Inverse space-dependent perfusion coefficient identification

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Abstract. The identification of the space-dependent perfusion coefficient in the one-dimensional transient bio-heat conduction equation is investigated. In this inverse coefficient identification problem, the additional measurement necessary to render a unique solution is a boundary temperature measurement. A numerical approach based on a Crank-Nicolson finite-difference scheme combined with Tikhonov’s regularization methods is developed. Numerical results are presented and discussed.

1. Introduction

In medical expertise the accurate evaluations of both temperature and blood perfusion rate through a certain region of tissue under investigation have always been an important task either before or during a surgical intervention, as well as in other thermo-regulatory tests. Further, the blood perfusion physiological importance is underlined by its property of providing the oxygen and nutrients necessary for life processes. The difficulty of estimating these two quantities depends on the place where this investigation, and eventually intervention, is required since, for certain parts of the body, only a restricted number of types of measurements can be carried out. Blood perfusion, which refers to the local, multidirectional flow through the living tissue, is defined as the blood volume exchanged per volume of tissue over time and has the units ml/ml/s, [1, 15]. However, at macroscopic level, the blood perfusion is considered to be a directionless quantity due to the very complex nature of the pathways through which it evolves.

In the one-dimensional case, the temperature and the spacewise-dependent blood perfusion coefficient \( P_f(x) > 0 \) are related through the Pennes bio-heat conduction equation, [2, 10, 13], which, in non-dimensional form, in the absence of sources, is given by:

\[
\frac{\partial^2 T}{\partial x^2}(x, t) - P_f(x)T(x, t) = \frac{\partial T}{\partial t}(x, t), \quad (x, t) \in (0, 1) \times (0, \infty).
\]  

(1)

In equation (1), \( T \) is the temperature of the tissue and the space-dependent perfusion coefficient \( P_f \) is defined by the following ratio

\[
P_f = \frac{w_b c_b L^2}{k},
\]

(2)

where \( w_b \) is the blood perfusion rate, \( c_b \) is the specific heat of the blood, \( L \) is the reference length of the biological body and \( k \) is the thermal conductivity of the tissue.
Since, in the inverse problems under investigation, both the temperature $T$ and the perfusion coefficient $P_f$ are unknown, the nonlinear second-order parabolic equation (1) will be solved under the prescription of initial and Neumann boundary conditions, when a particular type of additional information is provided. However, in general, the measurements accuracy, their invasiveness character, or the extent of practical use as well as the ability to take continuous measurements, create constraints over the range of types of possible additional information that can be considered. Several types of measurements and experimental work, that are modelled with equation (1), are described in [1, 5, 9, 14, 15, 16].

The cases when $P_f$ is assumed constant or time-dependent have been discussed elsewhere, [17, 18]. However, the blood perfusion coefficient need not be constant in space. Given the heterogeneity of the human body tissue, the coefficient $P_f$ will depend on the position in the region under investigation. This paper discusses the retrieval of the space-dependent coefficient $P_f(x) > 0$ along with the temperature $T(x, t)$ from measured exact and noisy boundary temperature measurements.

2. Mathematical Formulation
In this paper the considered space-dependent perfusion coefficient identification inverse problem states as follows:
Find the temperature $T(x,t)$ such that $T, T_x \in \mathcal{C}([0, \pi] \times [0, \infty])$, $T_t, T_{xx} \in \mathcal{C}((0, \pi) \times (0, \infty))$ and the space-dependent perfusion coefficient $P_f(x) \in \mathcal{C}([0, \pi])$, $P_f > 0$, satisfying the one-dimensional time-dependent bio-heat equation (1) subject to the initial condition
$$T(x,0) = 0, \quad x \in [0, \pi],$$
the Neuman boundary conditions
$$\frac{\partial T}{\partial x}(0,t) = 0, \quad t \geq 0, \quad \text{(4)}$$
$$\frac{\partial T}{\partial x}(\pi,t) = \mu(t), \quad t \geq 0, \quad \text{(5)}$$
where $\mu$ satisfies the following properties:
$$\mu \in \mathcal{C}^2([0, \infty)), \quad \mu(0) = \mu'(0) = 0, \quad \mu \neq 0,$$
and there exists $t_0 > 0$ such that $\mu(t) = 0$ for all $t \geq t_0$, *(6)*
and the additional boundary temperature measurement:
$$T(\pi,t) = g(t), \quad t > 0. \quad \text{(7)}$$

The uniqueness of solution for this inverse problem has been established in [6, pp.139-146], and is stated as follows:

**Theorem 2.1** Let $\mu(t)$ satisfy conditions (6). If $P_{f_1}(x), T_1(x,t), i = 1, 2$ are solutions, in the above regularity classes, of the inverse problem (1), (3)-(5) and (7), then $P_{f_1}(x) = P_{f_2}(x)$ for $x \in [0, \pi]$ and $T_1(x,t) = T_2(x,t)$ for $(x,t) \in [0, \pi] \times [0, \infty]$.

Note that if, instead of the boundary temperature measurement (7) at the active end $x = \pi$, where a non-zero heat flux is applied, we supply the additional boundary temperature measurement
$$T(0,t) = g_0(t), \quad t > 0,$$ *(8)*
at the inactive end \( x = 0 \), where no heat flux occurs, then, in order to obtain a unique solution for the inverse problem \((1),(3)-(5)\) and \((8)\), we further need to impose the condition

\[
P_f(x) = P_f(\pi - x) \quad \text{for} \quad x \in [0, \pi],
\]

see [6, p.144]. This is because the additional condition \((7)\) provides more information on \( P_f(x) \) than condition \((8)\). Furthermore, it is also possible to uniquely determine the triplet \((\hat{T}(x,t), P_f(x), \mu(t))\), satisfying \((1),(3)-(7)\), under the assumption that \( \mu \) is non-negative, see [6, p.145].

Note that instead of the Neumann boundary conditions \((4)\) and \((5)\) one can prescribe the Dirichlet boundary temperature conditions

\[
T(0,t) = 0, \quad t \geq 0
\]

and \((7)\). In this case, the additional measurement can be the heat flux \((5)\). Then, if \( g \neq 0 \) and there exists \( t_0 > 0 \) such that \( \int_{t_0}^{\infty} g(t) dt < \infty, \ g(t) = 0 \) for all \( t > t_0 \), then the inverse problem given by equation \((1),(3), (5), (7)\) and \((10)\) has a unique solution, see [11, 12]. The uniqueness of the solution of the problem given by equations \((1),(3)\) and \((7)\) also holds under the additional final temperature measurement, see [3, 8],

\[
T(x,t_0) = h(x), \quad x \in [0, \pi].
\]

### 3. Numerical Approach

As a first step, a finite-difference algorithm based on the Crank-Nicolson scheme, see [4, pp.387-389], is developed in order to solve the direct problem for the parabolic equation \((1)\), in which the coefficient \( P_f \) is considered known, subject to the initial and boundary conditions \((3)-(5)\). At this particular stage we only want to retrieve the temperature \( T(x,t) \) given the assumed knowledge of the positive entry \( P_f(x) \). Let us denote this particular computed solution by \( \hat{T}^{comp}(P_f; (x,t)) \).

A second step involves a gradient based optimisation procedure, supplied by the NAG routine E04FCF, which minimizes the order-0 and order-1 Tikhonov regularizations: \( F_0, F_1 : \{ P_f \mid P_f \in C([0, \pi]), \ P_f > 0 \} \to \mathbb{R}_+ \) defined by

\[
F_0(P_f) := \| T^{comp}(P_f; (x,t)) - g(t) \|_2^2 + \lambda \| P_f \|_2^2,
\]

\[
F_1(P_f) := \| T^{comp}(P_f; (x,t)) - g(t) \|_2^2 + \lambda \| P_f' \|_2^2,
\]

respectively, where \( \lambda > 0 \) is a regularization parameter to be prescribed. Remark, when \( \lambda = 0 \), expressions \((12)\) and \((13)\) coincide with the classical least-square functional, which produces an unstable solution.

The NAG routine E04FCF is a comprehensive algorithm for finding an unconstrained minimum of a sum of squares of \( m \) nonlinear functions in \( n \) variables \((m \geq n)\). Further, no derivatives are required to be supplied by the user, these being calculated internally by the routine using finite differences.

The minimization algorithm is initialized with a positive continuous function \( P_f \), which in our case is set to 1, i.e. \( P_{f, initial}^0(x) = 1 \). The constraint \( P_f > 0 \) cannot be imposed directly in the NAG routine, but, if in the iteration process some components of the discretized \( P_f \) happen to become negative, they are replaced by 1 at the next iteration level. Let us consider the following test example. Let us choose

\[
\mu(t) = \begin{cases} 
0 & \text{for } t = 0, \\
\frac{1}{e^{\frac{1}{2}(t-\frac{1}{2})^2}} & \text{for } t \in (0,1), \\
0 & \text{for } t \geq t_0 = 1,
\end{cases}
\]
which satisfies conditions (6) and seek to retrieve a positive continuous perfusion coefficient given by

\[ P_f(x) = 1 + x^2, \quad \text{for } x \in [0, \pi]. \]

The space interval \([0, \pi]\) is discretized into \(N_0 = 90\) uniform cells and we discretise a finite time interval \([0, t_f]\) into \(N = 100t_f\) uniform time intervals, where \(t_f \in \{1, 2, 4\}\).

Throughout the paper, all the computations are performed on a 64-bit ×86-Linux cluster architecture, with all the operations carried out in extended precision.

Figure 1 illustrates the results obtained with order-0 Tikhonov’s regularization for exact measurement data (7) used in each of the three time length intervals, where the appropriate choices for the values of the regularization parameter \(\lambda\) are considered, namely: \(\lambda = 10^{-20}\) for \(t_f = 1\), \(\lambda = 10^{-19}\) for \(t_f = 2\), and \(\lambda = 10^{-19}\) for \(t_f = 4\). In Figure 1(a), the natural logarithm of the functional given in (12), \(\ln(F_0)\), is represented as a function of the number of iterations obtained for \((t_f, \lambda) \in \{(1, 10^{-20}), (2, 10^{-19}), (4, 10^{-19})\}\). Figures 1(b)-(d) show the computed \(P_f\) in comparison to the exact solution given in (15) for \((t_f, \lambda) = (1, 10^{-20})\), \((t_f, \lambda) = (2, 10^{-19})\), and \((t_f, \lambda) = (4, 10^{-19})\), respectively. It should be noted that as we increase the time interval,
i.e. for \( t_f = 2 \) and then to \( t_f = 4 \), the results obtained improve, becoming more stable and at the same time increasing the accuracy, since more measurement information is added.

Figure 2 shows the results obtained for exact data when order-1 Tikhonov’s regularization is employed for each choice of the parameters \( (t_f, \lambda) \in \{(1, 10^{-18}), (2, 10^{-18}), (4, 10^{-18})\} \). In Figure 2(a), \( \ln(F_1) \) is represented as a function of the number of iterations for these three choices of \( (t_f, \lambda) \). Figures 2(b)-(d) represent the computed \( P_f \) in comparison to the exact solution given in (15) for \( (t_f, \lambda) = (1, 10^{-18}) \), \( (t_f, \lambda) = (2, 10^{-18}) \), and \( (t_f, \lambda) = (4, 10^{-18}) \), respectively. As expected, we can immediately observe that the results are significantly better than the ones obtained in the case of order-0 Tikhonov’s regularization, since more smoothness is imposed onto the numerical solution. Moreover, using order-1 regularization, the results obtained preserve the smoothness, the solution is stable and an accuracy of over 5 digits is achieved over all the space region, for \( (t_f, \lambda) = (4, 10^{-18}) \). On the other hand, we again observe that the quality of approximation increases as \( t_f \) increases. Also, we should mention that, for \( t_f \in \{2, 4\} \), very good approximation are achieved even for larger values of the regularization parameter \( \lambda \), namely for \( \lambda \in [10^{-18}, 10^{-14}] \).
(a) Logarithm of the objective functional $F_0$, for order-0 regularization, as a function of the number of iterations, and the numerically obtained $P_f(x)$ for (b) $t_f = 1$, (c) $t_f = 2$, and (d) $t_f = 4$, for 1% noisy data. In figures (b)-(d) the exact solution (15) is shown with dashed line.

Next, given the fact that real life measurements are inherently contaminated with errors, we test now the proposed algorithm on noisy data. Let us consider that the measurement $g(t)$ is perturbed by 1% of random multiplicative noise that is generated by a uniform distribution on the interval $[-1, 1]$, for each time node $t_j, j = \{1, \ldots, t_f \}$, and this noise is supplied by the NAG routine G05DAF. Figure 3 shows the results obtained with order-0 Tikhonov’s regularization, when the input measurement data $g$ in (7) is corrupted by 1% multiplicative noise, for each choice of the parameters couple $(t_f, \lambda) \in \{(1, 10^{-11}), (2, 10^{-11}), (4, 10^{-12})\}$. The functional $\ln(F_0)$ shown in Figure 3(a), as a function of the number of iterations, is monotonically decaying in this noisy case, for all couples $(t_f, \lambda)$. The results obtained for $P_f$ are highly sensitive to the level of noise in the data, as shown in the Figures 3(b)-(d) for $(t_f, \lambda) = (1, 10^{-11}), (t_f, \lambda) = (2, 10^{-11}),$ and $(t_f, \lambda) = (4, 10^{-12})$, respectively.

Figure 4 shows the results obtained when order-1 Tikhonov’s regularization is employed and the input measurement data $g$ in (7) is corrupted by 1% multiplicative noise, for each choice...
of the parameters couple \((t_f, \lambda) \in \{(1, 10^{-10}), (2, \frac{10^{-11}}{2}), (4, \frac{10^{-10}}{2})\}\). Again, as in the no noise situation, we can immediately see that when using the order-1 regularization in the noisy case we obtain a more accurate and stable numerical approximation than the one obtained when using the order-0 regularization.

4. Conclusions
The identification of the space-dependent perfusion coefficient in the bio-heat equation has been investigated. In the presence of initial and Neumann boundary conditions, with exact and noisy boundary temperature measurements taken into consideration, the inverse and ill-posed bio-heat conduction problem has been solved numerically. The numerical method that we have developed consists of two parts. In the first step we develop a direct solver based on the Crank-Nicolson finite-difference method, which is then coupled with the second step given by an optimization routine. In effect, the algorithm carries out a search over a class of continuous positive functions \(P_f\) in order to find a global minimum point for the nonlinear Tikhonov regularizing functional.
The Tikhonov zeroth- and first-order regularization procedures have been applied. Both in the case of exact and noisy boundary temperature measurements, the results obtained have shown that the first-order regularization is both stable and accurate and performs better that the zeroth-order regularization for all the values of the parameter $\lambda$ that were considered by inspection. More rigorous choices of the regularization parameter, based on the discrepancy principle, $L$-curve or generalized cross validation, will be reported in a future investigation.

The numerical results presented are obtained for the measurement data $g$ taken on the restricted time intervals $[0, t_f]$, for $t_f \in \{1, 2, 4\}$. As expected, more accurate and stable results can be obtained if a larger time interval of measurements is considered.

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