A new volume form on sub-Riemannian Heisenberg manifolds and their non-collapsed limits

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Abstract

In this paper, we introduce a new volume form to compute Riemannian and sub-Riemannian metrics simultaneously. By considering a uniform lower bound of the total measure induced from this volume, we show that the non-collapsed limit of a sequence of compact Heisenberg manifolds is isometric to again a compact Heisenberg manifold of the same dimension.

1 Introduction

A sub-Riemannian manifold is a triple $(M, E, g)$, where $M$ is a connected manifold, $E$ is a sub-bundle of the tangent bundle $TM$, and $g$ is a metric on $E$. If $E$ is the whole tangent bundle, then it is a Riemannian manifold.

A sub-Riemannian manifold is an example of the Gromov–Hausdorff limit of a sequence of Riemannian manifolds without curvature bounds. For example, let $g = g_h \oplus g_v$ be a left invariant Riemannian metric on the Heisenberg Lie group $H_n$, where $g_v$ is horizontal to a canonical contact 1-form. Consider a family $\{g_\epsilon = g_h \oplus (1 + \epsilon g_v)\}$ for $\epsilon > 0$. If $\epsilon$ goes to 0, then the sequence converges to the sub-Riemannian metric. For more detail, please see Example 3.

Although the curvature of such a sequence diverges, some analytic quantities converge. In [12], Juillet show that the heat flow and the gradient flow of the relative entropy coincide on the Heisenberg group. One of the key idea is to estimate the descending slope of the entropy and the Fisher’s information by the Riemannian approximation. This technique is generalized to all Carnot groups by Ambrosio–Stefani [2]. In [3], Balogh–Kristály–Sipos show a version of the Borell–Brascamp–Lieb inequality on the Heisenberg group. There the Riemannian approximation method is used to estimate the volume distortion coefficients. This result is generalized to ideal sub-Riemannian manifold by Barilari–Rizzi [5]. The method in [5] is different from that of [3], where they compute the intrinsic Jacobi fields on sub-Riemannian manifolds.

The above results assume a fixed reference measure on manifolds such as a Haar volume. The reason is that the canonical Riemannian volume form diverges when the sequence converges to a sub-Riemannian one. The motivation of this paper is to give an intrinsic definition which is compatible to
this convergence. Toward the goal, we introduce a new volume form \textit{minvol} called the minimal Popp’s volume. The definition is given in Definition 8. Roughly speaking, the minimal Popp’s volume form is the ‘minimum’ Popp’s volume of a sub-Riemannian Lie group \((G, E, g)\), where the minimum is chosen from all the Popp’s volume induced from the sub-Riemannian structure \((E', g_{E' \otimes E'}) \subset (E, g)\). It will be seen in Proposition 6 that it fills the gap between the canonical Riemannian volume form and the Popp’s volume on the Heisenberg group.

By using the minimal Popp’s volume, we study the Gromov–Hausdorff limits of a sequence of special class of sub-Riemannian manifolds, compact Heisenberg manifolds. In Riemannian geometry, the structure of the Gromov–Hausdorff limit of a sequence of Riemannian manifolds \(\{M_k\}\) is actively studied under the assumption of the sectional curvature or the Ricci curvature. The easiest case is flat manifolds, that is Riemannian manifolds with \(\text{Sec} \equiv 0\). Denote by \(\mathcal{M}(n, D, V)\) the set of \(n\)-dimensional compact flat Riemannian manifolds with \(\text{diam} \leq D < \infty\) and \(|\int \text{vol}| \geq V > 0\). If a sequence in \(\mathcal{M}(n, D, V)\) consists of flat tori, then the limit is again isometric to a flat tori of the same dimension. It is a consequence of the classical Mahler’s compactness theorem \([15]\). If a sequence consists of flat manifolds which are not necessarily tori, then the limit space may be a flat orbifold \([6]\).

We will study an analogy of these results for compact quotients of the Heisenberg Lie group. The Heisenberg Lie group \(H_n\) is the simply connected nilpotent Lie group diffeomorphic to \(\mathbb{C}^n \times \mathbb{R}\) with the law of group operation

\[(x_1, z_1) \cdot (x_2, z_2) = (x_1 + x_2, z_1 + z_2 + \frac{1}{2} \text{Im}(x_1 \cdot x_2)),\]

where we denote by the dot the hermitian product and \(\text{Im}(z)\) the imaginary part of a complex number \(z\). Let \(\Gamma\) be a lattice in \(H_n\). We call the quotient space \(\Gamma \backslash H_n\) a compact Heisenberg manifold. We say that a metric on \(\Gamma \backslash H_n\) is left invariant if the pullback metric on the universal cover \(\tilde{H}_n\) is left invariant. The Heisenberg Lie group plays a role of the Euclidean space for contact sub-Riemannnian manifolds. Indeed the tangent cone of a contact sub-Riemannian manifold is isometric to \(H_n\), as a consequence of Mitchell’s theorem \([16]\). Note that every tangent cone of a Riemannian manifold is isometric to the Euclidean space.

The main result of this paper is on non-collapsed Gromov–Hausdorff limits of a sequence of compact Heisenberg manifolds. From now on we say that a metric is sub-Riemannian if the rank of \(E\) is less than or equal to \(2n + 1\), and that it is properly sub-Riemannian if the rank of \(E\) is less than or equal to \(2n\).

**Theorem 1.** Let \(\{(\Gamma_k \backslash H_n, E_k, g_k)\}_{k \in \mathbb{N}}\) be a sequence of compact Heisenberg manifolds with left invariant sub-Riemannian metrics which converges in the Gromov–Hausdorff topology. Assume that there are positive constants \(D, V > 0\) such that

\[(D) \quad \text{diam}(\Gamma_k \backslash H_n, \text{dist}_k) \leq D,\]
Then we have a lattice $\Gamma_\infty < H_n$ and a left invariant sub-Riemannian metric $(E_\infty, g_\infty)$ such that the limit is isometric to $(\Gamma_\infty \setminus H_n, E_\infty, g_\infty)$.

Notice that if we replace the minimal Popp’s to the original Popp’s volume, then we obtain a sequence of Riemannian metrics which collapses to a flat tori although $|\int \text{vol}| > V > 0$, see Example [2].

It will be seen in Proposition 5 that if a metric is properly sub-Riemannian, then the minimal Popp’s volume coincides with the original Popp’s volume. This fact implies a similar statement for properly sub-Riemannian metrics with the original Popp’s volume, which is also new.

**Corollary 1.** Let $\{ (\Gamma_k \setminus H_n, E_k, g_k) \}_{k \in \mathbb{N}}$ be a sequence of compact Heisenberg manifolds with left invariant Riemannian or sub-Riemannian metrics which converges in the Gromov–Hausdorff topology. Assume that there are positive constants $D, V > 0$ such that

$(D) \text{ diam}(\Gamma_k \setminus H_n, \text{dist}_k) \leq D,$

$(V) |\int_{\Gamma_k \setminus H_n} \text{vol}_k| \geq V,$

where $\text{vol}_k$ denote the Popp’s volume.

Then we have a lattice $\Gamma_\infty < H_n$ and a left invariant sub-Riemannian metric $(E_\infty, g_\infty)$ such that the limit is isometric to $(\Gamma_\infty \setminus H_n, E_\infty, g_\infty)$.

The proof is done with the following two steps. First of all, we show the finiteness theorem for compact Heisenberg manifolds.

**Proposition 1** (Precisely in Proposition 8). Assume that a sub-Riemannian compact Heisenberg manifold $(\Gamma \setminus H_n, E, g_k)$ satisfies the assumption in Theorem 7. Then the number of isomorphism classes of lattices $\Gamma$ is less than a constant $C$ dependent only on $n, D$ and $V$.

By this proposition, we can assume that a sequence consists of diffeomorphic compact Heisenberg manifolds. For a fixed diffeomorphism type $\Gamma \setminus H_n$, we can compute the moduli space of isometry classes of left invariant sub-Riemannian metrics $M(\Gamma \setminus H_n)$, see Definition 9 and Theorem 13. It is similar to the well known moduli space of flat metrics on tori given as follows. For the $n$-dimensional Euclidean space $\mathbb{R}^n$ and the integer points $\mathbb{Z}^n$, the moduli space of flat Riemannian metrics $M(\mathbb{Z}^n \setminus \mathbb{R}^n)$ has a one-to-one correspondance with the double quotient space $O(n) \setminus GL_n(\mathbb{R}) / GL_n(\mathbb{Z})$. On the double coset space, we can endow the quotient topology induced from $GL_n(\mathbb{R})$. In a similar way to the moduli space of flat metrics on $\mathbb{T}^n$, we can define the canonical quotient topology $\mathcal{O}$ on the moduli space $M(\Gamma \setminus N)$.

Let $M_0(\Gamma \setminus H_n, \mathcal{O}_0)$ be the topological subspace of the moduli space $(M(\Gamma \setminus H_n), \mathcal{O})$. In [7], Boldt characterize precompact subsets in $(M_0(\Gamma \setminus H_n), \mathcal{O}_0)$, under the four conditions on metric tensors.
Theorem 2 (Corollary 3.14 in [7]). A subset $\mathcal{M}_0(\Gamma \setminus H_n)$ is precompact in $\mathcal{O}_0$ if and only if the conditions (A-1)-(A-4) hold.

We will give the explicit condition in Theorem 14. Actually it is easily checked that these conditions (A-1)-(A-4) is induced from the following geometric assumptions.

**Proposition 2** (Precisely in Proposition 7). Assume that a Riemannian compact Heisenberg manifold $(\Gamma \setminus H_n, g)$ satisfies the following;

(i) $\text{diam}(\Gamma \setminus H_n, g) \leq D$,

(ii) $\left| \int_{\Gamma \setminus H_n} \text{vol} \right| \geq V > 0$,

(iii) $\text{Ric}_g \geq -K$,

where $\text{vol}$ is the canonical Riemannian volume form. Then it satisfies the conditions (A-1)-(A-4).

We will generalize the previous facts to sub-Riemannian setting. In the following theorem, the conditions (A-1)-(A-3) is same to those of Theorem 14 and (A-4)' is a sub-Riemannian version of (A-4).

**Theorem 3** (Essentially due to Theorem 2). A subset in $\mathcal{M}(\Gamma \setminus H_n)$ is precompact in $\mathcal{O}$ if and only if the conditions (A-1)-(A-3), (A-4)' holds.

As in the Riemannian setting, we obtain the following proposition.

**Proposition 3** (Precisely in Proposition 9). Assume that a sub-Riemannian compact Heisenberg manifold $(\Gamma \setminus H_n, E, g)$ satisfies the condition (D) and (V) in Theorem 1. Then it satisfies the conditions (A-1)-(A-3) and (A-4)'.

Proposition 1, 3 and Theorem 3 show Theorem 1.

Proof of Theorem 1. By Proposition 1 we can assume $\Gamma_k \simeq \Gamma_l$ for all $k, l \in \mathbb{N}$. Theorem 3 and Proposition 3 show that the sequence has a convergent subsequence in the canonical quotient topology $\mathcal{O}$. Since the sequence converges in the Gromov–Hausdorff topology, the limit in $\mathcal{O}$ is unique. It also implies that the Gromov–Hausdorff limit is isometric to the limit in $\mathcal{O}$. \hfill \Box

Acknowledgement

The author thank to Professor Koji Fujiwara for helpful comments and careful check. The author also thank to Professor Enrico Le Donne and Gabriel Parrier for discussion on the relation to the Gromov–Hausdorff topology. This work was supported by JSPS KAKENHI Grant Number JP20J13261.

2 Preliminaries from sub-Riemannian geometry

In this section we prepare notations on sub-Riemannian geometry.
2.1 Sub-Riemannian structure

Let $M$ be a connected orientable smooth manifold, $(E, \tilde{g})$ a metric vector bundle on $M$, and $f : E \to TM$ be a fiberwise linear smooth map. For $x \in M$, denote by $D_x$ the image of $f|_{E_x}$. We call the collection of subspaces $\mathcal{D} = \{D_x\}_{x \in M}$ the distribution. On each subspace $D_x$ we define the inner product $g_x$ by

$$g_x(u, v) = \inf \{\tilde{g}(U, V) | u = f(U), v = f(V)\}.$$

**Definition 1** (Sub-Riemannian structure). A sub-Riemannian manifold is a triple $(M, \mathcal{D}, g)$. The pair $(E, f)$ (or $(\mathcal{D}, g)$) is called the sub-Riemannian structure on $M$.

We say that a vector field on $M$ is horizontal if it is a section of the distribution $\mathcal{D}$.

**Example 1.** Let $G$ be a connected Lie group, $\mathfrak{g}$ the associated Lie algebra, $\mathfrak{v} \subset \mathfrak{g}$ a subspace and $\langle \cdot, \cdot \rangle$ an inner product on $\mathfrak{v}$. For $x \in G$, denote by $L_x : G \to G$ the left translation by $x$. Define a sub-Riemannian structure on $G$ by

$$D_x = (L_x)_* \mathfrak{v}, \quad g_x(u, v) = \langle L_x^{-1} u, L_x^{-1} v \rangle.$$

Such a sub-Riemannian structure $(\mathcal{D}, g)$ is called left invariant. We sometimes write left invariant sub-Riemannian structure by $(\mathfrak{v}, \langle \cdot, \cdot \rangle)$.

The associated distance function is given in the same way to Riemannian distance. We say that an absolutely continuous path $c : [0, 1] \to M$ is admissible if $\dot{c}(t) \in D_{c(t)}$ a.e. $t \in [0, 1]$. We define the length of an admissible path by

$$\ell(c) = \int_0^1 \sqrt{g(\dot{c}(t), \dot{c}(t))} dt.$$

For $x, y \in M$, define the distance function by

$$d(x, y) = \inf \{\ell(c) | c(0) = x, c(1) = y, c \text{ is admissible}\}.$$

In general not every pair of points in $M$ is joined by an admissible path. This implies that the value of the function $d$ may be the infinity. The following bracket generating condition ensures that any two points are joined by an admissible path.

**Definition 2** (Bracket generating distribution). For every $i \in \mathbb{N}$, let $\mathcal{D}^i$ be the submodule in $\text{Vec}(M)$ inductively defined by

$$\mathcal{D}^1 = \mathcal{D}, \quad \mathcal{D}^{i+1} = \mathcal{D}^i + [\mathcal{D}, \mathcal{D}^i],$$

and set $\mathcal{D}^i_x = \{X(x) | X \in \mathcal{D}^i\}$ for $x \in M$.

We say that a distribution $\mathcal{D}$ is bracket generating if for all $x \in M$ there is $r = r(x) \in \mathbb{N}$ such that $\mathcal{D}^r_x = T_x M$.
Theorem 4 (Chow–Rashevskii’s theorem, Theorem 3.31 in [1]). Let \((M, D, g)\) be a sub-Riemannian manifold with a bracket generating distribution. Then the following two assertions hold.

1. \((M, d)\) is a metric space,
2. the topology induced by \((M, d)\) is equivalent to the manifold topology.

In particular, \(d : M \times M \to \mathbb{R}\) is continuous.

Assume that the metric space \((M, d)\) is proper, that is every closed ball is compact. With the help of the Ascoli–Alzera theorem, we can show the existence of a length minimizing path joining any two points (Theorem 3.43 in [1]).

2.2 Length minimizing paths

Let us study the structure of length minimizing paths on a sub-Riemannian manifold \((M, D, g)\).

For simplicity we assume that the dimension of \(D_x\) is equal to \(m\) for all \(x \in M\) and that we have a family of globally defined \(m\) smooth vector fields \(\{f_1, \ldots, f_m\}\) such that \(\{f_1(x), \ldots, f_m(x)\}\) is an orthonormal basis of \(D_x g_x\). We call such a family a generating family.

Let \(\sigma\) be the canonical symplectic form on \(T^* M\). For a given function \(h : T^* M \to \mathbb{R}\), there is a unique vector field \(\vec{h}\) on \(T^* M\) defined by

\[
\sigma(\cdot, \vec{h}(\lambda)) = dh|_\lambda \quad (\lambda \in T^* M).
\]

We call the vector \(\vec{h}\) the Hamiltonian vector field of \(h\).

Let \(h_i : T^* M \to \mathbb{R}\), \(i = 1, \ldots, m\) be the function defined by

\[
h_i(\lambda) = \langle \lambda | f_i(x) \rangle \quad (\lambda \in T^*_x M),
\]

where \(\langle \cdot | \cdot \rangle\) is the canonical pairing of covectors and vectors. Then length minimizing paths on a sub-Riemannian manifold are explained with the Hamiltonian vector fields of \(h_i\)’s as follows.

Theorem 5 (The Pontryagin maximal principle, Theorem 4.20 in [1]). Let \(\gamma : [0, T] \to M\) be a length minimizing path parametrized by constant speed, and write its differential by

\[
\dot{\gamma}(t) = \sum_{i=1}^m u_i(t) f_i(\gamma(t)).
\]

Then there is a Lipschitz curve \(\lambda : [0, T] \to T^* M\) such that

\[
\begin{cases}
\lambda(t) \in T^*_x M, \\
\dot{\lambda}(t) = \sum_{i=1}^m u_i(t) \vec{h}_i(\lambda(t)) \quad a.e. \ t \in [0, T],
\end{cases}
\]

and one of the following conditions satisfied:
\( (N) \quad h_i(\lambda(t)) = u_i(t), \quad i = 1, \ldots, m, \quad t \in [0, 1], \)

\( (A) \quad h_i(\lambda(t)) = 0, \quad i = 1, \ldots, m. \)

**Definition 3** (Normal extremal). The Lipschitz curve \( \lambda \) with the condition \( (N) \) is called a normal extremal, and its projection \( \gamma \) is called a normal trajectory.

**Definition 4** (Abnormal extremal). The Lipschitz curve \( \lambda \) with the condition \( (N) \) is called an abnormal extremal, and its projection \( \gamma \) is called an abnormal trajectory.

Let \( D_x^\perp \subset T^*_x M \) be the subspace defined by

\[
D_x^\perp = \{ \lambda \in T^*_x M \mid \langle \lambda | u \rangle = 0 \text{ for all } u \in D_x \}.
\]

With this notation, we can say that an extremal \( \lambda(t) = (x(t), p(t)) \) is abnormal if \( \lambda(t) \in D_x^\perp(t) \).

**Example 2.** Suppose that \( (M, D, g) \) is a Riemannian manifold. Then the condition \( (A) \) implies that \( \lambda(t) \in D_x^\perp = \{0\} \). Combined with the second equality in (1), such a Lipschitz curve \( \lambda \) is a constant curve. This argument shows that every abnormal trajectories on a Riemannian manifold is a constant curve.

**Remark 1.**
- An abnormal extremal and a normal extremal may project to the same trajectory. Hence a trajectory may be normal and abnormal simultaneously.
- On the Heisenberg group and its quotient spaces, there are no non-trivial abnormal extremals. Thus we compute only normal extremals in this paper.

A normal extremal is the solution to the differential equation, called the Hamiltonian system. Let \( H : T^* M \to \mathbb{R} \) be the function defined by

\[
H(\lambda) = \frac{1}{2} \sum_{i=1}^{m} h_i(\lambda)^2.
\]

This function is called the sub-Riemannian Hamiltonian.

**Remark 2.** If a manifold \( (M, D, g) \) is Riemannian, then we have the canonical metric \( g^* \) on \( T^* M \) induced by the musical isomorphism. The Riemannian Hamiltonian is defined by using this metric.

However, sub-Riemannian manifolds have no canonical isomorphism between the tangent bundle and cotangent bundle. Hence we use the canonical pairing of covectors and vectors instead of metrics on the cotangent bundle.

**Theorem 6** (Theorem 4.25 in [1]). A Lipschitz curve \( \lambda : [0, T] \to T^* M \) is a normal extremal if and only if it is a solution to the Hamiltonian system

\[
\dot{\lambda}(t) = \check{H}(\lambda(t)), \quad t \in [0, T],
\]
where $\vec{H}$ is the Hamiltonian vector field of $H$.

Moreover, the corresponding normal trajectory $\gamma$ is smooth and has a constant speed satisfying

$$\frac{1}{2} \| \dot{\gamma}(t) \|^2 = H(\lambda(t)).$$

A straightforward computation shows the local expression for $\lambda(t) = (x(t), p(t))$ written by

$$\begin{cases}
\dot{x}(t) = \frac{\partial H}{\partial p}, \\
\dot{p}(t) = -\frac{\partial H}{\partial x}.
\end{cases} \tag{2}$$

2.3 The Popp’s volume

On a Riemannian manifold, one has a canonical Riemannian volume form defined by

$$v = \nu_1 \wedge \cdots \wedge \nu_n,$$

where $\{\nu_1, \cdots, \nu_n\}$ is a dual coframe of an orthonormal basis. In sub-Riemannian geometry, we also have a canonical volume form, called Popp’s volume introduced in [17]. The Popp’s volume is defined under the following equiregular assumption.

**Definition 5** (Equiregular distribution). A sub-Riemannian manifold $(M, \mathcal{D}, g)$ is equiregular if for any $i \in \mathbb{N}$ the dimension of the subspaces $\mathcal{D}^i_x$ is independent of the choice of $x \in M$.

If $\mathcal{D}^r_x = T_x M$, we say that a sub-Riemannian manifold is $r$-step. For simplicity, we recall the definition of the Popp’s volume in the 2-step case.

**Definition 6** (Nilpotentization). The nilpotentization of $\mathcal{D}$ at the point $x \in M$ is the graded vector space

$$gr_x(\mathcal{D}) = \mathcal{D}_x \oplus \mathcal{D}^2_x/\mathcal{D}_x.$$

On the vector space $gr_x(\mathcal{D})$ we can define a new Lie bracket $[\cdot, \cdot]'$ by

$$\left[ X \ mod \ \mathcal{D}, Y \ mod \ \mathcal{D} \right]_x' = \left[ X, Y \right]_x \ mod \ \mathcal{D}_x,$$

The new Lie bracket rule induces a different Lie algebra structure from the original one.

From the inner product on $\mathcal{D}_x$, we obtain the inner product on the nilpotentization $gr_x(\mathcal{D})$ of $\mathcal{D}$. Let $\pi : \mathcal{D}_x \otimes \mathcal{D}_x \to \mathcal{D}^2_x/\mathcal{D}_x$ be the linear map given by

$$\pi(u \otimes v) = [U, V]_x \ mod \ \mathcal{D}_x,$$

where $U, V$ are horizontal extensions of $u, v$. Define the norm $\| \cdot \|_2$ on $\mathcal{D}^2_x/\mathcal{D}_x$ by

$$\| z \|_2 = \min \{ \| U(x) \|\| V(x) \| \mid [U, V]_x = z \ mod \ \mathcal{D}, U, V : \ horizontal \ vector \ fields \}.$$
This norm satisfies the parallelogram law, thus we obtain the inner product \( \langle \cdot, \cdot \rangle_2 \) on \( D^2_2/D_2 \). The direct sum of two inner product spaces \((D_2, g_2)\) and \((D^2_2/D_2, \langle \cdot, \cdot \rangle_2)\) gives the new inner product \( \langle \cdot, \cdot \rangle'_{x} \) on the nilpotization \( gr_x(D) \).

Let \( \omega_x \in \wedge^n gr_x(D)^* \) be the volume form obtained by wedging the elements of orthonormal dual basis in \((gr_x(D), \langle \cdot, \cdot \rangle'_{x})\). It is defined up to sign. By the following lemma, the volume \( \omega_x \in \wedge^n gr_x(D)^* \) is transported to the volume on \( \wedge^n T^*_x M \).

**Lemma 1** (Lemma 10.4 in [17]). Let \( E \) be a vector space of dimension \( n \) with a filtration by linear subspaces \( F_1 \subset F_2 \subset \cdots \subset F_l = E \). Let \( Gr(F) = F_1 \oplus F_2/F_1 \oplus \cdots F_l/F_{l-1} \) be the associated graded vector space. Then there is a canonical isomorphism \( \theta : \wedge^n E^* \simeq \wedge^n gr(F)^* \).

Let \( \theta : \wedge^n T^*_x M \to \wedge^n gr_x(D)^* \) be the isomorphism obtained by Lemma 1.

**Definition 7** (Popp’s volume). The Popp’s volume form \( vol(D, g) \) is defined by

\[
vol(D, g)_x = \theta^* \omega_x, \quad x \in M.
\]

Trivially the Popp’s volume of a Riemannian manifold is the canonical Riemannian volume form.

The Popp’s volume has a useful expression by using the structure constant. We say that a local frame \( X_1, \ldots, X_n \) is adapted if \( X_1, \ldots, X_m \) are orthonormal. Define the smooth functions \( c^l_{ij} \) on \( M \) by

\[
[X_i, X_j] = \sum_{l=1}^{n} c^l_{ij} X_l.
\]

We call them the structure constants. We define the \( n - m \) dimensional square matrix \( B \) by

\[
B_{hl} = \sum_{i,j=1}^{m} c^h_{ij} c^l_{ij}.
\]

**Theorem 7** (Theorem 1 in [4]). Let \( X_1, \ldots, X_n \) be a local adapted frame, and \( \nu^1, \ldots, \nu^n \) the dual coframe. Then the Popp’s volume \( vol(D, g) \) is written by

\[
vol(D, g) = (\det B)^{-\frac{1}{2}} \nu^1 \wedge \cdots \wedge \nu^n.
\]

### 2.4 The minimal Popp’s volume form on Lie groups

Let \( G \) be a connected Lie group, \( g \) its Lie algebra, and \((v, \langle \cdot, \cdot \rangle)\) a left invariant sub-Riemannian structure on \( G \). Moreover let \( F(v) \) be the set of bracket generating subspaces in \( v \). We define the minimal Popp’s volume on \((G, v, \langle \cdot, \cdot \rangle)\) as follows.

**Definition 8.** The minimal Popp’s volume is defined by

\[
mvol(v, \langle \cdot, \cdot \rangle) = \min \{ vol(w, \langle \cdot, \cdot \rangle) | w \subset F(v) \}.
\]
Here the order of volume forms is obtained from the coefficients of a fixed Haar volume $vol_0$. Hence we can take the infimum up to sign. The existence of the minimum is shown as follows. For a positive constant $C > 0$, define a closed subset $\mathcal{F}(\mathbf{v}, C) \subset \mathcal{F}(\mathbf{v})$ by

$$\mathcal{F}(\mathbf{v}, C) = \{ \mathbf{w} \subset \mathcal{F}(\mathbf{v}) \mid |\text{vol}(\mathbf{w}, \langle \cdot, \cdot \rangle|_{\mathbf{w} \otimes \mathbf{w}})| \leq |C\text{vol}_0|\}.$$  

From its definition, $\mathcal{F}(\mathbf{v}, C)$ is a closed subset of a finite union of the Grassmannians, thus the minimum exists from its compactness.

It is natural to ask how to define the minimal Popp’s volume on equiregular sub-Riemannian manifolds. In the case of Lie groups, we choose the minimum from the spans of left invariant vector fields $\mathcal{F}(\mathbf{v})$. Thus it is natural to consider the spans of every vector field, and consider their local minimum. However the following example shows that such definition does not work.

Let $(\mathbb{R}^3, g_0)$ be the 3-dimensional Euclidean space. For a given positive constant $R$, let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be functions such that

$$\begin{cases}
  f_1^2 + f_2^2 = 1, \\
  f_2(0) = 0, \\
  f_2'(0) = R.
\end{cases}$$

Consider the distribution $\mathcal{D} = \text{Span}\{\partial_x, f_1(x)\partial_y + f_2(x)\partial_z\}$. At the origin $(0, 0, 0)$, the adapted frame of the induced sub-Riemannian structure $(\mathcal{D}, g_0|_{\mathcal{D} \otimes \mathcal{D}})$ is $\{\partial_z, \partial_y, \partial_z\}$. By the formula of the Popp’s volume, we obtain

$$\text{vol}(\mathcal{D}, g_0|_{\mathcal{D} \otimes \mathcal{D}})(0, 0, 0) = \frac{1}{\sqrt{2R}}dxdydz.$$  

Since the choice of $R$ is arbitrary, the induced Popp’s volume form can be arbitrarily small at the origin.

The above example shows that the span of vector fields without any restriction may produce arbitrarily small Popp’s volume. Thus we need to restrict the choice of vector fields such as left invariant conditions in Lie group case. We leave the problem to the future researches.

### 3 Moduli space of left invariant metrics on compact nilmanifolds

Let $N$ be a simply connected nilpotent Lie group and $\Gamma < N$ a lattice. The quotient space $\Gamma \backslash N$ is called a compact nilmanifold. We say that a geodesic metric $\text{dist}$ on the quotient space $\Gamma \backslash N$ is left invariant if its lift $\tilde{\text{dist}}$ on $N$ is left invariant. In this section, we will study an explicit form of the moduli space of left invariant sub-Riemannian metrics on $\Gamma \backslash N$ defined as follows.
**Definition 9** (Moduli space). We denote by $M_k(\Gamma \backslash N)$ the set of isometry classes of left invariant sub-Riemannian metrics of rank $E = \dim N - k$ on $\Gamma \backslash N$. Moreover denote by $M(\Gamma \backslash N) = \bigcup_{k \geq 0} M_k(\Gamma \backslash N)$ the set of isometry classes of left invariant sub-Riemannian metrics of rank $E \leq \dim N$.

### 3.1 On general compact nilmanifolds

Let $\mathfrak{n}$ be the Lie algebra associated to $N$. We will give an explicit form of the moduli space $M(\Gamma \backslash N)$. It is easy to see that a sequence of Riemannian metric tensors $(g_{ij})$ diverges if it converges to a sub-Riemannian metric. However, the sequence of the associated cometric tensors $(g^{ij})$ converges to a degenerated tensor. Hence we give the parametrization via cometric tensors.

Let $(\Gamma_1 \backslash N_1, dist_1)$ be a compact nilmanifold with a left invariant sub-Riemannian metric. We have the sub-Riemannian lift $\tilde{dist}_1$ of $dist_1$ on $N_1$. It is well known that the sub-Riemannian distance $\tilde{dist}_1$ is induced from a left invariant sub-Riemannian structure $(\mathfrak{v}_1, \langle \cdot, \cdot \rangle_1)$, where $\mathfrak{v}_1$ is a bracket generating subspace in $\mathfrak{n}$ and $\langle \cdot, \cdot \rangle_1$ is an inner product on $\mathfrak{v}_1$. Let $(\Gamma_2 \backslash N_2, dist_2)$ be another compact nilmanifold isometric to $(\Gamma_1 \backslash N_1, dist_1)$, and $(\mathfrak{v}_2, \langle \cdot, \cdot \rangle_2)$ the associated left invariant sub-Riemannian structure. The following theorem shows how $(\mathfrak{v}_1, \langle \cdot, \cdot \rangle_1)$ and $(\mathfrak{v}_2, \langle \cdot, \cdot \rangle_2)$ are related.

**Theorem 8** (cf. Theorem 5.4 in [10]). $(\Gamma_1 \backslash N_1, dist_1)$ is isometric to $(\Gamma_2 \backslash N_2, dist_2)$ if and only if $N_1 \simeq N_2$, $\Gamma_1 \simeq \Gamma_2$ and there is an automorphism $\Phi \in Inn(N) \cdot Stab(\Gamma) < Aut(N)$ such that

- $\Phi^{-1}(\mathfrak{v}_1) = \mathfrak{v}_2$,
- $\Phi^*(\langle \cdot, \cdot \rangle_1) = \langle \cdot, \cdot \rangle_2$,

where we identify $N$ to $N_1 \simeq N_2$ and $\Gamma$ to $\Gamma_1 \simeq \Gamma_2$.

For its proof, we use the following fact.

**Theorem 9** (Theorem 2 in [14]). Let $(N_i, \tilde{dist}_i)$ $(i = 1, 2)$ be connected nilpotent Lie groups with left-invariant metrics which induce the manifold topologies. Then every isometry from $(N_1, \tilde{dist}_1)$ to $(N_2, \tilde{dist}_2)$ is affine.

Here we say that an isometry between Lie groups is *affine* if it is a composition of a Lie isomorphism and a left translation.

**Proof of Theorem 8** Let $F : (\Gamma_2 \backslash N_2, dist_2) \rightarrow (\Gamma_1 \backslash N_1, dist_1)$ be an isometry. Trivially $\Gamma_1$ is isomorphic to $\Gamma_2$. Since $\Gamma_1$ and $\Gamma_2$ are cocompact subgroup, their Mal’cev completions $N_1$ and $N_2$ are isomorphic. Thus we can write $\Gamma_1 \backslash N_1 = \Gamma_2 \backslash N_2 = \Gamma \backslash N$.

The isometry $F$ lifts to an isometry $\tilde{F} : (N, \tilde{dist}_2) \rightarrow (N, \tilde{dist}_1)$. By Theorem 9 there is an automorphism $\Phi_0 \in Aut(N)$ such that $\tilde{F} = L_{F(e)} \circ \Phi_0$. By left-invariance, the isomorphism $\Phi_0$ is also an isometry. Since every smooth
sub-Riemannian isometry preserves metric tensors, we have $\Phi^{-1}_0(v_1) = v_2$ and $\Phi^0(\langle \cdot, \cdot \rangle_1) = \langle \cdot, \cdot \rangle_2$.

The mapping $F \circ \Phi^{-1}_0$ is a self-isometry of $(N, \tilde{d}ist_1)$. Choose $x \in N$ so that $\sigma = L_x \circ F \circ \Phi^{-1}_0$ is an isometry of $(N, dist_1)$ preserving the identity. Again by Theorem 9, $\sigma \in Aut(N)$. Thus we obtain

$$R_x \circ \tilde{F} = L_x^{-1} \circ R_x \circ \sigma \circ \Phi_0 \in Aut(N).$$

The mapping $R_x \circ \tilde{F}$ is the lift of $R_x \circ F : \Gamma \setminus N \to \Gamma \setminus N$. Hence $R_x \circ F \in Stab(\Gamma)$ and $\sigma \circ \Phi_0 \in Inn(N) \cdot Stab(\Gamma)$. Since $\sigma$ is a self-isometry of $(N, dist_1)$,

$$\begin{align*}
(s \circ \Phi_0)^*(\langle \cdot, \cdot \rangle_1) &= \Phi_0^0 \sigma^*(\langle \cdot, \cdot \rangle_1) = \Phi_0^0 \langle \cdot, \cdot \rangle_1 = \langle \cdot, \cdot \rangle_2, \\
(s \circ \Phi_0)^{-1}(v_1) &= \Phi_0^{-1} \sigma^{-1}(v_1) = \Phi_0^{-1}(v_1) = v_2.
\end{align*}$$

Thus we obtain the desired automorphism by letting $\Phi = s \circ \Phi_0$.

Next we show the converse implication. As in the previous argument, we write $\Gamma_1 \setminus N_1 = \Gamma_2 \setminus N_2 = \Gamma \setminus N$. Let $\Phi = L_x \circ R_x^{-1} \circ \varphi \in Inn(N) \cdot Stab(\Gamma)$ be an automorphism of $N$ such that $\Phi^{-1}_1(v_1) = v_2$ and $\Phi^*(\langle \cdot, \cdot \rangle_1) = \langle \cdot, \cdot \rangle_2$, where $x \in N$ and $\varphi \in Stab(\Gamma)$. Since $\langle \cdot, \cdot \rangle_1$ and $v_1$ are left-invariant, we have $\Phi^{-1}_1(v_1) = (R_x^{-1} \circ \varphi)_1^{-1}(v_1)$ and $\Phi^*(\langle \cdot, \cdot \rangle_1) = (R_x^{-1} \circ \varphi)^*(\langle \cdot, \cdot \rangle_1)$. The mapping $R_x^{-1} \circ \varphi$ induces a diffeomorphism a $F : \Gamma \setminus N \to \Gamma \setminus N$. $(\Gamma \setminus N, dist_1)$ is isometric to $(\Gamma \setminus N, dist_2)$ via this $F$. \qed

Theorem 8 allows us to write the moduli space $\mathcal{M}(\Gamma \setminus N)$ by a double coset space of a set of matrices. Fix a basis $\{X_i\}_{i=1, \ldots, n}$ of $T_{e\Gamma}(\Gamma \setminus N) \simeq \mathfrak{n}$, and let $\langle \cdot, \cdot \rangle_0$ be the inner product on $\mathfrak{n}$ such that the basis $\{X_1, \ldots, X_n\}$ is orthonormal.

Define $A_k$ to be the family of linear transformations $A : \mathfrak{n} \to \mathfrak{n}$ of corank $k$ with the following two conditions:

1. There are $i_1, \ldots, i_k \in \{1, \ldots, n\}$ dependent on the choice of $A$ such that $KerA = \text{Span} \{X_{i_1}, \ldots, X_{i_k}\}$.

2. The subspace $ImA \subset \mathfrak{n}$ is bracket generating.

It is easy to see that $A_k$ is non-empty only if $k \geq \dim \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$. We sometimes identify a linear map $A \in A_k$ to a matrix in the basis $\{X_1, \ldots, X_n\}$.

From a linear transformation $A$, we obtain the inner product $\langle \cdot, \cdot \rangle_A$ on $ImA$ such that its orthonormal basis is $\{AX_1, \ldots, AX_n\}$. The pair $(ImA, \langle \cdot, \cdot \rangle_A)$ determines a left invariant sub-Riemannian structure of corank $k$ on $\Gamma \setminus N$.

Two maps $A, B \in A_k$ determine the same distribution if and only if $ImA = ImB$. Moreover $A$ and $B$ determine the same inner product if and only if there is $R \in O(n)$ such that $BR = A$. Indeed, if $\langle \cdot, \cdot \rangle_A = \langle \cdot, \cdot \rangle_B$, then for $u, v \in (KerA)^\perp$

$$\langle u, v \rangle_0 = \langle Au, Av \rangle_A = \langle Au, Av \rangle_B = \langle (B|_{KerB^\perp})^{-1}Au, (B|_{KerB^\perp})^{-1}Av \rangle_0.$$
Here the orthogonal complement is chosen via the inner product $⟨·, ·⟩_0$. Thus we have the orthogonal matrix $R$ such that

$$R|_{(\ker A)^⊥} = (B|_{(\ker B)^⊥})^{-1}A|_{(\ker A)^⊥}, \quad R(\ker A) = \ker B.$$  

This implies the equality $BR = A$. The converse implication trivially follows.

The above argument shows that the set of equivalence classes $\bigcup_{k=n/(n,n)}^{\mathbb{A}_k/O(n)}$ has one-to-one correspondence to the space of sub-Riemannian metrics on $\Gamma \setminus N$.

By using Theorem 8 we can classify isometry classes as follows.

**Theorem 10.** The moduli space of left invariant sub-Riemannian metrics on $\Gamma \setminus N$ is parametrized by

$$\mathcal{M}(\Gamma \setminus N) = (\text{Inn}(N) \cdot \text{Stab}(\Gamma)) \setminus \bigcup_{A_1} \mathbb{A}_k/O(n).$$

**Remark 3.** The cometric tensor associated to $A$ is $A^tA$ in the basis $\{X_1, \ldots, X_n\}$.

**Definition 10** (The canonical quotient topology). We call the topology induced from the quotient map $\mathbb{R}^{n^2} \supset \bigcup_{A_1} \mathbb{A}_k \to \mathcal{M}(\Gamma \setminus N)$ the canonical quotient topology.

### 3.2 The moduli space of compact Heisenberg manifolds

We give a specific computation of the moduli space of left invariant sub-Riemannian metrics on compact Heisenberg manifolds. Our argument is based on the construction of the moduli space of Riemannian metrics given by Gordon and Wilson in [9].

The Heisenberg Lie group is the $2n + 1$-dimensional Lie group given by

$$H_n = \{(x, z) \in \mathbb{C} \times \mathbb{R} \mid (x_1, z_1)(x_2, z_2) = (x_1 + x_2, z_1 + z_2 + \frac{1}{2} \text{Im}(x_1 \cdot x_2))\},$$

where the dot is the Hermitian product on $\mathbb{C}^n$. Let $\mathfrak{h}_n$ be the associated Lie algebra, and $\Gamma$ a lattice in $H_n$. For a parametrization $(x, z) = (x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ of $H_n \simeq \mathbb{C}^n \times \mathbb{R}$, we fix the associated basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$ of $\mathfrak{h}_n$ by

$$X_i(e) = \frac{\partial}{\partial x_i}, \quad Y_i(e) = \frac{\partial}{\partial y_i}, \quad Z(e) = \frac{\partial}{\partial z}.$$  

A straightforward computation shows that $[X_i, Y_i] = Z$ for all $i = 1, \ldots, n$ and the other brackets are zero.

Let $\exp : \mathfrak{h}_n \to H_n$ be the exponential map. It is well known that the exponential map is a diffeomorphism. The Campbell–Baker–Hausdorff formula asserts

$$\exp(U) \cdot \exp(V) = \exp(U + V + \frac{1}{2}[U, V]), \quad U, V \in \mathfrak{h}_n.$$

In particular we obtain

$$[\exp(U), \exp(V)] = \exp([U, V]),$$

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where the bracket in the left hand side is the commutator of the Lie group $H_n$. Hence we can identify the Heisenberg Lie group $H_n$ (resp. the commutator $\{\cdot,\cdot\}$) to its Lie algebra $h_n$ (resp. the Lie bracket $\{\cdot,\cdot\}$) via the exponential map.

We will identify a linear transformation $A \in \mathcal{A}_0 \cup \mathcal{A}_1$ to a matrix in the basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$. A matrix $A \in \mathcal{A}_0 \cup \mathcal{A}_1$ has the following canonical representative up to the left action of $\text{Inn}(H_n)_*$ and the right action of $O(2n+1)$. We call such a representative a matrix of weak canonical form.

**Lemma 2.** For $A \in \mathcal{A}_0 \cup \mathcal{A}_1$, there is $R \in O(2n+1)$ and $P \in \text{Inn}(H_n)_*$ such that

$$PAR = \begin{pmatrix} \tilde{A} & O \\ O & \rho_A \end{pmatrix},$$

where $\tilde{A}$ is a $2n \times 2n$ invertible matrix and $\rho_A \in \mathbb{R}$.

Moreover, let $P' \in \text{Inn}(H_n)_*$ and $R' \in O(2n+1)$ be other matrices such that (5) hold. Then they are unique in the following sense.

- There is an $(2n \times 2n)$-orthogonal matrix $\tilde{R}$ such that $R' = R \begin{pmatrix} \tilde{\tilde{R}} & O \\ O & \pm 1 \end{pmatrix}$,
- $P' = P$.

**Proof.** By multiplying an appropriate orthogonal group $R \in O(2n+1)$ from the right, we can assume that the matrix $AR$ maps $[h_n, h_n]$ onto itself. Then we can write the matrix $AR$ as

$$AR = \begin{pmatrix} \tilde{A} & O \\ \tilde{a} & \rho_A \end{pmatrix}$$

with an invertible matrix $\tilde{A} \in GL_{2n}(\mathbb{R})$, a vector $\tilde{a} \in \mathbb{R}^{2n}$ and $\rho_A \in \mathbb{R}$.

For $g = (x_1, \ldots, x_n, y_1, \ldots, y_n, z) \in H_n$, the matrix representation of the differential of the inner automorphism $P_g = (i_g)_*$ is written by

$$P_g = \begin{pmatrix} I_{2n} & O \\ \hat{g} & 1 \end{pmatrix},$$

where $\hat{g} = (-y_1, \ldots, -y_n, x_1, \ldots, x_n)$ and $I_{2n}$ is the identity matrix of rank $2n$.

With this terminology, we can write the matrix $P_gAR$ as

$$P_gAR = \begin{pmatrix} \tilde{A} & O \\ \tilde{a} + \hat{g} \tilde{A} & \rho_A \end{pmatrix},$$

Since $\tilde{A}$ is invertible, we can take a unique $\hat{g}$ such that $\tilde{a} + \hat{g} \tilde{A} = 0$.

Next we will prove the uniqueness of such matrices. Let $R' \in O(2n+1)$ be another orthogonal matrix such that

$$AR' = \begin{pmatrix} \tilde{A}' & O \\ \tilde{a}' & \rho_A' \end{pmatrix}.$$
We will write matrices $A$, $R$ and $R'$ as collections of vectors by

$$A = \begin{pmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_{2n+1} \end{pmatrix}, \quad R = (\vec{r}_1 \cdots \vec{r}_{2n+1}), \quad R' = (\vec{r}'_1 \cdots \vec{r}'_{2n+1}).$$

From the equality (6) and (8), we have

$$\begin{cases} \vec{a}_i \cdot \vec{r}_{2n+1} = 0 & \text{for } 1 \leq i \leq 2n, \\ \vec{a}_{2n+1} \cdot \vec{r}_{2n+1} = \rho_A, \\ \vec{a}_{2n+1} \cdot \vec{r}'_{2n+1} = \rho'_A. \end{cases}$$

The first equality implies that the two vectors $\vec{r}_{2n+1}$ and $\vec{r}'_{2n+1}$ are unit vector orthogonal to $\text{Span}\{\vec{a}_1, \ldots, \vec{a}_{2n}\}$. Hence we have $\vec{r}_{2n+1} = \pm \vec{r}'_{2n+1}$ and $\rho_A = \pm \rho'_A$.

Moreover, since $\{\vec{r}_1, \ldots, \vec{r}_{2n}\}$ and $\{\vec{r}'_1, \ldots, \vec{r}'_{2n}\}$ are orthonormal bases of the orthogonal complement of $\vec{r}_{2n+1}$, there is $\hat{R} \in O(2n)$ such that

$$'RR' = \begin{pmatrix} \hat{R} & O \\ O & \pm 1 \end{pmatrix}.$$ 

This shows the uniqueness of orthogonal matrices $R$.

We pass to the uniqueness of the inner product $P_g$. Let $g'$ be another element in $H_n$ and $R'$ the orthogonal matrix such that (8) holds. In the same way to (7), the matrix representation of $P_g'AR'$ is

$$P_g'AR' = \begin{pmatrix} \tilde{A} \hat{R} \\ \tilde{a} + g' \tilde{A} \end{pmatrix} \begin{pmatrix} \hat{R} & O \\ O & \pm \rho_A \end{pmatrix}.$$

Since $\tilde{A}$ and $\hat{R}$ are invertible, the matrix $P_g'AR'$ is weak canonical if and only if $g = g'$.

Next we compute the group $\text{Stab}(\Gamma)$. We recall the classification of isomorphism classes of lattices. Let $D_n$ be the set of $n$-tuples of positive integers $r = (r_1, \ldots, r_n)$ such that $r_i$ divides $r_{i+1}$ for all $i = 1, \ldots, n$. For $r \in D_n$, let $\Gamma_r < H_n$ be a subgroup defined by

$$\Gamma_r = \langle r_1 X_1, \ldots, r_n X_n, Y_1, \ldots, Y_n, Z \rangle.$$

**Theorem 11** (Theorem 2.4 in [9]). Any lattice $\Gamma < H_n$ is isomorphic to $\Gamma_r$ for some $r \in D_n$.

Moreover, $\Gamma_r$ is isomorphic to $\Gamma_s$ if and only if $r = s$. 

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We compute $Stab(\Gamma_r)$ for a fixed $n$-tuple $r \in D_n$. Let 
$$diag(r) = diag(r_1, \ldots, r_n, 1, \ldots, 1)$$
be the diagonal $2n \times 2n$ matrix, and define the skew-symmetric matrix 
$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$ 
Let $\tilde{Sp}(2n, \mathbb{R})$ be the union of symplectic and anti-symplectic matrices 
$$\tilde{Sp}(2n, \mathbb{R}) = \{ \beta \in GL_{2n}(\mathbb{R}) \mid \beta J_n \beta = \epsilon(\beta) J_n, \ \epsilon(\beta) = \pm 1 \}.$$ 
We embed $\tilde{Sp}(2n, \mathbb{R})$ into $GL_{2n+1}(\mathbb{R})$ via the mapping $\iota: \beta \mapsto (\beta \ 0 \ 0 \ 0 \ \epsilon(\beta)).$

With these notations, we give representations of $Stab(\Gamma_r)$ as follows.

**Theorem 12** (Theorem 2.7 in [9]). The matrix representation of $Stab(\Gamma_r)$ is given by 
$$\Pi_r = \iota (G_r \cap \tilde{Sp}(2n, \mathbb{R})), $$ 
where $G_r = diag(r)GL_{2n}(\mathbb{Z})diag(r)^{-1}$.

We obtain an explicit form of $M(\Gamma_r \setminus H_n)$ as follows.

**Theorem 13.** The moduli space $M(\Gamma_r \setminus H_n)$ has one-to-one correspondence 
$$M(\Gamma_r \setminus H_n) = \Pi_r \setminus (GL_{2n}(\mathbb{R}) \times \mathbb{R}) / (O(2n) \times \{\pm 1\}).$$

Here the action on the second factor only changes its sign. Moreover, any matrices acting upon the first factor has determinant $\pm 1$. These two facts show the following.

**Lemma 3.** For matrices $A, B \in A_0 \cup A_1$ with $[A] = [B] \in M(\Gamma_r \setminus H_n)$, we have 
- $|\det(A)| = |\det(B)|,$
- $|\rho_A| = |\rho_B|.$

**Remark 4.** The double coset space $G_r \setminus GL_{2n}(\mathbb{R})/O(2n)$ is homeomorphic to $GL_{2n}(\mathbb{Z})\setminus GL_{2n}(\mathbb{R})/O(2n)$, the moduli space of flat tori of dimension $2n$. However, the discrete subgroup $G_r \cap \tilde{Sp}(2n, \mathbb{R}) < G_r$ is infinite index for $n \geq 2$. Hence the classical Mahler’s compactness theorem does not directly imply the precompactness of subsets in $M(\Gamma_r \setminus H_n)$. Boldt’s work in [7] is to fulfill this gap by using the maximal eigenvalue of the $j$-operator defined in the next section.

### 4 Curvature, geodesics and volume form on the Heisenberg Lie group

In this section, we recall the Ricci curvature, geodesics and the Popp’s volume form on the Heisenberg Lie groups.
4.1 \(j\)-operator

Let \(v_0 \subset h_n\) be the subspace spanned by the \(2n\) vectors \(X_1, \ldots, X_n, Y_1, \ldots, Y_n\), and \(\omega\) the contact form on \(H_n\) such that \(\text{Ker}\omega = v_0\) and \(\omega(Z) = 1\). In other words, the contact form \(\omega\) is the dual covector of \(Z\), denoted by \(Z^*\). We sometimes write \(X_{n+i} = Y_i\) for abuse of notation. The following \(j\)-operator plays a central role in the study of the Heisenberg Lie group.

**Definition 11 (\(j\)-operator).** For a matrix \(A \in A_0 \cup A_1\) of weak canonical form, define the operator \(j_A : v_0 \to v_0\) by

\[
\langle j_A(U), V \rangle_A = -d\omega(U, V) = \omega([U, V]) \quad \text{for all } U, V \in v_0.
\]

The \(j\)-operator was used to describe the curvature tensor (8), the cut time of geodesics (11 and 18), and precompact subsets in the moduli space (7).

**Remark 5.** Our definition of the operator \(j_A\) is a special case of the original definition by Kaplan (13). For \(A \in A_0\), he defined the linear mapping \(j'_A : \mathfrak{h}_n \to \mathfrak{h}_n\) by

\[
\langle j'_A(W)U, V \rangle = \langle W, [U, V] \rangle_A.
\]

In this setting, our \(j\)-operator is written by \(j_A = j'_A(\rho_A^2 Z)\).

The following matrix representation of \(j_A\) is useful for later arguments.

**Lemma 4.** The matrix representation of \(j_A\) is \(A^t J_n A\) in the basis \(\{AX_1, \ldots, AX_{2n}\}\).

**Proof.** For \(i, j = 1, \ldots, 2n\), the \((i, j)\)-th element of the matrix \(A^t J_n A\) is

\[
(a_{i1} \cdots a_{i2n}) (a_{1j} \cdots a_{2nj}) = \sum_{k=1}^{2n} a_{ki}a_{k+nj} - a_{k+nj}a_{k+i}.
\]

It coincides with the \((i, j)\)-th element of the matrix representation of \(j_A\). Indeed for \(i, j = 1, \ldots, 2n\),

\[
\langle j_A(AX_i), AX_j \rangle = \omega([AX_i, AX_j])
\]

\[
= \omega \left( \sum_{k=1}^{n} a_k X_k + a_{k+nj} Y_k, \sum_{k=1}^{n} a_{k+i} X_k + a_{k+nj} Y_k \right)
\]

\[
= \sum_{k=1}^{n} a_{ki}a_{k+nj} - a_{k+nj}a_{k+i}.
\]

Since the operator \(j_A\) is skew symmetrizable, we can choose the representatives in the moduli space \(M(\Gamma \setminus H_n)\) so that the matrix representation of the \(j_A\) is simple.

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Definition 12. For a matrix $A \in A_0 \cup A_1$, define the positive numbers $d_1(A), \ldots, d_n(A)$ as follows.

- $0 < d_1(A) \leq \cdots \leq d_n(A),$
- $\pm \sqrt{-1}d_1(A), \ldots, \pm \sqrt{-1}d_n(A)$ are the eigenvalue of the $j_A$.

From the fundamental fact of matrix theory, there is an orthogonal matrix $R$ such that
\[
^tR^tJ_n\hat{A}R = \begin{pmatrix} O & \diag(d_1(A), \ldots, d_n(A)) \\ -\diag(d_1(A), \ldots, d_n(A)) & O \end{pmatrix},
\] (10)

Lemma 5. For matrices $A, B \in A_0 \cup A_1$ of weak canonical form with $[A] = [B]$ in $\mathcal{M}(\Gamma \setminus H_n)$, we have $d_i(A) = d_i(B)$ for all $i = 1, \ldots, n$.

Proof. From the construction of the moduli space, there is a matrix $P \in G_r \cap \tilde{Sp}(2n, \mathbb{R})$ and $R \in O(2n)$ such that $P\hat{A}R = \hat{B}$. Then we have
\[
^t\hat{B}J_n\hat{B} = ^tR^t\hat{A}^tPJ_nP\hat{A}R = \pm ^tR^t\hat{A}J_n\hat{A}R.
\]
This implies the coincidence of the eigenvalues $\pm \sqrt{-1}d_i$'s. □

Definition 13. We say that a matrix $A \in A_0 \cup A_1$ is of canonical form if it is weak canonical and
\[
^t\hat{A}J_n\hat{A} = \begin{pmatrix} O & \diag(d_1(A), \ldots, d_n(A)) \\ -\diag(d_1(A), \ldots, d_n(A)) & O \end{pmatrix}.
\]

Remark 6. The number $d_i(A)$ appeared in Boldt’s paper [7] as the eigenvalue of the matrix $^t\hat{A}J_n\hat{A}$. This matrix is similar to our matrix representation $\hat{A}J_n\hat{A}$, thus their definition coincides with ours.

We introduce another numerical data $\delta(A)$ to give an explicit form of the Popp’s volume form.

Definition 14. For a matrix $A \in A_0 \cup A_1$, denote by $\delta(A)$ the Hilbert–Schmidt norm of the matrix $^t\hat{A}J_n\hat{A}$.

Lemma 6. For two matrices $A, B \in A_0 \cup A_1$ of weak canonical form with $[A] = [B]$, we have $\delta(A) = \delta(B)$.

Proof. From the assumption, there are $P \in G_r \cap \tilde{Sp}(2n, \mathbb{R})$ and $R \in O(2n)$ such that $P\hat{A}R = \hat{B}$. Then we have
\[
\delta(B) = \| ^tR^t\hat{A}^tPJ_nP\hat{A}R \|_{HS}
= \| \pm ^t\hat{A}J_n\hat{A} \|_{HS}
= \delta(A),
\]
where we denote by $\| \cdot \|_{HS}$ the Hilbert–Schmidt norm of matrices. □
4.2 Ricci curvature

We recall an explicit formula of the Ricci curvature of the Heisenberg Lie group.

For a matrix $A \in \mathcal{A}_0$ of canonical form, let $Ric_A : \mathfrak{h}_n \otimes \mathfrak{h}_n \to \mathbb{R}$ be the Ricci tensor associated to the left invariant Riemannian metric $\langle \cdot, \cdot \rangle_A$.

**Proposition 4** (Special case of Proposition 2.5 in [8]). The Ricci tensor associated to the metric $\langle \cdot, \cdot \rangle_A$ is written by:

(a) $Ric_A(U, V) = 0$ for all $U \in \mathfrak{v}_0$ and $V \in [\mathfrak{h}_n, \mathfrak{h}_n]$,

(b) $Ric_A(U, V) = \left(\frac{1}{2\rho_A^2} j^2_A U, V\right)_A$ for all $U, V \in \mathfrak{v}_0$,

(c) $Ric_A(uZ, vZ) = -uv/4\rho_A^4 \operatorname{Tr}(j^2_A)$ for all $u, v \in \mathbb{R}$.

Let $S_A \subset \mathfrak{h}_n$ be the unit sphere associated to the inner product $\langle \cdot, \cdot \rangle_A$. For a unit vector $U \in S_A$, we write $Ric_A(U) = Ric_A(U, U)$. The function $Ric_A : S_A \to \mathbb{R}$ is called the Ricci curvature. Combined with the matrix representation of the operator $j_A$, we obtain the following lemma.

**Lemma 7.** For a matrix $A \in \mathcal{A}_0$ of canonical form, we have

$$
\begin{cases}
Ric_A(AX_i) = -\frac{1}{2\rho_A^2} d_i(A)^2 & \text{for all } i = 1, \ldots, 2n \\
Ric_A(AZ) = \frac{1}{2\rho_A^2} \sum_{k=1}^{n} d_k(A)^2.
\end{cases}
$$

**Proof.** By Lemma 4, the matrix representation of the operator $j^2_A$ is

$$\text{diag}(-d_1(A)^2, -d_2(A)^2, \ldots, -d_n(A)^2, -d_1(A)^2, \ldots, -d_n(A)^2)$$

in the basis $\{AX_1, \ldots, AX_{2n}\}$. Hence Proposition 4 and Remark 5 show that

$$Ric_A(AX_i) = \left(\frac{1}{2\rho_A^2} j^2_A AX_i, AX_i\right)_A = \left(-\frac{1}{2\rho_A^2} \sum_{k=1}^{n} d_k(A)^2(AX_k + AY_k), AX_i\right)_A = -\frac{1}{2\rho_A^2} d_i(A)^2.$$

for $i = 1, \ldots, 2n$. We also obtain $Ric_A(AZ)$ by a straightforward adoption of Proposition 4.

4.3 Geodesics

Let $A \in \mathcal{A}_0 \cup \mathcal{A}_1$ be a matrix of canonical form. For $i = 1, \ldots, n$, define the functions $h_{x_i}$ (resp. $h_{y_i}$ and $h_{z_i}$): $\mathcal{T}^* H_n \to \mathbb{R}$ by $h_{x_i}(p) = p(AX_i(x))$ (resp. $h_{y_i}(p) = p(AY_i(x))$ and $h_{z_i}(p) = p(Z(x))$) for $p \in \mathcal{T}^* H_n$. Suppose that an admissible path $\gamma : [0, T] \to H_n$ parametrized by $\gamma(t) = \exp(\sum_{i=1}^{n} x_i(t) AX_i +$
Proof. Let \( c \) the projection, and moreover the equality holds if and only if \( V \in \text{dist}_A \) (Proposition 3.11 in [8] for Riemannian case).

We will denote by \( \ell \) the lift of \( \ell \) for sub-Riemannian case) (Proposition 3.5 in [8] for Riemannian case and Lemma 14 in [18].

Set \( \gamma : [0, T] \rightarrow H_n \) be the geodesic issuing from the identity with the initial data of the extremal

\[
(h_{x_1}(0), \ldots, h_{y_1}(0), \ldots, h_z(0)) = (p_{x_1}, \ldots, p_{y_1}, \ldots, p_z).
\]

Set \( \xi_i = p_z d_i(A) \). Then \( \gamma \) is parametrized as follows.

- \( \text{If } p_z \neq 0, \text{ then } \)
  \[
  \left( \begin{array}{c}
  x_i(t) \\
  y_i(t)
  \end{array} \right) = \frac{1}{\xi_i} \left( \begin{array}{cc}
  \sin(\xi_i t) & \cos(\xi_i t) - 1 \\
  -\cos(\xi_i t) + 1 & \sin(\xi_i t)
  \end{array} \right) \left( \begin{array}{c}
  p_{x_i} \\
  p_{y_i}
  \end{array} \right),
  \]
  for each \( i = 1, \ldots, n \), and

  \[
  z(t) = \rho_A^2 p_z t + \frac{1}{2} \sum_{i=1}^n \left( \frac{\lambda_i t}{\xi_i} - \frac{\lambda_i}{\xi_i^2} \sin(\xi_i t) \right) \left( p_{x_i}^2 + p_{y_i}^2 \right).
  \]

- \( \text{If } p_z = 0, \text{ then } x_i(t) = p_{x_i} t, y_i(t) = p_{y_i} t \text{ and } z(t) \equiv 0. \)

For later arguments, we give an explicit distance from the identity to points in the horizontal direction and the vertical direction. Denote by \( \text{dist}_A \) the distance function on \( H_n \) associated to \( A \in A_0 \cup A_1 \).

**Lemma 9** (Proposition 3.11 in [8] for Riemannian case). For \( U \in \mathfrak{v}_0 \) and \( V \in [\mathfrak{h}_n, \mathfrak{h}_n] \), we have

\[
\overline{\text{dist}}_A(e, \exp(U + V)) \geq \|U\|_A.
\]

Moreover, the equality holds if and only if \( V = 0 \).

**Proof.** Let \( P : H_n \rightarrow \mathfrak{h}_n \rightarrow \mathfrak{v}_0 \) be the composition of the logarithm map and the projection, and \( c \) a length minimizing path from \( e \) to \( \exp(U, V) \). Then the composition \( P \circ c \) is a path in \( \mathfrak{v}_0 \) from 0 to \( U \). Moreover the length of \( c \) in
\((H_n, \im A, \langle \cdot, \cdot \rangle_A)\) is greater than or equal to that of \(P(c)\) in \((\nu_0, \langle \cdot, \cdot \rangle_A)\), thus we obtain the desired inequality.

The equality holds only when

\[
\text{length}(c) = \text{length}(P \circ c) = \|U\|_A.
\]  

(11)

From the uniqueness of geodesics in the inner product space \((\nu_0, \langle \cdot, \cdot \rangle_A)\), the second equality in (11) holds if and only if \(V = 0\). The first equality automatically follows provided \(V = 0\).

Lemma 10. For \(p \in \mathbb{R}\), the distance from \(e\) to \(\exp(pZ)\) is given as follows.

1. If \(|p| \leq \frac{2\pi \rho_A^2}{d_n(A)}\), then
   \[
   \tilde{\text{dist}}_A(e, \exp(pZ)) = \left| \frac{p}{\rho_A} \right|
   \]

2. If \(|p| \geq \frac{2\pi \rho_A^2}{d_n(A)}\), then
   \[
   \tilde{\text{dist}}_A(e, \exp(pZ)) = \frac{2}{d_n(A)} \sqrt{|p| \pi d_n(A) - \pi^2 \rho_A^2}.
   \]

Proof. From the symmetricity we can assume that \(p \geq 0\) and \(\rho_A \geq 0\). When \(\rho_A \neq 0\), the path \(\exp(t \rho_A Z)\) is a unit speed geodesic. It reaches the point \(\exp(pZ)\) at the time \(\frac{2\pi}{\rho_A}\).

Next we consider length minimizing geodesics of the initial data \((p_x, p_z)\) with \(p_x = (p_1, \ldots, p_{2n}) \neq (0, \ldots, 0)\). Such geodesics are length minimizing until the time \(\frac{2\pi}{\rho_A}\). It is a consequence of Theorem 2.9 in [11] for Riemannian case and Lemma 16 in [18] for sub-Riemannian case. Since the endpoint \(\exp(pZ)\) is in \([H_n, H_n]\), we obtain

\[
p_1 = \cdots = p_{m-1} = p_{n+1} = \cdots = p_{m-1+n} = 0,
\]

where \(m\) is the least integer such that \(d_m(A) = d_n(A)\). The endpoint condition and the initial data of the hamiltonian function \(h_1, \ldots, h_z\) of the geodesic give us the system

\[
\begin{cases}
\frac{p}{z} = \frac{2\pi \rho_A^2}{d_n(A)} + \frac{2\pi}{p^2 d_n(A)} |p_x|^2, \\
|p_x|^2 + \left(\frac{p_z}{\rho_A}\right)^2 = 1
\end{cases}
\]

(12)

This system has a solution only if \(p \geq \frac{2\pi \rho_A^2}{d_n(A)}\). This implies that if \(p < \frac{2\pi \rho_A^2}{d_n(A)}\), there is no geodesic from \(e\) to \(\exp(pZ)\) with the initial data \(p_x \neq 0\).

Conversely, assume that \(p\) is not less than \(\frac{2\pi \rho_A^2}{d_n(A)}\). Then we have the solution to (12), and its geodesic has length \(2\pi / \rho_A\). On the other hand, we have another straight segment \(\exp(t \rho_A Z)\) with length \(\frac{p}{\rho_A}\). It is easy to see that \(\frac{p}{\rho_A} \leq \frac{2\pi}{\rho_A}\) if and only if \(p = \frac{2\pi \rho_A^2}{d_n(A)}\). This shows the lemma.
4.4 Volume forms

In this section, we recall an explicit formula of the canonical Riemannian volume form and the Popp’s volume form on the Heisenberg Lie group.

First we consider a left invariant Riemannian metrics and its canonical Riemannian volume form. Let $A \in A_0$ be a matrix of canonical form. Denote by $v(h_n, A)$ the canonical Riemannian volume form. Since the Riemannian volume form is the wedge of the dual coframe of an orthonormal frame, we have

$$v(h_n, A) = (AX_1)^* \wedge \cdots \wedge (AX_{2n})^* \wedge (AZ)^*$$  \hspace{1cm} (13)

$$= \det A^{-1} X_1^* \wedge \cdots \wedge X_{2n}^* \wedge Z^*$$  \hspace{1cm} (14)

$$= (\det \tilde{A})^{-1} X_1^* \wedge \cdots \wedge X_{2n}^* \wedge Z^*.$$  \hspace{1cm} (15)

Next let us consider the Popp’s volume form for left invariant properly sub-Riemannian metrics. Let $A \in A_1$ be a matrix. For a later argument, we only assume that the kernel of $A$ is $[h_n, h_n]$. Then $A$ has a matrix representation

$$A = \begin{pmatrix} \tilde{A} & 0 \\ \hat{a} & 0 \end{pmatrix}.$$  \hspace{1cm}

Denote by $v(\text{Im}A, A)$ the Popp’s volume associated to the sub-Riemannian structure $(\text{Im}A, A)$. Note that if a matrix is weak canonical, then $\text{Im}A = v_0$.

Since $\tilde{A}J_n \tilde{A}$ is the matrix representation of $j_A$ in the basis $\{AX_1, \ldots, AX_{2n}\}$, its $(i, j)$-th entry coincides with the structure constant $c_{ij} = \omega([AX_i, AX_j])$. By Theorem 7, the Popp’s volume $v(\text{Im}A, A)$ is written by

$$v(\text{Im}A, A) = \delta(A)^{-1} (AX_1)^* \wedge \cdots \wedge (AX_{2n})^* \wedge Z^*$$  \hspace{1cm} (16)

$$= (\det \tilde{A})^{-1} \delta(A)^{-1} X_1^* \wedge \cdots \wedge X_{2n}^* \wedge Z^*.$$  \hspace{1cm} (17)

As we can see in the following example, there is a gap between the Riemannian volume form and the Popp’s volume form.

**Example 3.** Let $\Gamma_1 < H_1$ be the lattice generated by $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$. Let $\{A_k\}$ be a sequence of $3 \times 3$-matrices given by

$$A_k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{k} \end{pmatrix},$$

and $\langle \cdot, \cdot \rangle_{A_k}$ the associated inner product on $\mathfrak{h}_1$. The Gromov–Hausdorff limit of the sequence $\{([\Gamma_1 \setminus H_1, \langle \cdot, \cdot \rangle_{A_k})\}$ is isometric to a sub-Riemannian space $(\Gamma_1 \setminus H_1, v_0, \langle \cdot, \cdot \rangle_{A_\infty})$, where

$$A_\infty = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm}

It is easily to see that $j_{A_k} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $d_{\infty}(A_k) = 1$. By Lemma 7 the Ricci curvature diverges to $\pm \infty$.  \hspace{1cm} (22)
By the formula (15), the limit of the total measure with respect to the canonical Riemannian volume form is
\[ \left| \int_{\Gamma_1 \setminus H_1} v(h_1, A_k) \right| = |\det(A_k)|^{-1} = k \to \infty \quad (k \to \infty). \]

On the other hand, by the formula (17), the total measure of the limit space \((\Gamma_1 \setminus H_1, v_0, \langle \cdot, \cdot \rangle_{A_\infty})\) with respect to the Popp’s volume is
\[ \left| \int_{\Gamma_1 \setminus H_1} v(v_0, A_\infty) \right| = \frac{1}{\sqrt{2}}. \]

5 Minimal Popp’s volume form on the Heisenberg Lie group

In this section, we give an explicit formula of the minimal Popp’s volume on the Heisenberg group. For the sub-Riemannian structure \((v, \langle \cdot, \cdot \rangle_A)\) and \(w \subset v\), we will simply write the Popp’s volume of the restricted sub-Riemannian structure by \(v(w, A)\), and the minimal Popp’s volume by \(v(A)\).

**Proposition 5.** For a matrix \(A \in A_0 \cup A_1\) of canonical form,
\[ v(A) = \min\{|\rho_A|^{-1}, \delta(A)^{-1}\}|\det(\tilde{A})|^{-1}X_1^* \wedge \cdots \wedge Z^*, \]
where we write \(|\rho_A|^{-1} = \infty\) if \(\rho_A = 0\).

**Proof.** We consider the following three cases.

1. \(A \in A_0\) and \(\dim(w) = 2n + 1\),
2. \(A \in A_1\),
3. \(A \in A_0\) and \(\dim(w) = 2n\).

In case (1), a sub-Riemannian structure is uniquely determined by \((h_n, A)\). Hence the minimum is
\[ v(h_n, A) = |\det(\tilde{A})|^{-1}X_1^* \wedge \cdots \wedge Z^*. \]

In case (2), a sub-Riemannian structure is uniquely determined by \((v_0, A)\). Hence the minimum is
\[ v(v_0, A) = |\det(\tilde{A})|^{-1}X_1^* \wedge \cdots \wedge Z^*. \]

In case (3), there are uncountably many sub-Riemannian structures, thus we need to calculate their minimum.

Recall that the mapping \(P : h_n \to v_0\) is the projection. Since the subspace \(w\) is bracket generating, the restriction of the projection \(P|_w\) is a linear isomorphism from \(w\) to \(v_0\). Thus we have unique \(t_i\)'s in \(\mathbb{R}\) such that
(P_{w})^{-1}(AX_i) = AX_i + t_i Z. Since each AX_i is orthogonal to Z, the subset \( \left\{ \frac{AX_i + t_i Z}{\sqrt{1 + t_i^2}}, \ldots, \frac{AX_n + t_n Z}{\sqrt{1 + t_n^2}} \right\} \) becomes an orthonormal basis of \( (v, \langle \cdot, \cdot \rangle_{A\mid w}) \).

Since \([\mathfrak{h}_n, \mathfrak{h}_n]\) is the center of \(\mathfrak{h}_n\), we have

\[
\begin{bmatrix}
AX_i + t_i Z \\
\sqrt{1 + t_i^2}
\end{bmatrix}
\begin{bmatrix}
AX_j + t_j Z \\
\sqrt{1 + t_j^2}
\end{bmatrix} = c_{ij} Z.
\]

Combined with Theorem 7, the Popp’s volume form \(v(w, A)\) is written by

\[
v(w, A) = \left( \sum_{1 \leq i, j \leq 2n} \frac{c_{ij}^2}{(1 + t_i^2)(1 + t_j^2)} \right)^{\frac{1}{4}} \prod_{k=1}^{2n} \sqrt{1 + t_k^2} X^*_1 \wedge \cdots \wedge Z^*.
\]

For a fixed matrix \(A\), the Popp’s volume \(v(w, A)\) attains the minimum if and only if \(t_1 = \cdots = t_{2n} = 0\), that is, \(w = v_0\). By (17), the Popp’s volume form \(v(v_0, A)\) is written by

\[
v(v_0, A) = \delta(A)^{-1} \det A^{-1} X^*_1 \wedge \cdots \wedge Z^*.
\]

The above three cases show the conclusion.

It is easily checked that if a sequence of Riemannian metrics converges to sub-Riemannian one, then the canonical Riemannian volume diverges. However, the following proposition shows that the minimal Popp’s volume form is continuous under the canonical topology of the moduli space.

**Proposition 6.** Let \(\{[A_k]\} \subset \mathcal{M}(\Gamma_r \setminus H_n)\) be a sequence of metrics converging to \([A_0] \in \mathcal{M}(\Gamma_r \setminus H_n)\). Then there is measurable maps \(\phi_i : \Gamma_r \setminus H_n \to \Gamma_r \setminus H_n\) such that the push forward of the minimal Popp’s volume forms \(\phi^*(v(A_k))\) converges to \(v(A_0)\).

**Proof.** We can assume each \(A_k\) is of canonical form, the sequence \(\{\tilde{A}_k\}\) converges to \(\tilde{A}_0\), and the sequence \(\{\rho_{A_k}\}\) converges to \(\rho_{A_0}\). Since \(\tilde{A}_k\) converges to \(\tilde{A}_0\), we also have the continuity of \(\delta : \mathcal{M}(\Gamma_r \setminus H_n) \to \mathbb{R}\).

From the definition of the minimal Popp’s volume, it is trivial to see that \(v(A_k)\) converges to \(v(A_0)\). These canonical form of matrices are given by applying isomorphisms in \(\text{Inn}(H_n) \cdot \text{Stab}(\Gamma_r)\), hence we can pick measurable maps \(\phi_i\) from those isomorphisms.

The following example explains why we need to introduce the minimal Popp’s volume to characterize non-collapsed limits of compact Heisenberg manifolds.
Example 4. Let $\Gamma_1 < H_1$ be the lattice generated by $(\pm 1,0,0)$, $(0,\pm 1,0)$ and $(0,0,\pm 1)$. Let $\{B_k\}$ be a sequence of $3 \times 3$-matrices given by

$$B_k = \begin{pmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{k} \end{pmatrix},$$

and $g_k$ the associated Riemannian metrics on $\Gamma_1 \backslash H_1$. Then the sequence of compact Heisenberg manifolds $(\Gamma_1 \backslash H_1, g_k)$ converges to the flat torus $(S^1, g_{can})$, where $g_{can}$ is the canonical metric on $S^1$. This fact is explained as follows. The compact Heisenberg manifold has the fibration structure

$$S^1 \to \Gamma_1 \backslash H_1 \to T^2,$$

where the fiber $S^1$ has the representative $\{(0,0,t)\}$, the length of the fiber converges to zero as $k \to \infty$. Indeed the matrix representation of the $j$-operators are

$$(k \ 0) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix},$$

and by Lemma [10], the length of the fiber $S^1$ is

$$\tilde{\text{dist}}_{A_k}(e, \exp(Z)) = \frac{2}{k} \left( k\pi - \frac{\pi^2}{k^2} \right) \leq \frac{2\sqrt{\pi}}{\sqrt{k}} \to 0 \quad (k \to \infty).$$

This argument shows that the Gromov–Hausdorff limit of the sequence coincides with the limit of the base flat tori with the quotient metric $(T^2, g_{A_k})$. It is easy to see that the limit of these tori is isometric to $(S^1, g_{can})$, thus we omit its proof.

On the other hand, the total measure of the Riemannian compact Heisenberg manifold $(\Gamma_0 \backslash H_1, g_k)$ is

$$\int_{\Gamma_0 \backslash H_1} v(\eta_1, B_k) = |\det(B_k)|^{-1} = 1 \text{ under the original Popp's volume. These two calculation say that the topological collapse does not imply the volume collapse. However, if we calculate the total measure under the minimal Popp's volume, then}$$

$$\int_{\Gamma_0 \backslash H_1} v(B_k) = \frac{1}{\sqrt{2k^2}} \to 0.$$

6 Precompactness for Riemannian metrics

In this section, we recall Boldt’s condition for subsets in Riemannian moduli space $M_0(\Gamma \backslash H_n)$ being compact, and show that the conditions are equivalent to the bound on the diameter, the canonical Riemannian volume and the Ricci curvature.

The conditions for precompactness are given as follows.

**Theorem 14** (Corollary 3.14 in [7]). A subset $X \subset M_0(\Gamma \backslash H_n)$ is precompact in the canonical topology if and only if there are positive constants $C_1, C_2, C_3, C_-, C_+ > 0$ such that every matrix $A \in A_0$ with $[A] \in X$ satisfies the following four conditions:
(A-1) \( \min \{ \|X\|_A \mid X \in P(\Gamma_r) \} \geq C_1 \),

(A-2) \( |\det(\hat{A})| \geq C_2 \),

(A-3) \( d_n(A) \leq C_3 \),

(A-4) \( C_- \leq |\rho_A| \leq C_+ \).

Remark 7. For a matrix \( A \in \mathcal{A}_0 \) of canonical form, the associated metric tensor is \( 'A^{-1}A = \begin{pmatrix} 'A^{-1}A^{-1} & 0 \\ O & \rho_A^2 \end{pmatrix} \). The conditions (A-1)-(A-4) is obtained from Boldt’s original statement by adapting this matrix presentation.

The goal of this section is to show the following proposition.

Proposition 7. Assume that for a compact Heisenberg manifold \( (\Gamma_r \setminus H_n, g_A) \), there are positive constants \( D, V \) and \( K \) such that \( \text{diam}(\Gamma_r \setminus H_n, g_A) \leq d \), \( \left| \int_{\Gamma_r \setminus H_n} \text{vol}_{g_A} \right| \geq v \) and \( \text{Ric}_{g_A} \geq -K \). Then there are positive constants \( C_1, C_2, C_3, C_- , C_+ > 0 \) such that the conditions (A-1)-(A-4) hold.

We omit the converse statement since its proof duplicate with that of Theorem 14. Before the proof of Proposition 7, we prepare the following fundamental lemma.

Lemma 11. For any matrix \( A \in \mathcal{A}_0 \cup \mathcal{A}_1 \), we have

(1) \( \delta(A) = \sqrt{2 \sum_{i=1}^n d_i(A)^2} \),

(2) \( |\det(\hat{A})| = \prod_{i=1}^n d_i(A) \).

Proof. The first equality follows from the definition of the canonical form and Lemma 6. The second equality follows from

\[ \prod_{i=1}^n d_i(A)^2 = \det('A J_n \hat{A}) = \det(\hat{A})^2. \]

Proof of Proposition 7. First we show the inequality (A-2) with the constant \( C_2 = (4nD)^{-2n} \). Let \( \lambda_1(A), \ldots, \lambda_{2n}(A) \) be the eigenvalues of the metric tensor \( 'A^{-1}A^{-1} \) such that \( \lambda_1(A) \leq \cdots \leq \lambda_{2n}(A) \). We show that the maximal eigenvalue \( \lambda_{2n}(A) \) satisfies

\( \lambda_{2n}(A) \leq (4nD)^2. \) (18)

For \( i = 1, \ldots, n \), let \( \gamma_i = \exp(\frac{1}{2} X_i) \). It is easy to check that the length minimizing geodesic from \( \Gamma_r e \) to \( \Gamma_r \gamma_i^\frac{1}{2} \) is the projection of the geodesic in \( H_n \).
from e to $\gamma_i^{1/2}$. By Lemma 9

$$
\|\frac{\tilde{r}_i}{2}X_i\|_A = \text{dist}_A(e, \gamma_i^{1/2})
$$

$$
= \text{dist}_A(\Gamma_re, \Gamma_r\gamma_i^{1/2})
$$

$$
\leq \text{diam}(\Gamma_r \setminus H_n, \text{dist}_A) \leq D.
$$

Since $r_i \geq 1$,

$$
\|X_i\|_A \leq \frac{2D}{r_i} \leq 2D.
$$

(19)

In the same way, we obtain the inequality

$$
\|X_i+n\| \leq 2D
$$

(20)

for $i = 1, \ldots, n$.

Since $\{X_1, \ldots, X_{2n}\}$ is an orthonormal basis of $(v_0, \langle \cdot, \cdot \rangle_0)$, we can write

$$
X = \sum_i c_i X_i \text{ with } \sqrt{\sum_{i=1}^{2n} c_i^2} = \|X\|_0.
$$

By the triangle inequality, we have

$$
\|\hat{A}^{-1}X\|_0 = \|X\|_A \leq \sum_{i=1}^{2n} |c_i| \|X_i\|_A \leq \|X\|_0 \sum_{i=1}^{2n} 2D = 4nD\|X\|_0.
$$

Hence $\lambda_{2n}(A)$ is less than or equal to $(4nD)^2$ and

$$
|\det(\hat{A})| \geq (\lambda_1(A) \cdots \lambda_{2n}(A))^{-\frac{1}{2}} \geq \lambda_{2n}^{-n} \geq (4nD)^{-2n}.
$$

(21)

This shows the condition (A-2) with $C_2 = (4nD)^{-2n}$.

Next we show the upper bound of the inequality (A-4) with the constant $C_+ = \prod_{i=1}^n r_i (VC_2)^{-1}$. The lower bound of the total measure and the inequality (21) show the following inequality:

$$
V \leq \int_{\Gamma_r \setminus H_n} |\rho_A \det(\hat{A})|^{-1} X_1^* \wedge \cdots \wedge Z^* = \prod_{i=1}^n r_i |\rho_A \det(\hat{A})|^{-1} \leq \prod_{i=1}^n r_i |\rho_A|^{-1} C_2^{-1}.
$$

Hence we obtain

$$
|\rho_A| \leq \prod_{i=1}^n r_i (VC_2)^{-1}.
$$

(22)

Next we show the inequality (A-3) and the other side of (A-4) with the constants $C_3 = \sqrt{2K} C_+$ and $C_- = \frac{\sqrt{2K}}{4\sqrt{2K}}$ respectively.

By Lemma 7 and the lower bound of the Ricci curvature,

$$
\text{Ric}_A(AX_n) = -\frac{1}{2\rho_A^2} d_n(A)^2 \geq -K.
$$

(23)
Combined with the constant $C_+$, we obtain
\[ d_n(A) \leq \sqrt{2K} |\rho_A| \leq \sqrt{2K} C_+. \]
Combined with Lemma 11 and the constant $C_2$, we obtain
\[ |\rho_A| \geq \frac{1}{\sqrt{2K}} d_n(A) \geq \frac{1}{\sqrt{2K}} \sqrt{|\det(\tilde{A})|} \geq \frac{\sqrt{C_2}}{\sqrt{2K}}. \]

Finally we show the inequality (A-1) with the constant $C_1 = C_3^{-n}(4nD)^{-2n+1}$. It holds if $\lambda_1(A) \geq C_3^{-2n}(4nD)^{-4n+2}$, since
\[
\min \{ \|X\|_A \mid X \in P(\Gamma_r) \} \geq \min \{ \|X\|_A \mid \|X\|_0 \leq 1 \} = \sqrt{\lambda_1(A)}.
\]
By Lemma 11 the inequality (18) and (23), we obtain
\[
C_3^{-2n} \leq \prod_{i=1}^n d_i(A)^{-2} = \det(\tilde{A})^{-2} = \det('\tilde{A}'^{-1} \tilde{A}^{-1}) = \lambda_1(A) \cdots \lambda_{2n}(A) \leq \lambda_1(A) \lambda_{2n}(A)^{2n-1} \leq \lambda_1(A)(4nD)^{4n-2}.
\]
This gives the desired inequality. \hfill \Box

### 7 Proof of the main propositions

In this section, we show the two main propositions.

#### 7.1 Finite isomorphism classes of lattices

Recall that every lattice in $H_n$ is isomorphic to $\Gamma_r$ for $r = (r_1, \ldots, r_n) \in D_n$. We show that an upper bound of the diameter and a lower bound of the volume gives the finiteness of $r_n$. It also implies the finiteness of the diffeomorphism type under the conditions (D) and (V), since a homotopy equivalence of compact nilmanifolds gives a diffeomorphism.

**Proposition 8.** Let $(\Gamma_r \backslash H_n, dist_A, v(A))$ be a compact Heisenberg manifold with the sub-Riemannian distance and the minimal Popp’s volume associated to $A \in \mathcal{A}_0 \cup \mathcal{A}_1$. Assume $\text{diam}(\Gamma_r \backslash H_n, dist_A) \leq D$ and $\left| \int_{\Gamma_r \backslash H_n} v(A) \right| \geq V > 0$. Then there is a constant $C = C(n, D, V) > 0$ such that $r_n \leq C$. 

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Proof. A compact Heisenberg manifold \( \Gamma_r \backslash H_n \) has the canonical submersion \( \Gamma_r \backslash H_n \rightarrow P(\Gamma_r) \backslash v_0 \simeq T^{2n} \). From the sub-Riemannian metric \((ImA, \langle \cdot , \cdot \rangle_A)\) on \( \Gamma_r \backslash H_n \), we obtain the quotient flat Riemannian metric on \( P(\Gamma_r) \backslash v_0 \). From its definition, the quotient metric tensor \( g_A \) on \( T^{2n} \) is \( \widetilde{\rho}_A^{-1} \). In particular, the total measure of the torus is

\[
\left| \int_{P(\Gamma_r) \backslash v_0} \text{vol}_{g_A} \right| = \prod_{i=1}^{n} r_i |\det(\tilde{A})|^{-1}.
\]

Since the submersion is metric decreasing, the diameter of the torus \( (P(\Gamma_r) \backslash v_0, g_A) \) is less than or equal to \( D \). Thus we have

\[
\prod_{i=1}^{n} r_i |\det(\tilde{A})|^{-1} \leq |B^{2n}(D)|, \tag{24}
\]

where \( |B^{2n}(D)| \) is the volume of \( 2n \)-dimensional Euclidean ball of radius \( D \).

By the inequality \( (24) \), we obtain the upper bound of the total measure of the compact Heisenberg manifold by

\[
\left| \int_{\Gamma_r \backslash H_n} v(A) \right| = \prod_{i=1}^{n} r_i |\det(\tilde{A})|^{-1} \min \left\{ |\rho_A|^{-1}, \delta(A)^{-1} \right\} \leq |B^{2n}(D)| \min \left\{ |\rho_A|^{-1}, \delta(A)^{-1} \right\}. \tag{25}
\]

We will compare two values \( r_n \) and \( \min \{ |\rho_A|^{-1}, \delta(A)^{-1} \} \) to give the desired estimate for \( r_n \). By the inequality \( (19) \) and \( (20) \), we have

\[
\|X_n\|_A \leq \frac{2D}{r_n} \quad \text{and} \quad \|Y_n\|_A \leq 2D.
\]

Combined with the Campbell–Baker–Hausdorff formula \( (3) \) and the triangle inequality,

\[
\widetilde{\text{dist}}_A(e, Z) = \widetilde{\text{dist}}_A \left( e, \left[ \sqrt{r_n}X_n, \frac{1}{\sqrt{r_n}}Y_n \right] \right) \\
\leq 2\|\sqrt{r_n}X_n\|_A + 2\|\frac{1}{\sqrt{r_n}}Y_n\|_A \\
\leq 8D / \sqrt{r_n}
\]

On the other hand, by Lemma \( (10) \)

\[
\widetilde{\text{dist}}_A(e, Z) = \begin{cases} \frac{1}{\rho_A} \left( d_n(A) \leq 2\pi\rho_A^2 \right), \\ \frac{2}{\pi\rho_A} \sqrt{\pi d_n(A) - \pi^2\rho_A^2} \left( d_n(A) \geq 2\pi\rho_A^2 \right).
\end{cases}
\]

If \( d_n(A) \leq 2\pi\rho_A^2 \), then

\[
\min \{ |\rho_A|^{-1}, \delta(A)^{-1} \} \leq |\rho_A|^{-1} = \widetilde{\text{dist}}_A(e, Z) \leq \frac{8d}{\sqrt{r_n}}. \tag{27}
\]
If \( d_n(A) \geq 2\pi \rho_A^2 \), then

\[
\tilde{\text{dist}}_A(e, Z) = \frac{2}{d_n(A)} \sqrt{\pi d_n(A) - \pi^2 \rho_A^2} \geq \frac{2}{d_n(A)}
\]

and

\[
\min\{\rho_A^{-1}, \delta(A)^{-1}\} \leq \delta(A)^{-1} \leq d_n(A)^{-1} \leq \frac{\tilde{\text{dist}}_A(e, Z)^2}{4} \leq \frac{16D^2}{r_n}, \tag{28}
\]

where we use Lemma 11 in the inequality (*).

By the inequalities (26), (27) and (28), we have

\[
r_n \leq \max\{64D^2|B^{2n}(D)|^2 V^{-2}, 16D^2|B^{2n}(D)|V^{-1}\}.
\]

\[\square\]

### 7.2 Precompactness in the sub-Riemannian moduli space

In this section we fix an isomorphism class of lattice \( \Gamma_r \). We will generalize the arguments in Section 6 to the sub-Riemannian setting. As in the Riemannian setting, Boldt’s characterization of precompact subsets holds for the moduli space \( \mathcal{M}(\Gamma_r \setminus H_n) \).

**Theorem 15** (Essentially Theorem 3.13 in [7]). A subset \( \mathcal{X} \subset \mathcal{M}(\Gamma_r \setminus H_n) \) is precompact in the canonical topology if and only if there are positive constants \( C_1, C_2, C_3, C_+ > 0 \) such that every matrix \( A \in A_0 \) with \( [A] \in \mathcal{X} \) satisfies the following four conditions;

(A-1) \( \min\{\|X\|_A \mid X \in P(\Gamma_r)\} \geq C_1 \),

(A-2) \( |\det(\tilde{A})| \geq C_2 \),

(A-3) \( d_n(A) \leq C_3 \),

(A-4) \( |\rho_A| \leq C_+ \).

We omit the proof since it is same to that of Boldt. Our contribution is the following proposition.

**Proposition 9.** Let \( (\Gamma_r \setminus H_n, \text{dist}_A, v(A)) \) a compact Heisenberg manifold with the sub-Riemannian metric and the minimal Popp’s volume form associated to a matrix \( A \in A_0 \cup A_1 \). Assume that its diameter is less than or equal to \( D \) and the total measure is greater than or equal to \( V \) in the minimal Popp’s volume form. Then there are positive constants \( C_1, C_2, C_3, C_+ > 0 \) such that the conditions (A-1)-(A-4)” hold.

**Proof.** We obtain such a constant \( C_2 \) in the same way to Proposition 7. Moreover if the constant \( C_3 \) exists, then we also obtain the constant \( C_1 \) in the same
way to Proposition \[ \text{Hence we only need to show the existence of the constants } \]
\[ C_3 \text{ and } C_+. \]

To estimate the two constants \( C_3 \) and \( C_+ \), we need to consider the following two cases. First suppose that \( \delta(A) \) is less than or equal to \( |\rho_A| \). By Proposition \[ \text{and the assumption on the total measure, we have} \]
\[ V \leq \left| \int_{r_n \setminus H_n} v(A) \right| = \prod_{i=1}^{n} r_i |\rho_A \det(\tilde{A})|^{-1} \leq \prod_{i=1}^{n} r_i |\rho_A|^{-1} C_2^{-1}. \]
Then the condition (A-4)' holds with the constant \( C_+ = \prod_{i=1}^{n} r_i (vC_2)^{-1} \). Moreover, by using Lemma \[ \text{we also obtain the upper bound of } d_n(A) \text{ by} \]
\[ d_n(A) \leq \delta(A) \leq |\rho_A| \leq C_+, \]
thus we can choose \( C_3 = C_+ \).

Next suppose that \( \delta(A) \) is greater than or equal to \( |\rho_A| \). By Proposition \[ \text{and the assumption on the total measure, we have} \]
\[ V \leq \left| \int_{r_n \setminus H_n} v(A) \right| = \prod_{i=1}^{n} r_i |\delta(A)^{-1} \det(\tilde{A})|^{-1} \leq \prod_{i=1}^{n} r_i |\delta(A)|^{-1} C_2^{-1}. \]
Combined with Lemma \[ \text{the condition (A-3) hold with } C_3 = \prod_{i=1}^{n} r_i (vC_2)^{-1} \].
Again by Lemma \[ \text{we obtain the upper bound of } |\rho_A| \text{ by} \]
\[ |\rho_A| \leq \delta(A) \leq \sqrt{2n} d_n(A) \leq \sqrt{2n} C_3, \]
thus we can choose \( C_+ = \sqrt{2n} C_3 \).

\[ \square \]

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