Generation, Characterization and Manipulation of Quantum Correlations in Electron Beams

Shahaf Asban\textsuperscript{1} and Javier García de Abajo\textsuperscript{1,2}

\textsuperscript{1}ICFO - Institut de Ciencies Fotoniques, The Barcelona Institute of Science and Technology, 08860 Castelldefels (Barcelona), Spain
\textsuperscript{2}ICREA - Institució Catalana de Recerca i Estudis Avançats, Passeig Lluís Companys 23, 08010 Barcelona, Spain

Entanglement engineering plays a central role in quantum-enhanced technologies, with potential physical platforms that outperform their classical counterparts. However, free electrons remain largely unexplored despite their great capacity to encode and manipulate quantum information, due in part the lack of a suitable theoretical framework. Here we link theoretical concepts from quantum information to available free-electron sources. Specifically, we consider the interactions among electrons propagating near the surface of a polariton-supporting medium, and study the entanglement induced by pair-wise coupling. These correlations depend on controlled interaction interval and the initial electron bandwidth. We show that long interaction times of broadband electrons extend their temporal coherence. This in turn is revealed through a widened Hong-Ou-Mandel peak, and associated with an increased entanglement entropy. We then introduce a discrete basis of electronic temporal-modes, and discriminate between them via coincidence detection with a shaped probe. This paves the way for ultrafast quantum information transfer by means of free electrons, rendering the large alphabet that they span in the time domain accessible.

INTRODUCTION

Quantum degrees of freedom occupy a large parameter space compared with their classical counterparts. This property renders them challenging for simulation on classical computers. Nonetheless, it also endows them with a vast information capacity, useful for novel computational and metrologic paradigms \cite{1,2,3}. Entangled photon pairs have long been the work-horse of quantum enhancement demonstrations in the optical arena, with applications in metrology \cite{4,5}, imaging \cite{6,7,8,9,10} and spectroscopy \cite{11,12,13}. A key concept in generation of such useful states, is initiation of well-monitored interactions between continuous variables. The latter exhibit rich entanglement spectra and large state-space on which information can be recorded and accessed \cite{15,16,17,18,19}. These concepts have not yet been addressed in the well-established field of free-electron based techniques, such as spectroscopy and microscopy \cite{20}. Designing controlled entanglement of free-electron sources constitutes the main challenge, and this is precisely what we address here.

Extraordinary electron-beam-shaping capabilities have been recently demonstrated in electron microscopes combining light ultrafast optics elements \cite{21,22,23}. Revolutionary concepts such as free-electron qubits \cite{24} and cavity-induced quantum control \cite{25,26,27} are becoming available, pointing towards the emergence of next-generation quantum light-electron technologies. While photons maintain coherence over large distances, electrons decohere rapidly due to their strong environmental coupling. Combined with the control schemes above, this suggests electrons are valuable quantum probes. Lately, we have shown that electrons passing by polariton-supporting media, can experience geometrically controlled attractive interaction \cite{28}. This effect is closely related to Amperean pairing of electrons discussed in \cite{29,30,31}, which in turn induces an entangled EPR state in the long interaction time limit \cite{28}.

Here, we study the quantum correlations generated by abrupt interactions of electron pairs with a neighboring medium, as depicted in Fig. 1 for a controlled time interval $T_I$. We explore the transient state generated by abrupt interactions, as well as the steady state limit in the perturbative regime. By varying two control parameters - interaction time $T_I$ and initial electron bandwidth $\sigma_e$ - we effectively scan the degree of entanglement. The entanglement in the longitudinal dimension is characterized by the Schmidt decomposition of the wave-function. We then calculate the coincidence probability and display it versus the degree of entanglement. We denote the resulting eigenstates electronic temporal...
modes (ETMs) in analogy to their photonic counterparts. Finally, we propose a technique that is useful for real-time discrimination between ETMs, essential for state tomography and related quantum information processing applications.

Figure 1. **Physical platform for free-electron pair correlations.** (a) An uncorrelated electron pair $|\Psi_0\rangle$ propagates parallel to the planar surface of a polariton-supporting film of length $L$ along the propagation direction, transverse width $l_x \gg L$ and thickness $d = 1$ nm. (b) Initial distribution of the longitudinal component, centered around $k_0$ with $\sigma_x^2$ spread. (c) Spatial orientation of the variance in the transverse momentum spread $\sigma_{x,y}^2$. After an interaction time $T_I$, a correlated pair $|\Psi(T_I)\rangle$ is obtained.

**RESULTS**

**The pair-amplitude**

The electron-pair amplitude is obtained from the underlying electron-polariton coupling. We consider free electrons traveling with mean momentum $k_0$, as depicted in Fig. 1. The full Hamiltonian is given by three contributions: $H = H_e + H_\phi + H_{e-\phi}$. The electrons kinetic term is described by $H_e$, the electromagnetic field degrees of freedom combined with the surface polaritons are contained in $H_\phi$ and the electron-field coupling is $H_{e-\phi}$ (see Methods and Sec. S1 of SI). The electrons illustrated in Fig. 1 are assumed to be prepared in an initial state with statistical independence, reflected by the product state $|\Psi_0\rangle = \prod_{i=1}^2 \sum_{k_i} \alpha_{s_i}^{(i)} (k_i) |k_i, s_i\rangle$. Here $|k_i, s_i\rangle$ represents an electron state of momentum $k_i$ and spin $s_i$. The single-electron amplitude $\alpha_{s_i}^{(i)} (k_i)$ is determined by the preparation process, and we assume it to be a Gaussian centered around $k_0$ along the propagation axis in our calculations. As the electrons pass in vicinity to the film, they exchange energy with the medium. The interaction mediated by the polaritons decays exponentially with the distance from the medium, validating the use of perturbative approach. Expanding the evolution in the interaction picture to second order, we obtain the electron-pair wave-function in its generic form

$$|\Psi^{(2)}\rangle_\lambda = \sum_{k_1,k_2} \Phi_\lambda^{k_1,k_2} (k_1,k_2) |k_1,s_1; k_2,s_2\rangle,$$

where $\lambda$ labels a set of control parameters. In the present configuration, $\lambda$ parametrizes the dimensionless interaction time $T_I$ and the initial electron bandwidth $\sigma_e$. We are interested in the dynamics of the longitudinal component of the electron pair. By tracing the transverse momenta, we obtain an expression for $\Phi_\lambda^{k_1,k_2} (k_1,k_2)$, which we denote as the pair amplitude (see Eq. 4 in the Methods section). The pair amplitude exhibits continuous variable entanglement throughout most of the explored parameter space.

Entanglement spectrum and ETMs

It is useful to explore the parameter space of the pair amplitude by performing a Schmidt decomposition. The Schmidt-Mercer theorem allows us to express an inseparable state as a superposition of separable ones,

$$\Phi_\lambda^{k_1,k_2} (k_1,k_2) = \sum_n \sqrt{p_n} \psi_n (k_1) \phi_n (k_2),$$

where spin labels are omitted for brevity. The longitudinal eigenstates $\{\psi_n, \phi_n\}$ appear in pairs of ETMs. If the state $\psi_n$ is detected, its counterpart occupies the state $\phi_n$ with absolute certainty. The eigenvalues $p_n$ reflect the probability of detecting the $n$th mode.

The joint momentum-representation of the pair amplitude is displayed in Fig. 2, for selected values of the control parameters (i.e., a dimensionless interaction time $T_I$ and the electron bandwidth $\sigma_e$). The dimensionless interaction time

Figure 2. **Entanglement characterization.** (a) The bare pair amplitude $\Phi_{12}^{(1)}(k_1, k_2)$ is presented for selected control-parameter values. The amplitudes labeled $\Phi_i$ are calculated at the dimensionless interaction times $T_i = (10^{-2}, 10^{-3}, 10^{-5}, 10^{-5}, 10^{-5})$, with bandwidths $\sigma_c = \frac{2\pi}{\lambda_c} (2, 2, 2, 1/2, 1/20)$, respectively. (b) Collision entropy versus $T_i$ and $\sigma_c$. (c) Variation of the Schmidt number $\kappa$ along the dotted curve displayed in panel (b), exposing the short time mode meshing of narrow-band electrons. (d) Schmidt spectrum of the amplitude displayed in panel (e) ($\kappa \approx 6$). The corresponding eigenstates are displayed in the inset with matching colors. (e) Pair amplitude in joint momentum space, where we display the first (lowest-order) three modes.

**Coincidence detection**

A common approach to probe quantum correlations is by measuring the coincidence probability $P_{12}(\delta l) = \int dt d\tau \langle \Psi_1(t) \Psi_2(t+\tau) \Psi_1(t) \rangle$, assuming an experimental setup as sketched in Fig. 3. We consider balanced beam splitters (BSs) and obtain

$$P_{12}(\delta l) = 1 + \frac{1}{2} \sum_{n,m} \sqrt{p_np_m} |I_{nm}(\delta l)|^2,$$

where $n,m$ label ETM and $I_{nm}(\delta l) = \int dk e^{\phi_n^*(k) \phi_m(k)} e^{-\frac{\pi}{2} k^2 \delta l}$ (see Sec. S3 of the SI). Figure 3d displays $P_{12}(\delta l)$ as a function of the BS displacement, arranged in growing degree of entanglement. The probability ranges from $1/2$ (completely random) to unity (utterly antibunched) as expected from fermionic Hong-Ou-Mandel interference, usually revealed by a Pauli dip in matter systems [38–40]. In the left panel of Fig. 3d we scan $T_i$ while fixing $\sigma_c = 4\pi/\lambda_p$ in the broadband range. Interestingly, we find that the for higher degree of entanglement the probability peak extends over a wider range of $\delta l$. This can
be attributed to temporal expansion of the electron wave-function due to long interaction times. In the inset we see that $\kappa$ grows with $T_I$ in a piecewise linear manner. On the right panel $\sigma_e$ is varied while $T_I$ is fixed in the long-interaction range and a similar behavior is found. We find that $\kappa \propto \sigma_e^2$ and consider it a direct result of the initial Gaussian wave-packet, together with the emergent linear relations of $\kappa$ and the interaction time.

In Fig. 3a, the coincidence detection of the instantaneous incoming ETM with a known probe mode labeled $\phi_p$ is presented. The first three ETMs are extracted from the Schmidt decomposition of the amplitude displayed in Fig. 2. These modes are the eigenstates of the reduced single-electron density matrix, therefore in each realization one such mode is detected with probability $p_n$. When the incoming mode matches the shaped probe mode, the coincidence signal exhibits a peak for a vanishing path difference, as depicted in Fig. 3b. By counting the appearance rate of each mode separately we can deduce the probability vector $p_n$, and thus characterize the quantum state. Beyond state tomography this could also be used in coincidence with parallel operations on its ETM-twin, realizing more sophisticated information processing protocols.

Figure 3. **Coincidence detection.** (a) Incoming electron pairs are separated by an electron beam splitter (BS), and subsequently combined by another BS with controllably scanned position, providing a relative path difference $\delta l$. The two output ports $D_1$ and $D_2$ are measured in coincidence. (b) Coincidence probability $P_{12}$ for varying path difference $\delta l/\lambda_p$ and Schmidt number (degree of entanglement). The left panel corresponds to varying interaction time for fixed $\sigma_e = 4\pi/\lambda_p$. In the right panel the dimensionless time is fixed to $T_I = 5 \times 10^{-3}$ while the initial bandwidth is scanned. The insets show the relations between the control parameters and the Schmidt number.

Figure 4. **ETM discrimination.** (a) An incoming electron pair prepared in a superposition of ETMs is separated by a first BS, then combined with a (shaped) probe mode $\phi_p$ and finally measured in coincidence. (b) Coincidence outcomes of the probe with three possible incoming modes $n, p \in \{1, 2, 3\}$, as a function of path difference $\delta l$. The interference pattern displays increased response for identical probe and incoming ETM.

**DISCUSSION**

Near-fields evolving at the surface of polariton supporting materials provide a novel approach to generate and shape quantum correlations in charged particles, and in particular in free electrons. While such pairing mechanisms are suppressed in matter due to ambient noise (e.g., thermal), electrons structured in a beam undergo significantly less scattering events, thus enabling coherent interactions to persist over longer space-time intervals. We have shown
that electron pairs near polariton-supporting material boundaries undergo nontrivial coupling that generates entanglement. Such correlations are mathematically expressed by the apparent inseparability of the pair amplitude in Eq. 1, giving rise to the results displayed in Fig. 2. The Schmidt decomposition allows us to express the pair amplitude using a set of factorized states, providing useful measures for bipartite entanglement [6, 17, 41, 47]. This framework reveals simple relations between the control parameters and the resulting evolution of quantum correlations of the above setup. Such properties are desirable for entanglement engineering.

The large Hilbert-space dimensionality occupied by the ETMs, renders them as appealing ultrafast quantum information carriers. This has potential applications in quantum-enhanced electron metrology, as proposed using optical setups [14, 48]. For example, measuring the momentum of one of the electrons in the pair and the position of the other, one may obtain superresolved imaging. Such class of quantum enhancements benefits from the fact that single-particle (local) observables are not Fourier conjugates of the (extended) composite state. This work raises multiple open questions concerning vicarious temperature effects, the imprint of the medium topological entanglement spectrum, entanglement entropy, the imprint of the medium topological matter-electron quantum correlations. These are just a few examples of the emerging field of microscopic quantum electrodynamics [34, 35]. The electron-field coupling is expressed in terms of the Green tensor, encapsulating the geometric and spectral properties of the medium. Proceeding to calculate the first nontrivial order (second), we obtain the general form of Eq. 1. We consider Gaussian initial states of mean distance $y_0 = 5 \text{ nm}$ from the thin film and $\sigma_y = 0.5 \text{ nm}$ (see Fig. 1). Taking the long $l_z$ limit and choosing $\sigma_x = \frac{1}{X_p}$, we trace the transverse components and obtain

$$\Phi_{x_1, x_2}^\lambda (k_1, k_2) = N^{-1/2} \int dq \, \text{sinc} \left\{ \frac{hq}{m} (k_1 - k_2) T \right\} \alpha^{(1)}_{x_1} (k_1 - q) \chi (q) \alpha^{(2)}_{x_2} (k_2 + q).$$

Here, $N$ is a normalization constant, $T$ is the interaction time, and $\chi (q)$ is obtained by tracing the lateral wave vector $q_{\parallel} = (q_x, q_y)$ in the interaction picture (Sec. S1 of SI). Additionally, we have invoked the nonrecoil approximation for small momentum exchanges relative to $k_0$, resulting in a linear electron-energy exchange $\epsilon_{k + q} - \epsilon_k \approx hq \cdot v$, where $q$ is the polariton wave vector and $v$ is the electron velocity.

**METHODS**

*Pair-amplitude derivation*

The pair amplitude is obtained perturbatively in the interaction picture (Sec. S1 of the SI). The full Hamiltonian of the system contains three contributions: $H = H_e + H_\phi + H_{e-\phi}$. The electrons kinetic term is given by $H_e = \sum_{k, s} \epsilon_k c_{k, s}^\dagger c_{k, s}$, where the operator $c_{k, s}$ ($c_{k, s}^\dagger$) creates (annihilates) an electronic mode with momentum $k$ and spin $s$ obeying the anticommutation relations $\{c_{k, s}, c_{k', s'}^\dagger\} = \delta_{kk'} \delta_{ss'}$. The term $H_\phi$ describes the electromagnetic-field degrees of freedom combined with the surface polaritons in the framework of macroscopic quantum electrodynamics [34, 35]. The electron-field coupling is expressed using the Hamiltonian $[25, 35, 41]$

$$H_{e-\phi} = \frac{\epsilon_C \lambda}{2\pi} \sum_{k, q, s} c_{k+q, s}^\dagger c_{k, s} k \cdot A (q),$$

where $\lambda_C = \hbar/m_e c$ is the Compton wavelength of the electron, while $e$ and $m_e$ are its charge and mass, respectively. We have employed the Weyl-gauge, setting the scalar potential to zero and introduced $A (q)$, the vector field operator in momentum space. The vector-field in macroscopic quantum electrodynamics is expressed in terms of the Green tensor, encapsulating the geometric and spectral properties of the medium. Proceeding to calculate the first nontrivial order (second), we obtain the general form of Eq. 1. We consider Gaussian initial states of mean distance $y_0 = 5 \text{ nm}$ from the thin film and $\sigma_y = 0.5 \text{ nm}$ (see Fig. 1). Taking the long $l_z$ limit and choosing $\sigma_x = \frac{1}{X_p}$, we trace the transverse components and obtain

$$\Phi_{x_1, x_2}^\lambda (k_1, k_2) = N^{-1/2} \int dq \, \text{sinc} \left\{ \frac{hq}{m} (k_1 - k_2) T \right\} \alpha^{(1)}_{x_1} (k_1 - q) \chi (q) \alpha^{(2)}_{x_2} (k_2 + q).$$

Here, $N$ is a normalization constant, $T$ is the interaction time, and $\chi (q)$ is obtained by tracing the lateral wave vector $q_{\parallel} = (q_x, q_y)$ in the interaction picture (Sec. S1 of SI). Additionally, we have invoked the nonrecoil approximation for small momentum exchanges relative to $k_0$, resulting in a linear electron-energy exchange $\epsilon_{k + q} - \epsilon_k \approx hq \cdot v$, where $q$ is the polariton wave vector and $v$ is the electron velocity.

**ETMs calculation**

To find the set of ETMs $\{\psi_n, \phi_n\}$ and their weights $p_n$, we solve the integral eigenvalue equations $p_n \psi_n (k) = \int dk' K_1 (k, k') \psi_n (k')$ and $p_n \phi_n (k) = \int dk' K_2 (k, k') \phi_n (k')$ (Sec. S2 of SI). The kernels, which are found from the reductions

$$K_1 (k, k') = \int dk_1 \Phi^\lambda_{x_1, x_2} (k_1, k_2) \Phi^\lambda_{x_1, x_2} (k', k_2)$$

and

$$K_2 (k, k') = \int dk_1 \Phi^\lambda_{x_1, x_2} (k_1, k) \Phi^\lambda_{x_1, x_2} (k_1, k'),$$
can be interpreted as single-electron correlation functions. To obtain the Schmidt spectrum and characterize the degree of entanglement, we discretize the kernels and numerically solve the integral eigenvalue equations. We have used a $800 \times 800$ discretization of the kernel and repeated the procedure for each control parameter separately. The pair amplitude used for the generation of the kernels involves integration over the polariton degrees of freedom. We have done this numerically for each set of control parameters $\lambda$ using straightforward numerical integration of Eq. $\mathbf{8}$ on a uniform grid. The step size was varied to satisfy the convergence of the Schmidt number. The convergence criterion adopted in this scheme is $\max \{2 (k_{N+1} - k_N) / (k_{N+1} + k_N) \} \leq 0.05$, where $N$ is the number of data points within a constant range in the given kernel size.

**DATA AVAILABILITY**

The main results of this manuscript are composed of analytical and numerical calculations. All data generated, analyzed or required to reproduce the results of this study are included in this article and its Supplementary Information file.

[1] Carl W Helstrom. Quantum Detection and Estimation Theory. *J Stat Phys.*, 1(2):231–252, 1969.
[2] Carl W Helstrom. Resolution of Point Sources of Light as Analyzed by Quantum Detection Theory. *IEEE Trans. Inf. Theory*, I, 1973.
[3] Michael A Nielsen and Isaac L Chuang. Quantum Computation and Quantum Information: 10th Anniversary Edition. Cambridge University Press, 2010.
[4] Vittorio Giovannetti, Seth Lloyd, and Lorenzo Maccone. Advances in quantum metrology. *Nature Photonics*, 5(4):222–229, 2011.
[5] Jonathan P. Dowling. Quantum optical metrology—the lowdown on high-n00n states. *Contemporary Physics*, 49(2):125–143, 2008.
[6] Vladislav N Beskrovny and Mikhail I Kolobov. Quantum limits of super-resolution in reconstruction of optical objects. *Phys. Rev. A*, (December 2004):1–10, 2005.
[7] Mikhail I Kolobov. The spatial behavior of nonclassical light. *Reviews of Modern Physics*, 71(5):1539–1589, 1999.
[8] G. Brida, M. Genovese, and I. Ruo Berchera. Experimental realization of sub-shot-noise quantum imaging. *Nature Photonics*, 4(4):227–230, 2010.
[9] Lee A. Rozema, James D. Bateman, Dylan H. Mahler, Ryo Okamoto, Amir Feizpour, Alex Hayat, and Aephraim M. Steinberg. Scalable spatial superresolution using entangled photons. *Phys. Rev. Lett.*, 112:223602, Jun 2014.
[10] Yonatan Israel, Ron Tenne, Dan Oron, and Yaron Silberberg. Quantum correlation enhanced super-resolution localization microscopy enabled by a fibre bundle camera. *Nature Communications*, 8(1):14786, 2017.
[11] Shahaf Asban, Konstantin E. Dorfman, and Shaul Mukamel. Quantum phase-sensitive diffraction and imaging using entangled photons. *Proceedings of the National Academy of Sciences*, 116(24):11673–11678, 2019.
[12] Shaul Mukamel, Matthias Freyberger, Wolfgang P. Schleich, Marco Bellini, Alessandro Zavatta, Gerd Leuchs, Christine Silberhorn, Robert W Boyd, Luis Sanchez Soto, Andre Stefanov, Marco Barbieri, Anna Paterova, Leonid A Krivitskiy, Sharon Shwartz, Kenji Tamasaku, Konstantin Dorfman, Frank Schlamin, Vahid Sandoghdar, Michael G Raymer, Andrew H Marcus, Oleg Varavanski, Theodore Goodson III, Zhiyuan Zhou, Bao-Sen Shi, Shahaf Asban, Marlan O Scully, Girish S Agarwal, Tao Peng, Alejx V Sokolov, Zhedong Zhang, Ivan A Vartanians, Elena del Valle, and Fabrice P Laussy. Roadmap on quantum light spectroscopy. *Journal of Physics B: Atomic, Molecular and Optical Physics*, 2020.
[13] Konstantin E. Dorfman, Frank Schlamin, and Shaul Mukamel. Nonlinear optical signals and spectroscopy with quantum light. *Rev. Mod. Phys.*, 88:045008, Dec 2016.
[14] Frank Schlamin, Konstantin E. Dorfman, and Shaul Mukamel. Entangled two-photon absorption spectroscopy. *Accounts of Chemical Research*, 51(9):2207–2214, 2018. PMID: 30179458.
[15] C K Hong and L Mandel. Theory of parametric frequency down conversion of light. *Phys. Rev. A*, 31(4):2409–2418, apr 1985.
[16] C. K. Law, I. A. Walmsley, and J. H. Eberly. Continuous frequency entanglement: Effective finite hilbert space and entropy control. *Phys. Rev. Lett.*, 84:5304–5307, Jun 2000.
[17] C. K. Law and J. H. Eberly. Analysis and interpretation of high transverse entanglement in optical parametric down conversion. *Physical Review Letters*, 92(12):1–4, 2004.
[18] Alois Mair, Alipasha Vaziri, Gregor Weihs, and Anton Zeilinger. Entanglement of the orbital angular momentum states of photons.pdf. *Nature*, 412(6844):313, 2001.
[19] Robert Fickler, Mario Krenn, Radek Lapkiewicz, Sven Ramelow, and Anton Zeilinger. Real-time imaging of quantum entanglement. *Scientific Reports*, 3(Figure 1):1–5, 2013.

[20] F. J. García de Abajo. Optical excitations in electron microscopy. *Rev. Mod. Phys.*, 82:209–275, Feb 2010.

[21] G. M. Vanacore, I. Madan, G. Berruto, K. Wang, E. Pomarico, R. J. Lamb, D. McGrouther, I. Kaminer, B. Barwick, F. Javier García de Abajo, and F. Carbone. Attosecond coherent control of free-electron wave functions using semi-infinite light fields. *Nature Communications*, 9(1):2694, Jul 2018.

[22] G. M. Vanacore, G. Berruto, I. Madan, E. Pomarico, P. Biagoni, R. J. Lamb, D. McGrouther, O. Reinhardt, I. Kaminer, B. Barwick, H. Larocque, V. Grillo, E. Karimi, F. J. García de Abajo, and F. Carbone. Ultrafast generation and control of an electron vortex beam via chiral plasmonic near fields. *Nature Materials*, 18(6):573–579, 2019.

[23] I. Madan, G. M. Vanacore, E. Pomarico, G. Berruto, R. J. Lamb, D. McGrouther, T. T. A. Lummen, T. Latychevskaia, F. J. García de Abajo, and F. Carbone. Holographic imaging of electromagnetic fields via electron-light cíadeAbajo, and F. Carbone. *Holographic imaging of quantum entanglement*. *Science Advances*, 5(5), 2019.

[24] Ori Reinhardt, Chen Mechel, Morgan Lynch, and Ido Kaminer. Free-electron qubits, 2019.

[25] Ofer Kfir. Entanglements of electrons and cavity photons in the strong-coupling regime. *Phys. Rev. Lett.*, 123:103602, Sep 2019.

[26] Kangpeng Wang, Raphael Dahan, Michael Shentcis, Yaron Kauffmann, Adi Ben Hayun, Ori Reinhardt, Shai Tsesses, and Ido Kaminer. Coherent interaction between free electrons and a photonic cavity. *Nature*, 582(7810):50–54, Jun 2020.

[27] Ofer Kfir, Hugo Lourenço-Martins, Gero Storeck, Murat Sivis, Tyler R. Harvey, Tobias J. Kippenberg, Armin Feist, and Claus Ropers. Controlling free electrons with optical whispering-gallery modes. *Nature*, 582(7810):46–49, Jun 2020.

[28] S. Asban and F. Javier García de Abajo. Polariton-induced geometrically structured interactions of free-electron. *Submitted*, 2020.

[29] Sung-Sik Lee, Patrick A. Lee, and T. Senthil. Amperean pairing instability in the u(1) spin liquid state with fermi surface and application to κ−(BEDT−TTF)2Cu2(CN)3. *Phys. Rev. Lett.*, 98:067006, Feb 2007.

[30] Patrick A. Lee. Amperean pairing and the pseudogap phase of cuprate superconductors. *Phys. Rev. X*, 4:031017, Jul 2014.

[31] Frank Schlawin, Andrea Cavalleri, and Dieter Jaksch. Cavity-mediated electron-photon superconductivity. *Phys. Rev. Lett.*, 122:133602, Apr 2019.

[32] B. Brecht, Dileep V. Reddy, C. Silberhorn, and M. G. Raymer. Photon temporal modes: A complete framework for quantum information science. *Phys. Rev. X*, 5:041017, Oct 2015.

[33] Michael G Raymer and Ian A Walmsley. Temporal modes in quantum optics: then and now. *Physica Scripta*, 95(6):064002, mar 2020.

[34] Stefan Yoshi Buhmann. *Dispersion Forces I*. Springer-Verlag Berlin Heidelberg, 2012.

[35] Ho Trung Dung, Ludwig Knöll, and Dirk-Gunnar Welsch. Three-dimensional quantization of the electromagnetic field in dispersive and absorbing inhomogeneous dielectrics. *Phys. Rev. A*, 57:3931–3942, May 1998.

[36] Alfred Rényi. On measures of entropy and information. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*, pages 547–561, Berkeley, Calif., 1961. University of California Press.

[37] Martin Müller-Lennert, Frédéric Dupuis, Oleg Szehr, Serge Fehr, and Marco Tomamichel. On quantum rényi entropies: A new generalization and some properties. *Journal of Mathematical Physics*, 54(12):122203, 2013.

[38] E. Bocquillon, V. Freulon, J.-M Berroir, P. Degiovanni, B. Plaçais, A. Cavanna, Y. Jin, and G. Fève. Coherence and indistinguishability of single electrons emitted by independent sources. *Science*, 339(6123):1054–1057, 2013.

[39] Vittorio Giovannetti, Diego Frustaglia, Fabio Tesi, and Rosario Fazio. Electronic hong-ou-mandel interferometer for multimode entanglement detection. *Phys. Rev. B*, 74:115315, Sep 2006.

[40] I. Neder, N. Ofek, Y. Chung, M. Heiblum, and V. Umansky. Interference between two indistinguishable electrons from independent sources. *Nature*, 448(7151):333–337, Jul 2007.

[41] Artur Ekert and Peter L. Knight. Entangled quantum systems and the schmidt decomposition. *American Journal of Physics*, 63(5):415–423, 1995.

[42] S. Parker, S. Bose, and M. B. Plenio. Entanglement quantification and purification in continuous-variable systems. *Phys. Rev. A*, 61:032305, Feb 2000.

[43] G. Giedke, M. M. Wolf, O. Krüger, R. F. Werner, and J. I. Cirac. Entanglement of formation for symmetric gaussian states. *Phys. Rev. Lett.*, 91:107901, Sep 2003.

[44] S. S. Straupe, D. P. Ivanov, A. A. Kalinkin, I. B. Bobrov, S. P. Kulik, and D. Mogilevtsev. Self-calibrating tomography for angular schmidt modes in spontaneous parametric down-conversion. *Phys. Rev. A*, 87:042109, Apr 2013.
[45] Wiesław Laskowski, Daniel Richart, Christian Schwemmer, Tomasz Paterek, and Harald Weinfurter. Experimental schmidt decomposition and state independent entanglement detection. Phys. Rev. Lett., 108:240501, Jun 2012.

[46] Stefania Sciara, Rosario Lo Franco, and Giuseppe Compagno. Universality of schmidt decomposition and particle identity. Scientific Reports, 7(1):44675, Mar 2017.

[47] Steven B. Giddings and Massimiliano Rota. Quantum information or entanglement transfer between subsystems. Phys. Rev. A, 98:062329, Dec 2018.

[48] Vittorio Giovannetti, Seth Lloyd, and Lorenzo Maccone. Advances in quantum metrology. Nature Photonics, 5(4):222–229, Apr 2011.

[49] Martin Dressel and George GrÃŒner. Electrodynamics of Solids: Optical Properties of Electrons in Matter. Cambridge University Press, 2002.

ACKNOWLEDGMENTS

We gratefully acknowledge help from Noa Asban on the graphical illustrations. This work has been supported in part by the Spanish MINECO (MAT2017-88492-R and SEV2015-0522), the European Research Council (Advanced Grant 789104- eNANO), the European Commission (Graphene Flagship 696656), the Catalan CERCA Program, and Fundació Privada Cellex.