The phonon magnetochoiral effect consists of a nonreciprocity in the velocity or attenuation of acoustic waves when they propagate parallel and antiparallel to an external magnetic field. The first experimental observation of this effect has been reported recently in a chiral magnet and ascribed to the hybridization between acoustic phonons and chiral magnons. Here, we predict a potentially measurable phonon magnetochoiral effect of electronic origin in chiral Weyl semimetals. Caused by the Berry curvature and the orbital magnetic moment, this effect is enhanced for longitudinal phonons by the chiral anomaly.

Introduction.– In topological materials, the electronic energy bands and wave functions are characterized by nonzero integers known as topological invariants [1]. These invariants manifest themselves physically by virtue of peculiar electronic states localized at sample boundaries. To date, most experimental probes of topological invariants have concentrated on electronic transport and photoemission spectroscopy. Yet, developing alternative (possibly nonelectronic) ways to detect and exploit these invariants remains an active area of research [2]. Along this line of research, recent studies have shown that electronic topological phenomena can leave observable signatures in the properties of bulk atomic vibrations [3–13]. The latter are of interest because they are ubiquitous and can be accurately measured.

In the present work, we predict a new acoustic manifestation of the momentum-space geometry of electronic wave functions. We show that, in conducting crystals without inversion and mirror symmetries (chiral crystals), the electronic Berry curvature and orbital magnetic moment lead to the phonon magnetochoiral effect (PMCE). This is an effect whereby sound propagates with different speeds and attenuations in the directions parallel and antiparallel to an external magnetic field. PMCE is an elusive phenomenon: thus far, it has been observed only in Cu$_2$OSeO$_3$, an insulating chiral ferromagnet [14]. There, PMCE has been attributed to the hybridization between chiral magnons and acoustic branches of the phonon spectrum. In contrast, the PMCE we predict takes place in non-magnetic materials and relies on electron-phonon interactions.

For concreteness, we tailor our theory to chiral Weyl semimetals (WSM) [15–18], which possess attributes conducive to a significant PMCE. The minimal description of these WSM comprises two Weyl nodes with opposite chiralities and Berry curvatures, located at different energies. The energy dispersion around each Weyl node is linear and the system is doped so that the Fermi surface in the absence of a magnetic field consists of two disjointed Fermi spheres of different radii. The phonon dispersion of chiral WSM has been theoretically studied from first principles, though only in the absence of electron-phonon interactions and magnetic fields [19]. We find that longitudinal phonons propagating along the magnetic field pump charge back and forth between Weyl nodes of opposite chirality. The slow relaxation of this phonon-induced chiral charge imbalance amplifies the PMCE, rendering it potentially observable. Though our calculation focuses on WSM, it can be adapted to other chiral materials with nontrivial electronic band geometry, such as quantum anomalous Hall insulators [20], where a related PMCE may exist.

Formalism.– We adopt a semiclassical approach that combines the Boltzmann equation for electrons with Maxwell’s equations for the electromagnetic fields generated by the lattice vibrations, and the elasticity equations for the dynamics of the lattice. In this approach, the Berry curvature appears explicitly and disorder is included in the relaxation time approximation. Such semiclassical theory of sound waves was developed in the 1950s and 1960s, albeit for topologically trivial electron systems without Berry curvature [21]. The first attempt to augment it to topologically nontrivial materials was carried out in Ref. [6]. The authors of this work focused on the sound attenuation in nonchiral WSM, calculated from the entropy production rate. They compared the sound attenuation when the phonon wave vector is parallel and perpendicular to the external magnetic field, and ascribed the difference to the chiral anomaly. Below, we obtain the full phonon dispersion (including real and imaginary parts) by solving the elasticity equations in the presence of Weyl fermions.

The starting point is to calculate the distribution function $f_p(r,t)$ of electrons in a static and uniform magnetic field $\mathbf{B}$, in the presence of acoustic waves characterized by a displacement $\mathbf{u}(r,t)$ of the atomic positions with respect to equilibrium. Here, $\mathbf{p}$ is the electronic momentum whereas $r$ and $t$ are the space and time coordinates. The function $f$ is the solution of the Boltzmann equation

$$[\partial_t + \mathbf{v} \cdot \partial_r + \mathbf{p} \cdot \partial_p] f_p(r,t) = I_{\text{col}}[f_p(r,t)],$$

where $I_{\text{col}}[f]$ is the collision term to be discussed below.
and
\[ \dot{r} = \partial_p \varepsilon_p(r, t) + \hat{p} \times \Omega_p / \hbar \]
\[ \dot{p} = eE(t, t) + e\dot{r} \times B - \partial_r \varepsilon_p(r, t) \]  
(2)

are the group velocity of an electron and the force acting on it, respectively [22]. The electron’s charge is denoted as \( e \) and \( \Omega_p \) is the Berry curvature. The electric field \( E \) is internally produced by the lattice vibrations. In addition,

\[ \varepsilon_p(r, t) = \varepsilon_p^{(0)}(\lambda_{ij}(p) + p_i \dot{v}_j) u_{ij} + (p - m \dot{v}) \cdot \dot{u} \]  
(3)

is the energy of an electron in the presence of lattice vibrations. In Eq. (3), \( i, j \in \{x, y, z\} \), there is a sum over repeated indices, \( \varepsilon_p^{(0)} = \varepsilon_0(p) - m_p \cdot B \) is the energy of an electron in the absence of lattice vibrations, \( \dot{v} = \partial_p \varepsilon_p^{(0)} \) is the corresponding group velocity, \( \varepsilon_0(p) \) is the band energy for zero magnetic field, \( m_p \) is the orbital magnetic moment of an electron, \( \lambda_{ij}(p) \) is the acoustic deformation potential describing the electron-phonon coupling, and \( u_{ij} = (\partial r_i u_j + \partial r_j u_i)/2 \) is an element of the strain tensor.

We search for a solution of Eq. (1) in the form

\[ f_p(r, t) = f_p^{\text{eq}}(r, t) + \chi_p(r, t) \frac{\partial f_0(\varepsilon_p^{(0)})}{\partial \varepsilon_p^{(0)}} \]  
(4)

where \( f_0(x) = [\exp(x) + 1]^{-1} \) is the Fermi-Dirac distribution function. We limit ourselves to determining \( f \) up to first order in \( u \). In Eq. (4), we have defined the local equilibrium distribution function

\[ f_p^{\text{eq}}(r, t) \equiv f_0(\varepsilon_p(r, t) - p \cdot \dot{u}(r, t) - \mu_0 - \delta \mu(r, t)) \]  
(5)

where \( \mu_0 \) is the chemical potential of the electrons in the absence of lattice vibrations, and \( \delta \mu \) is the change in the chemical potential due to lattice vibrations. Because the local equilibrium is defined in the coordinate frame that moves with the lattice, the term \( p \cdot \dot{u} \) appears in it. The second term in the right hand side of Eq. (4) captures the phonon-induced deviations from the local Fermi-Dirac distribution, which occur near the Fermi energy (we assume that the temperature of the system is low compared to the Fermi temperature).

We evaluate \( \delta \mu \) via the “normalization condition” that the total electronic density be equal to the electronic density computed from the local equilibrium distribution [23, 24], i.e. \( \langle f(r, t) \rangle = \langle f_{\text{eq}}(r, t) \rangle \). Here, the notation \( \langle O \rangle \equiv \int d^3r / (2\pi \hbar)^3 O(1 + e\Omega_p / \hbar) \) stands for the integration of \( O \) over the Brillouin zone and includes the Berry curvature correction to the density of states [22]. This condition in turn implies \( \langle \chi_p(r, t) \rangle = 0 \), where the notation \( \langle O \rangle \equiv -\langle (O \partial f_0 / \partial \varepsilon_p^{(0)}) \rangle \) stands for the Fermi surface average of \( O \).

Thus far, the formalism described could be applied to an arbitrary electronic band. For a generic system, the simplest collision term to consider in Eq. (1) would be [21]

\[ I_{\text{coll}}[f_p(r, t)] = -\Gamma \chi_p(r, t) \frac{\partial f_0(\varepsilon_p^{(0)})}{\partial \varepsilon_p^{(0)}} \]  
(6)

\( \Gamma \) being a phenomenological relaxation rate, small compared to the Fermi energy. Hereafter, we concentrate in a minimal model of WSM, for which Eq. (6) is incomplete. Indeed, in a WSM with two valleys of opposite chirality, there exist two very different relaxation rates. First, intravalley scattering relaxes the nonequilibrium distribution function within each valley with a rate \( \Gamma_A \). Second, intervalley scattering relaxes nonequilibrium differences between the distribution functions of different valleys with a rate \( \Gamma_E \). It is commonly believed that \( \Gamma_E \ll \Gamma_A \), because intervalley relaxation involves relatively large scattering wave vectors.

In order to incorporate the two different relaxation rates in the problem (each of which plays a separate role in sound propagation), we use

\[ I_{\text{coll}}^{(\alpha)}[f_p^{(\alpha)}(r, t)] = -\Gamma_A \left( \chi_p^{(\alpha)}(r, t) - \frac{\langle \chi_p^{(\alpha)}(r, t) \rangle}{\langle 1^{(\alpha)} \rangle} \right) \]

\[ + \Gamma_E \left( \frac{\langle \chi_p^{(\alpha)}(r, t) \rangle}{\langle 1^{(\alpha)} \rangle} \right) \frac{\partial f_0^{(\alpha)}(\varepsilon_p^{(0)})}{\partial \varepsilon_p^{(0)}} \]  
(7)

where the superscript \( \alpha \in \{+,-\} \) indicates that the momentum \( p \) is taken near the Weyl node \( \alpha \), and \( \langle 1^{(\alpha)} \rangle \) corresponds to the density of states at the (unperturbed) Fermi level on node \( \alpha \). For \( \Gamma_E = \Gamma_A \), Eq. (7) reduces to Eq. (6), if we project the latter to the vicinity of node \( \alpha \). For \( u = 0 \), Eq. (7) reduces to the collision term used recently in Ref. [25] to describe purely electronic collective modes. The normalization condition imposes \( \sum_{\alpha}(\chi_{\alpha}^{(\alpha)}) = 0 \).

In Eq. (7), intravalley scattering relaxes the distribution of electrons in Weyl node \( \alpha \) towards a momentum-independent distribution with a local Fermi level \( \mu_0 + \delta \mu + \langle \chi_p^{(\alpha)}(r, t) \rangle / \langle 1^{(\alpha)} \rangle \), whereas intervalley scattering tends to equalize the electrochemical potentials at the two nodes. If \( \langle \chi_p^{(\alpha)}(r, t) \rangle = 0 \) (which implies that \( \langle \chi_p^{(-\alpha)}(r, t) \rangle = 0 \) through the normalization condition), the electrochemical potential is the same for the two Weyl nodes and the deviations from the Fermi-Dirac distribution happening on each node will relax through intravalley scattering alone. If \( \langle \chi_p^{(\alpha)}(r, t) \rangle = -\langle \chi_p^{(-\alpha)}(r, t) \rangle \neq 0 \), the electrochemical potential is not the same on the two nodes; this is a manifestation of the chiral anomaly produced by lattice vibrations. In that case, there will be an additional relaxation channel governed by intervalley scattering, which will tend to decrease \( \langle \chi_p^{(\alpha)}(r, t) \rangle \).

To solve Eq. (1), we first linearize it in \( u \) and then Fourier transform it from \( (r, t) \) to \( (q, \omega) \), where \( \omega \) and
as well as the frequency and wave vector of the lattice vibrations, respectively. We thus obtain two equations for 
\[ \lambda_{ij}^{(\alpha)}(\mathbf{q}, \omega) \] (one for each \( \alpha \)), which contain the additional unknowns \( E(\mathbf{q}, \omega) \), \( \delta \mu(\mathbf{q}, \omega) \), and \( \langle \chi(\mathbf{p}) \rangle \). These unknowns are related to one another via the normalization condition and Maxwell’s equations.

We solve Eq. (1) by expanding into a Taylor series in \( \partial^2 \mathbf{u} / \partial t^2 \), which we invoke for long wavelength acoustic phonons. Here, \( v_F^{(\alpha)} \) is the Fermi velocity at node \( \alpha \). Results can be further simplified by adopting the isotropic approximation for the deformation potential tensor \[ [21] \] in the vicinity of node \( \alpha \),

\[ \lambda_{ij}^{(\alpha)} \approx \lambda_{ij}^{(1)}(\alpha) \delta_{ij} + \lambda_{ij}^{(2)}(\alpha) p_i p_j / p^2, \] (8)

where \( \mathbf{p} \) is the momentum measured with respect to the node, and \( \lambda_{ij}^{(1/2)} \) are constants in units of energy. This approximation is motivated by the spherically symmetric energy dispersion around each Weyl node at \( B = 0 \). If the nodes \( \alpha = + \) and \( \alpha = - \) were related by a crystal symmetry, we would have \( \lambda_{ij}^{(+)} = \lambda_{ij}^{(-)} \). Yet, here we are interested in the situation where the two nodes are symmetry-unrelated.

Once the electronic distribution function is obtained, it is plugged into the elasticity equation describing the lattice vibrations \[ [21] \]:

\[ \rho \ddot{u}_i = \partial_r \sigma_{ij} + (\mathbf{j} \times \mathbf{B} + \mathbf{F})_i, \] (9)

where \( \rho \) is the mass density of the crystal, \( \sigma_{ij} \) is an element of the stress tensor in the absence of conduction electrons,

\[ \mathbf{j} = -e \langle \mathbf{f}_0 \rangle \dot{\mathbf{u}} + e \langle (\mathbf{f} \dot{\mathbf{f}}) \rangle \] (10)

is the total electric current (including the ionic and the electronic parts) evaluated to first order in \( \mathbf{u} \) and to zeroth order in \( \mathbf{B} \), and

\[ F_i = \partial_r \langle \langle \lambda_{ij} f \rangle \rangle \] (11)

is the \( i \) component of the “drag force” exerted by conduction electrons on the lattice. In Eq. (9), we have neglected the term \((m/e) \partial_0 \mathbf{j}) \), where \( m \) is the bare electron mass, because we are interested in values of the magnetic field \( (\geq 1 \text{ T}) \) such that the free electron cyclotron frequency greatly exceeds the frequency of sound waves.

The right hand side of Eq. (9) is (to leading order) linear in \( \mathbf{u} \). Thus, Eq. (9) can be recast as an eigenvalue problem. The corresponding eigenvectors give the polarization of the three sound waves, and the eigenvalues give their respective dispersion relations \( (\omega \sim \text{vs}) \).

Results. - Next, we summarize the main results of our calculation. We consider an electron-doped WSM, for which \( \Omega_{\mathbf{p}}^{(\pm)} = \pm |C| h^2 p / (2 p^3) = - m_p^{(\pm)} h / (e v_F^{(\pm)} p) \) \[ [27] \] and \(|C|\) is the absolute value of the Chern number at a node. Though equal to one in our model, we keep \( C \) as a bookkeeping parameter to track geometric effects in sound propagation. We focus on the change of sound velocity and attenuation due to magnetic field, to first order in \( B \) and in the diffusive regime. For simplicity, we fix the phonon propagation direction \( \hat{\mathbf{q}} = q \hat{z} \) along a high symmetry direction of a cubic chiral crystal.

We begin by considering the case \( \mathbf{B} = B \hat{z} \). In this configuration, Eq. (9) becomes \[ [26] \]

\[ \rho \omega^2 u_i \approx s_{ii z z} q_z^2 u_i + e \left( B_z q_z (u_i - u_z \delta_{iz}) \right) \langle \langle \Omega_{\mathbf{p}} \cdot \mathbf{f}_0 \rangle \rangle_0 \]

\[ - i q_z \delta \mu_i (\lambda_{zz} \omega \delta_{iz} + i q_z (\lambda_{iz} \chi_1) \rangle_0, \] (12)

where \( s_{ii z z} \) is an element of the stiffness tensor. The full analytical expressions for \( \chi_1 \) and \( \delta \mu_1 \) can be found in Ref. \[ [26] \].

For a given acoustic mode, the PMCE in the sound velocity is defined as

\[ v_{\text{MC}} \equiv \frac{c_s(q_z, B_z) - c_s(q_z, -B_z)}{c_s(q_z, 0)}, \] (13)

where \( c_s(q, \mathbf{B}) = \partial \omega_R(\mathbf{q}, \mathbf{B}) / \partial q \) is the sound velocity at wave vector \( \mathbf{q} \) and \( \omega_R(\mathbf{q}, \mathbf{B}) \) is the real part of the phonon dispersion (obtained from Eq. (12)). As the sound wave traverses a sample of thickness \( L \), its amplitude decays by a factor \( \exp(-\omega_t L / c_s) \), where \( \omega_t(\mathbf{q}, \mathbf{B}) \) is the imaginary part of the phonon frequency (obtained from Eq. (12)). Comparing the decay factors for opposite field orientations, we define the PMCE in sound attenuation as

\[ r_{\text{MC}} \equiv \frac{e^{-A(q_z, B_z)} - e^{-A(q_z, -B_z)}}{e^{-A(q_z, B_z)} + e^{-A(q_z, -B_z)}}, \] (14)

where \( A(\mathbf{q}, \mathbf{B}) = \omega_t(\mathbf{q}, \mathbf{B}) L / c_s(q_z, 0) \).

For longitudinal phonons \( (\mathbf{u} \cdot \mathbf{q} \neq 0) \), Eq. (12) yields \[ [26] \]
\[ v_{\text{MC}} \simeq \frac{e|C|}{\pi^2 \hbar^2 \rho c_s(q_z,0)} \frac{1}{\Gamma_E} \left( \langle 1^{(+)} \rangle_0 - \langle 1^{(-)} \rangle_0 \right) \left( \lambda_1^{(+)} - \lambda_1^{(-)} + \frac{\lambda_2^{(+)}}{3} - \frac{\lambda_2^{(-)}}{3} \right)^2 \]

\[ r_{\text{MC}} \simeq -\frac{7e|C|}{24\pi^2 \hbar^2 \rho c_s(q_z,0)^2 \Gamma_E} \frac{1}{(1)} \left( \lambda_1^{(+)} - \lambda_1^{(-)} + \frac{\lambda_2^{(+)}}{3} - \frac{\lambda_2^{(-)}}{3} \right), \quad (15) \]

where we have omitted terms that are smaller by at least a factor \( \Gamma_A/\Gamma_E \); some of these terms are intrinsic (independent of \( \Gamma_A \) and \( \Gamma_E \)) but quantitatively negligible. In strongly chiral WSM (where the deformation potentials and the Fermi level density of states differ strongly between the two nodes of opposite chirality), we have \( r_{\text{MC}}/v_{\text{MC}} \sim \Gamma_E L/c_s(q_z,0) \gg 1 \). For long wavelengths, no significant error is made by replacing \( c_s(q_z,0) \rightarrow c_s(0,0) \equiv c_s(0) \) in Eq. (15).

Figure 1: Magnetochiral effect in the velocity and attenuation of longitudinal sound waves (Eq. (15)), as a function of the frequency \( \omega \) of the sound. The figure is restricted to the regime \( \max(\omega, v_{\text{F}}^2 q^2/\Gamma_A) \ll \Gamma_E \ll \Gamma_A \ll v_{\text{F}} q \). The parameter values are indicated in the main text.

Equation (15) is the central result of this work. As expected for a magnetochiral effect, \( v_{\text{MC}} \) and \( r_{\text{MC}} \) are odd in \( q_z \) and \( B_z \). One remarkable aspect of \( v_{\text{MC}} \) and \( r_{\text{MC}} \) (which also applies to the omitted terms) is that they are proportional to \( |C| \): they thus originate entirely from the momentum-space geometry of electronic Bloch wave functions, i.e. from the Berry curvature and the orbital magnetic moment.

A second main feature is that \( v_{\text{MC}} \) and \( r_{\text{MC}} \) vanish in the presence of spatial inversion or a mirror symmetry. From group theory point of view, long wavelength acoustic phonons transform according to the vector representation of the crystal’s point group. Accordingly, in WSM possessing a mirror plane or an inversion center, the deformation potentials are identical in nodes of opposite chirality. Only chiral crystals (where the vector and pseudovector representations coincide) can have \( \lambda_1^{(+)} \neq \lambda_1^{(-)} \).

Another point to highlight is that \( v_{\text{MC}} \) and \( r_{\text{MC}} \) scale as \( 1/\Gamma_E^2 \) and \( 1/\Gamma_E \), respectively, when \( \max(\omega, v_{\text{F}}^2 q^2/\Gamma_A) \ll \Gamma_E \ll \Gamma_A \). The mechanism underlying this dependence is that longitudinal phonons propagating collinearly with the magnetic field generate a dynamical chiral population imbalance, whose magnitude is set by the intervalley relaxation rate. Because this relaxation rate is slow, the PMCE is enhanced.

A fourth important characteristic of Eq. (15) is that \( v_{\text{MC}} \) and \( r_{\text{MC}} \) are not negligible. Figure 1 displays Eq. (15) as a function of \( c_s(0)q \). For reasonable parameter values \((B = 1 T, \Gamma_E = 0.01 \text{ meV}, \lambda_1^{(+)} = 2.0 eV, \lambda_1^{(-)} = 1.0 eV, \rho = 10^4 \text{ kg/m}^3, c_s(0) = 2 \times 10^3 \text{ m/s}, L = 1 \text{ cm})\), \( v_{\text{MC}} \) and \( r_{\text{MC}} \) exceed the threshold of detectability (which is \( \sim 10^{-6} \) for \( v_{\text{MC}} \) [14], and \( \sim 10^{-3} - 10^{-2} \) for \( r_{\text{MC}} \) [28]). Clearly, the observability of \( v_{\text{MC}} \) and \( r_{\text{MC}} \) is aided by the slowness of the intervalley relaxation time. The value chosen here \((h/\Gamma_E \approx 50 \text{ ps})\) is within the range discussed in the literature [29–31].

The situation is significantly different for transverse phonons \((q \cdot u = 0)\). In our regime of interest, these phonons do not generate a chiral population imbalance [26]. Accordingly, the PMCE is much weaker (by at least a factor \( \Gamma_A/\Gamma_E \)) in that case. This statement also applies to all three phonon modes in the configuration \( q \cdot B = 0 \).

**Conclusions.** We have theoretically predicted a phonon magnetochiral effect in chiral Weyl semimetals, which originates from a nontrivial electronic band geometry. This effect is made potentially observable for longitudinal phonons by the slow relaxation rate of the dynamical valley imbalance induced by the lattice vibrations through the chiral anomaly. Future avenues of research include adapting our semiclassical theory to other topological materials and carrying out a fully quantum mechanical calculation in strong magnetic fields.

**Acknowledgements.** This research has been financed in part by the Canada First Research Excellence Fund, the Natural Science and Engineering Council of Canada, the Fonds de Recherche du Québec Nature et Technologies, and by the National Science Foundation under Grant No. NSF PHY-1748958. I.G. acknowledges the hospitality of the Kavli Institute for Theoretical Physics, where this work was finalized. We are grateful to O. An-tebi, D. Pesin, J. Quilliam and M. A. H. Vozmediano.
for informative discussions. We thank C. Ethier for her technical assistance in the early stages of this work.

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Supplemental material for “Phonon magnetochiral effect in chiral Weyl semimetals"

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In this supplemental material (SM) we provide additional information and details about the formalism described in the main text. The outline of the SM is as follows:

I. Equations of motion of an electron in the presence of lattice vibrations.
II. Boltzmann kinetic equation (BKE).
III. Solution of the BKE to zeroth order in the magnetic field.
IV. Solution of the BKE to first order in the magnetic field.
V. Elasticity equations for lattice vibrations in the presence of Weyl fermions.
VI. Velocity and attenuation of sound waves in Weyl semimetals: phonon magnetochiral effect.

I. Equations of motion of an electron in presence of lattice vibrations

The semiclassical equations of motion for an electron at position \( r \) with momentum \( p \) are [S22]

\[
\begin{align*}
\dot{r} &= \partial_p \varepsilon_p(r, t) + \dot{p} \times \Omega_p / \hbar \\
\dot{p} &= eE(r, t) + e\dot{r} \times B - \partial_r \varepsilon_p(r, t),
\end{align*}
\]

where \( \Omega_p \) is the Berry curvature,

\[
\varepsilon_p(r, t) = \varepsilon_p^{(0)} + \delta\varepsilon(r, t)
\]

is the energy of the electron in the presence of lattice vibrations and magnetic fields,

\[
\varepsilon_p^{(0)}(p) = \varepsilon_0(p) - m_p \cdot B
\]

is the energy in the absence of lattice vibrations, \( \varepsilon_0 \) is the band energy in absence of lattice vibrations and magnetic fields, \( m_p \) is the orbital magnetic moment and

\[
\delta\varepsilon_p(r, t) = (\lambda_{ij}(p) + p_i \tilde{v}_j) u_{ij}(r, t) + p_j \dot{u}_i - m \tilde{v}_i \dot{u}_i
\]

is the contribution of lattice vibrations to the electron’s energy (written in the lab frame) [S21]. In Eq. (S5), \( u \) is the displacement vector for the ion at position \( r \) and time \( t \), \( \lambda_{ij} \) is the \( (i,j) \) element of the deformation potential tensor \( (i,j \in \{x,y,z\}) \) and

\[
\ddot{v} = \partial_p \varepsilon^{(0)}_p = \partial_p \varepsilon_0 - \partial_p (m_p \cdot B) \equiv \ddot{v} - \partial_p (m_p \cdot B)
\]

is the electronic velocity in a magnetic field and in the absence of lattice vibrations. In the absence of a magnetic field, \( \ddot{v} \) becomes equal to \( v \). Note that \( v \) and \( \ddot{v} \) are functions of \( p \); for brevity, we omit the momentum subscript. Also for brevity, summations over repeated indices will be implicit throughout the text.

In a minimal model of a Weyl semimetal (WSM) containing two nodes of opposite chirality \( \alpha = \pm 1 \), separated in energy by \( 2\Delta \) \( (\Delta \neq 0 \text{ for chiral WSM}) \), we have

\[
\begin{align*}
\varepsilon_0^{(\alpha)}(p) &= v_F^{(\alpha)} p + \alpha \Delta \\
v^{(\alpha)} &= v_F^{(\alpha)} \hat{p} \\
\Omega_p^{(\alpha)} &= \left| C |\alpha| \hbar^2 \hat{p} \right| / 2p^2 \\
m_p^{(\alpha)} &= -|C| \alpha \hbar v_F^{(\alpha)} / 2p \\
\partial_p (m_p^{(\alpha)} \cdot B) &= -\alpha |C| \hbar v_F^{(\alpha)} / 2p \left( B - 2(B \cdot \hat{p}) \hat{p} \right),
\end{align*}
\]
where the superscript $\alpha$ indicates that the momentum $p$ is restricted to the vicinity of Weyl node $\alpha$ (not to be confused with the superscript 0 appearing elsewhere), $v^{(\alpha)}_F$ is the Fermi velocity describing the slope of the energy dispersion at node $\alpha$, and $|C| = 1$ is the Chern number. Although $|C| = 1$ for the minimal model, we will keep $|C|$ as a bookkeeping parameter for effects of geometric (Berry curvature, orbital magnetic moment) origin in the sound propagation. In Eqs. (S7), (S9), (S10) and (S11), $p$ is the momentum measured with respect to the Weyl node.

For latter reference, the space and momentum derivatives of the energy dispersion are given as

$$\partial_p \varepsilon_p(r,t) = \tilde{v} + u_{ij} \partial_p (\lambda_{ij} + p_i \tilde{v}_i) + \tilde{v}_i \partial_p p_i - m \dot{u}_i \partial_p \tilde{v}_i \tag{S12}$$

$$\partial_e \varepsilon_p(r,t) = \lambda_{ij} \partial_e u_{ij} + p_i \tilde{v}_j \partial_e u_{ij} + p_i \partial_e u_i - m \tilde{v}_i \partial_e \tilde{u}_i. \tag{S13}$$

Plugging Eqs. (S12) and (S13) in Eqs. (S1) and (S2), we get

$$\dot{p} = \frac{eE + ev \times B + \frac{e^2}{\hbar} \Omega_p (B \cdot E) - \partial_e \delta \varepsilon + e \partial_p \delta \varepsilon \times B - \frac{e}{\hbar} \Omega_p (B \cdot \partial_p \delta \varepsilon)}{1 + \frac{e}{\hbar} B \cdot \Omega_p} \tag{S14}$$

$$\dot{r} = \frac{\tilde{v} + \frac{e}{\hbar} E \times \Omega_p + \frac{e}{\hbar} B (\Omega_p \cdot \tilde{v}) + \partial_p \delta \varepsilon - \frac{1}{\hbar} \partial_e \delta \varepsilon \times \Omega_p + \frac{e}{\hbar} B (\Omega_p \cdot \partial_p \delta \varepsilon)}{1 + \frac{e}{\hbar} B \cdot \Omega_p} \tag{S15}$$

Next, we will use the equations of this section to set up the Boltzmann kinetic equation.

**II. Boltzmann kinetic equation (BKE)**

The Boltzmann kinetic equation for the electronic distribution function $f_p(r,t)$ has the following form,

$$\partial_t f + \dot{r} \cdot \partial_r f + \dot{p} \cdot \partial_p f = I_{\text{coll}}\{f\}, \tag{S16}$$

where $I_{\text{coll}}$ is the collision term. We seek a solution of the form

$$f_p(r,t) = f_p^{le}(r,t) + \chi_p(r,t) \frac{\partial f_0(\varepsilon_p^{(0)})}{\partial \varepsilon_p^{(0)}}, \tag{S17}$$

where

$$f_p^{le}(r,t) = f_0(\varepsilon_p(r,t) - p \cdot \dot{u} - \mu_0 - \delta \mu(r,t)) \tag{S18}$$

is the local equilibrium distribution function and $f_0$ is the Fermi-Dirac distribution. The local equilibrium has an effective chemical potential $\mu_0 + \delta \mu$, where $\mu_0$ is the chemical potential for $u = 0$ and $\delta \mu$ is the correction due to lattice vibrations. We will define $\delta \mu$ such that the following “normalization condition” is satisfied: the electron density calculated from the local equilibrium distribution is equal to the total electron density [S23, S24]. This condition implies a vanishing average of $\chi$ over the *total* Fermi surface of the system (which contains different pieces centered around different Weyl nodes) in the presence of a magnetic field.

The time derivative of the distribution function can be written as

$$\partial_t f = \frac{\partial f_0}{\partial \varepsilon_p^{(0)}} (\delta \varepsilon - p_i \dot{u}_i - \delta \mu + \chi) = \frac{\partial f_0}{\partial \varepsilon_p^{(0)}} (\lambda_{ij} \dot{u}_{ij} + p_i \tilde{v}_j \dot{u}_{ij} - m \tilde{v}_i \dot{u}_i - \delta \mu + \chi). \tag{S19}$$

Here and below, we omit the momentum subscript from $\chi$, for the sake of notational simplicity. The space derivative is given as

$$\partial_r f = \frac{\partial f_0}{\partial \varepsilon_p^{(0)}} (\partial_e \delta \varepsilon - p_i \partial_r \dot{u}_i - \delta \mu + \partial_r \chi) \tag{S20}$$

and the momentum derivative has the form

$$\partial_p f = \frac{\partial f_0}{\partial \varepsilon_p^{(0)}} \left[ \tilde{v} + \partial_p \delta \varepsilon - \dot{u}_i \partial_p p_i + \partial_p \chi \right] + \dot{v} \left[ \delta \varepsilon - p_i \dot{u}_i - \delta \mu + \chi \right] \frac{\partial f_0}{\partial \varepsilon_p^{(0)}}. \tag{S21}$$
For the collision integral, we use the relaxation time approximation with two different relaxation times for the intravalley and intervalley scattering (see main text for an explanation):

\[ f_{\text{coll}}^{(a)} f^{(a)} = - \left[ \Gamma_A \left( \chi^{(a)} - \frac{\langle \chi^{(a)} \rangle}{1^{(a)}} \right) + \Gamma_E \frac{\langle \chi^{(a)} \rangle_{1^{(a)}}}{1^{(a)}} \right] \partial_{\mu p}^{(a)} \left( \frac{\langle \chi^{(a)} \rangle_{p^0}}{1^{(a)}} \right), \]

where \( \alpha \) indicates that the momentum \( p \) is restricted to the vicinity of Weyl node \( \alpha \). As mentioned in the main text, the brackets \( \langle \ldots \rangle \) denote an average over the equilibrium \( (u = 0) \) Fermi surface (see also the next section of this supplemental material).

Plugging Eqs. \((S14)\), \((S15)\), \((S19)\), \((S20)\) and \((S21)\) in Eq. \((S16)\), keeping terms up to first order in \( u \) and restricting \( p \) to the vicinity of node \( \alpha \), we get

\[
D^{(a)} \partial_t \chi^{(a)} + \left( \psi^{(a)} + \frac{e}{\hbar} B (\Omega^{(a)} \cdot \vec{v}^{(a)}) \right) \cdot \partial_r \chi^{(a)} + e(\vec{v}^{(a)} \times B) \cdot \partial_p \chi^{(a)} + D^{(a)} \left[ \Gamma_A \left( \chi^{(a)} - \frac{\langle \chi^{(a)} \rangle}{1^{(a)}} \right) + \Gamma_E \frac{\langle \chi^{(a)} \rangle}{1^{(a)}} \right] \\
= -D^{(a)} \left( \lambda^{(a)}_{ij} u_{ij} + p_i \vec{v}^{(a)}_{ij} u_{ij} - m \vec{v}^{(a)}_{ij} u_{ij} - \delta \mu \right) + e(\vec{v}^{(a)} + \frac{e}{\hbar} (\Omega^{(a)} \cdot \vec{v}^{(a)}) B) \cdot \left( p_i \partial_r u_i + \partial_r \delta \mu \right) \\
- \psi^{(a)} \cdot \left( e + \frac{e^2}{\hbar} \Omega^{(a)} (B \cdot E) \right) + e(\vec{v}^{(a)} \times B) \cdot u, \tag{S23}
\]

where

\[ D^{(a)} \equiv 1 + \frac{e}{\hbar} B \cdot \Omega^{(a)} p. \tag{S24} \]

Once again, for brevity we omit the momentum subscripts from quantities such as \( \Omega \) and \( D \).

Using \( \partial_t \to -i\omega \), \( \partial_r \to i\mathbf{q} \) and \( \chi_p(r,t) \to \chi_p(q,\omega) \), the Fourier transform of Eq. \((S23)\) reads

\[
- \omega D^{(a)} \chi^{(a)} + i \left( \mathbf{q} \cdot \vec{v}^{(a)} + \frac{e}{\hbar} (\mathbf{q} \cdot B) (\Omega^{(a)} \cdot \vec{v}^{(a)}) \right) \chi^{(a)} + e(\vec{v}^{(a)} \times B) \cdot \partial_p \chi^{(a)} + D^{(a)} \left[ \Gamma_A \chi^{(a)} \right. \\
- \Gamma_A \left( \frac{\langle \chi^{(a)} \rangle}{1^{(a)}} \right) + \Gamma_E \left( \frac{\langle \chi^{(a)} \rangle_{1^{(a)}}}{1^{(a)}} \right) = -D^{(a)} \left[ -i \omega \left( \lambda^{(a)}_{ij} + p_i \vec{v}^{(a)}_{ij} \right) u_{ij} + m \vec{v}^{(a)} \omega^2 u_i + i \omega \delta \mu \right] \\
+ i(\delta \mu - \omega p_i u_i) \left( \mathbf{q} \cdot \vec{v}^{(a)} + \frac{e}{\hbar} (\mathbf{q} \cdot B) (\Omega^{(a)} \cdot \vec{v}^{(a)}) \right) - \psi^{(a)} \cdot \left( e + \frac{e^2}{\hbar} \Omega^{(a)} (B \cdot E) \right) - i \omega e(\vec{v}^{(a)} \times B) \cdot u, \tag{S25}
\]

where the complex number \( i \) should not be confused with the subscript \( i \) appearing in \( \lambda_{ij} \), \( u_{ij} \), \( u_i \) and \( p_i \). Also, \( u_{ij} = i(q_i u_j + q_j u_i)/2 \) is the Fourier transform of the strain tensor. The momentum appearing in terms such as \( p_i \vec{v}^{(a)}_{ij} \) and \( p_i u_i \) can be decomposed as \( p = \mathbf{P}^{(a)} + \delta \mathbf{p} \), where \( \mathbf{P}^{(a)} \) is the position of the Weyl node \( \alpha \) in momentum space and \( \delta \mathbf{p} \) is the momentum measured with respect to the node. In the case of time-reversal-symmetric WSM that we will concentrate on hereafter, the contribution from terms involving \( \mathbf{P}^{(a)} \) will be cancelled between time-reversed partners.

Hence, in \( p_i \vec{v}^{(a)}_{ij} \) and \( p_i u_i \), we can effectively think of \( \mathbf{p} \) as the momentum measured with respect to Weyl node \( \alpha \).

In the next section, we solve Eq. \((S25)\) perturbatively in the magnetic field. First, we will solve the BKE for \( B = 0 \) and then we will use this solution to derive an expression of \( \chi \) valid to linear order of magnetic field.

**III. Solution of the BKE to zeroth order in magnetic field**

The solution of Eq. \((S25)\) at \( B = 0 \) will be labeled as \( \chi_0 \). The electric field and the shift of the chemical potential at \( B = 0 \) will be likewise labelled as \( E_0 \) and \( \delta \mu_0 \), respectively. Also, we have \( D^{(a)} = 1 \) and \( \psi^{(a)} = \psi^{(a)} \). Therefore, Eq. \((S25)\) can be written as

\[
(-i \omega + i \mathbf{q} \cdot \mathbf{v}^{(a)} + \Gamma_A) \chi^{(a)}_0 - (\Gamma_A - \Gamma_E) \left( \frac{\langle \chi^{(a)}_0 \rangle}{1^{(a)}} \right) = -\lambda^{(a)}_{ij} \omega q_j u_i - m \vec{v}^{(a)} \omega^2 u_i - i \omega \delta \mu_0 - e E_0 \cdot v^{(a)} + i \mathbf{q} \cdot v^{(a)} \delta \mu_0, \tag{S26}
\]

where the subscript \( 0 \) in \( \langle \ldots \rangle_0 \) indicates that the average is taken on the \( B = 0 \) Fermi surface.
Equation (S26) contains multiple unknowns: $\chi_0$, $E_0$ and $\delta \mu_0$. We will see that they can be related to one another by virtue of Maxwell’s equations and the normalization condition. To do so, we begin by recognizing that the total charge density can be written as $Q + en$, where

$$Q = -en_0(1 - \partial_r \cdot u)$$

is the ionic charge (to first order in $u$), $n_0 = \langle (f_0) \rangle_0$ is the electron density in the absence of lattice vibrations and

$$n = \langle (f) \rangle_0$$

is the total electron density. Here, the double brackets $\langle (...) \rangle$ denote a momentum integral over the equilibrium ($u = 0$) Brillouin zone, and the subscript 0 in $\langle (...) \rangle_0$ is to remind that the integral is done for $B = 0$.

Therefore, the total charge density is

$$Q + en = -en_0(1 - \partial_r \cdot u) + e\langle (f) \rangle_0$$

\[= -en_0 + en_0\partial_r \cdot u + e\langle (f_0) \rangle_0 - e\langle \lambda_{ij}u_{ij} + p_i v_j u_{ij} - mv_i u_i - \delta \mu_0 + \chi_0 \rangle_0 \]

\[= en_0\partial_r \cdot u - e\langle \lambda_{ij}u_{ij} + (\partial_r \cdot u) n_0 - \delta \mu_0(1) \rangle_0 - e\langle \chi_0 \rangle_0 \]

\[= -e\langle \lambda_{ij}u_{ij} + e\delta \mu_0(1) \rangle_0 - e\langle \chi_0 \rangle_0, \]

where we have used $\langle O_p \rangle_0 = 0$ for any function $O_p$ that is odd in momentum, and $\langle p_i v_j \rangle_0 = \delta_{ij} n_0$. We remind the reader that the notation $\langle (...) \rangle$ implies an average over the full Fermi surface, while $\langle (...) \rangle_0$ indicates an average over the piece of the Fermi surface surrounding the node $\alpha$. Evidently, $\langle (...) \rangle = \sum_\alpha \langle (...)^{(\alpha)} \rangle$.

According to Gauss’s law,

$$\epsilon_{lat}\partial_r \cdot E_0 = Q + en,$$

where $\epsilon_{lat}$ is the high-frequency dielectric permittivity. Plugging Eq. (S30) in Eq. (S29), we get

$$\epsilon_{lat}\partial_r \cdot E_0 = -e\langle \lambda_{ij} \rangle_0 u_{ij} + e\delta \mu_0(1) - e\langle \chi_0 \rangle_0.$$

At zero magnetic field, the normalization condition implies $\langle \chi_0 \rangle_0 = 0$. Then,

$$\delta \mu_0 = \frac{\epsilon_{lat}\partial_r \cdot E_0}{e\langle (1) \rangle_0} + \frac{\langle \lambda_{ij} \rangle_0}{\langle (1) \rangle_0} u_{ij}. \tag{S32}$$

Fourier transforming,

$$\delta \mu_0 = \frac{i\mathbf{q} \cdot E_0\epsilon_{lat}}{e\langle (1) \rangle_0} + \frac{\langle \lambda_{ij} \rangle_0}{\langle (1) \rangle_0} i\mathbf{q} \cdot u_{ij}. \tag{S33}$$

Replacing Eq. (S33) in Eq. (S26) yields

$$\chi_0^{(\alpha)} = R^{(\alpha)} \left[ -\lambda_{ij}^{(\alpha)} \omega \delta_{ij} u_i + \frac{\langle \lambda_{ij} \rangle_0}{\langle (1) \rangle_0} \omega \delta_{ij} u_i - m\omega^2 v_i^{(\alpha)} u_i - (\mathbf{q} \cdot \mathbf{v}^{(\alpha)}) \frac{\langle \lambda_{ij} \rangle_0}{\langle (1) \rangle_0} q_j u_i - eE_0 \cdot \mathbf{v}^{(\alpha)} + \frac{\omega (\mathbf{q} \cdot E_0) \epsilon_{lat}}{e\langle (1) \rangle_0} \left[ \chi_0^{(\alpha)} \frac{\langle (1) \rangle_0}{\langle (1) \rangle_0} \right] \right],$$

where

$$R^{(\alpha)} = (-\omega + i\mathbf{q} \cdot \mathbf{v}^{(\alpha)} + \Gamma_A)^{-1}. \tag{S35}$$

We have thus removed one of the unknowns ($\delta \mu_0$). We still need to determine the electric field. To do so, we begin by taking the average of Eq. (S34) over the Fermi surface surrounding the node $\alpha$. The calculation becomes simple if the following conditions are simultaneously satisfied:

(i) $\Gamma_A \gg v_F^{(\alpha)} q$, (ii) $\Gamma_E \gg \max \left( \omega, \frac{(v_F^{(\alpha)})^2 q^2}{\Gamma_A} \right). \tag{S36}$

These conditions are realistic for long-wavelength acoustic phonons. For instance, for a phonon wave vector $q < 10^5 \text{ m/s}$, we have $heq \lesssim 10^{-2} \text{ meV}$ and $\hbar \omega \lesssim 10^{-4} \text{ meV}$. Both of these energy scales are small compared to $\hbar \Gamma_A$. 
in reasonably disordered WSM (where one may anticipate $\hbar \Gamma_A \approx 1 - 10 \text{meV at low temperatures}$). In contrast, $\hbar \Gamma_E$ has been estimated to be of the order of $10^{-2} \text{meV}$ because intervalley impurity scattering is suppressed with respect to intravalley impurity scattering. The fact that $\Gamma_A \gg \Gamma_E$ will be exploited in the next sections for further simplifications.

If the conditions (i) and (ii) above are realized, it is safe to approximate

$$R^{(\alpha)} \approx \frac{1}{\Gamma_A}.$$  \hspace{1cm} (S37)

With this proviso, Eq. (S34) becomes

$$\chi^{(\alpha)}_0 = -\frac{\lambda_{ij}}{\Gamma_A} + \frac{\xi_{ij}}{\Gamma_A} - \frac{m\omega^2 v_A^{(\alpha)} u_i}{\Gamma_A} + \frac{\omega e_{\text{lat}}(q \cdot E_0)}{\Gamma_A} \frac{q \cdot v (\xi_{ij})_0 - (q \cdot v)(q \cdot E_0)_{\text{lat}}}{e(1)_0 \Gamma_A} - \frac{\omega E_0 \cdot v}{\Gamma_A} \frac{q \cdot v (\xi_{ij})_0 - (q \cdot v)(q \cdot E_0)_{\text{lat}}}{e(1)_0 \Gamma_A}$$

$$\quad + \left(1 - \frac{\Gamma_E}{\Gamma_A} \right) \frac{\chi^{(\alpha)}_0}{\chi^{(1)}_0} \left(1 - \frac{\Gamma_E}{\Gamma_A} \right) \frac{\chi^{(\alpha)}_0}{\chi^{(1)}_0}.$$  \hspace{1cm} (S38)

In the limit $\Gamma_A \to \infty$, $\chi_0 \to \langle \chi^{(\alpha)}_0/(\xi^{(1)}_0) \rangle$ becomes independent of the direction of momentum. This limit could have been anticipated from the form of the collision term. Below, the limit $\Gamma_A \to \infty$ will be useful to extract the leading terms contributing to the phonon magnetochemical effect.

Now we take Fermi surface averages of Eq. (S38) around each node. It is here that the condition $\Gamma_E \gg \max(\omega, (v_F^{(\alpha)})^2 q^2/\Gamma_A)$ is invoked. Indeed, without any approximations, one has

$$\langle R^{(\alpha)} \rangle_0 = \frac{\langle 1^{(\alpha)} \rangle_0}{2q v_F^{(\alpha)}} \left[ \tan^{-1} \left( \frac{v_F^{(\alpha)} q - \omega}{\Gamma_A} \right) + \tan^{-1} \left( \frac{v_F^{(\alpha)} q + \omega}{\Gamma_A} \right) + i \frac{\langle 1^{(\alpha)} \rangle_0}{4q v_F^{(\alpha)}} \ln \frac{(q v_F^{(\alpha)} + \omega)^2 + \Gamma_A^2}{(q v_F^{(\alpha)} - \omega)^2 + \Gamma_A^2} \right],$$

which, for $\Gamma_A \gg v_F^{(\alpha)} q \gg \omega$, gives

$$\langle R^{(\alpha)} \rangle_0 \approx \frac{\langle 1^{(\alpha)} \rangle_0}{\Gamma_A} \left(1 - \frac{(v_F^{(\alpha)})^2 q^2}{3\Gamma_A^2} + i \frac{\omega}{\Gamma_A} \right).$$  \hspace{1cm} (S40)

The Fermi surface average of the last term in the right hand side of Eq. (S34) then gives

$$\langle \chi^{(\alpha)}_0 \rangle_0 \left(1 - \frac{\Gamma_E}{\Gamma_A} \right) \left(1 - \frac{(v_F^{(\alpha)})^2 q^2}{3\Gamma_A^2} + i \frac{\omega}{\Gamma_A} \right) \approx \langle \chi^{(\alpha)}_0 \rangle_0 \left(1 - \frac{\Gamma_E}{\Gamma_A} \right) \left(1 - \frac{(v_F^{(\alpha)})^2 q^2}{3\Gamma_A^2} + i \frac{\omega}{\Gamma_A} \right).$$  \hspace{1cm} (S41)

The first term in Eq. (S41) is clearly the largest; however, it cancels with the average of the left hand side of Eq. (S34). Doing the approximation in Eq. (S37) is tantamount to saying that, in Eq. (S41), $\Gamma_E/\Gamma_A \gg (v_F^{(\alpha)})^2 q^2/\Gamma_A^2$ and $\Gamma_E/\Gamma_A \gg \omega/\Gamma_A$, so that the last two terms in Eq. (S41) may be neglected. This then results in the conditions shown in Eq. (S36).

Taking the Fermi surface averages of Eq. (S38) around each node results in a system of two equations which, combined with the normalization condition $\langle \chi^{(\alpha)}_0 \rangle_0 + \langle \chi^{(-)}_0 \rangle_0 = 0$, gives

$$E_0 \parallel = 0.$$  \hspace{1cm} (S42)

for the longitudinal component of the electric field, and

$$\langle \chi^{(+)}_0 \rangle_0 = -\langle \chi^{(-)}_0 \rangle_0 = -\frac{\omega q_{ij} u_i}{E(1)_0} \left[ (\lambda^{(+)}_0 - 1^{(1)}_0) - (\lambda^{(-)}_0 - 1^{(-)}_0) \right].$$  \hspace{1cm} (S43)

We note that the longitudinal part of the electric field would not have vanished if we had not approximated $R^{(\alpha)}$ as $1/\Gamma_A$. In the isotropic approximation for the deformation potential tensor ($\lambda_{ij} = \lambda_1 \delta_{ij} + \lambda_2 p_i p_j/p^2$), we find

$$\langle \chi^{(+)}_0 \rangle_0 = -\langle \chi^{(-)}_0 \rangle_0 = -\frac{\omega q_{ij} (1^{(+)}_0 - 1^{(-)}_0)}{E(1)_0} \left( \frac{1}{3} \lambda^{(+)}_1 - \lambda^{(-)}_1 + \frac{1}{3} \lambda^{(+)}_2 - \frac{1}{3} \lambda^{(-)}_2 \right).$$  \hspace{1cm} (S44)
where we have used \((\lambda^{(\alpha)}_0)\)\(_0 = \delta_{ij}(\lambda^{(\alpha)}_0 + \lambda^{(\alpha)}_0/3(1^{(\alpha)}))\). We thus learn that, in the absence of a magnetic field, it is necessary to have a different deformation potentials on Weyl nodes of opposite chirality in order for lattice vibrations to induce an electrochemical potential difference between them. This occurs only in chiral WSM, where \(\lambda^{(+)}_{1(2)} \neq \lambda^{(-)}_{1(2)}\).

The transverse component of the electric field may be obtained by combining Faraday’s law with Ampère-Maxwell’s law, and by computing the current density from the electronic distribution function. For simplicity, we neglect the transverse electric fields produced by sound waves. This may be justified by the fact that the magnetic fields induced by lattice vibrations are small and vary slowly in time.

Therefore, Eq. (S38) becomes

\[
\chi^{(\alpha)}_0 = -\frac{\omega}{\Gamma_A} q_{ij} u_i \left( \lambda^{(\alpha)}_0 - \frac{\lambda^{(\alpha)}_0}{(1^{(\alpha)})_0} \right) - \frac{m\alpha^2 v^{(\alpha)}_1 u_i}{\Gamma_A} - \frac{\mathbf{q} \cdot \mathbf{v} (\lambda^{(\alpha)}_0)}{(1^{(\alpha)})_0} q_{ij} u_i + \frac{\lambda^{(\alpha)}_0}{(1^{(\alpha)})_0} \left( 1 - \frac{\Gamma_A}{\Gamma_A} \right)
\]  

(S45)

This completes the approximate solution of the BKE to zeroth order in \(B\). It contains no signatures of the Berry curvature. Such signatures will only appear when we turn on the magnetic field. Next, we will search for the solution of the BKE to first order in \(B\).

**IV. Solution of the BKE to first order in the magnetic field**

In this section, we derive the solution of the BKE to first order in a magnetic field. Before embarking on the subject, we present some mathematical preliminaries that will prove useful later on. We begin by recalling that, in a magnetic field, the expression for density of states in momentum space changes as

\[
\frac{1}{(2\pi \hbar)^3} \rightarrow \frac{1}{(2\pi \hbar)^3} (1 + \frac{e}{\hbar} \mathbf{B} \cdot \mathbf{\Omega_p}).
\]  

(S46)

Accordingly, the volume integral of a function \(\psi_p(B)\) over the Brillouin zone reads

\[
\langle \psi_p(B) \rangle = \int \frac{d^3p}{(2\pi \hbar)^3} \psi_p(B) \left( 1 + \frac{e}{\hbar} \mathbf{B} \cdot \mathbf{\Omega_p} \right).
\]  

(S47)

Similarly, the Fermi surface average of \(\psi_p(B)\) reads

\[
\langle \psi_p(B) \rangle = -\langle \psi_p(B) \frac{\partial f_0}{\partial \epsilon^{(0)}_p} \rangle = \int \frac{d^3p}{(2\pi \hbar)^3} \psi_p(B) \left( 1 + \frac{e}{\hbar} \mathbf{B} \cdot \mathbf{\Omega_p} \right) \delta(\epsilon^{(0)}_p - \mu_0).
\]  

(S48)

Below, we will be interested in evaluating volume and surface integrals to first order in magnetic field. We denote these quantities as \(\langle \psi_p(B) \rangle_1\) and \(\langle \psi_p(B) \rangle_1\), respectively. The formal expression for \(\langle \psi_p(B) \rangle_1\) can be rapidly obtained:

\[
\langle \psi_p(B) \rangle_1 \simeq \int \frac{d^3p}{(2\pi \hbar)^3} (\psi_p(B=0) + \mathbf{B} \cdot \partial\psi_p(B=0)) \left( 1 + \frac{e}{\hbar} \mathbf{B} \cdot \mathbf{\Omega_p} \right) \\
\simeq \langle \psi_p(B=0) \rangle_0 + \langle \mathbf{B} \cdot \partial\psi_p(B=0) \rangle_0 + \frac{e}{\hbar} \langle \psi_p(B=0) \rangle_0 \mathbf{\Omega_p} \cdot \mathbf{B} \rangle_0.
\]  

(S49)

The formal expression for \(\langle \psi_p(B) \rangle_1\) is slightly more cumbersome due to the presence of the Dirac delta and the fact that the energy of the electrons depends on the magnetic field via the magnetic moment \(\mathbf{m_p}\):

\[
\langle \psi_p^{(\alpha)}(B) \rangle_1 \simeq \int \frac{d^3p}{(2\pi \hbar)^3} \left( \psi^{(\alpha)}_p(B=0) + \mathbf{B} \cdot \partial\psi^{(\alpha)}_p(B=0) \right) \left( 1 + \frac{e}{\hbar} \mathbf{\Omega_p}^{(\alpha)} \cdot \mathbf{B} \right) \times \\
\times \left( \delta(\epsilon^{(\alpha)}_p(p) - \mu_0) - \frac{\mathbf{m_p}^{(\alpha)} \cdot \mathbf{B}}{v_F^{(\alpha)}} \partial_p \delta(\epsilon^{(\alpha)}_p(p) - \mu_0) \right).
\]  

(S50)

where, as usual, the subscript \(\alpha\) is to remind that the momentum integral in Eq. \((S50)\) is restricted to the vicinity of the Weyl node \(\alpha\). In the last line of Eq. \((S50)\), we have used

\[
\frac{\partial F(\epsilon^{(\alpha)}_0)}{\partial \epsilon^{(\alpha)}_0} = \left( \frac{\partial \epsilon^{(\alpha)}_0}{\partial p} \right)^{-1} \frac{\partial F(\epsilon^{(\alpha)}_0)}{\partial p} = \frac{1}{v_F^{(\alpha)}} \frac{\partial F(\epsilon^{(\alpha)}_0)}{\partial p},
\]  

(S51)
valid for any function \( F \) that depends on momentum only via \( \varepsilon_0^{(\alpha)} = v_F^{(\alpha)} p \). Here, \( v_F^{(\alpha)} \) is the Fermi velocity or the slope of the Weyl dispersion in the vicinity of node \( \alpha \), and \( p \) is the magnitude of the momentum measured from the node. Neglecting \( O(B^2) \) terms in Eq. (S50), we have

\[
\langle \psi_\alpha^{(\alpha)}(p) \rangle_1 = \int \frac{d^3p}{(2\pi\hbar)^3} \psi_\alpha^{(\alpha)}(p) \left[ \frac{\psi_\alpha^{(\alpha)}(0) + \mathbf{B} \cdot \partial_\mathbf{B} \psi_\alpha^{(\alpha)}(p)|_{\mathbf{B}=0} + \frac{e}{\hbar} \psi_\alpha^{(\alpha)}(p)|_{\mathbf{B}=0}(\mathbf{\Omega}_p^{(\alpha)} \cdot \mathbf{B})}{v_F^{(\alpha)}} \right] \delta(p - p_F^{(\alpha)}) 
\]

(S52)

where \( p_F^{(\alpha)} \) is the Fermi momentum measured from node \( \alpha \). From Eq. (S7), \( p_F^{(\alpha)} \) is defined via \( \mu_0 = v_F^{(\alpha)} p_F^{(\alpha)} + \alpha \Delta \). Using the identity

\[
\int f(p) \partial_p \delta(p - p_0) dp = -\partial_p f(p)|_{p=p_0},
\]

(S53)

we rewrite Eq. (S52) as

\[
\langle \psi_\alpha^{(\alpha)}(p) \rangle_1 = \int \frac{dS_F^{(\alpha)}}{(2\pi\hbar)^3} \psi_\alpha^{(\alpha)}(0) + \mathbf{B} \cdot \partial_\mathbf{B} \psi_\alpha^{(\alpha)}(p)|_{\mathbf{B}=0} + \frac{e}{\hbar} \psi_\alpha^{(\alpha)}(p)|_{\mathbf{B}=0}(\mathbf{\Omega}_p^{(\alpha)} \cdot \mathbf{B}) 
\]

(S54)

where \((\theta, \phi)\) are the polar and azimuthal angles in spherical coordinates and

\[
dS_F^{(\alpha)} = (p_F^{(\alpha)})^2 \sin \theta d\theta d\phi
\]

(S55)

is the surface area element on the Fermi surface near node \( \alpha \). In other words,

\[
\langle \psi_\alpha^{(\alpha)}(p) \rangle_1 = \langle \psi_\alpha^{(\alpha)}(0) \rangle_0 + \left\langle \frac{e}{\hbar} (\mathbf{B} \cdot \mathbf{\Omega}_p^{(\alpha)} \psi_\alpha^{(\alpha)}(p)|_{\mathbf{B}=0} \right\rangle_0 + \left\langle \mathbf{B} \cdot (\partial_\mathbf{B} \psi_\alpha^{(\alpha)}) \right\rangle_0 
\]

(S56)

where

\[
\langle \psi_\alpha^{(\alpha)} \rangle_0 = \frac{1}{(2\pi\hbar)^3} \int dS_F^{(\alpha)} \psi_\alpha^{(\alpha)}(p_F^{(\alpha)}).
\]

(S57)

In our minimal model of electron-doped WSM, for which \( \mathbf{m}_p^{(\alpha)} = -e v_F^{(\alpha)} \mathbf{p} \mathbf{\Omega}_p^{(\alpha)} \), Eq. (S56) can be further simplified as

\[
\langle \psi_\alpha^{(\alpha)}(p) \rangle_1 = \langle \psi_\alpha^{(\alpha)}(0) \rangle_0 + \left\langle \mathbf{B} \cdot (\partial_\mathbf{B} \psi_\alpha^{(\alpha)}) \right\rangle_0 + \frac{1}{v_F^{(\alpha)}} \left\langle \mathbf{B} \right\rangle_0 \left\langle \mathbf{m}_p^{(\alpha)} \cdot \mathbf{B} \right\rangle_0 
\]

(S58)

Armed with Eqs. (S49) and (S56), we now begin to derive the solution of the BKE in the presence of magnetic field. To linear order in \( B \), we can expand

\[
\chi = \chi_0 + \chi_1 \\
\delta \mu = \delta \mu_0 + \delta \mu_1
\]

(S59)

where the subscripts 0 and 1 have the meanings of zeroth order and linear order in \( B \), respectively. Using Eq. (S59) and Eq. (S20), we collect terms that are first order in \( B \) and arrive at
\frac{\chi_1^{(a)}}{R^{(a)}} = -\frac{e}{\hbar} (\mathbf{B} \cdot \mathbf{\Omega}^{(a)}) \left[ \lambda_{ij}^{(a)} \omega_0 \mathbf{u}_i \right] \left\{ \frac{\langle \lambda_{ij}^{(a)} \rangle_0}{\omega_0 \mathbf{u}_j} + \chi_0^{(a)} (\Gamma_A - i\omega) - (\Gamma_A - \Gamma_E) \frac{\langle \chi_1^{(a)} \rangle_0}{\langle 1^{(a)} \rangle_0} \right\}
+ i\mathbf{q} \cdot \left( \frac{e}{\hbar} (\mathbf{\Omega}^{(a)} \cdot \mathbf{v}^{(a)}) \mathbf{B} - \partial_p (\mathbf{m}^{(a)} \cdot \mathbf{B}) \right) \left( i\mathbf{q} \cdot \mathbf{u}_i \frac{\langle \lambda_{ij}^{(a)} \rangle_0}{\omega_0 \mathbf{u}_j} - \chi_0^{(a)} \right) - e(\mathbf{v}^{(a)} \times \mathbf{B}) \cdot (i\omega \mathbf{u} + \partial_p \chi_0^{(a)})
+ i(\mathbf{q} \cdot \mathbf{v}^{(a)} - \omega) \delta \mu_1 - e\mathbf{v}^{(a)} \cdot \mathbf{E}_1 + (\Gamma_A - \Gamma_E) \frac{\langle \chi_1^{(a)} \rangle_0}{\langle 1^{(a)} \rangle_0} \right\},
(S60)

where \chi_0 may be replaced by Eq. (S45). As expected, \chi_1^{(a)} \rightarrow \langle \chi_1^{(a)} \rangle_0 / \langle 1^{(a)} \rangle_0 when \Gamma_A \rightarrow \infty. In the derivation of Eq. (S60), we have used \mathbf{E}_0 \simeq 0 and \delta \mu_0 \approx i\mathbf{q} \cdot \mathbf{u}_i \langle \lambda_{ij}^{(a)} \rangle_0 / \langle 1^{(a)} \rangle_0. We have also used the relations

\langle \chi_0^{(a)} + \chi_1^{(a)} \rangle_1 \simeq \langle \chi_0^{(a)} \rangle_1 + \langle \chi_1^{(a)} \rangle_0 \simeq \langle \chi_0^{(a)} \rangle_0 + \langle \chi_1^{(a)} \rangle_0
\langle \chi_1^{(a)} \rangle_1 \simeq \langle \lambda_{ij}^{(a)} \rangle_0
\langle 1^{(a)} \rangle_1 \simeq \langle 1^{(a)} \rangle_0,
(S61)

which are valid to first order in \mathbf{B} and can be obtained from Eq. (S56). The last equality in the first line of Eq. (S61) relies on the fact that \partial_p \chi_0^{(a)} is even in momentum (note that \partial_p is the derivative with respect to the magnitude of the momentum). In addition, in Eq. (S60) we have omitted certain terms that are small and make a negligible contribution to the final results. These omissions rely on the fact that the following dimensionless ratios are very small for weakly doped semimetals:

\frac{p_F^{(a)} v_F^{(a)}}{\chi^{(a)}} \frac{m v_F^{(a)} \omega^2}{\chi^{(a)}}, \frac{c_s p_F^{(a)}}{\chi^{(a)}}, \frac{m v_F^{(a)} c_s^2}{\chi^{(a)}},
(S62)

Here, \textit{c}_s is the speed of sound in the absence of itinerant electrons and \omega \approx c_s q (modulo small corrections that we aim to calculate below).

Equation (S60) contains various unknowns: \chi_1^{(a)}, \delta \mu_1 and \mathbf{E}_1. They are related to one another by virtue of Maxwell’s equations and the normalization condition. The procedure to find these relations is akin to the one followed for the \mathbf{B} = 0 case. Like in that case, we will apply the approximation \Gamma^{(a)} \simeq 1/\Gamma_A, which is justified when \Gamma_A \gg v_F^{(a)}q and \Gamma_E \gg \max(\omega, (v_F^{(a)})^2 q^2 / \Gamma_A).

To first order in \mathbf{B} and \mathbf{u}, the total charge density reads

\mathbf{Q} + e\mathbf{n} = -e \langle f_0 \rangle_1 (1 - \partial_t \cdot \mathbf{u}) + e \langle f \rangle_1
= en_0 \partial_t \cdot \mathbf{u} - e \langle \chi_0 + \chi_1 \rangle_1.\, (S63)

The normalization condition implies that \langle \chi_0 + \chi_1 \rangle_1 = 0. Also, to first order in \mathbf{B}, \langle f_0 \rangle_1 \simeq \langle f \rangle_0, \langle p_i \mathbf{v}_j \rangle_1 \simeq \langle p_i \mathbf{v}_j \rangle_0 = \delta_{ij} \langle f_0 \rangle_0, \langle \lambda_{ij} \rangle_1 \simeq \langle \lambda_{ij} \rangle_0 and \langle 1 \rangle_1 \simeq \langle 1 \rangle_0. Using these relations, together with \mathbf{E}_0 \simeq 0 and \delta \mu_0 \approx \mu_0 \langle \lambda_{ij} \rangle_0 / \langle 1 \rangle_0, Gauss’ law can be written as

\epsilon_{lat} \partial_t \cdot \mathbf{E}_1 = e\delta \mu_1 \langle 1 \rangle_0 + em \mathbf{u}_i \langle \mathbf{v}_i \rangle_1,\, (S64)

which yields

\delta \mu_1 = \frac{i\mathbf{q} \cdot \mathbf{E}_1 \epsilon_{lat}}{e \langle 1 \rangle_0} + \frac{i m \omega \langle \mathbf{v}_i \rangle_1 \mathbf{u}_i}{\langle 1 \rangle_0}.\, (S65)

Next, we plug Eq. (S65) in Eq. (S60), thereby eliminating one of the unknowns. We still need to eliminate \mathbf{E}_1. The strategy to follow is to take the average of Eq. (S60) over the Fermi surface surrounding the node \alpha, and then apply the normalization condition \langle \chi_1^{(a)} \rangle_0 + \langle \chi_1^{(-)} \rangle_0 = 0. This gives the following expression for the longitudinal part of \mathbf{E}_1:

\mathbf{E}_{1||} = \frac{\epsilon}{\omega q \epsilon_{lat}} \left[ i \sum_{\alpha = \pm} \frac{e}{\hbar} \langle \mathbf{O}^{(a)} \rangle_0 \mathbf{v}^{(a)} \mathbf{B} - \langle \partial_p (\mathbf{m}^{(a)} \cdot \mathbf{B}) \rangle_0 \right] \left( 1 - \frac{\Gamma_E}{\Gamma_A} \right) \frac{\langle \chi_1^{(a)} \rangle_0}{\langle 1^{(a)} \rangle_0}
- i \sum_{\alpha = \pm} \left( \frac{e}{\hbar} \langle \mathbf{O}^{(a)} \rangle_0 \mathbf{v}^{(a)} \lambda_{ij}^{(a)} \rangle_0 \mathbf{B} - \langle \partial_p (\mathbf{m}^{(a)} \cdot \mathbf{B}) \lambda_{ij}^{(a)} \rangle_0 \right) \frac{\omega q \mathbf{u}_i}{\Gamma_A},\, (S66)
where we have used \( \langle \chi_0^{(+)} \rangle_0 = -\langle \chi_0^{(-)} \rangle_0 \), \( \langle \mathbf{v}(\alpha) \rangle_0 = 0 \), \( \langle (\mathbf{v}(\alpha) \times \mathbf{B}) \cdot \partial_p \chi_0^{(\alpha)} \rangle_0 = 0 \), \( \sum_i (\partial_i \mathbf{v}_j^{(\alpha)})_0 = 0 \) (for any \( i \) and \( j \)) and \( \sum_a (\partial_a (\mathbf{m}^{(\alpha)} \cdot \mathbf{B}))_0 = 0 \). In addition, in Eq. (S66) we have omitted terms that are proportional to the free electron mass \( m \). We have verified that the contribution of the latter to the phonon magnetochiral effect is intrinsic (i.e., independent of \( \Gamma_A \) and \( \Gamma_E \)) and geometric (i.e., proportional to \( |\mathcal{C}| \)), but quantitatively negligible.

We note that \( E_{1,\parallel} \) vanishes if the deformation potential has the same value in nodes of opposite chirality (which will be the case in nonchiral WSM). Likewise, \( E_{1,\parallel} \) vanishes if \( \mathbf{q} \perp \mathbf{B} \). Interestingly, \( E_{1,\parallel} \) is of purely geometrical origin (proportional to \( |\mathcal{C}| \)).

Much like in the \( B = 0 \) case, we will neglect the transverse component of \( \mathbf{E}_1 \). Then, using Eq. (S66), we get

\[
\langle \chi_1^{(\alpha)} \rangle_0 = \frac{3e}{\hbar} \frac{\langle (\mathbf{B} \cdot \mathbf{\Omega}^{(\alpha)}) \mathbf{a} \cdot \mathbf{v}(\alpha) \rangle_0}{\Gamma_E} \langle \lambda_{ij} \rangle_0 \frac{1}{(1)_{0}} q_j u_i - \frac{2e}{\hbar} \frac{\langle \mathbf{v}(\alpha) \cdot \mathbf{\Omega}^{(\alpha)} \rangle_0 (\mathbf{B} \cdot \mathbf{q}) \langle \lambda_{ij} \rangle_0}{(1)_{0}} q_j u_i
\]

\[- \frac{e}{\hbar} \langle (\mathbf{B} \cdot \mathbf{\Omega}^{(\alpha)}) \mathbf{a} \cdot \mathbf{v}(\alpha) \rangle_0 \frac{i\omega}{\Gamma_A \Gamma_E} \langle \lambda_{ij} \rangle_0 \frac{1}{(1)_{0}} q_j u_i - i\mathbf{q} \cdot \left( \frac{e}{\hbar} \langle \mathbf{\Omega}^{(\alpha)} \mathbf{a} \cdot \mathbf{v}(\alpha) \rangle_0 \mathbf{B} - \langle \partial_\mathbf{p} (\mathbf{m}^{(\alpha)} \cdot \mathbf{B}) \rangle_0 \right) \langle \chi_0^{(+)} \rangle_0 \frac{1}{(1)_{0}} \left( \frac{1}{\Gamma_E} - \frac{1}{\Gamma_A} \right)
\]

\[+ i\mathbf{q} \cdot \left( \frac{e}{\hbar} \langle \mathbf{\Omega}^{(\alpha)} \mathbf{a} \cdot \mathbf{v}(\alpha) \rangle_0 \mathbf{B} - \langle \partial_\mathbf{p} (\mathbf{m}^{(\alpha)} \cdot \mathbf{B}) \rangle_0 \right) \langle \lambda_{ij} \rangle_0 \frac{\omega q_j u_i}{(1)_{0} \Gamma_E \Gamma_A}
\]

\[+ i\mathbf{q} \cdot \sum_{\beta = +, -} \left( \frac{e}{\hbar} \langle \mathbf{\Omega}^{(\beta)} \mathbf{a} \cdot \mathbf{v}(\beta) \rangle_0 \mathbf{B} - \langle \partial_\mathbf{p} (\mathbf{m}^{(\beta)} \cdot \mathbf{B}) \rangle_0 \right) \langle \chi_0^{(+)} \rangle_0 \frac{\omega q_j u_i}{(1)_{0} \Gamma_E \Gamma_A}
\]

(S67)

This equation satisfies the normalization condition \( \langle \chi_1^{(+)} \rangle_0 + \langle \chi_1^{(-)} \rangle_0 = 0 \). Much like \( E_{1,\parallel} \), \( \langle \chi_1^{(1)} \rangle_0 \) is of geometrical origin and vanishes when \( \mathbf{q} \perp \mathbf{B} \). The fact that a collinear \( \mathbf{q} \) and \( \mathbf{B} \) induce an electrochemical potential difference between the nodes is related to the chiral anomaly.

Equations (S60), (S65), (S66) and (S67), together with the results from the previous section and Eq. (S59), complete the solution of the BKE to first order in \( B \). This solution will enable us to derive the expressions for the sound velocity and attenuation.

V. Elasticity equations for lattice vibrations in the presence of Weyl fermions

To calculate the velocity and attenuation of sound propagation, we use the elasticity equation for the lattice in the presence of conduction electrons

\[
\rho \ddot{\mathbf{u}} = \partial_r \sigma_{kk}^{\text{lat}} + \left( \mathbf{j}_{\text{el}}(\mathbf{r}, t) + \mathbf{j}_{\text{lat}}(\mathbf{r}, t) \right) \times \mathbf{B} + \mathbf{F}(\mathbf{r}, t),
\]

(S68)

where \( h \in \{x, y, z\} \), \( \rho \) is the mass density of the material, \( \sigma^{\text{lat}} \) is the stress tensor in the absence of conduction electrons, \( \mathbf{j}_{\text{el}} \) is the electronic current density, \( \mathbf{j}_{\text{lat}} \) is the ionic current density and \( \mathbf{F} \) is the drag force exerted by the electrons on the lattice. In Eq. (S68), we have neglected a term involving the time derivative of the total electric current density. This omission has been justified in the main text.

The stress tensor is related to strain through

\[
\sigma_{kk}^{\text{lat}} = s_{hkim} u_{im},
\]

(S69)

where \( s_{hkim} \) is the stiffness tensor whose general form depends on the crystal symmetry of the material.

The electronic current density is given as

\[
\mathbf{j}_{\text{el}}(\mathbf{r}, t) = e \int \frac{d^3 \mathbf{p}}{(2\pi \hbar)^3} \left( 1 + \frac{e}{\hbar} \mathbf{B} \cdot \mathbf{\Omega}_p \right) \mathbf{f}(\mathbf{r}, t),
\]

(S70)

while the lattice current to first order in \( \mathbf{u} \) reads

\[
\mathbf{j}_{\text{lat}}(\mathbf{r}, t) = -n_0 e \mathbf{u},
\]

(S71)
$e n_0 = e \langle f_0 \rangle$ being the ionic charge (to zeroth order in $u$). In linear response to the external magnetic field, it is sufficient to evaluate the total current density at zero field. The outcome reads

$$J = J_{\text{el}} + J_{\text{lat}} = -e \langle v \chi_0 \rangle_0 - \frac{e}{\hbar} \partial_r \times \hat{u} \langle \Omega_i (p_i - mv_i) f_0 \rangle_0. \quad (S72)$$

In the absence of the Berry curvature, the expression for $J$ reduces to that shown in Ref. [S23]. The part of the current coming from the Berry curvature is special in that it depends on all occupied electronic states and not just those at the Fermi surface. One can show that $\langle \Omega_p \cdot v f_0 \rangle_0 = 0$ because the contributions from the two nodes of opposite chirality cancel (regardless of the crystal being chiral or not). In contrast, $\langle \Omega_p \cdot p f_0 \rangle_0 \neq 0$. According to our estimates, this term is nonetheless small and its impact in the phonon magnetoelectric effect will turn out to be quantitatively unimportant.

Therefore, the Lorentz force acting on the current is

$$J \times B = \left[ -\frac{e}{\hbar} \langle v \chi_0 \rangle_0 - \frac{e}{\hbar} \partial_r \times \hat{u} \langle \Omega_i (p_i - mv_i) f_0 \rangle_0 \right] \times B \quad \simeq \frac{e}{V_A} \langle v \times B \rangle \langle q \cdot v \rangle_0 \frac{\langle \lambda_{ij} \rangle_0}{(1)_0} q_i u_i - \frac{e}{\hbar} \langle (B \cdot \partial_r) \hat{u} - \partial_r (\hat{u} \cdot B) \rangle \langle \Omega_i (p_i - mv_i) f_0 \rangle_0. \quad (S73)$$

To first order in $B$, the $h$ component of the drag force is given as

$$F_h(r, t) = \partial_r \langle \lambda_{hk} f \rangle_1 = \partial_r \int \frac{d^3 p}{(2\pi \hbar)^3} \left( 1 + \frac{e}{\hbar} B \cdot \Omega_p \right) \lambda_{hk} f(r, t). \quad (S74)$$

In sum, in order to compute the right hand side of Eq. (S68) to first order in $B$, we require the knowledge of the electronic distribution function $f$ to the same order in $B$. Following Eq. (S17),

$$f \approx f_0(\varepsilon_p^{(0)}) + (\lambda_{ij} u_{ij} + p_i \tilde{v}_j u_{ij} - m \tilde{v}_i \hat{u}_i - \delta \mu + \chi) \frac{\partial f_0}{\partial \varepsilon_p^{(0)}}. \quad (S75)$$

The first term in the right hand side of Eq. (S75) does not depend on space (it is the equilibrium distribution with chemical potential $\mu_0$) and thus it does not contribute to the drag force. The remaining terms lead to

$$F_h = -\partial_r \langle \lambda_{hk} \lambda_{ij} u_{ij} + p_i \tilde{v}_j u_{ij} - m \tilde{v}_i \hat{u}_i - \delta \mu_0 - \delta \mu_1 + \chi_0 + \chi_1 \rangle_1$$

$$\simeq -\partial_r \langle \lambda_{hk} \chi_1 \rangle_0 + \langle \partial_r \delta \mu_1 \rangle \langle \lambda_{hk} \rangle_0 - \partial_r \langle \frac{1}{V_F} (m_{\Omega p} \cdot B) \partial_p (\lambda_{hk} \chi_0) \rangle_0 + \text{terms independent of } B, \quad (S76)$$

where we have neglected an unimportant term proportional to the bare electron mass $m$. We do not explicitly write the zeroth order terms in the magnetic field, because we will be interested in predicting only the $B$-dependence of the sound velocity and attenuation. The third term in Eq. (S76) is proportional to the derivatives of the deformation potentials $\lambda_1$ and $\lambda_2$ with respect to the energy. We will hereafter ignore these terms (by implicitly assuming that $\lambda_1$ and $\lambda_2$ depend weakly on energy in the vicinity of the Fermi surface).

Upon Fourier transforming, we get

$$F_h \simeq -iq_k \langle \lambda_{hk} \chi_1 \rangle_0 + iq_k \delta \mu_1 \langle \lambda_{hk} \rangle_0 + \text{terms independent of } B. \quad (S77)$$

In order to connect with the results from the previous section, Eq. (S77) may be rewritten as

$$F_h \simeq -iq_k \sum_{\alpha = +, -} \left( \langle \lambda_{hk} \chi_1^{(\alpha)} \rangle_0 - \delta \mu_1 \langle \lambda_{hk} \rangle_0 \right) + \text{terms independent of } B. \quad (S78)$$

**VI. Velocity and attenuation of sound waves in WSM: phonon magnetoelectric effect**

Inserting the expressions for the drag force (Eq. (S77)) and the Lorentz force (Eq. (S73)) in Eq. (S68), we rewrite the elasticity equation as

$$\rho \omega^2 u_h = s_{hkim} q_k m u_i - \frac{e}{V_A} \langle v \times B \rangle h(q \cdot v) \langle \lambda_{ij} \rangle_0 (1)_0 q_j u_i + \frac{e}{\hbar} \langle (B \cdot q) \omega u_i \delta_{hi} - q_h (\omega u_i B_i) \rangle \langle \Omega_j (p_j - mv_j) f_0 \rangle_0 - iq_k \delta \mu_1 \langle \lambda_{hk} \rangle_0 + iq_k \langle \lambda_{hk} \chi_1 \rangle_0, \quad (S79)$$
where $h$ is a fixed index and the rest are summed over. Thus Eq. (S79) is a system of three equations (one for each value of $h$) that can be written in matrix form and solved as an eigenvalue problem. The eigenvalues give $\omega$ as a function of $q$ and $B$, while the eigenvectors give the direction of $u$ for the three acoustic modes. The terms of the drag force that are independent of the magnetic field renormalize the phonon frequency and render a nonzero linewidth; these effects will be implicitly absorbed through a redefinition of the stiffness tensor and by adding an imaginary $B$-independent part to the phonon frequency in Eq. (S79).

From here on we fix $B = B_\parallel \hat{z}$, where $\hat{z}$ is a high symmetry direction of a chiral cubic crystal (point group $O$ or $T$). We first consider the configuration in which sound propagates along $\hat{z}$, with a wave vector $q = q_\parallel \hat{z}$. Both $B_\parallel$ and $q_\parallel$ may be either positive or negative. For this configuration, Eq. (S79) becomes

\begin{align}
\rho \omega^2 u_z &= q_\parallel^2 s_{zzzz} u_z - i q_\parallel \delta \mu_1 \langle \lambda_{zz} \rangle_0 + i q_\parallel \langle \lambda_{zz} \chi_1 \rangle_0 \\
\rho \omega^2 u_x &= q_\parallel^2 s_{zzzz} u_x + \frac{e}{h} B_\parallel q_\parallel \omega u_x \langle (\Omega_p (p_j - m v_j) f_0) \rangle_0 + i q_\parallel \langle \lambda_{zz} \chi_1 \rangle_0 \\
\rho \omega^2 u_y &= q_\parallel^2 s_{yyzz} u_y + \frac{e}{h} B_\parallel q_\parallel \omega u_y \langle (\Omega_p (p_j - m v_j) f_0) \rangle_0 + i q_\parallel \langle \lambda_{zy} \chi_1 \rangle_0. 
\end{align}

(S80)

One can readily show that $\delta \mu_1$ and $\langle \lambda_{zz} \chi_1 \rangle_0$ involve only $u_z$. Likewise, it can be shown that the second and third lines in Eq. (S80) contain $u_x$ and $u_y$, but not $u_z$. Accordingly, the first line of Eq. (S80) describes a longitudinal acoustic phonon. The second and third line correspond to transverse acoustic phonons.

Let us first determine the dispersion of the longitudinal mode. On one hand, we need

\[ i q_\parallel \delta \mu_1 \langle \lambda_{zz} \rangle_0 = -\frac{q_\parallel^2}{2} \langle \lambda_{zz} \rangle_0 \epsilon_{\text{lat}} e E_{1,\parallel}, \]

(S81)

where we have used Eq. (S65) and have neglected an unimportant term proportional to the electron mass $m$. Plugging the value of electric field $E_{1,\parallel}$ from Eq. (S66), we reach

\[ i q_\parallel \delta \mu_1 \langle \lambda_{zz} \rangle_0 \simeq i u_\parallel q_\parallel^2 e B_\parallel |C| \frac{1}{4 \pi^2 \hbar^2} \frac{\langle \lambda_{zz} \rangle_0}{\langle \lambda_{zz} \rangle_0} \left( \frac{\langle \lambda_{zz}^{(+)} \rangle_0}{\langle \lambda_{zz}^{(+)} \rangle_0} - \frac{\langle \lambda_{zz}^{(-)} \rangle_0}{\langle \lambda_{zz}^{(-)} \rangle_0} \right), \]

(S82)

where we have used the expression of $\langle \lambda_0^{(+)} \rangle$ from Eq. (S43) and we have omitted terms that are a factor $\Gamma_A/\Gamma_E$ smaller than the terms shown. This omission is justified insofar as $\Gamma_A \gg \Gamma_E$, which is believed to hold in WSM. In addition, in the derivation of Eq. (S82), we have used the relations

\[ \langle \Omega^{(\alpha)} \cdot \psi^{(\alpha)} \rangle_0 = \frac{\alpha |C|}{4 \pi^2 \hbar}, \]

\[ \langle \partial_p, m_\parallel^{(\alpha)} \rangle_0 = -\frac{e \alpha |C|}{12 \pi^2 \hbar^2}. \]

(S83)

On the other hand, resorting to similar approximations (e. g., invoking $\Gamma_A \gg \Gamma_E$), we obtain

\[ -i q_\parallel \langle \lambda_{zz} \chi_1 \rangle_0 \simeq i u_\parallel q_\parallel^3 e B_\parallel |C| \frac{1}{4 \pi^2 \hbar^2} \frac{\langle \lambda_{zz} \rangle_0}{\langle \lambda_{zz} \rangle_0} \left( \frac{\langle \lambda_{zz}^{(+)} \rangle_0}{\langle \lambda_{zz}^{(+)} \rangle_0} - \frac{\langle \lambda_{zz}^{(-)} \rangle_0}{\langle \lambda_{zz}^{(-)} \rangle_0} \right) \\
- u_\parallel q_\parallel^3 e B_\parallel |C| \omega \left( \langle 1^{(+)} \rangle_0 - \langle 1^{(-)} \rangle_0 \right) \left( \frac{\langle \lambda_{zz}^{(+)} \rangle_0}{\langle \lambda_{zz}^{(+)} \rangle_0} - \frac{\langle \lambda_{zz}^{(-)} \rangle_0}{\langle \lambda_{zz}^{(-)} \rangle_0} \right)^2. \]

(S84)

Inserting Eq. (S82) and Eq. (S84) in the first line of Eq. (S80), we have

\begin{align}
0 &= q_\parallel^2 s_{zz} - \rho \omega^2 - i q_\parallel \frac{7e B_\parallel |C|}{12 \pi^2 \hbar^2} \frac{1}{\langle \lambda_{zz} \rangle_0} \left( \frac{\langle \lambda_{zz}^{(+)} \rangle_0}{\langle \lambda_{zz}^{(+)} \rangle_0} - \frac{\langle \lambda_{zz}^{(-)} \rangle_0}{\langle \lambda_{zz}^{(-)} \rangle_0} \right) \\
+ & q_\parallel^3 e B_\parallel |C| \omega \left( \langle 1^{(+)} \rangle_0 - \langle 1^{(-)} \rangle_0 \right) \left( \frac{\langle \lambda_{zz}^{(+)} \rangle_0}{\langle \lambda_{zz}^{(+)} \rangle_0} - \frac{\langle \lambda_{zz}^{(-)} \rangle_0}{\langle \lambda_{zz}^{(-)} \rangle_0} \right)^2, 
\end{align}

(S85)

where $s_{zz} = s_{zzzz}$. The magnetic-field corrections to the acoustic phonon dispersion are odd in $q_\parallel$ as well as in $B_\parallel$, and proportional to $|C|$. This confirms the existence of a phonon magnetochiral effect of purely band-geometric origin.
the magnetochiral effect will be more pronounced in the sound attenuation than in the sound velocity. From Eq. (S91) because it is relatively negligible when \( \lambda_{1(2)}^{(+)} + \lambda_{1(2)}^{(-)} \) is of the same order as \( \lambda_{1(2)}^{(+)} - \lambda_{1(2)}^{(-)} \), \( \delta \omega_R / \delta \omega_I \approx c_s(0)q_z \). Thus, we anticipate that the magnetochiral effect will be more pronounced in the sound attenuation than in the sound velocity.

The correction to the sound velocity due to the magnetic field is given by

\[
\delta c_s = \frac{\partial \delta \omega_R}{\partial |q_z|} = q_z |B_z| C \frac{1}{2\pi^2 \hbar^2 \rho} \Gamma_E^2 \left( \frac{\lambda_1^{(+)} - \lambda_1^{(-)} + \lambda_2^{(+)} - \lambda_2^{(-)}}{3} \right)^2,
\]

where we have used the relation

\[
\frac{\lambda_1^{(+)} - \lambda_1^{(-)} + \lambda_2^{(+)} - \lambda_2^{(-)}}{3} = \lambda_1^{(+)} - \lambda_1^{(-)}.
\]

Therefore, the phonon magnetochiral effect in the sound velocity is given as

\[
v_{MC} = c_s(B \parallel \hat{q}) - c_s(B \parallel -\hat{q})
\]

\[
\approx c_s |q_z| B_z |C| \frac{1}{\pi^2 \hbar^2 \rho s(0)} \Gamma_E^2 \left( \frac{\lambda_1^{(+)} - \lambda_1^{(-)} + \lambda_2^{(+)} - \lambda_2^{(-)}}{3} \right)^2.
\]

Clearly, broken inversion and mirror symmetries are required in order to have \( v_{MC} \neq 0 \). In order to proceed with a numerical estimate, we write

\[
\frac{\lambda_1^{(+)} - \lambda_1^{(-)} + \lambda_2^{(+)} - \lambda_2^{(-)}}{3} = 1 - \left( \frac{\epsilon_{F}^{(v)}}{v_F^{(v)}} \right)^2 \left( \frac{v_F^{(v)}}{v_F^{(v)}} \right)^3,
\]

where \( \epsilon_F^{(v)} = v_F^{(c)} p_F^{(c)} \) \( \epsilon_F^{(c)} \) is the distance in energy from the Weyl node \( \alpha \) to the equilibrium chemical potential. Using \( \epsilon_F^{(v)} = 20 \text{ meV}, \epsilon_F^{(c)} = 5 \text{ meV}, v_F^{(v)} = 10^5 \text{ m/s}, v_F^{(c)} = 1.5 \times 10^5 \text{ m/s}, \rho = 10^4 \text{ kg/m}^3, B_z = 1 \text{ T}, c_s(0) = 10^3 \text{ m/s}, q = 0.5 \times 10^6 \text{ m/s}, \lambda_1 = 1 ~ 2 \text{ eV}, we get \( v_{MC} \) of the order of a few parts per million. This is a priori measurable in state-of-the-art ultrasound velocity measurements, whose resolution is about one p.p.m.

Concerning the magnetochiral effect in the sound attenuation, it can be characterized by the dimensionless ratio

\[
r_{MC} \approx \frac{L \delta \omega_I}{c_s(0)}
\]

where \( L \) is the thickness of the sample traversed by the sound wave. The contribution from \( \delta c_s \) to \( r_{MC} \) has been omitted from Eq. (S91) because it is relatively negligible when \( \delta \omega_I \gg \delta \omega_R \) and \( \gamma(0) \ll c_s(0)q_z \). The ratio \( r_{MC} \) describes the relative change in the decay of the amplitude of the sound wave traversing the sample when the propagation direction is parallel and antiparallel to the magnetic field. For the numerical parameters presented above (in addition to \( L = 1 \text{ cm} \)), Eqs. (86) and (91) yield \( r_{MC} \approx 0.12 \), which is large and a priori easily detectable.

Having completed the discussion about the longitudinal mode, let us next investigate the transverse modes. From
Using the expression of produce any chirality imbalance (Eqs. (S43) and (S67) give zero). Solving the equations
Here, we have used
This set of equations can be rearranged in the form
Let us now look at the second equation in Eq. (S95). Here we will use the expressions for
\[ v_y = \frac{e}{h} (B_z \chi_z) \]
Thus, we learn that the transverse modes are circularly polarized in the presence of a magnetic field.

One important aspect of Eq. (S92) is that it does not contain \( \Gamma_E \). The reason is that transverse modes do not produce any chiral charge imbalance (Eqs. (S43) and (S67) both give zero). Solving the equations \( a + ib = 0 \) and \( a - ib = 0 \), we get the dispersion relations for the two transverse modes. These solutions do display a phonon magnetochiral effect. Yet, the effect is quantitatively negligible compared to the magnetochiral effect for the longitudinal mode. The latter is made larger by the fact that the chiral charge imbalance produced by longitudinal phonons relaxes slowly.

Finally let us look at the configuration \( \mathbf{q} \perp \mathbf{B} \). Plugging \( \mathbf{q} = q_x \mathbf{x} \) and \( \mathbf{B} = B_z \mathbf{z} \) in Eq. (S79), we have
Using the expression of \( \chi_1 \) from Eq. (S60), the first equation in Eq. (S95) takes the form

\[ \rho \omega^2 u_x = \mathbf{s}_{\mathbf{q} \mathbf{x} \mathbf{x}} q_x^2 u_x + \left[ \frac{q_x^3}{\Gamma_A} \left( \frac{\lambda_{xx} e}{\hbar} (B_z \Omega_z) u_x \right) + \frac{\omega}{\Gamma_A} q_x^2 \left( \frac{\lambda_{xx} e}{\hbar} B_z \Omega_z v_x \right) \right] u_x. \]

Let us now look at the second equation in Eq. (S95). Here we will use the expressions for \( \delta \mu_1, E_{1||} \) and \( \langle \chi_1^{(a)} \rangle \), such that

\[ \rho \omega^2 u_x = \mathbf{s}_{\mathbf{q} \mathbf{x} \mathbf{x}} q_x^2 u_x + \left[ \frac{q_x^3}{\Gamma_A} \left( \frac{\lambda_{xx} e}{\hbar} (B_z \Omega_z) u_x \right) + \frac{\omega q_x^3}{\Gamma_A} \left( \frac{\lambda_{xx} e}{\hbar} B_z \Omega_z v_x \right) \right] u_x. \]

For the third equation in Eq. (S95) we get

\[ \rho \omega^2 u_y = \mathbf{s}_{\mathbf{q} \mathbf{x} \mathbf{x}} q_x^2 u_y - 2 i q_x^2 \frac{e B_z (\lambda_2^+ - \lambda_2^-) \omega}{\Gamma_A} \left( \frac{\lambda_{xy} v_p u_x}{\rho^2} \right)_0. \]
The aforementioned three equations may be once again solved as an eigenvalue problem, leading to the dispersions of the three acoustic modes. Nevertheless, compared to the Eq. (S85), the contribution of the magnetic field to the dispersions is smaller by a factor $\Gamma_A/\Gamma_E$. Thus, the phonon magnetochiral effect is unlikely to be measurable in the $\mathbf{q} \perp \mathbf{B}$ configuration.