Effects of global charge conservation on time evolution of cumulants of conserved charges in relativistic heavy ion collisions

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We investigate the effect of the global charge conservation on the cumulants of conserved charges observed in relativistic heavy ion collisions in a finite rapidity window, $\Delta \eta$, with a special emphasis on the time evolution of fluctuations in the hadronic medium. It is argued that the experimental result of the net-electric charge fluctuation observed by ALICE does not receive effects from the global charge conservation, because of the finite diffusion distance of charged particles in the hadronic stage. We emphasize that the magnitude of the effect of the global charge conservation can be estimated experimentally by combining the information on the $\Delta \eta$ dependences of various cumulants of conserved charges, similarly to other dynamical properties of the hot medium.

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INTRODUCTION

Bulk fluctuations are macroscopic observables, which provide us various information on microscopic nature of the medium. In relativistic heavy ion collisions, it is believed that the bulk fluctuations observed by event-by-event analysis are useful observables, which enable us to characterize properties of the hot medium created by collision events and to find the QCD critical point [1-14]. Recently, experimental investigation of fluctuation observables in heavy ion collisions has been actively performed at the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC) [15-17]. Numerical analyses of higher order cumulants in equilibrium have also been carried out in lattice QCD Monte Carlo simulations [18].

Recently, among the fluctuation observables, especially higher-order cumulants of conserved charges have acquired much attention. One of the important properties of the conserved-charge fluctuations compared with non-conserving ones is that conserved charges can be defined unambiguously as Noether currents and as a result their fluctuations can be calculated without ambiguity in a given theory, for example, QCD [18]. Moreover, various ways to reveal the medium properties using conserved-charge fluctuations, especially non-Gaussianity and mixed cumulants, have been suggested [11].

Among the conserved-charge fluctuations, net baryon number [15] and net electric charge fluctuations are observable in relativistic heavy ion collisions. One of the important properties of these fluctuations is that their higher order cumulants normalized by a conserved quantity are suppressed in the deconfined medium reflecting the fact that the charges carried by elementary excitations are smaller in the deconfined medium [3-5]. Recent experimental result on the net electric charge fluctuation by the ALICE Collaboration at the LHC [17] shows that the value of the second order cumulant of the net electric charge $\langle (Q_{(net)}^2)_{c} \rangle$ normalized by the total number of charged particles $\langle Q_{(tot)} \rangle$,

$$\frac{\langle (Q_{(net)}^2)_{c} \rangle}{\langle Q_{(tot)} \rangle},$$

with $\delta Q = Q - \langle Q \rangle$ is significantly suppressed compared with the one in the equilibrated hadronic medium. This result is reasonably understood if one interprets the suppression as a survival of the small fluctuation generated in the primordial deconfined medium [3, 4, 11, 20, 21].

The experimental result of the ALICE Collaboration also shows that the suppression of $\langle (Q_{(net)}^2)_{c} \rangle$ becomes more prominent as the pseudo-rapidity window to count the number of particles, $\Delta \eta$, is taken to be larger. This $\Delta \eta$ dependence can also be reasonably explained with the above interpretation. This is because the approach of the magnitude of the fluctuation to the equilibrated values of the hadronic medium is slower as the volume to count the conserved-charge number is taken larger [11, 20]. In Ref. [11], it is also pointed out that the combined experimental information on the $\Delta \eta$ dependence of various cumulants of conserved charges enables us to verify the above picture on the second-order fluctuation.

However, there exists another mechanism called the global charge conservation (GCC) to cause the suppression of $\langle (\delta Q_{(net)}^2) \rangle$ compared with the thermal value. If one counts a conserved charge in the total system, created by the heavy-ion collisions, there are no event-by-event fluctuations because of the charge conservation. This fact is referred to as the GCC [1]. Owing to the finiteness of the hot medium generated in heavy ion collisions, the fluctuations in a finite $\Delta \eta$ range, $\langle (\delta Q_{(net)}^2)_{\Delta \eta} \rangle$, are also affected by the GCC. Moreover, this effect is more prominent for larger $\Delta \eta$. In Refs. [4, 22], on the assumption that the equilibration is established in the final state of the heavy ion collision, the magnitude of this effect at...
finite $\Delta \eta$ is estimated as

$$\langle (\delta Q^\text{(net)})^2 \rangle_{\Delta \eta} = \langle (\delta Q^\text{(net)})^2 \rangle_{\text{GC}} (1 - \Delta \eta/\eta_{\text{tot}}),$$

(2)

where $\langle (\delta Q^\text{(net)})^2 \rangle_{\text{GC}}$ is the fluctuation in the grand canonical ensemble and $\eta_{\text{tot}}$ denotes the total length of the system along the rapidity direction. Note that the system is assumed to be expanding longitudinally with the Bjorken scaling.

It, however, should be noted that the suppression of $\langle (\delta Q^\text{(net)})^2 \rangle_{\text{GC}}$ observed at ALICE is more significant than the one described solely by Eq. (2) as we will discuss in Sec. 2. This result shows that there exists another contribution for the suppression besides the GCC, such as the nonthermal effect as originally addressed in Refs. [3, 4]. Since Eq. (2) assumes the equilibration, when the fluctuation is not equilibrated the effect of the GCC at ALICE will be modified from this formula. The effect of the GCC at LHC energy, therefore, has to be revisited with the nonequilibrium effects incorporated.

In the present study, we investigate the effect of the GCC on cumulants of conserved charges under such nonequilibrium circumstances, by describing the time evolution of fluctuations in a system with finite volume. We extend the analyses in Refs. [11, 20] to the case with a finite volume with reflecting boundaries. We also discuss the effects of the GCC on higher-order cumulants of conserved charges for the first time.

By comparing our result with the experimental result at ALICE in Ref. [15], we find that the net electric charge fluctuation in the rapidity window observed by this experiment is hardly affected by the GCC. This result comes from the fact that the rapidity at which the fluctuations are affected by the GCC is approximately limited within the average diffusion distance of each particle from the boundaries. Accordingly, when $\eta_{\text{tot}}$ is sufficiently large the fluctuations observed at mid-rapidity region is not affected by the GCC. We also argue that the combination of the cumulants of conserved charges enables us to confirm this picture experimentally.

Because of the local charge conservation, the probability distribution of $n(\eta, \tau_0)$ at hadronization inherits from the one that existed in the deconfined medium [1]. After hadronization, particles diffuse and rescatter, and the distribution of $n(\eta, \tau)$ continues to approach the one of the equilibrated hadronic medium until kinetic freezeout at $\tau = \tau_0$. The particle number in a rapidity window $\Delta \eta$ at mid-rapidity is given by

$$Q(\Delta \eta, \tau) = \int_{-\Delta \eta/2}^{\Delta \eta/2} d\eta \, n(\eta, \tau).$$

(3)

In the following, we investigate the time evolution of the probability distribution of $Q(\Delta \eta, \tau)$ in hadronic medium with a finite volume until $\tau = \tau_0$. We then obtain the cumulants of the conserved charge at $\tau = \tau_0$, $\langle Q(\Delta \eta, \tau_0)^n \rangle_c$ as functions of $\Delta \eta$, of which the comparison with experiments will turn out to enable us to extract the diffusion constants of the hadronic medium and initial fluctuations at $\tau = \tau_0$, besides the effect of the GCC. Strictly speaking, the experiments measure the particle numbers in a pseudo-rapidity window, while in our study we investigate the distribution in the space-time rapidity. In the following, we assume the exact Bjorken scaling correspondence between the space-time rapidity of a particle and its momentum-space rapidity which is almost identical with the pseudo rapidity. The effects of the violation of the correspondence owing to thermal motion of particles and transverse expansion will be discussed in a future publication [29].

Here, we note that the second order cumulant of Eq. (3), $\langle (Q(\Delta \eta))^2 \rangle_c$ is directly related to the correlation function, $\langle \delta n(\eta_1) \delta n(\eta_2) \rangle$, as the latter is obtained by differentiating the former. The correlation function is further related to the balance function [24, 27]. The experimental information on the $\Delta \eta$ dependence of $\langle (Q(\Delta \eta))^2 \rangle_c$, therefore, is in principle the same as those obtained from these functions. On the other hand, higher order cumulants of $Q$ contain information which cannot be described by the two-point correlation function.

In this study, to describe the time evolution of fluctuation of $Q(\tau)$ we adopt the diffusion master equation [11]. In this model, we divide the system with the total rapidity length $\eta_{\text{tot}}$ into $M$ discrete cells with an equal finite length $a = \eta_{\text{tot}}/M$. We then consider a single species of particles for the moment, and denote the particle number existing in the $m$th cell as $n_m$ and the probability distribution that each cell contains $n_m$ particles as $P(n, \tau)$ with $n = (n_0, n_1, \cdots, n_m, \cdots, n_M-2, n_M-1)$. The model will be extended to the case of multi-particle species later. Finally, we assume that each particle moves to the adjacent cells with a probability $\gamma(\tau)$ per unit proper time, as a result of microscopic interactions. The probability

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Model

In the present study, we consider heavy ion collisions with sufficiently large $\sqrt{s_{\text{NN}}}$, at which the mid-rapidity region has an approximate boost invariance. Useful coordinates to describe such a system are the space-time rapidity $\eta$ and proper time $\tau$. We denote the net number of a conserved charge per unit $\eta$ as $n(\eta, \tau)$, and set $\tau = \tau_0$ at hadronization, which phenomenologically takes place at almost the same time with chemical freeze-out at sufficiently large $\sqrt{s_{\text{NN}}}$.
\(P(n, \tau)\) then obeys the differential equation

\[
\partial_\tau P(n, \tau) = \gamma(\tau) \sum_{m=0}^{M-1} [(n_m + 1)P(n + e_m - e_{m+1}, \tau) + P(n + e_m - e_{m-1}, \tau)] - 2n_m P(n, \tau)], \tag{4}
\]

where \(e_m\) is a unit vector whose all components are zero except for \(m\)th one, which takes unity. In order to take account of the finite size of the hot medium, we further require that the cells at both ends, at \(m = 0\) and \(M - 1\), exchange particles only with inner adjacent cells, \(m = 1\) and \(M - 2\), respectively. After solving Eq. (4) exactly, we take the continuum limit, \(a \to 0\). In this limit, each particle in this model behaves as a Brownian particle without correlations with one another [11, 29].

The average and the Gaussian fluctuation of \(n(\eta, \tau)\) in Eq. (1) in the continuum limit agree with the ones of the stochastic diffusion equation [20]

\[
\partial_\tau n(\eta, \tau) = D(\tau) \partial_\eta^2 n(\eta, \tau) + \partial_\eta \xi(\eta, \tau), \tag{5}
\]

with two reflecting boundaries when one sets \(D(\tau) = \gamma(\tau) a^2\) [11]. Here, \(\xi(\eta, \tau)\) is the temporarily-local stochastic force, whose property is determined by the fluctuation-dissipation relation. On the other hand, it is known from a general argument on the Markov process that the fluctuation of Eq. (5) in equilibrium becomes of Gaussian unless \(D(\tau)\) explicitly depends on \(n(\eta, \tau)\) [11, 30]. This property of Eq. (5) is not suitable to describe the higher order cumulants observed in relativistic heavy ion collisions, since the experimentally-measured cumulants [15, 17] take nonzero values close to the equilibrated ones. On the other hand, the diffusion master equation Eq. (4) can give rise to nonzero higher order cumulants in equilibrium, because of the discrete nature of the particle number [11]. This is the reason why we employ the diffusion master equation instead of the stochastic diffusion equation.

**Solving diffusion master equation**

Next, we determine the time evolution of cumulants by solving Eq. (4). The following numerical procedure is similar to the one in Ref. [11], while the introduction of boundaries gives rise to a new complexity. Determination of the initial condition also becomes more involved owing to the GCC.

We first consider the time evolution of the probability \(P(n, \tau)\) with a fixed initial condition

\[
P(n, 0) = \prod_{m=0}^{M-1} \delta_{n_m, N_m}, \tag{6}
\]

namely the initial particle numbers are fixed as \(n_m(\tau = \tau_0) = N_m\) for all \(m\) without fluctuations. By introducing the factorial generating function,

\[
G_t(s, \tau) = \sum_{n} \prod_{m=0}^{M-1} s_{n_m} P(n, \tau), \tag{7}
\]

Eq. (4) is transformed as

\[
\partial_\tau G_t(s, \tau) = \gamma(\tau) \left[ (s_1 - s_0) \partial_s s_0 + (s_{M-2} - s_{M-1}) \partial_{s_{M-1}} \right] G_t(s, \tau).
\]

Solving Eq. (8) with the method of characteristics, one obtains the solution with the initial condition Eq. (6) as

\[
G_t(s, \tau) = \prod_{m=0}^{M-1} \left( \sum_{k=0}^{M-1} r_k v_{km} e^{-\Omega_k(\tau)} \right)^{N_m}, \tag{9}
\]

where

\[
r_k = \sum_{m=0}^{M-1} s_m u_{mk}, \tag{10}
\]

with

\[
u_{mk} = \frac{1}{M} \cos \frac{\pi k (m + 1/2)}{M}, \tag{11}
\]

is the Fourier transform of \(s_m\) and

\[
\Omega_k(\tau) = \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_0} \gamma(\tau') \left( \frac{\pi k}{M} \right)^2.
\]

The inverse Fourier transform of Eq. (10) is given by

\[
s_m = \sum_{k=0}^{M-1} r_k v_{km} \quad \left( k = 0 \right)
\]

\[
u_{km} = \left\{ \begin{array}{cl}
1 & (k = 0) \\
2 \cos \frac{\pi k (m + 1/2)}{M} & (k \neq 0)
\end{array} \right.
\]

The cumulants of \(n_m\) are given by

\[
\langle n_{m_1} n_{m_2} \cdots n_{m_l} \rangle_c = \frac{\partial^l K}{\partial \theta_1 \cdots \partial \theta_l} \bigg|_{\theta = 0}, \tag{14}
\]

with \(K(\theta, \tau) = \log G_t(s, \tau)\) at \(s_m = e^{\nu_m}\). For example, the first and second order cumulants are calculated to be

\[
\langle n_m \rangle_c = \sum_{m' = 0}^{M-1} \sum_{k = 0}^{M-1} N_{m'} u_{mk} v_{km} e^{-\Omega_k(\tau)}, \tag{15}
\]

\[
\langle n_{m_1} n_{m_2} \rangle_c = \delta_{m_1 m_2} \langle n_{m_1} \rangle_c - \sum_{m' = 0}^{M-1} \sum_{k_1, k_2}^{M-1} N_{m'} u_{m_1 k_1} u_{m_2 k_2} \times v_{k_1 m_1} v_{k_2 m_2} e^{-\Omega_k(\tau) + \Omega_{k(\tau)}}, \tag{16}
\]

respectively.

Next, we take the continuum limit \(a \to 0\). We set the boundaries at \(\eta = \pm \eta_{hot}/2\). Then, the lower space-time
rapidity side of $m$th cell is located at $\eta = (m - M/2)a$. The particle number per unit rapidity is $n(\eta) = n_m/a$. The probability distribution $P(n, \tau)$ in Eq. (4) becomes a functional of the particle number density $n(\eta)$, which is denoted as $P[n(\eta), \tau]$. From Eq. (17), one finds that the average of $n(\eta)$ coincides with the solution of Eq. (5) with $D(\tau) = \gamma(\tau)a^2$. We thus take the continuum limit with fixed $D(\tau)$. Using Eq. (10), it is also confirmed that the Gaussian fluctuation in our model agrees with that of Eq. (5) for sufficiently smooth initial condition in this limit.

With the fixed initial condition $n(\eta, \tau_0) = N(\eta)$, the cumulants of Eq. (3) at proper time $\tau$ are calculated to be

$$\langle(Q(\Delta \eta, \tau))^n\rangle_{c} = \int_{-\eta_{tot}/2}^{\eta_{tot}/2} d\eta \ N(\eta)H_n(\eta), \quad (17)$$

where

$$H_1(\eta) = I(\eta),$$
$$H_2(\eta) = I(\eta) - I(\eta)^2,$$
$$H_3(\eta) = I(\eta) - 3I(\eta)^2 + 2I(\eta)^3,$$
$$H_4(\eta) = I(\eta) - 7I(\eta)^2 + 12I(\eta)^3 - 6I(\eta)^4,$$

with

$$I(\eta) = \frac{\Delta \eta}{\eta_{tot}} \sum_{k=-\infty}^{\infty} \cos \left( \frac{\pi k \eta}{\eta_{tot}} \right) \frac{\sin \left( \frac{\pi k \Delta \eta}{2\eta_{tot}} \right)}{\pi k} \frac{\sin \left( \frac{\pi k \Delta \eta}{2\eta_{tot}} \right)}{\eta_{tot}} \exp \left( -\frac{1}{2} \frac{(\pi k d(\tau))}{\eta_{tot}} \right)^2. \quad (22)$$

Here,

$$d(\tau) = \left[ 2 \int_{\tau_0}^{\tau} d\tau' D(\tau') \right]^{1/2} \quad (23)$$

is the average diffusion length of each Brownian particle.

Next, we extend the above results to the cases with general initial conditions with non-vanishing initial fluctuations. In the following, we also extend the formula to treat cumulants of the difference of densities of two particle species, $n_1(\eta, \tau)$ and $n_2(\eta, \tau)$.

$$Q_{(net)}(\Delta \eta, \tau) = \int_{-\Delta \eta/2}^{\Delta \eta/2} d\eta \ n_1(\eta, \tau) - n_2(\eta, \tau), \quad (24)$$

in order to consider cumulants of conserved charges, which are given by the difference of particle numbers; in the following we assume that the net number Eq. (24) is a conserved charge. If there exists the initial fluctuation $P[N_1(\eta), N_2(\eta), \tau_0] = J[N_1(\eta), N_2(\eta)]$, the probability distribution of a conserved charge $P[n_1(\eta), n_2(\eta), \tau]$ is given by the superposition of the solutions of fixed initial condition

$$P[n_1, n_2, \tau] = \sum_{\{N_1, N_2\}} J[N_1, N_2]P_{N_1}[n_1, \tau]P_{N_2}[n_2, \tau], \quad (25)$$

where $P_{N_1}[n, \tau]$ is the solution of Eq. (4) with the fixed initial condition $n(\eta, \tau_0) = N(\eta)$ and the sum runs over functional space of $N_1(\eta)$ and $N_2(\eta)$. Using Eq. (25) and the same technique used in Ref. [11], one can obtain the cumulants of Eq. (24).

**Initial condition**

Next, let us constrain the initial condition. Because we consider a finite system, the initial condition at $\tau = \tau_0$ should be determined in accordance with the GCC, i.e., the fluctuation of the net particle number in the total system should vanish in the initial condition. In the following, we constrain ourselves to the initial condition satisfying boost invariance between the two boundaries. In order to obtain the initial conditions which conform the GCC, here we model the initial configuration by an equilibrated free classical gas in a finite volume. In this system, using the fact that each particle can be observed at any spatial points in the system with the same probability, one can obtain the correlation functions by taking the continuum limit of the multinomial distribution. The result for the correlation functions of $N_{(net),(tot)}(\eta) = N_1(\eta) \mp N_2(\eta)$ up to the third order is given by

$$\langle N_i(\eta) \rangle_{c} = [N_i]_{c},$$
$$\langle N_{i_1}(\eta_1)N_{i_2}(\eta_2) \rangle_{c} = [N_{i_1}N_{i_2}]_{c}c(\delta(1,2) - 1/\eta_{tot}), \quad (27)$$
$$\langle N_{i_1}(\eta_1)N_{i_2}(\eta_2)N_{i_3}(\eta_3) \rangle_{c} = [N_{i_1}N_{i_2}N_{i_3}]_{c}\{c(\delta(1,2) - 1/\eta_{tot}) \}
- \{c[\delta(1,2) + \delta(2,3) + \delta(3,1)]/\eta_{tot} + 2(1/\eta_{tot})^2\} \quad (28)$$

when at least one of $i_n$ is (net) where the subscript $i_n$ denotes (net) or (tot). Here, $[N_{i_1} \cdots N_{i_n}]_{c}$ are susceptibilities of the initial condition in the grand canonical ensemble [11], and $\delta(j, k) = \delta(\eta_j - \eta_k)$. The expression for fourth and higher order terms are lengthy but obtained straightforwardly. The total number is not conserving and can have event-by-event fluctuation even when the total system is observed. Reflecting this property, we also assume that the number of $N_{(tot)}$ at $\tau = \tau_0$ in the total system obeys the one given by the grand canonical ensemble. One then finds that when all $i_n$ are (tot) in Eqs. (26) - (28) terms containing $\eta_{tot}$ do not appear. We note that Eq. (27) reproduces Eq. (12).

Using these initial conditions and Eq. (17), one obtains the first four cumulants of $Q_{(net)}(\Delta \eta, \tau)$ as
\[ \langle Q_{\text{net}}(\Delta \eta) \rangle_c = [N_{\text{net}}]_c \Delta \eta, \]
\[ \langle (Q_{\text{net}}(\Delta \eta))^2 \rangle_c = [N_{\text{tot}}]_c [\Delta \eta - F_2(\Delta \eta)] - [N_{\text{net}}^2]_c [\Delta \eta p - F_2(\Delta \eta)], \]
\[ \langle (Q_{\text{net}}(\Delta \eta))^3 \rangle_c = [N_{\text{net}}]_c [\Delta \eta - 3F_2(\Delta \eta) + 2F_3(\Delta \eta)] - 3[N_{\text{net}}]N_{\text{tot}}[\Delta \eta p - (1 + p) F_2(\Delta \eta) + F_3(\Delta \eta)] + [N_{\text{net}}^3]_c [2\Delta \eta p^2 - 3p F_2(\Delta \eta) + F_3(\Delta \eta)], \]
\[ \langle (Q_{\text{net}}(\Delta \eta))^4 \rangle_c = [N_{\text{net}}]_c [\Delta \eta - 7F_2(\Delta \eta) + 12F_3(\Delta \eta) - 6F_4(\Delta \eta)] + 3[N_{\text{net}}^2]_c [F_2(\Delta \eta) - 2F_3(\Delta \eta) + F_4(\Delta \eta)] - 4[N_{\text{net}}]_c [\Delta \eta p - (1 + 3p) F_2(\Delta \eta) + (3 + 2p) F_3(\Delta \eta) - 2F_4(\Delta \eta)] + [N_{\text{net}}^2]_c [2\Delta \eta p^2 - p \left( 3 + 2p - \frac{1}{\Delta \eta} F_2(\Delta \eta) \right) F_2(\Delta \eta) + (1 + 2p) F_3(\Delta \eta) - F_4(\Delta \eta)] - [N_{\text{net}}^3]_c \left[ 6\Delta \eta p^3 - 4p \left( 3p - \frac{1}{\Delta \eta} F_2(\Delta \eta) \right) F_2(\Delta \eta) + 3p F_3(\Delta \eta) - F_4(\Delta \eta) \right], \]

with
\[ F_n(\Delta \eta) = \int_{\eta_{\text{hot}}/2}^{\eta_{\text{tot}}/2} d\eta I(\eta)^n, \]

and \( p = \Delta \eta/\eta_{\text{hot}} \). Using \( \lim_{\tau \to \tau_0} F_n(\Delta \eta) = \Delta \eta \), one can check that only the first terms of Eqs. (29) - (32) survive at \( \tau = \tau_0 \). The initial condition Eqs. (20) - (28) thus are reproduced for \( \tau = \tau_0 \).

**Equilibration**

If \( D(\tau) \) is constant, although physically this is not the case, the system settles down to an equilibrium state in the large \( \tau \) limit. The cumulants in this limit is obtained by substituting \( \lim_{\tau \to \infty} F_n = \eta_{\text{hot}} \rho^n \) for \( n \geq 2 \) to Eqs. (29) - (32). One then obtains for second and third order
\[ \langle (Q_{\text{net}}(\Delta \eta))^2 \rangle_c = [N_{\text{tot}}]_c (1 - p), \]
\[ \langle (Q_{\text{net}}(\Delta \eta))^3 \rangle_c = [N_{\text{net}}]_c (1 - p)(1 - 2p), \]
respectively, with \( p = \Delta \eta/\eta_{\text{hot}} \). These \( p \) dependences are consistent with the binomial distribution functions \[19\]; in particular, Eq. (34) is nothing other than Eq. (2). Equation (35) is the generalization of Eq. (2) to the third order. On the other hand, the \( p \) dependence of the fourth order cumulant in the \( \tau \to \infty \) limit does not agree with the binomial form,
\[ \langle (Q_{\text{net}}(\Delta \eta))^4 \rangle_c \propto (1 - p)(1 - 6p + 6p^2), \]
if the initial fluctuation of the total number, \( [N_{\text{tot}}^2]_c \) has a nonzero value.

**EFFECTS OF THE GCC**

Next, let us study how the GCC affects the rapidity window dependence of the cumulants of conserved charges. Because the odd order cumulants are difficult to measure at LHC energy owing to their smallness, in this study we limit our attention only to the second and fourth order cumulants.

The cumulants of conserved charges Eqs. (29) - (32) are described by using three variables \( \Delta \eta, \eta_{\text{hot}}, \) and \( d(\tau) \), having the dimension of space-time rapidity. Among them, the diffusion length \( d(\tau) \) is an increasing function of the elapsed time \( \tau - \tau_0 \). When we describe the time evolution in what follows, we use the dimensionless parameters
\[ L \equiv \eta_{\text{hot}}/d(\tau), \quad T \equiv d(\tau)/\eta_{\text{hot}}. \]
Without initial fluctuation

First, we consider the $\Delta \eta$ dependence for the fixed initial condition, i.e., all fluctuations vanish at $\tau = \tau_0$,

$$[N_{(net)}^2]_c = [N_{(net)}^4]_c = [N_{(tot)}^2]_c = [N_{(tot)}^4]_c = 0.$$  

(38)

In Fig. 1, we show the $\Delta \eta$ dependence of $\langle (Q_{(net)}(\Delta \eta))^2 \rangle_c$, for several values of $L$. The horizontal axis is normalized by $d(\tau)$, while the vertical one is normalized by the equilibrated value in an infinite volume, $\langle Q_{(tot)}(\Delta \eta) \rangle_c$. For comparison, we also show the result with an infinite volume by the dashed and dotted lines. The figure shows that $\langle (Q_{(net)}(\Delta \eta))^2 \rangle_c$ vanishes at $\Delta \eta/d(\tau) = L$ for each $L$, namely $\Delta \eta = \eta_{tot}$, which is a trivial consequence of the GCC. On the other hand, as $\Delta \eta$ becomes smaller from this value, the result with finite $L$ approaches the one with infinite volume. The effect of $L$ vanishes almost completely except for the range

$$\Delta \eta/d(\tau) \gtrsim L - 2.$$  

(39)

This is somewhat an unexpected result compared with the previous estimate Eq. (2).

One can, however, give a physical interpretation to this result as follows. We first note that Eq. (39) is rewritten as

$$\frac{\eta_{tot} - \Delta \eta}{2} \lesssim d(\tau).$$  

(40)

In this expression, the left-hand side is the distance between the left (right) boundary and the left (right) edge of the rapidity window, while the right-hand side is the diffusion length of each Brownian particle. When Eq. (40) is satisfied, particles which are reflected by one of the boundaries at least once can enter the rapidity window. Therefore, the existence of the boundaries can affect the fluctuations of conserved charges in the rapidity window. On the other hand, when the condition Eq. (40) is not satisfied, particles inside the rapidity window do not know the existence of the boundaries, in other words, the fact that the system is finite. In the latter case, therefore, the fluctuations in the rapidity window are free from the effect of the GCC.

Figure 1 also shows that for $L = 3$ the $\Delta \eta$ dependence of $\langle (Q_{(net)}(\Delta \eta))^2 \rangle_c$ becomes almost a linear function; note that this is the behavior consistent with Eq. (2).

For $L = 3$, the diffusion length is almost comparable with the system size. When the condition $d(\tau) \gtrsim \eta_{tot}/2$ is satisfied, each particle can be anywhere in the system with almost an equal probability irrespective of its initial position at $\tau = \tau_0$. Because this is nothing but the condition for the establishment of the equilibration of the system, the estimate Eq. (2), which relies on the equilibration of the system, becomes applicable.

In this analysis, it is assumed that the particle current vanishes at the two boundaries. This assumption would not be suitable to describe the hot medium created by heavy ion collisions, since the hot medium does not have such hard boundaries but only baryon rich regions. From the above discussion, however, it is obvious that the effect of boundaries does not affect the fluctuations in the rapidity window unless the condition Eq. (40) is realized irrespective of the types of the boundaries.

In Figs. 2 and 3, we show $\langle (Q_{(net)}(\Delta \eta))^n \rangle_c/\langle Q_{(tot)}(\Delta \eta) \rangle_c$ for $n = 2$ and 4, respectively, as a function of $\Delta \eta/\eta_{tot}$ with the total
length $\eta_{\text{hot}}$ for five values of $T$ including infinity. When one can estimate the value of $\eta_{\text{hot}}$ in experiments the plots in Figs. 2 and 3 are suitable for comparison with the experiments [17]. The corresponding results in an infinite volume are also plotted for several values of $T$ by the dashed and dotted lines. With fixed $\eta_{\text{hot}}$, larger $T$ corresponds to larger $\tau$. Figure 2 shows that as $T$ increases, both the fluctuation increases and approaches a linear function representing the equilibration, Eq. (2) or (34). Moreover, the comparison of each result with the infinite-volume ones shows that the GCC can affect the fluctuations only for cases with large $\Delta \eta/\eta_{\text{hot}}$. As we shall see later, the largest rapidity windows covered by STAR at the top RHIC energy and ALICE correspond to $\Delta \eta/\eta_{\text{hot}} \approx 0.2$. Figure 2 suggests that for this rapidity coverage, when the value of $(\langle Q_{\text{net}}(\Delta \eta) \rangle)^2$ shows a suppression compared with Eq. (2), the effect of the GCC is negligible.

From Fig. 3 which shows the fourth order cumulants for several values of $T$, one obtains completely the same conclusion on the effect of the GCC. The figure shows that the effect of the GCC is visible only for large values of $\Delta \eta/\eta_{\text{hot}}$. We also notice that in Fig. 3 the $\Delta \eta/\eta_{\text{hot}}$ dependence in the large $\tau$ limit becomes of binomial form, Eq. (39). This limiting behavior, however, is not realized for initial conditions with $[N_{\text{tot}}^2]_c \neq 0$.

### Effect of initial fluctuation

Second, let us look at the $\Delta \eta$ dependence of cumulants with initial conditions having nonvanishing fluctuations. We first consider the effect of the fluctuations of total charge number $[N_{\text{tot}}^2]_c$, which is an observable proposed in Ref. [11] as a new probe for hadronization mechanism. To see the effect of $[N_{\text{tot}}^2]_c$, we set all the cumulants including net-charge at $\tau = \tau_0$ zero, $[N_{\text{net}}^2]_c = [N_{\text{net}}^2]_c = [N_{\text{net}}^2]_c = 0$. The second-order cumulant does not change from the previous result in Fig. 2 by including $[N_{\text{tot}}^2]_c$, but the fourth-order one does. In Fig. 4 we show the $\Delta \eta$ dependence of the fourth-order cumulant with

$$[N_{\text{tot}}^2]_c = [N_{\text{tot}}^2]_c,$$  (41)

which is realized in the Poissonian case. By comparing the results with the ones of the infinite volume shown by the dashed and dotted lines, one obtains the same conclusion on the effect of the GCC as in the previous subsection, i.e., the effect alters the fluctuation only for large $\Delta \eta/\eta_{\text{hot}}$.  

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**FIG. 4:** Fourth order cumulants of conserved charges under the initial condition [11] with five values of $T$.

**FIG. 5:** Second order cumulants of conserved charges under the initial condition [12] with five values of $T$. The experimental results in Ref. [17] with $\eta_{\text{hot}} = 8$ is also plotted by squares for comparison; see, Sec. for a detailed discussion.

**FIG. 6:** Fourth order cumulants of conserved charges under the initial condition [12] with five values of $T$. 


Figure 4 also shows that the $\Delta \eta$ dependence in Fig. 3 is qualitatively different from the one in Fig. 4. For example, although $\langle (Q_{(net)}(\Delta \eta))^4 \rangle_c$ is nonmonotonic and becomes negative in Fig. 4, such behaviors are not observed in Fig. 4, irrespective of the value of $T$. Moreover, these differences are observed even for small $\Delta \eta/\eta_{tot}$. This result indicates that the magnitude of $[N^2_{(tot)}]_c$ can be experimentally estimated by measuring the $\Delta \eta$ dependence of $\langle (Q_{(net)}(\Delta \eta))^4 \rangle_c$ [11].

Next, to see the effect of initial fluctuations of conserved charges, in Figs. 5 and 6 we show the second and fourth order cumulants with the initial condition

\[
\begin{align*}
[N^2_{(net)}]_c &= [N^4_{(net)}]_c = [N^2_{(net)}N_{(tot)}]_c = 0.5[N_{(tot)}], \\
[N^2_{(tot)}]_c &= [N]_c. 
\end{align*}
\]

The value $[N^2_{(net)}]_c = 0.5[N_{(tot)}]$ is taken from the estimate in Ref. [3]. By comparing the results with and without boundaries in these figures, one finds that the effect of GCC is observed already from small $\Delta \eta/\eta_{tot}$ in this case. This is because the initial condition determined in Eqs. (27) and (28) already include the effect of the GCC to some extent. These initial conditions are determined under an assumption that the medium just before the hadronization consists of equilibrated quarks. The effect of the GCC on the diffusion in the hadronic medium with small $\Delta \eta/\eta_{tot}$ is still almost invisible even in this case.

**Comparison with experimental result at ALICE**

We finally inspect the $\Delta \eta$ dependence of the net electric charge fluctuations observed at ALICE [17] in more detail. In order to estimate the effect of the GCC, we must first determine the magnitude of $\eta_{tot}$. From the pseudo-rapidity dependence of charged-particle yield at LHC energy, $\sqrt{s_{NN}} = 2.76$ TeV, in Ref. [31], we take the value $\eta_{tot} = 8$ in the following. Note that this value is considerably smaller than twice the beam rapidity $2\eta_{beam} \approx 16$. While the choice of $\eta_{tot}$ is ambiguous [31], the following discussion is not altered qualitatively by the choice of $\eta_{tot}$ in the range $8 \leq \eta_{tot} \leq 12$. With the choice $\eta_{tot} = 8$, the maximum rapidity coverage of the $2\pi$ detector, TPC, of ALICE, $\Delta \eta = 1.6$ [17], corresponds to $\Delta \eta/\eta_{tot} = 0.2$.

In Fig. 5 we overlay the experimental result of $\langle (Q_{(net)}(\Delta \eta))^2 \rangle_c/\langle (Q_{(tot)}(\Delta \eta))^2 \rangle$ in Fig. 3 of Ref. [17] for the centrality bin $0 - 5\%$ with $\eta_{tot} = 8$. From the figure, one can immediately conclude that the suppression of $\langle (Q_{(net)}(\Delta \eta))^2 \rangle_c$ in this experiment cannot be explained solely by the naive formula of the GCC, Eq. (2). From the discussion in the previous subsections, it is also concluded that the effect of the GCC on the diffusion in the hadronic stage is negligible in this experimental result.

From Fig. 5 one can also estimate that the value of $T$ is within the range $T \simeq 0.04 - 0.06$. Figures 3 and 4 show that with this value of $T$, the dependence of $\langle (Q_{(net)}(\Delta \eta))^4 \rangle_c/\langle (Q_{(tot)}(\Delta \eta))^2 \rangle$ on $\Delta \eta$ for $\Delta \eta/\eta_{tot} \lesssim 0.2$ is sensitive to the initial conditions, such as $[N^2_{(tot)}]_c$ and $[N^n_{(net)}]_c$. In particular, if the fluctuations at the hadronization are well suppressed the change of the sign of $\langle (Q_{(net)}(\Delta \eta))^4 \rangle_c/\langle (Q_{(tot)}(\Delta \eta))^2 \rangle$ will be observed experimentally as in Fig. 3. In this way, the combination of $\Delta \eta$ dependences of the higher order cumulants should be used as an experimental probe to investigate the primordial thermodynamics at LHC energy. Of course, the use of the $\Delta \eta$ dependences of the net baryon number cumulants [19] in addition will provide us more fruitful information.

Using the above estimate on $T$, the value of the diffusion length in the hadronic stage is also estimated as $\langle \tau \rangle = 0.32 - 0.48$. Since the diffusion length is directly related to the diffusion constant through Eq. (23), it is possible to make an estimate on the latter using this relation. First, we assume that the diffusion constant in the cartesian coordinate takes a constant value, $D_H$, in the hadronic medium. The diffusion constant in Eq. (23) is the one in the rapidity coordinate, which is related to $D_H$ as

\[
D(\tau) = D_H \tau^{-2}. \tag{43}
\]

By substituting Eq. (43) in Eq. (23) and carrying out the $\tau$ integral, we obtain

\[
D_H = \frac{d(\tau)^2}{2} \left[ \frac{1}{\tau_0} - \frac{1}{\tau_0} \right]^{-1}. \tag{44}
\]

Second, from the analysis of the dynamical models for LHC energy [32], we estimate the proper times of chemical and kinetic freezeout as $\tau_0 \approx 8 - 12$ fm and $\tau_0 \approx 20 - 30$ fm, respectively. Substituting these values in Eq. (44), we obtain

\[
D_H = 0.6 - 3.5. \tag{45}
\]

Although this is a rough estimate, it is notable that the value is not far from the ones estimated by combining the lattice simulations and the balance function [27].

Finally, we remark the following. In the above estimate we have implicitly assumed that the pseudo-rapidity dependence in Ref. [17] is identical with the one of the space-time rapidity $\eta$. These two rapidities, however, are not identical even under the assumption of Bjorken scaling because the momentum of a particle has a thermal distribution in the rest frame of the medium. Owing to this effect, the distribution $n(\eta)$ in space-time rapidity at $\tau_0$ is blurred in the experimental measurement by the pseudo-rapidity. This transformation acts to make the magnitude of $\langle (Q_{(net)}(\Delta \eta))^2 \rangle_c$ approach the Poissonian one. Since the estimate of the diffusion constant in Eq. (45) is made without this effect, the value
in Eq. (14) will become smaller when the effect is appropriately taken into account. One thus should regard Eq. (15) as the upper limit of the estimate of $D_H$. The effect of the rapidity blurring will be investigated elsewhere. It should also be remembered that the approximate correspondence between pseudo and space-time rapidities relies on the Bjorken scaling. Although the scaling is qualitatively valid at mid-rapidity region for LHC and top-RHIC energies, it is eventually violated as $\sqrt{s_{NN}}$ becomes smaller. When fluctuation observables at small $\sqrt{s_{NN}}$ are investigated, therefore, one should keep this effects in mind as well as the enhancement of the GCC effect owing to the decrease of $y_{tot}$ at small $\sqrt{s_{NN}}$.

SUMMARY

In the present study, we investigated the effect of the GCC on cumulants of conserved charges in heavy ion collisions by studying the time evolution of cumulants in a finite volume system with reflecting boundaries in the space-time rapidity space with the diffusion master equation. Our result shows that the effect of the GCC appears in the range of the diffusion length from the boundaries. This result suggests that the effects of the GCC must be investigated dynamically by taking account of the time evolution of the system generated in heavy ion collisions. By comparing our result with the $\Delta$ dependence of net electric charge fluctuation at ALICE [17], we showed that the effects of the GCC on the diffusion in the hadronic medium on cumulants of conserved charges are almost negligible in the rapidity window available at the ALICE detector.

We also emphasized that the $\Delta \eta$ dependence of fluctuations of conserved charges will tell us information on the properties and the time evolution of the hot medium generated in heavy ion collisions, namely the initial charge distribution, the mechanism of hadronization, and the diffusion constant. Up to now, the second order cumulant of the net electric charge at LHC has been the only fluctuation of conserved charges whose rapidity window dependence was measured. If the measurement of the rapidity window dependences of the higher order cumulants of net electric charge number, as well as those of net baryon number, are performed at both RHIC and LHC, it will make it possible to reveal various aspects of the hot medium created by heavy ion collisions.

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