Conformal Mechanics of Space Curves

Jemal Guven

Instituto de Ciencias Nucleares
Universidad Nacional Autónoma de México
Apdo. Postal 70-543, 04510 México, DF, MEXICO

Abstract

Any conformally invariant energy associated with a curve possess tension-free equilibrium states which are self-similar. When this energy is the three dimensional conformal arc-length, these states are the natural spatial generalizations of planar logarithmic spirals. In this paper, a geometric framework is developed to construct these states explicitly using the conservation laws associated with the symmetry. The tension along a curve, conserved in equilibrium, is first constructed. While the tension itself is not invariant, the statement of its conservation is. By projecting the conservation laws along the two orthogonal invariant normal directions, the Euler-Lagrange equations are reproduced in a manifestly conformally invariant form involving the conformal curvature and torsion. The conserved torque, scaling and special conformal currents are constructed explicitly by examining the behavior under rotations, rescaling and special conformal transformations respectively. Our specific interest will be the conditions under which the tension vanishes. When it does, the remaining conserved currents assume a strikingly simple form. In a companion paper, the self-similar spirals describing tension-free states will be constructed explicitly by integrating these currents.

Keywords: Conformal Invariance, Tension, Self-Similarity, Spirals

1 Introduction

Conformal invariance very often appears as a symmetry of physical processes, albeit only as an approximation or emergent within the process. This was already understood in the context of biological growth by D’Arcy Thompson more than a hundred years ago as is evident in his groundbreaking work *On Growth and Form* [1]. Its simplest manifestation is in a curve. This typically will be a one-dimensional subsystem of a larger system whose overall morphology is much more complicated. Thus the spiral arms in our galaxy, or the pattern of growth displayed by the chambers of a mollusk shell are self-similar, while the detailed morphology—of course—is not. For this to happen, the effective geometrical degrees of freedom describing the self-similar logarithmic spiral must somehow decouple from the rest of the system. The spiral itself need not be planar as D’Arcy Thompson himself appreciated.

D’Arcy Thompson worked without the benefit of a handle on the relevant mathematics to describe self-similar geometries and their connection to conformal symmetry. We would like to examine this question in a systematic way for curves in three-dimensional space, associating the simplest possible conformally invariant energy with the curve and exploring the equilibrium states consistent with it. We will see that the if the tension in these states vanishes, the length scale drops out with it and the resulting states will be self-similar. Because tension is not a conformal invariant, the conformal symmetry is broken. We make no attempt to understand the detailed interpolation between the fundamental physics describing the whole system and that of the subsystem exhibiting the symmetry, a difficult task at the best of times. As it turns out, however, a lot can be inferred from the symmetry and the pattern of its breaking.

The first task is to identify an appropriate energy consistent with conformal symmetry. This energy need not itself possess physical significance. Its role is to direct us to the simplest Euler-Lagrange exhibiting this symmetry. Scale invariance, alone, tends to be insufficient to pin-down the energy. Extending the symmetry to conformal...
symmetry narrows the possibilities, while expanding the patterns of its breaking. In three-dimensions, the conformal group is generated by similarities and inversions in spheres.

There are any number of local scale invariant energies one can associate with a curve, using the Frenet curvature and torsion, \( \kappa \) and \( \tau \). They will be of the form

\[
H_0 = \int ds \kappa \, G(\kappa/\tau),
\]

(1)

where \( G \) is any function, introduced first in a relativistic setting in \[2\]; but energies of this form do not admit self-similar geometries, unless trivially. Indeed it is relatively straightforward to show that their critical points are helices. Notably, none of these energies, except \( \int ds \tau \), is conformally invariant (modulo \( 2\pi \)); no others, apart perhaps from \( \int ds \kappa \), are natural in the context of curves. One needs to raise the number of derivatives to produce one. The intriguing point is that, up to an overall factor, there is a unique conformal invariant involving these variables and their first derivatives.

Conformal invariance crops up in the study of membranes. The symmetric bending or Willmore energy of a two-dimensional surface, quadratic in the extrinsic curvature \[3\], first identified by Sophie Germain \[4\], is a conformal invariant. The important role it plays in current membrane biophysics on the mesoscopic scales in which the membrane morphology comes into focus cannot be overstated. This was demonstrated spectacularly in a series of papers in the early nineties \[5\] \[6\] \[7\]. A well-known review, capturing the heady advances at that time, is provided in reference \[8\]. For a recent review, approaching the problem from a point of view not altogether different from the one used to approach the problem addressed here, see reference \[9\]. In contrast, the analog of the bending energy among curves, the Euler Elastic energy, quadratic in the Frenet curvature, is not even scale invariant never mind conformally invariant (the following references provide points of entry into a now vast literature \[10\] \[11\] \[12\] \[13\]). This dimensional distinction between surfaces and curves has significant physical consequences: indeed many of the peculiarities of fluid membranes can be traced back to the conformal invariance of the symmetric bending energy and the patterns of its breaking. There are, however, no local conformal invariants of a curve constructed using the undifferentiated curvature alone. The conformal arc-length, given by \[14\] \[15\],

\[
H = \int ds \left( \kappa^2 + \kappa^2 \tau^2 \right)^{1/4},
\]

(2)

is the simplest. Here \( s \) is arc-length and prime denotes a derivative with respect to \( s \). This is likely to be the simplest conformally invariant system there is. Yet there does not appear to have been any previous attempt made to examine the mechanics it implies.

It is significant that \( H \) possesses logarithmic spirals as planar critical points \[16\]. The planar limit has been examined from a mechanical point of view in \[17\].

Conformally equivalent curves are characterized by their conformal torsion and curvature, second and third order respectively in derivatives of their Frenet counterparts \[14\] \[15\].

\[
T = \frac{\kappa^3 \tau^2}{\nu^{3/2}} \left[ \frac{\tau}{\kappa} - \left( \frac{\kappa'}{\kappa^2 \tau} \right)' \right];
\]

(3a)

\[
\mathcal{K} = \frac{1}{8\nu^3} \left[ 4\nu (\nu'' - \kappa^2 \nu) - 5\nu^2 \right];
\]

(3b)

where \( \nu = \left( \kappa^2 + \kappa^2 \tau^2 \right)^{1/2} \). The former vanishes if the curve is planar or spherical. Logarithmic spirals and any curve connected to a logarithmic spiral by a conformal transformation are characterized by constant conformal curvature \( \mathcal{K} \) (with \( \tau \) set equal to zero in Eq. \[3\]).

Log spirals are obviously not the only planar critical points of the energy \[2\]. Their conformal descendants are also stationary, with the same energy; generically, these are double \( S \)-shaped spirals. It is thus clear that neither stationarity, nor constant conformal curvature selects the feature of logarithmic spirals that sets them apart from their conformal descendants: the conserved tension vanishes in logarithmic spirals whereas it does not in double spirals; the length scale associated with it is the distance between its poles \[2\]. In three (or indeed higher) dimensions, there are many more

\[2\] The possibility of non-trivial tension-free states in any physical system is unusual: one need only contrast this situation with that for Euler Elastica where the only tension-free open curves are straight lines.
Contrast the one parameter planar logarithmic spirals described in reference [17] with the zoo of self-similar space curves described in reference [18]. Unlike logarithmic spirals, they generally exhibit internal structure associated with non-vanishing torsion. Remarkably this structure is described by just two parameters.

The Euler-Lagrange equations, describing conformal geodesics, were first derived by Musso in 1994 [15]. He showed that these equations can be cast in terms of the conformal curvature and torsion:

\[
\begin{align*}
T^{**} - T^3 - 2K'T &= 0; \\
(K + 3T^2/2)' &= 0,
\end{align*}
\]

where the bullet represents a derivative with respect to conformal arc-length, \( \bullet = (1/\nu^{1/2})d/ds \) On a plane with \( T = 0 \), equilibrium states are described by curves with constant conformal curvature, described in reference [21].

More recently Magliaro et al. extended Musso’s analysis to higher dimensions [19]. The conformal symmetry is realized as the Lorentz group acting on a three-dimensional invariant subspace of a five dimensional Minkowski space; the question is approached by adapting the method of exterior differential systems (see, for example, reference [20]). This mathematics is very nice but—caveat lector—the demands placed on one’s preparation are significant.

Because conformally equivalent curves are completely characterized by their curvature and torsion, from a mathematical point of view Musso has solved the Euler-Lagrange equations. However, tension plays no role in Eqs. [4], nor is it clear how to isolate tension-free equilibrium states if the problem is approached this way, a necessity if the equilibrium is to be self-similar. An approach that places tension at its center is helpful. Such an approach, an extension of the method of auxiliary variables [22] developed by the author, is presented here. To accommodate the constraint on the tension, the conservation laws associated with conformal invariance are constructed. One can now focus on how the vanishing tension propagates through these conservation laws. Framed this way the geometric problem is also a mechanical one. The language of Euler elasticity is now appropriate, extended to accommodate the additional symmetry. This route builds on the approach adopted, in the context of planar curves, in reference [17].

The translational invariance of \( H \) permits the Euler-Lagrange equations to be recast as a conservation law: the Noether tension \( F \) is preserved along conformal geodesics. This tension is not the tension in an elastic rod. Because this approach breaks manifest conformal invariance, it is not obvious that these equations are equivalent to Eqs. [4]. The conservation law can be projected along two invariant normal directions which reproduce Eqs. [4]. Whereas tension is not itself conformally invariant, its conservation is. The two invariant directions do not to coincide with the Frenet normal and binormal. Indeed, it quickly becomes clear that the Frenet frame transforms in an unexpectedly complicated way under conformal transformations.

Rotational invariance identifies the conserved torque, \( M \). In the study of Euler elastic curves, the two Casimir invariants of the Euclidean group, \( \mathbb{F}^2 \) and \( \mathbb{F} \cdot M \) provide the constants of integration parametrizing solutions in terms of elliptic integrals. If the tension vanishes, as it does in self-similar equilibrium states, then so do both invariants. But, when the tension vanishes, the conserved magnitude of the torque is not only rotationally invariant, it also becomes translationally invariant. In a tension-free state, the torque will establish the spiral axis as well as provide a quadrature for the dimensionless variable, \( \kappa'/\kappa^2 \).

The method of auxiliary variables can be repurposed to identify the two additional conserved currents, the scaling and special conformal currents implied by the conformal invariance of the functional \( H \), without Euler elastic analogs. Scale invariance completely fixes the tangential tension, which itself determines the full tension in equilibrium states.

In the companion paper [18], these conservation laws are integrated to construct all three-dimensional self-similar spiral equilibrium states.

2 Conformal arc-length

The Frenet description of a space curve in terms of its acceleration and torsion may be intuitive, but as the number of derivatives increases, it becomes increasingly difficult to identify geometrically significant invariants built out of
them, never mind determining how they behave under deformation. The approach adopted in \cite{23} and then in \cite{25} works because the focus was on simple functionals of the curvature and the torsion which do not involve derivatives of $\kappa$ or $\tau$. To treat a functional involving higher derivatives, such as the conformal arc-length defined by Eq. \eqref{2} it is useful to introduce a one-dimensional covariant derivative that is invariant with respect to rotation of the two normals, treating the curvatures along the normal directions democratically. This approach was touched on in reference \cite{23}, its potential advantages suggested but not explored in depth.

Consider then an arc-length parametrized curve $s \rightarrow X(s)$ in three-dimensional Euclidean space with the inner product between two vectors denoted by a centeredot separating them.

Let prime denote a derivative with respect to arc-length, so that $t = X'$ is the unit tangent vector to the curve. Let $\{t, n^1, n^2\}$ be an orthonormal frame defined along it. The projection of the acceleration $t'$ onto each of the two normal vectors $n^I$ defines a curvature, $K^I$, $I = 1, 2$. As the curve is followed, these vectors will also rotate both into $t$ and among themselves. This behavior is summarized in the structure equations for the curve:\footnote{These equations are the direct analogues of the Gauss Weingarten equations for surfaces. The difference is that arc-length provides a privileged parametrization for curves. While this choice entails the surrender of manifest reparametrization invariance, it is straightforward to restore; the implications of this choice will be discussed in the context of the calculus of variations.}

\begin{align}
  t' &= -K^1 n^1; \\
  D n^I &= K^I t, \quad I = 1, 2. \tag{5a}
\end{align}

The one-dimensional covariant derivative $D$ appearing in Eq.\eqref{5b} is defined by $D = \partial_s \delta^I J + \omega^I J$, where $\omega^{IJ} = n^I \cdot n^J = -\omega^{JI}$ is a one-dimensional spin connection. Under a local rotation of the normals $D n^I$, like $n^I$, transforms as a vector; $n^I'$ does not. This framework can be extended in an obvious way to curves in higher dimensions.

Let $\{t, N, B\}$ denote the Frenet frame. The acceleration is directed along $N$, so that $K^1 = \kappa$, where $\kappa$ is the Frenet curvature; the curvature $K^2$ vanishes: $K^2 = 0$. The spin connection is now identified with the torsion: $\omega^{12} = -\tau = -\omega^{21}$. The well-known fundamental result is that, modulo Euclidean motions, the two independent scalars $\kappa$ and $\tau$ completely determine the curve \cite{24}. This is useful in principle; if one’s priority were to trace curves for art’s sake one could stop here; in practice, the conservation laws associated with the conformal symmetry of the problem addressed here will provide a much sharper characterization of the spiral geometry.

The only space curves of constant $\kappa$ and $\tau$ are helices. The analogue of this result when $s$ is replaced by $ln s$ is discussed in reference \cite{26}. In a sense, discussed in \cite{29}, any space curve is approximated locally by a helix almost everywhere.

Conformal invariants of curves must must first of all be Euclidean invariants. As such, they can be expressed in terms of the Frenet invariants and their derivatives. But this classification runs into difficulties as soon as higher derivatives are contemplated. For, whereas the curvature $\kappa$ is second order in derivatives, the torsion $\tau$ is of order three. A consequence is that it is not obvious what the natural scalars are using only these scalars as building blocks. For example, the oft-misused expression, $\kappa^2 + \tau^2$, may appear reasonable but it does not possess any geometrical or physical significance along curves that we know of. It does however play a role in an appropriate parametrization of developable strips \cite{27}.

This shortcoming never arises in the normal rotation covariant description introduced in Eq.\eqref{5b}, the relevant reparametrization invariant scalars are formed using $K^I$ and its covariant derivatives, $DK^I$, $D^2 K^I$ and so on, as building blocks. The simplest scalar in this approach, $K^I K_I/2$, coincides with the Euler energy density. Its analogue constructed using first derivatives is $DK^I DK_I/2$. To express this scalar in terms of Frenet variables, note first that $DK^I = K^I' + \omega^I J K^J$, so that with respect to the Frenet frame,

\begin{align}
  DK^1 &= \kappa'; \tag{6a} \\
  DK^2 &= \kappa \tau. \tag{6b}
\end{align}

It then follows that

\begin{equation}
  DK^I DK_I = \kappa'^2 + \kappa^2 \tau^2. \tag{7}
\end{equation}
This is none other than the sum of the Frenet scalars that appears in the conformal arc-length, Eq.\([2]\). This is not a coincidence.

### 2.1 Inversion of curves in spheres

Here a procedure to identify conformally invariant energies for curves will be sketched. Such invariants necessarily involve derivatives of the curvature. To see this, it is useful to step back and examine the behavior of curvatures and their covariant derivatives under conformal transformations.

Conformal transformations are the transformations of space that preserve angles. In all dimensions higher than two, any conformal transformation can be constructed by taking compositions of inversions in spheres, Euclidean motions, and scalings. The similarity transformations, Euclidean motions and scaling, act in an obvious way on Eq.\((5)\). The behavior under inversion in spheres is less obvious. But once it is understood how these equations transform behave under inversion in a sphere, determining the behavior under a more general conformal transformation becomes straightforward.

Under inversion in a sphere, any sphere maps to another sphere or (if the center of inversion sits on the surface of the sphere) to a plane. Modulo scaling and translation, what is left is inversion in a unit sphere centered at the origin. The point \(X\) on the curve embedded in the Euclidean space maps to the point \(\overline{X} = X/|X|^2\). Technically, the sphere should possess a radius to preserve dimensions. This is understood implicitly.

It is straightforward to describe how arc-length, the curvatures and their derivatives, defined in Eqs.\((5a)\) and \((5b)\), transform (see, for example, reference \([28]\) or \([9]\) for an analogous discussion for surfaces). Under inversion in a unit sphere located at the origin, the tangent and the normal vector transform as follows: \(\overline{t} \rightarrow |X|^2 R_X t\), whereas \(\overline{n}_I \rightarrow -R_X n_I\), where \(R_X\) is the linear operator,

\[
R_X = 1 - 2\hat{X} \otimes \hat{X},
\]

representing a reflection in the plane passing through the origin, orthogonal to \(\hat{X}\). Here 1 is the identity and \(X = X/|X|\). As a consequence of the behavior of the tangent vector, the arc-length transforms \(ds \rightarrow d\overline{s} = ds/|X|^2\).

As for the curvatures and connection, one finds that \(\overline{K}^I = -|X|^2 (K^I - 2 (X \cdot n^I)/|X|^2)\), \(\overline{\omega}^I_J = -|X|^2 \omega^I_J\).

The curvature transforms non-trivially. The connection transforms by a simple weight; as a consequence, the covariant derivative does also: \(D \rightarrow D_{\overline{s}} = -|X|^2 D\). A derivation of Eqs.\((9)\) is provided in Appendix A. As described in Appendix B, these expressions contrast favorably with their counterparts for the Frenet frame.

### 2.2 Conformal arc-length is the simplest conformal invariant

Using Eq.\((9)\), it is easy to see that

\[
D_s\overline{K}^I = -|X|^2 D \left( |X|^2 (K^I - 2 (X \cdot n^I)/|X|^2) \right)
= -|X|^4 DK^I.
\]

As a consequence, the scalar

\[
|DK|^2 = DK^I DK_I
\]

is a primary field, transforming with conformal weight \(|X|^4\), so that

\[
H = \int ds |DK|^1/2
\]

is a conformal invariant of curves, regardless of the dimension. Using Eq.\((7)\) this invariant is identified as the conformal arc-length, defined with respect to the Frenet frame in Eq.\((2)\).

Curiously, the covariant expression \((12)\) for the conformal arc-length, simple as it may be, does not appear in the
literature. Notice that whereas $DK^I$ transforms by a weight under conformal transformations, this is not true of either $\kappa'$ or $\kappa\tau$ separately. Confirming the conformal invariance of (2) using the Frenet frame is somewhat less than immediate. Having said this, it is only fair to bring the reader’s attention to the marvelously insightful construction of this invariant presented in reference [31].

3 Conformal Curvature

The transformation of the normal vector $DK_I$, given by Eq.(10) implies that $\nu = |DK|$ also satisfies

$$\bar{\nu} = |X|^4 \nu.$$  \hspace{1cm} (13)

Taking derivatives,

$$\frac{d\bar{\nu}}{d\bar{s}} = -\left(|X|^6 \nu' + 4|X|^4 (X \cdot t) \nu\right);$$

$$\frac{d^2\bar{\nu}}{d\bar{s}^2} = |X|^8 \nu'' + 10|X|^6 (X \cdot t) \nu'$$

$$+ 4|X|^4 \left[4(X \cdot t)^2 + |X|^2 \left[1 - K_J(X \cdot n^J)\right]\right] \nu.$$  \hspace{1cm} (14)

The $\nu\nu'$ terms generated under inversion of the scalar $\bar{\nu} = 4\nu

\nu'' - 5\nu'^2$ cancel, leaving

$$\bar{\nu} = |X|^12 \nu + 16|X|^8 \left[-4(X \cdot t)^2 + |X|^2 \left[1 - K_J(X \cdot n^J)\right]\right] \nu^2;$$  \hspace{1cm} (15)

The inhomogeneous term is identical to a term originating in the transformation of $\kappa^2 \nu^2$:

$$\tilde{\kappa}^2 \bar{\nu}^2 = |X|^8 \left(|X|^4 \kappa^2 \nu^2 + 4\left[-4(X \cdot t)^2 + |X|^2 \left[1 - K_J(X \cdot n^J)\right]\right]\right) \nu^2.$$  \hspace{1cm} (16)

As a consequence $4\nu(\nu'' - \kappa^2 \nu) - 5\nu'^2$ is a primary field, transforming with conformal weight $|X|^{12}$; the conformal curvature, defined by Eq.(3b), is thus identified as a conformal scalar [14]. $K$ depends on $|DK|$ and its first three derivatives. A more useful expression for $K$ is possible in terms of the variable

$$\mu = |DK|^{-1/2} = (\kappa^2 + \kappa^2 \tau^2)^{-1/4};$$  \hspace{1cm} (17)

$$K = -\mu (\partial^2 + \kappa^2 / 2) \mu + (\mu')^2 / 2,$$  \hspace{1cm} (18)

with the denominator suppressed. In this context, notice that one now identifies a conformally invariant bending energy, $\int ds \kappa^2 \mu^{-1}$.

Just as the normal vector $DK^I$ and the scalar $\nu$ transform under inversion with the same weight, one see that successive covariant derivatives are in correspondence with derivatives of $\nu$. It is thus evident that

$$K_0 = \frac{1}{8|DK|^3} \left[4DK_I(D^2 - \kappa^2)DK^I - 5(D^2 K)^2\right]$$  \hspace{1cm} (19)

is also a conformal invariant. While $K$ and $K_0$ coincide along planar curves with $\nu = \kappa'$, they differ in three or higher dimensions. To understand the relationship between them it is useful to examine the behavior of $K^I$ and its derivatives under conformal transformations a little more closely.

4 Conformal Torsion

The conformally transformed second covariant derivative is a linear combination of first and second covariant derivatives; similarly, the transformed third derivative is a linear combination of the first three derivatives and so on through higher derivatives.
If \( \varepsilon_{IJ} \) is the Levi-Civita tensor in the 2 dimensional normal space, then one can form both a pseudo-vector \( I_I = \varepsilon_{IJ} DK^J \) and a pseudo-scalar

\[
I_I = \varepsilon_{IJ} DK^I D^2 K^J ;
\]  

(20)

the later is also a conformal pseudo-scalar with conformal weight |X|\(^4\). The symmetric product of first derivatives appearing in the transformation of \( I \) vanishes on contraction with the Levi-Civita tensor\(^4\) It is also evident that one can construct higher-derivative analogues in higher-dimensional Euclidean spaces: in four Euclidean dimensions, the analogue of \( I \) is \( I = \varepsilon_{IJL} DK^I D^2 K^J D^3 K^L \), whereas \( \varepsilon_{IJL} DK^J D^2 K^L \) forms a pseudo-vector.

One can now construct a conformal invariant using \( I \) and the invariant one-form, \( \mu^{-1} ds \). At this point it is advantageous to introduce introduce the invariant unit vector in the normal space, \( U = U^I n_I \) where

\[
U_I = DK^I / |DK| .
\]

(21)

By construction \( U^I DU_I = 0 \), and \( DU^I \) is a primary field, and whereas \( D^2 U^I \) is not, it is straightforward to see how it will contribute to one:

\[
\begin{align*}
\bar{U}_I &= U_I \\
D_\bar{z} \bar{U}_I &= |X|^2 DU^I \\
D_\bar{z}^2 \bar{U}_I &= |X|^4 D^2 U^I + 2 |X|^2 (X \cdot t) DU^I .
\end{align*}
\]

(22)

More economically, define \( D_\mu = \mu D \). Now, under conformal inversion,

\[
\begin{align*}
D_\mu \bar{U}_I &= D_\mu U^I \\
D_\mu^2 \bar{U}_I &= D_\mu^2 U^I .
\end{align*}
\]

(23)

The conformal torsion (a conformal scalar of weight |X|\(^2\)) is given by

\[
\mathcal{T} = \varepsilon_{IJ} U^I D_\mu U^J = \varepsilon_{IJ} DK^I D^2 K^J / |DK|^{5/2} = \mu^5 I ,
\]

(24)

where \( I \) is defined by Eq.(20). The conformal invariant total torsion is then defined by

\[
J = \int ds \mu^{-1} \mathcal{T} = \int ds \varepsilon_{IJ} U^I DU^J .
\]

(25)

In three-dimensions, \( D_\mu U^I \) can be expressed in terms of \( U^I \) and \( \mathcal{T} \):

\[
D_\mu U_I = - \mathcal{T} \varepsilon_{IJ} U^J ;
\]

(26)

one can then expand \( D_\mu^2 U^I \) in terms of the two orthogonal normal vectors \( U^I \) and \( D_\mu U^I \):

\[
D_\mu^2 U_I = - \mathcal{T}^2 U^I + \frac{(\mathcal{T}')^\bullet}{2 \mathcal{T}^2} D_\mu U^I .
\]

(27)

Here the identity

\[
(D_\mu U)^2 = \mathcal{T}^2 ,
\]

(28)

following from Eq.(26), has been used. The bullet represents the derivative with respect to conformal arc-length. As a consequence of (27), the magnitude of the conformal second derivative can also be cast in terms of \( \mathcal{T} \) and its first conformal derivatives:

\[
(D_\mu^2 U)^2 = \mathcal{T}^4 + \mathcal{T}'^2 .
\]

(29)

It is now possible to answer the question posed earlier: what is the relationship between \( \mathcal{K}_0 \), defined by Eq.(19), and \( \mathcal{K} \) defined by Eq.(3 b). Note that \( (U^I = DK^I / \nu) \)

\[
\begin{align*}
\mathcal{K}_1 &= \frac{1}{8\nu^3} \left[ 4\nu U^I (D^2 - k^2) \nu U^I - 5 (D\nu U)^2 \right] \\
&= \mathcal{K} + \frac{1}{8\nu} \left[ 4U^I D^2 U^I - 5 (DU)^2 \right] \\
&= \mathcal{K} - \frac{9\mathcal{T}^2}{8} ,
\end{align*}
\]

(30)

\textsuperscript{4}This construction is analogous to that of the Frenet torsions in \cite{23} using \( DK^I \) and \( D^2 K^I \) instead of \( K^I \) and \( DK^I \).
where Eqs. (27) and (28) have been used. The cross terms vanish by the unitarity of $U_I$.

It is instructive to recast the conformal torsion in terms of the Frenet curvature and torsion and their derivatives. The efficient way to evaluate second (and higher) covariant derivatives in the Frenet gauge is to proceed iteratively. Thus, for $D^2 K^I$, use is made of the identity

$$D^2 K^I = (DK^I)' + \omega^I, \quad D^2 K^J,$$

(31)

where $DK^I$ are given by Eq. (6). Using the identities Eqs. (6) for the lower derivatives, one immediately identifies

$$D^2 K^1 = \kappa'' - \kappa \tau^2;$$

(32a)

$$D^2 K^2 = (\kappa \tau')' + \kappa' \tau.$$  

(32b)

Using Eqs. (6) and (32), one finds that $I$, defined by Eq. (20), can be factorized,

$$I = DK^1 D^2 K^2 - DK^2 D^2 K^1 = \kappa'' \left[ \frac{\tau}{\kappa} - \left( \frac{\kappa'}{\kappa^2 \tau} \right)' \right].$$

(33)

The conditions under which the conformal torsion $T$ vanishes are discussed in Appendix C. The integrated torsion can now also be cast in terms of the Frenet variables,

$$J = \int ds |DK|^1/2 = \int ds \frac{\kappa^3 \tau^2}{\kappa^2 + \kappa^2 \tau^2} \left[ \frac{\tau}{\kappa} - \left( \frac{\kappa'}{\kappa^2 \tau} \right)' \right].$$

(34)

Significant properties of $J$ are collected in Appendix C.

5 Critical points of curvature energies

One is now in a position to examine the behavior of the conformal arc-length under deformations of the curve: $X \rightarrow X + \delta X$. Any two curves related by a conformal transformation possess the same conformal arc-length. So if one of the two describes a critical point, or conformal geodesic, then so does the other.

In general, translational invariance of the energy identifies the Euler-Lagrange derivative with respect to $X$ with the divergence of a stress tensor, identified with the tension $F$. Conformal geodesics then satisfy $F' = 0$. Even though $F$ itself is not conserved under conformal transformations, the conservation law is. Whereas the energies of two curves related by a conformal transformation coincide, the tensions within them generally differ. Tension-free curves are necessarily in equilibrium. In contrast to the planar reduction of this problem, however, it is not obvious if every equilibrium state is conformally equivalent to a tension-free state [17].

Consider, more generally, a functional defined on an arc-length parametrized space curve, $H[X]$. This can always be cast in the form

$$H[X] = \int ds \mathcal{H}(K^I, DK^I),$$

(35)

where $DK^I = K^{I''} + \omega^I, J K^J$, and $\omega$ is the normal connection defined below Eq. (5b). The conformal arc-length depends only on $DK^I$. A dependence on $K^I$ will be admitted, not only because it involves no extra effort but also because it facilitates comparisons with Euler-Elastica and allows one to investigate the implications of replacing conformal invariance by scale invariance. For example, the energy

$$H[X] = \int ds (DK^I DK^I + \alpha (K^I K_I)^2)^{1/4}$$

(36)

is scale invariant for any choice of $\alpha$; but conformally invariant only if $\alpha = 0$.

For higher derivative energies, dismantling the covariant derivatives in favor of the Frenet scalars and their derivatives
is not an optimal strategy for turning the variational crank. Treating the normal directions democratically simplifies
the implementation of the calculus of variations.

The approach we adopt to examine the behavior of $H$ under small deformations, $X \to X + \delta X$, will be the method
of auxiliary variables, developed originally to examine surfaces [22] (see also [9] for a recent review, tailored to the
two-dimensional bending energy). This provides an efficient way to identify conserved currents associated with the
symmetries of any energy $H$ depending on geometric degrees of freedom [22, 17]. While originally developed for ener-
gies quadratic in curvature, there is no obstacle to considering energies involving higher derivatives (see, for example,
[32] or [33], a factor of two and a sign error in [32] were corrected discreetly in the more general treatment in [33]).
The implications of curvature derivatives on the boundary of contact in the adhesion of membranes are explored in [34].

The idea is to treat $K^I$ and $\omega^{IJ}$ as independent variables in $H$. To do this in a consistent way, it is necessary
to introduce Lagrange multipliers to impose the structure equations connecting them to $X$ as constraints. The func-
tional dependence on $X$ itself, as well as the intermediate variables $t$ and $n^I$ appears only within the constraints.

Thus one constructs the constrained functional

$$
H_C[X, t, n^I, K^I, \omega^{IJ}, \ldots] = H[K^I, \omega^{IJ}] + \int ds \left[ \frac{1}{2} T(1 - t \cdot t) - H_I(K^I - t \cdot Dn^I) - S_{IJ} (\omega^{IJ} - n^I \cdot n^J) \right] 
+ \int ds \left[ \frac{1}{2} \lambda_{IJ} (n^I \cdot n^J - \delta^{IJ}) - f_I(n^I \cdot t) + F \cdot (t - X^I) \right].
$$

The Euler-Lagrange equations for $K^I$ and $\omega^{IJ}$ identify $H_I$ and $S_{IJ}$ as the Euler Lagrange derivatives of the uncon-
strained functional $H[K^I, \omega^{IJ}]$ with respect to $K^I$ and $\omega^{IJ}$ respectively:

$$
H_I = \frac{\delta H}{\delta K^I}, \quad S_{IJ} = \frac{\delta H}{\delta \omega^{IJ}}.
$$

Explicit expressions for $H_I$ and $S_{IJ}$ for energies involving $K^I$ and $DK^I$ are

$$
H_I = \frac{\partial H}{\partial K^I} - D \left( \frac{\partial H}{\partial DK^I} \right), \quad S_{IJ} = \left( \frac{\partial H}{\partial DK^I} \right) K^J.
$$

The Euler-Lagrange equations for $t$ and $n^I$ identify the tension $F$ along the curve to be given by

$$
F = (T - H_I K^I) t - (DH_I - 2S_{IJ} K^J) n^I.
$$

To see this, notice that the Euler-Lagrange equation for $t$ gives (using the equations of structure [3b], which are
themselves implied by the constraints)

$$
F = T t - H_I Dn^I + f_I n^I = (T - H_I K^I) t + f_I n^I.
$$

The counterpart for $n$ implies

$$
f_I t = -D(H_I t) + 2S_{IJ} Dn^J + DS_{IJ} n^J + \lambda_{IJ} n^J,
$$

or, equivalently,

$$
(f_I + DH_I - 2S_{IJ} K^J) t - (H_I K^J + DS_{IJ} - \lambda_{IJ}) n^J = 0.
$$

The vanishing of the tangential component of this homogeneous equation implies $f_I = -(DH_I - 2S_{IJ} K^J)$, reprodu-
cing the normal projection in Eq. [40]. The vanishing of the antisymmetrized normal component implies the addition
kinematical constraint

$$
DS_{IJ} + H_{IJ} K^I = 0,
$$
which captures the normal rotational invariance of $H$. One can confirm that the identifications (39a) and (b) are consistent with Eq. (44).

In this framework, the functions $X$ appear only in the tangency constraint. Modulo boundary terms, examined in the next section,

$$\delta_X H_C[X, t, n^I, K^l, \omega^{lj}, \ldots] = \int ds \mathbf{F}' \cdot \delta X.$$  

(45)

But this implies that, modulo the boundary conditions dictated by these boundary terms,

$$\delta_X H[X] = \int ds \mathbf{F}' \cdot \delta X.$$  

(46)

It is thus clear that $H$ is stationary with respect to variations of $X$ when $\mathbf{F}$ is a constant vector along the curve:

$$\mathbf{F}' = 0.$$  

(47)

$\mathbf{F}$ is identified as the tension in the curve. To unpack the conservation law, collect the tangential and normal projections in Eq. (eq:Fdef):

$$\mathbf{F} = F_\parallel t + F_\perp n^I.$$  

(48)

Using the structure equations (5a) and (b), the normal and tangential projections of the conservation law (47) are given respectively by

$$E_\perp^I = n^I \cdot \mathbf{F}' = DF^I_\perp - K^l F_\parallel = 0, \quad I = 1, 2;$$  

(49a)

$$E_\parallel = t \cdot \mathbf{F}' = F_\parallel + F_\perp K^l = 0.$$  

(49b)

Notice that, on any curve, $S_{I,J} K^l K^j = 0$, so that $S_{I,J}$ never appears on the lhs of Eq. (49b).

In this framework, the one remaining unknown in the definition of $\mathbf{F}$ is $T$, the multiplier imposing the unitarity of $t$ or, equivalently, flagging $s$ as arc-length. Had the curve been parameterized arbitrarily, it would have been necessary to introduce a one-dimensional metric. In such an approach, $T$ is identified as the stress associated with this metric. The identity (49b) is then tautological: a consequence of the manifest reparametrization invariance of $H$. This approach is not generally optional for surfaces [22]. But here there is a privileged parametrization by arc-length parametrization. The price paid is the breaking of manifest reparametrization invariance. A consequence is that the tangential Euler-Lagrange equation is no longer satisfied identically. Its new role is to recover the multiplier $T$. Stationary states are then characterized by the 2 Euler-Lagrange equations: $E_I = 0, \quad I = 1, 2$.

To determine $T$, first recast Eq. (49b) in the form

$$(T - 2 H_1 K_1)' + H_1 DK^l = 0.$$  

(50)

Now note, using the definition of $H_1$ (39a), that

$$H_1 DK^l = \left[ \left( \frac{\partial H}{\partial K^l} \right)^{'} - D \left( \frac{\partial H}{\partial DK^l} \right) \right] DK^l$$

$$= H' - \left( \left( \frac{\partial H}{\partial DK^l} \right)^{'} \right) DK^l.$$  

(51)

Using this identity in Eq. (50), $T$ determined modulo a constant of integration, $\sigma$:

$$T = -H + 2 H_1 K_1 - \sigma + \left( \frac{\partial H}{\partial DK^l} \right) DK^l.$$  

(52)

The constant $\sigma$ is associated with the global constraint on the total arc-length implicit in this approach. If arc-length is not fixed, then $\sigma = 0$. The identification of $T$ restores the reparametrization invariance that was temporarily suspended in choosing $s$ to parametrize the curve. One can now express the tangential component of the tension appearing in the decomposition (48) completely in terms of the $K^l$ and their covariant derivatives:

$$F_\parallel = -H + H_1 K_1 + \left( \frac{\partial H}{\partial DK^l} \right) DK^l.$$  

(53)
In the Frenet gauge, Eqs. (49a) and (b) assume the strikingly simple form:

\begin{align*}
(F_1^\perp)' - \tau F_2^\perp - \kappa F_1^\parallel &= 0; \\
(F_2^\perp)' + \tau F_1^\perp &= 0; \\
F_1^\parallel + \kappa F_2^\perp &= 0.
\end{align*}

(54a)

(54b)

(54c)

Eq. (54c) determines $F_1^\perp$ completely in terms of $F_1^\parallel$; Eq. (54a) then determines $F_2^\perp$ in terms of $F_1^\parallel$. Finally, Eq. (54b) provides a third order equation for $F_1^\parallel$.

It is clear from this decomposition that a sufficient condition that $\mathbf{F} = 0$ in equilibrium is that the tangential component vanishes, or $F_1^\parallel = 0$. While this approach is of some interest in principle, it is not a very useful approach to solving these equations in practice.

6 The conformal tension

Let $\mathcal{H} = \mathcal{F}(\mathcal{U})$, where $\mathcal{U} = |DK|^2$. We have

\begin{align*}
H_I &= -2D(\mathcal{F}_I DK_I) \\
S_{IJ} &= 2\mathcal{F}_I DK_{IJ},
\end{align*}

(55a)

where $\mathcal{F}_I = \partial \mathcal{F} / \partial \mathcal{U}_I$. Thus $F_1^\parallel$, defined by Eq. (53), is given by

\begin{align*}
F_1^\parallel = -\mathcal{F} + 2\mathcal{F}_I \mathcal{U} - 2D(\mathcal{F}_I DK^I)K_I.
\end{align*}

(56)

The normal stress, $F_2^\parallel$, appearing in Eq. (40), is given by

\begin{align*}
F_2^\parallel = 2D^2(\mathcal{F}_I DK_I) + 2\mathcal{F}_I \kappa^2 P_K DK_I,
\end{align*}

(57)

where $P_K = \delta^i_j - \hat{K}^i \hat{K}_j$ is the projector on normal vectors orthogonal to $K^I$.

If, in particular, $\mathcal{F} = |DK|^{1/2}$, then

\begin{align*}
2\mathbf{F} &= -\left(\mu^3 \kappa \kappa'\right)' \mathbf{t} + \left(D^2(\mu U^I) + \mu \kappa^2 P_K U^I\right) \mathbf{n}_I,
\end{align*}

(58)

where $\mu$ is defined in Eq. (17), and $U^I$ is unit normal vector parallel to $DK^I$ (21). The remarkably simple expression for the tangential projection of $\mathbf{F}$ will be understood to be a consequence of the scale invariance of the energy. The first Casimir invariant of the Euclidean group is given by $F^2 = \mathbf{F} \cdot \mathbf{F}$; $F$ is not an invariant of the conformal group.

7 Recovery of the conformally invariant Euler-Lagrange equations

To reproduce the manifestly covariant Euler-Lagrange equations (4), let us first replace covariant derivatives with respect to arc-length in Eq. (58) by covariant derivatives with respect to conformal arc-length, defined above Eq. (23). We have for the normal tension:

\begin{align*}
2F_1^\perp &= \frac{1}{\mu} D^2_{\mu} \frac{1}{\mu} \mathcal{F}_I (\mu U^I) + \mu \kappa^2 P_K U^I \\
&= \frac{1}{\mu} D^2_{\mu} U^I + \frac{1}{\mu} [(D^2_{\mu} \ln \mu) U^I + (D_{\mu} \ln \mu) D_{\mu} U^I] + \mu \kappa^2 P_K U^I.
\end{align*}

(59)

Define $I_U = 2\mu U^IF_1^\parallel$; then

\begin{align*}
I_1 &= -T^2 + D^2_{\mu} \ln \mu + (\mu^2 \kappa^2 - (\mu K \cdot U)^2) \\
&= -T^2 - \kappa + (D_{\mu} \ln \mu)^2 / 2 + \left(\frac{1}{2} \mu^2 \kappa^2 - (\mu K \cdot U)^2\right),
\end{align*}

(60)
where the identity 27 is used to introduce the conformal torsion \( T \) in the first line and rewritten the identity 18 in the form,

\[
\mathcal{K} = -D_\mu^2 \ln \mu - k^2 \mu^2 / 2 + (D_\mu \ln \mu)^2 / 2,
\]

to introduce the conformal curvature \( \mathcal{K} \) on the second.

The normal projection of the tension orthogonal to \( U^I \), \( I_2 = 2\mu DU^I F_{\perp I} \), is

\[
I_2 = \mathcal{T}^* + (D_\mu \ln \mu) T^2 - (\mu K \cdot U)(\mu K \cdot D_\mu U).
\]

We now project the two normal Euler-Lagrange derivatives of \( H \) with respect to \( X \), given by Eqs. (49a), along \( U^I \) and \( DU^I \). For the projection of \( \mathcal{E}^I \) along \( U^I \), one has

\[
2\mu^2 U_I \mathcal{E}^I = 2D_\mu (\mu U^I F_{\perp I}^1) - 2(\mu D_\mu U_I + \mu^* U_I) F_{\perp I}^1 - 2\mu^2 (U \cdot K) F_{\parallel I}
\]

\[
= I_1 - D_\mu \ln \mu I_1 - I_2 + (\mu U \cdot K)^* (\mu U \cdot K),
\]

where the identity \( 2\mu F_{\parallel I} = -\mu (\mu K F_{\parallel I})^* = -\mu (\mu K \cdot U)^* \) is used in the last term. Using the definitions of \( I_1 \) and \( I_2 \), 60 and 62 respectively; collecting terms and making reuse of the identity, 61, this collapses

\[
\mathcal{E}_1 := 2\mu^2 U_I \mathcal{E}^I
\]

\[
= -\left( \frac{3}{2} T^2 + K \right)^* + L_1,
\]

where

\[
L_1 = \frac{1}{2} \left( \mu^2 k^2 \right)^* - (\mu K \cdot U)(\mu K \cdot U)^* - \left( \mu^2 k^2 - (\mu K \cdot U)^2 \right) D_\mu \ln \mu + (\mu K \cdot U)(\mu K \cdot D_\mu U); \tag{65}
\]

It is straightforward to show that \( L_1 \) vanishes. The projection of \( \mathcal{E}_I \) along \( DU^I \) can be written

\[
2\mu^2 DU_I \mathcal{E}^I = 2D_\mu (\mu DU^I F_{\perp I}) - 2(\mu^2 D_\mu U_I + \mu^* DU_I) F_{\perp I} + (\mu U \cdot K)^* (\mu DU \cdot K)
\]

\[
= I_2 - \mu^* I_2 - I_3 + (\mu U \cdot K)^* (DU \cdot K)
\]

\[
= I_2 - \mu^* I_3 + T^2 I_1 - T^* I_2 + (\mu U \cdot K)^* (DU \cdot K),
\]

where Eq. (27) has been used as well as the definition of \( T \) to express the projection of the normal tension along \( D_\mu U^I \) in terms of \( I_1 \) and \( I_2 \):

\[
I_3 := 2\mu (\mu D_\mu U^I F_{\parallel I}) = -T^2 I_1 + T^* I_2. \tag{67}
\]

Using the expressions 60 and 62 for \( I_1 \) and \( I_2 \), the manifestly conformally expression,

\[
2\mu^2 D_\mu U_I \mathcal{E}^I = -2\mathcal{K} T + T^* T - T^3, \tag{68}
\]

for \( DU_I \mathcal{E}^I \) follows. The Euler-Lagrange derivatives, \( \approx U_I \mathcal{E}^I \) and \( D_\mu U_I \mathcal{E}^I \), are themselves manifestly conformally invariant. The corresponding Euler-Lagrange equations, \( DU_I \mathcal{E}^I = 0 \) and \( U_I \mathcal{E}^I = 0 \), are also and they coincide with Eqs. (4a) and (b). The conservation of the tension (which is not itself rotationally invariant never mind conformally invariant) is equivalent to the conformally invariant Euler-Lagrange equations, derived first by Musso 15 using a very different approach, reflecting different objectives. Notably, the conserved tension plays no role.

### 8 Boundary variations and conservation laws

We now construct the torque associated with rotational invariance as well as the scalar and vector currents associated with conformal invariance. All three conserved currents will play a role in the construction of tension-free states.

First collect the boundary terms that have accumulated in the variation of \( H_C \) defined by Eq. (37). One has

\[
\delta H_C[X, \ldots] = \int ds \mathbf{F}' \cdot \delta X + \int ds \mathcal{J}', \tag{69}
\]
where
\[
\mathcal{J} = (H_1 t + S_{12} \mathbf{n}^t) \cdot \delta \mathbf{n}^t - F \cdot \delta \mathbf{X} + \left( \frac{\partial \mathcal{H}}{\partial DK^1} \right) \delta K^1.
\] (70)

Here \(H_1\) is defined by Eq. (59a) and \(S_{12}\) by Eq. (59b). The first three terms contributing to \(\mathcal{J}\) originate in the variations of \(X\) and \(n^t\) when derivatives are peeled off the variation and collected in a derivative. For the familiar Euler Elastic energy or any energy involving \(K^1\) alone, \(S_{12} = 0\) and \(\partial \mathcal{H}/\partial DK^1 = 0\) and the two surviving terms complete the specification of the boundary term. For conformal arc-length, there is an additional boundary term associated with the variation of \(K^1\). At this order, there is no boundary contribution associated with variations of the connection. This is easily understood: \(\omega^I\) always appears in the combination \(DK^1\), one derivative lower than \(K^1\). If however, an even higher order theory is contemplated—such as the conformal bending energy, quadratic in the conformal curvature—such terms will arise (albeit in a sense trivially) and will contribute to the special conformal current.

In equilibrium the first \textit{bulk} term appearing in Eq. (69) vanishes and only the boundary terms survive.

### 8.1 Rotational invariance and torque conservation

For rotations, defined by the axial vector \(\mathbf{b}\), \(\delta \mathbf{X} = \delta \mathbf{b} \times \mathbf{X}\) and \(\delta n^t = \delta \mathbf{b} \times n^t\), we have
\[
\delta_b H_C = \delta \mathbf{b} \cdot \int ds \left( \mathbf{M}' - \mathbf{X} \times \mathbf{F}' \right),
\] (71)

where the torque is defined by
\[
\mathbf{M} = \mathbf{X} \times \mathbf{F} - H_1 \mathbf{n}_2 + H_2 \mathbf{n}_1 - 2S_{12} \mathbf{t}.
\] (72)

In equilibrium, with \(\mathbf{F}' = 0\), \(\mathbf{M}\) is conserved, \(\mathbf{M}' = 0\). Notice that \(\mathbf{M}\) possesses the same dimensions as \(H\). As such it is dimensionless if \(H\) is scale invariant.

For the conformal arc-length, the Euler-Lagrange derivatives with respect to \(K^1\) and \(\omega_{12}\) are given by
\[
H_1 = -\frac{1}{2} D(\mu^3 DK_1),
\]
\[
2S_{12} = -\frac{1}{2} \mu^3 DK_2 K_1,
\] (73)

where \(\mu\) is defined by Eq. (17), and (from (58), we identify \(2F_\parallel = -\left(\mu^3 K^2\right)'\)). With respect to the Frenet frame they read
\[
2H_1 = -\left(\mu^3 K^2\right)' + \mu^3 K^2
\] (74a)
\[
2H_2 = -\left(\mu^3 K^2\right)' - \mu^3 K^2 \tau = -(\mu^3 K^2 \tau)' / \kappa
\] (74b)
\[
2S_{12} = -\mu^3 K^2 \tau / 2.
\] (74c)

The bending moment is given by the second Casimir invariant of the Euclidean group,
\[
\mathcal{M} = \mathbf{M} \cdot \hat{\mathbf{F}}.
\] (75)

One determines
\[
\mathcal{M} F = -H_1(DH_2 + 2S_{12} K^1) + H_2(DH_1 - 2S_{12} K^2) - 2F_\parallel S_{12}.
\] (76)

In contrast, for Euler Elastica, with \(\mathcal{H} = \kappa^2 / 2\), one determines \(S_{12} = 0\); \(H_1 = \kappa\), \(H_2 = 0\), and \(DH_2 = \kappa \tau\). As a result, \(\mathcal{M} F = -\kappa^2 \tau\). This determines \(\tau\) as a function of \(\kappa\).

\(\mathcal{M}\), like \(F\), is not a conformal invariant. But both Euclidean Casimir invariants vanish in tension-free states, so the issue is moot. In such states, \(\mathcal{M}\) is translationally invariant and \(M^2 = \mathbf{M} \cdot \mathbf{M}\) is a Euclidean invariant. This observation will play a significant role in the construction of tension-free states.

\[\text{To be technically correct, it possesses the dimensions of energy.}\]
8.2 Conformal invariance and its manifestations

The treatment of the calculus of variations, thus far, has not exploited the conformal invariance of the energy, and the additional conserved currents implied by this invariance. The task now is to identify them. Begin with scaling.

8.2.1 Scaling

Under rescaling, \( \delta_\lambda X = \lambda X \) and \( \delta_\lambda n^I = -\lambda n^I \), whereas \( \delta_\lambda K^I = -\lambda K^I \), \( \delta_\lambda \omega^{IJ} = -\lambda \omega^{IJ} \), and \( \delta_\lambda DK^I = -2\lambda DK^I \). Substituting into Eq. (70), one identifies \( J = \lambda S \), where the scaling current \( S \) is given by

\[
S = -F \cdot X + S_D,
\]

and

\[
S_D := -\partial\mathcal{H}/\partial K^I K^I.
\]

It follows from the identity (69) that the current \( S \) satisfies

\[
S' = -F' \cdot X,
\]

whenever the energy is scale invariant. \( S \) is conserved when \( F \) is. Note that, like the torque \( M \), \( S \) is dimensionless.

Eq. (79) is equivalent to the identity

\[
F|| = -S'_D,
\]

so that \( F|| \) is expressible as a derivative whenever \( H \) is scale invariant (whether in equilibrium or not). It is possible to show that the identity (80) is equivalent to the Euler scaling equation. This will be discussed in a more general setting elsewhere.

For the conformal arc-length, with \( \mathcal{H} = |DK|^1/2 \), \( \partial\mathcal{H}/\partial K^I = \mu^3 DK^I/2 \), so that

\[
S_D = -\mu^3 \kappa' / 2,
\]

where \( \mu \) is defined in Eq. (17). It is not obvious in the construction of \( F|| \) terminating in Eq. (53) that it is expressible as the derivative of \( S_D \) in any scale invariant theory.

In a tension-free state, \( S = S_D \) is a constant independent of \( X \). This equation places a constraint on the torsion \( \tau \) in terms of the dimensionless ratio \( \Sigma = -\kappa'/\kappa^2 \). On the other hand, modulo this constraint, the conservation of torque implies that \( \Sigma \) satisfies a quadrature, involving the addition parameter \( M \), the magnitude of the torque. This will be constructed explicitly in reference [18]. \( \Sigma \) can in turn be integrated to provide identify the Frenet curvature \( \kappa \). It is now possible to construct the self-similar spiral from its Frenet data [24]. The axis of this spiral is defined by the direction of \( M \). But it would be a mistake to stop here.

8.2.2 Special conformal current

The identification of the second conformal current involves examining the behavior of a curve under a special conformal transformation, the composition of an inversion with a translation followed by a second inversion, linearized in the intermediate translation \( c \), given by \( \delta_c x = |x|^2 R_x c \), where \( R_x \) is the same linear operator on 3-dimensional Euclidean space defined along curves in Eq. (8). The vector \( c \) has dimensions of inverse length squared.

In Appendix D, the two identities

\[
\delta_c n^I = 2(X \cdot n^I) c - 2(n^I \cdot c) X,
\]

and

\[
\delta_c K^I = 2[(X \cdot c) K^I - (n^I \cdot c)]
\]

are derived. Using Eq. (70), and the results just collected, the conformal current \( G \) associated with any conformally invariant energy involving three or less covariant derivatives \( (J = G \cdot c \) in the identity (69) is easily seen to be given by

\[
G = -2H_I F^I_0 + 2S_{IJ} V^{IJ} - |X|^2 R_X F - 2S_D X - 2 \partial\mathcal{H}/\partial K^I n^I.
\]
Here, the definition of $S_D$ in Eq. (78) has been used; $F_0^I$ is defined by
\[ F_0^I = (X \cdot t) n^I - (X \cdot n^I)t; \] (85)
we have also set $n^J \cdot \delta_c n^I = -V^{IJ} \cdot c$, where
\[ V^{IJ} = (n^J \cdot X) n^I - (X \cdot n^I) n^J. \] (86)

Using Eq. (77) to reassemble the scaling current $S$, we can also write
\[ G = -2H_I F_0^I + 2S_{IJ} V^{IJ} - |X|^2 F - 2S X - 2 \partial H / \partial D K^I n^I. \] (87)

Eq. (69) implies that, whenever $H$ is conformally invariant, $G$ satisfies
\[ G' = -|X|^2 R_X F' \]
\[ = 2F' \cdot XX - |X|^2 F' \]
\[ = -2S' X - |X|^2 F'. \] (88)

The last line is a consequence of Eq. (79). In particular, $G$ is conserved when $F$ is.

To verify this directly, the two identities,
\[ DF_0^I = n^I - V^{IJ} K^J; \]
\[ DV^{IJ} = -K^I F_0^J + K^J F_0^I, \] (89)
are useful. Together, they can be viewed as the higher co-dimensional analog for curves of an identity, describing the normal vector in terms of a potential, introduced in [35].

We can now use the definition of the torque given by Eq. (72), as well as the three obvious identities
\[ F_0^1 = (t \cdot X) N - (N \cdot X) t = X \times B; \]
\[ F_0^2 = (t \cdot X) B - (B \cdot X) t = -X \times N, \] (90)
and
\[ V^{12} = (B \cdot X) N - (N \cdot X) B = -X \times t, \] (91)
in the Frenet frame, to express the first two terms on the right hand side of Eq. (87) in terms of moment of the excess torque,
\[ -2H_I F_0^I + 2S_{IJ} V^{IJ} = 2X \times (M - X \times F). \] (92)

Finally, using the expression for $\partial H / \partial D K^I$ given at the end of subsection 8.2.1, the conserved conformal current for the conformal arc-length assumes the form
\[ G = 2X \times (M - X \times F) - |X|^2 F - 2S X - \mu \kappa' N + \kappa \tau B, \] (93)
involving the three conserved currents $F$, $M$, and $S$.

Suppose that $F = 0$. Then, under translation $G$ transforms by a constant vector. It is thus possible to choose the origin so that $G$, like $F$, vanishes. But if $G = 0$, all tension-free states are characterized completely by two independent parameters, the scaling rate $S$ and the torque magnitude, $M$.

## 9 Conclusions

In general, there will be conserved currents associated with the stationary states of a conformally invariant energy. For a conformally invariant energy defined on curves, these are identified as the tension and torque associated with the underlying Euclidean invariance; as well as the scaling and special conformal currents associated with the additional symmetry. We have provided an explicit construction of all four currents for the conformal arc-length.

We have shown how the conservation of the tension can be cast in a manifestly conformally invariant form, in
terms of the elementary conformal invariants, the conformal curvature and torsion and their derivatives with respect to conformal arc-length. Tension-free states necessarily involve the breaking of conformal symmetry. Neither the conformal curvature nor conformal torsion generally vanishes in these states.

In a companion paper, it will be shown how the conservation laws can be used to construct tension-free curves \[18\]. Show curves form self-similar spirals. Their significance stems from the fact that they represent the direct three-dimensional analogues of the logarithmic spiral. When the tension vanishes, the special conformal current does also if the origin is chosen appropriately. This turns out to be the spiral apex. It now becomes possible to construct a spherical polar coordinate system adapted to this origin and the direction of the torque, $\mathbf{M}[18]$. While it is not obvious in what order the conservation laws should be integrated, a natural order becomes apparent: the conserved scaling current determines the torsion in terms of the local curvature; the magnitude of the torque then provides a quadrature determining this curvature. This would appear to leave no significant role for the special conformal current. However, an intricate internal structure is revealed that would not be guessed at the level of the Frenet data implied by the torque quadrature. In logarithmic spirals, the two conserved currents $M$ and $S$ are not independent, constrained to satisfy $4MS = 1$. Now, however, if the torque is increased relative to the curvature $S$, such that with $4MS > 1$, however, the spiral will nutate between two fixed oppositely oriented circular cones, indicating the bounding polar angles with respect to the adapted coordinates. There will also be a pair of coaxial outer cones on which the spiral reverses direction along the $M$-axis. The spiral will oscillate with increasing amplitude along this axis in successive cycles of nutation, twisting and precessing about the axis as it grows, the torsion changes sign at the turning points where the growth along the axis is reversed. A deceptively simple single irreducible cycle in illustrated in Figure 1.

Unlike the logarithmic prototypes, with all points equivalent, their dimensional analogues exhibit internal structure which is captured by the nutating cycles; the points within each cycle inequivalent. Teasing out this structure involves somewhat different mathematics; it is not a short story and not without interest. The special conformal current plays an unexpected role. The subject will be taken up in a companion paper \[18\].

**Figure 1**: One complete cycle of a supercritical spiral, illustrating both the invariant cones restricting access to the poles, as well as the pair of coaxial outer cones on which the spiral reverses direction along the $M$-axis. The segments above (below) the mid-plane orthogonal to $M$ are colored orange (green). The angle through which the cycle precesses in a half-cycle is indicated. This particular spiral dilates by a factor of 38.15 over the course of this cycle. Rotating this cycle by $2.43\pi$ and rescaling by a factor of 38.15, and repeating generates the complete self-similar spiral.
**Acknowledgments**

Partial support from CONACyT grant no. 180901 is acknowledged.

**Appendix A  Derivations of Eqs. (9)**

The derivation of Eq. (9) involves the identity

\[ R_X R_X' = -2[1 - 2 \hat{X} \otimes \hat{X}] [\hat{X} \otimes \hat{X}' + \hat{X}' \otimes \hat{X}] \]

where \( P_X \) is the projection orthogonal to \( X \), which originates in the differentiation of the unit vectors appearing in \( R_X \):

\[ \hat{X}' = P_X t / |X|. \]  
(A.2)

Using Eq. (A.1) one has

\[ t \cdot R_X R_X' n^I = 2(n^I \cdot X) / |X|^2, \]  
(A.3)

and also

\[ n^I \cdot R_X R_X' n^J = 0. \]  
(A.4)

It is now clear that

\[ \hat{K}^I = \hat{t} \cdot \frac{d \hat{n}^I}{ds} = -|X|^2 \hat{K}^I + t \cdot R_X R_X' n^I, \]  
(A.5)

which coincides, on using Eq. (A.3), with the expression for \( \hat{K}^I \) given in Eq. (9). Similarly

\[ \hat{\omega}^{IJ} = \hat{n}^I \cdot \frac{d \hat{n}^J}{ds} = -|X|^2 \hat{\omega}^{IJ} + n^I \cdot R_X R_X' n^J. \]  
(A.6)

The second term on the right vanishes because of the identity Eq. (A.4). Eq. (A.6) thus reproduces the expression for \( \hat{\omega}^{IJ} \) given in Eq. (9).

**Appendix B  Conformally transformed Frenet frame**

As a consequence of Eq. (9), the Frenet curvature transforms as

\[ \bar{\kappa}^2 = \hat{K}_I \hat{K}^I = |X|^2 \kappa^2 - 2 \kappa (N \cdot X) + |X|^2 - (X \cdot t)^2. \]  
(B.1)

It does not transform as \( \kappa \rightarrow |X|^2 \kappa - 2(N \cdot X) \), as might naively have been expected. Gauge choices do not transform in a simple way under conformal transformations. The simplest way to determine how \( \tau \) transforms is to note that \( U = \kappa'^2 + \kappa^2 \tau^2 \rightarrow |X|^8 U \). Note in particular that \( \kappa' \) does not transform by a weight so neither does \( \tau \). It is evident that the Frenet frame transforms in a non-trivial way under conformal transformations, the more so the higher the dimension.

**Appendix C  The conformal torsion**

The term in square brackets appearing on the rhs of Eq. (34) vanishes when

\[ \left( \frac{\kappa'}{\kappa^2 \tau} \right)^{'} = \frac{\kappa'}{\kappa^2}, \]  
(C.1)

or

\[ \tau = \frac{\kappa'}{\kappa(C \kappa^2 - 1)^{1/2}}, \]  
(C.2)
where $C$ is a constant. Curves on spheres are conformally equivalent to curves on planes, where $\tau = 0$. Thus one should expect this invariant to vanish along such curves. It is well-known that on a unit sphere, the Frenet torsion is completely determined by the curvature and its first derivative through the relationship (C.2) (see, for example, [29]). On a unit sphere, $C = 1$.

On a sphere, $\tau$ can equivalently be cast in terms of the geodesic curvature, $\tau = \kappa g'/(\kappa^2 + 1)$, where we use the fact that the geodesic curvature $\kappa_g$ is related to $\kappa$ through the elementary relationship, $\kappa^2 = \kappa_g^2 + 1$, between Frenet curvature, geodesic and normal curvatures applied to a sphere.\footnote{This identity implies that the integrated torsion along a curve has the analytical expression in terms of the geodesic curvature, $\int ds \tau = \arctan \kappa_g + c$.}

We thus see that the rhs of Eq. (34) vanishes if and only if the curve lies on a sphere or a plane. Notice that neither $K$ nor $T$ possess a definite sign.

An alternative reorganization of $T$ points to a non-trivial result. Rewrite the middle line in (33):

$$I = \kappa' [\kappa \tau' + \kappa' \tau] - \kappa \tau \kappa'' + (\kappa'^2 + \kappa^2 \tau^2) \tau^2.$$  

(C.3)

This way we can rewrite

$$J = \int ds \frac{\kappa^2 \tau^2}{\kappa^2 + \kappa^2 \tau^2} \left( \frac{\kappa'}{\kappa} \right)' + \int ds \tau,$$

(C.4)

isolating the total torsion $\int ds \tau$ within the definition of $J$ (34). Let $\tan \Phi = \kappa'/\kappa \tau$. Now $J = \int d\Phi + \int ds \tau$, reproducing the well-known result that the total torsion is a conformal invariant mod $2\pi$ [30]. This has interesting consequences in the context of self-similar spirals [18]. The dimensionless variable $\Phi$ also plays an important role in the construction of these spirals.

### Appendix D Euler-Elastica

It is instructive to review the description of the Euler Elastic energy in this framework [23, 25]. Now $H = K^1 K_1/2 = \kappa^2/2$, so that $H_1 = K_1$; $S_{ij}$, of course, vanishes. From Eq.(53), one reads off $F_1 = \kappa^2/2 - \sigma$ (the constant of integration in Eq.(52) does not vanish, because arc-length is fixed). In addition, $F_1' = -DK_1'$, so Eq.(54b) reads $(\kappa^2 \tau)' = 0$, or $\tau = J/\kappa^2$, where $J$ is a constant of integration, identified as the Casimir invariant of the Euclidean group (defined by Eq.(75) in section 8). Modulo this equation, Eqs. (54a) and (c) together imply

$$\kappa' - \kappa^3/2 - J^2/\kappa^3 + \sigma \kappa = 0.$$

(D.1)

Integrating Eq.(D.1) yields the familiar quadrature [12],

$$\kappa'^2 + \left( \frac{1}{2} \kappa^2 - \sigma_0 \right)^2 + \frac{J^2}{\kappa^2} = F^2,$$

(D.2)

with constant $F^2 = F \cdot F$. The conserved tension itself, $F$, is given by

$$F = \left( \frac{1}{2} \kappa^2 - \sigma_0 \right) t - \kappa' N - \frac{J^2}{\kappa^2} B.$$

(D.3)

It vanishes only if $J = 0$, and $\kappa$ is constant. There are no (non-trivial) Euler elastic tension-free states.

### Appendix E Special Conformal Transformations

Transformation of arc-length:

In general,

$$\langle |X|^2 R_X c \rangle' = 2 (X \cdot t) c - 2 (X \cdot c) t - 2 (t \cdot c) X,$$

(E.1)
so that
\[ \delta_c ds = -2 (X \cdot c) \, ds. \]  
(E.2)

To see this, describe arc-length along a curve parametrized by a fixed parameter \( t \) so that \( ds^2 = |\dot{X}| dt^2 \) and \( \delta ds^2 = 2 (\dot{X} \cdot \delta X) dt^2 \), 
(E.3)

where the dot represents differentiation with respect to \( t \). In particular,
\[ \delta_c ds^2 = 2 \dot{t} \cdot (|X|^2 R_X c)' \, ds^2 \]
(E.4)

Transformation of basis vectors:
As a consequence of Eq.(E.2),
\[ \delta_c t = 2 (X \cdot t) c - 2 (t \cdot c) X; \]
(E.5)

Using \( t \cdot \delta_c n^I = -n^I \cdot \delta_c t \), Eq.(82) follows.

Transformation of curvature, its derivative, and the normal connection:
In general, using the definition of \( K^I \) and Eq.(5a), one has
\[ \delta K^I = -n^I \cdot \delta t + K_J n^J \cdot \delta n^I. \]  
(E.6)

Now use Eqs.(5b), (E.2), (E.5) and (82) to express the special conformally transformed acceleration appearing in the first term,
\[ \delta_c t' = 2c - 2K^I (X \cdot n_I) c + 2K^I (n_I \cdot c) X - 2(t \cdot c) t - 2K^I (X \cdot c) n_I. \]  
(E.7)

Using Eqs.(E.7) together with (82) in Eq.(E.6) reproduces Eq.(83).

Note that the contribution proportional to \( V_{IJ} K^J \), defined by Eq.(86) originating in the second term on the RHS of Eq.(E.6) cancels and identical term appearing in the first.

In the same way, it is found that
\[ \delta \omega^{IJ} = n^I \cdot \delta n^J + \delta n^I \cdot n^J = 2(X \cdot c) \omega^{IJ}. \]  
(E.8)

In conclusion,
\[ \delta_c DK^I = 4(X \cdot c) DK^I. \]  
(E.9)

From Eq.(E.9) and (E.2), follow the invariance of conformal arc-length, consistent with the finite transformation (10) under spherical inversion.

References
[1] D.W. Thompson, *On growth and form*, (Cambridge University Press, Cambridge, England, 1942).
[2] Y.A. Kuznetsov and M.S. Plyushchay, *Tachyonless models of relativistic particles with curvature and torsion*, Phys. Lett. B297 49-54 (1992)
[3] T.J. Willmore, *Total Curvature in Riemannian Geometry* (Ellis Horwood, Chichester, 1982)
[4] S. Germain, *Recherches sur la theorie des surfaces elasiques* (V. Courcier, Paris, 1821)

\[7 \delta_c K^I = 2(X \cdot c) DK^I. \]

\[8 \delta_c DK^I DK^J = 8(X \cdot c) DK^I DK^J, \]  
and \( \delta_c \int ds (DK^I DK^J)^{1/4} = 0. \]
[5] U. Seifert, *Conformal transformations of vesicle shapes* J. Phys. A: Math. Gen. 24 L573-L578 (1991)

[6] L. Hsu, R. Kusner and J. Sullivan, *Minimizing the Squared Mean Curvature Integral for Surfaces in Space Forms* Experimental Mathematics 1 191-207 (1992)

[7] F. Julicher, U. Seifert, and R. Lipowsky, *Conformal Degeneracy and Conformal Diffusion of Vesicles* Phys. Rev. Lett. 71 452 (1993)

[8] U. Seifert, *Configurations of fluid membranes and vesicles* Advances in Physics 46, 13-137 (1997)

[9] J. Guven and P. Vázquez-Montejo, *The geometry of fluid membranes: Variational principles, symmetries and conservation laws* in The Role of Mechanics in the Study of Lipid Bilayers, edited by D. J. Steigmann (Springer International Publishing, Cham, 2018) pp. 167-219.

[10] J. Langer and D. A. Singer, *The total squared curvature of closed curves* J. Diff. Geom. 20, 1-22 (1984)

[11] T. A. Ivey and D. A. Singer, *Knot types, homotopies and stability of closed elastic rods* Proc. Lond. Math. Soc. 79 429 (1999)

[12] D. A. Singer, *Lectures on Elastic Curves and Rods*, in Proceedings of Curvature and Variational Modelling in Physics and Biophysics, Edited by O. J. Garay, E. Garcia-Rio and R. Vázquez-Lorenzo, (American Institute of Physics, 2008)

[13] R. Levien, *The elastica: a mathematical history*, Technical Report No. UCB/EECS-2008-103, University of California at Berkeley (2008)

[14] G. Cairns, R. Sharpe, and L. Webb, *Conformal invariants for curves and surfaces in three dimensional space forms*, Rocky Mountain J. Math. 24 933-959 (1994); G. Cairns and R.W. Sharp, *On the inversive differential geometry of plane curves*, Enseign. Math. 36 (1990), 175-196 (1990).

[15] E. Musso, *The conformal arc-length functional* Math. Nachr. 165 107-131 (1994).

[16] M. Bolt *Extremal properties of logarithmic spirals* Beitrage zur Algebra und Geometrie 48 493-520 (2007).

[17] J. Guven and G Manrique *Conformal mechanics of planar curves* arXiv:1905.00488 (2019)

[18] J Guven *Conformal symmetry breaking and self-similar spirals* arXiv:1904.06876 (2019)

[19] M. Magliaro, L. Mari, and M. Rigoli, *On the geometry of curves and conformal geodesics in the Moebius space*, Ann. Global Anal. Geom. 40 133-165 (2011).

[20] Exterior Differential Systems and Euler-Lagrange Partial Differential Equations D. A. Grossman, P. Griffiths, and R. Bryant (Chicago University Press, 2003)

[21] R. Sulanke, *Submanifolds fo the Möbius space II. Frenet Formulas and Curves of Constant Curvatures*, Math. Nachr. 100 235?247 (1981).

[22] J. Guven *Membrane geometry with auxiliary variables and quadratic constraints* J. Phys. A: Math and Gen. 37 L313 (2004).

[23] G. Arreaga, R. Capovilla, and J. Guven *Frenet-Serret Dynamics* Class. Quantum Grav. 18 5065 (2001)

[24] M. Do Carmo, *Differential Geometry of Curves and Surface* (Prentice Hall, Upper Saddle River, 1976)

[25] R. Capovilla, C. Chryssomalakos, and J. Guven *Hamiltonians for curves* J. Phys. A35 6571-6587 (2002)

[26] J. Monterde, *Curves with constant curvature ratios* Boletin de la Sociedad Matemática Mexicana 13 177-186 (2007) arXiv:math/0412323v1

[27] W. Wunderlich Monatshefte für Mathematik 66, 276–289 (1962)

[28] J. Guven, P. Vázquez-Montejo *Force dipoles and stable local defects on fluid vesicles* Physical Review E 87 042710 (1913)
[29] A. Gray Modern Differential Geometry of Curves and Surfaces with Mathematica (Chapman and Hall/CRC; 3rd edition, 2006)

[30] T. Banchoff and J. White The behavior of the total twist and self-linking number of a closed space curve under inversions Math. Scand. 36 254262 (1975)

[31] A. Montesinos Amilibia, M.C. Romero Fuster, and E. Sanabria Codesal, Conformal curvatures of curves in Rn+1 Indag. Mathem. (N.S.) 12 3697382 (2001)

[32] J. Guven Conformally invariant bending energy for hypersurfaces J. Phys. A38 7943-7955 (2005).

[33] CR Graham, N Reichert Higher-dimensional Willmore energies via minimal submanifold asymptotics - arXiv preprint arXiv:1704.03852 2017

[34] M Deserno, MM Müller, J Guven Contact lines for fluid surface adhesion Physical Review E76 (1), 011605

[35] J. Guven, Laplace pressure as a surface stress in fluid vesicles J. Phys. A Math. Gen. 39(14), 3771 (2006).