Binary and ternary quasi-perfect codes with small dimensions

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1 Introduction

Let $F_q^n$ be the $n$-dimensional vector space over the finite field with $q$ elements $GF(q)$. A linear code $C$ is a $k$-dimensional subspace of $F_q^n$. For $x, y \in F_q^n$ let $d(x, y)$ denote the Hamming distance between $x$ and $y$, which is equal to the number of positions where $x$ and $y$ differ. The minimum Hamming distance for a code $C$ is defined by

$$d(C) = \min_{c_1, c_2 \in C, c_1 \neq c_2} d(c_1, c_2)$$

and the Hamming weight $w(x)$ of a vector $x \in F_q^n$ is defined by

$$w(x) = d(x, 0)$$

where 0 is the all zero vector. The packing radius $e(C)$ of the code is

$$e(C) = \left\lfloor \frac{d(C) - 1}{2} \right\rfloor$$

and this is the maximum weight of successfully correctable errors. The ball of radius $t$ around a word $y \in F_q^n$ is defined by

$$\{x | x \in F_q^n, d(x, y) \leq t \}.$$  

Then $e(C)$ is the largest possible integer number such that the balls of radius $e(C)$ around the codewords are disjoint. The covering radius $R(C)$ of a code $C$ is defined as the least possible integer number such that the balls of radius $R(C)$ around the codewords cover the whole $F_q^n$, i.e.

$$R(C) = \max_{x \in F_q^n} \min_{c \in C} d(x, c).$$
With these notations a $q$-ary linear code of length $n$, dimension $k$, minimum distance $d$ and covering radius $R$ is denoted by $[n, k, d]_q R$.

A coset of the code $C$ defined by the vector $x \in F_q^n$ is the set $x + C = \{ x + c \mid c \in C \}$. A coset leader of $x + C$ is a vector in $x + C$ of smallest weight. When the code is linear its covering radius is equal to the weight of the heaviest coset leader. The covering radius of a linear code can also be defined in terms of the parity check matrix.

**Theorem 1:** [23] Let $C$ be a $[n, k]$ code with parity check matrix $H$. The covering radius of $C$ is the smallest integer $R$ such that every $q$-ary $(n - k)$-tuple can be written as a linear combination of at most $R$ columns of $H$.

The special case are codes for which $R(C) = e(C)$ and such codes are called **perfect codes**. The problem of finding all perfect codes was begun by Go
tay in 1949 and completed in 1973 by Tietäväinen [11] and independently by Zinov'ev and Leont'ev [12]. The only perfect codes are: $[n, n, 1]_q$ codes for each $n \geq 1$; $[2s + 1, 1, 2s + 1]_2$ repetition codes for each $s \geq 1$; code of length $n$ containing only one codeword; $q$-ary codes with the parameters of Hamming codes; the $[23, 12, 7]_2$ binary Golay code; the $[11, 6, 5]_3$ ternary Golay code.

The next step in this direction is to consider codes for which packing and covering radii differ by 1, i.e. **quasi-perfect codes**. A code is called quasi-perfect (QP) if its packing radius is $e$ and its covering radius is $e + 1$, for some non-negative integer $e$. Clearly, the minimum distance of such a code is $2e + 1$ or $2e + 2$. Then a natural question is which codes are quasi-perfect? It is clear that any code with covering radius 1 and minimum distance 1 or 2 is quasi-perfect. Therefore, quasi-perfect codes with covering radius 1 are not interesting and we will focus on the investigation of quasi-perfect codes with covering radius greater than 1.

### 2 Known results about quasi-perfect codes with covering radius greater than 1

Quasi-perfect codes with covering radius 2 and 3 were extensively studied and many infinite families of binary, ternary and quaternary QP codes are known. In particular, codes with parameters $[n, k, d]_q 2 d = 3, 4$ are QP. These codes are connected with 1-saturating sets in projective spaces $PG(n - k - 1, q)$ and a lot of infinite families of such codes are described in the literature (see [1] - [10]). The following theorem leads to a chain of QP codes.

**Theorem 2:** Assume that an $[n, k, d]_q 2$ QP code with $n \leq \frac{q^{n-k}-1}{q-1} - 2$ and $3 \leq d \leq 4$ exists. Then an $[n + 1, k + 1, 3]_q 2$ QP code exists.

**Proof.** Let we add a column to a parity check matrix of a $[n, k, d]_q 2$ code to obtain a new $[n + 1, k + 1, d_{\text{new}}]_q R_{\text{new}}$ code. According to the Theorem 1 $R_{\text{new}} \leq 2$. As the new length $n + 1 \leq \frac{q^{n-k}-1}{q-1} - 1$, it is impossible that
3 Classification of binary and ternary quasi-perfect linear codes

$R_{new} = 1$. Also for any new column it holds that $d_{new} \leq 3$. If the new column is not obtained by multiplication of an old column by an element of the field $GF(q)$ then $d_{new} = 3$. The right choice of the new column is possible as $n \leq \frac{q^{n-k} - 1}{q-1} - 2$.

$[n, n-4, 5]_q$ QP codes correspond to complete arcs in the projective space $PG(3,q)$ and are investigated in [7], [8]. Survey and new results for the binary QP codes can be found in [13]. It appears that there is a great variety of QP codes of minimum distance up to 5.

Considerably less is known for $q$-ary QP codes with $q > 2$. One infinite family of ternary codes is known due to Gashkov and Sidel’nikov [1]. The family members are $[(3s + 1)/2, (3s + 1)/2 - 2s, 5]_3$ codes. Quasi-perfectness of two families of quaternary codes, namely $[(4s - 1)/3, (4s - 1)/3 - 2s, 5]_4$ and $[(2s^2 + 1)/3, (2s^2 + 1)/3 - 2s - 1, 5]_4$, presented in [15] and [16] was shown by Dodunekov [17, 18].

The first computer searches to find new quasi-perfect codes were by Wagner in 1966 [19]. He proposed a tree-search program which uses the properties of parity-check matrices of binary linear quasi-perfect codes to find such codes. Fixing the number of check digits and the number of errors to be corrected, the program finds one quasi-perfect code for each block length if such a code exists. Using this program 27 new binary linear QP codes were found [19, 20]. The codes have lengths between 19 and 55 and all have covering radius 3. Later Simonis [21] proved that one of Wagner’s codes, namely $[23, 14, 5]$, is unique.

Baicheva, Dodunekov and Kötter [22] investigated the weight structure and error-correcting performance of the ternary $[13, 7, 5]$ quadratic-residue code and showed that the covering radius of the code is equal to three, i.e. it is a quasi-perfect code. Recently Danev and Dodunekov [23] proved that this code is the first member of a family of ternary QP codes with parameters $[(3s - 1)/2, (3s - 1)/2 - 2s, 5]_3$ for all odd $s \geq 3$.

All these results lead to the following question: How restrictive is quasi-perfectness, i.e. are there inequivalent quasi-perfect codes? In our work we classify all binary of dimension up to 9 and ternary of dimension up to 6 linear QP codes as well as give some partial classifications for dimensions up to 14 and 13 respectively. It turned out that there are many cases where more than one QP code for fixed length and dimension exists. In this way we answer the above question.

3 Classification of binary and ternary quasi-perfect linear codes

The approach used in this work is based on the classification of codes with given parameters. First we fix the dimension of the code and determine the possible lengths and minimum distances of the codes which could be quasi-perfect. Then we classify all such codes and finally compute their covering radii. In this way we determine all quasi-perfect codes with the fixed parameters. In order to
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determine the parameters of possible candidates for quasi-perfect codes with
covering radius \( e + 1 \) we take into account that the minimum distance of these
codes can only be \( 2e + 1 \) or \( 2e + 2 \). Brouwer’s tables of bounds on the size of linear codes [24] and the tables for the least covering radius of binary [25] and
ternary linear codes [26] are used to find the possible lengths of QP codes when
the dimension is fixed. Once the parameters (length, dimension and minimum
distance) are determined all codes with these parameters are classified up to
equivalence using the approach of [27].

In the classification of the codes two main approaches were used. The first
one is based on puncturing, the second one on shortening. While in general
the dimension of the code is unchanged by puncturing, this is not true if all
non-zero positions of a codeword are deleted. Let \( G \) be a generator matrix of a
linear \([n, k, d]_q\) code \( C \). Then the residual code \( \text{Res}(C, c) \) of \( C \) with respect to a
codeword \( c \) is the code generated by the restriction of \( G \) to the columns where \( c \)
has a zero entry. A lower bound on the minimum distance of the residual code
is given by

**Lemma 1** [28] Suppose \( C \) is an \([n, k, d]_q\) code and suppose \( c \in C \) has weight \( w \), where \( d > w(q - 1)/q \). Then \( \text{Res}(C, c) \) is an \([n - w, k - 1, d']_q\) code with
\( d' \geq d - w + \lceil w/q \rceil \).

Inverting this operation, we search for an \([n, k, d]_q\) code on the basis of an
\([n - w, k - 1, d']_q\) code (its residual code with respect to a codeword of weight \( w \))
or an \([n - i, k, d']_q\) code (punctured on \( i \) coordinates code). We can apply the
same operation for the residual or punctured code respectively. This procedure
is repeated until we obtain as a start code a code with small parameters such
that all codes having these parameters can easily be classified. For example,
starting from the \([3, 2, 2]_2\) code, we obtain all \([8, 3, 5]_2\) codes, take only the
nonequivalent of them and then obtain all nonequivalent \([28, 4, 20]_2\) codes.

The second approach increases both the length and the dimension of the
code, i.e. we construct \([n, k, d]_q\) codes extending \([n - i, k - i, d]_q\) or \([n - i - 1, k - i, d]_q\) codes. The following result shows when the latter type of code can be used
[29] p. 592.

**Lemma 2** Let \( C \) be an \([n, k, d]_q\) code. If there exists a codeword \( c \in C^\perp \) with
\( wt(c) = i \), then there is an \([n - i, k - i + 1, d]_q\) code.

If \( G \) is a generator matrix for an \([n - i, k - i, d]_q\) or an \([n - i - 1, k - i, d]_q\) code, we extend it (in all possible ways) to

\[
\begin{pmatrix}
* & 1 & I_i \\
G & I & 0
\end{pmatrix}
\] or
\[
\begin{pmatrix}
* & 1 & I_i \\
G & I & 0
\end{pmatrix},
\]

respectively, where \( I_i \) is the \( i \times i \) identity matrix, \( 1 \) is an all-1 column vector,
and the starred submatrix is to be determined. If we let the matrix \( G \) be in
systematic form, we can fix \( k \) more columns to get

\[
\begin{pmatrix}
* & 0 & I_i \\
G_1 & I & 0
\end{pmatrix}
\] or
\[
\begin{pmatrix}
* & 0 & I_i \\
G_1 & I & 0
\end{pmatrix}.
\]

(2)
Again we can apply recursively the same approach to obtain $G_1$ while on the bottom of this hierarchy of extensions is the trivial $[k, k, 1]^q$ code.

In our investigation we are interested in codes with covering radius greater than 1, therefore they have minimum distance at least 3. Thus their dual codes are projective codes. To classify the binary codes with codimension $n-k$ up to 6 we use the results from [30] where all binary projective codes with dimensions up to 6 are classified. Then among the codes from [30] we consider only those having the necessary minimum distance of the dual code. For example, to classify all $[8, 2, 5]$ codes we consider 14 $[8, 6]$ projective codes. The dual code of only one of them has minimum distance 5 and thus we have only one $[8, 2, 5]$ code. In the same way using results from [31] where all ternary projective codes of dimension 4 are classified we classify the ternary linear codes of codimension 4 which could be quasi-perfect.

After the classification was completed we proceed with the determination of the covering radii of the codes. We recall that in the case of linear codes this is equivalent to the determination of the heaviest coset leader. To do this, we use the fact that if the code is in a systematic form, a representative of each coset can be found by generating all words of the form $(0, \ldots, 0, a)$, $a \in F^{n-k}_q$.

Taking into account that each vector of weight less than or equal to $e$ is a unique coset leader, we test only words of the above form and weight greater than $e$. Therefore we have to test at most $\sum_{i=e}^{n-k} \binom{n-k}{i+1} (q-1)^{i+1}$ words because if we obtain a coset leader of weight greater than $e+1$ we stop the check.

Remark. Let us denote by $\alpha_i$ for $i = 0, 1, \ldots, n$ the number of coset leaders of weight $i$. The set of the coset leaders of each QP code is known. All vectors of weights less than or equal to $e$ are coset leaders and thus $\alpha_i = \binom{n}{i} (q-1)^i$ for $i = 0, \ldots, e$. Then for $\alpha_{e+1}$ we get $\alpha_{e+1} = q^k - \sum_{i=0}^{e} \alpha_i$.

4 Results

By the approach described in the previous section all binary and ternary quasi-perfect codes of dimensions up to 9 and 6 correspondingly are determined, as well as some partial results for binary codes of dimensions up to 14 and ternary codes of dimensions up to 13 are obtained. The results are summarized in Table I.

Some of the codes from the table are not new and have already been constructed in previous works. We will note that QP codes with minimum distances 3 or 4 and covering radius 2 are connected with 1-saturating sets in projective spaces $PG(n-k-1, q)$ in the following way: the points of a 1-saturating $n$-set can be considered as $n-k$-dimensional columns of a parity-check matrix of an $[n, k]^q_2$ code. Also QP codes with minimum distance 4 are complete caps in $PG(n-k-1, q)$. Constructions of minimal 1-saturating sets and complete caps in binary projective spaces $PG(k-1, 2)$ are described in [4], [5], [6] - [9], [32]. Codes obtained in these works are marked with a *. Some of the marked codes are also obtained in [32] where recursive constructions of complete caps
Tab. 1: Binary and Ternary Quasi-perfect Codes

**Binary quasi-perfect codes**

| Code   | All | QP | Code   | All  | QP  |
|--------|-----|----|--------|------|-----|
| [5, 2, 3] | 1   | 1  | [14, 9, 3] | 126  | 113 |
| [6, 3, 3] | 1   | 1  | [15, 9, 3] | 11464 | 380 |
| [8, 2, 5]* | 1   | 1  | [17, 9, 5] | 1    | 1   |
| [7, 3, 3] | 3   | 2  | [14, 10, 3] | 1    | 1   |
| [8, 4, 4]* | 1   | 1  | [15, 10, 3] | 142  | 131 |
| [8, 4, 3] | 4   | 4  | [16, 10, 3] | 28900 | 2296 |
| [9, 4, 4]* | 4   | 1  | [19, 10, 5] | 31237 | 13  |
| [8, 4, 3] | 19  | 1  | [16, 11, 4]* | 1    | 1   |
| [11, 4, 5] | 1   | 1  | [16, 11, 3] | 143  | 143 |
| [9, 5, 3] | 5   | 5  | [17, 11, 4]* | 39   | 5   |
| [10, 5, 4]* | 4   | 1  | [17, 11, 3] | 70416 | 12221 |
| [10, 5, 3] | 37  | 12 | [20, 11, 5] | 13924 | 565 |
| [10, 6, 3] | 4   | 4  | [17, 12, 3] | 129  | 129 |
| [11, 6, 3] | 58  | 25 | [18, 12, 4]* | 33   | 1   |
| [14, 6, 5] | 11  | 1  | [21, 12, 5] | 2373 | 666 |
| [11, 7, 3] | 3   | 3  | [22, 12, 6] | 128  | 1   |
| [12, 7, 3] | 84  | 55 | [24, 12, 8] | 1    | 1   |
| [13, 7, 4]* | 45  | 1  | [24, 12, 7] | 11   | 11  |
| [13, 7, 3] | 1660| 7  | [25, 12, 8] | 7    | 2   |
| [15, 7, 5] | 6   | 4  | [18, 13, 3] | 113  | 113 |
| [12, 8, 3] | 2   | 2  | [19, 13, 3] | 366064 | 185208 |
| [13, 8, 3] | 109 | 88 | [22, 13, 5] | 128  | 120 |
| [14, 8, 3] | 4419| 65 | [19, 14, 3] | 91   | 91  |
| [13, 9, 3] | 1   | 1  | [20, 14, 4]* | 24   | 1   |

**Ternary quasi-perfect codes**

| Code   | All | QP | Code   | All  | QP  |
|--------|-----|----|--------|------|-----|
| [5, 2, 3]* | 2   | 2  | [11, 7, 3] | 339  | 319 |
| [6, 3, 3] | 1   | 1  | [12, 7, 3] | 60910 | 1   |
| [7, 4, 3] | 4   | 4  | [13, 7, 5] | 6    | 5   |
| [8, 4, 4]* | 3   | 2  | [11, 8, 3] | 1    | 1   |
| [8, 4, 3] | 37  | 5  | [12, 8, 3] | 805  | 753 |
| [8, 5, 3] | 3   | 3  | [14, 8, 5] | 1    | 1   |
| [9, 5, 3] | 87  | 23 | [12, 9, 3] | 1    | 1   |
| [9, 6, 3] | 3   | 3  | [13, 9, 3] | 1504 | 1479 |
| [10, 6, 4]* | 1   | 1  | [14, 10, 3] | 2695 | 2659 |
| [10, 6, 3] | 195 | 102| [15, 11, 3] | 4304 | 4304 |
| [12, 6, 6] | 1   | 1  | [16, 12, 3] | 6472 | 6472 |
| [12, 6, 5] | 36  | 18 | [17, 13, 3] | 8846 | 8846 |
| [10, 7, 3] | 2   | 2  |
in $PG(n - k - 1, 2)$ are given. Existing of codes with parameters $[10, 5, 3]_2^2$, $[14, 6, 5]_3^2$ and $[13, 7, 3]_2^2$ is shown in [1]. The $[17, 9, 5]_2^3$ code is the first representative from the infinite family of $[2^{2s} + 1, 2^{2s} + 1 - 4, 5]_2$, $s \geq 2$. The codes which are proved to be quasi-perfect by Dodunekov [17]. [19, 10, 5]_2^3, [20, 11, 5]_3^3, [23, 14, 5]_3^3 and [24, 14, 6]_3^3 are among the QP codes obtained by a computer search by Wagner. He obtained only one representative for each of the parameters. Our classification shows that QP codes with the first two parameters are not unique. There are additionally 12 $[19, 10, 5]_2^2$ and 564 $[20, 11, 5]_2^2$ quasi-perfect codes. $[24, 12, 8]_2$ is the well known extended Golay code which is also known to be a quasi-perfect one.

For a completeness of the classification results about QP codes, we will note some not classified in this work such codes. In [8] the unique $[6, 1, 5]_3^3$ and in [7] the unique $[5, 1, 5]_3^3$ codes are presented. As complete caps in $PG(4, 3)$ the $[16, 11, 4]_3^2$, $[17, 12, 4]_3^2$ and $[18, 13, 4]_3^2$ codes in [4] and in $PG(6, 2)$ the $[21, 14, 4]_2^2$ code in [2] are obtained. In [22] it is shown that there are 5 nonequivalent $[21, 14, 4]_2^2$ codes. Also applying Theorem 2 to codes from the Table the following chains of QP codes’ parameters can be obtained.

\[
[5, 2, 3]_2^2 \rightarrow [6, 3, 3]_2^2; \\
[8, 4, 4]_2^2 \rightarrow \ldots \rightarrow [14, 10, 3]_2^2; \\
[9, 4, 4]_2^2 \rightarrow \ldots \rightarrow [30, 25, 3]_2^2; \\
[13, 7, 4]_2^2 \rightarrow \ldots \rightarrow [18, 12, 3]_2^2 \rightarrow [19, 13, 3]_2^2 \rightarrow [20, 14, 3]_2^2 \rightarrow \ldots \rightarrow [62, 56, 3]_2^2; \\
[5, 2, 3]_3^2 \rightarrow [12, 3, 3]_3^2; \\
[8, 4, 4]_3^2 \rightarrow \ldots \rightarrow [17, 13, 3]_3^2 \rightarrow [18, 14, 3]_3^2 \rightarrow \ldots \rightarrow [40, 36, 3]_3^2; \\
[12, 7, 3]_3^2 \rightarrow [13, 8, 3]_3^2 \rightarrow \ldots \rightarrow [121, 116, 3]_3^2.
\]

Codes not classified in this work are boldfaced.

Until this work the only known examples of QP codes with minimum distance greater than 5 were binary repetition codes, the $[24, 12, 8]_2^4$ extended Golay code, the $[22, 12, 6]_2^3$ punctured Golay code, $[7, 1, 7]_3^4$ and $[8, 1, 7]_2^4$ codes classified in [8]. We provide examples of more such codes and in this way answer the first open question from the recent paper of Etzion and Mounits [13] where to find new or to prove the nonexistence of QP codes with $d > 5$ is suggested. The most interesting are $[24, 12, 7]_2^4$ and $[25, 12, 8]_2^4$ codes which are the first examples of quasi-perfect codes with $R = 4$ except the $[24, 12, 8]_2^4$ extended Golay and the $[8, 1, 7]_2^4$ repetition codes. The generator matrices of these codes are given in the Appendix. The codes are in a systematic form with generator matrix $G = [I_k | A]$ and the identity matrix $I_k$ is omitted in order to save space.

## 5 Conclusions

In this work classification results about binary and ternary linear quasi-perfect codes of small dimensions are obtained. More precisely, all binary QP codes of dimensions up to 9 and ternary QP codes of dimensions up to 6 are classified as well as some partial classifications about QP codes of dimensions up to 14 are got. The results show that for each dimension there are only few possible
lengths for which quasi-perfect codes exist. For some parameters hundreds and thousands of nonequivalent QP codes are found which means that quasi-perfectness is not so restrictive characteristic of the code. QP codes of minimum distance greater than 5 are obtained and therefore it could be expected that at greater dimensions QP codes with bigger covering radii exist. Thus it will be an interesting research problem to answer the following questions:

- Are there quasi-perfect codes with minimum distance greater than 8 except the binary repetition code?
- Is there an upper bound about minimum distance of a QP code?

At the end we will conclude with the observation that the classification of all parameters of QP codes would be much more difficult than the similar one for perfect codes.

6 Acknowledgement

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7 Appendix. Generator matrices of binary quasi-perfect codes with minimum distance 7 and 8

A. \([24, 12, 7], 4\) QP codes

\[
A_1 = \begin{pmatrix}
010010110101 \\
101000111001 \\
111010010010 \\
000110011011 \\
010101010110 \\
101100001110 \\
111111011101 \\
111010010010 \\
010101010111 \\
011000011111 \\
000111111110
\end{pmatrix}
\]

\[
A_2 = \begin{pmatrix}
110001001011 \\
001011000111 \\
100101100101 \\
010101010110 \\
101100001110 \\
011100100011 \\
111100110100 \\
100110110110 \\
110110000111 \\
111011100001 \\
000111111110
\end{pmatrix}
\]

\[
A_3 = \begin{pmatrix}
010101001011 \\
001100111001 \\
101110001110 \\
110110100001 \\
111110110111 \\
111100011111 \\
111011001101 \\
101010111001 \\
011001000111 \\
000111111110
\end{pmatrix}
\]

\[
A_4 = \begin{pmatrix}
100010111101 \\
001011010011 \\
101100001110 \\
110111010100 \\
111010110100 \\
111101111111 \\
111110100001 \\
010110010101 \\
110101101101 \\
100111111100
\end{pmatrix}
\]
| \( A_5 \)                  | \( A_6 \)                  |
|--------------------------|--------------------------|
| 000010111101             | 0000111011110            |
| 011100110100             | 101010101011             |
| 011001101001             | 101100101011             |
| 110100001101             | 010011110001             |
| 110000110001             | 010110101100             |
| 101101111111             | 111101111111             |
| 101110100001             | 111110000010             |
| 111100101010             | 111011001101             |
| 111101010001             | 111010111110             |
| 100110011011             | 010110011011             |
| 001111111000             | 001111111000             |
| 010111001111             | 000111100011             |
| 110001010101             | 000110101011             |
| 001011011001             | 001110010110             |
| 111010001001             | 100011001100             |
| 001111100010             | 100110110001             |
| 001111100010             | 101101111111             |
| 011011001100             | 101111001000             |
| 001110011001             | 101010101010             |
| 001110011001             | 101010111110             |
| 111110000001             | 100110001111             |
| 101101100010             | 001111100011             |
| 101101010111             | 010111111010             |
| 111110011100             | 010001111111             |
| 110011110101             | 010000111110             |
| 011000111011             | 000111111100             |

| \( A_7 \)                  | \( A_8 \)                  |
|--------------------------|--------------------------|
| 010110111110             | 110001010110             |
| 011011101100             | 111111011111             |
| 001111000111             | 101010111100             |
| 100011010111             | 010110101011             |
| 110111000110             | 010011010101             |
| 111100100001             | 101011001010             |
| 101011000101             | 011011110000             |
| 101101010111             | 111101000010             |
| 111110011100             | 101101010010             |
| 110011110101             | 010111000110             |
| 011000111011             | 010000111111             |
| 000111111100             | 000111111110             |

| \( A_9 \)                  | \( A_{10} \)                |
|--------------------------|---------------------------|
| 0001111000110            | 110001010110             |
| 101101010101             | 111111011111             |
| 101100101011             | 101010111100             |
| 110001010101             | 010110101010             |
| 100011010010             | 010110101010             |
| 101101010111             | 101010101010             |
| 111110011100             | 011011110000             |
| 110011110101             | 111101000010             |
| 011000111011             | 101101010010             |
| 000111111100             | 010111000110             |
| 000111001111             | 010001111111             |

| \( A_{11} \)             |
|--------------------------|
| 000111111100             |
| 101101010101             |
| 101100101011             |
| 110000110111             |
| 110011011100             |
| 111011100001             |
| 111110010010             |
| 011010111110             |
| 011001011011             |
| 010110110101             |
| 001111111000             |
| 000111001111             |
Appendix. Generator matrices of binary quasi-perfect codes with minimum distance 7 and 8

B. [25, 12, 8]_4 QP codes

\[ A_1 = \begin{pmatrix}
1101101100100 \\
1101001111001 \\
1110100001101 \\
1011001110010 \\
1011010101100 \\
1000100111110 \\
0111100101010 \\
0111101010001 \\
0100110011011 \\
0001111111000
\end{pmatrix} \quad A_2 = \begin{pmatrix}
1101101110000 \\
110001011110 \\
1001100101110 \\
1001101001101 \\
0101011010101 \\
0101010111010 \\
0100110001111 \\
0001111100011 \\
0010111111100
\end{pmatrix} \]

References

[1] R.L. Graham and N.J.A. Sloane, "On the covering radius of codes", *IEEE Trans. Inf. Theory*, vol. 31, No. 3, pp. 385-401, May, 1985.

[2] E.M. Gabidulin, A.A. Davydov, and L.M. Tombak, "Linear codes with covering radius 2 and other new covering codes", *IEEE Trans. Inf. Theory*, vol. 37, No. 1, pp. 219-224, 1991.

[3] J.W.P. Hirschfeld and L. Storme, "The packing problem in statistics, coding theory and finite projective spaces: Update 2001", in *Developments in Mathematics, vol. 3, Finite geometries*, A. Blokhuis, J.W.P. Hirschfeld, D. Jungnickel, and J.A. Thas, Eds. Dordrecht, The Netherlands: Kluwer, 2000, pp. 201-246.

[4] G. Faina, S. Marcugini, A. Milani and F. Pambianco, "The sizes \(k\) of complete \(k\)-caps in \(\text{PG}(n,q)\) for small \(q\) and \(3 \leq n \leq 5\), *Ars Combinatoria*, vol. 50, pp. 235-243, 1998.

[5] A.A. Davydov, "Constructions and families of nonbinary linear codes with covering radius 2", *IEEE Trans. Inf. Theory*, vol. 45, No. 5, pp. 1679-1686, July, 1999.

[6] A.A. Davydov, S. Marcugini, and F. Pambianco, "On saturating sets in projective spaces", *J. Combin. Theory, Ser. A*, vol. 103, pp.1-15, 2003.

[7] A.A. Davydov, S. Marcugini, and F. Pambianco, "Linear codes with covering radius 2,3 and saturating sets in projective geometry", *IEEE Trans. Inf. Theory*, vol. 50, pp. 537-541, March, 2004.

[8] A.A. Davydov, G. Faina, S. Marcugini, and F. Pambianco, "Locally optimal (non-shortening) linear covering codes and minimal saturating sets in projective spaces", *IEEE Trans. Inf. Theory*, vol. 51, No. 12, pp. 4378-4387, Dec., 2005.
7 Appendix. Generator matrices of binary quasi-perfect codes with minimum distance 7 and 8

[9] A.A. Davydov, S. Marcugini and F. Pambianco, Minimal 1-saturating sets and complete caps in binary projective spaces, J. Combinatorial Theory, Ser. A, vol. 113, pp. 647-663, 2006.

[10] M. Giulietti and F. Pasticci, "Quasi-perfect linear codes with minimum distance 4", IEEE Trans. Inf. Theory, vol. 53, No. 5, pp. 1928-1935, May, 2007.

[11] A. Tietäväinen, On the nonexistence of perfect codes over finite fields, SIAM J. Appl. Math., vol. 24, 1973, pp. 88-96.

[12] V. A. Zinov’ev and V. K. Leont’ev, On non-existence of perfect codes over Galois fields, Problems of Control and Information Theory/Problemy Upravlenija i Teorii Informazi, vol. 2, pp. 123-132, 1973.

[13] T. Etzion and B. Mounits, Quasi-perfect codes with small distance, IEEE Trans. Inf. Theory, vol.51, No 11, pp. 3938-3946, 2005.

[14] I.B. Gashkov and V.M. Sidel’nikov, Linear ternary quasiperfect codes correcting double errors, Problems of Information Transmission, vol. 22, No. 4, pp. 284-288, 1986.

[15] D.N. Gevorkijan, A.M. Avetisjan and G.A. Tigranjan, On the construction of codes correcting two errors in Hamming’s metric over Galois field, Vichislitel’naja technika, vol. 3, pp. 19-21, 1975 (in Russian).

[16] I.I. Dumer and V.A. Zinov’ev, Some new maximal codes over GF(4), Problems of Information Transmission, vol. 14, No. 3, pp. 174-181, 1978.

[17] S.M. Dodunekov, The optimal double-error correcting codes of Zetterberg and Dumer-Zinov’ev are quasiperfect, C. R. Acad. Bulgare Sci, vol.38, No 9, pp. 1121-1123, 1985.

[18] S.M. Dodunekov, Some quasiperfect double error correcting codes, Problems Control Inform. Theory/Problemy Upravl. Teor. Inform., vol. 15, No 5, pp. 367-375, 1986.

[19] T. Wagner, A search technique for quasi-perfect codes, Information and Control, vol. 9, pp. 94-99, 1966.

[20] T. Wagner, Some additional quasi-perfect codes, Information and Control, vol. 10, p. 334, 1967.

[21] J. Simonis, The [23,14,5] Wagner code is unique, Discrete Mathematics, vol. 213, pp. 269-282, 2000.

[22] T. Baicheva, S. Dodunekov and R. Kötter, On the Performance of the Ternary [13,7,5] Quadratic-Residue Codes, IEEE Trans. Inf. Theory, vol. 48, No. 2, pp. 562-564, 2002.
[23] D. Danev and S. Dodunekov, A family of ternary quasi-perfect codes, *Proc. International workshop on coding and cryptography*, Versailles, France, April 16-20, 2007, pp.109-115.

[24] A. E. Brouwer, Bounds on the size of linear codes, in *Handbook of Coding Theory*, V. S. Pless and W. C. Huffman, eds., Elsevier, Amsterdam, pp. 295-461, 1998.

[25] G. Cohen, I. Honkala, S. Litsyn and A. Lobstein, *Covering Codes*, North-Holland, Elsevier Science B.V., 1997.

[26] T. Baicheva and E. Velikova, Covering radii of ternary linear codes of small dimensions and codimensions, *IEEE Trans. Inf. Theory*, vol. 43, pp. 2057-2061, 1997.

[27] I. Boukliev, 'Q - EXTENSION’- strategy in algorithms, *Proc. of the International Workshop ACCT*, Bansko, Bulgaria, 2000, pp. 84-89.

[28] S.M. Dodunekov, Minimal block length of a q-ary code with prescribed dimension and code distance, *Probl. Inform. Transm.*, vol. 20, No 4, pp. 239-249, 1984.

[29] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting codes*, North-Holland Publishing Company, Amsterdam, London, New York, Tokyo, Ninth impression, 1996.

[30] I. Bouyukliev, On the binary projective codes with dimension 6, *Discrete Applied Mathematics*, vol. 154, pp. 1693-1708, 2006.

[31] T. Baicheva and I. Bouyukliev, On the ternary projective codes with dimensions 4 and 5, *Proc of the International Workshop on Algebraic and Combinatorial Coding Theory*, Kranevo, Bulgaria, 2004, pp. 34-39.

[32] M. Khatirinejad and P. Lisoněk, Classification and constructions of complete caps in binary spaces, *Designs, Codes and Cryptography*, vol. 39, pp. 17-31, 2006.