Density Wave Superradiance of Photonic Fluid in Frustrated Triangle Lattice Cavity Arrays

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The spontaneously broken of translational symmetry is usually due to the competition between local interactions and long-range interactions. However, in this paper, we show how a crystalline order can be generated by the competition between local interaction and long-range correlation by frustration. Here we propose a positive hopping Bose Hubbard model on triangle lattices with a pair creation term which comes from frustrated linked cavity arrays with degenerate quantum gases in them. We find by increasing the strength of pair creation term against local interaction strength, two kinds of density wave ordered superradiant photonic fluid phase can be realized and a first order transition between two different density wave ordered states is found. This proposal shows us a new way to produce coherent “solid” phase without the help of long range interactions.

Introduction. Recently, superradiance in a cavity has been realized experimentally \([1-3]\) due to the technique advances developed in strong atom-light coupling in cavity systems.\(^3\)\(^4\). Superradiance is a coherent state of strongly interacting atoms and light, which is originally proposed by Dicke in 1954.\(^7\) The superradiance transition in a single mode cavity spontaneously break a \(Z_2\) symmetry, where the phase of cavity field can only be 0 or \(\pi\) after condensation. The phase lock between cavity field and pumping field, together with the even odd lattice site switch in density pattern of atoms was observed in a following experiment.\(^3\) One can find the 0 and \(\pi\) phase of the cavity field is quite alike a ferromagnetic state with spin up and down. The only difference between a condensed cavity field and the spin is the magnitude of the cavity field is obtained by condensation, so it has self adjusted ability while the magnitude of spin is fixed without fluctuation.

When classical spins are put on three sites with anti-ferromagnetic coupling, strong frustration is generated and the energy of different classical spin configurations are degenerate. If one extend the three sites into a triangle lattice, there are infinite many degenerate classical spin configurations and the true ground state is chosen by quantum fluctuations.\(^8\)\(^-\)\(^\text{12}\) Similarly, we could add positive hopping between cavities to generate frustration between three superradiant condensates. However, in a superradiant condensate, the phase of the condensate is not exactly locked at 0 and \(\pi\) and the amplitude of condensate is not fixed as well. Comparing with spin model, the phase fluctuations and the magnitude fluctuations of the condensate loose the constraint of frustration a little bit. These photon density fluctuations and phase fluctuations are suppressed by atom-light coupling strength. For small atom-light coupling strength, the phase of condensate could be any value from 0 to 2\(\pi\), which is more similar to \(U(1)\) symmetry case. For extremely large atom-light coupling strength, the phase of the condensate is focused on 0 or \(\pi\), which is similar to Ising spin case \((Z_2)\). For this reason, by tuning the interaction strength of atom-light coupling, we have a symmetry crossover process from \(U(1)\) to \(Z_2\).

In this paper, we find the symmetry crossover can give rise to a density wave superradiance as the ground state of photons in frustrated triangle cavity arrays. To illustrate this point, an effective positive-hopping Bose-Hubbard model with pair generation term is introduced as our starting point.

\[
\hat{H} = J \sum_{\langle ij \rangle} \hat{a}_i^\dagger \hat{a}_j - \mu \sum_i \hat{n}_i + \frac{U}{2} \sum_i \hat{n}_i(\hat{n}_i - 1) - \chi \sum_i (\hat{a}_i + \hat{a}_i^\dagger)^2, \tag{1}
\]

where the system is put on a two-dimensional triangle lattice. \(J > 0\) is the hopping strength, \(\mu\) is the chemical potential, \(U\) is the onsite interaction energy and \(\chi\) is an induced interaction strength. \(\hat{a}_i\) is boson annihilation operator on cavity site \(i\). \(\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i\). Without the last term, this is a standard frustrated Bose Hubbard model, whose ground state is 120 degree ordered superfluid state.\(^\text{13}\)\(^-\)\(^\text{14}\). In \(U \to \infty\) limit, the frustrated Bose Hubbard model can be mapped into XY spin model, whose ground state is 120 degree magnetic order.\(^\text{13}\) However, when \(\chi\) term is added, photon condensate favors 0 or \(\pi\) phase, \(U(1)\) symmetry is explicitly broken to \(Z_2\), the frustration becomes Ising-like. But for a bosonic system, the condensate on every site can be adjusted self-consistently. This is quite different from spin systems. Then we find photon condensate could develop a density wave pattern to avoid strong frustration. Hence, a density wave superradiance is generated by competition between local interaction and correlation induced by frustration during a symmetry crossover from \(U(1)\) to \(Z_2\).

In the following, we will first discuss how to setup our system. Then we construct our solution from a three site...
show a typical phase diagram for varying $\chi/U$. In (a), we give a experimental set up to realize a

**FIG. 1:** In (a), we give a experimental set up to realize a
debug study of the frustrated lattice system for ground state to
state. The vertical axis gives density wave order

by strong atom-light coupling has been realized in ex-

where $\hat{\Theta} = \int d^{12} \tilde{n}_{\text{at}}(r)\eta(r)$ and $B = \int d^{12} \tilde{n}_{\text{at}}(r) U_0(r)$ are
two density orders of the atomic gases. $n_{\text{at}}(r)$ is atomic
density operator, $\eta(r)$ and $U_0(r)$ are two modes functions.
Terms like $-\chi(a^d + a)^2$ and interaction terms like $(U/2)\tilde{n}(-\tilde{n} - 1)$ can be generated by $(\hat{\Theta}\Theta)(a + a^d)^2$ and $(\hat{B}\tilde{B})a^d a^d a\tilde{a}\tilde{a}$, where $\langle \cdot \rangle$ are ensemble average. The strength of the $\chi$ and $U$ can then be tuned by the strength of $\eta(r)$ and $U_0(r)$ independently. Other terms are neglected for simplicity. By this setup, we can access the onsite terms in Eq. (1).

On the other hand, we design a scheme for cavity photons to tunnel between cavities with a phase shift. In Fig. (a), we set a half transmissive half reflective mirror in one cavity so that cavity photon can leave. There are other two mirrors which can produce a $\pi$ phase shift during the photon propagating by a half-wave loss mechanism. Finally, the photon enters another cavity by another half transmissive half reflective mirror. Through this setup we can have a positive hopping between two cavities ($J$ term), thus it is possible to realize Eq. (1).

Finally, to simplify our problem, we neglect the cavity decay rate $\kappa$ through out this paper.

**A three sites problem.** Here we try to construct our solution from few sites to many. Hence we start from a three site problem. Neglecting $J$, we find the cavity field $\hat{a}$ will condense into a superradiant state when $\chi$ is large enough to overcome the red detune of the cavity. We can find the condensate of $\langle \hat{a} \rangle$ has two equivalent phases to choose, 0 and $\pi$. Therefore when a anti-ferromagnetic coupling $J > 0$ is turned on between adjacent cavities, $\alpha_i = \langle \hat{a}_i \rangle$ favors anti-parallel configuration. Meanwhile, unlike the spin model which is completely a non-linear sigma model where the amplitude mode is infinite heavy, the magnitude of onsite condensation is variable in the present situation.

Here we apply a mean field theory with independent local order parameters $\alpha_i = \langle \hat{a}_i \rangle$ to Eq. (1) on three sites. Assuming the ground state is $|\Omega\rangle = |\alpha_1\rangle \otimes |\alpha_2\rangle \otimes |\alpha_3\rangle$, where $|\alpha_i\rangle$ is coherent state on site $i$. The self-consistent equations can be obtained by minimizing the ground state energy $E(\alpha_1, \alpha_2, \alpha_3) = \langle \Omega | \hat{H} | \Omega \rangle$ as

$$-\left(\mu + \chi + \frac{U}{2}\right)\alpha_2 + U|\alpha_3|^2\alpha_2 + J(\alpha_2 + \alpha_3) = 2\chi \text{Re}(\alpha_3).$$

where $(a,b,c) = (1,2,3), (2,3,1)$ and $(3,1,2), \text{Re}(\#)(\text{Im}(\#))$ takes the real part (imaginary part) of 

This solution requires $|\text{Re}(\alpha_2)| \leq |\alpha_2|$. At the point $|\text{Re}(\alpha_2)| = |\alpha_2|$, $\alpha_2$ becomes real. In this solution, we stress two points. 1) As long as $\chi \neq 0$, there is a density imbalance between different cavities. The condensate arranged a density order to reconcile with frustration and the density order is proportional to $\chi$. 2) The relative phase between different sites is not 180 degree, more like 120 degree order. For this reason we call this state a U(1) density wave superradiant state (U(1)-DW-SR).

In another possible solution, all the imaginary parts are order parameters are zero, that is $\text{Im}(\alpha_{a,b,c}) = 0$. The phases of every two neighboring sites are either parallel or anti-parallel. As the equations for $\alpha_2$ and $\alpha_3$ are symmetric, we assume $\alpha_2 = \alpha_3$. Then we have

$$(\alpha_1^2 - \mu - 1/2 - 3\chi)\alpha_1 = -2J\alpha_2$$

$$(\alpha_2^2 - \mu - 1/2 - 3\chi + J)\alpha_2 = -J\alpha_1$$

These two equations can be solved analytically and there are four solutions in total. These four solutions’ energy are shown in Fig. (b), and we find the lowest energy state takes one “spin up” large condensate, two small “spin
wave loss mechanism. Finally, the photon enters another

3E_G/N_\Lambda = 3J \sum_{a \neq b} \alpha_a \alpha_b - \left( \mu + \frac{U}{2} \right) \sum_a |\alpha_a|^2 + \frac{U}{2} \sum_a |\alpha_a|^4 - \chi \sum_a (\alpha_a + \alpha_a^*)^2. \quad (7)

FIG. 3: When we increase the site number to be 7, the possible solutions are given in (a) and (b), where the size of the spot represents the local density and the red and blue color means two nearly opposite phases. (b) is lowest energy configuration, (a) is a possible saddle point solution. Combining these two solutions, a lattice solution can be constructed by adding site one by one. In (c), we show the ground state configuration suggested by the few site solution in triangle lattices. A super unit cell can be found, which is circled by dashed lines in (c).
To minimize the ground state energy, we have

\[-\left(\mu + \chi + \frac{U}{2}\right) \alpha_a + U|\alpha_a|^2 \alpha_a + 3J(\alpha_b + \alpha_c) = 2\chi \text{Re}(\alpha_a),\]

where \((a, b, c) = (1, 2, 3), (2, 3, 1)\) and \((3, 1, 2)\). One can find this equation is the same as Eq. 3 when \(3J\) is replaced by \(J\). Therefore previous solutions for three site problem can be translated into phase diagram of the system. As we learn from the three site problem, there are two kind of solutions. The \(U(1)\) DW ordered state represents a density wave ordered superfluid phase with phase not exactly 180° \((U(1)-\text{DW-SR})\), and the \(\mathbb{Z}_2\) DW ordered state represents a density wave ordered superfluid phase with phase difference between large condensate site and small condensate site being 180°. According to Fig. 4(c), the mean field transition between these \(U(1)\)-DW-SR and \(\mathbb{Z}_2\)-DW-SR is a first order transition with a jump in both the superfluid order and the density order. We here present the phase diagram based on Eq. (2) and comparison of ground state energy. A density order is discovered when the original Hamiltonian is subject to a translational invariance and all interaction is local. A translational invariance is broken by a internal symmetry breaking. Interestingly, this density order is non-monotonously dependent on \(\chi/U\). When \(\chi/U\) is small, arbitrary small \(\chi\) brings the system a density order. When \(\chi/U\) is large, the symmetry of the system is fixed to \(\mathbb{Z}_2\) and the local density fluctuations are suppressed. As a result, density order is weakened for large \(\chi/U\).

There are two factors that are beyond a mean field theory. Here we analyze the impact of these two factors. The first factor is the spatial phase fluctuations of \(\alpha_{ia}\). We find there can be a uniform phase fluctuation in a super unit cell, \(\alpha_{ia} = \alpha_a e^{i\phi}\). One can find for a density operator, the local phase is always cancelled. Hence the density wave order is irrelevant to this phase fluctuation.

The second factor is quantum fluctuations between nearly degenerate configurations. As we learn in a three sites problem, when \(\chi/U\) is large, the phase fluctuations of local condensate is greatly suppressed, and the density wave order is also suppressed. For a large \(\chi/U\), the energy of different configurations in super unit cell is nearly degenerate. Hence it is more similar to frustrated Ising model. For a finite large \(\chi/U\), we start from a homogeneous configuration and ask how much a density order can lower the energy and how much a \(\mathbb{Z}_2\) spin liquid could lower the energy. Both two schemes lowers the energy from the highly degenerate starting point, therefore the true ground state should be the one lower more energy from this configuration. As the density order is vanishing in \(\chi/U \to \infty\) limit, so the energy gain by density order goes to zero in this limit. On the other hand, the contribution from quantum fluctuation is nonzero in the same limit. Therefore we expect a direct phase transition between these two states at finite \(\chi/U\). However, to verify the existence of \(\mathbb{Z}_2\) spin liquid and the phase transition requires methods which can properly count quantum fluctuations. This is beyond the scope of the present paper. We will leave this interesting phenomenon for the future study.

**Conclusion.** In this paper, we propose a positive hopping Bose-Hubbard model with a pair-generation term on a triangle lattice. Through a mean field study we extend our few site solutions to a lattice solution. We find the ground state of this model shows both off-diagonal long range order and crystalline diagonal long range order. There are two density wave coherent photonic fluid state phase transition to each other by a first order transition. The density wave order is non-monotonously dependent on the pair-generation term, which is small for both small and large pair-generation term. Thus we give an example to generate density by competition between local interactions and long range correlation induced by frustration where original model have only local interactions and is translational invariant.

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