GALOIS CLOSURES AND ELEMENTARY COMPONENTS OF HILBERT SCHEMES OF POINTS

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Abstract. Bhargava and the first-named author of this paper introduced a functorial Galois closure operation for finite-rank ring extensions, generalizing constructions of Grothendieck and Katz–Mazur. In this paper, we generalize Galois closures and apply them to construct a new infinite family of irreducible components of Hilbert schemes of points. We show that these components are elementary, in the sense that they parameterize algebras supported at a point. Furthermore, we produce secondary families of elementary components obtained from Galois closures by modding out by suitable socle elements.

1. Introduction

First introduced by Grothendieck [Gro95] in 1961, Hilbert schemes have played a central role in algebraic geometry, commutative algebra, and algebraic combinatorics. They are fundamental building blocks in the construction of many moduli spaces, and have seen numerous broad-ranging applications from the McKay correspondence [IN00, BKR01] to Haiman’s proof of the Macdonald positivity conjecture [Hai01]. While Hilbert schemes of points on smooth surfaces are irreducible [Fog68], in contrast, Iarrobino [Iar72, Iar73] and Iarrobino–Emsalem [IE78] showed that for \( n \geq 3 \) and \( d \) sufficiently large, the Hilbert scheme of points \( \text{Hilb}^d(\mathbb{A}^n) \) is reducible. It has remained a notoriously difficult problem to describe the structure of the irreducible components of \( \text{Hilb}^d(\mathbb{A}^n) \).

The study of all irreducible components of \( \text{Hilb}^d(\mathbb{A}^n) \) can be reduced to that of elementary components, namely irreducible components parameterizing subschemes supported at a point; this is due to the fact that, generically, every component is étale-locally the product of elementary ones. To date, relatively few constructions of irreducible components exist in the literature [Iar84, Sha90, IK99, CEVV09, EV10, Hui17, Jel19, Jel20, Hui21, SS23] and even less is known about elementary components. In fact, it was an open question for over 30 years to determine whether there exists an irreducible component of the punctual Hilbert scheme \( \text{Hilb}^d(O_{\mathbb{A}^n,0}) \) of dimension less than \( (n-1)(d-1) \), see [Iar87, p. 310], cf. [IE78, p. 186]; this question was only recently answered by the authors of this paper [SS23, Theorem 1.2].

In the current paper, we obtain a new infinite family of elementary components by applying a seemingly unrelated construction introduced by Bhargava and the first-named author in [BS14]: a functorial Galois closure operation for ring extensions that commutes with arbitrary base change. This Galois closure operation generalizes constructions appearing in work of Grothendieck [Che58, Chapter IV, Lemma 1] and Katz–Mazur [KM85, §1.8.2]; variants of the Galois closure were used in Bhargava’s groundbreaking work [Bha04, Bha08] and recently non-commutative generalizations of the Galois closure were applied by Ho and

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the first-named author [HS20] to uniformly construct many of the representations arising in arithmetic invariant theory, including many Vinberg representations.

Before stating our main results, let us elucidate the connection between Galois closures and Hilbert schemes of points. First, given any morphism of rings $B \rightarrow A$ that realizes $A$ as a free $B$-module of rank $n$, the Galois closure$^1$ $G(A/B)$ is a $B$-algebra which comes equipped with an action of the symmetric group $S_n$. Furthermore, if $A$ is a quotient of $B[x_1, \ldots, x_r]$, then $G(A/B)$ is naturally a quotient of $B[x_{i,j}]$, where $1 \leq i \leq r$ and $1 \leq j \leq n$. The ideal defining $G(A/B)$ contains the linear form $\sum_j x_{i,j}$ for every $i$, thus, upon eliminating the variables $x_{i,n}$, we see

$$[A] \in \text{Hilb}^n(\mathbb{A}^r_\mathbb{k}) \implies [G(A/\mathbb{k})] \in \text{Hilb}^{pts}(\mathbb{A}^{r(n-1)}_\mathbb{k})$$

where $\mathbb{k}$ is a field. For example, $G(\mathbb{k}^n/\mathbb{k}) \cong \mathbb{k}^n$ as a $\mathbb{k}$-algebra, and more specifically, is given by the regular representation. In particular, the Galois closure operation takes configurations of $n$ distinct points in $\mathbb{A}^r_\mathbb{k}$ to configurations of $n!$ distinct points in $\mathbb{A}^{r(n-1)}_\mathbb{k}$.

Of notable interest for us is that $G(A/B)$ need not be a flat $B$-module even though $A$ is flat. While this may seem unsightly at first, it turns out to be a useful feature in the study of Hilbert schemes; indeed, this gives us a method to take any easily understood algebra $A$ lying on the main component of the Hilbert scheme and produce a new algebra $G(A/\mathbb{k})$ which has the potential to lie off of the main component. This is particularly useful when one focuses attention on those algebras $A$ defined by monomial ideals $I$; here $G(A/\mathbb{k})$ carries a rich combinatorial structure coming both from the $S_n$-action and the fact that $I$ is monomial.

The algebras that featured prominently in [BS14], and those which play the most important role in our current paper, are

$$A_n := \mathbb{k}[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^2.$$

Within Poonen’s moduli space of rank $n$ rings [Poo08b], every algebra degenerates to $A_{n-1}$, and hence $G(A_{n-1}/\mathbb{k})$ has the largest possible dimension among Galois closures, see [BS14, Theorem 8]. One of the main theorems of [BS14], see Theorem 27 of (loc. cit.), gives the decomposition of $G(A_{n-1}/\mathbb{k})$ into irreducible $S_n$-representations, and in particular, gives a combinatorial formula for $d(n) := \dim_\mathbb{k} G(A_{n-1}/\mathbb{k})$.

It follows from the proof of [BS14, Theorem 5] that if $A$ is a flat $B$-algebra of rank $n \leq 3$, then $G(A/B)$ is a flat $B$-algebra of rank $n!$. In particular, if $A$ is a $\mathbb{k}$-algebra of rank at most 3 and $A$ lies on the main component of the Hilbert scheme, then $G(A/\mathbb{k})$ does as well. Thus, to find Galois closures living off of the main component, one must consider rings of rank $n \geq 4$.

Throughout this paper, we work over an algebraically closed field $\mathbb{k}$ of characteristic 0. Our main theorem is:

**Theorem 1.1.** If $n \geq 4$, then every irreducible component of $\text{Hilb}^{d(n)}(\mathbb{A}^{(n-1)^2}_\mathbb{k})$ containing $G(A_{n-1}/\mathbb{k})$ is elementary.

In fact, Theorem 1.1 follows from the more general Theorem 1.2, which we now describe.$^2$

By decoupling rank$_B(A)$ from $n$, we define a notion of higher Galois closure $G^{(n)}(A/B)$, see

$^1$also referred to as the $S_n$-closure

$^2$We wholeheartedly thank the anonymous referee for noting that Theorem 1.1 should hold in greater generality.
Section 2.2; the Galois closure from [BS14] is then given by $G(A/B) = G(\text{rank } A)(A/B)$. In our case of interest, we show (see Lemma 2.1) that

$$G^{(n)}(A_m/\mathbb{k}) = \mathbb{k}[x_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n]/I$$

where

$$I = (e_1(x_i) \mid 1 \leq i \leq m) + \sum_{j=1}^n (x_{1,j}, \ldots, x_{m,j})^2$$

and $e_1(x_i) := \sum_j x_{i,j}$. By eliminating $x_{i,n}$, we may naturally view $G^{(n)}(A_m/\mathbb{k})$ as a point in $\text{Hilb}^{\text{pts}}(\mathbb{A}^{m(n-1)})$.

A graded $\mathbb{k}$-algebra $R$ is said to have trivial negative tangents if the negatively graded pieces of the $T^1$-module $T^1(R/\mathbb{k}, R)_{<0}$ vanish; see Section 2.1 for a brief review of the $T^1$-modules. In [Jel19, Theorem 1.2], Jelsiejew proves that if $R$ has trivial negative tangents, then all components of the Hilbert scheme containing $[R]$ are elementary.

In what follows, we work throughout over $\mathbb{k}$ and so simply write $G^{(n)}(A_m)$ in place of $G^{(n)}(A_m/\mathbb{k})$. We prove:

**Theorem 1.2.** $G^{(n)}(A_m)$ has trivial negative tangents if and only if one of the following holds:

(i) $n = 1$,
(ii) $m = 1$ and $n \leq 2$,
(iii) $n = 3$ and $m \geq 3$,
(iv) $n \geq 4$ and $m \geq 2$.

For all $(n,m)$ in (i)–(iv) above, every irreducible component of $\text{Hilb}^{d(n,m)}(\mathbb{A}^{m(n-1)})$ containing $G^{(n)}(A_m)$ is elementary; see (2) for a combinatorial formula for $d(n,m)$.

**Remark 1.3.** In Theorem 4.10, we show that for $n = 3$ and $m \geq 11$, the higher Galois closure $G^{(3)}(A_m)$ lives on an irreducible locus $Z_m := \mathbb{A}^{2m} \times \text{Gr}(\binom{m}{2}, m^2 + m + 1)$ whose dimension is larger than that of the main component of the Hilbert scheme. However, showing that $G^{(3)}(A_m)$ lives on an elementary component is more involved. In fact, $Z_m$ is not generally equal to the full component containing $G^{(3)}(A_m)$ as one can check from a tangent space computation.

Furthermore, we prove that when $n = 3$, the higher Galois closure $G^{(3)}(A_m)$ always defines a smooth point of the Hilbert scheme. We do this by proving vanishing of the obstruction space for deformations. Combining Theorem 1.2 with Theorems 4.2 and 4.5 of [Jel19], we see that the obstruction lives in $T^2(G^{(n)}(A_m)/\mathbb{k}, G^{(n)}(A_m))_{\geq 0}$. We characterize precisely when this obstruction space vanishes.

**Theorem 1.4.** Let $G^{(n)}(A_m)$ have trivial negative tangents and assume $G^{(n)}(A_m) \not\cong \mathbb{k}$, i.e., $(n,m)$ is as in Theorem 1.2 (iii)–(iv). Then the obstruction space $T^2(G^{(n)}(A_m)/\mathbb{k}, G^{(n)}(A_m))_{\geq 0}$ vanishes if and only if

(a) $n = 3$ and $m \geq 3$, or
(b) $n = 4$ and $2 \leq m \leq 3$.

In particular, for all $(n,m)$ in (a)–(b) above, $G^{(n)}(A_m)$ is a smooth point of $\text{Hilb}^{d(n,m)}(\mathbb{A}^{m(n-1)})$ living on a unique elementary component.
Remark 1.5. One may wonder whether $G^{(n)}(A_m)$ defines a singular point of the scheme \(\text{Hilb}^{d(n,m)}(\mathbb{A}^{m(n-1)})\) for \((n, m)\) where the obstruction space is non-vanishing, i.e., for those \((n, m)\) not of the form (a)–(b). This appears to be a highly subtle question. See Remark 7.1 for more details.

Given the new elementary components we construct, one is particularly interested in having a formula for $d(n, m) := \dim_k G^{(n)}(A_m)$, and more generally, the Hilbert function of $G^{(n)}(A_m)$. We achieve this in Theorem 1.6 which gives much more refined information.

As mentioned above, one of the main theorems of [BS14] gives an explicit decomposition of $G(A_{n-1})$ into irreducible $S_n$-representations. We generalize this structure theorem to higher Galois closures. This result plays a key role in our proof of Theorem 1.4 as well as our analysis of the socle of $G^{(n)}(A_m)$. In what follows, $\mu$ denotes the irreducible representation (i.e., the Specht module) associated to a partition $\mu$, $K_{\mu\lambda}$ is the Kostka number [Sag01, Definition 2.11.1], and $\triangleright$ denotes dominance of partitions.

**Theorem 1.6.** For $m, n \geq 1$, we have an isomorphism of $S_n$-representations

$$G^{(n)}(A_m) \simeq \bigoplus_{\mu \triangleright \lambda \mu_1 = \lambda_1} \kappa_{\lambda} K_{\mu\lambda} V_{\mu},$$

where $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ are weakly decreasing sequences which run over partitions of $n$ into at most $M := \min(n, m + 1)$ parts, and

$$\kappa_{\lambda} := \binom{m}{k_0; \ldots; k_{M-1}}$$

is the multinomial coefficient with $k_j := \# \{i \neq 1 \mid \lambda_i = j \}$ for $j > 0$, and $k_0 := m - \sum_{j>0} k_j$. In particular,

$$d(n, m) = \sum_{\mu \triangleright \lambda \mu_1 = \lambda_1} \kappa_{\lambda} K_{\mu\lambda} \dim V_{\mu}$$

and the Hilbert function of $G^{(n)}(A_m)$ is given by

$$h_{G^{(n)}(A_m)}(i) = \sum_{\mu \triangleright \lambda \mu_1 = \lambda_1 = n-i} \kappa_{\lambda} K_{\mu\lambda} \dim V_{\mu}.$$
Theorem 1.7. Let $n \geq 4$ and $m \geq 2$. Choose $s_1, \ldots, s_r$ linearly independent homogeneous socle elements of minimal degree $D$, and assume $D \geq 3$. If $D \in \{ \lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil \}$, then let $s_i$ have multidegree $d_i := (d_{i,1}, \ldots, d_{i,m})$ and assume each $d_{i,j} \leq D - 2$. If

$$B := G^{(n)}(A_m)/(s_1, \ldots, s_r)$$

has no socle elements of degree strictly less than $D$, then $B$ has trivial negative tangents. In particular, every irreducible component of $\text{Hilb}^{d(n,m)-r}(\mathbb{A}_m^{m(n-1)})$ containing $B$ is elementary. Furthermore, if $m \geq n - 1$ and either:

(i) $n$ is even, or
(ii) $n$ is odd and $B$ has no socle elements of degree $D - 1$,

then $B$ automatically has no socle elements of degree strictly less than $D$, and hence has trivial negative tangents.

Remark 1.8. The condition that $B$ have no socle elements in degree $D - 1$ in Theorem 1.7(ii) is necessary, see Examples 6.1 and 6.2. This pair of examples shows that this condition is sometimes satisfied and sometimes not.

In Theorem 5.1, we analyze the $S_n$-representation structure of the socle of $G^{(n)}(A_m)$. Combining this analysis with Theorem 1.7(i), we give a combinatorial formula for the range of values that $r$ may assume. This formula is given in Corollary 1.9 below.

Let $n$ be even and let $\mathcal{P}$ be the set of partitions $\lambda$ of $n$ satisfying the following constraints: $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with $\lambda_1 = \lfloor \frac{n}{2} \rfloor$, $\lambda_2 \leq \lambda_1 - 2$, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$. Let $C_k$ denote the $k$th Catalan number and let

$$R(n) := C_{n/2} \sum_{\lambda \in \mathcal{P}} \kappa_\lambda$$

We then have:

Corollary 1.9. Let $n \geq 6$ be even and let $m \geq n - 1$. For all $0 \leq r \leq R(n)$, there exist socle elements $s_1, \ldots, s_r \in G^{(n)}(A_m)$ such that $B := G^{(n)}(A_m)/(s_1, \ldots, s_r)$ has trivial negative tangents. In particular, every irreducible component of $\text{Hilb}^{d(n,m)-r}(\mathbb{A}_m^{m(n-1)})$ containing $B$ is elementary.

Combining our formula (3) for the Hilbert function of $G^{(n)}(A_m)$ with Theorem 1.7 and Corollary 1.9, we obtain the following tables (for small $n, m$) recording the Hilbert functions of our algebras $G^{(n)}(A_m)/(s_1, \ldots, s_r)$ with trivial negative tangents. Table 1 is generated from Theorem 1.7 and computer calculations, and Table 2 is an immediate consequence of Corollary 1.9.

| $n$ | $G^{(n)}(A_2)$ | $G^{(n)}(A_3)$ | $G^{(n)}(A_4)$ | $G^{(n)}(A_6)$ |
|-----|----------------|----------------|----------------|----------------|
| 3   | N/A            | (1, 6, 3)     | (1, 8, 6 - r), 0 \leq r \leq 2 | (1, 10, 10 - r), 0 \leq r \leq 5 |
| 4   | (1, 6, 9 - r), 0 \leq r \leq 5 | (1, 9, 21 - r, 1), 0 \leq r \leq 12 | (1, 12, 38 - r, 4), 0 \leq r \leq 20 | (1, 15, 60 - r, 10), 0 \leq r \leq 30 |
| 5   | (1, 8, 21, 10 - r), 0 \leq r \leq 6 | (1, 12, 48, 44 - r), 0 \leq r \leq 27 | (1, 16, 86, 116 - r, 1), 0 \leq r \leq 40 | (1, 20, 135, 240 - r, 5), 0 \leq r \leq 100 |
Remark 1.10. Observe that the cases \((n, m) = (3, m), 3 \leq m \leq 5\), and \((n, m) = (4, 2)\) all define algebras of nilpotency class 2, in the sense of [Sha90]. We therefore recover Shafarevich’s examples with Hilbert functions \((1, d, e), 3 \leq e \leq \frac{(d-1)(d-2)}{6} + 2\). Note that our examples \((1, 6, 9 - r), 0 \leq r \leq 3\) provide new data for Shafarevich’s unexplored “middle interval” \(\frac{(d-1)(d-2)}{6} + 2 < e < \frac{d^2-1}{3}\) in the specific case \(d = 6\) (see [Sha90, p.179]).

Notice that \(G^{(4)}(A_3)\) modulo all of its socal elements of degree 2 is Gorenstein local with Hilbert function \((1, 9, 9, 1)\), and thus recovers an elementary component found by Iarrobino and Kanev. That is, this algebra is compressed (see [Iar84], [ET87, Section 3.3]), and is generic nonsmoothable by [IK99, Lemma 6.21], meaning it sits in the Hilbert scheme on a single component consisting generically of Gorenstein algebras with the same Hilbert function.

In light of Theorem 1.2, we end the introduction by posing the following general question.

Question 1.11. For which \(n\) and which finite rank \(\mathbb{k}\)-algebras \(A\) does the higher Galois closure \(G^{(n)}(A/\mathbb{k})\) have trivial negative tangents, or more generally, lie on an elementary component?

For small \(n\), Table 3 (in Section 8) lists those \(G^{(n)}(A/\mathbb{k})\) with trivial negative tangents for each isomorphism class of \(\mathbb{k}\)-algebras of rank at most \(m\). This makes use of Poonen’s work [Poo08a], which lists representatives of every such isomorphism class. It would be interesting to answer Question 1.11 even for these finitely many classes.

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2. Preliminaries

2.1. Review of the Truncated Cotangent Complex. We prove Theorems 1.1, 1.2, and 1.7 by showing that our algebras of interest have trivial negative tangents. In this section, we give a brief review of the truncated cotangent complex and the \(T^n\)-modules. For further details, we refer the reader to [Har10, §3] or [LS67].

Given any morphism \(B \rightarrow A\) of rings, we obtain a model of the truncated cotangent complex \(L_{A/B, \bullet}\) as follows. Let \(\pi: R \rightarrow A\) be a surjective map from a (possibly infinite type) polynomial ring \(R\) over \(B\). Let \(I = \ker(\pi), F'\) be a (possibly infinite rank) free \(R\)-module, and \(\pi': F \rightarrow I\) be a surjective \(R\)-module map. Let \(Q = \ker(\pi')\) and \(\text{Kos} \subseteq Q\) be the submodule of Koszul relations. The truncated cotangent complex of \(A\) over \(B\) is the following 3-term complex concentrated in homological degrees 0, 1, 2:

\[
L_{A/B, \bullet} : \Omega^1_{R/B} \otimes_R A \xrightarrow{d_1} F \otimes_R A \xrightarrow{d_2} Q/\text{Kos};
\]

Table 2. Examples of Hilbert functions of \(G^{(n)}(A_m)/(s_1, \ldots, s_r)\) with trivial negative tangents derived from Corollary 1.9
the map $d_2$ is induced by the inclusion $Q \subseteq F$ and $d_1$ is given by composing the map $F \otimes_R A \to I/I^2$ with the map induced by the derivation $R \to \Omega^1_{R/B}$. One then defines the $T^i$-modules to be

$$T^i(A/B, M) := H^i(\text{Hom}_A(L_{A/B}, M)),$$

for any $A$-module $M$ and $0 \leq i \leq 2$. Different choices of $R$, $F$, $\pi$, and $\pi'$ yield quasi-isomorphic elements $L_{A/B}$ of the derived category, see e.g. [Har10, Remark 3.3.1]; as a result, the $T^i$-modules depend only on $B \to A$ and $M$. Furthermore, when $A$ and $B$ carry gradings by an abelian group and $B \to A$ is a graded morphism, then the choices of $R$, $F$, $\pi$, and $\pi'$ can be made to respect the grading. As a result, if $M$ is a graded $A$-module, then $T^i(A/B, M)$ is also graded and the nine-term long exact sequences given in [Har10, Theorems 3.4-3.5] are graded morphisms.

Lastly, if $k$ is a field, then a (positively) $\mathbb{Z}$-graded $k$-algebra $A$ is said to have trivial negative tangents if the negatively graded pieces $T^1(A/\mathbb{k}, A)_{<0}$ vanish. Our primary case of interest is where $A = S/I$ with $S = \mathbb{k}[x_1, \ldots, x_n]$, $I$ is a homogeneous ideal, and each variable has degree 1. Then differentiation by $x_i$ yields an $S$-module map $\partial_i: I \to A$. In this case, $A$ has trivial negative tangents if and only if every negatively graded $S$-module map $\varphi: I \to A$ is a $k$-linear combination of $\partial_1, \ldots, \partial_n$.

2.2. Galois closures and higher Galois closures. We briefly review the Galois closure operation introduced in [BS14] and then introduce our higher Galois closure construction. Let $A$ be a (commutative) $B$-algebra which is free of rank $n < \infty$ as a $B$-module. (The Galois closure is, in fact, defined whenever $A$ is locally free of rank $n$, however we do not need this level of generality here.) For $a \in A$ and $1 \leq i \leq n$, let $a^{(i)} = 1 \otimes \cdots \otimes 1 \otimes a \otimes 1 \otimes \cdots \otimes 1 \in A^{\otimes n}$, where the $a$ is in the $i$th position and the tensor product is taken over $B$. The intuition behind the Galois closure is to treat the elements $a^{(1)}, \ldots, a^{(n)}$ as Galois conjugates. In particular, the elementary symmetric functions in the $a^{(i)}$ should be expressible as coefficients of an appropriate characteristic polynomial, as we now describe.

For every $a \in A$, we obtain a $B$-linear map $m_a: A \to A$ given by multiplication by $a$. Let $p_a(t) \in B[t]$ be the characteristic polynomial of $m_a$. Write $p_a(t) = \sum_{i=0}^{n}(-1)^i s_i(a)t^{n-i}$ and let $e_j(a)$ be the $j$th elementary symmetric function in $a^{(1)}, \ldots, a^{(n)}$. Then

$$G(A/B) := A^{\otimes n}/I, \quad \text{where} \quad I = \langle s_j(a) - e_j(a) \mid a \in A \rangle.$$ 

The $S_n$-action on $A^{\otimes n}$ given by permuting tensor factors descends to an action on $G(A/B)$. If $a_1, \ldots, a_n$ is a basis for $A$ as a $B$-module, then $I$ is generated by the expressions $s_j(a_i) - e_j(a_i)$ for $1 \leq i, j \leq n$, see [BS14, §2].

We define the higher Galois closure by allowing $n$ to be independent of rank$_B(A)$. We simply let

$$G^{(n)}(A/B) := A^{\otimes n}/I, \quad \text{where} \quad I = \langle s_j(a) - e_j(a) \mid a \in A \rangle.$$ 

Specializing to our case of interest, let $A_m = \mathbb{k}[x_1, \ldots, x_m]/(x_1, \ldots, x_m)^2$ where $\mathbb{k}$ is a characteristic 0 field. Let $x_{i,j} := x_i^{(j)}$, and for any linear form $g = \sum_{i=1}^{m} \lambda_i x_i$ with $\lambda_i \in \mathbb{k}$, let $g_j := \sum_{i=1}^{m} \lambda_i x_{i,j}$ and $e_i(g) := e_i(g_1, \ldots, g_n)$ denote the $i$th elementary symmetric function. We see then that

$$G^{(n)}(A_m/\mathbb{k}) = \mathbb{k}[x_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n]/I.$$ 

with
\[ I = \sum_{j=1}^{n} (x_{1,j}, \ldots, x_{m,j})^2 + \langle e_1(g), \ldots, e_n(g) \rangle \]
where \( g \) runs through all linear forms in the \( x_i \). In this notation, the \( S_n \)-action on \( G^{(n)}(A_m/k) \) is given by \( \sigma(x_{i,j}) = x_{i,\sigma(j)} \).

**Lemma 2.1.** We have
\[ I = \langle e_1(x_i) \mid 1 \leq i \leq m \rangle + \sum_{j=1}^{n} (x_{1,j}, \ldots, x_{m,j})^2. \]

**Proof.** We prove that \( I = \langle e_1(g) \mid g \rangle \) linear form in the \( x_i \) + \( \sum_{j=1}^{n} (x_{1,j}, \ldots, x_{m,j})^2 \). The result follows since \( \langle e_1(g) \mid g \rangle \) linear form in the \( x_i \) = \( \langle e_1(x_i) \mid 1 \leq i \leq m \rangle \).

We first note that all power sums \( P_\ell(g) := P_\ell(g_1, \ldots, g_n) := \sum_j g_j^\ell \) are contained in \( I \); indeed, \( P_1(g) \) is a linear combination of the \( e_1(x_i) \) which are in \( I \); for \( \ell > 1 \), the power sum \( P_\ell(g) \) is in the ideal generated by the \( g_j^2 \), which is contained in \( (x_{1,j}, \ldots, x_{m,j})^2 \). The Newton–Girard identities then express each \( e_k(g) \) as a \( \mathbb{Q} \)-linear combination of the \( P_\ell(g) \). Since \( \mathbb{Q} \subset k \), we have finished the proof. \( \Box \)

3. **Structure theorem for higher Galois closures: Theorem 1.6**

In this section, we prove Theorem 1.6 which generalizes one of the main results of [BS14] to higher Galois closures. This will play an important role in Sections 5 and 7, where we analyze the socle and obstruction space of \( G^{(n)}(A_m) \).

The following lemma will be useful throughout this section as well as Section 5.

**Lemma 3.1.** Let \( m \leq m' \), \( d = (d_1, \ldots, d_m) \in \mathbb{N}^m \), and \( d' = (d_1, \ldots, d_m, 0, \ldots, 0) \in \mathbb{N}^{m'} \). Then the natural projection map on multigraded pieces
\[ G^{(n)}(A_{m'})_{d'} \rightarrow G^{(n)}(A_m)_d \]
is an isomorphism of \( S_n \)-representations.

**Proof.** Letting \( I_{n,m} \subset A_{m}^{\otimes n} \) denote the defining ideal for \( G^{(n)}(A_m) \), simply observe that we have a commutative diagram
\[
\begin{array}{c}
0 \rightarrow (I_{n,m'})_{d'} \rightarrow (A_{m'}^{\otimes n})_{d'} \rightarrow G^{(n)}(A_{m'})_{d'} \rightarrow 0 \\
\uparrow \cong \uparrow \cong \uparrow \cong \\
0 \rightarrow (I_{n,m})_d \rightarrow (A_{m}^{\otimes n})_d \rightarrow G^{(n)}(A_m)_d \rightarrow 0
\end{array}
\]
where the rows are exact and the leftmost two vertical maps are isomorphisms of \( S_n \)-representations. \( \Box \)

**Lemma 3.2.** Let \( d = (d_1, \ldots, d_m) \in \mathbb{N}^m \).

(i) If \( n - \sum_i d_i < d_j \) for some \( j \), then \( G^{(n)}(A_m)_d = 0 \).

(ii) If \( |\{i \mid d_i \neq 0\}| \geq n \), then \( G^{(n)}(A_m)_d = 0 \).

**Proof.** Lemma 31 of [BS14] says that (i) holds when \( m = n - 1 \), however the proof applies verbatim when \( m \) is arbitrary: the proof uses an inclusion-exclusion argument on indices \( j \), which is applicable in our case as \( x_{i,j}x_{k,j} = 0 \).
For (ii), we first notice that if \(|\{i \mid d_i \neq 0\}| > n\), then \(G^{(n)}(A_m)_d = 0\) by the pigeonhole principle: for every monomial \(g \in G^{(n)}(A_m)_d\), there exists \(j\) and \(i \neq k\) such that \(x_{i,j}x_{k,j}\) divides \(g\). We may therefore assume \(|\{i \mid d_i \neq 0\}| = n\). By (i), after reordering variables, we know \(d = (1, \ldots, 1, 0, \ldots, 0)\). Then applying Lemma 3.1, we see \(G^{(n)}(A_m)_d \simeq G^{(n)}(A_n)_{(1,\ldots,1)}\) and the latter vanishes again by (i).

We now turn to Theorem 1.6.

**Proof of Theorem 1.6.** We have a decomposition

\[ G^{(n)}(A_m) = \bigoplus_{d \in \mathbb{Z}^m} G^{(n)}(A_m)_d \]

as \(S_n\)-representations. To determine \(G^{(n)}(A_m)_d\) as a representation, we may reorder variables in order to assume \(d = (d_1, \ldots, d_m)\) with \(n - \sum d_i \geq d_1 \geq \cdots \geq d_m\). If \(m \leq n - 1\), then let \(d' = (d_1, \ldots, d_m, 0, \ldots, 0) \in \mathbb{N}^{n-1}\) and, applying Lemma 3.1, we have an isomorphism

\[ G^{(n)}(A_{n-1})_{d'} \xrightarrow{\simeq} G^{(n)}(A_m)_d. \]

If \(m \geq n\), then by Lemma 3.2(ii), we know \(d_i = 0\) for \(i \geq n\) and so we may identify \(d\) with \(d' = (d_1, \ldots, d_{n-1}) \in \mathbb{N}^{n-1}\). Then again applying Lemma 3.1, we have an isomorphism

\[ G^{(n)}(A_m)_d \xrightarrow{\simeq} G^{(n)}(A_{n-1})_{d'}. \]

Let \(\lambda\) be the (ordered) partition of \(n\) given by \((n - \sum d_i, d'_1, \ldots, d'_{n-1})\); we refer to \(\lambda\) as the partition associated to \(d\). Theorem 27 of [BS14] shows

\[ G^{(n)}(A_{n-1})_{d'} = \bigoplus_{\mu \succ \lambda \atop \mu_1 = \lambda_1} K_{\mu\lambda} V_\mu. \]

Note that after removing zero entries from \(\lambda\), the result is a partition of \(n\) into at most \(M := \min(n, m+1)\) parts. Lastly, for any such partition \(\lambda\), the number of \(d \in \mathbb{Z}^m\) associated to \(\lambda\) is given by the multinomial coefficient \({m \choose k_0, \ldots, k_{n-1}}\) where \(k_j := \#\{i \neq 1 \mid \lambda_i = j\}\) for \(j > 0\), and \(k_0 := m - \sum_{j>0} k_j\).

4. **Trivial negative tangents for Galois closures**

We use the notation from equation (4) where \(I\) is given as in Lemma 2.1. We give \(G(A_n)\) the \(\mathbb{Z}^m\)-grading where \(x_{i,j}\) has degree \((0, \ldots, 0, 1, 0, \ldots, 0)\) with 1 in the \(i\)th place. This induces the \(\mathbb{Z}\)-grading where every \(x_{i,j}\) has degree 1.

Throughout the rest of this section, we let \(\mathcal{R} = \mathbb{k}[x_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n]\) and fix a graded \(\mathcal{R}\)-module map

\[ \varphi : I \to G^{(n)}(A_m) \]

of negative degree. The primary goal of this section is to prove \(G^{(n)}(A_m)\) has trivial negative tangents. To do so, we must show that \(\varphi\) is a \(\mathbb{k}\)-linear combination of the maps \(\partial_{i,j}\) given by differentiation by \(x_{i,j}\).

We prove Theorem 1.2 by first showing the result for \(n \geq 4\) even. We then reduce the case where \(n\) is odd to the case \(n - 1\). Since \(G^{(2)}(A_m)\) does not generally have trivial negative tangents, we must handle the case \(n = 3\) separately.
4.1. Tangents of degree at most minus two.

Proposition 4.1. Let $n, m \geq 2$. If $\varphi: I \to G^{(n)}(A_m)$ has degree at most $-2$, then $\varphi = 0$.

Proof. By Lemma 2.1, $I$ is generated by quadratic and linear forms. As a result, if $\deg(\varphi) < -2$, then $\varphi = 0$, so we need only consider the case where $\deg(\varphi) = -2$. Then $\varphi$ sends the quadratic generators of $I$ to constants, and for $i \neq k$, we have

$$x_{k,j}\varphi(x_{i,j}^2) = x_{i,j}\varphi(x_{i,j}x_{k,j}).$$

However, $x_{1,j}, \ldots, x_{m,j}$ are linearly independent elements of $G^{(n)}(A_m)$, and hence, $\varphi(x_{i,j}^2) = \varphi(x_{i,j}x_{k,j}) = 0$. □

4.2. Tangents of degree minus one: preliminaries. Having dispensed with the case where $\deg(\varphi) \leq -2$, we must now consider the case where $\deg(\varphi) = -1$. Showing triviality of such tangents will involve a careful analysis of the syzygies of $I$, and occupies the remainder of Section 4.

We begin by obtaining some preliminary information about the images under $\varphi$ of the quadratic generators of $I$.

Lemma 4.2. Let $\varphi: I \to G^{(n)}(A_m)$ have degree $-1$. Assume either (i) $n \geq 4$ and $m \geq 2$ or (ii) $n = 3$ and $m \geq 3$. Then $\varphi(x_{i,j}^2)$ and $\varphi(x_{i,j}x_{k,j})$ are in the $k$-linear span of $x_{1,j}, \ldots, x_{n-1,j}$.

Proof. Let $m \geq 2$. By symmetry and for ease of notation, it is enough to show the statement for $\varphi(x_{1,1}^2)$ and $\varphi(x_{1,1}x_{2,1})$. By degree constraints we know $\varphi(x_{1,1}^2)$ and $\varphi(x_{1,1}x_{2,1})$ are in the $k$-span of the $x_{i,j}$. Using that $e_1(x_i) = 0$ in $G^{(n)}(A_m)$, we may write

$$\varphi(x_{1,1}^2) = \sum_i \sum_{j \neq n} a_{i,j}x_{i,j} \quad \text{and} \quad \varphi(x_{1,1}x_{2,1}) = \sum_i \sum_{j \neq n} b_{i,j}x_{i,j},$$

where $a_{i,j}, b_{i,j} \in k$. Consider the syzygy

$$x_{2,1}\varphi(x_{1,1}^2) = x_{1,1}\varphi(x_{1,1}x_{2,1}).$$

Expanding, we see

$$\sum_i \sum_{j \neq 1, n} a_{i,j}x_{2,1}x_{i,j} = \sum_i \sum_{j \neq 1, n} b_{i,j}x_{1,1}x_{i,j}.$$

We must show that $a_{i,j} = b_{i,j} = 0$ for $j \neq 1, n$.

By considering the $\mathbb{Z}^m$-grading on $G^{(n)}(A_m)$, we see

$$\sum_{j \neq 1, n} a_{1,j}x_{2,1}x_{1,j} = \sum_{j \neq 1, n} b_{2,j}x_{1,1}x_{2,j};$$

for $i \neq 2$ we have

$$\sum_{j \neq 1, n} b_{i,j}x_{1,1}x_{i,j} = 0$$

and for $i \neq 1$ we have

$$\sum_{j \neq 1, n} a_{i,j}x_{2,1}x_{i,j} = 0.$$

First assume $n \geq 4$. We claim it is enough to prove linear independence of the set $S := \{x_{1,1}x_{2,j}, x_{2,1}x_{1,j} \mid j \neq 1, n\}$. Indeed, this would show that all coefficients $a_{1,j}$ and $b_{2,j}$ vanish in equation (7); furthermore, the transposition $(2, i) \in S_n$ acts on $G^{(n)}(A_m)$, sending $x_{1,1}x_{i,j}$...
to $x_{1,1}x_{2,j}$, so linear independence of $\mathcal{S}$ implies the vanishing of all coefficients $b_{i,j}$ in equation (8). A similar argument applies for equation (9).

Let $G^{(n)}(A_m)_2$ denote the $k$-vector space of elements of $G^{(n)}(A_m)$ whose total degree is 2; this is the vector space generated by elements of the form $x_{k,\ell}x_{i,j}$. To prove linear independence of $\mathcal{S}$, by symmetry of $x_1$ and $x_2$, it is enough show that for every $1 < j_0 < n$, there is a linear functional $f_{j_0}$ on $G^{(n)}(A_m)_2$ such that for all $1 < j < n$, we have $f_{j_0}(x_{2,1}x_{1,j}) = 0$ and $f_{j_0}(x_{1,1}x_{2,j}) = \delta_{j,j_0}$ with $\delta$ the Kronecker delta function. Since $n \geq 4$, we can choose $\ell_0 \neq 1, j_0, n$. Then we define

$$f_{j_0}(x_{1,1}x_{2,j_0}) = f_{j_0}(x_{1,\ell_0}x_{2,n}) = 1, \quad f_{j_0}(x_{1,1}x_{2,n}) = f_{j_0}(x_{1,\ell_0}x_{2,j_0}) = -1,$$

and let $f_{j_0}(x_{k,\ell}x_{i,j}) = 0$ for all other quadratics with $1 \leq i, k < n$ and $1 \leq \ell, j \leq n$. The functional $f_{j_0}$ is well-defined since it vanishes on all expressions of the form $x_{k,j}x_{i,j}$ and $x_{k,\ell}e_1(x_i)$ with $1 \leq i, k < n$ and $1 \leq \ell, j \leq n$.

When $n = 3$, the set $\mathcal{S}$ is no longer linearly independent. Eliminating the $x_{i,3}$ variables, we see

$$G^{(3)}(A_m) \simeq k[x_{i,1}, x_{i,2} \mid 1 \leq i \leq m]/(x_{i,j}x_{k,j}, x_{i,1}x_{k,2} + x_{k,1}x_{i,2}).$$

Thus, equations (8) and (9) tell us $a_{i,2} = b_{i,2} = 0$ for $i > 2$. Equation (7) tells us $a_{1,2} = -b_{2,2}$. By symmetry of the $x_{i,2}$ variables, we have therefore shown (with a slight change in notation) that

$$\varphi(x_{1,1}x_{k,1}) = \sum_i a_i^{(k)}x_{i,1} + b_1^{(k)}x_{1,2} + b_k^{(k)}x_{k,2}$$

where $b_k^{(k)} = -2b_1^{(1)}$. Then assuming $m \geq 3$, we may consider the syzygy

$$x_{3,1}\varphi(x_{1,1}x_{2,1}) = x_{2,1}\varphi(x_{1,1}x_{3,1}),$$

we find

$$b_1^{(2)}x_{1,2}x_{3,1} + b_{2}^{(2)}x_{2,2}x_{3,1} = b_1^{(3)}x_{2,1}x_{1,2} + b_3^{(3)}x_{2,1}x_{3,2}$$

which implies $b_1^{(2)} = b_1^{(3)} = 0$ and $b_2^{(2)} = -b_3^{(3)}$. Using that $b_k^{(k)} = -2b_1^{(1)}$, we see $b_2^{(2)} = b_3^{(3)} = 0$. Hence, $\varphi(x_{1,1}x_{2,1})$ is in the span of the $x_{i,1}$.

4.3. The case where $m \geq 2$ and $n \geq 4$ is even. The proof of Lemma 4.2 was obtained by considering syzygies among the quadratic generators of $I$. Next we consider syzygies involving quadratic generators as well as the linear generators $e_1(x_i)$. The form of these syzygies will depend on the parity of $n$. This subsection considers the case where $n$ is even; the case where $n$ is odd is handled in §4.4.

In Proposition 4.5, we will obtain certain non-vanishing results which we establish in Lemmas 4.3 and 4.4.

**Lemma 4.3.** Let $M \geq 1$ and

$$R = \mathbb{k}[w_1, \ldots, w_{2M}, z]/(w_1^2, \ldots, w_{2M}^2, \sum_i w_i, z^2).$$

Then

$$z \prod_{i=1}^M (w_{2i-1} - w_{2i})$$

is a non-zero element of $R$. 

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Proof. Consider the quotient $R' = R/J$ where $J = (w_1 + w_2, \ldots, w_{2M-1} + w_2M)$. Then $f := z \prod_{i=1}^{2M}(w_{2i-1} - w_{2i})$ has image in

$$R' \simeq k[w_1, w_3, \ldots, w_{2M-1}, z]/(z^2, w_1^2, w_3^2, \ldots, w_{2M-1}^2)$$

given by $2^Mzw_1w_3 \ldots w_{2M-1}$. Since this expression is square-free, it is non-zero in $R'$ and hence $f$ is non-zero in $R$.

**Lemma 4.4.** Let $M \geq 1$ and

$$R = k[w_1, \ldots, w_{2M}]/\left(w_1^2, \ldots, w_{2M}^2, \sum_i w_i\right).$$

If $c, a_1, \ldots, a_{2M} \in k$ and

$$c \prod_{i=1}^{2M}(w_{2i-1} - w_{2i}) = \sum_{j \text{ odd \atop 1 \leq j \leq 2M}} (a_{j}w_{j} - a_{j+1}w_{j+1}) \prod_{k \text{ odd \atop k \neq j}} (w_k - w_{k+1}),$$

then $c = \frac{1}{2} \sum_{j=1}^{2M} a_{j}$ holds.

**Proof.** Consider the quotient $R' = R/J$ where $J = (w_1 + w_2, \ldots, w_{2m-1} + w_{2m})$. Then

$$R' \simeq k[w_1, w_3, \ldots, w_{2m-1}]/(w_1^2, w_3^2, \ldots, w_{2m-1}^2).$$

In this quotient ring, we have

$$2^mcw_1w_3 \ldots w_{2m-1} = 2^{m-1} \sum_{j=1}^{2m} a_{j}w_{1}w_{3} \ldots w_{2m-1}.$$ 

Since $w_1w_3 \ldots w_{2m-1}$ is square-free, it is non-zero and so $c = \frac{1}{2} \sum_{j=1}^{2m} a_{j}$.

The following result greatly constrains the form $\varphi$ may take when $n$ is even.

**Proposition 4.5.** Let $m \geq 2$ and assume $n \geq 4$ is even. If $\varphi: I \to G(n)(A_m)$ has degree $-1$, then there exist $a_{i,j} \in k$ for which

$$\varphi(x_{i,j}^2) = a_{i,j}x_{i,j} \quad \text{and} \quad \varphi(e_1(x_i)) = \frac{1}{2} \sum_j a_{i,j}. $$

Furthermore, for any fixed $j$ and any linear form $g = \sum_i \lambda_i x_{i,j}$ with $\lambda_i \in k$, we have

$$\varphi(g^2) = \lambda g$$

for some $\lambda \in k$.

**Proof.** The “furthermore” statement follows immediately from the previous statements since one may use the $\text{GL}_m$-action on $G(n)(A_m)$ induced by the action on $A_m$. In other words, there exists $\alpha \in \text{GL}_m$ such that $\alpha(x_{1,j}) = g$. One may then apply the previous statements to the map $\alpha^{-1}\varphi\alpha$ which is also $k[x_{i,j}]$-linear of degree $-1$, thereby showing

$$\varphi(g^2) = \alpha^{-1}\varphi(\alpha(x_{1,j}^2)) = \alpha(\lambda x_{1,j}) = \lambda g.$$ 

Next, we show that $\varphi(x_{i,j}^2)$ is a scalar multiple of $x_{i,j}$. By symmetry, it is enough to do so when $i = 1$. By Lemma 4.2, we may write

$$\varphi(x_{1,j}^2) = \sum_i b_{i,j}x_{i,j}$$
with \( b_{i,j} \in \mathbb{k} \).

For any \( \sigma \in S_n \), by writing
\[
e_1(x_1) = \sum_{j \text{ odd}} (x_{1,\sigma(j)} + x_{1,\sigma(j+1)})
\]
and multiplying through by \( \prod_{k \text{ odd}} (x_{1,\sigma(k)} - x_{1,\sigma(k+1)}) \), we arrive at the following syzygy:
\[
\prod_{k \text{ odd}} (x_{1,\sigma(k)} - x_{1,\sigma(k+1)}) \varphi(e_1(x_1)) = \sum_{j \text{ odd} \, k \text{ odd} \, k \neq j} (x_{1,\sigma(k)} - x_{1,\sigma(k+1)}) (\varphi(x_{1,\sigma(j)}^2) - \varphi(x_{1,\sigma(j+1)}^2)).
\]
(11)

We next substitute \( \sum_i b_{i,j} x_{i,j} \) in place of \( \varphi(x_{1,j}^2) \) and compare the terms on both sides of the syzygy which are of multidegrees \((\frac{n}{2} - 1, 0, \ldots , 0, 1, 0, \ldots , 0)\), where the 1 is in the \( i \)-th place and \( i \neq 1 \). This yields
\[
\sum_{j \text{ odd} \, k \text{ odd} \, k \neq j} (x_{1,\sigma(k)} - x_{1,\sigma(k+1)}) (b_{i,j} x_{i,j} - b_{i,j+1} x_{i,j+1}) = 0.
\]
(12)

We must show \( b_{i,j} = 0 \) for all \( i \neq 1 \). By symmetry in \( i \), it is enough to prove \( b_{2,j} = 0 \) for all \( j \). Letting
\[
J_{\sigma} = (x_{1,\sigma(n-1)}, x_{1,\sigma(n)}) + (x_{2,\sigma(j)} \mid j \leq n-2) + (x_{i,j} \mid i \neq 1, 2),
\]
we see \( G^{(n)}(A_m)/J_{\sigma} \) is isomorphic to the ring \( R \) in the statement of Lemma 4.3, where \( M = \frac{n}{2} - 1 \) and the isomorphism is given by \( x_{1,\sigma(i)} \mapsto w_i, x_{2,\sigma(n-1)} \mapsto z, \) and \( x_{2,\sigma(n)} \mapsto -z \). Thus, in \( R \), equation (12) becomes
\[
(b_{2,\sigma(n-1)} + b_{2,\sigma(n)}) z \prod_{i=1}^{M} (w_{2i-1} - w_{2i}) = 0.
\]

By Lemma 4.3, we see \( b_{2,\sigma(n-1)} = -b_{2,\sigma(n)} \) for all \( \sigma \). So, for all \( j \), by choosing \( j, k, \ell \) distinct, we see that \( b_{2,j} = -b_{2,k}, b_{2,k} = -b_{2,\ell}, b_{2,j} = -b_{2,\ell} \) which implies \( b_{2,j} = 0 \).

To finish the proof, we must calculate \( \varphi(e_1(x_{i,j})) \). Substituting \( \varphi(x_{i,j}^2) = a_{i,j} x_{i,j} \) into (11) with \( \sigma \) equal to the identity permutation, and considering terms of multidegree \((\frac{n}{2}, 0, \ldots , 0)\), we see
\[
\prod_{k \text{ odd}} (x_{1,k} - x_{1,k+1}) \varphi(e_1(x_1)) = \sum_{j \text{ odd} \, k \text{ odd} \, k \neq j} (x_{1,k} - x_{1,k+1}) (a_{1,j} x_{1,j} - a_{1,j+1} x_{1,j+1}).
\]
(13)

Let \( J = (x_{i,j} \mid i \neq 1) \). Since \( G^{(n)}(A_m)/J \) is isomorphic to the ring \( R \) from Lemma 4.4, it follows that \( \varphi(e_1(x_1)) = \frac{1}{2} \sum_j a_{1,j} \). \( \square \)

The following result is the last ingredient we need to show \( G^{(n)}(A_m) \) has trivial negative tangents when \( n \) is even.

**Proposition 4.6.** Let \( n \geq 2 \) and \( \varphi : I \to G^{(n)}(A_m) \) have degree \(-1\). Suppose that for all linear forms \( g = \sum_i \lambda_i x_{i,j} \) with \( \lambda_i \in \mathbb{k} \), we have \( \varphi(g^2) = \lambda g \) for some \( \lambda \in \mathbb{k} \). If \( \varphi(x_{i,j}^2) = a_{i,j} x_{i,j} \) with \( a_{i,j} \in \mathbb{k} \), then for all distinct \( i,k \), we have
\[
\varphi(x_{i,j} x_{k,j}) = \frac{1}{2} a_{k,j} x_{i,j} + \frac{1}{2} a_{i,j} x_{k,j}.
\]
In particular, this holds when \( n \geq 4 \) and \( m \geq 2 \).
Proof. Proposition 4.5 shows that the hypotheses hold when \( n \geq 4 \) and \( m \geq 2 \). Let
\[
\varphi((x_{i,j} + x_{k,j})^2) = b_j^{i,k} (x_{i,j} + x_{k,j}) \quad \text{and} \quad \varphi((x_{i,j} - x_{k,j})^2) = c_j^{i,k} (x_{i,j} - x_{k,j})
\]
for \( b_j^{i,k}, c_j^{i,k} \in k \). Summing our two expressions for \((x_{i,j} \pm x_{k,j})^2\), we see
\[
b_j^{i,k} (x_{i,j} + x_{k,j}) + c_j^{i,k} (x_{i,j} - x_{k,j}) = \varphi((x_{i,j} + x_{k,j})^2) + \varphi((x_{i,j} - x_{k,j})^2)
= 2\varphi(x_{i,j}^2 + x_{k,j}^2) = 2(a_{i,j}x_{i,j} + a_{k,j}x_{k,j}).
\]
By linear independence of \( x_{i,j} \) and \( x_{k,j} \), we then have
\[
2b_j^{i,k} = a_{i,j} + a_{k,j} \quad \text{and} \quad 2c_j^{i,k} = a_{i,j} - a_{k,j}.
\]
Next, we see
\[
2\varphi(x_{i,j}x_{k,j}) = \varphi((x_{i,j} + x_{k,j})^2) - \varphi(x_{i,j}^2) - \varphi(x_{k,j}^2)
= b_j^{i,k} (x_{i,j} + x_{k,j}) - a_{i,j}x_{i,j} - a_{k,j}x_{k,j}
= a_{k,j}x_{i,j} + a_{i,j}x_{k,j}
\]
which finishes the proof. \( \square \)

We may now show triviality of negative tangents for \( m \geq 2 \) and \( n \geq 4 \) even.

**Proposition 4.7.** If \( m \geq 2 \) and \( n \geq 4 \) is even, then \( G^{(n)}(A_m) \) has trivial negative tangents.

*Proof.*** We write \( G^{(n)}(A_m) \) as the quotient of the ring \( \mathcal{R} = k[x_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n] \) by the ideal \( I \). Let \( \varphi : I \to G^{(n)}(A_m) \) be a negatively graded \( \mathcal{R} \)-module morphism. If \( \deg(\varphi) \leq -2 \), then Lemma 4.1 shows \( \varphi = 0 \). It remains to prove that if \( \deg(\varphi) = -1 \), then \( \varphi \) is in the \( k \)-span of the maps \( \partial_{i,j} \) given by differentiation by \( x_{i,j} \). Proposition 4.5 shows \( \varphi(x_{i,j}^2) = a_{i,j}x_{i,j} \) for \( a_{i,j} \in k \). Let \( \partial := \frac{1}{2} \sum_{i,j} a_{i,j} \partial_{i,j} \). Then \( \varphi(x_{i,j}^2) = \partial(x_{i,j}^2) \) and Proposition 4.6 shows that \( \varphi(x_{i,j}x_{k,j}) = \partial(x_{i,j}x_{k,j}) \), for distinct \( i \) and \( k \). Proposition 4.5 also shows \( \varphi(e_1(x_i)) = \partial(e_1(x_i)) \), and hence \( \varphi = \partial \). \( \square \)

4.4. **The case where** \( m \geq 2 \) and \( n \geq 5 \) **is odd.** In this section, we show that if \( n \) is odd, then triviality of negative tangents for \( G^{(n)}(A_m) \) follows from that of \( G^{(n-1)}(A_m) \). We first require a preliminary lemma.

**Lemma 4.8.** Let \( n \geq 3 \) be odd and
\[
R = k[w_1, \ldots, w_n, z_{n-2}, z_{n-1}, z_n]/J,
\]
where
\[
J = (w_1^2, \ldots, w_n^2, z_{n-2}^2, z_{n-1}^2, w_{n-2} + w_{n-1} + w_n, z_{n-2} + z_{n-1} + z_n)
+ (w_i z_i \mid i \geq n - 2) + (w_{2i-1} + w_{2i} \mid 1 \leq i \leq \frac{n-3}{2}).
\]

If \( f := \prod_{k \text{ odd} \atop k < n} (w_k - w_{k+1})b z_n = 0 \),
then \( b = 0 \).
Proof. We may eliminate all variables with even subscripts as follows: \( w_{n-1} = -(w_{n-2} + w_n) \), \( z_{n-1} = -(z_{n-2} + z_n) \), and \( w_{2i} = -w_{2i-1} \) for \( 1 \leq i \leq \frac{n-1}{2} \). Then \( w_{n-1}^2 = 0 \) tells us \( w_{n-2}w_n = 0 \) and similarly for the \( z \)-variables. Since \( w_{n-1}z_{n-1} = 0 \), we have \( w_{n-2}z_n = -z_{n-2}w_n \). Thus, we see \( R \cong R' \), where

\[
R' = \mathbb{K}[w_1, w_3, \ldots, w_n, z_{n-2}, z_n]/J'
\]

and

\[
J' = (w_1^2, w_3^2, \ldots, w_n^2, z_{n-2}^2, w_{n-2}z_{n-2}, w_nz_n, w_{n-2}w_n, z_{n-2}z_n, w_{n-2}z_n + z_{n-2}w_n).
\]

Notice that \( R' \) has a \( \mathbb{K} \)-basis given by \( w_1^{c_1}w_3^{c_3} \cdots w_{n-4}^{c_{n-4}}h \) with each \( c_i \in \{0, 1\} \) and \( h \) an element of \( \{1, w_{n-2}, z_{n-2}, w_n, z_n, w_{n-2}z_n\} \). In \( R' \), we have

\[
f = 2^{\frac{n-1}{2}} bw_1w_3\cdots w_{n-2}z_n.
\]

As a result, if \( f = 0 \), then \( b = 0 \). \( \square \)

Proposition 4.9. If \( m \geq 2 \) and \( n \geq 5 \) is odd, then \( G^{(n)}(A_m) \) has trivial negative tangents.

Proof. Let \( \mathcal{R} = \mathbb{K}[x_{i,j}] \). By Proposition 4.1, we need only consider maps \( \varphi : I \to G^{(n)}(A_m) = \mathcal{R}/I \) of degree \(-1\). By Lemma 4.2, we know \( \varphi \) sends the ideal \( K := (x_{1,n}, \ldots, x_{m,n}) \) to \( K/I \). Noting that

\[
G^{(n-1)}(A_m) \cong \mathcal{R}/(I + K),
\]

we obtain a commutative diagram

\[
\begin{array}{ccc}
I & \xrightarrow{\varphi} & G^{(n)}(A_m) \\
\downarrow & & \downarrow \\
I + K & \xrightarrow{\psi} & G^{(n-1)}(A_m)
\end{array}
\]

where we let \( \psi(K) = 0 \). Since \( n - 1 \) is even and at least 4, Proposition 4.7 tells us that there are \( c_{i,j} \in \mathbb{K} \) such that

\[
\psi = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} \partial_{i,j} =: \partial,
\]

where \( \partial_{i,j} \) denotes the derivative with respect to \( x_{i,j} \). We claim that \( \varphi = \partial \). Replacing \( \varphi \) by \( \varphi - \partial \), we may therefore assume \( \varphi \) has degree \(-1\) and \( \varphi(I) \subset K/I \).

We must show \( \varphi = 0 \). For \( j \neq n \), we see by Lemma 4.2 that \( \varphi(x_{i,j}x_{\ell,j}) \) is contained in \( ((x_{1,j}, \ldots, x_{m,j}) \cap K)/I = 0 \). Similarly, \( \varphi(e_1(x_i)) = 0 \) for all \( i \). It therefore remains to prove that \( \varphi(x_{i,n}x_{\ell,n}) = 0 \) for all \( i, \ell \).

For this, consider the following syzygy analogous to (11):

\[
x_{\ell,n} \prod_{k \text{ odd} \atop k < n} (x_{i,k} - x_{i,k+1}) \varphi(e_1(x_i)) =
\]

\[
x_{\ell,n} \sum_{j \text{ odd} \atop j < n} \prod_{k \text{ odd} \atop k \neq j} (x_{i,k} - x_{i,k+1})(\varphi(x_{i,j}^2) - \varphi(x_{i,j+1}^2)) + \prod_{k \text{ odd} \atop k < n} (x_{i,k} - x_{i,k+1}) \varphi(x_{i,n}x_{\ell,n}).
\]

In our case, (14) is particularly simple, and tells us

\[
\prod_{k \text{ odd} \atop k < n} (x_{i,k} - x_{i,k+1}) \varphi(x_{i,n}x_{\ell,n}) = 0.
\]
By Lemma 4.2, we may write \( \varphi(x_{i,n}x_{\ell,n}) = \sum_p b_p^{i,\ell} x_{p,n} \). Then taking into account multidegrees, the above equation tells us that for each \( p \),
\[
\prod_{k \text{ odd } k < n} (x_{i,k} - x_{i,k+1}) b_p^{i,\ell} x_{p,n} = 0,
\]
and so \( b_p^{i,\ell} \) vanishes by Lemma 4.8. \( \square \)

4.5. The case where \( n = 3 \). Our goal in this subsection is to prove the following:

**Theorem 4.10.** When \( m \geq 3 \), the ring \( G^{(3)}(A_m) \) has trivial negative tangents.

Moreover, \( G^{(3)}(A_m) \) lives on an irreducible locus \( Z_m := \mathbb{A}^{2m} \times \text{Gr}(\binom{m}{2}, m^2 + m + 1) \subset \text{Hilb}^{d(3,m)}(\mathbb{A}^{2m}) \) whose dimension is larger than that of the main component of \( \text{Hilb}^{d(3,m)}(\mathbb{A}^{2m}) \) for \( m \geq 11 \).

**Remark 4.11.** The ideal defining \( G^{(3)}(A_2) \) is given by
\[
I' = (z_1, z_2)^2 + (w_1, w_2)^2 + (z_1 w_2 + z_2 w_1) + (z_1 w_1, z_2 w_2).
\]
It is proved in [SS23, Theorem 1.3] that the ideal \( J = (z_1, z_2)^2 + (w_1, w_2)^2 + (z_1 w_2 + z_2 w_1) \) is the first member in an infinite family of smooth points on elementary components of \( \text{Hilb}^{pts}(\mathbb{A}^4) \), and that for most of these ideals, taking quotients by sufficiently general socle elements produces further examples of smooth points on elementary components [SS23, Theorem 1.5]. Here \( I' \) is obtained from \( J \) by adding socle elements, however \( I' \) cannot have trivial negative tangents, because its colength is 6 (see [CEVV09]).

Throughout this section, we use that \( G^{(3)}(A_m) \) has an explicit basis given by 1, the \( x_{i,j} \) for \( j = 1, 2 \) and all \( x_{i,1}x_{k,2} \) for \( i < k \). This follows from (10).

**Proposition 4.12.** If \( \varphi : I \to G^{(3)}(A_m) \) has degree \(-1\), then for any fixed \( j \) and any linear form \( g = \sum_i \lambda_i x_{i,j} \) with \( \lambda_i \in \mathbb{k} \), we have
\[
\varphi(g^2) = \lambda g
\]
for some \( \lambda \in \mathbb{k} \).

**Proof.** As in Proposition 4.5, it suffices to prove the result when \( g = x_{1,1} \). Note that
\[
2x_{1,1}x_{2,1} = (x_{1,1} + x_{1,2} - x_{1,3}) e_1(x_1) - x_{1,1}^2 - x_{1,2}^2 + x_{1,3}^2 \in I.
\]
Letting \( 1 < k \leq m \), and considering the multigraded pieces of the syzygy
\[
x_{k,1} \varphi(x_{1,1}x_{1,2}) = x_{1,2} \varphi(x_{1,1}x_{k,1})
\]
and using that \( \varphi(x_{1,1}x_{k,1}) \) only has \( x_{i,1} \) terms, we see that \( \varphi(x_{1,1}x_{1,2}) \) is in the span of \( x_{1,2}, x_{k,2}, \) and the \( x_{i,1} \). Since \( k \) is arbitrary and \( m \geq 3 \), this implies that \( \varphi(x_{1,1}x_{1,2}) \) is in the span of \( x_{1,2} \) and the \( x_{i,1} \).

We now consider the syzygy
\[
x_{1,2} \varphi(x_{1,1}^2) = x_{1,1} \varphi(x_{1,1}x_{1,2}).
\]
Using that \( \varphi(x_{1,1}^2) \) is in the span of the \( x_{i,1} \), we find \( \varphi(x_{1,1}^2) \) is in the span of \( x_{1,1} \). \( \square \)
Proof of Theorem 4.10. Proposition 4.1 tells us we may assume \(\deg(\varphi) = -1\). Proposition 4.12 shows \(\varphi(x_i^2) = a_{i,j} x_{i,j}\) for \(a_{i,j} \in \mathbb{k}\), and also tells us that the hypotheses of Proposition 4.6 are valid. As a result,

\[
\varphi(x_{i,j} x_{k,j}) = \frac{1}{2} a_{k,j} x_{i,j} + \frac{1}{2} a_{i,j} x_{k,j}.
\]

Thus, \(\varphi = \frac{1}{2} \sum_{i,j} a_{i,j} \partial x_{i,j}\) provided we can show \(\varphi(e_1(x_i)) = \frac{1}{2} \sum_j a_{i,j}\). By symmetry, it suffices to do so when \(i = 1\).

Returning to (16), we see that if we express \(\varphi(x_{1,1} x_{1,2})\) as a linear combination of the \(x_{i,1}\) and \(x_{i,2}\), then the \(x_{1,2}\)-coefficient of \(\varphi(x_{1,1} x_{1,2})\) is equal to the \(x_{k,1}\)-coefficient of \(\varphi(x_{1,1} x_{k,1})\), namely \(\frac{1}{2} a_{1,1}\). We next consider the syzygy

\[
2(x_{1,1} + x_{1,2}) \varphi(e_1(x_1)) = (x_{1,1} + x_{1,2} - x_{1,3}) \varphi(e_1(x_1))
\]

\[
= \varphi(x_{1,1}^2) + \varphi(x_{1,2}^2) - \varphi(x_{1,3}^2) + 2 \varphi(x_{1,1} x_{1,2})
\]

\[
= (a_{1,1} + a_{1,3}) x_{1,1} + (a_{1,2} + a_{1,3}) x_{1,2} + 2 \varphi(x_{1,1} x_{1,2}).
\]

Considering the \(x_{1,2}\)-coefficient, we find

\[
\varphi(e_1(x_1)) = \frac{1}{2} \sum_j a_{1,j}
\]

and hence \(G(3)(A_m)\) has trivial negative tangents.

Lastly, let \(\mathcal{R}' := \mathbb{k}[x_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq 2]\) and let \(M\) be the ideal generated by the \(x_{i,j} x_{k,\ell}\) for \(i = k\) or \(j = \ell\). Then \(\mathcal{R}' / M\) has basis given by 1, all \(x_{i,j}\), and all \(x_{i,1} x_{k,2}\) for \(i \neq k\).

Hence, \(D := \dim \mathcal{R}' / M = m^2 + m + 1\). From (10), we see \(G(3)(A_m)\) is obtained from \(\mathcal{R}' / M\) by additionally modding out by \((m)\) socle elements. Letting such socle elements vary over all possible choices yields a Grassmannian. Such points of this Grassmannian are all supported at the origin of \(\mathbb{A}^{2m}\), so after allowing for translation, we obtain the locus \(Z_m\). Computing, we find \(\dim Z_m = (D - (m) / (m) + 2m\) is strictly greater than \(2m(D + (m) / 2)\) when \(m \geq 11\). \(\square\)

4.6. Proof of Theorems 1.1 and 1.2. We prove Theorem 1.2 which implies Theorem 1.1.

Proof of Theorem 1.2. By [Jel19, Theorem 1.2], it suffices to prove \(G := G(n)(A_m)\) has trivial negative tangents. Note that in Theorems 1.1 and 1.2, we are viewing \(G\) as a point of \(\text{Hilb}^{d(n,m)}(\mathbb{A}^{m(n-1)})\) by eliminating the variables \(x_{i,n}\) for all \(i\), as explained in the introduction. The elimination of the \(x_{i,n}\) variables does not change the \(\mathbb{Z}\)-grading (or the \(\mathbb{Z}^m\)-grading) on \(G\). Since, \(T^1(G / \mathbb{k}, G_{<0}\) is independent of the choice of graded quotient presentation (see §2.1), to prove triviality of negative tangents, it suffices to express \(G\) as the quotient of the ring \(\mathcal{R} = \mathbb{k}[x_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n]\) by the ideal \(I\).

For \(n \geq 4\) and \(m \geq 2\), we see \(G\) has trivial negative tangents by Propositions 4.7 and 4.9. We know \(G\) has trivial negative tangents for \(n = 3\) and \(m \geq 3\) by Theorem 4.10.

It remains to consider the cases where \(n \leq 2\) or \(m = 1\) or \((n,m) = (3,2)\). When \(n = 1\), we see \(c_1(x_i) = x_{i,1}\) and so \(G(1)(A_m) = \mathbb{k}\), which has trivial negative tangents. When \(n = 2\), by eliminating all \(x_{i,1}\) variables, we find \(G(2)(A_m) = \mathbb{k}[x_{i,1}] / (x_{1,1}, \ldots, x_{m,1})^2 \simeq A_m\). When \(m \geq 2\), we see \(A_m\) is smoothable of dimension strictly greater than 1, hence does not have trivial negative tangents. When \(m = 1\), \(A_1 \simeq \mathbb{k}\) has trivial negative tangents.

For \(m = 1\) and \(n \geq 3\), consider the map \(\vphi: I \to G\) given by \(\varphi(x_{1,1}) = x_{1,2}\) and \(\varphi(g) = 0\) for all minimal generators of \(I\) with \(g \neq x_{1,1}\). Since \(x_{1,1}\) and \(x_{1,2}\) are linearly independent for
$n \geq 3$, we see $\varphi$ is not in the $\mathbb{k}$-span of the derivatives $\partial_{i,j}$. (Note that for $n=2$, $x_{1,1}$ and $x_{1,2}$ are linearly dependent.)

The remaining case $G^{(3)}(A_2)$ is checked by computer.

$$\square$$

5. Socle Elements of the Galois Closure

In this section, we give an explicit construction of the graded pieces of minimal degree in the socle of the Galois closure. This makes key use of our structure result Theorem 1.6. The main result of this section is:

**Theorem 5.1.** Let $m+1 \geq n \geq 4$ and $D = \lceil \frac{n}{2} \rceil$. The socle elements of minimal degree in $G^{(n)}(A_m)$ occur in degree $D$.

Specifically, suppose $d = (d_1, \ldots, d_m)$ has $\sum_i d_i = D$. Let $\mu = (D, D)$ if $n$ is even and $\mu = (D-1, D-1, 1)$ if $n$ is odd. Then every copy of $V_\mu \subset G^{(n)}(A_m)_d$ is contained in the socle.

The next result reduces Theorem 5.1 to the case of the Galois closure from [BS14].

**Proposition 5.2.** Theorem 5.1 holds provided it is true when $m = n - 1$.

*Proof.* Let $m \geq n - 1$. Let $d$ be as in the statement of the theorem and fix a copy $V_\mu \subset G^{(n)}(A_m)_d$. We show $V_\mu$ is contained in the socle. Multiplication by $x_{i,j}$ maps $G^{(n)}(A_m)_d$ to a multigraded piece $G^{(n)}(A_m)'_d$ where $\# \{i \mid d'_i \neq 0\} \leq [n/2] + 1 \leq n - 1$ since $n \geq 4$. So, after permuting the variables we can assume $d' = (d'_1, \ldots, d'_{n-1}, 0, \ldots, 0)$ which we can view as living in $\mathbb{N}^{n-1}$; we may therefore also view $d$ as living in $\mathbb{N}^{n-1}$. Then Lemma 3.1 tells us we have a commutative diagram

$$
\begin{array}{ccc}
G^{(n)}(A_m)_d & \xrightarrow{x_{i,j}} & G^{(n)}(A_m)'_d \\
\cong \downarrow{\pi_d} & & \downarrow{\pi_{d'}} \\
G^{(n)}(A_{n-1})_d & \xrightarrow{x_{i,j}} & G^{(n)}(A_{n-1})'_d
\end{array}
$$

with vertical maps being isomorphisms of $S_n$-representations. As a result, by the $m = n - 1$ case, we see

$$\pi_{d'}(x_{i,j}(V_\mu)) = x_{i,j}(\pi_d(V_\mu)) = 0$$

and so $x_{i,j}(V_\mu) = 0$.

Let us now show that if $\sum_i d_i < D$ and $0 \neq f \in G^{(n)}(A_m)_d$, then $f$ is not a socle. As in the previous paragraph, we may view $d$ as living in $\mathbb{N}^{n-1}$ and so we have an isomorphism $\pi_d: G^{(n)}(A_m)_d \xrightarrow{\cong} G^{(n)}(A_{n-1})_d$. By the $m = n - 1$ case, we know $\pi_d(f)$ is not a socle, so there exists $i,j$ with $x_{i,j}f \neq 0$. Since $i \leq n - 1 \leq m$, we may multiply by $x_{i,j}$ in $G^{(n)}(A_m)$ to obtain a commutative diagram as above. It follows that $\pi_{d'}(x_{i,j}f) = x_{i,j}\pi_d(f) \neq 0$ and hence $x_{i,j}f \neq 0$. $\square$

For the remainder of this section, we assume $m = n - 1$ and freely use the notation and terminology from [BS14, Section 12.2] and [Sag01, Chapter 2]. We write $G(A)$ in place of $G^{(n)}(A_{n-1})$ and write partitions of $n$ as $\mu = (\mu_1, \ldots, \mu_\ell)$ with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_\ell > 0$ and $\sum_i \mu_i = n$. To each such $\mu$, there is an associated $S_n$-representation $M_\mu$ with basis given by tabloids of shape $\mu$, see e.g., [Sag01, Definition 2.1.5]. We conflate $\mu$ with its Ferrers diagram, see [Sag01, Definition 2.1.1]. We label the boxes of $\mu$ from left-to-right and top-down, e.g. the $\mu_1$th box is the last box on the first row of $\mu$, and the $(\mu_1 + 1)$th box is the first box on the
second row of $\mu$. Let $\{t_\mu\}$ denote the standard generator of $M_\mu$, i.e. it is the row-equivalence class of the tableau $t_\mu$ of shape $\mu$ whose $i$th box has label $i$, i.e. $t_\mu(i) = i$.

We recall how the copies of $V_\mu \subset G(A)_d$ are constructed. Let $\lambda$ be the partition associated to $d$ and let $T$ be a (generalized) tableau of shape $\mu$ and content $\lambda$; this necessarily implies $\mu \triangleright \lambda$. Then there is a morphism $\theta_{T,\sigma}: M_\mu \rightarrow M_\lambda$, see [Sag01, Definition 2.9.3]. As shown in the proof of [BS14, Proposition 33] and the surrounding discussion, $G(A)_d = M_a/I_a \cong M_\lambda/I_\lambda$, where $a = (n - \sum d_i, d_1, \ldots, d_{n-1})$ is the ordered partition associated to $d$ and the isomorphism is induced by a permutation $\sigma \in S_n$ satisfying $\sigma(a) = (a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, \ldots, a_{\sigma^{-1}(n)}) = \lambda$. So we obtain a map

$$\theta_{T,\sigma}: M_\mu \xrightarrow{\theta_T} M_\lambda \xrightarrow{\sigma^{-1}} M_\mu \rightarrow G(A)_d.$$  

There is a canonical copy of $V_\mu \subset M_\mu$. If $T$ is semi-standard and $\mu_1 = \lambda_1$, then the image of $V_\mu$ under $\theta_{T,\sigma}$ yields a copy $V_\mu \subset G(A)_d$, see [Sag01, Theorem 2.10.1] and the proof of [BS14, Proposition 35]. The content of [BS14, Theorem 27] is that these are all the copies of $V_\mu$ in $G(A)_d$.

Explicitly, the map $\theta_{T,\sigma}$ can be described in the following way (see the proof of [BS14, Proposition 35]). Let $\sigma^{-1}T$ be the tableau of shape $\mu$ and content $a$ whose $i$th label is $(\sigma^{-1}T)(i) = \sigma^{-1}(T(i))$. To every tableau $S$ of shape $\mu$ and content $a$, we obtain an element $\alpha(S) \in G(A)_d$ given as follows: let $x_0 := 1$ and $S(i)$ denote the label in the $i$th box of $S$. Then

$$(17) \quad \alpha(S) := \prod_{i=1}^n x_{S(i)-1,i} \text{ and } \theta_{T,\sigma}(\{t_\mu\}) = \sum_{S \sim \sigma^{-1}T} \alpha(S),$$

where $S \sim \sigma^{-1}T$ means that $S$ and $\sigma^{-1}T$ are row equivalent, i.e. they are both of shape $\mu$ and content $a$, and they have the same labels in each row up to permutation.

Ultimately, our goal is to determine how multiplication by $x_{i,j}$ acts on the copies of $V_\mu \subset G(A)_d$. In order to do so, we begin by understanding how multiplication by $x_{i,j}$ relates to the maps $\theta_{T,\sigma}$. We make the following observations which will aid in our study of the socle. First, if $d = (d_1, \ldots, d_{n-1})$, then multiplication by $x_{i,j}$ sends $G(A)_d$ to $G(A)_{d'}$ where $d' = (d_1, \ldots, d_{i-1}, d_i + 1, d_{i+1}, \ldots, d_{n-1})$. We may assume $n - \sum d_i' \geq d_j'$ for all $j$, otherwise $G(A)_{d'} = 0$ by [BS14, Lemma 31]. Let $\lambda$ and $a$ be the partition and ordered partition associated to $d$, and let $\sigma(a) = \lambda$; let $k$ be the minimal index such that $\lambda_k = d_i$. Let $\lambda'$ and $a'$ be the partition and ordered partition associated to $d'$, so that

$$\lambda' = (\lambda_1 - 1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k + 1, \lambda_{k+1}, \ldots, \lambda_n).$$

Let $\mu \triangleright \lambda$ with $\mu_1 = \lambda_1$.

Let $T$ be a semi-standard tableau of shape $\mu$ and content $\lambda$. Since $T$ is semi-standard and $\mu_1 = \lambda_1 = a_1$, all boxes in the first row of $T$ and $\sigma^{-1}T$ are labelled 1 and no other 1 labels occur in either tableau. Let $\mu := (\mu_1, \mu_2, \ldots, \mu_\ell)$ and $\tilde{\mu'} := (\mu_1 - 1, \mu_2, \ldots, \mu_\ell, 1)$. Note that $\tilde{\mu'}$ might not be a partition shape since $\mu_1 - 1$ might be less than $\mu_2$. Let $T'$ be obtained from $T$ by removing a box from the first row and creating a new $(\ell + 1)$th row with a single box labelled $k$. Let $\mu'$ be the partition obtained from $\tilde{\mu'}$ by rearranging its rows in decreasing order. Correspondingly, we permute rows of $T'$ to obtain a tableau $T'$ of shape $\mu'$. Note that $T'$ has content $\lambda'$ but might not be semi-standard. Moreover, we may assume $\sigma(a') = \lambda'$ (namely, that $\sigma^{-1}(k) = i + 1$), so that $\theta_{T',\sigma}: M_{\mu'} \rightarrow G(A)_{d'}$ is determined by $\{t_{\mu'}\} \mapsto \sum_{S \sim \sigma^{-1}T} \alpha(S')$. In fact, the tableau $\sigma^{-1}T'$ has content $a'$ and, by construction, to
each $S \sim \sigma^{-1}T$ there is a uniquely corresponding tableau $S' \sim \sigma^{-1}T'$ whose rows are the same as in $S$, except for the shorter row of 1’s and the additional $(\ell + 1)$th row (the rows might have been rearranged). In particular, every $\alpha(S')$ is divisible by $x_{i,n}$ and so there exists $\tau \in S_n$ acting on $G(A)_d$ such that $\tau \alpha(S') = x_{i,j} \alpha(S)$, for all corresponding $S'$ and $S$. This shows that $x_{i,j} \theta_{T',\sigma}(t_{\mu'}) = \tau \theta_{T',\sigma}(t_{\mu'})$, and more generally, that if $\pi \in S_n$ and $x_{i,j} \theta_{T',\sigma}(\pi t_{\mu})$ is non-zero, then it is of the form $\rho \theta_{T',\sigma}(t_{\mu'})$ with $\rho \in S_n$. We therefore obtain a commutative diagram

$$
\begin{array}{ccc}
G(A)_d & \xrightarrow{x_{i,j}} & G(A)_{d'} \\
\downarrow \theta_{T',\sigma} & & \downarrow \tau \theta_{T',\sigma} \\
M_\mu & \xrightarrow{\chi_{i,j}} & M_{\mu'}.
\end{array}
$$

(18)

where $\chi_{i,j}(\pi t_{\mu}) = 0$ if $\pi^{-1}(j) > \lambda_1$, and $\chi_{i,j}(\pi t_{\mu}) = \tau^{-1} \rho \pi t_{\mu'}$ otherwise. Since $M_\mu$ has a basis given by the elements $\pi t_{\mu}$, this yields a well-defined map $\chi_{i,j}$. Note that $\chi_{i,j}(t_{\mu}) = t_{\mu'}$ and that $\chi_{i,j}$ is merely a vector space map which is not $S_n$-equivariant (just as multiplication by $x_{i,j}$ is not $S_n$-equivariant).

We now turn to the “moreover” statement in Theorem 5.1.

**Proposition 5.3.** Let $n$, $D$, $d$, and $\mu$ be as in Theorem 5.1. Then every copy of $V_\mu \subset G(A)_d$ is contained in the socle.

**Proof.** Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be the partition, and $a$ the ordered partition, associated to $d$. By definition of $d$ and $D$, we necessarily have $\mu_1 = \lambda_1 = a_1$. Then every copy of $V_\mu$ in $G(A)_d$ arises as the image $\theta_{T',\sigma}(V_\mu)$ under the map $\theta_{T',\sigma}: M_\mu \rightarrow G(A)_d$ for some semi-standard tableau $T$ of shape $\mu$ and content $\lambda$, and permutation $\sigma$ with $\sigma(a) = \lambda$. It is therefore enough to show that $\theta_{T',\sigma}(M_\mu)$ is contained in the socle.

Consider the commutative diagram (18). We may assume $n - \sum_k d'_k \geq d'_k$ for all $k$, otherwise $G(A)_{d'} = 0$. Letting $\lambda'$ be the partition associated to $d'$, we therefore have $\lambda' = (\lambda_1 - 1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k + 1, \lambda_{k+1}, \ldots, \lambda_\ell)$ where $k$ is the least integer for which $\lambda_k = d_i$. On the other hand, $\mu' = (D, D - 1, 1)$ if $n$ is even and $\mu' = (D - 1, D - 2, 1, 1)$ if $n$ is odd; note $D - 1 \geq 1$ if $n$ is even and $D - 2 \geq 1$ if $n$ is odd since $n \geq 4$. Thus, we must have $\mu'_k = \mu_1 = \lambda_1 > \lambda'_1$, which implies, by [BS14, Proposition 35] that the image of $\theta_{T',\sigma}$ is zero. We have therefore shown that $\theta_{T',\sigma}(M_\mu)$ lives in the socle. \qed

Let $m$ denote the ideal of $R$ generated by the $x_{i,j}$. The following proposition plays a key role in the remainder of this section.

**Proposition 5.4.** Let $d = (d_1, \ldots, d_{n-1})$ and suppose $d_p = d_{p+1} = \cdots = d_{n-1} = 0$. for some $1 \leq p < n$. Let $d' = (d_1, \ldots, d_{p-1}, 1, 0, \ldots, 0)$, let $\lambda$ be the partition associated to $d$ and $\lambda'$ be the partition associated to $d'$. Let $\nu = (\nu_1, \ldots, \nu_\ell)$ be a partition with $\nu \triangleright \lambda$ and $\lambda_1 = \nu_1 > \nu_2$. Let $T$ be a semi-standard tableau of shape $\nu$ and content $\lambda$, and let $T'$ be obtained from $T$ by removing a box from the first row and creating a new $(\ell + 1)$th row with a single box labelled $p + 1$.

Then $\nu' = (\nu_1 - 1, \nu_2, \ldots, \nu_\ell, 1)$ is a partition, $T'$ is semi-standard of shape $\nu'$ and content $\lambda'$, $\theta_{T',1}|_{V_{\nu'}}$ is injective, and $\theta_{T',1}(V_{\nu'}) \subset m \cdot \theta_{T,1}(V_{\nu}).$

**Proof.** Since $\nu_1 > \nu_2$, it is clear that $\nu'$ is a partition. Next, since $\nu \triangleright \lambda$ and $\lambda_1 = \nu_1 > \nu_2$, we also have $\lambda_1 > \lambda_2$. As a result, $\lambda'_1 = \lambda_1 - 1$. It is then immediate that $T'$ is semi-standard of
shape $\nu'$ and content $\lambda'$. Because $\lambda'_1 = \nu'_1$, the structure result [BS14, Theorem 27] tells us that $\theta_{T',1}|_{V'_{\nu'}}$ is injective.

Consider the maps $\chi_{p,j}$ constructed in (18). To finish the proof, it suffices to show that $V_{\nu'}$ is in the span of $\{\chi_{p,j}(V_{\nu'}) \mid j \geq 1\}$.

Let $\{t\} \in M_{\nu'}$ be a standard tabloid. Let $c_i$ denote the label appearing in the first box of the $i$th row of $t$ (so if $\{t\} = \{t_{\nu'}\}$ is the standard generator of $M_{\nu'}$, then $c_1 = 1$ and $c_i = \nu_1 + \cdots + \nu_{i-1}$, for $1 < i \leq \ell + 1$) and let $t_i$ be the tableau of shape $\nu$ defined as follows: its labels are the same as those of $t$ except in the first and last column; in the first column, the labels from top to bottom are $c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_{\ell+1}$ and the label in the unique box of the last column is given by $c_i$. Consider the polytabloid $e_t \in V_{\nu'}$ associated to $t$ (see [Sag01, Definition 2.3.2, Theorem 2.5.2]). Recall that $e_t$ is a signed sum of tabloids obtained from $t$ by permuting labels within columns. Extending the map $\alpha$ from (17) by linearity, it makes sense to evaluate $\alpha$ on a polytabloid. Let $C_i$ and $C_{t_i}$ denote the column-stabilizers of $t$ and $t_i$, respectively. We obtain the equality

$$\theta_T(e_t) = \theta_T \left( \sum_{x \in C_t} (\text{sign } \pi) \pi \{t\} \right) = \sum_{x \in C_t} (\text{sign } \pi) \pi \theta_T(t) = \sum_{x \in C_t} (\text{sign } \pi) \pi \left( \sum_{S' \sim T'} S' \right)$$

and thus

$$\theta_{T',1}(e_t) = \sum_{x \in C_t} \sum_{S' \sim T'} (\text{sign } \pi) \pi \alpha(\pi \cdot S') \in G(A)_{d'},$$

where $\pi \in S_n$ acts on a generalized tableau $S'$ here by permuting positions, as in [Sag01, Proposition 2.9.2]. There is a natural bijection $C_i = \{\pi \in C_t \mid \pi(n) = c_i\} \cong C_{t_i}$, $\pi \leftrightarrow \rho$, obtained by requiring that the labels of $\pi t$ and $\rho t_i$ coincide where their shapes intersect (in the sense of [Sag01, Definition 5.1.2]). Breaking up this sum according to which $c_i$ appears in the $n$th box of $\nu'$, we obtain

$$\theta_{T',1}(e_t) = \sum_{i=1}^{\ell+1} \sum_{x \in C_t} \sum_{S' \sim T'} (\text{sign } \pi) x_{p,c_i} \prod_{j=1}^{n} x_{S'_{\pi(j-1)}} \cdot x_{S'_{\pi(-1)j}}$$

$$= \sum_{i=1}^{\ell+1} \sum_{x \in C_t} \sum_{S' \sim T'} (\text{sign } \pi) x_{p,c_i} \prod_{j=1}^{n} x_{S'_{\pi(-1)j}}$$

On the other hand, we have

$$\theta_{T,1}(e_t) = \sum_{\rho \in C_{t_i}} (\text{sign } \rho) \alpha(\rho \cdot S) = \sum_{\rho \in C_t} (\text{sign } \rho) \prod_{j=1}^{n} x_{S_{\pi(-1)j}} \in G(A)_{d'}.$$
Using the obvious bijection \( \{S' \mid S' \sim T' \} \cong \{S \mid S \sim T \} \) and applying the appropriate permutation \( \tau \) to reposition labels finally gives

\[
\tau \theta_{T',1}(e_i) = \sum_{i=1}^{\ell+1} (-1)^{\ell+1-i} x_{p,\tau(e_i)} \theta_{T,1}(e_{\ell+1-i}).
\]

Hence, this shows that \( e_i \) is in the span of \( \{\chi_{p,j}(V_\nu) \mid j \geq 1 \} \), finishing the proof. \( \square \)

Theorem 5.1 subsequently follows from the corollary below.

**Corollary 5.5.** Let \( n \) and \( D \) be as in Theorem 5.1. If \( d = (d_1, \ldots, d_{n-1}) \) satisfies the inequality \( \sum_k d_k < D \), then \( G(A)_d \) contains no non-trivial socle elements.

**Proof.** Since the socle is an \( S_n \)-subrepresentation of \( G(A) \), it suffices to prove that for all partitions \( \nu \), no copy of \( V_\nu \) in \( G(A)_d \) is contained in the socle. Suppose to the contrary. Let \( \lambda = (\lambda_1, \ldots, \lambda_p) \) be the partition associated to \( d \); note that \( p-1 \leq \sum_i d_i < D \). Because \( G(A)_d \cong G(A)(\lambda_2, \ldots, \lambda_n) \) as \( S_n \)-representations, we may assume that the partitions associated to \( d \) satisfy \( \lambda = a \). Suppose there exists some \( \nu \triangleright \lambda \) with \( \nu_1 = \lambda_1 \) and some semi-standard tableau \( T \) of shape \( \nu \) and content \( \lambda \) such that \( \theta_{T,1}(V_\nu) \subset G(A)_d \) is contained in the socle. We will examine multiplication by \( x_{p,j} \), where \( p \) is the number of rows of \( \lambda \).

Let \( \nu = (\nu_1, \ldots, \nu_\ell) \) and \( d' = (d_1, \ldots, d_{\ell+1}, 1, 0, \ldots, 0) \). Since \( \lambda_1 > n - D \), we have \( \lambda_1 \geq \frac{n+1}{2} \) regardless of whether \( n \) is even or odd. This implies \( \lambda_1 > \lambda_2 \); indeed, otherwise \( \lambda_1 + \lambda_2 = 2 \lambda_1 > n \), a contradiction. Therefore, the partition \( \lambda' \) associated to \( d' \) satisfies \( \lambda'_1 = \lambda_1 - 1 \). Similarly, \( \nu_1 = \lambda_1 > n - D \) and so \( \nu_1 > \nu_2 \). We may therefore apply Proposition 5.4. By our assumption that \( \theta_{T,1}(V_\nu) \subset G(A)_d \) is contained in the socle, we see that \( x_{p,j} \theta_{T,1}(V_\nu) = 0 \) and so \( \theta_{T,1}(\chi_{p,j}(V_\nu)) = 0 \). However, Proposition 5.4 tells us every non-zero element \( v \in V_\nu \) is in the span of \( \{\chi_{p,j}(V_\nu) \mid j \geq 1 \} \). This implies that \( \theta_{T,1}(v) = 0 \), contradicting the fact that \( \theta_{T,1}|_{V_\nu'} \) is injective. \( \square \)

The following corollary characterizes the annihilator of the square of the maximal ideal \( \mathfrak{m} \). This will play an important role in Theorem 1.7 below.

**Corollary 5.6.** Let \( n \) and \( D \) be as in Theorem 5.1. Let \( d = (d_1, \ldots, d_{n-1}) \) satisfy the inequality \( \sum_k d_k < D \) and let \( \lambda \) and \( a \) be the partitions associated to \( d \). If \( n \) is even, then \( G(A)_d \cap \text{Ann}(\mathfrak{m}^2) = 0 \) holds. If \( n \) is odd, let \( \varepsilon = (D, D-1) \); then we have \( G(A)_d \cap \text{Ann}(\mathfrak{m}^2) \subseteq \sum_T \theta_{T,\sigma}(V_\varepsilon) \), where \( T \) runs over semi-standard Young tableaux of shape \( \varepsilon \) and content \( \lambda \), and \( \sigma(a) = \lambda \).

**Proof.** Let \( \lambda = (\lambda_1, \ldots, \lambda_p) \). As \( G(A)_d \cap \text{Ann}(\mathfrak{m}^2) \) is a subrepresentation, it suffices to determine which \( V_\nu \)'s are contained in the intersection; again, we may assume \( a = \lambda \). Let \( \nu = (\nu_1, \ldots, \nu_\ell) \triangleright \lambda \) with \( \nu_1 = \lambda_1 \) and let \( T \) be semi-standard of shape \( \nu \) and content \( \lambda \). Because \( \sum_k d_k < D \), we have \( \nu_1 > \nu_2 + 1 \), unless \( n \) is odd and \( \nu = \varepsilon \).

If \( \nu_1 > \nu_2 + 1 \), then \( \lambda_1 > \lambda_2 + 1 \) as well. In this case, noting that \( n \geq 4 \) and \( p-1 < D \), we see \( \lambda' = (\lambda_1 - 2, \lambda_2, \lambda_3, \ldots, \lambda_p, 1, 1) \) and \( \nu' = (\nu_1 - 2, \nu_2, \nu_3, \ldots, \nu_\ell, 1, 1) \) are partitions. Let \( T' \) be the semi-standard tableau obtained from \( T \) by labelling the boxes on the \((\ell+1)\)th and \((\ell+2)\)th rows with \( p+1 \) and \( p+2 \), respectively. Applying Proposition 5.4 twice, we see that \( \theta_{T,1}(V_\nu) \subseteq \mathfrak{m}^2 \theta_{T,1}(V_\nu) \). Because \( \theta_{T,1}|_{V_\nu'} \) is injective, we find that \( \theta_{T,1}(V_\nu) \) cannot be contained in \( \text{Ann}(\mathfrak{m}^2) \). \( \square \)
6. TRIVIAL NEGATIVE TANGENTS FOR QUOTIENTS BY MINIMAL DEGREE SCLES

In this section, we prove Theorem 1.7 and Corollary 1.9, giving a formula for how large we may take the value of \( r \). We begin with two examples showing that condition (ii) in Theorem 1.7 is neither vacuous or superfluous.

**Example 6.1.** One checks via computer that we may quotient \( G(A_4) = G(5)(A_4) \) by all copies of \( V_\mu \) in \( G(A_4)_d \) for all \( d \) whose associated partition is \( (1,1,1) \), i.e., in Theorem 1.7, we may take \( n = 5, m = 4, \) and \( r = 40 \).

**Example 6.2.** Let \( J \) be the ideal of \( G(A_6) = G(7)(A_6) \) generated by all socle elements in degree \( D = 4 \). Let \( T \) be the tableau of shape \( \varepsilon = (4,3) \) and content \( \lambda = (4,1,1,1) \), so the entries of \( T \) are \( 1,1,1,2,3,4 \) read left-to-right and top-down. Let \( t \) be the standard tableau of shape \( \varepsilon \) whose \( i \)th box has label \( i \). One can check by computer that \( \theta_{T,1}(e_t) \) yields a non-zero socle element of \( B \) in degree 3.

In order to prove that \( B \) has trivial negative tangents, we compare \( T^1(B/\mathbb{k}, B) \) to \( T^1(G(n)(A_m)/\mathbb{k}, B) \) and \( T^1(B/G^{(n)}(A_m), B) \). The following gives the appropriate vanishing result we need for \( T^1(G^{(n)}(A_m)/\mathbb{k}, B)_{<0} \).

**Proposition 6.3.** Let \( n \geq 4 \) and \( m \geq 2 \). Let \( s_1, \ldots, s_r \in G^{(n)}(A_m) \) be socle elements and assume each \( s_i \) is contained in a multigraded component \( G^{(n)}(A_m)_{d_i} \) where \( d_i = (d_{i,1}, \ldots, d_{i,m}) \) satisfies:

- \( \sum_j d_{i,j} \geq 3 \).
- if \( D_i := \sum_j d_{i,j} \in \{ [\frac{n}{2}], [\frac{n}{2}] \} \), then every \( d_{i,j} \leq D_i - 2 \).

If \( J = (s_1, \ldots, s_r) \), then \( B := G^{(n)}(A_m)/J \) satisfies \( T^1(G^{(n)}(A_m)/\mathbb{k}, B)_{<0} = 0 \).

**Proof.** Let \( \mathcal{R} = \mathbb{k}[x_{i,j}] \) and let \( \varphi : I \rightarrow B \) be a graded \( \mathcal{R} \)-module map of negative degree.

First suppose \( \deg(\varphi) \leq -2 \). Then (5) holds modulo \( J \); the terms in the equation have degree at most 1 and none of the \( s_i \) have degree 1, so we see (5) remains true. Thus, the proof of Proposition 4.1 shows \( \varphi = 0 \).

Now suppose \( \deg(\varphi) = -1 \). Note first that (6) holds modulo \( J \) and since all terms in the ensuing expansion have degree 2 < \( D_i \) for all \( i \), the equations following (6) also remain true. As a result, the conclusion of Lemma 4.2 is still valid.

To handle the case where \( n \) is even, it therefore suffices to show that the proof of Proposition 4.7 still applies in our setting. For this, it is enough to show that the conclusions of Proposition 4.5 and Proposition 4.6 remain valid. Note that (11) holds modulo \( J \). By assumption, \( J \) contains no elements of multidegree \( (\frac{n}{2} - 1,0,\ldots,0,1,0,\ldots,0) \), where the 1 is in the \( i \)th place and \( i \neq 1 \). As a result, (12) holds. Similarly, \( J \) contains no elements of multidegree \( (\frac{n}{2},0,\ldots,0) \), and so (13) holds. Hence, the conclusion of Proposition 4.5 remains valid. Finally, Proposition 4.6 still holds since the equations considered in the proof only involve syzygies of degree at most 2. We conclude that \( \varphi \) is a \( \mathbb{k} \)-linear combination of derivatives \( \partial_{i,j} \).

We now turn to the case where \( n \) is odd. We must show the proof of Proposition 4.9 applies. As shown above, the conclusions of Proposition 4.1 and Lemma 4.2 hold so we again have a
commutative diagram

\[
\begin{array}{ccc}
I & \xrightarrow{\varphi} & G^{(n)}(A_m)/J \\
\downarrow & & \downarrow \\
I + K & \xrightarrow{\psi} & G^{(n-1)}(A_m)/J'
\end{array}
\]

where \(J'\) is the ideal generated by the images of the \(s_i\) in \(G^{(n-1)}(A_m)\), and where we let \(\psi(K) = 0\). Since \(n - 1 \geq 4\) is even, there are \(c_{i,j} \in \mathbb{k}\) such that \(\psi = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} \partial_{i,j} =: \partial\). Replacing \(\varphi\) by \(\varphi - \partial\), we may assume \(\varphi(I) \subset K + K'\), where \(K'\) is generated by all \(s_i\) mapping to 0 in \(G^{(n-1)}(A_m)\). We must prove \(\varphi = 0\). Since all \(D_i \geq 3\), we see \(\varphi(x_{i,j}x_{l,j}) = \varphi(e_1(x_i)) = 0\) for \(j \neq n\). Using that (14) holds and using our hypothesis on the \(d_{i,j}\), we see (15) holds and so we again find \(\varphi(x_{i,n}x_{l,n}) = 0\).

We have therefore shown that \(T^1(G^{(n)}(A_m)/\mathbb{k}, B)_{<0} = 0\). 

We next give a general criterion to show \(T^1(B'/A', B')_{<0} = 0\), where \(A' \to B'\) is a surjection of rings. (We add superscript primes so as not to conflict with our running notation \(G^{(n)}(A_m) = \mathcal{R}/I, B = G^{(n)}(A_m)/J, \) etc.) In the proof of Theorem 1.7 we apply this criterion to show that \(T^1(B/B'(A)^m, B)_{<0} = 0\).

**Proposition 6.4.** Let \(I' \subset S' = \mathbb{k}[y_1, \ldots, y_n] \) be a graded ideal. Let \(A' = S'/I'\) and \(D\) be the minimal degree of a socle element of \(A'\). If \(s_1, \ldots, s_r\) are linearly independent degree \(D\) socle elements modulo \(I'\), \(J' = (s_1, \ldots, s_r) + I'\), and \(B' = S'/J'\) has no socle elements in degrees less than \(D\), then \(T^1(B'/A', B')_{<0} = 0\).

**Proof.** We first construct the truncated cotangent complex \(L_{B'/A'}\). We have a surjective map \(\pi: A' \to B'\), from which we see that \(L_{B'/A'} = \Omega^1_{A'/A} \otimes_A B' = 0\). Next, \(\ker \pi = J'/I'\) which, as a graded \(A'\)-module, is isomorphic to \(\mathbb{k}(-D)^{\oplus r}\). We then have a short exact sequence

\[0 \to m_{A'}(-D)^{\oplus r} \to A'(-D)^{\oplus r} \to J'/I' \to 0\]

and we let \(\text{Kos} \subset m_{A'}(-D)^{\oplus r}\) denote the Koszul syzygies. From this data, we see the truncated cotangent complex is given by

\[L_{B'/A'} : 0 \leftarrow B'(-D)^{\oplus r} \leftarrow m_{A'}(-D)^{\oplus r} / \text{Kos} .\]

Every element of \(T^1(B'/A', B')\) is represented by a \(B'\)-module map \(\varphi: B'(-D)^{\oplus r} \to B'\) which vanishes when precomposed with \(m_{A'}(-D)^{\oplus r} / \text{Kos} \to B'(-D)^{\oplus r}\); in particular, it vanishes when precomposed with \(m_{A'}(-D)^{\oplus r} \to B'(-D)^{\oplus r}\), and hence must send the generators of \(B'(-D)^{\oplus r}\) to \(\text{Soc}(B')\). However, there are no non-zero socle elements of degree less than \(D\), so if \(\varphi\) is negatively graded, it must vanish. \(\square\)

Combining Propositions 6.3 and 6.4 will yield all statements of Theorem 1.7, except for (i) and (ii); we address (i) and (ii) in the result below

**Proposition 6.5.** Let \(m \geq n - 1\), and keep the notation and hypotheses of Theorem 1.7. If (i) or (ii) hold, then \(B\) has no socle elements of degree less than \(D\).

**Proof.** Any such socle element of \(B\) would yield a non-zero element in \(G^{(n)}(A_m) \cap \text{Ann}(m^2)\) of degree less than \(D\). By Corollary 5.6, \(n\) must be odd and all such elements live in copies of \(V_\varepsilon\) in degree \(D - 1\). \(\square\)
Proof of Theorem 1.7. The pair of natural ring maps \( k \to G^{(n)}(A_m) \to B \) yields a long exact sequence containing the following portion, see e.g., [Har10, Theorem 3.5]:
\[
\cdots \longrightarrow T^1(B/G^{(n)}(A_m), B) \longrightarrow T^1(B/k, B) \longrightarrow T^1(G^{(n)}(A_m)/k, B) \longrightarrow \cdots.
\]
We know \( T^1(G^{(n)}(A_m)/k, B)_{< 0} \) vanishes by Proposition 6.3 and \( T^1(B/G^{(n)}(A_m), B)_{< 0} \) vanishes by Proposition 6.4. Thus, \( T^1(B/k, B)_{< 0} = 0 \). It follows from [Jel19, Theorem 1.2] that all components of the Hilbert scheme containing \([B]\) are elementary. The remaining statements of Theorem 1.7 are handled by Proposition 6.5.

Lastly, we turn to Corollary 1.9, giving a formula for the value of \( r \) appearing in Theorem 1.7.

Proof of Corollary 1.9. Let \( D \) and \( \mu \) be as in Theorem 5.1. If \( d = (d_1, \ldots, d_m) \) with \( \sum_i d_i = D \), then the socle of \( G^{(n)}(A_m) \) contains every copy of \( V_\mu \) in \( G^{(n)}(A_m)_d \). When \( n \geq 6 \) is even, Theorem 1.7 tells us we may take the quotient by all such copies \( V_\mu \) and maintain triviality of negative tangents, provided that every \( d_i \leq \lfloor \frac{n}{2} \rfloor - 2 \). Letting \( \lambda \) be the partition associated to \( d \), the above condition exactly means that \( \lambda \) is in the set \( P \) defined before the statement of Corollary 1.9. By Theorem 1.6, all such copies of \( V_\mu \) together have dimension
\[
R_n := \sum_{\lambda \in P} m_\lambda K_{\mu \lambda} \dim V_\mu.
\]
It therefore suffices to show that \( R_n \) agrees with the function \( R(n) \).

Using the hook length formula [Sag01, Theorem 3.10.2], one computes
\[
\dim V_\mu = \frac{n!}{(\frac{n}{2})!(\frac{n}{2} + 1)!} = C_{n/2},
\]
for even \( n \). Since \( \lambda_1 = \mu_1 \), it is easy to see that when \( n \) is even, there is precisely one semi-standard Young tableau of shape \( \mu \) and content \( \lambda \), hence \( K_{\mu \lambda} = 1 \). This establishes that \( R_n = R(n) \). \( \square \)

7. Smoothness and obstructions for \( G^{(n)}(A_m) \)

We now turn to Theorem 1.4, which addresses precisely when the obstruction space for \( G^{(n)}(A_m) \) vanishes. In particular, this shows \( G^{(n)}(A_m) \) is smooth for \( n = 3 \) and for some sporadic cases where \( n = 4 \).

Proof of Theorem 1.4. We first handle the case when \( n = 3 \). By (10), we see the ideal defining \( G^{(3)}(A_m) \) is generated by quadratics. So all syzygies live in degree at least 3. Thus, any map \( \psi \colon L_G^{(3)}(A_m),2 \to G^{(3)}(A_m) \) of non-negative degree must have image in degree at least 3, however \( G^{(3)}(A_m) \) is concentrated in degree at most 2. Thus, \( T^2(G^{(3)}(A_m)/k, G^{(3)}(A_m))_{\geq 0} \) vanishes.

The remaining arguments proceed as follows. We choose a homogeneous element \( s \in L_{2,G^{(n)}(A_m)} \) corresponding to a minimal syzygy of \( I \), and choose a homogeneous non-zero socle element \( f \in G^{(n)}(A_m) \) with \( \deg(f) \geq \deg(s) \). We then let
\[
\psi : L_{2,G^{(n)}(A_m)} \to G^{(n)}(A_m)
\]
send \( s \) to \( f \) and send all other minimal syzygies to 0. The map \( \psi \) is well-defined since \( f \) is in the socle; note that \( \deg(\psi) = \deg(f) - \deg(s) \geq 0 \). We then show \( [\psi] \neq 0 \).
Consider first the case where \( m \geq 2 \) and \( n \geq 8 \). Let \( \ell = 2 \) (resp. \( \ell = 3 \)) if \( n \) is even (resp. odd), and let

\[
f = \prod_{j=1}^{\ell} x_{2,j} \cdot \prod_{j=\ell+1}^{(n+\ell)/2} x_{1,j}.
\]

We see that the multidegree \((2-\ell, \ell, 0, \ldots, 0)\) piece of \( G(n)(A_m) \) is generated as an \( S_n \)-representation by \( f \). Since this multidegree piece is non-zero by Theorem 1.6, we see \( f \neq 0 \). Furthermore, \( f \) is in the socle by Lemma 3.2(i). Now, if \( n \) is even, let \( s \) be the syzygy among \( e_1(x_2) \) and \( x_{2,1}^2, \ldots, x_{2,n}^2 \) given by equation (8) with \( \sigma = \text{id} \). If \( n \) is odd, let \( s \) be the syzygy among \( e_1(x_2), x_{2,1}^2, \ldots, x_{2,n-1}^2 \), and \( x_{2,n} \) given by equation (14). Then we consider the map \( \psi \) sending \( s \) to the socle \( f \), and sending all other syzygies to zero; note that

\[
\deg(\psi) = \deg(s) - \deg(f) = 0.
\]

If \( [\psi] = 0 \), then \( \psi \) extends to a map \( \varphi: L_{G(n)(A_m)/S_n} \rightarrow G(n)(A_m) \). This implies that \( f \) is expressible as an \( \mathbb{k}[x_{i,j}] \)-linear combination of monomials all of which have multidegree \((d_1, \ldots, d_m)\) with \( d_2 \geq (n+\ell)/2 - 2 \), e.g., if \( n \) is even, we then have

\[
f = \prod_{k \text{ odd}} (x_{2,k} - x_{2,k+1}) \varphi(e_1(x_1)) - \sum_{j \text{ odd}} \prod_{k \text{ odd}, k \neq j} (x_{2,k} - x_{2,k+1}) (\varphi(x_{2,j}^2) - \varphi(x_{2,j+1}^2)).
\]

Since \( n \geq 8 \), we see \( (n+\ell)/2 - 2 > \ell \), which is a contradiction. Thus, \( [\psi] \neq 0 \).

It remains to consider the cases \( 4 \leq n \leq 7 \). Assuming \( m \geq n + 1 \), we take

\[
f = x_{1,1} \cdots x_{n-1,n-1}
\]

and \( s \) to be the cubic syzygy between \( x_{n,n}^2 \) and \( x_{n,n} x_{n+1,n} \). Again by Theorem 1.6, \( f \) is non-zero since it generates the multidegree piece \((1, \ldots, 1, 0, \ldots, 0)\) of \( G(n)(A_m) \) which is the sign representation; it is a socle by Lemma 3.2(i). We see \( \deg(\psi) = n - 4 \geq 0 \). If \( [\psi] \neq 0 \), then an argument analogous to the above shows that \( f \in (x_{n,n}, x_{n+1,n}) \) which is not the case, as one sees from multidegrees.

We have now reduced to finitely many cases, which one may check by computer. Alternatively, one may further reduce the number of computer checks by using arguments analogous to the above. For \( m \geq 3 \) and \( n \geq 5 \), one may take \( s \) to be the cubic syzygy between \( x_{2,n}^2 \) and \( x_{2,n} x_{3,n} \), and

\[
f = x_{1,1} x_{1,2} \cdots x_{1,\lfloor n/2 \rfloor}
\]

for \( n \) even and

\[
f = x_{1,1} x_{1,2} \cdots x_{1,\lfloor n/2 \rfloor} x_{2,\lfloor n/2 \rfloor}.
\]

for \( n \) odd. Thus, one need only check the cases \( (n, m) \in \{(4, m) \mid 2 \leq m \leq 4\} \cup \{(n, 2) \mid 4 \leq n \leq 7\} \) by computer.

**Remark 7.1.** Determining whether or not \( G(n)(A_m) \) is smooth appears to be a subtle question. Outside of specialized cases, there are few techniques to show \( G(n)(A_m) \) is smooth when the obstruction space is non-vanishing. Most relevant to the current paper, in [Jel19], Jelisiejew constructs an infinite family of algebras which have trivial negative tangents and define smooth points of the Hilbert scheme, yet their obstruction spaces do not vanish, see [SS23, §1.1] for a detailed overview of Jelisiejew’s method of proof. Unfortunately, Jelisiejew’s techniques are not applicable in our setting. Indeed, when \( m \geq 3 \), [Jel19, Theorem 4.12] does not apply since the monomial ideal \( M \) generated by quadratics of \( I \) does not define a smooth
point of the Hilbert scheme. On the other hand, when $m = 2$, the ideal $M$ does define a smooth point, however [Jel19, Theorem 4.12] is still not applicable since the map $\partial_{\geq 0}$ need not be surjective, e.g., $\partial_0$ is not surjective for $(m,n) = (2,5)$. It is especially worth noting that when $M$ is not smooth, Jelisiejew’s strategy to show triviality of negative tangents also does not apply in our setting, hence the need for our explicit computations in Section 4.

In light of Remark 7.1, we pose:

**Question 7.2.** For $n \geq 4$ and $(n,m) \notin \{(4,2), (4,3)\}$, does $G^{(n)}(A_m)$ define a singular point of $\text{Hilb}^d(n,m)(A^{n(m-1)})$?

8. Data for rings of rank at most 6

We conclude by gathering data on Question 1.11 for small $n$ and all isomorphism classes of small rank rings. As mentioned in the introduction, in [Poo08a, Table 1], Poonen lists all isomorphism classes of $k$-algebras of rank $m \leq 6$; for any $m > 6$, it is well-known that there are infinitely many isomorphism classes of rings of ranks $m$. Table 3 summarizes computations done in Macaulay2; a check mark (✓) indicates that $G^{(n)}(A)$ has trivial negative tangents, × indicates it does not, and ? indicates that our computation ran for at least 12 hours without terminating. Note that $G^{(1)}(A) \simeq k$ always trivially has trivial negative tangents, while $G^{(2)}(A) \cong A/(x_1^2, \ldots, x_m^2)$ never has trivial negative tangents when $2 \leq \text{rank } A \leq 6$, as the Hilbert scheme is irreducible in these cases. Thus, Table 3 considers the values $n \geq 3$.

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\[3\text{calculated over } k = \mathbb{Z}/101\mathbb{Z}\]
Table 3. A list of all isomorphism classes of \( \mathbb{k}\)-algebras \( A = \mathcal{S}/I \) of low rank where \( G^{(n)}(A) \) has trivial negative tangents

| \( S \) | \( I \) | \( G^{(n)}(A) \) has TNT |
|-----|-----|-----------------|
| \( \mathbb{k}[x] \) | \( (x^2) \) | \( n = 3 \) | \( 4 \) | \( 5 \) | \( 6 \) |
| \( \mathbb{k}[x] \) | \( (x^3) \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| \( \mathbb{k}[x, y] \) | \( \{x^2, xy, y^2\} \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| \( \mathbb{k}[x, y] \) | \( \{x^3, xy^2, y^3\} \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| \( \mathbb{k}[x, y, z] \) | \( \{x^4, x^2y, y^4\} \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| \( \mathbb{k}[x, y, z, w] \) | \( \{x^5, x^3y, y^5\} \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| \( \mathbb{k}[x, y, z, w, v] \) | \( \{x^6, x^4y, y^6\} \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| \( \mathbb{k}[x, y, z, w, v] \) | \( \{x^7, x^5y, y^7\} \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| \( \mathbb{k}[x, y, z, w, v] \) | \( \{x^8, x^6y, y^8\} \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| \( \mathbb{k}[x, y, z, w, v] \) | \( \{x^9, x^7y, y^9\} \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| \( \mathbb{k}[x, y, z, w, v] \) | \( \{x^{10}, x^8y, y^{10}\} \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |

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