Abstract

Anyons in one spatial dimension can be defined by correctly identifying the configuration space of indistinguishable particles and imposing Robin boundary conditions. This allows an interpolation between the bosonic and fermionic limits. In this paper, we study the quantum entanglement between two one-dimensional anyons on a real line as a function of their statistics.

1 Introduction

It is well-known that, in quantum mechanics, the indistinguishability of particles forces the multiparticle wavefunctions to be either symmetric (bosonic) or antisymmetric (fermionic) under the exchange of any pair of particles. In the last few decades it has emerged that in low dimensions it is possible to have more general quantum statistics. The classical roots for this can be traced to the non-trivial topology of the associated configuration space. Particles which obey these generalised statistics are called anyons, and they interpolate between bosons and fermions. Interestingly, these particles appear as collective excitations in fractional quantum Hall systems. In view of this, the quantum mechanical and thermodynamic properties of anyons have been extensively studied [1] [2].

The interest in anyons has been revived recently because of their potential application in topological quantum computation [3]. In topological quan-
tum computation, instead of using qubits one uses anyons to store information in their non-trivial wavefunctions. Since these are topologically protected, it is hoped that a topological quantum computer leads to fault-tolerant and decoherence-free computation.

However, a completely robust, fault-tolerant physical system is not desirable because it does not allow us to store any information, let alone manipulate or extract it. In view of this, it is important to allow the system to interact with the apparatus (environment) in a controlled manner.

This motivates us to revisit the old problems of anyon quantum mechanics, and study them in the framework of open quantum systems. In particular, we are interested in knowing how the entanglement between two anyons depends on the statistics parameter when one of them is considered to be the system, and the other, the environment.

There are two complexities associated with this problem. First, it is well-known that for indistinguishable particles, the standard methods used to quantify the entanglement, like finding the Schmidt rank, taking a partial trace, and finding the von Neumann entropy, etc. fail to work. The main reason for this is the non-factorizability of the multi-particle Hilbert space of indistinguishable particles. Various approaches has been proposed to circumvent this problem [4] [5] [6] [7] [8]. Second, these approaches mostly restrict their attention to bosons and fermions.

Returning to our problem, we find it useful to follow the information theoretic approach to quantum entanglement developed by Franco and Compagno [8]. In their work they show how it is possible to define the reduced density matrix in a system of indistinguishable particles by defining an inner product between states belonging to Hilbert spaces with different dimensionalities. It is straightforward to recast this method in the language of second quantization [9] [10], which is especially suited for our purposes. Within this framework, we show how the results can be generalised to anyons by the simple prescription of using the anyonic algebra for the creation and annihilation operators instead of the bosonic and fermionic algebras which are recovered as special cases.
The rest of the paper is organized as follows.

In section 2, we review the information theoretic approach developed by Franco and Compagno, with special emphasis on its reformulation in the language of second quantization.

In section 3, we review the model of indistinguishable particles on a real line, first studied by Leinaas and Myrheim [11]. In this model they first construct the classical configuration space by identifying different configurations which can be obtained by permutations of particle positions, and then quantize the system to obtain a wavefunction that interpolates between the bosonic and fermionic limits through a statistics parameter $\eta$ coming from the Robin boundary conditions. A second quantization of this model [12] gives rise to an $\eta$-dependent algebra for the creation and annihilation operators, which reduces to the usual bosonic and fermionic algebras as limiting cases.

In section 4, we use the above results to compute the reduced density matrix and the von Neumann entropy of a system of two anyons on a line.

In section 5 we conclude by giving a summary and an outlook.

### 2 Information Theoretic Approach to Indistinguishable Particles

In the usual approach, a state of a system of indistinguishable particles is obtained by first quantizing the system as if the particles were distinguishable, by labelling them. We then apply the symmetrization postulate on the product wavefunctions to get bosonic and fermionic states.

It is instructive to restate this in the language of transition amplitudes. For example, a two-particle state is simply written as $|\psi, \phi\rangle$, where $\psi$ and $\phi$ are generic labels, not particle labels, and the symmetrization postulate is not invoked. Quantum statistics enters through the definition of the inner product of these states.

For distinguishable particles, an initial state $|\phi, \psi\rangle$ can only evolve into the
final state, say, $|\varphi, \zeta\rangle$ for which we compute the amplitude. But when the parti-
cles are indistinguishable, both the final states $|\varphi, \zeta\rangle$ and $|\zeta, \varphi\rangle$ contribute to the
amplitude. For the case of bosons and fermions, the simple recipe of introducing
the right sign to account for the exchanges takes care of this complication.

This ad hoc procedure does not easily generalise to anyons. It is therefore
desirable to have a more fundamental approach to the problem where the in-
distinguishability of the particles is maintained throughout. This is the idea
behind the information theoretic approach developed in [8].

If $|\varphi, \zeta\rangle$ and $|\phi, \psi\rangle$ denote two two-particle states, their inner product is,$$
\langle \varphi, \zeta | \phi, \psi \rangle = \langle \varphi | \phi \rangle \langle \zeta | \psi \rangle + \eta \langle \varphi | \psi \rangle \langle \zeta | \phi \rangle.
$$

(1)

where $\eta = 1$ for bosons and $\eta = -1$ for fermions.

The inner product between states belonging to Hilbert spaces of different
dimensionality can also be defined. If we consider an unnormalized two-particle
state, $|\Phi\rangle = |\varphi_1, \varphi_2\rangle$, the inner product with a single-particle state $|\psi\rangle$ is

$$
\langle \psi | \cdot | \varphi_1, \varphi_2 \rangle \equiv \langle \psi | \varphi_1, \varphi_2 \rangle = \langle \psi | \varphi_1 \rangle | \varphi_2 \rangle + \eta \langle \psi | \varphi_2 \rangle | \varphi_1 \rangle.
$$

(2)

This is a projective measurement on a single particle, where the unnormalized
two-particle state is projected on to $|\psi\rangle$. In a similar manner, the inner product
between an $N$-particle state and a single-particle state is also defined. This defi-
nition of inner product between states belonging to Hilbert spaces with different
dimensions can be used to define the reduced density matrix as shown below.

Let $|\Phi\rangle$ be a normalized $N$-particle state. To perform the partial trace we
choose a basis $\{|\psi_k\rangle\}$ for the single-particle Hilbert space. The normalized pure
state after projecting on to a state $|\psi_k\rangle$ is

$$
|\phi_k\rangle = \frac{\langle \psi_k | \varphi_1, \varphi_2 \rangle}{\sqrt{\langle \Pi^{(1)}_k \rangle_\Phi}}.
$$

(3)

where $\Pi^{(1)}_k = |\psi_k\rangle \langle \psi_k|$. Define a one-particle identity operator as $\mathbb{I}^{(1)} = \sum_k \Pi^{(1)}_k$. Then the proba-
bility of finding a single particle in the state $|\psi_k\rangle$ is

$$p_k = \frac{\langle \Pi^{(1)}_k \rangle_{\Phi}}{\langle \Pi^{(1)} \rangle_{\Phi}}$$

(4)

With the knowledge of $|\phi_k\rangle$ and the corresponding probabilities $p_k$, the reduced density matrix is defined as follows.

$$\rho^{(1)} = \text{Tr}^{(1)} |\Phi\rangle \langle \Phi| = \sum_k p_k |\phi_k\rangle \langle \phi_k|$$

(5)

After obtaining the reduced density matrix, the von Neumann entropy can be calculated as usual,

$$S(\rho^{(1)}) = -\text{Tr} \left( \rho^{(1)} \log \rho^{(1)} \right) = -\sum_i \lambda_i \log \lambda_i,$$

where $\lambda_i$ is an eigenvalue of the reduced density matrix.

**Second quantization formalism**

We can recast the above idea in the language of second quantization. If $|\Phi\rangle$ is an $N$-particle state, its inner product with a single-particle state $|\psi_k\rangle$ is

$$a_{\psi_k} |\Phi\rangle \equiv \langle \psi_k | \cdot | \Phi \rangle$$

Note that since $a_{\psi_k}$ is an annihilation operator, the left hand side of the above equation represents an $(N-1)$-particle state which, by definition, is the inner product on the right hand side. As mentioned earlier, this simple expedient allows us to go beyond bosons and fermions by suitably generalising the operator algebra. We present this in the next section.

We conclude this section by noting that the expression for the reduced density matrix in the second quantization formalism is

$$\rho^{(1)} = \text{Tr}^{(1)} |\Phi\rangle \langle \Phi| = \frac{\sum_k a_{\psi_k} |\Phi\rangle \langle \Phi| a_{\psi_k}^\dagger}{\langle \Phi| \hat{n} |\Phi \rangle}$$

(6)

Here $\hat{n} = \sum_k a_{\psi_k}^\dagger a_{\psi_k}$ is the total number operator. The details are given in A.
3 Anyons

The Symmetrization Postulate in quantum mechanics says that the wave function of a system of indistinguishable particles is (anti-)symmetric with respect to the exchange of any two particles. This follows from attaching labels to the particles, as if they were distinguishable, and (anti-)symmetrizing the product wave function with respect to these labels. But, labelling indistinguishable particles is intrinsically contradictory. So, it is desirable to look beyond this ad hoc prescription.

In a seminal paper, Leinaas and Myrheim trace the origin of the Symmetrization Postulate to the non-trivial topology of the underlying classical configuration space of indistinguishable particles. As a spin-off of this insight, they show that, in low dimensions, it is possible to have objects which are more general than bosons and fermions. These are called anyons.

Let us consider a system of \( N \) particles in \( d \) dimensions. Let \( X = \mathbb{R}^d \) be the configuration space of a single particle. If the particles are distinguishable, the configuration space of the system is \( X_N = X^N \) where \( X^N \) denotes an \( N \)-fold tensor product of the single-particle space \( X \). A point in the space \( \mathbf{x} = (x_1, x_2, ..., x_N) \) represents a physical configuration of the system.

If the \( N \) particles are indistinguishable, the configuration space is \( Y_N = (X^N - D)/S_N \) where \( D \) denotes the set of points which correspond to two or more particles occupying the same position, and \( S_N \) is the permutation group on \( N \) elements. The deletion of \( D \) is in anticipation of the Pauli Exclusion Principle for fermions, and \( S_N \) ensures that the points \( \mathbf{x} = (x_1, x_2, ..., x_N) \) and \( \mathbf{x'} = (x_{P(1)}, x_{P(2)}, ..., x_{P(N)}) \) which represent the same physical configuration are identified. Here \( P \) represents an arbitrary permutation. The wave function of the system is determined by the one-dimensional unitary representations of the fundamental group \( \pi_1(Y_N) \) of the configuration space. For the case of indistinguishable particles, this turns out to be the permutation group in dimensions \( d \geq 3 \), allowing only bosons and fermions. In two dimensions, the fundamental group of the system is \( \pi_1(Y_N) = B_N \), where \( B_N \) is the braid group on \( N \) strings,
whose one dimensional unitary representations allow the wavefunction to pick up a phase $e^{i\theta}$, where $\theta$ is a real parameter, under an exchange. This is the underlying reason for the possibility of having anyons in low dimensions.

**Indistinguishable Particles On the Real Line**

In the case of indistinguishable particles on a real line, it is not possible to perform an exchange without taking the particles through each other: an exchange gets inextricably linked with scattering. It is nevertheless possible to define quantum statistics by following the Leinaas-Myrheim prescription, as shown below in the specific example of two indistinguishable particles on a real line. If $x_1$ and $x_2$ are the positions of the particles, we observe that the points $x = (x_1, x_2)$ and $x' = (x_2, x_1)$ represent the same configuration, and hence need to be identified. The identification is done by folding the $(x_1 x_2)$ plane along the line $x_1 = x_2$ which represents the diagonal points. Without loss of generality, we choose to work with the half plane $x_1 < x_2$. The problem can be solved by prescribing appropriate boundary conditions along the diagonal.

We choose the free particle Hamiltonian for the system

$$H = -\frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)$$  \hspace{1cm} (7)

where we use the units $\hbar = c = 1$ and set mass equal to one. To ensure that particles remain bounded in the region $x_1 < x_2$, we impose the boundary condition that the normal component of the probability current vanishes at the boundary. That is,

$$\left. \left( \psi^*(x) \left( -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \psi(x) - \psi(x) \left( -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \psi^*(x) \right) \right|_{x_1 = x_2} = 0$$  \hspace{1cm} (8)

Note that above equation also ensures self-adjointness of the Hamiltonian. The general solution of the above equation is given by,

$$\left. \left( -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \psi(x) \right|_{x_1 = x_2} = \eta \psi(x) \right|_{x_1 = x_2}$$  \hspace{1cm} (9)
where $\eta$ is a real parameter. The eigenstates of the Hamiltonian are

$$
\psi(x) = e^{i(k_1 x_1 + k_2 x_2)} + e^{-i(\phi_\eta(k_2 - k_1))} e^{i(k_2 x_1 + k_1 x_2)}
$$

(10)

where,

$$
\phi_\eta(k_2 - k_1) = 2 \tan^{-1}\left(\frac{\eta}{k_2 - k_1}\right)
$$

Note that $\eta = 0$ and $\eta = \infty$ correspond to Neumann and Dirichlet boundary conditions respectively on the diagonal i.e. the set of coincident points $x_1 = x_2$. The former gives a symmetric wavefunction, while the latter gives an antisymmetric wavefunction which also enforces the Pauli Exclusion Principle. Arbitrary values of $\eta$ correspond to Robin boundary conditions, with the corresponding wavefunctions being neither symmetric nor antisymmetric. These are, by definition, one-dimensional anyons.

For $\eta < 0$, it is easy to see that the system admits one bound state. This follows from the requirement that the wavefunction is well-behaved at $\pm \infty$, which in turn implies that the momentum of the centre of mass coordinate is purely real, and the momentum of the relative coordinate is purely imaginary.

We mention in passing that for the case of three or more particles, there are several diagonals corresponding to coincident points; but the Robin boundary conditions can be generalized in a straightforward manner as shown in the next section.

**$N$ particles on the real line**

In the case of $N$ identical particles on a real line the configuration space can be constructed in a similar way and is chosen to be the region where $\mathcal{R} = \{x|x_1 < x_2 < x_3 < ... < x_N\}$. The Hamiltonian is again the free particle Hamiltonian

$$
H = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}
$$

(11)

and the Robin boundary conditions are

$$
\left.\left(\frac{\partial}{\partial x_{j+1}} - \frac{\partial}{\partial x_j}\right)\psi(x)\right|_{x_{j+1}=x_j} = \eta \psi(x)\left|_{x_{j+1}=x_j}
$$

(12)
The corresponding anyonic wavefunctions are obtained by solving the Schrodinger equation for which we employ the ansatz \( \psi(x) = \int_{k \in \mathbb{C}} dk \, \alpha(k) e^{ikx} \). The coefficients \( \alpha(k) \) satisfy,

\[
\alpha(k) = \begin{cases} 
  e^{-i(\phi(\eta(k_{j+1} - k_j)))} \alpha(P_j k) & \text{if } k_{j+1} - k_j \neq i\eta \\
  0 & \text{if } k_{j+1} - k_j = i\eta 
\end{cases}
\]

(13)

where an elementary permutation \( P_j \) permutes the \( j \)th and \( (j + 1) \)th elements and

\[
\phi(\eta(k_{j+1} - k_j)) = 2 \tan^{-1}\left(\frac{\eta}{k_{j+1} - k_j}\right)
\]

(14)

The basis functions are of the form \( \psi_k(x) \propto \sum_{P \in S_n} e^{i\phi_P(\eta)k} \alpha(Pk) e^{i(Pk)x} \). As in the two-particle case, only special values of \( k \) are permitted when \( \eta < 0 \). In contrast to the two-particle case, however, we can have bound states with different number of particles.

**Second quantization**

As already mentioned in the Introduction, we find it useful to recast the above results in the language of second quantization, as was done in [12]. We use the following generalised \( \eta \)-dependent algebra for the second quantized anyonic field operators.

\[
[\Psi(x), \Psi^\dagger(y)] = \delta(x - y) - 2\eta \int_0^\infty dz \, e^{-z\eta} \Psi^\dagger(y - z) \Psi(x - z)
\]

\[
[\Psi^\dagger(x), \Psi^\dagger(y)] = -2\eta \int_0^\infty dz \, e^{-z\eta} \Psi^\dagger(y + z) \Psi^\dagger(x - z).
\]

(16)
Note that this algebra reduces to the standard bosonic and fermionic limits for 
\( \eta \to 0 \) and \( \eta \to \infty \) respectively. Also note that this algebra is slightly different 
from the one presented in [12]. As shown in B.1 the above equations can be 
derived starting from the corresponding algebra for the creation and annihilation 
operators for momentum states, related to the second quantized fields through 
the usual relations 
\[
\Psi^\dagger(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dke^{ikx}a^\dagger_k.
\]

Defining the number operator, \( \hat{N} \), in the usual manner, 
\( \hat{N} = \int_{-\infty}^{\infty} dx \, \Psi^\dagger(x)\Psi(x) \), 
it is straightforward to see 
\[
\left[ \hat{N}, \Psi^\dagger(y) \right] = \Psi^\dagger(y) \\
\left[ \hat{N}, \Psi(y) \right] = -\Psi(y)
\]

Thus, although the algebra for the anyonic fields is more complicated than the 
bosonic and fermionic cases, the number operator can be defined in the usual 
fashion, and satisfies the standard commutation relation with the second quantized fields. This allows us to interpret the matrix elements of the fields in the 
number operator basis as operators which transform multiparticle wavefunctions 
into other wavefunctions with more or fewer number of particles as explained 
by Fock [13]. In B.2 we explicitly verify that the modified algebra satisfies the 
conditions derived by Fock.

4 Entropy of Two Identical Particles

We consider two indistinguishable particles on the real line. We assume that the 
statistics parameter \( \eta \) is non-negative, so that the particles are anyons. Note 
that the bosonic and fermionic limits can be retrieved from the general case as 
special cases.

The field operator \( \Psi^\dagger(x) \) acting on the vacuum creates a particle localised at 
\( x \). Rather than dealing with these localised states, it is convenient for our pur-
poses to work with smeared fields defined as follows: 
\( \Psi^\dagger_f = \int_{-\infty}^{\infty} dx \, f(x)\Psi^\dagger(x) \), 
where \( f(x) \in \mathcal{S}(\mathbb{R}) \), is a function in the Schwartz space [14]. The algebra of the
smeared fields is readily obtained to be

\[
\left[\Psi_f, \Psi_g^\dagger\right] = \langle f | g \rangle - 2\eta \int_0^\infty dz \int_{-\infty}^\infty dx \int_{-\infty}^\infty dy f^*(x)g(y)e^{-z\eta} \Psi^\dagger(y-z)\Psi(x-z)
\]

\[
\left[\Psi_f^\dagger, \Psi_g^\dagger\right] = -2\eta \int_0^\infty dz \int_{-\infty}^\infty dx \int_{-\infty}^\infty dy f^*(x)g(y)e^{-z\eta} \Psi^\dagger(y+z)\Psi^\dagger(x-z)
\]  

(17)

where the inner product \(\langle f | g \rangle = \int_{-\infty}^\infty dx \, f^*(x)g(x)\). We use the following notation to denote the states \(|f\rangle \equiv \Psi_f^\dagger |0\rangle\). If we choose a set of orthonormal functions \(\{f_n(x)\}\), the corresponding set of states \(\{|f_n\rangle\}\) will form a basis for the single-particle Hilbert space. For our purpose we chose \(f_n(x) = h_n(x)\), where

\[
h_n(x) = \frac{1}{\sqrt{\sqrt{n\pi}}} H_n(x) e^{-\frac{x^2}{2}}
\]

is \(n\)-th eigenstate of the harmonic oscillator.

Let the two-particle state be

\[
|\Phi_{j,i}\rangle = \frac{1}{\mathcal{N}} \Psi_{h_j}^\dagger \Psi_{h_i}^\dagger |0\rangle
\]  

(18)

Here \(\mathcal{N} = \langle 0 | \Psi_{h_j} \Psi_{h_i} \Psi_{h_j}^\dagger \Psi_{h_i}^\dagger |0\rangle\) is the normalization constant. We use the one-particle basis \(\{|h_n\rangle\}\) as the basis to calculate both the partial trace and the eigenvalues of the reduced density matrix. The one-particle reduced density matrix \(\rho^{(1)}\) is obtained from the two-particle state as follows:

\[
\rho^{(1)} = \sum_{k=0}^\infty \Psi_{h_k}^\dagger \Psi_{h_k} \langle \Phi_{j,i} | \hat{n} | \Phi_{j,i}\rangle
\]

\(\hat{n} = \sum_{k=0}^\infty \Psi_{h_k}^\dagger \Psi_{h_k}\) is the total number operator. A matrix element of the reduced density matrix is given by

\[
\rho^{(1)}_{m,n} = \langle h_m | \rho^{(1)} | h_n \rangle = \frac{\sum_{k=0}^\infty \langle 0 | \Psi_{h_m} \Psi_{h_k} \Psi_{h_j}^\dagger \Psi_{h_i}^\dagger |0\rangle \langle \Phi_{j,i} | \hat{n} | \Phi_{j,i}\rangle}{\langle \Phi_{j,i} | \hat{n} | \Phi_{j,i}\rangle}
\]

The expressions for the matrix element can be obtained analytically. They are given by an infinite series involving parabolic cylinder functions. They depend on \(\eta\). The detailed calculations are given in [C].

Since the expressions for the reduced density matrix are cumbersome, we resort to calculating the eigenvalues numerically through Mathematica, by using the formula

\[
\sum_{m=0}^\infty \rho^{(1)}_{m,n} g(n) = \lambda_n g(m)
\]  

11
where $\lambda_n$ is an eigenvalue. The von Neumann entropy is then given by the usual formula

$$S(\rho^1) = -\text{Tr}(\rho^1 \log(\rho^1)) = -\sum_i \lambda_i \log(\lambda_i)$$

The dependence of the von Neumann entropy on the statistics parameter $\eta$ is plotted in the following figures for different choices of the initial two-particle state.

Figure 1: Plot of entropy vs statistics parameter $\eta$ for the initial two-particle state $|\Phi_{0,0}\rangle$.

In the above plot, the two-particle state is taken to be $|\Phi_{0,0}\rangle$. It is worth noting that for $\eta = 0$, both the particles are in the same state. The entropy is zero, consistent with what is expected of bosons.
Figure 2: Plot of entropy vs statistics parameter $\eta$ for the initial state two-particle state $|\Phi_{1,0}\rangle$.

In the above plot, the two-particle state is taken to be $|\Phi_{1,0}\rangle$. In this case, it is worth noting that for both $\eta = 0$ and $\eta \to \infty$, the entropy is equal to unity, consistent with the results of [8].

In order to get a better insight into what the above plots mean, it is useful to compare our results with [8]. Franco and Compagno consider a model of two indistinguishable qubits in an asymmetric double-well potential. In particular, they study the spin correlations between the qubits in the same spatially localised state, namely the left trough. It is important to note that the potential acts as a crutch to produce various states for the qubits, namely, states which are localised either on the left side, or the right side, or those which are in a superposition of the left and right sides. Once a state is specified, only the finite-dimensional Hilbert spaces associated with the qubits play a role. For example, they show that when both the qubits are localised in the left well, the state $|L \uparrow, L \uparrow\rangle$ is not entangled, whereas, the state $|L \uparrow, L \downarrow\rangle$ is maximally entangled analogous to the Bell state for distinguishable qubits. In arriving at this result the one-particle basis used is finite-dimensional, because only the spin degrees of freedom of the qubits are considered.

In our model, the state $|\Phi_{0,0}\rangle$ corresponds to the state $|L \uparrow, L \uparrow\rangle$, and the
state $|\Phi_{1,0}\rangle$ corresponds to the state $|L \uparrow, L \downarrow\rangle$. But there are crucial differences. The states in our model represent not two indistinguishable qubits, but two indistinguishable particles, hence the one-particle basis over which we sum is infinite-dimensional. Moreover, it depends on the statistics parameter $\eta$. Our primary goal was to establish the dependence of the entropy on the statistics parameter. That is what is displayed in the above plots. It is gratifying to note that in the limiting cases of bosons and fermions given by $\eta = 0$ and $\eta \to \infty$, our results are in complete agreement with [8].

The other results that Franco and Compagno obtain regarding non-local entanglement use superpositions of states localised in the left and right wells, and are beyond the scope of the present work.

5 Conclusions

The problem of studying the entanglement between indistinguishable particles in quantum mechanics is tricky. A naive usage of the usual measures like the Schmidt rank and the von Neumann entropy leads to wrong results.

A way to bypass these problems, restricted to bosons and fermions, was developed by Franco and Compagno [8] by using ideas coming from information theory.

In this paper we use their results, in the second quantized formulation, to study the entanglement between two one-dimensional anyons. The generalised algebra of one-dimensional anyons obtained from a second quantization of the Leinaas-Myrheim model [12] plays a crucial role in our analysis.

We succeed in obtaining the dependence of the von Neumann entropy on the statistics parameter qualitatively.

The calculations presented in this paper are readily generalizable to studying entanglement between two clusters of anyons with an arbitrary number of particles. Other one-dimensional models admitting anyonic statistics like indistinguishable particles on a ring and the Calogero model are also worth investigating.
The most interesting problem will, of course, be to investigate the entangle-
ment between anyons in two dimensions, both in the abelian and non-abelian
cases, because of their direct relevance to topological quantum computation.

We hope to address these questions in our future work.

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A The reduced density matrix in the second
quantization formalism

In the second quantization language the N-particle state $|\Phi\rangle$ is obtained by
acting with a suitable combination of creation operators on the vacuum state.
Let the set of states \{ $|\psi_k\rangle \equiv a_{\psi_k}^\dagger |0\rangle$ \} form a basis for the single particle Hilbert
space. In analogy with equation 3, the state $|\phi_k\rangle$ is defined in the second
quantization formalism as follows.

\[
|\phi_k\rangle = \frac{a_{\psi_k} |\Phi\rangle}{\sqrt{\langle \Phi | a_{\psi_k}^\dagger a_{\psi_k} | \Phi \rangle}}
\]

The corresponding probabilities are

\[
p_k = \frac{\langle \Phi | a_{\psi_k}^\dagger a_{\psi_k} | \Phi \rangle}{\langle \Phi | \hat{n} | \Phi \rangle}
\]

where $\hat{n} = \sum_k a_{\psi_k}^\dagger a_{\psi_k}$ is the total number operator. Then, the one-particle
reduced density matrix is

\[
\rho^{(1)} = \frac{\sum_k a_{\psi_k} |\Phi\rangle \langle \Phi | a_{\psi_k}^\dagger}{\langle \Phi | \hat{n} | \Phi \rangle}
\]
B Real space algebra

B.1 Derivation of the real space algebra

The algebra of creation and annihilation operators of momentum states obtained in [12] is

\begin{align*}
a_p^\dagger a_q^\dagger &= e^{i\phi(p-q)}a_q^\dagger a_p^\dagger \\
a_p a_q^\dagger &= e^{-i\phi(p-q)}a_q^\dagger a_p + \delta(p-q)
\end{align*}

where the phase \( e^{i\phi(p-q)} = \frac{p+q-i\eta}{p-q-i\eta} \). The above relations may be rewritten in a slightly modified way as follows:

\begin{align*}
a_p^\dagger a_q^\dagger &= \left(\frac{p+q+i\eta}{p-q+i\eta}\right)^\dagger a_q^\dagger a_p^\dagger \\
a_p a_q^\dagger &= \left(\frac{p+q-i\eta}{p-q+i\eta}\right)^\dagger a_q^\dagger a_p + \delta(p-q)
\end{align*}

Note that the creation and annihilation operators for the momentum states are related to the second quantized fields through the relations

\[
\Psi^\dagger(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ikx} a_k^\dagger.
\]

To obtain the algebra of field operators we calculate the commutator between field operators.

\[
\Psi(x)\Psi^\dagger(y) - \Psi^\dagger(y)\Psi(x) = \int_{-\infty}^{\infty} dp dq e^{-ipx+iqy} (a_p a_q^\dagger - a_q^\dagger a_p)
\]

Substituting for \( a_p a_q^\dagger \) from the algebra of creation and annihilation operators for momentum states,

\[
\Psi(x)\Psi^\dagger(y) - \Psi^\dagger(y)\Psi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp dq e^{-ipx+iqy} a_q^\dagger a_p \left( \left(\frac{p-q+i\eta}{p-q+i\eta}\right) - 1 \right) + \delta(p-q)
\]

\[
= \delta(x-y) - \frac{\eta}{\pi} \int_{-\infty}^{\infty} dp dq e^{-ipx+iqy} a_q^\dagger a_p \left( \frac{1}{-ip+iq+\eta} \right)
\]

\[
= \delta(x-y) - \frac{\eta}{\pi} \int_{0}^{\infty} dz e^{-z\eta} \int_{-\infty}^{\infty} dp dq e^{-ip(x-z)+iq(y-z)} a_q^\dagger a_p
\]

\[
= \delta(x-y) - 2\eta \int_{0}^{\infty} dz e^{-z\eta} \Psi^\dagger(y-z)\Psi(x-z)
\]
Similarly, if we look at the commutator \([\Psi^\dagger(x), \Psi^\dagger(y)]\), we obtain:

\[
\Psi^\dagger(x)\Psi^\dagger(y) - \Psi^\dagger(y)\Psi^\dagger(x) = \int_{-\infty}^{\infty} dp dq e^{ipx+iqy} (a_p^\dagger a_q^\dagger - a_q^\dagger a_p^\dagger)
\]

Substituting for \(a_p^\dagger a_q^\dagger\),

\[
\Psi^\dagger(x)\Psi^\dagger(y) - \Psi^\dagger(y)\Psi^\dagger(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp dq e^{ipx+iqy} a_q^\dagger a_p^\dagger \left( \frac{p - q - i\eta}{p - q - i\eta} - 1 \right)
\]
\[
= -\frac{\eta}{\pi} \int_{-\infty}^{\infty} dp dq e^{ipx+iqy} a_q^\dagger a_p^\dagger \int_{0}^{\infty} dz e^{-z(p-q+i\eta)}
\]
\[
= -\frac{\eta}{\pi} \int_{0}^{\infty} dz e^{-z\eta} \int_{-\infty}^{\infty} dp dq e^{ip(x-z)+iq(y+z)} a_q^\dagger a_p^\dagger
\]
\[
= -2\eta \int_{0}^{\infty} dz e^{-z\eta} \Psi^\dagger(y+z)\Psi^\dagger(x-z)
\]

B.2 Check on the algebra

The symmetrization postulate for multiparticle wavefunctions in the first quantized formalism has an intimate connection with the algebra of creation and annihilation operators in the second quantized formalism. This was clearly explained in very general terms by Fock for the case of bosons and fermions in [13]. In this appendix we verify the consistency of the anyonic algebra we use along similar lines.

The field operator \(\Psi(x)\) acts on the sequence of functions

\[
\left( \begin{array}{c} 
\text{const.} \\
\psi(x_1) \\
\psi(x_1, x_2) \\
\psi(x_1, x_2, x_3) \\
\vdots 
\end{array} \right)
\]

as follows

\[
\Psi(x) \left( \begin{array}{c} 
\text{const.} \\
\psi(x_1) \\
\psi(x_1, x_2) \\
\psi(x_1, x_2, x_3) \\
\vdots 
\end{array} \right) = \left( \begin{array}{c} 
\text{const.} \\
\sqrt{2}\psi(x, x_1) \\
\sqrt{3}\psi(x, x_1, x_2) \\
\sqrt{4}\psi(x, x_1, x_2, x_3) \\
\vdots 
\end{array} \right)
\]

(21)
where the functions $\psi(x_1), \psi(x_1, x_2), \psi(x_1, x_2, x_3), \ldots$ are interpreted as Schrödinger wave functions [13].

Applying the operator $\Psi(x')\Psi(x)$ on the sequence of functions, we obtain

$$
\Psi(x') \Psi(x) \begin{pmatrix}
\text{const.} \\
\psi(x_1) \\
\psi(x_1, x_2) \\
\psi(x_1, x_2, x_3) \\
\ldots
\end{pmatrix} = \begin{pmatrix}
\sqrt{2.1}\psi(x, x') \\
\sqrt{3.2}\psi(x, x', x_1) \\
\sqrt{4.3}\psi(x, x', x_1, x_2) \\
\sqrt{5.4}\psi(x, x', x_1, x_2, x_3) \\
\ldots
\end{pmatrix}.
$$

(23)

Similarly, applying the operator $\Psi(x)\Psi(x')$ on the sequence of functions, we get

$$
\Psi(x) \Psi(x') \begin{pmatrix}
\text{const.} \\
\psi(x_1) \\
\psi(x_1, x_2) \\
\psi(x_1, x_2, x_3) \\
\ldots
\end{pmatrix} = \begin{pmatrix}
\sqrt{1.2}\psi(x', x) \\
\sqrt{2.3}\psi(x', x, x_1) \\
\sqrt{3.4}\psi(x', x, x_1, x_2) \\
\sqrt{4.5}\psi(x', x, x_1, x_2, x_3) \\
\ldots
\end{pmatrix}.
$$

(24)

In the case of bosons, the right hand side of Eq.23 and Eq.24 are the same because the bosonic wavefunction is symmetric under the exchange of any pair of coordinates. This implies that the field operators $\Psi(x)$ and $\Psi(x')$ commute with each other. In the case of fermions, using the same argument and by noting that the fermionic wavefunctions are anti-symmetric, one can obtain the usual anti-commutation relation between $\Psi(x)$ and $\Psi(x')$.

In our case the field operators satisfy the following algebra

$$
[\Psi(x), \Psi^\dagger(y)] = \delta(x - y) - 2\eta \int_0^\infty dz \ e^{-\eta z} \Psi^\dagger(y - z) \Psi(x - z)
$$

$$
[\Psi^\dagger(x), \Psi^\dagger(y)] = -2\eta \int_0^\infty dz \ e^{-\eta z} \Psi^\dagger(y + z) \Psi^\dagger(x - z)
$$

The consistency of the algebra requires that the following equation holds

$$
\left(\Psi(x) \Psi(y) - \Psi(y) \Psi(x) - 2\eta \int_0^\infty dz e^{-\eta z} \Psi(x - z) \Psi(y + z)\right) \begin{pmatrix}
\text{const.} \\
\psi(x_1) \\
\psi(x_1, x_2) \\
\psi(x_1, x_2, x_3) \\
\ldots
\end{pmatrix} = 0
$$

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\[
\begin{pmatrix}
\sqrt{2.1} (\psi(y, x) - \psi(x, y) - 2\eta \int_0^\infty dz e^{-z\eta}\psi(y + z, x - z)) \\
\sqrt{3.2} (\psi(y, x, x_1) - \psi(x, y, x_1) - 2\eta \int_0^\infty dz e^{-z\eta}\psi(y + z, x - z, x_1)) \\
\sqrt{4.3} (\psi(y, x, x_1, x_2) - \psi(x, y, x_1, x_2) - 2\eta \int_0^\infty dz e^{-z\eta}\psi(y + z, x - z, x_1, x_2))
\end{pmatrix} = 0
\]

where \(\psi(x_1, x_2, ..., x_N)\) is the \(N\)-anyon wave function. Let the wave function be

\[
\psi(x_1, ..., x_N) = \sum_{P \in S_N} \alpha(k_{P(1)}, ..., k_{P(N)}) e^{i(k_{P(1)}x_1 + ... + k_{P(N)}x_N)}
\]

where the coefficients satisfy

\[
\alpha(...k_j, ..., k_i, ...) = \left(\frac{k_j - k_i - i\eta}{k_j - k_i + i\eta}\right) \alpha(...k_i, ..., k_j, ...)
\]

We have to calculate

\[
\left(\psi(y, x, x_3, ..., x_N) - \psi(x, y, x_3, ..., x_N) - 2\eta \int_0^\infty dz e^{-z\eta}\psi(y + z, x - z, x_3, ..., x_N)\right)
\]

Substituting the expression for the wave function,

\[
\sum_{P \in S_N} \left(\alpha(k_{P(1)}, ..., k_{P(N)}) e^{i(k_{P(1)}y+k_{P(2)}x+k_{P(3)}x_3+...+k_{P(N)}x_N)}
\right.
\]

\[
- \alpha(k_{P(1)}, ..., k_{P(N)}) e^{i(k_{P(1)}x+k_{P(2)}y+k_{P(3)}x_3+...+k_{P(N)}x_N)}
\]

\[
- 2\eta \int_0^\infty dz e^{-z\eta}\alpha(k_{P(1)}, ..., k_{P(N)}) e^{i(k_{P(1)}(y+z)+k_{P(2)}(x-z)+k_{P(3)}x_3+...+k_{P(N)}x_N)}
\]

\[
= \sum_{P \in S_N} \left(\alpha(k_{P(1)}, ..., k_{P(N)}) e^{i(k_{P(1)}y+k_{P(2)}x+k_{P(3)}x_3+...+k_{P(N)}x_N)}
\right.
\]

\[
- \alpha(k_{P(1)}, ..., k_{P(N)}) e^{i(k_{P(1)}x+k_{P(2)}y+k_{P(3)}x_3+...+k_{P(N)}x_N)}
\]

\[
- \frac{2i\eta}{k_{P(1)} - k_{P(2)} + i\eta} \alpha(k_{P(1)}, ..., k_{P(N)}) e^{i(k_{P(1)}y+k_{P(2)}x+k_{P(3)}x_3+...+k_{P(N)}x_N)}
\]

we find that the coefficient of the term \(e^{i(k_{P(1)}x+k_{P(2)}y+k_{P(3)}x_3+...+k_{P(N)}x_N)}\) is

\[
\alpha(k_{P(2)}, k_{P(1)}, ..., k_{P(N)}) - \left(\frac{k_{P(1)} - k_{P(2)} - i\eta}{k_{P(1)} - k_{P(2)} + i\eta}\right) \alpha(k_{P(1)}, k_{P(2)}, ..., k_{P(N)})
\]

Using the relation among coefficients, it is easy to see that above term is zero, as expected.
C Calculation of the one-particle reduced density matrix

The matrix elements of the one-particle reduced density matrix are

$$\rho_{m,n}^{(1)} = \frac{\sum_{k=0}^{\infty} \langle 0 | \psi_{h_m} \psi_{h_k} | \Phi_{j,i} \rangle \langle \Phi_{j,i} | \psi_{h_k}^\dagger \psi_{h_m} | 0 \rangle}{\langle \Phi_{j,i} | n | \Phi_{j,i} \rangle}.$$ 

Using the definition of the state $|\Phi_{j,i}\rangle$, it is rewritten as,

$$\rho_{m,n}^{(1)} = \sum_k \langle 0 | \psi_{h_m} \psi_{h_k} \psi_{h_j}^\dagger | 0 \rangle \langle 0 | \psi_{h_i} \psi_{h_j} \psi_{h_k}^\dagger | 0 \rangle \langle \psi_{h_k} \psi_{h_m} | \psi_{h_i} \psi_{h_j}^\dagger | 0 \rangle$$

$$2 \langle 0 | \psi_{h_i} \psi_{h_j} \psi_{h_k}^\dagger | 0 \rangle$$

To obtain the expression for the one-particle reduced density matrix a generic term of the following form is calculated.

$$\langle 0 | \psi_{h_m} \psi_{h_k} \psi_{h_j}^\dagger | 0 \rangle = \langle h_k | h_j \rangle \langle h_m | h_i \rangle + \langle h_k | h_i \rangle \langle h_m | h_j \rangle$$

$$\quad- \int_0^\infty dz \; 2\eta e^{-z\eta} \int_{-\infty}^\infty dx dy \; e_m^*(y-z)h_j^*(x)h_i(y-h_i(x-z))$$

Using the above formula, the denominator of the one-particle reduced density matrix can be obtained by setting $m = i$ and $k = j$. The numerator is calculated below.

$$\sum_k \langle 0 | \psi_{h_m} \psi_{h_k} \psi_{h_j}^\dagger | 0 \rangle \langle 0 | \psi_{h_i} \psi_{h_j} \psi_{h_k}^\dagger | 0 \rangle$$

$$= \langle h_m | h_i \rangle \langle h_n | h_i \rangle + \langle h_i | h_j \rangle \langle h_m | h_i \rangle \langle h_n | h_j \rangle$$

$$+ \langle h_j | h_i \rangle \langle h_m | h_j \rangle \langle h_i | h_n \rangle + \langle h_m | h_j \rangle \langle h_j | h_n \rangle$$

$$- 2\eta \langle h_m | h_i \rangle \int_0^\infty dz \int_{-\infty}^\infty dx dy \; e^{-z\eta}h_n(y-z)h_j^*(x)h_i^*(y)h_i^*(x-z)$$

$$\quad- 2\eta \langle h_m | h_j \rangle \int_0^\infty dz \int_{-\infty}^\infty dx dy \; e^{-z\eta}h_n(y-z)h_i(x)h_j^*(y)h_i^*(x-z)$$

$$\quad- 2\eta \langle h_i | h_n \rangle \int_0^\infty dz \int_{-\infty}^\infty dx dy \; e^{-z\eta}h_m^*(y-z)h_j^*(x)h_i(y)h_i(x-z)$$

$$\quad- 2\eta \langle h_j | h_n \rangle \int_0^\infty dz \int_{-\infty}^\infty dx dy \; e^{-z\eta}h_m^*(y-z)h_i(x)h_j(y)h_i(x-z)$$

$$\quad+ 4\eta^2 \int_0^\infty dz dz' \int_{-\infty}^\infty dx dy \; e^{-(z+z')\eta}h_m^*(y-z)h_j(y)h_i(x-z)$$

$$\quad\times h_n(y'-z')h_j^*(y')h_i^*(x-z')$$

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To calculate the one-particle reduced density matrix, we use the following integrals \[15\].

\[
\int_{-\infty}^{\infty} dz \ e^{-\frac{z^2}{2} + \frac{1}{2} (z-\zeta)^2} H_n(z) H_p(z-\zeta) = \frac{1}{\Gamma(n+1)} \sqrt{\pi} e^{-\frac{\zeta^2}{4}} \sqrt{2^{n+1} n!} \sqrt{2^{p+1} p!} (-\zeta)^{p-n} \\
\times \sqrt{2^{n-p} \Gamma(n+1) \Gamma(p+1)} \ {}_1\tilde{F}_1\left(-n; -n+p+1; \frac{\zeta^2}{2}\right), \ n, p \in \mathbb{N}
\]

\[
\int_{0}^{\infty} x^{\nu-1} e^{-\beta x^2 - \gamma x} = (2\beta)^{-\frac{\nu}{2}} \Gamma(\nu) e^{\frac{\gamma^2}{4\beta}} D_{-\nu}\left(\frac{\gamma}{\sqrt{2\beta}}\right), \ \nu > -1
\]

Here \(_{1}\tilde{F}_1(a; b; z)\) denotes the regularized confluent hypergeometric function and \(D_{-\nu}(z)\) denotes the parabolic cylinder function. The matrix elements of the one-particle reduced density matrix obtained from the initial state \(|\Phi_{0,0}\rangle\) are given below.

\[
\left(\rho_{0,0}^{(1)}\right)_{m,n} = \frac{1}{d_1} \left( 4\delta_{m0}\delta_{n0} - 4\eta\delta_{m0} (-1)^n \frac{1}{\sqrt{2^n n!}} D(n+1, \eta) \\
- 4\eta\delta_{n0} (-1)^m \frac{1}{\sqrt{2^m m!}} D(m+1, \eta) \\
+ (-1)^{m+n} \frac{4\eta^2}{\sqrt{2^{n+m} n! m!}} \sum_{l=0}^{\infty} \left( \frac{1}{2^l l!} D(m+l+1, \eta) D(n+l+1, \eta) \right) \right)
\]

where \(D(\nu, x) = \Gamma(\nu) e^{\frac{x^2}{2}} D_{-\nu}(x)\) and

\[
d_1 = 4(1 - \eta D(-1, \eta))
\]

The matrix elements of the one-particle reduced density matrix obtained from the initial state \(|\Phi_{1,0}\rangle\) are given below.

\[
\left(\rho_{1,0}^{(1)}\right)_{m,n} = \frac{1}{d_2} \left( \delta_{m1}\delta_{n1} + \delta_{m0}\delta_{n0} - \delta_{m1} (-1)^{n+1} \frac{\sqrt{2}\eta}{\sqrt{2^n n!}} D(n+2, \eta) \\
- \delta_{m0} \eta (-1)^n \frac{1}{\sqrt{2^n n!}} (2D(n+1, \eta) - D(n+3, \eta)) \\
- \delta_{n1} (-1)^{m+1} \frac{\sqrt{2}\eta}{\sqrt{2^m m!}} D(m+2, \eta) \\
- \delta_{n0} \eta (-1)^m \frac{1}{\sqrt{2^m m!}} (2D(m+1, \eta) - D(m+3, \eta)) \right)
\]
\[ + \frac{4\eta^2(-1)^{m+n}}{2^{m+n}m!n!} \sum_{l=0}^{\infty} \frac{1}{2^l l!} \left( 2\mathcal{D}(n + l + 1, \eta)\mathcal{D}(m + l + 1, \eta) 
 - \mathcal{D}(n + l + 1, \eta)\mathcal{D}(m + l + 3, \eta) + 2\mathcal{D}(n + l + 2, \eta)\mathcal{D}(m + l + 2\eta) 
 - \mathcal{D}(n + l + 3, \eta)\mathcal{D}(m + l + 1, \eta) \right) \]

where

\[ d_2 = 2 \left( 1 + \frac{\eta}{2} \mathcal{D}(3, \eta) \right) \]

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