A CANTOR DYNAMICAL SYSTEM IS SLOW IF AND ONLY IF ALL ITS FINITE ORBITS ARE ATTRACTING

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Abstract. In this paper we completely solve the problem of when a Cantor dynamical system $(X, f)$ can be embedded in $\mathbb{R}$ with vanishing derivative. For this purpose we construct a refining sequence of marked clopen partitions of $X$ which is adapted to a dynamical system of this kind. It turns out that there is a huge class of such systems.

1. Introduction

We say that a dynamical system $(X, f)$ can be embedded in the real line with vanishing derivative when there exists a differentiable function $g: \mathbb{R} \to \mathbb{R}$, a closed set $Z \subset \mathbb{R}$ invariant for $g$ and a homeomorphism $\psi: X \to Z$ such that $g|_Z = \psi \circ f \circ \psi^{-1}$ and $g'|_Z \equiv 0$. In this paper we completely solve the following problem:

Question 1.1. What maps acting on a Cantor set can be embedded in the real line with vanishing derivative?

The well known Banach fixed point theorem implies that a contraction on compact metric space must have a fixed point. Therefore searching for an example of map $f$ on the real line with a closed invariant set without fixed points on which $f' \equiv 0$ seems a task doomed to failure. In fact, it was suspected by Edrei in his paper from 1952 [6] that even a weaker condition cannot be satisfied. Generally speaking, Edrei conjectured that any map on a compact set which locally does not increase distances must be an isometry. Soon after [13] Williams provided examples of maps which are not local isometries at some points, answering the original question of Edrei. However all these examples have isolated points and some of them also have fixed points. Since then relations between local shrinking and periodic points remained unclear, until very recently. In 2016 Jasiński and Ciesielski constructed in [12] an embedding of 2-adic odometer in the real line. The embedding was obtained by direct application of Jarník theorem together with very delicate construction of a metric on Cantor set leading to derivative zero. This technique was in huge part relying on a clever representation of the 2-adic group defining odometer. This technique was then extended in [2] to all odometers, together with some other carefully constructed examples such as an attractor-repellor pair or transitive non-minimal Cantor system. Still, it was not clear how much vanishing derivative is correlated with the existence of a metric making the map an isometry or at least having entropy zero. At this point it was expected that such an embedding may not exist for expansive maps (in particular subshifts) since these systems have divergence of orbits hidden in the dynamics (see discussion in [3]). Then in [3], J.P.Boroński, J.Kupka and P.Oprocha brought a surprising answer to this, showing that every minimal dynamical system on a Cantor set can be embedded in the real line with vanishing derivative everywhere. The proof of this result makes use of J.-M.Gambaudo and M.Martens [10] representation of minimal dynamical systems on the Cantor set with graph coverings which satisfy certain properties. These representations are to some extent similar to representations of odometers, sharing
the property that there is a unique vertex at which all the cycles intersect. This property was crucial in the main proof.

It was clear from examples in [2] that minimality does not characterize the class of Cantor systems which can be embedded in the real line with vanishing derivative. On the other hand, it was known that periodic points are in many cases an obstacle for the required embedding. As a consequence the natural class to consider was the one of aperiodic systems. It is known that aperiodic systems on a Cantor set can be represented with Bratelli-Vershik diagrams [11]. There are also techniques (e.g. see [9]) which allow to obtain graph coverings representation out of Kakutani-Rohlin towers, rooting the Bratelli-Vershik representation for minimal systems on the Cantor set. Despite the fact that representations of aperiodic systems do not have the convenient structure of Gambaudo-Martens representations and can consist of cycles with numerous points of intersections, we managed to describe them in a way suitable for our needs. One valuable tool in our research was a deep understanding of aperiodic systems reflected in recent results (e.g. see [5]). The paper is almost completely devoted to proving the following theorem, providing a complete answer to Question 1.1.

Theorem 1.2. A dynamical system \((X, f)\) on a Cantor set \(X\) can be embedded in the real line with vanishing derivative if and only if all finite orbits of \((X, f)\) are attractors.

In fact it is possible to state this result in a slightly more general way. If \(X\) is a finite set then it is obvious that it can be embedded in the real line with vanishing derivative. If \(X\) is infinite but zero dimensional then we can present \(X\) as \(X = C \cup R\) where \(C\) is a Cantor set and \(R\) is at most countable and consists of isolated points. We can then replace each isolated point \(x\) by a Cantor set \(C_x\) and define \(g(y) = f(x)\) for every \(y \in C_x\). Additionally, the limit set of any \(y \in C_x\) is the same as the one of \(x\). This way we obtain a map \(g\) acting on Cantor set such that \(g|X = f\). We can then apply Theorem 1.2 to \(g\), obtaining the following.

Theorem 1.3. Any dynamical system \((X, f)\) on a zero-dimensional compact set \(X\) can be embedded in the real line with vanishing derivative if and only if all finite orbits of \((X, f)\) are attractors.

In [12] the authors introduced the property of locally radially shrinking defined as follows:

(LRS) for every \(x \in X\) there exists an \(\epsilon_x > 0\) such that \(d(x, y) < \epsilon_x\) implies \(d(f(x), f(y)) < d(x, y)\) for all \(y \neq x\).

It is clear from this definition that if a map \(f\) with (LRS) has a periodic point, then its orbit is attracting. This combined with Theorem 1.2 shows that on zero-dimensional compact sets, the class of maps with (LRS) and the class of maps which can be embedded in real line with vanishing derivative are exactly the same. This reveals another unexpected connection between shrinking and dynamics. It is also worth emphasizing that being attractor is a topological property, while (LRS) depends on metrics.

This article is organized as follows: Section 2 is an exposition of the graph coverings representation of Cantor dynamical systems. For the reader’s convenience we also provide proofs of some elementary facts, in order for the article to be self-contained. Section 3 contains a construction of graph covering representations adapted to Cantor dynamical systems which have only attracting finite orbits, and Section 4 contains a proof of Theorem 1.2.
2. Zero-dimensional dynamical systems and graph coverings

Let us start with basic definitions. A topological space is called zero-dimensional when it has a base which consists of clopen (open and closed) sets. A Cantor set is any zero-dimensional compact metric space without isolated points. It is well known that all Cantor sets are homeomorphic.

In this paper $(X,d)$ is a fixed Cantor set and $f : X \to X$ is a continuous function. We will later impose additional conditions on $f$. For short, the pair $(X,f)$ will be called a Cantor system or a dynamical system (on the Cantor set $X$).

Notation 2.1. For every set $U$ of open subsets of $X$ (in particular a partition or a subset of a partition), we denote $E(U)$ the union of its elements.

Notation 2.2. For the remainder of this paper, we fix a sequence of partitions $(U^0_n)_{n \geq 1}$ of $X$ into clopen sets satisfying the condition:

\[
\forall n \geq 1, \forall u \in U^0_n, \ u = E(\{v \in U^0_{n+1} : v \subset u\}) \quad \text{and} \quad \lim_{n \to \infty} \text{mesh}(U^0_n) = 0.
\]

Remark 2.3. For every finite clopen partition $V$ of $X$, there exists some integer $n \geq 1$ such that every element of $V$ is the union of some elements in $U^0_n$.

We reproduce in Section 2.1, for completeness, the characterization of Cantor systems in terms of graph coverings that one can find in [8, Theorem 3.9]. We prove a characterization of these representations for aperiodic Cantor systems in Section 2.2.

2.1. General formulation.

2.1.1. Clopen partitions and graphs. In the following we will use finite clopen partitions and finite graphs in order to represent the behavior of the dynamical system $(X,f)$. Preliminary notations are introduced in this section.

Notation 2.4. For every finite clopen partition $U$ of $X$, we will denote by $G(U)$ the finite directed graph whose vertex set is $U$ and whose edges are the pairs $(u,v) \in U^2$ such that there exists $x \in u$ with $f(x) \in v$.

Definition 2.5. Let us consider two directed graphs $G$ and $G'$ whose vertex sets are respectively $V$ and $V'$ and whose edges are the pairs $(u,v) \in E$, $u \in U$ and $v \in V$ such that there exists $x \in u$ with $f(x) \in v$.

Definition 2.6. For two finite clopen partitions $U,V$, we say that $V$ refines $U$ when for all $v \in V$ there exists $u \in U$ such that $v \subset u$. This relation will be denoted $V \prec U$.

For two finite clopen partitions $U,V$, we will also denote

\[
U \lor V = \{u \cap v : u \in U, v \in V\},
\]

which is another finite clopen partition refining both $U$ and $V$.

Notation 2.7. Let us consider two finite clopen partitions $U,V$ of $X$ such that $V \prec U$. We will denote $\pi_U^V$ the graph morphism from $G(V)$ to $G(U)$ such that for all $u \in V$, $u \subset \pi_U^V(u)$.
2.1.2. From zero-dimensional systems to graph coverings.

Definition 2.8. The graph covering representation of the dynamical system $(X,f)$ relative to a sequence of finite clopen partitions $\mathcal{U} = (\mathcal{U}_n)_{n \geq 1}$ which satisfy the condition (2.1), denoted by $G(\mathcal{U}, f)$, is the data of two sequences $(G_n)_{n \geq 1}$ and $(\pi_n)_{n \geq 1}$ such that:

(i) for all $n \geq 1$, $G_n = G(\mathcal{U}_n)$;

(ii) for all $n \geq 1$, $\pi_n = \pi_{\mathcal{U}_n}$.

For all $m \geq n$ we will also set $\pi_{n,m} = \pi_n \circ \ldots \circ \pi_m$. We denote by $V_{\mathcal{U},f}$ the following set:

$$V_{\mathcal{U},f} = \left\{(u_n)_{n \geq 1} \in \prod_{n \geq 1} V_n : \forall n \geq 1, \pi_n(u_{n+1}) = u_n \right\}.$$  

We may view $V_{\mathcal{U},f}$ as an inverse limit defined by the spaces $V_n$ together with bonding maps $\pi_n$.

Lemma 2.9. For each $u \in V_{\mathcal{U},f}$, there exists a unique $v \in V_{\mathcal{U},f}$ such that for all $n \geq 1$, $(u_n, v_n) \in E_n$. (We will set $f_{\mathcal{U},f}(u) := v$).

Proof. The proof is standard. We give it for completeness.

- **Existence:** Let us consider $u \in V_{\mathcal{U},f}$, and denote $x$ the unique element of $X$ such that for all $n \geq 1$, $x \in u_n$. Let us fix $v$ such that for all $n \geq 1$, $f(x) \in v_n$. By definition for all $n \geq 1$, $(u_n, v_n) \in E_n$ and since $f(x) \in v_{n+1}$, we have $v_{n+1} \subseteq v_n$ and so $\pi_n(v_{n+1}) = v_n$. In particular $v \in V_{\mathcal{U},f}$.

- **Uniqueness:** Moreover let us consider $v' \in V_{\mathcal{U},f}$ such that for all $n \geq 1$, $(u_n, v'_n) \in E_n$. For all $n \geq 1$, by definition of the graph $G_n$, there exists $x_n \in u_n$ such that $f(x_n) \in v'_n$. The sequence $(x_n)_n$ converges towards $x$ and since $f$ is continuous $f(x_n)$ converges towards $f(x)$. As a consequence for all $k \geq 1$ there exists some $l$ such that for all $m \geq l$, $v'_m \subseteq v_k$. In particular $v'_k \cap v_k = \pi_{k,m}(v'_{m+1}) \cap v_k \neq \emptyset$, which implies $v'_k = v_k$. This proves $v' = v$, completing the proof.

Definition 2.10. Let us denote by $\varphi_{\mathcal{U},f} : X \to V_{\mathcal{U},f}$ (or simply $\varphi$ when there is no ambiguity) the function such that for all $x \in X$, $\varphi_{\mathcal{U},f}(x)$ is the unique (since $\text{mesh}(\mathcal{U}_n) \to 0$) sequence $u \in V_{\mathcal{U},f}$ such that for all $n \geq 1$, $x \in u_n$.

The following is straightforward:

Proposition 2.11. The map $\varphi_{\mathcal{U},f}$ is a homeomorphism and $f_{\mathcal{U},f} = \varphi_{\mathcal{U},f} \circ f \circ \varphi_{\mathcal{U},f}^{-1}$.

2.1.3. From graph coverings to zero-dimensional systems. Reciprocally, let us consider a sequence of finite graphs $\mathcal{G} = (G_n)_{n \geq 1}$ and a sequence $\pi = (\pi_n)_{n \geq 1}$ of surjective graph morphisms $\pi_n : V_{n+1} \to V_n$, and assume that for all $n \in V_n$ there exists $v \in V_n$ such that $(u, v) \in E_n$, where $V_n$ is the vertex set of $G_n$ and $E_n$ its edge set. Let us denote $V_{\mathcal{G},\pi}$ the set

$$V_{\mathcal{G},\pi} = \left\{(u_n) \in \prod_{n \geq 1} V_n : \forall n \geq 1, \pi_n(u_{n+1}) = u_n \right\}.$$  

This set is a metric space with the metrization of Tychonoff product of discrete topologies on the finite sets $V_n$, $n \geq 1$. Let us also assume that for all $u \in V_{\mathcal{G},\pi}$, there exists a unique $v \in V_{\mathcal{G},\pi}$ such that $(u_n, v_n) \in E_n$ for all $n \geq 1$ (it is in particular the case when for every $n$ that if $(u, v), (u, w) \in E_{n+1}$ then $\pi_n(v) = \pi_n(w)$). Denoting $f_{\mathcal{G},\pi}(u)$ this sequence $v$, we have the following:
Lemma 2.12. The map \( f_{G, \pi} : V_{G, \pi} \rightarrow V_{G, \pi} \) is continuous (as a consequence \((V_{G, \pi}, f_{G, \pi})\) is a dynamical system on a zero-dimensional compact metric space).

Proof. Let us consider some \( u \in V_{G, \pi} \), a sequence \((u^k)_{k \geq 0}\) such that \( u^k \rightarrow u \), and a subsequence \((v^k)_{k \geq 0}\) of this one such that \((f_{G, \pi}(v^k))_k\) converges towards some \( v \). For all \( n \geq 1 \) there exists some \( k_n \) such that for all \( k \geq k_n \) and all \( l \leq n \), \( u^k_l = u_l \) and \( f_{G, \pi}(u^k) = v_l \). As a consequence for all \( l \leq n \), \((u_l, v_l) \in E_l \). Since this is satisfied for all \( n \geq 1 \), we have \( v = f_{G, \pi}(u) \). As a consequence every subsequence of \((f_{G, \pi}(u^k))_k\) converges towards \( f_{G, \pi}(u) \), so \((f_{G, \pi}(u^k))_k\) converges towards \( f_{G, \pi}(u) \). We proved that \( f_{G, \pi} \) is continuous.

□

Definition 2.13. A telescoping of some \((G, \pi)\) is a tuple \((G', \pi')\) such that there exists a sequence of integers \( n = (n_k)_{k \geq 0} \) with \( G'_k = G_{n_k} \) and \( \pi'_k = \pi_{n_k, n_{k+1}} \) for all \( k \geq 0 \).

From this definition it is straightforward that for all \( u \in V_{G', \pi'} \), there is a unique \( v \) such that \((u_k, v_k)\) is an edge of \( G'_k \) for all \( k \geq 0 \), provided that uniqueness of edges holds for \( V_{G, \pi} \). The following is also obvious.

Lemma 2.14. For each telescoping \((G', \pi')\) of \((G, \pi)\), \((V_{G', \pi'}, f_{G', \pi'})\) is conjugated to \((V_{G, \pi}, f_{G, \pi})\).

2.2. Aperiodic Cantor systems. Let \( G(U, f) = ((G_n)_{n \geq 1}, (\pi_n)_{n \geq 1}) \) be a graph coverings representation of \( f \) associated with a sequence of partitions \( U \). We assume that for all \( n \geq 1 \), if \((u, v) \in E_{n+1} \) and \((u^\prime, v^\prime) \in E_{n+1} \), then \( \pi_n(v) = \pi_n(v^\prime) \). Such a sequence of partitions always exists, since the space is zero-dimensional.

In the following, for every graph considered we will designate by circuit any cycle in this graph which is minimal for the inclusion (considering a cycle as a sequence of edges). The length of a circuit is the number of edges it contains.

Lemma 2.15. A point \( x \in X \) is periodic for \( f \) with period \( k \geq 1 \) \((f^k(x) = x)\) if and only if there exists some \( m \geq 1 \) and a sequence \((c_n)_{n \geq m} \) such that for all \( n \geq m \), \( c_n \) is a circuit of length \( k \) in \( G_n \), \( \pi_n(c_{n+1}) = c_n \) and \( \varphi(x)_n \) is a vertex in \( c_n \).

Proof. \((\Rightarrow)\): Let us consider some \( x \in X \) such that \( f^k(x) = x \). Then Proposition 2.11 implies that \( \frac{1}{k} f^l(x) = \varphi(x) \). Moreover there exists some \( m \geq 1 \) such that for all \( n \geq m \), the points \( f^l(x), l < k \), belong to distinct elements of the partition \( U_n \) which form a cycle \( c_n \) of length \( k \) in \( G_n \). We have directly that \( c_{n+1} \) is mapped to \( c_n \) by \( \pi_n \). Moreover by definition \( \varphi(x)_n \) is an element of \( c_n \) for all \( n \geq m \). If cycles \( c_n \) are not minimal cycles, that is, they contain a cycle of smaller period, then it is not hard to see that the period of \( x \) is smaller than \( k \), which is a contradiction.

\((\Leftarrow)\): Reciprocally considering some sequence of circuits such as in the statement of the lemma, let us consider \( u^0_n, \ldots, u^k_n \) elements of the circuit \( c_n \) such that \( u^0_n = \varphi(x)_n \) for all \( l \leq k - 1 \), \((u^0_n, u^{l+1}) \in E_n \) and \((u^k_n, u^0_n) \in E_n \). Let \( w = \varphi(f(x)) \) and observe that \((u_n^0, w_n) \in E_n \) which implies (by hypothesis) that

\[
\pi_{n-1}(u^1_n) = \pi_{n-1}(w_n) = w_{n-1}.
\]

Therefore \( w = u^1 \) and by the same reasoning we have that \( u^{l+1} = \varphi(f^l(x)) \) for all \( l \leq k \). Since \( \varphi(x)_n = u^k_n \) for all \( n \), \( \frac{1}{k} f^l(x) = \varphi(x) \), which implies that \( f^k(x) = x \). Since each \( c_n \) is a circuit, the period of \( x \) cannot be smaller than \( k \).

□
Theorem 2.17. Assume that a sequence of partitions $\mathcal{U}$ satisfies the condition (2.1). Then the dynamical system $(X, f)$ is aperiodic if and only if $\nu_n(\mathcal{U}, f) \to +\infty$.

Proof. $(\Rightarrow)$: Using Lemma 2.14 it is sufficient to prove that for all $n$ there exists $m \geq n$ such that all the circuits in $G_{m+1}$ are mapped through $\pi_{n,m}$ to a concatenation of at least two circuits in $G_n$. Let us assume ad absurdum that there exists $n$ such that for all $m \geq n$, there exists a circuit in $G_m$ which is mapped injectively onto a circuit in $G_n$, i.e. its image is not a concatenation of two or more circuits in $G_n$. For this $n$ enumerate circuits in $G_n$, say $c^n_1, \ldots, c^n_r$ and let $r$ be the length of the longest of them. Then for all $m > n$ enumerate circuits in $G_m$ of length at most $r$, say $c^m_1, \ldots, c^m_k$. For each $m > n$ there exists a sequence $i^m_1, \ldots, i^m_m$ such that for all $s \in \{n, \ldots, m-1\}$, $\pi_{s,m-1}(c^m_{i^m_s}) = c^n_{i^m_s}$. Denote by $x^{(m)} \in \Pi^\infty_{i=n}\{1, \ldots, k_i\}$ the sequence such that $(x^{(m)})_s = i^m_s$ for $s < m$ and $(x^{(m)})_s = 1$ for $s \geq m$. By extracting a subsequence if necessary, we may assume that $(x^{(m)})$ converges to some sequence $x$. Then for every $s > n$ and $s' > s$ sufficiently large we have
\[
\pi_{s-1}(c_{x_s}) = \pi_{s-1}(\pi_{s,s'-1}(c_{x_{s'}})) = \pi_{s-1,s'-1}(c_{x_{s'}}) = c_{x_{s-1}}.
\]
We constructed an inverse sequence of circuits, hence by Lemma 2.15 the system $(X, f)$ has a periodic point, which was assumed to be false.

$(\Leftarrow)$: Reciprocally, if $\nu_n(f) \to +\infty$, it is clear that the conditions of Lemma 2.15 can not be satisfied for any sequence of circuits $(c_n)$, thus $(X, f)$ has no periodic point.

\[\square\]

Remark 2.18. If $(X, f)$ is aperiodic then there is a subsequence of the sequence of partitions $\mathcal{U}$ such that in the associated telescoping $(G', \pi')$ of $(G, \pi)$, the image $\pi'_n(c)$ of any circuit $c$ in $G'_{n+1}$ is a concatenation of at least two circuits in $G_n$.

Remark 2.19. Similar to the graph coverings representation of Gambaudo and Martens [10], we just proved a characterization for aperiodicity. While in general we may not hope for a representation with all cycles intersecting at a unique vertex as in [10], we will show that some “special” vertices still exists, and some other useful properties can be required.

3. Purely attracting zero-dimensional systems

This section contains the main changes compared to the initial construction of [9]. In order to prove the Theorem 1.2 we will consider graph coverings representations in the case of zero-dimensional systems with only attracting finite orbits, which is more complex than the minimal ones. In particular we will need to define particular finite clopen partitions - that we call supercyclical - which are adapted to the case considered, by discriminating two disjoint parts, one of which consists in a neighborhood of all the finite orbits whose length is smaller than a certain integer - called attracted part. Moreover we will need to enrich the graph coverings representation - exposed in Section 2.1 - on sequences of refining supercyclical partitions with some markers which satisfy some structural conditions.

We define attracting orbits in Section 3.1 and supercyclical partitions in Section 3.2. We define then some operations on supercyclical partitions which act separately on the attracted part and the other one - called supercyclical. In Section 3.3 we deal with the attracted part. In the following sections we deal with the
supercyclical part: in Section 3.4 we expose the construction of Krieger markers in the supercyclical part and in Section 3.5 we expose how to mark supercyclical partitions such that the set of markers forms an acyclical cut of the corresponding graph using Krieger markers, in a way that markers can not be mapped to other markers by iteration of the graph morphisms. Section 3.6 contains a description of a procedure in order to ensure that in every partition of the sequence, the divergent points - which have at least two outgoing edges in the graph - coincide with markers.

Before entering into the exposition of this part, let us make some remark on the graph coverings representations of Cantor systems. It can happen that some of the edges between vertices of a certain level $G_n$ are not representing accurately the dynamical behavior of the system. Let us illustrate this on an example: consider for instance the system which has a graph coverings representation as on Figure 1. Although in the first level $G_1$ the two circuits have nonempty intersection, one can see that the system still consists in two disjoint periodic orbits. In fact one could separate the two circuits in the first level while the graph coverings would still represent the same system. It is possible to construct a procedure that transforms a graph coverings representation of a system into a similar representation of the same system but without this phenomenon, which can be inconvenient in the proofs using graph coverings (in particular for Theorem 1.2 in the present text). We had to deal with a couple other problems of this type. Although it is in principle possible to deal with them by working directly on a graph coverings representation obtained with a simple sequence of refining clopen partitions, it appeared a lot more efficient and simple to work on constructing a suitable sequence of partitions before considering the graph coverings representation associated with it.

![Figure 1](image)

**Figure 1.** In this representation, the two periodic orbits of the systems are not distinguished in the first graph.

### 3.1. Attracting orbits.

**Definition 3.1.** We say that a subset $u \subset X$ is **stable** (by $f$) when $f(u) \subset u$. A finite orbit $p$ of the system $(X, f)$ is called **attracting** whenever there exists a stable clopen set $u$ containing $p$ such that

$$\bigcap_{n \geq 0} f^n(u) = p.$$  

We also say that $u$ is **attracted** by $p$ or that $u$ is an **attracting neighborhood** of $p$.  

Note that if \( p \) is attracting, then for every open set \( v \) containing \( p \) there exists some integer \( n \geq 1 \) such that \( f^n(u) \subseteq v \). It is also straightforward that \( f^n(u) \) is stable for every \( n \geq 1 \). In the following, whenever considering a finite orbit \( p \), we will denote by \( |p| \) its cardinality. The following is straightforward:

**Lemma 3.2.** Let us consider two distinct (and thus disjoint) finite orbits \( p \) and \( p' \) and two stable clopen sets \( u, u' \) which are respectively attracted by \( p, p' \). The sets \( u \) and \( u' \) are disjoint.

**Notation 3.3.** In the present and following sections we consider a zero-dimensional dynamical system \((X, f)\) whose finite orbits are all attracting. We also assume that \( X \) is a Cantor set. For simplicity we will designate such a system as purely attracting one.

**Lemma 3.4.** For all \( n \geq 1 \), a purely attracting system \((X, f)\) has only finitely many periodic orbits of period \( n \).

**Proof.** Since \( X \) is compact, any infinite sequence of periodic orbits of period \( n \) has a subsequence which converges, relatively to Hausdorff distance, to a finite orbit \( p \) whose period is a divisor of \( n \). Since this finite orbit is attracting, this is not possible.

**Remark 3.5.** Lemma 3.4 can be seen as a consequence of the statement [12, Lemma 12] that locally radially shrinking systems have a finite number of finite orbits of length \( n \) for all integers \( n \geq 1 \).

### 3.2. Supercyclical partitions.

**Definition 3.6.** A supercyclical partition of the system \((X, f)\) is a pair \((\mathcal{U}, n)\), where \( \mathcal{U} \) is a finite clopen partition of \( X \) and \( n \geq 1 \) is an integer, such that there exists a sequence \((\mathcal{U})_{|p| \leq n}\) of subsets of \( \mathcal{U} \), where the indexes are periodic orbits, such that for every periodic orbit \( p \) for which \(|p| \leq n\):

\[(S1) \quad \mathcal{E}(\mathcal{U}) \text{ is stable, contains } p \text{ and is attracted by it.}\]

For a finite clopen partition \( \mathcal{U} \), the largest integer \( n \) (or infinity if there is no upper bound) such that \((\mathcal{U}, n)\) is a supercyclical partition is called the supercyclical order of \( \mathcal{U} \), and is denoted by \( o(\mathcal{U}) \). By convention, if there is no \( n \geq 1 \) such that \((\mathcal{U}, n)\) is supercyclical, we set \( o(\mathcal{U}) = 0 \). Furthermore, by extension, we say that a finite clopen partition is supercyclical when \( o(\mathcal{U}) \geq 1 \).

**Remark 3.7.** This definition should not introduce any confusion, as we specify each time if the supercyclical partition is attached with an integer \( n \) or not.

**Remark 3.8.** Observe that if \((X, f)\) has only finitely periodic orbits whose periods are not larger than an integer \( l \) and \( l \leq o(\mathcal{U}) \) then \( o(\mathcal{U}) = +\infty \).

Let us note that in Definition 3.6 as a consequence of Lemma 3.2 for two different periodic orbits \( p \) and \( p' \), the sets \( \mathcal{E}(p) \) and \( \mathcal{E}(p') \) are disjoint. The following proposition shows furthermore that there is a canonical choice for the sets \( \mathcal{U} \):

**Proposition 3.9.** Let us consider \( \mathcal{U} \) a supercyclical partition of the system \((X, f)\), and \( p \) a finite orbit such that \(|p| \leq o(\mathcal{U}) \). The set of subsets \( U \) of \( \mathcal{U} \) such that \( \mathcal{E}(U) \) is attracted by \( p \) admits a maximum for the inclusion relation.

**Proof.** Indeed, it is sufficient to consider the union of all these sets \( U \).

**Notation 3.10.** We will denote \( \mathcal{U} \) the maximum provided by Proposition 3.9. The collection of the sets \( \mathcal{U} \) for \(|p| \leq n \) makes \((\mathcal{U}, n)\) a supercyclical partition. We will call attracted part of \((\mathcal{U}, n)\) the union of the sets \( \mathcal{U} \) for \(|p| \leq n \). The remainder
of the partition is called its \textbf{supercyclical part}. For simplicity we will also call - without introducing ambiguity - attracted part and supercyclical part the images by \( E \) of these respective sets. For the same reason, for a finite clopen partition \( \mathcal{U} \), we will call supercyclical part and attracted part of \( \mathcal{U} \) respectively the supercyclical part and attracted part of \((\mathcal{U}, o(\mathcal{U}))\).

\textbf{Remark 3.11.} The attachment of an integer \( n \) to the definition of supercyclical partition will be used in the following in order to specify what we consider in the context to be the attracted part of the supercyclical partition. The purpose of doing so is to prevent modifications on the supercyclical part to affect the attracted part.

\textbf{Remark 3.12.} When \( f \) is surjective, since \( X \) is without isolated points, the supercyclical part of every supercyclical partition \( \mathcal{U} \) is non-empty. Moreover if \((X, f)\) is aperiodic then every clopen partition \( \mathcal{U} \) of \( X \) is supercyclical, \( o(\mathcal{U}) = +\infty \) and the attracted part \( \mathcal{U} \) is empty.

The following is straightforward:

\textbf{Lemma 3.13.} Let us consider \( \mathcal{U} \) a supercyclical partition for \((X, f)\) and \( \mathcal{V} \) another finite clopen partition such that \( \mathcal{V} \prec \mathcal{U} \). Then the partition \( \mathcal{V} \) is supercyclical and its order is at least \( o(\mathcal{U}) \). Furthermore for all \( p \) such that \( |p| \leq o(\mathcal{U}) \), we have that \( \mathcal{E}(\mathcal{U}^p) \subset \mathcal{E}(\mathcal{V}^p) \).

\textbf{Lemma 3.14.} Let us consider \( \mathcal{U} \) a supercyclical partition for \((X, f)\) and assume that \( o(\mathcal{U}) < +\infty \). There exists another supercyclical partition \( \mathcal{V} \) of order at least \( o(\mathcal{U}) + 1 \) which refines \( \mathcal{U} \).

\textbf{Proof.} Every finite orbit \( p \) such that \( |p| \geq o(\mathcal{U}) + 1 \) is contained in the supercyclical part of \( \mathcal{U} \). Since \( o(\mathcal{U}) < +\infty \) there exists at least one such orbit \( p \). Let \( l \geq o(\mathcal{U}) + 1 \) be the smallest possible period of these orbits. The supercyclical part of \( \mathcal{U} \) is a clopen set, hence for every periodic orbit \( p \) of period \( l \) there exists a clopen attracted neighborhood \( u_p \) of \( p \) which is contained in the supercyclical part of \( \mathcal{U} \). There exists \( m > 0 \) such that each of the sets \( u_p \) is a union of elements of \( \mathcal{U}_m^0 \) (this sequence was fixed at the beginning of Section 2), and the same holds for the complement, in the supercyclical part of \( \mathcal{U} \), of the union of these sets. We then define \( \mathcal{V} \) by collecting the following elements: the elements of \( \mathcal{U} \) contained in the attracted part of \( \mathcal{U} \) and the elements of \( \mathcal{U}_m^0 \) contained in the supercyclical part of \( \mathcal{U} \). For every finite orbit \( p \) with \( |p| \leq o(\mathcal{U}) \), we consider \( \mathcal{V}^p = \mathcal{U}^p \) and when \( |p| = l \), \( \mathcal{V}^p \) consists of the sets \( u \in \mathcal{U}_m^0 \) such that \( u \subset u_p \). Since these sets sum up to \( u_p \), which is attracted to \( p \), the sequence \( (\mathcal{V}^p)_{|p| \leq l} \) makes the partition \( \mathcal{V} \) supercyclical of order at least \( o(\mathcal{U}) + 1 \). \( \square \)

Let us consider \( \mathcal{U} \) a supercyclical partition for \((X, f)\). For all finite orbit \( p \) such that \( |p| \leq o(\mathcal{U}) \), the subgraph of \( G(\mathcal{U}) \) which corresponds to \( \mathcal{U}^p \) does not contain edges from a vertex in this part to another vertex outside of it since \( \mathcal{U}^p \) is a stable set (see Figure 2 for an illustration). While \( \mathcal{E}(\mathcal{U}^p) \) contains a unique periodic orbit, the graph associated with \( \mathcal{U}^p \) may contain other circuits than the one corresponding to the finite orbit, although they do not represent finite orbits but are traces of infinite orbits. We will thus refine the sets \( \mathcal{U}^p \), as partitions of the respective sets \( \mathcal{E}(\mathcal{U}) \), so that the associated graph contains a unique circuit.

\subsection{3.3. Refinement of the attracted part.}

In the following, we will need to have the following additional properties on the graph of each constructed supercyclical partition:

(S2) each of the subgraphs of the attracted part corresponding to some \( \mathcal{U}^p \) contains a unique circuit.
Figure 2. Illustration of the graph $G(\mathcal{U})$ for a supercyclical partition $\mathcal{U}$ of $(X, f)$. The dashed regions correspond to the attracted part of the partition, the remainder corresponds to the supercyclical part.

(S3) in the attracted part there is not divergent vertex (a vertex with at least two outgoing edges).

We would like to refine supercyclical partitions in order to remove these vertices, as illustrated on Figure 3. As a byproduct, we will also remove "artificial" circuits. This is the purpose of the present section.

Figure 3. Removing the divergent vertices in the attracted part.

Let us consider $\mathcal{U}$ a supercyclical partition. In this section we define a sequence $(\kappa_n(\mathcal{U}))_{n \geq 0}$ of supercyclical partitions such that for all $n$, $\kappa_n(\mathcal{U})$ has the same order as $\mathcal{U}$ and such that:

(i) for all $n \geq 0$, $\kappa_{n+1}(\mathcal{U}) \prec \kappa_n(\mathcal{U})$;

(ii) $\kappa_0(\mathcal{U}) = \mathcal{U}$;

(iii) for all $n \geq 1$ the supercyclical part of $\kappa_n(\mathcal{U})$ is identical to the one of $\mathcal{U}$.

We then prove some properties of these partitions related to the graph representation.

Let us recall that we fixed a sequence of clopen partition $(\mathcal{U}_n^0)_{n \geq 1}$ which satisfies the condition (2.1). Let us define the sequence $(\kappa_n(\mathcal{U}))_{n \geq 1}$ by separately defining the supercyclical part and the attracted part. For all $n \geq 1$, the partition $\kappa_n(\mathcal{U})$ has the following elements: the ones of $\mathcal{U}$ that are in its supercyclical part; and the elements of some sets $\mathcal{H}_n(p)$ defined below, for $|p| \leq o(\mathcal{U})$, such that for all $p$, the elements of $\mathcal{H}_n(p)$ form a partition of $E^n(\mathcal{U})$.

Let us define the sequence $(\mathcal{H}_n(p))_{n \geq 1}$ recursively for all $p$ such that $|p| \leq o(\mathcal{U})$. The principle of the definition is to "track back" how the points in $E^n(\mathcal{U})$ approach
the orbit \( p \). Let us fix one periodic orbit \( p = \{p_1, \ldots, p_k\}, \ k \leq o(U) \). Assume standard ordering on the orbit, that is \( p_l = f^{l-1}(p_1) \) for all \( l \leq k \). Let \( n_0 \) be the smallest integer \( s \) such that:

1. there exists a partition of \( \mathcal{E}(\mathcal{U}) \) with elements of \( \mathcal{U}_0^0 \);
2. this partition refines \( \mathcal{U} \) as a partition of \( \mathcal{E}(\mathcal{U}) \);

Before going further, let us prove that there exist \( \hat{u}_1, \hat{u}_2, \ldots, \hat{u}_k \) disjoint clopen sets such that for all \( l \leq k \), \( \hat{u}_l \) contains \( p_1 \), and \( f(\hat{u}_l) \subset \hat{u}_{l+1}, \) where for convenience we set \( \hat{u}_{k+1} := \hat{u}_1 \). Since \( p_1 \) is periodic orbit, there is an open set \( u \supseteq p_1 \) such that \( f^i(\mathcal{P}) \cap f^j(\mathcal{P}) = \emptyset \) for \( i \neq j \). By the well-known property of attracting sets (e.g. see Proposition V.15 in [1]), there is an open set \( w \supseteq p_1 \) such that \( \mathcal{W} \subset u \) and \( f^k(\mathcal{W}) \subset w \). Taking a finite cover of \( \mathcal{W} \) by clopen sets of sufficiently small diameter, we obtain a clopen set \( \hat{u}_1 \supseteq p_1 \) such that \( \mathcal{W} \subset \hat{u}_1 \subset u \) and \( f^k(\hat{u}_1) \subset w \subset \hat{u}_1 \). Since \( f(\hat{u}_1) \) is closed and \( \hat{u}_1 \) is open, we can find again a sufficiently small clopen neighborhood \( \hat{u}_2 \supset f(\hat{u}_1) \) such that \( f^{k+1}(\hat{u}_2) \subset \hat{u}_1 \). In the same way we find clopen sets \( \hat{u}_2, \ldots, \hat{u}_k \). By construction we have that \( f(\hat{u}_l) \subset \hat{u}_{l+1} \) for all \( l \). Furthermore the disjointness of these sets derives from \( f^i(\mathcal{W}) \cap f^j(\mathcal{W}) = \emptyset \) for \( i \neq j \).

For all \( n \geq 0 \) we denote by \( \mathcal{H}_n(p) \) the partition of \( \mathcal{E}(\mathcal{U}) \) defined as:

\[
\mathcal{H}_n(p) = \bigcup_{m \geq 0} S^{(n)}_m(p),
\]

where the sequence \( (S^{(n)}_m(p))_{m \geq 0} \) is constructed as follows:

(i) consider \( w^n \) some clopen set contained in \( \bigcup_{l \leq k} \hat{u}_l \) and which consists in some \( \bigcup_{l \leq k} \hat{u}_l \) union of elements of \( \mathcal{U}_{n+n_0} \) and minimal such that \( w^n \) is stable under \( f \);
(ii) define \( S^{(n)}_0(p) = \{w^n_1, \ldots, w^n_k\} \), where \( w^n_l = w^n \cap \hat{u}_l \);
(iii) \( S^{(n)}_1(p) = \{v \cap f^{-1}(w^n) : 1 \leq l \leq k, \ v \in \mathcal{U}_{n+n_0}, \ v \subset \mathcal{E}(U)\} \);
(iv) for all \( m \geq 1 \), \( S^{(n)}_m(p) = \{v \cap f^{-1}(w) : w \in S^{(n)}_{m-1}(p), \ v \in \mathcal{U}_{n+n_0}, \ v \subset \mathcal{E}(U)\} \).

In words the points in the union of elements of \( S^{(n)}_m(p) \) are the ones that arrive in some \( w^n_l \) after \( m \) iterations of \( f \). This idea is similar to the one used in the construction of Gambaud-Martens representation [10], although resulting graphs are quite different.

It is straightforward to see that the elements of \( \mathcal{H}_n(p) \) are disjoint and edges starting in \( S^{(n)}_{m+1}(p) \) have to end in \( S^{(n)}_m(p) \) for all \( m \geq 0 \). Moreover, for each \( x \in \mathcal{E}(\mathcal{U}) \), there exists some \( m \) such that \( f^m(x) \in w^n \). As a consequence \( \mathcal{H}_n(p) \) covers \( \mathcal{E}(\mathcal{U}) \), which is compact, meaning that only finitely many sets \( S^{(n)}_m(p) \) are not empty. Thus \( \kappa_n(U) \) is indeed a finite partition of \( X \). Schematic illustration of next Lemma can be found on the right part of Figure 3. It shows that our construction indeed leads to conditions (S2) and (S3).

**Remark 3.15.** In order to obtain (S2) and (S3) and get rid of divergent vertices, it is enough to use only \( \kappa_1(U) \). However we present the construction with \( \kappa_n \) since it may be of independent interest for further research, since it allows to refine attracted part without changing supercyclic part.

**Lemma 3.16.** Consider a supercyclical partition \( \mathcal{U} \). For all \( n \geq 1 \), the restriction of the graph \( G(\kappa_n(U)) \) to any \( \kappa_n(U) \) with \( |p| \leq o(U) \) consists in a unique circuit, together with a finite number of paths whose intersection with the circuit is reduced to the endpoint of this path and if two such paths coincide at some point, they coincide until their endpoint.
Proof. Let us fix a finite orbit $p$ such that $|p| \leq o(\mathcal{U})$. It is straightforward that the elements of $S_{m}^{(n)}(p)$ form a circuit in the graph $G(\kappa_{n}(\mathcal{U}))$. Let us consider some $u \in \mathcal{H}_{n}(p)$ which is not in this circuit. By construction there exists $m \geq 1$ such that $u \in S_{m}^{(n)}(p)$. There is a sequence of vertices $(u_{l})_{0 \leq l \leq m}$ such that for all $l$, $u_{l} \in S_{m-l}^{(n)}$ and for $l \leq m - 1$, $(u_{l}, u_{l+1})$ is an edge of the graph $G(\kappa_{n}(\mathcal{U}))$. By definition, each element of some $S_{m}^{(n)}$, $m \geq 1$ is connected to a unique element of $S_{m-1}^{(n)}$. This implies the lemma.

3.4. Markers compatible with supercyclical partitions. In the following, we will use an adaptation of Krieger’s notion of markers in order to mark partitions.

**Definition 3.17.** Let us consider $\mathcal{U}$ a supercyclical partition for $(X, f)$ and $1 \leq n \leq o(\mathcal{U})$, and let us denote by $\mathcal{S}$ the supercyclical part of $\mathcal{U}$. A $(n, t, N)$-marker for $\mathcal{U}$ is clopen set $F \subset E(\mathcal{S})$ such that

1. the set $F$ is $(n+1)$-separated, meaning that the sets $F, f^{-1}(F), \ldots, f^{-n}(F)$ are pairwise disjoint;
2. the sets $F, f^{-1}(F), \ldots, f^{-N}(F)$ cover $f^{-t}(E(\mathcal{S}))$.

**Theorem 3.18.** Fix $n \geq 1$ and let us consider $\mathcal{U}$ a supercyclical partition of order at least $n$ for $(X, f)$. There exists some integer $m \geq 1$ such that $\mathcal{U}$ admits an $(n, (n+1)m+n, 2n+1)$-marker.

Proof. The proof is very similar to the one presented in [5] for aperiodic Cantor systems. We adapt it here to a purely attracting system, with appropriate changes to our slightly more general setting.

For each $x$ in the supercyclical part of $\mathcal{U}$ there exists $u_{x}$ clopen neighborhood of $x$ which is included in the supercyclical part and which is $(n+1)$-separated. Since the attracted part of $\mathcal{U}$ is stable by $f$, for all $k \leq n$, $f^{-k}(u_{x})$ is included in the supercyclical part. The clopen sets $u_{x}$ for $x$ in the supercyclical part form an open cover of this part, which is a compact set. As a consequence there is a finite subcover of it with these clopen sets. Let us denote $u_{j}, j \in \{1, \ldots, m\}$ the elements of this subcover. The sets $u_{j}' = f^{-(n+1)m}(u_{j})$ are all $(n+1)$-separated, and they cover $f^{-(n+1)m}(E(\mathcal{S}))$.

Let us define $F_{1} = u_{1}'$ and recursively:

$$F_{j+1} := F_{j} \bigcup \left( \bigcup_{-(n+1) \leq i \leq (n+1)} f^{-i}(F_{j}) \right).$$

Finally set $F = F_{m}$. Let us prove that $F$ is a $(n, (n+1)m+n, 2n+1)$-marker:

1. **The set $F$ is $(n+1)$-separated.** Let us prove inductively that each $F_{j}$ is $(n+1)$-separated. This is immediate for $F_{1}$. Assume the property is verified for $F_{j-1}$ and that, ad absurdum, that $F_{j}$ is not $(n+1)$-separated. As a direct consequence, there exist two $i', i''$ such that $0 \leq i' < i'' < n+1$ such that $f^{-i'}(F_{j}) \cap f^{-i''}(F_{j}) \neq \emptyset$. Thus we have $F_{j} \cap f^{-i}(F_{j}) \neq \emptyset$ for some $0 < i < n+1$, and therefore there exists $x$ such that $x \in F_{j}$ and $f^{i}(x) \in F_{j}$. We have, by definition of $F_{j}$, that $F_{j} \subset F_{j-1} \cup u_{j}'$. Since both of these sets are $(n+1)$-separated, the two points $x$ and $f^{i}(x)$ belong to different of them. The point which belongs to $u_{j}'$ also belongs to $F_{j-1}$, a contradiction.
(2) Every point of $f^{-(n+1)m-n}(E(S))$ visits $F$ at least once in $2n + 1$ iterations. Consider a point $x$ in the set $f^{-(n+1)m-n}(S)$. Then $f^n(x)$ belongs to some $u_j$. If $f^n(x) \in F_j$, then $f^n(x) \in F$. The only way that $f^n(x)$ may not belong to $F_j$ is if it belongs to

$$\bigcup_{-(n+1)<i<(n+1)} F^{-i}(F_{j-1}).$$

Then we obtain that $x \in f^{-n-1}(F)$ with $-n-1 < i < n+1$. This concludes the proof.

\[\square\]

3.5. Well marked partitions. In this section we define marked partitions, as well as constraints on the marker system that we will need to be satisfied when constructing a suitable sequence of partitions for the system $(X,f)$.

**Definition 3.19.** A marked partition is a pair $(\mathcal{U}, \tau, \chi)$ where $(\mathcal{U}, \tau)$ is a supercyclical partition for $(X,f)$, and $\chi : \mathcal{U} \rightarrow \{\uparrow, \downarrow, *, 0\}$ an element $u$ of $\mathcal{U}$ such that $\chi(u) = *$ is called a marker. When $\chi(u) = \uparrow$, $u$ is called a potential.

**Definition 3.20.** We say that a marked partition $(\mathcal{U}, \tau, \chi)$ is well marked when the function $\chi$ has the following properties:

1. For all $u$ in the attracted part of $(\mathcal{U}, \tau)$, $\chi(u) = 0$; for all other $u$, $\chi(u) \neq 0$.
2. The circuits whose vertices are all contained in the supercyclical part of $(\mathcal{U}, \tau)$ all contain at least one marker and one potential.

The following is straightforward:

**Lemma 3.21.** Let us consider a well marked supercyclical partition $(\mathcal{U}, \tau, \chi)$ and another finite clopen partition $(\mathcal{V}, \tau')$ where $\mathcal{V}$ refines $\mathcal{U}$ and $\tau' \geq \tau$. Let us denote $\chi' : \mathcal{V} \rightarrow \{\uparrow, \downarrow, *, 0\}$ whose value is 0 on the attracted part of $(\mathcal{V}, \tau')$ and such that for all $v \in \mathcal{V}$ in the supercyclical part of $(\mathcal{V}, \tau')$, $\chi(\pi_{\mathcal{U}}(v)) = \chi'(v)$. Then $(\mathcal{V}, \tau', \chi')$ is well-marked.

**Definition 3.22.** Let us consider two well-marked partitions $(\mathcal{U}, \tau, \chi)$ and $(\mathcal{V}, \tau', \chi')$ such that $\mathcal{V} \prec \mathcal{U}$. We say that $(\mathcal{V}, \tau', \chi')$ is well-marked relatively to $(\mathcal{U}, \tau, \chi)$ when for all $u$ in the supercyclical part of $(\mathcal{V}, \tau')$, the word $\chi'(u)\chi(\pi_{\mathcal{U}}(u))$ is in $\{\uparrow, \downarrow, \uparrow\uparrow, \downarrow\downarrow, *\}$. In particular, Definition 3.22 means that for a position in the supercyclical part of $(\mathcal{V}, \tau)$, if it is marked by $\pi_{\mathcal{U}}$ to a potential then it is a marker, a potential or is marked with $\downarrow$. Otherwise it is marked with $\uparrow$.

**Notation 3.23.** For a finite clopen supercyclical partition $\mathcal{U}$, we denote by $\eta(\mathcal{U})$ the maximal length of a circuit in $G(\mathcal{U})$ whose vertices are all contained in the supercyclical part of $\mathcal{U}$.

**Lemma 3.24.** Let us consider a well-marked partition $(\mathcal{U}, \tau, \chi)$ and some integer $n$ larger or equal to 1. There exists a well marked partition $(\mathcal{V}, \tau', \chi')$, where $\mathcal{V}$ is of order at least $\max(2\eta(\mathcal{U}), o(\mathcal{U}) + 1)$ such that $\mathcal{V} \prec \mathcal{U}$, $\mathcal{V} \prec \mathcal{U}_n$, $\tau' \geq \tau + 1$, and $(\mathcal{V}, \tau', \chi')$ is well-marked relatively to $(\mathcal{U}, \tau, \chi)$.

**Proof.** (1) Setup: We first consider $\mathcal{V}_0$ to be the refinement $\mathcal{U} \cup \mathcal{U}_{n+1}$, for $t$ large enough so that $o(\mathcal{V}_0) \geq o(\mathcal{U}) + 1$. If $o(\mathcal{V}_0) < 2\eta(\mathcal{U})$, we use Lemma 3.11 in order to refine this partition into another supercyclical partition $\mathcal{V}_1$ of order at least $2\eta(\mathcal{U})$. Otherwise we set $\mathcal{V}_1 := \mathcal{V}_0$. The partition $\mathcal{V}_1$ refines $\mathcal{U}$ and $\mathcal{U}_n$, and is of order at least $\max(2\eta(\mathcal{U}), o(\mathcal{U}) + 1)$. Since these properties are stable by refinement, the partitions constructed later in this proof $(\mathcal{V}_2$ and $\mathcal{V})$ will also have this property. As
a consequence of Theorem 10.18, the partition \( V_1 \) admits some \((s, (s+1)m+s, 2s+1)\)-marker denoted by \( F \), where \( m \geq 1 \) and \( s = 2\eta(U) \).

2) **Further refinements:** (i) There exists an integer \( n_1 \geq n + t \) such that for \( l \leq s \) each of the sets \( f^{-l}(F) \) is the union of some elements in \( U_{n_1}^0 \). Since \( F \) is clopen, the set \( K = \bigcup_{i=0}^n f^{-i}(F) \) is also clopen, and as a consequence there exists some \( \zeta > 0 \) such that if \( d(x, f^{-l}(F)) < \zeta \) for some \( l \leq s \) then \( x \in f^{-l}(F) \).

We may also impose that \( n_1 \) is large enough so that for every \( u \in U_{n_1}^0 \) we have \( \text{diam} f^l(u) < \zeta/(2s+2) \) for \( l = 0, \ldots, 2s+1 \). We then consider \( V_2 = V_1 \cup U_{n_1} \).

(ii) We perform one more modification on \( V_2 \). By definition of a marker, if we denote by \( S_1 \) the supercyclical part of \( V_1 \) then \( F, f^{-1}(F), \ldots, f^{-2s-1}(F) \) cover the set \( f^{-(s+1)m-s}(E(S_1)) \). Since \( E(S_2) \subset E(S_1) \) (by definition, the attracted part of \( V_2 \) contains the attracted part of \( V_1 \)), we have:

\[
f^{-(s+1)m-s}(E(S_2)) \subset f^{-(s+1)m-s}(E(S_1)).
\]

Furthermore, every point in \( E(S_2) \setminus f^{-(s+1)m-s}(E(S_2)) \) have to enter the attracted part of \( V_2 \) after at most \((s+1)m+s\) iterations. Therefore, for sufficiently large integer \( n_2 \), the continuity of \( f \) implies that any vertex \( u \in U_{n_2}^0 \) either satisfies the inclusion \( u \in f^{-(s+1)m-s}(E(S_2)) \) or \( f^{s+1m+s}(u) \subset E(V_2 \setminus S_2) \). In particular, considering \( V = V_2 \cup U_{n_2}^0 \), we have that \( E(S) \subset f^{-(s+1)m-s}(E(S_2)) \), and thus \( E(S) \subset f^{-(s+1)m-s}(E(S_1)) \), where \( S \) is the supercyclical part of \( V \).

We set \( \tau^* := \text{diam}(V) \). Since \( \text{diam}(V) \geq \text{diam}(U) + 1 \geq \tau + 1 \), we have \( \tau^* \geq \tau + 1 \).

Furthermore, the supercyclical part of \( V, \tau^* \) is equal to the one of \( V \), which is included in the supercyclical part of \( U \) and thus of \( (U, \tau) \).

3) **Definition of \( \chi' \) on the attracted part of \( V \):** We claim that there exists \( \chi' \) such that \((V, \tau^*, \chi')\) is well-marked and well-marked relatively to \((U, \tau, \chi)\). We set \( \chi'(u) = 0 \) for all \( u \) in the attracted part of \( V \). For every \( u \) in the supercyclical part of \( V \), if \( \chi'(\pi_U^V(u)) \in \{*, 
\} \) or if \( u \) is not in any circuit contained in the supercyclical part of \( V \), set \( \chi'(u) = _1 \). We are left to define \( \chi' \) on preimages of potentials by \( \pi_U^V \) which are in a circuit of vertices in the supercyclical part of \( V \).

4) **Circuits of the supercyclical part of \( V \) follow the sequence \( f^{-(s+1)}(F), f^{-1}(F), \ldots, F \) on some segment:** (i) **Claim:** By construction, for each of the circuits of vertices in the supercyclical part of \( V \), its vertices are included in \( f^{-(s+1)m-s}(E(S_1)) \) (since \( E(S) \subset f^{-(s+1)m-s}(E(S_1)) \)). As a consequence (by the definition of marker) each of its vertices is included in some \( f^{-k}(F) \) for \( k \leq 2s+1 \).

We claim that each circuit has some sequence of consecutive vertices respectively included in \( f^{-(s+1)}(F), f^{-1}(F), \ldots, F \) (it is possible to have other vertices outside of these sets).

(ii) **When the circuit intersects \( F \):** Indeed, for any vertex \( u \in F \) such that there exists an edge \((v, u)\), there exists \( v \in v \) such that \( f(x) \in U \subset F \).

This implies that \( v \cap f^{-1}(F) \neq \emptyset \) and since \text{diam} \( v < \zeta \) we have \( v \subset f^{-1}(F) \).

We apply a similar reasoning to \( u \subset f^{-(s)+1}(F) \) for \( l \leq (s-1) \): as a consequence the claim holds for any circuit which has a vertex included in \( F \). (iii) **General case:**

Let \( c \) be a circuit whose vertices are included in the supercyclical part of \( V \). Since it is covered by the sets \( f^{-1}(F) \) for \( l \leq 2s+1 \), there is some \( l \leq 2s+1 \) and a vertex \( u_0 \) in the circuit \( c \) such that \( u_0 \cap f^{-1}(F) \neq \emptyset \). Let also \( u_1, \ldots, u_l \) be the next \( l \) consecutive vertices in the circuit \( c \). For all \( i \leq (l-1) \), there exists \( x_i \in u_i \) such that \( f(x_i) \in u_{i+1} \). Fix also a point \( x \in u_0 \) and \( x_l \in u_l \). Since \( x, x_l \in u_0 \) we have \( d(f^{-1}(x), f^{-1}(x_l)) < \zeta/(2s+2) \). Similarly for all \( i \leq (l-1) \), since \( f(x_i), x_{i+1} \in u_{i+1} \) we have \( d(f^{1-i}(x_i), f^{l-i-1}(x_{i+1})) < \zeta/(2s+2) \). As a consequence we have:

\[
d(f(x), x_l) \leq d(f(x), f^{-1}(x_l)) + \sum_{i=0}^{l-1} d(f^{1-i}(x_i), f^{l-i-1}(x_{i+1})) \leq \frac{\zeta(l+1)}{2s+2} \leq \zeta.
\]
This implies that the distance between \( x_l \) and \( F \) is smaller than \( \zeta \), and thus that \( u_l \subset F \), which means that the claim holds for \( c \).

**5) Choosing markers and potentials in the preimages of potentials:** Let us consider a circuit \( c \) in the supercyclical part of \( V \) and a sequence of consecutive vertices of \( c \) contained respectively in \( f^{-s+1}(F), f^{-1}(F), \ldots, F \) - note that any two such sequences cannot have any common element. The first \( \eta(U) \) of these vertices are mapped altogether via \( \pi\) to a path which contains at least one circuit in \( G(U) \) whose vertices are in the supercyclical part of \( U \), by definition of \( \eta(U) \). Since \( (U, \tau, \chi) \) is well-marked, each circuit in \( G(U) \) whose vertices are in the supercyclical part of \( (U, \tau) \) contains at least one potential. Thus one of these vertices \( \eta(U) \) is mapped to a potential. This is true also for the last \( \eta(U) \) of these vertices (let us recall that \( s \geq 2\eta(U) \)).

Let us list all the circuits that are in the supercyclical part of \( V \), say \( c_1, \ldots, c_t \). Consider successively each of these circuits. Each time do the following steps:

1. If the first set of \( \eta(U) \) vertices defined above contains one for which \( \chi' \) is already defined to be \( * \), then pass.

2. Otherwise pick one preimage \( u \) of a potential in this set and define \( \chi'(u) = * \).

3. If the second set of vertices defined above contains one for which \( \chi' \) is already defined to be \( \uparrow \), then pass.

4. Otherwise pick one preimage \( u \) of a potential in this set and define \( \chi'(u) = \uparrow \).

For each vertex \( u \) for which \( \chi' \) is still undefined at the end of this process, set \( \chi'(u) = \downarrow \). By construction, since sets \( f^{-s+1}(F), f^{-1}(F), \ldots, F \) are pairwise disjoint, it is straightforward that \( (V, \chi') \) is well-marked and well-marked relatively to \( (U, \tau, \chi) \).

As a direct consequence of the construction in the proof of Lemma \( 3.24 \) we also have the following.

**Lemma 3.25.** There exists a well-marked partition \( (U, \tau, \chi) \) for \( (X, f) \) such that \( U \prec U_0^1 \).

**Remark 3.26.** Before going further, let us explain why we needed to use Krieger’s markers in order to obtain Lemma \( 3.24 \).

![Figure 4](image-url)  
**Figure 4.** Two finite directed graphs \( G = (V, E) \) (left) and \( G' = (V', E') \) (right) and \( \pi : V \to V' \) morphism, sending vertices of \( G \) to the one of the same color in \( G' \). The graph \( G' \) can correspond to a well-marked partition \( (V, \tau, \chi) \). However it is not possible to find a partition whose graph is \( G \) and which would be well-marked relatively to \( (V, \tau, \chi) \). Indeed, whatever the way we mark the red vertices, there will be at least one circuit left with no marker or no potential.

Intuitively we could mark the partitions using a simple argument which consists in noticing that it is possible to refine supercyclic partitions so that circuits in the
supercyclic part are uniformly arbitrarily large, and thus are mapped by the graph
morphism to a concatenation of at least two circuits of the initial partition’s graph.
This way each circuit contains two preimages of potentials, one of which we could
define to be a marker and the other one a potential. The problem with this reasoning
however is that circuits may have a lot of intersections, and marking vertices in a
circuit could prevent the possibility of similar markings for other circuits. This
appears clear for the two graphs $G, G'$ on Figure 2 and the morphism $\pi$ from $G$
to $G'$. In this figure $G$ could be replaced with some graphs whose circuits could be
taken uniformly arbitrarily long (and having many more intersections).

3.6. Rectification of a partition well marked relatively to another one. In
this section we define the last operation on marked partitions in order to ensure
some other properties that will be needed for the graph coverings representation
used in the proof of Theorem 1.2.

**Definition 3.27.** Let us consider $G$ a finite directed graph. We will call acyclic
cut of $G$ any set of its vertices such that by removing from $G$ all the edges that are
pointing at a vertex in this set, we obtain an acyclic graph.

**Definition 3.28.** Let us consider $G$ a finite directed graph. A divergent vertex
of this graph is some vertex $u$ which has at least two outgoins edges.

**Notation 3.29.** Let us consider a finite directed graph $G$ and $\chi: V \to \{\downarrow, \uparrow, 0, *\}$,
where $V$ is the set of vertices of $G$. Let us denote by $R(G, \chi)$ the set whose elements
are the following ones: every vertex $u \in V$ such that $\chi(u) = 0$ and such that there
is no divergent vertex $v \in V$ with $\chi(v) \neq 0$ with an edge from $v$ to $u$; the edges
of $G$ pointing at any of the vertices in $R(G, \chi)$, and the edges from $u$ to $v$ with
$\chi(v) = \chi(u) = 0$.

Let us denote by $I(G, \chi)$ the graph obtained from $G$ by removing the vertices
and edges which are in $R(G, \chi)$. We will also denote $A(G, \chi)$ the graph obtained
from $I(G, \chi)$ by adding, for each vertex $u$ of this graph such that $\chi(u) = *$ (the
function $\chi$ is defined on vertices of $I(G, \chi)$ by restriction) a copy $c_u$ of this vertex,
and changing the edges pointing at $u$ so that they point at $c_u$ without changing their
origin. We also denote $S(G, \chi)$ the set of pairs $(u, c_u)$ for $u$ vertex of $I(G, \chi)$ such
that $\chi(u) = *$. The purpose of this set is to keep record of which vertex is a copy of
which vertex, for the reason that we will have to conflate $u$ and $c_u$ at some point.
This construction is illustrated on Figure 3.

**Remark 3.30.** Considering a well-marked partition $(U, \tau, \chi)$, the set of its markers
forms an acyclic cut of $I(G(U), \chi)$. Thus the graph $A(G(U), \chi)$ is acyclic.

For a supercyclical partition $U$ and any integer $n \geq 1$, every divergent element
of $G(\kappa_n(U))$ belongs to the supercyclical part of $\kappa_n(U)$. The specificity of purely
attracting Cantor systems, besides the fact that supercyclical partitions segregate
circuits (no circuit crosses both the attracted part and the supercyclical one) lies
in this property, which will allow us, by refining partitions, to "move" all divergent
elements so that they coincide with markers.

**Definition 3.31.** For an acyclic directed graph, we call initial vertex any vertex
of this graph which has no edge pointing at it.

**Notation 3.32.** Let us consider $A$ an acyclic directed graph. We will denote $V(A)$
its vertex set. Consider two vertices $v$ and $v'$ of $A$ such that there is a path from $v$
to $v'$ in $A$. We call distance - which is not a distance in the usual sense - between
$v$ and $v'$ the minimal length of a path from $v$ to $v'$. We will denote $\delta(A)$ the
maximal distance between an initial point and a divergent point of $A$, and we use
the convention $\delta(A) = 0$ when there is no divergent point in $A$. We will also denote
$\mu(A)$ the number of divergent vertices which realize this maximum.
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\( (G, \chi) \)

\[ \mathcal{I}(G, \chi) \]

\[ \mathcal{A}(G, \chi) \]

Figure 5. Illustration on an example of the definition of the graphs \( \mathcal{I}(G, \chi) \) and \( \mathcal{A}(G, \chi) \) for the graph \( G = G(U) \) and \( \chi : U \to \{0, \downarrow, \uparrow, *\} \) where \( (U, \chi) \) is a well-marked partition; the function \( \chi \) is partially represented (for simplicity): only markers and vertices with \( \chi(u) = 0 \) (the ones in dashed regions) are represented.

Let us observe that for an acyclic directed graph \( A \), this graph possesses a divergent point which is not initial if and only if \( \delta(A) > 0 \).

**Definition 3.33.** Let us consider a directed acyclic graph \( A \) such that \( \delta(A) > 0 \) and \( v \) a divergent vertex realizing the maximum in the definition of \( \delta(A) \). We consider the acyclic graph \( \hat{A} \) obtained from \( A \) after the following modifications: (i) First replace the vertex \( v \) with the set of vertices \( v \cap f^{-1}(w) \), where there is an edge pointing from \( v \) to \( w \); (ii) Replace all the edges pointing to \( v \) by a set of edges pointing at the constructed vertices \( v \cap f^{-1}(w) \) according to the behaviour of the system \( (X, f) \).

Let us denote \( \leq_{\text{lex}} \) the lexicographic order on \( \mathbb{R}^2 \). The following is straightforward:

**Lemma 3.34.** With notations from Definition 3.33, we have that:

\[ (\delta(\hat{A}), \mu(\hat{A})) <_{\text{lex}} (\delta(A), \mu(A)) \]

**Notation 3.35.** For an acyclic graph such that \( \delta(A) > 0 \) and \( \chi : V(A) \to \{0, *, \uparrow, \downarrow\} \), we denote \( \hat{\chi} : V(\hat{A}) \to \{0, *, \uparrow, \downarrow\} \) such that for every vertex created in the definition of \( \hat{A} \), the value of \( \hat{\chi} \) is equal to the value of \( \chi \) on the vertex that they replace. On the other vertices the function \( \hat{\chi} \) coincides with \( \chi \).

**Notation 3.36.** For simplicity, for a well-marked partition \( (V, \tau, \chi) \), we will denote \( \delta(V, \chi) \) and \( \mu(V, \chi) \) the respective numbers \( \delta(A(G(V), \chi)) \) and \( \mu(A(G(V), \chi)) \).

### 3.6.1 Decreasing the value of \( (\delta, \mu) \) for a well-marked partition

Let us consider a well-marked partition \( (U, \tau, \chi) \) and \( A \) the acyclic graph \( A(G(U), \chi) \), and assume that \( \delta(A) > 0 \). Let us denote \( \chi_A \) the function \( V(A) \to \{\uparrow, \downarrow, 0, *\} \) such that:

- \( \chi_A(u) = \chi(u) \) for all \( u \in V(A) \) except for vertices created by the replacement process.
- For newly created vertices, \( \chi_A(u) \) is defined according to the behaviour of the system \( (X, f) \).
(i) \( \chi_A \) is identical to \( \chi \) on the vertices of \( \mathcal{I}(G(U), \chi) \) on which \( \chi \) is defined;
(ii) \( \chi_A(c_u) = * \) whenever \((u, c_u) \in S(G(U), \chi)\).

In other words we add mark \(*\) to all newly created vertexes \( c_u \).

Consider \( v \) a vertex which realizes the maximum in the definition of \( \delta(A) \). Let us modify the graph \( G(U) \) using \( A \). Strictly speaking, we construct a new graph \( G \) as follows:

1. For all \((u, c_u) \in S(G(U), \chi)\), remove from \( A \) the vertex \( c_u \) from the graph and change the edges pointing at \( c_u \) so that they point at \( u \), without changing their origin (this is possible because divergent vertexes are left unchanged in the transformation of \( A \) into \( \hat{A} \)).
2. Add the vertices and edges of the set \( R(G(U), \chi) \) (this operation is possible since the origin vertexes of these edges are unchanged by the transformation of \( A \) into \( \hat{A} \)).

In other words we glue back vertexes marked \(*\) who were split before according to the relation \( S(G(U), \chi) \). We also have to define a modification of function \( \chi \), since the set of vertexes changed. Let \( V \) denote the set of vertexes of \( G \). Define \( \chi': V \to \{\uparrow, \downarrow, *, 0\} \), such that \( \chi' \) coincide with \( \chi_A \) on the graph obtained out of \( \hat{A} \) is step (1), and with value 0 on the vertices added in step (2).

**Lemma 3.37.** There exists a partition \( V \) which refines \( U \) such that \( G(V) = G \). Moreover \((V, \tau, \chi') \) is well-marked and for every vertex \( u \) of \( G(V) \), we have \( \chi'(u) = \chi(\pi^V_U(u)) \). Furthermore:

\[
(\delta(V, \chi'), \mu(V, \chi')) < \text{lex} (\delta(U, \chi), \mu(U, \chi)).
\]

**Proof.** Indeed the vertices created in the transformation of \( A \) into \( \hat{A} \) form a partition of the vertex they replace. The collection of vertexes of the graph \( G \) thus forms a finite clopen partition \( V \) of \( X \) and this partition refines \( U \). Hence \( G = G(V) \). By construction of \( \chi_A \), we have \( \chi'(u) = \chi(\pi^V_U(u)) \) for all \( u \) vertex of \( G \). We have that \((V, \tau, \chi') \) is well-marked. This comes from the two facts that the image of every circuit of \( G(V) \) by \( \pi^V_U \) contains at least one circuit of \( G(U) \), and that for all \( u \), \( \chi'(u) = \chi(\pi^V_U(u)) \). The consequence of these facts is that every circuit in the supercyclical part of \((V, \tau) \) contains at least one marker and one potential. The last part of the lemma is a direct consequence of Lemma 3.34 \( \square \)

**Theorem 3.38.** Let us consider a well marked partition \((U, \tau, \chi)\). There exists another well marked partition \((V, \tau, \chi')\) such that \( V \) refines \( U \) and such that divergent points of \( V \) are markers for \((V, \tau, \chi')\) and all these markers are mapped to markers of \((U, \tau, \chi)\) by \( \pi^V_U \).

**Proof.** Let us construct a sequence of well-marked partitions \((U_m, \tau, \chi_m)_{m \geq 0}\) recursively as follows:

1. The first element of this sequence is given by \( U_0 = \kappa_1(U) \), which refines \( U \) and \( \chi_0 : U_0 \to \{\uparrow, 0, *, \downarrow\} \) which coincides with \( \chi \) on the supercyclical part of \( U_0 \) and takes constant value 0 on the attracted part of \( U_0 \). We have directly that \((U_0, \tau, \chi_0) \) is well marked.
2. For all \( m \geq 0 \), if \( \delta(U_m, \chi_m) > 0 \), \((U_{m+1}, \chi_{m+1}) \) is the well-marked partition obtained from \((U_m, \tau, \chi_m) \) by Lemma 3.37. In this case we have that

\[
(\delta(U_{m+1}, \chi_{m+1}), \mu(U_{m+1}, \chi_{m+1})) < \text{lex} (\delta(U_m, \chi_m), \mu(U_m, \chi_m)).
\]

Otherwise set \((U_{m+1}, \tau, \chi_{m+1}) \) equal to \((U_m, \tau, \chi_m) \). By this construction for all \( m \), \( U_{m+1} \) refines \( U_m \) and \((U_m, \tau, \chi_m) \) is well marked.

By infinite descent argument, there exists some \( m_0 \geq 0 \) (minimal) such that \( \delta(U_{m_0}, \chi_{m_0}) = 0 \). This means that every divergent vertex in the graph of \( U_{m_0} \) is
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initial in this graph or is a marker. By refining one more time we can remove the initial divergent vertices by splitting them in as many vertices as outgoing arrows. We denote \( V \) the obtained partition, and \( \chi^* : V(V) \to \{ \uparrow, \downarrow, 0, * \} \) obtained by attributing, for each split vertex, its value for \( \chi_{m_0} \) to the vertices introduced after splitting.

The marked partition \( (V, \tau, \chi^*) \) is well-marked (indeed \( (U_{m_0}, \tau, \chi_{m_0}) \) is well-marked and the construction of \( (V, \tau, \chi^*) \) from it does not modify the circuits) and \( V \ll U \). By construction every divergent vertex of the graph \( G(V) \) is a marker of \( (V, \tau, \chi^*) \).

Moreover Lemma 3.37 implies that for each vertex \( u \) of \( G(V) \),

\[
\chi^*(u) = \chi \left( \prod_{k=0}^{n} \pi_{U_k} \circ \pi_{U_{m_0}} \circ \ldots \circ \pi_{U_{m_0-1}} (u) \right) = \chi(\pi_U^V(u)).
\]

This implies that markers of \( (V, \tau, \chi^*) \) are mapped to markers of \( (U, \tau, \chi) \) by \( \pi_U^V \).

This concludes the proof. \( \square \)

The partition obtained by the above theorem has the following property:

(S4) the partition is well marked and all divergent vertexes are markers.

**Theorem 3.39.** The system \( (X, f) \) admits a sequence of finite clopen partitions \( U^n = (U_n)_{n \geq 1} \) such that \( \lim_{n \to \infty} \text{mesh}(U_n) = 0 \) and for all \( n \geq 1 \) and for all \( u \in U_n, u \) is equal to the set \( E(\{ v \in U_{n+1} : v \subset U \}) \), and a sequence of functions \( \chi_n : U_n \to \{ \downarrow, 0, *, \uparrow \} \) and a non-decreasing sequence of integers \( \tau_n \geq n \) such that for all \( n \geq 0 \), \( (U_n, \tau_n, \chi_n) \) is a well-marked partition, and \( (U_{n+1}, \tau_{n+1}, \chi_{n+1}) \) is well marked relatively to \( (U_n, \tau_n, \chi_n) \). Moreover every divergent point of \( U_n \) is a marker of \( (U_n, \tau_n, \chi_n) \).

**Proof.** This result can be derived from all the previous steps leading to conditions (S1)-(S4) together with keeping that each of the partition is well marked with respect to the previous one. Formally, it is obtained as follows.

First we use Lemma 3.22 to get a well marked partition for \( (X, f) \), and define \( (U_1, \tau_1, \chi_1) \) to be this partition. Let us assume that we have constructed \( (U_k, \tau_k, \chi_k) \) for all \( k \leq n \) for some \( n \geq 1 \). We then apply Theorem 3.38 on the well-marked partition obtained from Lemma 3.22 with integer \( n+1 \), and set \( (U_{n+1}, \tau_{n+1}, \chi_{n+1}) \) to be the obtained marked partition. By construction this partition is well-marked and well-marked relatively to \( (U_n, \tau_n, \chi_n) \), and \( \tau_{n+1} \geq \tau_n + 1 \), which implies that \( \tau_n \geq n \) for all \( n \). Also every divergent point of \( U_{n+1} \) is a marker of \( (U_{n+1}, \tau_{n+1}, \chi_{n+1}) \). Moreover we have that for all \( n \geq 1 \), \( U_n \ll U_0 \) and \( U_{n+1} \ll U_n \), which implies that \( U^* \) satisfies the condition (2.1). \( \square \)

A direct consequence of properties of \( U^* \) constructed in the proof of Theorem 3.39 is that for every \( v \in V_{U^*} \), there exists at most one integer \( n \geq 1 \) such that \( \chi_n(v_n) = * \). Moreover if \( v_n \) is divergent in \( G(U_n) \), then \( \chi_n(v_n) = * \). These two properties will be crucial in the proof of main theorem.

4. ON THE EMBEDDING PROBLEM : PROOF OF THEOREM 1.2

In this section we will provide a proof of Theorem 1.2 that we recall here (the notion of attracting finite orbit is provided in Definition 3.1):

**Theorem 1.2.** Any Cantor dynamical system \( (X, f) \) can be embedded in the interval \([0, 1]\) with vanishing derivative if and only if it is purely attracting.

The implication \((\Leftarrow)\) builds on all previous construction and its proof is presented in Section 1.1. The converse is an immediate consequence of the following simple observations.
Lemma 4.1. Let us assume that $X$ is a Cantor set and $f$ is differentiable on $X$. Any finite orbit $p$ of the system $(X, f)$ such that for all $x \in p$, $f'(x) = 0$ is attracting.

**Corollary 4.2.** If a Cantor dynamical system $(X, f)$ can be embedded in the real line with vanishing derivative then it is purely attracting.

**Proof.** Assume that $(X, f)$ can be embedded in the real line with vanishing derivative. This means that there exist $Z \subset \mathbb{R}$, $g: Z \to Z$ with $g' \equiv 0$ and $\psi: X \to Z$ an embedding which conjugates $(Z, g)$ and $(X, f)$. Since $g' \equiv 0$, Lemma 4.1 implies that all periodic orbits of $g$ are attracting. Attracting orbits are preserved under conjugacy, thus $(X, f)$ is purely attracting, which completes the proof. □

4.1. **Sufficiency of the condition.** The embedding required in Theorem 4.2 will be a consequence of the following theorem obtained first in [7] (see [4] for a contemporary proof of this result). It will allow us to reduce the problem to construction of a metric on Cantor set.

**Theorem 4.3** (Jarník). Let $X \subset \mathbb{R}$ be a perfect set and $f: X \to \mathbb{R}$ differentiable. Then there exists a differentiable extension $\tau: \mathbb{R} \to \mathbb{R}$ of $f$.

Set a purely attracting Cantor dynamical system $(X, f)$. Let us consider $\mathbb{U}^* = (\mathcal{U}_n)_{n \geq 1}$ and $(\tau_n)_{n \geq 1}$ obtained with Theorem 3.39 for this system. Let us also consider $(\mathbf{G}, \pi)$ the graph covering representation which corresponds to $\mathbb{U}^*$ for $(X, f)$.

For each $n \geq 1$, we define an acyclic graph $A_n$ obtained from $G_n = G(\mathcal{U}_n)$ by:

(i) removing edges that are pointing at a marker (a vertex $u$ of $G_n$ such that $\chi_n(u) = \ast$);

(ii) removing the vertices in circuits included in the attracted part of $(\mathcal{U}_n, \tau_n)$ and edges pointing at any of these vertices.

We also fix a total order on every set $\pi_n^{-1}(v)$ for $v \in V_n$ and $n \geq 1$, where $V_n$ is the vertex set of $G_n$ and $\pi_n: V_{n+1} \to V_n$ is associated bonding map.

4.1.1. **Contraction rates.** Fix $n \geq 1$ and a finite orbit $p$ in the attracted part of $\mathcal{U}_n$. By Lemma 3.10 for all $u \in \mathcal{P}_n$, there is a unique path in $G(\mathcal{U}_n)$ starting from $u$ and ending on an element of the circuit corresponding to $p$. We will denote by $\delta_n(u)$ the number of edges in this path.

The attracted part of $(\mathcal{U}_n, \tau_n)$ contains the attracted part of $(\mathcal{U}_{n-1}, \tau_{n-1})$. Indeed, let us remind that the attracted part of $(\mathcal{U}_n, \tau_n)$ is the union of the sets $\mathcal{U}_n$ for $|p| \leq \tau_n$. Since $\tau_n \geq \tau_{n-1} + 1$, it contains in particular the sets $\mathcal{U}_n$ for $|p| \leq \tau_{n-1}$. Since $\mathcal{U}_n$ refines $\mathcal{U}_{n-1}$, for $|p| \leq \tau_{n-1}$, $\mathcal{U}_n \subset \mathcal{U}_{n-1}$. This implies that the attracted part of $\mathcal{U}_n$ is contained in the attracted part of $\mathcal{U}_n$.

As a consequence every preimage by $\pi_{n-1}$ of a vertex in the circuit corresponding to a finite orbit $p$ in the attracted part of $(\mathcal{U}_{n-1}, \tau_{n-1})$ is contained in the attracted part of $(\mathcal{U}_n, \tau_n)$. Thus $\delta_n$ is defined for these vertices. We will denote by $\omega_n(p)$ the maximum of $\delta_n$ on $\mathcal{P}_n$.

For all $n \geq 1$ we define the shrinking rate $\lambda_n \in (0, 1)$ by:

$$\lambda_n = \frac{1}{2^n} \cdot \frac{1}{|V_{n+1}| + 1}$$

Additionally, we define an auxiliary length $\epsilon_n$ by putting $\epsilon_1 = 1$ and then inductively for all $n \geq 1$:

$$\epsilon_{n+1} = \frac{1}{4 \cdot 2^n |E_n|} \cdot \frac{\lambda_n |E_n|}{|V_{n+1}| + 1} \cdot \epsilon_n$$
4.1.2. From the graph representation to interval partitions. In the following, we will denote by \( \ell(I) \) the length of a compact interval \( I \) and by \( \rho(I) \) its middle point. Let us notice that the data of \( \ell(I) \) together with \( \rho(I) \) determines completely the interval \( I \).

We construct a function \( \iota: V_{G_0} \to \mathcal{I}^{\mathbb{N}} \), where \( \mathcal{I} \) is the set of compact intervals of \( \mathbb{R} \), by defining functions \( \iota_n: V_n \to \mathcal{I} \) such that \( \iota(v)_n = \iota_n(v_n) \) for all \( n \geq 1 \) and such that for each \( v \in V_{n+1} \) we have inclusion \( \iota_{n+1}(v) \subseteq \iota_n(\pi_n(v)) \). We define this sequence of functions recursively. In order to define \( \ell(\iota_n(v)) \) for all \( v \) and then the middle point \( \rho(\iota_n(v)) \).

If \( v \) is a vertex of \( G_n \) which is an initial vertex in the acyclic graph \( A_n \), we set \( \ell(\iota_n(v)) = \epsilon_n \), and determine \( \ell(\iota_n(v)) \) on all the other vertices \( v \) of \( G_n \) that are in \( A_n \) by imposing that for two vertices \( u, v \in V_n \) such that \( (u, v) \) is an edge in \( A_n \) we denote by \( S_v = \{ w : (w, v) \text{ is an edge of } A_n \} \) and

\[
\ell(\iota_n(v)) = \min_{w \in S_v} \lambda_n \cdot \ell(\iota_n(w)).
\]

This condition can be easily ensured by recursion. Roughly speaking, the length of the interval \( \iota_n(v) \) “shrinks” as one goes along any path in \( A_n \) with shrinking rate at least \( \lambda_n \). On every vertex \( v \) which is in a circuit \( c \) corresponding to a finite orbit \( p \) included in the attracted part of \( U_n \), we set \( \ell(\iota_n(v)) \) to be \( \lambda_n \cdot m_n(p) \), where \( m_n(p) \) is the minimum of the numbers \( \ell(\iota_n(u)) \) where the vertex \( u \) does not belong to the circuit \( c \) but there is \( w \) in \( c \) and edge \( (u, w) \) belongs to \( G_n \).

For \( n = 1 \), we can choose freely the middle points \( \rho(\iota_1(v)) \) for \( v \in V_1 \). We only have to ensure that any two intervals \( \iota_1(v) \) have empty intersection. When \( n \geq 2 \), in order to define the middle point of the interval \( \iota_n(v) \) for each vertex \( v \), we will determine its relative position in the set \( \iota_{n-1}(\pi_{n-1}(v)) \). It will to some extent rely on the ordering in the set \( \pi_{n-1}^{-1}(w) \) where \( w = \pi_{n-1}(v) \in V_{n-1} \).

Let us fix any \( w \in V_{n-1} \). We distinguish two cases:

1. **When \( w \) is outside of any circuit included in the attracted part of \( U_{n-1} \):**

   For \( k \) such that \( v \) is the \( k \)-th elements of \( \pi_{n-1}^{-1}(w) \), see Figure 6

   \[
   \rho(\iota(v)) = \left( \rho(\iota(w)) - \frac{\ell(\iota(w))}{2} \right) + k \cdot \frac{\ell(\iota(w))}{|\pi_{n-1}^{-1}(w)| + 1}.
   \]

   ![Figure 6](image.png)

   **Figure 6.** Illustration of the definition of \( \iota_n \) on preimages of some vertex in \( G_{n-1} \) when this vertex is not in a circuit corresponding to a finite orbit.

   The choice of \( \lambda_n \) and \( \epsilon_n \) ensures that these intervals are disjoint and included in \( \iota_{n-1}(w) \). Indeed the length of each of the intervals \( \iota_n(v) \) for \( v \in \pi_{n-1}^{-1}(w) \) is of the form \( \lambda_n^l \cdot \epsilon_n \) with \( l \geq 0 \) and thus is smaller than...
When \( 1 \) is in the intervals \( I_k \subset \omega \), in other words, this means that for \( v \in \pi^{-1}_{n-1}(w) \)

\[
\ell(v) \leq \frac{1}{2} \cdot \frac{\ell(t_{n-1}(w))}{|V_n| + 1} < \frac{1}{2} \cdot \frac{\ell(t_{n-1}(w))}{|\pi^{-1}_{n-1}(w)| + 1}.
\]

This implies that these intervals are disjoint and included in \( \tau_{n-1}(w) \).

(2) **When \( w \) is in a circuit included in the attracted part of \( \mathcal{U}_{n-1} \):**

Let us denote by \( p \) the finite orbit corresponding to this circuit. Let us consider the intervals \( I^{(n)}_k(w), 0 \leq k \leq \omega_n(p) \) such that for \( k < \omega_n(p) \):

\[
\rho(I^{(n)}_k(w)) = \rho(t_{n-1}(w)) - \frac{\ell(t_{n-1}(w))}{2} + \frac{1}{2} \cdot \frac{\ell(t_{n-1}(w))}{2}.
\]

In other words, this means that for \( k < \omega_n(p) \) the distance between the center of the interval \( I^{(n)}_k(w) \) and the leftmost point of the interval \( t_{n-1}(w) \) is \( \frac{1}{2} \cdot \frac{\ell(t_{n-1}(w))}{2} \).

When \( k = \omega_n(p) \):

\[
\rho(I^{(n)}_k(w)) = \rho(t_{n-1}(w)) - \frac{\ell(t_{n-1}(w))}{2} + \frac{1}{8} \cdot \frac{\ell(t_{n-1}(w))}{2}.
\]

Moreover for all \( k \leq \omega_n(p) \), \( \ell(I^{(n)}_k(w)) = \frac{1}{2} \cdot \frac{\ell(t_{n-1}(w))}{2} \). These intervals are used as "containers" for the intervals \( \tau_n(v) \) for \( v \in \pi^{-1}_{n-1}(w) \).

In each of the intervals \( I^{(n)}_k(w) \) we place the intervals \( \tau_n(v) \) for \( v \) preimages of \( w \) which are distance \( k \) from the circuit, as follows. For \( j \) such that \( v \) is the \( j \)th of elements of the set \( S_k(w) = \{ v \in \pi^{-1}_{n-1}(w) : \delta_n(v) = \omega_n(p) - k \} \):

\[
\rho(\tau(v)) = \left( \rho(I^{(n)}_k(w)) - \frac{\ell(I^{(n)}_k(w))}{2} \right) + j \cdot \frac{\ell(I^{(n)}_k(w))}{|S_k(w)| + 1}.
\]

See an illustration on Figure 7.

**Figure 7.** Illustration of the definition of \( \tau_n(v) \) for \( v \) a preimage by \( \pi_{n-1} \) of \( w \) which is in a circuit of the attracted part of \( \mathcal{U}_{n-1} \).

For similar reasons as in the first case, the intervals \( I^{(n)}_k(w) \) are pairwise disjoint and included in \( \tau_{n-1}(w) \), and the intervals \( \tau_n(v) \) for \( v \in \pi^{-1}_{n-1}(w) \) such that \( \delta_n(w) = \omega_n(p) - k \) are disjoint and included in \( I^{(n)}_k(w) \).
4.1.3. Embedding with vanishing derivative. Since for all $n \geq 1$ and $v \in V_n$,

$$\ell(\nu_n(v)) \leq \epsilon_n \leq \frac{1}{4^n},$$

and that for all $v$, $\nu_n(v) \subset \nu_{n-1}(\pi_{n-1}(v))$, for all $v \in V_{G,\pi}$, the intersection

$$\bigcap_{n \geq 1} \nu_n(v_n)$$

is reduced to a point in $\mathbb{R}$. Let us denote this point by $\psi(x)$, where

$x$ is equal to $\varphi_{U,f}^{-1}(v)$. By construction, $\psi$ is continuous and injective and thus a homeomorphism onto its image. Let us denote $Z = \psi(X)$. As a consequence $\psi$ conjugates $f$ with a map $\sigma : Z \to Z$.

We are going to prove that for all $z \in Z$, $\sigma'(z)$ is defined and equal to 0. Then a direct application of Theorem 4.3 will end the proof of the theorem.

Let us set $z = \psi(x)$. We will prove that for all $n \geq 1$ and $v \in V_{G,\pi}$, for $v'$ sufficiently close to $v$, we have:

$$\left| \psi(f \circ \varphi_{U,f}^{-1}(v)) - \psi(f \circ \varphi_{U,f}^{-1}(v')) \right| \leq \frac{1}{2^n} \left| \psi(\varphi_{U,f}^{-1}(v)) - \psi(\varphi_{U,f}^{-1}(v')) \right|$$

which implies that

$$\sigma'(z) = \sigma'(\psi(x)) = \sigma'(\psi(\varphi_{U,f}^{-1}(v))) = 0.$$ 

Let us consider some $v \in V_{G,\pi}$ and $n \geq 1$. By construction there exists $m > n + 3$ such that for all $k \geq m$, $\chi_k(v_k) \neq *$. Since divergent vertices coincide with markers, for all $k \geq m$, $v_k$ is not divergent. Let us prove the above inequality when $v'$ coincides with $v$ on the $m$ first elements. Let us consider such sequence $v'$ and denote by $l$ the smallest among integers $k > m$ such that $v_k \neq v'_k$. From this point we distinguish two cases:

1. The point $v$ is not periodic for the map $f_{U,f}$: Thus we can assume without loss of generality that $v_{{l-1}} = v'_{{l-1}}$ is not in a circuit included in the attracted part of $(U_{l-1}, \tau_{l-1})$. As a consequence, and since $v_{{l-1}}$ is not divergent, $\psi(f \circ \varphi_{U,f}^{-1}(v))$ and $\psi(f \circ \varphi_{U,f}^{-1}(v'))$ both lie in the same interval and this interval has (by definition of $\epsilon_{l-1}$ length bounded from above by $\lambda_{l-1} \cdot \ell(\nu_{l-1}(v_{{l-1}}))$). As a consequence:

$$\left| \psi(f \circ \varphi_{U,f}^{-1}(v)) - \psi(f \circ \varphi_{U,f}^{-1}(v')) \right| \leq \lambda_{l-1} \cdot \ell(\nu_{l-1}(v_{{l-1}})).$$

By definition of $\lambda_{l-1}$ and $\rho$ we thus have:

$$\lambda_{l-1} \cdot \ell(\nu_{l-1}(v_{{l-1}})) \leq \frac{1}{2^{l-1}} \cdot \frac{\ell(\nu_{l-1}(v_{{l-1}}))}{|V_l| + 1} \leq \frac{1}{2^{l-1}} \cdot |\rho(\nu(\nu_l)) - \rho(\nu(\nu'_l))|$$

As a direct consequence:

$$\left| \psi(f \circ \varphi_{U,f}^{-1}(v)) - \psi(f \circ \varphi_{U,f}^{-1}(v')) \right| \leq \frac{1}{2^{l-1}} \cdot |\rho(\nu(\nu_l)) - \rho(\nu(\nu'_l))|.$$ 

Since $\psi(\varphi_{U,f}^{-1}(v)) \in \nu(\nu_l)$ and $\psi(\varphi_{U,f}^{-1}(v')) \in \nu(\nu'_l)$ these points are at distance less than $\frac{1}{2} \ell(\nu(\nu_l)) = \frac{1}{2} \ell(\nu(\nu'_l))$ from the respective points $\rho(\nu(\nu_l))$ and $\rho(\nu(\nu'_l))$. Recall that by the definition we have

$$|\rho(\nu(\nu_l)) - \rho(\nu(\nu'_l))| \geq \frac{\ell(\nu_{l-1}(v_{{l-1}}))}{|V_l| + 1} \geq 2\ell(\nu_l))$$

and therefore

$$\left| \psi(\varphi_{U,f}^{-1}(v)) - \psi(\varphi_{U,f}^{-1}(v')) \right| \geq \frac{1}{2} |\rho(\nu(\nu_l)) - \rho(\nu(\nu'_l))| = \frac{1}{2} \ell(\nu_l))$$

and this implies that

$$\sigma'(z) = \sigma'(\psi(x)) = \sigma'(\psi(\varphi_{U,f}^{-1}(v))) = 0.$$ 

This gives
\[
\left| \psi(f \circ \varphi_{U,f}^{-1}(v)) - \psi(f \circ \varphi_{U,f}^{-1}(v')) \right| \leq \frac{1}{2^l-2} \left| \psi(\varphi_{U,f}^{-1}(v)) - \psi(\varphi_{U,f}^{-1}(v')) \right|
\]
completing the proof of this case, since \( l > m > n \), in particular \( l - 2 \geq n \), and so (4.2) holds in this case.

(2) The point \( v \) is periodic for the map \( f_{U,f} \): In this case we can assume without loss of generality that \( v_{l-1} = v'_{l-1} \) is in a circuit corresponding to a finite orbit \( p \) included in the attracted part of \( (U_{l-1}, \tau_{l-1}) \), and so is \( v_l \).

As a consequence \( v'_l \) is not in this circuit (otherwise we would have \( v_l = v'_l \) which contradicts the choice of \( l \)).

This implies that the point \( \psi(\varphi_{U,f}^{-1}(v')) \) lies in some interval \( f_{k}^{(l)}(v_{l-1}) \) constructed in Section 4.1.2 for \( w = v_{l-1} \) and the integer \( l \), for \( k < \omega_{l}(p) \), where \( p \) is the circuit to which \( v_l \) belongs. The point \( \psi(\varphi_{U,f}^{-1}(v)) \) is the left extreme point of the interval \( f_{k}^{(l)}(v_{l-1}) \). As a consequence the distance between the two points \( \psi(\varphi_{U,f}^{-1}(v)) \) and \( \psi(\varphi_{U,f}^{-1}(v')) \) is larger than:
\[
\left( \frac{1}{2^{l-1}} - \frac{1}{4 \cdot 2^{\omega_{l}(p)}} - \frac{1}{8 \cdot 2^{\omega_{l}(p)}} \right) \ell(t_{l-1}(v_{l-1})) \geq \frac{1}{8} \cdot \frac{1}{2^{l-1}} \ell(t_{l-1}(v_{l-1})).
\]

With a similar reasoning the distance between \( \psi(f \circ \varphi_{U,f}^{-1}(v)) \) and \( \psi(f \circ \varphi_{U,f}^{-1}(v')) \) is smaller than:
\[
\left( \frac{1}{2^{l-1}} + \frac{1}{4 \cdot 2^{\omega_{l}(p)}} + \frac{1}{8 \cdot 2^{\omega_{l}(p)}} \right) \ell(t_{l-1}(v_{l-1})) \leq \frac{5}{8} \cdot \frac{1}{2^{l-1}} \ell(t_{l-1}(v_{l-1})).
\]

By combining the equations we have that:
\[
\left| \psi(f \circ \varphi_{U,f}^{-1}(v)) - \psi(f \circ \varphi_{U,f}^{-1}(v')) \right| \leq \frac{5}{2^{l-1}} \cdot \left| \psi(\varphi_{U,f}^{-1}(v)) - \psi(\varphi_{U,f}^{-1}(v')) \right|.
\]

As a consequence, since \( l > n + 3 \):
\[
\left| \psi(f \circ \varphi_{U,f}^{-1}(v)) - \psi(f \circ \varphi_{U,f}^{-1}(v')) \right| \leq \frac{1}{2^{l-1}} \cdot \left| \psi(\varphi_{U,f}^{-1}(v)) - \psi(\varphi_{U,f}^{-1}(v')) \right|
\]
which is the equation (4.2).

This concludes the proof of Theorem 1.2.

**Remark 4.4.** In the first case of the above enumeration, we used critically the fact that divergent points coincide with markers. Without this, \( \psi(f \circ \varphi_{U,f}^{-1}(v)) \) and \( \psi(f \circ \varphi_{U,f}^{-1}(v')) \) could actually belong to two different intervals. In this case we would not have a sufficiently small upper bound on their distance.

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