SYMPLECTIC TOPOLOGY OF SU(2)-REPRESENTATION VARIETIES AND LINK HOMOLOGY, I: SYMPLECTIC BRAID ACTION AND THE FIRST CHERN CLASS

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Abstract. There are some similarities between cohomology of SU(2)-representation varieties of the fundamental group of some link complements and the Khovanov homology of the links. We start here a program to explain a possible source of these similarities. We introduce a symplectic manifold $\mathcal{M}$ with an action of the braid group $B_{2n}$ preserving the symplectic structure. The action allows to associate a Lagrangian submanifold of $\mathcal{M}$ to every braid. The representation variety of a link can then be described as the intersection of such Lagrangian submanifolds, given a braid presentation of the link. We expect this to go some way in explaining the similarities mentioned above.

1. Introduction

Let $C$ be the conjugacy class of traceless matrices in the Lie group $SU(2)$. The origin of this paper is the following observation.

Let $L$ be a link in $\mathbb{R}^3$. Consider the space $\tilde{J}_C(L)$ of representations of the fundamental group of the link complement in $SU(2)$ satisfying the condition that they send meridians to the conjugacy class $C$. Typically $\tilde{J}_C(L)$ has several connected components $J^r_C(L)$. Let

$$H^*(\tilde{J}_C(L); \mathbb{Z}) = \bigoplus_r H^*(J^r_C(L); \mathbb{Z})$$

be the singular cohomology of this space.

Let, on the other hand, $Kh^{i,j}(L)$ denote the integral Khovanov homology of the link $L$ and let $Kh^k(L)$ denote the singly graded homology theory obtained by collapsing the grading along $k = i - j$.

1.1. Observation. For every prime knot $L$ with seven crossings or less, and for every $(2,n)$-torus link, there are integers $N_r = N_r(L)$ such that

$$Kh^*(L) = \bigoplus_r H^*(J^r_C(L); \mathbb{Z})\{N_r\}.$$  \hspace{1cm} (1.1)

That is, the Khovanov homology consists of pieces, which are isomorphic to the cohomology of the components of the representation variety $\tilde{J}_C(L)$, but with each such component shifted by some integer.

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To prove this observation let us recall that the conjugacy class $C$ is an example of a topological quandle. In [25], given a topological quandle $Q$ and an oriented link $L$ in $\mathbb{R}^3$, the second author constructed an invariant space $J_Q(L)$ of the link. When the quandle $Q$ is a conjugacy class $K$ in a topological group $G$ (or a union of conjugacy classes), the invariant space $J_K(L)$ can be identified with the space of homomorphisms from the fundamental group of the link complement $\pi_1(\mathbb{R}^3 - L)$ to the group $G$ mapping the positively oriented meridians into $K$ (see [25], Lemma 4.6). In particular, if $K$ is the conjugacy class $C$ of traceless matrices in $SU(2)$, the representation variety $\tilde{J}_C(L)$ can be identified with the invariant space $J_C(L)$ of [25]. From now on we shall use the symbol $J_C(L)$ for both of them. The invariant spaces $J_C(L)$ of all prime knots with up to seven crossings and of all $(2,n)$-torus links can be easily determined by the techniques used in Section 5 of [25]. The Khovanov homology of such knots, on the other hand, can be computed by using the Mathematica package JavaKh [16]. The isomorphism (1.1) then follows.

We leave details to the reader but we illustrate with

1.2. Example. The singly graded Khovanov homology $Kh^*(T_{2,n})$ for $(2,n)$-torus knots can be written as

$$Kh^*(T_{2,n}) = H^*(S^2\{n-2\}) \oplus \bigoplus_{i=1}^{\frac{n-1}{2}} H^*(\mathbb{R}P^3)\{2i-2+n\},$$

see [13]. And, as we compute in Section [9]

$$J_C(T_{2,n}) = S^2 \cup \bigcup_{j=1}^{\frac{n-1}{2}} \mathbb{R}P^3.$$

For more examples, see Section [9].

One of the future aims of our work is to enhance the link invariant $J_C(L)$ by introducing a grading on its connected components. The grading should explain the shifts in our examples.

Note that the group $SU(2)$ acts on $J_C(L)$ by conjugation. If the space $J_C(L)$ is factored out by this action it becomes the moduli space of flat $SU(2)$-connections on the link complement with holonomies along meridians belonging to the conjugacy class $C$.

In [8], Guruprasad, Huebschmann, Jeffrey and Weinstein study the moduli space $\mathcal{M}$ of flat $SU(2)$-connections on a surface with $n$ marked points. They express $\mathcal{M}$ by symplectic reduction of a certain finite-dimensional symplectic manifold $\mathcal{A}$ called an “extended moduli space”. The space $\mathcal{M}$ plays a crucial role in the quantized Chern-Simons gauge theory, which is the 3-dimensional origin of the Jones polynomial, which in turn is the Euler characteristic of Khovanov homology.

The main idea of our paper is to express the representation variety $J_C(L)$ of the link $L$ as an intersection of two Lagrangian submanifolds of a symplectic manifold. We achieve this in the following way. The link $L$ can be represented as a closure of a braid $\sigma$ on $n$ strands. The braid group $B_n$ on $n$ strands acts on the product $P_n$ of $n$ copies of $C$. The invariant space $J_C(L)$ is defined as the subspace of fix-points of $\sigma$ acting on $P_n$ [25]. We can also describe $J_C(L)$ as an intersection of a “twisted” diagonal of $P_n$ in $P_{2n} = P_n \times P_n$ with a “twisted” graph of $\sigma$. We shall use this description of $J_C(L)$. Thus we consider the product $P_{2n}$ of $2n$ copies of the
conjugacy class $C$. Following Guruprasad, Huebschmann, Jeffreys and Weinstein [8] we interpret $P_{2n}$ as a space naturally associated to the moduli space of flat $SU(2)$-connections on the 2-sphere $S^2$ with $2n$ punctures and with monodromies around the punctures belonging to $C$. As shown in [8], $P_{2n}$ contains the extended moduli space $\mathcal{M}$ mentioned above as an open submanifold with a symplectic structure. We show that the action of the braid group $B_{2n}$ on $P_{2n}$ maps the submanifold $\mathcal{M}$ to itself and preserves its symplectic form. (It should, perhaps, be stressed that the symplectic structure on $\mathcal{M}$ is not the restriction of the product symplectic structure on $P_{2n} = \prod C$. That one would not have been preserved by the action of $B_{2n}$.)

Furthermore, to every braid $\sigma$ on $n$ strands we associate a Lagrangian submanifold $\Gamma_\sigma$ in $\mathcal{M}$. In particular, the identity braid defines a Lagrangian submanifold $\Lambda$. Finally, $\Gamma_\sigma$ and $\Lambda$ intersect in a space homeomorphic to $J_C(L)$.

This picture fits the general scheme of a classical field theory. One may think of $L$ as composed of two braids; one inside a 2-sphere, piercing it in $2n$ points, and one outside (usually, but not necessarily, the identity braid). The theory associates a symplectic manifold to the 2-sphere with marked points and Lagrangian submanifolds to the 3-discs with braids which are bounded by it.

In good cases (when the intersection is clean) we hope to be able to use the Lagrangians to define an index associated to the components of $J_C(L)$. This will be the topic of our next paper. In later work we plan to extend this to a Floer-type Lagrangian homology theory based on the invariant spaces $J_C(L)$.

(The idea of describing $J_C(L)$ as the intersection of $\Lambda$ and $\Gamma_\sigma$ goes back to X-S.Lin, [20]. However, no symplectic topology was involved in his considerations.)

It should be emphasized that there are examples where the cohomology of $J_C(L)$ does not give back Khovanov homology groups of $L$. The first such that we know of is the knot $9_{42}$ in Rolfsen’s table. In section 10 we determine the representation variety $J_C(L)$ for $L = 9_{42}$ and point out the difference between its cohomology and the collapsed Khovanov homology of the knot.

As already indicated, this paper is the first one in a series. It is devoted to the construction of the symplectic structure on the manifold $\mathcal{M}$, to the proof of its invariance under the braid group action, to the definition of the Lagrangian manifolds $\Gamma_\sigma$ and to the study of the first Chern class of $\mathcal{M}$.

The contents of the present paper are as follows.

In Section 2 we study the topology of certain spaces $K_{2n}$. These spaces are closely related to the moduli spaces of flat $SU(2)$-connections on the 2-dimensional sphere $S^2$ with $2n$ holes with holonomy around the holes belonging to the conjugacy class $C$ of traceless matrices in $SU(2)$. To be exact, $K_{2n}$ is the space $\text{Hom}(\pi, SU(2))_C$ of representations of the fundamental group $\pi$ of the punctured sphere with values in $SU(2)$ such that the standard generators of $\pi$ are mapped to $C$. The space $K_{2n}$ is a manifold with singularities. Observe that we do not remove singularities. The space $K_{2n}$ is simply connected and we determine its second homotopy group $\pi_2(K_{2n})$ (Proposition 2.9). In order to do that we study the structure of the singularities of $K_{2n}$. These are completely described in Appendix A.

The space $K_{2n}$ is a subspace of the product $P_{2n}$ of $2n$ copies of $S^2$, $P_{2n} = \prod_{i=1}^{2n} S^2$. (Observe that the conjugacy class $C$ can be identified with $S^2$.). In Section
3 we recall the construction of Guruprasad, Huebschmann, Jeffrey and Weinstein, [8], of a symplectic structure on an open neighbourhood $\mathcal{M}$ of $K_{2n}$ in $P_{2n}$.

The space $P_{2n}$ admits an action of the braid group $B_{2n}$ on $2n$ strands. The manifold $\mathcal{M}$ can be chosen to be invariant under the action of $B_{2n}$. In Section 4 we show that the braid group $B_{2n}$ acts on $\mathcal{M}$ by symplectomorphisms (see Theorem 4.1).

In Section 5 we introduce Lagrangian submanifolds $\sigma(\Lambda)$, $\sigma \in B_{2n}$, of the symplectic manifold $\mathcal{M}$. Some of these are then used to express the invariant spaces $J_C(L)$ of links as intersections of two Lagrangian submanifolds in $\mathcal{M}$.

In Section 6 we study the first Chern class $c_1(\mathcal{M})$ of $\mathcal{M}$. We consider first the evaluation of $c_1(\mathcal{M})$ on the homology classes of maps $\gamma_{k,\varepsilon} : S^2 \to K_{2n}$ given by $\gamma_{k,\varepsilon}(A) = (J, \ldots, J, A, \varepsilon A, J, \ldots, (1)^n \varepsilon J)$, where $A \in S^2$ and $J$ is a fixed base point of $S^2$ (here $A$ appears on the right hand side in the $k$-th factor). We formulate Theorem 6.1 which says that the evaluation $\langle c_1(\mathcal{M}) \mid \gamma_{k,\varepsilon} \rangle$ is equal to $0$. Theorem 6.1 is proven in Appendix B.

We construct then some special mapping $f_n : S^2 \to K_{2n}$ $(n \geq 2)$ and formulate Theorem 6.2 which says that the evaluation $\langle c_1(\mathcal{M}) \mid \{f_n\} \rangle$ is equal to $-2$ for all $n \geq 2$. We prove Theorem 6.2 in Appendix C.

Finally, we show that the subgroup $G$ of $\pi_2(K_{2n})$ generated by the homotopy classes $[f_n]$ and $[\gamma_{k,\varepsilon}]$, $\varepsilon = \pm 1$, $k = 1, \ldots, 2n - 1$, is of index at most 2.

In Section 7 we apply the results of Section 6 to show that the symplectic manifold $\mathcal{M}$ is monotone (Theorem 7.4). It is perhaps worth mentioning that our proof of this fact is carried out completely in a finite-dimensional context.

In Section 8 we point to the connection between the Lagrangian submanifolds constructed in Section 5 and the invariant spaces (representation varieties) $J_C(L)$ of links and indicate the subject of study which will be described in the second part of this paper.

In Section 9 we discuss several examples of links $L$ whose singly graded Khovanov homology $Kh^*(L)$ is isomorphic to the cohomology $H^*(J_C(L))$ of its invariant space. In particular, we show this to be true for all two-bridge knots, at least over $\mathbb{Q}$, and point to the arguments which make us strongly believe this to be true even over the integers $\mathbb{Z}$.

In Section 10 we determine the invariant space $J_C(L)$ for the knot $L = 9_{42}$. We show that it is a disjoint union of a 2-sphere $S^2$ and of seven copies of $\mathbb{R}P^3$ and point out the difference between its cohomology and the collapsed Khovanov homology $Kh^*(9_{42})$ of the knot.

There are three appendices. In Appendix A we analyse the singularities of the space $K_{2n}$. There are several components of the singular locus, all isomorphic to each other. Every component has a neighbourhood which is identified with a fibre bundle over the 2-dimensional sphere $S^2$ with the fiber being the cone $C(S^{2n-3})$ over a product of two $(2n-3)$-dimensional spheres (see Theorem A.7).

In Appendix B we prove Theorem 6.1 and in Appendix C Theorem 6.2.

The research described in this paper was initiated by the observation of similarities between the cohomology of the $SU(2)$-representation varieties $J_C(L)$ and the Khovanov homology $Kh^*(L)$ of certain links $L$, the first instances of which we made in September 2006.
The recently published paper of P.B.Kronheimer and T.S.Mrowka, [18], has also the same observation as the starting point. The approach and the setting of our paper is however different as we work with Lagrangian intersections while the authors of [18] construct a link homology based on the instanton Floer homology.

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2. Topology of the space of representations

2.1. The group action and the differential of $p$.

Let $G = SU(2)$ be the Lie group of $2 \times 2$ unitary matrices with determinant 1. Elements of $SU(2)$ are matrices $\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$, $a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$. The center of $SU(2)$ is $Z = \{I, -I\}$, where $I$ is the identity matrix.

Let us identify the 2-dimensional sphere $S^2$ with the subset $S$ of $SU(2)$ consisting of matrices of trace 0 i.e. with the set $S$ of matrices of the form $\begin{pmatrix} it & z \\ -\overline{z} & -it \end{pmatrix}$, $t \in \mathbb{R}$, $z \in \mathbb{C}$, $t^2 + |z|^2 = 1$. The group $SU(2)$ acts transitively on the set $S$ by conjugation and $S$ is a conjugacy class of $SU(2)$. Observe that for any matrix $A \in S$ one has $A^2 = -I$.

For $A \in S$ denote by $G_A$ the isotropy subgroup of $A$ under that action of $G$. The matrix $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ belongs to $S$. The isotropy subgroup $G_J$ of $J$ consists of matrices $\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$, $t \in \mathbb{R}$, and is the unique maximal torus of $G$ containing $J$. The isotropy subgroups $G_A$ of other matrices $A \in S$ are conjugates of $G_J$ and are the unique maximal tori of $G$ containing $A$. Observe that for $A, B \in S$ with $A \neq \pm B$ one has $G_A = G_{-A}$ and $G_A \cap G_B = \{I, -I\}$.

Let $P_{2n}$ be the product of $2n$ copies of $S^2$, $P_{2n} = S^2 \times \ldots \times S^2$. Denote by $p_{2n} : P_{2n} \rightarrow SU(2)$ the map defined by $p_{2n}(A_1, \ldots, A_{2n}) = A_1 \cdot A_2 \cdot \ldots \cdot A_{2n}$ for $A_j \in SU(2)$, $tr(A_j) = 0$, $j = 1, \ldots, 2n$. We shall denote the mapping $p_{2n}$ by $p$ whenever it will be clear from the context which $n$ is meant.

Let $K_{2n} = p^{-1}_{2n}(I)$, $\tilde{K}_{2n} = p^{-1}_{2n}(-I)$.

The group $SU(2)$ acts diagonally (by conjugation) on $P_{2n}$ and on itself (again by conjugation) and the map $p_{2n}$ is equivariant. (Both actions are from the left.) Since $I, -I \in SU(2)$ are fixed points of the action on $SU(2)$, the spaces $K_{2n} = p^{-1}_{2n}(I)$ and $\tilde{K}_{2n} = p^{-1}_{2n}(-I)$ are equivariant subsets of $P_{2n}$ and inherit an action of $SU(2)$.
Let $\epsilon_1, \ldots, \epsilon_{2n} \in \{1, -1\}$. Denote by $\Delta_{\epsilon_1, \ldots, \epsilon_{2n}}$ the closed equivariant submanifold of $P_{2n}$ defined by

$$\Delta_{\epsilon_1, \ldots, \epsilon_{2n}} = \{ (\epsilon_1 A, \ldots, \epsilon_{2n} A) \in P_{2n} \mid A \in \mathcal{S} \}.$$ 

Each submanifold $\Delta_{\epsilon_1, \ldots, \epsilon_{2n}}$ is diffeomorphic to $S^2$. The isotropy subgroup of a point $(\epsilon_1 A, \ldots, \epsilon_{2n} A)$ in $\Delta_{\epsilon_1, \ldots, \epsilon_{2n}}$ is equal to the isotropy subgroup $G_A$ of the matrix $A$ in $SU(2)$. Two subsets $\Delta_{\epsilon_1, \ldots, \epsilon_{2n}}$ and $\Delta_{\delta_1, \ldots, \delta_{2n}}$ are equal if $\delta_i = \epsilon_i$ for some $\epsilon = \pm 1$ and all $i$, and disjoint otherwise.

Every subset $\Delta_{\epsilon_1, \ldots, \epsilon_{2n}}$ is contained either in $K_{2n}$ or in $\bar{K}_{2n}$ depending on whether $(-1)^n(\prod \epsilon_i)$ is equal to 1 or to $-1$. Let us denote by $\Sigma$ the union of these subsets $\Delta_{\epsilon_1, \ldots, \epsilon_{2n}}$ which are contained in $K_{2n}$ and by $\bar{\Sigma}$ the union of those $\Delta_{\epsilon_1, \ldots, \epsilon_{2n}}$ which are contained in $\bar{K}_{2n}$.

Observe that the multiplication by $-1$ on, say, the first coordinate of $P_{2n}$ gives a diffeomorphism of $P_{2n}$ onto itself which maps $K_{2n}$ onto $\bar{K}_{2n}$ and vice-versa, and maps $\Sigma$ onto $\bar{\Sigma}$.

Let $W_{2n} = P_{2n} - (\Sigma \cup \bar{\Sigma})$. It is an open equivariant subset of $P_{2n}$.

**Lemma 2.1.** For every point $(A_1, \ldots, A_{2n}) \in W_{2n}$, $n \geq 1$, the isotropy subgroup of the action of $SU(2)$ is equal to $Z = \{1, -1\}$.

**Proof.** The center $Z$ is contained in isotropy subgroups of all points of $P_{2n}$. Since $(A_1, \ldots, A_{2n}) \in W_{2n}$ there is a pair of indices $1 \leq i, j \leq 2n$ such that $A_i \neq \pm A_j$. Let $G_{A_i}$ and $G_{A_j}$ be the isotropy subgroups of the matrices $A_i$ and $A_j$ in $SU(2)$. The isotropy subgroup of $(A_1, \ldots, A_{2n})$ is contained in $G_{A_i} \cap G_{A_j} = \{1, -1\}$ and thus equal to it.

Thus the group $SO(3) = SU(2)/\{\pm I\}$ acts freely on $W_{2n}$.

**Lemma 2.2.** The mapping $p_{2n} : P_{2n} \rightarrow SU(2)$ is a submersion at every point of $W_{2n}$.

**Proof.** It is enough to prove the lemma in the case when $n = 1$. Indeed, if $(A_1, \ldots, A_{2n}) \in W_{2n}$ then there is an index $i$ such that $A_i \neq \pm A_{i+1}$. Consider the mapping $j : W_2 \rightarrow W_{2n}$ given by $j((B_1, B_2)) = (A_1, \ldots, A_{i-1}, B_1, B_2, A_{i+2}, \ldots, A_{2n})$ and the diffeomorphism $f : SU(2) \rightarrow SU(2)$ given by $f(X) = A_1 \cdot \ldots \cdot A_{i-1} \cdot X \cdot A_{i+2} \cdot \ldots \cdot A_{2n}$. The diagram

$$\begin{array}{ccc}
W_2 & \xrightarrow{j} & W_{2n} \\
\downarrow{p_2} & & \downarrow{p_{2n}} \\
SU(2) & \xrightarrow{f} & SU(2)
\end{array}$$

commutes. Hence, if $p_2$ is a submersion at the point $(A_i, A_{i+1})$ in $W_2$ then $p_{2n}$ is a submersion at the point $j((A_i, A_{i+1})) = (A_1, \ldots, A_{i-1}, A_i, A_{i+1}, A_{i+2}, \ldots, A_{2n})$ in $W_{2n}$.

We assume now that $n = 1$. Let $(A_1, A_2)$ be a point in $W_2$. We have $A_1 \neq \pm A_2$. Conjugating by an element of $SU(2)$ if necessary, we can assume that $A_1 = J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Then $A_2 = \begin{pmatrix} is & v \\ -\overline{v} & -is \end{pmatrix}$ with $s \in \mathbb{R}$, $v \in \mathbb{C}$, $s^2 + |v|^2 = 1$ and $v \neq 0$. 

For $t \in \mathbb{R}$, $-1 < t < 1$, define matrices $A_1(t) = \begin{pmatrix} i\sqrt{1-t^2} & -t \\ t & -i\sqrt{1-t^2} \end{pmatrix}$, $\tilde{A}_1(t) = \begin{pmatrix} i\sqrt{1-t^2} & -it \\ -it & -i\sqrt{1-t^2} \end{pmatrix}$ and $A_2(t) = \begin{pmatrix} is & ve^{-it} \\ -ve^{it} & -is \end{pmatrix}$.

Observe that $A_1(t), \tilde{A}_1(t), A_2(t) \in \mathcal{S}$ for $-1 < t < 1$ and $A_1(0) = \tilde{A}_1(0) = A_1$ while $A_2(0) = A_2$. Consider the smooth curves $h_1(t) = (A_1(t), A_2)$, $h_2(t) = (\tilde{A}_1(t), A_2)$ and $h_3(t) = (A_1, A_2(t))$ in $P_2$. We have $h_i(0) = (A_1, A_2)$ for $1 \leq i \leq 3$ and

$$\frac{d}{dt} (p_2(h_1(t)))|_{t=0} = \frac{d}{dt} (A_1(t) \cdot A_2)|_{t=0} = \begin{pmatrix} -i\sqrt{1-t^2} & -1 \\ 1 & i\sqrt{1-t^2} \end{pmatrix}_{t=0} \begin{pmatrix} i & v \\ -v & -is \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} is & v \\ -v & -is \end{pmatrix} = \begin{pmatrix} \overline{v} & is \\ is & v \end{pmatrix},$$

$$\frac{d}{dt} (p_2(h_2(t)))|_{t=0} = \frac{d}{dt} (\tilde{A}_1(t) \cdot A_2)|_{t=0} = \begin{pmatrix} -i\sqrt{1-t^2} & -i \\ -it & i\sqrt{1-t^2} \end{pmatrix}_{t=0} \begin{pmatrix} i & v \\ -v & -is \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} is & v \\ -v & -is \end{pmatrix} = \begin{pmatrix} is & v \\ -v & -is \end{pmatrix}$$

and

$$\frac{d}{dt} (p_2(h_3(t)))|_{t=0} = \frac{d}{dt} (A_1 \cdot A_2(t))|_{t=0} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & -iv \\ -iv & 0 \end{pmatrix} = \begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix}.$$ 

The three matrices $\begin{pmatrix} \overline{v} & is \\ is & v \end{pmatrix}$, $\begin{pmatrix} iv & -s \\ s & -iv \end{pmatrix}$, $\begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix}$ with $s^2 + |v|^2 = 1, v \neq 0$ are linearly independent over real numbers. Indeed, if

$$\alpha \cdot \begin{pmatrix} \overline{v} & is \\ is & v \end{pmatrix} + \beta \cdot \begin{pmatrix} iv & -s \\ s & -iv \end{pmatrix} + \gamma \cdot \begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix} = 0,$$ 

then $(\alpha + i\beta)\overline{v} = 0$ and $(i\alpha - \beta)s + \gamma v = 0$. Since $v \neq 0$ we get $\alpha = \beta = \gamma = 0$.

Thus the rank of the differential $d(p_2)(A_1, A_2)$ is equal to 3 and the mapping $p_2$ is a submersion at the point $(A_1, A_2) \in W_2$.

Let $\tilde{B} = SU(2) - \{\pm I\}$ and $\tilde{E} = p_{2n}^{-1}(B) = P_{2n} - (K_{2n} \cup \tilde{K}_{2n})$. Denote by $\tilde{p}$ restriction of the mapping $p_{2n}$ to $\tilde{E}$. Then $\tilde{p} : \tilde{E} \to \tilde{B}$ is a proper map. Since $\tilde{E} \subset W_{2n}$, Lemma 2.2 implies that $\tilde{p} : \tilde{E} \to \tilde{B}$ is a submersion. Hence, we get

**Corollary 2.3.** The mapping $\tilde{p} : \tilde{E} \to \tilde{B}$ is a locally trivial fibration.

The subsets $K_{2n}$ and $\tilde{K}_{2n}$ are real algebraic subvarieties of $P_{2n}$. Let $E = P_{2n} - K_{2n}$. We have $K_{2n} \subset E$. 

\[ \square \]
Corollary 2.4. The subspace $K_{2n}$ is a strong deformation retract of $E$.

Proof. Let $B = SU(2) - \{-I\}$. We identify $B$ with a 3-dimensional disc with center at $I$. Since $K_{2n}$ is a real algebraic subvariety of $P_{2n}$ there exists an open neighbourhood $U$ of $K_{2n}$ in $P_{2n}$ such that $K_{2n}$ is a strong deformation retract of $U$, (see [1]). There exists a closed concentric subdisc $D$ of $B$ with center at $I$ such that $p^{-1}_2(D) \subset U$. Since $\overline{\partial} : \overline{E} \to \overline{B}$ is a locally trivial fibration (Corollary 2.3) the radial strong deformation retraction of $B$ onto $D$ can be lifted to $\overline{E}$. That deformation followed by the deformation retraction of $U$ onto $K_{2n}$ gives a strong deformation retraction of $E$ onto $K_{2n}$. □

Remark 2.5. It follows from Lemma 2.2 that the singularities of the real algebraic varieties $K_{2n}$ and $\tilde{K}_{2n}$ (if any) are contained in the subsets $\Sigma$ respectively $\tilde{\Sigma}$.

Corollary 2.6. The spaces $K_{2n}$ and $\tilde{K}_{2n}$ are path-connected.

Proof. Let $x$ and $y$ be two points in $K_{2n}$. Choose a smooth path $\sigma$ in $P_{2n}$ from $x$ to $y$. We can choose $\sigma$ to be transversal to, hence disjoint from, the 2-dimensional submanifold $\Sigma$. Once that has been done, we can further deform $\sigma$ to be transversal to, hence disjoint from, the submanifold $\tilde{K}_{2n} - \Sigma$ which is of codimension 3. As a result the path $\sigma$ lies in $E$. Applying the strong deformation retraction of $E$ onto $\tilde{K}_{2n}$ to $\sigma$ we get a path $\sigma'$ in $K_{2n}$ from $x$ to $y$. Hence $K_{2n}$ is path-connected. Since $\tilde{K}_{2n}$ is homeomorphic to $K_{2n}$, it is also path-connected. □

2.2. Singularities of $K_{2n}$ and $\tilde{K}_{2n}$

The main aim of this subsection is to prove that the space $\tilde{K}_{2n} - \Sigma$ is path-connected. This claim will be a consequence of a much stronger result which we formulate as Proposition 2.7 and which in turn will follow from a detailed analysis of the singularities of $K_{2n}$ and $\tilde{K}_{2n}$.

It follows from Lemma 2.2 that the singularities of the real algebraic varieties $K_{2n}$ and $\tilde{K}_{2n}$ (if any) are contained in the subsets $\Sigma$ respectively $\tilde{\Sigma}$. We shall now study the nature of these singularities.

It is enough to study the structure of $K_{2n}$ in a neighbourhood of one subset $\Delta_{\epsilon_1,...,\epsilon_{2n}}$ with $(-1)^n(\prod \epsilon_i) = 1$. Indeed, let $f_j : P_{2n} \to P_{2n}$ be the homomorphism $f_j(A_1,...,A_j,...,A_{2n}) = (A_1,...,-A_j,...,A_{2n})$ of multiplication of the $j$-th coordinate of $P_{2n}$ by $-I$. Then $f_j$ exchanges $K_{2n}$ with $\tilde{K}_{2n}$ and exchanges different subsets $\Delta_{\epsilon_1,...,\epsilon_{2n}}$. Actually, the mappings $f_j$ define a smooth action of the group $(\mathbb{Z}/2\mathbb{Z})^{2n}$ on $P_{2n}$ which acts transitively on the family of all subsets $\Delta_{\epsilon_1,...,\epsilon_{2n}}$. It follows that the structure of $K_{2n}$ in a neighbourhood of every subset $\Delta_{\epsilon_1,...,\epsilon_{2n}}$ contained in $K_{2n}$ is the same.

Consider a sequence $(\epsilon_1,...,\epsilon_{2n}) \in \{\pm 1\}^{2n}$ with $(-1)^{2n} \prod_{j=1}^{2n} \epsilon_j = 1$. Then $\Delta = \Delta_{\epsilon_1,...,\epsilon_{2n}} \subset K_{2n}$ and $\Delta$ is diffeomorphic to $S = S^2$. 
Proposition 2.7. There is an open neighbourhood $U$ of $\Delta$ in $K_{2n}$ and a continuous mapping $\xi : \overline{U} \to \Delta$ whose restriction to $\Delta$ is the identity and which is a locally trivial fibration with a fibre homeomorphic to the cone $C(S^{2n-3} \times S^{2n-3})$ over the product of two $(2n-3)$-dimensional spheres.

Proof of Proposition 2.7 is deferred to the Appendix A. (See Theorem A.7.)

By the remarks above Proposition 2.7 is also valid for $\Delta = \Delta_{\epsilon_1, \ldots, \epsilon_{2n}}$ with $(-1)^n(\prod \epsilon_i) = -1$ and with $K_{2n}$ replaced by $\tilde{K}_{2n}$.

Corollary 2.8. If $n \geq 2$ then the spaces $K_{2n} - \Sigma$ and $\tilde{K}_{2n} - \tilde{\Sigma}$ are path-connected.

Proof. It is enough to prove the statement for $K_{2n} - \Sigma$. Let $x$ and $y$ be points in $K_{2n} - \Sigma$. By Corollary 2.6 there is a path $\sigma(t), 0 \leq t \leq 1$, in $K_{2n}$ from $x$ to $y$. Let $k$ be the number of components $\Delta_{\epsilon_1, \ldots, \epsilon_{2n}}$ which the path $\sigma$ meets. The proof is by induction on $k$. Suppose $\Delta$ is a component of $\Sigma$ met by $\sigma$. Choose a neighbourhood $U$ of $\Delta$ in $K_{2n}$ as in Proposition 2.7. We can assume that $U$ does not meet other components of $\Sigma$. Let $t_1$ be the smallest value of $t$ such that $\sigma(t) \in \overline{U}$ and let $t_2$ be the largest one. Since $n \geq 2$, the base $S^{2n-3} \times S^{2n-3}$ of the cone $C(S^{2n-3} \times S^{2n-3})$ is path-connected and, therefore, there exists a path $\tau$ in $\overline{U} - \Delta$ from $\sigma(t_1)$ to $\sigma(t_2)$. Then the path $\sigma(t)$ from $\sigma(0)$ to $\sigma(t_1)$ followed by the path $\tau$ and then followed by path $\sigma(t)$ from $\sigma(t_2)$ to $\sigma(1)$ gives a path in $K_{2n}$ from $x$ to $y$ with a smaller number $k$. That proves Corollary.

2.2. The first and the second homotopy groups of $K_{2n}$.

The first two homotopy groups of the space $P_{2n}$ are $\pi_1(P_{2n}) = 0$ and $\pi_2(P_{2n}) \cong \bigoplus_{i=1}^{2n} \mathbb{Z}_i$, where all $\mathbb{Z}_i \cong \mathbb{Z}$. Let $j : K_{2n} \to P_{2n}$ be the inclusion. The first two homotopy groups of $K_{2n}$ are given by

Proposition 2.9. If $n \geq 2$, then

(i) $\pi_1(K_{2n}) = 0$.

(ii) There is a short split exact sequence

$$
0 \longrightarrow \mathbb{Z} \longrightarrow \pi_2(K_{2n}) \longrightarrow \pi_2(P_{2n}) \longrightarrow 0.
$$

Proof. Recall that $K_{2n}$ is a strong deformation retract of $E = P_{2n} - \tilde{K}_{2n}$ (see Corollary 2.4). Therefore it is enough to prove the Proposition for the space $E$ instead of $K_{2n}$. Let $\tilde{j} : E \to P_{2n}$ be the inclusion.

(i) Let $\gamma : S^1 \to E$ be an arbitrary loop in $E$. Then $\tilde{j} \circ \gamma$ is a loop in $P_{2n}$. Since the space $P_{2n}$ is 1-connected there is an extension of $\tilde{j} \circ \gamma$ to a map $\Gamma : D^2 \to P_{2n}$. The submanifold $\Sigma$ is of codimension $4n - 2 \geq 6$ in $P_{2n}$. Therefore the mapping $\Gamma$ can be chosen to be disjoint from $\Sigma$. One gets $\Gamma : D^2 \to P_{2n} - \tilde{\Sigma}$. It follows from Lemma 2.2 that $\tilde{K}_{2n} - \tilde{\Sigma}$ is a submanifold of $P_{2n} - \tilde{\Sigma}$ of codimension 3. Therefore $\Gamma$ can deformed rel $\partial D^2$ even further and chosen to be disjoint from $\tilde{K}_{2n} - \tilde{\Sigma}$ and hence from $\tilde{K}_{2n}$. In that way we get a map $\Gamma : D^2 \to P_{2n} - \tilde{K}_{2n} = E$ extending the loop $\gamma$. Hence the homotopy class of $\gamma$ in $\pi_1(E)$ is trivial, showing that $\pi_1(E) = 0$.
(ii) Let $\zeta : S^2 \to P_{2n}$ be arbitrary. The same dimensional argument as in the proof of part (i) shows that $\zeta$ can be deformed to a map $\zeta' : S^2 \to E$. This proves that $\tilde{j}_* : \pi_2(E) \to \pi_2(P_{2n})$ is surjective.

Let $H \subset \pi_2(E)$ be the kernel of $\tilde{j}_* : \pi_2(E) \to \pi_2(P_{2n})$. We shall now construct an isomorphism $\chi : H \to \mathbb{Z}$.

We choose an orientation of $P_{2n}$ and of $SU(2)$. That gives an orientation of $P_{2n} - \Sigma$. By Lemma 2.2 the subset $\overline{K}_{2n} - \Sigma = \overline{p}_{2n}^{-1}(I) - \Sigma$ is a submanifold of $P_{2n} - \Sigma$ and we give it an orientation such that locally at its points the orientation of $P_{2n}$ is equal to that of $\overline{K}_{2n} - \Sigma$ followed by that of $SU(2)$.

Let $\varphi : S^2 \to E$ be a smooth map representing an element of $H$. Since $\tilde{j}_*([\varphi]) = 0$, there exists a smooth extension $\Phi : D^3 \to P_{2n}$ of $\varphi$. Again, since the codimension of the submanifold $\Sigma$ of $P_{2n}$ is $4n - 2 \geq 6$ we can choose $\Phi$ whose image is disjoint from $\Sigma$. Moreover, we can choose $\Phi$ to be transversal to the oriented submanifold $\overline{K}_{2n} - \Sigma$. We define $\chi(\varphi) \in \mathbb{Z}$ to be the oriented intersection number of $\overline{K}_{2n} - \Sigma$ with $\Phi : D^3 \to P_{2n} - \Sigma$ in $P_{2n} - \Sigma$. This number is independent of the choice of $\varphi$ in its homotopy class and of the choice of its extension $\Phi : D^3 \to P_{2n} - \Sigma$. Indeed, let $\varphi' : S^2 \to E$ be homotopic to $\varphi$ with $F : S^2 \times I \to E$ being a homotopy between them. Choose an extension $\Phi' : D^3 \to P_{2n}$ of $\varphi'$ with properties as those of $\Phi$. Identify the oriented 3-dimensional sphere $S^3$ with the union $D^3 \cup S^2 \times I \cup (-D^3)$, where $-D^3$ denotes the standard 3-dimensional disc with the reversed orientation and where one component of the oriented boundary of $S^2 \times I$ has been identified with the boundary of $D^3$ while the other component with the boundary of $-D^3$. The mappings $\Phi$ on $D^3$, $F$ on $S^2 \times I$ and $\Phi'$ on $-D^3$ fit together and define a mapping $G : S^3 \to P_{2n}$ which is transversal to $\overline{K}_{2n}$ and does not meet $\Sigma$. (Observe that the restriction of $G$ to $S^2 \times I$ is given by $F$ whose image lies in $E$ and hence does not intersect $\overline{K}_{2n}$). The oriented intersection number $m$ of $\overline{K}_{2n} - \Sigma$ with $G$ is equal to $\chi(\varphi) - \chi(\varphi')$. Here $\chi(\varphi)$ is defined by the choice of the extension $\Phi$ and $\chi(\varphi')$ by the choice of the extension $\Phi'$.

On the other hand, the oriented real subvariety $\overline{K}_{2n}$ of $P_{2n}$ with singularities of codimension at least 6 defines a homology class in $H_{2n-3}(P_{2n}; \mathbb{Z})$ and the mapping $G$ defines a homology class in $H_3(P_{2n}; \mathbb{Z})$. Both these classes are equal to 0 since $P_{2n}$ has only trivial homology in odd degrees. At the same time, the intersection number $m$ is equal to the homological intersection number of those two homology classes and hence equal to 0. Therefore $\chi(\varphi) - \chi(\varphi') = m = 0$ and, finally, $\chi(\varphi) = \chi(\varphi')$.

Thus, we get a well defined integer $\chi([\varphi]) \in \mathbb{Z}$ depending only on the homotopy class of $\varphi : S^2 \to E$ for $[\varphi] \in H = ker(j_*)$. It is obvious from the construction that in that way we get a group homomorphism $\chi : H \to \mathbb{Z}$.

Let us choose a point $x$ in $\overline{K}_{2n} - \Sigma$ and a normal space $N_x$ to $\overline{K}_{2n}$ in $P_{2n}$ at $x$ (a fiber of a regular neighbourhood). We have $\dim N_x = 3$. A sphere $S_x$ in $N_x$ with centrum at $x$ and a small radius gives us a mapping $\varphi : S^2 \to E$ with $\chi([\varphi]) = \pm 1$. Thus $\chi$ is an epimorphism.

To show that $\chi : H \to \mathbb{Z}$ is a monomorphism let us assume that $\varphi : S^2 \to E$ is a map such that $[\varphi] \in H$ and $\chi([\varphi]) = 0$. It follows that $\varphi$ extends to a map $\Phi : D^3 \to P_{2n}$ whose image does not meet $\Sigma$ and intersects $\overline{K}_{2n}$ transversally in an even number of points, half of them with index +1 and half with index −1.
According to Corollary 2.8, the manifold $\tilde{K}_{2n} - \Sigma$ is path-connected. Thus, applying the same arguments on cancellation of intersection points in pairs as in the proof of the theorem of Whitney (see [23], Theorem 6.6.) one proves that the map $\Phi$ can be deformed rel. $\partial D^3$ to a map whose image does not intersect $\tilde{K}_{2n}$ at all. This new deformed map is of the form $\Psi : D^3 \to E$ with $\Psi|\partial D^3 = \varphi$. We conclude that the homotopy class of $[\varphi] \in \pi_2(E)$ is trivial. Hence $\chi : H \to \mathbb{Z}$ is a monomorphism and therefore an isomorphism.

That proves the second part of Proposition 2.9 since $H$ was the kernel of $j_* : \pi_2(E) \to \pi_2(P_{2n})$. The short exact sequence is split because $\pi_2(P_{2n})$ is a free $\mathbb{Z}$-module.

Let $h_n : P_{2n} \to P_{2n+2}$ be the map defined by

$$h_n(A_1, ..., A_{2n}) = (A_1, ..., A_{2n}, -J, J).$$

(2.1)

The diagram

$$
\begin{array}{ccc}
P_{2n} & \xrightarrow{h_n} & P_{2n+2} \\
\downarrow P_{2n} & & \downarrow P_{2n+2} \\
SU(2) & & SU(2)
\end{array}
$$

(2.2)

commutes. Hence $h_n$ maps $K_{2n} = p_{2n}^{-1}(I)$ into $K_{2n+2} = p_{2n+2}^{-1}(I)$. We denote the restriction of $h_n$ to $K_{2n}$ by the same symbol, $h_n : K_{2n} \to K_{2n+2}$.

Consider the homomorphism induced by $h_n$ on the second homotopy groups

$$h_n^* : \pi_2(K_{2n}) \to \pi_2(K_{2n+2}).$$

(We do not specify the base points as it is inessential. Whenever necessary we can take $x_0 = (-J, J, ..., -J, J) \in K_{2n}$ as a base point.)

**Proposition 2.10.** If $n \geq 2$ then the homomorphism $h_n^* : \pi_2(K_{2n}) \to \pi_2(K_{2n+2})$ is a monomorphism mapping the kernel of $j_* : \pi_2(K_{2n}) \to \pi_2(P_{2n})$ isomorphically onto the kernel of $j_* : \pi_2(K_{2n+2}) \to \pi_2(P_{2n+2})$.

**Proof.** The diagram

$$
\begin{array}{cccc}
0 & \to & \mathbb{Z} & \to & \pi_2(K_{2n}) & \xrightarrow{j_*} & \pi_2(P_{2n}) & \to & 0 \\
\downarrow h_n^* & & \downarrow & & \downarrow h_n^* & & \downarrow & & \downarrow h_n^* \\
0 & \to & \pi_2(K_{2n+2}) & \xrightarrow{j_*} & \pi_2(P_{2n+2}) & \to & 0
\end{array}
$$

(2.3)

commutes. Hence the homomorphism $h_n^* : \pi_2(K_{2n}) \to \pi_2(K_{2n+2})$ maps the kernel $\mathbb{Z}$ of $j_* : \pi_2(K_{2n}) \to \pi_2(P_{2n})$ into the kernel $\mathbb{Z}$ of $j_* : \pi_2(K_{2n+2}) \to \pi_2(P_{2n+2})$.

Let $\gamma : D^3 \to P_{2n}$ be an embedding of a small 3-dimensional disc which intersects $\tilde{K}_{2n} = p_{2n}^{-1}(-I)$ transversally in one non-singular point with the intersection number 1. Then the homotopy class of the restriction of $\gamma$ to the boundary $\partial D^3$ of the disc, $\gamma|_{\partial D^3} : \partial D^3 \to (P_{2n} - \tilde{K}_{2n}) \simeq K_{2n}$, gives a generator $\xi$ of the kernel of $j_* : \pi_2(K_{2n}) \to \pi_2(P_{2n})$.

Since the diagram (2.2) commutes, it follows that the composition $h_n \circ \gamma$ is an embedding of $D^3$ into $P_{2n+2}$ which again intersects $\tilde{K}_{2n+2} = p_{2n+2}^{-1}(-I)$ transversally in one non-singular point with the intersection number 1. The homotopy class
of its restriction \((h_n \circ \gamma)|_{\partial D^3}\) to the boundary of the disc gives a generator \(\xi'\) of the kernel of \(j_* : \pi_2(K_{2n+2}) = \pi_2(P_{2n+2} - K_{2n+2}) \to \pi_2(P_{2n+2})\).

Thus \(h_{n*} : \pi_2(K_{2n}) \to \pi_2(K_{2n+2})\) maps the generator \(\xi\) of the kernel of \(j_* : \pi_2(K_{2n}) \to \pi_2(P_{2n})\) to the generator \(\xi'\) of the kernel of \(j_* : \pi_2(K_{2n+2}) \to \pi_2(P_{2n+2})\) and, hence, maps the first of the kernels isomorphically onto the second one. (Actually, the restriction of \(h_{n*}\) to the kernels is the identity homomorphism, when both are identified with \(\mathbb{Z}\) in the way indicated in the proof of Lemma 2.9). As the right-hand vertical arrow in the diagram (2.3) is obviously a monomorphism, so is the homomorphism \(h_{n*} : \pi_2(K_{2n}) \to \pi_2(K_{2n+2})\).

\[\square\]

3. THE SYMPLECTIC STRUCTURE

Let \(G\) be a compact Lie group and \(\mathfrak{g}\) its Lie algebra. We choose an invariant \(\mathbb{R}\)-valued positive definite inner product \(\cdot\) on \(\mathfrak{g}\).

Let \(\mathcal{C} = (C_1, \ldots, C_m)\) be a sequence of \(m\) conjugacy classes in \(G\) (not necessarily distinct).

Denote by \(F\) a free group on \(m\) generators \(z_1, \ldots, z_m\).

Let \(\text{Hom}(F, G)\) be the set of all group homomorphisms \(\varphi : F \to G\) such that \(\varphi(z_j) \in C_j\), \(1 \leq j \leq m\). We identify \(\text{Hom}(F, G)\) with the smooth manifold \(C_1 \times \cdots \times C_m\) by identifying \(\varphi\) with \((g_1, \ldots, g_m) = (\varphi(z_1), \ldots, \varphi(z_m))\). The group \(G\) acts on \(\text{Hom}(F, G)\), on the conjugacy classes \(C_j\) and on itself by conjugation.

Denote by \(f_j : \text{Hom}(F, G) \to C_j\), \(j = 1, \ldots, m\), the projection on the \(j\)th factor and by \(r : \text{Hom}(F, G) \to G\) the mapping \(r(\varphi) = \varphi(z_1) \cdot \cdots \cdot \varphi(z_m)\). All these are \(G\)-equivariant maps.

Let \(K = r^{-1}(e) \subset \text{Hom}(F, G)\), where \(e \in G\) is the unit element.

According to \([8]\) there exists a neighbourhood \(\mathcal{M}\) of \(K\) in \(\text{Hom}(F, G)\) and a closed 2-form \(\omega_C\) on \(\mathcal{M}\) which is non-degenerate and hence gives a symplectic structure on \(\mathcal{M}\). The form \(\omega_C\) is denoted by \(\omega_{C, \varphi, C}\) in \([8]\). We shall now recall its definition.

The form \(\omega_C\) is a sum of three parts.

The first part: Let \(C_* (F)\) be the chain complex of the nonhomogeneous reduced normalised bar resolution of \(F\), see \([21]\). (We use the notation of that reference.)

The form \(\omega_C\) is not uniquely defined. Its definition depends on a choice of a 2-chain \(c\) in \(C_2(F)\) satisfying

\[\partial c = [z_1 \cdot z_2 \cdot \ldots \cdot z_m] - [z_1] - \ldots - [z_m],\]

where, for any \(x \in F\), the symbol \([x]\) denotes the corresponding generator of \(C_1(F)\). As pointed out in \([8]\), page 391, one such possible choice is

\[c = -[z_1 \cdot \ldots \cdot z_{m-1} | z_m] - [z_1 \cdot \ldots \cdot z_{m-2} | z_{m-1}] - \ldots - [z_1 | z_2].\]

(3.1)

We shall stay with this choice of \(c\) for the rest of the paper.

Let us first recall some definitions from \([28]\) and \([10]\). (We follow the convention used in \([10]\) that if \(\alpha\) is a \(p\)-form and \(\beta\) is a \(q\)-form then

\[(\alpha \wedge \beta)(v_1, \ldots, v_{p+q}) = \sum_{p, q \text{ shuffles}} (\text{sgn} \pi)\alpha(v_{\pi(1)}, \ldots, v_{\pi(p)}) \beta(v_{\pi(p+1)}, \ldots, v_{\pi(p+q)}),\]

for tangent vectors \(v_1, \ldots, v_{p+q}\).}
We start by defining a 3-form $\lambda$ on $G$ and a 2-form $\Omega$ on $G \times G$.

Denote by $\omega$ the $g$-valued, left-invariant 1-form on $G$ which maps each tangent vector to the left-invariant vector field having that value. The corresponding right-invariant form will be denoted by $\bar{\omega}$.

The tripiple product

$$\tau(x, y, z) = \frac{1}{2} [x, y] \cdot z, \quad x, y, z \in g$$

yields an alternating trilinear form on $g$. Denote by $\lambda$ the left translate of $\tau$. It is a closed invariant 3-form on $G$ and it satisfies

$$\lambda = \frac{1}{12} [\omega, \omega] \cdot \omega.$$ 

(Note that by the convention used $[\omega, \omega](X, Y) = 2[X, Y]$ for arbitrary left-invariant vector fields $X$ and $Y$ on $G$.)

For any differential form $\alpha$ on $G$ denote by $\alpha_j$ the pullback of $\alpha$ to $G \times G$ by the projection $p_j$ to the $j$th factor. Let

$$\Omega = \frac{1}{2} \omega_1 \cdot \bar{\omega}_2.$$ 

This is a real-valued 2 form on $G \times G$. According to the convention

$$(\omega_1 \cdot \bar{\omega}_2)(U, V) = \omega_1(U) \cdot \bar{\omega}_2(V) - \bar{\omega}_2(U) \cdot \omega_1(V).$$  \hspace{1cm} (3.2)

Let us now consider the evaluation map

$$E : F^2 \times \text{Hom}(F, G)_C \to G^2,$$

and, for every $x, y \in F$, let us denote by $E_{[x, y]} : \text{Hom}(F, G)_C \to G^2$ the map given by $E_{[x, y]}(\phi) = (\phi(x), \phi(y))$. Denote by $\omega_{[x, y]}$ the 2-form on $\text{Hom}(F, G)_C$ which is the pullback of $\Omega$ by the map $E_{[x, y]}$

$$\omega_{[x, y]} = E_{[x, y]}^*(\Omega).$$

Finally, for $c = -\sum_{j=1}^{m-1} [z_1 \cdot ... \cdot z_j | z_{j+1}] \in C_2(F)$, let

$$\omega_c = -\sum_{j=1}^{m-1} \omega_{[z_1 \cdot ... \cdot z_j | z_{j+1}]}.  \hspace{1cm} (3.3)$$

**The second part:** Let $\tilde{\mathcal{H}}$ be an open $G$-equivariant subset of the Lie algebra $g$ containing 0 and such that (i) the exponential mapping is a diffeomorphism of $\tilde{\mathcal{H}}$ onto $\exp(\tilde{\mathcal{H}})$, and (ii) $\tilde{\mathcal{H}}$ is star-shaped w.r.t. 0.

Let $\mathcal{H} = r^{-1}(\exp(\tilde{\mathcal{H}}))$. It is an open subset of $\text{Hom}(F, G)_C$ containing $K$.

Denote by $\eta$ the embedding of $\mathcal{H}$ into $\text{Hom}(F, G)_C$ and by $\tilde{r} : \mathcal{H} \to \tilde{\mathcal{H}}$ the composition of the map $r : \text{Hom}(F, G)_C \to G$ restricted to $\mathcal{H}$ with the map $\exp^{-1} : \exp(\tilde{\mathcal{H}}) \to \tilde{\mathcal{H}}$. Also denote by $\tilde{f}_j : \mathcal{H} \to C_j, j = 1, ..., m$, the restrictions of $f_j : \text{Hom}(F, G)_C \to C_j$ to $\mathcal{H}$.

Let $h : \Omega^*(g) \to \Omega^{*-1}(g)$ be the standard homotopy operator given by integration of forms along linear paths in $g$, see [24], Lemma 2.13.1. Then $\beta = h(\exp^*(\lambda))$ is a 2-form on $g$ and we define a 2-form $\omega_{c, \mathcal{H}}$ on $\mathcal{H}$ by

$$\omega_{c, \mathcal{H}} = \eta^* \omega_c - \tilde{r}^* \beta.$$  \hspace{1cm} (3.4)
The third part: Let $C$ be a conjugacy class in $G$. For a point $p$ of $C$, an arbitrary tangent vector is of the form

$$X_p - pX = (X - \text{Ad}(p)X)p \in T_p C,$$

where $X$ is an element of the Lie algebra $\mathfrak{g}$ identified with the tangent space $T_e G$ of $G$ at $e$ and where $X_p$ and $pX$ are the right and the left translation of $X$ by $p$ respectively. The formula

$$\tau(X_p - pX, Y_p - pY) = \frac{1}{2}(X \bullet \text{Ad}(p)Y - Y \bullet \text{Ad}(p)X), \quad p \in C,$$

yields an equivariant 2-form $\tau$ on $C$, [8], Sec.6.

For the conjugacy classes $C_1, ..., C_m$ in $G$ denote by $\tau_1, ..., \tau_m$ the corresponding 2-forms on them.

Finally define a 2-form $\omega_C$ on $\mathcal{H}$ by

$$\omega_C = \omega_{c, P} + f_1^* \tau_1 + ... + f_m^* \tau_m .$$

(The form $\omega_C$ is denoted by $\omega_{c, P, C}$ in [8].)

One of the main results of [8], in the special case considered in this Section (this is the case of a surface of genus $g = 0$), is Theorem 3.1 ([8]; Thm. 8.12). There is a $G$-invariant neighbourhood $\mathcal{M}$ of $K$ in $\mathcal{H}$ where the 2-form $\omega_C$ is symplectic.

For the proof of the Theorem we refer the reader to [8].

We conclude this Section with a straightforward observation.

For some special conjugacy classes $C_1, ..., C_m$ the 2-form $\omega_C$ simplifies somewhat. That holds in particular for the conjugacy classes of interest in this paper.

Let us suppose that $C$ is a conjugacy class in $G$ such that for every $p \in C$ one has

$$p^2 \in Z(G) ,$$

where $Z(G)$ is the center of $G$.

We have

Lemma 3.2. If $C$ is a conjugacy class in $G$ satisfying the condition (3.8) then the 2-form $\tau$ vanishes on $C$.

Proof. Since the inner product $\bullet$ on $\mathfrak{g}$ is invariant with respect to the adjoint action of $G$ and since, as a consequence of the condition (3.8), $\text{Ad}(p^2) = \text{Id}$ for every $p \in C$, it follows that

$$X \bullet \text{Ad}(p)Y = \text{Ad}(p)X \bullet \text{Ad}(p)Y = \text{Ad}(p^2)X \bullet Y = \text{Ad}(p)X \bullet Y = Y \bullet \text{Ad}(p)X$$

for all $X, Y \in \mathfrak{g}$.

Therefore the expression on the right hand side of (3.6) vanishes and so does the 2-form $\tau$ on $C$. 

Corollary 3.3. If the conjugacy classes in $C = (C_1, ..., C_m)$ all satisfy the condition (3.8) then the 2-form $\omega_C$ on $\mathcal{H}$ is given by

$$\omega_C = \omega_{c, P} = \eta^* \omega_c - \tilde{\nu}^* \beta .$$
Observe that the conjugacy class $\mathcal{S}$ in $SU(2)$ considered in the previous Sections does satisfy the condition (3.8) as $A^2 = -I$ for every $A \in \mathcal{S}$.

We shall abuse the notation and write $\omega_c$ for the form $\eta^* \omega_c$ on $\mathcal{H}$.

**Corollary 3.4.** If the conjugacy classes in $\mathcal{C} = (C_1, ..., C_m)$ all satisfy the condition (3.8) then at all points of $K = r^{-1}(c) = \hat{r}^{-1}(0)$ the 2-form $\omega_C$ is given by

$$\omega_C = \omega_c .$$

**Proof.** The mapping $\hat{r} : \mathcal{H} \to \mathcal{O}$ maps $K$ to $0 \in \mathcal{O} \subset \mathfrak{g}$. Since the 2-form $\beta$ on $\mathfrak{g}$ vanishes at 0, the 2-form $\hat{r}^* \beta$ vanishes over $K$. Hence $\omega_C = \eta^* \omega_c = \omega_c$. $\square$

4. THE ACTION OF THE BRAID GROUP

We continue with the notation of Section 3.

Let $B_m$ be the braid group on $m$ strands. It is generated by $m - 1$ elementary braids $\sigma_k$, $k = 1, ..., m - 1$ subject to relations

$$\sigma_k \sigma_j = \sigma_j \sigma_k \quad \text{if} \quad |k - j| > 1,$$

$$\sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1} \quad \text{for} \quad k = 1, ..., m - 2.$$

Let $F_m$ be the free group on $m$ generators $z_1, ..., z_m$. The braid group $B_m$ acts from the right on $F_m$. The action of the $k$-th elementary braid $\sigma_k$ is given by the automorphism $\sigma_k : F_m \to F_m$ defined by

$$\sigma_k(z_j) = \begin{cases} z_j & j \neq k, k + 1, \\ z_k z_{k+1} z_k^{-1} & j = k, \\ z_k & j = k + 1, \end{cases} \quad (4.1)$$

see [3], Cor. 1.8.3.

The right action of $B_m$ on $F_m$ induces a left action of $B_m$ on the space of all group homomorphisms $\text{Hom}(F_m, G)$ given by $\sigma(\varphi)(f) = \varphi(\sigma(f))$ for any $\sigma \in B_m$, $\varphi \in \text{Hom}(F_m, G)$ and $f \in F_m$.

Let $C$ be a conjugacy class in $G$ and let $\mathcal{C} = (C_1, ..., C_m)$ be the sequence of $m$ copies of $C$. Recall that $\text{Hom}(F_m, G)_C$ is the subspace of $\text{Hom}(F_m, G)$ consisting of those homomorphisms $\varphi : F_m \to G$ which map all generators $z_j$ of $F_m$ to $C$.

As a consequence of the definition (4.1) we get that for any braid $\sigma \in B_m$ and any generator $z_j \in F_m$ the element $\sigma(z_j)$ is conjugate to some generator $z_i$ of $F_m$. It follows that if $\varphi \in \text{Hom}(F_m, G)_C$ then, for any braid $\sigma$, the homomorphism $\sigma(\varphi)$ also belongs to $\text{Hom}(F_m, G)_C$ i.e. that $\text{Hom}(F_m, G)_C$ is a $B_m$-invariant subspace of $\text{Hom}(F_m, G)$. Therefore the braid group $B_m$ acts also on $\text{Hom}(F_m, G)_C$. (This action is a special case of actions of $B_m$ studied in [25].)

If we identify $\text{Hom}(F_m, G)_C$ with the product $\prod_{j=1}^m C$ of $m$ copies of $C$ by identifying $\varphi \in \text{Hom}(F_m, G)_C$ with $(\varphi(z_1), ..., \varphi(z_m)) \in \prod_{j=1}^m C$ then the action of the $k$-th elementary braid $\sigma_k$ on $g = (g_1, ..., g_m) \in \prod C$ is given by

$$(\sigma_k(g))_j = \begin{cases} g_j & j \neq k, k + 1, \\ g_k g_{k+1} g_k^{-1} & j = k, \\ g_k & j = k + 1, \end{cases} \quad (4.2)$$
where \((,)\) stands for the projection on the \(j\)-th factor of \(\prod C\).

Recall the mapping \(r : \text{Hom}(F_m, G)_C \rightarrow G\) defined by \(r(g_1, ..., g_m) = g_1 \cdot ... \cdot g_m\).

It follows from (4.2) that for every elementary braid \(\sigma_k\) and \(g = (g_1, ..., g_m) \in \prod C\)
\[r(\sigma_k(g)) = g_1 \cdot ... \cdot g_m = r(g)\.

Therefore, for every braid \(\sigma \in B_m\), we have
\[r \circ \sigma = r\quad (4.3)\]

It follows from (4.3) that the subset \(\mathcal{H} = r^{-1}(\exp(\tilde{\beta}))\) of \(\text{Hom}(F_m, G)_C\) is invariant with respect to the action of \(B_m\). Moreover, since \(\tilde{r} : \mathcal{H} \rightarrow \tilde{\beta}\) is the composition of \(r\) and \(\exp^{-1}\), we have
\[\tilde{r} \circ \sigma = \tilde{r}\quad (4.4)\]

and, consequently,
\[\sigma^*(\tilde{r}^* \beta) = \tilde{r}^* \beta\quad (4.5)\]

for every braid \(\sigma \in B_m\).

Finally, consider the open subset \(\mathcal{M}\) of \(\mathcal{H}\) the existence of which is ascertained in Theorem 3.1. Since the space \(\prod C\) is compact and the group \(G\) is Hausdorff, we can assume (possibly choosing a smaller \(\mathcal{M}\)) that \(\mathcal{M}\) is of the form \(\mathcal{M} = r^{-1}(U)\) for some open \(Ad\)-invariant neighbourhood \(U\) of the identity element \(e\) of the group \(G\). It follows then from (4.3) that such a subset \(\mathcal{M}\) of \(\mathcal{H}\) is not only \(G\)-invariant but also \(B_m\)-invariant and inherits the action of \(B_m\). It follows also from (4.3) that \(K = r^{-1}(e)\) is a \(B_m\)-invariant subset of \(\prod C\) and of \(\mathcal{M}\).

The main aim of this section is to prove

**Theorem 4.1.** If the conjugacy class \(C\) satisfies the condition (3.8) then the symplectic 2-form \(\omega_C\) on \(\mathcal{M}\) satisfies
\[\sigma^*(\omega_C) = \omega_C\]

for every braid \(\sigma \in B_m\), i.e. the braid group \(B_m\) acts on \(\mathcal{M}\) through symplectomorphisms.

**Proof.** According to Corollary 3.3 the 2-form \(\omega_C\) on \(\mathcal{H}\) is given by
\[\omega_C = \eta^* \omega_c - \tilde{r}^* \beta\.

By (4.5) we know already that \(\sigma^*(\tilde{r}^* \beta) = \tilde{r}^* \beta\) for any braid \(\sigma \in B_m\). Hence it is enough to show that \(\sigma^*(\eta^* \omega_c) = \eta^* \omega_c\).

The embedding \(\eta\) of \(\mathcal{H}\) into \(\text{Hom}(F_m, G)_C = \prod C\) commutes, by the definition, with the action of \(B_m\). Therefore, it will be enough to prove that
\[\sigma^*(\omega_c) = \omega_c\quad (4.6)\]

for any braid \(\sigma \in B_m\).

According to (3.3) we have
\[\omega_c = - \sum_{j=1}^{m-1} \omega_{[z_1, ..., z_j | z_{j+1}]}.\quad (4.7)\]

Recall that \(\omega_{[z_1, ..., z_j | z_{j+1}]} = E_j^*(\Omega)\), where \(E_j : \prod_{j=1}^m C \rightarrow G \times G\) is given by
\[E_j(g_1, ..., g_m) = (g_1 \cdot ... \cdot g_j, g_{j+1})\]

and where \(\Omega\) is the 2-form on \(G \times G\) defined by (3.2).
Denote by $F_k : \prod C \to \prod C$ the diffeomorphism of $\prod C$ given by the action of the $k$-th elementary braid $\sigma_k$ and described in (4.2).

Thus $\sigma_k^*(\omega_{[z_1,\ldots,z_j|z_{j+1}]}) = F_k^*(\omega_{[z_1,\ldots,z_j|z_{j+1}]}) = F_k^*E_j^*(\Omega) = (E_j \circ F_k)^*(\Omega)$.

Observe that if $j + 1 \neq k, k + 1$ then the mapping $E_j \circ F_k : \prod C \to G \times G$ satisfies $E_j \circ F_k = E_j$. Hence

$$\sigma_k^*(\omega_{[z_1,\ldots,z_j|z_{j+1}]}) = (E_j \circ F_k)^*(\Omega) = E_j^*(\Omega) = \omega_{[z_1,\ldots,z_j|z_{j+1}]}$$

(4.8)

if $j + 1 \neq k, k + 1$.

We shall now show that

$$\sigma_k^*(\omega_{[z_1,\ldots,z_{k-1}|z_k]} + \omega_{[z_1,\ldots,z_k|z_{k+1}]}) = \omega_{[z_1,\ldots,z_{k-1}|z_k]} + \omega_{[z_1,\ldots,z_k|z_{k+1}]}$$

(4.9)

on $\prod C = \text{Hom}(F_m, G)c$.

Before we start let us observe that the 2-forms $\omega_{[x|y]} = E_{[x|y]}^*(\Omega)$ are defined not only on $\prod C = \text{Hom}(F_m, G)c$ but also on the whole manifold $\prod_{i=1}^m G = \text{Hom}(F_m, G)$.

Let $g = (g_1, \ldots, g_m) \in \prod_{i=1}^m C$. We represent tangent vectors to the conjugacy class $C$ at the point $g_i \in C$ in the form $X_i g_i$ with $X_i \in g$. Let $X = (X_1 g_1, \ldots, X_m g_m)$ and $Y = (Y_1 g_1, \ldots, Y_m g_m)$ be two arbitrary tangent vectors to $\prod C$ at the point $g$. Here $X_1, \ldots, X_m, Y_1, \ldots, Y_m \in g$.

As $E_{k-1} : \prod C \to G \times G$, $E_{k-1}(g_1, \ldots, g_m) = (g_1 \cdots g_{k-1}, g_k)$ and $E_k : \prod C \to G \times G$, $E_k(g_1, \ldots, g_m) = (g_1 \cdots g_k, g_{k+1})$, we have

$$dE_{k-1}(X) = dE_{k-1}(X_1 g_1, \ldots, X_m g_m) =$$

$$= \left( \sum_{j=1}^{k-1} g_1 \cdots g_{j-1} X_j g_j \cdots g_{k-1}, X_k g_k \right) =$$

$$= \left( g_1 \cdots g_{k-1} \sum_{j=1}^{k-1} \text{Ad}(g_j \cdots g_{k-1})^{-1}(X_j), X_k g_k \right)$$

and

$$dE_k(X) = \left( g_1 \cdots g_k \sum_{j=1}^k \text{Ad}(g_j \cdots g_k)^{-1}(X_j), X_{k+1} g_{k+1} \right)$$

and similarly for the tangent vector $Y$.

Hence, according to (3.2),

$$\omega_{[z_1,\ldots,z_{k-1}|z_k]}(X, Y) = E_{k-1}^*(\Omega)(X, Y) = \Omega(dE_{k-1}(X), dE_{k-1}(Y)) =$$

$$= \frac{1}{2} \left[ \omega_1(dE_{k-1}(X)) \bullet \bar{\omega}_2(dE_{k-1}(Y)) - \omega_1(dE_{k-1}(Y)) \bullet \bar{\omega}_2(dE_{k-1}(X)) \right]$$

(4.10)

and

$$\omega_{[z_1,\ldots,z_k|z_{k+1}]}(X, Y) = \frac{1}{2} \left[ \left( \sum_{j=1}^k \text{Ad}(g_j \cdots g_k)^{-1}(X_j) \right) \bullet Y_{k+1} - \text{Symm} \right]$$

(4.11)
where the term “Symm” means “the same terms with the rôles of X and Y exchanged”.

On the other hand

\[ dF_k(X) = dF_k(X_1 g_1, \ldots, X_m g_m) = \]

\[ = ((X_1 g_1, \ldots, X_{k-1} g_{k-1}, X_k g_k g_{k+1} g_k^{-1} + g_k X_k g_{k+1} g_k^{-1} - g_k g_{k+1} g_k^{-1} X_k, X_k g_k, X_{k+2} g_{k+2}, \ldots, X_m g_m) = \]

\[ = ((X_1 g_1, \ldots, X_{k-1} g_{k-1}, (X_k + Ad(g_k)(X_{k+1}) - Ad(g_k g_{k+1} g_k^{-1})(X_k)) g_k g_{k+1} g_k^{-1}, X_k g_k, X_{k+2} g_{k+2}, \ldots, X_m g_m) \]

and similarly for Y.

Therefore, by the same argument as in the derivation of (4.10) and (4.11), we get

\[ (\sigma_k^* \omega_{z_1 \ldots z_{k-1} z_k})(X, Y) = \omega_{z_1 \ldots z_{k-1} z_k}(dF_k(X), dF_k(Y)) = \]

\[ = \frac{1}{2} \left[ \left( \sum_{j=1}^{k-1} Ad(g_j \ldots g_{k-1})^{-1}(X_j) \right) \cdot (Y_k + Ad(g_k)(Y_{k+1}) - Ad(g_k g_{k+1} g_k^{-1})(Y_k)) - \text{Symm} \right]. \]

(4.12)

Similarly,

\[ (\sigma_k^* \omega_{z_1 \ldots z_k z_{k+1}})(X, Y) = \omega_{z_1 \ldots z_k z_{k+1}}(dF_k(X), dF_k(Y)) = \]

\[ = \frac{1}{2} \left[ \left( \sum_{j=1}^{k-1} Ad(g_j \ldots g_{k-1} g_k g_{k+1} g_k^{-1})^{-1}(X_j) + Ad(g_k g_{k+1} g_k^{-1})^{-1}(X_k + Ad(g_k)(X_{k+1}) - Ad(g_k g_{k+1} g_k^{-1})(X_k)) \right) \cdot (Y_k - \text{Symm}) \right] = \]

\[ = \frac{1}{2} \left[ \left( \sum_{j=1}^{k-1} Ad(g_j \ldots g_k)^{-1}(X_j) + Ad(g_k)^{-1}(X_k + Ad(g_k)(X_{k+1}) - Ad(g_k g_{k+1} g_k^{-1})(X_k)) \right) \cdot Ad(g_{k+1} g_k^{-1})(Y_k) - \text{Symm} \right] = \]

\[ = \frac{1}{2} \left[ \left( \sum_{j=1}^k Ad(g_j \ldots g_k)^{-1}(X_j) + X_{k+1} - Ad(g_{k+1} g_k^{-1})(X_k) \right) \cdot Ad(g_{k+1} g_k^{-1})(Y_k) - \text{Symm} \right]. \]

(4.13)
The next to the last equality follows from the invariance of the inner product \( \cdot \) with respect to \( \text{Ad}(g_{k+1} g_k^{-1}) \).

It follows from (4.10) and (4.12) that

\[
A := \left( \sigma_k \omega_{[z_1, \ldots, z_{k-1}, z_k]} \right)(X, Y) - \omega_{[z_1, \ldots, z_{k-1}, z_k]}(X, Y) = \\
= \frac{1}{2} \left[ \left( \sum_{j=1}^{k-1} \text{Ad}(g_{j+1} g_k^{-1})(X_j) \right) \cdot \left( \text{Ad}(g_k)(Y_{k+1}) - \text{Ad}(g_{k+1} g_k^{-1})(Y_k) \right) - \text{Symm} \right] = \\
= \frac{1}{2} \left[ \left( \sum_{j=1}^{k-1} \text{Ad}(g_{j+1} g_k^{-1})(X_j) \right) \cdot \left( Y_{k+1} - \text{Ad}(g_{k+1} g_k^{-1})(Y_k) \right) - \text{Symm} \right].
\]

The last equality holds by the invariance of \( \cdot \) with respect to \( \text{Ad}(g_k)^{-1} \).

Similarly, it follows from (4.11) and (4.13) that

\[
B := \left( \sigma_k \omega_{[z_1, \ldots, z_k, z_{k+1}]} \right)(X, Y) - \omega_{[z_1, \ldots, z_k, z_{k+1}]}(X, Y) = \\
= \frac{1}{2} \left[ \left( \sum_{j=1}^{k} \text{Ad}(g_{j+1} g_k^{-1})(X_j) \right) \cdot \left( \text{Ad}(g_k)(Y_{k+1}) - Y_{k+1} \right) + \\
+ \left( X_{k+1} - \text{Ad}(g_{k+1} g_k^{-1})(X_k) \right) \cdot \left( \text{Ad}(g_{k+1} g_k^{-1})(Y_k) \right) - \text{Symm} \right].
\]

Consequently,

\[
A + B = \frac{1}{2} \left[ \left( \text{Ad}(g_k)^{-1}(X_k) \right) \cdot \left( \text{Ad}(g_{k+1} g_k^{-1})(Y_k) - Y_{k+1} \right) + \\
+ \left( X_{k+1} - \text{Ad}(g_{k+1} g_k^{-1})(X_k) \right) \cdot \left( \text{Ad}(g_{k+1} g_k^{-1})(Y_k) \right) - \text{Symm} \right] = \\
= \frac{1}{2} \left[ \left( \text{Ad}(g_k)^{-1}(X_k) \right) \cdot \left( \text{Ad}(g_{k+1} g_k^{-1})(Y_k) \right) - \left( Y_{k+1} - \text{Ad}(g_{k+1} g_k^{-1})(Y_k) \right) \right] - \\
- \left( \text{Ad}(g_k)^{-1}(Y_k) \right) \cdot \left( \text{Ad}(g_{k+1} g_k^{-1})(X_k) \right) - \\
- \left( \text{Ad}(g_k)^{-1}(X_k) \right) \cdot \text{Y}_{k+1} - \left( \text{Ad}(g_k)^{-1}(Y_k) \right) \cdot X_{k+1} + \\
+ \left( X_{k+1} - \text{Ad}(g_{k+1} g_k^{-1})(X_k) \right) - \text{Y}_{k+1} - \left( \text{Ad}(g_{k+1} g_k^{-1})(Y_k) \right) \right] - \\
- X_k \cdot Y_k + Y_k \cdot X_k.
\]
Rearranging the terms and using the invariance of the inner product one gets

\[ A + B = \frac{1}{2} \left[ \left( (\text{Ad}(g_{k+1}^{-1}g_k^{-1})(X_k)) \cdot (\text{Ad}(g_k^{-1})(Y_k)) - \right. \right. \\
\left. \left. - (\text{Ad}(g_k)^{-1}(Y_k)) \cdot (\text{Ad}(g_{k+1}g_k^{-1})(X_k)) \right) - \right. \right. \\
\left. \left. \left( (\text{Ad}(g_k)^{-1}(X_k)) \cdot Y_{k+1} + (\text{Ad}(g_{k+1}^{-1})(Y_{k+1})) \cdot (\text{Ad}(g_k^{-1})(X_k)) \right) + \right. \right. \\
\left. \left. + \left( (\text{Ad}(g_{k+1}^{-1})(X_{k+1})) \cdot (\text{Ad}(g_k^{-1})(Y_k)) + (\text{Ad}(g_k)^{-1}(Y_k)) \cdot X_{k+1} \right) \right] = \right. \right. \\
\left. \left. \frac{1}{2} \left[ \left( (\text{Ad}(g_{k+1}^{-1}g_k^{-1})(X_k)) - (\text{Ad}(g_{k+1}g_k^{-1})(X_k)) \right) \cdot (\text{Ad}(g_k^{-1})(Y_k)) - \right. \right. \\
\left. \left. - \left( Y_{k+1} + Ad(g_{k+1}^{-1})(Y_{k+1}) \right) \cdot (\text{Ad}(g_k^{-1})(X_k)) + \right. \right. \\
\left. \left. + \left( X_{k+1} + Ad(g_{k+1}^{-1})(X_{k+1}) \right) \cdot (\text{Ad}(g_k^{-1})(Y_k)) \right] \right. \right. \\
\left. \left. \right) . \right) \\
\right) \right) \\
(4.16) \]

Since the conjugacy class \( C \) satisfies the condition (3.8) we have \( g_{k+1}^2 = d \) with \( d \) belonging to the center \( Z(G) \) of \( G \). Hence \( g_{k+1}^{-1} = d^{-1}g_{k+1} \) and \( \text{Ad}(g_{k+1}^{-1}) = \text{Ad}(d^{-1}g_{k+1}) = \text{Ad}(g_{k+1}) \). Therefore

\[ \text{Ad}(g_{k+1}^{-1}g_k^{-1})(X_k) = \text{Ad}(g_{k+1}g_k^{-1})(X_k) \] (4.17)

and the first term in the last expression of (4.16) vanishes.

Moreover, if \( X_jg_j \) is a tangent vector to the conjugacy class \( C \) at the point \( g_j \) then there exists \( \bar{X}_j \in \mathfrak{g} \) such that \( X_jg_j = \bar{X}_jg_j - g_j\bar{X}_j = (\bar{X}_j - \text{Ad}(g_j)(\bar{X}_j))g_j \). Therefore \( X_j = \bar{X}_j - \text{Ad}(g_j)(\bar{X}_j) \). Applying this observation with \( j = k + 1 \) we get

\[ X_{k+1} + Ad(g_{k+1}^{-1})(X_{k+1}) = \bar{X}_{k+1} - Ad(g_{k+1})(\bar{X}_{k+1}) + \]
\[ + Ad(g_{k+1}^{-1})(\bar{X}_{k+1} - Ad(g_{k+1})(\bar{X}_{k+1})) = \]
\[ = -Ad(g_{k+1})(\bar{X}_{k+1}) + Ad(g_{k+1}^{-1})(\bar{X}_{k+1}) = \]
\[ = -Ad(g_{k+1}^{-1})(\bar{X}_{k+1}) + Ad(g_{k+1}^{-1})(\bar{X}_{k+1}) = \]
\[ = 0 \] (4.18)

and, similarly,

\[ Y_{k+1} + Ad(g_{k+1}^{-1})(Y_{k+1}) = 0 . \] (4.19)

Hence the second and the third term in the last expression of (4.16) vanish.

The equality (4.16) together with (4.17), (4.18) and (4.19) gives us now

\[ (\sigma_k^r(\omega[z_1\ldots z_{k-1}z_k] + \omega[z_1\ldots z_kz_{k+1}]))(X, Y) - \]
\[ - (\omega[z_1\ldots z_{k-1}z_k] + \omega[z_1\ldots z_kz_{k+1}])\right) \]
\[ (X, Y) = \]
\[ = A + B = \]
\[ = 0 \]
for all vectors $X$ and $Y$ tangent to $\prod C$ at a point $g = (g_1, \ldots, g_m)$. Therefore
\[
\sigma_k^* \omega_{[z_1, \ldots, z_{k-1}, z_k]} + \sigma_k^* \omega_{[z_1, \ldots, z_{k+1}]} = \omega_{[z_1, \ldots, z_{k-1}, z_k]} + \omega_{[z_1, \ldots, z_{k+1}]}.
\]
Together with the identities (4.8) that proves
\[
\sigma_k^* \omega_c = \sigma_k^* \left( - \sum_{j=1}^{m-1} \omega_{[z_1, \ldots, z_{j+1}]} \right) = - \sum_{j=1}^{m-1} \omega_{[z_1, \ldots, z_{j+1}]} = \omega_c
\]
for all elementary braids $\sigma_k$, $k = 1, \ldots, m - 1$. Therefore
\[
\sigma^* \omega_c = \omega_c
\]
for all braids $\sigma \in B_m$. According to (4.6) that concludes the proof of Theorem 4.1.

5. The Lagrangian submanifolds

We continue with the notation of Section 3.

We shall now consider the case when $m = 2n$.

Let $C_1, \ldots, C_n$ be conjugacy classes in the compact Lie group $G$. For $i = 1, \ldots, n$ define conjugacy classes $C_{2n-i+1}$ in $G$ by
\[
C_{2n-i+1} = C_i^{-1} = \{ g \in G \mid g^{-1} \in C_i \}
\]
and let $C = (C_1, \ldots, C_n, C_{n+1}, \ldots, C_{2n})$. Consider a submanifold $\Lambda$ of $\text{Hom}(F_{2n}, G)_C = C_1 \times \ldots \times C_{2n}$,
\[
\Lambda = \{ (p_1, \ldots, p_{2n}) \in C_1 \times \ldots \times C_{2n} \mid p_{2n-i+1} = p_i^{-1}, \ i = 1, \ldots, n \}.
\]
Observe that the projection $C_1 \times \ldots \times C_{2n} \to C_1 \times \ldots \times C_n$ onto the first $n$ factors gives a diffeomorphism between $\Lambda$ and the product $C_1 \times \ldots \times C_n$ with the inverse mapping given by $(p_1, \ldots, p_{2n}) \mapsto (p_1, \ldots, p_n, p_n^{-1}, \ldots, p_1^{-1})$.

Recall the mapping $r : C_1 \times \ldots \times C_{2n} \to G$, $r(g_1, \ldots, g_{2n}) = g_1 \cdot \ldots \cdot g_{2n}$. Note that $r$ maps all points of $\Lambda$ onto the identity $I \in G$. Hence $\Lambda \subset K = r^{-1}(I) \subset \mathcal{M}$. According to Theorem 3.1, $\mathcal{M}$ equipped with the 2-form $\omega_c$ is a symplectic manifold.

We assume for the rest of this section that all the conjugacy classes $C_1, \ldots, C_n$ satisfy the condition (3.8).

Proposition 5.1. $\Lambda$ is a Lagrangian submanifold of $\mathcal{M}$.

Proof. Let us consider a point $x = (p_1, \ldots, p_n, p_n^{-1}, \ldots, p_1^{-1}) \in \Lambda$. Let $i$ and $j$ be a pair of integers such that $1 \leq i, j \leq n$ and let $X_i, Y_j$ be elements of the Lie algebra $\mathfrak{g}$ such that $X_i : p_i \in T_{p_i}G$ and $Y_j : p_j \in T_{p_j}G$ are vectors tangent to $C_i$ at $p_i$ and to $C_j$ at $p_j$ respectively. Then $U = \oplus U_k$, $V = \oplus V_k \in T_x(C_1 \times \ldots \times C_{2n})$ given by
\[
U_k = \begin{cases} X_i : p_i & \text{if } k = i, \\ -p_i^{-1} \cdot X_i = (-Ad(p_i^{-1})(X_i)) \cdot p_i^{-1} & \text{if } k = 2n - i + 1, \\ 0 & \text{otherwise} \end{cases}
\]
and

$$V_k = \begin{cases} Y_j \cdot p_j & \text{if } k = j, \\ -p_j^{-1} \cdot Y_j = (-\text{Ad}(p_j^{-1})(Y_j)) \cdot p_j^{-1} & \text{if } k = 2n - j + 1, \\ 0 & \text{otherwise} \end{cases} \quad (5.2)$$

are tangent to the submanifold $\Lambda$ of $C_1 \times \ldots \times C_{2n}$ at the point $x$ (and the whole tangent space $T_x(\Lambda)$ is spanned by such vectors).

We shall now show that $\omega_C(U, V) = 0$. Since the conjugacy classes $C_1, \ldots, C_n$ (and, hence, their inverses $C_{n+1}, \ldots, C_{2n}$) satisfy the condition (3.8) and since $\Lambda \subset K$, we have $\omega_C(U, V) = \omega_c(U, V)$ (see Corollary 3.4).

Let us recall that $\omega_c = -\sum_{k=1}^{2n-1} \omega[z_1, \ldots, z_k[z_{k+1}]$.

We shall consider two cases.

(i) Let us suppose that $i = j$. Then, by the definition of $\omega[z_1, \ldots, z_k[z_{k+1}]$, we have $\omega[z_1, \ldots, z_k[z_{k+1}](U, V) = 0$ for all $k \neq 2n - i$. For $k = 2n - i$ we get

$$\omega[z_1, \ldots, z_{2n-i}[z_{2n-i+1}](U, V) = \frac{1}{2} \left\{ \omega(df_{z_1, \ldots, z_{2n-i}}(U)) \cdot \overline{\omega}(df_{z_{2n-i+1}}(V)) - \omega(df_{z_1, \ldots, z_{2n-i}}(V)) \cdot \overline{\omega}(df_{z_{2n-i+1}}(U)) \right\} =$$

$$= \frac{1}{2} \left\{ \omega(p_{1 \ldots p_{i-1} X_i p_{1 \ldots p_{2n-i}}}) \cdot \overline{\omega}(-\text{Ad}(p_i^{-1})(Y_i) \cdot p_i^{-1}) - \omega(p_{1 \ldots p_{i-1} Y_i p_{1 \ldots p_{2n-i}}}) \cdot \overline{\omega}(\text{Ad}(p_i^{-1})(X_i)) \cdot p_i^{-1} \right\} =$$

$$= \frac{1}{2} \left\{ (\text{Ad}(p_{1 \ldots p_{2n-i}}^{-1})(X_i)) \cdot (-\text{Ad}(p_i^{-1})(Y_i)) - (\text{Ad}(p_{1 \ldots p_{2n-i}}^{-1})(Y_i)) \cdot (-\text{Ad}(p_i^{-1})(X_i)) \right\} =$$

$$= \frac{1}{2} \left\{ (\text{Ad}(p_{1 \ldots p_{2n-i} \ldots p_i^{-1}}^{-1})(X_i)) \cdot (-Y_i) - (\text{Ad}(p_{1 \ldots p_{2n-i} \ldots p_i^{-1}}^{-1})(Y_i)) \cdot (-X_i)) \right\}$$

Since for every point $x = (p_1, \ldots, p_n, p_{n-1}, \ldots, p_1^{-1})$ in $\Lambda$ one has $p_{1 \ldots p_{2n-i} \ldots p_i^{-1}} = I$, we get

$$\omega[z_1, \ldots, z_{2n-i}[z_{2n-i+1}](U, V) = \frac{1}{2} \left\{ X_i \cdot (-Y_i) - Y_i \cdot (-X_i) \right\} = 0. \quad (5.3)$$

Thus $\omega_C(U, V) = \omega_c(U, V) = 0$.

(ii) Let us now suppose that $i < j$. Observe first of all that $\omega[z_1, \ldots, z_k[z_{k+1}](U, V) = 0$ for all $k \neq j - 1, 2n - j$ and $2n - i$. That follows from the definition of $\omega[z_1, \ldots, z_k[z_{k+1}]$. Then we have

$$\omega[z_1, \ldots, z_{j-1}[z_j](U, V) = \frac{1}{2} \left\{ \omega(df_{z_1, \ldots, z_{j-1}}(U)) \cdot \overline{\omega}(df_{z_j}(V)) - \omega(df_{z_1, \ldots, z_{j-1}}(V)) \cdot \overline{\omega}(df_{z_j}(U)) \right\} =$$

$$= \frac{1}{2} \left\{ \omega(p_{1 \ldots p_{j-1} X_i p_{1 \ldots p_{j-1}}}) \cdot \overline{\omega}(Y_j \cdot p_j) \right\} =$$

$$= \frac{1}{2} \left\{ (\text{Ad}(p_{1 \ldots p_{j-1}}^{-1})(X_i)) \cdot Y_j \right\}, \quad (5.4)$$
where the second equality follows from $df_z(U) = 0$. Similarly
\[
\omega[\{z_1\ldots z_{2n-j} \mid z_{2n-j+1}\}](U, V) = \frac{1}{2} \left\{ \omega(df_{z_1\ldots z_{2n-j}}(U)) \cdot \overline{\omega(df_{z_{2n-j+1}}(V))} - \omega(df_{z_1\ldots z_{2n-j}}(V)) \cdot \overline{\omega(df_{z_{2n-j+1}}(U))} \right\} = \\
= \frac{1}{2} \left\{ \omega(p_1\ldots p_{i-1} X_i p_i \ldots p_{2n-j}) \cdot \overline{(-p_j^{-1} \cdot Y_j)} \right\} = \\
= \frac{1}{2} \left\{ \omega(p_1\ldots p_{i-1} X_i p_i \ldots p_{2n-j}) \cdot \overline{(-Ad(p_j^{-1})(Y_j) \cdot p_j^{-1})} \right\} = \\
= \frac{1}{2} \left\{ Ad((p_i \ldots p_{2n-j}^{-1})(X_i)) \cdot (-Ad(p_j^{-1})(Y_j)) \right\} = \\
= -\frac{1}{2} \left\{ (Ad((p_i \ldots p_{2n-j}^{-1})(X_i)) \cdot Y_j \right\}.
\]
(5.5)

Now, for every point $x = (p_1, \ldots, p_n, p_n^{-1}, \ldots, p_1^{-1}) \in \Lambda$, one has $p_i \ldots p_{2n-j}^{-1} = p_i \ldots p_{j-1}$, provided $i < j$. Hence
\[
\omega[\{z_1\ldots z_{2n-i} \mid z_{2n-i+1}\}](U, V) = -\frac{1}{2} \left\{ (Ad((p_i \ldots p_{j-1}^{-1})(X_i)) \cdot Y_j \right\}.
\]
(5.6)

Finally, observe that
\[
df_{z_1\ldots z_{2n-i}}(V) = p_1 \ldots p_{j-1} Y_j p_j \ldots p_{2n-i} - p_1 \ldots p_{2n-j}^{-1} Y_j p_{2n-j+2} \ldots p_{2n-i} = \\
p_1 \ldots p_{j-1} (Y_j p_j \ldots p_{2n-j+1} - p_j \cdots p_{2n-j+1} Y_j) p_{2n-j+2} \ldots p_{2n-i} = \\
= 0,
\]
because for all points $x = (p_1, \ldots, p_n, p_n^{-1}, \ldots, p_1^{-1}) \in \Lambda$ one has $p_j \cdots p_{2n-j+1} = I \in G$. Since, according to (5.2), we also have $df_{z_{2n-i+1}}(V) = 0$, it follows that
\[
\omega[\{z_1\ldots z_{2n-i} \mid z_{2n-i+1}\}](U, V) = \frac{1}{2} \left\{ \omega(df_{z_1\ldots z_{2n-i}}(U)) \cdot \overline{\omega(df_{z_{2n-i+1}}(V))} - \omega(df_{z_1\ldots z_{2n-i}}(V)) \cdot \overline{\omega(df_{z_{2n-i+1}}(U))} \right\} = \\
= 0.
\]
(5.7)

Therefore,
\[
\omega_C(U, V) = -\sum_{k=1}^{2n-1} \omega[\{z_1\ldots z_k \mid z_{k+1}\}](U, V) = \\
= -\omega[\{z_1\ldots z_{j-1} \mid z_j\}](U, V) - \omega[\{z_1\ldots z_{2n-j} \mid z_{2n-j+1}\}](U, V) - \\
= 0.
\]

That proves Proposition 5.1.

Let now $C$ be a conjugacy class in $G$ satisfying the condition (3.8) and such that
\[
C = C^{-1} = \{ g \in G \mid g^{-1} \in C \}.
\]
(5.8)

(An example of such a class is the conjugacy class $S$ in $G = SU(2)$ consisting of trace-free matrices considered in Section 2.)

Let $C_1 = \ldots = C_n = C$ and let $\mathcal{C} = (C, \ldots, C)$ ($2n$ copies of $C$). Under these assumptions the braid group $\mathcal{B}_{2n}$ on $2n$ strands acts on $\mathcal{M}$ and on $K = r^{-1}(I)$ (see Section 4). According to Theorem 4.1, the action of $\mathcal{B}_{2n}$ on $\mathcal{M}$ preserves the symplectic structure. The following corollary is then an immediate consequence of Proposition 5.1.
Corollary 5.2. For every \( \tau \in B_{2n} \), \( \tau(\Lambda) \) is a Lagrangian submanifold of \( \mathcal{M} \).

Observe that both \( \Lambda \) and \( \tau(\Lambda) \) are contained in \( K = r^{-1}(I) \).

Let us now consider the embedding of the braid group \( B_n \) on \( n \) strands into \( B_{2n} \) extending a braid \( \sigma \in B_n \) to a braid \( \bar{\sigma} \in B_{2n} \) which is trivial on the first \( n \) strands. Let us denote by \( \Gamma_{\sigma} \) the submanifold \( \bar{\sigma}(\Lambda) \).

Corollary 5.3. For every \( \sigma \in B_n \), \( \Gamma_{\sigma} = \bar{\sigma}(\Lambda) \) is a Lagrangian submanifold of \( \mathcal{M} \).

Again, for every \( \sigma \in B_n \), the manifold \( \Gamma_{\sigma} \) is contained in the subspace \( K = r^{-1}(I) \) of \( \mathcal{M} \).

The manifolds \( \Gamma_{\sigma} \) will play an important rôle in later parts of the paper.

6. The almost complex structure and the first Chern class

We shall now apply the constructions of Section 3 to the situation considered in Section 2.

We assume that the Lie group \( G \) is equal to \( SU(2) \) and that the conjugacy classes \( C_j \) are all equal to \( \mathcal{S} = \{ A \in SU(2) \mid \text{trace}(A) = 0 \} \). We consider the constructions of Section 3 in that special case.

Let \( F \) be the free group on \( 2n \) generators \( z_1, \ldots, z_{2n} \), \( n \geq 1 \), and let \( \mathcal{C} = (C_1, \ldots, C_{2n}) = (\mathcal{S}, \ldots, \mathcal{S})(2n \text{ elements}) \).

Consider the manifold \( \text{Hom}(F, G)_\mathcal{C} \) of all group homomorphisms \( \varphi : F \to G \) such that \( \varphi(z_j) \in \mathcal{S}, 1 \leq j \leq m \), and the map \( r : \text{Hom}(F, G)_\mathcal{C} \to G, r(\varphi) = \varphi(z_1) \cdot \ldots \cdot \varphi(z_m) \). In Section 2 the manifold \( \text{Hom}(F, G)_\mathcal{C} \) was denoted by \( P_{2n} \) and the map \( r \) by \( p_{2n} \).

Choose the open subset \( \tilde{\mathcal{O}} \) in \( \mathfrak{g} = su(2) \) to be the connected component of \( \exp^{-1}(SU(2) - \{-I\}) \) containing 0 and let \( \mathfrak{H} = r^{-1}(\exp(\tilde{\mathcal{O}})) = r^{-1}(SU(2) - \{-I\}) \).

The subspace \( \mathfrak{H} \) is an open neighbourhood of \( K = r^{-1}(I) \) in \( \text{Hom}(F, G)_\mathcal{C} \). Again, recall that in Section 2 the subset \( \mathfrak{H} \) was denoted by \( E \) and the subset \( K \) by \( K_{2n} \).

According to Theorem 3.1 there exists an open neighbourhood \( \mathcal{M} \) of \( K = K_{2n} \) in \( \mathfrak{H} = E \) such that the 2-form \( \omega_\mathfrak{C} \) is symplectic on \( \mathcal{M} \). We can assume that \( \mathcal{M} \) has been chosen so that it is homotopy equivalent to \( K_{2n} \). From now on we shall consider \( \mathcal{M} \) as a symplectic manifold equipped with the form \( \omega_\mathfrak{C} \).

Let \( T\mathcal{M} \) be the tangent bundle of \( \mathcal{M} \). We choose a complex structure on \( T\mathcal{M} \) compatible with \( \omega_\mathfrak{C} \). Let \( c_1(\mathcal{M}) = c_1(T\mathcal{M}) \) be the first Chern class of \( T\mathcal{M} \), \( c_1(\mathcal{M}) \in H^2(\mathcal{M}; \mathbb{Z}) \).

The main aim of this Section is to determine how \( c_1(\mathcal{M}) \) evaluates on some elements of \( \pi_2(K) \). As a consequence, we shall prove in the next Section that the symplectic manifold \( \mathcal{M} \) is monotone.

For every integer \( k \) such that \( 1 \leq k \leq 2n - 1 \) and for \( \epsilon = \pm 1 \) let us consider elements of \( \pi_2(K) \) represented by the maps \( \gamma_{k, \epsilon} : S^2 \to K \) given by

\[
\gamma_{k, \epsilon}(A) = (J, J, \ldots, J, A, \epsilon A, J, \ldots, (-1)^n \epsilon J), \quad \text{for } A \in S^2, \quad (6.1)
\]
where, as in Section 2, $S^2$ has been identified with $\mathcal{S}$, $J = \left( \begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix} \right)$ and the first factor of $A$ on the RHS is in the $k$-th place. In the case when $k = 2n - 1$ the sign $(-1)^n \epsilon$ is placed at the first factor of $J$. The maps $\gamma_{k,\epsilon}$ are embeddings of $S^2$ into $K$ and, hence, into $M$ and $P_{2n}$.

Let $[\gamma_{k,\epsilon}] \in H_2(\mathcal{M}; \mathbb{Z})$ be the homology classes represented by the corresponding mappings.

**Theorem 6.1.** For all integers $k$ such that $1 \leq k \leq 2n - 1$ and $\epsilon = \pm 1$, the evaluation of the first Chern class $c_1(\mathcal{M})$ on the homology classes $[\gamma_{k,\epsilon}]$ is equal to $0$,

$$\langle c_1(\mathcal{M}) | [\gamma_{k,\epsilon}] \rangle = 0.$$

Proof of Theorem 6.1 is deferred to the Appendix B. (See Theorem B.1.)

We shall now, for every integer $n \geq 2$, define a mapping $f_n : S^2 \to K_{2n}$.

We identify the 2-dimensional sphere $S^2$ with a manifold $\tilde{S}$ which is a union of two hemispheres $U_1$ and $U_2$ and of the cylinder $U_3$ joining the boundaries of the hemispheres as in Figure 6.1.

![Diagram](image)

**Fig. 6.1**

We look upon the cylinder $U_3$ as the product $[0, \pi] \times S^1$ of the interval $[0, \pi]$ and of the circle $S^1 = \mathbb{R}/2\pi \mathbb{Z}$. The identification of $S^2$ with $\tilde{S}$ is such that the orientation of $S^2$ corresponds to the product orientation on the cylinder $U_3 = [0, \pi] \times S^1 = [0, \pi] \times \mathbb{R}/2\pi \mathbb{Z}$.

Moreover, we identify the hemispheres $U_1$ and $U_2$, each one separately, with subspaces of the conjugacy class $\mathcal{S}$. The hemisphere $U_1$ is identified with the subset of $\mathcal{S}$ containing matrices $A = \left( \begin{smallmatrix} i & -\epsilon \\ -\epsilon & -i \end{smallmatrix} \right)$ with $\text{Im}(z) \geq 0$ and the hemisphere $U_2$ with the subset containing matrices $A$ with $\text{Im}(z) \leq 0$.

The points $(0, \theta_2)$ in $U_3$ are identified with matrices $\left( \begin{smallmatrix} i \cos \theta_2 & \sin \theta_2 \\ -\sin \theta_2 & -i \cos \theta_2 \end{smallmatrix} \right)$ in $U_1$. The points $(\pi, \theta_2)$ in $U_3$ are identified with matrices $\left( \begin{smallmatrix} i \cos \theta_2 & \sin \theta_2 \\ -\sin \theta_2 & -i \cos \theta_2 \end{smallmatrix} \right)$ in $U_2$.

We define the mapping $f_n : \tilde{S} \to K_{2n}$ by

$$f_n(A) = \begin{cases} (J, J, A, -J, J, ..., -J, J) & \text{if } A \in U_1, \\ (-J, J, A, -A, -J, J, ..., -J, J) & \text{if } A \in U_2. \end{cases} \quad (6.2)$$

To define the restriction of $f_n$ to the cylinder $U_3 = [0, \pi] \times S^1 = [0, \pi] \times \mathbb{R}/2\pi \mathbb{Z}$, let us denote by $A_{\theta}$ the matrix $\left( \begin{smallmatrix} i \cos \theta & \sin \theta \\ -\sin \theta & -i \cos \theta \end{smallmatrix} \right) \in \mathcal{S}$. Then, for $x = [\theta_1, \theta_2] \in$
$U_3 = [0, \pi] \times \mathbb{R}/2\pi\mathbb{Z}$, we set

$$f_n(x) = f_n([\theta_1, \theta_2]) = (A_{\theta_1}, J, A_{\theta_2}, A_{\theta_1+\theta_2}, -J, J, \ldots, -J, J). \quad (6.3)$$

As defined above, the mapping $f_n$ apriori takes its values in $P_{2n}$. Let $p_j : P_{2n} \to S$ be the projection on the $j$-th factor, $j = 1, \ldots, 2n$, and let $f_{n,j} = p_j \circ f_n : \tilde{S} \to S$ be the coordinate maps of $f_n$. By the definition

$$f_{n,j}(x) = (-1)^j J \quad \text{for all } x \in \tilde{S} \text{ and } 5 \leq j \leq 2n.$$

Let $r : P_{2n} \to SU(2)$ be the mapping $r(A_1, \ldots, A_{2n}) = A_1 \cdot \ldots \cdot A_{2n}$. One can check directly

$$f_{n,1}(x) \cdot f_{n,2}(x) \cdot f_{n,3}(x) \cdot f_{n,4}(x) = I \quad \text{for all } x \in \tilde{S}.$$

It follows that $r(f(x)) = f_{n,1}(x) \cdot f_{n,2}(x) \cdot \ldots \cdot f_{n,2n}(x) = f_{n,1}(x) \cdot f_{n,2}(x) \cdot f_{n,3}(x) \cdot f_{n,4}(x) \cdot (-JJ)^{n-2} = I \cdot I^{n-2} = I$ for all $x \in \tilde{S}$. Hence the image of $f_n$ lies in $K_{2n} = r^{-1}(I) \subset P_{2n}$ and we get the mapping $f_n : \tilde{S} \to K_{2n}$.

The mapping $f_n$ is smooth away from the boundaries of the subsets $U_i$, $i = 1, 2, 3$. One can see directly that $f_n$ can be smoothed in small collars of $\partial U_i$ without changing the image $f_n(\tilde{S})$. Whenever necessary we shall assume, without further mentioning it, that such a smoothing has been done.

We choose an orientation of $\tilde{S}$ so that it coincides on $U_3 = [0, \pi] \times \mathbb{R}/2\pi\mathbb{Z}$ with the product of the standard orientations of $[0, \pi]$ and of $\mathbb{R}/2\pi\mathbb{Z}$. Let us then choose an orientation of $S$ so that under the identification of $U_1$ and $U_2$ with subsets of $S$ the orientations of $S$ and of $\tilde{S}$ coincide.

The degrees of the coordinate maps $f_{n,j} = p_j \circ f_n : \tilde{S} \to S$ are

$$\deg(f_{n,j}) = \deg(p_j \circ f_n) = \begin{cases} 0 & j \neq 3, \ 1 \leq j \leq 2n, \\ 1 & j = 3. \end{cases}$$

(6.4)

Indeed, the mappings $f_{n,j} : \tilde{S} \to S$ are not surjective for $j \neq 3$, while $\deg(f_{n,3}) = 1$ follows immediately from the construction.

Let $f_n : H_2(\tilde{S}, \mathbb{Z}) \to H_2(K_{2n}, \mathbb{Z})$ be the homomorphism induced by $f_n$ on the second homology groups and let $[\tilde{S}] \in H_2(\tilde{S}, \mathbb{Z})$ be given by the orientation of $\tilde{S}$. Consider the element $f_n \cdot [\tilde{S}] \in H_2(K_{2n}, \mathbb{Z})$.

**Theorem 6.2.** For every $n \geq 2$

$$\langle c_1(\mathcal{M}) \mid f_n \cdot [\tilde{S}] \rangle = -2.$$

Proof of Theorem 6.2 is deferred to the Appendix C. (See Theorem C.1.)

We shall now consider the subgroup of $\pi_2(K_{2n})$ generated by the elements $[f_n]$ and $[\gamma_{k,\epsilon}]$ with $1 \leq k \leq 2n - 1$, $\epsilon = \pm 1$. We shall show that this subgroup is of index 2 in $\pi_2(K_{2n})$.

Let $j : K_{2n} \to P_{2n}$ be the inclusion. The product structure of $P_{2n} = \bigsqcup_{i=1}^{2n} S$ gives an identification $\pi_2(P_{2n}) = \bigsqcup_{i=1}^{2n} \mathbb{Z}$. It follows directly from the definition of the mappings $\gamma_{k,\epsilon}$ that under the induced homomorphism $j_* : \pi_2(K_{2n}) \to \pi_2(P_{2n})$
the homotopy class \([\gamma_k, \epsilon] \in \pi_2(K_{2n})\) is mapped to \(j_*(\gamma_k, \epsilon) = \bigoplus_{i=1}^{2n}(j_*([\gamma_k, \epsilon]))_i \in \pi_2(P_{2n})\) with

\[
(j_*(\gamma_k, \epsilon))_i = \begin{cases} 
1 & \text{if } i = k, \\
\epsilon & \text{if } i = k+1, \\
0 & \text{otherwise.}
\end{cases} \tag{6.5}
\]

It follows from (6.5) that the homotopy class \([\gamma_k] \in \pi_2(K_{2n})\) is mapped to

\[
(j_*(\gamma_k)) = \bigoplus_{i=1}^{2n}(j_*([\gamma_k]))_i \in \pi_2(P_{2n})
\]

with

\[
(j_*(\gamma_k))_i = \begin{cases} 
1 & \text{if } i = 3, \\
0 & \text{otherwise.}
\end{cases} \tag{6.6}
\]

Observe that if \(h_n : K_{2n} \to K_{2n+2}\) is the mapping \(h_n(A_1, ..., A_{2n}, -J, J)\) as in (2.1) then

\[
h_n \circ f_n = f_{n+1}\tag{6.7}
\]

for \(n \geq 2\).

Let us consider the element

\[
\alpha_n = 2[f_n] - [\gamma_{3,1}] - [\gamma_{3,-1}] \in \pi_2(K_{2n}). \tag{6.8}
\]

Because of (6.7) and of the definition of \(\gamma_{k,\epsilon}\) one has

\[
h_{n*}(\alpha_n) = \alpha_{n+1}. \tag{6.9}
\]

It follows from (6.3) and (6.6) that

\[
j_*(\alpha_n) = 0. \tag{6.10}
\]

Let \(\xi_n\) be a generator of \(\text{Ker}(j_* : \pi_2(K_{2n}) \to \pi_2(P_{2n})) \cong \mathbb{Z}\).

**Lemma 6.3.** For \(n \geq 2\),

\[
\alpha_n = k\xi_n \quad \text{with} \quad k \in \{\pm 1, \pm 2\}
\]

in \(\pi_2(K_{2n})\).

**Proof.** According to Lemma 2.10, the homomorphism \(h_{n*} : \pi_2(K_{2n}) \to \pi_2(K_{2n+2})\) maps \(\text{Ker}(j_* : \pi_2(K_{2n}) \to \pi_2(P_{2n}))\) isomorphically onto \(\text{Ker}(j_* : \pi_2(K_{2n+2}) \to \pi_2(P_{2n+2}))\). Thus \(h_{n*}(\xi_n) = \pm \xi_{n+1}\). Since, by (6.3), \(h_{n*}(\alpha_n) = \alpha_{n+1}\), in order to prove the Lemma, it is enough to show that

\[
\alpha_2 = k\xi_2 \quad \text{with} \quad k \in \{\pm 1, \pm 2\} \tag{6.11}
\]

in \(\pi_2(K_4)\).

Let us now consider the quotient projection \(q : K_4 \to K_4/G\). The space \(K_4/G\) is known as the “pillow-case”, Figure 6.2, see [20].

It can be described as the space \([0, \pi] \times S^1/\sim\), where \(S^1 = \mathbb{R}/2\pi\mathbb{Z}\),
with the identifications $(0, \theta_2) \sim (0, -\theta_2)$ and $(\pi, \theta_2) \sim (\pi, -\theta_2)$ for $\theta_2 \in \mathbb{R}/2\pi \mathbb{Z}$.

A point $[\theta_1, \theta_2]$ in $K_4/G = [0, \pi] \times S^1/\sim$ is represented by the point

\[
\begin{pmatrix}
\cos \theta_1 & \sin \theta_1 \\
-\sin \theta_1 & -\cos \theta_1
\end{pmatrix}
\begin{pmatrix}
\cos \theta_2 & \sin \theta_2 \\
-\sin \theta_2 & -\cos \theta_2
\end{pmatrix}
\begin{pmatrix}
\cos(\theta_1 + \theta_2) & \sin(\theta_1 + \theta_2) \\
-\sin(\theta_1 + \theta_2) & -\cos(\theta_1 + \theta_2)
\end{pmatrix}
\]

in $K_4$, (see [20], p.347).

Therefore $K_4/G$ is homeomorphic to a 2-dimensional sphere and we orient it by the product of the natural orientations of $[0, \pi]$ and of $S^1 = \mathbb{R}/2\pi \mathbb{Z}$. The choice of orientation gives us an isomorphism $\pi_2(K_4/G) = \mathbb{Z}$.

Let us consider the induced homomorphism $q_*: \pi_2(K_4) \to \pi_2(K_4/G) = \mathbb{Z}$.

According to Lemma 2.10, $q_*([\gamma_3,1]) = q_*([\gamma_3,-1]) = 0$. Thus $q_*(\alpha_2) = 2q_*([f_2]) = 2[q \circ f_2]$.

Now, the mapping $f_2 : S^2 = \tilde{S} \to K_4$ was devised in such a way that the composition $q \circ f_2 : S^2 \to K_4/G \cong S^2$ has degree 1. Indeed, $\tilde{S} = U_1 \cup U_2 \cup U_3$ and $q \circ f_2$ maps the interior of $U_3$ homeomorphically and preserving the orientations onto the open subset of the “pillow-case” corresponding to $0 < \theta_1 < \pi$, while it maps $U_1$ into the closed subset given by $\theta_1 = 0$ and maps $U_2$ into the closed subset given by $\theta_1 = \pi$.

Therefore $[q \circ f_2] = 1$ in $\pi_2(K_4/G) = \mathbb{Z}$ and

\[q_*(\alpha_2) = 2, \tag{6.12}\]

As $\alpha_2$ belongs to the kernel of $j_*: \pi_2(K_4) \to \pi_2(P_4)$ one has $\alpha_2 = k \xi_2$

for some $k \in \mathbb{Z}$. Applying $q_*$ we get

\[2 = q_*(\alpha_2) = k \cdot q_*(\xi_2)\]

in $\pi_2(K_4/G) = \mathbb{Z}$. Thus $k$ divides 2. That proves Lemma 6.3.

Let $G$ be the subgroup of $\pi_2(K_{2n})$ generated by the homotopy classes $[f_n]$ and $[\gamma_k, \epsilon]$, $k = 1, \ldots, 2n - 1$, $\epsilon = \pm 1$.

**Lemma 6.4.** Let $n \geq 2$. $G$ is a subgroup of index at most 2 in $\pi_2(K_{2n})$. 

Proof. It follows directly from (6.5) and (6.6) that the homomorphism $j_* : \pi_2(K_{2n}) \to \pi_2(P_{2n})$ maps the subgroup $\mathcal{G}$ surjectively onto $\pi_2(P_{2n})$. Moreover, since $\alpha_n = 2[f_n - [\gamma_3,1] - [\gamma_3,1]] \in \mathcal{G}$, it follows from Lemma 5.3 that $\mathcal{G}$ intersects the kernel of $j_*$ in a subgroup of index 1 or 2. Hence $\mathcal{G}$ is a subgroup of index 1 or 2 in $\pi_2(K_{2n})$. \hfill \Box

7. The manifold $\mathcal{M}$ is monotone

The aim of this Section is to prove that the symplectic manifold $\mathcal{M} \subset P_{2n}$ is monotone. Denote $K = K_{2n} \subset \mathcal{M} \subset P_{2n}$.

Let $\omega_C$ be the symplectic 2-form on $\mathcal{M}$. According to Corollary 3.4, the restriction of the 2-form $\omega_C$ to the bundle $\mathcal{T}|_{K_{2n}}$ is equal to the 2-form $\omega_c$, the definition of which we shall now recall.

The 2-form $\omega_c$ is defined on the whole manifold $P_{2n}$. Let $\mathfrak{g}$ be the Lie algebra of the Lie group $G = SU(2)$ equipped with an invariant positive definite inner product •. Denote by $\omega$ the $\mathfrak{g}$-valued, left-invariant 1-form on $G$ which maps each tangent vector to the left-invariant vector field having that value. The corresponding right-invariant form will be denoted by $\bar{\omega}$.

For any differential form $\alpha$ on $G$ denote by $\alpha_j$ the pullback of $\alpha$ to $G \times G$ by the projection to the $j$th factor. Let

$$\Omega = \frac{1}{2} \omega_1 \bullet \bar{\omega}_2 .$$

This is a real-valued 2-form on $G \times G$. According to the convention

$$(\omega_1 \bullet \omega_2)(U,V) = \omega_1(U) \bullet \bar{\omega}_2(V) - \omega_2(U) \bullet \omega_1(V) .$$ (7.1)

For any integer $j$ such that $1 \leq j \leq 2n - 1$, let $E_j : P_{2n} \to G \times G$ be the mapping $E_j(A_1, ..., A_{2n}) = (A_1 \cdot ... \cdot A_j, A_{j+1})$ and let

$$\omega_{1}^{j} \cdot z_{j}^{j+1} = E_j^{*}(\Omega) .$$

The 2-form $\omega_c$ on $P_{2n}$ is defined by

$$\omega_c = - \sum_{j=1}^{2n-1} \omega_{1}^{j} \cdot z_{j}^{j+1} .$$

Let us again consider the mappings $\gamma_{k,\epsilon} : S^2 \to K$, $1 \leq k \leq 2n - 1$, $\epsilon = \pm 1$, defined in (6.1).

Lemma 7.1. For all $1 \leq k \leq 2n - 1$, $\epsilon = \pm 1$, the pull-back of the form $\omega_C$ by the mapping $\gamma_{k,\epsilon}$ vanishes,

$$\gamma_{k,\epsilon}^{*}(\omega_C) = 0 .$$

Proof. The restriction of the symplectic 2-form $\omega_C$ to the bundle $\mathcal{T}|_{K}$ is equal to $\omega_c = - \sum_{j=1}^{2n-1} \omega_{1}^{j} \cdot z_{j}^{j+1}$ (Corollary 3.4). Since the image of $\gamma_{k,\epsilon}$ lies in $K$, we have to show that $\gamma_{k,\epsilon}^{*}(\omega_c) = 0$.

Let us denote $p = A \in \mathcal{S}$ and $(p_1, ..., p_{2n}) = (J, J, ..., J, A, \epsilon A, J, ..., (-1)^{n} \epsilon J)$, $p_j \in \mathcal{S}$, $p = p_k = \epsilon p_{k+1}$. Every tangent vector $v \in T_p\mathcal{S}$ is of the form $v = Xp - pX = \bar{X}p$ for some $X \in \mathfrak{su}(2)$ and $\bar{X} = X - \text{Ad}(p)(X) \in \mathfrak{su}(2)$.
Let $X, Y \in \mathfrak{su}(2)$. We have $\tilde{X}p, \tilde{Y}p \in T_{p}S$. We shall now calculate the value $\omega_{c}(d\gamma_{k,e}(\tilde{X}p), d\gamma_{k,e}(\tilde{Y}p))$ of the 2-form $\gamma_{k,e}^{*}(\omega_{c})$ on the pair $(\tilde{X}p, \tilde{Y}p)$.

Let $\pi_{1}, \pi_{2}: G \times G \to G$ be the projections onto the first and the second factor. Since for $j \neq k$ at least one of the compositions $\pi_{1} \circ E_{j} \circ \gamma_{k,e}$ or $\pi_{2} \circ E_{j} \circ \gamma_{k,e}$ is a constant mapping into a point, it follows from the definition (7.1) of $\Omega$ that

$$\gamma_{k,e}^{*}(\omega_{|z_{1},\ldots,z_{j}+1}) = \gamma_{k,e}^{*}(E_{j}^{*}(\Omega)) = 0 \quad \text{for} \quad j \neq k.$$  

For $j = k$ we have $d(E_{k} \circ \gamma_{k,e})(\tilde{X}p) = (p_{1} \cdots p_{k-1} \cdot \tilde{X}p_{k}, \tilde{X}p_{k+1}) \in T_{E_{k}(\gamma_{k,e}(p))}(G \times G)$. It follows that

$$\omega_{c}(d\gamma_{k,e}(\tilde{X}p), d\gamma_{k,e}(\tilde{Y}p)) = -\omega_{|z_{1},\ldots,z_{k+1}}(d\gamma_{k,e}(\tilde{X}p), d\gamma_{k,e}(\tilde{Y}p)) =$$

$$= -(E_{k}^{*}(\Omega))(d\gamma_{k,e}(\tilde{X}p), d\gamma_{k,e}(\tilde{Y}p)) =$$

$$= -\Omega(d(E_{k} \circ \gamma_{k,e})(\tilde{X}p), d(E_{k} \circ \gamma_{k,e})(\tilde{Y}p)) =$$

$$= -\Omega((p_{1} \cdots p_{k-1} \cdot \tilde{X}p_{k}, \tilde{X}p_{k+1}), (p_{1} \cdots p_{k-1} \cdot \tilde{Y}p_{k}, \tilde{Y}p_{k+1})) =$$

$$= -(p_{1} \cdots p_{k-1} \cdot p_{k} \cdot Ad(p_{k}^{-1})(\tilde{X}), \tilde{X}p_{k+1})$$

$$= \left(\frac{1}{2} \left(\omega(p_{1} \cdots p_{k-1} \cdot p_{k} \cdot Ad(p_{k}^{-1})(\tilde{X})) \cdot \tilde{Y}p_{k+1}\right) - \omega(p_{1} \cdots p_{k-1} \cdot p_{k} \cdot Ad(p_{k}^{-1})(\tilde{Y})) \cdot \tilde{X}p_{k+1}\right) =$$

$$= \left(\frac{1}{2} \left(Ad(p_{k}^{-1})(\tilde{X})\right) \cdot \tilde{Y} - (Ad(p_{k}^{-1})(\tilde{Y})) \cdot \tilde{X}\right) =$$

$$= \frac{1}{2} \left(\tilde{X} \cdot (Ad(p_{k})(\tilde{Y})) - (Ad(p_{k})(\tilde{Y})) \cdot \tilde{X}\right) =$$

$$= 0.$$  

The next to the last equality follows from the invariance of the inner product $\bullet$ and from the fact that $Ad(p_{k}^{-1}) = Ad(-p_{k}) = Ad(p_{k})$.

The equality (7.2) means that $\gamma_{k,e}^{*}(\omega_{c}) = 0$, as claimed. \hfill \Box

Let us denote by $\nu_{k}: P_{2n} \to P_{2n}$ the multiplication by $-I$ in the $k$-th factor, $\nu_{k}(A_{1}, \ldots, A_{k-1}, A_{k}, A_{k+1}, \ldots, A_{2n}) = (A_{1}, \ldots, A_{k-1}, -A_{k}, A_{k+1}, \ldots, A_{2n})$, $1 \leq k \leq 2n$. Let $\nu_{kl} = \nu_{k} \circ \nu_{l}: P_{2n} \to P_{2n}$, $1 \leq k, l \leq 2n$.

**Lemma 7.2.** For any pair of integers $k, l$ such that $1 \leq k, l \leq 2n$,

$$\nu_{kl}^{*}(\omega_{c}) = \omega_{c}.$$  

**Proof.** Let $\eta: G \to G$ be the multiplication by $-I$, $\eta(A) = -A$. Since $-I$ belongs to the center of $G$, we have $\eta^{*}(\omega) = \omega$ and $\eta^{*}(\bar{\omega}) = \bar{\omega}$. Therefore both maps $\eta \times id: G \times G \to G \times G$ and $id \times \eta: G \times G \to G \times G$ satisfy

$$\eta \times id)^{*}(\Omega) = \Omega \quad \text{and} \quad (id \times \eta)^{*}(\Omega) = \Omega.$$  

(7.3)

Now, $E_{j} \circ \nu_{kl} = (\psi_{1} \times \psi_{2}) \circ E_{j}$, where $\psi_{1}, \psi_{2}: G \to G$ are equal to $\eta$ or to $id$. It follows from (7.3) that $\psi_{1} \times \psi_{2})^{*}(\Omega) = \Omega$. Therefore

$$\nu_{kl}^{*}(\omega_{|z_{1}\ldots z_{j+1}}) = \nu_{kl}^{*}E_{j}^{*}(\Omega) = (E_{j} \circ \nu_{kl})^{*}(\Omega) = \big((\psi_{1} \times \psi_{2}) \circ E_{j}\big)^{*}(\Omega) =$$

$$= E_{j}^{*}(\psi_{1} \times \psi_{2})^{*}(\Omega) = E_{j}^{*}(\Omega) =$$

$$= \omega_{|z_{1}\ldots z_{j+1}}.$$
for $1 \leq j \leq 2n - 1$. Consequently

$$\nu_{kl}^j(\omega_c) = \nu_{kl}^j \left( - \sum_{j=1}^{2n-1} \omega_{[z_1 \ldots z_j]} \right) = - \sum_{j=1}^{2n-1} \omega_{[z_1 \ldots z_j]} = \omega_c.$$

\[\Box\]

Observe that $\nu_{kl}^j(K_{2n}) = K_{2n}$ for all pairs $(k, l)$. We denote by $V$ the finite group of diffeomorphisms of $P_{2n}$ generated by $\nu_{kl}$, $1 \leq k, l \leq 2n$. $V$ is isomorphic to $\mathbb{Z}_2^{2n-1}$.

Consider again the mapping $f_n : S^2 \to K_{2n}$ defined in (6.2) - (6.3). Recall that for the purpose of this definition the 2-sphere $S^2$ has been identified with the manifold $\mathcal{S} = U_1 \cup U_2 \cup U_3$ (see Figure 7.1),

Fig. 7.1

where $U_1, U_2, U_3$ are submanifolds (with boundary) of $\mathcal{S}$, intersecting only along boundaries. The identification of $S^2$ with $\mathcal{S}$ maps the standard orientation of $S^2$ to the product orientation of $U_3 = [0, \pi] \times (\mathbb{R}/2\pi\mathbb{Z})$.

Let $[S^2] \in H_2(S^2, \mathbb{Z})$ be the fundamental class given by the standard orientation of $S^2$ and let $[\omega_c] \in H^2(\mathcal{M}, \mathbb{R})$ be the deRham cohomology class of the closed form $\omega_c$. Let $f_n^* : H_2(S^2, \mathbb{R}) \to H_2(\mathcal{M}, \mathbb{R})$ be the homomorphism induced by $f_n : S^2 \to K_{2n} \subset \mathcal{M}$.

The symplectic 2-form $\omega_c$ depends on a choice of the invariant symmetric inner product $\bullet$ on the Lie algebra $\mathfrak{so}(2)$, which we have so far not specified in this paper. From now on we choose $\bullet$ to be given by $X \bullet Y = -\frac{1}{2} \text{Trace}(XY)$ for $X, Y \in \mathfrak{so}(2)$.

**Lemma 7.3.** The evaluation $\langle [\omega_c] \mid f_n^*([S^2]) \rangle = -\pi^2$.

**Proof.** It is clear from (6.2) and (6.3) that the mapping $f_n : \mathcal{S} \to \mathcal{M}$ is piece-wise smooth, being smooth on each of the pieces $U_1, U_2$ and $U_3$ separately. We have then

$$\langle [\omega_c] \mid f_n^*([S^2]) \rangle = \sum_{j=1}^{3} \int_{U_j} (f_n |_{U_j})^* (\omega_c). \quad (7.4)$$

As the image of $f_n$ lies in $K_{2n}$, we have $(f_n |_{U_j})^* (\omega_c) = (f_n |_{U_j})^* (\omega_c) = 0$. Indeed, this can be proven in exactly the same way as Lemma 7.2 (c) and can be shown as follows. We see from (6.2) that $f_n |_{\overline{U}_1} = \nu' \circ \gamma_{3, 1} |_{\overline{U}_1}$ and $f_n |_{\overline{U}_2} = \nu'' \circ \gamma_{3, -1} |_{\overline{U}_2}$ for some diffeomorphisms $\nu', \nu'' \in \mathcal{V}$, where $\mathcal{V}$ is the group of diffeomorphisms of $P_{2n}$ introduced above. According to Lemma 7.2 we have $\nu'(\omega_c) = \nu''(\omega_c) = \omega_c$. Thus $(f_n |_{U_j})^* (\omega_c) = (f_n |_{U_j})^* (\omega_c) = (\gamma_{3, 1} |_{U_j})^* (\omega_c) = (\gamma_{3, 1} |_{U_j})^* (\omega_c) = (\gamma_{3, 1} |_{U_j})^* (\omega_c) = 0$, the
The last equality following from Lemma 7.1. The proof that $(f_n \mid_{U_2})^*(\omega_C) = 0$ is similar.

We shall now calculate $(f_n \mid_{U_3})^*(\omega_C)$. Let $\partial_{\theta_1} = \frac{\partial}{\partial \theta_1}$ and $\partial_{\theta_2} = \frac{\partial}{\partial \theta_2}$ be the coordinate vector fields given by the coordinates $(\theta_1, \theta_2)$ on $U_3 = [0, \pi] \times (\mathbb{R}/2\pi\mathbb{Z})$. Let $B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Recall that $A_\theta = \begin{pmatrix} i \cos \theta & \sin \theta \\ -\sin \theta & -i \cos \theta \end{pmatrix}$ and observe that

$$\frac{dA_\theta}{d\theta} = B \cdot A_\theta.$$ 

Let us denote

$$f_n(\{\theta_1, \theta_2\}) = (p_1, \ldots, p_{2n})$$

with $p_j = p_j(\theta_1, \theta_2) \in \mathcal{S}$, $j = 1, \ldots, 2n$. Thus $p_1 = A_{\theta_1}$, $p_2 = J$, $p_3 = A_{\theta_2}$, $p_4 = A_{\theta_1+\theta_2}$ and $p_j = (-1)^j J$ for $j \geq 5$. We have

$$df_n(\partial_{\theta_1}) = (B \cdot A_{\theta_1}, 0 \cdot J, 0 \cdot A_{\theta_2}, B \cdot A_{\theta_1+\theta_2}, 0 \cdot p_5, \ldots, 0 \cdot p_{2n}) =$$

$$= (B \cdot p_1, 0 \cdot p_2, 0 \cdot p_3, B \cdot p_4, 0 \cdot p_5, \ldots, 0 \cdot p_{2n})$$

and

$$df_n(\partial_{\theta_2}) = (0 \cdot A_{\theta_1}, 0 \cdot J, B \cdot A_{\theta_2}, B \cdot A_{\theta_1+\theta_2}, 0 \cdot p_5, \ldots, 0 \cdot p_{2n}) =$$

$$= (0 \cdot p_1, 0 \cdot p_2, B \cdot p_3, B \cdot p_4, 0 \cdot p_5, \ldots, 0 \cdot p_{2n}).$$

It follows that $\omega_{[z_1, \ldots, z_j]}(df_n(\partial_{\theta_1}), df_n(\partial_{\theta_2})) = 0$ for $j = 1$ and $j \geq 4$. Furthermore,

$$\omega_C(df_n(\partial_{\theta_1}), df_n(\partial_{\theta_2})) = \omega_C(df_n(\partial_{\theta_1}), df_n(\partial_{\theta_2})) =$$

$$= \sum_{j=1}^{2n-1} \omega_{[z_1, \ldots, z_j]}(df_n(\partial_{\theta_1}), df_n(\partial_{\theta_2})) =$$

$$= \sum_{j=2}^3 \omega_{[z_1, \ldots, z_{j+1}]}(df_n(\partial_{\theta_1}), df_n(\partial_{\theta_2})) =$$

$$= (\Omega((B \cdot p_1 p_2, 0 \cdot p_3), (0 \cdot p_1 p_2, B \cdot p_3)) +$$

$$+ \Omega((B \cdot p_1 p_2 p_3, B \cdot p_4), (p_1 p_2 \cdot B \cdot p_3, B \cdot p_4))) =$$

$$= \frac{1}{2}(\omega(B \cdot p_1 p_2) \cdot \omega(B \cdot p_3) +$$

$$+ \omega(B \cdot p_1 p_2 p_3) \cdot \omega(B \cdot p_4) - \omega(p_1 p_2 \cdot B \cdot p_3) \cdot \omega(B \cdot p_4)) =$$

$$= \frac{1}{2}\left((\text{Ad}(p_1 p_2^{-1})(B)) \cdot B + (\text{Ad}(p_1 p_2 p_3^{-1})(B) - \text{Ad}(p_3^{-1})(B)) \cdot B\right).$$

(7.5)

Since $A_\theta B A_{\theta}^{-1} = -B$ for all $\theta \in \mathbb{R}$, we have $\text{Ad}(p_j)(B) = -B$ for $j = 1, 2, 3$. From (7.5) we get then that

$$\omega_C(df_n(\partial_{\theta_1}), df_n(\partial_{\theta_2})) = \frac{1}{2} (B \cdot B) = \frac{1}{4} \text{Trace}(BB) = -\frac{1}{2}.$$ 

Therefore $(f_n \mid_{U_3})^*(\omega_C) = -\frac{1}{2} d\theta_1 \wedge d\theta_2$ on $U_3$ and, consequently,

$$\int_{U_3} (f_n \mid_{U_3})^*(\omega_C) = -\pi^2.$$ 

Lemma 7.3 follows now from (7.3) due to the fact that $(f_n \mid_{U_1})^*(\omega_C) = (f_n \mid_{U_2})^*(\omega_C) = 0$. □
Let us recall that a symplectic manifold \((M, \omega)\) is called monotone if the cohomology class \([\omega] \in H^2(M, \mathbb{R})\) is a positive multiple of the first Chern class \(c_1(M)\).

**Theorem 7.4.** The symplectic manifold \(\mathcal{M} \subset P_{2n}\) is monotone,

\[
[\omega_C] = \frac{\pi^2}{2} c_1(\mathcal{M}).
\]

**Proof.** Recall that \(K_{2n}\) is a strong deformation retract of \(\mathcal{M}\) and that it is 1-connected (Proposition 2.9). According to Lemma 6.3, the subgroup \(G\) of \(H_2(\mathcal{M}, \mathbb{Z})\) generated by the homotopy classes \([\gamma_j, \epsilon]\), \(j = 1, \ldots, 2n-1, \epsilon = \pm 1\), and by \([f_n]\) is of index at most 2 in \(H_2(\mathcal{M})\). According to Theorems 6.1 and 6.2 one has

\[
\langle c_1(\mathcal{M}) \mid [f_n] \rangle = -2 \quad \text{and} \quad \langle c_1(\mathcal{M}) \mid [\gamma_j, \epsilon] \rangle = 0
\]

for \(j = 1, \ldots, 2n-1, \epsilon = \pm 1\).

Now, Lemma 7.1 shows that \(\langle [\omega_C] \mid [\gamma_j, \epsilon] \rangle = 0\) for \(j = 1, \ldots, 2n-1, \epsilon = \pm 1\).

Lemma 7.3 shows that \(\langle [\omega_C] \mid [f_n] \rangle = -\pi^2\). Thus the identity

\[
[\omega_C] = \frac{\pi^2}{2} c_1(\mathcal{M})
\]

holds when evaluated on the elements of the subgroup \(G\) of index at most 2 in \(H_2(\mathcal{M}, \mathbb{Z})\). Therefore it holds when evaluated on all elements of \(H_2(\mathcal{M}, \mathbb{Z})\) and thus holds in \(H^2(\mathcal{M}, \mathbb{R})\). \(\square\)

## 8. Topological quandles and link invariants

Every conjugacy class in a topological group is a topological quandle. In [25] the second author introduced invariants of oriented links in \(\mathbb{R}^3\) derived from topological quandles. If \(L\) is an oriented link in \(\mathbb{R}^3\) and \(Q\) is a topological quandle one defines an invariant \(J_Q(L)\) which is a topological space well-defined up to a homeomorphism. If \(Q\) is a conjugacy class \(C\) in a Lie group \(G\) and \(L\) is the closure of a braid \(\sigma\) on \(n\) strands then the invariant space \(J_Q(L)\) is, by definition, equal to the subspace of fixed points \(Fix(\sigma)\) of the action of the braid \(\sigma\) on the product \(\prod_{i=1}^n C\) of \(n\) copies of \(C\).

However, it is immediate that the space \(Fix(\sigma)\) and, hence, the invariant space \(J_Q(L)\) is also homeomorphic to the intersection \(\Lambda \cap \Gamma_\sigma\) in \(\prod_{i=1}^n C \times \prod_{i=1}^n C^{-1}\) (see Section 5) of the “twisted diagonal” \(\Lambda\) with the “twisted graph” \(\Gamma_\sigma\) of \(\sigma\).

In the present paper we concentrate on the case when \(Q = C\) is the conjugacy class \(S\) in the group \(G = SU(2)\). Note that the quandle structure on \(S\) given by the conjugation in \(SU(2)\) is the same as the quandle structure obtained from looking at \(S\) as the symmetric Riemannian manifold isometric to the standard 2-dimensional sphere \(S^2\).
9. Examples

Let $G$ be a Lie group. We shall now recall the action of the braid group on $n$ strands $B_n$ on $G^\times n$ used in [25].

Let $\sigma$ be an $n$-strand braid. To describe the image of $(g_1, \ldots, g_n)$ under $\sigma$ we use braid diagrams with edges labelled by elements of $G$ as in Figure 9.1. This figure describes how to label the two outgoing half-edges in a crossing given labels on the incoming ones. Label the bottom edges of $\sigma$ by $(g_1, \ldots, g_n)$. Tracing through the braid, the rules in Figure 9.1 then determine another $n$-tuple of $G$-elements $\sigma(g_1, \ldots, g_n)$ on the top edges of the braid. This defines the braid action.

Let $L$ be the oriented link in $S^3$ obtained as the closure of the braid $\sigma$. From the above description it is easy to see that the fixed point set

$$J_G(L) := \text{Fix}(\sigma) = \{ \rho \in G^\times n | \sigma(\rho) = \rho \},$$

can be identified with the space $\text{Hom}(\pi_1(S^3 \setminus L), G)$ of homomorphisms of the fundamental group of the complement of $L$ into $G$. (This observation implies, in particular, that the space $J_G(L)$ is a link invariant.) Indeed, a necessary and sufficient condition for the labelling of the diagram of $\sigma$ by group elements satisfying the rules of Figure 9.1 to extend over the “closing arcs” of the closure of $\sigma$ is that $\sigma(\rho) = \rho$. Then the labelling describes a homomorphism of the Wirtinger presentation of the link complement into $G$ (see [25], Lemma 4.6). Let us call such a labelling admissible.

All of the above arguments continue to hold if we replace the group $G$ by some specified conjugacy class $C$ in $G$. In this case $J_C(L) := \text{Fix}(\sigma) = \{ \rho \in C^\times n | \sigma(\rho) = \rho \}$ can be identified with $\text{Hom}(\pi_1(S^3 \setminus L), G)_C$ - the space of representations which send the positively oriented link meridians of $L$ into the conjugacy class $C$. For the rest of this chapter, unless explicitly stated otherwise, we will be concerned with the case $G = SU(2)$ and $C = \{ A \in SU(2) | \text{tr}(A) = 0 \}$.

Before going on to the examples, let us recall the behaviour of the conjugation action of $C$ on itself. Viewing $SU(2)$ as the unit 3-sphere in the space of quaternions, $C$ is the unit 2-sphere in the space of pure quaternions $\mathbb{R}^3$. If $g$ and $h$ are in $C$, the element $g^{-1}hg$ is obtained by rotating $h$ by the angle $\pi$ around $g$.

Equivalently, $g$ acts on $h$ by the involutive isometry $i_g$ (reflection in $g$) associated to $g$ if we regard $C \cong S^2$ as a symmetric Riemannian manifold.
9.1. \((2,n)\)-torus links. A \((2,n)\)-torus link \(T = T_{2,n}\) is the closure of a 2-strand braid \(\sigma = \sigma_1^n\) (where \(\sigma_1\) is the elementary braid on 2 strands generating the braid group \(B_2\)). If \(n\) is odd, \(T_{2,n}\) is a knot, if \(n\) is even it is a link of two components. The space \(J_C(T_{2,n})\) is the space of fixed points of the action of the braid \(\sigma\) on \(C \times C\).

Let us compute \(\sigma(a,b)\) for two arbitrary points \(a,b\) on the 2-sphere \(C\). Choose a great circle \(S\) containing \(a\) and \(b\). If \(a\) and \(b\) are neither equal nor antipodal, there is a unique choice. Recall that the isometry \(a \mapsto i_b(a)\) associated to \(b\) is given by reflecting \(a\) in \(b\) and that for the braid generator \(\sigma_1\) we have

\[
\sigma_1(a,b) = (b,i_b(a)).
\]

We parametrise \(S\) starting at \(b\), with \(a\) at the angle \(\alpha\). Thus \((a,b) = (\alpha,0)\). Then it is easy to see that

\[
\sigma(\alpha,0) = (-n-1)\alpha, n\alpha).
\]

Hence, \((a,b)\) is a fix-point of \(\sigma\) iff \(n\alpha \equiv 0 \mod 2\pi\), that is if \(\alpha = \pm \frac{2\pi j}{n}\), \(0 \leq j \leq \frac{n-1}{2}\).

Consider the projection on the second factor \(p : J_C(T_{2,n}) \to C\). For \(n\) odd, the fibre over \(b\) is the disjoint union of \(b\) and \(\frac{n+1}{2}\) circles, each in a plane perpendicular to \(b\), and consisting of the \(a\) which have distance \(\frac{2\pi j}{n}\) to \(b\). It is clear from this description that \(J_C(T_{2,n})\) is the disjoint union of an \(S^2\) and \(\frac{n-1}{2}\) copies of the unit tangent bundle to \(S^2\) (which is the same as \(\mathbb{R}P^3\)).

If \(n\) is even, the fibre is the disjoint union of two points (corresponding to the angles \(0\) and \(\pi\), respectively) and \(\frac{n}{2}\) circles. Thus in this case, \(J_C(T_{2,n})\) is diffeomorphic to the union of two copies of \(S^2\) and \(\frac{n}{2}\) copies of \(\mathbb{R}P^3\) unit tangent bundle of \(S^2\).

In other words, for \((2,n)\)-torus links we have

\[
J_C(T_{2,n}) = S^2 \cup S^2 \cup \bigcup_{j=1}^{\frac{n-1}{2}} \mathbb{R}P^3, \tag{9.1}
\]

where one of the spheres appears only if \(n\) is even.

The Khovanov homology \(Kh^{i,j}(T_{2,n})\) was computed already in the seminal paper \[13\]. The non-zero groups are as follows. First, there is a pair of groups each isomorphic to the integers.

\[
Kh^{0,-n}(T_{2,n}) = \mathbb{Z}
\]

\[
Kh^{0,2-n}(T_{2,n}) = \mathbb{Z}
\]

Then there is a number of “triples” of groups, one triple for each \(1 \leq k \leq \frac{n-1}{2}\):

\[
Kh^{-2k-1,-4k-2-n}(T_{2,n}) = \mathbb{Z}
\]

\[
Kh^{-2k,-4k-n}(T_{2,n}) = \mathbb{Z}_2
\]

\[
Kh^{-2k,-4k+2-n}(T_{2,n}) = \mathbb{Z}
\]

If \(n\) is odd and these are placed on the \((i,2j+1)\)-lattice (if \(n\) is even, the \((i,2j)\)-lattice), they form a chess-board “knight move” pattern (for more on this see below, Remark \[9,2\].

Finally, for even \(n\), there is another pair
\[
Kh^{-n,-3n}(T_{2,n}) = \mathbb{Z} \\
Kh^{-n,-2-3n}(T_{2,n}) = \mathbb{Z}
\]

If we collapse the bigrading in these formulas to a single grading \( k = i - j \), the first and last groups become copies of the cohomology of the 2-sphere \( S^2 \), with grading shifted by \( n - 2 \) and \( 2n - 2 \) respectively. Similarly, each “triple” as above becomes a copy of the cohomology of real projective space \( \mathbb{R}P^3 \), shifted by \( 2k - 2 + n \). Calling the resulting groups \( Kh^* \), we obtain

\[
Kh^*(T_{2,n}) \cong H^*(S^2)\{n - 2\} \oplus H^*(S^2)\{2n - 2\} \oplus \bigoplus_{k=1}^{n-1} H^*(\mathbb{R}P^3)\{2k - 2 + n\},
\]

where the second term appears only when \( n \) is even. We note that exactly the same spaces as in (9.1) appear here, only shifted by certain integers. All the cohomology here and below is with coefficients in the integers \( \mathbb{Z} \).

9.2. The figure eight knot \( 4_1 \). The invariant space for the figure eight knot was computed in [25]. The result is

\[
J_C(4_1) = S^2 \cup \mathbb{R}P^3 \cup \mathbb{R}P^3.
\]

Consulting JavaKh, [10], collapsed Khovanov homology for the figure eight knot is

\[
Kh^*(4_1) = H^*(S^2)\{-1\} \oplus H^*(\mathbb{R}P^3)\{-3\} \oplus H^*(\mathbb{R}P^3)\{0\}.
\]

9.3. The knot \( 5_2 \). For the five crossing knot which is not a torus knot,

\[
J_C(5_2) = S^2 \cup \mathbb{R}P^3 \cup \mathbb{R}P^3 \cup \mathbb{R}P^3.
\]

This can be easily obtained by using the techniques of [25], Section 5. The details are left to the reader. JavaKh tells us that Khovanov homology of the knot \( 5_2 \) is

\[
H^*(S^2)\{1\} \oplus H^*(\mathbb{R}P^3)\{5\} \oplus H^*(\mathbb{R}P^3)\{7\} \oplus H^*(\mathbb{R}P^3)\{11\}
\]

9.4. The knot \( 6_1 \). This knot has

\[
J_C(6_1) = S^2 \cup \mathbb{R}P^3 \cup \mathbb{R}P^3 \cup \mathbb{R}P^3 \cup \mathbb{R}P^3
\]

and Khovanov homology, again by JavaKh,

\[
H^*(S^2)\{-1\} \oplus H^*(\mathbb{R}P^3)\{-3\} \oplus H^*(\mathbb{R}P^3)\{-1\} \oplus H^*(\mathbb{R}P^3)\{0\} \oplus H^*(\mathbb{R}P^3)\{2\}.
\]

Remark 9.1. The same pattern as above recurs for all seven crossing knots and for at least some eight-crossing knots. The invariant spaces of those knots can be computed by techniques of [25], Section 5. The limits are set by our capacity for solving the equations for the invariant spaces, which become very hard to solve when the number of labels used is more than three. We have been able to calculate one example which does not produce the Khovanov homology groups, namely the knot \( 9_{42} \), in Rolfsen’s table. The calculation is presented in Section [11].

Remark 9.2. In the bigraded Khovanov homology of knots, there is the well-known pattern of “knight moves”: two \( \mathbb{Z} \)'s and one \( \mathbb{Z}_2 \) arranged in the chess board constellation of a knight move. In our examples, these are the source of the \( H^*(\mathbb{R}P^3) \)'s after collapse of the bigrading. There are also two additional copies of \( \mathbb{Z} \) sitting in \( i \)-degree 0. They correspond to \( H^*(S^2) \) after collapse. It has been conjectured
by Alexander Shumakovitch [26]. Conjecture 3, that for alternating knots all the homology groups are arranged in knight moves and a "sphere pair". (The same is conjectured for non-split alternating links, and even more generally for H-thin links, except that there are more sphere pairs. Compare for example \(T_2,n\)-links with even \(n\) above.) With rational coefficients this conjecture (originally due to Bar-Natan, Garoufalidis and Khovanov [2, 7]) is a theorem due to Eun Soo Lee [19].

9.5. Reduced homology. Consider \(J_C(L) = Fix(\sigma)\) as the set of admissible labelings of the link diagram of \(L\) by elements from \(C\) as above. Then the choice of an arc in the link diagram determines a map from \(J_C(L)\) to \(C\). This map is a fibration. One may consider the cohomology of its fibre for a fixed point in \(C\). In all the examples above (and all the ones we have computed), the fibre projection is the identity on the sphere components and the natural projection on the \(\mathbb{R}P^3\)s. We get the following set of examples in which we see that this gives the “reduced” \([14]\) (collapsed) Khovanov homology.

9.6. \(T_{2,n}\)-torus links - reduced case. The fibre projection to \(C\) is the identity on the \(S^2\)-components and the natural projection on the \(\mathbb{R}P^3\)s. Thus the fibre is

\[
\{\ast\} \cup \{\ast\} \cup \bigcup_{i=0}^{\frac{n-1}{2}} S^1,
\]

where the second term appears only if \(n\) is even. Reduced collapsed Khovanov homology is

\[
H^*(\ast)\{n-1\} \oplus H^*(\ast)\{2n-1\} \oplus \bigoplus_{i=0}^{\frac{n-1}{2}} H^*(S^1)\{2i-1+n\}.
\]

9.7. Two-bridge links. All of the above examples are special cases of two-bridge links. For two-bridge knots and with rational coefficients the observation \([1,1]\) of the Introduction holds true. Indeed, we have

**Theorem 9.3.** Let \(K\) be a two-bridge knot and let \(\{J_C^L(K)\}_{r \in R}\) be the set of connected components of the representation variety \(J_C(K)\). Then there are integers \(N_r = N_r(K), r \in R\), such that

\[
Kh^*(K; \mathbb{Q}) = \bigoplus_{r \in R} H^*(J_C^L(K); \mathbb{Q})\{N_r\}.
\]

**Proof.** Let \(G = SU(2)\). A two-bridge knot group \(\pi = \pi_1(S^3\setminus K)\) can be presented by two meridional generators \(a,b\) and one relation. For any representation in \(J_C(K) = Hom(\pi, G)_C\), the images of the generators will lie on some great circle, and the single equation in the angle \(\alpha\) between them can be solved just as in the case of \((2,n)\)-torus links. The result is that \(J_C(K)\) is the disjoint union of a copy of \(S^2\) and a number of \(\mathbb{R}P^3\)s. These connected components are exactly the \(G\)-conjugacy classes of these representations.

Let

\[
S^1_A = \{a + bi : a^2 + b^2 = 1\} \subset G
\]

and

\[
S^1_B = \{cj + dk : c^2 + d^2 = 1\} \subset C \subset G.
\]
The disjoint union of these circles in \( G = SU(2) \) is called the binary dihedral group \( N \), see [15].

The inclusion of \((N, S_B^1, B)\) into \((G, C)\) induces an obvious map

\[
\text{Hom}(\pi, N)_{S_B^1}/N \xrightarrow{\cong} \text{Hom}(\pi, G)_{C}/G.
\]

This map is surjective since any representation in \( \text{Hom}(\pi, G)_{C} \) can be \( G \)-conjugated into \( N \). It is also injective. Indeed, let \( \psi \) and \( \phi \) be two representations in \( \text{Hom}(\pi, N)_{S_B^1} \) and suppose they are conjugate modulo \( G \). This amounts to saying that the pair \( \phi(a), \phi(b) \) of points in \( S_B^1 \) can be rotated and reflected into \( \psi(a), \psi(b) \). But in fact any rotation or reflection of \( S_B^1 \) can already be effected by conjugation by elements of \( N \).

According to [15], Theorem 10, the number \( n \) of non-abelian conjugacy classes of binary dihedral representations is given by the formula

\[
n = \frac{|\Delta_{-1}(K)| - 1}{2}
\]

where \( \Delta_{-1}(K) \) is the Alexander polynomial evaluated at \(-1\). Furthermore, all the non-abelian representations are contained in \( \text{Hom}(\pi, N)_{S_B^1} \).

A representation in \( \text{Hom}(\pi, N)_{S_B^1} \) is non-abelian if and only if it maps the two generators to distinct and non-antipodal elements, that is, if it lies in one of the \( \mathbb{R}P^3 \) components of \( \text{Hom}(\pi, G)_{C} \). Thus \( n \) is also the number of \( \mathbb{R}P^3 \)'s. Taking into account also the one component of abelian representations (the sphere), we get

\[
\dim(H^*(J_C(K); \mathbb{Q})) = 2(n + 1),
\]

since every connected component contributes two dimensions to cohomology.

Now, \( \Delta_{-1}(K) \) is known as the determinant of \( K \), and it is a classical theorem (see e.g. [5]) that for alternating knots \( L \) the determinant is the number of spanning trees in the Tait (checkerboard) graph of \( L \). Furthermore, a theorem of Champanerkar and Kofman [6] says that the rank of reduced Khovanov homology for an alternating knot is exactly this number of spanning trees. Finally, the ranks of unreduced and reduced Khovanov homologies are related by

\[
\dim(Kh(K); \mathbb{Q}) = \dim(Kh_{\text{red}}(K); \mathbb{Q}) + 1.
\]

It follows that

\[
\dim(H^*(J_C(K); \mathbb{Q})) = 2(n + 1) = |\Delta_{-1}(K)| + 1 = \dim(Kh(K); \mathbb{Q})).
\]

\[ \square \]

**Remark 9.4.** Theorem 9.3 is probably true also with integral coefficients. For example, it would follow from the conjecture mentioned in Remark 9.2 since all two-bridge knots are alternating.

9.8. **Connected sum of knots - The square knot.** It is a result of the second author (unpublished) that if \( K_1 \) and \( K_2 \) are knots and we form their connected sum \( K_1 \# K_2 \) then the invariant space \( J_C(K_1 \# K_2) = J_C(K_1) \times_C J_C(K_2) \) is the pullback of the fibrations \( J_C(K_1) \to C \) and \( J_C(K_2) \to C \) over the base arcs. (This result holds for any conjugation quandle \( Q \).)

In the case of the connected sum of the trefoil knot with its mirror image, a.k.a. the square knot \( Sq \), we get

\[
J_C(Sq) = S^2 \cup \mathbb{R}P^3 \cup \mathbb{R}P^3 \cup \mathbb{R}P^3 \times S^1.
\]
Again, a consultation of JavaKh gives

\[ Kh^*(Sq) = H^*(S^2)\{-1\} \oplus H^*(\mathbb{RP}^3)\{-4\} \oplus H^*(\mathbb{RP}^3)\{1\} \oplus H^*(\mathbb{RP}^3 \times S^1)\{-2\}. \]

10. AN EXAMPLE WITH NON-ISOMORPHIC COHOMOLOGY GROUPS

In this section we calculate, as promised in Remark 9.1, the representation variety \( J_C(L) \) of the knot \( L = 9_{42} \) and show that its cohomology groups do not coincide with the Khovanov homology groups. Cognoscenti will note that the knot in question is non-alternating and that it has all the possible “card symbols” in Bar-Natan’s table [2].

**Proposition 10.1.** \( J_C(9_{42}) = S^2 \cup \bigcup_{i=1}^{7} \mathbb{RP}^3. \)

**Proof.** We consider the knot diagram of 9_{42} pictured in Figure 10.1 and regard the representation variety \( J_C(9_{42}) \) as the space of admissible colorings of the diagram in that figure, [25]. As we can see we need to use three colors \( a, b, c \). For simplicity of notation let us use the functional notation for conjugation, so that

\[ a(b) = a^{-1}ba \quad (= aba^{-1}). \]

![Fig. 10.1](image-url)

The three circles in the figure encircle the crossings which give the three equations to be satisfied by the colors \( a, b, c \).

It is easy to verify that the equations become

\[ c(b(a(b))) = a(b(a)), \quad (1) \]
Hence Equation 2 becomes

\[ a(b(a(b(a)))) = c(b(c)) \]  

(2)

and

\[ c(b(c(b))) = a(b(a(b(a(c))))) \]  

(3)

10.1. **Case 1:** \( a = \pm b \). If \( a \) and \( b \) coincide, then the first equation says that that \( c(a) = a \). Thus, \( c = \pm a \). The second equation, however, then becomes \( a = c \), so only the solution \( a = b = c \) survives. It trivially satisfies the last equation, too. The solutions of this form constitute the diagonal \( S^2 \) in \( S^2 \times S^2 \times S^2 \). If, instead, \( a \) and \( b \) are antipodal, then equation 1 says that \( c \) should take the antipode of \( a \) to \( a \). Thus \( c \) should be on the equator midway between \( a \) and \( b = -a \). Then \( b(c) = -c \), so that equation 2 becomes \( a = -c \), which is a contradiction. Thus, there are no solutions of the form \((a, -a, c)\). This case contributes one copy of \( S^2 \) to \( J_C(9_{42}) \).

10.2. **Case 2:** \( a \neq \pm b \). If \( a \) and \( b \) are not parallel then they lie on a unique great circle \( \Gamma \subset S^2 \). We will parametrize \( \Gamma \) by arc-length starting at \( a \) with \( b \) corresponding to angle \( \theta \). Consider the expressions \( \alpha = a(b(a)) \) and \( \beta = b(a(b)) \). They are both certain points on \( \Gamma \). Equation 1 says that reflection in \( c \) takes \( \alpha \) to \( \beta \). We will consider different subcases, based on the relative position of \( \alpha \) and \( \beta \).

10.2.1. **Case 2a:** \( \alpha = \beta \). In this case, Equation 1 says that \( c \) fixes \( \alpha \), that is, \( c = \pm \alpha \).

In terms of the angle \( \theta \), it is easy to see that \( \alpha = \beta \) means \( 5\theta = 0 \mod 2\pi \), so that \( a \) and \( b \) are vertices of a regular pentagon \( P \) on \( \Gamma \). Clearly \( \alpha \) is also on the pentagon.

Since \( c(a) = \alpha \), \( c \) must be a point on the regular decagon \( D \) containing \( P \). (So far there are two antipodal possibilities for \( c \).) Next, note that the left hand side of Equation 2 collapses under the given assumption:

\[ a(b(a(b(a)))) = a(b(a(b(b)))) = a(a(b)) = b. \]

Hence Equation 2 becomes

\[ b = c(b(c)), \]

which has solutions \( 3\theta = 0 \mod 2\pi \) in terms of the angle \( \theta \) between \( b \) and \( c \). If \( \theta \neq 0 \) this is clearly incompatible with \( c \) lying on \( D \). If \( \theta = 0 \), so that \( b = c \), then it is easy to see that Equation 1 reduces to \( a = b \), so that we are back in Case 1. Hence there are no new solutions from this case.

10.2.2. **Case 2b:** \( \alpha = -\beta \). In this case, Equation 1 says that \( c \) takes \( \alpha \) to its antipode. Hence \( c \) will lie on the great equatorial circle \( \Delta \) all of whose points are equidistant from \( \alpha \) and \( \beta \). In terms of the angle \( \theta \) between \( a \) and \( b \), Equation 1 says \( 5\theta + \pi = 0 \mod 2\pi \). So, \( b \) and the antipode of \( a \) are vertices of a regular pentagon \( P' \) on \( \Gamma \). The regular pentagon \( P \) with \( a \) as a vertex is the complement of \( P' \) in a regular decagon \( D \) on \( \Gamma \). Since reflection in a vertex of \( D \) preserves \( P \) and \( P' \) respectively, it follows that \( \alpha \) is in \( P \) and \( \beta \) is in \( P' \). Next we observe that the right hand side of Equation 3 equals \( c(c) \), which is \(-c\), since \( \alpha \) is perpendicular to \( c \). Denote by \( \tau \) the angle between \( b \) and \( c \) on the great circle \( E \) that they span. \( \Delta \) contains no vertices of \( D \), hence \( b \neq \pm c \). The equation 3 then has the solutions \( \tau = \frac{\pi}{3} + 2\frac{\pi}{3} \), that is \( c \) is a vertex of an equilateral triangle on \( E \), which also contains as a vertex the antipode of \( b \). Assume first that \( c \) is antipodal to \( b \). In particular, \( c \) is a vertex of \( P \). Hence, \( c(\alpha) \) is again in \( P \). But by Equation 1, \( c(\alpha) = \beta \) which is in \( P' \). Contradiction. Assume instead that \( c \) is not antipodal to \( b \). Then there are exactly two possibilities for \( c \): the two points on \( \Delta \) at a distance \( \frac{\pi}{3} \) from \( b \). That these two points exist is clear from the relative position of \( \Gamma \) and \( \Delta \): \( b \) cannot be
\(\beta\), since it would mean that \(baba^{-1}b^{-1} = b\) which is equivalent to \(a(b) = b\) and, hence, to \(a = \pm b\), which is excluded by assumption. Therefore the distance of \(b\) to \(\Delta\) along \(\Gamma\) is at most \(\frac{3\pi}{10} < \frac{\pi}{3}\).

One readily checks from this geometric description that both choices of \(c\) give solutions \((a, b, c)\) also to Equation 2. Indeed, one checks that \(c(b(c)) = -b\) while \(a(b(a(b(a))))\) is the point on \(\Gamma\) corresponding to the angle \(-2\theta = \theta + \pi \mod 2\pi\) which also gives \(-b\).

Hence given \(a\), there are two circles worth of choices of \(b\) at angle \(\theta\) from \(a\), and then exactly two choices of \(c\) for each such pair \(a, b\). These solutions give a compact subspace of the space of all solutions which is a union of some connected components of that space. The two choices of \(c\) give points \((a, b, c)\) which lie in different connected components. To see that we observe that the continuous function \(\det(a, b, c)\), where \(a, b, c\) are interpreted as vectors in \(\mathbb{R}^3\), does not vanish on the subspace of solutions considered here and that it differs in sign for the two different choices of \(c\), given the pair \(a, b\). Thus, points \((a, b, c_1)\) and \((a, b, c_2)\) with \(c_1 \neq c_2\) lie in different connected components. It follows that the solutions of this form constitute 4 copies of \(\mathbb{R}P^3\).

10.2.3. \textbf{Case 2c:} \(\alpha \neq \pm \beta\). In the last case, where \(\alpha\) and \(\beta\) are non-parallel, Equation 1 forces \(c\) to be on \(\Gamma\). Denote, as before, the angle between \(a\) and \(b\) by \(\theta\). Then \(\alpha\) corresponds to \(-2\theta\) and \(\beta\) to \(3\theta\). Equation 1 says that \(c\) must be on the line bisecting the angle between \(\alpha\) and \(\beta\), that is at an angle \(\vartheta = \frac{\theta}{2} + k\pi\) from \(a\). With this notation we have \(\theta = 2\vartheta\) and it is straightforward to compute that the equations 2 and 3 are both equivalent to \(7\vartheta = 0 \mod 2\pi\), where we should disregard the trivial solution \(\vartheta = 0\) since it belongs to a case already studied. Denote the regular heptagon containing \(a\) and \(c\) by \(H\). The equality \(\theta = 2\vartheta\) tells us that \(b\) is uniquely determined given \(a\) and \(c\). Once \(a\) is given, there are three circles worth of choices of \(c\) at angle \(\vartheta\) from \(a\) and then exactly one \(b\) for every such pair \(a, c\). The solutions of this form constitute 3 copies of \(\mathbb{R}P^3\).

Thus Case 1 contributes the component \(S^2\), Case 2a nothing, Case 2b four copies of \(\mathbb{R}P^3\) and Case 2c three copies of \(\mathbb{R}P^3\) to the solution space \(J_C(9_{42})\). Proposition \textbf{10.1} follows.

\makelabel{Comparison to Khovanov homology}{Comparison to Khovanov homology}

According to Proposition \textbf{10.1},

\[
J_C(9_{42}) = S^2 \cup \bigcup_{i=1}^7 \mathbb{R}P^3. \tag{10.1}
\]

On the other hand, consultation of JavaKh gives for collapsed Khovanov homology,

\[
Kh^*(9_{42}) = H^*(S^2)\{-1\} \oplus H^*(\mathbb{R}P^3)\{-5\} \oplus H^*(\mathbb{R}P^3)\{-3\} \oplus H^*(\mathbb{R}P^3)\{-2\} \oplus H^*(\mathbb{R}P^3)\{0\}.
\]

Hence the two homology groups differ by three copies of \(\mathbb{R}P^3\).
In this appendix we shall give a proof of Proposition 2.7 describing structure of the singularities of the spaces \( K_{2n} \) and \( \bar{K}_{2n} \).

As pointed out in Subsection 2.2 the singularities of \( K_{2n} \) and of \( \bar{K}_{2n} \) are contained in the subsets \( \Sigma \) and \( \bar{\Sigma} \) respectively. The sets \( \Sigma \) and \( \bar{\Sigma} \) are disjoint unions of subsets \( \Delta_{\epsilon_1,...,\epsilon_{2n}} \) and there is a smooth action of the group \((\mathbb{Z}/2\mathbb{Z})^{2n}\) on \( P_{2n} \) mapping \( K_{2n} \cup \bar{K}_{2n} \) to itself and acting transitively on the family of all subsets \( \Delta_{\epsilon_1,...,\epsilon_{2n}} \). Hence the structure of \( K_{2n} \) and of \( \bar{K}_{2n} \) in a neighbourhood of each \( \Delta_{\epsilon_1,...,\epsilon_{2n}} \) is the same and it is enough to study it for just one \( \Delta = \Delta_{\epsilon_1,...,\epsilon_{2n}} \). We choose \( \Delta = \Delta_{\epsilon_1,...,\epsilon_{2n}} \), with \( \epsilon_j = (-1)^j \).

Thus \( \Delta = \{ (A, -A, A, -A, ..., A, -A) \mid A \in S \} \). Recall that \( S \subset SU(2) \) consists of the matrices \( \left( \begin{array}{cc} u & z \\ -z & u \end{array} \right) \) with \( t \in \mathbb{R}, z \in \mathbb{C} \) and \( t^2 + |z|^2 = 1 \).

We shall now equip \( P_{2n} \) with a Riemannian metric.

Let us first choose a left- and right-invariant Riemannian metric on the Lie group \( SU(2) \). The geodesics \( \gamma \) in \( SU(2) \) with \( \gamma(0) = I \) are precisely the one-parameter subgroups of \( SU(2) \), see [22], Lemma 21.2.

Let \( U \) be the linear subspace of the Lie algebra \( su(2) \) consisting of the matrices \( \bar{u} = \left( \begin{array}{cc} -u & 0 \\ 0 & u \end{array} \right) \), \( u \in \mathbb{C} \). The tangent space \( T_J S \) to the submanifold \( S \) of \( SU(2) \) at the point \( J = \left( \begin{array}{cc} 0 & 0 \\ 0 & -1 \end{array} \right) \) consists of the matrices \( \left( \begin{array}{cc} 0 & z \\ -\bar{z} & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & iz \\ iz & 0 \end{array} \right) \cdot J \), \( z \in \mathbb{C} \), and, hence, it is equal to \( UJ = \{ \bar{u}J \mid \bar{u} \in U \} \).

Whenever convenient we shall identify \( U \) with \( \mathbb{C} \) via \( \bar{u} \leftrightarrow u \). Under this identification the restriction of the Riemannian metric to \( U \) is a positive multiple of the standard real inner product on \( \mathbb{C} = \mathbb{R}^2 \).

For any matrix \( \bar{u} = \left( \begin{array}{cc} u & 0 \\ 0 & u \end{array} \right) \), \( u \in \mathbb{C} \), one has \( \bar{u}^2 = -|u|^2 I \). Therefore

\[
e^{-\bar{u}} = \left( \sum_{j=0}^{\infty} \frac{(-|u|^2)^j}{(2j)!} \right) I + \left( \sum_{j=0}^{\infty} \frac{(-|u|^2)^j}{(2j + 1)!} \right) \bar{u} = \cos(|u|) I + \frac{\sin(|u|)}{|u|} \bar{u},
\]

where \( \frac{\sin(z)}{z} \) is to be interpreted as an entire analytic function. It follows that, if \( z \in \mathbb{C}, |z| = 1, B = \left( \begin{array}{cc} 0 & iz \\ iz & 0 \end{array} \right) \), then

\[
\gamma(t) = e^{tB} \cdot J = \left( \begin{array}{cc} i \cos(t) & \sin(t) z \\ -\sin(t) \bar{z} & -i \cos(t) \end{array} \right), \quad t \in \mathbb{R}, \quad (A.1)
\]
is the geodesic in \( SU(2) \) with the starting point at \( J \) and corresponding to the tangent vector \( \left( \begin{array}{cc} 0 & z \\ -\bar{z} & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & iz \\ iz & 0 \end{array} \right) \cdot J \in T_J S \).

Observe also that, since \( \text{Tr}(e^{tB} \cdot J) = 0 \), one has \( \gamma(t) = e^{tB} \cdot J \in S \) for all \( t \in \mathbb{R} \), showing that \( S \) is a totally-geodesic submanifold of \( SU(2) \).

We equip \( S \) with the Riemannian metric being the restriction of that of \( SU(2) \) and give \( P_{2n} = \prod_{1}^{2n} S \) the product Riemannian metric. Geodesics in \( P_{2n} \) are then products of geodesics in the factors. The group \( SU(2) \) acts on \( P_{2n} \) through isometries. (The Riemannian metric we have introduced on \( S \) is, of course, nothing but the ordinary metric on the 2-dimensional sphere. It will however be convenient to phrase it that way.)
Recall that we have the $SU(2)$-equivariant mapping $p_{2n} : P_{2n} \rightarrow SU(2)$, $p_{2n}(A_1, \ldots, A_{2n}) = A_1 \cdot \ldots \cdot A_{2n}$ and that $K_{2n} = p_{2n}^{-1}(I)$ and $\hat{K}_{2n} = p_{2n}^{-1}(-I)$.

From now on we denote $p_{2n}$ by $p$.

Let us consider the point $\mathcal{J} = (J, -J, -J, \ldots, J, -J) \in \Delta \subset P_{2n}$. The orbit of $\mathcal{J}$ under the action of $SU(2)$ is equal to $\Delta$. The isotropy subgroup of $\mathcal{J}$ is the maximal torus $T$ consisting of the diagonal matrices in $SU(2)$. We identify the tangent space $T_{\mathcal{J}} P_{2n}$ to $P_{2n}$ at the point $\mathcal{J}$ with $U^{2n} = \bigoplus_{j=1}^{2n} U \cong \mathbb{C}^{2n}$ through $T_{\mathcal{J}} P_{2n} \ni (\bar{u}_1 \cdot J, \bar{u}_2 \cdot (-J), \ldots, \bar{u}_{2n} \cdot (-J)) \mapsto (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{2n}) \in U^{2n}$.

We have $p(\mathcal{J}) = I$. The derivative of $p$ at $\mathcal{J}$ is a $T$-equivariant linear map $dp_{\mathcal{J}} : T_{\mathcal{J}} P_{2n} \rightarrow su(2)$ given by

$$dp_{\mathcal{J}} : U^{2n} \rightarrow su(2),$$

$$dp_{\mathcal{J}}(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{2n}) = \sum_{j=1}^{2n} (-1)^{j-1} \bar{u}_j.$$  \hfill (A.2)

The last claim follows from the fact that the conjugation by $J$ acts on $U$ as multiplication by $-1$. The image of $dp_{\mathcal{J}}$ is equal to $U$.

The tangent space to $\Delta$ at the point $\mathcal{J}$ is a subspace of $T_{\mathcal{J}} P_{2n}$ which corresponds to the diagonal $\Delta$ of $U^{2n}$, $\Delta = \{ (\bar{u}, \bar{u}, \ldots, \bar{u}) \in U^{2n} \mid \bar{u} \in U \}$. It is a subspace of $\text{Ker}(dp_{\mathcal{J}})$ (as follows also directly from the fact that $p(\Delta) = \{ I \}$). As the circle $T$ acts on $\Delta$ with $\mathcal{J}$ being a fixed point, the tangent space $\hat{\Delta} = T_{\mathcal{J}} \Delta$ is an invariant subspace of $T_{\mathcal{J}} P_{2n}$. Let $V$ be its orthogonal complement in $T_{\mathcal{J}} P_{2n}$. Since $T$ acts on $T_{\mathcal{J}} P_{2n}$ through isometries, $V$ is a $T$-invariant subspace as well.

We shall denote $SU(2)$ by $G$ to simplify notation.

The $G$-submanifold $\Delta$ of $P_{2n}$ has an equivariant regular neighbourhood $G$-diffeomorphic to a neighbourhood of the zero-section in the normal bundle $G \times_T V$ of $\Delta$ in $P_{2n}$ (see [17] or [12], Thm.1.3, p.3). The $G$-diffeomorphism between them can be obtained as follows.

Given a tangent vector $v \in V$ let $\gamma_v$ be the geodesic in $P_{2n}$ starting at the point $\mathcal{J}$ and such that $\frac{d\gamma_v}{dt}(0) = v$. We define $\phi : G \times_T V \rightarrow P_{2n}$ by $\phi([g, v]) = g(\gamma_v(1))$. The mapping $\phi$ is $G$-invariant and, when restricted to a small enough neighbourhood $Y$ of the zero-section of the normal bundle $G \times_T V$, it is a diffeomorphism of $Y$ onto a neighbourhood of $\Delta$ in $P_{2n}$, [12], p.3-4.

Hence, in order to describe the singularities of $K_{2n} = p^{-1}(I)$ along $\Delta$, it is enough to look at the restriction of the mapping $p$ to the neighbourhood $\phi(G \times_T V)$ of $\Delta$ or, in other words, to study the singularities of the composed mapping $\tilde{p} = p \circ \phi : G \times_T V \rightarrow SU(2)$ along the zero-section of $G \times_T V$. Since the mapping $\tilde{p}$ is $G$-equivariant, it will be enough to study the singularity of the restricted mapping $\tilde{p} : V \rightarrow SU(2)$, $\tilde{p}(v) = \tilde{p}([I, v]) = p(\gamma_v(1))$ at the origin.

Since $\phi$ is nothing but a restriction of the exponential map of the Riemannian manifold $P_{2n}$, its derivative at the zero-section of $G \times_T V$ is equal to the identity. Therefore the derivative $dp_0 : V \rightarrow su(2)$ of $\tilde{p}$ at the point $0$ is given by the restriction of (A.2) to $V$. We want to determine the structure of the subspace $\tilde{p}^{-1}(I)$ of $V$ locally in a neighbourhood of the point $0$.

Since $V$ is an orthogonal complement in $T_{\mathcal{J}} P_{2n}$ of a subspace contained in $\text{Ker}(dp_{\mathcal{J}})$ it follows from (A.2) that the image of the derivative $dp_0$ is equal to $U$. Thus $dp_0$ is of rank 2 and $0 \in V$ is a singular point of $\tilde{p} : V \rightarrow SU(2)$. 

The circle $T$ consisting of the diagonal matrices is a submanifold of $SU(2)$. Its tangent space $T_I \mathbb{T}$ at $I \in \mathbb{T}$ is transversal (and complementary) to $U$ in $su(2)$. Therefore there is an open neighbourhood $\tilde{V}$ of $0$ in $V$ such that the restriction of $\tilde{\rho}$ to $\tilde{V}$, $\tilde{\rho}|_{\tilde{V}} : \tilde{V} \to SU(2)$, is transversal to the submanifold $T$.

Let $M$ be the pre-image of $\mathbb{T}$ under $\tilde{\rho}|_{\tilde{V}}$, $M = \tilde{\rho}^{-1}(\mathbb{T}) \cap \tilde{V}$. It is a smooth submanifold of $\tilde{V}$ of codimension $2$ containing the point $0$. The tangent space $T_0M$ is equal to $(d\tilde{\rho}_0)^{-1}(T_I \mathbb{T})$. As $d\tilde{\rho}_0(V) = U$ and $U \cap T_I \mathbb{T} = \{0\}$, it follows that $T_0M$ is also equal to the kernel of $d\tilde{\rho}_0 : V \to su(2)$.

Let us consider the restriction of $\tilde{\rho}$ to the submanifold $M$, $\hat{f} = \tilde{\rho}|_M : M \to \mathbb{T}$. The space $\hat{\rho}^{-1}(I) \cap \tilde{V}$ which we want to study is a subspace of $M$ equal to $\hat{f}^{-1}(I)$.

Let $W$ be a neighbourhood of $I$ in $SU(2)$ which is a domain of the chart $\text{Log} : W \to su(2)$, ( $\text{Log}$ being the inverse of the exponential mapping $\exp : su(2) \to SU(2)$). We can assume that $\hat{\rho}(\tilde{V}) \subset W$.

Let $T$ be the linear subspace of $su(2)$ which is the tangent space of the circle $T$ at $I$. Then $T$ consists of diagonal matrices in $su(2)$ and it is the orthogonal complement of $U$. Let $\pi : su(2) \to T$ be the orthogonal projection of $su(2)$ onto $T$ and let $\iota : T \to \mathbb{R}$ be the isomorphism $\iota (\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}) = s$. Finally, define mappings

$$
\overline{\pi} : \tilde{V} \to T \quad \text{and} \quad F : \tilde{V} \to \mathbb{R}.
$$

to be $\overline{\pi} = \pi \circ \text{Log} \circ (\hat{\rho}|_{\tilde{V}})$ and $F = \iota \circ \overline{\pi}$.

Observe that since $\hat{\rho}(M) \subset \mathbb{T}$ and $\text{Log}(T) \subset T$, the restriction of $\overline{\pi}$ to $M$ is equal to $\text{Log} \circ \hat{f}$, $\overline{\pi}|_M = \text{Log} \circ \hat{f}$. Therefore $\hat{\rho}^{-1}(I) \cap \tilde{V} = \hat{f}^{-1}(I) = (F|_M)^{-1}(0)$.

As the derivative of $\text{Log}$ at $I$ is the identity map, we have $d\hat{\rho}_0 = \pi \circ d\tilde{\rho}_0$ and $dF_0 = \iota \circ \pi \circ d\tilde{\rho}_0$. Since the image of $d\tilde{\rho}_0$ is equal to $U$ which is the kernel of $\pi$, we get that $d\overline{\pi}_0 = 0$ and $dF_0 = 0$. Thus the origin $0 \in \tilde{V}$ is a critical point of the function $F$.

Let us denote by $f$ the restriction of $F$ to $M$,

$$
f = F|_M = \iota \circ \overline{\pi}|_M = \iota \circ \text{Log} \circ \hat{f} : M \to \mathbb{R}. \quad (A.3)
$$

The origin $0 \in M$ is a critical point of $f$.

We want to determine the Hessian (quadratic form) of $f$ at the point $0$. Let $\hat{N}$ be the kernel of $d\overline{\rho}_0 : V \to su(2)$. It is a (linear) submanifold of codimension $2$ in $V$ containing $0$. The tangent spaces of $M$ and $\hat{N}$ at $0$ coincide. Let $N = \hat{N} \cap V$ and $g = F|_N : N \to \mathbb{R}$. The origin $0$ is a critical point of $g$ and we have

**Lemma A.1.** The Hessian quadratic forms of $g$ and of $f$ at $0$ coincide,

$$
\text{Hess}_0(f) = \text{Hess}_0(g).
$$

**Proof.** The equality of both Hessians depends on the fact that $0 \in \tilde{V}$ is a critical point of the function $F$ (and it would not need to hold otherwise).

Let $X_0, Y_0 \in T_0M = T_0N$ be two tangent vectors to $M$, and hence to $N$, at $0$.

Let us extend $Y_0$ to a vector field $Y$ on $M$ and then extend $Y$ to a vector field $\tilde{Y}$ on $\tilde{V}$. Let $\text{Hess}_0(F)$ be the Hessian bilinear form of $F : V \to \mathbb{R}$ at $0$. Then $\tilde{Y}(F)|_M = Y(f)$ and

$$
\text{Hess}_0(f)(X_0, Y_0) = X_0(Y(f)) = X_0(\tilde{Y}(F)|_M) = X_0(\tilde{Y}(F)) = \text{Hess}_0(F)(X_0, Y_0).
$$
(See [22], p.4, for the definition of the Hessian bilinear form.)

Since \( g = F |_N \), we get in a similar way that \( \text{Hess}_0(g)(X_0, Y_0) = \text{Hess}_0(F)(X_0, Y_0) \).
Therefore \( \text{Hess}_0(f)(X_0, Y_0) = \text{Hess}_0(g)(X_0, Y_0) \), proving the Lemma.

Hence, in order to describe the Hessian of \( f \) at \( 0 \), it is enough to describe the Hessian of \( g \) at \( 0 \), \( g \) being the restriction of \( F \) to the linear submanifold \( N \) of \( V \).

The identification of \( U \) with \( \mathbb{C} \) chosen before gives \( U \) a structure of a complex vector space. The action of the circle \( T \) on \( U \) preserves that structure. The derivative \( dp_\mathcal{J} : U^{2n} \to U \) given by (A.2) is a complex linear map. As \( \Delta = T_\mathcal{J} \Delta \) is a \( \mathbb{C} \)-linear subspace of \( U^{2n} \) (being the diagonal), so is its orthogonal complement \( V \). The map \( dp_\mathcal{J} : V \to U \) is \( \mathbb{C} \)-linear as a restriction of \( dp_\mathcal{J} \).

The tangent space \( T_0N \) to \( N \) at \( 0 \) is equal to \( \tilde{N} = \ker(dp_\mathcal{J} : V \to U) \). Since \( dp_\mathcal{J} \) is \( \mathbb{C} \)-linear and surjective, \( \tilde{N} \) is a \( \mathbb{C} \)-linear subspace of \( \tilde{V} \) of complex codimension 1. As \( V \subset U^{2n} \) is the orthogonal complement of the diagonal \( \Delta \subset U^{2n} \), a \( \mathbb{C} \)-basis of \( V \) is given by vectors \( e_j = e_j - e_{j+1}, \ j = 1, ..., 2n-1 \), where \( e_1, ..., e_{2n} \) is the standard \( \mathbb{C} \)-basis of \( U^{2n} = \mathbb{C}^{2n} \). Since, according to (A.2), \( dp_\mathcal{J}(e_j) = dp_\mathcal{J}(e_j - e_{j+1}) = (-1)^{j-1} - (-1)^j = (-1)^{j-1} \cdot 2 \in U = \mathbb{C} \), it follows that the vectors

\[
w_j = e_j + e_{j+1} = e_j - e_{j+1} \in U^{2n} = \mathbb{C}^{2n}, \quad j = 1, ..., 2n-2,
\]

form a \( \mathbb{C} \)-basis of the tangent space \( T_0N = \ker(dp_\mathcal{J}) \) and an \( \mathbb{R} \)-basis of \( T_0N \) is given by the vectors \( v_k, \ \kappa = 1, ..., 4n-4 \),

\[
v_k = \begin{cases} 
  w_j & \text{if } k = 2j-1, \\
  iw_j & \text{if } k = 2j,
\end{cases} \quad j = 1, ..., 2n-2.
\]

Let \( (x_1, ..., x_{4n-4}) \) be the system of coordinates on \( N = T_0N \cap \tilde{V} \) given by the \( \mathbb{R} \)-basis \( \{v_k\} \). We shall compute the Hessian matrix of the function \( g \) at 0 w.r.t. these coordinates.

Let \( 1 \leq k \leq l \leq 4n-4 \). We consider the mapping \( g_{k,l} : \mathbb{R}^2 \to \mathbb{R}, \ g_{k,l}(s,t) = g(sv_k + tv_l) \), in order to compute

\[
\frac{\partial^2 g}{\partial x_k \partial x_l}(0,0) = \frac{\partial^2 g_{k,l}}{\partial t \partial s}(0,0).
\]

Let \( \exp : V \to P_{2n} \) be the exponential mapping of the Riemannian manifold \( P_{2n} \) restricted to the normal space \( V \) of the orbit \( \Delta \). We have \( g = F |_N = I \circ \exp |_N \). The mappings \( g \) and \( g_{k,l} \) can be factorized through the mappings \( \tilde{g} = \exp |_N : N \to \mathfrak{su}(2) \) and \( \tilde{g}_{k,l} : \mathbb{R}^2 \to \mathfrak{su}(2), \tilde{g}_{k,l}(s,t) = \tilde{g}(sv_k + tv_l) \) respectively.

We shall first compute \( \frac{\partial^2 g_{k,l}}{\partial t \partial s}(0,0) \in \mathfrak{su}(2) \). Let us recall a result of the Lie theory, [27],

**Theorem A.2.** Let \( H \) be a Lie group and \( \mathfrak{h} \) its Lie algebra. There is a neighbourhood \( W \) of 0 in \( \mathfrak{h} \) such that for \( X, Y \in W \subset \mathfrak{h} \) one has

(i) \( \exp X \cdot \exp Y = \exp (X + Y + \frac{1}{2}[X,Y] + O(\rho^3)) \);

(ii) \( \exp X \cdot \exp (-X) \cdot \exp (-Y) = \exp ([X,Y] + O(\rho^3)) \).

(Here \( \exp \) is the exponential mapping of the Lie group \( H \) and \( \rho \) is a length of the vector \( (X,Y) \) in \( \mathfrak{h} \times \mathfrak{h} \).)
Recall also that for \((u_1, \ldots, u_{2n}) \in V \subset U^{2n}\) we have
\[
\bar{g}(u_1, \ldots, u_{2n}) = \log(p(\exp(u_1, \ldots, u_{2n}))) = \\
= \log(p(e^{u_1}, e^{u_2}, \ldots, e^{u_{2n}}(-J))) = \\
= \log(e^{u_1} \cdot e^{u_2}(-J) \cdots e^{u_{2n}}(-J)) = \\
= \log(e^{u_1} \cdot e^{-u_2} \cdots e^{u_{2n-1}} \cdot e^{-u_{2n}}).
\]
In the following \(r = \sqrt{s^2 + t^2}\). We consider several cases.
(1) If \(k = l = 2j - 1\) then
\[
\bar{g}_{k,l}(s, t) = \bar{g}(sv_k + tv_k) = \bar{g}(s + t)(w_j) = \bar{g}(s + t)(e_j - e_{j+2}) = \\
= \log\left(e^{(-1)^{j-1}(s+t)}, e^{(-1)^{j+1}(s+t)}\right) = \log(I) = \\
= 0.
\]
We get the same result in cases when
(2) \(k = 2j - 1\) and \(l = k + 1, k + 2, k + 4\) or \(l \geq k + 6\),
(3) \(k = l = 2j\),
(4) \(k = 2j\) and \(l \geq k + 4\).
In all those cases one gets, of course, \(\frac{\partial^2 \bar{g}_{k,l}}{\partial t \partial s}(0, 0) = 0\).
(5) If \(k = 2j - 1, l = k + 3\) then
\[
\bar{g}_{k,l}(s, t) = \bar{g}(sv_k + tv_{k+3}) = \bar{g}(s(e_j - e_{j+3}) + t(ie_{j+1} - ie_{j+3})) = \\
= \log\left(e^{(-1)^{j-1}s}, e^{(-1)^{j+3}(s+t)}\right) = \\
= [(-1)^{j-1}s, (-1)^{j+3}(t)] + O(r^3) = \text{(by Theorem A.2 (ii))} \\
= -st[\bar{I}, \bar{i}] + O(r^3) = \\
= -2st\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) + O(r^3)
\]
The last equality follows from \([\bar{I}, \bar{i}] = [\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)] = 2\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)\).
Consequently, \(\frac{\partial^2 \bar{g}_{k,l}}{\partial t \partial s}(0, 0) = -2\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)\) in that case.
(6) If \(k = 2j - 1, l = k + 5\) then
\[
\bar{g}_{k,l}(s, t) = \bar{g}(sv_k + tv_{k+5}) = \bar{g}(s(e_j - e_{j+3}) + t(ie_{j+2} - ie_{j+4})) = \\
= \log\left(e^{(-1)^{j-1}s}, e^{(-1)^{j+4}(-s+ti)}\cdot e^{(-1)^{j+3}(\bar{t})}\right) = \\
= [\text{by applying Theorem A.2 (i) twice }] = \\
= (-1)^{j-1}s + (-1)^{j+1}(-s + ti) + \frac{1}{2}[\bar{s}, (-s + ti)] - (-1)^{j+3}(\bar{t}) + \\
+ \frac{1}{2}[(-1)^{j-1}\bar{s} + (-1)^{j+1}(-s + ti), -(s - t)] + O(r^3) = \\
= \frac{1}{2}[\bar{s}, (\bar{t})] + O(r^3) = \frac{1}{2}st[\bar{I}, \bar{i}] + O(r^3) = \\
= st\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) + O(r^3).
\]
Therefore \(\frac{\partial^2 \bar{g}_{k,l}}{\partial t \partial s}(0, 0) = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)\) in that case.
(7) If \( k = 2j, \ l = k + 1 \) we get
\[
\overline{g}_{k,l}(s,t) = 2st \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + O(r^3),
\]
and, consequently, \( \frac{\partial^2 \overline{g}_{k,l}}{\partial t \partial s}(0,0) = 2 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \). The proof is the same as in the case (5).

Finally,
(8) if \( k = 2j, \ l = k + 3 \) we get
\[
\overline{g}_{k,l}(s,t) = -st \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + O(r^3),
\]
and, hence, \( \frac{\partial^2 \overline{g}_{k,l}}{\partial t \partial s}(0,0) = - \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \). The proof is the same as in the case (6).

The results of the cases (1) - (8) give a proof of

Lemma A.3. If \( 1 \leq k \leq l \leq 4n - 4 \), then
\[
\frac{\partial^2 \overline{g}_{k,l}}{\partial t \partial s}(0,0) = \begin{cases} -2 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \text{for } k = 2j - 1, \ l = k + 3, \\
\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \text{for } k = 2j - 1, \ l = k + 5, \\
2 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \text{for } k = 2j, \ l = k + 1, \\
- \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \text{for } k = 2j, \ l = k + 3, \\
0 & \text{otherwise}. \end{cases}
\]

As an immediate consequence of Lemma A.3 we get

Corollary A.4. The Hessian matrix \( H \) of the function \( g : N \to \mathbb{R} \) at the point \( \vec{0} \) is given by
\[
H_{k,l} = \frac{\partial^2 g}{\partial x_k \partial x_l} (\vec{0}) = \begin{cases} -2 & \text{for } k = \text{odd}, \ l = k + 3, \\
1 & \text{for } k = \text{odd}, \ l = k + 5, \\
2 & \text{for } k = \text{even}, \ l = k + 1, \\
-1 & \text{for } k = \text{even}, \ l = k + 3, \\
0 & \text{otherwise}, \end{cases}
\]
for \( 1 \leq k \leq l \leq 4n - 4 \).

Proof. The function \( g : N \to \mathbb{R} \) is a composition of the mapping \( \overline{g} : N \to \mathfrak{su}(2) \) and of the linear function \( \iota \circ \pi : \mathfrak{su}(2) \to \mathbb{R} \). Since \( \iota \circ \pi \) is linear,
\[
\frac{\partial^2 g}{\partial x_k \partial x_l} (\vec{0}) = (\iota \circ \pi) \left( \frac{\partial^2 \overline{g}}{\partial x_k \partial x_l} (\vec{0}) \right) = (\iota \circ \pi) \left( \frac{\partial^2 \overline{g}_{k,l}}{\partial t \partial s}(0,0) \right).
\]
Corollary follows now from Lemma A.3 since \( (\iota \circ \pi) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = 1 \).

Thus the (symmetric) Hessian matrix \( H \) of \( g \) (and hence of \( f \)) at \( \vec{0} \) is a \((4n - 4) \times (4n - 4)\)-matrix in a block form
\[
H = \begin{pmatrix}
C & A \\
B & C & A \\
& B & C & A \\
& & B \\
& & & C & A \\
& & & B & C
\end{pmatrix}.
\]
with all blocks of the size $4 \times 4$ and of the form

$$C = \begin{pmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 \\ -2 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = A^T. \quad (A.7)$$

Recall that the signature of a real symmetric matrix is defined as the number of its positive eigenvalues minus the number of the negative eigenvalues.

**Proposition A.5.** The Hessian matrix $H$ is invertible and its signature is equal to 0.

**Proof.** Let $P$ be the $(4n - 4) \times (4n - 4)$ permutation matrix of the permutation exchanging the first coordinate with the second one, the third coordinate with the fourth one, and so on, i.e. of the permutation $((1, 2)(3, 4)\ldots(4n - 5, 4n - 4))$. One checks directly that $P^{-1}HP = PHP = -H$. This implies that the number of positive eigenvalues of $H$ is equal to the number of its negative eigenvalues. Thus the signature of $H$ is equal to 0.

It remains to show that the matrix $H$ is invertible.

It follows from Corollary A.4 that $H_{k,l} \neq 0$ implies that $k - l$ is odd.

Let us consider the matrix $\tilde{H} = PH$ which is obtained from $H$ by exchanging the first row with the second one, the third row with the fourth one, and so on. Then $\tilde{H}_{k,l} \neq 0$ implies that $k - l$ is even. The matrix $\tilde{H}$ is skew-symmetric. Indeed, $\tilde{H}^T = (PH)^T = H^TP^T = HP = P(P^{-1}HP) = P(-H) = -\tilde{H}$. It has the block form

$$\tilde{H} = \begin{pmatrix} \tilde{C} & \tilde{A} & \tilde{B} & \tilde{C} \tilde{A} \\ \tilde{B} & \tilde{C} & \tilde{A} & \tilde{B} \\ \tilde{B} & \tilde{C} & \tilde{A} & \tilde{B} \end{pmatrix}$$

with $\tilde{C} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \\ -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$, $\tilde{A} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \end{pmatrix}$ and $\tilde{B} = -\tilde{A}^T$. \quad (A.9)

Denote by $H'$ the $(2n - 2) \times (2n - 2)$-matrix obtained from $\tilde{H}$ by removing all rows and all columns with an even index. Similarly, denote by $H''$ the $(2n - 2) \times (2n - 2)$-matrix obtained from $\tilde{H}$ by removing all rows and all columns with an odd index. Then

$$\det(H) = \det(\tilde{H}) = \det(H') \cdot \det(H'') \quad (A.10)$$

and both $H'$ and $H''$ are skew-symmetric matrices. Moreover, on inspection of (A.9) we see that $H'' = -H'$. Thus $\det(H') = \det(H'')$ and $\det(H) = \det(H')^2$. 


The skew-symmetric $(2n - 2) \times (2n - 2)$-matrix $H' = H'(n)$ has the form

$$H'(n) = \begin{pmatrix}
0 & 2 & -1 \\ -2 & 0 & -2 & 1 \\ 1 & 2 & 0 & 2 & -1 \\ -1 & -2 & 0 & -2 & 1 \\
\end{pmatrix}.$$

Furthermore, $\det(H'(n)) = (Pf(H'(n)))^2$, where $Pf(H'(n))$ is the Pfaffian of $H'(n)$, see [1], Thm.3.27.

If $A$ is a skew-symmetric matrix let us denote by $A_{i\bar{i}}$, $i \geq 2$, the matrix obtained from $A$ by removing the first and the $i$-th row and the first and the $i$-th column. Observe that $H'(n + 1)_{i \bar{i}} = H'(n)$.

Let us denote $Pf(n) = Pf(H'(n))$, $n \geq 2$. We have

**Lemma A.6.** $Pf(n + 2) = 2Pf(n + 1) + Pf(n)$ for $n \geq 2$.

**Proof.** If $A = (a_{ij})$ is a skew-symmetric $2m \times 2m$-matrix then

$$Pf(A) = \sum_{i=2}^{2m} (-1)^{i}a_{1i}Pf(A_{i\bar{i}}),$$

see [1], p.142. Applying this to the matrix $H'(n + 2)$ and using $H'(n + 2)_{i \bar{i}} = H'(n + 1)$ we get

$$Pf(H'(n + 2)) = 2Pf(H'(n + 1)) + Pf(H'(n + 2)_{i \bar{i}}),$$

where

$$H'(n + 2)_{i \bar{i}} = \begin{pmatrix}
0 & 1 & 0 & 0 \\ -1 & 0 & -2 & 1 \\ 0 & 2 & 0 & 2 & -1 \\ 0 & -1 & -2 & 0 & -2 \\
1 & 2 \\
\end{pmatrix}.$$}

Observe that $(H'(n + 2)_{i \bar{i}})_{i \bar{i}} = H'(n)$. Using (A.12) again we get

$$Pf(H'(n + 2)_{i \bar{i}}) = Pf((H'(n + 2)_{i \bar{i}})_{i \bar{i}}) = Pf(H'(n)),$$

and, finally,

$$Pf(H'(n + 2)) = 2Pf(H'(n + 1)) + Pf(H'(n)),$$

as claimed in the Lemma. $\square$

Now, if $A = (a_{ij})$ is a skew-symmetric $2m \times 2m$-matrix then

for $m = 1$, $Pf(A) = a_{12}$;

for $m = 2$, $Pf(A) = a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23}$,

see [1], p.142.
Proof. Proposition A.5 and Lemma A.1 imply that the function with the section of the bundle given by the vertex of the cone. 

\[ H'(2) = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \quad \text{and} \quad H'(3) = \begin{pmatrix} 0 & 2 & -1 & 0 \\ -2 & 0 & -2 & 1 \\ 1 & 2 & 0 & 2 \\ 0 & -1 & -2 & 0 \end{pmatrix}, \]

we get \( Pf(H'(2)) = 2 \) and \( Pf(H'(3)) = 5 \).

It follows now from Lemma A.6, by induction on \( n \), that \( Pf(H'(n)) > 0 \) for all \( n \geq 2 \). Consequently, \( \det(H'(n)) = (Pf(H'(n)))^2 > 0 \) and 

\[ \det(H) = (\det(H'(n)))^2 > 0 \quad \text{for all} \quad n \geq 2. \]

The Hessian matrix \( H \) is therefore invertible. That completes the proof of Proposition A.5. \( \square \)

Let us recall that \( p : P_{2n} \to SU(2) \) is the mapping \( p(A_1, \ldots, A_{2n}) = A_1 \cdots A_{2n} \), that \( K_{2n} = p^{-1}(I) \subset P_{2n} \) and that \( \Delta = \{(A, -A, A, -A, \ldots, A, -A) \in P_{2n} \mid A \in SU(2), Tr(A) = 0 \} \subset K_{2n} \).

**Theorem A.7.** There is a neighbourhood \( U \) of \( \Delta \) in \( K_{2n} \) which is homeomorphic to the bundle \( G \times_{T} C(S^{2n-3} \times S^{2n-3}) \), where \( C(S^{2n-3} \times S^{2n-3}) \) is the cone over the product of two \((2n-3)\)-dimensional spheres. The homeomorphism identifies \( \Delta \) with the section of the bundle given by the vertex of the cone.

Proof. Proposition A.5 and Lemma A.1 imply that the function \( f : M \to \mathbb{R} \) has at \( 0 \in M \) a non-degenerate critical point of index equal to \( \frac{1}{2} \dim M = 2n - 2 \). It follows that there is a \( T \)-equivariant neighbourhood \( W \) of the point \( 0 \) in \( M \) such that the intersection of the level set \( f^{-1}(0) \) with \( W \) is \( T \)-homeomorphic to the cone \( C(S^{2n-3} \times S^{2n-3}) \) over the product \( S^{2n-3} \times S^{2n-3} \) of two spheres of dimension \( 2n - 3 \).

Recall that the level set \( f^{-1}(0) \) is equal to \( \tilde{p}^{-1}(I) \cap \tilde{V} \), where \( \tilde{p} : V \to SU(2) \) is the restriction of the mapping \( p : P_{2n} \to SU(2) \) to the normal subspace of the orbit \( \Delta \) in \( P_{2n} \) and \( \tilde{V} \) is a small neighbourhood of \( 0 \) in \( V \). (Choosing \( \tilde{V} \) small enough we can assume that \( W = M \).)

In a neighbourhood \( Y \) of \( \Delta \) in \( P_{2n} \), the mapping \( p \) is equivalent to the \( G \)-equivariant extension \( \tilde{p} : G \times_{T} V \to SU(2) \) of the \( T \)-equivariant mapping \( \tilde{p} : V \to SU(2) \). Therefore, \( K_{2n} \cap Y = p^{-1}(I) \cap Y \) is homeomorphic to \( G \times_{T} f^{-1}(0) \) and, hence, to \( G \times_{T} C(S^{2n-3} \times S^{2n-3}) \). Under this homeomorphism, the inclusion of \( \Delta \) into \( p^{-1}(I) \cap Y \) corresponds to the inclusion of the “zero-section” \( \Delta = G \times_{T} \{0\} \cong G/T \) into the bundle \( G \times_{T} C(S^{2n-3} \times S^{2n-3}) \), the point \( 0 \) being the vertex of the cone. Finally, put \( U = K_{2n} \cap Y \). \( \square \)

**APPENDIX B. EVALUATION OF THE FIRST CHERN CLASS I**

In this appendix we shall give a proof of Theorem [6.1]

Recall that in Section [6] for every integer \( k \) such that \( 1 \leq k \leq 2n - 1 \) and for \( \epsilon = \pm 1 \) we have defined the mappings \( \gamma_{k,\epsilon} : S^{2} \to K \) by

\[ \gamma_{k,\epsilon}(A) = (J, J, \ldots, J, A, \epsilon A, J, \ldots, (-1)^{n} \epsilon J), \quad \text{for} \quad A \in S^{2}, \]
where, as in Section 2, $S^2$ has been identified with $S$, $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and the first factor of $A$ on the RHS is in the $k$-th place. In the case when $k = 2n - 1$ the sign $(-1)^n\epsilon$ is placed at the first factor of $J$. The mappings $\gamma_{k,\epsilon}$ are embeddings of $S^2$ into $K$ and, hence, into $\mathcal{M}$ and $P_{2n}$.

Let $[\gamma_{k,\epsilon}] \in \text{H}_2(\mathcal{M};\mathbb{Z})$ be the homology classes represented by the corresponding mappings.

Theorem B.1. For all integers $k$ such that $1 \leq k \leq 2n - 1$ and $\epsilon = \pm 1$, the evaluation of the first Chern class $c_1(\mathcal{M})$ on the homology classes $[\gamma_{k,\epsilon}]$ is equal to 0,

$$
\langle c_1(\mathcal{M}) | [\gamma_{k,\epsilon}] \rangle = 0.
$$

Proof. Let $G_{2n}$ be a product of $2n$ copies of $G$. $P_{2n}$ is a submanifold of $G_{2n}$. The formulas (3.2) and (3.3) make sense and define a smooth 2-form on $G_{2n}$. We denote this form by $\tilde{\omega}_c$, it is an extension of the 2-form $\omega_c$ on $P_{2n}$.

For an integer $k$ such that $1 \leq k \leq 2n - 1$ let $G_{2n}^k$ be the submanifold of $G_{2n}$ defined by

$$
G_{2n}^k = \{(g_1, ..., g_{2n}) \in G_{2n} \mid g_j \in S \text{ for } j \neq k, k + 1\}.
$$

Then $P_{2n}$ is a submanifold of $G_{2n}^k$ and, allowing for an abuse of notation, we consider $\gamma_{k,\epsilon}$ also as an embedding of $S^2$ into $G_{2n}^k$, $\gamma_{k,\epsilon} : S^2 \to G_{2n}^k$.

Let $\xi_{k,\epsilon}$ be the restriction of the tangent bundle $TG_{2n}^k$ to the embedded sphere $\gamma_{k,\epsilon}(S)$, $\xi_{k,\epsilon} = TG_{2n}^k|_{\gamma_{k,\epsilon}(S)}$.

Let $\tilde{\omega}_c$ be the restriction of the 2-form $\tilde{\omega}_c$ to the bundle $\xi_{k,\epsilon}$. The vector bundle $\tau_{k,\epsilon} = TP_{2n} |_{\gamma_{k,\epsilon}(S)}$ is a subbundle of $\xi_{k,\epsilon}$ and the restriction of $\tilde{\omega}_c$ to this subbundle is equal to the restriction of $\omega_c$, $\tilde{\omega}_c |_{\tau_{k,\epsilon}} = \omega_c |_{\tau_{k,\epsilon}}$.

The factorization of $G_{2n}^k$ as the product $G_{2n}^k = \bigoplus_{j=1}^{2n} X_j$ with $X_k = X_{k+1} = G$ and $X_j = S$ for $j \neq k, k + 1$ gives us a splitting of the tangent bundle $TG_{2n}^k$ as a direct sum

$$
TG_{2n}^k = \bigoplus_{j=1}^{2n} \pi_j^*(TX_j),
$$

where $\pi_j : G_{2n}^k \to X_j$ is the projection onto the $j$th factor. When restricted to the embedded sphere $\gamma_{k,\epsilon}(S) \subset G_{2n}^k$ it gives us a corresponding splitting

$$
\xi_{k,\epsilon} = \bigoplus_{j=1}^{2n} \zeta_j,
$$

with

$$
\zeta_j = \pi_j^*(TX_j) |_{\gamma_{k,\epsilon}(S)} = \begin{cases} 
\gamma_{k,\epsilon}(S) \times T_j(S) & \text{if } j \neq k, k + 1, \\
\gamma_{k,\epsilon}(S) \times \mathfrak{g} & \text{if } j = k, k + 1.
\end{cases}
$$

(B.2)

Here the identification of $\zeta_k$ and $\zeta_{k+1}$ with $\gamma_{k,\epsilon}(S) \times \mathfrak{g}$ is done via the identification of $TG$ with $G \times \mathfrak{g}$ through the left translations in $G$. That means that every element of $TG$ is written uniquely in the form $g \cdot v$ with $g \in G$ and $v \in \mathfrak{g}$. (In cases when we shall need to use the identification of $TG$ with $G \times \mathfrak{g}$ through the right translations we shall indicate that.)
Let us extend the invariant inner product $\langle \cdot, \cdot \rangle$ from the Lie algebra $\mathfrak{g}$ to the tangent bundle $TG$ by translation. Let $s$ be a normal vector field to $S$ in $G$ of norm 1 w.r.t. that inner product. One can, for example, take $s(J) = (0, 1, 0) = J \cdot (0, 0, 1)$, extend it to the whole sphere $S$ by conjugation (this will give a well-defined non-vanishing normal field) and then normalise it. The normal field $s$ on $S$ gives us two linearly independent sections $s_1$ and $s_2$ of the bundle $\xi_k$ defined by $s_1(A_1, \ldots, A_{2n}) = s(A_k)$ and the section $s_2$ in $\xi_{k+1}$ defined by $s_2(A_1, \ldots, A_{2n}) = s(A_{k+1})$.

Let $A \in S$ and let $v_i$ and $w_m$ belong to the fibers of the bundles $\xi_i$ and $\xi_m$ over the point $\gamma_k(A)$ respectively, $v_i \in \xi^{-1}_i(\gamma_k(A))$, $w_m \in \xi^{-1}_m(\gamma_k(A))$. We shall now compute the values $\widetilde{\omega}(v_i, w_m)$ of the 2-form $\widetilde{\omega}$ on these elements. As $\widetilde{\omega}$ is anti-symmetric, we can assume that $1 \leq i \leq m \leq 2n$.

Recall that, according to (3.3),

$$\widetilde{\omega} = -\sum_{j=1}^{2n-1} \omega_{[z_1 \ldots z_j | z_{j+1}]} \quad \text{(B.3)}$$

Let $f_{z_1 \ldots z_j} : G_{2n}^k \to G$ be a mapping defined by $f_{z_1 \ldots z_j}(A_1, \ldots, A_{2n}) = A_1 \cdot A_2 \cdot \ldots \cdot A_j$ and let $f_{z_{j+1}} : G_{2n}^k \to G$ be the projection onto the $(j+1)$-st coordinate. For any two tangent vectors $x$ and $y$ in the same fiber of the tangent bundle $TG_{2n}$ we have

$$\omega_{[z_1 \ldots z_j | z_{j+1}]}(x, y) = \Omega((df_{z_1 \ldots z_j}(x), df_{z_{j+1}}(x)), (df_{z_1 \ldots z_j}(y), df_{z_{j+1}}(y)))
= \frac{1}{2} \Omega(\omega(df_{z_1 \ldots z_j}(x)) \cdot \omega(df_{z_{j+1}}(y)) - \omega(df_{z_1 \ldots z_j}(y)) \cdot \omega(df_{z_{j+1}}(x))). \quad \text{(B.4)}$$

**Lemma B.2.** Let $1 \leq i \leq m \leq 2n$.

(i) If $j + 1 \neq m$ then $\omega_{[z_1 \ldots z_j | z_{j+1}]} (v_i, w_m) = 0$.

(ii) If $i = m$ then $\omega_{[z_1 \ldots z_j | z_{j+1}]} (v_i, w_m) = 0$.

**Proof.** Since $df_{z_{j+1}}(v_i) = 0$ unless $j + 1 = i$ and $df_{z_{j+1}}(w_m) = 0$ unless $j + 1 = m$, it follows from (B.4) that $\omega_{[z_1 \ldots z_j | z_{j+1}]} (v_i, w_m) = 0$ for $j + 1 \neq i, m$. If $j + 1 = i$ then $df_{z_{j+1}}(v_i) = df_{z_{j+1}}(w_m) = 0$ and again $\omega_{[z_1 \ldots z_j | z_{j+1}]} (v_i, w_m) = 0$. The claims of the Lemma follow. \hfill \Box

Thus, if $1 \leq i \leq m \leq 2n$, then

$$\widetilde{\omega}(v_i, w_m) = -\omega_{[z_1 \ldots z_{m-1} | z_m]} (v_i, w_m) = -\frac{1}{2} \omega(df_{z_1 \ldots z_{m-1}}(v_i)) \cdot \omega(df_{z_m}(w_m)). \quad \text{(B.5)}$$

Let $U = \{ (0, 0, u) \in \mathfrak{g} \mid u \in \mathfrak{c} \}$. Then $T_J(S) = J \cdot U$.

Let $\chi = \epsilon(-1)^n$ if $k = 2n - 1$ and $\chi = 1$ if $k \neq 2n - 1$ and let $\alpha = \epsilon(-1)^n$ if $k < 2n - 1$ and $\alpha = 1$ if $k = 2n - 1$.

Since $\gamma_{k,\epsilon}(A) = (\chi J, \ldots, J, A, \epsilon A, J, \ldots, \alpha J)$, we have, according to the identifications $\text{B.2}$,

$$v_i = \begin{cases} 
(\gamma_{k,\epsilon}(A), \mu J \cdot v) & \text{with } v \in U \text{ if } i \neq k, k + 1 \\
(\gamma_{k,\epsilon}(A), A \cdot v) & \text{with } v \in \mathfrak{g} \text{ if } i = k \\
(\gamma_{k,\epsilon}(A), \epsilon A \cdot v) & \text{with } v \in \mathfrak{g} \text{ if } i = k + 1,
\end{cases} \quad \text{(B.6)}$$
by
Proof. Observe that \[\langle 1 \rangle \in \mathfrak{g} \] as claimed. The last equality follows by invariance of the inner product \(\cdot\), the previous one from the fact that \(Ad(A) = Ad(-A) = Ad(A^{-1})\) for all \(A \in S\).

Lemma B.3. Let \(1 \leq i \leq m \leq 2n\). Then

\[
\tilde{\omega}_c(v_i, w_m) = (-\frac{1}{2}) \begin{cases} 
0 & \text{if } i = m, \\
-(-1)^{m-i-1} v \cdot (Ad(A)(w)) & \text{if } i < m = k, \\
(Ad(A)(v)) \cdot w & \text{if } i = k, m = k + 1, \\
-(Ad(J^{m-1}) Ad(A)(v)) \cdot w & \text{if } i = k < m - 1, \\
-(Ad(J^{m-1})(v)) \cdot w & \text{otherwise.} 
\end{cases}
\]

Here \(v\) and \(w\) are as in (B.6) resp. (B.7).

Proof. The claim of the Lemma follows from (B.3) by a direct check of nine cases. Observe that \(Ad(J)\) acts on \(U \subset \mathfrak{g}\) by multiplication by \(-1\).

Case 1: if \(i = m\) then \(\tilde{\omega}_c(v_i, w_m) = 0\) by Lemma B.2 (ii).

Case 2: if \(i < m < k\) then \(v_i = (\gamma_{k,i}(A), \mu J \cdot v)\) with \(v \in U \subset \mathfrak{g}\) and \(df_{z_{1} \cdots z_{m-1}}(v_i) = \chi J^{m-1} \cdot (Ad(J^{m-1-i})(v)) = \chi J^{m-1} \cdot (Ad(J^{m-1-i})(v))\). Similarly, \(w_m = (\gamma_{k,i}(A), \lambda J \cdot w)\) with \(w \in U\) and \(df_{z_{m}}(w_m) = \lambda J \cdot w = (Ad(J)(w)) \cdot \lambda J = (-w) \cdot J\). (Here we use the right translation by \(J\), and the fact that \(\lambda = 1\) in this case.) It follows that \(\tilde{\omega}_c(v_i, w_m) = -\frac{1}{2} \omega(df_{z_{1} \cdots z_{m-1}}(v_i)) \cdot \omega(df_{z_{m}}(w_m)) = -\frac{1}{2}(-1)^{m-i-1} v \cdot (-w) = -\frac{1}{2}(-1)(Ad(J^{m-1-i})(v)) \cdot w\), as claimed.

Case 3: if \(i < m = k\) then \(df_{z_{1} \cdots z_{m-1}}(v_i) = \chi J^{m-1} \cdot (Ad(J^{m-1-i})(v)) = \chi J^{m-1} \cdot ((-1)^{m-i-1}) \cdot v\) with \(v \in U\) just as in Case 2, while \(w_m = (\gamma_{k,i}(A), A \cdot w)\) with \(w \in \mathfrak{g}\), \(df_{z_{m}}(w_m) = A \cdot w = (Ad(A)(w)) \cdot A\) (again, here we use the right translation by \(A\)) and \(\tilde{\omega}_c(v_i, w_m) = -\frac{1}{2} \omega(df_{z_{1} \cdots z_{m-1}}(v_i)) \cdot \omega((Ad(A)(w)) \cdot A) = (-\frac{1}{2})(-1)^{m-i-1} v \cdot (Ad(A)(w))\) as claimed.

Case 4: if \(i < k, m = k + 1\) then \(df_{z_{1} \cdots z_{m-1}}(v_i) = \chi J^{m-2} \cdot (Ad(J^{m-2-i})(v))\) with \(v \in U\) and \(df_{z_{m}}(w_m) = \epsilon A \cdot w = (Ad(A)(w)) \cdot (\epsilon A)\) with \(w \in \mathfrak{g}\). Consequently,

\[
\tilde{\omega}_c(v_i, w_m) = (-\frac{1}{2})(Ad(J^{m-2-i})(v)) \cdot (Ad(A)(w)) = \\
= (-\frac{1}{2})(Ad(-A) \cdot Ad(J^{m-2-i})(v)) \cdot (Ad(A)(w)) = \\
= (-\frac{1}{2})(Ad(A) \cdot Ad(J^{m-2-i})(v)) \cdot (Ad(A)(w)) = \\
= (-\frac{1}{2})(-1)(Ad(J^{m-1-i})(v)) \cdot w, 
\]

as claimed. The last equality follows by invariance of the inner product \(\cdot\), the previous one from the fact that \(Ad(A) = Ad(-A) = Ad(A^{-1})\) for all \(A \in S\).
Case 5: if \( i = k, \ m = k + 1 \) then \( df_z_{1,...,z_{m-1}}(v_i) = \chi J^{m-2}A \cdot v \) with \( v \in g \) and \( df_z_{m}(w_m) = \epsilon A \cdot w = (Ad(A)(w)) \cdot (\epsilon A) \) with \( w \in g \). Thus
\[
\bar{\omega}_c(v_i, w_m) = (-\frac{1}{2})\omega(\chi J^{m-2}A \cdot v) \bullet \bar{\omega}(Ad(A)(w)) \cdot (\epsilon A)) = \\
= (-\frac{1}{2})v \bullet Ad(A)(w) = \\
= (-\frac{1}{2})(Ad(A^{-1})(v) \bullet w) \quad \text{(by invariance of } \bullet) \\
= (-\frac{1}{2})(Ad(A)(v) \bullet w) \quad \text{(since } Ad(A^{-1}) = Ad(-A) = Ad(A))
\]
as claimed.

Case 6: if \( i < k < m - 1 \) then \( df_z_{1,...,z_{m-1}}(v_i) = \chi \epsilon J^{m-1} \cdot (Ad(J^{m-1-i})(v)), \ v \in U \) and \( df_z_{m}(w_m) = \lambda J \cdot w = Ad(J)(w) \cdot (\lambda J) = (-w) \cdot (\lambda J), \ w \in U \). Thus
\[
\bar{\omega}_c(v_i, w_m) = (-\frac{1}{2})(Ad(J^{m-1-i})(v)) \bullet (-w) \\
= (-\frac{1}{2})(-1)(Ad(J^{m-2-i} - 1(Ad(A^{-1})(v)) \bullet w = \\
= (-\frac{1}{2})(-1)(Ad(J^{m-2-i})(Ad(A)(v))) \bullet w,
\]
as claimed.

Case 7: if \( i = k < m - 1 \) then \( df_z_{1,...,z_{m-1}}(v_i) = \chi \epsilon J^{m-1} \cdot (Ad(AJ^{m-2-i})(v)), \ v \in g \) and \( df_z_{m}(w_m) = (-w) \cdot (\lambda J), \ w \in U \). Thus
\[
\bar{\omega}_c(v_i, w_m) = (-\frac{1}{2})(Ad(AJ^{m-2-i}-1(v)) \bullet (-w) = \\
= (-\frac{1}{2})(-1)(Ad(J^{m-2-i} - 1(Ad(A^{-1})(v))) \bullet w = \\
= (-\frac{1}{2})(-1)(Ad(J^{m-2-i})(Ad(A)(v))) \bullet w,
\]
as claimed.

Case 8: if \( i = k + 1 < m \) then \( df_z_{1,...,z_{m-1}}(v_i) = \epsilon J^{m-1} \cdot (Ad(J^{m-1-i})(v)), \ v \in g \) and \( df_z_{m}(w_m) = \lambda J \cdot w = (-w) \cdot (\lambda J), \ w \in U \). Therefore
\[
\bar{\omega}_c(v_i, w_m) = (-\frac{1}{2})(Ad(J^{m-1-i})(v)) \bullet (-w),
\]
as claimed.

Case 9: if \( k + 1 < i < m \) then \( df_z_{1,...,z_{m-1}}(v_i) = \epsilon J^{m-1} \cdot (Ad(J^{m-1-i})(v)), \ v \in U \) and \( df_z_{m}(w_m) = \lambda J \cdot w = (-w) \cdot (\lambda J), \ w \in U \). Therefore
\[
\bar{\omega}_c(v_i, w_m) = (-\frac{1}{2})(Ad(J^{m-1-i})(v)) \bullet (-w),
\]
again as claimed.

This concludes the proof of Lemma B.3.

Let \( W \) be the fibre of the vector bundle \( \xi_{k,\epsilon} \) over the point \( \gamma_{k,\epsilon}(J) \). Then \( W = \bigoplus_{j=1}^{2n} V_j \) is the direct sum of real vector spaces
\[
V_j = \begin{cases} 
T_j(S) & j \neq k, k + 1, \\
g & j = k, k + 1.
\end{cases} \quad (B.8)
\]
The identities (B.1) and (B.2) yield a trivialization of the bundle \( \xi_{k,\epsilon} \)
\[
\psi_{k,\epsilon}: \xi_{k,\epsilon} \xrightarrow{\cong} \gamma_{k,\epsilon}(S) \times W
\]
which is the identity on the fiber over $\gamma_{k,\epsilon}(J)$.

Let $\Phi_k$ be the automorphism of the bundle $\gamma_{k,\epsilon}(S) \times W$ given by

$$\Phi_k(\gamma_{k,\epsilon}(A), \bigoplus_{j=1}^{2n} v_j) = (\gamma_{k,\epsilon}(A), \bigoplus_{j=1}^{2n} \phi_k(A, v_j))$$

with

$$\phi_k(A, v_j) = \begin{cases} v_j & \text{if } j \neq k, \\ Ad(-JA)(v_k) & \text{if } j = k, \end{cases}$$

for $A \in S$, $v_j \in V_j$.

Denote by $\Phi_k$ the corresponding automorphism of the bundle $\xi_{k,\epsilon}$. Denote by $\omega_0$ its restriction to the fiber $W$ over the point $\gamma_{k,\epsilon}(J)$. The form $\omega_0$ gives us the product form $\tilde{\omega}_0$ on the trivial bundle $\gamma_{k,\epsilon}(S) \times W$ and, via the isomorphism $\psi_{k,\epsilon}$, a 2-form $\tilde{\omega}_0$ on the bundle $\xi_{k,\epsilon}$.

Let $(\xi_{k,\epsilon}, \tilde{\omega}_c)$ and $(\xi_{k,\epsilon}, \tilde{\omega}_0)$ denote the real vector bundle $\xi_{k,\epsilon}$ equipped with the 2-forms $\tilde{\omega}_c$ and $\tilde{\omega}_0$ respectively.

**Lemma B.4.** (i) The automorphism $\Phi_k : \xi_{k,\epsilon} \to \xi_{k,\epsilon}$ of the real vector bundle $\xi_{k,\epsilon}$ satisfies $\Phi_k^*(\tilde{\omega}_0) = \tilde{\omega}_c$.

(ii) $\tilde{\omega}_0$ is a non-degenerate 2-form on $\xi_{k,\epsilon}$.

(iii) $\tilde{\omega}_c$ is a non-degenerate 2-form on $\xi_{k,\epsilon}$.

**Proof.** (i) Observe that when $A = J$ the expressions of Lemma B.3 give us a formula for the 2-form $\omega_0$. Moreover, Lemma B.3 shows that for any $A \in S$ and $v, w \in \xi_{k,\epsilon}^{-1}(\gamma_{k,\epsilon}(A))$ one has

$$\tilde{\omega}_c(v, w) = \tilde{\omega}_0(\Phi_k(v), \Phi_k(w)).$$

This shows that $\Phi_k^*(\tilde{\omega}_0) = \tilde{\omega}_c$.

(ii) The vector bundle $\xi_{k,\epsilon}$ contains as a subbundle of real codimension 2 the restriction $\tau_{k,\epsilon} = TP_{2n} |_{\gamma_{k,\epsilon}(S)}$ of the tangent bundle $TP_{2n}$ of $P_{2n}$ to the sphere $\gamma_{k,\epsilon}(S)$. The restriction of the 2-form $\tilde{\omega}_c$ to $\tau_{k,\epsilon}$ is equal to the restriction of the symplectic form $\omega_c$ to $\tau_{k,\epsilon}$ and, hence, is non-degenerate by Theorem 3.1 and Corollary 3.4.

Let $s_1(\gamma_{k,\epsilon}(J))$, $s_2(\gamma_{k,\epsilon}(J)) \in W$ be the values of the sections $s_1$, $s_2$ at the point $\gamma_{k,\epsilon}(J)$. In the decomposition (B.8) the value $x_1 = s_1(\gamma_{k,\epsilon}(J))$ is contained in $V_k = g$ while $x_2 = s_2(\gamma_{k,\epsilon}(J))$ is contained in $V_{k+1} = g$ and both are orthogonal to $U \subset g$ w.r.t. the inner product $\cdot$.

All the elements of the fibre of $\tau_{k,\epsilon}$ over the point $\gamma_{k,\epsilon}(J)$ belong to $\bigoplus_{j=1}^{2n} U$ in the decomposition (B.8). Moreover, the subspace $U \subset g$ is invariant under the action of $Ad(J)$ and is orthogonal to $s(J)$ w.r.t. the inner product $\cdot$. It follows then from the formulas of Lemma B.3 that $x_1$ and $x_2$ are orthogonal w.r.t. the 2-form $\tilde{\omega}_c$ (and hence w.r.t. the 2-form $\tilde{\omega}_0$) to the fibre of $\tau_{k,\epsilon}$ over $\gamma_{k,\epsilon}(J)$. Furthermore, again from Lemma B.3 case $i = k$, $m = k + 1$, we have

$$\tilde{\omega}_0(x_1, x_2) = \tilde{\omega}_c(x_1, x_2) = (Ad(J)(x_1)) \cdot x_2.$$
Let us choose complex structures on \( S \) and on \( \eta \) compatible with their symplectic forms. The isomorphism (B.10) equips then \( \xi_{k,\epsilon} \) with a complex structure compatible with the 2-form \( \tilde{\omega}_c \) and

\[
c_1(\xi_{k,\epsilon}, \tilde{\omega}_c) = c_1(\xi_{k,\epsilon}) = c_1(\tau_{k,\epsilon}) + c_1(\eta). \tag{B.11}
\]

On the other hand the values of the section \( s_1 \) of \( \xi_{k,\epsilon} \) never lie in \( \tau_{k,\epsilon} \). Therefore the decomposition of \( s_1 \) in the direct sum (B.10) gives us a nowhere vanishing section \( \tilde{s} \) of \( \eta \). Since \( \eta \) is a 1-dimensional complex bundle it follows that as a complex bundle \( \eta \) is trivial and \( c_1(\eta) = 0 \). Therefore, from (B.9) and (B.11) we get

\[
c_1(\tau_{k,\epsilon}) = c_1(\xi_{k,\epsilon}, \tilde{\omega}_c) - c_1(\eta) = 0.
\]

As \( \langle c_1 | [\gamma_{k,\epsilon}] \rangle = c_1(\tau_{k,\epsilon}) \), that proves Theorem B.1. \( \square \)

The subspace \( K_{2n} \) of \( P_{2n} \) is \( G \)-invariant, thus \( G = SU(2) \) acts on \( K_{2n} \). Let us consider the projection onto the orbit space \( q : K_{2n} \to K_{2n}/G \).

**Lemma B.5.** Let \( n \geq 2 \). The composed mappings \( q \circ \gamma_{k,\epsilon} : S^2 \to K_{2n}/G \) are contractible for all \( 1 \leq k \leq 2n-1 \) and \( \epsilon = \pm 1 \).

**Proof.** The mapping \( \gamma_{k,\epsilon} \) is an embedding of the 2-dimensional sphere \( S^2 \) into \( K_{2n} \) with the image

\[
\gamma_{k,\epsilon}(S^2) = \{ (J, ..., J, A, \epsilon A, J, ..., (-1)^n \epsilon J) \in P_{2n} \mid A \in S \}.
\]

Let \( T \) be the 1-dimensional torus in \( SU(2) \) which is the isotropy subgroup of the point \( J \) in \( G \) under the conjugacy action of \( G \) on itself. \( T \) consists of the
diagonal matrices in $SU(2)$. The torus $T$ is the largest subgroup of $G$ keeping the subset $\gamma_{k,\epsilon}(S^2)$ invariant and no element of $G - T$ maps any point of $\gamma_{k,\epsilon}(S^2)$ into $\gamma_{k,\epsilon}(S^2)$. Hence, the image of $\gamma_{k,\epsilon}(S^2)$ in $K_{2n}/G$ under the quotient projection $q$ is equal to $\gamma_{k,\epsilon}(S^2)/T$. But, $\gamma_{k,\epsilon}(S^2)/T$ is homeomorphic to $S/T$. The space $S/T$ is homeomorphic to an interval and hence is contractible. Thus the image of $\gamma_{k,\epsilon}(S^2)$ under $q$ is a contractible subspace of $K_{2n}/G$. Consequently, the composition $q \circ \gamma_{k,\epsilon} : S^2 \to K_{2n}/G$ is a contractible mapping.

\[ \square \]

**APPENDIX C. Evaluation of the first Chern class II**

Let $\tilde{S}$ be the smooth manifold diffeomorphic to the 2-dimensional sphere defined in Section 6 (see Figure 6.1). Let $f = f_n : \tilde{S} \to K_{2n}$ be the mapping defined by (6.2) and (6.3).

Consider the homomorphism $f_* : H_2(\tilde{S}, \mathbb{Z}) \to H_2(K_{2n}, \mathbb{Z})$ induced by $f$ on the second homology groups. The orientation of $\tilde{S}$ gives us $[\tilde{S}] \in H_2(\tilde{S}, \mathbb{Z})$. Let us consider the element $f_*([\tilde{S}]) \in H_2(K_{2n}, \mathbb{Z})$.

Let $\mathcal{M} \subset P_{2n}$ be the open neighbourhood of $K_{2n}$ given by Theorem 3.1. $\mathcal{M}$ is a symplectic manifold with the symplectic form $\omega_{\mathcal{C}}$. We assume that $\mathcal{M}$ has been chosen so that it is homotopy equivalent to $K_{2n}$. Let $c_1(\mathcal{M}) \in H^2(\mathcal{M}, \mathbb{Z}) = H^2(K_{2n}, \mathbb{Z})$ be the first Chern class of the symplectic structure on $\mathcal{M}$.

**Theorem C.1.** For every $n \geq 2$

$$ (c_1(\mathcal{M}) | f_*[\tilde{S}]) = -2. $$

**Proof.** Let us denote by $K'_4$ the subspace of $K_{2n}$ consisting of points $(A_1, ..., A_{2n}) \in K_{2n}$ such that $A_j = (-1)^j J$ for $j \geq 5$. Observe that such points satisfy the condition

$$ A_1 \cdot A_2 \cdot A_3 \cdot A_4 = I. \quad (C.1) $$

By the definition of the mapping $f$, its image is contained in $K'_4$.

Let $\xi$ be the restriction of the tangent bundle of $P_{2n}$ to $K'_4$, $\xi = TP_{2n}|_{K'_4} = T\mathcal{M}|_{K'_4}$. It is a symplectic vector bundle over $K'_4$ with the symplectic form $\omega_{\mathcal{C}} = \omega_{\mathcal{C}}$ (see Corollary 3.4). The product decomposition of $P_{2n} = \prod S$ gives us a canonical identification of vector bundles

$$ \xi \cong \bigoplus_{j=1}^{2n} p_j^*(TS)|_{K'_4}. \quad (C.2) $$

We denote by $dp_j : \xi \to p_j^*(TS)|_{K'_4}$ the projection onto the $j$-th summand.

Let us consider the vector subbundle $\xi_1 = \bigoplus_{j=5}^{2n} p_j^*(TS)|_{K'_4}$ of $\xi$. Since $p_j(x) = (-1)^j J$ for every $j \geq 5$ and $x \in K'_4$, we have a trivialization

$$ \Phi : \xi_1 \to K'_4 \times \bigoplus_{j=5}^{2n} V_j, \quad (C.3) $$

where $V_j = T_{(-1)^j J}S$ is the tangent space to $S$ at the point $(-1)^j J$.

Let us denote $V = \bigoplus_{j=5}^{2n} V_j$. We identify $\xi_1$ with $K'_4 \times V$ through $\Phi$. 
Lemma C.2. The restriction of $\omega_c$ to $\xi_1$ is a non-degenerate 2-form on $\xi_1$ and it does not depend on the first coordinate $x \in K'_4$ in $\xi_1 = K'_4 \times V$.

Proof. There is an obvious identification of the vector space $V$ with the tangent space $T_Y P_{2n-4}$ to $P_{2n-4}$ at the point $Y = (-J, J, -J, ..., -J, J) \in P_{2n-4}$. It follows from the definition (3.3) of the 2-form $\omega_c = -\sum \omega_{[z_1, ..., z_j, z_{j+1}]}$ and from the identity (C.1) that, for every point $x \in K'_4$, this identification maps the restriction of $\omega_c$ to the fiber of $\xi_1$ over $x$ to the 2-form $\omega_c$ on $T_Y P_{2n-4}$. Therefore the restriction of $\omega_c$ to $\xi_1$ does not depend on $x \in K'_4$. Moreover, since the 2-form $\omega_c$ on $T_Y P_{2n-4}$ is non-degenerate (Theorem 3.1), so is the restriction of $\omega_c$ to $\xi_1$. □

It follows that $\xi_1$ is a symplectic subbundle of $\xi$ and, as a symplectic bundle, $\xi_1$ is trivial. Therefore

$$c_1(\xi_1) = 0.$$  \hspace{1cm} (C.4)

Let $\xi_2$ be the symplectic orthogonal complement of $\xi_1$ in $\xi$. Thus $\xi_2$ is a symplectic subbundle of $\xi$, of real dimension 8, and

$$\xi = \xi_2 \oplus \xi_1$$  \hspace{1cm} (C.5)

as symplectic bundles. Therefore

$$c_1(\xi) = c_1(\xi_2) + c_1(\xi_1) = c_1(\xi_2).$$  \hspace{1cm} (C.6)

Let $\zeta$ be the vector subbundle of $\xi$ given by

$$\zeta = \bigoplus_{j=1}^4 p_j^*(TS)|_{K'_4}$$  \hspace{1cm} (C.7)

in the decomposition (C.2). Then $\xi = \zeta \oplus \xi_1$ as vector bundles. Let $\varphi : \zeta \to \xi_2$ be the projection of $\zeta$ onto $\xi_2$ in the decomposition (C.5). It is an isomorphism of vector bundles. See Figure C.1.

We shall now describe the isomorphism $\varphi$ explicitly. Let $U$ be the subspace of $g = su(2)$ consisting of matrices of the form $\begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}$, $u \in \mathbb{C}$, and let $\text{proj} : g \to U$ be the orthogonal projection of $g$ onto $U$ w.r.t. the inner product $\cdot$. As $A \cdot B = \frac{-1}{2} \alpha \text{tr}(AB)$ for some $\alpha \in \mathbb{R}$, $\alpha > 0$, we see that $\text{proj}(\begin{pmatrix} 0 & z \\ -\bar{z} & -t \end{pmatrix}) = \begin{pmatrix} 0 & \bar{z} \\ z & 0 \end{pmatrix}$ for $t \in \mathbb{R}$, $z \in \mathbb{C}$.

![Fig. C.1](image-url)
Let \( x = (A_1, A_2, A_3, A_4, -J, J, ..., -J, J) \) be an arbitrary point in \( K'_4 \), with \( A_i \in S \), \( A_1A_2A_3A_4 = I \), and let

\[
v = v_1 \oplus v_2 \oplus v_3 \oplus v_4 \in \zeta_x
\]

be an arbitrary vector in the fiber \( \zeta_x \) of \( \zeta \) over \( x \). Here \( v_j \in T_{A_j}S \) and therefore \( v_j = v_j \cdot A_j \) for some \( v_j \in g \), \( j = 1, ..., 4 \).

Since \( \varphi : \zeta \to \xi_2 \subset \xi = 2n \bigoplus_{j=1}^{2n} p_j^*(TS)|_{K'_4} \), we have

\[
\varphi(v) = \varphi(v_1 \oplus v_2 \oplus v_3 \oplus v_4) = \bigoplus_{j=1}^{2n} w_j,
\]

with \( w_j = w_j \cdot A_j \in T_{A_j}S \), some \( w_j \in g \), \( j = 1, ..., 2n \) (and where \( A_j = (-1)^j J \) for \( j \geq 5 \)).

The explicit description of the isomorphism \( \varphi \) is given by

**Lemma C.3.**

\[
w_j = \begin{cases} 
    v_j & \text{if } j = 1, ..., 4, \\
    -\sum_{i=1}^{4} \text{proj}(\text{Ad}(A_4A_3...A_i)(v_i)) & \text{if } j = 5, ..., 2n.
\end{cases}
\]

**Remark C.4.** Observe that \( w_5 = w_6 = ... = w_{2n} \in U \).

**Proof.** Let \( w_j \in g \) be given by the formulas of Lemma C.3. Since \( w_5, w_6, ..., w_{2n} \in U \), we have \( w_j = w_j \cdot (-1)^j J \in T_{(-1)^j J}(S) \) for \( j = 5, ..., 2n \). Therefore \( \bigoplus_{j=5}^{2n} w_j \in \xi_1 \).

For \( i = 1, ..., 4 \) we have \( w_i = v_i \) and, hence, \( w_i = w_i \cdot A_i = v_i \cdot A_i = v_i \in T_{A_i}(S) \).

It follows that

\[
\bigoplus_{j=1}^{2n} w_j - v = \bigoplus_{j=1}^{2n} w_j - \bigoplus_{i=1}^{4} v_i = \bigoplus_{j=5}^{2n} w_j \in \xi_1.
\]

To prove that \( \varphi(v) = \bigoplus_{j=1}^{2n} w_j \) it is therefore enough to show that \( \bigoplus_{j=1}^{2n} w_j \in \xi_2 \)

i.e. to show that \( \bigoplus_{j=1}^{2n} w_j \) is symplectic orthogonal to \( \xi_1 \) w.r.t. the 2-form \( \omega_c \).
Let \( u \in U \) and let \( u_k = u \cdot (-1)^k J \in \mathcal{T}_{A_k}(S) \), \( k = 5, \ldots, 2n \). Then, for \( i = 1, \ldots, 4 \),

\[
\omega_c(v_i, u_k) = -\omega|_{z_1 \ldots z_{k-1} \mid z_k}(v_i, u_k) = -\frac{1}{2} \omega(dz_{z_1 \ldots z_{k-1}}(v_i)) \cdot \omega(f_k(u_k)) = -\frac{1}{2} \omega(A_1 \ldots A_{i-1} \cdot v_i \cdot A_i \ldots A_4(-1)^{k-1}J \cdot (-1)^{k-1}J \cdot Ad((A_i \ldots A_4J^{j-5})^{-1})(v_i)) = -\frac{1}{2} \omega(A_1 \ldots A_4(-1)^k J \cdot Ad((A_i \ldots A_4J^{j-5})^{-1})(v_i)) \cdot \omega(u \cdot (-1)^{k}J) = -\frac{1}{2} \omega(A_1 \ldots A_4(v_i)) \cdot Ad(J^{j-5})(u) = (-1)^k \frac{1}{2} \omega(A_1 \ldots A_4(v_i)) \cdot u = (-1)^k \frac{1}{2} \omega(u \cdot (-1)^{k}J) = (-1)^k \frac{1}{2} \omega(u \cdot (-1)^{k}J)\]

(C.8)

Furthermore, for \( 5 \leq j \leq k \) and \( w_j = w_j \cdot A_j = w_j \cdot (-1)^j J \),

\[
\omega_c(w_j, u_k) = -\omega|_{z_1 \ldots z_{k-1} \mid z_k}(w_j, u_k) = \begin{cases} 
-\frac{1}{2} \omega(dz_{z_1 \ldots z_{k-1}}(w_j)) \cdot \omega(f_k(u_k)) & \text{if } 5 \leq j < k, \\
0 & \text{if } j = k,
\end{cases}
\]

\[
= \begin{cases} 
-\frac{1}{2} \omega(A_1 \ldots A_4(-1)^{k-1}J \cdot Ad(J^{j-5})(w_j)) \cdot \omega(u \cdot (-1)^{k}J) & \text{if } 5 \leq j < k, \\
0 & \text{if } j = k,
\end{cases}
\]

\[
= \begin{cases} 
\frac{1}{2} Ad(J^{j-5})(w_j) \cdot u & \text{if } 5 \leq j < k, \\
0 & \text{if } j = k,
\end{cases}
\]

\[
= \begin{cases} 
\frac{1}{2} (-1)^{k-j-1} w_j \cdot u & \text{if } 5 \leq j < k, \\
0 & \text{if } j = k.
\end{cases}
\]

Exchanging the roles of \( w_j \) and \( u_k \) we get, for \( 5 \leq j, k \leq 2n \) and \( w_j = w_j \cdot (-1)^j J \),

\[
\omega_c(w_j, u_k) = \begin{cases} 
\frac{1}{2} (-1)^{k-j-1} w_j \cdot u & \text{if } j < k, \\
0 & \text{if } j = k,
\end{cases}
\]

\[
= \begin{cases} 
\frac{1}{2} (-1)^{k-j} w_j \cdot u & \text{if } j < k.
\end{cases}
\]

Hence, with \( w_5 = w_6 = \ldots = w_{2n} \), we have

\[
\omega_c(\sum_{j=5}^{2n} w_j, u_k) = \sum_{j=5}^{2n} \omega_c(w_j, u_k) = \frac{1}{2} \sum_{j=5}^{k-1} (-1)^{k-j-1} \omega_c(w_5, u_k) = \frac{1}{2} (-1)^{k} w_5 \cdot u.
\]

(C.9)
Corollary C.5. There exists an isotropic subbundle \( \eta \) of \( \xi_1 \) such that for every \( v \in \zeta \)

\[ \varphi(v) - v \in \eta. \]

Proof. Let \( \eta = \{ \sum_{j=5}^{2n} w_j \in \xi_1 \mid w_j = w_j \cdot (-1)^j J \text{ with } w_5 = w_6 = \ldots = w_{2n} \in U \} \).

Then, according to Lemma C.3 and Remark C.4

\[ \varphi(v) - v \in \eta \]

for all \( v \in \zeta \). Furthermore, if \( \sum_{j=5}^{2n} w_j, \sum_{k=5}^{u_k} u_k \in \eta \) with \( w_j = w_j \cdot (-1)^j J, u_k = u_k \cdot (-1)^k J \) and \( w_j, u_k \in U \) then \( w_5 = \ldots = w_{2n}, u_5 = \ldots = u_{2n} \) and, as follows from (C.7)

\[ \omega_c \sum_{j=5}^{2n} w_j, \sum_{k=5}^{u_k} u_k \] = \[ \sum_{j=5}^{2n} \omega_c \left( \sum_{j=5}^{2n} w_j, u_k \right) \]
= \[ \frac{1}{2} \left( \sum_{k=5}^{2n} (-1)^k \right) (w_5 \bullet u_5) \]
= \[ 0. \]

Thus \( \eta \) is an isotropic subbundle of \( \xi_1 \).

Corollary C.6. The vector bundle isomorphism \( \varphi : \zeta \to \xi_2 \) satisfies

\[ \omega_c (\varphi(v), \varphi(w)) = \omega_c (v, w) \]

for all \( v, w \in \zeta_\pi \) and all \( x \in K'_4 \). Thus \( \zeta \) and \( \xi_2 \) are isomorphic symplectic bundles.

Proof. Let \( x \in K'_4 \) and let \( v, w \in \zeta_\pi \). Let \( \eta \) be the isotropic subbundle of Corollary C.5. Then \( \varphi(v) - v \in \eta_\pi, \varphi(w) - w \in \eta_\pi \) and, therefore,

\[ \omega_c (v - \varphi(v), w - \varphi(w)) = 0. \]

Moreover,

\[ \omega_c (v - \varphi(v), \varphi(w)) = 0 = \omega_c (\varphi(v), w - \varphi(w)) \]
since $\varphi(v), \varphi(w) \in \xi_2$, and $v - \varphi(v), w - \varphi(w) \in \xi_1$, and $\xi_2$ and $\xi_1$ are symplectic orthogonal w.r.t. $\omega_c$ to each other by the definition of $\xi_2$.

Consequently,

$$\omega_c(v, w) = \omega_c((v - \varphi(v)) + \varphi(v), (w - \varphi(w)) + \varphi(w)) =$$

$$= \omega_c(v - \varphi(v), w - \varphi(w)) + \omega_c(v - \varphi(v), \varphi(w)) +$$

$$+ \omega_c(\varphi(v), w - \varphi(w)) + \omega_c(\varphi(v), \varphi(w)) =$$

$$= \omega_c(\varphi(v), \varphi(w)).$$

□

Corollary C.6 implies that $c_1(\xi_2) = c_1(\zeta)$ and, following (C.6), that $c_1(\xi_3) = c_1(\xi_2) = c_1(\zeta)$. (C.10)

We identify the space $K_4$ with $K_4'$ by identifying points $(A_1, A_2, A_3, A_4) \in K_4$ with points $(A_1, A_2, A_3, A_4, -J, J, ..., -J, J) \in K_4'$. The bundle $\zeta$ over $K_4'$ is then identified with the restriction $TP_4|_{K_4}$ of the tangent bundle of $P_4$ to $K_4$. We denote even this bundle by $\zeta$.

Recall that the mapping $f: \tilde{S} \to K_{2n}$ factorizes as

$$\tilde{S} \xrightarrow{f} K_{2n}$$

$$f\downarrow$$

$$K_4 \xrightarrow{=} K_4'$$

We shall now study the symplectic bundle $f^*(\zeta)$ over $\tilde{S}$.

Let $B$ be a 1-dimensional manifold diffeomorphic to a circle and described as a union of three subspaces $B_1, B_2$ and $B_3$, as in Fig.C.2

with $B_1$ and $B_2$ being arcs and $B_3$ being a disjoint union of two intervals. We think of $B_1$ as the interval $\{1\} \times [0, \pi]$, of $B_2$ as the interval $\{2\} \times [\pi, 2\pi]$ and of $B_3$ as the product $[0, \pi] \times \{\pm 1\}$ and we identify $(1, 0) \in B_1 = \{1\} \times [0, \pi]$ with $(0, 1) \in B_3$, $(1, \pi) \in B_1 = \{1\} \times [0, \pi]$ with $(0, -1) \in B_3$, $(2, \pi) \in B_2 = \{2\} \times [\pi, 2\pi]$ with $(\pi, -1) \in B_3$ and $(2, 2\pi) \in B_2$ with $(\pi, 1) \in B_3$, see Fig. C.2.

Let $D = B \times I$, $I = [0, \pi]$, and $D_j = B_j \times I$, $j = 1, 2, 3$, see Fig. C.3.
$D$ is a 2-manifold with the boundary consisting of two circles $C_0$ and $C_1$, 
$\partial D = C_0 \cup C_1$, where $C_0 = B \times \{0\}$ and $C_1 = B \times \{\pi\}$.

We define a smooth mapping $h : D \to \tilde{S}$ such that $h(D_j) \subset U_j$ by

$$h(j, \theta, t) = \left(\begin{array}{cc}
i \cos(t) & \sin(t)e^{i\theta} \\
-\sin(t)e^{-i\theta} & -i \cos(t)\end{array}\right) \in U_j$$

if $j = 1$ and $(j, \theta, t) \in D_1 = \{1\} \times [0, \pi] \times I$ or $j = 2$ and $(j, \theta, t) \in D_2 = \{2\} \times [\pi, 2\pi] \times I$, and define

$$h(\theta, \epsilon, t) = (\theta, [\epsilon t]) \in U_3$$

if $j = 3$ and $(\theta, \epsilon, t) \in D_3 = [0, \pi] \times \{\pm 1\} \times I$.

Observe that $h$ maps the boundary circles $C_k$, $k = 0, 1$, to the subsets $\tilde{C}_k$ of $U_3$ consisting of the points $(\theta, [k\pi])$, $\theta \in [0, \pi]$. The subsets $\tilde{C}_0$ and $\tilde{C}_1$ are disjoint and each one is diffeomorphic to an interval, see Fig. C.4.

$$\tilde{C}_0$$

$\tilde{C}_1$

Fig. C.4

The composition $f \circ h$ maps $D$ into $K_4$. By the definition of $f$, for every point of $U_3$ of the form $(\theta, [k\pi])$, $k = 0, 1$, we have

$$f(\theta, [k\pi]) = \left(\begin{array}{cc}
i \cos \theta & \sin \theta \\
-\sin \theta & -i \cos \theta\end{array}\right), J, (-1)^k J, (-1)^k\left(\begin{array}{cc}
i \cos \theta & \sin \theta \\
-\sin \theta & -i \cos \theta\end{array}\right)$$

(C.11)

Let us denote by $\tilde{C}_k$, $k = 0, 1$, the subsets of $K_4$ consisting of all the points of the form (C.11) with $0 \leq \theta \leq \pi$. Again, both $\tilde{C}_0$ and $\tilde{C}_1$ are diffeomorphic to a compact interval and they are disjoint. The composition $f \circ h : D \to K_4$ maps the boundary circles $C_k$ into the subsets $\tilde{C}_k$, $k = 0, 1$.

We shall construct a trivialization of the symplectic bundles $\zeta|_{\tilde{C}_k}$, $k = 0, 1$, and exhibit a compatible complex structure on them.

Let us first consider the restriction of the bundle $\zeta$ to $\tilde{C}_0$. Since $\zeta = \bigoplus_{j=1}^4 p_j^*(T_S)$, at a point

$$x = (A_1, A_2, A_3, A_4) = \left(\begin{array}{cc}
i \cos \theta & \sin \theta \\
-\sin \theta & -i \cos \theta\end{array}\right), J, J, \left(\begin{array}{cc}
i \cos \theta & \sin \theta \\
-\sin \theta & -i \cos \theta\end{array}\right)$$

of $\tilde{C}_0$ one has a basis $v_{j,k} = v_{j,k}(x)$, $j = 1, \ldots, 4$, $k = 1, 2$, of the real 8-dimensional fiber $\zeta_x$ given by

$$v_{j,k}(x) = v_{j,k} \cdot A_j \in p_j^*(T_{A_j} S) \quad \text{with} \quad v_{j,k} = v_{j,k}(x) \in g$$

(C.12)
For example, the entry in the second row and the fourth column is equal to

\[
\omega_c(v_{1,2}, v_{2,2}) = -\frac{1}{2} \omega(z_{1}|z_{2}) (v_{1,2}, v_{2,2}) = \\
= -\frac{1}{2} \omega \left( \begin{pmatrix} i \sin \theta & -\cos \theta \\ \cos \theta & -i \sin \theta \end{pmatrix} \right) \cdot A_1 \cdot \overline{A}_2 = \\
= -\frac{1}{2} \omega \left( A_1 \cdot \begin{pmatrix} -i \sin \theta & \cos \theta \\ -\cos \theta & i \sin \theta \end{pmatrix} \right) \cdot \overline{A}_2 = \\
= -\frac{1}{2}(-\frac{1}{2}) \text{tr} \left( \begin{pmatrix} -i \sin \theta & \cos \theta \\ -\cos \theta & i \sin \theta \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \right) = \frac{1}{2} \cos \theta = \\
= \frac{1}{2} c .
\]

We shall now construct a complex structure in the bundle $\zeta|\hat{C}_0$ compatible with the symplectic form $\omega_c$. 

The basis vectors are ordered lexicographically in $(j, k)$. The basis $\{v_{j,k}\}$ gives a trivialization of the restriction of the bundle $\zeta$ to $\hat{C}_0$.

Since $\hat{C}_0 \subset K_4$, we have $\omega_c = \omega_c = - \sum_{j=1}^{3} \omega_{|z_1 \cdots z_j | z_{j+1}}$ on $\zeta$ (see Corollary 3.4).

It follows that $\omega_c(v_{j,k_1}, v_{j,k_2}) = 0$ for $k_1, k_2 = 1, 2$, and, if $j_1 < j_2$, then

\[
\omega_c(v_{j_1,k_1}, v_{j_2,k_2}) = -\omega_{|z_1 \cdots z_{j_2-1} | z_{j_2}} (v_{j_1,k_1}, v_{j_2,k_2}) = \\
= -\frac{1}{2} \omega(df_{\overline{z}_{j_1}})(v_{j_1,k_1}) \cdot \overline{\omega}(df_{\overline{z}_{j_2}})(v_{j_2,k_2}) = \\
= -\frac{1}{2} \omega(A_1 \cdots A_{j_2-1} \cdot \text{Ad}(A_1 \cdots A_{j_2-1})^{-1})(v_{j_1,k_1}) \cdot \overline{\omega}(v_{j_2,k_2} \cdot A_{j_2}) = \\
= -\frac{1}{2} (\text{Ad}(A_1 \cdots A_{j_2-1})^{-1})(v_{j_1,k_1}) \cdot v_{j_2,k_2} .
\]

(C.14)

Let us, from now on, choose the inner product $\bullet$ to be $a \bullet b = \frac{1}{4} \text{tr}(ab)$ for $a, b \in \mathfrak{g} = su(2)$. A direct calculation, left to the reader, using (C.14), shows that the matrix of the symplectic form $\omega_c$ on $\zeta_{c}$ in the basis $\{v_{j,k}\}$ is equal to

\[
A_x = A(\theta) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & c & 0 & -c & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -c & 0 & 0 & 0 & 1 & 0 & -c \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & c & 0 & -1 & 0 & 0 & 0 & c \\ -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & c & 0 & -c & 0 & 0 \end{pmatrix}
\]

with $c = \cos \theta$.

(C.15)

For example, the entry in the second row and the fourth column is equal to

\[
\omega_c(v_{1,2}, v_{2,2}) = -\omega_{|z_1 | z_2} (v_{1,2}, v_{2,2}) = \\
= -\frac{1}{2} \omega \left( \begin{pmatrix} i \sin \theta & -\cos \theta \\ \cos \theta & -i \sin \theta \end{pmatrix} \right) \cdot A_1 \cdot \overline{A}_2 = \\
= -\frac{1}{2} \omega \left( A_1 \cdot \begin{pmatrix} -i \sin \theta & \cos \theta \\ -\cos \theta & i \sin \theta \end{pmatrix} \right) \cdot \overline{A}_2 = \\
= -\frac{1}{2}(-\frac{1}{2}) \text{tr} \left( \begin{pmatrix} -i \sin \theta & \cos \theta \\ -\cos \theta & i \sin \theta \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \right) = \frac{1}{2} \cos \theta = \\
= \frac{1}{2} c .
\]
Let us consider the subbundle $W'$ of $\zeta|_{\hat{C}_0}$ spanned by $v_{1,1}$, $v_{1,2}$, $v_{4,1}$ and $v_{4,2}$. According to (C.15)
\begin{align*}
\omega_C(v_{1,1}, v_{4,1}) &= \omega_C(v_{1,2}, v_{4,2}) = 1 \\
\omega_C(v_{1,1}, v_{1,2}) &= \omega_C(v_{1,1}, v_{4,2}) = \omega_C(v_{1,2}, v_{4,1}) = 0.
\end{align*}
(C.16)
Thus $v_{1,1}$, $v_{1,2}$, $v_{4,1}$, $v_{4,2}$ is a symplectic basis of $W'$.

Let $W''$ be the symplectic orthogonal complement of $W'$ in $\zeta|_{\hat{C}_0}$. We define sections $u_1$, $u_2$, $u_3$ and $u_4$ of $\zeta|_{\hat{C}_0}$ to be
\begin{align*}
u_1 &= v_{1,1} + v_{2,1} - v_{4,1}, & u_2 &= c v_{1,2} + v_{2,2} - c v_{4,2}, \\
u_3 &= -v_{1,1} + v_{3,1} + v_{4,1}, & u_4 &= -c v_{1,2} + v_{3,2} + c v_{4,2},
\end{align*}
(C.17)
where $c = \cos \theta$. A direct check, using (C.15) shows that the sections $u_1$, $u_2$, $u_3$, $u_4$ belong to and form a symplectic basis of $W''$.

It follows that an isomorphism $\mathcal{J} : \zeta|_{\hat{C}_0} \rightarrow \zeta|_{\hat{C}_0}$ defined by
\begin{align*}
\mathcal{J}(v_{1,1}) &= v_{4,1}, & \mathcal{J}(v_{4,1}) &= -v_{1,1}, \\
\mathcal{J}(v_{1,2}) &= v_{4,2}, & \mathcal{J}(v_{4,2}) &= -v_{1,2},
\end{align*}
(C.18)
is a complex structure on $\zeta|_{\hat{C}_0}$ compatible with the 2-form $\omega_C$. It follows also that the sections
\begin{align*}
w_1 &= v_{1,1}, & w_2 &= v_{1,2}, & w_3 &= u_1, & w_4 &= u_2,
\end{align*}
(C.19)
of $\zeta|_{\hat{C}_0}$ form a basis over $\mathbb{C}$ of the fibres of $\zeta|_{\hat{C}_0}$ w.r.t. this complex structure.

In terms of that basis one has
\begin{align*}
v_{1,1} &= w_1, & v_{1,2} &= w_2, \\
v_{2,1} &= (1 + \mathcal{J})w_1 + w_3, & v_{2,2} &= c(1 + \mathcal{J})w_2 + w_4, \\
v_{3,1} &= (1 - \mathcal{J})w_1 + \mathcal{J}w_3, & v_{3,2} &= c(1 - \mathcal{J})w_2 + \mathcal{J}w_4, \\
v_{4,1} &= \mathcal{J}w_1, & v_{4,2} &= \mathcal{J}w_2.
\end{align*}
(C.20)
A similar discussion concerning the restriction of the bundle $\zeta$ to the subspace $\hat{C}_1$ differs from the case of $\hat{C}_0$ only in the fact that $A_3 = -\mathcal{J}$ and $A_4 = -\begin{pmatrix}i \cos \theta & \sin \theta \\ -\sin \theta & -i \cos \theta \end{pmatrix}$. Everything else follows through in exactly the same way and with the same results.

We now choose and fix for the rest of the paper a complex structure $\mathcal{J}$ on the bundle $\zeta$ compatible with the symplectic 2-form $\omega_C$ and such that it coincides over $\hat{C}_0 \cup \hat{C}_1$ with the complex structure $\mathcal{J}$ described in (C.18).

We have $f^*(\zeta) = f^*(TP_4) = f^*(TP_M)$, where $M = \mathcal{M} \subset P_4$ is an open neighbourhood of $K_4$ given by Theorem 3.1 in case $n = 2$.

The subspace $K_4 = \tilde{r}^{-1}(0)$ is a $G$-invariant subspace of $P_4$. As observed in Section 2, $K_4$ contains four exceptional orbits $\Delta_{1,1,1,1}$, $\Delta_{1,1,1,1}$, $\Delta_{1,1,1,1}$, $\Delta_{1,1,1,1}$, $\Delta_{1,1,1,1}$, $\Delta_{1,1,1,1}$ of the action of $G$. All these orbits are diffeomorphic to the 2-sphere $S$ and the isotropy subgroups at their points are isomorphic to circles.

Let us denote by $L_A$ the complement of these four orbits in $K_4$. It is a smooth $G$-invariant manifold. The isotropy subgroups at the points of $L_A$ are all equal to $\{\pm I\}$ and the group $SO(3) = SU(2)/\{\pm I\}$ acts freely on $L_A$. 


The matrices \( X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ X_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ X_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) form a basis of \( \mathfrak{g} \). Through the action of \( G \) they induce tangent vector fields \( \tilde{X}_1, \tilde{X}_2, \tilde{X}_3 \) on \( P_4 \) which, at every point of \( L_4 \), are tangent to \( L_4 \) and linearly independent. As \( f^*(\zeta) \) is a pullback of the tangent bundle \( TP_4 \) the vector fields \( \tilde{X}_1, \tilde{X}_2, \tilde{X}_3 \) induce sections \( \tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3 \) of the bundle \( f^*(\zeta) \).

Let us now consider the pull-back \( \vartheta = (f \circ h)^*(\zeta) \) of \( \zeta \) to \( D \). It is a symplectic vector bundle of the real fiber dimension 8 equipped with a compatible complex structure \( \mathcal{J} \). The boundary \( \partial D \) is the union of the two circles \( C_0 \) and \( C_1 \) and \((f \circ h)(C_1) = \overline{C}_1 \) in \( K_4 \). Therefore the restriction of \( \vartheta \) to the boundary \( \partial D \) is equipped with the basic sections \( v_{i,k}, j = 1, \ldots, 4, k = 1, 2 \), over \( \mathbb{R} \) defined by (C.12 - C.13), with the complex structure \( \mathcal{J} \) defined by (C.18) and also with the basic sections \( w_1, \ldots, w_4 \) over \( C \) chosen as in (C.19). Since \( \vartheta = h^*f^*(TP_4) \) is a pull-back of \( TP_4 \), the vector fields \( \tilde{X}_1, \tilde{X}_2, \tilde{X}_3 \) induce sections \( Y_1, Y_2, Y_3 \) of \( \vartheta \).

As the composition \( f \circ h \) maps the interior \( D - \partial D \) of \( D \) into \( L_4 \), the sections \( Y_1, Y_2, Y_3 \) are linearly independent over every point of \( D - \partial D \).

Let again \( x \) be a point of \( \overline{C}_k \), \( k = 0, 1 \),

\[
x = \left( \begin{array}{cc} i \cos \theta & \sin \theta \\ -\sin \theta & -i \cos \theta \end{array} \right), \quad J, (-1)^k J, (-1)^k \left( \begin{array}{cc} i \cos \theta & \sin \theta \\ -\sin \theta & -i \cos \theta \end{array} \right)\]

with \( 0 \leq \theta \leq \pi \). We shall determine the values of the vector fields \( \tilde{X}_1, \tilde{X}_2, \tilde{X}_3 \) at \( x \).

Let \( X \in \mathfrak{g} \) and let \( g(t) \) be a 1-parameter subgroup of \( G = SU(2) \) such that \( g'(0) = X \). Let \( G \) act on itself by conjugation and let \( \tilde{X} \) be the vector field on \( G \) induced by \( X \) through this action. If \( A \in G \) then the value of \( \tilde{X} \) at \( A \) is

\[
\tilde{X}(A) = \frac{d}{dt}(g(t)A g(t)^{-1})|_{t=0} = XA - AX = (X - Ad(A)X) \cdot A . \tag{C.21}
\]

If \( A = \left( \begin{array}{cc} i \cos \theta & \sin \theta \\ -\sin \theta & -i \cos \theta \end{array} \right) \) then

\[
X_1 - Ad(A)X_1 = 2 \sin \theta \left( \begin{array}{cc} \sin \theta & -\cos \theta \\ \cos \theta & -i \sin \theta \end{array} \right),
\]

\[
X_2 - Ad(A)X_2 = 2 \left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right),
\]

\[
X_3 - Ad(A)X_3 = 2 \cos \theta \left( \begin{array}{cc} \sin \theta & -\cos \theta \\ \cos \theta & -i \sin \theta \end{array} \right) . \tag{C.22}
\]

Applying now (C.22) to each one of the four coordinates of \( x \) and using (C.12), (C.13) and (C.20) we obtain

\[
\tilde{X}_1(x) = 2 \sin \theta \cdot (v_{1,2} + v_{4,2}) = 2 \sin \theta \cdot (1 + \mathcal{J})w_2 ,
\]

\[
\tilde{X}_2(x) = 2(v_{1,1} + v_{2,1} + v_{3,1} + v_{4,1}) = 2(1 + \mathcal{J})(w_1 + w_3) ,
\]

\[
\tilde{X}_3(x) = 2(\cos \theta \cdot v_{1,2} + \cos \theta \cdot v_{2,2} + \cos \theta \cdot v_{3,2} + \cos \theta \cdot v_{4,2}) = 2(1 + \mathcal{J})(c w_2 + w_4) . \tag{C.23}
\]

That gives also values of the sections \( \widetilde{Y}_1, \widetilde{Y}_2, \widetilde{Y}_3 \) at the points of \( \overline{C}_0 \) and \( \overline{C}_1 \).

We shall now define a fourth section of the bundle \( \vartheta = (f \circ h)^*(\zeta) \) over \( D \). Let \( X_4 \) be the tangent field on \( D = B \times I \) parallel to the factor \( I \), \( X_4(b,t) = -\frac{d}{dt} \), see Fig[C]5.
Lemma C.7. Values of the sections $Y_1, Y_2, Y_3, Y_4$ of the bundle $\vartheta$ are $\mathbb{R}$-linearly independent over every point of the interior $D - \partial D$.

Proof. Suppose that at a point $(b, t) \in D = B \times I$ with $0 < t < \pi$ (i.e. belonging to the interior of $D$) one has

$$\sum_{i=1}^{4} \lambda_i Y_i(b, t) = 0$$

for some $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, 4$.

Let $p_j : P_4 \rightarrow \mathcal{S}$, $j = 1, \ldots, 4$, be the projection on the $j$-th coordinate. As $p_2(f(y)) = J \in \mathcal{S}$ for all $y \in \mathcal{S}$, we get that $p_2(f \circ h)(z) = J \in \mathcal{S}$ for all $z \in D$. Therefore the derivative $d(p_2 \circ f \circ h)$ maps $X_4(b, t)$ to 0. It follows that $\varphi = dp_2 \circ f \circ h$ maps $Y_4(b, t)$ to 0 and that it maps $\sum_{i=1}^{4} \lambda_i Y_i(b, t)$ to $\sum_{i=1}^{3} \lambda_i \varphi(Y_i(b, t)) = \sum_{i=1}^{3} \lambda_i dp_2(\bar{X}_i(x))$ with $x = f(h(b, t))$. Hence (C.24) implies

$$\sum_{i=1}^{3} \lambda_i dp_2(\bar{X}_i(x)) = 0 .$$

However, at the point $p_2(x) = p_2(f \circ h)(b, t) = J$ we have $dp_2(\bar{X}_1(x)) = (X_1 - Ad(J)X_1) \cdot J = 0$ while $dp_2(\bar{X}_2(x)) = (X_2 - Ad(J)X_2) \cdot J = 2\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \cdot J$ and $dp_2(\bar{X}_3(x)) = (X_3 - Ad(J)X_3) \cdot J = 2\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \cdot J$ are linearly independent in $T_J \mathcal{S}$. (See (C.22)). Therefore $\lambda_2 = \lambda_3 = 0$.

Furthermore, for every point $(b, t) \in D - \partial D$ one has $(p_3 \circ f \circ h)(b, t) \neq \pm J$. It follows then directly from the definition of the tangent field $X_4$ on $D$ and of the map $f \circ h$ that $(dp_3 \circ f \circ h)(Y_4(b, t))$ is a non-zero tangent vector to $\mathcal{S}$ at the point $A = (p_3 \circ f \circ h)(b, t)$ and that this vector is tangent to the great circle joining $A$ and $J$ in $\mathcal{S}$. On the other hand, the tangent vector $(dp_3 \circ f \circ h)(Y_1(b, t)) = dp_3(\bar{X}_1((f \circ h)(b, t))) = (X_1 - Ad(A)X_1) \cdot A$ is transversal to that great circle at the point $A$. Therefore $(dp_3 \circ f \circ h)(Y_1(b, t))$ and $(dp_3 \circ f \circ h)(Y_4(b, t))$ are $\mathbb{R}$-linearly independent vectors in $T_A \mathcal{S}$. Consequently, $\lambda_1 = \lambda_4 = 0$. It follows that the values of $Y_1, Y_2, Y_3, Y_4$ are $\mathbb{R}$-linearly independent over every point $(b, y)$ in the interior of $D$.

Lemma C.8. For every point $y \in D - \partial D$ the values $Y_j(y)$, $j = 1, \ldots, 4$, span a Lagrangian subspace of the fiber $\vartheta_y$. 

Fig. C.5
Corollary C.9. For every point compatible with \( \vartheta \) is embedded as the vector \( a \) basis of the complex vector space \( g \) embedded in \( \hat{\vartheta} \) as embedded in the vector space \( g \).

Proof. The statement of the Lemma is equivalent to the claim that, for \( x = (f \circ h)(y) \), the values \( \hat{X}_1(x), \hat{X}_2(x), \hat{X}_3(x) \) and \( \hat{X}_4(x) = d((f \circ h)(X_4(y))) \) span a Lagrangian subspace of \( \zeta_x \). Now, \( \zeta_x = T_x P_4 = T_x \mathcal{M}_4 \) and \( \hat{r} : \mathcal{M}_4 \to g \), \( \hat{r}(A_1, A_2, A_3, A_4) = \exp^{-1}(A_1A_2A_3A_4) \), is a momentum mapping for the symplectic manifold \( \mathcal{M}_4 \) with the diagonal action of \( G \) by the conjugation.

As the vector fields \( \hat{X}_1, \hat{X}_2, \hat{X}_3 \) are induced by \( X_1, X_2, X_3 \in g \) under that action of \( G \) and the point \( x = (f \circ h)(y) \) belongs to the \( G \)-invariant subspace \( K_4 = \hat{r}^{-1}(0) \), we have \( \hat{X}_i(x) \in \text{Ker}(d\hat{r} : T_x \mathcal{M}_4 \to g) \), \( i = 1, 2, 3 \).

Similarly, since \( f \circ h \) maps \( D \) into \( K_4 = \hat{r}^{-1}(0) \), we have \( \hat{X}_4(x) \in \text{Ker}(d\hat{r} : T_x \mathcal{M}_4 \to g) \).

Therefore, by the momentum map property,

\[
\omega_C(\hat{X}_j(x), \hat{X}_k(x)) = (d\hat{r}(\hat{X}_k(x))) \cdot X_j = 0 \quad \text{for } j = 1, 2, 3 \text{ and } k = 1, \ldots, 4.
\]

Hence \( \hat{X}_1(x), \ldots, \hat{X}_4(x) \) span an isotropic subspace of \( \zeta_x \). By Lemma \([C,7]\) that subspace has real dimension 4 and, therefore, is Lagrangian. \( \square \)

Let us recall that the bundle \( \vartheta \) is equipped with the complex structure \( J \) compatible with \( \omega_C \).

Corollary C.9. For every point \( y \in D - \partial D \) the values \( Y_j(y), \ j = 1, \ldots, 4 \), form a basis of the complex vector space \( \vartheta_y \).

Proof. The values \( Y_j(y), \ j = 1, \ldots, 4 \), are linearly independent over \( \mathbb{R} \) by Lemma \([C,7]\) and belong to a Lagrangian subspace of \( \vartheta_x \) by Lemma \([C,8]\). Hence, they are linearly independent over \( \mathbb{C} \). As \( \dim \vartheta_x = 8 \), Corollary \([C,9]\) follows. \( \square \)

In order to compute \( c_1(\zeta) \) we shall now study the behaviour of the sections \( Y_1, \ldots, Y_4 \) in a neighbourhood of the boundary \( \partial D = C_0 \cup C_1 \). We shall deform continuously some of the sections.

The mapping \( f \circ h \) maps \( C_k \) onto \( \hat{C}_k \subset K_4, \ k = 0, 1 \). By the definition the values of \( Y_1, Y_2, Y_3 \) at a point \( y \in \partial D \) are given by \( \hat{X}_1(x), \hat{X}_2(x), \hat{X}_3(x) \in \zeta_x \) with \( x = (f \circ h)(y) \). According to \([C,23]\), both \( Y_2(y) = \hat{X}_2(x) \) and \( Y_3(y) = \hat{X}_3(x) \) are non-zero (and actually \( \mathbb{C} \)-linearly independent) at all points \( y \in \hat{D} \). By Corollary \([C,9]\) the same is true for \( y \in D - \partial D \), hence for all \( y \in D \). We shall leave the sections \( Y_2 \) and \( Y_3 \) unchanged.

In order to describe the deformations of \( Y_1 \) and \( Y_4 \) it will be useful to see the sphere \( S \) as embedded in \( G = SU(2) \), to see the tangent bundle \( TS \) of the sphere as embedded in the tangent bundle \( TG \) of \( G \), to identify \( TG \) with \( g \times G \) through the right translations, in that way to look upon every fiber of the bundle \( TS \) as embedded in \( g \) and, finally, to look upon every fiber of the bundle \( TP_4 = (TS)^4 \) as embedded in the vector space \( g^4 = g \oplus g \oplus g \oplus g \). Thus a tangent vector \( (v_1, v_2, v_3, v_4) \in T_{(A_1, A_2, A_3, A_4)}P_4 \) with \( v_j = v_j \cdot A_j, \ A_j \in S, \ v_j \in g, \ j = 1, \ldots, 4 \), is embedded as the vector \( v_1 \oplus v_2 \oplus v_3 \oplus v_4 \) in \( g^4 \).

The value \( Y_1((b, t)) \) of the section \( Y_1 \) at the point \( (b, t) \in D \) can be identified with the value of the section \( \tilde{Y}_1 = \hat{X}_1 \circ f \) of the bundle \( f^*\zeta \) at the point \( h(b, t) \in \hat{S} \). According to \([C,23]\) and Lemma \([C,7]\) the section \( \tilde{Y}_1 \) is non-zero at all except
four points of $\tilde{S}$. The four exceptional points are: $z_1 = (0, 0) \in U_3 = [0, \pi] \times S^1$ identified with $J \in U_1$, $z_2 = (\pi, 0) \in U_3$ identified with $J \in U_2$, $z_3 = (0, \pi) \in U_3$ identified with $-J \in U_1$ and $z_4 = (\pi, \pi) \in U_3$ identified with $-J \in U_2$. Of these four points $z_1, z_2 \in \tilde{C}_0$ while $z_3, z_4 \in \tilde{C}_1$.

We shall now calculate directional derivatives of $\tilde{Y}_1 = \tilde{X}_1 \circ f$ at the points $z_1, \ldots, z_4$. The values of $\tilde{Y}_1$ are seen as elements of the vector space $\mathfrak{g}_4$ and so will be their derivatives. Observe that since the section $\tilde{Y}_1 = \tilde{X}_1 \circ f$ vanishes at all $z_i$, to calculate its directional derivatives amounts to changing its length and taking a limit. Hence, when pulled back over $D$, the section can be continuously deformed, by changing only its length over $D - \partial D$, to a section which has the directional derivatives as its values along parts of $\partial D$.

The directional derivatives of $\tilde{Y}_1$ at $z_1$ and $z_2$

![Diagram](Fig. C.6)

Figure C.6 describes a neighbourhood of $\tilde{C}_0$ in $\tilde{S}$. It is a union of a neighbourhood of $\tilde{C}_0$ in $U_3$ (the middle strip $0 \leq \theta_1 \leq \pi$), of a neighbourhood of $z_1$ in $U_1$ (in the left half-plane $\theta_1 \leq 0$) and of a neighbourhood of $z_2$ in $U_2$ (in the right half-plane $\theta_1 \geq \pi$). In the middle strip

$$f(\theta_1, \theta_2) = \left( \begin{array}{cc} i \cos \theta_1 & \sin \theta_1 \\ -\sin \theta_1 & -i \cos \theta_1 \end{array} \right), \quad J = \left( \begin{array}{cc} i \cos \theta_2 & \sin \theta_2 \\ -\sin \theta_2 & -i \cos \theta_2 \end{array} \right),$$

$$\left( \begin{array}{cc} i \cos(\theta_1 + \theta_2) & \sin(\theta_1 + \theta_2) \\ -\sin(\theta_1 + \theta_2) & -i \cos(\theta_1 + \theta_2) \end{array} \right) \right) \in P_4$$

(see the definition of $f$). Hence, according to (C.22),

$$\tilde{Y}_1(\theta_1, \theta_2) = (\tilde{X}_1 \circ f)(\theta_1, \theta_2) =$$

$$= \left( \begin{array}{cc} 2 \sin \theta_1 & \left( \begin{array}{cc} i \sin \theta_1 & -\cos \theta_1 \\ \cos \theta_1 & -i \sin \theta_1 \end{array} \right), \quad 0, \quad 2 \sin \theta_2 & \left( \begin{array}{cc} i \sin \theta_2 & -\cos \theta_2 \\ \cos \theta_2 & -i \sin \theta_2 \end{array} \right), \quad 2 \sin(\theta_1 + \theta_2) & \left( \begin{array}{cc} i \sin(\theta_1 + \theta_2) & -\cos(\theta_1 + \theta_2) \\ \cos(\theta_1 + \theta_2) & -i \sin(\theta_1 + \theta_2) \end{array} \right) \right) \right) \in \mathfrak{g}_4.$$
Thus the directional derivative of $\bar{Y}_1$ at the point $(\theta_1, \theta_2) = (0, 0)$ in the direction of the vector $u = (-\sin(t), \cos(t))$, $\pi \leq t \leq 2\pi$, is

$$D_u(\bar{Y}_1)(0, 0) = 2\left(-\sin(t) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0, \cos(t) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right),$$

$$(\cos(t) - \sin(t)) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) \in g^4,$$

which, according to (C.13) and (C.20), is equivalent to

$$D_u(\bar{Y}_1)(0, 0) = -2\sin(t) (v_{1,2} + v_{4,2}) + 2\cos(t) (v_{3,2} + v_{4,2}) =$$

$$= -2\sin(t) (1 + \mathcal{J})w_2 + 2\cos(t) (c(1 - \mathcal{J}) + \mathcal{J})w_2 + \mathcal{J}w_4 =$$

$$= -2\sin(t) (1 + \mathcal{J})w_2 + 2\cos(t) (w_2 + \mathcal{J}w_4).$$

(C.25)

The last equality holds because $c = \cos \theta_1 = 1$ at $(\theta_1, \theta_2) = (0, 0)$.

Let now $k(t)$ be the real analytic function $k(t) = \frac{\sin(t)}{t} = \sum_{i=0}^{\infty} (-1)^i \frac{t^{2i}}{(2i + 1)!}$ and let $r = \sqrt{\theta_1^2 + \theta_2^2}$. Define $k(\theta_1, \theta_2)$ to be $k(\theta_1, \theta_2) = k(r) = \sum_{i=0}^{\infty} (-1)^i \frac{(\theta_1^2 + \theta_2^2)^i}{(2i + 1)!}$.

We have $k(0, 0) = 1$ and $\frac{\partial k}{\partial \theta_1}(0, 0) = 0 = \frac{\partial k}{\partial \theta_2}(0, 0)$.

To describe $f$ in the left half-plane we can choose coordinates $(\theta_1, \theta_2)$ there in such a way that

$$f(\theta_1, \theta_2) = (J, J, A, A)$$

(C.26)

with $A = \left( \begin{array}{cc} is & v \\ -\bar{v} & -is \end{array} \right)$, $s = \cos(r) \in \mathbb{R}$ and $v = v(\theta_1, \theta_2) = k(\theta_1, \theta_2) \cdot (\theta_2 - i\theta_1) \in \mathbb{C}$.

According to (C.21) we get

$$\bar{Y}_1(\theta_1, \theta_2) = (B, B, C, C)$$

with $B = X_1 - \text{Ad}(J)X_1 = 0$ and

$$C = X_1 - \text{Ad}(A)X_1 = \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right) - A \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right) A^{-1} =$$

$$= \left( \begin{array}{cc} i(1 + |v|^2 - s^2) & -2sv \\ 2sv & -i(1 + |v|^2 - s^2) \end{array} \right).$$

We have $v(0, 0) = 0$, $\frac{\partial v}{\partial \theta_1}(0, 0) = -i$, $\frac{\partial v}{\partial \theta_2}(0, 0) = 1$, $s(0, 0) = 1$, $\frac{\partial s}{\partial \theta_1}(0, 0) = 0 = \frac{\partial s}{\partial \theta_2}(0, 0)$. Thus

$$\left. \frac{\partial C}{\partial \theta_1} \right|_{(\theta_1, \theta_2) = (0, 0)} = 2 \left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right)$$

and

$$\left. \frac{\partial C}{\partial \theta_2} \right|_{(\theta_1, \theta_2) = (0, 0)} = 2 \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right).$$
It follows that the directional derivative of $\tilde{Y}_1$ at the point $(\theta_1, \theta_2) = (0, 0)$ in the direction of the vector $u = (-\sin(t), \cos(t))$, $0 \leq t \leq \pi$, is

$$D_u(\tilde{Y}_1)(0,0) = -\sin(t) \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ \end{pmatrix} +$$

$$+ \cos(t) \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ \end{pmatrix} =$$

$$= -2 \sin(t) (v_{3,1} + v_{4,1}) + 2 \cos(t) (v_{3,2} + v_{4,2}) =$$

$$= -2 \sin(t) (w_1 + Jw_3) + 2 \cos(t) (w_2 + Jw_4).$$

Thus, for $u = (-\sin(t), \cos(t))$, we have at the point $(\theta_1, \theta_2) = (0, 0)$

$$D_u(\tilde{Y}_1)(0,0) = \begin{cases} 
-2 \sin(t) (w_1 + Jw_3) + 2 \cos(t) (w_2 + Jw_4) & \text{if } 0 \leq t \leq \pi, \\
-2 \sin(t) (1 + J)w_2 + 2 \cos(t) (w_2 + Jw_4) & \text{if } \pi \leq t \leq 2\pi.
\end{cases}$$

(C.28)

A similar calculation performed at the point $(\theta_1, \theta_2) = (\pi, 0)$ in the strip $0 \leq \theta_1 \leq \pi$ shows that the directional derivative of $\tilde{Y}_1$ at $(\theta_1, \theta_2) = (\pi, 0)$ in a direction of $u = (-\sin(t), \cos(t))$, $0 \leq t \leq \pi$, is equal to

$$D_u(\tilde{Y}_1)(\pi,0) = -\sin(t) \begin{pmatrix} 2 & 0 & 1 & -1 \\ 1 & 0 & 2 & 1 \\ \end{pmatrix} +$$

$$+ \cos(t) \begin{pmatrix} 2 & 0 & 1 & -1 \\ 1 & 0 & 2 & 1 \\ \end{pmatrix},$$

which, according to (C.13), is equivalent at $\theta = \theta_1 = \pi$ to

$$D_u(\tilde{Y}_1)(\pi,0) = 2 \sin(t) (v_{1,2} + v_{4,2}) + 2 \cos(t) (v_{3,2} - v_{4,2}) =$$

$$= 2 \sin(t) (1 + J)w_2 + 2 \cos(t) ((c(1 - J) - J)w_2 + Jw_4) =$$

$$= 2 \sin(t) (1 + J)w_2 + 2 \cos(t) (-w_2 + Jw_4).$$

(Note that at $\theta = \theta_1 = \pi$ one has $c = \cos\theta_1 = -1$.)

A calculation which follows exactly the case considered in the left half-plane $\theta_1 \leq 0$ shows that in the right half-plane $\theta_1 \geq \pi$ and for $u = (-\sin(t), \cos(t))$ with $\pi \leq t \leq 2\pi$ one has

$$D_u(\tilde{Y}_1)(\pi,0) = -\sin(t) \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ \end{pmatrix} +$$

$$+ \cos(t) \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ \end{pmatrix},$$

which, according to (C.13), is equivalent at $\theta = \theta_1 = \pi$ to

$$D_u(\tilde{Y}_1)(\pi,0) = -2 \sin(t) (v_{3,1} + v_{4,1}) + 2 \cos(t) (v_{3,2} - v_{4,2}) =$$

$$= -2 \sin(t) (w_1 + Jw_3) + 2 \cos(t) ((-w_2 + Jw_4).$$

Thus, for $u = (-\sin(t), \cos(t))$, we have at the point $(\theta_1, \theta_2) = (\pi, 0)$

$$D_u(\tilde{Y}_1)(\pi,0) = \begin{cases} 
2 \sin(t) (1 + J)w_2 + 2 \cos(t) (-w_2 + Jw_4) & \text{if } 0 \leq t \leq \pi, \\
-2 \sin(t) (w_1 + Jw_3) + 2 \cos(t) (-w_2 + Jw_4) & \text{if } \pi \leq t \leq 2\pi.
\end{cases}$$

(C.29)

Let $W$ be an open contractible neighbourhood of the subset $\tilde{C}_0$ in $\tilde{S}$ as indicated in Figure C.7.
Let $W_1$ and $W_2$ be two small open discs around the point $(\theta_1, \theta_2) = (0, 0)$ and around $(\theta_1, \theta_2) = (\pi, 0)$ respectively. We introduce a cut in the $(\theta_1, \theta_2)$-plane from $W_1$ to $W_2$ along the $\theta_1$-axis. Let $\mathcal{K}$ be the contour in the $(\theta_1, \theta_2)$-plane consisting of four paths: $\tau_1 = \partial W_1$, $\tau_2$ = lower edge of the cut, $\tau_3 = \partial W_2$ and $\tau_4$ = upper edge of the cut. The contour $\mathcal{K}$ is oriented counter-clock-wise. See Figure C.7.

What we have done above shows that the section $\tilde{Y}_1$ of the bundle $f^*(\zeta)$ can be deformed continuously on a compact subset of $W$, by changing only its length and taking limits, to a section $\tilde{Y}_1'$ defined in the complement of $W_1 \cup W_2$ and such that the values of $\tilde{Y}_1'$ on $\tau_1$ are given by (C.28), its values on $\tau_3$ are given by (C.29) and that on the both edges $\tau_2$, $\tau_4$ of the cut $\tilde{Y}_1'$ is equal to

$$\tilde{Y}_1' = 2(1 + J) w_2$$

(see (C.23)). We change the length of the section $\tilde{Y}_1 = \tilde{X}_1 \circ f$ along the cut by dividing it by $\sin(\theta_1)$. The reason for introducing the cut will be explained below.

We shall now consider the values of the section $Y_4$ over the points of the contour $\mathcal{K}$. Let us denote by $\tilde{Y}_4$ the section of the bundle $f^*(\zeta)$ which corresponds to $Y_4$ over $W$ with $W_1 \cup W_2$ removed and with the cut between $W_1$ and $W_2$. The reason for introducing the cut is the fact that the values of $\tilde{Y}_4$ over both its edges $\tau_3$ and $\tau_4$ are different.

By the definition of the vector field $X_4$ on $D$ and of the mapping $f \circ h$ we get that the values of $\tilde{Y}_4$ are tangent vectors to the curves $f(j(s))$ in $P_4$, where the curves $j(s)$ are given as in the Figure C.8.
The curves \( j(s) \) are half-lines parametrised linearly, orthogonal to \( \theta_1 \)-axis in the strip \( 0 \leq \theta_1 \leq \pi \) and converging radially to \((\theta_1, \theta_2) = (0, 0)\) in the half-plane \( \theta_1 \leq 0 \), and to \((\theta_1, \theta_2) = (\pi, 0)\) in the half-plane \( \theta_1 \geq \pi \). All the curves \( j(s) \) are oriented towards the segment \( \{ 0 \leq \theta_1 \leq \pi, \theta_2 = 0 \} \).

To find the values of \( \tilde{Y}_4 \) at the point \((\theta_1, 0_+)\) of the upper edge \( \tau_4 \) of the cut we take \( j(s) = (\theta_1, -s) \), \(-\infty < s \leq 0\). Then

\[
f(j(s)) = \begin{pmatrix} i \cos \theta_1 & \sin \theta_1 \\ -\sin \theta_1 & -i \cos \theta_1 \end{pmatrix}, \quad J, \begin{pmatrix} i \cos(s) & -\sin(s) \\ \sin(s) & -i \cos(s) \end{pmatrix}, \begin{pmatrix} i \cos(\theta_1 - s) & \sin(\theta_1 - s) \\ -\sin(\theta_1 - s) & -i \cos(\theta_1 - s) \end{pmatrix} \in P_4
\]

and

\[
\tilde{Y}_4(\theta_1, 0_+) = \frac{d}{ds}(f(j(s))) \bigg|_{s=0} = \begin{pmatrix} 0, 0, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} i \sin \theta_1 & -\cos \theta_1 \\ \cos \theta_1 & -i \sin \theta_1 \end{pmatrix} \end{pmatrix} \in TP_4.
\]

At the point

\[
f(\theta_1, 0_+) = \begin{pmatrix} i \cos \theta_1 & \sin \theta_1 \\ -\sin \theta_1 & -i \cos \theta_1 \end{pmatrix}, J, J, \begin{pmatrix} i \cos \theta_1 & \sin \theta_1 \\ -\sin \theta_1 & -i \cos \theta_1 \end{pmatrix}
\]

of \( C_0 \) the last expression, according to (C.31), is equivalent to

\[
\tilde{Y}_4(\theta_1, 0_+) = -v_{3,1} - v_{4,1}.
\]

By (C.31) we get

\[
\tilde{Y}_4(\theta_1, 0_+) = -(w_1 + Jw_3).
\]

A similar calculation at the lower edge \( \tau_2 \) of the cut gives

\[
\tilde{Y}_4(\theta_1, 0_-) = v_{3,1} + v_{4,1} = w_1 + Jw_3.
\]

Observe that (C.31) gives also values of \( \tilde{Y}_4 \) at those parts of \( \partial W_1 \) and \( \partial W_2 \) which lie in the domain \( 0 \leq \theta_1 \leq \pi, \theta_2 \geq 0 \), while (C.32) gives values of \( \tilde{Y}_4 \) at the parts of \( \partial W_1 \cup \partial W_2 \) lying in the domain \( 0 \leq \theta_1 \leq \pi, \theta_2 \leq 0 \). Observe also that, since the RHS of (C.31) and (C.32) are non-vanishing vectors at all points of \( C_0 \) (including end points), we do not take directional derivatives here.

To find the value of \( \tilde{Y}_4 \) at a point \((u(t), 0, 0)\) of \( \partial W_1 \) lying in the domain \( \theta_1 \leq 0 \) and corresponding to the vector \( u(t) = (-\sin(t), \cos(t)) \), \( 0 \leq t \leq \pi \), we take \( j(s) = (s \sin(t), -s \cos(t)) \) and get

\[
f(j(s)) = (J, J, A, A) \in P_4
\]

with

\[
A = \begin{pmatrix} i \cos(s) & v \\ -\overline{v} & -i \cos(s) \end{pmatrix}, \quad v = -sk(-s \sin(t), s \cos(t)) \cdot (\cos(t) + i \sin(t)) \in \mathbb{C}.
\]
(see (C.26)). Thus

\[
\frac{d}{ds} \left( f(j(s)) \right) \bigg|_{s=0} = \left( 0, 0, \left( \begin{array}{cc} 0 & z \\ \frac{z}{2} & 0 \end{array} \right), \left( \begin{array}{cc} 0 & z \\ \frac{z}{2} & 0 \end{array} \right) \right) \in T_{(J,J,J,J)} P_4
\]

with \( z = -\cos(t) - i \sin(t) \in \mathbb{C} \). According to (C.12, C.13) this equality, at the point \( (\theta_1, \theta_2) = (0, 0) \) i.e. with \( \theta = 0 \), is equivalent to

\[
\tilde{Y}_4(u(t), 0, 0) = -\cos(t) \left( \begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & 0 \end{array} \right) - \sin(t) \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) = -\cos(t) (v_{3,1} + v_{4,1}) - \sin(t) (v_{3,2} + v_{4,2}), \quad 0 \leq t \leq \pi.
\]

That in turn, by (C.19)-(C.20), with \( c = \cos(0) = 1 \) is equivalent to

\[
\tilde{Y}_4(u(t), 0, 0) = -\cos(t) (w_1 + J w_3) - \sin(t) (w_2 + J w_4).
\]

For a similar calculation for points \((u(t), \pi, 0)\) of \( \partial W_2 \) lying in the domain \( \theta_1 \geq \pi \) and corresponding to the vector \( u(t) = (-\sin(t), \cos(t)) \) with \( \pi \leq t \leq 2\pi \), we take \( j(s) = (\pi + s \sin(t), -s \cos(t)) \) and get

\[
f(j(s)) = (-J, J, A, -A) \in P_4
\]

with \( A \) as in (C.33). Consequently,

\[
\tilde{Y}_4(u(t), \pi, 0) = \frac{d}{ds} \left( f(j(s)) \right) \bigg|_{s=0} = \left( 0, 0, \left( \begin{array}{cc} 0 & z \\ \frac{z}{2} & 0 \end{array} \right), \left( \begin{array}{cc} 0 & z \\ \frac{z}{2} & 0 \end{array} \right) \right) \in T_{(-J,J,J,J)} P_4
\]

with \( z = -\cos(t) - i \sin(t) \in \mathbb{C} \). According to (C.12, C.13) this equality, at the point \( (\theta_1, \theta_2) = (\pi, 0) \) i.e. with \( \theta = \pi \), is equivalent to

\[
\tilde{Y}_4(u(t), \pi, 0) = -\cos(t) \left( \begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & 0 \end{array} \right) - \sin(t) \left( \begin{array}{ccc} 0 & i & 0 \\ -i & 0 \end{array} \right) = -\cos(t) (v_{3,1} + v_{4,1}) - \sin(t) (v_{3,2} - v_{4,2}), \quad \pi \leq t \leq 2\pi.
\]

By (C.19)-(C.20) with \( c = \cos(\pi) = -1 \), we get

\[
\tilde{Y}_4(u(t), \pi, 0) = -\cos(t) (w_1 + J w_3) + \sin(t) (w_2 - J w_4).
\]

for \( \pi \leq t \leq 2\pi \).

Let us now calculate the determinant

\[
D = \left| \begin{array}{cccc} -\tilde{Y}_1' & -\tilde{Y}_2' & -\tilde{Y}_3' & -\tilde{Y}_4' \\ -\tilde{Y}_1 & -\tilde{Y}_2 & -\tilde{Y}_3 & -\tilde{Y}_4 \\ \end{array} \right|
\]

of the sections \( \tilde{V}_1', \tilde{V}_2', \tilde{V}_3', \tilde{V}_4' \) in the \( \mathbb{C} \)-basis \( w_1, ..., w_4 \) over points of the contour \( \mathcal{K} \). We first subdivide \( \mathcal{K} \) into paths as in Figure C.9.
Over \( \tau'_1 \) we have, according to (C.23), (C.28) and (C.34),

\[
\begin{align*}
\tilde{Y}'_1(u(t), 0, 0) &= -2 \sin(t) (w_1 + Jw_3) + 2 \cos(t) (w_2 + Jw_4), \\
\tilde{Y}'_2(u(t), 0, 0) &= 2(1 + J) (w_1 + w_3), \\
\tilde{Y}'_3(u(t), 0, 0) &= 2(1 + J) (w_2 + w_4), \\
\tilde{Y}'_4(u(t), 0, 0) &= -\cos(t) (w_1 + Jw_3) - \sin(t) (w_2 + Jw_4),
\end{align*}
\]

with \( 0 \leq t \leq \pi \). Hence

\[
\mathcal{D} = \mathcal{D}(t) = \begin{bmatrix} -2 \sin(t) & 2 \cos(t) & -2i \sin(t) & 2i \cos(t) \\ 2(1 + i) & 0 & 2(1 + i) & 0 \\ 0 & 2(1 + i) & 0 & 2(1 + i) \\ -\cos(t) & -\sin(t) & -i \cos(t) & -i \sin(t) \end{bmatrix} = 2^5
\]

for \( 0 \leq t \leq \pi \) over the path \( \tau'_1 \).

Over \( \tau''_1 \), according to (C.23), (C.28) and (C.32), together with the remark following it, one has

\[
\begin{align*}
\tilde{Y}''_1(u(t), 0, 0) &= -2 \sin(t) (1 + J)w_2 + 2 \cos(t) (w_2 + Jw_4), \\
\tilde{Y}''_2(u(t), 0, 0) &= 2(1 + J) (w_1 + w_3), \\
\tilde{Y}''_3(u(t), 0, 0) &= 2(1 + J) (w_2 + w_4), \\
\tilde{Y}''_4(u(t), 0, 0) &= w_1 + Jw_3,
\end{align*}
\]

for \( \pi \leq t \leq \frac{3}{2} \pi \). Hence

\[
\mathcal{D} = \mathcal{D}(t) = \begin{bmatrix} 0 & 2(\cos(t) - \sin(t)(1 + i)) & 0 & 2i \cos(t) \\ 2(1 + i) & 0 & 2(1 + i) & 0 \\ 0 & 2(1 + i) & 0 & 2(1 + i) \\ 1 & 0 & i & 0 \end{bmatrix} = 2^5
\]

for \( \pi \leq t \leq \frac{3}{2} \pi \) over \( \tau''_1 \).

Over \( \tau'''_1 \) the only difference compared to (C.35) is that, according to (C.31) and the remark following it, \( \tilde{Y}_4(u(t), 0, 0) = -(w_1 + Jw_3) \). Hence

\[
\mathcal{D} = \mathcal{D}(t) = 2^5(\cos(t) - i \sin(t)) \quad \text{for} \quad \frac{3}{2}\pi \leq t \leq 2\pi
\]

over \( \tau'''_1 \).

Over \( \tau_2 \), according to (C.23), (C.30) and (C.32), one has

\[
\begin{align*}
\tilde{Y}''_1(\theta_1, 0) &= 2(1 + J)w_2, \\
\tilde{Y}''_2(\theta_1, 0) &= 2(1 + J) (w_1 + w_3), \\
\tilde{Y}''_3(\theta_1, 0) &= 2(1 + J) (w_2 + w_4), \\
\tilde{Y}''_4(\theta_1, 0) &= w_1 + Jw_3,
\end{align*}
\]
with $0 \leq \theta_1 \leq \pi$ and $c = \cos(\theta_1)$. Therefore
\[
\mathcal{D} = \mathcal{D}(\theta_1) = \begin{vmatrix}
0 & 2(1 + i) & 0 & 0 \\
2(1 + i) & 0 & 2(1 + i) & 0 \\
0 & 2c(1 + i) & 0 & 2(1 + i) \\
1 & 0 & i & 0
\end{vmatrix} = -2^5 i \quad \text{(C.42)}
\]
for $0 \leq \theta_1 \leq \pi$ over the path $\tau_2$.

The case of $\tau_4$ differs from that of $\tau_2$ only in $\tilde{Y}_4(\theta_1, 0) = -(\mathbf{w}_1 + \mathcal{J}\mathbf{w}_3)$. Thus, over $\tau_4$ we have
\[
\mathcal{D} = \mathcal{D}(\theta_1) = 2^5 i \quad \text{(C.43)}
\]
for $0 \leq \theta_1 \leq \pi$.

Over $\tau_3'$, according to (C.23), (C.29), and (C.32), one has
\[
\begin{align*}
\tilde{Y}_1'(u(t), \pi, 0) &= 2 \sin(t)(1 + \mathcal{J})\mathbf{w}_2 + 2 \cos(t)(-\mathbf{w}_2 + \mathcal{J}\mathbf{w}_4), \\
\tilde{Y}_2'(u(t), \pi, 0) &= 2(1 + \mathcal{J})(\mathbf{w}_1 + \mathbf{w}_3), \\
\tilde{Y}_3'(u(t), \pi, 0) &= 2(1 + \mathcal{J})(-\mathbf{w}_2 + \mathbf{w}_4), \\
\tilde{Y}_4'(u(t), \pi, 0) &= \mathbf{w}_1 + \mathcal{J}\mathbf{w}_3,
\end{align*}
\]
with $\frac{\pi}{2} \leq t \leq \pi$. Therefore
\[
\mathcal{D} = \mathcal{D}(t) = \begin{vmatrix}
0 & 2(-\cos(t) + \sin(t)(1 + i)) & 0 & 2i \cos(t) \\
2(1 + i) & 0 & 2(1 + i) & 0 \\
0 & -2(1 + i) & 0 & 2(1 + i) \\
1 & 0 & i & 0
\end{vmatrix} = 2^5 (\cos(t) - i \sin(t)), \quad \text{(C.45)}
\]
for $\frac{\pi}{2} \leq t \leq \pi$ over $\tau_3'$.

Over $\tau_3''$ the only difference compared to (C.44) is that, according to (C.31),
\[
\tilde{Y}_4'(u(t), \pi, 0) = -(\mathbf{w}_1 + \mathcal{J}\mathbf{w}_3). \quad \text{Hence}
\]
\[
\mathcal{D} = \mathcal{D}(t) = -2^5(\cos(t) - i \sin(t)) \quad \text{for} \quad 0 \leq t \leq \frac{\pi}{2} \quad \text{(C.46)}
\]
over $\tau_3''$.

Finally, over $\tau_3'''$, according to (C.23), (C.29), and (C.35), one has
\[
\begin{align*}
\tilde{Y}_1'(u(t), \pi, 0) &= -2 \sin(t)(\mathbf{w}_1 + \mathcal{J}\mathbf{w}_3) + 2 \cos(t)(-\mathbf{w}_2 + \mathcal{J}\mathbf{w}_4), \\
\tilde{Y}_2'(u(t), \pi, 0) &= 2(1 + \mathcal{J})(\mathbf{w}_1 + \mathbf{w}_3), \\
\tilde{Y}_3'(u(t), \pi, 0) &= 2(1 + \mathcal{J})(-\mathbf{w}_2 + \mathbf{w}_4), \\
\tilde{Y}_4'(u(t), \pi, 0) &= -\cos(t)(\mathbf{w}_1 + \mathcal{J}\mathbf{w}_3) + \sin(t)(\mathbf{w}_2 - \mathcal{J}\mathbf{w}_4)
\end{align*}
\]
for $\pi \leq t \leq 2\pi$. Hence
\[
\mathcal{D} = \mathcal{D}(t) = \begin{vmatrix}
-2 \sin(t) & -2 \cos(t) & -2i \sin(t) & 2i \cos(t) \\
2(1 + i) & 0 & 2(1 + i) & 0 \\
0 & -2(1 + i) & 0 & 2(1 + i) \\
-\cos(t) & \sin(t) & -i \cos(t) & -i \sin(t)
\end{vmatrix} = -2^5, \quad \text{(C.47)}
\]
for $\pi \leq t \leq 2\pi$ over $\tau_3'''$.

It follows from (C.37) - (C.47) that the values of $\mathcal{D}$ along the contour $\mathcal{K}$ lie in $\mathbb{C} - \{0\}$ and transverse once the circle of radius $2^5$ with center at 0 in the clock-wise direction. Hence, the winding number of $\mathcal{D}$ with respect to 0 along $\mathcal{K}$ is equal to $-1$,
\[
\text{Ind}_{\mathcal{K}}(\mathcal{D}, 0) = -1 \quad \text{(C.48)}
\]
Let us recall that the sphere $\tilde{S}$ is oriented in such a way that the orientation of $U_3 = [0, \pi] \times \mathbb{R}/2\pi\mathbb{Z}$ is equal to the product of the standard orientations of $[0, \pi]$ and of $\mathbb{R}/2\pi\mathbb{Z}$. The contour $K$ is then positively oriented.

We need also to study the sections $\tilde{Y}_1, \ldots, \tilde{Y}_4$ in a neighbourhood of the subset $\tilde{C}_1$. Let $W'$ be a neighbourhood of $\tilde{C}_1$ in $\tilde{S}$ analogous to the neighbourhood $W$ of $\tilde{C}_0$. Let $K'$ be a contour in $W'$ analogous to the contour $K$ in $W$. Let $K'$ be oriented in the same way with respect to the orientation of $\tilde{S}$ as $K$ is. Let $D'$ be the corresponding determinant function. By calculations analogous to those in the case of $D$ one proves that also in that case

$$\text{Ind}_{K'}(D', 0) = -1.$$  \hfill (C.49)

(Actually, if $K'$ is obtained from $K$ through a translation along the $\theta_2$-axis then we get $D' = -D$. That proves (C.49).)

As $\tilde{Y}_1, \ldots, \tilde{Y}_4$ are sections of the bundle $f^*(\zeta)$ which are $\mathbb{C}$-linearly independent over $\tilde{S} - (\tilde{C}_0 \cup \tilde{C}_1)$ we get

$$\langle [\tilde{S}], c_1(f^*(\zeta)) \rangle = \text{Ind}_{K}(D, 0) + \text{Ind}_{K'}(D', 0) = -2.$$  \hfill (C.10)

Since, according to (C.10), $c_1(\zeta) = c_1(\xi)$, one has finally

$$\langle f_*(\tilde{S}), c_1 \rangle = \langle f_*(\tilde{S}), c_1(\mathcal{M}) \rangle = \langle f_*(\tilde{S}), c_1(\xi) \rangle =$$

$$\langle f_*(\tilde{S}), c_1(\zeta) \rangle = \langle [\tilde{S}], c_1(f^*(\zeta)) \rangle =$$

$$= -2.$$  \hfill (C.49)

That proves Theorem C.1. \hfill \square

**References**

[1] E. Artin, *Geometric Algebra*, Interscience Publ., New York 1957.

[2] D. Bar-Natan, *On Khovanov’s categorification of the Jones polynomial*, Alg. Geom. Topol. 2 (2002) 337-370.

[3] J. Birman, *Braids, Links, and Mapping Class Groups*, Princeton Univ. Press, 1975.
[4] J. Bochnak, M. Coste, M. F. Roy, *Real Algebraic Geometry*, Springer-Verlag, Berlin 1987.
[5] G. Burde, H. Zieschang, *Knots*, Walter de Gruyter, Berlin 1985.
[6] A. Champanerkar, I. Kofman, *Spanning trees and Khovanov homology*, arXiv:math.GT/0607510.
[7] S. Garoufalidis, *A conjecture on Khovanov's invariants*, preprint, 2001.
[8] K. Guruprasad, J. Huebschmann, L. Jeffrey, A. Weinstein, *Group systems, groupoids, and moduli spaces of parabolic bundles*, Duke Math. Journal 89 (1997), 377-412.
[9] V. Guillemin, *Clean intersection theory and Fourier integrals*, in Fourier Integral Operators and Partial Differential Equations, Lecture Notes in Mathematics vol. 459, Springer Berlin/Heidelberg 1975, 23-35.
[10] J. Huebschmann, *Symplectic and Poisson structures of certain moduli spaces*, I, Duke Math. Journal 80 (1995), 737-756.
[11] P. Hilton, U. Stammbach, *A Course in Homological Algebra*, Springer-Verlag, New York 1971.
[12] K. Jänich, *Differenzierbare G-mannigfaltigkeiten*, Lecture Notes in Mathematics vol. 59, Springer-Verlag, Berlin 1968.
[13] M. Khovanov, *A categorification of the Jones polynomial*, Duke. Math. Journal 2000.
[14] M. Khovanov, *Patterns in knot cohomology*, Experimental Mathematics 12 (2003).
[15] E. Klassen, *Representations of Knot Groups in SU(2)*, Trans. Am. Math. Soc. vol. 326 (1991), 795-828.
[16] KnotTheory, a knot theory Mathematica package, http://katlas.math.toronto.edu/wiki/KnotTheory
[17] J. L. Koszul, *Sur certain groupes de transformations de Lie*, Coll. Int. Centre Nat. Rech. Sci. 52 Géométrie Différentielle (1953), 137-142.
[18] P. B. Kronheimer, T. S. Mrowka, *Knot homology groups from instantons*, preprint arXiv:math.GT/0806.1053, p. 1-119.
[19] E. S. Lee, *An endomorphism of the Khovanov invariant*, Adv. Math. 197 (2005), 554-586.
[20] X. S. Lin, *A knot invariant via representation spaces*, J. Differential Geometry 35 (1992), 337-357.
[21] S. Mac Lane, *Homology*, Springer-Verlag, Berlin 1963.
[22] J. Milnor, *Morse Theory*, Princeton Univ. Press, 1963.
[23] J. Milnor, *Lectures on h-cobordism theorem*, Princeton Univ. Press, 1965.
[24] R. Narasimhan, *Analysis on real and complex manifolds*, North-Holland, Amsterdam, 1968.
[25] R. Rubinsztein, *Topological quandles and invariants of links*, J. Knot Theory and its Ramifications 16 (2007), 789-808.
[26] A. Shumakovitch, *Torsion of the Khovanov homology*, preprint arXiv:math.GT/0405474, p. 1-18.
[27] V. S. Varadarajan, *Lie groups, Lie algebras and their representations*, Prentice-Hall, Englewood Cliffs, New Jersey, 1974.
[28] A. Weinstein, *"The symplectic structure on moduli spaces"*, in The Floer Memorial Volume, ed. by H. Hofer, C. Taubes, A. Weinstein and E. Zehnder, Progr. Math. 133, Birkhäuser Verlag, Basel, 1995, 627-635.

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