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Shearing Coordinates and Convexity of Length Functions on Teichmüller Space

By M. Bestvina, K. Bromberg, K. Fujiwara, and J. Souto

Abstract. We prove that there are Fenchel-Nielsen coordinates for the Teichmüller space of a finite area hyperbolic surface with respect to which the length functions are convex.

1. Introduction. Let $X$ be a complete hyperbolic surface with finite area and $\mathcal{T}(X)$ be its Teichmüller space. Recall that points in $\mathcal{T}(X)$ are equivalence classes of marked hyperbolic surfaces, i.e. equivalence classes of pairs $(Y, f)$ where $Y$ is a hyperbolic surface and $f : X \rightarrow Y$ is a quasi-conformal homeomorphism.

The goal of this paper is to describe certain coordinates of $\mathcal{T}(X)$ with respect to which the length functions are convex. Given a homotopically essential, non-peripheral curve $\gamma \subset X$ and a point $(Y, f) \in \mathcal{T}(X)$ the curve $f(\gamma)$ is freely homotopic to a unique geodesic in the hyperbolic surface $Y$. Denoting by $\ell_\gamma(Y, f)$ the length of this geodesic we obtain a well-defined function

$$\ell_\gamma : \mathcal{T}(X) \rightarrow \mathbb{R}_+.$$ 

This is the length function associated to the curve $\gamma$.

Of the many well-known ways to parametrize Teichmüller space perhaps the most classical is attributed to Fenchel and Nielsen [8]. Fix a pants decomposition $\mathcal{P}$ of $X$, i.e. a multicurve such that $X \setminus \mathcal{P}$ is homeomorphic to the disjoint union of thrice punctured spheres. A simple computation shows that $\mathcal{P}$ has $|\mathcal{P}| = 3g + n - 3$ components where $g$ is the genus and $n$ the number of cusps of $X$. The Fenchel-Nielsen coordinates

$$\Phi : \mathcal{T}(X) \sim \rightarrow \mathbb{R}_+^{3g+n-3} \times \mathbb{R}_+^{3g+n-3}$$

associate to each point in $\mathcal{T}(X)$ the length and the twist for each component of the pants decomposition $\mathcal{P}$.

While there is a canonical way to define the length function, in constrast, to define the “twist parameter” it has to be decided what a twist of zero represents. There

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is no canonical way to determine this so we refer to any of the possible length-twist coordinates as Fenchel-Nielsen coordinates associated to the pants decomposition $\mathcal{P}$. We prove:

**Theorem 1.1.** Let $X$ be a complete, finite area, hyperbolic surface of genus $g$ and with $n$ cusps, and fix a pants decomposition $\mathcal{P}$ of $X$. There are Fenchel-Nielsen coordinates $\Phi : \mathcal{T}(X) \sim \rightarrow \mathbb{R}^{3g+n-3} \times \mathbb{R}^{3g+n-3}$ associated to $\mathcal{P}$ such that for any essential curve $\gamma$ in $X$ the function

$$l_\gamma \circ \Phi^{-1} : \mathbb{R}^{3g+n-3} \times \mathbb{R}^{3g+n-3} \rightarrow \mathbb{R}^+$$

is convex. If moreover the curve $\gamma$ intersects all the components of $\mathcal{P}$ then $l_\gamma \circ \Phi$ is strictly convex.

As far as the authors are aware the first convexity result of this type is due to Douady (Exposé 7 [7]) who proved that length functions are convex along earthquakes on simple closed curves. Our proof could be considered as a generalization of Douady’s methods. Other convexity results of the length functions $l_\gamma$ are due to Kerckhoff [11] and Wolpert [19]. They proved respectively that the length functions are convex along earthquake paths and Weil-Petersson geodesics. Both authors derived from their results proofs of the so-called Nielsen realization problem; so do we.

**Theorem 1.2.** (Kerckhoff) The action of every finite subgroup of the mapping class group on $\mathcal{T}(X)$ has a fixed point.

Tromba [17] gave a different proof of Theorem 1.2 using the convexity of the energy functional along Weil-Petersson geodesics. Proofs of this theorem in a completely different spirit are due to Gabai [9] and Casson-Jungreis [6].

In order to prove Theorem 1.1 we follow a slightly indirect path. We will associate a continuous map $s_\lambda : \mathcal{T}(X) \rightarrow \mathbb{R}^{|\lambda|}$ to every maximal, finite leaved lamination $\lambda$ of $X$, where $|\lambda|$ is the number of leaves of $\lambda$. The image $T_\lambda = s_\lambda(\mathcal{T}(X))$ of $s_\lambda$ is an open convex subset of a linear subspace of the correct dimension $6g+2n-6$. We refer to

$$s_\lambda : \mathcal{T}(X) \sim \rightarrow T_\lambda$$

as shearing coordinates associated to the lamination $\lambda$. Thurston [16] and Bonahon [2] defined shearing coordinates for a general maximal lamination. When the lamination is finite leaved their coordinates are equivalent with ours.

Given a pants decomposition $\mathcal{P}$ of $X$, we choose a maximal lamination $\lambda$ containing $\mathcal{P}$ and describe Fenchel-Nielsen coordinates associated to $\mathcal{P}$ in such a way so that the map

$$\Phi \circ s_\lambda^{-1} : T_\lambda \sim \rightarrow \mathbb{R}^{3g+n-3} \times \mathbb{R}^{3g+n-3}$$
is linear. In particular, Theorem 1.1 follows immediately from the following more general result:

**Theorem 1.3.** Let $X$ be a complete, finite area, hyperbolic surface with genus $g$ and with $n$ cusps. Let $\lambda$ be a maximal lamination in $X$ with finitely many leaves and let $s_\lambda : \mathcal{T}(X) \to T_\lambda$ be the shearing coordinates associated to $\lambda$. For any essential curve $\gamma$ in $X$ the function

$$l_\gamma \circ s_\lambda^{-1} : T_\lambda \to \mathbb{R}_+$$

is convex. If moreover the curve $\gamma$ intersects all the leaves of $\lambda$ then $l_\gamma \circ s_\lambda^{-1}$ is strictly convex.

The paper is organized as follows. After a few preliminaries in Section 2, we introduce in Section 3 the shearing coordinates and reduce the proof of Theorem 1.3 to Proposition 3.7, our main technical result. In Section 4 we relate Fenchel-Nielsen coordinates to shearing coordinates and prove Theorem 1.1. In Section 5 we study the length function on the Teichmüller space of the annulus and finally in Section 6 we prove Proposition 3.7. The proofs are, once one is accustomed to the notation, elementary.

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2. **Teichmüller space and other important objects.** We refer to [5, 10] for facts and definitions in this section.

Let $X$ be a complete, orientable hyperbolic surface with possibly infinite area. Then $X = \mathbb{H}^2/\Gamma$ where $\Gamma$ is a discrete group in $\text{Isom}^+(\mathbb{H}^2)$. There is a natural compactification of $\mathbb{H}^2$ by $S^1$ and the action of $\Gamma$ extends continuously to an action on $S^1$. The domain of discontinuity of this action is the largest open subset of $S^1$ where $\Gamma$ acts properly discontinuously. If $\Omega$ is the domain of discontinuity for $\Gamma$ then $\bar{X} = (\mathbb{H}^2 \cup \Omega)/\Gamma$ is a surface with boundary. If $X$ has finite area then $\Omega$ is empty but the converse does not always hold.

A marked hyperbolic surface is a pair $(Y,f)$ where $Y$ is a hyperbolic surface and $f : X \to Y$ is a quasi-conformal homeomorphism. Two marked surfaces $(Y_0,f_0)$ and $(Y_1,f_1)$ are equivalent if there is an isometry $\phi : Y_0 \to Y_1$ such that $\phi \circ f_0$ and $f_1$ are isotopic via an isotopy that is the identity on $\partial X = \bar{X}\setminus X$.

The Teichmüller space $\mathcal{T}(X)$ is the set of equivalence classes of marked hyperbolic surfaces. We give $\mathcal{T}(X)$ a metric (and topology) as follows. The distance, $d_T((Y_0,f_0),(Y_1,f_1))$, between two pairs is the infimum of the logarithm of the quasi-conformal constant of all maps $\phi : Y_0 \to Y_1$ with $\phi \circ f_0$ isotopic to $f_1$ via a map that is the identity on $\partial X$.

**Remark.** We will be mostly interested in Teichmüller spaces of finite area surfaces, although the ambitious reader can easily verify that our results extend to
a more general setting. In particular, as we will see in Section 5 the \( \text{Teichm"uller} \) space of a hyperbolic annulus plays a key role in our work.

Continuing with the same notation as above, we can identify the universal covers of \( X \) and \( Y \) with the hyperbolic plane \( \mathbb{H}^2 \); these identifications are unique up to composition with an isometry of \( \mathbb{H}^2 \). Let \( \tilde{f} : \mathbb{H}^2 \to \mathbb{H}^2 \) be the lift of the quasi-conformal homeomorphism \( f : X \to Y \). It is a classical result that the quasi-conformal map \( \tilde{f} \) extends continuously to a quasi-symmetric map of \( \partial \tilde{f} : \partial \mathbb{H}^2 \to \partial \mathbb{H}^2 \) [1]; here \( \partial \mathbb{H}^2 = S^1 \) is the boundary at infinity of \( \mathbb{H}^2 \). Moreover, lifts of isotopic maps that are the identity on \( \partial X \) have extensions which differ by composition with (the boundary extensions of) isometries of \( \mathbb{H}^2 \).

A lamination on \( X \) is a closed, but perhaps not compact, subset of \( X \) which is foliated by geodesics. If the surface \( X \) has finite area, we will be only be interested in laminations \( \lambda \) with finitely many leaves. Recall that the only non-isolated leaves of such a lamination are simple closed geodesics in \( X \) (see e.g. [4, Lemma 4.2.2 and Theorem 4.2.8]).

Let \( f : X \to Y \) be a quasi-conformal homeomorphism between hyperbolic surfaces and let \( \tilde{f} : \mathbb{H}^2 \to \mathbb{H}^2 \) and \( \partial \tilde{f} : \partial \mathbb{H}^2 \to \partial \mathbb{H}^2 \) be as above. The maps \( f \) and \( \tilde{f} \) need not take geodesics to geodesics. In order to by-pass this problem, we associate to any geodesic \( \gamma \subset \mathbb{H}^2 \) the unique geodesic \( \bar{f}(\gamma) \subset \mathbb{H}^2 \) which has the same endpoints on \( \partial \mathbb{H}^2 \) as the arc \( \tilde{f}(\gamma) \). If \( \gamma \) is a closed geodesic on \( X \) then its pre-image \( \tilde{\gamma} \) in \( \mathbb{H}^2 \) will be equivariant and therefore \( \bar{f}(\tilde{\gamma}) \) will also be equivariant and descend to a geodesic \( \bar{f}(\gamma) \) on \( Y \). If \( \gamma \) is also simple then \( \bar{f}(\gamma) \) is simple as well. For a lamination we apply \( \bar{f} \) to each geodesic in the lamination.

3. Shearing coordinates. The goal of this section is to define shearing coordinates for the \( \text{Teichm"uller} \) space \( \mathcal{T}(X) \) of a finite area hyperbolic surface.

3.1. Ideal triangulations. Before setting up our coordinates we need some definitions and notation. An ideal triangle on a hyperbolic surface \( X \) is the image of an injective, local isometry from an ideal triangle in \( \mathbb{H}^2 \) to the surface. Recall that any two ideal triangles in \( \mathbb{H}^2 \) are isometric and that ideal triangles have the same isometry group as Euclidean equilateral triangles. When \( \Delta \) is an ideal triangle in \( \mathbb{H}^2 \) and \( \tilde{f} : \mathbb{H}^2 \to \mathbb{H}^2 \) a lift of a quasi-conformal homeomorphism \( f : Y \to X \) as in the previous section, we denote by \( \bar{f}(\Delta) \) the ideal triangle whose boundary is the \( \tilde{f} \)-image of the geodesics bounding \( \Delta \).

An ideal triangulation of \( X \) is a lamination with finitely many leaves whose complementary components are ideal triangles. An ideal triangle has a unique inscribed disk that is tangent to all three sides of the triangle. The midpoints of the sides are three tangency points; compare with Figure 1(a).

Let \( \gamma_0 \) and \( \gamma_1 \) be geodesics in \( \mathbb{H}^2 \) that are asymptotic to a point \( p_\infty \in \partial \mathbb{H}^2 \). For each \( p_0 \in \gamma_0 \) there is a unique \( p_1 \in \gamma_1 \) such that the horocycle based at \( p_\infty \) through \( p_0 \) intersects \( \gamma_1 \) at \( p_1 \). We define a map \( h_{\gamma_0,\gamma_1} : \gamma_0 \to \gamma_1 \) by \( h_{\gamma_0,\gamma_1}(p_0) = p_1 \). We
allow the possibility that $\gamma_0 = \gamma_1$ in which case $h_{\gamma_0, \gamma_1}$ is the identity map; compare with Figure 1(b).

Let $\Delta^a$ and $\Delta^b$ be ideal triangles in $\mathbb{H}^2$ with disjoint interiors and let $\gamma$ be a geodesic separating the two triangles. We also assume that both $\Delta^a$ and $\Delta^b$ are asymptotic to $\gamma$; that is, there are sides $\gamma^a$ and $\gamma^b$ of $\Delta^a$ and $\Delta^b$ that are asymptotic to $\gamma$. Let $m^a$ and $m^b$ be the midpoints of $\gamma^a$ and $\gamma^b$.

We define $s(\Delta^a, \Delta^b, \gamma)$ to be the signed distance from $h_{\gamma^a, \gamma}(m^a)$ to $h_{\gamma^b, \gamma}(m^b)$ where the sign is determined by orienting $\gamma$ such that $\Delta^a$ is on the left of $\gamma$; compare with Figure 2. If $\Delta^a$ and $\Delta^b$ have a common boundary edge there is only one choice for $\gamma$ so we will sometimes write $s(\Delta^a, \Delta^b) = s(\Delta^a, \Delta^b, \gamma)$.

The following lemma is a collection of simple facts on $s(\cdot, \cdot, \cdot)$ whose proof we leave to the interested reader.

**Lemma 3.1.** Let $\Delta^a$, $\Delta^b$ and $\gamma$ be as above, all three sharing an ideal vertex, and $\gamma$ separating $\Delta^a$ and $\Delta^b$.

- $s(\Delta^b, \Delta^a, \gamma) = s(\Delta^a, \Delta^b, \gamma)$.
- If $\phi$ is an orientation preserving isometry of $\mathbb{H}^2$ then $s(\Delta^a, \Delta^b, \gamma) = s(\phi(\Delta^a), \phi(\Delta^b), \phi(\gamma))$.
- $s(\Delta^a, \Delta^b, \gamma') = s(\Delta^a, \Delta^b, \gamma)$ for any other geodesic $\gamma'$ separating $\Delta^a$ and $\Delta^b$ and asymptotic to $\gamma$. 

If moreover $\Delta^0, \Delta^1, \ldots, \Delta^k$ is a chain of ideal triangles with disjoint interior and a common ideal vertex, and $\gamma_i$ is a geodesic separating $\Delta^0, \ldots, \Delta^{i-1}$ from $\Delta^i, \ldots, \Delta^k$ for $i = 1, \ldots, k$ then we have

$$s(\Delta^0, \Delta^k, \gamma) = s(\Delta^0, \Delta^1, \gamma_1) + \cdots + s(\Delta^{k-1}, \Delta^k, \gamma_k)$$

for any geodesic $\gamma$ separating $\Delta^0$ and $\Delta^k$.

**Remark.** Working in say the upper half-plane model, assume that two ideal triangles $\Delta_a = [\theta_1, \theta_2, \theta_3]$ and $\Delta_b = [\theta_1, \theta_3, \theta_4]$ have a common boundary edge $\gamma = [\theta_1, \theta_3]$; the cyclic ordering of the vertices of $\Delta_a$ and $\Delta_b$ is counterclockwise. Let

$$\kappa = [\theta_1, \theta_2, \theta_3, \theta_4] = \frac{(\theta_1 - \theta_3)(\theta_2 - \theta_4)}{(\theta_1 - \theta_4)(\theta_2 - \theta_3)}$$

be the cross-ratio of the four vertices $\theta_1, \theta_2, \theta_3, \theta_4$. We have then the formula

$$s(\Delta^a, \Delta^b, \gamma) = \log(\kappa - 1).$$

Decomposing the general picture into adjacent ideal triangles and using the last claim of Lemma 3.1, it is easy to also express $s(\Delta^a, \Delta^b, \gamma)$ as sums of logarithms of algebraic expressions in cross ratios when $\Delta^a$ and $\Delta^b$ do not have a common boundary edge, but have a common ideal vertex. If the two triangles do not share any vertices, as in Figure 3, the computation can be made by considering the auxiliary triangle $\Delta$:

$$s(\Delta^a, \Delta^b, \gamma) = s(\Delta^a, \Delta, \gamma') + s(\Delta, \Delta^b, \gamma).$$

**3.2. The coordinates.** Let $\lambda$ be from now on an ideal triangulation of the complete finite area hyperbolic surface $X$. We now define a coordinate map $s_\lambda : \mathcal{T}(X) \to \mathbb{R}^{[\lambda]}$ by defining a coordinate function $s_\gamma$ for each leaf $\gamma$ of $\lambda$. We need
to make some choices in the definition of \( s_{\lambda} \) but the map will be unique up to post-composition with a linear map.

Assume \( \gamma \) is an isolated leaf of \( \lambda \) and let \( \tilde{\gamma} \) be a component of the pre-image of \( \gamma \) in the universal cover. Let \( \tilde{\lambda} \) be the pre-image of \( \lambda \) in the universal cover. Let \( \Delta^a_\gamma \) and \( \Delta^b_\gamma \) be the two ideal triangles in the complement of \( \tilde{\lambda} \) whose boundary contains \( \tilde{\gamma} \). Then we define \( s_{\gamma}(Y,f) = s(\tilde{f}(\Delta^a_\gamma),\tilde{f}(\Delta^b_\gamma),\tilde{f}(\tilde{\gamma})) \). Here \( \tilde{f} \) is as in the end of Section 2. It is clear that \( s_{\gamma}(Y,f) \) is independent of the choice of the lift \( \tilde{\gamma} \).

Now assume that \( \gamma \) is a closed curve in \( \lambda \). To define \( s_{\gamma} \) we need to make some arbitrary choices. Again let \( \tilde{\gamma} \) be a component of the pre-image of \( \gamma \) in the universal cover. In this case there will not be ideal triangles whose boundary contains \( \tilde{\gamma} \). Instead we choose ideal triangles \( \Delta^a_\gamma \) and \( \Delta^b_\gamma \) such that \( \tilde{\gamma} \) separates the triangles and they are both asymptotic to \( \tilde{\gamma} \). We then define \( s_{\gamma}(Y,f) = s(\tilde{f}(\Delta^a_\gamma),\tilde{f}(\Delta^b_\gamma),\tilde{f}(\tilde{\gamma})) \).

**Remark.** Suppose \( \hat{\Delta}^a_\gamma \) and \( \hat{\Delta}^b_\gamma \) are different choices of triangles, yielding a function \( \hat{s}_{\gamma}(Y,f) = s(\tilde{f}(\hat{\Delta}^a_\gamma),\tilde{f}(\hat{\Delta}^b_\gamma),\tilde{f}(\tilde{\gamma})) \). By the last claim of Lemma 3.1 and the remark following it, the difference \( \hat{s}_{\gamma}(Y,f) - s_{\gamma}(Y,f) \) is a linear combination of the functions \( s_{\eta}(Y,f) \) corresponding to the isolated leaves \( \eta \) separating \( \Delta^a_\gamma \) from \( \hat{\Delta}^a_\gamma \) and \( \Delta^b_\gamma \) from \( \hat{\Delta}^b_\gamma \).

We now define our coordinate map by

\[
(2) \quad s_{\lambda} : \mathcal{T}(X) \rightarrow \mathbb{R}^{|\lambda|}
\]

by \( s_{\lambda}(Y,f) = (s_{\gamma_1}(Y,f),\ldots,s_{\gamma_n}(Y,f)) \) where \( \gamma_1,\ldots,\gamma_n \) are the components of \( \lambda \). By the remark above, different choices amount to postcomposing \( s_{\lambda} \) with an invertible linear transformation of \( \mathbb{R}^{|\lambda|} \).

To see that \( s_{\lambda} \) is continuous we fix an identification of \( \tilde{X} = \mathbb{H}^2 \) with the upper half-plane. Given a quadruple \( (\theta_1,\theta_2,\theta_3,\theta_4) \) with \( \theta_i \in \partial \mathbb{H}^2 = \mathbb{R} \cup \{\infty\} \) we then define a function

\[
\mathcal{T}(X) \rightarrow \mathbb{R}, \quad (Y,f) \mapsto [\partial \tilde{f}(\theta_1),\partial \tilde{f}(\theta_2),\partial \tilde{f}(\theta_3),\partial \tilde{f}(\theta_4)].
\]

Since the cross-ratio is invariant under isometries of \( \mathbb{H}^2 \) this function is well-defined. Given four points in \( S^1 \) the cross ratio defines a map to \( \mathbb{R} \) on the space of quasi-symmetric maps. By [18] a quasi-symmetric map is quasi-Möbius and therefore this map is continuous and it follows that the above map on \( \mathcal{T}(X) \) is continuous. Also see [12]. Since \( s_{\lambda}(Y,f) \) can be expressed as a continuous function of cross-ratios it follows that \( s_{\lambda} \) is continuous.

*3.3. Image of \( s_{\lambda} \).* Our next goal is to determine the image of \( s_{\lambda} \). Before doing so we need some more notation.

Assume that \( \gamma \) is a closed leaf of \( \lambda \). Our choice of \( \Delta^a_\gamma \) and \( \Delta^b_\gamma \) determines an \( a \)-side and a \( b \)-side of \( \gamma \). In particular if \( C_\gamma \) is a collar neighborhood of \( \gamma \) then \( C_\gamma \setminus \gamma \) has two components which we call the *sides* of \( \gamma \). If \( \tilde{C}_{\gamma} \) is the component of the
pre-image of $C_γ$ that contains $\tilde{γ}$ then $Δ^a_γ$ will intersect one of the components of $\tilde{C}_γ \setminus \tilde{γ}$. The image of this component in $X$ will be one of the components of $C_γ \setminus γ$. This is the $a$-side. Then $Δ^b_γ$ will intersect the other component of $\tilde{C}_γ \setminus \tilde{γ}$ and this component will map to the $b$-side of $γ$.

We need to assign a sign to each side of $γ$ that will be determined by the direction $Δ^a_γ$ and $Δ^b_γ$ spiral around $γ$. As above, orient $\tilde{γ}$ so that $Δ^a_γ$ is on the left. Then $σ^a_γ = -1$ if $Δ^a_γ$ is asymptotic to the forward end of $\tilde{γ}$ and $σ^a_γ = +1$ if $Δ^a_γ$ is asymptotic to the negative end. We make a similar definition for $σ^b_γ$. Note that since $\tilde{γ}$ separates $Δ^a_γ$ from $Δ^b_γ$ we need to change the orientation of $\tilde{γ}$ when we define $σ^b_γ$.

Still assuming that $γ$ is a closed leaf of $λ$, we will now express its length in terms of our coordinates. The intersection of each leaf $β$ of $λ$ with the $a$-side of $γ$ will have 0, 1 or 2 components of infinite length. Let $n^a_γ(β)$ be this number and similarly define $n^b_γ(β)$. Note that if $β$ is a closed leaf then the number will always be zero. The content of the following lemma is that the length function $ℓ_γ$ is given by either of the following two linear, and hence convex, functions

$$
ℓ^a_γ : \mathbb{R}^{|λ|} \longrightarrow \mathbb{R}, \quad ℓ^a_γ(\mathbf{x}) = σ^a_γ \sum_{i=1}^{|λ|} n^a_γ(γ_i)x_i
$$

$$
ℓ^b_γ : \mathbb{R}^{|λ|} \longrightarrow \mathbb{R}, \quad ℓ^b_γ(\mathbf{x}) = σ^b_γ \sum_{i=1}^{|λ|} n^b_γ(γ_i)x_i.
$$

The following lemma completes the discussion begun in Section 9 of Chapter 3 of [14]. See also [13].

**Lemma 3.2.** If $γ$ is a closed curve in $λ$ and $ℓ_γ : \mathcal{T}(X) \rightarrow \mathbb{R}$ is its length function then

$$
ℓ_γ = ℓ^a_γ \circ s_λ = ℓ^b_γ \circ s_λ.
$$
Proof. Let \( (Y, f) \) be a marked hyperbolic structure in \( T(X) \). We will calculate \( \ell^a_\gamma \circ s_\lambda(Y, f) \). Working in the universal cover \( \mathbb{H}^2 \) using the upper half space model we can assume that \( \bar{f}(\tilde{\gamma}) \) is the vertical line at \( x = 0 \).

If we assume that \( \sigma^a_\gamma = -1 \) then we can also choose \( \tilde{f} \) so that the geodesics in \( \bar{f}(\tilde{\lambda}) \) that intersect that \( a \)-side of \( \bar{f}(\tilde{\gamma}) \) are vertical geodesics with negative \( x \)-coordinates. Label these \( x \)-coordinates \( x_i \) with \( x_i + 1 < x_i \). By our normalization the subgroup of the deck group for \( Y \) that fixes \( \bar{f}(\tilde{\gamma}) \) will be generated by the isometry \( z \mapsto e^{\ell_\gamma(Y, f)} z \). The set of vertical geodesics will be invariant under this isometry and we will have \( x_{i+k} = e^{\ell_\gamma(Y, f)} x_i \) where

\[
    k = \sum_{\beta \in \lambda} n^a_\gamma(\beta)
\]

is the number of components of infinite length in the intersection of \( \lambda \) with the \( a \)-side of \( \gamma \). Therefore \( \ell_\gamma(Y, f) = \log x_{i+k}/x_i \).

Let \( \Delta_i \) be the ideal triangle that has two vertical sides with \( x \)-coordinate \( x_i \) and \( x_{i+1} \). Note that the midpoints of the two vertical sides will have the same \( y \)-coordinate which we label \( m_i \) and that \( x_{i+1} = x_i - m_i \). We also observe that \( s(\Delta_i, \Delta_{i+1}) = -\log(x_{i+1}/m_i) \) so

\[
    \sum_{i=0}^{k-1} s(\Delta_i, \Delta_{i+1}) = -\sum_{i=0}^{k-1} \log(m_{i+1}/m_i) = -\log m_k + \log m_0
\]

\[
    = -\log(x_k - x_{k+1}) + \log(x_0 - x_1)
\]

\[
    = -\log(x_k/x_0)
\]

\[
    = -\ell_\gamma(Y, f).
\]

The second to last equality follows from the fact that \( \ell_\gamma(Y, f) = \log(x_k/x_0) = \log(x_{k+1}/x_1) \) and therefore \( x_{k+1}/x_k = x_1/x_0 \).

To finish the proof we need to write the sum on the left in terms of our coordinates. To do so we note that each vertical geodesic with \( x \)-coordinate \( x_i \) maps to an isolated component \( \beta \) of \( \lambda \) and \( s_\beta(Y, f) = s(\Delta_{i-1}, \Delta_i) \). Furthermore this geodesic will intersect the \( a \)-side of \( \gamma \) so \( n^a_\gamma(\beta) \) is positive. In fact each component \( \beta \) of \( \lambda \) that intersects the \( a \)-side of \( \gamma \) will have exactly \( n^a_\gamma(\beta) \) pre-images among the vertical geodesics with \( x \)-coordinates \( x_1, \ldots, x_k \). Therefore

\[
    \ell^a_\gamma \circ s_\lambda(Y, f) = \sigma^a_\gamma \sum_{i=1}^{\lambda} n^a_\gamma(\gamma_i) s_\gamma_i(Y, f)
\]

\[
    = -\sum_{\beta \in \lambda} n^a_\gamma(\beta) s_\beta(Y, f)
\]
which completes the proof when $\sigma^a_\gamma = -1$.

When $\sigma^a_\gamma = +1$ the proof is exactly the same except the vertical geodesics on the $a$-side of $\bar{f}(\tilde{\gamma})$ have positive $x$-coordinate. If we label the coordinates $x_i$ with $x_i < x_{i+1}$ then $s(\Delta_i, \Delta_{i+1}) = \log m_{i+1}/m_i$. The rest of the proof of the proof is exactly the same so this accounts for the $\sigma^a_\gamma = +1$ in the definition of $\ell^a_\gamma$.

The proof for $\ell^b_\gamma$ is obviously the same. □

Now let $c$ be a cusp of $X$. We let $n_c(\beta)$ be the number of components of infinite length of the intersection of $\beta$ with a horospherical neighborhood of $c$ and define

$$\ell_c(x) = \sum_{i=1}^{\mid \lambda \mid} n_c(\gamma_i) x_i.$$ We then have the following lemma.

**LEMMA 3.3.** If $c$ is a cusp of $X$ then $\ell_c \circ s_\lambda(Y, f) = 0$ for all $(Y, f)$ in $T(X)$.

**Proof.** The proof follows the same basic idea as the proof of Lemma 3.2. We can assume that a component of the pre-image of the horosphere neighborhood of the cusp is a horosphere neighborhood of infinity in the upper half-space model of $\mathbb{H}^2$. Then the geodesics in $\bar{f}(\tilde{\lambda})$ that intersect this neighborhood will be vertical geodesics with $x$-coordinates $x_i$ labeled such that $x_i < x_{i+1}$. The neighborhood will be invariant under the deck transformation that fixes infinity and we can assume that it is of the form $z \mapsto z + 1$. In particular $x_{i+k} = x_i + 1$ where

$$k = \sum_{\beta \in \lambda} n_c(\beta).$$

The set of midpoints $m_i$ will also be invariant under the action of $z \mapsto z + 1$ so we also have $m_i = m_{i+k}$. Repeating the calculations from Lemma 3.2 we see that

$$\sum_{i=0}^{k-1} s(\Delta_i, \Delta_{i+1}) = \log x_k/x_0 = 0$$

and that

$$\ell_c \circ s_\lambda(Y, f) = \sum_{i=0}^{k-1} s(\Delta_i, \Delta_{i+1})$$

and the lemma is proven. □

The functions $\ell^a_\gamma, \ell^b_\gamma$ and $\ell_c$ are linear functions defined on all of $\mathbb{R}^{\mid \lambda \mid}$. 

Lemmas are statements that are proven to be true. In this case, Lemma 3.4 states that the collection of functions \( \ell_a^\gamma, \ell_b^\gamma \) and \( \ell_c \) are linearly independent.

Proof. To visualize the lamination, remove the \( \epsilon \)-neighborhood of each closed leaf, for a small \( \epsilon > 0 \), and also a small horoball neighborhood of each cusp, to obtain a compact surface \( X' \). Each complementary component of the lamination gets truncated from an ideal triangle to a hexagon, with every other side on the boundary of \( X' \) and we will call these boundary arcs. The remaining sides are arcs of non-closed leaves, and we will call them leaves.

Consider the vector space \( V \) of formal linear combinations of non-closed leaves. Each boundary component of \( X' \) yields a vector in \( V \), namely the sum of leaves intersecting it (with multiplicity 2 if both endpoints are on this boundary component). We will show that the collection of these vectors is linearly independent. This implies the lemma, since each \( \ell_c \) corresponds to one of the vectors associated with a cusp and \( \ell_a^\gamma, \ell_b^\gamma \) correspond to the two components of the boundary of \( \epsilon \)-neighborhood of \( \gamma \).

Consider a linear combination of the above vectors, given by choosing a coefficient for each boundary component of \( X' \). This linear combination is 0 precisely when the sum of the two coefficients at the ends of each leaf is 0. Given a truncated triangle, we have three coefficients \( u, v, w \) for the 3 boundary arcs and the equations for the sides are \( u + v = u + w = v + w = 0 \). Thus \( u = v = w = 0 \), so all coefficients are 0.

Corollary 3.5. The image of \( s_\lambda \) is contained in \( T_\lambda \subset \mathbb{R}^{|\lambda|} \).

3.4. The coordinates are coordinates. We prove now the central result of this section:

Theorem 3.6. The map \( s_\lambda \) is a homeomorphism from \( \mathcal{T}(X) \) to \( T_\lambda \).

We will derive Theorem 3.6 from some standard results about Teichmüller space and the following proposition.

Proposition 3.7. Let \( \alpha \) be a closed curve on \( X \) and \( \ell_\alpha : \mathcal{T}(X) \to \mathbb{R}^+ \) its length function. Then there is a convex function \( \bar{\ell}_\alpha : \mathbb{R}^{|\lambda|} \to \mathbb{R} \) such that \( \ell_\alpha = \bar{\ell}_\alpha \circ s_\lambda \). Furthermore the \( \bar{\ell}_\alpha \)-image of \( T_\lambda \) is in \( \mathbb{R}^+ \) and if \( \alpha \) intersects every leaf of \( \lambda \) then \( \bar{\ell}_\alpha \) is strictly convex.

Proposition 3.7 is the main technical result of this paper; we defer its proof to the final section.

Proof of Theorem 3.6. As mentioned before, the fact that \( s_\lambda \) can be expressed in terms of cross ratios shows that it is continuous. By Corollary 3.5 the image
lies in $T_\lambda$ which is a convex subset of a linear subspace of $\mathbb{R}^{|\lambda|}$. The number of ideal triangles in $X \setminus \lambda = -2\chi(X) = \operatorname{area}(X)/\pi$ and therefore $|\lambda| = -3\chi(X) + n_c$ where $n_c$ is the number of closed curves in $\lambda$. In the definition of $T_\lambda$ there is one equation for each closed curve and one equation for each cusp. By Lemma 3.4 these equations are also linearly independent so $T_\lambda$ is an open subset of a linear subspace of dimension $|\lambda| - n_c - n_p$ where $n_p$ is the number of punctures (or cusps). Note that $\dim T(X) = -3\chi(X) - n_p = |\lambda| - n_c - n_p$.

If $s_\lambda(Y_0) = s_\lambda(Y_1)$ then by Proposition 3.7, $\ell_\alpha(Y_0) = \ell_\alpha(Y_1)$ for all closed curves $\alpha$ and therefore $Y_0 = Y_1$. This implies that $s_\lambda$ is injective. By invariance of domain $s_\lambda$ is a local homeomorphism and its image an open subset of $T_\lambda$.

To see that the image is closed take a finite collection of closed curves $\Gamma = \{\alpha_1, \ldots, \alpha_k\}$ such that the complementary pieces are disks or punctured disks. Such a collection binds the surface. Let $\ell_\Gamma : T(X) \to \mathbb{R}^+$ be the sum of the length functions $\ell_{\alpha_i}$ and similarly define $\bar{\ell}_\Gamma : \mathbb{R}^{|\lambda|} \to \mathbb{R}$ as the sum of the $\bar{\ell}_{\alpha_i}$.

Let $Y_i$ be a sequence in $T(X)$ such that $s_\lambda(Y_i) \to x$ in $T_\lambda$. By Proposition 3.7 $\bar{\ell}_\Gamma(x) < \infty$ and since $\ell_\Gamma(Y_i) = \bar{\ell}_\Gamma \circ s_\lambda(Y_i) \to \bar{\ell}_\Gamma(x)$ we see that $\ell_\Gamma(Y_i)$ is uniformly bounded. By Lemma 3.1 of [11] or Proposition 2.4 of [15], $\ell_\Gamma$ is a proper function and therefore the $Y_i$ lie in a compact subset of $T(X)$. In particular we can find a subsequence such that $Y_{i_k}$ converges to some $Y \in T(X)$ and therefore $\lim s_\lambda(Y_i) = \lim s_\lambda(Y_{i_k}) = s_\lambda(Y)$ in the image of $s_\lambda$. Therefore the image is a closed set.

Since the image is open and closed it must be all of $T_\lambda$.  

Before moving on we observe that Theorem 1.3 in an immediate consequence of Proposition 3.7 and Theorem 3.6:

**Theorem 1.3.** Let $X$ be a complete hyperbolic surface with finite area of genus $g$ and with $n$ cusps. Let $\lambda$ be a maximal lamination in $X$ with finitely many leaves and let $s_\lambda : T(X) \acute{\to} T_\lambda$ be the shearing coordinates associated to $\lambda$. For any essential curve $\gamma$ in $X$ the function

$$l_\gamma \circ s_\lambda^{-1} : T_\lambda \longrightarrow \mathbb{R}_+$$

is convex. If moreover the curve $\gamma$ intersects all the leaves of $\lambda$ then $l_\gamma \circ s_\lambda^{-1}$ is strictly convex.

**3.5. The Nielsen realization conjecture.** Kerckhoff proved the follow theorem using the convexity of length functions along earthquake paths. We give a similar proof using our convexity result.

**Theorem 3.8.** (Kerckhoff) If $\Gamma$ binds then $\ell_\Gamma$ has a unique minimum on $T(X)$.

**Proof.** Fix an ideal triangulation $\lambda$. Since $\Gamma$ binds there must be some curve in $\Gamma$ that is not in $\lambda$ so by Proposition 3.7 the length function $\bar{\ell}_\Gamma$ is strictly convex on
As we noted in the proof of Theorem 3.6 the function $\ell_\Gamma$ is proper and since $\ell_\Gamma = \bar{\ell}_\Gamma \circ s_\lambda$ the function $\bar{\ell}_\Gamma$ is also proper. A proper, strictly convex function that is bounded below has a unique minimum so $\bar{\ell}_\Gamma$, and therefore $\ell_\Gamma$, has a unique minimum. \qed

Kerckhoff used this result to prove the Nielsen realization conjecture. As the proof is short we include it here.

**Theorem 1.2.** (Kerckhoff) The action of every finite subgroup of the mapping class group on $T(S)$ has a fixed point.

**Proof.** Let $G$ be the finite subgroup. The $G$ orbit of any finite set of binding curves will still bind and will be $G$-invariant. Let $\Gamma$ be such a $G$-invariant binding set and let $X$ be the unique minimum of $\ell_\Gamma$. Clearly $X$ is fixed by $G$. \qed

### 3.6. Surfaces with geodesic boundary.

We conclude this section with a few remarks on surfaces with geodesic boundary. Here it is natural to use a slightly different definition of Teichmüller space. Let $X$ be a finite area hyperbolic surface with boundary components $\beta_1, \ldots, \beta_k$. For each boundary component $\beta_i$ choose an interval $I_i$ of $(0, \infty)$ and let $T(X; I_1, \ldots, I_k)$ be the Teichmüller space of marked hyperbolic surfaces where the length of $\beta_i$ is in the interval $I_i$. We allow the $I_i$ to be open, closed, half-open or a point. As above, two marked surfaces $(Y_0, f_0)$ and $(Y_1, f_1)$ are equivalent if there is an isometry $\phi : Y_0 \to Y_1$ such that $\phi \circ f_0$ and $f_1$ are isotopic. The difference with our previous definition is that here the isotopy does not need to fix the boundary. We similarly modify the definition of the distance between two marked hyperbolic surfaces. This weaker notion of equivalence will give a finite dimensional Teichmüller space.

An ideal triangulation of $X$ is still a finite leaved geodesic lamination whose complement is the union of open ideal triangles; in particular, $\partial X \subset \lambda$. Let $\hat{\lambda} = \lambda \setminus \partial X$ be the union of the all interior leaves of $\lambda$. The definition of the coordinate map $s_\lambda : T(X; I_1, \ldots, I_k) \to \mathbb{R}^{\hat{\lambda}}$ still makes sense. However, we need to make a slightly more involved argument for the continuity of $s_\lambda$ as the universal cover of $X$ is not $\mathbb{H}^2$ but rather a proper subset of $\mathbb{H}^2$. We can resolve this issue as follows. If $f : X \to Y$ is a $k$-quasiconformal homeomorphism then we can extend the lift $\tilde{f} : \tilde{X} \to \tilde{Y}$ to all of $\mathbb{H}^2$ by doubling both $X$ and $Y$ along their geodesic boundaries and extending $f$ in the obvious way to the doubled surfaces. The lift of this doubled map to the universal cover will be a $k$-quasi-conformal extension of $\tilde{f}$ to all $\mathbb{H}^2$. We can then repeat the argument for the continuity of $s_\lambda$ as in Section 3.2.

For each boundary component $\beta$ of $X$ we have a function $\ell_\beta : \mathbb{R}^{\hat{\lambda}} \to \mathbb{R}$ where the composition $\ell_\beta \circ s_\lambda$ is the length function for the boundary component. The definition of $\ell_\beta$ is exactly the same as the definition of the functions $\ell_\gamma^a$ and $\ell_\gamma^b$. Let $T_{\lambda, I_1, \ldots, I_k}$ be the subset of $T_\lambda \subset \mathbb{R}^{\hat{\lambda}}$ satisfying $\ell_\beta(x) \in I_i$ for $i = 1, \ldots, k$. 


THEOREM 3.9. The map $s_\lambda$ is a homeomorphism from $T(X; I_1, \ldots, I_k)$ to $T_{\lambda, I_1, \ldots, I_k}$.

Proof. Note that if each $I_i$ is a point then $T_\lambda$ is an open subset of a linear subspace of $\mathbb{R}^{\hat{\lambda}}$ and the proof is the same as the proof Theorem 3.6. The general case follows from this observation. \hfill $\square$

4. Fenchel-Nielsen coordinates. In this section we prove Theorem 1.1. We will assume knowledge of some basic form of the Fenchel-Nielsen coordinates; see for example [3].

Let $X$ be a hyperbolic surface and $\gamma$ a non-peripheral simple closed curve. Fenchel and Nielsen defined for $t \in \mathbb{R}$ the twist deformation $T^t_\gamma(X)$ of $X$ along $\gamma$ with twist parameter $t$ as follows. First isotope $\gamma$ to the geodesic in its free homotopy class, also denoted by $\gamma$, and let $g : \mathbb{R} \to \gamma$ be a parametrization by arc-length. Let $\gamma_1$ and $\gamma_2$ be the two boundary components obtained after cutting $X$ along $\gamma$ and for $i = 1, 2$ let $g_i : \mathbb{R} \to \gamma_i$ be the induced parametrization. Up to relabeling, we may assume that $g_1$ is orientation preserving and $g_2$ is orientation reversing with respect to the induced orientation of the boundary curves $\gamma_1$ and $\gamma_2$. The hyperbolic surface $T^t_\gamma(X)$ is obtained from the cut open surface by identifying the points $g_1(s)$ with $g_2(s + t)$ for all $s \in \mathbb{R}$. Compare with Figure 5.

The surface $T^t_\gamma(X)$ is just a hyperbolic surface; in particular it has so far no marking. However, it is well-known that there is flow, the Fenchel-Nielsen Dehn-twist flow

$$\tau_\gamma : \mathbb{R} \times T(X) \longrightarrow T(X), \quad (t, (Y, f)) \longmapsto \tau^t_\gamma(Y, f)$$

such that for $(Y, f) \in T(X)$, the surface $\tau^t_\gamma(Y)$ is the hyperbolic surface associated to the point $\tau^t_\gamma(Y, f)$. 

Figure 5. Twist deformation for $t$ equal to the length of the bold printed arc.
If $\gamma$ and $\gamma'$ are disjoint curves, then the flows $\tau_\gamma$ and $\tau_{\gamma'}$ commute. In particular, labeling by $\gamma_1, \ldots, \gamma_{|P|}$ the components of some pants decomposition $P$ of $X$ we have a free and proper action

$$\tau_P : \mathbb{R}^{|P|} \times \mathcal{T}(X) \to \mathcal{T}(X), \quad ((t_1, \ldots, t_{|P|}), (Y, f)) \mapsto \tau^{(t_1, \ldots, t_{|P|})}_P(Y, f)$$

where $\tau^{(t_1, \ldots, t_{|P|})}_P = \tau^{t_1}_{\gamma_1} \circ \cdots \circ \tau^{t_{|P|}}_{\gamma_{|P|}}$.

To the pants decomposition $P$ of $X$ we can also associate the function

$$\ell_P : \mathcal{T}(X) \to (\mathbb{R}^+)^{|P|}$$

which assigns to each point in $\mathcal{T}(X)$ the $|P|$-tuple of lengths of curves in the pants decomposition. Observe that the $\mathbb{R}^{|P|}$-action $\tau_P$ preserves by definition the function $\ell_P$ and hence acts on the fibers.

In fact, any existence results for Fenchel-Nielsen coordinates implies that the $\tau_P$-orbits actually coincide with the fibers of $\ell_P$. We can summarize this discussion as follows:

**Proposition 4.1.** Let $P$ be a pants decomposition of a finite area surface $X$. Then

$$\ell_P : \mathcal{T}(X) \to (\mathbb{R}^+)^{|P|}$$

has a natural structure as a $\mathbb{R}^{|P|}$-principal bundle.

Every principal bundle over a contractible space is trivial but not canonically trivialized. In fact, any choice of a section yields a trivialization and vice-versa. From the point of view of the authors, all coordinates for Teichmüller space obtained by trivializing the principal bundle $\ell_P : \mathcal{T}(X) \to (\mathbb{R}^+)^{|P|}$ deserve to be referred to as Fenchel-Nielsen coordinates. We are now ready to prove Theorem 1.1.

**Theorem 1.1.** Let $X$ be a complete, finite area, hyperbolic surface of genus $g$ and with $n$ cusps, and fix a pants decomposition $P$ of $X$. There are Fenchel-Nielsen coordinates $\Phi : \mathcal{T}(X) \sim \mathbb{R}^{3g+n-3}_+ \times \mathbb{R}^{3g+n-3}$ associated to $P$ such that for any essential curve $\gamma$ in $X$ the function

$$l_\gamma \circ \Phi^{-1} : \mathbb{R}^{3g+n-3}_+ \times \mathbb{R}^{3g+n-3} \to \mathbb{R}_+$$

is convex. If moreover the curve $\gamma$ intersects all the components of $P$ then $l_\gamma \circ \Phi$ is strictly convex.

**Proof.** To begin, extend $P$ to an ideal triangulation $\lambda$ and let

$$s_\lambda : \mathcal{T}(X) \to T_\lambda$$
be the shearing coordinates associated to $\lambda$. Theorem 1.1 will follow immediately from Theorem 1.3 when we interpret the shearing coordinates $s_\lambda$ as Fenchel-Nielsen coordinates.

Denote by $\lambda_0 = \lambda \setminus P$ the set of isolated leaves in $\lambda$, recall the definition of the convex polytope $T_\lambda \subset \mathbb{R}^{|\lambda|} = \mathbb{R}^{|P|} \times \mathbb{R}^{|\lambda_0|}$ and observe that the factor $\mathbb{R}^{|P|} \times \{0\}$ in the above splitting of $\mathbb{R}^{|\lambda|}$ is contained in $T_\lambda$. In particular, the canonical $\mathbb{R}^{|P|}$-principal bundle structure on

$$\pi: \mathbb{R}^{|\lambda|} = \mathbb{R}^{|P|} \times \mathbb{R}^{|\lambda_0|} \longrightarrow \mathbb{R}^{|\lambda_0|}$$

induces a $\mathbb{R}^{|P|}$-principal bundle structure on

$$\pi: T_\lambda \longrightarrow \pi(T_\lambda).$$

Here $\pi$ is the projection to the second factor of the splitting of $\mathbb{R}^{|\lambda|}$.

It follows directly from the definition just before Lemma 3.2 that for all $\gamma \in P$ the linear form $\ell_\gamma: T_\lambda \rightarrow \mathbb{R}^+$ is independent of the $\mathbb{R}^{|P|}$-factor and hence induces a well-defined linear function $\hat{\ell}_\gamma$ on $\pi(T_\lambda)$. Observe that this function is positive by Lemma 3.2. Denoting by $\hat{\ell}_P: \pi(T_\lambda) \rightarrow (\mathbb{R}^+)^{|P|}$ the function whose $\gamma$-coordinate is $\hat{\ell}_\gamma$ we have from Lemma 3.2 that the following diagram commutes

$$\begin{array}{ccc}
T(X) & \xrightarrow{s_\lambda} & T_\lambda \\
\downarrow \ell_P & & \downarrow \pi \\
(\mathbb{R}^+)^{|P|} & \xrightarrow{\hat{\ell}_P} & \pi(T_\lambda).
\end{array}$$

(3)

The map $\hat{\ell}_P$ is then linear and surjective. Since

$$\dim(\pi(T_\lambda)) \leq \dim(T_\lambda) - \dim(\mathbb{R}^{|P|}) \leq |P|$$

we obtain that $\hat{\ell}_P$ is a linear isomorphism. In particular, the bundles $T(X) \rightarrow (\mathbb{R}^+)^{|P|}$ and $T_\lambda \rightarrow \pi(T_\lambda)$ are isomorphic as fiber bundles. Moreover, it follows directly from the definition of the shearing coordinates that they are also equivalent as $\mathbb{R}^{|P|}$-principal bundles. Compare with Figure 6.

The projection $T_\lambda \rightarrow (\mathbb{R}^+)^{|P|}$ is linear and has kernel of dimension $|P|$. In particular, there is a linear map $L: T_\lambda \rightarrow \mathbb{R}^{|P|} \times (\mathbb{R}^+)^{|P|}$ such that the following diagram commutes:

$$\begin{array}{ccc}
T_\lambda & \xrightarrow{\pi} & \mathbb{R}^{|P|} \times (\mathbb{R}^+)^{|P|} \\
\downarrow L & & \downarrow \pi \\
(\mathbb{R}^+)^{|P|} & \xrightarrow{\hat{\ell}_P} & \pi(T_\lambda).
\end{array}$$

(4)

Here the unlabeled arrow is the projection on the first factor.
Combining (3) and (4) we obtain a trivialization of the principal bundle \( \ell_P : \mathcal{T}(X) \to (\mathbb{R}^+)^{|\mathcal{P}|} \), i.e. Fenchel-Nielsen coordinates, which differ from the shearing coordinates \( s_\lambda \) by a linear map. Since by Theorem 1.3 the length functions are convex with respect to the shearing coordinates and convexity is preserved by linear maps, the same result holds for this choice of Fenchel-Nielsen coordinates. This concludes the proof of Theorem 1.1. \( \square \)

5. **Length functions on** \( \mathcal{T}(A) \). Let \( A \) be a complete hyperbolic annulus whose core curve \( \alpha \) is isotopic to a geodesic, and let \( \mathcal{T}(A) \) be the Teichmüller space of \( A \). Observe that according to the definition above, \( \mathcal{T}(A) \) is infinite dimensional. Fix also an ideal triangulation \( \tilde{\lambda} \) of \( A \) (without closed leaves) and let \( \lambda \) be the sublamination of \( \tilde{\lambda} \) obtained by deleting all leaves of \( \lambda \) which are disjoint from the core geodesic of \( A \). Assume that \( \lambda \) contains only finitely many non-isolated leaves. In this section we construct a map

\[
s_\lambda : \mathcal{T}(A) \to \mathbb{R}^{|\lambda|}
\]

and a convex function

\[
\bar{\ell} : \mathbb{R}^{|\lambda|} \to (0, \infty]
\]

such that \( \ell = \bar{\ell} \circ s_\lambda \). Here

\[
\ell : \mathcal{T}(A) \to (0, \infty)
\]

is the function which assigns to each hyperbolic annulus the length of its core curve.
5.1. A treatise on wedges. An ideal wedge is the region bounded by two asymptotic geodesics in $H^2$. If $W$ is an ideal wedge in $H^2$ or $A$ then there is a unique ideal triangle $\Delta(W)$ two of whose boundary components are the boundary components of $W$. In particular, the two boundary edges $\gamma_1$ and $\gamma_2$ of the ideal wedge $W$ inherit midpoints from the triangle $\Delta(W)$. Orienting both boundary edges towards their shared ideal point, we parametrize $\gamma_1$ and $\gamma_2$ by $\mathbb{R}$ via the signed distance to the midpoint. To $(x,y) \in \mathbb{R}^2$ we associate the pair of points in $\partial W$ corresponding to $x$ in $\gamma_1$ and $y$ in $\gamma_2$ respectively; let $d(x,y)$ be the distance in $H^2$ of these two points in $\partial W$.

We can describe the function $d : \mathbb{R}^2 \rightarrow (0,\infty)$ more explicitly using the upper half space model of $H^2$. Namely, define $d(x,y)$ to be the distance in $H^2$ between the points $ie^x$ and $1+ie^y$.

**Lemma 5.1.** The function $d$ is strictly convex and

$$|x - y| \leq d(x,y) \leq |x - y| + \frac{1}{\max\{e^x, e^y\}}.$$

**Proof.** This follows from the fact that $H^2$ is negatively curved and hence that the distance function is convex. It can also be proved by direct computation. $\square$

An injective isometric immersion of a wedge into a hyperbolic annulus $A$ will also be called an ideal wedge; all the definitions above carry over without difficulties.

An ideal wedgelation is a geodesic lamination $\lambda$ on $A$ whose complement is a disjoint union of ideal wedges, and such that $\lambda$ contains only finitely many non-isolated leaves. For the sake of concreteness we will also assume that every leaf of $\lambda$ which is isolated to one side is actually isolated.

**Lemma 5.2.** Let $S$ be a hyperbolic surface, $A$ an annulus and $\pi : A \rightarrow S$ a covering. Suppose that $\lambda$ an ideal triangulation of $S$ whose preimage $\pi^{-1}(\lambda)$ does
not have closed leaves. The set of leaves of $\pi^{-1}(\lambda)$ which intersect every curve in $A$ homotopic to the core curve is an ideal wedgelation.

On a finite area surface it is not possible to consistently orient an ideal triangulation. On an annulus, an ideal wedgelation can be consistently oriented and this will be important in the work below. To do so we fix an orientation on $A$ and of its core geodesic $\alpha$. Observe that all the leaves of an ideal wedgelation $\lambda$ of $A$ intersect $\alpha$ exactly once. We then orient every geodesic in $\lambda$ so that if $\gamma$ is a geodesic in $\lambda$ the orientation of $A$ at $\alpha \cap \gamma$ is given by the ordered pair of the oriented tangent vectors to $\alpha$ and $\gamma$.

Observe that if $W$ is an ideal wedge of in $A \setminus \lambda$, then the orientations of the boundary edges of $W$ determined by $W$ and the orientations determined by $\lambda$ agree if and only if the vertex of $W$ is to the left of the core curve $\alpha$.

5.2. The shearing map. Recall that for every ideal wedge $W$ in $H^2$ or $A$ we have associated an ideal triangle $\Delta(W)$. If $W^a$ and $W^b$ are disjoint wedges in $H^2$ and $\gamma$ is a geodesic in $H^2$ separating $W^a$ from $W^b$ and asymptotic to some boundary edge of $W^a$ and some boundary edge $W^b$ we define

$$s(W^a, W^b, \gamma) = s(\Delta(W^a), \Delta(W^b), \gamma).$$

The claims of Lemma 3.1 hold in this setting as well.

An ideal wedgelation of $A$ yields now a map

$$s_\lambda : T(A) \longrightarrow \mathbb{R}^{[\lambda]}$$

in the same way as it did for an ideal triangulation on a surface except we replace ideal triangles with ideal wedges $W^\pm_\gamma$ and $W^\pm_\eta$. We remark that $s_\lambda$ does not yield coordinates of $T(A)$. We refer to the map $s_\lambda$ as the shearing map.
5.3. The function $L$. Let $\lambda$ be an ideal wedgelation of $A$ transversal to the core geodesic, $\lambda_0 \subset \lambda$ the collection of isolated leaves of $\lambda$ and $W$ the set of ideal wedges in $A \setminus \lambda$. Recall that $\lambda \setminus \lambda_0$ consists, by assumption, of only finitely many leaves and that we are also working under the additional assumption that the leaves in $\lambda \setminus \lambda_0$ are non-isolated on both sides.

For each $W \in W$ let $\gamma^-_W$ be the left boundary edge and $\gamma^+_W$ be the right boundary edge where left and right is defined with respect to the orientation of the core curve $\alpha$. Observe that both $\gamma^-_W$ and $\gamma^+_W$ belong to $\lambda$. If the two boundary components $\gamma^-_W$ and $\gamma^+_W$ of $W$ are asymptotic to the left $\alpha$ then define $d_W(x,y) = d(x,y)$ for $(x,y) \in \mathbb{R}^2$. If they are asymptotic to the right of $\alpha$ then define $d_W(x,y) = d(-x,-y)$. Here left and right are defined with respect to the orientation of $\alpha$.

Define a function $L : \mathbb{R}^{|\lambda|} \times \mathbb{R}^{|\lambda_0|} \to (0, \infty]$ by

$$L(x,y) = \sum_{W \in W} d_W(\gamma^-_W(y),\gamma^+_W(y) + \gamma^+_W(x)).$$

Here $\gamma(x)$ is the $\gamma$-coordinate of $x \in \mathbb{R}^{\lambda}$. Note that $L$ does not depend on the coordinates of the non-isolated leaves in $\lambda$.

Meaning of $L$. The definition of $L$ is a bit obscure. Before moving on we explain its meaning in the particular case that $\lambda$ is a finite wedgelation. In particular $\lambda_0 = \lambda$ and the sum in the definition of $L$ is finite. Given a point $(B,f) \in \mathcal{T}(A)$, its image $x = s_\lambda(B,f)$ under the shearing map encodes how the different wedges are glued in $X$. The element $y \in \mathbb{R}^{\lambda_0}$ picks a point in each leaf of the (finite) wedgelation $\lambda$. In particular, $y$ picks in every wedge $W$ two different points in the boundary. Let $\alpha_W$ be the geodesic segment in $W$ joining these two points. The juxtaposition of all the segments $\alpha_W$ is a closed loop in $(B,f)$ homotopic to the core curve; $L(x,y)$ is the length of this loop.

5.4. Piecewise geodesic segments. Our next aim is to generalize the preceding discussion on the meaning of the function $L$. Assume now that $\lambda$ is a general wedgelation and recall that $\lambda_0$ is the collection of isolated leaves of $\lambda$.

For a marked hyperbolic annulus $(B,f) \in \mathcal{T}(A)$ let $\mathcal{P}(B,f)$ be the set of closed continuous loops $\beta$ on $B$ homotopic to $f(\alpha)$ and such that the intersection of $\beta$ with each wedge in $B \setminus \bar{f}(\lambda)$ is a geodesic segment. In particular, any $\beta \in \mathcal{P}(B,f)$ intersects each geodesic in $\bar{f}(\lambda)$ exactly once. We need to define a set of coordinates for $\mathcal{P}(B,f)$.

For each non-isolated leaf $\gamma$ in $\lambda_0$ let $m_{\bar{f}(\gamma)}$ be the midpoint of $\bar{f}(\gamma)$ coming from the ideal triangle $\Delta(\bar{f}(W^-_\gamma))$ where, as above, $W^-_\gamma$ is the wedge to the left of $\gamma$. We let

$$p_\lambda : \mathcal{P}(B,f) \to \mathbb{R}^{\lambda_0}$$
be the map defined by the property that $\gamma(p_\lambda(\beta))$ is the signed distance between $m\bar{f}(\gamma)$ and the intersection of $\beta$ with $\bar{f}(\gamma)$. As always, the sign is determined by the orientation induced by the core geodesic $\alpha$.

The map $p_\lambda$ is clearly injective. However, if $\lambda_0 \neq \lambda$ it is easy to see that it is not surjective. Our next goal is to determine the image of $p_\lambda$. In order to do so we define two quantities $a^-_\gamma(x, y)$ and $a^+_\gamma(x, y)$ for every non-isolated leaf $\gamma$; recall that we are assuming that leaves which are isolated to one side are actually isolated. We first define $a^-_\gamma$. Label the geodesics between $\gamma$ and $W^-\gamma_1$, the (arbitrarily) chosen wedge to the left of $\gamma$, by $\gamma_0^- = \gamma_{W^-\gamma_1}^-, \gamma_1^-, \ldots$ so that they are positively ordered with respect to $\alpha$. Define

\begin{equation}
(5) \quad a^-_\gamma(x, y) = \lim_{n \to \infty} \left( \gamma_n^- (y) + \sum_{i=0}^{n} \gamma_i^- (x) \right)
\end{equation}

if the limit exists. The definition of $a^-_\gamma(x, y)$ is very similar with small changes. Label the geodesics between $W^+\gamma_1$ and $\gamma$ by $\gamma_0^+ = \gamma_{W^+\gamma_1}^+, \gamma_1^+, \ldots$ so that they are negatively ordered with respect to $\gamma$ and define

\begin{equation}
(6) \quad a^+_\gamma(x, y) = \lim_{n \to \infty} \left( \gamma_n^+ (y) - \sum_{i=0}^{n-1} \gamma_i^+ (x) \right)
\end{equation}

if the limit exist and $a^+_\gamma(x, y) = \infty$ otherwise. To understand the meaning of the sums in the definition we suggest to the reader to compare with the last claim in Lemma 3.1.

Given $x \in \mathbb{R}^{\lambda}$ define $P_x \subset \mathbb{R}^{\lambda_0}$ to be the set

$$P_x = \left\{ y \in \mathbb{R}^{\lambda_0} \mid \text{the limits (5) and (6) exist, and} \quad a^-_\gamma(x, y) = a^+_\gamma(x, y) + \gamma(x) < \infty \text{ for all } \gamma \in \lambda \setminus \lambda_0 \right\}.$$

We prove:

**Lemma 5.3.** Let $(B, f)$ be a marked hyperbolic annulus in $T(A)$. If $x = s_\lambda(B, f)$ then the $p_\lambda$-image of $\mathcal{P}(B, f)$ is $P_x$ and

$$\text{length}(\beta) = L(x, p_\lambda(\beta)).$$

**Proof.** Let $\gamma$ be a geodesic in $\lambda \setminus \lambda_0$. As above we label the geodesics on the left of $\gamma$ by $\gamma_0^- , \gamma_1^- , \ldots$ and on the right by $\gamma_0^+, \gamma_1^+, \ldots$. If $\beta$ is a curve in $\mathcal{P}(B, f)$ let $p_i^\pm$ be the point of intersection of $\beta$ with $\bar{f}(\gamma_i^\pm)$. Note that the sequence $p_i^-$ and $p_i^+$ will limit to the same point $p$ on $\gamma$ since $\beta$ is a continuous path. Therefore both $h_{f(\gamma_i^-), f(\gamma)}(p_i^-)$ and $h_{f(\gamma_i^+), f(\gamma)}(p_i^+)$ will limit to $p$, where $h_{s, t}(\cdot)$ as is defined in Section 3.1. Let $d_i^-$ be the signed distance between $h_{f(\gamma_0^-), f(\gamma)}(m_{\bar{f}(\gamma_0^-)})$ and $h_{f(\gamma_i^-), f(\gamma)}(p_i^-)$ and similarly define $d_i^+$. Then $d_i^-$ will limit to the signed distance between $h_{f(\gamma_0^-), f(\gamma)}(m_{\bar{f}(\gamma_0^-)})$ and $p$ which we label $d^-$. Similarly $d^+$, the
limit of $d_i^+$, will be the signed distance between $h_{f(γ_0^+)}, f(γ) (m_{f(γ_0^+)})$ and $p$. Since
the signed distance between $h_{f(γ_0)}, f(γ) (m_{f(γ_0)}$ and $h_{f(γ_0^+)}, f(γ) (m_{f(γ_0^+)}$ is $γ(x)$
we have
\[d^- - d^+ = γ(x).\]

Since the $h$-maps are isometries and $h_{f(γ_0^+)}, f(γ) (m_{f(γ_0^+)} = h_{f(γ_0)}, f(γ)$ we have
that $d_i^+$ is also equal to the signed distance between $h_{f(γ_0)}, f(γ) (m_{f(γ_0)}$ and $p_i^-$. By definition $γ_n^−(p_λ(β))$ is the signed distance between $m_{f(γ_n)}$ and $p_n^−$ so we have
\[d_n^- = γ_n^−(p_λ(β)) + \sum_{i=0}^{n} γ_i^−(x)\]

by the last claim of Lemma 3.1. Therefore $d^- = a^-_γ(x, p_λ(β))$ and similarly $d^+ = a^+_γ(x, p_λ(β))$. Rearranging (7) we have
\[a^-_γ(x, p_λ(β)) = a^+_γ(x, p_λ(β)) + γ(x)\]

and therefore $p_λ(β)$ is in $P_x$.

For every $y \in P_x$ we need to build a curve $β$ on $B$ such that $p_λ(β) = y$. Given a
geodesic $α$ in $λ_0$ let $p_α$ be the point on $f(α)$ whose signed distance from $m_{f(α)}$ is $α(y)$. Then on each of the finitely many components of $B \setminus f(λ \setminus λ_0)$ there is an arc
that intersects each $α$ in $λ_0$ at $p_α$ and is a geodesic in each wedge. For these arcs
to complete to a simple closed curve in $P(B, f)$ we need that for each geodesic $γ$ in $λ \setminus λ_0$ the limit of the points $p_γ^−$ is equal to the limit of $p_γ^i$. From the first
paragraph of this proof we see that this holds when $y$ is in $P_x$.

Finally we note that the length of $β$ is the sum of the lengths of the restriction
of $β$ to each ideal wedge. This is exactly the sum $L(x, p_λ(β))$. \(\Box\)

Before moving on to more interesting topics, we observe:

**Lemma 5.4.** The set $P = \{(x, y) \in \mathbb{R}^{|λ|} \times \mathbb{R}^{|λ_0|}, y \in P_x\}$ is a linear subspace
and the projection $P \to \mathbb{R}^{|λ|}$ is surjective.

### 5.5. Convexity

We restrict from now on the function $L$ to the linear subspace $P$. The function $L$ is an infinite sum of convex functions. This sum will not be finite everywhere so we need an extended notion of convexity. If $V$ is vector
space and $f : V \to (−∞, \infty]$ is a function then $f$ is convex if
\[f\left(\frac{x_0 + x_1}{2}\right) \leq \frac{f(x_0) + f(x_1)}{2}\]
whenever $f(x_0)$ and $f(x_1)$ are finite. The function is strictly convex if the inequality
is always strict. We need the following general lemma.
Lemma 5.5. Let $V_0$ and $V_1$ be vector spaces and $P$ a subspace of $V_0 \times V_1$ whose projection onto $V_0$ is onto. Let $P_x = \{ y \in V_1 | (x, y) \in P \}$. If $F : V_0 \times V_1 \rightarrow (0, \infty]$ is convex then

$$f(x) = \inf_{y \in P_x} F(x, y)$$

is convex. If the infimum is realized for all $x$ and if $F$ is strictly convex on $P$, then $f$ is strictly convex as well.

Proof. Let $x_0$ and $x_1$ be points in $V_0$. We can assume that both $f(x_0)$ and $f(x_1)$ are finite for otherwise the lemma is trivial. For any $\epsilon > 0$ we can choose a $y_0 \in P_{x_0}$ and $y_1 \in P_{x_1}$ such that $f(x_i) > F(x_i, y_i) - \epsilon$. Since $P$ is a subspace we have that $(y_0 + y_1)/2$ is in $P_{(x_0 + x_1)/2}$ and therefore

$$f \left( \frac{x_0 + x_1}{2} \right) \leq F \left( \frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2} \right) \leq \frac{F(x_0, y_0) + F(x_1, y_1)}{2} \leq \frac{f(x_0) + f(x_1)}{2} + \epsilon.$$

Since the $\epsilon$ is arbitrary we have that

$$f \left( \frac{x_0 + x_1}{2} \right) \leq \frac{f(x_0) + f(x_1)}{2}$$

and $f$ is convex.

If $f$ is realized by a minimum then there are $y_0$ and $y_1$ such that $f(x_i) = F(x_i, y_i)$. If $F$ is also strictly convex we have

$$f \left( \frac{x_0 + x_1}{2} \right) \leq F \left( \frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2} \right) \leq \frac{F(x_0, y_0) + F(x_1, y_1)}{2} = \frac{f(x_0) + f(x_1)}{2}$$

which implies that $f$ is strictly convex. \qed

Theorem 5.6. Let $\bar{\ell} : \mathbb{R}^{|\lambda|} \rightarrow (0, \infty]$ be defined by

$$\bar{\ell}(x) = \inf_{y \in P_x} L(x, y).$$

Then $\bar{\ell}(x)$ is realized for all $x$. Furthermore $\bar{\ell}$ is strictly convex and $\ell = \bar{\ell} \circ s_\lambda$. 
Furthermore the length function. Then there is a convex function \( R \) on \( P \) all \( y \) therefore if \( \alpha \) is strictly convex along any line through distinct points \((\gamma_0, \gamma_1)\) we must have \( \beta(\gamma_0) = \beta(\gamma_1) \) and \( \beta(\gamma_1) = \beta(\gamma_1) \) for all \( \beta \in \lambda_0 \). For all closed curves \( \gamma \) in \( \lambda \) we then have \( \alpha \Big((x_0, y_0) = \alpha \Big(x_0, y_1) \Big) \) and therefore if \( y_0 \in \beta \) and \( y_1 \in \beta \), we also have \( \gamma(x_0) = \gamma(x_1) \). Therefore if \( L \) is not strictly convex along the line it cannot lie in \( \beta \); equivalently \( L \) is strictly convex on \( \beta \). Lemma 5.5 then implies that \( \bar{\ell} \) is strictly convex.

6. Proof of Proposition 3.7. We now return to a finite area hyperbolic surface \( X \) with an ideal triangulation \( \lambda \) and recall the proposition.

**Proposition 3.7.** Let \( \alpha \) be a closed curve on \( X \) and \( \ell_\alpha : \mathcal{T}(X) \to \mathbb{R}^+ \), its length function. Then there is a convex function \( \bar{\ell}_\alpha : \mathbb{R}^{|\lambda|} \to \mathbb{R} \) such that \( \ell_\alpha = \bar{\ell}_\alpha \circ s_\lambda \). Furthermore the \( \bar{\ell}_\alpha \)-image of \( \mathcal{T}_\lambda \) is in \( \mathbb{R}^+ \) and if \( \alpha \) intersects every leaf of \( \lambda \) then \( \bar{\ell}_\alpha \) is strictly convex.

Let \( \pi_\alpha : X_\alpha \to X \) be the annular covering such that \( \alpha \) lifts to \( X_\alpha \). If \( (Y, f) \) is a pair in \( \mathcal{T}(X) \) then \( Y_{f(\alpha)} \) is the annulus cover of \( Y \) such that \( f(\alpha) \) lifts to \( Y_{f(\alpha)} \). The map \( f : X \to Y \) lifts to a quasi-conformal homeomorphism \( f_\alpha : X_\alpha \to Y_{f(\alpha)} \). This induces a map \( \Pi_\alpha : \mathcal{T}(X) \to \mathcal{T}(X_\alpha) \). Let \( \ell_\alpha : \mathcal{T}(X_\alpha) \to (0, \infty) \) be the geodesic length function for \( \alpha \) and let \( \ell : \mathcal{T}(X_\alpha) \to (0, \infty) \) be the geodesic length function for the core curve of the annulus. Then \( \ell_\alpha = \ell \circ \Pi_\alpha \).

Let \( \lambda_\alpha \) be the set of geodesics in \( \pi_\alpha^{-1}(\lambda) \) that intersect the core curve \( \alpha \). Then \( \lambda_\alpha \) is an ideal wedgelation of \( X_\alpha \) by Lemma 5.2. The covering map \( \pi_\alpha \) defines a map \( \bar{\pi}_\alpha \) from the leaves of \( \lambda_\alpha \) to the leaves of \( \lambda \) which induces a map \( \bar{\Pi}_\alpha : \mathbb{R}^{|\lambda|} \to \mathbb{R}^{|\lambda_\alpha|} \).

**Proposition 6.1.** We can choose the coordinates for \( \mathcal{T}(A) \) and \( \mathcal{T}(S) \) such that \( s_{\lambda_\alpha} \circ \Pi_\alpha = \bar{\Pi}_\alpha \circ s_\lambda \).
Proof. We will in fact show that for any choice of coordinate map \( s_\lambda \), coordinates \( s_\lambda \alpha \) can be chosen such that the proposition holds. Assuming \( s_\lambda \) has been chosen we will construct \( s_\lambda \alpha \) and let the reader check that the desired relation holds.

The covering map \( \pi : \mathbb{H}^2 \to X \) factors through \( \pi_\alpha \). In particular \( X \) and \( X_\alpha \) have the same universal cover and the pre-image \( \tilde{\lambda} \) of \( \lambda_\alpha \) is a subset of the pre-image \( \tilde{\lambda} \) of \( \lambda \). Let \( \gamma' \) be a geodesic in \( \lambda_\alpha \) and let \( \gamma = \pi_\alpha(\gamma') \) be its image in \( \lambda \). Let \( \tilde{\gamma}' \) be a component of the pre-image of \( \gamma' \) in \( \mathbb{H}^2 \). In our choice of coordinate map \( s_\lambda \) we have chosen a pre-image \( \tilde{\gamma} \) in \( \mathbb{H}^2 \) of \( \gamma \) and ideal triangles \( \Delta_{\gamma}^a \) and \( \Delta_{\gamma}^b \). Then there is a deck transformation \( g \in \pi_1(X) \) such that \( g(\tilde{\gamma}) = \tilde{\gamma}' \). Since \( \tilde{\gamma} \) separates \( \Delta_{\gamma}^a \) from \( \Delta_{\gamma}^b \) we have that \( \tilde{\gamma}' \) separates \( g(\Delta_{\gamma}^a) \) from \( g(\Delta_{\gamma}^b) \). These ideal triangles will also be asymptotic to \( \tilde{\gamma}' \) so each must have exactly two edges that intersect \( \tilde{\alpha} \), the pre-image of \( \alpha \) in \( \mathbb{H}^2 \) and these edges will be in \( \tilde{\lambda} \). Exactly one of these ideal triangles, say \( g(\Delta_{\gamma}^a) \), will be on the left of \( \tilde{\gamma}' \). Since it has two edges in \( \tilde{\lambda} \) there will be a wedge \( W \) in \( \mathbb{H}^2 \backslash \tilde{\lambda} \) with \( \Delta(W) = g(\Delta_{\gamma}^a) \). We let \( \tilde{W}_\gamma = W \). Similarly the ideal triangle \( g(\Delta_{\gamma}^b) \) determines a wedge in \( \mathbb{H}^2 \backslash \tilde{\lambda} \) which we define to be \( \tilde{W}_\gamma^+ \). If \( g(\Delta_{\gamma}^b) \) is on the left we reverse the labels. This completes the construction of \( s_\lambda \alpha \). \( \square \)

Let \( \bar{\ell} : \mathbb{R} |\lambda_\alpha| \to [0, \infty] \) be the factorization of \( \ell \) given in the previous subsection so that \( \ell = \bar{\ell} \circ s_\lambda \). Define \( \bar{\ell}_\alpha : \mathbb{R} |\lambda| \to [0, \infty] \) by \( \bar{\ell}_\alpha = \bar{\ell} \circ \bar{\Pi}_\alpha \). We then have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{T}(X) & \xrightarrow{\sigma_\lambda} & \mathbb{R} |\lambda| \\
\downarrow{\Pi_\alpha} & \downarrow{\bar{\ell}_\alpha} & \downarrow{\bar{\Pi}_\alpha} \\
\mathcal{T}(X_\alpha) & \xrightarrow{\sigma_{\lambda_\alpha}} & \mathbb{R} |\lambda_\alpha|.
\end{array}
\]

The first claim of Proposition 3.7 follows directly from the commutativity of the triangle on the top. The convexity of \( \bar{\ell}_\alpha \) follows from the strict convexity of \( \bar{\ell} \) and the fact that \( \bar{\Pi}_\alpha \) is linear. In the case that \( \alpha \) intersects every leaf of \( \lambda \), the map \( \bar{\Pi}_\alpha \) is also injective and hence \( \bar{\ell}_\alpha \) is convex as claimed. The following lemma now concludes the proof of Proposition 3.7:

**Lemma 6.2.** The function \( \bar{\ell}_\alpha \) is finite on \( T_\lambda \).

If we knew at this stage that \( s_\lambda \) is a homeomorphism onto its image \( T_\lambda \), the claim of Lemma 6.2 would follow also from the commutativity of the diagram above and the fact that \( \ell_\alpha \) is obviously finite. Unfortunately, we derived this fact from Proposition 3.7.
**Proof.** Let \( x \) be in \( T_\lambda \) and let \( x_\alpha = \tilde{\Pi}_\alpha(x) \). We will choose a \( y \in R^{(\lambda_\alpha)_0} \) such that \( y \in P_{x_\alpha} \) and \( L(x_\alpha, y) < \infty \). Let \( \gamma_\alpha \) be a non-isolated leaf of \( \lambda_\alpha \). Then \( \tilde{\pi}_\alpha(\gamma_\alpha) \) is a closed leaf \( \gamma \) of \( \lambda \). Let \( \gamma_i^+ \) be the leaves on the right of \( \gamma_\alpha \) labeled as they were in the previous section. Assume that the right side of \( \gamma_\alpha \) maps to the \( a \)-side of \( \gamma \). Then \( \tilde{\pi}_\alpha(\gamma_i^+) = \tilde{\pi}_\alpha(\gamma_i^+ + k) \) where

\[
k = \sum_{\beta \in \lambda} n_\beta^a(\beta).
\]

Therefore \( \gamma_i^+(x_\alpha) = \gamma_i^+(x_\alpha) \) and there is a constant \( c \) such that

\[
c = \sum_{i=j}^{j+k-1} \gamma_i^+(x_\alpha)
\]

for all \( j \geq 0 \). It follows that

\[
\sum_{i=0}^{nk+j} \gamma_i^+(x_\alpha) = nc + \sum_{i=0}^{j} (x_\alpha).
\]

Choose the \( \gamma_i^+ \)-coordinates of \( y \) such that

\[
\gamma_n^+(y) = \sum_{i=0}^{n-1} \gamma_i^+(x_\alpha) = \gamma_{n-1}^+(y) + \gamma_{n-1}^+(x_\alpha).
\]

Let \( W_{\gamma_\alpha}^+ \) be the set of wedges in \( W \) whose boundary edges are in \( \gamma_i^+ \) and \( \gamma_i^+ \) for some \( i \geq 0 \). Then

\[
\sum_{W \in W_{\gamma_\alpha}^+} d_W(\gamma_W^-, y, \gamma_W^+(y) + \gamma_W^+(x_\alpha))
\]

\[
= \sum_{j=0}^{k-1} \sum_{i=0}^{\infty} d(\gamma_{j+k+1}^+(y), \gamma_{j+k+1}^+(y) + \gamma_{j+k+1}^+(x_\alpha))
\]

\[
= \sum_{j=0}^{k-1} \sum_{i=0}^{\infty} d(ic + \gamma_j^+(x_\alpha), ic + \gamma_j^+(x_\alpha))
\]

\[
\leq \sum_{j=0}^{k-1} \sum_{i=0}^{\infty} e^{-ic - \gamma_j^+(x_\alpha)}
\]

\[
< \infty
\]

where the inequality on the third line follows from Lemma 5.1.
We now take the leaves $\gamma_i$ on the left of $\gamma_\alpha$. Similarly, choose the $\gamma_i^-$ coordinates but so that we have $a^+_{\gamma_\alpha}(x_\alpha, y) + \gamma_\alpha(x_\alpha) = a^-_{\gamma_\alpha}(x_\alpha, y)$ and define
\[
\gamma_n^-(y) = \gamma_\alpha(x_\alpha) - \sum_{i=0}^n \gamma_i^-(x_\alpha).
\]

Similarly define $W^-_{\gamma_\alpha}$ to be the set of wedges in $W$ whose boundary edges are $\gamma_i^-$ and $\gamma_{i+1}^-$ for some $i \geq 0$. In the same way we find that
\[
\sum_{W \in W^-_{\gamma_\alpha}} d_W(\gamma_W^-(y), \gamma_W^+(y) + \gamma_W^+(x_\alpha)) < \infty.
\]

We repeat the above construction for each of the finitely many non-isolated leaves of $\lambda_\alpha$. Note that each wedge $W \in W$ is asymptotic to at most one of the non-isolated leaves $\gamma_\alpha$ and therefore is in at most one of the sets $W^+_{\gamma_\alpha}$ or $W^-_{\gamma_\alpha}$ that contains $W$ and all but finitely many of the $W$ are in one of these sets. This defines $y$ for all but finitely many coordinates. The remaining we choose freely. We can then split the infinite sum $L(x_\alpha, y)$ into finitely many sums each of which is finite. Therefore $\bar{\ell}_\alpha(x) \leq L(x_\alpha, y) < \infty$. 

This concludes the proof of Proposition 3.7 and with it the whole paper.

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