Dominions in decomposable varieties*

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Abstract. Dominions, in the sense of Isbell, are investigated in the context of decomposable varieties of groups. An upper and lower bound for dominions in such a variety is given in terms of the two varietal factors, and the internal structure of the group being analyzed. Finally, the following result is established: If a variety \( \mathcal{N} \) has instances of nontrivial dominions, then for any proper subvariety \( \mathcal{Q} \) of \( \text{Group} \), \( \mathcal{NQ} \) also has instances of nontrivial dominions.

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Section 1. Introduction

Suppose that a group \( G \), a subgroup \( H \) of \( G \), and a class \( \mathcal{C} \) of groups containing \( G \) are given. Are there any elements \( g \in G \setminus H \) such that any two morphisms between \( G \) and a \( \mathcal{C} \)-group which agree on \( H \) must also agree on \( g \)?

To put this question in a more general context, let \( \mathcal{C} \) be a full subcategory of the category of all algebras (in the sense of Universal Algebra) of a fixed type which is closed under passing to subalgebras. Let \( A \in \mathcal{C} \), and let \( B \) be a subalgebra of \( A \). Recall that, in this situation, Isbell [1] defines the \textit{dominion of} \( B \) \textit{in} \( A \) (in the category \( \mathcal{C} \)) to be the intersection of all equalizer subalgebras of \( A \) containing \( B \).

Explicitly,

\[
\text{dom}_A^\mathcal{C}(B) = \left\{ a \in A \mid \forall C \in \mathcal{C}, \forall f, g : A \rightarrow C, \text{ if } f|_B = g|_B \text{ then } f(a) = g(a) \right\}.
\]

Therefore, the question with which we opened this discussion may be rephrased in terms of the dominion of \( H \) \textit{in} \( G \) in the category of context.

If \( H = \text{dom}_G^\mathcal{C}(H) \) we say that the dominion of \( H \) \textit{in} \( G \) (in the category \( \mathcal{C} \)) is \textit{trivial}, and say it is \textit{nontrivial} otherwise.
In this work we will study dominions when the category $C$ is a product of two proper nontrivial varieties of groups. In Section 2 we will recall the basic properties of varieties that we will need; we refer the reader to Hanna Neumann’s excellent book [7] for more information on varieties of groups. Since wreath products are closely related to products of varieties, and are used in the proofs of our results, we will also recall some of their properties. In Section 3 we will state and prove the main results of this work, which give an upper and lower bounds for the dominion of a subgroup in a product variety. Finally, in Section 4 we will use this result to prove that if a variety $N$ has instances of nontrivial dominions, so will $NQ$ for any variety $Q \neq Group$. Along the way we will obtain some other results which are of interest in their own right.

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**Section 2. Preliminary results and wreath products**

All groups will be written multiplicatively, and all maps will be assumed to be group morphisms, unless otherwise specified. Given a group $G$, the identity element of $G$ will be denoted by $e_G$, although we will omit the subscript if it is understood from context. Given a group $G$ and a subgroup $H$, $N_G(H)$ denotes the normalizer of $H$ in $G$.

Recall that a variety of groups is a full subcategory of $Group$ which is closed under taking arbitrary direct products, quotients, and subgroups. Alternatively, it is the collection of all groups (and all group morphisms between them) which satisfy a given set of identities.

We direct the reader to [5] for the basic properties of dominions. I mention the most important of them, which are not hard to verify: $dom_G^C(−)$ is a closure
operator on the lattice of subgroups of $G$; if $\mathcal{C}$ is closed under quotients, then normal subgroups are dominion-closed; the dominion construction respects finite direct products; and the dominions construction respects quotients in a variety. That is, if $\mathcal{V}$ is a variety of groups, $G \in \mathcal{V}$, and $H$ is a subgroup of $G$, $N$ a normal subgroup of $G$ contained in $H$, then
\[
\text{dom}_{G/N}(H/N) = \text{dom}_{G}(H) / N.
\]

Given a variety $\mathcal{V}$, a group $G \in \mathcal{V}$ and a subgroup $H$ of $G$, the *amalgamated coproduct (in $\mathcal{V}$) of $G$ with itself over $H$*, denoted by $G \amalg_H^\mathcal{V} G$ is the universal $\mathcal{V}$-group, equipped with embeddings $\lambda, \rho: G \to G \amalg_H^\mathcal{V} G$ such that $\lambda|_H = \rho|_H$. If $H$ is the trivial subgroup, the resulting object is the usual coproduct in $\mathcal{V}$, which we denote simply by $G \amalg^\mathcal{V} G$. It is not hard to verify that in a variety, the dominion of $H$ is the equalizer of the two canonical embeddings into the amalgamated coproduct.

Given two varieties of groups, $\mathcal{N}$ and $\mathcal{Q}$, recall that the *product variety* $\mathcal{V} = \mathcal{N} \mathcal{Q}$ is the variety of all groups which are extensions of a group in $\mathcal{N}$ by a group in $\mathcal{Q}$; that is, it consists of all groups $G$ which have a normal subgroup $N \in \mathcal{N}$, such that $G/N \in \mathcal{Q}$.

We will say that a variety $\mathcal{V}$ is *nontrivial* iff $\mathcal{V} \neq \mathcal{G}$ and $\mathcal{V} \neq \mathcal{E}$, and we will call it *trivial* otherwise. A variety $\mathcal{V}$ factors nontrivially (or is decomposable) if it can be expressed as the product of two nontrivial varieties.

The semigroup of varieties of groups has the structure of a cancellation semigroup with 0 and 1. The zero element is the variety of all groups (denoted by $\mathcal{G}$), while the identity is the trivial variety, consisting only of the trivial group and denoted by $\mathcal{E}$. Furthermore, every variety other than $\mathcal{G}$ can be uniquely factored as a product of a finite number of indecomposable varieties (in a unique order), with $\mathcal{E}$ having the empty factorization, so that the semigroup with neutral element of varieties other than $\mathcal{G}$ is freely generated by the indecomposable varieties. See Theorems 21.72, 23.32 and 23.4 in [7].
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Given a variety $V$ and a group $G$ (not necessarily in $V$), we will denote by $V(G)$ the verbal subgroup of $G$ associated to $V$. This is the subgroup generated by all values of the words $w$ which are laws of $V$. In particular, $G \in V$ if and only if $V(G) = \{e\}$. We also note the universal property associated to $V(G)$: for any normal subgroup $N \triangleleft G$, $G/N \in V$ if and only if $V(G) \subseteq N$.

The term $n$-generator group will mean that the group in question can be generated by $n$ elements, but may in fact need less. The term $n$-variable word will refer to a word in $x_1, \ldots, x_n$. This convention relies on the fact that in a law involving $n$ variables, the name of these variables is immaterial.

We also recall the definition of the wreath product of two groups. Given groups $N$ and $K$, the regular (unrestricted) wreath product of $N$ by $K$, denoted by $N \wr K$, is constructed as follows:

We take $N^K$, the cartesian power of $N$ consisting of all functions $\phi: K \to N$ (not necessarily group morphisms) multiplied componentwise. For each element $k \in K$, we let $\beta_k: N^K \to N^K$ be the mapping that takes $\phi \in N^K$ to $\phi^k$, by
\[
\phi^k(y) = \phi(yk^{-1}) \quad \text{for all } y \in K.
\]

Then $\beta_k$ is an automorphism of $N^K$, and (if $N \neq \{e\}$) the set of these automorphisms is a group isomorphic to $K$. Let $P$ be the semidirect product of $N^K$ by this group of automorphisms: that is, we take pairs $(k, \phi)$, with $k \in K$ and $\phi \in N^K$, with multiplication
\[
(k, \phi)(\ell, \psi) = (k\ell, \phi^k \psi),
\]
and identify the group of elements $(k, e)$ with $K$, and the group of elements $(e, \phi)$ with $N^K$ so that elements of $P$ become products $k\phi$, and we have $\phi^k = k^{-1}\phi k$. In this notation, $P$ is also called the complete (or unrestricted) wreath product.

If we replace $N^K$ with $N^{(K)}$, the subgroup of functions $\phi: K \to N$ such that $\phi(k) = e$ for almost all $k \in K$, the resulting group is called the restricted wreath product, and we shall denote it by $N \wr r K$. Clearly, $N \wr r K$ is a subgroup of $N \wr K$. 

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In general, if $\Omega$ is a $K$-set, we may form the product of $|\Omega|$ copies of $N$, and then let $K$ act on this product by permuting the components. This is called the $\Omega$-wreath product of $N$ by $K$, and is denoted by $N \wr_{\Omega} K$. We see that the regular wreath product is nothing more than a special case of this situation, where $\Omega$ is $K$ itself, with the action being given by the right regular action.

Next we recall some basic results about product varieties and wreath products. Their proofs are straightforward and therefore we omit them, although the reader will be able to provide them herself if she so desires.

**Proposition 2.1.** (Proposition 21.12 in [7]) Let $N$ and $Q$ be two varieties of groups. Given a group $G$, the verbal subgroup corresponding to the variety $N^Q$ is given by $N(Q(G))$. \hfill $\square$

**Lemma 2.2.** (Lemma 16.1 in [7]) An $n$-generator group $G$ belongs to the variety $V$ if and only if it satisfies the $n$-variable laws of $V$. \hfill $\square$

We let $V^{(n)}$ denote the variety defined by the set of $n$-variable laws of $V$.

**Theorem 2.3.** (Theorems 16.21 and 16.23 in [7]) The variety $V^{(n)}$ consists of all groups whose $n$-generator subgroups belong to $V$. Therefore, $V^{(n)} \supseteq V$, and $V^{(n)} = V$ if and only if $V$ can be defined by $n$-variable laws. Also $V^{(1)} \supseteq V^{(2)} \supseteq \cdots \supseteq V^{(n)} \supseteq \cdots \supseteq V = \bigcap_{i \geq 1} V^{(i)}$. \hfill $\square$

**Theorem 2.4.** Let $N \in \mathcal{N}$ and $Q \in \mathcal{Q}$. Then $N \wr Q \in \mathcal{N} \mathcal{Q}$ and $N \wr Q \in \mathcal{N} \mathcal{Q}$. \hfill $\square$

Proof: $N \wr Q$ is an extension of $N^Q$ by $Q$. $N^Q$ is an $\mathcal{N}$-group, since it is the direct product of copies of $N$, and $N \in \mathcal{N}$. Therefore, $N \wr Q$ is the extension of an $\mathcal{N}$-group by a $\mathcal{Q}$-group, hence an element of $\mathcal{N} \mathcal{Q}$.

Finally, since $N \wr Q$ is a subgroup of $N \wr Q$, it is also an element of $\mathcal{N} \mathcal{Q}$.

Let $N$ and $K$ be two groups, and let $f: N \to M$ be a group morphism. Given an element $k\phi \in N \wr K$, we may obtain an element of $M \wr K$ induced by $f$, namely $k(f \circ \phi)$; note that $\phi$ is a map from $K$ to $N$, so $f \circ \phi$ is a map from $K$ to $M$, that
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is an element of $M^K$. We let the induced map from $N \wr K$ to $M \wr K$ be denoted by $f^\ast$. With a straightforward verification we obtain:

**Theorem 2.5.** (See 22.21–22.14 in [7]) Let $N$ and $K$ be two groups, and let $f : N \to M$ be a group morphism. Then the induced map $f^\ast : N \wr K \to M \wr K$ is a group homomorphism. Furthermore, if $f$ is an embedding, then so is $f^\ast$. 

**Theorem 2.6.** (Kaloujnine and Krasner [3]) The complete wreath product $A \wr B$ contains an isomorphic copy of every group that is an extension of $A$ by $B$.

*Proof:* Let

$$1 \longrightarrow A \overset{\alpha}{\longrightarrow} G \overset{\pi}{\longrightarrow} B \longrightarrow 1$$

be an exact sequence of groups, and let $\tau$ be a transversal of $\pi$ (that is, $\pi \circ \tau = \text{id}_B$; note that $\tau$ is not necessarily a group morphism). Let $T$ be the image of $\tau$. Thus, $T$ is a set of coset representatives for $\alpha(A)$ in $G$. We now define a mapping $\gamma : G \to A \wr B$ as follows: for $g \in G$, let

$$\gamma(g) = \pi(g) \varphi_g, \quad \text{where for all } y \in B$$

$$\varphi_g(y) = \alpha^{-1} \left( \tau(y \pi(g)^{-1}) g \tau(y)^{-1} \right);$$

one can check that the right hand side is meaningful. A straightforward computation will confirm that $\gamma$ is a monomorphism. 

**Remark 2.7.** Note that the embedding $\gamma$ obtained in Theorem 2.6 depends on the choice of transversal $\tau$ (or equivalently, on the choice of coset representatives $T$).

**Remark 2.8.** Given a fixed group $B$, we can make the set of groups with maps to $B$ into a category, using for morphisms the group homomorphisms that make commuting triangles with the maps into $B$. Given $G$ as an extension of $A$ by $B$, we have a given map $\pi : G \to B$, and we also have a canonical map from $A \wr B$ to $B$. It is easy to verify that the embedding of $G$ into $A \wr B$ given in Theorem 2.6 is actually a morphism in this category.
Section 3. Upper and lower bounds for the dominion in a product variety

We now turn our attention to the behavior of dominions in product varieties. Throughout this section, we will work inside a variety of groups $\mathcal{V}$ which can be decomposed as a product $\mathcal{V} = \mathcal{N}\mathcal{Q}$ of two nontrivial varieties $\mathcal{N}$ and $\mathcal{Q}$, unless otherwise specified.

**Theorem 3.9.** (See [4]) Suppose that $\mathcal{V} = \mathcal{N}\mathcal{Q}$, where $\mathcal{N}$ is a nontrivial variety. Let $G \in \mathcal{Q}$, and $H$ a subgroup of $G$. Then $\text{dom}^\mathcal{V}_{G}(H) = H$.

*Proof:* Let $G$ and $H$ be as in the statement. The key idea is that if one can find a group $N \in \mathcal{N}$ and an action of $G$ on $N$ such that the stabilizer in $G$ of some $n \in N$ is exactly $H$, then we may form the semidirect product $N \rtimes G$ determined by this action, which will lie in $\mathcal{V}$. In this semidirect product, $H$ becomes the centralizer in $G$ of the element $n$ of the semidirect product. Then the equalizer of the inclusion of $G$ into the semidirect product, and the composite of this inclusion with conjugation by $n$, is precisely $H$, which would prove that $H$ is equal to its own dominion in $G$ in the variety $\mathcal{V}$.

We can obtain such an $\mathcal{N}$-group by letting $N$ be a direct product of $|G/H|$ copies of a single nontrivial group $M \in \mathcal{N}$, with the action of $G$ permuting the copies by left multiplication on the left cosets of $H$. We then let $n$ be an element of $N \subseteq M \wr G/H G$ with support equal to $\{H\}$, which finishes the proof. \[\square\]

What about groups in $\mathcal{V}$ but not $\mathcal{Q}$? We have a partial answer in the following three results:

**Theorem 3.10.** Let $\mathcal{V} = \mathcal{N}\mathcal{Q}$ be a variety, where $\mathcal{N}$ is a nontrivial variety. Let $G \in \mathcal{V}$, and let $N \triangleleft G$, with $N \in \mathcal{N}$ and $G/N \in \mathcal{Q}$. Then for all subgroups $H$ of $G$, $\text{dom}^\mathcal{V}_{G}(H) \subseteq NH$. In particular,

$$\text{dom}^\mathcal{V}_{G}(H) \subseteq \mathcal{Q}(G)H.$$ 

*Proof:* Since $H \subseteq NH$, we have that $\text{dom}^\mathcal{V}_{G}(H) \subseteq \text{dom}^\mathcal{V}_{G}(NH)$. Also,

$$NH/N \subseteq G/N \in \mathcal{Q}.$$
By Theorem 3.9, \( \text{dom}_{G/N}^V(NH/N) = NH/N \). Since dominions respect quotients, we know that \( \text{dom}_{G/N}^V(NH/N) = (\text{dom}_{G}^V(NH))/N \). Therefore,

\[
\text{dom}_{G}^V(H) \subseteq \text{dom}_{G}^V(NH) = NH
\]

completing the proof. \( \square \)

**Lemma 3.11.** Let \( V = N \mathcal{Q} \) be a nontrivial factorization of \( V \), and let \( G \in V \), \( H \) a subgroup of \( G \), and \( N = \mathcal{Q}(G) \). Let \( D = \text{dom}_N^V(N \cap H) \). Then

\[
HD = \langle H, D \rangle \subseteq \text{dom}_G^{N \mathcal{Q}}(H).
\]

**Proof:** Let \( K \in V \) and let \( f, g: G \to K \) be two group morphisms, such that \( f|_H = g|_H \).

Since \( N = \mathcal{Q}(G) \), we must have \( f(N), g(N) \subseteq \mathcal{Q}(K) \). In particular, \( f|_N \) and \( g|_N \) are actually maps of \( N \)-groups. We also have \( f|_{H \cap N} = g|_{H \cap N} \), so we must have that \( f|_N \) and \( g|_N \) agree on \( \text{dom}_N^V(N \cap H) = D \). In particular, \( f \) and \( g \) agree on \( D \). Therefore,

\[
\langle H, D \rangle \subseteq \text{dom}_G^{N \mathcal{Q}}(H).
\]

Next we claim that \( \langle H, D \rangle = HD \). Indeed, \( H \cap N \) is normal in \( H \), and since the dominion certainly respects isomorphisms of group and subgroup pairs, the \( N \)-dominion \( D \) of \( H \cap N \) in \( N \) is likewise fixed by the automorphisms of \( N \) induced by elements of \( H \). Thus \( H \subseteq N_G(D) \), so \( \langle H, D \rangle = HD \), as claimed. \( \square \)

With slightly more information, we can obtain a stronger result:

**Theorem 3.12.** Let \( V = N \mathcal{Q} \) be a nontrivial factorization of \( V \), and let \( G \in V \), \( H \) a subgroup of \( G \), and \( N = \mathcal{Q}(G) \). Let \( D = \text{dom}_N^V(N \cap H) \). If \( N_G(D)N = G \), then

\[
\text{dom}_G^{N \mathcal{Q}}(H) = HD.
\]

**Proof:** First note that \( \langle H, D \rangle = HD \) as in the proof of Lemma 3.11.
By Lemma 3.11, we already know that $HD \subseteq \text{dom}_{G}^{N}(Q)(H)$. To prove that under the added hypothesis that $N_{G}(D)N = G$ we also get the inclusion $\text{dom}_{G}^{N}(Q)(H) \subseteq HD$, we need to define a transversal of $N$ in $G$. We claim there is a transversal $\tau: G/N \to G$ with the following properties:

(3.13) $\tau(N) = e$.

(3.14) $\tau(yN) \in N_{G}(D)$ for all $yN \in G/N$.

(3.15) For every $h \in H$, $y \in G$, there exists an element $h' \in H$ such that $\tau(yh^{-1}N) = \tau(yN)h'^{-1}$.

Since $N_{G}(D)N = G$, conditions (3.13) and (3.14) are easy to achieve. The reason for (3.15) will become apparent during the course of the proof.

We now establish the existence of a transversal satisfying (3.13)–(3.15). Let us look at the left action of $H$ on the set of cosets of $N$, under which $h \in H$ takes the coset $tN$ to the coset $tNh^{-1} = th^{-1}N$. Since $N$ is normal, this is a well defined action. Thus, the set of cosets of $N$ is divided into orbits by the $H$-action.

For each $H$-orbit, we first define $\tau$ to take some arbitrary coset $tN$ in that orbit to a representative in $N_{G}(D)$, which we now choose once and for all, making sure to select $e$ as a representative for $N$ to satisfy (3.13). For any other coset $t'N$ in the same orbit, there exists an element $h \in H$ such that $t' \equiv \tau(tN)h^{-1}$ (mod $N$), because this corresponds to the $H$-action on the set of cosets. Choose such an $h$ for each coset (the choice of $h$ is only determined up to congruence modulo $H \cap N$), and define $\tau(t'N) = \tau(tN)h^{-1}$. Since $H \subseteq N_{G}(D)$, this guarantees that both (3.14) and (3.15) are satisfied by $\tau$. Note that the above choices require the use of the Axiom of Choice if the number of orbits is not finite.

Let $\pi: G \to G/N$ be the canonical projection onto the quotient. For simplicity, we will now write the cosets using their chosen representatives; that is, whenever we write such a coset as $tN$ it will be understood that $t$ is the chosen representative.
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of that coset. If we wish to represent the coset of an arbitrary element \( y \in G \), where \( y \) is not the chosen representative of its coset, we will call this \( \pi(y) \).

Since \( G \) is an extension of \( N \) by \( G/N \), we have an embedding defined as in Theorem 2.6 (and depending on our choice of \( \tau \)), \( \gamma: G \rightarrow N \vartriangleleft G/N \); since \( N \in \mathcal{N} \), and \( N = \mathcal{Q}(G) \), we have \( G/N \in \mathcal{N} \mathcal{Q} \).

Consider the group \( N \vartriangleleft N \). Let \( \lambda, \rho: N \rightarrow \vartriangleleft N \) be the two immersions of \( N \) into \( \vartriangleleft N \). Since \( D \) is its own dominion in \( N \) (in the variety \( \mathcal{N} \)), we have \( \lambda(n) = \rho(n) \) if and only if \( n \in D \).

By Theorem 2.5, the maps \( \lambda \) and \( \rho \) induce two maps

\[
\lambda^*: \rho^*: N \vartriangleleft (G/N) \rightarrow (N \vartriangleleft D \vartriangleleft N) \vartriangleleft (G/N).
\]

We now consider the composite maps \( \lambda^* \circ \gamma \) and \( \rho^* \circ \gamma \).

Let \( n \in N \). By definition of \( \gamma \), we have \( \gamma(n) = \varphi_n \), where \( \varphi_n: G/N \rightarrow N \) is given by

\[
\varphi_n(yN) = \tau\left( yN\pi(n)^{-1}\right)n\tau(yN)^{-1} = \tau(yN)n\tau(yN)^{-1} = yny^{-1}.
\]

Therefore, \( \lambda^* \circ \varphi_n(yN) = \lambda(yny^{-1}) \), and \( \rho^* \circ \varphi_n(yN) = \rho(yny^{-1}) \). Comparing the values at \( y = e \), we see that \( \lambda^* \circ \gamma(n) = \rho^* \circ \gamma(n) \) only if \( n \in D \). We claim that in fact \( \lambda^* \circ \gamma(n) = \rho^* \circ \gamma(n) \) if and only if \( n \in D \).

To establish this claim, let \( d \in D \), and let \( yN \in G/N \). Then we have that \( \varphi_d(yN) = ydy^{-1} \). By choice of \( \tau \), we must have \( y \in N_G(D) \), so \( ydy^{-1} \in D \). In particular,

\[
\lambda(ydy^{-1}) = \rho(ydy^{-1}).
\]

This proves the claim.

Therefore, \( (\rho^* \circ \gamma)|_N \) and \( (\lambda^* \circ \gamma)|_N \) agree exactly on \( D \). We claim that \( \rho^* \circ \gamma \) and \( \lambda^* \circ \gamma \) also agree on \( H \).

Let \( h \in H \). Then \( \gamma(h) = \pi(h)\varphi_h \), as in Theorem 2.6. Since \( \lambda^* \) and \( \rho^* \) leave the \( G/N \) component unchanged, we may concentrate on \( \varphi_h \).
Let \( yN \in G/N \). By definition of \( \gamma \) we have

\[
\varphi_h(yN) = \tau(yN\pi(h)^{-1}) h\tau(yN)^{-1} = yh'^{-1}hy^{-1}
\]
where \( yh'^{-1} = \tau(yN\pi(h)^{-1}) \). In particular, we have \( h'^{-1} \equiv h^{-1} \pmod{N} \), so that \( h'^{-1}h \in H \cap N \). Therefore, \( h'^{-1}h \in D \), and \( y \in N_G(D) \) by choice of \( \tau \), so \( \varphi_h(yN) \in D \) for all \( yN \in G/N \). Therefore, \( \lambda^* \circ \varphi_h = \rho^* \circ \varphi_h \). So we have that \( \lambda^* \circ \gamma|_H = \rho^* \circ \gamma|_H \). In particular, the dominion is contained in the equalizer of these two maps.

We therefore have that

\[
(H, D) \subseteq \text{dom}_{N\mathbb{Q}}^N(H) \subseteq HN,
\]
and \( N \cap \text{dom}_{N\mathbb{Q}}^N(H) = D \). Now let \( d \) be an element of \( \text{dom}_{N\mathbb{Q}}^N(H) \). Since \( \text{dom}_{N\mathbb{Q}}^N(H) \subseteq HN \) by Theorem 3.10, we can write \( d = hn \) for some \( h \in H \), \( n \in N \). In particular, \( h^{-1}d = n \in \text{dom}_{N\mathbb{Q}}^N(H) \cap N \). Since \( D = \text{dom}_{N\mathbb{Q}}^N(H) \cap N \), we must have \( n \in D \), so \( \text{dom}_{N\mathbb{Q}}^N(H) \subseteq \langle H, D \rangle \), as claimed. This proves the equality and concludes the proof.

Remark 3.16. Note that we used the amalgamated coproduct \( N \coprod_D N \) in the construction. The properties that were used of this coproduct were that there are two maps from our group \( N \) into the amalgamated coproduct, which agree on \( D \) and disagree elsewhere. We could replace the amalgamated coproduct with any group with this property. For example, if \( \mathcal{N} = \mathcal{A} \), the variety of abelian groups, we could take the group \( N/D \) and the canonical projection and the zero map from \( N \) into \( N/D \).

Remark 3.17. We might wonder whether the condition that \( N_G(D)N = G \) is really necessary. It is certainly used in the construction given in the proof. For suppose that some coset \( tN \in G/N \) does not intersect \( N_G(D) \). When we try to prove that \( \lambda^*\gamma|_D = \rho^*\gamma|_D \), we run into a problem, for there exists some \( d \in D \) such that \( \tau(tN)d\tau(tN)^{-1} \notin D \), whatever the definition of \( \tau(tN) \) is, and this
means that $\lambda \circ \varphi_d(tN) \neq \rho \circ \varphi_d(tN)$. In any case, we already have examples where $\text{dom}_{N/Q}^G(H) \subseteq H$. Namely, letting $N = Q = A$, $V$ is the variety of metabelian groups; since $D = H \cap Q(G)$, the equality given in the theorem would imply that dominions are trivial in $A^2$. However, there are nontrivial dominions in this variety (see [5]), so the equality does not hold in general.

**Corollary 3.18.** Let $\mathcal{V} = NQ$ be a nontrivial factorization of $\mathcal{V}$, and let $G \in \mathcal{V}$, $H$ a subgroup of $G$, and $N = Q(G)$. Let $D = \text{dom}_{N}^N(N \cap H)$, and let $D'$ be a subgroup of $N$ such that $D \subseteq D'$, $\text{dom}_{N}^N(D') = D'$, $HD' = D'H$ and $N_G(D')N = G$. Then

$$\langle H, D \rangle \subseteq \text{dom}_{N}^{NQ}(H) \subseteq HD'.$$

**Proof:** Let $H' = \langle H, D' \rangle$. Since $HD' = D'H$, it follows that $H' = HD'$. First we claim that $H' \cap N = D'$. Indeed, if $hd' \in H' \cap N$, with $h \in H$ and $d' \in D'$, then since $D' \subseteq N$, we must have $h \in N$. But $H \cap N = D \subseteq D'$, so $hd' \in D'$, as claimed.

By hypothesis, $\text{dom}_{N}^N(D') = D'$, so we apply Theorem 3.12 to $H'$ which tells us that $H' = \text{dom}_{G}^{NQ}(H')$. Since $H \subseteq H'$, it follows that $\text{dom}_{G}^{NQ}(H) \subseteq H'$, as claimed.

**Corollary 3.19.** Let $\mathcal{V} = NQ$ be a nontrivial factorization of $\mathcal{V}$, let $G \in \mathcal{V}$, $H$ a subgroup of $G$, $N = Q(G)$, and let $D'$ be the normal closure of $H \cap N$ in $G$. Then

$$(3.20)\quad \langle H, \text{dom}_{N}^N(H \cap N) \rangle \subseteq \text{dom}_{G}^{NQ}(H) \subseteq \langle H, D' \rangle.$$

**Proof:** This is just a special case of Corollary 3.18, since the normality of $D'$ implies that $HD' = D'H$. 

**Corollary 3.21.** Let $\mathcal{V} = NQ$ be a nontrivial factorization of $\mathcal{V}$, and let $G \in \mathcal{V}$, $H$ a subgroup of $G$, and $N = Q(G)$. If $H \cap N \trianglelefteq G$, then $\text{dom}_{G}^{NQ}(H) = H$. In particular, if $H \cap N = \{e\}$, then $\text{dom}_{G}^{NQ}(H) = H$.

**Proof:** If $H \cap N \trianglelefteq G$, then $H \cap N$ is also normal in $N$, hence equals its own dominion in $N$. The rest now follows from Corollary 3.18.
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It is not hard to show that in the context of Corollary 3.19, either, neither, or both of the inclusions in (3.20) can be proper.

Section 4. Applications: varieties with nontrivial dominions

First, we make a few remarks for the extreme cases, when $\mathcal{N} \subseteq \mathcal{Q}$, and when $\mathcal{N} \cap \mathcal{Q} = \mathcal{E}$. In the latter case, we say that $\mathcal{N}$ and $\mathcal{Q}$ are disjoint.

Theorem 4.22. Let $\mathcal{N}$ and $\mathcal{Q}$ be two varieties, such that $\mathcal{N} \subseteq \mathcal{Q}$, and let $G \in \mathcal{N}$. Then for any subgroup $H$ of $G$, $\text{dom}^{N,Q}_G(H) = H$.

Proof: Follows from Theorem 3.9, since $G \in \mathcal{N}$ implies $G \in \mathcal{Q}$. \qed

Before going to the second case, let us characterize the pairs of varieties $\mathcal{N}$ and $\mathcal{Q}$ which are disjoint. Recall that we say that a variety $\mathcal{N}$ has exponent $n$ if the variety satisfies the law $x^n$.

Lemma 4.23. Two varieties $\mathcal{N}$ and $\mathcal{Q}$ are disjoint if and only if they are of relatively prime exponents.

Proof: If $\mathcal{N}$ and $\mathcal{Q}$ are of relatively prime exponents, then any group $G \in \mathcal{N} \cap \mathcal{Q}$ satisfies the laws $x^n$ and $x^m$, where $\gcd(n, m) = 1$. Therefore, it satisfies the law $x$, so $G = \{e\}$. Hence $\mathcal{N}$ and $\mathcal{Q}$ are disjoint. Now suppose that $\mathcal{N}$ and $\mathcal{Q}$ are not of relatively prime exponents. Then there must be a nontrivial factor group of the free $\mathcal{N}$-group of rank 1 which is isomorphic to some nontrivial factor group of the free $\mathcal{Q}$-group of rank 1, so the varieties are not disjoint. \qed

Theorem 4.24. Let $\mathcal{V} = \mathcal{N} \mathcal{Q}$, where $\mathcal{N}$ and $\mathcal{Q}$ are disjoint nontrivial varieties of groups. Let $G \in \mathcal{N}$, and let $H$ be a subgroup of $G$. Then

$$\text{dom}^{N,Q}_G(H) = \text{dom}^N_G(H).$$

Proof: Let $D = \text{dom}^N_{\mathcal{Q}(G)}(H \cap \mathcal{Q}(G))$. Since $G \in \mathcal{N}$, it follows that $\mathcal{Q}(G) = G$, so $N_G(D) \mathcal{Q}(G) = G$. The conditions of Theorem 3.12 are thus satisfied. Since $\mathcal{Q}(G) = G$, $H \cap \mathcal{Q}(G) = H$, so $D = \text{dom}^N_G(H)$; since $H \subseteq D$, it follows that

$$HD = D = \text{dom}^{N,Q}_G(H)$$

as claimed. \qed
Recall as well that a group $G$ in a variety $V$ is said to be absolutely closed (in $V$) if for any group $K \in V$, with $G$ a subgroup of $K$, we have $\text{dom}^V_K(G) = G$.

**Theorem 4.25.** Let $V = NQ$, where $N$ and $Q$ are disjoint nontrivial varieties of groups, and let $G \in Q$. Then $G$ is absolutely closed in $V$.

**Proof:** Let $K$ be a group in $V$ such that $G$ is a subgroup of $K$. Consider the subgroup $G \cap Q(K)$. Since $Q(K)$ is an $N$-group and $G \in Q$, it follows that $G \cap Q(K)$ is trivial. By Corollary 3.21, $\text{dom}^V_K(G) = G$. □

**Theorem 4.26.** There are uncountably many varieties of solvable groups with instances of nontrivial dominions. Moreover, there are uncountably many varieties of groups with instance of nonsurjective epimorphisms.

**Proof:** In [5] we established that the varieties of nilpotent groups of class at most 2 and exponent $\ell^2$ (for an odd prime $\ell$) have instances of nontrivial dominions. Also, Ol’šanskiı (see [9]) has shown that for any two relatively prime odd numbers $p$ and $q$, there exist uncountably many varieties of solvable groups with exponent $8pq$; and Vaughan-Lee (see [10]) has established the existence of uncountably many varieties of solvable groups with exponent 16. Theorem 4.24 now gives uncountably many varieties of solvable groups with nontrivial dominions, by taking products of suitably chosen subvarieties of $N_2^\ell$ and exponent $\ell^2$, with solvable varieties of exponent relatively prime to $\ell$.

An example of B.H. Neumann in [8] shows that $\text{Var}(A_5)$ has instances of non-surjective epimorphisms, namely the immersion of $A_4$ into $A_5$. Now consider the product of $V = \text{Var}(A_5)$ with any variety of solvable groups $S$; by the theorems of Ol’šanskiı and Vaughan-Lee quoted above, there are uncountably many such varieties. Since $A_5$ is simple and nonabelian, it follows that $S(A_5) = A_5$ for all such $S$, so that Theorem 3.12 applies to give

$$\text{dom}^{V\Sigma}_{A_5}(A_4) = \text{dom}^V_{A_5}(A_4) = A_5;$$

so $V\Sigma$ has instances of nonsurjective epimorphisms, giving the result. □
For the property of having nontrivial dominions, we can show even more. Using Lemma 3.11, we can establish that if $N$ is a variety with this property, and $Q$ is any nontrivial variety, then $NQ$ also has instances of nontrivial dominions.

The idea is to find a group in $NQ$ whose $Q$-verbal subgroup is isomorphic to a group in $N$ which has a subgroup with a nontrivial dominion. Then we can pick the corresponding subgroup and apply Lemma 3.11. To prove this, we first need a few technical lemmas. Along the way we will also remark on a couple of consequences of these lemmas that are interesting in their own right.

Some of the following lemmas are stated in generality, with $V$ representing an arbitrary variety of algebras of a given type. In any case, we drop for now the assumption that $V$ can be written as a product $V = NQ$.

The first lemma is due to Isbell, and it is valid not only for varieties but also for what he calls “right closed categories”:

**Lemma 4.27.** (Theorem 1.2 in [1]) Let $V$ be a variety of algebras, and suppose that $V$ has instances of nontrivial dominions. Then there exists a finitely generated algebra $G \in V$ and a finitely generated subalgebra $H$ such that

\[ H \subseteq \text{dom}^V_G(H). \]

**Proof:** We sketch the argument, and refer the reader to Isbell’s paper for details. Let $G'$ be an algebra, and $H'$ a subalgebra such that the dominion of $H'$ in $G'$ is nontrivial; let $d \in G' \setminus H'$ be an element of the dominion not in $H'$. One considers the coproduct $G' \amalg V G'$, the amalgamated coproduct $G' \amalg^V_H G'$, and the natural quotient map $\pi$ between them. The fact that $d$ is in the dominion means that the elements $\lambda(d)$ and $\rho(d)$ of $G' \amalg^V G'$ have the same image under $\pi$. This can be expressed as a finite sequence of elements $(w_0, \ldots, w_n)$ of $G' \amalg^V G'$, starting with $w_0 = \lambda(d)$, ending with $w_n = \rho(d)$, and such that in $(G' \amalg^V G') \times (G' \amalg^V G')$, each element $(w_j, w_{j+1})$ lies in the subalgebra generated by all elements of the form $(x, x)$, $(\lambda(h), \rho(h))$, and $(\rho(h), \lambda(h))$, with $h \in H'$. We let $H$ be the algebra generated by the $h$’s used in this finite expression, and let $G$ be obtained by throwing in all other elements that we need.  

\[ \square \]
Remark 4.28. Note also that at most finitely many identities of \( \mathcal{V} \) are needed to express the fact that \( \lambda(d) \) and \( \rho(d) \) have the same image under \( \pi \). So the argument above shows that if the variety \( \mathcal{V} \) is not finitely based (that is, it is not defined by a finite number of identities), then there exists a finitely based variety \( \mathcal{V}' \), with \( \mathcal{V} \subseteq \mathcal{V}' \), and a finitely generated algebra \( G \in \mathcal{V} \) with a finitely generated subalgebra \( H \), such that the dominion of \( H \) in \( G \) (in the variety \( \mathcal{V}' \)) is nontrivial. Namely, we let \( \mathcal{V}' \) be the variety defined only by the identities used in the process described.

Remark 4.29. The argument may also be adapted to show that one can take \( H \) finitely generated, and \( G \) not merely finitely generated, but also finitely presented.

Remark 4.30. The statement analogous to Lemma 4.27 is not true if we replace the variety \( \mathcal{V} \) with an arbitrary category of algebras, and not even in the case of pseudovarieties. Explicitly, Example 8.87 in [5] shows that there are non-trivial dominions in the category of all nilpotent groups (which is a pseudovariety). However, Theorem 3.11 in [6] shows that dominions of subgroups of finitely generated nilpotent groups are trivial in the category of all nilpotent groups. Thus Lemma 4.27 does not hold in general.

Recall that the variety \( \mathcal{V}^{(m)} \) is the variety defined by the \( m \)-variable identities of \( \mathcal{V} \), or equivalently, the variety of all algebras whose \( m \)-generator algebras belong to \( \mathcal{V} \).

Lemma 4.31. Let \( \mathcal{V} \) be a variety of algebras with instances of nontrivial dominions. Then there exists an \( n > 1 \) such that for all \( m \geq 2n \), the variety \( \mathcal{V}^{(m)} \) also has instances of nontrivial dominions.

Proof: By Lemma 4.27, there is a finitely generated algebra \( G \) with a subalgebra \( H \) which has nontrivial dominion. Say that \( G \) is an \( n \)-generated algebra. Let \( m \geq 2n \), and consider the algebra

\[
K = G \Join_{\mathcal{V}^{(m)}} G.
\]

Since \( K \) is generated by two copies of \( G \), it is a \( 2n \)-generated algebra, hence lies in \( \mathcal{V} \). In particular, since the two embeddings of \( G \) into \( K \) agree on \( H \), they agree on \( \text{dom}_G^V(H) \), which properly contains \( H \). Hence \( H \) is properly contained in \( \text{dom}_G^{(m)}(H) \), as claimed. \( \square \)
Remark 4.32. We remark, however, that we may have $\mathcal{V}^{(m)} = \mathcal{V}$ for sufficiently large $m$ (for example, if $\mathcal{V}$ is finitely based), so Lemma 4.31 may not yield any new information.

We now return our attention to varieties of groups.

Lemma 4.33. Let $\mathcal{V}$ be a variety of groups with instances of nontrivial dominions. Then there exists an $n > 1$, and a subgroup $H$ of $F_n = F_n(\mathcal{V})$, the relatively free $\mathcal{V}$-group of rank $n$, such that $H \subseteq \text{dom}^\mathcal{V}_{F_n}(H)$.

Proof: By Lemma 4.27, there exists a finitely generated group $G \in \mathcal{V}$, and a subgroup $K$ of $G$, such that $K \subseteq \text{dom}^\mathcal{V}_G(H)$.

Let $g_1, \ldots, g_n$ be generators for $G$, and let $\pi : F_n(\mathcal{V}) \to G$ be the map from the relatively free group of rank $n$ to $G$ sending the free generators of $F_n(\mathcal{V})$, $x_1, \ldots, x_n$ to $g_1, \ldots, g_n$, respectively. Let $N = \ker(\pi)$.

Let $H$ be the subgroup of $F_n(\mathcal{V})$ containing $N$ which corresponds, under $\pi$, to the subgroup $K$ of $G$. Since dominions respect quotients in varieties,

$$\text{dom}^\mathcal{V}_{F_n}(K) = \text{dom}^\mathcal{V}_{F_n/N}(H/N) = \left(\text{dom}^\mathcal{V}_{F_n}(H)\right)/N.$$ 

Since $K \subseteq \text{dom}^\mathcal{V}_G(K)$, it follows that $H \subseteq \text{dom}^\mathcal{V}_{F_n}(H)$, as desired. \hfill $\blacksquare$

Remark 4.34. The proof of Lemma 4.33 does not easily generalize to an arbitrary variety of algebras; the main problem lies in the argument regarding the quotient. It is easy to see that, given an algebra $A$, subalgebra $B$, and congruence relation $R$, we have an inclusion

$$\text{dom}^\mathcal{V}_A(H) / R \subseteq \text{dom}^\mathcal{V}_B(H/R);$$

in the case of varieties of groups, the reverse inclusion also holds, but this may not be true in general (it may not even be true of arbitrary classes of groups, although it is true of unions of pseudovarieties [5]). In the argument above, we know that the subalgebra on the right hand side is bigger than $H/R$, but if we simply pull back to $G$ we could, conceivably, end up with a subalgebra which is equal to $H$. I do not know if the general result itself is true, using a different proof.
Lemma 4.35. If $\mathcal{V}$ is a variety of groups with instances of nontrivial dominions, then there exists a subgroup $H$ of the countably generated relatively free $\mathcal{V}$-group $F = F_\infty(\mathcal{V})$ such that $H \subseteq \text{dom}^\mathcal{V}_F(H)$.

Proof: By Lemma 4.33, there exists an $n > 1$, and a subgroup $H$ of $F_n = F_n(\mathcal{V})$ such that $H \subseteq \text{dom}^\mathcal{V}_{F_n}(H)$. Let $i: F_n(\mathcal{V}) \to F$ be the immersion of $F_n$ into $F$ sending the free generators of $F_n$ to the first $n$ free generators of $F$. Then

$$i(H) \subseteq \text{dom}^\mathcal{V}_{i(F_n)}(i(H)) \subseteq \text{dom}^\mathcal{V}_F(i(H)).$$

This proves the lemma.

Lemma 4.36. Let $F = F_\infty(\mathcal{G})$ be the absolutely free group on infinitely many generators $x_1, x_2, \ldots, x_n, \ldots$, and let $\mathcal{V}$ be a nontrivial set of words. Then $\mathcal{V}(F)$ is free of countably infinite rank.

Proof: $\mathcal{V}(F)$ is a subgroup of an absolutely free group, so by a classical theorem of Schreier it is itself free. It remains to show that it is not finitely generated.

Suppose that $\mathcal{V}(F)$ is finitely generated. Then there is a bound to the subscripts of the free generators of $F$ that occur in the generators of $\mathcal{V}(F)$, so there exists an $n_0 > 0$ such that all generators of $\mathcal{V}(F)$ involve only $x_1, x_2, \ldots, x_{n_0}$.

Now let $v(z_1, \ldots, z_r) \in \mathcal{V}$ be a nonidentity element; note that the element

$$v(x_{n_0+1}, \ldots, x_{n_0+r})$$

is a nonidentity element of $\mathcal{V}(F)$, expressible in terms of a set of generators disjoint from $x_1, \ldots, x_{n_0}$, which contradicts the assumption that all elements of $\mathcal{V}(F)$ are expressible in terms of $x_1, \ldots, x_{n_0}$ only. We conclude that $\mathcal{V}(F)$ is of countably infinite rank, as claimed.

We can now prove the result we mentioned above:

Theorem 4.37. Let $\mathcal{N}$ be a variety of groups with instances of nontrivial dominions, and let $\mathcal{Q}$ be any variety of groups different from $\mathcal{G}$. Then $\mathcal{N} \mathcal{Q}$ also has instances of nontrivial dominions.
Proof: Let $F = F_{\infty}(\mathcal{N}\mathcal{Q})$ be the countably generated relatively free group in $\mathcal{N}\mathcal{Q}$. By Proposition 2.1 $F$ is given by $F_{\infty}(\mathcal{G})/\mathcal{N}(Q(F_{\infty}(\mathcal{G})))$, and the $Q$-verbal subgroup of $F$ is given by

$$Q(F) = Q(F_{\infty}(\mathcal{G}))/\mathcal{N}(Q(F_{\infty}(\mathcal{G}))).$$

Since $Q \neq \mathcal{G}$, we have $Q(F_{\infty}) \neq \{e\}$. We claim that $Q(F) \cong F_{\infty}(\mathcal{N})$. Indeed, $Q(F_{\infty}(\mathcal{G}))$ is isomorphic to $F_{\infty}(\mathcal{G})$ by Lemma 4.36. Therefore,

$$Q(F) = Q(F_{\infty}(\mathcal{G}))/\mathcal{N}(Q(F_{\infty}(\mathcal{G})))$$
$$\cong F_{\infty}(\mathcal{G})/\mathcal{N}(F_{\infty}(\mathcal{G}))$$
$$\cong F_{\infty}(\mathcal{N}).$$

By Lemma 4.35, since $\mathcal{N}$ has instances of nontrivial dominions, there exists a subgroup $H$ of $F_{\infty}(\mathcal{N})$ such that

$$H \subseteq \text{dom}^{\mathcal{N}}_{F_{\infty}(\mathcal{N})}(H).$$

Now consider $H$ as a subgroup of $Q(F)$, and hence as a subgroup of $F$. By Lemma 3.11, if we let $G = F$, we have that

$$\left< H, \text{dom}^{\mathcal{N}}_{Q(F)}(H \cap Q(F)) \right> \subseteq \text{dom}^{\mathcal{N}\mathcal{Q}}_{F}(H).$$

But $H \cap Q(F) = H$, and by choice of $H$, $H \subseteq \text{dom}^{\mathcal{N}}_{Q(F)}(H)$. In particular,

$$H \subseteq \left< H, \text{dom}^{\mathcal{N}}_{Q(F)}(H) \right> = \text{dom}^{\mathcal{N}}_{Q(F)}(H) \subseteq \text{dom}^{\mathcal{N}\mathcal{Q}}_{F}(H)$$

so $\mathcal{N}\mathcal{Q}$ has instances of nontrivial dominions, as claimed.

Remark 4.38. Theorem 4.37 now provides an alternative proof to the first sentence of Theorem 4.26, simply by taking any solvable variety $\mathcal{N}$ with instances of nontrivial dominions (e.g. the variety of nilpotent groups of class 2, $A^2$, etc.), and looking at all possible products $\mathcal{N}\mathcal{Q}$ with $Q$ solvable and nontrivial.
Remark 4.39. We might ask whether there is a converse to Theorem 4.37. That is, given a nontrivial factorization $\mathcal{V} = \mathcal{N}\mathcal{Q}$, can we deduce that $\mathcal{V}$ has trivial dominions if $\mathcal{N}$ (or maybe, if both $\mathcal{N}$ and $\mathcal{Q}$) has trivial dominions? The answer to this question is no, as we have already noted that there are nontrivial dominions in the variety of metabelian groups, which has nontrivial factorization $\mathcal{V} = \mathcal{A}\mathcal{A}$, and all dominions are trivial in the variety of abelian groups.

Corollary 4.40. Let $\ell \geq 2$, and let $\mathcal{S}_\ell = \mathcal{A}^\ell$ be the variety of all solvable groups of solvability length at most $\ell$. Then $\mathcal{S}_\ell$ has instances of nontrivial dominions.

Proof: We write $\mathcal{S}_\ell = \mathcal{A}^2\mathcal{A}^{\ell-2}$. As we have already mentioned, there are instances of nontrivial dominions in $\mathcal{A}^2$. The result now follows from Theorem 4.37.

Remark 4.41. On the other hand, if $\mathcal{C}_{S}$ is the category of all solvable groups, then dominions are trivial in $\mathcal{C}_{S}$. For given $G \in \mathcal{C}_{S}$, and $H$ a subgroup of $G$, we know that $G \in \mathcal{S}_\ell$ for some $\ell > 0$. Hence $G \in \mathcal{S}_{\ell+1} = \mathcal{S}\mathcal{S}_\ell$. By Theorem 3.9, $\text{dom}_{\mathcal{C}_{S}}^{\mathcal{S}_{\ell+1}}(H) = H$. In particular, $\text{dom}_{\mathcal{C}_{S}}^{\mathcal{S}_{\ell}}(H) = H$.

Corollary 4.40 shows the converse of Theorem 4.37 is false. This suggests the following questions: Given a nontrivial variety $\mathcal{N}$, does $\mathcal{N}^2$ always have instances of nontrivial dominions? More generally, given a variety $\mathcal{V} = \mathcal{N}\mathcal{Q}$, with $\mathcal{N}, \mathcal{Q}$ nontrivial, will $\mathcal{V}$ necessarily have instances of nontrivial dominions? I do not know the answer to these two questions.

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