Abstract

In 2001, Károlyi, Pach and Tóth introduced a family of point sets to solve an Erdős-Szekeres type problem; which have been used to solve several other Erdős-Szekeres type problems. In this paper we refer to these sets as nested almost convex sets. A nested almost convex set \( X \) has the property that the interior of every triangle determined by three points in the same convex layer of \( X \), contains exactly one point of \( X \). In this paper, we introduce a characterization of nested almost convex sets. Our characterization implies that there exists at most one (up to order type) nested almost convex set of \( n \) points. We use our characterization to obtain a linear time algorithm to construct nested almost convex sets of \( n \) points, with integer coordinates of absolute values at most \( O(n \log n) \). Finally, we use our characterization to obtain an \( O(n \log n) \)-time algorithm to determine whether a set of points is a nested almost convex set.

1 Introduction.

We say that a set of points in the plane is in general position if no three of them are collinear. Throughout this paper all points sets are in general position. In [9], Erdős asked for the minimum integer \( E(s,l) \) that satisfies the following. Every set of at least \( E(s,l) \) points, contains \( s \) points in convex position and at most \( l \) points in its interior. A \( k \)-hole of \( X \) is a polygon with \( k \) vertices, all of which belong to \( X \) and has no points of \( X \) in its interior; the polygon may be convex or non-convex. In 1983, Horton surprised the community with a simple proof that \( E(s,l) \) does not exist for \( l = 0 \) and \( s \geq 7 \) [12]. Horton constructed arbitrarily large point set with no convex 7-holes. Note that for \( l = 0, E(s,l) \) is the minimum integer such that every set of at least \( E(s,0) \) points contains at least one \( s \)-hole.

In 2001 [14] Károlyi, Pach and Tóth introduce a family of sets that, although was not given a name, it was used in other works related to the original question of Erdős. In this paper we refer the elements of this family as nested almost convex sets. They have been used in the following problems.

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A modular version of the Erd˝ os problem. In 2001 [14] K´ arolyi, Pach and Tóth use the nested almost convex sets to prove that, for any \( s \geq 5l/6 + O(1) \), there is an integer \( B(s, l) \) with the following property. Every set of at least \( B(s, l) \) points in general position contains \( s \) points in convex position such that the number of points in the interior of their convex hull is 0, modulo \( (l) \). This “modular” version of the Erd˝ os problem was proposed by Bialostocki, Dierker, and Voxman [5]. This was proved for \( s \geq l + 2 \) by Bialostocki et al. The original upper bound on \( B(s, l) \) was later improved by Caro in [7].

A version of the Erd˝ os problem in almost convex sets. We say that \( X \) is an almost convex set if every triangle with vertices in \( X \) contains at most one point of \( X \) in its interior. Let \( N(s) \) be the smallest integer such that every almost convex set of at least \( N(s) \) points contains an \( s \)-hole. In 2007 [17] Valtr Lippner and K´ arolyi use the nested almost convex sets to prove that:

\[
N(s) = \begin{cases} 
2^{(s+1)/2} - 1 & \text{if } s \geq 3 \text{ is odd} \\
2^{s/2} - 1 & \text{if } s \geq 4 \text{ is even.} 
\end{cases} 
\]  

The authors use the nested almost convex sets to attain the equality in (1). The existence of \( N(s) \) was first proved by K´ arolyi, Pach and Tóth in [14]. The upper bound for \( N(s) \) was improved by Kun and Lippner in [15], and it was improved again by Valtr in [10].

Maximizing the number of non-convex 4-holes. In 2014 [1] Aichholzer, Fabila-Monroy, González-Aguilar, Hackl, Heredia, Huemer, Urrutia and Vogtenhuber prove that the maximum number of non-convex 4-holes in a set of \( n \) points is at most \( n^3/2 - \Theta(n^2) \). The authors use the nested almost convex sets to prove that some sets have \( n^3/2 - \Theta(n^2 \log(n)) \) non-convex 4-holes.

Blocking 5-holes. A set \( B \) blocks the convex \( k \)-holes in \( \mathcal{X} \), if any \( k \)-hole of \( \mathcal{X} \) contains at least one element of \( B \) in the interior of its convex hull. In 2015 [6] Cano, Garcia, Hurtado, Sakai, Tejel and Urritia use the nested almost convex sets to prove that: \( n/2 - 2 \) points are always necessary and sometimes sufficient to block the 5-holes of a point set with \( n \) elements in convex position and \( n = 4k \). The authors use the nested almost convex sets as an example of a set for which \( n/2 - 2 \) points are sufficient to block its 5-holes.

We now define formally the nested almost convex sets.

Definition 1.1. Let \( \mathcal{X} \) be a point set; let \( k \) be the number of convex layers of \( \mathcal{X} \); and for \( 1 \leq j \leq k \), let \( R_j \) be the set of points in the \( j \)-th convex layer of \( \mathcal{X} \). We say that \( \mathcal{X} \) is a nested almost convex set if:

1. \( \mathcal{X}_j := R_1 \cup R_2 \cup \cdots \cup R_j \) is in general position,
2. the vertices in the convex hull of \( \mathcal{X}_j \) are the elements of \( R_j \), and
3. any triangle determined by three points of \( R_j \) contains precisely one point of \( \mathcal{X}_{j-1} \) in its interior.

In this paper, we give a characterization of when a set of points is a nested almost convex set. This is done by first defining a family of trees. If there exists a map, that satisfies certain properties, from the point set to the nodes of a tree.
in the family, then the point set is a nested almost convex set. This map encodes a lot of information about the point set. For example, it determines the location of any given point with respect to the convex hull; we use this information to obtain an $O(n \log n)$-time algorithm to decide whether a set of points is a nested almost convex set. This map also determines the orientation of any given triplet of points. This implies that for every $n$ there exists essentially at most one nested almost convex set. We further apply this information to obtain a linear-time algorithm that produces a representation of a nested almost convex set of $n$ points on a small integer grid of size $O(n^{\log_2 5})$.

The order type of a point set $X = \{x_1, x_2, \ldots, x_n\}$ is a mapping that assigns to each ordered triplet $(x_i, x_j, x_k)$ an orientation. If $x_k$ is to the left of the directed line from $x_i$ to $x_j$, the orientation of $(x_i, x_j, x_k)$ is counterclockwise. If $x_k$ is to the right of the directed line from $x_i$ to $x_j$, the orientation of $(x_i, x_j, x_k)$ is clockwise. We say that two set of points have the same order type, if there exist a bijection between these sets that preserves the orientation of all triplets.

The order type was introduced by Goodman and Pollack in [10], and it has been widely used in Combinatorial Geometry to classify point sets; two sets of points are essentially the same if they have the same order type. As a consequence of the characterization of nested almost convex sets presented in Section 2, we have the following.

**Theorem 1.2.** If $n = 2^k - 2$ or $n = 3 \cdot 2^k - 2$ there is exactly one order type that correspond to a nested almost convex set with $n$ points; for other values of $n$, nested almost convex sets with $n$ points do not exist.

In previous papers, two constructions of nested almost convex sets have been presented. The first construction was introduced by Károlyi, Pach and Tóth in [14]. The second construction was introduced by Valtr, Lippner and Károlyi in [17], six years later.

**Construction 1:** Let $X_1$ be a set of two points. Assume that $j > 0$ and that $X_j$ has been constructed. Let $z_1, \ldots, z_r$ denote the vertices of $R_j$ in clockwise order. Let $P_j$ be the polygon with vertices in $R_j$. Let $\varepsilon_j, \delta_j > 0$. For any $1 \leq i \leq r$, let $\ell_i$ denote the line through $z_i$ orthogonal to the bisector of the angle of $P_j$ at $z_i$. Let $z'_i$ and $z''_i$ be points in $\ell_i$ at distance $\varepsilon_j$ of $z_i$. Finally, move $z'_i$ and $z''_i$ away from $P_j$ at distance $\delta_j$, in the direction orthogonal to $\ell_i$, and denote the resulting points by $u'_i$ and $u''_i$, respectively. Let $R_{j+1} = \{u'_i, u''_i : i = 1 \ldots r\}$ and $X_{j+1} = X_j \cup R_{j+1}$. It is easy to see that if $\varepsilon_j$ and $\delta_j$ are sufficiently small, then $X_{j+1}$ is an almost convex set. See Figure 1.

**Construction 2:** Let $X_1$ be a set of one point. Let $R_2$ be a set of three points such that, the point in $X_1$ is in the interior of the triangle determined by $R_2$. Let $X_2 = X_1 \cup R_2$. Now recursively, suppose that $X_j$ and $R_j$ have been constructed and construct the next convex layer $R_{j+1}$ as in Construction 1. See Figure 1.

Computers are frequently used to decide whether particular sets satisfy some properties. Thus, a representation of large nested almost convex sets could be necessary. Construction 1 and Construction 2 provide such representations; however, the coordinates of the points in those constructions are not integers or are too large with respect to the value of $n$. This is prone to rounding errors.
or increases the cost of computation. Thus, it is better if the coordinates of the points in the representations are small integers.

A drawing of $X$ is a set of points with integer coordinates and with the same order type than $X$. The size of a drawing is the maximum of the absolute values of its coordinates. Other works on point sets drawings are [3, 4, 11, 13].

In Section 3, we prove that a nested almost convex set of $n$ points (if it exists), can be drawn in an integer grid of size $O(n^{\log_2 5}) \simeq O(n^{2.322})$. Furthermore, we provide a linear time algorithm to find this drawing. A lower bound of $\Omega(n^{1.5})$ on the size of any drawing of a nested almost convex set of $n$ points can be derived from the following observations. Any drawing of an $n$-point set in convex position has size $\Omega(n^{1.5})$ [13]; and every nested almost convex set of $n$ points has a $\Theta(n)$ points in convex position. This is presented in detail in Section 2.

In Section 4, we are interested in finding an algorithm to decide whether a given point set is a nested almost convex set. A straightforward $O(n^4)$-time algorithm for this problem can be given using Definition 1.1. This can be improved to $O(n^2)$ as follows. Using the algorithm presented in Section 3 an instance of nested almost convex set can be constructed. Recently in [2], Aloupis, Iacono, Langerman, Óskaan and Wuhrer gave an $O(n^2)$-time algorithm to decide whether two given sets of $n$ points have the same order type. Thus, using their algorithm and our instance solves the decision problem in $O(n^2)$ time. We further improve on this by presenting $O(n \log n)$ time algorithm.

### 2 Characterization of Nested Almost Convex Sets.

In this section we prove Theorem 2 in which the nested almost convex sets are characterized. First we introduce some definitions.

Throughout this section: $X$ will denote a set of $n$ points in general position; $k$ will denote the number of convex layers of $X$; $R_j$ will denote the set of points in the $j$-th convex layer of $X$, $R_1$ being the most internal; and $X_j$ will denote the set of points in $X$, that are in $R_j$ or in the interior of its convex hull.

$T_1(k)$: We define $T_1(k)$ as the complete binary tree with $2^{k+1} - 1$ nodes. The $j$-level of $T_1(k)$ is defined as the set of the nodes at distance $j$ from the root.

**Type 1:** We say that $X$ is of type 1 if $|R_j| = 2^j$ for $1 \leq j \leq k - 1$. Note that if $X$ is of Type 1, then for every $1 \leq j \leq k$, the number of points in $R_j$ is
equal to the number of nodes in the \( j \)-level of \( T_1(k) \).

**Type 1 labeling:** An injective function \( \psi : \mathcal{X} \rightarrow T_1(k) \) is a type 1 labeling, if \( \mathcal{X} \) is Type 1 and \( \psi \) labels the nodes (different to the root) of \( T_1(k) \) with different points of \( \mathcal{X} \).

**Type 2 labeling:** We define \( T_2(k) \) as the tree that, its root has three children, and each child is the root of a complete binary tree with \( 2^{k-1} - 1 \) nodes. The \( j \)-level of \( T_2(k) \) is defined as the set of the nodes at distance \( j - 1 \) from the root.

**Type 2:** We say that \( \mathcal{X} \) is of type 2 if \( |R_1| = 1 \) and \( |R_j| = 3 \cdot 2^{j-2} \) for \( 2 \leq j \leq k \). Note that if \( \mathcal{X} \) is of Type 2, the for every \( 1 \leq j \leq k \), the number of points in \( R_j \) is equal to the number of nodes in the \( j \)-level of \( T_2(k) \).

**Type 2 labeling:** An injective function \( \psi : \mathcal{X} \rightarrow T_2(k) \) is a Type 2 labeling, if \( \mathcal{X} \) is Type 2 and \( \psi \) labels the nodes (also the root) of \( T_2(k) \) with different points of \( \mathcal{X} \).

**Labeling:** Let \( T \) be equal to \( T_1(k) \) or \( T_2(k) \). We say that a map \( \psi : \mathcal{X} \rightarrow T \) is a labeling, if \( \psi \) is a Type 1 labeling or a Type 2 labeling. Note that, if \( \mathcal{X} \) admits a labeling then \( n = 2^{k-1} - 2 \) or \( n = 3 \cdot 2^{k-1} - 2 \).

In the following, when the map \( \psi : \mathcal{X} \rightarrow T \) is clear from the context, we say that a point is the label of a node of \( T \) if the point is mapped to the node by \( \psi \). This way, given a node \( u \) of \( T \), we denote by \( x_u \) its label. We denote by \( u(l) \) and \( u(r) \) the left and right children of \( u \) in \( T \), respectively.

**Nested:** We say that a labeling is nested if, for \( 1 \leq j \leq k \), the left to right order of labels of the nodes in the \( j \)-level of \( T \), corresponds to the counterclockwise order of the points in \( R_j \).

**Adoptable:** Given a point \( p \) in \( R_j \) and two points \( q_1, q_2 \) in \( R_{j+1} \), we say that \( q_1, q_2 \) are adoptable from \( p \) if, for every other point \( q_3 \) in \( R_{j+1} \), \( p \) is in the interior of the triangle determined by \( q_1, q_2, q_3 \). We say that a nested labeling is adoptable if, for every node \( u \) in \( T \), \( x_u(l) \) and \( x_u(r) \) are adoptable from \( x_u \).

We denote by \( R_j(u) \) the set of points in \( R_j \) that label a descendant of \( u \). With respect to the counterclockwise order, we denote by: first\([R_j(u)]\), the first point in \( R_j(u) \); last\([R_j(u)]\), the last point in \( R_j(u) \); previous\([R_j(u)]\), the point in \( R_j \) previous to first\([R_j(u)]\); and next\([R_j(u)]\), the point in \( R_j \) next to last\([R_j(u)]\). See figure 2.

**Well laid:** We say that a nested labeling is well laid if, for every \( u \) in \( T \), \( x_u \) is in the intersection of the triangle determined by previous\([R_k(u)]\), first\([R_k(u)]\), last\([R_k(u)]\) and the triangle determined by first\([R_k(u)]\), last\([R_k(u)]\), next\([R_k(u)]\).

Let \( u \) be a node of \( T \). We denote by \( \mathcal{X}_u \) the set of points \( x_v \) such that \( v \) is descendant of \( u \) in \( T \). We denote by \( \overline{\mathcal{X}_u} \) the set \( \mathcal{X}_u \cup \{x_u\} \). Given two sets of points \( A \) and \( B \), we call any directed line from a point in \( A \) to a point in \( B \), an \((A,B)\)-line.
Internal separation: We say that a nested labeling is an internal separation if for every node $u$ of $T$, every point in $\mathcal{X}/\mathcal{X}_u$ is to the left of every $(\mathcal{X}_u(l), \mathcal{X}_u(r))$-line $\ell$.

External separation: We say that a nested labeling is an external separation if for every node $u$ of $T$, every point in $\mathcal{X}/\mathcal{X}_u$ is to the left of every $(\mathcal{X}_u(l), \{x_u\})$-line and to the left of every $(\{x_u\}, \mathcal{X}_u(r))$-line.

Theorem 2.1. Let $\mathcal{X}$ be a point set in general position. Then the following statements are equivalent:

1. $\mathcal{X}$ is a nested almost convex set.
2. $\mathcal{X}$ admits a labeling that is nested, adoptable and well laid.
3. $\mathcal{X}$ admits a labeling that is an internal separation and an external separation.

Proof of Theorem 2.1

The proof of Theorem 2.1 is divided into three parts: first we prove that $1 \implies 2$; afterwards we prove that $2 \implies 3$; and finally we prove that $3 \implies 1$.

1 $\implies$ 2

In this part we assume that $\mathcal{X}$ is a nested almost convex set, and we introduce a labeling $\psi'$ that is nested, adoptable and well laid.

It is clear from the definition of labeling that a necessary condition for $\mathcal{X}$ to admit a labeling is that $\mathcal{X}$ must be type 1 or type 2. In the following lemma we prove that, if $\mathcal{X}$ is a nested almost convex, then $\mathcal{X}$ is type 1 or type 2.

Lemma 2.2. If $\mathcal{X}$ is a nested almost convex set then we have one of the following cases:

1. $|R_j| = 2^j$ for $1 \leq j \leq k - 1$.
2. $|R_1| = 1$ and $|R_j| = 3 \cdot 2^{j-2}$ for $2 \leq j \leq k$. 
Proof. Suppose that \( R_1 \) has three or more points. In this case, the interior of the convex hull of \( R_1 \) has at least one point of \( \mathcal{X} \); this contradicts that \( R_1 \) is the first convex layer of \( \mathcal{X} \). Thus \( R_1 = \mathcal{X}_1 \), and \( \mathcal{X}_1 \) has one or two points. This proves the lemma for \( j = 1 \).

Any triangulation of \( R_{j+1} \), has \(|R_{j+1}| - 2\) triangles and each triangle has exactly one point of \( \mathcal{X}_j \) in its interior; thus \(|\mathcal{X}_j| = |R_{j+1}| - 2\). In particular, if \(|\mathcal{X}_j| = 2\) or \(|\mathcal{X}_j| = 1\) then \(|\mathcal{X}_2| = 4\) or \(|\mathcal{X}_2| = 3\), respectively. This proves the lemma for \( j = 2 \).

For the other cases, note that

\[
|R_{j+1}| = |\mathcal{X}_j| + 2 = |R_j| + |\mathcal{X}_{j-1}| + 2 = 2|R_j|.
\]

Now we define \( \psi' \) on a subset of nodes of \( T \) depending on whether \( \mathcal{X} \) is of type 1 or type 2.

- If \( \mathcal{X} \) is of type 1: \( \psi' \) labels the two nodes in the 1-level of \( T_1(k) \), with the two points in \( R_1 \).
- If \( \mathcal{X} \) is of type 2: \( \psi' \) labels the node in the 1-level of \( T_2(k) \), with the point in \( R_1 \); \( \psi' \) labels the three nodes in the 1-level of \( T_2(k) \), with the three points in \( R_2 \) (such that, the left to right order of labels of the nodes in the 2-level of \( T \), coincides to the counterclockwise order of the points in \( R_2 \)).

To define \( \psi' \) on the other nodes of \( T \), we use the following Lemma.

**Lemma 2.3.** Let \( p_0, \ldots, p_t \) be the set of points in \( R_j \) in counterclockwise order. Then, the points in \( R_{j+1} \) can be listed in counterclockwise order as \( q_0, q_1, \ldots, q_{2t+1} \), where the points \( q_{2i}, q_{2i+1} \) are adoptable from \( p_i \) for \( 0 \leq i \leq t \).

**Proof.** Let \( \mathcal{T} \) be the set of triangles determined by three consecutive points of \( R_{j+1} \) in counterclockwise order. We first show that:

**Claim 2.3.1.** Each point of \( R_j \) is in exactly two consecutive triangles of \( \mathcal{T} \).

Assume that \( j \geq 2 \) (and note that Claim 2.3.1 holds for \( j = 1 \)). Let \( \triangle \) be the interior of a triangle of \( \mathcal{T} \). By the almost convex set definition, there is one point of \( \mathcal{X}_j \) in \( \triangle \). This point must be in \( R_j \), since the convex hull of \( R_{j+1} \) without \( \triangle \) (and its boundary) is convex. Thus, there is one point of \( \mathcal{X}_j \) in the interior of each triangle of \( \mathcal{T} \). As the triangles of \( \mathcal{T} \) are defined by consecutive points of \( R_{j+1} \), each point of \( R_j \) is in at most two triangles of \( \mathcal{T} \). Thereby Claim 2.3.1 follows from \(|\mathcal{T}| = |R_{j+1}| = 2|R_j|\).

The two triangles of \( \mathcal{T} \) that contain \( p_0 \), are defined by four consecutive points of \( R_{j+1} \); let \( q_0 \) be the second of these points. Let \( q_0, q_1, \ldots, q_{2t+1} \) be the points of \( R_{j+1} \) in counterclockwise order. Note that, for each \( p_i \), the middle two points of the four points that define the two triangles that contain \( p_i \), are \( q_{2i} \) and \( q_{2i+1} \). Thus \( q_{2i} \) and \( q_{2i+1} \) are adoptable from \( p_i \).

Now we define \( \psi' \) on the other nodes of \( T \) recursively. For each labeled node \( u \), \( \psi' \) labels \( u(l) \) and \( u(r) \) with the two points adoptable from the label of \( u \). We do this so that, the left to right order of the labels of the nodes in the \((j+1)\)-level of \( T \), correspond to the counterclockwise order of the points in \( R_{j+1} \). Note that...
ψ′ is nested and adoptable. It remains to prove that ψ′ is well laid. We prove this in Lemma 2.5.

**Lemma 2.4.** If u is a node of T, the label of every descendant of u is contained in the convex hull of R_k(u).

**Proof.** We claim that every set R_{j-1}(u), with at least two points, is contained in the convex hull of R_j(u). Let p be a point in R_{j-1}(u) and let q and q′ be the labels of the children of the node labeled by p. By construction of ψ′, q and q′ are adoptable from p. As R_{j-1}(u) has at least two points, R_j(u) has at least four points. Let △ be a triangle determined by q, q′ and another point of R_j(u). By definition of adoptable, p is in the interior of △ and in consequence in the interior of the convex hull of R_j. An inductive application of the previous claim proves this lemma.

**Lemma 2.5.** Let u be a node of T. Then x_u is in the intersection of the triangle determined by previous(R_k(u)), first(R_k(u)) and last(R_k(u)) and the triangle determined by first(R_k(u)), last(R_k(u)) and next(R_k(u)).

**Proof.** Let j be the index such that the j-level of T contains u. Let R_j′ be the set that contains first(R_k(v)) and last(R_k(v)) for all nodes v in the j-level of T. Let T be the set of triangles determined by three consecutive points of R_j′ in counterclockwise order. We first show the following claim.

**Claim 2.5.1.** Each point of R_j is in exactly two consecutive triangles of T.

Note that every point of \( \mathcal{X} \setminus \mathcal{X}_j \), is the label of some descendant of a node v in the j-level of T. Thus, by Lemma 2.4 every point of \( \mathcal{X} \setminus \mathcal{X}_j \) is in the convex hull of R_k(v) for some node v in the j-level of T. Let A be the region obtained from the convex hull of \( \mathcal{X} \), by removing the convex hull of R_k(v) for each v in the j-level of T. Note that the set of points of \( \mathcal{X} \) that are in A is \( \mathcal{X}_j \).

Let △ be the interior of a triangle of T. By the nested almost convex set definition, there is one point of \( \mathcal{X} \) in △. As △ is contained in A, this point must be in \( \mathcal{X}_j \). This point must also be in R_j, since A without △ (and its boundary) is convex. Thus, there is one point of R_j in the interior of each triangle of T. As the triangles of T are defined by consecutive points of R_k, each point of R_j is in at most two triangles of T. Thereby Claim 2.5.1 follows from |T| = |R_j′| = 2|R_j|.

Let △′ be the intersection of the triangle determined by previous(R_{j+1}(u)), first[R_{j+1}(u)] and last(R_{j+1}(u)), with the triangle determined by first[R_{j+1}(u)], last[R_{j+1}(u)] and next(R_{j+1}(u)). Note that first[R_{j+1}(u)] and last[R_{j+1}(u)] are the labels of the children of u. By definition of ψ′, x_u is in the interior of every triangle determined by first[R_{j+1}(u)], last[R_{j+1}(u)] and every other point of R_{j+1}; thus x_u is in △′. By Claim 2.5.1, x_u is in the interior of two triangles of T, but there are only two triangles of T that intersect △′; these are the triangles determined by previous(R_k(u)), first[R_k(u)] and last[R_k(u)], and the triangle determined by first[R_k(u)], last[R_k(u)], next[R_k(u)].

\[\Rightarrow\]

[2] \Rightarrow [3]

In this part we assume that there is a labeling ψ′ of \( \mathcal{X} \) that is nested, adoptable and well laid; and we prove that ψ′ is an internal separation and an external separation.
Lemma 2.6. \( \psi' \) is an internal separation.

Proof. Let \( u \) be a node of \( T \) and recall that \( u(l), u(r) \) are the left and right children of \( u \), respectively. We need to prove that every point in \( \mathcal{X}/\mathcal{X}_u \) is to the left of every \((\mathcal{X}_{u(l)}, \mathcal{X}_{u(r)})\)-line.

Let \( \ell \) be the directed segment from \( \text{first}[R_k(u(l))] \) to \( \text{last}[R_k(u(r))] \). By Lemma 2.5, each point in \( \mathcal{X}/\mathcal{X}_u \) is in the interior of a triangle whose vertices are to the left of, or on \( \ell \); thus every point in \( \mathcal{X}/\mathcal{X}_u \) is to the left of \( \ell \). By Lemma 2.4, every point in \( \mathcal{X}_{u(l)} \cup \mathcal{X}_{u(r)} \) is to the right of \( \ell \). We claim that:

Claim 2.6.1. No \((\mathcal{X}_{u(l)}, \mathcal{X}_{u(r)})\)-line intersects \( \ell \).

As the end points of \( \ell \), \( \text{first}[R_k(u(l))] \) and \( \text{last}[R_k(u(r))] \), are in the boundary of the convex hull of \( \mathcal{X} \); to prove that every point in \( \mathcal{X}/\mathcal{X}_u \) is to the left of every \((\mathcal{X}_{u(l)}, \mathcal{X}_{u(r)})\)-line, it is enough to show Claim 2.6.1.

Let \( P_1 \) be the polygonal chain that starts at \( q_1 \) := \( \text{first}[R_k(u(l))] \), follows the points of \( R_k(u(l)) \) in counterclockwise order, and ends at \( q_2 := \text{last}[R_k(u(l))] \). Similarly, let \( P_2 \) be the polygonal chain that starts at \( q_3 := \text{first}[R_k(u(r))] \), follows the points of \( R_k(u(r)) \) in counterclockwise order, and ends at \( q_4 := \text{last}[R_k(u(r))] \). To prove Claim 2.6.1, it is enough to show that every \((\mathcal{X}_{u(l)}, \mathcal{X}_{u(r)})\)-line intersects both \( P_1 \) and \( P_2 \).

Let \( q \) be the intersection point of the diagonals of the quadrilateral defined by \( q_1, q_2, q_3 \) and \( q_4 \). By Lemma 2.4 and Lemma 2.5, \( \mathcal{X}_{u(l)} \) is contained in the convex hull of \( P_1 \cup \{q\} \), and \( \mathcal{X}_{u(r)} \) is contained in the convex hull of \( P_2 \cup \{q\} \). Let \( \ell' \) be an \((\mathcal{X}_{u(l)}, \mathcal{X}_{u(r)})\)-line. Note that the slope of \( \ell' \), is in the range from the slope of the line defined by \( q_1 \) and \( q_3 \), to the slope of the line defined by \( q_2 \) and \( q_4 \), in counterclockwise order. Thus \( \ell' \) intersects both \( P_1 \) and \( P_2 \). □

Lemma 2.7. \( \psi' \) is an external separation.

Proof. Let \( u \) be a node of \( T \). We need to prove that every point in \( \mathcal{X}/\mathcal{X}_u \) is to the left of every \((\mathcal{X}_{u(l)}, \{x_u\})\)-line and to the left of every \((\mathcal{X}_{u(r)}, \{x_u\})\)-line. We prove that every point in \( \mathcal{X}/\mathcal{X}_u \) is to the left of every \((\mathcal{X}_{u(l)}, \{x_u\})\)-line. That every point in \( \mathcal{X}/\mathcal{X}_u \) is to the left of every \((\mathcal{X}_{u(r)}, \{x_u\})\)-line can be proven in a similar way.

Let \( P \) be the polygonal chain that starts at \( \text{next}[R_k(u)] \), follows the points of \( R_k \) in counterclockwise order, and ends at \( \text{previous}[R_k(u)] \). Note that, by Lemma 2.5, \( \mathcal{X}/\mathcal{X}_u \) is contained in the convex hull of \( P \). Thus, to prove that every point in \( \mathcal{X}/\mathcal{X}_u \) is to the left of every \((\mathcal{X}_{u(l)}, \{x_u\})\)-line, it is enough to show that \( x_u \) is to the right of the directed line from \( \text{last}[R_k(u(l))] \) to \( \text{next}[R_k(u)] \). See Figure 3

Let \( j \) be the index such that the \( j \)-level of \( T \) contains \( u \). For \( j < i \leq k \), let \( \ell_i \) be the directed line from \( \text{last}[R_i(u(l))] \) to \( \text{next}[R_i(u)] \). We show that \( x_u \) is to the right of \( \ell_i \) by induction. As \( x_{u(l)} \) and \( x_{u(r)} \) are adoptable from \( x_u \), and \( x_{u(l)} = \text{last}[R_{j+1}(u(l))]; x_u \) is in the interior of the triangle determined by \( \text{last}[R_{j+1}(u(l))], x_{u(r)} \) and \( \text{next}[R_{j+1}(u)] \). Thus the induction holds for \( i = j + 1 \). Suppose that \( x_u \) is to the right of \( \ell_i \). Let \( \text{last}[R_{i+1}(u(l))] \) and \( p \) be the two children of \( \text{last}[R_i(u(l))] \). Let \( \text{next}[R_{i+1}(u)] \) and \( q \) be the two children of \( \text{next}[R_i(u)] \). Let \( \square \) be the quadrilateral determined by \( \text{last}[R_{i+1}(u(l))] \), \( p \), \( q \) and \( \text{next}[R_{i+1}(u)] \). As \( \text{last}[R_{i+1}(u(l))] \), \( p \), \( q \) and \( \text{next}[R_{i+1}(u)] \) are in \( R_{i+1} \), and any
triangulation of □ has two triangles; there are two points of $X'$ in □. As those points are last[$R_i(u(l))$] and next[$R_i(u)$], $x_u$ is not in the interior of □. Thus $x_u$ is not between $\ell_i$ and $\ell_{i+1}$, and therefore $x_u$ is to the right of $\ell_{i+1}$.

3 $\implies$ 1

In this part we finish the proof of Theorem 2.1. We assume that there is a labeling $\psi'$ of $X$ that is an internal separation and an external separation, and we prove that $X$ is a nested almost convex set. For this it is enough to prove Lemma 2.8. As consequence of Lemma 2.8 and Theorem 2.1, Theorem 1.2 holds.

**Lemma 2.8.** Let $X$ be an $n$-point set that admits a labeling $\psi : X \to T$ that is an internal separation and an external separation. Then the order type of $X$ is determined by $T$ and:

- If $n = 2^{k-1} - 2$, then $X$ has the same order type than any $n$-point set obtained from Construction 1.
- If $n = 3 \cdot 2^{k-1} - 2$, then $X$ has the same order type than any $n$-point set obtained from Construction 2.

**Proof.** The labeling that $X$ admits can be a type 1 labeling or a type 2 labeling. If $X$ admits a type 1 labeling, $|X| = 2^{k+1} - 2$ for some integer $k$; in this case, an almost convex set with the same cardinality than $X$ can be obtained using Construction 1. If $X$ admits a type 2 labeling, $|X| = 3 \cdot 2^{k-1} - 2$ for some integer $k$; in this case, an almost convex set with the same cardinality than $X$ can be obtained using Construction 2. Let $Y$ be an almost convex set with $|X|$ points obtained from Construction 1 or Construction 2. We prove that $X$ and $Y$ have the same order type, and that this order type is determined by $T$.

Assume that $X$ admits a type 1 labeling. The case when $X$ admits a type 2 labeling can be proven in a similar way. As $Y$ is an almost convex set, $Y$ admits a labeling that is an internal separation and an external separation. Let $\psi_Y : Y \to T$ be such type 1 labeling.

Let $f := \psi_Y^{-1}(\psi')$. We prove that $f : X \to Y$ is a bijection that preserves the orientation of all triplets. Let $x_1, x_2, x_3$ be different points in $X$, let $u_1, u_2, u_3$ be the nodes of $T$ that $x_1, x_2, x_3$ label in $\psi'$, and let $y_1, y_2, y_3$ be the labels of $u_1, u_2, u_3$ in $\psi_Y$. Note that $f(x_1) = y_1$, $f(x_2) = y_2$ and $f(x_3) = y_3$. To prove
that \((x_1, x_2, x_3)\) and \((y_1, y_2, y_3)\) have the same orientation, we show that the position of \(u_1, u_2\) and \(u_3\) in \(T\) determines the orientation of any labeling of \(u_1, u_2\) and \(u_3\).

Given a node \(w\) of \(T\), denote by \(T_w\) the subtree of \(T\) that contains every descendant of \(w\). Let \(w\) be the farthest node from the root of \(T\), such that at least two of \(u_1, u_2, u_3\) are in \(T_w\). If two of \(u_1, u_2, u_3\) are in the left subtree of \(T_w\) or, two of \(u_1, u_2, u_3\) are in the right subtree of \(T_w\); the orientation of the labels of \(u_1, u_2, u_3\) is determined by an external separation. If there are not two of \(u_1, u_2, u_3\) in the left subtree of \(T_w\) or in the right subtree of \(T_w\); there is one of \(u_1, u_2, u_3\) in the left subtree of \(T_w\), one \(u_1, u_2, u_3\) in the right subtree of \(T_w\), and the other one is not in the left or right subtree of \(T_w\). In this case, the orientation of the labels of \(u_1, u_2, u_3\) is determined by an internal separation.

\[\Box\]

3 Drawings of Nested Almost Convex Sets with Small Size.

Let \(X'\) be a nested almost convex set with \(n\) points, and let \(k\) be the number of convex layers of \(X'\). In this section we construct a drawing of \(X'\) of size \(O(n^{\log_2 5})\). This section is divided into three parts. First, we construct a \(2^{k+1} - 2\) point set \(X\) with integer coordinates and size \(2 \cdot 5^{k+1}\). Afterwards, we prove that \(X\) is a nested almost convex set. Finally, we obtain a subset of \(X\) that is a drawing of \(X'\).

Construction of \(X\).

Recall that \(T_1(k)\) is the complete binary tree with \(2^{k+1} - 1\) nodes, and the \(j\)-level of \(T_1(k)\) is the set of nodes at distance \(j\) from the root of \(T_1(k)\). Before defining \(X\), we will construct a point set \(Y\) in convex position, and for each node \(u\) in \(T_1(k)\), we will define a set \(Y_u \subset Y\) of consecutive points of \(Y\) in counterclockwise order. The point \(x_u\) will denote the midpoint between the first and last points of \(Y_u\) in counterclockwise order. The set \(X\) will be the set of points \(x_u\) such that \(u\) is a node of \(T_1(k)\) different from the root.

Let \(p, o\) and \(q\) be points in the plane and let \(c \in [0, 1]\). We denote by \(\overline{pq}\) and \(\overline{po}\) the segments from \(o\) to \(p\) and from \(o\) to \(q\), respectively. We say that \(\alpha = (q, o, p)\) is a corner, if the angle from \(\overline{pq}\) to \(\overline{po}\) counterclockwise is less than \(\pi\). Let \(\alpha := (q, o, p)\) be a corner. We denote by \(\text{LeftPoint}(\alpha, c)\) the point in the segment \(\overline{po}\) at distance \(c|\overline{po}|\) from \(o\). We denote by \(\text{RightPoint}(\alpha, c)\) the point in the segment \(\overline{po}\) at distance \(c|\overline{po}|\) from \(o\). See Figure 3.

Recursively, we define a corner \(\alpha_u\) for each node \(u\) of \(T_1(k)\). The corner of the root of \(T_1(k)\) is defined as \(((0, 2 \cdot 5^{k+1}), (0, 0), (2 \cdot 5^{k+1}, 0))\). Let \(u\) be a node for which its corner \(\alpha_u\) has been defined; the corners of its left and right children, \(u(l)\) and \(u(r)\), are defined as follows (See Figure 4):

\[
\begin{align*}
\alpha_{u(l)} &= (\text{LeftPoint}(\alpha_u, 2/5), \text{LeftPoint}(\alpha_u, 1/5), \text{RightPoint}(\alpha_u, 1/5)) \\
\alpha_{u(r)} &= (\text{LeftPoint}(\alpha_u, 1/5), \text{RightPoint}(\alpha_u, 1/5), \text{RightPoint}(\alpha_u, 2/5))
\end{align*}
\]
Figure 4: Illustration of corners $\alpha_u$, $\alpha_u(l)$ and $\alpha_u(r)$, where $\alpha_u = (q, o, p)$.

Let $v$ be a leaf of $T_1(k+1)$. Note that $v$ is a child of a leaf $u$ of $T_1(k)$. If $v$ is the left child of $u$, let $y_v := \text{LeftPoint}(\alpha_u, 1/5)$. If $v$ is the right child of $u$, let $y_v := \text{RightPoint}(\alpha_u, 1/5)$. We define $Y$ as the set of points $y_v$ such that $v$ is a leaf of $T_1(k+1)$. Given a node $u$ of $T_1(k)$, we define $Y_u$ as the set of points $y_v$ such that $v$ is a descendant of $u$, and $v$ is a leaf of $T_1(k+1)$. With respect to the counterclockwise order, we denote by: first$[Y_u]$, the first point in $Y_u$; last$[Y_u]$, the last point in $Y_u$; previous$[Y_u]$, the point in $Y_u$ previous to first$[Y_u]$; and next$[Y_u]$, the point in $Y_u$ next to last$[Y_u]$.

Lemma 3.1. Let $u$ be a node of $T_1(k)$. Let $v_1, v_2, \ldots, v_t$ be the leaves of $T_1(k+1)$, that are descendant of $u$, ordered from left to right. Then $y_{v_1}, y_{v_2}, \ldots, y_{v_t}$ are in convex position, and are the points in $Y_u$ in counterclockwise order.

Proof. Let $(q, o, p) := \alpha_u$; $q' := \text{LeftPoint}(\alpha_u, 2/5)$; and $p' := \text{RightPoint}(\alpha_u, 2/5)$. Let $\triangle(u)$ be the triangle determined by $q'$, $o$ and $p'$. Inductively from the leaves to the root of $T_1(k)$, it can be proven that:

1. The set of points of $Y$ in $\triangle(u)$ is $Y_u$; from which: first$[Y_u]$ is on the segment from $o$ to $q'$, last$[Y_u]$ is on the segment from $o$ to $p'$, and the other points are in the interior of $\triangle(u)$.

2. The points $q', y_{v_1}, y_{v_2}, \ldots, y_{v_t}, p'$ are in convex position, and appear in this order counterclockwise.

This proof follows from 2.

By Lemma 3.1, $Y$ is in convex position, and for each node $u$ in $T_1(k)$, $Y_u$ is a subset of consecutive points of $Y$ in counterclockwise order. We denote by $x_u$ the midpoint between first$[Y_u]$ and last$[Y_u]$. Let $X$ be the set of points $x_u$ such that $u$ is a node of $T_1(k)$ different from the root.

Let $u$ be a node of $T_1(k)$ at distance $j$ from the root, let $(q, o, p) := \alpha_u$ and let $v$ be a leaf of $T_1(k+1)$. Recursively note that, the coordinates of $q$, $o$ and $p$ are divisible by $2 \cdot 5^{k+1-j}$. Thus, the coordinates of $y_v$ are divisible by 2, $x_u$ has integer coordinates, and $X$ has size $2 \cdot 5^{k+1}$.

$X$ is a nested Almost Convex Set.

In this subsection we prove that $X$ is a nested almost convex set. By Theorem 2.1 it is enough to prove that $X$ admits a labeling that is an internal
separation and an external separation. Let \( \psi : \mathcal{X} \to T_1(k) \) be the type 1 labeling that labels each node \( u \) of \( T_1(k) \) different from the root, with \( x_u \). We prove that \( \psi \) is both an internal separation and an external separation.

**Lemma 3.2.** If \( u \) is a node of \( T_1(k) \) at distance \( j \) from the root, then \( \text{first}[\mathcal{Y}_u] = \text{LeftPoint}(\alpha_u, c_j) \) and \( \text{last}[\mathcal{Y}_u] = \text{RightPoint}(\alpha_u, c_j) \), where

\[
c_j = \frac{1}{4} \left( 1 - 5^{j - k - 1} \right).
\]

**Proof.** Note that \( Y \) the convex hull of \( \bigcup_{i=k}^{j} \ell \) is to the left of every \( (\mathcal{X}_{\alpha(l)}, \mathcal{X}_{\alpha(r)}) \)-line. If \( j = k \), then: \( u \) is a leaf of \( T_1(k) \); \( c_j = 1/5 \); and \( \text{first}[\mathcal{Y}_u] = \text{LeftPoint}(\alpha_u, c_j) \) and \( \text{last}[\mathcal{Y}_u] = \text{RightPoint}(\alpha_u, c_j) \). Suppose that \( j < k \); and that this lemma holds for larger values of \( j \). Let \( u(l) \) and \( u(r) \) be the left and right children of \( u \). Note that by induction,

\[
\text{first}[\mathcal{Y}_u] = \text{LeftPoint}(\alpha_u(l), c_{j+1}) = \text{LeftPoint}(\alpha_u, c_{\ast}),
\]

where \( c_{\ast} = (1/5)c_{j+1}+1/5 = c_j \); thus \( \text{first}[\mathcal{Y}_u] := \text{LeftPoint}(\alpha_u, c_j) \). In a similar way \( \text{last}[\mathcal{Y}_u] := \text{RightPoint}(\alpha_u, c_j) \).

**Lemma 3.3.** \( \psi \) is an internal separation.

**Proof.** Let \( u \) be a node of \( T_1(k) \) different from the root, and let \( u(l) \), \( u(r) \) be the left and right children of \( u \), respectively. We need to prove that every point in \( \mathcal{X}/\mathcal{X}_u \) is to the left of every \( (\mathcal{X}_{\alpha(l)}, \mathcal{X}_{\alpha(r)}) \)-line.

Let \( \ell \) be the directed segment from \( \text{first}[\mathcal{Y}_{u(l)}] \) to \( \text{last}[\mathcal{Y}_{u(r)}] \). As each point in \( \mathcal{X}/\mathcal{X}_u \) is the midpoint between two points that are not to the right of \( \ell \), every point in \( \mathcal{X}/\mathcal{X}_u \) is not to the right of \( \ell \). As every point in \( \mathcal{X}_{\alpha(l)} \cup \mathcal{X}_{\alpha(r)} \), is the midpoint between a point to the right of \( \ell \) and a point that is not to the left of \( \ell \), every point in \( \mathcal{X}_{\alpha(l)} \cup \mathcal{X}_{\alpha(r)} \) is to the right of \( \ell \). We claim that:

**Claim 3.3.1.** No \( (\mathcal{X}_{\alpha(l)}, \mathcal{X}_{\alpha(r)}) \)-line intersects \( \ell \).

As the endpoints of \( \ell \), \( \text{first}[\mathcal{Y}_{u(l)}] \) and \( \text{last}[\mathcal{Y}_{u(r)}] \), are in the boundary of the convex hull of \( \mathcal{Y} \); to prove that every point in \( \mathcal{X}/\mathcal{X}_u \) is to the left of every \( (\mathcal{X}_{\alpha(l)}, \mathcal{X}_{\alpha(r)}) \)-line, it is enough to show Claim 3.3.1.

Let \( P_1 \) be the polygonal chain that starts at \( \text{first}[\mathcal{Y}_{u(l)}] \), follows the points of \( \mathcal{Y}_{u(l)} \) in counterclockwise order, and ends at \( \text{last}[\mathcal{Y}_{u(l)}] \). Similarly, let \( P_2 \) be the polygonal chain that starts at \( \text{first}[\mathcal{Y}_{u(r)}] \), follows the points of \( \mathcal{Y}_{u(r)} \) in counterclockwise order, and ends at \( \text{last}[\mathcal{Y}_{u(r)}] \). To prove Claim 3.3.1 it is enough to show that every \( (\mathcal{X}_{\alpha(l)}, \mathcal{X}_{\alpha(r)}) \)-line intersects \( P_1 \) and \( P_2 \). This follows from the fact that \( \mathcal{X}_{\alpha(l)} \) is contained in the convex hull of \( P_1 \), and \( \mathcal{X}_{\alpha(r)} \) is contained in the convex hull of \( P_2 \).

**Lemma 3.4.** Let \( u \) be a node of \( T_1(k) \) at distance \( j \) from the root, and let \((q,o,p) := \alpha_u \). Suppose that the nodes in the \( j \)-level of \( T_1(k) \), are ordered from left to right.

1. If \( u \) is not the first node, then the points \( o, \text{first}[\mathcal{Y}_u], \text{previous}[\mathcal{Y}_u] \) and \( q \) are collinear, and \( \text{previous}[\mathcal{Y}_u] = \text{LeftPoint}(u, c) \) for some \( c > 3/5 \).
2. If $u$ is not the last node, then the points $o, \text{last}[\mathcal{Y}_u], \text{next}[\mathcal{Y}_u]$ and $p$ are collinear, and $\text{next}[\mathcal{Y}_u] = \text{RightPoint}(u, c)$ for some $c > 3/5$.

Proof. To prove 1 and 2 note that, for any two consecutive nodes in the $j$-level of $T_1(k)$, there is a segment that contains one side of each the corners corresponding to these nodes; then apply Lemma 3.2 \hfill \Box

Lemma 3.5. $\psi$ is an external separation.

Proof. Let $u$ be a node of $T_1(k)$ and $u(l), u(r)$ be the left and right children of $u$, respectively. We need to prove that every point in $\mathcal{X}/\mathcal{X}_u$ is to the left of every $(\mathcal{X}_u(l), \{x_u\})$-line and to the left of every $(\{x_u\}, \mathcal{X}_u(r))$-line. We prove that every point in $\mathcal{X}/\mathcal{X}_u$ is to the left of every $(\mathcal{X}_u(l), \{x_u\})$-line. That every point in $\mathcal{X}/\mathcal{X}_u$ is to the left of every $(\{x_u\}, \mathcal{X}_u(r))$-line can be proven in a similar way.

Let $P$ be the polygonal chain that starts at next $\mathcal{Y}_u$, follows the points of $\mathcal{Y}$ in counterclockwise order, and ends at previous $\mathcal{Y}_u$. Note that $\mathcal{X}/\mathcal{X}_u$ is contained in the convex hull of $P$. Thus, to prove that every point in $\mathcal{X}/\mathcal{X}_u$ is to the left of every $(\mathcal{X}_u(l), \{x_u\})$-line, it is enough to show that next $\mathcal{Y}_u$ is to the left of the directed line from last $\mathcal{Y}_u(l)$ to $x_u$.

Let $\ell$ be the directed line from last $\mathcal{Y}_u(l)$ to $x_u$ and let $(q, o, p) := (u, o, p)$. Note that $x_u$ and last $\mathcal{Y}_u(l)$ are in the interior of the wedge determined by $o_u$ from $o$ to $p$ in counterclockwise order. By Lemma 3.2 next $\mathcal{Y}_u$ is on $op$ and next $\mathcal{Y}_u = \text{RightPoint}(u, c)$ for some $c > 3/5$. To finish this proof we show that $\ell$ intersects $op$ at a point $\text{RightPoint}(u, c')$ for some $c' < 3/5$.

Consider the following coordinate system, $o$ is the origin, $p$ has coordinates $(1,0)$ and $q$ has coordinates $(0,1)$. Assume that this is the new coordinate system. Let $t$ be such that the intersection point between $\ell$ and the abscissa is the point $(t,0)$; thereby, we need to prove that $t < 3/5$.

By Lemma 3.2 first $\mathcal{Y}_u$ and last $\mathcal{Y}_u$ have coordinates $(0,c_j)$ and $(c_j,0)$; thus, $x_u$ has coordinates $(c_j/2, c_j/2)$. By construction of $o_u(l)$ and Lemma 3.2 last $\mathcal{Y}_u(l)$ is in the segment from $(0,1/5)$ to $(1/5,0)$ in RightPoint$(u(l), c_j+1)$. Thus last $\mathcal{Y}_u(l)$ has coordinates $(\frac{1}{5}c_j+1, \frac{1}{5}(1-c_j+1))$ and the equation of $\ell$ is:

$$x = \frac{c_j+1/5-c_j/2}{(1-c_j+1)/5-c_j/2}(y-c_j/2) + c_j/2$$

taking $y = 0$, $s = k - j$, and replacing $c_j$ and $c_{j+1}$, we have that:

$$t = -\frac{1}{40 \cdot 5^s} - \frac{1}{40(1 + 3/5^s)} - \frac{1}{40(3 \cdot 5^s + 5^{2s})} + \frac{1}{8(3/5^s + 1)} + \frac{1}{8(3 + 5^s)} + \frac{1}{8}$$

Finally, as $5^s \geq 1$

$$t < \frac{3}{8} + \frac{1}{8(4)} + \frac{1}{8} = \frac{17}{32} < \frac{3}{5}.$$

\hfill \Box

Construction of a Drawing of $\mathcal{X}$.

In this subsection we find a subset of $\mathcal{X}$ that is a drawing of $\mathcal{X}'$. By Theorem 1.2 there are two cases: $\mathcal{X}'$ is of type 1 and has $n = 2^{k+1}$ points; or $\mathcal{X}'$ is of type 2 and has $n = 3 \cdot 2^{k+1}$ points. By Theorem 1.2 if $\mathcal{X}'$ is type 1, $\mathcal{X}'$ and $\mathcal{X}$ have the same order type and $\mathcal{X}$ is a drawing of $\mathcal{X}'$. Assume that $\mathcal{X}'$ is type 2.
Let $w$ be the root of $T_1(k)$; $u$ and $u'$ be the children of $w$: $u(l)$ and $u(r)$ be the children of $u$; and $u'(l)$ and $u'(r)$ be the children of $u'$. We define $T$ as the tree obtained from $T_1(k)$, by making $u'(l)$ the third child of $u$ and removing $w$, $u'$, $u'(r)$ and every descendant of $u'(r)$. Recall that $T_2(k)$ is a tree such that, its root has three children, and each child is the root of a complete binary tree with $2^{k-1} - 1$ points. Note that $T$ and $T_2(k)$ are isomorphic.

Let $X_2$ be the set of points $x_u$ such that $u$ is in $T$. Let $\psi' : X_2 \to T$ be such that $\psi'(x_u) = u$. Note that: as $\psi$ is an internal separation, $\psi'$ is an internal separation; and as $\psi$ is an external separation, $\psi'$ is external separation. Thus by Theorem 2.1, $X_2$ is a nested almost convex set.

By Theorem 1.2, as $X_2$ has $3 \cdot 2^{k-1} - 2$ points, $X_2$ and $X'$ have the same order type and $X_2$ is a drawing of $X'$.

4 Decision Algorithm for Nested Almost Convexity.

Let $X$ be a set of $n$ points. In this section, we present an $O(n \log n)$ time algorithm, to decide whether $X$ is a nested almost convex set. This algorithm is based in Theorem 2.1 and consists of four steps. At each step, it is verified if $X$ satisfies a certain property; $X$ is a nested almost convex set if and only if $X$ satisfies each of these properties.

By Theorem 1.2, if $X$ is a nested almost convex set, then $n = 2^{k-1} - 2$ or $n = 3 \cdot 2^{k-1} - 2$ for some integer $k$. The first step is to verify whether $X$ has one of those cardinalities. If $n = 2^{k-1} - 2$ let $T := T_1(k)$. If $n = 3 \cdot 2^{k-1} - 2$ let $T := T_2(k)$. Recall that: the $j$-level of $T_1(k)$ is defined as the set of the nodes at distance $j$ from the root; and the $j$-level of $T_2(k)$ is defined as the set of the nodes at distance $j - 1$ from the root.

By Lemma 2.2, if $X$ is a nested almost convex set then: for $1 \leq j \leq k$, the number of nodes in the $j$-level of $T$ is equal to the number of nodes in the $j$-th convex layer of $X$. The second step is to verify whether $X$ satisfies Lemma 2.2. Chazelle [8] showed that, the convex layers of a given an $n$-point set can be found in $O(n \log n)$ time; thus the second step can be done in $O(n \log n)$ time. We denote by $R_j$ the set of points in the $j$-th convex layer of $X$.

The third step is to verify whether $X$ satisfies Lemma 2.3. For $1 \leq j \leq k - 1$, we do the following. Let $p_0, \ldots, p_l$ be the points in $R_j$ in counterclockwise order. We search for two consecutive points in $R_{j+1}$ that are adoptable by $p_0$. If those points exist, they are the only pair of consecutive points in $R_{j+1}$ that are adoptable by $p_0$. Let $q_0, q_1, \ldots, q_{2t+1}$ be the points in $R_{j+1}$ in counterclockwise order, such that $q_0$ and $q_t$ are adoptable by $p_0$. Then we verify whether $q_{2i}, q_{2i+1}$ are adoptable by $p_i$ for $0 \leq i \leq t$.

Let $p$ be in $R_j$, and let $q_r, q_{r+1}, q_{r+2}, q_{r+3}$ be four consecutive points in $R_{j+1}$. Note that $q_{r+1}$ and $q_{r+2}$ are adoptable by $p$, if and only if, $p$ is in the intersection of the triangle determined by $q_r, q_{r+1}$ and $q_{r+2}$, and the triangle determined by $q_{r+1}, q_{r+2}$ and $q_{r+3}$. Thus, we can verify whether $q_{2i}, q_{2i+1}$ are adoptable by $p_i$ in constant time; the third step hence requires linear time.

If $X$ satisfies Lemma 2.3 we can define a labeling $\psi : X \to T$ like the one defined in Section 2.4. The fourth step is to verify if $\psi$ is well laid, this requires linear time.
According to the proof of Theorem 2.1, $\mathcal{X}$ is a nested almost convex set if and only if $\mathcal{X}$ verifies the properties in previous four steps. This can be done in $O(n \log n)$ time.

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