EXISTENCE OF INFINITELY MANY FREE BOUNDARY MINIMAL HYPERSURFACES

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Abstract. In this paper, we prove that in any compact Riemannian manifold with smooth boundary, of dimension at least 3 and at most 7, there exist infinitely many almost properly embedded free boundary minimal hypersurfaces. This settles the free boundary version of Yau’s conjecture. The proof uses adaptions of A. Song’s work and the early works by Marques-Neves in their resolution to Yau’s conjecture, together with Li-Zhou’s regularity theorem for free boundary min-max minimal hypersurfaces.

1. Introduction

1.1. Motivation from closed Riemannian manifolds. Finding out minimal submanifolds has always been an important theme in Riemannian geometry. In 1960s, Almgren [1,2] initiated a variational theory to find minimal submanifolds in any compact Riemannian manifolds (with or without boundary). He proved that weak solutions, in the sense of stationary varifolds, always exist. About twenty years later, the interior regularity theory for codimension one hypersurfaces was developed by Pitts [22] and Schoen-Simon [23]. As a consequence, they showed that in any closed manifold \((M^{n+1}, g)\), there exists at least one embedded closed minimal hypersurface, which is smooth except possibly along a singular set of Hausdorff codimension at least 7. Then Yau conjectured the following:

Conjecture 1.1 (S.-T. Yau [31]). Every closed three-dimensional Riemannian manifold \((M^3, g)\) contains infinitely many (immersed) minimal surfaces.

The first progress of this Yau’s Conjecture was made by Marques-Neves in [20], where they proved the existence of infinitely many embedded minimal hypersurfaces for closed manifolds with positive Ricci curvature, or more generally, for closed manifolds satisfying the “Embedded Frankel Property”. Using the Weyl Law for the volume spectrum [18], Irie-Marques-Neves [13] proved Yau’s conjecture for generic metrics. Recently, in a remarkable work [26], A. Song completely solved the Conjecture building on the methods developed by Marques-Neves [19,20]. Such a method also helped Song give a much stronger theorem: every closed Riemannian manifold \((M^{n+1}, g)\) of dimension \(3 \leq (n + 1) \leq 7\) contains infinitely many embedded minimal hypersurfaces.

1.2. Questions and Main results in compact Riemannian manifolds with boundary. In this paper, we consider compact manifolds with boundary \((M, \partial M, g)\), which is the program set out by Almgren in the hypersurface case [12]. Then each critical point of the area functional is so called a free boundary minimal hypersurface, which is a hypersurface with vanished mean curvature and meeting \(\partial M\) orthogonally along its boundary. Based on previous works [22,24],
Li-Zhou [17] proved the regularity on the free boundary, which implies the existence of free boundary minimal hypersurfaces in general compact manifolds with boundary.

Based on this regularity result, it is natural to raise a question bringing free boundary version of Yau’s conjecture:

**Question 1.2.** Does every compact Riemannian manifold with smooth boundary of dimension $3 \leq (n + 1) \leq 7$ contain infinitely many free boundary minimal hypersurfaces?

Inspired by [13, 19], the author together with Guang, Li and Zhou proved the denseness of free boundary minimal hypersurfaces in compact manifolds with smooth boundary for generic metrics in [8]. Moreover, the author also proved that those free boundaries are dense in the boundary of the manifold; see [27]. In this paper, we settle Question 1.2 by adapting the arguments in [26].

**Theorem 1.3.** In any compact Riemannian manifold with boundary $(M^{n+1}, \partial M, g)$, of dimension $3 \leq (n + 1) \leq 7$, there exist infinitely many almost properly embedded free boundary minimal hypersurfaces.

In this paper, we also use the growth of min-max width, which was firstly studied by Gromov [7] and quantified by Liokumovich-Marques-Neves in [18]. According to the regularity theory in [17, 22, 23], each width is associated with an almost properly embedded free boundary minimal hypersurfaces with multiplicities; see [8, Proposition 7.3]. If each multiplicity is one, then since the widths are a sequence of real numbers going to infinity, it would lead to a direct proof of Yau’s conjecture in the generic case. This is conjectured by Marques-Neves [19], and has been completely proven by Zhou [32] for closed manifolds; see also Chodosh-Mantoulidis [5] for three-manifolds of the Allen-Cahn version. However, such a kind of question remains open for compact manifolds with boundary.

We also mention there are other approaching to Question 1.2 in some special compact Riemannian manifolds with boundary. In the three dimensional round ball $\mathbb{B}^3$, Fraser-Schoen [6] obtained the free boundary minimal surface with genus 0 and arbitrary many boundary components. By desingularization of the critical catenoid and the equatorial disk, Kapouleas-Li [14] constructed infinitely many new free boundary minimal surfaces which have large genus in $\mathbb{B}^3$. We refer to [15] for more results in $\mathbb{B}^3$.

### 1.3. Difficulties.

Compared to closed manifolds, the new main challenge is that in compact Riemannian manifolds with boundary, the free boundary minimal hypersurfaces may have non-empty touching sets (see Definition 2.2). Such touching phenomena always bring the main difficulties in the study of related problems; see [8, 10, 11, 17, 28, 33, 34]. Precisely, if cutting a manifold along an almost free boundary minimal hypersurface with non-empty touching set, the result would never be a manifold even in the topological sense. In this paper, we come up with several new concepts (see Section 2) and develop the “embedded Frankel property” in several ways (see Subsection 2.2 and Theorem 4.1) which may be helpful in the further studies.

Another challenge is the regularity of free boundary minimal hypersurfaces produced by min-max theory in compact manifolds whose boundaries are not smooth. We mention that there is no such regularity even for minimizing problems, which would be quite crucial for the smoothness of replacements (see [17, Proposition 6.3]). Nevertheless, we get the full regularity in our situation (see Theorem 3.8) by noticing that Li-Zhou’s [17] result holds true for all smooth boundary points.
1.4. Outline of the proof. Let \((M^{n+1}, \partial M, g)\) be a compact Riemannian manifold with non-empty boundary, of \(3 \leq (n+1) \leq 7\). Assume that \((M, \partial M, g)\) contains only finitely many almost properly embedded free boundary minimal hypersurfaces. Borrowing the idea from Song \[26\], we notice that there are two key points:

- cutting along stable free boundary minimal hypersurfaces to get a connected component \(N\) so that the free boundary minimal hypersurfaces in \(N \setminus T\) (here \(T\) is the new boundary part from cutting process) satisfy the Frankel property;
- producing almost properly embedded free boundary minimal hypersurfaces in \(N \setminus T\) by using min-max theory for \(C(N)\), which is a non-compact manifold by gluing to \(N\) the cylindrical manifold \(T \times [0, +\infty)\) under the conformal metric.

For the first part, we have to cut along the improper hypersurfaces, which would never lead the new thing to be a manifold even in the topological sense. To overcome this, we choose an order of those hypersurfaces carefully so that every time there is a connected component which is a compact manifold with piecewise smooth boundary satisfying our condition. Precisely, we cut along stable, properly embedded free boundary minimal hypersurfaces first and take a connected component \((N_1, \partial N_1, T_1, g)\) \((T_1\) is the new boundary part from cutting process) so that there is no stable properly embedded one in \(N_1 \setminus T_1\). Then each almost properly embedded free boundary minimal hypersurfaces in \(N_1 \setminus T_1\) generically separates \(N_1\) (see Subsection 2.2). If \(N_1\) doesn’t satisfy the Frankel property, then we prove that there exists a free boundary minimal hypersurface \(\Sigma\) so that one of the connected component of \(T_1 \setminus \Sigma\) is good enough for us; see Lemma \[2.11\].

For the second part, we approach \(C(N)\) by a sequence of compact manifold with piecewise smooth boundary \(N_\epsilon\). The key observation is that Li-Zhou’s regularity holds true for all smooth boundary points. Hence we can use the monotonicity formula \[9\] Theorem 3.4; \[24\] §17.6\] to show that for any \(p\) fixed, any \(\epsilon > 0\) small enough, the width \(\omega_p(N_\epsilon)\) is associated with a properly embedded free boundary minimal hypersurface whose boundary lies on \(N_\epsilon \cap \partial M\); see Theorem \[3.8\] for details.

This paper is organized as follows. In Section 2 we give basic definitions and prove a generalized Frankel property for free boundary minimal hypersurfaces in the end. Then in Section 3 we prove a min-max theory for a non-compact manifold with boundary. Finally, we prove the main theorem in Section 4. In Appendix A we state a strong maximum principle for stationary varifolds in compact manifolds with boundary and also sketch the proof. Appendix B contains the collection of the calculation in Theorem 3.10.

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2. Preliminary for free boundary minimal hypersurfaces

In this section, we give the basic notations and some lemmas about constructing area minimizers in compact manifolds with boundary.

Throughout this paper, \((M^{n+1}, \partial M, g)\) is always a compact Riemannian manifold with smooth boundary and \(3 \leq (n+1) \leq 7\). Generally, \((M, \partial M, g)\) can be regarded as a domain of a closed
Riemannian manifold \((\widetilde{M}, g)\). We also need to consider compact manifold with piecewise smooth boundary.

**Definition 2.1** ([10 Definition 2.2]). For a manifold with piecewise smooth boundary, \(N\) is called a manifold with boundary \(\partial N\) and portion \(T\) if

- \(\partial N\) and \(T\) are smooth, which may be disconnected;
- \(\partial N \cup T\) is the (topological) boundary of \(N\) and \(\partial N \cap T = \partial T\).

We will denote it by \((N, \partial N, T, g)\).

We remark that in the above definition, the interior of \(\partial N\) and \(T\) are disjoint.

**Definition 2.2** ([17 Definition 2.6]). Let \((N^{n+1}, \partial N, T, g)\) be a compact Riemannian manifold with boundary and portion. Let \(\Sigma^n\) be a smooth \(n\)-dimensional manifold with boundary \(\partial \Sigma\). We say that a smooth embedding \(\phi: \Sigma \rightarrow N\) is an almost properly embedding of \(\Sigma\) into \(N\) if \(\phi(\Sigma) \subset N\) and \(\phi(\partial \Sigma) \subset \partial N\). We say that \(\Sigma\) is an almost properly embedded hypersurface in \(N\).

For an almost properly embedded hypersurface \((\Sigma, \partial \Sigma)\), we allow the interior of \(\Sigma\) to touch \(\partial N\). That is to say: \(\text{Int}(\Sigma) \cap \partial N\) may be non-empty. We usually call \(\text{Int}(\Sigma) \cap \partial N\) the touching set of \(\Sigma\).

**Definition 2.3** ([17 Section 2.3]). Let \((\Sigma, \partial \Sigma)\) be an almost properly embedded hypersurface in \((N, \partial N, T, g)\). Then \(\Sigma\) is called a free boundary minimal hypersurface if the mean curvature vanishes everywhere and \(\Sigma\) meets \(\partial N\) orthogonally along \(\partial \Sigma\).

We also use the term of free boundary hypersurface if \(\Sigma\) only meets \(\partial N\) orthogonally along \(\partial \Sigma\).

In this paper, we also need to deal with free boundary hypersurfaces which have touching sets from only one side.

**Definition 2.4.** A two-sided embedded free boundary hypersurface \((\Sigma, \partial \Sigma)\) in \((N, \partial N, T, g)\) is half-properly embedded if it is almost properly embedded and has a unit normal vector field \(n\) so that \(n = \nu_{\partial M}\) along the touching set of \(\Gamma\).

### 2.1. Neighborhoods foliated by free boundary hypersurfaces

Given a metric on \(N\), \((N, \partial N, T, g)\) can always be isometrically embedded into a compact Riemannian manifold with smooth boundary \((M, \partial M, g)\). Also, we embed \((M, \partial M, g)\) isometrically into a smooth Riemannian manifold \((\widetilde{M}, g)\) which has the same dimension with \(M\) and \(N\). Let \(\Gamma\) be a two-sided, almost properly embedded, free boundary hypersurface in \((N, \partial N, T, g)\), Then \(X \in X(\widetilde{M})\) is called an admissible vector field on \(\widetilde{M}\) for \(\Gamma\) if \(X|_{\Gamma}\) is a normal vector field of \(\Gamma\) and \(X(p) \in T_p(\partial \tilde{M})\) for \(p\) in some neighborhood of \(\partial \Gamma\) in \(\partial \tilde{M}\). Note that such an admissible vector field is always associated with a family of diffeomorphisms of \(\widetilde{M}\).

**Lemma 2.5.** Let \(\Gamma\) be an almost properly embedded, two-sided non-degenerate free boundary minimal hypersurface in \((N, \partial N, T, g)\) and \(n\) a choice of unit normal vector on \(\Gamma\). Let \(\{\Phi(t)\}_{-1 \leq t \leq 1}\) be a family of diffeomorphisms of \(\widetilde{M}\) associated to an admissible vector field on \(\widetilde{M}\) for \(\Gamma\) so that \(\frac{\partial \Phi(x, t)}{\partial t} \big|_{t=0, x \in \Gamma} = n(x)\). Then there exist a positive number \(\delta_1\) and a smooth map \(w: \Gamma \times (-\delta_1, \delta_1) \rightarrow \mathbb{R}\) with the following properties:

1. For each \(x \in \Gamma\), we have \(w(x, 0) = 0\) and \(\phi := \frac{\partial}{\partial t} w(x, t)\big|_{t=0}\) is a positive function which is the first eigenfunction of the second variation of area on \(\Gamma\);
2. For each \(t \in (-\delta_1, \delta_1)\), we have \(\int_{\Gamma} (w(\cdot, t) - t\phi) \phi = 0\);
(3) for each \( t \in (-\delta_1, \delta_1) \setminus \{0\} \), \( \{\Phi(x, w(x,t)) : x \in \Gamma\} \) is an embedded hypersurface in \( \widetilde{M} \) with free boundary on \( \partial M \) and mean curvature either positive or negative.

Lemma 2.6 follows from the implicit function theorem. With more effort, we have a similar result for degenerate stable free boundary minimal hypersurfaces.

**Lemma 2.6.** Let \( \Gamma \) be an almost properly embedded, two-sided degenerate stable free boundary minimal hypersurface in \((M, \partial M, g)\) and \( n \) a choice of unit normal vector on \( \Gamma \). Let \( \{\Phi(\cdot, t)\}_{-1 \leq t \leq 1} \) be a family of diffeomorphisms of \( \tilde{M} \) associated to an admissible vector field on \( \tilde{M} \) for \( \Gamma \) so that \( \frac{\partial \Phi(x, t)}{\partial t}\big|_{t=0, x \in \Gamma} = n(x) \). Then there exist a positive number \( \delta_1 \) and a smooth map \( w : \Gamma \times (-\delta_1, \delta_1) \to \mathbb{R} \) with the following properties:

1. For each \( x \in \Gamma \), we have \( w(x, 0) = 0 \) and \( \phi := \frac{\partial}{\partial t} w(x, t)|_{t=0} \) is a positive function in the kernel of the Jacobi operator of \( \Gamma \);
2. For each \( t \in (-\delta_1, \delta_1) \), we have \( \int_{\Gamma} (w(\cdot, t) - t\phi) \phi = 0 \);
3. For each \( t \in (-\delta_1, \delta_1) \), \( \{\Phi(x, w(x,t)) : x \in \Gamma\} \) is an embedded hypersurface in \( \tilde{M} \) with free boundary on \( \partial M \) and mean curvature either positive or negative or identically zero.

**Proof.** The proof here is similar to [4, Proposition 5; 26, Lemma 10].

Denote the space

\[
Y := \{f \in C^\infty(\Gamma) : \int_{\Gamma} f \phi = 0\}.
\]

Define a map \( \Psi : Y \times \mathbb{R} \to Y \times C^\infty(\partial \Gamma) \) by

\[
\Psi(f, t) = \left( \phi^{-1}[H(\Phi(x, f + t\phi)) - \frac{1}{\text{Area}(\Gamma)} \int_{\Gamma} H(\Phi(x, f + t\phi))], \langle n(\Phi(x, f + t\phi)), \nu_{\partial M}\rangle|_{\partial \Gamma}\right).
\]

Then the first derivative (see [10, Lemma 2.5]) is

\[
D \Psi_{(0,0)}(f, 0) = \left( \phi^{-1}(L f - \frac{1}{\text{Area}(\Gamma)} \int_{\Gamma} L f), f h^{\partial M}(n, n) - \langle \nabla f, \nu_{\partial M}\rangle|_{\partial \Gamma}\right)
\]

Here \( L = \Delta + \text{Ric}(n, n) + |A|^2 \) is the Jacobi operator. Hence \( D_1 \Psi_{(0,0)} f = 0 \) is equivalent to \( L f = c \) and \( \frac{\partial f}{\partial t} = h^{\partial M}(n, n)f \) (where \( \eta \) is the co-normal of \( \Gamma \)), which implies that \( f = 0 \). Then by the implicit function theorem, for each \( t \in (-\delta_1, \delta_1) \), there exists a function \( u(\cdot, t) \in Y \) so that \( \Psi(u, t) = (0, 0) \). Now define \( w(x, t) = u(x, t) + t\phi \). Clearly, \( w \) satisfies (2) and (3).

It remains to verify (1). Indeed, according to the implicit function theorem, we also have

\[
D_1 \Psi_{(0,0)} \left( \frac{\partial u}{\partial t}\right)|_{(0,0)} + D_2 \Psi_{(0,0)} \left( \frac{\partial}{\partial t}\right)|_{(0,0)} = 0.
\]

By the direct computation, \( D_2 \Psi_{(0,0)} \left( \frac{\partial}{\partial t}\right)|_{(0,0)} = 0 \). Recall that \( D_1 \Psi_{(0,0)} \) is nondegenerate. Hence \( \frac{\partial u}{\partial t}|_{t=0} = 0 \), which implies the desired result. \qed

Let \( S \) be a two-sided free boundary minimal hypersurface in an \((n + 1)\)-dimensional compact manifold \((\tilde{M}, \partial \tilde{M})\) (possibly with portion). Let \( \tilde{M} \) be a closed Riemannian manifold so that \( \tilde{M} \) is a compact domain of \( M \). Let \( \mu > 0 \); consider a neighborhood \( \mathcal{N} \) of \( S \) in \( \tilde{M} \) and a diffeomorphism

\[
\tilde{F} : S \times (-\mu, \mu) \to \mathcal{N}
\]

such that \( \tilde{F}(x, 0) = x \) for \( x \in S \). We define the following (cf. [26, Section 3]):
• \( S \) has a contracting neighborhood if there are such \( \mu, \mathcal{N} \) and \( \tilde{F} \) such that for all \( t \in [-\mu, \mu] \setminus \{0\} \), \( \tilde{F}(S \times \{t\}) \) has free boundary and mean curvature vector pointing towards \( S \);
• \( S \) has an expanding neighborhood if \( S \) is unstable or there are such \( \mu, \mathcal{N} \) and \( \tilde{F} \) such that for all \( t \in [-\mu, 0) \) (resp. \( t \in (0, \mu] \), \( \tilde{F}(S \times \{t\}) \) has free boundary and mean curvature vector pointing away from \( S \);
• \( S \) has a mixed neighborhood if there are such \( \mu, \mathcal{N} \) and \( \tilde{F} \) such that for all \( t \in (0, \mu) \) (resp. \( t \in (0, \mu] \), \( \tilde{F}(S \times \{t\}) \) has free boundary and mean curvature vector pointing away from (resp. pointing towards) \( S \);
• \( S \) has a contracting neighborhood in one side if there are such \( \mu, \mathcal{N} \) and \( \tilde{F} \) such that for all \( t \in (0, \mu] \), \( \tilde{F}(S \times \{t\}) \) has free boundary and mean curvature vector pointing towards \( S \); such a neighborhood in one side is said to be proper if \( \tilde{F}(S \times \{t\}) \subset \hat{M} \) for \( t \in (0, \mu) \);
• \( S \) has an expanding neighborhood in one side if \( S \) is unstable or there are such \( \mu, \mathcal{N} \) and \( \tilde{F} \) such that for all \( t \in (0, \mu) \), \( \tilde{F}(S \times \{t\}) \) has free boundary and mean curvature vector pointing away from \( S \); such a neighborhood in one side is said to be proper if \( \tilde{F}(S \times \{t\}) \subset \hat{M} \) for \( t \in (0, \mu) \).

Let \( S \) be a one-sided free boundary minimal hypersurface in \((\hat{M}, \partial \hat{M}, g)\). Denote by \( \tilde{S} \) the double cover of \( S \). Consider the double cover \((\hat{M}', \partial \hat{M}', g')\) of \((\hat{M}, \partial \hat{M}, g)\) so that \( \tilde{S} \) is a two-sided free boundary minimal hypersurface in it. Then we say that \( S \) has a contracting (resp. an expanding) neighborhood if \( \tilde{S} \) has a contracting (resp. an expanding) neighborhood.

Remark 2.7. Let \( S \) be a two-sided free boundary minimal hypersurface and \( \tilde{F} \) be the diffeomorphism as above. \( S \) is called to have no proper contracting neighborhood in one side provided that each neighborhood in one side is not contracting or non-proper, i.e. there exist two sequences of real numbers \( t_i^+ \to 0^+ \) and \( t_i^- \to 0^- \) so that for each \( t_i = t_i^+ \) or \( t_i^- \),

- either \( \tilde{F}(S \times \{t_i\}) \) has mean curvature vector pointing away from \( S \);
- or \( \tilde{F}(S \times \{t_i\}) \setminus \hat{M} \neq \emptyset \).

2.2. Construction of area minimizers. Let \((N, \partial N, T, g)\) be a connected compact manifold with boundary and portion. Let \((\Sigma, \partial \Sigma)\) be an almost properly embedded hypersurface in \((N, \partial N, T, g)\). Recall that \( \Sigma \) generically separates \( N \) (see [10, Section 5]) if there is a cut-off function \( \phi \) defined on \( \Sigma \) satisfying the following:

- \( \phi \) is compactly supported in \( \Sigma \setminus \partial \Sigma \) such that \( \langle \phi \mathbf{n} \rangle_{\nu_{\partial M}} < 0 \) on the touching set, where \( \mathbf{n} \) is the normal vector field of \( \Sigma \);
- \( \Sigma_{t \phi} := \{ \exp_x(t \phi \mathbf{n}) : x \in \Sigma \} \) separates \( N \) for all sufficiently small \( t > 0 \).

If \( \Sigma \) generically separates \( N \), then \( N \setminus \Sigma \) can be divided into two part by the signed distance function to \( \Sigma \). These two parts are called the generic components.

In this section, we consider the following conditions of \((N, \partial N, T, g)\):

A) the portion \( T \) is a free boundary minimal hypersurface in \((N, \partial N, g)\) and has a contracting neighborhood in one side;
B) each two-sided free boundary minimal hypersurface generically separates \( N \);
C) any properly embedded, two-sided, free boundary minimal hypersurface in \( N \setminus T \) has a neighborhood which is either contracting or expanding or mixed;
D) any half-properly embedded, two-sided, free boundary minimal hypersurface in \( N \setminus T \) has a proper neighborhood in one side which is either contracting or expanding;
E) each properly embedded, one-sided, free boundary minimal hypersurface has an expanding neighborhood;
F) at most one connected component of $\partial N$ is a closed minimal hypersurface, and if it happens, it has an expanding neighborhood in one side in $N$.

Let $\Gamma_1$ and $\Gamma_2$ be two disjoint, connected free boundary minimal hypersurface in $(N, \partial N, T, g)$ with $\Gamma_j \subset N \setminus T$ ($j = 1, 2$).

**Proposition 2.8.** Assume that $(N, \partial N, T, g)$ satisfies \( A \), \( B \) and \( F \). Suppose that $\Gamma_j$ ($j = 1, 2$) is two-sided, non-degenerate and has no proper contracting neighborhood in one side (see Remark 2.7). Then there exists a properly embedded free boundary minimal hypersurface in $N \setminus T$ which is an area minimizer.

**Proof.** We first consider that $\Gamma_j$ is not contained in $\partial N$. Since $\Gamma_j$ is non-degenerate, then $\Gamma_j$ a contracting or expanding neighborhood (see Lemma 2.5), i.e. there exist $\mu > 0$, a neighborhood $N_j$ of $\Gamma_j$ in $\tilde{M}$, and a diffeomorphism

$$F^j : \Gamma_j \times (-\mu, \mu) \to N_j$$

such that $F^j(x, 0) = x$ for $x \in \Gamma_j$ and for each $t \in (-\mu, \mu) \setminus \{0\}$, $F^j(\Gamma_j \times \{t\})$ has free boundary and mean curvature vector pointing towards or away from $\Gamma_j$. By assumption \( B \), $\Gamma_1$ and $\Gamma_2$ generically separates $N$. Hence $N \setminus (\Gamma_1 \cup \Gamma_2)$ has three generic components. Let $N'$ be the closure of the generic component of $N \setminus (\Gamma_1 \cup \Gamma_2)$ that contains $\Gamma_1$ and $\Gamma_2$. Without loss of generality, we assume that for $t > 0$, $F^j(\Gamma_j \times \{t\})$ intersects $N' \setminus (\Gamma_1 \cup \Gamma_2)$.

Now take $\epsilon \in (0, \mu)$ so that $F^j(\Gamma_j \times \{\pm \epsilon\})$ meets $\partial N$ transversally for $j = 1, 2$.

**Case 1:** Both $\Gamma_1$ and $\Gamma_2$ have expanding neighborhoods.

In this case, we consider

$$N_1 := N' \setminus \bigcup_{j=1}^2 F^j(\Gamma_j \times [0, \epsilon)), \quad \partial N_1 := \partial N \cap N_1,$$

$$T_1 := \big[ \bigcup_{j=1}^2 F^j(\Gamma_j \times \{\epsilon\}) \cup T \big] \cap N'.$$

Clearly, $(N_1, \partial N_1, T_1, g)$ is a compact manifold with boundary and portion. Moreover, $F^1(\Gamma_1 \times \{\epsilon\})$ represents a non-zero relative homology class in $(N_1, \partial N_1)$. By minimizing the area of this class, we obtain a stable free boundary minimal hypersurface and a connected component $S$ is properly embedded in $N \setminus T$, which is the desired hypersurface since it is obtained by a minimizing procedure.

**Case 2:** Both $\Gamma_1$ and $\Gamma_2$ have contracting neighborhoods.

In this case, we consider

$$N_2 := \bigcup_{j=1}^2 F^j(\Gamma_j \times [-\epsilon, 0)) \cup N',$$

$$\partial N_2 := (\partial N \cap N') \cup \bigcup_{j=1}^2 F^j(\partial \Gamma_j \times [-\epsilon, 0)),$$

$$T_2 := \bigcup_{j=1}^2 F^j(\Gamma_j \times \{-\epsilon\}) \cup (T \cap N').$$

Clearly, $(N_2, \partial N_2, T_2, g)$ is a compact manifold with boundary and portion (see Figure II). We can minimize the area of the relative homology class represented by $F^1(\Gamma_1 \times \{-\epsilon\})$ to get a free boundary minimal hypersurface. Particularly, one connected component is stable and properly embedded in $N \setminus T$ and is an area minimizer.

**Case 3:** $\Gamma_1$ has a contracting neighborhood and $\Gamma_2$ has an expanding neighborhood.
In this case, we consider
\[ N_3 := N' \cup \bar{F}^1(\Gamma_1 \times [-\epsilon, 0)) \setminus \bar{F}^2(\Gamma_2 \times [0, \epsilon)), \]
\[ \partial N_3 := (\partial N \cap N') \cup \bar{F}^1(\partial \Gamma_j \times [-\epsilon, 0)) \setminus \bar{F}^2(\Gamma_2 \times [0, \epsilon)), \]
\[ T_3 := \bar{F}^1(\Gamma_1 \times \{-\epsilon\}) \cup [(\bar{F}^2(\Gamma_2 \times \{\epsilon\}) \cup T) \cap N']. \]

By the same argument in the first two cases, we then obtain the desired hypersurface.

To complete the proof, it suffices to consider \( \Gamma_1 \subset \partial N \). Then by assumption (F), \( \Gamma_1 \) has an expanding neighborhood in one side. Then it is just a subcase of Case 1 or Case 3. In either case, we can find a properly embedded, stable free boundary minimal hypersurface having a contracting neighborhood. \( \square \)

Remark 2.9. In both Case 1 and 3, we used the expanding neighborhood in one side to be a barrier for minimizing problems even if such a neighborhood is not proper. The key observation here is that the interior of \( \bar{F}^j(\Gamma_j \times \{\epsilon\}) \) intersects \( N' \) with angles less than \( \pi/2 \). We refer to \([10, Lemma 4.13]\) for details.

We now give a stronger proposition by a perturbation argument.

**Proposition 2.10.** Suppose that \((N, \partial N, T, g)\) satisfies (A–F). Suppose that \( \Gamma_j \ (j = 1, 2) \) is two-sided and has no proper contracting neighborhood in one side (see Remark 2.7). Then there exists a two-sided, properly embedded, stable, free boundary minimal hypersurface having a contracting neighborhood.

**Proof of Proposition 2.10.** Firstly, we consider that \( \Gamma_j \) is not part of \( \partial N \) for \( j = 1, 2 \). Denote by \( N' \) the closure of the generic component of \( N \setminus (\Gamma_1 \cup \Gamma_2) \) that contains \( \Gamma_1 \) and \( \Gamma_2 \). Let \( N_j \) be a neighborhood of \( \Gamma_j \) in \( \bar{M} \) and
\[ \bar{F}^j : \Gamma_j \times (-\mu, \mu) \to N_j \]
be the map constructed by Lemma 2.6 for \( \Gamma_j \) and \( \mu > 0 \). Without loss of generality, we assume that for \( t < 0 \),
\[ \bar{F}^j(\Gamma_j \times \{t\}) \cap N' = \emptyset. \]
Since $\Gamma_j$ does not have a proper and contracting neighborhood in one side, then by (C) and (D), each neighborhood in one side is expanding if it is proper. Hence in both cases, we can always take $\epsilon > 0$ so that for $j = 1, 2$,
\[
\text{Area}(\tilde{F}^j(\Gamma_j \times \{\epsilon\}) \cap N) < \text{Area}(\Gamma_j).
\]
Denote by
\[
A_1 := \min_{j \in \{1, 2\}} \text{Area}(\tilde{F}^j(\Gamma_j \times \{\epsilon\})).
\]
Then we can take $r_k \rightarrow 0, q_j \in \Gamma_j \setminus \partial N$ so that
\[
B_{r_k}(q_j) \cap \partial N = \emptyset \quad \text{and} \quad B_{r_k}(q_j) \cap \tilde{F}^j(\Gamma_j \times \{\epsilon\}) = \emptyset.
\]
By [13, Proposition 2.3] (see also [10, Remark 5.5]), there exists a sequence of perturbed metrics $g_k \rightarrow g$ on $\tilde{M}$ so that
\begin{itemize}
  \item $g_k(x) = g(x)$ for all $x \in \tilde{M} \setminus (B_{r_k}(q_1) \cup B_{r_k}(q_2))$;
  \item both $\Gamma_1$ and $\Gamma_2$ are non-degenerate free boundary minimal hypersurfaces in $(N, \partial N, T, g_k)$.
\end{itemize}
Clearly, $(N, \partial N, T, g_k)$ satisfies (A), (B) and (F). Applying Proposition 2.8, there exists a properly embedded free boundary minimal hypersurface $S_k$ which is an area minimizer. Moreover, by the argument in Proposition 2.8,
\[
\text{Area}_{g_k}(S_k) < A_1.
\]
Letting $k \rightarrow \infty$, by the compactness for stable free boundary minimal hypersurfaces [9], $S_k$ smoothly converges to a stable free boundary minimal hypersurface $S \subset N'$ in $(N, \partial N, T, g)$ with $\text{Area}(S) \leq A_1$. Such an area upper bound gives that $S$ is not $\Gamma_1$ or $\Gamma_2$. Then by the maximum principle, $S \cap B_{r_k}(q_j) = \emptyset$ for $j = 1, 2$. From the smooth convergence and the fact of $r_k \rightarrow 0$, $S_k \cap B_{r_k}(q_j) = \emptyset$ for large $k$. Hence for large $k$, $S_k$ is an area minimizer in $(N, \partial N, T, g)$. By the assumption (F), $S_k$ must be two-sided. Therefore, it is the desired free boundary minimal hypersurface in $(N, \partial N, T, g)$.

It remains to consider $\Gamma_1 \subset \partial N$. By assumption (F), $\Gamma_1$ has an expanding neighborhood in one side in $N$. Then we just need to perturb the metric slightly near $\Gamma_2$. By a similar argument in Proposition 2.8, we can also obtain a stable, properly embedded, free boundary minimal hypersurface with respect to perturbed metrics. Then the process above also gives a desired hypersurface. \hfill \Box

Now we are ready to state the main result in this section, which is a generalized Frankel property and will be used in the proof of Theorem 4.1.

**Lemma 2.11.** Suppose that $(N, \partial N, T, g)$ satisfies (A) and contains two disjoint connected free boundary minimal hypersurfaces in $N \setminus T$. Then $N \setminus T$ contains a two-sided, free boundary minimal hypersurface with a proper and contracting neighborhood in one side.

**Proof.** Let $\Gamma_1$ and $\Gamma_2$ be two disjoint, free boundary minimal hypersurfaces in $(N, \partial N, T, g)$.

**Case 1:** $\Gamma_1$ and $\Gamma_2$ are both two-sided.

Without loss of generality, we assume that $\Gamma_1$ and $\Gamma_2$ have no proper contracting neighborhood in one side. Then this lemma follows from Proposition 2.10.

**Case 2:** $\Gamma_1$ is two-sided and $\Gamma_2$ is one-sided.

Without loss of generality, we assume that $\Gamma_1$ has no proper contracting neighborhood in one side and $\Gamma_2$ is not properly embedded or has an expanding neighborhood. Now consider the
double cover \((N_1, \partial N_1, T_1, g)\) of \((N, \partial N, T, g)\) so that the double cover \(\tilde{\Gamma}_2\) of \(\Gamma_2\) is a two-sided free boundary minimal hypersurface in \((N_1, \partial N_1, T_1, g)\). Then applying Proposition 2.10 again, we obtain a two-sided, properly embedded, free boundary minimal hypersurface \(S\) having a contracting neighborhood so that \(S \subset N_1 \setminus (\Gamma_1 \cup \tilde{\Gamma}_2)\). Clearly, \(S\) is the desired hypersurface in \(N \setminus T\).

Case 3: Both \(\Gamma_1\) and \(\Gamma_2\) are one-sided.

Consider the double cover \((N_2, \partial N_2, T_2, g)\) of \((N, \partial N, T, g)\) so that the double cover \(\tilde{\Gamma}_1\) of \(\Gamma_1\) is a two-sided free boundary minimal hypersurface in \((N_2, \partial N_2, T_2, g)\). Then the desired result follows from Case 2.

We finish this section by giving an area lower bound for the free boundary minimal hypersurfaces. This can also be seen as an application of the construction of area minimizer in Proposition 2.10; cf. [26, Lemma 12].

**Lemma 2.12.** Suppose that \((N, \partial N, T, g)\) satisfies [A–F]. Let \(T_1, \cdots, T_q\) be the connected components of \(T\). Assume that

(i) every properly embedded free boundary minimal hypersurface in \(N \setminus T\) has an expanding neighborhood;
(ii) every half-properly embedded free boundary minimal hypersurface in \(N \setminus T\) has an expanding neighborhood in one side which is proper.

Then for any free boundary minimal hypersurface \(\Gamma\) in \(N \setminus T\):

1. if \(\Gamma\) is two-sided,
   \[ \text{Area}(\Gamma) > \max\{\text{Area}(T_1), \cdots, \text{Area}(T_q)\}; \]
2. if \(\Gamma\) is one-sided,
   \[ 2\text{Area}(\Gamma) > \max\{\text{Area}(T_1), \cdots, \text{Area}(T_q)\}. \]

**Proof.** We prove (1) and then (2) follows by considering the double cover. Without loss of generality, we assume that \(T_1\) is the connected component of \(T\) that has maximal area.

Assume on the contrary that \(\Gamma\) is a two-sided free boundary minimal hypersurface in \(N \setminus T\) so that

\[ \text{Area}(\Gamma) \leq \max\{\text{Area}(T_1), \cdots, \text{Area}(T_q)\}. \]

Denote by \(N'\) the closure of the generic component of \(N \setminus \Gamma\) that contains \(T_1\) and \(\Gamma\). We divide the proof into two cases by considering whether \(\Gamma\) has a proper neighborhood in \(N'\) or not.

If \(\Gamma\) has a proper neighborhood in one side in \(N'\), then such a neighborhood is expanding. Denote by

\[ \partial N' = \partial N \cap N' \quad \text{and} \quad T' = (T \cap N') \cup \Gamma. \]

Then \((N', \partial N', T', g)\) is a compact manifold with boundary and portion. Clearly, \(\Gamma\) represents a non-trivial relative homology class in \((N', \partial N')\). Using the argument in Proposition 2.10 we obtain a two-sided, properly embedded, free boundary boundary minimal hypersurface \(S\) having a contracting neighborhood. Note that \(S\) does not contain \(T'\) since

\[ \text{Area}(S) < \text{Area}(\Gamma) \leq \text{Area}(T_1) \leq \text{Area}(T'). \]

Then \(S\) has a connected component in \(N' \setminus T'\), which contradicts the assumption (i).

If \(\Gamma\) has no proper neighborhood in one side in \(N'\), then we can use a perturbation argument in Lemma 2.11 to construct an area minimizer \(S'\) having \(\text{Area}(S') < \text{Area}(\Gamma)\). Such an area
bound also implies that $S$ contains a two-sided free boundary minimal hypersurface in $N' \setminus T$. This also contradicts (1).

\[ \square \]

2.3. **No mass concentration at the corners.** In a compact manifold $(N^{n+1}, \partial N, g)$ with smooth boundary (without portion), then the monotonicity formula in [9] gives that a stationary $n$-varifold $V$ with free boundary can not support on an $(n-1)$-dimensional submanifold. In this subsection, we generalize this result directly in a compact manifold with boundary and non-empty portion.

**Lemma 2.13.** Let $(N, \partial N, T, g)$ be an $(n+1)$-dimensional compact manifold with boundary and portion. Let $V$ be an $n$-varifold such that the first variation vanishes along each vector field $X$ satisfying that $X$ is tangential when restricted on $\partial N$ and $T$. Denote by $S_V$ the support of $V$. Supposing that $T$ is a stable free boundary minimal hypersurface and $S_V \cap \partial T \neq \emptyset$, then for any neighborhood $U$ of $\partial T$ in $N$, $S_V$ intersects $U \setminus \partial T$.

**Proof of Lemma 2.13.** Recall that by Lemmas 2.5 and 2.6 there exists a neighborhood $\mathcal{N}$ of $T$ and a diffeomorphism $F : T \times [0, \epsilon) \rightarrow \mathcal{N}$ so that $F(T \times \{t\})$ is an embedded free boundary hypersurface.

Assume on the contrary that there exists a neighborhood $U$ of $\partial T$ so that $S_V \cap U \subset \partial T$. Without loss of generality, we assume that $U \subset \mathcal{N}$. Then $t\nabla t$ is tangential when restricted on $\partial N$ and $T$ in $U$. Hence the first variation of $V \setminus U$ vanishes along $t\nabla t$, i.e.

\[ (2.1) \quad 0 = \int \text{div}_{S}(t\nabla t)dV_{U}(x, S) = \int |p_{S}\nabla t|^{2}dV_{U}(x, S). \]

Here $p_{S}(\cdot)$ is the projection to $S$.

Let $\partial$ be the distance to $\partial T$ in $T$. We now extend it to be a function defined in a neighborhood of $T$ by setting

\[ \partial(F(x, t)) := \partial(x). \]

Then $\partial \nabla \partial$ is tangential when restricted on $\partial N$ and $T$ in $U$. Hence we also have

\[ (2.2) \quad 0 = \int \text{div}_{S}(\partial \nabla \partial)dV_{U}(x, S) = \int |p_{S}\nabla \partial|^{2}dV_{U}(x, S). \]

Note that $S$ is an $n$-dimensional hyperplane in $T_{x}N$ and $\nabla s \perp \nabla \partial$ at $x \in \partial T$. Hence there exists a unit vector $a$ so that

\[ a \in S \setminus \{c_1 \nabla s + c_2 \nabla \partial : c_1, c_2 \in \mathbb{R}\}. \]

Then

\[ |p_{S}\nabla t|^{2} + |p_{S}\nabla \partial|^{2} \geq |g(\nabla s, a)|^{2} + |g(\nabla \partial, a)|^{2} \geq \min\{ |\nabla t|^{2}, |\nabla \partial|^{2} \}. \]

Denote by $c_{0} = \min_{q \in F(T \times [0, \epsilon])} \min\{|\nabla t|^{2}, 1\} > 0$. Note that $|\nabla \partial| = 1$ on $\partial T$ by definition. Hence we have that for $x \in \partial T$,

\[ |p_{S}\nabla s|^{2} + |p_{S}\nabla \partial|^{2} \geq c_{0}. \]

However, this contradicts (2.1) and (2.2). Hence Lemma 2.13 is proved. \[ \square \]

We remark that in the Lemma 2.13, the stability and minimality of $T$ are both redundant. It is only used to construct a neighborhood foliated by free boundary hypersurfaces. Such a result may hold true for any embedded hypersurface with free boundary by using the implicit function theorem.
3. Confined min-max free boundary minimal hypersurfaces

3.1. Construction of non-compact manifold with cylindrical ends. In this part, we define the manifold with boundary and cylindrical ends. Then we will construct a sequence of compact manifold with boundary and portion converging to this non-compact manifold in some sense. The construction here is similar to [26, Section 2.2] with necessary modifications.

Let \((N, \partial N, T, g)\) be a connected compact Riemannian manifold with boundary and portion endowed with a metric \(g\). Suppose that \(T\) is a free boundary minimal hypersurface. Then by Lemmas 2.5 and 2.6, there is a neighborhood of \(T\) which is smoothly foliated with properly embedded leaves. In other words, there exist a neighborhood \(\mathcal{N}\) of \(T\) and a diffeomorphism

\[ F: T \times [0, \hat{t}) \to \mathcal{N}, \]

where \(F(T \times \{0\}) = T\) and for all \(t \in (0, \hat{t})\), \(F(T \times \{t\})\) is a properly embedded hypersurface with free boundary. Moreover, there exists a positive function \(\phi\) on \(T\) so that

\[ F_*(\frac{\partial}{\partial t})|_T = \phi n \text{ and } g(F_*(\frac{\partial}{\partial t}), n) > 0 \text{ on } F(T \times [0, \hat{t}]), \]

where \(n\) is the unit inward normal vector field of \(F(T \times \{t\})\) in \(N\).

In general, \(F_*(\frac{\partial}{\partial s})\) may not be a tangential vector field around \(\partial T\) along \(\partial N\). Fortunately, we can amend the diffeomorphism \(F\) to overcome this (cf. [19, Proof of Deformation Theorem C]).

Note that \(t\) can be used to define a function on \(F(T \times [0, \hat{t}])\) by setting

\[ t(F(x, a)) := a. \]

Taking a vector field \(X\) of \(N\) so that it is an extension of \(\nabla t/|\nabla t|^2\) and \(X|_{\partial N} \in T(\partial N)\). Let \((\mathcal{F}_t)_{t \geq 0}\) be the one-parameter family of diffeomorphisms generated by \(X\). We now claim that \(\mathcal{F}_t(T) = F(T \times \{t\})\) for any \(t \in [0, \hat{t}]\). Namely, for any \(x \in T\), we have

\[ \frac{d}{ds} t(\mathcal{F}_s(x)) = \langle \nabla t, X \rangle = 1. \]

Hence \(t(\mathcal{F}_s(T)) = s\) and the claim is proved. Note that \(\mathcal{F}\) can also be regarded as a diffeomorphism

\[ \mathcal{F}: T \times [0, \hat{t}) \to \mathcal{N} \]

by setting \(\mathcal{F}(x, t) = \mathcal{F}_t(x)\). By the definition of \(\mathcal{F}\) and \(X\), we have that

\[ \mathcal{F}_*(\frac{\partial}{\partial t}) = \nabla t/|\nabla t|^2 \]

for \(t \in [0, \hat{t}]\). From (3.1), we also have

\[ \mathcal{F}_*(\frac{\partial}{\partial s})|_T = \phi n \text{ and } g(\mathcal{F}_*(\frac{\partial}{\partial s}), n) > 0 \text{ on } \mathcal{F}(T \times [0, \hat{t}]). \]

We also use \(\frac{\partial}{\partial s}\) to denote \(\mathcal{F}_*(\frac{\partial}{\partial s})\) for simplicity.

Clearly, there exists a positive smooth function \(f\) on \(\mathcal{F}(T \times [0, \hat{t}])\) so that the metric \(g\) can be written as

\[ g = g_t(q) \oplus (f(q) dt)^2, \quad \forall q \in \mathcal{F}(T \times \{t\}). \]

Here \(g_t = g_{\mathcal{F}(T \times \{t\})}\) is the restricted metric and it can be extend to define a 2-form over \(TN\). Namely, a vector field \(X\) can be decomposed to

\[ X = X_\perp + X_\parallel, \]
where \( X_\perp \) is normal to \( \frac{\partial}{\partial t} \) and \( X_\parallel \) is a multiple of \( \frac{\partial}{\partial t} \). Then for any two vector fields \( X, Y \), we define
\[
g_t(X, Y) := g_t(X_\perp, Y_\perp).
\]

We also remark that \( f|_T = \phi \) by (3.3).

Now for any \( \epsilon < 1 \), define on \( N \) the following metric \( h_\epsilon \):
\[
h_\epsilon(q) := \begin{cases} 
g_t(q) \oplus (\partial_\epsilon(t) f(q) dt)^2 & \text{for } q \in \mathcal{F}(T \times [0, \epsilon]) \\
g(q) & \text{for } q \in N \setminus \mathcal{F}(T \times [0, \epsilon]).\end{cases}
\]

Here \( \partial_\epsilon \) is chosen to be a smooth function on \([0, \epsilon]\) so that

- \( 1 \leq \partial_\epsilon \) and \( \frac{\partial}{\partial t} \partial_\epsilon \leq 0 \);
- \( \partial_\epsilon \equiv 1 \) in a neighborhood of \( \epsilon \);
- \( \lim_{\epsilon \to 0} \int_{\epsilon/2}^\epsilon \partial_\epsilon = +\infty \);

Obviously, we have the following lemma.

**Lemma 3.1** (cf. [26, Lemma 4]). Suppose that the leaf \( \mathcal{F}(T \times \{t\}) \) has free boundary on \( \partial N \) and its non-zero mean curvature vector points towards \( T \) for all \( t \in (0, \epsilon) \) in \((N, \partial N, T, g)\). Then each slice \( \mathcal{F}(T \times \{t\}) \) is a free boundary hypersurface and satisfies the following with respect to the new metric \( h_\epsilon \):

- (1) it has non-zero mean curvature vector pointing in the direction of \(-\frac{\partial}{\partial t}\);
- (2) its mean curvature goes uniformly to zero as \( \epsilon \) converges to 0;
- (3) its second fundamental form is bounded by a constant \( C \) independent of \( \epsilon \).

Let \( \varphi : T \times \{0\} \to T \) be the canonical identifying map. Define the following non-compact manifold with cylindrical ends:
\[
C(N) := N \cup_\varphi (T \times [0, +\infty)).
\]

We endow it with the metric \( h \) such that \( h = g \) on \( N \) and
\[
h = g_\perp T \oplus (f_0 dt)^2
\]
on \( T \times [0, +\infty) \). Here \( g_\perp T \) is the restriction of \( g \) to the tangent bundle of \( T \) and
\[
f_0(x, t) = f(\varphi(x, 0)) = \phi(\varphi(x));
\]
see (3.4) for the definition of \( f \). We remark that under the metric \( h \), each slice \( T \times \{t\} \) is totally geodesic. We define the homeomorphism \( \gamma : T \times (-\hat{t}, \hat{t}) \to \mathcal{F}(T \times [0, \hat{t})) \cup_\varphi T \times (-\hat{t}, 0) \) by
\[
\gamma(x, t) = \mathcal{F}(x, t) \text{ for } t \in [0, \hat{t}); \quad \gamma(x, t) = (x, t) \text{ for } t \in (-\hat{t}, 0).
\]

The following lemma gives that \((C(N), h)\) is a \( C^1 \) manifold.

**Lemma 3.2.** With the differential structure associated with \( h \) on \( N \) and \( T \times (0, +\infty) \), \( C(N) \) is a \( C^1 \) manifold with boundary. Moreover, the metric is Lipschitz continuous.

**Proof.** To complete the proof, it suffices to prove that \( \gamma \) is a \( C^1 \) map. Note that \( \gamma \) is smooth for \( x \) everywhere. Since \( \mathcal{F} \) and \( \varphi \) are diffeomorphisms, \( \gamma(x, \cdot) \) is smooth for \( t \neq 0 \). Now we consider its behavior at \( t = 0 \). On the one hand, by (3.3),
\[
\lim_{t \to 0^+} \gamma_*(\frac{\partial}{\partial t}) = \lim_{t \to 0^+} \mathcal{F}_*(\frac{\partial}{\partial t}) = \phi n.
\]
On the other hand, for \( t < 0 \), \( \varphi_*(\partial_t) \) is parallel to the tangent space the level set \( \varphi(T \times \{t\}) \) and it has the norm
\[
\left| \varphi_*(\frac{\partial}{\partial t}) \right|_h = f_0,
\]
which implies that
\[
\lim_{t \to 0^+} \gamma_*(\frac{\partial}{\partial t}) = \lim_{t \to 0^-} \varphi_*(\frac{\partial}{\partial t}) = f_0 n = \phi n.
\]
Thus we conclude that \( C(N) \) is \( C^1 \).

Note that the metric is smooth on \( N \) and \( T \times [0, +\infty) \). Hence it is Lipschitz on \( C(N) \). This completes the proof of Lemma 3.2.

The following Lemma shows that \((N, \partial N, T, h_\epsilon)\) converges to the non-compact manifold with cylindrical ends \( C(N) \) and the convergence is smooth away from \( \mathcal{F}(T \times \{0\}) \).

**Lemma 3.3** (cf. [26] Lemma 5). Let \( q \) be a point of \( N \setminus T \). Then \((N, h_\epsilon, q)\) converges geometrically to \((C(N), h, q)\) in the \( C^0 \) topology as \( \epsilon \to 0 \). Moreover, the geometric convergence is smooth outside of \( T \subset C(N) \) in the following sense:

1. Let \( q \in N \setminus \mathcal{F}(T \times [0, \hat{t}]) \). Then as \( \epsilon \to 0 \),
\[
(N \setminus \mathcal{F}(T \times [0, \epsilon]), h_\epsilon, q)
\]
converges geometrically to \((N \setminus T, g, q)\) in the \( C^\infty \) topology;
2. Fix any connected component \( T_1 \) of \( T \). Let \( q_\epsilon \in \mathcal{F}(T_1 \times [0, \epsilon]) \) be a point at fixed distance \( \hat{d} > 0 \) from \( \mathcal{F}(T_1 \times \{\epsilon\}) \) for the metric \( h_\epsilon, \hat{d} \) being independent of \( \epsilon \). Then
\[
(\mathcal{F}(T_1 \times [0, \epsilon]), h_\epsilon, q_\epsilon)
\]
subsequently converges geometrically to \((T_1 \times (0, +\infty), h, q_{\infty})\) in the \( C^\infty \) topology, where \( h \) is defined as \((3.5)\), and \( q_{\infty} \) is a point of \( T_1 \times (0, +\infty) \) at distance \( \hat{d} \) from \( T_1 \times \{0\} \).

**Proof.** The first item follows from that \( h_\epsilon = g \) on \( N \setminus \mathcal{F}(T \times [0, \epsilon]) \). We now prove the second one. Define the coordinate \( s \) by
\[
s(\mathcal{F}(x, t)) := -\int_t^\epsilon \varphi_*(u) du.
\]
Then \((\partial^2 s/dt)^2 = ds^2\) and \(|\nabla^e s|_{h_\epsilon} = |\nabla t|_g\). Hence
\[
h_\epsilon(q) = g_s(q) \oplus (f(q)ds)^2.
\]
Note that \( g_t(\mathcal{F}(x, t)) \to g_0(\mathcal{F}(x, 0)) = g_{\infty} T \) and \( f(\mathcal{F}(x, t)) \to f(\varphi(x, 0)) \) as \( \epsilon \to 0 \). Thus we conclude that
\[
h_\epsilon(\mathcal{F}(x, s)) \to g_{\infty} T \oplus (f_0 ds)^2 = h.
\]

In the following lemma, we describe more about the convergence in a neighborhood of \( \mathcal{F}(T \times \{0\}) \) by finding explicit local charts.

**Lemma 3.4** (cf. [26] Lemma 6). There exists \( \eta > 0 \) such that for each \( \epsilon \in (0, \hat{t}/2) \) small, there is an embedding \( \sigma_\epsilon : T \times [-\hat{t}/2, \hat{t}/2] \to N \) satisfying the following properties:

1. \( \sigma_\epsilon(T \times \{0\}) = \mathcal{F}(T \times \{\epsilon\}) \) and
\[
\sigma_\epsilon(T \times [-\hat{t}/2, \hat{t}/2]) = \{q \in N : |s(q)| \leq \hat{t}/2\};
\]
Then we define the embedding map

\[ \sigma^*_\epsilon h_\epsilon \|_{C^1(T \times [-\epsilon/2, \epsilon/2])} < \eta, \text{ where } \| \cdot \|_{C^1(T \times [-\epsilon/2, \epsilon/2])} \text{ is computed under the product metric} \]

\[ h' = g_T \oplus ds^2; \]

(3) the metrics \( \sigma^*_\epsilon h_\epsilon \) converge in the \( C^0 \) topology to \( \gamma^* h_\epsilon T \times [-\epsilon/2, \epsilon/2] \) (see (3.6) for the definition of \( \gamma \)).

Proof. Given \( \epsilon > 0 \), recall that

\[ s(t) = - \int_0^t \vartheta_s(u)du. \]

Then we define the embedding map \( \sigma_\epsilon : T \times [-\epsilon/2, \epsilon/2] \to N \) by

\[ \sigma_\epsilon(x, u) = \mathcal{F}(x, s^{-1}(u)). \]

The first item follows immediately. The second one comes from the fact that \( \vartheta_\epsilon \geq 1 \). The last one follows from Lemmas 3.2 and 3.3 and the fact that \( g = h_\epsilon \) on \( N \setminus \mathcal{F}(T \times [0, \epsilon]) \).

\[ \square \]

3.2. Notations from geometric measure theory. We now recall the formulation in [18].

Let \( (M, \partial M, g) \subset \mathbb{R}^L \) be a compact Riemannian manifold with piecewise smooth boundary. Let \( \mathcal{R}_k(M; \mathbb{Z}_2) \) (resp. \( \mathcal{R}_k(\partial M) \)) be the space of \( k \)-dimensional rectifiable currents in \( \mathbb{R}^L \) with coefficients in \( \mathbb{Z}_2 \) which are supported in \( M \) (resp. \( \partial M \)). Denote by \( \mathcal{M} \) the mass norm. We now recall the formulation in [17] using equivalence classes of integer rectifiable currents. Let

(3.7)

\[ Z_k(M, \partial M; \mathbb{Z}_2) := \{ T \in \mathcal{R}_k(M; \mathbb{Z}_2) : spt(\partial T) \subset \partial M \}. \]

We say that two elements \( S_1, S_2 \in Z_k(M, \partial M; \mathbb{Z}_2) \) are equivalent if \( S_1 - S_2 \in \mathcal{R}_k(\partial M; \mathbb{Z}_2) \).

Denote by \( Z_k(M, \partial M; \mathbb{Z}_2) \) the space of all such equivalence classes. For any \( \tau \in Z_k(M, \partial M; \mathbb{Z}_2) \), we can find a unique \( T \in \tau \) such that \( T, \partial M = 0 \). We call such \( T \) the canonical representative of \( \tau \) as in [17]. For any \( \tau \in Z_k(M, \partial M; \mathbb{Z}_2) \), its mass and flat norms are defined by

\[ \mathcal{M}(\tau) := \inf \{ \mathcal{M}(S) : S \in \tau \} \quad \text{and} \quad \mathcal{F}(\tau) := \inf \{ \mathcal{F}(S) : S \in \tau \}. \]

The support of \( \tau \in Z_k(M, \partial M; \mathbb{Z}_2) \) is defined by

\[ spt(\tau) := \bigcap_{S \in \tau} spt(S). \]

By [17] Lemma 3.3, we know that for any \( \tau \in Z_k(M, \partial M; \mathbb{Z}_2) \), we have \( \mathcal{M}(S) = \mathcal{M}(\tau) \) and \( spt(\tau) = spt(S) \), where \( S \) is the canonical representative of \( \tau \).

Recall that the varifold distance function \( \mathcal{F} \) on \( \mathcal{V}_k(M) \) is defined in [22], (2.1 (19)), which induces the varifold weak topology on the set \( \mathcal{V}_k(M) \cap \{ V : \| V \|(M) \leq c \} \) for any \( c \). We also need the \( \mathcal{F} \)-metric on \( Z_k(M, \partial M; \mathbb{Z}_2) \) defined as follows: for any \( \tau, \sigma \in Z_k(M, \partial M; \mathbb{Z}_2) \) with canonical representatives \( S_1 \in \tau \) and \( S_2 \in \sigma \), the \( \mathcal{F} \)-metric of \( \tau \) and \( \sigma \) is

\[ \mathcal{F}(\tau, \sigma) := \mathcal{F}(\tau - \sigma) + \mathcal{F}(|S_1|, |S_2|), \]

where \( \mathcal{F} \) on the right hand side denotes the varifold distance on \( \mathcal{V}_k(M) \).

For any \( \tau \in Z_k(M, \partial M; \mathbb{Z}_2) \), we define \( |\tau| \) to be \( |S| \), where \( S \) is the unique canonical representative of \( \tau \) and \( |S| \) is the rectifiable varifold corresponding to \( S \).

We assume that \( Z_k(M, \partial M; \mathbb{Z}_2) \) has the flat topology induced by the flat metric. With the topology of mass norm or the \( \mathcal{F} \)-metric, the space will be denoted by \( Z_k(M, \partial M; \mathcal{M}; \mathbb{Z}_2) \) or \( Z_k(M, \partial M; \mathcal{F}; \mathbb{Z}_2) \).

Let \( X \) be a finite dimensional simplicial complex. Suppose that \( \Phi : X \to \mathcal{Z}_n(M, \partial M; \mathcal{F}; \mathbb{Z}_2) \) is a continuous map with respect to the \( \mathcal{F} \)-metric. We use \( \Pi \) to denote the set of all continuous
maps $\Psi : X \to \mathcal{Z}_n(M, \partial M; F; Z_2)$ such that $\Phi$ and $\Psi$ are homotopic to each other in the flat topology. The *width* of $\Pi$ is defined by

$$L(\Pi) = \inf_{\Phi \in \Pi} \sup_{x \in X} M(\Phi(x)).$$

Given $p \in \mathbb{N}$, a continuous map in the flat topology

$$\Phi : X \to \mathcal{Z}_n(M, \partial M; Z_2)$$

is called a *$p$-sweepout* if the $p$-th cup power of $\lambda = \Phi^*(\lambda)\tilde{\lambda} X$ is non-zero in $H^p(X; Z_2)$ where $0 \neq \tilde{\lambda} \in H^1(\mathcal{Z}_n(M, \partial M; Z_2); Z_2) \cong Z_2$. Denote by $\mathcal{P}_p(M)$ the set of all $p$-sweepouts that are continuous in the flat topology and have no concentration of mass ([20 §3.7]), i.e.

$$\lim_{r \to 0} \sup \{M(\Phi(x) \cap B_r(q)) : x \in X, q \in M\} = 0.$$

In [20] and [18], the $p$-width is defined as

$$(3.8) \quad \omega_p(M; g) := \inf_{\Phi \in \mathcal{P}_p} \sup \{M(\Phi(x)) : x \in \mathrm{dmn}(\Phi)\}.$$ 

**Remark 3.5.** In this paper, we used the integer rectifiable currents, which is the same with [17]. However, the formulations are equivalent to that in [18]; see [8, Proposition 3.2] for details.

For the non-compact setting, the following definition does not depend on the choice of the exhaustion sequences by [18, Lemma 2.15 (1)].

**Definition 3.6 ([20 Definition 7]).** Let $(\mathcal{N}^{n+1}, g)$ be a complete non-compact Lipschitz manifold. Let $K_1 \subset K_2 \subset \cdots \subset K_i \subset \cdots$ be an exhaustion of $\mathcal{N}$ by compact $(n+1)$-submanifolds with piecewise smooth boundary. The $p$-width of $(\mathcal{N}, g)$ is the number

$$\omega_p(\mathcal{N}; g) = \lim_{i \to \infty} \omega_p(K_i; g) \in [0, +\infty].$$

3.3. **Min-max theory for manifolds with boundary and ends.** Let $(N, \partial N, T, g)$ be a compact manifold with boundary and portion such that $T$ is a free boundary minimal hypersurface in $(N, \partial N, g)$ with a contracting neighborhood in one side of $N$. Let $T_1, \cdots, T_m$ be the connected components of $T$ and suppose that $T_1$ has the largest area among their components:

$$\text{Area}(T_1) \geq \text{Area}(T_j) \quad \text{for all} \quad j \in \{1, \cdots, m\}.$$ 

The purpose of this subsection is to prove the $p$-width $\omega_p(\mathcal{C}(N))$ is associated with almost properly embedded free boundary minimal hypersurfaces with multiplicities.

We give the upper and lower bounds for $\omega_p(\mathcal{C}(N); h)$.

**Lemma 3.7 (cf. [20 Theorem 8]).** There exists a constant $C$ depending on $h$ such that for all $p \in \{1, 2, 3, \cdots\}$:

$$\omega_{p+1}(\mathcal{C}(N)) - \omega_p(\mathcal{C}(N)) \geq \text{Area}(T_1);$$

$$p \cdot \text{Area}(T_1) \leq \omega_p(\mathcal{C}(N)) \leq p \cdot \text{Area}(T_1) + C p^{\frac{n+1}{n}}.$$

**Proof.** The proof here actually is the same with [20 Theorem 8], which is an application of Lusternick-Schnirelman Inequalities in [18 Section 3.1]. We sketch the idea here.

Firstly, we know that $\omega_1(T \times [-R, R]; h)$ is realized by a varifold $V_R$. Then by [17], $V_R$ is a free boundary minimal hypersurface when restricted outside $\partial T \times \{-R\}$. Moreover, by [17] again, the first variation of $V_R$ vanishes along each vector field $X$ satisfying that $X$ is tangential on $T \times \{-R\}$ and $\partial T \times (R, R)$. Since each slice of $T \times \{t\}$ is minimal and stable, then by Lemma...
for any neighborhood $U$ of $\partial T \times \{ \pm R \}$, the support of $V_R$ intersects $U \setminus \partial T \times \{ \pm R \}$ if it intersects $\partial T \times \{ \pm R \}$. Together with the monotonicity formula and maximum principle (see [26] Theorem 8 for details), we always have
\[
\omega_1(T \times [-R, R]) = \text{Area}(T_1)
\]
for sufficiently large $R$. Letting $R \to \infty$, we conclude that
\[
\omega_1(T \times \mathbb{R}; h) = \text{Area}(T_1).
\]
Then by Lusternick-Schnirelman Inequalities,
\[
\omega_{p+1}(T_1 \times [0, 2R]; h) \geq \omega_p(T_1 \times [0, R]; h) + \omega_1(T_1 \times (R, 2R]; h).
\]
Letting $R \to \infty$,
\[
\omega_{p+1}(T \times \mathbb{R}; h) \geq \omega_p(T_1 \times \mathbb{R}; h) + \text{Area}(T_1).
\]
By induction, $\omega_p(T_1 \times \mathbb{R}) \geq p \cdot \text{Area}(T_1)$. On the other hand, by direct construction, we have that $\omega_p(T_1 \times \mathbb{R}) \leq p \cdot \text{Area}(T_1)$. Therefore,
\[
\omega_p(T_1 \times \mathbb{R}) = p \cdot \text{Area}(T_1).
\]
We now prove (3.9). Fix $q \in N$ and take $R$ large enough so that $B(q, 3R)$ contains two disjoint part $B(q, R)$ and $T_1 \times [0, R]$. Then by Lusternick-Schnirelman Inequalities,
\[
\omega_{p+1}(B(q, 3R); h) \geq \omega_p(B(q, R); h) + \omega_1(T_1 \times [0, R]; h).
\]
Letting $R \to 0$, then we have
\[
\omega_{p+1}(C) \geq \omega_p(C) + \text{Area}(T_1),
\]
which is exactly the desired inequality.

In the next, we prove (3.10). Clearly, the first half follows from (3.9). Using [18] Lemma 4.4 (see also [26] Proof of Theorem 8),
\[
\omega_p(C) \leq \omega_p(N) + \omega_p(T \times \mathbb{R}) \leq p \cdot \text{Area}(T_1) + C \cdot p^{n+1}.
\]
Here the last inequality we used the Weyl Law of $\omega_p(N)$ by Liokumovich-Marques-Neves [18] §1.1. This finishes the proof.

Let $h_\epsilon$ be the metric constructed in Subsection 3.1. Denote by $N_\epsilon = N \setminus \mathcal{F}(T \times \{0, \epsilon/2\})$, which is a compact manifold with piecewise smooth boundary $\partial N_\epsilon$. For simplicity, denote by $T_\epsilon = \mathcal{F}(T \times \{\epsilon/2\})$. Although there is no general regularity for min-max theory in such a space, we can use the uniform upper bound of the width and the monotonicity formulas of [9] Theorem 3.4; [24] §17.6 to prove that $\omega_p(N_\epsilon; h_\epsilon)$ is realized by embedded free boundary minimal hypersurfaces.

**Theorem 3.8.** Fix $p \in \mathbb{N}$. For $\epsilon > 0$ small enough, there exist disjoint, connected, almost properly embedded, free boundary minimal hypersurfaces $\Gamma_1, \ldots, \Gamma_N$ contained in $N_\epsilon \setminus \mathcal{F}(T \times \{\epsilon/2\})$ and positive integers $m_1, \ldots, m_N$ such that
\[
\omega_p(N_\epsilon; h_\epsilon) = \sum_{j=1}^N m_j \cdot \text{Area}(\Gamma_j) \quad \text{and} \quad \sum_{j=1}^N \text{Index}(\Gamma_j) \leq p.
\]
Proof. Choose a sequence \( \{ \Phi_i \} \in \mathcal{P}_p(N_\epsilon) \) such that
\[
\lim_{i \to \infty} \sup \{ M(\Phi_i(x)) : x \in X_i = \text{dmm}(\Phi_i) \} = \omega_k(N_\epsilon; g).
\]
Without loss of generality, we can assume that the dimension of \( X_i \) is \( p \) for all \( i \) (see [19, §1.5] or [13, Proof of Proposition 2.2]).

By the Discretization Theorem [17, Theorem 4.12] and the Interpolation Theorem [8, Theorem 4.4], we can assume that \( \Phi_i \) is a continuous map to \( Z_n(N_\epsilon, \tilde{\partial}N_\epsilon; \mathbb{Z}_2) \) in the \( \mathbf{F} \)-metric. Denote by \( \Pi_i \) the homotopy class of \( \Phi_i \). By [8, Proposition 7.3, Claim 1],
\[
\lim_{i \to \infty} L(\Pi_i) = \omega_p(N_\epsilon; h_\epsilon).
\]
For any \( p \in \{ 1, 2, 3, \cdots \} \), by [21, Lemma 1] and Lemma [3.3]
\[
\lim_{i \to \infty} \omega_p(N_\epsilon; h_\epsilon) = \omega_p(C(N); h).
\]
Hence we can assume \( L(\Pi_i) \) has a uniform upper bound not depending on \( i \) or \( \epsilon \).

We first prove that \( L(\Pi_i) \) is realized by free boundary minimal hypersurfaces. Without loss of generality, we assume that
\[
L(\Pi_i) < \omega_p(N_\epsilon; h_\epsilon) + 1.
\]
By the work of Li-Zhou [17, Theorem 4.21], there exists a varifold \( V^i_\epsilon \) so that
- \( L(\Pi_i) = M(V^i_\epsilon) \);
- with respect to metric \( h_\epsilon \), \( V^i_\epsilon \) is stationary in \( N_\epsilon \setminus \partial T_\epsilon \) with free boundary; moreover, the first variation vanishes along each vector field \( X \) satisfying that \( X \) is tangential when restricted on \( \partial N \cap N_\epsilon \) and \( T_\epsilon \);
- with respect to metric \( h_\epsilon \), \( V^i_\epsilon \) is almost minimizing in small annuli with free boundary for any \( q \in N_\epsilon \setminus \partial T_\epsilon \).

Denote by \( S^i_\epsilon \) the support of \( \| V^i_\epsilon \| \). Also, by the regularity theorem given by Li-Zhou [17, Theorem 5.2], when restricted in \( N_\epsilon \setminus \partial T_\epsilon \), \( S^i_\epsilon \) is a free boundary minimal hypersurface.

Now we are going to prove that \( S^i_\epsilon \) does not intersect \( \partial T_\epsilon \) for \( \epsilon \) small enough. Suppose not, then by Lemma [2.13] for any neighborhood \( U \) of \( \partial T_\epsilon \) in \( N \), \( S^i_\epsilon \) intersects \( U \setminus \partial T_\epsilon \). \( S^i_\epsilon \) intersects \( U \setminus \partial T_\epsilon \) for all neighborhood of \( T_\epsilon \). By the maximum principle, \( \Sigma^i_\epsilon \) also has to intersect \( \mathcal{F}(T \times \{ \hat{t} \}) \). Note that \( M(V^i_\epsilon) \) is uniformly bounded from above for \( i \) since \( L(\Pi_i) \) is uniformly bounded. This contradicts the monotonicity formula [9, Theorem 3.4; 24, §17.6]. Hence \( S^i_\epsilon \) is almost properly embedded free boundary minimal hypersurface in \( N_\epsilon \setminus \partial T_\epsilon \).

Next we prove the index bound for \( S^i_\epsilon \). Such a bound follows from the argument in [19] (see also [8, Theorem 6.1] for free boundary minimal hypersurfaces) if we can construct a sequence of metrics \( h^i_\epsilon \to h_\epsilon \) in the \( C^\infty \) topology on \( N \) so that all the free boundary minimal hypersurface in \((N, \partial N, T, h^i_\epsilon)\) is countable.

To do this, we first embed \((N, \partial N, T, h_\epsilon)\) isometrically into a compact manifold with boundary \((\hat{N}, \partial \hat{N}, g_\epsilon)\). By [3], we can get a sequence of smooth metrics \( h^i_\epsilon \to g_\epsilon \) on \( \hat{N} \) so that every finite cover of free boundary minimal hypersurface in \((\hat{N}, \partial \hat{N}, h^i_\epsilon)\) is non-degenerate. Then using the argument in [8, Proposition 5.3] (see also [28]), the free boundary minimal hypersurfaces in \((\hat{N}, \partial \hat{N}, T, h^i_\epsilon)\) is countable.

Now we have proved that for \( \epsilon \) small enough, there exists \( V^i_\epsilon \) so that \( L(V^i_\epsilon) = L(\Pi_i) \) and the support of \( V^i_\epsilon \) is a free boundary minimal hypersurface \( S^i_\epsilon \) with \( \text{Index}(S^i_\epsilon) \leq p \). Letting \( i \to \infty \), this theorem follows from the compactness for free boundary minimal hypersurfaces in [14]. \( \square \)
Remark 3.9. Furthermore, the monotonicity formulas [9, Theorem 3.4; 24, §17.6] and mean convex foliation also indicate that there is $R > 0$ and a point $q_0 \in N \setminus \mathcal{F}(T \times [0, \hat{t}])$ such that for all $\epsilon$ small enough, $S_\epsilon^i$ is contained in the ball $B_{h_\epsilon}(q_0, R)$.

Now we can prove the main result in this section, which can been seen as an analog of [26, Theorem 9].

**Theorem 3.10.** Let $(N, \partial N, T, g)$ be a compact manifold with boundary and portion in Theorem 3.8. Let $(\mathcal{C}(N), h)$ be as in Subsection 3.1. For all $p \in \{1, 2, 3, \ldots \}$, there exist disjoint, connected, embedded free boundary minimal hypersurfaces $\Gamma_1, \ldots, \Gamma_N$ contained in $N \setminus T$ and positive integers $m_1, \ldots, m_N$ such that

$$\omega_p(\mathcal{C}(N); h) = \sum_{j=1}^N m_j \text{Area}(\Gamma_j).$$

Besides, if $\Gamma_j$ is one-sided then the corresponding multiplicity $m_j$ is even.

**Proof.** We follow the steps given by Song in [26].

Recall that $N_\epsilon = N \setminus \mathcal{F}(T \times [0, \epsilon/2])$. By Theorem 3.8 and Remark 3.9 we obtain a varifold $V_\epsilon$ so that

- $\text{M}(V_\epsilon) = \omega_p(N_\epsilon; h_\epsilon)$;
- the support of $V_\epsilon$ is an almost properly embedded free boundary minimal hypersurface, denoted by $S_\epsilon$;
- for fixed $p > 0$, there exist $R > 0$ and a point $q_0 \in N \setminus \mathcal{F}(T \times [0, \hat{t}])$ such that for all $\epsilon$ small enough, $S_\epsilon^i$ is contained in the ball $B_{h_\epsilon}(q_0, R)$;
- Index(support of $V_\epsilon$) $\leq p$.

The next step is to take a limit as a sequence $\epsilon_k \to 0$. Note that $\omega_p(N_\epsilon; h_\epsilon)$ converges to $\omega_p(\mathcal{C}(N); h)$. Thus $V_{\epsilon_k}$ subsequently converges to a varifold $V_\infty$ in $\mathcal{C}(N)$ of total mass $\omega_p(\mathcal{C}(N); h)$, whose support is denoted by $S_\infty$.

Using the compactness again, $S_\infty \setminus (\mathcal{C}(N) \setminus T)$ is an almost properly embedded free boundary minimal hypersurface since $h_\epsilon$ converges smoothly in this region. Then by the maximum principle again, $S_\infty$ is contained in the compact set $(N, g)$. Furthermore, we will prove that $V_\infty$ is $g$-stationary with free boundary on $\partial N$. Once this has been proven, then applying [26, Proposition 3], $V_\infty$ is actually a $g$-stationary integral varifold with free boundary. Recall that each connected component intersects $F(T \times \{\hat{t}\})$. Hence no component of $S_\infty$ is contained in $T$. Then by the strong maximum principle in Lemma 3.1, $S_\infty \subset N \setminus T$. Therefore, from the compactness [11], $S_\infty$ is a free boundary minimal hypersurface in $N \setminus T$, and we also conclude that the one-sided components of $S_\infty$ have even multiplicities.

It remains to show that $V_\infty$ is $g$-stationary with free boundary in $(N, \partial N, T, g)$. For $\epsilon \geq 0$, we will denote by $\nabla^\epsilon$ and $\text{div}^\epsilon$ the connection and divergence computed in the metric $h_\epsilon$ (by convention $h_0 = g$). Let $\mathcal{X}(N, \partial N)$ be the collection of vector fields $X$ so that

- $X(x) \in T_x N$ for any $x \in N$;
- $X$ can be extended to a smooth vector field on $\tilde{N}$;
- $X(x) \in T_x(\partial N)$ for any $x \in \partial N$;

Our goal is to prove that the first variation along $X \in \mathcal{X}(N, \partial N)$ vanishes:

$$\delta V_\infty(X) = \int \text{div}^\infty S X(x) dV_\infty(x, S) = 0.$$  

(3.12)
We use the same strategy with [26, Proof of Theorem 8]. In the following, we give the necessary modification and put the computation in Appendix [23].

**Part I: Normalize the coordinate function with respect to \( h_\epsilon \).**

Recall that for \( \epsilon > 0 \) small enough, the map
\[
F : T \times [0, \hat{t}] \to N
\]
is a diffeomorphism onto its image. Note that the support of \( V_\infty \) restricted to \( N \setminus T \) is an almost properly embedded free boundary minimal hypersurface. Hence we can assume that the vector field \( X \) is supported in \( F(T \times [0, \hat{t}/2]) \). Thus for all \( \epsilon \) small enough, the vector field \( X \) restricted to \( N_\epsilon := N \setminus F(T \times (0, \epsilon/2)) \) can be decomposed into two components
\[
X = X_\perp + X_\parallel,
\]
where \( X_\perp \) is orthogonal to \( \nabla^\epsilon t \) and \( X_\parallel \) is a multiple of \( \nabla^\epsilon t \).

For \( q = F(x, t) \), denote
\[
n(q) := f(q) \partial_\epsilon(t) \nabla^\epsilon t,
\]
which is a unit vector field with respect to the metric \( h_\epsilon \). Recall that the coordinate \( s \) is defined by
\[
s(F(x, t)) := - \int_t^\epsilon \partial_\epsilon(u) du.
\]
Then for the points where the metric is changed, \( s \) is negative. Clearly,
\[
\nabla^\epsilon s = \partial_\epsilon(t) \nabla^\epsilon t = (\partial_\epsilon(t))^{-1} \nabla t,
\]
which implies that
\[
|\nabla^\epsilon s|_{h_\epsilon} = (f(q))^{-1} = |\nabla t|_g.
\]
We use \( \partial / \partial s \) and \( \partial / \partial t \) to denote \( \mathcal{F}_s(\partial / \partial s) \) and \( \mathcal{F}_s(\partial / \partial t) \), respectively. Then we also have
\[
\frac{\partial}{\partial s} = (\partial_\epsilon(t))^{-1} \frac{\partial}{\partial t}.
\]
Recall that the map \( F \) is defined by the first eigenfunction in Lemma [26] and \( |\nabla t|_T = \phi^{-1} \) \( n \). Then we can normalize and fix such a positive function so that \( \max_{\{x \in T\}} \phi = 1 \). Since \( \nabla t \) is a smooth vector field, then for \( \epsilon \) small enough,
\[
2 \max_{x \in \mathcal{T}} \phi^{-1} \geq |\nabla t|_g \geq 1/2, \text{ for } x \in \mathcal{F}(T \times [0, 2\epsilon]).
\]
Let \( (\gamma(u))_{0 \leq u \leq r} \) be a geodesic in \( (N_\epsilon, h_\epsilon) \) with \( \gamma(0) \in \mathcal{F}(T \times \{\epsilon\}) \). Then
\[
s(\gamma(r)) - s(\gamma(0)) = \int_0^r h_\epsilon(\nabla^\epsilon s, \gamma'(u)) du \geq - \int_0^r |\nabla^\epsilon s|_{h_\epsilon} du \geq -2r \max_{x \in \mathcal{T}} \phi^{-1}.
\]
If we take \( C_0 = 2 \max_{x \in \mathcal{T}} \phi^{-1} \), then
\[
B_{h_\epsilon}(q_0, R) \subset \left[ N \setminus \mathcal{F}(T \times [0, \epsilon]) \right] \cup \{ q \in \mathcal{F}(T \times [0, \epsilon]) : s \geq -C_0 R \}.
\]

**Part II: The uniform upper bound for points with non-parallel normal vector field.**

Let \((y, S)\) be a point of the Grassmannian bundle of \( N \) and let \((e_1, \cdots, e_n)\) be an \( h_\epsilon \)-orthonormal basis of \( S \) so that \( e_1, \cdots, e_{n-1} \) are \( h_\epsilon \)-orthogonal to \( \nabla^\epsilon t \). Denote by \( \hat{n} \) the unit normal vector of \( S \) under the metric \( h_\epsilon \). Let \( e_n^* \) be a unit vector such that \((e_1, \cdots, e_n^*)\) is an \( h_\epsilon \)-orthonormal basis of the \( n \)-plane \( h_\epsilon \)-orthogonal to \( \nabla^\epsilon t \) at \( y \).
The main result in this part is that for any\( b > 0, \)
\[
\lim_{\epsilon \to 0} \int_{\mathcal{F}(T \times [0,2\epsilon]) \times \mathbf{G}(n+1,n)} \chi(\{|h_\epsilon(e_n,n)| > b\}) dV_\epsilon(x,S) = 0.
\]
In particular,
\[
V_\infty \{ (x,S) : x \in T, S \neq T_x T \} = 0.
\]

The proof is similar to Song [26, (11)]. We postpone the proof of (3.15) to Subsection B.1 in Appendix B.

We now explain how to deduce (3.12) from the previous estimates. Take a sequence \( \epsilon_k \to 0. \) Consider
\[
A_k := \mathcal{F}(T \times [0,2\epsilon_k]) \quad \text{and} \quad B_k := N \setminus \mathcal{F}(T \times [0,2\epsilon_k]).
\]
Then by taking a subsequence (still denoted by \( A_k \) and \( B_k \)), we can assume that there are two varifolds \( V'_\infty \) and \( V''_\infty \) in \( N \) so that as \( k \to \infty \), the following convergences in the varifolds sense take place:
\[
V_k := V_{\epsilon_k} \rightharpoonup V_\infty,
\]
\[
V'_k := V_{\epsilon_k} \mathcal{L}(A_k \times \mathbf{G}(n+1,n)) \rightharpoonup V'_\infty,
\]
\[
V''_k := V_{\epsilon_k} \mathcal{L}(B_k \times \mathbf{G}(n+1,n)) \rightharpoonup V''_\infty.
\]
Recall that we decomposed \( X = X_\perp + X_\parallel. \)

**Part III:** We will show first that
\[
\int \text{div}^0 X_\perp^0 dV_\infty = \lim_{k \to \infty} \int \text{div}^\epsilon_k X_\perp^k dV_k = 0.
\]

Let \((x,S)\) and \(e_1, \ldots, e_n, e_n^*\) be defined as before and let \( S_\perp \) denote the \( n \)-plane at \( x \) orthogonal to \( \nabla s \). By the construction of \( h_\epsilon \), we have that for any \( e' \in S_\perp, \)
\[
h_\epsilon(\nabla_{e'} X_\perp^\epsilon, e') = g(\nabla_{e'} X_\perp^\epsilon, e').
\]
Then a direct computation gives that
\[
\text{div}_S X_\perp^\epsilon = \text{div}_{S_\perp} X_\perp^\epsilon + \Upsilon(\epsilon, x, S, X),
\]
where
\[
\Upsilon(\epsilon, x, S, X) = h_\epsilon(\nabla_{e_n}^\epsilon X_\perp^\epsilon, e_n) - h_\epsilon(\nabla_{e_n}^\epsilon X_\parallel^\epsilon, e_n^*) \leq 2|\nabla^\epsilon X_\parallel^\epsilon|_{h_\epsilon} |e_n - e_n^*|_{h_\epsilon}.
\]
By the construction of \( h_\epsilon \), we have that \(|\nabla^\epsilon X_\parallel^\epsilon|_{h_\epsilon}\) is uniformly bounded in \( \epsilon > 0. \) Together with (3.15), we in fact have (see Subsection B.2 for details)
\[
\lim_{k \to \infty} \int \text{div}^\epsilon_k X_\perp^k dV_k(x,S) = \int \text{div}^0 X_\perp^0 dV_\infty.
\]
On the other hand, using the facts that \( h_\epsilon = g \) and \( X_\parallel^\epsilon \) smoothly converges to \( X_\perp^0 \) in \( B_k \), we have
\[
\int \text{div}^0 X_\perp^0 dV_\infty = \lim_{k \to \infty} \int \text{div}^0 X_\perp^0 dV_k = \lim_{k \to \infty} \int \text{div}^\epsilon_k X_\perp^k dV_k = \lim_{k \to \infty} \int \text{div}^\epsilon_k X_\parallel^\epsilon dV_k.
\]
Then (3.17) follows immediately.
Part IV: Finally, we prove that
\[ \int \text{div}^0 X^0 \| dV = 0. \]

By the definition of \( X^\| \), there exists \( \varphi \) so that \( X^0 = \varphi \nabla t \). Now define
\[ Z^\varepsilon := \varphi \nabla^\varepsilon s. \]

Then the most important thing is that \( |\nabla^\varepsilon Z^\varepsilon|_{h^\varepsilon} \) is uniformly bounded (see Subsection 3.3). Using the same argument in [26, Theorem 9], such a property enables us (see Subsection 3.4) to prove that
\[ \lim_{k \to \infty} \int S_k \text{div}^k X^k \| dV''(x, S) = 0. \]

Using the facts that \( h^\varepsilon = g \) and \( X^\varepsilon \) smoothly converges to \( X^0 \) in \( B_k \), we have
\[ \int \text{div}^0 X^0 \| dV'' = \lim_{k \to \infty} \int \text{div}^0 X^0 \| dV'' = \lim_{k \to \infty} \int \text{div}^k X^k \| dV'' = 0. \]

On the other hand, the minimality of \( T \) and (3.16) give that
\[ \int \text{div}^0 X^0 \| dV' = 0. \]

Therefore,
\[ \int \text{div}^0 X^0 \| dV = \int \text{div}^0 X^0 \| dV' + \int \text{div}^0 X^0 \| dV'' = 0. \]

The desired equality (3.12) follows from Part III and IV. \( \square \)

4. Proof of main theorem

Now we are ready to prove our main theorem. The conditions (A–F) defined in Subsection 2.2 will be used frequently.

Theorem 4.1. Let \((M^{n+1}, \partial M, g)\) be a connected compact Riemannian manifold with smooth boundary and \(3 \leq (n + 1) \leq 7\). Then there exist infinitely many almost properly embedded free boundary minimal hypersurfaces.

Proof. Assume on the contrary that \((M, \partial M, g)\) contains only finitely many free boundary minimal hypersurfaces. Then by the construction in Lemma 2.6, (C) and (D) hold true.

Now we prove that by cutting along free boundary minimal hypersurfaces in finite steps, we can construct a compact manifold with boundary and portion satisfying Frankel property and each free boundary minimal hypersurface that does not intersect the portion must have area larger than each connected component of the portion.

Let \( T_0^0 \) be the union of the connected components of \( \partial M \) which is a closed minimal hypersurface having a contracting neighborhood in one side in \( M \). Denote by \( M_0^0 := M \) and \( \partial M_0^0 = \partial M \setminus T_0^0 \). Then \((M_0^0, \partial M_0^0, T_0^0, g)\) is a compact manifold with boundary and portion satisfying (A), (C) and (D).

Firstly, cut \( M_0^0 \) along a one-sided properly embedded free boundary minimal hypersurface \( \Gamma_0 \) of \((M_0^0, \partial M_0^0, T_0^0, g)\) in \( M_0^0 \setminus T_0^0 \) having a contracting neighborhood. Denote by \( M_1^0 \) the closure of \( M_0^0 \setminus \Gamma_0 \) and define
\[ \partial M_1^0 := M_1^0 \cap \partial M_0^0 \quad \text{and} \quad T_1^0 := T_0^0 \cup \Gamma_0, \]
where $\bar{\Gamma}_0$ is the double cover of $\Gamma_0$ in $M_0^2$. Then repeat this procedure by cutting $M_1^0$ along a one-sided free boundary minimal hypersurface $\Gamma_1 \subset M_1^0 \setminus \bar{\Gamma}_0$. Thus we construct a finite sequence $(M_0^0, \partial M_0^0, T_0^0, g), (M_1^0, \partial M_1^0, T_1^0, g), \ldots, (M_j^0, \partial M_j^0, T_j^0, g)$ by successive cuts. Then after finitely many times (denoted by $J$), $M_j^0 \setminus T_j^0$ does not contain any one-sided properly embedded free boundary minimal hypersurfaces having a contracting neighborhood. Denote by $M_j$ clearly, $(M_j^0, \partial M_j^0, T_j^0, g)$ satisfies $[A], [C], [D]$ and $[E]$.

Secondly, we cut $M_0^1$ along a two-sided, properly embedded, free boundary minimal hypersurface $\Gamma_0'$ in $(M_0^1, \partial M_0^1, T_0^1, g)$ that has a contracting neighborhood. Denote by $M_1^1$ the closure of one of the connected components of $M_1^0 \setminus \Gamma_0'$ and define

$$\partial M_1^1 := M_1^1 \cap \partial M_0^1 \text{ and } T_1^1 := M_1^1 \cap (T_0^1 \cup \Gamma_{0,1}^0 \cup \Gamma_{0,2}^0),$$

where $\Gamma_{0,1}$ and $\Gamma_{0,2}$ are the two free boundary minimal hypersurfaces that are both isometric to $\Gamma_0$. Then after finitely many times, we obtain a compact manifold with boundary and portion (denoted by $(M_0^2, \partial M_0^2, T_0^2, g)$) that every properly embedded free boundary minimal hypersurface in $M_0^2 \setminus T_0^2$ has an expanding neighborhood. Moreover, we have that:

**Claim 1.** Every two-sided properly embedded free boundary minimal hypersurface of $(M_0^2, \partial M_0^2, T_0^2, g)$ in $M_0^2 \setminus T_0^2$ separates $M_0^2$.

**Proof of Claim 1**. If not, there is a two-sided free boundary hypersurface $\Sigma$ in $(M_0^2, \partial M_0^2, T_0^2, g)$ does not separate $M_0^2$. Then $\Sigma$ represents a nontrivial relative homology class in $(M_0^2, \partial M_0^2)$. Then we can obtain an area minimizer, which contains a component $S$ in $M_0^2 \setminus T_0^2$. In particular, $S$ is properly embedded and has a contracting neighborhood, which contradicts $[E]$ and the fact that every properly embedded free boundary minimal hypersurface in $(M_0^2, \partial M_0^2, T_0^2, g)$ has an expanding neighborhood. □

Similarly, we have the following:

**Claim 2.** At most one connected component of $\partial M_0^2$ is a closed minimal hypersurface, and if it happens, it has an expanding neighborhood in one side in $M_0^2$.

**Proof of Claim 2**. We argue by contradiction. Assume there are two disjoint connected components $\Gamma_0'$ and $\Gamma_0''$ in $\partial M_0^2$ are closed minimal hypersurfaces. Then by the definition of $T_0^2$, both $\Gamma_0'$ and $\Gamma_0''$ have expanding neighborhoods in one side in $M_0^2$. Then $\Gamma_0'$ represents non-trivial relative homology class in $(M_0^2, \partial M_0^2 \setminus (\Gamma_0' \cup \Gamma_0''))$. By minimizing the area of this class, we obtain a properly embedded free boundary minimal hypersurface having a contracting neighborhood, which leads to a contradiction. □

Claim 1 gives that each two-sided free boundary minimal hypersurface generically separates $M_0^2$ (see Subsection 2.2). Claim 2 implies that $(M_0^2, \partial M_0^2, T_0^2, g)$ satisfies $[E]$. Therefore, $(M_0^2, \partial M_0^2, T_0^2, g)$ satisfies $[A], [E]$.

Thirdly, we cut $(M_0^2, \partial M_0^2, T_0^2, g)$ along a two-sided, half-properly embedded free boundary minimal hypersurface $\Gamma'' \subset M_0^2 \setminus T_0^2$ which has a proper and contracting neighborhood in one side. By Claim 1, $\Gamma''$ generically separates $M_0^2$. Denote by $M_2^1$ the closure of the generic component containing the proper neighborhood in one side. Define

$$\partial M_1^2 := (M_1^2 \cap \partial M_0^2) \setminus \Gamma'' \text{ and } T_1^2 = (T_0^2 \cap M_0^2) \cup \Gamma''.$$
Figure II. Cutting half-properly embedded hypersurfaces.

Then \((M^2, \partial M^2, T_1^2, g)\) is a compact manifold with boundary and portion (see Figure II). By successive cuts in finitely many times, we obtain a compact manifold with boundary and portion (denoted by \((N, \partial N, T, g)\)) so that each two-sided, half-properly embedded, free boundary minimal hypersurface has a proper and expanding neighborhood in one side. By Lemma 2.11 every two almost properly embedded free boundary minimal hypersurfaces of \((N, \partial N, T, g)\) in \(N \setminus T\) intersect with each other. Without loss of generality, let \(T_1\) be the connected component of \(T\) so that

\[
\text{Area}(T_1) = \max\{\text{Area}(T') : T' \text{ is a connected component of } T\}.
\]

Then by Lemma 2.12 each free boundary minimal hypersurface \(\Sigma\) in \((N, \partial N, T, g)\) satisfies that

- if \(\Sigma\) is two-sided, \(\text{Area}(\Sigma) > \text{Area}(T_1)\);
- if \(\Sigma\) is one-sided, \(2\text{Area}(\Sigma) > \text{Area}(T_1)\).

Thus we get the desired compact manifold with boundary and portion.

We now proceed the proof of Theorem 4.1. Let \(\mathcal{C}(N)\) be the construction in Subsection 3.1. Theorem 3.10 gives that \(\omega_p(\mathcal{C}(N); h)\) is realized by free boundary minimal hypersurfaces in \(N \setminus T\). Moreover, since every two free boundary minimal hypersurfaces of \((N, \partial N, T, g)\) in \(N \setminus T\) intersect each other, there exist integers \(\{m_p\}\) and free boundary minimal hypersurfaces \(\{\Sigma_p\}\) so that

\[
\omega_p(\mathcal{C}(N)) = m_p \cdot \text{Area}(\Sigma_p).
\]

By Lemma 3.7 the width of \(\mathcal{C}(N)\) satisfies

\[
\omega_{p+1}(\mathcal{C}(N)) - \omega_p(\mathcal{C}(N)) \geq \text{Area}(T_1);
\]

\[
p \cdot \text{Area}(T_1) \leq \omega_p(\mathcal{C}(N)) \leq p \cdot \text{Area}(T_1) + C p^{\frac{2}{p+1}}.
\]

Together with (4.1) we get a contradiction to [26, Lemma 13].

Appendix A. A strong maximum principle

In [31, Theorem 4], White gave a strong maximum principle for varifolds in closed Riemannian manifolds. Using the same spirit, Li-Zhou proved a maximum principle in compact manifolds with boundary, which played an important role in their regularity theorem for min-max minimal
hence, we give a strong maximum principle, which is used in Theorem 3.10.

Lemma A.1 (cf. [16] Theorem 1.4; [30] Theorem 4]). Let $(N, \partial N, T, g)$ be a compact manifold with boundary and portion so that $T$ is a free boundary minimal hypersurface. Let $V$ be a $g$-stationary varifold with free boundary in $\partial N$, i.e. for any $X \in \mathfrak{X}(N, \partial N)$,

$$\delta V(X) := \int \text{div} X dV = 0.$$ 

(1) If the support of $V$ (denoted by $S$) contains any point of a connected component of $T$, then $S$ contains the whole connected component;

(2) If $V$ is a $g$-stationary integral varifold with free boundary, then $V$ can be written as $W + W'$, where the support of $W$ is the union of several connected components of $T$ and the support of $W'$ is disjoint from $T$.

Proof. Without loss of generality, we assume that $T$ is connected and non-degenerate. We first prove (1) by contradiction. Assume that $S$ does not contain $T$. By [25] Theorem], $S$ does not intersect the interior of $T$. We now prove that $S \cap \partial T = \emptyset$.

In this lemma, we always embed $N$ isometrically into a smooth, compact $(n+1)$-Riemannian manifold with boundary $(M, \partial M, g)$. We also fix a diffeomorphism $\Phi : T \times (-\delta, \delta) \rightarrow M$ which is associated with an extension of $n$ in $\mathfrak{X}(N, \partial N)$. Here $n$ is the unit outward normal vector field of $T$ in $N$.

We argue by contradiction. Assume that $p \in S \cap \partial T$. Firstly, we use [25] Theorem, Step A) to construct a free boundary hypersurface outside $S$ near $p$ so that it has mean curvature vector field pointing towards $S$. To do this, we take $U \subset T$ be the neighborhood of $p$ from Proposition A.2 and $w|_{\Gamma_2} = \theta \eta$, where $\eta$ is a non-trivial and non-positive function supported in the interior of $\Gamma_2$ and $\theta > 0$ is a constant. Note that $\Gamma_2 = \text{Closure}(\partial U \cap \text{Int} T)$. Note that $S$ does not intersect the interior of $T$. Then we can take $\theta > 0$ sufficiently small so that if $\Phi(x, y) \in S$, then $y \leq \theta \eta(x)$. Fix this value $\theta$.

For simplicity, denote by $v_{s,t}$ the constructed graph function $v_t$ for $h = s$ and $w|_{\Gamma_2} = \theta \eta$ in Proposition A.2. Then by the maximum principle, $v_{0,0}(p) < 0$. Hence for $s > 0$ small enough, we always have $v_{s,0}(p) < 0$. Fix such $s$. Let $t_0$ be the largest $t$ so that $v_{s,t}$ intersects $S$. It follows that $t_0 > 0$, which implies that $S$ does not intersect $\Phi(\Gamma_2, \theta \eta + t_0)$.

We now proceed our argument. Note that $v_{s,t_0}$ is a graph function of a free boundary hypersurface with mean curvature vector pointing towards $T$. Then by the strong maximum principle [30], $S$ can not touch the interior of $\Phi(U, v_{s,t_0})$. Using the free boundary version maximum principle [16], $S$ can not touch $\Phi(\partial T \cap U, v_{s,t_0})$. Then this contradicts the construction of $v_{s,t_0}$.

Now (2) follows from (1) and a standard argument in [30] Theorem 4]. Indeed, set

$$d := \inf \{ \{ \Theta(x, V) : x \in \text{Int} T \} \cup \{ 2\Theta(x, V) : x \in \partial T \} \}.$$

Then $V - d[T]$ is still a $g$-stationary integral varifold with free boundary, where $\{ T \}$ is the the varifold associated to $T$. Then $V - d[T]$ does not contain $T$. Hence it does not intersect $T$. The proof is finished.

Proposition A.2. Let $(M^{n+1}, \partial M, g)$ be a compact Riemannian manifold with boundary, and let $(\Sigma, \partial \Sigma) \subset (M, \partial M)$ be an embedded, free boundary minimal hypersurface. Given a point $p \in \partial \Sigma$, there exist $\epsilon > 0$ and a neighborhood $U \subset M$ of $p$ such that if $h : U \rightarrow \mathbb{R}$ is a smooth
function with \( \|h\|_{C^{2,\alpha}} < \epsilon \) and

\[ w : \Sigma \cap U \to \mathbb{R} \] satisfies \( \|w\|_{C^{2,\alpha}} < \epsilon, \)

then for any \( t \in (-\epsilon, \epsilon), \) there exists a \( C^{2,\alpha} \)-function \( v_t : U \cap \Sigma \to \mathbb{R}, \) whose graph \( G_t \) meets \( \partial M \) orthogonally along \( U \cap \partial \Sigma \) and satisfies:

\[ H_{G_t} = h|_{G_t}, \]

(where \( H_{G_t} \) is evaluated with respect to the upward pointing normal of \( G_t \)), and

\[ v_t(x) = w(x) + t, \] if \( x \in \partial(U \cap \Sigma) \cap \text{Int} \).

Furthermore, \( v_t \) depends on \( t, h, w \) in \( C^1 \) and the graphs \( \{ G_t : t \in [-\epsilon, \epsilon] \} \) forms a foliation.

**Proof.** The proof follows from \([29, \text{Appendix}]\) together with the free boundary version \([3, \text{Section 3}]\). The only modification is that we need to use the following map to replace \( \Phi \) in \([3, \text{Section 3}]\):

\[ \Psi : \mathbb{R} \times X \times Y \times Y \to Z_1 \times Z_2 \times Z_3. \]

The map \( \Psi \) is defined by

\[ \Psi(t, g, h, w, u) = (H_g(t + w + u) - h, g(N_g(t + w + u), \nu_g(t + w + u)), u|_{\Gamma_2}); \]

here all the notions are the same as \([3, \text{Section 3}]\). We remark that \( \Gamma_2 = \text{Closure}(\partial(U \cap \Sigma) \cap \text{Int} \).

**Appendix B. Computation in the Proof of Theorem 3.10**

In this appendix, we collect the computation in Theorem 3.10.

**B.1. Proof of (3.15).** Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a non-negative function. Then it can also be seen as a function on \( M \) by

\[ \varphi(\mathcal{F}(x, t)) := \varphi(s(\mathcal{F}(x, t))). \]

Let \( H^e \) (resp. \( A^e \)) denote the mean curvature (resp. second fundamental form) at \( y \) of \( \mathcal{F}(T \times \{t\}). \)

Let \( n := \nabla^e s/|\nabla^e s|_{h_e} = \nabla t/|\nabla t|_{h_e}. \) Then we have

\[ \frac{\partial}{\partial s} = f n, \]

where \( \frac{\partial}{\partial s} = \mathcal{F}_*(\frac{\partial}{\partial s}). \) We can compute the divergence as follows:

**(B.1)**

\[
\begin{align*}
\text{div}_M^e \left( \frac{\partial}{\partial s} \right) &= \text{div}_{h_e}^e (\varphi \frac{\partial}{\partial s}) - h_e (\nabla^e_n (\varphi f n), \bar{n}) \\
&= \varphi'(s)|h_e(e_n, n)|^2 + \varphi h_e(\nabla^e f, n) + \varphi H^e f - \varphi h_e(\nabla^e f, \bar{n})h_e(n, \bar{n}) - \varphi f h_e(\nabla^e n, \bar{n}) \\
&= \varphi'(s) \cdot |h_e(e_n, n)|^2 + \varphi h_e(\nabla^e f, n) + \varphi H^e f - \varphi h_e(\nabla^e f, \bar{n})h_e(n, \bar{n}) - \varphi f h_e(\nabla^e n, \bar{n})h_e(n, \bar{n}) \\
&\quad - \varphi f h_e(\nabla^e n, e_n^e) \cdot |h_e(n, e_n^e)|^2 \\
&= [\varphi'(s) - \varphi f A^e(e_n^e, e_n^e)] \cdot |h_e(e_n, n)|^2 + \varphi H^e f - \varphi h_e(\nabla^e f + f \nabla^e n, e_n^e)h_e(n, e_n^e)h_e(n, \bar{n}) + \\
&\quad + \varphi h_e(\nabla^e f, n) \cdot |h_e(e_n, n)|^2.
\end{align*}
\]

Note that by \( \frac{\partial}{\partial s} = f n, \)

**(B.2)**

\[ h_e(\nabla^e_n n, e_n^e) = -h_e(n, \nabla^e_n \frac{\partial}{\partial s}) = -h_e(\nabla^e f, e_n^e), \]
and by \( \frac{\partial}{\partial t} = (f \vartheta_\epsilon)^{-1} n \),
\[
h_\epsilon(\nabla^\epsilon f, n) = h_\epsilon(\nabla^\epsilon f, (f \vartheta_\epsilon)^{-1} \frac{\partial}{\partial t}) = (f \vartheta_\epsilon)^{-1} \frac{\partial f}{\partial t}.
\]

Hence we conclude that (B.1) becomes
\[
(\text{B.3}) \quad \text{div}_S(\varphi \frac{\partial}{\partial s}) = \left[ \varphi'(s) - \varphi f A^\epsilon(e_n^*, e_n^*) + \varphi(f \vartheta_\epsilon)^{-1} \frac{\partial f}{\partial t} \right] \cdot |h_\epsilon(e_n, n)|^2 + \varphi H^\epsilon f.
\]

If we define the vector field (\( \beta \) is to be specified later)
\[
Y^\epsilon := (1 - \beta(s)) \exp(-Cs) \frac{\partial}{\partial s},
\]
then from (B.4), we have
\[
(\text{B.4}) \quad \text{div}_S Y^\epsilon \\
\leq \left( \frac{\partial}{\partial s} \left[ (1 - \beta(s)) \exp(-Cs) \right] + (1 - \beta(s)) \exp(-Cs) \left[ h_\epsilon(\nabla^\epsilon f, n) - f A^\epsilon(e_n^*, e_n^*) \right] \right) \cdot |h_\epsilon(e_n, n)|^2 + \\
+ (1 - \beta(s)) \exp(-Cs) \cdot |H^\epsilon f| \\
\leq -\beta'(s) \exp(-Cs) |h_\epsilon(e_n, n)|^2 + (|H^\epsilon f| + |h_\epsilon(\nabla^\epsilon f, n)|).
\]

For the second inequality, we used that
\[
\left| (f \vartheta_\epsilon)^{-1} \frac{\partial f}{\partial t} - f A^\epsilon(e_n^*, e_n^*) \right| \leq C.
\]

Since the varifold \( V_\epsilon \) is \( h_\epsilon \)-stationary with free boundary, for all \( \epsilon > 0 \) small:
\[
\delta V_\epsilon(Y^\epsilon) = \int \text{div} Y^\epsilon dV_\epsilon = 0.
\]

Now we consider \( \beta(s) : \mathbb{R} \rightarrow [0, 1] \) to be a non-decreasing function such that
- \( \beta(s) \equiv 0 \) (resp. 1) when \( s \leq -\bar{R} \) (resp. \( s \geq 2\epsilon \));
- on \([-\bar{R}, \epsilon]\), \( \frac{\partial \beta}{\partial s} \geq 1/(2\bar{R}) \).

Here \( \bar{R} \) is large enough so that spt \( V_\epsilon \) does not intersect \( \{ s < -\bar{R} \} \); see (3.14).

By the computation in (B.3), for any \( b > 0 \), we obtain the main result in this part:
\[
\int_{\mathcal{F}(T \times [0, 2\epsilon]) \times \mathcal{G}(n+1, n)} \chi \{(h_\epsilon(e_n, n)) > b\} dV_\epsilon(x, S) \\
\leq 2\bar{R} \exp(C\bar{R}) b^{-2} \int_{F(T \times [0, 3\epsilon]) \times \mathcal{G}(n+1, n)} |H^\epsilon| \cdot f \ dV_\epsilon(x, S) \\
\rightarrow 0, \quad \text{as} \quad \epsilon \rightarrow 0.
\]
B.2. Proof of (3.20).

\[
\lim_{k \to \infty} \left| \int \text{div}_S^k X_{\perp}^k dV'_{k}(x, S) - \int \text{div}_0^k X_{\perp}^0 dV'_{\infty} \right|
\]

\[
= \lim_{b \to 0} \lim_{k \to \infty} \left| \int \chi_{\{|h_{e_k}(e_n, n)| \leq b\}} \text{div}_S^k X_{\perp}^k dV'_{k}(x, S) - \int \text{div}_0^k X_{\perp}^0 dV'_{\infty} \right|
\]

\[
\leq \lim_{b \to 0} \lim_{k \to \infty} \left| \int \chi_{\{|h_{e_k}(e_n, n)| \leq b\}} \text{div}_0^k X_{\perp}^k dV'_{k}(x, S) - \int \text{div}_0^k X_{\perp}^0 dV'_{\infty} \right|
\]

\[
+ \lim_{b \to 0} \lim_{k \to \infty} \int \chi_{\{|h_{e_k}(e_n, n)| \leq b\}} 2|\nabla^k e_{\perp} X_{\perp}^k|_{h_{e_k}} \cdot |e_n - e^*_n| dV'_{k}(x, S)
\]

\[
= \lim_{k \to \infty} \int \text{div}_0^k X_{\perp}^k dV'_{k}(x, S) - \int \text{div}_0^k X_{\perp}^0 dV'_{\infty} = 0.
\]

Here the inequality is from (3.19).

B.3. $|\nabla^\epsilon Z^\epsilon|_{h_{e}}$ is uniformly bounded. Recall that

\[
Z^\epsilon := \varphi \nabla^\epsilon s = \varphi f^{-1} n.
\]

Then for $1 \leq i, j \leq n - 1$,

\[
|h_{\epsilon}(\nabla^\epsilon e_i, e_j)| \leq |\varphi f^{-1}| \cdot |A^\epsilon(e_i, e_j)| \leq |X^0||_g,
\]

\[
|h_{\epsilon}(\nabla^\epsilon Z, n)| \leq |(\nabla^\epsilon(\varphi f^{-1}))|_{h_{\epsilon}} = |(\nabla^0(\varphi f^{-1}))|_{h_{\epsilon}}.
\]

\[
h_{\epsilon}(\nabla^\epsilon Z, e_i) = h_{\epsilon}(\nabla^\epsilon(\varphi f^{-1}), e_i) = h_{\epsilon}(\nabla^\epsilon(\varphi f^{-1}), \varphi^{-1} f^{-1} \frac{\partial}{\partial t}) = \varphi^{-1} f^{-1} \frac{\partial}{\partial t} (\varphi f^{-1}),
\]

\[
|h_{\epsilon}(\nabla^\epsilon Z, e_j)| \leq |h_{\epsilon}(f^{-1} \nabla^\epsilon(\varphi f^{-1}), e_j)| = |h_{\epsilon}(f^{-1} Z, \nabla^\epsilon e_j)| \leq |\varphi f^{-2}(\nabla f)|_{g},
\]

B.4. Proof of (3.21). Let $H^\epsilon$ be the mean curvature as above. Recall that

\[
Z^\epsilon := \varphi \nabla^\epsilon s.
\]

Then the divergence is

(B.5) \[
\text{div}_S^\epsilon Z^\epsilon = \text{div}_S^\epsilon Z^\epsilon + h_{\epsilon}(\nabla^\epsilon e_n, e_n) - h_{\epsilon}(\nabla^\epsilon e_n, N^\epsilon, e^*_n)
\]

\[
= h_{\epsilon}(Z^\epsilon, n) \cdot H^\epsilon + \Upsilon^\epsilon(\epsilon, x, S, X),
\]

where

\[
|\Upsilon^\epsilon(\epsilon, x, S, X)| = |h_{\epsilon}(\nabla^\epsilon e_n, e_n) - h_{\epsilon}(\nabla^\epsilon e_n, Z^\epsilon, e^*_n)| \leq 2|\nabla^\epsilon Z^\epsilon|_{h_{\epsilon}} \cdot |e_n - e^*_n|_{h_{\epsilon}}.
\]

Recall that $h_{e_k} = g$ on $B_k$. Then we have

\[
\lim_{k \to \infty} \left| \int \text{div}_S^k X_{\parallel}^k dV''_{k}(x, S) \right| = \lim_{k \to \infty} \left| \int \text{div}_S^k Z_{\perp}^k dV'_{k}(x, S) \right| = \lim_{k \to \infty} \left| \int \text{div}_S^k Z_{\perp}^k dV'_{k}(x, S) \right|
\]

\[
\leq \lim_{b \to 0} \lim_{k \to \infty} \int \chi_{\{|h_{e_k}(e_n, n)| \leq b\}}|h_{e_k}(Z^\epsilon, n) \cdot H^\epsilon| + 2|\nabla^k e_{\perp} Z^\epsilon|_{h_{e_k}} \cdot |e_n - e^*_n|_{h_{e_k}} dV'_{k}(x, S)
\]

\[
= 0.
\]

Here the first equality comes from the fact that $X^\epsilon_{\parallel} = Z^\epsilon_{\parallel}$ as in $B_k$; the second equality follows from that $V_k$ is stationary with free boundary; the last equality comes from Lemma 3.1.
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