On determinacy/indeterminacy of Moment Problems

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0 Introduction

This paper treat determinacy of strong moment problems in part I and indeterminacy of strong moment problems in part II. This paper is a summary of the following papers:

[1] Aldén, E., Determinancy of Strong Moment Problems. Department of Mathematics, Umeå University. ISSN 0345-3928 1987:10, S-901 87 Umeå, Sweden.

[2] Aldén, E., On Indeterminancy of Strong Moment Problems. Department of Mathematics, Umeå University. ISSN 0345-3928 1988:2, S-901 87 Umeå, Sweden.

[3] Aldén, E., Indeterminancy of Strong Moment Problems. Department of Mathematics, Umeå University. ISSN 1103-6540 1995:7, S-901 87 Umeå, Sweden.

This paper will treat determinacy/indeterminacy of the strong Stieltjes and Hamburger moment problems, part I. Indeterminacy, part II, for certain class of distribution functions. We conclude by proving a theorem for indeterminacy of the strong problems above for general distribution functions.

Definition 0.1. A function $\alpha$ is called a distribution function, if $\alpha(x)$ is real-valued, bounded and non-decreasing, on some interval $I \in \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. We also require $\alpha$ to have infinitely many points of increase in $I$.

Definition 0.2. By a strong moment problem we mean: Given a sequence $\{\mu_n\}_{n \in \mathbb{Z}}$ of real numbers, find a distribution function $\alpha(x)$ on the interval $I$ such that

$$\mu_n = \int_I x^n d\alpha(x), \ n \in \mathbb{Z},$$

where $\mathbb{Z}$ is the set of all integers. The number $\mu_n$ is called the moment of $\alpha$ of order $n$, $n \in \mathbb{Z}$. 
If we take $n \in \mathbb{Z}_+$ in (1) we get the classical moment problem, where $\mathbb{Z}_+$ is the set of all non-negative integers.

The table below explains the various types of moment problems. We may consider an arbitrary bounded interval $[a, b]$ in the Hausdorff case.

| Hausdorff    | Stieltjes    | Hamburger    |
|--------------|--------------|--------------|
| classical    |              |              |
| $n \in \mathbb{Z}_+, I = [0, 1]$ | $n \in \mathbb{Z}_+, I = [0, \infty]$ | $n \in \mathbb{Z}_+, I = [- \infty, \infty]$ |
| strong       |              |              |
| $n \in \mathbb{Z}, I = [0, 1]$ | $n \in \mathbb{Z}, I = [0, \infty]$ | $n \in \mathbb{Z}, I = [- \infty, \infty]$ |

Table 1

Throughout this paper $\alpha$, $\beta$ and $\sigma$ denote distribution functions.

Definition 0.3. A moment problem (1) is said to be determinate if it has at most one solution $\alpha$. Otherwise it is indeterminate.

Definition 0.4. Two distribution functions $\alpha_1$ and $\alpha_2$ are said to be equal, $\alpha_1(x) = \alpha_2(x)$, if

$$\int_I f(x)d\alpha_1(x) = \int_I f(x)d\alpha_2(x)$$

for all continuous functions $f$ with compact support.

In the classical Hamburger case,

$$\int_{-\infty}^{\infty} \frac{\log \sigma'(u)}{1 + u^2} du > -\infty,$$

(2)

is a sufficient condition for indeterminacy of (1), see Akhiezer [1], p.87.

We have the corresponding condition for indeterminacy in the classical Stieltjes case,

$$\int_0^{\infty} \frac{\log \sigma'(u^2)}{1 + u^2} du > -\infty,$$

(3)

which was proved by the author [3], Theorem 5. We call conditions (2) and (3) Krein conditions.
In this paper we will prove that the conditions (2) and (3) are sufficient for indeterminacy also of the strong Hamburger and Stieltjes moment problems, provided that the distribution function $\sigma(u)$ has the following symmetry property.

**Definition 0.5.** A distribution function $\sigma$ on $I$, where $I = [0, \infty[$ or $I = ]-\infty, \infty[$ is called symmetric if

$$d\sigma(u) = -d\sigma\left(\frac{1}{u}\right), \quad u \in I \setminus \{0\}. \quad (4)$$

For any symmetric distribution function $\sigma$ we have $\mu_n = \mu_{-n}$, $n \in \mathbb{Z}$, provided that $\sigma$ has moments of all orders.

Carleman [7a,7b,7c] gave sufficient conditions for determinacy of the classical Stieltjes and Hamburger moment problems. These conditions are

$$\sum_{n=0}^{\infty} \mu_n^{(\frac{1}{2})} = \infty \quad (5)$$

and

$$\sum_{n=0}^{\infty} \mu_{2n} = \infty, \quad (6)$$

for the strong moment problems the corresponding results are replaced by

$$\sum_{n \in \mathbb{Z}, n \neq 0} \mu_n^{(\frac{1}{2})} = \infty \quad (7)$$

and

$$\sum_{n \in \mathbb{Z}, n \neq 0} \mu_{2n}^{(\frac{1}{2})} = \infty \quad (8)$$

respectively.

The conditions (5) and (6) are called Carleman conditions. Analogously we call (7) and (8) conditions of Carleman type.

In (1981-84), W.B. Jones, O. Njåstad, W.J. Thron and H. Waadeland [8,9] have stated and proved necessary and sufficient conditions for the existence for solutions and determinacy. This has been done for both the strong Stieltjes and Hamburger moment problems.

Two families of strong moment problems are given in section 2. The purpose is to study the sharpness in the sufficient conditions for determinacy.
Definition 0.6 Let \( \{\mu_n\}_{-\infty}^{\infty} \) be a sequence of non-negative numbers.

We call the condition

\[
\sum_{\substack{n \in \mathbb{Z} \setminus \{0\}}} \mu_n \left( \frac{(-1)^n}{|n|} \right) w_n = \infty,
\]

a weighted Carleman condition for the strong Stieltjes moment problem (1).

We prove in section 3, that the weighted Carleman condition above with the weight \( \xi^{[n]} \) is not sufficient for determinacy of the strong Stieltjes moment problem.

A family of strong Stieltjes moment problems

\[
\mu_n(d) = \int_0^{\infty} x^n d\alpha_d(x), \quad n \in \mathbb{Z}_+
\]

is a limiting case of the condition of Carleman type (7). If

\[
\sup_d \left\{ \sum_{\substack{n \in \mathbb{Z} \setminus \{0\}}} (\mu_n(d)) \left( \frac{(-1)^n}{|n|} \right) \right\} = \infty,
\]

 Analogously for the conditions (2), (3) and (8).

In Theorem 2.1 we will drop the condition of symmetric distribution function and prove that conditions (2) and (3) also are sufficient conditions for indeterminacy in the general strong Hamburger and Stieltjes moment problems respectively.

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**Key words** Classical moment problem, strong moment problem, distribution function, determinacy, indeterminacy, symmetric distribution.
1. Determinacy of the strong Stieltjes moment problem.

The title above may seem a bit obscure, since we shall start to consider indeterminacy. However, it will become clear that indeed we deal with determinacy.

This section will deal with a class of strong moment problems. We will prove that this is a limiting case for the conditions of Carleman type as well the condition for indeterminacy, the Krein type condition.

In [3] Example 5, p.25, we considered the following family of strong Stieltjes moment problems. Given \( \{ \mu_n(d) \}_{n=-\infty}^{\infty} = \{ \mu_n \}_{n=-\infty}^{\infty} \) according to

\[
\mu_n = \int_0^1 x^n e^{-\left[ \frac{1}{1+x+d} \right]} \, dx + \int_1^{\infty} x^n e^{-x(1+d)} \, dx, \quad n \in \mathbb{Z}, \quad 0 < d \leq 1.
\]

We found that the moments \( \mu_n \) are bounded from above and below by constant multiples of \( |n|^{-1/(1+d)} \). The condition of Carleman type (7) is not fulfilled for the moment problem above but the sums

\[
\sum_{n=0}^{\infty} \mu_n \left( \frac{-1}{n+1} \right), \quad \sum_{n=-\infty}^{-3} \mu_n \left( \frac{-1}{n+1} \right),
\]

of the positive and negative moments become arbitrarily large when \( d \) tends to zero, since for \( d = 0 \) we have the divergent harmonic series. Hence (9) is determinate for \( d = 0 \).

When \( d > 0 \), we can not make any conclusion about determinacy or indeterminacy.

From this example we get the idea to construct a family of strong Stieltjes moment problems, by defending a distribution function \( \alpha \) with the symmetry property (4) below.

If \( \alpha(x) \) is a distribution function defined on \([0, \infty]\) then \( -\alpha \left( \frac{1}{x} \right) \) also is a distribution function on \([0, \infty]\).

**Definition 1.1.** Suppose \( \alpha(x) \) is a distribution function defined on \([1, \infty]\) and define \( \tilde{\alpha}(x) \) according to

\[
\tilde{\alpha}(x) = \begin{cases} 
\alpha \left( \frac{1}{x} \right) + m, & x \in [0,1[, \\
\alpha(x), & x \in [1, \infty[,
\end{cases}
\]

where \( m \) is a real constant such that \( \tilde{\alpha}(x) \) becomes non-decreasing. It follows from the construction of \( \tilde{\alpha}(x) \) above that \( \tilde{\alpha} \) is a symmetric distribution function.

We are now in a position to construct a family of strong Stieltjes moment problems from a family of classical Stieltjes moment problems.
Let $\alpha_1$ be a distribution function on $[1, \infty]$ with moments of all non-negative orders. Consider the following classical Stieltjes moment problem and corresponding strong moment problem

$$\mu_n = \int_0^1 x^n d(-\alpha(\frac{1}{x})) + \int_1^\infty x^n d\alpha_1(x), \quad n \in \mathbb{Z}_+ \tag{10}$$

and

$$\mu_n = \int_0^1 x^n d(-\alpha(\frac{1}{x})) + \int_1^\infty x^n d\alpha_1(x), \quad n \in \mathbb{Z} \tag{11}$$

respectively.

We now claim that:

i) $\mu_{-n} = \mu_n$, $n \in \mathbb{Z}$,

ii) if the classical moment problem (10) is determinate, then the corresponding strong Stieltjes moment problem (11) also is determinate.

iii) If the classical moment problem (10) is indeterminate and has another symmetric distribution function $\alpha_2$, then the strong Stieltjes moment problem (11) is also indeterminate.

To prove (i) we note that

$$\mu_{-n} = \int_0^1 x^{-n} d(-\alpha_1(\frac{1}{x})) + \int_1^\infty x^{-n} d\alpha_1(x) = \int_0^1 x^n d(-\alpha(\frac{1}{x})) + \int_1^\infty x^n d\alpha_1(x), \quad n \in \mathbb{Z},$$

hence (i) follows. We omit the proofs of (ii) and (iii).
Part II

2. Indeterminacy of the strong Hamburger moment problem.

In this section we will prove that the Krein condition (2) is a sufficient condition for indeterminacy in the strong Hamburger case for a symmetric distribution function.

Definition 2.1. Let, $1 \leq p < \infty$, and let $f$ be a real or complex valued and measurable function on $\mathbb{R}$. Let $\sigma(u), \ -\infty < u < \infty$, be a distribution function on $\mathbb{R}$. We define

$$\|f\|_{p,\sigma} = \left(\int_{-\infty}^{\infty} |f(u)|^p d\sigma(u)\right)^{\frac{1}{p}}$$

and let $L^p_{\sigma}$ consist of all $f$ for which $\|f\|_{p,\sigma} < \infty$.

We now claim the following result.

Theorem 2.1. Let $\sigma(u), \ -\infty < u < \infty$, be a symmetric distribution function, and suppose also that $\sigma(u)$ generates finite moments of all orders. Let

$$\mu_n = \int_{-\infty}^{\infty} u^n d\sigma(u), \ n \in \mathbb{Z}. \quad (12)$$

If

$$\int_{-\infty}^{\infty} \log \left(\frac{\sigma'(u)}{1+u^2}\right) du > -\infty, \quad (13)$$

then the strong Hamburger moment problem (13) is indeterminate and $\mu_n = \mu_{-n}, \ n \in \mathbb{Z}$. Here $\sigma'(u)$ is the derivative of the absolutely continuous part of the function $\sigma(u)$.

The proof is based on density in $L^p_{\sigma}$ of the set of all rational functions $u^n, n \in \mathbb{Z}$. We can achieve a condition for the density in $L^p_{\sigma}$ of the set of all rational functions $u^{-k}, k \in \mathbb{Z}_+$, by considering the linear hull of the functions $e^{i\alpha u}, \alpha \geq 0$. This is done in the following lemma, which is the analogue to the lemma in the classical Hamburger case concerning the linear hull of the functions $e^{i\alpha u}, \alpha > 0$, see [5].
Lemma 1. Suppose that $\sigma(u), -\infty < u < \infty$, is a symmetric distribution function. The linear hull of the functions $e^{i\alpha \frac{u}{2}}, \alpha \geq 0$, is dense in $L^p_\sigma$, $p \geq 1$ if and only if

$$\int_{-\infty}^{\infty} \frac{\log \sigma'(u)}{1+u^2} du = -\infty,$$  

(14)

where $\sigma'(u)$ is the derivative of the absolutely continuous part of the function $\sigma(u)$.

Proof. In view of Krein’s theorem [1], p.87, it suffices to prove that the linear hull of $e^{i\alpha \frac{u}{2}}, \alpha \geq 0$, is dense in $L^p_\sigma$ if and only if this holds for the linear hull of $e^{i\alpha u}, \alpha \geq 0$. First assume that the linear hull of $e^{i\alpha \frac{u}{2}}, \alpha \geq 0$, is dense in $L^p_\sigma$. Let $f \in L^p_\sigma, \frac{1}{p} + \frac{1}{q} = 1$ be such that

$$\int_{-\infty}^{\infty} f(u)e^{i\alpha \frac{u}{2}}d\sigma(u) = 0, \alpha \geq 0.$$  

(15)

A change of variable gives

$$\int_{-\infty}^{\infty} f\left(\frac{1}{u}\right)e^{i\alpha \frac{u}{2}}d\sigma(u) = 0, \alpha \geq 0.$$  

It now follows that the linear hull of $e^{i\alpha \frac{u}{2}}, \alpha \geq 0$, is dense in $L^p_\sigma$ from the Hahn-Banach Theorem and the fact that $f\left(\frac{1}{u}\right) \in L^p_\sigma$. The converse is proved analogously. This proves Lemma 1.

Proof of Theorem 2.1. Let $\sigma$ be a distribution function on $R$ satisfying condition (14). Let us first prove that the set of all rational functions $u^{-k}, k \in \mathbb{Z}_+$ is not dense in $L^p_\sigma$, $p \geq 1$. By Lemma 1 and the Hahn-Banach Theorem there exists a function $f(u) \in L^p_\sigma, f \neq 0$, where $\frac{1}{p} + \frac{1}{q} = 1$, such that

$$\int_{-\infty}^{\infty} f(u)e^{i\alpha \frac{u}{2}}d\sigma(u) = 0, \alpha \geq 0.$$  

(15)

Now differentiating (15) $k$ times with respect to $\alpha$ and putting $\alpha = 0$ yields

$$\int_{-\infty}^{\infty} f(u)u^{-k}d\sigma(u) = 0, k \in \mathbb{Z}_+.$$  

Whence it follows that the set of all rational functions $u^{-k}, k \in \mathbb{Z}_+$ is not dense in $L^p_\sigma$, $p \geq 1$, by Krein’s theorem, see [1], p.87. We conclude that the set of all rational functions $u^n, n \in \mathbb{Z}$, is not dense in $L^p_\sigma, p \geq 1$.
By the same argument as in the proof in the classical Hamburger case, it follows we have an indeterminate strong Hamburger moment problem, see [1], p.47-49. This completes the proof of Theorem 2.1.

**Example 1.** Consider the following family of strong Hamburger moment problems.

\[
\mu_n = \mu_n(c,d) = \int_{-\infty}^{\infty} u^n d\sigma(u) = \int_{-\infty}^{\infty} u^n \left( \frac{1}{2} \sqrt{\frac{d}{\pi}} \cdot \frac{e^{-d(|u|)^2}}{|u|^n} \right) du,
\]

(16) where \( d \in ]0, \infty[ , c \in \mathbb{R} , n \in \mathbb{Z} \), see [4], p.14.

(i) The family (16) of strong Hamburger moment problems is indeterminate for all \( d \neq 0 , c \in \mathbb{R} \), but the distribution function \( \sigma \) is symmetric only for \( c = 1 \).

(ii) \[ \mu_n = \begin{cases} 
\mu_{2k} = \frac{e^{(2k+1-c)^2}}{4d}, & n = 2k, \ k \in \mathbb{Z}, \\
\mu_{2k+1} = 0, & n = 2k + 1, \ k \in \mathbb{Z}
\end{cases} \]

(iii) the condition (13) holds for all \( d > 0 \) and \( c \in \mathbb{R} \).

From this example and Example 2 in the next section, we conclude that there exist indeterminate strong moment problems for which the Krein conditions (2) and (3) are fulfilled, but the distribution function is not symmetric.
3. Indeterminacy of the strong Stieltjes moment problem.

In this section we will prove the corresponding indeterminacy theorem for the strong Stieltjes moment problem for a symmetric distribution function $\sigma$.

**Theorem 3.1.** Let $\sigma(u), 0 \leq u < \infty$, be a symmetric distribution function, and suppose also that $\sigma(u)$ possesses finite moments of all orders. Let

$$\mu_n = \int_0^\infty u^n d\sigma(u), n \in \mathbb{Z}. \tag{17}$$

If

$$\int_0^\infty \frac{\log \sigma'(u^2)}{1 + u^2} du > -\infty, \tag{18}$$

then the strong Stieltjes moment problem (17) is indeterminate and $\mu_n = \mu_{-n}, n \in \mathbb{Z}$.

Theorem 3.1. is proved analogously as in the classical Stieltjes case, with only minor changes, see [5], Theorem 5. We omit the details.

**Example 2.** (See [4], p-11) Consider the following family of strong Stieltjes moment problems. Let $d \in [0, \infty[, c \in \mathbb{R}$ and

$$\mu_n = \mu_n(c,d) = \int_0^\infty u^n d\sigma(u) = \int_0^\infty u^n \left( \frac{1}{\sqrt{\pi d}} \cdot \frac{e^{-d(\log u)^2}}{u^c} \right) du, n \in \mathbb{Z}. \tag{19}$$

Then

(i) the family of strong stieltjes moment problem (19) is indeterminate for all $d > 0, c \in \mathbb{R}$, but the distribution function $\sigma$ is symmetric only for $c = 1$.

(ii) $\mu_n = e \left( \frac{(n+1-c)^2}{4d} \right), n \in \mathbb{Z}.$

(iii) condition (18) holds for all $d > 0$ and $c \in \mathbb{R}$. 

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4. Indeterminacy of the strong Hamburger and Stieltjes moment problems for a distribution function \( \sigma \) that is not necessarily of “symmetry type”

In this section we will prove that the Krein condition (2) and (3) are sufficient conditions for indeterminacy of the strong Hamburger and Stieltjes moment problems. Let us state Definition 2.1 once again.

\textit{Definition 2.1.} Let \( 1 \leq p < \infty \), and let \( f \) be a real or complex valued and measurable function on \( \mathbb{R} \). Let \( \sigma(u), -\infty < u < \infty \), be a distribution function on \( \mathbb{R} \). We define

\[ \|f\|_{p,\sigma} = \left( \int_{-\infty}^{\infty} |f(u)|^p d\sigma(u) \right)^{1/p} \]

We will now state our main results.

\textbf{Theorem 4.1.} Let \( \sigma(u), -\infty < u < \infty \), be a distribution function, and suppose also that \( \sigma(u) \) generates finite moments of all orders. Let

\[ \mu_n = \int_{-\infty}^{\infty} u^n d\sigma(u), n \in \mathbb{Z}. \] \hspace{1cm} (20)

If

\[ \int_{-\infty}^{\infty} \log \frac{\sigma'(u)}{1+u^2} du > -\infty, \] \hspace{1cm} (21)

then the strong Hamburger moment problem (13) is indeterminate and. Here \( \sigma'(u) \) is the derivative of the absolutely continuous part of the function \( \sigma(u) \).

The proof is based on non-density in \( L_p^\sigma \) of the set of all rational functions \( \{u^n, n \in \mathbb{Z}\} \).

\textbf{Proof of Theorem 4.1.} Suppose that

\[ \int_{-\infty}^{\infty} \log \frac{\sigma'(u)}{1+u^2} du > -\infty, \] \hspace{1cm} (22)

This is equivalent to the fact that \( e^{iau}, \alpha \geq 0 \) is not dense in \( L_p^\sigma \), see Aachieser [1], p. 87. From the Hahn-Banach Theorem there exists a function \( g \in L_p^\sigma \) such that

\[ \int_{-\infty}^{\infty} e^{iau} g(u) d\sigma(u) = 0, \alpha \geq 0. \] \hspace{1cm} (23)
Differentiating \((23)\) \(n\) times with respect to \(\alpha\) and putting \(\alpha = 0\) yields
\[
\int_{-\infty}^{\infty} u^n g(u) d\sigma(u)
\]
for \(n \in \mathbb{Z}_+\). This means that the polynomials \(\{u^n, n \geq 0\}\) are non-dense in \(L^p_\sigma, p \geq 1\). To achieve non-denseness of \(u^n\) for negative values of \(n\) we proceed as follows. Consider the function \(F_1(u)\) defined by
\[
F_1(u) = \int_{-\infty}^{\infty} \frac{e^{i\alpha u}}{iu} g(u) d\sigma(u),
\]
which is well defined since \(\sigma\) has finite moments of all orders. Differentiation of \(F_1(\alpha)\) with respect to \(\alpha\) gives \(F_1(\alpha) = 0\), according to \((24)\). From this we conclude that \(F_1(\alpha)\) is constant and from the Riemann-Lebesgue lemma the Fourier transform \(F_1\) has the property
\[
\lim_{n \to \infty} F_1(\alpha) = 0
\]
hence \(F_1(\alpha) = 0\). Hence \(F_1(\alpha) = 0\). Hence \(F_1(\alpha)\) is identically zero and letting \(\alpha\) tending to zero gives
\[
\int_{-\infty}^{\infty} \frac{1}{u^2} g(u) d\sigma(u) = 0,
\]
which is \((24)\) for \(n = -1\).
Now define the function \(F_2(\alpha)\) by
\[
F_2(u) = \int_{-\infty}^{\infty} \frac{e^{i\alpha u}}{(iu)^2} g(u) d\sigma(u),
\]
which is also well defined. Then \(F_2'(\alpha) = F_1(\alpha) \equiv 0\) and \(F_2(\alpha) \equiv 0\) by the same argument as above. Letting \(\alpha \to 0\) yields
\[
\int_{-\infty}^{\infty} \frac{1}{u^2} g(u) d\sigma(u) = 0,
\]
We continue this process by defining
\[
F_k(\alpha) = \int_{-\infty}^{\infty} \frac{e^{i\alpha u}}{(iu)^k} g(u) d\sigma(u), k \geq 1.
\]
\(F_k(\alpha)\) is well defined, since \(\sigma\) has finite moments of all orders, and \(F_{k+1}'(\alpha) = F_k(\alpha)\)
By an induction argument we can prove that \(F_k(\alpha) \equiv 0\) all \(k \geq 1\).
Now letting \( \alpha \) tending to zero we have proved the expression (24) for \( \alpha \) for \( n < 0 \). Since now (24) holds for all \( n \in \mathbb{Z} \), the set of all rational functionns \( \{u^n, n \in \mathbb{Z}\} \) is not dense in \( L^p_\sigma, p \geq 1 \). We are now in a position that we can prove that the strong Hamburger moment problem (25) is indeterminate.

Let us suppose that \( \|f\|_{\infty, \sigma} \leq 1 \). Consider the distribution function \( \beta(u) \) defined by

\[
\beta(u) = \sigma(u)[1 + s \cdot g(u)], \ u \in \mathbb{R}, \ s \in [-1, 1].
\]

It follows (24), for \( n \in \mathbb{Z} \), that \( \beta(u) \) has the same moments as \( \sigma(u) \), and since \( g \not\equiv 0 \), \( \beta(u) \) and \( \sigma(u) \) are different distribution functions possessing the same moments \( \mu_n, n \in \mathbb{Z} \). The proof is finished.

The corresponding result for the Stieltjes moment problem is as follows.

**Theorem 4.2.** Let \( \sigma(u), -\infty < u < \infty \), be a distribution function, and suppose also that \( \sigma(u) \) possesses finite moments of all orders.

Let

\[
\mu_n = \int_0^\infty u^nd\sigma(u), \ n \in \mathbb{Z}.
\]  \hspace{1cm} (25)

If

\[
\int_0^\infty \frac{\log \sigma'(u^2)}{1 + u^2} du > -\infty,
\]  \hspace{1cm} (26)

then the strong Stieltjes moment problem (17) is indeterminate and. Here \( \sigma'(u) \) is the derivative of the absolutely continuous part of the function \( \sigma(u) \).

Theorem 4.2 is proved as in the strong Hamburger case with only minor changes. See also the proof in the classical Stieltjes case, in [3], p. 10-15.

**Proof of Theorem 4.2.** From [3], p. 12-13, we have that (26) implies that \( \{e^{i\alpha u}, \alpha \geq 0\} \) is not dense in \( L^\infty_\sigma([0, \infty[) \) since \( L^2_\sigma \subset L^1_\sigma \). From the Hahn-Banach Theorem there exists a function \( g \in L^\infty_\sigma([0, \infty[) \) such that

\[
\int_0^\infty e^{i\alpha u} g(u) d\sigma(u), \ \alpha \geq 0.
\]  \hspace{1cm} (27)

Now we define the function \( F_k(\alpha) \) by
\[
F_k(\alpha) = \int_{0}^{\infty} \frac{e^{iu\alpha}}{(iu)^k} g(u) \, d\sigma(u), \quad k \in \mathbb{Z}_+.
\]

which is well defined since all moments are finite. First by differentiation of (27) with respect to \(\alpha\), \(k\) times \(k \geq 0\), and putting \(\alpha = 0\) yields that

\[
\int_{0}^{\infty} u^k g(u) \, d\sigma(u), \quad k \in \mathbb{Z}_+.
\]

By precisely the same argument as in the strong Hamburger case we get that

\[
\int_{0}^{\infty} \frac{1}{u^k} g(u) \, d\sigma(u), \quad k \in \mathbb{Z}_+.
\]

and hence \(\left\{ \frac{1}{u^k}, k \in \mathbb{Z}_+ \right\}\) is not dense in \(L_1^\sigma([0, \infty[)\). From this we conclude that the set of all rational functions \(\{u^n, n \in \mathbb{Z}\}\) is not dense in \(L_1^\sigma([0, \infty[)\).

Now we may conclude, by the same argument as in the strong Hamburger case, that the Krein condition (26) is a sufficient condition for indeterminacy of the strong Stieltjes moment problem (25) and the proof is finished.

**Remark 4.1.** Berg [6], Theorem 5.1, p. 28, has a sufficient condition

\[
2 \int_{0}^{\infty} \frac{u \log \sigma'(u^2)}{1 + u^2} \, du = \int_{0}^{\infty} \frac{\log \sigma'(u)}{1 + u^2} \, du > -\infty, \quad (28)
\]

for indeterminacy of the classical Stieltjes moment problem.

This is proved by a more direct approach that the proof the classical Hamburger case in Achiezer but it not known if the condition of Berg above also is a sufficient condition for indeterminacy of the strong Stieltjes moment problem (25). It is easy to see that the Krein condition (26) implies (28) but it not known if (28) is a sufficient for the polynomials (rational functions) \(\{u^n, n \in \mathbb{Z}_+(n \in \mathbb{Z})\}\) to be non-dense in \(L_1^\sigma(R)\).

Our main result is that conditions (21) and (26) are in fact sufficient conditions for indeterminacy of the strong Hamburger and Stieltjes moment problems, respectively. This is proved in Theorem 4.1 and Theorem 4.2 (Section 4).

In section 5 we shall define a closely related moment problem, where we construct a symmetric distribution function \(\hat{\sigma}(u)\) from a distribution function \(\sigma(u)\).
5. The symmetrized strong Hamburger moment problem

Let us first define the following class of distribution functions.

**Definition 5.1.** A distribution function \( \sigma \) on \( I \), where \( I = [0, \infty[ \) or \( I = ]-\infty, \infty[ \), is called symmetric if

\[
d\sigma(u) = d\sigma\left(\frac{1}{u}\right), \quad u \in I \setminus \{0\}.
\]

(29)

For any symmetric distribution function \( \sigma \) we have \( \mu_{-n} = \mu_n, n \in \mathbb{Z} \), provided \( \sigma \) has moments of all orders.

Now consider the strong Hamburger moment problem

\[
\mu_n = \int_{-\infty}^{\infty} u^n d\sigma(u), n \in \mathbb{Z}.
\]

(30)

Suppose also that \( \sigma(u) \) possesses finite moments of all orders (hence \( \sigma \) has no mass at the origin). Now define the symmetrized strong Hamburger moment problem (31) according to

\[
\tilde{\mu}_n = \int_{-\infty}^{\infty} u^n d\tilde{\sigma}(u), n \in \mathbb{Z},
\]

(31)

where

\[
\tilde{\mu} = \int_{-\infty}^{\infty} d\tilde{\sigma}(u) = \int_{-\infty}^{\infty} d\sigma(u) - \int_{-\infty}^{\infty} \frac{1}{u} d\sigma(u), u \in \mathbb{R} \setminus \{0\},
\]

(32)

where \( Y_0 \) is the Heaviside function located at the origin. Then \( \tilde{\sigma} \) is a symmetric distribution function, in the sense of definition (30) with finite moments of all orders, see [3] p. 9-11. The moments \( \tilde{\mu}_n \) are easily calculated.

From definition (32) we get

\[
\tilde{\mu}_n = \frac{1}{2} (\mu_n + \mu_{-n}), n \in \mathbb{Z}.
\]

If \( \sigma \) is already symmetric then \( \tilde{\sigma} = \sigma \) and \( \tilde{\mu}_n = \mu_n, n \in \mathbb{Z} \). From Section 4 we know that (31) is indeterminate if

\[
\int_{-\infty}^{\infty} \log \tilde{\sigma}'(u) \frac{1}{1 + u^2} du > -\infty,
\]

(33)
The condition (32) is equivalent to
\[
\int_{-\infty}^{\infty} \log \left[ \sigma' (u) + \frac{1}{u} \sigma' \left( \frac{1}{u} \right) \right] \frac{1}{1 + u^2} du > -\infty.
\] (34)

**Remark 5.1.** Since we immediately get the following estimate

\[
\int_{-\infty}^{\infty} \log \left[ \sigma' (u) + \frac{1}{u} \sigma' \left( \frac{1}{u} \right) \right] \frac{1}{1 + u^2} du \leq \int_{-\infty}^{\infty} \log \left[ \sigma' (u) + \frac{1}{u} \sigma' \left( \frac{1}{u} \right) \right] \frac{1}{1 + u^2} du,
\] (35)

It is obvious that if the Krein condition is fulfilled for the moment problem (30) and hence (30) is indeterminate, then the strong moment symmetrized Hamburger moment problem (31) is also indeterminate.

Except for the relation stated above, in Remark 5.1, it is unknown if an indeterminate strong Hamburger moment problem generally implies that the strong symmetrized Hamburger moment problem is indeterminate.

It is also not known if the converse is true.
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