On Stable Hypersurfaces with Vanishing Scalar Curvature

Gregório Silva Neto

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Abstract

We will prove that there are no stable complete hypersurfaces of $\mathbb{R}^4$ with zero scalar curvature, polynomial volume growth and such that $\frac{(-K)}{H^3} \geq c > 0$ everywhere, for some constant $c > 0$, where $K$ denotes the Gauss-Kronecker curvature and $H$ denotes the mean curvature of the immersion. Our second result is the Bernstein type one there is no entire graphs of $\mathbb{R}^4$ with zero scalar curvature such that $\frac{(-K)}{H^3} \geq c > 0$ everywhere. At last, it will be proved that, if there exists a stable hypersurface with zero scalar curvature and $\frac{(-K)}{H^3} \geq c > 0$ everywhere, that is, with volume growth greater than polynomial, then its tubular neighborhood is not embedded for suitable radius.

1 Introduction

Let $x : M^3 \to \mathbb{R}^4$ be an isometric immersion. If $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of the second fundamental form, then the scalar curvature $R$, the non-normalized mean curvature $H$, and the Gauss-Kronecker curvature $K$ are given, respectively, by

\[ R = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3, \quad H = \lambda_1 + \lambda_2 + \lambda_3 \quad \text{and} \quad K = \lambda_1 \lambda_2 \lambda_3. \]  

(1.1)
In 1959, Hartman and Nirenberg, cf. [8], have shown that the only surfaces with zero Gaussian curvature in three-dimensional Euclidean space are planes and cylinders.

Generalizing this fact, in 1977, Cheng and Yau, cf. [16], showed that the only complete non-compact hypersurfaces with constant scalar curvature and non-negative sectional curvature in the Euclidean space \( \mathbb{R}^{n+1} \) are the generalized cylinders \( S^{n-p} \times \mathbb{R}^p \).

Let \( D \subset M^3 \) be a regular domain, i.e., a domain with compact closure and piecewise smooth boundary. A \textit{compact supported variation} of the immersion \( x \) is a differentiable map \( X : (\epsilon, \epsilon) \times D \to \mathbb{R}^4 \), \( \epsilon > 0 \), such that, for each \( t \in (-\epsilon, \epsilon) \), \( X_t : D \to \mathbb{R}^4 \), \( X_t(p) = X(t, p) \) is an immersion, \( X_0 = x|_D \) and \( X_1|_{\partial D} = X_0|_{\partial D} \). We recall that hypersurfaces of \( \mathbb{R}^4 \) with zero scalar curvature are critical points of the functional

\[
\mathcal{A}_1(t) = \int_M H(t)dM_t
\]

under all variations compactly supported in \( D \) (see [13], [1], [14], [4]).

Following Alencar, do Carmo, and Elbert, cf. [2], let us define the concept of stability for immersions with zero scalar curvature. Let \( A : TM \to TM \) the linear operator associated to the second fundamental form of immersion \( x \). We define the \textit{first Newton transformation} \( P_1 : TM \to TM \) by

\[
P_1 = HI - A,
\]

where \( I \) denotes identity operator. We now introduce a second order differential operator which will play a role similar to that of Laplacian in the minimal case:

\[
L_1(f) = \text{div}(P_1(\nabla f)),
\]

where \( \text{div} X \) denotes the divergence of vector field \( X \), and \( \nabla f \) denotes the gradient of the function \( f \) in the induced metric. In [9], Hounie and Leite showed that \( L_1 \) is elliptic if and only if \( \text{rank} \ A = 1 \). Thus, \( K \neq 0 \) everywhere implies \( L_1 \) is elliptic, and if \( H > 0 \), then \( P_1 \) is a positive definite linear operator.

Computing the second derivative of functional \( \mathcal{A}_1 \) we obtain

\[
\frac{d^2\mathcal{A}_1}{dt^2} \bigg|_{t=0} = -2 \int_M f(L_1 f - 3Kf)dM,
\]

where \( f = \langle \frac{dX}{dt}(0), \eta \rangle \), and \( \eta \) is the normal vector field of the immersion.
Since \( H^2 = |A|^2 + 2R \), if \( R = 0 \) then \( H^2 = |A|^2 \), i.e., if \( K \neq 0 \) everywhere, then \( H^2 = |A|^2 \neq 0 \) everywhere. It implies that \( H > 0 \) everywhere or \( H < 0 \) everywhere. Hence, unlike minimal case, the sign of functional \( A_1 \) depends on choice of orientation of \( M^3 \). Following Alencar, do Carmo and Elbert, see [2], if we choose an orientation such that \( H > 0 \) everywhere, then the immersion will be stable if \( \frac{d^2 A_1}{dt^2} \bigg|_{t=0} > 0 \) under all compact support variations. Otherwise, i.e., if we choose an orientation such that \( H < 0 \), then \( x \) is stable if \( \frac{d^2 A_1}{dt^2} \bigg|_{t=0} < 0 \). For more details, see [2].

In the pursuit of this subject, Alencar, do Carmo and Elbert, cf. [2], have posed the following:

**Question.** Is there any stable complete hypersurface \( M^3 \) in \( \mathbb{R}^4 \) with zero scalar curvature and everywhere non-zero Gauss-Kronecker curvature?

The goal of this paper is to give some partial answers to this question. Let \( B_r(p) \) be the geodesic ball with center \( p \in M \) and radius \( r \). We say that a Riemannian manifold \( M^3 \) has *polynomial volume growth*, if there exists \( \alpha \in [0, 4] \) such that

\[
\frac{\text{vol}(B_r(p))}{r^{\alpha}} < \infty,
\]

for all \( p \in M \).

A well known inequality establishes that

\[
HK \leq \frac{1}{2} R^2. \tag{1.4}
\]

If \( R = 0 \) and \( K \neq 0 \) everywhere, then the quotient \( \frac{K}{H^3} \) is always negative, independent on choice of orientation. Furthermore, considering \( K \) and \( H^3 \) as functions of the eigenvalues of second fundamental form, we can see that

\[
0 < \frac{(-K)}{H^3} \leq \frac{4}{27},
\]

provided \( K \) and \( H^3 \) are homogeneous polynomials of degree 3. For details, see Appendix.

The first result is
Theorem A. There is no stable complete hypersurface $M^3$ of $\mathbb{R}^4$ with zero scalar curvature, polynomial volume growth and such that
\[ \frac{(-K)}{H^3} \geq c > 0 \]
everywhere, for some constant $c > 0$. Here $H$ denotes the mean curvature and $K$ denotes the Gauss-Kronecker curvature of the immersion.

As a consequence of Theorem A, we obtain the following Bernstein type result.

Theorem B. There are no entire graphs $M^3$ of $\mathbb{R}^4$ with zero scalar curvature and such that
\[ \frac{(-K)}{H^3} \geq c > 0 \]
everywhere, for some constant $c > 0$. Here $H$ denotes the mean curvature and $K$ denotes the Gauss-Kronecker curvature of the immersion.

Following Nelli and Soret, cf. [12], in section 5 we show that, if $M^3$ is a stable complete hypersurface of $\mathbb{R}^4$ with zero scalar curvature and such that $\frac{(-K)}{H^3} \geq c > 0$ everywhere, then the tube around $M$ is not embedded for suitable radius. Precisely, we define the tube of radius $h$ around $M$ the set
\[ T(M, h) = \{ x \in \mathbb{R}^4; \exists p \in M, x = p + t\eta, t \leq h(p) \} \]
where $\eta$ is the normal vector of second fundamental form of the immersion and $h : M \to \mathbb{R}$ is an everywhere non-zero smooth function. We prove

Theorem C. Let $M^3$ be a stable complete hypersurface of $\mathbb{R}^4$ with vanishing scalar curvature. Suppose that the second fundamental form of the immersion is bounded and there exists a constant $c > 0$ such that $\frac{(-K)}{H^3} \geq c > 0$ everywhere. Then, for constants $0 < b_1 \leq 1$, $b_2 > 0$, and for any smooth function $h : M \to \mathbb{R}$ satisfying
\[ h(p) \geq \min \left\{ \frac{b_1}{|A(p)|}, b_2 \rho(p)^{\delta} \right\}, \quad p \in M, \delta > 0, \]
the tube $T(M, h)$ is not embedded. Here, $\rho(p)$ denotes the intrinsic distance in $M$ to a fixed point $p_0 \in M$.  

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2 Preliminary Results

Let $B(X,Y) = \nabla_X Y - \nabla_X Y$ be the second fundamental form of immersion $x$, where $\nabla$ and $\bar{\nabla}$ are the connections of $M^3$ and $\mathbb{R}^4$, respectively. The shape operator is the only symmetric linear operator $A : TM \rightarrow TM$ such that

$$B(X,Y) = \langle A(X), Y \rangle \eta, \ \forall \ X,Y \in TM,$$

where $\eta$ is the normal field of the immersion $x$.

Denote by $|A|^2 = \text{tr}(A^2)$ the matrix norm of second fundamental form. Since $H^2 = |A|^2 + 2R$, if $R = 0$, then $H^2 = |A|^2$. Hence, $K \neq 0$ everywhere implies $H = |A| \neq 0$ everywhere, and we can choose an orientation of $M$ such that $H > 0$ everywhere.

 Remark 2.1. From now on, let us fix an orientation of $M^3$ such that $H > 0$ everywhere.

A well known inequality establishes that

$$HK \leq \frac{1}{2}R^2.$$

Therefore, by using inequality above, $R = 0$ and $H > 0$ everywhere implies $K < 0$ everywhere.

Define $P_1 : TM \rightarrow TM$ by $P_1 = HI - A$ the first Newton transformation. If $R = 0$ and $H > 0$, then $P_1$ is positive definite. It was proved by Hounie and Leite in a general point of view, see [9]. In fact, $P_1$ positive definite implies $L_1(f) = \text{div}(P_1(\nabla f))$ is an elliptic differential operator. Let us give here a proof for sake of completeness. It suffices to prove that $H - \lambda_i > 0, \ i = 1, 2, 3$. In fact,

$$\lambda_i^2(H - \lambda_1) = \lambda_i^2(\lambda_2 + \lambda_3) = \lambda_i^2\lambda_2 + \lambda_i^2\lambda_3.$$
Since \( R = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 = 0 \), we have
\[
0 = \lambda_1 R = \lambda_1 (\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3) = \lambda_1^2 \lambda_2 + \lambda_1^2 \lambda_3 + \lambda_1 \lambda_2 \lambda_3,
\]
i.e.,
\[
\lambda_1^2 \lambda_2 + \lambda_1^2 \lambda_3 = -\lambda_1 \lambda_2 \lambda_3 = -K > 0.
\]
Thus,
\[
\lambda_1^2 (H - \lambda_1) = \lambda_1^2 \lambda_2 + \lambda_1^2 \lambda_3 = -\lambda_1 \lambda_2 \lambda_3 > 0,
\]
and then, \( H - \lambda_1 > 0 \). The other cases are analogous.

Our choice of orientation, i.e., that one such that \( H > 0 \) everywhere, implies stability condition is equivalent to
\[
-3 \int_M K f^2 dM \leq \int_M \langle P_1 (\nabla f), \nabla f \rangle dM. \tag{2.1}
\]
The inequality (2.1) is known as stability inequality.

**Remark 2.2.** When \( H < 0 \), then \( K > 0 \) and \( P_1 \) is negative definite. In this case, stability condition is equivalent to
\[
3 \int_M K f^2 dM \leq \int_M \langle (-P_1) (\nabla f), \nabla f \rangle dM.
\]

Let \( \nabla A(X, Y, Z) := \langle \nabla_Z (A(X)) - A(\nabla_Z X), Y \rangle \) be the covariant derivative of operator \( A \). The following proposition will play an important role in the proof of main theorems.

In [6], do Carmo and Peng showed a very similar inequality for minimal hypersurfaces.

**Proposition 2.1.** If \( R = 0 \), then
\[
|\nabla A|^2 - |\nabla H|^2 \geq \frac{2}{3} |\nabla H|^2,
\]
where \( \nabla H \) denotes the gradient of \( H \).

**Proof.** Let us fix \( p \in M \) and choose \( \{e_1(p), e_2(p), e_3(p)\} \) an orthonormal basis of \( T_p M \) such that \( h_{ij}(p) = \lambda_i(p) \delta_{ij} \), where \( h_{ij} = \langle A(e_i), e_j \rangle \), \( \lambda_i(p) \) denotes the eigenvalues of \( A \) in \( p \) and \( \delta_{ij} \) is the Kronecker delta
\[
\delta_{ij} = \begin{cases} 
1 & \text{if } i = j; \\
0 & \text{if } i \neq j.
\end{cases}
\]
Extending this basis by parallel transport along geodesics starting on $p$, to a referential in a neighbourhood of $p$, we have $\nabla_{e_i(p)} e_j(p) = 0$, for all $i, j = 1, 2, 3$. This is called geodesic referential at $p$.

Let us denote by $h_{ij;k} = (h_{ij})_k := e_k(h_{ij})$ the covariant derivatives of function $h_{ij}$, and by $h_{ijk}$ the components of tensor $\nabla A$ in the referential $\{e_1, e_2, e_3\}$, i.e., $h_{ijk} = \nabla A(e_i, e_j, e_k)$. Since $\{e_1, e_2, e_3\}$ is a geodesic referential, we have

$$ h_{ijk} = \nabla A(e_i, e_j, e_k) = \langle \nabla_{e_k}(A(e_i)) - A(\nabla_{e_k} e_i), e_j \rangle = \langle \nabla_{e_k}(A(e_i)), e_j \rangle $$

$$ = e_k(\langle A(e_i), e_j \rangle) - (A(e_i), \nabla_{e_k} e_j) = e_k(\langle A(e_i), e_j \rangle) = e_k(h_{ij}) $$

$$ = h_{ij;k}. $$

Since $R = 0$, then $H^2 = |A|^2$. By using this fact, we have

$$ 4H^2 |\nabla H|^2 = |\nabla (H^2)|^2 = |\nabla (|A|^2)|^2 = \sum_{k=1}^{3} \left[ \left( \sum_{i,j=1}^{3} h_{ij}^2 \right)_k \right]^2 $$

$$ = \sum_{k=1}^{3} \left( \sum_{i,j=1}^{3} 2h_{ij}h_{ij;k} \right)^2 = 4 \sum_{k=1}^{3} \left( \sum_{i=1}^{3} h_{ii}h_{ii;k} \right)^2. $$

Now, by using Cauchy-Schwarz inequality, we obtain

$$ 4 \sum_{k=1}^{3} \left( \sum_{i=1}^{3} h_{ii}h_{ii;k} \right)^2 \leq 4 \sum_{k=1}^{3} \left[ \left( \sum_{i=1}^{3} h_{ii}^2 \right) \left( \sum_{i=1}^{3} h_{ii;k}^2 \right) \right] $$

$$ = 4|A|^2 \left( \sum_{i=1}^{3} h_{ii;k}^2 \right) = 4H^2 \left( \sum_{i=1}^{3} h_{ii;k}^2 \right). $$

Therefore,

$$ |\nabla H|^2 \leq \sum_{i,k=1}^{3} h_{ii;k}^2, \quad (2.2) $$

On the other hand,

$$ \nabla H = \sum_{k=1}^{3} \left( \sum_{i=1}^{3} h_{ii;k} \right) e_k. $$
By using Codazzi equations for immersions in Euclidean space, i.e., $h_{ijk} = h_{ikj}$, we have

$$|\nabla H|^2 = \sum_{k=1}^{3} \left( \sum_{i=1}^{3} h_{iik} \right)^2 \leq 3 \sum_{i,k=1}^{3} h_{iik}^2 = \frac{3}{2} \left[ \sum_{i,k=1}^{3} h_{ik}^2 + \sum_{i,k=1}^{3} h_{kii}^2 \right].$$

Therefore

$$\left(1 + \frac{2}{3}\right) |\nabla H|^2 \leq \sum_{i,k=1}^{3} h_{iik}^2 + \sum_{i,k=1}^{3} h_{kii}^2 \leq \sum_{i,j,k=1}^{3} h_{ijk}^2 = |\nabla A|^2.$$  

\[\square\]

3 Main Theorems

Hereafter, we will fix a point $p_0 \in M$ and denote by $B_r$ the geodesic (intrinsic) ball of center $p_0$ and radius $r$.

The main tool to prove Theorem A stated in the Introduction is the following

**Proposition 3.1.** Let $x : M^3 \to \mathbb{R}^4$ be a stable isometric immersion with zero scalar curvature and such that $K$ is nowhere zero. Then, for all smooth function $\psi$ with compact support in $M$, for all $\delta > 0$ and $0 < q < \sqrt{1/3}$, there exists constants $\Lambda_1(q), \Lambda_2(q) > 0$ such that

$$\int_M H^{5+2q} \left( \frac{(-K)}{H^3} - \Lambda_1 \delta^{\frac{5+2q}{2+q}} \right) \psi^{5+2q} dM \leq \Lambda_2 \delta^{-\frac{5+2q}{2+q}} \int_M |\nabla \psi|^{5+2q} dM. \quad (3.1)$$

**Proof.** Let us choose an orientation such that $H > 0$ and apply the corresponding stability inequality

$$3 \int_M (-K) f^2 dM \leq \int_M \langle P_i(\nabla f), \nabla f \rangle dM, \quad (3.2)$$

for $f = H^{1+q}\varphi$, where $q > 0$, and $\varphi$ is a smooth function compactly supported on $M$.

First note that

$$\nabla f = \nabla (H^{1+q}\varphi) = (1 + q)H^q \varphi \nabla H + H^{1+q} \nabla \varphi.$$
It implies
\[
\langle P_1(\nabla f), \nabla f \rangle = (1 + q)^2 H^{2q} \varphi^2 \langle P_1(\nabla H), \nabla H \rangle \\
+ 2(1 + q) H^{1+2q} \varphi \langle P_1(\nabla H), \nabla \varphi \rangle \\
+ H^{2+2q} \langle P_1(\nabla \varphi), \nabla \varphi \rangle.
\]

Since $H > 0$, then $P_1$ is positive definite. Now, let us estimate the second term in the right hand side of identity above. By using Cauchy-Schwarz inequality followed by inequality $xy \leq \frac{x^2}{2} + \frac{y^2}{2}$, for all $x, y \in \mathbb{R}$, we obtain
\[
H^{1+2q} \varphi \langle P_1(\nabla H), \nabla \varphi \rangle = H^{2q} \varphi \sqrt{\beta \varphi} \sqrt{P_1(\nabla H)} \langle (1/\sqrt{\beta}) H \sqrt{P_1(\nabla \varphi)} \rangle \\
\leq H^{2q} \varphi \sqrt{\beta \varphi} \sqrt{P_1(\nabla H)} \frac{\| (1/\sqrt{\beta}) H \sqrt{P_1(\nabla \varphi)} \|^2}{2} + \frac{\| (1/\sqrt{\beta}) H \sqrt{P_1(\nabla \varphi)} \|^2}{2} \\
\leq \frac{\beta}{2} H^{2q} \varphi^2 \langle P_1(\nabla H), \nabla H \rangle + \frac{1}{2\beta} H^{2+2q} \langle P_1(\nabla \varphi), \nabla \varphi \rangle, \tag{3.3}
\]
for any constant $\beta > 0$. Then stability inequality (3.2) becomes
\[
3 \int_M (-K) H^{2+2q} \varphi^2 dM \leq (1 + q)^2 \int_M H^{2q} \varphi^2 \langle P_1(\nabla H), \nabla H \rangle dM \\
+ 2(1 + q) \int_M H^{1+2q} \varphi \langle P_1(\nabla H), \nabla \varphi \rangle dM \\
+ \int_M H^{2+2q} \langle P_1(\nabla \varphi), \nabla \varphi \rangle dM \tag{3.4}
\]
\[
\leq \left( (1 + q)^2 + (1 + q) \beta \right) \int_M H^{2q} \varphi^2 \langle P_1(\nabla H), \nabla H \rangle dM \\
+ \frac{1 + (1 + q)}{\beta} \int_M H^{2+2q} \langle P_1(\nabla \varphi), \nabla \varphi \rangle dM.
\]
Let us estimate $\int_M H^{2q} \varphi^2 \langle P_1(\nabla H), \nabla H \rangle dM$. By using identity
\[
L_1(fg) = \text{div}(P_1(\nabla (fg))) = \text{div}(f P_1(\nabla g)) + g L_1 f + \langle P_1(\nabla f), \nabla g \rangle,
\]

we have
\[ L_1(H^{2+2q} \varphi^2) = \text{div}(HP_1(\nabla(H^{1+2q} \varphi^2))) + H^{1+2q} \varphi^2 L_1(H) \]
\[ + \langle P_1(\nabla H), \nabla(H^{1+2q} \varphi^2) \rangle \]
\[ = \text{div}(HP_1(\nabla(H^{1+2q} \varphi^2))) + H^{1+2q} \varphi^2 L_1(H) \]
\[ + (1 + 2q) H^{2q} \varphi^2 \langle P_1(\nabla H), \nabla H \rangle + 2H^{1+2q} \varphi \langle P_1(\nabla H), \nabla \varphi \rangle. \]

Integrating both sides of the identity above and by using Divergence Theorem, we obtain
\[ (1 + 2q) \int_M H^{2q} \varphi^2 \langle P_1(\nabla H), \nabla H \rangle dM = - \int_M H^{1+2q} \varphi^2 L_1(H) dM \]
\[ - 2 \int_M H^{1+2q} \varphi \langle P_1(\nabla H), \nabla \varphi \rangle dM. \]

By using inequality (3.3), we have
\[ (1 + 2q) \int_M H^{2q} \varphi^2 \langle P_1(\nabla H), \nabla H \rangle dM \leq - \int_M H^{1+2q} \varphi^2 L_1(H) dM \]
\[ + \beta \int_M H^{2q} \varphi^2 \langle P_1(\nabla H), \nabla H \rangle \]
\[ + \frac{1}{\beta} \int_M H^{2+2q} \langle P_1(\nabla \varphi), \nabla \varphi \rangle dM, \]
i.e.,
\[ (1 + 2q - \beta) \int_M H^{2q} \varphi^2 \langle P_1(\nabla H), \nabla H \rangle dM \leq - \int_M H^{1+2q} \varphi^2 L_1(H) dM \]
\[ + \frac{1}{\beta} \int_M H^{2+2q} \langle P_1(\nabla \varphi), \nabla \varphi \rangle dM. \]

On the other hand, is well known, see [1], Lemma 3.7, that
\[ -L_1(H) = |\nabla H|^2 - |\nabla A|^2 - 3HK. \]

Since \( P_1 \) is positive definite, we have
\[ \langle P_1(\nabla H), \nabla H \rangle \leq (\text{tr} P_1)|\nabla H|^2 = 2H|\nabla H|^2, \]

i.e.,
\[ (1 + 2q - \beta) \int_M H^{2q} \varphi^2 \langle P_1(\nabla H), \nabla H \rangle dM \leq - \int_M H^{1+2q} \varphi^2 L_1(H) dM \]
\[ + \frac{1}{\beta} \int_M H^{2+2q} \langle P_1(\nabla \varphi), \nabla \varphi \rangle dM. \]
\[|\nabla H|^2 \geq \frac{1}{2H} \langle P_1(\nabla H), \nabla H \rangle.\]

By using Proposition 2.1 and inequality above, we obtain

\[-L_1(H) \leq -\frac{2}{3} |\nabla H|^2 - 3HK \leq -\frac{1}{3H} \langle P_1(\nabla H), \nabla H \rangle - 3HK.\]

Then

\[
\left( \frac{4}{3} + 2q - \beta \right) \int_M H^{2q} \varphi^2 \langle P_1(\nabla H), \nabla H \rangle dM \leq 3 \int_M H^{2+2q} (-K) \varphi^2 dM
\]

\[+ \frac{1}{\beta} \int_M H^{2+2q} \varphi \langle P_1(\nabla \varphi), \nabla \varphi \rangle dM.\]

Replacing last inequality in (3.4), stability inequality becomes

\[3 \int_M (-K) H^{2+2q} \varphi^2 dM \leq 3C_1 \int_M H^{2+2q} (-K) \varphi^2 dM
\]

\[+ C_2 \int_M H^{2+2q} \langle P_1(\nabla \varphi), \nabla \varphi \rangle dM,\]

i.e.,

\[3(1 - C_1) \int_M H^{2+2q} (-K) \varphi^2 dM \leq C_2 \int_M H^{2+2q} \langle P_1(\nabla \varphi), \nabla \varphi \rangle dM.\]

where

\[C_1 = \frac{(1 + q)^2 + \beta(1 + q)}{\frac{4}{3} + 2q - \beta}, \quad C_2 = 1 + \frac{(1 + q)}{\beta} \frac{(1 + q)^2 + (1 + q)\beta}{\beta (\frac{4}{3} + 2q - \beta)},\]

\[0 < q < \sqrt{1/3} \text{ by hypothesis, and } \beta \text{ is taken such that } 0 < \beta < \frac{1/3 - q^2}{q + 2}. \text{ This choice of } \beta \text{ is necessary to have } C_1 < 1. \text{ In fact,}

\[\beta < \frac{1/3 - q^2}{q + 2} \Rightarrow q^2 + \beta q + 2\beta < \frac{1}{3}\]

\[\Rightarrow (1 + q)^2 + \beta(1 + q) < \frac{4}{3} + 2q - \beta\]

\[\Rightarrow C_1 = \frac{(1 + q)^2 + \beta(1 + q)}{\frac{4}{3} + 2q - \beta} < 1.\]
Therefore,
\[ \int_M H^{2+2q}(-K)\varphi^2 dM \leq \frac{C_2}{3(1 - C_1)} \int_M H^{2+2q} \langle P_1(\nabla \varphi), \nabla \varphi \rangle dM. \]

On the other hand, since \( P_1 \) is positive definite, we have
\[ \langle P_1(\nabla \varphi), \nabla \varphi \rangle \leq (\text{tr } P_1)|\nabla H|^2 \leq 2H|\nabla \varphi|^2. \]

Denoting by \( C_3 = \frac{2C_2}{3(1 - C_1)} \), we have
\[
\int_M H^{2+2q}(-K)\varphi^2 dM \leq \frac{C_3}{2} \int_M H^{2+2q} \langle P_1(\nabla \varphi), \nabla \varphi \rangle dM \leq C_3 \int_M H^{3+2q} |\nabla \varphi|^2 dM.
\]

Letting \( \varphi = \psi^p \), where \( 2p = 5 + 2q \), we obtain
\[
\int_M H^{2+2q}(-K)\psi^{5+2q} dM \leq C_3 p^2 \int_M H^{3+2q} \psi^{3+2q} |\nabla \psi|^2 dM. \tag{3.5}
\]

By using Young’s inequality, i.e.,
\[ xy \leq \frac{x^a}{a} + \frac{y^b}{b}, \quad \frac{1}{a} + \frac{1}{b} = 1 \]

with
\[ x = \delta H^{3+2q} \psi^{3+2q}, \quad y = \frac{|\nabla \psi|^2}{\delta}, \quad a = \frac{5 + 2q}{3 + 2q}, \quad b = \frac{5 + 2q}{2}, \quad \text{and} \quad \delta > 0, \]
we have
\[ H^{3+2q} \psi^{3+2q} |\nabla \psi|^2 \leq \frac{3 + 2q}{5 + 2q} \delta^{\frac{5+2q}{5+2q}} H^{5+2q} \psi^{5+2q} + \frac{2}{5 + 2q} \delta^{-\frac{5+2q}{5+2q}} |\nabla \psi|^{5+2q}. \]

Replacing last inequality in inequality (3.5), we obtain
\[
\int_M H^{2+2q}(-K)\psi^{5+2q} dM \leq \frac{3 + 2q}{5 + 2q} p^2 C_3 \delta^{\frac{5+2q}{5+2q}} \int_M H^{5+2q} \psi^{5+2q} dM
\]
\[
+ \frac{2}{5 + 2q} p^2 C_3 \delta^{-\frac{5+2q}{2}} \int_M |\nabla \psi|^{5+2q} dM,
\]

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i.e.,

\[
\int_M H^{5+2q} \left( \frac{(-K)}{H^3} - \Lambda_1 \delta^{5+2q} \right) \psi^{5+2q} dM \leq \Lambda_2 \delta^{-\frac{5+2q}{2}} \int_M |\nabla \psi|^{5+2q} dM, \tag{3.6}
\]

where \( \Lambda_1 = \frac{3 + 2q}{5 + 2q} p^2 C_3 \) and \( \Lambda_2 = \frac{2p^2}{5 + 2q} C_3 \).

**Remark 3.1.** In [15], Schoen, Simon, and Yau obtained the following Sobolev type inequality for minimal hypersurfaces \( M^n \) immersed in \( \mathbb{R}^{n+1} \):

\[
\int_M |A|^{2p} \psi^{2p} dM \leq C(n, p) \int_M |\nabla \phi|^{2p} dM, \tag{3.7}
\]

for \( p \in [2, 2+\sqrt{2/n}) \), and for all function \( \psi : M \to \mathbb{R} \) compactly supported on \( M \). By using inequality of Proposition 3.1, we obtain a similar result for hypersurfaces \( M^3 \) immersed in \( \mathbb{R}^4 \) with zero scalar curvature. In fact, if \( R = 0 \), then \( H^2 = |A|^2 \). Choosing an orientation such that \( H > 0 \), we have \( H = |A| \). In this case, we have

**Corollary 3.1** (Sobolev type inequality). Let \( x : M^3 \to \mathbb{R}^4 \) be a stable isometric immersion with zero scalar curvature and such that \( \frac{(-K)}{H^3} \geq c > 0 \) everywhere. Then, for all smooth function \( \psi \) with compact support in \( M \), for all \( \delta > 0 \) and \( p \in (5/2, 5/2 + \sqrt{1/3}) \), there exists a constant \( C(p) > 0 \) such that

\[
\int_M |A|^{2p} \psi^{2p} dM \leq C(p) \int_M |\nabla \psi|^{2p} dM. \tag{3.8}
\]

**Remark 3.2.** In the recent article [11], Ilias, Nelli, and Soret, obtained results in this direction for hypersurfaces with constant mean curvature.

Now let us prove the Theorem A stated in the Introduction.

**Theorem A.** There is no stable complete hypersurface \( M^3 \) of \( \mathbb{R}^4 \) with zero scalar curvature, polynomial volume growth and such that

\[
\frac{(-K)}{H^3} \geq c > 0
\]

everywhere, for some constant \( c > 0 \).
Proof. Suppose by contradiction there exists a complete stable hypersurface attending conditions of Theorem A. Then we can apply Proposition 3.1. Choose the compact supported function \( \psi : M \to \mathbb{R} \) defined by

\[
\psi(p) = \begin{cases} 
1 & \text{if } p \in B_r; \\
\frac{2r - \rho(p)}{r} & \text{if } p \in B_{2r} \setminus B_r; \\
0 & \text{if } p \in M \setminus B_{2r},
\end{cases}
\]

(3.9)

where \( \rho(p) = \rho(p, p_0) \) is the distance function of \( M \). By using this function \( \psi \) in the inequality of Proposition 3.1, we have

\[
\int_{B_r} H^{5+2q} \left( \frac{(-K)}{H^3} - \Lambda_1 \delta^{\frac{5+2q}{3+2q}} \right) dM \leq \int_{B_{2r}} H^{5+2q} \left( \frac{(-K)}{H^3} - \Lambda_1 \delta^{\frac{5+2q}{3+2q}} \right) \psi^{5+2q} dM \\
\leq \Lambda_2 \delta^{-\frac{5+2q}{2}} \int_{B_{2r}} |\nabla \psi|^{5+2q} dM \\
\leq \Lambda_2 \delta^{-\frac{5+2q}{2}} \operatorname{vol} B_{2r} \\
\leq \Lambda_2 \delta^{-\frac{5+2q}{2}} \cdot \lim_{r \to \infty} \frac{\operatorname{vol}(B_r)}{r^\alpha} \cdot \lim_{r \to \infty} \frac{1}{r^{5+2q-\alpha}} = 0
\]

for \( 0 < q < \sqrt{1/3} \). Taking \( \delta > 0 \) sufficiently small and since, by hypothesis, \( \frac{(-K)}{H^3} \geq c > 0 \), we get

\[
\left( \frac{(-K)}{H^3} - \Lambda_1 \delta^{\frac{5+2q}{3+2q}} \right) > 0.
\]

By hypothesis, \( M \) has polynomial volume growth. It implies that

\[
\lim_{r \to \infty} \frac{\operatorname{vol}(B_r)}{r^\alpha} < \infty, \quad \alpha \in (0, 4].
\]

Letting \( r \to \infty \) in the inequality (3.10), we obtain

\[
\lim_{r \to \infty} \int_{B_r} H^{5+2q} \left( \frac{(-K)}{H^3} - \Lambda_1 \delta^{\frac{5+2q}{3+2q}} \right) dM \leq \Lambda_2 \lim_{r \to \infty} \frac{\operatorname{vol}(B_{2r})}{r^\alpha} \cdot \lim_{r \to \infty} \frac{1}{r^{5+2q-\alpha}} = 0.
\]

Therefore \( H \equiv 0 \), and this contradiction finishes the proof of the theorem. \( \square \)

Remark 3.3. In the proof of Theorem A, \( M \) need not even be properly immersed, since we are taking intrinsic (geodesic) balls. Since \( M \) is complete, we have \( M = \bigcup_{n=1}^\infty B_{r_n} \) for some sequence \( r_n \to \infty \), and thus we can take \( r \to \infty \) in the estimate.
Remark 3.4. By using their Sobolev inequality (3.7), Schoen, Simon, and Yau gave a new proof of Bernstein’s Theorem for dimension less than or equal to 5, namely, that the only entire minimal graphs \( M^n \) in \( \mathbb{R}^{n+1} \), \( n \leq 5 \) are hyperplanes. By using our version of Sobolev inequality (3.8), we prove the following Bernstein type result.

As a corollary of Theorem A, we have the following result.

**Theorem B.** There are no entire graphs \( M^3 \) of \( \mathbb{R}^4 \) with zero scalar curvature and such that

\[
\frac{(-K)}{H^3} \geq c > 0
\]

everywhere, for some constant \( c > 0 \).

**Proof.** Suppose there exists an entire graph \( M \) satisfying the conditions of corollary. In [3], Proposition 4.1, p. 3308, Alencar, Santos, and Zhou showed that entire graphs with zero scalar curvature and whose mean curvature does not change sign are stable. Since \( R = 0 \) by hypothesis, we have \( H^2 = |A|^2 \). Provided \( K \neq 0 \) everywhere, we have \( H^2 = |A|^2 > 0 \) which implies that \( H \) does not change sign. Thus, the entire graph \( M \) is stable. On the other hand, it is well known that graphs satisfy \( \text{vol}(B_r) \leq Cr^4, C > 0 \). Therefore, by using the hypothesis \( \frac{(-K)}{H^3} \geq c > 0 \), inequality (3.10) in the proof of Theorem A, p.14, and taking \( r \to \infty \) we obtain the same contradiction. \( \square \)

### 4 Examples

The class of hypersurfaces treated here is non-empty, as shown in the following example. It can be found in [10], Lemma 2.1, p. 400. See also [2], p. 213 – 214 and [7], p. 161.

**Example 4.1.** Let \( M^3 \hookrightarrow \mathbb{R}^4 \) the rotational hypersurface parametrized by

\[
X(t, \theta, \varphi) = (f(t) \sin \theta \cos \varphi, f(t) \sin \theta \sin \varphi, f(t) \cos \theta, t),
\]

where \( f(t) = \frac{t^2}{4m} + m \) and \( m \) is a non-negative constant. The principal curvatures are

\[
\lambda_1 = \lambda_2 = \frac{m^{1/2}}{f^{3/2}}, \quad \lambda_3 = -\frac{1}{2} \frac{m^{1/2}}{f^{3/2}}.
\]
Then $R = 0$ and $\frac{-K}{H^3} = \frac{4}{27}$ everywhere. Since $M^3$ is a rotational hypersurface and its profile curve is quadratic, it has polynomial volume growth. Then by Theorem A the immersion is unstable.

This example appears in the Theory of Relativity as the embedding of the space-like Schwarzschild manifold of mass $m/2 > 0$, see Introduction of [5], for details.

The following class of hypersurfaces are well known, see [2], p. 214, and they are the classical examples of stable hypersurfaces with zero scalar curvature. This class show us that some condition over nullity of Gauss-Kronecker curvature are needed.

Example 4.2. Let $M^3 \subset \mathbb{R}^4$ be the cylinder parametrized by

$$x(u,v,t) = (u,v,\alpha(t),\beta(t)); \quad u,v,t \in \mathbb{R},$$

where $c(t) := (\alpha(t),\beta(t))$ is a parametrized curve with positive curvature $k(t)$ at every point. In this case, principal curvatures are

$$\lambda_1 = 0, \lambda_2 = 0, \text{ and } \lambda_3 = k(t).$$

Thus $R = 0$, $H > 0$ and $K = 0$ everywhere. Then, $M^3$ is stable, see [2].

Observe that if $c(t) = (t,f(t))$, the cylinder $M$ is the graph of the smooth function $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ given by $F(u,v,t) = f(t)$. In particular, taking $f(t) = t^2$ or $f(t) = \sqrt{1+t^2}$ we obtain an entire graph with polynomial volume growth, $R = 0, H > 0$ and $K = 0$ everywhere.

5 Non-embedded Tubes

Let $x : M^3 \rightarrow \mathbb{R}^4$ be an isometric immersion. Following Nelli and Soret, see [12], we define the tube of radius $h$ around $M$ the set

$$T(M,h) = \{ x \in \mathbb{R}^4; \exists p \in M, x = p + t\eta, t \leq h(p) \},$$

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where $\eta$ is the normal vector of the second fundamental form of $x$, and $h : M \to \mathbb{R}$ is an everywhere non-zero smooth function. If $|A| \neq 0$ everywhere, we define the subfocal tube the set

$$T \left( M, \frac{\epsilon}{|A|} \right), \ 0 < \epsilon \leq 1.$$  

Denote by $T(r, h)$ the tube of radius $h$ around $B_r \subset M$, i.e., considering $M = B_r$ in the above definition, and let

$$V(r, h) = \int_{T(r, h)} dT,$$

where $dT$ denotes the volume element of the tube. If $R = 0$, and choosing an orientation such that $H > 0$, we have $H = |A|$. Under the conditions of Proposition 3.1, and assuming that $\frac{(-K)}{H^3} \geq c > 0$, then there exists a constant $C(q)$ depending only on $0 < q < \sqrt{1/3}$ such that

$$\int_{B_r} |A|^{5+2q} \psi^{5+2q} dM \leq C(q) \int_{B_r} |\nabla \psi|^{5+2q} dM. \quad (5.1)$$

Choosing the same function with compact support used in the proof of Theorem A (see (3.9), p. 14), we obtain

$$\int_{B_r} |A|^{5+2q} dM \leq C(q) \frac{\text{vol}(B_r)}{r^{5+2q}}.$$

The following lemma is essentially the same Lemma 1 of [12], p. 496, and the proof will be omitted here.

**Lemma 5.1.** Let $M^3$ be a complete, stable hypersurface of $\mathbb{R}^4$ satisfying $R = 0$ and $\frac{(-K)}{H^3} \geq c > 0$ everywhere.

(a) For $r > 0$ sufficiently large, there exists a constant $\alpha(q)$, depending only on $0 < q < \sqrt{1/3}$ such that

$$\text{vol}(B_r) > \alpha(q)r^{5+2q}. \quad (5.2)$$

(b) For each $\beta > 1$, $0 < q < \sqrt{1/3}$, and $r > 0$ satisfying inequality (5.2) above, there exists a sufficiently large $\tilde{r} > r$ such that

$$\text{vol}(B_{\tilde{r}}) - \text{vol}(B_{\beta r}) > \alpha(q)r^{5+q}.$$
The next result is a vanishing scalar curvature version of Theorem 1, p. 499 of [12].

**Theorem C.** Let $M^3$ be a stable complete hypersurface of $\mathbb{R}^4$ with vanishing scalar curvature. Suppose that the second fundamental form of the immersion is bounded and there exists a constant $c > 0$ such that $\frac{(-K)}{H^3} \geq c > 0$ everywhere. Then, for constants $0 < b_1 \leq 1$, $b_2 > 0$, and for any smooth function $h : M \to \mathbb{R}$ satisfying

$$h(p) \geq \inf \left\{ \frac{b_1}{|A(p)|}, b_2 \rho(p) \delta \right\}, \quad \delta > 0,$$

(5.3)

the tube $T(M, h)$ is not embedded. Here, $\rho(p)$ denotes the intrinsic distance in $M$ to a fixed point $p_0 \in M$.

**Proof.** In [12], Nelli and Soret showed that

$$V(r, h) = \int_{B_r} h(p) dM - \frac{1}{2} \int_{B_r} h(p)^2 H(p) dM - \frac{1}{4} \int_{B_r} h(p)^4 K(p) dM.$$

By using the classical inequality between geometric and quadratic means, one finds that

$$K = \lambda_1 \lambda_2 \lambda_3 \leq \lambda_1 \lambda_2 \lambda_3 |A|$$

$$\leq \left( \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}{3} \right)^{3/2}$$

$$= \frac{1}{3\sqrt{3}} |A|^3,$$

i.e.,

$$K(p) \leq \frac{1}{3\sqrt{3}} |A(p)|^3.$$

(5.4)

Let $B^+_r$ the set where $\frac{b_1}{|A(p)|}$ is the infimum and $B^-_r = B_r \setminus B^+_r$. Then

$$V(r, h) \geq b_1 \int_{B^+_r} \frac{1}{|A|} dM - \frac{b_1^2}{2} \int_{B^+_r} \frac{1}{|A|^2} H dM - \frac{b_1^4}{4} \int_{B^+_r} \frac{1}{|A|^3} K dM$$

$$+ b_2 \int_{B^-_r} \rho^\delta dM - \frac{b_2^2}{2} \int_{B^-_r} \rho^{2\delta} H dM - \frac{b_2^4}{4} \int_{B^-_r} \rho^{4\delta} K dM.$$
Since $H = |A|$, we have

$$V(r, h) \geq \left( b_1 - \frac{b_1^2}{2} - \frac{b_1^4}{12\sqrt{3}} \right) \int_{B_r^+} \frac{1}{|A|} dM + b_2 \int_{B_r^-} \rho^\delta dM - \frac{b_2^2}{2} \int_{B_r^-} \rho^{2\delta} H dM - \frac{b_2^4}{4} \int_{B_r^-} \rho^{4\delta} K dM.$$  

Let us estimate the integrals over $B_r^-$. By using inequality (5.4) above, we get

$$-K \geq -\frac{1}{3\sqrt{3}} |A|^3 \geq -\frac{b_1}{b_2} \frac{1}{3\sqrt{3}} \rho^{-3\delta}$$  

and

$$-H = -|A| \geq \frac{b_1}{b_2} \rho^{-\delta}.$$  

By hypothesis, $|A|$ is bounded, then there exists $a := \inf_{M} \frac{1}{|A|}$. Therefore

$$V(r, h) \geq \left( b_1 - \frac{b_1^2}{2} - \frac{b_1^4}{12\sqrt{3}} \right) \int_{B_r^+} \frac{1}{|A|} dM + \left( b_2 - \frac{b_2 b_1}{2} - \frac{b_2^4 b_1}{12\sqrt{3}} \right) \int_{B_r^-} \rho^\delta dM$$

$$\geq a \left( b_1 - \frac{b_1^2}{2} - \frac{b_1^4}{12\sqrt{3}} \right) \text{vol}(B_r^+) + \left( b_2 - \frac{b_2 b_1}{2} - \frac{b_2^4 b_1}{12\sqrt{3}} \right) \int_{B_r^-} \rho^\delta dM.$$  

On the other hand, for $r$ sufficiently large,

$$\int_{B_r^-} \rho^\delta dM = \int_{B_r^- \setminus B_{\beta^{-1}r}} \rho^\delta dM + \int_{B_{\beta^{-1}r}} \rho^\delta dM \geq \int_{B_r^- \setminus B_{\beta^{-1}r}} \rho^\delta dM$$

$$\geq \left( \frac{r}{\beta} \right)^\delta [\text{vol}(B_r^-) - \text{vol}(B_{\beta^{-1}r})]$$

$$\geq [\text{vol}(B_r^-) - \text{vol}(B_{\beta^{-1}r})].$$

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Then
\[ V(r, h) \geq a \left( b_1 - \frac{b_1^2}{2} - \frac{b_1^4}{12\sqrt{3}} \right) \left( \text{vol}(B_r^+) - \text{vol}(B_{\beta^{-1}r}^+) + \text{vol}(B_{\beta^{-1}r}^+) \right) + \left( b_2 - \frac{b_2^2b_1}{2} - \frac{b_2^2b_1^2}{12\sqrt{3}} \right) \left( \text{vol}(B_r^-) - \text{vol}(B_{\beta^{-1}r}^-) \right) \]
\[ \geq C \left( \text{vol}(B_r) - \text{vol}(B_{\beta^{-1}r}) \right). \]

By using Lemma 5.1, item (b), there exists \( \tilde{r} > r \) such that
\[ V(\tilde{r}, h) \geq C\tilde{r}^{5+q}. \] (5.5)

The Euclidean distance is less than or equal to the intrinsic distance. It implies
\[ B_r(p) \subset B(p, r), \]
where \( B_r(p) \equiv B_r \) and \( B(p, r) \) denotes the intrinsic and the Euclidean ball of center \( p \) and radius \( r \). By using (5.3), we have
\[ h(q) \geq \min \left\{ b_1, b_2\rho(q)^\delta \right\} \geq \min \left\{ \inf_M \frac{b_1}{|A|}, b_2\rho(q)^\delta \right\} = \inf_M \frac{b_1}{|A|} = b_1 a, \]
for \( 0 < b_1 \leq 1 \) and \( \rho \) sufficiently large, then
\[ T(r, b_1a) \subset T(r, h). \]

Suppose, by contradiction, that \( T(r, b_1a) \) is embedded. Since
\[ T(r, b_1a) \subset B(p, r + 2b_1a), \]
then its volume \( V(r, b_1a) \) satisfies
\[ V(r, b_1a) \leq \text{vol}(B(p, r + 2b_1a)) = \omega_4(r + 2b_1a)^4, \]
where \( \omega_4 \) is the volume of \( B(p, 1) \). Let us consider two different cases. First, if \( M \) is not contained in any ball, above inequality is a contradiction with (5.5) for \( r \) sufficiently large. Therefore, \( T(r, b_1a) \), and thus \( T(r, h) \), is not embedded for \( r \) sufficiently large. In the second case, if \( M \) is contained in some ball, then \( T(M, h) \) has finite volume (since \( T(M, h) \) is embedded) and it is also a contradiction with (5.5). \( \square \)
6 Appendix

Let us prove the following fact established in the Introduction:

Let \( x : M^3 \to \mathbb{R}^4 \) be an isometric immersion with zero scalar curvature. If \( H \) and \( K \) denotes the mean curvature and Gauss-Kronecker curvature, respectively, then

\[
0 \leq -\frac{K}{H^3} \leq \frac{4}{27} \quad \text{everywhere on } M.
\]

Figure 1: Representation of the domain \( N_\omega \) of \( \frac{K}{H^3} \) over \( S^2 \), considering this function as an algebraic function of the eigenvalues. This domain is the intersection of one of the plane \( \lambda_1 + \lambda_2 + \lambda_3 = 1 \) with \( S^2 \). The hypothesis cuts off only three small neighbourhoods around the coordinate axis.

In fact, let \((\lambda_1, \lambda_2, \lambda_3) = t\omega\) where \( \omega \in S^2 \). By using (1.1), we can see that \( R, H \) and \( K \) are homogeneous polynomials. It implies \( H(t\omega) = tH(\omega), \ R(t\omega) = t^2R(\omega), \ K(t\omega) = t^3K(\omega) \) and hence

\[
\frac{K}{H^3}(t\omega) = \frac{K}{H^3}(\omega).
\]

Then the behavior of \( \frac{K}{H^3} \) depends only of its values on the sphere \( S^2 \). Since \( N := \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3; R = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 0\} \) is closed and \( S^2 \) is compact, we obtain that \( N_\omega = N \cap S^2 \) is compact, see figure 6. Then, \( \frac{K}{H^3} : N_\omega \to \mathbb{R} \) is a continuous
function with compact domain. The claim then follow from the Weierstrass maxima and minima theorem. Upper bound $\frac{1}{27}$ can be found by using Lagrange multipliers method.

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Gregório Silva Neto
Universidade Federal de Alagoas,
Instituto de Matemática,
57072-900, Maceió, Alagoas, Brazil.
gregorio@im.ufal.br