THE CAUCHY PROBLEM FOR NON-ISENTRPIC COMPRESSIBLE
NAVIER-STOKES/ALLEN-CAHN SYSTEM WITH DEGENERATE
HEAT-CONDUCTIVITY

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Abstract. The Cauchy problem for non-isentropic compressible Navier-Stokes/Allen-
Cahn system with degenerate heat-conductivity \( \kappa(\theta) = \tilde{\kappa} \theta^\beta \) in 1-d is discussed in this
paper. This system is widely used to describe the motion of immiscible two-phase flow in
numerical simulation. The wellposedness for strong solution of this problem is established
with the \( H^1 \) initial data for density, temperature, velocity, and the \( H^2 \) initial data for
phase field. The result shows that no discontinuity of the phase field, vacuum, shock
wave, mass or heat concentration will be developed at any finite time in the whole space.
From the hydrodynamic point of view, this means that no matter how complex the
interaction between the hydrodynamic and phase-field effects, phase separation will not
occur, but the phase transition is possible.

1. Introduction

Coolant in nuclear power systems is usually operated at high temperature and pressure,
therefore, compressible two-phase flow is often encountered, for example, the saturated
water-vapor systems, etc. The similar phenomenon occurs in the flow of multipolymers
in chemical industries. In this paper, we study the Cauchy problem for compressible non-
isentropic Navier-Stokes/Allen-Cahn system proposed by Heida-Málek-Rajagopal [10] (or
see Blesgen [2]). This system is an important mathematical model for the numerical
simulation of compressible immiscible two-phase flow, an important feature of this model
is that the diffusion interface between phases represents the free boundary between phases,
described as following:

\[
\begin{align*}
\rho_t + \text{div}(\rho \mathbf{u}) &= 0, \\
(\rho \mathbf{u})_t + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) &= \text{div} \mathbb{T}, \\
(\rho \phi)_t + \text{div}(\rho \mathbf{u} \phi) &= -\mu, \\
\mu &= \rho \frac{\partial f}{\partial \phi} - \text{div}(\rho \frac{\partial f}{\partial \nabla \phi}), \\
(\rho E)_t + \text{div}(\rho E \mathbf{u}) &= \text{div}(\mathbb{T} \mathbf{u} + \kappa(\theta) \nabla \theta - \mu \frac{\partial f}{\partial \nabla \phi}),
\end{align*}
\]

where \( \tilde{x} \in \mathbb{R}^N \) is the spatial variable, \( \tilde{t} \) the time, \( N \) the spatial dimension. The unknown
functions \( \rho(\tilde{x}, \tilde{t}), \mathbf{u}(\tilde{x}, \tilde{t}), \theta(\tilde{x}, \tilde{t}), \phi(\tilde{x}, \tilde{t}) \) denote the total density, the velocity, the absolute
temperature and the phase field of the immiscible two-phase flow respectively. \( \mu(\tilde{x}, \tilde{t}) \) is the
chemical potential, $f$ the phase-phase interfacial free energy density, here we consider its common form as following (see Lowengrub-Truskinovsky \([24]\), Heida-Málek-Rajagopal \([10]\))

$$f(\rho, \phi, \nabla \phi) \overset{\text{def}}{=} \frac{1}{4\epsilon}(1 - \phi^2)^2 + \frac{\epsilon}{2\rho} |\nabla \phi|^2, \quad (1.2)$$

where $\epsilon > 0$ the thickness of the interface between the phases. The Cauchy stress-tensor $\mathbf{T}$ is represented by

$$\mathbf{T} = 2\tilde{\nu}\mathbb{D}(\mathbf{u}) + \tilde{\lambda}(\text{div}\mathbf{u})\mathbb{I} - p\mathbb{I} - \rho\nabla \phi \otimes \frac{\partial f}{\partial \nabla \phi}. \quad (1.3)$$

In the expression \((1.3)\), $\mathbb{I}$ is the unit matrix, $\mathbb{D}\mathbf{u}$ is the so-called deformation tensor

$$\mathbb{D}\mathbf{u} = \frac{1}{2}(\nabla \mathbf{u} + \nabla^\top \mathbf{u}), \quad (1.4)$$

here the superscript $\top$ denotes the transpose and all vectors are column ones. The positive constants $\tilde{\nu} > 0, \tilde{\lambda} > 0$ are viscosity coefficients, satisfying

$$\tilde{\nu} > 0, \quad \tilde{\lambda} + \frac{2}{N}\tilde{\nu} \geq 0. \quad (1.5)$$

The total energy density $\rho E$ is given by

$$\rho E = \rho e + \rho f + \frac{1}{2}\rho u^2, \quad (1.6)$$

where $\rho e$ is the internal energy, $\frac{1}{2}\rho u^2$ the kinetic energy. The pressure $p$, entropy $s$ and $\rho, e, f$ obey the second law of thermodynamics

$$ds = \frac{1}{\theta}(d(e + f) + pd(\frac{1}{\rho})), \quad (1.7)$$

which implies that

$$p = \rho^2 \frac{\partial(e + f)}{\partial \rho} + \theta^2 \frac{\partial p}{\partial \theta} = \rho^2 \frac{\partial e(\rho, \theta)}{\partial \rho} - \frac{\epsilon}{2} |\nabla \phi|^2 + \theta \frac{\partial p}{\partial \theta}. \quad (1.8)$$

Throughout this paper, we consider the ideal polytropic gas, that is,

$$p(\rho, \theta) = R\rho \theta - \frac{\epsilon}{2} |\nabla \phi|^2, \quad (1.9)$$

and $\epsilon$ satisfies

$$\epsilon = c_v \theta + \text{constant}, \quad (1.10)$$

where $c_v$ is the specific heat capacity. $\kappa(\theta)$ is the heat conductivity satisfying

$$\kappa(\theta) = \tilde{\kappa} \theta^\beta, \quad (1.11)$$

with positive constants $\tilde{\kappa} > 0$ and $\beta > 0$.

**Remark 1.1.** The assumption \((1.11)\) is based on the following reason: the heat conductivity $\kappa$ of compressible immiscible two-phase flow vary with temperature under very high temperature and density environment. Strictly speaking, the Chapman-Enskog expansion for the first order approximation tells us, the coefficient of heat conduction depends on temperature (see Chapman-Colwing \([3]\)).
Substituting (1.2), (1.3), (1.6), (1.9) and (1.10) into (1.1), then (1.1) is simplified as
\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
\rho u_t + \rho (u \cdot \nabla u) - 2\nu \text{div}D - \lambda \nabla \text{div} u &= -\text{div}\left(\epsilon \nabla \phi \otimes \nabla \phi - \frac{\epsilon}{2} |\nabla \phi|^2 + \theta \frac{\partial p}{\partial \theta}\right), \\
\rho \phi_t + \rho u \cdot \nabla \phi &= -\mu, \\
\rho \mu &= \frac{\rho}{\epsilon} (\phi^3 - \phi) - \epsilon \Delta \phi, \\
c_v (\rho \phi_t + \rho u \cdot \nabla \theta) + \theta \rho \theta \text{div} u - \text{div}(\kappa(\theta \nabla \theta) = 2\nu \|D_u\|^2 + \lambda (\text{div} u)^2 + \mu^2.
\end{align*}
\] (1.12)

In order to study the wellposedness of Cauchy problems for (1.12), we start with the one-dimensional case for the system (1.12) in this paper. Without loss of generality, we define
\[
v = \frac{1}{\rho}, \quad \nu = 2\nu + \hat{\lambda},
\] (1.13)
and we assume that \(\nu = \hat{R} = c_v = \hat{k} = 1\). By using Lagrange coordinate transformation
\[
x = \int_0^{\tilde{x}} \rho(\xi, \hat{t})d\xi, \quad t = \hat{t},
\] (1.14)
then the Cauchy problem for system (1.12) in 1-D can be rewritten as
\[
\begin{align*}
v_t - u_x &= 0, \\
u_t + \left(\frac{\theta}{v}\right)_x &= \left(\frac{u_x}{v}\right)_x - \frac{\epsilon}{2} \left(\frac{\phi_x}{v^2}\right)_x, \\
\phi_t &= -v \mu, \\
\mu &= \frac{1}{\epsilon} \left(\phi^3 - \phi\right) - \epsilon \left(\frac{\partial \phi}{\partial x}\right)_x, \\
\theta_t + \frac{\theta}{v} u_x - \left(\frac{\theta^3}{v^2}\right)_x &= \frac{u_x^2}{v} + v \mu^2,
\end{align*}
\] (1.15)
with initial condition
\[
(v, u, \theta, \phi)(x, 0) = (v_0, u_0, \theta_0, \phi_0)(x), \quad (v_0, u_0, \theta_0, |\phi_0|)(x) \xrightarrow{x \to \pm \infty} (1, 0, 0, 1). \] (1.16)

**Remark 1.2.** What we want to point out here is that, (1.16) implies that initial concentration difference \(\phi_0\) of the immiscible two-phase flow is 1 or -1 at far fields.

Before giving the main theorem, we will briefly review the relevant work of (1.15) that has been done. For the Cauchy problem of compressible non-isentropic Navier-Stokes equations with constant coefficient of heat conduction \((\beta = 0)\) in (1.11)) in 1-d (that is \(\phi \equiv 1\) in (1.15)), Kanel [19] obtained the solvability and the behavior of the solution as \(t \to \infty\) under the assumption that the initial functions are small perturbations of the constants. Kazhikhov [18] presented the global existence of solutions without any restrictions on the smallness of the initial data. Jiang [17] proved that in any bounded interval, the specific volume is pointwise bounded from below and above for all \(t \geq 0\) and a generalized solution is convergent as \(t \to \infty\). Li-Liang [22] got the boundedness of the temperature from below and above independently of both time and space, and they shown that the global solution is asymptotically stable as time tends to infinity for large initial perturbations.

For higher dimensional problems of non-isentropic compressible Navier-Stokes equations, Matsumura-Nishada [25] proved that there exists a unique global solution for the small initial data in 3-d. For the isentropic case, Lions [23] obtained global existence
Moreover, there exists a positive constant $C$ such that

$$C^{-1} \leq v(x,t) \leq C, \quad C^{-1} \leq \theta(x,t) \leq C, \quad \phi(x,t) \in [-1, 1], \quad (x,t) \in \mathbb{R} \times [0,T].$$

Theorem 1.1. Assume that

$$(v_0 - 1, u_0, \theta_0 - 1) \in H^1(\mathbb{R}), \quad \phi_0^2 - 1 \in L^2(\mathbb{R}), \quad \phi_{0x} \in H^1(\mathbb{R}),$$

and

$$\inf_{x \in \mathbb{R}} v_0(x) > 0, \quad \inf_{x \in \mathbb{R}} \theta_0(x) > 0, \quad \phi_0(x) \in [-1, 1].$$

Then, the Cauchy problem (1.15)-(1.16) has a unique strong solution $(v, u, \theta, \phi)$ such that for fixed $T > 0$, satisfying

$$\begin{cases}
v - 1, u, \theta - 1 \in L^\infty(0,T; H^1(\mathbb{R})), \quad \phi^2 - 1 \in L^\infty(0,T; L^2(\mathbb{R})), \\
\phi_x \in L^\infty(0,T; H^1(\mathbb{R})), \quad (\phi^2 - 1)_x \in L^2(0,T; L^2(\mathbb{R})), \\
v_x \in L^2(0,T; L^2(\mathbb{R})), \quad u_x, \theta_x \in L^2(0,T; H^1(\mathbb{R})), \quad \phi_{xx} \in L^2(0,T; H^1(\mathbb{R})),
\end{cases}$$

Moreover, there exists a positive constant $C$ depending on the initial data and $T$, satisfying

$$C^{-1} \leq v(x,t) \leq C, \quad C^{-1} \leq \theta(x,t) \leq C, \quad \phi(x,t) \in [-1, 1], \quad (x,t) \in \mathbb{R} \times [0,T].$$
Remark 1.3. Theorem 1.1 shows that no discontinuity of the phase field, vacuum, shock wave, mass or heat concentration will be developed in finite time as the initial data \((v_0, u_0, \theta_0, |\phi_0|)(x) \rightarrow x \rightarrow \pm \infty \Rightarrow (1, 0, 1, 1)\). Which means that no matter how complex the interaction between the hydrodynamic and phase-field effects, as well as the the motion of the compressible two-phase immiscible flow has large oscillations, there is no separation of the phase field in the finite time.

Now we briefly describe some key points of proof for Theorem 1.1. The most important of the proof is to get the positive upper bound and the lower bound of \(v, \theta\) and \(\phi\). Otherwise the system (1.15) will degenerate. For this purpose, firstly, inspired by the idea of Kazhikhov [18] and Jiang [17], we obtain a key expression of \(v\) (see (2.11)). Secondly, using the expression of \(v\), combining with the basic energy estimate (2.2), the truncation function method, and the convexity of \(y - \ln y - 1\), we get the lower bound of \(v\) and \(\theta\) (see (2.18)). Further, after observing the key inequality (see (2.34))

\[
\sup_{x \in \mathbb{R}} \left( \frac{\phi_x}{v} \right)^2(x, t) \leq C \left( \sup_{x \in \mathbb{R}} (\theta - 1) + 1 + V(t) \right),
\]

the upper bound of \(v\) can be derived. Finally, with the help of the key inequality as following (see (2.43)),

\[
\int_0^T \sup_{x \in \mathbb{R}} (\theta - 1)^2 dt \leq C.
\]

(1.21)

the higher order energy estimates for \(\phi\) and \(v\) can be achieved through the tedious energy estimates, (see (2.37), (2.54)). In particular, the upper bound of \(\|\theta_v\|_{L^\infty(0,T;L^2)}\) is derived (see (2.69)), and thus the upper bound of temperature \(\theta\) is achieved. The whole procedure of the proof will be carried out in the next section.

2. The Proof of Theorem

The local existence and uniqueness for strong solutions of (1.15)-(1.16) is presented as following which can be proved by the fixed point method, the details are omitted here.

Lemma 2.1. Let (1.17) and (1.18) hold, then there exists some \(T_*>0\) such that, the Cauchy problem (1.15)-(1.16) has a unique strong solution \((v, u, \theta, \phi)\) satisfying

\[
\begin{aligned}
&v - 1, u, \theta - 1 \in \mathcal{L}^\infty(0, T^*; H^1(\mathbb{R})), \phi^2 - 1 \in \mathcal{L}^\infty(0, T^*; L^2(\mathbb{R})), \\
&\phi_x \in \mathcal{L}^\infty(0, T^*; H^1(\mathbb{R})), \quad (\phi^2 - 1)_x \in \mathcal{L}^2(0, T^*; L^2(\mathbb{R})), \\
&v_x \in \mathcal{L}^2(0, T^*; L^2(\mathbb{R})), u_x, \theta_x \in \mathcal{L}^2(0, T^*; H^1(\mathbb{R})), \phi_{xx} \in \mathcal{L}^2(0, T^*; H^1(\mathbb{R})),
\end{aligned}
\]

(2.1)

With the existence of a local solution, Theorem 1.1 can be achieved by extending the local solutions globally in time from the following series of prior estimates. Without loss of generality, in the following prior estimates, we assume that \(\nu = R = c_v = \tilde{\kappa} = 1\), and

\[
\int_{-\infty}^{+\infty} (v_0 - 1) dx = 1, \quad \int_{-\infty}^{+\infty} \left( \frac{v_0^2}{2} + (\theta_0 - 1) + \frac{1}{4\epsilon}(\phi_0^2 - 1)^2 + \frac{\epsilon}{2\nu_0} \phi_0 dx \right) dx = 1.
\]

(2.2)

From here to the end of this paper, \(C > 0\) denotes the generic positive constant depending only on \(\|v_0 - 1, u_0, \theta_0 - 1\|_{H^1(\mathbb{R})}, \|\phi_0^2 - 1\|_{L^2(\mathbb{R})}, \|\phi_0\|_{H^1(\mathbb{R})}, \inf_{x \in \mathbb{R}} v_0(x), \text{ and } \inf_{x \in \mathbb{R}} \theta_0(x)\).
Lemma 2.2. Let \((v, u, \theta, \phi)\) be a smooth solution of \((1.15)-(1.16)\) on \((-\infty, +\infty) \times [0, T]\). Then it holds
\[
\sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} \left( \frac{u^2}{2} + \frac{1}{4e} (\phi^2 - 1)^2 + \frac{\epsilon \phi^2}{2} + (v - \ln v - 1) + (\theta - \ln \theta - 1) \right) dx \\
+ \int_0^T V(t) dx dt \leq E_0,
\]
where
\[
E_0 \overset{\text{def}}{=} \int_{-\infty}^{+\infty} \left( \frac{u^2}{2} + \frac{1}{4e} (\phi_0^2 - 1)^2 + \frac{\epsilon \phi_0^2}{2} + (v_0 - \ln v_0 - 1) + (\theta_0 - \ln \theta_0 - 1) \right) dx,
\]
and
\[
V(t) = \int_{-\infty}^{+\infty} \left( \frac{\theta^2}{v \theta^2} + \frac{u^2}{v \theta} + \frac{v \mu^2}{\theta} \right) dx.
\]
Proof. From \((1.15)\), \((1.16)\) and \((2.2)\), we have
\[
\int_{-\infty}^{+\infty} (v - 1) dx = 1, \quad \int_{-\infty}^{+\infty} \left( \frac{u^2}{2} + (\theta - 1) + \frac{1}{4e} (\phi^2 - 1)^2 + \frac{\epsilon \phi^2}{2} \right) dx = 1.
\]
Multiplying \((1.15)_1\) by \(1 - \frac{1}{v}\), \((1.15)_2\) by \(u\), \((1.15)_3\) by \(\mu\), \((1.15)_5\) by \(1 - \frac{1}{v}\), adding them together, we get
\[
\left( \frac{u^2}{2} + \frac{1}{4e} (\phi^2 - 1)^2 + \frac{\epsilon \phi^2}{2} + (v - \ln v - 1) + (\theta - \ln \theta - 1) \right) + \left( \frac{\theta^2}{v \theta^2} + \frac{u^2}{v \theta} + \frac{v \mu^2}{\theta} \right)
= u_x + \left( \frac{uu_x}{v} - \frac{u \theta}{v} \right)_x + \left( \frac{\theta \theta_x}{v} \right)_x + \epsilon \left( \frac{\phi_x \phi_t}{v} \right)_x - \epsilon \left( \frac{\phi_x u}{v^2} \right)_x.
\]
Integrating \((2.7)\) over \((-\infty, +\infty) \times [0, T]\) by parts, \((2.3)\) is obtained, the proof of Lemma 2.2 is finished. \(\Box\)

Lemma 2.3. Let \((v, u, \theta, \phi)\) be a smooth solution of \((1.15)-(1.16)\) on \((-\infty, +\infty) \times [0, T]\), then \(\forall n = 0, \pm 1, \pm 2, \cdots\), there are points \(a_n(t), b_n(t)\) on the interval \([n, n + 1]\), such that
\[
v(a_n(t), t) \overset{\text{def}}{=} \bar{v}_n(t) = \int_n^{n+1} v(x, t) dx \in [\alpha_1, \alpha_2],
\]
\[
\theta(b_n(t), t) \overset{\text{def}}{=} \bar{\theta}_n(t) = \int_n^{n+1} \theta(x, t) dx \in [\alpha_1, \alpha_2],
\]
where \(0 < \alpha_1 < \alpha_2\) are the two roots of the following algebraic equation
\[
y - \ln y - 1 = E_0.
\]
Proof. By using the convexity of the function \(y - \ln y - 1\) and the Jensen’s inequality, then
\[
\int_n^{n+1} \theta dx - \ln \int_n^{n+1} \theta dx - 1 \leq \int_n^{n+1} (\theta - \ln \theta - 1) dx,
\]
\[
\int_n^{n+1} v dx - \ln \int_n^{n+1} v dx - 1 \leq \int_n^{n+1} (v - \ln v - 1) dx.
\]
Combining with the inequality \((2.3)\), using the convexity of the function \(y - \ln y - 1\) once again, \((2.8)\) is obtained immediately. The proof of Lemma 2.3 is finished. \(\Box\)
Lemma 2.4. Let \((v, u, \theta, \phi)\) be a smooth solution of (1.15)-(1.16) on \((-\infty, +\infty) \times [0, T]\), then \(\forall n = 0, \pm 1, \pm 2, \ldots\), it has the following expression of \(v\)

\[
v(x, t) = D(x, t)Y(t) + \int_{0}^{t} D(x, t)Y(t) \left( \theta(x, \tau) + \frac{\epsilon}{2} \frac{\phi_{x}^{2}(x, \tau)}{v(x, \tau)} \right) d\tau, \quad x \in [n, n + 1],
\]

where

\[
D(x, t) = v_{0}(x)e^{\int_{a_{n}(t)}^{t} (u(y, t) - u_{0}(y)) dy},
\]

and

\[
Y(t) = \frac{v(a_{n}(t), t)}{v_{0}(a_{n}(t))} e^{-\int_{0}^{a_{n}(t)} (\frac{\epsilon}{2} + \frac{\phi_{x}^{2}}{v^{2}})(a_{n}(s), s) ds}.
\]

Proof. We rewrite (1.15) as

\[
(ln v)_{xt} = \left( \frac{\theta}{v} + \frac{\epsilon \phi_{x}^{2}}{2 v^{2}} \right)_{x} + u_{t},
\]

where we have used (1.15). Integrating the above equation over \((0, t)\), we obtain

\[
(ln v)_{x} = \left( \int_{0}^{t} \left( \frac{\theta}{v} + \frac{\epsilon \phi_{x}^{2}}{2 v^{2}} \right) d\tau \right)_{x} + u - u_{0} + (ln v_{0})_{x},
\]

For \(x \in [n, n + 1]\), integrating (2.15) from \(a_{n}(t)\) to \(x\) by parts, we have

\[
v(x, t) = D(x, t)Y(t) e^{\int_{a_{n}(t)}^{t} \left( \frac{\theta}{v} + \frac{\epsilon \phi_{x}^{2}}{2 v^{2}} \right)(x, s) ds},
\]

with \(D(x, t)\) and \(Y(t)\) as defined in (2.12) and (2.13) respectively. Now we introduce the function \(g(x, t)\) as following

\[
g(x, t) = \int_{0}^{t} \left( \frac{\theta}{v} + \frac{\epsilon \phi_{x}^{2}}{2 v^{2}} \right)(x, s) ds,
\]

by using (2.16), we get the following ordinary differential equation for \(g(x, t)\)

\[
g_{t} = \frac{\theta(x, t) + \frac{\epsilon \phi_{x}^{2}(x, t)}{2 v(x, t)}}{v(x, t)} = \frac{\theta(x, t) + \frac{\epsilon \phi_{x}^{2}(x, t)}{v(x, t)}}{D(x, t)Y(t) e^{\theta}},
\]

and this gives

\[
e^{\theta} = 1 + \int_{0}^{t} \frac{\theta(x, \tau) + \frac{\epsilon \phi_{x}^{2}(x, \tau)}{D(x, \tau)Y(\tau) v(x, \tau)}}{D(x, \tau)Y(\tau)} d\tau,
\]

substituting the expression above into (2.16), we have (2.11). Thus the proof of Lemma 2.4 is finished.

\[
Lemma 2.5. Let \((v, u, \theta, \phi)\) be a smooth solution of (1.15)-(1.16) on \((-\infty, +\infty) \times [0, T]\), then it holds that for
\]

\[
v(x, t) \geq C^{-1}, \quad \theta(x, t) \geq C^{-1}, \quad x \in (-\infty, +\infty) \times [0, T], \quad \forall (x, t) \in (-\infty, +\infty) \times [0, T],
\]

\[
\int_{0}^{T} \int_{-\infty}^{+\infty} \frac{\theta^{3} \phi_{x}^{2}}{v^{2\alpha + 1}} \varphi_{n} dx dt \leq C, \quad \forall ~ 0 < \alpha < 1,
\]

where \(\varphi_{n}\) is defined in (2.23).
Proof. Firstly, from (1.17), (2.6) and the definition (2.12) of $D$, we have
\[ C^{-1} \leq D(x,t) \leq C, \quad \forall (x,t) \in (-\infty, +\infty) \times [0, T]. \] (2.19)
Moreover, by using (2.11), we have
\[ Y^{-1}(t) \int_n^{n+1} v(x,t)dx = \int_n^{n+1} D(x,t)dx + \int_0^t \int_n^{n+1} \frac{D(x,t)(\theta(x,\tau) + \frac{\phi^2(x,\tau)}{v(x,\tau)})}{D(x,\tau)Y(\tau)}dxdt, \]
Applying the inequality (2.8) and (2.19) to the result above, there exists a positive constant $C$, satisfying
\[ C^{-1}Y^{-1}(t) \leq 1 + \int_0^t Y^{-1}(s)ds \leq CY^{-1}(t), \] (2.20)
which implies that
\[ 0 < C^{-1} \leq Y(t) \leq C < +\infty, \quad \forall (x,t) \in (-\infty, +\infty) \times [0, T]. \] (2.21)
From (2.16), (2.19) and (2.21), we obtain the lower bound of $v$ as following
\[ v(x,t) \geq C^{-1}, \quad \forall (x,t) \in (-\infty, +\infty) \times [0, T]. \] (2.22)
Secondly, for $\forall n = 0, \pm1, \pm2, \cdots$, let us consider the cut-off function:
\[ \varphi_n(x) = \begin{cases} e^{\frac{1}{n}(x-n)}, & x \leq n, \\ 1, & n \leq x \leq n+1, \\ e^{\frac{1}{n}(n+1-x)}, & x \geq n+1. \end{cases} \] (2.23)
For $\forall p > 2$, multiplying (1.15) by $\theta^{-p}\varphi_n$, integrating over $(-\infty, +\infty)$ with respect to $x$, by using (2.22), we have
\[
\frac{1}{p-1} \frac{d}{dt} \int_{-\infty}^{+\infty} (\theta^{-1})^{p-1} \varphi_n dx + \int_{-\infty}^{+\infty} \frac{u_x^2}{v\theta^p} \varphi_n dx + \int_{-\infty}^{+\infty} \frac{v\mu^2}{\theta^p} \varphi_n dx + p \int_{-\infty}^{+\infty} \frac{\theta^p \theta_x^2}{v\theta^{p+1}} \varphi_n dx
\leq \int_{-\infty}^{+\infty} \frac{u_x}{v\theta^{p-1}} \varphi_n dx + \int_{-\infty}^{+\infty} \frac{\theta^p \theta_x}{v\theta^p} \varphi_n dx
\leq \frac{1}{2} \int_{-\infty}^{+\infty} \frac{u_x^2}{v\theta^p} \varphi_n dx + \frac{1}{2} \int_{-\infty}^{+\infty} \frac{1}{v\theta^{p-2}} \varphi_n dx + \frac{p}{2} \int_{-\infty}^{+\infty} \frac{\theta^p \theta_x^2}{v\theta^{p+1}} \varphi_n dx + \frac{1}{2p} \int_{-\infty}^{+\infty} \theta^p \varphi_n dx + C(\|\theta^{-1} \varphi \|_{L^{p-2}}^{p-2} + \|\theta^{-1} \varphi \|_{L^{p-1}}^{p-1}).
\]
Applying Gronwall’s inequality to the above result, we obtain
\[
\sup_{0 \leq t \leq T} \|\theta^{-1}(\cdot,t)\|_{L^{p-1}(\mathbb{R}^n)} \leq C, \quad \forall p > \max\{2, \beta + 1\},
\]
where $C$ is independent of $n$, and further, letting $p$ tends to infinity, we do eventually get the lower bound of $\theta$ on $(-\infty, +\infty)$.

Finally, for $0 < \alpha < 1$, multiplying (1.15) by $\theta^{-\alpha}$, integrating over $(-\infty, +\infty) \times [0, T]$ by parts, repeating the above analysis steps, we have
\[
\int_{-\infty}^{+\infty} \frac{\theta^p \theta_x^2}{v\theta^{p+1}} \varphi_n dxdt \leq C.
\] (2.24)
The proof of Lemma 2.5 is completed. □
Lemma 2.6. Let \((v, u, \theta, \phi)\) be a smooth solution of (1.15)-(1.16) on \((-\infty, +\infty) \times [0, T]\), then \(\forall n = 0, \pm 1, \pm 2, \cdots\), it has the following inequalities:

\[
|\phi(x, t)| \leq C, \quad v(x, t) \leq C, \quad \forall (x, t) \in (-\infty, +\infty) \times [0, T],
\]

where \(C\) is only dependent of \(\epsilon, E_0\).

**Proof.** Firstly, \(\forall n = 0, \pm 1, \pm 2, \cdots\), from (2.3), we have

\[
\frac{\epsilon}{4} \int_n^{n+1} \phi^4(x, t) dx \leq \frac{\epsilon}{2} \int_n^{n+1} \phi^2(x, t) dx + \frac{\epsilon}{4} + E_0
\]

\[
\leq \frac{\epsilon}{8} \int_n^{n+1} \phi^4(x, t) dx + 3\epsilon + E_0,
\]

which implies that

\[
\int_n^{n+1} \phi^4(x, t) dx \leq 6 + \frac{8E_0}{\epsilon},
\]

and therefore

\[
\int_n^{n+1} \phi(x, t) dx \leq C.
\]

where \(C\) is independent of \(n\). Now \(\forall (x, t) \in [n, n+1) \times [0, T]\),

\[
|\phi(x, t)| \leq \left| \int_n^{n+1} (\phi(x, t) - \phi(y, t)) dy \right| + \int_n^{n+1} |\phi(y, t)| dy
\]

\[
\leq \left| \int_n^{n+1} \left( \int_y^x \phi(x) dx \right) dy \right| + C
\]

\[
\leq \left( \int_n^{n+1} \frac{\phi_x^2}{\nu} \right) \frac{1}{2} + C
\]

\[
\leq \frac{(2E_0)}{\epsilon} \frac{1}{2} + C.
\]

Secondly, \(\forall n = 0, \pm 1, \pm 2, \cdots\), for \(\alpha = \min\{1, \beta\}/2\), using (2.18), we get

\[
\int_0^T \max_{x \in [n, n+1]} (\theta - 1) dt \leq C + C \int_0^T \int_n^{n+1} |\theta_x| dx dt
\]

\[
\leq C + C \int_0^T \int_n^{n+1} \frac{\theta^\beta \theta_x^2}{\nu \theta^1 + \alpha} dx dt + C \int_0^T \int_n^{n+1} \frac{\nu \theta^1 + \alpha}{\theta^\beta} dx dt
\]

\[
\leq C + C \int_0^T \left[ \int_0^{+\infty} \frac{\theta^\beta \theta_x^2}{\nu \theta^1 + \alpha} \varphi_n dx dt + \frac{1}{2} \int_0^T \max_{x \in [n, n+1]} (\theta - 1) dt \right].
\]

which together with (2.29) yields that

\[
\int_0^T \sup_{x \in \mathbb{R}} (\theta - 1) dt \leq C.
\]

Finally, we can give the upper bounds of \(v\). In fact, combining the expression of \(v\) (2.11) with the upper and lower bound estimates (2.19)–(2.22), we have the following inequality

\[
v(x, t) = D(x, t)Y(t) + \int_0^t D(x, t)Y(t) \left( \theta(x, \tau) + \frac{\epsilon \phi_x^2(x, \tau)}{2 v(x, \tau)} \right) d\tau
\]
\[ \leq C + C \int_0^t \left( \sup_{x \in \mathbb{R}} \theta(x, \tau) + \sup_{x \in \mathbb{R}} \left( \frac{\phi_x(x, \tau)}{v(x, \tau)} \right)^2 \sup_{x \in \mathbb{R}} v(x, \tau) \right) d\tau. \]

Deriving from (1.15)_4, we have
\[ \epsilon \left( \frac{\phi_x}{v} \right)_x = -\mu + \frac{1}{\epsilon} (\phi^3 - \phi), \quad (2.32) \]
and then from (2.3), (2.6), (2.18), (2.28), we obtain
\[ \int_{-\infty}^{+\infty} \left( \frac{\phi_x}{v} \right)^2 x^2 dx \leq C (1 + V(t)). \quad (2.33) \]

Using (2.3),(2.6),(2.22),(2.28), (2.33), we get
\[ \sup_{x \in \mathbb{R}} \left( \frac{\phi_x}{v} \right)^2 (x, t) \leq C \int_{-\infty}^{+\infty} \frac{\phi_x}{v} \left( \frac{\phi_x}{v} \right) x dx \leq C \int_{-\infty}^{+\infty} \theta \frac{\phi_x^2}{v^2} dx + \int_{-\infty}^{+\infty} \left( \frac{\phi_x}{v} \right)^2 x^2 dx \leq C \left( \sup_{x \in \mathbb{R}} (\theta - 1) + 1 + V(t) \right). \quad (2.34) \]

Substituting (2.34) into (2.31), we achieve
\[ v(x, t) \leq C + C \int_0^t \left( \sup_{x \in \mathbb{R}} (\theta - 1) + 1 + V(t) \right) \sup_{x \in \mathbb{R}} v(x, \tau) d\tau. \quad (2.35) \]

Applying the Gronwall inequality to the above (2.35), we get
\[ v(x, t) \leq C, \quad \forall (x, t) \in (-\infty, +\infty) \times [0, T]. \quad (2.36) \]

The proof of Lemma 2.6 is finished.

**Lemma 2.7.** Let \((v, u, \theta, \phi)\) be a smooth solution of (1.15)-(1.16) on \((-\infty, +\infty) \times [0, T]\), then for \(\forall (x, t) (-\infty, +\infty) \times [0, T]\), the following inequalities hold
\[ \sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} v_x^2 dx \leq C, \quad \int_0^T \int_{-\infty}^{+\infty} \left( (\phi^2 - 1)_x^2 + \phi_x^2 + \phi_t^2 \right) dx dt \leq C. \quad (2.37) \]

**Proof.** Firstly, we rewrite (1.15)_2 as following
\[ \left( u - \frac{v_x}{v} \right)_t = -\left( \frac{\theta}{v} + \frac{\epsilon}{2} \left( \frac{\phi_x}{v} \right)^2 \right) \quad (2.38) \]

Multiplying (2.38) by \(u - \frac{v_x}{v}\), integrating by parts over \((-\infty, +\infty) \times [0, T]\), we have
\[
\frac{1}{2} \int_{-\infty}^{+\infty} \left( u - \frac{v_x}{v} \right)_x^2 (x, t) dx - \frac{1}{2} \int_{-\infty}^{+\infty} \left( u_0 - \frac{v_x}{v}(x, 0) \right)_x^2 dx \\
= \int_0^T \int_{-\infty}^{+\infty} \left( \frac{v_x}{v^2} \frac{\theta}{v} - \frac{\phi_x}{v} \frac{\phi_x}{v} \right)_x \left( u - \frac{v_x}{v} \right) dx dt \\
= -\int_0^T \int_{-\infty}^{+\infty} \frac{\theta v_x^2}{v^3} dx dt + \int_0^T \int_{-\infty}^{+\infty} \frac{\theta u v_x}{v^2} dx dt \\
- \int_0^T \int_{-\infty}^{+\infty} \frac{\theta}{v} \left( u - \frac{v_x}{v} \right) dx dt - \int_0^T \int_{-\infty}^{+\infty} \frac{\epsilon \phi_x}{v} \left( \frac{\phi_x}{v} \right)_x \left( u - \frac{v_x}{v} \right) dx dt. \quad (2.39) \]

\]
Moreover, integrating \((2.39)\), we have
\[
\left| \int_{-\infty}^{T} \int_{-\infty}^{+\infty} \frac{\theta u v_x}{v^2} dx dt \right| \leq \frac{1}{2} \int_{0}^{T} \int_{-\infty}^{+\infty} \frac{\theta v_x^2}{v^3} dx dt + \frac{1}{2} \int_{0}^{T} \int_{-\infty}^{+\infty} \frac{u^2 \theta}{v} dx dt
\]
\[
\leq \frac{1}{2} \int_{0}^{T} \int_{-\infty}^{+\infty} \frac{\theta v_x^2}{v^3} dx dt + C \int_{0}^{T} \sup_{x} \theta dt
\]
\[
\leq \frac{1}{2} \int_{0}^{T} \int_{-\infty}^{+\infty} \frac{\theta v_x^2}{v^3} dx dt + C. \tag{2.40}
\]

Next, by using \((2.3)\) and \((2.18)\) we obtain
\[
\left| \int_{0}^{T} \int_{-\infty}^{+\infty} \frac{\theta_x}{v} (u - v_x) dx dt \right| \leq \int_{0}^{T} \int_{-\infty}^{+\infty} \frac{\theta^2 \theta_x^2}{v^2} dx dt + \frac{1}{2} \int_{0}^{T} \int_{-\infty}^{+\infty} \frac{u^2 (u - v_x)^2}{v^3} dx dt
\]
\[
\leq C + C \int_{0}^{T} (\sup_{x} (\theta - 1)^2 + 1) \int_{-\infty}^{+\infty} (u - v_x)^2 dx dt. \tag{2.41}
\]

Moreover, it follows from \((2.3)\), \((2.18)\), \((2.25)\) and \((2.30)\) that, for \(\alpha < \beta\) and small enough \(\varepsilon > 0\), \(\forall n = 0, \pm 1, \pm 2, \cdots\), we get
\[
\int_{0}^{T} \max_{x \in [n, n+1]} (\theta - 1)^2 dt
\]
\[
\leq C \int_{0}^{T} \max_{x \in [n, n+1]} |(\theta - 1)^2 - \int_{n}^{n+1} (\theta - 1)^2 dx| dt + C \int_{0}^{T} \max_{x \in [n, n+1]} (\theta - 1) dt
\]
\[
\leq \int_{0}^{T} \int_{n}^{n+1} |\theta - 1| |\theta_x| dx dt + C
\]
\[
\leq C(\varepsilon) \int_{0}^{T} \int_{-\infty}^{+\infty} \frac{\theta^2 \theta_x^2}{v^2} v^1 \alpha^1 \beta^1 \alpha^1 dx dt + \varepsilon \int_{0}^{T} \int_{n}^{n+1} \frac{v (\theta - 1)^2}{\theta^2 - \alpha^1 \beta^1 \alpha^1} dx dt + C
\]
\[
\leq C + C \varepsilon \int_{0}^{T} \max_{x \in [n, n+1]} (\theta - 1)^2 dt, \tag{2.42}
\]
which follows that
\[
\int_{0}^{T} \sup_{x \in R} (\theta - 1)^2 dt \leq C. \tag{2.43}
\]

Moreover, integrating \((1.15)\) over \((-\infty, +\infty) \times [0, T]\), combining with \((2.22)\), \((2.36)\), \((2.43)\), we obtain
\[
\int_{0}^{T} \int_{-\infty}^{+\infty} \frac{u_x^2 + (v \mu)^2}{v} dx dt = \int_{-\infty}^{+\infty} (\theta - 1) dx - \int_{-\infty}^{+\infty} (\theta_0 - 1) dx + \int_{0}^{T} \int_{-\infty}^{+\infty} \frac{\theta}{v} u_x dx dt
\]
\[
\leq C + \frac{1}{2} \int_{0}^{T} \int_{-\infty}^{+\infty} \frac{u_x^2}{v} dx dt, \tag{2.44}
\]
which implies that
\[
\int_{0}^{T} \int_{-\infty}^{+\infty} (u_x^2 + (v \mu)^2) dx dt \leq C. \tag{2.45}
\]
Furthermore, from (2.18), (2.33), (2.36), we have
\[
\left| \int_0^T \int_{-\infty}^{+\infty} \frac{\phi_x}{v} (\phi_x) \frac{d}{dx} \left( u - \frac{v_x}{v} \right) dx dt \right| \\
\leq C \int_0^T \int_{-\infty}^{+\infty} \left( \left| \frac{\phi_x}{v} \right|^2 + \left| \frac{\phi_x}{v} \right|^2 (u - \frac{v_x}{v})^2 \right) dx dt \\
\leq C + C \int_0^T \sup_{x \in \mathbb{R}} \left| \frac{\phi_x}{v} \right|^2 \int_{-\infty}^{+\infty} (u - \frac{v_x}{v})^2 dx dt
\]
(2.46)

Substituting (2.40), (2.41), (2.46) into (2.39), from (2.34), (2.43) and Gronwall’s inequality, we obtain
\[
\sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} \left( u - \frac{v_x}{v} \right)^2 dx + \int_0^T \int_{-\infty}^{+\infty} \frac{\theta v_x^2}{v^3} dx dt \leq C. \tag{2.47}
\]
Together with (2.3), (2.18) and (2.25), we achieve
\[
\sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} v_t^2 dx \leq C.
\]

Secondly, rewriting (1.15)_{3,4} as follows
\[
\phi_t - \epsilon \phi_{xx} = -\frac{\phi_x v_x}{v} - \frac{v}{\epsilon} (\phi^3 - \phi). \tag{2.48}
\]

Multiplying (2.48) by $\phi_{xx}$, integrating the resultant over $(-\infty, +\infty)$, with respect to $x$, combining with (2.18), (2.25), (2.34) and (2.28), we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{+\infty} \phi_x^2 dx + \epsilon \int_{-\infty}^{+\infty} \phi_{xx}^2 dx + \frac{1}{2} \epsilon \int_{-\infty}^{+\infty} (\phi^2 - 1)_x^2 dx \\
= \epsilon \int_{-\infty}^{+\infty} \phi_x v_x \phi_{xx} dx + \frac{1}{\epsilon} \int_{-\infty}^{+\infty} (1 - \phi^2) \phi_x^2 dx \\
\leq C \left( \sup_{x \in \mathbb{R}} \phi_x^2(x, t) \int_{-\infty}^{+\infty} v_x^2 dx + \int_{-\infty}^{+\infty} \phi_x^2 dx \right) + \frac{1}{2} \epsilon \int_{-\infty}^{+\infty} \phi_{xx}^2 dx \\
\leq C \left( \sup_{x \in \mathbb{R}} (\theta - 1) + 1 + V(t) + \int_{-\infty}^{+\infty} \phi_x^2 dx \right) + \frac{1}{2} \epsilon \int_{-\infty}^{+\infty} \phi_{xx}^2 dx, \tag{2.49}
\]
by using (2.3) and Gronwall inequality, we get
\[
\sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} \phi_x^2 dx + \int_0^T \int_{-\infty}^{+\infty} \left( (\phi^2 - 1)_x^2 + \phi_{xx}^2 \right) dx dt \leq C. \tag{2.50}
\]

Finally, from (2.48), we have
\[
\phi_t = \epsilon \phi_{xx} - \epsilon \frac{\phi_x v_x}{v} - \frac{v}{\epsilon} (\phi^3 - \phi), \tag{2.51}
\]
integrating (2.51) over $(-\infty, +\infty)$, we obtain
\[
\int_{-\infty}^{+\infty} \phi_t^2 dx \leq C \left( \int_{-\infty}^{+\infty} \phi_{xx}^2 dx + \int_{-\infty}^{+\infty} \phi_x^2 v_x^2 dx + \int_{-\infty}^{+\infty} (\phi^3 - \phi)^2 dx \right)
\]
The proof of Lemma 2.7 is completed. \QED

**Lemma 2.8.** Let \((v, u, \theta, \phi)\) be a smooth solution of (1.15)-(1.16) on \((-\infty, +\infty) \times [0, T]\), then for \(v(x, t) \in (-\infty, +\infty) \times [0, T]\), the following inequality holds

\[
\sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} \phi_{xx}^2 \, dx + \int_{0}^{T} \int_{-\infty}^{+\infty} \left( \phi_{xt}^2 + \left( \frac{\phi_x}{v} \right)^2 \right) \, dx \
\leq C \left( \int_{-\infty}^{+\infty} \phi_{xx}^2 \, dx + 1 \right), \tag{2.52}
\]

then, from (2.50), we achieve

\[
\int_{0}^{T} \int_{-\infty}^{+\infty} \phi_t^2 \, dx \, dt \leq C. \tag{2.53}
\]

**Proof.** Rewriting (2.48) as

\[
\frac{\phi_t}{v} - \epsilon \left( \frac{\phi_x}{v} \right)_x = -\frac{1}{\epsilon} (\phi^3 - \phi), \tag{2.55}
\]
differentiating (2.55) with respect to \(x\), we obtain

\[
\left( \frac{\phi_x}{v} \right)_t - \epsilon \left( \frac{\phi_x}{v} \right)_{xx} = -\frac{1}{\epsilon} (\phi^3 - \phi) + \frac{\phi_t v_x}{v^2} - \frac{\phi_x u_x}{v^2}, \tag{2.56}
\]

multiplying (2.56) by \(\left( \frac{\phi_x}{v} \right)_t\), integrating the resultant over \((-\infty, +\infty)\), from (2.3), (2.18), (2.25), (2.37), (2.45), (2.52), we have

\[
\int_{-\infty}^{+\infty} \left( \frac{\phi_x}{v} \right)_t^2 \, dx + \frac{\epsilon}{2} \frac{d}{dt} \int_{-\infty}^{+\infty} \left( \frac{\phi_x}{v} \right)^2 \, dx
\]

\[
= -\frac{1}{\epsilon} \int_{-\infty}^{+\infty} (\phi^3 - \phi)_x \left( \frac{\phi_x}{v} \right) \, dx + \int_{-\infty}^{+\infty} \phi_t v_x \left( \frac{\phi_x}{v} \right)_t \, dx - \int_{-\infty}^{+\infty} \frac{\phi_x u_x}{v^2} \left( \frac{\phi_x}{v} \right)_t \, dx
\]

\[
\leq C \left( \int_{-\infty}^{+\infty} (3 \phi^2 - 1) \phi_x^2 \, dx + \int_{-\infty}^{+\infty} \phi_t^2 \phi_x^2 \, dx + \int_{-\infty}^{+\infty} \phi_x^2 u_x^2 \, dx + \frac{1}{3} \int_{-\infty}^{+\infty} \left( \frac{\phi_x}{v} \right)_t^2 \, dx \right)
\]

\[
\leq C \left( 1 + \|\phi_t\|_{L^\infty} \int_{-\infty}^{+\infty} u_x^2 \, dx + \|\phi_x\|_{L^2}^2 \int_{-\infty}^{+\infty} u_x^2 \, dx \right) + \frac{1}{3} \int_{-\infty}^{+\infty} \left( \frac{\phi_x}{v} \right)_t^2 \, dx
\]

\[
\leq C \left( 1 + \int_{-\infty}^{+\infty} \phi_x^2 \, dx + \epsilon \int_{-\infty}^{+\infty} \phi_{xx}^2 \, dx + \int_{-\infty}^{+\infty} \left( \frac{\phi_x}{v} \right)^2 \, dx \int_{-\infty}^{+\infty} u_x^2 \, dx + \frac{1}{3} \int_{-\infty}^{+\infty} \left( \frac{\phi_x}{v} \right)_t^2 \, dx \right)
\]

\[
\leq C \left( 1 + \int_{-\infty}^{+\infty} \phi_{xx}^2 \, dx + \epsilon \int_{-\infty}^{+\infty} \phi_{xx}^2 \, dx + \int_{-\infty}^{+\infty} \left( \frac{\phi_x}{v} \right)^2 \, dx \int_{-\infty}^{+\infty} u_x^2 \, dx + \frac{1}{2} \int_{-\infty}^{+\infty} \left( \frac{\phi_x}{v} \right)_t^2 \, dx \right)
\]

where in the last inequality \(\phi_{xt} = \left( \frac{\phi_x}{v} \right)_t v + \frac{\phi_x u_x}{v^2}\) is used. Therefore, from Gronwall’s inequality, we get

\[
\sup_{t \in [0, T]} \int_{-\infty}^{+\infty} \left( \frac{\phi_x}{v} \right)_t^2 \, dx + \int_{0}^{T} \int_{-\infty}^{+\infty} \left( \frac{\phi_x}{v} \right)_t^2 \, dx \, dt \leq C. \tag{2.58}
\]
Combining with (2.3), (2.37), we have
\[
\begin{align*}
\sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} \phi_x^2 dx + \int_0^T \int_{-\infty}^{+\infty} \phi_t^2 dx dt & \leq C. \tag{2.59}
\end{align*}
\]
Furthermore, using the Sobolev embedding theorem, follows from (2.3) and (2.58), we obtain
\[
\sup_{(x,t) \in \mathbb{R} \times [0,T]} \left| \frac{\phi_x}{v} \right| \leq \sqrt{2} \left\| \frac{\phi_x}{v} \right\| \left\| \left( \frac{\phi_x}{v} \right)_x \right\| \leq C. \tag{2.60}
\]
Moreover, from (2.56) and the inequalities obtained above, we achieve
\[
\int_0^T \int_{-\infty}^{+\infty} \left( \frac{\phi_x}{v} \right)_x^2 dx dt \leq C, \tag{2.61}
\]
the proof of Lemma 2.8 is finished. \(\square\)

**Lemma 2.9.** Let \((v,u,\theta,\phi)\) be a smooth solution of (1.15)-(1.16) on \((-\infty, +\infty) \times [0,T] \), then for \(\forall (x,t) \in (-\infty, +\infty) \times [0,T] \), the following inequality holds
\[
\sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} u_x^2 dx + \int_0^T \int_{-\infty}^{+\infty} (u_t^2 + u_{xx}^2) dx dt \leq C. \tag{2.62}
\]

**Proof.** Multiplying (1.15)2 by \(u_{xx}\) and integrating the resultant over \((-\infty, +\infty) \times (0,T) \), by using (2.18), (2.25), (2.37), (2.43), (2.58), (2.60), we obtain
\[
\begin{align*}
\frac{1}{2} \int_{-\infty}^{+\infty} u_x^2 dx + \int_0^T \int_{-\infty}^{+\infty} u_{xx}^2 dx dt & \leq C + \frac{1}{2} \int_0^T \int_{-\infty}^{+\infty} u_{xx}^2 dx dt + C \int_0^T \int_{-\infty}^{+\infty} (\theta_x^2 + \theta^2 v_x^2 + \left| \frac{\phi_x}{v} \right| \left( \frac{\phi_x}{v} \right)_x^2 + u_x^2 v_x^2) dx dt \\
& \leq C + \frac{1}{2} \int_0^T \int_{-\infty}^{+\infty} u_{xx}^2 dx dt + C \int_0^T \int_{-\infty}^{+\infty} \theta_x^2 dx dt + C \int_0^T \sup_{x \in \mathbb{R}} \theta_x^2 \int_{-\infty}^{+\infty} v_x^2 dx dt \\
& + C \sup_{(x,t) \in \mathbb{R} \times [0,T]} \left| \frac{\phi_x}{v} \right| \int_0^T \int_{-\infty}^{+\infty} \left| \left( \frac{\phi_x}{v} \right)_x \right|^2 dx dt + C \int_0^T \sup_{x \in \mathbb{R}} u_x^2 \int_{-\infty}^{+\infty} v_x^2 dx dt \\
& \leq C + \frac{3}{4} \int_0^T \int_{-\infty}^{+\infty} \left[ \frac{\theta_x^2}{v} \right] dx dt + C_1 \int_0^T \int_{-\infty}^{+\infty} \frac{\theta_x^2}{v} dx dt, \tag{2.63}
\end{align*}
\]
where the following inequality are used
\[
\begin{align*}
\int_0^T \sup_{x \in \mathbb{R}} u_x^2 dx dt & \leq C(\delta) \int_0^T \int_{-\infty}^{+\infty} u_x^2 dx dt + \delta \int_0^T \int_{-\infty}^{+\infty} u_{xx}^2 dx dt \\
& \leq C(\delta) + \delta \int_{-\infty}^{+\infty} u_{xx}^2 dx dt. \tag{2.64}
\end{align*}
\]
Multiplying (1.15)5 by \(\theta - 1\), and integrating the resultant over \(\mathbb{R} \times (0,T) \), by using (2.43), (2.45) and (2.64), we obtain
\[
\begin{align*}
\frac{1}{2} \int_{-\infty}^{+\infty} (\theta - 1)^2 dx + \int_0^T \int_{-\infty}^{+\infty} \frac{\theta_x^2}{v} dx dt
\end{align*}
\]
\[
\leq C + C \int_0^T \int_{-\infty}^{+\infty} \theta(\theta - 1)|u_x|dxdt + C \int_0^T \int_{-\infty}^{+\infty} (u_x^2 + \mu^2)(\theta - 1)dxdt
\]
\[
\leq C + C \int_0^T \int_{-\infty}^{+\infty} \theta u_x^2 dxdt + C \int_0^T \int_{-\infty}^{+\infty} \theta(\theta - 1)^2 dxdt + \int_0^T \sup_{x \in \mathbb{R}}(\theta - 1)dt
\]
\[
\leq C + C \int_0^T \sup_{x \in \mathbb{R}} u_x^2 dt + C \int_0^T \sup_{x \in \mathbb{R}}(\theta - 1)^2 dt
\]
\[
\leq C(\delta) + C\delta \int_0^T \int_{-\infty}^{+\infty} \frac{u_x^2}{v} dxdt. \quad (2.65)
\]

From (2.63) and (2.65), for \( \delta \) small enough, we have
\[
\sup_{t \in [0, T]} \int_{-\infty}^{+\infty} ((\theta - 1)^2 + u_x^2) dx + \int_0^T \int_{-\infty}^{+\infty} \theta \beta \theta_x^2 dxdt + \int_0^T \int_{-\infty}^{+\infty} u_x^2 dxdt \leq C. \quad (2.66)
\]

Rewriting (1.15) as
\[
u_t = -\left(\frac{\theta}{v}\right)_x + \frac{u_x v_x}{v^2} - \frac{\phi_x}{v} \left(\frac{\phi}{v}\right)_x, \quad (2.67)
\]

from (2.37), (2.43), (2.60), (2.64), (2.54) and (2.66), we get
\[
\int_0^T \int_{-\infty}^{+\infty} u_x^2 dxdt \leq C \int_0^T \int_{-\infty}^{+\infty} \left(u_{xx}^2 + u_x^2 v_x^2 + \theta_x^2 + \theta v_x^2 + \left|\frac{\phi_x}{v}\right| \left|\left(\frac{\phi}{v}\right)_x\right|ight) dxdt
\]
\[
\leq C. \quad (2.68)
\]

Together with (2.66), the energy inequality (2.62) is achieved. The proof of Lemma 2.9 is completed. \( \Box \)

**Lemma 2.10.** Let \((v, u, \theta, \phi)\) be a smooth solution of (1.15)-(1.16) on \((-\infty, +\infty) \times [0, T]\), then for \(\forall (x, t) \in (-\infty, +\infty) \times [0, T]\), the following inequality holds
\[
\sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} \theta_x^2 dx + \int_0^T \int_{-\infty}^{+\infty} (\theta_t^2 + \theta_{xx}^2) dxdt \leq C. \quad (2.69)
\]

**Proof.** Multiplying (1.15)\(_3\) by \(\theta \beta \theta_t\) and integrating the resultant over \((0, 1)\), by using (2.18), (??), (2.66), we have
\[
\frac{1}{2} \frac{d}{dt} \left( \int_{-\infty}^{+\infty} \frac{(\theta \beta \theta_t)^2}{v} dx \right) + \int_{-\infty}^{+\infty} \theta \beta \theta_t dx
\]
\[
= -\frac{1}{2} \int_{-\infty}^{+\infty} \frac{(\theta \beta \theta_x)^2}{v} u_x dx + \int_{-\infty}^{+\infty} \theta \beta \theta_t \left(-\theta u_x + u_x^2 + v^2 \mu^2\right) dx
\]
\[
\leq C \sup_{x \in \mathbb{R}} |u_x| \theta^2 \int_{-\infty}^{+\infty} \theta \beta \theta_x^2 dx + \frac{1}{2} \int_{-\infty}^{+\infty} \theta \beta \theta_t dx + C \int_{-\infty}^{+\infty} \theta \beta + u_x^2 dx + C \int_{-\infty}^{+\infty} \theta (u_x^4 + \mu^4) dx
\]
\[
\leq C \int_{-\infty}^{+\infty} \theta \beta \theta_x^2 dx \int_{-\infty}^{+\infty} (\theta \beta \theta_x)^2 dx + \frac{1}{2} \int_{-\infty}^{+\infty} \theta \beta \theta_t dx + C \sup_{x \in \mathbb{R}} ((\theta - 1)^{2\beta+2} + u_x^4 + \mu^4) + C. \quad (2.70)
\]
Now we deal with the term \( \sup_{x \in \mathbb{R}} \left( (\theta - 1)^{2\beta + 2} + u_x^4 + \mu^4 \right) \) in the last inequality of (2.70). Applying Lemma 2.9, direct computation shows that
\[
\int_0^T \sup_{x \in \mathbb{R}} u_x^4 \, dt \leq C \int_0^T \int_{-\infty}^{+\infty} |u_x^3 u_{xx}| \, dx \, dt \\
\leq C \int_0^T \sup_{x \in \mathbb{R}} \left( \int_{-\infty}^{+\infty} u_x^2 \, dx \right)^{\frac{1}{2}} \left( \int_{-\infty}^{+\infty} u_{xx}^2 \, dx \right)^{\frac{1}{2}} \, dt \\
\leq \frac{1}{2} \int_0^T \sup_{x \in \mathbb{R}} u_x^4 \, dt + C \int_0^T \int_{-\infty}^{+\infty} (u_x^2 + u_{xx}) \, dx \, dt \\
\leq \frac{1}{2} \int_0^T \sup_{x \in \mathbb{R}} u_x^4 \, dt + C,
\] (2.71)
then
\[
\int_0^T \sup_{x \in \mathbb{R}} u_x^4 \, dt \leq C.
\] (2.72)
Combining with (2.54), by the same way above, we have
\[
\int_0^T \sup_{x \in \mathbb{R}} \mu^4 \, dt \leq C.
\] (2.73)
Moreover, by using Sobolev embedding theorem and (2.66), we get
\[
\sup_{x \in \mathbb{R}} (\theta - 1)^{2\beta + 2} \leq C(\delta) \int_{-\infty}^{+\infty} (\theta - 1)^{2\beta + 2} \, dx + \delta \int_{-\infty}^{+\infty} (\theta - 1)^{2\beta} \theta_x^2 \, dx \\
\leq \frac{1}{2} \sup_{x \in \mathbb{R}} (\theta - 1)^{2\beta + 2} + C(\delta) + C\delta \int_{-\infty}^{+\infty} (\theta^3 \theta_x)^2 \, dx.
\] (2.74)
Substituting (2.72), (2.73), (2.74) into (2.70), by using Gronwall’s inequality and (??), we obtain
\[
\sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} (\theta^3 \theta_x)^2 \, dx + \int_0^T \int_{-\infty}^{+\infty} \theta^3 \theta_x^2 \, dx \, dt \leq C.
\] (2.75)
Therefore, in view of (2.74), we have
\[
\sup_{(x,t) \in (-\infty, +\infty) \times [0,T]} \theta \leq C.
\] (2.76)
Thus, both (2.74) and (2.75) lead to
\[
\sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} \theta_x^2 \, dx + \int_0^T \int_{-\infty}^{+\infty} \theta_t^2 \, dx \, dt \leq C.
\] (2.77)
From (1.15) again, we also have
\[
\frac{\theta^3 \theta_{xx}}{v} = \theta_t - \frac{\beta \theta^2 \theta_x^2}{v^2} + \frac{\theta^3 \theta_x v_x}{v^2} + \frac{\theta u_x}{v} - \frac{u_x^2}{v} + \frac{(\mu^4 + u_x^4 + u_{xx}^2 + \theta_t^2)}{v},
\] (2.78)
which yields that
\[
\int_0^T \int_{-\infty}^{+\infty} \theta_{xx}^2 \, dx \, dt \leq C \int_0^T \int_{-\infty}^{+\infty} \left( \theta_x^2 v_x^2 + \theta_x^4 + u_x^4 + u_{xx}^2 + \theta_t^2 \right) \, dx \, dt.
\]
\[
\leq C(\delta) + C\delta \int_0^T \sup_{x \in \mathbb{R}} \theta_x^2 dt
\]
\[
\leq C(\delta) + C\delta \int_0^T \int_{-\infty}^{+\infty} \theta_{xx}^2 dx dt.
\] (2.79)

Furthermore, by using maximum principle, we obtain \(-1 \leq \phi \leq 1\). The proof of Lemma 2.10 is completed. \(\square\)

So far, from the a priori estimates of solutions (see Lemma 2.1-Lemma 2.10), Theorem 1.1 can be obtained by extending the local solutions globally in time. For details, please refer to [6], [11], and the references therein.

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