Status of the lower spins in the Rarita-Schwinger four-vector spinor $\psi_\mu$ within the method of the combined Lorentz- and Poincaré invariant projectors

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We investigate the status of the lower spin-$1/2$ companions to spin-$3/2$ within the four-vector spinor, $\psi_\mu$. According to its reducibility, $\psi_\mu \rightarrow [(1/2, 1) \oplus (1, 1/2)] \oplus [(1/2, 0) \oplus (0, 1/2)]$ this representation space contains two spin-$1/2$ sectors, the first one transforming as a genuine Dirac-spinor, $(1/2, 0) \oplus (0, 1/2)$, and the second as the companion to spin-$3/2$ in $(1/2, 1) \oplus (1, 1/2)$.

In order to correctly identify the covariant spin-$1/2$ degrees of freedom in the Rarita-Schwinger field of interest we exploit the properties of the Casimir invariants of the Lorentz algebra to distinguish between the irreducible Dirac- and $(1/2, 1) \oplus (1, 1/2)$ representation spaces and construct corresponding momentum-independent (static) projectors which we then combine with a dynamical spin-$1/2$ Poincaré covariant projector, based on the two Casimir invariants of the Poincaré algebra—the squared momentum, and the squared Pauli-Lubanski vector. In so doing we obtain two spin-$1/2$ wave equation, and prove them to describe causal propagation of the wave fronts within an electromagnetic field. We furthermore calculate Compton scattering off each one of the above states, and find that the amplitudes corresponding to the first spin-$1/2$ are identical to those of a Dirac particle and conclude on the observability of this state. Also for the second spin-$1/2$ we find finite cross sections in all directions in the ultrarelativistic limit, and conclude that its observability is not excluded neither by causality of propagation within an electromagnetic environment, nor by unitarity of the Compton scattering amplitudes in the ultraviolet. Finally, we notice that the method of the combined Lorentz- and Poincaré invariant projectors could be instrumental in opening a new avenue toward the consistent description of any spin by means of second order Lagrangians written in terms of sufficiently large reducible representation spaces equipped with separate Lorentz– and Dirac indices. Specifically, the antisymmetric Lorentz tensor of second rank with either Dirac-spinor components, $\psi_{[\mu\nu]}$, or, with four-vector spinor components, $\psi_{[\mu\nu]}\eta$, can be employed in the description of single spin $(3/2, 0) \oplus (0, 3/2)$, or spin-5/2 as part of $(2, 1/2) \oplus (1/2, 2)$, respectively.

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I. INTRODUCTION

The theory of fields with spins $s \geq 1$ is mainly based on Lorentz-algebra representation spaces of multiple spins (and parities) which are of the type $(j,j)$ for bosons with spin $s = 2j$, and of the type $(j,j) \otimes [(1/2,0) \oplus (0,1/2)]$ for fermions with spin $s = (2j + 1/2)$ [1]. The particles of interest are associated with the highest spins in the spaces under discussion, while their lower spin companions have to be projected out in order to ensure the correct spins as well [2]. In supergravity one encounters a metric and a four-vector-spinor Rarita-Schwinger field that can contain fields of lower spins within this context has been noticed in the dressing of the spin-3/2 propagator within the framework of the Schwinger-Dyson equation [2], [3]. Moreover, in Compton scattering off the $\Delta(1232)$ resonance [4], they seem to provide contributions to the non-resonant background and give rise to interference effects. Finally, in theories of supergravity one encounters a metric and a four-vector-spinor Rarita-Schwinger field that can contain fields of lower spins as well [5]. For all these reasons clarifying the status of the lower spin sector in the spin-3/2 Rarita-Schwinger field appears timely. It is the goal of the present work to contribute to that clarification.

The spin-3/2 Rarita-Schwinger field is Lorentz transformed as a four-vector with Dirac-spinor components, $\psi_\mu \sim (1/2,1/2) \otimes [(1/2,0) \oplus (0,1/2)]$. This Lorentz invariant representation space is reducible according to

$$
(1/2,1/2) \otimes [(1/2,0) \oplus (0,1/2)] \rightarrow [(1/2,0) \oplus (0,1/2)] \oplus [(1/2,1) \oplus (1,1/2)],
$$

and the challenge is to construct the space under discussion in such a way that it guarantees that the first and second spin-1/2 companions to spin-3/2 correctly transform as $(1/2,0) \oplus (0,1/2)$, and $(1/2,1) \oplus (1,1/2)$, respectively. To achieve our goal we exploit the properties of the Casimir invariants of the Lorentz algebra to distinguish between the above irreducible representation spaces [6] and construct corresponding projectors which bring the great advantage of being momentum-independent, and which efficiently separate the pure-spin-1/2 sector from $(1/2,1) \oplus (1,1/2)$. Then the spin-1/2 residing in the latter representation space is separated from its spin-3/2 companion by means of the second order Poincaré covariant projector from [7], an operator based on the two Casimir invariants of the Poincaré algebra, the squared momentum, and the squared Pauli-Lubanski vector. In so doing, we find two spin-1/2 wave equations which we show to describe causal propagation of the respective wave fronts in the presence of an electromagnetic field. We furthermore calculate Compton scattering off both states, and in finding that the amplitudes corresponding to the first spin-1/2 are identical to those of a genuine Dirac particle, conclude on the observability of this very state. Also for the second one we find finite cross sections in all directions in the ultrarelativistic limit, and conclude that its observability is not excluded neither by causality of propagation within an electromagnetic environment, nor by unitarity of the Compton scattering amplitudes.

The paper is organized as follows. In the next section we present the construction of the momentum-independent Lorentz projectors and briefly review the recently developed Poincaré covariant projector method that has proved successful in the consistent description of various processes [8]–[9] including particles with spins ranging from 1/2 to 3/2. There we furthermore present the combined Lorentz- and Poincaré invariant projectors, develop the formalism
for spin-1/2 description in $\psi_{\mu}$ and obtain the corresponding wave equations. In section III we solve the spin-
1/2 wave equations from the previous section. In section IV we test the causality of the emerging wave equations
describing the spin-1/2 degrees of freedom in $\psi_{\mu}$, obtain the associated Lagrangians and describe the electromagnetic
properties of the lower spins in the four-vector spinor. Section V is devoted to the calculation of Compton scattering
off spin-1/2 transforming as $(1/2, 0) \oplus (0, 1/2)$, or $(1/2, 1) \oplus (1, 1/2)$. The paper closes with brief conclusions.

II. THE METHOD OF THE COMBINED LORENTZ- AND POINCARÉ INVARIANT PROJECTORS

Equations of motion for particles transforming according to a given Lorentz-algebra representation space, \{\kappa\} spanned by the generic degrees of freedom $\psi(p, \lambda)$, with $p$ denoting the three-dimensional momentum, and $\lambda$ standing for the set of quantum numbers characterizing the degrees of freedom under consideration, are straightforwardly constructed from a projector (see [8] for more details), call it $\Pi_{\{\kappa\}}(p)$, as

$$\Pi_{\{\kappa\}}(p)\psi(p, \lambda) = \psi(p, \lambda). \tag{2.1}$$

In single-spin valued representation spaces, such as the Dirac one, it is customary to use for $\Pi_{\{\kappa\}}(p, \lambda)$ one of the parity projectors, $(\pm p^2 + m^2)/2m$. On the contrary, in order to track down the desired spin-$s$, within an irreducible representation space containing $N$ spin-sectors, $s_1, ..., s_N$, the projector $\Pi_{\{\kappa\}}^{+}(p, \lambda)$ has to be taken as the product of $(N - 1)$ covariant spin-projectors of the type designed in refs. [8]-[9] on the basis of the two Casimir invariants of the Poincaré group, the squared four-momentum, $P^2$, and the squared Pauli-Lubanski vector, $W^2$. The disadvantage is that in so doing one ends up with uncomfortable to deal with equations of the order $P^2(N - 1)$. In the present study we reveal the advantages of describing spin-$s$ in terms of sufficiently large reducible representation spaces equipped by separate Lorentz- and Dirac spinor indices and such that the desired spin is contained either within a single-, or within a maximally two-spin valued irreducible subspace. The two spins within the latter subspace can be separated by a Poincaré covariant projector, which is second order in the momenta. All the remaining subspaces can be distinguished by the Casimir invariants of the Lorentz algebra and removed by properly constructed momentum-independent projectors. In effect, $\Pi_{\{\kappa\}}(p, \lambda)$ can be furnished as a product of a single Poincaré covariant projector that is of second order in the momenta and several momentum independent Lorentz projectors which results in quadratic wave equations for any spin. Below we shall illustrate the above concept on the example of the four-vector spinor Rarita-Schwinger representation space that is of prime interest to the present study. Our point is that the Lorentz-invariant projectors are indispensable for the correct identification of the spin-1/2 degrees of freedom in $\psi_{\mu}$ that transform irreducibly under Lorentz transformations.

A. Poincaré covariant spin-$s$ projectors

The Poincaré covariant projector method [7] relies upon the two Casimir operators of the Poincaré algebra, the squared four-momentum, $P^2$, and the squared Pauli-Lubanski vector, $W^2$. These operators fix in their turn the mass-$m$, and the spin-$s$ quantum numbers of the states, here denoted by $w_{(m,s)}$, transforming according to the representation space of interest. One has

$$P^2 w_{(m,s)} = m^2 w_{(m,s)}, \tag{2.2}$$

$$W^2 w_{(m,s)} = -p^2 s(s + 1) w_{(m,s)}. \tag{2.3}$$

The $W^2$ Casimir invariant is constructed from the elements, $M_{\mu\nu}$, of the Lorentz algebra in the representation of interest. Specifically for the four-vector spinor one finds,

$$[W^2]_{\alpha\beta} = [W^\mu]_{\alpha\gamma}[W_{\mu}]_{\gamma\beta} = [T_{\mu\nu}]_{\alpha\beta}p^\mu p^\nu, \tag{2.4}$$

$$[T_{\mu\nu}]_{\alpha\beta} = \frac{1}{4} \epsilon^{\alpha\beta\gamma\delta}p_\gamma p_\delta [M^\lambda]_{\alpha\beta} [M^\kappa]_{\delta\gamma}, \tag{2.5}$$

$$[M_{\mu\nu}]_{\alpha\beta} = [M^V_{\mu\nu}]_{\alpha\beta} + g_{\alpha\beta} M_{\mu\nu}^S. \tag{2.6}$$

where $[M^V_{\mu\nu}]_{\alpha\beta}$ and $[M^S_{\mu\nu}]$ represent the Lorentz-group generators within the respective four-vector–, and the Dirac-spinor building blocks. Their explicit forms read

$$[M^V_{\mu\nu}]_{\alpha\beta} = i(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}), \tag{2.7}$$

$$M_{\mu\nu}^S = \frac{1}{2} \sigma_{\mu\nu} = i[\gamma_{\mu}, \gamma_{\nu}], \tag{2.8}$$
with $\gamma_\mu$ being the standard Dirac matrices. Notice that the operators in (2.4),(2.5) and (2.6) are $16 \times 16$ matrices just as are the generators in $\psi_\mu$, and consequently carry next to the Lorentz indices, also spinor indices, here suppressed for the sake of simplifying notations. The Rarita-Schwinger representation contains only two spin sectors, namely $s_1 = 1/2$ and $s_2 = 3/2$, and one can construct projectors, $P_{W^2}^{(m,s)}$, over spin and mass in terms of the $P^2$ and $W^2$ operators as follows (see [7] for the details):

$$P_{W^2}^{(m,1/2)} w^{(m,1/2)} = \frac{P^2}{m^2} \left( \frac{W^2 - \epsilon_{3/2}}{\epsilon_{1/2} - \epsilon_{3/2}} \right) w^{(m,1/2)} = w^{(m,1/2)},$$

(2.9)

$$P_{W^2}^{(m,3/2)} w^{(m,3/2)} = \frac{P^2}{m^2} \left( \frac{W^2 - \epsilon_{1/2}}{\epsilon_{3/2} - \epsilon_{1/2}} \right) w^{(m,3/2)} = w^{(m,3/2)}.$$  

(2.10)

Here, $\epsilon_s = -p^2 s(s + 1)$ is the $W^2$ eigenvalue corresponding to $w^{(m,s)}$, the mass-$m$ and spin-$s$ eigenstates to the operators $P^2$ and $W^2$. With the aid of (2.4)–(2.8) we find

$$[P_{W^2}^{(m,1/2)}(p)]_{\alpha\beta} = \frac{1}{3m^2} \left[ (T_{\mu\nu})_{\alpha\beta} + \frac{15}{4} g_{\alpha\beta} g_{\mu\nu} \right] p^\mu p^\nu = \frac{p^2}{m^2} [P^{(1/2)}(p)]_{\alpha\beta},$$

(2.11)

$$[P_{W^2}^{(m,3/2)}(p)]_{\alpha\beta} = -\frac{1}{3m^2} \left( (T_{\mu\nu})_{\alpha\beta} + \frac{3}{4} g_{\alpha\beta} g_{\mu\nu} \right) p^\mu p^\nu = \frac{p^2}{m^2} [P^{(3/2)}(p)]_{\alpha\beta},$$

(2.12)

where, as it will be made explicit later, the operators $[P^{(s)}(p)]_{\alpha\beta}$ project over states of spin-$s$. Expressions for the latter can be found, among others, in Ref. [10] and read:

$$\left[ P^{(1/2)}(p) \right]_{\alpha\beta} = \frac{1}{3} \gamma_\alpha \gamma_\beta + \frac{1}{3p^2} (\not{p} \gamma_\alpha p_\beta + p_\alpha \gamma_\beta \not{p}),$$

(2.13)

$$\left[ P^{(3/2)}(p) \right]_{\alpha\beta} = g_{\alpha\beta} - \frac{1}{3} \gamma_\alpha \gamma_\beta - \frac{1}{3p^2} (\not{p} \gamma_\alpha p_\beta + p_\alpha \gamma_\beta \not{p}).$$

(2.14)

The equation of motion for spin-$s$ in $\psi_\mu$, resulting from (2.1), (2.9), and (2.10) is

$$\left( (T_{\mu\nu})_{\alpha\beta} p^\mu p^\nu - m^2 g_{\alpha\beta} \right) [w^{(s)}]_{\beta} = 0, \quad s = \frac{1}{2}, \frac{3}{2},$$

(2.15)

where

$$[T^{(1/2)}_{\mu\nu}]_{\alpha\beta} = \frac{1}{3} \left( (T_{\mu\nu})_{\alpha\beta} + \frac{15}{4} g_{\alpha\beta} g_{\mu\nu} \right),$$

(2.16)

$$[T^{(3/2)}_{\mu\nu}]_{\alpha\beta} = -\frac{1}{3} \left( (T_{\mu\nu})_{\alpha\beta} + \frac{3}{4} g_{\alpha\beta} g_{\mu\nu} \right).$$

(2.17)

Carrying out the contractions amounts to the following free equations of motion:

$$\left( \frac{1}{3} \gamma_\alpha \gamma_\beta p^2 + \frac{1}{3} (\not{p} \gamma_\alpha p_\beta + p_\alpha \gamma_\beta \not{p}) - m^2 g_{\alpha\beta} \right) [w^{(1/2)}]_{\beta} = 0,$$

(2.18)

$$\left( \frac{1}{3} \gamma_\alpha \gamma_\beta p^2 - \frac{1}{3} (\not{p} \gamma_\alpha p_\beta + p_\alpha \gamma_\beta \not{p}) - (m^2 - p^2) g_{\alpha\beta} \right) [w^{(3/2)}]_{\beta} = 0.$$  

(2.19)

Notice that the Poincaré covariant projector method designs the spin-$s$ description on the basis of the transformation properties of the free particle. It therefore leaves their electromagnetic properties unspecified and allows to replace $[T^{(s)}_{\mu\nu}]_{\alpha\beta}$ by tensors which are equivalent on-shell, though become distinct upon introducing interactions. The electromagnetic constants have to be fixed at a later stage by some properly chosen dynamical constraints. As already announced in the previous section, from now on we focus on the two spin-$1/2$ sectors in $\psi_\mu$.

### B. Momentum independent Lorentz invariant projectors on irreducible representation spaces

The spin-$1/2$ projectors composed in the preceding section from the Casimir invariants of the Poincaré algebra are indifferent to the reducibility of the Lorentz representation and one can not expect that their eigenvectors transform
irreducibly. To ensure the correct Lorentz transformation properties of the two spin-1/2 degrees of freedom in \( \psi_\mu \), it is necessary to construct additional projectors based upon the Casimir invariants of the Lorentz algebra itself. The Lorentz algebra has two Casimir operators, usually denoted by \( F \) and \( G \), and given by \( [6] \)

\[
[F]_{\alpha\beta} = \frac{1}{4} [M^{\mu\nu}]_{\alpha} \gamma^\nu [M_{\mu\nu}]_{\gamma\beta} = \frac{9}{4} g_{\alpha\beta} + \frac{i}{2} \sigma_{\alpha\beta}, \tag{2.20}
\]

\[
[G]_{\alpha\beta} = \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} [M^{\mu\nu}]_{\alpha} \gamma^\rho [M_{\rho\sigma}]_{\gamma\beta} = \gamma^5 \sigma_{\alpha\beta} - \frac{3}{2} i \gamma^5 g_{\alpha\beta}, \tag{2.21}
\]

where use has been made of (2.6) to simplify the expressions. Their respective eigenvalue problems read,

\[
F w_{(j_1,j_2)} = \frac{1}{2} (K(K + 2) + M^2) w_{(j_1,j_2)},
\]

\[
G w_{(j_1,j_2)} = i M (K + 1) w_{(j_1,j_2)}, \tag{2.22}
\]

where \( w_{(j_1,j_2)} \) are the states transforming irreducibly as \((j_2, j_1) \oplus (j_1, j_2)\), and

\[
K = j_1 + j_2, \quad M = |j_1 - j_2|. \tag{2.23}
\]

The eigenvectors are of well defined parities, while those of \( G \) are chiral states. In the following we choose to work with the \( F \) invariant. We have verified that the \( F \) invariant commutes with the \( W^2 \)-operator of the squared Pauli-Lubanski vector, thus providing the possibility to design a Lorentz projector whose eigenvectors are simultaneously eigenvectors to the Poincaré covariant spin-1/2 projector in (2.13). Such projectors, here denoted by \( P^{(j_1)}_F \), with \( j_1 = 0, 1 \), select \( w^{(j_1)} \) states transforming according to the \((1/2, j_1) \oplus (j_1, 1/2)\) sectors of the Rarita-Schwinger representation space:

\[
P^{(0)}_F w^{(0)} = \left( \frac{F - \lambda_1}{\lambda_0 - \lambda_1} \right) w^{(0)} = w^{(0)}, \tag{2.24}
\]

\[
P^{(1)}_F w^{(1)} = \left( \frac{F - \lambda_0}{\lambda_1 - \lambda_0} \right) w^{(1)} = w^{(1)}. \tag{2.25}
\]

Here \( \lambda_{j_1} \) are the \( F \) eigenvalues

\[
\lambda_0 = \frac{3}{4}, \quad \lambda_1 = \frac{11}{4}, \tag{2.26}
\]

corresponding to \( j_1 = 0, 1 \) and \( j_2 = 1/2 \). The projector operators are then easily calculated as,

\[
[P^{(0)}_F]_{\alpha\beta} = \frac{1}{4} \gamma_\alpha \gamma_\beta, \tag{2.27}
\]

\[
[P^{(1)}_F]_{\alpha\beta} = g_{\alpha\beta} - \frac{1}{4} \gamma_\alpha \gamma_\beta. \tag{2.28}
\]

The orthogonality and completeness properties of the above set of operators are easily seen. Compared to the Poincaré covariant projectors, the Lorentz ones have the advantage to be momentum-independent, which will allow us to remove redundant irreducible sectors of a reducible representation space without increasing the power of the momentum dependence of the wave equation. Finally, it is not difficult to verify that

\[
[P^{(0)}_F]_{\alpha\gamma} [P^{(m,3/2)}_W(p)]_{\gamma\beta} = 0, \tag{2.29}
\]

\[
[P^{(1)}_F]_{\alpha\gamma} [P^{(m,3/2)}_W(p)]_{\gamma\beta} = [P^{(m,3/2)}_W(p)]_{\alpha\beta}, \tag{2.30}
\]

which confirms the correct assignment of spin-3/2 to the \((1/2, 1) \oplus (1, 1/2)\) invariant subspace in the four-vector-spinor.

The next section is devoted to the solution of the above Eqs. (2.24) and (2.25).

### C. Combined Lorentz– and Poincaré invariant projectors

The idea of the present work is to identify the lower spin degrees of freedom in \( \psi_\mu \) by means of the following combined Lorentz- and Poincaré invariant projectors:

\[
\Pi_{(1/2,j_1) \oplus (j_1, 1/2)} w^{(m,s)}_{(j_1)} = w^{(m,1/2)}_{(j_1)}, \quad \Pi_{(1/2,j_1) \oplus (j_1, 1/2)} = P^{(j_1)}_F P^{(m,1/2)}_W, \tag{2.31}
\]
which translates into the following two equations of motion (compressed in one):

\[
\left( \Gamma^{(j_i)}_{\mu\nu} \right)_{\alpha\beta} p^\mu p^\nu - m^2 g_{\alpha\beta} \right) w^{(m,1/2)}_{(j_i)} = 0, \quad j_i = 0, 1. \tag{2.32}
\]

Here, the Lorentz- and Poincaré- projectors enter the \( \Gamma^{(j_i)}_{\mu\nu} \)\_\( \alpha\beta \) tensors as

\[
\left[ \Gamma^{(j_i)}_{\mu\nu} \right]_{\alpha\beta} p^\mu p^\nu = m^2 \left[ P^{(j_i)}_F \right]_{\alpha\gamma} \left[ P^{(m,1/2)}_{W^2} \right]_{\gamma\beta} = \left[ P^{(j_i)}_F \right]_{\alpha\gamma} \left[ T^{(1/2)}_{\mu\nu} \right]_{\gamma\beta} p^\mu p^\nu, \tag{2.33}
\]

with \( \left[ T^{(1/2)}_{\mu\nu} \right]_{\gamma\beta} \) taken from (2.16). Carrying out the contraction in the \( \mu \) and \( \nu \) indices, the equation for \((1/2, 0) \oplus (0,1/2)\) emerges particularly simple as,

\[
\left( \frac{1}{4} \gamma_\alpha \gamma_\beta p^2 - m^2 g_{\alpha\beta} \right) w^{(m,1/2)}_{(0)} = 0, \tag{2.34}
\]

while the one for \((1/2, 1) \oplus (1,1/2)\) results as,

\[
\left[ \frac{4}{3} \left( p_\alpha - \frac{1}{4} \gamma_\alpha \not{p} \right) \left( p_\beta - \frac{1}{4} \not{p} \gamma_\beta \right) - m^2 g_{\alpha\beta} \right] w^{(m,1/2)}_{(1)} = 0. \tag{2.35}
\]

The latter equation incorporates an auxiliary condition as visible by contracting it by \( \gamma^\alpha \), finding

\[
-m^2 \gamma_\beta w^{(m,1/2)}_{(1)} = 0. \tag{2.36}
\]

The meaning of (2.36) is that \( w^{(m,1/2)}_{(1)} \) does not have any projection on \((1/2, 0) \oplus (0,1/2)\), as it should be, and in accord with the established reduction of the Rarita-Schwinger space, discussed in the introduction. We furthermore define the following pair of orthogonal matrices,

\[
[f^{(0)}(p)]^\alpha = \frac{1}{2m} \gamma^\alpha \not{p}, \tag{2.37}
\]

\[
[f^{(1)}(p)]^\alpha = \frac{2}{\sqrt{3m}} \left( p^\alpha - \frac{1}{4} \gamma^\alpha \not{p} \right), \tag{2.38}
\]

which are orthonormalized on mass-shell according to,

\[
\left[ \tilde{f}^{(j)}(p) \right]^\alpha \left[ f^{(j')}(p) \right]_{\alpha} = \delta_{jj'} \frac{p^2}{m^2}, \tag{2.39}
\]

with \( \left[ \tilde{f}^{(j)}(p) \right]^\alpha = \gamma^0 \left( [f^{(j)}(p)]^\alpha \right)^{j_0} \). In terms of the \( [f^{(j)}(p)]^\alpha \) matrices, the kinetic terms of the equations of motion are bi-linearized according to,

\[
\left[ [f^{(j)}(p)]_{\alpha} \left[ f^{(j)}(p) \right]_{\beta} - g_{\alpha\beta} \right] w^{(m,1/2)}_{(j_i)} = 0. \tag{2.40}
\]

In combination with the Eq. (2.39) one sees that the \( [f^{(j)}(p)]^\alpha \) matrices take the part in (2.34) and (2.35), i.e. in the spin-1/2 sector of \( \psi_\mu \), of the Feynman slash, \( \not{p} \), in the bi-linearization of the Klein-Gordon equation where \( \not{p} \cdot \not{p} = p^2 \). Now the propagator for each \( j_i \)-value is then the inverse of the respective equation operator,

\[
[S^{(j)}(p)]_{\alpha\beta} = \left( \left[ \Gamma^{(j)}_{\mu\nu} \right]_{\alpha\beta} p^\mu p^\nu - m^2 g_{\alpha\beta} \right)^{-1}, \tag{2.41}
\]

and given by

\[
[S^{(j)}(p)]_{\alpha\beta} = \frac{[\Delta^{(j)}(p)]_{\alpha\beta}}{p^2 - m^2 + i\epsilon}, \tag{2.42}
\]

where

\[
[\Delta^{(0)}(p)]_{\alpha\beta} = \frac{1}{m^2} \left( \frac{1}{4} p^2 \gamma_\alpha \gamma_\beta - (p^2 - m^2) g_{\alpha\beta} \right), \tag{2.43}
\]

\[
[\Delta^{(1)}(p)]_{\alpha\beta} = \frac{1}{m^2} \left[ \frac{4}{3} \left( p_\alpha - \frac{1}{4} \gamma_\alpha \not{p} \right) \left( p_\beta - \frac{1}{4} \not{p} \gamma_\beta \right) + (m^2 - p^2) g_{\alpha\beta} \right]. \tag{2.44}
\]
In terms of the Lorentz- and Poincaré-projectors this is equivalent to,

\[ [\Delta^{(j_1)}(p)]_{\alpha\beta} = \frac{p^2}{m^2} [p F^{(j_1)}]_{\alpha\gamma} [p^{(1/2)}(p)]_{\gamma\beta} + \frac{(p^2 - m^2)}{m^2} g_{\alpha\beta}, \]  

(2.45)

while in terms of the \([f^{(j_1)}(p)]^\alpha\) matrices this is just

\[ [\Delta^{(j_1)}(p)]_{\alpha\beta} = [f^{(j_1)}(p)]_{\alpha}[f^{(j_1)}(p)]_{\beta} + \frac{(p^2 - m^2)}{m^2} g_{\alpha\beta}. \]  

(2.46)

### III. SOLUTIONS TO THE SPIN-1/2 EQUATIONS OF MOTION

In the current section we compare the spin-1/2 solutions of the pure Poincaré covariant projector in (2.18) to those of the combined Lorentz- and Poincaré projectors in (2.34)–(2.35) and draw some non-trivial conclusions regarding their space-time properties.

A. The eigenvectors to the Poincaré covariant spin-1/2 projection

The solutions of the eigenvalue problem of the spin-1/2 Poincaré projector in (2.9) are no more but the spin-1/2 eigenvectors of \(W^2\), which we here denote by \([w(p, \lambda)]^\alpha\). They emerge in the direct product of the four vectors in \((1/2, 1/2)\) and the spinors in \((1/2, 0) \oplus (0, 1/2)\). The \((1/2, 1/2)\) representation is spanned by one scalar and three vectorial degrees of freedom. The spin-1/2 four-vector spinors emerging from the coupling of the scalar in \((1/2, 1/2)\) to the Dirac spinor will be termed as scalar-spinors \((SS)\),

\[ [w_{SS}^\alpha(p, \lambda)] = \phi^\alpha(p) u_\pm(p, \lambda) = \frac{p^\alpha}{m} u_\pm(p, \lambda). \]  

(3.1)

Here, \(\phi^\alpha(p) = p^\alpha/m\) is the only spin-0 vector in \((1/2, 1/2)\), while \(u_\pm(p, \lambda)\) are the usual Dirac spinors in \((1/2, 0) \oplus (0, 1/2)\), of positive (+) and negative (−) parities, i.e., \(u\) and \(\bar{u}\) spinors in the terminology of [11]), and whose polarizations are \(\lambda = 1/2, 1/2\). The spin-1/2 vectors emerging of the coupling of the spin-1/2 vectors in \((1/2, 1/2)\), in the cartesian basis denoted by \(\eta^\alpha(p, \ell)\) (with \(\ell = -1, 0, 1\) [7], to the Dirac spinor will be termed as vector-spinors \((VS)\). They are constructed within the ordinary angular momentum coupling scheme in terms of appropriate Clebsch-Gordan coefficients, and read

\[ [w_{VS}^\alpha(p, 1/2)] = -\sqrt{\frac{1}{3}} [\eta(p, 0)]^\alpha u_+(p, 1/2) + \sqrt{\frac{2}{3}} [\eta(p, 1)]^\alpha u_+(p, -1/2), \]  

(3.2)

\[ [w_{VS}^\alpha(p, -1/2)] = \sqrt{\frac{1}{3}} [\eta(p, 0)]^\alpha u_+(p, -1/2) - \sqrt{\frac{2}{3}} [\eta(p, -1)]^\alpha u_+(p, 1/2). \]  

(3.3)

Notice that while \(\phi^\alpha(p)\) is of positive parity, the parity of \([\eta(p, \ell)]^\alpha\) is negative. In consequence, the positive- (negative-) parity scalar-spinors are made up of positive- (negative-) parity Dirac spinors, while the positive- (negative-) parity vector-spinors are made up of negative- (positive-) parity Dirac spinors. Alternatively, the above vector-spinors can be derived in exploiting the following relationship,

\[ [W^2(p)]_{\alpha\beta} [W^S(p)]_{\beta} = -p^2 \frac{1}{2} \left( \frac{1}{2} + 1 \right) [W^S(p)]_{\alpha}, \]  

(3.4)

where \([W^S(p)]_{\alpha}\) is the Pauli-Lubanski operator in the Dirac-spinor representation,

\[ [W^S(p)]^\mu = \frac{1}{2} \epsilon^{\mu\nu\sigma\rho} M^{S}_{\sigma\rho} p_\nu = -\frac{i}{2} \gamma^5 \sigma^{\mu\nu} p_\nu. \]  

(3.5)

The relationship in (3.4) suggests that one can construct spin-1/2 eigenstates to \([W^2(p)]_{\alpha\beta}\) by the aid of the operator \([W^S(p)]_{\alpha}\) of the Pauli-Lubanski vector. Indeed, one can verify that the vector-spinors in the equations (3.2) and (3.3) are equivalent to

\[ [w_{VS}^\alpha(p, \lambda)] = \frac{2}{\sqrt{3}m} [W^S(p)]_{\alpha} \gamma^5 u_\pm(p, \lambda). \]  

(3.6)
The definitions in (3.1) and (3.6) will prove very useful in the following. Specifically, the demonstration of the orthogonality between scalar- and vector-spinors will benefit from the well known property of the Pauli-Lubanski vector of being divergence-less,

\[ p^\alpha [W^S(p)]_\alpha = 0, \quad (3.7) \]

which implies the following condition satisfied by the vector-spinors:

\[ p^\alpha [w^V_\pm (p, \lambda)]_\alpha = 0. \quad (3.8) \]

In effect, one first finds the four independent scalar-spinors, and then as another set, the four independent vector-spinors, summing up to eight spin-1/2 states, as it should be. Together with the eight independent spin-3/2 states (not considered here) the total of sixteen independent degrees of freedom in the four-vector-spinor is recovered.

1. Parity of the states diagonalizing the Poincaré covariant spin-1/2 projectors

As long as the spin-1/2 states in (3.1) and (3.6) have well defined parities, one can construct from them the corresponding parity projectors. Each spin-1/2 is normalized according to

\[ [\hat{\mathbb{P}}^\alpha_\pm (p, \lambda)]^\alpha [w^\alpha_\pm (p, \lambda)]_\alpha = \pm 1, \quad (3.9) \]

where use has been made of

\[ [\hat{\mathbb{P}}^\alpha_\pm (p, \lambda)]^\alpha = p^\alpha \gamma^\pm (p, \lambda), \quad (3.10) \]
\[ [\hat{\mathbb{P}}^\alpha (p, \lambda)]^\alpha = \gamma^\pm (p, \lambda)(p^\alpha W^S(p)]_\alpha, \quad (3.11) \]

with \( \gamma^\pm (p, \lambda) = [\gamma^0 u^\pm (p, \lambda)]^\dagger \) as customary. The parity projectors are then obtained from (3.9)-(3.11) according to

\[ [\hat{\mathbb{P}}^\alpha_\pm (p, \lambda)]^\alpha = \sum \lambda [w^\alpha_\pm (p, \lambda)]_\alpha \hat{\mathbb{P}}^\alpha_\pm (p, \lambda)]^\alpha, \quad (3.12) \]

which amounts to the following explicit expressions,

\[ [\hat{\mathbb{P}}^\alpha_\pm (p, \lambda)]^\alpha = \pm \frac{\hat{p} + m}{2m} \frac{1}{m^2} p^\alpha p^\beta, \quad (3.13) \]
\[ [\hat{\mathbb{P}}^\alpha (p, \lambda)]^\alpha = \pm \frac{\hat{p} + m}{2m} \frac{1}{3m^2} p^\mu p^\nu. \quad (3.14) \]

where we have used

\[ \sum \lambda u(p, \lambda) \bar{u}(p, \lambda) = \frac{\hat{p} + m}{2m}. \quad (3.15) \]

The projectors in (3.13) and (3.14) can also be cast in terms of the more familiar spin-1/2 projectors, \( [\mathbb{P}^{(1/2)}_\alpha](p)]_\alpha \beta \) from (2.13) and [10]:

\[ [\mathbb{P}^{(1/2)}_1 (p)]_\alpha \beta = \frac{- p^\alpha p^\beta}{p^2} + \frac{1}{3} \gamma^\alpha \gamma^\beta + \frac{1}{p^2} (\hat{p} \gamma^\alpha p^\beta + p^\alpha \gamma^\beta \hat{p}), \quad (3.16) \]
\[ [\mathbb{P}^{(1/2)}_2 (p)]_\alpha \beta = [\mathbb{P}^{(1/2)} (p)]_\alpha \beta - [\mathbb{P}^{(1/2)}_1 (p)]_\alpha \beta = \frac{p^\alpha p^\beta}{p^2}, \quad (3.17) \]

in so doing, we establish the following relationships between parity- and spin projector operators:

\[ [\mathbb{P}^{SS}_\alpha (p)]_\alpha \beta = \frac{\pm \hat{p} + m}{2m} [\mathbb{P}^{(1/2)}_1 (p)]_\alpha \beta, \quad (3.18) \]
\[ [\mathbb{P}^{SV}_\alpha (p)]_\alpha \beta = \frac{\pm \hat{p} + m}{2m} [\mathbb{P}^{(1/2)}_2 (p)]_\alpha \beta. \quad (3.19) \]

Notice that we use the boldface \( (p) \) when the mass-shell condition \( p^2 = m^2 \) holds valid. We can also construct the so called switch projectors by making combinations between scalar-spinors and vector-spinors. However, we
prefer to sum up all four projectors in (3.18) and (3.19) with the aim to recover \([\mathbb{P}^{(1/2)}_0(p)]_{\alpha\beta}\) in (2.11) as previously obtained from the Poincaré covariant spin-1/2 projector as

\[
\frac{1}{m^2} \left( \frac{1}{3} \sigma_{\alpha\mu} \sigma_{\beta\nu} + g_{\alpha\mu} g_{\beta\nu} \right) p^\mu p^\nu = [\mathbb{P}^{(1/2)}_0(p)]_{\alpha\beta},
\]  

(3.20)

where we have made use of (3.1) and (3.6). The procedure of constructing \([\mathbb{P}^{(1/2)}_0(p)]_{\alpha\beta}\) from the parity eigenstates happens to prescribe the correct \(p^\mu p^\nu\), ordering in the momentum dependence of the equation of motion, a circumstance that will prove crucial upon gauging. As a next step we shall obtain the explicit expressions for the states that diagonalize the combined Lorentz- and Poincaré invariant projectors.

B. The spin-1/2 eigenstates to the combined Lorentz- and Poincaré invariant projectors

None of the \(w^{\text{SS}}_\pm(p, \lambda)\) and \(w^{\text{VS}}_\pm(p, \lambda)\) four-vector spinors from above is an eigenstate to the \(F\) Casimir invariant of the Lorentz algebra in (2.22). However, the commutation of the spin-1/2 projector \(\mathbb{P}^{(1/2)}_0\) in (3.20) with the Lorentz projectors \(\mathbb{P}^{(j_1)}_F\) in (2.27), permits their diagonalizing in the same basis and allows us to construct \(\mathbb{P}^{(1/2)}\) eigenstates that simultaneously transform according to one of the two \((1/2, j_1) \oplus (j_1, 1/2)\) sectors \((j_1 = 1, 0)\) in \(\psi_\mu\). These are the states \([w(p, \lambda)^{(m,1/2)}_\pm]_{\alpha\beta}\) from (2.32). In the following however we will need also the quantum number of parity, \(\pm\), as a label of the states. For the sake of simplifying notation from now onward we will drop the \((m, 1/2)\) label and re-denote the above states by \([w^{(j_1)}_\pm(p, \lambda)]_{\alpha\beta}\).

\[
[w^{(j_1)}_\pm(p, \lambda)]_{\alpha\beta} = N[\mathbb{P}^{(j_1)}_F]_{\alpha\beta}[w^{\text{SS/SV}}_\pm(p, \lambda)],
\]  

(3.21)

where \([w^{\text{SS/SV}}_\pm(p, \lambda)]_{\beta}\) is either a scalar- or a vector-spinor and \(N\) is a normalization factor.

We can now benefit from our knowledge on \(w^{\text{SS}}_\pm(p, \lambda)\) in (3.1) and \(w^{\text{VS}}_\pm(p, \lambda)\) in (3.6) and find the following \(j_1 = 0\) projections:

\[
[\mathbb{P}^{(0)}_F]_{\alpha\beta}[w^{\text{SS}}_\pm(p, \lambda)]_{\beta} = \frac{1}{4m} \gamma^\alpha \slashed{p} u_\pm(p, \lambda),
\]  

(3.22)

\[
[\mathbb{P}^{(0)}_F]_{\alpha\beta}[w^{\text{VS}}_\pm(p, \lambda)]_{\beta} = \frac{\sqrt{3}}{4m} \gamma^\alpha \slashed{p} u_\pm(p, \lambda).
\]  

(3.23)

In this manner the \((1/2, 0) \oplus (0, 1/2)\) components of \(\omega^{\text{SS}}(p, \lambda)\) and \(\omega^{\text{VS}}(p, \lambda)\) are identified. In a way similar, their \((1/2, 1) \oplus (1, 1/2)\) components are identified as

\[
[\mathbb{P}^{(1)}_F]_{\alpha\beta}[w^{\text{SS}}_\pm(p, \lambda)]_{\beta} = \frac{1}{m} \left( p^\alpha - \frac{1}{4} \gamma^\alpha \right) u_\pm(p, \lambda),
\]  

(3.24)

\[
[\mathbb{P}^{(1)}_F]_{\alpha\beta}[w^{\text{VS}}_\pm(p, \lambda)]_{\beta} = -\frac{1}{\sqrt{3}m} \left( p^\alpha - \frac{1}{4} \gamma^\alpha \right) u_\pm(p, \lambda).
\]  

(3.25)

Taking care of the proper normalizations, and in terms of the \([f^{(j_1)}(p)]^\alpha\) matrices from (2.37) and (2.38), the eigenstates \([w^{(j_1)}_\pm(p, \lambda)]^\alpha\) to the combined Lorentz- and Poincaré invariant projectors can finally be cast into the following forms:

\[
[w^{(j_1)}_\pm(p, \lambda)]^\alpha = [f^{(j_1)}(p)]^\alpha u_\pm(p, \lambda).
\]  

(3.26)

This is

\[
[w^{(0)}_\pm(p, \lambda)]^\alpha = \frac{1}{2m} \gamma^\alpha \slashed{p} u_\pm(p, \lambda),
\]  

(3.27)

\[
[w^{(1)}_\pm(p, \lambda)]^\alpha = \frac{2}{\sqrt{3}m} \left( p^\alpha - \frac{1}{4} \gamma^\alpha \right) u_\pm(p, \lambda).
\]  

(3.28)

Our first observation concerns the simplicity of the \(j_1 = 0\) solutions. Next we realize that the \(j_1 = 1\) solutions obey same relationship,

\[
\gamma^\alpha[w_\pm^{(1)}(p, \lambda)]_{\alpha} = 0,
\]  

(3.29)

as the one already reported in (2.36). There are two polarizations available for each parity and two possible parities for each \(j_1\) value, making a total of eight spin-1/2 independent states which are now classified according to two distinct irreducible Lorentz invariant representation spaces.
1. Parity projectors from the states

The eigenstates of the combined Lorentz-and Poincaré invariant projector in (3.27)-(3.28) are also of well defined parities, and are normalized according to (3.26) as,

\[ |w_{\pm}^{(j_1)}(p, \lambda)\rangle \alpha = \pm 1. \]  

(3.30)

Their conjugates are defined as,

\[ \overline{w}_{\pm}^{(j_1)}(p, \lambda) = [\gamma^0 w_{\pm}^{(j_1)}(p, \lambda)]^\dagger. \]  

(3.31)

Then the parity projectors constructed from (3.12) emerge as,

\[ \left[ P_{\pm}^{(j_1)}(p) \right]_{\alpha\beta} = [f^{(j_1)}(p)]_{\alpha} \left( \pm \frac{\not{p} + m}{2m} \right) [\overline{f}^{(j_1)}(p)]_{\beta}, \]  

meaning that

\[ \left[ P_{\pm}^{(0)}(p) \right]_{\alpha\beta} = \frac{1}{2m} \left( \frac{1}{4m^2} \gamma_\alpha (\not{p} + m) \not{p} \gamma_\beta, \right) \]  

(3.33)

\[ \left[ P_{\pm}^{(1)}(p) \right]_{\alpha\beta} = \frac{1}{2m} \left( \frac{1}{4} \gamma_\alpha (\not{p} + m) \not{p} \gamma_\beta, \right) \]  

(3.34)

IV. ELECTROMAGNETIC INTERACTION

Having appropriately constructed the free equations of motion in (2.32) does not necessarily guarantee that they describe physically observable particles. Also if we expect to describe electromagnetically interacting particles, then our equations of motion have also to remain consistent upon the electromagnetic gauging. The gauging procedure is very sensitive to the momentum dependence of the wave equations, especially when they are of second order as are ours. For this reason before proceeding further we have to attend several details.

We begin with casting the free equations of motion (2.32) for spin-1/2 transforming in \((1/2, j_1) \oplus (j_1, 1/2)\) as

\[ \left( \Gamma_{\mu\nu}^{(j_1)} \right)_{\alpha\beta} p^\mu p^\nu - m^2 g_{\alpha\beta} \} |w^{(j_1)}\rangle_{\beta} = 0. \]  

(4.1)

Here

\[ \left[ \Gamma_{\mu\nu}^{(j_1)} \right]_{\alpha\beta} p^\mu p^\nu = \left[ P_F^{(j_1)} \right]_{\alpha} \gamma^\gamma \left[ T_{\mu\nu}^{(1/2)} \right]_{\gamma\beta} \} p^\mu p^\nu, \]  

(4.2)

\[ \left[ T_{\mu\nu}^{(1/2)} \right]_{\alpha\beta} p^\mu p^\nu = m^2 \left[ F^{(1/2)}(p) \right]_{\alpha\beta}. \]  

(4.3)

There is certain ambiguity regarding the momentum dependence of the \(T_{\mu\nu}^{(1/2)}\) tensor in so far its antisymmetric part is not uniquely fixed within the method. Due to the commutativity of the four-momenta \(p^\mu\) and \(p^\nu\), it is obvious, that for free particles contributions of the type \(\left[ T_{\mu\nu}^{(1/2)} \right]_{\gamma\beta} [p^\mu p^\nu] \) nullify. However, upon gauging, \(p^\mu \rightarrow \pi^\mu = p^\mu - eA^\mu\), the commutator between the gauged momenta gives rise to the electromagnetic field tensor,

\[ \left[ \pi^\mu, \pi^\nu \right] = -ieF^{\mu\nu}. \]  

(4.4)

We here require the antisymmetric part of \(T_{\mu\nu}^{(1/2)}\) to coincide with the one emerging from the Lorentz- and Poincaré covariant projector as constructed from the states in (3.27)-(3.28). In so doing, the \(\Gamma_{\mu\nu}^{(j_1)}\) tensor is found as,

\[ \left[ \Gamma_{\mu\nu}^{(j_1)} \right]_{\alpha\beta} = \left[ P_F^{(j_1)} \right]_{\alpha} \gamma^\gamma \left( \frac{1}{3} \sigma_{\gamma\mu} \sigma_{\gamma\nu} + g_{\gamma\mu} g_{\gamma\nu} \right), \]  

(4.5)

where use of (3.20) has been made. Explicitly for each \(j_1\)-value we have,

\[ \left[ \Gamma_{\mu\nu}^{(0)} \right]_{\alpha\beta} = \frac{1}{4} \gamma_\alpha \gamma_\mu \gamma_\nu \gamma_\beta, \]  

(4.6)

\[ \left[ \Gamma_{\mu\nu}^{(1)} \right]_{\alpha\beta} = \frac{4}{3} \left( g_{\alpha\mu} - \frac{1}{4} \gamma_\mu \gamma_\alpha \right) \left( g_{\nu\beta} - \frac{1}{4} \gamma_\nu \gamma_\beta \right), \]  

(4.7)

and in reference to (2.27) and (2.28). With these definitions, both equations of motion in (4.1) will be shown in the subsequent section to pass the causality test, and thus qualify for the description of electromagnetically interacting spin-1/2 particles transforming as \((1/2, j_1) \oplus (j_1, 1/2)\).
A. The causality test

The hyperbolicity and causality of the equations of motion of order \( \leq 2 \) in the derivatives can be tested using the Courant-Hilbert method, which requires us to calculate the characteristic determinant of the gauged equations. In order to obtain the gauged equations, we first switch in (4.1) from momentum to position space using \( \left[ \psi^{(j_2)} \right]^{\alpha} = [u^{(j_2)}(p, \lambda)]^{\alpha} e^{-ixp} \) as

\[
\left( [\Gamma_{\mu\nu}]_{\alpha\beta} \partial^\mu \partial^\nu + m^2 g_{\alpha\beta} \right) [\psi^{(j_1)}]^{\beta} = 0. \tag{4.8}
\]

This equation is in reality a 16 \( \times \) 16 dimensional matrix equation for the 16-component state \( [\psi^{(j_1)}]^{\beta} \). However, considering only the relevant degrees of freedom of a spin-1/2 particle (regardless of its parity) we have to arrange the above equation as a 4 \( \times \) 4 matrix equation acting on a 4-component state vector as indicated by the explicit form of the solutions in terms of 4-component spinors. Then, according to the gauge principle, we couple this equations minimally to an electromagnetic field according to

\[
\partial \to D = \partial + ieA, \tag{4.9}
\]

where \( e \) is the electric charge of the particle. The characteristic determinant is then found by replacing the highest order derivatives by the components of the vector \( n^\mu \), normal to the characteristic surfaces, and which characterizes the propagation of the (classical) wave fronts of the gauged equation. If the vanishing of the characteristic determinant demands to have a real-valued time-like component \( n^0 \), then the equation is hyperbolic. If this determinant nullifies as \( n^\mu n_\mu = 0 \), then the equation is in addition causal [12].

1. Gauging the wave equation for the \((1/2, 0) \oplus (0, 1/2)\) sector

In order to find the explicit form of the gauged equation for the case under consideration, we first substitute \( \Gamma_{\mu\nu}^{(0)} \) in (4.8) by its definition in (4.6), and then, in making use of (3.27), and \( \psi = u_{\pm}(p, \lambda)e^{-ip\cdot x} \), we arrive at

\[
\left( \frac{1}{4} \gamma_\alpha \gamma_\mu \gamma_\nu \gamma_\beta \partial^\mu \partial^\nu + m^2 g_{\alpha\beta} \right) \left( -\frac{i}{2m} \gamma^\beta \partial_D \right) \psi = 0. \tag{4.10}
\]

This equation arranges to a 4 \( \times \) 4 matrix equation upon contraction by \( \gamma^\alpha \) from the left, then the factorization of a \( \gamma_\beta \)-matrix becomes possible, with the result,

\[
(\gamma_\mu \gamma_\nu \partial^\mu \partial^\nu + m^2) \gamma_\beta \left( -\frac{i}{2m} \gamma^\beta \partial_D \right) \psi = 0. \tag{4.11}
\]

Carrying out now the \( \gamma_\beta \gamma_\beta \) contraction amounts to,

\[
\frac{2}{m} \left( \gamma_\mu \gamma_\nu \partial^\mu \partial^\nu + m^2 \right) (-i \partial_D) \psi = 0, \tag{4.12}
\]

which leads to the following gauged equation,

\[
\frac{2}{m} \left( \gamma_\mu \gamma_\nu D^\mu D^\nu + m^2 \right) (-i D) \psi = 0. \tag{4.13}
\]

This equation is of third order in derivatives, but it favorably factorizes into a quadratic and a linear equation, thus allowing us to apply the Courant-Hilbert criterion to each one of the factors separately [13]. In so doing, the characteristic determinant also factorizes into two characteristic determinants of the respective quadratic and linear equations. In this fashion, it becomes possible to test the causality of the wave equation for the \( j_1 = 0 \). We calculate

\[
D^{(0)}(n) = D^{(0)}_1(n)D^{(0)}_2(n) = \left( \frac{2}{m} \right)^4 (n^2)^4 (n^2)^2, \tag{4.14}
\]

where

\[
D^{(0)}_1(n) = \left| -\frac{2}{m} \gamma_\mu \gamma_\nu n^\nu n^\mu \right| = \left| -\frac{2}{m} n^2 \right| = \left( \frac{2}{m} \right)^4 (n^2)^4, \tag{4.15}
\]\n
\[
D^{(0)}_2(n) = \left| \vec{f}_i \right| = (n^2)^2. \tag{4.16}
\]

Nullifying \( D^{(0)}(n) \) in (4.14) amounts to the condition \( n^2 = n^\mu n_\mu = 0 \), which is the accepted indicator of causal propagation.
Using (4.7) to substitute for $\Gamma^{(1)}_{\mu
u}$ in (4.8) together with the explicit forms of the (4.8) solutions from (3.28) amounts to,

$$\frac{4}{3}\left[ g_{\alpha\mu} - \frac{1}{4}\gamma_\alpha\gamma_\mu \right] \left( g_{\nu\beta} - \frac{1}{4}\gamma_\nu\gamma_\beta \right) \partial^\mu \partial^\nu + m^2 g_{\alpha\beta} \right] \frac{2}{\sqrt{3m}} \left( -i\partial^\beta + \frac{1}{4}\gamma^\beta \not{p} \right) \psi = 0, \quad (4.17)$$

with $\psi = u_\pm(p, \lambda)e^{-ix\cdot p}$. Then by virtue of the auxiliary condition (3.29) the equation (4.17) simplifies as,

$$\frac{4}{3} \left[ \left( g_{\alpha\mu} - \frac{1}{4}\gamma_\alpha\gamma_\mu \right) \partial^\mu \partial^\beta + m^2 g_{\alpha\beta} \right] \frac{-2i}{\sqrt{3m}} \left( \partial^\beta - \frac{1}{4}\gamma^\beta \not{p} \right) \psi = 0. \quad (4.18)$$

In order to obtain a $4 \times 4$ matrix equation we perform a contraction by $\partial^\alpha$ arriving at,

$$\frac{4}{3} \left[ \left( \partial_\mu - \frac{1}{4}\not{p}\gamma_\mu \right) \partial_\mu + m^2 \right] \left[ \partial_\beta \left( \partial^\beta - \frac{1}{4}\gamma^\beta \not{p} \right) \right] \psi = 0. \quad (4.19)$$

This equation can then be factorized into two quadratic equations,

$$\frac{4}{3} \left[ \frac{-2i}{\sqrt{3m}} \left( D_\mu - \frac{1}{4}\not{D}\gamma_\mu \right) D_\mu + m^2 \right] \left[ D_\beta \left( D^\beta - \frac{1}{4}\gamma^\beta \not{D} \right) \right] \psi = 0. \quad (4.20)$$

Notice that we have not made any use of the commutativity of the $\partial$-derivatives, so that the gauged equation can be written as

$$\frac{4}{3} \left[ \frac{-2i}{\sqrt{3m}} \left( D_\mu - \frac{1}{4}\not{D}\gamma_\mu \right) D_\mu + m^2 \right] \left[ D_\beta \left( D^\beta - \frac{1}{4}\gamma^\beta \not{D} \right) \right] \psi = 0. \quad (4.21)$$

The characteristic determinant for $j_1 = 1$ is then found as the product of the following two determinants:

$$D^{(1)}(n) = \left( \frac{4}{3} \frac{-2i}{\sqrt{3m}} \right)^4 D_1^{(1)}(n) D_2^{(1)}(n) = \left( \frac{4}{3} \frac{2}{\sqrt{3m}} \right)^4 \left( \frac{3}{4} \right)^8 (n^2)^4 = \left( \frac{\sqrt{3}}{2m} \right)^4 (n^2)^4, \quad (4.22)$$

where

$$D_1^{(1)}(n) = \left| \left( -n_\mu + \frac{1}{4} \not{n}\gamma_\mu \right) n^\mu \right| = \left| \frac{3}{4} n^2 \right| = \left( \frac{3}{4} \right)^4 (n^2)^4, \quad (4.23)$$

$$D_2^{(1)}(n) = \left| n_\beta \left( -n^\beta + \frac{1}{4} \gamma^\beta \not{n} \right) \right| = \left| \frac{3}{4} n^2 \right| = \left( \frac{3}{4} \right)^4 (n^2)^4. \quad (4.24)$$

Again, the condition for the determinant (4.22) to be zero, holds valid for $n^2 = n^\mu n_\mu = 0$, thus ensuring causality. This result completes the proof of the causality of the equations of motion for both spin-1/2 sectors in $\psi_\mu$, transforming in $(1/2, j_1) \oplus (j_1, 1/2)$ with $j_1 = 0, 1$.

### B. The Lagrangians for the lower spin-1/2 sectors of the Rarita-Schwinger four-vector field

For practical calculations it is advantageous to have at ones disposal a gauged Lagrangian, out of which one can deduce the Feynman rules of the theory. The Lagrangians for positive and negative parity states usually differ only by an overall sign, it compensates for the different normalizations of states of opposite parities. With this in mind and for the sake of concreteness, in the following we shall only deal with Lagrangians written in terms of the positive parity states. The free equations of motion (4.8) for the positive parity states relate to the Lagrangian by the Euler-Lagrange equations. For second order equations of motion the corresponding Lagrangians are of the form,

$$\mathcal{L}^{(j_1)}_{\text{free}} = (\partial^\mu [\psi^{(j_1)}]^{\alpha}[\Gamma_{\mu\nu}^{(j_1)}]_{\alpha\beta} \partial^\nu [\psi^{(j_1)}]^{\beta} - m^2 [\psi^{(j_1)}]^{\alpha}[\psi^{(j_1)}]_{\alpha}, \quad (4.25)$$
\[i[S^{(j_1)}(p)]_{\alpha\beta} \]
\[\beta \rightarrow \alpha\]

Here, \(L\) is the Lagrangian, obtained as the inverse of the equations of motion, the explicit form of \(S^{(j_1)}(p)\) for each \(j_1\)-value is given in (2.41)-(2.44).

\[\epsilon^\nu(q, \ell)\]
\[\epsilon^\nu(q, \ell)^*\]
\[\epsilon^\nu(q, \ell) = \epsilon^\nu(q, \ell)^*, \quad \epsilon^\nu(q, \ell)^* = \epsilon^\nu(q, \ell)\]
\[\epsilon^\nu(q, \ell) \quad \epsilon^\nu(q, \ell)^*\]

Now \(L^{(j_1)}\) can be decomposed into free and interaction Lagrangians as,

\[L^{(j_1)} = L_{\text{free}}^{(j_1)} + L_{\text{int}}^{(j_1)}\] \hspace{1cm} (4.26)
\[L_{\text{int}}^{(j_1)} = -\frac{1}{2} j_{\mu}^{(j_1)} A^\mu + k_{\mu\nu}^{(j_1)} A^\mu A^\nu.\] \hspace{1cm} (4.27)

Here, \(j_{\mu}^{(j_1)}\) is the electromagnetic current, while \(k_{\mu\nu}^{(j_1)}\) is the structure of a two-photon coupling. In momentum space, and for the positive parity states \(\{w^{(j_1)}(p, \lambda)\}_{\beta}\) we find,

\[j_{\mu}^{(j_1)} = \epsilon_{\alpha\beta}^{(j_1)}(p', \lambda')^{\alpha} \langle [V_{\mu}^{(j_1)}(p', p)]_{\alpha\beta} \rangle \langle [w^{(j_1)}(p, \lambda)]_{\beta}\rangle,\] \hspace{1cm} (4.28)
\[k_{\mu\nu}^{(j_1)} = \epsilon_{\alpha\beta}^{(j_1)}(p', \lambda')^{\alpha} \langle [C_{\mu\nu}^{(j_1)}(p', p)]_{\alpha\beta} \rangle \langle [w^{(j_1)}(p, \lambda)]_{\beta}\rangle,\] \hspace{1cm} (4.29)

where \([V_{\mu}^{(j_1)}(p', p)]_{\alpha\beta}\) and \([C_{\mu\nu}^{(j_1)}(p', p)]_{\alpha\beta}\) are the one- and two-photon vertexes which, together with the propagators \(2.42\), determine the Feynman rules. The latter are depicted on the Figs. 1, 2, 3.

\[\langle [V_{\mu}^{(j_1)}(p', p)]_{\alpha\beta}\rangle = \langle [\Gamma_{\nu}^{(j_1)}]_{\alpha\beta} p^\nu + [\Gamma_{\nu}^{(j_1)}]_{\alpha\beta} p^\nu\rangle,\] \hspace{1cm} (4.30)
\[\langle [C_{\mu\nu}^{(j_1)}(p', p)]_{\alpha\beta}\rangle = \frac{1}{2} \langle [\Gamma_{\mu}^{(j_1)}]_{\alpha\beta} + [\Gamma_{\nu}^{(j_1)}]_{\alpha\beta}\rangle.\] \hspace{1cm} (4.31)

In particular, \([V_{\mu}^{(j_1)}(p', p)]_{\alpha\beta}\) obeys the Ward-Takahashi identity,

\[(p' - p)^\mu [V_{\mu}^{(j_1)}(p', p)]_{\alpha\beta} = \langle [S^{(j_1)}(p')]_{\alpha\beta}^{-1} - \langle [S^{(j_1)}(p)]_{\alpha\beta}^{-1}\rangle,\] \hspace{1cm} (4.32)

where \([S^{(j_1)}(p)]_{\alpha\beta}\) are the propagators in (2.42). This relationship leads to gauge invariance of the amplitudes which define the Compton scattering process. However, before evaluating this process it is very instructive to figure out the values of the magnetic dipole moments of the particles under consideration prescribed by the currents in (4.29).
C. Magnetic dipole moments

We begin with the currents in momentum space in (4.29) and for positive parity states with polarization \( \lambda \), which are given in terms of Dirac's \( u \)-spinors by,

\[ j^{(j_1)}_{\mu}(p', p) = e \overline{\Psi}(p', \lambda) \tilde{V}^{(j_1)}_{\mu}(p', p) u(p, \lambda). \]  

(4.34)

Here we have used (3.26) and found,

\[ \tilde{V}^{(j_1)}_{\mu}(p', p) = [\tilde{T}^{(j_1)}(p')]^{\alpha}[\gamma_{\mu}^{(j_1)}(p', p)]^{\alpha \beta}[f^{(j_1)}(p)]^{\beta}. \]

(4.35)

Incorporation of the mass-shell condition amounts to,

\[ j^{(0)}_{\mu}(p', p) = e \overline{\Psi}(p', \lambda) (2m \gamma_\mu) u(p, \lambda), \]

(4.36)

\[ j^{(1)}_{\mu}(p', p) = e \overline{\Psi}(p', \lambda) \left( \frac{4}{3} (p' + p)_{\mu} - \frac{2m}{3} \gamma_\mu \right) u(p, \lambda). \]

(4.37)

One immediately notices that for \( j_1 = 0 \), the textbook Dirac current is recovered, as it should be and in accord with the reducibility of the the four-vector spinor discussed in the introduction. We now make use of the Gordon decomposition of the Dirac current,

\[ 2m e \overline{\Psi}(p', \lambda) \gamma_\mu u(p, \lambda) = e \overline{\Psi}(p', \lambda) \left[ (p' + p)_{\mu} + 2i M^S_{\mu \nu} (p' - p)^\nu \right] u(p, \lambda). \]

(4.38)

Here \( M^S_{\mu \nu} \) are the elements of the Lorentz algebra in \((1/2, 0) \oplus (0, 1/2)\) in (2.8), while the factor 2 in front of them stands for the gyromagnetic ratio. As a result, the currents in (4.36) and (4.37) take the form

\[ j^{(j_1)}_{\mu}(p', p) = e \overline{\Psi}(p', \lambda) \left[ (p' + p)_{\mu} + ig^{(j_1)} M^S_{\mu \nu} (p' - p)^\nu \right] u(p, \lambda), \]

(4.39)

where

\[ g^{(0)} = 2, \]

(4.40)

\[ g^{(1)} = -\frac{2}{3}. \]

(4.41)

The above eqs. (4.40) and (4.41) show that the electromagnetic currents for particles transforming in \((1/2, j_1) \oplus (j_1, 1/2)\) are characterized by different magnetic dipole moments for different \( j_1 \) values. The gauged Lagrangian corresponding to the combined Lorentz- and Poincaré invariant projector, that describes particles of charge \( e \) transforming in \((1/2, 0) \oplus (0, 1/2)\) predicts the following magnetic moment,

\[ \mu^{(0)}(\lambda) = \frac{2 \lambda e}{2m}. \]

(4.42)

The latter coincides with the standard value for a Dirac particle of polarization \( \lambda \). Instead, the Lagrangian of same type predicts for particles of charge \( e \) transforming in \((1/2, 1) \oplus (1, 1/2)\) a magnetic dipole moment of

\[ \mu^{(1)}(\lambda) = -\frac{2 \lambda e}{3 2m}. \]

(4.43)

This dependence of the magnetic dipole moment on the space-time transformation properties of the particle, is similar to the one found for high-spin states, where particles with equal spins, transforming in different representation spaces of the Lorentz algebra, have also been observed to be characterized by different sets of electromagnetic multipole moments [14]. An electromagnetic process however is not entirely determined by the electromagnetic multipole moments of the particles, which by definition are associated with the on-shell states. It is basically determined by the complete gauged Lagrangian. Remarkable, different Lagrangians can lead to the same multipole moments [15], [14]. The knowledge on the electromagnetic multipole moments is therefore not sufficient to completely characterize a theory. The more profound test for a Lagrangian regards processes involving off-shell states. One such process, the Compton scattering, is considered in detail the next section.
respectively, while using the following we use the symbols $\text{(j)}$ each $M_{\text{(j)}}^{(j)}$ to the Compton scattering amplitudes are constructed in terms of the Feynman rules (shown in the Figs. 4,5,6) (see for example [1]). In the following we present the calculation for each $j_1$-value separately.

V. COMPTON SCATTERING OFF SPIN-1/2 IN $(1/2,j_1) \oplus (j_1,1/2)$

The Compton scattering amplitudes are constructed in terms of the Feynman rules (shown in the Figs. 1,2,3) for each $j_1$-value, giving

$$M^{(j_1)} = M_A^{(j_1)} + M_B^{(j_1)} + M_C^{(j_1)},$$

(5.1)

where $M_A^{(j_1)}$, $M_B^{(j_1)}$, $M_C^{(j_1)}$ correspond to the amplitudes for direct, exchange and point scatterings, respectively. In the following we use the symbols $p$ and $p'$ to denote the momentum of the incident and scattered spin-1/2 particles respectively, while using $q$ and $q'$ to denote the momentum of the incident and scattered photons respectively, so that

$$iM_A^{(j_1)} = e^2[w^{(j_1)}(p',\lambda')]_0^\alpha [U_{\mu\nu}^{(j_1)}(p',Q,p)]_{\alpha\beta}[w(p,\lambda)^{(j_1)}]_0^\beta [\epsilon^\mu(q',\ell')]^* \epsilon^\nu(q,\ell),$$

(5.2)

$$iM_B^{(j_1)} = e^2[w^{(j_1)}(p',\lambda')]_0^\alpha [U_{\mu\nu}^{(j_1)}(p',R,p)]_{\alpha\beta}[w(p,\lambda)^{(j_1)}]_0^\beta [\epsilon^\mu(q',\ell')]^* \epsilon^\nu(q,\ell),$$

(5.3)

$$iM_C^{(j_1)} = -e^2[w^{(j_1)}(p',\lambda')]_0^\alpha [C^0_{\mu\nu} + C^0_{\nu\mu}]_{\alpha\beta}[w(p,\lambda)^{(j_1)}]_0^\beta [\delta^\mu(q',\ell')]^* \epsilon^\nu(q,\ell),$$

(5.4)

where $Q = p + p' = q + q'$ and $R = p' - q = p - q'$ stand for the momentum of the intermediate states and

$$[U_{\mu\nu}^{(j_1)}(p',Q,p)]_{\alpha\beta} = [V_{\mu\nu}^{(j_1)}(p',Q)]_{\alpha\beta} = [V_{\nu\mu}^{(j_1)}(Q,p)]_{\alpha\beta}.$$
A. Compton scattering off particles in the single-spin \((1/2, 0) \oplus (0, 1/2)\) sector of the four-vector–spinor

For the case under investigation it is very useful to write the amplitudes in terms of \(u\)-spinors:

\[
i\mathcal{M}_A^{(0)} = e^2 \pi(p', \lambda') \tilde{U}_{\mu
u}^{(0)}(p', Q, p) u(p, \lambda)[\epsilon^\mu(q', \ell')][\epsilon^\nu(q, \ell)],
\]

\[
i\mathcal{M}_B^{(0)} = e^2 \pi(p', \lambda') \tilde{U}_{\nu\mu}^{(0)}(p', R, p) u(p, \lambda)[\epsilon^\mu(q', \ell')][\epsilon^\nu(q, \ell)],
\]

\[
i\mathcal{M}_C^{(0)} = -e^2 \pi(p', \lambda') \tilde{C}_{\mu\nu}^{(0)} u(p, \lambda)[\epsilon^\mu(q', \ell')][\epsilon^\nu(q, \ell)],
\]

where

\[
\tilde{U}_{\mu
u}^{(j_1)}(p', Q, p) = \mathcal{J}^{(j_1)}(p')^\alpha[U^{(j_1)}(p', Q, p)]_{\alpha\beta}[f^{(j_1)}(p)]^\beta,
\]

\[
\tilde{C}_{\mu\nu}^{(j_1)} = \mathcal{J}^{(j_1)}(p')^\alpha[C_{\mu\nu}^{(j_1)} + C_{\nu\mu}^{(j_1)}]_{\alpha\beta}[f^{(j_1)}(p)]^\beta
\]

with the \([f^{(0)}(p)]^\alpha\) matrices taken from (2.37). Making use of the explicit form of the propagator in (2.42), on the one side, and the vertices in (4.31), (4.32) with \(j_1 = 0\), one the other side, we find,

\[
\tilde{U}_{\mu
u}^{(0)}(p', Q, p) = 2m\gamma_{\mu} \frac{Q + m}{Q^2 - m^2} \gamma_{\nu} + \gamma_{\mu}\gamma_{\nu},
\]

\[
\tilde{U}_{\nu\mu}^{(0)}(p', R, p) = 2m\gamma_{\nu} \frac{R + m}{R^2 - m^2} \gamma_{\mu} + \gamma_{\nu}\gamma_{\mu},
\]

\[
\tilde{C}_{\mu\nu}^{(0)} = 2g_{\mu\nu}.
\]

Here, we achieved some simplifications by replacing by \(m\) all appearances of \(p\), and \(p\) as well on the left as on the right, respectively, in noticing that these always act on \(u\)-spinors of positive parity only. The complete amplitude then emerges as,

\[
i\mathcal{M}_A^{(0)} = 2m e^2 \pi(p', \lambda') \left( \gamma_{\mu} \frac{Q + m}{Q^2 - m^2} \gamma_{\nu} + \gamma_{\mu} \frac{R + m}{R^2 - m^2} \gamma_{\nu} \right) u(p, \lambda)[\epsilon^\mu(q', \ell')][\epsilon^\nu(q, \ell)],
\]

which is just the Compton scattering amplitude associated with the Dirac Lagrangian for states normalized to unity. This expression can be further extended toward an arbitrary magnetic dipole moment, a subject of the next section.

1. Allowing for an arbitrary \(g\)-factor for \((1/2, 0) \oplus (0, 1/2)\) particles

The factorization in (4.11) and the calculation of the characteristic determinant in (4.15), together with the freedom of choice of the antisymmetric part of the \([\Gamma_{\mu\nu}^{(0)}]_{\alpha\beta}\) tensor admitted by the Poincaré covariant projector method, allows us to make the following extension:

\[
[\Gamma_{\mu\nu}^{(0)}]_{\alpha\beta} \rightarrow [\Gamma_{\mu\nu}^{(0)}(g)]_{\alpha\beta} = [\Gamma_{\mu\nu}^{(0)}]_{\alpha\beta} + \frac{i}{4}(2-g)\gamma_{\alpha}M_{\mu\nu}^g\gamma_{\beta}.
\]

The magnetic dipole moment of a particle with polarization \(\lambda\) corresponding to this extension is then given by,

\[
\mu^{(0)}(g, \lambda) = \frac{g e^2}{2m},
\]

thus leading to an arbitrary \(g\)-factor counterpart to our previous magnetic dipole moment of the fixed value \(g^{(0)} = 2\) in (4.42). The Compton scattering calculations for this extension requires to incorporate the replacement in (5.15) into the Feynman rules for \(j_1 = 0\). In the calculation of the squared amplitude we use the following formulas for any \(j_1\):

\[
|\mathcal{M}^{(j_1)}|^2 = \frac{1}{4} \sum_{\lambda, \lambda', \ell, \ell'}|\mathcal{M}^{(j_1)}|_\lambda[\mathcal{M}^{(j_1)}]^\dagger_{\lambda'} = Tr \left[\mathcal{M}^{(j_1)}\mathcal{M}^{(j_1)*}(p', Q, R, p)\right]
\]
where we have defined
\[ M^{(j_1)}_{\mu\nu}(p', Q, R, p) = \frac{e^2}{2} \left( \frac{b' + m}{2m} \right) U^{(j_1)}_{\mu\nu}(p', Q, R, p), \tag{5.19} \]
\[ U^{(j_1)}_{\mu\nu}(p', Q, R, p) = U^{(j_1)}_{\nu\mu}(p', Q, p) + \bar{U}^{(j_1)}_{\nu\mu}(p', R, p) - \bar{C}^{(j_1)}_{\mu\nu}, \tag{5.20} \]
with \( \bar{U} \) and \( \bar{C} \) taken from (5.9) and (5.10). Here we have also used (3.12) and:
\[ [\mathcal{P}^{(j_1)}_+(p)]_{\alpha\beta} = [f^{(j_1)}(p)]_{\alpha} \left( \frac{b + m}{2m} \right) f^{(j_1)}(p)]_{\beta}, \tag{5.21} \]
for the spin-1/2 target particles of positive parity and
\[ \sum_{\ell} \epsilon^\mu(q, \ell)[\epsilon^\nu(q, \ell)]^* = -g^{\mu\nu}, \tag{5.22} \]
for the polarization vectors of the photons. The result for the averaged squared amplitude for \( j_1 = 0 \) with an arbitrary \( g \)-factor ends up being
\[ |M^{(0)}(g^{(0)})|^2 = f_0 + f_D + \frac{e^4(2m^2 - s - u)}{16m^2(m^2 - s)^2(m^2 - u)^2} \sum_{k=1}^{4} (g^{(0)} - 2)^k a_k, \tag{5.23} \]
where \( s, u \) are the standard Mandelstam variables and we are using the notations
\[ f_0 = \frac{4e^4(5m^8 - 4(s + u)m^6 + (s^2 + u^2) + s^2u^2)}{(m^2 - s)^2(m^2 - u)^2}, \tag{5.24} \]
\[ f_D = -\frac{2e^4(-2m^2 + s + u)^2}{(m^2 - s)(m^2 - u)}. \tag{5.25} \]
Here, \( f_0 \) is the Compton scattering squared amplitude corresponding to spin-0 particles \([16]\) and \( (f_0 + f_D) \) is the standard averaged squared amplitude for Compton scattering coming from the Dirac Lagrangian. We furthermore have defined:
\[ a_1 = -32m^2(m^2 - s)(m^2 - u)(2m^2 - s - u), \tag{5.26} \]
\[ a_2 = -4 \left( 13m^8 - 17(s + u)m^6 + (6(s^2 + u^2) + 20us) \right) m^4 - 7su(s + u)m^2 + 3s^2u^2, \tag{5.27} \]
\[ a_3 = -8 \left( m^2 - s \right)^2 \left( m^2 - u \right)^2, \tag{5.28} \]
\[ a_4 = (m^2 - s)(m^2 - u)(m^2(s + u) - 2su). \tag{5.29} \]
The expression (5.23) coincides with the one previously reported in \([17]\) where the spin-1/2 particles in the \((1/2, 0) \oplus (0, 1/2)\) representation space have been allowed to be of an arbitrary \( g \)-factor. The result of the current section, in combination with the causality proof of the relevant wave equation delivered above, completes the consistency proof of the combined- Lorentz-and Poincaré invariant projector method in its application to the \((1/2, 0) \oplus (0, 1/2)\) sector of the four-vector spinor.

### B. Compton scattering off spin-1/2 particles in the two-spin valued \((1/2, 1) \oplus (1, 1/2)\) sector of the four-vector–spinor

The averaged squared amplitude in this case is elaborated applying the method already presented in the previous section. Using again (5.17)-(5.22), now for \( j_1 = 1 \), gives,
\[ |M^{(1)}(g^{(1)})|^2 = f_0 + f_D + \frac{e^4(2m^2 - s - u)}{16m^2(m^2 - s)^2(m^2 - u)^2} \sum_{k=1}^{4} (g^{(1)} - 2)^k a_k, \tag{5.30} \]
with \( g^{(1)} = -2/3 \), same as before in (4.41), and with the same \( a_k \) coefficients in (5.26)-(5.29). The above expression coincides in form with (5.23) and with the result previously reported in \([17]\), in the particular case of \( g = -2/3 \).
Obtaining the differential cross-section in the laboratory frame from squared amplitudes of the types in (5.23) and (5.30) is straightforward (see [17] for details). After some algebraic manipulations one arrives at,

$$
\frac{d\sigma(g^{(j_1)}, \eta, x)}{d\Omega} = z_0 + z_D + \frac{(x-1)^2 r_0^2}{64((x-1)\eta-1)^3} \sum_{k=1}^{4} (g^{(j_1)} - 2)^k b_k,
$$

(5.31)

where \( r_0 = e^2/(4\pi m) = \alpha m, \eta = \omega/m \) where \( \omega \) is the energy if the incident photon and \( x = \cos \theta \), being \( \theta \) the scattering angle in the laboratory frame. In (5.31) \( z_0 \) denotes the standard differential cross-section for Compton scattering off spin-0 particles and \( (z_0 + z_D) \) is the standard differential cross-section for Compton scattering off Dirac particles, this is

$$
z_0 = \frac{(x^2 + 1) r_0^2}{2((x-1)\eta-1)^2},
$$

(5.32)

$$
z_D = -\frac{(x-1)^2 \eta^2 r_0^2}{2((x-1)\eta-1)^3}.
$$

(5.33)

We further have introduced the following notations,

$$
b_1 = -32(x-1)\eta^2,
$$

(5.34)

$$
b_2 = 4(x^2 - 3x + 8) \eta^2,
$$

(5.35)

$$
b_3 = 16\eta^2,
$$

(5.36)

$$
b_4 = (x + 3)\eta^2.
$$

(5.37)

The differential cross-section (5.31) has the following properties:

$$
\lim_{x \to 1} \frac{d\sigma(g^{(j_1)}, \eta, x)}{d\Omega} = r_0^2,
$$

(5.38)

$$
\lim_{\eta \to 0} \frac{d\sigma(g^{(j_1)}, \eta, x)}{d\Omega} = \frac{r_0^2}{2}(x^2 + 1),
$$

(5.39)

$$
\lim_{\eta \to \infty} \frac{d\sigma(g^{(j_1)}, \eta, x)}{d\Omega} = 0,
$$

(5.40)

meaning that in the forward direction \( (x = \cos \theta = 1) \) it takes the \( r_0^2 \) value. In the classical \( \eta \to 0 \) limit the differential cross section is symmetric with respect to the scattering angle \( \theta \), while in the high energy \( \eta \to \infty \) limit it vanishes independently of the \( g^{(j_1)} \) factor value. This observation applies to each one of the two \( j_1 = 0 \), and \( j_1 = 1 \) sectors of \( \psi_\mu \) considered here, and the related \( g^{(0)} = 2 \) and \( g^{(1)} = -2/3 \) values. The behavior of the differential cross-section is displayed in Fig. 7, which is a plot of,

$$
d\tilde{\sigma}^{(j_1)} = \frac{1}{r_0^2} \frac{d\sigma(g^{(j_1)}, \eta, x)}{d\Omega}, \quad j_1 = 0, 1.
$$

(5.41)

and \( g^{(0)} = 2, g^{(1)} = -2/3 \). Integration of (5.31) over the solid angle leads to the total cross-sections,

$$
\sigma(g^{(j_1)}, \eta) = s_0 + s_D + \sum_{k=1}^{4} (g^{(j_1)} - 2)^k \left( \frac{c_k}{128\eta(2\eta + 1)^2} + \frac{\log(2\eta + 1) h_k}{256\eta^2} \right) 3\sigma_T,
$$

(5.42)

where \( \sigma_T \) stands for the Thompson cross section \( \sigma_T = (8/3)\pi r_0^2 \). The following notations have been used,

$$
s_0 = \frac{3(\eta + 1)\sigma_T(2\eta(\eta + 1) - (2\eta + 1) \log(2\eta + 1))}{4\eta^3(2\eta + 1)},
$$

(5.43)

$$
s_D = \frac{3\sigma_T((2\eta + 1)^2 \log(2\eta + 1) - 2\eta(3\eta + 1))}{8\eta(2\eta + 1)^2},
$$

(5.44)
where \( s_0 \) and \( (s_0 + s_D) \) are the standard cross-sections for Compton scattering off spin-0 and spin-1/2 Dirac particles, while the \( c \) and \( h \) coefficients stand for the following quantities,

\[

c_1 = -32\eta(3\eta + 1), \\
c_2 = 4 \left( 6\eta^3 + \eta^2 + 8\eta + 3 \right), \\
c_3 = 16\eta^3, \\
c_4 = \eta \left( 4\eta^2 + 3\eta + 1 \right), \\
h_1 = 32\eta, \\
h_2 = 4(\eta - 3), \\
h_3 = 0, \\
h_4 = -\eta.
\]

The total cross section (5.42) has the following limits,

\[
\lim_{\eta \to 0} \sigma(g^{(j_1)}, \eta) = \sigma_T, \\
\lim_{\eta \to \infty} \sigma(g^{(j_1)}, \eta) = \frac{3}{128}(g^{(j_1)} - 2)^2((g^{(j_1)})^2 + 2)\sigma_T.
\]

Consequently, while in the \( g^{(j_1)} = 2 \) case the cross section was vanishing, for \( g^{(j_1)} = -2/3 \), it approaches \( \frac{11\pi}{27} \). In Fig. 8 the following quantity is plotted,

\[
\tilde{\sigma}(g^{(j_1)}, \eta) \equiv \frac{1}{\sigma_T}\sigma(g^{(j_1)}, \eta).
\]

For \( g^{(j_1)} = 2 \) one observes the usual decreasing behavior of the Dirac cross section with energy increase, while for \( g^{(j_1)} = -2/3 \) i.e. for spin-1/2 in \((1/2, 1) \oplus (1, 1/2)\), the cross section \( \tilde{\sigma}(g^{(j_1)}, \eta) \) at high energy approaches the fixed value of \( \frac{11\pi}{27} \) as one can see in the Fig. 8.

1. **Allowing for an arbitrary \( g \)-factor for spin-1/2 in \((1/2, 1) \oplus (1, 1/2)\)**

Testing causality of the wave equation for spin-1/2 in \((1/2, 1) \oplus (1, 1/2)\) was possible not only because of the specifically chosen antisymmetric part of \([\Gamma_{\mu\nu}^{(1)}]_{\alpha\beta} \) tensor, but also because of the auxiliary condition (3.29),
which takes its origin from the relation

\[ \gamma^\alpha [f^{(1)}](\rho)_\alpha = 0. \]  

\[ (5.58) \]

In being momentum independent, \( (5.56) \) ensures that the particle always belongs to \((1/2, 1) \oplus (1, 1/2)\). There is a number of antisymmetric structures available in the four-vector spinor representation space (see for example [16] for details) which we can employ to build up an extension of the \([\Gamma^{(1)}_{\mu\nu}]_{\alpha\beta}\) tensor toward an arbitrary \(g\) value. However, the majority of these extensions do not allow us to perform the factorization \((4.19)\) (essential for providing the causality proof of the propagation within an electromagnetic field) nor would they satisfy \((5.56)\). In fact there is only one acceptable tensor for this purpose and it reads,

\[ -\gamma_\alpha \gamma_\mu \gamma_\nu \gamma_\beta + 2\gamma_\nu \gamma_\beta g_{\alpha\mu} - 2\gamma_\mu \gamma_\beta g_{\alpha\nu} + \gamma_\alpha \gamma_\beta g_{\mu\nu}. \]

\[ (5.57) \]

Notice however that, by virtue of

\[ \gamma^\alpha [f^{(1)}(p)]_\alpha = 0, \]

\[ (5.58) \]

an extension based on \((5.57)\) does not provide any contribution neither to the on-shell current of the type of \((4.34)\), nor to the related magnetic dipole moment. A similar situation is observed with regard to the Compton scattering amplitude \((5.1)\) where also there, all contributions coming from the extension are vanishing, ultimately because of \((5.58)\). From this we conclude that any extension toward an arbitrary \(g\)-factor that is compatible with our causality testing procedure, is irrelevant to both the magnetic dipole moment, and the Compton scattering cross section values. For this reason we restrict ourselves to the calculation of Compton scattering off spin-1/2 in \((1/2, 1) \oplus (1, 1/2)\) presented above in reference to the \(g^{(1)} = -2/3\) value following from \((4.41)\).

VI. CONCLUSIONS

In the present work we studied the status of the lower-spin components of the Rarita-Schwinger four-vector spinor \(\psi_\mu\), a Lorentz-invariant representation space reducible according to \((1.1)\). The three criteria we proposed to qualify representations of the Lorentz algebra for the description of physical and observable particles of spin-\(s\) are:

(i) Irreducibility,

(ii) Hyperbolicity and causality of the related wave equations,

(iii) Finiteness of the Compton scattering cross sections in all directions and in the ultra relativistic limit.
In order to fulfill the first criterion we extended the method of the Poincaré covariant spin-\(s\) and mass-\(m\) projectors \[7\] to include momentum independent Lorentz-invariant projectors, i.e. projectors constructed from the parity conserving Casimir invariant of the Lorentz-algebra. In so doing we found two quadratic in the momenta wave equations for the two spin-1/2 sectors in \(\psi_{\mu}\), bi-linearized by properly constructed \(4 \times 4\) matrices in (2.40), the first associated with the single-spin Dirac representation space, \((1/2, 0) \oplus (0, 1/2)\), and the second for the spin-1/2 companion to spin-3/2 in \((1/2, 1) \oplus (1, 1/2)\). We demonstrated hyperbolicity and causality of both equations. We then showed that the electromagnetic current and the Compton scattering amplitudes of the first lower spin coincide with those of a genuine Dirac particle, and are characterized by a \(g = 2\) value, as it should be, and concluded on its observability. Finally we calculated Compton scattering off the second spin-1/2 in \(\psi_{\mu}\), and for a gyromagnetic ratio of \(g = −2/3\) could find finite cross sections in all directions and the ultraviolet limit. Therefore, the observability of the latter state is not excluded by none of the above three criteria. As long as the spin-1/2 under discussion is the companion to the observable spin-3/2 of equal rights within the irreducible representation space \((1/2, 1) \oplus (1, 1/2)\) (the two states are related by ladder operators), we conclude that all its properties strongly point towards its physical nature.

We furthermore notice that the method of the combined Lorentz-and Poincaré invariant projectors is suitable for the description of fermions of any spin by quadratic equations for sufficiently large Lorentz algebra representations equipped by separate Lorentz- and Dirac spinor indices. For example, pure spin-3/2 can be embedded into the totally antisymmetric tensor of second rank with Dirac spinor components, \(\Psi_{[\mu\nu]}\), a representation space that is reducible according to

\[
\Psi_{[\mu\nu]} \sim [(1, 0) \oplus (0, 1)] \otimes \left[\frac{1}{2} , 0 \right] \oplus \left[0 , \frac{1}{2} \right]
\]

\[
\rightarrow \left[\frac{1}{2} , 0 \right] \oplus \left[0 , \frac{1}{2} \right] \oplus \left[1 , \frac{1}{2} \right] \oplus \left[\frac{1}{2} , 1 \right] \oplus \left[\frac{3}{2} , 0 \right] \oplus \left[0 , \frac{3}{2} \right]
\].

(6.1)

The two redundant irreducible subspaces accompanying the single spin-3/2 in \(\Psi_{[\mu\nu]}\) can be projected out by momentum independent Lorentz-invariant projectors constructed along the lines of section II B, while the \((3/2, 0) \oplus (0, 3/2)\) subspace can be identified by the Poincaré covariant projector which is second order in the momenta. Similarly, spin-5/2 can be embedded in the totally antisymmetric Lorentz tensor of second rank with four-vector-spinor components, \(\Psi_{[\mu\nu]_\eta}\), a representation space reducible according to

\[
\Psi_{[\mu\nu]_\eta} \sim [(1, 0) \oplus (0, 1)] \otimes \left[\frac{1}{2} \right] \otimes \left[\frac{1}{2} \right] \otimes \left[0 , \frac{1}{2} \right] \oplus \left[\frac{1}{2} , 0 \right] \oplus \left[0 , \frac{1}{2} \right]
\]

\[
\rightarrow 2 \left[\frac{1}{2} , 0 \right] \oplus \left[0 , \frac{1}{2} \right] \oplus 3 \left[\frac{1}{2} , \frac{1}{2} \right] \oplus \left[\frac{3}{2} , \frac{1}{2} \right] \oplus \left[\frac{1}{2} , \frac{3}{2} \right] \oplus \left[\frac{3}{2} , \frac{3}{2} \right] \oplus \left[\frac{1}{2} , 0 \right] \oplus \left[0 , \frac{1}{2} \right]
\].

(6.2)

In this case, all the invariant subspaces beyond the two-spin sector \((2, 1/2) \oplus (1, 2/2)\) can be projected out by properly constructed momentum independent Lorentz-invariant projectors while spin-3/2, and spin-5/2 in \((2, 1/2) \oplus (1, 2/2)\) can be separated by means of a Poincaré covariant projector. In this fashion, a second order Lagrangian for spin-5/2 description in terms of a representation space of separate Lorentz and Dirac spinor indices can be furnished.

We expect relevance of our observations in processes which are sensitive to the irreducibility of the Lorentz representations.

[1] S. Weinberg, \textit{The Quantum Theory of Fields. Vol. 1: Foundations} (Cambridge University Press, Cambridge, 1995).
[2] A. E. Kaloshin and V. P. Lomov, Part. Nucl. Lett. \textbf{8}, 868 (2011).
[3] A. E. Kaloshin and V. P. Lomov, Mod. Phys. Lett. \textbf{19}, 135 (2004).
[4] V. Paskalutsa, and O. Scholten, Nucl. Phys. A \textbf{591}, 658 (1995).
[5] Daniel F. Freed, \textit{Five Lectures on Supersymmetry} (American Mathematical Society, 1999).
[6] Brain G. Wyborne, \textit{Group theory for physicists} (Wiley & Sons, N.Y., 1974).
[7] M. Napsuciale, M. Kirchbach and S. Rodriguez, Eur. Phys. J. A \textbf{29}, 289 (2006).
[8] E.G. Delgado-Acosta, M. Kirchbach, M. Napsuciale, S. Rodriguez, Phys. Rev. D \textbf{87}, 096010 (2013).
[9] M. Napsuciale, S. Rodriguez, E.G. Delgado-Acosta and M. Kirchbach, Phys. Rev. D \textbf{77}, 014009 (2008).
[10] P. Van Nieuwenhuizen, Phys. Rept. \textbf{68} (1981) 189.
[11] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1998).
[12] G. Velo and D. Zwanziger, Phys. Rev. **188**, 2218 (1969).
[13] L. M. Rico and M. Kirchbach, Mod. Phys. Lett. A **21**, 2961 (2006).
[14] E. G. Delgado-Acosta, M. Kirchbach, M. Napsuciale and S. Rodriguez, Phys. Rev. D **85**, 116006 (2012).
[15] C. Lorcé, Phys. Rev. D **79**, 113011 (2009).
[16] E. G. Delgado-Acosta and M. Napsuciale, Phys. Rev. D **80**, 054002 (2009).
[17] E. G. Delgado-Acosta, M. Napsuciale, and S. Rodriguez, Phys. Rev. D **83**, 073001 (2011).