MARTINGALE CONVERGENCE THEOREMS FOR TENSOR SPLINES

MARKUS PASSENBRUNNER

Abstract. In this article we prove martingale type pointwise convergence theorems pertaining to tensor product splines defined on \(d\)-dimensional Euclidean space (\(d\) is a positive integer), where conditional expectations are replaced by their corresponding tensor spline orthoprojectors. Versions of Doob’s maximal inequality, the martingale convergence theorem and the characterization of the Radon-Nikodým property of Banach spaces \(X\) in terms of pointwise \(X\)-valued martingale convergence are obtained in this setting. Those assertions are in full analogy to their martingale counterparts and hold independently of filtration, spline degree, and dimension \(d\).

1. Introduction

In this article we prove pointwise convergence theorems pertaining to tensor product splines defined on \(d\)-dimensional Euclidean space in the spirit of the known results for martingales. We begin by discussing the situation for martingales and, subsequently, for one-dimensional splines. For martingales, we use [13] and [4] as references. Let \((\Omega, (\mathcal{F}_n), \mathbb{P})\) be a filtered probability space. A sequence of integrable functions \((f_n)_{n \geq 1}\) is a martingale if \(\mathbb{E}(f_{n+1}|\mathcal{F}_n) = f_n\) for any \(n\), where we denote by \(\mathbb{E}(\cdot|\mathcal{F}_n)\) the conditional expectation operator with respect to the \(\sigma\)-algebra \(\mathcal{F}_n\). This operator is the orthoprojector onto the space of \(\mathcal{F}_n\)-measurable \(L^2\)-functions and it can be extended to act on the Lebesgue-Bochner space \(L^1_X\) for any Banach space \((X, \|\cdot\|)\). Observe that if \(f \in L^1_X\), the sequence \((\mathbb{E}(f|\mathcal{F}_n))\) is a martingale. In this case, we have that \(\mathbb{E}(f|\mathcal{F}_n)\) converges almost surely to \(\mathbb{E}(f|\mathcal{F})\) with \(\mathcal{F} = \sigma(\cup_n \mathcal{F}_n)\). A crucial step in the proof of this convergence theorem is Doob’s maximal inequality

\[
\mathbb{P}(\sup_n \|f_n\| > t) \leq \frac{\sup_n \|f_n\|_{L^1_X}}{t}, \quad t > 0,
\]

which states that the martingale maximal function \(\sup \|f_n\|\) is of weak type \((1,1)\). For general scalar-valued martingales, we have the following convergence theorem: any martingale \((f_n)\) that is bounded in \(L^1\) has an almost sure limit function contained in \(L^1\). This limit can be identified as the Radon-Nikodým derivative of the \(\mathbb{P}\)-absolutely continuous part of the measure \(\nu\) defined by

\[
(1.1) \quad \nu(A) = \lim_m \int_A f_m \, d\mathbb{P}, \quad A \in \cup_n \mathcal{F}_n.
\]

This limit exists because of the martingale property of \((f_n)\). The same convergence theorem as above holds true for \(L^1_X\)-bounded \(X\)-valued martingales \((f_n)\), provided there exists a Radon-Nikodým derivative of the \(\mathbb{P}\)-absolutely continuous part of the now \(X\)-valued measure \(\nu\) in (1.1). Banach spaces \(X\) where this is always possible are said to have the

2010 Mathematics Subject Classification. 41A15, 42B25, 46B22, 42C10, 60G48.
Key words and phrases. Tensor product spline orthoprojectors, Almost everywhere convergence, Maximal functions, Radon-Nikodým property, Martingale techniques.
Radon-Nikodým property (RNP) (see Definition 2.3). The RNP of a Banach space is even characterized by martingale convergence meaning that in any Banach space $X$ without RNP, we can find a non-convergent and $L^1_X$-bounded martingale.

Consider now the special case where each $\sigma$-algebra $\mathcal{F}_n$ is generated by a partition of a bounded interval $I \subset \mathbb{R}$ into finitely many subintervals $\left(I_{n,i}\right)_i$ of positive length as atoms of $\mathcal{F}_n$. In this case, $(\mathcal{F}_n)$ is called an interval filtration on $I$. Then, the characteristic functions $\left(\chi_{I_{n,i}}\right)_i$ of those atoms are a sharply localized orthogonal basis of $L^2(\mathcal{F}_n)$ w.r.t. (with respect to) Lebesgue measure $\lambda = |\cdot|$. If we want to preserve the localization property of the basis functions, but at the same time consider spaces of functions with higher smoothness, a natural candidate is the space of piecewise polynomial functions of order $k$, given by

$$S^k(\mathcal{F}_n) = \{ f : I \to \mathbb{R} \mid f \text{ is } k-2 \text{ times continuously differentiable and}$$

$$a \text{ polynomial of order } k \text{ on each atom of } \mathcal{F}_n \},$$

where $k$ is an arbitrary positive integer. One reason for this is that $S^k(\mathcal{F}_n)$ admits a special basis, the so called B-spline basis $(N_{n,i})_i$, that consists of non-negative and localized functions $N_{n,i}$. Here, the term “localized” means that the support of each function $N_{n,i}$ consists of at most $k$ neighbouring atoms of $\mathcal{F}_n$. A second reason is that if $(\mathcal{F}_n)$ is an increasing sequence of interval $\sigma$-algebras, then the sequence of corresponding spline spaces $S^k(\mathcal{F}_n)$ is increasing as well. Note that the aforementioned properties of the B-spline functions $(N_{n,i})_i$ imply that they do not form an orthogonal basis of $S^k(\mathcal{F}_n)$ for $k \geq 2$. For more information on spline functions, see e.g. [19]. Let $P^k_n$ be the orthogonal projection operator onto $S^k(\mathcal{F}_n)$ with respect to the $L^2$ inner product on $I$ equipped with the Lebesgue measure. Since the space $S^1(\mathcal{F}_n)$ consists of piecewise constant functions, $P^1_n$ is the conditional expectation operator with respect to the $\sigma$-algebra $\mathcal{F}_n$ and the Lebesgue measure. In general, the operator $P^k_n$ can be written in terms of the B-spline basis $(N_{n,i})_i$ as

$$P^k_n f = \sum_i \int_I f N_{n,i} d\lambda \cdot N^*_n \cdot N_{n,i},$$

where the functions $(N^*_n)_i$, contained in the spline space $S^k(\mathcal{F}_n)$, are the biorthogonal (or dual) system to the B-spline basis $(N_{n,i})_i$. Due to the uniform boundedness of the B-spline functions $N_{n,i}$, we are able to insert functions $f$ in formula (1.2) that are contained not only in $L^2$, but in the Lebesgue-Bochner space $L^1_X$, thereby extending the operator $P^k_n$ to $L^1_X$.

Similarly to the definition of martingales, we adopt the following notion introduced in [13]: let $(f_n)_{n \geq 1}$ be a sequence of functions in the space $L^1_X$. We call this sequence a $k$-martingale spline sequence (adapted to $(\mathcal{F}_n)$) if

$$P^k_n f_{n+1} = f_n, \quad n \geq 1.$$  

The local nature of the B-splines and the nestedness of the spaces $(S^k(\mathcal{F}_n))_n$ ultimately allow us to transfer the classical martingale theorems discussed above to $k$-martingale spline sequences adapted to arbitrary interval filtrations $(\mathcal{F}_n)$ and for any positive integer $k$, just by replacing conditional expectation operators with the spline projection operators $P^k_n$. Indeed, for any positive integer $k$, we have the following results.
(i) (Shadrin’s theorem)
There exists a constant \( C \) (depending only on \( k \) and not on \( (\mathcal{F}_n) \)) such that
\[
\sup_n \| P^k_n : L^1_X \rightarrow L^1_X \| \leq C.
\]

(ii) (Doob’s inequality for splines)
There exists a constant \( C \) such that for any \( k \)-martingale spline sequence \( (f_n) \),
\[
|\{ \sup_n \| f_n \| > t \} | \leq C \sup_n \| f_n \|_{L^1_X}, \quad t > 0.
\]

(iii) (Pointwise convergence of spline projections)
For any Banach space \( X \) and any \( f \in L^1_X \), the sequence \( (P^k_n f) \) converges almost everywhere to some \( L^1_X \)-function.

(iv) (RNP characterization by pointwise spline convergence)
For any Banach space \( X \), the following statements are equivalent:
(a) \( X \) has RNP,
(b) every \( k \)-martingale spline sequence that is bounded in \( L^1_X \) converges almost everywhere to an \( L^1_X \)-function.

We give a few comments regarding the proofs of the statements (i)–(iv) above. Property (i), for arbitrary \( k \), was proved by A. Shadrin in the groundbreaking paper [20]. We also refer to the article [9] by M. v. Golitschek, who gave a substantially shorter proof of (i). It should be noted that in the case \( k = 1 \), due to Jensen’s inequality for conditional expectations, we can choose \( C = 1 \) in (i). Property (ii) was proved in [16]. By a standard argument for passing from a weak type \((1,1)\) inequality of a maximal function to a.e. convergence for \( L^1_X \)-functions (see for instance Chapter 1 of [5]), item (iii) was proved in [16] in the case that \( \cup_n \mathcal{F}_n \) generates the Borel-\( \sigma \)-algebra on \( I \) and in [12] in general. We also identify the limit of \( P^k_n f \) as \( P_\infty f \), where \( P_\infty \) is (the \( L^1_X \)-extension of) the orthogonal projector onto the \( L^2 \)-closure of \( \cup_n S^k(\mathcal{F}_n) \). The implication (a) \( \implies \) (b) in item (iv) was also proved in [12], whereas the reverse implication (b) \( \implies \) (a) was shown in [14] by constructing a non-convergent, \( L^\infty_X \)-bounded \( k \)-martingale spline sequence with values in Banach spaces \( X \) without RNP for any positive integer \( k \).

Almost everywhere convergence of orthogonal Franklin (i.e. the piecewise linear case) and spline series has a long history: It was proved by Z. Ciesielski in [1] that orthonormal spline expansions of \( L^1 \)-functions with respect to the dyadic partition on the interval converge almost everywhere. Z. Ciesielski and A. Kamont [2] showed that Franklin series of integrable functions corresponding to arbitrary partitions converge almost everywhere for every possible partition.

Let us also mention a few results in a slightly different direction. Note that in the underlying manuscript, we consider the question under which conditions martingale spline sequences \( (f_n) \) converge almost everywhere to some function \( f \). As the \( L^1 \)-bounded martingale \( f_n = 2^n \mathbb{1}_{[0,2^{-n}]} \) on the unit interval shows, the a.e. limit \( f \) (which is zero in this case) cannot be used to recover the sequence \( (f_n) \). One can then ask for conditions so that \( (f_n) \) actually is (uniquely) determined by the limit function \( f \). Such conditions were given for the piecewise linear dyadic case by G. G. Gevorkyan [6]. There are many generalizations of this result, see for instance M. Pohosyan [17] or G. G. Gevorkyan, K. A. Navasardyan [8] for more general partitions, G. G. Gevorkyan [7] for the multivariate case, and K. Keryan,
A. Khachatryan [11] for higher order splines. For more information regarding such so-called uniqueness results, the interested reader should consult the references cited in the aforementioned articles.

In this article we are concerned with pointwise convergence of multivariate martingale spline sequences. Let \( d \) be a positive integer and, for \( j = 1, \ldots, d \), let \((\mathcal{F}_n^j)\) be an interval filtration on the interval \( I \subset \mathbb{R} \). Filtrations \((\mathcal{F}_n)\) of the form \( \mathcal{F}_n = \mathcal{F}_n^1 \otimes \cdots \otimes \mathcal{F}_n^d \) will be called an \emph{interval filtration} on the cube \( I^d \). Then, the atoms of \( \mathcal{F}_n \) are of the form \( A_1 \times \cdots \times A_d \) with atoms \( A_j \) in \( \mathcal{F}_n^j \). For a tuple \( k = (k_1, \ldots, k_d) \) consisting of \( d \) positive integers, denote by \( P_k^n \) the orthogonal projector with respect to \( d \)-dimensional Lebesgue measure \( | \cdot | = \lambda^d \) onto the tensor product spline space \( S^{k_1}(\mathcal{F}_n^1) \otimes \cdots \otimes S^{k_d}(\mathcal{F}_n^d) \). The tensor product structure of \( P_k^n \) immediately allows us to conclude (i) in this case, i.e., \( P_k^n \) is bounded on \( L^1_X(I^d) \) by a constant depending only on \( k \) (cf. also [13 Corollary 3.1]).

Similarly to the one-dimensional case above, we then introduce the following notion:

**Definition 1.1.** Let \((\mathcal{F}_n)\) be an interval filtration on a \( d \)-dimensional cube \( I^d \). A sequence of functions \( (f_n)_{n \geq 1} \) in the space \( L^1_X(I^d) \) is a \emph{k-martingale spline sequence} (adapted to \((\mathcal{F}_n)\)) if

\[
P_k^n f_{n+1} = f_n, \quad n \geq 1.
\]

The implication (b) \( \Longrightarrow \) (a) in item (iv) for martingale spline sequences on \( I^d \) can easily be deduced from its one-dimensional version as well. Indeed, for Banach spaces \( X \) without RNP we get, for any positive integer \( k_1 \), a non-convergent \( X \)-valued \( k_1 \)-martingale spline sequence \( (f_n^1) \) on \( I \). Then, \( f_n(x_1, \ldots, x_d) = f_n^1(x_1) \) is a non-convergent \( X \)-valued \((k_1, \ldots, k_d)\)-martingale spline sequence on \( I^d \) for any choice of positive integers \( k_2, \ldots, k_d \).

The main objective of this article is to prove the remaining assertions (iii), (iv) and the implication (a) \( \Longrightarrow \) (b) in item (iv) for martingale spline sequences on \( I^d \). The basic idea in the proof of (ii) for \( d = 1 \) (see [16 Proposition 2.3]) is the pointwise bound

\[
\|P_k^n f(x)\| \leq C_{k,\mathcal{M}_{HL}} f(x)
\]

of \( P_k^n \) by the \emph{Hardy-Littlewood maximal function}

\[
\mathcal{M}_{HL} f(x) = \sup_{J \ni x} \frac{1}{|J|} \int_{J} \|f(y)\| \, dy,
\]

where the supremum is taken over all intervals \( J \) that contain the point \( x \). This is enough to imply (ii) for \( d = 1 \) as it is a well known fact that \( \mathcal{M}_{HL} \) itself satisfies the weak type (1,1) bound

\[
|\{\mathcal{M}_{HL} f > t\}| \leq \frac{3}{t} \|f\|_{L^1_{\lambda}}, \quad t > 0.
\]

In dimensions \( d > 1 \), by using this ad-hoc approach (see [15 Proposition 3.3]) one would need the \emph{strong maximal function} \( \mathcal{M}_S f(x) \) on the right hand side of (1.3), where \( \mathcal{M}_S f(x) \) is defined by the same formula (1.4) as \( \mathcal{M}_{HL} f(x) \), but where the supremum is taken over all \( d \)-dimensional axis-parallel rectangles \( J \subset I^d \) containing the point \( x \). As a matter of fact, this is not enough to derive (iii), since the best possible weak type inequality for \( \mathcal{M}_S \) is true only in the Orlicz space \( L(\log L)^{d-1} \) (see [3,11,18]), which is a strict subset of \( L^1 \).

Here we show how to employ the martingale spline structure, especially nestedness of atoms, to avoid the usage of the strong maximal function \( \mathcal{M}_S \) altogether and replace it by an intrinsic maximal function that is (as we will show) of weak type (1,1). This is crucial in the proof of the statements (ii), (iii), (iv) for any dimension \( d \). Those statements are in
full analogy to the martingale and one-dimensional spline results. The validity of (iii) and (iv) for martingale spline sequences on $I^d$ solves a problem stated in [15].

The organization of this article is as follows. In Section 2 we collect a few basic facts about vector measures needed in the sequel. In Section 3, we prove items (ii) and (iii) for martingale spline sequences on $I^d$ (Proposition 3.1 and Theorem 3.3 respectively). In Section 4, the implication (a) $\implies$ (b) of item (iv) is proved in this case (Theorem 4.1) under the restriction that $\cup_n \mathcal{F}_n$ generates the Borel-$\sigma$-algebra on $I^d$. In Section 5, we show this assertion for general interval filtrations on $I^d$ and give an explicit formula for the pointwise limit of martingale spline sequences.

2. Preliminaries

We refer to the book [4] by J. Diestel and J.J. Uhl for basic facts on vector valued integration, martingales, vector measures and the results that follow.

Let $\Omega$ be a set, $\mathcal{A}$ an algebra of subsets of $\Omega$ and $(X, \| \cdot \|)$ a Banach space. A function $\nu : \mathcal{A} \to X$ is a (finitely additive) vector measure if, whenever $E_1, E_2 \in \mathcal{A}$ are disjoint, we have $\nu(E_1 \cup E_2) = \nu(E_1) + \nu(E_2)$. If, in addition, $\nu(\bigcup_{n=1}^\infty E_n) = \sum_{n=1}^\infty \nu(E_n)$ in the norm topology of $X$ for all sequences $(E_n)$ of mutually disjoint members of $\mathcal{A}$ such that $\bigcup_{n=1}^\infty E_n \in \mathcal{A}$, then $\nu$ is a countably additive vector measure. The variation $|\nu|$ of a finitely additive vector measure $\nu$ is the set function

$$|\nu|(E) = \sup_{\pi} \sum_{A \in \pi} \|\nu(A)\|,$$

where the supremum is taken over all partitions $\pi$ of $E$ into a finite number of mutually disjoint members of $\mathcal{A}$. If $\nu$ is a finitely additive vector measure, then the variation $|\nu|$ is monotone and finitely additive. The measure $\nu$ is of bounded variation if $|\nu|(\Omega) < \infty$. If $\mu : \mathcal{A} \to [0, \infty)$ is a finitely additive measure and $\nu : \mathcal{A} \to X$ is a finitely additive vector measure, $\nu$ is $\mu$-continuous if $\lim_{\mu(E) \to 0} \nu(E) = 0$. If $\mu_1, \mu_2 : \mathcal{A} \to [0, \infty)$ are two finitely additive measures on $\mathcal{A}$, $\mu_1$ and $\mu_2$ are mutually singular if for each $\varepsilon > 0$ there exists a set $A \in \mathcal{A}$ so that

$$\mu_1(A^c) + \mu_2(A) \leq \varepsilon.$$

**Theorem 2.1** (Lebesgue decomposition of vector measures). Let $\mathcal{A}$ be an algebra of subsets of the set $\Omega$. Let $\nu : \mathcal{A} \to X$ be a finitely additive vector measure of bounded variation. Let $\mu : \mathcal{A} \to [0, \infty)$ be a finitely additive measure.

Then there exist unique finitely additive vector measures of bounded variation $\nu_c, \nu_s$ so that

1. $\nu = \nu_c + \nu_s$, $|\nu| = |\nu_c| + |\nu_s|$,  
2. $\nu_c$ is $\mu$-continuous,  
3. $|\nu_s|$ and $\mu$ are mutually singular.

This theorem can be found in [4, Theorem 9 on p. 31]. The following theorem is part of [4, Theorem 2 on p. 27] after using [4, Proposition 15 on p. 7].

**Theorem 2.2** (Extension theorem). Let $\mathcal{A}$ be an algebra of subsets of a set $\Omega$ and let $\mathcal{F}$ be the $\sigma$-algebra generated by $\mathcal{A}$. Let $\nu : \mathcal{A} \to X$ be a countably additive vector measure of bounded variation.

Then, $\nu$ has a unique countably additive extension $\tilde{\nu} : \mathcal{F} \to X$. 


**Definition 2.3** ([4 Definition 3, p. 61]). A Banach space $X$ admits the *Radon-Nikodým property (RNP)* if for every measurable space $((\Omega, \mathcal{F}), \mu)$, for every positive, finite, countably additive measure $\mu$ on $((\Omega, \mathcal{F})$ and for every $\mu$-continuous, countably additive vector measure $\nu$ of bounded variation, there exists a function $f \in L^1_X(\Omega, \mathcal{F}, \mu)$ such that

$$\nu(A) = \int_A f \, d\mu, \quad A \in \mathcal{F}.$$ 

**3. Maximal functions of Tensor spline projectors**

Let $d$ be a positive integer and let $(\mathcal{F}_n) = (\mathcal{F}_n^1 \otimes \cdots \otimes \mathcal{F}_n^d)$ be an interval filtration on $I^d$ for some interval $I = (a, b)$ with $a < b$ and $a, b \in \mathbb{R}$. Each $\sigma$-algebra $\mathcal{F}_n$ is then generated by a finite, mutually disjoint family $\{I_{n,i} : i \in \Lambda\}$, $\Lambda \subset \mathbb{Z}^d$, of $d$-dimensional rectangles of the form $I_{n,i} = \prod_{\ell=1}^d (a_{\ell}, b_{\ell}]$ for some $a \leq a_{\ell} < b_{\ell} \leq b$. We write $\mathcal{A}(\mathcal{F}_n) = \{I_{n,i} : i \in \Lambda\}$ to denote this collection of atoms of the $\sigma$-algebra $\mathcal{F}_n$. We assume that $\Lambda$ is of the form $\Lambda^1 \times \cdots \times \Lambda^d$ where for each $\ell = 1, \ldots, d$, $\Lambda^\ell$ is a finite set of consecutive integers and the rectangles $I_{n,i}$ have the property that they are ordered in the same way as $\mathbb{R}^d$, i.e., if $i, j \in \Lambda$ with $i_{\ell} < j_{\ell}$ then the projection of $I_{n,i}$ onto the $\ell$th coordinate axis lies to the left of the projection of $I_{n,j}$ onto the $\ell$th coordinate axis. For $x \in I^d$, let $A_n(x)$ be the uniquely determined atom (rectangle) $A \in \mathcal{F}_n$ so that $x \in A$. For two atoms $A, B \in \mathcal{F}_n$, define $d_n(A, B) := |i - j|_1$ if $A = I_{n,i}$ and $B = I_{n,j}$ and where $|w|_1 = \sum_{\ell=1}^d |w_{\ell}|$ denotes the $\ell^1$ norm of the vector $w$. If $U = \cup_{\ell} A_{\ell}$ and $V = \cup_{\ell} B_{\ell}$ are (finite) unions of atoms in $\mathcal{F}_n$, we set $d_n(U, V) = \min_{n,m} d_n(A_{\ell}, B_{\ell})$. Additionally, for a non-negative integer $s$, define $A_{n,s}(x)$ to be the union of all atoms $A$ in $\mathcal{F}_n$ with $d_n(A, A_n(x)) \leq s$. Moreover, for a Borel set $B \subset I^d$, let $A_{n,s}(B) = \cup_{x \in B} A_{n,s}(x)$.

For each $\ell = 1, \ldots, d$, let $k_{\ell}$ be a positive integer. Define the tensor product spline space of order $k = (k_1, \ldots, k_d)$ associated to $\mathcal{F}_n$ as

$$S_n := S^{k_1}(\mathcal{F}_n^1) \otimes \cdots \otimes S^{k_d}(\mathcal{F}_n^d).$$

The space $S_n$ admits the tensor product B-spline basis $(N_{n,i})_i$ defined by

$$N_{n,i} = N_{n,i_1}^1 \otimes \cdots \otimes N_{n,i_d}^d,$$

where $(N_{n,i_1}^1)_i$, $\ldots$, $(N_{n,i_d}^d)_i$ denotes the B-spline basis of $S^{k_1}(\mathcal{F}_n^1)$ that forms a partition of unity. The support $E_{n,i} = \text{supp} N_{n,i}$ of $N_{n,i}$ is composed of at most $k_1 \cdots k_d$ neighbouring atoms of $\mathcal{F}_n$. Consider the orthogonal projection operator $P_n = P^k$ onto $S_n$ with respect to $d$-dimensional Lebesgue measure $| \cdot | = \lambda^d$. Using the B-spline basis and its biorthogonal system $(N_{n,i}^*)$, the orthogonal projector $P_n$ is given by

$$P_n f = \sum_i \int_{I^d} f N_{n,i}^* d\lambda^d \cdot N_{n,i}^*, \quad f \in L^1_X(I^d).$$

The dual B-spline functions $N_{n,i}^*$ admit the following crucial geometric decay estimate

$$|N_{n,i}^*(x)| \leq C \frac{q^{d_n(E_{n,i}, A_n(x))}}{|\text{re}(E_{n,i} \cup A_n(x))|}, \quad x \in I^d,$$

for some constants $C$ and $q \in [0, 1]$ that depend only on $k$, where $\text{re}(S)$ denotes the smallest, axis-parallel rectangle containing the set $S$. This inequality was shown in [16 Theorem 1.2] for $d = 1$ and if $d > 1$, (3.2) is a consequence of the fact that $N_{n,i}^*$ is the tensor product of one-dimensional dual B-spline functions. Inserting this estimate in
formula (3.1) for $P_n f$ and as $E_{n,i}$ consists of at most $k_1 \cdots k_d$ neighbouring atoms of $F_n$, setting $C_k := C(k_1 \cdots k_d)q^{-|k|}$, we get the pointwise estimate

$$\|P_n f(x)\| \leq C_k \sum_{A \in A(F_n)} b_n(q, \|f\| d\lambda^d, A, x), \quad f \in L^1_X(I^d)$$

introducing the expression

$$b_n(q, \theta, A, x) = q^{d_n(A,A_n(x))} \theta(A), \quad A \in A(F_n), x \in I^d$$

for a positive, finitely additive measure $\theta$ on the algebra $\mathcal{A} = \bigcup_n \mathcal{F}_n$. In view of inequality (3.3), it suffices to consider, instead of the maximal function of the projection operators $P_n$, the maximal functions given by

$$\mathcal{M}_K \theta(x) = \sup_{n \geq K} \sum_{A \in A(F_n)} b_n(q, \theta, A, x), \quad x \in I^d$$

for any positive integer $K$ and some fixed parameter $q \in [0, 1)$.

If we abbreviate by $\mathcal{M} f$ the maximal function $\mathcal{M}_1(|f| d\lambda^d)$, we have the following weak type $(1,1)$ result.

**Proposition 3.1.** The maximal function $\mathcal{M}$ is of weak type $(1,1)$, i.e. there exists a constant $C$ depending only on the dimension $d$ and on the parameter $q \in [0, 1)$, so that we have the inequality

$$|\{\mathcal{M} f > t\}| \leq \frac{C}{t} \|f\|_{L^1}, \quad t > 0, \quad f \in L^1(I^d).$$

**Proof.** Set $B = I^d, K = 1$ and $\theta = |f| d\lambda^d$ in Theorem 3.2 below and observe that the geometric series in equation (3.6) converges. \hfill \Box

The following result about the maximal operators $\mathcal{M}_K$ is the focal point in our investigations.

**Theorem 3.2.** Let $(\mathcal{F}_n)$ be an interval filtration on $I^d$ and let $\theta$ be a non-negative, finitely additive measure on the algebra $\mathcal{A} = \bigcup_n \mathcal{F}_n$.

Then, for any Borel set $B \subset I^d$ and any positive integer $K$,

$$|B \cap \{\mathcal{M}_K \theta > t\}| \leq \frac{C}{t} \sum_{s=0}^{\infty} q^{s/2} (s + 1)^{d-1} \theta(A_{K,s}(B)), \quad t > 0$$

for some constant $C$ depending only on $q$ and $d$.

**Proof.** Set $G_t = B \cap \{\mathcal{M}_K \theta > t\}$ and let $x \in G_t$. Then, there exists an index $n \geq K$ so that

$$\sum_{A \in A(F_n)} b_n(q, \theta, A, x) > t.$$

Letting $c = (2 \sum_{\ell=0}^{\infty} \rho^\ell d = (2/(1 - \rho)) d < \infty$ with $\rho = q^{1/2}$, we obtain that there exists at least one atom $F$ of the $\sigma$-algebra $\mathcal{F}_n$ so that

$$b_n(\rho, \theta, F, x) > t/c.$$

Therefore, for $x \in G_t$, we choose $n_x < \infty$ to be the minimal index $n \geq K$ so that there exists an atom $F$ of $\mathcal{F}_{n_x}$ satisfying inequality (3.7). We choose a particular atom $F$ of $\mathcal{F}_{n_x}$
with this property which will be denoted by $F_x$. The collection of atoms \( \{A_{n_x}(x) : x \in G_t\} \) is nested and covers the set $G_t$. Thus, it is possible to choose a countable subset $\Gamma \subset G_t$ such that the corresponding collection \( \{A_{n_x}(x) : x \in \Gamma\} \) consists only of maximal and mutually disjoint atoms, still covering the set $G_t$. Perform the following estimate using inequality (3.7):

\[
|G_t| \leq \sum_{x \in \Gamma} |A_{n_x}(x)| \leq \frac{c}{t} \cdot \sum_{x \in \Gamma} \rho^{d_{n_x}(F_x, A_{n_x}(x))} \theta(F_x)
\]

(3.8)

where for $m \in \mathbb{Z}^d$, $\Gamma_m$ is the set of all $x \in \Gamma$ so that, if $A_{n_x}(x) = I_{n_x,i}$ and $F_x = I_{n_x,j}$ for some $i, j \in \mathbb{Z}^d$, we have $i - j = m$.

Next, we show that for each $m \in \mathbb{Z}^d$, the collection \( \{F_x : x \in \Gamma_m\} \) consists of mutually disjoint sets. Assume the contrary, i.e. for some $m \in \mathbb{Z}^d$ there exist two points $x, y \in \Gamma_m$ that are different from each other with $F_x \cap F_y \neq \emptyset$. For definiteness, assume that $n_x \geq n_y$, and thus the nestedness of the $\sigma$-algebras $(\mathcal{F}_n)$ implies $F_x \subseteq F_y$. Assume that $i, i', j, j' \in \mathbb{Z}^d$ are such that

\[
I_{n_x,i} = A_{n_x}(x), \quad I_{n_y,i'} = A_{n_y}(y), \quad I_{n_x,j} = F_x, \quad I_{n_y,j'} = F_y.
\]

Since $x, y \in \Gamma_m$, we know that $i - j = m = i' - j'$. Therefore, since $\mathcal{F}_{n_x}$ is finer than $\mathcal{F}_{n_y}$ and by the inclusion $F_x \subseteq F_y$, we have

(3.9) \quad \text{re}(F_x \cup A_{n_x}(x)) \subseteq \text{re}(F_y \cup A_{n_y}(x)) \subseteq \text{re}(F_y \cup A_{n_y}(y)).

Moreover, this and the definition of the distance $d_{n_y}$ implies

(3.10) \quad d_{n_y}(F_y, A_{n_y}(y)) \geq d_{n_y}(F_y, A_{n_y}(x)).

Combining (3.9) and (3.10) yields $b_{n_y}(\rho, \theta, F_y, x) \geq b_{n_y}(\rho, \theta, F_y, y)$; additionally, by definition of $n_y, F_y$ we have the inequality $b_{n_y}(\rho, \theta, F_y, y) > t/c$. Together, this implies

\[
b_{n_y}(\rho, \theta, F_y, x) > t/c.
\]

As $n_x \geq K$ is the minimal index so that such an inequality at the point $x$ is possible and $n_x \geq n_y$ we get that $n_x = n_y =: n$. Since $A_n(x) \cap A_n(y) = \emptyset$ we know that in this case $i \neq i'$ and $x, y \in \Gamma_m$ implies $i - j = m = i' - j'$. Together, this yields $j \neq j'$ which means $F_x \cap F_y = \emptyset$, contradicting the assumption $F_x \subseteq F_y$. Therefore, $F_x$ and $F_y$ are disjoint, concluding the proof of the fact that \( \{F_x : x \in \Gamma_m\} \) consists of mutually disjoint sets for each $m \in \mathbb{Z}^d$.

If $(U_j)$ is a countable collection of disjoint members of $\mathcal{A}$ and if $U \in \mathcal{A}$ with $\bigcup_{j=1}^{\infty} U_j \subset U$, then $\sum_{j=1}^{\infty} \theta(U_j) \leq \theta(U)$, since for finite sums this is clear by finite additivity and positivity of $\theta$ and the general case follows by passing to infinity. We apply this simple fact to the sum $\sum_{x \in \Gamma_m} \theta(F_x)$ with $U = A_{K, |m|_1}(B)$ to obtain from (3.8)

\[
|G_t| \leq \frac{c}{t} \cdot \sum_{s=0}^{\infty} \rho^s \theta(A_{K,s}(B)) \left( \sum_{|m|_1=s} 1 \right) \leq \frac{2^d c}{t} \sum_{s=0}^{\infty} \rho^s (s + 1)^{d-1} \theta(A_{K,s}(B)),
\]

which is the conclusion of the theorem. \qed
Combining Proposition 3.1 with the bound (3.3) on the operators \( P_n \), we obtain that the maximal function of the spline projectors \( P_n \) also satisfies a weak type \((1,1)\) inequality
\[
\left| \{ \sup_n \|P_n f\| > t \} \right| \leq \frac{C\|f\|_{L_1(X)}}{t}, \quad t > 0, \quad f \in L_1^d(I^d),
\]
for some constant \( C \) depending only on \( k \). This proves Doob’s inequality (ii) on page 3 for martingale spline sequences on \( I^d \). Indeed, given a martingale spline sequence \( (f_n) \) on \( I^d \), apply (3.11) to the function \( f = f_m \) for a fixed positive integer \( m \) and pass \( m \to \infty \) to get (ii) for martingale spline sequences on \( I^d \).

As a corollary, we have the following result about almost everywhere convergence of \( P_n f \) for \( f \in L_1^d(I^d) \), proving (iii) for tensor spline projections.

**Theorem 3.3.** Let \( X \) be any Banach space and let \( f \in L_1^d(I^d) \). Then, there exists \( g \in L_1^d(I^d) \) such that
\[
P_n f \to g \quad \lambda^d\text{-almost everywhere.}
\]

**Remark.** (i) The proof of Theorem 3.3 follows along the same lines as the proof of the one-dimensional case [12, Theorem 3.2] and uses standard arguments for passing from a weak type maximal inequality of the form (3.11) to almost everywhere convergence of \( P_n f \) for \( L^1 \)-functions \( f \). For this argument, a dense subset of \( L^1 \) is needed, for which it is “clear” that pointwise convergence takes place. In [12, Lemma 3.1], for one-dimensional splines, this dense set is chosen to be the space of continuous functions \( C(I) \) on the closure of the interval \( I \). For an arbitrary dimension \( d \), we can use \( C(I) \otimes \cdots \otimes C(I) \) as dense subset of \( L^1 \), for which it is a consequence of the one-dimensional convergence result [12, Lemma 3.1] and its tensor product structure that \( P_n f \) converges pointwise for \( f \in C(I) \otimes \cdots \otimes C(I) \).

(ii) As in the one-dimensional case, the limit function \( g \) in Theorem 3.3 can be identified explicitly as the \((L_1^d\text{-extension of the})\) orthogonal projection of the function \( f \) onto the closure of \( \cup_n S_n \), which, in the particular case that \( \cup_n \mathcal{F}_n \) generates the Borel-\( \sigma \)-algebra on \( I^d \), coincides with the function \( f \).

We also note another immediate corollary of Theorem 3.2 that will be used later.

**Corollary 3.4.** Let \((\mathcal{F}_n)\) be an interval filtration on \( I^d \) and let \( \theta \) be a non-negative, finitely additive measure on the algebra \( \mathcal{A} = \cup_n \mathcal{F}_n \). Let \( D \in \mathcal{A} \) be arbitrary and set
\[
L_t := \left\{ x \in I^d : \limsup_n \sum_{A \in \mathcal{A}(\mathcal{F}_n)} b_n(q, \theta, A, x) > t \right\}.
\]
Let \( R \) be a non-negative integer.

If \( B \subset D \) is a Borel set such that \( A_{K,R}(B) \subset D \) for some \( K \), we have
\[
|B \cap L_t| \leq \frac{C}{t} \left( \theta(D) + \sum_{s>R} q^{s/2}(s + 1)^{d-1} \theta(I^d) \right), \quad t > 0
\]
for some constant \( C \) depending only on \( d \) and \( q \).

**Proof.** This just follows from Theorem 3.2 by noting that \( L_t \subset \{ M_K \theta > t \} \) for any positive integer \( K \). \( \Box \)

**Remark.** Assume that in Corollary 3.4 the measure \( \theta \) is a \( \sigma \)-additive Borel measure on \( I^d \) and replace the term \( \theta(A) \) in the definition (3.4) of \( b_n \) by the term \( \theta(\bar{A}) \) with the closure \( \bar{A} \) of \( A \) in \( \bar{I}^d \). Then, the assertion of Corollary 3.4 still holds if we replace \( \theta(D) \) and \( \theta(I^d) \) on
the right hand side of (3.12) by $\theta(D)$ and $\theta(I^d)$ respectively. Indeed, the only modification in the proof of Theorem 3.2 is that we have to replace $F_x$ by $F_x$ in (3.8), but this only gives an additional factor of $2^d$ on the right hand side of (3.6) and (3.12) since each point of $I^d$ is contained in at most $2^d$ closures of disjoint rectangles.

4. The Convergence Theorem for dense filtrations

In this section, we show the remaining implication $(a) \implies (b)$ of item (iv) on page 3 for martingale spline sequences on $I^d$ in the case where $\mathcal{A} := \cup_n \mathcal{F}_n$ generates the Borel-$\sigma$-algebra on $I^d$. We restrict ourselves to this special setting in this section to present the crucial arguments in a concise form. In order to lift the subsequent result from this hypothesis, we use technical arguments in the spirit of those in the proof of the one-dimensional result [12, Sections 4 and 6]. This will be presented in detail in Section 5.

Theorem 4.1. Let $(\mathcal{F}_n)$ be an interval filtration on $I^d$ so that $\mathcal{A} = \cup_n \mathcal{F}_n$ generates the Borel-$\sigma$-algebra and let $X$ be a Banach space with RNP. Let $(g_n)$ be an $X$-valued martingale spline sequence adapted to $(\mathcal{F}_n)$ with $\sup_n \|g_n\|_{L^1_X} < \infty$.

Then, there exists $g \in L^1_X(I^d)$ so that $g_n \to g$ almost everywhere with respect to Lebesgue measure $\lambda^d$.

Remark. As for martingales (see [4]), the basic proof idea of this result is to define a vector measure $\nu$ based upon the martingale spline sequence $(g_n)$, whose absolutely continuous part with respect to Lebesgue measure $\lambda^d$ has a density $g \in L^1_X$ by the RNP of $X$, which is then the a.e. limit of $g_n$.

Proof. Part I: The limit operator $T$.

For $f \in S_m$ and $n \geq m$, since the operator $P_n$ is selfadjoint and using the martingale spline property of the sequence $(g_n)$,

$$\int_{I^d} g_n \cdot f \, d\lambda^d = \int_{I^d} g_n \cdot P_m f \, d\lambda^d = \int_{I^d} P_m g_n \cdot f \, d\lambda^d = \int_{I^d} g_m \cdot f \, d\lambda^d.$$

This means in particular that for all $f \in \cup_m S_m$, the limit of $\int_{I^d} g_n \cdot f \, d\lambda^d$ exists, so we define the linear operator

$$T : \cup_m S_m \to X, \quad f \mapsto \lim_n \int_{I^d} g_n \cdot f \, d\lambda^d.$$

We can write $g_n$ in terms of this operator $T$. Indeed, by the martingale spline property of $(g_n)$ again,

$$(4.1) \quad g_n = P_n g_n = \sum_i \int_{I^d} g_n N_{n,i} \, d\lambda^d \cdot N_{n,i}^* = \sum_i \lim_m \int_{I^d} g_m N_{n,i} \, d\lambda^d \cdot N_{n,i}^* = \sum_i (TN_{n,i}) N_{n,i}^*.$$

By Alaoglu’s theorem, we may choose a subsequence $\ell_n$ such that the bounded sequence of measures $\|g_{\ell_n}\|_X \, d\lambda^d$ converges in the weak*-topology on the space of Radon measures on the closure $\bar{I}^d$ of $I^d$ to some finite scalar measure $\mu$ on $\bar{I}^d$, i.e.

$$\lim_{n \to \infty} \int_{I^d} f \|g_{\ell_n}\| \, d\lambda^d = \int_{I^d} f \, d\mu, \quad f \in C(\bar{I}^d).$$
For a fixed positive integer \( m \), we then get another subsequence of \((\ell_n)\), again denoted by \((\ell_n)\), so that for each atom \( A \) of \( \mathcal{F}_m \), the sequence \( \|g_{\ell_n}\| \, d\lambda^d \) weak*-converges to some Radon measure \( \mu_A \) on the closure \( \overline{A} \) of \( A \) satisfying \( \mu = \sum_{A \in A(\mathcal{F}_m)} \mu_A \). Each function \( f \in S_m \) is continuous and a polynomial in the interior \( A^\circ \) of each atom \( A \in \mathcal{F}_m \). Denote by \( f_A \) the continuous function on the closure \( \overline{A} \) of \( A \) that coincides with \( f \) on \( A^\circ \). Then, for \( \ell_n \geq m \) and \( f \in S_m \)

\[
\|T f\| = \left\| \int_{I^d} f g_{\ell_n} \, d\lambda^d \right\| \leq \int_{I^d} |f| \|g_{\ell_n}\| \, d\lambda^d \\
= \sum_{A \in A(\mathcal{F}_m)} \int_A |f_A| \|g_{\ell_n}\| \, d\lambda^d \rightarrow \sum_{A \in A(\mathcal{F}_m)} \int_{I^d} |f_A| \, d\mu_A \\
\leq \sum_{A \in A(\mathcal{F}_m)} \int_{I^d} \limsup_{s \rightarrow y} |f(s)| \, d\mu_A(y) = \int_{I^d} \limsup_{s \rightarrow y} |f(s)| \, d\mu(y).
\]

For \( f \in \cup_n S_n \) define

\[
(4.2) \quad \|f\| := \int_{I^d} \limsup_{s \rightarrow y} |f(s)| \, d\mu(y),
\]

which is a seminorm on \( \cup_n S_n \). As for \( L^p \)-spaces, we factor out the functions \( f \in \cup_n S_n \) with \( \|f\| = 0 \) in order to get a norm. Then, denote by \( W \) the completion of \( \cup_n S_n \) in this norm and extend the operator \( T \) to \( W \) continuously.

**Part II:** Representing \( T \) in terms of a vector measure \( \nu \).

Let \( Q = \prod_{\ell=1}^d (a_\ell, b_\ell] \) be an arbitrary atom of the \( \sigma \)-algebra \( \mathcal{F}_n \) for some positive integer \( n \). Let \( \ell \in \{1, \ldots, d\} \) be an arbitrary coordinate direction. If the order of the polynomials \( k_\ell \) in direction \( \ell \) equals 1 (piecewise constant case), we set \( f_\ell^m = \mathbb{1}_{(a_\ell, b_\ell]} \) for \( m \geq n \), which satisfies \( f_\ell^m \in S^{k_\ell}(F_m^\ell) \). If \( k_\ell > 1 \), we first choose an open interval \( O \) and a closed interval \( C \) (both in \( I \)) so that \( C \subseteq (a_\ell, b_\ell] \subseteq O \) and \( |O \setminus C| \leq 1/m \). The sets \( C \) and \( O \) are chosen so that as many endpoints of \( C \) and \( O \) coincide with the corresponding endpoints of \( (a_\ell, b_\ell] \) as possible. Then, let \( f_\ell^m \in \cup_j S^{k_\ell}(F_j^\ell) \) be a non-negative function that is bounded by 1 and satisfies

\[
\text{supp } f_\ell^m \subset O \quad \text{and} \quad f_\ell^m \equiv 1 \text{ on } C.
\]

Such a function exists since \( \mathcal{A} \) generates the Borel-\( \sigma \)-algebra if one additionally notices the facts that B-splines form a partition of unity and have localized support. If we define \( f_m = f_1^m \otimes \cdots \otimes f_d^m \), the sequence \( (f_m) \) is Cauchy in \( \cup_j S_j \) with respect to the norm in (4.2) and we let \( I_Q \) be the limit in \( W \) of the sequence \( (f_m) \) satisfying

\[
\|T I_Q\| = \lim_{m \rightarrow \infty} \|T f_m\| \leq \mu(Q)
\]

(here, the closure of \( Q \) is taken in \( I^d \)). This definition of \( I_Q \) also has the property that if \( Q \) is an atom in \( \mathcal{F}_n \) and \( (Q_j)_{j=1}^\ell \) is a finite sequence of disjoint atoms \( Q_j \) in \( \mathcal{F}_{n_j} \) with \( n_j \geq n \) and \( Q = \cup_{j=1}^\ell Q_j \), we have \( I_Q = \sum_{j=1}^\ell I_{Q_j} \). Therefore, if \( \mathcal{F}_n \ni A = \cup_{j=1}^\ell Q_j \) for some disjoint atoms \( (Q_j)_{j=1}^\ell \) in \( \mathcal{F}_n \), it is well defined to set

\[
I_A = \sum_{j=1}^\ell I_{Q_j} \in W.
\]
Based upon that, we define the finitely additive vector measure $\nu$ on $(I^d, \mathcal{A})$ with values in $X$ by

$$\nu(A) := T(I_A), \quad A \in \mathcal{A}. \tag{4.3}$$

This vector measure $\nu$ is of bounded variation, since if $\pi$ is a finite partition of $I^d$ into sets of $\mathcal{A}$ and if $m < \infty$ is the minimal index so that $A \in \mathcal{F}_m$ for all $A \in \pi$, we have

$$\sum_{A \in \pi} \|T(I_A)\| \leq \sum_{Q \in A(\mathcal{F}_m)} \|T(I_Q)\| \leq \sum_{Q \in A(\mathcal{F}_m)} \mu(Q) \leq 2^d \mu(\bar{I}^d),$$

as each point in $\bar{I}^d$ is contained in at most $2^d$ closures of atoms of $\mathcal{F}_m$.

Observe that for all $f \in \bigcup_n S_n$, we have

$$\int_{I^d} f \, d\nu = T(f). \tag{4.4}$$

Indeed, each $f \in \bigcup_n S_n$ can be approximated uniformly by linear combinations of characteristic functions of atoms of the form $\chi_m := \sum_{Q \in A(\mathcal{F}_m)} \alpha_Q 1_Q$ as $m \to \infty$, which then also has the property that $f_m := \sum_{Q \in A(\mathcal{F}_m)} \alpha_Q 1_Q \to f$ in $W$ as $m \to \infty$ and thus also $Tf_m \to Tf$ in $X$ by the continuity of the operator $T$. As, by definition (4.3) of $\nu$, we have $\int \chi_m \, d\nu = Tf_m$, equation (4.4) follows by letting $m \to \infty$.

**PART III: Conclusion.**

Continuing the calculation in equation (4.1), using the measure $\nu$ and (4.4),

$$g_n = \sum_i \int_{I^d} N_{n,i} \, d\nu \cdot N_{n,i}^*. \tag{4.5}$$

Apply Lebesgue’s decomposition Theorem 2.1 to the measure $\nu$ with respect to $\lambda^d$ to get two finitely additive measures $\nu_c, \nu_s$ of bounded variation with

$$\nu = \nu_c + \nu_s, \tag{4.6}$$

where $\nu_c$ is $\lambda^d$-continuous and $|\nu_s|$ is singular to $\lambda^d$. As $\lambda^d$ is countably additive, so is the $\lambda^d$-continuous measure $\nu_c$ and by the extension theorem (Theorem 2.2) extends uniquely to a countably additive vector measure $\nu_c$ on the Borel-\(\sigma\)-algebra on $I^d$, which, by the RNP of $X$ can be written as $d\nu_c = g \, d\lambda^d$ for some $g \in L^1_X$. Therefore,

$$g_n = \sum_i \int_{I^d} N_{n,i} g \, d\lambda^d \cdot N_{n,i}^* + \sum_i \int_{I^d} N_{n,i} \, d\nu_s \cdot N_{n,i}^*. \tag{4.6}$$

The first part on the right hand side of this equation equals $P_n g$ for the $L^1_X$ function $g$ and this converges a.e. to $g$ by Theorem 3.3 and the remark following it.

The second part, denoted by $P_n \nu_s$, converges to 0 almost everywhere, which we will now show. Let $t > 0$ be arbitrary and define

$$G_t := \{ y \in I^d : \limsup_n \| P_n \nu_s(y) \| > t \}.$$

Then, let $\varepsilon > 0$ be arbitrary and choose $D \in \mathcal{A}$ with the property

$$\lambda^d(D^c) + |\nu_s|(D) \leq \varepsilon,$$

which is possible since $|\nu_s|$ is singular to $\lambda^d$. By (3.3), replacing $\|f\| \, d\lambda^d$ with $|\nu_s|$, $\|P_n \nu_s(y)\| \leq C_k \sum_{A \in A(\mathcal{F}_n)} b_n(q, |\nu_s|, A, y)$
for some constants \( C_k \) and \( 0 < q < 1 \) depending only on \( k \) with \( b_n \) as in (3.4). Therefore, \( G_t \subset L_{t/C_k} \) with

\[
L_u = \left\{ y \in I^d : \limsup_n \sum_{A \in \mathcal{A}(\mathcal{F}_n)} b_n(q, |\nu_s|, A, y) > u \right\}.
\]

We apply Lemma 5.1 below (with \( Y = I^d \)) to the measure \( \theta = |\nu_s| \) on \( \mathcal{A} \) and the set \( D \) to get, for any \( u > 0 \), the estimate

\[
|L_u| = |D^c \cap L_u| + |D \cap L_u| \leq \varepsilon + C\varepsilon / u.
\]

Since this is true for any \( \varepsilon > 0 \), we obtain \( |L_u| = 0 \) for any \( u > 0 \). Thus,

\[
|\{ y \in I^d : \limsup_n \| P_n \nu_s(y) \| > 0 \}| = \left| \bigcup_{r=1}^{\infty} G_{1/r} \right| \leq \left| \bigcup_{r=1}^{\infty} L_{1/(C_k r)} \right| = \lim_{r \to \infty} |L_{1/(C_k r)}| = 0,
\]

which completes the proof of the theorem. \( \square \)

5. THE CONVERGENCE THEOREM FOR ARBITRARY FILTRATIONS

Now we discuss the necessary modifications in the proof of Theorem 4.1 when the interval filtration \( (\mathcal{F}_n) \) is allowed to be arbitrary. Assume for some Banach space \( X \) with RNP, \( (g_n) \) is an \( X \)-valued martingale spline sequence adapted to \( (\mathcal{F}_n) \) with \( \sup_n \| g_n \|_{L^\infty} < \infty \).

Part I of the proof of Theorem 4.1 does not use the density of the filtration \( (\mathcal{F}_n) \) in \( I^d \), which means that we get an operator \( T : \cup_n S_n \rightarrow X \) and a finite measure \( \mu \) on \( I^d \) satisfying

\[
\| T f \| \leq \int_{I^d} \limsup_{s \to y} |f(s)| \, d\mu(y), \quad f \in \cup_n S_n.
\]

The operator \( T \) is then extended continuously to the completion \( W \) of \( \cup_n S_n \) w.r.t. the norm on the right hand side of (5.1). With the aid of this operator, the martingale spline sequence \( (g_n) \) is written as

\[
g_n = \sum_i (T N_{n,i}) N_{n,i}^*.
\]

We distinguish the analysis of the convergence of \( g_n(y) \) depending on in which coordinate direction the filtration \( (\mathcal{F}_n) \) is dense at the point \( y \). To this end, for \( \ell = 1, \ldots, d \), we define \( \Delta_{n}^\ell \subset I \) to be the set of all endpoints of atoms in the \( \sigma \)-algebra \( \mathcal{F}_n^\ell \). Next, let \( U^\ell \) be the complement (in \( I \)) of the set of all accumulation points of \( \cup_n \Delta_{n}^\ell \). Note that \( U^\ell \) is open (in \( I \)), thus it can be written as a countable union of disjoint open intervals \( (U_j^\ell) \). Let

\[
B_j^\ell = \{ a \in \partial U_j^\ell : \text{ there is no sequence of points in } U_j^\ell \cap (\cup_n \Delta_{n}^\ell) \text{ that converges to } a \}
\]

and define \( V_j^\ell := U_j^\ell \cup B_j^\ell \) and \( V^\ell := \cup_j V_j^\ell \).

**Lemma 5.1.** Let \( (\mathcal{F}_n) \) be an interval filtration on \( I^d \) and let \( \theta \) be a non-negative, finitely additive and finite measure on \( \mathcal{A} \). For \( \varepsilon > 0 \), let \( D \in \mathcal{A} \) with \( \theta(D) \leq \varepsilon \) and

\[
L_t := \left\{ x \in I^d : \limsup_n \sum_{A \in \mathcal{A}(\mathcal{F}_n)} b_n(q, \theta, A, x) > t \right\}.
\]
Then, there exists a finite constant $C$, depending only on $q$ and on $d$ so that

$$|D \cap L_t \cap Y| \leq \frac{C\varepsilon}{t}, \quad t > 0,$$

with $Y = (V^1)^c \times \cdots \times (V^d)^c$.

**Proof.** We shrink the set $D$ properly to then apply Corollary 3.4. This is done as follows. Since $D \in \mathcal{A}$, we can write it as $D = \bigcup_{j=1}^s Q_j$ for disjoint atoms $(Q_j)$ of some $\sigma$-algebra $\mathcal{F}_n$. For each $j$, we have $Q_j = Q_j^1 \times \cdots \times Q_j^d$ for some intervals $Q_j^\ell$, $\ell = 1, \ldots, d$. Assume without restriction that for all $\ell \in \{1, \ldots, d\}$, the interior of the interval $Q_j^\ell$ contains at least two points from $(V^\ell)^c$, since otherwise we would have $|Q_j \cap L_t \cap Y| \leq |Q_j \cap Y| = 0$. Fix $\ell \in \{1, \ldots, d\}$, set $\eta = \varepsilon/(tL|I|^{d-1}d)$ and define the interval $J^\ell \subset Q_j^\ell$ such that

1. $Q_j^\ell \setminus J^\ell$ has two connected components and in each one there exists a point of $(V^\ell)^c$ that has positive distance to $J^\ell$ and to the boundary of $Q_j^\ell$.
2. $|Q_j^\ell \setminus J^\ell| \cap (V^\ell)^c \leq \eta$.

This is possible since $Q_j^\ell \cap (V^\ell)^c$ contains at least two points. Then, set $Q_j' = J^1 \times \cdots \times J^d$ and $B = \bigcup_{j=1}^s Q_j'$ and we get, by the choice of $\eta$,

$$\left| D \cap L_t \cap Y \right| \leq \left| (D \setminus B) \cap Y \right| + \left| B \cap L_t \right| \leq \varepsilon/t + \left| B \cap L_t \right|.$$

Choose the positive integer $R$ sufficiently large so that

$$\sum_{s>R} (s+1)^{d-1}q^{s/2} \theta(t^d) \leq \varepsilon.$$

Then, there exists an integer $K$ so that $A_{K,R}(B) \subset D$, which is true by construction of $B$. Apply now Corollary 3.4 to get $\left| B \cap L_t \right| \leq C\varepsilon/t$, which together with (5.3) implies the assertion of the lemma. \qed

**Remark.** Assume that in Lemma 5.1 the measure $\theta$ is a $\sigma$-additive Borel measure on $\overline{I}^d$ and replace the term $\theta(A)$ in the definition (3.4) of $b_n$ by the term $\theta(\overline{A})$ with the closure $\overline{A}$ of $A$ in $\overline{I}^d$. Then, the assertion of Lemma 5.1 still holds with an additional factor of $2^d$ on the constant $C$, since the same is true for Corollary 3.4.

For a point $y = (y^1, \ldots, y^d) \in I^d$, each coordinate $y^\ell$ is either contained in some set $V^\ell_{ji}$ or in $(V^\ell)^c$. After rearranging the coordinates, we assume that $y \in F$, where $F = F^1 \times \cdots \times F^d$ with $F^\ell = V^\ell_{ji}$ if $\ell \leq s$ and $F^\ell = (V^\ell)^c$ if $\ell > s$ for some $s \in \{0, \ldots, d\}$. We want to split $g_n(y) = \sum_i(TN_{n,i})N^*_{n,i}(y)$ into the parts where $T$ acts on functions restricted to the set $F_\delta$ for $\delta \in \{0, 1\}^d$ with $F_\delta = E^1 \times \cdots \times E^d$ where $E^\ell = F^\ell$ if $\delta_\ell = 0$ and $E^\ell = (F^\ell)^c$ if $\delta_\ell = 1$. In order to construct elements in $W$ that correspond to the functions $N_{n,i} 1_{F_\delta}$, we need the following lemma.

**Lemma 5.2.** For any $\ell \in \{1, \ldots, d\}$, let $f \in S^k(\mathcal{F}^\ell_n)$ for some $n$. For any interval $V^\ell_{ji}$, there exists a sequence $(h_m)$ of functions $h_m \in S^k(\mathcal{F}^\ell_m)$, open intervals $O_m$ and closed intervals $C_m$ (both in $\overline{I}$) satisfying

1. $O_m \to V^\ell_{ji}$ as $m \to \infty$,
2. $\text{supp} h_m \subset O_m$,
3. $h_m \equiv f$ on $C_m \cap I$,
4. The closure of $O_m \setminus C_m$ converges to the empty set as $m \to \infty$. 


Proof. Without loss of generality, assume that \( f = N^\ell_{n,i} \) for some integer \( i \). For \( m \geq n \), we can write
\[
N^\ell_{n,i} = \sum_{r} \lambda_{m,r} N^\ell_{m,r},
\]
where the absolute value of each coefficient \( \lambda_{m,r} \) is \( \leq 1 \). Set
\[
h_m = \sum_{r \in \Lambda_m} \lambda_{m,r} N^\ell_{m,r},
\]
where the set \( \Lambda_m \) is defined to contain precisely those indices \( r \) so that the support of \( N^\ell_{m,r} \) intersects \( V^\ell_j \) but the (Euclidean) distance between the support of \( N^\ell_{m,r} \) and \( \partial U^\ell_j \setminus B^\ell_j \) is positive. The function \( h_m \) is then contained in \( S^\ell_{\ell}(\mathcal{F}^\ell_m) \) and satisfies \( |h_m| \leq 1 \). With this setting, the support of \( h_m \) is contained in \( O_m \) for some open interval \( O_m \) and \( h_m \equiv N^\ell_{n,i} \) on some closed interval \( C_m \subset O_m \). Since the endpoints of \( V^\ell_j \) are accumulation points of \( \cup n \Delta^\ell_n \) or endpoints of \( I \), the intervals \( O_m \) and \( C_m \) can be chosen to satisfy items (1) and (4).

Let now \( (h^\ell_{j,m})_m \) be the sequence of functions from Lemma 5.2 corresponding to a function \( f^\ell \in S^\ell_{\ell}(\mathcal{F}^\ell_{n_i}) \) for some positive integer \( n_i \) and the set \( V^\ell_j \).

1. If \( E^\ell = V^\ell_j \), set \( h_m = h^\ell_{j,m} \).
2. If \( E^\ell = (V^\ell_j)^c \), set \( h_m = f^\ell - h^\ell_{j,m} \).
3. If \( E^\ell = V^\ell_i \), set \( h_m = \sum_{j=1}^{m} h^\ell_{j,K_m} \).
4. If \( E^\ell = (V^\ell)^c \), set \( h_m = f^\ell - \sum_{j=1}^{m} h^\ell_{j,K_m} \).

Then, define \( h_m = h^1_m \otimes \cdots \otimes h^d_m \). Since \( \cup_{j \geq m} V^\ell_j \) tends to the empty set as \( m \to \infty \) for each \( \ell \), and due to the properties guaranteed by Lemma 5.2 of the functions \( (h^\ell_n) \), if \( K_m \) is chosen sufficiently large, \( h_m \in S_{K_m} \) is Cauchy in the Banach space \( W \) and its limit will be denoted by \( (f^1I_{E_1}) \otimes \cdots \otimes (f^dI_{E_d}) \). If \( f^\ell = N^\ell_{n,i} \) is some B-spline function for all \( \ell \) and some positive integer \( n \), we will also write \( N^\ell_{n,i}I_{F_\delta} \) for this limit in \( W \), which (by \( (5.1) \)) satisfies
\[
\|T(N^\ell_{n,i}I_{F_\delta})\| = \lim_m \|Th_m\| \leq \liminf_m \mu(\text{supp}h_m) \leq \mu(F_\delta \cap \text{supp}N^\ell_{n,i}).
\]

This construction allows us to decompose the martingale spline sequence \( g_n \) into
\[
g_n = \sum_{\delta \in \{0,1\}^d} g_{n,\delta}, \quad \text{with} \quad g_{n,\delta} = \sum_i \mu(T(N^\ell_{n,i}I_{F_\delta})N^\ell_{n,i}) \text{ for } \delta \in \{0,1\}^d.
\]
We treat the sequence \( (g_{n,\delta})_n \) for each fixed \( \delta \in \{0,1\}^d \) separately.

**Case 1:** We begin by considering the case where one of the first \( s \) coordinates of \( \delta \) equals one. Without restriction assume that the first coordinate of \( \delta \) equals one. Write \( N^\ell_{n,i} = N^\ell_{n,i_1} \otimes N^\ell_{n,i_2} \), with \( i = (i_1,i_2) \) for an integer \( i_1 \) and a \( (d-1) \)-tuple of integers \( i_2 \), thus \( g_{n,\delta} \) can be written as
\[
g_{n,\delta}(y_1,y_2) = \sum_{i_2} \left( \sum_i T(N^\ell_{n,i}I_{F_\delta})N^\ell_{n,i_1}(y_1) \right) N^\ell_{n,i_2}(y_2), \quad (y_1,y_2) \in F.
\]
Fix \( y_1 \in U^1_{j_1} \) and \( t > 0 \). Let \( \varepsilon > 0 \) and denote by \( A^1_n(y_1) \) the atom in \( \mathcal{F}^1_n \) that contains the point \( y_1 \). Then, \( \beta := \inf_n |A^1_n(y_1)| > 0 \) since \( U^1_{j_1} \) does not contain accumulation points of \( \cup_n \Delta^1_n \). Choose an open interval \( O \supseteq V^1_{j_1} \) so that \( \mu((O \setminus V^1_{j_1}) \times \bar{I}^{d-1}) \leq \varepsilon \mu(\bar{I}^d) \). Then,
choose $M$ sufficiently large so that for all $n \geq M$, we have $q_{n}(\mathcal{A}_{n}(y_{1}),B_{n}) \leq \varepsilon$ for all atoms $B_{n}$ in $\mathcal{F}_{n}^{1}$ with $B_{n} \cap O^{c} \neq \emptyset$. This is possible since the endpoints of $V_{j_{1}}^{1}$ are accumulation points of $\bigcup_{n} \Delta_{n}^{1}$. Split the sum over $i_{1}$ in (5.5) into indices $i_{1}$ so that supp $N_{n,i_{1}}^{1} \subseteq O$ and its complement and use the geometric decay estimate (3.2) for the dual B-splines $N_{n,i_{1}}^{1*}$ and $N_{n,i_{2}}^{1*}$ and estimate (5.4). With the measures

$$
\theta_{1}(A) = \frac{1}{\beta} \mu((O \setminus V_{j_{1}}^{1}) \times A), \quad \theta_{2}(A) = \varepsilon \frac{1}{\beta} \mu(I \times A)
$$

satisfying max\{\theta_{1}(\bar{I}^{d-1}), \theta_{2}(\bar{I}^{d-1})\} \leq \varepsilon \mu(\bar{I}^{d})/\beta$ and the notation $\mathcal{F}_{n}^{1} = \mathcal{F}_{n}^{2} \otimes \cdots \otimes \mathcal{F}_{n}^{d}$, we then obtain for $n \geq M$

$$
\|g_{n,\delta}(y_{1}, y_{2})\| \leq C \sum_{A \in \mathcal{A}(\mathcal{F}_{n}^{1}^{*})} \left( b_{n}(q, \theta_{1}, A, y_{2}) + b_{n}(q, \theta_{2}, A, y_{2}) \right),
$$

where the expressions $b_{n}(q, \theta, A, y_{2})$ are defined as in (3.4), but with $\theta(A)$ replaced by $\theta(A)$. Here and in the following, the letter $C$ denotes a constant that depends only on $k$, $d$, $q$ and that may change from line to line. Then, applying Corollary 3.4 (also using the remark succeeding it) in dimension $d-1$ with $B = D = \bar{I}^{d-1}$, we estimate

$$
|\{y_{2} : \limsup \|g_{n,\delta}(y_{1}, y_{2})\| > t\}| \leq C \varepsilon \mu(\bar{I}^{d})/t\beta.
$$

We have this inequality for any $\varepsilon > 0$, which implies $|\{y_{2} : \limsup \|g_{n,\delta}(y_{1}, y_{2})\| > t\}| = 0$. As this is true for any $t > 0$ and any $y_{1} \in U_{j_{1}}^{1}$, we get $g_{n,\delta} \to 0$ almost everywhere on $F$.

**Case 2:** Next, consider the case where $\delta \neq 0$ but the first $s$ coordinates of $\delta$ equal 0. Write $N_{n,i}^{*} = N_{n,i_{1}}^{*s} \otimes N_{n,i_{2}}^{*s}$ where $i = (i_{1}, i_{2})$ for an $s$-tuple of integers $i_{1}$ and a $(d-s)$-tuple of integers $i_{2}$, thus, $g_{n,\delta}$ can be written as

$$
ge_{n,\delta}(y_{1}, y_{2}) = \sum_{i_{1}} \left( \sum_{i_{2}} T(N_{n,i}I_{F_{\delta}})N_{n,i_{1}}^{*s}(y_{1}) \right)N_{n,i_{2}}^{*s}(y_{2}), \quad (y_{1}, y_{2}) \in F.
$$

Denote by $A^{s}_{\mu}(y_{1})$ the atom $A$ in $\mathcal{F}_{m}^{1} \otimes \cdots \otimes \mathcal{F}_{m}^{s}$ with $y_{1} \in A$. If we fix $y_{1} \in U_{j_{1}}^{1} \times \cdots \times U_{j_{s}}^{s}$, we know that $\beta := \inf_{m} |A^{s}_{\mu}(y_{1})| > 0$. Next, define $Y = F_{m}^{s+1} \otimes \cdots \otimes \mathcal{F}_{m}^{d}$ and $Z = E_{m}^{s+1} \otimes \cdots \otimes E_{m}^{d}$, and define the measure $\theta(A) = \mu(I^{s} \times (A \cap Z))$. Observe that $\theta(Y) = 0$, since $E_{m}^{\ell} \cap E_{m}^{t} = \emptyset$ for some $\ell > s$ by the form of $\delta$. Using estimate (3.2) for the dual B-spline functions and estimate (5.4) bounding the operator $T$ in terms of $\mu$,

$$
\|g_{n,\delta}(y_{1}, y_{2})\| \leq C \sum_{A \in \mathcal{A}(\mathcal{F}_{n}^{*s})} b_{n}(q, \theta, A, y_{2})
$$

where the expression $b_{n}(q, \theta, A, y_{2})$ is defined as in (3.4), but with $\theta(A)$ replaced by $\theta(\overline{A})$. Approximate $Y$ by a sequence of sets $Y_{m} \in \mathcal{F}_{m}^{*s}$ with $Y_{m} \to Y$. Then, for each $\varepsilon > 0$, there exists a positive integer $m(\varepsilon)$ with $|Y \setminus Y_{m(\varepsilon)}| \leq \varepsilon$ and $\theta(Y_{m(\varepsilon)}) \leq \varepsilon$. For $t > 0$, apply Lemma 5.1 (and the remark succeeding it) in dimension $d-s$ with $D = Y_{m(\varepsilon)}$ to deduce

$$
|L_{t} \cap Y| \leq |Y_{m(\varepsilon)} \cap L_{t} \cap Y| + |Y \setminus Y_{m(\varepsilon)}| \leq \frac{C \varepsilon}{t} + \varepsilon.
$$
with
\[ L_t = \left\{ y_2 \in I^{d-s} : \limsup_n \sum_{A \in \mathcal{A}(\mathcal{F}^{d-s})} b_n(q, \theta, A, y_2) > t \right\}. \]

Since (5.6) holds for any \( \varepsilon > 0 \), we obtain \( |L_t \cap Y| = 0 \) for any \( t > 0 \), which gives that for any fixed \( y_1, g_{n, \delta}(y_1, y_2) \) converges to 0 a.e. in \( y_2 \in Y \). Summarizing and combining this with Case 1 for \( \delta \), we have \( g_{n, \delta} \to 0 \) a.e. on \( F \) as \( n \to \infty \) if one of the coordinates of \( \delta \) equals 1.

**Case 3:** It remains to consider the case \( g_{n,0} \), i.e. the choice \( \delta = 0 \).

For each \( \ell \leq s \), the B-splines \( N_{n,r}^\ell \), whose supports intersect \( V_{j\ell}^r \) can be indexed in such a way that for each fixed \( r \), the function \( N_{n,r}^\ell \chi_{V_{j\ell}^r} \) converges uniformly to a function \( \bar{N}_r^\ell \) as \( n \to \infty \) (cf. [12, Section 4]). This is the case since the interior of \( \mathcal{F}_{n,r}^\ell \). Depending on whether the endpoints of \( V_{j\ell}^r \) can be approximated from inside of \( V_{j\ell}^r \) by points in \( \mathcal{F}_{n,r}^\ell \), there are different possibilities for the index set \( \Lambda^\ell \) of the functions \( \bar{N}_r^\ell \). It can either be finite, infinite on one side or bi-infinite.

We have the following biorthogonal functions to \( \bar{N}_r^\ell \) that admit the same geometric decay estimate (3.2) than the dual B-spline functions \( N_{n,r}^{\ell^*} \). This result is similar to [12, Lemma 4.2].

**Lemma 5.3.** Let \( \ell \in \{1, \ldots, d\} \). For each \( r \in \Lambda^\ell \), the sequence \( N_{n,r}^{\ell^*} \) converges uniformly on each atom of \( \mathcal{A}^\ell = \cup_{y \in \mathcal{F}_n^\ell} U_{y\ell}^r \) contained in \( V_{j\ell}^r \) to some function \( \bar{N}_r^{\ell^*} \) satisfying the estimate
\[
|\bar{N}_r^{\ell^*}(y)| \leq C \frac{q^{d(A(y), E_r)}}{|\mathrm{re}(A(y) \cup E_r)|}, \quad y \in U_{y\ell}^r,
\]
denoting by \( A(y) \) the atom of \( \mathcal{A}^\ell \) that contains the point \( y \), by \( E_r \) the support of \( \bar{N}_r^\ell \) and by \( d(A(y), E_r) \) the number of atoms in \( \mathcal{A}^\ell \) between \( A(y) \) and \( E_r \).

**Proof.** Fix the index \( r \in \Lambda^\ell \), the point \( y \in U_{y\ell}^r \) and \( \varepsilon > 0 \). Since \( r \in \Lambda^\ell \) is fixed, the support \( E_{n,r} \) of \( N_{n,r}^{\ell^*} \) intersects \( U_{y\ell}^r \) for some index \( n \) and we know that \( \beta = \inf_m |E_{m,r}| > 0 \). Additionally, set \( \gamma = |A(y)| \). Without restriction, we assume that \( \beta, \gamma \leq 1 \). Next, we choose \( L \) sufficiently large so that \( Lq^{\ell^*} \leq \varepsilon \beta \gamma \) and, for any positive integer \( n, d_n(A_n(y), E_{n,r}) \leq L \). Moreover, choose an open interval \( O \supset V_{j\ell}^r \) satisfying \( |O \setminus U_{y\ell}^r| \leq \varepsilon \beta \gamma / L \). Based on that, choose \( M \) sufficiently large so that each of the intervals \( (\inf O, y) \) and \( (y, \sup O) \) contains at least \( L \) points of \( \Delta_{M}^\ell \) and so that, for indices \( \nu \) with \( |\nu - r| \leq 2L \) we have
\[
\|N_{n,r}^{\ell^*} - N_{m,r}^{\ell^*}\|_{L^\infty(U_{y\ell}^r)} \leq \varepsilon \beta \gamma / L, \quad m, n \geq M.
\]

For \( n \geq m \geq M \), expand the function \( N_{n,r}^{\ell^*} \) in the basis \( (N_{n,r}^{\ell^*})_\nu \) as
\[
N_{n,r}^{\ell^*} = \sum_\nu \alpha_{r\nu} N_{n,r}^{\ell^*}. 
\]
The coefficients \( \alpha_{r\nu} \) are bounded by a constant independently of \( r, \nu \) and \( m, n \) as we will now see. To this end, we use the geometric decay inequality (3.2) for the dual B-spline functions \( N_{n,r}^{\ell^*} \) to obtain
\[
|\alpha_{r\nu}| = \left| \int_I N_{n,r}^{\ell^*} N_{n,r}^{\ell} \, d\lambda \right| \leq C \sum_{A \in \mathcal{A}(\mathcal{F}^{d-s}_n)} \frac{q^{d_m(A, E_{m,r})}}{|\mathrm{re}(A \cup E_{m,r})|} \int_A N_{n,r}^{\ell^*} \, d\lambda
\]
Denoting $f_{\nu} = N_{m,\nu}^f - N_{n,\nu}^f$, whose absolute value is bounded by 1,
\[
\delta_{\nu} = \int I_{m,\nu}^f N_{m,\nu}^f d\lambda = \int I_{m,\nu}^f N_{n,\nu}^f d\lambda + \int I_{n,\nu}^f f_{\nu} d\lambda = \alpha_{\nu} + \int I_{n,\nu}^f f_{\nu} d\lambda.
\]
For indices $\nu$ with $|\nu - r| \leq 2L$, we now estimate this last integral, by decomposing it into the integrals $I_1, I_2, I_3$ over $U_{j_{r+1}}, O \setminus U_{j_{r+1}}$, and $O^c$, respectively. By estimate (5.8) and the fact that the integral of $N_{n,r}^{f_{\nu}}$ is smaller than a constant $C$ by (3.2), the integral $|I_1|$ can be bounded by $C \varepsilon \beta / L$. For the second integral, we use the fact that the integrand is bounded by $C/\beta$ and the measure estimate for $O \setminus U_{j_{r+1}}$ to deduce $|I_2| \leq C \varepsilon / L$. For the remaining integral $I_3$, we note that on $O^c$, the function $N_{n,r}^{f_{\nu}}$ is bounded by $C q^f / \beta$, which, together with estimate (3.2) and the choice of $L$ gives $|I_3| \leq C \varepsilon / L$.

Summarizing,
\[
|\alpha_{\nu} - \delta_{\nu}| \leq C \varepsilon / L, \quad |\nu - r| \leq 2L.
\]
This can be used to estimate the difference between $N_{m,r}^{f_{\nu}}$ and $N_{n,r}^{f_{\nu}}$ for $n \geq m \geq M$ pointwise as follows
\[
|N_{m,r}^{f_{\nu}}(y) - N_{n,r}^{f_{\nu}}(y)| = \left| \sum_{\nu} (\alpha_{\nu} - \delta_{\nu}) N_{n,\nu}^{f_{\nu}}(y) \right| \leq C \varepsilon + \sum_{\nu:|\nu - r| > 2L} |\alpha_{\nu} N_{n,\nu}^{f_{\nu}}(y)|,
\]
by using the bound $|N_{n,\nu}^{f_{\nu}}(y)| \leq C / \gamma$. By the choice of $L$, the inequality $|\nu - r| > 2L$ yields $d_n(A_n(y), E_{n,\nu}) > L$ and thus, the geometric decay estimate for $N_{n,\nu}^{f_{\nu}}$, the boundedness of $\alpha_{\nu}$, and the choice of $L$ implies that the latter sum is bounded by $C \varepsilon$. This, in turn, leads to the estimate $|N_{m,r}^{f_{\nu}}(y) - N_{n,r}^{f_{\nu}}(y)| \leq C \varepsilon$ and thus the convergence of $N_{m,r}^{f_{\nu}}(y)$, which is uniform in $A(y)$ since all the estimates above only depend on $A(y)$ and not on the particular point $y$. Now, estimate (5.7) follows from the corresponding estimate of $N_{n,r}^{f_{\nu}}$ by letting $n \to \infty$. 

Write $F = Z \times Y$ with $Z = F^1 \times \cdots \times F^s = V_{j_1}^1 \times \cdots \times V_{j_s}^s$ and $Y = F^{s+1} \times \cdots \times F^d$. For an $s$-tuple of integers $i_1 = (r_1, \ldots, r_s)$ and a $(d - s)$-tuple of integers $i_2 = (r_{s+1}, \ldots, r_d)$, set $N_{m,i_1}^{\leq s} I_Z = N_{m,1}^{i_1} I_{F^1} \otimes \cdots \otimes N_{m,s}^{i_s} I_{F^s}$ and $N_{n,i_2}^{\leq s} I_Y = N_{n,1}^{i_2} I_{F^{s+1}} \otimes \cdots \otimes N_{n,d}^{i_d} I_{F^d}$. The uniform convergence of $N_{m,r}^{\ell} I_{V_{j_\ell}^i}$ to $N_{m,r}^{\ell} I_{V_{j_\ell}^i}$ for $\ell \leq s$ as $m \to \infty$ implies that for fixed $n$ and $i_1$, the sequence $(N_{m,i_1}^{\leq s} I_Z \otimes N_{n,i_2}^{> s} I_Y)$ converges in $W$ to some element as $m \to \infty$, which we denote by $N_{i_1}^{\leq s} \otimes N_{n,i_2}^{> s} I_Y$. By the continuity of $T$, we also have $T(N_{m,i_1}^{\leq s} I_Z \otimes N_{n,i_2}^{> s} I_Y) \to T(N_{i_1}^{\leq s} \otimes N_{n,i_2}^{> s} I_Y)$ in $X$ as $m \to \infty$. Using the expressions $T(N_{i_1}^{\leq s} \otimes N_{n,i_2}^{> s} I_Y)$ and the dual functions $N_{i_1}^{\leq s} = N_{i_1}^{1*} \otimes \cdots \otimes N_{i_1}^{r_s}$ to $N_{i_1}^{\leq s}$ given by Lemma 5.3, define
\[
(5.10) \quad u_n = \sum_{i_1,i_2} T(N_{i_1}^{\leq s} \otimes N_{n,i_2}^{> s} I_Y)(N_{i_1}^{\leq s} \otimes N_{n,i_2}^{> s} I_Y).
\]
Next, we show that the sequence $(g_{n,0})$ and the sequence $(u_n)$ have the same a.e. limit on $F$. Indeed, for fixed $y_1 \in U_{j_1}^{1*} \times \cdots \times U_{j_s}^{s*}$, the difference of those two functions has the form
\[
g_{n,0}(y_1, \cdot) - u_n(y_1, \cdot) = \sum_{i_2} N_{n,i_2}^{> s} \sum_{i_1} T((N_{n,i_1}^{\leq s} I_Z - \tilde{N}_{i_1}^{\leq s}) \otimes N_{n,i_2}^{> s} I_Y) N_{n,i_1}^{\leq s}(y_1).
\]
\[ + \sum_{i_1} T(\overline{N}_{i_1}^{\leq s} \otimes N_{n,i_2}^{> s} I_Y) (N_{n,i_1}^{\leq s}(y_1) - \overline{N}_{i_1}^{\leq s}(y_1)) \].

Denote $\mathcal{F}_n^{> s} = \mathcal{F}_n^{s+1} \otimes \cdots \otimes \mathcal{F}_n^{d}$. Using (3.2) for $N_{n,i_2}^{> s}$ and $N_{n,i_1}^{\leq s}$, Lemma 5.3 the uniform boundedness and the localized support of $\overline{N}_{i_1}^{\leq s}$, and the bound (3.4) of the operator $T$ in terms of the measure $\mu$, we obtain for all $\varepsilon > 0$ an index $M$ so that for $n \geq M$

\[
\|g_{n,0}(y_1, y_2) - u_n(y_1, y_2)\| \leq \sum_{A \in A(\mathcal{F}_n^{> s})} b_n(q, \theta, A, y_2),
\]

where $\theta$ is the measure given by $\theta(A) = \varepsilon \mu(\overline{I}^s \times (A \cap Y))$ and the expression $b_n(q, \theta, A, y_2)$ is defined as in (3.3), but with $\theta(A)$ replaced with $\theta(A)$. By Corollary 3.4 (and the remark succeeding it) with $B = D = I^{d-s}$ we obtain $|L_t| \leq C \theta(I^{d-s})/t \leq C \varepsilon \mu(I^d)/t$ with

\[
L_t = \{ y_2 \in I^{d-s} : \limsup_n \sum_{A \in A(\mathcal{F}_n^{> s})} b_n(q, \theta, A, y_2) > t \}.
\]

This implies, using also (5.11),

\[
|\{ y_2 \in I^{d-s} : \limsup_n \|g_{n,0}(y_1, y_2) - u_n(y_1, y_2)\| > t \}| = 0
\]

for any $t > 0$, i.e., $g_{n,0}$ and $u_n$ have the same a.e. limit on $F$.

Therefore, in order to identify the a.e. limit of $(g_{n,0})$ on $F$ (which, by Cases 1 and 2, is also the a.e. limit of $(g_n)$), we identify the a.e. limit of $(u_n)$. Similar to Part II in the proof of Theorem 4.4, we want to construct, for each $i_1$, a vector measure $\nu_{i_1}$ on $\mathcal{F}_n^{> s} = \cup_n \mathcal{F}_n^{> s}$ based on the expressions $T(\overline{N}_{i_1}^{\leq s} \otimes N_{n,i_2}^{> s} I_Y)$. The aim is to have, for each B-spline function $N_{n,i_2}^{> s}$, the representation

\[
\int N_{n,i_2}^{> s} d\nu_{i_1} = T(\overline{N}_{i_1}^{\leq s} \otimes N_{n,i_2}^{> s} I_Y).
\]

To this end, let $Q = \prod_{s=1}^d (a_\ell, b_\ell]$ be an atom of the $\sigma$-algebra $\mathcal{F}_n^{> s}$ for some positive integer $n$. For the definition of the measure $\nu_{i_1}(Q)$, we approximate the characteristic function $1_Q$ of $Q$ by spline functions $f_{m+1}^{s+1} \otimes \cdots \otimes f_m^d$ contained in $\cup_j (S_{k+1}^{s+1}(\mathcal{F}_j^{> s}) \otimes \cdots \otimes S_{k}^d(\mathcal{F}_j^d))$, which will be done as follows. Let $\ell \in \{ s+1, \ldots, d \}$. If the order of the polynomials $k_\ell$ in direction $\ell$ equals 1 (piecewise constant case), we set $f_m^\ell = 1_{[a_\ell, b_\ell]}$ for $m \geq n$, which satisfies $f_m^\ell \in S_{k_\ell}(\mathcal{F}_m^\ell)$. If $k_\ell > 1$, we apply the following construction of the approximation $f_m^\ell$ of the characteristic function of the interval $(a_\ell, b_\ell]$.

If $a_\ell$ is contained in the countable set $\cup_j \partial U_j^{\ell}$ and if $a_\ell$ is not an endpoint of $I$, we choose $c \in (a_\ell, a_\ell + 1/m)$ that is not contained in $\cup_j \partial U_j^{\ell}$. Otherwise, set $c = a_\ell$. Similarly, if $b_\ell$ is contained in $\cup_j \partial U_j^{\ell}$ and if $b_\ell$ is not an endpoint of $I$, we choose $d \in (b_\ell, b_\ell + 1/m)$ that is not contained in $\cup_j \partial U_j^{\ell}$. Otherwise, set $d = b_\ell$. Put

\[
J(x) = \begin{cases} V_j^{\ell}, & \text{if } x \in U_j^{\ell}, \\ \emptyset, & \text{otherwise}, \end{cases}
\]

and define the interval

\[
J = \left( (c, d] \setminus J(c) \right) \cup J(d),
\]

which has the property that $J \cap (V^{\ell})^c = (c, d] \cap (V^{\ell})^c$. We then choose a closed interval $C$ and an open interval $O$ (both in $I$) with $C \subseteq J \subseteq O$ and the property $|O \setminus C| \leq 1/m$. 


The sets $C$ and $O$ are chosen so that as many endpoints of $C$ and $O$ coincide with the corresponding endpoints of $(c, d]$ as possible. Then, let $f_m \in \cup_j S^k(\mathcal{F}_j^f)$ be a non-negative function that is bounded by 1 and satisfies
\[
\text{supp } f_m^\ell \subseteq O \quad \text{and} \quad f_m^\ell \equiv 1 \text{ on } C.
\]
This is possible since if $c$ or $d$ are endpoints of $J$, they are contained in $(\cup_j \overline{U}_j^f)^c$ and thus can be approximated from both sides with grid points $\cup_j \Delta_j^\ell$. Otherwise, the endpoints of $J$ are also endpoints of some set $V_j^f$, which are accumulation points of $\cup_j \Delta_j^\ell$ as well.

Then, define $f_m = f_m^{s+1} \otimes \cdots \otimes f_m^d$ which gives, for each index $i_1$, a Cauchy sequence $N_{i_1}^{\leq s} \otimes f_m^i Y$ in $W$. Its limit will be written as $N_{i_1}^{\leq s} \otimes (I_{Q} \cdot I_Y)$. Then, continuing in a similar fashion as in Part II of the proof of Theorem 4.1, we make sense of the expression $T(N_{i_1}^{\leq s} \otimes (I_{A} \cdot I_Y))$ for any $A \in \mathcal{A}^{>s}$ and define the measure $\nu_{i_1}(A) = T(N_{i_1}^{\leq s} \otimes (I_{A} \cdot I_Y))$ for $A \in \mathcal{A}^{>s}$ whose total variation satisfies $|\nu_{i_1}((I_{d-s} - 2^d - s) \mu(\text{supp } N_{i_1}^{\leq s} \times Y))$. Additionally, for any B-spline function $N_{n,i_1}^{>s}$, we have equation (5.12). Now, as in Part III of the proof of Theorem 4.1 denoting by $w_{i_1}$ the Radon-Nikodym density of the absolutely continuous part of $\nu_{i_1}$ with respect to Lebesgue measure $\lambda^{d-s}$,

\[
\sum_{i_1} N_{i_1}^{\leq s}(y_1)w_{i_1}(y_2) = \sum_{i_1} N_{i_1}^{\leq s}(y_1)w_{i_1}(y_2)
\]
as $n \to \infty$ for almost every $(y_1, y_2) \in F$. Using the estimate from Lemma 5.3 for $N_{i_1}^{\leq s}$ and the above estimate for the total variation of the measures $\nu_{i_1}$, we obtain that $\|g\|_{L^1_X(F)} \leq C \cdot \mu(F)$.

Thus, we have proven the following theorem:

**Theorem 5.4.** Let $(\mathcal{F}_n)$ be an interval filtration on $I^d$ and let $X$ be a Banach space with RNP. Let $(g_n)$ be an $X$-valued martingale spline sequence adapted to $(\mathcal{F}_n)$ with $\sup_n \|g_n\|_{L^1_X} < \infty$.

Then, there exists $g \in L^1_X(I^d)$ so that $g_n \to g$ almost everywhere with respect to Lebesgue measure $\lambda^d$.

**Remark.** Employing the notation developed in this section, we emphasize that the pointwise limit $g$ has the explicit representation
\[
g(y_1, y_2) := \sum_{i_1} N_{i_1}^{\leq s}(y_1)w_{i_1}(y_2), \quad (y_1, y_2) \in F,
\]
where $N_{i_1}^{\leq s}$ are the functions given by Lemma 5.3 corresponding to $F^1 \times \cdots \times F^s = V_{j_1}^1 \times \cdots \times V_{j_s}^s$ and the function $w_{i_1}$ is the Radon-Nikodym density of the absolutely continuous part (w.r.t Lebesgue measure $\lambda^{d-s}$) of the measure $A \mapsto T(N_{i_1}^{\leq s} \otimes (I_{A} \cdot I_Y))$ with $Y = (V^{s+1})^c \times \cdots \times (V^d)^c$.

**Acknowledgements.** The author is supported by the Austrian Science Fund FWF, project P32342.

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Institute of Analysis, Johannes Kepler University Linz, Austria, 4040 Linz, Altenberger Strasse 69

Email address: markus.passenbrunner@jku.at