$ and $X^+$ are the negative space and positive space of the quadratic form
\[
\mathcal{B}(v) = \frac{1}{2} \int (|\nabla v|^2 + V(x)v^2)
\]
respectively, note that $\dim X^- = \ell$. Moreover, there is a constant $\eta > 0$ such that
\[
\pm \mathcal{B}(v) \geq \eta \|v\|^2, \quad v \in X^\pm.
\]
Lemma 3.3. The functional $\Phi$ has a local linking at 0 with respect to decomposition $X = X^- \oplus X^+$.

Proof. Unlike in semilinear problems, because the principle part of $\Phi$, denoted by $Q : X \to \mathbb{R}$,
\[
Q(v) = \frac{1}{2} \int (|\nabla v|^2 + V(x)f^2(v)),
\]
is not a quadratic form on $v$, it seems difficult to verify the local linking property of $\Phi$. Fortunately, by the nice properties of the map $f$ given by Proposition 2.1, we can easily overcome this difficulty.

By direct computation, we find that
\[
\dot{f}(t) = -\frac{f(t)\ddot{f}(t)}{(1 + 2f^2(t))^{3/2}}.
\]
By Proposition 2.1, it is easy to see that $\dot{f}$ is bounded. Hence, although $\Phi$ is only of class $C^1$, its principle part $Q$ is a $C^2$-functional on $X$, with derivatives given by
\[
\langle Q'(v), \phi \rangle = \int (\nabla \cdot \nabla \phi + V(x)f(v)\dot{f}(v)\phi),
\]
\[
\langle Q''(v)\phi, \psi \rangle = \int \{\nabla \phi \cdot \nabla \psi + V(x)[\dot{f}^2(v) + f(v)\dot{f}(v)]\phi \psi\}
\]
for all $v, \phi, \psi \in X$. In particular, since $f(0) = 0$ and $\dot{f}(0) = 1$, we have $Q'(0) = 0$ and
\[
\langle Q''(0)\phi, \psi \rangle = \int (\nabla \phi \cdot \nabla \psi + V(x)\phi \psi).
\]
Now applying the Taylor formula we get
\[
Q(v) = Q(0) + \langle Q'(0), v \rangle + \frac{1}{2!}\langle Q''(0)v, v \rangle + o(\|v\|^2)
\]
\[
= \frac{1}{2} \int (|\nabla v|^2 + V(x)v^2) + o(\|v\|^2), \quad \text{as } \|v\| \to 0. \tag{3.2}
\]
On the other hand, by the properties of $f$ again, we deduce from (1.4) that
\[
\lim_{t \to 0} \frac{G(f(t))}{t^2} = \lim_{t \to 0} \frac{G(f(t))f^2(t)}{f^2(t)}t^2 = 0.
\]
From this and condition $(g_0)$, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that
\[
|G(f(t))| \leq \varepsilon t^2 + C_\varepsilon |t|^{p/2}.
\]
Therefore, since $p > 4$, as $\|v\| \to 0$ we have
\[
\int G(f(v)) = o(\|v\|^2).
\]
Using this and (3.2), as \(\|v\| \to 0\) we have
\[
\Phi(v) = Q(v) - \int G(f(v)) = \frac{1}{2} \int (|\nabla v|^2 + V(x)v^2) + o(||v||^2).
\]
From this and (3.1), it is easy to see that the conclusion of the lemma is true.

**Remark 3.4.** If the quadratic form
\[
\mathcal{B}(v) = \int (|\nabla v|^2 + V(x)v^2)
\]
is positive definite, then \(X^- = \{0\}\) and the zero function 0 is a strict local minimizer of \(\Phi\). Under some additional assumptions on the nonlinearity \(g(u)\), we could deduce that \(\Phi\) satisfies the mountain pass geometry. This approach seems simpler than those presented in [34, 37, 38].

To apply Proposition 3.1, in what follows, we shall investigate the critical groups of \(\Phi\) at infinity. Firstly, we prove a crucial property of the transformation \(f\).

**Lemma 3.5.** Suppose that \(\tilde{g} : \mathbb{R} \to \mathbb{R}\) satisfies \(\tilde{g}(s)s \geq 0\), then for all \(s \in \mathbb{R}\) we have
\[
\tilde{g}(f(s))\dot{f}(s)s \geq \frac{1}{2} \tilde{g}(f(s))f(s). \tag{3.3}
\]

**Proof.** If \(s \geq 0\) then \(\tilde{g}(f(s)) \geq 0\), by Proposition 2.1 (3) we get (3.3). If \(s < 0\), then \(\tilde{g}(f(s)) \leq 0\) and since \(\dot{f}\) is an even function, by Proposition 2.1 (3) we have
\[
-\dot{f}(s)s = \dot{f}(-s)(-s) \geq \frac{f(-s)}{2}.
\]
Multiplying both side by \(-\tilde{g}(f(s))\) and noting that \(f\) is odd, we get
\[
\tilde{g}(f(s))\dot{f}(s)s \geq -\tilde{g}(f(s))\frac{f(-s)}{2} = \frac{1}{2} \tilde{g}(f(s))f(s).
\]

**Lemma 3.6.** There exists \(A > 0\) such that, if \(\Phi(v) \leq -A\), then
\[
\left. \frac{d}{dt} \Phi(tv) \right|_{t=1} < 0.
\]

**Proof.** Otherwise, there exists a sequence \(\{v_n\} \subset X\) such that \(\Phi(v_n) \leq -n\) but
\[
\langle \Phi'(v_n), v_n \rangle = \left. \frac{d}{dt} \right|_{t=1} \Phi(tv_n) \geq 0. \tag{3.4}
\]
Note that, since \(\tilde{g}(s)s \geq 0\), by Proposition 2.1 (4) and Lemma 3.5, we have
\[
\langle \Phi'(v_n), v_n \rangle = \int (|\nabla v_n|^2 + \tilde{V}(x)f(v_n)\dot{f}(v_n))v_n - \int \tilde{g}(f(v_n))\dot{f}(v_n)v_n
\leq \int (|\nabla v_n|^2 + \tilde{V}(x)f^2(v_n)) - \frac{1}{2} \int \tilde{g}(f(v_n))f(v_n). \tag{3.5}
\]
Combining (3.4), (3.5) and using (2.1), we have
\[
0 \geq \frac{\mu}{2} \Phi(v_n) - \langle \Phi'(v_n), v_n \rangle
\geq \left(\frac{\mu}{4} - 1\right) \int (|\nabla v_n|^2 + \tilde{V}(x)f^2(v_n)) + \frac{1}{2} \int (\tilde{g}(f(v_n))f(v_n) - \mu \tilde{g}(f(v_n)))
\geq \left(\frac{\mu}{4} - 1\right) \int (|\nabla v_n|^2 + \tilde{V}(x)f^2(v_n)) - \frac{1}{2} \left(\frac{\mu}{2} - 1\right) m \int f^2(v_n). \tag{3.6}
\]
We denote
\[ \rho_n = \left\{ \int \left( |\nabla v_n|^2 + \tilde{V}(x)f^2(v_n) \right) \right\}^{1/2}. \]

Then we must have \( \rho_n \to +\infty \). Otherwise, by the argument of Wu [37, Page 2626–2628] mentioned in the proof of Lemma 2.2, \( \{v_n\} \) is a bounded sequence in \( X \) and \( \Phi(v_n) \) is also bounded, contradicting the fact that \( \Phi(v_n) \leq -n \).

Set \( h_n = f(v_n)/\rho_n \). As in (2.3), \( \{h_n\} \) is a bounded sequence in \( X \). Up to a subsequence, we may assume that
\[ h_n \to h \text{ in } X, \quad h_n \to h \text{ in } L^2, \quad h_n \to h \text{ a.e. in } \mathbb{R}^N. \]  
(3.7)

Multiplying both sides of (3.6) by \( \rho_n^2 \), we obtain
\[ \frac{1}{2} \left( \frac{\mu}{2} - 1 \right) m|h_n|^2 \geq \frac{\mu}{4} - 1. \]
(3.8)

Then, since \( \mu > 4 \), (3.7) and (3.8) imply that \( h \neq 0 \).

Now, because \( \rho_n \to +\infty \), similar to (2.4) we have
\[ \frac{1}{\rho_n^2} \int \tilde{g}(f(v_n))f(v_n) \to +\infty. \]

Consequently, by (3.4) and (3.5),
\[ 0 \leq \Phi(v_n), v_n) \leq \int \left( |\nabla v_n|^2 + \tilde{V}(x)f^2(v_n) \right) - \frac{1}{2} \int \tilde{g}(f(v_n))f(v_n) \]
\[ = \rho_n^2 \left( 1 - \frac{1}{2\rho_n^2} \int \tilde{g}(f(v_n))f(v_n) \right) \to -\infty. \]

This is impossible. Thus the conclusion of the lemma must be true.

**Lemma 3.7.** Under our assumptions, \( C_i(\Phi, \infty) \cong 0 \) for \( i \in \mathbb{N} \).

**Proof.** Let \( B \) be the unit ball in \( X \), \( S = \partial B \) the unit sphere. Let \( A > 0 \) be the number given in Lemma 3.6. Without lose of generality we assume that
\[ -A < \inf_{||v|| \leq 2} \Phi(v). \]

Then for \( w \in S \), since \( |f(t)| \leq |t| \), reasoning as in (2.4) we deduce
\[ \Phi(sw) \leq \frac{1}{2} \int \left( |\nabla(sw)|^2 + \tilde{V}(x)(sw)^2 \right) - \int \tilde{G}(f(sw)) \]
\[ = s^2 \left( \frac{1}{2} - \frac{1}{s^2} \int \tilde{G}(f(sw)) \right) \to -\infty \]
as \( s \to +\infty \).

Consequently, there exists \( s_w > 0 \) such that \( \Phi(s_w w) = -A \). Set \( v = s_w w \), a direct computation and Lemma 3.6 gives
\[ \frac{d}{ds} \bigg|_{s=s_w} \Phi(sw) = \frac{1}{s_w} \frac{d}{dt} \bigg|_{t=1} \Phi(tv) < 0. \]

By the implicit function theorem, \( w \mapsto s_w \) is a continuous function on \( S \). Using this function and a standard argument (see, e.g., [23, 36]), we can construct a deformation from \( X \backslash B \) to \( \Phi_{-A} = \Phi^{-1}(-\infty, -A) \), and deduce via the homotopic invariance of singular homology
\[ C_i(X, \Phi_{-A}) \cong H_i(X, X \backslash B) = 0, \quad \text{for } i \in \mathbb{N}. \]

**Proof (Proof of Theorem 1.1).** We have verified that \( \Phi \) satisfies the Cerami condition. By Lemma 3.3, \( \Phi \) has a local linking at 0 with respect to the decomposition \( X = X^- \oplus X^+ \), hence by Proposition 3.2, for \( \ell = \dim X^- \), we have
\[ C_i(\Phi, 0) \neq 0. \]
On the other hand, Lemma 3.7 says that $C_7(\Phi, \infty) = 0$. By Proposition 3.1, $\Phi$ has a nonzero critical point $v$. Now $u = f(v)$ is a nontrivial solution of Problem (1.1).

4. Multiplicity result

To prove Theorem 1.2, we will apply the following symmetric mountain pass theorem due to Ambrosetti-Rabinowitz [1].

**Proposition 4.1** ([31, Theorem 9.12]). Let $X$ be an infinite dimensional Banach space, $\Phi \in C^2(X, \mathbb{R})$ be even, satisfies Cerami condition and $\Phi(0) = 0$. If $X = Y \oplus Z$ with $\dim Y < \infty$, and $\Phi$ satisfies

(I) there are constants $\rho, \alpha > 0$ such that $\Phi|_{\partial B_{\rho}} \geq \alpha$,

(II) for any finite dimensional subspace $W \subset X$, there is an $R = R(W)$ such that $\Phi \leq 0$ on $W \setminus B_R$

then $\Phi$ has a sequence of critical values $c_j \to +\infty$.

**Lemma 4.2.** Let $W$ be a finite dimensional subspace of $X$, then $\Phi$ is anti-coercive on $W$, that is

$$\Phi(v) \to -\infty, \quad \text{as } ||v|| \to \infty, \quad v \in W.$$  

**Proof.** For any $\{v_n\} \subset W$ with $||v_n|| \to \infty$, set $h_n = ||v_n||^{-1}v_n$. Then $\{h_n\}$ is a bounded sequence in $W$. Because $\dim W < \infty$, there exists $h \in W \setminus \{0\}$ such that

$$||h_n - h|| \to 0, \quad h_n(x) \to h(x) \quad \text{a.e. } x \in \mathbb{R}^N.$$  

For $x \in \{h \neq 0\}$, we have $|v_n(x)| \to \infty$ and $|f(v_n(x))| \to \infty$. Therefore, if $n$ large enough we have $|v_n(x)| \geq 1$. By Proposition 2.1 (5), we have

$$\frac{\tilde{G}(f(v_n(x)))}{||v_n||^2} = \frac{\tilde{G}(f(v_n(x)))}{f^4(v_n(x))} \frac{f^4(v_n(x))}{v_n^2(x)} h_n^2(x) \geq k^4 \frac{\tilde{G}(f(v_n(x)))}{f^4(v_n(x))} h_n^2(x) \to +\infty.$$  

By the Fatou lemma, we have

$$\int \frac{\tilde{G}(f(v_n))}{||v_n||^2} \geq \int_{h \neq 0} \frac{\tilde{G}(f(v_n))}{||v_n||^2} \to +\infty$$  

and

$$\Phi(v_n) = ||v_n||^2 \left( \frac{1}{2||v_n||^2} \int (|\nabla v_n|^2 + \tilde{V}(x)f^2(v_n)) - \frac{\tilde{G}(f(v_n))}{||v_n||^2} \right)$$  

$$\leq ||v_n||^2 \left( \frac{1}{2} - \int \frac{\tilde{G}(f(v_n))}{||v_n||^2} \right) \to -\infty,$$  

the desired result follows.

**Proof** (Proof of Theorem 1.2). Under the assumptions of Theorem 1.2, we know that $\Phi$ is an even functional satisfying the Cerami condition. Lemma 4.2 implies that $\Phi$ satisfies condition (II) of Proposition 4.1.

By condition $(g_0)$ and Proposition 2.1, there exist positive constants $C_1$ and $C_2$ such that

$$|G(f(t))| \leq C_1|t|^2 + C_2|t|^{p/2}. \quad (4.1)$$  

For $i \geq \ell$, let $Z_i = \text{span} \{\phi_i, \phi_{i+1}, \ldots\}$. Then

$$\beta_i = \sup_{v \in Z_i, ||v||=1} |v|_2 \to 0, \quad \text{as } i \to \infty,$$  

see e.g. [8, Lemma 2.5]. Therefore, there exists $k > \ell$ such that

$$\lambda = \eta - C_1 \beta_k^2 > 0.$$  

Let

$$Y = \text{span} \{\phi_1, \ldots, \phi_{k-1}\}, \quad Z = \text{span} \{\phi_k, \phi_{k+1}, \ldots\}.$$
Then $Z \subset X^+$ and $X = Y \oplus Z$.

For $v \in Z$, using (4.1) and Taylor expansion as in the proof of Lemma 3.3, and noting that $p > 4$, we have

$$\Phi(v) = \frac{1}{2} \int (|\nabla v|^2 + V(x)f^2(v)) - \int G(f(v))$$
$$= \frac{1}{2} \int (|\nabla v|^2 + V(v)v^2) - \int G(f(v)) + o(\|v\|^2)$$
$$\geq \eta \|v\|^2 - C_1 \|v\|^2 - C_2 \|v\|^p/2 + o(\|v\|^2)$$
$$\geq \eta \|v\|^2 - C_1 \beta^2 \|v\|^2 + o(\|v\|^2)$$
$$= \lambda \|v\|^2 + o(\|v\|^2).$$

Now it is easy to see that condition $(I_1)$ of Proposition 4.1 is verified.

By Proposition 4.1, $\Phi$ has a sequence of critical points $\{v_n\}$ such that $\Phi(v_n) \to +\infty$. Since $\Phi(v) = J(f(v))$, let $u_n = f(v_n)$. Then $\{u_n\}$ is a sequence of solutions for (1.1) such that $J(u_n) \to +\infty$.

References

[1] A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Functional Analysis 14 (1973) 349–381.
[2] T. Bartsch, S. Li, Critical point theory for asymptotically quadratic functionals and applications to problems with resonance, Nonlinear Anal. 28 (1997) 419–441.
[3] T. Bartsch, Z. Q. Wang, Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$, Comm. Partial Differential Equations 20 (1995) 1725–1741.
[4] L. Brizhik, A. Eremko, B. Piette, W. Zakrzewski, Electron self-trapping in a discrete two-dimensional lattice, Physica D 159 (2001) 71–90.
[5] L. Brizhik, A. Eremko, B. Piette, W. J. Zakrzewski, Static solutions of a $D$-dimensional modified nonlinear Schrödinger equation, Nonlinearity 16 (2003) 1481–1497.
[6] K.-c. Chang, Infinite-dimensional Morse theory and multiple solution problems, Progress in Nonlinear Differential Equations and their Applications, 6, Birkhäuser Boston, Inc., Boston, MA, 1993.
[7] H. Chen, S. Liu, Standing waves with large frequency for 4-superlinear Schrödinger-Poisson systems, Ann. Mat. Pura Appl. (4) 194 (2015) 43–53.
[8] S.-J. Chen, C.-L. Tang, High energy solutions for the superlinear Schrödinger-Maxwell equations, Nonlinear Anal. 71 (2009) 4927–4934.
[9] Y. Chen, X. Wu, Existence of nontrivial solutions and high energy solutions for a class of quasilinear Schrödinger equations via the dual-perturbation method, Abstr. Appl. Anal. (2013) Art. ID 256324, 13.
[10] M. Colin, L. Jeanjean, Solutions for a quasilinear Schrödinger equation: a dual approach, Nonlinear Anal. 56 (2004) 213–226.
[11] J. M. do Ó, U. Severo, Quasilinear Schrödinger equations involving concave and convex nonlinearities, Commun. Pure Appl. Anal. 8 (2009) 621–644.
[12] X.-D. Fang, Z.-Q. Han, Existence of nontrivial solutions for a quasilinear Schrödinger equations with sign-changing potential, Electron. J. Differential Equations (2014) No. 05, 8.
[13] X.-D. Fang, A. Szulkin, Multiple solutions for a quasilinear Schrödinger equation, J. Differential Equations 254 (2013) 2015–2032.
[14] M. F. Furtado, E. D. Silva, M. L. Silva, Quasilinear elliptic problems under asymptotically linear conditions at infinity and at the origin, Z. Angew. Math. Phys. 66 (2015) 277–291.
[15] H. Hartmann, W. Zakrzewski, Electrons on hexagonal lattices and applications to nanotubes, Phys. Rev. B 68 (2003) 184–302.
[16] W. Kryszewski, A. Szulkin, Generalized linking theorem with an application to a semilinear Schrödinger equation, Adv. Differential Equations 3 (1998) 441–472.
[17] S. Kurihara, Large-amplitude quasi-solitons in superfluid films, J. Phys. Soc. Jpn 50 (1981) 3262–3267.
[18] S. J. Li, M. Willem, Applications of local linking to critical point theory, J. Math. Anal. Appl. 189 (1995) 6–32.
[19] J. Liu, Z.-Q. Wang, Soliton solutions for quasilinear Schrödinger equations. I, Proc. Amer. Math. Soc. 131 (2003) 441–448 (electronic).
[20] J. Q. Liu, The Morse index of a saddle point, Systems Sci. Math. Sci. 2 (1989) 32–39.
[21] J. Q. Liu, S. J. Li, An existence theorem for multiple critical points and its application, Kexue Tongbao (Chinese) 29 (1984) 1025–1027.
[22] J.-q. Liu, Y.-q. Wang, Z.-Q. Wang, Soliton solutions for quasilinear Schrödinger equations. II, J. Differential Equations 187 (2003) 473–493.
[23] S. Liu, Existence of solutions to a superlinear $p$-Laplacian equation, Electron. J. Differential Equations (2001) No. 66, 6.
[24] S. Liu, On superlinear Schrödinger equations with periodic potential, Calc. Var. Partial Differential Equations 45 (2012) 1–9.
[25] S. Liu, Y. Wu, Standing waves for 4-superlinear Schrödinger-Poisson systems with indefinite potentials, Bull. Lond. Math. Soc. 49 (2017) 226–234.
[26] L. A. Maia, J. C. Oliveira Junior, R. Ruviaro, A quasi-linear Schrödinger equation with indefinite potential, Complex Var. Partial Differential Equations 61 (2016) 574–586.
[27] J. Mawhin, M. Willem, Critical point theory and Hamiltonian systems, Applied Mathematical Sciences, vol. 74, Springer-Verlag, New York, 1989.
[28] O. m. H. Miyagaki, S. I. Moreira, Nonnegative solution for quasilinear Schrödinger equations that include supercritical exponents with nonlinearities that are indefinite in sign, J. Math. Anal. Appl. 421 (2015) 643–655.
[29] A. Moameni, Soliton solutions for quasilinear Schrödinger equations involving supercritical exponent in $\mathbb{R}^N$, Commun. Pure Appl. Anal. 7 (2008) 89–105.
[30] M. Poppenberg, K. Schmitt, Z.-Q. Wang, On the existence of soliton solutions to quasilinear Schrödinger equations, Calc. Var. Partial Differential Equations 14 (2002) 329–344.
[31] P. H. Rabinowitz, Minimax methods and their application to partial differential equations, in Seminar on nonlinear partial differential equations (Berkeley, Calif., 1983), Math. Sci. Res. Inst. Publ., vol. 2, Springer, New York, 1984, 307–320.
[32] D. Ruiz, G. Siciliano, Existence of ground states for a modified nonlinear Schrödinger equation, Nonlinearity 23 (2010) 1221–1233.
[33] E. A. B. Silva, G. F. Vieira, Quasilinear asymptotically periodic Schrödinger equations with critical growth, Calc. Var. Partial Differential Equations 39 (2010) 1–33.
[34] E. A. B. Silva, G. F. Vieira, Quasilinear asymptotically periodic Schrödinger equations with subcritical growth, Nonlinear Anal. 72 (2010) 2935–2949.
[35] Y. Wang, W. Zou, Bound states to critical quasilinear Schrödinger equations, NoDEA Nonlinear Differential Equations Appl. 19 (2012) 19–47.
[36] Z. Q. Wang, On a superlinear elliptic equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 8 (1991) 43–57.
[37] X. Wu, Multiple solutions for quasilinear Schrödinger equations with a parameter, J. Differential Equations 256 (2014) 2619–2632.
[38] M. Yang, Existence of solutions for a quasilinear Schrödinger equation with subcritical nonlinearities, Nonlinear Anal. 75 (2012) 5362–5373.
[39] J. Zhang, X. Lin, X. Tang, Ground state solutions for a quasilinear Schrödinger equation, Mediterr. J. Math. 14 (2017) Art. 84, 13.
[40] J. Zhang, X. Tang, W. Zhang, Infinitely many solutions of quasilinear Schrödinger equation with sign-changing potential, J. Math. Anal. Appl. 420 (2014) 1762–1775.