Optimal Young’s inequality and its converse: 
a simple proof

by
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Abstract. We give a new proof of the sharp form of Young’s inequality for convolutions, first proved by Beckner [Be] and Brascamp-Lieb [BL]. The latter also proved a sharp reverse inequality in the case of exponents less than 1. Our proof is simpler and gives Young’s inequality and its converse altogether.

The classical convolution inequality of Young asserts that for all functions \(f \in L^p(\mathbb{R})\) and \(g \in L^q(\mathbb{R})\) we have
\[
\|f * g\|_r \leq \|f\|_p \|g\|_q,
\]
where \(p, q, r \geq 1\) and \(1/p + 1/q = 1 + 1/r\). This inequality is sharp only when \(p\) or \(q\) is one. The best constants in Young’s inequality were found by Beckner [Be], using tensorisation arguments and rearrangements of functions. In [BL], Brascamp and Lieb derived them from a more general inequality, which we will refer to as the Brascamp-Lieb inequality; this Brascamp-Lieb inequality was also successfully applied to several problems in convex geometry by K. Ball (see [B] for one example). The expression of the best constant for Young’s inequality is rather complicated but can be easily memorized via a simple principle: it is obtained when \(f\) and \(g\) are Gaussian functions on the real line, \(f(x) = \exp(-p'x^2)\) and \(g(x) = \exp(-q'x^2)\), where \(p'\) is the conjugate exponent of \(p\). This principle has been largely developed by Lieb in the more recent paper [Li]; among many other results, this paper contains a new proof of the Brascamp-Lieb inequality (let us also mention [Ba] where we give yet another proof).

A reverse form of Young’s inequality was found by Leindler [Le]: for \(0 < p, q, r \leq 1\) and \(f, g\) non-negative,
\[
\|f * g\|_r \geq \|f\|_p \|g\|_q.
\]
Again these inequalities are sharp only when \(p\) or \(q\) is one. The sharp reverse inequalities were obtained by Brascamp and Lieb in the same paper. It is also shown in [BL] that the reverse Young inequalities imply another important inequality, the inequality of Leindler and Prekopa, a close relative of the Brunn-Minkowski inequality ([Le], [Pr]). As far as we know, the proof from [BL] is the only proof available for this sharp reverse Young inequality; in our opinion, it is both rather mysterious and complicated, and uses many
ingredients: tensorisation, Schwarz symmetrisation, Brunn-Minkowski and some not so intuitive phenomenon for the measure in high dimension. To the contrary, our argument is elementary and gives a unified treatment of both cases, the Young inequality and the reverse inequality.

It is well known that tensorisation arguments allow to deduce the multidimensional case from the one-dimensional (see [Be] for example): if the best constant is $C$ for the real line, it will be $C^N$ in the case of $\mathbb{R}^N$. We state now the precise results. For every $t > 0$, we define $t'$ by $1/t + 1/t' = 1$ (notice that $t'$ is negative when $t < 1$). Let us introduce for every $t > 0$

$$C_t = \sqrt{\frac{t^{1/t}}{|t'|^{1/t'}}}.$$

The general multi-dimensional result is as follows:

**Theorem 1.** Let $p, q, r > 0$ satisfy $1/p + 1/q = 1 + 1/r$, and let $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$ be non-negative functions.

If $p, q, r \geq 1$ then

$$\|f * g\|_r \leq \left(\frac{C_p C_q}{C_r}\right)^N \|f\|_p \|g\|_q. \quad (1)$$

If $p, q, r \leq 1$ then

$$\|f * g\|_r \geq \left(\frac{C_p C_q}{C_r}\right)^N \|f\|_p \|g\|_q. \quad (2)$$

It is easy to check that when $N = 1$ and $p, q \neq 1$, there is equality in (1) or (2) for the functions $f(x) = \exp(-|p'| x^2)$ and $g(x) = \exp(-|q'| x^2)$. As was said above, it is enough to prove the inequalities when $N = 1$. We will prove this case in a modified form (Theorem 2) for which we introduce some notation. The condition $1/p + 1/q = 1 + 1/r$ is equivalent to the relation $1/p' + 1/q' = 1/r'$ for the conjugates, and $r'$, $p'$ and $q'$ have the same sign if $p, q, r > 1$ or $p, q, r < 1$. We set

$$c = \sqrt{r'/q'} \quad \text{and} \quad s = \sqrt{r'/p'}.$$ 

Notice that $c^2 + s^2 = 1$. We also introduce the constant

$$K(p, q, r) = \frac{p^{\frac{1}{p'}} q^{\frac{1}{q'}}}{r^{\frac{1}{r'}}}$$

that will appear several times in the rest of this paper. We can now state an equivalent form of Theorem 1. Indeed, a simple change of variables shows that the following Theorem 2 is equivalent to Theorem 1 when $N = 1$, provided $p, q$ and $r$ are different from 1.
Theorem 2. Let $p, q, r > 0$ satisfy $1/p + 1/q = 1 + 1/r$ and either $p, q, r > 1$ or $p, q, r < 1$. Let $c = \sqrt{r'/q'}$, $s = \sqrt{r'/p'}$, and let $f, g$ be non-negative functions in $L^1(\mathbb{R})$.

If $p, q, r > 1$ then

$$
\left( \int_\mathbb{R} \left( \int_\mathbb{R} f^{1/p} (cx - sy) g^{1/q} (sx + cy) \, dx \right)^r \, dy \right)^{1/r} \leq K(p, q, r) \left( \int_\mathbb{R} f \right)^{1/p} \left( \int_\mathbb{R} g \right)^{1/q}.
$$

If $p, q, r < 1$ then

$$
\left( \int_\mathbb{R} \left( \int_\mathbb{R} f^{1/p} (cx - sy) g^{1/q} (sx + cy) \, dx \right)^r \, dy \right)^{1/r} \geq K(p, q, r) \left( \int_\mathbb{R} f \right)^{1/p} \left( \int_\mathbb{R} g \right)^{1/q}.
$$

In both cases, there is equality when $f(x) = \exp(-px^2)$ and $g(x) = \exp(-qx^2)$.

By the monotone convergence theorem, it is enough to prove Theorem 2 for functions on $\mathbb{R}$ that are dominated by some centered Gaussian function. Next, it suffices to prove it for continuous, positive functions; indeed, assume that $f \leq G$, where $G(x) = M \exp(-\varepsilon x^2)$ is a centered Gaussian function, for some $\varepsilon > 0$ and $M > 0$; if $G_1(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$ and $G_n(x) = nG_1(nx)$ then $f_n = \min(f \ast G_n, G)$ tends to $f$ in $L^p$-norm for every $p \geq 1$. Each function $f_n$ is continuous, positive (it vanishes at some $x \in \mathbb{R}$ only if $f$ is the zero function in $L^1$). Let $(g_n)$ be an approximating sequence for $g$, built in the same way. If Theorem 2 holds for $f_n$ and $g_n$ for all $n$, then it is true for $f$ and $g$ by the dominated convergence theorem: we first pass to the limit for the inside integral of the expression at the left side of the inequality; then, the domination condition is satisfied for the function of $y$ defined by the inside integral, and we can conclude. Computations are especially nice if we assume that $f(x) \leq M \exp(-p\varepsilon x^2)$, $g(x) \leq M \exp(-q\varepsilon x^2)$ for some $\varepsilon > 0$ and $M > 0$.

We state now a lemma that is the real crux of the matter.

Lemma 1. Assume that $p, q, r > 1$ and that $1/p + 1/q = 1 + 1/r$. Let $f$, $g$, $F$ and $G$ be continuous positive functions in $L^1(\mathbb{R})$, such that $\int f = \int F$ and $\int g = \int G$. We have

$$
\left( \int \left( \int f^{1/p} (cx - sy) g^{1/q} (sx + cy) \, dx \right)^r \, dy \right)^{1/r} \leq \int \left( \int F^{r/p} (cX - sY) G^{r/q} (sX + cY) \, dY \right)^{1/r} \, dX.
$$

Let us first comment about this Lemma. The two numbers $r/p$ and $r/q$ are larger than one, and will play the role of $1/P$ and $1/Q$ for some $P, Q < 1$. Letting also $1/R = r$, we get $1/P + 1/Q = 1 + 1/R$, so that the right-hand side of inequality (3) is similar to the left-hand side, but for exponents less than 1. An easy computation will convince the reader that the two sides of (3) are equal when $f(x) = F(x) = \exp(-px^2)$ and $g(x) = G(x) = \exp(-qx^2)$. These facts imply that Lemma 1 contains both Young’s inequality and the reverse Young’s inequality with optimal constant. Of course, Lemma 1 is also valid for non-negative $L^1$ functions by approximation, as explained before.
Proof of Lemma 1. The proof is based on a parametrization of functions which was used in [HM] and was suggested by Brunn’s proof of the Brunn-Minkowski inequality. We assume that \( f, F, g \) and \( G \) are continuous and positive functions in \( L^1(\mathbb{R}) \), such that \( \int f = \int F \) and \( \int g = \int G \). We may also assume that the left-hand integral in (3) is finite (using monotone convergence). Since \( \int f = \int F \) and \( \int g = \int G \), there exist two functions \( u \) and \( v \) from \( \mathbb{R} \) to \( \mathbb{R} \) such that for all \( t \)

\[
\int_{-\infty}^{t} f = \int_{-\infty}^{t} F \quad \text{and} \quad \int_{-\infty}^{t} g = \int_{-\infty}^{t} G.
\]

Since \( f, g, F \) and \( G \) are continuous and never vanish, \( u \) and \( v \) are increasing bijections of \( \mathbb{R} \) and are continuously differentiable. For all \( t \),

\[
(4) \quad u'(t) f(u(t)) = F(t) \quad \text{and} \quad v'(t) g(v(t)) = G(t).
\]

The mapping \( T \) defined by \( T(x, y) = (u(x), v(y)) \) is a bijection of \( \mathbb{R}^2 \). Let \( R \) be the rotation

\[
\begin{pmatrix}
  c & -s \\
  s & c 
\end{pmatrix}
\]

in \( \mathbb{R}^2 \). We consider the change of variable \((x, y) = \Theta(X, Y)\) in \( \mathbb{R}^2 \) given by the mapping \( \Theta = T R T \); this means that

\[
x = cu(cX - sY) + sv(sX + cY), \quad y = -su(cX - sY) + cv(sX + cY).
\]

It is clear that \( \Theta \) is a differentiable bijection of \( \mathbb{R}^2 \). Its jacobian \( J \Theta \) at a point \((X, Y)\) is equal to

\[
J \Theta(X, Y) = u'(cX - sY)v'(sX + cY).
\]

We want an upper estimate for the integral (finite by assumption)

\[
I = \left( \int \left( \int f^{1/p}(cx - sy)g^{1/q}(sx + cy) \, dx \right)^r \, dy \right)^{1/r}.
\]

Using the \((L^r, L^{r'})\)-duality, there exists a positive function \( h \) such that \( \|h\|_{r'} = 1 \) and

\[
I = \int \int f^{1/p}(cx - sy)g^{1/q}(sx + cy)h(y) \, dx \, dy.
\]

By the change of variable \((x, y) = \Theta(X, Y)\), we see that \( I \) is equal to

\[
\int \int f^{1/p}(u(cX - sY))g^{1/q}(v(sX + cY))h(-su(cX - sY) + cv(sX + cY)) u'(cX - sY)v'(sX + cY) dX dY.
\]
In order to shorten the formulas, let us write

\[ U = u(cX - sY), \quad V = v(sX + cY) \]
\[ U' = u'(cX - sY), \quad V' = v'(sX + cY). \]

From the relations (4) we get

\[ I = \int \int f^{1/p}(u(cX - sY))g^{1/q}(v(sX + cY))h(-sU + cV)U'V' \, dX \, dY \]
\[ = \int \left( \int F^{1/p}(cX - sY)G^{1/q}(sX + cY)h(-sU + cV)(U')^{1/p'}(V')^{1/q'} \, dY \right) \, dX. \]

Using Hölder’s inequality for the integral in \( Y \) with parameters \( r \) and \( r' \), we obtain

\[ I \leq \int \left( \int F^{r/p}(cX - sY)G^{r/q}(sX + cY)h(-sU + cV)(U')^{r'/p'}(V')^{r'/q'} \, dY \right)^{1/r} \, dX. \]

Let \( H(X) = \int h^{r'}(-sU + cV)(U')^{r'/p'}(V')^{r'/q'} \, dY \), then

\[ H(X) = \int h^{r'}(a(X,Y))(u'(cX - sY))s^2(v'(sX + cY))c^2 \, dY, \]

where

\[ a(X,Y) = -s\, u(cX - sY) + c\, v(sX + cY). \]

We have

\[ \frac{\partial a}{\partial Y}(X,Y) = s^2u'(cX - sY) + c^2v'(sX + cY). \]

By the arithmetic-geometric inequality \((U')^{s^2}(V')^{c^2} \leq s^2U' + c^2V'\), hence

\[ H(X) \leq \int h^{r'}(a(X,Y))\frac{\partial a}{\partial Y}(X,Y) \, dY = \int h^{r'} = 1. \]

This proves that

\[ I \leq \int \left( \int F^{r/p}(cX - sY)G^{r/q}(sX + cY) \right)^{1/r} \, dX \]

and this ends the proof of Lemma 1.
Proof of Theorem 2. If we apply Lemma 1 with

\[ f(x) = F(x) = \sqrt{p/\pi} \exp(-px^2), \quad g(x) = G(x) = \sqrt{q/\pi} \exp(-qx^2), \]

there is equality in (3) and both members are equal to \( K(p,q,r) \). Applying (3) with any \( f \) and \( g \) such that \( \int f = \int g = 1 \) and with the preceding \( F \) and \( G \) gives Theorem 2 for \( p,q,r > 1 \). Suitably read from right to left, inequality (3) gives Theorem 2 when the indices are less than 1. Indeed, let \( p_1, q_1, r_1 < 1 \) be such that \( 1/p_1 + 1/q_1 = 1 + 1/r_1 \), and assume that \( \int f = \int F = \int g = \int G = 1 \). If we define the triple \( (p,q,r) \) by \( p = p_1/r_1 \), \( q = q_1/r_1 \) and \( r = 1/r_1 \), then \( p,q,r > 1 \) and \( 1/p + 1/q = 1 + 1/r \). So inequality (3) is valid for this triple. A straightforward computation gives that

\[
c_1 = \frac{r_1}{q_1} = \sqrt{\frac{r'}{p'}} = s \quad \text{and} \quad s_1 = \frac{r_1}{p_1} = \sqrt{\frac{r'}{q'}} = c.
\]

Thus, inequality (3) raised to the power \( r \) becomes

\[
\left( \int \left( \int f^{r_1/p_1} (s_1 x - c_1 y) g^{r_1/q_1} (c_1 x + s_1 y) \, dx \right)^{1/r_1} \, dy \right)^{1/r_1} \leq \left( \int \left( \int F^{1/p_1} (s_1 X - c_1 Y) G^{1/q_1} (c_1 X + s_1 Y) \, dY \right)^{r_1} \, dX \right)^{1/r_1}.
\]

This is exactly the reverse version of (3) for \( p_1, q_1, r_1 < 1 \) applied to the functions \( \tilde{F}, \tilde{G}, \tilde{f}, \tilde{g} \) where \( \tilde{f}(x) = f(-x) \). As before, choosing \( f(x) = \sqrt{p_1/\pi} \exp(-p_1 x^2) \), and \( g(x) = \sqrt{q_1/\pi} \exp(-q_1 x^2) \) implies Theorem 2 when the parameters are less than 1.

By the previous argument, there is equality in Theorem 2 when \( f(x) = \exp(-px^2) \), and \( g(x) = \exp(-qx^2) \). We prove now that up to scalar multiplication, translation and dilatation, this is the only equality case.

Theorem 3. Let \( p,q,r > 0 \) be such that \( 1/p + 1/q = 1 + 1/r \) and either \( p,q,r > 1 \) or \( p,q,r < 1 \). Let \( c = \sqrt{r'/q'} \), \( s = \sqrt{r'/p'} \), and let \( f,g \) be two non-negative functions in \( L^1(\mathbb{R}) \). Then

\[
(5) \quad \left( \int \left( \int f^{1/p} (cx - sy) g^{1/q} (sx + cy) \, dx \right)^{r} \, dy \right)^{1/r} = K(p,q,r) \left( \int f \right)^{1/p} \left( \int g \right)^{1/q}
\]

if and only if there exist \( a, b \geq 0, \lambda > 0 \) and \( y, z \in \mathbb{R} \) such that for all \( x \)

\[
(6) \quad f(x) = a \exp(-\lambda p(x - y)^2) \quad g(x) = b \exp(-\lambda q(x - z)^2).
\]

Proof. Using a simple change of variables, one can check that functions of the form (6) satisfy equality (5). We show now that only these functions do. We give the proof for
Let us assume first that \( f \) and \( g \) are continuous, positive and satisfy equality (5). We may assume that \( \int f = \int g = 1 \). If we set \( F(x) = \sqrt{p/\pi} \exp(-px^2) \) and \( G(x) = \sqrt{q/\pi} \exp(-qx^2) \), we get equality in (3). We follow the proof of Lemma 1 step by step. First, we know here that the integral \( I \) is finite by equality (5). There must be equality everywhere in the proof of inequality (3) for \( f, g, F \) and \( G \). In particular the equality when the arithmetic-geometric inequality was applied implies that for all \( X, Y \) (with the notation from the proof of Lemma 1)

\[
u'(cX - sY) = \nu'(sX + cY).
\]

So there exists \( \mu > 0 \) such that \( \nu' = \nu' = \mu \). Therefore \( u(t) = \mu(t - x_0) \) for some \( x_0 \). Formula (4) implies that

\[
\mu f(\mu(t - x_0)) = \sqrt{p/\pi} \exp(-pt^2),
\]

so \( f \) is Gaussian with variance \( \mu/p \). By the same method we show that \( g \) is Gaussian with variance \( \mu/q \).

For general \( f \) and \( g \), we need the following lemma, which was communicated to me by K. Ball (the reader will recognize in (7) the form of the Brascamp-Lieb inequality; the next Lemma tells something about maximizers for this inequality). We denote by \( \langle ., \cdot \rangle \) the scalar product in \( \mathbb{R}^n \).

**Lemma 2.** Let \( m \geq n \) be integers and \( \alpha_i > 0 \), \( u_i \in \mathbb{R}^n \), \( i = 1 \ldots m \). Assume that there exists \( M > 0 \) such that for all non-negative integrable functions \( f_i, i = 1, \ldots, m \) on \( \mathbb{R} \), one has

\[
\int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i^{\alpha_i}(\langle x, u_i \rangle) \, dx \leq M \prod_{i=1}^{m} \left( \int_{\mathbb{R}} f_i \right)^{\alpha_i}
\]

and assume that \( M \) is the smallest possible constant for which this is true. Let us call maximizer a \( m \)-tuple \((f_1, \ldots, f_m)\) of non zero functions for which inequality (7) is an equality.

If \((f_1, \ldots, f_m)\) and \((g_1, \ldots, g_m)\) are maximizers, then so is \((f_1 \ast g_1, \ldots, f_m \ast g_m)\).

**Proof.** We may assume that \((f_1, \ldots, f_m)\) and \((g_1, \ldots, g_m)\) are maximizers and \( \int f_i = \int g_i = 1 \), for \( i = 1, \ldots, m \). We define two functions \( F, G \) on \( \mathbb{R}^n \) by

\[
F(x) = \prod_{i=1}^{m} f_i^{\alpha_i}(\langle x, u_i \rangle) \quad \text{and} \quad G(x) = \prod_{i=1}^{m} g_i^{\alpha_i}(\langle x, u_i \rangle).
\]

We know that \( \int F = \int G = M \). So

\[
M^2 = \left( \int F \right) \left( \int G \right) = \int F \ast G
\]

\[
= \int \int \prod_{i=1}^{m} f_i^{\alpha_i}(\langle x, u_i \rangle) \prod_{i=1}^{m} g_i^{\alpha_i}(\langle y, u_i \rangle) \, dy \, dx
\]

\[
= \int \left( \int \prod_{i=1}^{m} [f_i(\langle x, u_i \rangle - \langle y, u_i \rangle)g_i(\langle y, u_i \rangle)]^{\alpha_i} \, dy \right) \, dx.
\]
Applying inequality (7) to the functions \( k_i(t) = f_i(\langle x, u_i \rangle - t) g_i(t) \), we get

\[
M^2 \leq M \int \prod_{i=1}^{m} \left( \int f_i(\langle x, u_i \rangle - t) g_i(t) \, dt \right)^{\alpha_i} \, dx = M \int \prod_{i=1}^{m} \left( f_i * g_i \right)^{\alpha_i} (\langle x, u_i \rangle) \, dx.
\]

It follows that \((f_1 * g_1, \ldots, f_m * g_m)\) is a maximizer.

Now we can finish the proof of Theorem 3. The functions \( f \) and \( g \) satisfy (5). As the functions \( \Gamma_p(x) = \exp(-px^2) \) and \( \Gamma_q(x) = \exp(-qx^2) \) have the same extremal property, the preceding lemma implies that \( f * \Gamma_p \) and \( g * \Gamma_q \) have it too. But these functions are positive and continuous; by the previous argument they are of the form (6). Using the Fourier transform, one obtains that \( f \) and \( g \) are of the form (6).

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