Unified $\ell_{2\to\infty}$ Eigenspace Perturbation Theory for Symmetric Random Matrices

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Abstract

Modern applications in statistics, computer science and network science have seen tremendous values of finer matrix spectral perturbation theory. In this paper, we derive a generic $\ell_{2\to\infty}$ eigenspace perturbation bound for symmetric random matrices, with independent or dependent entries and fairly flexible entry distributions. In particular, we apply our generic bound to binary random matrices with independent entries or with certain dependency structures, including the unnormalized Laplacian of inhomogenous random graphs and $m$-dependent matrices. Through a detailed comparison, we found that for binary random matrices with independent entries, our $\ell_{2\to\infty}$ bound is tighter than all existing bounds that we are aware of, while our condition is weaker than most of them with only one exception in a special regime. We employ our perturbation bounds in three problems and improve the state of the art: concentration of the spectral norm of sparse random graphs, exact recovery of communities in stochastic block models and partial consistency of divisive hierarchical clustering. Finally we discuss the extensions of our theory to random matrices with more complex dependency structures and non-binary entries, asymmetric rectangular matrices and induced perturbation theory in other metrics.

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1 Introduction

Matrix spectral perturbation theory is one of the most fundamental and powerful tools in various areas including statistics, network sciences and computer sciences. In many problems, it is important to understand how eigenvalues and eigenvectors change when the underlying matrix $A^*$ is perturbed into $A$. Focusing on symmetric matrices, Weyl's inequality provides a simple bound for eigenvalues [Weyl, 1912] and Davis-Kahan Theorem provides a surprisingly clean bound for eigenspaces in terms of any unitarily invariant norm [Davis and Kahan, 1970]. We refer to interested readers to Stewart [1990], Kato [2013] and the appendix of Bai and Silverstein [2010] for fruitful results over the past century.

Modern applications are posing new challenges for this long-standing area. In these problems, the unitarily invariant norm, such as Frobenius norm or operator norm, in the classical eigenvector perturbation theory may be too coarse to achieve the goal. It is then crucial to derive eigenvector/eigenspace perturbation bounds in terms of finer norms that are not unitarily invariant. Among others, one important norm is the $\ell_2^{\rightarrow \infty}$ norm, which yields the row-wise perturbation bound on the eigenvector matrix. To be specific, let $A$ and $A^*$ be two symmetric matrices with

$$E = A - A^*.$$  \hfill (1)

Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $\lambda_1^* \geq \lambda_2^* \geq \ldots \geq \lambda_n^*$ be the eigenvalues of $A$ and $A^*$, respectively. Given positive integers $s$ and $r$, let

$$A = \text{diag}(\lambda_{s+1}, \lambda_{s+2}, \ldots, \lambda_{s+r}), \quad A^* = \text{diag}(\lambda_{s+1}^*, \lambda_{s+2}^*, \ldots, \lambda_{s+r}^*).$$ \hfill (2)

Let $U, U^* \in \mathbb{R}^{n \times r}$ be a matrix of eigenvectors such that

$$AU = U\Lambda, \quad A^*U^* = U^*\Lambda^*.$$ \hfill (3)

The $\ell_2^{\rightarrow \infty}$ perturbation theory is seeking for bounds on the $\ell_2^{\rightarrow \infty}$ distance between $U$ and $U^*$, defined as

$$d_{2^{\rightarrow \infty}}(U, U^*) \triangleq \inf_{O \in \mathbb{R}^{r \times r}, O^TO = I} \|OU - U^*\|_{2^{\rightarrow \infty}}.$$  

where $\|V\|_{2^{\rightarrow \infty}} = \max_{i \in [n]} \|V_i\|_2$ and $V_i$ is the $i$-th row of $V$. The $\ell_2^{\rightarrow \infty}$ norm is not invariant to left unitary transformation and thus not supported by classical Davis-Kahan Theorem. When $r = 1$, $d_{2^{\rightarrow \infty}}(U, U^*)$ reduces to the entrywise perturbation of the eigenvector. Compared to perturbation bounds in operator norm, it provides much finer information.

1.1 Existing works

Early efforts on $\ell_2^{\rightarrow \infty}$ perturbation was motivated by the stability of Markov chain [e.g. O’cinneide, 1993, Ipsen and Meyer, 1994]. The focus was on the stationary distribution, which is the first eigenvector of the transition matrix of a finite state Markov chain. The investigation in random graph theory can be dated back to Mitra [2009] which studied the entrywise perturbation for the leading eigenvector of the adjacency matrix of an Erdős-Rényi graph $\mathbb{G}(n, p)$. In particular, Mitra [2009] considered the sparse graph regime allowing the parameter $p$ to decay with $n$ as long as $p \geq (\log n)^6/n$. He further studied the entrywise perturbation of the second eigenvector for dense planted partition models where the within-block average degree is $\sqrt{n}$. This is the first work proving that spectral clustering can achieve exact recovery, namely
Table 1: The regime that each method works on for binary random matrices with independent entries. The √* symbol in the third column means that the bound is derived for both full and top-r eigenspace recovery but only the former is available for binary random matrices. $p^*$ denotes the maximum entry of $A^*$ and $\kappa^*$ denotes the condition number of $\Lambda^*$. The constraints may not be explicitly mentioned in those papers but can be derived from their conditions. They are only necessary conditions and the real constraint may be more stringent. See Appendix E for details.

| Method                | Full | Top-r | Partial | $np^*$ | $r$    | $\kappa^*$ | $\|U^*\|_{2\rightarrow\infty}$ |
|-----------------------|------|-------|---------|--------|--------|------------|------------------------------|
| Abbe et al. [2017]    | ✓    | ✓     | ✓       | $\geq \log n/\log \log n$ | No     | $\leq \log(np^*)$ | No                           |
| Eldridge et al. [2017]| ✓    | ✓     | ✓       | $\geq (\log n)^{2+\epsilon}$ | $= 1$  | $= 1$      | $\leq 1/\sqrt{n}$          |
| Mao et al. [2017]     | ✓    | ✓     | x       | $\geq (\log n)^{2+\epsilon}$ | No     | No         | $\leq \sqrt{rp^*}$         |
| Fan et al. [2018]     | ✓    | ✓*    | x       | $\geq \log n$               | $\leq 1$ | No         | $\leq 1/\sqrt{n}$          |
| Cape et al. [2019a]   | ✓    | x     | x       | $\geq (\log n)^{2+\epsilon}$ | $\leq (\log n)^{2+\epsilon}$ | $\leq 1$ | No               |
| Cape et al. [2019b]   | ✓    | ✓*    | x       | $\geq \log n$               | No     | No         | No                           |
| This paper            | ✓    | ✓     | ✓       | $\geq \log n/\log \log n$   | No     | No         | No                           |

Recent years has seen a surge of interest in $\ell_{2\rightarrow\infty}$ perturbation theory especially for random matrices. This includes robust covariance estimation and robust principal component analysis for heavy-tailed data [Fan et al., 2018], community detection [Balakrishnan et al., 2011, Eldridge et al., 2017, Mao et al., 2017, Abbe et al., 2017, Cape et al., 2019a], multiple graph inference [Cape et al., 2019b], phase synchronization [Zhong and Boumal, 2018, Abbe et al., 2017], matrix completion [Abbe et al., 2017]. It also serves as a powerful tool to push forward other theoretical works such as random graph theory [e.g. Lugosi et al., 2018].

Despite the great success in different applications, the existing $\ell_{2\rightarrow\infty}$ perturbation theory is not satisfactory in that the bounds work under rather different regimes. To better describe each work, we classify the applicability into three categories:

- **Full eigenspace recovery:** $s = 0$ and $\lambda^*_{r+1} = \ldots = \lambda^*_n = 0$. In this case, $A^*$ has to be low rank.

- **Top-r eigenspace recovery:** $s = 0$ and there is no restriction on other eigenvalues.

- **Partial eigenspace recovery:** There is no restriction on $s$ or other eigenvalues.

We consider binary random matrices with independent entries for illustration. Table 1 summarizes the necessary conditions for each bound to work, where $p^*$ denotes the maximum entry of $A^*$, $\kappa^* = \lambda^*_{s+1}/\lambda^*_s$ denotes the condition number of $\Lambda^*$, and $\preceq$ ($\succeq$) denotes smaller (larger) in order. See Appendix E for the derivation of these claims. As with Davis-Kahan theorem, $\ell_{2\rightarrow\infty}$ perturbation theory also requires sufficient eigen-gap. Since these bounds work under different regimes, we consider the intersection of them.
Table 2: Comparison of the conditions on the eigen-gap and the bounds in each work under the regime $np^* \geq (\log n)^{2+\epsilon}$, $r \leq 1$, $\kappa^* \leq 1$, $\|U^*\|_{2 \to \infty} \leq 1/\sqrt{n}$, $A_{i,j}^* \sim p^*$.

| Work                  | Condition on the Eigen-Gap | Bound on $\sqrt{n}d_{2 \to \infty}(U, U^*)$ |
|-----------------------|-----------------------------|---------------------------------------------|
| Abbe et al. [2017]    | $np^*/\log(np^*)$           | $\leq 1$                                   |
| Eldridge et al. [2017]| $np^*$                      | $\leq \sqrt{(\log n)^{2+\epsilon}/np^*}$  |
| Mao et al. [2017]     | $\sqrt{np^*(\log n)^{1+\epsilon}/2}$ | $\leq \sqrt{(\log n)^{2+\epsilon}/np^*}$  |
| Fan et al. [2018]     | $np^*$                      | $\leq 1$                                   |
| Cape et al. [2019a]   | $np^*$                      | $\leq \sqrt{(\log n)^{2+\epsilon}/np^*}$  |
| Cape et al. [2019b]   | $np^*$                      | $\leq 1$                                   |
| This paper            | $\sqrt{np^*}$               | $\geq \sqrt{\log n/np^*}$                 |

For comparison, namely the regime $np^* \geq (\log n)^{2+\epsilon}$, $r \leq 1$, $\kappa^* \leq 1$, $\|U^*\|_{2 \to \infty} \leq 1/\sqrt{n}$, $A_{i,j}^* \sim p^*$. To simplify we also assume that all entries of $A^*$ are in the same order of $p^*$. Table 2 summarizes the conditions on eigen-gaps as well as the bounds for each work in this special case.

From Table 1, we can see that the bounds of Eldridge et al. [2017], Mao et al. [2017], Cape et al. [2019a] require the maximum degree $np^*$ to grow faster than $(\log n)^{2+\epsilon}$. However, the critical regime of $np^*$ in random graph theory is typically $\log n$, under which phase transition occurs. Therefore, although $(\log n)^{2+\epsilon}$ appears to be close to the critical regime, the extra $(\log n)^{1+\epsilon}$ term is too artificial to explain interesting phenomena. Abbe et al. [2017]'s work is the first to remove this extra logarithmic terms using an ingenious leave-one-out argument. Nonetheless, it imposes a stringent assumption on the condition number: in the critical regime it only allows the condition number to grow as $\log \log n$. The bound of Fan et al. [2018] also removes the artificial log-factors but it requires the number of eigenvectors to be bounded and the eigenvectors have low coherence. From Table 2, we can see that all works require a stringent assumption on the eigen-gap except Mao et al. [2017]. Indeed, it is easy to show that the eigen-gap is always upper bounded by $np^*$. As a consequence, the four works except Mao et al. [2017] require the eigen-gap to be nearly the largest achievable value even in this special regime. By contrast, in Davis-Kahan Theorem, the lower bound on the eigen-gap is simply the operator norm of the perturbation, which is of order $\sqrt{np^*}$ in this case. Although Mao et al. [2017] gets rid of the $np^*$ lower bound, it still involves extra log-factors. Finally, in terms of the perturbation bound, we see that the bounds of Abbe et al. [2017], Fan et al. [2018] and Cape et al. [2019b] stay the same as $np^*$ increases. They are clearly inferior to the others that are decaying with $np^*$.

The real problems may not lie in the nice intersection regime as above. The existing bounds work in different regimes and there is no one dominating all others in terms of either applicability or tightness. This creates hurdles to choose which bound to use. Furthermore, it provides evidence that none of the existing bounds is tight and there is still much room for improvement and unification.

1.2 This paper

In this paper, we derive generic $d_{2 \to \infty}$ bounds that work for random matrices with independent entries or with dependent entries with certain dependency structures (Theorem 2.3 - Theorem 2.6). As with Abbe et al. [2017] and Cape et al. [2019a], our theory not only covers $d_{2 \to \infty}(U, U^*)$ but also covers
\(d_{2 \to \infty}(U, U^* + V)\) for some \(V \in \mathbb{R}^{n \times r}\) that yields a better approximation of \(U\). In our theory, the entry distribution can be arbitrary provided that some characteristics of the perturbation \(E\) can be controlled, e.g. operator norm and linear contrasts of rows. This includes but not limited to Bernoulli, Gaussian, sub-Gaussian and sub-exponential distributions. For the sake of length, we only discuss binary random matrices, because it is arguably the most challenging yet most common case calling for \(\ell_{2 \to \infty}\) perturbation theory, and provide a brief discussion in Section 7.2 for other entry distributions.

For binary random matrices, we derive the \(d_{2 \to \infty}\) bounds for the case with independent entries, or equivalently the adjacency matrix of random graphs (Theorem 3.4). In this case, our bound works in a broader regime than all of existing bounds as shown in the last row of Table 1. Under the special regime for Table 2, our result has the least stringent condition on the eigen-gap that matches Davis-Kahan Theorem while our bound is strictly sharper than all others. Through a more detailed comparison in Appendix E, we found that our bound is tighter than all existing bounds that we are aware of in all regimes, except in some corner cases where our bound is equivalent to some of the others. In addition, our eigen-gap condition is weaker than all others except in the case of full eigenspace recovery with \(np^* \geq (\log n)^{2+\epsilon}\), \(\min\{\kappa^*, r\} \geq (\log n)^{1+\epsilon/2}\) for which Mao et al. [2017]’s condition is weaker than ours.

In addition, we derive the inequality for unnormalized Laplacian of random graphs (Theorem 3.11) and discuss the case for binary matrices with certain \(m\)-dependence structure (Section 7.1). The former is particularly useful for community detection problems in network science. Both cases are straightforward applications of our generic bounds.

Our bounds can be applied to the problems considered in the works listed in Table 1. In this paper, we apply our bounds to three other problems. The first one is to bound the variance and derive the concentration of the spectral norm of sparse random graphs. This is a fundamental and long-standing problem in random graph theory. Classical theory shows that, for Erdős-Rényi graphs, the variance is bounded by a universal constant regardless of the graph sparsity and the spectral norm is sub-gaussian with an \(O(1)\) parameter. The recent work by Lugosi et al. [2018] drastically improves the bound of variance and sub-gaussian parameters to \(O(p^*)\) for Erdős-Rényi graphs using the \(\ell_{2 \to \infty}\) perturbation theory. However, they only prove the result for \(p^* \geq (\log n)^{3}\) and conjecture that it carries over to the critical regime \(p^* \geq \log n/n\). We prove this conjecture using our new \(\ell_{2 \to \infty}\) perturbation theory. Moreover, we extend the result to general inhomogeneous random graphs.

The second problem is the strong consistency of spectral clustering for community detection in sparse stochastic block models (SBM). Despite the substantial literature on this topic, existing algorithms are rarely as easy-to-implement and computationally efficient as the standard spectral clustering algorithm [e.g. Von Luxburg, 2007], which is simply a singular value decomposition plus a \(K\)-means algorithm. However, perhaps surprisingly, the strong consistency of spectral clustering in sparse graphs is not established until recently [Abbe et al., 2017, Su et al., 2019]. In this paper, we apply our \(\ell_{2 \to \infty}\) bounds to prove the strong consistency of spectral clustering algorithms, using the adjacency matrix or the unnormalized Laplacian, for general SBMs in the critical regime with average degree \(O(\log n)\). We also study the SBMs with growing number of communities and obtain the best available dependence on it for spectral algorithms.

The third problem is the partial consistency of divisive hierarchical clustering on binary tree stochastic block models (BTSBM), proposed by Li et al. [2018a]. BTSBMs are special SBMs that embed communities
into a tree-like hierarchy in order to bring interpretability. It is also a useful framework to analyze divisive hierarchical clustering algorithms such as iterative spectral bi-partitioning [e.g. Spielman and Teng, 1996, Balakrishnan et al., 2011]. Li et al. [2018a] derived sufficient conditions that certain divisive hierarchical clustering algorithms are able to exactly recover the whole or a part of the hierarchy in the regime with the average degree $O((\log n)^{2+\epsilon})$. In this paper, we extend the result to the critical regime where each connection can be written as $a_j \log n/n$ and establish the sufficient condition to recover any part of the hierarchy in terms of the coefficients $a_j$’s. We found an unexpected connection between BTSBMs and misspecified SBMs. In addition we accurately quantify an observation in Li et al. [2018a] that certain partial structure may still be recovered even if the communities are information theoretically unrecoverable.

The rest of the article is organized as follows: Section 2 presents the generic bounds and Section 3 presents the bounds for binary random matrices. Three examples are collected in Section 4 - Section 6. Section 7 discusses several extensions, including binary random matrices with $m$-dependence structure, random matrices with non-binary entry distribution, singular space perturbation for asymmetric matrices and perturbation bounds in other metrics. Most technical proofs are relegated into Appendix. Appendix A establishes the proof of our main generic bound. The proof is quite involved so we parse it into six steps. All other proofs related to the generic bounds are presented in Appendix B. Appendix C contains all technical proofs for binary random matrices. The miscellaneous proofs in Section 4 - Section 6 are collected in Appendix D. Appendix E provides a detailed comparison between our bounds and all existing ones that we are aware of. This includes the justification of Table 1 and Table 2. Finally, Appendix F presents useful concentration inequalities for binary random variables which are useful in Section 3.

2 An Generic $\ell_{2\to\infty}$ Bound for Symmetric Random Matrices

2.1 Assumptions

Throughout the paper we consider the setup (1) - (3). We denote by $[n]$ the set $\{1, \ldots, n\}$ and by $1_n$ the $n$-dimensional vector with all entries 1. For any vector $x$, let $\|x\|_p$ denotes its $p$-norm. For any matrix $M$, let $M^T_k$ denote the $m$-th row of $M$, $\|M\|_{op}$ denote its operator norm and $\|M\|_F$ denote its Frobenius norm. Further we denote by $\lambda_{\text{max}}(M)$ (resp. $\lambda_{\text{min}}(M)$) the largest (resp. the smallest) eigenvalue of $M$ in absolute values, by $\kappa(M)$ the condition number $\lambda_{\text{max}}(M)/\lambda_{\text{min}}(M)$. In particular we write $\lambda_{\text{min}}^*(\Lambda^*)$ as $\lambda_{\text{min}}^*$ for short.

To state our generic bound, we need to define the following quantities.

- **Effective eigen-gap $\Delta^*$:**
  \[ \Delta^* \triangleq \min\{ \text{sep}_{s+1, s+r}(A^*), \lambda_{\text{min}}^* \}, \tag{4} \]
  where $\text{sep}_{s+1, s+r}(A^*) = \min\{ \lambda_s^* - \lambda_{s+1}^*, \lambda_{s+r}^* - \lambda_{s+r+1}^* \}$ with $\lambda_0^* = \infty$ and $\lambda_{n+1}^* = -\infty$. Note that $\Delta^* = \text{sep}_{s+1, s+r}(A^*)$ except when $\Lambda^*$ includes both positive and negative eigenvalues but 0 is not an eigenvalue of $\Lambda^*$. Therefore $\Delta^*$ is essentially the eigen-gap in the conventional sense.

- **Effective condition number $\bar{\kappa}^*$:**
  \[ \bar{\kappa}^* \triangleq \min\{ \kappa(\Lambda^*), 2r \}, \tag{5} \]
Note that the effective condition number is never larger than 2r unless the problem is ill-conditioned. On the other hand, if $\Lambda^*$ is well-conditioned but $r$ is large, $\bar{\kappa}^*$ can also be small.

- full eigenspace $\bar{U}^*$ of $A^*$, i.e.
  \[
  A^*\bar{U}^* = \bar{U}^*\bar{\Lambda}^*,
  \]
  where $\bar{\Lambda}^*$ includes all non-zero eigenvalues of $A^*$. Note that the number of columns $\bar{U}^*$ may significantly differ from that of $U^*$.

Our generic $\ell_{2\rightarrow\infty}$ bound requires the following two assumptions.

**A1** For any $\delta \in (0,1)$, there exists a random matrix $A^{(k)} \in \mathbb{R}^{n \times n}$ such that
\[
d_{TV}(P_{A^{(k)}}, P_{A_k} \times P_{A^{(k)}}) \leq \delta / n,
\]
where $d_{TV}$ denotes the total variation distance and it holds simultaneously for all $k$ and all contiguous subsets $S \subset [r]$ that
\[
\|A^{(k)} - \Lambda\|_{op} \leq L_1(\delta), \quad \frac{\|((A^{(k)} - \Lambda)U_S)\|_{op}}{\lambda_{\min}(\Lambda^*_S)} \leq (\kappa(\Lambda^*_S)L_2(\delta) + L_3(\delta))\|U_S\|_{2\rightarrow\infty},
\]
with probability at least $1 - \delta$ for some deterministic functions $L_1(\delta), L_2(\delta), L_3(\delta)$, where $U_S \in \mathbb{R}^{n \times |S|}$ denotes the matrix formed by columns of $U$ with indices in $S$.

**A2** There exists deterministic functions $\lambda_-(\delta), E_+(\delta), \bar{E}_+(\delta), E_\infty(\delta)$, such that for any $\delta \in (0,1)$, the following event occurs with probability at least $1 - \delta$:
\[
\|\Lambda - \bar{\Lambda}^*\|_{max} \leq \lambda_-(\delta), \quad \|EU^*\|_{op} \leq E_+(\delta), \quad \|E\bar{U}^*\|_{op} \leq \bar{E}_+(\delta), \quad \|W\|_{2\rightarrow\infty} \leq E_\infty(\delta).
\]

**A3** There exist deterministic functions $b_\infty(\delta), b_2(\delta) > 0$, such that for any $\delta \in (0,1), k \in [n], r' \leq r$ and fixed matrix $W \in \mathbb{R}^{n \times r'}$,
\[
\|W^T W\|_2 \leq b_\infty(\delta)\|W\|_{2\rightarrow\infty} + b_2(\delta)\|W\|_{op}, \quad \text{with probability at least } 1 - \delta / n,
\]

**A4** $\bar{\kappa}_* \geq 4(\sigma(\delta) + L_1(\delta) + \lambda_-(\delta))$ where
\[
\eta(\delta) = E_\infty(\delta) + b_\infty(\delta) + b_2(\delta), \quad \sigma(\delta) = \{\bar{\kappa}^*L_2(\delta) + L_3(\delta) + 1\}\eta(\delta) + E_+(\delta),
\]
Assumption A1 is worth some discussion. Roughly speaking, A1 controls the amount and the structure of entry dependence. The following proposition gives three cases where $L_1(\delta), L_2(\delta)$ and $L_3(\delta)$ can be exactly characterized. The proof is relegated to Appendix B.

**Proposition 2.1.** Assume that

(a) If $A_{ij}$’s are independent random variables, then there exists $A^{(k)}$ such that A1 is satisfied with
\[
L_1(\delta) = \sqrt{2}(\|A^*\|_{2\rightarrow\infty} + E_\infty(\delta)), \quad L_2(\delta) = 1, \quad L_3(\delta) = \frac{E_\infty(\delta) + \lambda_-(\delta) + \|A^*\|_{2\rightarrow\infty}}{\lambda_{\min}}.
\]

(b) Assume that for any $k$, there exists a subset $N_k \subset [n]$, such that $A_k$ is independent of $\{A_i : i \notin N_k\}$. Let $m = \max_k |N_k|$, then there exists $A^{(k)}$ such that A1 is satisfied with
\[
L_1(\delta) = \sqrt{2m}(\|A^*\|_{2\rightarrow\infty} + E_\infty(\delta)), \quad L_2(\delta) = m, \quad L_3(\delta) = \frac{m(E_\infty(\delta) + \lambda_-(\delta) + \|A^*\|_{2\rightarrow\infty})}{\lambda_{\min}}.
\]
In literature [e.g. Abbe et al., 2017], it is typically assumed that
\[ \| E \|_{op} \leq E_2(\delta) \] with probability at least \( 1 - \delta \). \hfill (9)

Assumption A2 is satisfied under (9) if
\[ \lambda_-(\delta) = E_+(\delta) = \bar{E}_+(\delta) = E_\infty(\delta) = E_2(\delta). \] \hfill (10)

This is because \( \| A\Lambda - \Lambda^*\|_{\max} \leq \| E \|_{op} \) by Weyl’s inequality, \( \| EU^*\|_{op} \leq \| E \|_{op}, \| EU^*\|_{op} \leq \| E \|_{op} \) and \( \| E \|_{2 \to \infty} \leq \| E \|_{op} \) by definition. In general, A2 can be strictly weaker than (9).

Assumption A3 requires a concentration inequality on linear transforms of rows of \( E \). A similar version is considered in Abbe et al. [2017] except that the operator norm is replaced by the Frobenius norm. We emphasize that our assumption A3 can yield tighter result when \( r > 1 \). Using a standard \( \epsilon \)-net argument, A3 can be verified by considering vectors \( W \) only. The following proposition summarizes the result with the proof relegated to Appendix C.

**Proposition 2.2.** Suppose that for any \( \delta \in (0, 1) \) and vector \( w \in \mathbb{R}^n \), there exists \( a_\infty(\delta), a_2(\delta) > 0 \) such that for each \( k \)
\[ E_m^T w \leq a_\infty(\delta)\| w \|_{\infty} + a_2(\delta)\| w \|_{2}, \]
with probability at least \( 1 - \delta \). Then assumption A3 holds with
\[ b_\infty(\delta) = 2a_\infty \left( \frac{\delta}{5^r n} \right), \quad b_2(\delta) = 2a_2 \left( \frac{\delta}{5^r n} \right). \]

Assumption A4 guarantees sufficient eigen-gap. In classical perturbation theory [e.g. Davis and Kahan, 1970] based on operator norm or Frobenius norm, it is necessary to assume \( \Delta^* \geq \lambda_-(\delta) \). Nonetheless, we will show that A4 is equivalent to \( \Delta^* \geq \lambda_-(\delta) \) in many applications.

### 2.2 Main results

Based on our assumptions, we first derive an bound for \( d_{2 \to \infty}(U, AU^*(\Lambda^*)^{-1}) \). As shown in Abbe et al. [2017] and Cape et al. [2019b], \( A U^*(\Lambda^*)^{-1} \) is a better approximation of \( U \) than \( U^* \). The proof is quite involved and thus presented step by step in Appendix A.

**Theorem 2.3.** Given any \( \delta \in (0, 1) \). Let \( \Delta^*, \kappa^* \) and \( \bar{U}^* \) be defined in (4) - (6). Then under assumptions A1-A4,
\[ d_{2 \to \infty}(U, AU^*(\Lambda^*)^{-1}) \leq \frac{C}{\Delta^*} \left\{ \sigma(\delta) \left( \| U^* \|_{2 \to \infty} + \frac{\| EU^* \|_{2 \to \infty}}{\lambda^*_{\min}} \right) + \frac{E_+(\delta)b_2(\delta)}{\lambda^*_{\min}} \right. \]
\[ \left. + \min \left\{ E_+(\delta) \xi_1, \bar{E}_+(\delta)\| \kappa^* \|_{\max}\xi_2, \bar{E}_+(\delta)\kappa^*\xi_3 \right\} \right\}, \]
with probability at least \( 1 - B(r)\delta \), where \( C \) is a universal constant (that can be chosen as 72),
\[ B(r) = 10 \min \{ r, 1 + \log_2 \kappa^* \}, \] \hfill (11)

and
\[ \xi_1 = \frac{\| A^* \|_{2 \to \infty}}{\lambda^*_{\min}}, \quad \xi_2 = \frac{\sqrt{\| A^* \|_{\max}}}{\sqrt{\lambda^*_{\min} I(A^* \text{ is psd})}}, \quad \xi_3 = \| \bar{U}^* \|_{2 \to \infty}. \] \hfill (12)
Remark 2.1. The last term is the minimum of three fundamentally different bounds, which are obtained from Kato’s integral [e.g. Kato, 1949]; see Appendix A.4 for details. The second term kicks in only when $A^*$ is positive semi-definite. This is true in many cases such as spiked wigner ensemble and phase synchronization [e.g. Abbe et al., 2017]. The third term becomes useful for full eigenspace recovery. This is common in the cases of community detection (Section 5).

Remark 2.2. All terms in the bound are deterministic except for $\|EU^*\|_{2 \rightarrow \infty}$. Although assumption $A3$ directly yields a bound as follows:

$$\|EU^*\|_{2 \rightarrow \infty} \leq b_\infty(\delta)\|U^*\|_{2 \rightarrow \infty} + b_2(\delta)\|U^*\|_{\text{op}} = b_\infty(\delta)\|U^*\|_{2 \rightarrow \infty} + b_2(\delta),$$

we found it can be sharpened in some applications; see Section 3 for instance. For this reason we keep this term in the bound and derive its upper tail case by case.

By the triangle inequality, we can directly obtain the following perturbation bound for $d_{2 \rightarrow \infty}(U, U^*)$ by $d_{2 \rightarrow \infty}(U, AU^*(\Lambda^*)^{-1}) + d_{2 \rightarrow \infty}(AU^*(\Lambda^*)^{-1}, U^*)$. We leave the proof in Appendix B.

**Theorem 2.4.** Under the same setting of Theorem 2.3,

$$d_{2 \rightarrow \infty}(U, U^*) \leq \frac{C\|EU^*\|_{2 \rightarrow \infty}}{\lambda^*_\min} + \frac{C}{\Lambda^*} \left\{ \sigma(\delta)\|U^*\|_{2 \rightarrow \infty} + \frac{E_+(\delta)b_2(\delta)}{\lambda^*_\min} + \min \left\{ E_+(\delta)\xi_1, \frac{E_+(\delta)}{\sqrt{\kappa^*}}\xi_2, \frac{E_+(\delta)}{\kappa^*}\xi_3 \right\} \right\},$$

with probability at least $1 - B(r)\delta$, where $C$ is a universal constant (that can be chosen as 72).

In many cases, the first term dominates the second term, in which cases Theorem 2.4 essentially implies that $d_{2 \rightarrow \infty}(U, U^*) \approx \|EU^*\|_{2 \rightarrow \infty}/\lambda^*_\min$.

### 2.3 Sharpening the bound via diagonal surgery

In this subsection we discuss a trick, referred to as “diagonal surgery”, to further improve the $\ell_{2 \rightarrow \infty}$ bound. The motivation is from matrices with large diagonal elements. If $A^*$ has high rank or even full rank, the last term in Theorem 2.3 and 2.4 reduce to $\min\{\xi_1, \sqrt{\kappa^*}\xi_2\}$. However, both $\xi_1$ and $\xi_2$ will be large when $A^*$ has large diagonal elements. For instance, when $A$ is the unnormalized graph Laplacian, defined in Section 3.2 later, of an Erdős-Renyi graph with edge connection probability $p$, then $\|A^*\|_{2 \rightarrow \infty} \geq \|A^*\|_{\text{max}} = (n - 1)p$ while $\|A^*\|_{2 \rightarrow \infty} = p\sqrt{n - 1}, \|A^*\|_{\text{max}} = p$. If we were to use Theorem 2.3 or 2.4, the bound applied to graph Laplacian would be significantly worse than that applied to the adjacency matrix. To overcome this shortcoming, we found a modification of the proof that allows us to replace $A$ by $A - \Sigma$ where $\Sigma$ is a possibly random diagonal matrix, provided that all diagonal elements of $\Sigma$ are well separated from $\Lambda^*$.

**A0’** For any $\delta \in (0, 1)$, i

$$\frac{\min_{j \in [s+1, s+r]} |\Lambda^*_jj|}{\min_{j \in [s+1, s+r], k \in [n]} |\Lambda^*_jj - \Sigma^*kk|} \leq \Theta(\delta),$$

with probability at least $1 - \delta$ for some deterministic function $\Theta(\delta) > 0$. 

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Let

\[ A = A - \Sigma, \quad \hat{A} = \mathbb{E}A, \quad \hat{E} = \hat{A} - \hat{A}^*. \]

All other assumptions need to be slightly modified.

**A’1** The same as A1 except that (7) is replaced by

\[ d_{TV} \left( \mathbb{P}(\hat{A}, \mathbb{E}A), \mathbb{P}(\hat{A}, \mathbb{E}A) \right) \leq \delta/n. \]

**A’2** There exists deterministic functions \( \lambda_-(\delta), E_+^{\mathbb{E}}(\delta), \tilde{E}_\infty(\delta) \), such that for any \( \delta \in (0, 1) \), the following event holds with probability at least \( 1 - \delta \):

\[ \|A - A^*\|_{\max} \leq \lambda_-(\delta), \quad \|EU^*\|_{op} \leq E_+(\delta), \quad \|\tilde{E}\|_{2\to\infty} \leq \tilde{E}_\infty(\delta). \]

**A’3** There exists deterministic functions \( \tilde{b}_\infty(\delta), \tilde{b}_2(\delta) > 0 \), such that for any \( \delta \in (0, 1) \), \( k \in [n] \), \( r' \leq r \) and fixed matrix \( W \in \mathbb{R}^{n \times r'} \),

\[ \|\tilde{E}_k \|_2 \leq \tilde{b}_\infty(\delta)\|W\|_{2\to\infty} + \tilde{b}_2(\delta)\|W\|_{op}, \text{with probability at least } 1 - \delta/n. \]

**A’4** \( \Delta^* \geq 4 (\Theta(\delta)\tilde{\sigma}(\delta) + L_1(\delta) + \lambda_-(\delta) + E_+(\delta)) \) where

\[ \tilde{\eta}(\delta) = \tilde{E}_\infty(\delta) + \tilde{b}_\infty(\delta) + \tilde{b}_2(\delta), \quad \tilde{\sigma}(\delta) = \{\tilde{\kappa}^*L_2(\delta) + L_3(\delta) + 1\}\tilde{\eta} + E_+(\delta). \]

Unlike Theorem 2.3, \( AU^*(\Lambda^*)^{-1} \) is no longer an approximation of \( U \) after “diagonal surgery”. In fact, the approximation becomes \( U^* + V \) where

\[ V_k^T = E_k^T U^*(\Lambda^* - \Sigma_{kk}I)^{-1}. \]

Note that when \( \Sigma_{kk} \equiv 0 \),

\[ U^* + V = U^* + EU^*(\Lambda^*)^{-1} = AU^*(\Lambda^*)^{-1}, \]

which recovers the case in Section 2.2.

**Theorem 2.5.** Given any \( \delta \in (0, 1) \). Let \( \Delta^* \) be defined in (4) and \( \tilde{\kappa}^* \) be defined in (5). Then under assumptions A’0 - A’4,

\[ d_{2\to\infty}(U, U^* + V) \leq \mathbb{C}\left\{ \left( \frac{E_+^2}{(\Delta^*)^2} + \frac{\Theta(\delta)\tilde{\sigma}(\delta)}{\Delta^*} \right) \left( \|U^*\|_{2\to\infty} + \Theta(\delta)\|EU^*\|_{2\to\infty} \right) \right. \]

\[ + \left. \frac{\Theta(\delta)(\tilde{b}_2(\delta) + \|\hat{A}\|_{2\to\infty})E_+}{\lambda_{\min}\Delta^*} \right\}, \]

with probability at least \( 1 - B(r)\delta \), where \( B(r) \) is defined in (11) and \( C \) is a universal constant (that can be chosen as 136).

Similar to Theorem 2.4, we can derive a bound for \( d_{2\to\infty}(U, U^*) \) using the triangle inequality.

**Theorem 2.6.** Under the same setting of Theorem 2.5,

\[ d_{2\to\infty}(U, U^*) \leq \mathbb{C}\left\{ \frac{\Theta(\delta)}{\lambda_{\min}}\|EU^*\|_{2\to\infty} + \left( \frac{E_+^2}{(\Delta^*)^2} + \frac{\Theta(\delta)\tilde{\sigma}(\delta)}{\Delta^*} \right) \|U^*\|_{2\to\infty} \right. \]

\[ + \left. \frac{\Theta(\delta)(\tilde{b}_2(\delta) + \|\hat{A}\|_{2\to\infty})E_+}{\lambda_{\min}\Delta^*} \right\}, \]

with probability at least \( 1 - B(r)\delta \), where \( C \) is a universal constant (that can be chosen as 136).
3 \( \ell_{2 \to \infty} \) Perturbation Theory for Binary Random Matrices

Throughout this section we will ignore the universal constant terms for notational convenience. In particular, we use the symbol \( \leq \) (resp. \( \geq \)) to hide universal constants. We say a constant is universal if it does not depend on any quantity in the problem (e.g. \( A^* \), \( n \), \( \delta \), etc.). Specifically, we say \( A \preceq B \) (resp. \( A \succeq B \)) with probability \( 1 - \delta \) for two variables \( A \) and \( B \), stochastic or deterministic, if and only if \( A \preceq CB \) (resp. \( A \succeq CB \)) with probability \( 1 - \delta \) for some universal constant \( C \). All proofs in this section are relegated to Appendix C.

3.1 Binary random matrices with independent entries

In this subsection, we consider \( A \) as a binary random matrix with independent entries with \( A^* \) being its expectation, i.e.

\[
A^*_{ij} = A_{ij}^* = p_{ij}, \quad A_{ij} = A_{ji} \sim \text{Ber}(p_{ij}), \quad (A_{ij})_{1 \leq i \leq j \leq n} \text{ are independent.} \tag{16}
\]

Note that we allow \( p_{ii} > 0 \). Let

\[
p^* = \max_{ij} p_{ij}, \quad \bar{p}^* = \max_i \frac{1}{n} \sum_{j=1}^{n} p_{ij}, \quad R(\delta) = \log(n/\delta) + r. \tag{17}
\]

**Lemma 3.1.** Under the setting (16), given any \( \alpha > 0 \), assumption A3 is satisfied with

\[
b_{\infty}(\delta) \leq \frac{R(\delta)}{\alpha \log R(\delta)}, \quad b_2(\delta) \leq \frac{\sqrt{p^*} R(\delta)^{(1+\alpha)/2}}{\alpha \log R(\delta)}.
\]

**Lemma 3.2.** Under the setting (16), assumption A2 is satisfied with

\[
\lambda_-(\delta), E_+(\delta), E_+(\delta), E_\infty(\delta) \leq \sqrt{np^*} + \sqrt{\log(n/\delta)},
\]

**Lemma 3.3.** Under the setting (16),

\[
\|EU^*\|_2 \to \infty \preceq R(\delta)\|U^*\|_2 \to \infty + \sqrt{R(\delta)p^*},
\]

It is then easy to derive the \( \ell_{2 \to \infty} \) perturbation theory for binary random matrices with independent entries from Theorem 2.3 and 2.4 based on Lemma 3.1 - 3.3 and Proposition 2.1 on \( L_1(\delta), L_2(\delta), L_3(\delta) \).

**Theorem 3.4.** Fix any \( \delta \in (0, 1) \), \( \alpha \in (0, 1) \). Let

\[
g(\delta) = \sqrt{np^*} + \frac{R(\delta)}{\alpha \log R(\delta)}, \tag{18}
\]

and assume that

\[
\Delta^* \succeq C\bar{r}^* g(\delta), \tag{19}
\]

for some universal constant \( C \) that is large enough. Then with probability at least \( 1 - (B(\bar{r}) + 1)\delta \),

\[
d_{2 \to \infty}(U, AU^*(\Lambda^*)^{-1}) \preceq \frac{1}{\Delta^*} \left\{ \bar{r}^* g(\delta) \left( 1 + \frac{R(\delta)}{\lambda_{\min}^*} \right) \|U^*\|_2 \to \infty \right. + \frac{\sqrt{R(\delta)p^*}}{\lambda_{\min}^*} \left( \bar{r}^* g(\delta) + \frac{\sqrt{np^*} R(\delta)^{\alpha}}{\alpha \log R(\delta)} \right) \left\} + \frac{\sqrt{np^*} + \sqrt{\log(n/\delta)}}{\Delta^*} \min \left\{ \|A^*\|_2 \to \infty, \frac{\sqrt{\bar{r}^* p^*}}{\sqrt{\lambda_{\min}^*} I(A^* \text{ is psd})}, \bar{r}^* \|\bar{U}^*\|_2 \to \infty \right\},
\]

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then with probability at least $1 - (B(r) + 1)\delta$,

$$d_{2\to\infty}(U, AU^* (\Lambda^*)^{-1}) \leq \frac{1}{\Delta^*} \left( \tilde{\kappa}^* g(\delta) \left( 1 + \frac{R(\delta)}{\lambda_{\min}^*} \right) \|U^*\|_{2\to\infty} + \frac{\sqrt{R(\delta)p^*}}{\lambda_{\min}^*} \left( \tilde{\kappa}^* g(\delta) + \frac{\sqrt{np^* R(\delta)\alpha}}{\alpha \log R(\delta)} \right) \right),$$

$$d_{2\to\infty}(U, U^*) \leq \left( \frac{\tilde{\kappa}^* g(\delta)}{\Delta^*} + \frac{R(\delta)}{\lambda_{\min}^*} \right) \|U^*\|_{2\to\infty} + \frac{\sqrt{R(\delta)p^*}}{\lambda_{\min}^*} \left( 1 + \frac{\sqrt{np^* R(\delta)\alpha}}{\alpha \Delta^* \log R(\delta)} \right).$$

The second case we consider imposing a lower bound for $\lambda_{\min}^*$.

**Corollary 3.6.** Under the settings of Theorem 3.4, if

$$\lambda_{\min}^* \geq \frac{np^*}{\sqrt{n}\|U^*\|_{2\to\infty}}, \quad (20)$$

then with probability at least $1 - (B(r) + 1)\delta$,

$$d_{2\to\infty}(U, AU^* (\Lambda^*)^{-1}) \leq \frac{\tilde{\kappa}^* g(\delta)}{\Delta^*} \left( 1 + \frac{R(\delta)}{\lambda_{\min}^*} \right) \|U^*\|_{2\to\infty},$$

$$d_{2\to\infty}(U, U^*) \leq \left( \frac{\tilde{\kappa}^* g(\delta)}{\Delta^*} + \frac{R(\delta)}{\lambda_{\min}^*} \right) \|U^*\|_{2\to\infty} + \frac{\sqrt{R(\delta)p^*}}{\lambda_{\min}^*}.$$

### 3.2 Unnormalized graph laplacian

Let $A$ be a binary matrix with independent entries as defined in (16) and $L$ be the unnormalized graph Laplacian of $A$,$$
L = D - A, \; \text{where} \; D = \text{diag}(D_{11}, \ldots, D_{nn}), \; D_{ii} = \sum_{j=1}^{n} A_{ij}. \quad (21)
$$

We use the notation $L, D, A$ by convention. We assume $A_{ii} = 0$ without loss of generality (because $L$ does not depend on $A_{ii}$). Throughout this section we will treat $L$ as $A$ and $L^*$ as $A^*$. Similar to binary matrices with independent entries, we derive the bound for quantities involved in Theorem 2.3. In particular, we apply the diagonal surgery technique with $\Sigma = \text{diag}(L_{11}, L_{22}, \ldots, L_{nn})$. Let

$$\tilde{L} = L - \Sigma, \; E = L - \Sigma E, \; \tilde{E} = \tilde{L} - \Sigma \tilde{L}.$$
Apart from the notation in (17), we also define the following quantity
\[ M(\delta) = \sqrt{n\tilde{p}^* \log(n/\delta) + \log(n/\delta)}. \]  

Lemma 3.7. Under the setting (21), given any \( \alpha > 0 \), assumption A’3 is satisfied with
\[ \tilde{b}_\infty(\delta) \leq \frac{R(\delta)}{\alpha \log R(\delta)}, \quad \tilde{b}_2(\delta) \leq \frac{\sqrt{p^*} R(\delta)^{(1+\alpha)/2}}{\alpha \log R(\delta)}. \]

Lemma 3.8. Under the setting (21), assumption A’2 is satisfied with
\[ \tilde{E}_\infty(\delta) \leq \sqrt{n\tilde{p}^* + \log(n/\delta)}, \quad E_+, \lambda_-(\delta) \leq M(\delta). \]

Lemma 3.9. Under the setting (21),
\[ \|EU^*\|_{2\to\infty} \leq (M(\delta) + r)\|U^*\|_{2\to\infty} + \sqrt{R(\delta)p^*}. \]

Lemma 3.10. Let \( \mathcal{L} \) be the unnormalized Laplacian of \( \mathcal{A} \) where \( \mathcal{A} \) is a binary random matrix that satisfies the condition in part (b) of Proposition 2.1. Then there exists \( \mathcal{L}^{(1)}, \ldots, \mathcal{L}^{(n)} \) satisfying A’1 with
\[ L_1(\delta) \leq \sqrt{nM(\delta) + m(\sqrt{n\tilde{p}^* + \log(n/\delta))}, \quad L_2(\delta) = m, \quad L_3(\delta) \leq m \left( n\tilde{p}^* + \sqrt{\log(n/\delta)} \right). \]

In particular, the setting considered in this subsection is a special case with \( m = 1 \). Putting the pieces together we deduce the following theorem.

Theorem 3.11. Fix any \( \delta \in (0, 1) \) and \( \alpha > 0 \). Let \( R(\delta), g(\delta) \) and \( M(\delta) \) be defined as in (17), (18) and (22), respectively. Further let
\[ \bar{\kappa} = \tilde{\kappa}^* + n\tilde{p}^*/\lambda_{\min}^*, \]
and assume that
\[ \Delta^* \geq C\{\Theta(\delta)\bar{\kappa}^* g(\delta) + (\Theta(\delta) + 1)M(\delta)\}, \]  

for some universal constant \( C \) that is large enough. Then with probability at least \( 1 - (B(r) + 1)\delta \),
\[ d_{2\to\infty}(U, U^* + V) \leq \left( \frac{M(\delta)^2}{(\Delta^*)^2} + \frac{\Theta(\delta)\bar{\kappa}^* g(\delta) + M(\delta)}{\Delta^*} \right) \left( 1 + \frac{\Theta(\delta)\bar{\kappa}^*}{\lambda_{\min}^*} \right) \|U^*\|_{2\to\infty} + \frac{\Theta(\delta)\sqrt{R(\delta)p^*}}{\lambda_{\min}^*} \]
\[ + \frac{\Theta(\delta)M(\delta)\sqrt{p^*}}{\Delta^*\lambda_{\min}^*} \left( \sqrt{n\tilde{p}^* + \frac{\sqrt{R(\delta)^{1+\alpha}}}{\alpha \log R(\delta)}} \right), \]
and
\[ d_{2\to\infty}(U, U^*) \leq \left( \frac{M(\delta)^2}{(\Delta^*)^2} + \frac{\Theta(\delta)\bar{\kappa}^* g(\delta) + M(\delta)}{\Delta^*} + \frac{\Theta(\delta)\bar{\kappa}^*}{\lambda_{\min}^*} \right) \|U^*\|_{2\to\infty} + \frac{\Theta(\delta)\sqrt{Rp^*}}{\lambda_{\min}^*} \]
\[ + \frac{\Theta(\delta)M(\delta)\sqrt{p^*}}{\Delta^*\lambda_{\min}^*} \left( \sqrt{n\tilde{p}^* + \frac{\sqrt{R(\delta)^{1+\alpha}}}{\alpha \log R(\delta)}} \right). \]

Note that
\[ \frac{|\Lambda_{jj}^*|}{|\Lambda_{jj}^* - \mathcal{L}_{kk}|} \leq \frac{\Lambda_{jj}^*}{\max\{0, |\Lambda_{jj}^* - \mathcal{L}_{kk}^*| - |\mathcal{L}_{kk} - \mathcal{L}_{kk}^*|\}}. \]
It is easy to derive a concentration inequality for \( \max_k |\mathcal{L}_{kk} - \mathcal{L}_{kk}^*| \), This suggests the following bound for \( \Theta(\delta) \).
Lemma 3.12. Let

\[ \Theta^* = \frac{\min_{j \in [s+1,s+r]} |\Lambda^*_j|}{\min_{j \in [s+1,s+r], k \in [n]} |\Lambda^*_j - \mathcal{L}_{kk}|}. \]

Then \( \Theta(\delta) \leq 5\Theta^* \) if

\[ \min_{j \in [s+1,s+r], k \in [n]} |\Lambda^*_j - \mathcal{L}_{kk}| \geq 5M(\delta). \]

4 Concentration of The Spectral Norm of Random Graphs

4.1 Background

Erdős-Rényi graph is the most fundamental object in random graph theory. The probability connection matrix \( A^* \) of an Erdős-Rényi graph has

\[ A^*_{ii} = 0, \quad A^*_{ij} = p, \quad \forall i \neq j. \]

Concentration of the spectral norm or the extreme eigenvalues of Erdős-Rényi graphs has received considerable attention. Adapted from the proof of Boucheron et al. [2013] (Example 3.14), based on Efron-Stein inequality, we can show that

\[ \text{Var} (\|A\|_{\text{op}}) \leq 2. \]

Alon et al. [2002] proved the sub-gaussian behavior of \( \|A\|_{\text{op}} \) in the sense that

\[ P (\|A\|_{\text{op}} - E\|A\|_{\text{op}} > t) \leq 2e^{-t^2/32}, \quad \forall t > 0. \]

See also [Boucheron et al., 2013, Example 8.7] for a simplified proof. The key argument underlying the above results is the Efron-Stein-type inequalities that involve the leave-one-out behavior of \( \|A\|_{\text{op}} \) as a function of independent random variables \( (A_{ij})_{i<j} \). Specifically, write \( \|A\|_{\text{op}} \) as \( Z \) for convenience and denote by \( Z_{ij} \) the operator norm of matrix \( A^{(ij)} \), which equals to \( A \) except that the \((i,j)\)-th entry is replaced by an independent copy \( A'_{ij} \) drawn from a Bernoulli distribution with parameter \( p \). Then Efron-Stein inequality implies that

\[ \text{Var}(Z) \leq EV_+, \quad \text{where} \quad V_+ = E' \sum_{i<j} (Z - Z_{ij})^2. \]

Here \((x)_+ \) denotes \( \max\{x,0\} \) by convention and \( E' \) denotes the expectation over \( (A'_{ij})_{i<j} \) (while conditioning on \( (A_{ij})_{i<j} \)). Using the variational representation of operator norm, we can rewrite \( Z - Z_{ij} \) as follows:

\[ Z - Z_{ij} = \sup_{u:||u||=1} |u^T A u| - \sup_{u:||u||=1} |u^T A^{(ij)} u|. \]

Let \( u_1 \) be the eigenvector of \( A \) corresponding to its largest eigenvalue in absolute values, then

\[ \sup_{u:||u||=1} |u^T A u| = \|A\|_{\text{op}}, \quad \sup_{u:||u||=1} |u^T A^{(ij)} u| \geq |u_1^T A^{(ij)} u_1| \geq \pm u_1^T A^{(ij)} u_1. \]

If \( u_1^T A u_1 \geq 0 \), then

\[ Z - Z_{ij} \leq u_1^T (A - A^{(ij)}) u_1 \leq |u_1^T (A - A^{(ij)}) u_1|; \]

if \( u_1^T A u_1 < 0 \), then

\[ Z - Z_{ij} \leq -u_1^T (A - A^{(ij)}) u_1 \leq |u_1^T (A - A^{(ij)}) u_1|. \]
Putting two pieces together, we conclude that 
\[(Z - Z_{ij})_+ \leq |u_1^T (A - A^{(ij)}) u_1| = 2|u_1| |u_{ij}| |A_{ij} - A'_{ij}|.\] (28)

As a result, 
\[V_+ \leq 4 \sum_{i<j} u_{ij}^2 u_{ij}' (A_{ij} - A'_{ij})^2 = 4 \sum_{i<j} u_{ij}^2 u_{ij}' (p + (1 - 2p) A_{ij}).\] (29)

To prove (25), one can simply use the naive bound that 
\[p + (1 - 2p) A_{ij} \leq p + 1 - 2p \leq 1.\]

Therefore, we obtain an almost sure bound for \(V_+\): 
\[V_+ \leq 4 \sum_{i<j} u_{ij}^2 u_{ij}' = 2 \sum_{i<j} u_{ij}^2 u_{ij}' \leq 2 \left( \sum_{i=1}^n u_{i1}^2 \right)^2 = 2.\] (30)

Then Efron-Stein inequality implies that 
\[\text{Var}(|A|_{op}) \leq \mathbb{E} V_+ \leq 2.\]

However, this bound is loose for small \(p\) in which case the term \((p + (1 - 2p) A_{ij})\) is most likely equal to \(p\) instead of the conservative bound 1. However, the complicated dependence between \(u_1\) and \(A\) makes it hard to operationalize this intuition. Recently, an interesting work by Lugosi et al. [2018] showed that 
\[\text{Var}(|A|_{op}) \leq c_1 p, \quad \text{if } p \geq c_2 (\log n)^3 / n.\] (31)

for some universal constant \(c_1, c_2 > 0\). Furthermore, they proved the partial sub-gaussian behavior of \(|A|_{op}\) in the sense that 
\[\mathbb{P}(|A|_{op} - \mathbb{E}|A|_{op} > \sqrt{p} t) \leq c_3 e^{-t^2/c_4}, \quad \forall t \leq c_5 \sqrt{np \log np} / (\log n \log(1/p)),\] (32)

for some universal constants \(c_3, c_4, c_5 > 0\) under the condition that \(p \geq c_2 (\log n)^3 / n\). Under this regime, the upper bound for \(t\) in (32) is diverging, implying that the tail probability holds for values much larger than \(\sqrt{p}\).

The idea is based on the phenomenon called "eigenvector delocalization" that all entries of \(u_1\) are small [e.g. Mitra, 2009, Erdős et al., 2013, Lugosi et al., 2018]. Intuitively, \(u_1\) should be close to \(u_1^*\), the eigenvector of \(A^*\) corresponding to the largest eigenvalue in absolute values. A simple calculation shows that \(u_1^* = 1_n / \sqrt{n}\). In particular, Lugosi et al. [2018] improved the previous results and proved that with high probability, \(|u_1|_\infty \leq 1/\sqrt{n}\) if \(p \geq (\log n)^3 / n\). This motivates the following bound for \(V_+\) in (29) 
\[V_+ \leq \|u_1\|_\infty^4 W, \quad \text{where } W = 4 \sum_{i<j} (p + (1 - 2p) A_{ij}).\] (33)

Using the concentration that \(|u|_\infty \approx 1/\sqrt{n}\) and \(W \approx \mathbb{E} W = 4n(n - 1)p(1 - p)\) and by Efron-Stein inequality (27), they proved that \[\text{Var}(|A|_{op}) \leq \mathbb{E} V_+ \leq \frac{n(n - 1)p(1 - p)}{n^2} \leq p.\]

The bound of tail probability in (32) can be derived similarly using higher moments of \(V_+\) and the moment concentration inequalities derived by Boucheron et al. [2005].
Lugosi et al. [2018] conjectured that the requirement \( p \geq (\log n)^3/n \) is artifical and the critical regime should be \( p \geq \log n/n \). In this section, we will close this gap by using our \( \ell_2 \to \infty \) bound in Section 3.1. Furthermore, we will discuss the behavior of \( \|A\|_{\text{op}} \) when \( p \leq \log n/n \) but \( p \geq \log/(n \log \log n) \). Throughout the rest of this section we assume that

\[
\frac{1}{2} \geq p \geq \frac{C_0 \log n}{n \log \log n} \tag{34}
\]

for some universal constant \( C_0 > 0 \). Similar to Section 3, we use the notation \( \preceq \) and \( \succeq \) to hide universal constants.

### 4.2 Improved results for Erdős-Rényi graphs

To start with, we state the moment concentration inequality by Boucheron et al. [2005].

**Proposition 4.1** (Theorem 15.6 and 15.7 of Boucheron et al. [2013]). Let \( Z = f(A) \) and \( Z_{ij} = f(A^{ij}) \). Further let \( V^+ \) be defined in (27) and

\[ M = \max_{i<j} (Z - Z_{ij})^+ . \]

Then for any \( k \geq 2 \),

\[
(\mathbb{E}(Z - EZ)^k)^{1/k} \leq \sqrt{3k} \left( \mathbb{E}[V^+]^{k/2} \right)^{1/k} ,
\]

\[
(\mathbb{E}|Z - EZ|^k)^{1/k} \leq 4.16 k \left( \mathbb{E}[V^+]^{k/2} \right)^{1/k} + \sqrt{k} (\mathbb{E}[M^k])^{1/k} .
\]

As a result,

\[
(\mathbb{E}|Z - EZ|^k)^{1/k} \leq 4 \sqrt{k} \left( \mathbb{E}[V^+]^{k/2} \right)^{1/k} + 4k (\mathbb{E}[M^k])^{1/k} .
\]

It is well-known [e.g. Vershynin, 2010] that \( Z \) is sub-gaussian with parameter \( \sigma^2 \) iff \( (\mathbb{E}|Z - EZ|^k)^{1/k} \leq c \sqrt{k} \sigma \) for any \( k \geq 2 \) with some universal constant \( c \). Suppose this is true for \( k \leq k_0 \), then we can still derive the sub-gaussian behavior of \( Z \).

**Lemma 4.2.** If \( 2 \leq k \leq k_0 \),

\[ (\mathbb{E}|Z - EZ|^k)^{1/k} \leq \sqrt{k} \sigma, \]

for some \( \sigma > 0 \), then for \( t \leq \sqrt{k_0} \sigma \),

\[ P(|Z - EZ| \geq t) \leq \exp \left\{ 1 - \frac{t^2}{2k \sigma^2} \right\} . \]

Although Lemma 4.2 does not hold for all \( t \), it is desirable if \( k_0 \) is large because \( \sigma \) is the scale of \( |Z - EZ| \). In the following, we will bound the higher order moments of \( V^+ \) and \( M \), thereby bounding the tail probability of \( \|A\|_{\text{op}} \). Recalling from (28) and (33) that

\[ M \leq 2 \|u_1\|_\infty^2 , \quad V^+ \leq \|u_1\|_\infty^4 W . \]

Both involve the moments of \( \|u_1\|_\infty \). Our theory in Section 3 gives the \((1 - \delta)\) upper tail bound for \( \|u_1 - u_1^*\|_\infty \) in the form of

\[ P(\|u_1 - u_1^*\|_\infty \geq A_1 + A_2 \sqrt{\log \left( \frac{1}{\delta} \right) } + A_3 \log \frac{1}{\delta} ) \leq \delta . \tag{35} \]
If (35) holds for all $\delta$ then it directly yields a moment bound using Fubini’s formula. However, $\delta$ cannot be arbitrarily small otherwise the condition (19) in Theorem 2.3 may be violated so that (35) may fail. So we first characterize the minimal $\delta$ such that (35) remains valid.

**Lemma 4.3.** Under condition (34), (19) holds for all $\delta > \delta^*$ where

$$\delta^* = \exp \left\{ - \frac{np \log(np)}{2C} \right\},$$

and $C$ is the constant in (19) in Theorem 3.4, if $C_0$ and $n$ are bounded below by a universal constant (which can be chosen as $\max\{64C^2, 12C, 16\}$).

Based on Lemma 4.3 and Corollary 3.6, we can derive the moment bound for $V_+$ and $M$.

**Lemma 4.4.** Under the assumptions of Lemma 4.3, for any $k > 0$,

$$\left( \mathbb{E} V_+^k \right)^{1/k} \leq \sqrt{p} \left( 1 + \frac{(k \vee \log n)^2}{(np)^2} \right) + \exp \left\{ - \frac{np \log(np)}{2Ck} \right\},$$

and

$$\left( \mathbb{E} M^k \right)^{1/k} \leq \frac{1}{n} \left( 1 + \frac{(k \vee \log n)^2}{(np)^2} \right) + \exp \left\{ - \frac{np \log(np)}{2Ck} \right\},$$

where $C$ is the constant in (19) in Theorem 3.4.

**Proof.** In this case, note that $A^* = pI_{1_n}I_n^T - pI_{n \times n}$. Then

$$p^* = p, \quad \lambda_1^* = (n-1)p, \quad \lambda_2^* = \ldots = \lambda_n^* = -p, \quad \text{and} \quad u_1^* = 1_n/\sqrt{n}.$$

Let $\Lambda^* = \lambda_1^*$, then

$$\Delta^* = \lambda_{\min}^* = (n-1)p, \quad \|U^*\|_2 \rightarrow \infty = \|u_1^*\|_\infty = \frac{1}{\sqrt{n}}, \quad \delta^* = 1.$$

Thus,

$$\lambda_{\min}^* \gtrsim \frac{np\delta^*}{\sqrt{n}\|U^*\|_2 \rightarrow \infty}.$$

By Lemma 4.3 below, the condition (19) is satisfied for all $\delta \geq \delta^*$. Then by Corollary 3.6 with $\alpha = 0.5$ and $\delta \geq \delta^*$, with probability $1 - \delta$,

$$d_{2 \rightarrow \infty}(u_1, u_1^*) \leq \left( \sqrt{\frac{np}{p} + \log(n/\delta) / \log(n/\delta)} + \frac{\log(n/\delta)}{np} \right) \frac{1}{\sqrt{n}} + \sqrt{\frac{\log(n/\delta)}{np}}$$

$$\leq \frac{1}{\sqrt{n}} \left( \sqrt{np} + \frac{\log(n/\delta)}{np} + \frac{\log(n/\delta)}{np} \right)$$

$$\leq \frac{1}{\sqrt{n}} \left( \log(n/\delta) + \frac{\log(n/\delta)}{np} \right)$$

By the triangle inequality,

$$\|u_1\|_\infty \leq d_{2 \rightarrow \infty}(u_1, u_1^*) + \|u_1^*\|_\infty.$$

Using the fact that $2\sqrt{y} \leq y + 1$ and $\sqrt{n}\|u_1^*\|_\infty = 1$, there exists a universal constant $C_1$ such that for each $\delta \geq \delta^*$,

$$\sqrt{n}\|u_1\|_\infty \leq C_1 \left( 1 + \frac{\log(n/\delta)}{np} \right) \leq C_1 \left( 1 + \frac{\log n}{np} + \frac{\log(1/\delta)}{np} \right),$$

(37)
with probability $1 - \delta$. Denote by $B_u$ the RHS of (37) with $\delta = \delta^*$ and by $\mathcal{V}_1$ the event that $\sqrt{n} \|u_1\|_{\infty} \leq B_u$.

Then
\[
P(\mathcal{V}_1) \geq 1 - \delta^* = 1 - \exp\left\{ - \frac{np \log(np)}{2C} \right\}.
\]

On the other hand, note that
\[
\mathbb{E} W = 4 \sum_{i < j} (p + (1 - 2p)p) = 4n(n - 1)p(1 - p) \leq 4n^2 p,
\]
and
\[
W - \mathbb{E} W = 4(1 - 2p) \sum_{i < j} (A_{ij} - p).
\]

By Lemma F.1 with $w = 1_{n(n-1)/2}$ and $\delta = \delta' = \exp\{-e/2\Omega\} = \exp\{-en(n-1)/4\}$,
\[
W - \mathbb{E} W \leq 8 \log \left( \frac{1}{\delta} \right) = 2en(n-1)p \leq 6n^2 p
\]
with probability $1 - \delta$. Let $\mathcal{V}_2$ denote the event that $W \leq 10n^2 p$, then
\[
P(\mathcal{V}_2) \geq 1 - \exp\{-en(n-1)/4\} \geq 1 - \exp\{-n^2 p/3\}. \tag{38}
\]

Let $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$. Then
\[
P(\mathcal{V}^c) \leq \exp\left\{ - \frac{np \log(np)}{2C} \right\} + \exp\left\{ - \frac{n^2 p}{3} \right\} \leq C_2 \exp\left\{ - \frac{np \log(np)}{2C} \right\},
\]
where $C_2$ is a universal constant. By definition, on $\mathcal{V}$,
\[
\|u_1\|_{\infty}^4 W \leq 10p(\sqrt{n} \|u_1\|_{\infty})^4.
\]

In addition, by (30) we have
\[
V_+ I_{\mathcal{V}^c} \leq 2 I_{\mathcal{V}^c}.
\]

Therefore,
\[
\left( \mathbb{E} V_+^{k/2} \right)^{1/k} = \left( \mathbb{E} V_+^{k/2} I_+ + \mathbb{E} V_+^{k/2} I_{\mathcal{V}^c} \right)^{1/k} \leq \left( \mathbb{E} V_+^{k/2} I_+ \right)^{1/k} + \left( \mathbb{E} V_+^{k/2} I_{\mathcal{V}^c} \right)^{1/k} \leq \sqrt{nk} \mathbb{E} \left( \left( \sqrt{n} \|u_1\|_{\infty}^2 I_+ \right)^k \right)^{1/k} + \exp\left\{ - \frac{np \log(np)}{2Ck} \right\} \tag{39}
\]

By (37),
\[
P \left( Y \leq \frac{\log(1/\delta)}{np} \right) \leq \delta, \quad \forall \delta \geq \delta^*, \quad \text{where} \quad Y = \left[ \frac{\sqrt{n} \|u_1\|_{\infty}}{C_1} - \left( 1 + \frac{\log a}{np} \right) \right]^+. \]

Denote by $B_Y$ the upper bound with $\delta = \delta^*$. This can be written equivalently as
\[
P(Y \geq y) \leq \exp\{-npy\}, \quad \forall y \leq B_Y.
\]

By definition, $Y I_\mathcal{V} \leq YI(Y \leq B_Y)$. Using Fubini’s theorem, for any $k > 0$,
\[
\mathbb{E} V_2^k I(Y \leq B_Y) = \int_0^{B_Y} 2ky^{2k-1} P(Y \geq y)dy \leq \int_0^{B_Y} 2ky^{2k-1} \exp\{-npy\} dy
\]
\[
\leq \int_0^{\infty} 2ky^{2k-1} \exp\{-npy\} dy = \frac{\Gamma(2k + 1)}{(np)^{2k}}.
\]
By Stirling’s formula, we have
\[(EY^{2k}I_Y)^{1/k} \leq (EY^{2k}I(Y \leq B_Y))^{1/k} \leq \frac{k^2}{(np)^2}.\]

As a result,
\[
\left( E(\sqrt{n}\|u_1\|_\infty)^{2k}I_Y \right)^{1/k} \leq \left( E \left( C_1 Y + 1 + \frac{\log n}{np} \right)^{2k}I_Y \right)^{1/k}
\]
\[
\leq 2^{2k}E(C_1 Y)^{2k}I_Y + 2^{2k} \left( 1 + \frac{\log n}{np} \right)^{2k} \leq 4C_1^2 \left( EY^{2k}I_Y \right)^{1/k} + 4 \left( 1 + \frac{\log n}{np} \right)^2
\]
\[
\leq \frac{k^2}{(np)^2} + 1 + \frac{(\log n)^2}{(np)^2} \leq 1 + \frac{(k \lor \log n)^2}{(np)^2}.
\]

(40)

The bound of \( \left( E[V^{k/2}_+] \right)^{1/k} \) is then proved by plugging this into (39).

Recalling (28), we have
\[ M \leq 2\|u_1\|_\infty^2. \]
It is easy to see that \( M \leq 2 \). Similar to (39),
\[
(EM^k)^{1/k} \leq \left( EM^k I_Y + EM^{k/2} I_{V^c} \right)^{1/k} \leq \left( EM^k I_Y \right)^{1/k} + \left( EM^k I_{V^c} \right)^{1/k}
\]
\[
\leq \frac{1}{n} E \left( ([\sqrt{n}\|u_1\|_\infty]^{2k}I_Y) \right)^{1/k} + \exp \left\{ \frac{-np\log(np)}{2Ck} \right\}.
\]

(41)

The bound of \( (EM^k)^{1/k} \) is then completed by (40).

**Theorem 4.5.** Under condition (34) with \( C_0 \) sufficiently large,
\[ \text{Var}(\|A\|_{op}) \preceq p \left( 1 + \frac{(\log n)^4}{(np)^4} \right). \]

In particular, if \( p \geq \log n/n \), then
\[ \text{Var}(\|A\|_{op}) \preceq p. \]

**Remark 4.1.** This closed the gap conjectured by Lugosi et al. [2018]. Moreover, our result shows that when \( np \sim \log n/\log \log n \),
\[ \text{Var}(\|A\|_{op}) \preceq p(\log \log n)^4. \]

**Proof.** By Efron-Stein inequality (27),
\[ \text{Var}(\|A\|_{op}) \leq EV_. \]

For sufficiently large \( n \), by Lemma 4.4 with \( k = 2 \),
\[ EV_+ \leq p \left( 1 + \frac{(\log n)^4}{(np)^4} \right) + \exp \left\{ \frac{-np\log(np)}{4C} \right\}. \]

Since \( np \geq C_0 \log n/\log \log n \), when \( C_0 \) is sufficiently large,
\[ \exp \left\{ \frac{-np\log(np)}{4C} \right\} \leq n^{-2} \leq p. \]

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For small $n$, $\text{Var}(\|A\|_{\text{op}}) \leq 2$ as shown in (25). In summary,
\[
\text{Var}(\|A\|_{\text{op}}) \leq p \left( 1 + \frac{(\log n)^4}{(np)^4} \right).
\]

\[\square\]

**Lemma 4.6.** Under condition (34),
\[(E|Z - EZ|^k)^{1/k} \leq c' \sqrt{kp} \left( 1 + \frac{(\log n)^2}{(np)^2} \right), \quad \forall k \leq k_0,
\]
where $c'$ is a universal constant and
\[
k_0 = \frac{np}{2C' \vee 1} \min \left\{ 1, \frac{\log(np)}{\log(1/p)} \right\}.
\]

**Proof.** Write $\|A\|_{\text{op}}$ as $Z$ when no confusion can arise. By Proposition 4.1 and Lemma 4.4,
\[
(E|Z - EZ|^k)^{1/k} \leq \left( \sqrt{kp} + \frac{k}{n} \right) \left( 1 + \frac{(k \vee \log n)^2}{(np)^2} \right) + k \exp \left\{ - \frac{np \log(np)}{2Ck} \right\}.
\]
For $k \leq k_0$, then the first term has the order of $\sqrt{kp}$ because under (34),
\[
k = \sqrt{kp} \frac{1}{\sqrt{n^2 p}} \leq \sqrt{kp}, \quad 1 + \frac{(k \vee \log n)^2}{(np)^2} \leq 1 + \frac{(\log n)^2}{(np)^2}
\]
To bound the second term is dominated by $\sqrt{kp}$ in order, it is left to show that
\[
\sqrt{\frac{k}{p}} \exp \left\{ - \frac{np \log(np)}{2Ck} \right\} = \exp \left\{ - \frac{np \log(np)}{2Ck} + \frac{1}{2} \log k + \frac{1}{2} \log \left( \frac{1}{p} \right) \right\} \leq 1.
\]
Since $k \leq k_0 \leq np/(2C \vee 1)$,
\[
k \log k \leq \frac{np \log(np)}{2C} \implies \log k \leq \frac{np \log(np)}{2Ck}.
\]
Further, since $k \leq k_0 \leq np \log(np)/2C \log(1/p)$,
\[
\log \left( \frac{1}{p} \right) \leq \frac{np \log(np)}{2Ck}.
\]
Thus,
\[
- \frac{np \log(np)}{2Ck} + \frac{1}{2} \log k + \frac{1}{2} \log \left( \frac{1}{p} \right) \leq 0,
\]
and (43) is proved. The proof is then completed. \[\square\]

Together with Lemma 4.2, Lemma 4.6 implies that partial sub-gaussian behavior of $\|A\|_{\text{op}}$.

**Theorem 4.7.** Under condition (34), there exists universal constants $C_1, C_2 > 0$ such that
\[
\mathbb{P}(|Z - EZ| \geq t) \leq \exp \left\{ 1 - \frac{t^2}{C_1 \sigma^2} \right\}, \quad \forall t \leq C_2 \sigma \sqrt{np} \min \left\{ 1, \sqrt{\frac{\log(np)}{\log(1/p)}} \right\},
\]
where
\[
\sigma = \sqrt{p} \left( 1 + \frac{(\log n)^2}{(np)^2} \right).
\]

**Remark 4.2.** When $p \geq \log n/n$, $\sigma \approx \sqrt{p}$. This proves the conjecture of Lugosi et al. [2018]. On the other hand, it is worth comparing the range of $t$ with the sub-gaussian behaviors. In Lugosi et al. [2018] (equation (2.1)), the multiplicative factor of $\sqrt{p}$ in the upper bound of $t$ is $\sqrt{np \log(np)}/\log n \log(1/p)$ while that of ours is $np \min\{1, \log(np)/\log(1/p)\}$, ignoring the constants. When $np = \text{PolyLog}(n)$, ours reduces to $np \log(np)/\log(1/p)$ which is $\sqrt{np \log n}$ larger than the one in Lugosi et al. [2018].
4.3 Extension to inhomogeneous graphs

The results can be directly extended to general inhomogenous graphs because the proof carries over if both \( \|u_1 - u_2\|_\infty \) and \( \|u_1\|_\infty \) have small moments. To be specific we consider a random graph with an adjacency matrix under the setting of Section 3.1. Apart from \( p^* \) and \( \bar{p}^* \) defined in (17), we further defined

\[
\bar{p} = \frac{1}{n(n-1)} \sum_{i \neq j} p_{ij}.
\]

Note that \( \bar{p} \leq \bar{p}^* \leq p^* \) and in many applications they differ in small multiplicative factors. We distinguish them to cover the cases where many \( p_{ij} \)'s are tiny. Then we can apply Theorem 3.4 to achieve the task. Here for convenience we assume that \( |\lambda^*_1| \geq np^*/\sqrt{n}u_1^* \|_\infty \) so that the simplified bound in Corollary 3.6 can be applied. Further we consider the case with \( p^* \leq 1/2 \) for convenience. The proof is more technical but qualitatively the same as the results for Erdős-Rényi graphs so we present it in Appendix D.1.

**Theorem 4.8.** Let \( |\lambda^*_1| \geq |\lambda^*_2| \geq \ldots \geq |\lambda^*_n| \) be eigenvalues of \( A^* \). Suppose \( |\lambda^*_1| > |\lambda^*_2| \) and let \( u_1^* \) be the eigenvector corresponding to \( \lambda^*_1 \). Let

\[
\zeta = \sqrt{n}u_1^* \|_\infty.
\]

Assume that

\[
|\lambda^*_1| \geq C_0 \frac{np^*}{\zeta}, \quad \Delta^* = \min \left\{ |\lambda^*_1|, \min_j |\lambda^*_1 - \lambda^*_j| \right\} \geq C_0 \left( \sqrt{np^*} + \frac{\log n}{\log \log n} \right), \quad n^2 \bar{p} \geq C_0.
\]

Then

\[
\text{Var}(\|A\|_{op}) \leq \bar{p} \left\{ 1 + \left( \frac{\sqrt{np^*} + \log n}{\Delta^*} \right)^4 \right\} \zeta^4 + \exp \left\{ -\frac{\Delta^* \log \Delta^* \cdot n^2 \bar{p}}{2C \sqrt{3}} \right\}
\]

where \( C \) is the universal constant in (19) in Theorem 3.4. In addition, there exists universal constants \( C_1, C_2 > 0 \) such that

\[
\mathbb{P}(|Z - EZ| \geq t) \leq \exp \left\{ 1 - \frac{t^2}{C_1 \sigma^2} \right\}, \quad \forall t \leq C_2 \sigma \min \left\{ \Delta^*, \frac{\Delta^* \log \Delta^*}{\log(1/p^2)}, \frac{n^2 \bar{p}}{\log(1/p^2)}, \frac{n^2 \bar{p}}{\log(1/p^2)} \right\}^{1/2},
\]

where

\[
\sigma = \sqrt{\bar{p}} \left\{ 1 + \left( \frac{\sqrt{np^*} + \log n}{\Delta^*} \right)^2 \right\}^2 \zeta^2.
\]

5 Exact Recovery of Spectral Clustering

5.1 Background

Stochastic block model (SBM) is a popular model to analyze community detection algorithms. The probability matrix of an SBM with \( K \) blocks is given by \( A^*_{ij} = B_{c_i c_j} \) for some matrix \( B \in [0, 1]^{K \times K} \) where \( c_i \) denotes the cluster label of \( i \). Equivalently,

\[
A^* = \hat{A}^* - \text{diag}(\hat{A}^*), \quad \text{where} \quad \hat{A}^* = ZBZ^T,
\]

where \( Z \in \mathbb{R}^{n \times K} \) denotes the membership matrix whose \( i \)-th row \( Z_i = e_{c_i} \), the \( c_i \)-th canonical basis of \( R^K \). The goal is to recover the cluster labels \( c = (c_1, \ldots, c_n) \), up to label permutation. Let \( \hat{c} \) denote the estimated cluster labels via an algorithm. We say that the algorithm achieves exact recovery if

\[
\mathbb{P} \left( \exists \text{ permutation } \pi, \ \hat{c}_{\pi(i)} = c_i \ \forall i \in [n] \right) = 1 - o(1).
\]

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The problem has been widely studied and can be solved using different algorithms [e.g. Bui et al., 1987, Boppana, 1987, Dyer and Frieze, 1989, Snijders and Nowicki, 1997, Jerrum and Sorkin, 1998, Condon and Karp, 2001, Carson and Impagliazzo, 2001, McSherry, 2005, Shamir and Tsur, 2007, Bickel and Chen, 2009, Coja-Oghlan, 2010, Rohe et al., 2011, Oymak and Hassibi, 2011, Balakrishnan et al., 2011, Choi et al., 2012, Mossel et al., 2014, Abbe and Sandon, 2015, Abbe et al., 2015, Chin et al., 2015, Hajek et al., 2016, Guédon and Vershynin, 2016, Yun and Proutiere, 2016, Agarwal et al., 2017, Gao et al., 2017, Amini and Levina, 2018, Bandeira, 2018, Chen et al., 2018, Vu, 2018, Fei and Chen, 2018, Li et al., 2018b,a, Su et al., 2019]; see Abbe [2017] for a nice review of this topic.

Spectral clustering algorithms are appealing due to the straightforward implementation and computational efficiency compared to other algorithms. In this section we consider the standard spectral clustering algorithm [e.g. Von Luxburg, 2007], which embeds each observation into the subspace spanned by $K$ eigenvectors of some operators and applies $K$-means or $K$-medians algorithm on embedded vectors.

Specifically, we consider the following procedure:

**Step 1** Compute the $K$ eigenvectors of the adjacency matrix corresponding to the $K$ largest eigenvalues in absolute values, or the $K$ eigenvectors of the unnormalized Laplacian corresponding to the $K$ smallest eigenvalues, denoted by $U$;

**Step 2** Perform $K$-medians algorithm on $U$ to get the estimates of cluster membership.

The intuition underlying the algorithm is that $U$ should approximate $U^*$, the eigenvector matrix of the expectation of the adjacency matrix or the Laplacian, which identifies the cluster labels using $K$-medians algorithm. Consider the adjacency matrix as an example. Let $n_i$ denote the number of units in cluster $i$ and without loss of generality assume that

$$Z = \begin{bmatrix} 1_{n_1} & 0 & \cdots & 0 \\ 0 & 1_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1_{n_K} \end{bmatrix}.$$ 

Let $M = \text{diag}(\sqrt{n_1}, \ldots, \sqrt{n_K})$ and $Q = ZM^{-1}$. Then $Q^TQ = I$ and

$$\hat{A}^* = Q(MBM)Q^T. \quad (45)$$

Let $VAV^T$ be the spectral decomposition of $MBM$. Then $QVB(QV)^T$ is the spectral decomposition of $\hat{A}^*$ since $QU_0$ is an orthogonal matrix. As a result, the eigenvector matrix of $\hat{A}^*$ is $\tilde{U}^* = QV$. By definition,

$$\tilde{U}^* = \begin{bmatrix} \frac{1_{n_1}V_1^T}{\sqrt{n_1}} \\ \frac{1_{n_2}V_2^T}{\sqrt{n_2}} \\ \vdots \\ \frac{1_{n_K}V_K^T}{\sqrt{n_K}} \end{bmatrix},$$

where $V_i^T$ is the $i$-th row of $V$. It is clear that $\tilde{U}^*_i = \tilde{U}^*_i$ iff $i$ and $i'$ belong to the same cluster. Thus $K$-medians can perfectly identify the clusters using $\tilde{U}^*$. Since $A^*$ approximates $\hat{A}^*$, $\tilde{U}^* \approx U^* \approx U$. This intuitively justifies the spectral clustering algorithm.
Early works investigated the weak recovery of spectral clustering, meaning that the misclassification error is vanishing with high probability [e.g. Rohe et al., 2011, Lei and Rinaldo, 2015, Joseph and Yu, 2016]. This is weaker than exact recovery which requires the misclassification error to be zero with high probability. The exact recovery was proved for dense graphs of which the average degree is polynomial in the graph size [e.g. McSherry, 2001, Balakrishnan et al., 2011]. Perhaps surprisingly, for sparse graphs of which the average degree is $O(\log n)$, the exact recovery was not proved until Abbe et al. [2017]. In particular, they proved that the spectral clustering is information theoretically optimal for two-block SBMs with equal block sizes: with within-block probability $p = a \log n/n$ and between-block probability $q = b \log n/n$ where $a > b$ are two constants, spectral clustering on the adjacency matrix achieves the exact recovery iff $\sqrt{a} - \sqrt{b} > \sqrt{2}$. Later Su et al. [2019] extended the result to spectral clustering on normalized Laplacians for general $K$-block SBMs when the average degree is of order $\log n$.

In this section, we derive the exact recovery of spectral clustering on adjacency matrices or unnormalized Laplacians for general SBMs with fixed or growing $K$ through a simple analysis based on our $\ell_2 \to \infty$ bounds in Section 3.1. The results can be directly extended to SBMs with dependent entries using the results from Section 7.1 but we leave them to interested readers as the derivation is almost identical. Before delving into the analysis, we prove a useful lemma that connects the $K$-medians algorithm and the $\ell_2 \to \infty$ perturbation bounds. In particular, the $K$-medians algorithm applied on $U$ will return cluster labels as

$$\hat{c}_i = \arg\min_r \|U_i - \hat{v}_r\|_2,$$  
(46)

where $U^T_i$ is the $i$-th row of $U$ and

$$(\hat{v}_1, \ldots, \hat{v}_K) = \arg\min_{(v_1, \ldots, v_K) \in [K]} \frac{1}{n} \sum_{i=1}^{n} \min_{r \in [K]} \|U_i - v_r\|_2$$

Lemma 5.1. Let $U, \tilde{U}^* \in \mathbb{R}^{n \times k}$ be two matrices and $C_1, \ldots, C_K$ be a partition of $[n]$ with $|C_s| = n \pi_s$. Assume that

$$\tilde{U}^*_i = v^*_s, \quad \forall i \in C_s,$$

and $v^*_s \neq v^*_{s'}$ for any pair $s \neq s'$. Then the $K$-medians algorithm exactly recovers $C_1, \ldots, C_K$ if

$$d_{2 \to \infty}(U, \tilde{U}^*) \leq \frac{\min_{s \neq s'} \|v^*_s - v^*_{s'}\|_2}{6}.$$ 

5.2 Exact recovery of SBM with fixed $K$

In this subsection, we consider a standard asymptotic setting where $K$ is held fixed,

$$B = \rho_n B_0, \quad n_r/n \to \pi_r,$$

for some rate function $\rho_n$, fixed matrix $B_0$ and fixed numbers $\pi_1, \ldots, \pi_K > 0$ that sum up to 1. For this model, it is known that the information theoretic lower bound for $\rho_n$ is $\log n/n$, in the sense that if $\rho_n/(\log n/n) \to 0$ then no algorithm can achieve exact recovery [e.g. Abbe et al., 2015, Abbe, 2017]. In this subsection we will show that spectral clustering can achieve exact recovery if $\rho_n > c \log n/n$ for a sufficiently large constant $c$ via either the adjacency matrix or the unnormalized Laplacian separately.

Theorem 5.2 (exact recovery via adjacency matrix). Assume that $B_0$ is full rank. Fix any $q > 0$. Then there exists constants $c$ and $n_0$ that only depends on $B_0$, $\pi_r$’s and $q$ such that if $n \geq n_0$ and

$$n \rho_n \geq c \log n,$$  
(48)
then spectral clustering using the adjacency matrix achieves exact recovery with probability at least $1 - n^{-q}$.

Proof. Let $\Lambda^* \in \mathbb{R}^{K \times K}$ denote the diagonal matrix of all non-zero eigenvalues of $A^*$ and $U^* \in \mathbb{R}^{n \times K}$ be the corresponding eigenvector matrix. Let $R_0 = \text{diag}(\sqrt{\pi_1}, \ldots, \sqrt{\pi_K})$. Recalling the decomposition (45) and let $R = M/\sqrt{n}$, we have

$$\hat{A}^* = n\rho_n Q(RB_0R)Q^T, \quad \hat{U}^* = QV,$$

where $V \in \mathbb{R}^{K \times K}$ is the matrix formed by all eigenvectors of $RB_0R$. By definition of $Q$, $\hat{U}^*_i = \frac{V_i}{\sqrt{\pi_i}}$ for any $i \in C_s$ where $V_i^T$ is the $s$-th row of $V$. Let $v_s^* = \hat{U}^*_i$ for $i \in C_s$. Using the fact that $V$ is an orthogonal matrix, we have

$$\|v_s^* - v_s^*\|_2 = \sqrt{\|v_s^*\|_2^2 + \|v_s^*\|_2^2 - 2(v_s^*, v_s^*)} = \sqrt{\frac{1}{n_s} + \frac{1}{n_{s'}}} \geq \frac{1}{\min_{s \in [K]} \sqrt{\pi_s} \sqrt{n}}.$$

By Lemma 5.1, it is left to prove that

$$d_{2 \to \infty}(U, \hat{U}^*) \leq \frac{\min_{s \in [K]} \sqrt{\pi_s}}{6 \sqrt{n}} \leq \frac{2c_1}{\sqrt{n}}. \quad (49)$$

By the triangle inequality,

$$d_{2 \to \infty}(U, \hat{U}^*) \leq d_{2 \to \infty}(U, U^*) + d_{2 \to \infty}(U^*, \hat{U}^*).$$

By definition,

$$d_{2 \to \infty}(U^*, \hat{U}^*) = \inf_{O \in O_K} \|U^*O - \hat{U}^*\|_{2 \to \infty} \leq \inf_{O \in O_K} \|U^*O - \hat{U}^*\|_{\text{op}}.\leq$$

By Davis-Kahan Theorem [Yu et al., 2014, Theorem 2],

$$\inf_{O \in O_K} \|U^*O - \hat{U}^*\|_{\text{op}} \leq \frac{\sqrt{8K}\|A^* - \hat{A}^*\|_{\text{op}}}{\lambda_K - \hat{\lambda}_{K+1}} \leq \frac{\sqrt{8K}\rho_n}{\lambda_K},$$

where we use the fact that $\hat{A}^* - A^* = \text{diag}(\hat{A}^*)$ and $\text{rank}(\hat{A}^*) = K$. Since

$$\hat{\lambda}_{K} = n\rho_n\lambda_{\min}(RB_0R) \geq n\rho_n \left( \min_{r \in [K]} \sqrt{\frac{n_r}{n}} \right) \lambda_{\min}(B_0).$$

Thus, when $n$ is sufficiently large,

$$d_{2 \to \infty}(U^*, \hat{U}^*) \leq \inf_{O \in O_K} \|U^*O - \hat{U}^*\|_{\text{op}} \leq \frac{\sqrt{16K}}{\lambda_{\min}(B_0) \min_{r \in [K]} \sqrt{\pi_r} n} \frac{1}{n} \leq \frac{c_1}{\sqrt{n}}, \quad (50)$$

Combined with (49), it is left to prove that

$$d_{2 \to \infty}(U, U^*) \leq \frac{c_1}{\sqrt{n}}. \quad (51)$$

Throughout the rest of the proof we treat $B_0$, $q$ and $\pi_r$’s as constants. Note that $R \to R_0$ and hence

$$RB_0R \to R_0B_0R_0.$$

In addition, since $B_0$ is full-rank and $\pi_r > 0$, $R_0B_0R_0$ is full rank. Then there exists $n_0$ that only depends on $B$ and $R_0$ such that whenever $n \geq n_0$,

$$\lambda_{\min}(\hat{A}^*) > \frac{2n\rho_n}{3} \lambda_{\min}(R_0B_0R_0).$$
By Weyl's inequality,
\[ |\lambda_{\text{min}}(\tilde{\Lambda}^*) - \lambda^*_{\text{min}}| \leq \|\tilde{A}^* - A^*\|_{\text{op}} \leq \rho_n. \]
Thus for sufficiently large \( n \),
\[ \lambda^*_{\text{min}} > \frac{n\rho_n}{2} \lambda_{\text{min}}(R_0B_0R_0). \] (52)

Let \( \Delta^* \) be the counterpart of \( \Delta^* \) for \( \tilde{A}^* \). Then by Weyl's inequality
\[ |\Delta^* - \Delta^*| \leq 2\rho_n. \]
Since \( \Delta^* = \lambda_{\text{min}}(\tilde{\Lambda}^*) \), for sufficiently large \( n \),
\[ \Delta^* > \frac{n\rho_n}{2} \lambda_{\text{min}}(R_0B_0R_0). \] (53)

On the other hand, it is easy to see that
\[ \|\tilde{U}^*\|_{2 \to \infty} = \frac{1}{\min_{s \in [K]} \sqrt{n_s}} = \frac{1}{\min_{s \in [K]} \sqrt{\pi_s}} \sqrt{n}. \]

By (50),
\[ \|\tilde{U}^*\|_{2 \to \infty} - \|U^*\|_{2 \to \infty} \leq d_{2 \to \infty}(U^*, \tilde{U}^*) \leq \frac{c_1}{\sqrt{n}}. \]
Thus,
\[ \|U^*\|_{2 \to \infty} \leq \frac{1}{\sqrt{n}}. \] (54)

Set \( \delta = n^{-q} \) and \( \alpha = 0.5 \) in Corollary 3.6. By (52) and (54),
\[ \lambda^*_{\text{min}} \geq \frac{np^*}{\sqrt{n}\|U^*\|_{2 \to \infty}}. \]

Moreover, \( \bar{\kappa}^* \leq 2K \leq 1 \), \( p^* \leq \rho_n \) and
\[ R(\delta) \leq \log n, \quad g(\delta) \leq \sqrt{n\rho_n} + \frac{\log n}{\log \log n}. \]

By (53), for sufficiently large \( n \) and \( c \) in the condition (48),
\[ \Delta^* > \frac{n\rho_n}{2} \lambda_{\text{min}}(R_0B_0R_0) \geq C\bar{\kappa}^* g(\delta) \]
where \( C \) is the universal constant in Theorem 3.4. Thus, the conditions of Corollary 3.6 are satisfied. As a result,
\[ d_{2 \to \infty}(U, U^*) \leq \left( \sqrt{n\rho_n} + \frac{\log n}{\rho_n} \log \log n + \frac{\log n}{\rho_n} \right) \frac{1}{\sqrt{n}} + \sqrt{\frac{\log n}{\rho_n} \frac{1}{\sqrt{n}}} \]
Equivalently, there exists a constant \( c_2 \) that only depends on \( B_0 \), \( q \) and \( \pi_r \)'s such that
\[ d_{2 \to \infty}(U, U^*) \leq \sqrt{\frac{\log n}{n\rho_n}} \frac{c_2}{\sqrt{n}}. \]
By condition (48),
\[ d_{2 \to \infty}(U, U^*) \leq \frac{c_2}{c} \frac{1}{\sqrt{n}}. \]
Therefore, (51) follows if \( c > c_2^2/c_1^2 \). The proof is then completed. \( \square \)
Theorem 5.3 (exact recovery via unnormalized Laplacian). Let \( R_0 = \text{diag}(\sqrt{\pi_1}, \ldots, \sqrt{\pi_K}) \), \( \tilde{d}_0 = B_0 R_0^2 I_K \) and \( \tilde{D}_0 = \text{diag}(\tilde{d}_{01}, \ldots, \tilde{d}_{0K}) \). Further let

\[
\tilde{L}_0 = \tilde{D}_0 - R_0 B_0 R_0.
\]

Assume that

\[
\text{rank}(\tilde{L}_0) = K - 1, \quad \lambda_{\text{max}}(\tilde{L}_0) < \lambda_{\text{min}}(\tilde{D}_0).
\]  

(55)

Fix any \( q > 0 \). Then there exists constants \( c \) and \( n_0 \) that only depends on \( B_0, \pi_i \)'s and \( q \) such that if \( n \geq n_0 \) and

\[
n\rho_n \geq c \log n.
\]  

(56)

then spectral clustering using the unnormalized Laplacian achieves exact recovery with probability at least \( 1 - n^{-q} \).

The condition (55) is motivated by the following result on the eigen-structure of population Laplacian \( L^* \). The proof is relegated to Appendix D.2.

**Lemma 5.4.** Let \( R = \text{diag}(\sqrt{n_1/n}, \ldots, \sqrt{n_K/n}) \), \( \tilde{d} = B_0 R^2 I_K \) and \( \tilde{D} = \text{diag}(\tilde{d}_1, \ldots, \tilde{d}_K) \). Further let

\[
\tilde{L} = \tilde{D} - RB_0 R.
\]

Let \( \tilde{L} = V \Sigma V^T \) be the spectral decomposition. Then the spectral decomposition of \( L^* \) can be written as

\[
L^* = \begin{bmatrix} U^* & \tilde{U}^* \end{bmatrix} \begin{bmatrix} \Lambda^* & 0 \\ 0 & \tilde{\Lambda}^* \end{bmatrix} \begin{bmatrix} U^* & \tilde{U}^* \end{bmatrix}^T
\]

where

\[
U^* = QV, \quad \Lambda^* = n\rho_n \Sigma, \quad \tilde{U}^* = \begin{bmatrix} Q_1 & 0 & \cdots & 0 \\ 0 & Q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q_K \end{bmatrix}, \quad \tilde{\Lambda}^* = n\rho_n \begin{bmatrix} \tilde{d}_1 I_{n_1-1} & 0 & \cdots & 0 \\ 0 & \tilde{d}_2 I_{n_2-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{d}_K I_{n_K-1} \end{bmatrix}
\]

and \( Q_j \in \mathbb{R}^{n_j \times (n_j-1)} \) can be any matrix such that \( Q_j Q_j^T = I_{n_j} - I_{n_j} I_{n_j}^T / n_j \).

Lemma 5.4 implies that \( U^* \) can be used to identify the clusters. Since \( R \to R_0 \), we have \( \tilde{L}_0 \to \tilde{L} \) and \( \tilde{D}_0 \to \tilde{D} \). Under condition (55), when \( n \) is large enough,

\[
\lambda_{\text{min}}(\tilde{\Lambda}^*) > \lambda_{\text{max}}^*.
\]

As a result, \( U^* \) corresponds to the \( K \) smallest eigenvalues of \( L^* \). This is almost necessary for the algorithm described in section 5.1 to work. The proof of Theorem 5.3 is relegated to Appendix D.2 since it is similar to that of Theorem 5.2.

5.3 Exact recovery of SBM with growing \( K \)

The last subsection discusses the case with fixed \( K \) and the analyses hide all constants that depend on \( K \). It is also of interest to investigate the tolerance on \( K \) in applications where \( K \) tends to be large.
Most of work tackling with this question focuses on computationally less efficient algorithms such as likelihood method [e.g. Choi et al., 2012] or semidefinite programming (SDP) method [e.g. Amini and Levina, 2018, Chen et al., 2018, Fei and Chen, 2018]. By contrast, the results on the vanilla spectral clustering algorithms, described in Section 5.1, are rather limited [e.g. Rohe et al., 2011, Su et al., 2019].

Equipped with the $\ell_2 \to \infty$ perturbation theory in this paper, we can easily derive the results for general SBMs with growing $K$. To keep the exposition clear, we focus on the balanced assortative four-parameter model with

$$B = \rho_n B_0, \quad B_0 = (a - b) I + b 1_K 1_K^T, \quad a > b > 0, \quad n_1 = \ldots = n_K = m \triangleq n/K,$$

(57)

where $a$ and $b$ are two constants. This is a widely studied special SBM in literature. Under this model, the best available dependence on $K$ is given by SDP methods [e.g. Fei and Chen, 2018] with

$$n\rho_n \geq K^2 + K \log n.$$

(58)

For spectral clustering [Su et al., 2019, Example 2.1], the dependence becomes much worse

$$n\rho_n \geq K^7 \log n.$$

(59)

In this subsection, we analyze the spectral clustering algorithms, described in Section 5.1, using the adjacency matrix and the unnormalized Laplacian, respectively. For both algorithms, we show that the dependence on $K$ is better than (59), although it still leaves a gap to (58).

**Theorem 5.5.** Fix any $q > 0$ Under the model (57), there exists a constant $c$ which only depends on $(a, b, q)$ such that

1. the spectral clustering algorithm using the adjacency matrix achieves exact recovery with probability $1 - n^{-q}$ if $n\rho_n \geq c(K^4 + K^3 \log n)$;
2. the spectral clustering algorithm using the unnormalized Laplacian achieves exact recovery with probability $1 - n^{-q}$ if $n\rho_n \geq cK^3 \log n$;

**Proof.** For the sake of length, we only present the proof for part (1) and leave part (2) to Appendix D.2. By (45) and (57), it is clear that

$$A^* = \tilde{A}^* - \rho_n a I_n, \quad \tilde{A}^* = \frac{n\rho_n}{K} Q B_0 Q^T.$$

Thus, $A^*$ and $\tilde{A}^*$ have the same eigenvectors and $U^* = \tilde{U}^*$. Let $V \Sigma V^T$ be the spectral decomposition of $B_0$. Then it is easy to see that

$$\Sigma = \text{diag} \left( a + (K - 1)b, a - b, \ldots, a - b \right) \text{ (K-1 copies)}.$$

Thus, the top-$K$ eigenvalue matrix $\Lambda^*$ of $A^*$ is $(n\rho_n/K) \Sigma - \rho_n a I_n = (m - a) \rho_n \Sigma$ and the corresponding eigenvector matrix $U^*$ can be written as $U^* = QV$. As a result,

$$\min_{s \neq s'} \|\nu^*_s - \nu^*_{s'}\|_2 = \sqrt{\frac{2}{m}},$$

(60)
By Lemma 5.1, it is left to prove that

\[ d_{2 \to \infty}(U, U^*) \leq \frac{1}{6K} \min_{s \neq s'} \| \nu_s^* - \nu_{s'}^* \|_2 = \frac{\sqrt{2}}{6K \sqrt{m}}. \] 

(61)

We split \( U^* \) into two parts \( U_1^* \in \mathbb{R}^{n \times 1} \) and \( U_2^* \in \mathbb{R}^{n \times (K-1)} \) where \( U_1^* \) corresponds to the largest eigenvalue of \( A^* \) while \( U_2^* \) gives other eigenvectors in \( U^* \). The same split is applied to \( U \) which yields \( U_1 \in \mathbb{R}^{n \times 1} \) and \( U_2 \in \mathbb{R}^{n \times (K-1)} \). We also split \( \Lambda \) and \( \Lambda^* \) similarly. It is easy to see that \( d_{2 \to \infty}(U, U^*) \leq d_{2 \to \infty}(U_1, U_1^*) + d_{2 \to \infty}(U_2, U_2^*) \). Thus (61) is true provided that

\[ d_{2 \to \infty}(U_1, U_1^*) \leq \frac{\sqrt{2}}{12K \sqrt{m}}, \quad d_{2 \to \infty}(U_2, U_2^*) \leq \frac{\sqrt{2}}{12K \sqrt{m}} \]

(62)

By definition,

\[ \lambda_{\min}(\Lambda_1^*) \geq \Delta_1^* = (m-a)\rho_n Kb, \quad \bar{\kappa}_1 = \kappa_1^* = 1, \]

and

\[ \lambda_{\min}(\Lambda_2^*) \geq \Delta_2^* = (m-a)\rho_n \min \{ Kb, a-b \}, \quad \bar{\kappa}_1 = \kappa_1^* = 1, \]

Set \( \delta = n^{-q} \) and \( \alpha = 1/\log R(\delta) \) in Corollary 3.5. Note that this choice of \( \alpha \) implies that

\[ \frac{R(\delta)}{\alpha \log R(\delta)} = R(\alpha), \quad R(\delta)^{\alpha} = \exp\{ \alpha \log R(\delta) \} = e. \]

Moreover, \( p^* \leq \rho_n \), and

\[ R(\delta) \leq \log n + K, \quad g(\delta) \leq \sqrt{m} + \log n + K. \]

Since \( n\rho_n > c(K^4 + K^3 \log n) \), for sufficiently large \( n \) and \( c \),

\[ \Delta_1^* \geq \Delta_2^* = (m-a)\rho_n \min \{ Kb, a-b \} = (m-a)K \rho_n \min \{ Kb, a-b \} \leq C\kappa_1^* g(\delta) = C\kappa_1^* g(\delta), \]

where \( C \) is the universal constant in (19). Thus the condition of Corollary 3.5 is satisfied for both \( (\Lambda_1^*, U_1^*) \) and \( (\Lambda_2^*, U_2^*) \). Note that \( m-a \geq m \). Thus, \( \lambda_{\min}(\Lambda_1^*), \lambda_{\min}(\Lambda_2^*), \Delta_1^*, \Delta_2^* \geq m\rho_n \). By Corollary 3.5, for both \( j = 1, 2 \),

\[ d_{2 \to \infty}(U_j, U_j^*) \leq \left( \frac{\sqrt{\rho_n} + \log n + K}{m \rho_n} + \frac{\log n + K}{m \rho_n} \right) \frac{1}{\sqrt{m}} + \sqrt{\frac{\log n + K}{m \rho_n}} \left( 1 + \frac{\sqrt{\rho_n}}{m \rho_n} \right) \]

\[ \leq \left( \frac{\sqrt{\rho_n}}{m \rho_n} + \frac{\log n + K}{m \rho_n} + \sqrt{\frac{\log n + K}{m \rho_n}} \right) \frac{1}{\sqrt{m}}, \]

\[ \leq \frac{\log n + K}{m \rho_n} \frac{1}{\sqrt{m}}, \]

where (i) uses the fact that

\[ \frac{\sqrt{\rho_n}}{m \rho_n} = \frac{K}{\sqrt{\rho_n}} \leq 1, \]

and (ii) uses the fact that

\[ \frac{\log n + K}{m \rho_n} = \sqrt{\frac{K \log n}{\rho_n}} \leq 1, \quad \frac{\sqrt{\rho_n}}{m \rho_n} = \sqrt{\frac{K}{m \rho_n}} \leq \sqrt{\frac{\log n + K}{m \rho_n}}. \]

Equivalently, there exists a constant \( c_1 \) that only depends on \( a, b \) and \( q \) such that

\[ d_{2 \to \infty}(U_j, U_j^*) \leq \frac{\log n + K}{m \rho_n} \frac{c_1}{\sqrt{m}} = \frac{K \log n + K^2}{n \rho_n} \frac{c_1}{\sqrt{m}} \leq \frac{c_1}{\sqrt{2K \sqrt{m}}}. \]

As a result, (62) is true if \( c > 1/72c_2^2 \). This completes the proof. \( \square \)
6 Partial Consistency of Divisive Hierarchical Clustering

6.1 Background

Hierarchical community detection is widely used in practice. As opposed to agglomerative (bottom-up) hierarchical clustering, divisive (top-down) hierarchical clustering starts from the whole network, tests if there are at least two communities, and then divides it into a few mega-communities if so and stops splitting otherwise. The procedure proceeds recursively for each of mega-community until none of the mega-communities at hand passes the test for more than one community. Unlike agglomerative clustering, divisive clustering is scalable for giant networks with large number of clusters in terms of both computation and storage cost. The idea emerged in machine learning problems such as graph partitioning and image segmentation, referred to as graph bi-partitioning [Spielman and Teng, 1996, Shi and Malik, 2000, Kannan et al., 2004]. Despite the empirical success in various applications, the theoretical analysis is challenging. The existing analyses either require complicated but artificial modification of the algorithms [e.g. Dasgupta et al., 2006] or only hold for dense networks (with polynomial average degree) [e.g. Balakrishnan et al., 2011].

A recent work by Li et al. [2018a] established a framework to study divisive hierarchical clustering. They proposed the Binary Tree SBM (BTSBM) as the basis for analysis. A BTSBM is an SBM, described in Section 5.1, with $K = 2^d$ clusters, embedded into the leaf nodes of a full binary tree with $d + 1$ layers. The $\ell$-th layer is equipped with a parameter $p_{d-\ell+1}$ and each cluster is encoded as a length-$d$ binary string. We illustrate it in Figure 1 with $d = 3$. The connection probability matrix $B$ is then decided as follows: for any two clusters $c$ and $c'$, let $x_d \ldots x_1$ and $x'_d \ldots x'_1$ be their binary representation, then

$$B_{c,c'} = p_{D(c,c')}, \quad \text{where } D(c,c') = \max\{i : x_i \neq x'_i\}I(c \neq c').$$
For instance, for the BTSBM in Figure 1, the block matrix $B$ is

$$
B = \begin{bmatrix}
p_0 & p_1 & p_2 & p_2 & p_3 & p_3 & p_3 \\
p_1 & p_0 & p_2 & p_2 & p_3 & p_3 & p_3 \\
p_2 & p_2 & p_0 & p_1 & p_3 & p_3 & p_3 \\
p_3 & p_3 & p_3 & p_0 & p_1 & p_2 & p_2 \\
p_3 & p_3 & p_3 & p_3 & p_1 & p_0 & p_2 \\
p_3 & p_3 & p_3 & p_3 & p_2 & p_2 & p_0 \\
p_3 & p_3 & p_3 & p_3 & p_2 & p_2 & p_1 \\
p_3 & p_3 & p_3 & p_3 & p_2 & p_2 & p_1 & p_0
\end{bmatrix}.
$$

Assuming equal block sizes for all communities, Li et al. [2018a] analyzed the HCD-Sign algorithm, which splits the network into two mega-communities according to the sign of the second eigenvector of the adjacency matrix, under both assortative BTSBM ($p_0 > p_1 > \ldots > p_d$) and dis-assortative BTSBM ($p_0 < p_1 < \ldots < p_d$). They provided explicit conditions under which all mega-communities in the first $\ell$ layers can be exactly recovered for any $\ell \leq d$. Unlike the $K$-way spectral clustering, described in Section 5.1, which requires knowing $K$ exactly, HCD-Sign can be completely agnostic to $K$ while only requires a “consistent” stopping rule, such as the one based on non-backtracking operator [Le and Levina, 2015].

The proof relies on the nice eigen-structure of BTSBM as stated below.

**Proposition 6.1.** [Theorem 5 of Li et al. [2018a]] Under either assortative or dis-assortative BTSBM,

1. the first and the second largest eigenvalue (in absolute value) of $A^*$ are both unique, given by
   $$
   \lambda_1^* = (m-1)p_0 + m \sum_{i=1}^{d-1} 2^{i-1}p_i + m2^{d-1}p_d, \quad \lambda_2^* = (m-1)p_0 + m \sum_{i=1}^{d-1} 2^{i-1}p_i - m2^{d-1}p_d;
   $$

2. the eigen-gap $\Delta^*$ between $\lambda_2^*$ and others is
   $$
   \Delta^* = \begin{cases} 
   n \min\{p_d, |p_{d-1} - p_d|/2\} & \text{(for assortative BTSBM)} \\
   n|p_{d-1} - p_d|/2 & \text{(for dis-assortative BTSBM)}
   \end{cases}
   $$

3. the eigenvector corresponding to $\lambda_2^*$ is
   $$
   u_2 = \frac{1}{\sqrt{n}} \begin{bmatrix} 1_{n/2} \\ -1_{n/2} \end{bmatrix};
   $$

4. rank($A^*$) = $K$. $A^*$ is psd under assortative BTSBM, but not psd under dis-assortative BTSBM.

Proposition 6.1 implies that the second eigenvector of $A^*$ perfectly identifies the binary split. They proceed by showing that $||u_2 - u_2^*||_\infty \ll 1/\sqrt{n}$ thereby proving the exact recovery for each split. The technique underlying their analysis is derived by Eldridge et al. [2017].

Although BTSBM is still restrictive, it is much more general than the typical four-parameter model described in Section 5.3. More importantly, BTSBM captures the multi-scale nature of real-world networks: the clusters are not treated as “exchangeable” but encoding different granularity of similarity. Furthermore, divisive hierarchical clustering algorithms are able to achieve partial recovery, i.e. recovering mega-communities up to layer $\ell + 1$ without recovering the communities at the finest level, imply that
the stable structure (i.e. mega-communities) may be recovered even if the finest communities cannot, depending on how similar different clusters are.

With the new $\ell_2 \to \infty$ perturbation bounds in this paper, we can refine Li et al. [2018a]'s results and obtain a more accurate characterization of the partial exact recovery phenomenon. We start from a generic sufficient condition for recovering one split and then apply it to analyze the case with fixed $K$. The case with growing $K$ can be analyzed similarly but we leave it to interested readers for the sake of length.

### 6.2 A generic sufficient condition for recovering one split

The exact recovery for the first split is achieved by HCD-Sign iff there exists $s \in \{-1, +1\}$ such that

$$\text{sign}(u_{2i}^*) = \text{sign}(u_{2i}) s, \quad \forall i \in [n].$$  \hfill (64)

As observed by Abbe et al. [2017] as well as our theory, $u_2$ is closer to $Au_2^*/\lambda_2^*$ than to $u_2^*$. The following lemma provides a sufficient condition for (64) based on $Au_2^*/\lambda_2^*$.

**Lemma 6.2.** (64) holds if there exists $s \in \{-1, +1\}$ such that for all $i \in [n]$,

$$\text{sign}(u_{2i}^*) = \text{sign}(A_i^T(\sqrt{n}u_2^*)) s,$$  \hfill (65)

and

$$|A_i^T(\sqrt{n}u_2^*)| > \left(\sqrt{n} \frac{|u_2^* - Au_2^*|}{\lambda_2^*}\right) |\lambda_2^*|.$$  \hfill (66)

**Proof.** We only prove the assortative case and the proof for the dissortative case is similar. Assume $s = 1$ without loss of generality. The condition (66) can be rewritten as

$$\left|\frac{A_i^T u_2^*}{\lambda_2^*}\right| > \left(\frac{u_2^* - Au_2^*}{\lambda_2^*}\right) |\lambda_2^*|.$$  \hfill (67)

By Proposition 6.1, $\lambda_2^* > 0$ and

$$\text{sign}(u_{2i}) = \text{sign}\left(\frac{A_i^T u_2^*}{\lambda_2^*}\right) = \text{sign}(A_i^T(\sqrt{n}u_2^*)) = \text{sign}(A_i^T(\sqrt{n}u_2^*)).$$  

Together with condition (65), we complete the proof. \hfill \square

By Proposition 6.1,

$$A_i^T(\sqrt{n}u_2^*) = \sum_{j=1}^{n/2} A_{ij} - \sum_{j=n/2+1}^{n} A_{ij}. \hfill (68)$$

Let

$$Z_i = (-1)^{I(i>n/2)} A_i^T(\sqrt{n}u_2^*)$$

Under BTSBM, it is not hard to see that $Z_1, \ldots, Z_n$ are i.i.d.. Note that $\text{sign}(u_{2i}^*) = (-1)^{I(i>n/2)}$. By Lemma 6.2, (65) and (66) are both satisfied if

$$\min_{i \in [n]} Z_i > \left(\sqrt{n} \frac{|u_2^* - Au_2^*|}{\infty}\right) |\lambda_2^*| \quad \text{or} \quad \max_{i \in [n]} Z_i < - \left(\sqrt{n} \frac{|u_2^* - Au_2^*|}{\infty}\right) |\lambda_2^*|.$$  \hfill (69)

It is left to show the above event occurs with high probability.

The following lemma provides a tail probability estimate for $Z_i$. The proof is relegated to Appendix D.3.
Lemma 6.3.

(1) In the assortative case, for any $t > 0$,
\[
\log \mathbb{P}(Z_t \leq t) \leq \frac{t}{2} \log \left( \frac{\lambda_1^* + \lambda_2^*}{\lambda_1^* - \lambda_2^*} \right) - \frac{1}{2} \left( \sqrt{\lambda_1^* + \lambda_2^*} - \sqrt{\lambda_1^* - \lambda_2^*} \right)^2.
\]

(2) In the dis-assortative case, for any $t > 0$,
\[
\log \mathbb{P}(Z_t \geq -t) \leq \frac{t}{2} \log \left( \frac{\lambda_1^* - \lambda_2^*}{\lambda_1^* + \lambda_2^*} \right) - \frac{1}{2} \left( \sqrt{\lambda_1^* + \lambda_2^*} - \sqrt{\lambda_1^* - \lambda_2^*} \right)^2.
\]

Combined with the high probability upper bound for $\|u_2 - Au_2^*/\lambda_2^*\|_\infty$ obtained by Theorem 3.4, we can derive the following result for recovering the first split exactly. The proof is straightforward so we relegate it into Appendix D.3.

Theorem 6.4. Consider a BTSBM that is either assortative or dis-assortative. Let $\Delta^*$ be defined as in Proposition 6.1 and $p^* = \max\{p_0, p_d\}$. Suppose there exists $\alpha > 0$ and $q > 0$ such that
\[
\Delta^* > (q + 1)C\left( \sqrt{\lambda_1^*} + \frac{\log n}{\alpha \log \log n} \right),
\]
where $C$ is the universal constant in (19). Fix any $\delta > n^{-q}$. Then the first split can be recovered with probability $1 - 2\delta$ if
\[
\frac{1}{2} \left( \sqrt{\lambda_1^* + \lambda_2^*} - \sqrt{\lambda_1^* - \lambda_2^*} \right)^2 - \log n - C' \frac{\lambda_1^*}{\Delta^*} \left( \lambda_1^* + \lambda_2^* \right) \left( \xi_{n1} + \xi_{n2} \right) \geq \log \left( \frac{1}{\delta} \right),
\]
where $C'$ is a universal constant and
\[
\xi_{n1} = \left( \sqrt{\lambda_1^*} + \frac{\log n}{\alpha \log \log n} \right) \left( 1 + \frac{\log n}{|\lambda_2^*|} \right) + \sqrt{(\log n)np^*} \frac{\log n + \sqrt{\lambda_1^* (\log n)^\alpha}}{\alpha \log \log n},
\]
and
\[
\xi_{n2} = \begin{cases} \min \left\{ \sqrt{np^*}, \sqrt{\lambda_2^* K} \right\} & \text{(assortative)} \\ \min \left\{ \sqrt{\lambda_1^* np^*/|\lambda_2^*|}, \sqrt{\lambda_2^* K} \right\} & \text{(dis-assortative)} \end{cases}
\]

6.3 Partial exact recovery for BTSBM

Consider the setting where $K$ is fixed and
\[
p_j = \rho_n a_j,
\]
for a set of constants $(a_0, \ldots, a_d)$. Li et al. [2018a] proves the exact recovery in the regime $np_n \geq (\log n)^{2+\epsilon}$. On the other hand, if $K$ is known, the information theoretic lower bound for recovering all communities (not including mega-communities) is $\rho_n = \log n/n$ [e.g. Abbe and Sandon, 2015]. The extra logarithmic factors in Li et al. [2018a] is simply an artifact of using the non-tight $\ell_\infty$ perturbation bound by Eldridge et al. [2017]. With Theorem 6.4 derived from our $\ell_\infty$ perturbation theory, we can prove the exact recovery in the regime $\rho_n = \log n/n$ and provide precise condition on the constants $(a_0, \ldots, a_d)$.

Theorem 6.5. Assume that $\rho_n = \log n/n$ and either $a_0 > a_1 > \ldots > a_d > 0$ (assortative) or $0 < a_0 < a_1 < \ldots < a_d$ (dis-assortative). Fix any $\ell \in [d]$. If further
\[
|\sqrt{a_r} - \sqrt{a_r}| > \sqrt{2^{2\ell-1-1}}, \quad r = d, d-1, \ldots, d-\ell + 1,
\]

(72)
where
\[ \bar{a}_r = a_0 + \sum_{j=1}^{r-1} 2^{j-1} a_j, \]  
(73)
then all mega-communities up to layer \( \ell + 1 \) can be exactly recovered with probability \( 1 - o(1) \) as \( n \) tends to infinity.

**Remark 6.1.** The quantity \( \bar{a}_r \) is essentially the average connection probability in each mega-community at \((d-r+2)\)-th layer.

The condition (72) has an interesting implication. Take \( \ell = d \), it is equivalent to
\[ |\sqrt{\bar{a}_d} - \sqrt{a_d}| > \sqrt{2}. \]
Further take \( d = 3 \) for illustration and recall (63). Consider the following hypothetical SBM with connection probability matrix
\[ B' = \rho_n \begin{bmatrix} \bar{a}_3 & \bar{a}_3 & \bar{a}_3 & \bar{a}_3 & a_3 & a_3 & a_3 & a_3 \\ \bar{a}_3 & \bar{a}_3 & \bar{a}_3 & \bar{a}_3 & a_3 & a_3 & a_3 & a_3 \\ \bar{a}_3 & \bar{a}_3 & \bar{a}_3 & \bar{a}_3 & a_3 & a_3 & a_3 & a_3 \\ \bar{a}_3 & \bar{a}_3 & \bar{a}_3 & \bar{a}_3 & a_3 & a_3 & a_3 & a_3 \\ a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 \\ a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 \\ a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 \\ a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 \end{bmatrix}. \]  
(74)
This is essentially a 2-block SBM with parameter \( \bar{a}_3 \log n/n \) and \( a_3 \log n/n \) where \( \bar{a}_3 \) is the average probability in the mega-community. It is well-known that (74) can be exactly recovered if and only if
\[ |\sqrt{a_3} - \sqrt{a_3}| > \sqrt{2}. \]
In other words, recovering the first split of the BTSBM with connection probability matrix (63) is indistinguishable from recovering the blocks if the induced 2-block model, which replaces the connection probabilities by the within-mega-community average. This is an unexpected robustness result for mis-specified SBM models.

On the other hand, as mentioned earlier, we want to investigate the possibility that the finest communities cannot be recovered but higher-level mega-communities can. To investigate this, we first derive the necessary condition for the exact recovery of \( K \) communities in the leaf nodes. This is a simple consequence of the existing results on general SBMs [e.g. Abbe and Sandon, 2015].

**Lemma 6.6.** No algorithm can recover all of \( K \) communities in the leaf nodes of a BTSBM, that is either assortative or dis-assortative, with high probability if\[ |\sqrt{a_0} - \sqrt{a_1}| < \sqrt{K}. \]  
(75)
Note that \( \bar{a}_3 = a_0 \). The necessary condition (75) is essentially the negation of (72) with \( \ell = d \) and \( r = 1 \). If \( |\sqrt{a_0} - \sqrt{a_1}| < \sqrt{K} \), Lemma 6.6 implies that the finest communities cannot be recovered exactly by any algorithm, including HCD algorithms. However, if (72) holds for \( \ell = d - 1 \), we may recover all all mega-communities up to the second last layers. This is true, for instance, if \( a_0 \approx a_1 \) and
\[ |\sqrt{\bar{a}_r} - \sqrt{a_r}| > \sqrt{2^{d-r+1}}, \quad r = d, d-1, \ldots, 2, \]
where
\[ a'_r = a_1 + \sum_{j=1}^{r-1} 2^{j-1} a_j. \]

Therefore, Theorem 6.5 provides a precise characterization of the partial exact recovery phenomenon.

7 Extensions

7.1 Random matrices with other dependency structure

In Section 3.2 we discuss the unnormalized Laplacian as an example of random matrices with dependent entries. From assumption A1, it is not hard to see that our generic bounds allow much more flexible dependency structure. As shown in part (b) of Proposition 2.1, A1 is satisfied if the rows are m-dependent. If we can further derive bounds for \( \|E\|_{\op} \) and \( E_k^T W \) as in A2 and A3, Theorem 2.3 would yield an \( \ell_2 \to \infty \) perturbation bound.

Concentration inequalities for both quantities have been investigated for various dependency structures. We consider a slightly artificial one, motivated by Paulin [2012], just to illustrate the possibility to handle complex dependency structure. In particular, we assume that \( A_{ij} \)'s can be partitioned into M subsets such that the entries within the same block are independent while the blocks can be arbitrarily dependent. In this case, \( E \) can be decomposed as the sum of M matrices \( \{E^{(\ell)} : \ell \in [M]\} \) where \( E^{(\ell)}_{ij} = E_{ij} \) if \( (i, j) \) belongs to the \( \ell \)-th block and \( E^{(\ell)}_{ij} = 0 \) otherwise. By Lemma 3.2 and a union bound,
\[ \lambda_-(\delta), E_+(\delta), \tilde{E}_+(\delta), E_\infty(\delta) \preceq E_2(\delta) \preceq M \left( \sqrt{n \bar{p}^*} + \sqrt{\log(nM/\delta)} \right). \]

On the other hand, we apply the same decomposition on \( E_k^T W \) and Lemma 3.1 implies that
\[ b_\infty(\delta) \preceq \frac{MR(\delta/M)}{\alpha \log R(\delta/M)}, \quad b_2(\delta) \preceq \frac{M \sqrt{p^*} R(\delta/M)^{(1+\alpha)/2}}{\alpha \log R(\delta/M)}. \]

Given these bounds, it is a simple exercise to derive the condition on \( \Delta^* \) through A4 as well as the \( \ell_2 \to \infty \) bound for \( d_2 \to \infty(U, AU^*(\Lambda^*)^{-1}) \) and \( d_2 \to \infty(U, U^*) \) by Theorem 2.3 and Theorem 2.4. It is also straightforward to derive the bounds for the unnormalized Laplacian using the results in Section 2.3.

The above case is by no means the end of the story. More complicated dependency structures can be handled similarly using more delicate bounds [e.g. Paulin, 2012]. The punchline is that our theory reduces the less tractable \( \ell_2 \to \infty \) perturbation bound to the more tractable concentration bounds on \( E \).

7.2 Non-binary random matrices

Our result can also be easily extended to non-binary random matrices. For instance, for Gaussian random matrices with \( A_{ij} \overset{\text{indep.}}{\sim} N(\mu_{ij}, \sigma^2_{ij}) \), Corollary 3.9 of Bandeira and Van Handel [2016] implies that
\[ E_2(\delta) \preceq \bar{\sigma}^* \sqrt{n} + \sigma^* \sqrt{\log \left( \frac{n}{\delta} \right)}, \quad \text{where} \quad \bar{\sigma}^* = \max_i \sqrt{\frac{1}{n} \sum_{j=1}^n \sigma_{ij}^2}, \quad \sigma^* = \max_i \sigma_{ij}. \]

This establishes the bounds for quantities in assumption A2. By Propostion 2.1,
\[ L_1(\delta) \leq \|A^*\|_{2 \to \infty} + E_2(\delta), \quad L_2(\delta) \leq 1, \quad L_3(\delta) \leq (\|A^*\|_{2 \to \infty} + E_2(\delta))/\lambda_{\text{min}}^*. \]
On the other hand, for any $W \in \mathbb{R}^{n \times r'}$ with $r' \leq r$, $E_k^TW \sim N(0, W^TD_kW)$ where $D_k = \text{diag}(\sigma_{k_j}^2)_{j=1}^n$. Thus, $E_k^TW \overset{d}{=} (W^TD_kW)^{1/2}X$ where $X \sim N(0, I_{r'})$. Since the mapping $f : x \mapsto ||((W^TD_kW)^{1/2})x||_2$ is $||W^TD_kW||_\text{op}$-Lipschitz and $||W^TD_kW||_\text{op} \leq \sigma^*||W||_\text{op}$, by Gaussian concentration inequality,

$$
P \left( ||E_k^TW||_2 \geq E||E_k^TW||_2 + t \right) \leq \exp\left\{ -\frac{t^2}{\sigma^2||W||_\text{op}^2} \right\}.
$$

Taking $t = \sigma^*||W||_\text{op}\sqrt{\log(n/\delta)}$, we know that

$$
||E_k^TW||_2 \leq E||E_k^TW||_2 + t \leq \sqrt{E||E_k^TW||^2_2} + t \leq \sqrt{\text{tr}(W^TD_kW)} + t
\leq ||W||_\text{op}\sigma^* \left( \sqrt{r} + \sqrt{\log(n/\delta)} \right) \leq \sigma^* R(\delta)||W||_\text{op}.
$$

As a result, we have $b_\infty(\delta) = 0$ and $b_2(\delta) \leq \sigma^* R(\delta)$ in Assumption A3. As commented in Remark 2.2,

$$
||EU^*||_{2\to\infty} \leq \sigma^* R(\delta).
$$

Putting pieces together, we can derive the $\ell_{2\to\infty}$ bound in this case by Theorem 2.3 and Theorem 2.4.

Following the same strategy, we can extend the results to other entry distributions. The bound for $||E||_\text{op}$ can be found in Bandeira and Van Handel [2016], Latala et al. [2018], Rebrova [201] for sub-gaussian, sub-exponential, heavy-tailed, symmetric random variables. The row-wise concentration inequality can be obtained from standard moment generating function arguments [e.g. Vershynin, 2010].

### 7.3 Asymmetric random matrices

Our perturbation bound can be extended to asymmetric and rectangular matrices using the Hermitian dilation trick [e.g. Paulsen, 2002]. Given a rectangular matrix $A \in \mathbb{R}^{m \times n}$ with $m \geq n$, the Hermitian dilation of $A$ is defined as

$$
\tilde{A} = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}.
$$

Let $\tilde{U}\tilde{\Sigma}\tilde{V}^T$ be the singular value decomposition (SVD) of $\tilde{A}$ where $\tilde{U} \in \mathbb{R}^{m \times n}, \tilde{V} \in \mathbb{R}^{n \times n}$ are two orthogonal matrices and $\tilde{\Sigma} \in \mathbb{R}^{n \times n}$ is a diagonal matrix. Then $\tilde{U}\tilde{\Sigma}\tilde{U}^T$ is the SVD of $\tilde{A}$ where

$$
\tilde{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{U} \\ \tilde{V} \end{bmatrix}, \quad \tilde{\Sigma} = \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}.
$$

Given a pair of asymmetric matrices $A, A^* \in \mathbb{R}^{m \times n}$, their left singular spaces $U, U^*$ and corresponding right singular spaces $V, V^*$, our bound can be applied to their Hermitian dilation to yield a bound for

$$
d_{2\to\infty}\left( \begin{bmatrix} U \\ V \end{bmatrix}, \begin{bmatrix} U^* \\ V^* \end{bmatrix} \right).
$$

This provides an upper bound for both $d_{2\to\infty}(U, U^*)$ and $d_{2\to\infty}(V, V^*)$.

### 7.4 Perturbation in other metrics

Given an $\ell_{2\to\infty}$ bound, we can derive the perturbation bound in other metrics. One example is the $\ell_{2\to\infty}$ bound for projection matrices, namely $||UU^T - U^*(U^*)^T||_{2\to\infty}$, which is studied in Mao et al. [2017]. Note
that for any $O \in \mathcal{O}^r$, 

$$
UU^T - U^*(U^*)^T = U(O(O)^T - U^*(U^*)^T = (U(O - U^*)((O)^T + U^*(O - U^*)^T).
$$

Then

$$
\|(U(O - U^*)((O)^T\|_{2 \to \infty} \leq \|U(O - U^*)\|_{2 \to \infty}\|O\|_{op} = \|U(O - U^*)\|_{2 \to \infty},
$$

and

$$
\|U^*(O(O - U^*)^T\|_{2 \to \infty} \leq \|U^*\|_{2 \to \infty}\|O - U^*\|_{op} \leq \sqrt{n}\|U^*\|_{2 \to \infty}\|O - U^*\|_{2 \to \infty}.
$$

Taking $O$ as the orthogonal matrix that minimizes $\|U(O - U^*)\|_{2 \to \infty}$, we conclude that

$$
\|UU^T - U^*(U^*)^T\|_{2 \to \infty} \leq (\sqrt{n}\|U^*\|_{2 \to \infty} + 1)\|U(O - U^*)\|_{2 \to \infty} \leq \sqrt{n}\|U^*\|_{2 \to \infty}\|U(O - U^*)\|_{2 \to \infty}.
$$

Another example is the entrywise bound for $UU^T - U^*(U^*)^T$. For any $O \in \mathcal{O}^r$, we have

$$
UU^T - U^*(U^*)^T = (U(O - U^*)((O)^T - U^*(O - U^*)^T + (U(O - U^*)((O)^T).
$$

Taking $O \in \mathcal{O}^r$ that minimizes $\|U(O - U^*)\|_{2 \to \infty}$ and using the fact that $\|AB\|_{max} \leq \|A\|_{2 \to \infty}\|B\|_{2 \to \infty}$, we have

$$
\|UU^T - U^*(U^*)^T\|_{max} \leq d_{2 \to \infty}(U, U^*)^2 + \|U^*\|_{2 \to \infty}d_{2 \to \infty}(U, U^*).
$$

Finally, we can derive an entry-wise bound for $U(AU^T - U^*(U^*)^T$, which is of interest if the goal is to recover the low-rank component. Similar to the derivation for projection matrices, by Weyl’s inequality,

$$
\|U(AU^T - U^*(U^*)^T\|_{max} \leq d_{2 \to \infty}(U, U^*)\|U^*\|_{2 \to \infty}\lambda_{max} + \|U^*\|_{2 \to \infty}\|E\|_{op}
$$

$$
+ d_{2 \to \infty}(U, U^*)^2\lambda_{max} + d_{2 \to \infty}(U, U^*)^2\|E\|_{op}.
$$

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A  Proof of Theorem 2.3

The proof is very involved, so we split the proof into six steps.

A.1  Notation

Let $O^r$ denote the space of all $r \times r$ orthogonal matrices and $1_n$ denote a $n$-dimensional vector with all entries 1. For any vector $x$, let $\|x\|_p$ denotes its $p$-norm. For any matrix $M$, denote by $M^T_k$ the $m$-th row of $M$, by $\|M\|_op$ its operator norm and by $\|M\|_F$ its Frobenius norm. Moreover, for any $p, q \in [1, \infty]$, let

$$\|M\|_{p \rightarrow q} = \sup_{\|\omega\|_p = 1} \|M\omega\|_q.$$  

In particular,  

$$\|M\|_{2 \rightarrow \infty} = \max_k \|M_k\|_2.$$  

Suppose $U \Sigma V^T$ is the singular value decomposition of $M$. When $M$ is a square matrix, we define the matrix sign as  

$$\text{sign}(M) = UV^T.$$  

By definition, $\text{sign}(M)$ is orthogonal. When $n = 1$, $M$ is a scalar and $\text{sign}(M)$ reduces to the classical sign of scalars. Further we denote by $\lambda_{\text{max}}(M)$ (resp. $\lambda_{\text{min}}(M)$) the largest (resp. the smallest) eigenvalue of $M$ in absolute values and by $\kappa(M)$ the condition number $\lambda_{\text{max}}(M)/\lambda_{\text{min}}(M)$. We say a square matrix $M$ positive semi-definite (psd) if all eigenvalues of $M$ are non-negative. In particular, we write $\lambda_{\text{max}}(\Lambda^*_{\ast})$ (resp. $\lambda_{\text{min}}(\Lambda^*_{\ast})$) as $\lambda^*_{\text{min}}$ (resp. $\lambda^*_{\text{max}}$) and $\kappa(\Lambda^*_{\ast})$ as $\kappa^*$ for short.

For any matrices $U, Z \in \mathbb{R}^{n \times r}$ with orthonormal columns, let $\Theta$ denote the principal angle matrix between the two subspaces spanned by $U$ and $Z$, such that $U^T Z$ has the singular value decomposition $U^T Z = \bar{U}(\cos(\Theta))\bar{V}^T$ where $\Theta = \text{diag}(\theta_1, \ldots, \theta_r)$ with $\theta_j \in [0, \frac{\pi}{2}]$.

For any Hermitian matrices $B_1, B_2 \in \mathbb{R}^{n \times n}$, let $\lambda_1(B_j) \geq \lambda_2(B_j) \geq \ldots \geq \lambda_n(B_j)$ be the eigenvalues of $B_j (j = 1, 2)$. Let  

$$\text{sep}_{s+1, s+r}(B_1, B_2) = \min\{|\lambda_i(B_1) - \lambda_j(B_2)| : i \notin \{s + 1, \ldots, s + r\}, j \in \{s + 1, \ldots, s + r\}\}.$$  

Note that $\text{sep}_{s+1, s+r}$ is not symmetric in the sense that $\text{sep}_{s+1, s+r}(B_1, B_2) \neq \text{sep}_{s+1, s+r}(B_2, B_1)$. When $B_1 = B_2 = B$, we write it as $\text{sep}_{s+1, s+r}(B)$ for short.

A.2  Preparation: preliminary properties

When $r = 1$, $U$ and $U^*$ are vectors and it is straightforward to show that  

$$d_{2 \rightarrow \infty}(U, U^*) = \|U \text{sign}(U^T U^*) - U^*\|_{2 \rightarrow \infty}.$$  

This motivates us to consider an upper bound of $d_{2 \rightarrow \infty}(U, U^*)$ as  

$$d_{2 \rightarrow \infty}(U, U^*) \leq \|U \text{sign}(H) - U^*\|_{2 \rightarrow \infty},$$  

where  

$$H = U^T U^*.$$
Similarly for distance between $U$ and $AU^*(A^*)^{-1}$, we consider the upper bound
\[
d_{2\to\infty}(U, AU^*(A^*)^{-1}) \leq \|U\text{sign}(H) - AU^*(A^*)^{-1}\|_{2\to\infty}. \tag{78}
\]
This was also considered in Abbe et al. [2017]. Our goal is to derive upper bounds for (78) and (76).

Finally, let $A^{(1)}, \ldots, A^{(n)}$ be $n$ auxiliary matrices that satisfy the following condition, as the deterministic analogue of assumption $A1$ with $S = [r]$.

**C0** There exists $L_1, L_2, L_3$ such that for all $k$,
\[
\|A^{(k)} - A\|_{\text{op}} \leq L_1, \quad \frac{\|(A^{(k)} - A)U\|_{\text{op}}}{\lambda_{\min}^*} \leq (\kappa^* L_2 + L_3) \|U\|_{2\to\infty}.
\]

Similarly we define $\Lambda^{(k)}$ as the diagonal matrix given by the $(s + 1)$-th to the $(s + r)$-th largest eigenvalues and $U^{(k)} \in \mathbb{R}^{n \times r}$ as a matrix of eigenvectors corresponding to $\Lambda^{(k)}$ i.e.
\[
A^{(k)}U^{(k)} = U^{(k)}\Lambda^{(k)}.
\]

Further let
\[
H^{(k)} = (U^{(k)})^TU^*.
\]

The following proposition provides a simple yet important property of eigen-separation.

**Proposition A.1.** For any Hermitian matrices $B_1, B_2 \in \mathbb{R}^{n \times n}$,
\[
\text{sep}_{s + 1, s + r}(B_1, B_2) \geq \max\{\text{sep}_{s + 1, s + r}(B_1), \text{sep}_{s + 1, s + r}(B_2)\} - \max_{i \in [s + 1, s + r]} \|\lambda_i(B_1) - \lambda_i(B_2)\|.
\]

**Proof.** For any $i, j$,
\[
|\lambda_i(B_1) - \lambda_j(B_2)| \geq |\lambda_i(B_2) - \lambda_j(B_2)| - |\lambda_i(B_1) - \lambda_i(B_2)|.
\]
The proof is completed by considering all pairs of $i$ and $j$. \qed

Based on Proposition A.1, we can derive the eigen-separation among $A$, $A^*$ and $A^{(k)}$.

**Lemma A.2.** Let $E$ be defined as in (1). Under condition $C0$,
\[
\text{sep}_{s + 1, s + r}(A, A^*) \geq \text{sep}_{s + 1, s + r}(A^*) - \|\Lambda - \Lambda^*\|_{\text{max}},
\]
and for any $k$,
\[
\text{sep}_{s + 1, s + r}(A^{(k)}, A) \geq \text{sep}_{s + 1, s + r}(A^*) - \|A^{(k)} - A\|_{\text{op}}.
\]

**Proof.** The first part is a direct result of Proposition A.1. By definition,
\[
\|A^{(k)} - A\|_{\text{op}} \leq L_1.
\]
The second part is then proved by noting that
\[
\text{sep}_{s + 1, s + r}(A^{(k)}, A) \geq \text{sep}_{s + 1, s + r}(A) - \|A^{(k)} - A\|_{\text{op}}
\]
where the last inequality uses Weyl’s inequality and
\[
\text{sep}_{s + 1, s + r}(A) \geq \text{sep}_{s + 1, s + r}(A, A^*) - \|\Lambda - \Lambda^*\|_{\text{max}} \geq \text{sep}_{s + 1, s + r}(A^*) - 2\|\Lambda - \Lambda^*\|_{\text{max}}.
\]
\qed
Recall the definition of $\Delta^*$ in (4) and let

$$
\Gamma = \frac{\Delta^* - L_1}{2}.
$$

(79)

Note that the first term of $\Gamma$ is essentially the half eigen-gap if $0$ is an eigenvalue but not in $\Lambda^*$. Under assumption $A4$, $\Gamma$ has the same order as $\Delta^*$. Throughout the rest of this section, we assume the following condition:

C1. $\|\Lambda - \Lambda^*\|_{\text{max}} \leq \Gamma/2$ where $E$ is defined in (1) and $\Gamma$ is defined in (79).

**Corollary A.3.** Under condition C0 and C1,

$$
\min\{\text{sep}_{s+1,s+r}(A, A^*), \text{sep}_{s+1,s+r}(A^{(k)}, A)\} \geq \Gamma.
$$

The above results on eigen-gaps allow us to apply Davis-Kahan Theorem [Davis and Kahan, 1970] to bound the discrepancy between the eigenspaces of $A$ and $A^*$. In particular, we use the following version of Davis-Kahan Theorem.

**Proposition A.4.** [Chap. V, Theorem 3.6 Stewart, 1990] For any Hermitian matrix $B \in \mathbb{R}^{n \times n}$, $M \in \mathbb{R}^{r \times r}$ and any matrix $Z \in \mathbb{R}^{n \times r}$ with orthonormal columns, let $B$ have the spectral decomposition

$$
\begin{bmatrix}
U^T \\
\tilde{U}^T
\end{bmatrix}
B
\begin{bmatrix}
U \\
\tilde{U}
\end{bmatrix} =
\begin{bmatrix}
\Lambda & 0 \\
0 & \tilde{\Lambda}
\end{bmatrix}.
$$

Assume that there exists some $\omega > 0$ and $a, b \in \mathbb{R}$,

$$
eig(M) \subset [a, b], \quad \eig(\tilde{\Lambda}) \subset \mathbb{R} \setminus [a - \omega, b + \omega],
$$

where $\eig(\cdot)$ denote the set of all eigenvalues. Further let

$$R = BZ - ZM$$

and $\Theta$ be the principal angle matrix between $U$ and $Z$. Then for any unitarily invariant norm $\| \cdot \|$, 

$$
\|\sin \Theta\| \leq \frac{\|R\|}{\omega}.
$$

**Proposition A.5.** [Chap. I, Theorem 5.5 Stewart, 1990] Let $\Theta$ be the principal angle (matrix) between $U \in \mathbb{R}^{n \times r}$ and $Z \in \mathbb{R}^{n \times r}$, then

$$
\|UU^T - ZZ^T\|_{\text{op}} = \|\sin \Theta\|_{\text{op}}.
$$

**Lemma A.6.** Let $E$ be defined as in (1). Under condition C0 and C1,

$$
\|UU^T - U^*(U^*)^T\|_{\text{op}} \leq \frac{\|EU^*\|_{\text{op}}}{\Gamma},
$$

(80)

and for any $k$,

$$
\|U^{(k)}(U^{(k)})^T - UU^T\|_{\text{op}} \leq \frac{\lambda_{\min}^2(k^*L_2 + L_3)}{\Gamma} \|U\|_{2 \rightarrow \infty}.
$$

(81)

**Proof.** First let $B = A, Z = U^*, M = \Lambda^*$ in Proposition A.4. Then by Proposition A.4 and Proposition A.5,

$$
\|UU^T - U^*(U^*)^T\|_{\text{op}} \leq \frac{\|AU^* - U^*\Lambda^*\|_{\text{op}}}{\text{sep}_{s+1,s+r}(A, A^*)} = \frac{\|AU^* - A^*U^*\|_{\text{op}}}{\text{sep}_{s+1,s+r}(A, A^*)} = \frac{\|EU^*\|_{\text{op}}}{\text{sep}_{s+1,s+r}(A, A^*)}.
$$

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The proof of the first part is completed by Corollary A.3.

For the second part, let $B = A^{(k)}, Z = U, M = \Lambda$. Then by Proposition A.5 and Proposition A.5,

$$
\|U^{(k)}(U^{(k)})^T - UU^T\|_{op} \leq \frac{\|A^{(k)}U - U\Lambda\|_{op}}{\text{sep}_{s+1,s+r}(A^{(k)}, A)} = \frac{\|A^{(k)} - A\|_{op}}{\text{sep}_{s+1,s+r}(A^{(k)}, A)}.
$$

The proof is completed by condition C0 and Corollary A.3. \hfill \Box

## A.3 Step I: a preliminary deterministic bound

Throughout this subsection we assume that all eigenvalues are of the same sign, i.e. $\lambda_{s+1}^* \lambda_{s+r}^* > 0$. In step V we deal with the general case.

**Lemma A.7.** Assume that $\lambda_{s+1}^* \lambda_{s+r}^* > 0$. Under condition C0 and C1,

$$
\|(U\text{sign}(H) - AU^*(\Lambda^*)^{-1})\|_{2 \to \infty} \leq \beta \|U\|_{2 \to \infty} + \frac{\|A^*(UH - U^*)\|_{2 \to \infty}}{\lambda_{\min}^*} + \max_{k} \frac{\|E_k^T(U^{(k)}H^{(k)} - U^*)\|_{2}}{\lambda_{\min}^*},
$$

where

$$
\beta \triangleq \frac{EU^*_{op}^2}{\Gamma^2} + \frac{\|EU^*\|_{op}}{\lambda_{\min}^*} + \frac{\|E\|_{2 \to \infty}(\kappa^* L_2(\delta) + L_3(\delta))}{\Gamma}.
$$

**Proof.** Without loss of generality we assume that $\lambda_{s+1}^* \geq \lambda_{s+r}^* > 0$. Otherwise we replace $A$ (resp. $A^*, \Lambda, \Lambda^*$) by $-A$ (resp. $-A^*, -\Lambda, -\Lambda^*$).

Applying the triangle inequality, we have

$$
\|(U\text{sign}(H) - AU^*(\Lambda^*)^{-1})\|_2 \leq \|U_k^T(\text{sign}(H) - H)\|_2 + \|UH - AU^*(\Lambda^*)^{-1}\|_2.
$$

The second term can be further bounded as follows:

$$
(UH - AU^*(\Lambda^*)^{-1})_k^T
= (U\Lambda^*(\Lambda^*)^{-1} - U\Lambda^*(\Lambda^*)^{-1} + (U\Lambda^*(\Lambda^*)^{-1} - AU^*(\Lambda^*)^{-1} - EU^*(\Lambda^*)^{-1} - AU^*(\Lambda^*)^{-1} + EU^*(\Lambda^*)^{-1} - AU^*(\Lambda^*)^{-1} - EU^*(\Lambda^*)^{-1} + EU^*(\Lambda^*)^{-1}) - (U\Lambda^*(\Lambda^*)^{-1} - AU^*(\Lambda^*)^{-1} + EU^*(\Lambda^*)^{-1} - AU^*(\Lambda^*)^{-1} + EU^*(\Lambda^*)^{-1} - AU^*(\Lambda^*)^{-1} + EU^*(\Lambda^*)^{-1})
\begin{align*}
\leq & \left\{ U_k^T((H\Lambda^* - \Lambda H) - (U\Lambda^*(\Lambda^*)^{-1} - AU^*(\Lambda^*)^{-1} + EU^*(\Lambda^*)^{-1} - AU^*(\Lambda^*)^{-1} + EU^*(\Lambda^*)^{-1} - AU^*(\Lambda^*)^{-1} + EU^*(\Lambda^*)^{-1}) \right\} (\Lambda^*)^{-1} \\
= & \left\{ U_k^T((H\Lambda^* - \Lambda H) + EU^*(\Lambda^*)^{-1} - AU^*(\Lambda^*)^{-1} + EU^*(\Lambda^*)^{-1} - AU^*(\Lambda^*)^{-1} + EU^*(\Lambda^*)^{-1} - AU^*(\Lambda^*)^{-1} + EU^*(\Lambda^*)^{-1}) \right\} (\Lambda^*)^{-1} \\
= & \left\{ U_k^T((H\Lambda^* - \Lambda H) + EU^*(\Lambda^*)^{-1} - AU^*(\Lambda^*)^{-1} + EU^*(\Lambda^*)^{-1} - AU^*(\Lambda^*)^{-1} + EU^*(\Lambda^*)^{-1} - AU^*(\Lambda^*)^{-1} + EU^*(\Lambda^*)^{-1}) \right\} (\Lambda^*)^{-1} \\
\end{align*}
$$

where (i) uses the fact that $U\Lambda = AU$. Applying the triangle inequality again we obtain that

$$
\|(UH - AU^*(\Lambda^*)^{-1})_k\|_2 \leq \frac{1}{\lambda_{\min}^*} \left\{ \|U_k^T(\text{sign}(H) - H)\|_2 + \|E_k^T(UH - U^{(k)}H^{(k)})\|_2 \\
+ \|E_k^T(U^{(k)}H^{(k)} - U^*)\|_2 + \|(AU^*_k)^T(UH - U^*)\|_2 \right\}.
$$
We will derive bounds for $J_1$, $J_2$ and $J_3$ separately in the rest of the proof.

**Step 1: Bounding $J_1$.** Let $H$ have the singular value decomposition $H = \hat{U}(\cos \Theta)\hat{V}^T$. Then

$$\|H - \text{sign}(H)\|_{\text{op}} = \|\hat{U}(I - \cos \Theta)\hat{V}^T\|_{\text{op}} \leq \|I - \cos \Theta\|_{\text{op}}.$$  

For any $\theta \leq \pi/2$,

$$1 - \cos \theta \leq 1 - \cos^2 \theta = \sin^2 \theta.$$  

Thus,

$$\|I - \cos \Theta\|_{\text{op}} \leq \|\sin \Theta\|_{\text{op}}^2.$$  

By Proposition A.5,

$$\|\sin \Theta\|_{\text{op}}^2 = \|UU^T - U^*(U^*)^T\|_{\text{op}}^2.$$  

Finally by Lemma A.6 we obtain that

$$J_1 \leq \|U_k\|_2 \|H - \text{sign}(H)\|_{\text{op}} \leq \frac{\|EU^*\|_{\text{op}}^2}{\Gamma^2} \|U_k\|_2 \leq \frac{\|EU^*\|_{\text{op}}^2}{\Gamma^2} \|U\|_{2 \to \infty}.$$  

(85)

**Step 2: Bounding $J_2$.** By definition, $U^T A = (A^T U)^T = (AU)^T = (UA^T) = \Lambda U^T$ and $U^* A^* = A^* U^*$. Thus,

$$HA^* - \Lambda H = \Lambda U^T U^* - U^T U^* \Lambda = U^T AU^* - U^T A^* U^* = U^T EU^*.$$  

Since $U$ and $U^*$ have orthonormal columns,

$$\|HA^* - \Lambda H\|_{\text{op}} \leq \|EU^*\|_{\text{op}}.$$  

Thus,

$$J_2 \leq \|EU^*\|_{\text{op}} \|U_k\|_2 \leq \|EU^*\|_{\text{op}} \|U\|_{2 \to \infty}.$$  

(86)

**Step 3: Bounding $J_3$.** Since $H^{(k)} = (U^{(k)})^T U^*$ and $U^*$ has orthonormal columns,

$$\|U^{(k)} H^{(k)} - U H\|_{\text{op}} = \|U^{(k)} (U^{(k)})^T U^* - U U^T U^*\|_{\text{op}} \leq \|U^{(k)} (U^{(k)})^T - U U^T\|_{\text{op}}.$$  

(87)

By Lemma A.6,

$$J_3 \leq \frac{\|E_k\|_2 \lambda_{\text{min}}^*(\kappa^* L_2 + L_3)}{\Gamma} \|U_k\|_2 \leq \frac{\|E\|_{2 \to \infty} \lambda_{\text{min}}^*(\kappa^* L_2 + L_3)}{\Gamma} \|U\|_{2 \to \infty}.$$  

(88)

The proof is then completed by combining (85), (86) and (88).

**A.4 Step II: deterministic bound for $\|A^*(UH - U^*)\|_2$ via Kato’s integral**

**Lemma A.8.** Assume that $\lambda_{s+1}^* \lambda_{s+r}^* > 0$. Under condition $C1$,

- It always holds that

$$\|A^*(UH - U^*)\|_{2 \to \infty} \leq \frac{\|EU^*\|_{\text{op}} \|A^*\|_{2 \to \infty}}{\Gamma}.$$  

(89)
• If $A^*$ is positive semidefinite, then
\[
\|A^*(UH - U^*)\|_{2 \to \infty} \leq 3.61 \frac{\|E \bar{U}^*\|_{\text{op}} \sqrt{\lambda_{\text{max}}}}{\Gamma} A^\star \|_{\text{max}}.
\]

• If $A^*$ is low-rank (with rank $K$) with $\bar{U}^*$ being defined in (6) in page 8, then
\[
\|A^*(UH - U^*)\|_{2 \to \infty} \leq 3.84 \frac{\|E \bar{U}^*\|_{\text{op}} \lambda^*}{\Gamma} \|\bar{U}^\star\|_{2 \to \infty}.
\]

Proof. First we notice that
\[
\|A^*(UH - U^*)\|_{2 \to \infty} = \|A^*(UU^T - U^*(U^*)^T)U^\star\|_{2 \to \infty} \leq \|A^*(UU^T - U^*(U^*)^T)\|_{2 \to \infty}.
\]

We derive bounds for $\|A^*(UU^T - U^*(U^*)^T)\|_{2 \to \infty}$ in each case separately.

Case 1: $A^*$ has no constraint

By Lemma A.6,
\[
\|UU^T - U^*(U^*)^T\|_{\text{op}} \leq \frac{\|E \bar{U}^*\|_{\text{op}}}{\Gamma}.
\]

Thus,
\[
\|A^*(UU^T - U^*(U^*)^T)\|_{2 \to \infty} \leq \frac{\|UU^T - U^*(U^*)^T\|_{\text{op}} \|A^\star\|_{2 \to \infty}}{\Gamma} \leq \frac{\|E \bar{U}^*\|_{\text{op}} \|A^\star\|_{2 \to \infty}}{\Gamma}.
\]

Case 2: $A^*$ is positive semidefinite

Recall that $\bar{U}^* \bar{\Lambda}^* (\bar{U}^*)^T$ is the singular value decomposition of $A^*$. Then
\[
\|A^*(UU^T - U^*(U^*)^T)\|_{2 \to \infty} \leq \|\bar{U}^* (\bar{\Lambda}^*)^{1/2}\|_{2 \to \infty} (\bar{\Lambda}^*)^{1/2} (\bar{U}^*)^T (UU^T - U^*(U^*)^T) \|_{\text{op}}.
\]

Let $Q = \bar{U}^* (\bar{\Lambda}^*)^{1/2}$ with $Q_i$ being the $i$-th row. Then $A^* = QQ^T$ and hence
\[
\max_i \|Q_i\|_2 = \max_i \sqrt{A_{ii}^*} = \sqrt{\|A^\star\|_{\text{max}}}.
\]

As a result,
\[
\|A^*(UU^T - U^*(U^*)^T)\|_{2 \to \infty} \leq \sqrt{\|A^\star\|_{\text{max}}} (\bar{\Lambda}^*)^{1/2} (\bar{U}^*)^T (UU^T - U^*(U^*)^T) \|_{\text{op}}.
\]

Thus it is left to bound $(\bar{\Lambda}^*)^{1/2} \bar{U}^* (UU^T - U^*(U^*)^T) \|_{\text{op}}$.

WLOG, we assume that all eigenvalues are positive. For convenience, write
\[
a = \lambda^\star_{\text{min}}, \quad b = \lambda^\star_{\text{max}}, \quad h = \frac{\Gamma}{2} = \frac{\Delta^\star - L_1}{4}.
\]

Further we write
\[
a' = a - 2h, \quad b' = b + 2h.
\]

Note that
\[
h \leq \frac{1}{4} \Delta^\star = \frac{1}{4} \min \{ \text{sep}_{s+1,s+r}(A^\star), \lambda^\star_{\text{min}} \},
\]

and thus all eigenvalues of $A^\star$, as well as 0, are at least $2h$ apart from $a'$ and $b'$. Fix any $\gamma > 0$. Let $C$ be a positively oriented rectangular contour on the complex plane with corners $a' \pm \gamma \sqrt{-1}$ and $b' \pm \gamma \sqrt{-1}$.
Then all eigenvalues in $\Lambda^*$ are inside $\mathcal{C}$ while all other eigenvalues are outside $\mathcal{C}$. By Assumption C1, $\mathcal{C}$ also separates the eigenvalues in $\Lambda$. The famous Kato’s integral [Kato, 1949] implies that

$$UU^T = \frac{1}{2\pi\sqrt{-1}} \oint_{\mathcal{C}} (A - zI)^{-1}dz, \quad U^*(U^*)^T = \frac{1}{2\pi\sqrt{-1}} \oint_{\mathcal{C}} (A^* - zI)^{-1}dz.$$  

Noting that $C^{-1} - B^{-1} = -B^{-1}(C - B)C^{-1}$, we have

$$UU^T - U^*(U^*)^T = -\frac{1}{2\pi\sqrt{-1}} \oint_{\mathcal{C}} (A^* - zI)^{-1}E(A - zI)^{-1}dz.$$  

Let $A = \bar{U}\bar{\Lambda}\bar{U}^T$ be the SVD of $A$ where $\bar{U} \in \mathbb{R}^{n \times n}$ be an orthogonal matrix. Then

$$(\bar{\Lambda}^*)^{1/2}U^* (UU^T - U^*(U^*)^T) = -\frac{1}{2\pi\sqrt{-1}} (\bar{\Lambda}^*)^{1/2}U^* \oint_{\mathcal{C}} (A^* - zI)^{-1}E(A - zI)^{-1}dz$$

$$= -\frac{1}{2\pi\sqrt{-1}} (\bar{\Lambda}^*)^{1/2} (U^*)^T \oint_{\mathcal{C}} \bar{U}^*(\bar{\Lambda}^* - zI)^{-1}(\bar{U}^*)^T E\bar{U}(\bar{\Lambda} - zI)^{-1}\bar{U}^T dz$$

$$= -\frac{1}{2\pi\sqrt{-1}} \left( \oint_{\mathcal{C}} (\bar{\Lambda}^*)^{1/2}(\bar{\Lambda}^* - zI)^{-1}(\bar{U}^*)^T E\bar{U}(\bar{\Lambda} - zI)^{-1}dz \right) \bar{U}^T.$$  

Let $q(z)$ be the integrand and $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ be the four edges of $\mathcal{C}$, i.e.

$$\mathcal{C}_1 = \{a' + x\sqrt{-1} : x \in [-\gamma, \gamma]\}, \quad \mathcal{C}_2 = \{y - \gamma\sqrt{-1} : y \in [a', b']\},$$

$$\mathcal{C}_3 = \{b' + x\sqrt{-1} : x \in [-\gamma, \gamma]\}, \quad \mathcal{C}_4 = \{y + \gamma\sqrt{-1} : y \in [a', b']\}.$$  

Then

$$\| (\bar{\Lambda}^*)^{1/2}U^* (UU^T - U^*(U^*)^T) \|_{op} \leq \frac{1}{2\pi} \left( \left\| \oint_{\mathcal{C}_1} q(z)dz \right\|_{op} + \left\| \oint_{\mathcal{C}_2} q(z)dz \right\|_{op} + \left\| \oint_{\mathcal{C}_3} q(z)dz \right\|_{op} + \left\| \oint_{\mathcal{C}_4} q(z)dz \right\|_{op} \right).$$  

We note that Kato’s integral is also deployed by [Oliveira, 2009, Lemma A.2] and [Mao et al., 2017, Section 5]. However, they directly bound the above quantity by

$$\max_{z \in \mathcal{C}} |q(z)| \times (\text{the perimeter of } \mathcal{C}).$$

This turns out to be loose. Instead we will bound each term in (97) separately.

We start from $\left\| \oint_{\mathcal{C}_1} q(z)dz \right\|_{op}$. Since $\mathcal{C}_1$ is a vertical line with intercept $\alpha'$, we have

$$\left\| \oint_{\mathcal{C}_1} q(z)dz \right\|_{op} = \left\| \int_{\gamma}^{\gamma} q(\alpha' + x\sqrt{-1})dx \right\|_{op} \leq \int_{\gamma}^{\gamma} \| q(\alpha' + x\sqrt{-1}) \|_{op} dx.$$  

Because $\| \cdot \|_{op}$ is sub-multiplicative,

$$\| q(\alpha' + x\sqrt{-1}) \|_{op} \leq \| (\bar{\Lambda}^*)^{1/2} \|_{op} \| (\bar{\Lambda}^* - (\alpha' + x\sqrt{-1})I)^{-1} \|_{op} \| (\bar{U}^*)^T E\bar{U} \|_{op} \| (\bar{\Lambda} - (\alpha' + x\sqrt{-1})I)^{-1} \|_{op}$$

$$\leq \| E\bar{U}^* \|_{op} \max_{\mathcal{C}[n]} \frac{1}{\sqrt{(\lambda_1 - \alpha')^2 + x^2}} \max_{\mathcal{C}[n]} \sqrt{\frac{|\lambda_1^*|}{(\lambda_1^* - \alpha')^2 + x^2}}$$

(98)

We emphasize that the maximum is taken over all eigenvalues instead of just the eigenvalues in $\Lambda$ and $\Lambda^*$. By Assumption C1,

$$|\lambda_1 - \lambda_1^*| \leq \| \Lambda - \Lambda^* \|_{max} \leq \frac{\Gamma}{2} = h.$$  

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By construction, for any \(i\),
\[
|\lambda_i^* - a'| = |\lambda_i^* - a + 2h| \geq \min\{2h, \text{sep}_{s+1,s+r}(A^*) - 2h\} = 2h.
\]

By the triangle inequality we find that for any \(i\),
\[
|\lambda_i - a'| \geq h.
\]

Therefore,
\[
\max_{i \in [n]} \frac{1}{\sqrt{(\lambda_i^* - a')^2 + x^2}} \leq \frac{1}{\sqrt{h^2 + x^2}}.
\] (99)

On the other hand, let
\[
g(z; x, a') = \frac{|z + a'|}{\sqrt{z^2 + x^2}}.
\] (100)

Then
\[
\max_{i \in [n]} \sqrt{\frac{|\lambda_i^*|}{(\lambda_i^* - a')^2 + x^2}} \leq \max_{i \in [n]} \sqrt{g(\lambda_i^* - a'; x, a')} \max_{i \in [n]} \frac{1}{((\lambda_i^* - a')^2 + x^2)^{1/4}}
\]
\[
\leq \max_{i \in [n]} \sqrt{g(\lambda_i^* - a'; x, a')} \frac{1}{(h^2 + x^2)^{1/4}}
\]
\[
\leq \sup_{z: |z| \geq h} \sqrt{g(z; x, a')} (h^2 + x^2)^{1/4}.
\] (101)

where the last step uses the fact that \(|\lambda_i^* - a'| \geq 2h \geq h\). Now we study the properties of \(g(z; x, a')\). Note that when \(z \neq -a'\),
\[
\frac{d}{dz} (\log g(z; x, a')) = \frac{1}{z + a'} - \frac{z}{z^2 + x^2} = \frac{x^2 - a'z}{(z^2 + x^2)(z + a')}.
\] (102)

Since \(a' = a - 2h \geq 2h > 0\), \(g(z; x, a')\) is decreasing on \((-\infty, -a']\), increasing on \((-a', \frac{x^2}{a'})\) and decreasing on \([\frac{x^2}{a'}, \infty)\). As a result,
\[
\sup_{z: |z| \geq h} g(z; x, a') \leq \begin{cases} \max(g(-\infty; x, a'), g(\pm h; x, a')) & \left(\frac{x^2}{a'} \leq h\right) \\ g \left(\frac{x^2}{a'}; x, a'\right) & \left(\frac{x^2}{a'} > h\right) \end{cases} = \begin{cases} \frac{a' + h}{\sqrt{a' + \frac{x^2}{a'}}} & \left(\frac{x^2}{a'} \leq h\right) \\ \frac{a' + h}{\sqrt{a' + \frac{x^2}{a'}}} & \left(\frac{x^2}{a'} > h\right) \end{cases}.
\] (103)

When \(x^2/a' > h\), \(\frac{\sqrt{x^2 + x^2}}{x} \leq \sqrt{\frac{a' + h}{h}}\). Therefore
\[
\sup_{z: |z| \geq h} g(z; x, a') \leq \frac{a' + h}{\sqrt{h^2 + x^2}} I(|x| \leq \sqrt{a'h}) + \sqrt{\frac{a' + h}{h}} I(|x| > \sqrt{a'h}).
\] (104)

Putting (98), (99), (101) and (104) together, we obtain that
\[
\|q(a' + x\sqrt{-1})\|_{op} \leq \|E\hat{U}^*\|_{op} \frac{\sqrt{a' + h}}{h^2 + x^2} I(|x| \leq \sqrt{a'h}) + \frac{1}{(h^2 + x^2)^{3/4}} I(|x| > \sqrt{a'h}).
\] (105)

As a consequence,
\[
\left\| \int_{C_1} q(z) dz \right\|_{op} \leq \int_{-\infty}^{\infty} \|q(a' + x\sqrt{-1})\|_{op} dx
\]
Similarly,

$$\left\| \int_{C_3} q(z)dz \right\|_{\text{op}} \leq \frac{\| E\bar{U}^* \|_{\text{op}}}{h} \left( \pi \sqrt{b' + h} + \frac{4}{b'\sqrt{b}} \right).$$

Since $x \mapsto (x + h)^{1/2}$ is concave and also $a' < a \leq b < b'$, we have

$$\sqrt{a' + h} + \sqrt{b' + h} \leq \sqrt{a + h} + \sqrt{b + h} \leq 2\sqrt{b + h}. \quad (106)$$

Since $h \leq a/4 \leq b/4$, we have $h \leq a'/2$, $h \leq b'/6$. Then

$$\left\| \int_{C_1} q(z)dz \right\|_{\text{op}} + \left\| \int_{C_3} q(z)dz \right\|_{\text{op}} \leq \frac{\| E\bar{U}^* \|_{\text{op}}}{h} \left( 2\pi \sqrt{b' + h} + \frac{4}{a'\sqrt{a}} \right) + \frac{4}{b'\sqrt{b}}$$

$$\leq \frac{\| E\bar{U}^* \|_{\text{op}}}{h} \left( 2\pi \sqrt{b^* + \frac{4}{\sqrt{b}} \left( \frac{3}{2} \right)^{1/4} \sqrt{h} + 4 \frac{7}{6} \right)^{1/4} \right)$$

$$\leq \frac{\| E\bar{U}^* \|_{\text{op}}}{h} \left( 2\pi \sqrt{b^* + \frac{4}{\sqrt{b}} \left( \frac{3}{2} \right)^{1/4} \sqrt{h} + 4 \frac{7}{6} \right)^{1/4} \right)$$

$$\leq 11.32 \frac{\| E\bar{U}^* \|_{\text{op}}}{h} \sqrt{b}. \quad (107)$$

Note that (107) is independent of $\gamma$. For the integral on $C_2$, we use the crude bound that

$$\left\| \int_{C_2} q(z)dz \right\|_{\text{op}} \leq |C_2| \max_{x \in C_2} \| q(x) \|_{\text{op}} = (b' - a') \max_{x \in C_2} \| q(x) \|_{\text{op}}.$$ 

By (98), for any $y \in \mathbb{R}$,

$$\| q(y + \gamma \sqrt{-1}) \|_{\text{op}} \leq \frac{\| E\bar{U}^* \|_{\text{op}}}{\gamma^2} \sqrt{\Lambda^*}. \quad (99)$$

Letting $\gamma \to \infty$,

$$\left\| \int_{C_2} q(z)dz \right\|_{\text{op}} \to 0. \quad (108)$$

Putting (97), (107) and (108) together and recalling that $\Gamma = 2h$, we conclude that

$$\left\| (\Lambda^*)^{1/2} \bar{U}^* \left( UU^T - U^*(U^*)^T \right) \right\|_{\text{op}} \leq \frac{11.32 \| E\bar{U}^* \|_{\text{op}} \sqrt{b}}{2\pi h} \leq 3.61 \frac{\| E\bar{U}^* \|_{\text{op}} \sqrt{\Lambda_{\max}}}{\Gamma}.$$ 

The proof is then completed by (92).

**Case 3: $A^*$ is low rank**

Note that

$$\| A^* (UU^T - U^*(U^*)^T) \|_{2 \to \infty} \leq \| \bar{U}^* \|_{2 \to \infty} \| \Lambda^* (\bar{U}^*)^T (UU^T - U^*(U^*)^T) \|_{\text{op}}.$$ 

$$\| A^* (UU^T - U^*(U^*)^T) \|_{2 \to \infty} \leq \| \bar{U}^* \|_{2 \to \infty} \| \Lambda^* (\bar{U}^*)^T (UU^T - U^*(U^*)^T) \|_{\text{op}}.$$ 

(109)

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Thus it is left to bound $\|\tilde{A}*(\tilde{U}^*)^T (UU^T - U^*(U^*)^T)\|_{op}$.

Similar to (96), we deduce that

$$\tilde{A}^* \tilde{U}^* (UU^T - U^*(U^*)^T) = -\frac{1}{2\pi \sqrt{-1}} \left( \oint_{\lambda} \tilde{A}^*(\tilde{A}^* - zI)^{-1} \tilde{U}^* \tilde{U} \tilde{A}^* \tilde{U} (\tilde{A}^* - zI)^{-1} \right) \tilde{U}.$$  \hfill (110)

Let $\tilde{q}(z)$ be the integrand and $a', b', C_1, C_2, C_3, C_4$ and $g(z; x, a')$ be defined as in (100) in Case 2. Throughout this part we assume that $\gamma \geq h$. Similar to (98) and by (104),

$$\|\tilde{q}(a' + x\sqrt{-1})\|_{op} \leq \|E\tilde{U}^*\|_{op} \max_{i\in\mathbb{R}^2} \frac{1}{\sqrt{\lambda_i - a'^2 + x^2}} \max_{i\in\mathbb{R}^2} \frac{1}{\sqrt{\lambda_i - a'^2 + x^2}} \|\lambda_i\|$$

$$\leq \|E\tilde{U}^*\|_{op} \sup_{z:|z|\geq h} g(z; x, a')$$

$$\leq \|E\tilde{U}^*\|_{op} \frac{a' + h}{h^2 + x^2} I(|x| \leq \sqrt{a'h}) + \|E\tilde{U}^*\|_{op} \sqrt{\frac{a' + h}{h^2 + x^2}} I(|x| > \sqrt{a'h}).$$  \hfill (111)

As a consequence,

$$\left\| \int_{-\gamma}^{\gamma} \tilde{q}(a' + x\sqrt{-1}) \right\|_{op} dx$$

$$\leq \|E\tilde{U}^*\|_{op} (a' + h) \int_{|x| \leq \sqrt{a'h}} \frac{dx}{h^2 + x^2} + \|E\tilde{U}^*\|_{op} \sqrt{\frac{a' + h}{h}} \int_{\gamma \geq |x| > \sqrt{a'h}} \frac{dx}{h^2 + x^2}$$

$$= \frac{2(a' + h)}{h} \|E\tilde{U}^*\|_{op} \int_{0}^{\sqrt{a'h}} \frac{dy}{1 + y^2} + \frac{2\sqrt{a' + h}}{h} \|E\tilde{U}^*\|_{op} \int_{\gamma}^{1} \frac{dy}{1 + y^2}$$

$$\leq \frac{2(a' + h)}{h} \|E\tilde{U}^*\|_{op} \left( \pi(a' + h) + 2\sqrt{a' + h} \log(\gamma + \sqrt{1 + y^2}) \right)$$

$$= \frac{\|E\tilde{U}^*\|_{op}}{h} \left( \pi(a' + h) + 2\sqrt{a' + h} \log(\gamma + \sqrt{1 + \gamma^2/h^2}) - \log(1 + \sqrt{2}) \right).$$

Since $\gamma \geq h$, we have

$$\log(\gamma/h + \sqrt{1 + \gamma^2/h^2}) \leq \log(\gamma/h + \sqrt{2\gamma^2/h^2}) = \log\left(\frac{\gamma}{h}\right) + \log(1 + \sqrt{2}).$$

Thus we have

$$\left\| \int_{-\gamma}^{\gamma} \tilde{q}(a' + x\sqrt{-1}) \right\|_{op} dx \leq \frac{\|E\tilde{U}^*\|_{op}}{h} \left( \pi(a' + h) + 2\sqrt{a' + h} \log\left(\frac{\gamma}{h}\right) \right).$$

Similarly,

$$\left\| \int_{-\gamma}^{\gamma} \tilde{q}(a' + x\sqrt{-1}) \right\|_{op} dx \leq \frac{\|E\tilde{U}^*\|_{op}}{h} \left( \pi(b' + h) + 2\sqrt{b' + h} \log\left(\frac{\gamma}{h}\right) \right).$$

We recall (106) that

$$\sqrt{a' + h} + \sqrt{b' + h} \leq \sqrt{a + h} + \sqrt{b + h} \leq 2\sqrt{b + h}.$$

Also recalling that $h \leq a/4 \leq b/4$, we have that

$$\left\| \int_{-\gamma}^{\gamma} \tilde{q}(a' + x\sqrt{-1}) \right\|_{op} + \left\| \int_{-\gamma}^{\gamma} \tilde{q}(a' + x\sqrt{-1}) \right\|_{op} \leq \frac{\|E\tilde{U}^*\|_{op}}{h} \left( \pi(a' + b' + 2h) + 4\sqrt{b + h} \log\left(\frac{\gamma}{h}\right) \right).$$
Putting (113), (114) and (115) together, we have that
\[
\begin{align*}
\gamma = C & \quad \text{In summary, then (110), (112) and (116) together yield}\n\end{align*}
\]
To bound the integrals on \(C_2\) and \(C_4\), we use the same strategy as in Case 2,
\[
\left\| \oint_{C_2} \hat{q}(z)dz \right\|_\infty + \left\| \oint_{C_4} \hat{q}(z)dz \right\|_\infty \leq 2(b' - a') \max_{z \in C_2 \cup C_4} \| \hat{q}(z) \|_\infty
\]
\[
\leq 2b \max_{w \in [a', b]} \| \hat{q}(w + \gamma \sqrt{-1}) \|_\infty
\]
where the last step uses the fact that \(b' - a' = b - a + 4b \leq b\) and \(\hat{q}(w + \gamma \sqrt{-1}) = \hat{q}(w - \gamma \sqrt{-1})\). Then
\[
\| \hat{q}(w + \gamma \sqrt{-1}) \|_\infty \leq \| EU^* \|_{op} \max_{i \in [\pi]} \frac{1}{\sqrt{(\lambda_i^* - w)^2 + \gamma^2}} \max_{i \in [\pi]} \frac{|\lambda_i^*|}{\sqrt{(\lambda_i^* - w)^2 + \gamma^2}}
\]
\[
\leq \| EU^* \|_{op} \max_{i \in [\pi]} \frac{|\lambda_i^*|}{\gamma} \leq \frac{\| EU^* \|_{op} \sqrt{b^2 + \gamma^2}}{\gamma}.
\]
To bound the last term we distinguish two cases:

- If \(\lambda_i^* \in [a', b']\), then
  \[
  \sup_{w \in [a', b']} \frac{|\lambda_i^*|}{\sqrt{(\lambda_i^* - w)^2 + \gamma^2}} \leq g(|\lambda_i^*| - a'; \gamma, a') \leq \sup_{z} g(z; \gamma, a') \leq \frac{\sqrt{a'^2 + \gamma^2}}{\gamma}.
  \]

- When \(\lambda_i^* \in (b', \infty)\), using a similar argument as above, we obtain that
  \[
  \sup_{w \in [a', b']} \frac{|\lambda_i^*|}{\sqrt{(\lambda_i^* - w)^2 + \gamma^2}} \leq \frac{\sqrt{b'^2 + \gamma^2}}{\gamma}.
  \]

In summary,
\[
\max_{i \in [\pi]} \sup_{w \in [a', b']} \frac{|\lambda_i^*|}{\sqrt{(\lambda_i^* - w)^2 + \gamma^2}} \leq \frac{\sqrt{b'^2 + \gamma^2}}{\gamma}.
\]
Putting (113), (114) and (115) together, we have that
\[
\left\| \oint_{C_2} \hat{q}(z)dz \right\|_\infty + \left\| \oint_{C_4} \hat{q}(z)dz \right\|_\infty \leq 2b \| EU^* \|_{op} \sqrt{b'^2 + \gamma^2},
\]
Then (110), (112) and (116) together yield
\[
\| \tilde{A}^* \hat{U}^* (UU^T - U^*(U^*)^T) \|_{2 \to \infty} \leq \frac{b \| EU^* \|_{op} \sqrt{b'^2 + \gamma^2}}{2\pi h}
\]
Let \(\gamma = b\). Then
\[
\| \tilde{A}^* \hat{U}^* (UU^T - U^*(U^*)^T) \|_{2 \to \infty} \leq \frac{b \| EU^* \|_{op} \sqrt{b'^2 + b^2 \frac{b}{b^2} \frac{5}{6}}}{2\pi h}
\]
(117)
Let \( m(x) = x/\exp(x) \). Note that \( \frac{d}{dx}(\log m(x)) = \frac{1}{x} - 1 \). Thus \( m(x) \) reaches its maximum at \( x = 1 \). Then
\[
\sqrt{\frac{b}{h}} \log \left( \sqrt{\frac{b}{h}} \right) = m \left( \log \sqrt{\frac{b}{h}} \right) \leq m(1) = \frac{1}{e}.
\]
On the other hand,
\[
\frac{b^2 + b^2}{b} = \left( \frac{b + 2h}{b} \right)^2 + 1 \leq \frac{9}{4} + 1 = \frac{13}{4}.
\]
Thus, (117) implies that
\[
\|A^* U^* (UU^T - U^* (U^*)^T)\|_{op} \leq \frac{b \|EU^*\|_{op}}{2\pi h} \left( \frac{5\pi}{2} + \frac{8}{e} \sqrt{\frac{5}{4}} + \frac{1}{2} \sqrt{\frac{13}{4}} \right) \leq \frac{1.918b \|EU^*\|_{op}}{h} \leq \frac{3.84b \|EU^*\|_{op}}{\Gamma},
\]
where the last inequality uses the fact that \( h = \Gamma/2 \). The proof is then completed by (109).

**A.5 Step III: stochastic bound for \( \|E_k^T (U^{(k)} H^{(k)} - U^*)\|_2 \)**

From this subsection we will derive the stochastic bound by taking the randomness of \( A \) into account. We assume \textbf{A1} and \textbf{A3} hold. In particular, we choose \( A^{(1)}, \ldots, A^{(n)} \) that satisfy \textbf{A1} with the subset \( S = [r] \), i.e.
\[
d_{TV}(P_{(A_k, A^{(k)})}, P_{A_k} \times P_{A^{(k)}}) \leq \delta/n,
\]
and it holds with probability at least \( 1 - \delta \) that
\[
\|A^{(k)} - A\|_{op} \leq L_1(\delta), \quad \| (A^{(k)} - A)U \|_{op} \leq (\kappa^* L_2(\delta) + L_3(\delta)) \| U \|_{2 \rightarrow \infty},
\]
simultaneously for all \( k \). In addition, we re-define \( \Gamma \) as follows by setting \( L_1 = L_1(\delta) \),
\[
\Gamma(\delta) = \frac{\Delta^* - L_1(\delta)}{2}.
\]
We start from a concentration bound.

**Lemma A.9.** Given any \( \delta \in (0, 1) \) and \( W^{(k)} \in \mathbb{R}^{n \times r} \) that only depends on \( A^{(k)} \). Then under assumptions \textbf{A1} and \textbf{A3}, it holds simultaneously for all \( k \) that
\[
\| E_k^T W^{(k)} \|_2 \leq b_\infty(\delta) W^{(k)} \|_{2 \rightarrow \infty} + b_2(\delta) \| W^{(k)} \|_{op}
\]
with probability at least \( 1 - 2\delta \) where \( b_\infty(\delta), b_2(\delta) \) are defined in assumption \textbf{A3}.

**Proof.** Using the representation of total variation distance, there exists a coupling \( \hat{E}_k \) of \( E_k \) for each \( k \) such that
\( \hat{E}_k \) is independent of \( A^{(k)} \), \( \mathbb{P}(E_k \neq \hat{E}_k) \leq \delta/n \).

The lemma follows if we can prove that
\[
\| \hat{E}_k^T W^{(k)} \|_2 \leq b_\infty(\delta) W^{(k)} \|_{2 \rightarrow \infty} + b_2(\delta) \| W^{(k)} \|_{op}
\]
holds simultaneously for all \( k \) with probability at least \( 1 - \delta \). Since \( \hat{E}_k \) is independent of \( W^{(k)} \), the above inequality is guaranteed by assumption \textbf{A3}. \qed
Lemma A.10. Assume that $\lambda_{s+1} > 0$. Fix any $\delta \in (0,1)$. Under assumptions A1 and A3, it holds simultaneously for all $k$ with probability at least $1 - 3\delta$,

$$\max_k \| E_k^T (U_k^T H^k - U^*) \|_2 \leq b_\infty(\delta) \| U \text{sign}(H) - AU^*(\Lambda^*)^{-1} \|_{2 \to \infty} + \frac{b_\infty(\delta) \| EU^* \|_{2 \to \infty}}{\Lambda_{\text{min}}^s} + \frac{b_2(\delta) \| EU^* \|_{\text{op}}}{\Gamma(\delta)} + \left( \frac{\Lambda_{\text{min}}^s (b_\infty(\delta) + b_2(\delta)) (\kappa^* L_2(\delta) + L_3(\delta))}{\Gamma(\delta)} + \frac{b_\infty(\delta) \| EU^* \|_{\text{op}}^2}{\Gamma(\delta)^2} \right) \| U \|_{2 \to \infty}. $$

Proof. For notational convenience, we will suppress the notation $(\delta)$ for all quantities that involve it. Let $W(k) = U(k) H(k) - U^*$ and let $V_1$ denote the event that

$$\| E_k^T W(k) \|_2 \leq b_\infty \| W(k) \|_{2 \to \infty} + b_2 \| W(k) \|_{\text{op}}$$

simultaneously for all $k$, and $V_2$ denote the event that

$$\| A^k - A \|_{\text{op}} \leq L_1, \quad \frac{\| (A^k - A)U \|_{\text{op}}}{\Lambda_{\text{min}}^s} \leq (\kappa^* L_2 + L_3) \| U \|_{2 \to \infty},$$

simultaneously for all $k$.

Then Lemma A.9 and assumption A1 implies that

$$\mathbb{P}(V_1) \geq 1 - 2\delta, \quad \mathbb{P}(V_2) \geq 1 - \delta.$$ 

A simple union bound implies that

$$\mathbb{P}(V) \geq 1 - 3\delta, \quad \text{where} \quad V = V_1 \cap V_2.$$ 

Throughout the rest of the proof we restrict the attention into $V$. On $V$, for all $k$,

$$\| E_k^T (U(k) H(k) - U^*) \|_2 \leq b_\infty \| U(k) H(k) - U^* \|_{2 \to \infty} + b_2 \| U(k) H(k) - U^* \|_{\text{op}}. \quad (119)$$

First we bound $\| U(k) H(k) - U^* \|_{2 \to \infty}$. Applying the triangle inequality,

$$\| U(k) H(k) - U^* \|_{2 \to \infty} \leq \| U(k) H(k) - U H \|_{2 \to \infty} + \| U H - U^* \|_{2 \to \infty}. \quad (120)$$

Note that $\| B \|_{2 \to \infty} \leq \| B \|_{\text{op}}$ for any Hermitian matrix $B$ and thus

$$\| U(k) H(k) - U^* \|_{2 \to \infty} \leq \| U(k) H(k) - U H \|_{\text{op}} \leq \| (U(k)(U(k))^T - U U^T) U^* \|_{\text{op}} \leq \| U(k)(U(k))^T - U U^T \|_{\text{op}}. \quad (121)$$

On the other hand,

$$\| U H - U^* \|_{2 \to \infty} \leq \| U H - AU^*(\Lambda^*)^{-1} \|_{2 \to \infty} + \| AU^*(\Lambda^*)^{-1} - U^* \|_{2 \to \infty} \leq \| U H - AU^*(\Lambda^*)^{-1} \|_{2 \to \infty} + \| AU^*(\Lambda^*)^{-1} - A^* U^*(\Lambda^*)^{-1} \|_{2 \to \infty} \leq \| U H - A^* U^*(\Lambda^*)^{-1} \|_{2 \to \infty} + \| EU^*(\Lambda^*)^{-1} \|_{2 \to \infty} \leq \| U H - A^* U^*(\Lambda^*)^{-1} \|_{2 \to \infty} + \frac{\| EU^* \|_{2 \to \infty}}{\Lambda_{\text{min}}^s} \leq \| U \text{sign}(H) - AU^*(\Lambda^*)^{-1} \|_{2 \to \infty} + \| U(H - \text{sign}(H)) \|_{2 \to \infty} + \frac{\| EU^* \|_{2 \to \infty}}{\Lambda_{\text{min}}^s} \leq \| U \text{sign}(H) - AU^*(\Lambda^*)^{-1} \|_{2 \to \infty} + \frac{\| EU^* \|_{\text{op}}^2}{\Gamma^2} \| U \|_{2 \to \infty} + \frac{\| EU^* \|_{2 \to \infty}}{\Lambda_{\text{min}}^s}, \quad (122)$$

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where the last line uses (85) in page 47. Putting (120)–(122) together, we obtain that
\[
\|U^{(k)}H^{(k)} - U^*\|_2 \leq \|U^{(k)}(U^{(k)})^T -UU^T\|_\text{op}
\]
\[
+ \|\text{U} \text{sign}(H) - AU^*(\Lambda^*)^{-1}\|_2 \epsilon + \frac{\|E U^*\|_\text{op}^2}{\Gamma^2} \|U\|_2 \epsilon + \frac{\|E U^*\|_2}{\lambda_\text{min}}.
\]  
(123)

Next we bound \(\|U^{(k)}H^{(k)} - U^*\|_\text{op}\). Applying the triangle inequality,
\[
\|U^{(k)}H^{(k)} - U^*\|_\text{op} \leq \|U^{(k)}H^{(k)} - UH\|_\text{op} + \|UH - U^*\|_\text{op}.
\]
It has been proved in (121) that
\[
\|U^{(k)}H^{(k)} - UH\|_\text{op} \leq \|U^{(k)}(U^{(k)})^T -UU^T\|_\text{op}.
\]
Similarly,
\[
\|UH - U^*\|_\text{op} \leq \|UU^T - U^*(U^*)^T\|_\text{op}.
\]
Thus,
\[
\|U^{(k)}H^{(k)} - U^*\|_\text{op} \leq \|U^{(k)}(U^{(k)})^T -UU^T\|_\text{op} + \|UU^T - U^*(U^*)^T\|_\text{op}.  
\]  
(124)

Putting (119), (123) and (124) together, we conclude that
\[
\|E_k^T(U^{(k)}H^{(k)} - U^*)\|_2
\]
\[
\leq b_\infty \left( \|\text{U} \text{sign}(H) - AU^*(\Lambda^*)^{-1}\|_2 \epsilon + \frac{\|E U^*\|_\text{op}^2}{\Gamma^2} \|U\|_2 \epsilon + \frac{\|E U^*\|_2}{\lambda_\text{min}} \right)
\]
\[
+ (b_\infty + b_2)\|U^{(k)}(U^{(k)})^T -UU^T\|_\text{op} + b_2\|UU^T - U^*(U^*)^T\|_\text{op}.
\]

The proof is then completed by Lemma A.6.

\[\square\]

A.6 Step IV: summarizing Step I – Step III

Putting Lemma A.7 – A.10 together, we arrive at our first bound.

Lemma A.11. Assume that \(\lambda_{s+1}^* \lambda_{s+r} > 0\). Under assumptions A1 - A3 and
\[
\Gamma(\delta) \geq 2 \max\{E_+(\delta), \lambda_-(\delta)\},
\]

it holds with probability at least \(1 - 4\delta\) that
\[
\left(1 - \frac{(\kappa^* L_2(\delta) + L_3(\delta) + 1)\eta(\delta) + E_+(\delta)}{\Gamma(\delta)}\right) \|\text{U} \text{sign}(H) - AU^*(\Lambda^*)^{-1}\|_2 \epsilon \leq \left(\frac{\kappa^* L_2(\delta) + L_3(\delta) + 1)\eta(\delta) + E_+(\delta)}{\Gamma(\delta)}\right) \left(\frac{\|U^*\|_2 \epsilon + \frac{\|E U^*\|_2}{\lambda_\text{min}}}{\lambda_\text{min}} + \frac{1}{\Gamma(\delta)} \left(\frac{E_+(\delta)b_2(\delta)}{\lambda_\text{min}} + \xi(\delta)\right)\right),
\]

where \(\Gamma(\delta)\) is defined in (79),
\[
\xi(\delta) = \min \left\{E_+(\delta)\xi_1, 3.61E_+(\delta)\kappa^* \xi_2, 3.84E_+(\delta)\kappa^* \xi_3\right\}
\]

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Proof. Without loss of generality we assume that $\lambda_{s+1}^* \geq \lambda_{s+r}^* > 0$. Otherwise we replace $A$ (resp. $A^*,\Lambda,\Lambda^*$) by $-A$ (resp. $-A^*,-\Lambda,-\Lambda^*$). Let $\mathcal{V}$ be the event in Lemma A.10 and $\mathcal{V}'$ be the event in assumption A2. Then

$$P(\mathcal{V}) \geq 1 - 4\delta, \quad \text{where } \mathcal{V} = \mathcal{V} \cap \mathcal{V}'.$$ 

Throughout the rest of the proof we restrict the attention onto $\mathcal{V}$. For notational convenience, we will suppress the notation ($\delta$) for all quantities that involve it.

Since $\Gamma \geq 2\lambda_-$, condition C1 is satisfied on event $\mathcal{V}$. By Lemma A.8, on $\mathcal{V}$,

$$\frac{\|A^*(UH - U^*)\|_2}{\lambda_{s+1}^*} \leq \frac{\xi}{\Gamma}.$$ 

By Lemma A.7 and Lemma A.10, on $\mathcal{V}$,

$$\frac{\|(U\text{sign}(H) - AU^*(\Lambda^*)^{-1})_k\|_2}{\lambda_{s+1}^*} \leq \frac{b_\infty}{\lambda_{s+1}^*} \frac{\|U\text{sign}(H) - AU^*(\Lambda^*)^{-1}\|_{2\to\infty}}{\lambda_{s+1}^*} + \frac{1}{\Gamma} \left( \frac{E_+ b_2}{\lambda_{s+1}^*} + \xi \right)$$

$$+ \frac{b_\infty \|EU^*\|_{2\to\infty}}{\lambda_{s+1}^*} + \left( \beta + \frac{b_\infty + b_2}{\lambda_{s+1}^*} (\kappa^* L_2 + L_\beta) + \frac{b_\infty E_+^2}{\lambda_{s+1}^* \Gamma^2} \right) \frac{\|U\|_{2\to\infty}}{\lambda_{s+1}^*}. \quad (125)$$

Recalling the definition of $\beta$ in (83), on event $\mathcal{V}$,

$$\beta \leq \frac{E_+^2}{\Gamma^2} + \frac{E_+}{\lambda_{s+1}^*} + \frac{E_\infty (\kappa^* L_2 + L_\beta)}{\Gamma}.$$ 

Then (125) implies that

$$\left( 1 - \frac{b_\infty}{\lambda_{s+1}^*} \right) \frac{\|U\text{sign}(H) - AU^*(\Lambda^*)^{-1}\|_{2\to\infty}}{\lambda_{s+1}^*} \leq \tilde{\beta} \frac{\|U\|_{2\to\infty}}{\lambda_{s+1}^*} + \frac{b_\infty \|EU^*\|_{2\to\infty}}{\lambda_{s+1}^*} + \frac{1}{\Gamma} \left( \frac{E_+ b_2}{\lambda_{s+1}^*} + \xi \right), \quad (126)$$

where

$$\tilde{\beta} = \beta + \frac{b_\infty + b_2}{\lambda_{s+1}^*} (\kappa^* L_2 + L_\beta) + \frac{b_\infty E_+^2}{\lambda_{s+1}^* \Gamma^2}. \quad (127)$$

On the other hand, since $\text{sign}(H)$ is orthogonal,

$$\|U\|_{2\to\infty} = \|U\text{sign}(H)\|_{2\to\infty} \leq \|U\text{sign}(H) - AU^*(\Lambda^*)^{-1}\|_{2\to\infty} + \|AU^*(\Lambda^*)^{-1}\|_{2\to\infty}$$

$$\leq \|U\text{sign}(H) - AU^*(\Lambda^*)^{-1}\|_{2\to\infty} + \|A^* U^*(\Lambda^*)^{-1}\|_{2\to\infty} + \frac{\|EU^*\|_{2\to\infty}}{\lambda_{s+1}^*}$$

$$= \|U\text{sign}(H) - AU^*(\Lambda^*)^{-1}\|_{2\to\infty} + \|U^*\|_{2\to\infty} + \frac{\|EU^*\|_{2\to\infty}}{\lambda_{s+1}^*}. \quad (128)$$

Combining (128) with (126), we obtain that

$$\left( 1 - \tilde{\beta} - \frac{b_\infty}{\lambda_{s+1}^*} \right) \frac{\|U\text{sign}(H) - AU^*(\Lambda^*)^{-1}\|_{2\to\infty}}{\lambda_{s+1}^*} \leq \tilde{\beta} \frac{\|U\|_{2\to\infty}}{\lambda_{s+1}^*} + \frac{\|EU^*\|_{2\to\infty}}{\lambda_{s+1}^*} + \frac{1}{\Gamma} \left( \frac{E_+ b_2}{\lambda_{s+1}^*} + \xi \right)$$

$$\leq \left( \tilde{\beta} + \frac{b_\infty}{\lambda_{s+1}^*} \right) \left( \|U^*\|_{2\to\infty} + \frac{\|EU^*\|_{2\to\infty}}{\lambda_{s+1}^*} \right) + \frac{1}{\Gamma} \left( \frac{E_+ b_2}{\lambda_{s+1}^*} + \xi \right). \quad (129)$$

By definition of $\beta$ in (83), that of $\tilde{\beta}$ in (127) and that of $\eta$ in (8),

$$\tilde{\beta} + \frac{b_\infty}{\lambda_{s+1}^*} = \left( 1 + \frac{b_\infty}{\lambda_{s+1}^*} \right) \frac{E_+^2}{\Gamma^2} + \frac{b_\infty + E_+}{\lambda_{s+1}^*} + \frac{(E_\infty + b_\infty + b_2)(\kappa^* L_2 + L_\beta)}{\Gamma}$$

and

$$\|U\text{sign}(H) - AU^*(\Lambda^*)^{-1}\|_{2\to\infty} \leq \frac{\xi}{\Gamma}. \quad (130)$$
Lemma A.12. Therefore, Lemma A.11 implies the following result.

Thus under $\tilde{\eta} = \eta = \frac{\eta}{\lambda_{\min}}$, AU

Since $\Gamma \geq 2E_+$,

Similarly,

Therefore, (130) implies that

The proof is then completed by (129).

If $(\kappa^* L_2 + L_3 + 1)\eta + E_+ < \Gamma$, we can use a self-bounding argument to derive the bound for $\|U \text{sign}(H) - AU^*(\Lambda^*)^{-1}\|_{2 \to \infty}$. In particular, this is true if the following stronger version of A4 is satisfied:

The bound in Lemma A.12 involves the condition number $\kappa^*$, which can be ineffective in ill-conditioned cases. Fortunately, we can remove this dependence by appropriately partitioning the columns $U$ and $U^*$ into blocks and applying Lemma A.12 separately on each block. This idea is also proposed in Mao et al.
Let $O$ has a smaller norm. A sufficient condition for $\tilde{\Delta}_j$ holds simultaneously for all blocks with probability at least $1 - 4B\delta$ that

$$
\Delta_j^* \geq \Delta^*.
$$

(134)

Let $O_j = \text{sign}(H_j)$ and $O = \text{diag}(O_1, \ldots, O_B)$. By definition,

$$
UO = AU^*(A^*)^{-1} = (U_1O_1 - AU^*_1(A_1^*)^{-1}; U_2O_2 - AU^*_2(A_2^*)^{-1}; \ldots; U_BO_B - AU^*_B(A_B^*)^{-1}).
$$

Thus by (133),

$$
\|UO - AU^*(A^*)^{-1}\|_{2 \to \infty} \leq \sqrt{\sum_{j=1}^{B} \|U_jO_j - AU^*_j(A^*_j)^{-1}\|_{2 \to \infty}^2} \leq \sum_{j=1}^{B} \|U_jO_j - AU^*_j(A^*_j)^{-1}\|_{2 \to \infty}
$$

(135)

Since $O^T O = \text{diag}(O_j^T O_j) = I$, $O \in \mathcal{O}^\circ$. Therefore,

$$
\frac{3}{16} d_{2 \to \infty}(U, AU^*(A^*)^{-1})
$$

$$
\leq \sum_{j=1}^{B} \frac{1}{\Delta_j} \left\{ \left( \left\{ \kappa_j^* L_2(\delta) + L_3(\delta) + 1 \right\} \eta(\delta) + E_+(\delta) \right) \left( \|U^*\|_{2 \to \infty} + \frac{\|EU^*\|_{2 \to \infty}}{\lambda_{\min,j}} \right) + \frac{E_+(\delta)b_2(\delta)}{\lambda_{\min,j}} + \xi_j \right\}
$$

$$
= L_2(\delta)\eta(\delta) \left( \sum_{j=1}^{B} \frac{\kappa_j^*}{\Delta_j} \right) \|U^*\|_{2 \to \infty} + \left( \sum_{j=1}^{B} \frac{1}{\Delta_j} \right) \|U^*\|_{2 \to \infty}
$$

$$
+ L_2(\delta)\eta(\delta) \left( \sum_{j=1}^{B} \frac{\kappa_j^*}{\Delta_j \lambda_{\min,j}} \right) \|EU^*\|_{2 \to \infty} + \left( \sum_{j=1}^{B} \frac{1}{\Delta_j \lambda_{\min,j}} \right) \|EU^*\|_{2 \to \infty}
$$

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By definition of $\zeta_j$, we deduce that
\[
\sum_{j=1}^{B} \frac{\xi_j(\delta)}{\Delta_j^*} \leq \min \left\{ \left\| A^* \right\|_{2 \rightarrow \infty} \left( \sum_{j=1}^{B} \frac{1}{\Delta_j^* \lambda_{\min,j}^*} \right), \right. \\
3.61 \sqrt{\frac{\left\| A^* \right\|_{\text{max}}}{I(A^* \text{ is psd})}} \left( \sum_{j=1}^{B} \frac{\sqrt{\kappa_j^*}}{\lambda_{\min,j}^*} \right), \left. 3.84 \left\| \bar{U}^* \right\|_{2 \rightarrow \infty} \left( \sum_{j=1}^{B} \frac{\kappa_j^*}{\Delta_j^*} \right) \right\}. \tag{137}
\]

The final task is to find a desirable partition of eigenvalues. In particular, we propose a generic partition that is a modification of the one in Definition 5.1 of Mao et al. [2017] which yields a better pre-conditioning.

**Warm-up: eigen-partition for positive eigenvalues**

Since the description is rather technical, we start from a simple case where all eigenvalues in $\Lambda^*$ are strictly positive.

**Definition A.1** (a pre-conditioned eigen-partition for positive eigenvalues). Assume that $\lambda_{s+r}^* > 0$. Let
\[
g_t^* = \lambda_t^* - \max\{\lambda_{t+1}^*, 0\}, \quad t = s + r, s + r - 1, \ldots, s + 1 \quad \text{and} \quad \lambda_0^* = \infty.
\]
Let $t_0 = s + r$ and define $t_1, t_2, \ldots$ recursively as
\[
t_\ell = \max\{s < t < t_{\ell-1} : g_t^* > 2g_{t_{\ell-1}}^*, \lambda_t^* > 2\lambda_{t_{\ell-1}}^* \}.
\]
Let $B = \min\{\ell : t_\ell \text{ does not exist}\}$ and let $t_B = s$. Finally we define the partition as
\[
S_j = \{t_{j-1}, t_{j-1} - 1, \ldots, t_j + 1\}, \quad j = 1, 2, \ldots, B.
\]

We use the following example to illustrate these quantities.

**Example A.1.** Let $s = 0, r = 10$. Table 3 gives the values of $\lambda_t^*, g_t^*$ and $t_\ell$. This setting gives four blocks:

| index $k$ | 11 | 10 | 9  | 8  | 7  | 6  | 5  | 4  | 3  | 2  | 1  | 0  |
|-----------|----|----|----|----|----|----|----|----|----|----|----|----|
| $\lambda_t^*$ | -1 | 1  | 3  | 5  | 8  | 15 | 16 | 23 | 25 | 40 | 55 |    |
| $g_t^*$ | 1  | 2  | 2  | 3  | 7  | 1  | 7  | 2  | 15 | 15 |    |    |
| $t_\ell$ | $t_0$ | $t_1$ | $t_2$ | $t_3$ | $t_4$ |    |    |    |    |    |    |    |

Table 3: An illustrating example of eigen-separation (Example A.1).

$S_1 = \{10, 9, 8\}, S_2 = \{7, 6, 5\}, S_3 = \{4, 3, 2\}, S_4 = \{1\}.$

Roughly speaking, the first step guarantees that each block has sufficient eigen-gap with other eigenvalues and the second step guarantees the condition number within each block and the number of blocks are both small. The following lemma gives the property of the partition.
Lemma A.13. Assume that $\lambda^*_s > 0$. Let $S_1, \ldots, S_B$ be the partition generated in Definition A.1. Then

$$B \leq \min\{r, 1 + \log_2 \kappa^*\}, \quad \kappa^*_j \leq 2|S_j|, \quad \Delta^*_j \geq \Delta^*, \quad \lambda_{\min,j}^* \geq \lambda_{\min}^*,$$

and for any $\gamma, \gamma' > 0$,

$$\sum_{j=1}^B \frac{1}{\Delta^*_j \lambda_{\min,j}^*} \leq \frac{H(\gamma, \gamma')}{\Delta^* \lambda_{\min}^*},$$

where

$$H(\gamma, \gamma') = \begin{cases} \frac{1}{a} + \frac{\gamma}{\gamma + \gamma'} & , \quad (\gamma' > 0) \\ \frac{1}{a} + 1 & , \quad (\gamma' = 0) \end{cases}$$

$$a = 1 - 2^{-(\gamma + \gamma')} \quad (38)$$

Proof. We use the notation in Definition A.1. To prove the first result, note that

$$\kappa^* = \frac{\lambda^*_{s+1}}{\lambda^*_{s+r}} = \frac{\lambda^*_{b+1}}{\lambda^*_0} \geq \frac{\lambda^*_{b-1}}{\lambda^*_0} = \prod_{j=1}^{B-1} \frac{\lambda^*_j}{\lambda^*_{j-1}} > 2^{B-1}.$$

This implies that $B \leq 1 + \log_2 \kappa^*$. Since all blocks are non-empty, we also have $B \leq r$.

To prove the second result, let

$$t'_j = \min\{t_j < t < t_{j-1} : g_t > 2g_{t_{j-1}}, \lambda^*_t \leq 2\lambda^*_{t_{j-1}}\}.$$

In other words, $t'_j$ is the point that is closest to $t_j$ in $j$-th block such that the eigengap is sufficiently large but the corresponding eigenvalue is small. Note that $t'_j$ may not exist. In Example A.1, it is easy to see that $t'_1$ does not exist, $t'_2 = 6$ and $t'_3 = 2$. We distinguish three cases:

- If $t'_j$ does not exist, by definition of $t_j$ in Definition A.1, we know that

$$g_t \leq 2g_{t_{j-1}}, \quad \forall \ t \in (t_j, t_{j-1}).$$

Then

$$\kappa^*_j = \frac{\lambda^*_{t_{j+1}}}{\lambda^*_{t_{j-1}}} \leq \frac{\lambda^*_{t_{j+1}}}{\lambda^*_0} \leq \frac{\lambda^*_{t_{j-1}}}{\lambda^*_0} \leq 2|S_j|.$$

- if $t'_j = t_j + 1$, then

$$\kappa^*_j = \frac{\lambda^*_{t_{j+1}}}{\lambda^*_{t_{j-1}}} \leq \frac{\lambda^*_t}{\lambda^*_0} \leq 2|S_j|.$$

- if $t'_j > t_j + 1$, by definition of $t'_j$,

$$\lambda^*_j \leq 2\lambda^*_{t_{j-1}}, \quad g_t \leq 2g_{t_{j-1}} < g_{t'_j}, \quad \forall \ t \in (t'_j, t_{j-1}).$$

Using a similar argument as in the above case we can show that

$$\frac{\lambda^*_{t_{j+1}}}{\lambda^*_j} \leq \frac{\lambda^*_{t_{j+1}}}{\lambda^*_{t_{j-1}}} \leq \frac{\lambda^*_t}{\lambda^*_0} \leq t'_j - t_j \leq |S_j|.$$

Therefore,

$$\kappa^*_j = \frac{\lambda^*_{t_{j+1}}}{\lambda^*_j} \leq \frac{\lambda^*_{t_{j+1}}}{\lambda^*_0} \leq 2|S_j|.$$
To prove the last three results, note that \( t_0 = s + r \) and \( t_B = s \),

\[
\text{sep}(A^*) = \min \left\{ \lambda^*_{j-1} - \lambda^*_{j-1+1}, \lambda^*_j - \lambda^*_{j+1} \right\} = \min \{ g_{t_{j-1}}, g_j \},
\]

and

\[
\text{sep}_{s+1,s+r}(A^*) = \min \left\{ \lambda^*_s - \lambda^*_{t_{0}+1}, \lambda^*_s - \lambda^*_{t_{B}+1} \right\} = \min \{ g_{t_0}, g_{t_B} \}.
\]

We distinguish two cases:

- If \( j < B \), the definition of \( t_j \) guarantees that

\[
\text{sep}(A^*) = g_{t_{j-1}} > 2g_{t_{j-2}} = \text{sep}(A^*).
\]

Also noticing that \( \lambda_{\min,j}^* = \lambda_{t_{j-1}}^* \geq 2\lambda_{t_{j-2}}^* = 2\lambda_{\min}(A_{j-1}^*) \), we have

\[
\Delta_j^* \geq 2\Delta_{j-1}^*.
\]

- If \( j = B \), then

\[
\text{sep}(A^*) = \min \{ g_{t_B}, g_{t_B} \} \geq \min \{ g_{t_0}, g_{t_B} \} = \text{sep}_{s+1,s+r}(A^*),
\]

and \( \lambda_{\min}(A_B^*) \geq \lambda_{\min}^* \). Thus,

\[
\Delta_B^* \geq \Delta^*.
\]

In summary,

\[
\lambda_{\min,j}^* \geq 2^{j-1}\lambda_{\min}^* \quad (j = 1, \ldots, B)
\]

and

\[
\Delta_j^* \geq 2^{j-1}\Delta^* \quad (j = 1, \ldots, B - 1), \quad \Delta_B^* \geq \Delta^*.
\]

As a result,

\[
\sum_{j=1}^{B} \frac{1}{\Delta_j^* \lambda_{\min,j}^*} \leq \frac{1}{\Delta^* \lambda_{\min}^*} \left( 2^{-(B-1)\gamma'} + \sum_{j=1}^{B-1} 2^{-(j-1)(\gamma + \gamma')} \right) = \frac{1}{\Delta^* \lambda_{\min}^*} \left( 2^{-(B-1)\gamma'} + \frac{1 - 2^{-(B-1)(\gamma + \gamma')}}{1 - 2^{-(\gamma + \gamma')}} \right).
\]

Let \( x = 2^{-(B-1)\gamma'} \) and \( a = 1 - 2^{-(\gamma + \gamma')} \). If \( \gamma' = 0 \), then

\[
2^{-(B-1)\gamma'} + \frac{1 - 2^{-(B-1)(\gamma + \gamma')}}{1 - 2^{-\gamma}} = 1 + \frac{1 - 2^{-(B-1)\gamma}}{a} \leq 1 + \frac{1}{a} = H(\gamma, 0).
\]

If \( \gamma' > 0 \), then

\[
2^{-(B-1)\gamma'} + \frac{1 - 2^{-(B-1)(\gamma + \gamma')}}{1 - 2^{-\gamma}} = x + \frac{1 - x^{(\gamma + \gamma')/\gamma'}}{a} \leq h(x; \gamma, \gamma').
\]

Note that

\[
\frac{d}{dx} h(x; \gamma, \gamma') = 1 - \frac{\gamma + \gamma'}{a\gamma'} x^{\gamma'/\gamma'}
\]

and thus \( h(x; \gamma, \gamma') \) reaches its maximum at \( x_* = (a\gamma'/(\gamma + \gamma'))^{\gamma'/\gamma} \). Then

\[
h(x; \gamma, \gamma') \leq h(x_*; \gamma, \gamma') = \frac{1}{a} + x_* \left( 1 - \frac{x_*^{\gamma'/\gamma}}{a} \right) = \frac{1}{a} + \frac{\gamma x_*}{\gamma + \gamma'} = H(\gamma; \gamma').
\]
Eigen-partition in general cases

Suppose $A^*$ contains both positive and negative eigenvalues, then we can first split them into the positive and negative blocks and partition each according to Definition A.1.

Definition A.2 (a pre-conditioned eigen-partition in general cases).

1. If $\lambda_{s+r} > 0$, define the partition $S_1, \ldots, S_B$ by Definition A.1:
2. If $\lambda_{s+1} < 0$, define the partition $S_1, \ldots, S_B$ on $(-\lambda_{s+r}, \ldots, -\lambda_{s+1})$ by Definition A.1;
3. If $\lambda_{s+r} < 0 < \lambda_{s+1}$, let $b$ be the integer such that $\lambda_{s+b} < 0 < \lambda_{s+b+1}$. Define $S^+_1, \ldots, S^+_B$ on $(\lambda_{s+b+1}, \ldots, \lambda_{s+1})$ and $S^-_1, \ldots, S^-_B$ on $(\lambda_{s+r}, \ldots, \lambda_{s+b})$ by Definition A.1. Finally re-index the subsets as $S_1, \ldots, S_B$ with $B = B^+ + B^-$ with any ordering.

It is straightforward to derive the following counterpart result of Lemma A.13.

Lemma A.14. Let $S_1, \ldots, S_B$ be the partition generated in Definition A.2. Then

$$B \leq \min \{r, 2 + 2 \log_2 \kappa^* \}, \quad \kappa^*_j \leq 2|S_j|, \quad \Delta^*_j \geq \Delta^*, \quad \lambda^*_{\min,j} \geq \lambda^*_{\min},$$

and for any $\gamma, \gamma' > 0$,

$$\sum_{j=1}^B \frac{1}{\Delta^*_j \lambda^*_{\min,j}} \leq \frac{2H(\gamma, \gamma')}{\Delta^* \lambda^*_{\min}},$$

where $H(\gamma, \gamma')$ is defined in (138).

Removing the dependence on the condition number via eigen-partition

Let $S_1, \ldots, S_B$ be the partition generated in Definition A.2. By Lemma A.14,

$$\sum_{j=1}^B \frac{\kappa^*_j}{\Delta^*_j} \leq \sum_{j=1}^B \frac{2|S_j|}{\Delta^*} = \frac{2r}{\Delta^*},$$

$$\sum_{j=1}^B \frac{1}{\Delta^*_j} \leq \frac{2H(1, 0)}{\Delta^*} = \frac{6}{\Delta^*}$$

$$\sum_{j=1}^B \frac{\kappa^*_j}{\Delta^*_j \lambda^*_{\min,j}} \leq \sum_{j=1}^B \frac{2|S_j|}{\Delta^* \lambda^*_{\min}} \leq \frac{2r}{\Delta^* \lambda^*_{\min}}$$

$$\sum_{j=1}^B \frac{1}{\Delta^*_j \lambda^*_{\min,j}} \leq \frac{2H(1, 1)}{\Delta^* \lambda^*_{\min}} \leq \frac{3.06}{\Delta^* \lambda^*_{\min}}$$

$$\sum_{j=1}^B \frac{\sqrt{\kappa^*_j}}{\Delta^*_j \lambda^*_{\min,j}} \leq \sqrt{2r \frac{2H(1, 0.5)}{\Delta^* \sqrt{\lambda^*_{\min}}}} \leq \sqrt{2r} \frac{3.72}{\Delta^* \sqrt{\lambda^*_{\min}}}$$

To apply (136), we still need $A4$ holds for each block. By Lemma A.14, $\kappa^*_j \leq 2r$ for all $j$, thus it is sufficient to assume the following stronger version of $A4$:

$$\hat{A}4 \quad \Delta^* \geq 4 \left(2r L_2(\delta) + L_3(\delta) + 1 \right) \eta(\delta) + E_+(\delta) + L_1(\delta) + \lambda_-(\delta).$$

Combining the bounds with (133), (136) and (137), we reach the following result.
Lemma A.15. Under assumptions A1 - A3 and $\mathring{A}_4$,
\[
d_{2\to\infty}(U, AU^*(\Lambda^*)^{-1}) \leq C \left\{ \left( \{2rL_2(\delta) + L_3(\delta) + 1\} \eta(\delta) + E_+(\delta) \right) \left( \|U^*\|_{2\to\infty} + \frac{\|EU^*\|_{2\to\infty}}{\lambda_{\min}^*} \right) + \frac{E_+(\delta)b_2(\delta)}{\lambda_{\min}^*} + \min \left\{ E_+(\delta)\xi_1, \sqrt{2r}\bar{E}_+(\delta)\xi_2, 2r\bar{E}_+(\delta)\xi_3 \right\} \right\},
\]
with probability at least $1 - 4\min\{r, 2 + 2\log_2 \kappa^*\} \delta$, where $C$ is a universal constant (that can be chosen as 72).

A.8 Final step

When $\kappa^* \ll r$, Lemma A.12 yields better results than Lemma A.15. However, it has an extra assumption that $\lambda_{\min}^* > 0$. Fortunately, this condition can be removed by partitioning the eigenvalues into 2 blocks, with all positive and negative eigenvalues in $\Lambda^*$, respectively. Using the same argument as (133) and noting that $A4$ holds for both blocks, we can prove the following result.

Lemma A.16. Under assumptions A1 - A3 and $\mathring{A}_4$,
\[
d_{2\to\infty}(U, AU^*(\Lambda^*)^{-1}) \leq C \left\{ \left( \{\kappa^*L_2(\delta) + L_3(\delta) + 1\} \eta(\delta) + E_+(\delta) \right) \left( \|U^*\|_{2\to\infty} + \frac{\|EU^*\|_{2\to\infty}}{\lambda_{\min}^*} \right) + \frac{E_+(\delta)b_2(\delta)}{\lambda_{\min}^*} + \min \left\{ E_+(\delta)\xi_1, \sqrt{\kappa^*}\bar{E}_+(\delta)\xi_2, \kappa^*\bar{E}_+(\delta)\xi_3 \right\} \right\},
\]
with probability at least $1 - 8\delta$, where $C$ is a universal constant (that can be chosen as 41).

Finally, if $\kappa^* > 2r$, A4 is equivalent to $\mathring{A}_4$ and we can apply Lemma A.16; otherwise, $A4$ is equivalent to $\mathring{A}_4$ and we can apply Lemma A.15. Theorem 2.3 is then proved by noticing that
\[
\min\{1 - 4\min\{r, 2 + 2\log_2 \kappa^*\} \delta, 1 - 8\delta\} \geq 1 - B(r)\delta,
\]

B Proof of Other Results in Section 2

Proof of Proposition 2.1. Assume $S = [r]$ without loss of generality. Let $V$ denote the event that
\[
\|A - \Lambda^*\|_{\max} \leq \lambda_-(\delta), \quad \|EU^*\|_{\op} \leq E_+(\delta), \quad \|E\|_{2\to\infty} \leq E_\infty(\delta).
\]
Then by definition,
\[
P(V) \geq 1 - \delta.
\]

We prove each case separately.

(a) Let $A^{(k)}$ be defined as
\[
[A^{(k)}]_{ij} = A_{ij}I(i \neq k, j \neq k).
\]
Then since $A_{ij}$’s are independent, $A^{(k)}$ is independent of $A_k$. It is left to prove the deterministic inequalities on the event $V$. First,
\[
\|A^{(k)} - A\|_{\op} \leq \|A^{(k)} - A\|_F \leq \sqrt{\sum_{j=1}^n A_{jk}^2 + A_{kj}^2} = \sqrt{2}\|A_k\|_2 \leq \sqrt{2}\|A\|_{2\to\infty}.
\]
Note that on the event \( \mathcal{V} \),
\[
\|A\|_{2 \to \infty} \leq \|A^*\|_{2 \to \infty} + \|E\|_{2 \to \infty} \leq \|A^*\|_{2 \to \infty} + E_\infty(\delta). 
\]
(139)
Thus, \( L_1(\delta) \) can be chosen as \( \sqrt{2}(\|A^*\|_{2 \to \infty} + E_\infty(\delta)) \). On the other hand, let \( A_{ki}' = A_{ki}I(i \neq k) \). Then
\[
\|(A^{(k)}) - A\|_{\text{op}} \leq \|A^T_kU\|_2 + \|A_{ki}'U_i^T\|_{\text{op}} = \|(AU)\|_2 + \|A_k'\|_2\|U_k\|_2 \\
= \|\|U\|k\|_2 + \|A_k'\|_2\|U_k\|_2 = \|U_k\|_2 \|A\|_2 + \|A_k'\|_2\|U_k\|_2 \\
\leq (\lambda_{\max}(A) + \|A_k\|_2)\|U_k\|_2.
\]
By definition,
\[
|\lambda_{\max}(A) - \lambda_{\max}^*| \leq \lambda-(\delta).
\]
Then by (139), we conclude that on the event \( \mathcal{V} \),
\[
\|(A^{(k)}) - A\|_{\text{op}} \leq (\lambda_{\max}^* + E_\infty(\delta) + \lambda-(\delta) + \|A^*\|_{2 \to \infty})\|U\|_{2 \to \infty}
\]
Thus, \( L_2(\delta) = 1 \) and \( L_3(\delta) = \frac{E_\infty(\delta) + \lambda-(\delta) + \|A^*\|_{2 \to \infty}}{\lambda^*_{\min}} \).
(b) This is a generalized version of part (a) and the proof strategy is almost the same. Let
\[
([A^{(k)}]_{ij} = A_{ij}I(i \notin N_k, j \notin N_k).
\]
(140)
Then \( A^{(k)} \) is independent of \( A_k \). It is left to prove the deterministic inequalities on the event \( \mathcal{V} \). First,
\[
\|(A^{(k)}) - A\|_{\text{op}} \leq \|A^{(k)} - A\|_F \leq \sqrt{\sum_{i \in N_k} \sum_{j=1}^n (A_k^2_{ij} + A_{ij}^2)} = \sqrt{2 \sum_{i \in N_k} \|A_i\|_2^2} \leq \sqrt{2|N_k|\|A\|_{2 \to \infty}}.
\]
Since \( |N_k| \leq m, L_1(\delta) \) can be taken as \( \sqrt{2m}(\|A^*\|_{2 \to \infty} + E_+(\delta)) \) by (139). On the other hand, let \( \tilde{A}_{ij} = A_{ij}I(j \notin N_k) \) for \( i \in N_k \). Then on event \( \mathcal{V} \),
\[
\|(A^{(k)}) - A\|_{\text{op}} \leq \sum_{i \in N_k} \left( \|A_i^T U\|_2 + \|\tilde{A}_i U_i^T\|_{\text{op}} \right) = \sum_{i \in N_k} \left( \|(AU)\|_2 + \|\tilde{A}_i\|_2\|U_i\|_2 \right) \\
= \sum_{i \in N_k} \left( \|U_i A\|_2 + \|\tilde{A}_i\|_2\|U_i\|_2 \right) = \sum_{i \in N_k} \|U_i^T A\|_2 + \|\tilde{A}_i\|_2\|U_i\|_2 \\
\leq \sum_{i \in N_k} (\lambda_{\max}(A) + \|A_i\|_2)\|U_i\|_2 \\
\leq |N_k|(\lambda_{\max}^* + E_\infty(\delta) + \lambda-(\delta) + \|A^*\|_{2 \to \infty})\|U\|_{2 \to \infty}.
\]
Since \( |N_k| \leq m \), we can take \( L_2(\delta) = m \) and \( L_3(\delta) = \frac{m(E_\infty(\delta) + \lambda-(\delta) + \|A^*\|_{2 \to \infty})}{\lambda^*_{\min}} \).

\[\square\]

**Proof of Proposition 2.2.** Let \( S^{r-1} \) be the \( r \)-dimensional unit sphere and \( M(\epsilon) \) be a minimal \( \epsilon \)-net of \( S^{r-1} \), i.e. \( \forall \zeta \in S^{r-1}, \) there exists \( \zeta' \in M(\epsilon) \) such that \( ||\zeta - \zeta'||_2 \leq \epsilon \). It is well-known that
\[
|M(\epsilon)| \leq \left(1 + \frac{2}{\epsilon}\right)^r.
\]
(141)
Then for any vector $x \in \mathbb{R}^r$,
\[
\|x\|_2 = \sup_{\zeta \in \mathcal{S}^{-1}} \zeta^T x = \sup_{\zeta \in \mathcal{S}^{-1}, \zeta' \in \mathcal{M}(\epsilon)} (\zeta^T x + (\zeta - \zeta')^T x) \\
\leq \max_{\zeta' \in \mathcal{M}(\epsilon)} (\zeta')^T x + \|x\|_2 \epsilon.
\]
This implies that
\[
\|x\|_2 \leq \frac{1}{1 - \epsilon} \max_{\zeta \in \mathcal{M}(\epsilon)} x^T \zeta.
\]
Applying (142) with $x = E_k^T W$, we have
\[
\|E_k^T W\|_2 \leq \frac{1}{1 - \epsilon} \max_{\zeta \in \mathcal{M}(\epsilon)} E_k^T (W \zeta).
\]
Let $\delta' = \delta/(1 + 2/\epsilon)^n$, $a'_\infty = a_\infty(\delta')/(1 - \epsilon)$ and $a'_2 = a_2(\delta')/(1 - \epsilon)$. Further, (141) implies that $\delta' \leq \delta/|\mathcal{M}(\epsilon)|n$ and $A_1$ implies that for each given $\zeta \in \mathcal{M}(\epsilon)$,
\[
E_k^T (W \zeta) \leq (1 - \epsilon) (a'_\infty \|W \zeta\|_\infty + a'_2 \|W \zeta\|_2) \leq (1 - \epsilon) (a'_\infty \|W\|_{2 \to \infty} + a'_2 \|W\|_{\text{op}}).
\]
with probability $1 - \delta'$. Applying the union bound implies that
\[
\|E_k^T W\|_2 \leq \frac{1}{1 - \epsilon} \max_{\zeta \in \mathcal{M}(\epsilon)} E_k^T (W \zeta) \leq a'_\infty \|W\|_{2 \to \infty} + a'_2 \|W\|_{\text{op}}
\]
holds simultaneously for all $\zeta \in \mathcal{M}(\epsilon)$ with probability at least $1 - |\mathcal{M}(\epsilon)|\delta' \geq 1 - \delta/n$. The proof is then completed by (142) and taking $\epsilon = 0.5$. \qed

**Proof of Theorem 2.4.** By the triangle inequality,
\[
d_{2 \to \infty}(U, U^*) \leq d_{2 \to \infty}(U, AU^*(\Lambda^*)^{-1}) + d_{2 \to \infty}(AU^*(\Lambda^*)^{-1}, U^*). \tag{143}
\]
By definition,
\[
d_{2 \to \infty}(AU^*(\Lambda^*)^{-1}, U^*) \leq \|AU^*(\Lambda^*)^{-1} - U^*\|_{2 \to \infty} = \|EU^*(\Lambda^*)^{-1}\|_{2 \to \infty} \leq \frac{\|EU^*\|_{2 \to \infty}}{\lambda_{\text{min}}^*}. \tag{144}
\]
By Theorem 2.3,
\[
d_{2 \to \infty}(U, AU^*(\Lambda^*)^{-1}) \leq \frac{\|EU^*\|_{2 \to \infty}}{\lambda_{\text{min}}^*} + \frac{C}{\Delta^*} \left\{ \bar{r}^* L_2(\delta) + L_3(\delta) + 1 \right\} \eta(\delta) \left( \|U^*\|_{2 \to \infty} + \frac{\|EU^*\|_{2 \to \infty}}{\lambda_{\text{min}}^*} \right) + \frac{E_+ (\delta) b_2(\delta)}{\lambda_{\text{min}}^*} + \min \left\{ E_+ (\delta) \xi_1, \bar{E}_+ (\delta) \sqrt{\bar{r}^*} \xi_2, \overline{E}_+ (\delta) \bar{r}^* \xi_3 \right\}.
\]
By assumption $A_4$, the coefficient of $\|EU^*\|_{2 \to \infty}/\lambda_{\text{min}}^*$ can be bounded by
\[
1 + \frac{C \left[ \bar{r}^* L_2(\delta) + L_3(\delta) + 1 \right] \eta(\delta)}{\Delta^*} \leq 1 + \frac{C}{4}.
\]
When $C$ is chosen as 72, $1 + C/4 \leq 72$. The proof is then completed. \qed

**Proof of Theorem 2.5.** We only need to modify the six steps in the proof of Theorem 2.3 in Appendix A. In particular, Step I and Step III need to be substantially modified while all other steps remain almost the same. Let $\gamma_0$ be the event given by $A_0' - A_2$, i.e.
\[
\min_{j \in [s+1, s+r]} |\Lambda_j^*| \leq \Theta(\delta),
\]
\[\|A^{(k)} - A\|_{\text{op}} \leq L_1(\delta), \quad \| (A^{(k)} - A) U \|_{\text{op}} \leq (\kappa^* L_2(\delta) + L_3(\delta)) \| U \|_{2 \to \infty},\]

where \(A^{(k)}\) satisfies the total variation condition in \(A' 1\) and
\[\| \Lambda - \Lambda^* \|_{\text{max}} \leq \lambda_\ldots, \quad \| E U^* \|_{\text{op}} \leq E_\ldots, \quad \| E \|_{2 \to \infty} \leq \tilde{E}_\ldots(\delta).\]

Then
\[\mathbb{P}(\mathcal{V}_0) \geq 1 - 3\delta.\] (145)

Throughout the proof we will restrict the attention into \(\mathcal{V}_0\) and suppress the notation \((\delta)\) for all quantities that involve it.

**Step I:** assume \(C_1\) hold as in Appendix A.2. Again we start by assuming that all eigenvalues are of the same sign, i.e. \(\lambda_{s+1}^* \lambda_{s+r}^* > 0\). In step V we deal with the general case.

Recalling that
\[\hat{A} = A - \Sigma, \quad \hat{A}^* = E \hat{A}, \quad \hat{E} = \hat{A} - \hat{A}^*,\]
we have
\[\begin{align*}
(UH - U^*)^T_k &= (UH - AU^*(\Lambda^*)^{-1})^T_k + (EU^*(\Lambda^*)^{-1})^T_k \\
&= \{U_k^T (H \Lambda^* - \Lambda H) + A_k^T (UH - U^*)\} (\Lambda^*)^{-1} + \frac{E_k^T U^*(\Lambda^*)^{-1}}{E_k^T} \\
&= \{U_k^T (H \Lambda^* - \Lambda H) + \tilde{A}_k^T (UH - U^*)\} (\Lambda^*)^{-1} + \Sigma_{kk} (UH - U^*)^T_k (\Lambda^*)^{-1} + \frac{E_k^T U^*(\Lambda^*)^{-1}}{E_k^T} \\
&= \left\{U_k^T (H \Lambda^* - \Lambda H) + \tilde{E}_k^T (UH - U^*)^{(k)} H^{(k)} + \tilde{E}_k^T (U^*)^{(k)} H^{(k)} - U^* + (\tilde{\Lambda}_k^*)^T (UH - U^*)\right\} (\Lambda^*)^{-1} \\
&\quad + \Sigma_{kk} (UH - U^*)^T_k (\Lambda^*)^{-1} + \frac{E_k^T U^*(\Lambda^*)^{-1}}{E_k^T}.
\end{align*}\]

Rearranging the second last term to the left handed side and multiplying both sides by \(\Lambda^*\) and recalling that \(V_k^T = E_k^T U^*(\Lambda^* - \Sigma_{kk} I)\), we obtain that
\[\begin{align*}
(UH - U^* - V_k)^T_k (\Lambda^* - \Sigma_{kk} I) &= U_k^T (H \Lambda^* - \Lambda H) + \tilde{E}_k^T (UH - U^*)^{(k)} H^{(k)} + \tilde{E}_k^T (U^*)^{(k)} H^{(k)} - U^* + (\tilde{\Lambda}_k^*)^T (UH - U^*) \\
&\quad + \Sigma_{kk} (UH - U^*)^T_k (\Lambda^*)^{-1} + \frac{E_k^T U^*(\Lambda^*)^{-1}}{E_k^T}.
\end{align*}\]

By the triangle inequality and the definition of \(\Theta\), on event \(\mathcal{V}\) we obtain that
\[\begin{align*}
(UH - U^* - V_k)^T_k \leq \frac{\Theta}{\lambda_{\min}^*} \left\{ ||U_k^T (H \Lambda^* - \Lambda H)||_2 + ||\tilde{E}_k^T (UH - U^*)^{(k)} H^{(k)}||_2 \\
&\quad + ||\tilde{E}_k^T (U^*)^{(k)} H^{(k)} - U^*||_2 + ||(\tilde{\Lambda}_k^*)^T (UH - U^*)||_2 \right\}.
\end{align*}\]

By (86) and (88) in page 47, on event \(\mathcal{V}_0\) defined at the beginning of the proof,
\[\| (UH - U^* - V_k) \|_2 \leq \Theta \left( \frac{E_\ldots + \tilde{E}_\ldots (L_2 \kappa^* + L_3)}{\Gamma} \right) \| U \|_{2 \to \infty} \]
\[\quad + \frac{\Theta}{\lambda_{\min}^*} \left\{ ||\tilde{E}_k^T (U^*)^{(k)} H^{(k)} - U^*||_2 + ||(\tilde{\Lambda}_k^*)^T (UH - U^*)||_2 \right\},\]
where
\[\Gamma = \frac{1}{2} (\Delta^* - L_1).\]
By (85),
\[
\|(U \text{sign}(H) - U^*-V)_k\|_2 \leq \|(UH - U^*-V)_k\|_2 + \|U(\text{sign}(H) - H)\|_{2\to\infty}
\]
\[
\leq \beta \|U\|_{2\to\infty} + \frac{\Theta}{\lambda_{\min}} \left\{ \|\tilde{E}_k^T (U^{(k)} H^{(k)} - U^*)\|_2 + \|\tilde{A}^*(UH - U^*)\|_{2\to\infty} \right\},
\]
where
\[
\beta \triangleq \frac{E^2 + \Theta E_+}{\Gamma^2} + \frac{\Theta E_+}{\lambda_{\min}} + \frac{\Theta \lambda_{\max} (\kappa_2 + L_3)}{\Gamma}.
\]

**Step II:** since \(\tilde{A}^*\) may have very different eigenvalues and eigenvectors from \(A^*\), the Kato's integral cannot be directly applied here. For this reason, we only consider the bound (89). The same proof shows that
\[
\|\tilde{A}^*(UH - U^*)\|_{2\to\infty} \leq \frac{E_+ \|\tilde{A}^*\|_{2\to\infty}}{\Gamma}.
\]

**Step III:** assuming \(\lambda_{s+1} \lambda_{s+r} > 0\). We can follow the proof of Lemma A.10 to derive a bound for
\[
\max_k \|\tilde{E}_k^T (U^{(k)} H^{(k)} - U^*)\|_2.
\]
Since \(\tilde{E}_k\) is a function of \(A_k\),
\[
d_T V(\tilde{E}_k, A^{(k)}) \leq \delta/n,
\]
we can still apply Lemma A.9 with \(W^{(k)} = U^{(k)} H^{(k)} - U^*\). Let \(\mathcal{V}_1\) denote the event that
\[
\|\tilde{E}_k^T W^{(k)}\|_2 \leq \tilde{b}_2 \|W^{(k)}\|_{2\to\infty} + \tilde{b}_2 \|W^{(k)}\|_{op} \text{ simultaneously for all } k.
\]
Then Lemma A.9 implies that
\[
P(\mathcal{V}_1) \geq 1 - 2\delta.
\]
A simple union bound implies that
\[
P(\mathcal{V}) \geq 1 - 5\delta, \quad \text{where } \mathcal{V} = \mathcal{V}_0 \cap \mathcal{V}_1.
\]
Throughout the rest of the proof we will restrict the attention into \(\mathcal{V}\). By (120) and (121) in page 55, we have
\[
\|W^{(k)}\|_{2\to\infty} \leq \|U^{(k)} (U^{(k)})^T - UU^T\|_{op} + \|UH - U^*\|_{2\to\infty}.
\]
By (124),
\[
\|W^{(k)}\|_{op} \leq \|U^{(k)} (U^{(k)})^T - UU^T\|_{op} + \|UU^T - U^* (U^*)^T\|_{op}.
\]
Putting pieces together, we know that on event \(\mathcal{V}\),
\[
\|\tilde{E}_k^T (U^{(k)} H^{(k)} - U^*)\|_2
\]
\[
\leq (\tilde{b}_2 + \tilde{b}_2) \|U^{(k)} (U^{(k)})^T - UU^T\|_{op} + \tilde{b}_2 \|UU^T - U^* (U^*)^T\|_{op} + \tilde{b}_2 \|UH - U^*\|_{2\to\infty}
\]
\[
\leq (\tilde{b}_2 + \tilde{b}_2) \|U^{(k)} (U^{(k)})^T - UU^T\|_{op} + \tilde{b}_2 \|UU^T - U^* (U^*)^T\|_{op}
\]
\[
+ \tilde{b}_2 \|U \text{sign}(H) - U^*\|_{2\to\infty} + \tilde{b}_2 \|U (\text{sign}(H) - H)\|_{2\to\infty}.
\]
By Lemma A.6 and (85),
\[
\|\tilde{E}_k^T (U^{(k)} H^{(k)} - U^*)\|_2 \leq \tilde{b}_2 \|U \text{sign}(H) - U^*\|_{2\to\infty}
\]
\[
+ \left( \lambda_{\min} \left( \frac{\lambda_{\min} \tilde{b}_2 + \tilde{b}_2 (\kappa_2 + L_3)}{\Gamma} \right) + \frac{\tilde{b}_2 E_+ + \Gamma^2}{\Gamma^2} \right) \|U\|_{2\to\infty} + \frac{\tilde{b}_2 E_+}{\Gamma}.
\]
Step IV: assuming $\lambda_{*+1} > 0$. Putting (146), (148) and (151) together, we obtain that

$$
\left(1 - \frac{\Theta \tilde{b}_\infty}{\lambda_{*_{\text{min}}}}\right) \| (U \text{sign}(H) - U^* - V) \|_2 \\
\leq \tilde{\beta}\|U\|_{2\to\infty} + \frac{\Theta}{\lambda_{*_{\text{min}}}} \left(\frac{b_2 E_+}{\Gamma} + \frac{E_+\|\tilde{A}^*\|_{2\to\infty}}{\Gamma}\right).
$$

(152)

where

$$
\tilde{\beta} = \beta + \frac{\Theta (\tilde{b}_\infty + \tilde{b}_2)(\kappa^* L_2 + L_3)}{\Gamma} + \frac{\Theta \tilde{b}_\infty^2 E_+}{\lambda_{*_{\text{min}}} \Gamma^2}
$$

(153)

and $\beta$ is defined in (147). On the other hand, since $\text{sign}(H)$ is orthogonal,

$$
\|U\|_{2\to\infty} = \|U \text{sign}(H)\|_{2\to\infty} \leq \|U \text{sign}(H) - U^* - V\|_{2\to\infty} + \|U^*\|_{2\to\infty} + \|V\|_{2\to\infty}.
$$

Plugging this into (152), we have

$$
\left(1 - \tilde{\beta} - \frac{\Theta \tilde{b}_\infty}{\lambda_{*_{\text{min}}}}\right) \| (U \text{sign}(H) - U^* - V) \|_2 \\
\leq \tilde{\beta} (\|U^*\|_{2\to\infty} + \|V\|_{2\to\infty}) + \frac{\Theta (\tilde{b}_2 + \|\tilde{A}^*\|_{2\to\infty}) E_+}{\lambda_{*_{\text{min}}} \Gamma}.
$$

(154)

By definition,

$$
\tilde{\beta} + \frac{\Theta \tilde{b}_\infty}{\lambda_{*_{\text{min}}}}
\leq \left(1 + \frac{\Theta \tilde{b}_\infty}{\lambda_{*_{\text{min}}}}\right) \frac{E_+^2}{\Gamma^2} + \frac{\Theta (\tilde{b}_\infty + E_+)}{\lambda_{*_{\text{min}}}} + \frac{\Theta (\tilde{E}_\infty + \tilde{b}_\infty + \tilde{b}_2)(\kappa^* L_2 + L_3)}{\Gamma}
\leq \left(1 + \frac{E_+^2}{\Gamma^2}\right) \frac{\Theta \tilde{b}_\infty}{\lambda_{*_{\text{min}}}} + \frac{\Theta (\tilde{\eta}(\kappa^* L_2 + L_3) + E_+)}{\Gamma}.
$$

(155)

Similar to Step IV in Appendix A, we start from a stronger version of assumption A4:

$$
\tilde{A}'^4 \Delta^* \geq 4 \left(\Theta (\{\kappa^* L_2 + L_3 + 1\} \tilde{\eta} + E_+) + L_1 + \lambda_- + E_+\right).
$$

Under assumption $\tilde{A}'^4$, C1 holds and

$$
\Gamma \geq 2E_+.
$$

As a result,

$$
\left(1 + \frac{E_+^2}{\Gamma^2}\right) \frac{b_\infty}{4\lambda_{*_{\text{min}}}} \leq \frac{5\tilde{\eta}}{8\Gamma} \leq \tilde{\eta}.
$$

By (155),

$$
\tilde{\beta} + \frac{\Theta \tilde{b}_\infty}{\lambda_{*_{\text{min}}}} \leq \frac{E_+^2}{\Gamma^2} + \frac{\Theta (\{\kappa^* L_2 + L_3 + 1\} \tilde{\eta} + E_+)}{\Gamma}.
$$

On the other hand, assumption $\tilde{A}'^4$ implies that

$$
\Gamma \geq 2\Theta (\{\kappa^* L_2 + L_3 + 1\} \tilde{\eta} + E_+).
$$

Then

$$
\tilde{\beta} + \frac{\Theta \tilde{b}_\infty}{\lambda_{*_{\text{min}}}} \leq \frac{1}{4} + \frac{3}{4} = \frac{3}{4}.
$$
By (154), we deduce that
\[
\|U \text{sign}(H) - U^* - V\|_2 \\
\leq 4 \left( \frac{E^2}{\Gamma} + \Theta(\{\kappa L_2 + L_3 + 1\} \tilde{\eta} + E_+) \right) \left( \|U^*\|_2 \to \infty + \|V\|_2 \to \infty \right) + \frac{4\Theta(\tilde{b}_2 + \|\hat{A}^*\|_2 \to \infty) E_+}{\lambda^*_\min \Gamma} \\
\leq 29 \left( \frac{E^2}{(\Delta^*)^2} + \Theta(\{\kappa L_2 + L_3 + 1\} \tilde{\eta} + E_+) \right) \left( \|U^*\|_2 \to \infty + \|V\|_2 \to \infty \right) + \frac{11\Theta(\tilde{b}_2 + \|\hat{A}^*\|_2 \to \infty) E_+}{\lambda^*_\min \Delta^*},
\]
(156)
where the last inequality uses the fact that
\[\Gamma = \frac{1}{2}(\Delta^* - L_1) \geq \frac{3}{8} \Delta^*.
\]
On the other hand,
\[
\|V\|_2 \to \infty = \max_k \|V_k\|_2 \leq \frac{\Theta \|EU^*\|_2 \to \infty}{\lambda^*_\min}.
\]
(157)
By (156) we obtain that
\[
\|U \text{sign}(H) - U^* - V\|_2 \to \infty \\
\leq 29 \left( \frac{E^2}{(\Delta^*)^2} + \Theta(\{\kappa L_2 + L_3 + 1\} \tilde{\eta} + E_+) \right) \left( \|U^*\|_2 \to \infty + \frac{\Theta \|EU^*\|_2 \to \infty}{\lambda^*_\min} \right) + \frac{11\Theta(\tilde{b}_2 + \|\hat{A}^*\|_2 \to \infty) E_+}{\lambda^*_\min \Delta^*}.
\]
(158)
Recall that this is true on \(V\), which has probability at least \(1 - 5\delta\) according to (150).

**Step V:** let \(S_1, \ldots, S_B\) be the partition given by Lemma A.14. As in Appendix A.7, let
\[
\text{sep}_j(A^*) = \text{sep}_{S_j}(A^*), \quad \Delta^*_j \triangleq \min \{\text{sep}_j(A^*), \lambda^*_\min,j\}.
\]
(159)
Then with probability at least \(1 - 5B\delta\), it holds simultaneously for all blocks that
\[
\|U_j \text{sign}(H_j) - U^*_j - V_j\|_2 \to \infty \\
\leq 29 \left( \frac{E^2}{(\Delta^*_j)^2} + \Theta(\{\kappa L_2 + L_3 + 1\} \tilde{\eta} + E_+) \right) \left( \|U^*_j\|_2 \to \infty + \frac{\Theta \|EU^*_j\|_2 \to \infty}{\lambda^*_\min,j} \right) + \frac{11\Theta(\tilde{b}_2 + \|\hat{A}^*_j\|_2 \to \infty) E_+}{\lambda^*_\min,j \Delta^*_j} \\
\leq 29 \left( \frac{E^2}{(\Delta^*_j)^2} + \Theta(\{\kappa L_2 + L_3 + 1\} \tilde{\eta} + E_+) \right) \left( \|U^*_j\|_2 \to \infty + \frac{\Theta \|EU^*_j\|_2 \to \infty}{\lambda^*_\min,j} \right) + \frac{11\Theta(\tilde{b}_2 + \|\hat{A}^*_j\|_2 \to \infty) E_+}{\lambda^*_\min,j \Delta^*_j},
\]
where the last inequality uses the fact that \(U^*_j\) (resp. \(EU^*_j\)) is a sub-block of \(U^*\) (resp. \(EU^*\)), and thus has a smaller norm. Recalling 135 in page 59, we have
\[
d_{2 \to \infty}(U, U^* + V) \leq 29 \sum_{j=1}^{B} \left( \frac{E^2}{(\Delta^*_j)^2} + \Theta(\{\kappa L_2 + L_3 + 1\} \tilde{\eta} + E_+) \right) \left( \|U^*_j\|_2 \to \infty + \frac{\Theta \|EU^*_j\|_2 \to \infty}{\lambda^*_\min,j} \right) + \frac{11\Theta(\tilde{b}_2 + \|\hat{A}^*_j\|_2 \to \infty) E_+}{\lambda^*_\min,j \Delta^*_j},
\]
(160)
By Lemma A.14,
\[
\sum_{j=1}^{B} \frac{1}{(\Delta^*_j)^2} \leq \frac{2H(2, 0)}{(\Delta^*)^2} = \frac{14}{3(\Delta^*)^2}
\]
70
To apply (160), we still need $A_4$ holds for each block. By Lemma A.14, $\kappa_j^* \leq 2r$ for all $j$, thus it is sufficient to assume the following stronger version of $A_4$:

$$\tilde{A}'_4 \geq 4 \left( \Theta \left( \{2rL_2 + L_3 + 1\} \tilde{\eta} + E_+ \right) + L_1 + \lambda_- + E_+ \right).$$

Then under assumptions $A'1 - A'3$ and $\tilde{A}'_4$, we have

$$d_{2\to\infty}(U,U^* + V) \leq C \left\{ \frac{E_+^2}{(\Delta^*)^2} + \Theta \left( \{2rL_2 + L_3 + 1\} \tilde{\eta} + E_+ \right) \right\} \left( \|U^*\|_{2\to\infty} + \Theta \|EU^*\|_{2\to\infty} \right) \frac{\Theta(b_2 + \|\tilde{A}^*\|_{2\to\infty})}{\lambda^*_\min \Delta^*} + \Theta \frac{\|V\|_{2\to\infty}}{\lambda^*_\min \Delta^*}. \right\} \right\} \right\}$$

with probability at least $1 - 10\delta$, where $C$ is a universal constant that can be chosen as 58.

**Final step:** when $\kappa^* \ll r$, we should use (158) instead of (162). We can split the eigenvalues into 2 blocks, with all positive and negative eigenvalues in $\Lambda^*$, respectively. Then similar to Appendix A.8, we have

$$d_{2\to\infty}(U,U^* + V) \leq C \left\{ \frac{E_+^2}{(\Delta^*)^2} + \Theta \left( \{2rL_2 + L_3 + 1\} \tilde{\eta} + E_+ \right) \right\} \left( \|U^*\|_{2\to\infty} + \Theta \|EU^*\|_{2\to\infty} \right) \frac{\Theta(b_2 + \|\tilde{A}^*\|_{2\to\infty})}{\lambda^*_\min \Delta^*} + \Theta \frac{\|V\|_{2\to\infty}}{\lambda^*_\min \Delta^*} \right\} \right\}$$

with probability at least $1 - 10\delta$, where $C$ is a universal constant that can be chosen as 58.

The proof of Theorem 2.5 is then completed by considering two cases $\kappa^* > 2r$ and $\kappa^* \leq 2r$ separately as in Appendix A.8.

**Proof of Theorem 2.6.** By (157),

$$\|V\|_{2\to\infty} \leq \frac{\Theta \|EU^*\|_{2\to\infty}}{\lambda^*_\min \Delta^*}.$$

The proof is then completed by assumption $A'_4$.

**C Proofs of Results in Section 3**

**C.1 Proofs for Section 3.1**

The proofs heavily exploit concentration inequalities for binary random variables derived in Appendix F.
Proof of Lemma 3.1. Setting
\[2\gamma = (\log(1/\delta))^{-(1-\alpha)}\]
in Lemma F.3 yields that the condition of Proposition 2.2 holds with
\[a_\infty(\delta) = \frac{2\log(1/\delta)}{F^{-1}(2\gamma \log(1/\delta))}, \quad a_2(\delta) = a_\infty(\delta) \sqrt{\frac{p^*}{2(\log(1/\delta))^{1-\alpha}}}.
\]
By Lemma F.2, \(F^{-1}(x) \geq \log x/2\) and thus
\[a_\infty(\delta) \leq \frac{4\log(1/\delta)}{\alpha \log \log(1/\delta)}, \quad a_2(\delta) \leq \frac{\sqrt{8p^*(\log(1/\delta))^{1+\alpha}}}{\alpha \log \log(1/\delta)}.
\]
By Proposition 2.2,
\[b_\infty(\delta) = 2a_\infty \left(\frac{\delta}{5r n}\right), \quad b_2(\delta) = 2a_2 \left(\frac{\delta}{5r n}\right).
\]
The proof is completed by the fact that \(x \mapsto \log x/\log \log x\) is increasing in \(x\) and
\[\log(5r n/\delta) \preceq R(\delta).
\]
\qed

To prove Lemma 3.2, we need the following concentration inequality.

Proposition C.1. [Latała et al., 2018, Remark 4.12] There exists a universal constant \(C\) such that for any \(\epsilon \in [0,1]\) and \(t \geq 0\),
\[\mathbb{P} \left( \|E\|_{\infty} \geq 2(1+\epsilon) \max_i \sqrt{\sum_j \mathbb{E}[E_{ij}^2]} + t \right) \leq n \exp \left\{ -\frac{\epsilon t^2}{C} \right\}.
\]

Proof of Lemma 3.2. For our purpose, we let \(\epsilon = 1, \ t = \sqrt{C \log (n/\delta)}\), then
\[\|E\|_{\infty} \leq 2(1+\epsilon) \max_i \sqrt{\sum_j \mathbb{E}[E_{ij}^2]} + t = 4 \max_i \sqrt{\sum_j \mathbb{E}[E_{ij}^2]} + \sqrt{C (\log (n/\delta))},
\]
with probability at least \(1 - \delta\). Since \(\mathbb{E}[E_{ij}^2] = p_{ij}(1-p_{ij}) \leq p_{ij}\), we have
\[E_2(\delta) \leq \sqrt{\bar{p}^*} + \sqrt{\log(n/\delta)}.
\]
The result is proved by (10).

Proof of Lemma 3.3. Note that \(F^{-1}(\epsilon) = 1\). Setting
\[2\gamma = \frac{\epsilon}{\log(1/\delta)},
\]
in Lemma F.3 yields that the condition of Proposition 2.2 holds with
\[a_\infty(\delta) = \frac{2\log(1/\delta)}{F^{-1}(\epsilon)} = 2 \log(1/\delta), \quad a_2(\delta) = a_\infty(\delta) \sqrt{\frac{ep^*}{2 \log(1/\delta)}} = \sqrt{2ep^* \log(1/\delta)}.
\]
Note that they are different from the ones in Lemma 3.1. By Proposition 2.2,
\[\|EU^*\|_{2 \to \infty} \leq 2a_\infty(\delta/5r n)\|U^*\|_{2 \to \infty} + 2a_2(\delta/5r n).
\]
where we use the fact that $\|U^*\|_{op} = 1$. Then

$$log(5^r n/\delta) = log(n/\delta) + (log 5)r \leq R(\delta).$$

The proof is then completed. $\square$

**Proof of Theorem 3.4.** Let $\mathcal{V}$ be the intersection of the events in Theorem 2.3 and Lemma 3.3. Then a union bound implies that

$$P(\mathcal{V}) \geq 1 - (B(r) + 1)\delta.$$ 

Throughout the rest of the proof we restrict the attention on $\mathcal{V}$. For notational convenience we will suppress the notation $(\delta)$ for all quantities that involve it.

By Lemma 3.2

$$\eta \leq \sqrt{np^*} + \sqrt{\log(n/\delta)} + \frac{R}{\alpha \log R} + \frac{\sqrt{R^{1+\alpha}p^*}}{\alpha \log R} \leq g. \tag{163}$$

By part (a) of Proposition 2.1 and Lemma 3.2,

$$L_1 \leq \|A^*\|_{2\to\infty} + E_\infty \leq g, \tag{164}$$

where we use the fact that

$$\|A^*\|_{2\to\infty} = \max_i \left[ \sum_{j=1}^n p_{ij}^2 \leq \sqrt{np^*} \right] \leq \sqrt{np^*}.$$ 

In addition,

$$L_2 = 1, \quad L_3 \leq \|A^*\|_{2\to\infty} + E_\infty + \lambda_\min \leq \frac{g}{\Delta^*} \leq 1. \tag{165}$$

First we verify that assumption $A_4$ holds in this case. By (19) in page 12, (163) and (165),

$$\Delta^* \geq C \bar{\kappa}^* g \geq CC'(\bar{\kappa}^* L_2 + L_3 + 1)\eta$$

where $C'$ is a universal constant. In addition, by (19), (164) and (165),

$$\Delta^* \geq C \bar{\kappa}^* g \geq CC''(E_+ + L_1 + \lambda_-).$$

By taking $C = 4/C' + 4/C''$, we prove that

$$\Delta^* \geq 4(\sigma + L_1 + \lambda_-).$$

This validates assumption $A_4$. Since assumptions $A_1 - A_4$ hold, by Theorem 2.3, we obtain that

$$d_{2\to\infty}(U, AU^*(\Lambda^*))^{-1} \leq (\frac{1}{\Delta^*}) \left\{ (\bar{\kappa}^* \eta + E_+) \left[ \|U^*\|_{2\to\infty} + \|EU^*\|_{2\to\infty} \right] + \left( E_+ b_2 \lambda_\min + \min\{E_+ \xi_1, \bar{E}_+ \sqrt{\bar{\kappa}^*} \xi_2, \bar{E}_+ \bar{\kappa}^* \xi_3 \} \right) \right\}$$

$$\leq (\frac{1}{\Delta^*}) \left\{ \bar{\kappa}^* g \left[ \|U^*\|_{2\to\infty} + \frac{R}{\lambda_\min} \|U^*\|_{2\to\infty} + \frac{\sqrt{Rp^*}}{\lambda_\min} \right] \right\}$$

$$+ (\sqrt{np^*} + \sqrt{\log(n/\delta)}) \left( \frac{\sqrt{R^{1+\alpha}p^*}}{\alpha (\log R) \lambda_\min} + \min\{\xi_1, \sqrt{\bar{\kappa}^*} \xi_2, \bar{\kappa}^* \xi_3 \} \right) \right\}$$
\[
\frac{1}{\Delta^*} \left\{ \bar{\kappa}^* g \left( 1 + \frac{R}{\lambda_{\min}} \right) \| U \|_2 \rightarrow \infty + \sqrt{R p^*} \left( \bar{\kappa}^* g + \frac{\sqrt{n p^*} + \sqrt{\log(n/\delta)}}{\alpha \log R} \right) \right\} \\
+ \frac{\sqrt{n p^*} + \sqrt{\log(n/\delta)}}{\Delta^*} \min\{\xi_1, \sqrt{\bar{\kappa}^* \xi_2}, \bar{\kappa}^* \xi_3\}
\]
\[
\leq \frac{1}{\Delta^*} \left\{ \bar{\kappa}^* g \left( 1 + \frac{R}{\lambda_{\min}} \right) \| U \|_2 \rightarrow \infty + \sqrt{R p^*} \left( \bar{\kappa}^* g + \frac{\sqrt{n p^*} R^*}{\alpha \log R} \right) \right\} \\
+ \frac{\sqrt{n p^*} + \sqrt{\log(n/\delta)}}{\Delta^*} \min\{\xi_1, \sqrt{\bar{\kappa}^* \xi_2}, \bar{\kappa}^* \xi_3\},
\]
where (i) uses the fact that \(\bar{\kappa}^* L_2 + L_3 + 1 \geq \bar{\kappa}^*\), (ii) uses Lemma 3.2 and (163), and (iii) uses the fact that
\[
\frac{\sqrt{\log(n/\delta)} R^*}{\alpha \log R} \leq \frac{R}{\alpha \log R} \leq \bar{\kappa}^* g.
\]
The proof of this inequality is completed by plugging in the definition of \(\xi_1, \xi_2\) and \(\xi_3\).

By Lemma 3.3 we obtain that
\[
d_{2 \rightarrow \infty}(U, U^*) \leq \left\{ \bar{\kappa}^* g \left( 1 + \frac{R}{\lambda_{\min}} \right) + \frac{R}{\lambda_{\min}} \right\} \| U^* \|_2 \rightarrow \infty + \sqrt{R p^*} \left( \bar{\kappa}^* g + \sqrt{n p^*} \right) \left( \sqrt{\frac{\lambda_{\min}}{\alpha \log R}} \right) + 1 \\
+ \frac{\sqrt{n p^*} + \sqrt{\log(n/\delta)}}{\Delta^*} \min\left\{ \frac{\| A^* \|_2 \rightarrow \infty}{\lambda_{\min}}, \frac{\sqrt{n p^*} \sqrt{\bar{\kappa}^* R^*}}{\lambda_{\min} \sqrt{\min(I(A^* \text{ is psd})}, \bar{\kappa}^* \| \bar{U}^* \|_2 \rightarrow \infty} \right\}.\]
By (19), \(\bar{\kappa}^* g/\Delta^* \leq 1\) and thus the above bound can be simplified as the one in Theorem 3.4.  

**Proof of Corollary 3.5.** In this case, we only keep the third term \(\bar{\kappa}^* \| \bar{U}^* \|_2 \rightarrow \infty\) in the minimum. Then
\[
(\sqrt{n p^*} + \sqrt{\log(n/\delta)}) \bar{\kappa}^* \| \bar{U}^* \|_2 \rightarrow \infty \leq (\sqrt{n p^*} + \sqrt{R}) \bar{\kappa}^* \| \bar{U}^* \|_2 \rightarrow \infty \leq \bar{\kappa}^* g \| \bar{U}^* \|_2 \rightarrow \infty \leq \bar{\kappa}^* g \| U^* \|_2 \rightarrow \infty.
\]
This can be incorporated into the first term \(\bar{\kappa}^* g \left( 1 + \frac{R}{\lambda_{\min}} \right) \| U^* \|_2 \rightarrow \infty\). The proof is then completed.  

**Proof of Corollary 3.6.** In this case, we only keep the first term \(\frac{\| A^* \|_2 \rightarrow \infty}{\lambda_{\min}}\) in the minimum. Then
\[
(\sqrt{n p^*} + \sqrt{\log(n/\delta)}) \frac{\| A^* \|_2 \rightarrow \infty}{\lambda_{\min}} \leq g \frac{\sqrt{n p^*}}{n p^*} \sqrt{n p^*} \| U^* \|_2 \rightarrow \infty = g \| U^* \|_2 \rightarrow \infty \leq \bar{\kappa}^* g \| U^* \|_2 \rightarrow \infty.
\]
This term can also be incorporated into the first term \(\bar{\kappa}^* g \left( 1 + \frac{R}{\lambda_{\min}} \right) \| U^* \|_2 \rightarrow \infty\). Thus, we obtain that
\[
d_{2 \rightarrow \infty}(U, AU^*(\Lambda^*)^{-1}) \leq \frac{1}{\Delta^*} \left\{ \bar{\kappa}^* g \left( 1 + \frac{R}{\lambda_{\min}} \right) \| U^* \|_2 \rightarrow \infty + \frac{\sqrt{R p^*}}{\lambda_{\min}^*} \left( \bar{\kappa}^* g + \frac{\sqrt{n p^*} R^*}{\alpha \log R} \right) \right\}.
\]
Note that \(\sqrt{n} \| U^* \|_2 \rightarrow \infty \geq \sqrt{p^*} \geq 1\). By (20),
\[
np^* \leq \lambda_{\min}^* \sqrt{n} \| U^* \|_2 \rightarrow \infty \leq \lambda_{\min}^* (\sqrt{n} \| U^* \|_2 \rightarrow \infty)^2.
\]
As a result,
\[
\sqrt{\frac{R p^*}{\lambda_{\min}^*}} \leq 1 + \sqrt{n} \frac{R}{\lambda_{\min}^*} \leq \sqrt{\frac{R}{\lambda_{\min}^*}} \| U^* \|_2 \rightarrow \infty \\
\leq \left( 1 + \frac{R}{\lambda_{\min}^*} \right) \| U^* \|_2 \rightarrow \infty.
\]
On the other hand,
\[
\frac{\sqrt{R\bar{p}}}{\lambda_{\min}} \sqrt{n \bar{p}} R^{\alpha} \leq \frac{1}{\sqrt{n}} \frac{n \bar{p}^*}{\lambda_{\min}^*} \frac{\sqrt{R^{1+\alpha}}}{\alpha \log R}
\]
\[
\leq \|U^*\|_{2 \to \infty} \frac{\sqrt{R^{1+\alpha}}}{\alpha \log R} \leq \|U^*\|_{2 \to \infty} g
\]
where the last inequality uses the fact that \(\alpha < 1\). Therefore, we conclude that
\[
d_{2 \to \infty}(U, AU^*(\Lambda^*)^{-1}) \leq \bar{\kappa}^* g \left(1 + \frac{R}{\lambda_{\min}^*}\right) \|U^*\|_{2 \to \infty}.
\]
Finally, by the triangle inequality and Lemma 3.3,
\[
d_{2 \to \infty}(U, U^*) \leq d_{2 \to \infty}(U, AU^*(\Lambda^*)^{-1}) + \frac{\|EU^*\|_{2 \to \infty}}{\lambda_{\min}^*}
\]
\[
\leq \left\{ \bar{\kappa}^* g \left(1 + \frac{R}{\lambda_{\min}^*}\right) \|U^*\|_{2 \to \infty} + \frac{\sqrt{R\bar{p}}}{\lambda_{\min}^*} \right\} \|U^*\|_{2 \to \infty} + \frac{\sqrt{R\bar{p}}}{\lambda_{\min}^*}.
\]

C.2 Proofs for Section 3.2

Note that \(\tilde{E} = \tilde{L} - E\tilde{L} = -(\tilde{A} - E\tilde{A})\) where \(\tilde{A}\) is a binary matrix with independent entries. Thus Lemma 3.7 is a direct consequence of Lemma 3.1 and the bound for \(E_\infty(\delta)\) in Lemma 3.8 is a direct consequence of Lemma 3.2. For other results we need the following lemma.

Lemma C.2. For any \(\delta \in (0, 1)\), it holds with probability \(1 - \delta\) that
\[
\max_k |\mathcal{L}_{kk} - \mathcal{L}_{kk}^*| \leq 4M(\delta).
\]
Moreover,
\[
E_2(\delta) \leq M(\delta), \quad \tilde{E}_2(\delta) \leq \sqrt{n \bar{p}} + \sqrt{\log(n/\delta)}.
\]

Proof. By Lemma F.3 with \(w = 1_n\) and \(\gamma = c/2 \log(1/\delta')\), it holds with probability \(1 - \delta'\) that
\[
\mathcal{L}_{kk} - \mathcal{L}_{kk}^* \leq 2 \log(1/\delta')(1 + \sqrt{\gamma n \bar{p}^*}) \leq 2 \log(1/\delta') + \sqrt{2en\bar{p}^* \log(1/\delta')},
\]
Similarly, with probability \(1 - \delta'\),
\[
\mathcal{L}_{kk}^* - \mathcal{L}_{kk} \leq 2 \log(1/\delta') + \sqrt{2en\bar{p}^* \log(1/\delta')}.
\]
Letting \(\delta' = \delta/2n\) and applying the union bound, we obtain that
\[
\max_k |\mathcal{L}_{kk} - \mathcal{L}_{kk}^*| \leq 2 \log(2n/\delta) + \sqrt{2en\bar{p}^* \log(2n/\delta)} \leq 4 \log(n/\delta) + \sqrt{4en\bar{p}^* \log(n/\delta)} \leq 4M(\delta). \quad (166)
\]
The result on \(\tilde{E}_2(\delta)\) can be obtained from Lemma 3.2. By Weyl's inequality,
\[
\|E\|_{op} \leq \|\tilde{E}\|_{op} + \max_k |\mathcal{L}_{kk} - E\mathcal{L}_{kk}|.
\]
Thus,
\[
E_2(\delta) \leq M(\delta) + \sqrt{n \bar{p}^*} + \sqrt{\log(n/\delta)} \leq M(\delta).
\]
Proof of Lemma 3.8. Since $\lambda_-(\delta), E_+(\delta) \leq E_2(\delta)$. This is a direct consequence of Lemma C.2.

Proof of Lemma 3.9. By Lemma 3.3,

$$\|\tilde{E}U^*\|_{2\to\infty} \leq R(\delta)\|U^*\|_{2\to\infty} + \sqrt{R(\delta)p^*}.$$  

By the triangle inequality and Lemma C.2,

$$\|EU^*\|_{2\to\infty} \leq \|\tilde{E}U^*\|_{2\to\infty} + \|(E-\tilde{E})U^*\|_{2\to\infty} \leq \|\tilde{E}U^*\|_{2\to\infty} + \max_k |\mathcal{L}_{kk} - \mathcal{L}_{kk}^*|\|U^*\|_{2\to\infty}$$

$$\leq \|\tilde{E}U^*\|_{2\to\infty} + M(\delta)\|U^*\|_{2\to\infty} \leq (M(\delta) + R(\delta))\|U^*\|_{2\to\infty} + \sqrt{R(\delta)p^*}.$$ 

Proof of Lemma 3.10. Let $A^{(k)}$ be defined as in (140) in page 65 and $D^{(k)} = \text{diag}(A^{(k)}1_n)$. Construct $\mathcal{L}^{(k)}$ as

$$\mathcal{L}^{(k)} = D^{(k)} - A^{(k)} + \sum_{i \in \mathcal{N}_k} \mathcal{L}_{ii}^* e_i e_i^T,$$

where $e_i$ is the $i$-th canonical basis in $\mathbb{R}^n$. Then $\mathcal{L}^{(k)}$ is independent of $\mathcal{L}_k$ because $\mathcal{L}^{(k)}$ only depends on $(A_{ij})_{i,j \notin \mathcal{N}_k}$, which are independent of $A_k$, and $\mathcal{L}_k$ is a function of $A_k$. Let $\mathcal{V}(\delta)$ denote the event that $\|E\|_{op} \leq E_2(\delta/2), \|\tilde{E}\|_{op} \leq \tilde{E}_2(\delta/2)$. Then $\mathbb{P}(\mathcal{V}(\delta)) \geq 1 - \delta$ and it is left to prove the deterministic inequalities on the event $\mathcal{V}(\delta)$. First,

$$\|\mathcal{L}^{(k)} - \mathcal{L}\|_{op}^2 \leq \|\mathcal{L}^{(k)} - \mathcal{L}\|_{F}^2,$$

$$= \sum_{i \in \mathcal{N}_k, j \neq i} (\mathcal{L}_{ij}^{(k)} - \mathcal{L}_{ij})^2 + \sum_{i \in \mathcal{N}_k} (\mathcal{L}_{ii}^{(k)} - \mathcal{L}_{ii})^2 + \sum_{i \in \mathcal{N}_k} (\sum_{j \in \mathcal{N}_k} A_{ij})^2$$

$$= \sum_{i \in \mathcal{N}_k, j \neq i} (A_{ij}^2 + A_{ji}^2) + \sum_{i \in \mathcal{N}_k} (\mathcal{L}_{ii}^{(k)} - \mathcal{L}_{ii})^2 + \sum_{i \in \mathcal{N}_k} \left(\sum_{j \in \mathcal{N}_k} A_{ij}\right)^2$$

$$\stackrel{(i)}{=} \sum_{i \in \mathcal{N}_k, j \neq i} (A_{ij}^2 + A_{ji}^2) + \sum_{i \in \mathcal{N}_k} E_{ii}^2 + |\mathcal{N}_k| \sum_{i \in \mathcal{N}_k} |\mathcal{N}_k| A_{ij}^2$$

$$\stackrel{(ii)}{=} 2 \sum_{i \in \mathcal{N}_k, j \neq i} A_{ij}^2 + \sum_{i \in \mathcal{N}_k} E_{ii}^2 + |\mathcal{N}_k| \sum_{i \in \mathcal{N}_k} A_{ij}^2$$

$$\leq (m + 2) \sum_{i \in \mathcal{N}_k} \|A_i\|_{2\to\infty}^2 + \sum_{i \in \mathcal{N}_k} E_{ii}^2$$

$$\leq (m + 1)^2 \|A\|_{2\to\infty}^2 + m \|E\|_{op}^2$$

where (i) uses the Cauchy-Schwarz inequality, (ii) uses the symmetry of $A$ and (iii) uses the fact that $(m + 2)m \leq (m + 1)^2$ and that $|E_{ii}| \leq \|E\|_{op}$. As a result,

$$\|\mathcal{L}^{(k)} - \mathcal{L}\|_{op} \leq (m + 1)\|A\|_{2\to\infty} + \sqrt{m}\|E\|_{op}$$

$$\leq (m + 1)\|A^*\|_{2\to\infty} + (m + 1)\|A - A^*\|_{2\to\infty} + \sqrt{m}\|E\|_{op}$$

$$= (m + 1)\|A^*\|_{2\to\infty} + (m + 1)\|\tilde{E}\|_{2\to\infty} + \sqrt{m}\|E\|_{op}$$

$$\leq m\|A^*\|_{2\to\infty} + m\tilde{E}_2(\delta/2) + \sqrt{m}\tilde{E}_2(\delta/2)$$
Then

By definition,

with

By Lemma 3.8 and Lemma 3.10,

\[ L \]

On the other hand, we can derive a decomposition of \( L^{(k)} - L \). Let \( \tilde{L}_1, \tilde{L}_2, \tilde{L}_3 \in \mathbb{R}^n \) be three matrices with

\[ \tilde{L}_{1,ij} = L_{ij} I(i \in N_k, j \in [n]), \quad \tilde{L}_{2,ij} = L_{ij} I(i \in N_k^c, j \in N_k), \quad \tilde{L}_3 = \sum_{i \in N_k} L_i^* e_i e_i^T. \]

By definition,

\[ L - L^{(k)} = \tilde{L}_1 + \tilde{L}_2 - \tilde{L}_3. \]

Then

\[
\| (L^{(k)} - L)U \|_{\text{op}} \leq \| \tilde{L}_1 U \|_{\text{op}} + \| \tilde{L}_2 U \|_{\text{op}} + \| \tilde{L}_3 U \|_{\text{op}} \\
\leq \sum_{i \in N_k} \left( \| L_i^T U \|_2 + \| \tilde{L}_{2i} U_i \|_2 \right) + \sum_{i \in N_k} \| L_i^* e_i U_i \|_{\text{op}} \\
\leq \sum_{i \in N_k} \left( \| (LU)_i \|_2 + \| \tilde{L}_{2i} U_i \|_2 \right) + \sum_{i \in N_k} \| L_i^* U_i \|_2 \\
= \sum_{i \in N_k} \left( \| (U \Lambda) U_i \|_2 + \| \tilde{L}_{2i} U_i \|_2 \right) + \sum_{i \in N_k} \| L_i^* U_i \|_2 \\
= \sum_{i \in N_k} \| U_i^T \Lambda \|_2 + \| \tilde{L}_{2i} U_i \|_2 + \sum_{i \in N_k} \| L_i^* U_i \|_2 \\
\leq \sum_{i \in N_k} (\lambda_{\text{max}}(\Lambda) + \| \Lambda \|_2) \| U_i \|_2 + \sum_{i \in N_k} \| \tilde{L}_{2i} \| \| U_i \|_2 \\
\leq \left( m \lambda_{\text{max}}(\Lambda) + m \| \Lambda \|_2 + m \max_i \| \tilde{L}_{2i} \| \right) \| U \|_{2 \to \infty}. 
\]

By Weyl’s inequality and the triangle inequality, we arrive at

\[
\| (L^{(k)} - L)U \|_{\text{op}} \leq m \left( \lambda_{\text{max}}^* + \| \Lambda^* \|_{2 \to \infty} + 2 \| E \|_{\text{op}} + \max_i \| \tilde{L}_{2i} \| \right) \| U \|_{2 \to \infty}. 
\]

Thus on event \( \mathcal{V}(\delta) \), by Lemma C.2, we have \( L_2(\delta) = m \) and \( L_3(\delta) \leq m(\sqrt{n \delta} + \sqrt{\log(n/\delta)}) \).

\[ \square \]

**Proof of Theorem 3.11.** For notational convenience we will suppress the notation \( (\delta) \) for all quantities that involve it. First we prove that assumption A’4 is satisfied. By Lemma 3.7 and Lemma 3.8,

\[ \hat{\eta} \leq \sqrt{m \hat{p}}^* + \sqrt{\log(n/\delta)} + \frac{R}{a \log R} \leq g. \]

By Lemma 3.10 and the fact that \( \lambda_{\text{max}}^* \geq \Delta^* \geq g \geq \sqrt{\log(n/\delta)} \),

\[ \hat{\kappa}^* L_2 + L_3 + 1 \leq \hat{\kappa}^* + \frac{n \hat{p}^* + \sqrt{\log(n/\delta)}}{\lambda_{\text{min}}^*} \leq \hat{\kappa}'. \]

By Lemma 3.8 and Lemma 3.10,

\[ L_1 + \lambda - E_+ \leq M. \]

Putting pieces together, we have

\[ \Theta \hat{\delta} \leq \Theta(\hat{\kappa}'g + M), \quad \Theta \hat{\delta} + L_1 + \lambda - E_+ \leq \Theta \hat{\kappa}'g + (\Theta + 1)M \leq \Theta \hat{\kappa}'g + (\Theta + 1)M. \]
Thus, when $C$ is large enough, assumption $A^4$ is satisfied.

Next we prove the bound for $d_{2 \to \infty}(U, U^* + V)$. We bound each term in Theorem 2.5 separately. By Lemma 3.8 and (167),

$$\frac{E^2}{(\Delta^*)^2} + \frac{\Theta \tilde{\sigma}}{\Delta^*} \leq \frac{M^2}{(\Delta^*)^2} + \frac{\Theta (\tilde{k}' g + M)}{\Delta^*},$$

(168)

By Lemma 3.9,

$$\|U^*\|_{2 \to \infty} + \frac{\Theta \|E U^*\|_{2 \to \infty}}{\lambda^*_{\min}} \leq \left(1 + \frac{\Theta (M + r)}{\lambda^*_{\min}}\right) \|U^*\|_{2 \to \infty} + \frac{\Theta \sqrt{R p^*}}{\lambda^*_{\min}}$$

$$\leq \left(1 + \frac{\Theta r}{\lambda^*_{\min}}\right) \|U^*\|_{2 \to \infty} + \frac{\Theta \sqrt{R p^*}}{\lambda^*_{\min}},$$

(169)

where the last line uses the condition that $\lambda^*_{\min} \geq \Delta^* \geq \Theta M$. By Lemma 3.7 and Lemma 3.8,

$$\frac{\Theta (\tilde{b}_2 + \|\tilde{A}^i\|_{2 \to \infty}) E_+}{\Delta^* \lambda^*_{\min}} \leq \Theta M \sqrt{p^*} \left(\sqrt{n p^*} + \frac{\sqrt{R^2 + \alpha}}{\alpha \log R}\right).$$

(170)

By (168), (169), (170) and Theorem 2.5, we have

$$d_{2 \to \infty}(U, U^* + V)$$

$$\leq \left(\frac{M^2}{(\Delta^*)^2} + \frac{\Theta (\tilde{k}' g + M)}{\Delta^*}\right) \left(1 + \frac{\Theta r}{\lambda^*_{\min}}\right) \|U^*\|_{2 \to \infty} + \frac{\Theta \sqrt{R p^*}}{\lambda^*_{\min}} \left(\sqrt{n p^*} + \frac{\sqrt{R^2 + \alpha}}{\alpha \log R}\right).$$

This completes the proof of the first bound. For the second one, we apply Theorem 2.6. By Lemma 3.9,

$$\frac{\Theta \|E U^*\|_{2 \to \infty}}{\lambda^*_{\min}} \leq \frac{\Theta (M + r)}{\lambda^*_{\min}} \|U^*\|_{2 \to \infty} + \frac{\Theta \sqrt{R p^*}}{\lambda^*_{\min}}.$$

By the triangle inequality,

$$d_{2 \to \infty}(U, U^*)$$

$$\leq \left\{ \left(\frac{M^2}{(\Delta^*)^2} + \frac{\Theta (\tilde{k}' g + M)}{\Delta^*}\right) \left(1 + \frac{\Theta r}{\lambda^*_{\min}}\right) + \frac{\Theta (M + r)}{\lambda^*_{\min}} \right\} \|U^*\|_{2 \to \infty} + \frac{\Theta M \sqrt{p^*}}{\lambda^*_{\min}} \sqrt{n p^*} + \frac{\sqrt{R^2 + \alpha}}{\alpha \log R}.$$

The proof is then completed by assumption $A^4$ that

$$\frac{M^2}{(\Delta^*)^2} + \frac{\Theta (\tilde{k}' g + M)}{\Delta^*} \leq 1.$$

Proof of Lemma 3.12. By (166), with probability $1 - \delta$,

$$\max_k |\mathcal{L}_{kk} - \mathcal{L}_{kk}^*| \leq 4 M (\delta) \leq \frac{4}{5} \min_j \min_{k \in [n]} \{ |\Lambda_{jj}^* - \mathcal{L}_{kk}^*| \}.$$

As a result,

$$\Theta(\delta) \leq \frac{\min_j \min_{k \in [n]} \{ |\Lambda_{jj}^*| \}}{\min_j \min_{k \in [n]} \{ |\Lambda_{jj}^* - \mathcal{L}_{kk}^*| - 4 M (\delta) \}} \leq 5 \Theta^*. $$

□
D Other Proofs

D.1 Proofs in Section 4

Proof of Lemma 4.2. By Markov inequality, for any \( k \leq k_0 \),

\[
\Pr(|Z - EZ| \geq t) \leq t^{-k} \mathbb{E}|Z - EZ|^k = \left( \frac{\sigma \sqrt{k}}{t} \right)^k = \exp \left\{ -k \log \left( \frac{t}{\sigma} \right) + k \log k \right\} \triangleq \exp(g(k; t)).
\]

Note that \( g'(k; t) = \frac{\log k}{2} - \log(t/\sigma) \) and

\[
\log[(g')^{-1}(0; t)] = 2 \log(t/\sigma) - 1.
\]

When \( \sqrt{2}e \leq t/\sigma \leq \sqrt{k_0}e \), \( 2 \leq (g')^{-1}(0; t) \leq k_0 \). Thus if we let \( k = (g')^{-1}(0; t) = t^2/e\sigma^2 \), then

\[
\Pr(|Z - EZ| \geq t) \leq \exp \left\{ -\frac{t^2}{2c^2} \right\}.
\]

When \( t/\sigma < \sqrt{2}e \),

\[
\Pr(|Z - EZ| \geq t) \leq 1 \leq \exp \left\{ 1 - \frac{t^2}{2c^2} \right\}.
\]

\( \blacksquare \)

Proof of Lemma 4.3. Assume that \( C_0 > \max\{64C^2, 12C, e\} \) and \( n \geq 16 \). Let \( h_1(y) = y - \log y \), \( h_2(y) = \log y / \log \log y \). It is easy to see that \( h_1 \) is increasing on \([1, \infty)\) and \( h_1(y) \geq 1 \). Since \( h_2(y) = \exp\{h_1(\log \log y)\} \), \( h_2 \) is increasing on \([e^e, \infty)\). Since \( n \geq 16 > e^e \), \( g(\delta) \) is decreasing. Noting that \( \bar{\kappa}^* = 1 \) and \( \Delta^* = (n - 1)p \), it is left to prove that

\[
g(\delta^*) = \sqrt{np} + \frac{\log(n/\delta^*)}{\log \log(n/\delta^*)} \leq \Delta^*/C.
\]

Since \( p \geq C_0 \log n / (n \log \log n) \), we have \( np > C_0 h_2(n) > C_0 \). As a result,

\[
\sqrt{np} \leq \frac{np}{\sqrt{C_0}} \leq \frac{2(n - 1)p}{\sqrt{C_0}} \leq \frac{2\Delta^*}{\sqrt{C_0}} \leq \frac{\Delta^*}{4C}.
\]

Thus it is left to show that

\[
h_2(n/\delta^*) = \frac{\log(n/\delta^*)}{\log \log(n/\delta^*)} \leq \frac{3\Delta^*}{4C}.
\]

By definition,

\[
\log(n/\delta^*) = \frac{np \log(np)}{2C} + \log n.
\]

By (34),

\[
np \log(np) \geq C_0 \log n \log \log n (\log C_0 + \log \log n - \log \log \log n) \geq C_0 \log(n) (1 - \exp\{-h_1(\log \log \log n)\}) \geq C_0 \log(n) (1 - \exp\{-1\}) \geq \frac{C_0}{2} \log n > 6C \log n.
\]

As a result,

\[
\log(n/\delta^*) \leq \frac{2}{3C} np \log(np)
\]

On the other hand, recalling that \( np > C_0 > e \),

\[
\log \log(n/\delta^*) \geq \log \log(1/\delta^*) = \log(np) + \log \log(np) > \log(np).
\]

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The proof is then completed by
\[
\frac{2n}{3C} \leq \frac{2n}{3(n-1)C} \Delta^* \leq \frac{32}{45C} \Delta^* \leq \frac{3}{4C} \Delta^*.
\]

**Proof of Theorem 4.8.** We follow the same pipeline as the proof for Erdős-Rényi graphs. First it is easy to see by modifying (33) that
\[
V_+ \leq \|u_1\|_\infty^4 \tilde{W}, \quad \text{where } \tilde{W} = 4 \sum_{i<j} (p_{ij} + (1 - 2p_{ij})A_{ij}).
\]
In addition, \(V_+ \leq 2\) almost surely. Similarly we have \(\|u_1\|_\infty^2 \leq 2\).

Next, we derive the minimal \(\delta\) for Corollary 3.6 to work. Since \(\bar{\kappa}^* = 1\) and \(C_0\) is sufficiently large, it is easy to see that \(\delta^*\), as in Lemma 4.3, can be chosen as follows
\[
\delta^* = \exp \left\{ -\Delta^* \log \frac{\Delta^*}{2C} \right\}.
\]
By Corollary 3.6 with \(\alpha = 0.5\) and \(\delta \geq \delta^*\),
\[
d_{2 \to \infty}(u_1, u^*_1) \leq \left( \frac{\sqrt{np} + \log(n/\delta)}{\Delta^*} + \frac{\log(n/\delta)}{|\lambda_1^*|} \right) \frac{\zeta}{\sqrt{n}} + \frac{\sqrt{\log(n/\delta)p^*}}{\lambda_1^*}
\]
\[
\leq \frac{\sqrt{np} + \log(n/\delta)}{\Delta^*} \frac{\zeta}{\sqrt{n}} + \frac{\sqrt{\log(n/\delta)p^*}}{\lambda_1^*}
\]
\[
\leq \frac{\sqrt{np} + \log(n/\delta)}{\Delta^*} \frac{\zeta}{\sqrt{n}} + \frac{\log(n/\delta)}{np^*} \frac{\zeta}{\sqrt{n}}
\]
\[
\leq \left( \frac{\sqrt{np} + \log(n/\delta)}{\Delta^*} + \sqrt{\frac{\log(n/\delta)}{\Delta^*}} \right) \frac{\zeta}{\sqrt{n}}
\]
\[
\leq \left( 1 + \sqrt{\frac{np}{\Delta^*}} + \log(n/\delta) \right) \frac{\zeta}{\sqrt{n}},
\]
where (i) uses the fact that \(|\lambda_1^*| \geq \Delta^*\), (ii) uses the assumption that \(\lambda_1^* \geq np^*/\zeta\), (iii) applies the inequality that \(\Delta^* \leq |\lambda_1^*| \leq np^*\) and (iv) applies the simple inequality that \(2 \sqrt{y} \leq y + 1\). By the triangle inequality and noting that \(\sqrt{n}\|u_1\|_\infty = \zeta\), there exists a universal constant \(C_1\) such that for each \(\delta \geq \delta^*\),
\[
\sqrt{n}\|u_1\|_\infty \leq C_1 \left( 1 + \sqrt{\frac{np}{\Delta^*} + \log(n/\delta)} \right) \frac{\zeta}{\sqrt{n}} = C_1 \left( 1 + \frac{\sqrt{np} + \log n}{\Delta^*} \right) \frac{\zeta}{\sqrt{n}} + C_1 \frac{\zeta}{\Delta^*} \log \left( \frac{1}{\delta} \right),
\]
with probability \(1 - \delta\). Denote by \(B_u\) the RHS of (171) with \(\delta = \delta^*\) and by \(V_1\) the event that \(\sqrt{n}\|u_1\|_\infty \leq B_u\). Then
\[
\mathbb{P}(V_1) \geq 1 - \delta^* = 1 - \exp \left\{ -\frac{\Delta^* \log \Delta^*}{2C} \right\}.
\]
Let \(V_2\) denote the event that \(\tilde{W} \leq 10n^2\bar{p}\). Using the same argument above (38) in the proof of Lemma 4.4, we can show that
\[
\mathbb{P}(V_2) \geq 1 - \exp\{-n^2\bar{p}/3\}.
\]
Let \(V = V_1 \cap V_2\). Then
\[
\mathbb{P}(V) \leq \exp \left\{ -\frac{\Delta^* \log \Delta^* \wedge n^2\bar{p}}{2C \vee 3} \right\}.
\]

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Using the same argument as in the proof of Lemma 4.4, it is easy to derive the following analogue of (40):

$$\left( \mathbb{E}(\sqrt{n}\|u_1\|_\infty)^{2k} \right)^{1/k} \leq \zeta^2 \left\{ 1 + \left( \frac{k \vee (\sqrt{n+p^2} + \log n)}{\Delta^*} \right)^2 \right\}. $$

Thus,

$$ \left( \mathbb{E}V_{k/2}^k \right)^{1/k} = \left( \mathbb{E}V_{k/2}^k + \mathbb{E}V_{k/2}^k_{\nu} \right)^{1/k} \leq \left( \mathbb{E}V_{k/2}^k \right)^{1/k} + \left( \mathbb{E}V_{k/2}^k_{\nu} \right)^{1/k} $$

$$ \leq \sqrt{p}\mathbb{E} \left( (\sqrt{n}\|u_1\|_\infty)^{2k} \right)^{1/k} + \exp \left\{ - \frac{\Delta^* \log \Delta^* \wedge n^2 \bar{p}}{2C \vee 3} \right\}.$$

Using the same argument for (40) in the proof of Lemma 4.4, we can show that

$$ \left( \mathbb{E}(\sqrt{n}\|u_1\|_\infty)^{2k} \right)^{1/k} \leq \left\{ 1 + \left( \frac{k \vee (\sqrt{n+p^2} + \log n)}{\Delta^*} \right)^2 \right\} \zeta^2. $$

As a result,

$$ \left( \mathbb{E}[V_{k/2}] \right)^{1/k} \leq \bar{p} \left\{ 1 + \left( \frac{k \vee (\sqrt{n+p^2} + \log n)}{\Delta^*} \right)^2 \right\} \zeta^2 + \exp \left\{ - \frac{\Delta^* \log \Delta^* \wedge n^2 \bar{p}}{2C \vee 3} \right\}. $$

Let $k = 2$ and by Efron-Stein inequality, we obtain that

$$ \text{Var}(\|A\|_{op}) \leq \mathbb{E}[V_2] \leq \bar{p} \left\{ 1 + \left( \frac{\sqrt{n+p^2} + \log n}{\Delta^*} \right)^4 \right\} \zeta^4 + \exp \left\{ - \frac{\Delta^* \log \Delta^* \wedge n^2 \bar{p}}{2C \vee 3} \right\}. $$

On the other hand, recalling that $M \leq 2\|u_1\|_\infty^2$, similarly to (172) we have

$$ \left( \mathbb{E}[M^k] \right)^{1/k} \leq \frac{1}{n} \left\{ 1 + \left( \frac{k \vee (\sqrt{n+p^2} + \log n)}{\Delta^*} \right)^2 \right\} \zeta^2 + \exp \left\{ - \frac{\Delta^* \log \Delta^* \wedge n^2 \bar{p}}{2C \vee 3} \right\}. $$

Then by Proposition 4.1, we obtain that

$$ \left( \mathbb{E}[\|A\|_{op} - \mathbb{E}[\|A\|_{op}]^k \right)^{1/k} \leq \left( \sqrt{\bar{p}} + \frac{k}{n} \right) \left\{ 1 + \left( \frac{\sqrt{n+p^2} + \log n}{\Delta^*} \right)^2 \right\} \zeta^2 + k \exp \left\{ - \frac{\Delta^* \log \Delta^* \wedge n^2 \bar{p}}{2C \vee 3} \right\}. $$

Let

$$ k_0 = \frac{1}{2C \vee 3} \min \left\{ \Delta^*, \frac{\Delta^* \log \Delta^*}{\log(1/p\zeta^2)}, \frac{n^2 \bar{p}}{\log(n^2 \bar{p})}, \frac{n^2 \bar{p}}{\log(1/p\zeta^2)} \right\}. $$

Consider any $k \leq k_0$. Since $n^2 \bar{p} \geq 1$, $k_0 \leq n^2 \bar{p}$. Thus,

$$ \sqrt{\bar{p}} + \frac{k}{n} \leq \sqrt{k \bar{p}}. $$

For the second term,

$$ k \exp \left\{ - \frac{\Delta^* \log \Delta^* \wedge n^2 \bar{p}}{(2C \vee 3) \bar{p}} \right\} = \sqrt{k \bar{p} \zeta} \exp \left\{ - \frac{\Delta^* \log \Delta^* \wedge n^2 \bar{p}}{2C \vee 3} + \frac{1}{2} \log k + \frac{1}{2} \log \left( \frac{1}{p\zeta^2} \right) \right\}. $$

Since $k \leq \frac{\Delta^* \log \Delta^* \wedge n^2 \bar{p}}{2(2C \vee 3) \log(1/p\zeta^2)}$,

$$ \frac{1}{2} \log \left( \frac{1}{p\zeta^2} \right) \leq \frac{\Delta^* \log \Delta^* \wedge n^2 \bar{p}}{2(2C \vee 3) \bar{p}}. $$

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Since $k \leq \frac{\Delta^* \wedge n^2 \bar{p}}{2C \sqrt{3}}$,

$$k \log k \leq \frac{\Delta^* \log \Delta^*}{2C \sqrt{3}},$$

and

$$k \log k \leq \frac{1}{2C \sqrt{3}} \frac{n^2 \bar{p} \left( \log(n^2 \bar{p} / (2C \sqrt{3})) - \log(n^2 \bar{p} / (2C \sqrt{3})) \right)}{\log(n^2 \bar{p} / (2C \sqrt{3}))} \leq \frac{n^2 \bar{p}}{2C \sqrt{3}},$$

where we use the condition that $n^2 \bar{p}$ is sufficiently large. Thus,

$$\frac{1}{2} \log k \leq \frac{\Delta^* \log \Delta^* \wedge n^2 \bar{p}}{2(2C \sqrt{3})k}.$$

$$\sqrt{k \bar{p} \zeta} \exp \left\{ -\frac{\Delta^* \log \Delta^* \wedge n^2 \bar{p}}{(2C \sqrt{3})k} + \frac{1}{2} \log k + \frac{1}{2} \log \left( \frac{1}{\bar{p} \zeta^2} \right) \right\} \leq \sqrt{k \bar{p} \zeta}.$$

Putting pieces together into (174), we conclude that

$$(\mathbb{E}\|A\|_{op} - \mathbb{E}\|A\|_{op})^{1/k} \leq \sqrt{k \bar{p}} \left\{ 1 + \left( \frac{\sqrt{n \bar{p}} \xi + \log n}{\Delta^*} \right)^2 \right\} \zeta^2.$$

Finally, the proof is completed by Lemma 4.2. \hfill \Box

### D.2 Proofs in Section 5

**Proof of Lemma 5.1.** First we prove that $\hat{v}_r$’s defined in (47) are distinct. By definition,

$$\frac{1}{n} \sum_{i=1}^{n} \min_{r \in [K]} \|U_i - \hat{v}_r\|_2 \leq \frac{1}{n} \sum_{i=1}^{n} \min_{r \in [K]} \|U_i - v'_r\|_2 = \frac{1}{n} \sum_{i=1}^{n} \min_{j \in [n]} \|U_i - \hat{U}_j\|_2$$

$$\leq \frac{1}{n} \sum_{i=1}^{n} \|U_i - \hat{U}_i\|_2 \leq \max_i \|U_i - \hat{U}_i\|_2.$$

By the triangle inequality,

$$\frac{1}{n} \sum_{i=1}^{n} \min_{r \in [K]} \|U_i - \hat{v}_r\|_2 \geq \frac{1}{n} \sum_{i=1}^{n} \min_{r \in [K]} \|\hat{U}_i\|_2 - \hat{v}_r\|_2 - \max_i \|U_i - \hat{U}_i\|_2.$$

The above two inequalities imply that

$$\frac{1}{n} \sum_{i=1}^{n} \min_{r \in [K]} \|\hat{U}_i\|_2 - \hat{v}_r\|_2 \leq 2 \max_i \|U_i - \hat{U}_i\|_2. \quad (175)$$

For each $i$, let $r_i = \arg \min_{r \in [K]} \|\hat{U}_i - \hat{v}_r\|_2$. Since $\hat{U}_i = \hat{U}_i$ if $c_i = c_{i'}$, it must be true that $r_i = r_{i'}$. Write $r_i$ as $r_s$ for $i \in C_s$. Then

$$\frac{1}{n} \sum_{i=1}^{n} \min_{r \in [K]} \|\hat{U}_i - \hat{v}_r\|_2 = \sum_{s=1}^{K} \pi_s \|v^*_s - \hat{v}_{r_s}\|_2.$$

For any $s$, (175) implies that

$$\pi_s \|v^*_s - \hat{v}_{r_s}\|_2 \leq 2 \max_i \|U_i - \hat{U}_i\|_2 \implies \|v^*_s - \hat{v}_{r_s}\|_2 \leq \frac{2}{\min_{r \in [K]} \pi_r} \max_i \|U_i - \hat{U}_i\|_2. \quad (176)$$

By the triangle inequality,

$$\|\hat{v}_{r_s} - \hat{v}_{r_{s'}}\|_2 \geq \|v^*_s - v^*_s\|_2 - \|v^*_s - \hat{v}_{r_s}\|_2 - \|v^*_s - \hat{v}_{r_{s'}}\|_2.$$
On the other hand, for any
This proves (177) and hence completes the proof.

On the other hand, for any $i \in C_s$,

$$\arg\min_{r \in [K]} \|U_i - \hat{v}_r\|_2 = s. \quad (177)$$

By the triangle inequality and (176),

$$\|U_i - \hat{v}_s\|_2 \leq \|U_i - \hat{U}_s\|_2 + \|\hat{v}_s - \hat{v}_s\|_2 \leq \max_i \|U_i - \hat{U}_s\|_2 + \frac{2}{\min_{r \in [K]} \pi_r} \max_i \|U_i - \hat{U}_s\|_2$$

On the other hand, for any $s' \neq s$,

$$\|U_i - \hat{v}_{s'}\|_2 \geq -\|U_i - \hat{U}_{s'}\|_2 + \|\hat{v}_{s'} - \hat{v}_{s'}\|_2 \geq -\|U_i - \hat{U}_{s'}\|_2 + \|\hat{v}_{s'} - \hat{v}_{s'}\|_2 - \|\hat{v}_{s'} - \hat{v}_{s'}\|_2$$

$$\geq \|\hat{v}_{s'} - \hat{v}_{s'}\|_2 - \frac{3}{\min_{r \in [K]} \pi_r} \max_i \|U_i - \hat{U}_{s'}\|_2$$

This proves (177) and hence completes the proof. \qed

**Proof of Lemma 5.4.** Let $V^* = \begin{bmatrix} U^* & \hat{U}^* \end{bmatrix}$. Then

$$V^*(V^*)^T = U^*(U^*)^T + \hat{U}^*(\hat{U}^*)^T = QVV^TQ^T + \begin{bmatrix} Q_1 \text{T} Q_1^T & 0 & \ldots & 0 \\ 0 & Q_2 \text{T} Q_2^T & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & Q_K \text{T} Q_K^T \end{bmatrix}$$

$$= QQ^T + \begin{bmatrix} I_{n_1} - 1_{n_1} & 0 \& \ldots \& 0 \\ 0 & I_{n_2} - 1_{n_2} & \ldots \& 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I_{n_K} - 1_{n_K} \end{bmatrix} = I_n.$$

Thus $V^*$ is orthogonal. Then it is left to prove that

$$\mathcal{L}^* U^* = U^* \Lambda^*, \quad \mathcal{L}^* \hat{U}^* = \hat{U}^* \hat{\Lambda}^*.$$  

(178)

The first equation is equivalent to

$$\mathcal{L}^* Q V = n \rho_n Q V \Sigma \iff \mathcal{L}^* Q = n \rho_n Q \bar{\Sigma}$$

Note that

$$\bar{D}^* = \begin{bmatrix} d_1^* I_{n_1} & 0 & \ldots & 0 \\ 0 & d_2^* I_{n_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_K^* I_{n_K} \end{bmatrix}$$

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where
\[ d_i^* = \sum_{j=1}^{n} n_j (\rho_n B_{0,ij}) = n\rho_n \sum_{j=1}^{n} \pi_j B_{0,ij} = n\rho_n \hat{d}_i. \]

Since $L^*$ does not depend on the diagonal elements of $A^*$, we have $L^* = \hat{D}^* - \hat{A}^*$ where $\hat{A}^*$ is defined in (45) and $\hat{D}^* = \text{diag}(\hat{A}^*1_n)$. Then
\[
\hat{D}^*Q = \begin{bmatrix}
\frac{d_1^*}{\sqrt{n_1}}1_{n_1} & 0 & \ldots & 0 \\
0 & \frac{d_2^*}{\sqrt{n_2}}1_{n_2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \frac{d_K^*}{\sqrt{n_K}}1_{n_K}
\end{bmatrix} = n\rho_n Q \hat{D}.
\]

On the other hand,
\[ \hat{A}^*Q = n\rho_n Q(RB_0R)Q^TQ = n\rho_n Q(RB_0R). \]
As a result,
\[ L^*Q = \hat{D}^*Q - \hat{A}^*Q = n\rho_n Q \hat{L}. \]
This proves the first equation of (178). To prove the second one, notice that
\[ Q^T\hat{U}^* = 0 \implies \hat{A}^*\hat{U}^* = 0. \]
Thus,
\[ L^*\hat{U}^* = \hat{D}^*\hat{U}^* = \begin{bmatrix}
d_1^*Q_1 & 0 & \ldots & 0 \\
0 & d_2^*Q_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & d_K^*Q_K
\end{bmatrix} = \hat{U}^*\hat{A}^*. \]
This proves the second equation of (178) and thus completes the proof.

**Proof of Theorem 5.3.** First we note that $L^*$ does not depend on the diagonal elements of $A^*$. Thus we can pretend $A^* = \hat{A}^*$ without loss of generality. Next we note that the smallest eigenvalue of $L^*$ is 0 with an eigenvector $1_n$. Since it is a constant for all units, the output of $K$-medians is not affected if it is removed. Thus, we can take $\Lambda^* \in \mathbb{R}^{(K-1) \times (K-1)}$ as the diagonal matrix of the second to the $K$-th smallest eigenvalues of $L^*$ and $U^* \in \mathbb{R}^{n \times (K-1)}$ as the corresponding eigenvector matrix.

Let $\lambda_{(2)}(\cdot)$ denote the second smallest eigenvalue and
\[ \beta = \frac{1}{2} \min\{\lambda_{(2)}(\tilde{L}_0), \lambda_{\min}(\tilde{D}_0) - \lambda_{\max}(\tilde{L}_0)\}. \]
Define $R, \tilde{D}$ and $\tilde{L}$ as in Lemma 5.4. Then
\[ \tilde{L} \rightarrow \tilde{L}_0, \quad \tilde{D} \rightarrow \tilde{D}_0. \]
Thus there exists a constant $n_0$ that only depends on $B_0$ and $\pi_r$'s such that
\[ \min\{\lambda_{(2)}(\tilde{L}), \lambda_{\min}(\tilde{D}) - \lambda_{\max}(\tilde{L})\} > \beta. \quad (179) \]
By Lemma 5.4,
\[ \lambda_{\min} \geq \Delta^* \geq n\rho_n \beta. \quad (180) \]
Furthermore, the matrix $U^*$ in this proof differs from the one in Lemma 5.4 by just a column of $1_n$. By Lemma 5.4, 
\[
\begin{bmatrix}
1_n & U^*
\end{bmatrix} = QV.
\]
It is easy to see that $U_i^* = \nu_i^*$ if $i \in C_s$ and thus,
\[
\|\nu_s^* - \nu_s'\|_2 = \left\| \frac{1}{\sqrt{n_s}} V_s - \frac{1}{n_s'} \right\|_2 = \sqrt{\frac{1}{n_s} + \frac{1}{n_s'}} \geq \frac{1}{\min_{s \in [K]} \sqrt{s}} \frac{1}{\sqrt{n}}.
\]
Moreover,
\[
\|U^*\|_2 \rightarrow \infty \leq \|QV\|_2 \rightarrow \infty \leq \frac{1}{\min_{s \in [K]} \sqrt{s}} \leq \frac{1}{\min_{s \in [K]} \sqrt{s}} \frac{1}{\sqrt{n}}.
\]
By Lemma 5.1, it is left to prove
\[
d_2 \rightarrow \infty(U, U^*) \leq \min_{s \in [K]} \sqrt{s} \frac{1}{\sqrt{n}} \leq \frac{c_1}{\sqrt{n}}.
\]  
(181)

Set
\[
\delta = n^{-q}, \quad \alpha = 1/\log R(\delta)
\]
in Theorem 3.11. Note that this choice of $\alpha$ implies that
\[
\frac{R(\delta)}{\alpha \log R(\delta)} = R(\alpha), \quad R(\delta)^{1+\alpha} = R(\delta) \exp \{\alpha \log R(\delta)\} = eR(\delta).
\]
Then $\bar{\kappa}^* \leq 2(K - 1) \leq 1$, $p^* \leq \rho_n$,
\[
\bar{\kappa}' \leq 1 + \frac{np_n}{\rho_n \beta} \leq 1, \quad R(\delta) \leq \log n, \quad g(\delta) \leq \sqrt{n\rho_n + \log n}, \quad M(\delta) \leq \sqrt{n\rho_n \log n}.
\]
By (180), when $c$ in the condition (56) and $n$ are sufficiently large,
\[
\Delta^* \geq C(\bar{\kappa}' \gamma(\delta) + M(\delta)),
\]
where $C$ is the universal constant in (19). On the other hand, consider $\Theta^*$ in Lemma 3.12. By definition,
\[
\Lambda^*_{jj} = n\rho_n \Sigma_{jj}, \quad \Lambda^*_{kk} = n\rho_n \tilde{D}_{kk}
\]
where $\Sigma$ and $\tilde{D}$ are defined in Lemma 5.4. Then
\[
\frac{\mid \Lambda^*_{jj} \mid}{\mid \Lambda^*_{jj} - \Lambda^*_{kk} \mid} = \frac{\Sigma_{jj}}{\mid \Sigma_{jj} - \tilde{D}_{kk} \mid}.
\]
By (179),
\[
\tilde{D}_{kk} \geq \lambda_{\min}(\tilde{D}) > \lambda_{\max}(\tilde{L}) + \beta = \lambda_{\max}(\Sigma) + \beta \geq \Sigma_{jj} + \beta,
\]
and
\[
\Sigma_{jj} \leq \lambda_{\max}(\tilde{L}) \rightarrow \lambda_{\max}(\tilde{L}_0).
\]
As a result,
\[
\Theta^* = \min_{j \in [s+1,s+r]} \frac{\mid \Lambda^*_{jj} \mid}{\mid \Lambda^*_{jj} - \Lambda^*_{kk} \mid} \leq \frac{\lambda_{\max}(\tilde{L}_0)}{\beta} \leq 1,
\]
and
\[
\min_{j \in [s+1,s+r], k \in [n]} \mid \Lambda^*_{jj} - \Lambda^*_{kk} \mid \geq n\rho_n \beta \geq 5M(\delta)
\]
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When $c$ in the condition (56) and $n$ are sufficiently large. By Lemma 3.12, $\Theta(\delta) \lesssim 1$.

In summary, both conditions of Theorem 3.11 are satisfied. Then by Theorem 3.11, we have

$$d_{2 \to \infty}(U, U^*) \leq \left( \frac{n \rho_n \log n}{(n \rho_n)^2} + \frac{\sqrt{n \rho_n} + \sqrt{n \log n} + \sqrt{n \rho_n \log n}}{n \rho_n} + \frac{1}{n \rho_n} \right) \frac{1}{\sqrt{n}} + \frac{\sqrt{(\log n) \rho_n}}{n \rho_n}$$

$$+ \frac{\sqrt{n \rho_n \log n}}{(n \rho_n)^2} \left( \sqrt{n \rho_n} + \sqrt{n \log n} \right)$$

$$\lesssim \sqrt{\log n} \frac{1}{n \rho_n \sqrt{n}}.$$

Equivalently, there exists a constant $c_2$ that only depends on $B_0$, $q$ and $\pi_i$'s such that

$$d_{2 \to \infty}(U, U^*) \leq \sqrt{\log n} \frac{c_2}{n \rho_n}.$$

By condition (56),

$$d_{2 \to \infty}(U, U^*) \leq \frac{c_2}{\sqrt{c_1}} \frac{1}{\sqrt{n}}.$$

Therefore, (181) follows if $c > c_2^2 / c_1^2$. The proof is then completed.

Proof of Theorem 5.5 part (2). First we note that $L^*$ does not depend on the diagonal elements of $A^*$. Thus we can pretend $A^* = \tilde{A}^*$. In this case, $D^* = m \rho_n (a + (K - 1)b) I_n$. Thus, $L^*$ and $A^*$ have the same eigen-structure except that the eigenvalues of $L^*$ are equal to $m \rho_n (a + (K - 1)b)$ minus those of $A^*$. Similar to the proof of Theorem 5.3, we can ignore the first eigenvector of $L^*$ in the analysis and focus on the second to the $K$-th eigenvectors.

Equivalently, $U^*$ is taken as $U_2^*$ in part (1) and

$$\Lambda^* = m \rho_n (a + (K - 1)b) I_n - m \rho_n (a - b) I_n = m \rho_n K b I_n = n \rho_n b I_n.$$

As a result,

$$\|U^*\|_{2 \to \infty} \leq \frac{1}{\sqrt{m}}, \quad \lambda^*_{\min} = n \rho_n b, \quad \Delta^* = m \rho_n \min\{Kb, a - b\}, \quad \kappa^* = 1. \quad (182)$$

Using the same argument as (61), it is left to show that

$$d_{2 \to \infty}(U, U^*) \leq \frac{\sqrt{2}}{6K \sqrt{m}}. \quad (183)$$

Set $\delta = n^{-q}$ and $\alpha = 1 / \log R(\delta)$ in Theorem 3.11. Then $p^* \lesssim \rho_n$,

$$R(\delta) \leq \log n + K, \quad g(\delta) \leq \sqrt{n \rho_n} + \log n + K, \quad M(\delta) \leq \sqrt{n \rho_n \log n}, \quad \kappa' \leq 1 + \frac{n \rho_n}{n \rho_n b} \lesssim 1.$$

On the other hand,

$$\Lambda^*_{jj} = m \rho_n (a - b), \quad L^*_{kk} = D^*_{kk} = m \rho_n (a + (K - 1)b).$$

Let $\Theta^*$ be defined in Lemma 3.12. Then

$$\Theta^* = \frac{a - b}{K b} \leq 1,$$

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and for sufficiently large \( n \) and \( c \),
\[
|\Lambda^*_j - L^*_k| = m \rho_n K b = n \rho_n b \geq 5M(\delta).
\]

By Lemma 3.12, we have
\[
\Theta \preceq 1/K. \quad (184)
\]

Since \( n \rho_n > cK^3 \log n \), for sufficiently large \( n \) and \( c \),
\[
\Delta^* = m \rho_n \min\{Kb, a - b\} = n \rho_n \frac{\min\{Kb, a - b\}}{K} \geq C(\Theta(\delta)k'(\delta) + (\Theta(\delta) + 1)M(\delta)),
\]
where \( C \) is the universal constant in (19).

Thus both conditions of Theorem 3.11 are satisfied. By (182), (184) and Theorem 3.11,
\[
d_{2 \rightarrow \infty}(U, U^*) \leq \left( \frac{m \rho_n \log n}{(m \rho_n)^2} + \frac{\sqrt{n \rho_n} + \log n + K + \sqrt{n \rho_n \log n}}{K m \rho_n} + \frac{1}{n \rho_n} \right) \frac{1}{\sqrt{m}} + \frac{\sqrt{\log n \rho_n}}{K n \rho_n}
\]
\[
+ \frac{\sqrt{n \rho_n \log n \rho_n}}{K (m \rho_n)(n \rho_n)} \left( \sqrt{n \rho_n} + \sqrt{\log n} \right)
\]
\[
\leq \left( \frac{K^2 \log n}{n \rho_n} + \frac{\log n}{n \rho_n} \right) \frac{1}{\sqrt{m}}.
\]

It is straightforward to show that each term is bounded by \( 1/36K \) for sufficiently large \( c \). This proves (183) and hence the theorem.

\[\square\]

### D.3 Proofs in Section 6

**Proof of Lemma 6.3.** First we prove part (1). Let \( m = n/K \). By definition,
\[
Z_i = \sum_{d=1}^{d-1} X_{i0} + \sum_{j=1}^{d-1} \sum_{i=1}^{2^{j-1}} X_{ij} - \sum_{i=1}^{2^{d-1}} X_{id}
\]
where \( X_{ij} \overset{i.i.d.}{\sim} \text{Ber}(p_j) \). Then for any \( \nu > 0 \),
\[
\log \mathbb{E}[e^{-\nu Z_i}] = (m - 1) \log (p_0 e^{-\nu} + 1 - p_0) + m \sum_{j=1}^{d} 2^{j-1} \log (p_j e^{-\nu} + 1 - p_j) + m^{2d-1} \log (p_d e^{\nu} + 1 - p_d)
\]
\[
\leq (m - 1)p_0 (e^{-\nu} - 1) + m \sum_{j=1}^{d} 2^{j-1} p_j (e^{-\nu} - 1) + m^{2d-1} p_d (e^{\nu} - 1)
\]
\[
= \left( (m - 1)p_0 + m \sum_{j=1}^{d} 2^{j-1} p_j \right) (e^{-\nu} - 1) + m^{2d-1} p_d (e^{\nu} - 1)
\]
\[
= \frac{\lambda^*_1 + \lambda^*_2}{2} (e^{-\nu} - 1) + \frac{\lambda^*_1 - \lambda^*_2}{2} (e^{\nu} - 1),
\]
where the last line uses Proposition 6.1. Note that \( \lambda^*_2 > 0 \). Let
\[
\nu = \frac{1}{2} \log \frac{\lambda^*_1 + \lambda^*_2}{\lambda^*_1 - \lambda^*_2}.
\]

Then \( \nu > 0 \) and
\[
\log \mathbb{E}[e^{-\nu Z_i}] = -\frac{1}{2} \left( \sqrt{\lambda^*_1 + \lambda^*_2} - \sqrt{\lambda^*_1 - \lambda^*_2} \right)^2.
\]

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By Markov’s inequality,

\[
\log P(Z_i \leq t) = \log P(e^{-\nu Z_i} \geq e^{-\nu t}) \leq \frac{t}{2} \log \left( \frac{\lambda^*_1 + \lambda^*_2}{\lambda^*_1 - \lambda^*_2} \right) - \frac{1}{2} \left( \sqrt{\lambda^*_1 + \lambda^*_2} - \sqrt{\lambda^*_1 - \lambda^*_2} \right)^2.
\]

This proves the part (1). For part (2), as in part (1), for any \( \nu > 0, \)

\[
\log E[e^{\nu Z_i}] = \frac{\lambda^*_1 + \lambda^*_2}{2} (e^\nu - 1) + \frac{\lambda^*_1 - \lambda^*_2}{2} (e^{-\nu} - 1).
\]

Note that \( \lambda^*_2 < 0. \) Let

\[
\nu = \frac{1}{2} \log \frac{\lambda^*_1 - \lambda^*_2}{\lambda^*_1 + \lambda^*_2}.
\]

Then \( \nu > 0 \) and

\[
\log E[e^{\nu Z_i}] = -\frac{1}{2} \left( \sqrt{\lambda^*_1 + \lambda^*_2} - \sqrt{\lambda^*_1 - \lambda^*_2} \right)^2.
\]

By Markov’s inequality,

\[
\log P(Z_i \geq -t) = \log P(e^{\nu Z_i} \geq e^{-\nu t}) \leq \frac{t}{2} \log \left( \frac{\lambda^*_1 - \lambda^*_2}{\lambda^*_1 + \lambda^*_2} \right) - \frac{1}{2} \left( \sqrt{\lambda^*_1 + \lambda^*_2} - \sqrt{\lambda^*_1 - \lambda^*_2} \right)^2.
\]

This completes the proof.

**Proof of Theorem 6.4.** Throughout the proof we use the notation of Theorem 3.4. Then by Proposition 6.1,

\[
n \bar{p}^* = (m - 1)p_0 + m \sum_{i=1}^d 2^{i-1}p_i = \lambda^*_1, \quad \bar{r}^* = 1.
\]

Since \( \delta \geq n^{-\gamma}, \)

\[
\log n \leq R(\delta) \leq (q + 1) \log n \implies g(\delta) \leq \frac{(q + 1) \log n}{\alpha \log \log n}.
\]

Then (70) implies that

\[
\Delta^* > C \bar{r}^* g(\delta).
\]

Thus the condition of Theorem 3.4 is satisfied. By Theorem 3.4, with probability \( 1 - \delta \),

\[
\left\| u_2 - A u_2^* \right\|_\infty \leq \frac{1}{\Delta^*} \left\{ \left( \sqrt{\lambda^*_1} + \frac{\log n}{\alpha \log \log n} \right) \left( 1 + \frac{\log n}{|\lambda^*_2|} \right) \left\| u_2^* \right\|_{2 \rightarrow \infty}
\]
\[
+ \sqrt{\frac{(\log n)p^*}{|\lambda^*_2|}} \left( \sqrt{\lambda^*_1} + \frac{\log n}{\alpha \log \log n} + \frac{\sqrt{\lambda^*_1}(\log n)^\alpha}{\alpha \log \log n} \right)
\]
\[
+ \left( \sqrt{|\lambda^*_2|} + \sqrt{\log n} \right) \min \left\{ \frac{\sqrt{\lambda^*_1} p^*}{|\lambda^*_2|}, \frac{\sqrt{p^*}}{|\lambda^*_2| I(A^* \text{ is psd}) \sqrt{K}} \right\} \right\}
\]
\[
\overset{(i)}{\leq} \frac{1}{\sqrt{n \Delta^*}} \left\{ \left( \sqrt{\lambda^*_1} + \frac{\log n}{\alpha \log \log n} \right) \left( 1 + \frac{\log n}{|\lambda^*_2|} \right) + \sqrt{\frac{(\log n)p^*}{|\lambda^*_2|}} \log n + \frac{\sqrt{\lambda^*_1}(\log n)^\alpha}{\alpha \log \log n}
\]
\[
+ \sqrt{|\lambda^*_2|} \min \left\{ \frac{\sqrt{\lambda^*_1} p^*}{|\lambda^*_2|}, \frac{\sqrt{n p^*}}{|\lambda^*_2| I(A^* \text{ is psd})} \sqrt{K} \right\} \right\}
\]
\[
\overset{i}{\leq} \frac{\xi_{n1} + \xi_{n2}}{\sqrt{n \Delta^*}},
\]

where (i) uses the fact that

\[
\sqrt{\log n} \min \left\{ \frac{\sqrt{\lambda^*_1} p^*}{|\lambda^*_2|}, \frac{\sqrt{n p^*}}{|\lambda^*_2| I(A^* \text{ is psd})} \sqrt{K} \right\} \leq \sqrt{\log n} \frac{\sqrt{\lambda^*_1} p^*}{|\lambda^*_2|} = \frac{\sqrt{(\log n)p^*}}{|\lambda^*_2|} \sqrt{\lambda^*_1}
\]

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and
\[
\sqrt{\lambda_1^* + \frac{\lambda_1^*(\log n)^2}{\alpha \log \log n}} \leq \sqrt{\frac{\lambda_1^*(\log n)^2}{\alpha \log \log n}}.
\]
Equivalently, there exists a universal constant \( C' \) such that
\[
\sqrt{n} \left\| u_2 - \frac{Au_2^*}{\lambda_2^*} \right\|_\infty \leq C' \frac{\xi_{n1} + \xi_{n2}}{\Delta^*}
\]
with probability \( 1 - \delta \).

On the other hand, by Lemma 6.3, in the assortative case we have
\[
\log P \left( Z_i \leq C' \frac{\xi_{n1} + \xi_{n2}}{\Delta^*} \lambda_2^* \right) \leq C' \frac{\lambda_2^*}{\Delta^*} \left( \log \frac{\lambda_1^* + \lambda_2^*}{\lambda_1^* - \lambda_2^*} \right) \left( \xi_{n1} + \xi_{n2} \right) - \frac{1}{2} \left( \sqrt{\lambda_1^* + \lambda_2^*} - \sqrt{\lambda_1^* - \lambda_2^*} \right)^2.
\]
Under condition (70),
\[
P \left( Z_i \leq C' \frac{\xi_{n1} + \xi_{n2}}{\Delta^*} \lambda_2^* \right) \leq \exp \left\{ -\log n - \log \left( \frac{1}{\delta} \right) \right\} \leq \frac{\delta}{n}
\]
A simple union bound then implies that
\[
P \left( \min_{i \in [n]} Z_i \leq C' \frac{\xi_{n1} + \xi_{n2}}{\Delta^*} \lambda_2^* \right) \leq \delta.
\]
Finally, by (185),
\[
P \left( \min_{i \in [n]} Z_i \leq \left( \sqrt{n} \left\| u_2 - \frac{Au_2^*}{\lambda_2^*} \right\|_\infty \right) \lambda_2^* \right) \leq 2\delta.
\]
The proof for assortative BTSBM is then completed by (69). Similarly we can prove it for dis-assortative BTSBM.

\[\square\]

**Proof of Theorem 6.5.** It is left to show that for any node at \( r \)-th layer \((r \leq \ell)\), its first split can be exactly recovered with probability \( 1 - o(1) \) as \( n \) tends to infinity. Assume \( r = \ell \) without loss of generality. Note that this node corresponds to a BTSBM with size \( n' = n/2^{d-1} \) and parameters \((a_0, a_1, \ldots, a_{d-\ell+1})\). Throughout the rest of the proof, all symbols (e.g. \( \lambda_1^*, \lambda_2^*, \Delta^* \)) are defined for this sub-model.

Let \( \epsilon \in (0, 1) \) be any constant such that
\[
\frac{1}{2^{d-\ell+1}} \left( \sqrt{\bar{a}_d} - \sqrt{\bar{a}_d} \right)^2 > 1 + 3\epsilon.
\]
By Proposition 6.1 and definition of \( \bar{a}_\ell \),
\[
\lambda_1^* = m \rho_n \left( 2^{d-1} \bar{a}_\ell + 2^{d-1} \bar{a}_\ell \right) - \rho_n a_0 = \log n \left( \bar{a}_\ell + \frac{a_0}{2^{d-\ell+1}} - \frac{a_0}{n} \right),
\]
and
\[
\lambda_2^* = m \rho_n \left( 2^{d-1} \bar{a}_\ell - 2^{d-1} \bar{a}_\ell \right) - \rho_n a_0 = \log n \left( \bar{a}_\ell - \frac{a_0}{2^{d-\ell+1}} - \frac{a_0}{n} \right).
\]
As a result,
\[
\frac{1}{2} \left( \sqrt{\lambda_1^* + \lambda_2^*} - \sqrt{\lambda_1^* - \lambda_2^*} \right)^2 = \frac{1}{2^{d-\ell+1}} \left( \sqrt{\bar{a}_\ell - \frac{2^{d-\ell+1} a_0}{n}} - \sqrt{\bar{a}_\ell} \right)^2
\]
\[
= \frac{\log n}{2^{d-\ell+1}} \left( \sqrt{\bar{a}_\ell} - \sqrt{\bar{a}_\ell} - \frac{2^{d-\ell+1} a_0}{n \left( \sqrt{\bar{a}_\ell} - \frac{2^{d-\ell+1} a_0}{n} + \sqrt{\bar{a}_\ell} \right)} \right)^2.
\]
Then for sufficiently large $n$,

$$\frac{1}{2} \left( \sqrt{\lambda_1^2 + \lambda_2^2} - \sqrt{\lambda_1^2 - \lambda_2^2} \right)^2 \geq \frac{1 + 2e \log n}{1 + 3e} \left( \sqrt{a - \sqrt{a}} \right)^2 \geq (1 + 2e) \log n. \quad (186)$$

Since $(a_0, \ldots, a_t)$ and $K$ are all constants, $n \sim n'$ and $\log n \geq \lambda_1^* \geq \lambda_2^* \geq \log n$ and $\Delta^* \geq \log n$. Let $\alpha = 0.5$ in Theorem 6.4. Then for sufficiently large $n$, the condition 70 is satisfied since the RHS is

$$\sqrt{\lambda_1^2 + \frac{\log n'}{\alpha \log \log n'}} \leq \sqrt{\log n} + \frac{\log n}{\log \log n} = o(\log n).$$

In addition, it is easy to see that

$$\xi_{n1} \leq \frac{\log n'}{\log \log n'} = o(\log n), \quad \xi_{n2} \leq \sqrt{\log n'} = o(\log n),$$

and

$$\frac{\lambda_2^*}{\Delta^*} \leq 1, \quad \left| \log \left( \frac{\lambda_1^* + \lambda_2^*}{\lambda_1^* - \lambda_2^*} \right) \right| \leq 1.$$

Thus, for sufficiently large $n$,

$$C' \frac{\lambda_2^*}{\Delta^*} \left| \log \left( \frac{\lambda_1^* + \lambda_2^*}{\lambda_1^* - \lambda_2^*} \right) \right| (\xi_{n1} + \xi_{n2}) \leq \epsilon \log n.$$

Combined with (186), we have

$$\frac{1}{2} \left( \sqrt{\lambda_1^2 + \lambda_2^2} - \sqrt{\lambda_1^2 - \lambda_2^2} \right)^2 - \log n - C' \frac{\lambda_2^*}{\Delta^*} \left| \log \left( \frac{\lambda_1^* + \lambda_2^*}{\lambda_1^* - \lambda_2^*} \right) \right| (\xi_{n1} + \xi_{n2}) \geq \epsilon \log n.$$

Let $\delta = n^{-\epsilon}$ in Theorem 6.4. Then the first split is exactly recovered with probability $1 - 2n^{-\epsilon} = 1 - o(1)$. This completes the proof. \(\square\)

**Proposition D.1 (Theorem 1 of Abbe and Sandon [2015]).** For a general SBM with connection probability matrix $B = B_0 \log n n$ where $B_0 \in \mathbb{R}^{K \times K}$ is a fixed matrix. Further let $\Pi = \text{diag}(\pi_1, \ldots, \pi_K)$. Then exact recovery is achievable iff

$$\min_{\ell, i \in [K]} D_+((\Pi B)_{i, i}, (\Pi B)_{j, j}) \geq 1,$$

where $D_+: \mathbb{R}^K \times \mathbb{R}^K \to \mathbb{R}$ with

$$D_+(\theta, \psi) = \max_{t \in [0, 1]} \sum_{i=1}^{K} \left( t \theta_i + (1 - t) \psi_i - \theta_i \psi_i^1_{t-\ell} \right).$$

**Proof of Lemma 6.6.** Using the notation of Proposition D.1, we have $\Pi = (1/K) I_K$. Thus,

$$D_+((\Pi B)_{i, i}, (\Pi B)_{j, j}) = \frac{1}{K} D_+(B_{i, i}, B_{j, j}).$$

We prove that

$$\min_{i \neq j} D_+(B_{i, i}, B_{j, j}) = (\sqrt{a_0} - \sqrt{a_1})^2. \quad (187)$$

When $i = 1$ and $j = 2$, only the first two entries differ and thus,

$$D_+(B_1, B_2) = \max_{t \in [0, 1]} \left( ta_0 + (1 - t)a_1 - a_0^t a_1^{-t} \right) + (ta_1 + (1 - t)a_0 - a_0^t a_0^{-t})$$

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\[ a_0 + a_1 - \min_{t \in [0,1]} (a_0 a_1^{1-t} + a_1 a_0^{1-t}) \]
\[ = a_0 + a_1 - 2\sqrt{a_0 a_1} = (\sqrt{a_0} - \sqrt{a_1})^2, \]
where the second last line uses the convexity and the symmetry of \( t \mapsto a_0 a_1^{1-t} + a_1 a_0^{1-t}. \) It is left to prove that for any \( i \neq j, \)
\[ D_+ (B_i, B_j) \geq (\sqrt{a_0} - \sqrt{a_1})^2. \]
By definition, \( B_{i,i} = B_{j,j} = a_0 \) and \( B_{i,j} = B_{j,i} = a_k \) for some \( k \neq 0. \) Ignoring all other entries,
\[ D_+ (B_i, B_j) \geq \max_{t \in [0,1]} (ta_0 + (1-t)a_k - a_0(1-t) + (1-t)a_0 - a_k a_0^{1-t}). \]
Using the same argument as above, we have
\[ D_+ (B_i, B_j) \geq (\sqrt{a_0} - \sqrt{a_k})^2 \geq (\sqrt{a_0} - \sqrt{a_1})^2. \]
Thus (187) is proved.

If \(|\sqrt{a_0} - \sqrt{a_1}| < \sqrt{K}|, \) then
\[ \min_{i,j \in [K]} D_+ ((\Pi B)_i, (\Pi B)_j) = \frac{(\sqrt{a_0} - \sqrt{a_1})^2}{K} < 1. \]
By Proposition D.1, it is impossible to achieve exact recovery. \( \square \)

E Comparison With Other Bounds on Binary Random Matrices

E.1 Comparison with Abbe et al. [2017]

The assumptions they required are the following:

**B3** Suppose \( \phi(x) \) is continuous and non-decreasing on \( \mathbb{R}_+ \) with \( \phi(0) = 0 \) and \( \phi(x)/x \) being non-increasing.

For any \( \delta \in (0,1) \) and matrix \( W \in \mathbb{R}^{n \times r}, \) it holds with probability at least \( 1 - \delta \) simultaneously for all \( k \in [n] \) that
\[ \|E_k^T W\|_2 \leq \Delta^* \|W\|_{2 \to \infty} \phi \left( \frac{\|W\|_F}{\sqrt{n} \|W\|_{2 \to \infty}} \right); \]

**B4** \( \Delta^* \geq \gamma^{-1} \max\{\|A^*\|_{2 \to \infty}, \|E\|_{op}\} \) with probability \( 1 - \delta \) for any \( \gamma > 0 \) such that
\[ \kappa^* \leq \frac{1}{32 \max\{\gamma, \phi(\gamma)\}}. \] (188)

Under their assumption **B3**, as shown in their proof of Lemma 6 (equation (59)) in Section A.2,
\[ \|E_k^T W\|_2 \leq \phi(\gamma) \left( \|W\|_{2 \to \infty} + \frac{\|W\|_F}{\sqrt{n} \gamma} \right) \]
for any \( \gamma > 0. \) This corresponds to our assumption **A3** with
\[ b_\infty(\delta) = \phi(\gamma), \quad b_2(\delta) = \frac{\phi(\gamma)}{\sqrt{n} \gamma}, \]
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There are two points that are worth made here for clarity. Firstly, they treated $\delta$ as a constant and hence did not explicitly specify the dependence on $\delta$. This essentially sets $\delta = O(1/n)$ as in our assumption $A_3$. Specifying the dependence on $\delta$ yields a tighter moment bound for $d_{2\to\infty}(U,U^*)$, which is useful in our first example on concentration of spectral norm of random graphs (Section 4). Secondly, the $\phi$ function may implicitly depend on $r$. For instance, in the binary case, our Proposition 2.2 shows that $b_\infty(\delta)$ scales linearly with $r$. Abbe et al. [2017] did not specify the dependence on $r$ because they either consider the Gaussian case where $b_\infty(\delta) = 0$ or the general case with $r = O(1)$. So in our comparison we will also keep these settings. For simplicity we only consider the regime
\[ np^* \geq C \log n, \] 
for some universal constant $C > 0$.

**Comparison of assumptions**

First we compare the assumptions of Abbe et al. [2017] and our theory. As stated in their Section 1.2, for binary matrices with independent entries and $r = O(1)$,
\[ \phi(x) \preceq \frac{np^*}{\Delta^* \max\{1, \log(1/x)\}} \] 
Then the condition (188) on $\gamma$ reads as
\[ \kappa^* \max\left\{ \gamma, \frac{np^*}{\Delta^* \max\{1, \log(1/\gamma)\}} \right\} \leq \frac{1}{32}. \]

The first term implies that
\[ \gamma \leq (32\kappa^*)^{-1} \leq 1/32 \implies \max\{1, \log(1/\gamma)\} \geq \log(1/\gamma). \]

The second term then implies that
\[ \Delta^* \geq \frac{32\kappa^* \log(1/\gamma)}{np^*}. \]
Putting the pieces together, $B_4$ implies that
\[ \Delta^* \geq \max\left\{ \frac{\|A^*\|_{2\to\infty}}{\gamma}, \frac{\|E\|_{\text{op}}}{\gamma}, \frac{\kappa^* np^*}{\log(1/\gamma)} \right\}. \]
This implies that
\[ \Delta^* \geq \min_{\gamma} \max\left\{ \frac{\|E\|_{\text{op}}}{\gamma}, \frac{\kappa^* np^*}{\log(1/\gamma)} \right\}. \]
The first part is decreasing in $\gamma$ while the second part is increasing $\gamma$. Thus, the minimum is achieved at $\gamma^*$ such that two terms are equal, i.e.
\[ \frac{1}{\gamma^*} \log\left( \frac{1}{\gamma^*} \right) = \frac{\kappa^* np^*}{\|E\|_{\text{op}}} \]
Under the regime (189), $1 \leq \|E\|_{\text{op}} \leq \sqrt{np^*}$ and thus $np^*/\|E\|_{\text{op}} \geq \sqrt{np^*}$. As a result,
\[ \frac{1}{\gamma^*} \log\left( \frac{1}{\gamma^*} \right) \sim \frac{\kappa^* np^*}{\|E\|_{\text{op}}} \iff \frac{1}{\gamma^*} \sim \frac{\kappa^* np^*/\|E\|_{\text{op}}}{\log \kappa^* + \log(np^*)}. \]
Therefore, their assumptions $B_3$ and $B_4$ hold only if
\[ \Delta^* \geq \max\left\{ \frac{\|E\|_{\text{op}}}{\gamma^*}, \frac{\kappa^* np^*}{\log(1/\gamma^*)} \right\} \geq \frac{\kappa^* np^*}{\log \kappa^* + \log(np^*)}. \]
By contrast, our theory (condition (19) in Theorem 3.4) only requires
\[
\Delta^* \succeq \kappa^* g(\delta) = g(\delta) = \sqrt{np^*} + \frac{\log n}{\log \log n}. \tag{193}
\]
This is always less stringent than (192). The only case it is equivalent to (192) is when \(np^* \sim \log n\) and \(\kappa^* \leq 1\). When \(np^* \geq (\log n)^2\), our condition is \(\kappa^* \sqrt{np^*}\) better than (192).

Furthermore, note that
\[
\Delta^* \leq \lambda^*_\min \leq \lambda^*_\max \leq np^*. \tag{194}
\]
By (194), their condition (192) can hold only if \(\kappa^* \preceq \log (np^*)\).

When \(np^* \preceq (\log n)^b\) for some \(b > 0\) as typically studied for random graphs, (192) only permits well-conditioned case with \(\kappa^* \leq \log \log n\). Even in the case of dense graphs where \(np^*\) grows polynomially, the condition number should be no larger than \(\log n\). By contrast, our theory allows the condition number to be arbitrarily large.

On the other hand, even in the well-conditioned case \(\kappa^* \leq 1\), (192) can only hold if \(\Delta^* \geq np^*/\log(np^*)\). Therefore, the minimal eigen-gap for which their theory works is only \(\log \log n\) smaller than the upper bound (194) when \(np^*\) grows poly-logarithmically and is \(\log n\) smaller than the upper bound when \(np^*\) grows polynomially. By contrast, our theory allows the eigen-gap to be much smaller than the upper bound.

**Comparison of bounds**

By (190) and (192),
\[
\frac{\|A^*\|_{2\to\infty}}{\Delta^*} \leq \frac{\sqrt{np^*}}{\Delta^*} \leq \frac{\phi(1)}{\sqrt{n}} \leq \phi(1)\|U^*\|_{2\to\infty}. \tag{195}
\]

In their Theorem 2.1, they showed,
\[
d_{2\to\infty}(U, AU^*(A^*)^{-1}) \preceq \kappa^*(\kappa^* + \phi(1)(\gamma + \phi(\gamma))) \|U^*\|_{2\to\infty} + \gamma \frac{\|A^*\|_{2\to\infty}}{\Delta^*} \tag{196}
\]
\[
d_{2\to\infty}(U, U^*) \preceq (\phi(1) + \kappa^*(\kappa^* + \phi(1)(\gamma + \phi(\gamma))) \|U^*\|_{2\to\infty} + \gamma \frac{\|A^*\|_{2\to\infty}}{\Delta^*}
\]

By (191) and (194),
\[
\gamma \succeq \frac{\|E\|_{op}}{\Delta^*} \geq \frac{1}{np^*}.
\]

As a result,
\[
\gamma + \phi(\gamma) \succeq \frac{np^*}{\Delta^* \log(np^*)}.
\]

Thus, their bounds (195) and (196) are at least
\[
\frac{np^* \kappa^*/2}{\Delta^* \log(np^*)} \|U^*\|_{2\to\infty} \quad \text{and} \quad \frac{np^*}{\Delta^*} \left(1 + \frac{\kappa^*/2}{\log(np^*)}\right) \|U^*\|_{2\to\infty}. \tag{197}
\]
For simple comparison, we assume $\Delta^* \sim np^*$. Then their bounds (197) are at least
\[
\kappa^* \frac{2}{\log(np^*)} \|U^*\|_{2 \to \infty} \quad \text{and} \quad \left(1 + \frac{\kappa^*}{\log(np^*)}\right) \|U^*\|_{2 \to \infty}.
\] (198)

Turning to our bound. Since $\lambda^*_{\min} \geq \Delta^* \sim np^*$, the condition of Corollary 3.6 is satisfied. By Corollary 3.6 and (189),
\[
d^*_{2 \to \infty}(U, AU^*(\Lambda^*)^{-1}) \leq \left(\frac{1}{\sqrt{np^*}} + \frac{\log n}{np^* \log n}\right) \|U^*\|_{2 \to \infty};
\]
(199)
\[
d^*_{2 \to \infty}(U, U^*) \leq \left(\sqrt{np^* + \log n}\right) \|U^*\|_{2 \to \infty} + \sqrt{\frac{\log n}{np^*}} \frac{1}{\sqrt{n}} \leq \sqrt{\frac{\log n}{np^*}} \|U^*\|_{2 \to \infty}.
\]
(200)

It is easy to see that both bounds dominate (198) except in the case $\kappa^* \leq 1$, $np^* \sim \log n$ where two bounds are equivalent.

More importantly, (198) does not improve in order when $np^*$ increases except when $np^*$ grows from $(\log n)^b$ to $n^b$ so that the bound grows from $\kappa^*/\log \log n \|U^*\|_{2 \to \infty}$ to $\kappa^*/\log n \|U^*\|_{2 \to \infty}$. By contrast, our bounds improves constantly as $np^*$ grows in order. For instance, when $np^* \gg (\log n)^2$, (199) is better than (198) with $p^* \sim 1$.

### E.2 Comparison with Eldridge et al. [2017]

**Comparison of assumptions**

Eldridge et al. [2017] considered the case where
\[
r = 1, \quad np^* \gtrsim (\log n)^{2+\epsilon},
\]
for some $\epsilon > 0$. They also implicitly assumed
\[
\lambda^*_{\min} \geq \Delta^* \geq np^*, \quad \max_{i \in [s+1, s+r]} \|U_i^*\|_\infty \leq \frac{1}{\sqrt{n}}.
\]

This is at least as strong as the condition of Corollary 3.6, itself being the most special case of our general theory in Section 3. Although the bound on a single eigenvector can yield bounds for eigenspaces, it requires the multiplicity of each eigenvalue to be 1 and sufficient eigen-gap for each eigenvalue. This cannot be applied to problems in Section 5.3.

**Comparison of bounds**

In this setting they proved that with high probability,
\[
\|U - U^*\|_\infty \lesssim \sqrt{\frac{(\log n)^{2+\eta}}{np^*}} \|U^*\|_\infty,
\]
for any $\eta \in (0, \epsilon/2)$. By contrast, as shown in (200), our bound implies that
\[
\|U - U^*\|_\infty \lesssim \sqrt{\frac{\log n}{np^*}} \|U^*\|_\infty.
\]

Thus, our bound is at least $\sqrt{(\log n)^{1+\epsilon}}$ better than their bound even in this special setting. In order for $\|U - U^*\|_\infty \lesssim \|U^*\|_\infty$ as in most applications, our bound only requires $np^* \gtrsim \log n$ while their bound requires $np^* \gtrsim (\log n)^{2+\epsilon}$.
E.3 Comparison with Cape et al. [2019a]

Comparison of assumptions

Cape et al. [2019a] considers the full recovery for low-rank matrices, i.e.

\[ s = 0, \quad \lambda^*_r = \cdots = \lambda^*_n = 0. \]

They further assume that

\[ np^* \geq (\log n)^{2+\epsilon}, \quad r \leq (\log n)^{2+\epsilon} \]

for some \( \epsilon > 0 \) and

\[ \kappa^* \leq 1, \quad \lambda^*_\min = \Delta^* \geq np^*. \]

This is a highly specialized setting and cannot be applied to problems in Section 4, Section 5.3 and Section 6.

Comparison of bounds

Under their assumptions, they proved that

\[ d_{2 \to \infty} (U, U^*) \leq \frac{\sqrt{r(\log n)^{2+\epsilon}}}{\sqrt{np^*}} \|U^*\|_{2 \to \infty}. \]

As shown in (200), our bound reads as

\[ d_{2 \to \infty} (U, U^*) \leq \frac{\sqrt{\log n \log \log n}}{np^*} \|U^*\|_{2 \to \infty}. \]

It is clear that our bound is \( \sqrt{r(\log n)^{1+\epsilon}} \) better than their bound. More importantly, our bound is not affected by the number of eigenvectors to recover and thus allow \( r \) to be as large as \( n \), in which case their bound is not informative.

E.4 Comparison with Mao et al. [2017]

Comparison of assumptions

As in Cape et al. [2019a], Mao et al. [2017] considers the full recovery problem for low-rank matrices. They assumed that

\[ np^* \geq (\log n)^{2+\epsilon}, \quad \lambda^*_\min = \Delta^* \geq \sqrt{np^* (\log n)^{1+\epsilon/2}}, \quad \max_{j \in [r]} \|U^*_j\|_{\infty} \leq \sqrt{p^*}, \]

for some \( \epsilon > 0 \). The assumption on \( \|U^*_j\|_{\infty} \) forces the eigenvectors to be diffused and the matrix \( A^* \) to have low coherence [Candes and Recht, 2009]. By contrast, we do not have any assumption on \( U^* \). Moreover, our assumption on the eigen-gap is

\[ \Delta^* \geq \min\{\kappa^*, r\} \left( \sqrt{np^*} + \frac{\log n}{\log \log n} \right). \]

Under their regime \( np^* \geq (\log n)^{2+\epsilon} \), this is weaker than their condition if \( \min\{\kappa^*, r\} \leq (\log n)^{1+\epsilon/2} \). However, their condition can be weaker in the ill-conditioned case with many eigenvectors to be recovered.

Comparison of bounds
Under their assumptions, they proved that
\begin{equation}
\|UU^T - U^*(U^*)^T\|_{2\to\infty} \leq \frac{\hat{r}^* \sqrt{np^*}}{\Delta^*} \left( \hat{r}^* + (\log n)^{1+\epsilon/2} \right) \left( \sqrt{r} \max_{i \in [r]} \|U_i^*\|_\infty \right). \tag{201}
\end{equation}
They show that the same bound holds for $d_{2\to\infty}(U,U^*)$. By contrast, our Corollary 3.5 with $\alpha = 0.5$ implies that
\begin{equation}
d_{2\to\infty}(U,U^*) \leq \frac{\hat{r}^* \sqrt{np^*}}{\Delta^*} \left( \frac{\log n}{\sqrt{n\|U^*\|_{2\to\infty}}} + \sqrt{\log(n)p^*} \left( 1 + \sqrt{\frac{np^*(\log n)^{1/2}}{\Delta^* \log n}} \right) \right).
\end{equation}
Under their assumptions, it is not hard to see that the above bound simplifies as below:
\begin{equation}
d_{2\to\infty}(U,U^*) \leq \frac{\hat{r}^* \sqrt{np^*}}{\Delta^*} \left( \frac{\log n}{\sqrt{n\|U^*\|_{2\to\infty}}} \right) \|U^*\|_{2\to\infty} \leq \frac{\hat{r}^* \sqrt{np^*}}{\Delta^*} \left( 1 + \sqrt{n\|U^*\|_{2\to\infty}} \right) \|U^*\|_{2\to\infty}. \tag{202}
\end{equation}
Since $\|U^*\|_{2\to\infty} \leq \sqrt{r \max_{i \in [r]} \|U_i^*\|_{\infty}}$, our bound is always better than (201). If $r \leq \log n$, our bound is $\sqrt{r(\log n)^{1+\epsilon}}$ better than theirs; if $r \geq \log n$, our bound is $\hat{r}^* + (\log n)^{1+\epsilon/2}$ better than theirs.

E.5 Comparison with other deterministic bounds

In literature there are also several deterministic $\ell_{2\to\infty}$ bounds that do not depend on the random structure of the matrices. Because of the generality, they are typical much weaker than those tailored for random matrices. Although it is unfair to compare two types of bounds, we discuss the comparison here for completeness.

We are aware of three purely deterministic $\ell_{2\to\infty}$ bounds by Fan et al. [2018], Cape et al. [2019b] and Damle and Sun [2019]. The first two are derived for rectangular matrices and the last one is derived for symmetric matrices. When applied to symmetric matrices, all of the above works only consider top-$r$ recovery.

Fan et al. [2018] assumes
\begin{equation}
\lambda_{\min}^* \geq r^3(\sqrt{n}\|U^*\|_{2\to\infty})^2\|E\|_{\infty} + \|A^* - U^*\Lambda^*U^T\|_{\infty}. \tag{203}
\end{equation}
The second term is hard to bound in general except in the full recovery problem where the second term vanishes. In this case, (203) can be simplified as
\begin{equation}
\lambda_{\min}^* \geq \frac{r^3(\sqrt{n}\|U^*\|_{2\to\infty})^2\|E\|_{\infty}}.
\end{equation}
Note that
\begin{equation}
\lambda_{\min}^* \leq \left( \frac{1}{\sqrt{n}} \right)^T A^* \left( \frac{1}{\sqrt{n}} \right) = \frac{1}{n} \sum_{i,j=1}^n A_{ij}^* \leq np^*,
\end{equation}
and Bernstein’s inequality implies that
\begin{equation}
\|E\|_{\infty} \geq np^* + \log n.
\end{equation}
As a result, (204) holds only if
\begin{equation}
np^* \geq \log n, \quad r \leq 1, \quad \sqrt{n}\|U^*\|_{2\to\infty} \leq 1, \quad \Delta^* = \lambda_{\min}^* \geq np^*.
\end{equation}
Under these conditions, they prove that
\[ d_{2 \to \infty}(U, U^*) \leq \frac{r^{5/2}(\sqrt{n}\|U^*\|_{2 \to \infty})^2\|E\|_{\infty}}{\lambda_{\min}^2\sqrt{n}} \leq \|U^*\|_{2 \to \infty}. \]

This bound matches our bound (200) only when \( np^* \sim \log n \), but is \( \sqrt{np^*}/\log n \) worse than ours when \( np^* \gg \log n \).

The condition was improved by Cape et al. [2019b] into
\[ \lambda_{\min} \geq \|E\|_{\infty}. \]

This eliminates the constraint on \( r \) and \( \sqrt{n}\|U^*\|_{2 \to \infty} \) but still requires \( n \bar{p}^* \geq \log n \). For full recovery problem, Cape et al. [2019b] obtained essentially the same bound as below:
\[ d_{2 \to \infty}(U, U^*) \leq \frac{\|E\|_{\infty}}{\lambda_{\min}^2} \|U^*\|_{2 \to \infty} \leq \|U^*\|_{2 \to \infty}. \]

On the other hand, for top-\( r \) problem, Damle and Sun [2019] requires
\[ \min\{\Delta^*, \text{sep}_{2 \to \infty, \hat{U}^*}(\Lambda^*, A^* - U^* \Lambda^* (U^*)^T) \} \geq \sqrt{np^*}, \]
where \((\hat{U}^*, U^*)\) forms an orthonormal basis in \( \mathbb{R}^n \) and
\[ \text{sep}_{2 \to \infty, W}(B, C) = \inf\{\|ZB - CZ\|_{2 \to \infty} : Z \in \text{ran} W, ||Z||_{2 \to \infty} = 1\}. \]

However this condition is hard to parse except in the full recovery problem for which it is shown that
\[ \min\{\Delta^*, \text{sep}_{2 \to \infty, \hat{U}^*}(\Lambda^*, A^* - U^* \Lambda^* (U^*)^T) \} = \Delta^*. \]

In this case, their condition reads as \( \Delta^* \geq \sqrt{np^*} \). This is the weakest one among all aforementioned works. However, their bound is also the weakest for binary random matrices with independent entries, although it is tight for some deterministic matrices. Indeed, their bound is
\[ d_{2 \to \infty}(U, U^*) \leq \left( \frac{\|E\|_{\text{op}}}{\Delta^*} \right)^2 \|U^*\|_{2 \to \infty} + \frac{\|\hat{U}^* E_{2,1}\|_{2 \to \infty}}{\Delta^*} + \frac{\|\hat{U}^* (\hat{U}^*)^T E\|_{2 \to \infty}}{(\Delta^*)^2}. \]

The third term is large in general as \( \hat{U}^* \) includes all other eigenvectors include those corresponding to the zero eigenvalue. For instance consider an Erdős-Rényi graph with self-loop, i.e. \( A^* = p^*1_n 1_n^T \). In this case \( U^* = 1_n/\sqrt{n} \) and thus
\[ \hat{U}^* (\hat{U}^*)^T = I_n - \frac{1}{n} 1_n 1_n^T. \]

As a result,
\[ \|\hat{U}^* (\hat{U}^*)^T E\|_{2 \to \infty} = \|\left( I_n - \frac{1}{n} 1_n 1_n^T \right) E\|_{2 \to \infty} \sim \|E\|_{2 \to \infty} \sim \sqrt{np^*}. \]

Therefore, when \( p^* \geq \log n/n \), the third term alone is lower bounded by
\[ \frac{\sqrt{np^*} \sqrt{np^*}}{(np^*)^2} = \frac{1}{np^*}. \]

This is too loose compared to all aforementioned bounds.
F Concentration Inequalities for Binary Random Variables

Lemma F.1. Let \((X_i)_{i=1}^n\) be independent Bernoulli variables with \(\mathbb{E}X_i = p_i\). Given any vector \(w \in \mathbb{R}^n\), let
\[
S_n = \sum_{i=1}^n w_i(X_i - p_i).
\]
Then for any \(\delta \in (0, 1)\), it holds with probability \(1 - \delta\) that
\[
S_n \leq \frac{2 \log(1/\delta)}{F^{-1}(2\Omega \log(1/\delta))} \|w\|_\infty.
\]
where
\[
\Omega = \frac{\|w\|_2}{\sum_{i=1}^n p_i w_i^2}, \quad F(x) = x^2 e^x.
\]

Proof. Without loss of generality we assume \(\|w\|_\infty = 1\). For any \(\lambda > 0\) and \(t > 0\), by Markov’s inequality
\[
\log \mathbb{P}(S_n \geq t) \leq -\lambda t + \log \mathbb{E}[e^{\lambda S_n}].
\]
By definition,
\[
\log \mathbb{E}[e^{\lambda S_n}] = \sum_{i=1}^n (\log(1 - p_i + p_i e^{\lambda w_i}) - \lambda p_i)
\]
\[
\leq \sum_{i=1}^n p_i \left( e^{\lambda w_i} - \lambda w_i - 1 \right)
\]
\[
\leq \sum_{i=1}^n p_i \frac{(\lambda w_i)^2}{2} e^{|w_i|}
\]
\[
\leq \sum_{i=1}^n p_i \frac{(\lambda w_i)^2}{2} e^\lambda = \frac{F(\lambda)}{2\Omega},
\]
where (i) uses the inequality that \(\log(1+x) \leq x\) for all \(x > -1\), (ii) uses the inequality that \(e^x - x - 1 \leq \frac{x^2 e^{|x|}}{2}\) and (iii) uses the fact that \(|w_i| \leq \|w\|_\infty = 1\).

Fix any \(\lambda\), let \(t = \frac{F(\lambda)}{\lambda \Omega}\). Then (206) implies that
\[
\log \mathbb{P}(S_n \geq \frac{F(\lambda)}{\lambda \Omega}) \leq -\frac{F(\lambda)}{2\Omega}.
\]
Let \(\lambda = F^{-1}(2\Omega \log(1/\delta))\). Then we obtain that with probability \(1 - \delta\),
\[
S_n \leq \frac{2 \log(1/\delta)}{F^{-1}(2\Omega \log(1/\delta))}.
\]

Lemma F.2. \(F^{-1}(x)\) is increasing and \(F^{-1}(x)/\sqrt{x}\) is decreasing. For any \(x_0 \geq e\),
\[
F^{-1}(x) \geq \begin{cases}
\sqrt{x/x_0} & \text{for any } x \leq x_0 \\
\log x - 2 \log \log x & \text{for any } x > e \\
\log x/2 & \text{for any } x > 0
\end{cases}
\]

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Proof. Notice that $F(\lambda)$ is increasing. This implies that $F^{-1}(x)$ is increasing. On the other hand, let $F^{-1}(x)/\sqrt{x} = v(x), \text{ then by definition}$

$$v(x)^2 e^{\sqrt{x} v(x)} = 1.$$ 

This implies that $v(x)$ is decreasing. Since $F(\lambda)$ is increasing and $F(\log x_0) = x_0(\log x_0)^2 \geq x_0$, for any $x \leq x_0$

$$F^{-1}(x) \leq \log x_0.$$ 

By definition, 

$$x = (F^{-1}(x))^2 e^{F^{-1}(x)} \leq (F^{-1}(x))^2 x_0 \implies F^{-1}(x) \geq \sqrt{\frac{x}{x_0}}.$$ 

On the other hand, for any $x > e$,

$$F(\log x - 2 \log \log x) = (\log x - 2 \log \log x)^2 e^{\log x - 2 \log \log x} = x \frac{(\log x - 2 \log \log x)^2}{(\log x)^2} \leq x.$$ 

Thus,

$$F^{-1}(x) \geq \log x - 2 \log \log x.$$ 

Finally, noting that for any $\lambda > 0$,

$$e^\lambda = \sum_{n \geq 0} \frac{\lambda^n}{n!} = \geq \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{6} = \lambda^2 + 2 \sqrt{\lambda \frac{\lambda^3}{6}} \geq \lambda^2,$$ 

we have

$$F(\lambda) \leq e^{2\lambda} \implies F^{-1}(x) \geq \frac{\log x}{2}.$$

Lemma F.3. Under the same setting of Lemma F.1, for any $\gamma > 0$, it holds with probability $1 - \delta$ that

$$S_n \leq \frac{2 \log(1/\delta)}{F^{-1}(2\gamma \log(1/\delta))} \left( ||w||_\infty + \sqrt{\gamma \sum_{i=1}^n p_i w_i^2} \right) \leq \frac{2 \log(1/\delta)}{F^{-1}(2\gamma \log(1/\delta))} \left( ||w||_\infty + \min\{\sqrt{\gamma} p^* ||w||_2, \sqrt{\gamma n p^*} ||w||_\infty\} \right),$$

where

$$\bar{p} = \frac{1}{n} \sum_{i=1}^n p_i, \quad p^* = \max_i p_i.$$  \hspace{1cm} (207)

Proof. If $\Omega \geq \gamma$, since $F^{-1}(x)$ is increasing,

$$F^{-1}(2\Omega \log(1/\delta)) \geq F^{-1}(2\gamma \log(1/\delta));$$

If $\Omega < \gamma$, since $F^{-1}(x)/\sqrt{x}$ is decreasing,

$$\frac{F^{-1}(2\Omega \log(1/\delta))}{\sqrt{2\Omega \log(1/\delta)}} \geq \frac{F^{-1}(2\gamma \log(1/\delta))}{\sqrt{2\gamma \log(1/\delta)}}$$

$$\implies F^{-1}(2\Omega \log(1/\delta)) \geq F^{-1}(2\gamma \log(1/\delta)) \sqrt{\frac{\Omega}{\gamma}}.$$ 

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By Lemma F.1, with probability $1 - \delta$,

$$S_n \leq \frac{2 \log(1/\delta)}{F^{-1}(2\gamma \log(1/\delta))} \|w\|_\infty \left(1 + \sqrt{\frac{\gamma}{\Omega}}\right)$$

$$\leq \frac{2 \log(1/\delta)}{F^{-1}(2\gamma \log(1/\delta))} \left(\|w\|_\infty + \sqrt{\frac{\gamma}{\sum_{i=1}^{n} p_i w_i^2}}\right).$$

\[\square\]