The feedback control of the fractional Mackey-Glass system with monotone production rate

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Abstract. This work is devoted to analyze the stability of the feedback control of the fractional Mackey-Glass system. Both the stability at the zero equilibrium point and the positive equilibrium point of the control system was obtained. Given small control gain for system, the stability interval about time delay and the stability region about destruction rate and time delay was obtain. Finally, an example was given to illustrate the result of paper.

Keywords: Feedback control, Stability switch, Time delay, fractional order, Mackey–Glass system.

1. Introduction (Heading 1)

Recent years have seen remarkable developments in the research of Mackey-glass systems and its modified systems in [1-5] the reference in it. The Mackey-glass system was pioneered by Mackey and Glass in [1] to explain regulation and control mechanisms in physiological systems. And the system is the following model

\[
\frac{dN(t)}{dt} = \frac{\beta \theta^n N(t-\tau)}{\theta^n + N^n(t-\tau)} - ax(t),
\]

(1)

Where \( N(t) \) is the density of cells in the circulation, \( \tau \geq 0 \) is the maturation delay between the start of the production of immature cells in the bone marrow and their release into circulation, \( \beta \) is the feedback rate of the system, \( \theta \) is called a shape parameter, \( a \) is the destruction rate, \( n \) is a positive constant. The detailed biological meaning of these parameters refer to references [1-2]. In[2], system (1) was called Hematopoiesis model with Unimodal Production Rate, and the Stability, Hopf bifurcation, control and other dynamic behaviors of the system were studied in [2,3] and its references.

In [4, 5], the model (1) was extended to the case of fractional order. The discretization schemes was used to study the stability and chaos control of the system without time delay in[4], and the method was also used to study the chaotic of the system with cell loss delay and cell loss term coefficient \( r = 1 \) in [5].

Delay feedback control is one of the most effective methods of chaos control. It is to apply a feedback signal to realize the control of the dynamic system. Pyragas type time-delay feedback control is adopted...
usually to ensure the equilibrium point of the controlled system does not change. The control strategy was pioneered in [6] and has been effective in different systems [7-9].

In the present paper, Consider applying Pyragas type delay feedback control to the system in reference [4, 5], we have the following closed-loop system

\[ C \frac{D_t^\alpha}{\alpha} x(t) = -\alpha x(t) + \frac{\beta x(t - \tau_1)}{1 + x^n(t)} + k(x(t - \tau_2) - x(t)), \] (2)

Where \( k \) is the feedback control gain, \( \tau_2 \) is the time delay in the feedback controller. And \( \beta, a, n, \tau_1 \) has the same meaning with model (1). \( \frac{D_t^\alpha}{\alpha} x(t) \) is the Caputo’s fractional derivative of \( x(t) \) with the order \( \alpha \in (0,1] \) and the detailed definition we refer the reader to [10]. The present work is devoted to stability of the fractional Mackey-Glass control system (2) for the different control gain value \( k \).

The paper is organized as following. In Section 2, the stability of zero equilibrium of the fractional Mackey-glass model is studied; in Section 3, the stability of positive equilibrium of the model is obtained; in Section 4, the stability switches of the fractional Mackey-Glass system is discussed. And finally, an example is given to illustrate the conclusion in the present paper.

2. The Stability of zero Equilibrium Point of the control system

The linearization equation of the control system (2) and the characteristic equation is

\[ s^\alpha + a + k - \beta e^{-\alpha \tau_1} - ke^{-\alpha \tau_2} = 0. \] (3)

One has the conclusion of the stability of zero equilibrium point of system (2).

**Theorem 2.1** The system (2) is unstable for all the time delay \( \tau_2 \) and gain \( k \) at the zero equilibrium point if \( a > 0 \) any \( k < \frac{\beta + a}{2} \).

**Proof** Let \( s = p + iq = re^{i\theta} \) is the root of (3), where

\[ p = \text{Re}(s), \quad q = \text{Im}(s), \quad r = \sqrt{p^2 + q^2}, \quad \theta = \text{Arg}(s), \quad i^2 = 1. \]

Then \( s^\alpha = r^\alpha e^{i\alpha \theta}, e^{-s\tau_2} = e^{-p\tau_2} (\cos(\tau_2q) - i \sin(\tau_2q)) \), Substitute it into (3) and separate the real and the imaginary part, it gets

\[ \begin{cases} r^\alpha \cos(\alpha \theta) + a + k - \beta e^{-p\tau_1} \cos(\tau_1 p) - ke^{-p\tau_2} \cos(\tau_2 p) = 0, \\ r^\alpha \sin(\alpha \theta) + \beta e^{-p\tau_1} \sin(\tau_1 q) + ke^{-p\tau_2} \sin(\tau_2 q) = 0. \end{cases} \] (4)

Obviously, \( q = 0 \) is equivalent to \( \theta = 0 \), and \( q = 0, \theta = 0 \) satisfy the second equation of (4). Substitute \( q = 0 \) and \( \theta = 0 \) into left side of the first equation of (4), and let

\[ f(p) = p^\alpha + a - \beta e^{-p\tau_1} + k(1 - e^{-p\tau_2}), \quad p_0 = \beta^\frac{1}{\alpha}. \]

Then \( p_0 > 0 \) and

\[ f(0) = a - \beta < 0, \quad f(p_0) = a + \beta(1 - e^{-p_0\tau_1}) + k(1 - e^{-p_0\tau_2}) > 0. \]

For the function \( f(p) \) is continuous, There is at least one point \( p \in (0, p_0) \) such that \( f(p) = 0 \), that is the characteristic equation (3) has at least one positive real part characteristic root, and the system (2) is unstable at the zero equilibrium point. The proof is end.
However, given $a > \beta > 0$, one has

**Theorem 2.2** The system (2) is stable for all the time delay $\tau_1$ and $\tau_2$ at the zero equilibrium point if $a > \beta > 0$ any $k < \frac{\beta + a}{2}$.

**Proof** Let $s = i\omega(\omega > 0)$ satisfy (3), Substitute $s^\alpha$ and $e^{i\omega t}$ into (3) and separate the real and the imaginary part, it obtain

$$\omega^\alpha \cos \frac{\pi \alpha}{2} + a = \beta \cos \omega \tau_1 + k \cos \omega \tau_2,$$

$$\omega^\alpha \sin \frac{\pi \alpha}{2} = -\beta \sin \omega \tau_1 - k \sin \omega \tau_2.$$

Then

$$\omega^\alpha = -(a + k) \cos \frac{\pi \alpha}{2} + \sqrt{\beta^2 + k^2 + 2\beta k \cos(\omega(\tau_1 - \tau_2)) - (a + k)^2 \sin^2 \frac{\pi \alpha}{2}}.$$ Furthermore

$$\beta^2 + k^2 + 2\beta k \cos(\omega(\tau_1 - \tau_2)) - (a + k)^2 > 0.$$ And it is equivalent to

$$\beta^2 - a^2 + 2\beta k \cos(\omega(\tau_1 - \tau_2)) > 2ak.$$ For $2ak \geq 2ak \cos(\omega(\tau_1 - \tau_2))$, one has

$$(\beta - a)(\beta + a + 2k \cos(\omega(\tau_1 - \tau_2))) > 0.$$ Then $\beta > a$ for $\beta + a + 2k \cos(\omega(\tau_1 - \tau_2)) > 0$ which obtained by the assumptions of $k < \frac{\beta + a}{2}$.

And $\beta > a$ is contradicts with the assumptions of the theorem. Thus the characteristic root curve of the characteristic equation (3) has never intersected with the imaginary axis.

On the other hand, all characteristic roots of (3) have negative real part if $\tau_1 = \tau_2 = 0$, system (2) is stable at its zero equilibrium point by the results of [11].

Therefore, the system (2) is stable for all the time delay $\tau_1$ and $\tau_2$ at the zero equilibrium point under the conditions of the theorem. The proof is end.

3. The Stability of positive Equilibrium Point of the control system

The point $x^* = \left(\frac{\beta}{a} - 1\right)^{\frac{1}{n}}$ is the positive equilibrium point of the control system. The characteristic equation of the system (2) at $x^*$ is

$$s^\alpha + a + k + be^{-\tau_1} - ke^{-\tau_2} = 0,$$ \hspace{1cm} (5)

Where

$$b = a\left(n - 1 - \frac{na}{\beta}\right).$$ \hspace{1cm} (6)

In the following, the stability of the positive equilibrium point of the system is discussed if the delay $\tau_1 = \tau_2 = \tau$. One has the following equation by (5)
Theorem 3.1 The system (2) is stable for all the time delay $\tau$ at the positive equilibrium point if $\beta > a > 0$ and the control gain $k > \frac{a}{2} \left( n - 2 - \frac{na}{\beta} \right)$.

Proof It is easily to get $s^r = -(a+b)$ if $\tau = 0$ and $b + a = nr \left( 1 - \frac{r}{\beta} \right) > 0$ if $\beta > a$. All characteristic roots of the equation have a negative real part and the system is stable.

In the case of $\tau > 0$, Substitute $s = i\omega$ into (7) and separate the real and the imaginary part, and gets

$$
\begin{align*}
\omega^2 \cos \frac{\pi \alpha}{2} + a + k &= -(b - k) \cos \omega \tau, \\
\omega^2 \sin \frac{\pi \alpha}{2} &= (b - k) \sin \omega \tau.
\end{align*}
$$

Then

$$
\left( \omega^2 + (a + k) \cos \frac{\pi \alpha}{2} \right)^2 = (b - k)^2 - (a + k)^2 \sin^2 \frac{\pi \alpha}{2}.
$$

It can be obtained that Eq. (7) has no pure imaginary roots in the case of $(b - k)^2 < (a + k)^2$. Moreover, $(b - k)^2 < (a + k)^2$ is equivalent to $b - a < 2k$ for $b + a > 0$. Furthermore, $b - a < 2k$ is equivalent to $\frac{na - 2(k + a)}{na} < \frac{a}{\beta}$ by Eq. (6). The proof is end.

From the proof of theorem 3.1, one has the following result if $(b - k)^2 > (a + k)^2$.

Lemma 3.1 The characteristic equation (6) has pure imaginary roots if $\beta > a > 0$ and the control gain $k < \frac{a}{2} \left( n - 2 - \frac{na}{\beta} \right)$.

Under the assumptions of lemma 3.1, one has the result as following.

Theorem 3.2 Under the assumptions of lemma 3.1, there exists a critical delay $\tau_*$ such that the system (2) is stable as $\tau \in [0, \tau_*)$ an unstable at as $\tau \in (\tau_*, +\infty)$ the positive equilibrium point. Hopf bifurcation occurs as $\tau = \tau_*$.

Proof By (9), it arrives

$$
\omega^2 = -(a + k) \cos \frac{\pi \alpha}{2} + \sqrt{(b - k)^2 - (a + k)^2 \sin^2 \frac{\pi \alpha}{2}}.
$$

(10)
From (6), one can get $b - k > 0$ if $k < \frac{a}{2} \left( n - \frac{na}{\beta} \right)$. Thus $\omega \tau$ is in the second quadrant and
\[
\tan \omega \tau = -\frac{\omega^a \sin \frac{\pi \alpha}{2}}{\omega^a \cos \frac{\pi \alpha}{2} + a + k}.
\]
By (8), Let $\theta \in (\frac{\pi}{2}, \pi)$, and
\[
\theta = \pi - \arctan \frac{\omega^a \sin \frac{\pi \alpha}{2}}{\omega^a \cos \frac{\pi \alpha}{2} + a + k}
\]
Then $\omega \tau = \theta + 2j\pi$, $j = 0, 1, 2 \cdots$ and
\[
\tau_j = \frac{\theta + 2j\pi}{\omega}, j = 0, 1, 2 \cdots
\]
Where $\omega$ satisfies (10).

In (7), let $P(s^\alpha) = s^\alpha + a + k, Q(s^\alpha) = b - k$, and $s = i\omega$. Then the real part and imaginary part of $P(i\omega^\alpha)$ and $Q(i\omega^\alpha)$ are the following respectively,
\[
P_R = \omega^a \cos \frac{\pi \alpha}{2} + a + k, P_I = \omega^a \sin \frac{\pi \alpha}{2}, Q_R = b - k, Q_I = 0.
\]
And one can get the auxiliary function
\[
F(\omega, \tau) = (P_R)^2 + (P_I)^2 - (Q_R)^2 + (Q_I)^2
\]
\[
= \omega^{2\alpha} + 2\omega^a (a + k) \cos \frac{\pi \alpha}{2} + (a + k)^2 - (b - k)^2
\]
\[
G(\omega, \tau) = \omega \tau + \arctan \frac{P_I}{P_R} - \arctan \frac{Q_I}{Q_R} = \omega \tau + \arctan \left( \frac{\omega^a \sin \frac{\pi \alpha}{2}}{\omega^a \cos \frac{\pi \alpha}{2} + (a + k)} \right)
\]
Where $b$ satisfies (6).

From the result in [12] one can get
\[
\text{Re} \left( \frac{d\alpha}{d\tau} \right) = \text{sign}(J) = \text{sign} \left( \frac{\partial F}{\partial \omega} \cdot \frac{\partial G}{\partial \tau} \right) = \text{sign} \left( 2\alpha \omega^\alpha (\omega^\alpha + \cos \frac{\pi \alpha}{2} ) \right).
\]
If given $\tau_*$ is the critical delay and $s = i\omega(\tau_*)$ is the pure imaginary root of eq. (7), then
\[
\text{Re} \left( \frac{d\alpha}{d\tau} \right) \bigg|_{(\tau_*, \omega(\tau_*))} > 0 \text{ for } 0 < \alpha \leq 1, \omega^\alpha > 0.
\]
Thus the characteristic root curve crosses the imaginary axis from the complex plane to the right of the complex plane at the point $(\tau_*, \omega(\tau_*))$ as the delay increases across the critical time delay $\tau_*$. As a result, the system is ultimately unstable as time increases. And the characteristic roots of (7) all have negative real parts as $\tau \in [0, \tau_*)$ if $k < \frac{a}{2} \left( n - \frac{na}{\beta} \right)$. The stability switch of the system occurs when $\tau = \tau_*$, where $\tau_* = \min \tau_j$. The proof is end.

From theorem 3.2, the following conclusions can be drawn directly

**Theorem 3.3** Under the assumptions of lemma 3.1, stability region of the positive equilibrium point of the system can be determined uniquely by $\tau = 0$ and $\tau = \tau_*$ on the $\tau-a$ plane.
4. Example

Let $\alpha = 0.9$, $r = 0.1$, $\beta = 2$, $n = 10$, the equilibrium point of (3) is $x_0 = 1.3424$. If $k = 0$, that is the system is uncontrolled, one can get the minimum critical delay $\tau_* = 2.2728$, and $\omega_* = 0.8115$ correspondingly. And the system is asymptotic stable as $\tau \in [0, \tau_*)$, is unstable as $\tau > \tau_*$. As it is shows in Fig.1 that the system is unstable at $\tau_* = 2.28$.

Given the control gain $k = 0.1$, that is the system is controlled, one can get the minimum critical delay $\tau_* = 3.0013$, and $\omega_* = 0.6645$ correspondingly. The control system is asymptotic stable as $\tau \in [0, 3.0013)$, is unstable as $\tau > 3.0013$. It is shows in Fig.2 that the system is asymptotic stable as $\tau = 2.9$. And it is shows in Fig.3 that the system is unstable at $\tau = 3.1$, Hopf bifurcation occurs and there is a periodic solution.

The uncontrolled system is unstable as $\tau \in (2.2728, 3.0013)$, while the control system with control gain $k = 0.1$ is stable if the time delay in the same interval. It can be seen that the system instability is delayed by applying Pyragas type delay feedback control to the system.

**Figure 1.** The Phase diagram of system (2) under the initial condition $x(t) = 0.1, t \in [-\pi, 0]$ with $\alpha = 0.9, r = 0.1, \beta = 2, n = 10, \tau = 2.28, k = 0$.

**Figure 2.** The time history of system (2) under the initial condition $x(t) = 0.1, t \in [-\pi, 0]$ with $\alpha = 0.9, r = 0.1, \beta = 2, n = 10, \tau = 2.9, k = 0.1$. 
Figure 3. The Phase diagram of system (3) under the initial condition\( x(t) = 0.1, t \in [-\pi, 0] \) with\( \alpha = 0.9, r = 0.1, \beta = 2, n = 10, \tau = 3.1, k = 0.1 \).

5. Conclusion
In this paper, we studied the stability of the fractional Mackey-Glass system with Pyrages type delay feedback control. Whether the control system is stable at the zero equilibrium point depends on the value between the feedback rate \( \beta \) of the system and the destruction rate \( a \) of the system for given control gain \( k < \frac{\beta + a}{2} \). If the system delay is equal to the control delay and the feedback rate of the system is greater than the destruction rate, under the condition of large control gain, the control system can be stabilized with all-time delay at the positive equilibrium point. On the other hand, if small control gain is used in the control system, the stability interval of the system with respect to time delay was obtained. Also the stability region about destruction rate and time delay was got.

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