Abstract  We study conditions of Hörmander’s $L^2$-estimate and the Ohsawa-Takegoshi extension theorem. Introducing a twisted version of the Hörmander-type condition, we show a converse of Hörmander’s $L^2$-estimate under some regularity assumptions on an $n$-dimensional domain. This result is a partial generalization of the one-dimensional result obtained by Berndtsson (1998). We also define new positivity notions for vector bundles with singular Hermitian metrics by using these conditions. We investigate these positivity notions and compare them with classical positivity notions.

Keywords  $L^2$-estimate, singular Hermitian metrics, Ohsawa-Takegoshi $L^2$-extension theorems

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1 Introduction

There are many curvature-positivity notions for Hermitian holomorphic vector bundles on complex manifolds. The situation is simple for line bundles: a Hermitian metric $h = e^{-\phi}$ on a line bundle has positive curvature if and only if the corresponding local weight $\phi$ is plurisubharmonic. When we consider a singular Hermitian metric, its curvature defined as a current is positive.

For general vector bundles, the situation is much complicated. There are some positivity notions which are not equivalent to each other. When we consider a singular metric on vector bundles, we cannot even define its curvature (see [14, Theorem 1.5]). Therefore, in order to generalize curvature-positivity concepts of vector bundles for singular metrics, we have to seek their characterizations without using curvature tensors.

For the Griffiths semi-positivity, such a characterization has been obtained (see [8, 9, 11, 13, 14]). On the other hand, we do not know such a characterization for the Nakano semi-positivity, which is stronger than the Griffiths positivity and a natural setting for using Hörmander’s $L^2$-methods. This is one of the difficulties in the study of singular Hermitian metrics on vector bundles.

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1) Very recently, a characterization of the Nakano semi-positivity was obtained by Deng et al. [7]. Their result states that Hörmander-type $L^2$-estimates with an optimal coefficient is equivalent to the Nakano semi-positivity. We hope that we can study properties of “Nakano semi-positive singular Hermitian metrics” by this characterization.
In [8], a new characterization of a plurisubharmonicity was obtained. Namely, the possibility of the $L^2$-extension with a certain condition on its estimate is equivalent to the plurisubharmonicity. Here, we state the precise condition in the trivial line bundle case (for more general settings, see Definition 3.2).

**Definition 1.1** (See [8]). Let $\Omega \subset \mathbb{C}^n$ be a domain and $\phi$ be an upper semi-continuous function. We say that $(\Omega, \phi)$ satisfies the multiple $L^2$-extension property if there exists a number $C_m > 0$ for each $m$ such that

- for every $p \in \Omega$ with $\phi(p) \neq -\infty$, there exists a holomorphic function $f$ satisfying $f(p) = 1$ and
  \[
  \int_{\Omega} |f|^2 e^{-m \phi} \leq C_m e^{-m \phi(p)},
  \]

and

- (growth condition) $C_m$ satisfies the condition $\lim_{m \to +\infty} \frac{\log C_m}{m} = 0$.

For vector bundles, it is proved in [8] that the multiple $L^2$-extension property implies the Griffiths positivity.

In [1], it is proved that for a continuous function on a one-dimensional domain, the availability of the Hörmander estimate implies the subharmonicity. This kind of study has been applied to various fields (see [10]). In [8], Deng et al. asked a problem if one can extend this result to higher-dimensional cases. In Section 2, we show a partial converse of Hörmander’s $L^2$-estimate for line bundles on an $n$-dimensional domain. We define the twisted Hörmander condition as follows.

**Definition 1.2.** Let $\Omega \subset \mathbb{C}^n$ be a domain and $\phi$ be an upper semi-continuous function. We say that $(\Omega, \phi)$ satisfies the twisted Hörmander condition if, for every positive integer $m$, the smooth strictly plurisubharmonic function $\psi$ on $\Omega$, and the smooth $\bar{\partial}$-closed $(0,1)$-form $\alpha$ with compact support and finite norm

\[
\int_{\Omega} |\alpha|^2 e^{-m \phi + \psi} \leq \int_{\Omega} |\alpha|^2 e^{-\phi + \psi},
\]

there exists a smooth function $u$ such that

- $\bar{\partial} u = \alpha$, and

- $\int_{\Omega} |u|^2 e^{-m \phi + \psi} \leq \int_{\Omega} |\alpha|^2 e^{-\phi + \psi}$.

Using this, we state a converse of Hörmander’s $L^2$-estimate as follows.

**Theorem 1.3.** Let $\Omega \subset \mathbb{C}^n$ be a domain. Assume that $\text{Pole}(\phi)$ is closed and $\phi$ is a locally Hölder continuous function on $\Omega \setminus \text{Pole}(\phi)$. If the twisted Hörmander condition is satisfied for $(\Omega, \phi)$, $\phi$ is plurisubharmonic.

Here, we consider a twisted version of Hörmander conditions. Our proof is an application of the theorem in [8]. Twisting with an additional weight $\psi$ enables us to prove the multiple $L^2$-extension property and thus the plurisubharmonicity of $\phi$.

We shall also prove a partial converse of Hörmander’s $L^2$-estimate for vector bundles (see Theorem 3.5). We also introduce several positivity notions for vector bundles. To be precise, we study a singular Hermitian metric which is positive in the sense of twisted Hörmander (see Definition 3.7) and a singular Hermitian metric which is positive in the sense of multiple $L^2$-extension (see Definition 3.8).

It is important to consider the sheaf of square integrable holomorphic sections of vector bundles with respect to positively curved singular Hermitian metrics. For line bundles, such a sheaf is called the multiplier ideal sheaf. Multiplier ideal sheaves are proved to be coherent by $L^2$-estimates. For general vector bundles, little is known about the coherence of such sheaves, because of the lack of $L^2$-estimates for general singular Hermitian metrics. Hosono [12, Theorem 1.1] has proved the coherence of such sheaves for singular Hermitian metrics induced by holomorphic sections. We prove that the sheaf of locally square integrable holomorphic sections with respect to a metric which is positive in the sense of twisted Hörmander is coherent.

**Theorem 1.4.** Let $(E, h)$ be positively curved in the sense of twisted Hörmander. Assume that $|u|_h$, is upper semi-continuous for any local holomorphic section $u \in \mathcal{O}(E^*)$. Then $\mathcal{E}(h)$ is a coherent subsheaf.
of $\mathcal{O}(E)$, where $\mathcal{O}(h)$ is the sheaf of germs of locally square integrable holomorphic sections of $E$ with respect to $h$.

We also study metrics which is positive in the sense of multiple $L^2$-extension. We show that the positivity in this sense is strictly weaker than the Nakano semi-positivity.

**Theorem 1.5.** There exists a positively curved vector bundle $(E, h)$ in the sense of multiple $L^2$-extension such that $(E, h)$ is not Nakano semi-positive.

This is a partial answer to the question proposed by Deng et al. [8] (see Question 3.10 below). We also propose some questions about the above new positivity notions.

### 2 A converse of Hörmander’s $L^2$-estimate for line bundles

In this section, we formulate a Hörmander-type condition and prove the equivalence to plurisubharmonicity under some regularity assumption.

**Definition 2.1.** Let $\Omega \subset \mathbb{C}^n$ be a domain and $\phi : \Omega \to [-\infty, +\infty)$ be an upper semi-continuous function. We say that $(\Omega, \phi)$ satisfies the twisted Hörmander condition if, for every positive integer $m$, the smooth strictly plurisubharmonic function $\psi$ on $\Omega$, and the smooth $\bar{\partial}$-closed $(0, 1)$-form $\alpha$ with compact support and finite norm $\int_{\Omega} |\alpha|^2 \sqrt{-1} \bar{\partial} \bar{\partial} \psi e^{-(m\phi + \psi)} < +\infty$, there exists a smooth function $u$ such that

- $\bar{\partial} u = \alpha$, and
- $\int_{\Omega} |u|^2 e^{-(m\phi + \psi)} \leq \int_{\Omega} |\alpha|^2 \sqrt{-1} \bar{\partial} \bar{\partial} \psi e^{-(m\phi + \psi)}$.

**Remark 2.2.**

(1) To formulate the condition, we used an additional weight function $\psi$. This enables us to prove the multiple $L^2$-extension property (under some regularity assumption) and therefore the plurisubharmonicity of $\phi$.

(2) We clearly see that on a domain in $\mathbb{C}^n$, an upper semi-continuous function $\phi$ is plurisubharmonic if and only if for every smooth strictly plurisubharmonic function $\psi$, $\phi + \psi$ is plurisubharmonic. Hence, it is worth considering a twisted version of Hörmander’s $L^2$-estimate.

(3) If $\phi$ is a plurisubharmonic and $\Omega$ is pseudoconvex, the twisted Hörmander condition is automatically satisfied. Indeed, the standard estimate gives the following:

\[
\int_{\Omega} |u|^2 e^{-(m\phi + \psi)} \leq \int_{\Omega} |\alpha|^2 \sqrt{-1} \bar{\partial} \bar{\partial} \psi e^{-(m\phi + \psi)}.
\]

Since $\phi$ is plurisubharmonic, we have $\sqrt{-1} \bar{\partial} \bar{\partial} (m\phi + \psi) \geq \sqrt{-1} \bar{\partial} \bar{\partial} \psi$ (in the sense of currents). Then we have

\[
|\cdot|^2 \sqrt{-1} \bar{\partial} \bar{\partial} (m\phi + \psi) \leq |\cdot|^2 \sqrt{-1} \bar{\partial} \bar{\partial} \psi
\]

and thus

\[
\int_{\Omega} |u|^2 e^{-(m\phi + \psi)} \leq \int_{\Omega} |\alpha|^2 \sqrt{-1} \bar{\partial} \bar{\partial} (m\phi + \psi) e^{-(m\phi + \psi)} \leq \int_{\Omega} |\alpha|^2 \sqrt{-1} \bar{\partial} \bar{\partial} \psi e^{-(m\phi + \psi)}.
\]

(4) One may formulate, as in Definition 1.1, the twisted Hörmander condition with constants $C_m$ and the same growth condition $\lim_{m \to \infty} \frac{\log C_m}{m} = 0$.

We will show that under some continuity assumption, the Hörmander condition implies the multiple $L^2$-extension property. We set $\text{Pole}(\phi) := \{\phi^{-1}(-\infty)\}$, the set of poles of $\phi$.

**Theorem 2.3 (= Theorem 1.3).** Let $\Omega \subset \mathbb{C}^n$ be a domain. Assume that $\text{Pole}(\phi)$ is closed and $\phi$ is a locally Hölder continuous function on $\Omega \setminus \text{Pole}(\phi)$, i.e., for every $\Omega' \subset \Omega \setminus \text{Pole}(\phi)$, there exist constants $\alpha = \alpha_{\Omega'} \in [0, 1]$ and $C = C_{\Omega'} > 0$ such that $|\phi(z) - \phi(w)| \leq C|z - w|^\alpha$ for every $z, w \in \Omega'$. If the twisted Hörmander condition (see Definition 2.1) is satisfied for $(\Omega, \phi)$, $\phi$ is plurisubharmonic.

**Proof.** Fix a domain $\Omega' \subset \Omega \setminus \text{Pole}(\phi)$. We will show that $(\Omega', \phi)$ satisfies the multiple $L^2$-extension property (see Definition 1.1).
Fix a point \( w \in \Omega' \) with \( \phi(w) > -\infty \) and an integer \( m > 0 \). We will construct a holomorphic function \( f \in \Omega' \) such that \( f(w) = 1 \) and
\[
\int_{\Omega'} |f|^2 e^{-m\phi} \leq C_m e^{-m\phi(w)},
\]
where \( C_m \) is a constant independent of the choice of \( w \in \Omega' \).

Taking \( \chi = \chi(t) \) to be a smooth function on \( \mathbb{R} \) such that
- \( \chi(t) = 1 \) for \( t \leq 1/2 \),
- \( \chi(t) = 0 \) for \( t \geq 1 \), and
- \( |\chi'(t)| \leq 3 \) on \( \mathbb{R} \).

Define a \((0,1)\)-form \( \alpha \) by
\[
\alpha := \partial \chi \left( \frac{|z-w|^2}{\epsilon^2} \right) = \chi' \left( \frac{|z-w|^2}{\epsilon^2} \right) \sum_j \frac{z_j - w_j}{\epsilon^2} d\bar{z}_j.
\]
We apply the twisted Hörmander condition for the weight \( \psi_{\epsilon, \delta} := \log(|z-w|^2 + \epsilon^2) + n \log(|z-w|^2 + \delta^2) \), where \( \epsilon \) and \( \delta \) are positive parameters.

Then we obtain a smooth function \( u_{\epsilon, \delta} \) on \( \Omega' \) such that
- \( \partial u_{\epsilon, \delta} = \alpha \) and
- \[
\int_{\Omega'} |u_{\epsilon, \delta}|^2 e^{-(m\phi + \psi_{\epsilon, \delta})} \leq \int_{\Omega'} |\alpha|^2 e^{-m\phi} e^{-(m\phi + \psi_{\epsilon, \delta})}.
\]
Since
\[
|\alpha|^2 = \left| \chi' \left( \frac{|z-w|^2}{\epsilon^2} \right) \right|^2 \cdot \frac{1}{\epsilon^4} \left| \sum_j (z_j - w_j) d\bar{z}_j \right|^2 e^{-(m\phi + \psi_{\epsilon, \delta})}
\]
and the support of \( \chi' \left( \frac{|z-w|^2}{\epsilon^2} \right) \) is in \( \{1/2 \leq |z-w|^2/\epsilon^2 \leq 1\} \), we have

\begin{align*}
\text{(Right-hand side of (2.1))} \\
= \int_{\{1/2 \leq |z-w|^2/\epsilon^2 \leq 1\}} \left| \chi' \left( \frac{|z-w|^2}{\epsilon^2} \right) \right|^2 \cdot \frac{1}{\epsilon^4} \left| \sum_j (z_j - w_j) d\bar{z}_j \right|^2 e^{-(m\phi + \psi_{\epsilon, \delta})} \\
\leq \frac{9}{\epsilon^4} \int_{\{1/2 \leq |z-w|^2/\epsilon^2 \leq 1\}} \left| \sum_j (z_j - w_j) d\bar{z}_j \right|^2 e^{-(m\phi + \psi_{\epsilon, \delta})}.
\end{align*}

Letting \( \omega := i\partial\bar{\partial}|z|^2 \), we have
\[
\sqrt{-1} \partial\bar{\partial} \psi_{\epsilon, \delta} \geq \sqrt{-1} \partial\bar{\partial} \log(|z-w|^2 + \epsilon^2) \geq \frac{\epsilon^2}{(|z-w|^2 + \epsilon^2)^{2n}} \omega,
\]
and thus
\[
| \cdot |^2 e^{-m\phi} \leq \frac{1}{(|z-w|^2 + \epsilon^2)^{2n}} \omega.
\]
Combining (2.2) and (2.3), we have the following:
\[
\begin{align*}
\text{(2.2)} \leq \frac{9}{\epsilon^4} \int_{\{1/2 \leq |z-w|^2/\epsilon^2 \leq 1\}} |z-w|^2 \left( \frac{|z-w|^2 + \epsilon^2}{\epsilon^2} \right)^2 e^{-(m\phi + \psi_{\epsilon, \delta})} \\
= \frac{9}{\epsilon^4} \int_{\{1/2 \leq |z-w|^2/\epsilon^2 \leq 1\}} \left( \frac{|z-w|^2}{\epsilon^2} \right)^2 \left( \frac{1}{|z-w|^2 + \epsilon^2} \right)^{2n} \frac{1}{(|z-w|^2 + \delta^2)^n} e^{-m\phi} \\
= \frac{9}{\epsilon^4} \int_{\{1/2 \leq |z-w|^2/\epsilon^2 \leq 1\}} |z-w|^2 \left( \frac{|z-w|^2 + \epsilon^2}{\epsilon^2} \right)^2 \left( \frac{1}{|z-w|^2 + \delta^2} \right)^n e^{-m\phi}
\end{align*}
\]
Thus for every sequence, finally we can obtain a sequence where

\[
C_n = \frac{9}{\epsilon^2} e^{2n - 2 \frac{2n}{\epsilon^2}} e^{-m \inf_{B(w, \epsilon)} \phi}
\]

where \(C_n\) is a positive constant depending only on \(n\).

To summarize, we have obtained a smooth function \(u_{\epsilon, \delta}\) on \(\Omega'\) such that

- \(\partial u_{\epsilon, \delta} = \alpha\), and
- the following estimate holds:

\[
\int_{\Omega'} |u_{\epsilon, \delta}|^2 e^{-(m \phi + \psi_{\epsilon, \delta})} \leq 9C_n e^{2n - 2 \frac{2n}{\epsilon^2}} e^{-m \inf_{B(w, \epsilon)} \phi}. \tag{2.4}
\]

Let \(\delta \to 0\). The right-hand side of (2.4) is increasing to

\[
9C_n e^{2n - 2 \frac{2n}{\epsilon^2}} e^{-m \inf_{B(w, \epsilon)} \phi} = 9C_n \frac{2n}{\epsilon^2} e^{-m \inf_{B(w, \epsilon)} \phi}.
\]

Let us show the convergence of \(u_{\epsilon, \delta}\). We have

\[
\int_{\Omega'} |u_{\epsilon, \delta}|^2 e^{-(m \phi + \psi_{\epsilon, \delta})} \leq 9C_n \frac{2n}{\epsilon^2} e^{-m \inf_{B(w, \epsilon)} \phi}.
\]

Note that the weight function \(\psi_{\epsilon, \delta}\) is decreasing when \(\delta \to 0\). Therefore, \(e^{-\psi_{\epsilon, \delta}}\) is increasing when \(\delta \to 0\).

Fix \(\delta_0 > 0\). Then for \(\delta < \delta_0\), (since \(e^{-\psi_{\epsilon, \delta}} > e^{-\psi_{\epsilon, \delta_0}}\) we have

\[
\int_{\Omega'} |u_{\epsilon, \delta}|^2 e^{-(m \phi + \psi_{\epsilon, \delta})} \leq \int_{\Omega'} |u_{\epsilon, \delta}|^2 e^{-(m \phi + \psi_{\epsilon, \delta_0})} \leq 9C_n \frac{2n}{\epsilon^2} e^{-m \inf_{B(w, \epsilon)} \phi}.
\]

Thus \(\{u_{\epsilon, \delta}\}_{\delta < \delta_0}\) forms a bounded sequence in \(L^2(\Omega', e^{-(m \phi + \psi_{\epsilon, \delta_0})})\). We can choose a weakly convergent sequence \(\{u_{\epsilon, \delta_0}\}_k\) in \(L^2(\Omega', e^{-(m \phi + \psi_{\epsilon, \delta_0})})\). Since the \(L^2\)-norm is lower semi-continuous under weak limits, the (weak) limit function \(u_{\epsilon, \delta} \to 0\) weakly convergent in \(L^2(\Omega', e^{-(m \phi + \psi_{\epsilon, \delta_0})})\) for every \(j\).

Then we have

\[
\int_{\Omega'} |u_{\epsilon, \delta}|^2 e^{-(m \phi + \psi_{\epsilon, \delta})} \leq 9C_n \frac{2n}{\epsilon^2} e^{-m \inf_{B(w, \epsilon)} \phi}
\]

for every \(j\), and by the monotone convergence theorem,

\[
\int_{\Omega'} |u_{\epsilon, \delta}|^2 e^{-(m \phi + \psi_{\epsilon, \delta})} \leq 9C_n \frac{2n}{\epsilon^2} e^{-m \inf_{B(w, \epsilon)} \phi},
\]

where \(\psi_{\epsilon, \delta} := \psi_{\epsilon, \delta_0}\).

Since differential operators are continuous under weak limits, we have \(\partial u_{\epsilon, \delta} = \alpha\).

The integral

\[
\int_{\Omega'} |u_{\epsilon, \delta}|^2 e^{-(m \phi + \psi_{\epsilon, \delta})} = \int_{\Omega'} e^{m \phi} \frac{1}{\epsilon^2} \frac{1}{|z - w|^2} \frac{1}{2n} e^{-m \phi}
\]

is finite while the weight \(\frac{1}{|z - w|^2}\) is not integrable near \(w\), and thus \(u_{\epsilon, \delta}(w)\) must be 0.

Let \(f_{\epsilon} := \chi(|z - w|^2/\epsilon^2) - u_{\epsilon, \delta}\). Then \(f_{\epsilon}(0) = 1\) and

\[
\left( \int_{\Omega'} |f_{\epsilon}|^2 e^{-m \phi} \right)^{1/2} \leq \left( \int_{\Omega'} \chi \left( \frac{|z - w|^2}{\epsilon^2} \right) \frac{1}{2n} e^{-m \phi} \right)^{1/2} + \left( \int_{\Omega'} |u_{\epsilon, \delta}|^2 e^{-m \phi} \right)^{1/2}. \tag{2.5}
\]
We will estimate each term of the right-hand side of (2.5). Since $\chi \leq 1$ and the support of $\chi(|z-w|^2/\epsilon^2)$ is contained in $\{|z-w|^2 \leq \epsilon^2\}$, the first term can be bounded by 

$$C_n \epsilon^{2n} e^{-m \inf_B (w, \epsilon) \phi} \frac{1}{\epsilon^2}.$$ 

Next, we consider the second term. We have 

$$\int_{\Omega'} |u|^2 e^{-m\phi} \leq \left[ \sup_{z \in \Omega'} |z-w|^{2n} (|z-w|^2 + \epsilon) \right] \int_{\Omega'} |u|^2 e^{-m\phi} \frac{1}{(|z-w|^2 + \epsilon^2)} \frac{1}{|z-w|^{2n}}$$

$$\leq \left[ \sup_{z \in \Omega'} |z-w|^{2n} (|z-w|^2 + \epsilon) \right] \cdot 9C_n \epsilon^{2n} e^{-m \inf_B (w, \epsilon) \phi}.$$ 

Assuming $\epsilon < 1$, we can bound the sup by $(R+1)^{2n+2}$, where $R$ is the radius of $\Omega'$. Therefore, the second term is bounded by $C'_n \epsilon^{2n} e^{-m \inf_B (w, \epsilon) \phi}$.

Therefore, we have 

$$\int_{\Omega'} |f|^2 e^{-m\phi} \leq C' \left( \epsilon^n + \frac{1}{\epsilon} \right)^2 e^{-m \inf_B (w, \epsilon) \phi},$$

where $C'$ is a constant independent of $m$.

By the assumption that $\phi$ is locally Hölder continuous on $\Omega \setminus \text{Pole}(\phi)$, we have $|\phi(z) - \phi(w)| \leq C_\Omega |z-w|^\alpha$ for every $z \in \Omega'$. Let $\epsilon := 1/\{(mC_\Omega)^{1/\alpha}\}$. Then we have $|m\phi(z) - m\phi(w)| \leq 1$ for $|z-w| \leq \epsilon$. Thus,

$$\int_{\Omega'} |f|^2 e^{-m\phi} \leq C' \left( \frac{1}{m^\alpha C_\Omega} + m^\frac{\alpha}{2} C_\Omega^{\frac{\alpha}{2}} \right)^2 e^{-m\phi(w)+1}$$

$$= C' \epsilon \left( \frac{1}{m^\alpha C_\Omega} + m^\frac{\alpha}{2} C_\Omega^{\frac{\alpha}{2}} \right)^2 e^{-m\phi(w)}. \quad (2.6)$$

The coefficient of (2.6) satisfies the growth condition $\frac{\log C_\omega}{m} \to 0$, and thus we have verified the multiple $L^2$-extension property for $(\Omega', \phi)$.

Then, by [8], it follows that $\phi$ is plurisubharmonic on $\Omega \setminus \text{Pole}(\phi)$. Since $\phi$ takes the value $-\infty$ on $\text{Pole}(\phi)$, $\phi$ is also plurisubharmonic on $\Omega$.

**Remark 2.4.** In the last part of the proof, we used the assumption that $\phi$ is locally Hölder continuous. After the first version of the manuscript was completed, a proof only assuming continuity of $\phi$ was obtained in [6]. They just used the uniform convergence $\inf_B (w, \epsilon) \phi \to \phi$ on compact subsets. This was also pointed out by a referee.

### 3 A converse of Hörmander’s $L^2$-estimate for vector bundles and relations to various positivity notions

#### 3.1 A converse of Hörmander’s $L^2$-estimate for vector bundles

In this subsection, we prove a version of Theorem 2.3 for vector bundles. First of all, we introduce the definition of singular Hermitian metrics on vector bundles. Throughout this section, $E \to \Omega$ denotes a holomorphic vector bundle over a domain $\Omega \subset \mathbb{C}^n$, $\omega$ denotes the standard Kähler metric on $\Omega$, $dV_\omega$ is the volume form determined by $\omega$, $h$ denotes a singular Hermitian metric on $E$, and $h^*$ denotes the dual metric on the dual vector bundle $E^*$. 

**Definition 3.1** (See [11, Definition 17.1] and [14, Definition 1]). A singular Hermitian metric $h$ on $E$ is a measurable map from $\Omega$ to the space of non-negative Hermitian forms on the fibers, i.e., $h$ satisfies the following conditions:

1. $h$ is finite and positive definite almost everywhere on each fiber.
2. The function $|s|_h : U \to [0, +\infty]$ is measurable whenever $U \subset \Omega$ is an open set and $s \in H^0(U, E)$.

The Ohsawa-Takegoshi type condition, which is called the multiple $L^p$-extension property for vector bundles is introduced in [8]. The precise definition is as follows.
Definition 3.2 (Multiple $L^p$-extension property [8]). Let $p > 0$ be a fixed constant. Assume that for any $z \in \Omega$, any non-zero element $a \in E_z$ with finite norm $|a|_h < +\infty$, and any $m \geq 1$, there is a holomorphic section $f_m \in H^p(\Omega, E_{\otimes m})$ such that $f_m(z) = a^{\otimes m}$ and $f_m$ satisfies the following condition:

$$
\int_{\Omega} |f_m|_{h^{\otimes m}}^p \omega \leq C_m |a^{\otimes m}|_{h^{\otimes m}} = C_m |a|^m_p,
$$

where $C_m$ are constants independent of $z \in \Omega$ and satisfy the growth condition $\lim_{m \to \infty} \frac{1}{m} \log C_m = 0$. Then $(E, h)$ is said to have the multiple $L^p$-extension property.

In this paper, we only consider the multiple $L^2$-extension property. We also define the Hörmander-type condition for vector bundles.

Definition 3.3. We say that $(E, h)$ satisfies the twisted Hörmander condition on $\Omega$ if, for every positive integer $m$, the smooth strictly plurisubharmonic function $\psi$ on $\Omega$, and the smooth $\partial$-closed $E_{\otimes m}$-valued $(n, 1)$-form $\alpha = \sum_j \alpha_j dz \wedge d\bar{\sigma}_j$ ($\alpha_j$ is a smooth section of $E_{\otimes m}$) with compact support and finite norm

$$
\int_{\Omega} \frac{1}{2} \sum_{1 \leq i, j \leq n} (\psi^{(ij)}_{\alpha_i, \alpha_j})_{h^{\otimes m}} e^{-\psi} \omega \leq |\alpha|_{h^{\otimes m}},
$$

there exists a smooth $E_{\otimes m}$-valued $(n, 0)$ form $u$ such that

- $\partial u = \alpha$, and
- $\int_{\Omega} |u|_{h^{\otimes m}, \omega}^2 \omega \leq \int_{\Omega} \frac{1}{2} \sum_{1 \leq i, j \leq n} (\psi^{(ij)}_{\alpha_i, \alpha_j})_{h^{\otimes m}} e^{-\psi} \omega$,

where $dz := dz_1 \wedge \cdots \wedge dz_n$ and $(\psi^{(ij)}_{\alpha_i, \alpha_j})_{1 \leq i, j \leq n}$ is the inverse matrix of $(\frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j})_{1 \leq i, j \leq n}$.

Remark 3.4. (1) $(\psi^{(ij)}_{\alpha_i, \alpha_j})_{1 \leq i, j \leq n}$ in the above definition corresponds to the inverse operator of $[\sqrt{-1}\partial \bar{\partial} \psi \otimes Id_{E_{\otimes m}}, \Lambda_{\omega}]$. Here, $[\cdot, \cdot]$ denotes a graded Lie bracket, and $\Lambda_{\omega}$ denotes an adjoint of the operator $L := \omega \wedge \partial$.

(2) When $h$ is smooth and Nakano positive and $\Omega$ is pseudoconvex, we can show that $h$ satisfies the twisted Hörmander condition.

We will show that under some regularity condition, the twisted Hörmander condition implies the multiple $L^2$-extension property for vector bundles.

Theorem 3.5. We fix a bounded domain $\Omega' \subset \Omega$. Assume that $|u|_h$ is upper semi-continuous for any local holomorphic section $u \in \mathcal{O}(E')$. Moreover, we assume that $\log |u|_h$ is locally Hölder continuous on $\Omega$ for any non-zero local holomorphic section $u$, i.e., for every $\Omega' \subset \Omega$, $|\log |u(z)|_{h(z)} - \log |u(w)|_{h(w)}| \leq C_{\Omega'}|z - w|^\alpha$ for positive constants $\alpha \in (0, 1]$ and $C_{\Omega'} > 0$ independent of $u \in H^0(\Omega', E)$. If $(E, h)$ satisfies the twisted Hörmander condition (see Definition 3.3) on $\Omega$, $(E, h)$ has the multiple $L^2$-extension property on $\Omega'$. Then $(E, h)$ is also Griffiths semi-positive.

Proof. For any $w \in \Omega'$, we take any non-zero element $a \in E_w$. Taking the smooth function $\chi$ in the proof of Theorem 2.3, we define a $\partial$-closed $E_{\otimes m}$-valued $(n, 1)$-form $\alpha$ by

$$
\alpha := \partial \left( a^{\otimes m} \frac{1}{\epsilon} \left( \frac{|z - w|^2}{\epsilon^2} \right) dz \right) = a^{\otimes m} \frac{1}{\epsilon} \sum_j \left( \frac{z_j - w_j}{\epsilon} \right) d\sigma_j.
$$

We also take the weight function

$$
\psi_{\epsilon, \delta} := \log(|z - w|^2 + \epsilon^2) + n \log(|z - w|^2 + \delta^2)
$$

for $\epsilon, \delta > 0$. Since

$$
\sum_{1 \leq i, j \leq n} \left( \psi^{(ij)}_{\epsilon, \delta} \frac{(z_i - w_i)\chi'}{\epsilon^2} a^{\otimes m}, \frac{(z_j - w_j)\chi'}{\epsilon^2} a^{\otimes m} \right)_{h^{\otimes m}} \leq \frac{1}{\epsilon^4} |a^{\otimes m}_{h^{\otimes m}}| \sum_{1 \leq i, j \leq n} \psi^{(ij)}_{\epsilon, \delta} (z_i - w_i)(\sigma_j - \bar{\sigma}_j)
$$

$$
\leq \frac{1}{\epsilon^4} |a^{\otimes m}_{h^{\otimes m}}| \frac{|z - w|^2(|z - w|^2 + \epsilon^2)^2}{\epsilon^2},
$$
we have
\[
\int_\Omega \sum_{1 \leq i, j \leq n} \left( \frac{\psi_{i, \delta}^n (z_i - w_i) \chi_{j} \zeta_{i,m} (z_j - w_j) \chi_{j} \zeta_{j,m}}{\epsilon^2} \right) e^{-\psi_{i, \delta} \epsilon} dV_w \\
\leq \frac{1}{\epsilon^2} \int_\Omega |x|^{2} |u_{i,m}|^{2} |w| |z - w|^2 \left\{ |z - w|^2 + \epsilon^2 \right\} e^{-\psi_{i, \delta} \epsilon} dV_w \\
\leq \frac{9}{\epsilon^2} \int_{(\epsilon^2/2)|z - w|^2 \leq \epsilon^2} \frac{|z - w|^2 \left\{ |z - w|^2 + \epsilon^2 \right\}}{(|z - w|^2 + \epsilon^2)^n} |a|^{2m} dV_w \\
\leq 9C_n \epsilon^{2n-2} \frac{1}{(\epsilon^2/2 + \epsilon^2)^n} \sup_{|z - w| \leq \epsilon} |a|^{2m}_{h(z)}
\]
for some positive constant $C_n$ depending only on $n$. Repeating the same argument of the proof of Theorem 2.3, we find a smooth $E_{\phi}^{\omega}$-valued $(n, 0)$ form $u_{i}$ such that

- $\partial u_{i} = \alpha$, and
- $\int_\Omega |u_{i}|^{2} dV_{\omega} \leq \frac{9C_n \epsilon^{2n}}{\epsilon} \sup_{|z - w| \leq \epsilon} |a|^{2m}_{h(z)}$.

Since $\log |a|_{h(z)}$ is Hölder continuous,

$$
\sup_{|z - w| \leq \epsilon} |a|^{2m}_{h(z)} \leq \epsilon^{2n \alpha} |a|^{2m}_{h(w)}
$$

for constants $\alpha \in (0, 1]$ and $C_{\alpha} > 0$.

Let $f_{\epsilon} := a^{m} \chi dz - u_{i}$. Then $f_{\epsilon}$ is a holomorphic $(n, 0)$-form, $f_{\epsilon}(w) = a^{m}$, and satisfies the following inequality:

$$
\int_\Omega |f_{\epsilon}|^{2} dV_{\omega} \leq C \left( \epsilon^{2n} + \frac{1}{\epsilon^2} \right) \epsilon^{2mC \alpha} |a|^{2m}_{h(w)},
$$

where $C > 0$ is a constant depending on $\Omega'$. Taking $\epsilon = 1/(mC_{\alpha})^{1/\alpha}$, we have

$$
\int_\Omega |f_{\epsilon}|^{2} dV_{\omega} \leq 2C \left( \frac{1}{(mC_{\alpha})^{\frac{2n}{\alpha}}} + (mC_{\alpha})^{\frac{2}{\alpha}} \right) |a|^{2m}_{h(w)}.
$$

Since the above coefficient satisfies the growth condition $\log C_n / m \to 0$, we can conclude that $(E, h)$ has the multiple $L^{2}$-extension property on $\Omega'$. We also see that $(E, h)$ is Griffiths semi-positive from the results of [8, Theorem 6.4].

**Remark 3.6.** Continuity of $\log |u|_{h}$ for any non-zero local holomorphic section $u$ implies that $h$ is positive definite and finite on $\Omega$. In Theorem 2.3, $\phi$ has a plurisubharmonic extension from $\Omega \setminus \text{Pole}(\phi)$ to $\Omega$. However, “the singular set” of singular Hermitian metrics on vector bundles is much more complicated. In the case where $h$ has general singularity, we are not sure that Theorem 3.5 holds.

### 3.2 New positivity notions for vector bundles

In this subsection, we introduce various positivity notions for vector bundles and compare them. Throughout this section, we let $(X, \omega)$ be a smooth Hermitian manifold, $E \to X$ be a holomorphic vector bundle of rank $r$ over $X$, and $h$ be a singular Hermitian metric on $E$.

Firstly, we define a positively curved singular Hermitian metric in the sense of twisted Hörmander.

**Definition 3.7.** We say that $(E, h)$ is positively curved in the sense of twisted Hörmander if for any point $x \in X$, there exists an open neighborhood $U$ of $x$ such that $(E, h)$ satisfies the twisted Hörmander condition (see Definition 3.3) on $U$.

We also define a positively curved metric in the sense of multiple $L^{2}$-extension.

**Definition 3.8.** We say that $(E, h)$ is positively curved in the sense of multiple $L^{2}$-extension if for any point $x \in X$, there exists an open neighborhood $U$ of $x$ such that $(E, h)$ has the multiple $L^{2}$-extension property (see Definition 3.2) on $U$.

Theorem 3.5 implies the following theorem.
Theorem 3.9. Let \((E, h)\) be positively curved in the sense of twisted Hörmander. Assume that \(|u|_{h^k}\) is upper semi-continuous for any local holomorphic section \(u \in \mathcal{O}(E^*)\). If \(\log |u|_{h^k}\) is locally Hölder continuous for any non-zero local holomorphic section \(u\), \((E, h)\) is positively curved in the sense of multiple \(L^2\)-extension.

In [8], Deng et al. proved that a vector bundle which has the multiple \(L^2\)-extension property is positively curved in the sense of Griffiths. Moreover, they proposed the next question.

Question 3.10. Is the multiple \(L^2\)-extension property stronger than Griffiths positivity? Is it more or less equivalent to Nakano positivity?

In this subsection, we give a partial answer to Question 3.10. To be precise, we show the following theorem.

Theorem 3.11 (= Theorem 1.5). There is a positively curved vector bundle \((E, h)\) in the sense of multiple \(L^2\)-extension such that \((E, h)\) is not Nakano semi-positive.

To prove this theorem, we prepare the following essential lemma.

Lemma 3.12. Let \((E, h)\) be positively curved in the sense of multiple \(L^2\)-extension, and \((Q, h_Q)\) be the quotient bundle and quotient metric of \((E, h)\). Then \((Q, h_Q)\) is also positively curved in the sense of multiple \(L^2\)-extension.

Proof. Let \(\beta : E \to Q\) be the quotient map. For any point \(x \in X\), there exists an open neighborhood \(U\) of \(x\) such that \((E, h)\) has the multiple \(L^2\)-extension property on \(U\). For any \(z \in U\) and a nonzero element \(a \in Q_z\) with finite norm \(|a|_{h_Q} < +\infty\), there is a nonzero element \(b \in E_z\) such that \(\beta(b) = a\) and \(|a|_{h_Q} = |b|_{h}\). Since \((E, h)\) has the multiple \(L^2\)-extension property on \(U\), for any \(m \in \mathbb{N}\), there is a holomorphic section \(f_m \in H^0(U, E^{\otimes m})\) such that \(f_m(z) = b^\otimes m\) and

\[
\int_U |f_m|^2_{h^{\otimes m}} dV_\omega \leq C_m |b^2|^m_{h^{\otimes m}},
\]

where \(C_m\) are constants satisfying the growth condition \(\lim_{m \to \infty} \frac{1}{m} \log C_m = 0\). Therefore, we have

\[
\int_U |\beta^\otimes m \circ f_m|^2_{h_Q^{\otimes m}} dV_\omega \leq \int_U |f_m|^2_{h^{\otimes m}} dV_\omega \leq C_m |b|^2m_{h} = C_m |a|^2m_{h_Q}.
\]

Consequently, we can conclude that \((Q, h_Q)\) has the multiple \(L^2\)-extension property on \(U\). Hence, \((Q, h_Q)\) is positively curved in the sense of multiple \(L^2\)-extension. \(\square\)

Here, we give the following example.

Example 3.13. Let \(Q\) be the vector bundle of rank \(n\) over \(\mathbb{P}^n\) defined by

\[
0 \to \mathcal{O}(-1) \to \mathbb{C}^{n+1} \to Q \to 0,
\]

where \(\mathbb{C}^{n+1}\) is the trivial vector bundle of rank \(n + 1\) over \(\mathbb{P}^n\) and \(\mathcal{O}(-1)\) is the tautological line bundle. It is known that \(Q\) is not Nakano semi-positive (see [3, Chapter VII, Example 6.8]).

Lemma 3.12 and Example 3.13 imply Theorem 3.11.

Proof of Theorem 3.11. We consider the vector bundles \(\mathbb{C}^{n+1}\) and \(Q\) over \(\mathbb{P}^n\) in Example 3.13. Let \(h_0\) be the standard Euclidean metric on \(\mathbb{C}^{n+1}\) and \(h_Q\) be the quotient metric of \(h_0\) on \(Q\). Since \((\mathbb{C}^{n+1}, h_0)\) is positively curved in the sense of multiple \(L^2\)-extension, \((Q, h_Q)\) is also positively curved in the sense of multiple \(L^2\)-extension. However, \((Q, h_Q)\) is not Nakano semi-positive. Hence, \((Q, h_Q)\) satisfies the conclusion of Theorem 3.11. \(\square\)

Now we consider the coherence of higher rank analogue of multiplier ideal sheaves. For a line bundle with a singular Hermitian metric, the sheaf of locally square integrable holomorphic sections is called the multiplier ideal sheaf. It is known that multiplier ideal sheaves are coherent for positively curved singular Hermitian metrics. For vector bundles, the coherence of these sheaves is not known in general due to the lack of results like \(L^2\)-estimates. Here, we prove that the twisted Hörmander condition implies the coherence of the sheaf of square integrable holomorphic sections.
Theorem 3.14 (= Theorem 1.4). Let $(E, h)$ be positively curved in the sense of twisted Hörmander. Assume that $|u|_{h^*}$ is upper semi-continuous for any local holomorphic section $u \in \mathcal{O}(E^*)$. Then $\mathcal{E}(h)$ is a coherent subsheaf of $\mathcal{O}(E)$, where $\mathcal{E}(h)$ is the sheaf of germs of locally square integrable holomorphic sections of $E$ with respect to $h$.

Proof. The following proof is based on the proof of [4, Proposition 5.7].

For any point $x \in X$, there exists an open neighborhood $\Omega$ of $x$ such that $(E, h)$ satisfies the twisted Hörmander condition on $\Omega$. Since coherence is a local property, we can assume that $\Omega$ is a bounded domain in $\mathbb{C}^n$, $E = \Omega \times \mathbb{C}^r$ is the trivial bundle over $\Omega$, and each element of $h^*$ is bounded on $\Omega$. Let $H^0_{(2, h)}(\Omega, \mathbb{C}^r)$ be the square integrable $\mathbb{C}^r$-valued holomorphic functions with respect to $h$ on $\Omega$. By the strong Noetherian property of coherent sheaves, $H^0_{(2, h)}(\Omega, \mathbb{C}^r)$ generates a coherent ideal sheaf $\mathcal{F} \subset \mathcal{O}(E) = \mathcal{O}(\mathbb{C}^r)$. First of all, we will show that

$$\mathcal{F}_x + \mathcal{E}(h)_x \cap m_x^{k+1} \cdot \mathcal{O}(\mathbb{C}^r)_x = \mathcal{E}(h)_x,$$

where $m_x$ is a maximal ideal of $\mathcal{O}_{\Omega, x}$ and $k$ is any positive integer. It is enough to show that

$$\mathcal{E}(h)_x \subset \mathcal{F}_x + \mathcal{E}(h)_x \cap m_x^{k+1} \cdot \mathcal{O}(\mathbb{C}^r)_x. \tag{3.2}$$

We take any element $f = (f_1, \ldots, f_r) \in \mathcal{E}(h)_x$, where $f_i$ is a holomorphic function defined in a neighborhood $U$ of $x$. Let $\theta$ be a cut-off function with support in $U$ such that $\theta = 1$ in a neighborhood of $x$. We define a $\bar{\partial}$-closed $\mathbb{C}^r$-valued $(n, 1)$-form $\alpha$ as

$$\alpha = \bar{\partial}(\theta f dz).$$

We also take a weight function

$$\psi_\delta(z) = (n + k) \log(|z - x|^2 + \delta^2) + |z|^2.$$

Solving a $\bar{\partial}$-equation, we get a smooth $\mathbb{C}^r$-valued $(n, 0)$-form $u_\delta$ such that

- $\bar{\partial} u_\delta = \alpha$, and
- $\int_{\Omega} |u_\delta|^2 e^{-\psi_\delta} dV = \int_{\Omega} \sum_{1 \leq i, j \leq n} (\psi_\delta(\alpha_i, \alpha_j)) e^{-\psi_\delta} dV < +\infty$.

The right-hand side of the above inequality has an upper bound independent of $\delta$. Taking limits $\delta \to 0$ and repeating the argument of the proof of Theorem 3.5, we obtain a smooth $\mathbb{C}^r$-valued $(n, 0)$-form $udz$ such that

- $\bar{\partial}(udz) = \alpha$, and
- $\int_{\Omega} \frac{|u|^2}{|z - x|^2 + \delta^2} dV < +\infty$.

Since each element of $h^*$ is bounded, there exists a positive constant $C$ such that

$$|g|^2_{h^*} \geq C|g|^2 = C(|g_1|^2 + \cdots + |g_r|^2)$$

for any $\mathbb{C}^r$-valued smooth function $g$. Hence we get

$$\int_{\Omega} \frac{|u|^2}{|z - x|^{2(n+k)}} dV < +\infty.$$  

Letting $F := \theta f - u$, we obtain $F \in H^0_{(2, h)}(\Omega, \mathbb{C}^r)$ and $f_x - F_x = u_x \in \mathcal{E}(h)_x \cap m_x^{k+1} \cdot \mathcal{O}(\mathbb{C}^r)_x$. This proves (3.2).

Finally, we will prove $\mathcal{F}_x = \mathcal{E}(h)_x$. The Artin-Rees lemma implies that there exists an integer $l \geq 1$ such that

$$m_x^{k+1} \cdot \mathcal{O}(\mathbb{C}^r)_x \cap \mathcal{E}(h)_x = m_x^{k-l+1} (m_x^l \cdot \mathcal{O}(\mathbb{C}^r)_x \cap \mathcal{E}(h)_x)$$

holds for any $k \geq l - 1$. Therefore, for $k > l - 1$, we have

$$\mathcal{E}(h)_x = \mathcal{F}_x + \mathcal{E}(h)_x \cap m_x^{k+1} \cdot \mathcal{O}(\mathbb{C}^r)_x$$

$$= \mathcal{F}_x + m_x^{k-l+1} (m_x^l \cdot \mathcal{O}(\mathbb{C}^r)_x \cap \mathcal{E}(h)_x).$$
\[ \mathcal{F}_x + m_x \cdot \mathcal{E}(h)_x \]
\[ \subset \mathcal{E}(h)_x. \]

By Nakayama’s lemma, we obtain \( \mathcal{F}_x = \mathcal{E}(h)_x \). Thus we can conclude that \( \mathcal{E}(h) \) is coherent. \( \square \)

Finally, we give a simple example of a singular Hermitian metric satisfying the twisted Hörmander condition.

**Theorem 3.15.** Let \( h \) be a Griffiths semi-positive singular Hermitian metric. Then \( (E \otimes \det E, h \otimes \det h) \) is positively curved in the sense of twisted Hörmander.

**Proof.** We set \( G := E \otimes \det E \) and \( g := h \otimes \det h \) for simplicity. For any point \( x \in X \), we take an open Stein neighborhood \( U \) of \( x \) such that \( E \) is trivial over \( U \). It is easy to show that \( (G, g) \) satisfies the twisted Hörmander condition on \( U \). Let \( m \) be a positive integer, \( \psi \) be a smooth strictly plurisubharmonic function on \( U \), and \( \alpha = \sum_j \alpha_j dz \wedge d\bar{z}_j \) be a smooth \( \partial \)-closed \( G^{\otimes m} \)-valued \((n, 1)\)-form with compact support and finite norm
\[ \int_U \sum_{1 \leq i, j \leq n} (\psi^{(i)}(\bar{\alpha}_i, \alpha_j)_{g^{\otimes m}} e^{-\psi} dV_\omega < +\infty. \]

We take a Stein exhaustion \( \{U_\lambda\}_\lambda \) of \( U \) such that \( U_\lambda \Subset U_{\lambda+1} \Subset \cdots \Subset U \) and each \( U_\lambda \) is a Stein domain. It is known that there exists a sequence of the smooth Hermitian metrics \( \{h_\nu\}_{\nu=1}^\infty \) with positive Griffiths curvature, increasing pointwise to \( h \) on each \( U_\lambda \) (see [2, Proposition 3.1]). From the results of Demailly and Skoda [5], we see that \( g_\nu \) is Nakano semi-positive for each \( \nu \), where \( g_\nu := h_\nu \otimes \det h_\nu \). Since \( g_\nu^{\otimes m} e^{-\psi} \) is Nakano positive, we obtain a smooth \( G^{\otimes m} \)-valued \((n, 0)\)-form \( u_{\lambda,\nu} \) such that \( \partial u_{\lambda,\nu} = \alpha \) and
\[ \int_{U_\lambda} |u_{\lambda,\nu}|^2 (g^{\otimes m}) e^{-\psi} dV_\omega \leq \int_{U_\lambda} (|\sqrt{-1} \Theta (g^{\otimes m} e^{-\psi}, \Lambda_\omega)|^{-1} \alpha, \alpha)_{(g^{\otimes m} e^{-\psi})} dV_\omega \]
\[ \leq \int_{U_\lambda} \sum_{1 \leq i, j \leq n} (\psi^{(i)}(\bar{\alpha}_i, \alpha_j)_{g^{\otimes m}} e^{-\psi} dV_\omega \]
\[ \leq \int_{U} \sum_{1 \leq i, j \leq n} (\psi^{(i)}(\bar{\alpha}_i, \alpha_j)_{g^{\otimes m}} e^{-\psi} dV_\omega \]
by using Hörmander’s \( L^2 \)-estimate. Using a diagonal argument and taking weak limits \( \lambda \to \infty \) and \( \nu \to \infty \), we get a smooth solution \( u \) such that

• \( \partial u = \alpha \), and
• \( \int_U |u|^2 (g^{\otimes m}) e^{-\psi} dV_\omega \leq \int_U \sum_{1 \leq i, j \leq n} (\psi^{(i)}(\bar{\alpha}_i, \alpha_j)_{g^{\otimes m}} e^{-\psi} dV_\omega. \)

Therefore, we can conclude that \((G, g)\) satisfies the twisted Hörmander condition on \( U \). \( \square \)

Inspired by Theorems 3.14 and 3.15, we propose the following question in relation to Question 3.10.

**Question 3.16.** Is positivity in the sense of twisted Hörmander more or less equivalent to Nakano positivity?

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