Numerical model for calculating displacements near a crack

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Abstract. The paper presents a two-dimensional model of elastic deformations. An isotropic medium region in near a crack propagating along the surface is considered. The results correlate well with an analytical solution.

1. Introduction
The model of calculation deformations and mechanical stresses near a crack formed after a pulsed heat load on material is presented in paper [1]. The application of the model for description of experimental results is sufficiently limited by the assumption of plastic deformation absence. Plastic deformation can be taken into account in a two-dimensional numerical model, while the discussed model is based on analytical reduction of two-dimensional equations. The paper is aimed at development of the two-dimensional numerical model of deformations near a crack. The specific objective of the paper is the calculation of the deformations of a quarter space with forces in the form of a derivative of the delta function and comparison of the results with the analytical results [2], which are the basis of the model in [1, 3, 4].

2. Problem definition
We consider the problem of finding displacements around a crack at the surface of a rectangular sample. Let us assume that the crack is located along the axis x (Figure 1). The equations of the linear theory of elasticity of isotropic medium without volume forces and temperature changes have the following form [2]:

\[(1 - 2\nu)\triangle \mathbf{u} + \text{grad} \text{ div} \mathbf{u} = 0,\]

where \(\mathbf{u} = (u, v)\) is the displacement and \(\nu\) is the Poisson ratio. We find a stationary solution for the system of equations of elliptic type via a steady-state solution of the following evolutionary system:
\[
\begin{aligned}
(1 - 2\nu) \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial z} \right) &= \frac{\partial u}{\partial \tau}, \\
(1 - 2\nu) \left( \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \frac{\partial}{\partial z} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial z} \right) &= \frac{\partial v}{\partial \tau},
\end{aligned}
\]

where \( \tau \) is the parameter of the steady state. In the calculation domain (Figure 2), let us set the location of the crack at the point \( z_0 \) on the axis \( z \).

Figure 1. Quarter space filled with an elastic medium.

Figure 2. Coordinate system indicates to boundaries of computational domain.

3. Boundary condition

Let \( \sigma \) be the stress tensor with the components \( \sigma_{ij} \) and \( f = (f_i) = (\sigma_{ij} n_i) \) be the surface forces, where the indexes correspond to the directions of the axes \( y, z \). The relation between the stresses and displacements is expressed by the Hooke law:

\[
\sigma_{ij} = E \left( \frac{1}{1 + \nu} \varepsilon_{ij} + \frac{\nu}{(1 + \nu)(1 - 2\nu)} (\varepsilon_{yy} + \varepsilon_{zz}) \delta_{ij} \right),
\]

where \( E \) is the Young modulus, \( \delta_{ij} \) is the Kronecker delta, and \( \varepsilon_{ij} \) is the strain tensor:

\[
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).
\]

The Poisson ratio and the Young modulus fully characterize the elastic properties of the isotropic material.

We denote the boundaries as \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \), as in Figure 2. At the boundaries \( \gamma_1, \gamma_3 \) the surface forces are zero. The crack at the point \( z_0 \) is given through the surface forces on the boundary \( \gamma_3 \) by the derivative of the delta function. We fix the boundary \( \gamma_4 \) by the conditions \( u = 0 \). Eventually, the boundary conditions can be written as:

\[
\begin{align*}
\gamma_1, \gamma_3 : & \quad \begin{cases} 
\sigma_{yz} = 0, \\
\sigma_{zz} = 0,
\end{cases} \quad \Rightarrow \begin{cases}
\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y} = 0, \\
\nu \frac{\partial u}{\partial y} + (1 - \nu) \frac{\partial v}{\partial z} = 0,
\end{cases} \\
\gamma_2 : & \quad \begin{cases} 
\sigma_{yz} = 0, \\
\sigma_{yy} = \delta'(z - z_0),
\end{cases} \quad \Rightarrow \begin{cases}
\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y} = 0, \\
(1 - \nu) \frac{\partial u}{\partial y} + \nu \frac{\partial v}{\partial z} = \frac{(1 - 2\nu)(1 + \nu)}{E} \delta'(z - z_0),
\end{cases} \\
\gamma_4 : & \quad u = 0, \quad v = 0.
\end{align*}
\]
4. Numerical simulation

We introduce in the calculation domain a square grid with a step $h$. Let us denote by $u_{km}^n$ and $v_{km}^n$ the values of $u$ and $v$ at the node $(y_k,z_m), \; k = 1, N_y, \; m = 1, N_z$, at the $n$-th iteration.

The numerical implementation is based on the Douglas–Rachford scheme and the run method [5]. The finite difference scheme for the presented system of equations has the following form:

$$
\begin{align*}
\frac{u_{km}^{n+1/2} - u_{km}^n}{\tau} &= (1 - 2\nu) \left( \Delta_y^{n+1/2} u + \Delta_z^{n+1/2} u \right) + \Delta_y^{n+1/2} u + \Delta_y^{n+1/2} v, \\
\frac{u_{km}^{n+1} - u_{km}^{n+1/2}}{\tau} &= (1 - 2\nu) \left( \Delta_z^{n+1} u + \Delta_y^{n+1} u \right), \\
\frac{v_{km}^{n+1/2} - v_{km}^n}{\tau} &= (1 - 2\nu) \left( \Delta_y^{n+1/2} v + \Delta_y^{n+1/2} v \right) + \Delta_y^{n+1/2} u + \Delta_y^{n+1/2} v, \\
\frac{v_{km}^{n+1} - v_{km}^{n+1/2}}{\tau} &= (1 - 2\nu) \left( \Delta_y^{n+1} u + \Delta_z^{n+1} u \right),
\end{align*}
$$

where

$$
\Delta_y^n f = f_{k+1,m}^n - 2f_{km}^n + f_{k-1,m}^n, \quad \Delta_z^n f = f_{km+1}^n - 2f_{km}^n + f_{km-1}^n,
$$

$$
\Delta_{yz}^n f = \frac{f_{k+1,m+1}^n - f_{k+1,m-1}^n - f_{k-1,m+1}^n + f_{k-1,m-1}^n}{4h^2}.
$$

For the numerical implementation by the prediction method, the system of difference equations is rewritten in the canonical form. Each equation is supplemented by a pair of conditions at the boundary:

$$
\begin{align*}
\frac{u_{1m}^{n+1/2} - u_{2m}^{n+1/2}}{\nu} &= u_{2m-1}^{n+1/2} - \nu v_{2m-1}^{n+1/2} + \frac{(1 - 2\nu)(1 + \nu)}{(1 - \nu)E} f_m, \\
u_{N_y m}^{n+1/2} &= 0.
\end{align*}
$$

$$
\begin{align*}
u_{k1}^{n+1/2} &= \nu_{k2}^{n+1/2} + \frac{\nu}{2(\nu - 1)} \left( \nu_{k+12}^{n} - \nu_{k-12}^{n} \right), \\
u_{kN_z}^{n+1/2} &= \nu_{k1}^{n+1/2} - \frac{\nu}{2(\nu - 1)} \left( \nu_{k1N_z}^{n} - \nu_{k1N_z-1}^{n} \right), \\
u_{k1}^{n+1/2} &= \nu_{k2}^{n+1/2} + \frac{\nu}{2(\nu - 1)} \left( \nu_{k12}^{n} - \nu_{k12}^{n} \right), \\
u_{kN_z}^{n+1/2} &= \nu_{k1}^{n+1/2} - \frac{\nu}{2(\nu - 1)} \left( \nu_{k1N_z}^{n} - \nu_{k1N_z-1}^{n} \right), \\
u_{k1}^{n+1} &= \nu_{k2}^{n+1} + \frac{\nu}{2} \left( \nu_{k2}^{n+1} - \nu_{k2}^{n+1} \right), \\
u_{kN_z}^{n+1} &= \nu_{k1}^{n+1} - \frac{\nu}{2} \left( \nu_{k1}^{n+1} - \nu_{k1}^{n+1} \right),
\end{align*}
$$

We used the following approximation $f_m$ of the derivative of the delta function at the boundary:

$$
f_m = \begin{cases} 1, & \text{if } m = m_z, \\ -1, & \text{if } m = m_z + 1, \\ 0, & \text{else}, \end{cases}
$$

where the node number $m_z = \left[ \frac{z_k}{h} \right]$ corresponds to the location of the crack.
5. Simulation results
In the calculations for the problem with simplified conditions on the boundary \( f_0 = \text{constant} \), the first order of convergence of the scheme was obtained.

The results of the calculation of the two components of the displacement vector in the vicinity of the crack are presented (Figure 3, 4). The calculations were carried out on a uniform grid of 401x401 nodes, with \( h = 0.025 \), \( \tau = 0.01 \). 15000 iterations were made. The crack location \( z_0 \) corresponds to nodes 40 and 41 at the boundary \( \gamma_2 \), which corresponds to 1.

Below is a comparison of the obtained calculations with the analytical solution [1, 6] for the \( u \) component on the boundary \( \gamma_2 \) (Figure 5) and \( v \) component on the boundary \( \gamma_1 \) (Figure 6). The calculations were made with the same parameters.

The small discrepancy between the numerical experiment and the analytical solution is presumably because of the distance \( h \) between the points of application of the force in calculations, while in reality this distance is infinitely small, and the fact that the infinite area is replaced by a finite square in the numerical calculation.
6. Conclusion
A model of elastic deformations was implemented, which describes displacements in the material around a crack propagating along the surface. The model problem in a two-dimensional formulation was solved numerically. The two components of the displacement vector were obtained. The calculated data correlate well with the analytical solution. In the future, it is planned to use the resulting model to calculate the conditions for the formation of cracks in metals under strong heat load.

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