Strongly Embedded Subgroups of Groups of Odd Type

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Abstract
In this paper we prove that any strongly embedded subgroup of a $K^*$-group $G$ of finite Morley rank and odd type that does not interpret any bad field and where $pr(G) \geq 2$ has to be solvable. If $n(G) \geq 3$ this has two important consequences. If there exists an involution $i \in G$ such that $C_G(i)$ is not solvable, then $G$ does not contain any proper 2-generated core and centralisers of involutions have trivial cores.

1 Introduction

One of the great open problems in model theory is the Cherlin-Zil’ber conjecture that states that an infinite simple group of finite Morley rank is isomorphic, as an abstract group, to an algebraic group over an algebraically closed field. This paper belongs to the series of publications on the classification of tame groups of finite Morley rank. We call a group of finite Morley rank tame if none of its proper sections is a bad group and if it does not interpret a bad field. Here a bad group is a non-solvable group of finite Morley rank all of whose proper definable and connected subgroups are nilpotent and a bad field is a ranked structure of the form $\langle K, +, \cdot, A \rangle$, where $K$ is an algebraically closed field and $A$ is a proper infinite multiplicative subgroup of $K^*$. Bad groups and bad fields are assumed not to exist, but the proof is likely to require model theoretic methods. However it is hoped that the classification of tame simple groups can be achieved using mainly ideas from finite group theory. The first of these is to analyse a minimal counterexample. A group of finite Morley rank is called a $K$-group, if every infinite simple definable and connected section of the group is an algebraic group over an algebraically closed field. A $K^*$-group is a group of finite Morley rank in which every proper definable subgroup is a $K$-group. In this inductive setting the conjecture reduces to the following:
Conjecture 1 An infinite simple tame $K^*$-group is isomorphic to an algebraic group over an algebraically closed field.

It is known \cite{2} that an infinite simple tame $K^*$-group is either of even type, meaning that its Sylow 2-subgroups are nilpotent, definable and of bounded exponent, or of odd type, meaning that the Sylow 2-subgroups are divisible abelian by finite. In this paper we are dealing with the classification of tame simple $K^*$-groups of odd type. For some general background concerning the classification of tame simple groups of odd type see \cite{9}. We need to introduce some terminology. If $E$ is a finite elementary abelian 2-group, its 2-rank $m(E)$ is the minimal number of generators of $E$. If $H$ is any subgroup of a group $G$ of finite Morley rank and odd type, its 2-rank $m(H)$ is the maximum of the 2-ranks of elementary abelian subgroups in $H$. If $H$ is definable and $S$ is a Sylow 2-subgroup of $H$, then the normal 2-rank $n(H)$ is the maximum of the 2-ranks of normal elementary abelian subgroups in $S$.

Let $S$ be a Sylow 2-subgroup in a group $G$ of finite Morley rank. We define the 2-generated core $\Gamma_{S,2}(G)$ as the definable closure of the group generated by all normalizers $N_G(U)$ of all subgroup $U \leq S$ with $m(U) \geq 2$.

A group $G$ of finite Morley rank will be called quasi-simple, if $G = G'$ and $G/Z(G)$ is non-abelian simple. By a component of $G$ we mean a connected definable subnormal quasisimple subgroup. The layer $L(G)$ of $G$ is the product of all quasisimple subnormal subgroups of $G$. It is definable and normal in $G$. The Fitting subgroup $F(G)$ is the group generated by all normal nilpotent subgroups of $G$. $F(G)$ can be shown to be nilpotent. The generalized Fitting subgroup $F^*(G)$ is taken to be $F^*(G) \ast L^*(G)$. Then $C^*_G(F^*(G)) \leq F^*(G)$. $G$ satisfies the B-conjecture, if $G \triangleleft F^*(C_G(i))$ for any involution $i \in G$. Simple algebraic groups over an algebraically closed field of characteristic not 2 satisfy the B-conjecture. Let $i \in G$ be an involution. A component $A \triangleleft L(C_G(i))$ is called intrinsic if $i \in Z(A)$. An involution is called classical if its centraliser contains an intrinsic component isomorphic to SL(2, $K$) for an algebraically closed field $K$.

Fact 1 Let $G$ be a simple tame $K^*$-group of odd type. Then one of the following statements is true.

- $n(G) \leq 2$.
- $G$ has a proper 2-generated core.
- $G$ satisfies the B-conjecture and contains a classical involution.

In this paper we show that the second case cannot occur, if $n(G) \geq 3$ and $G$ contains a non-solvable centraliser of an involution. For partial results on the first case compare \cite{5} and \cite{6}. Berkman \cite{7} has furthermore done a nearly complete analysis of the third case.

Let $G$ be a group of finite Morley rank. A proper definable subgroup $M$ of $G$ is said to be strongly embedded if it contains involutions and for every
$g \in G \setminus M$, $M \cap M^g$ does not contain involutions. Altínel has shown in [1], that any simple tame $K^*$-group of even type that has a strongly embedded subgroup is isomorphic to $\text{SL}_2(K)$ for an algebraically closed field $K$ of characteristic 2. If Conjecture [3] is true, then this is the only place, where strongly embedded subgroups appear in the theory of simple tame $K^*$-groups. The main aim of this section is to prove the following partial result.

**Theorem 2** Let $G$ be a simple $K^*$-group of odd type that does not interpret any bad field and such that $\text{pr}(G) \geq 2$. Let $M < G$ be a strongly embedded subgroup. Then $M^\circ$ is solvable.

**Corollary 3** Let $G$ be a simple tame $K^*$-group with a strongly embedded subgroup $M$. Then $M$ is solvable or $G$ is of odd type and Prüfer 2-rank 1.

Proof. $G$ is either of odd or of even type by [2]. If $G$ is of odd type and $\text{pr}(G) \geq 2$, then $M$ is solvable by Theorem 2. If $G$ is of odd type, then $M$ is solvable by [1]. $\square$

Groups of finite Morley rank with a strongly embedded subgroup have only one conjugacy class of involutions by [10, 10.19]. Furthermore a proper subgroup $M$ of a group $G$ of finite Morley rank is strongly embedded if and only if $M$ contains involutions, $C_G(t) \leq M$ for any $t \in I(M)$ and $N_G(S) \leq M$ for any Sylow 2-subgroup $S$ of $M$ by [1]. Their structural properties are discussed in [1] and slightly further explored in [4].

## 2 Strongly embedded subgroups

We can describe strongly embedded subgroups differently, if $G$ is a connected $K^*$-group of finite Morley rank and odd type such that $\text{pr}(G) \geq 2$.

**Proposition 4** Let $G$ be a connected $K^*$-group of finite Morley rank and odd type such that $\text{pr}(G) \geq 2$. Assume that $M < G$ is a strongly embedded subgroup of $G$. Then

1. $M^\circ = \langle C_G(t)^\circ \mid t \in D^* \rangle$ for any four-subgroup $D$ of $M$ and
2. $M$ is a maximal proper definable subgroup of $G$.

Thus, if $G$ is a $K$-group, then it cannot contain a strongly embedded subgroup.

Proof. Any Sylow 2-subgroup $S$ of $M$ is already a Sylow 2-subgroups of $G$ by [1]. Thus there exists a four-subgroup $D \leq M$, as $\text{pr}(G) \geq 2$. Then $M^\circ = \langle C_M(t)^\circ \mid t \in D^* \rangle$ by [1, 5.14]. On the other hand $C_G(t)^\circ \leq M^\circ$ for all $t \in D^*$ as $M$ is strongly embedded, which gives us that $M^\circ = \langle C_G(t)^\circ \mid t \in D^* \rangle$.

We show that $M$ is a maximal proper definable subgroup of $G$. Let $N$ be a definable subgroup of $G$ such that $M \leq N < G$. Then $N$ is strongly embedded as well by [4]. Hence $N^\circ = \langle C_G(t)^\circ : t \in D^* \rangle = M^\circ$ by [4, 5.14].
again. Especially $N \leq N_G(M^o)$. Let $g \in N_G(M^o)$. Then $M^o \leq M \cap M^g$ contains involutions as $I(M) = I(M^g)$ by [1]. Since $M$ is strongly embedded, this implies that $N_G(M^g) \leq M$ and hence $N = N_G(M^o) = M$.

As finally $G = \langle C_G(t)^o \mid t \in D^o \rangle$ for any four-subgroup $D \leq G$, if $G$ is a $K$-group by [1, 5.14], $G$ cannot contain a strongly embedded subgroup in this case. □

The reverse direction is true as well.

**Proposition 5** Let $G$ be a connected $K^*$-group of finite Morley rank and odd type such that $pr(G) \geq 2$. Let $N := \langle C_G(t)^o \mid l \in D^o \rangle$ for a four-subgroup $D$ of $G$. Then either $N$ is normal in $G$ or $N_G(N)$ is a strongly embedded subgroup of $G$.

**Proof.** Assume that $M := N_G(N) < G$. Then $M$ is a $K$-group which contains $D$ and $M^o = N$ by [1, 5.14]. We are going to show that $C_G(t) \leq M$ for any $t \in I(M)$ and that $N_G(S) \leq M$ for any Sylow 2-subgroup $S$ of $M$.

Let $t \in I(M)$ be any involution. Then $t$ is contained in Sylow 2-subgroup $S$ of $M$. As $Z(S) \neq 1$ by [1, 6.22], there exists $s \in S$ such that $D_1 := \langle t, s \rangle$ is a four-subgroup of $S$ with $D_1 \cap Z(S) \neq 1$. Thus $M^o = \langle C_M(k)^o \mid k \in D_1^o \rangle$ by [1, 5.14]. Let $L := \langle C_G(k) \mid k \in D_1^o \rangle$. Then $C_G(t) \leq L$. We show that $L \leq M$. As $Z(S) \cap D_1 \neq 1$, $S \leq L$. Let $g \in M$ such that $D^o \leq S$. Then

$$L^o = \langle C_L(l)^o \mid l \in (D^o)^o \rangle \leq N^o = N.$$  

by [1, 5.14] again. Thus $M^o \leq L^o \leq N = M^o$, $L^o = N$ and $L \leq N_G(N) = M$. Hence $C_G(t) \leq M$ for any involution $t \in M$.

Let furthermore $S$ be any Sylow 2-subgroup of $M$. As $C_G(t) \leq M$ for all $t \in M$, $pr(M) = 2$. As $N_G(S) \leq N_G(S^o)$, it is sufficient to show that $N_G(S^o) \leq M$. Let $g \in N_G(S^o)$. Then either $g \in C_G(s) \leq M$ for some involution $s \in S^o$ or there exists an elementary abelian subgroup $E \leq S^o$ of order at least 4 such that $g \in N_G(E)$. Since $M^o = \langle C_G(l)^o : l \in E \setminus \{1\} \rangle$ by the first part of the proof, this implies that $g \in N_G(M^o) = M$. □

### 3 Examples of strongly embedded subgroups

We are now able to exhibit two possible examples of strongly embedded subgroups.

**Remark 6** Let $H$ be a $K$-group of odd type that contains an elementary abelian 2-subgroup $E$ of order 8. Then

$$H^o = \langle C_H(D)^o \mid D \leq E, |E : D| = 2 \rangle.$$  

**Proof.** Let $D_1 \leq E$ be a four-subgroup. Then $H^o = \langle C_H(i)^o \mid i \in D_1^o \rangle$ by [1, 5.14]. However, $E \leq C_H(i)$ for any $i \in D_1^o$ and thus

$$C_H(i)^o = \langle C_H(D)^o \mid i \in D, |E : D| = 2 \rangle$$  

by [1, 5.14] again. □
Proposition 7 Let G be a simple $K^*$-group of odd type such that $pr(G) \geq 2$ and assume that G contains an elementary abelian subgroup E of order 8. If G contains a proper 2-generated core $\Gamma$, then G contains a strongly embedded subgroup $M := N_G(\Gamma^\circ)$.

Proof: We may assume that $\Gamma = \Gamma_{S,2}$, where S is a Sylow 2-subgroup of G that contains E. As $pr(G) \geq 2$, $S \leq N_G(S^\circ) \leq \Gamma$ by definition and $\Gamma^\circ = \langle C_T(D)^\circ \mid D \leq E, [E : D] = 2 \rangle$ by Remark 8. Since $N_G(D) \leq \Gamma$ for all four-subgroups $D \leq E$,

$$\Gamma^\circ = \langle C_G(D)^\circ \mid D \leq E, [E : D] = 2 \rangle = \langle C_G(i)^\circ \mid i \in D_1^\circ \rangle$$

for any four-subgroup $D_1 \leq E$. The claim now follows by Proposition 8. \qed

To give the second example, we need some more definitions. Let G be a group of finite Morley rank and $\theta$ a function from I(G) into the set of definable subgroups of G. We say that $\theta$ is a signalizer functor for G if $\theta(s) \leq O(C_G(s))$ is a connected definable normal subgroup of $C_G(s)$ for all $s \in I(G)$ and if furthermore for any commuting involutions $s, t \in G$

$$\theta(t) \cap C_G(s) = \theta(s) \cap C_G(t).$$

In particular, $\theta$ is a signalizer functor for any simple $K^*$-group G of odd type if $\theta(t) = O(C_G(t))$ for any involution $t \in G$ by [14, B.29]. A signalizer functor $\theta$ is called complete, if for any elementary abelian subgroup $E \leq G$ of order $\geq 8$ the subgroup $\theta(E) = \{\theta(t), t \in E \setminus \{1\}\}$ is a connected $2^\perp$-subgroup and $C_{\theta(E)}(s) = \theta(s)$ for any $s \in E^\ast$. We call $\theta$ nilpotent if all the subgroups $\theta(t)$ for $t \in I(H)$ are nilpotent.

Fact 8 ([8]) Any nilpotent signalizer functor $\theta$ on a group G of finite Morley rank is complete.

For the following proposition compare [14, B.?] and [8, 9].

Proposition 9 Let G be a simple $K^*$-group of odd type that does not interpret a bad field. Assume that $pr(G) \geq 2$ and that G contains an elementary abelian subgroup E of order 8. Let $\theta$ be the signalizer functor defined by $\theta(s) = O(C_G(s))$ for all $s \in I(G)$. Then either $O(C_G(s)) = 1$ for all $s \in E^\ast$ or $N_G(N_G(\theta(E)))$ is a strongly embedded subgroup of G.
Proof. As $G$ does not interpret a bad field, $\theta$ is a nilpotent signalizer functor and thus complete by Fact 8. Hence $\theta(E)$ is nilpotent. Let $D$ be any four-subgroup of $E$. As $D$ acts on $\theta(E)$ by a definable group automorphism

$$\theta(E) = \langle C_{\theta(E)}(t), \ t \in D^* \rangle$$

by [9, 4.6]. Since $\theta$ is complete this implies that

$$\theta(E) = \langle O(C_G(t)), \ t \in D^* \rangle$$

In particular $N_G(D) \leq N_G(\theta(E))$. Thus

$$N_G(\theta(E)) = \langle C_G(D) \cap O(C_G(D)) \mid D \leq E, [E : D] = 2 \rangle = \langle C_G(i) \cap O(C_G(i)) \mid i \in D^*_1 \rangle$$

for any four-subgroup $D_1 \leq E$ by Remark 8. The claim now follows by Proposition 8. $\square$

4 Centralisers of involutions

To prove Theorem 3, we need the following crucial fact about centralizers of involutions, which is also one of the main tools to classify groups of small Prüfer 2-rank.

Fact 10 (9) Let $G$ be a simple $K^*$-group of finite Morley rank and odd type that does not interpret a bad field. Let $i \in I(G)$. Then $C_G(i) \cap O(C_G(i))$ is a central product of an abelian divisible group $T$ and a semisimple group $H$ all of whose components are simple algebraic groups over algebraically closed fields of characteristic different from 2.

Lemma 11 Let $H$ be a group of finite Morley rank. For any $X \leq H$ and $h \in H$ let $\overline{X} := XO(H)/O(H)$ and $\overline{t} := tO(H)$.

(i) If $t \in I(H)$ then $\overline{t} \in I(\overline{H})$ and if $\overline{s} \in I(\overline{H})$ then there exists an involution in $sO(H)$.

(ii) $O(\overline{H}) = 1$.

(iii) Let $t, s \in I(H)$. If $[\overline{t}, \overline{s}] = 1$, then there exists $u \in I(sO(H))$ such that $[t, u] = 1$.

(iv) Let $t \in I(H)$. Then $I(tO(H)) = t'O(H)$. Especially, $t, s \in I(H)$ are conjugate in $H$ if and only if $\overline{t}, \overline{s}$ are conjugate in $\overline{H}$.

Proof. For the proof of (i), (iii) and (iv) compare [5, 5]. To show (ii), let $B \geq O(H)$ be such that $B = O(\overline{H})$. Then $B$ is a definable, connected normal subgroup of $H$ since $O(H)$ and $O(\overline{H})$ are definable and connected. Furthermore $B$ cannot contain an involution by (i) and $B = O(H)$ by maximality of $O(H)$. $\square$
Lemma 12 Let $H$ be a group of finite Morley rank and odd type. Let $S$ be a Sylow 2-subgroup of $H$ and $N \triangleleft H$ a definable solvable subgroup. Then $S^o N/N$ is the connected component of a Sylow 2-subgroup of $H/N$.

Proof. Let $P$ be a Sylow 2-subgroup of $H/N$ that contains $S^o N/N$. Let furthermore $F$ be a subgroup of $H$ which contains $N$ such that $F/N = d(P^o)$. Then $F$ is a definable solvable group and $S^o \leq F$. Thus $S^o$ is the connected component of a Sylow 2-subgroup of $F$ and $S^o N/N$ is the connected component of a Sylow 2-subgroup of $F/N = d(P^o)$ by \cite{3}. Hence $S^o N/N = P^o$ and the claim follows. \hfill \Box 

Lemma 13 Let $G$ be a simple $K^*$-group of finite Morley rank and odd type that does not interpret a bad field. Let $i \in G$ be an involution and $C := C_G(i)^o/O(C_G(i)) = H * T$ as in Fact \cite{14}. If $Q$ is a Sylow 2-subgroup of $H$ and $S$ a Sylow 2-subgroup of $C$ that contains $Q$, then $S^o = Q^o R$ where $R$ is the Sylow 2-subgroup of $T$. Especially $pr(C) = pr(H) + pr(T)$, were either $T$ is trivial or $pr(T) \geq 1$.

Proof. Since $C$ is the central product of $H$ and $T$, $H \cap T \leq Z(H) \leq Z(C)$ and $H \cap T$ is finite, as $H$ is semisimple. Let $Q$ be a Sylow 2-subgroup of $H$ and $S$ a Sylow 2-subgroup of $C$ that contains $Q$. Since $T$ is a central subgroup of $C$, the Sylow 2-subgroup $R$ of $T$ is contained in $S$ and $Q * R \leq S$. $T$ is as a divisible group connected by \cite{14} ex. 3, p. 78]. Thus $R$ is connected as well by \cite{11} 9.29 and $Q^o R \leq S^o$.

As $T$ is abelian $S^o T/T$ is the connected component of a Sylow 2-subgroup of $C/T$ by Lemma \cite{12} where $S^o T/T \cong S^o / (S^o \cap T) = S^o / R$. On the other hand $Q^o (H \cap T)/(H \cap T) \cong Q^o / (Q^o \cap T)$ is the connected component of a Sylow 2-subgroup of $H/(H \cap T)$ by Lemma \cite{12} again. As $C/T = HT/T \cong H/(H \cap T)$ actually $S^o / R \cong Q^o / (Q^o \cap T) = Q^o / (Q^o \cap R) \cong Q^o R / R$ and $S^o = Q^o R$. As $pr(C) = pr(S)$, $pr(H) = pr(Q)$ and $pr(T) = pr(R)$ this implies that $pr(C) = pr(H) + pr(T)$.

Assume now that $pr(T) = 0$. As $R$ is connected, $R = 1$. Especially $T \leq O(C)$. However $O(C) = 1$ by Lemma \cite{13} and $T$ is trivial. \hfill \Box 

Proposition 14 Let $G$ be a simple $K^*$-group of odd type that does not interpret a bad field. Then $N_G(P^o)$ is solvable-by-finite for any Sylow 2-subgroup $P$ of $G$.

Proof. Since $P^o$ is a 2-torus $|N_G(P^o) / C_G(P^o)| < \infty$ by \cite{10} 6.16. Hence it is enough to show that $C_G(P^o)^o$ is solvable.

Let $i \in P^o$ be any involution. As $P$ is infinite by \cite{8}, $i$ exists. We write $X = XO(C_G(i))/O(C_G(i))$ for any subgroup $X \leq C_G(i)$. If $C := C_G(i)^o$ is solvable, the claim follows, since $C_G(P^o) \leq C_G(i)$. Assume now that $C$ is not solvable. Then $C = H * T$ by Fact \cite{14} where $T$ is an abelian divisible group and $H$ is a nontrivial semisimple group all of whose components $H_n$ for $1 \leq n \leq k$ are simple algebraic groups over algebraically closed fields of characteristic different from 2. $\mathcal{P}$ is the connected component of a Sylow 2-subgroup of $C$ by Lemma \cite{12}. Thus $\mathcal{P}^o = Q^o R$, where $Q$ is a Sylow 2-subgroup of $H$ and $R$ the Sylow
2-subgroup of $T$ by Lemma 13. Let $Q_n$ for $1 \leq n \leq k$ be the projections of $Q^o$ onto $H_n$. As $Q^o$ consists of commuting 2-elements, $Q_n$ is an abelian 2-group and hence contained in a maximal torus $T_n$ of $H_n$ for all $1 \leq n \leq k$ by [15, 15.4]. As $Q_n$ are the connected components of Sylow 2-subgroups of the simple algebraic groups $H_n$ for all $1 \leq n \leq k$, $C_H(Q^o) = T$, where $T := T_1 \cdots T_k$. Thus $C_H(T^o) = T$ as well and $C_G(i)^o \leq C_C(T^o)$ is abelian. Hence $C_G(i)^o \leq C_G(P^o)^o$ is a connected solvable group. However, $C_G(P^o)^o \leq C_G(i)^o$ and $C_G(P^o)^o \leq C_G(i)^o \leq C_G(P^o)^o$ is solvable.

We need one more result to prove Theorem 3.

Lemma 15 Let $N$ be a group of finite Morley rank and $H \triangleleft N$ a definable normal subgroup such that $H^o$ is a $2^+$-group. Then $N^o = C_N(i)^oH^o$ for all involutions $i \in H$.

Notice that the Lemma reduces to the Frattini argument [10, 10.12], if $(i)$ is a Sylow 2-subgroup of $H$.

Proof. Let $i \in I(H)$. Then $H^o = C_{H^o}(i)H^o$ where $H^o$ is the set of elements of $H^o$ inverted by $i$ by [10, ex. 14, p. 73]. Furthermore

\[(*) \quad \rk(H^o) = \rk(C_{H^o}(i)) + \rk(H^o).
\]

We claim that $H^o = ii^H = i[iH^o]$. Let $h \in H^o$. As $H^o$ is as a $2^+$-group 2-divisible by [10, ex. 11, p. 72], there exists an element $d \in H^o$ such that $h = d^2$. On the other hand $h^i = h^{-1}$ and thus already $d = d^{-1}$ by [10, ex. 12, p. 72]. Thus $d \in H^o$ and $t = d^2 = [i, d] \in ii^H$. Furthermore for all $d \in H^o$, $i^d = i[iH^o]$ as $i^d = [i, d] \in iH^o$. The claim follows since $iI(iH^o)$ is a subset of $H^o$ inverted by $i$.

$i^N$ on the other hand is the disjoint union of finitely many sets $I(j_kH^o)$ for $1 \leq k \leq d$ where $d \leq |H/H^o|$ and $j_k \in i^N$. Thus we may assume that

\[(**) \quad \rk(N) = \rk(C_N(i)) + \rk(i^N) = \rk(C_N(i)) + \rk(H^o).
\]

(1) and (**) imply that

\[\rk(N) = \rk(C_N(i)) + \rk(H^o) - \rk(C_N(i) \cap H^o) = \rk(C_N(i)H^o).
\]

Hence $N^o = C_N(i)^oH^o$ for any involution $i \in N$. \hfill \Box

5 Proof of the theorem

Let $\sigma$ be the solvable radical of $M^o$. Since $M$ does not interpret a bad field $O(M^o)$ is nilpotent and hence, as a definable connected subgroup, contained in
\(\sigma^o\). Notice that, since \(O(M^o)\) is characteristic in \(M^o\), \(O(M^o) \leq O(M) \leq M^o\) and thus \(O(M^o) = O(M)\). As furthermore \(M\) is a strongly embedded \(K\)-group

\[O(C_G(t)) = O(C_M(t)) \leq O(M) \leq \sigma^o\]

for all involutions \(t \in M\) by \([1]\). \(\sigma^o\) is a characteristic subgroup of \(M^o\) and hence normal in \(M\). Since \(M\) contains one conjugacy class of involutions as a strongly embedded subgroup by \([10, 10.19]\), this implies that either \(I(M) \subseteq \sigma^o\) or \(\sigma^o\) does not contain involutions.

(a) If \(\sigma^o\) contains involutions, then \(M^o\) is solvable.

Proof. Assume that \(I(M) \subseteq \sigma^o\). Let \(S_1\) be a Sylow 2-subgroup of \(\sigma^o\) and \(S \supseteq S_1\) a Sylow 2-subgroup of \(M\). Then \(S\) is a Sylow 2-subgroup of \(G\) by \([1]\). Furthermore \(S_1\) is connected by \([10, 9.29]\) and thus a divisible subgroup of the abelian group \(S^o\). Hence \(S_1\) has a complement \(B\) in \(S^o\) by the theorem of Baer. As, however, all involutions are contained in \(\sigma^o\), \(B\) to be trivial, \(S_1 = S^o\) and \(S^o\) is a Sylow 2-subgroup of \(\sigma^o\). By the Frattini argument \([10, 10.12]\) \(M = N_M(S^o)\sigma^o\). Since \(N_M(S^o)/C_M(S^o)\) is finite by \([10, 6.16]\), \(M^o = C_M(S^o)\sigma^o\). However \(C_G(S^o)\sigma^o\) is solvable by Corollary \([14]\). Thus \(M\) is already solvable and \(M^o = \sigma^o\).

(b) If \(\sigma\) contains involutions, then either \(I(M)\) is finite or \(M^o\) is solvable.

Proof. We may assume by (a) that \(\sigma^o\) is a 2\(^-\)-group and that \(I(M) \subseteq \sigma\) as \(\sigma\) is normal in \(M\). As \(M^o = C_G(i)^o\sigma^o\) for any involution \(i \in M\) by Lemma \([13]\),

\[\mathcal{M}^o := M^o/\sigma^o \cong C_G(i)^o/(C_G(i)^o \cap \sigma^o) = C_G(i)^o/O(C_G(i))\]

for any involution \(i \in M\). Since \(I(M) = I(M^o)\) by \([1]\), \(\{i\sigma^o|i \in I(M)\} \subseteq Z(\mathcal{M})\). If \(I(M)\) is infinite, then \(M^o = \sigma\), since \(M^o\) would otherwise contain infinitely many commuting involutions by Lemma \([14]\). Hence \(M^o\) is solvable in this case.

(c) If \(I(M)\) is finite, then either \(pr(G) = 1\) or \(M^o\) is solvable.

Proof. If \(I(M)\) is finite, then \(M^o = C_G(i)^o\) for any involution \(i \in M\) as \(rk(M) = rk(C_G(i)) + rk(I(M))\). Hence \(M^o/O(M)\) is a central product of an abelian divisible group \(T\) and a semisimple group \(H\) all of whose components are algebraic groups over algebraically closed fields of characteristic not 2 by Fact \([11]\). We assume that \(M^o\) is not solvable. Then \(O(M) = \sigma^o\) by (a) and \(T = 1\), as \(T\) is connected. Let \(L_1, \ldots, L_k\) be the components of \(M^o/O(M) = H\). By Lemma \([11]\) all involutions of \(H\) are central in \(H\). Especially \(|I(L_i)| = 2^{pr(L_i)-1}\), as \(Z(L_i)\) is the intersection of all maximal tori in \(L_i\) for \(1 \leq i \leq k\) by \([13]\) ex. 2, p. 162 and \(|I(H)| = 2^{pr(H)-1}\), as \(Z(H)\) is the intersection of all maximal tori in \(H\) by \([13]\) ex. 2, p. 162) again. Let \(g \in M\). As \(g\) acts as automorphism on \(M^o/O(M)\), for all \(1 \leq i \leq k\) there exists \(1 \leq s \leq k\) such that \(L_i^g = L_s\) by \([11, 7.1]\) and thus \(I(L_i)^g = I(L_s)\). If \(i_1 \in I(L_1)\), then \(I(M^o/O(M)) = \{I(L_1), \ldots, I(L_k)\}\) since \(M\),
and hence $M/O(M)$ by Lemma 11 contains one conjugacy class of involutions. Set $p := pr(L_1)$. Then $pr(H) = kp$ and $|\{I(L_1), \cdots , I(L_k)\}| \leq k^{2p-1}$. This can only happen for $k = 1$, since $|I(H)| = 2^{k_p-1} > k^{2p-1} \geq \|M\|$ for $k > 1$. Hence $H$ is already quasi-simple. However, the only quasi-simple algebraic group in which all involutions are central is $SL_2(K)$ for an algebraically closed field of characteristic not 2. (Compare e.g. [11].) As $pr(H) = pr(M^\circ/O(M)) = pr(M^\circ)$ by Lemma 11, the claim follows.

Suppose that $\sigma$ does not contain involutions. Then $M^\circ$ is not solvable since $pr(M) = pr(G) \geq 2$ and $1 \neq M := M/\sigma$ contains one conjugacy class of involutions as in the proof of Lemma 11. As furthermore $M$ is a $K$-group $\overline{M^\circ}$ is the direct sum of simple algebraic groups over algebraically closed fields of characteristic not 2 by [11]. Let $L_1, \cdots , L_k$ be the components of $\overline{M^\circ}$. As $L_1$ contains involutions for any $1 \leq l \leq k$, $M$ acts transitively on the components and all components are isomorphic by [11] 7.10.

We claim that $L_1$ contains one conjugacy class of involutions. For assume that $i, j$ in $I(L_1)$ are not conjugate in $L_1$. Then there exists $m \in M$ such that $i^m = j$. As $\overline{M^\circ}$ is the direct product of conjugates of $L_1$, this implies that $L_1^m = L_1$. Thus $C_{L_1}(i)^m = C_{L_1}(j)$. However, non-conjugate involutions of simple algebraic groups have non-isomorphic centralizers of involutions by [11] 4.3. Contradiction.

The only simple algebraic groups with one conjugacy class of involutions are $PSL_2(K), PSL_3(K)$ or $G_2(K)$ for an algebraically closed field $K$ of characteristic not 2. Hence we may assume that $\overline{M^\circ}$ is isomorphic to a direct sum of copies of these groups. On the other hand $M^\circ = C_G(i)^\circ H$ for a connected $2^+$-group $H$ which can be chosen independently from $i \in I(M)$ by [11] 3.10 since $M$ is strongly embedded. Hence $\overline{M^\circ}$ is the product of the centralizer of the involution $\tau$ and a Borel subgroup $B$ of $\overline{M^\circ}$ in the sense of algebraic group theory which contains the connected nilpotent group $H/\sigma$. As $H$ is independent from $i \in I(M)$, we can choose $i$ such that $\tau \in B$. Let $B = U \rtimes T$, where $U$ is the unipotent radical of $B$ and $T$ a maximal torus which contains $\tau$ by [11] 19.3. Let $U_l$ be the projections of $U$ into $L_l$ and $T_l$ be the projections of $T$ into $L_l$ for $1 \leq l \leq k$. Then $\tau = i_1 \cdots i_k$ where $i_l \in I(T_l)$ and thus $T_l \leq C_{L_l}(i_l)$ for $1 \leq l \leq k$. Hence $\dim L_l \leq \dim C_{L_l}(i_l) + \dim U_l$ for $1 \leq l \leq k$. We may assume that $i_1 \neq 1$.

If $L_1 \cong PSL_2(K)$, then $\dim L_1 = 3$, $\dim C_{L_1}(i_1) = 1$ and $\dim U_1 = 1$.

If $L_1 \cong PSL_3(K)$, then $\dim L_1 = 8$, $\dim C_{L_1}(i_1) = 4$ and $\dim U_1 = 3$.

If $L_1 \cong G_2(K)$, then $\dim L_1 = 14$, $\dim C_{L_1}(i_1) = 6$ and $\dim U_1 = 6$.

Thus none of these cases can occur. Contradiction.

\section{Corollaries}
Corollary 16 Let $G$ be a simple $K^*$-group of odd type that does not interpret a bad field. Assume that $\text{pr}(G) \geq 2$ and that $G$ contains an elementary abelian subgroup of order 8. If there exists an involution $i \in G$ such that $C_G(i)$ is not solvable, then $G$ does not contain a proper 2-generated core.

Proof. Assume that $G$ does contain a proper 2-generated core. Then $G$ contains a strongly embedded subgroup $M$ by Proposition 7. If there exists an involution $i \in G$ such that $C_G(i)$ is not solvable, we may assume that $i \in M$, as $G$ contains only one conjugacy class of involutions. Then $C_G(i) \leq M$ since $M$ is strongly embedded and $M$ is not solvable. Contradiction to Theorem 2. $\blacksquare$

Corollary 17 Let $G$ be a simple $K^*$-group of odd type that does not interpret a bad field. Assume that $\text{pr}(G) \geq 2$ and that $G$ contains an elementary abelian subgroup of order 8. If there exists an involution $i \in G$ such that $C_G(i)$ is not solvable, then $O(C_G(t)) = 1$ for all $t \in E^*$.

Proof. By Proposition 9 either $O(C_G(t)) = 1$ for all $t \in E^*$ or $G$ contains a strongly embedded subgroup $M$. The second case cannot occur, however, as in the proof of Corollary 16, which implies the claim. $\blacksquare$

Corollary 18 Let $G$ be a simple $K^*$-group of odd type that does not interpret a bad field. Assume that $\text{pr}(G) \geq 2$. If there exists an involution $i \in G$ such that $C_G(i)$ is not solvable, then $G = \langle C_G(k)^0 \mid k \in D_1^* \rangle$ for any four-subgroup $D_1 \leq G$.

Proof. $D_1 \leq G$ be a four-subgroup of $G$ and set $M := \langle C_G(k)^0 \mid k \in D_1^* \rangle$. By Proposition 9, either $G = N_G(M)$ or $M$ is a strongly embedded subgroup. Since $G$ cannot contain a strongly embedded subgroup as in the proof of Corollary 16, $G = M$ as $G$ is simple. $\blacksquare$

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References

[1] T. Altınel, Groups of Finite Morley Rank with Strongly Embedded Subgroups, J. Algebra 180 (1996), 778-807.

[2] T. Altınel, A. Borovik and G. Cherlin, Groups of mixed type, J. Algebra 192 (1997), 524-571.

[3] T. Altınel / G. Cherlin / L.-J. Corredor / A. Nesin, A Hall theorem for $\omega$-stable groups, to appear in J. London Math. Soc.
[4] C. Altseimer, “Bender-Gruppen von endlichem Morley Rang”, Diplomarbeit an der Eberhard-Karls-Universität Tübingen, 1995.

[5] C. Altseimer, A Characterization of PSp(4, K), to appear in Comm. Algebra.

[6] C. Altseimer, A Characterization of $G_2(K)$, in preparation.

[7] A. Berkman, The Classical Involution Theorem for Tame Groups of Finite Morley Rank, in preparation.

[8] A. Borovik, On signalizer functors for groups of finite Morley rank in “Soviet-French Colloquium on Model Theory”, Karaganda, 1990, 11.

[9] A. Borovik, Simply locally finite groups of finite Morley Rank and odd type, in “Proceedings of NATO ASI on Finite and Locally Finite Groups”, Istanbul, 1994, 248-284.

[10] A. Borovik / A. Nesin, “Groups of Finite Morley Rank”, Oxford University Press, 1994.

[11] D. Gorenstein / R. Lyons / R. Solomon, “The classification of the finite simple groups 3”, Mathematical surveys and Monographs 40, AMS, Providence, 1998.

[12] L. Fuchs, “Abelian Groups”, 3rd ed., Pergamon Press, Oxford, 1967.

[13] J. Humphreys, “Linear Algebraic Groups”, Springer-Verlag, New York Inc., 1975.