Abstract

A b-coloring of a graph is a coloring of its vertices such that every color class contains a vertex that has a neighbor in all other classes. The b-chromatic number of a graph is the largest integer \( k \) such that the graph has a b-coloring with \( k \) colors. We show how to compute in polynomial time the b-chromatic number of a graph of girth at least 9. This improves the seminal result of Irving and Manlove on trees.

1 Introduction

Let \( G \) be a simple graph. A proper coloring of \( G \) is an assignment of colors to the vertices of \( G \) such that no two adjacent vertices have the same color. The chromatic number of \( G \) is the minimum integer \( \chi(G) \) such that \( G \) has a proper coloring with \( \chi(G) \) colors. Suppose that we have a proper coloring of \( G \) and there exists a color \( h \) such that every vertex \( x \) with color \( h \) is not adjacent to at least one other color (which may depend on \( x \)); then we can change the color of these vertices and thus obtain a proper coloring with fewer colors. This heuristic can be applied iteratively, but we cannot expect to reach the chromatic number of \( G \), since the coloring problem is \( \mathcal{NP} \)-hard. On the basis of this idea, Irving and Manlove introduced the notion of b-coloring in [15]. Intuitively, a b-coloring is a proper coloring that cannot be improved by the above heuristic, and the b-chromatic number measures the worst possible such coloring. More formally, consider any vertex coloring of \( G \). A vertex \( u \) is said to be a b-vertex (for this coloring) if \( u \) has a neighbor colored with each color different from its own color. A b-coloring of \( G \) is a proper coloring of \( G \) such that each color class contains a b-vertex. A basis of a b-coloring is a set of b-vertices, one for each color class. The b-chromatic number of \( G \) is the largest integer \( k \) such that \( G \) has a b-coloring with \( k \) colors. We denote it by \( \chi_b(G) \).

Naturally, a proper coloring of \( G \) with \( \chi(G) \) colors is a b-coloring of \( G \), since it cannot be improved. Hence, \( \chi(G) \leq \chi_b(G) \). For an upper bound, observe that if \( G \) has a b-coloring with \( k \) colors, then \( G \) has at least \( k \) vertices with

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degree at least \( k - 1 \) (a basis of the b-coloring). Thus, if \( m(G) \) is the largest integer such that \( G \) has at least \( m(G) \) vertices with degree at least \( m(G) - 1 \), we know that \( G \) cannot have a b-coloring with more than \( m(G) \) colors, i.e.,
\[
\chi_b(G) \leq m(G).
\]

This upper bound was introduced by Irving and Manlove in [15]. They showed that the difference between \( \chi_b(G) \) and \( m(G) \) can be arbitrarily large for general graphs. They proved that \( \chi_b(G) \) is equal to \( m(G) \) or \( m(G) - 1 \) when \( G \) is a tree, and provided a polynomial time algorithm that computes \( \chi_b(G) \) for every tree.

In addition, the problem was proved to be NP-hard in general graphs [15], and remains so even when restricted to bipartite graphs [22]. These concepts have received much attention recently; for example, see [1] to [27].

Many of these works investigate the b-chromatic number of graphs under assumptions that involve the existence of large cycles. For example, Irving and Manlove’s algorithm for trees can actually work on graphs with girth at least 11, as noticed by A. Silva in [26]. Also, there are a number of results about \( d \)-regular graphs with girth at least 5 [3, 6, 18, 22, 23]. In this paper we improve Irving and Manlove’s result for graphs with large girth; more specifically, we prove the following.

**Theorem 1.1.** If \( G \) is a graph with girth at least 9, then \( \chi_b(G) \geq m(G) - 1 \).

Here is an outline of the proof of Theorem 1.1. A special set of vertices, called a good set of vertices, is defined and graphs are distinguished between having a good set and not having a good set. Next, we state some results by Irving and Manlove [15] and by A. Silva [26] that say that a graph \( G \) with girth \( (G) \geq 8 \) that does not have a good set cannot be b-colored with \( m(G) \) colors and has a b-coloring with \( m(G) - 1 \) colors (hence, \( \chi_b(G) = m(G) - 1 \)); also, A. Silva proved that if \( G \) with girth at least 8 has a good set, then one can be found in polynomial time. Finally, and this is the original part of the paper, it is shown that if \( G \) with girth at least 9 has a good set, then \( \chi_b(G) = m(G) \). The proof of Theorem 1.1 yields a polynomial time algorithm that finds an optimal b-coloring of graphs with girth at least 9.

## 2 Definitions and partial results

In this section, we present some necessary definitions and the results by Irving and Manlove [15] and A. Silva [26] that complement our proof. The graph terminology used in this paper follows [4].

Let \( G \) be a simple graph. We denote by \( V(G) \) and \( E(G) \) the sets of vertices and edges of \( G \), respectively. If \( X \subseteq V(G) \), then \( N^X(u) \) represents the set \( N(u) \cap X \). The girth of \( G \) is the size of a shortest induced cycle of \( G \).

Recall that \( m(G) \) is the largest integer \( k \) such that \( G \) has at least \( k \) vertices with degree at least \( k - 1 \). We say that a vertex \( u \in V(G) \) is dense if \( d(u) \geq m(G) - 1 \); and we denote the set of dense vertices of \( G \) by \( M(G) \).
Let $W$ be a subset of $M(G)$, and let $u$ be any vertex in $V(G) \setminus W$. If $u$ is such that every vertex $v \in W$ is either adjacent to $u$ or has a common neighbor $w \in W$ with $u$ such that $d(w) = m(G) - 1$, then it is said that $W$ encircles vertex $u$ (or that $u$ is encircled by $W$). A subset $W$ of $M(G)$ of size $m(G)$ is a good set if (our definition is slightly different from the one given by Irving and Manlove):

(a) $W$ does not encircle any vertex, and
(b) Every vertex $x \in V(G) \setminus W$ with $d(x) \geq m(G)$ is adjacent to a vertex $w \in W$.

**Theorem 2.1** (15). Let $G$ be any graph and $W$ be a subset of $M(G)$ with $m(G)$ vertices. If $W$ encircles some vertex $v \in V(G) \setminus W$, then $W$ is not a basis of a $b$-coloring with $m(G)$ colors.

**Theorem 2.2** (26). If $G$ is a graph with girth at least 8, then $G$ does not have a good set if and only if $|M(G)| = m(G)$ and $M(G)$ encircles a vertex in $V(G) \setminus M(G)$. Moreover, a good set of $G$ (if any exists) can be found in polynomial time.

A part of the proof of Theorem 2.1 consists of the following theorem:

**Theorem 2.3** (26). Let $G$ be a graph with girth at least 8. If $G$ has no good set, then $\chi_b(G) = m(G) - 1$.

Now, all we need to prove is that if $G$ does have a good set, then $G$ can be $b$-colored with $m(G)$ colors, which is done in the next section.

## 3 Coloring graphs with a good set

In this section we prove the second part of the main theorem, namely:

**Theorem 3.1.** Let $G$ be a graph with girth at least 9. If $G$ has a good set, then $\chi_b(G) = m(G)$.

Let $W = \{v_1, \ldots, v_{m(G)}\}$ be a good set of $G$. Our aim is to construct a $b$-coloring of $G$ with $m(G)$ colors such that, for each $i \in \{1, \ldots, m(G)\}$, vertex $v_i$ is a $b$-vertex of color $i$. We start by assigning color $i$ to $v_i$, for each $i \in \{1, \ldots, m(G)\}$. Next, we extend this partial coloring to the rest of the graph in several steps. Before explaining each step, we need to introduce some other terminology and notation.

A link is any path of length two or three whose extremities are in $W$ and whose internal vertices are not in $W$. Any interior vertex of a link is called a link vertex. Let $L$ be the set of all link vertices.

We first color $G[W \cup L]$ in a way not to repeat too many colors in $N(w)$, for all $w \in W$, and at the end we extend the obtained partial coloring to a $b$-coloring of $G$ with $m(G)$ colors. Let $G' = G[W \cup L]$, $L_1$ be the set of vertices of $L$ that have at least one neighbour in $L$ and $L_2$ be the set of vertices in $L$ that have at least two neighbours in $W$. The steps below are followed in order in such a way that we only move on to the next step when all the possible vertices are iterated.
1. For each \( x \in L_1 \), let \( x' \in N^L(x) \). Since \( x' \in L \), there must exist \( v_i \in N^W(x') \); color \( x \) with \( i \);

2. For each \( v_i \in W \), let \( N^*_i = N(v_i) \cap L_2 = \{x_1, \ldots, x_q\} \). Also, let \( v_{ij} \in N^W(x_j) \setminus \{v_i\} \). If \( q > 1 \), then use colors \( i_1, \ldots, i_q \) to color the uncolored vertices in \( N^*_i \) in a way that \( x_j \) is not colored with \( i_j \) (it suffices to make a derangement of those colors on the vertices);

3. Let \( x \in L_2 \) still uncolored be such that there exists \( v_i \in N^W(x) \) that has some neighbor \( y \in L_1 \). Let \( c \) be the color of \( y \); color \( x \) with \( c \) and recolor \( y \) with \( j \), for any \( v_j \in N^W(x) \setminus \{v_i\} \);

4. Finally, if \( x \in L_2 \) is still uncolored, we know that \( N^L(v_i) = \{x\} \), for all \( v_i \in N^W(x) \). Since \( N^L(x) = \emptyset \), we can color \( x \) with \( i \), for any \( v_i \) that is not adjacent to \( x \) and has no common neighbor with \( x \) in \( W \) of degree \( m(G) - 1 \), which exists as \( x \) is not encircled by \( W \).

Suppose that the algorithm above produces a partial coloring that colors every vertex in \( L \) in such a way that, at the end, each \( v_i \in W \) has at least as many uncolored neighbors as missing colors in its neighborhood. Since \( L \) is colored, we know that the uncolored neighbors of \( W \) form a stable set. Thus, we can independently color \( N(v_i) \) in such a way that \( v_i \) sees every other color, for all \( v_i \in W \). By the definition of a good set, we know that if \( d(v) \geq m(G) \), then \( v \) is already colored; hence, the partial coloring can be greedily transformed into a \( b \)-coloring with \( m(G) \) colors. Now, to prove that the algorithm works, we show that after these steps the obtained partial coloring \( \psi \) satisfies:

\( P1 \) \( \psi \) is proper; and

\( P2 \) the number of uncolored neighbors of \( v_i \) is at least the number of missing colors in \( N(v_i) \), for each \( v_i \in W \).

**Proof of Theorem 3.1**

First, we make some observations concerning the coloring procedure. Note that \( L_1 \cap L_2 \) is not necessarily empty, but all vertices in this subset are colored in Step 1. However, a vertex \( x \in L_1 \cap L_2 \) may play a role in Step 2 in the following way: if \( x \in N(v_i) \) and there exists \( x' \in N^{L_2}(v_i) \setminus L_1 \), then \( x' \) may be colored with color \( j \) for some \( v_j \in N^W(x) \setminus \{v_i\} \), while the color of \( x \) remains unchanged. Also, note that, in Step 3, since \( N^{L_2}(v_i) = \{x\} \), we have \( y \in L_1 \setminus L_2 \). Hence, \( N^W(y) = \{v_i\} \) and, consequently, the color of \( y \) cannot be changed again. Thus (*) the color of \( y \) is changed at most once, for every \( y \in L_1 \). Finally, if \( x \) receives color \( i \) in Step 1, 2 or 3, then one of the following holds (fact (iii) holds because of (*)):

(i) \( x \) receives color \( i \) in Step 1 and there exists a path \( \langle x, x', v_i \rangle \), for some \( x' \in L_1 \); or

(ii) \( x \) receives color \( i \) in Step 2 and there exists a path \( \langle x, v_j, x', v_i \rangle \), for some \( v_j \in W \) and \( x' \in L_2 \); or
(iii) $x$ receives color $i$ in Step 3 and there exists a path $\langle x, v_j, y, y', v_i \rangle$, for some $v_j \in W$, $y \in L_1 \setminus L_2$ and $y' \in L_1$; or

(iv) $x$ is recolored with color $i$ in Step 3 and there exists a path $\langle x, v_j, x', v_i \rangle$, for some $v_j \in W$ and $x' \in L_2 \setminus L_1$.

We first prove that P1 holds after Step 3. Suppose that there exists an edge $wz$ such that $\psi(w) = \psi(z) = i$. Since $G$ has no cycle of length at most 7, the paths defined in (i)-(iv) are shortest paths. Therefore, vertex $v_i$ has no neighbor colored $i$ and hence, $w, z \in L$. Also, as $wz \in E(G)$, we have $w, z \in L_1$ and they are colored in Step 1 and maybe recolored in Step 3. By (i) and (iv), there exist a $w, v_i$-path $P_w$ and a $z, v_i$-path $P_z$, both of length at most 3. Note that either $P_w + P_z + wz$ contains a cycle of length at most 7 or one of these paths consists of the edge $wz$ followed by the other path. Because $G$ has girth at least 9, the latter case occurs. We get as contradiction as this implies that at least one path is defined by (i) and, thus, vertex $v_i$ has a neighbor colored $i$.

Now, we prove that P2 also holds after Step 3. We actually prove that, after Step 3, no color is repeated in $N(v_i)$, for each $v_i \in W$. Suppose there exist a vertex $v_j \in W$ and $w, z \in N(v_j)$ such that $\psi(w) = \psi(z) = i$. First, consider the case $v_i \in \{w, z\}$. Since the paths defined by (i)-(iv) are shortest paths, we have that (i) occurs for the vertex in $\{w, z\} \setminus \{v_i\}$. We get a contradiction as this implies $G$ has a cycle of length 4. Therefore we may assume $v_j \notin \{w, z\}$.

Now, by (i)-(iv), there exist a $w, v_i$-path $P_w$ and a $z, v_i$-path $P_z$. Let $\ell_w$ and $\ell_z$ be the length of $P_w$ and $P_z$, respectively. Clearly $\ell_w, \ell_z \leq 4$. Note that either $P_w + P_z + \langle w, v_j, z \rangle$ contains a cycle of length at most $\ell_w + \ell_z + 2$ or either $P_w$ or $P_z$ consists of the path $\langle w, v_j, z \rangle$ followed by the other path. Since both $w$ and $z$ are at distance at least 2 from $v_i$ and $\ell_w, \ell_z \leq 4$, the latter can only occur if one of the paths is defined by (i), say $P_w$, and the other is defined by (iii), say $P_z$. We get a contradiction as $P_z = \langle z, v_j, w, y, v_i \rangle$ implies $w$ is recolored in Step 3 and therefore, $P_w$ must be defined by (iv). Now, suppose that the former occurs, i.e., $P_w + P_z + \langle w, v_j, z \rangle$ contains a cycle of length at most $\ell_w + \ell_z + 2$. Because $G$ has girth at least 9, we have $\ell_w + \ell_z \geq 7$. This implies that at least one of $P_w$ and $P_z$, say $P_z$, is defined by (iii), and the other is not defined by (i). Therefore $z$ is colored in Step 3 and $N^{L_z}(v_j) = \{z\}$. Furthermore, $w \in L_1 \setminus L_2$ and $N^W(w) = \{v_j\}$. Therefore, since (i) does not occur for $w$, we have that $P_w$ must be defined by (iv). Thus the only choice for $P_w$ is $\langle w, v_j, z, v_i \rangle$, a contradiction as P1 holds.

Finally, consider $x$ to be colored during Step 4 with color $i$. By the choice of $i$ we know that $v_i \notin N(x)$. Thus, since $N^L(x) = \emptyset$, Property P1 holds. Now, suppose that some $v_j \in N(x)$ is such that color $i$ already appears in $N(v_j)$. Since $N^L(v_j) = \{x\}$ we must have $v_i \in N(v_j)$ and, by the choice of $i$, $d(v_j) > m(G) - 1$. Property P2 thus follows as $i$ is the only repeated color in the neighborhood of $v_j$. \qed
4 Conclusion

We showed that if $G$ is a graph with girth at least 9, then $\chi_b(G) \geq m(G) - 1$, improving the result by Irving and Manlove [15]. We also give an algorithm that finds the $b$-chromatic number of $G$ in polynomial time.

In [25], Maffray and Silva conjecture that any graph $G$ with no $K_{2,3}$ as subgraph has $b$-chromatic number at least $m(G) - 1$. Observe that these graphs contain all graphs with girth at least 9; thus, we have given a partial answer to their conjecture. Actually, if their conjecture holds, then $\chi_b \geq m(G) - 1$ holds for every $G$ with girth at least 5. However, a different approach is needed as our proof strongly relies on the fact that girth($G$) $\geq 9$. Moreover, Theorem 3.1 does not hold for an infinite family of cacti with girth 5, as can be seen in [7]. This means that the situation where $G$ has no good set is not the only situation where a graph $G$ with girth at least 5 cannot be $b$-colored with $m(G)$ colors.

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References

[1] R. Balakrishnan and S. Francis Raj. Bounds for the $b$-chromatic number of vertex-deleted subgraphs and the extremal graphs (extended abstract). Electron. Notes Discrete Math. 34, 353–358, 2009.

[2] D. Barth, J. Cohen and T. Faik. On the $b$-continuity property of graphs. Discrete Appl. Math. 155, 1761–1768, 2007.

[3] M. Blidia, F. Maffray and Z. Zemir. On $b$-colorings in regular graphs. Discrete Appl. Math. 157 (8), 1787–1793, 2009.

[4] A. Bondy and U.S.R. Murty. Graph Theory. Spring-Verlag Press, 2008.

[5] F. Bonomo, G. Duran, F. Maffray, J. Marenco and M. Valencia-Pabon. On the $b$-coloring of cographs and $P_4$-sparse graphs. Graphs and Combin. 25 (2), 153–167, 2009.

[6] S. Cabello and M. Jakovac. On the $b$-chromatic number of regular graphs. Discrete Appl. Math. 159 (3), 1303–1310, 2011.

[7] V. Campos, C. Linhares Sales, F. Maffray, and A. Silva. $b$-chromatic number of cacti. Electron. Notes Discrete Math. 35, 281–286, 2009.

[8] F. Chaouche and A. Berrachedi. Some bounds for the $b$-chromatic number of a generalized Hamming graphs. Far East J. Appl. Math. 26, 375–391, 2007.
[9] S. Corteel, M. Valencia-Pabon and J-C. Vera. On approximating the b-chromatic number. Discrete Appl. Math. 146, 106–110, 2005.

[10] B. Effantin. The b-chromatic number of power graphs of complete caterpillars. J. Discrete Math. Sci. Cryptogr. 8, 483–502, 2005.

[11] B. Effantin and H. Kheddouci. The b-chromatic number of some power graphs. Discrete Math. Theor. Comput. Sci. 6, 45–54, 2003.

[12] B. Effantin and H. Kheddouci. Exact values for the b-chromatic number of a power complete k-ary tree. J. Discrete Math. Sci. Cryptogr. 8, 117–129, 2005.

[13] H. Hajiabolhassan. On the b-chromatic number of Kneser graphs. Discrete Appl. Math. 158, 232–234, 2010.

[14] C. T. Hoang and M. Kouider. On the b-dominating coloring of graphs. Discrete Appl. Math. 152, 176–186, 2005.

[15] R.W. Irving and D.F. Manlove. The b-chromatic number of a graph. Discrete Appl. Math. 91, 127–141, 1999.

[16] R. Javadi and B. Omoomi. On b-coloring of the Kneser graphs. Discrete Math. 309, 4399–4408, 2009.

[17] M. Jakovac and S. Klavzar. The b-chromatic number of cubic graphs. Graphs and Combin. 26, 107–118, 2010.

[18] M. Kouider. b-chromatic number of a graph, subgraphs and degrees. Technical Report 1392, Université Paris Sud, 2004.

[19] M. Kouider and M. Maheo. Some bounds for the b-chromatic number of a graph. Discrete Math. 256, 267–277, 2002.

[20] M. Kouider and M. Maheo. The b-chromatic number of the cartesian product of two graphs. Studia Sci. Math. Hungar. 44, 49–55, 2007.

[21] M. Kouider and M. Zaker. Bounds for the b-chromatic number of some families of graphs. Discrete Math. 306, 617–623, 2006.

[22] J. Kratochvíl, Zs. Tuza, and M. Voigt. On the b-chromatic number of graphs. Lecture Notes In Computer Science 2573, 310–320, 2002.

[23] M. Kouider and A.E. Sahili. About b-colouring of regular graphs. Technical Report 1432, Université Paris Sud, 2006.

[24] Saaed Shaebani. On the b-chromatic number of regular graphs without 4-cycles. Submitted to Discrete Appl. Math..

[25] F. Maffray and A. Silva. b-colouring outerplanar graphs with large girth. To appear in Discrete Math., 2012.
[26] A. Silva. The b-chromatic number of some tree-like graphs. PhD Thesis, Université de Grenoble, 2010.

[27] C. I. B. Velasquez, F. Bonomo and I. Koch. On the b-coloring of P4-tidy graphs. *Discrete Appl. Math.* 159, 60–68, 2011.