An explicit lifting construction of CAP forms on $O(1,5)$

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We explicitly construct non-tempered cusp forms on the orthogonal group $O(1,5)$ of signature $(1+, 5-)$. Given a definite quaternion algebra $B$ over $\mathbb{Q}$, the orthogonal group is attached to the indefinite quadratic space of rank 6 with the anisotropic part defined by the reduced norm of $B$. Our construction can be viewed as a generalization of the previous work by the first two authors joint with Masanori Muto to the case of any definite quaternion algebras, for which we note that the work just mentioned takes up the case where the discriminant of $B$ is two. Unlike the previous work the method of the construction is to consider the theta lifting from Maass cusp forms to $O(1,5)$, following the formulation by Borcherds. The cuspidal representations generated by our cusp forms are studied in detail. We determine all local components of the cuspidal representations and show that our cusp forms are CAP forms.

Keywords: Theta lifting; non-tempered cusp forms; CAP forms.

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1. Introduction
Since the discovery of counterexamples to the Ramanujan conjecture by Saito-Kurokawa [20] and Howe-Piatetskii-Shapiro [11] et al. we have known that one has to take into consideration the existence of cuspidal representations with a non-tempered local component towards the classification of cuspidal representations. We call such cusp forms non-tempered. The representation theoretic study of [20] and
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[11] by Piatetskii Shapiro [27] leads to the notion of CAP representations, namely cuspidal representations nearly equivalent to irreducible constituents of parabolic inductions (see Definition 8.5). There have been active representation theoretic studies on CAP representation (cf. Soudry [37], Gelbart-Rogawski [7], Rallis-Schiffmann [31], Ginzburg [8], Ginzburg-Rallis-Soudry [10], Ginzburg-Jiang-Soudry [9], et al). The CAP representations are expected to exhaust a large class of non-tempered cusp forms.

We are motivated by a non-holomorphic real analytic construction of non-tempered cusp forms. Our study began with [23], which provided a non-tempered cusp form on $GL_2(B)$ for a division quaternion algebra $B$ over $Q$ with discriminant 2. This was inspired by the paper [28] of the second named author, whose tool is the converse theorem by Maass [22]. We have also constructed non-tempered cusp forms on the orthogonal group $O(1, 5)$ or $O(1, 5)$ as $Q$-algebraic groups (cf. Section 2.3), where $SO(1, 5)$ and $O(1, 5)$ are attached to the quadratic form of signature $(1, 5)$ whose anisotropic part is defined by the reduced norm of $B$. Following the approach of [21] this paper constructs non-tempered cusp forms on $O(1, 5)$ for the case of any definite quaternion algebra $B$, namely with no restriction on the discriminants of $B$. They turn out to satisfy the CAP properties. The lifting constructions from smaller groups are typical ways to find examples of non-tempered cusp forms. For references in this direction we cite Oda [26], Rallis-Schiffmann [30], Ikeda [12,13], Ikeda-Yamana [14], Yamana [40,41] and Kim-Yamauchi [18] et al.

Let us now describe the main results of the paper. Let $d_B$ be the discriminant of a definite quaternion algebra $B$ over $Q$. For a maximal order $\mathcal{O}$ of $B$, let $\mathcal{O}'$ be the dual lattice of $\mathcal{O}$ with respect to the reduced trace of $B$. We denote by $Q_{A_0}$ (cf. Sections 2) the quadratic form attached to the reduced norm of $B$. Let $\Gamma$ be the stabilizer of the lattice $\mathcal{O} \otimes \mathbb{Z}^2$ in the $Q$-rational points of the orthogonal group defined by $Q_A = Q_{A_0} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (cf. Sections 2). The space of modular forms on the 5-dimensional hyperbolic space with respect to $\Gamma$ (respectively the space of Maass cusp forms of level $d_B$) is denoted by $M(\Gamma, \sqrt{-1} \Gamma)$ (respectively $S(\Gamma_0(d_B), r)$) and any $F \in M(\Gamma, \sqrt{-1} \Gamma)$ has the Fourier expansion (see Section 2 for details on notations)

$$F(n(x)a_y) = \sum_{\beta \in \mathcal{O}'} A(\beta)y^2K_{\sqrt{-1} \Gamma}(4\pi \sqrt{Q_{A_0}(\beta)}y)e^{(i \beta A_0 x)}. \quad (1.1)$$

Our construction is given by a theta lift $F_I$ from Maass cusp forms $f$ of level $d_B$, with Fourier coefficients $A(\beta)$ explicitly described in terms of Fourier coefficients $c(m)$ of $f$.

To describe the formula for $A(\beta)$, let us introduce the set of the primitive elements as follows:

$$\mathcal{O}_{prim}' := \{\beta \in \mathcal{O}' : \frac{1}{n} \beta \not\in \mathcal{O}' \text{ for all positive integers } n > 1\}.$$
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Write $\beta \in \mathcal{O}'$ as

$$\beta = \prod_{p|d_B} p^{n_p} \beta_0, \quad u_p \geq 0, n > 0, \gcd(n, d_B) = 1 \text{ and } \beta_0 \in \mathcal{O}'_{\text{prim}}.$$ 

Let $q_{\beta_0} = \frac{q_{\beta_0}}{n_{\beta_0}}$ be the denominator of the simple fraction for the reduced norm of $\beta_0$ (cf. Section 3.3), which is a divisor of $d_B$. For $p|d_B$, set

$$\delta_p = \begin{cases} 0 & \text{if } p|q_{\beta_0}; \\ 1 & \text{if } p \nmid q_{\beta_0}. \end{cases}$$

Let us assume that the Maass cusp form $f$ has the Atkin-Lehner eigenvalue $\epsilon_p$ at $p|d_B$ and has the trivial central character. Define

$$A(\beta) := \sqrt{Q_{A_0}(\beta)} \sum_{p|d_B} \sum_{t_p=0}^{2u_{\epsilon_p}+\delta_p} \sum_{d|n} c \left( \frac{-Q_{A_0}(\beta)}{\prod_{p|d_B} p^{t_p} d^2} \right) \prod_{p|d_B} (\epsilon_p)^{t_p-1}. \quad (1.2)$$

Putting together the results obtained in Theorem 4.4, Proposition 4.5, Proposition 5.2, Theorem 6.2, Theorem 6.5 and Theorem 7.1 we have the following result.

**Theorem 1.1.** Let $B$ be a definite quaternion division algebra with discriminant $d_B$, which is square-free by definition, and let $\mathcal{O}$ be any maximal order of $B$. Let $f \in S(\Gamma_0(d_B), \mathcal{r})$, be an Atkin-Lehner eigenfunction with eigenvalues $\epsilon_p$ for $p|d_B$. Let $F_f$ be a function on the 5-dimensional hyperbolic space given by the Fourier expansion (1.1) with coefficients $A(\beta)$ given in (1.2). Then the following is true:

1) $F_f$ is a non-zero, cusp form in $\mathcal{M}(\Gamma, \sqrt{-1} \mathcal{r})$ for all non zero $f$.
2) Suppose further that $f$ is a Hecke eigenform with eigenvalues $\lambda_p$ for all $p|d_B$. Then $F_f$ is also an eigenfunction for the Hecke algebra $\mathcal{H}_p$ for all primes $p$.
3) For $p|d_B$, let $\mu_i$, $i = 1, 2, 3$ be the Hecke eigenvalues for $F_f$ corresponding to the three generators $C_3^{(1)}, i = 1, 2, 3$ of $\mathcal{H}_p$. Then we have

$$\mu_1 = p^2(\lambda^2 - 2) + pf_{2,1} = p^2(\lambda^2 + p + p^{-1})$$

$$\mu_i = |R_2^{(i-1)}| \left( \mu_1 - \frac{p^{i-1} - 1}{p^i - 1} f_{3,1} \right), (i = 2, 3)$$

See (6.3) and (6.4) for the definition of $f_{2,1}, f_{3,1}$ and $|R_2^{(i-1)}|$.
4) Suppose that $f$ is a newform. For $p|d_B$, let $\mu$ be the Hecke eigenvalue of $F_f$ for the Hecke operator $C_3^{(1)}$, which generates $\mathcal{H}_p$. Then we have

$$\mu = p^3 + p^2 - p + 1.$$ 

By adelizing our explicit lifts in terms of their Fourier expansion we can develop their Hecke theory to obtain the theorem above. This also enables us to understand the cuspidal representations generated by the lifts explicitly.
Theorem 1.2 (Theorem 8.4, Proposition 8.6, Proposition 8.7). Suppose that the Maass cusp form $f$ is a newform with the trivial central character, and Hecke eigenvalues $\lambda_p$ for primes $p \nmid d_B$. Let $\pi$ be the cuspidal representation of $O(1,5)(\mathbb{A})$ generated by the lift $F_f$ from $f$.

1. The representation $\pi$ is irreducible and decomposes into the restricted tensor product $\pi = \otimes_v \pi_v$ of irreducible admissible representations $\pi_v$.

2. For $v = p < \infty$, if $p \nmid d_B$ then $\pi_p$ is the spherical constituent of the unramified principal series representation of $O(1,5)(\mathbb{Q}_p) \simeq O(3,3)(\mathbb{Q}_p)$ with the Satake parameter

$$\text{diag} \left( \frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}, p, 1, p^{-1}, \frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2} \right).$$

3. For $v = p < \infty$, if $p \mid d_B$ then $\pi_p$ is the spherical constituent of the spherical representation $L(\chi)$ of $O(1,5)(\mathbb{Q}_p)$ induced from the unramified character $\chi$ of the split torus of $O(1,5)(\mathbb{Q}_p)$ isomorphic to $\mathbb{Q}_p^* \setminus \mathbb{Q}_p$ with $\chi(p) = p$.

4. For every finite prime $p$, $\pi_p$ is non-tempered. Suppose that the Selberg conjecture on the maximal Laplace eigenvalue holds for $f$. Then $\pi_\infty$ is tempered.

5. The cuspidal representation is a CAP representation associated with some explicit parabolic induction of $O(3,3)(\mathbb{A})$.

6. Let $\sigma$ denote the cuspidal representation of $GL_2(\mathbb{A})$ generated by $f$. Let $\Pi = \text{Ind}_{P_{2,2}(\mathbb{A})}^{GL_2(\mathbb{A})}[\det |^{1/2}_\mathbb{A}] \times [\det |^{1/2}_\mathbb{A}]$, with the parabolic subgroup $P_{2,2}$ of $GL_4$ with Levi part $GL_2 \times GL_2$. By $L(F_f, \text{std}, s)$ (respectively $L(\Pi, \wedge, s)$) we denote the standard $L$-function for the lift $F_f$ (respectively exterior square $L$-function of $\Pi$). We have

$$L(F_f, \text{std}, s) = L(\Pi, \wedge, s) = L(\text{sym}^2(f), s)\zeta(s-1)\zeta(s)\zeta(s+1).$$

Let us note that a priori, the lift $F_f$ depends on the discriminant $d_B$ of the quaternion algebra $B$, the Atkin-Lehner eigenform $f \in S(\Gamma_0(d_B), r)$ and the maximal order $\mathcal{O}$ in $B$. The above theorem shows that the local components of the representation $\pi$ generated by $F_f$ are in fact independent of the maximal order $\mathcal{O}$ and the Atkin-Lehner eigenvalues of $f$. It is interesting that the explicit Fourier coefficients $A(\beta)$ clearly depend on the maximal order $\mathcal{O}$ and the Atkin-Lehner eigenvalues $\epsilon_p$ for $p\mid d_B$, while the local components of the cuspidal automorphic representation do not. A multiplicity one theorem for $O(1,5)$ would imply that different maximal orders would give lifts which are different vectors in the same cuspidal automorphic representations. Such a multiplicity one theorem is not currently available but is expected since we have the multiplicity one theorem by Badulescu and Renard [1] for the group $PGL_2(B)$.

There are a few significant differences between the results and methods of this paper as compared to our previous work in [21,23]. In [23], we restricted ourselves to the case $d_B = 2$. Here the discrete group $\Gamma$ was generated by translations and
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An inversion. The Maass converse theorem [22] gives a criterion for modularity with respect to such groups, and we used it to show that the proposed lift in [23] is a modular form. A situation for which we know that the discrete subgroup \(\Gamma\) has such generators is when \(O\) satisfies the Euclidean property with respect to the reduce norm. Using [25] we can prove that this happens only when the discriminant \(d_B\) equals 2, 3 or 5. In this case, we have obtained the proof of modularity of our lift using the Maass converse theorem, but we have not included it in this article since we are interested in general \(B\)s. Instead, to prove modularity, we have used the more general method of Borcherds theta lifts as in [21].

In [21], we were constructing lifts to modular forms on \(O(1, 8n + 1)\) starting from Maass forms of full level. For the lifting in Theorem 1.1 above, we need to consider Maass forms with square-free level \(d_B\). For the Borcherds theta lift method to work, an initial step is to transition from scalar valued Maass forms with level \(d_B\) to vector valued modular forms with respect to the Weil representation of \(SL_2(\mathbb{Z})\). We work out the corresponding vector valued modular forms and obtain explicit formulas for their Fourier coefficients. This is an active area of research, and the explicit formulas for the Fourier coefficients in the square-free case might be of independent interest.

The explicit formula (1.2) for the Fourier coefficients \(A(\beta)\) in Proposition 4.5 needs subtle understanding of the structure of the discriminant form \(O'/O\) to determine which elements of \(O'\) correspond to which cusps of \(\Gamma_0(d_B)\). Furthermore, we remark that, in [21] and [23], we showed the non-vanishing of the lifts by reducing the non-vanishing of \(A(\beta)\) to that of \(c(-M)\) for a suitable positive integer \(M\). For the proof we used the explicit formula for \(A(\beta)\) together with the surjectivity for the norm map of some special lattice to the set of non-negative integers. However, since the maximal order \(O\) is arbitrary and such surjectivity is not always true for a general \(O\), even the explicit nature of the formula for \(A(\beta)\) is not sufficient to obtain non-vanishing. For the non-vanishing of the lift from Theorem 1.1, we could perhaps use Bhargava’s 15 Theorem [2] to show that the norm map is surjective for special cases of maximal orders. But for obtaining the theorem in full generality we use another approach using a simple idea from linear algebra. This requires us to first show that the map \(f \mapsto F_f\) takes Hecke eigenforms to Hecke eigenforms. We then see the non-vanishing of \(A(1)\) of \(F_f\) for a non-zero Hecke eigenform \(f\). For the non-vanishing, it turns out to be enough to show that, when \(f\) runs over a Hecke eigenbasis of \(S(\Gamma_0(d_B), r)\), the lifts \(F_f\) distinguish each other by their Hecke eigenvalues at just one prime \(p \nmid d_B\), which lead us to prove that the map \(f \mapsto F_f\) is a linear injection from \(S(\Gamma_0(d_B), r)\). We should remark that there is a well known approach to the non-vanishing of theta lifts using the inner product formula initiated by Rallis [29]. Our method is very different and elementary.

To obtain the Hecke theory, we use the work of Sugano [38]. The case of \(p \nmid d_B\) follows directly as in [21]. The Hecke theory for primes \(p|d_B\) requires a detailed analysis of the non-split group and makes use of the explicit formula of the Fourier coefficients \(A(\beta)\).
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Let us explain the outline of the paper. In Section 2 we begin with the review on the orthogonal groups over which we work. This section includes fundamental facts on definite quaternion algebras and accidental isomorphisms necessary for the coming discussion. Section 3 is devoted to a detailed study on vector-valued modular forms. This section includes an explicit description of vector-valued forms lifted from Maass cusp forms with square-free levels, which is indispensable for deducing an explicit formula for Fourier coefficients of our lifts. In Section 4 we formulate our lifts as the theta lifts to \(O(1,5)\) in the non-adelic setting and provide their explicit formula for the Fourier coefficients. The lifts are proved to be cuspidal in Section 5.

To obtain the representation theoretic aspect of our lifts we adelize them and discuss their Hecke theory in Section 6. In Section 7 we obtain the non-vanishing of our lifts by virtue of the study on the Hecke theory. In Section 8 we have a detailed understanding of the cuspidal representations generated by our lifts, all of whose local components are determined explicitly. As a result our lifts are non-tempered at every non-archimedean place while they are tempered at the archimedean place under the assumption that the Selberg conjecture on the minimal Laplace eigenvalue holds for Maass cusp forms \(f\). The lifts are then proved to be CAP forms attached to some explicitly given parabolic induction for the split orthogonal group \(O(3,3)\). Section 8 ends with an explicit formula for the global standard \(L\)-functions of the lifts from Maass cusp forms, whose statement is given as Proposition 8.7. The definition follows Sugano [38, Section 7 (7.6)]. Proposition 8.7 also shows that our global standard \(L\)-function coincides with the exterior square \(L\)-function for some parabolic induction of \(GL_4\).

2. Preliminaries

In this section, we give the definitions of orthogonal groups, modular forms and quaternion algebras. We also give details on certain accidental isomorphisms.

2.1. Orthogonal groups and modular forms

Let \(A_0 \in M_4(\mathbb{Q})\) be a positive definite symmetric matrix, and put \(A = \begin{bmatrix} 1 & -A_0 \\ A_0 & 1 \end{bmatrix}\).

By \(\mathcal{G}\) and \(\mathcal{H}\) we denote the \(\mathbb{Q}\)-algebraic groups defined by

\[
\mathcal{G}(\mathbb{Q}) = \{g \in \text{GL}_6(\mathbb{Q}) \mid {}^t g A g = A\}, \quad \mathcal{H}(\mathbb{Q}) = \{h \in \text{GL}_4(\mathbb{Q}) \mid {}^t h A_0 h = A_0\}
\]

respectively. Both \(\mathcal{G}\) and \(\mathcal{H}\) are referred to as orthogonal groups. We introduce the standard proper \(\mathbb{Q}\) parabolic subgroup \(\mathcal{P}\) of \(\mathcal{G}\) defined by the Levi decomposition.
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$\mathcal{P} = N \mathcal{L}$ with

$$\mathcal{N}(Q) = \left\{ n(x) = \begin{pmatrix} 1 & x A_0 & \frac{1}{4} x A_0 x \\ 1 & x \\ 1 \end{pmatrix} \mid x \in \mathbb{Q}^4 \right\},$$

$$\mathcal{L}(Q) = \left\{ a_\alpha = \begin{pmatrix} \alpha \\ h \\ \alpha^{-1} \end{pmatrix} \mid \alpha \in \mathbb{Q}^\times, h \in \mathcal{H}(Q) \right\}.$$

Assume that $L_0$ is a maximal even integral lattice in $\mathbb{Q}^4$ with respect to $A_0$. We put

$$L := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x, z \in \mathbb{Z}, y \in L_0 \right\} = L_0 \oplus \mathbb{Z}^2.$$

This is a maximal lattice with respect to $A$. We let $\Gamma := \{ \gamma \in \mathcal{G}(Q) \mid \gamma L = L \}$.

Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$ and $\mathbb{A}_f$ be the set of finite adeles in $\mathbb{A}$. We consider the adelic extensions of the $\mathbb{Q}$-algebraic groups above, denoted by $\mathcal{G}(\mathbb{A})$, $\mathcal{H}(\mathbb{A})$, $\mathcal{P}(\mathbb{A})$, $\mathcal{N}(\mathbb{A})$ and so on. Let $L_p := L \otimes \mathbb{Z}_p$ and $L_{0,p} := L_0 \otimes \mathbb{Z}_p$ and we put $K_f := \prod_{p < \infty} K_p$ and $U_f := \prod_{p < \infty} U_p$ with

$$K_p := \{ k \in \mathcal{G}(\mathbb{Q}_p) \mid kL_p = L_p \}, \quad U_p := \{ u \in \mathcal{H}(\mathbb{Q}_p) \mid uL_{0,p} = L_{0,p} \}$$

for each finite prime $p < \infty$. Let $K_\infty$ be the maximal compact subgroup of $\mathcal{G}(\mathbb{R})$ given by

$$\left\{ g \in \mathcal{G}(\mathbb{R}) \mid {}^t g \begin{pmatrix} 1 \\ A_0 \\ 1 \end{pmatrix} g = \begin{pmatrix} 1 \\ A_0 \\ 1 \end{pmatrix} \right\}.$$

With $A_\infty := \{ a_y = \begin{pmatrix} y \\ 1 \\ y^{-1} \end{pmatrix} \mid y \in \mathbb{R}_+ \}$ the Iwasawa decomposition $\mathcal{G}(\mathbb{R}) = \mathcal{N}(\mathbb{R})A_\infty K_\infty$ gives us the 5-dimensional hyperbolic space $\mathbb{H}_5$ as follows.

$$\mathbb{R}^4 \times \mathbb{R}_+ \ni (x, y) \mapsto n(x)a_y \in \mathcal{G}(\mathbb{R})/K_\infty.$$

**Definition 2.1.** For $r \in \mathbb{C}$ we denote by $\mathcal{M}(\Gamma, r)$ the space of smooth functions $F$ on $\mathcal{G}(\mathbb{R})$ satisfying the following conditions:

i) $\Omega \cdot F = \frac{1}{8} (r^2 - 4) F$, where $\Omega$ is the Casimir operator defined in [21, (2.3)],

ii) for any $(\gamma, g, k) \in \Gamma \times \mathcal{G}(\mathbb{R}) \times K_\infty$, we have $F(\gamma g k) = F(g)$,

iii) $F$ is of moderate growth.

As usual we say that $F \in \mathcal{M}(\Gamma, r)$ is a cusp form if it vanishes at all the cusps of $\Gamma$. 
From Proposition 2.3 of [21], we see that a cusp form $F$ in $M(\Gamma, r)$ has the Fourier expansion

$$F(n(x) a_y) = \sum_{\beta \in L_0 \setminus \{0\}} A(\beta) y^2 K_0(4\pi \sqrt{Q_{A_0}(\beta)} y) e(\beta A_0 x),$$

with the dual lattice $L_0'$ of $L_0$. Here, $Q_{A_0}$ is the quadratic form corresponding to $A_0$.

### 2.2. Quaternion algebras

We want to restrict to the case where the lattice $L_0$ from the previous section corresponds to maximal orders in division quaternion algebras. In this section, we will provide the relevant information about quaternion algebras, maximal orders and their duals. A good reference is the book [39] by Jon Voight. Let $B$ be a definite division quaternion algebra over $\mathbb{Q}$, given by $\mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$, with $i^2 = a, j^2 = b, ij = ji = k$. Let us denote the standard involution on $B$ by $\alpha \mapsto \bar{\alpha}$. Let the trace and norm be defined by $\text{tr}(\alpha) = \alpha + \bar{\alpha}$ and $\text{Nrd}(\alpha) = \alpha \bar{\alpha}$. Assume that $B$ has discriminant $d_B = N$. Hence, $N$ is a square-free integer with an odd number of prime factors.

Let $\mathcal{O}$ be any maximal order in $B$. Let $A_0$ be the gram matrix of $\mathcal{O}$ with respect to some basis, so that $\mathcal{O} \simeq (\mathbb{Z}^4, A_0)$. Let $Q_{A_0}$ be the quadratic form given by $Q_{A_0}(x) = \frac{1}{2} x A_0 x$ for $x \in \mathbb{Z}^4$, and $B_{A_0}$ be the corresponding bilinear form. Note that if $\alpha, \beta \in \mathcal{O}$ get mapped to $x, y \in \mathbb{Z}^4$, then $\text{Nrd}(\alpha) = Q_{A_0}(x)$ and $\text{tr}(\alpha \bar{\beta}) = B_{A_0}(x, y)$. Let

$$L = \{ \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} : a, b \in \mathbb{Z}, \alpha \in \mathcal{O} \}.$$ 

Then $L \simeq (\mathbb{Z}^6, A)$, with $A = \begin{bmatrix} a & 1 \\ -A_0 & 1 \end{bmatrix}$. Then $Q_A(a, x, b) = ab - Q_{A_0}(x)$. Hence, the signature of $L$ is $(1, 5)$. The bilinear form $B_A$ on $L$ is given by

$$B_A(x, x) = 2Q_A(x),$$

and

$$B_A(x, y) = \frac{1}{2}(B_A(x + y, x + y) - B_A(x, x) - B_A(y, y)) = 'x A y$$

for $x, y \in L$. We will be considering the orthogonal groups $\mathcal{G}$ and $\mathcal{H}$ with respect to the above matrices $A$ and $A_0$. Define the dual of $\mathcal{O}$ by

$$\mathcal{O}' := \{ \alpha \in B(\mathbb{Q}) : \text{tr}(\alpha \mathcal{O}) \subset \mathbb{Z} \}.$$ 

Let us collect some facts about $\mathcal{O}$ and $\mathcal{O}'$. 

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i) Since $O$ is maximal, we can see that $O = \{ \alpha \in O' : \text{Nrd}(\alpha) \in \mathbb{Z} \}$.

ii) Let the discriminant $\text{disc}(O)$ be as in [39, (15.1.2)]. We have $\text{disc}(O) = N^2$, since $O$ is a maximal order [39, Theorem 15.5.5]. We also have [39, Lemma 15.6.17]

\[ \text{disc}(O) = [O' : O] = N^2. \]

iii) Define

$$(O')^{-1} := \{ \alpha \in B(Q) : O'\alpha O' \subset O' \}.$$  

By [39, Proposition 16.5.8], we have $(O')^{-1}O' = O$. Further, we also have [39, Equation 16.8.4]

$\text{Nrd}((O')^{-1}) = \text{ideal generated by Nrd}(\alpha)$ for all $\alpha \in (O')^{-1} = N\mathbb{Z}$.

This gives us

$$\text{Nrd}(O') = \frac{1}{N} \mathbb{Z}.$$  

iv) For a prime number $p$, let $O_p = O \otimes \mathbb{Z}_p$ and $O'_p = O' \otimes \mathbb{Z}_p$. It is known that $O_p$ is a maximal order in $B_p = B \otimes \mathbb{Q}_p$. For $p \nmid N$, $B_p$ is isomorphic to $M_2(Q_p)$. Up to conjugation, there is a unique maximal order in $B_p$ given by $M_2(Z_p)$, which is its own dual.

v) For $p | N$, $B_p$ is a division algebra. From [36, Theorem 5.13], we have the following information on the local maximal order and its dual.

- We have a unique maximal order $O_p$ in $B_p$ given by $\{ \alpha \in B_p : \text{Nrd}(\alpha) \in \mathbb{Z}_p \}$.
- Let $\mathfrak{P} := \{ \alpha \in B_p : \text{Nrd}(\alpha) \in p\mathbb{Z}_p \}$. Then we have

$$\mathfrak{P}^m = \{ \alpha \in B_p : \text{Nrd}(\alpha) \in p^m\mathbb{Z}_p \}$$

for $m \in \mathbb{Z}$, $pO_p = \mathfrak{P}^2$, and $O'_p = \mathfrak{P}^{-1}$.

- Let $K_p \subset B_p$ be the unique non-trivial unramified extension of $\mathbb{Q}_p$. We have

$$K_p = \begin{cases} \mathbb{Q}_2(\sqrt{5}) & \text{if } p = 2; \\
\mathbb{Q}_p(\sqrt{-1}) & \text{if } p \equiv 3, 7 \pmod{8}; \\
\mathbb{Q}_p(\sqrt{2}) & \text{if } p \equiv 5 \pmod{8}; \\
\mathbb{Q}_p(\sqrt{q}) & \text{if } p \equiv 1 \pmod{4}, q \equiv 3 \pmod{4}, \left(\frac{q}{p}\right) = -1. \end{cases}$$

Let $O_{K_p}$ be the ring of integers of $K_p$. Then there exists $w_p \in B_p$ such that $w_p^2 = p$ and $B_p = K_p + w_pK_p$, $O_p = O_{K_p} + w_pO_{K_p}$ and $\mathfrak{P} = w_pO_p$. Hence, $O'_p = w_p^{-1}O_p = O_{K_p} + w_p^{-1}O_{K_p}$.

- We have

$$O'_p/O_p \simeq \langle w_p^{-1}\rangle \simeq \langle w_p^{-1}\rangle \times \langle uw_p^{-1}\rangle \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z},$$
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where

$$u = \begin{cases} \sqrt{5} & \text{if } p = 2; \\ -1 & \text{if } p \equiv 3 \pmod{4}; \\ \sqrt{2} & \text{if } p \equiv 5 \pmod{8}; \\ \sqrt{q} & \text{if } p \equiv 1 \pmod{8}. \end{cases}$$

2.3. Accidental isomorphisms

For a quaternion algebra $E$ over $\mathbb{Q}$ with the reduced norm $N_E$ we view $(E, N_E)$ as a rank 4 quadratic space over $\mathbb{Q}$. This gives rise to the rank 6 quadratic space $V_E := (E, N_E) \oplus \mathbb{H}$ with the hyperbolic space $\mathbb{H}$. For the subsequent argument we will need the two well-known accidental isomorphisms

$$E^\times \times E^\times / \{(z, z) \mid z \in GL_1\} \simeq GSO(E, N_E),$$

$$GL_2(E) \times GL_1 / \{(z \cdot 14, z^{-2}) \mid z \in GL_1\} \simeq GSO(V_E)$$

as $\mathbb{Q}$-algebraic groups (cf. [5, Section 3]).

Let $E := M_2$ be the matrix algebra of degree two over $\mathbb{Q}$. The group on the right hand side of the first isomorphism is the similitude group defined by the determinant form of $M_2$. We denote this by $GSO(2,2)$ in view of the signature of the quadratic space at the archimedean place. The isomorphism is induced by

$$GL_2 \times GL_2 \ni (h_1, h_2) \mapsto M_2 \ni X \mapsto h_1 X h_2^{-1} \in M_2.$$

Let $t$ be the main involution of $M_2$. This induces the outer automorphism

$$GL_2 \times GL_2 \ni (h_1, h_2) \mapsto (t(h_1)^{-1}, t(h_2)^{-1}) \in GL_2 \times GL_2.$$

We denote this by $t$. With this $t$ we have an isomorphism

$$GO(2, 2) \simeq GSO(2, 2) \rtimes \langle t \rangle.$$

Regarding the second isomorphism the similitude group on the right hand side is defined by the quadratic form $ab - N_E(X)$ defined on the $\mathbb{Q}$-vector space

$$V_E := \left\{ \begin{pmatrix} a & x \\ \iota(x) & b \end{pmatrix} \mid a, b \in \mathbb{Q}, x \in M_2 \right\}.$$ 

Since the signature of this quadratic space is $(3+, 3-)$ this group can be denoted by $GSO(3,3)$. The isomorphism is given by

$$GL_4 \times GL_1 \ni (g, z) \mapsto V_E \ni X \mapsto z \cdot g X \iota(g) \in V_E,$$

where we put $\iota(g) := \begin{pmatrix} \iota(x) & \iota(y) \\ \iota(z) & \iota(w) \end{pmatrix}$ for $g = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ with $x, y, z, w \in M_2$.

Of course, we are interested in the case of $E = B$. For this case the similitude groups can be denoted by $GSO(4)$ and $GSO(1,5)$ for the first and second isomorphisms respectively.
3. Vector valued modular form

In this section, we will start with a weight 0 Maass form for $\Gamma_0(N)$ and construct a weight $(0, 0)$ vector-valued modular form for the Weil representation of $SL_2(\mathbb{Z})$ on a group algebra of a discriminant form. The main reference for this section is [34].

3.1. The discriminant form

As in the previous section, let $B$ be a definite quaternion algebra over $\mathbb{Q}$ with discriminant $d_B = N$, a square-free integer. Let $\mathcal{O}$ be any maximal order of $B$ with $\mathcal{O} \simeq (\mathbb{Z}^4, A_0)$. Let $Q_{A_0}$, $L$ and $A$ be as in Section 2.2.

Let $\mathcal{O}'$ and $L'$ be the dual of $\mathcal{O}$ and $L$ respectively with respect to bilinear forms $B_{A_0}$ and $B_A$. We have described the dual $\mathcal{O}'$ in the previous section. We have

$$L' = \{ \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} : a, b \in \mathbb{Z}, \alpha \in \mathcal{O}' \}.$$ 

Define the discriminant form $D$ by $D = L'/L$. From the description of $L'$ above, we have $D = L'/L = \mathcal{O}'/\mathcal{O}$. This $D$ inherits the quadratic form $Q_D$ and bilinear form $B_D$ (with values in $\mathbb{Q}/\mathbb{Z}$) from those of $\mathcal{O}'$ considered modulo 1. The level of $D$ is the smallest positive integer $n$ such that $nQ_D(\mathbf{1}) \equiv 0 \pmod{1}$ for all $\mathbf{1} \in D$. Since $\text{Nrd}(\mathcal{O}') = \frac{1}{N} \mathbb{Z}$, we see that the level of $D$ is $N$. Every discriminant form is an orthogonal direct sum of basic discriminant forms, which are described in Section 3 of [34].

Let us write $D = \prod_{p \mid N} D_p$, where by Section 2.2, we have

$$D_p = \langle w_p^{-1} \rangle \times \langle uw_p^{-1} \rangle.$$

We have $Q_D(w_p^{-1}) = -1/p$ and $Q_D(uw_p^{-1}) = u^2/p$. When $p = 2$, we see that $Q_D(w_2^{-1}) = Q_D(uw_2^{-1}) = B_D(w_2^{-1}, uw_2^{-1}) = 1/2$. Hence, in the notation of Section 3 of [34], we have $D_2 = 2\mathbb{H}^2$.

Next, suppose $p$ is an odd prime. Since $Q_D(w_p^{-1}) = -1/p$, the basic discriminant form corresponding to $\langle w_p^{-1} \rangle$ is $p^\epsilon$, where $\epsilon = \left( \frac{-2}{p} \right)$. On the other hand

$$Q_D(uw_p^{-1}) = \begin{cases} -1/p & \text{if } p \equiv 3 \pmod{4}; \\ 2/p & \text{if } p \equiv 5 \pmod{8}; \\ q/p & \text{if } p \equiv 1 \pmod{8}. \end{cases}$$

If $Q_D(uw_p^{-1}) = a/p$, then $\langle uw_p^{-1} \rangle$ corresponds to the discriminant form $p'^\epsilon$, where $\epsilon' = \left( \frac{2a}{p} \right)$. Hence, by Section 3 of [34], we have

$$D_p = \begin{cases} \langle p^{+1} \times p^{+1} \rangle = p^{+2} & \text{if } p \equiv 3 \pmod{8}; \\ \langle p^{-1} \times p^{-1} \rangle = p^{-2} & \text{if } p \equiv 7 \pmod{8}; \\ \langle p^{+1} \times p^{-1} \rangle = p^{-2} & \text{if } p \equiv 1, 5 \pmod{8}. \end{cases} (3.1)$$

We have the following relevant information about $D$. 

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will often use the notation $(\text{anti-linear in the second argument})$ is defined by $\langle \cdot, \cdot \rangle$. The conjecture on the minimal Laplace eigenvalue for Maass cusp forms, $r$ on the generators of $\text{SL}_2$, is given in the next lemma.

To construct a vector valued modular form for $\text{SL}_2$, we start with a scalar valued modular form of level divisible by the level of $D$.

**Lemma 3.1.** Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and $\mu \in D$. Then 
\[ \rho_D(M)e_\mu = e(bdQ_D(\mu))e_{d\mu}. \]
In particular, we have $\rho_D(M)e_0 = e_0$ for all $M \in \Gamma_0(N)$.

**Proof.** From equation (4.1) of [34] we get, for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and $\mu \in D$, 
\[ \rho_D(M)e_\mu = \chi_D(a)e(bdQ_D(\mu))e_{d\mu}, \]
where $\chi_D(a) = \left( \frac{a}{|D|} \right)e((a - 1) \cdot \text{oddity}(D)/8)$.

Note that $|D| = N^2$ and oddity$(D) = 4$ if $N$ is even and 0 if $N$ is odd. Hence, for all $D$, we have that $\chi_D$ is the trivial character. This gives the lemma.

**3.3. Scalar to vector valued modular form**

To construct a vector valued modular form for $\text{SL}_2$, with values in $\mathbb{C}[D]$, one has to start with a scalar valued modular form of level divisible by the level of $D$. If such a scalar valued modular form exists for a character $\chi_D$, the corresponding vector valued modular forms can be constructed using the Weil representation.

**Theorem 3.1.** Let $D$ be a squarefree integer. Then there exists a vector valued modular form for $\text{SL}_2$ with values in $\mathbb{C}[D]$ and character $\chi_D$, if and only if $D = \{ p \mid N \}$, where $p$ are prime numbers dividing $N$.

**Proof.** The proof follows from the fact that the scalar valued modular forms for $\text{SL}_2$ correspond to the minimal Laplace eigenvalue for Maass cusp forms, $r$ on the generators of $\text{SL}_2$. According to the Selberg conjecture on the minimal Laplace eigenvalue for Maass cusp forms, $r$ should be...
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The Fourier expansion of $f \in S(\Gamma_0(N), r)$ is given by

$$f(u + iv) = \sum_{n \neq 0} c(n) W_0 \frac{1}{\sqrt{n}} (4\pi |n| v) e(nu).$$

for $h := \{u + iv \in \mathbb{C} : v > 0\}$. Define $L_D(f) : h \to \mathbb{C}[D]$ by

$$L_D(f) = \sum_{M \in \Gamma_0(N) \setminus \text{SL}_2(\mathbb{Z})} f|M\rho_D(M)^{-1}e_0,$$

where $(f|M)(\tau) = f(M \cdot \tau) := f((a\tau + b)/(c\tau + d))$ for $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{R})$.

**Proposition 3.2.** Let $f \in S(\Gamma_0(N), r)$. The function $L_D(f)$ is well-defined and satisfies

$$L_D(f)|\gamma = \rho_D(\gamma) L_D(f),$$

for all $\gamma \in \text{SL}_2(\mathbb{Z})$.

**Proof.** The well-definedness of $L_D(f)$ follows from the $\Gamma_0(N)$-invariance of $f$ and Lemma 3.1. The automorphy condition follows from a simple change of variable.

Let us remark here that if $H$ is an isotropic subgroup of $D$, then the $e_0$ term in the definition of $L_D(f)$ can be replaced by a sum over $H$. In our case, the only isotropic subgroup of $D$ is the trivial one.

In the remainder of the section, we will obtain a formula for the Fourier expansion of $L_D(f)$. From page 660 of [33], we have

$$L_D(f)(\tau) = \sum_{c|N} \sum_{\mu \in \mathbb{D} \mathbb{Z}} \xi_c \sqrt{|D_c|} N \frac{q^N \mu_x (\tau)}{\sqrt{|D|}} e_\mu.$$

Let us explain the terms appearing in the formula above.

i) For any integer $t$, set $D_t := \{\mu \in D : t\mu = 0\}$. In our case, for every $t|N$, we have $D_t = \oplus_{p|D} D_p$. Hence, $|D_t| = t^2$ for $t|N$.

ii) We have

$$\xi_c := \left( \frac{-c}{|D|} \right) \prod_{p \nmid D} \gamma_p(D),$$

with

$$\gamma_p(p^{k+2}) = e(-p\text{-excess}(p^{k+2})/8) \text{ if } p \text{ is odd,}$$

$$\gamma_2(p^{k+2}) = e(\text{oddity}(2^{k+2})/8).$$

We have $p\text{-excess}(p^{k+2}) = 2(p - 1) + k \pmod{8}$ where $k = 4$ if the sign is $-$ and $k = 0$ if the sign is $+$. By (3.1), we have $\gamma_p(D_p) = -1$ for all primes $p$. Hence

$$\xi_c = \prod_{p \nmid D} (-1).$$
iii) Finally, let us describe the functions $g_{\pm,j}$. For every $c|N$, choose $M_c = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z})$ such that $d \equiv 1 \pmod{c}$ and $d \equiv 0 \pmod{N/c}$. As in page 658 of [33], we have, for $0 \leq j \leq N/c$,

$$g_{\pm,j}(\tau) = \frac{1}{N/c} \sum_{k \equiv j \pmod{N/c}} e\left(-\frac{jk}{N/c}\right) (f|M_c T^k)(\tau).$$

The integer $j_{\mu,N/c}$ is defined by $(j_{\mu,N/c})/(N/c) \equiv -Q_D(\mu) \pmod{1}$.

Putting all this together, we see that (3.3) now gives us

$$\mathcal{L}_D(f)(\tau) = \sum_{c|N} \prod_{p|\frac{N}{c}} (-1) \sum_{k \equiv 0 \pmod{\frac{N}{c}}} (f|M_c T^k)(\tau) \sum_{\mu \in D} e(kQ_D(\mu)) \epsilon_\mu. \quad (3.4)$$

To simplify this further, we will assume that $f$ is an eigenfunction of all the Atkin-Lehner operators. For every $c|N$, the Atkin-Lehner operator corresponds to the action on $f$ by the matrix $W_\infty^c \in \text{SL}_2(\mathbb{Z})$ given by

$$W_\infty^c = \begin{bmatrix} N/c & 0 \\ 0 & 1 \end{bmatrix} \text{ with } \text{det}(W_\infty^c) = \frac{N}{c}.$$

Note that $W_\infty^c \in \text{Z}(\mathbb{Q})\Gamma_0(N)$ with $\text{Z}(\mathbb{Q}) := \{ z \cdot 1_2 \mid z \in \mathbb{Q} \}$. Now set $\widehat{W}_c := W_\infty^c \Gamma_0(1)$.

Since $N$ is square-free, the cusps of $\Gamma_0(N)$ are given by $1/c$ where $c$ runs over all divisors of $N$. The cusp $1/N$ corresponds to infinity. Given a matrix $M = \begin{bmatrix} \gamma' & \delta' \\ \nu' & \mu' \end{bmatrix} \in \text{SL}_2(\mathbb{Z})$, it is well known that $M(\infty)$ contains the representative $1/c$, where $c = \gcd(c',N)$. Hence, we have $M_c(\infty) = \widehat{W}_c(\infty)$, which implies that there is a $\gamma_c \in \Gamma_0(N)$ such that $M_c = \gamma_c \widehat{W}_c$.

**Proposition 3.3.** Let $f \in S(\Gamma_0(N),r)$ be a Maass cusp form of weight $0$ with respect to $\Gamma_0(N)$ with Laplace eigenvalue $(r^2+1)/4$. Assume that $f$ is an eigenfunction of the Atkin-Lehner operators and let $f|W_\infty^c = \epsilon_\infty f$. Then, we have

$$\mathcal{L}_D(f)(\tau) = \sum_{c|N} \epsilon_\infty \prod_{p|\frac{N}{c}} (-1) \sum_{a \equiv 0 \pmod{\frac{N}{c}}} \sum_{\nu \neq 0 \pmod{\frac{N}{c}}} (c(n)W_0, e(nu) (4\pi |n| c |N| ) e(nu\frac{c}{N}) \times \sum_{\mu \in D \mod{\frac{N}{c}}} \epsilon_\mu)$$

$$\sum_{\nu+c \equiv 0 \pmod{\frac{N}{c}}} \sum_{\nu \neq 0 \pmod{\frac{N}{c}}} \sum_{\mu \in D \mod{\frac{N}{c}}} \epsilon_\mu.$$
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Note that we have

$$\sum_{k \mod N} e(nk \frac{c}{N}) e(kQ_D(\mu)) e_\mu = \sum_{a \mod N} \sum_{\mu \in D} e(kc \frac{n+a}{N}) e_{\mu}.$$ 

Here, we have used that $\frac{N}{c} Q_D(D_N) \subset \mathbb{Z}$. We have

$$\sum_{k \mod N} e(kc \frac{n+a}{N}) = \left\{ \begin{array}{ll} \frac{N}{c} & \text{if } n+a \equiv 0 \pmod{\frac{N}{c}}; \\ 0 & \text{otherwise.} \end{array} \right.$$ 

Substituting these in (3.4) gives us the formula in the statement of the proposition.

We want to rewrite the formula for $L_D(f(\tau))$ in Proposition 3.3 in the form $\sum_{\mu \in D} f_\mu(\tau) e_\mu$. For this, let us first associate to every $2$-dimensional integer $q$ as follows. Since $NQ_D(D_N) \subset \mathbb{Z}$, write $Q_D(D_N) = b = \frac{a}{\gcd(a,q)}$, where $\gcd(a,q) = 1$. Observe that $2D_N \equiv q \pmod{N}$ for every $c$ satisfying $q|\frac{N}{c}|$. Hence, we have

$$L_D(f(\tau)) = \sum_{\mu \in D} \left( \sum_{\epsilon \equiv \frac{N}{c} \pmod{\frac{N}{c}}} \prod_{p \nmid \frac{N}{c}} (-1) \right. 
\times \sum_{n \not\equiv -Q_D(\mu) \pmod{1}} c(n) W_0(-4\pi |n| \frac{c}{N}) e(nu \frac{c}{N}) e_\mu. \quad (3.5)$$

Observe that the coefficient $f_\mu(\tau)$ of $e_\mu$ above depends only on $Q_D(\mu)$. Hence, for any $c \in \mathbb{G}(\mathbb{Q})$, we have

$$L_{cD}(f) = \sum_{\mu \in cD} f'_\mu(\tau) e_\mu \text{ with } f'_\mu = f_{c^{-1}\mu}. \quad (3.6)$$

4. Theta lifts

In this section, we will construct the theta lift of $f \in S(\Gamma_0(N), r)$, $N$ square-free, to an automorphic form on 5-dimensional hyperbolic space as in [3]. Also see [21].

4.1. Real hyperbolic space as a Grassmanian manifold

We will follow the construction of the theta lift in Section 3 of [21]. We recall from Section 2 that if $g \in \mathbb{G}(\mathbb{R})$, then we can write

$$g = n(x) a_y k, \text{ where } n(x) = \begin{bmatrix} 1 & x A_0 & \frac{1}{2} x A_0 x \\ 0 & 1 & x \\ 0 & 1 & 1 \end{bmatrix}, \quad x \in \mathbb{R}^4, \quad a_y = \begin{bmatrix} y \\ 1 \end{bmatrix},$$

for $y \in \mathbb{R}^+$, $k \in K_\infty$ where $K_\infty$ is the maximal compact subgroup of $\mathbb{G}(\mathbb{R})$ (cf. Section 2.1) and that

\[ \mathbb{R}^4 \times \mathbb{R}^+ \ni (x,y) \mapsto n(x) a_y \in \mathbb{G}(\mathbb{R})/K_\infty \]
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gives the 5-dimensional hyperbolic space $\mathbb{H}_5$. Let $V_5 := (\mathbb{R}^6, Q_A)$ and let $D$ be the Grassmanian of positive oriented lines in the quadratic space $V_5$. Note that $V_5 = L \otimes \mathbb{R}$, where $L$ was the lattice defined in Section 2.2. We will identify $\mathbb{H}_5$ with a connected component of $D$ as follows.

$$G := \{ (x, y) \mapsto \nu(x, y) := \frac{1}{\sqrt{2}}(y + y^{-1}Q_{A_0}(x), -y^{-1}x, y^{-1}) \in V_5 \}$$

satisfying $B_A(\nu(x, y), \nu(x, y)) = 1$. It generates the positive, oriented line $\mathbb{R} \cdot \nu(x, y)$, which is an element in $D$. In fact, we see that $D^+ := \{ \mathbb{R} \cdot \nu(x, y) \mid (x, y) \in \mathbb{H}_5 \}$ is one of the two connected components of $D$. We now note that the quadratic space $V_5$ is isometric to $\mathbb{R}^{1,5}$, where $\mathbb{R}^{1,5}$ denotes the real vector space $\mathbb{R}^6$ with the quadratic form

$$Q_{1,5}(x_1, x_2, \cdots, x_6) := \frac{1}{2} \left( x_1^2 - \sum_{j=2}^{6} x_j^2 \right).$$

We slightly abuse the notation by using $\nu$ to represent the line generated by $\nu(x, y)$. Every line $\nu \in D^+$ induces an isometry

$$\iota_\nu : V_5 \rightarrow \mathbb{R} \cdot \nu \oplus (\nu^\perp, Q_{A_0}|_{\nu^\perp}) \cong \mathbb{R}^{1,5}$$

$$\lambda \mapsto (\iota_\nu^+(\lambda), \iota_\nu^-(\lambda)),$$

where

$$\iota_\nu^+(\lambda) := B_A(\lambda, \iota_\nu\lambda), \quad \iota_\nu^-(\lambda) := \lambda - \iota_\nu^+(\lambda) \in \nu^\perp$$

are the components of $\lambda$. Let us remark here that, if we fix $(x, y) \in \mathbb{H}_5$, then we get a corresponding isometry of $V_5$ into $\mathbb{R}^{1,5}$ where the one dimensional positive definite subspace is the line generated by $\nu(x, y)$.

Note that $\iota_\nu^+(\gamma \cdot \lambda) = \gamma \cdot \iota_\nu^+(\lambda)$ for any $\gamma \in \mathbb{G}(\mathbb{R})$ and $\lambda \in V_5$. Next, we collect some facts about the distinguished elements

$$z := \iota(1, 0_4, 0), \quad z' := \iota(0, 0_4, 1),$$

$$\mu_0 := -z' + \frac{z_{\nu^+}}{2B_A(z_{\nu^+}, z_{\nu^+})} + \frac{z_{\nu^-}}{2B_A(z_{\nu^-}, z_{\nu^-})}$$

and their properties. These will be useful later in the Fourier expansion of the theta lift.

**Lemma 4.1.**

i) We have $B_A(z, z) = B_A(z', z') = 0$ and $B_A(z, z') = 1$.

ii) Let $z = (z_{\nu^+}, z_{\nu^-})$ where $z_{\nu^+} = \iota_\nu^+(z)$ and $z_{\nu^-} = \iota_\nu^-(z)$. Then

$$z_{\nu^+} = B_A(z, \nu)\nu = \frac{1}{\sqrt{2y}}\nu, \quad z_{\nu^-} = z - z_{\nu^+},$$

$$B_A(z_{\nu^+}, z_{\nu^+}) = \frac{1}{2y}, \quad B_A(z_{\nu^-}, z_{\nu^-}) = -\frac{1}{2y^2}.$$
iii) We have \( \mu_0 = -z' + y^2(2z_{\nu'} - z) \).

iv) Let \( \lambda \in \mathcal{O}' \) and consider it as an element of \( L' \). Then
\[
B_A(\lambda, \mu_0) = \lambda_A \mu_0.
\]

**Proof.** Part i) follows from the definition of \( B_A \). For part ii) use \( B_A(\nu, \nu) = 1 \) and part i). Part iii) follows from part ii). For part iv), use \( B_A(\lambda, z) = B_A(\lambda, z') = 0. \square \)

4.2. The theta kernel

Let \( w^+ \) (respectively \( w^- \)) be the orthogonal complement of the line generated by \( z_{\nu^+} \) (respectively \( z_{\nu^-} \)) in \( i_V^-(V_5) \) (respectively \( i_V^+(V_5) \)). For \( \lambda \in V_5 \), let \( \lambda_{w^+} \) and \( \lambda_{w^-} \) be the projection of \( \lambda \) to \( w^+ \) and \( w^- \) respectively. We define the linear map \( w : V_5 \to \mathbb{R}^{1,5} \) by \( w(\lambda) = (\lambda_{w^+}, \lambda_{w^-}) \), so that \( w \) is an isomorphism from \( w^+ \) and \( w^- \) to their images and \( w \) vanishes on \( z_{\nu^+} \) and \( z_{\nu^-} \). For our special case, \( w^+ \) is trivial, the image of \( w \) is 4-dimensional, and the first coordinate of \( w(\lambda) \) is 0.

If \( p \) is a polynomial on \( \mathbb{R}^{1,5} \), we say that \( p \) has homogeneous degree \((m^+, m^-)\) if it is homogeneous of degree \( m^+ \) in the first variable and homogeneous of degree \( m^- \) in the last 5 variables. For \( h^+, h^- \) integers satisfying \( 0 \leq h^+ \leq m^+ \) and \( 0 \leq h^- \leq m^- \) define polynomials \( p_{w, h^+, h^-} \) on \( w(V_5) \) of homogeneous degree \((m^+ - h^+, m^- - h^-)\) by
\[
p(t_{\nu}(\lambda)) = \sum_{h^+, h^-} B_A(\lambda, z_{\nu^+})^{h^+} B_A(\lambda, z_{\nu^-})^{h^-} p_{w, h^+, h^-}(w(\lambda)). \tag{4.1}
\]

Let \( p : \mathbb{R}^6 \to \mathbb{R} \) be the polynomial given by \( p(x_1, \cdots, x_6) = -2^{-2}x_1^2 \). We get a polynomial on \( V_5 \) defined by \( p \circ t_{\nu} \) given by the formula
\[
p(t_{\nu}(\lambda)) = -2^{-2} B_A(\lambda, \nu)^2 = -2^{-1} y^2 B_A(\lambda, z_{\nu^+})^2.
\]

By (4.1), we have
\[
p_{w, h^+, h^-} = \begin{cases} -2^{-1} y^2 & \text{if } (h^+, h^-) = (2, 0); \\ 0 & \text{otherwise}. \end{cases} \tag{4.2}
\]

Note that the polynomial \( p_{w, h^+, h^-} \) is a constant in this case.

Let \( \Delta \) be the Laplacian on \( \mathbb{R}^{1,5} \). For \( \tau \in \mathfrak{h} \), \( (x, y) \in \mathbb{H}_5 \) and \( \mu \in D = L'/L \), define
\[
\theta^L_\mu(\tau, (x, y), p) := \sum_{\lambda \in L + \mu} \left( \exp\left( \frac{-\Delta}{8\pi \nu} \right)(p)(t_{\nu}(\lambda)) \right) \times \exp(2\pi \sqrt{-1} \left( Q_A(t_{\nu}^+(\lambda))(\tau + Q_A(t_{\nu}^-(\lambda))(\bar{\tau}) \right)),
\]
\[
\Theta_L(\tau, (x, y), p) := \sum_{\mu \in D} \epsilon_\mu \theta^L_\mu(\tau, (x, y), p).
\]

**Proposition 4.2.** For \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}) \), we have
\[
\Theta_L\left( \frac{a\tau + b}{c\tau + d}, (x, y), p \right) = |c\tau + d|^5 \rho_D\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \Theta_L(\tau, (x, y), p).
\]
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Proof. The transformation formula in the \( \tau \) variable follows from Theorem 4.1 of [3] by noticing that \( b^+ = 1, b^- = 5, m^+ = 2, \) and \( m^- = 0 \).

4.3. The theta lift

Let \( f \in S(\Gamma_0(N), \tau), N \) square-free, be an Atkin-Lehner eigenform with eigenvalues \( \varepsilon_c \) for all \( c \mid N \). Let \( \mathcal{L}_D(f) \) be the \( \mathbb{C}[D] \) valued modular form as defined in (3.2). Let \( \Theta_L(\tau, \nu(x, y), p) \) be the theta function defined in the previous section. Define

\[
\Phi_L(\nu(x, y), p, f) := \int_{\text{SL}_2(\mathbb{Z}) \backslash } (\mathcal{L}_D(f))(\tau)\Theta_L(\tau, \nu(x, y), p)\nu^\frac{z}{2d} du dv.
\]

Here, complex conjugation on \( \mathbb{C}[D] \) is given by \( \overline{\epsilon}_\mu := \epsilon_{-\mu} \). In the product of \( \Theta_L \) and \( \mathcal{L}_D(f), \) we are taking the inner product in \( \mathbb{C}[D] \) to get a \( \mathbb{C}[D] \)-valued function.

By Propositions 3.2 and 4.2, we see that the integrand is indeed invariant under \( \text{SL}_2(\mathbb{Z}) \).

Lemma 4.3. Let \( \gamma \in \Gamma = \{ \gamma \in G(\mathbb{Q}) : \gamma \mathbf{L} = \mathbf{L} \} \). Then

\[
\Phi_L(\gamma \nu(x, y), p, f) = \Phi_L(\nu(x, y), p, f).
\]

Proof. Using the definition of \( \mathbf{L}' \), it is easy to see that \( \Gamma \) acts on \( \mathbf{L}' \) and hence on \( \mathcal{V}_5 \) as well. Every element of \( \Gamma \) fixes \( 0 \) and, for \( 0 \neq \mu \in \mathcal{D} \) with \( \gamma \in \Gamma \), we have \( Q_\mathcal{D}(\mu) = Q_\mathcal{D}(\gamma^{-1} \mu) \). Let us observe that \( p(\nu(\lambda)) = -2^{-2}B_A(\gamma^{-2} \lambda, \nu) \) and \( B_A(\lambda, \gamma \nu) = B_A(\gamma^{-1} \lambda, \nu) \). Now, a change of variable gives us \( \theta_{\mathcal{D}}^L(\tau, \gamma \nu, p) = \theta_{\mathcal{D}}^L(\tau, \nu, p) \).

Hence

\[
\Theta_L(\tau, \gamma \nu(x, y), p) = \sum_{\mu \in \mathcal{D}} \epsilon_{\mu} \theta_{\gamma^{-1} \mu}^L(\tau, \nu(x, y), p).
\]

From (3.5), we know that the \( \epsilon_{\mu} \)-component of \( \mathcal{L}_D(f) \) depends only on \( Q_\mathcal{D}(\mu) \). As seen above, \( Q_\mathcal{D}(\mu) = Q_\mathcal{D}(\gamma^{-1} \mu) \) for all \( \mu \neq 0 \) in \( \mathcal{D} \). Upon integration, we get the result.

Let \( z^\perp \) be the orthogonal complement of the line \( z \) generates in \( V_5 \). By part i) of Lemma 4.1, we see that \( z \in z^\perp \). Let \( K := (\mathbb{L} \cap z^\perp)/\mathbb{Z} \). By the definition of \( B_A \), we can see that the lattice \( K \) is isomorphic to \( \mathcal{O} \). Given \( \mathcal{L}_D(f) \) as above, we can define a \( \mathbb{C}[K'/K] \)-valued function \( (\mathcal{L}_D(f))_K \) which is a modular form for \( \text{SL}_2(\mathbb{Z}) \) with respect to \( \rho_{K'/K} \). In our case, since \( K = \mathcal{O} \), we have \( K'/K = \mathcal{D} \) and hence \( (\mathcal{L}_D(f))_K = \mathcal{L}_D(f) \). We want to show that the Fourier expansion of \( \Phi_L(\nu(x, y), f) \) is of the form (2.1). By Theorem 7.1 of [3], the Fourier expansion of \( \Phi_L(\nu(x, y), f) \) is given by
\[ \Phi_L(v(x,y), f) = \frac{1}{2\sqrt{Q_A(z_{v^+})}} \sum_{h \geq 0} h! \left( \frac{Q_A(z_{v^+})}{2\pi} \right)^h \Phi_K(w, p_{w,h}, (L_D(f))_K) \]

\[ + \frac{1}{\sqrt{Q_A(z_{v^+})}} \sum_{h \geq 0} \sum_{h^+, h^-} \frac{h! (-2Q_A(z_{v^+}))^h}{(2i)^{h^++h^-}} \left( \frac{1}{h} \right)^{h^+} \left( \frac{1}{h} \right)^{h^-} \times \sum_j \sum_{\lambda \in K'} (-\Delta)^j \left( p_{w,h^+, h^-} \right)(w(\lambda)) \]

\[ \times \sum_{n>0} e(B_A(n\lambda, \mu_0)) h^{h^+} \sum_{\mu \in D} e(nB_A(\mu, z)) \]

\[ \times \int_{v>0} c_{\mu, Q_A(\lambda)}(v) \exp \left( -\frac{\pi n^2}{4 e Q_A(z_{v^+})} - 2\pi v (Q_A(\lambda_{w^+}) - Q_A(\lambda_{w^-})) \right) \times v^{h^--h^- - j - \frac{1}{2}} dv. \]

Here

\[ v^2 (L_D(f))(u + iv) = \sum_{\mu \in D} c_{\mu} \sum_{m \in \mathbb{Q}} c_{\mu, m} \epsilon(m u). \]

Also, \( \mu_0 \) is as defined in part iii) of Lemma 4.1. In addition, \( \lambda_{w^+} \) and \( \lambda_{w^-} \) are defined in the beginning of Section 4.2. Let us now apply this formula to our particular situation.

i) We have \( K = O \), hence \( K' = O' \).

ii) By (4.2), we have

\[ p_{w,h^+, h^-} = \begin{cases} -2^{-1} y^2 & \text{if } (h^+, h^-) = (2,0); \\ 0 & \text{otherwise.} \end{cases} \]

Hence, the first sum is zero. In the second sum over \( h, h^+, \) and \( h^- \), we only have the case \( h = 0, h^+ = 2, \) and \( h^- = 0 \). Since \( p_{w,h^+, h^-} \) is a constant function, the sum over \( j \) vanishes for all \( j > 0 \). Hence, we only get \( j = 0 \).

iii) By Lemma 4.1, we have \( Q_A(z_{v^+}) = (4y^2)^{-1} \) and \( B_A(\lambda, \mu_0) = \ell A \mu_0 \).

iv) Since \( D = O'/O \), we can see that, for \( \mu \in D \), we have \( B_A(\mu, z) \in \mathbb{Z} \). Hence, \( e(nB_A(\mu, z')) = 1 \) for all \( \mu \in D \). Furthermore, for each \( \lambda \), there is exactly one \( \mu \in D \) such that \( \mu|_{L_{v^+}} = \lambda \).

v) For \( \lambda \in O' \), we have \( Q_A(\lambda) = -Q_{A_0}(\lambda) \).

vi) For \( \lambda \in O' \), we have \( \lambda_{w^+} = 0 \) since \( w^+ \) is the trivial space in our case. We have \( Q_A(\lambda_{w^-}) = -Q_{A_0}(\lambda) \).

vii) By (3.5), for \( \mu \in D, \lambda \in O' \), and \( v \in \mathbb{R}_{>0} \), we have

\[ c_{\mu, Q_A(\lambda)}(v) = c_\mu(\lambda) \frac{1}{\sqrt{\pi}} (4\pi Q_{A_0}(\lambda) v)^{\frac{1}{2}}, \]
An explicit lifting construction for $O(1,5)$

Let $\mu_\lambda \in D$ be such that $\lambda \in \mu_\lambda + O$. Then we see that $c_\mu(\lambda)$ is non-zero only if $\mu = \mu_\lambda$.

viii) We have

\[
\int_{v>0} v^{2} W_{0, \frac{\sqrt{\pi}}{2}} (4\pi |Q_{A_0}(\lambda)|v) \exp(-\frac{\pi n^2 y^2}{v} - 2\pi v Q_{A_0}(\lambda))v^{-2 - \frac{1}{2}} dv
\]

\[
= \int_{v>0} W_{0, \frac{\sqrt{\pi}}{2}} (4\pi |Q_{A_0}(\lambda)|v) \exp(-\frac{\pi n^2 y^2}{v} - 2\pi Q_{A_0}(\lambda))dv \quad \text{by } (v \mapsto \frac{1}{v})
\]

\[
= 4 \sqrt{Q_{A_0}(\lambda)} K_{\sqrt{\pi}}(4\pi \sqrt{Q_{A_0}(\lambda)n}y).
\]

We have used

\[
\int_{0}^{\infty} \exp(-pt - \frac{a}{2t})W_{0, \frac{\sqrt{\pi}}{2}} (\frac{a}{t})dt = 2 \sqrt{\frac{a}{p}} K_{\sqrt{\pi}} (2\sqrt{ap})
\]

(cf. [4, 4.22 (22)]) with $a = 4\pi Q_{A_0}(\lambda)$ and $p = \pi n^2 y^2$.

Putting this together, we see that the Fourier expansion is given by

\[
2y \left(\frac{-1}{4}\right) \sum_{\lambda \in O'} \frac{-y^2}{2} \sum_{n>0} c(n\lambda A_0 x) n^2 c_{\mu_\lambda}(\lambda) 4 \sqrt{Q_{A_0}(\lambda) ny} K_{\sqrt{\pi}}(4\pi \sqrt{Q_{A_0}(\lambda)n}y)
\]

\[
= \sum_{\lambda \in O'} \sum_{n>0} n \sqrt{Q_{A_0}(\lambda)} c_{\mu_\lambda}(\lambda) y^2 K_{\sqrt{\pi}}(4\pi \sqrt{Q_{A_0}(\lambda)n}y) c(n\lambda A_0 x)
\]

\[
= \sum_{\beta \in O'} \sqrt{Q_{A_0}(\beta)} \left( \sum_{\frac{d}{n} \beta \in O'} c_{\mu_\frac{\beta}{d}} \left(\frac{\beta}{d}\right) \right) y^2 K_{\sqrt{\pi}}(4\pi \sqrt{Q_{A_0}(\beta)y}) e(\beta A_0 x).
\]

From Lemma 4.3 and the above Fourier expansion (compare to (2.1)), we get

**Theorem 4.4.** $\Phi_L(\nu(x, y), f)$ belongs to $\mathcal{M}(\Gamma, \sqrt{-1\tau})$.

We will use the remaining section to obtain a formula for the Fourier coefficients of $\Phi_L(\nu(x, y), f)$ in terms of the Fourier coefficients of $f$. For $\beta \in O'$, set

\[
A(\beta) = \sqrt{Q_{A_0}(\beta)} \sum_{\frac{d}{n} \beta \in O'} c_{\mu_\frac{\beta}{d}} \left(\frac{\beta}{d}\right). \quad (4.4)
\]

We will now obtain a formula for $A(\beta)$ in terms of the Fourier coefficients $c(n)$ of $f$. Let us define the primitive elements of $O'$ by

\[
O'_{\text{prim}} := \{ \beta \in O' : \frac{1}{n} \beta \notin O' \text{ for all positive integers } n > 1 \}.
\]
Proposition 4.5. Write $\beta \in \mathcal{O}'$ as

$$
\beta = \prod_{p|N} p^{u_p} n \beta_0, \quad u_p \geq 0, n > 0, \gcd(n, N) = 1 \text{ and } \beta_0 \in \mathcal{O}'_{\text{prim}}.
$$

Let $q_{\beta_0} = q_{\mu_{\beta_0}}$. For $p | N$, set

$$
\delta_p = \begin{cases} 
0 & \text{if } p | q_{\beta_0}; \\
1 & \text{if } p \nmid q_{\beta_0}.
\end{cases}
$$

Then

$$
A(\beta) = \sqrt{Q_{A_0}(\beta)} \sum_{p | N} \sum_{t_p = 0}^{2u_p + \delta_p} \sum_{d(n|p^t)} c(\prod_{p | N} p^{u_p}) \prod_{p | N} (\varepsilon_p) p^{t_p - 1}.
$$

(4.5)

Proof. From (4.3) and (4.4), it is clear that we can take $n = 1$ above. Let $S_0$ be the set of primes dividing $q_{\beta_0}$ and $S'$ be the subset of $S_0$ with $u_p = 0$. For any set of primes $S$, denote by $N_S$ the product of all primes in $S$. From (4.3) and (4.4), we have

$$
A(\beta) = \sqrt{Q_{A_0}(\beta)} \sum_{p | N} c(p^{\beta_{\mu_{p/N}}^{\mu_{p/N}}}) \prod_{p | N} p^{u_p}.
$$

Then

$$
A(\beta) = \sqrt{Q_{A_0}(\beta)} \sum_{S' \subset S \subset S_0} \sum_{a_p = 0}^{u_p} \sum_{a_p = 0}^{u_p - 1} c(p^{\beta_{p^{\mu_{p/N}}^{\mu_{p/N}}}} \prod_{p | N} p^{u_p} \prod_{p | N} p^{a_p})
$$

$$
\times \left( \prod_{p | N} p^{a_p} \prod_{p | N} p^{u_p} \right).
$$

We are essentially splitting up the sum according to which $a_p = u_p$ for $p \in S_0$ so that, for all the $\beta'$ appearing in the sum above, we have $q_{\beta'} = N_S$. Hence applying (4.3), we have

$$
A(\beta) = \sqrt{Q_{A_0}(\beta)} \sum_{S' \subset S \subset S_0} \sum_{a_p = 0}^{u_p} \sum_{a_p = 0}^{u_p - 1} \sum_{a_p = 0}^{u_p - 1} (\varepsilon_p) c(p^{\beta_{p^{\mu_{p/N}}^{\mu_{p/N}}}} \prod_{p | N} p^{a_p})
$$

$$
\times \prod_{p \mid \beta'} (-\varepsilon_p) c(p^{\beta_{p^{\mu_{p/N}}^{\mu_{p/N}}}} \prod_{p \mid \beta'} p^{a_p} (N_S) \prod_{p \mid \beta'} p^{u_p} / c).
$$
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Here, if $p|N_S$ then we have $p|(N/c)$ for all $c|(N/N_S)$. Hence, we have

$$A(\beta) = \sqrt{Q_{A_0}(\beta)} \sum_{S' \subset S \subset S_0} \sum_{\substack{V \in 2^S \setminus \{\emptyset\} \\ a_p = 0}} \sum_{\substack{U \in 2^{S_0} \setminus \{\emptyset\} \\ a_p = 0}} \prod_{p} (-\varepsilon_p)$$

$$\times \prod_{p|\frac{N}{N_S}} (\varepsilon_p) \left( \prod_{p|N_S} \frac{Q_{A_0}(\beta)}{p^{2u_p-1}} \prod_{p|c} \frac{Q_{A_0}(\beta)}{p^{2u_p-1}} \right)$$

$$= \sqrt{Q_{A_0}(\beta)} \sum_{S' \subset S \subset S_0} \sum_{\substack{V \in 2^S \setminus \{\emptyset\} \\ a_p = 0}} \sum_{\substack{U \in 2^{S_0} \setminus \{\emptyset\} \\ a_p = 0}} \prod_{p} (-\varepsilon_p)$$

$$\times \prod_{p|\frac{N}{N_S}} (\varepsilon_p)^{2u_p-1} \prod_{p|c} (-\varepsilon_p)^{2u_p} e\left( \frac{-Q_{A_0}(\beta)}{\prod_{p|N_S} p^{2u_p-1} \prod_{p|c} p^{2u_p}} \right).$$

Further, we can divide the set $\{c|(N/N_S)\}$ into a set of pairs $(c,c')$ such that $cc' = N/N_S$. Note that for any prime $p|(N/N_S)$ and any pair $(c,c')$ as above, $p$ divides exactly one of $N/c$ or $N/c'$. Also note, in the denominator in the last term above, the possible exponents of $p$ are as follows: if $p|q_{3_b}$, then the exponents vary from 0 to $2u_p-1$, and if $p \nmid q_{3_b}$, then the exponents vary from 0 to $2u_p$. In addition, a term for $-\varepsilon_p$ appears in the product if and only if the exponent of $p$ in the denominator is odd. Changing variable from $a_p$ to $t_p$, and noting the definition of $\delta_p$ in the statement of the proposition, we finally get the formula in (4.5).

\[\square\]

5. The cuspidality of the theta lifts

We show the cuspidality of our theta lifts

$$\Phi_{L}(\nu(x,y),p,f) := \int_{SL_2(\mathbb{Z}) \backslash H} \mathcal{L}_D(f) \check{\Phi}_L(\tau,\nu(x,y),p) \frac{dv}{v^2}. $$

The first step is to understand the action of $c \in \mathcal{G}(\mathbb{Q})$ on $\Phi_{L}(\nu(x,y),p,f)$.

**Lemma 5.1.** For any $c \in \mathcal{G}(\mathbb{Q})$, we have

$$\Phi_{L}(c\nu(x,y),p,f) = \Phi_{c^{-1}}(\nu(x,y),p,f).$$

**Proof.** Recall that $\iota_{g,\nu}(g \cdot \lambda) = g \cdot \iota_{\nu}(\lambda)$ for $(g,\lambda,\nu) \in \mathcal{G}(\mathbb{R}) \times \mathbb{R}^6 \times D^+$, which yields

$$\iota_{g,\nu}^+(g \cdot \lambda) = g \cdot \iota_{\nu}^+(\lambda), \ \iota_{g,\nu}^-(g \cdot \lambda) = g \cdot \iota_{\nu}^-(\lambda).$$

In addition, we note that

$$p(\iota_{g,\nu}(\lambda)) = p(\nu(g^{-1} \cdot \lambda))$$
for \((g, \lambda) \in G(\mathbb{R}) \times \mathbb{R}^6\). For \((g, \mu) \in G(\mathbb{R}) \times L'\) we thereby have
\[
\sum_{\lambda \in \mu + L} \exp(( -\frac{\Delta}{8\pi y} (p)) (\iota_{g, \nu}(\lambda))\mathfrak{e}(Q(\iota_{g, \nu}(\lambda))\tau + Q(\iota_{g, \nu}(\lambda))\bar{\tau})) = \sum_{\lambda \in \mu + L} \exp(( -\frac{\Delta}{8\pi y} (p)) (\iota_{\nu}(g^{-1} \cdot \lambda))\mathfrak{e}(Q(g \cdot \iota_{\nu}^\dagger (g^{-1} \cdot \lambda))\tau + Q(g \cdot \iota_{\nu}^\dagger (g^{-1} \cdot \lambda))\bar{\tau})) = \sum_{\lambda \in \mu + L} \exp(( -\frac{\Delta}{8\pi y} (p)) (\iota_{\nu}(\lambda))\mathfrak{e}(Q(\iota_{\nu}^\dagger (\lambda))\tau + Q(\iota_{\nu}^\dagger (\lambda))\bar{\tau})) = \theta_{g^{-1}}^{L}(\tau, \nu(x, y), p).
\]
Writing \(L_D(f) = \sum_{\mu \in D} f_\mu e_\mu\) and using (3.6), we get
\[
L_D(f) \overline{\Theta}_{L}(\tau, \nu, p) = \sum_{\mu \in D} f_\mu \bar{\theta}_{g^{-1}}^{L}(\tau, \nu(x, y), p)
= \sum_{\mu \in D} f_\mu \bar{\theta}_{g^{-1}}^{L}(\tau, \nu(x, y), p)
= \sum_{\mu \in D} f_\mu \bar{\theta}_{g^{-1}}^{L}(\tau, \nu(x, y), p)
= L_{g^{-1}}(f) \overline{\Theta}_{g^{-1}}^{L}(\tau, \nu, p).
\]
Here we put \(g^{-1}D := c^{-1}L'/c^{-1}L\), which is isomorphic to \(D\). Upon integration, we get the result of the lemma. \(\square\)

For any cusp \(c \in \mathcal{P}(Q)\backslash G(\mathbb{Q})/\Gamma\), we see that \(Q_A(x) = Q_A(c^{-1}x)\) for all \(x \in \mathbb{Q}^6\). Hence, the lattice \(c^{-1}L\) has the same associated quadratic form as \(L\). Therefore, the discriminant form \(c^{-1}D = c^{-1}L'/c^{-1}L\) is isomorphic to \(D = L'/L\) as quadratic modules and hence Proposition 4.5 applies to the Fourier expansion of \(\Phi_{c^{-1}L}(\nu(x, y), p, f)\). In particular, \(\Phi_{c^{-1}L}(\nu(x, y), p, f)\) has no constant term and therefore we get the following.

**Proposition 5.2.** For each representative \(c\) of the \(\Gamma\)-cusps, \(\Phi_{L}(\nu(x, y), p, f)\) has no constant term. Namely, our lifts \(\Phi_{L}(\nu(x, y), p, f)\) are cuspidal.

6. **Hecke Theory**

6.1. **Adelization of automorphic forms**

To study the action of the Hecke operators on our cusp forms constructed by the lift, we need the adelic as well as non-adelic treatment of automorphic forms.

For \(h \in H(\mathbb{A})\), we have the decomposition \(h = au^{-1}\) with \((a, u) \in \text{GL}_4(\mathbb{Q}) \times (\Pi_{p<\infty} \text{SL}_4(\mathbb{Z}_p) \times \text{SL}_4(\mathbb{R}))\). Let \(O_h := (\Pi_{p<\infty} h_p \mathbb{Z}_p^4 \times \mathbb{R}^4) \cap \mathbb{Q}^4\) for \(h = (h_v)_{v \leq \infty} \in H(\mathbb{A})\). Then, we have \(O_h = aO\) (c.f. [21, Section 3.3]). The dual lattice \(O_h'\) is then equal to \(a^{-1}O'\). Here note that we regard \(O\) and \(O'\) as \(\mathbb{Z}^4\) equipped with the quadratic forms induced by the reduced norm.
To obtain an adelic Fourier expansion, let \( f \in S(\Gamma_0(N), r) \) be a Maass cusp form with the Fourier expansion \( f(z) = \sum_{n \neq 0} c(n) W_n \overline{e(4\pi n y) e(x)} \). Let \( \Lambda \) be the standard additive character of \( \mathbb{A}/\mathbb{Q} \). We introduce the following Fourier series
\[
F_f(n(x)a_ykg) := \sum_{\lambda \in \mathbb{Q}^\times \setminus \{0\}} F_{f, \lambda}(n(x)a_ykg) \quad \forall (x, y, k, g) \in \mathbb{A}^4 \times \mathbb{R}_+^\times \times K_\infty \times \mathcal{G}(\mathbb{A}_f)
\]
with
\[
F_{f, \lambda}(n(x)a_ykg) := A_\lambda(g) y^2 K_{\sqrt{-r}}(4\pi |\lambda| A y) \Lambda(\lambda A x),
\]
where \( A_\lambda(g) \) is defined by the following conditions:
\[
A_\lambda \left( \begin{pmatrix} 1 & h \\ s & 1 \end{pmatrix} \right) := \begin{cases} \sqrt{Q_{A_\lambda}(\lambda)} \sum_{p | N} \sum_{\tau_p = 0} \sum_{d | n} c_{\frac{-Q_{A_\lambda}(\lambda)}{p^r}} \prod_{p | N} (\epsilon_p)^{r_{p-1}} & (\lambda \in \mathcal{O}_{\mathbb{A}}) \\ 0 & (\lambda \in \mathbb{Q}^4 \setminus \mathcal{O}_{\mathbb{A}}) \end{cases}
\]
\[
A_\lambda \left( \begin{pmatrix} s \\ h \end{pmatrix} \right) := ||s||_{\mathbb{A}}^2 A_{||s||_{\mathbb{A}}^{-1} \lambda} \left( \begin{pmatrix} 1 \\ h \end{pmatrix} \right)
\]
\[
A_\lambda(n(x)gk) := \Lambda(\lambda A x) A_\lambda(g) \quad \forall (x, g, k) \in \mathbb{A}_f^4 \times \mathcal{G}(\mathbb{A}_f) \times K_f.
\]
Here
1. \( u_p, \delta_p \) and \( n \) are as defined in Proposition 4.5 for \( \beta = h^{-1} \lambda \).
2. \( (s, h) \in \mathbb{A}_f^4 \times \mathcal{H}(\mathbb{A}_f) \) and \( ||s||_{\mathbb{A}} \) denotes the idele norm of \( s \).

Note, the definitions of \( u_p, \delta_p \) and \( n \) do not depend on the decomposition \( h = au^{-1} \), which was essentially pointed out in the proof of [21, Lemma 3.2]. The following lemma is settled by the same reasoning as [21, Lemma 3.2].

For \( r \in \mathbb{C} \), let \( M(\mathcal{G}(\mathbb{A}), r) \) denote the space of smooth functions \( F \) on \( \mathcal{G}(\mathbb{A}) \) satisfying the following conditions:
1. \( \Omega \cdot F = \frac{1}{N}(r^2 - 4) F \), where \( \Omega \) is the Casimir operator defined in [21].
2. For any \( (\gamma, g, k) = \mathcal{G}(\mathbb{Q}) \times \mathcal{G}(\mathbb{A}) \times K_f \), we have \( F(\gamma g k) = F(g) \).
3. \( F \) is of moderate growth.

Note that \( F \in M(\mathcal{G}(\mathbb{A}), r) \) has the Fourier expansion
\[
F(g) = \sum_{\lambda \in \mathbb{Q}^4} F_{\lambda}(g), \quad F_{\lambda}(g) := \int_{\mathbb{A}_f^4} F(n(x)g) \Lambda(\lambda A x) dx,
\]
where \( dx \) is the invariant measure normalized so that the volume of \( \mathbb{A}_f^4 / \mathbb{Q}^4 \) is one.

The adelic function \( F \) is called a cusp form if \( F_0 \equiv 0 \) in the Fourier expansion.

**Proposition 6.1.** The adelic function \( F_f \) is a cusp form belonging to \( M(\mathcal{G}(\mathbb{A}), \sqrt{-r}) \).

**Proof.** By the argument similar to [21, Theorem 3.3] this follows from the Fourier expansion discussed in Section 4.3. \( \square \)
6.2. Sugano Theory

We will show that if \( f \) is a Hecke eigenform then \( F_f \) is an Hecke eigenform by using the non-archimedean local theory of Sugano [38, Section 7]. For a prime \( p \), let \( F = \mathbb{Q}_p \) with the ring of integers \( \mathbb{Z}_p \). Let \( n_0 \leq 4 \) and let \( S_0 \in M_{n_0}(F) \) be an anisotropic even symmetric matrix of degree \( n_0 \). For the \( m \times m \) matrix \( J_m = \left( \begin{smallmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{smallmatrix} \right) \), let \( G_m \) denote the group of \( F \)-valued points of the orthogonal group of degree \( 2m + n_0 \), defined by the matrix \( Q = \left( \begin{smallmatrix} J_m \\ S_0 \end{smallmatrix} \right) \). Denote by \( L_m := \mathbb{Z}_p^{2m+n_0} \) the maximal lattice with respect to \( Q_m \) and let \( K_m \) be the maximal compact open subgroup of \( G_m \) defined by the lattice

\[
K_m := \{ g \in G_m \mid g L_m = L_m \}.
\]

(6.2)

Let \( \mathcal{H}_m \) be Hecke algebra for \( (G_m, K_m) \) and define \( C_m^{(r)} \in \mathcal{H}_m \) to be the double cosets \( K_m c_m^{(r)} K_m \), where

\[
c_m^{(r)} := \text{diag}(p, \ldots, p, 1, \ldots, 1, p^{-1}, \ldots, p^{-1}) \in G_m
\]

which is a diagonal matrix whose first \( r \) and last \( m-r \) entries are \( p \) and \( p^{-1} \) respectively. By [38, Section 7], \( \{ C_m^{(r)} \mid 1 \leq r \leq m \} \) forms generators of the Hecke algebra \( \mathcal{H}_m \).

We embed \( G_i \) for \( i \leq m \) in \( G_m \) as a subgroup by the middle \((2i+n_0) \times (2i+n_0)\) block. We regard \( K_i \) as a subgroup of \( K_m \) similarly. The invariant measure of \( G_m \) is normalized so that the volume of \( K_i \) is one for each \( i \leq m \).

For a prime \( p \nmid N \), we have \( n_0 = 0 \) and \( m = 3 \). In this case, the lattice \( L_3 \) is self-dual. For a non-negative integer \( k \), let

\[
f_{k,j} := \frac{p^{j-1}(p^{k-j+1}-1)(p^{k-j}+1)}{p^j-1} \quad (\forall j \in \mathbb{Z} \setminus \{0\}),
\]

(6.3)
a special case of [38, 7.11] for \( n_0 = \delta = 0 \). For positive integers \( k, r \), set \( R_k^{(r)} := K_k/(K_k \cap c_k^{(r)} K_k (c_k^{(r)})^{-1}) \), and let \( |R_k^{(r)}| \) denote the cardinality of \( R_k^{(r)} \). We have

\[
|R_k^{(r)}| = \begin{cases} 
\prod_{j=1}^{r} f_{k,j} & (1 \leq r \leq k); \\
1 & (r = 0).
\end{cases}
\]

(6.4)

Following the methods in Section 4 of [21], we get the following theorem (essentially Theorem 4.11 of [21] for \( n = 1/2 \)).

**Theorem 6.2.** Suppose that \( f \) is a Hecke eigenform and let \( \lambda_p \) be the Hecke eigenvalue of \( f \) at \( p < \infty \) with \( p \nmid N \). Then the following holds.

i) \( F_f \) is a Hecke eigenform.

ii) Let \( \mu_i \) be the Hecke eigenvalue with respect to the Hecke operator \( C^{(i)}_{-1} \) for \( 1 \leq i \leq 3 \). We have

\[
\mu_1 = p^2(\lambda_p^2 - 2) + pf_{2,1} = p^2(\lambda_p^2 + p + p^{-1});
\]
6.3. The case $p \mid N$

When $p \mid N$, we have $m = 1$ and $n_0 = 4$. Hence, the Hecke algebra $\mathcal{H}_1$ is generated by $C_1^{(1)}$ which is the double coset $K_1c_1^{(1)}K_1$ as defined in Section 6.2. Let $n(x) \in G_1$ be as defined in Section 2.1 and let $(t, g) := \text{diag}(t, g, t^{-1}) \in G_1$ for $t \in \mathbb{Q}_p^*$ and $g \in G_0$.

**Lemma 6.3.**

$$C_1^{(1)} = \bigcup_{x \in \mathcal{X}_1} (p, 1_4)n(x)K_1 \cup \bigcup_{x \in \mathcal{X}_3} (1, 1_4)n(x)K_1 \cup (p^{-1}, 1_4)K_1$$

where

$$\mathcal{X}_1 = \{x \in p^{-1}\mathcal{O}/\mathcal{O}\}, \mathcal{X}_3 = \{x \in (\mathcal{O}/\mathcal{O})/\mathcal{O}\}.$$  

**Proof.** This is a direct result of [38, Lemma 7.1] for $v = 0$ and $r = 1$. Note, $c_0^{(1)} = 1_4$ and hence, $R_0^{(1)} = 1$ and $u = 1_4$. $\mathcal{X}_{0,1}^{(1)}$ and $\mathcal{X}_{0,3}^{(1)}$ simplify as above whereas $\mathcal{X}_{0,2}^{(1)}$ and $\mathcal{X}_{0,4}^{(1)}$ are empty since $r = 1$ and $v = 0$ respectively. $\square$

We can now describe the action of $C_1^{(1)}$ with the invariant measure $dx$ of $G_1$ normalized so that the volume $\int_{K_1} dx = 1$. Define

$$(C_1^{(1)} \cdot \Phi)(g) := \int_{G_1} \text{char}_{K_1c_1^{(1)}K_1}(x)\Phi(gx)dx$$

for $\Phi \in \mathcal{M}(\mathcal{G}(\mathbb{A}), r)$. The following proposition derives the action of $C_1^{(1)}$ on Fourier coefficients of $\Phi$.

**Proposition 6.4.** Let $\Phi \in \mathcal{M}(\mathcal{G}(\mathbb{A}), \sqrt{-1}r)$ be a lift. Then

$$(C_1^{(1)} \cdot \Phi)(n(x)a_y) = \sum_{\lambda \in \mathcal{O} \setminus \{0\}} A_\lambda' \|g\|^2 \sqrt{-1}r(4\pi \sqrt{Q_{A_\lambda}(\lambda)g})A(\lambda A_0(x)),$$

where

$$A_\lambda' = \begin{cases} p^2A_{p\lambda}(1) - A_\lambda(1) + p^2A_\lambda(1) + p^2A_{p^{-1}\lambda}(1) & \text{if } \lambda \in p\mathcal{O} \setminus \{0\}; \\ p^2A_{p\lambda}(1) - A_\lambda(1) + p^2A_\lambda(1) & \text{if } \lambda \in \mathcal{O} \setminus p\mathcal{O}; \\ p^2A_{p\lambda}(1) - A_\lambda(1) & \text{if } \lambda \in \mathcal{O} \setminus \mathcal{O}. \\ \end{cases}$$

**Proof.** Since $\int_{K_1} dx = 1$, Lemma 6.3 implies that the action of $C_1^{(1)}$ on $\Phi$ can be expressed as

$$(C_1^{(1)} \cdot \Phi)(g) = \sum_{x \in \mathcal{X}_1} \Phi(g(p, 1_4)n(x)) + \sum_{x \in \mathcal{X}_3} \Phi(gn(x)) + \Phi(g(p^{-1}, 1_4)).$$
Here, we are using the fact that \( \Phi \in \mathcal{M}(\mathcal{G}(\mathbb{A}, \sqrt{-1}r)) \) is right invariant under \( K_1 \).

Let \( g = n(x_0)a_y \) with \( x_0 \in \mathbb{A}^d \) and \( y \in \mathbb{R}_+ \). Let \( a_y^\# := \text{diag}(p, 1_4, p^{-1}) \) embedded diagonally in \( \mathcal{G}(\mathbb{Q}) \). We will abuse the notation to denote \((1_{\infty}, \ldots, (p, 1_4), \ldots)\) and \((1_{\infty}, \ldots, n(x), \ldots)\) by \((p, 1_4)\) and \(n(x)\) respectively, where the nontrivial terms are at the \( p \)-th place. Hence,

\[
(C_1^{(1)} \cdot \Phi)(n(x_0)a_y) = \sum_{x \in \mathcal{X}_1} \Phi(n(x_0)a_y(p, 1_4)n(x)) + \sum_{x \in \mathcal{X}_3} \Phi(n(x_0)a_y n(x)) + \Phi(n(x_0)a_y(p^{-1}, 1_4)).
\]

Note,

\[
\Phi(n(x_0)a_y(p, 1_4)n(x)) = \Phi(a_y^\#, n(x_0)a_y(p, 1_4)n(x)) \\
= \Phi(n(p^{-1}x_0)a_{p^{-1}y}(1_{\infty}, (p^{-1}, 1_4), \ldots, (1, 1_4), (p^{-1}, 1_4), \ldots)n(x)) \\
= \Phi(n(p^{-1}x_0)a_{p^{-1}y}n(x)(1_{\infty}, (p^{-1}, 1_4), \ldots, (1, 1_4), (p^{-1}, 1_4), \ldots)) \\
= \Phi(n(p^{-1}x_0)n(x)a_{p^{-1}y}).
\]

We obtain the last equality as \(n(x)\) and \(a_{p^{-1}y}\) commute, and \((1_{\infty}, (p^{-1}, 1_4), \ldots, (1, 1_4), (p^{-1}, 1_4), \ldots)\) belongs to the maximal compact \( K_f K_\infty \).

By similar computation for other terms, we obtain

\[
(C_1^{(1)} \cdot \Phi)(n(x_0)a_y) = \sum_{x \in \mathcal{X}_1} \Phi(n(p^{-1}x_0)n(x)a_{p^{-1}y}) + \sum_{x \in \mathcal{X}_3} \Phi(n(x_0)n(x)a_y) + \Phi(n(px_0)a_{py}) \\
= \sum_{Q \in \mathcal{Q}\backslash\{0\}} A_\lambda(1)(p^{-1}y)^2 K_{1r}(4\pi \sqrt{Q A_0(\lambda)p^{-1}}y) \sum_{x \in \mathcal{X}_1} \Lambda(\lambda A_0(p^{-1}x_0)p, x) \\
+ \sum_{Q \in \mathcal{Q}\backslash\{0\}} A_\lambda(1)(y)^2 K_{1r}(4\pi \sqrt{Q A_0(\lambda)y}) \sum_{x \in \mathcal{X}_3} \Lambda(\lambda A_0(0)x, p, x) \\
+ \sum_{Q \in \mathcal{Q}\backslash\{0\}} A_\lambda(1)(py)^2 K_{1r}(4\pi \sqrt{Q A_0(\lambda)py}) \Lambda(\lambda A_0(px)). \tag{6.5}
\]

Here, \((p^{-1}x_0)_p, x\) is \(p^{-1}x_{0_v}\) at all places \(v \neq p\) and is \(p^{-1}x_{0_p} + x\) at the place \(p\). Similarly, \((x_0)_p, x\) is \(x_{0_v}\) at all places \(v \neq p\) and \(x_{0_p} + x\) at the place \(p\). Note,

\[
\sum_{x \in \mathcal{X}_1} \Lambda(\lambda A_0(p^{-1}x_0)p, x) = \Lambda(\lambda A_0(p^{-1}x_0)) \sum_{x \in \mathcal{X}_1} \Lambda(\lambda A_0x) \tag{6.6}
\]

with the summation over \(x \in \mathcal{X}_1\) happening only at the \(p\)-th place. As \(\Lambda\) is an additive character being summed over a group \(\mathcal{X}_1 = \{ x \in p^{-1}\mathcal{O}/\mathcal{O} \}\), we get

\[
\sum_{x \in \mathcal{X}_1} \Lambda(\lambda A_0x) = \begin{cases} p^d & p^{-1}\lambda \in \mathcal{O}'; \\ 0 & \text{otherwise.} \end{cases} \tag{6.7}
\]
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Similarly,
\[
\sum_{x \in X_3} \Lambda(t \lambda A_0(x_0, x)) = \Lambda(t \lambda A_0) \sum_{x \in X_3} \Lambda(t \lambda A_0) x
\]
being summed over $X_3 = \{x \in (O' - O)/O\}$. Note,
\[
\sum_{x \in X_3} \Lambda(t \lambda A_0) = \sum_{x \in O'/O} \Lambda(t \lambda A_0) - 1.
\]
Hence, using that $\Lambda$ is an additive character being summed over a group $O' = Og$, we get
\[
\sum_{x \in X_3} \Lambda(t \lambda A_0 x) = \sum_{x \in O'/O} \Lambda(t \lambda A_0) - 1.
\]
Therefore, substituting (6.6) in (6.5) we get the formula for $A'_\lambda(1)$ as defined in the statement of the proposition.

To write the action of the Hecke operator in terms of Fourier coefficients given in Proposition 4.5, we write $A_\lambda(1) = A(\beta)$ where $\beta = \prod p^{u_p} \alpha^n \beta_0$ as in the proposition.

Note, for $\lambda \in O'$ and $\beta \in O'$ the conditions for $A'_\lambda(1)$ on $\lambda$ from Proposition 6.4 above translate to conditions on $\beta$ as follows:
\[
\begin{align*}
\lambda \in pO' \setminus \{0\} & \iff u_p \geq 1; \\
\lambda \in O \setminus pO' & \iff u_p = 0, \delta_p = 1; \\
\lambda \in O' \setminus O & \iff u_p = 0, \delta_p = 0.
\end{align*}
\]
Then, as
\[
A_{p \lambda}(1) = A(p \beta); \quad A_{p^{-1} \lambda}(1) = A(p^{-1} \beta)
\]
we can rewrite the $A'_\lambda(1)$ in terms of $\beta$ as
\[
A'_\lambda(1) = \begin{cases} p^2 A(p \beta) + (p^2 - 1) A(\beta) + p^2 A(p^{-1} \beta) & \text{if } u_p \geq 1; \\
p^2 A(p \beta) + (p^2 - 1) A(\beta) & \text{if } u_p = 0, \delta_p = 1; \\
p^2 A(p \beta) - A(\beta) & \text{if } u_p = 0, \delta_p = 0.
\end{cases}
\]
Let $f \in S(\Gamma_0(N), r)$ be a newform with Hecke eigenvalue $\lambda_p$ for the operator defined by the action of the double coset $\Gamma_0(N)[1 \over p] \Gamma_0(N)$ at prime $p$. Assuming it is an Atkin Lehner eigenform with eigenvalue $\epsilon_p$, it can be shown that
\[
\lambda_p = -\epsilon_p.
\]
Using the single coset decomposition
\[
\Gamma_0(N) \left[ \begin{array}{c} 1 \\ p \end{array} \right] \Gamma_0(N) = \bigsqcup_{b=0}^{p-1} \Gamma_0(N) \left[ \begin{array}{c} 1 \\ b \end{array} \right] \]
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([19, Lemma 9.14]) we have

$$\sum_{b=0}^{p-1} f\left(\frac{z+b}{p}\right) = \lambda_p f(z).$$

In terms of Fourier coefficients, using (6.11), we get

$$c(pm) = \frac{\lambda_p}{p} c(m) = \frac{-\epsilon_p}{p} c(m) \quad \forall m \in \mathbb{Z}.$$ 

Therefore,

$$c(m) = \frac{p}{-\epsilon_p} c(pm) \quad \forall m \in \mathbb{Z},$$

and

$$c\left( \frac{-Q_{A_0}(\beta)}{p^2 p^{-1}} \prod_{\ell \mid N, \ell \neq p} (\ell^t - 1)^{-1} (-\epsilon_\ell)^{t_\ell - 1} \right) \prod_{\ell \mid N} (\ell^t - 1)^{-1} (-\epsilon_\ell)^{t_\ell - 1} (-\epsilon_p)^{t_p - 1}$$

$$= \left( \frac{p}{-\epsilon_p} \right)^{t_p} c\left( \frac{-Q_{A_0}(\beta)}{p^{-1}} \prod_{\ell \mid N, \ell \neq p} (\ell^t - 1)^{-1} (-\epsilon_\ell)^{t_\ell - 1} \right)$$

$$= p^{t_p} c\left( \frac{-Q_{A_0}(\beta)}{p^{-1}} \prod_{\ell \mid N, \ell \neq p} (\ell^t - 1)^{-1} (-\epsilon_\ell)^{t_\ell - 1} \right) (-\epsilon_p)^{-1}. \quad (6.12)$$

Hence, as $(-\epsilon_p)^{-1} = -\epsilon_p$, we have

$$\sum_{\ell=0}^{2u_\beta + \delta_\beta} \sum_{d \mid n} c\left( \frac{-Q_{A_0}(\beta)}{\ell^t - 1} \prod_{\ell \mid N} (\ell^t - 1)^{-1} \right) \prod_{\ell \mid N} (\ell^t - 1)^{-1}$$

$$= p^{2u_\beta + \delta_\beta + 1} - 1 \sum_{d \mid n} \frac{c\left( \frac{-Q_{A_0}(\beta)}{p^{-1}} \prod_{\ell \mid N, \ell \neq p} (\ell^t - 1)^{-1} \right) \prod_{\ell \mid N} (\ell^t - 1)^{-1} (-\epsilon_p)}{p - 1}. \quad (6.13)$$

**Theorem 6.5.** Let $f \in S(\Gamma_0(N), r)$ be a newform and eigenfunction of the Atkin Lehner involution with eigenvalue $\epsilon_p$ at each $p \mid N$. Let $F_f$ be the lift of $f$ defined in (6.1). Then $F_f$ is a Hecke eigenform with

$$C_1^{(1)} \cdot F_f = (p^3 + p^2 + p - 1) F_f.$$

**Proof.** We shall prove the Hecke eigenvalue for the most general case of $\beta$ with $u_\beta \geq 1$. The proof for the cases $u_\beta = 0$ with $\delta_\beta \in \{0, 1\}$ is similar and follows immediately.
after substituting for \( u_p \) and \( \delta_p \). Using (6.13) and \( Q_{\ell_0}(a\beta) = a^2 Q_{\ell_0}(\beta) \), we have

\[
p^2 A(p\beta) = p^3 \sqrt{Q_{\ell_0}(\beta)} \sum_{\ell \mid N, \ell \neq p} \sum_{t \geq 0} \frac{p^{2u_p + \delta_p + 3} - 1}{p - 1} \times \sum_{d \mid \ell_0} c\left(-p^2 Q_{\ell_0}(\beta) \prod_{\ell \mid N, \ell \neq p} \ell \right)^{t_\ell - 1} (-\epsilon_\ell) \]

\[
A(\beta) = \sqrt{Q_{\ell_0}(\beta)} \sum_{\ell \mid N, \ell \neq p} \sum_{t \geq 0} \frac{p^{2u_p + \delta_p + 1} - 1}{p - 1} \times \sum_{d \mid \ell_0} c\left(-Q_{\ell_0}(\beta) \prod_{\ell \mid N, \ell \neq p} \ell \right)^{t_\ell - 1} (-\epsilon_\ell) \]

\[
p^2 A(p^{-1}\beta) = p \sqrt{Q_{\ell_0}(\beta)} \sum_{\ell \mid N, \ell \neq p} \sum_{t \geq 0} \frac{p^{2u_p + \delta_p - 1} - 1}{p - 1} \times \sum_{d \mid \ell_0} c\left(-p^{-2} Q_{\ell_0}(\beta) \prod_{\ell \mid N, \ell \neq p} \ell \right)^{t_\ell - 1} (-\epsilon_\ell) \]

Note, \( A(p^{-1}\beta) = 0 \) if \( u_p = 0 \). By (6.12) and the fact that \((-\epsilon_p)^2 = 1\), we have

\[
p^2 A(p\beta) + (p^2 - 1) A(\beta) + p^2 A(p^{-1}\beta) = p \sqrt{Q_{\ell_0}(\beta)} \sum_{\ell \mid N, \ell \neq p} \sum_{t \geq 0} \frac{p^{2u_p + \delta_p + 3} - 1}{p - 1} \times \sum_{d \mid \ell_0} c\left(-Q_{\ell_0}(\beta) \prod_{\ell \mid N, \ell \neq p} \ell \right)^{t_\ell - 1} (-\epsilon_\ell) \]

\[
+ (p^2 - 1) \sqrt{Q_{\ell_0}(\beta)} \sum_{\ell \mid N, \ell \neq p} \sum_{t \geq 0} \frac{p^{2u_p + \delta_p + 1} - 1}{p - 1} \times \sum_{d \mid \ell_0} c\left(-Q_{\ell_0}(\beta) \prod_{\ell \mid N, \ell \neq p} \ell \right)^{t_\ell - 1} (-\epsilon_\ell) \]

\[
+ p^3 \sqrt{Q_{\ell_0}(\beta)} \sum_{\ell \mid N, \ell \neq p} \sum_{t \geq 0} \frac{p^{2u_p + \delta_p - 1} - 1}{p - 1} \times \sum_{d \mid \ell_0} c\left(-Q_{\ell_0}(\beta) \prod_{\ell \mid N, \ell \neq p} \ell \right)^{t_\ell - 1} (-\epsilon_\ell) \]
Hence,

\[
p^2 A(\beta) + (p^2 - 1)A(\beta) + p^2 A(p^{-1}\beta) = \sqrt{Q_{A_0}(\beta)} \sum_{t|N} \sum_{\ell \neq p} \left( \frac{p(p^{2u_\ell + \delta_\ell} - 1)}{p-1} + \frac{(p^2 - 1)(p^{2u_\ell + \delta_\ell + 1} - 1)}{p-1} \right)
+ \frac{p^3(p^{2u_\ell + \delta_\ell - 1} - 1)}{p-1} \sum_{d|n} c \left( \frac{-Q_{A_0}(\beta)}{p-1} \prod_{\ell \neq p} \epsilon(\ell) \epsilon^{\ell-1}(-\epsilon_p) \prod_{\ell | N} (-\epsilon_t) \epsilon^{\ell-1}(-\epsilon_p) \right)
\]

\[
= (p^3 + p^2 + p - 1) \sqrt{Q_{A_0}(\beta)} \sum_{t|N} \sum_{\ell \neq p} \frac{p^{2u_\ell + \delta_\ell + 1} - 1}{p-1}
\times \sum_{d|n} c \left( \frac{-Q_{A_0}(\beta)}{p-1} \prod_{\ell \neq p} \epsilon(\ell) \epsilon^{\ell-1}(-\epsilon_p) \prod_{\ell | N} (-\epsilon_t) \epsilon^{\ell-1}(-\epsilon_p) \right)
\]

\[
= (p^3 + p^2 + p - 1) A(\beta). \square
\]

7. Non-vanishing of the lift

In this section, we will obtain the non-vanishing of the map \( f \rightarrow F_f \) constructed in Section 4. Let us start by observing that the proof of Lemma 4.5 of [23] can be used to conclude that there exists \( M > 0 \) such that the Fourier coefficient \( c(-M) \) of \( f \) is non-zero. If \( f \) is a Hecke eigenform, then this implies that \( c(-1) \neq 0 \). Using the explicit formula (4.5) for the Fourier coefficients for \( F_f \), we can see that in this case we get \( A(1) \neq 0 \). Hence, the map \( f \rightarrow F_f \) is injective when restricted to Hecke eigenforms \( f \). We will now prove the injectivity for all \( f \).

Consider a basis of Hecke eigenforms \( \{ f_1, \ldots, f_k \} \) of \( S(\Gamma_0(N), r) \). Since this is a finite set, we can find a prime \( p \mid N \) such that the Hecke eigenvalues \( \lambda_p^{(i)} \) of \( f_i \) for \( i = 1, \ldots, k \) satisfy \( |\lambda_p^{(i)}| \neq |\lambda_p^{(j)}| \) for all \( i \neq j \). This follows from Corollary 4.1.3 of [32]. Let \( F_{f_1}, \ldots, F_{f_k} \) be the lifts of \( f_1, \ldots, f_k \). By Theorem 6.2, we know that \( F_i \) are Hecke eigenforms with eigenvalues \( \mu_{p,1,i} = p^2 (\lambda_p^{(i)})^2 + p + p^{-1} \). Because of the choice of \( p \), we again see that \( \mu_{p,1,i} \neq \mu_{p,1,j} \) for all \( i \neq j \).

**Theorem 7.1.** The map \( f \rightarrow F_f \) is an injective linear map on \( S(\Gamma_0(N), r) \).

**Proof.** Let notations be as above the statement of the theorem. Suppose there exist complex numbers \( c_1, \ldots, c_k \) such that \( c_1 F_{f_1} + \cdots + c_k F_{f_k} = 0 \). Applying the
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Hecke operator $C_{3,p}^{(1)}$ $k - 1$ times, we get

$$c_1F_1 + c_2F_2 + \cdots + c_kF_k = 0$$
$$\mu_{p,1,1}c_1F_1 + \mu_{p,1,2}c_2F_2 + \cdots + \mu_{p,1,k}c_kF_k = 0$$
$$\mu_{p,1,1}^2c_1F_1 + \mu_{p,1,2}^2c_2F_2 + \cdots + \mu_{p,1,k}^2c_kF_k = 0$$
$$\cdots = \ldots$$
$$\mu_{p,1,1}^{k-1}c_1F_1 + \mu_{p,1,2}^{k-1}c_2F_2 + \cdots + \mu_{p,1,k}^{k-1}c_kF_k = 0.$$

This can be rewritten as

$$\begin{bmatrix}
1 & 1 & \cdots & 1 \\
\mu_{p,1,1} & \mu_{p,1,2} & \cdots & \mu_{p,1,k} \\
\mu_{p,1,1}^2 & \mu_{p,1,2}^2 & \cdots & \mu_{p,1,k}^2 \\
\cdots & \cdots & \cdots & \cdots \\
\mu_{p,1,1}^{k-1} & \mu_{p,1,2}^{k-1} & \cdots & \mu_{p,1,k}^{k-1}
\end{bmatrix}
\begin{bmatrix}
c_1F_1 \\
c_2F_2 \\
\vdots \\
c_kF_k
\end{bmatrix}
= 0.$$

The matrix on the left hand side is a Vandermonde matrix, with determinant

$$\prod_{1 \leq i < j \leq k} (\mu_{p,1,i} - \mu_{p,1,j}) \neq 0,$$

since all the $\mu_{p,1,j}$’s are distinct. Hence the matrix is invertible, which implies that $c_iF_i = 0$ for all $i$. But all the $F_i$ are non-zero, so all the $c_i = 0$. This completes the proof of the theorem.

**Remark 7.2.** Here, without assuming that $f$ is a Hecke eigenform, we cannot get the non-vanishing as in [23] only using the explicit formula (4.5) for the Fourier coefficients of $F_f$. The reason is that even though we can find an integer $M > 0$ such that $c(-M) \neq 0$, there is no guarantee that, for an arbitrary maximal order $O$, there exists $\beta \in O'$ such that $Q_{\lambda_0}(\beta) = M$.

8. CAP representation associated to the lift

Assume that $f \in S(\Gamma_0(N), r)$ is a newform, and let $F_f \in \mathcal{M}(\mathcal{G}(\mathbb{A}), \sqrt{-1}r)$ be the corresponding lift defined in (6.1). Let $\pi_F$ be the representation of $\mathcal{G}(\mathbb{A})$ generated by $F_f$.

8.1. Local components of the representation

8.1.1. The archimedean component

Let

$$N_\infty := \{n(x) \mid x \in \mathbb{R}^1\}, \quad A_\infty := \{a_y \mid y \in \mathbb{R}^+\}$$

for $n(x)$ and $a_y$ as defined in Section 4.1. Let $\delta_s : A_\infty \to \mathbb{C}^\times$ be a quasi-character given by $\delta_s(y) = y^s$ for a parameter $s \in \mathbb{C}$. We can trivially extend $\delta_s$ to the parabolic subgroup $P_\infty$ with Langlands decomposition $P_\infty = N_\infty A_\infty M_\infty$.
for $M_\infty := \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \mid m \in \mathcal{H}(\mathbb{R}) \right\}$. We define the normalized parabolic induction induced from $\delta_s$ by $I_{\mathfrak{p}_\infty}^{G_\infty}(\delta_s)$. Proposition 5.5 of [21] for $N = 4$ gives us

**Proposition 8.1.** The archimedean component of $\pi_F$ is isomorphic to $I_{\mathfrak{p}_\infty}^{G_\infty}(\delta_s)$ as admissible $G_\infty$ module, and irreducible. If $r$ is real, namely, $f$ satisfies the Selberg conjecture on the minimal eigenvalue of the hyperbolic Laplacian, $F$ is tempered at the archimedean place.

Using Theorem 3.1 of [24] and Proposition 6.1, we see that $F$ is irreducible. Since $F$ is a cusp form, we can conclude that $\pi_F$ is an irreducible, cuspidal representation of $G(\mathbb{A})$. Hence, we can decompose $F = \bigoplus \chi$, where $\chi$ is an irreducible, admissible representation of $G(\mathbb{Q})$. We have obtained the description of $\pi_\infty$ above. Next we will describe $\pi_p$ for finite primes $p$.

**8.1.2. Non-archimedean component: $p \nmid N$ case**

Let $p$ be a prime with $p \nmid N$. Let $\chi_1, \chi_2, \chi_3$ be unramified characters of $\mathbb{Q}_p^\times$. We get a character $\chi$ of the split torus of $G(\mathbb{Q}_p)$ via

$$\text{diag}(a_1, a_2, a_3, a_3^{-1}, a_2^{-1}, a_1^{-1}) \rightarrow \chi_1(a_1)\chi_2(a_2)\chi_3(a_3).$$

Extend this to a character of the minimal parabolic subgroup of $G(\mathbb{Q}_p)$ by setting it to be trivial on the unipotent radical. By unramified principal series representation of $G(\mathbb{Q}_p)$ we mean the normalized parabolic induction $I(\chi)$ of $G(\mathbb{Q}_p)$ induced from $\chi$, the character of the minimal parabolic subgroup.

The argument of the proof of [21, Theorem 5.6] works also for our setting. From Theorem 6.2 we thus deduce the following:

**Proposition 8.2.** For primes $p \nmid N$, the local component $\pi_p$ of $\pi_F$ is the spherical constituent of the unramified principal series representation $I(\chi)$ of $G(\mathbb{Q}_p)$ where the character $\chi$ corresponds to the three unramified characters $\chi_1, \chi_2, \chi_3$ given by

$$\chi_1(\varpi_p) = \left( \frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2} \right)^2, \chi_2(\varpi_p) = p, \chi_3(\varpi_p) = 1.$$ 

Here, $\varpi_p$ is an uniformizer in $\mathbb{Q}_p$. Hence, $\pi_p$ is non-tempered for every $p \nmid N$.

**8.1.3. Non-archimedean component: $p|N$ case**

Let $p$ be a prime with $p|N$. For an unramified character $\chi$ of $\mathbb{Q}_p^\times$, we get a character of the torus of $G(\mathbb{Q}_p)$ via

$$\text{diag}(y, 1, 1, 1, y^{-1}) \rightarrow \chi(y).$$

We can extend this to a character of the maximal parabolic subgroup $P$ by setting it to be trivial on the unipotent radical. The modulus character is given by

$$\delta_P(a_y n(x)) = |y|^4.$$

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Define the normalized unramified principal series $I(\chi)$ consisting of all smooth functions $f : G(\mathbb{Q}_p) \to \mathbb{C}$ satisfying
\[ f(a_p n(x)g) = |y|^2 \chi(y) f(g) \quad \text{for all } g \in \mathbb{Q}_p^\times, x \in \mathbb{Q}_p^4, g \in G(\mathbb{Q}_p). \]

If $f_1$ is an unramified vector in $I(\chi)$, then the Hecke operator $C_i^{(1)}$ acts on $f_1$ by a constant. To obtain the constant, using Lemma 6.3, we see that
\[
(C_i^{(1)} f_1)(1) = \int_{G(\mathbb{Q}_p)} \text{char}_{K_{i+1} K_i}(x) f_1(x) dx
= \sum_{x \in X_1} f_1(a_p n(x)) + \sum_{x \in X_1} f_1(n(x)) + f_1(a_{p-1})
= p^4 |\chi(p)| f_1(1) + (p^2 - 1) f_1(1) + |p^{-1}|^2 \chi(p^{-1}) f_1(1)
= (p^2 \chi(p) + p^2 - 1 + p^2 \chi(p^{-1})) f_1(1).
\]

Proposition 8.3. Let $p | N$. The local representation $\pi_p$ is the spherical constituent of the unramified principal series $I(\chi)$ with $\chi(\varpi_p) = p$. The representation $\pi_p$ is non-tempered.

Proof. $F_f$ is right invariant under the maximal compact $K_p$. Hence, $\pi_p$ is the spherical constituent of an unramified principal series. Comparing (8.1) with the Hecke eigenvalue from Theorem 6.5 we get
\[ p^3 + p^2 + p - 1 = p^2 \chi(\varpi_p) + p^2 - 1 + p^2 \chi^{-1}(\varpi_p) \]
implying
\[ \chi(\varpi_p) = p \text{ or } p^{-1}. \]
In view of the conjugation by the Weyl group we can take $\chi$ so that $\chi(\varpi_p) = p$.

Let us show that $\pi_p$ is non-tempered. We remark that [21, Theorem 5.2] is not applicable to this case since the assumption “$m \geq 2$” does not hold. If $\pi_p$ is tempered the matrix coefficient $\langle \pi_p(g)v_0, v_0 \rangle$ with a spherical vector $v_0$ should belong to $L^{2+\epsilon}(G(\mathbb{Q}_p))$ for any $\epsilon > 0$. However, calculate the integral of $|\langle \pi_p(g)v_0, v_0 \rangle|^{2+\epsilon}$ over the open domain of $G(\mathbb{Q}_p)$ as follows:
\[
\bigcup_{m \in \mathbb{Z}} \langle p^m, 1 \rangle K_1.
\]
This yields a divergent series $\sum_{m \in \mathbb{Z}} p^{-m(2+\epsilon)} |\langle v_0, v_0 \rangle|^{2+\epsilon}$ and hence $\langle \pi_p(g)v_0, v_0 \rangle$ is not $2 + \epsilon$-integrable for any $\epsilon > 0$, as required.

8.2. Cuspidal representation generated by $F_f$ and its CAP property

Following the description of the local components, we can now state the result for the explicit determination of the cuspidal representation generated by $F_f$.

Theorem 8.4. Let $f$ be a newform in $S(\Gamma_0(N), r)$ and let $\pi_F$ be the cuspidal representation generated by $F_f$. Then,
i) π_F is irreducible and decomposes into the restricted tensor product π_F = \otimes'_{v \leq \infty} \pi_v of irreducible admissible representations π_v of \mathcal{G}(\mathbb{Q}_v).

ii) For v = p < \infty, if p \nmid N then π_p is the spherical constituent of the unramified principal series representation of \mathcal{G}_p with the Satake parameters

\begin{align*}
\text{diag} \left( \left( \frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2} \right)^2, p, 1, p^{-1}, \left( \frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2} \right)^{-2} \right). 
\end{align*}

iii) For v = p < \infty, if p \mid N then π_p is the spherical constituent of the parabolic induction I(\chi) of \mathcal{G}(\mathbb{Q}_p) defined by

χ(p) = p.

iv) For every finite prime p, π_p is non-tempered. Suppose that the Selberg conjecture holds for f, namely r is a real number for the Laplace eigenvalue for f. Then \pi_\infty is tempered.

**Proof.** This follows from Proposition 8.1, Proposition 8.2 and Proposition 8.3. □

We now review the definition of a CAP representation from [23, Definition 6.6].

**Definition 8.5.** Let \( G_1 \) and \( G_2 \) be two reductive algebraic groups over a number field \( F \) such that \( G_{1,v} \simeq G_{2,v} \) for almost all places \( v \), where \( G_{i,v} = G_i(F_v) \) \((i = 1, 2)\) is the group of \( F_v \)-points of \( G_i \) for the local field \( F_v \) at \( v \). Let \( P_2 \) be a parabolic subgroup of \( G_2 \) with Levi decomposition \( P_2 = M_2 N_2 \). An irreducible cuspidal automorphic representation \( \pi = \otimes' \pi_v \) of \( G_1(\mathbb{A}) \) is called **cuspidal associated to parabolic (CAP)** \( P_2 \), if there exists an irreducible cuspidal automorphic representation \( \sigma \) of \( M_2 \) such that \( \pi_v \simeq \pi'_v \) for almost all places \( v \), where \( \pi'_v = \otimes' \pi'_v \) is an irreducible constituent of \( \text{Ind}_{P_2(\mathbb{A})}^{G_2(\mathbb{A})}(\sigma) \).

For our case \( G_1 = G = \text{O}(1, 5) \) and \( G_2 = \text{O}(3, 3) \). We have \( G_{1,p} = G_{2,p} \) for all \( p \nmid N \). Let \( \sigma \) be a cuspidal representation of \( \text{GL}_2 \) generated by a Maass cusp form \( f \) with the trivial central character. Assume that \( f \) is a newform. We want to regard the representation \( |\det|_{\mathbb{A}}^{1/2} \times |\det|_{\mathbb{A}}^{1/2} \) of \( \text{GL}_2(\mathbb{A}) \times \text{GL}_2(\mathbb{A}) \) (cf.
[23, Section 6.2]) as the representation of \( \mathbb{A}^\times \times O(2, 2)(\mathbb{A}) \), which is isomorphic to a Levi subgroup of a maximal parabolic subgroup \( P(\mathbb{A}) \) of \( O(3, 3)(\mathbb{A}) \). Recall that our previous work [23] introduced the parabolic induction from the representation \( |\det|_{\mathbb{A}}^{-1/2} \times |\det|_{\mathbb{A}}^{1/2} \) of \( \text{GL}_2(\mathbb{A}) \times \text{GL}_2(\mathbb{A}) \) to discuss the CAP property of our lifting for the case of \( d_B = 2 \) in the setting of \( \text{GL}_2 \) over \( B \). In the present setting we consider the parabolic induction from the aforementioned representation of \( \mathbb{A}^\times \times O(2, 2)(\mathbb{A}) \) instead and can show that \( \pi_F \) is a CAP representation attached to this parabolic induction.

To see this we start with recalling the following two isomorphisms (cf. Section 2.3)

\[ \text{GL}_2 \times \text{GL}_2 / \{(z, z) \mid z \in \text{GL}_1 \} \simeq \text{GSO}(2, 2), \quad \text{GO}(2, 2) = \text{GSO}(2, 2) \times \langle t \rangle. \]
We now note that the representation \(| \det \frac{1}{\lambda} \frac{1}{\lambda} \otimes \det \frac{1}{\lambda} \frac{1}{\lambda} \sigma \) of \(GL_2(\mathbb{A}) \times GL_2(\mathbb{A})\) can be regarded as the representation of \(GSO(2, 2)(\mathbb{A})\) since the central character of \(\sigma\) is trivial. We construct a representation of \(GO(2, 2)(\mathbb{A})\) by considering its induced representation from \(GSO(2, 2)(\mathbb{A})\) to \(GO(2, 2)(\mathbb{A})\). Furthermore consider the pullback of the representation of \(GO(2, 2)(\mathbb{A})\) to \(A^\times \times O(2, 2)(\mathbb{A})\) via the surjection \(A^\times \times O(2, 2)(\mathbb{A}) \rightarrow GO(2, 2)(\mathbb{A})\). We denote the resulting representation simply by \(\sigma\) and introduce the normalized parabolic induction \(\text{Ind}^{O(3, 3)(\mathbb{A})}_{P(\mathbb{A})}\), where \(P\) is the maximal parabolic subgroup with Levi subgroup isomorphic to \(GL(1) \times O(2, 2)\) and the abelian unipotent radical. Then we have the following:

**Proposition 8.6.** Let \(\pi_F\) be as above and recall that we have assumed that the Maass cusp form \(f\) is a newform. The cuspidal representation \(\pi_F\) is CAP to the parabolic induction \(\text{Ind}^{O(3, 3)(\mathbb{A})}_{P(\mathbb{A})}\).

**Proof.** We first review the accidental isomorphism \((GL_4 \times GL_1)/\{(z \cdot 1_4, z^{-2}) \mid z \in GL_1\} \simeq GSO(3, 3)\) (see Section 2.3). The restriction of this isomorphism to the \(GL_4\)-factor gives rise to the isomorphism of the maximal split tori of the \(GL_4\)-factor and \(SO(3, 3)\) induced by

\[
\text{diag}(x_1, x_2, x_3, x_4) \mapsto \text{diag}(x_1x_2, x_1x_3, x_2x_4, x_2x_3, x_3x_4)
\]

for \(x_i \in GL_1, 1 \leq i \leq 4\), where note that \(SO(3, 3) = \{(g, z) \in GSO(3, 3) \mid \det(g) z^2 = 1\}\) (cf. [5, Section 3]). In [23, Section 6.1 (6.6), Theorem 6.7], for the \(GL_4\)-setting, we have \(\text{diag}(a_1, a_2, a_3, a_4)\) with

\[
a_1 = p^{1/2} \lambda_p + \sqrt{\lambda_p^2 - 4} \frac{2}{2}, \quad a_2 = p^{1/2} \lambda_p - \sqrt{\lambda_p^2 - 4} \frac{2}{2}, \quad a_3 = p^{-1/2} \lambda_p + \sqrt{\lambda_p^2 - 4} \frac{2}{2},
\]

\[
a_4 = p^{-1/2} \lambda_p - \sqrt{\lambda_p^2 - 4} \frac{2}{2},
\]

as the Satake parameter of the parabolic induction from \(| \det \frac{1}{\lambda} \frac{1}{\lambda} \otimes \det \frac{1}{\lambda} \frac{1}{\lambda} \sigma\) at a prime \(p \nmid N\). Now note that \(O(3, 3)\) and \(SO(3, 3)\) has the same maximal split torus. In view of the isomorphism of the split tori for \(PGL_4\) and \(O(3, 3)\) the corresponding Satake parameter for the \(O(3, 3)\)-setting is

\[
\text{diag}(p, 1, \left(\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}\right)^2, \left(\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}\right)^2, 1, p^{-1}),
\]

which is conjugate to the Satake parameter as in Theorem 8.4 under the action of the Weyl group.

We now prove that the parabolic induction \(\text{Ind}^{O(3, 3)(\mathbb{A})}_{P(\mathbb{A})}\) \(\sigma\) has the Satake parameter above. We note that by the accidental isomorphism \((GL_2 \times GL_2)/\{(z, z) \mid z \in GL_1\} \simeq GSO(2, 2)\), the Satake parameter \(\text{diag}(a_1, a_2, a_3, a_4) = \text{diag}(a_1, a_2) \times \)
diag$(a_3, a_4)$ is mapped to that of GSO$(2, 2)$ given by
\[
\text{diag}\left(\frac{a_1}{a_3}, \frac{a_1}{a_4}, \frac{a_2}{a_3}, \frac{a_2}{a_4}\right) = \text{diag}(p, p, \lambda_p + \frac{\sqrt{\lambda^2_p - 4}}{2}, \lambda_p + \frac{\sqrt{\lambda^2_p - 4}}{2}, \frac{\lambda^2_p - 4}{2}, \lambda_p + \frac{\sqrt{\lambda^2_p - 4}}{2}, p).
\]

In addition, we remark that \(\text{diag}(p, p, p, p)\) corresponds to the character of the similitude factor of GSO$(2, 2) (\mathbb{Q}_p)$. We thereby see that \(\text{Ind}_{P(A)}^{O(3,3)}(\sigma)\) has the desired Satake parameter at \(p \nmid N\) since the representation \(\sigma\), viewed as that of the Levi subgroup of \(O(3,3)\), has the same Satake parameter.

To conclude the proof, as has been pointed out in [21, Section 5.1], we remark that it is valid for the non-connected group \(O(3,3)\) that conjugacy classes of the Satake parameters by the Weyl group classify irreducible unramified principal series, up to isomorphisms. We therefore see that \(\pi_F\) is nearly equivalent to an irreducible constituent of \(\text{Ind}_{P(A)}^{O(3,3)}(\sigma)\) as required.

8.3. Global standard L-function for \(F_f\)

We define the standard \(L\)-function of the orthogonal group \(G\), following Sugano [38, Section 7, (7.6)]. The local factors for places \(p \nmid d_B\) are well known. We find them in [38, Section 7, (7.6)]. For places \(p|d_B\), the case of \((a_0, \partial) = (4, 2)\) in [38, Section 7 (7.6)] is valid. We define the standard \(L\)-function by the Euler product over all finite primes. Putting the local datum of Theorem 8.4 (ii) and (iii) together, we have the following:

**Proposition 8.7.** Suppose that a Maass cusp form \(f\) is a newform in \(S(\Gamma_0(N), r)\) and recall that \(\sigma\) denotes the cuspidal representation of \(GL_2(A)\) generated by \(f\). Let \(\Pi\) be the irreducible constituent of \(\text{Ind}_{P(2,2)}^{GL_2(A)}(|\det|^{-1/2} \sigma \times |\det|^{1/2} \sigma)\) with Satake parameters as in the proof of Proposition 8.6, where \(P_{2,2}\) is the parabolic subgroup of \(GL_4\) with Levi part \(GL_2 \times GL_2\). By \(L(F_f, \text{std}, s)\) (respectively \(L(\Pi, \wedge, s)\)) we denote the standard \(L\)-function for the lift \(F_f\) (respectively exterior square \(L\)-function of \(\Pi\)). We have
\[
L(F_f, \text{std}, s) = L(\Pi, \wedge, s) = L(\text{sym}^2(f), s)\zeta(s-1)\zeta(s)\zeta(s+1),
\]
where the Riemann zeta function \(\zeta(s)\) is defined by the Euler product over all finite primes.

**Proof.** We explain only how to get the equality for the local factors for \(p \mid N\) since the local factors at \(p \nmid N\) are calculated in a formal manner by using the explicit formula for the Satake parameters of \(F_f\) and \(\Pi\), where we see the proof of Proposition 8.6 for the Satake parameter of \(\Pi\).

According to [38, Section 7 (7.6)] the local factors of \(L(F_f, \text{std}, s)\) are written as
\[
(1 - \chi(p)p^{-s})^{-1}(1 - \chi(p)^{-1}p^{-s})^{-1}(1 - p^{-s})^{-1}(1 - p^{N-1}-1)^{-1}.
\]
Now note that, for \(p|N\), the local component of the cuspidal representation generated by \(f\) is a (twisted) Steinberg representation. From [6, p485] we then know that
the local symmetric square $L$-function $L_p(\text{sym}^2(f), s)$ is $(1 - p^{-(s+1)})^{-1}$ for $p|N$. We thereby obtain the local factors of $L(F_f, \text{std}, s)$ at $p|N$.

We are left with the proof of $L(F_f, \text{std}, s) = L(\Pi, \wedge, s)$ at $p|N$. We use the recent result by Y. Jo [17, Theorem 5.7] to see that the local factor of $L(\Pi, \wedge, s)$ at $p|N$ admits a decomposition into the product

$$L_p(\sigma, \wedge, s + 1)L_p(\sigma, \wedge, s - 1)L_p(\sigma \times \sigma, s)$$

of the local exterior square $L$-function and the local Rankin-Selberg $L$-function for $\sigma$. We can verify that the local exterior $L$-functions of $\sigma$ at finite primes are nothing but the local Riemann zeta function (cf. [16, Proposition 4.1]). From [6, (1.4.3)] we deduce $\zeta_p(s) \zeta_p(s + 1)$ as a result we obtain the desired coincidence $L(F_f, \text{std}, s) = L(\Pi, \wedge, s)$.

\text{Remark 8.8.} The above coincidence of the two $L$-functions is expected in the framework of the Langlands $L$-functions (for instance see [5, Section 4]). We remark that our example is given for non-generic representations while the case of generic representations is known to be proved by Shahidi’s theory [35, Theorem 3.5] (see [5, Lemma 4.1]).

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