UNFOLDED SEIBERG–WITTEN FLOER SPECTRA, II: RELATIVE INVARIANTS AND THE GLUING THEOREM

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Abstract. We use the construction of unfolded Seiberg–Witten Floer spectra of general 3-manifolds defined in our previous paper to extend the notion of relative Bauer–Furuta invariants to general 4-manifolds with boundary. One of the main purposes of this paper is to give a detailed proof of the gluing theorem for the relative invariants.

1. Introduction

The Bauer–Furuta invariant, which was introduced in [2], can be regarded a stable homotopy refinement of the Seiberg–Witten invariants [20] for closed 4-manifolds. The invariant takes value in equivariant stable cohomotopy group of spheres and can give interesting applications in 4-manifold theory, such as the 10/8-theorem [7]. On the other hand, the Seiberg–Witten Floer spectrum, which was first introduced by Manolescu for rational homology 3-spheres [13], can be regarded as a stable homotopy refinement of monopole Floer homology [10]. Using this Seiberg–Witten Floer spectrum, Manolescu extended the notion of the Bauer–Furuta invariant to 4-manifolds whose boundary are rational homology spheres. This “relative” invariant takes value in a stable cohomotopy group of the Seiberg–Witten Floer spectrum of the boundary manifold.

In the previous paper [9], we have constructed the “unfolded” version of Seiberg–Witten Floer spectrum for general 3-manifolds. It is then natural to extend Manolescu’s construction of relative Bauer–Furuta invariant to arbitrary 4-manifolds with boundary. Recall that the unfolded spectrum comes with two variations: type-A and type-R. Consequently, the unfolded relative Bauer–Furuta invariant will also come with type-A and type-R variations. Here the letters “A” and “R” stand for the notions “attractor” and “repeller” in dynamical system. The type-A invariant is an object in a category $\mathcal{G}$ of ind-spectra, and the type-R invariant is an object in a category $\mathcal{G}^*$ of pro-spectra.

Definition 1.1. Let $(Y_{\text{in}}, s_{\text{in}})$ and $(Y_{\text{out}}, s_{\text{out}})$ be two oriented (but not necessarily connected) spin$^c$ 3–manifolds. A spin$^c$ cobordism from $(Y_{\text{in}}, s_{\text{in}})$ to $(Y_{\text{out}}, s_{\text{out}})$ is a compact, oriented spin$^c$ 4–manifold $(X, \hat{s})$ together with a fixed diffeomorphism

$$(\partial X, \hat{s}|_{\partial X}) \cong (\overset{\cdot}{-} Y_{\text{in}}, s_{\text{in}}) \sqcup (Y_{\text{out}}, s_{\text{out}})$$
between spin$^c$ 3–manifolds. Here we use the canonical identification between the set of spin$^c$ structures on $Y_{in}$ and the corresponding set for $-Y_{in}$, the orientation reversal of $Y_{in}$. When $Y_{in} = \emptyset$, we also call $(X, \hat{s})$ a spin$^c$ manifold with boundary $(Y_{out}, s_{out})$.

Let $(X, \hat{s})$ be a connected spin$^c$ cobordism from $(Y_{in}, s_{in})$ to $(Y_{out}, s_{out})$. We equip $X$ with a Riemannian metric $\hat{g}$ and a spin$^c$ connection $\hat{A}_0$. Denote the restriction of $\hat{A}_0, \hat{g}$ to $Y_{in}$ (resp. $Y_{out}$) by $A_{in}$ and $g_{in}$ (resp. $A_{out}$ and $g_{out}$). Then the type-A unfolded relative Bauer–Furuta invariant of $X$ is constructed as a morphism in the stable category $\mathcal{S}$

\[
\mathbf{bf}^A(X, \hat{s}; S^1) : \Sigma^-(V_X^+ \oplus V_{in}) T(X, \hat{s}; S^1) \wedge \text{swf}^A(Y_{in}, s_{in}, A_{in}, g_{in}; S^1) \\
\rightarrow \text{swf}^A(Y_{out}, s_{out}, A_{out}, g_{out}; S^1).
\]

Here $\text{swf}^A(Y, s, A, g; S^1)$ is the type-A unfolded Seiberg-Witten Floer spectrum defined in [9, Definition 5.7].

The type-R unfolded relative Bauer–Furuta invariant of $X$ is constructed analogously as a morphism in the stable category $\mathcal{S}^*$

\[
\mathbf{bf}^R(X, \hat{s}; S^1) : \Sigma^-(V_X^+ \oplus V_{out}) T(X, \hat{s}; S^1) \wedge \text{swf}^R(Y_{in}, s_{in}, A_{in}, g_{in}; S^1) \\
\rightarrow \text{swf}^R(Y_{out}, s_{out}, A_{out}, g_{out}; S^1).
\]

Here $\text{swf}^R(Y, s, A, g; S^1)$ is the type-R unfolded Seiberg-Witten Floer spectrum defined in [9, Definition 5.9]. The object $T(X, \hat{s}; S^1)$ is the Thom spectrum of virtual index bundle associated to a family of Dirac operators. See Lemma 5.18 below for the precise definition. We also denote by $V_X^+$ a maximal positive subspace of $\text{Im}(H^2(X, \partial X; \mathbb{R}) \rightarrow H^2(X; \mathbb{R}))$ with respect to the intersection form, $V_{in}$ the cokernel of $\nu^*: H^1(X; \mathbb{R}) \rightarrow H^1(Y_{in}; \mathbb{R})$, and $V_{out}$ similarly. We refer readers to Section 5.4 and Definition 5.21 for more detailed descriptions of these objects.

In Section 5.5 we show that these are invariants of the 4-manifold with boundary in the following sense.

**Theorem 1.2.** As one varies $(\hat{g}, \hat{A}_0)$, both domain and target of $\mathbf{bf}^A(X, \hat{s}; S^1)$ are changed by suspending or desuspending with the same number of copies of $\mathbb{C}$; the morphism $\mathbf{bf}^A(X, \hat{s}; S^1)$ is invariant as a stable homotopy class. The same result holds for $\mathbf{bf}^R(X, \hat{s}; S^1)$. Moreover, when $c_1(s|_Y)$ is torsion, one can construct further normalizations:

\[
\text{BF}^A(X, \hat{s}; S^1) : \Sigma^-(V_X^+ \oplus V_{in}) T(X, \hat{s}; S^1) \wedge \text{SWF}^A(Y_{in}, s_{in}; S^1) \\
\rightarrow \text{SWF}^A(Y_{out}, s_{out}; S^1).
\]

\[
\text{BF}^R(X, \hat{s}; S^1) : \Sigma^-(V_X^+ \oplus V_{out}) T(X, \hat{s}; S^1) \wedge \text{SWF}^R(Y_{in}, s_{in}; S^1) \\
\rightarrow \text{SWF}^R(Y_{out}, s_{out}; S^1),
\]

which are completely independent of metrics and base connections.
See [9, Definition 5.10] for the definitions of the normalized unfolded Seiberg-Witten Floer spectra $\text{SWF}^A(Y, s; S^1)$ and $\text{SWF}^R(Y, s; S^1)$, and the normalized unfolded relative Bauer-Furuta invariants $\text{BF}^A(X, \hat{s}; S^1)$ and $\text{BF}^R(X, \hat{s}; S^1)$ will be defined in Definition 5.23 and Definition 5.24 below.

**Remark 1.3.** First, we emphasize that our unfolded relative invariant is defined over the relative Picard torus

$$\text{Pic}^0(X, Y) \cong \ker(H^1(X; \mathbb{R}) \to H^1(Y; \mathbb{R}))/\ker(H^1(X; \mathbb{Z}) \to H^1(Y; \mathbb{Z})).$$

Secondly, the choice of labeling each boundary component corresponds to which side its unfolded spectrum will appear in the morphism. Essentially, $\text{swf}^A(Y)$ is the Spanier–Whitehead dual of $\text{swf}^R(-Y)$ and $\text{bf}^A(X)$ is the same as $\text{bf}^R(X^\dagger)$ where $X^\dagger: -Y_{\text{out}} \to -Y_{\text{in}}$ is the adjoint cobordism of $X: Y_{\text{in}} \to Y_{\text{out}}$. Finally, both $\text{BF}^A$ and $\text{BF}^R$ agree with Manolescu’s construction when $b_1(Y) = 0$. In this case, we denote $\text{BF}^A = \text{BF}^R$ by BF.

**Example 1.4.** Let us consider the case when the 4-manifold $X$ is $S^2 \times D^2$ or $D^3 \times S^1$ and $\hat{s}_0$ is the unique torsion spin$^c$ structure. Its boundary is $S^2 \times S^1$ with torsion spin$^c$ structure, whose unfolded spectrum is $S^0$ by calculation in [9]. In this case, the type-A invariant $\text{bf}^A(X, \hat{s}_0)$ is a stable homotopy class in $[S^0, S^0]_{S^1} \cong \mathbb{Z}$. By classical Hodge theory and [2, Lemma 3.8], we can conclude that $\text{bf}^A(X, \hat{s}_0) = 1$, the identity element.

On the other hand, $\text{bf}^R(S^2 \times D^2, \hat{s}_0)$ is a stable homotopy class in $[S^{-1}, S^0]_{S^1}$ as $V_{\text{out}} = \mathbb{R}$. The group $[S^{-1}, S^0]_{S^1}$ is trivial, therefore $\text{bf}^R(S^2 \times D^2, \hat{s}_0) = 0$, the trivial element.

One of the main goals of the paper is to prove the gluing theorem for unfolded relative Bauer–Furuta invariants. When decomposing a 4-manifold $X$ to two pieces along a 3-manifold $Y$, the gluing theorem can express the (relative) Bauer–Furuta invariant of $X$ in terms of a “product” of relative invariants of the two pieces. The case when $Y = S^3$ was first proved by Bauer [11] using only invariants of closed 4-manifolds and the positive scalar curvature metric of $S^3$. The case when $Y$ is a homology 3-sphere was proved by Manolescu [13]. Our setup and argument closely follow and generalize those of Manolescu.

Generally, our gluing theorem works when $Y$ is any 3-manifold. Some mild homological assumptions will be made. These conditions are not essential in the sense that they can be removed under more generalized notion of category and unfolded spectrum (see the upcoming remark for more explanation). We now state the gluing theorem which will reappear in Section 6.1 with more details.

**Theorem 1.5.** Let

$$(X_0, \hat{s}_0): (Y_0, s_0) \to (Y_2, s_2), \ (X_1, \hat{s}_1): (Y_1, s_1) \to (-Y_2, s_2)$$

be connected, spin$^c$ cobordisms and

$$(X, \hat{s}): (Y_0, s_0) \sqcup (Y_1, s_1) \to \emptyset$$
be the glued cobordism along $Y_2$. If the following conditions hold

(i) $Y_2$ is connected,
(ii) $b_1(Y_0) = b_1(Y_1) = 0$,
(iii) $\text{im}(H^1(X_0; \mathbb{R}) \to H^1(Y_2; \mathbb{R})) \subset \text{im}(H^1(X_1; \mathbb{R}) \to H^1(Y_2; \mathbb{R}))$,

then, under the natural identification between domains and targets, one has

$$BF(X, \hat{s})|_{\text{Pic}^0(X,Y_2)} = \tilde{\varepsilon}(\text{bf}^A(X_0, \hat{s}_0), \text{bf}^R(X_1, \hat{s}_1)),$$

(1)

where $\tilde{\varepsilon}(\cdot, \cdot)$ is the Spanier-Whitehead duality operation defined in Section 4.5 and the relative Picard torus $\text{Pic}^0(X, Y_2)$ is given by

$$\ker(H^1(X; \mathbb{R}) \to H^1(Y_2; \mathbb{R}))/\ker(H^1(X; \mathbb{Z}) \to H^1(Y_2; \mathbb{Z})).$$

**Remark 1.6.** The main limitation of the unfolded construction is that one can only recover the partial Bauer–Furuta invariant of $X$ on $\text{Pic}^0(X, Y_2)$ rather than on the full Picard torus. Regarding the hypotheses of the theorem,

- Condition (iii) is to avoid dealing with type-A and type-R of $\text{bf}(Y_0)$, $\text{bf}(Y_1)$, and $BF(X)$. If one tries to extend this direction, a category containing more general kinds of diagrams in $C$ will be required.
- Condition (iii) is to control the harmonic action of the relative gauge groups on $Y_2$. Otherwise, a more generalized version of unfolded spectrum such as mixture of type-A and type-R will be needed.

Many classical operations (e.g. log transformation, Fintushel–Stern surgery, fiber sum) in 4-dimensional topology involve gluing and pasting 4-manifolds along 3-manifolds with $b_1 > 0$ (especially the 3-torus). Our gluing theorem provides a tool to study the Bauer–Furuta invariant under these operations. The long term goal is to use this idea to compute the Bauer–Furuta invariant for interesting examples of irreducible 4–manifolds and draw new results beyond the reach of classical Seiberg–Witten invariant. Here, we mention one of the consequences on classical surgery.

**Corollary 1.7.** Let $(X, \hat{s})$ be a smooth, oriented, spin$^c$ 4-manifold with $b_1(\partial X) = 0$ or $\partial X = \emptyset$. Let $\gamma$ be an embedded loop with $[\gamma] \neq 0 \in H_1(X; \mathbb{R})$. If $X'$ is the 4-manifold obtained by surgery along $\gamma$ and $\hat{s}'$ is the unique spin$^c$ structure that is isomorphic to $\hat{s}$ on $X \setminus \gamma$, then we have

$$BF(X', \hat{s}') = BF(X, \hat{s})|_{\text{Pic}^0(X, Y)},$$

where $Y = S^2 \times S^1$ and $H^1(X) \to H^1(Y)$ is induced by inclusion of $\gamma$.

**Proof.** Let $N \cong D^3 \times S^1$ be a tubular neighborhood of $\gamma$ and denote by $Y \cong S^2 \times S^1$ its boundary. Since $\gamma$ is homologically essential, we can apply the gluing theorem for $X = N \cup_Y (X \setminus N)$ and $X' = (S^2 \times D^2) \cup_Y (X \setminus N)$ to obtain

$$BF(X, \hat{s})|_{\text{Pic}^0(X, Y)} = \tilde{\varepsilon}(\text{bf}^A(D^3 \times S^1, \hat{s}_0), \text{bf}^R(X \setminus N, \hat{s}|_{X \setminus N}))$$

and

$$BF(X', \hat{s})|_{\text{Pic}^0(X', Y)} = \tilde{\varepsilon}(\text{bf}^A(S^2 \times D^2, \hat{s}'_0), \text{bf}^R(X \setminus N, \hat{s}|_{X \setminus N})).$$
Here $\hat{s}_0$ and $\hat{s}_0'$ are torsion spin$^c$ structures. By Example 1.4 we have
\[ \text{bf}^A(D^3 \times S^1, \hat{s}_0) = \text{bf}^A(S^2 \times D^2, \hat{s}_0') = 1, \]
so $\text{BF}(X, \hat{s})|_{\text{Pic}^0(X,Y)} = \text{BF}(X', \hat{s}')|_{\text{Pic}^0(X',Y)}$. In addition, one can check that $\text{Pic}^0(X',Y) = \text{Pic}^0(X',Y)$ and we recover the full Bauer–Furuta invariant on $(X', \hat{s}')$.

□

Remark 1.8. Recently, there has been significant amount of attentions [21, 16, 11, 19] on using Floer homology to study ribbon cobordance and ribbon homology cobordisms (homology cobordisms with only 1-handles and 2-handles). In particular, Daemi-Lidman-Vela-Vick-Wong [5] proved that the Heegaard Floer homology $\hat{HF}(Y_1)$ is a summand of $\hat{HF}(Y_2)$ if there is a ribbon homology cobordism from $Y_1$ to $Y_2$. A central tool in their proof is a surgery formula that describes cobordism maps under surgery along essential loops. We expect that our Corollary 1.7 can be useful in proving a parallel result for Manolescu’s Seiberg-Witten Floer spectrum.

As another consequence, we can give a vanishing result for the relative Bauer-Furuta invariant. The case of closed manifolds was proved by Frøyshov [6, Theorem 1.1.1]. See also [17, Proposition 4.6.5] for an analogous result for the Seiberg–Witten invariant.

Corollary 1.9. Let $X$ be a smooth, oriented, 4-manifold containing an embedded 2-sphere $S$ which has trivial self-intersection and is homologically essential (i.e., $[S] \neq 0 \in H_2(X; \mathbb{R})$). Suppose either $X$ is closed or $b_1(\partial X) = 0$. Then its (relative) Bauer–Furuta invariant for any spin$^c$ structure is trivial.

Proof. We focus on the case $\partial X \neq \emptyset$. The closed case is similar but simpler. Since its self-intersection is trivial, the sphere $S$ has a neighborhood diffeomorphic to $S^2 \times D^2$, denoted by $X_1$. We then have a decomposition $X = X_0 \cup Y_0 \cup X_1$ where $X_0$ is $X \setminus X_1$ and $Y_0 \cong S^2 \times S^1$.

For a spin$^c$ structure $\hat{s}$ on $X$, there are two possibilities:

Suppose $\langle c_1(\hat{s}), [S] \rangle \neq 0$. Then the conclusion follows from standard neck-stretching argument: Let $g_0$ be a positive scalar curvature metric on $Y_0$. We equip $X$ with a Riemannian metric $\hat{g}$ whose restriction to a collar neighborhood of $Y_0$ equals the product metric $[-1,1] \times g_0$. By stretching the neck of $Y_0$, we get a family of Riemannian metrics $\{\hat{g}_T \mid T \geq 1\}$ on $X$. Since the 3-dimensional Seiberg-Witten equations on $(Y_0, \hat{s}|_{Y_0})$ has no solution, there are no finite type $X$-trajectories for the metric $g_T$ when $T$ is large enough (see Definition 5.10). By the converging result Lemma 5.13 there are no $(n, \epsilon)$-approximated $X$-trajectory of length $L$ when $n, L$ are large and $\epsilon$ is small (see Definition 5.13). Therefore, by construction of the relative Bauer-Furuta invariant [13, Page 918] (see also [30]), we see that it is vanishing in this case.
Suppose \(\langle c_1(\hat{s}), [S]\rangle = 0\). Then the conclusion follows from our gluing theorem: Since \(S\) is essential, the condition
\[
\text{im}(H^1(X_0; \mathbb{R}) \to H^1(Y; \mathbb{R})) \subset \text{im}(H^1(X_1; \mathbb{R}) \to H^1(Y; \mathbb{R}))
\]
holds because both images are zero. From the Mayer–Vietoris sequence, we have \(\text{Pic}^0(X, Y) = \text{Pic}^0(X)\). We now apply the gluing theorem
\[
\text{BF}(X; \hat{s}) = \tilde{\epsilon}(\text{bf}^A(X_0, \hat{s}|X_0), \text{bf}^R(X_1, \hat{s}, \hat{s}|X_1)).
\]
Since the type-R relative Bauer–Furuta invariant \(\text{bf}^R(S^2 \times D^2, \hat{s}|X_0) = 0\) by Example 1.4, we conclude that \(\text{BF}(X, \hat{s})\) is trivial.

\[\Box\]

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## 2. Summary of constructions and proofs

Most of required background in Conley theory is contained in Section 3. Background for our stable categories and Spanier–Whitehead duality is contained in Section 4. We summarize the major constructions here.

### 2.1. Unfolded Seiberg–Witten Floer spectra

Here we will recall the construction and definition of the unfolded Seiberg–Witten Floer spectrum [9]. Let \(Y\) be a closed spin\(^c\) 3-manifold (not necessarily connected) with a spinor bundle \(S_Y\).

We always work on a Coulomb slice \(\text{Coul}(Y) = \{(a, \phi) \in i\Omega^1(Y) \oplus \Gamma(S_Y) \mid d^*a = 0\}\) with Sobolev completion. With a basepoint chosen on each connected component, we identify the residual gauge group with the based harmonic gauge group \(G_{Y, \phi}^h \approx H^1(Y; \mathbb{Z})\) acting on \(\text{Coul}(Y)\). We consider a strip of balls in \(\text{Coul}(Y)\) translated by this action
\[
\text{Str}(R) = \{x \in \text{Coul}(Y) \mid \exists h \in G_{Y, \phi}^h \text{ s.t. } \|h \cdot x\|_{L^2_k} \leq R\}. \tag{2}
\]

Recall from [9] Definition 3.1] that a Seiberg-Witten trajectory is called “finite type” if it is contained in a bounded region of \(\text{Coul}(Y)\) in the \(L^2_k\)-norm. The boundedness result for 3-manifolds [9] Theorem 3.2] states that all finite-type Seiberg-Witten trajectories are contained in \(\text{Str}(R)\) for \(R\) sufficiently large.

The basic idea of unfolded construction is to consider increasing sequences of bounded regions in the Coulomb slice. To do this, we choose a basis for \(H^1(Y; \mathbb{R})\) and use it to identify \(i\Omega^h(Y)\), the space of imaginary valued harmonic 1-forms, with \(\mathbb{R}^{b_1(Y)}\). Under this isomorphism, we let
\[
p_H = (p_{H, 1}, \cdots, p_{H, b_1(Y)}) : \text{Coul}(Y) \to \mathbb{R}^{b_1(Y)}
\]
be the $L^2$-orthogonal projection. Let $\tilde{g} : \mathbb{R} \to \mathbb{R}$ be a certain “step function” with small derivative (see [9, Figure 1]). We consider the function

$$
g_{j, \pm} = \tilde{g} \circ p_{H,j} \pm \mathcal{L} : \text{Coul}(Y) \to \mathbb{R}$$

for $1 \leq j \leq b_1(Y)$. Here $\mathcal{L}$ denotes the balanced-perturbed Chern-Simons-Dirac functional. These functions $g_{j, \pm}$ are constructed in such a way that for any real number $\theta$ and any integer $m$, the region $J_{m, \pm} := \text{Str}(\tilde{R}) \cap \bigcap_{1 \leq j \leq b_1} (-\infty, \theta + m]$

is bounded. We pick a sequence of finite-dimensional subspaces $V_{\lambda_n}^\mu$ coming from eigenspaces of the operator $(\ast d, /D)$ and define $J_{m, \pm} := J_{m, \mp} \cap V_{\lambda_n}^\mu$.

The main point is that when we choose a generic $\theta$, the region $J_{m, \pm}$ becomes an isolating neighborhood with respect to the approximated Seiberg–Witten flow $\varphi^*_n$ on $V_{\lambda_n}^\mu$ when $n$ is large relative to $m$. This is essentially because the perturbations we add on $\pm \mathcal{L}$ have small derivatives. We can now define desuspended Conley indices

$$I_{m, \pm}^+ = \Sigma^{-V_{\lambda_n}^0} I(\text{inv}(J_{m, \pm}^+), \varphi_n),$$

$$I_{m, \pm}^- = \Sigma^{-V_{\lambda_n}^0} I(\text{inv}(J_{m, \pm}^-), \varphi_n)$$

as objects in the stable category $\mathcal{C}$ (see Section 4). Here $V_{\lambda_n}^0$ is the orthogonal complement of the space of harmonic 1-forms in $V_{\lambda_n}^\mu$. Note that these objects do not depend on $n$ up to canonical isomorphism of the form $\bar{\rho}_{n, \pm} : I_{m, \pm}(Y) \to I_{m+1, \pm}(Y)$.

The unfolded Seiberg-Witten Floer spectra are represented by direct and inverse systems in the stable category $\mathcal{C}$ as follows

$$\text{swf}^A(Y) : I_1^+ \xrightarrow{\bar{\rho}_1^+} I_2^+ \xrightarrow{\bar{\rho}_2^+} \cdots$$

$$\text{swf}^R(Y) : I_1^- \xleftarrow{\bar{\rho}_1^-} I_2^- \xleftarrow{\bar{\rho}_2^-} \cdots$$

Connecting morphisms in the diagram for $\text{swf}^A(Y)$ are induced by attractor relation while morphisms in $\text{swf}^R(Y)$ are induced by repeller relation. More precisely, we have morphisms between desuspended Conley indices

$$\bar{\iota}_{m, \pm}^+ : I_{m, \pm}^+(Y) \to I_{m+1, \pm}^+(Y)$$

$$\bar{\iota}_{m-1, \pm}^- : I_{m-1, \pm}^-(Y) \to I_{m-1, \pm}^-(Y).$$

Then, the morphisms $\bar{j}_m, \bar{\iota}_m$ in (3) are given by composition of $\bar{\rho}_{n, \pm}$’s and $\bar{\iota}_{n, \pm}$ appropriately.

2.2. Unfolded Relative Bauer–Furuta invariants. Let $X$ be a compact, connected, oriented, Riemannian 4–manifold with boundary $Y = -Y_{\text{in}} \sqcup Y_{\text{out}}$. To define the invariant, we pick auxiliary homological data which corresponds to a choice of basis of $H^1(X; \mathbb{R})$ and keeps track of both kernel and image of $\iota^* : H^1(X; \mathbb{R}) \to H^1(Y; \mathbb{R})$ (see the list at beginning of Section 5.1).
In this construction, we use the double Coulomb slice $\text{Coul}^{CC}(X)$ as a gauge fixing. The main idea is to find suitable finite-dimensional approximations for the Seiberg–Witten map together with the restriction map

$$(SW,r): \text{Coul}^{CC}(X) \to L^2_{k-1/2}(i\Omega^2_+(X) \oplus \Gamma(S^2_S)) \oplus \text{Coul}(Y).$$

Note that there is an action of $H^1(X;\mathbb{Z})$ on both sides with restriction on $\text{Coul}(Y)$. Compactness of solutions can only be achieved modulo this action. However, the construction of the unfolded spectra does not behave well under the action of $H^1(X;\mathbb{Z})$ on $\text{Coul}(Y)$. This is essentially the reason why we can define the unfolded relative invariant only on the relative Picard torus induced from $\ker \iota^*$. As one can see in the basic boundedness result (Theorem 5.11), we need a priori bound on the $\text{im} \ i^*$-part quantified by the projection $\hat{p}_{\beta}$. We will focus on type-A relative invariant $bf^A(X)$. Although it is formulated as a morphism from $\text{swf}^A(Y_{in})$ to $\text{swf}^A(Y_{out})$, the main part of the construction is to obtain maps of the form

$$B(W_{n,\beta})/S(W_{n,\beta})$$

$$\to (B(U_n)/S(U_n)) \wedge I(\text{inv}(J_{m_0}^-(-Y_{in}))) \wedge I(\text{inv}(J_{m_1}^+(Y_{out}))). \quad (4)$$

The left hand side is the Thom space of a finite-dimensional subbundle $W_{n,\beta}$ of the Hilbert bundle

$$W_X = \text{Coul}^{CC}(X)/\ker(H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})).$$

while $B(U_n)/S(U_n)$ is a sphere. We point out that the right hand side is intuitively $\text{swf}^R(-Y_{in}) \wedge \text{swf}^A(Y_{out})$, which may be viewed as a ‘mixed’-type unfolded spectrum of $Y$. It is possible to formally consider this in a larger category containing both $\mathcal{S}$ and $\mathcal{S}^*$, but we will not pursue this in this paper. Another remark is that $W_{n,\beta}$ has extra constraint $\hat{p}_{\beta,\text{out}} = 0$ to control the $\text{im} \ i^*$-part mentioned earlier. The reason we only need the part on $Y_{out}$ is because we start with a fixed $m_0$ and then choose sufficiently large $m_1$. The order of dependency of parameters is established at the beginning of Section 5.4.

A notion of pre-index pair (see Section 3.2) is also required to define the map (4). This part closely resembles original Manolescu’s construction [13] in the case $b_1(Y) = 0$. The last step to apply Spanier–Whitehead duality (see Section 4.3) between $\text{swf}^R(-Y_{in})$ and $\text{swf}^A(Y_{in})$ and define the relative invariant as a morphism in $\mathcal{S}$.

2.3. The Gluing theorem. Let $X_0: Y_0 \to Y_2$ and $X_1: Y_1 \to -Y_2$ be connected, oriented cobordisms. We consider the composite cobordism $X = X_0 \sqcup Y_2 X_1$ glued along $Y_2$ from $Y_0 \sqcup Y_1$ to the empty manifold.

The main technical difficulty of the proof of the gluing theorem is that two different kinds of index pairs arise in the construction. On one hand, to define the relative invariant, we require an index pair $(N_1, N_2)$ to contain a certain pre-index pair $(K_1, K_2)$. On the other hand, we need a manifold
isolating block when dealing with duality morphisms. In general, a canonical homotopy equivalence between index pairs can be given by flow maps (Theorem 3.4), but the formula can sometimes be inconvenient to work with and the common squeeze time $T$ can be arbitrary.

This is the reason we introduce the concept of $T$-tameness, which is a quantitative refinement of notions in Conley theory (see Section 3.2 and 3.4). The flow maps from $T$-tame index pairs can be simplified (Lemma 3.13). Most boundedness results in this paper are stated for trajectories with finite length. As a result, the time parameter $T$, which also corresponds to the length of a cylinder, has a uniform bound during the construction.

The proof of the gluing theorem can be divided into two major parts. The first part, contained in Section 6.2, involves simplifying the flow maps and duality morphisms. We carefully set up all the parameters needed to explicitly write down $\tilde{\epsilon}(\mathcal{F}^A(\mathcal{X}_0), \mathcal{F}^B(\mathcal{X}_1))$. For instance, we can represent Conley index part of the map as a composition of smash product of flow maps and Spanier–Whitehead duality map

$$\tilde{\epsilon}(t_0, t_1): K_0/S_0 \wedge K_1/S_1 \to \tilde{N}_0/\tilde{N}_0^+ \wedge \tilde{N}_1/\tilde{N}_1^+ \wedge B^+(V_2^0, \epsilon)$$

given by formula (44). (Here $V_2^0$ is a finite dimensional subspace of $\text{Coul}(Y)$ coming from eigenspaces of $(d^*, \mathcal{D})$.) After two steps, we deform the formula to the one given in Proposition 6.10.

The second part of the proof of the gluing theorem, contained in Section 6.4, is to deform Seiberg–Witten maps on $X_0$ and $X_1$ to the Seiberg–Witten map on $X$. Many of the arguments here will be similar to Manolescu’s proof [14] when $b_1(Y) = 0$. The crucial part is to deform gauge fixing with boundary conditions and harmonic gauge groups on $X_0$ and $X_1$ to those on $X$. For clarity, we subdivide the deformation to seven steps. A recurring technique is to move between maps and conditions on the domain (Lemma 6.13). Other ingredients such as stably $c$-homotopic pairs are contained in Section 6.3.

3. Conley Index

In this section, we recall basic facts regarding the Conley index theory and develop some further properties we need. Without any modification, all the results and constructions of this section can be adapted to the $G$-equivariant theory, when $G$ is a compact Lie group. See [31] and [18] for more details.

3.1. Conley theory: definition and basic properties. Let $\Omega$ be a finite dimensional manifold and $\varphi$ be a smooth flow on $\Omega$, i.e. a $C^\infty$-map $\varphi: \Omega \times \mathbb{R} \to \Omega$ such that $\varphi(x, 0) = x$ and $\varphi(x, s + t) = \varphi(\varphi(x, s), t)$ for any $x \in \Omega$ and $s, t \in \mathbb{R}$. We often denote by $\varphi(x, I) := \{\varphi(x, t) \mid t \in I\}$ for a subset $I \subset \mathbb{R}$.

Definition 3.1. Let $A$ be a compact subset of $\Omega$.

1. The maximal invariant subset of $A$ is given by $\text{inv}(\varphi, A) := \{x \in A \mid \varphi(x, \mathbb{R}) \subset A\}$. We simply write $\text{inv}(A)$ when the flow is clear from the context.
(2) A is called an *isolating neighborhood* if \( \text{inv}(A) \) is contained in the interior \( \text{int}(A) \).

(3) A compact subset \( S \) of \( \Omega \) is called an *isolated invariant set* if there is an isolating neighborhood \( \tilde{A} \) such that \( \text{inv}(\tilde{A}) = S \). In this situation, we also say that \( \tilde{A} \) is an isolating neighborhood of \( S \).

A central idea in Conley index theory is a notion of index pairs.

**Definition 3.2.** For an isolated invariant set \( S \), a pair \((N,L)\) of compact sets \( L \subset N \) is called an *index pair* of \( S \) if the following conditions hold:

(i) \( \text{inv}(N \setminus L) = S \subset \text{int}(N \setminus L) \);

(ii) \( L \) is an exit set for \( N \), i.e. for any \( x \in N \) and \( t > 0 \) such that \( \varphi(x,t) \notin N \), there exists \( \tau \in [0,t) \) with \( \varphi(x,\tau) \in L \);

(iii) \( L \) is positively invariant in \( N \), i.e. if \( x \in L \), \( t > 0 \), and \( \varphi(x,\{0,t\}) \subset N \), then we have \( \varphi(x,\{0,t\}) \subset L \).

We state two fundamental facts regarding index pairs:

- For an isolated invariant set \( S \) with an isolating neighborhood \( A \), there always exists an index pair \((N,L)\) of \( S \) such that \( L \subset N \subset A \).
- For any two index pairs \((N,L)\) and \((N',L')\) of \( S \), there is a natural homotopy equivalence \( N/L \to N'/L' \).

These lead to definition of the Conley index.

**Definition 3.3.** Given an isolated invariant set \( S \) of a flow \( \varphi \) with an index pair \((N,L)\), we denote by \( I(\varphi,S,N,L) \) the space \( N/L \) with \([L] \) as the base-point. The *Conley index* \( I(\varphi,S) \) can be defined as a collection of pointed spaces \( I(\varphi,S,N,L) \) together with natural homotopy equivalences between them. We sometimes write \( I(S) \) when the flow is clear from the context.

Given two index pairs, a canonical homotopy equivalence between them was constructed by Salamon \[18\].

**Theorem 3.4** \[18\] (Lemma 4.7). If \((N,L)\) and \((N',L')\) are two index pairs for the same isolated invariant set \( S \), then there exists \( T > 0 \) such that

- \( \varphi(x,[-T,T]) \subset N' \setminus L' \) implies \( x \in N \setminus L \);
- \( \varphi(x,[-T,T]) \subset N \setminus L \) implies \( x \in N' \setminus L' \).

Moreover, for any \( T \geq \bar{T} \), the map \( s_{T,(N,L),(N',L')} : N/L \to N'/L' \) given by

\[
\begin{align*}
 s_{T,(N,L),(N',L')}([x]) := \\
 & \begin{cases} 
 [\varphi(x,3T)] & \text{if } \varphi(x,\{0,2T\}) \subset N \setminus L \text{ and } \varphi(x,\{T,3T\}) \subset N' \setminus L' \\
 [L'] & \text{otherwise}
\end{cases}
\end{align*}
\]

is well-defined and continuous. The maps \( s_{T,(N,L),(N',L')} \) are all homotopic to each other for different \( T \geq \bar{T} \) and they give an isomorphism between \( N/L \) and \( N'/L' \) in the homotopy category of pointed spaces. These isomorphisms satisfy the following properties

- For any index pair \((N,L)\), the map \( s_{T,(N,L),(N,L)} \) is homotopic to the identity map on \( N/L \);
• For any index pairs \((N, L), (N', L')\) and \((N'', L'')\), the composition
\[ s_{T,(N',L'),(N'',L'')} \circ s_{T,(N,L),(N',L')} : N/L \to N''/L'' \]

is homotopic to \(s_{T,(N,L),(N',L')}\).

We call \(s_{T,(N,L),(N',L')}\) the flow map at time \(T\). We sometimes also write \(s_T\) when the index pairs are clear from the context.

Through the rest of the paper, we will be always working in the homotopy category when talking about maps between Conley indices. Namely, all maps should be understood as homotopy equivalent classes of maps and all commutative diagrams only hold up to homotopy. More precisely, a map \(f : I(\varphi_1, S_1) \to I(\varphi_2, S_2)\) between two Conley indices mean a collection of maps (in the homotopy category)
\[ \{f_{(N_1,L_1),(N_2,L_2)} : I(\varphi_1, S_1, N_1, L_1) \to I(\varphi_2, S_2, N_2, L_2)\}, \]
from any representative of \(I(\varphi_1, S_1)\) to any representative of \(I(\varphi_2, S_2)\), such that the following diagram commutes up to homotopy:
\[
\begin{array}{ccc}
I(\varphi_1, S_1, N_1, L_1) & \xrightarrow{f_{(N_1,L_1),(N_2,L_2)}} & I(\varphi_2, S_2, N_2, L_2) \\
\downarrow s_{T,(N_1,L_1),(N_1',L_1')} & & \downarrow s_{T,(N_2,L_2),(N_2',L_2')} \\
I(\varphi_1, S_1, N_1', L_1') & \xrightarrow{f_{(N_1',L_1'),(N_2',L_2')}} & I(\varphi_2, S_2, N_2', L_2')
\end{array}
\]
In the language of \([18]\), this collection of maps gives a morphism between two connected simple systems \(I(\varphi_1, S_1)\) and \(I(\varphi_2, S_2)\). Note that to define such a collection \(\{f_{\ast \ast}\}\), we only need to specify a single map
\[ f_{(N_1,L_1),(N_2,L_2)} : I(\varphi_1, S_1, N_1, L_1) \to I(\varphi_2, S_2, N_2, L_2) \]
for specific choices of \((N_1, L_1)\) and \((N_2, L_2)\). This is because all the other maps can be obtained by composing it with flow maps.

Next, we consider a situation when an isolated invariant set can be decomposed to smaller isolated invariant sets.

**Definition 3.5.**

1. For a subset \(A\), we define the \(\alpha\)-limit set and respectively \(\omega\)-limit set as following
\[ \alpha(A) = \bigcap_{t<0} \varphi(A, (-\infty, t]) \quad \text{and} \quad \omega(A) = \bigcap_{t>0} \varphi(A, [t, \infty)). \]

2. Let \(S\) be an isolated invariant set. A compact subset \(T \subset S\) is called an attractor (resp. repeller) if there exists a neighborhood \(U\) of \(T\) in \(S\) such that \(\omega(U) = T\) (resp. \(\alpha(U) = T\)).

3. When \(T\) is an attractor in \(S\), we define the set \(T^* := \{x \in S \mid \omega(x) \cap T = \emptyset\}\), which is a repeller in \(S\). We call \((T, T^*)\) an attractor-repeller pair in \(S\).

Note that an attractor and a repeller are isolated invariant sets. We state an important result relating Conley indices of an attractor-repeller pair.
Proposition 3.6 ([18] Theorem 5.7]). Let $S$ be an isolated invariant set with an isolating neighborhood $A$ and $(T, T^*)$ be an attractor-repeller pair in $S$. Then there exist compact sets $\tilde{N}_3 \subset \tilde{N}_2 \subset \tilde{N}_1 \subset A$ such that the pairs $(\tilde{N}_2, \tilde{N}_3), (\tilde{N}_1, \tilde{N}_3), (\tilde{N}_1, \tilde{N}_2)$ are index pairs for $T, S$ and $T^*$ respectively. The maps induced by inclusions give a natural coexact sequence of Conley indices

\[
I(\varphi, T) \xrightarrow{i} I(\varphi, S) \xrightarrow{r} I(\varphi, T^*) \rightarrow \Sigma I(\varphi, T) \rightarrow \Sigma I(\varphi, S) \rightarrow \cdots .
\]

We call the triple $(\tilde{N}_3, \tilde{N}_2, \tilde{N}_1)$ an index triple for the pair $(T, T^*)$ and call the maps $i$ and $r$ the attractor map and the repeller map respectively.

3.2. $T$-tame pre-index pair and $T$-tame index pair. For a set $A$ and $I \subset \mathbb{R}$, let us denote

\[
A^I := \{x \in \Omega \mid \varphi(x, I) \subset A\}.
\]

We also write $A^{[0, \infty]}$ and $A^{[-\infty, 0]}$ as $A^+$ and $A^-$ respectively. The following notion of pre-index pair was introduced by Manolescu [13].

Definition 3.7. A pair $(K_1, K_2)$ of compact subsets of an isolating neighborhood $A$ is called a pre-index pair in $A$ if

(i) For any $x \in K_1 \cap A^+$, we have $\varphi(x, (0, \infty)) \subset \text{int}(A)$;

(ii) $K_2 \cap A^+ = \emptyset$.

We have two basic results regarding pre-index pairs.

Theorem 3.8 ([13] Theorem 4]). For any pre-index pair $(K_1, K_2)$ in an isolating neighborhood $A$, there exists an index pair $(N, L)$ satisfying

\[
K_1 \subset N \subset A, \ K_2 \subset L.
\]

Theorem 3.9 ([8] Proposition A.5]). Let $(K_1, K_2)$ be a pre-index pair and $(N_1, L_1), (N_2, L_2)$ be two index pairs containing $(K_1, K_2)$. Denote by $\iota_j: K_1/K_2 \rightarrow N_j/L_j$ the map induced by inclusion. Let $s_T: N_1/L_1 \rightarrow N_2/L_2$ be the flow map for some large $T$. Then, the composition $s_T \circ \iota_1$ is homotopic to $\iota_2$.

Consequently, when $(K_1, K_2)$ is a pre-index pair in an isolating neighborhood $A$, we have a canonical map to the Conley index

\[
\iota: K_1/K_2 \rightarrow I(S),
\]

where $S = \text{inv}(A)$ and the map is induced by inclusion.

Next, we discuss the quantitative refinement of Theorem [8,8] which will be useful in our formulation of relative Bauer–Furuta invariant and the gluing theorem. Let us consider the following definition.

Definition 3.10. Let $A$ be an isolating neighborhood. For a positive real number $T$, a pair $(K_1, K_2)$ of compact subsets of $A$ is called a $T$-tame pre-index pair in $A$ if it satisfies the following conditions:

(i) There exists a compact set $A' \subset \text{int}(A)$ containing $A^{[-T, T]}$ such that, if $x \in K_1 \cap A^{[0, T']}$ for some $T' \geq T$, then $\varphi(x, [0, T' - T]) \subset A'$.
Definition 3.12. For a positive real number \( t \), an index pair \((N, L)\) contained in an isolating neighborhood \( A \) is called a **T-tame index pair** in \( A \) if it satisfies the following conditions:

(i) Both \( N \) and \( L \) are positively invariant in \( A \);
(ii) \( A^{[-T, T]} \subset N \);
(iii) \( A^{[0, T]} \cap L = \emptyset \).

A subset \( A \) is also called a **T-tame isolating neighborhood** if \( A^{[-T, T]} \subset \text{int}(A) \).

One important reason why we are interested in \( T \)-tame index pairs is that the definition of the flow maps can be simplified when one of the index pairs is \( T \)-tame.

Lemma 3.13. Let \((N, L)\) and \((N', L')\) be two index pairs in an isolating neighborhood \( A \). Let \( T \) be a sufficiently large number so that the flow map \( s_T: N/L \rightarrow N'/L' \) is well-defined. If the index pair \((N, L)\) is \( T \)-tame, then flow map \( s_T \) can be given by a formula

\[
s_T([x]) = \begin{cases} 
\varphi(x, 3T) & \text{if } \varphi(x, [0, 3T]) \subset A \text{ and } \varphi(x, [T, 3T]) \subset N' \setminus L', \\
[L'] & \text{otherwise}.
\end{cases}
\]

Proof. We only need to show that the following two conditions are equivalent for \( x \in N \):

(ii) \( K_2 \cap A^{[0, T]} = \emptyset \).

It is straightforward to see that a \( T \)-tame pre-index pair in \( A \) is pre-index pair in \( A \). The converse also holds.

Lemma 3.11. Let \((K_1, K_2)\) be a pre-index pair in an isolating neighborhood \( A \). Then, there exists \( \bar{T} > 0 \) such that \((K_1, K_2)\) is a \( T \)-tame pre-index pair in \( A \) for any \( T \geq \bar{T} \).

Proof. It is straightforward to see that \( K_2 \cap A^{[0, +\infty)} = \emptyset \) implies \( K_2 \cap A^{[0, T]} = \emptyset \) for a sufficiently large \( T > 0 \). We are left with checking that condition (ii) of Definition 3.10 holds for a sufficiently large \( T > 0 \).

Suppose that the condition does not hold for \( T \) such that \( T_j \to \infty \). Then we can find sequences \( \{x_{j, k}\} \) and \( \{T_{j, k}'\} \) where \( x \in K_1 \cap A^{[0, T_j]} \) and \( 0 \leq T_{j, k}' \leq T_{j, k} - T_j \) such that \( \varphi(x_{j, k}, T_{j, k}') \to y \in \partial A \) as \( k \to \infty \). Now assume that there is a sequence of such \( \{T_j\} \) with \( T_j \to \infty \). Passing to a subsequence, one can find a sequence \( \{k_j\} \) such that \( x_{j, k_j} \to x_\infty \in K_1 \cap A^+ \) and \( \varphi(x_{j, k_j}, T_{j, k_j}') \to y \in \partial A \). If \( T_{j, k}' \to T'' \), we see that \( \varphi(x_\infty, T'') = y \). This contradicts with definition of the pre-index pair \((K_1, K_2)\). On the other hand, we observe that \( \varphi(x_{j, k_j}, T_{j, k_j}'') \in A^{[-T_{j, k_j}', T]} \). If \( \{T_{j, k_j}'\} \) goes to infinity, we obtain that \( y \in \text{inv}(A) \). This is a contradiction because \( A \) is an isolating neighborhood, i.e. \( \text{inv}(A) \cap \partial A = \emptyset \).

\( \square \)

We next consider the \( T \)-tame version of index pairs.
Let us suppose that

\[ \varphi(x, [0, 3T]) \subset A \quad \text{and} \quad \varphi(x, [T, 3T]) \subset N' \setminus L'; \]

(2) \[ \varphi(x, [0, 2T]) \subset N \setminus L \quad \text{and} \quad \varphi(x, [T, 3T]) \subset N' \setminus L'. \]

It is straightforward to see that (2) implies (1) since \( N \subset A \) and \( N' \subset A \).

Let us suppose that \( \varphi(x, [0, 3T]) \subset A \). Since \( N \) is positively invariant in \( A \), we have \( \varphi(x, [0, 3T]) \subset N \). By property of \( T \)-tame index pair, we have \( \varphi(x, [0, 2T]) \cap L = \emptyset \) and we are done.

We now show a quantitative refinement of [13, Theorem 4].

**Theorem 3.14.** For any \( T > 1 \), let \( A \) be a \((T - 1)\)-tame isolating neighborhood and \( (K_1, K_2) \) be a \((T - 1)\)-tame pre-index pair in \( A \). Then, there exists a \( T \)-tame index pair in \( A \) which contains \( (K_1, K_2) \).

**Proof.** The proof is an adaption of the arguments in [13] to the \( T \)-tame setting. Denote by \( \tilde{K}_1 = K_1 \cup A^{-T,T} \). We claim that \( (\tilde{K}_1, K_2) \) is a pre-index pair in \( A \). Since \( (K_1, K_2) \) is already a pre-index pair in \( A \), it suffices to check that \( \varphi(y, [0, \infty)) \subset \text{int}(A) \) for any \( y \in A^{-T,T} \cap A^+ = A^{-T,\infty} \).

This is straightforward since \( A \) is \((T - 1)\)-tame.

By Theorem 3.8 there exists an index pair containing \( (\tilde{K}_1, K_2) \). From the argument of the proof, one could pick a compact subset \( C \subset A \) and as well as an open neighborhood \( V \) of \( C \) such that the following conditions hold:

(I) \( C \) is a compact neighborhood of \( A^+ \cap \partial A \) in \( A \);

(II) \( C \cap A^- = \emptyset \);

(III) \( C \cap P_A(\tilde{K}_1) = \emptyset \);

(IV) \( V \) is an open neighborhood of \( A^+ \) in \( A \);

(V) \( \overline{V \setminus C} \subset \text{int}(A) \);

(VI) \( K_2 \cap V = \emptyset \).

Recall that, for a subset \( K \) of \( A \), the set \( P_A(K) \) is given by

\[ P_A(K) := \{ \varphi(x, t) \mid x \in K, \ t \geq 0 \ \text{and} \ \varphi(x, [0, t]) \subset A \}. \]

Let us say that a pair \( (C, V) \) is good if it satisfies all of the above conditions.

After specifying a good pair \( (C, V) \), a compact subset \( B \) can be chosen so that

\[ (N, L) = (P_A(B) \cup P_A(A \setminus V), P_A(A \setminus V)) \]

is an index pair containing \( (\tilde{K}_1, K_2) \).

Our strategy is to carefully choose a good pair \( (C, V) \) so that the induced \( (N, L) \) is a \( T \)-tame index pair in \( A \) which contains \((K_1, K_2)\).

Since \( (K_1, K_2) \) is a \( T \)-tame pre-index pair (as \( T > T - 1 \)), we can take a compact set \( A' \) in \( A \) satisfying condition (1) of Definition 3.10. Fix a compact set \( A'' \) in \( A \) such that

\[ A' \subset \text{int}(A''), \ A'' \subset \text{int}(A) \]

and pick a real number \( T' \in (T - 1, T) \). Consider a pair

\[ (C_0, V_0) = ((A \setminus \text{int}(A'')) \cap A^{[0,T']}, A^{[0,T']}) \]
Note that $V_0$ is closed. We have the following observations:

- $A^+ \cap \partial A \subset C_0$: This is obvious as $A'' \subset \text{int}(A)$ and $A^+ \subset A^{[0,T']}$. 
- The distance between $C_0$ and $A^-$ is positive: Observe that

$$C_0 \cap A^- = (A \setminus \text{int}(A'')) \cap A^{(-\infty,T']}
\subset (A \setminus \text{int}(A'')) \cap A^{[-T+1,T-1]}
= \emptyset,$$

where we have used the fact that $A^{[-T+1,T-1]} \subset A' \subset \text{int}(A'')$. Since $C_0$ and $A^-$ are compact, the distance between them is positive.
- The distance between $C_0$ and $P_A(\tilde{K}_1)$ is positive: Suppose that this is not true. Since $C_0$ is compact, there would be a sequence $\{x_j\}$ of points in $\tilde{K}_1$ and a sequence of nonnegative number $\{t_j\}$ such that $\varphi(x_j, [0,t_j]) \subset A$ and $y_j = \varphi(x_j, t_j)$ converges to a point $y$ in $C_0$.

If $t_j \to \infty$, we would have $\varphi(y, (-\infty,0]) \subset A$, which means that $y \in A^-$. This is a contradiction since $C_0 \cap A^- = \emptyset$.

After passing to a subsequence, we now assume that $(x_j, t_j) \to (x, t)$ a point in $\tilde{K}_1 \times [0, \infty)$. If $x \in K_1$, then $x \in K_1 \cap A^{[0,t+T']}$ because $\varphi(x, [0,t]) \subset A$ and $y = \varphi(x, t) \in C_0 \subset A^{[0,T']}$.

By the property of $A'$, we have

$$\varphi(x, [0, t + T' - (T - 1)]) \subset A',
$$

which implies that $y \in A'$. This is a contradiction since $C_0 \cap A' = \emptyset$.

If $x \in A^{[-T,T]}$, then $y \in A^{[-T+1,T'-1]} \subset A^{[-T+1,T-1]}$. This is also a contradiction since $C_0 \cap A^{[-T+1,T-1]} = \emptyset$.
- $A^+ \subset V_0$: This is clear from the definition of $V_0$.
- $V_0 \setminus C_0 \subset \text{int}(A)$: We will actually prove that $V_0 \setminus C_0 \subset A''$. Since $A''$ is closed, it is sufficient to show that $V_0 \setminus C_0 \subset A''$. It is then straightforward to see that $V_0 \setminus C_0 = A^{[0,T']} \cap \text{int}(A'') \subset A''$.
- The distance between $K_2$ and $V_0$ is positive: Since $(K_1, K_2)$ is $(T-1)$-tame, we have $K_2 \cap A^{[0,T-1]} = \emptyset$, and consequently $K_2 \cap V_0 = \emptyset$. Since $K_2$ and $V_0$ are compact, the distance between them is positive.

For a sufficiently small positive number $d$, we define

$$C := \{x \in A | \text{dist}(x, C_0) \leq d\},
V := \{x \in A | \text{dist}(x, \tilde{V}_0) < d\}.$$

From the above observations, one can check that $(C, V)$ is a good pair.

We finally check that $(N, L) = (P_A(B) \cup P_A(A \setminus V), P_A(A \setminus \tilde{V}))$ is $T$-tame.

(i) Notice that $P_A(S)$ is positively invariant for any subset $S \subset A$ and that the union of two positively invariant sets in $A$ is again positively invariant in $A$. Thus, $N, L$ are positively invariant in $A$.

(ii) From our construction, we have $A^{[-T,T]} \subset \tilde{K}_1 \subset N$.

(iii) We are left to show that $A^{[0,T]} \cap L = \emptyset$. Suppose that there is an element $x \in A^{[0,T]} \cap L$. From the definition, we obtain $y \in A \setminus V$ and $t \geq 0$ such that $\varphi(y, [0, t]) \subset A$ and $x = \varphi(y, t)$. It follows that
$y \in A^{[0,T+t]}$. On the other hand, we have $A^{[0,T+t]} \subset A^{[0,T']} = V_0 \subset V$. This is a contradiction since $y \not\in \tilde{V}$.

\[\square\]

3.3. The attractor-repeller pair arising from a strong Morse decomposition. In many situations, we obtain an attractor-repeller pair by decomposing an isolating neighborhood to two parts. We introduce the following notion, which arises in many situations.

**Definition 3.15.** Let $(A_1, A_2)$ be a pair of compact subsets of an isolating neighborhood $A$. We say that $(A_1, A_2)$ is a strong Morse decomposition of $A$ if

- $A = A_1 \cup A_2$;
- For any $x \in A_1 \cap A_2$, there exists $\epsilon > 0$ such that $\varphi(x, (0, \epsilon)) \cap A_1 = \emptyset$ and $\varphi(x, (-\epsilon, 0)) \cap A_2 = \emptyset$.

Simply speaking, the flow leaves $A_1$ immediately and enters $A_2$ immediately at any point on $A_1 \cap A_2$ (see Figure 1). A strong Morse decomposition naturally occurs when we split $A$ by a level set of some function transverse to the flow. Let us summarize some basic properties of a strong Morse decomposition in the following lemma. The proofs are straightforward and we omit them.

![Figure 1. A strong Morse decomposition](image)

**Lemma 3.16.** Let $(A_1, A_2)$ be a strong Morse decomposition of an isolating neighborhood $A$. Then, we have the following results.

1. $A_1$ (resp. $A_2$) is negatively (resp. positively) invariant in $A$;
2. $A_1 \cap A_2 = \partial A_1 \cap \partial A_2$ and $\partial A_i = (\partial A \cap A_i) \cup (A_1 \cap A_2)$ for $i = 1, 2$;
3. $A_1$ and $A_2$ are isolating neighborhoods;
4. $(\text{inv}(A_2), \text{inv}(A_1))$ is an attractor-repeller pair in $\text{inv}(A)$.

When an attractor-repeller pair comes from a strong Morse decomposition, we have an extra property for index triples as follows.

**Lemma 3.17.** Let $(A_1, A_2)$ be a strong Morse decomposition of $A$. Suppose that $(\tilde{N}_3, \tilde{N}_2, \tilde{N}_1)$ is an index triple for $(\text{inv}(A_2), \text{inv}(A_1))$ and denote by $\tilde{N}_2' = \tilde{N}_2 \cup (\tilde{N}_1 \cap A_2)$. Then, $(\tilde{N}_3, \tilde{N}_2', \tilde{N}_1)$ is again an index triple for $(\text{inv}(A_2), \text{inv}(A_1))$. In particular, we can always pick an index triple $(\tilde{N}_3, \tilde{N}_2, \tilde{N}_1)$ of $(\text{inv}(A_2), \text{inv}(A_1))$ satisfying $\tilde{N}_1 \cap A_2 \subset \tilde{N}_2$. 
Proof. We simply check each condition of index pairs one by one.

- $\tilde{N}_2'$ is positively invariant in $\tilde{N}_1$: Since $A_2$ is positively invariant in $A$, $A_2 \cap \tilde{N}_1$ is positively invariant in $\tilde{N}_1$. The set $\tilde{N}_2$ is also positively invariant in $\tilde{N}_1$ because $(\tilde{N}_1, \tilde{N}_2)$ is an index pair. It is straightforward to see that the union of two positively invariant sets is a positively invariant set.

- $\tilde{N}_2'$ is an exit set for $\tilde{N}_1$ because $\tilde{N}_2'$ contains $\tilde{N}_2$, which is an exit set for $\tilde{N}_1$.

- $\text{inv}(A_1) = \text{inv}(\tilde{N}_1 \setminus \tilde{N}_2') \subset \text{int}(\tilde{N}_1 \setminus \tilde{N}_2')$: Consider an element $x \in \text{inv}(\tilde{N}_1 \setminus \tilde{N}_2) = \text{inv}(A_1)$. Then, $\varphi(x, (-\infty, \infty))$ is contained in $(\tilde{N}_1 \setminus \tilde{N}_2) \cap \text{int}(A_1)$. Since $\text{int}(A_1) \cap A_2 = \emptyset$, we see that $\varphi(x, (-\infty, \infty)) \subset \tilde{N}_1 \setminus (\tilde{N}_2 \cup (\tilde{N}_1 \cap A_2))$. Thus, $x \in \text{inv}(\tilde{N}_1 \setminus \tilde{N}_2')$ and $\text{inv}(\tilde{N}_1 \setminus \tilde{N}_2) \subset \text{int}(\tilde{N}_1 \setminus \tilde{N}_2')$. Since $\tilde{N}_1 \setminus \tilde{N}_2' \subset \tilde{N}_1 \setminus \tilde{N}_2$, we have $\text{inv}(\tilde{N}_1 \setminus \tilde{N}_2') = \text{inv}(\tilde{N}_1 \setminus \tilde{N}_2) = \text{inv}(A_1)$. Note that $\text{inv}(A_1) \subset \text{int}(\tilde{N}_1 \setminus \tilde{N}_2')$ because $\text{inv}(A_1) \subset \text{int}(\tilde{N}_1 \setminus \tilde{N}_2)$ and $\text{inv}(A_1) \cap A_2 = \emptyset$.

- $\tilde{N}_3$ is positively invariant in $\tilde{N}_2'$ because $\tilde{N}_3$ is positively invariant in $\tilde{N}_1$, which contains $\tilde{N}_2'$.

- $\tilde{N}_3$ is an exit set for $\tilde{N}_2'$: We only have to check that $\tilde{N}_3$ is an exit set for $\tilde{N}_1 \cap A_2$. Suppose that $x \in \tilde{N}_1 \cap A_2$ but $\varphi(x, t) \notin \tilde{N}_1 \cap A_2$ for some $t > 0$. Notice that a flow cannot go from $A_2$ to $A_1$ since $(A_1, A_2)$ is a strong Morse decomposition. If $\varphi(x, t) \in \tilde{N}_1$, we would have $\varphi(x, t) \notin A_2$ which implies $\varphi(x, t) \notin A$ a contradiction. When $\varphi(x, t) \notin \tilde{N}_1$, we can use the fact that $\tilde{N}_3$ is an exit set for $\tilde{N}_1$.

- $\text{inv}(A_2) = \text{inv}(\tilde{N}_2' \setminus \tilde{N}_3) \subset \text{int}(\tilde{N}_2' \setminus \tilde{N}_3)$: Suppose that we have $x \in \text{inv}(\tilde{N}_2' \setminus \tilde{N}_3)$ such that $\varphi(x, t) \notin \tilde{N}_2 \setminus \tilde{N}_3$ for some $t \in \mathbb{R}$. Since $\varphi(x, (-\infty, \infty))$ does not intersect $\tilde{N}_3$, which is an exit set for both $\tilde{N}_2$ and $\tilde{N}_1 \cap A_2$, one can deduce that $\varphi(x, (-\infty, \infty)) \subset \tilde{N}_1 \cap A_2$. This implies $x \in \text{inv}(A_2) = \text{inv}(\tilde{N}_2 \setminus \tilde{N}_3)$ which is a contradiction. Therefore, $\text{inv}(\tilde{N}_2' \setminus \tilde{N}_3) \subset \text{inv}(\tilde{N}_2 \setminus \tilde{N}_3)$ while the converse is trivial. Consequently, $\text{inv}(\tilde{N}_2' \setminus \tilde{N}_3) = \text{inv}(A_2)$ is contained in $\text{int}(\tilde{N}_2 \setminus \tilde{N}_3) \subset \text{int}(\tilde{N}_2' \setminus \tilde{N}_3)$.

When an attractor-repeller pair arises from a strong Morse decomposition, we will show that canonical maps from pre-index pairs are compatible with the attractor and repeller maps.

**Proposition 3.18.** Let $(A_1, A_2)$ be a strong Morse decomposition of $A$ and let $(K_1, K_2)$ be a pre-index pair in $A_2$. Then, we have the following:

1. $(K_1, K_2)$ is also a pre-index pair in $A$;
(2) We have a commutative diagram

\[
\begin{array}{ccc}
K_1/K_2 & \xrightarrow{i_2} & I(\text{inv}(A_2)) \\
\downarrow{i_1} & & \downarrow{i} \\
I(\text{inv}(A)) & & &
\end{array}
\]

where \(i, i_2\) are the canonical maps and \(i: I(\text{inv}(A_2)) \to I(\text{inv}(A))\) is the attractor map.

Proof.

(1) Consider \(x \in K_1\) satisfying \(\varphi(x, [0, \infty)) \subset A\). Since \(A_2\) is positively invariant in \(A\) and \(x \in K_1 \subset A_2\), we have \(\varphi(x, [0, \infty)) \subset A_2\). Consequently, we see that \(\varphi(x, [0, \infty)) \subset \text{int}(A_2)\) because \((K_1, K_2)\) is an pre-index pair in \(A_2\). Now, consider \(x \in K_2 \cap A^+\). Again, since \(A_2\) is positively invariant in \(A\), we have \(\varphi(x, [0, \infty)) \subset A_2\). This is impossible because \(K_2 \cap A^+_2 = \emptyset\).

(2) Let \(\tilde{N}_3 \subset \tilde{N}_2 \subset \tilde{N}_1 \subset A\) be an index triple for \((\text{inv}(A_2), \text{inv}(A_1))\) such that \(\tilde{N}_1 \cap A_2 \subset \tilde{N}_2\) (cf. Lemma 3.17) and let \(L \subset N \subset A\) (resp. \(L_2 \subset N_2 \subset A_2\)) be an index pair for \(\text{inv}(A)\) (resp. \(\text{inv}(A_2)\)) that contains \((K_1, K_2)\). By Theorem 3.14 we may also assume that both \((N, L)\) and \((N_2, L_2)\) are \(T\)-tame. By possibly increasing \(T\), we also assume that we have flow maps \(s_T: N/L \to \tilde{N}_1/\tilde{N}_3\) and \(s'_T: N_2/L_2 \to \tilde{N}_2/\tilde{N}_3\). Then, the map \(i \circ i\) is represented by a composition

\[
K_1/K_2 \xrightarrow{i_2} N_2/L_2 \xrightarrow{s'_T} \tilde{N}_2/\tilde{N}_3 \xrightarrow{i} \tilde{N}_1/\tilde{N}_3
\]

while the map \(i\) is represented by the composition

\[
K_1/K_2 \xrightarrow{i} N/L \xrightarrow{s_T} \tilde{N}_1/\tilde{N}_3.
\]

We will show that these two compositions are in fact the same map.

Applying Lemma 3.13 one can check that \(i \circ s'_T \circ i_2\) sends \([x]\) to \([\varphi(x, 3T)]\) if

\[
\varphi(x, [0, 3T]) \subset A_2, \quad \varphi(x, [T, 3T]) \subset \tilde{N}_2 \setminus \tilde{N}_3
\]

and to the basepoint otherwise. On the other hand, \(s_T \circ i\) sends \([x]\) to \([\varphi(x, 3T)]\) if

\[
\varphi(x, [0, 3T]) \subset A, \quad \varphi(x, [T, 3T]) \subset \tilde{N}_1 \setminus \tilde{N}_3
\]

and to the basepoint otherwise. It is obvious that condition (8) implies (9). On the other hand, condition (9) implies (8) for \(x \in K_1 \subset A_2\) simply because \(A_2\) is positively invariant in \(A\) and \(\tilde{N}_1 \cap A_2 \subset \tilde{N}_2\).

\(\square\)
Proposition 3.19. Let \((A_1, A_2)\) be a strong Morse decomposition of \(A\) and let \((K_3, K_4)\) be a pre-index pair in \(A\). Consider a pair \((K_3', K_4') := (K_3 \cap A_1, (K_4 \cap A_1) \cup (K_3 \cap A_1 \cap A_2))\). Then, we have the followings:

1. The pair \((K_3', K_4')\) is a pre-index pair in \(A_1\):
2. A map \(q: K_3/K_4 \to K_3'/K_4'\) given by

\[
q([x]) = \begin{cases} [x] & \text{if } x \in K_3', \\ [K_4'] & \text{otherwise}, \end{cases}
\]

is well-defined and continuous;
3. We have a commutative diagram

\[
\begin{array}{ccc}
K_3/K_4 & \xrightarrow{i} & I(\text{inv}(A)) \\
\downarrow q & & \downarrow r \\
K_3'/K_4' & \xrightarrow{i'} & I(\text{inv}(A_1))
\end{array}
\]

where \(i, i'\) are the canonical maps and \(r: I(\text{inv}(A)) \to I(\text{inv}(A_1))\) is the repeller map.

Proof.

1. We will check the two conditions of pre-index pair directly. Suppose that \(x \in K_3'\) and \(\varphi(x, [0, \infty)) \subset A_1\). We can see that \(\varphi(x, [0, \infty)) \cap (A_1 \cap A_2) = \emptyset\) from the property (II) of strong Morse decomposition. Since \((K_3, K_4)\) is a pre-index pair in \(A\) and \(x \in K_3 \cap A^+\) we have \(\varphi(x, [0, \infty)) \cap \partial A = \emptyset\). Consequently, we can deduce that \(\varphi(x, [0, +\infty)) \cap \partial A_1 = \emptyset\) because \(\partial A_1 = (\partial A \cap A_1) \cup (A_1 \cap A_2)\).

Since \((K_3, K_4)\) is a pre-index pair in \(A\), we have \(K_4 \cap A^+ = \emptyset\). It follows directly that \((K_4 \cap A_1) \cap A^+_1 = \emptyset\). On the other hand, we can see that \((K_3 \cap A_1 \cap A_2) \cap A^+_1 = \emptyset\) as a point on \(A_1 \cap A_2\) leaves \(A_1\) immediately. Therefore, \(K_3'\) has empty intersection with \(A^+_1\).

2. Note that \(q\) is continuous because \((K_3 \setminus K_3') \cap K_3' = K_3 \cap A_1 \cap A_2 \subset K_4\). For \(x \in K_4 \cap K_3' \subset K_4 \cap A_1 \subset K_4'\), we see that \(q\) is well-defined.

3. As in the proof of Proposition 3.18 let \(\tilde{N}_3 \subset \tilde{N}_2 \subset \tilde{N}_1 \subset A\) be an index triple for \((\text{inv}(A_2), \text{inv}(A_1))\) with \(\tilde{N}_1 \cap A_2 \subset \tilde{N}_2\) and let \(L \subset N \subset A\) (resp. \(L_1 \subset N_1 \subset A_1\)) be an index pair for \(A\) (resp. for \(A_1\)) that contains \((K_3, K_4)\) (resp. \((K_3', K_4')\)). By Theorem 3.14, we can assume that \((N, L)\) and \((N_1, L_1)\) are both \(T\)-tame. By possibly increasing \(T\), we also assume that we have flow maps \(s_T: N/L \to \tilde{N}_1/\tilde{N}_3\) and \(s'_T: N_1/L_1 \to \tilde{N}_1/\tilde{N}_2\). Then, the map \(q \circ i'\) is represented by

\[
K_3/K_4 \xrightarrow{q} K_3'/K_4' \xrightarrow{i'} N_1/L_1 \xrightarrow{s'_T} \tilde{N}_1/\tilde{N}_2,
\]

and the map \(r \circ i\) is represented by

\[
K_3/K_4 \xrightarrow{i} N/L \xrightarrow{s_T} \tilde{N}_1/\tilde{N}_3 \xrightarrow{r} \tilde{N}_1/\tilde{N}_2.
\]

We will show that these two compositions are in fact the same maps.
Applying Lemma 3.13 one can check that $s'_{T} \circ \iota' \circ q$ sends $[x]$ to $[\varphi(x, 3T)]$ if
\[\varphi(x, [0, 3T]) \subset A_{1} \quad \text{and} \quad \varphi(x, [T, 3T]) \subset \tilde{N}_{1} \setminus \tilde{N}_{2}\] (10)
and to the basepoint otherwise. On the other hand, $r \circ s_{T} \circ \iota$ sends $[x]$ to $[\varphi(x, 3T)]$ if
\[\varphi(x, [0, 3T]) \subset A, \quad \varphi(x, [T, 3T]) \subset \tilde{N}_{1} \setminus \tilde{N}_{3}\] (11)
and to the basepoint otherwise. Clearly, condition (10) implies condition (11). We will check that the two conditions are the same.

**3.4. T-tame manifold isolating block for Seiberg-Witten flow.**

**Definition 3.20.** For a compact set $N$ in $\Omega$, we consider the following subsets of its boundary:
\[n^{+}(N) := \{ x \in \partial N \mid \exists \epsilon > 0 \text{ s.t. } \varphi(-\epsilon, 0) \cap N = \emptyset \},\]
\[n^{-}(N) := \{ x \in \partial N \mid \exists \epsilon > 0 \text{ s.t. } \varphi(0, \epsilon) \cap N = \emptyset \} .\]

A compact set $N$ is called an **isolating block** if $\partial N = n^{+}(N) \cup n^{-}(N)$.

It is straightforward to verify that an isolating block $N$ is an isolating neighborhood and that $(N, n^{-}(N))$ is an index pair.

**Definition 3.21.** Let $S$ be a compact subset of $\Omega$. If $N$ is a compact submanifold of $\Omega$ and is also an isolating block with $\text{inv } N = S$, we call $N$ a **manifold isolating block** of $S$.

In [4], it is proved that, for any isolating neighborhood $A$, we can always find a manifold isolating block $N$ of $\text{inv } A$ with $N \subset A$. We also introduce a notion of tameness for an isolating block as quantitative refinement as in Section 3.2.

**Definition 3.22.** Let $A$ be an isolating neighborhood and $T$ be a positive number. An isolating block $N$ in $A$ is called $T$-tame if $A[-T,T] \subset \text{int}(N)$.

We turn into special situation involving construction of the spectrum invariants a $3$-manifold $Y$: $\text{swf}^{A}(Y, s, A_{0}, g; S^{1})$ and $\text{swf}^{R}(Y, s, A_{0}, g; S^{1})$. Let $R_{0}$ be the universal constant from [9, Theorem 3.2] such that all finite-type Seiberg-Witten trajectories are contained in $\text{Str}(R_{0})$ (see (2)). Take
a positive number $\tilde{R}$ with $\tilde{R} > R_0$, sequences $\lambda_n \to -\infty$, $\mu_n \to \infty$ and consider the sets

$$J^\pm_m := \text{Str}(\tilde{R}) \cap \bigcap_{1 \leq j \leq b_1} g_{j,\pm}^{-1}(\infty, \theta + m),$$

$$J^m_n := J^\pm_m \cap V^{\mu_n}_{\lambda_n},$$

defined in Section 2.1 (see also [9, Section 5.1] for more details.)

**Lemma 3.23.** For each positive integer $m$, there is a positive number $T_m$ independent of $n$ such that

$$(J^m_n)^{[-2T,2T]} \subset \text{int}\left\{(J^m_n)^{[-T,T]}\right\},$$

for all $T > T_m$ and $n$ sufficiently large. In particular, $(J^m_n)^{[-2T,2T]} \subset \text{int}(J^m_n)$. Similar results hold for $J^m_n$.

**Proof.** If the statement is not true, we have a sequence $T_n \to \infty$ such that we can take elements

$$x_n \in (J^m_n)^{[-2T_n,2T_n]} \cap \partial\left\{(J^m_n)^{[-T_n,T_n]}\right\}.$$  

In particular, we would have

$$\varphi^m_n(x_n, [-2T_n, 2T_n]) \subset J^m_n$$

and $\varphi^m_n(x_n, t_n) \in \partial J^m_n$ for some $t_n \in [-T_n, T_n]$, which implies

$$\varphi^m_n(x_n, t_n) \in (J^m_n)^{[-T_n,T_n]}.$$  

On the other hand, by [9, Lemma 5.4], we must have

$$\varphi^m_n(x_n, t_n) \in \partial \text{Str}(\tilde{R}).$$

This is a contradiction to [9, Lemma 5.5 (a)].

We now state the main result of this section.

**Proposition 3.24.** Let $T_m$ be the constant from Lemma 3.23. When $T > 4T_m$ and $n$ is sufficiently large, we can always find a $T$-tame manifold isolating block $N^{m,+}_n$ of $\text{inv}(J^{m,+}_n)$ with $N^{m,+}_n \subset J^{m,+}_n$. A similar result holds for $J^{m,-}_n$.

**Proof.** Fix $m$ and suppose that $n$ is sufficiently large so that the statement of Lemma 3.23 holds. Take a positive number $T$ with $T > 4T_m$. By Lemma 3.23 we have

$$(J^{m,+}_n)^{[-T,T]} \subset \text{int}\left\{(J^{m,+}_n)^{[-T/2,T/2]}\right\}$$

and

$$(J^{m,+}_n)^{[-T/2,T/2]} \subset \text{int}\left\{(J^{m,+}_n)^{[-T/4,T/4]}\right\}.$$  

We can take a smooth function $\tau : V^{\mu_n}_{\lambda_n} \to [0, 1]$ such that $\tau = 0$ on $(J^{m,+}_n)^{[-T,T]}$, and $\tau = 1$ on $V^{\mu_n}_{\lambda_n} \setminus (J^{m,+}_n)^{[-T/2,T/2]}$. 


and a smooth bump function \( t_m : \text{Conv}(Y) \to [0, 1] \) such that
\[
\iota_m^{-1}((0, 1]) \text{ is bounded, and } \iota_m = 1 \text{ in a neighborhood of } J_{m+1}^+.
\]

Let \( \tilde{\varphi}_m^n \) be the flow on \( V_{\lambda_n}^m \) generated by \( \tau \cdot t_m \cdot (l + p_{\lambda_n} \circ c) \). We will prove that \( J_{m+1}^+ \) is an isolating neighborhood of \( \text{inv}(\tilde{\varphi}_m^n, J_{m+1}^+) \). If this is not true, we can take
\[
x \in \partial J_{m+1} \cap \text{inv}(\tilde{\varphi}_m^n, J_{m+1}^+).
\]

Put
\[
\begin{align*}
P^+(x) &:= \{ \varphi^n_m(x, t)|t \geq 0, \varphi^n_m(x, [0, t]) \subset J_{m+1}^+ \}, \\
P^-(x) &:= \{ \varphi^n_m(x, t)|t \leq 0, \varphi^n_m(x, [t, 0]) \subset J_{m+1}^+ \}.
\end{align*}
\]

Suppose that \( P^+(x) \cap (J_{m+1}^+)^{[-T/2, T/2]} = \emptyset \). This means a forward \( \varphi^n_m \)-trajectory of \( x \) inside \( J_{m+1}^+ \) lie outside \( (J_{m+1}^+)^{[-T/2, T/2]} \), so that a forward \( \varphi^n_m \)-trajectory agrees with a forward \( \tilde{\varphi}_m^n \)-trajectory. Consequently, we have \( \varphi^n_m(x, [0, \infty)) = \tilde{\varphi}_m^n(x, [0, \infty)) \subset J_{m+1}^+ \). Hence \( \varphi^n_m(x, T/2) \in P^+(x) \) and \( \varphi^n_m(x, T/2) \in (J_{m+1}^+)^{[-T/2, T/2]} \) which is a contradiction. We can now conclude that \( P^+(x) \cap (J_{m+1}^+)^{[-T/2, T/2]} \neq \emptyset \) and, in particular, \( x \in (J_{m+1}^+)^{[0, T/2]} \).

Similarly we can deduce that \( x \in (J_{m+1}^+)^{[-T/2, 0]} \). These facts imply that
\[
x \in (J_{m+1}^+)^{[-T/2, T/2]} \cap \partial J_{m+1}^+,
\]
which is a contradiction because
\[
(J_{m+1}^+)^{[-T/2, T/2]} \subset \text{int} \left\{ (J_{m+1}^+)^{[-T/4, T/4]} \right\} \subset \text{int}(J_{m+1}^+)
\]

Therefore \( J_{m+1}^+ \) is an isolating neighborhood of \( \text{inv}(\tilde{\varphi}_m^n, J_{m+1}^+) \). By the result of Conley and Easton [4], we can find a manifold isolating block \( N_{m+}^+ \) of \( \text{inv}(\tilde{\varphi}_m^n, J_{m+1}^+) \) with \( N_{m+}^+ \subset J_{m+1}^+ \). Note that
\[
(J_{m+1}^+)^{[-T, T]} \subset \text{inv}(\tilde{\varphi}_m^n, J_{m+1}^+) \subset \text{int}(J_{m+1}^+)
\]

Since the directions of the flows \( \varphi^n_m \) and \( \tilde{\varphi}_m^n \) coincide on \( \partial N_{m+}^+ \subset J_{m+1}^+ \) \( \tau^{-1}(0) \), we see that \( N_{m+}^+ \) is also a manifold isolating block of \( \text{inv}(\varphi^n_m, J_{m+1}^+) \).

Thus \( N_{m+}^+ \) is a \( T \)-tame manifold isolating block of \( \text{inv}(\varphi^n_m, J_{m+1}^+) \) in \( J_{m+1}^+ \).

\[ \square \]

4. Stable homotopy categories

4.1. Summary. In this section, we will discuss the stable homotopy categories \( \mathcal{C}, \mathcal{G}, \mathcal{G}^* \). The discussion in this section will be needed to construct the gluing formula in Theorem 6.1.

First let us briefly recall the definition of the categories. (See [9] for the details.) An object of \( \mathcal{C} \) is a triple \( (A, m, n) \), where \( A \) is a pointed topological space with \( S^1 \)-action which is \( S^1 \)-homotopy equivalent to a finite \( S^1 \)-CW complex (see [15] Chapter I, Section 3) for the definition of equivariant CW
complex), $m$ is an integer and $n$ is a rational number. The set of morphisms between $(A_1, m_1, n_1)$ and $(A_2, m_2, n_2)$ is given by

$$
\text{mor}_\mathcal{C}((A_1, m_1, n_1), (A_2, m_2, n_2)) = \lim_{u,v \to \infty} [(\mathbb{R}^u \oplus \mathbb{C}^v)^+ \land A_1, (\mathbb{R}^{u+m_1-m_2} \oplus \mathbb{C}^{v+n_1-n_2})^+ \land A_2]_{S^1}
$$

if $n_1 - n_2$ is an integer, and we define $\text{mor}_\mathcal{C}((A_1, m_1, n_1), (A_2, m_2, n_2))$ to be the empty set if $n_1 - n_2$ is not an integer. Here $[\cdot, \cdot]_{S^1}$ is the set of pointed $S^1$-homotopy classes, $\mathbb{R}$ is the one dimensional trivial representation of $S^1$ and $\mathbb{C}$ is the standard two dimensional representation of $S^1$. The category $\mathcal{S}$ is the category of direct systems

$$
Z : Z_1 \overset{j_1}{\to} Z_2 \overset{j_2}{\to} \cdots
$$

in $\mathcal{C}$. Here $Z_m$ and $j_m$ are an object and morphism in $\mathcal{C}$ respectively. For objects $Z, Z'$ in $\mathcal{S}$, the set morphism is defined by

$$
\text{mor}_\mathcal{S}(Z, Z') = \lim_{\infty \leftarrow m} \lim_{n \to \infty} \text{mor}_\mathcal{C}(Z_m, Z'_n).
$$

The category $\mathcal{S}^*$ is the category of inverse systems

$$
\tilde{Z} : \tilde{Z}_1 \overset{\tilde{j}_1}{\leftarrow} \tilde{Z}_2 \overset{\tilde{j}_2}{\leftarrow} \cdots
$$

in $\mathcal{C}$. Here $\tilde{Z}_m$ and $\tilde{j}_m$ are an object and morphism in $\mathcal{C}$ respectively. For objects $\tilde{Z}, \tilde{Z}'$ in $\mathcal{S}^*$, the set of morphisms is defined by

$$
\text{mor}_{\mathcal{S}^*}(\tilde{Z}, \tilde{Z}') = \lim_{\infty \leftarrow n} \lim_{m \to \infty} \text{mor}_\mathcal{C}(\tilde{Z}_m, \tilde{Z}_n').
$$

In Section 4.2, we will define the smash product in the category $\mathcal{C}$ and prove that $\mathcal{C}$ is a symmetric, monoidal category (Lemma 4.1). In Section 4.3, we will introduce the notion of the $S^1$-equivariant Spanier-Whitehead duality between the categories $\mathcal{S}$ and $\mathcal{S}^*$. We will say that $Z \in \text{ob} \mathcal{S}$ and $\tilde{Z} \in \text{ob} \mathcal{S}^*$ are $S^1$-equivariant Spanier-Whitehead dual to each other if there are elements

$$
\epsilon \in \lim_{\infty \leftarrow n} \lim_{m \to \infty} \text{mor}_\mathcal{C}(\tilde{Z}_n \land Z_m, S), \eta \in \lim_{\infty \leftarrow n} \lim_{m \to \infty} (S, Z_m \land \tilde{Z}_n),
$$

which satisfy certain conditions (Definition 4.3). Here $S = (S^0, 0, 0) \in \mathcal{C}$. The elements $\epsilon, \eta$ are called duality morphisms. In Section 4.4, we will prove that the Seiberg-Witten Floer stable spectra $\text{swf}^A(Y) \in \text{ob} \mathcal{S}$ and $\text{swf}^R(-Y) \in \text{ob} \mathcal{S}^*$ are $S^1$-equivariant Spanier-Whitehead dual to each other (Proposition 4.11). We will construct natural duality morphisms for $\text{swf}^A(Y)$ and $\text{swf}^R(-Y)$ which will be needed for the gluing formula of the Bauer-Furuta invariants (Theorem 6.1).

We will focus on the $S^1$-equivariant stable homotopy categories. But the statements can be proved for the $\text{Pin}(2)$-equivariant stable homotopy categories in a similar way.
4.2. Smash product. In this subsection, we establish the symmetric monoidal structure on the category \( \mathcal{C} \). To do this, we will define the smash product as a bifunctor \( \wedge : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \). First we define the smash product of two objects \((A_1, m_1, n_1), (A_2, m_2, n_2) \in \mathcal{C}\). Here \( A_i \) is an \( S^1 \)-topological space, \( m_i \in 2\mathbb{Z}, n_i \in \mathbb{Q} \). We define the smash product by

\[
(A_1, m_1, n_1) \wedge (A_2, m_2, n_2) := (A_1 \wedge A_2, m_1 + m_2, n_1 + n_2),
\]

where \( A_1 \wedge A_2 \) denotes the classical smash product on pointed topological spaces. Next we define the smash product of morphisms. Suppose that for \( i = 1, 2 \) a map

\[
f_i : (\mathbb{R}^{k_i} \oplus \mathcal{C}^{l_i})^+ \wedge A_i \rightarrow (\mathbb{R}^{k_i+m_i-m_i'} \oplus \mathcal{C}^{l_i+n_i-n_i'})^+ \wedge A_i'
\]

represents a morphism \([f_i] \in \text{mor}_\mathcal{C}((A_i, m_i, n_i), (A_i', m_i', n_i'))\). We may suppose that \( k_i \) is even. We define a map

\[
f_1 \wedge f_2 : (\mathbb{R}^{k_1} \oplus \mathbb{R}^{k_2} \oplus \mathcal{C}^{l_1} \oplus \mathcal{C}^{l_2})^+ \wedge A_1 \wedge A_2 \rightarrow
(\mathbb{R}^{k_1+m_1-m_1'} \oplus \mathbb{R}^{k_2+m_2-m_2'} \oplus \mathcal{C}^{l_1+n_1-n_1'} \oplus \mathcal{C}^{l_2+n_2-n_2'})^+ \wedge A_1' \wedge A_2'
\]

by putting the suspension indices for \( f_1 \) on the left and those for \( f_2 \) on the right. We define \([f_1] \wedge [f_2]\) to be the morphism represented by \( f_1 \wedge f_2 \). To prove that this operation is well defined, we need to check that for \( a, b \in \mathbb{Z}_{>0} \), we have

\[
\Sigma^{(\mathbb{R}^a \oplus \mathcal{C}^b)^+} (f_1 \wedge f_2) \cong (\Sigma^{(\mathbb{R}^a \oplus \mathcal{C}^b)^+} f_1) \wedge f_2 \cong f_1 \wedge (\Sigma^{(\mathbb{R}^a \oplus \mathcal{C}^b)^+} f_2),
\]

where \( \cong \) means \( S^1 \)-equivariant stably homotopic. The first equivalence is obvious. The second equivalence follows from the fact that the following diagram is commutative up to homotopy for \( u_1 = k_1, k_1 + m_1 - m_1', u_2 = k_2, k_2 + m_2 - m_2', v_1 = l_1, l_1 + n_1 - n_1', v_2 = l_2, l_2 + n_2 - n_2':

\[
\begin{array}{ccc}
(\mathbb{R}^a \oplus \mathbb{R}^{a_1} \oplus \mathbb{R}^{a_2})^+ \wedge (\mathcal{C}^b \oplus \mathcal{C}^{n_1 \oplus \mathcal{C}^{n_2}})^+ & \xrightarrow{\text{id}} & (\mathbb{R}^u)^+ \wedge (\mathcal{C}^v)^+
\
(\gamma_{\mathbb{R}^a, \mathbb{R}^{a_1} \oplus \mathbb{R}^{a_2}})^+ \wedge (\gamma_{\mathcal{C}^b, \mathcal{C}^{n_1} \oplus \mathcal{C}^{n_2}})^+ & \xrightarrow{\text{id}} & (\gamma_{\mathbb{R}^u, \mathbb{R}^v})^+
\end{array}
\]

Here \( u = a+u_1+u_2, v = b+v_1+v_2 \) and \( \gamma_{\mathbb{R}^a, \mathbb{R}^{a_1}} \) is the map which interchange \( \mathbb{R}^a \) and \( \mathbb{R}^{a_1} \). Similarly for \( \gamma_{\mathcal{C}^b, \mathcal{C}^{n_1}} \). Note that \( u_1 \in 2\mathbb{Z} \) by the assumption on \( k_1, m_1, m_1' \).

There is an isomorphism

\[
\gamma_{(A_1, m_1, n_1), (A_2, m_2, n_2)} : (A_1, m_1, n_1) \wedge (A_2, m_2, n_2) \rightarrow (A_2, m_2, n_2) \wedge (A_1, m_1, n_1)
\]

represented by the obvious homeomorphism \( A_1 \wedge A_2 \rightarrow A_2 \wedge A_1 \). It is not difficult to see that \( \gamma \) is natural in \( (A_i, m_i, n_i) \). That is, the following
diagrams are commutative for $f_i \in \text{mor}_C((A_i, m_i, n_i), (A'_i, m'_i, n'_i))$:

$$
\begin{array}{c}
(A_1, m_1, n_1) \wedge (A_2, m_2, n_2) \xrightarrow{\gamma} (A_2, m_2, n_2) \wedge (A_1, m_1, n_1) \\
\vspace{0.2cm}
\downarrow f_1 \wedge f_2 \\
\gamma \\
(A'_1, m'_1, n'_1) \wedge (A'_2, m'_2, n'_2) \xrightarrow{\gamma} (A'_2, m'_2, n'_2) \wedge (A'_1, m'_1, n'_1).
\end{array}
$$

(Again, we need the assumption that $m_i$ is even here.) Once the well-definedness of $\wedge$ and the naturality are established we can prove the following lemma easily by checking the axioms at the level of topological spaces.

**Lemma 4.1.** The category $C$ equipped with $\wedge$ and $\gamma$ is a symmetric monoidal category with unit $S = (S^0, 0, 0)$.

We briefly mention the $Pin(2)$-case. The smash product $\wedge$ and the interchanging operation $\gamma$ can be defined on the category $C_{Pin(2)}$ in exactly the same way as before. As a result, the category $C_{Pin(2)}$ is also a symmetric monoidal category.

### 4.3. Equivariant Spanier-Whitehead duality

In this subsection we will set up the equivariant Spanier-Whitehead duality between the categories $\mathcal{C}$ and $\mathcal{S}^*$. Although we will mostly focus on the $S^1$-case for simplicity, all definitions and proofs can be easily adapted to the $Pin(2)$-case. As a result, a duality between $\mathcal{S}_{Pin(2)}$ and $\mathcal{S}^*_{Pin(2)}$ can also be set up in a similar way.

The following definition is motivated by [12, Chapter III] and [15, Chapter XVI Section 7].

**Definition 4.2.** Let $U, W$ be objects of $C$ and put $S = (S^0, 0, 0) \in \text{ob } C$. Suppose that there exist morphisms

$$
\epsilon : W \wedge U \to S, \eta : S \to U \wedge W
$$

such that the compositions

$$
U \cong S \wedge U \xrightarrow{\eta \wedge \text{id}} U \wedge W \wedge U \xrightarrow{\text{id} \wedge \epsilon} U \wedge S \cong U
$$

and

$$
W \cong W \wedge S \xrightarrow{\text{id} \wedge \eta} W \wedge U \wedge W \xrightarrow{\epsilon \wedge \text{id}} S \wedge W \cong W
$$

are equal to the identity morphisms respectively. Then we say that $U$ and $W$ are Spanier-Whitehead dual to each other and call $\epsilon$ and $\eta$ duality morphisms.

We generalize this definition to the duality between $\mathcal{S}$ and $\mathcal{S}^*$.

**Definition 4.3.** Let

$$
Z : Z_1 \to Z_2 \to Z_3 \to \cdots
$$

be an object of $\mathcal{S}$ and

$$
\bar{Z} : \bar{Z}_1 \leftarrow \bar{Z}_2 \leftarrow \bar{Z}_3 \leftarrow \cdots
$$
be an object of $\mathcal{S}^*$. Suppose that we have an element

$$\epsilon \in \lim_{\infty \leftarrow m} \lim_{n \to \infty} \text{mor}_S(\tilde{Z}_n \wedge Z_m, S)$$

represented by a collection $\{\epsilon_{m,n} : \tilde{Z}_n \wedge Z_m \to S\}_{m>0, n \gg m}$ and an element

$$\eta \in \lim_{\infty \leftarrow n} \lim_{m \to \infty} \text{mor}_S(S, Z_m \wedge \bar{Z}_n)$$

represented by a collection $\{\eta_{m,n} : S \to Z_m \wedge \bar{Z}_n\}_{n>0, m \gg n}$ which satisfy the following conditions:

(i) For any $m > 0$ there exists $n$ large enough relative to $m$ and $m'$ large enough relative to $n$ such that the composition

$$Z_m \cong S \wedge Z_m \xrightarrow{\eta_{m',n} \wedge \text{id}} Z_{m'} \wedge \bar{Z}_n \wedge Z_m \xrightarrow{\text{id} \wedge \epsilon_{m,n}} Z_{m'} \wedge Z_m \wedge S \cong Z_m'$$

is equal to the connecting morphism $Z_m \to Z_{m'}$ of the inductive system $Z$.

(ii) For any $n > 0$, there exists $m$ large enough relative to $n$ and $n'$ large enough to $m$ such that the composition

$$\bar{Z}_{n'} \cong \bar{Z}_{n'} \wedge S \xrightarrow{\text{id} \wedge \eta_{m,n}} \bar{Z}_{n'} \wedge Z_m \wedge \bar{Z}_n \xrightarrow{\epsilon_{m,n} \wedge \text{id}} S \wedge \bar{Z}_n \cong \bar{Z}_{n}$$

is equal to the connecting morphism $\bar{Z}_{n'} \to \bar{Z}_n$ of the projective system $\bar{Z}$.

Then we say that $Z$ and $\bar{Z}$ are $S^1$-equivariant Spanier-Whitehead dual to each other and we call $\epsilon$ and $\eta$ duality morphisms.

We end this subsection with introducing a smashing operation $\tilde{\epsilon}$, which will be used to give the statement of the gluing theorem for the Bauer-Furuta invariant.

**Definition 4.4.** Let $Z \in \text{ob} \mathcal{S}$ and $\bar{Z} \in \text{ob} \mathcal{S}^*$ be objects that are $S^1$-equivariant Spanier-Whitehead dual to each other with duality morphisms $\epsilon, \eta$. Suppose that we have objects $W \in \text{ob} \mathcal{C}$ ($\subset \text{ob} \mathcal{S}$), $\bar{W} \in \text{ob} \mathcal{C}$ ($\subset \text{ob} \mathcal{S}^*$) and morphisms

$$\rho \in \text{mor}_\mathcal{S}(W, Z), \ \bar{\rho} \in \text{mor}_{\mathcal{S}^*}(\bar{W}, \bar{Z}).$$

Choose a morphism $\rho_m : W \to Z_m$ which represents $\rho$ and let $\{\bar{\rho}_n : \bar{W} \to \bar{Z}_n\}_{n>0}$ be the collection which represents $\bar{\rho}$. We define the morphism $\tilde{\epsilon}(\rho, \bar{\rho}) \in \text{mor}_\mathcal{E}(W \wedge \bar{W}, S)$ by the composition

$$\bar{W} \wedge W \xrightarrow{\bar{\rho}_n \wedge \rho_m} \bar{Z}_n \wedge Z_m \xrightarrow{\epsilon_{m,n}} S.$$
4.4. Spanier-Whitehead duality of the unfolded Seiberg-Witten Floer spectra. Let $Y$ be a closed, oriented 3-manifold with a Riemannian metric $g$ and spin$^c$ structure $s$, and let $-Y$ be $Y$ with opposite orientation. As in Section 2.1, the unfolded Seiberg-Witten Floer spectrum $\text{swf}^A(Y, s, A_0, g; S^1) \in \text{ob} \mathcal{G}$ is represented by

$$\text{swf}^A(Y) : I_1 \xrightarrow{j_1} I_2 \xrightarrow{j_2} \ldots$$

with $I_n := \Sigma^{-V_{\lambda_n}} I(\text{inv}(V_{\lambda_n}^\mu \cap J_n^+), \varphi_n)$. It is not hard to see that the unfolded spectrum $\text{swf}^R(-Y, s, A_0, g; S^1) \in \text{ob} \mathcal{G}^*$ can be represented by

$$\text{swf}^R(-Y) : \bar{I}_1 \xleftarrow{\bar{j}_1} \bar{I}_2 \xleftarrow{\bar{j}_2} \ldots,$$

where $\bar{I}_n := \Sigma^{-V_{\lambda_n}^\mu} I(\text{inv}(V_{\lambda_n}^\mu \cap J_n^+), \bar{\varphi}_n)$ and $\bar{\varphi}_n$ is the reverse flow of $\varphi_n$.

For integers $m, n$ with $m < n$ we also write $j_{m,n}, \bar{j}_{m,n}$ for the compositions

$$I_m \xrightarrow{j_{m,n}} I_{m+1} \xrightarrow{j_{n-1}} \ldots \xrightarrow{j_{n-1}} I_n,$$

$$\bar{I}_n \xleftarrow{\bar{j}_{n-1}} \bar{I}_{n-1} \xleftarrow{\bar{j}_{n-2}} \ldots \xleftarrow{\bar{j}_{n-1}} \bar{I}_m.$$

We will define duality morphisms $\epsilon$ and $\eta$ between $\text{swf}^A(Y, s, A_0, g; S^1)$ and $\text{swf}^R(-Y, s, A_0, g; S^1)$, as follows. Take an $S^1$-equivariant manifold isolating block $N_n$ for $\text{inv}(V_{\lambda_n}^\mu \cap J_n^+)$. That is, $N_n$ is a compact submanifold of $V_{\lambda_n}^\mu$ of codimension 0 and there are submanifolds $L_n, \bar{L}_n$ of $\partial N_n$ of codimension 0 such that

$$L_n \cup \bar{L}_n = \partial N_n, \quad \partial L_n = \partial \bar{L}_n = L_n \cap \bar{L}_n$$

and that $(N_n, L_n), (N_n, \bar{L}_n)$ are index pairs for $\text{inv}(V_{\lambda_n}^\mu \cap J_n^+, \varphi_n), \text{inv}(V_{\lambda_n}^\mu \cap J_n^+, \bar{\varphi}_n)$ respectively. Fix a small positive number $\delta > 0$. For a subset $P \subset V_{\lambda_n}^\mu$ we write $\nu_\delta(P)$ for

$$\{x \in V_{\lambda_n}^\mu | \text{dist}(x, P) \leq \delta\}.$$ 

Choose $S^1$-equivariant homotopy equivalences

$$a_n : N_n \to N_n \setminus \nu_\delta(\bar{L}_n), \quad b_n : N_n \to N_n \setminus \nu_\delta(L_n)$$

such that

$$\|a_n(x) - x\| < 2\delta \quad \text{for} \ x \in N_n,$$

$$a_n(L_n) \subset L_n, \quad a_n(x) = x \quad \text{for} \ x \in N_n \setminus \nu_{3\delta}(\partial N_n),$$

$$\|b_n(y) - y\| < 2\delta \quad \text{for} \ y \in N_n,$$

$$b_n(\bar{L}_n) \subset \bar{L}_n, \quad b_n(y) = y \quad \text{for} \ y \in N_n \setminus \nu_{3\delta}(\partial N_n).$$

Put $B_\delta = \{x \in V_{\lambda_n}^\mu | \|x\| \leq \delta\}$ and $S_\delta = \partial B_\delta$. Define

$$\epsilon_{n,n} : (N_n/\bar{L}_n) \land (N_n/L_n) \to (V_{\lambda_n}^\mu)^+ = B_\delta/S_\delta.$$
by the formula

\[
\epsilon_{n,n}([y] \land [x]) = \begin{cases} 
[b_n(y) - a_n(x)] & \text{if } ||b_n(y) - a_n(x)|| < \delta \\
* & \text{otherwise}
\end{cases}
\]

It is straightforward to see that \( \epsilon_n \) is a well-defined, continuous \( S^1 \)-equivariant map. Taking the desuspension by \( V_{\lambda_n}^{\mu_n} \) we get a morphism

\[
\epsilon_{n,n} : \tilde{I}_n \land I_n \to S.
\]

For \( m, n \) with \( m < n \), we define a morphism \( \epsilon_{m,n} : \tilde{Z}_n \land Z_m \to S \) to be the composition

\[
\tilde{I}_n \land I_m \xrightarrow{\text{id} \land j_{m,n}} \tilde{I}_n \land I_n \xrightarrow{\epsilon_{n,n}} S.
\]

**Lemma 4.5.** With the above notation, the morphism \( \epsilon_{m,n} \in \text{mor}_\epsilon(\tilde{I}_n \land I_m, S) \) is independent of the choices of \( N_n, a_n, b_n \) and \( \delta \).

**Proof.** The proof of the independence from \( \delta \) is straightforward. We prove the independence from \( N_n, a_n \) and \( b_n \). Fix an isolating neighborhood \( A(\subset V_{\lambda_n}^{\mu_n} \cap J_n^n) \) of \( \text{inv}(V_{\lambda_n}^{\mu_n} \cap J_n^n) \). Take two manifold isolating blocks \( N_n, N'_n \) for \( \text{inv}(V_{\lambda_n}^{\mu_n} \cap J_n^n) \) included in \( \text{int} \ N \). Then we get two maps

\[
\tilde{\epsilon}_{n,n} : (N_n/\mathcal{T}_n) \land (N_n/L_n) \to B_\delta/S_\delta, \quad \tilde{\epsilon}'_{n,n} : (N'_{n}/\mathcal{T}_n) \land (N'_{n}/L'_n) \to B_\delta/S_\delta.
\]

It is sufficient to show that the following diagram is commutative up to \( S^1 \)-equivariant homotopy:

\[
\begin{array}{ccc}
(N_n/\mathcal{T}_n) \land (N_n/L_n) & \xrightarrow{\tilde{\epsilon}_{n,n}} & B_\delta/S_\delta \\
\downarrow{s \land s} & & \\
(N'_{n}/\mathcal{T}_n) \land (N'_{n}/L'_n) & \xrightarrow{\tilde{\epsilon}'_{n,n}} & B_\delta/S_\delta
\end{array}
\]

Here \( s = s_T : N_n/L_n \to N'_n/L'_n, \bar{s} = \bar{s}_T : N_n/\mathcal{T}_n \to N'_n/\mathcal{T}_n \) are the flow maps with large \( T > 0 \):

\[
s([x]) = \begin{cases} 
[\varphi(x, 3T)] & \text{if } \varphi(x, [0, 2T]) \subset N_n \setminus L_n, \varphi(x, [T, 3T]) \subset N'_n \setminus L'_n, \\
* & \text{otherwise}
\end{cases}
\]

\[
\bar{s}([y]) = \begin{cases} 
[\varphi(y, -3T)] & \text{if } \varphi(y, [-2T, 0]) \subset N_n \setminus \mathcal{T}_n, \varphi(y, [-3T, -T]) \subset N'_n \setminus \mathcal{T}_n, \\
* & \text{otherwise}
\end{cases}
\]

The proof can be reduced to the case \( N'_n \subset \text{int} \ N_n \) since we can find a manifold isolating block \( N''_n \) with \( N''_n \subset \text{int} \ N_n \), \( \text{int} \ N'_n \). Assume that \( N'_n \subset \text{int} \ N_n \). Taking sufficiently large \( T > 0 \) we have

\[
A^{[-T,T]} \subset (N'_n \setminus \nu_{3\delta}(\partial N'_n)) \subset (N_n \setminus \nu_{3\delta}(\partial N_n)). \quad (13)
\]
It is straightforward to see that $\hat{\epsilon}_{n,n}$ is homotopic to a map $\hat{\epsilon}_{n,n}^{(0)} : (N_n/L_n) \cap (N_n/L_n) \to B_\delta/S_\delta$ defined by

$$\hat{\epsilon}_{n,n}^{(0)}([y] \land [x]) = \begin{cases} [\Delta_{T,n}(y, x)] & \text{if } \begin{cases} \varphi(x, [0, 3T]) \subset N_n \setminus L_n, \\ \varphi(y, [-3T, 0]) \subset N_n \setminus L_n, \\ \|\Delta_{T,n}(y, x)\| < \delta \end{cases} \\ * & \text{otherwise}. \end{cases}$$

Here

$$\Delta_{T,n}(y, x) = b_n(\varphi(y, -3T)) - a_n(\varphi(x, 3T)).$$

Suppose that $\epsilon^{(0)}([y] \land [x]) \neq *$. Then

$$\varphi(x, 3T) \in N_n^{[-3T,0]}, \varphi(y, -3T) \in N_n^{[0,3T]}, \|\varphi(y, -3T) - \varphi(x, 3T)\| < 5\delta.$$

Taking small $\delta > 0$ and the using the fact that $N_n \subset \text{int } A$, we may suppose that

$$\varphi(x, 3T), \varphi(y, -3T) \in A^{[-3T,3T]},$$

which implies

$$a_n(\varphi(x, 3T)) = \varphi(x, 3T), \ b_n(\varphi(y, -3T)) = \varphi(y, -3T).$$

Here we have used (12) and (13). We can assume that $\delta$ is independent of $x, y$ since $N_n$ is compact. So we have

$$\hat{\epsilon}_{n,n}^{(0)}([y] \land [x]) = \begin{cases} [\varphi(y, -3T) - \varphi(x, 3T)] & \text{if } \begin{cases} \varphi(x, [0, 3T]) \subset N_n \setminus L_n, \\ \varphi(y, [-3T, 0]) \subset N_n \setminus L_n, \\ \|\varphi(y, -3T) - \varphi(x, 3T)\| < \delta, \end{cases} \\ * & \text{otherwise}. \end{cases}$$

On the hand, we can write

$$\hat{\epsilon}'_{n,n} \circ (s \land \bar{s})([y] \land [x]) = \begin{cases} [\Delta'_{T,n}(y, x)] & \text{if } \begin{cases} \varphi(x, [0, 2T]) \subset N_n \setminus L_n, \\ \varphi(x, [T, 3T]) \subset N_n' \setminus L_n, \\ \varphi(y, [-2T, 0]) \subset N_n \setminus L_n, \\ \varphi(y, [-3T, -T]) \subset N_n' \setminus L_n, \\ \|\Delta'_{T,n}(y, x)\| < \delta, \end{cases} \\ * & \text{otherwise}. \end{cases}$$

Here

$$\Delta'_{T,n}(y, x) = b'_n(\varphi(y, -3T)) - a'_n(\varphi(x, 3T)).$$

As before, if $\hat{\epsilon}'_{n,n} \circ (s \land \bar{s})([y] \land [x]) \neq *$ we have

$$\varphi(x, 3T), \varphi(y, -3T) \in A^{[-3T,3T]}$$
and we can write
\[
\epsilon'_{n,n} \circ (s \wedge \tilde{s})([y] \wedge [x]) = \begin{cases} \\
\varphi(y, -3T) - \varphi(x, 3T) & \text{if } x, 3T \\
\varphi(x, [0, 2T]) \subset N_n \setminus L_n, \\
\varphi(x, [T, 3T]) \subset N'_n \setminus L'_n, \\
\varphi(y, [-2T, 0]) \subset N_n \setminus T_n, \\
\varphi(y, [-3T, -T]) \subset N'_n \setminus T'_n, \\
\|\varphi(y, -3T) - \varphi(x, 3T)\| < \delta, \\
\end{cases}
\]
otherwise.

We will show that \( \epsilon^{(0)}_{n,n} = \epsilon'_{n,n} \circ (s \wedge \tilde{s}) \). It is sufficient to prove that \( \epsilon^{(0)}_{n,n}([y] \wedge [x]) \neq * \) if and only if \( \epsilon'_{n,n} \circ (s \wedge \tilde{s})([y] \wedge [x]) \neq * \). It is straightforward to see that if \( \epsilon'_{n,n} \circ (s \wedge \tilde{s})([y] \wedge [x]) \neq * \) then \( \epsilon^{(0)}_{n,n}([y] \wedge [x]) \neq * \) using the assumption that \( N'_n \subset \text{int } N_n \). Conversely, suppose that \( \epsilon^{(0)}_{n,n}([y] \wedge [x]) \neq * \). Then \( \varphi(x, 3T), \varphi(y, -3T) \in A^{[-3T, 3T]} \) and we have
\[
\varphi(x, [2T, 3T]) = \varphi(\varphi(x, 3T), [-T, 0]) \subset A^{[-2T, 2T]} \subset \text{int } N_n, \\
\varphi(y, [-3T, 2T]) = \varphi(\varphi(y, -3T), [0, T]) \subset A^{[-2T, 2T]} \subset \text{int } N_n.
\]
This implies that \( \epsilon^{(0)}_{n,n}([y] \wedge [x]) \neq * \).

A calculation similar to that in the proof of Lemma [12] proves the following:

**Lemma 4.6.** Suppose that \( \lambda < \lambda_n, \mu > \mu_n \). Take \( S^1 \)-equivariant manifold isolating blocks \( N_n, N'_n \) for \( \text{inv}(V^\lambda_{\lambda_n} \cap J^+_n), \text{inv}(V^\mu_{\mu_n} \cap J^+_n) \). Note that we have canonical homotopy equivalences
\[
\Sigma^\lambda_{\lambda_n} (N_n/L_n) \cong N'_n/L'_n, \quad \Sigma^\mu_{\mu_n} (N_n/T_n) \cong N'_n/T'_n.
\]
See Proposition 5.6 of [9]. The following diagram is commutative up to \( S^1 \)-equivariant homotopy:
\[
\begin{diagram}
\Sigma^\mu_{\mu_n} (N_n/T_n) \wedge \Sigma^\lambda_{\lambda_n} (N_n/L_n) \\
\cong \epsilon'_{n,n} \Rightarrow \Sigma^W \epsilon_{n,n} \\
\Rightarrow \epsilon_{n,n} \\
(N'_n/T'_n) \wedge (N'_n/L'_n)
\end{diagram}
\]
Here \( W = V^\lambda_{\lambda_n} \oplus V^\mu_{\mu_n} \).

This lemma implies that the morphism \( \epsilon_{n,n} \) (and hence \( \epsilon_{m,n} \)) is independent of the choices of \( \lambda_n, \mu_n \).
We have obtained a collection \( \{ \epsilon_{m,n} : \bar{I}_n \wedge I_m \to S \}_{n \geq m} \) of morphisms. Since \( j_{m,n} = j_{m+1,n} \circ j_{m,m+1} \), the following diagram is commutative:

\[
\begin{array}{ccc}
\bar{I}_n \wedge I_m \\
\downarrow_{\text{id} \wedge j_{m,m+1}} \\
\bar{I}_n \wedge I_{m+1}
\end{array}
\xrightarrow{\epsilon_{m,n}}
\begin{array}{ccc}
S \\
\downarrow_{\epsilon_{m+1,n}} \\
S
\end{array}
\]

(14)

**Lemma 4.7.** For \( m < n \), the following diagram is commutative:

\[
\begin{array}{ccc}
\bar{I}_n \wedge I_m \\
\downarrow_{\hat{j}_{n,n+1} \wedge \text{id}} \\
\bar{I}_{n+1} \wedge I_m
\end{array}
\xrightarrow{\epsilon_{m,n}}
\begin{array}{ccc}
S \\
\downarrow_{\epsilon_{m,n+1}} \\
S
\end{array}
\]

(15)

**Proof.** We have to prove that the following diagram is commutative up to \( S^1 \)-equivariant homotopy:

\[
\begin{array}{ccc}
(N_n/\bar{T}_n) \wedge (N_m/L_m) \\
\downarrow_{\hat{\epsilon}_{n,n+1} \wedge \text{id}} \\
(N_{n+1}/\bar{T}_{n+1}) \wedge (N_m/L_m)
\end{array}
\xrightarrow{\hat{\epsilon}_{n,m}}
\begin{array}{ccc}
B_\delta/S_\delta \\
\downarrow_{\epsilon_{m,n+1}} \\
B_\delta/S_\delta
\end{array}
\]

(16)

By Lemma 4.3 we can use the following specific manifold isolating blocks (with corners). First take a manifold isolating block \( N_{n+1} \) for \( \text{inv}(V_{\lambda_{n+1}}^{\mu_{n+1}} \cap J_{n+1}^+ \cup J_{n+1}^-) \). We have compact submanifolds \( L_{n+1}, \bar{T}_{n+1} \) in \( \partial N_{n+1} \) with

\[
\partial N_{n+1} = L_{n+1} \cup \bar{T}_{n+1}, \quad \partial L_{n+1} = \partial \bar{T}_{n+1} = L_{n+1} \cap \bar{T}_{n+1}.
\]

Moreover \( (N_{n+1}, L_{n+1}) \) is an index pair for \( (\text{inv}(V_{\lambda_{n+1}}^{\mu_{n+1}} \cap J_{n+1}^+, \varphi_{n+1}) \) and \( (N_{n+1}, \bar{T}_{n+1}) \) is an index pair for \( (\text{inv}(V_{\lambda_{n+1}}^{\mu_{n+1}} \cap J_{n+1}^+, \varphi_{n+1}) \), where \( \varphi_{n+1} \) is the reverse flow of \( \varphi_{n+1} \). Put

\[
N_m := N_{n+1} \cap J_m^+ = N_{n+1} \cap \bigcap_{j=1}^{b_1} g_{j,+}^{-1}((-\infty,m + \theta]),
\]

\[
L_m := L_{n+1} \cap N_m,
\]

\[
\bar{T}_m := (\bar{T}_{n+1} \cap N_m) \cup \bigcup_{j=1}^{b_1} N_m \cap g_{j,+}^{-1}(m + \theta),
\]

\[
N_n := N_{n+1} \cap J_n^+ = N_{n+1} \cap \bigcap_{j=1}^{b_1} g_{j,+}^{-1}((-\infty,n + \theta]),
\]

\[
L_n := L_{n+1} \cap N_n,
\]

\[
\bar{T}_n := (\bar{T}_{n+1} \cap N_n) \cup \bigcup_{j=1}^{b_1} N_n \cap g_{j,+}^{-1}(n + \theta).
\]
Then $N_m, N_n$ are isolating blocks for $\text{inv}(V^{j+}_{\lambda_n+1} \cap J^+_m)$, $\text{inv}(V^{j+}_{\lambda_n+1} \cap J^+_n)$ and $N_m, N_n, L_m, \overline{T}_m, L_n, \overline{T}_n$ are manifolds with corners (for generic $\theta$). Moreover $(N_m, L_m), (N_n, L_n), (N_n, L_n)$ are index pairs for $(\text{inv}(V^{j+}_{\lambda_n+1} \cap J^+_m), \varphi_{n+1}), (\text{inv}(V^{j+}_{\lambda_n+1} \cap J^+_n), \varphi_{n+1}), (\text{inv}(V^{j+}_{\lambda_n+1} \cap J^+_n), \varphi_{n+1})$, $(\text{inv}(V^{j+}_{\lambda_n+1} \cap J^+_n), \varphi_{n+1})$ respectively. Also we have

\[
\begin{align*}
L_m \cup \overline{T}_m &= \partial N_m, \quad \partial L_m = \partial \overline{T}_m = L_m \cap \overline{T}_m, \\
L_n \cup \overline{T}_n &= \partial N_n, \quad \partial L_n = \partial \overline{T}_n = L_n \cap \overline{T}_n.
\end{align*}
\]

The connecting morphisms $j_{n,m} : I_m \to I_n, j_{n,m+1} : I_m \to I_{n+1}$ and $\tilde{j}_{n,m+1} : I_{n+1} \to I_n$ are induced by the inclusions

\[
i_{m,n} : N_m/L_m \to N_n/L_n, \quad i_{m,n+1} : N_m/L_m \to N_{n+1}/L_{n+1}
\]

and projection

\[
\tilde{i}_{n,m+1} : N_{n+1}/\overline{T}_{n+1} \to N_{n+1}/\overline{T}_n.
\]

With the index pairs we have taken above, for $x \in N_m, y \in N_{n+1}$ we can write

\[
\tilde{\epsilon}_{m,n+1}([y] \wedge [x]) = \begin{cases} 
[b_{n+1}(y) - a_{n+1}(x)] & \text{if } \|b_{n+1}(y) - a_{n+1}(x)\| < \delta, \\
\ast & \text{otherwise}.
\end{cases}
\]

Also we have

\[
\tilde{\epsilon}_{m,n} \circ (\tilde{i}_{n,m+1} \wedge \text{id})([y] \wedge [x]) = \begin{cases} 
[b_n(y) - a_n(x)] & \text{if } y \in N_n, \|b_n(y) - a_n(x)\| < \delta, \\
\ast & \text{otherwise}.
\end{cases}
\]

We may suppose that $a_n(x) = a_{n+1}(x)$ for $x \in N_m$. Note that if $\tilde{\epsilon}_{m,n+1}([y] \wedge [x]) \neq \ast$ or $\tilde{\epsilon}_{m,n} \circ (\tilde{i}_{n,m+1} \wedge \text{id})([y] \wedge [x]) \neq \ast$ we have $y \in \nu_{5\delta}(N_m)$. For small $\delta > 0$ we can suppose that $\nu_{5\delta}(N_m) \cap N_{n+1} \subset N_n$ and that $b_{n+1}(y) = b_n(y)$ for $y \in \nu_{5\delta}(N_m) \cap N_{n+1}$. This implies that (16) commutes.

\[
\square
\]

The commutativity of the diagrams (14) and (15) means that the collection $\{\epsilon_{m,n}\}_{m,n}$ defines an element $\epsilon$ of $\lim_{\infty \leftarrow m,n} \text{mor}_\epsilon(I_n \wedge I_m, S)$. Next we will define $\eta \in \lim_{\infty \leftarrow m,n} \text{mor}(S, I_m \wedge I_n)$. Take a manifold isolating block $N_n(\subset V^{b_n}_{\lambda_n})$ of $\text{inv}(V^{j+}_{\lambda_n} \cap J^+_n)$. As usual we have compact submanifolds $L_n, \overline{T}_n$ of $\partial N_n$ such that

\[
\partial N_n = L_n \cup \overline{T}_n, \quad \partial L_n = \partial \overline{T}_n = L_n \cap \overline{T}_n
\]

and that $(N_n, L_n), (N_n, \overline{T}_n)$ are index pairs for $(\text{inv}(V^{j+}_{\lambda_n} \cap J^+_n), \varphi_n)$, $(\text{inv}(V^{j+}_{\lambda_n} \cap J^+_n), \varphi_n)$ respectively. Taking a large positive number $R > 0$ we may suppose
that $N_n \subset B_{R/2}$, where $B_{R/2} = \{x \in V_{\lambda_n}^m ||x|| \leq R/2\}$. We define

$$\tilde{\eta}_{n,n} : (V_{\lambda_n}^m)^+ = B_R/S_R \to (N_n/L_n) \wedge (N_n/T_n)$$

by

$$\tilde{\eta}_{n,n}([x]) = \begin{cases} [x] \wedge [x] & \text{if } x \in N_n, \\ * & \text{otherwise.} \end{cases}$$

We can see that $\tilde{\eta}_{n,n}$ is a well-defined continuous map and induces a morphism

$$\eta_{n,n} : S \to I_n \wedge \tilde{I}_n.$$ 

For $m > n$, we define $\eta_{m,n} : S \to I_m \wedge \tilde{I}_n$ to be the composition

$$S \eta_{m,n} \leftarrow I_m \wedge \tilde{I}_n \overset{j_{m,n}}{\longrightarrow} I_m \wedge \tilde{I}_n.$$ 

**Lemma 4.8.** The morphism $\eta_{m,n} \in \text{mor}_S(S, I_m \wedge \tilde{I}_n)$ is independent of the choices of $R$ and $N_n$.

**Proof.** The independence from $R$ is straightforward. We prove the independence from the choice of $N_n$. Take another manifold isolating block $N'_n$ of inv($V_{\lambda_n}^m \cap J_n^+$). We may assume that $N_n, N'_n \subset A$ for an isolating neighborhood $A$ of inv($V_{\lambda_n}^m \cap J_n^+$). It is sufficient to show that the following diagram is commutative up to $S^1$-equivariant homotopy:

$$\begin{array}{ccc} B_R/S_R & \xrightarrow{\tilde{\eta}_{n,n}} & (N_n/L_n) \wedge (N_n/T_n) \\
\downarrow \tilde{\eta}_{n,n} & & \downarrow s \wedge \tilde{s} \\
(N'_n/L'_n) \wedge (N'_n/T'_n) & \end{array}$$

Here $s = s_T, \tilde{s} = \tilde{s}_T$ are the flow maps with $T \gg 0$. For $x \in B_R$ we have

$$(s \wedge \tilde{s}) \circ \tilde{\eta}_{n,n}([x]) = \begin{cases} \varphi(x, 3T) \wedge \varphi(x, -3T) & \text{if } \varphi(x, [0, 2T]) \subset N_n \setminus L_n, \\
\varphi(x, [T, 3T]) \subset N'_n \setminus L'_n, \\
\varphi(x, [-2T, 0]) \subset N_n \setminus L_n, \\
\varphi(x, [-3T, -T]) \subset N'_n \setminus L'_n, \\
* & \text{otherwise.} \end{cases}$$

and

$$\tilde{\eta}'_{n,n}([x]) = \begin{cases} [x] \wedge [x] & \text{if } x \in \text{int } N'_n, \\ * & \text{otherwise.} \end{cases}$$

We can reduce the proof to the case $N_n \subset \text{int } N'_n$. Suppose $N_n \subset \text{int } N'_n$. Also we may assume that $A^{-T,T} \subset \text{int } N_n$, choosing a sufficiently large $T$.

If $(s \wedge \tilde{s}) \circ \tilde{\eta}_{n,n}([x]) \neq *$, we have

$$\varphi(x, [-3T, 3T]) \subset \text{int } N'_n.$$ 

Conversely, suppose that $\varphi(x, [-3T, 3T]) \subset \text{int } N'_n$. Then we have $x \in A^{-3T, 3T}$. Hence

$$\varphi(x, [-2T, 2T]) \subset A^{-T,T} \subset \text{int } N_n.$$
Therefore \( \varphi(x, [0, 2T]) \subset \mathcal{N}_n \setminus L_n \), \( \varphi(x, [-2T, 0]) \subset \mathcal{N}_n \setminus \overline{L}_n \). Thus \( (s \land s) \circ \eta_{n,n}([x]) \neq \ast \). We have obtained:

\[
(s \land s) \circ \eta_{n,n}([x]) =
\begin{cases}
[\varphi(x, 3T)] \land [\varphi(x, -3T)] & \text{if } \varphi(x, [-3T, 3T]) \subset \text{int } \mathcal{N}_n', \\
\ast & \text{otherwise}.
\end{cases}
\]

This is homotopic to \( \eta_{n,n}' \) through a homotopy \( H \) which maps \(([x], s)\) to

\[
[\varphi(x, 3(1 - s)T)] \land [\varphi(x, -3(1 - s)T)]
\]

if \( \varphi(x, [-3(1 - s)T, 3(1 - s)T]) \subset \text{int } \mathcal{N}_n' \)

and to the basepoint otherwise.

\[
\square
\]

**Lemma 4.9.** Let \( \lambda < \lambda_n, \mu > \mu_n \). Take manifolds index pairs \( \mathcal{N}_n, \mathcal{N}_n' \) for \( \text{inv}(J_n \cap V^\mu_n), \text{inv}(J_n \cap V^\lambda_n) \). Then we have the canonical \( S^1 \)-equivariant homotopy equivalence:

\[
\Sigma V^\lambda_n (\mathcal{N}_n/L_n) \cong \mathcal{N}_n'/L_n', \quad \Sigma V^\mu_n (\mathcal{N}_n/\overline{L}_n) \cong \mathcal{N}_n'/\overline{L}_n'.
\]

See Proposition 5.6 of \( [9] \). The following diagram is commutative up to \( S^1 \)-equivariant homotopy:

\[
\begin{array}{ccc}
(V^\mu_n)_+ & \xrightarrow{\Sigma W \eta_{n,n}} & \Sigma W (\mathcal{N}_n/L_n) \land (\mathcal{N}_n/\overline{L}_n) \\
\downarrow & & \downarrow \\
(N_n'/L_n') \land (N_n'/\overline{L}_n') & \xrightarrow{\eta_{n,n}'} & \mathcal{N}_n'/L_n' \land \mathcal{N}_n'/\overline{L}_n'
\end{array}
\]

Here \( W = V^\lambda_n \oplus V^\mu_n \).

This lemma implies that \( \eta_{n,n} \) (and hence \( \eta_{m,n} \)) is independent of the choice of \( \lambda_n, \mu_n \).

Since \( j_{n,m+1} = j_{m,n+1} \circ j_{n,m} \) for \( m \geq n \), the following diagram is commutative:

\[
\begin{array}{ccc}
S & \xrightarrow{\eta_{m,n}} & I_m \land \overline{I}_n \\
\eta_{m+1,n} & \downarrow & \downarrow j_{m,m+1} \land \text{id} \\
I_{m+1} \land \overline{I}_n & \xrightarrow{j_{n,m+1}} & I_m \land \overline{I}_{n+1}
\end{array}
\]

\[
(17)
\]

**Lemma 4.10.** For \( m \geq n + 1 \), the following diagram is commutative:

\[
\begin{array}{ccc}
S & \xrightarrow{\eta_{m,n}} & I_m \land \overline{I}_n \\
\eta_{m+1,n} & \downarrow & \downarrow \text{id} \land j_{n,n+1} \\
I_m \land \overline{I}_{n+1} & \xrightarrow{j_{n,m+1}} & I_m \land \overline{I}_{n+1}
\end{array}
\]

\[
(18)
\]
Proof. Let \( m \geq n + 1 \). We have to show that the following diagram is commutative up to \( S^1 \)-equivariant homotopy:

\[
\begin{array}{c}
B_R/S_R \xrightarrow{\tilde{\eta}_{m,n}} \left( N_m/L_m \right) \wedge \left( N_n/T_n \right) \\
\downarrow \tilde{\eta}_{m,n+1} \quad \downarrow \text{id} \wedge \tilde{\eta}_{n,n+1}
\end{array}
\]

(\ref{equation})

By Lemma \[4.8\] we can use the following specific manifold isolating blocks \( N_m, N_n, N_{n+1} \) (with corners). Fix a manifold isolating block \( N_m \) for \( \text{inv}(V_{\lambda_m}^+ \cap J_m^+) \). Then we have compact submanifolds \( L_m, T_m \) in \( \partial N_m \) such that

\[
\partial N_m = L_m \cap T_m, \quad \partial L_m = \partial T_m = L_m \cap T_m.
\]

Moreover \( (N_m, L_m) \) is an index pair for \( (\text{inv}(V_{\lambda_m}^+ \cap J_m^+)), \varphi_m) \) and \( (N_m, T_m) \) is an index pair for \( (\text{inv}(V_{\lambda_m}^+ \cap J_m^+)), \varphi_m) \). Put

\[
\begin{align*}
N_{n+1} &:= N_m \cap J_{n+1}^+ = N_m \cap \bigcap_{j=1} g_{j,1}^{-1}((-\infty, n + 1 + \theta]), \\
L_{n+1} &:= N_{n+1} \cap L_m, \\
T_{n+1} &:= (T_m \cap N_{n+1}) \cup \bigcup_{j=1} b_1 (N_{n+1} \cap g_{j,1}^{-1}(n + 1 + \theta)).
\end{align*}
\]

Then \( N_{n+1}, L_{n+1} \) and \( T_{n+1} \) are manifolds with corners (for generic \( \theta \)), and \( (N_{n+1}, L_{n+1}), (N_{n+1}, T_{n+1}) \) are index pairs for \( (\text{inv}(V_{\lambda_m}^+ \cap J_{n+1}^+), \varphi_m), \) \( (\text{inv}(V_{\lambda_n}^+ \cap J_{n+1}^+), \varphi_m) \) respectively. We define \( N_n, L_n, T_n \) similarly.

The attractor maps \( i_{n,m} : N_n/L_n \rightarrow N_m/L_m, i_{n+1,m} : N_{n+1}/L_{n+1} \rightarrow N_m/L_m \) are the inclusions. The repeller map \( \tilde{i}_{n,n+1} : N_{n+1}/T_{n+1} \rightarrow N_n/T_n \) is the projection:

\[
N_{n+1}/T_{n+1} \rightarrow \begin{cases} \\
N_{n+1} / \left( T_{n+1} \cup \bigcup_{j=1} b_1 (N_{n+1} \cap g_{j,1}^{-1}([n + \theta, \infty])) \right) & \text{if } x \in N_n, \\
\end{cases}
\]

With these index pairs, for \( x \in B_R \) we can write

\[
\tilde{\eta}_{m,n}([x]) = \begin{cases} \\
[x] \wedge [x] & \text{if } x \in N_n, \\
* & \text{otherwise},
\end{cases}
\]

and

\[
(\text{id} \wedge \tilde{\eta}_{n,n+1}) \circ \tilde{\eta}_{m,n+1}([x]) = \begin{cases} \\
[x] \wedge [x] & \text{if } x \in N_n, \\
* & \text{otherwise}.
\end{cases}
\]

Thus the diagram \( \ref{equation} \) is commutative. \( \square \)

The commutativity of the diagrams \( \ref{equation1}, \ref{equation2} \) implies that the collection \( \{\eta_{m,n}\}_{m,n} \) defines an element \( \eta \in \lim_{\infty \leftarrow n \rightarrow \infty} \text{mor}_\xi(S, I_m \wedge I_n) \).
Proposition 4.11. The morphisms $\epsilon$ and $\eta$ are duality morphisms between $\text{swf}^A(Y)$ and $\text{swf}^R(-Y)$.

Proof. Fix positive numbers $R, \delta$ with $0 < \delta \ll 1 \ll R$. Let $\pi : B_{R}/S_{R} \to B_{\delta}/S_{\delta}$ be the projection

$$B_{R}/S_{R} \to B_{R}/(B_{R} \setminus \text{int} B_{\delta}) = B_{\delta}/S_{\delta},$$

which is a homotopy equivalence. We have to prove that the diagrams (20) below is commutative for $m \ll n \ll m'$ and that the diagram (21) below is commutative up to $S^{1}$-equivariant homotopy for $n \ll m \ll n'$. (See Lemma 3.5 of [12].)

$$\begin{array}{ccc}
(B_{R}/S_{R}) \land (N_{m}/L_{m}) & \xrightarrow{\eta_{m',n} \land \text{id}} & (N_{m'}/L_{m'}) \land (N_{n}/\bar{L}_{n}) \land (N_{m}/L_{m}) \\
\gamma \circ (\pi \land \iota_{m,m'}) & \xrightarrow{} & \text{id} \land \iota_{m,n} \\
\end{array} \quad (20)$$

Here $B_{R} = B(V_{\lambda_{m'}}^{m'}), S_{R} = \partial B(V_{\lambda_{m'}}^{m'}), N_{m}, N_{n}, N_{m'}$ are isolating blocks for $\text{inv}(V_{\lambda_{m'}}^{m'} \cap J_{m'}^{+}), \text{inv}(V_{\lambda_{m'}}^{m'} \cap J_{n}^{+}), \text{inv}(V_{\lambda_{m'}}^{m'} \cap J_{n'}^{+})$ and $\gamma$ is the interchanging map $(B_{\delta}/S_{\delta}) \land (N_{m'}/L_{m'}) \to (N_{m'}/L_{m'}) \land (B_{\delta}/S_{\delta})$.

$$\begin{array}{ccc}
(N_{n'}/\bar{L}_{n'}) \land (B_{R}/S_{R}) & \xrightarrow{\text{id} \land \iota_{n,n'}} & (N_{n'}/\bar{L}_{n'}) \land (N_{m}/L_{m}) \land (N_{n}/\bar{L}_{n}) \\
\gamma \circ (\iota_{n,n'} \land \pi) & \xrightarrow{} & \iota_{m,n'} \land \text{id} \\
(B_{\delta}/S_{\delta}) \land (N_{n}/\bar{L}_{n}) & \xrightarrow{\sigma \land \text{id}} & (B_{\delta}/S_{\delta}) \land (N_{n}/\bar{L}_{n}) \\
\end{array} \quad (21)$$

Here $B_{R} = B(V_{\lambda_{m'}}^{m'}), S_{R} = \partial B(V_{\lambda_{m'}}^{m'}), N_{m}, N_{n}, N_{m'}$ are isolating blocks for $\text{inv}(V_{\lambda_{m'}}^{m'} \cap J_{m'}^{+}), \text{inv}(V_{\lambda_{m'}}^{m'} \cap J_{n}^{+}), \text{inv}(V_{\lambda_{m'}}^{m'} \cap J_{n'}^{+})$, $\gamma$ is the interchanging map $(N_{n}/L_{n}) \land (B_{\delta}/S_{\delta}) \to (B_{\delta}/S_{\delta}) \land (N_{n}/L_{n})$ and $\sigma : B_{\delta}/S_{\delta} \to B_{\delta}/S_{\delta}$ is defined by $\sigma(v) = -v$.

First we consider (20). Let $m \ll n \ll m'$. Take a manifold isolating block $N_{m'}$ for $\text{inv}(V_{\lambda_{m'}}^{m'} \cap J_{m'}^{+})$. As in the proof of Lemma 1.7 from $N_{m'}$ and the functions $g_{j,+}$, we get index pairs

$$(N_{n}, L_{n}), (N_{n}, \bar{L}_{n}), (N_{m}, L_{m}), (N_{m}, \bar{L}_{m})$$

for

$$\text{inv}(V_{\lambda_{m'}}^{m'} \cap J_{m'}^{+}), \varphi_{m'}), (\text{inv}(V_{\lambda_{m'}}^{m'} \cap J_{n}^{+}), \varphi_{m}), (\text{inv}(V_{\lambda_{m'}}^{m'} \cap J_{n'}^{+}), \varphi_{m'}).$$

The attractor map

$$i_{m,n} : N_{m}/L_{m} \to N_{n}/L_{n}, \quad i_{n,m} : N_{n}/L_{n} \to N_{m}/L_{m}$$

are the injections, and the repeller maps

$$\bar{i}_{m,n'} : N_{m'}/\bar{L}_{m'} \to N_{n}/\bar{L}_{n}, \quad \bar{i}_{m,n} : N_{n}/\bar{L}_{n} \to N_{m}/\bar{L}_{m}$$

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are the projections.

For \( x \in N_m \) and \( y \in B_R(= B(V^R_m', R)) \), we can write

\[
(id \wedge \hat{e}_{m, n}) \circ (\hat{\eta}_{m', n} \wedge id)([y] \wedge [x]) =
\begin{cases}
[y] \wedge [b_n(y) - a_n(x)] & \text{if } \{ y \in N_n, \|b_n(y) - a_n(x)\| < \delta, \\
* & \text{otherwise}.
\end{cases}
\]

Note that if \( \|b_n(x) - a_n(x)\| < \delta \) for some \( x \in N_m \) we have \( y \in \nu_{5\delta}(N_m) \).

Fix an \( S^1 \)-equivariant homotopy equivalence

\( r : \nu_{5\delta}(N_m) \to N_m \)

which is close to the identity such that

\[
\begin{align*}
& r(\nu_{5\delta}(L_n) \cap \nu_{5\delta}(N_m)) \subset L_m, \quad r(\nu_{5\delta}(L_m)) \subset L_m.
\end{align*}
\]

Then \( (id \wedge \hat{e}_{m, n}) \circ (\hat{\eta}_{m', n} \wedge id) \) is homotopic to a map

\( f : (B_R/S_R) \wedge (N_m/L_m) \to (N_{m'}/L_{m'}) \wedge (B_{\delta}/S_{\delta}) \)

defined by

\[
\begin{cases}
[r(y)] \wedge [b_n(y) - a_n(x)] & \text{if } \{ x \in N_m, y \in N_n \cap \nu_{5\delta}(N_m), \|b_n(y) - a_n(x)\| < \delta, \\
* & \text{otherwise}.
\end{cases}
\]

Define

\[
H : (B_R/S_R) \wedge (N_m/L_m) \times [0, 1] \to (N_{m'}/L_{m'}) \wedge (B_{\delta}/S_{\delta})
\]

by

\[
\begin{cases}
[r((1 - s)y + sx)] \wedge [b_n(y) - a_n(x)] & \text{if } \{ x \in N_m, y \in N_n \cap \nu_{5\delta}(N_m), \|b_n(y) - a_n(x)\| < \delta, \\
* & \text{otherwise}.
\end{cases}
\]

We can easily see that \( H \) is well-defined. We will show that \( H \) is continuous.

It is sufficient to show that if we have a sequence \( (x_j, y_j, s_j) \) in \( N_m \times N_m \times [0, 1] \) with \( y_j \to y \in \partial N_m = L_n \cup \overline{L}_n \) we have \( H([y_j] \wedge [x_j], s_j) \to * \). If \( y \in \overline{L}_n \), we have \( \|b_n(y_j) - a_n(x_j)\| \geq \delta \) for large \( j \). Hence \( H([y_j] \wedge [x_j], s_j) \to * \). Consider the case \( y \in L_n \). Assume that \( \lim_{j \to \infty} H([y_j] \wedge [x_j], s_j) \neq * \). After passing to a subsequence we may suppose that \( H([y_j] \wedge [x_j], s_j) \neq * \) for all \( j \). Then \( \|y_j - x_j\| < 5\delta \) for all \( j \). For large \( j \) we have \( (1 - s_j)y_j + s_jx_j \in \nu_{5\delta}(L_n) \cap \nu_{5\delta}(N_m) \). Hence \( r((1 - s_j)y_j + s_jx_j) \in L_m \subset L_{m'} \), which implies \( H([y_j] \wedge [x_j], s_j) = * \). This is a contradiction. Therefore \( H \) is continuous.
We have $H(\cdot, 0) = f$ and

$$H([y] \wedge [x], 1) = \begin{cases} 
[r(x)] \wedge [b_n(y) - a_n(x)] & \text{if } x \in N_m, y \in N_n, \\
* & \text{otherwise}.
\end{cases}$$

Fix a positive number $\delta' > 0$ with $0 \ll \delta' \ll \delta$. Take an $S^1$-equivariant continuous map $a'_n : N_n \to N_n$ such that

$$\|a'_n(x) - a_n(x)\| < 2\delta', \quad a'_n(N_n) \subset N_n \setminus \nu_{\delta}(\partial N_n).$$

Through the homotopy equivalence

$$B_\delta/S_\delta = V_{\lambda_m}^{\mu_{\mu'}}/(V_{\lambda_m}^{\mu_{\mu'}} - \text{int } B_\delta) \to V_{\lambda_m}^{\mu_{\mu'}}/(V_{\lambda_m}^{\mu_{\mu'}} - \text{int } B_{\delta'}) = B_{\delta'}/S_{\delta'},$$

$H(\cdot, 1)$ is homotopic to a map

$$f' : (B_R/S_R) \wedge (N_m/L_m) \to (N_{m'}/L_{m'}) \wedge (B_{\delta'}/S_{\delta'})$$

defined by

$$f'([y] \wedge [x]) = \begin{cases} 
[r(x)] \wedge [b_n(y) - a'_n(x)] & \text{if } x \in N_m, y \in N_n, \\
* & \text{otherwise}.
\end{cases}$$

There is a homotopy $h : N_n \times [0, 1] \to N_n$ from $b_n$ to the identity such that

$$h(T_n, s) \subset \overline{T}_n, \quad \|h(y, s) - y\| < 2\delta$$

for all $y \in N_n$ and $s \in [0, 1]$. Then $h$ naturally induces a homotopy

$$H' : (B_R/S_R) \wedge (N_m/L_m) \to (N_{m'}/L_{m'}) \wedge (B_{\delta'}/S_{\delta'})$$

defined by

$$H'([y] \wedge [x], s) = \begin{cases} 
[r(x)] \wedge [h(y, s) - a'_n(x)] & \text{if } x \in N_m, y \in N_n, \\
* & \text{otherwise}.
\end{cases}$$

It is straightforward to see that $H'$ is well-defined. To show that $H'$ is continuous, it is sufficient to prove that if we have a sequence $(x_j, y_j, s_j)$ in $N_m \times N_n \times [0, 1]$ with $y_j \to y \in \partial N_n = L_n \cup \overline{T}_n$ then $H'([y_j] \wedge [x_j], s_j) \to *$. Suppose that $y \in \overline{T}_n$. Then for large $j$ we have $\|h(y_j, s) - a'_n(x)\| \geq \delta'$. Thus $H([y_j] \wedge [x_j], s_j) \to *$.

It is straightforward to see that $H'$ is well-defined. To show that $H'$ is continuous, it is sufficient to prove that if we have a sequence $(x_j, y_j, s_j)$ in $N_m \times N_n \times [0, 1]$ with $y_j \to y \in \partial N_n = L_n \cup \overline{T}_n$ then $H'([y_j] \wedge [x_j], s_j) \to *$. Suppose that $y \in L_n$. If $\lim_{j \to \infty} H'([y_j] \wedge [x_j], s_j) \neq *$, after passing to a subsequence, we may assume that $H([y_j] \wedge [x_j], s_j) \neq *$ for all $j$, which implies that $\|y_j - x_j\| < 5\delta$. So we have $x_j \in \nu_{\delta}(L_n) \cap \nu_{\delta}(N_m)$ for large $j$. Hence $r(x_j) \in N_m$, which means $H''([y_j] \wedge [x_j], s_j) = *$. This is contradiction. Therefore $H'$ is continuous.

We can see that $H'$ is a homotopy from $f'$ to a map $f'' : (B_R/S_R) \wedge (N_m/L_m) \to (N_{m'}/L_{m'}) \wedge (B_{\delta'}/S_{\delta'})$ defined by

$$f''([y] \wedge [x]) = \begin{cases} 
[r(x)] \wedge [y - a'_n(x)] & \text{if } x \in N_m, y \in B_R, \\
* & \text{otherwise}.
\end{cases}$$
Note that for \( y \in B_R \setminus N_n \) we have \( f''([y] \wedge [x]) = * \) since \( \|y - a'_n(x)\| \geq \delta' \).

Define
\[
H'' : (B_R/S_R) \wedge (N_m/L_m) \times [0, 1] \to (N_{m'}/L_{m'}) \wedge (B_{y'}/S_{y'})
\]
by
\[
H''([y] \wedge [x], s) = \begin{cases} [r(x)] \wedge [y - (1 - s)a'_n(x)] & \text{if } x \in N_m, y \in B_R, \\
* & \text{if } \|y - (1 - s)a'_n(x)\| < \delta', \\
\end{cases}
\]
otherwise.

It is straightforward to see that \( H'' \) is well-defined and continuous. We have
\[
H''([y] \wedge [x], 1) = \begin{cases} [r(x)] \wedge [y] & \text{if } x \in N_m, y \in B_R, \\
* & \text{if } \|y\| < \delta', \\
\end{cases}
\]
otherwise.

We can easily show that \( H''(\cdot, 1) \) is homotopic to \( \gamma \circ (\pi \wedge i_{m,m'}) \).

We have proved that \( (\text{id} \wedge \hat{\epsilon}_{m,n}) \circ (\hat{\eta}_{m',n} \wedge \text{id}) \) is \( S^1 \)-equivariantly homotopic to \( \gamma \circ (\pi \wedge i_{m,m'}) \), which implies that the diagram \( (20) \) is commutative up to \( S^1 \)-equivariant homotopy.

Let us consider \( (21) \). We have to prove that for \( n \ll m \ll n' \) the composition
\[
(N_{n'}/\overline{T}_{n'}) \wedge (B_R/S_R) \xrightarrow{\text{id} \wedge \hat{\eta}_{m,n}} (N_{n'}/\overline{T}_{n'}) \wedge (N_m/L_m) \wedge (N_n/\overline{L}_n)
\]
\[
\xrightarrow{\hat{\epsilon}_{m,n'} \wedge \text{id}} (B_{\delta}/S_{\delta}) \wedge (N_n/\overline{L}_n)
\]
is \( S^1 \)-equivariantly homotopic to \( (\sigma \wedge \text{id}) \circ \gamma \circ (i_{m',n} \wedge \pi) \).

For \( x \in B_R = B(V_{n'_{m,n'}}, R), y \in N_{n'} \) we have
\[
(\hat{\epsilon}_{m,n'} \wedge \text{id}) \circ (\text{id} \wedge \hat{\eta}_{m,n})([y] \wedge [x]) =
\begin{cases} [b_n'(y) - a_n'(x)] \wedge [x] & \text{if } x \in N_n, \\
* & \text{if } \|b_n'(y) - a_n'(x)\| < \delta, \\
\end{cases}
\]
otherwise.

Take a homotopy equivalence \( \tilde{r} : \nu_{5\delta}(N_n) \to N_n \) which is close to the identity such that
\[
\tilde{r}(\nu_{5\delta}(\overline{T}_{n'}) \cap \nu_{5\delta}(N_n)) \subset \overline{T}_{n'}, \tilde{r}(\nu_{5\delta}(\overline{L}_n)) \subset \overline{L}_n.
\]
(22)

Note that if \( \|b_n'(y) - a_n'(x)\| < \delta \) for some \( x \in N_n \) we have \( y \in \nu_{5\delta}(N_n) \).

It is straightforward to see that \( (\hat{\epsilon}_{m,n'} \wedge \text{id}) \circ (\text{id} \wedge \hat{\eta}_{m,n}) \) is homotopic to a map
\[
f : (N_{n'}/\overline{T}_{n'}) \wedge (B_R/S_R) \to (B_{\delta}/S_{\delta}) \wedge (N_n/L_n)
\]
defined by

\[ f([y] \wedge [x]) = \begin{cases} 
  [b_{n'}(y) - a_{n'}(x)] \wedge [\bar{\varphi}(x)] & \text{if } \begin{cases} 
    x \in N_n, \\
    y \in N_{n'} \cap \nu_{5\delta}(N_n), \\
    \|b_{n'}(y) - a_{n'}(x)\| < \delta,
  \end{cases} \\
  * & \text{otherwise.}
\end{cases} \]

Define a homotopy \( H : (N_{n'}/L_{n'}) \wedge (B_R/S_R) \times [0, 1] \to (B_\delta/S_\delta) \wedge (N_n/L_n) \) by

\[ H([y] \wedge [x], s) = \begin{cases} 
  [b_{n'}(y) - a_{n'}(x)] \wedge [\bar{\varphi}((1-s)x + sy)] & \text{if } \begin{cases} 
    x \in N_n, \\
    y \in N_{n'} \cap \nu_{5\delta}(N_n), \\
    \|b_{n'}(y) - a_{n'}(x)\| < \delta
  \end{cases}, \\
  * & \text{otherwise.}
\end{cases} \]

Then \( H \) is a well-defined and continuous homotopy from \((\text{id} \wedge \hat{\epsilon}_{m,n'}) \circ (\text{id} \wedge \hat{\eta}_{m,n})\) to \( H(\cdot, 1) \). We have

\[ H([y] \wedge [x], 1) = \begin{cases} 
  [b_{n'}(y) - a_{n'}(x)] \wedge [\bar{\varphi}(y)] & \text{if } \begin{cases} 
    x \in N_n, \\
    y \in N_{n'} \cap \nu_{5\delta}(N_n), \\
    \|b_{n'}(y) - a_{n'}(x)\| < \delta
  \end{cases}, \\
  * & \text{otherwise.}
\end{cases} \]

Fix a positive number \( \delta' \) with \( 0 < \delta' \ll \delta \). Take a continuous map \( b_{n'} : N_n \to N_n \) such that

\[ b_{n'}(N_n) \subset N_n \setminus \nu_{\delta'}(\partial N_n), \quad \|b_{n'}(y) - b_{n'}(x)\| < 2\delta' \quad \text{(for } x \in N_n). \]

Then through the homotopy equivariance \( B_\delta/S_\delta \to B_{\delta'}/S_{\delta'}, f \) is homotopic to a map

\[ f' : (N_{n'}/L_{n'}) \wedge (B_R/S_R) \to (B_{\delta'}/S_{\delta'}) \wedge (N_n/L_n) \]

defined by

\[ f'([y] \wedge [x]) = \begin{cases} 
  [b_{n'}(y) - a_{n'}(x)] \wedge [\bar{\varphi}(y)] & \text{if } \begin{cases} 
    x \in N_n, \\
    y \in N_{n'} \cap \nu_{5\delta}(N_n), \\
    \|b_{n'}(y) - a_{n'}(x)\| < \delta',
  \end{cases} \\
  * & \text{otherwise.}
\end{cases} \]

There is a homotopy \( h : N_{n'} \times [0, 1] \to N_{n'} \) from \( a_{n'} \) to the identity such that

\[ h(L_{n'}) \subset L_{n'}, \quad \|h(y, s) - y\| < 2\delta. \]
We can see that \( h \) induces a homotopy \( H' \) from \( f' \) to a map \( f'' \) defined by

\[
f''([y] \land [x]) = \begin{cases} 
[b'_n(y) - x] \land [\bar{r}(y)] & \text{if } x \in B_R, \\
\ast & \text{otherwise.}
\end{cases}
\]

Note that for \( x \in B_R \setminus \text{int } N_n \) we have \( f'([y] \land [x]) = \ast \) since \( ||b'_n(y) - x|| \geq \delta' \).

Define a homotopy \( H'' : (N_n/\mathcal{L}_n') \land (B_R/S_R) \times [0,1] \to (B_{S'}/S_{S'}) \land (N_n/\mathcal{L}_n) \) by

\[
H''([y] \land [x], s) := \begin{cases} 
[(1 - s)b'_n(y) - x] \land [\bar{r}(y)] & \text{if } x \in B_R, \\
\ast & \text{otherwise.}
\end{cases}
\]

Then \( H'' \) is well-defined and continuous, and we have

\[
H''([y] \land [x], 1) = \begin{cases} 
[-x] \land [\bar{r}(y)] & \text{if } x \in B_R, \\
\ast & \text{otherwise.}
\end{cases}
\]

It is straightforward to see that \( H''(\cdot, 1) \) is homotopic to \( (\sigma \land \text{id}) \circ \gamma \circ (\tilde{i}_{m,n} \land \pi) \). Thus the diagram (21) is commutative up \( S^1 \)-equivariant homotopy.

5. Relative Bauer-Furuta invariants for 4-manifolds

5.1. Setup. Let \( X \) be a compact, connected, oriented, Riemannian 4–manifold with nonempty boundary \( \partial X := Y \) not necessarily connected. Equip \( X \) with a spin\(^c\) structure \( \hat{s} \) which induces a spin\(^c\) structure \( s \) on \( Y \). Denote by \( S_X = S^+ \oplus S^- \) the spinor bundle of \( X \) and denote by \( \hat{\rho} \) the Clifford multiplication. Choose a metric \( \hat{g} \) on \( X \) so that a neighborhood of the boundary is isometric to the cylinder \( [-3,0] \times Y \) with the product metric and \( \partial X \) identified with \( \{0\} \times Y \). To make some distinction, we will often decorate notations associated to \( X \) with hat. For instance, let \( g \) be the Riemannian metric on \( Y \) restricted from \( \hat{g} \) on \( X \). Let \( S_Y \) be the associated spinor bundle on \( Y \) and \( \rho : TY \to \text{End}(S_Y) \) be the Clifford multiplication.

We write \( Y = \bigsqcup Y_j \) as a union of connected component. From now on, we will treat \( X \) as a spin\(^c\) cobordism, i.e. we label each connected component of \( Y \) as either incoming or outgoing satisfying \( Y = -Y_{\text{in}} \sqcup Y_{\text{out}} \).

We sometimes write this cobordism as \( X : Y_{\text{in}} \to Y_{\text{out}} \). Denote by \( \iota : Y \hookrightarrow X \) the inclusion map. We also choose the following auxiliary data when defining our invariants

- A basepoint \( \hat{o} \in X \).
- A set of loops \( \{\alpha_1, \ldots, \alpha_{b_{1,n}}\} \) in \( X \) representing a basis of the cokernel of the induced map \( \iota_* : H_1(Y; \mathbb{R}) \to H_1(X; \mathbb{R}) \).
• A set of loops \( \{ \beta_1, \ldots, \beta_{b_0} \} \) in \( Y_{in} \) representing a basis of a subspace complementary to kernel of the induced map \( t_*: H_1(Y_{in}; \mathbb{R}) \to H_1(X; \mathbb{R}) \).
• A set of loops \( \{ \beta_{b_0+1}, \ldots, \beta_{b_1,0} \} \) in \( Y_{out} \) such that \( \{ \beta_1, \ldots, \beta_{b_1,0} \} \)
  represents a basis of a subspace complementary to the kernel of the induced map \( t_*: H_1(Y; \mathbb{R}) \to H_1(X; \mathbb{R}) \).
• A based path data \( [\eta] \), whose definition is given below.

**Definition 5.1.** A based path data is an equivalence class of paths \( (\eta_1, \eta_2, \ldots, \eta_{b_0}(Y)) \)
where each \( \eta_j \) is a path from \( \hat{o} \) to a point in \( Y_j \). We say that \( (\eta_1, \ldots, \eta_{b_0}(Y)) \)
and \( (\eta_1', \ldots, \eta_{b_0}(Y)) \) are equivalent if the composed path \( \eta_j^*(-\eta_j) \) represents the zero class in \( H_1(X,Y; \mathbb{R}) \) for each \( j = 1, \ldots, b_0(Y) \).

**Remark 5.2.** (i) The set of loops \( \{ \alpha_1, \ldots, \alpha_{b_1,0} \} \) corresponds to a dual basis of kernel of \( t^*: H^1(X; \mathbb{R}) \to H^1(Y; \mathbb{R}) \).
(ii) The set of loops \( \{ \beta_1, \ldots, \beta_{b_1,0} \} \) corresponds to a dual basis of image of \( t^*: H^1(X; \mathbb{R}) \to H^1(Y; \mathbb{R}) \).
(iii) It follows that \( b_{1,0} = \dim \ker t^*, \ b_{1,\beta} = \dim \text{im} t^*, \) and \( b_{1,0} + b_{1,\beta} = b_{1}(X) \).

As usual, we will set up the Seiberg–Witten equations on a particular slice of the configuration space. For the manifold with boundary \( X \), we will consider the double Coulomb condition introduced by the first author \[^8\] rather than the classical Coulomb–Neumann condition. Let us briefly recall the definition.

**Definition 5.3.** For a 1-form \( \hat{a} \) on \( X \), we have a decomposition \( \hat{a}|_Y = t\hat{a} + n\hat{a} \)
on the boundary, where \( t\hat{a} \) and \( n\hat{a} \) are the tangential part and the normal part respectively. When \( Y = \bigsqcup Y_i \) has several connected components, we denote by \( t_i\hat{a} \) and \( n_i\hat{a} \) the corresponding parts of \( \hat{a}|_Y \). We say that a 1-form \( \hat{a} \) satisfies the double Coulomb condition if:

1. \( \hat{a} \) is coclosed, i.e. \( d^*\hat{a} = 0 \);
2. Its restriction to the boundary is coclosed, i.e. \( d^*(t\hat{a}) = 0 \);
3. For each \( j \), we have \( \int_{Y_j} t_j(*\hat{a}) = 0 \).

We denote by \( \Omega_{CC}^1(X) \) the space of 1-forms satisfying the double Coulomb condition.

As a consequence of \[^8\] Proposition 2.2, we can identify \( H^1(X; \mathbb{R}) \) with the space of harmonic 1-forms satisfying double Coulomb condition

\[ H^1(X; \mathbb{R}) \cong \mathcal{H}_{CC}^1(X) := \{ a \in \Omega_{CC}^1(X) \mid d\hat{a} = 0 \} \]

Since \( X \) is connected, we observe that the cohomology long exact sequence of the pair \( (X,Y) \) gives rise to a short exact sequence

\[ 0 \to \mathbb{R}^{b_0(Y)-1} \to H^1(X,Y; \mathbb{R}) \to \ker t^* \to 0. \]

By the classical Hodge Theorem, each element of the relative cohomology group \( H^1(X,Y; \mathbb{R}) \) is represented by a harmonic 1-form with Dirichlet
boundary condition. Since condition (3) from Definition 5.3 is of codimension \( b_0(Y) - 1 \), we can conclude that the space of harmonic 1-forms satisfying both Dirichlet boundary condition and condition (3) from Definition 5.3 is isomorphic to \( \ker \iota^* \). Notice that such 1-forms trivially satisfy other double Coulomb conditions. Hence, we make an identification

\[
\ker \iota^* \cong \mathcal{H}_{DC}^1(X) := \{ \hat{a} \in \Omega^1_{CC}(X) \mid d\hat{a} = 0, \ t\hat{a} = 0 \}. \tag{23}
\]

The double Coulomb slice \( Coul^{CC}(X) \) is defined as

\[
Coul^{CC}(X) := L^2_{k+1/2}(i\Omega^1_{CC}(X) \oplus \Gamma(S^+)),
\]

where \( k \) is an integer greater than 4 fixed throughout the paper. Next, we introduce projections from \( Coul^{CC}(X) \) related to the chosen loops \( \{\alpha_1, \ldots, \alpha_{b_1,\alpha}\} \) and \( \{\beta_1, \ldots, \beta_{b_1,\beta}\} \). We define a (nonorthogonal) projection

\[
\hat{p}_\alpha : Coul^{CC}(X) \to \mathcal{H}_{DC}^1(X) \tag{24}
\]

by sending \( (\hat{a}, \hat{\phi}) \) to the unique element in \( \mathcal{H}_{DC}^1(X) \) satisfying

\[
\int_{\alpha_j} \hat{a} = i \int_{\alpha_j} \hat{p}_\alpha(\hat{a}, \hat{\phi}) \text{ for every } j = 1, 2, \ldots, b_1,\alpha.
\]

On the other hand, we define a map

\[
\hat{p}_\beta : Coul^{CC}(X) \to \mathbb{R}^{b_1,\beta}
\]

\[
(\hat{a}, \hat{\phi}) \mapsto (-i \int_{\beta_1} t\hat{a}, \ldots, -i \int_{\beta_{b_1,\beta}} t\hat{a}).
\]

Note that \( \hat{p}_\alpha \) and \( \hat{p}_\beta \) together keep track of the \( H^1(X; \mathbb{R}) \)-component of \( (\hat{a}, \hat{\phi}) \). We have a decomposition

\[
\hat{p}_\beta = \hat{p}_{\beta,in} \oplus \hat{p}_{\beta,out},
\]

where

\[
\hat{p}_{\beta,in}(\hat{a}, \hat{\phi}) = (-i \int_{\beta_1} t\hat{a}, \ldots, -i \int_{\beta_{b_1,n}} t\hat{a}),
\]

\[
\hat{p}_{\beta,out}(\hat{a}, \hat{\phi}) = (-i \int_{\beta_{b_1,n+1}} t\hat{a}, \ldots, -i \int_{\beta_{b_1,\beta}} t\hat{a}).
\]

We now proceed to describe the group of gauge transformations. Denote by \( G_X \) the \( L^2_{k+3/2} \)-completion of \( \text{Map}(X, S^1) \). The action of an element \( \hat{u} \in \text{Map}(X, S^1) \) is given by

\[
\hat{u} \cdot (\hat{a}, \hat{\phi}) = (\hat{a} - \hat{u}^{-1} d\hat{u}, \hat{u}\hat{\phi}).
\]

The proof of the following lemma is a slight adaptation of [8, Proposition 2.2] and we omit it.
Lemma 5.4. Inside each connected component of $\mathcal{G}_X$, there is a unique element $\hat{u} : X \to S^1$ satisfying
\[ \hat{u}(\hat{o}) = 1, \quad u^{-1} du \in i\Omega^1_{CC}(X). \]
These elements form a subgroup, denoted by $\mathcal{G}^{h,\hat{o}}_X$, of harmonic gauge transformation with double Coulomb condition.

Consequently, there is a natural isomorphism
\[ \mathcal{G}^{h,\hat{o}}_X \cong \pi_0(\mathcal{G}_X) \cong H^1(X; \mathbb{Z}). \]
We also denote by $\mathcal{G}^{h,\hat{o}}_{X,Y}$ the subgroup of $\mathcal{G}^{h,\hat{o}}_X$ that corresponds to the subgroup $\ker(H^1(X; \mathbb{Z}) \to H^1(Y; \mathbb{Z}))$ of $H^1(X; \mathbb{Z})$. Observe that each element in $\mathcal{G}^{h,\hat{o}}_{X,Y}$ restricts to a constant function on each component of $Y$.

Now we define the relative Picard torus
\[ \text{Pic}^0(X,Y) := H^1_{DC}(X)/\mathcal{G}^{h,\hat{o}}_{X,Y}, \]
This is a torus of dimension $b_1,\alpha$. The double Coulomb slice $\text{Coul}^{CC}(X)$ is preserved by $\mathcal{G}^{h,\hat{o}}_X$ and thus $\mathcal{G}^{h,\hat{o}}_{X,Y}$.

Our main object of interest will be the quotient space $\text{Coul}^{CC}(X)/\mathcal{G}^{h,\hat{o}}_{X,Y}$ regarded as a Hilbert bundle over $\text{Pic}^0(X,Y)$ with bundle structure induced by the projection $\hat{p}_\alpha$. The bundle will be denoted by
\[ \mathcal{W}_X := \text{Coul}^{CC}(X)/\mathcal{G}^{h,\hat{o}}_{X,Y}. \]
Remark 5.5. A different Hilbert bundle structure of $\mathcal{W}_X$ can be induced by the orthogonal projection
\[ \hat{p}_\perp : \text{Coul}^{CC}(X) \to H^1_{DC}(X). \]
However, we prefer $\hat{p}_\alpha$ because $\hat{p}_\alpha$ behaves better than $\hat{p}_\perp$ and simplifies our argument in the proof of gluing theorem for relative Bauer-Furuta invariants.

Definition 5.6. For a pair $(\hat{a},\hat{o}) \in \text{Coul}^{CC}(X)$, we denote by $[\hat{a},\hat{o}]$ the corresponding element in the Hilbert bundle $\mathcal{W}_X$. We write $\| \cdot \|_F$ for the fiber-direction norm on $\mathcal{W}_X$. Note that the norm $\| \cdot \|_F$ is not directly given by the restriction of the $L^2_{k+1/2}$-norm on $\text{Coul}^{CC}(X)$ because the latter is not invariant under $\mathcal{G}^{h,\hat{o}}_{X,Y}$. However, we can construct $\| \cdot \|_F$ as follows: We cover $\text{Pic}^0(X,Y)$ by finitely many small balls $\{B_i\}$. Each $B_i$ can be lifted as a subset of $H^1_{DC}$. With such a lift chosen, we can identify the total space of $\mathcal{W}_X|_{B_i}$ as a subset of $\text{Coul}^{CC}(X)$. Then we use the restriction of the $L^2_{k+1/2}$-norm on $\text{Coul}^{CC}(X)$ to define the fiber direction norm on $\mathcal{W}_X|_{B_i}$. Using a partition of unity, we patch these norms together to form $\| \cdot \|_F$.

Let us fix a fundamental domain $\mathcal{D} \subset H^1_{DC}(X)$ throughout this section. The following equivalence of norms is a consequence of the compactness of $\mathcal{D}$. 
Lemma 5.7. There exists a positive constant $C$ such that for any $(\hat{a}, \hat{\phi}) \in \text{Coul}^{CC}(X)$ such that $\hat{p}_\alpha(\hat{a}, \hat{\phi}) \in \mathfrak{D}$, we have

$$\frac{\|[[\hat{a}, \hat{\phi}]\|_F}{C} \leq ||(\hat{a}, \hat{\phi})\|_{L^2_k} \leq C \cdot (\|[\hat{a}, \hat{\phi}]\|_F + 1).$$

Lastly, we will consider some restriction maps on the bundle. Recall that the Coulomb slice on 3-manifolds is given by

$$\text{Coul}(Y) := \{(a, \phi) \in L^2_k(i\Omega^1(Y) \oplus \Gamma(S_Y)) \mid d^*a = 0\}.$$  

From the definition of double Coulomb slice, we obtain a natural restriction map

$$r : \text{Coul}^{CC}(X) \to \text{Coul}(Y),$$

$$(\hat{a}, \hat{\phi}) \mapsto (t\hat{a}, \hat{\phi}|_Y).$$

We would want to also define a restriction map from $W_X$ to $\text{Coul}(Y)$. Notice that $r(\hat{u} \cdot (\hat{a}, \hat{\phi}))$ might not be equal to $r(\hat{a}, \hat{\phi})$ even if $\hat{u} \in \mathcal{G}^{h,\hat{\phi}}_{X,Y}$ because $\hat{u}|_Y \neq 1$ in general. This is where we use the based path data $[\tilde{\eta}]$ to define a “twisted” restriction map

$$r' = r'_{[\tilde{\eta}]} : \text{Coul}^{CC}(X) \to \prod_{j=1}^{b_0(Y)} \text{Coul}(Y_j) = \text{Coul}(Y),$$

$$(\hat{a}, \hat{\phi}) \mapsto \prod_{j=1}^{b_0(Y)} (t_j\hat{a}, e^{i\int_{\eta_j} \hat{p}_\alpha(\hat{a}, \hat{\phi}) \cdot \hat{\phi}|_{Y_j}}).$$

The following result can be verified by a simple calculation.

**Lemma 5.8.** For each $\hat{u} \in \mathcal{G}^{h,\hat{\phi}}_{X,Y}$, we have $r'(\hat{u} \cdot (\hat{a}, \hat{\phi})) = r'(\hat{a}, \hat{\phi})$. Moreover, the twisted restriction map $r'$ does not depend on the choice of the representative $\tilde{\eta}$ in its equivalent class.

As a result, we can define the induced twisted restriction map

$$\tilde{r} = \tilde{r}_{[\tilde{\eta}]} : W_X \to \text{Coul}(Y).$$

Note that $\tilde{r}$ is fiberwise linear since $\hat{p}_\alpha(\hat{a}, \hat{\phi})$ is constant on each fiber. Moreover, there is a decomposition $(\tilde{r}_{\text{in}}, \tilde{r}_{\text{out}}) : W_X \to \text{Coul}(-Y_{\text{in}}) \times \text{Coul}(Y_{\text{out}})$

### 5.2. Seiberg–Witten maps and finite-dimensional approximation.

On the boundary 3-manifold $Y$, we fix a base spin connection $A_0$. We require that the induced curvature $F_{A_0}$ on $\text{det}(S_Y)$ equals $2\pi \nu_0$, where $\nu_0$ is the harmonic 2-form representing $-c_1(s)$. Furthermore, we pick a good perturbation $f = (\bar{f}, \delta)$ where $\bar{f}$ is an extended cylinder function and $\delta$ is a real number (see [9, Definition 2.3] for details). Auxiliary choices in the construction of the unfolded spectrum $\text{SWF}(Y,s)$ will be made but not mentioned at this point.
On the 4-manifold $X$, we fix a base spin${}^c$ connection $\hat{A}_0$ such that $\nabla_{\hat{A}_0} = \frac{df}{dt} + \nabla_{A_0}$ on $[-3,0] \times Y$. As usual, the space of spin${}^c$ connections on $S_X$ can be identified with $i\Omega^1(X)$ via the correspondence $\hat{A} \mapsto \hat{A} - \hat{A}_0$. For a 1-form $\hat{a} \in i\Omega^1(X)$, we let $\mathcal{D}_a^+ : \Gamma(S^+) \to \Gamma(S^-)$ be the Dirac operator associated to the connection $\hat{A}_0 + \hat{a}$. We also denote by $\mathcal{D}^+: = \mathcal{D}_0^+$ the Dirac operator corresponding to the base connection $\hat{A}_0$, so we can write $\mathcal{D}_a^+ = \mathcal{D}^+ + \hat{\rho}(\hat{a})$.

On $Y$, we denote by $\mathcal{D}_{A_0 + a}$ the Dirac operator associated to the connection $A_0 + a$ where $a \in i\Omega^1(Y)$ and denote by $\mathcal{D}_a := \mathcal{D}_{A_0 + a}$.

Furthermore, we perturb the Seiberg–Witten map by choosing the following data:

- Pick a closed 2-form $\omega_0 \in i\Omega^2(X)$ such that $\omega_0|_{[-3,0] \times Y} = \pi^*\nu_0$.
- Pick a bump-function $\iota : [-3,0] \to [0,1]$ satisfying $\iota \equiv 0$ on $[-3,-2]$ and $\iota \equiv 1$ on $[-1,0]$ and $0 \leq \iota'(x) \leq 2$. For $t \in [-3,0]$, denote by $a_t$ the pull back of $\hat{a}$ by the inclusion $\{t\} \times Y \to X$ and let $\phi_t = \hat{\phi}_{|\{t\} \times Y}$.

We define a perturbation on $X$ supported in the collar neighborhood of $Y$ by

$$q(\hat{a}, \hat{\phi}) := \iota(t)(dt \wedge \text{grad}^1 f(a_t, \phi_t) + \ast \text{grad}^1 f(a_t, \phi_t)), \text{grad}^2 f(a_t, \phi_t).$$

The (perturbed) Seiberg–Witten map is then given by

$$\text{SW} : \text{Coul}^{CC}(X) \to L^2_{k - 1/2}(i\Omega^2_+(X) \oplus \Gamma(S_X^-))$$

$$\text{SW}(\hat{a}, \hat{\phi}) = (d^+ \hat{a}, \mathcal{D}_a^+(\hat{\phi})) + \left(\frac{1}{2} \mathcal{D}_a^+ \hat{\rho}^{-1}(\hat{\phi}^*\hat{\phi})_0 - \hat{\omega}_a^+ - \hat{\rho}(\hat{a})\hat{\phi} + q(\hat{a}, \hat{\phi}),$$

where $(\hat{\phi}^*\hat{\phi})_0$ denotes the trace-free part of $\hat{\phi}^*\hat{\phi} \in \Gamma(\text{End}(S_X^+))$.

We consider a decomposition

$$\text{SW} = L + Q$$

where

$$L(\hat{a}, \hat{\phi}) = (d^+ \hat{a}, \mathcal{D}_a^+(\hat{\phi}))$$

and $Q = \text{SW} - L$.

By computation similar to that in the proof of Proposition 11.4.1 of [10], making use of the tameness condition on $\text{grad}^f$ (see [10] Definition 10.5.1), we can deduce the following lemma:

**Lemma 5.9.** For any number $j \geq 2$, if a subset $U \subset \text{Coul}^{CC}(X)$ is bounded in $L^2_j$, then the set $Q(U)$ is also bounded in $L^2_j$.

We will next consider Seiberg–Witten maps on the Hilbert bundle $\mathcal{W}_X$. Notice that the map

$$\mathcal{SW} : \text{Coul}^{CC}(X) \to L^2_{k - 1/2}(i\Omega^2_+(X) \oplus \Gamma(S_X^-)) \times \mathcal{H}^1_{DC}(X)$$

is equivariant under the action of $\mathcal{G}^{h,0}_X$, where the action on the target space is given by

$$\hat{u} \cdot ((\omega, \hat{\phi}), \hat{h}) := ((\omega, \hat{u}\hat{\phi}), \hat{h} - \hat{u}^{-1}d\hat{u}).$$

Consequently, $\mathcal{SW} : \hat{\mathcal{W}}_X$ induces a bundle map over $\text{Pic}^0(X, Y)$ denoted by

$$\mathcal{SW} : \mathcal{W}_X \to (L^2_{k - 1/2}(i\Omega^2_+(X) \oplus \Gamma(S_X^-)) \times \mathcal{H}^1_{DC}(X)) / \mathcal{G}^{h,0}_X.$$
By Kuiper’s theorem, the Hilbert bundle $(L_{k-1/2}^2(i\Omega^2_+(X) \oplus \Gamma(S_X^\wedge)) \times \mathcal{H}_{DC}(X))/\mathcal{G}_{X,Y}^{h,\hat{o}}$ can be trivialized. We fix a trivialization and consider the induced projection from this bundle to its fiber $L_{k-1/2}^2(i\Omega^2_+(X) \oplus \Gamma(S_X^\wedge))$. Composing the map $SW$ with this projection, we obtain a map $\tilde{SW}: W_X \to L_{k-1/2}^2(i\Omega^2_+(X) \oplus \Gamma(S_X^\wedge))$.

As the map $(L, \hat{p}_\alpha)$ is also equivariant under the action of $\mathcal{G}_{X,Y}^{h,\hat{o}}$, the decomposition (25) induces a decomposition $\tilde{SW} = \tilde{L} + \tilde{Q}$, where $\tilde{L}$ is a fiberwise linear map.

On the 3-dimensional Coulomb slice $Coul(Y)$, a Seiberg–Witten trajectory is a trajectory $\gamma: I \to Coul(Y)$ on some interval $I \subset \mathbb{R}$ satisfying an equation

$$-\frac{d\gamma}{dt}(t) = (l + c)(\gamma(t)),$$

where $l + c$ comes from gradient of the perturbed Chern–Simons–Dirac functional $CSD_{\mu_0,f}$ (cf. [9, Section 2]). Recall that $l = (\ast d, /D)$ and $c$ has nice compactness properties.

Let $V^\mu_\lambda \subset Coul(Y)$ be the span of eigenspaces of $l$ with eigenvalues in the interval $(\lambda, \mu]$ and let $p_{\mu_\lambda}^\alpha$ be the $L^2$-orthogonal projection onto $V^\mu_\lambda$. An approximated Seiberg–Witten trajectory is a trajectory on a finite-dimensional subspace $\gamma: I \to V^\mu_\lambda$ satisfying an equation

$$-\frac{d\gamma}{dt}(t) = (l + p_{\mu_\lambda}^\alpha \circ c)(\gamma(t)).$$

From now on, let us fix a decreasing sequence of negative real numbers $\{\lambda_n\}$ and an increasing sequence of positive real numbers $\{\mu_n\}$ such that $-\lambda_n, \mu_n \to \infty$. As a consequence of [8, Proposition 3.1], the linear part $(\tilde{L}, p_{-\infty}^{\mu_n} \circ \tilde{r}): W_X \to L^2_{k-1/2}(i\Omega^2_+(X) \oplus \Gamma(S_X^\wedge)) \oplus V_{-\infty}^{\mu_n}$ is fiberwise Fredholm. Now we choose an increasing sequence $\{U_n\}$ of finite-dimensional subspaces of $L^2_{k-1/2}(i\Omega^2_+(X) \oplus \Gamma(S_X^\wedge))$ with the following two properties:

(i) As $n \to \infty$, the orthogonal projection $P_{U_n}: L^2_{k-1/2}(i\Omega^2_+(X) \oplus \Gamma(S_X^\wedge)) \to U_n$ converges to the identity map pointwisely.

(ii) For any point $p \in \text{Pic}^0(X,Y)$ and any $n$, the restriction of $(\tilde{L}, p_{-\infty}^{\mu_n} \circ \tilde{r})$ to the fiber over $p$ is transverse to $U_n$.

Note that $\hat{p}_\alpha(\hat{a}) = 0$ on $\partial X$ and hence the family of the Dirac operators $D_{\hat{p}_\alpha(\hat{a})}$ has no spectral flow. Consequently, we see that

$$W_n := (\tilde{L}, p_{-\infty}^{\mu_n} \circ \tilde{r})^{-1}(U_n \times V_{-\infty}^{\mu_n})$$
is a finite-dimensional vector bundle over the Picard torus \( \text{Pic}^0(X, Y) \). We define an approximated Seiberg-Witten map as 
\[
\tilde{SW}_n := \tilde{L} + P_{U_n} \circ \tilde{Q} : W_n \to U_n.
\]

5.3. Boundedness results. In this section, we will establish analytical results needed to set up our construction of the relative Bauer–Furuta invariants. Uniform boundedness of the following objects and their approximated analogues will be our main focus here.

**Definition 5.10.** A finite type \( X \)-trajectory is a pair \((\tilde{x}, \gamma)\) such that
- \( \tilde{x} \in W_X \) satisfying \( \tilde{SW}(\tilde{x}) = 0 \);
- \( \gamma : [0, \infty) \to \text{Coul}(Y) \) is a finite type Seiberg–Witten trajectory;
- \( \tilde{r}(\tilde{x}) = \gamma(0) \).

Recall that a smooth path in \( \text{Coul}(Y) \) is called finite type if it is contained in a fixed bounded set (in the \( L^2 \)-norm).

With a basepoint chosen on each connected component \( Y_j \), we recall that we can define the based harmonic gauge group \( G^{h,o}_Y \cong H^1(Y; \mathbb{Z}) \). The group \( G^{h,o}_Y \) has a residual action on \( \text{Coul}(Y) \). Then we consider a strip of balls in \( \text{Coul}(Y) \) translated by this action
\[
\text{Str}(R) = \{ x \in \text{Coul}(Y) | \exists h \in G^{h,o}_Y \text{ s.t. } \| h \cdot x \|_{L^2} \leq R \}.
\]

Loosely speaking, a finite type \( X \)-trajectory corresponds to a Seiberg–Witten solution on \( X^* := X \cup ([0, \infty) \times Y) \). The following result resembles [8, Corollary 4.3] but we give a more direct proof without relying on broken trajectories and regular perturbations.

**Theorem 5.11.** For any \( M > 0 \), there exists a constant \( R_0(M) > 0 \) such that for any finite type \( X \)-trajectory \((\tilde{x}, \gamma)\) satisfying
\[
\hat{p}_\beta(\tilde{x}) \in [-M, M]^{b_1,\beta} \tag{26}
\]
we have
\[
\| \tilde{x} \|_F < R_0(M) \text{ and } \gamma([0, \infty)) \subset \text{int(\text{Str}(R_0(M)))}.
\]

**Proof.** Let \( \{ (\tilde{x}_n, \gamma_n) \} \) be a sequence of finite type \( X \)-trajectories satisfying \( (26) \). Without loss of generality, we may pick a representative \( \tilde{x}_n = [(\hat{a}_n, \hat{\phi}_n)] \) such that
\[
\hat{p}_\alpha(\hat{a}_n, \hat{\phi}_n) \in \mathfrak{D} \tag{27}
\]
where \( \mathfrak{D} \) is the fundamental domain fixed before Lemma 5.7.

Since \( \gamma_n \) is finite type, we see that the energy of \( \gamma_n|_{[t-1, t+1]} \) goes to 0 as \( t \to \infty \) for any \( n \). In particular, the energy of \( \gamma_n|_{[t-1, t+1]} \) is bounded above by 1 for any \( n \) and any \( t \) large enough compared to \( n \). Then, it is not hard to show that there is a constant \( R' \) such that \( \gamma_n(t) \in \text{int(\text{Str}(R'))} \) for any \( n \) and any \( t \) large enough compared to \( n \). Since \( \text{CSD}_{v_0,t} \) is bounded on \( \text{int(\text{Str}(R'))} \) and \( \text{CSD}_{v_0,t} \) is decreasing along \( \gamma_n \), we can obtain a uniform lower bound \( C_1 \) of \( \text{CSD}_{v_0,t}(\gamma_n(t)) \) for any \( n \in \mathbb{N}, t \geq 0 \).
We now consider solutions on $X' := X \cup ([0,1] \times Y)$ obtained by gluing together $(\hat{a}_n, \hat{\phi}_n)$ and $\gamma_n|_{[0,1]}$. Remark that the condition $\hat{r}(\hat{x}) = \gamma(0)$ from the twisted restriction is slightly different from the setup in \cite{8} Corollary 4.3. However, we can still glue in a controlled manner since we control $\hat{p}_\alpha(\hat{a}_n, \hat{\phi}_n)$ in (26). The uniform lower bound $C_1$ of $\text{CSD}_{\alpha_0, f}(\gamma_n(t))$ implied that the energy of these solutions on $X'$ (see \cite{10} (4.21),(24.25) for definition) has a uniform upper bound. We now apply the compactness theorem \cite{10} Theorem 24.5.2 adapted to the balanced situation; after passing to a subsequence and applying suitable gauge transformations, the solution on $X'$ converges in $C^\infty$ on the interior domain $X$. In particular, we can find $\hat{u}_n \in \mathcal{G}_{X}^{h,\sigma}$ such that $\hat{u}_n \cdot (\hat{a}_n, \hat{\phi}_n)$ converges in $L^2_{k+1/2}$ to some $(\hat{a}_\infty, \hat{\phi}_\infty) \in \text{Coul}^{CC}(X)$.

By (26) and (27), we have controlled values of $\hat{p}_\alpha$ and $\hat{p}_\beta$ of $(\hat{a}_n, \hat{\phi}_n)$. This implies that $\{\hat{u}_n\}$ takes only finitely many values in $\mathcal{G}_{X}^{h,\sigma}$. After passing to a subsequence, we can assume that $\hat{u}_n$ does not depend on $n$ and $(\hat{a}_n, \hat{\phi}_n)$ converges in $L^2_{k+1/2}$.

On the collar neighborhood $[-1,0] \times Y$ of $X$, the solution $(\hat{a}_n, \hat{\phi}_n)$ can be transformed to a Seiberg–Witten trajectory in a controlled manner. We subsequently glue this part together with $\gamma_n$ to obtain a Seiberg–Witten trajectory

$$\gamma'_n: [-1, \infty) \to \text{Coul}(Y).$$

Since $(\hat{a}_n, \hat{\phi}_n)$ converges in $L^2_{k+1/2}$, we have a uniform upper bound $C_2$ on $\text{CSD}_{\alpha_0, f}(\gamma'_n(-1))$. As a result, the energy of a trajectory $\gamma'_n|_{[t-1,t+1]}$ is bounded above by $C_2 - C_1$ for any $t \geq 0$ and $n \in \mathbb{N}$. We can then conclude that there is a constant $R''$ such that $\gamma_n(t) \in \text{int}(\text{Str}(R''))$ for any $t \geq 0$ and $n \in \mathbb{N}$. This finishes the proof. \hfill $\square$

**Corollary 5.12.** There exists a uniform constant $R_1$ such that for any finite type $X$-trajectory $(\hat{x}, \gamma)$, we have $\gamma(t) \in \text{Str}(R_1)$ for any $t \in [0, \infty)$.

**Proof.** By looking at the lattice induced by the chosen basis on $\text{im} \ i^*$, there is a constant $C$ such that, for any $\hat{x} \in \mathcal{W}_X$, one can find a gauge transformation $\hat{u} \in \mathcal{G}_{X}^{h,\sigma}$ satisfying $\hat{p}_\beta(\hat{u} \cdot \hat{x}) \in [-C, C]$.

Let $(\hat{x}, \gamma)$ be an arbitrary finite type X-trajectory. We then apply Theorem 5.11 to $(\hat{u} \cdot \hat{x}, (\hat{u}|_Y) \cdot \gamma)$ with $M = C$ and $\hat{u}$ chosen as in the previous paragraph. Consequently, we may set $R_1 = R_0(C)$ so that $(\hat{u}|_Y) \cdot \gamma(t) \in \text{int}(\text{Str}(R_1))$ for any $t \in [0, \infty)$. This implies $\gamma(t) \in \text{int}(\text{Str}(R_1))$ for any $t \in [0, \infty)$. \hfill $\square$

Now we consider an approximated version of X-trajectories.

**Definition 5.13.** For $n \in \mathbb{N}$, $\epsilon \in [0, \infty)$, and $T \in (0, \infty]$, a finite type $(n, \epsilon)$-approximated X-trajectory of length $T$ is a pair $(\hat{x}, \gamma)$ such that

- $\hat{x} \in \mathcal{W}_n$ satisfies $\|\text{SW}_n(\hat{x})\|_{L^2_{k-1/2}} \leq \epsilon$;
There exists a constant $\theta$.

**Proposition 5.16.**

The proof of the following convergence of approximated trajectories is a slight adaption of [8, Lemma 4.4] and we omit it.

**Lemma 5.14.** Let $\tilde{S}, S$ be bounded subsets of $W_X$ and $\text{Coul}(Y)$ respectively. Let $\{(\tilde{x}_j, \gamma_j)\}$ be a sequence of finite type $(n_j, \epsilon_j)$-approximated $X$-trajectory of length $T_j$ such that $\tilde{x}_j \in \tilde{S}, \gamma_j \subset S$ for any $j$ and $(n_j, \epsilon_j, T_j) \to (\infty, 0, \infty)$. Then there exists a finite type $X$-trajectory $(\tilde{x}_\infty, \gamma_\infty)$ such that, after passing to a subsequence, we have

- $\tilde{x}_j$ converges to $\tilde{x}_\infty$ in $W_X$;
- $\gamma_j$ converges to $\gamma_\infty$ uniformly in $L^2_k$ on any compact subset of $[0, \infty)$.

As a result of this lemma, we can deduce boundedness for approximated $X$-trajectories.

**Proposition 5.15.** Let $M \geq 0$ be a fixed number. For any bounded subsets $\tilde{S} \subset W_X$ and $S \subset \text{Coul}(Y)$, there exist $\epsilon_0, N, T \in (0, \infty)$ such that: for any finite type $(n, \epsilon)$-approximated $X$-trajectory $(\tilde{x}, \gamma)$ of length $T \geq \bar{T}$ satisfying

$$n \geq N, \epsilon \leq \epsilon_0, \tilde{x} \in \tilde{S}, \gamma \subset S \text{ and } \tilde{p}_\beta(\tilde{x}) \in [-M, M]^{b_1, \beta},$$

we have $\|\tilde{x}\|_F < R_0(M)$ where $R_0(M)$ is the constant from Theorem 5.11.

**Proof.** Suppose that the result is not true for some $\tilde{S}, S$. There would be a sequence $\{(\tilde{x}_j, \gamma_j)\}$ of finite type $(n_j, \epsilon_j)$-approximated $X$-trajectory of length $T_j$ with $\tilde{x}_j \in \tilde{S}, \gamma_j \subset S$ and $(n_j, \epsilon_j, T_j) \to (\infty, 0, \infty)$ such that $\|\tilde{x}_j\|_F \geq R_0(M)$ and $\tilde{p}_\beta(\tilde{x}) \in [-M, M]^{b_1, \beta}$.

By Lemma 5.14, after passing to a subsequence, we can find a finite type $X$-trajectory $(\tilde{x}_\infty, \gamma_\infty)$ such that $\tilde{x}_j \to \tilde{x}_\infty$ in $W_X$. In particular, this implies

$$\|\tilde{x}_\infty\|_F = \lim_{j \to \infty} \|\tilde{x}_j\|_F \geq R_0(M) \text{ and}$$

$$\tilde{p}_\beta(\tilde{x}_\infty) = \lim_{j \to \infty} \tilde{p}_\beta(\tilde{x}_j) \in [-M, M]^{b_1, \beta},$$

which is a contradiction with Theorem 5.11. $\square$

**Proposition 5.16.** There exists a constant $R_2$ with the following significance: for any bounded subsets $\tilde{S} \subset W_X$ and $S \subset \text{Coul}(Y)$, there exist $\epsilon_0, N, \bar{T} \in (0, +\infty)$ such that for any finite type $(n, \epsilon)$-approximated $X$-trajectory $(\tilde{x}, \gamma)$ of length $T \geq \bar{T}$ satisfying

$$n \geq N, \epsilon \leq \epsilon_0, \tilde{x} \in \tilde{S} \text{ and } \gamma \subset S$$

We have $\gamma|_{[0, T - \bar{T}]} \subset \text{Str}(R_2)$. 
Proof. Recall that there is a universal constant $R_0$ such that any sufficiently approximated Seiberg–Witten trajectory $\gamma' : [-T, T] \to V^k$ with sufficiently long length $T$ and with $\gamma' \subset S$ must satisfy $\gamma(0) \in \text{Str}(R_0)$ (cf. the constant $R_0$ from [9, Corollary 3.7]). We pick $R_2 = \max\{R_0, R_1\}$ where $R_1$ is the constant from Corollary 5.12.

Suppose the result is not true for some $\tilde{S}, S$. Then we can find sequences $n_j, \epsilon_j, T_j, \tilde{T}_j$ with $n_j \to \infty, \tilde{T}_j \leq T_j, \tilde{T}_j \to \infty$ such that there is a sequence $\{(\tilde{x}_j, \gamma_j)\}$ of finite type $(n_j, \epsilon_j)$-approximated $X$-trajectory of length $T_j$ with $\tilde{x}_j \subset \tilde{S}, \gamma_j \subset S$ and with $\gamma_j([0, T_j - \tilde{T}_j]) \not\subset \text{Str}(R_2)$.

We have a number $t_j \in [0, T_j - \tilde{T}_j]$ such that $\gamma_j(t_j) \not\in \text{Str}(R_2)$. The property of $R_0$ forces $t_j$ to converge to a finite number $t_\infty$ after passing to a subsequence.

By Lemma 5.14, there exists an finite type $X$-trajectory $(\tilde{x}_\infty, \gamma_\infty)$ such that, after passing to a subsequence, $\gamma_j$ converges to $\gamma_\infty$ uniformly in $L^2_k$ on any compact subset of $[0, \infty)$. In particular, $\gamma_j(t_j) \to \gamma_\infty(t_\infty)$ which contradicts with Corollary 5.12.

\[ \square \]

5.4. Construction. The majority of this section, in fact, is dedicated to construction of type-A unfolded relative invariant. The construction of type-R invariant can be obtained almost immediately after applying duality argument.

Let us pick $\tilde{R}$ a number greater than the constant $R_2$ from Proposition 5.16. Recall that the unfolded spectra $\text{swf}^A(Y_{out})$ and $\text{swf}^R(-Y_{in})$ are obtained by cutting the unbounded set $\text{Str}_Y(\tilde{R})$ into bounded subsets and applying finite dimensional approximations. With a choice of cutting functions, we obtain increasing sequences of bounded sets $\{J_m^{\pm}(-Y_{in})\}_m$ contained in $\text{Str}_{Y_{in}}(\tilde{R})$ and $\{J_m^{\pm}(Y_{out})\}_m$ contained in $\text{Str}_{Y_{out}}(\tilde{R})$ for each positive integer $n$. See Section 2.1 for brief summary.

For a normed vector bundle $V$, we will denote by $B(V, r)$ the disk bundle of radius $r$ and denote by $S(V, r)$ the sphere bundle of radius $r$. We will consider a subbundle of $W_X$ given by

$$W_{X, \beta} := \{ \tilde{x} \in W_X | \hat{p}_{\beta, out}(\tilde{x}) = 0 \}.$$ 

We also denote $W_{n, \beta} = W_n \cap W_{X, \beta}$ and let $\overline{SW}_{n, \beta}$ be the restriction of $\overline{SW}_n$ on $W_{n, \beta}$.

For a fixed positive integer $m_0$, since $\{J_m^{\pm}(-Y_{in})\}$ is bounded, we can find a number $M(m_0)$ such that $|\int_{\beta_j} a \beta| \leq M(m_0)$ for all $(a, \phi) \in J_m^{\pm}(-Y_{in})$ and $j = 1, \ldots, b_n$. We then choose a number $R$ greater than $R_0(M(m_0))$ the constant from Theorem 5.11. Since $\tilde{r}_{out}(B(W_X, R))$ is bounded, we can find a positive integer $m_1$ such that

$$\tilde{r}_{out}(B(W_X, R)) \cap \text{Str}_{Y_{out}}(\tilde{R}) \subset J_{m_1}^{+}(Y_{out}).$$
For $\epsilon > 0$, $n \in \mathbb{N}$, we consider the following subsets of $V^{\mu_n}_{\lambda_n}$:

\[
K_1(n,m_0,R,\epsilon) = \left( J^{m_0}_{m_0}(-Y_{\text{in}}) \times \text{Str}_{\text{out}}(\tilde{R}) \right) \cap p^{\mu_n}_{-\infty} \circ \tilde{r} \left( \widehat{SW}_{n,\beta}(B(U_n,\epsilon)) \cap B(W_{n,\beta},R) \right),
\]

\[
K_2(n,m_0,R,\epsilon) = \left\{ \left( J^{m_0}_{m_0}(-Y_{\text{in}}) \times \text{Str}_{\text{out}}(\tilde{R}) \right) \cap p^{\mu_n}_{-\infty} \circ \tilde{r} \left( \widehat{SW}_{n,\beta}(B(U_n,\epsilon)) \cap S(W_{n,\beta},R) \right) \right\} \cup \left\{ \partial \left( J^{m_0}_{m_0}(-Y_{\text{in}}) \times \text{Str}_{\text{out}}(\tilde{R}) \right) \cap \left( p^{\mu_n}_{-\infty} \circ \tilde{r} \left( \widehat{SW}_{n,\beta}(B(U_n,\epsilon)) \cap B(W_{n,\beta},R) \right) \right) \right\}.
\]

Notice that $K_1(n,m_0,R,\epsilon) \subset J^{m_0}_{m_0}(-Y_{\text{in}}) \times J^{m_1}_{m_1}(Y_{\text{out}})$ from our choice of $m_1$ and $K_2(n,m_0,R,\epsilon)$ plays a role of a boundary of $K_1(n,m_0,R,\epsilon)$.

The following is the key result of this section (cf. [8 Proposition 4.5]).

**Proposition 5.17.** For a choice of $m_0, m_1$ and $R$ chosen above, there exist $N \in \mathbb{N}$ and $\bar{T}, \epsilon_0 > 0$ such that, for any $n \geq N$ and $\epsilon \leq \epsilon_0$, the pair $(K_1(n,m_0,R,\epsilon), K_2(n,m_0,R,\epsilon))$ is a $\bar{T}$-tame pre-index pair in an isolating neighborhood $J^{m_0}_{m_0}(-Y_{\text{in}}) \times J^{m_1}_{m_1}(Y_{\text{out}})$.

**Proof.** We choose numbers $(N, \bar{T}, \epsilon_0)$ satisfying both Proposition 5.15 and Proposition 5.16 with $\hat{S} = B(W_X,R)$, $S = J^{m_0}_{m_0}(-Y_{\text{in}}) \times J^{m_1}_{m_1}(Y_{\text{out}})$, and $M = M(m_0)$. Moreover, we may pick a larger $N$ so that $J^{m_0}_{m_0}(-Y_{\text{in}}) \times J^{m_1}_{m_1}(Y_{\text{out}})$ is an isolating neighborhood for all $n > N$ (cf. [9 Lemma 5.5 and Proposition 5.8]). We will check directly that

\[
(K_1(n,m_0,R,\epsilon), K_2(n,m_0,R,\epsilon))
\]

is a $\bar{T}$-tame pre-index pair.

Suppose that $y \in K_1(n,m_0,R,\epsilon)$ and $\varphi_n(y,[0,T]) \subset J^{m_0}_{m_0}(-Y_{\text{in}}) \times J^{m_1}_{m_1}(Y_{\text{out}})$ with $T \geq \bar{T}$. From definition, there is $\tilde{x} \in W_{n,\beta}$ such that $\|\widehat{SW}_n(\tilde{x})\| \leq \epsilon$ and $p^{\mu_n}_{-\infty} \circ \tilde{r}(\tilde{x}) = y$. These give rise to a finite type $(n,\epsilon)$-approximated $X$-trajectory $(\tilde{x},\gamma)$ of length $T$. By Proposition 5.16, we have $\varphi_n(y,[0,T-\bar{T}]) \subset \text{Str}(R_2) \subset \text{int}(\text{Str}(\tilde{R}))$. From our choices of $J^{m_0}_{m_0}, J^{m_1}_{m_1}$, it is not hard to check that $\varphi_n(y,[0,T-\bar{T}])$ lies in some compact subset inside the interior of $J^{m_0}_{m_0}(-Y_{\text{in}}) \times J^{m_1}_{m_1}(Y_{\text{out}})$.

For the second pre-index pair condition, let us assume that $y \in K_2(n,m_0,R,\epsilon)$ and $\varphi_n(y,[0,T]) \subset J^{m_0}_{m_0}(-Y_{\text{in}}) \times J^{m_1}_{m_1}(Y_{\text{out}})$. This also gives rise to a finite type $(n,\epsilon)$-approximated $X$-trajectory $(\tilde{x},\gamma)$ of length $T$. Since $p^{\mu_n}_{-\infty} \circ \tilde{r}_y(\tilde{x}) \in J^{m_0}_{m_0}(-Y_{\text{in}})$ and $\tilde{x} \in W_{X,\beta}$, we can see that $\hat{p}_\beta(\tilde{x}) \in [-M(m_0),M(m_0)]^{b_1,\beta}$. 

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By Proposition 5.15, we have \( \|x\|_F < R_0(M) < R \), which implies that 
\[
y \in \partial \left( J_{m_0}^n (Y_{in}) \times Str_{Y_{out}}(\tilde{R}) \right).
\]
Again, from Proposition 5.16, we must have 
\[
y \in \left\{ \partial J_{m_0}^n (Y_{in}) \setminus \partial Str_{Y_{in}}(\tilde{R}) \right\} \times Str_{Y_{out}}(\tilde{R}).
\]
This is impossible because the approximated trajectories on 
\[
\partial J_{m_0}^n (Y_{in}) \setminus \partial Str_{Y_{in}}(\tilde{R})
\]
immediately leave \( J_{m_0}^n (Y_{in}) \).

The proposition allows us to consider a map 
\[
v(n, m_0, R, \epsilon) : B(W_n, R)/S(W_n, R) \to (B(U_n, \epsilon)/S(U_n, \epsilon)) \wedge (K_1(n, m_0, R, \epsilon)/K_2(n, m_0, R, \epsilon))
\]
given by 
\[
v(n, m_0, R, \epsilon)(x) := \begin{cases} 
(SW_{n, \beta}(x), [p^\mu_{-\infty} \circ \tilde{r}(x)]) & \text{if } \|\tilde{SW}_{n, \beta}(x)\|_{L^2_k}^{1/2} \leq \epsilon, \\
* & \text{otherwise.}
\end{cases}
\]
It follows from our construction that this map is well-defined and continuous. By Proposition 5.17 and Theorem 3.8, we have a canonical map from \( K_1(n, m_0, R, \epsilon)/K_2(n, m_0, R, \epsilon) \) to the Conley index of \( J_{m_0}^n (Y_{in}) \times J_{m_1}^n (Y_{out}) \). This gives a map 
\[
\tilde{v}(n, m_0, R, \epsilon) : B(W_n, R)/S(W_n, R) \to (B(U_n, \epsilon)/S(U_n, \epsilon)) \wedge I(\inv(J_{m_0}^n (Y_{in}))) \wedge I(\inv(J_{m_1}^n (Y_{out}))).
\]
It is a standard argument to check that \( \tilde{v}(n, m_0, R, \epsilon) \) does not depend on \( R \) or \( \epsilon \) as long as they satisfy all the requirements to define \( v(n, m_0, R, \epsilon) \).

Before proceeding, let us describe the Thom space 
\[
B(W_n, R)/S(W_n, R)
\]
in term of index bundle. Consider a family of Dirac operators 
\[
D : L^2_{k+1/2}(S^+_X) \times H^1_{DC}(X) \to L^2_{k-1/2}(S^-_X) \times H^1_{Dir} \times H^1_{DC}(X)
\]
\[
(\hat{\phi}, h) \mapsto (\hat{\Phi}^+_h \hat{\phi}, \Pi_{Dir}(\hat{\phi}|_Y), h),
\]
where \( H^1_{Dir} \) is the closure in \( L^2_{k}(\Gamma(S_Y)) \) of the subspace spanned by the eigenvectors of \( \hat{\Phi}^+_h \) with nonpositive eigenvalues and let \( \Pi_{Dir} \) be the orthogonal projection. As in Section 5.2, this map is equivariant under an action by
\( \mathcal{G}_X^h,\hat{\delta} \). We then take the quotient to obtain a map between Hilbert bundles over \( \text{Pic}^0(X,Y) \) and trivialize the right hand side so that we have

\[
D : (L_{k+1/2}^2(S_X^+) \times H^1_{DC}(X))/\mathcal{G}_X^h,\hat{\delta} \to L_{k-1/2}^2(S_X^-) \times H^-_{Dir}.
\]

Since \( D \) is fiberwise Fredholm, the preimage \( D^{-1}(U) \) is a finite dimensional subbundle for a finite dimensional subspace \( U \subset L_{k-1/2}^2(S_X^-) \times H^-_{Dir} \) transverse to the image of the restriction of \( D \) to any fiber. Here we use the fact that the rank of \( D^{-1}(U) \) is constant because \( h|_Y = 0 \) and there is no spectral flow.

We consider the desuspension \( \Sigma^{-U}B(\tilde{D}^{-1}(U), R)/S(\tilde{D}^{-1}(U), R) \) of the Thom space in the stable category \( \mathcal{C} \). The following lemma follows from a standard homotopy argument.

**Lemma 5.18.** The object \( \Sigma^{-U}B(\tilde{D}^{-1}(U), R)/S(\tilde{D}^{-1}(U), R) \) does not depend on any choice in the construction given that \( g|_Y = g \) and \( \hat{A}_0|_Y = A_0 \). We will call this object Thom spectrum of virtual index bundle associated to the Dirac operators, denoted by \( T(X, \hat{s}, A_0, g, \hat{o}; S^1) \).

**Remark 5.19.** For different choices of base points, one can construct an isomorphism by choosing a path between them. However, isomorphisms given by different paths are different unless they are homotopic relative to \( Y \).

Recall from Section 2.1 that we have desuspended Conley indices

\[
\begin{align*}
I_{m_0}^{n,-}(Y_{in}) &= \Sigma^{-V_{\lambda_n}^0(-Y_{in})}I(\text{inv}(J_{m_0}^{n,-}(Y_{in}))), \\
I_{m_1}^{n,+}(Y_{out}) &= \Sigma^{-V_{\lambda_n}^0(Y_{out})}I(\text{inv}(J_{m_1}^{n,+}(Y_{out}))).
\end{align*}
\]

We see that if we desuspend the map \( \tilde{v}(n, m_0, R, \epsilon) \) by \( V_{\lambda_n}^0(-Y_{in}) \oplus V_{\lambda_n}^0(Y_{out}) \oplus U_n \), the right hand side will become \( I_{m_0}^{n,-}(Y_{in}) \land I_{m_1}^{n,+}(Y_{out}) \). As a consequence of Lemma 5.18 we can also identify the left hand side after desuspension as follows:

**Lemma 5.20.** Let \( V_X^+ \) be a maximal positive subspace of

\[
\text{im}(H^2(X, \partial X; \mathbb{R}) \to H^2(X; \mathbb{R}))
\]

with respect to the intersection form and let \( V_{in} \) be the cokernel of \( \iota^* : H^1(X; \mathbb{R}) \to H^1(Y_{in}; \mathbb{R}) \). Then, we have

\[
\Sigma^{-V_{\lambda_n}^0(-Y_{in}) \oplus V_{\lambda_n}^0(Y_{out}) \oplus U_n} B(W_n, \beta, R)/S(W_n, \beta, R) \cong \Sigma^{-V_X^+ \oplus V_{in}}T(X, \hat{s}, A_0, g, \hat{o}; S^1).
\]

**Proof.** This is a bundle version of index computation in \cite{S} Proposition 3.1. From there, we are only left to keep track of \( H^1(X; \mathbb{R}) \) and \( H^1(Y; \mathbb{R}) \) as we pass to bundle and subspace, i.e. the base of the bundle is the torus of dimension \( b_1, \alpha \) and we take a slice of codimension \( b_1, \beta - b_{in} \). Note that we desuspend by \( V_{\lambda_n}^0(Y_{out}) \), the orthogonal complement of \( H^1(Y_{out}; \mathbb{R}) \) in
induced by repeller and attractor respectively. To have a morphism to the
bundle of the real part of \( \tilde{C} \) in the stable category
have canonical isomorphisms

The desired isomorphism follows in the same manner.

Consequently, we obtain a morphism

\[
\psi_{m_0, m_1}^n : \Sigma^{-\langle X \oplus V \rangle} T(X, \hat{s}, A_0, g, \hat{o}; S^1) \to I_{m_0}^{n-}(-Y_{\text{in}}) \wedge I_{m_1}^{n+}(Y_{\text{out}}) \quad (31)
\]

in the stable category \( \mathfrak{C} \). Note that such a morphism is defined for any positive integer \( m_0 \) with \( m_1 \) large relative to \( m_0 \) and \( n \) large relative to \( m_0, m_1 \).

Recall that, to define unfolded spectra \( \text{swf}^A(Y_{\text{out}}) \) and \( \text{swf}^R(-Y_{\text{in}}) \), we have canonical isomorphisms

\[
\tilde{\rho}_{m_0}^{n-}(-Y_{\text{in}}) : I_{m_0}^{n-}(-Y_{\text{in}}) \to I_{m_0}^{n+1,-}(-Y_{\text{in}}),
\]

\[
\tilde{\rho}_{m_1}^{n+}(Y_{\text{out}}) : I_{m_1}^{n+}(Y_{\text{out}}) \to I_{m_1}^{n+1,+}(Y_{\text{out}})
\]

and also morphisms

\[
\tilde{\tau}_{m_0-1}^{n-} : I_{m_0-1}^{n-}(-Y_{\text{in}}) \to I_{m_0}^{n-1}(-Y_{\text{in}})
\]

\[
\tilde{\tau}_{m_1}^{n+} : I_{m_1}^{n+}(Y_{\text{out}}) \to I_{m_1+1}^{n+}(Y_{\text{out}})
\]

induced by repeller and attractor respectively. To have a morphism to the unfolded spectra, we have to check that the maps \( \{\psi_{m_0, m_1}^n\} \) are compatible with all such morphisms.

**Lemma 5.21.** When \( n \) is large enough relative to \( m_0, m_1 \), we have the following:

1. \( (\tilde{\rho}_{m_0}^{n-}(-Y_{\text{in}}) \wedge \tilde{\rho}_{m_1}^{n+}(Y_{\text{out}})) \circ \psi_{m_0, m_1}^n = \psi_{m_0+1, m_1}^n \);
2. \( (\tilde{\tau}_{m_0-1}^{n-} \wedge \text{id}_{I_{m_1}^{n+}(Y_{\text{out}})}) \circ \psi_{m_0, m_1}^n = \psi_{m_0-1, m_1}^n \);
3. \( (\text{id}_{I_{m_0}^{n-}(-Y_{\text{in}})} \wedge \tilde{\tau}_{m_1}^{n+}) \circ \psi_{m_0, m_1}^n = \psi_{m_0, m_1+1}^n \).

**Proof.** The proof of (1) can be given by standard homotopy arguments similar to [43 Section 9] and [9 Proposition 5.6]. Whereas (2) and (3) follow from Proposition 3.18 and 3.19 respectively.

The last step is to apply Spanier–Whitehead duality between \( I_{m_0}^{n+}(Y_{\text{in}}) \) and \( I_{m_0}^{n-}(-Y_{\text{in}}) \) (see Section 3.3 and 4.4 for details). As a result, we can turn the morphism \( \psi_{m_0, m_1}^n \) to a morphism

\[
\tilde{\psi}_{m_0, m_1}^n : \Sigma^{-\langle X \oplus V \rangle} T(X, \hat{s}, A_0, g, \hat{o}; S^1) \wedge I_{m_0}^{n+}(Y_{\text{in}}) \to I_{m_1}^{n+}(Y_{\text{out}}),
\]

which will define the relative Bauer–Furuta invariant.
\textbf{Definition 5.22.} For the cobordism \(X: Y_{\text{in}} \rightarrow Y_{\text{out}}\), the collection of morphisms \(\{\tilde{\psi}_{m_0,m_1}^n \mid m_0 \in \mathbb{N}, \ m_1 \gg m_0, n \gg m_0,m_1\}\) in \(\mathcal{C}\) gives rise to a morphism

\[
\begin{align*}
\mathbb{H}^A(X, \hat{s}, A_0, g, \hat{o}; \eta, [\tilde{\eta}]; S^1): & \quad \Sigma^-(V_X^+ \oplus V_{\text{in}})T(X, \hat{s}, A_0, g, \hat{o}; S^1) \wedge \mathsf{swf}^A(Y_{\text{in}}, s_{\text{in}}, A_{\text{in}}, g_{\text{in}}; S^1) \\
& \rightarrow \mathsf{swf}^A(Y_{\text{out}}, s_{\text{out}}, A_{\text{out}}, g_{\text{out}}; S^1)
\end{align*}
\]

in \(\mathcal{S}\). This will be called the type-A unfolded relative Bauer–Furuta invariant of \(X\).

Note that Lemma \[5.21\] and compatibility of the dual maps ensure that \(\{\tilde{\psi}_{m_0,m_1}^n\}\) are compatible with the direct systems. When \(s = \tilde{s}|_Y\) is torsion, we can also define the normalized relative Bauer–Furuta invariant. In this torsion case, let us define the normalized Thom spectrum

\[
\tilde{T}(X, \hat{s}, \hat{o}; S^1) := (T(X, \hat{s}, A_0, g, \hat{o}; S^1), 0, n(Y, s, A_0, g)),
\]

where \(n(Y, s, A_0, g)\) is given by \(\frac{1}{2}(\eta(\partial) - \dim_{\mathbb{C}}(\ker \partial) + \frac{\text{dim}_{\mathbb{C}}}{4})\) (see (21) of \[9\]).

\textbf{Definition 5.23.} When \(s = \tilde{s}|_Y\) is torsion, the normalized type-A unfolded relative Bauer–Furuta invariant of \(X\)

\[
\begin{align*}
\mathbb{B}^A(X, \hat{s}, \hat{o}; [\tilde{\eta}]; S^1): & \quad \Sigma^{-}(V_X^+ \oplus V_{\text{in}})\tilde{T}(X, \hat{s}, \hat{o}; S^1) \wedge \mathsf{SWF}^A(Y_{\text{in}}, s_{\text{in}}; S^1) \\
& \rightarrow \mathsf{SWF}^A(Y_{\text{out}}, s_{\text{out}}; S^1)
\end{align*}
\]

is given by desuspending \(\mathbb{H}^A(X, \hat{s}, A_0, g, \hat{o}; [\tilde{\eta}]; S^1)\) by \(n(Y, s, A_0, g)\).

We then define the type-R invariant by simply considering the dual of type-A invariant of the adjoint cobordism \(X^+: -Y_{\text{out}} \rightarrow -Y_{\text{in}}\). In particular, the dual of the morphism

\[
\tilde{\psi}_{m_1,m_0}^n(X^+): \quad \Sigma^{-}(V_{X^+}^+ \oplus V_{\text{in}}(X^+))T(X^+, \hat{s}, A_0, g, \hat{o}; S^1) \wedge I_{m_1}^n(-Y_{\text{out}}) \rightarrow I_{m_0}^n(-Y_{\text{in}}),
\]

gives a morphism

\[
\psi_{m_0,m_1}^n: \quad \Sigma^{-}(V_X^+ \oplus V_{\text{out}})T(X, \hat{s}, A_0, g, \hat{o}; S^1) \wedge I_{m_0}^n(Y_{\text{in}}) \rightarrow I_{m_1}^n(Y_{\text{out}}).
\]

Note that \(V_{\text{in}}(X^+) := V_{\text{out}}\) is the cokernel of \(\iota^* : H^1(X; \mathbb{R}) \rightarrow H^1(Y_{\text{out}}; \mathbb{R})\). The morphism \(\psi_{m_0,m_1}^n\) is defined for any positive integer \(m_1\) with \(m_0\) large relative to \(m_1\) and \(n\) large relative to \(m_0, m_1\). We can now give a definition in a similar fashion.
**Definition 5.24.** For the cobordism \( X : Y_{in} \to Y_{out} \), the type-R unfolded relative Bauer–Furuta invariant of \( X \) is a morphism

\[
bf^R(X, \hat{s}, A_0, g, \hat{o}, [\hat{n}]; S^1):
\]

\[
\Sigma^{- (V_X \oplus \hat{V}_{out})} T(X, \hat{s}, A_0, g, \hat{o}; S^1) \wedge \swf^R(Y_{in}, s_{in}, A_{in}, g_{in}; S^1)
\]

\[
\to \swf^R(Y_{out}, s_{out}, A_{out}, g_{out}; S^1)
\]

in \( \mathcal{S}^* \) given by the collection of morphisms \( \{ \tilde{\psi}_{n_0, m_1}^n \mid m_1 \in \mathbb{N}, m_0 \gg m_1, n \gg m_0, m_1 \} \). When \( s = \hat{s}|_Y \) is torsion, one can also desuspend \( bf^R(X, \hat{s}, A_0, g, \hat{o}, [\hat{n}]; S^1) \) by \( n(Y, s, A_0, g) \) to obtain the normalized type-R unfolded relative Bauer–Furuta invariant of \( X \)

\[
BF^R(X, \hat{s}, \hat{o}, [\hat{n}]; S^1):
\]

\[
\Sigma^{- (V_X \oplus \hat{V}_{out})} \tilde{T}(X, \hat{s}, \hat{o}; S^1) \wedge \swf^R(Y_{in}, s_{in}; S^1) \to \swf^R(Y_{out}, s_{out}; S^1).
\]

**Remark 5.25.** One can also construct the maps \( \tilde{\psi}_{n_0, m_1}^n \) directly by replacing \( (Y_{in}, Y_{out}) \) with \( (Y_{out}, Y_{in}) \) in the construction throughout this section.

### 5.5. Invariance of the relative invariants

In this subsection, we will show that the morphism \( bf^A = bf^A(X, \hat{s}, A_0, g, \hat{o}, [\hat{n}]; S^1) \) and \( BF^R(X, \hat{s}, \hat{o}, [\hat{n}]; S^1) \) depends only on \( A_0, g, \hat{o}, [\hat{n}] \). We have to check that they are independent of the choices of

1. cutting function \( \tilde{g} \), cutting value \( \theta \), harmonic 1-forms \( \{ h_j \}_{j=1}^{b_1} \) representing generators of \( \text{im}(H^1(Y; \mathbb{Z}) \to H^1(Y; \mathbb{R})) \),
2. Riemannian metric \( \tilde{g} \), connection \( \hat{A}_0 \) on \( X \) with \( \tilde{g}|_Y = g, \hat{A}_0|_Y = A_0 \),
3. perturbation \( f : \text{Coul}(Y) \to \mathbb{R} \).

Moreover when \( c_1(s) \) is torsion, we will show that \( BF^A(X, \hat{s}, \hat{o}, [\hat{n}]; S^1) \) and \( BF^R(X, \hat{s}, \hat{o}, [\hat{n}]; S^1) \) are independent of \( A_0, g \) too.

Choose two cutting functions \( \tilde{g}, \tilde{g}' \), cutting values \( \theta, \theta' \) and sets of harmonic 1-forms \( \{ h_j \}_{j=1}^{b_1}, \{ h'_j \}_{j=1}^{b_1} \) representing generators of

\[
2\pi i \text{ im}(H^1(Y; \mathbb{Z}) \to H^1(Y; \mathbb{R})).
\]

We get two inductive systems

\[
\swf^A(Y, \{ h_j \} \downarrow \tilde{g}, \theta) = (I_1 \to I_2 \to \cdots),
\]

\[
\swf^A(Y, \{ h'_j \} \downarrow \tilde{g}', \theta') = (\tilde{I}_1 \to \tilde{I}_2 \to \cdots)
\]

in \( \mathcal{C} \). Here \( I_m, \tilde{I}_m \) are the desuspension of the Conley indices

\[
I_{S^1}(\varphi^n, \text{inv}(J_{m,+}^n)), \ I_{S^1}(\varphi^n, \text{inv}(J_{m,+}^n))
\]

for \( n \gg m \) by \( V^0_{\lambda_m} \), and \( J_{m,+}^n, J_{m,+}^n \) are the bounded sets in \( \text{Str}(\hat{R}) \) defined by using \( \{ h_j \} \downarrow \tilde{g}, \theta \), \( \{ h'_j \} \downarrow \tilde{g}', \theta' \).
Choosing integers $m_j \ll \tilde{m}_j \ll m_{j+1}$, we can assume that $\text{inv}(\tilde{J}_{m_j}^{n,+})$ is an attractor in $\text{inv}(J_{m_j}^{n,+})$ and we have the attractor map

$$I_{S^n}(\text{inv}(J_{m_j}^{n,+})) \to I_{S^n}(\text{inv}(\tilde{J}_{m_j}^{n,+}))$$

which induces a morphism

$$I_{m_j} \to \tilde{I}_{m_j}.$$ 

Similarly we have a morphism

$$\tilde{I}_{m_j} \to I_{m_{j+1}}.$$ 

These morphisms induce an isomorphism between $\text{swf}^A(Y, \{h_j\}_{j \leq}, \tilde{g}, \theta)$ and $\text{swf}^A(Y, \{h'_j\}_{j \leq}, \tilde{g}', \theta')$. The isomorphism between

$$\text{swf}^R(Y, \{h_j\}_{j \leq}, \tilde{g}, \theta) \text{ and } \text{swf}^R(Y, \{h'_j\}_{j \leq}, \tilde{g}', \theta')$$

is obtained similarly. The morphisms in $\text{[31]}$ inducing the relative invariants $bf^A, bf^R$ are compatible with the attractor maps and repeller maps as in stated in Lemma $5.21$. It means that $bf^A, bf^R$ are independent of the choices of $\{h_j\}_{j \leq}, \tilde{g}, \theta$ up to the canonical isomorphisms.

Choose connections $\hat{A}_0, \hat{A}_0'$ on $X$ with $\hat{A}_0|_Y = \hat{A}_0'|_Y = A_0$ and Riemannian metrics $\hat{g}, \hat{g}'$ on $X$ with $\hat{g}|_Y = \hat{g}'|_Y = g$. Then the homotopies

$$\hat{A}_0(s) = (1 - s)\hat{A}_0 + s\hat{A}_0', \hat{g}(s) = (1 - s)\hat{g} + s\hat{g}'$$

naturally induce the homotopy between the maps $v, v'$ defined in $\text{[29]}$ associated with $(\hat{g}_o, \hat{A}_0), (\hat{g}', \hat{A}_0')$. Hence $bf^A, bf^R$ are independent of $A_0, \hat{g}$.

Take sequences $\lambda_n, \lambda'_n, \mu_n, \mu'_n$ with $-\lambda_n, -\lambda'_n, \mu_n, \mu'_n \to \infty$. Then we get objects

$$I^{n,-}_{m_0}(-Y_{in}), I^{n,+}_{m_1}(Y_{out}), \tilde{I}^{n,-}_{m_0}(-Y_{in}), \tilde{I}^{n,+}_{m_1}(Y_{out}).$$

We have canonical isomorphisms

$$I^{n,-}_{m_0}(-Y_{in}) \cong \tilde{I}^{n,-}_{m_0}(-Y_{in}), I^{n,+}_{m_1}(Y_{out}) \cong \tilde{I}^{n,+}_{m_1}(Y_{out})$$

for $n$ large relative to $m_0, m_1$. The morphisms $\psi^{n}_{m_0,m_1}$ are compatible with these isomorphisms as stated in Lemma $5.21$. Therefore $bf^A, bf^R$ are independent of $\lambda_n, \mu_n$ up to canonical isomorphisms.

Let us consider the invariance of $bf^A, bf^R$ with respect to the perturbation $f$. Take two perturbations $f_1, f_2 : \text{Coul}(Y) \to \mathbb{R}$. Then we obtain two inductive systems

$$\text{swf}^A(Y, f_1) = (I_1 \to I_2 \to \cdots)$$

$$\text{swf}^A(Y, f_2) = (\tilde{I}_1 \to \tilde{I}_2 \to \cdots)$$

in the category $\mathcal{C}$, which are isomorphic to each other. Let us recall how to get the isomorphism briefly. (See Section 6.3 of $\text{[9]}$ for the details.) The perturbations $f_1, f_2$ define the functionals $\mathcal{L}_1, \mathcal{L}_2$, which induce the flows

$$\varphi^n(\mathcal{L}_1), \varphi^n(\mathcal{L}_2) : V^{\mu_n} \times \mathbb{R} \to V^{\mu_n}.$$
The objects $I_m, \tilde{I}_m$ are the desuspensions by $V^0_{\lambda_n}$ of the Conley indices

$$I_{S^1}(\varphi^n(\mathcal{L}_1), \text{inv}(J^{n+}_m)), \quad I_{S^1}(\varphi^n(\mathcal{L}_2), \text{inv}(\tilde{J}^{n+}_m)).$$

Choose integers $k_m, \tilde{k}_m$ with $0 \ll k_m \ll \tilde{k}_m \ll k_{m+1}$. Then we have

$$J^{\pm}_m \subset \tilde{p}_k^{-1}([-e_m + 1, e_m - 1)^{[b_h]} \cap \text{Str}(\hat{R})$$

$$\subset p_k^{-1}([-e_m, e_m)^{[b_h]} \cap \text{Str}(\hat{R})$$

for some large positive number $e_m$. We have a map

$$i^n_m : I_{S^1}(\varphi^n(\mathcal{L}_1), \text{inv}(J^{n+}_m)) \rightarrow I_{S^1}(\varphi^n(\mathcal{L}_2), \text{inv}(\tilde{J}^{n+}_m)),$$

which induces the isomorphism between $\text{swf}^A(Y, f_1)$ and $\text{swf}^A(Y, f_2)$. The map $i^n_m$ is the composition $\rho_1 \circ \rho_2$ of

$$\rho_1 : I_{S^1}(\varphi^n(\mathcal{L}^0_{em}), \text{inv}(\tilde{J}^{n+}_{km})) \rightarrow I_{S^1}(\varphi^n(\mathcal{L}_2), \text{inv}(\tilde{J}^{n+}_{km}))$$

and

$$\rho_2 : I_{S^1}(\varphi^n(\mathcal{L}_1), \text{inv}(J^{n+}_m)) \rightarrow I_{S^1}(\varphi^n(\mathcal{L}^0_{em}), \text{inv}(\tilde{J}^{n+}_{km})).$$

Here $\mathcal{L}^0_{em}$ is a functional on $Coul(Y)$ such that

$$\mathcal{L}^0_{em} = \mathcal{L}_1 \text{ on } p^{-1}_k([-e_m + 1, e_m - 1)^{[b_h]}),$$

$$\mathcal{L}^0_{em} = \mathcal{L}_2 \text{ on } p^{-1}_k(\mathbb{R}^{[b_h]} \setminus [-e_m, e_m)^{[b_h]}).$$

The map $\rho_1$ is the homotopy equivalence induced by a homotopy

$$\{\varphi(\mathcal{L}^s_{em})\}^{0 \leq s \leq 1},$$

where $\mathcal{L}^s_{em} = s\mathcal{L}_1 + (1-s)\mathcal{L}^0_{em}$. Note that

$$\text{inv}(J^{n+}_{km}, \varphi^n(\mathcal{L}^s_{em}))(= \text{inv}(J^{n+}_{km}, \varphi^n(\mathcal{L}_1)))$$

is an attractor in $\text{inv}(\tilde{J}^{n+}_{km}, \varphi^n(\mathcal{L}^s_{em})))$. The map $\rho_2$ is the attractor map.

Similarly the isomorphism between $\text{swf}^R(Y, f_1)$ and $\text{swf}^R(Y, f_2)$ is induced by the composition of the repeller map and the homotopy equivalence induced by the homotopy of the flows.

To prove the invariance of $bf^A, bf^R$ with respect to perturbation $f$, we need to show that the morphisms $\rho_{1,2}$ are compatible with the attractor maps, the repeller maps and the homotopy equivalence induced by the homotopy of the flows. The compatibility with the attractor maps and the repeller maps is already stated in Lemma 5.21. We will show the compatibility with the homotopy equivalence induced by the homotopy of the flows.

Take perturbations $f_0, f_1 : Coul(-Y_{in}) \coprod Coul(Y_{out}) \rightarrow \mathbb{R}$. Let us consider the flow

$$\bar{\varphi}^n : V^{\mu_n}_{\lambda_n} \times [0, 1] \times \mathbb{R} \rightarrow V^{\mu_n}_{\lambda_n} \times [0, 1]$$
on $V_{0n}^\mu \times [0, 1]$, induced by the homotopy

$$\mathcal{L}_{V_{in}^\mu m_0}^s \prod \mathcal{L}_{V_{out}^\mu m_1}^s \colon Coul(-Y_{in}) \prod Coul(Y_{out}) \to \mathbb{R} \quad (0 \leq s \leq 1).$$

(33)

We also have the Seiberg-Witten map on $X$ induced by the homotopy:

$$Coul^{CD}(X) \times [0, 1] \to L_{k-1}^2(i\Omega^+(X) \oplus S_{\bar{X}}) \oplus V_{\infty}^\mu \times [0, 1].$$

Using the flow and the Seiberg-Witten map, for a small positive number $\epsilon > 0$, we define

$$\tilde{K}_1 = \tilde{K}_1(n, m_0, \epsilon), \quad \tilde{K}_2 = \tilde{K}_2(n, m_0, \epsilon) \subset B(V_{0n}^\mu, \tilde{R}) \times [0, 1]$$

as in [2]. As before we can show that $(\tilde{K}_1, \tilde{K}_2)$ is a pre-index pair and can find an index pair $(\tilde{N}, \tilde{L})$ such that

$$\tilde{K}_1(n, m_0, \epsilon) \subset \tilde{N}, \quad \tilde{K}_2(n, m_0, \epsilon) \subset \tilde{L}.$$

For $s \in [0, 1]$, put

$$K_{1,s}(n, m_0, \epsilon) := \tilde{K}_1(n, m_0, \epsilon) \cap (V_{0n}^\mu \times \{s\}),$$

$$K_{2,s}(n, m_0, \epsilon) := \tilde{K}_2(n, m_0, \epsilon) \cap (V_{0n}^\mu \times \{s\}),$$

$$N_s := \tilde{N} \cap (V_{0n}^\mu \times \{s\}),$$

$$L_s := \tilde{L} \cap (V_{0n}^\mu \times \{s\}).$$

We get the map

$$v_s : B(W_{n, \beta}, R)/S(W_{n, \beta}, R)$$

$$\to (B(U_n, \epsilon)/S(U_n, \epsilon)) \wedge (K_{1,s}(n, m_0, \epsilon)/K_{2,s}(n, m_0, \epsilon))$$

$$\leftarrow (B(U_n, \epsilon)/S(U_n, \epsilon)) \wedge (N_s/L_s).$$

The maps $v_0, v_1$ induce morphisms

$$\psi_0 : \Sigma^{-((V_{0n}^\mu + V_{in})T} \to I_{m_0}^{n+}(-Y_{in})_0 \wedge I_{m_1}^{n+}(Y_{out})_0,$$

$$\psi_1 : \Sigma^{-((V_{0n}^\mu + V_{in})T} \to I_{m_0}^{n-}(-Y_{in})_1 \wedge I_{m_1}^{n+}(Y_{out})_1$$

for $0 \ll m_0 \ll m_1 \ll n$ as before. We have to check that the following diagram is commutative:

$$\Sigma^{-((V_{0n}^\mu + V_{in})T \xrightarrow{v_0} I_{m_0}^{n-}(-Y_{in})_0 \wedge I_{m_1}^{n+}(Y_{out})_0 \xrightarrow{\cong}$$

$$I_{m_0}^{n-}(-Y_{in})_1 \wedge I_{m_1}^{n+}(Y_{out})_1$$

(34)

Here $I_{m_0}^{n-}(-Y_{in})_0 \wedge I_{m_1}^{n+}(Y_{out})_0 \cong I_{m_0}^{n-}(-Y_{in})_1 \wedge I_{m_1}^{n+}(Y_{out})_1$ is the isomorphism induced by the homotopy [33]. Consider the inclusion

$$i_s : N_s/L_s \hookrightarrow \tilde{N}/\tilde{L}$$
for \( s \in [0, 1] \). By Theorem 6.7 and Corollary 6.8 of [18], \( i_s \) is a homotopy equivalence and the following diagram is commutative up to homotopy:

\[
\begin{array}{ccc}
N_0/L_0 & \xrightarrow{i_0} & \tilde{N}/\tilde{L} \\
\cong & \downarrow & \\
N_1/L_1 & \xrightarrow{i_1} & 
\end{array}
\]  \hspace{1cm} (35)

Here \( N_0/L_0 \cong N_1/L_1 \) is the homotopy equivalence induced by the homotopy (33). With the homotopy

\[ i_s \circ v_s : B(W_{n,\beta}, R)/S(W_{n,\beta}) \to (B(U_n, \epsilon)/S(U_n, \epsilon)) \land (\tilde{N}/\tilde{L}) \]

between \( i_0 \circ v_0 \) and \( i_1 \circ v_1 \) and the commutativity of the diagram (33), we can see that the diagram (34) is commutative. The invariance of \( BF^A, BF^R \) with respect to perturbation \( f \) has been proved.

Assume that \( c_1(s) \) is torsion. We will prove that the normalized invariants \( BF^A, BF^R \) are independent of Riemannian metric \( g \) and base connection \( A_0 \) on \( Y \). Take Riemannian metrics \( g, g' \) and connections \( A_0, A'_0 \) on \( Y \). Let us consider the homotopy

\[ A_0(s) = (1 - s)A_0 + sA'_0, \quad g(s) = (1 - s)g + sg' \quad (s \in [0, 1]). \]

Choose continuous families of Riemannian metrics \( \hat{g}(s) \) and connections \( \hat{A}_0(s) \) on \( X \) with \( \hat{g}(s)|_Y = g(s), \hat{A}_0(s)|_Y = A_0(s) \). Splitting the interval \([0, 1]\) into small intervals \([0, 1] = [0, t_1] \cup \cdots \cup [t_{N-1}, t_N]\), the discussion is reduced to the case when \( \lambda_n, \mu_n \) (for some fixed, large number \( n \)) are not eigenvalues of the Dirac operators \( D_s \) on \( Y \) associated to \( g(s), A(s) \). In this case, the dimension of \( W_{n,\beta}(s) \) is constant, where

\[ W_{n,\beta}(s) := (L_s, p_{\infty}^{\mu_n})^{-1}(U_n \times V_{\lambda_n}^{\mu_n}(s)) \cap W_{X,\beta}(s). \]

Then we can mimic the discussion about the invariance with respect to perturbation \( f \) to get a homotopy \( v_s \) between \( v_0 \) and \( v_1 \) which are the maps in [29] associated \((\hat{g}, \hat{A}_0), (\hat{g}' , \hat{A}'_0)\). Therefore the morphisms \( \psi_{m_0, m_1}^n \) associated with \((\hat{g}_0, \hat{A}_0)\) and \((\hat{g}_1, \hat{A}_1)\) are the same. Note that the objects \((V_{\lambda_n}^0(s) \oplus C^n(Y, g_s, A_s))^+ \) of \( \mathcal{C} \) for \( s = 0, 1 \) are isomorphic to each other. Taking the desuspension by \( V_{\lambda_n}^0(s) \oplus C^n(Y, g_s, A_s) \), we conclude that \( BF^A, BF^R \) are independent of \( g, A_0 \) up to canonical isomorphisms.

6. The Gluing Theorem

6.1. Statement and setup of the gluing theorem. In this section, let \( X_0 : Y_0 \to Y_2 \) and \( X_1 : Y_1 \to -Y_2 \) be connected, oriented cobordisms with the following properties:

- \( Y_2 \) is connected;
- \( Y_0, Y_1 \) may not be connected but \( b_1(Y_0) = b_1(Y_1) = 0. \)
By gluing the two cobordisms along $Y_2$, we obtain a cobordism $X: Y_0 \cup Y_1 \to \emptyset$. As in Section 5, we choose the following data when defining the relative Bauer–Furuta invariants:

- A spin$^c$ structure $\hat{s}$ on $X$.
- A Riemannian metric $\hat{g}$ on $X$, we require it equals the product metric near $Y_i$.
- A base connection $\hat{A}_0$ on $X$;
- A base point $\hat{o} \in Y_2$ and a based path data $[\vec{\eta}_i]$ on $X_i$ for $i = 0, 1$.
- The path from $\hat{o}$ to $Y_2$ is chosen to be the constant path. By patching $[\vec{\eta}_1]$ and $[\vec{\eta}_2]$ together in the obvious way, we get a based path data $[\vec{\eta}]$ on $X$;
- Denote the restriction of $\hat{s}$ (resp. $\hat{g}$ and $\hat{A}_0$) to $X_i$ by $\hat{s}_i$ (resp. $\hat{g}_i$ and $\hat{A}_0$) and the restriction to $Y_j$ by $s_j$ (resp. $g_j$ and $A_0^j$).

With the above data chosen, we obtain the invariants

$$\text{bf}^A(X_0, \hat{s}_0, \hat{A}_0^0, \hat{o}, [\vec{\eta}_0]; S^1),$$

$$\text{bf}^R(X_1, \hat{s}_1, \hat{A}_1^1, \hat{g}_1, \hat{o}, [\vec{\eta}_1]; S^1),$$

and

$$\text{BF}(X, \hat{s}, \hat{o}, [\vec{\eta}]).$$

For shorthand, we write them as $\text{bf}^A(X_0)$, $\text{bf}^R(X_1)$ and $\text{BF}(X)$ respectively throughout this section.

**Theorem 6.1 (The Gluing Theorem).** With the above setup, if we assume further that

$$\text{im}(H^1(X_0; \mathbb{R}) \to H^1(Y_2; \mathbb{R})) \subset \text{im}(H^1(X_1; \mathbb{R}) \to H^1(Y_2; \mathbb{R})),$$

then, under the natural identification between domains and targets, we have

$$\text{BF}(X)|_{\text{Pic}^0(X, Y_2)} = \overline{\varepsilon}(\text{bf}^A(X_0), \text{bf}^R(X_1)),$$

where $\overline{\varepsilon}(\cdot, \cdot)$ is the Spanier-Whitehead duality operation defined in Section 4.3.

**Corollary 6.2.** When the map $H^1(X_0; \mathbb{R}) \to H^1(Y_2; \mathbb{R})$ is trivial, we recover the full Bauer–Furuta invariant

$$\text{BF}(X) = \overline{\varepsilon}(\text{bf}^A(X_0), \text{bf}^R(X_1)).$$

**Corollary 6.3.** When $s_2$ is torsion and (36) is satisfied, we also have analogous result for the normalized version

$$\text{BF}(X)|_{\text{Pic}^0(X, Y_2)} = \overline{\varepsilon}(\text{BF}^A(X_0), \text{BF}^R(X_1)).$$

We begin by setting up some notations. Let $\iota_i: Y_2 \to X_i$ be the inclusion map. We pick a set of loops

$$\{\alpha_0^0, \ldots, \alpha_{b_0^0, \alpha}^0\}, \{\alpha_1^1, \ldots, \alpha_{b_1^1, \alpha}^1\}, \{\beta_1, \ldots, \beta_{b_1, \beta}\}$$

with the following properties:
under the assumption (36), the above properties further imply the following two properties:

- The set
  \[ \{ \alpha^0_1, \ldots, \alpha^0_{b_{1,\alpha}} \} \cup \{ \alpha^1_1, \ldots, \alpha^1_{b_{1,\alpha}} \} \cup \{ \beta_1, \ldots, \beta_{b_{1,\beta}} \} \]
  represents a basis of \( H_1(X; \mathbb{R}) \);
- \( \{ \alpha^0_1, \ldots, \alpha^0_{b_{1,\alpha}} \} \cup \{ \alpha^1_1, \ldots, \alpha^1_{b_{1,\alpha}} \} \) represent a basis of \( H_1(X_2; \mathbb{R}) \).

Under the assumption (36), the above properties further imply the following two properties:

- For \( i = 0, 1 \), the set \( \{ \alpha^i_1, \ldots, \alpha^i_{b_{i,\alpha}} \} \) is contained in the interior of \( X_i \)
  and represents a basis of the induced map
  \[ (\iota_i)_*: H_1(Y_2; \mathbb{R}) \to H_1(X_i; \mathbb{R}) \].
- \( \{ \beta_1, \ldots, \beta_{b_{1,\beta}} \} \subset Y_2 \) represents a basis for a subspace complementary to the kernel of \((\iota_0)_*: H_1(Y_2; \mathbb{R}) \to H_1(X_0; \mathbb{R})\). Note that \( \langle \rangle \).

As before, we use \( \mathcal{G}_{X_i}^{h,\alpha} \) to denote the group of harmonic gauge transformations \( u \) on \( X_i \) such that \( u(\hat{\alpha}) = 1 \) and \( u^{-1} du \in \Omega^1(X_i) \), and let \( \mathcal{G}_{X_i,\partial X_i}^{h,\alpha} \) be the subgroup of \( \mathcal{G}_{X_i}^{h,\alpha} \) corresponding to \( \ker(H^1(X_i; \mathbb{Z}) \to H^1(\partial X_i; \mathbb{Z})) \).

Note that \( \mathcal{G}_{X_i,\partial X_i}^{h,\alpha} \cong H^1(X_i, Y_2; \mathbb{Z}) \).

As in Section 5 for \( i = 0, 1 \), we consider the bundles
\[
\mathcal{W}_{X_i} = \text{Coul}^{CC}(X_i)/\mathcal{G}_{X_i,\partial X_i}^{h,\alpha},
\]
over \( \text{Pic}^0(X_i, \partial X_i) \) and the subbundle
\[
\mathcal{W}_{X_0,\beta} := \{ x \in \mathcal{W}_{X_0} | \hat{p}_{\beta}(x) = 0 \},
\]
where the projection \( \hat{p}_{\beta}: \text{Coul}^{CC}(X_0) \to \mathbb{R}^{b_{1,\beta}} \) is given by
\[
\hat{p}_{\beta}(\hat{\alpha}, \hat{\phi}) = (-i \int_{\beta_1} \hat{t}_\alpha, \ldots, -i \int_{\beta_{b_{1,\beta}}} \hat{t}_\alpha).
\]

**Remark 6.4.** Recall that \( \mathcal{G}_{X_0}^{h,\alpha} \) is constructed from the bundle \( \mathcal{W}_{X_0} \),
while \( \mathcal{G}_{X_0}^{h,\alpha} \) is constructed from a smaller bundle \( \mathcal{W}_{X_0,\beta} \). As a consequence, the decomposition (37) is essential in proving the pairing theorem involving the A-invariant for \( X_0 \) and R-invariant for \( X_1 \). Since existence of this asymmetric decomposition is implied by Condition (36), one can not switch \( X_0 \) and \( X_1 \) in this condition (without changing types of the relative invariants).

We have a basic boundedness result for glued trajectories:

**Proposition 6.5.** There exists a universal constant \( R_3 \) with the following significance: For any tuple \((\tilde{x}_0, \tilde{x}_1, \gamma_0, \gamma_1, \gamma_2, T)\) satisfying the following conditions:

- \( (\tilde{x}_0, \tilde{x}_1) \in \mathcal{W}_{X_0,\beta} \times \mathcal{W}_{X_1} \) satisfies \( \mathcal{S}W(\tilde{x}_j) = 0 \);
- \( \gamma_i: (-\infty, 0] \to \text{Coul}(Y_i) \) (\( i = 0, 1 \)) and \( \gamma_2: [-T, T] \to \text{Coul}(Y_2) \) are finite type Seiberg-Witten trajectories.
• \( \tilde{r}_0(\tilde{x}_0) = \gamma_0(0), \tilde{r}_2(\tilde{x}_0) = \gamma_2(-T), \tilde{r}_2(\tilde{x}_1) = \gamma_2(T) \) and \( \tilde{r}_1(\tilde{x}_1) = \gamma_1(0) \), where \( \tilde{r}_j \) denotes the twisted restriction map to \( \text{Coul}(Y_j) \);

one has \( \|\tilde{x}_i\|_F \leq R_3 \) for \( i = 0, 1 \) and \( \gamma_j \in \text{Str}_Y(R_3) \) for \( j = 0, 1, 2 \).

**Proof.** Suppose there exists a sequence not satisfying such uniform bounds. We also assume that \( T \to +\infty \) as the case when \( T \) is uniformly bounded is trivial. From the condition \( \hat{p}_\beta(x_0) = 0 \), the norm of \( \gamma_0 \) and the norm \( \|\tilde{x}_0\|_F \) is controlled by Theorem 5.11. Notice that the solutions converge to a broken trajectory on the \( Y_2 \)-neck, which is contained in \( \text{Str}_Y(R) \) for some universal constant \( R \) by [9, Theorem 3.2]. As in the construction, of \( \text{swf}(Y_2) \), we consider a sequence of bounded subset \( \{J^+_m(Y_2)\} \) of \( \text{Str}_Y(R) \) (cf. [9, Definition 5.3]). Since \( \|\tilde{x}_0\|_F \) is uniformly bounded, \( \tilde{r}_2(\tilde{x}_0) \) is contained in \( J^+_m(Y_2) \) for some fixed \( m \). From the fact that \( J^+_m(Y_2) \) is an attractor with respect to the Seiberg–Witten flow, we see that the whole broken trajectory is contained in \( J^+_m(Y_2) \). In particular, \( \tilde{r}_2(\tilde{x}_1) \) also belongs to \( J^+_m(Y_2) \). We then apply Theorem 5.11 again on \( X_1 \) to control \( \|\tilde{x}_1\|_F \) and the norm of \( \gamma_1 \). \( \square \)

Following Section 5.4 we will start to consider finite-dimensional approximation of the Seiberg–Witten map on both \( X_0 \) and \( X_1 \). Let us fix an increasing sequence of positive real numbers \( \{\mu_n\} \) such that \( \mu_n \to \infty \). For \( i = 0, 1, 2 \), let \( V^i_n \subset \text{Coul}(Y_i) \) be the span of eigenspaces with respect to \( (*d, \mathcal{D}) \) with eigenvalues in the interval \([-\mu_n, \mu_n]\). For \( i = 0, 1 \), we choose appropriate finite-dimensional subspaces \( U^i_n \subset L^{2,1}(\Omega^2(X_i) \oplus \Gamma(S_X^{-})) \). The preimages of \( U^i_n \times V^i_n \times V^2_n \) under \( (L, p^\mu_\infty \circ \tilde{r}) \) give rise to finite-dimensional subbundles \( W^{0}_{n,\beta} \subset \mathcal{W}_{X_0,\beta} \) and \( W^1_n \subset \mathcal{W}_{X_1} \).

We now state the boundedness result for approximated solutions.

**Proposition 6.6.** For any \( R > 0 \) and \( L \geq 0 \) and any bounded subsets \( S_i \) of \( \text{Coul}(Y_i) \) \( (i = 0, 1, 2) \), there exist constants \( \varepsilon, N, T > 0 \) with the following significance: For any tuple \( (\tilde{x}_0, \tilde{x}_1, \gamma_0, \gamma_1, \gamma_2, n, T, T') \) satisfying

1. \( n > N, T' > T, \) and \( T \leq L, \)
2. \( (\tilde{x}_0, \tilde{x}_1) \in B(W^{0}_{n,\beta}, R) \times B(W^1_n, R) \) such that \( \|\text{SW}_n(\tilde{x}_j)\|_{L^2_{k-1/2}} < \varepsilon \) \( (j = 0, 1) \),
3. \( \gamma_i: [-T', 0] \to \gamma_i \cap S_i \) \((i = 0, 1)\) and \( \gamma_2: [-T, T] \to V^2_n \cap S_2 \) are finite type approximated Seiberg-Witten trajectories,
4. \( p^\mu_\infty \circ \tilde{r}_0(\tilde{x}_0) = \gamma_0(0), p^\mu_\infty \circ \tilde{r}_2(\tilde{x}_0) = \gamma_2(-T), p^\mu_\infty \circ \tilde{r}_2(\tilde{x}_1) = \gamma_2(T) \) and \( p^\mu_\infty \circ \tilde{r}_1(\tilde{x}_1) = \gamma_1(0) \),

one has the following estimate

1. \( \|\tilde{x}_i\|_F \leq R_3 + 1 \) for \( i = 0, 1 \);
2. \( \gamma_2 \subset \text{Str}_Y(R_3 + 1); \)
3. \( \gamma_i([-T', -T], 0) \subset B_Y(R_3 + 1) \) for \( i = 0, 1 \).

Here \( R_3 \) is the constant from Proposition 6.5.

**Proof.** The proof is analogous to that of Proposition 5.15 and Proposition 5.16 where one applies Proposition 6.5 instead. \( \square \)
Recall that, for \( i = 0, 1 \), the manifold \( Y_i \) is a rational homology sphere and one can find a sufficiently large ball \( B_Y(\tilde{R}_0) \) in the Coulomb slice containing all finite type Seiberg-Witten trajectories (cf. \([13]\)). On \( Y_2 \), an unbounded subset \( \text{Str}_{Y_2}(\tilde{R}_2) \) contains all finite type Seiberg-Witten trajectories when \( \tilde{R}_2 \) is sufficiently large. With a choice of cutting functions, we obtain an increasing sequence of bounded sets \( \{ J_m^+(Y_2) \} \) contained in \( \text{Str}_{Y_2}(\tilde{R}_2) \). Note that we can identify \( J_m^{i-}(-Y_2) = J_m^{i+}(Y_2) \).

Throughout the rest of the section, we will fix the following parameters in order of dependency carefully.

(i) Pick \( \hat{R}_0 > R_3 \) such that any finite type \( X_0 \)-trajectories \((x, \gamma)\) with \( x \in W_{X_0, \beta} \) satisfies \( \|x\|_F \leq \hat{R}_0 \) (cf. Theorem 5.11).

(ii) Pick \( \hat{R}_0, \hat{R}_2 > R_3 + 2 \) such that \( \tilde{r}(B(W_{X_0, \beta}, \hat{R}_0)) \subset B_{Y_0}(\hat{R}_0) \times \text{Str}_{Y_2}(\hat{R}_2) \) and also \( B_{Y_0}(\hat{R}_0 - 1) \times \text{Str}_{Y_2}(\hat{R}_2 - 1) \) contains all finite type Seiberg-Witten trajectories.

(iii) Choose a positive integer \( m \) such that

\[
\tilde{r}_2(B(W_{X_0, \beta}, \hat{R}_0)) \subset J_{m-1}^+(Y_2).
\]

(iv) Pick \( \hat{R}_1 > R_3 + 1 \) such that any finite type \( X_1 \)-trajectory \((x, \gamma)\) with \( \tilde{r}_2(x) \in J_m^+(Y_2) \), one has \( \|x\|_F < \hat{R}_1 \).

(v) Choose a positive number \( \tilde{R}_1 \) such that \( \tilde{r}_2(B(W_{X_1}, \tilde{R}_1)) \subset B_{Y_1}(\tilde{R}_1) \) and \( B_{Y_1}(\tilde{R}_1 - 1) \) contains all finite type Seiberg-Witten trajectory on \( Y_1 \).

6.2. Deformation of the duality pairing. In this section, we will focus on describing the right hand side of the gluing theorem

\[
\bar{\epsilon}(bf^{A}(X_0), bf^{R}(X_1))
\]

and its deformation. As in Section 5.4, we write down the following subsets in order to define \( bf^{A}(X_0) \) and \( bf^{R}(X_1) \):

\[
K_0 = \rho_{-}\infty \circ \tilde{r}(SW_n^{-1}(B(U_n^{0}, \epsilon)) \cap B(W_{n, \beta}, \hat{R}_0)),
\]

\[
S_0 = \rho_{-}\infty \circ \tilde{r}(SW_n^{-1}(B(U_n^{0}, \epsilon)) \cap S(W_{n, \beta}, \hat{R}_0)),
\]

\[
K_1 = \rho_{-}\infty \circ \tilde{r}(SW_n^{-1}(B(U_n^{1}, \epsilon)) \cap B(W_n^1, \hat{R}_1)) \cap (V_n^1 \times J_{m}^{n-}(-Y_2)),
\]

\[
S_1 = \{ \rho_{-}\infty \circ \tilde{r}(SW_n^{-1}(B(U_n^{1}, \epsilon)) \cap S(W_n^1, \hat{R}_1)) \cap (V_n^1 \times J_{m}^{n-}(-Y_2)) \}
\]

\[
\cup \{ K_1 \cap (V_n^1 \times \partial J_{m}^{n-}(Y_2)) \}.
\]

Note that some of the subsets are simpler because \( b_1(Y_0) = b_1(Y_1) = 0 \).

The parameters \((\hat{R}_0, \hat{R}_1, \tilde{R}_0, \tilde{R}_1, \tilde{R}_2, m)\) are selected earlier. Subsequently, we will also fix a large number \( L_0 \) with the following property and then proceed to \( n \) and \( \epsilon \).

(vi) Choose a positive number \( L_0 \) such that, for any large \( n \) and small \( \epsilon \), one has
(a) \((K_0, S_0)\) and \((K_1, S_1)\) are \(L_0\)-tame pre-index pairs. This follows from Proposition 5.17 red by applying it to \(X_0\) and \(X_1\);
(b) The pair \((K^0, S^0)\), as defined below
\[ K^0 = \{(y_0, y_1) \mid (y_0, y) \times (y_1, y) \in K_0 \times K_1 \text{ for some } y\} \]
\[ S^0 = \{(y_0, y_1) \mid (y_0, y) \times (y_1, y) \in S_0 \times K_1 \cup K_0 \times S_1 \text{ for some } y\}, \]
is an \(L_0\)-tame pre-index pair for \(B(V_n^0, \tilde{R}_0) \times B(V_n^1, \tilde{R}_1)\). This follows from Proposition 6.6 with \(L = 0\).
(c) Pick a slightly smaller closed subset \(J'_m \subset \text{int}(J_m^+(Y_2))\) such that for any approximated trajectory \(\gamma: [-L_0, L_0] \to B(V_n^0, \tilde{R}_0) \times B(V_n^1, \tilde{R}_1) \times J_m^+(Y_2)\), one has \(\gamma(0) \in B(V_n^0, \tilde{R}_0 - 1) \times B(V_n^1, \tilde{R}_1 - 1) \times J'_m\) (cf. [9, Lemma 5.5]).
(d) \(L_0 > 4T_m(j)\) where \(T_m(j)\) is the constant which appeared in Lemma 3.24, applying to the manifold \(Y_j\).
(vii) Finally, we pick a large positive integer \(n\) and a small positive real number \(\epsilon\) so that
(a) The above assertions for \(L_0\) hold;
(b) Proposition 6.6 holds for \(L = 3L_0\), \(R = \max(\tilde{R}_0, \tilde{R}_1)\), \(S_0 = B_{Y_0}(\tilde{R}_0)\), \(S_1 = B_{Y_1}(\tilde{R}_1)\) and \(S_2 = J'_m(Y_2)\).

With all the above parameters fixed, we have canonical maps to Conley indices
\[ \iota_0: K_0/S_0 \to I(B(V_n^0, \tilde{R}_0)) \wedge I(J_m^{n,+}(Y_2)), \]
\[ \iota_1: K_1/S_1 \to I(B(V_n^1, \tilde{R}_1)) \wedge I(J_m^{n,-}(Y_2)). \]

For simplicity, we will write \(A_j = B(V_n^j, \tilde{R}_j)\) and \(A'_j = B(V_n^j, \tilde{R}_j - 1)\) for \(j = 0, 1\). We also let \(A_2\) denote \(J_m^{n,+}(Y_2)\) and let \(A'_2\) be a closed subset satisfying
\[ (\text{Str}_Y(\tilde{R}_2 - 1) \cap J_m^{n,+}(Y_2)) \cup (J'_m \cap V_n^2) \subset \text{int}(A'_2) \subset A'_2 \subset \text{int}(A_2). \]

By our choice of \(L_0\) and Proposition 3.24, there exists a manifold isolating block \(\tilde{N}_j\) satisfying
\[ A_j^{[-L_0,L_0]} \subset \text{int}(\tilde{N}_j) \subset \tilde{N}_j \subset A'_j. \tag{38} \]

Let \(\varphi^j\) be the approximated Seiberg–Witten flow on \(A_j\). Denote by \(\tilde{N}_j^-\) (resp. \(\tilde{N}_j^+\)) be the submanifold of \(\partial \tilde{N}_j\) where \(\varphi^j\) points outward (resp. inward).

By the choice of \(L_0\) and Lemma 3.13 and Theorem 3.14, we can express the smash product of canonical maps
\[ \iota_0 \wedge \iota_1: K_0/S_0 \wedge K_1/S_1 \to \tilde{N}_0/\tilde{N}_0^- \wedge \tilde{N}_2/\tilde{N}_2^- \wedge \tilde{N}_1/\tilde{N}_1^- \wedge \tilde{N}_2/\tilde{N}_2^- \]
as a map sending \((y_0, y_2, y_1, y_2')\) to
\[ (\varphi^0(y_0, 3L_0), \varphi^2(y_2, 3L_0), \varphi^1(y_1, 3L_0), \varphi^2(y_2, -3L_0)) \]
Lemma 6.7. There exists a positive constant $\bar{\varepsilon}_0$ such that one can find a closed subset $B_0 \subset \text{int}(\bar{N}_2)$ with the following property: For any $(y_2, y_2')$ satisfying (40) and

$$||\varphi^j(y_2, 3L) - \varphi(y_2', -3L)|| \leq 5\bar{\varepsilon}_0,$$

one has

$$\varphi^2(y_2, [0, 3L_0]) \subset B_0 \text{ and } \varphi^2(y_2', [-L_0, -3L_0]) \subset B_0.$$

In particular, $(y_2, y_2')$ will satisfy (41).

Proof. From (38), we see that one can choose $\hat{\varepsilon}_0$ such that one can find a closed subset $B_0 = A^{-L_0,L_0}_2$ if we consider the case $\hat{\varepsilon}_0 = 0$. For positive $\hat{\varepsilon}_0$, we pick $B_0$ to be a slightly larger closed subset containing $A^{-L_0,L_0}_2$ and then apply a continuity argument.

To deform our maps, we also consider a variation of the above lemma.

Lemma 6.8. There exists a positive constant $\bar{\varepsilon}_1$ such that for any $L \in [0, L_0]$ and any $(y_0, y_2, y_2', y_1) \in K_0 \times K_1$ satisfying (39) and

$$\varphi^2(y_2, [0, 3L]) \subset A_2 \text{ and } \varphi^2(y_2', [-3L, 0]) \subset A_2;$$

we have

$$||\varphi^2(y_2, 3L) - \varphi^2(y_2', -3L)|| \leq \bar{\varepsilon}_1.$$

Proof. We first consider the case $\bar{\varepsilon}_1 = 0$. Then, by Proposition 6.6 and our choice of $(n, \varepsilon)$, we have $\varphi^2(y_2, [0, 6L]) \subset \text{Str}_{m_1}(\bar{R}_2 - 1)$. From our choice, we also have $y_2 \in J^{n_{m-1}^+}(Y_2) \subset V^2_n$. Since $J^{n_{m-1}^+}(Y_2)$ is an attractor in $J^{n_{m-1}^+}(Y_2)$, we have $\varphi^2(y_2, [0, 6L]) \subset J^{n_{m-1}^+}(Y_2)$. Thus

$$\varphi^2(y_2, [0, 6L]) \subset (\text{Str}_{m_1}(\bar{R}_2 - 1) \cap J^{n_{m-1}^+}(Y_2)) \subset \text{int}(A'_2).$$

The general case follows from a continuity argument.

We will also consider the following subsets enlarging $(K^0, S^0)$

$$K^\varepsilon := \{ (y_0, y_1) \mid (y_0, y_2) \times (y_1, y_2') \in K_0 \times K_1$$

for some $y_2, y_2'$ with $||y_2 - y_2'|| \leq \varepsilon$,

$$S^\varepsilon := \{ (y_0, y_1) \mid (y_0, y_2) \times (y_1, y_2') \in (S_0 \times K_1) \cup (K_0 \times S_1)$$

for some $y_2, y_2'$ with $||y_2 - y_2'|| \leq \varepsilon$. 

When the following conditions are all satisfied

$$\varphi^j(y_j, [0, 3L_0]) \subset A_j \text{ and } \varphi^j(y_j, [L_0, 3L_0]) \subset \bar{N}_j \setminus \bar{N}_j^- \text{ for } j = 0, 1; \tag{39}$$

$$\varphi^2(y_2, [0, 3L_0]) \subset A_2 \text{ and } \varphi^2(y_2', [-3L_0, 0]) \subset A_2; \tag{40}$$

$$\varphi^2(y_2, [L_0, 3L_0]) \subset \bar{N}_2 \setminus \bar{N}_2^- \text{ and } \varphi^2(y_2', [-L_0, -3L_0]) \subset \bar{N}_2 \setminus \bar{N}_2^+ \tag{41}.$$
Since \((K^0, S^0)\) is an \(L_0\)-tame pre-index pair, the following can be obtained by a continuity argument.

**Lemma 6.9.** There exists a positive constant \(\bar{\epsilon}_2\) such that the pair \((K^\bar{\epsilon}, S^\bar{\epsilon})\) is an \(L_0\)-tame pre-index pair for any \(0 \leq \bar{\epsilon} \leq \bar{\epsilon}_2\).

For a vector space or a vector bundle, denote by \(B^+(V, R)\) the sphere \(B(V, R)/S(V, R)\). Recall that the Spanier-Whitehead duality map (see Section 4.4)

\[
\bar{\epsilon}: \tilde{N}_2/\tilde{N}_2^- \wedge \tilde{N}_2/\tilde{N}_2^+ \to B^+(V_2^\bar{\epsilon}, \bar{\epsilon})
\]

can be given by

\[
\bar{\epsilon}(y_2, y'_2) = \begin{cases} 
\eta_-(y_2) - \eta_+(y'_2) & \text{if } \|\eta_-(y_2) - \eta_+(y'_2)\| \leq \bar{\epsilon}, \\
* & \text{otherwise.}
\end{cases}
\]

Here we pick \(\bar{\epsilon} < \min\{\epsilon_0, \epsilon_1, \epsilon_2\}\) and \(\eta_+: \tilde{N}_2 \to \tilde{N}_2\) are homotopy equivalences which are identity on \(B_0 \subset \text{int}(\tilde{N}_2)\) and satisfy \(\|\eta_\pm(x) - x\| \leq 2\bar{\epsilon}\) for any \(x\). Here \(B_0\) is the closed set in Lemma 6.7.

Consequently we can write down the composition of \(\iota_0 \wedge \iota_1\) and \(\bar{\epsilon}\) as a map

\[
\bar{\epsilon}(\iota_0, \iota_1): K_0/S_0 \wedge K_1/S_1 \to \tilde{N}_0/\tilde{N}_0^- \wedge \tilde{N}_1/\tilde{N}_1^- \wedge B^+(V_2^\bar{\epsilon}, \bar{\epsilon})
\]

given by

\[
(y_0, y_2, y_1, y'_2) \mapsto (\varphi^0(y_0, 3L_0), \varphi^1(y_1, 3L_0), \varphi^2(y_2, 3L_0) - \varphi^2(y'_2, -3L_0)) \quad (44)
\]

if (39) and (40) and

\[
\|\varphi^2(y_2, 3L_0) - \varphi^2(y'_2, -3L_0)\| \leq \bar{\epsilon} \quad (45)
\]

are satisfied. This follows from Lemma 6.7 and our choice of \(\bar{\epsilon}\) and \(\eta_\pm\).

We now begin to deform the map \(\bar{\epsilon}(\iota_0, \iota_1)\).

**Step 1.** We will deform the map so that \(L_0\) in the last term of (44) goes from \(L_0\) to 0. To achieve this, we consider a family of maps

\[
K_0/S_0 \wedge K_1/S_1 \to \tilde{N}_0/\tilde{N}_0^- \wedge \tilde{N}_1/\tilde{N}_1^- \wedge B^+(V_2^\bar{\epsilon}, \bar{\epsilon})
\]

\[
(y_0, y_2, y_1, y'_2) \mapsto (\varphi^0(y_0, 3L_0), \varphi^1(y_1, 3L_0), \varphi^0(y_2, 3L) - \varphi^2(y'_2, -3L))
\]

if (39) together with the conditions

\[
\varphi^2(y_2, [0, 3L]) \subset A_2, \ \varphi^2(y'_2, [-3L, 0]) \subset A_2
\]

\[
\|\varphi^2(y_2, 3L) - \varphi^2(y'_2, -3L)\| \leq \bar{\epsilon}
\]

are all satisfied. Lemma 6.8 guarantees that this is a continuous family. Thus, \(\bar{\epsilon}(\iota_0, \iota_1)\) is homotopic to the map \(\bar{\epsilon}_0(\iota_0, \iota_1)\) at \(L = 0\), which is given by

\[
(y_0, y_2, y'_2, y_1) \mapsto (\varphi^0(y_0, 3L_0), \varphi^1(y_1, 3L_0), y_2 - y'_2) \quad (46)
\]

if (39) and \(\|y_2 - y'_2\| \leq \bar{\epsilon}\) are satisfied.
Step 2. By Lemma 6.9 \((K^\ell, S^\ell)\) is an \(L_0\)-tame pre-index pair and we have a canonical map
\[
\iota^\ell : K^\ell / S^\ell \to I(B(V^0_n, \tilde{R}_0)) \land I(B(V^1_n, \tilde{R}_1)).
\]
It is not hard to check that the map given by
\[
K_0/S_0 \land K_1/S_1 \to I(B(V^0_n, \tilde{R}_0)) \land I(B(V^1_n, \tilde{R}_1)) \land (V^2_n)^+.
\]
(47)
\[
(y_0, y_2, y_1, y'_2) \mapsto \begin{cases} 
(\iota^\ell(y_0, y_1), y_2 - y'_2) & \text{if } \|y_2 - y'_2\| \leq \bar{\epsilon}, \\
\ast & \text{otherwise}
\end{cases}
\]
is well-defined and continuous. From Lemma 3.13 we can represent \(\iota^\ell\) by a map
\[
K^\ell / S^\ell \to \tilde{N}_0 / \tilde{N}_0^- \land \tilde{N}_1 / \tilde{N}_1^-;
\]
\[
(y_0, y_1) \mapsto (\varphi^0(y_0, 3L_0), \varphi^1(y_1, 3L_0)),
\]
if (39) is satisfied. Consequently, we are able to replace the first two components of the map (46) by \(\iota^\ell\) and the map \(\tilde{\boldsymbol{e}}_0(\iota_1, \iota_2)\) by (47).

Finally, recall that the relative Bauer–Furuta invariant \(\text{bf}^A(X_0)\) is obtained from composition of a map
\[
B^+(W^0_{n, \beta}, \tilde{R}_0) \to B^+(U^0_n, \epsilon) \land K_0 / S_0
\]
and the canonical map \(\iota_0\). The invariant \(\text{bf}^R(X_1)\) is obtained similarly. Then, \(\tilde{\boldsymbol{e}}(\text{bf}^A(X_0), \text{bf}^R(X_1))\) is given by applying Spanier–Whitehead dual to their smash product. From previous steps, we can conclude the following result

**Proposition 6.10.** The morphism \(\tilde{\boldsymbol{e}}(\text{bf}^A(X_0), \text{bf}^R(X_1))\) can be represented by suitable desuspension of the map
\[
B^+(W^0_{n, \beta}, \tilde{R}_0) \land B^+(W^1_n, \tilde{R}_1)
\]
\[
\rightarrow B^+(U^0_n, \epsilon) \land B^+(U^1_n, \epsilon) \land B^+(V^2_n, \epsilon) \land I^n(-Y_0) \land I^n(-Y_1)
\]
defined by
\[
(\tilde{x}_0, \tilde{x}_1) \mapsto (\tilde{S}W_n(\tilde{x}_0), \tilde{S}W_n(\tilde{x}_1), r_2(\tilde{x}_0) - r_2(\tilde{x}_1), \iota^\ell(r_0(\tilde{x}_0), r_1(\tilde{x}_1)))
\]
if \(\|\tilde{S}W_n(\tilde{x}_i)\| \leq \epsilon\) and \(\|r_2(\tilde{x}_0) - r_2(\tilde{x}_1)\| \leq \bar{\epsilon}\) and sending \((\tilde{x}_0, \tilde{x}_1)\) to the base point otherwise. Here \(I^n(-Y_i)\) denotes \(I(B(V^i_n, \tilde{R}_i))\) for \(i = 0, 1\).

6.3. **Stably c-homotopic pairs.** In this subsection, we recall notions of stably c-homotopy and SWC triples which were originally introduced by Manolescu [14]. These provide a convenient framework when deforming stable homotopy maps coming from construction of Bauer–Furuta invariants. Although most of the definitions are covered in [14], we rephrase them in a slightly more general setting which is easier to apply in our situation. We also give some details for completeness and concreteness.

Let \(p_i : E_i \to B\) (\(i = 1, 2\)) be Hilbert bundles over some compact space \(B\). We denote by \(\| \cdot \|_i\) the fiber-direction norm of \(E_i\). Let \(E_1\) be the fiberwise...
completion of $E_1$ using a weaker norm, which we denote by $|\cdot|_1$. We also assume that for any bounded sequence $\{x_n\}$ in $E_1$, there exist $x_\infty \in E_1$ such that after passing to a subsequence, we have

- $\{x_n\}$ converge to $x_\infty$ weakly in $E_1$.
- $\{x_n\}$ converge to $x_\infty$ strongly in $E_1$.

**Definition 6.11.** A pair $l, c: E_1 \to E_2$ of bounded continuous bundle maps is called an admissible pair if it satisfies the following conditions:

- $l$ is a fiberwise linear map;
- $c$ extends to a continuous map $\tilde{c}: \tilde{E}_1 \to E_2$.

At this point, we will focus on the context of the gluing theorem as in Section 6.1. Let $V = \text{Coul}(Y_0) \times \text{Coul}(Y_1)$ with $b_1(Y_0) = b_1(Y_1) = 0$. As before, denote by $V^\mu_\lambda$ the subspace spanned by the eigenvectors of $(\lambda, \mu)$ with eigenvalue in $(\lambda, \mu]$ and denote the projection $V \to V^\mu_\lambda$ by $p^\mu_\lambda$. Motivated by the Seiberg-Witten map on 4-manifolds with boundary, we give the following definition.

**Definition 6.12.** Let $(l, c)$ be an admissible pair from $E_1$ to $E_2$ and let $r: E_1 \to V$ be a continuous map which is linear on each fiber. We call $(l, c, r)$ an **SWC-triple** (which stands for Seiberg–Witten–Conley) if the following conditions are satisfied:

1. The map $l \oplus (p^0_{-\infty} \circ r): E_1 \to E_2 \oplus V^0_{-\infty}$ is fiberwise Fredholm.
2. There exists $M' > 0$ such that for any pair of $x \in E_1$ satisfying $(l + c)(x) = 0$ and a half-trajectory of finite type $\gamma: (-\infty, 0] \to V$ with $r(x) = \gamma(0)$, we have $\|x\|_1 < M'$ and $\|\gamma(t)\| < M'$ for any $t \geq 0$.

Two SWC-triples $(l_i, c_i, r_i)$ ($i = 0, 1$) (with the same domain and targets) are called **c-homotopic** if there is a homotopy between them through a continuous family of SWC triples with a uniform constant $M'$.

Two SWC-triples $(l_i, c_i, r_i)$ ($i = 0, 1$) (with possibly different domain and targets) are called **stably c-homotopic** if there exist Hilbert bundles $E_3, E_4$ such that $((l_1 \oplus \text{id}_{E_3}, c_1 \oplus 0_{E_3}), r_1 \oplus 0_{E_3})$ is c-homotopic to $((l_2 \oplus \text{id}_{E_4}, c_2 \oplus 0_{E_4}), r_2 \oplus 0_{E_4})$.

For any SWC triple $(l, c, r)$, we can define a relative Bauer–Furuta type invariant as a pointed stable homotopy class

$$BF(l, c, r) \in \{\Sigma^n \mathcal{T}(\text{ind}(l, p^0_{-\infty} \circ r)), \text{SWF}(-Y_0) \wedge \text{SWF}(-Y_1)\},$$

where

$$n = n(Y_0, s_{Y_0}, g_{Y_0}) + n(Y_1, s_{Y_1}, g_{Y_1})$$

by so called “SWC-construction” analogous to the construction in Section 5 described below.

Let us pick a trivialization $E_2 \cong F_2 \times B$ with a projection $q: E_2 \to F_2$, an increasing sequence of real numbers $\lambda_n \to \infty$ and a sequence of increasing finite-dimensional subspaces $\{F_n\} \subset F_2$ such that the projections $p_n: F_2 \to
Given $F^n_n$ converge pointwisely to the identity map and $q^{-1}(F^n_n) \times V_{-\lambda_n}^\lambda \subset E_2 \times V_{-\lambda_n}^\lambda$ is transverse to the image of $(l, p_{-\lambda_n}^\lambda \circ r)$ on each fiber. Let $E^n_1$ be the preimage $(l, p_{-\lambda_n}^\lambda \circ r)^{-1}(q^{-1}(F^n_n) \times V_{-\lambda_n}^\lambda)$ which is a finite rank subbundle.

Consider an approximated map

$$f_n = p_n \circ q \circ (l + c): E^n_1 \to F^n_2.$$  

From the definition of the SWC triple, one can deduce the following result in the same manner as the construction of relative invariants for Seiberg-Witten maps: for any $R', R \gg 0$ satisfying $r(B(E_1, R)) \subset B(V, R')$, there exist $N, \epsilon_0$ such that for any $n \geq N$ and $\epsilon < \epsilon_0$, the pair of subsets

$$(p_{-\lambda_n}^\lambda \circ r(f_n^{-1}(B(F^n_n, \epsilon)) \cap B(E_1, R)), p_{-\lambda_n}^\lambda \circ r(f_n^{-1}(B(F^n_n, \epsilon)) \cap S(E_1, R)))$$

is a pre-index pair in the isolating neighborhood $B(V_{-\lambda_n}^\lambda, R')$.

From this, we can find an index pair $(N, L)$ containing the above pre-index pair, which allows us to define an induced map

$$B(E^n_1, R)/S(E^n_1, R) \to B(F^n_2, \epsilon)/S(F^n_2, \epsilon) \land N/L.$$  

After desuspension, we obtain a stable map

$$h: \Sigma^n C \text{T}(\text{ind}(l, p_{-\lambda_n}^\lambda \circ r)) \to \text{SWF}(-Y_0) \land \text{SWF}(-Y_1).$$

By standard homotopy arguments, the stable homotopy class $[h]$ does not depend on auxiliary choices. As a result, we define the stable homotopy class $[h]$ to be the relative invariant $BF(l, c, r)$ for this SWC triple.

It is straightforward to prove that two stably $c$-homotopic SWC triples give the same stable homotopy class. This is the main point of introducing SWC construction. We end with a very useful lemma which is a generalization of Observation 1 in [14, Section 4.1] and allows us to move between maps and conditions on the domain.

**Lemma 6.13.** Let $(l, c)$ be an admissible pair from $E_1$ to $E_2$ and let $r: E_1 \to V$ be a continuous map which is linear on each fiber. Suppose that we have a surjective bundle map $g: E_1 \to E_3$. Then the triple $(l \oplus g, c \oplus 0_{E_3}, r)$ is an SWC triple if and only if the triple $(l|_{\ker g}, c|_{\ker g}, r|_{\ker g})$ is an SWC triple. In the case that such two triples are SWC triples, they are stably $c$-homotopic to each other.

### 6.4. Deformation of the Seiberg-Witten map.

Throughout this section, we will denote by

$$G = H^1(X, Y_2; \mathbb{Z}) \cong H^1(X_0, Y_2; \mathbb{R}) \times H^1(X_1, Y_2; \mathbb{Z})$$

and fix such an identification. Furthermore, we introduce the notation

$$\Omega^1(X_1, Y_1, \alpha^1) := \left\{ \hat{a} \in \Omega^1(X_1) \mid \text{d}^* t_{Y_1} (\hat{a}) = 0, \int_{Y_1^j} (\ast \hat{a}) = 0, \int_{\alpha^1_k} \hat{a} = 0, \forall j, k \right\}.$$
and define $\Omega^1(X_0, Y_0, \alpha^0 \cup \beta)$ and $\Omega^1(X, Y_0 \cup Y_1, \alpha^0 \cup \alpha^1 \cup \beta)$ similarly. Let us also denote all the relevant Hilbert spaces

- $V_{X_0} := L^2(X_0, Y_0, \alpha^0 \cup \beta) \oplus \Gamma(S^+_X)$;
- $V_{X_1} := L^2(X_1, Y_1, \alpha^1) \oplus \Gamma(S^+_X)$;
- $V_X := L^2(X, Y_0 \cup Y_1, \alpha^0 \cup \alpha^1 \cup \beta) \oplus \Gamma(S^+_X)$;
- $V := \text{Coul}(Y_0) \times \text{Coul}(Y_1)$;
- $U_{X_i} := L^2(X_i, i\Omega^0(X_i) \oplus \Omega^1(X_i) \oplus \Gamma(S^+_X))$ for $i = 0, 1$;
- $U_X := L^2(X, i\Omega^0(X) \oplus \Omega^1(X) \oplus \Gamma(S^+_X))$;
- $H^1(X, Y_2; \mathbb{R})$, where $X_\bullet$ stands for $X_0, X_1$ or $X$.

Here $\Omega^0_0(X)$ denotes the space of functions on $X$ which integrate to zero.

Note that $G$ acts on all these spaces as following:

- On differential forms, the action is trivial.
- On spinors, we use the identification

\[ G \cong \mathfrak{g}^{b,\hat{o}}_{X,Y_2}, \tag{48} \]

where $\mathfrak{g}^{b,\hat{o}}_{X,Y_2}$ denotes the group of harmonic gauge transformations $u$ on $X$ such that $u^{-1}du \in i\Omega^1_{CC}(X)$ and $u|_{Y_2} = e^f$ with $f(\hat{o}) = 0$. The action is by gauge transformation. Note that we will use the restriction of $\mathfrak{g}^{b,\hat{o}}_{X,Y_2}$ on $X_0$ and $X_1$ instead of the harmonic gauge transformation satisfying boundary condition on $X_0$ or $X_1$.

- On the homology $H^1(X_\bullet, Y_2; \mathbb{R})$, the action is given by negative translation.

We consider Hilbert bundles

\[ \tilde{V}_X = (V_X \times H^1(X, Y_2; \mathbb{R}))/G, \]
\[ \tilde{U}_X = (U_X \times H^1(X, Y_2; \mathbb{R}))/G \]

over $\text{Pic}^0(X, Y_2)$ and a pair of maps $l_X, c_X: V_X \times H^1(X, Y_2; \mathbb{R}) \to L^2(X, i\Omega^0(X) \oplus \Omega^1(X) \oplus \Gamma(S^+_X)) \times H^1(X, Y_2; \mathbb{R})$ given by

\[ l_X(\hat{a}, \phi, h) := (d^+\hat{a}, \mathcal{P}_{A_0+i\tau(h)}^+ \phi, h), \quad c_X := (F^+_{\hat{A}_0} - \rho^{-1}(\phi^\ast)_0, \rho(\hat{a})\phi, h), \]

where $\tau(h)$ is the unique harmonic 1-form $u$ on $X$ representing $h$ such that $t_{Y_2}(\tau(h))$ is exact and $\tau(h) \in i\Omega^1_{CC}(X)$. It is straightforward to see that $l_X$ and $c_X$ are equivariant under the $G$-action. Thus, we can take the quotient and obtain bundle maps

\[ (d^* \oplus \tilde{l}_X), (0 \oplus \tilde{c}_X): \tilde{V}_X \to \tilde{U}_X. \]

Observe that the double Coulomb condition on $V_X$ is simplified to just $d^*(\hat{a}) = 0$. It then follows that $(\tilde{l}_X|_{\ker d^*}, \tilde{c}_X|_{\ker d^*}, (\tilde{r}_0, \tilde{r}_1)|_{\ker d^*})$ is an SWC-triple and $BF(X)|_{\text{Pic}^0(X, Y_2)}$ is precisely obtained from the SWC-construction.
of this triple, where \( \tilde{r}_i : \tilde{V}_X \to \text{Coul}(Y_i) \) denotes the twisted restriction map as in Section 5.

The goal of this section is to deform \( BF(X)|_{\text{Pic}^0(X,Y_2)} \) to the map \( \tilde{\epsilon}(b^L(X_0), b^R(X_1)) \) represented as in Proposition 6.10. There will be several steps.

Step 1. We move the gauge fixing condition \( d^* = 0 \) to stably c-homotopic maps. Since

\[
d^* : i\Omega^1(X,Y_0 \cup Y_1, \alpha^0 \cup \alpha^1 \cup \beta) \to i\Omega^0_0(X)
\]

is surjective, we directly apply Lemma 6.13 and obtain the following:

**Lemma 6.14.** The relative Bauer–Furuta invariant \( BF(X)|_{\text{Pic}^0(X,Y_2)} \) is obtained by the SWC construction on the triple \((d^* \oplus \tilde{l}_X, 0 \oplus \tilde{c}_X, (\tilde{r}_0, \tilde{r}_1))\).

Step 2. We begin to glue configurations on \( X_0 \) and \( X_1 \) to obtain configurations on \( X \). Let us consider a Sobolev space of configurations on the boundary

\[
V_{Y_2}^{k-m} := L^2_{k-m}(i\Omega^1(Y_2) \oplus i\Omega^0(Y_2) \oplus \Gamma(SY_2)).
\]

for \( 0 \leq m \leq k \).

For any 1-form \( \hat{b} \) on \( X \), we can combine the Levi–Civita connection on \( \Lambda^* T^*(X_i) \) and the spin\(^c \) connection \( \hat{A}_0|_{X_i} + \hat{b} \) to obtain a connection on \( \Lambda^* T^*(X_i) \oplus S_{X_i} \). We use \( \nabla^b \) to denote the corresponding covariant derivative. Consider a map

\[
D^{(m)} : V_{X_0} \times V_{X_1} \times H^1(X,Y_2; \mathbb{R}) \to V_{Y_2}^{k-m} \times H^1(X,Y_2; \mathbb{R})
\]

\[
(x_0, x_1, h) \mapsto ((\nabla^\tau(h)|_{X_0})^m x_0)|_{Y_2} - ((\nabla^\tau(h)|_{X_1})^m x_1)|_{Y_2}, h),
\]

where \( \tilde{n} \) is the outward normal direction of \( Y_2 \subset X_0 \). Here, we apply standard bundle isomorphisms \( T^*(X_i)|_{Y_2} \cong T^*Y_2 \oplus \mathbb{R} \) and \( S_{X_i}|_{Y_2} \cong SY_2 \).

It is clear that the map \( D^{(m)} \) is equivariant under the action of \( G \). As a result, we can take the quotient and obtain a map

\[
\tilde{D}^{(m)} : \tilde{V}_{X_0,X_1} \to \tilde{V}_{Y_2}^{k-m},
\]

where we set

\[
\tilde{V}_{X_0,X_1} := (V_{X_0} \times V_{X_1} \times H^1(X,Y_2; \mathbb{R}))/G
\]

\[
\tilde{V}_{Y_2}^{k-m} := (V_{Y_2}^{k-m} \times H^1(X,Y_2; \mathbb{R}))/G.
\]

We state the gluing result for these spaces, which is a variation of [14, Lemma 3]. The proof is only local near \( Y_2 \) and can be adapted without change.

**Lemma 6.15.** The bundle map

\[
(\tilde{D}^{(k)}, \cdots, \tilde{D}^{(0)}) : \tilde{V}_{X_0,X_1} \to \bigoplus_{m=0}^{k} \tilde{V}_{Y_2}^{k-m}
\]

is fiberwise surjective and the kernel can be identified with the bundle \( \tilde{V}_X \).
Analogous to the maps $d^* \oplus l_X$ and $0 \oplus c_X$, we define the map

$$l_{X_0,X_1} : V_{X_0} \times V_{X_1} \times H^1(X,Y_2;\mathbb{R}) \to U_{X_0} \times U_{X_1} \times H^1(X,Y_2;\mathbb{R})$$

$$((\hat{a}_0, \phi_0), (\hat{a}_1, \phi_1), h)$$

$$\mapsto ( (d^* \hat{a}_0, d^* \hat{a}_1, D^+_{(\hat{a}_0 + i\tau(h))|X_0} \phi_0), (d^* \hat{a}_1, d^* \hat{a}_1, D^+_{(\hat{a}_1 + i\tau(h))|X_1} \phi_1), h)$$

and the map

$$c_{X_0,X_1} : V_{X_0} \times V_{X_1} \times H^1(X,Y_2;\mathbb{R}) \to U_{X_0} \times U_{X_1} \times H^1(X,Y_2;\mathbb{R})$$

$$((\hat{a}_0, \phi_0), (\hat{a}_1, \phi_1), h)$$

$$\mapsto ( (0, F^+_{A_0^0}|_{X_0} - \rho^{-1}(\phi_0 \phi_0^*)_{0}, \rho(\hat{a}_0)\phi_0), (0, F^+_{A_0^0}|_{X_1} - \rho^{-1}(\phi_1 \phi_1^*)_{1}, \rho(\hat{a}_1)\phi_1), h).$$

Then, by taking quotient, we get bundle maps

$$\tilde{l}_{X_0,X_1}, \tilde{c}_{X_0,X_1} : \tilde{V}_{X_0,X_1} \to \tilde{U}_{X_0,X_1},$$

where $\tilde{U}_{X_0,X_1} := (U_{X_0} \times U_{X_1} \times H^1(X,Y_2;\mathbb{R}))/G$. By gluing of Sobolev spaces, the bundle $\tilde{U}_X$ can be identified as a subbundle of $\tilde{U}_{X_0,X_1}$. Let $pj$ be the orthogonal projection to this subbundle. The following result is then a consequence of Lemma 6.15 and Lemma 6.13.

**Lemma 6.16.** The triple

$$(\text{pj} \circ \tilde{l}_{X_0,X_1}, \tilde{D}^{(k)}, \cdots, \tilde{D}^{(0)}), (\text{pj} \circ \tilde{c}_{X_0,X_1}, 0, \cdots, 0), (\tilde{r}_0, \tilde{r}_1))$$

(49)

is an SWC-triple and is stably c-homotopic to $(d^* \oplus \tilde{l}_X, 0 \oplus \tilde{c}_X, (\tilde{r}_0, \tilde{r}_1))$.

**Step 3.** Next, we will glue the Sobolev spaces of the target. Let us consider a map

$$E^{(m)} : U_{X_0} \times U_{X_1} \times H^1(X,Y_2;\mathbb{R}) \to V_{Y_2}^{k-1-m} \times H^1(X,Y_2;\mathbb{R})$$

$$E^{(m)}(y_0, y_1, h) = (((\nabla^{\tau(h)}|_{X_0})|_{Y_2} y_0) - ((\nabla^{\tau(h)}|_{X_1})|_{Y_2} y_1), h),$$

where we also apply standard bundle isomorphisms

$$\Lambda^2_+(X_i)|_{Y_2} \cong T^* Y_2, \quad S_{X_i}|_{Y_2} \cong S Y_2.$$ 

By taking quotient with respect to the action of $G$, we obtain bundle maps

$$\tilde{E}^{(m)} : \tilde{U}_{X_0,X_1} \to \tilde{V}_{Y_2}^{k-1-m}.$$ 

**Proposition 6.17.** The triple

$$(\text{pj} \circ \tilde{l}_{X_0,X_1}, \tilde{E}^{(k-1)} \circ \tilde{l}_{X_0,X_1}, \cdots, \tilde{E}^{(0)} \circ \tilde{l}_{X_0,X_1}, \tilde{D}^{(0)}),$$

$$(\text{pj} \circ \tilde{c}_{X_0,X_1}, \tilde{E}^{(k-1)} \circ \tilde{c}_{X_0,X_1}, \cdots, \tilde{E}^{(0)} \circ \tilde{c}_{X_0,X_1}, 0, (\tilde{r}_0, \tilde{r}_1))$$

(50)

is an SWC-triple and is c-homotopic to the triple (49).
Proof. We simply consider a linear c-homotopy between them as follows: For $1 \leq m \leq k$ and $0 \leq t \leq 1$, define a map

$$\tilde{D}_t^{(m)} = (1 - t) \cdot \tilde{D}^{(m)} + t \cdot \tilde{E}^{(m-1)} \circ \tilde{l}_{X_0,X_1}$$

and the following maps from $\tilde{V}_{X_0,X_1}$ to $\tilde{U}_X \oplus \left( \bigoplus_{m=0}^{k} V_{Y_2}^{k-m} \right)$

$$l_t := (\pi \circ \tilde{l}_{X_0,X_1}, \tilde{D}_t^{(k)}, \ldots, \tilde{D}_t^{(1)}, \tilde{D}_t^{(0)}),$$

$$c_t := (\pi \circ \tilde{c}_{X_0,X_1}, t \cdot \tilde{E}^{(k-1)} \circ \tilde{c}_{X_0,X_1}, \ldots, t \cdot \tilde{E}^{(0)} \circ \tilde{c}_{X_0,X_1}, 0).$$

This will give a c-homotopy as a result of the following lemma. \qed

Lemma 6.18. For any $0 \leq t \leq 1$, the map

$$(l_t, \tilde{p}_t^{0,\infty} \circ (\tilde{r}_0, \tilde{r}_1)) : \tilde{V}_{X_0,X_1} \to \tilde{U}_X \oplus \left( \bigoplus_{m=0}^{k} V_{Y_2}^{k-m} \right) \oplus V_{-\infty}^{0}(-Y_0 \cup -Y_1)$$

is fiberwise Fredholm. Moreover, the zero set $(l_t + c_t)^{-1}(0) \subset \tilde{V}_{X_0,X_1}$ is independent of $t$ and can be described as

$$\{(\hat{a}, \phi, h) \in \tilde{V}_X | d^* \hat{a} = 0 \text{ and } (\tilde{A}_0 + i \tau(h) + \hat{a}, \phi) \text{ is a Seiberg-Witten solution} \}.$$

Proof. The key observation is that $E^{(m)} \circ l_{X_1,X_2} - \tilde{D}^{(m+1)}$ contains at most $m$-th derivative in the normal direction. Then, one can prove inductively that

$$(\tilde{D}_t^{(k)}, \ldots, \tilde{D}_t^{(1)}, \tilde{D}_t^{(0)})(x_0, x_1) = 0 \implies (\tilde{D}_t^{(k)}, \ldots, \tilde{D}_t^{(0)})(x_0, x_1) = 0,$$

so that the kernel of $l_t$ does not depend on $t$. Similarly, one can show that $(\tilde{D}_t^{(k)}, \ldots, \tilde{D}_t^{(1)}, \tilde{D}_t^{(0)})$ is fiberwise surjective for all $t$. Since $t = 0$ is the map from Lemma 6.16, the map $(l_t, \tilde{p}_t^{0,\infty} \circ (\tilde{r}_0, \tilde{r}_1))$ is fiberwise Fredholm for all $t$.

The second part was essentially proved in [14, Section 4.11] using similar inductive argument. \qed

Step 4. We now make the following identification:

Lemma 6.19. The bundle map (over Pic^0(X, Y_2))

$$(\pi \circ \tilde{E}^{(k)} \cdot \ldots \cdot \tilde{E}^{(0)}, \xi) : \tilde{U}_{X_0,X_1} \to \tilde{U}_X \oplus \left( \bigoplus_{m=0}^{k-1} V_{Y_2}^{k-1-m} \right) \oplus \mathbb{R}$$

is an isomorphism. The map $\xi$ is given by $\xi(x_1, x_2, h) = \int_{X_0} f_0 + \int_{X_1} f_1$, where $f_i$ is the 0-form component of $x_i$.

Proof. This also follows from gluing result of Sobolev spaces [14, Lemma 3]. The only difference here is that the 0-form component $\tilde{U}_X$ consists of functions which integrate to 0. From the standard decomposition $\Omega^0(X) = \Omega_0^0(X) \oplus \mathbb{R}$, we can see that the projection onto $\mathbb{R}$ is given by the map $\xi$. \qed
On the other hand, we decompose $\tilde{D}^{(0)}$ from the following decomposition of the Hilbert spaces:

$$V_{Y_2}^k = Coul(Y_2) \oplus H \oplus \mathbb{R} \text{ with } H = L_k^2(i\Omega^0(Y_2) \oplus \Omega^0_0(Y_2)).$$

(51)

We denote the corresponding components of $D^{(0)}$ (resp. $\tilde{D}^{(0)}$) by $D_{Y_2}$, $D_H$ and $\tilde{D}_\mathbb{R}$ (resp. $\tilde{D}_{Y_2}$, $\tilde{D}_H$ and $\tilde{D}_\mathbb{R}$).

We make an observation that the SWC-triple (50) in Proposition 6.17 arises from a composition

$$\tilde{V}_{X_0, X_1} \to \tilde{U}_{X_0, X_1} \oplus Coul(Y_2) \oplus H$$

$$\to \tilde{U}_X \oplus \left( \bigoplus_{m=0}^{k-1} \tilde{V}_{Y_2}^{k-1-m} \right) \oplus \mathbb{R} \oplus Coul(Y_2) \oplus H,$$

where the first arrow is $(i\tilde{I}_{X_0, X_1} + \tilde{c}_{X_0, X_1}, \tilde{D}_{Y_2}, \tilde{D}_H)$ and the second arrow is the isomorphism $(pj, \tilde{E}^{(k-1)} \cdots \tilde{E}^{(0)}, \tilde{\xi}, id, id)$. The only thing we need to check is that $\tilde{D}_\mathbb{R} = \xi \circ \tilde{I}_{X_0, X_1}$ on the 1-form component, which follows from the Green-Stokes formula

$$\int_{Y_2} t(\ast \tilde{a}_0) - \int_{Y_2} t(\ast \tilde{a}_1) = \int_{X_0} d^* \tilde{a}_0 + \int_{X_1} d^* \tilde{a}_1.$$

Thus, we can conclude

**Lemma 6.20.** The SWC-triple (50) can be identified with the triple

$$( (i\tilde{I}_{X_0, X_1}, \tilde{D}_{Y_2}, \tilde{D}_H), (\tilde{c}_{X_0, X_1}, 0, 0), (\tilde{r}_0, \tilde{r}_1)).$$

(52)

**Step 5.** In this step, we focus on deforming the $\tilde{D}_H$-component which corresponds to boundary conditions for gauge fixing. We sometimes omit spinors from expressions in this step.

For $\tilde{a}_j \in i\Omega^1(X_j)$, we have a Hodge decomposition $t_{Y_2}(\tilde{a}_j) = a_j + bj$ on $Y_2$ with $a_j \in \ker d^*$ and $b_j \in \text{im } d$. We also denote by $e_j := c_j - \frac{L_c \text{dvol}(Y_2)}{\text{vol}(Y_2)} \in i\Omega_0^0(Y_2)$, where $\tilde{a}_j|_{Y_2} = t_{Y_2}(\tilde{a}_j) + c_j dt$. With this formulation, we see that

$$D_H(\tilde{a}_0, \tilde{a}_1) = (b_0 - b_1, e_0 - e_1).$$

Let us consider an isomorphism

$$\tilde{d} : L_k^2(i\Omega^0_0(Y_2)) \to L_k^2(i\Omega^0(Y_2))$$

defined by $\tilde{d}f := \lambda^{-1} df$ for any $f \in i\Omega^0_0(Y_2)$ with $d^* df = \lambda^2 f$ with $\lambda > 0$ using the spectral decomposition of $d^* d$. We let

$$d^* : L_k^2(i\Omega^0(Y_2)) \to L_k^2(i\Omega^0_0(Y_2))$$

be its formal adjoint. Note that $d^*$ can also be obtained directly by $d^*\alpha := \lambda f$ for $\alpha = df$ satisfying $dd^* \alpha = \lambda^2 \alpha$ with $\lambda > 0$ and $\int_{Y_2} f = 0$. We then define a family of maps

$$D_{H,t} : V_{X_0} \times V_{X_1} \to H$$

given by

$$D_{H,t}(\tilde{a}_0, \tilde{a}_1) := (b_0 - b_1, t \cdot d^* (b_0 + b_1) + (1-t) \cdot (e_0 - e_1)).$$
The main point here is to establish that the gauge fixing conditions
$D_{H,t} = 0$ are isomorphic and vary continuously. In particular, we will find
a harmonic gauge transformation in the identity component to relate them.
For a pair of coclosed 1-forms $(a_0, a_1) \in \Omega^1(X_0, Y_0, \alpha^0 \cup \beta) \times \Omega^1(X_1, Y_1, \alpha^1)$
with $b_0 = b_1$, finding such a transformation amounts to solving for a pair of
functions $(f_0, f_1) \in \Omega^0(X_0) \times \Omega^0(X_1)$ such that
\[
2t \cdot \bar{\partial} \cdot d(f_0|_{Y_2}) + (1-t)(\partial \bar{\pi} f_0|_{Y_2} - \partial \bar{\pi} f_1|_{Y_2})
\]
\[
= 2t \cdot \bar{\partial} \cdot d(b_0) + (1-t)(e_0 - e_1)
\]
and also satisfies other gauge fixing conditions. We have the following existence
and uniqueness result.

**Lemma 6.21.** Let $W \subset L^2_{k+3/2}(X_0; \mathbb{R}) \times L^2_{k+3/2}(X_1; \mathbb{R})$ be the subspace
containing all functions $(f_0, f_1)$ satisfying the following conditions:

1. $\Delta f_i = 0$;
2. $f_i(\hat{o}) = 0$;
3. $f_0|_{Y_2} = f_1|_{Y_2}$;
4. $f_i$ is a constant on each component of $Y_i$, $i = 0, 1$;
5. $\partial \bar{\pi} f_i$ integrates to zero on each component of $Y_i$, $i = 0, 1$.

Then the map $\rho_t: W \rightarrow L^2_k(\Omega^0_0(Y_2))$ defined by
\[
\rho_t(f_0, f_1) = 2t \cdot \bar{\partial} \cdot d(f_0|_{Y_2}) + (1-t)(\partial \bar{\pi} f_0|_{Y_2} - \partial \bar{\pi} f_1|_{Y_2})
\]
is an isomorphism.

**Proof.** We first show that $\rho_t$ is an isomorphism when $t = 1$. For $\xi \in L^2_k(i\Omega^0_0(Y_2))$, we want to find $f_2$ such that $f_2|_{Y_2} = \frac{\xi}{2} - \frac{\xi(\hat{o})}{2}$ and satisfies
the other conditions. The existence and uniqueness of such functions follow from the same argument as in the double Coulomb condition (cf. [1], Proposition 2.2).

Since each $\rho_t$ corresponds to Laplace equation with mixed Dirichlet and
Neumann boundary condition, it is Fredholm with index zero (from $t = 1$).
Thus, for $t < 1$, we are left to show that $\rho_t$ is injective. Suppose $\rho_t(f_0, f_1) = 0$. Then by Green’s formula, we have
\[
(1-t) \left( \int_{X_0} \langle df_0, df_0 \rangle + \int_{X_1} \langle df_1, df_1 \rangle \right) = (1-t) \int_{Y_2} f_0(\partial \bar{\pi} f_0 - \partial \bar{\pi} f_1)
\]
\[
= -2t \int_{Y_2} f_0 \cdot (\bar{\partial} \cdot d(f_0|_{Y_2}))
\]
The first expression is nonnegative but the expression
\[
\int_{Y_2} f_0(\bar{\partial} \cdot d(f_0|_{Y_2})) = \int_{Y_2} (f_0)^2 - \frac{1}{\text{vol}(Y_2)}(\int_{Y_2} f_0)^2
\]
is also nonnegative by Cauchy–Schwartz inequality. Hence both $f_0$ and $f_1$
must be constant and are in fact identically zero because $f_i(\hat{o}) = 0$. \qed
As $D_{H,t}$ is equivariant, we can form bundle maps $\tilde{D}_{H,t}$ and obtain a $c$-homotopy.

**Proposition 6.22.** For any $t \in [0,1]$, the triple 

$$(\tilde{l}_{X_0,X,t}, \tilde{D}_{Y_2}, \tilde{D}_{H,t}), (\tilde{c}_{X_0,X_1}, \tilde{0}, (\tilde{r}_0, \tilde{r}_1))$$

is an SWC-triple. Consequently, this provides a $c$-homotopy between the triples at $t = 0$ and $t = 1$.

**Proof.** The statement for $t = 0$ follows from Lemma 6.20. For each element in the kernel of $(\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_{H,0})$ there is a unique gauge transformation to an element in the kernel of $(\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_{H,0})$ as a result of Lemma 6.21. This provides a linear bijection, so the kernel of $(\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_{H,t})$ is also finite-dimensional.

The map $(\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_{H,t})$ differs from the map $(\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_{H,0})$ only at the $\Omega^0_0(Y_2)$-component. By Lemma 6.21, the map $\rho_t$ is surjective, so the map $(\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_{H,t})$ is surjective on the $\Omega^0_0(Y_2)$-component. This implies that the cokernels at each $t$ are in fact the same. Therefore, $(\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_{H,t})$ are Fredholm.

Applying Lemma 6.21 again, one can see that there is a unique gauge transformation from a solution of $((\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_{H}), (\tilde{c}_{X_0,X_1}, \tilde{0}, (\tilde{r}_0, \tilde{r}_1)))$ to a solution of $((\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_{H,t}), (\tilde{c}_{X_0,X_1}, \tilde{0}, (\tilde{r}_0, \tilde{r}_1)))$ which depends continuously. This provides a homeomorphism between them. Then the boundedness result follows from the case $t = 0$ and compactness of $[0,1]$.

---

**Step 6.** Here, we will basically change the action of $G$ by identifying it with a group of harmonic gauge transformations with different boundary conditions. Recall from our setup that $\tau(h)$ for $h \in H^1(X; Y_2; \mathbb{R})$ is the unique harmonic 1-form on $X$ representing $h$ such that $\tau_2(\tau(h))$ is exact and $\tau(h) \in i\Omega^\ast_{CC}(X)$. Note that for $t \in [0,1]$, 

$$D_{H,t}(\tau(h)|_{X_0}, \tau(h)|_{X_1}) = (0, 2t\tilde{d}^* (\tau_2(\tau(h))))$$

We put

$$(\xi_{0,t}(h), \xi_{1,t}(h)) := \rho_t^{-1}(2t\tilde{d}^* (\tau_2(\tau(h))))$$

We then apply gauge transformation to $\tau(h)$ and define

$$\tau_t = (\tau_{X_0,t}, \tau_{X_1,t}) : H^1(X; Y_2; \mathbb{R}) \to \Omega^1_{h}(X_0) \times \Omega^1_{b}(X_1)$$

$$h \mapsto (\tau(h)|_{X_0} - d\xi_{0,t}(h), \tau(h)|_{X_1} - d\xi_{1,t}(h)).$$

From our construction, we have $D_{H,t}(\tau_t(h)) = 0$ and $d\xi_{i,0} = 0$.

We will consider harmonic gauge transformations corresponding to boundary condition $D_{H,0} = 0$. For $h \in G$, we define

$$u_t(h) := (u_{X_0,t}(h), u_{X_1,t}(h))$$

such that $u_{X_i,t}(h)$ is the unique gauge transformation on $X_i$ satisfying

$$u_{X_i,t}(h)(\partial) = 1, \quad u_{X_i,t}^{-1} du_{X_i,t} = \tau_{X_i,t}(h).$$
Notice that for $u_{X_i,0}$ is the restriction of $u \in \mathcal{G}^{h,\alpha}_{X,Y}$ and $u_{X_i,t}(h) = e^{-\xi_{i,t}(h)}u_{X_i,0}(h)$.

Consider a new action $\varphi_t$ of $G$ on the spaces $V_{X_i}, U_{X_i}, H^1(X,Y;\mathbb{R})$, $\text{Coul}(Y_t)$ and $H$ such that the action on spinors is given by the gauge transformations $(u_{X_o,t}(h), u_{X_i,t}(h))$ instead of restriction of $u \in \mathcal{G}^{h,\alpha}_{X,Y}$. We also consider a map

$$l^t_{X_0,X_1}, c^t_{X_0,X_1}: V_{X_0} \times V_{X_1} \times H^1(X,Y;\mathbb{R}) \to U_{X_0} \times U_{X_1} \times H^1(X,Y;\mathbb{R})$$

by replacing the term $\tau(h)|_{X_i}$ in the definition (cf. (6.4)) with $\tau_{X_i,t}(h)$.

It is not hard to check that the maps $l^t_{X_0,X_1}, c^t_{X_0,X_1}$, $D_{Y_2} \times \text{id}_{H^1(X,Y;\mathbb{R})}$ and $D_{H,t} \times \text{id}_{H^1(X,Y;\mathbb{R})}$ are all equivariant under the action $\varphi_t$. By taking quotients, we obtain bundles

$$\tilde{V}^t_{X_0,X_1} := (V_{X_0} \times V_{X_1} \times H^1(X,Y;\mathbb{R}))/\langle G, \varphi_t \rangle;$$

$$\tilde{U}^t_{X_0,X_1} := (U_{X_0} \times U_{X_1} \times H^1(X,Y;\mathbb{R}))/\langle G, \varphi_t \rangle$$

and bundle maps $\tilde{l}^t_{X_0,X_1}$, $\tilde{c}^t_{X_0,X_1}$, $\tilde{D}_{Y_2,t}$, $\tilde{D}_{H,t}$. We can consider an obvious bundle isomorphism from $\tilde{V}_{X_0,X_1}$ (resp. $\tilde{U}_{X_0,X_1}$) to $\tilde{V}^t_{X_0,X_1}$ (resp. $\tilde{U}^t_{X_0,X_1}$) by sending $(a_i, \phi_i, h)$ to $(a_i, e^{i\xi_{i,t}(h)}\phi_i, h)$. All of the above maps fit in a commutative diagram.

We can conclude:

**Lemma 6.23.** The triple $((\tilde{l}^t_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_H),(\tilde{c}^t_{X_0,X_1}, 0, 0), (\tilde{r}_0, \tilde{r}_1))$ is an SWC triple and is c-homotopic to the triple

$$(\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_H), (c^t_{X_0,X_1}, 0, 0), (r_0, r_1)).$$

Let us take a closer look at the SWC triple

$$(\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_H), (c^t_{X_0,X_1}, 0, 0), (r_0, r_1)).$$

Observe that the boundary condition $b_0 - b_1 = 0$ and $\tilde{d}^t_{b_0 + b_1} = 0$ implies $b_0 = b_1 = 0$. This allows us to recover the double Coulomb condition on $X_i$.

**Lemma 6.24.** The operator

$$(d^*_{X_0}, d^*_{X_1}, D_{H,1}): V_{X_1} \times V_{X_2} \to L^2_{k-1/2}(i\Omega^0(X_0) \oplus i\Omega^0(X_1)) \oplus H$$

is surjective and its kernel can be written as

$L^2_{k+1/2}(i\Omega^1_{CC}(X_0, \alpha^0 \cup \beta) \oplus \Gamma(S^+_0)) \times L^2_{k+1/2}(i\Omega^1_{CC}(X_1, \alpha^1) \oplus \Gamma(S^+_1))$.

**Proof.** We consider a pair of exact forms $(df_0, df_1)$. Then, surjectivity reduces to finding a solution of Poisson equation with Dirichlet boundary condition on $X_0$ and $X_1$. 

□
Note that we can identify
\[ L^2_{k+1/2}(i\Omega^1_{CC}(X_0, \alpha^0 \cup \beta) \oplus \Gamma(S^+_0)) \times H^1(X_0, Y_2; \mathbb{R}) \cong \text{Coul}^{CC}(X_0, \beta) \] (53)
by sending \((\hat{a}_0, \phi)\) to \((\hat{a}_0 + \hat{a}_h, \phi)\), where \(\hat{a}_h\) is the element in \(\mathcal{H}^1_{DC}(X_0)\) corresponding to \(h\) (cf. (23) from Section 5). Under this identification, the natural projection to \(H^1(X_0, Y_2; \mathbb{R})\) becomes the map \(\tilde{p}_{\alpha, X_0}\) (cf. (24)). Similarly, we have an isomorphism
\[ L^2_{k+1/2}(i\Omega^1_{CC}(X_1, \alpha^1) \oplus \Gamma(S^+_1)) \times H^1(X_1, Y_2; \mathbb{R}) \cong \text{Coul}^{CC}(X_1). \]
As a result, the action \(\varphi^*\) provides an action on \(\text{Coul}^{CC}(X_0, \beta) \times \text{Coul}^{CC}(X_1)\) via an identification
\[ G = H^1(X_0, Y_2) \times H^1(X_1, Y_2) \cong \mathcal{G}_{X_0, \partial X_0}^{h, \delta} \times \mathcal{G}_{X_1, \partial X_1}^{h, \delta}. \]
This holds because \(Y_0\) and \(Y_1\) are homology spheres.

As in Section 5 we have Seiberg–Witten maps
\[ \overline{SW}_{X_0} = \bar{L}_{X_0} + \bar{Q}_{X_0} : \text{Coul}^{CC}(X_0, \beta)/\mathcal{G}_{X_0, \partial X_0}^{h, \delta} \rightarrow \]
\[ (L^2_{k-1/2}(i\Omega^1_{CC}(X_0) \oplus \Gamma(S^-_0)) \times \mathcal{H}^1_{DC}(X_0))/\mathcal{G}_{X_0, \partial X_0}^{h, \delta}, \]
\[ \overline{SW}_{X_1} = \bar{L}_{X_1} + \bar{Q}_{X_1} : \text{Coul}^{CC}(X_1)/\mathcal{G}_{X_1, \partial X_1}^{h, \delta} \rightarrow \]
\[ (L^2_{k-1/2}(i\Omega^1_{CC}(X_1) \oplus \Gamma(S^-_1)) \times \mathcal{H}^1_{DC}(X_1))/\mathcal{G}_{X_1, \partial X_1}^{h, \delta}. \]
Since an element of \(\mathcal{G}_{X_1, \partial X_1}^{h, \delta}\) takes value 1 on \(Y_2\), there are well-defined restriction maps \(r_2\) from \(\text{Coul}^{CC}(X_0, \beta)/\mathcal{G}_{X_0, \partial X_0}^{h, \delta}\) and \(\text{Coul}^{CC}(X_1)/\mathcal{G}_{X_1, \partial X_1}^{h, \delta}\) to \(\text{Coul}(Y_2)\). We then consider a map
\[ \bar{D}_Y : \text{Coul}^{CC}(X_0, \beta)/\mathcal{G}_{X_0, \partial X_0}^{h, \delta} \times \text{Coul}^{CC}(X_1)/\mathcal{G}_{X_1, \partial X_1}^{h, \delta} \rightarrow \text{Coul}(Y_2) \]
\[ (x_0, x_1) \mapsto r_2(x_0) - r_2(x_1). \]

**Corollary 6.25.** The triple \((\bar{L}_{X_0}, \bar{L}_{X_1}, \bar{D}_Y), (\bar{Q}_{X_0}, \bar{Q}_{X_1}, 0), (\bar{r}_0, \bar{r}_1)\) is an SWC triple stably \(\epsilon\)-homotopic to the triple
\[(\overline{(i^1_{X_0, X_1}, \bar{D}_{Y_2}, \bar{D}_H^1)}, (\epsilon_{X_0, X_1}^1, 0, 0), (\bar{r}_0, \bar{r}_1)). \]

**Proof.** This follows by applying Lemma 6.13 to the triple
\[(\overline{(i^1_{X_0, X_1}, \bar{D}_{Y_2}, \bar{D}_H^1)}, (\epsilon_{X_0, X_1}^1, 0, 0), (\bar{r}_0, \bar{r}_1))\]
with \(g = (d_{X_0}^*, d_{X_1}^*, D_{H, 1})\) as in Lemma 6.24 \(\square\)

**Step 7.** This is the final step. Recall from Section 6.1 that we chose finite dimensional subspaces \(U_n^i\) of \(L^2_{k-1/2}(i\Omega^1_{CC}(X_i) \oplus \Gamma(S^-_i))\) and eigen-spaces \(V_n^i\) of \(\text{Coul}(Y_i)\). In the SWC construction of the triple
\[(\bar{L}_{X_0}, \bar{L}_{X_1}, \bar{D}_Y), (\bar{Q}_{X_0}, \bar{Q}_{X_1}, 0), (\bar{r}_0, \bar{r}_1),\]
the subbundles involved are preimages of the map \((\bar{L}_{X_0}, \bar{L}_{X_1}, \bar{D}_Y, \tilde{p}_{-\infty} \circ \bar{r}_0, \tilde{p}_{-\infty} \circ \bar{r}_1)\) rather than preimages of the product map \((\bar{L}_{X_0}, \tilde{p}_{-\infty} \circ \bar{r}_0, \tilde{p}_{-\infty} \circ \bar{r}_1)\).
Lemma 1] to control \( \tilde{\tau}_2 \times (\tilde{L}_{\mu_n} \circ \tilde{\tau}_1, \tilde{L}_{-\mu_n} \circ \tilde{\tau}_2) \) in the construction of relative Bauer–Furuta invariants. Note that there is a choice of trivialization but we do not emphasize this here.

Using the spectral decomposition, we see that \( r_2(x_0) - r_2(x_1) \in V_{-\mu_n} \) if and only if

\[
\begin{align*}
    p_{\mu_n}^\infty \circ r_2(x_0) &= p_{\mu_n}^\infty \circ r_2(x_1), \\
    p_{-\mu_n}^\infty \circ r_2(x_1) &= p_{-\mu_n}^\infty \circ r_2(x_0).
\end{align*}
\]

We introduce a family of subbundles: for \( t \in [0, 1] \),

\[
W_{n,t}^{n_0, X_0, X_1} := \{(x_0, x_1) \in (\text{Coul}^{CC}(X_0, \beta)/G_{X_0, \beta X_0}^{h, \hat{h}}) \times (\text{Coul}^{CC}(X_1)/G_{X_1, \beta X_1}^{h, \hat{h}}) | \\
    p_{-\mu_n}^\infty \tilde{\tau}_i(x_i) \in V_n, \quad \tilde{L}_{n, X_i}(x_i) \in U_n, \\
    p_{\mu_n}^\infty \circ r_2(x_0) = tp_{\mu_n}^\infty \circ r_2(x_1), \\
    p_{-\mu_n}^\infty \circ r_2(x_1) = tp_{-\mu_n}^\infty \circ r_2(x_0) \}.
\]

We have a boundedness result for this family.

Lemma 6.26. For any \( R > 0 \), there exist \( N, \epsilon_0 \) with the following significance: For any \( n > N, t \in [0, 1) \), \( (x_0, x_1) \in B^+(W_{n,t}^{n_0, X_0, X_1}, R) \) and \( \gamma_i : (-\infty, 0] \to B(V_{-\lambda_n}^{n_0, X_0, X_1}, R) \) where \( i = 0, 1 \) satisfying

\[
\begin{align*}
    &\bullet |p_{-\mu_n}^\infty (r_2(x_0) - r_2(x_1))|_{L_2^\epsilon} \leq \epsilon, \\
    &\bullet |p_{\mu_n}^\infty \circ \tilde{SW}_{X_i}(x_i)|_{L_2^{k-1/2}} \leq \epsilon, \\
    &\bullet \gamma_i \text{ is an approximated trajectory with } \gamma_i(0) = p_{-\mu_n}^\infty \circ \tilde{\tau}_i(x_i),
\end{align*}
\]

one has \( \|x_i\|_{F} \leq R_3 + 1 \) and \( \|\gamma_i(t)\|_{L_2^k} \leq R_3 + 1 \), where \( R_3 \) is the constant in Proposition 6.5.

Proof. The proof is essentially identical to Proposition 6.6 by using [14, Lemma 1] to control \( |p_{\mu_n}^\infty \circ r_2(x_0)|_{L_2^k} \) (resp. \( |p_{-\mu_n}^\infty \circ r_2(x_1)|_{L_2^k} \)) in terms of \( \|\tilde{L}_{X_0}(x_0)\|_{L_2^{k-1/2}} \) (resp. \( \|\tilde{L}_{X_1}(x_1)\|_{L_2^{k-1/2}} \)).

As a result, we obtain a family of maps as \( t \in [0, 1] \). When \( t = 1 \), this is the same as the SWC construction for the original triple. When \( t = 0 \), we have

\[
W_{n,t}^{n_0, X_0, X_1} = W_{n, 0}^{0, X_0, X_1} \times W_{n}^1
\]

and we then recover the homotopy class in Proposition 6.10. The proof of the gluing theorem is finished.

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