Conformal field theory of critical Casimir interactions in 2D

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Abstract – Thermal fluctuations of a critical system induce long-ranged Casimir forces between objects that couple to the underlying field. For two-dimensional (2D) conformal field theories (CFT) we derive an exact result for the Casimir interaction between two objects of arbitrary shape, in terms of 1) the free energy of a circular ring whose radii are determined by the mutual capacitance of two conductors with the objects' shape; and 2) a purely geometric energy that is proportional to the conformal charge of the CFT, but otherwise super-universal in that it depends only on the shapes and is independent of boundary conditions and other details.

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Objects embedded in a medium limit its natural fluctuations, resulting in fluctuation-induced forces [1]. The most naturally occurring examples result from modification of electromagnetic fluctuations, that are manifested variously from van der Waals interactions [2] (between atoms and molecules) to Casimir forces (between conducting plates) [3]. While fluctuations of the latter are primarily quantum in origin, thermal fluctuations of correlated fluids lead to similar interactions, most notably at a critical point (where correlation lengths are macroscopic) [4,5]. Critical fluctuation-induced forces have been observed in helium [6] and in binary liquid mixtures [7–9]. Critical fluctuations of a binary mixture were recently employed to manipulate and assemble colloidal particles [10].

Biological membranes are mainly composed of mixtures of lipid molecules, and could potentially be poised close to a critical point demixing point [11,12], in the two-dimensional Ising universality class. It has been suggested that membrane concentration fluctuations could thus lead to critical Casimir forces between inclusions on such membranes, motivating the computation of such forces between discs embedded in the critical Ising model [13]. Membranes (and interfaces) also undergo thermal shape fluctuations governed by the energy costs of bending (and surface tension) [14]. Modification of these fluctuations have also been proposed as a source of interactions amongst inclusions on membranes [15,16], possibly accounting for patterns of colloidal particles at an interface [17]. There is extensive literature on this topic, and the interested reader can consult recent publications [18,19]. Yet another entropic force is proposed to act between surface-membrane bio-adhesion bonds [20].

Conformal field theories (CFTs) have proved highly successful in studies of two-dimensional (2D) systems at criticality [21,22]. (For a general overview see ref. [23].) CFTs are a class of field theories that are invariant under conformal transformations, which are position-dependent dilations. The number of parameters needed to specify a global conformal transformation is finite (6 in 2D), and consequently global conformal invariance constrains the form of field correlators just slightly more than the mere invariance under rotations and dilations. However, in two spatial dimensions any locally analytic function provides a conformal mapping, which may not be regular at all points. The class of such local conformal mappings in 2D is thus infinite, and is the key reason for the great success of CFT in the study of critical phenomena in 2D, both in unbounded and bounded systems. Various boundary conditions have been examined for Ising (or 3-state Potts) model on a cylinder [24].

CFT methods have also been used to compute Casimir forces between parallel plates [25,26], and for
spheres [27,28]. Non-spherical particles at large separations have been studied with the small-particle operator expansion [29,30]. However, a general formulation for interactions between two (or more) objects of arbitrary shape embedded in a CFT appears to be lacking. Some special cases recently studied include interactions between two spherical holes in a free field [31], between circular inclusions [13] and needles [32] in a critical Ising system. (We note in passing exact solutions for Casimir interactions between spheres in three dimensions [27,28,33].) Our approach relies on two important ingredients: i) the existence of (local) conformal mappings of 2D regions of basically any shape to simple standard regions [34]; and ii) a simple transformation law for the stress tensor under any such transformation. The Casimir force between two objects can be expressed as a contour integral of the stress tensor along any contour surrounding either one of the two objects. A priori, the stress tensor of a CFT is only known for simple regions of the complex plane, like a cylinder or circular annulus. From ii) we conclude that the stress can be computed if we find a (local) conformal transformation that maps the region outside the two objects to some standard region of the plane (we shall consider a circular annulus below) for which the stress tensor is known. Hence, i) implies that we can compute the Casimir interaction between two objects of practically any shape embedded in any CFT.

We consider a general two-dimensional classical field theory with an energy that is invariant under conformal transformations. Examples include free theories, such as the capillary-wave Hamiltonian that describes deformations with small gradients around a flat interface, and interacting theories, like the Ising model at its critical point. The corresponding CFT is assumed to couple to two compact objects covering areas $S_1$ and $S_2$ via conformally invariant boundary conditions on the boundaries $\partial S_\alpha$ ($\alpha = 1$ or 2). Examples include Dirichlet or Neumann conditions for a free field, and pinned or free conditions for the Ising model. In the following we assume that the boundaries $\partial S_\alpha$ are Jordan curves.

Before explaining the main steps of the derivation, and presenting examples, we summarize our main result: The doubly connected domain bounded by $\partial S_1$ and $\partial S_2$ can be conformally mapped to the surface of a cylinder with unit radius and length $\ell$, or alternatively to an annulus with outer and inner radii of 1 and $e^{-\ell}$, respectively, see fig. 1. The map $w(z)$ to the cylinder has an electrostatic interpretation: The real (and imaginary) part of the map $\tilde{w}(z)$ is the electrostatic potential (and its conjugate function) outside the objects with the potential set to $-1$ for $\partial S_1$, and to 0 for $\partial S_2$, with net charges of $-Q$ and $+Q$, respectively [34]. The cylinder length $\ell = 2\pi/C$ is then given by the mutual capacitance $C$ of two cylindrical conducting surfaces in 3D that have the areas $S_\alpha$ as their cross-section. The map to the annulus is then $\tilde{w}(z) = \exp[\pi i/2]$. Our main result is that the $x$ and $y$ components of the Casimir force acting on object 2 are combined into the complex force

$$F \equiv \frac{F_x - iF_y}{2} = -\partial_\zeta F_{\text{ann}} - \frac{ie}{24\pi} \oint_{\partial S_2} \{\tilde{w}, z\} dz, \quad (1)$$

where $\partial_\zeta = (\partial_x - i\partial_y)/2$. In the first contribution above, $F_{\text{ann}}$ is the free energy of the CFT on the annulus with the boundary conditions of $\partial S_1$ ($\partial S_2$) on the inner (outer) circle (see below for examples); and the derivative is with respect to $\zeta = (x_2 - x_1) + i(y_2 - y_1)$, the distance in the complex plane between two origins $(x_\alpha, y_\alpha)$ on the objects. (Note that throughout the paper we set $k_B T = 1$, such that $F = -\ln Z$.) The second term is proportional to $c$, the conformal charge of the CFT, and involves the integral of the Schwarzian derivative [23] of the conformal map $\tilde{w}$, $\{\tilde{w}, z\} \equiv (\partial_2^2 \tilde{w}/\partial z \partial_\zeta^2 \tilde{w} - (3/2)(\partial_2^2 \tilde{w}/\partial z^2 \partial_\zeta \tilde{w}))^2$, along the boundary $\partial S_2$ performed counterclockwise. The Schwarzian derivative vanishes precisely for global conformal maps and hence can be considered as a measure of the deviation of a general conformal map from a global conformal map. This second contribution to the force can be written in terms of a “geometric” free energy as $F_{\text{geo}} = -\partial_\zeta F_{\text{geo}} \cdot F_{\text{ann}}$ varies with the CFT but depends on geometry only through the capacitance via $\ell = 2\pi/C$. By contrast $F_{\text{geo}}$ is fully determined by the shape of the objects, independently of the CFT. In this sense, $F_{\text{geo}}$ is super-universal as it is the same for all CFTs (up to a factor of $c$). It vanishes if and only if $\tilde{w}$ is a global conformal map, i.e., when the objects $S_\alpha$ are circular. This follows
as the Schwarzian derivative measures the deviation of the map from being global.

**Sketch of proof.** We begin by relating the change in the cylinder length \( \ell \) with the objects separation \( \zeta \), to the map \( w(z) \). After a small displacement of \( S_z \), the electrostatic energy is modified by

\[
\delta E_{el} = \frac{1}{2\pi i} \oint_{\partial S_z} \alpha(z) T_{el}(z) dz + c.c.,
\]

where \( \alpha(z) \) reverses the motion. The electrostatic stress tensor is well known and can be expressed in terms of the cylinder map \( w(z) \) by \( T_{el}(z) = -(\pi/2)(\partial_z w)^2 \). Since at fixed charges \( Q = \pm 2\pi \), \( \delta E_{el} = -(2\pi)^2 \delta(1/2C) \), and \( \ell = 2\pi/C \), \( \delta \ell = -\delta E_{el}/\pi \). By applying eq. (2) within \( S_z \), with \( \alpha = -\delta x \) and \( \alpha = -i\delta y \), and setting \( \partial_t \ell = (\partial_x \ell - i\partial_y \ell)/2 \), we then find \( \partial_t \ell = (i/\pi) \oint_{\partial S_z} (\partial_z w)^2 dz \).

The displacement of \( S_z \) changes the Casimir free energy by an amount \( \delta F \), also given by eq. (2) with \( \delta F \) replaced by the stress tensor \( T(z) \) of the CFT outside the objects. To obtain a simple expression for \( T(z) \) in terms of the above maps we proceed as follows: As in eq. (2), the stress tensor for the cylinder can be expressed in terms of the derivative of the free energy with respect to its length by

\[
\partial^2 F = \frac{\delta^2 F_{\text{bound}}}{\delta x^2} \frac{\delta^2 F_{\text{bound}}}{\delta y^2} \frac{\delta^2 F_{\text{bound}}}{\delta z^2}.
\]

Asymptotic limits of the annulus free energy. The scaling of \( F_{\text{ann}} \) for small and large \( \ell = 2\pi/C \) (and hence short and large separations \( |\zeta| \)) can be obtained from two equivalent representations of one-dimensional quantum field theories (QFTs) (p. 423 of ref. [23]).

First, consider the QFT on a circle of circumference \( \delta \) with Hamiltonian \( H = (2\pi/\delta)(\hat{L}_0 + \hat{L}_0 - c)/12 \), where \( \hat{L}_0, \hat{L}_0 \) are Virasoro generators in the plane. The Euclidean space-time of the QFT forms a cylinder with length \( \ell \) in the time direction, whose classical free energy is \( F_{\text{cyl}} = -c(\pi/6)(\ell/\delta) + F_{\text{ann}} \), with \( F_{\text{ann}} = -\ln|\zeta|\exp\left(-2\pi i(\ell/\delta)(\hat{L}_0 + \hat{L}_0)\right|b) \) and boundary states \( |a) \) and \( |b) \). The boundary term \( \sim \ell \) is the extensive part of the cylinder energy, given by the ground state of the QFT. If the lowest eigenvalue of \( \hat{L}_0 + \hat{L}_0 \) is zero (e.g. in unitary CFTs), for \( \ell \gg \delta \equiv 2\pi one has \( F_{\text{ann}} \sim e^{-\eta/2} \ell^2 \), where \( \eta/2 \) is the smallest positive eigenvalue of \( \hat{L}_0 + \hat{L}_0 \) that couples to \( |a) \) and \( |b) \) (see footnote 2). The decay of the two-point correlation function of the corresponding scaling field in unbounded


distance is dominated by points of closest approach. In the so-called proximity force approximation (PFA) [2,37], the force between smoothly varying surfaces is obtained by integrating the pressure for parallel plates, evaluated at local separations. This procedure is indeed consistent with the short-distance contribution from \( F_{\text{ann}} = -\partial_\ell F_{\text{ann}} \) that follows from eq. (3). However, there is no corresponding “parallel-plane pressure” for the geometric force, since the
\(\tilde{w}(z)\) is now a global conformal map with \(\{\tilde{w}, z\} = 0\). (For the same reason \(F_{\text{geo}} = 0\) between two circles.) For PFA to remain valid, any contribution of \(F_{\text{geo}}\) should be subleading to \(F_{\text{ann}}\) as \(d \to 0\), and we believe that \(F_{\text{geo}}\) approaches a shape-dependent constant in this limit. PFA is not expected to hold for non-smooth surfaces, such as those with sharp corners or tips. Indeed, for the case of needles (discussed below), we find that both \(F_{\text{geo}}\) and \(F_{\text{ann}}\) scale as \(1/d\) for \(d \to 0\).

### Free energy of the annulus for specific models

The free energy for an annulus is known exactly for certain CFTs. For the free Gaussian field of a surface-tension-dominated interface (with infinite capillary length), the free energy \(F_{\text{ann}}\) on the annulus can be expressed in terms of the Dedekind eta function \(\eta(\tau) = e^{i\pi\tau/12}\prod_{n=1}^{\infty}(1 - e^{2i\pi n\tau})\), which is defined on the upper complex \(\tau\)-plane.

One then obtains

\[
\begin{align*}
F_{\text{ann},D} & = \frac{\pi}{6C} + \frac{1}{2} \ln \left( \frac{2\pi}{C} \right) + \ln \eta \left( \frac{2i}{C} \right), \\
F_{\text{ann},N} & = \frac{\pi}{6C} + \ln \eta \left( \frac{2i}{C} \right),
\end{align*}
\]

for Dirichlet and Neumann conditions, respectively, and dropping an unimportant constant for the former. For small \(C\) (large separations), this leads to

\[
F_{\text{ann},D} \approx F_{\text{ann},D,\text{small}} C = (1/2) \ln(2\pi/C) \quad \text{and} \quad F_{\text{ann},N} \approx F_{\text{ann},N,\text{small}} C = -e^{-4\pi/C}. 
\]

This distinct behavior at large separations follows from the absence of monopoles for Neumann boundary conditions. The Neumann result corresponds to \(\eta = 4\) and thus \(F_{\text{ann}}\) scales the same way at large separations as \(F_{\text{geo}}\). For large \(C\) (small separations) an expansion to all orders yields the simple forms

\[
\begin{align*}
F_{\text{ann},D,\text{large}} C & = \ln \left( \frac{\pi}{2} - \frac{\pi}{24} C + \frac{\pi}{6} \right), \\
F_{\text{ann},N,\text{large}} C & = \frac{1}{2} \ln \left( \frac{\pi}{2} - \frac{\pi}{24} C + \frac{\pi}{6} \right).
\end{align*}
\]

The accuracy of the approximations for small and large \(C\) is remarkable, with maximum errors of roughly 0.32% and 1.6% for Dirichlet and Neumann cases, respectively.

For \(c = 1/2\), CFT describes the continuum limit of the critical Ising model. The free energy of the annulus depends on the boundary conditions. For fixed spins on the boundaries, one has

\[
F_{\text{ann},\pm} = \frac{\pi}{12C} - \ln \left[ \chi_0 \left( \frac{2i}{C} \right) + \chi_1 \left( \frac{2i}{C} \right) \pm \sqrt{2}\chi_{\pm} \left( \frac{2i}{C} \right) \right],
\]

with upper (lower) sign for like (unlike) boundary conditions and with Virasoro characters

\[
\begin{align*}
\chi_0(\tau) & = \left[ \sqrt{\theta_3(\tau)/\eta(\tau)} + \sqrt{\theta_4(\tau)/\eta(\tau)} \right]/2, \\
\chi_1(\tau) & = \left[ \sqrt{\theta_3(\tau)/\eta(\tau)} - \sqrt{\theta_4(\tau)/\eta(\tau)} \right]/2, \\
\chi_{\pm}(\tau) & = \sqrt{\theta_2(\tau)/(2\eta(\tau))},
\end{align*}
\]

where \(\theta_j(\tau) \equiv \theta_j(0|\tau)\) are Jacobi theta functions [23]. At large distance \(\ell\) one has \(F_{\text{ann},\pm} \to \mp \sqrt{2}\pi\ell/8\), and for vanishing \(\ell\) the limits \(F_{\text{ann},+} \to -\pi^2/(24\ell^2)\) and \(F_{\text{ann},-} \to 23\pi^2/(24\ell^2)\). Both are consistent with the predicted asymptotic behaviors with \(\eta = 1/4, \tilde{\eta}_+ = 0, \text{and} \tilde{\eta}_- = 1\).

### Examples

We illustrate the power of our general result with two examples for the free Gaussian field of the interface (capillary wave) Hamiltonian for which \(c = 1\). The first case consists of two circles of equal radii \(R\) and center-to-center separation \(D\), as in fig. 2(a). This is the only (compact) geometry for which the geometric free energy \(F_{\text{geo}}\) vanishes. The mutual capacitance is \(C = 2\pi/\arccosh(1/2(D/R)^2 - 1)\) [38], and substitution into eq. (6) yields, at small surface-to-surface separation \(d = D - 2R \ll R\), the Dirichlet Casimir free energy

\[
F_D = -\frac{\pi^2}{24\sqrt{x}} \left[ 1 + \frac{1}{24} - \frac{4}{\pi^2} x + \left( \frac{1}{6\pi^2} - \frac{17}{5760} \right) x^2 + \cdots \right],
\]

with \(x = d/R\), where we have dropped a distance-independent constant. At large distance one has

\[
F_D = \frac{1}{2} \ln \left( \frac{\ln D}{R} \right) - \frac{1}{2(D/R)^2 \ln D/R} + \cdots,
\]

which is in agreement with ref. [39].

Next consider two aligned needles of length \(L\) and tip-to-tip distance \(d\), as in fig. 2(b). The conformal map \(w(z)\) can be constructed by the Schwarz-Christoffel transformation for polygons [38], and the mutual capacitance is \(C = K(\sqrt{1 - k^2})/K(k)\), where \(K(k)\) is the complete elliptic integral of the first kind, with \(k = d/(2L + d)\). Contrary to smooth surfaces, \(F_{\text{geo}}\) does not go to a constant at short distances for needles which have singular curvature. In this limit, both \(F_{\text{ann}}\) and \(F_{\text{geo}}\) scale logarithmically with separation for D and N conditions. At large separation, the geometric component contributes to leading order only for N conditions. The total Casimir force for Dirichlet conditions is given by

\[
\begin{align*}
2LF_D & = \frac{1}{2\pi \ln(8x)} + \frac{1 + \ln(8x)}{4x^2 \ln^2(8x)} + O(x^{-3}), \\
2LF_D & = \frac{1}{8x} - \frac{1}{8} + \frac{x}{4} + O(x^{-2}),
\end{align*}
\]

for large and small \(x = d/(2L)\), respectively. For Neumann conditions the two limits read

\[
\begin{align*}
2LF_N & = -\frac{1}{512x^5} + \frac{5}{1024x^6} + O(x^{-7}),
\end{align*}
\]
The capacitance $C$ can be computed with high precision numerically; the asymptotic forms of $F_{\text{ann}}$ corresponding to small and large separations of the objects are known for all CFTs. We find that the geometric contribution to the force falls off as $|c|^{-5}$ for large separations $|c|$, while its short-distance behavior is non-trivially dependent on smoothness and other characteristics of the shape of the objects. To clarify this intricate shape dependence, calculations for other geometries are on the way. In particular, while not presented here for brevity, we have confirmed the $1/d$ divergence of the geometric force for finite wedges of arbitrary opening angle.

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