Zero-Divisor Graph of Real-Valued Continuous Functions on a Frame

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Abstract. The main object of this paper is to study the zero-divisor graph $\Gamma(\mathcal{R}L)$ of the ring $\mathcal{R}L$. Using the properties of the lattice $\text{Coz}L$, we associate the ring properties of $\mathcal{R}L$, the graph properties of $\Gamma(\mathcal{R}L)$, and the properties of a completely regular frame $L$. Paths in $\Gamma(\mathcal{R}L)$ are investigated, and it is shown that the diameter of $\Gamma(\mathcal{R}L)$ and the girth of $\Gamma(\mathcal{R}L)$ coincide whenever $L$ has at least 5 elements. Cycles in $\Gamma(\mathcal{R}L)$ are surveyed, a ring-theoretic and a frame-theoretic characterizations are provided for the graph $\Gamma(\mathcal{R}L)$ to be triangulated or be hypertriangulated. We show that $\Gamma(\mathcal{R}L)$ is complemented if and only if the space of minimal prime ideals of $\mathcal{R}L$ is compact. The relation between the clique number of $\Gamma(\mathcal{R}L)$, the cellularity of $L$ and the dominating number of $\Gamma(\mathcal{R}L)$ is given. Finally, we prove that if $\Gamma(\mathcal{R}L)$ is not triangulated, then the set of centers of $\Gamma(\mathcal{R}L)$ is a dominating set if and only if the socle of $\mathcal{R}L$ is an essential ideal.

1. Introduction

Let $\mathcal{R}$ be a commutative ring with identity. As in [1] and [16], by the zero-divisor graph $\Gamma(\mathcal{R})$ of $\mathcal{R}$ we mean the (simple) graph with vertices nonzero zero-divisors of $\mathcal{R}$ such that there is an edge between vertices $x$ and $y$ if and only if $x \neq y$ and $xy = 0$.

Let $\mathcal{C}(X)$ be the ring of all real-valued continuous functions on a completely regular Hausdorff space $X$. The zero-divisor graph $\Gamma(\mathcal{C}(X))$ has been studied by Azarpanah and Motamedi in [3]. They have investigated the relations between ring properties of $\mathcal{C}(X)$, graph properties of $\Gamma(\mathcal{C}(X))$ and topological properties of the space $X$.

The ring of real-valued continuous functions on a frame $L$ is the set of all frame homomorphisms $\alpha : L(\mathbb{R}) \to L$, where $L(\mathbb{R})$ is the frame of reals, that is, the frame of open subsets of $\mathbb{R}$. This ring is denoted by $\mathcal{R}L$ (see [4, 5] for details). Our main purpose in this article is to study the relations between the ring properties of $\mathcal{R}L$, the graph properties of $\Gamma(\mathcal{R}L)$ and the frame-theoretic properties of the frame $L$. Our characterizations extend similar ones for $\Gamma(\mathcal{C}(X))$ given in [3]. Although, in the statements of the characterizations we give verbatim, literal translations of those in $\Gamma(\mathcal{C}(X))$, our proofs are, of necessity, entirely different in that the proofs in [3] use points of spaces involved while our proofs rely heavily on the properties of the cozero part of frames.

Section 3 commences with a description that the concept of distance in $\Gamma(\mathcal{R}L)$ is captured in pointfree topology (Proposition 3.3). We then determine the diameter, girth and the radius of $\Gamma(\mathcal{R}L)$. It turns out that the diameter, the girth and the radius of $\Gamma(\mathcal{R}L)$ are 2 or 3, 2 or 3 and 2 or 3, respectively.

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In Section 4, we study cycles in $\Gamma(\mathcal{RL})$. It turns out that the cycles in $\Gamma(\mathcal{RL})$ have only length 3 or 4. Graphical characterizations of regularity and almost regularity of the ring $\mathcal{RL}$, and pointfree characterizations for the graph $\Gamma(\mathcal{RL})$ to be triangulated, hypertriangulated and complemented are provided in this section. In the paper [15], the concept of a middle $P$-frame has been introduced by Ighedo. In Proposition 4.5, we show that $\Gamma(\mathcal{RL})$ is a hypertriangulated graph if and only if $L$ is a connected middle $P$-frame.

We introduce the cellularity in Definitions 5.1. This definition and the weight of a frame enable us to study the dominating number and the clique number of $\Gamma(\mathcal{RL})$ in Section 5. A pointfree characterization and an algebraic characterization for the set of centers of $\Gamma(\mathcal{RL})$ to be a dominating set are given in Theorem 5.7.

2. Preliminaries

2.1. Frames

For general facts concerning pointfree functions rings, general topology, the ring $C(X)$, and frames see [4, 5, 13, 14, 17]. Here, we recall a few definitions and results that will be relevant for our discussion.

A frame is a complete lattice for which finite meets distribute over arbitrary joins. Let $L$ be a frame. We denote the top element and the bottom element of $L$ by $\top$ and $\bot$ respectively. Throughout this context $L$ will denote a frame. The frame of open subsets of a topological space $X$ is denoted by $\mathcal{C}(X)$.

The pseudocomplement of an element $a \in L$, denoted $a^*$, is the element

$$a^* = \bigvee \{x \in L \mid a \land x = \bot\}.$$

We recall that:

1. if $a \leq b$, then $b^* \leq a^*$.
2. $a \leq a^{**}$ and $a^* = a^{**}$.
3. $(a \lor b)^* = a^* \land b^*$ and $(a \land b)^{**} = a^{**} \land b^{**}$.

An element $a$ of $L$ is said to be complemented if $a \lor a^* = \top$, and dense if $a^* = \bot$. We call the set of all complemented elements of $L$ the Boolean part of $L$, and denote it by $BL$. For any frame $L$, we have $BL = \{x \in L : x \lor x^* = \top\}$ and $BL$ is a sublattice of $L$. Notice that every element $x$ of $BL$ has a unique complement, which is denoted by $x^*$.

An element $p \in L$ is said to be an atom if $p \neq \bot$ and there exists no element $x$ with $\bot < x < p$.

A frame $L$ is said to be completely regular if, for each $a \in L$, $a = \bigvee \{x \in L \mid x \ll a\}$, where $x \ll a$ means that there are elements $(c_q)$ indexed by the rational numbers $\mathbb{Q} \cap [0, 1]$ such that $c_0 = x$, $c_1 = a$, and $c_p < c_q$ for $p < q$. Note that $x \ll a$ means that there is an element $b$ such that $x \land b = \bot$ and $b \lor a = \top$, or equivalently, $x^* \lor a = \top$. Throughout, all frames under consideration are assumed to be completely regular.

2.2. The ring $\mathcal{RL}$

Regarding the frame of reals $L(\mathbb{R})$ and the $f$-ring $\mathcal{RL}$ of continuous real functions on $L$, we use the notation of [5]. We freely use the properties of the cozero map $\text{coz}: \mathcal{RL} \to L$, given by

$$\text{coz} \alpha = \bigvee \{\alpha(p, q) \mid q < 0 \text{ or } p > 0\},$$

and those of $\text{Coz}L = \{\text{coz} \alpha \mid a \in \mathcal{RL}\}$, the cozero part of $L$. Note that $\text{Coz}L$ is a regular sub-$\sigma$-frame of $L$; and a frame is completely regular if and only if it is generated by its cozero part. We refer to [4–6] for general properties of cozero elements and cozero parts of frames.

3. Paths in $\Gamma(\mathcal{RL})$

To begin with, we note that $\alpha \in \mathcal{RL}$ is a zero-divisor if and only if $\text{coz} \alpha$ is not dense (see [9, Corollary 4.2] for details). Hence, $0 \neq a \in \Gamma(\mathcal{RL})$ if and only if $(\text{coz} \alpha)^* \neq \top$. Also if $a \in L$ is different from the top or the bottom, then there is $a \in \Gamma(\mathcal{RL})$ such that $\text{coz} \alpha \ll a$. To see this, by complete regularity, take $a \in \mathcal{RL}$ such that $\bot \neq \text{coz} \alpha \ll a$. That is to say $(\text{coz} \alpha)^* \lor a = \top$, implying that $(\text{coz} \alpha)^* \neq \bot$. Thus $a \in \Gamma(\mathcal{RL})$. 


Remark 3.1. If $|L| = 2$, that is, $L = 2$, then $\mathcal{RL}$ is isomorphic with the field of real numbers, that is, $\mathcal{RL} \cong \mathbb{R}$. On the other hand, the three-element chain $3 = \langle \bot, m, \top \rangle$ is not completely regular. Thus for studying the zero-divisor graph of $\mathcal{RL}$, we should consider $|L| \geq 4$. Next, by [1, Theorem 2.2], it is easy to see that $\Gamma(\mathcal{RL})$ is always infinite.

Recall that for two vertices $\alpha$ and $\beta$ of $\Gamma(\mathcal{RL})$, $d(\alpha, \beta)$ is the length of the shortest path from $\alpha$ to $\beta$. The diameter of $\Gamma(\mathcal{RL})$ is denoted by $\text{diam}(\Gamma(\mathcal{RL}))$ and is defined by $\text{diam}(\Gamma(\mathcal{RL})) = \sup\{d(\alpha, \beta) | \alpha, \beta \in \Gamma(\mathcal{RL})\}$. The girth of $\Gamma(\mathcal{RL})$, denoted $\text{gr}(\Gamma(\mathcal{RL}))$, is defined as the length of the shortest cycle in $\Gamma(\mathcal{RL})$.

The following proposition characterizes the concept of distance in $\Gamma(\mathcal{RL})$ using cozero elements of $L$.

First, we need the following lemma. In order to state this lemma, we need some background. As in [7], if $\alpha : L(\mathbb{R}) \to L$ and $a \in L$, then $\alpha a$ denotes the composite $\alpha : L(\mathbb{R}) \to L \to L a$. Recall also from [7, Lemma 1] that if $a \ll b$ in $L$, then there exists $\varphi \in \mathcal{RL}$ such that $0 \leq \varphi \leq 1$, $\varphi|a = 1$, and $\varphi|b = 0$.

Lemma 3.2. For every $\alpha, \beta \in \Gamma(\mathcal{RL})$, there exists a vertex $\varphi \in \Gamma(\mathcal{RL})$ adjacent to both $\alpha$ and $\beta$ if and only if $(\text{coz} \alpha)^* \land (\text{coz} \beta)^* \neq \bot$.

Proof. We begin with the sufficiency. Let $(\text{coz} \alpha)^* \land (\text{coz} \beta)^* \neq \bot$. Then, by complete regularity, there exists $\gamma \in \mathcal{RL}$ such that $\bot \neq \text{coz} \gamma \ll (\text{coz} \alpha)^* \land (\text{coz} \beta)^*$. Now, take $\varphi \in \mathcal{RL}$ such that $\varphi|\text{coz} \gamma = 1$ and $\varphi((\text{coz} \alpha)^* \land (\text{coz} \beta)^*) = 0$. The latter implies that $\text{coz} \varphi \land ((\text{coz} \alpha)^* \land (\text{coz} \beta)^*) = \bot$ and hence $\text{coz} \varphi \leq ((\text{coz} \alpha)^* \land (\text{coz} \beta)^*)^* = (\text{coz} \alpha)^{**} \land (\text{coz} \beta)^{**} = (\text{coz} \alpha)^* \land (\text{coz} \beta)^*$.

In consequence,

$$
\text{coz}(\varphi \alpha) = \text{coz} \varphi \land \text{coz} \alpha \leq ((\text{coz} \alpha)^* \land (\text{coz} \beta)^*) \land \text{coz} \alpha
$$

$$
= ((\text{coz} \alpha)^* \land \text{coz} \alpha) \land (\text{coz} \beta)^* = \bot.
$$

Therefore $\varphi \alpha = 0$, similarly $\varphi \beta = 0$. Consequently, $\varphi \in \Gamma(\mathcal{RL})$ and $\varphi$ adjacent to both $\alpha$ and $\beta$. Conversely, if there exists $\varphi \in \Gamma(\mathcal{RL})$ adjacent to both $\alpha$ and $\beta$, then $\varphi \alpha = \varphi \beta = 0$. This implies that $\text{coz} \varphi \leq (\text{coz} \alpha)^* \land (\text{coz} \beta)^*$ and therefore $(\text{coz} \alpha)^* \land (\text{coz} \beta)^* \neq \bot$, since $\text{coz} \varphi \neq \bot$.

Proposition 3.3. Let $\alpha, \beta \in \Gamma(\mathcal{RL})$. Then the following statements hold.
1. $d(\alpha, \beta) = 1$ if and only if $\text{coz} \alpha \land \text{coz} \beta = \bot$.
2. $d(\alpha, \beta) = 2$ if and only if $\text{coz} \alpha \land \text{coz} \beta \neq \bot$ and $(\text{coz} \alpha)^* \land (\text{coz} \beta)^* \neq \bot$.
3. $d(\alpha, \beta) = 3$ if and only if $\text{coz} \alpha \land \text{coz} \beta \neq \bot$ and $(\text{coz} \alpha)^* \land (\text{coz} \beta)^* = \bot$.

Proof. (1) Trivial.

To prove (2), first suppose that $d(\alpha, \beta) = 2$. Then, by part (1), $\text{coz} \alpha \land \text{coz} \beta \neq \bot$ and there exists $\varphi \in \Gamma(\mathcal{RL})$ such that $\varphi$ is adjacent to both $\alpha$ and $\beta$. Therefore, by Lemma 3.2, $(\text{coz} \alpha)^* \land (\text{coz} \beta)^* \neq \bot$. Conversely, let $\text{coz} \alpha \land \text{coz} \beta \neq \bot$ and $(\text{coz} \alpha)^* \land (\text{coz} \beta)^* \neq \bot$. Then, by part (1), $d(\alpha, \beta) > 1$ and, by Lemma 3.2, there is a vertex adjacent to both $\alpha$ and $\beta$. These imply $d(\alpha, \beta) = 2$.

To show (3), let $d(\alpha, \beta) = 3$. Clearly, by parts (1) and (2), $\text{coz} \alpha \land \text{coz} \beta \neq \bot$ and $(\text{coz} \alpha)^* \land (\text{coz} \beta)^* = \bot$. Conversely, suppose that $\text{coz} \alpha \land \text{coz} \beta \neq \bot$ and $(\text{coz} \alpha)^* \land (\text{coz} \beta)^* = \bot$. By parts (1) and (2), $d(\alpha, \beta) > 2$. Now, if a vertex $\delta$ is adjacent to $\alpha$ and a vertex $\gamma$ is adjacent to $\beta$, then $\alpha \delta = \beta \gamma = 0$. In consequence, $\text{coz} \delta \land \text{coz} \gamma \leq (\text{coz} \alpha)^* \land (\text{coz} \beta)^* = \bot$, implying that $\text{coz}(\delta \gamma) = \bot$ and hence $\delta \gamma = 0$, this means that $\delta$ is adjacent to $\gamma$. Therefore $d(\alpha, \beta) = 3$.

Note that if $\alpha \in \Gamma(\mathcal{RL})$, then $\text{coz} \alpha \land \text{coz} \alpha \neq \bot$ and $(\text{coz} \alpha)^* \land (\text{coz} \alpha)^* \neq \bot$. Now as a consequence, by part (2) of Proposition 3.3, we have the following.

Corollary 3.4. Whenever $|L| \geq 4$, then $\text{diam} \Gamma(\mathcal{RL}) \geq 2$.

We intend to show that $\text{diam} \Gamma(\mathcal{RL}) = \text{gr} \Gamma(\mathcal{RL}) = 3$ for when $|L| \neq 4$. For this we shall need a series of results. We begin with a lemma. Before the following lemma is presented, let us recall that a graph $G$ is connected if there is a path between any two distinct vertices. Note that $\Gamma(\mathcal{RL})$ is always connected (see [1, Theorem 2.3]).
Lemma 3.5. Let $\alpha, \beta \in \Gamma(\mathcal{R}L)$ be such that $\text{coz} \, \alpha \land \text{coz} \, \beta = \top$ and $(\text{coz} \, \alpha)^* \land (\text{coz} \, \beta)^* \neq \bot$. Then $\text{gr} \, \Gamma(\mathcal{R}L) = 3$.

Proof. By hypothesis, we have $\alpha \beta = 0$ and $\alpha^2 + \beta^2$ is a nonzero zero-divisor. Hence, there exists $\gamma \in \Gamma(\mathcal{R}L)$ such that $\gamma(\alpha^2 + \beta^2) = 0$, implying that $\gamma^2(\alpha^2 + \beta^2) = 0$. This shows that $\gamma \alpha = \gamma \beta = 0$. Therefore $\text{gr} \, \Gamma(\mathcal{R}L) = 3$. \hfill $\Box$

Recall that if $a$ is an atom of $L$, then, by complete regularity, it is complemented and so $a \in \text{Coz} \, L$. For the proof of the next corollary, we shall use the fact that if $a$ and $b$ are two atoms of $L$ such that $a' \neq b$, then

$$a \land b = \bot \Rightarrow a \lor b \neq \top \Rightarrow a' \land b' \neq \bot.$$

Corollary 3.6. Whenever $L$ has at least 3 atoms, then $\text{diam} \, \Gamma(\mathcal{R}L) = \text{gr} \, \Gamma(\mathcal{R}L) = 3$.

Proof. Whenever $L$ has at least 3 atoms, then there exist $\alpha, \beta \in \Gamma(\mathcal{R}L)$ such that $\text{coz} \, \alpha \land \text{coz} \, \beta = \bot$ and $(\text{coz} \, \alpha)^* \land (\text{coz} \, \beta)^* = \bot$. Now, part (3) of Proposition 3.3 implies that $\text{diam} \, \Gamma(\mathcal{R}L) = 3$. To prove the second part it is easy to see that there exist $\delta, \rho \in \mathcal{R}L$ such that $\text{coz} \, \delta \land \text{coz} \, \rho = \bot$ and $(\text{coz} \, \delta)^* \land (\text{coz} \, \rho)^* = \bot$. Now, Lemma 3.5 shows that $\text{gr} \, \Gamma(\mathcal{R}L) = 3$. \hfill $\Box$

Recall from [8, Lemma 3.3] that if $\text{coz} \, \alpha \ll \text{coz} \, \beta$ for some $\alpha, \beta \in \mathcal{R}L$, then there exists $\delta \in \mathcal{R}L$ such that $\alpha = \delta \beta$.

Example 3.7. Suppose that $|L| = 4$. Since the four-element chain $4 = \{\bot, m < n, \top\}$ is not completely regular, we can conclude that $L = \{\bot, a, b, \top\}$, where $b = a'$. Now, let $\alpha, \beta \in \mathcal{R}L$ with $\text{coz} \, \alpha = a$, $\text{coz} \, \beta = b$. We put

$A = \{\delta \in \mathcal{R}L \mid \text{coz} \, \delta = \text{coz} \, \alpha\}$ \quad and \quad $B = \{\gamma \in \mathcal{R}L \mid \text{coz} \, \gamma = \text{coz} \, \beta\}$.

It is easy to see that the zero-divisor graph of $\mathcal{R}L$ is a graph where its vertices are two disjoint nonempty sets $A$ and $B$ such that two vertices $\delta$ and $\gamma$ are adjacent if and only if $\delta \in A$ and $\gamma \in B$. This means that $\Gamma(\mathcal{R}L)$ is a bipartite complete graph. Consequently, $\text{diam} \, \Gamma(\mathcal{R}L) = 2$ and $\text{gr} \, \Gamma(\mathcal{R}L) = 4$.

Before proving the next result let us notice the following about cozero elements. If $a \ll b$ in $L$, then there is $c \in \text{Coz} \, L$ such that $a \ll c \ll b$ (see [6, Corollary 1]). As a consequence, there exists $d \in \text{Coz} \, L$ such that $a \land d = \bot$ and $d \lor b = \top$.

Lemma 3.8. Let $\alpha \in \Gamma(\mathcal{R}L)$. Then the following statements hold.

1. If $\text{coz} \, \alpha \not\in \mathcal{R}L$, then there exists $\beta \in \Gamma(\mathcal{R}L)$ such that $d(\alpha, \beta) = 3$.
2. Let $\text{coz} \, \alpha \in \mathcal{R}L$. If there exists $\gamma \in \Gamma(\mathcal{R}L)$ such that $\text{coz} \, \gamma \not\leq \text{coz} \, \alpha$ and $\text{coz} \, \gamma \in \mathcal{R}L$, then there exists $\beta \in \Gamma(\mathcal{R}L)$ such that $d(\alpha, \beta) = 3$.

Proof. (1). Since $\text{coz} \, \alpha \not\in \mathcal{R}L$, there exists $\delta \in \mathcal{R}L$ such that $\bot \neq \text{coz} \, \delta \ll \text{coz} \, \alpha$. This means that $\text{coz} \, \delta \land \text{coz} \, \beta = \bot$ and $\text{coz} \, \alpha \lor \text{coz} \, \beta = \top$ for some $\beta \in \mathcal{R}L$. In consequence,

$$(\text{coz} \, \beta)^* \neq \bot, \text{coz} \, \alpha \land \text{coz} \, \beta \neq \bot \text{ and } (\text{coz} \, \alpha)^* \land (\text{coz} \, \beta)^* = \bot.$$ 

Therefore, part (3) of Proposition 3.3 implies that $d(\alpha, \beta) = 3$.

(2). Putting $\text{coz} \, \beta = (\text{coz} \, \gamma)^*$, we then have

$$\text{coz} \, \alpha \land \text{coz} \, \beta \neq \bot \text{ and } (\text{coz} \, \alpha)^* \land (\text{coz} \, \beta)^* = \bot.$$ 

Therefore, part (3) of Proposition 3.3 implies that $d(\alpha, \beta) = 3$. \hfill $\Box$

Proposition 3.9. Whenever $L$ has at least 5 elements, then $\text{diam} \, \Gamma(\mathcal{R}L) = 3$.

Proof. We consider two cases. 

Case 1: Suppose that $\text{Coz} \, L = \mathcal{R}L$ (note that $L$ can be finite or infinite). Then $\text{Coz} \, L$ has at least 3 elements other than the top or the bottom. When $\text{Coz} \, L$ has at least 3 atoms, Corollary 3.6 implies that $\text{diam} \, \Gamma(\mathcal{R}L) = 3$. Now, suppose $\text{coz} \, \alpha$ is not an atom for some $\alpha \in \mathcal{R}L$. Then there exists $\delta \in \mathcal{R}L$ such that $\bot \neq \text{coz} \, \delta \leq \text{coz} \, \alpha$. Therefore, part (2) of Lemma 3.8 implies that $\text{diam} \, \Gamma(\mathcal{R}L) = 3$.
Case 2: Suppose $BL \not\subseteq Coz L$ (note that $L$ and $Coz L$ are infinite). If there exists $a \in \Gamma(\mathcal{RL})$ such that $coz a \in Coz L \setminus BL$, then part (1) of Lemma 3.8 implies that $gim diam \Gamma(\mathcal{RL}) = 3$. Otherwise, for every $a \in \mathcal{RL}$ with $(coz a)^* \neq \bot$, we have $coz a \in BL$. This means that we can choose $\delta \in \Gamma(\mathcal{RL})$ with $coz \delta \in Coz L$. So, take $\rho \in \mathcal{RL}$ such that $coz \rho = (coz \delta)^*$. In case $coz \delta$ or $coz \rho$ is not an atom, part (2) of Lemma 3.8 implies that $gim diam \Gamma(\mathcal{RL}) = 3$. Now, let $coz \delta$ and $coz \rho$ be two atoms. If there is $\gamma \in \Gamma(\mathcal{RL})$ such that $coz \gamma \neq coz \delta$ and $coz \gamma \neq coz \rho$, then, by either Corollary 3.6 or part (1) of Lemma 3.8, we can conclude that $gim diam \Gamma(\mathcal{RL}) = 3$. Otherwise, by complete regularity, it is easy to show $L = \{\bot, coz \delta, coz \rho, \top\}$ which is a contradiction. Therefore, the proof is complete. □

The combination of this proposition with Example 3.7 imply the following corollary.

**Corollary 3.10.** The $gim diam \Gamma(\mathcal{RL}) = 2$ if and only if $L = \{\bot, a, b, \top\}$, where $b = a'$.

Next, we are going to discuss the girth of $\Gamma(\mathcal{RL})$. We begin with the following lemma. For the proof of this lemma, we shall use the following fact: If $a, b \in L$ and $a \ll b$, then $b' \ll a'$.

**Lemma 3.11.** Let $\alpha \in \Gamma(\mathcal{RL})$. Then the following statements hold.

1. Let $coz a \in BL$. If there exists $\gamma \in \Gamma(\mathcal{RL})$ such that $coz \gamma \not\subseteq coz a$ and $coz \gamma \in BL$, then $gim gr \Gamma(\mathcal{RL}) = 3$.

2. If $coz a \not\in BL$, then $gim gr \Gamma(\mathcal{RL}) = 3$.

**Proof.** (1). Putting $coz \beta = (coz a)^*$, we then have

$$coz \gamma \land coz \beta \leq coz \alpha \land coz \beta = \bot, \text{ in consequence } coz \gamma \land coz \beta = \bot;$$

and also we claim that $(coz \gamma)^* \land (coz \beta)^* \neq \bot$. To see this, suppose, by way of contradiction, that $(coz \gamma)^* \land (coz \beta)^* = \bot$. Then $(coz \gamma)^* \land coz \gamma = \bot$, implying that $coz \gamma \leq coz \gamma \leq coz \alpha$ and hence $coz \gamma = coz \gamma$ which is a contradiction. Therefore, by Lemma 3.5, $gim gr \Gamma(\mathcal{RL}) = 3$.

(2). Since $coz a \not\in BL$, there exists $\delta \in \mathcal{RL}$ such that $\bot \neq coz \delta \ll coz a$ and so $(coz a)^* \ll (coz \delta)^*$. Take $\rho \in \mathcal{RL}$ such that $(coz a)^* \ll coz \rho \ll (coz \delta)^*$. This show that $(coz \rho)^* \neq \bot$ and $coz \rho \land coz \delta \leq coz \rho \land (coz \delta)^* = \bot$, that is, $coz \rho \land coz \delta = \bot$. Now, if $(coz \rho)^* \land (coz \delta)^* = \bot$, then Lemma 3.5 shows that $gim gr \Gamma(\mathcal{RL}) = 3$. Otherwise, let $(coz \rho)^* \land (coz \delta)^* = \bot$. On the other hand, $coz \rho \ll (coz \delta)^*$ implies that $(coz \rho)^* \lor (coz \delta)^* = \top$. Therefore $(coz \delta)^* \in BL$. Now, we consider two cases.

Case 1: Suppose $(coz \alpha)^* \in BL$. Then since $coz \delta \ll coz a, (coz \alpha)^* \lor coz a = \top$, showing that $(coz \delta)^* \neq (coz \alpha)^*$. Therefore, by (1), $gim gr \Gamma(\mathcal{RL}) = 3$ since $(coz \alpha)^* \not\subseteq (coz \alpha)^*$, $(coz \alpha)^* \not\subseteq \bot$, and $(coz \delta)^* \neq \bot$.

Case 2: Suppose $(coz \alpha)^* \not\in BL$. Pick $\varphi \in \mathcal{RL}$ such that $(coz \varphi)^* \neq \bot$ and $\bot \neq coz \varphi \ll (coz \alpha)^*$, implying that $coz \varphi \land coz \delta = \bot$. Now, if $(coz \varphi)^* \land (coz \delta)^* \neq \bot$, then, by Lemma 3.5, $gim gr \Gamma(\mathcal{RL}) = 3$. Otherwise, $(coz \varphi)^* \land (coz \delta)^* = \bot$ implies that $(coz \varphi)^* \leq (coz \delta)^*$, which is $gim gr \Gamma(\mathcal{RL}) = 3$. Consequently, $(coz \delta)^* = (coz \alpha)^*$, implying $(coz \delta)^* = (coz \alpha)^*$, a contradiction because $(coz \delta)^* \in BL$ implies $(coz \alpha)^* \in BL$. □

**Proposition 3.12.** Whenever $L$ has at least 5 elements, then $gim gr \Gamma(\mathcal{RL}) = 3$.

**Proof.** We consider two cases.

Case 1: Suppose $Coz L = BL$. Then $Coz L$ has at least 3 elements other than the top or bottom. Whenever $Coz L$ has at least 3 atoms, then Corollary 3.6 implies $gim gr \Gamma(\mathcal{RL}) = 3$. Now, suppose $coz a$ is not an atom for some $a \in \mathcal{RL}$. Then there exists $\tau \in \mathcal{RL}$ such that $\bot = coz \tau \not\subseteq coz a$. Therefore, part (1) of Lemma 3.11 implies that $gim gr \Gamma(\mathcal{RL}) = 3$.

Case 2: Suppose $BL \not\subseteq Coz L$. If there exists $a \in \Gamma(\mathcal{RL})$ such that $coz a \in Coz L \setminus BL$, then part (2) of Lemma 3.11 implies that $gim gr \Gamma(\mathcal{RL}) = 3$. Otherwise, for every $a \in \mathcal{RL}$ with $(coz a)^* \neq \bot$, we have $coz a \in BL$. This means that we can choose $\delta \in \Gamma(\mathcal{RL})$ with $coz \delta \in Coz L$. Hence, take $\rho \in \mathcal{RL}$ such that $coz \rho = (coz \delta)^*$. In case $coz \delta$ or $coz \rho$ is not an atom, part (1) of Lemma 3.11 implies that $gim gr \Gamma(\mathcal{RL}) = 3$. Now, suppose $coz \delta$ and $coz \rho$ are atoms. If there is $\gamma \in \Gamma(\mathcal{RL})$ such that $coz \gamma \neq coz \delta$ and $coz \gamma \neq coz \rho$, then, by either Corollary 3.6 or part (1) of Lemma 3.11, we can conclude that $gim gr \Gamma(\mathcal{RL}) = 3$. Otherwise, by complete regularity, it is easy to show $L = \{\bot, coz \delta, coz \rho, \top\}$ which is a contradiction. Therefore, the proof is complete. □
An immediate consequence of Example 3.7 and the previous proposition, is the following corollary.

**Corollary 3.13.** The gr $\Gamma(\mathcal{RL}) = 4$ if and only if $L = \{\bot, a, b, \top\}$ where $b = a'$.

Combining Propositions 3.9 and 3.12, we have the following theorem.

**Theorem 3.14.** The diameter of $\Gamma(\mathcal{RL})$ and the girth of $\Gamma(\mathcal{RL})$ coincide whenever $L$ has at least 5 elements.

In what follows, we intend to study the radius of the zero-divisor graph of $\Gamma(\mathcal{RL})$. Let us recall the definition of the radius of a graph $G$. The associated number of a vertex $x$ of a graph $G$ denoted by $e(x)$ is defined as $e(x) = \sup \{d(x, y) \mid x \neq y \in G\}$. A center of $G$ is defined to be a vertex $t$ with the smallest associated number. The associated number $e(t)$ of any center $t$ is said to be the radius of $G$ and is denoted by $\rho(G)$.

**Proposition 3.15.** Suppose $a \in \Gamma(\mathcal{RL})$, then

$$e(a) = \begin{cases} 2 & \text{if } \coz a \text{ is an atom} \\ 3 & \text{otherwise.} \end{cases}$$

**Proof.** First, let $\coz a$ be an atom. Consider $\beta \in \Gamma(\mathcal{RL})$. Then $\coz a \land \coz \beta = \coz a$ or $\coz a \land \coz \beta = \bot$. The latter implies that $d(\alpha, \beta) = 1$. By the former case, we have $\coz a \leq \coz \beta$, showing that $(\coz a)^* \land (\coz \beta)^* = (\coz \beta)^* = \top$. Hence, by part (2) of Proposition 3.3, $d(a, \beta) = 2$. Therefore $e(a) = 2$ since $\coz a \land \coz 2a = \coz a$. Now, suppose $\coz a$ is not an atom. Then we consider two cases.

Case 1: If $\coz a \notin \mathcal{BL}$, then part (1) of Lemma 3.8 implies that $e(a) = 3$.

Case 2: Suppose $\coz a \in \mathcal{BL}$. Since $\coz a$ is not an atom, there exists $\delta \in \mathcal{RL}$ such that $\bot \neq \coz \delta \ll \coz a$ and $(\coz \delta)^* \neq \bot$. In case $\coz \delta \in \mathcal{BL}$, part (2) of Lemma 3.8 implies that $e(a) = 3$. Otherwise, by Case 1, $e(\delta) = 3$. Take $\beta \in \Gamma(\mathcal{RL})$ such that $d(\delta, \beta) = 3$, and so, by part (2) of Proposition 3.3, we have $\coz \delta \land \coz \beta \neq \bot$ and $(\coz \delta)^* \land (\coz \beta)^* = \bot$. This implies quickly that $\coz a \land \coz \beta \neq \bot$ and $(\coz a)^* \land (\coz \beta)^* = \bot$. Again, by part (2) of Proposition 3.3, $d(a, \beta) = 3$, which shows that $e(a) = 3$. $\square$

As an immediate consequence, we now have the following corollary.

**Corollary 3.16.** If $L$ has an atom, then $\rho(\Gamma(\mathcal{RL})) = 2$; otherwise $\rho(\Gamma(\mathcal{RL})) = 3$.

By this corollary and the definition of center, if $L$ has no atoms, that is, $\rho(\Gamma(\mathcal{RL})) = 3$; then every vertex is a center. But whenever $L$ has at least one atom, that is, if $\rho(\Gamma(\mathcal{RL})) = 2$, then the set of centers of $\Gamma(\mathcal{RL})$ is the set of all vertices $a \in \Gamma(\mathcal{RL})$ such that $\coz a$ is an atom.

4. Cycles in $\Gamma(\mathcal{RL})$

A graph $G$ is called triangulated (hypertriangulated) if each vertex (edge) of $G$ is a vertex (edge) of a triangle. In the next proposition, we show that every vertex of $\Gamma(\mathcal{RL})$ is a cycle vertex, that is, every vertex of $\Gamma(\mathcal{RL})$ belongs to a cycle. In fact, it turns out that for every vertex $a$ in $\Gamma(\mathcal{RL})$, there exists a 4-cycle (quadrangle) containing $a$ and whenever $L$ has no atoms, then for every vertex $a$ of $\Gamma(\mathcal{RL})$, there exists a 3-cycle (triangle) containing $a$.

**Proposition 4.1.** For a frame $L$, every vertex of $\Gamma(\mathcal{RL})$ is a 4-cycle vertex.

**Proof.** For every vertex $a$, there exists a vertex $\beta$ such that $a\beta = 0$ since $\Gamma(\mathcal{RL})$ is always connected. Therefore $a\beta = (2a)\beta = (2a)(2\beta) = a(2\beta) = 0$, that is, the path with vertices $a, \beta, 2a$ and $2\beta$ is a cycle with length 4 containing $a$. $\square$

By the above proposition, every vertex in $\Gamma(\mathcal{RL})$ is a vertex of a cycle. It is also easy to see that every edge in $\Gamma(\mathcal{RL})$ is an edge of a cycle.

In the following theorem, we give the frame-theoretic property of $L$ and the ring-theoretic property of $\mathcal{RL}$ for which the graph $\Gamma(\mathcal{RL})$ is triangulated. We begin with the following lemma.
Lemma 4.2. If $\alpha, \beta \in \Gamma(\mathcal{RL})$ such that $\text{coz} \alpha \not\subseteq \text{coz} \beta$ and $\text{coz} \beta \in BL$, then $\alpha$ is a vertex of a triangle.

Proof. Putting $\text{coz} \gamma = (\text{coz} \beta)^*$, we then have $\text{coz} \alpha \wedge \text{coz} \gamma = \bot$. Now, $(\text{coz} \alpha)^* \wedge (\text{coz} \gamma)^* = \bot$ implies that $\text{coz} \alpha = \text{coz} \beta$ which is a contradiction. In consequence, $(\text{coz} \alpha)^* \wedge (\text{coz} \gamma)^* \neq \bot$. Therefore, by Lemma 3.2, there exists a vertex $\delta$ adjacent to both $\alpha$ and $\gamma$, showing $\alpha$ is a vertex of the triangle with vertices $\alpha, \gamma$ and $\delta$. \hfill $\square$

Recall that an ideal of a ring is essential if it meets every nonzero ideal non-trivially. By Lemma 4.3 in [9], an ideal $I$ in $\mathcal{RL}$ is essential if and only if $\bigvee_{I \neq 0} \text{coz} \delta$ is dense.

Theorem 4.3. The following are equivalent for a frame $L$.

1. $\Gamma(\mathcal{RL})$ is a triangulated graph.
2. $L$ has no atoms.
3. There is no maximal ideal in $\mathcal{RL}$ generated by an idempotent.

Proof. (1) $\Rightarrow$ (2). Let $\Gamma(\mathcal{RL})$ be a triangulated graph and suppose $L$ has an atom $a$. Consider $a \in \mathcal{RL}$ such that $\text{coz} a = a'$, clearly $a \in \Gamma(\mathcal{RL})$. Since $\Gamma(\mathcal{RL})$ is triangulated, then there are $\gamma, \delta \in \Gamma(\mathcal{RL})$ such that $a \gamma = a \delta = a \delta = 0 = 0$. This shows that $\text{coz} \gamma \leq a$, $\text{coz} \delta \leq a$, and $\text{coz} \gamma \wedge \text{coz} \delta = \bot$. Therefore $\gamma = 0$ or $\delta = 0$, which is a contradiction.

(2) $\Rightarrow$ (1). Suppose that $L$ does not contain atoms and take $a \in \Gamma(\mathcal{RL})$. Then there exists $0 \neq \beta \in \mathcal{RL}$ such that $\text{coz} \beta \ll (\text{coz} \alpha)^*$ and $\text{coz} \beta \neq (\text{coz} \alpha)^*$. Now, we consider two cases.

Case 1: Suppose $(\text{coz} \beta)^* \wedge (\text{coz} \alpha)^* \neq \bot$. Then, by Lemma 3.2, there exists a vertex $\gamma$ adjacent to both $\alpha$ and $\beta$, which is a contradiction.

Case 2: Assume $(\text{coz} \beta)^* \wedge (\text{coz} \alpha)^* = \bot$. Pick $\delta \in \mathcal{RL}$ such that $\text{coz} \delta \ll \text{coz} \beta$. If $(\text{coz} \delta)^* \wedge (\text{coz} \alpha)^* \neq \bot$, then, similar to Case 1, $a$ is a vertex of a triangle. Otherwise, let $(\text{coz} \delta)^* \wedge (\text{coz} \alpha)^* = \bot$. Then since $(\text{coz} \beta)^* \vee (\text{coz} \alpha)^* = \top$ and $(\text{coz} \delta)^* \vee (\text{coz} \alpha)^* = \top$, we can conclude that $(\text{coz} \alpha)^* = (\text{coz} \beta)^* = (\text{coz} \delta)^*$. This shows that $\text{coz} \beta \vee (\text{coz} \alpha)^* = \top$ and so $\text{coz} \beta \in BL$. On the other hand, $\text{coz} \beta \ll (\text{coz} \alpha)^*$ implies that $\text{coz} a \leq (\text{coz} \alpha)^* \ll (\text{coz} \beta)^*$, showing $\text{coz} a \leq (\text{coz} \beta)^*$. Therefore if $\text{coz} a \neq (\text{coz} \beta)^*$, then, by Lemma 4.2, $a$ is a vertex of a triangle. Otherwise, $\text{coz} a = (\text{coz} \beta)^*$, which is a contradiction.

(2) $\Rightarrow$ (3). Let $M$ be a maximal ideal of $\mathcal{RL}$ generated by an idempotent. Take an idempotent $\eta \in \mathcal{RL}$ such that $M = \{\eta\}$. Then, $\{1 - \eta\}$ is a minimal ideal generated by the idempotent $1 - \eta$, and hence, by the proof of Lemma 3.4 in [11], $\text{coz}(1 - \eta)$ is an atom.

(3) $\Rightarrow$ (2). Let $a$ be an atom of $L$. Again, by Lemma 3.4 in [11], the ideal $M_a = \{\delta \in \mathcal{RL} : \text{coz} \delta \leq a\}$ is a minimal ideal and $\bigvee M_a = a$. Hence, Lemma 4.3 in [9] shows that $M_a$ is a non-essential ideal of $\mathcal{RL}$. Thus, Lemma 4.5 in [11] implies that $M_a = \{\eta\}$ for some idempotent $\eta \in \mathcal{RL}$. Now, since $\mathcal{RL}$ is a reduced ring, $M = \{1 - \eta\}$ is a maximal ideal of $\mathcal{RL}$ generated by an idempotent. \hfill $\square$

Recall from [3] that a zero set $Z$ in $X$ is said to be a middle zerounset if there exist two proper zerounsets $E$ and $F$ such that $Z = E \cap F$ and $E \cup F = X$. A space $X$ is called a middle $P$-space if every non-top middle zerounset in $X$ has a nonempty interior. Clearly, every almost $P$-space is a middle $P$-space but not conversely (see [3] for details). Now, adapting this to frames, Ighedo [15] has introduced the following definition.

Definition 4.4. (1) A cozero element $c \in L$ is said to be middle cozero element if there exist two cozero elements $a$ and $b$ other than the bottom such that $c = a \vee b$ and $a \wedge b = \bot$.

(2) A frame $L$ is called a middle $P$-frame if every non-top middle cozero element in $L$ is not dense. This is equivalent to saying $L$ is a $P$-frame if and only if the only dense middle cozero element of $L$ is $\top$.

Clearly, a topological space $X$ is a middle $P$-space if and only if the frame $\mathcal{S}X$ is a middle $P$-frame. For more details about middle $P$-frames see [15].

In the following proposition, we consider frame-theoretic properties of $L$ for which the graph $\Gamma(\mathcal{RL})$ is hypertriangulated. Recall from [4] that a frame is disconnected if there is at least one non-trivial complemented element. A frame is connected if it is not disconnected, or equivalently, if $a \wedge b = \bot$ and $a \vee b = \top$ imply $a = \top$ or $b = \top$. We conclude that (coz $\alpha$) $\subseteq$ coz $\beta$ and coz $\beta \in BL$, then $\alpha$ is a vertex of a triangle.
Proposition 4.5. For a frame $L$, $\Gamma(L)$ is a hypertriangulated graph if and only if $L$ is a connected middle $P$-frame.

Proof. Let $\Gamma(L)$ be a hypertriangulated graph. If $L$ is not connected, then there exists a complemented element $\perp \neq a$ in $L$. Take $a, b$ in $RL$ such that $\text{coz} a = a$ and $\text{coz} b = b'$. Now, $\text{coz} a \wedge \text{coz} b = \perp$ implies that $a$ is adjacent to $b$ and since $(\text{coz} a)^* \wedge (\text{coz} b)^* = \perp$, then by Lemma 3.2, there is no vertex adjacent to both $a$ and $b$. So the edge $a - b$ does not belong to a triangle, which is a contradiction; therefore $L$ is connected.

Now, let $\text{coz} a$ be a middle cozero element. Then $\text{coz} a = \text{coz} b \vee \text{coz} \gamma$ and $\text{coz} b \wedge \text{coz} \gamma = \perp$ for some cozero elements $\text{coz} b$ and $\text{coz} \gamma$. Since $\Gamma(L)$ is hypertriangulated, then $b - \gamma$ is an edge of a triangle, that is, there exists a vertex $\delta$ such that $\beta \delta = \gamma \delta = 0$. This implies that $\text{coz} a = \text{coz} b \vee \text{coz} \gamma \leq (\text{coz} \delta)^*$, showing $\perp \neq \text{coz} \delta \leq (\text{coz} \delta)^* \leq (\text{coz} a)^*$ which means that $(\text{coz} a)^* \neq \perp$. Consequently, $L$ is a middle $P$-frame.

Conversely, let $L$ be a connected middle $P$-frame and $a - b$ be an edge in $\Gamma(L)$. Since $\text{coz} a \wedge \text{coz} b = \perp$ and $L$ is connected, then $\text{coz} a \vee \text{coz} b \neq \top$. This shows that $(\text{coz} a)^* \wedge (\text{coz} b)^* \neq \perp$, because $L$ is a middle $P$-frame. Now, by Lemma 3.2 there exists a vertex adjacent to both $a$ and $b$. This means that $\Gamma(L)$ is a hypertriangulated graph. \qed

If $a$ and $b$ are two vertices in $\Gamma(L)$, by $c(a, b)$, we mean the length of the smallest cycle containing $a$ and $b$. For every two vertices $a$ and $b$, all possible cases for $c(a, b)$ are provided in the next proposition.

Proposition 4.6. Let $a, b \in \Gamma(L)$. Then the following statements hold.

1. $\text{c}(a, b) = 3$ if and only if $\text{coz} a \wedge \text{coz} b = \perp$ and $(\text{coz} a)^* \wedge (\text{coz} b)^* \neq \perp$.
2. $\text{c}(a, b) = 4$ if and only if either $\text{coz} a \wedge \text{coz} b \neq \perp$ and $(\text{coz} a)^* \wedge (\text{coz} b)^* \neq \perp$ or $\text{coz} a \wedge \text{coz} b = \perp$ and $(\text{coz} a)^* \wedge (\text{coz} b)^* = \perp$.
3. $\text{c}(a, b) = 6$ if and only if $\text{coz} a \wedge \text{coz} b \neq \perp$ and $(\text{coz} a)^* \wedge (\text{coz} b)^* = \perp$.

Proof. First of all, by Lemma 3.2 and Proposition 3.3, it is easily checked that parts (1) and (2) are true. To prove part (3), if $\text{c}(a, b) = 6$, then parts (1) and (2) imply that $\text{coz} a \wedge \text{coz} b \neq \perp$ and $(\text{coz} a)^* \wedge (\text{coz} b)^* = \perp$. Conversely, since $\text{coz} a \wedge \text{coz} b \neq \perp$ and $(\text{coz} a)^* \wedge (\text{coz} b)^* = \perp$, then by part (3) of Proposition 3.3, we have $d(a, b) = 3$. Thus, there exist vertices $\gamma$ and $\delta$ such that $\alpha \gamma = \gamma \delta = \delta \beta = 0$. Now, if some vertex $\varphi$ is adjacent to $\beta$, then $\varphi \beta = 0$. In consequence, $\gamma \varphi \leq (\text{coz} b)^*$ and $\gamma \gamma \leq (\text{coz} a)^*$ implying that $\text{coz} \gamma \wedge \text{coz} \varphi \leq (\text{coz} a)^* \wedge (\text{coz} b)^* = \perp$, and so $\varphi$ is adjacent to $\gamma$. But this simply means that $c(a, b) \geq 5$. On the other hand, $d(a, b) = 3$ implies that $a$ is not adjacent to $\varphi$. Therefore, $c(a, b) \geq 6$.

As an immediate consequence, we now have the following result.

Corollary 4.7. For a frame $L$, the following statements hold.

1. Every cycle in $\Gamma(L)$ has length 3 or 4.
2. Every edge of $\Gamma(L)$ is an edge of a cycle with length 3 or 4.

A frame $L$ is said to be a $P$-frame if for every $a \in RL$, $\text{coz} a$ is complemented. It is well known that $L$ is a $P$-frame if and only if the ring $RL$ is regular (that is, for every $a \in RL$, there exists $b \in RL$ such that $a^2 b = a$).

A frame $L$ is called almost $P$-frame if every cozero element in $L$ is regular (or equivalently, every nonunit element of $RL$ is zero-divisor). Whenever $L$ is an almost $P$-frame, we call the ring $RL$ almost regular. We refer the reader to [8] and [9] for more details and properties of $P$-frames and almost $P$-frames.

Proposition 4.8. For a frame $L$, the ring $RL$ is almost regular if and only if for every nonunit $a \in RL$, there exists $1 \neq b \in RL$ such that $a = ab$.

Proof. To prove the ‘if’ part, let $a \in RL$ be a nonunit element. We can assume that $a$ is a nonzero-nonunit element since otherwise there is nothing to prove. Then $\perp \neq (\text{coz} a)^* \neq \top$, and so there exists $\gamma \in RL$ such that $\text{coz} \gamma \ll (\text{coz} a)^*$, implying that $\text{coz} a \leq (\text{coz} a)^* \ll (\text{coz} \gamma)^*$. It follows that $\text{coz} a \ll \text{coz} \rho \ll (\text{coz} \gamma)^*$ for some $\rho \in RL$. Therefore, by [8, Lemma 3.3], there exists $1 \neq b \in RL$ such that $a = ab$.

To prove the ‘only if’ part, it is enough to show that every nonunit element of $RL$ is zero-divisor. Let $a \in RL$ be a nonunit element. Then, by the hypothesis, there exists $1 \neq b \in RL$ such that $a = ab$. This shows that $\text{coz} a \wedge \text{coz} (1 - b) = \perp$, implying that $\text{coz} (1 - b) \leq (\text{coz} a)^*$. In consequence $(\text{coz} a)^* \neq \perp$, since $\perp \neq \text{coz} (1 - b)$. Therefore $a$ is zero-divisor. \qed
In the following theorem, the almost regularity of the ring $RL$ is characterized graphically. We begin with the following lemma.

**Lemma 4.9.** Let $L$ be an infinite frame. Then there exist $\delta, \gamma \in \Gamma(L)$ such that $\text{coz} \delta \lor \text{coz} \gamma = \top$ and $\text{coz} \delta \land \text{coz} \gamma \neq \bot$.

**Proof.** We consider two cases.

Case 1. Suppose that for every $a \in \Gamma(L)$, $\text{coz} a \in BL$. Now, if there exists $a \in \Gamma(L)$ such that $\text{coz} a$ is not an atom, then we have $\text{coz} b \ll \text{coz} a$ for some $b \in \Gamma(L)$ with $\text{coz} b \neq \text{coz} a$. Putting $\text{coz} \delta = \text{coz} a$ and $\text{coz} \gamma = (\text{coz} \beta)^\ast$, we then have $\text{coz} \delta \lor \text{coz} \gamma = \top$ and $\text{coz} \delta \land \text{coz} \gamma \neq \bot$. Otherwise, for every $a \in \Gamma(L)$, $\text{coz} a$ is an atom. Then it is easy to see that $|L| = 4$ which is a contradiction.

Case 2. Let $\text{coz} a \notin BL$ for some $a \in \Gamma(L)$. Then there exists $\beta \in \Gamma(L)$ with $\text{coz} \beta \ll \text{coz} a$. Take $\text{coz} \delta \in \text{Coz} L$ such that $\text{coz} \beta \land \text{coz} \delta = \bot$ and $\text{coz} a \lor \text{coz} \delta = \top$. Putting $\gamma = a$, we then have $\text{coz} \delta \lor \text{coz} \gamma = \top$ and $\text{coz} \delta \land \text{coz} \gamma \neq \bot$. $\square$

**Theorem 4.10.** For a frame $L$, $RL$ is an almost regular ring if and only if for every pair of vertices $\beta$ and $\gamma$ of $\Gamma(L)$ and every nonunit $a \in RL$, $c(a\beta, a\gamma) \leq 4$.

**Proof.** Necessity. Let $RL$ be an almost regular ring. Then for any nonunit $a \in RL$, we have $(\text{coz} a)^\ast \neq \bot$. Since $(\text{coz} a)^\ast \leq (\text{coz} a\beta)^\ast \land (\text{coz} a\gamma)^\ast$, then by parts (1) and (2) of Proposition 4.6 $c(a\beta, a\gamma) \leq 4$.

Sufficiency. Suppose that $a \in RL$ is a nonunit and $c(a\beta, a\gamma) \leq 4$ for all vertices $\beta$ and $\gamma$ of $\Gamma(L)$. If $L$ is finite, then it is easy to show that $RL$ is almost regular. Otherwise, suppose that $L$ is infinite. Then, by Lemma 4.9, there exist $\delta, \rho \in \Gamma(L)$ such that $\text{coz} \delta \lor \text{coz} \rho = \top$ and $\text{coz} \delta \land \text{coz} \rho \neq \bot$. If $\text{coz} a\delta \land \text{coz} a\rho = \bot$, then $\text{coz} a \land \text{coz} \delta \land \text{coz} \rho = \bot$. This implies that $\bot \neq \text{coz} \delta \land \text{coz} \rho \leq (\text{coz} a)^\ast$. In consequence, $(\text{coz} a)^\ast \neq \bot$. Now, suppose that $\text{coz} a\delta \land \text{coz} a\rho \neq \bot$. Then since $c(\text{coz} a\delta, \text{coz} a\rho) \leq 4$, by part (2) of Proposition 4.6, we have $(\text{coz} a\delta)^\ast \land (\text{coz} a\rho)^\ast \neq \bot$. On the other hand,

$$\text{coz} a = \text{coz} a \land \top = \text{coz} a \land \text{coz} \delta \lor \text{coz} \rho = \text{coz} a\delta \lor \text{coz} a\rho,$$

and so $(\text{coz} a)^\ast = (\text{coz} a\delta)^\ast \land (\text{coz} a\rho)^\ast$. Consequently, $(\text{coz} a)^\ast \neq \bot$. Therefore $a$ is zero-divisor and the proof is complete. $\square$

In the next theorem, the regularity of the ring $RL$ is characterized graphically.

**Theorem 4.11.** For a frame $L$, $RL$ is a regular ring if and only if $RL$ is an almost regular ring and for every vertex $a$ of $\Gamma(L)$, there exists a vertex $\beta$ of $\Gamma(L)$ adjacent to $a$ such that $c(a, \beta) = 4$.

**Proof.** Necessity. If $RL$ is regular, then clearly it is almost regular. Since for every vertex $a$, $\text{coz} a$ is complemented, then $(\text{coz} a)^\ast$ is also a cozero element. Putting $\text{coz} \beta = (\text{coz} a)^\ast$, we then have $\text{coz} a \land \text{coz} \beta = \bot$ and $(\text{coz} a)^\ast \land (\text{coz} \beta)^\ast = \bot$. Now, part (2) of Proposition 4.6 implies that $c(a, \beta) = 4$.

Sufficiency. Let $0 \neq a \in RL$ be a nonunit. Then $a \in \Gamma(L)$ since $RL$ is an almost regular ring. Now, the hypothesis implies that $a$ is adjacent to $\beta$ for some $\beta \in \Gamma(L)$, such that $c(a, \beta) = 4$. Hence, by part (2) of Proposition 4.6, we have $(\text{coz} a)^\ast \land (\text{coz} \beta)^\ast = \bot$. This shows that $(\text{coz} a \lor \text{coz} \beta)^\ast = \bot$ and hence $(\text{coz} a \lor \text{coz} \beta)^\ast = \top$, implying that $\text{coz} a \lor \text{coz} \beta = \top$ since $RL$ is almost regular. Therefore $\text{coz} a$ is complemented, that is, $RL$ is regular. $\square$

As defined in [16], for distinct vertices $x$ and $y$ in a graph $G$, we say that $x$ and $y$ are orthogonal, written $x \perp y$, if $x$ and $y$ are adjacent and there is no vertex $z$ of $G$ which is adjacent to both $x$ and $y$. A graph $G$ is said to be complemented if for each vertex $x$ of $G$, there is a vertex $y$ of $G$, called a complement of $x$, such that $x \perp y$, and that $G$ is uniquely complemented if $G$ is complemented and whenever $x \perp y$ and $x \perp z$, then $y \approx z$, this means that $y$ and $z$ are adjacent to exactly the same vertices.

By Lemma 3.2, for every two vertices $a$ and $\beta$ in $\Gamma(L)$, $a \perp \beta$ if and only if $\text{coz} a \land \text{coz} \beta = \bot$ and $(\text{coz} a)^\ast \land (\text{coz} \beta)^\ast = \bot$. Hence, $\Gamma(L)$ is complemented if and only if for every vertex $a$, there exists a vertex $\delta$ such that $\text{coz} a \land \text{coz} \delta = \bot$ and $(\text{coz} a)^\ast \land (\text{coz} \delta)^\ast = \bot$. 

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For our next characterization, we let \( \text{Min}(\mathcal{R}L) \) be the space of minimal prime ideals of \( \mathcal{R}L \). A frame \( L \) has been defined in [9] to be cozero-complemented if for each \( a \in \text{Coz} L \), there exists \( b \in \text{Coz} L \) such that \( a \land b = 1 \) and \( a \lor b \) is dense. The space \( \text{Min}(\mathcal{R}L) \) is compact if and only if \( L \) is cozero-complemented (see [9, Proposition 4.7]). Now, the following corollary is obvious.

**Corollary 4.12.** For a frame \( L \), \( \Gamma(\mathcal{R}L) \) is complemented if and only if \( L \) is cozero-complemented if and only if the space \( \text{Min}(\mathcal{R}L) \) is compact.

We close this section by giving a direct proof for the next proposition which is also true for every reduced ring, see [2, Theorem 3.5].

**Proposition 4.13.** For a frame \( L \), \( \Gamma(\mathcal{R}L) \) is complemented if and only if it is uniquely complemented.

**Proof.** To prove the nontrivial part of the proposition, let \( \alpha, \beta \) and \( \gamma \) be vertices of \( \Gamma(\mathcal{R}L) \) such that \( \alpha \not\sim \beta \) and \( \alpha \not\sim \gamma \). So \( \text{Coz} \alpha \land \text{Coz} \beta = \perp, \) \( \text{Coz} \alpha' \land (\text{Coz} \beta') = \perp, \) \( \text{Coz} \alpha \land \text{Coz} \gamma = \perp \) and \( (\text{Coz} \alpha') \land (\text{Coz} \gamma') = \perp \). In consequence, \( \text{Coz} \beta \leq (\text{Coz} \alpha')' \leq (\text{Coz} \gamma)' \) and \( \text{Coz} \gamma \leq (\text{Coz} \alpha')' \leq (\text{Coz} \beta')'' \), showing that \( (\text{Coz} \beta)' = (\text{Coz} \gamma)' \). This means that \( \beta \) and \( \gamma \) are adjacent to exactly the same vertices, that is, \( \beta \sim \gamma \). Therefore \( \Gamma(\mathcal{R}L) \) is uniquely complemented. \( \square \)

5. Dominating sets

A dominating set of a graph \( G \) is a set of vertices \( V \) such that every vertex outside \( V \) is adjacent to at least one vertex in \( V \). The dominating number of \( G \) denoted by \( dtG \) is the smallest cardinal number of the form \( |V| \), where \( V \) is a dominating set. A complete subgraph of \( G \) is a subgraph in which every vertex is adjacent to every other vertex. The smallest cardinal number \( a \) such that every complete subgraph \( G \) has cardinality \( \leq a \), denoted by \( \omega G \), is called the clique number of \( G \).

The cellularity of the space \( X \) is denoted by \( c(X) \) and is the smallest cardinal number \( n \geq \aleph_0 \) such that every family of pairwise disjoint of nonempty open subsets of \( X \) has cardinality \( \leq n \). This motivates the following definition.

**Definition 5.1.** The cellularity of a frame \( L \) is denoted by \( c(L) \) and is the smallest cardinal number \( n \geq \aleph_0 \) such that every family of pairwise disjoint of nonzero elements of \( L \) has cardinality \( \leq n \).

Clearly, for a topological space \( X \), we have \( c(X) = c(\mathcal{C}X) \).

**Proposition 5.2.** For a frame \( L \), \( \omega \Gamma(\mathcal{R}L) = c(L) \). In particular, whenever \( L \) is a boolean frame, then \( \omega \Gamma(\mathcal{R}L) = |L| \).

**Proof.** We first show that \( \omega \Gamma(\mathcal{R}L) \leq c(L) \). Let \( A \subseteq \Gamma(\mathcal{R}L) \) be a complete subgraph. Then for every \( \alpha, \beta \in A \), \( \alpha \beta = 0 \) implies that \( \text{Coz} \alpha \land \text{Coz} \beta = \perp \). Thus \( S = \{ \text{Coz} \gamma \mid \gamma \in A \} \) is a family of pairwise disjoint nonzero elements of \( L \). This establishes that \( \omega \Gamma(\mathcal{R}L) \leq c(L) \). To show the other containment, suppose that \( S \) is a collection of pairwise disjoint nonzero elements of \( L \). For every \( s \in S \), pick \( \alpha_s \in \mathcal{R}L \) such that \( \text{Coz} \alpha_s \ll s \) and \( (\text{Coz} \alpha_s)' \neq \perp \). Now, for every \( s, t \in S \), we have \( \alpha_s \alpha_t = 0 \). So \( A = |\alpha_s \mid s \in S| \) is a complete subgraph of \( \Gamma(\mathcal{R}L) \). This shows that \( \omega \Gamma(\mathcal{R}L) \geq c(L) \) and therefore \( \omega \Gamma(\mathcal{R}L) = c(L) \). The second part is obvious. \( \square \)

The set of all cardinal numbers of the form \( |S| \), where \( S \) is a base for a frame \( L \), has a smallest element; this cardinal number is called the weight of the frame \( L \) and is denoted by \( \omega(L) \) (see [12]). Clearly, for a topological space \( X \), we have \( \omega(X) = \omega(\mathcal{C}X) \).

**Proposition 5.3.** For a frame \( L \), \( dt \Gamma(\mathcal{R}L) \leq \omega(L) \).

**Proof.** Suppose \( S \) is a base for \( L \). Then for every \( s \in S \), take \( \alpha_s \in \mathcal{R}L \) such that \( \text{Coz} \alpha_s \ll s \). We claim that \( A = \{ \alpha_s \mid s \in S \} \) is a dominating set of \( \Gamma(\mathcal{R}L) \). To see this, let \( \beta \in \Gamma(\mathcal{R}L) \), then there exists \( s_0 \in S \) such that \( s_0 \leq (\text{Coz} \beta)' \). This implies that \( \text{Coz} \alpha_{s_0} \leq (\text{Coz} \beta)' \) and hence \( \text{Coz} \alpha_{s_0} \land (\text{Coz} \beta)' = \perp \). Therefore \( \text{Coz} \alpha_{s_0} \land \text{Coz} \beta = \perp \), that is, \( \alpha_{s_0} \beta = 0 \) and consequently \( A \) is a dominating set. Now, \( dt \Gamma(\mathcal{R}L) \leq |A| \leq |S| \) for every base \( S \) of \( L \). But this means that \( dt \Gamma(\mathcal{R}L) \leq \omega(L) \). \( \square \)
In the following example, we show that there is a frame \( L \) for which the dominating number of \( \Gamma(\mathcal{RL}) \) is strictly less than the weight of \( L \).

**Example 5.4.** Consider \( \beta \mathbb{N} \), the Stone–Čech compactification of \( \mathbb{N} \). By Example 3.3 in [3], \( d\Gamma(C(\beta \mathbb{N})) = \mathbb{N}_0 \leq c = \omega(\beta \mathbb{N}) \). Putting \( L = \mathcal{C}(\beta \mathbb{N}) \), we then have \( d\Gamma(\mathcal{RL}) = d\Gamma(C(\beta \mathbb{N})) \leq \omega(\beta \mathbb{N}) = \omega(L) \), since \( C(\beta \mathbb{N}) \equiv \mathcal{R}C(\beta \mathbb{N}) \).

Next, we intend to give a frame-theoretic characterization and an algebraic characterization for the set of centers of \( \Gamma(\mathcal{RL}) \) to be a dominating set. Before we give these characterizations, we first give the two propositions below. Recall that a frame is *atomic* if below every nonzero element there is an atom. We denote the set of centers of \( \Gamma(\mathcal{RL}) \) by \( C(\Gamma(\mathcal{RL})) \).

**Proposition 5.5.** For a frame \( L \), \( \Gamma(\mathcal{RL}) \) is not triangulated and the set of centers of \( \Gamma(\mathcal{RL}) \) is a dominating set containing at least two elements if and only if \( L \) is atomic.

**Proof.** \((\Rightarrow)\) Since \( \Gamma(\mathcal{RL}) \) is not triangulated, then by Theorem 4.3, \( L \) has at least one atom and so Proposition 5.5 implies that \( C(\Gamma(\mathcal{RL})) = \{ \varphi \in \Gamma(\mathcal{RL}) \mid \text{coz } \varphi \text{ is an atom} \} \). Now, let \( a \in L \) be different from the top and the bottom, and let \( a \in \Gamma(\mathcal{RL}) \) be such that \( \text{coz } a \ll a \). If \( \text{coz } a \) is an atom, then there is nothing to prove. Otherwise, pick \( \beta \in \mathcal{RL} \) such that \( \text{coz } \beta \ll \text{coz } a \) and so \( (\text{coz } a)^{-} \ll (\text{coz } \beta)^{-} \). Now, we consider two cases.

Case 1. Suppose that \( (\text{coz } a)^{-} \) is an atom. Since, by our supposition, \( C(\Gamma(\mathcal{RL})) \) has at least two elements, let \( c \in L \) be an atom such that \( c \neq (\text{coz } a)^{-} \). Then \( c \wedge \text{coz } a = c = c \wedge \text{coz } a = \bot \). The latter is not possible, lest we have \( c \leq (\text{coz } a)^{-} \), implying \( (\text{coz } a)^{-} = c \). Therefore \( c \wedge \text{coz } a = c \), showing that \( c \ll \text{coz } a \leq a \).

Case 2. Suppose that \( (\text{coz } a)^{-} \) is not an atom. Take \( \gamma \in \mathcal{RL} \) such that \( \gamma \wedge \text{coz } a \preccurlyeq (\text{coz } a)^{-} \preccurlyeq \gamma \wedge (\text{coz } \beta)^{-} \). Clearly \( \gamma \in \Gamma(\mathcal{RL}) \setminus C(\Gamma(\mathcal{RL})) \), so we can choose \( \delta \in \Gamma(\mathcal{RL}) \) such that \( \text{coz } \delta \wedge \gamma = \bot \) because \( C(\Gamma(\mathcal{RL})) \) is a dominating set. This shows that \( \text{coz } \delta \wedge (\text{coz } a)^{-} = \bot \), that is, \( \text{coz } \delta \leq (\text{coz } a)^{-} \). Next, since \( \text{coz } \delta \) is an atom, we have \( \text{coz } a \wedge \text{coz } \delta = \text{coz } \delta \) or \( \text{coz } a \wedge \text{coz } \delta = \bot \). The latter is not possible, lest we have \( \text{coz } \delta \leq (\text{coz } a)^{-} \), implying \( \text{coz } \delta = \bot \) which is a contradiction. Therefore \( \text{coz } \delta \leq \text{coz } a \leq a \).

\((\Leftarrow)\) Suppose that \( L \) is atomic. Then by Theorem 4.3 \( \Gamma(\mathcal{RL}) \) is not triangulated. For the second part, first note that if \( L \) has exactly one atom, then the present hypothesis implies that \( L = 2 \), a contradiction since \( |L| \geq 4 \). Next, let \( a \in \Gamma(\mathcal{RL}) \setminus C(\Gamma(\mathcal{RL})) \). Then there exists an atom \( \tilde{a} \) such that \( \tilde{a} \leq (\text{coz } a)^{-} \). Putting \( a = \text{coz } \beta \), we then have \( \beta \in C(\Gamma(\mathcal{RL})) \) with \( \text{coz } \beta \leq (\text{coz } a)^{-} \). Thus \( \text{coz } \beta \wedge (\text{coz } a)^{-} = \bot \), implying that \( \text{coz } \beta \wedge \text{coz } a = \bot \), that is, \( \beta \) adjacent to \( \beta \).

Before proving the next proposition, we recall some definitions. A frame \( L \) is *basically disconnected* if \( c \vee c' = \top \) for all \( c \in \text{Coz} \, L \), and \( L \) is *zero dimensional* if it has a base of complemented elements (see [4] for details).

**Proposition 5.6.** Let \( \Gamma(\mathcal{RL}) \) not be triangulated. If the set of centers of \( \Gamma(\mathcal{RL}) \) is a dominating set, then it has at least two elements.

**Proof.** Since \( \Gamma(\mathcal{RL}) \) is not triangulated, then by Theorem 4.3, \( L \) has at least one atom. Then Proposition 5.5 implies that \( C(\Gamma(\mathcal{RL})) = \{ \varphi \in \Gamma(\mathcal{RL}) \mid \text{coz } \varphi \text{ is an atom} \} \). Now suppose, by way of contradiction, that \( |C(\Gamma(\mathcal{RL}))| = 1 \), that is, \( L \) has exactly one atom, say \( a \). Take \( a, \beta \in \mathcal{RL} \) such that \( a = \text{coz } a \) and \( a' = \text{coz } \beta \). Now, we continue the proof in three stages.

The first stage: We show that if \( \gamma \in \Gamma(\mathcal{RL}) \), then \( (\text{coz } \gamma)^{-} = \text{coz } a \) or \( (\text{coz } \gamma)^{-} = \text{coz } \beta \). We claim that for every \( \gamma \in \Gamma(\mathcal{RL}) \) with \( \gamma \neq \text{coz } a \), \( (\text{coz } \gamma)^{-} = \text{coz } a \). It suffices to prove that for every \( \gamma \in \Gamma(\mathcal{RL}) \) with \( \gamma \neq \text{coz } a \) and \( \gamma \neq \text{coz } \beta \), \( (\text{coz } \gamma)^{-} = \text{coz } a \). To see this, take \( \gamma \in \Gamma(\mathcal{RL}) \) such that \( \gamma \neq \text{coz } a \) and \( \gamma \neq \text{coz } \beta \). Then \( \text{coz } a \wedge (\text{coz } \gamma) = \bot \) since \( C(\Gamma(\mathcal{RL})) = \{ a \} \) is a dominating set. This shows that \( \text{coz } \gamma \vee \text{coz } a = \top \) because \( \text{coz } \gamma \neq \text{coz } \beta \). Pick \( \varphi \in \mathcal{RL} \) such that \( \text{coz } \varphi = \text{coz } \gamma \vee \text{coz } a \). If \( (\text{coz } \varphi)^{-} \neq \bot \), then \( \text{coz } \varphi \wedge \text{coz } a = \bot \), showing \( \text{coz } \varphi \leq \text{coz } \beta \), that is, \( \text{coz } a \leq \text{coz } \beta \) which is a contradiction. Consequently, \( (\text{coz } \varphi)^{-} = \bot \), this means that \( (\text{coz } \gamma)^{-} \wedge \text{coz } \beta = \bot \), implying that \( (\text{coz } \gamma)^{-} \leq \text{coz } a \). Therefore \( (\text{coz } \gamma)^{-} = \text{coz } a \) since \( \text{coz } a \leq (\text{coz } \gamma)^{-} \).

The second stage: We show that \( L \) is a zero dimensional frame. By [4, Proposition 8.4.4], it suffices to prove that \( L \) is a basically disconnected frame. To see this, let \( c \in \text{Coz} \, L \). Then, by the first stage, we have \( c' = \bot \) or \( c' = \text{coz } a \) or \( c' = \text{coz } \beta \). This shows that \( c' \vee c'' = \top \), that is, \( L \) is basically disconnected.

The third stage: We argue to arrive at a contradiction. By the second stage, \( L \) has a base of complemented elements, say \( S \). If \( S = \{ \text{coz } a, \text{coz } \beta \} \), then \( \text{coz } \beta \) is an atom which is a contradiction. Otherwise, there exists
δ ∈ Γ(RL) such that coz δ ∈ BL, coz δ ≠ coz α and coz δ ≠ coz β. Since C(Γ(RL)) is a dominating set, coz δ ∧ coz α = ⊥ and (coz δ)∗ ∧ coz α = ⊥, showing that coz δ ≤ coz β and (coz δ)∗ ≤ coz β. In consequence, coz β = ⊤ which is a contradiction.

We conclude the article with the following theorem. Before the theorem is presented, let us recall that the socle of a ring R is the ideal generated by minimal ideals of R. In [10, 11], the socle of RL is characterized as the ideal consisting of functions each of which has cozero equal to a join of finitely many atoms. The equivalence of parts (2) and (3) of the following theorem is shown in [10]. Now combining Propositions 5.5 and 5.6, we obtain the following result.

**Theorem 5.7.** The following statements are equivalent for a frame L.

1. Γ(RL) is not triangulated and the set of centers of Γ(RL) is a dominating set.
2. L is atomic.
3. The socle of RL is an essential ideal.

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