Koebe and Carathéodory type boundary behavior results for harmonic mappings

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ABSTRACT

We study the behavior of the boundary function of a harmonic mapping from global and local points of view. Results related to the Koebe lemma are proved, as well as a generalization of a boundary behavior theorem by Bshouty, Lyzzaik and Weitsman. We also discuss this result from a different point of view, from which a relation between the boundary behavior of the dilatation at a boundary point and the continuity of the boundary function of our mapping can be seen.

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1. Introduction

In this paper we consider certain questions related to the boundary behavior of harmonic mappings. More precisely, we first give results analogous to classical Koebe’s lemma. In the last part, we present a new approach and a generalized version of [1, Theorem 6].

In 1915, Koebe [2] proved that if a bounded analytic function tends to zero on a sequence of arcs in the unit disk $\mathbb{D}$, which approach a subarc of the unit circle $\partial \mathbb{D}$, then it must be identically zero. In [3], Rung studied the behavior of the analytic functions when the arcs are of positive hyperbolic diameter. Results related to the Koebe lemma for quasiconformal mappings can also be found in [4]. Our first aim is to obtain a similar result for harmonic mappings.

Another classical boundary behavior result is the following theorem of Lindelöf [5, p.259]:

**Theorem A:** Suppose that $\gamma$ is a parametric curve, with parameter interval $[0, 1]$, such that $|\gamma(t)| < 1$ if $t < 1$ and $\gamma(1) = 1$. If $f$ is a bounded analytic function in $\mathbb{D}$, and

$$\lim_{t \to 1} f(\gamma(t)) = \alpha,$$

then $f$ has a non-tangential limit $\alpha$ at $1$.  

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For generalized versions of this result for different classes of mappings, see e.g. [4,6]. Clearly, the harmonic mapping \( f(z) = \arg(1 - z) + i\Re(1 - z) \) shows that the result as such does not hold for harmonic mappings in general. However, in [7], Ponnusamy and Rasila showed a connection between the multiplicity of the zeros of a harmonic mapping, as they tend to a boundary point along a line, and the existence of the angular limit at this point. These results show that under certain additional assumptions, the boundary behavior can be controlled, which serves as a motivation for this investigation. Secondly, we discuss the Carathéodory–Osgood–Taylor theorem (also known as Carathéodory’s extension theorem for conformal mappings), which was proved by Carathéodory and independently by Osgood and Taylor in 1913. This result allows extension of a conformal mapping, obtained for example from the Riemann Mapping Theorem, to the boundary of a Jordan domain.

**Theorem B**: Let \( \Omega \) be a Jordan domain and \( f \) be a conformal mapping from \( \mathbb{D} \) onto \( \Omega \). Then \( f \) can be extended to a homeomorphism of \( \overline{\mathbb{D}} \) onto \( \overline{\Omega} \) (the closure of \( \Omega \)).

An analogue of the above theorem also holds for the quasiconformal mappings in the plane (see [8, p.42–44]). This shows that conformal and quasiconformal mappings of \( \mathbb{D} \) onto Jordan domains behave similarly at \( \partial \mathbb{D} \).

However, the harmonic case is more complicated, because such a result does not hold. We have the following well-known counterexample. Recall that for a function \( f : \mathbb{D} \to \mathbb{C} \) the **cluster set** of \( f \) at the point \( e^{it} \), is the quantity

\[
C(f, e^{it}) = \bigcap_U f(U \cap \mathbb{D}),
\]

where the intersection is taken over all neighborhoods \( U \) of \( e^{it} \).

**Example 1.1**: Let \( f \) be the function given by the Poisson formula

\[
f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} e^{i\theta(t)} \, dt, \quad z \in \mathbb{D},
\]

where

\[
\theta(t) = \begin{cases} 
0, & \text{if } 0 \leq t \leq 2\pi/3, \\
2\pi/3, & \text{if } 2\pi/3 \leq t \leq 4\pi/3, \\
4\pi/3, & \text{if } 4\pi/3 \leq t < 2\pi.
\end{cases}
\]

Then \( f \) is a univalent harmonic mapping from \( \mathbb{D} \) onto the triangle with vertices at the points \( 1, e^{2\pi i/3}, e^{4\pi i/3} \). In addition, it can be shown that, as \( z \) approaches \( \partial \mathbb{D} \), the cluster sets of \( f \) at the points \( 1, e^{2\pi i/3} \) and \( e^{4\pi i/3} \) are the three sides of the triangle and the circular arcs (separated by those points) are mapped to the three vertices. Clearly, the boundary function of \( f \) is not a homeomorphism.

**Continuity on the boundary.** The above discussions lead to the question of continuity of the boundary function. Recall the next result of Hengartner and Schober [9] which is concerned with the behavior of the boundary function of a harmonic mapping in \( \mathbb{D} \).
Theorem C: Let $\Omega$ be a bounded and simply connected Jordan domain, whose boundary is locally connected. Suppose that $a$ is analytic in $\mathbb{D}$ with $a(\mathbb{D}) \subset \mathbb{D}$ and $w_0$ is a fixed point in $\Omega$. Then there exists a univalent function $f$ such that $f_\Omega(z) = a(z)f_\Omega(z)$, i.e. $a(z) = a_f(z)$, with the following properties:

(a) $f(0) = w_0$, $f_\Omega(0) > 0$ and $f(\mathbb{D}) \subset \Omega$;
(b) There is a countable set $E \subset \partial \mathbb{D}$ such that the unrestricted limits $f^*(e^{it}) = \lim_{z \to e^{it}} f(z)$ exist on $\partial \mathbb{D} \setminus E$ and they are on $\partial \Omega$;
(c) The functions

$$f^*(e^{it}) = \text{ess lim}_{s \downarrow t} f^*(e^{is}) \quad \text{and} \quad f^*(e^{it}) = \text{ess lim}_{s \uparrow t} f^*(e^{is})$$

exist on $\partial \mathbb{D}$ and are equal on $\partial \mathbb{D} \setminus E$;
(d) The cluster set of $f$ at $e^{it} \in E$ is the straight line segment joining $f^*(e^{it})$ to $f^*(e^{it})$.

For $e^{it} \in E$, we say that $f^*$ has a jump discontinuity at $e^{it}$. Otherwise $f^*$ is said to be continuous. When $\|a_f\|_\infty = 1$, where $\|a_f\|_\infty = \sup\{|a_f(z)| : z \in \mathbb{D}\}$, then $E$ can be non-empty. If $|a_f(z)| < k < 1$, then $f$ is quasiconformal in $\mathbb{D}$ and hence it can be extended to a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{\Omega}$.

In the last part of the paper, we present a generalized version of a theorem proved by Bshouty, Lyzzaik and Weitsman (see [1, Theorem 6]). There, we define Jordan curves in $\mathbb{D}$, which end at a point $\zeta \in \partial \mathbb{D}$. Then we show that a condition on the integrals of a quantity which contains the second complex dilatation of a harmonic mapping $f$, over these arcs, implies the continuity of the boundary function $f^*$ at $\zeta$. This observation gives an extra tool that can be used in order for us to examine the behavior of the boundary function of a harmonic mapping. Moreover, it can be considered as a Carathéodory type result for a class of non-quasiconformal harmonic mappings.

2. Preliminaries

A continuous mapping $f = u + iv$ is a complex-valued harmonic mapping in a domain $\Omega \subset \mathbb{C}$, if both $u$ and $v$ are real harmonic functions in $\Omega$. Then we write $\Delta f = 0$, where $\Delta$ is the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\partial \bar{z}}.$$ 

In any simply connected domain $\Omega$, $f$ has the unique representation $f = h + \bar{g}$ with $g(0) = 0$, where $h$ and $g$ are analytic (see [10,11]). A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $\Omega$ is that $|h'(z)| > |g'(z)|$ for all $z \in \Omega$. Let

$$D_f := \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} = 1 + \frac{|a_f|}{1 - |a_f|}, \quad a_f := \frac{f_{\bar{z}}}{f_z}. $$

The quantity $D_f := D_f(z)$ is called the dilatation of $f$ at the point $z$. Clearly, $1 \leq D_f(z) < \infty$. If $D_f(z)$ is bounded throughout a given region $\Omega$ by a constant $K \in [1, \infty)$, then the sense-preserving diffeomorphism $f$ is said to be $K$-quasiconformal in $\Omega$. A mapping that is $K$-quasiconformal for some $K \geq 1$ is simply called quasiconformal. In addition, the ratio
$\nu_f = f_2/f_z$ is called the complex dilatation of $f$ and $a_f = f_2/f_z$ is said to be the second complex dilation of $f$. Thus, $0 \leq |a_f| = |\nu_f| < 1$ if and only if $f$ is sense-preserving.

Let $D(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| < r \}$ and $\overline{D}(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| \leq r \}$. In particular, we write $D(r) = D(0, r)$ and $\overline{D}(r) = \overline{D}(0, r)$. In this paper, we consider the harmonic mappings in $D$ or in $H = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$. Let $A \subset \mathbb{C}$ be a non-empty set. We denote the diameter of $A$ by $d(A)$, where $d(A) = \sup \{|z - w| : z, w \in A\}$. The distance between a point $z_0 \in \mathbb{C} \setminus A$ and the set $A$ is given by $d(z_0, A) = \inf \{|z_0 - w| : w \in A\}$.

3. Conformal invariants

In this section, we recall some useful definitions and results we need for the proofs.

3.1. Modulus of a path family

A path in $\mathbb{C}$ is a continuous mapping $\gamma : I \to \mathbb{C}$, where $I$ is an (possibly unbounded) interval in $\mathbb{R}$.

Let $\Gamma$ be a path family in $\mathbb{C}$ and $\mathcal{F}(\Gamma)$ be the set of all Borel functions $\rho : \mathbb{C} \to [0, \infty]$ such that

$$\int_{\gamma} \rho(z)|dz| \geq 1,$$

for every locally rectifiable path $\gamma \in \Gamma$. The functions in $\mathcal{F}(\Gamma)$ are called admissible functions for $\Gamma$. We define the conformal modulus of $\Gamma$ to be

$$\mathcal{M}(\Gamma) = \inf_{\rho \in \mathcal{F}(\Gamma)} \int_{\mathbb{C}} \rho^2(z) \, dx \, dy,$$

where $z = x + iy$.

Let $\Gamma_1$ and $\Gamma_2$ be two path families in $\mathbb{C}$. We say that $\Gamma_2$ is minorized by $\Gamma_1$ and write $\Gamma_1 < \Gamma_2$ if every $\gamma \in \Gamma_2$ has a subpath in $\Gamma_1$. If $\Gamma_1 < \Gamma_2$, then $\mathcal{F}(\Gamma_1) \subset \mathcal{F}(\Gamma_2)$, and hence $\mathcal{M}(\Gamma_1) \geq \mathcal{M}(\Gamma_2)$. Suppose that $A$ and $B$ are contained in a domain $D$. We use $\Delta(A, B; D)$ to denote the family of curves, connecting $A$ and $B$ in $D$. When $D$ is the whole plane, we write $\Delta(A, B)$. For more information about the modulus of a path family, see e.g. [12–14].

A domain $D$ in $\overline{\mathbb{C}} = \mathbb{C} \cup \{ \infty \}$ is called a ring, if $\overline{\mathbb{C}} \setminus D$ has exactly two components. If the components are $E$ and $F$, we denote the ring by $R(E, F)$.

**Example 3.1:** We consider the annulus $D = \{ z \in \mathbb{C} : 0 < r_1 \leq |z| \leq r_2 \}$. Then,

$$\mathcal{M}(\Delta(\partial \overline{D}(r_1), \partial \overline{D}(r_2), D)) = \frac{2\pi}{\log \frac{r_2}{r_1}}.$$

By [15, Theorem 34.3], we have the following result.

**Lemma 3.1:** A sense-preserving homeomorphism $f : D \to D'$ is $K$-quasiconformal if and only if

$$\mathcal{M}(\Gamma)/K \leq \mathcal{M}(f(\Gamma)) \leq K \mathcal{M}(\Gamma)$$

for every path family $\Gamma$ in $D$. 

3.2. Canonical ring domains

The complementary components of the Grötzsch ring $R_{G}(s)$ in $\mathbb{C}$ are $\overline{D}$ and $[s, \infty)$, $s > 1$, and those of the Teichmüller ring $R_{T}(s)$ are $[-1, 0]$ and $[s, \infty)$, $s > 0$. We define two special functions $\gamma_2(s)$, $s > 1$ and $\tau_2(s)$, $s > 0$ by

$$
\gamma_2(s) = M(\Delta(\overline{D}, [s, \infty])), \\
\tau_2(s) = M(\Delta([-1, 0], [s, \infty])),
$$

respectively. We shall refer to those functions as the Grötzsch capacity function and the Teichmüller capacity function. It is well-known [14, Lemma 5.53] that for all $s > 1$,

$$
\gamma_2(s) = \frac{1}{2} \tau_2(s^2 - 1),
$$

and that $\tau_2 : (0, \infty) \to (0, \infty)$ is a decreasing homeomorphism. Our notation is based on [12].

The lemma given below is a useful tool in proving Theorem 4.1 (see [4, Lemma 5.33]).

**Lemma 3.2:** Let $C \subset D$ be a continuum with $0 < d(C) \leq 1$. Then, if $d(0, C) > 0$, we have that

$$
M\left(\Delta(\overline{D}(1/2), C; D)\right) \geq \frac{1}{4} \tau_2 \left(\frac{d(0, C)}{d(C)}\right).
$$

3.3. Hyperbolic distance

Let $\Omega$ be a simply connected domain in $\mathbb{C}$. If $a, b \in \Omega$, then the hyperbolic distance between $a$ and $b$ in $\Omega$ is denoted by $\rho_{\Omega}(a, b)$ (cf. [14, p.19]). For $a \in \Omega$ and $M > 0$, the hyperbolic disk $\{x \in \Omega : \rho_{\Omega}(a, x) < M\}$ is denoted by $D_{\Omega}(a, M)$. If $z_1, z_2 \in \mathbb{H}$, then (cf. [14, Equality (2.8)])

$$
cosh(\rho_{\mathbb{H}}(z_1, z_2)) = 1 + \frac{|z_1 - z_2|^2}{2\text{Im}(z_1) \text{Im}(z_2)}.
$$

An overview on hyperbolic geometry can be found in [16].

4. Koebe type results

In this section, we shall show two Koebe type results for sense-preserving harmonic mappings with different conditions.

The following result is an analogue of Koebe’s lemma for univalent harmonic mappings in $D$.

**Theorem 4.1:** Suppose that $f : D \to \mathbb{C}$ is either a univalent sense-preserving harmonic mapping satisfying $f_{z}(0) = 0$ or a constant function. Let $\alpha \in \mathbb{C}$ and $\{C_{j}\}$ be a sequence of non-degenerate continua such that $C_{j} \subset D(r_{j})$, $C_{j} \to \partial D$ as $j$ goes to infinity, and $|f(z) - \alpha| < M_{j}$ for $z \in C_{j}$, where $\lim_{j \to \infty} r_{j} = 1$ and $\lim_{j \to \infty} M_{j} = 0$. If

$$
\limsup_{j \to \infty} \tau_2 \left(\frac{1}{d(C_{j})}\right) \left(\log \frac{1}{M_{j}}\right) \frac{1 - r_{j}}{1 + r_{j}} = \infty,
$$

where $\tau_2$ is the Teichmüller capacity function, defined in Section 3.2, then $f \equiv \alpha$ in $D$. 

In particular, if

$$\limsup_{j \to \infty} \left( \log \frac{1}{d(C_j)} \right)^{-1} \left( \log \frac{1}{M_j} \right) \frac{1 - r_j}{1 + r_j} = \infty,$$

then \( f \equiv \alpha \).

Let \( f \) be analytic at \( z_0 \) and suppose that \( f(z_0) = 0 \), but \( f \) is not identically zero. Then \( f \) has a zero of order or multiplicity \( n \) at \( z_0 \) if

$$f(z_0) = f'(z_0) = \cdots = f^{(n-1)}(z_0) = 0, \quad \text{and} \quad f^{(n)}(z_0) \neq 0.$$  

If \( f \) is analytic at \( z_0 \), and has zero of order \( n \) at \( z_0 \), we write \( \mu(z_0, f) = n \). For a sense-preserving harmonic mapping \( f \) in \( \mathbb{D} \), the multiplicity can be obtained from the decomposition \( f = h + \overline{g} \). Suppose that \( h \) and \( g \) have, respectively, multiplicity \( n \) and \( m \) at \( 0 \) with \( n \leq m \). We say that \( f \) has zero of order \( n \) at \( z_0 \) and write \( \mu(z_0, f) = n \).

Next, we state a result concerning the boundary behavior of a sense-preserving harmonic mapping, defined in \( \mathbb{H} \), which has a condition involving multiplicity of the zeroes.

**Theorem 4.2:** Suppose that \( f = h + \overline{g} \) is a sense-preserving harmonic mapping in \( \mathbb{H} \) with \(|f(z)| < 1\). Let \( \{b_k\} \) be a sequence of points in \( \mathbb{H} \) such that \( f(b_k) = 0 \) and \( \lim_{k \to \infty} b_k = 0 \). If

$$\lim_{k \to \infty} \left( \frac{4 + \epsilon}{4 + \epsilon} - \text{Im}\{b_k\} \right)^{\mu(b_k f)} = 0,$$  

then \( f \equiv 0 \) in \( \mathbb{H} \).

### 4.1. Proof of Theorem 4.1

Clearly, for a constant function \( f \equiv \alpha \), the condition \(|f(z) - \alpha| = 0 < M_j \), \( z \in C_j \) is always satisfied for any sequence \( M_j \) of positive numbers, and these numbers can be chosen so that the condition (1) holds.

We assume that \( f \) is univalent, i.e. not a constant function in \( \mathbb{D} \), and prove a contradiction. We may also assume that \( 0 < d(C_j) \leq \frac{1}{4} \). Let \( \Gamma_j = \Delta(\overline{\mathbb{D}}(\frac{1}{2}), C_j; \mathbb{D}(r_j)) \). Then by Lemma 3.2, we have

$$\mathcal{M}(\Gamma_j) \geq \frac{1}{4} \tau_2 \left( \frac{d(0, C_j)}{d(C_j)} \right) \geq \frac{1}{4} \tau_2 \left( \frac{1}{d(C_j)} \right).$$  

(3)

Now, let \( w = d(f(\overline{\mathbb{D}}(\frac{1}{2})), \alpha) > 0 \). Because \( \lim_{j \to \infty} M_j = 0 \), without loss of generality, we may assume that \( M_j < \min\{w^2, 1/w\} \). Then,

$$f(\Gamma_j) > \Delta(\partial \mathbb{D}(\alpha, M_j), \partial \mathbb{D}(\alpha, w); \mathbb{D}(\alpha, w) \setminus \mathbb{D}(\alpha, M_j)),$$

and hence,

$$\mathcal{M}(f(\Gamma_j)) \leq 2\pi \left( \log \frac{w}{M_j} \right)^{-1} \leq 2\pi \left( \frac{1}{2} \log \frac{1}{M_j} \right)^{-1}.$$  

(4)

Because \( f \) is a univalent sense-preserving harmonic mapping with \( f_z(0) = 0 \), we have that \( I_f(z) = |f_z(z)|^2 - |f\overline{z}(z)|^2 > 0 \), and thus, \( a_f \) is analytic in \( \mathbb{D} \), with \(|a_f| = |v_f| < 1 \) and
$a_f(0) = 0$. By the classical Schwarz lemma it follows that $|a_f(z)| \leq |z|$ in $\mathbb{D}$ and

$$D_f = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} = \frac{1 + |a_f|}{1 - |a_f|} \leq \frac{1 + |z|}{1 - |z|}.$$ 

Therefore, for any $z \in \mathbb{D}(r_j)$,

$$D_f \leq \frac{1 + |z|}{1 - |z|} \leq \frac{1 + r_j}{1 - r_j} = K_j,$$

which implies that $f$ is $K_j$-quasiconformal in $\mathbb{D}(r_j)$. Then Lemma 3.1 leads to

$$\mathcal{M}(\Gamma_j) \leq K_j\mathcal{M}(f(\Gamma_j)). \quad (5)$$

It follows from (3) $\sim$ (5), we have

$$\tau_2 \left( \frac{1}{d(C_j)} \right) \leq 16\pi \frac{1 + r_j}{1 - r_j} \left( \log \frac{1}{M_j} \right)^{-1}.$$

That is

$$\tau_2 \left( \frac{1}{d(C_j)} \right) \left( \log \frac{1}{M_j} \right) \frac{1 - r_j}{1 + r_j} \leq 16\pi,$$

which is a contradiction with the assumption (1). The proof of the first part is finished.

By the same argument as in the second part of [4, Theorem 5.34], we have that

$$\tau_2 \left( \frac{1}{d(C_j)} \right) \left( \log \frac{1}{M_j} \right) \geq \frac{\pi}{2} \left( \log \frac{1}{d(C_j)} \right)^{-1},$$

which gives the second part of the theorem.

Recall the following result, which is originally from an unpublished manuscript of Mateljević and Vuorinen (see [7, Lemma 4.3]):

**Lemma 4.3:** Let $f$ be a sense-preserving harmonic mapping of $\mathbb{D}$ such that $f(0) = 0$ and $f(\mathbb{D}) \subset \mathbb{D}$. Then

$$|f(z)| \leq \frac{4}{\pi} \arctan |z|^{\mu(0,f)} \leq \frac{4}{\pi} |z|^{\mu(0,f)} \quad \text{for} \ z \in \mathbb{D}.$$ 

**4.2. Proof of Theorem 4.2**

Fix $\epsilon > 0$. Assume that $f : \mathbb{H} \to \mathbb{D}$ is a sense-preserving harmonic mapping with $f(b_k) = 0$. Let $\phi_k : \mathbb{D} \to \mathbb{H}$ be the Möbius transformation, with $\phi_k(0) = b_k$, $k = 1, 2, \ldots$. Then, the function $\psi_k = f \circ \phi_k$ is a sense-preserving harmonic mapping from $\mathbb{D}$ into itself with $\psi_k(0) = 0$. Therefore, from Lemma 4.3, we have that

$$|\psi_k(w)| = |(f \circ \phi_k)(w)| \leq \frac{4}{\pi} \arctan |w|^{\mu(0,\psi_k)} \leq \frac{4}{\pi} |w|^{\mu(b_k,f)}, \quad w \in \mathbb{D}.$$ 

For $w \in \mathbb{D}$, let $z = \phi_k(w)$. Because the Möbius transformations from $\mathbb{D}$ onto $\mathbb{H}$ are isometries of the hyperbolic distance, then $z \in D_{\mathbb{H}}(b_k, R_k)$ if and only if $w \in D_{\mathbb{D}}(0, R_k)$, which
is equivalent to the condition $|w| < \tanh \frac{R_k}{2}$, where $R_k > 0$ (cf. [14, p.24]). Thus, for all $z \in D_{\mathbb{H}}(b_k, R_k)$, we have

$$|f(z)| = |\psi_k(w)| \leq \frac{4}{\pi} \left( \tanh \frac{R_k}{2} \right)^{\mu(b_k, f)}.$$

**Claim 4.4:** There exists an index $k_0$ such that for $k \geq k_0$, $i \in D_{\mathbb{H}}(b_k, R_k)$, where $R_k = \log 4 + \varepsilon \Im\{b_k\}$.

It follows from the assumption $\lim_{k \to \infty} b_k = 0$ that there exists an index $k_1$, such that, for $k \geq k_1$ the sequence $\{b_k\}$ is contained in $\mathbb{D}(i, 2)$. It also implies that, for the fixed $\varepsilon$, $\Im\{b_k\} < \varepsilon$ for all $k \geq k_2$, for a certain $k_2$. Let $k_0 = \max(k_1, k_2)$. Then, for every $k \geq k_0$,

$$\cosh(\rho_{\mathbb{H}}(i, b_k)) = 1 + \frac{|i - b_k|^2}{2\Im\{b_k\}} \leq 1 + \frac{2}{\Im\{b_k\}}.$$

Because the number of the terms of $\{b_k\}$ which are outside of $\mathbb{D}(i, 2)$ is finite, we may assume that every term is contained in it. Therefore,

$$\frac{e^{\rho_{\mathbb{H}}(i, b_k)} + e^{-\rho_{\mathbb{H}}(i, b_k)}}{2} = \cosh(\rho_{\mathbb{H}}(i, b_k)) \leq 1 + \frac{2}{\Im\{b_k\}}$$

or

$$\log(e^{2\rho_{\mathbb{H}}(i, b_k)} + 1) \leq \log \left( e^{\rho_{\mathbb{H}}(i, b_k)} \left( 2 + \frac{4}{\Im\{b_k\}} \right) \right).$$

(7) implies that

$$\rho_{\mathbb{H}}(i, b_k) < \log \frac{4 + \varepsilon}{\Im\{b_k\}}.$$

Hence, Claim 4.4 is true.

Now, it follows from (6) and Claim 4.4 that

$$|f(i)| \leq \frac{4}{\pi} \left( \tanh \frac{\log \frac{4 + \varepsilon}{\Im\{b_k\}}}{2} \right)^{\mu(b_k, f)}.$$

Moreover,

$$\tanh \left( \frac{1}{2} \log \frac{4 + \varepsilon}{\Im\{b_k\}} \right) = \frac{(4 + \varepsilon) - \Im\{b_k\}}{(4 + \varepsilon) + \Im\{b_k\}},$$

and thus, our initial assumption (2) implies that

$$|f(i)| \leq \lim_{k \to \infty} \left( \frac{4}{\pi} \left( \tanh \frac{\log \frac{4 + \varepsilon}{\Im\{b_k\}}}{2} \right)^{\mu(b_k, f)} \right) = 0.$$

The same argument can be applied at any point in the disk $D_{\mathbb{H}}(b_k, R_k)$, and hence, $f(z) = 0$ for all $z \in D_{\mathbb{H}}(b_k, R_k)$. 
Because the line segment \([i/2, i]\) is contained in each disk \(D_{\mathbb{H}}(b_k, R_k)\) for sufficiently large \(k\), by applying the Uniqueness (or Identity) Theorem at both the analytic and the anti-analytic parts of \(f\), we obtain that \(f \equiv 0\) in \(\mathbb{H}\).

5. Dilatation and the boundary function

The next result is a generalization of a result by Bshouty, Lyzzaik and Weitsman (see [1]). It removes the assumption that the dilatation has to be a Blaschke product. It connects the boundary behavior of the dilatation \(a_f\) of a univalent harmonic mapping \(f\) with the continuity of its boundary function \(f^*\) at a point of \(\partial \mathbb{D}\). Here \(f^*\) is a complex Lebesgue integrable function on \(\partial \mathbb{D}\), which has \(f\) as its Poisson integral.

Let \(f = h + \overline{g}\) be a harmonic mapping. Its complex dilatation is given by \(a_f(z) = \frac{g'(z)}{h'(z)}\). One cannot expect that the dilatation itself can decide whether \(f^*(\xi)\) is continuous without involving both \(h\) and \(g\) themselves, where \(\xi \in \partial \mathbb{D}\) and \(f^*\) is the boundary function of \(f\). If we assume \(f\) is sense preserving, then \(|a_f(z)| < 1\) which ensures that \(|g'(z)| < |h'(z)|\). This may reduce the problem to \(h\). Indeed this is the case, Lemma 2 in [1] states in particular:

If \(f\) is a sense-preserving harmonic mapping and \(f^*\) is of bounded variation then for \(\xi \in \partial \mathbb{D}\), \(f^*\) is continuous at \(\xi\) if, and only if,

\[
\lim_{r \to 1^-} (1 - r)h'(r\xi) = 0.
\]

In the next theorem, we allow ourselves to connect the continuity of \(f^*\) with \(a\) by tying them to area of \(f(\mathbb{D})\).

Let \(\zeta = e^{i\theta_0} \in \partial \mathbb{D}\) and \(0 < m < 1/\pi\). For \(0 < |\theta - \theta_0| \leq \min\{\pi, 1/m\}\), we define the curve \(\Gamma_{\zeta, m}(\theta) = (1 - m|\theta - \theta_0|)e^{i\theta}\). It is clear that \(\Gamma_{\zeta, m}\) is completely included in \(\mathbb{D}\), tending to \(\zeta\) as \(\theta \to \theta_0\).

In the case \(\zeta = 1\), we have the curve \(\Gamma_m(\theta) = (1 - m|\theta|)e^{i\theta}\), which is symmetric with respect to the real axis, meeting it at the point \(m\pi - 1\). We denote by \(R_m\) the bounded region which has the boundary \(\Gamma_m\).

**Theorem 5.1:** Suppose that \(\Omega\) is a bounded, convex and simply connected domain. Let \(a_f\) be an analytic function from \(\mathbb{D}\) into itself and \(f\) be a univalent solution of \(\overline{f} = a_f z\) from \(\mathbb{D}\) onto \(\Omega\). If for each \(0 < m < 1/\pi\)

\[
\mathcal{L}(m) := \int_{\Gamma_{\zeta, m}} \frac{1 - |a_f(z)|^2}{1 - |z|^2} |dz| = \infty,
\]

then the boundary function \(f^*\) is continuous at \(\xi \in \partial \mathbb{D}\).

**Proof:** Without loss of generality, we can assume that \(\zeta = 1\). Suppose that \(\mathcal{L}(m) = \infty\), for every \(m, 0 < m < 1/\pi\), and assume on the contrary that \(f^*\) has a jump at 1. Then [1, Theorem 3] implies that the angular limit \(\lim_{\zeta \to 1} (1 - |z|)|h'(z)|\) is equal to a constant \(c > 0\). Because \(|a_f(z)| < 1\) and \(f\) is univalent in \(\mathbb{D}\), we have that \(|h'(z)| > 0\) in \(\mathbb{D}\) and the same is true for \((1 - |z|)|h'(z)|\).

Fix \(m = m_0 < 1/\pi, 0 < |\theta| \leq \pi\) and \(z \in \Gamma_{m_0}(\theta)\), we have

\[
\left| \frac{dz}{d\theta} \right| = \sqrt{m_0^2 + (1 - m_0|\theta|)^2} \leq m_0 + 1 - m_0|\theta| < 2.
\]
Thus,
\[ \infty = \mathcal{L}(m_0) = \int_{\Gamma_{m_0}} \frac{(|h'(z)|^2 - |g'(z)|^2)(1 - |z|)}{|h'(z)|^2(1 + |z|)^2} \, |dz| \]
\[ \leq \int_{\Gamma_{m_0}} c'(|h'(z)|^2 - |g'(z)|^2)m_0|\theta| \, |dz| \]
\[ < c_1 \int_{-\pi}^{\pi} (|h'(z)|^2 - |g'(z)|^2)|\theta| \, d\theta, \quad (8) \]
where \(c', c_1\) are two positive constants and \(z = (1 - m_0|\theta|)e^{i\theta}\).

In the final step of the proof we will study the behavior of the curve \(\Gamma_m\) (see Figure 1) as a mapping from \(R_{m_0} = [(0, m_0) \times (-\pi, \pi)] \setminus \{(m, 0), 0 < m < m_0\}\) onto the region \(Q_{m_0} = (\mathbb{D} \setminus R_{m_0}) \setminus (-1, -1/2)\). Observe that \(\Gamma_m\) tends to \(\partial \mathbb{D}\) as \(m \to 0\). In order to make it clear, we write
\[ \Gamma_m(\theta) = (1 - m|\theta|) e^{i\theta} = r e^{i\theta}. \]
Then \(r = (1 - m|\theta|)\) and the Jacobian \(|\frac{D(r, \theta)}{D(m, \theta)}| = |\theta|\). Therefore, for the area \(A\) of \(f(\mathbb{D})\) we have that
\[
A = \iint_{\mathbb{D}} (|h'(z)|^2 - |g'(z)|^2) \, dx \, dy = \iint_{R_{m_0}} (|h'(z)|^2 - |g'(z)|^2) r \, dr \, d\theta \\
= \iint_{R_{m_0}} (|h'(z)|^2 - |g'(z)|^2)(1 - m|\theta|)|\theta| \, dm \, d\theta \\
> \int_0^{m_0} (1 - m\pi) \left( \int_{-\pi}^{\pi} (|h'(z)|^2 - |g'(z)|^2)|\theta| \, d\theta \right) \, dm = \infty,
\]
where the last equality follows from (8). This contradicts the initial assumption that \(f\) is bounded, and hence, the theorem is proved.

\[ \blacksquare \]

**Remark 5.1:** Let \(\Omega\) be as above, \(a_f : \mathbb{D} \to \mathbb{D}\) analytic and \(f\) be a univalent solution of the equation \(f_{\zeta} = a_{f_{\zeta}}\) from \(\mathbb{D}\) onto \(\Omega\). If there exists an analytic function \(F : \mathbb{D} \to \mathbb{D}\) such that
\[
\int_{\Gamma_{m, \zeta}} \frac{1 - |F(z)|^2}{1 - |z|^2} \, |dz| = \infty,
\]
for any \(m \in (0, \frac{1}{\pi})\), and \(a_f\) is majorized by \(F\) (denoted by \(a_f \ll F\)) in \(\mathbb{D}\), then the boundary function of \(f\) is continuous at \(\zeta \in \mathbb{D}\).

Indeed, majorization implies that there exists an analytic function \(\phi : \mathbb{D} \to \mathbb{D}\) such that \(a_f(z) = \phi(z) F(z)\) for all \(z \in \mathbb{D}\). Hence,
\[
\int_{\Gamma_{m, \zeta}} \frac{1 - |a_f(z)|^2}{1 - |z|^2} \, |dz| = \int_{\Gamma_{m, \zeta}} \frac{1 - |\phi(z)|^2 |F(z)|^2}{1 - |z|^2} \, |dz| \\
\geq \int_{\Gamma_{m, \zeta}} \frac{1 - |F(z)|^2}{1 - |z|^2} \, |dz| = \infty.
\]
The previous result concludes the proof.
Figure 1. The curve $\Gamma_m$ for $m = 0.2$ and $|\theta| \in (0, \pi]$, contained in $\mathbb{D}$.

Example 5.1: If $a_f(z) = \alpha z$ with $|\alpha| < 1$, then $\mathcal{L}(m)$ is infinite at every point, and hence, the boundary function $f^*$ is a continuous function. When $|\alpha| = 1$ then $\mathcal{L}(m)$ is finite, in one case when $\Omega$ is a quadrilateral triangle, $f^*$ has jump continuity between every two vertices and in another case when $\Omega$ is a strictly concave triangle, i.e. a triangle with three strictly concave sides with respect to its interior and three cusps of zero angles, $f^*$ is continuous at every point.

Theorem 5.1 states that if $\mathcal{L}(m) = \infty$ for all accessible $m$ at some point $\zeta$ and $(1 - |z|)|h'(z)| > C$ with constant $C > 0$, then the area of $\Omega$ is infinite. In the next theorem we study the converse theorem.

Theorem 5.2: Suppose that $\Omega$ is a convex and simply connected domain. Let $a_f$ be an analytic function from $\mathbb{D}$ into itself and $f$ be a univalent solution of $\overline{f} = a_fz$ from $\mathbb{D}$ onto $\Omega$. Fix $\zeta \in \partial \mathbb{D}$ where $f(\zeta)$ is finite, and assume that

$$(1 - |z|)|h'(z)| \leq H$$

in any compact set in $\mathbb{D} \cup \{\zeta\}$. Then

$$\text{Area}(\Omega) < \frac{\pi H^2}{0.075} \int_0^{1/\pi} \mathcal{L}(m) \, dm.$$ 

Remark 5.2: It follows immediately that if $\text{Area}(\Omega) = \infty$, then at least for $m \in E_\zeta$, where $E_\zeta$ is a set of positive measure, $\mathcal{L}(m) = \infty$ at $\zeta$. 
Proof: Without loss of generality we can assume that $\zeta = 1$. Let $0 < m < \frac{1}{\pi}$. Then on $\Gamma_m$ we have $|\frac{dz}{d\theta}| = |m + i(1 - m|\theta|)| > 0.3$, so that,

$$\mathcal{L}(m) = \int_{\Gamma_m} \frac{1 - |a_f(z)|^2}{1 - |z|^2} |dz| = \int_{\Gamma_m} \frac{(|h'(z)|^2 - |g'(z)|^2)(1 - |z|)}{|h'(z)|^2(1 - |z|^2)(1 + |z|)} |dz|$$

$$\geq \int_{\Gamma_m} \frac{(|h'(z)|^2 - |g'(z)|^2)m|\theta|}{2H^2} |dz|$$

$$> \frac{0.3m}{2H^2} \int_{-\pi}^{\pi} (|h'(z)|^2 - |g'(z)|^2)|\theta| d\theta.$$ 

Thus

$$\int_0^{1/\pi} \mathcal{L}(m) \, dm \geq \frac{0.075}{\pi^2 H^2} \int_{-\pi}^{\pi} (|h'(z)|^2 - |g'(z)|^2)|\theta| d\theta.$$ 

On the other hand,

$$\text{Area}(\Omega) = \int\int_D (|h'(z)|^2 - |g'(z)|^2)r \, dr \, d\theta$$

$$= \int_dm \int_{\frac{1}{2\pi}} d\theta \int_D (|h'(z)|^2 - |g'(z)|^2)(1 - m|\theta|)|\theta| \, dm \, d\theta$$

$$< \int_0^{1/\pi} \int_{-\pi}^{\pi} (|h'(z)|^2 - |g'(z)|^2)|\theta| \, d\theta \, dm$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (|h'(z)|^2 - |g'(z)|^2)|\theta| \, d\theta.$$ 

Therefore,

$$\text{Area}(\Omega) < \frac{\pi H^2}{0.075} \int_0^{1/\pi} \mathcal{L}(m) \, dm,$$

completing the proof.

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