The second generalized Hamming weight of some evaluation codes arising from a projective torus

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Abstract

In this paper we find the second generalized Hamming weight of some evaluation codes arising from a projective torus, and it allows to compute the second generalized Hamming weight of the codes parameterized by the edges of any complete bipartite graph. Also, at the beginning, we obtain some results about the generalized Hamming weights of some evaluation codes arising from a complete intersection when the minimum distance is known and they are non–degenerate codes. Finally we give an example where we use these results to determine the complete weight hierarchy of some codes.

I. INTRODUCTION

The main results of this paper solve the following problem: if we have the system of two polynomial equations

\begin{align}
F_1(X_1, \ldots, X_s) &= 0, \\
F_2(X_1, \ldots, X_s) &= 0,
\end{align}

where \(F_1\) and \(F_2\) are two linearly independent homogeneous polynomials in \(s\) variables over a finite field \(F_q\) with \(q\) elements, and with degree \(d\), then, what is the maximum possible number of solutions of system (1) in a projective torus \(T_{s-1}\) (see Definition (3)?)

In [2] Boguslavsky answered this question when we change the torus by the projective space \(\mathbb{P}^{s-1}\):

**Theorem 1:** The maximum possible number of solutions of system (1) in the projective space \(\mathbb{P}^{s-1}\) when \(d < q - 1\) is given by

\[(d - 1)q^{s-2} + p_{s-3} + q^{s-3},\]

where \(p_m = |\mathbb{P}^m| = q^m + q^{m-1} + \cdots + q + 1\). When \(d \geq q + 1\) it is known that this number is \(p_{s-1}\) [28].

In our case we obtain (see Theorems 13 and 16) that the maximum possible number of solutions of system (1) in \(T_{s-1}\) when \(d < q - 1\) is given by

\[(q - 1)^{s-2}(d - 1) + (q - 1)^{s-3},\]

and when \(d \geq q - 1\) we can find polynomials that vanish on the complete projective torus, that is, this number is \(|T_{s-1}| = (q - 1)^{s-1}\). Moreover if we choose \(F_1\) and \(F_2\) such that they do not vanish on the torus \(T_{s-1}\), the maximum number of solutions is given by (see Remark 5)

\[(q - 1)^{s-1} - \left(\frac{(q - 1)^{s-(k+2)}(q - 1 - l)}{(q - 1)^{s-(k+3)} + (q - 2)}\right),\]

where \(s \geq 2\), \(d \geq 1\), and \(k, l\) are the unique integers such that \(d = k(q - 2) + l\), \(k \geq 0\), \(1 \leq l \leq q - 2\). These results allow to compute the second generalized Hamming weight of some evaluation codes arising from the projective torus (see Theorem 17). This weight should not be confused with the second Hamming weight, also called next–to–minimal weight, which was computed by Carvalho in [3] Theorem 2.4] in a more general case (affine cartesian codes) when \(2 \leq d < q - 1\), \(s > 3\).

Actually the generalized Hamming weights of a linear code were introduced in [19], [20], and rediscovered by Wei in [30]. The study of these weights is related to trellis coding, \(t\)–resilient functions, and it was motivated from some applications from cryptography. The weight hierarchy of a code has been analyzed in several cases (see [29] and [15]), for example:

1) Golay code.
2) Product codes.

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3) Codes from classical varieties: Reed–Muller codes, codes from quadrics, Hermitian varieties, Grassmannians, Del Pezzo surfaces.

4) Algebraic geometric codes.

5) Cyclic and trace codes: BCH, Melas.

6) Codes parameterized by the edges of simple graphs.

Furthermore some evaluation codes are the main object of study in this work. They have been studied since many years ago. At the very beginning they were called Reed–Muller–Type codes. These codes are obtained by evaluating the linear space of homogeneous $d$-forms on a subset of points $X$ of a projective space $\mathbb{P}^{s-1}$ over a finite field with $q$ elements, $K = \mathbb{F}_q$. We denote this linear code by $C_X(d)$ (see Definition 2). When $X$ is the whole projective space we obtain the Projective Reed–Muller codes (see [28]). The main parameters of $C_X(d)$ were computed in [6] when $X$ is the Segre variety (its rational points).

Some results were described in [24] when $X$ is the Veronese variety. The main characteristics of $C_X(d)$ were studied in [4] and [17] when $X$ is a complete intersection. In spite of the minimum distance in this case remains unknown, in [1] and [5] there are lower bounds for this parameter and in [27] there is a generalization of these results. Although we do not know the value of the minimum distance in the case of complete intersections in the general case, some particular situations have been determined (cartesian codes, see [21], codes parameterized by a projective torus, see [26], codes parameterized by a degenerate projective torus, see [12]). When the minimum distance is known, we can compute some generalized Hamming weights, as Theorem 9 shows.

On the other hand the notion of codes parameterized by a set of monomials, which are evaluation codes where $X$ is a subgroup of the projective torus $\mathbb{T}_{s-1}$, was introduced in [25]. Moreover if $G$ is a simple graph (we only consider finite undirected graphs without loops or multiple edges) and $X$ is the set parameterized by its edges, the code $C_X(d)$ has been studied in several cases (see [9], [10], [11], [13], [14], [15], [22], [23], [25]).

The contents of this paper are as follows. In section II we introduce the definitions needed to understand the main results of this paper. In section III we obtain Theorem 9 which gives the value of some generalized Hamming weights in the case of evaluation codes arising from complete intersections if the minimum distance is known. In section IV we obtain Theorem 11 which generalizes [25] Proposition 5.2 and shows that the generalized Hamming weights have the same behavior that the minimum distance in these evaluation codes. In section V we give the second generalized Hamming weight of the codes parameterized by the projective torus (see Theorem 17 and it allows to compute the second generalized Hamming weight of the codes parameterized by the edges of a complete bipartite graph. Finally we give Example 18 where, using the results obtained in the previous sections, we find the complete weight hierarchy of the codes $C_T(d)$ in a finite field with 5 elements.

II. Preliminaries

Let $K = \mathbb{F}_q$ be a finite field with $q$ elements and let $\mathbb{P}^{s-1}$ be a projective space over $K$. Let $X = \{P_1, \ldots, P_{|X|}\}$ be a subset of $\mathbb{P}^{s-1}$ where $|X|$ is the cardinality of the set $X$. Let $S = K[X_1, \ldots, X_s] = \oplus_{d \geq 0} S_d$ be a polynomial ring with the natural grading. It is easy to see that for each $i$ there is $f_i \in S_d$ such that $f_i(P_i) \neq 0$. Consider the evaluation map

$$ev_d : S_d \to K^{|X|},$$

$$f \mapsto \left( \frac{f(P_1)}{f_1(P_1)}, \ldots, \frac{f(P_{|X|})}{f_{|X|}(P_{|X|})} \right).$$

This evaluation map is a linear map between the $K$–vector spaces $S_d$ and $K^{|X|}$.

**Definition 2:** The evaluation code of order $d$ associated to the set $X$ is the image of the evaluation map $ev_d$, and it is denoted by $C_X(d)$. Therefore

$$C_X(d) = \left\{ \left( \frac{f(P_1)}{f_1(P_1)}, \ldots, \frac{f(P_{|X|})}{f_{|X|}(P_{|X|})} \right) : f \in S_d \right\}. $$

Notice that $C_X(d)$ is a linear subspace of $K^{|X|}$ and its main characteristics have been related with some algebraic invariants of the quotient ring $S/I_X$, where $I_X$ is the vanishing ideal of $X$ (see for example [13] or [22]). The dimension of $C_X(d)$ is given by the Hilbert function of $S/I_X$, that is, $\dim_K(C_X(d)) = H_X(d)$. The length of $C_X(d)$, $|X|$, is given by the degree, or the multiplicity, of $S/I_X$. Moreover the regularity index is the Castelnuovo–Mumford regularity of $S/I_X$. We recall that, if $a_X$ is the $a$–invariant of $S/I_X$, then $a_X$ is its regularity index minus $1$. Unfortunately we could not find some algebraic invariants that match with the generalized Hamming weights of $C_X(d)$.

Now it is important to introduce an Abelian group under componentwise multiplication that is a subset of the projective space $\mathbb{P}^{s-1}$.

**Definition 3:** The projective torus $\mathbb{T}_{s-1}$ is given by

$$\mathbb{T}_{s-1} = \{ [t_1 : \cdots : t_s] \in \mathbb{P}^{s-1} : t_i \in K^* \},$$

where $K^* = K \setminus \{0\}$.
When $X$ is the toric set parameterized by $s$ monomials (see [13 Equation (1)]), $X$ is a subgroup of $\mathbb{T}_{s-1}$ and we can study the code $C_X(d)$ given in Definition 2 if $f_i = \pi^d_i$ for all $i = 1, \ldots, |X|$. In this situation $C_X(d)$ is called a code parameterized by these monomials. In the case that the monomials are given by the edges of a simple graph $\mathcal{G}$ (see [18] for the basic definitions about graphs) we say that $C_X(d)$ is parameterized by the edges of $\mathcal{G}$ (see [14 Definitions (2) and (4)]).

Because of one of the cases studied here is when $X$ is a complete intersection, we recall this definition.

Definition 4: A set $X \subseteq \mathbb{P}^{s-1}$ is called a (zero–dimensional ideal–theoretic) complete intersection if the vanishing ideal $I_X$ is generated by a regular sequence of $s – 1$ elements.

On the other hand we need the definition of the generalized Hamming weights, introduced in [19], [20], [30], and also known as higher weights, effective lengths or Wei weights.

Definition 5: If $B$ is subset of $K^{|X|}$, the support of this set is

$$\text{supp}(B) = \{i : \text{there exists } (w_1, \ldots, w_{|X|}) \in B \text{ such that } w_i \neq 0\}.$$ 

The $r$th generalized Hamming weight of the code $C_X(d)$ is given by

$$d_r(C_X(d)) = \min \{|\text{supp}(D)| : D \text{ is a subcode of } C_X(d) \text{ and } \dim_K D = r\},$$

for $1 \leq r \leq H_X(d)$. The weight hierarchy of the code $C_X(d)$ is the set of integers $\{d_r(C_X(d)) : 1 \leq r \leq H_X(d)\}$.

It is a well known fact that $d_1(C_X(d))$ is precisely the minimum distance of the evaluation code $C_X(d)$. Moreover we say that $C_X(d)$ is an $r$–MDS code if the Singleton–Type bound (see [29 Corollary 3.1]) is attained, that is

$$d_r(C_X(d)) = |X| - H_X(d) + r.$$

Also we need the notion of non–degenerate codes.

Definition 6: If $C \subseteq K^{|X|}$ is a linear code, we define

$$\pi_i : C \to K,$$

$$\pi_i(v_1, \ldots, v_{|X|}) = v_i,$$

for $i = 1, \ldots, |X|$. We say that $C$ is degenerate if for some $i$ the image of $\pi_i$ is zero. Otherwise it is called non–degenerate. If $C$ is non–degenerate then $d_k(C) = |X|$, where $k$ is the dimension of $C$ as a linear subspace of $K^{|X|}$.

III. Evaluation codes associated to complete intersections

Lemma 7: Let $X \subseteq \mathbb{P}^{s-1}$ be a complete intersection and let $a_X$ be the $a$–invariant of $S/I_X$. Then

1) $d_1(C_X(a_X)) = 2$.
2) $H_X(d) + H_X(a_X - d) = |X|.$

Proof: [4, Proposition 2.7] and [8 Lemma 3].

Proposition 8: Let $X \subseteq \mathbb{P}^{s-1}$ be a complete intersection and $a_X$ be the $a$–invariant of $S/I_X$. Then the $r$th generalized Hamming weight of $C_X(a_X)$ is

$$d_r(C_X(a_X)) = r + 1,$$

for $r = 1, \ldots, H_X(a_X) = |X| - 1$.

Proof: We set $d_r := d_r(C_X(a_X))$. The claim follows immediately from Lemma 7 and the fact that

$$2 = d_1 < d_2 < \cdots < d_{|X| - 1} \leq |X|.$$

Remark 1: Let $d \geq a_X + 1$. In these cases $C_X(d) = K^{|X|}$ and therefore $d_r(C_X(d)) = r$ for all $r = 1, \ldots, H_X(d) = |X|$.

Theorem 9: Let $X$ be a complete intersection and $a_X$ be its $a$–invariant. We assume that the codes $C_X(d)$ and $C_X(a_X - d)$ are non–degenerate codes for all $1 \leq d < a_X$. Moreover let $\beta := H_X(d)$, $\beta \geq d_1(C_X(a_X - d))$. Then

$$d_{\beta-i}(C_X(d)) = |X| - i,$$

for $i = 0, \ldots, d_1(C_X(a_X - d)) - 2$.

Proof: Notice that $d_{\beta}(C_X(d)) = |X|$, because $C_X(d)$ is non–degenerate. Moreover $C_X(d)^{\perp}$, the dual code of $C_X(d)$, and $C_X(a_X - d)$ have the same weight hierarchy (see [8 Theorem 2]). By using [29 Corollary 4.1] we obtain that $C_X(d)$ is an $r$–MDS code if $r$ is given by

$$r = |X| + 2 - H_X(a_X - d) - d_1(C_X(a_X - d)) = |X| + 2 - (|X| - \beta) - d_1(C_X(a_X - d)) = \beta + 2 - d_1(C_X(a_X - d)).$$

Actually for this $r$ we get

$$d_r(C_X(d)) = |X| + 2 - d_1(C_X(a_X - d)).$$
As $C_X(d)$ is $s$–MDS for any $s \geq r$ (see [29 Corollary 3.2]), we conclude that
\[ d_{r+j}(C_X(d)) = d_r(C_X(d)) + j, \]  
for $j = 0, \ldots, \beta - r$. If we take $i := \beta - r - j$, we notice that $0 \leq i \leq \beta - r = d_1(C_X(a_X - d)) - 2$. By using Equations (2) and (3) we get
\[ d_{\beta-i}(C_X(d)) = d_r(C_X(d)) + \beta - r - i = |X| - i, \]
and the claim follows. 

**Corollary 1:** With the same notation of Theorem 9 Let $\alpha = H_X(d) - d_1(C_X(a_X - d)) + 2$. Then $C_X(d)$ is an $r$–MDS code for $\alpha \leq r \leq |X|$. In fact
\[ d_{\alpha+j}(C_X(d)) = |X| - d_1(C_X(a_X - d)) + 2 + j, \]  
for all $j = 0, \ldots, d_1(C_X(a_X - d)) - 2$.

**Proof:** The claim follows immediately from Equations (2) and (3) in the proof of Theorem 9. 

**Example 10:** We use the information given in [21, Example 4.4]. Let $K = \mathbb{F}_{181}$ be a finite field with 181 elements. Let $X$ be the following projective degenerate torus, which is a complete intersection (see [12, Theorem 1]).

\[ X = \{ [1 : t_1^{36} : t_2^{30} : t_3^0] \in \mathbb{P}^3 : t_1, t_2, t_3 \in K^* \}. \]

Thus $|X| = 90$ and $a_X = 12$. If we take $d = 5$, $H_X(5) = 35$ and $d_1(C_X(7)) = 7$. Therefore we get the last six generalized Hamming weights of $C_X(5)$:
\[ d_{35-i}(C_X(5)) = 90 - i \]  
for all $i = 0, \ldots, 5$, and $C_X(5)$ is $r$–MDS for $30 \leq r \leq 35$. In a similar way, by taking $d = 7$, $H_X(C_X(7)) = 55$ and $d_1(C_X(5)) = 9$, we obtain the last eight generalized Hamming weights of $C_X(7)$:
\[ d_{55-i}(C_X(7)) = 90 - i \]  
for all $i = 0, \ldots, 7$, and $C_X(7)$ is $r$–MDS for $48 \leq r \leq 55$.

**IV. CODES PARAMETERIZED BY A SET OF MONOMIALS**

Let $X \subseteq \mathbb{T}_{n-1}$ be a toric set parameterized by a set of monomials and $C_X(d)$ its associated code. The following result generalizes [25 Proposition 5.2] and shows that the generalized Hamming weights have the opposite behavior that the Hilbert function.

**Theorem 11:** Let $C_X(d)$ be a parameterized code and let $1 \leq r \leq H_X(d)$. Therefore $C_X(d)$ is non–degenerate and if $d_r(C_X(d)) = r$ then $d_r(C_X(d+1)) = r$. Otherwise if $d_r(C_X(d)) > r$ then $d_r(C_X(d)) > d_r(C_X(d+1))$.

**Proof:** By taking $f = X^1_t \in S_d$, we conclude that $(1,1,\ldots,1) \in C_X(d)$, and then this code is non–degenerate. Let $X = \{ P_1, \ldots, P_{|X|} \}$ and $\Lambda_f \in C_X(d)$ with
\[ \Lambda_f := \left( \frac{f(P_1)}{X_1^d(P_1)}, \ldots, \frac{f(P_{|X|})}{X_{|X|}^d(P_{|X|})} \right), \]  
where $f \in S_d$. Thus
\[ \Lambda_f = \left( \frac{(X_1f)(P_1)}{X_1^{d+1}(P_1)}, \ldots, \frac{(X_{|X|}f)(P_{|X|})}{X_{|X|}^{d+1}(P_{|X|})} \right), \]
and then $\Lambda_f \in C_X(d+1)$. This shows that $C_X(d) \subseteq C_X(d+1)$ for all $d$. Therefore
\[ d_r(C_X(d+1)) \leq d_r(C_X(d)), \]  
for all $r = 1, \ldots, H_X(d)$. If $d_r(C_X(d)) = r$ then, by inequality (6), we get
\[ r \leq d_r(C_X(d+1)) \leq d_r(C_X(d)) = r, \]
and the claim follows in this case. Now we consider $d_r(C_X(d)) > r$. Let $D$ be a subspace of $C_X(d)$ with dimension $r$ and such that $d_r(C_X(d)) = |\supp(D)|$. Let $B = \{ \Lambda_{f_1}, \ldots, \Lambda_{f_s} \}$ be a basis of $D$, where we use the notation of the Equation (5) to $\Lambda_{f_i}$. Therefore
\[ r < d_r(C_X(d)) = |\supp(D)| = |\supp(B)|. \]

Let $i,j \in \supp(B)$ with $i \neq j$. Then there exists $f_{i_1}, f_{i_2} \in S_d$ (not necessarily different) such that $f_{i_1}(P_i) \neq 0$ and $f_{i_2}(P_j) \neq 0$, where $i_1, i_2 \in \{ 1, \ldots, r \}$. As $P_i \neq P_j$ we can take, without loss of generality, $P_i = [1, a_{i_1}, \ldots, a_{i_1}]$, $P_j = [1, b_{i_2}, \ldots, b_{i_2}]$ and $a_{k_1} \neq b_{k_2}$ for some $k \in \{ 2, \ldots, s \}$. We define the following homogeneous polynomials $g_m \in S_{d+1}$:
\[ g_m := (a_{k_1}X_1 - X_k)f_m, \]
for all \( m = 1, \ldots, r \). Notice that \( g_m(P_1) = 0 \) for all \( m = 1, \ldots, r \), but \( g_0(P_1) \neq 0 \). Furthermore let \( B' = \{ \Lambda_0, \ldots, \Lambda_\gamma \} \). \( B' \) is a linearly independent set (because \( B \) is also a linearly independent set) and if \( D' \) is the subspace of \( C_X(d + 1) \) generated by \( B' \), we conclude that (because \( i \in \text{supp}(B) \setminus \text{supp}(B') \))

\[
d_r(C_X(d + 1)) \leq |\text{supp}(D')| = |\text{supp}(B')| < |\text{supp}(B)| = d_r(C_X(d)),
\]

and the claim follows.

Remark 2: The behavior of the generalized Hamming weights given in Theorem 11 is shown in the Tables 4 and 5 of the Example 18.

V. CODES PARAMETERIZED BY THE PROJECTIVE TORUS

If we take \( X = T_{s-1} \), it is a complete intersection and the length, dimension, minimum distance and \( \alpha \)-invariant are known (see [4, 7], and [26]). Furthermore when \( s = 2 \) the code \( C_{T_r} \), is MDS and its complete weight hierarchy is given in [15]. Actually Theorem 9 can be used to find some generalized Hamming weights for the codes \( C_{T_{s-1}}(d) \), as the Example 18 shows. Moreover in order to prove the theorem that gives the second generalized Hamming weight of \( C_{T_{s-1}}(d) \) we use the following notation. If \( f \in S_d \) then

\[
Z_{T_{s-1}}(f) := \{ [P] \in T_{s-1} : f(P) = 0 \}.
\]

Also from now on we use \( \beta \) as a generator of the cyclic group \( (K^*, \cdot) \). Notice that if \( q = 2 \) then \( |T_{s-1}| = 1 \) and \( C_{T_{s-1}}(d) = K \). Thus in this section we assume \( q \geq 3 \).

Lemma 12: Let \( s = 3 \) and let \( d \in \mathbb{N} \), \( 1 \leq d \leq q - 2 \). Then we can find \( f_1, f_2 \in S_d \) such that

\[
|Z_{T_{s-1}}(f_1) \cap Z_{T_{s-1}}(f_2)| = 1 + (d - 1)(q - 1).
\]

Proof: We notice that \( T_2 = \{ [1 : t_1 : t_2] \in \mathbb{F}^2 : t_1, t_2 \in K^* \} \). If \( d = 1 \) then we take \( f_1 = X_1 - X_3 \) and \( f_2 = X_2 - X_3 \). Therefore

\[
Z_{T_{s-1}}(f_1) = \{ [1 : \alpha : 1] \in \mathbb{F}^2 : \alpha \in K^* \}, \quad \text{and} \quad Z_{T_{s-1}}(f_2) = \{ [1 : \alpha : \alpha] \in \mathbb{F}^2 : \alpha \in K^* \}.
\]

Thus \( |Z_{T_{s-1}}(f_1) \cap Z_{T_{s-1}}(f_2)| = 1 \), and the claim follows for \( d = 1 \). Now let \( 2 \leq d \leq q - 2 \). Furthermore let \( f_1 = (\beta^{q-2}X_1 - X_3) \prod_{j=1}^{d-1}(\beta^jX_1 - X_3) \) and \( f_2 = (X_2 - X_3) \prod_{j=1}^{d-1}(\beta^jX_1 - X_3) \). Notice that \( f_1, f_2 \in S_d \) and if \( [P] = [1 : t_1 : t_2] \in T_2 \) then \( f_1(P) = (\beta^{q-2} - t_2) \prod_{j=1}^{d-1}(\beta^j - t_2) \) and \( f_2(P) = (t_1 - t_2) \prod_{j=1}^{d-1}(\beta^j - t_2) \). Moreover if \( B_j := \{ [1 : \beta^j : \alpha] \in K^* \} \) for \( 1 \leq j \leq d - 1 \) then

\[
Z_{T_{s-1}}(f_1) = \{ [1 : \alpha : \beta^{q-2} : \alpha] \in K^* \} \cup \bigcup_{j=1}^{d-1} B_j, \quad \text{and} \quad Z_{T_{s-1}}(f_2) = \{ [1 : \alpha : \alpha] \in K^* \} \cup \bigcup_{j=1}^{d-1} B_j.
\]

As \( B_j \cap B_j' = \emptyset \) if \( j \neq j' \), \( \{ [1 : \alpha : \beta^{q-2} : \alpha] \in K^* \} \cap \{ [1 : \alpha : \alpha] \in K^* \} = \{ [1 : \beta^{q-2} : \beta^{q-2} = \alpha] \} \), and \( \{ [1 : \beta^{q-2} : \beta^{q-2} - \alpha] \} \notin \bigcup_{j=1}^{d-1} B_j \), we conclude that

\[
|Z_{T_{s-1}}(f_1) \cap Z_{T_{s-1}}(f_2)| = 1 + (d - 1)(q - 1),
\]

and the claim follows.

Theorem 13: Let \( s \geq 3 \), \( \eta = (q - 2)(s - 2) \) and \( r = (q - 2)(s - 1) \). Then we can find \( F, G \in S_d \) such that

\[
|Z_{T_{s-1}}(F) \cap Z_{T_{s-1}}(G)| = \begin{cases} (q - 1)^{s-3}(q + 1) & \text{if } l \leq \eta \\ (q - 1)^{s-3}(q + l) & \text{if } \eta < l \leq r \end{cases}
\]

where \( k \) and \( l \) are the unique integers such that \( d = k(q - 2) + l \), \( k \geq 0 \) and \( 1 \leq l \leq q - 2 \).

Proof: Case I: Let \( 1 \leq d \leq q - 2 \). Let \( F(X_1, \ldots, X_s) := f_1(X_1, X_2, X_3) \) and \( G(X_1, \ldots, X_s) := f_2(X_1, X_2, X_3) \), where \( f_1 \) and \( f_2 \) are the polynomials given in the Lemma 12. It is easy to see that \( [t_1 : t_2 : t_3 : t_4 : \cdots : t_s] \in Z_{T_{s-1}}(F) \cap Z_{T_{s-1}}(G) \) if and only if \( [t_1 : t_2 : t_3 : t_4 : \cdots : t_s] \in Z_{T_{s-1}}(F') \cap Z_{T_{s-1}}(G) \) for any \( t_4, \ldots, t_s \in K^* \). Therefore

\[
|Z_{T_{s-1}}(F) \cap Z_{T_{s-1}}(G)| = (q - 1)^{s-3}[1 + (d - 1)(q - 1)],
\]

and due to the fact that in this case \( l = d \) and \( k = 0 \), the claim follows.

Case II: Let \( q - 2 < d \leq (q - 2)(s - 2) \). Notice that in this case \( 1 \leq k \leq s - 3 \). We define the following polynomials:

\[
H_k := \prod_{j=1}^{k} \prod_{j=1}^{q-2}(\beta^jX_1 - X_j), \quad F_k := \prod_{j=1}^{k} \prod_{j=1}^{q-2}(\beta^jX_1 - X_j + 1), \quad G_k := (X_{k+2} - X_{k+3}) \prod_{j=1}^{k}(\beta^jX_1 - X_{k+2}).
\]

Moreover we take \( F := H_k \cdot f_{k,t} \) and \( G := H_k \cdot g_{k,t} \). We notice that \( F, G \in S_d \), where \( d = k(q - 2) + l \). Obviously \( Z_{T_{s-1}}(H_k) \subseteq Z_{T_{s-1}}(F) \cap Z_{T_{s-1}}(G) \). Let \( [P] = [1 : \alpha_2, \ldots, \alpha_s] \in T_{s-1} \). If there exists \( i \in \{ 2, \ldots, k + 1 \} \) such that \( \alpha_i \neq 1 \), we can say \( \alpha_i = \beta^r \) with \( r \leq q - 2 \), then \( (\beta^rX_1 - X_i)(P) = 0 \), but \( \beta^rX_1 - X_i \) is a factor of \( H_k \), thus \( [P] \in Z_{T_{s-1}}(F) \cap Z_{T_{s-1}}(G) \). Actually if

\[
A = \{ [1 : 1 : \cdots : 1 : \alpha_{k+2} : \cdots : \alpha_s] : \alpha_i \in K^* \}
\]

then \( T_{s-1} \setminus A \subseteq Z_{T_{s-1}}(F) \cap Z_{T_{s-1}}(G) \) and \( |T_{s-1} \setminus A| = (q - 1)^{s-3} - (q - 1)^{s-3}(k+1) \). Now we need to find out the number of zeroes of \( F \) and \( G \) that are in \( A \). If \( [P] \in A \) then \( H_k(P) \neq 0 \); thus we need to analyze just the zeroes of \( f_{k,t} \) and \( g_{k,t} \).
that are in A. But $f_{k,l}$ and $g_{k,l}$ are of degree $l$, and if we use the proof of Lemma 12 and the Case I above (we consider the entries $1, k + 2, k + 3, \ldots, s$ of the points of $A$, that is $s - k$ entries) we conclude that

$$|Z_{T_{s-1}}(f_{k,l}) \cap Z_{T_{s-1}}(g_{k,l}) \cap A| = (q - 1)^{s-(k+3)}[(q - 1)(l - 1) + 1].$$

Therefore

$$|Z_{T_{s-1}}(F) \cap Z_{T_{s-1}}(G)| = (q - 1)^{s-1} - (q - 1)^{s-(k+1)} + (q - 1)^{s-(k+3)}[(q - 1)(l - 1) + 1]$$

$$= (q - 1)^{s-(k+3)}[(q - 1)k^2 - (q - 1)^2 + (q - 1)(l - 1) + 1]$$

$$= (q - 1)^{s-(k+3)}[(q - 1)k^2 - (q - 1)(q - l) + 1],$$

and the claim follows.

**Case III:** Let $(q - 2)(s - 2) < d \leq (q - 2)(s - 1)$. In this case $k = s - 2$. We use the polynomials $f_{s-2,l} = (\beta q - 2)X_1 - X_s \prod_{i=2}^{s-1}(\beta X_1 - X_s)$ and $g_{s-2,l} = (X_1 - X_s) \prod_{i=2}^{s-1}(\beta X_1 - X_s)$. We continue using the notation introduced above for the remaining polynomials. Its is immediate that $Z_{T_{s-1}}(H_{s-2}) = T_{s-1} \setminus B$, where $B = \{1 : 1 : \cdots : 1 : \alpha_s : \alpha_s \in K^*\}$.

Therefore

$$|Z_{T_{s-1}}(H_{s-2})| = (q - 1)^{s-1} - (q - 1).$$

Moreover

$$Z_{T_{s-1}}(f_{s-2,l}) \cap Z_{T_{s-1}}(g_{s-2,l}) \cap B = \{1 : 1 : \cdots : 1 : \beta^i : i = 1, \ldots, l - 1\}.$$

Thus $|Z_{T_{s-1}}(f_{s-2,l}) \cap Z_{T_{s-1}}(g_{s-2,l}) \cap B| = l - 1$. We conclude that

$$|Z_{T_{s-1}}(F) \cap Z_{T_{s-1}}(G)| = (q - 1)^{s-1} - (q - 1) + l - 1 = (q - 1)^{s-1} - q + l,$$

and the claim follows.

**Remark 3:** The formula for the minimum distance of the codes $C_{T_{s-1}}(d)$ was found in [26, Theorem 3.5]:

$$d_1(C_{T_{s-1}}(d)) = \begin{cases} (q - 1)^{s-(k+2)}(q - 1 - l) & \text{if } 1 \leq d < r \\ 1 & \text{if } d \geq r, \end{cases}$$

(7)

where $k$ and $l$ are the unique integers such that $d = k(q - 2) + l$, $k \geq 0$, $1 \leq l \leq q - 2$, and $r = (q - 2)(s - 1)$. It is easy to see that Equation (7) can be reduced to

$$d_1(C_{T_{s-1}}(d)) = \left[(q - 1)^{s-(k+2)}(q - 1 - l)\right],$$

(8)

for all $d \geq 1$.

From now on we use the following notation:

$$Z_1(s, d) := (q - 1)^{s-1} - (q - 1)^{s-(k+2)}(q - 1 - l),$$

where $k$ and $l$ are the unique integers such that $d = k(q - 2) + l$, $k \geq 0$ and $1 \leq l \leq q - 2$. It is immediate that if $f \in S_d \setminus T_{s-1}(d)$ then

$$|Z_{T_{s-1}}(f)| \leq Z_1(s, d).$$

Also from now on we use $Z_2(s, d)$ as the right hand side of the equation involved in Theorem 13 that is

$$Z_2(s, d) := \begin{cases} (q - 1)^{s-(k+3)}[(q - 1)k^2 - (q - 1)(q - l) + 1] & 1 \leq d \leq \eta \\ (q - 1)^{s-1} - q + l & \eta < d \leq r, \end{cases}$$

where $k$ and $l$ are the unique integers such that $d = k(q - 2) + l$, $k \geq 0$ and $1 \leq l \leq q - 2$.

**Lemma 14:** With the notation introduced above. Let $s, d \in \mathbb{N}$, $s \geq 3$.

1) If $1 \leq d' < d \leq (q - 2)(s - 1)$ then
   a) $Z_1(s, d') \leq Z_1(s, d)$.
   b) $Z_2(s, d') \leq Z_2(s, d)$.
   c) $Z_2(s, d') \leq Z_1(s, d') \leq Z_2(s, d) \leq Z_1(s, d)$.

2) a) $(q - 1)Z_1(s, d) = Z_1(s + 1, d)$.
   b) $(q - 1)Z_2(s, d) \leq Z_2(s + 1, d)$.

**Proof:**

1) a) This result is an obvious consequence of the fact that $Z_1(s, d) = (q - 1)^{s-1} - d_1(C_{T_{s-1}}(d))$, and [25, Proposition 5.2].
b) Let \( 2 \leq d \leq (q - 2)(s - 2) \). If \( l > 1 \) then
\[
\mathcal{Z}_2(s, d - 1) = (q - 1)^{s-1} - (q - 1)^{s-(k+2)}(q + 1 - l) + (q - 1)^{s-(k+3)}
\]
\[
\leq (q - 1)^{s-1} - (q - 1)^{s-(k+2)}(q - l) + (q - 1)^{s-(k+3)} = \mathcal{Z}_2(s, d).
\]
If \( l = 1 \) we obtain that
\[
\mathcal{Z}_2(s, d - 1) = (q - 1)^{s-1} - 2(q - 1)^{s-(k+1)} + (q - 1)^{s-(k+2)}
\]
\[
= (q - 1)^{s-1} - (q - 1)^{s-(k+2)}(q - l) + (q - 1)^{s-(k+3)} = \mathcal{Z}_2(s, d).
\]
Let \((q - 2)(s - 2) < d \leq (q - 2)(s - 1)\). If we take \( l > 1 \) then
\[
\mathcal{Z}_2(s, d - 1) = (q - 1)^{s-1} - q + l - 1 \leq (q - 1)^{s-1} - q + l = \mathcal{Z}_2(s, d).
\]
If \( l = 1 \) then
\[
\mathcal{Z}_2(s, d - 1) = (q - 1)^{s-1} - 2(q - 1) + 1 \leq (q - 1)^{s-1} - q + 1 = \mathcal{Z}_2(s, d),
\]
and the claim follows.

c) It is immediate that for any \( d, \mathcal{Z}_2(s, d) \leq \mathcal{Z}_1(s, d) \). Let \( 1 \leq d' < d \leq (q - 2)(s - 2) \). If \( l > 1 \) then
\[
\mathcal{Z}_1(s, d') \leq \mathcal{Z}_1(s, d - 1) = (q - 1)^{s-1} - (q - 1)^{s-(k+2)}(q - l)
\]
\[
\leq (q - 1)^{s-1} - (q - 1)^{s-(k+2)}(q - l) + (q - 1)^{s-(k+3)} = \mathcal{Z}_2(s, d).
\]
If \( l = 1 \) then
\[
\mathcal{Z}_1(s, d') \leq \mathcal{Z}_1(s, d - 1) = (q - 1)^{s-1} - (q - 1)^{s-(k+1)}
\]
\[
\leq (q - 1)^{s-1} - (q - 1)^{s-(k+2)}(q - l) + (q - 1)^{s-(k+3)} = \mathcal{Z}_2(s, d).
\]
Let \((q - 2)(s - 2) < d \leq (q - 2)(s - 1)\) and we suppose that \( d' < d \). If \( l > 1 \) then
\[
\mathcal{Z}_1(s, d') \leq \mathcal{Z}_1(s, d - 1) = (q - 1)^{s-1} - (q - 1)^{s-(k+2)}(q - l + 1)
\]
\[
= (q - 1)^{s-1} - (q - 1)^{s-(k+2)}(q - l) + (q - 1)^{s-(k+3)} = \mathcal{Z}_2(s, d).
\]
If \( l = 1 \) then
\[
\mathcal{Z}_1(s, d') \leq \mathcal{Z}_1(s, d - 1) = (q - 1)^{s-1} - (q - 1) = \mathcal{Z}_2(s, d),
\]
and the claim follows.

2) a) It is an obvious conclusion because of the definition of \( \mathcal{Z}_1(s, d) \).
b) Let \( 1 \leq d \leq (q - 2)(s - 2) \). Then
\[
(q - 1)\mathcal{Z}_2(s, d) = (q - 1)^{s} - (q - 1)^{s-(k+1)}(q - l) + (q - 1)^{s-(k+2)} = \mathcal{Z}_2(s + 1, d).
\]
If \((q - 2)(s - 2) < d \leq (q - 2)(s - 1)\) then
\[
(q - 1)\mathcal{Z}_2(s, d) = (q - 1)^{s} - (q - 1)(q - l) \leq (q - 1)^{s} - (q - l) = \mathcal{Z}_2(s + 1, d),
\]
and the whole claim follows.

Moreover we need the following definition. If we take \( f \in K[X_1, \ldots, X_s] \) and \( a \in K^* \) then
\[
f_a(X_1, \ldots, X_s-1) := f(X_1, \ldots, X_s-1, aX_1).
\]

Notice that \( f_a \in K[X_1, \ldots, X_s-1] \). We assume that the points in the projective space are in standard position, that is, the first non–zero entry from the left is 1. If \([Q] \in \mathbb{P}^{s-2} \), \([Q] = [t_1 : \cdots : t_{s-1}]\) we denote by \([Q_a] := [t_1 : \cdots : t_{s-1} : at_1] \in \mathbb{P}^{s-1} \).

Lemma 15: With the notation introduced above.

1) \( f_a = 0 \) (the zero polynomial) if and only if \( aX_1 - X_s \) divides \( f \).
2) Let \( P = [1 : b_2 : \cdots : b_s] \in T_{s-1} \) and \( Q = [1 : b_2 : \cdots : b_{s-1}] \in T_{s-2} \) (obviously \([P] = [Q_{b_s}]\)). Then \( f(P) = 0 \) if and only if \( f_{b_s}(Q) = 0 \).

Proof: The second part of the proof is immediate from the definitions. In order to prove the first assertion we suppose that \( f_a = 0 \). Fix a monomial ordering where \( X_s > \cdots > X_1 \). By using the division algorithm there are \( g, r \in K[X_1, \ldots, X_s] \) such that \( f = (aX_1 - X_s)g + r \), where \( r = 0 \) or \( r \) is a \( K \)-linear combination of monomials, none of which is divisible by \( X_s \). Therefore \( r \in K[X_1, \ldots, X_s] \). But
\[
0 = f_a(X_1, \ldots, X_s-1) = f(X_1, \ldots, X_s-1, aX_1) = r(X_1, \ldots, X_s-1).
\]
Thus \( r = 0 \) and \( aX_1 - X_s \) divides \( f \). The converse is obvious.
Theorem 16: Let \( s, d \in \mathbb{N}, 1 \leq d \leq (q-2)(s-2), \) and let \( f, g \) two linearly independent polynomials on \( S_d \setminus \mathcal{I}_{s-1}(d) \). Then
\[
|Z_{d-1}(f) \cap Z_{d-1}(g)| \leq (q-1)^{s^2 - (k+3)}[(q-1)^{k+2} - (q-1)(q-l) + 1],
\]
where \( k \) and \( l \) are the unique integers such that \( d = k(q-2) + l, k \geq 0, 1 \leq l \leq q-2 \).

Proof: Notice that if \( f \in K[X_1, \ldots, X_s]_d \),
\[
f = \sum_{i=1}^{r_1} (a_i X_1 - X_s)^n f',
\]
where \( a_1, \ldots, a_{r_1} \) are different non-zero elements of \( K \), \( f'_a \neq 0 \) for all \( a \in K^* \), and
\[
\tilde{f} = \sum_{i=1}^{r_1} (a_i X_1 - X_s)^n f',
\]
then \( |Z_{d-1}(f)| = |Z_{d-1}(\tilde{f})| \). Thus there is no loss of generality if we assume that any polynomial of the form \( (9) \) can be studied as if it were of the form \( (10) \). We proceed by induction on \( s \) (the number of variables). Let \( s = 3 \) and \( f, g \) be two linearly independent polynomials in \( K[X_1, X_2, X_3]|_{1 \leq d \leq (3-2)(q-2) = q-2} \). For all \( a \in K^* \), \( f_a, g_a \in K[X_1, X_2]_d \). Let \( \mathcal{A} := \{ a \in K^* : f_a = 0 \} \), and \( \mathcal{B} := \{ b \in K^* : g_b = 0 \} \). We consider the following cases.

Case A: \( \mathcal{A} = \mathcal{B} = \emptyset \). Thus \( f_a \neq 0, g_a \neq 0 \) for all \( a \in K^* \). By using \( (15) \) Equation (6) for \( r = 2 \) we obtain that
\[
|Z_{d-1}(f_a) \cap Z_{d-1}(g_a)| = (q-1) - (q-d) = d - 1 = l - 1.
\]
If \( a \) runs over all the elements of \( K^* \) and we use Lemma \( (15) \) then
\[
|Z_{d-1}(f) \cap Z_{d-2}(g)| \leq (q-1)(l-1) \leq (q-1)(l-1) + 1,
\]
and the case A follows.

Case B: \( \mathcal{A} = \{ a_1, \ldots, a_{r_1} \} \neq \emptyset, \mathcal{B} = \{ b_1, \ldots, b_{r_2} \} \neq \emptyset, \) and \( \mathcal{A} \cap \mathcal{B} = \emptyset \). In this case we can write \( f = f'H_1 \), where \( H_1 = \prod_{i=1}^{r_1} (a_i X_1 - X_3) \) and \( f'_a \neq 0 \) for all \( a \in K^* \). Also \( g = g'H_2 \) with \( H_2 = \prod_{i=1}^{r_2} (b_i X_1 - X_3) \) and \( g'_a \neq 0 \) for all \( a \in K^* \). As \( Z_{d-1}(H_1) \cap Z_{d-1}(H_2) = \emptyset \), we obtain that
\[
|Z_{d-1}(f) \cap Z_{d-1}(g)| = |Z_{d-1}(H_1) \cap Z_{d-1}(g)| + |Z_{d-1}(H_2) \cap Z_{d-1}(f')| + |Z_{d-1}(f') \cap Z_{d-1}(g') \cap Z_{d-1}(H_1) \cap Z_{d-1}(H_2)|.
\]
Notice that \( (H_1)_{a_i} = 0 \) for all \( i = 1, \ldots, r_1 \). As \( Z_{d-1-1}(f) \leq Z_1(s, d) \) and \( \deg g'_a - d - r_2 \) we get
\[
|Z_{d-1}(g'_a)| \leq l - r_2.
\]
As \( i \in \{1, \ldots, r_1\} \) we obtain that
\[
|Z_{d-1}(H_1) \cap Z_{d-1}(g)| \leq r_1(l - r_2).
\]
In the same way
\[
|Z_{d-1}(H_2) \cap Z_{d-1}(f')| \leq r_2(l - r_1).
\]
Let \( r_3 = \min \{ r_1, r_2 \} \) and consider the polynomials \( f'' := X_1^{r_1-r_3} f \) and \( g'' := X_1^{r_2-r_3} g \). Notice that \( Z_{d-1}(f'') = Z_{d-1}(f'), \) \( Z_{d-1}(g'') = Z_{d-1}(g'), \) and \( \deg f'' = \deg g'' = d - r_3. \) If we proceed similarly to case A then
\[
|Z_{d-1}(f'') \cap Z_{d-1}(g'') \cap Z_{d-1}(H_1) \cap Z_{d-1}(H_2)| \leq (q-1 - r_1 - r_2)(l-1 - r_3) \leq (q-1 - r_1 - r_2)(l-1 - r_3),
\]
and by using Equation \( (11) \) we conclude that
\[
|Z_{d-1}(f) \cap Z_{d-1}(g)| \leq r_1(l - r_2) + r_2(l - r_1) + (q-1 - r_1 - r_2)(l-1 - r_3)
\]
\[
= 2r_1 r_2 + (q-1)(l-r_3) + r_1 r_3 + r_2 r_3 \leq (q-1)(l-r_3)
\]
\[
= (q-1)(l-1) - (q-1)(r_3-1) \leq (q-1)(l-1) + 1,
\]
and the case B follows.

Case C: \( \mathcal{A} = \emptyset, \mathcal{B} = \{ b_1, \ldots, b_{r_2} \} \neq \emptyset \). Thus \( g = g'H_2 \), where \( g' \) and \( H_2 \) were defined in case B, \( f_a \neq 0 \) and \( g'_a \neq 0 \) for all \( a \in K^* \). Therefore, as above,
\[
|Z_{d-1}(H_2) \cap Z_{d-1}(f)| \leq r_2 l.
\]
If we define the polynomial \( X_1^{r_2} g' \), then this polynomial has the same zeroes than \( g' \) and its degree is \( d \). Moreover
\[
|Z_{d-1}(X_1^{r_2} g' \cap Z_{d-1}(f)| = |Z_{d-1}(g' \cap Z_{d-1}(f)| \leq |Z_{d-1}(g')| \leq l - r_2.
\]
Due to the fact that \( i = 1, \ldots, r_2 \), we get
\[
|Z_{d-1}(f) \cap Z_{d-1}(g)| \leq (q-1 - r_2)(l - r_2),
\]
and then
\[
|Z_{T_2}(f) \cap Z_{T_2}(g)| = |Z_{T_2}(H_2) \cap Z_{T_2}(f)| + |Z_{T_2}(f) \cap Z_{T_2}(g') \setminus Z_{T_2}(H_2)| \leq r_2l + (q - 1 - r_2)(l - r_2)
\]
\[
= (q - 1)(l - r_2) + r_2^2 = (q - 1)(l - 1) - (r_2 - 1)(q - 1) + r_2^2
\]
\[
= (q - 1)(l - 1) + 1 - (q - 2 - r_2)(r_2 - 1) \leq (q - 1)(l - 1) + 1,
\]
because \(1 \leq r_2 \leq q - 2\), and the case C follows.

**Case D:** \(A \cap B = \{c_1, \ldots, c_r\} \neq \emptyset\). Then \(f = f'H, g = g'H\) with \(H = \prod_{i=1}^r (c_iX_1 - X_3)\), and \(f_i' \neq 0, g_i' \neq 0\) for all \(i = 1, \ldots, r_4\). Notice that \(1 \leq r_4 \leq l = d\) and if \(r_4 = l\) then \(f\) and \(g\) are linearly independent polynomials, which is wrong. That is, \(r_4 < l\). Notice that, by using [15, Equation (6)] with \(r = 2\),
\[
|Z_{T_1}(f_{c_i}') \cap Z_{T_1}(g_{c_i}')| \leq l - 1 - r_4.
\]
Therefore
\[
|Z_{T_2}(f') \cap Z_{T_2}(g') \setminus Z_{T_2}(H)| \leq |Z_{T_2}(f') \cap Z_{T_2}(g')| \leq (q - 1 - r_4)(l - 1 - r_4),
\]
and thus
\[
|Z_{T_2}(f) \cap Z_{T_2}(g)| = |Z_{T_2}(H)| + |Z_{T_2}(f') \cap Z_{T_2}(g') \setminus Z_{T_2}(H)| \leq r_4(q - 1) + (q - 1 - r_4)(l - 1 - r_4)
\]
\[
= (q - 1)(l - 1) - r_4(l - 1 - r_4) \leq (q - 1)(l - 1) + 1,
\]
and the case D follows.

Cases A, B, C, and D prove the claim for \(s = 3\). We assume that the result follows for \(s\) and we will prove it for \(s + 1\). Let \(f\) and \(g\) be two linearly independent polynomials in \(K[X_1, \ldots, X_{s+1}]_d \setminus I_{T_s}(d)\). We continue using the notation for the sets \(A\) and \(B\) introduced above. Although this proof is quite similar to the case \(s = 3\) there are some details that must be explained. We divide the proof in the following cases.

**Case I:** \(A = B = \emptyset\). Thus \(f_a \neq 0, g_a \neq 0\) for all \(a \in K^*\). By the inductive hypothesis we know that
\[
|Z_{T_{s-1}}(f_a) \cap Z_{T_{s-1}}(g_a)| \leq Z_2(s, d).
\]
Therefore, by using Lemmas 14 and 15 we get
\[
|Z_{T_s}(f) \cap Z_{T_s}(g)| \leq (q - 1)Z_2(s, d) \leq Z_2(s + 1, d),
\]
and the case I follows.

**Case II:** \(A = \{a_1, \ldots, a_{r_1}\} \neq \emptyset, B = \{b_1, \ldots, b_{r_2}\} \neq \emptyset, \) and \(A \cap B = \emptyset\). In this case we can write \(f = f'H_1\), where \(H_1 = \prod_{i=1}^{r_1} (a_iX_1 - X_{s+1})\) and \(f_a' \neq 0\) for all \(a \in K^\ast\). Also \(g = g'H_2\) with \(H_2 = \prod_{i=1}^{r_2} (b_iX_1 - X_{s+1})\) and \(g_a' \neq 0\) for all \(a \in K^\ast\). We observe that \(f' \in K[X_1, \ldots, X_{s+1}]_{d-r_2}\), and \(g' \in K[X_1, \ldots, X_{s+1}]_{d-r_2}\). As \(Z_{T_s}(H_1) \cap Z_{T_s}(H_2) = \emptyset\), we obtain that
\[
|Z_{T_s}(f) \cap Z_{T_s}(g)| = |Z_{T_s}(H_2) \cap Z_{T_s}(f')| + |Z_{T_s}(H_1) \cap Z_{T_s}(g')| + |Z_{T_s}(f') \cap Z_{T_s}(g') \setminus Z_{T_s}(H_1H_2)|.
\]
Also, by the definition of \(Z_1(s, d)\), we get
\[
|Z_{T_{s-1}}(g_a')| \leq Z_1(s, d - r_2),
\]
for all \(i = 1, \ldots, r_1\). By using the fact that \((H_1)_{a_i} = 0\) for all \(i = 1, \ldots, r_1\) and Lemma 14 we obtain that
\[
|Z_{T_s}(g') \cap Z_{T_s}(H_1)| \leq r_1Z_1(s, d - r_2) \leq r_1Z_2(s, d).
\]
In exactly the same way we get
\[
|Z_{T_s}(f') \cap Z_{T_s}(H_2)| \leq r_2Z_1(s, d - r_1) \leq r_2Z_2(s, d).
\]
Let \(r_3 = \min\{r_1, r_2\}\) and consider the polynomials \(f'' := X_1^{r_1-r_3}f'\) and \(g'' := X_1^{r_2-r_3}g'\). Notice that \(Z_{T_s}(f'') = Z_{T_s}(f')\), \(Z_{T_s}(g'') = Z_{T_s}(g')\), and \(deg f'' = deg g'' = d - r_3\). Thus, by the inductive hypothesis,
\[
|Z_{T_{s-1}}(f_a') \cap Z_{T_{s-1}}(g_a')| = |Z_{T_{s-1}}(f_a'') \cap Z_{T_{s-1}}(g_a'')| \leq Z_2(s, d - r_3).
\]
As \(K^\ast - (A \cup B)| = q - 1 - r_1 - r_2\), then
\[
|Z_{T_s}(f') \cap Z_{T_s}(g') \setminus Z_{T_s}(H_1H_2)| \leq (q - 1 - r_1 - r_2)Z_2(s, d - r_3) \leq (q - 1 - r_1 - r_2)Z_2(s, d).
\]
By using Equations 12, 13, 14, and Lemma 14, we conclude that
\[
|Z_{T_s}(f) \cap Z_{T_s}(g)| \leq (q - 1)Z_2(s, d) \leq Z_2(s + 1, d),
\]
and the case II follows.
\textbf{Case III:} \( A = \emptyset, B = \{b_1, \ldots, b_{r_2}\} \neq \emptyset \). Thus \( g = g'H_2 \), where \( g' \) and \( H_2 \) were defined in case II, \( f_a \neq 0 \) for all \( a \in K^* \), and \( g'_b \neq 0 \) for all \( i = 1, \ldots, r_2 \). In a similar way to the previous cases we get
\[
|Z_{\mathcal{T}_s}(f) \cap Z_{\mathcal{T}_s}(g)| \leq r_2 Z_1(s, d) + (q - 1 - r_2) Z_1(s, d - r_2).
\]
If \( r_2 < l \) then
\[
r_2 Z_1(s, d) + (q - 1 - r_2) Z_1(s, d - r_2) = Z_2(s + 1, d) - (q - 1)^s - (k + 1)(q - 2 - r_2)(r_2 - 1) \leq Z_2(s + 1, d).
\]
If \( r_2 \geq l \), as \( r_2 = |B| \leq |K^*| = q - 1 \), then \( 1 \leq l \leq r_2 \leq q - 1 \). If \( r_2 = q - 1 \), then \( g_a = 0 \) for all \( a \in K^* \). Thus \( g \in \mathcal{T}_s(d) \), which is false. Therefore \( 1 \leq l \leq r_2 \leq q - 2 \). Moreover
\[
r_2 Z_1(s, d) + (q - 1 - r_2) Z_1(s, d - r_2) = Z_2(s + 1, d) - (q - 1)^s - (k + 1)[q_2(q - 1 - l) + 1]
\]
and the case III follows.

\textbf{Case IV:} \( A \cap B = \{c_1, \ldots, c_{r_4}\} \neq \emptyset \). Thus \( f = f'H, g = g'H \) with \( H = \prod_{i=1}^{r_4} (c_i X_1 - X_3) \), and \( f'_{c_i} \neq 0, g'_{c_i} \neq 0 \) for all \( i = 1, \ldots, r_4 \). We know that
\[
|Z_{\mathcal{T}_s}(f) \cap Z_{\mathcal{T}_s}(g)| = |Z_{\mathcal{T}_s}(H)| + |Z_{\mathcal{T}_s}(f') \cap Z_{\mathcal{T}_s}(g') \setminus Z_{\mathcal{T}_s}(H)|.
\]
By the inductive hypothesis,
\[
|Z_{\mathcal{T}_{s-1}}(f'_{c_i}) \cap Z_{\mathcal{T}_{s-1}}(g'_{c_i})| \leq Z_2(s, d - r_4),
\]
for all \( i = 1, \ldots, r_4 \). Therefore
\[
|Z_{\mathcal{T}_{s-1}}(f') \cap Z_{\mathcal{T}_{s-1}}(g') \setminus Z_{\mathcal{T}_s}(H)| \leq (q - 1 - r_4) Z_2(s, d - r_4).
\]
Also, as \( H_{c_i} = 0 \) for all \( i = 1, \ldots, r_4 \), we get that \( |Z_{\mathcal{T}_s}(H)| \leq r_4(q - 1)^{s-1} \), and thus
\[
|Z_{\mathcal{T}_s}(f) \cap Z_{\mathcal{T}_s}(g)| \leq r_4(q - 1)^{s-1} + (q - 1 - r_4) Z_2(s, d - r_4).
\]
If \( r_4 \leq l \) then
\[
r_4(q - 1)^{s-1} + (q - 1 - r_4) Z_2(s, d - r_4) = (q - 1) Z_2(s, d) - r_4(q - 1)^s - (k + 1)[q_2(q - 1 - l)]
\]
\[
\leq (q - 1) Z_2(s, d) - r_4(q - 1)^s - (k + 1) = Z_2(s + 1, d).
\]
If \( r_4 = r_4 = (k - 1)(q - 2) + q - 2 + l - r_4 \). Therefore it is easy to see that
\[
r_4(q - 1)^{s-1} + (q - 1 - r_4) Z_2(s, d - r_4) = (q - 1)^s - (q - 1)^{s-(k+1)}(r_4 + 2 - l)(q - 1 - r_4) + (q - 1 - r_4)(q - 1)^{s-(k+2)}
\]
\[
= Z_2(s + 1, d) + (q - 1)^{s-(k+2)}[(q - 1)(q - l) - (q - 1)(q - l - 2)] \leq Z_2(s + 1, d).
\]
Thus case IV follows and so does the claim.

\textbf{Remark 4:} If \( s = 2 \) and \( q = 3 \) then (see \([15]\)) \( d_2(C_{T_1}(d)) = 2 \) for all \( d \geq 1 \). Moreover if \( q > 3 \) the second generalized Hamming weight of \( C_{T_1}(d) \) is given by (see \([15]\) Equation (6)))
\[
d_2(C_{T_1}(d)) = \begin{cases} 
q - d & \text{if } 1 \leq d \leq q - 3, \\
2 & \text{if } d > q - 3.
\end{cases}
\]
\textbf{Theorem 17:} The second generalized Hamming weight of the code \( C_{T_{s-1}}(d) \), \( s \geq 3, d \geq 1 \) is given by
\[
d_2(C_{T_{s-1}}(d)) = \begin{cases} 
(q - 1)^{s-(k+3)}[(q - 1)(q - l) - 1] & \text{if } 1 \leq d \leq \eta,
q - l & \text{if } \eta < d < r,
2 & \text{if } d \geq r.
\end{cases}
\]
where \( k \) and \( l \) are the unique integers such that \( d = k(q - 2) + l, k \geq 0, 1 \leq l \leq q - 2, \eta = (q - 2)(s - 2) \) and \( r = (q - 2)(s - 1) \).
\textbf{Proof: Case I:} \( d \geq (q - 2)(s - 1) \). As the regularity index in this case is \((q - 2)(s - 1)\), then \( C_{T_{s-1}}(d) = K^{(q - 1)^{s-1}} \).

Therefore \( d_2(C_{T_{s-1}}(d)) = 2 \), and the claim follows.

\textbf{Case II:} \((q - 2)(s - 2) < d < (q - 2)(s - 1) \). In this case \( k = s - 2 \). Let \( F \) and \( G \) be the polynomials given in the Case III of the proof of \([13]\) Clearly the corresponding codewords \( \Lambda_F \) and \( \Lambda_G \) (we use the notation given in the proof of \([17]\) with \( X = T_{s-1} \)) are linearly independent (because it is easy to find \( [P] \in T_{s-1} \) such that \( f_{s-2,l}(P) = 0 \), but \( g_{s-2,l}(P) \neq 0 \), for example \( [P] = [1 : 1 : \cdots : \beta^{q-2}] \)). Let \( W \) be the subspace of \( C_{T_{s-1}}(d) \) generated by \( \Lambda_F \) and \( \Lambda_G \). Thus \( \dim W = 2 \) and, by using \([13]\) we obtain that
\[
|\operatorname{supp}(W)| = |\operatorname{supp} \{\Lambda_F, \Lambda_G\}| = |T_{s-1} - |Z_{T_{s-1}}(F) \cap Z_{T_{s-1}}(G)| = (q - 1)^{s-1} - [(q - 1)^{s-1} - q + l] = q - l.
\]
Therefore $d_2(C_{T_{s-1}}(d)) \leq q - l$. But for these values of $d$, $d_1(C_{T_{s-1}}(d)) = q - 1 - l$ (see Equation (12)). As $d_2(C_{T_{s-1}}(d)) > d_1(C_{T_{s-1}}(d))$ we conclude that $d_2(C_{T_{s-1}}(d)) \geq q - l$. Then $d_2(C_{T_{s-1}}(d)) = q - l$, and the claim follows.

**Case III:** Let $1 \leq d \leq (q - 2)(s - 2)$. In this case $k \leq s - 3$. If we take $F$ and $G$ as the polynomials defined in the Cases I or II of the proof of Theorem 15 (depending on the value of $d$) and, in a similar way of the Case II above, $U$ is the subspace of $C_{T_{s-1}}(d)$ generated by $\Lambda_F$ and $\Lambda_G$, then
\[
|\text{supp}(U)| = |\text{supp} \{\Lambda_F, \Lambda_G\}| = |T_{s-1}| - |Z_{T_{s-1}}(F) \cap Z_{T_{s-1}}(G)|
= (q - 1)^s - [(q - 1)^s - (q - 1)^{k + 2} - (q - 1)(q - l) + 1]
= (q - 1)^s - [(q - 1)^{s - 1} - (q - 1)^{s - (k + 3)}[(q - 1)(q - l) - 1]]
= (q - 1)^s - (k + 3)[(q - 1)(q - l) - 1].
\]

Therefore
\[
d_2(C_{T_{s-1}}(d)) \leq |\text{supp}(U)| = (q - 1)^s - (k + 3)[(q - 1)(q - l) - 1].
\]

(17)

On the other hand let $\mathcal{D}$ be a subspace of $C_{T_{s-1}}(d)$ with $\dim K \mathcal{D} = 2$. If $\{\Lambda_F, \Lambda_G\}$ is a $K$–basis of $\mathcal{D}$ then, by using Theorem 16 we obtain that
\[
|\text{supp}(\mathcal{D})| \geq (q - 1)^s - (k + 3)[(q - 1)(q - l) - 1].
\]

Therefore
\[
d_2(C_{T_{s-1}}(d)) \geq (q - 1)^s - (k + 3)[(q - 1)(q - l) - 1].
\]

(18)

Equations (17), and (13) prove case III. The claim follows from cases I, II, and III.

**Remark 5:** It is easy to see that the formulae for the second generalized Hamming weight of the codes $C_{T_{s-1}}(d)$ given by Equations (15) and (16) can be reduced to
\[
d_2(C_{T_{s-1}}(d)) = d_1(C_{T_{s-1}}(d)) + [(q - 1)^s - (k + 3)(q - 2)],
\]
where $s \geq 2$, $d \geq 1$, $k$ and $l$ are the unique integers such that $d = k(q - 2) + l$, $k \geq 0$, $1 \leq l \leq q - 2$, and $d_1(C_{T_{s-1}}(d))$ is given by Equation (8).

**Example 18:** Let $K = \mathbb{F}_5$ be a finite field with 5 elements. For the codes $C_{T_{s}}(d)$ with $1 \leq d \leq 6$ (because the $a$–invariant is 5, see [7] Lemma 1, (II)) we obtain the complete weight hierarchy by using Proposition 8 Theorem 9 Theorem 17 and [80] Theorem 3 (see Tables I and II).

| $d$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | $d_6$ | $d_7$ | $d_8$ |
|-----|------|------|------|------|------|------|------|------|
| 1   | 12   | 15   | 16   | –    | –    | –    | –    | –    |
| 2   | 8    | 11   | 12   | 14   | 15   | 16   | –    | –    |
| 3   | 4    | 7    | 8    | 10   | 11   | 12   | 13   | 14   |
| 4   | 3    | 4    | 6    | 7    | 8    | 9    | 10   | 11   |
| 5   | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    |
| 6   | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    |

**TABLE I**

THE FIRST EIGHT GENERALIZED HAMMING WEIGHTS FOR $C_{T_{s}}(d)$, $q = 5$, $1 \leq d \leq 6$.

| $d$ | $d_9$ | $d_{10}$ | $d_{11}$ | $d_{12}$ | $d_{13}$ | $d_{14}$ | $d_{15}$ | $d_{16}$ |
|-----|------|----------|----------|----------|----------|----------|----------|----------|
| 1   | –    | –        | –        | –        | –        | –        | –        | –        |
| 2   | –    | –        | –        | –        | –        | –        | –        | –        |
| 3   | 15   | 16       | –        | –        | –        | –        | –        | –        |
| 4   | 12   | 13       | 14       | 15       | 16       | –        | –        | –        |
| 5   | 10   | 11       | 12       | 13       | 14       | 15       | 16       | –        |
| 6   | 9    | 10       | 11       | 12       | 13       | 14       | 15       | 16       |

**TABLE II**

THE REMAINING GENERALIZED HAMMING WEIGHTS FOR $C_{T_{s}}(d)$, $q = 5$, $1 \leq d \leq 6$.

For example if we take $d = 2$, $a_{T_{s}} = d = 3$, $d_1(C_{T_{s}}(3)) = 4$ (see [26] Theorem 3.5), and $\beta = H_{T_{s}}(2) = 6$ (see [7] Lemma 1, (III)). Then, by Theorem 9

\[
d_6-i(C_{T_{s}}(2)) = 16 - i \text{ for all } i = 0, \ldots, 2.
\]

Moreover if we take $d = 3$, $d_1(C_{T_{s}}(2)) = 8$ then

\[
d_{10-i}(C_{T_{s}}(3)) = 16 - i \text{ for all } i = 0, \ldots, 6.
\]
Notice that \( C_{T_2}(3) \) is equivalent (we use the definition given in [8] Remark 1) to the dual code of \( C_{T_2}(2) \). Therefore if we use Theorem 17] and the Duality Theorem (see [30, Theorem 3]) we obtain that 
\[
d_2(C_{T_2}(2)) = 11 \quad \text{and} \quad d_3(C_{T_2}(2)) = 12, 
\]
and thus we get the six Hamming weights of \( C_{T_2}(2) \). It is important to comment that we use Macaulay2 [16] to check some computations.

**Remark 6:** Let \( X \) be the toric set parameterized by the edges of the complete bipartite graph \( K_{m,n} \). Equations (8) and (16) allow to compute the second generalized Hamming weight of the code \( C_X(d) \), because this code is \( C_{T_{m-1}}(d) \otimes C_{T_{n-1}}(d) \) (see [9]). Actually
\[
d_2(C_X(d)) = \min\{d_1(C_{T_{m-1}}(d)) \cdot d_2(C_{T_{n-1}}(d)), d_2(C_{T_{m-1}}(d)) \cdot d_1(C_{T_{n-1}}(d))\}.
\]

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