Operators generated by Morse-Smale mappings

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Abstract

Weighted shift operators $B$ in space $L^2(X, \mu)$ that are induced by Morse-Smale type of mappings are considered. A description of the properties of $B - \lambda I$ for $\lambda$ belonging to spectrum $\Sigma(B)$ is given. In particular, there is the necessary and sufficient condition that $B - \lambda I$ be a one-sided invertible and the condition that set $\text{Im}(B - \lambda I)$ be non-closed. These conditions use a new notation: an oriented decomposition of oriented graph $G(X, \alpha)$ generated by mapping $\alpha$.

1 Introduction

A bounded linear operator $B$ acting in Banach space $F(X)$ of functions (or vector-valued functions) defined in a set $X$ is called a weighted shift operator (WSO) if it can be represented as

$$Bu(x) = a_0(x)u(\alpha(x)), \quad x \in X,$$

where $\alpha : X \to X$ is a given mapping and $a_0$ is a scalar or matrix-valued function on $X$. 

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Such operators, the operator algebras generated by them and the functional equations related to such operators have been studied by many authors in various functional spaces as independent objects and in relation to various applications. The properties of WSO depend on: the space of the functions under investigation, the form of the coefficients and on map \( \alpha \). The main problem, as pertains to this subject, is related to establishing the relations between the spectral properties of weighted shift operators and the behavior of the trajectories, i.e. the dynamic properties of mapping \( \alpha \) generating the operator.

In fact, a description of spectra \( \Sigma(B) \) in classical spaces was obtained in a rather general setting. These results give us the invertibility conditions for operator \( B - \lambda I \). Apart from the invertibility conditions, some subtle properties of operator \( B - \lambda I \) are of considerable interest, such as closedness of the range and one-sided invertibility.

We restrict ourselves below to operators in space \( L^2(X, \mu) \), so we formulate results known only for this case without dealing with their generalizations to other classes of spaces.

The relationship between the dynamics of \( \alpha \) and the subtle spectral properties of a WSO has not been investigated yet. Earlier, such properties were studied only for some special classes of maps \( \alpha \). One of these classes are the diffeomorphisms of a closed interval where only two fixed points exist: those attracting and repelling (\([6], [9]\)). In a general situation the relationship between the dynamics of \( \alpha \) and the spectral properties of \( B - \lambda I \) has not been investigate yet. In \([4], [5]\), as model examples, we shall consider mapping \( \alpha \) on the \( m \)-dimensional simplex

\[
X = \{ x \in [0, 1]^m : 0 \leq x_1 \leq x_2 \leq \ldots \leq x_m \leq 1 \},
\]

where \((m + 1)\) fixed points exist:

\[
F(k) = (0, 0, \ldots, 0, 1, 1, \ldots, 1, \underbrace{1}_{k}), \quad k = 0, 1, \ldots, m.
\]

Point \( F(0) \) is repelling, point \( F(m) \) is attracting and all of the other points \( F(2), F(3), \ldots, F(m-1) \) are saddle points. In the present paper we investigate WSO generated by Morse-Smale type mappings \( \alpha \). A mapping \( \alpha \) will be called a map of Morse-Smale type if for \( \alpha \) a set \( Fix(\alpha) = \{ F_i : \alpha(F_i) = F_i \} \) of periodic points is finite and the trajectory of each point tends to a trajectory of one of the periodic points when \( j \to +\infty \), and tends to a trajectory of periodic points (possibly different ones) when \( j \to -\infty \). Mappings \( \alpha \) from the model examples are particular cases of Morse-Smale type of mappings.
2 The spectrum of weighted shift operator

For spaces $L^2(X, \mu)$ operator $B = a_0 T_\alpha$ can be conveniently written in the form of $B = a \tilde{T}_\alpha$ where

$$\tilde{T}_\alpha u(x) = |\rho(x)|^{\frac{1}{2}} u(\alpha(x))$$

is a unitary operator in $L^2(X, \mu)$ and $\rho(x) = \frac{du(\alpha^{-1}(x))}{d\mu(x)}$ is the Radon-Nikodym derivative. This function $\rho$ exists if and only if map $\alpha$ preserves the class of the measure $\mu$ (i.e. if for a measurable set $E$ equality $\nu((\alpha^{-1}(E))) = 0$ holds, if and only if $\mu(E) = 0$). When writing the operator in the form of $B = a \tilde{T}_\alpha$, the properties of operator $B$ are more simply expressed using the reduced coefficient $a(x) = |a_0(x)| |\rho(x)|^{-\frac{1}{2}}$. A description of the spectrum in space $L^2(X, \mu)$ uses the following notions. A measure $\mu$ on $X$ is called $\alpha$-invariant, if $\mu(\alpha^{-1}(\omega)) = \mu(\omega)$, for all measurable sets $\omega$. A measure $\mu$ on $X$ is called ergodic with respect to $\alpha$, if from equality $\alpha^{-1}(\omega) = \omega$ follows either $\mu(\omega) = 0$ or $\mu(X \setminus \omega) = 0$. For a given map $\alpha : X \to X$, a topological space $X$ is called $\alpha$-connected if cannot be decomposed into two nonempty closed subsets invariant with respect to $\alpha$.

**Theorem 2.1** ([7],[8],[10]). Let $X$ be a compact space, $\mu$ be a measure on $X$, such that its support coincides with the whole space. Let $\alpha : X \to X$ be an invertible continuous mapping, preserving the class of measure $\mu$. Let $a \in C(X)$. Let $R(B)$ be the spectral radius of the operator $B = a \tilde{T}_\alpha$. Then the following relation holds in $L^2(X, \mu)$:

$$R(B) = \max_{\nu \in M_\alpha(X)} \exp \left[ \int_X \ln |a(x)| d\nu \right],$$

where $a$ is the reduced coefficient and $M_\alpha(X)$ is the set of probability measures on $X$, invariant and ergodic with respect to the mapping $\alpha$. If the set of nonperiodic point of the map $\alpha$ is everywhere dense in $X$, and the space $X$ is $\alpha$-connected and $a(x) \neq 0$ for all $x$, then the spectrum $\Sigma(B)$ is a subset of the ring

$$\Sigma(B) = \{ \lambda \in \mathbb{C} : r(B) \leq |\lambda| \leq R(B) \}$$

where

$$r(B) = \min_{\nu \in M_\alpha(X)} \exp \left[ \int_X \ln |a(x)| d\nu \right].$$

For a Morse-Smale type mapping $\alpha$ a set $M_\alpha(X)$ consists of finitely many measures, each of which is concentrated on the trajectory of one of
the periodic points
\[ M_\alpha(X) = \{ \delta_F, F \in Fix(\alpha) \} \]
where
\[ \forall_{\omega \in \Phi} \quad \delta_F(\omega) = \begin{cases} 1, & \text{for } F \in \omega; \\ 0, & \text{for } F \in X \setminus \omega. \end{cases} \]
In addition, for such a mapping the set of nonperiodic points is dense in \( X \) (with the exception of degenerate cases, when the fixed points are isolated points of space \( X \)). Therefore, for maps of the Morse-Smale type, Theorem 2.1 gives an explicit description of the spectrum.

**Corollary 2.2.** If \( \alpha : X \to X \) is a Morse-Smale type of mapping, \( a \in C(X) \) and \( a(x) \neq 0 \ \forall_x \), then the spectrum \( \Sigma(B) \) of operator \( B = aT_\alpha \) in space \( L^2(X, \mu) \) is a subset of the ring (2.1) where
\[ R(B) = \max_{F \in Fix(\alpha)} |a(F)|, \quad r(B) = \min_{F \in Fix(\alpha)} |a(F)|. \]

### 3 Model examples. WSO on \( m \)-dimensional simplex \( X \subset [0, 1]^m \)

We shall consider the class of mappings \( \alpha : X \to X \) of the form
\[ \alpha(x_1, x_2, \ldots, x_n) = (\gamma(x_1), \gamma(x_2), \ldots, \gamma(x_n)). \] (3.1)

Where \( \gamma : [0, 1] \to [0, 1] \) be a diffeomorphism with only two fixed points: 0 and 1. To be exact, we suppose that \( \gamma(x) > x \) for \( 0 < x < 1 \).

Our map \( \alpha \) has in \( X \) only \( m + 1 \) fixed points \( F(k), \ k \in \{0, 1, \ldots, m\} \). Let us arrange the numbers \( |a(F(k))| \) in the increasing order and thus we obtain a transposition \( \sigma \) of the index-set \( \{0, 1, 2, \ldots, m\} \) such that
\[ |a(F(\sigma(0)))| < |a(F(\sigma(1)))| < \ldots < |a(F(\sigma(m)))|. \] (3.2)

The main results of this situation.

**Theorem 3.1** ([5], Theorem 4.2). Let \( X \) be simplex (1.2). Let \( \alpha : X \to X \) be a mapping given by (3.1) and \( B = aT_\alpha \) is WSO in \( L^2(X, \mu) \). Assume that \( a \in C(X) \), \( a(x) \neq 0 \ \forall x \in X \), and all the numbers \( |a(F(k))| \) are different. Let \( \sigma \) be the transposition such that (3.2) holds. Assume also that
\[ |a(F(0))| < |a(F(m))|. \]
The properties of the operator \( B - \lambda I \) depend on the disposition of the numbers \( |a(F(k))| \) and \( |\lambda| \) in the following way.
I. Let
\[ |\lambda| = |a(F(m))| \quad \text{or} \quad |\lambda| = |a(F(0))|. \]
Then the range \( \text{Im}(B - \lambda I) \) is nonclosed subspace.

II. Let
\[ |a(F(0))| < |\lambda| < |a(F(m))|. \]
Then \( \dim \ker(B - \lambda I) = \infty \) and the range \( \text{Im}(B - \lambda I) \) is an everywhere dense subspace in \( L^2(X, \mu) \).

The operator \( B - \lambda I \) is right-sided invertible if and only if
\[ |\lambda| \neq |a(F(k))| \]
and the set \( \{0, 1, 2, \ldots, k_0\} \) is invariant under the transposition \( \sigma \), where \( k_0 \) is such a number that
\[ |a(F(\sigma(k_0) - 1))| < |\lambda| < |a(F(\sigma(k_0)))|. \]

III. Let
\[ |a(F(\sigma(0)))| \leq |\lambda| < |a(F(0))| \quad \text{or} \quad |a(F(m))| < |\lambda| \leq |a(F(\sigma(m)))|. \]
Then \( \ker(B - \lambda I) = 0 \) and the range \( \text{Im}(B - \lambda I) \) is a nonclosed everywhere dense subspace in \( L^2(X, \mu) \).

4 Operators generated by Morse-Smale type mappings

The main object of investigation in the present paper is a class of WSO \( B = a\tilde{T}_\alpha \) in space \( L^2(X, \mu) \) where \( X \) is \( \alpha \)-conected and a compact topological space, \( \alpha : X \to X \) is an invertible, measurable Morse-Smale type mapping and a reduced coefficient \( a \in C(X) \).

According to Corollary 2.2, for \( a(x) \neq 0 \) spectrum of the operator \( B = a\tilde{T}_\alpha \) is a ring
\[ \Sigma(B) = \{ \lambda \in \mathbb{C} : \min_{F \in \text{Fix}(\alpha)} |a(F)| \leq |\lambda| \leq \max_{F \in \text{Fix}(\alpha)} |a(F)| \}. \]

Circles
\[ S_k = \{ \lambda : |\lambda| = |a(F)|, F \in \text{Fix}(\alpha) \} \]
belong to \( \Sigma(B) \) and split the spectrum into smaller subrings. The spectral properties of \( B - \lambda I \) are different for \( \lambda \) belonging to every such subring and
every circle \( S_k \). In paper \([4]\) the model example of operators induced by a map of Morse-Smale type has been studied. Maps \( \alpha \) have three fixed points, where the point \( F_1 \) is repelling, point \( F_2 \) is a saddle point and point \( F_3 \) is attracting. In this example the spectrum decomposes into two subrings and the properties of operator \( B - \lambda I \) depend on the domination/ minoration relationships between numbers \( |a(F_k)| \) and \( \lambda \). The form of these inequalities is determined by the dynamics of map \( \alpha \). Formulating the results even in that relatively simple case is very cumbersome - it involves 18 different subcases. It turns out that the subtle properties of operator \( B - \lambda I \) have a different dependence on value \( a(F_k) \) in fixed points of various types. In order to describe the dynamics of \( \alpha \) we use the structure of the oriented graph \( G(X, \alpha) \). The vertices of \( G(X, \alpha) \) are all fixed points \( F \in \text{Fix}(\alpha) \) and the edge \( F_j \to F_k \) belongs to \( G(X, \alpha) \) if and only if there is a point \( x \in X \) such that its trajectory \( \alpha^n(x) \) tends to \( F_k \) when \( n \to +\infty \) and tends to \( F_j \) when \( n \to -\infty \). In these terms the oriented edge \( F_j \to F_k \) is included in the graph if and only if \( \Omega^+_k \cap \Omega^-_j \neq \emptyset \), where

\[
\Omega^+_k = \{ x : \lim_{n \to +\infty} \alpha^n(x) = F_k \} \cap \Omega^-_j = \{ x : \lim_{n \to -\infty} \alpha^n(x) = F_j \}.
\]

Useing a structured graph we can simply formulate the important properties of the weighted shift operators. By using the coefficient \( a \in C(X) \) and the number \( \lambda \in \mathbb{C} \) we form two subsets of the set of vertices of the graph

\[
G^-(\lambda, a) = \{ F \in \text{Fix}(\alpha) : |a(F)| < |\lambda| \},
G^+(\lambda, a) = \{ F \in \text{Fix}(\alpha) : |\lambda| < |a(F)| \}.
\] (4.1)

We say that \((G^-(\lambda, a), G^+(\lambda, a))\) give a decomposition of the graph, if the condition

\[
G(X, \alpha) = G^-(\lambda, a) \cup G^+(\lambda, a)
\]

holds, i.e. if

\[
|\lambda| \neq |a(F)|, \forall F \in \text{Fix}(\alpha).
\] (4.2)

The graph decomposition we call oriented to the right, if any edge connecting the point \( F_j \in G^-(a, \lambda) \) to the point \( F_k \in G^+(a, \lambda) \) is oriented from \( F_k \) to \( F_j \).

The decomposition we call oriented to the left, if any edge connecting the point \( F_j \in G^-(a, \lambda) \) to the point \( F_k \in G^+(a, \lambda) \) is oriented from \( F_j \) to \( F_k \).

The basic result in this direction is the following.
Main Theorem 4.1. Let $\alpha$ be a Morse-Smale type of mapping. Let $X$ be a compact space and $a \in C(X)$. Let $B$ be a weighted shift operator. The operator $B - \lambda I$ is invertible from the right (left) if and only if the graph decomposition $(G^-(a, \lambda), G^+(a, \lambda))$ is oriented to the right (to the left).

Corollary 4.2. If there are $j$ and $k$, such that the set $\Omega_k^+ \cap \Omega_j^-$ is dense in $X$, then the image of the operator is closed if and only if the operator is one-sided invertible.

5 Steps of proof of the Main Theorem.

The Main Theorem may be proved in much the same way as Theorem 4.2 in [5]. Here are some important sketches of these concepts.

5.1 Reduction to operators $B_{kj}$.

Consider a family of subsets $\Omega_{kj}$ in $X$, such that $\mu(\Omega_{kj}) > 0$. Then

$$X = \bigcup_{k,j} \Omega_{kj} \text{ almost everywhere},$$

and

$$L^2(X) = \oplus_{kj} L^2(\Omega_{kj}).$$

Let $B_{kj}$ denote the weighted shift operator in space $L^2(\Omega_{kj})$. It follows that

$$B = \oplus_{kj} B_{kj}.$$ 

Therefore, an investigation of $B$ can reduced to an investigation of $B_{kj}$. The structures of operators $B_{kj}$ are similar, and it is sufficient to consider only one of them. We will consider operator $B_{kj}$ in $L^2(\Omega_{kj}, \mu_{kj})$ where $\Omega_{kj} := X_0$ is compact space and measure $\mu_{kj}$ is such that

$$\forall E \subset \Omega_{kj}, \mu_{kj}(E) = \mu(E \cap \Omega_{kj}),$$

then

$$L^2(X, \mu) = \oplus_{kj} L^2(\Omega_{kj}, \mu_{kj}).$$

This reduction is useful because the dynamics of $\alpha$ on $X_0$ is simpler than the one on $X$. Map $\alpha : X_0 \to X_0$ has one repelling point, $F_j$, and one attracting point $F_k$, all other fixed points are saddle points.
5.2 Isomorphism of the spaces $L^2(X_0, \mu)$ and $L^2(\Theta, l^2(\mathbb{Z}))$

First of all we construct a representation of $L^2(X_0, \mu)$ as a space $L^2(\Theta, l^2(\mathbb{Z}))$ of vector-valued functions dependent on a parameter $\tau \in \Theta$, where $\Theta$ is a subset of $X_0$. In this representation $B$ takes the form of an operator-valued function $B = B(\tau)$, where

$$(B(\tau)u)(k) = a(\alpha^k(\tau))u(k + 1),$$

are weighted shift operators in the space $l^2(\mathbb{Z})$. Thus our problem is reduced to the consideration of a family of discrete weighted shift operators which can be studied in detail.

Let us choose a fundamental set for the map $\alpha$, i.e. such a measurable set $\Theta$ that the sets $\Theta_k = \alpha_k(\Theta)$ are disjoint and

$$\mu(X_0 \setminus \bigcup_{k \in \mathbb{Z}} \alpha_k(\Theta)) = 0.$$ 

For an arbitrary map $\alpha$ it can happen that the fundamental set does not exist, but for Morse-Smale type of mapping such sets exists.

**Lemma 5.1.** Let $\alpha : X_0 \to X_0$ be a map of the Morse-Smale type. Then the fundamental set $\Theta$ exists.

**Proof.** $V_j, V_k$ are open neighbourhoods of points $F_j, F_k$ such that $V_j \cap V_k = \emptyset$. Let $W = \bigcup_{j \geq 1} \alpha^j(V_k)$ satisfy $\alpha(W) \subset W$ and $W \cap V_j = \emptyset$. If $x_0 \in W$, then $\alpha(x_0) \in W$, $\alpha^2(x_0) \in W$ and so on. Let us denote $\Theta$ as a set of points from trajectory, which is the first in neighbourhood $W$, i.e.

$$\Theta := (W \setminus \alpha(W)) \cap X_0.$$ 

From this construction set $\Theta$ is a measurable set, $\mu(\Theta) > 0$ and

$$\alpha^j(\Theta) \cap \alpha^k(\Theta) = \emptyset, \ j \neq k.$$ 

□

The proofs of the assertions formulated below are in [4], [5].

**Lemma 5.2** (Lemma 5.2 in [5]). If the image $\text{Im}(B - \lambda I)$ is closed then the image $\text{Im}[B(\tau) - \lambda I]$ is closed for almost all $\tau \in \Theta$.

**Lemma 5.3** (Lemma 5.5 in [5]). The operator $B - \lambda I$ is right-sided invertible if and only if the operators $B(\tau) - \lambda I$ are right-sided invertible for almost all $\tau \in \Theta$ and for every such $\tau$ a right-sided inverse $R(\tau)$ to $B(\tau) - \lambda I$ can be chosen so that the family $R(\tau)$ will be bounded with respect to operator norm.
From these lemmas we should find a family of the right-sided inverses operator to discreet operators $B(\tau)$ and estimates their norms.

5.3 WSO in $l^2(\mathbb{Z})$.

Let $W u(k) = u(k + 1)$ be the shift operator defined in the space $l^2(\mathbb{Z})$, $a = (a(k))$ be a sequence and $aW$ be the weighted shift operator in $l^2(\mathbb{Z})$. We assume that $a(k) \neq 0$ for all $k$, and there exist the limits

$$\lim_{k \to +\infty} a(k) := a(+\infty) \neq 0, \quad \lim_{k \to -\infty} a(k) := a(-\infty) \neq 0.$$ 

Then spectrum $\Sigma(aW)$ is the ring

$$\Sigma(aW) = \{\lambda : r \leq |\lambda| \leq R\},$$

where

$$R = \max\{|a(+\infty)|, |a(-\infty)|\}, \quad r = \min\{|a(+\infty)|, |a(-\infty)|\}.$$ 

Assuming that

$$|a(-\infty)| < |\lambda| < |a(+\infty)|$$

the operator $aW - \lambda I$ is right-sided invertible. In [4] a family of the right-sided inverses to $aW - \lambda I$ was constructed and estimates of their norms were obtained in Lemma 7.1 and Lemma 7.2.

5.4 The dynamics of Morse-Smale type mappings.

Let $\alpha : X_0 \to X_0$ be a Morse-Smale type mapping and $V_i$ an open neighbourhood of the fixed point $F_i$.

Lemma 5.4. Let

$$\widetilde{X}_0 := X_0 \setminus \bigcup_i V_i.$$ 

There exists a number $N \in \mathbb{N}$ such that for every $\tau \in \Theta$ the number of the indeks where $\alpha^n(\tau) \in \widetilde{X}_0$ is no larger than $N$, i.e.

$$\text{card}(\{\alpha^n(\tau)\} \cap \widetilde{X}_0) \leq N \ \forall \tau \in \Theta.$$ 

Proof. Let us denote $V := \bigcup_i V_i$. Set $V$ is open, then $\widetilde{X}_0 = X_0 \setminus V$ is closed. Set $X_0$ can be represented in the form

$$X_0 = \bigcup_{n=1}^{\infty} \bigcup_i \alpha^{-n}(V_i).$$
This is an open cover of \( X_0 \). While \( \tilde{X}_0 \subset X_0 \) is closed and compact, there exists a finite subcover, i.e

\[
\tilde{X}_0 \subset \bigcup_{n=1}^{N} \bigcup_i \alpha^{-n}(V_i) \setminus \bigcup_i V_i.
\]

We have \( \text{card}\{\alpha^n(\tau)\} \cap \tilde{X}_0 \leq N \). \hfill \Box

**Lemma 5.5.** Let \( S \) be an arbitrary natural number, \( F_a, F_b \in \text{Fix}(\alpha) \) and exist edge \( F_a \rightarrow F_b \). There exists an open set \( W \subset X_0 \), such that for every \( x \in W \)

\[
\text{card}(\{\alpha^0(x)\} \cap V_a) > S, \quad \text{card}(\{\alpha^0(x)\} \cap V_b) > S,
\]

and the points of trajectory \( \alpha^j(x) \) after they leave neighbourhood \( V_a \) but do not yet come to neighbourhood \( V_b \) do not lay on the other neighbourhoods of points \( F_i \in \text{Fix}(\alpha) \).

**Proof.** Let \( \text{Fix}(\alpha) = \{F_a, F_b\} \) then \( F_a \) is attracting point and \( F_b \) is repelling point. There exists an open set \( W = \bigcup_{l \geq 1} \alpha^{-l}(V_b) \) that condition (5.2) holds.

For \( x \in W \) we have:

\[
\lim_{n \to \infty} \alpha^n(x) = F_a \quad \text{and} \quad \alpha^{-n}(W) \subset W \quad \text{and} \quad \lim_{n \to \infty} \alpha^{-n}(x) = F_b.
\]

Let \( F_a, F_b \in \text{Fix}(\alpha) \) be saddle points. Choose open sets \( W_a, W_b \) such that \( W_a \subset V_a, \ W_b \subset V_b \) and

\[
\forall x \in W_a \quad \text{card}(\{\alpha^0(x)\} \cap V_a) > S, \quad \forall x \in W_b \quad \text{card}(\{\alpha^0(x)\} \cap V_b) > S.
\]

There exists an open set \( W = W_a \cap \alpha^{-1}(W_b) \) that condition (5.2) holds. \hfill \Box

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