BROWNIAN SHEET AND TIME INVERSION
FROM $G$-ORBIT TO $L(G)$-ORBIT

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Abstract. We have proved in a previous paper that a space-time Brownian motion conditioned to remain in a Weyl chamber associated to an affine Kac–Moody Lie algebra is distributed as the radial part process of a Brownian sheet on the compact real form of the underlying finite dimensional Lie algebra, the radial part being defined considering the coadjoint action of a loop group on the dual of a centrally extended loop algebra. We present here a very brief proof of this result based on a time inversion argument and on elementary stochastic differential calculus.

1. Introduction

We propose here a short proof of the main result of [5]. Let us briefly recall this result. For this we need to consider a connected simply connected simple compact Lie group $G$ and its Lie algebra $\mathfrak{g}$ equipped with an invariant scalar product for the adjoint action of $G$ on $\mathfrak{g}$. One considers a standard Brownian sheet $\{x_{s,t}, s \in [0,1], t \geq 0\}$ with values in $\mathfrak{g}$ and for each $t > 0$, the process $\{Y_{s,t}, s \in [0,1]\}$ starting from the identity element of $G$ and satisfying the stochastic differential equation (in $s$)

$$tdY_{s,t} = Y_{s,t} \circ dx_{s,t},$$

where $\circ$ stands for the Stratonovitch integral. This is a $G$-valued process. The adjoint orbits in $G$ are in correspondence with an alcove which is a fundamental domain for the action on a Cartan subalgebra $\mathfrak{t}$ of the extended Weyl group associated the roots of $G$. We have proved in [5] that if for any $t > 0$ one denotes by $O(Y_{1,t})$ the element in the alcove corresponding to the orbit of $Y_{1,t}$ then the random process

$$\{(t,t\circ Y_{1,t}) : t \geq 0\}$$

is a space-time brownian motion in $\mathbb{R} \times t$ conditioned in Doob’s sense to remain in a Weyl chamber which occurs in the framework of affine Kac–Moody algebras [8]. The proof of [5] rests on a Kirillov–Frenkel character formula [7] from which follows an intertwining relation between the transition probability semi-group of the Brownian sheet and the one of the conditioned process. Then a Rogers and Pitman’s criteria [11] can be applied, which provides the result. The conditioned process obtained when $G = SU(2)$ plays a crucial role in [2] where a Pitman type theorem is proved for a real Brownian motion in the unit interval. Time inversion is a key ingredient to get the Pitman type theorem in this case. In the present communication a new proof of the main result of [5] is proposed, which rests on such a time inversion firstly and secondly on an elementary but nice property of the Brownian sheet on $\mathfrak{g}$ and its wrapping on $G$.

\[1\]with the convention that $Y_{1,0}$ is the identity element of $G$
The results presented are valuable for themselves rather than for their proofs which are rudimentary. We present them in section 2 before giving the precise definitions of the objects that they involve. The rest of the communication is organized as follows. In section 3 we recall the general framework of \cite{5}. In particular we describe the coadjoint orbits of the loop group $L(G)$ in the dual of the centrally extended loop algebra $L(g)$ and the Weyl chamber associated to such an infinite dimensional Lie algebra which is an affine Kac–Moody algebra. In section 4 we define the radial process associated to the Brownian sheet on $g$ and recall the main theorem of \cite{5}. In section 5 we define two Doob conditioned processes living respectively in an alcove or in an affine Weyl chamber, and prove that the two processes are equal up to a time inversion. Finally in section 6 we propose a brief proof of the main result of \cite{5}.

## 2. Statement of the results

Let us fix $\gamma$ in a fixed alcove associated to $G$ and consider $\{X_{s,t}^\gamma : s \in [0,1], t \geq 0\}$ a random sheet with values in $G$, such that for any $t \geq 0$,

$$
\begin{align*}
X_{s,t}^\gamma &= X_{s,t} \circ d(x_{s,t} + \gamma s) \\
X_{0,t}^\gamma &= e,
\end{align*}
$$

where $e$ is the identity element of $G$. Then one has the three following statements, the second one being an immediate consequence of the first, and the last one being deduced from the second by a time inversion argument.

**Statement 1**: The random process $\{X_{1,t}^\gamma : t \geq 0\}$ is a standard Brownian motion on $G$ starting from $\exp(\gamma)$.

**Statement 2**: The random process $\{O(X_{1,t}^\gamma) : t \geq 0\}$ is a standard Brownian motion starting from $\gamma$ conditioned to remain in the alcove.

**Statement 3**: The radial part process $\{\text{rad}(t \Lambda_0 + \int_0^1 (\cdot | d(x_{s,t} + \gamma st)) : t \geq 0\}$ is a space-time Brownian motion with drift $\gamma$ conditioned to remain in an affine Weyl chamber.

## 3. Loop group and its orbits

In this part we fix succinctly the general framework of the results. One can find more details in \cite{5} and references therein for instance. Let $G$ be a connected simply connected simple compact Lie group and $g$ its Lie algebra equipped with a Lie bracket denoted by $[\cdot, \cdot]_g$. We choose a maximal torus $T$ in $G$ and denote by $t$ its Lie algebra. By compacity we suppose without loss of generality that $G$ is a matrix Lie group. We denote by $\text{Ad}$ the adjoint action of $G$ on itself or on its Lie algebra $g$ which is equipped with an $\text{Ad}(G)$-invariant scalar product $(\cdot | \cdot)$. We consider the real vector space $L(g)$ of smooth loops defined on the unit circle $S^1$ with values in $g$, $S^1$ being identified with $[0,1]$. We equip $L(g)$ with an
Ad(G)-invariant scalar product also denoted by \((-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\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Real roots. We consider the complexified Lie algebra $C \otimes_R g$ of $g$ that we denote by $g_C$. The set of real roots is

$$\Phi = \{ \alpha \in t^* : \exists X \in g_C \setminus \{0\}, \forall H \in t, [H, X] = 2i\pi \alpha(H).X \}.$$

Suppose that $g$ is of rank $n$ and choose a set of simple real roots $\Pi = \{ \alpha_k, k \in \{1, \ldots, n\} \}$. We denote by $\Phi^+$ the set of positive real roots. The half sum of positive real roots is denoted by $\rho$. Letting for $\alpha \in \Pi$, $g_\alpha = \{ X \in g : \forall H \in t, [H, X] = 2i\pi \alpha(H).X \}$, the coroot $\alpha^\vee$ of $\alpha \in \Phi$ is defined as the only vector of $t$ in $[g_\alpha, g_{-\alpha}]$ such that $\alpha(\alpha^\vee) = 2$.

One considers the Weyl group $W^\vee$ and the group $\Gamma^\vee$ respectively generated by reflections $s_{\alpha^\vee}$ and translations $t_{\alpha^\vee}$, for $\alpha \in \Pi$, and the extended Weyl group $\Omega$ generated by $W^\vee$ and $\Gamma^\vee$. Actually $\Omega$ is the semi-direct product $W^\vee \ltimes \Gamma^\vee$. A fundamental domain for its action on $t$ is

$$A = \{ x \in t : \forall \alpha \in \Phi^+, 0 \leq \alpha(x) \leq 1 \}.$$

Adjoint $G$-orbit. The group $G$ being simply connected, the conjugaison classes $G/\text{Ad}(G)$ is in correspondence with the fundamental domain $A$. Actually for every $u \in G$, there exists a unique element $x \in A$ such that $u \in \text{Ad}(G)\{ \exp(x) \}$.

For $\tau \in \mathbb{R}_+$, one defines the alcove $A_\tau$ of level $\tau$ by

$$A_\tau = \{ x \in t : \forall \alpha \in \Phi^+, 0 \leq \alpha(x) \leq \tau \},$$

i.e. $A_\tau = \tau A$. In particular $A_1 = A$.

Alcoves and coadjoint $L(G)$-orbit. For a positive real number $\tau$ and a linear form $\xi \in \widetilde{L}(g)^*$ written as in (3) there is a unique element in $a \in A_\tau$ such that $X_1 \in \text{Ad}(G)\{ \exp(a/\tau) \}$ where $X = \{ X_s : s \in [0, 1] \}$ starts from the identity element $e$ of $G$ and satisfies

$$\tau dX = Xdx.$$

Discussion above ensures that the pair $(\tau, a)$ determines the orbit of $\xi$. Thus coadjoint orbits in the subspace of linear forms in $\mathbb{R}_+^* \Lambda_0 + L(g)^*$ written like in (3) are in one-to-one correspondence with

$$\{(\tau, a) \in \mathbb{R}_+^* \times t : a \in A_\tau \}.$$
**Affine Weyl chamber.** From now on the scalar product on $\mathfrak{g}$ is normalized such that $(\theta|\theta) = 2$. We denote by $\theta$ the highest real root and we let $\alpha_0^\vee = c - \theta^\vee$.

We consider

$$\widehat{\mathfrak{h}} = \text{Vect}_\mathbb{C}\{\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_n^\vee, d\} \quad \text{and} \quad \widehat{\mathfrak{h}}^* = \text{Vect}_\mathbb{C}\{\alpha_0, \alpha_1, \ldots, \alpha_n, \Lambda_0\},$$

where $\alpha_0 = \delta - \theta$ and for $i \in \{0, \ldots, n\}$

$$\alpha_i(d) = \delta_0, \quad \delta(\alpha_i^\vee) = 0, \quad \Lambda_0(\alpha_i^\vee) = \delta_0, \quad \Lambda_0(d) = 0.$$

We let

$$\widehat{\Pi} = \{\alpha_i : i \in \{0, \ldots, n\}\} \quad \text{and} \quad \widehat{\Pi}^\vee = \{\alpha_i^\vee : i \in \{0, \ldots, n\}\}.$$ 

Then $(\widehat{\mathfrak{h}}, \widehat{\Pi}, \widehat{\Pi}^\vee)$ is a realization of a generalized Cartan matrix of affine type.

These objects are studied in details in [8]. The following definitions mainly come from chapters 1 and 6. We consider the restriction of $(\cdot|\cdot)$ to $t$ and extend it to $\widehat{\mathfrak{h}}$ by $\mathbb{C}$–linearity and by letting

$$(Cc + Cd|t) = 0, \quad (c|c) = (d|d) = 0, \quad (c|d) = 1.$$

Then the linear isomorphism

$$\nu : \widehat{\mathfrak{h}} \to \widehat{\mathfrak{h}}^*$$

$$h \mapsto (h|\cdot)$$

identifies $\widehat{\mathfrak{h}}$ and $\widehat{\mathfrak{h}}^*$. We still denote $(\cdot|\cdot)$ the induced bilinear form on $\widehat{\mathfrak{h}}^*$. We record that

$$(\delta|\alpha_i) = 0, \quad i = 0, \ldots, n, \quad (\delta|\delta) = 0, \quad (\delta|\Lambda_0) = 1.$$ 

Due to the normalization we have $\nu(\theta^\vee) = \theta$ and $(\theta^\vee|\theta^\vee) = 2$. We define the affine Weyl group $\widehat{W}$ as the subgroup of $\text{GL}(\widehat{\mathfrak{h}}^*)$ generated by fundamental reflections $s_\alpha$, $\alpha \in \widehat{\Pi}$, defined by

$$s_\alpha(\beta) = \beta - \beta(\alpha^\vee)\alpha, \quad \beta \in \widehat{\mathfrak{h}}^*.$$

The bilinear form $(\cdot|\cdot)$ is $\widehat{W}$-invariant. The affine Weyl group $\widehat{W}$ is equal to the semi-direct product $W \ltimes \Gamma$, where $W$ is the Weyl group of $G$ generated by $s_{\alpha_i}$, $i \in \{1, \ldots, n\}$, and $\Gamma$ the group of translations $t_\alpha$, $\alpha \in \nu(Q^\vee)$, defined by

$$t_\alpha(\lambda) = \lambda + \lambda(c)\alpha - \left[(\lambda|\alpha) + \frac{1}{2}(\alpha|\alpha)\lambda(c)\right]\delta, \quad \lambda \in \widehat{\mathfrak{h}}^*.$$

Identification of $\widehat{\mathfrak{h}}$ and $\widehat{\mathfrak{h}}^*$ via $\nu$ allows to define an action of $\widehat{W}$ on $\widehat{\mathfrak{h}}$. One lets $wx = \nu^{-1}w\nu x$, for $w \in \widehat{W}$, $x \in \widehat{\mathfrak{h}}$. Then the action of $\widehat{W}$ on $\Lambda_0 \oplus \mathfrak{t}^* \oplus \mathbb{R}\delta/\mathbb{R}\delta$ or $d \oplus \mathfrak{t} \oplus \mathbb{R}\Lambda_0/\mathbb{R}\Lambda_0$ is identified to the one of $\Omega$ on $t$. Moreover a fundamental domain for the action of $\widehat{W}$ on the quotient space $(\mathbb{R}^+\Lambda_0 + t^* + \mathbb{R}\delta)/\mathbb{R}\delta$ is

$$\{\lambda \in \mathbb{R}\Lambda_0 + t^* : \lambda(\alpha^\vee) \geq 0, \alpha \in \widehat{\Pi}\},$$

and for $\tau \geq 0$, $\tau\Lambda_0 + \phi_a$, with $\phi_a = (a|\cdot)$, is in this fundamental domain if and only if $a \in A_\tau$. Then we consider the following domain which is identified with the fundamental affine Weyl chamber viewed in the quotient space

$$C_W = \{(\tau, x) \in \mathbb{R}^+ \times t : x \in A_\tau\}.$$
4. COADJOINT $L(G)$-ORBIT AND BROWNIAN MOTION

When \( \{x_s : s \in [0,1]\} \) is a continuous semi-martingale with values in \( \mathfrak{g} \), then for \( \tau > 0 \) the stochastic differential equation

\[
\tau \, dX = X \circ dx,
\]

where \( \circ \) stands for the Stratonovitch integral, has a unique solution starting from \( e \). Such a solution is a \( G \)-valued process, that we denote by \( \epsilon(\tau,x) \) \cite{9,10}. This is the Stratonovitch stochastic exponential of \( \psi \). The previous discussion leads naturally to the following definition.

**Definition 4.1.** For \( \tau \in \mathbb{R}_+ \), and \( x = \{x_s : s \in [0,1]\} \) a \( \mathfrak{g} \)-valued continuous semi-martingale, we defines the radial part of \( \tau \Lambda_0 + \int_0^1 (\cdot | dx_s) \) that we denote by \( \text{rad}(\tau \Lambda_0 + \int_0^1 (\cdot | dx_s)) \) by\(^2\)

\[
\text{rad}(\tau \Lambda_0 + \int_0^1 (\cdot | dx_s)) = (\tau, a),
\]

where \( a \) is the unique element in \( A_\tau \) such that \( \epsilon(\tau,x) \) \( \in \text{Ad}(G)\{\exp(a/\tau)\} \).

We have proved in \cite{5} the following theorem, where the conditioned space-time Brownian motion is the one defined in section \[5.2\] This is this theorem for which we propose a new proof.

**Theorem 4.2.** If \( \{x_{s,t}: s \in [0,1], t \geq 0\} \) is a Brownian sheet with values in \( \mathfrak{g} \) such that for any \( a, b \in \mathfrak{g}, s_1, s_2 \in [0,1], t_1, t_2 \in \mathbb{R}_+^* \),

\[
\mathbb{E}(\langle a|x_{s_1,t_1}\rangle(b|x_{s_2,t_2}\rangle) = \min(s_1, s_2) \min(t_1, t_2) (a|b),
\]

then

\[
\{\text{rad}(t \Lambda_0 + \int_0^1 (\cdot | dx_{s,t})) : t \geq 0\}
\]

is a space-time Brownian motion in \( \mathbb{R} \times t \) conditioned to remain in the affine Weyl chamber \( C_W \).

5. CONDITIONED BROWNIAN MOTIONS

In whole the communication, when we write \( f_t(x) \propto g_t(x) \) for \( f_t(x), g_t(x) \in \mathbb{C} \), we mean that \( f_t(x) \) and \( g_t(x) \) are equal up to a multiplicative constant independent of the parameters \( t \) and \( x \).

5.1. A Brownian motion conditioned to remain in an alcove. There is a common way to construct a Brownian motion conditioned in Doob sense to remain in an alcove, which is to consider at each time the \( \text{Ad}(G) \)-orbit of a brownian motion in \( G \). The brownian motion on \( G \) is left Levy process. Its transition probability densities \( (p_s)_{s>0} \) with respect to the Haar measure on \( G \) can be expanded as a sum of characters of highest-weight complex representations of \( G \). These representations are in correspondence with

\[
P_+ = \{ \lambda \in \mathfrak{t}^* : \lambda(\alpha_i^\vee) \in \mathbb{N}, i \in \{0, \ldots, n\}\}.
\]

\(^2\)We do not specify in which space lives this distribution. We use this notation here just to keep track of the fact that when \( x \) is a Brownian motion the Wiener measure provides a natural measure on a coadjoint orbit in the original work of I. B. Frenkel.
One has for $s \geq 0$, $u, v \in G$,
\[
p_s(u, v) = p_s(e, u^{-1}v) = \sum_{\lambda \in P_+} \chi_\lambda(e)\chi_\lambda(u^{-1}v)e^{-\frac{(2\pi)^2}{2}((||\lambda + \rho||^2 - ||\rho||^2)},
\]
where $\chi_\lambda$ is the character of the irreducible representation of highest weight $\lambda$ (see for instance [6]). By the Weyl character formula one has
\[
\chi_\lambda(e^h) = \frac{\sum_{w \in W} \det(w)e^{2\pi i \langle w(\lambda + \rho), h \rangle}}{\sum_{w \in W} \det(w)e^{2\pi i \langle \rho, h \rangle}}.
\]
We let
\[
\pi(h) = \prod_{\alpha \in \Phi_+} \sin \pi \alpha(h),
\]
which is the denominator in (6). Such a process starting from $u \in G$ can be obtained considering a standard Brownian motion $\{x_s : s \geq 0\}$ with values in $g$, and the solution $\{X_s : s \geq 0\}$ of the stochastic differential equation
\[
dX = X \circ dx
\]
with initial condition $X_0 = u$. Then $\{X_s : s \geq 0\}$ is a standard Brownian motion on $G$ starting from $u$. If $u = \exp(\gamma)$ with $\gamma \in A$ then the process $\{r_\gamma^s : s \geq 0\}$ such that for any $s \geq 0$, $r_\gamma^s$ is the unique element in $A$ such that $X_s \in \text{Ad}(G)\{\exp(r_\gamma^s)\}$,
is a Markov process starting from $\gamma$ with transition probability densities $(q_t)_{t \geq 0}$ with respect to the Haar measure on $G$ given by
\[
q_t(x, y) \propto \pi(y)^2 \sum_{\lambda \in P_+} \chi_\lambda(e^{-x})\chi_\lambda(e^y)e^{-\frac{(2\pi)^2}{2}((||\lambda + \rho||^2 - ||\rho||^2)},
\]
for $t \geq 0$, $x, y \in A$. This is obtained integrating over an $\text{Ad}(G)$-orbit (see (4.3.3) in [7] for instance) and using the Weyl integration formula. This Markov process is actually a Brownian motion killed on the boundary of $A$ conditioned never to die. In fact if we denote by $(u_t)_{t \geq 0}$ the transition densities of the standard Brownian motion on $t$ killed on the boundary of $A$, a reflection principle gives that for $t > 0$, $x, y \in A$,
\[
u_t(x, y) = \sum_{w \in \Omega} \det(w)p_t(x, w(y)),
\]
where $p_t$ is the standard heat kernel on $t$ and $\det(w)$ is the determinant of the linear part of $w$. A Poisson summation formula (see [4] for general results, and [7] or [5] for this particular case) then shows that
\[
q_t(x, y) \propto \frac{\pi(y)}{\pi(x)}e^{2\pi^2(\rho|\rho|)t}u_t(x, y),
\]
which is the transition probability of the killed Brownian motion conditioned in the sense of Doob to remain in $A$.

3The presence of a factor $2\pi$ is due to the fact that we have considered the real roots rather than the infinitesimal ones.
5.2. A space-time Brownian motion conditioned to remain in an affine Weyl chamber. We define a space-time Brownian motion conditioned to remain in an affine Weyl chamber as it has been defined in [5] and also in [2] when \( G = SU(2) \). It is defined as an \( h \)-process, with the help of an anti-invariant classical theta function. For \( \tau \in \mathbb{R}_+^*, b \in t, a \in A_\tau \), we define \( \hat{\psi}_b(\tau, a) \) by

\[
\hat{\psi}_b(\tau, a) = \frac{1}{\pi(b)} \sum_{w \in W} \det(w) e^{(w(\tau A_0 + \phi_0), d + b)}.
\]

From now on we fix \( \gamma \in A \). One considers a standard Brownian motion \( \{b_t : t \geq 0\} \) with values in \( t \), the space-time Brownian motion \( \{B^\gamma_t = (t, b_t + \gamma t) : t \geq 0\} \), and the stopping time \( T = \inf\{t \geq 0 : B^\gamma_t \notin C_W\} \). One defines a function \( \Psi_\gamma \) on \( C_W \) by

\[
\Psi_\gamma : (t, x) \in C_W \rightarrow e^{-(\gamma|x|)} \hat{\psi}_\gamma(t, x).
\]

Identity [8] and decomposition \( \widetilde{W} = W \rtimes \Gamma \) implies that

\[
\Psi_\gamma(t, x) \pi(\gamma) \propto t^{-n/2} u_{\gamma}(\gamma, x/t) e^{\frac{b}{2}||\gamma - x/t||^2}
\]

**Proposition 5.1.** The function \( \Psi_\gamma \) is a constant sign harmonic function for the killed process \( \{B^\gamma_{t \wedge T} : t \geq 0\} \), vanishing on the boundary of \( C_W \).

**Proof.** The fact that \( \Psi_\gamma \) is harmonic and satisfies the boundary conditions is clear from (11). It is non negative by (12). \( \square \)

**Definition 5.2.** We define \( \{A^\gamma_t = (t, a^\gamma_t) : t \geq 0\} \) as the killed process \( \{B^\gamma_{t \wedge T} : t \geq 0\} \) starting from \( (0, 0) \) conditioned in Doob’s sense not to die, via the harmonic function \( \Psi_\gamma \).

More explicitly, if we let for \( t \geq 0 \), \( K^\gamma_t = B^\gamma_{t \wedge T} \), and \( K^\gamma_t = (t, k^\gamma_t) \), then \( \{A^\gamma_t = (t, a^\gamma_t) : t \geq 0\} \) is a Markov process starting from \( (0, 0) \) such that for \( r, t > 0 \), the probability density of \( a^\gamma_{t-r} \) given that \( a^\gamma_t = x \), with \( x \in A_r \), is

\[
s^\gamma_t((r, x), (r + t, y)) = \frac{\Psi_\gamma(r + t, y) w^\gamma_t((r, x), (r + t, y))}{\Psi_\gamma(r, x)},
\]

where \( w^\gamma_t((r, x), (r + t, y)) \) is the probability density of \( k^\gamma_{t+r} \) given that \( k^\gamma_t = x \), and the probability density of \( a^\gamma_t \) is given by

\[
s^\gamma_t((0, 0), (t, y)) = C_t \Psi_\gamma(t, y) \pi(\frac{y}{t}) e^{-\frac{1}{2t}||y-\gamma t||^2}, \ y \in A_t,
\]

where \( C_t \) is a normalizing constant depending on \( t \).

5.3. The two conditioned processes and time inversion. Actually the two Doob transformations previously defined are equal up to a time inversion. We prove this property as it is done in [2] for the Brownian motion in the unit interval. The following lemma is immediately deduced from (12) and (14).

**Lemma 5.3.** For \( t > 0, x \in A \), one has

\[
s^\gamma_{1/t}((0, 0), (1/t, x/t)) = q_t(\gamma, x).
\]

**Lemma 5.4.** For \( 0 < r \leq t, x \in A_r, y \in A_t \)

\[
e^{-\frac{1}{2t}||y||^2} u_{\gamma - \frac{1}{t}}(y/t, x/r) = e^{-\frac{1}{2r}||x||^2} w^0_{t-r}((r, x), (t, y)).
\]
Proof. Using expression (8) and the time inversion invariance property for the standard heat kernel on $t$, one obtains that
\[ e^{-\frac{1}{2t}||y||^2}u_{\frac{t}{r}}(y/t, x/r) = e^{-\frac{1}{2r}||x||^2} \sum_{w \in \Omega} e^{-\frac{1}{2r}(||y||^2-||yw(y/t)||^2)}p_{t-r}(x, tw(y/t)). \]
The sum on the right-hand side of the identity is exactly $w_0^t-1(r, x)$ according to lemma 6.3 of [4], which achieves the proof. □

In the following proposition \( \{r^\gamma_t : t \geq 0\} \) is the conditioned process defined in section 5.1 and \( \{a^\gamma_t : t \geq 0\} \) is the one defined in section 5.2.

**Proposition 5.5.** One has in distribution
\[ \{ta^\gamma_{1/t} : t \geq 0\} \overset{d}{=} \{r^\gamma_t : t \geq 0\}. \]
Proof. It follows immediately from the two previous lemmas and identity (12). □

6. A new proof of Theorem 4.2

For every $t > 0$ one considers the diffusion process \( \{Y^\gamma_{s,t} : s \in [0, 1]\} \) starting from the identity element $e$ of $G$ satisfying the EDS (in $s$)
\[ tdY^\gamma_{s,t} = Y^\gamma_{s,t} \circ d(x_{s,t} + \gamma st). \]
For $u \in G$ one denotes by $O(u)$ the unique element in $A$ such that
\[ u \in \text{Ad}(G) \{\exp(O(u))\}. \]
We have proved in [5] that the random process \( \{(t, tO(Y^0_{1,t})) : t \geq 0\} \) is distributed as \( \{A^0_t : t \geq 0\} \). As $Y^\gamma$ satisfies
\[ dY^\gamma_{s,t} = Y^\gamma_{s,t} \circ d\left(\frac{1}{t}x_{s,t} + \gamma s\right), \]
and \( \{\frac{1}{t}x_{s,t} : s, t > 0\} \overset{d}{=} \{x_{s,1/t} : s, t > 0\} \), one could deduce from [5], with the help of a Kirillov-Frenkel character formula from [7] and a Cameron–Martin theorem, that the result remains true for any $\gamma \in A$.

We propose here a brief proof of the theorem, which is valid for every $\gamma$. For every $t \geq 0$, one considers the diffusion process \( \{X^\gamma_{s,t} : s \in [0, 1]\} \) starting from $e \in G$ satisfying the stochastic differential equation (in $s$)
\[ (15) \quad dX^\gamma_{s,t} = X^\gamma_{s,t} \circ d(x_{s,t} + \gamma s). \]

**Proposition 6.1.**

(1) For $t, t' \geq 0$, the random process \( \{X^\gamma_{s,t+t'}(X^\gamma_{s,t})^{-1} : s \in [0, 1]\} \) has the same law as \( \{X^0_{s,t'} : s \in [0, 1]\} \).

(2) For $t, t' \geq 0$, the random process \( \{X^\gamma_{s,t+t'}(X^\gamma_{s,t})^{-1} : s \in [0, 1]\} \) is independent of \( \{X^\gamma_{s,r} : s \in [0, 1], r \leq t\} \).

(3) The random process \( \{X^\gamma_{1,t} : t \geq 0\} \) is a standard Brownian motion in $G$ starting from $\exp(\gamma)$. 
Proof. For the first point, we let $Z_s = X^\gamma_{s,t+t'}(X^\gamma_{s,t})^{-1}$, $s \in [0,1]$. The process $\{(X^\gamma_{s,t})^{-1} : s \in [0, 1]\}$ satisfies the EDS (in $s$)

$$d(X^\gamma_{s,t})^{-1} = -d(x_{s,t} + \gamma s) \circ (X^\gamma_{s,t})^{-1}$$

from which we immediately deduce that $Z$ satisfies

$$dZ_s = Z_s \circ X^\gamma_{s,t} d(x_{s,t+t'} - x_{s,t})(X^\gamma_{s,t})^{-1}.$$ 

As $\{\int_0^s X^\gamma_{t,t'} d(x_{t,t'+t'} - x_{t,t})(X^\gamma_{t,t'})^{-1} : s \in [0,1]\}$ has the same law as $\{x_{s,t'} : s \in [0,1], t < t'\}$, and is independent of $\{x_{s,r} : s \in [0,1], r \leq t\}$, one gets the first two points, which imply in particular that $\{X^\gamma(t) : t \geq 0\}$ is a right Levy process. The Ad$(G)$-invariance of the increments law implies that it is also a left Levy process. As for any $t > 0$, $X^0_{1,t}$ and $X^0_{0,1}$ are equal in distribution, the third point follows.

□

Proposition 6.1 has the two following corollaries, the second one being deduced from the first by proposition 5.5.

**Corollary 6.2.** The random process $\{O(X^\gamma_{1,t}) : t \geq 0\}$ is a standard Brownian motion starting from $\gamma$ killed on the boundary of $A$ conditioned in Doob’s sense to remain in $A$.

**Corollary 6.3.** The random process $\{(t,t')O(X^\gamma_{1,1/t}) : t \geq 0\}$ has the same distribution as the conditioned process $\{A^\gamma_t : t \geq 0\}$.

As the two processes $\{x_{s,1/t} : s, t > 0\}$ and $\{\frac{1}{t}x_{s,t} : s, t > 0\}$ are equal in distribution, Theorem 4.2 follows from corollary 6.3 with $\gamma = 0$. For any $\gamma \in A$, one has under the same hypothesis as in the theorem the following one.

**Theorem 6.4.** The radial part process

$$\{\text{rad}(t\Lambda_0 + \int_0^1 (\cdot) d(x_{s,t} + \gamma st)) : t \geq 0\}$$

is distributed as the Doob conditioned process $\{A^\gamma_t : t \geq 0\}$.

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