Prime filtrations of the powers of an ideal

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ABSTRACT

We prove that for all \( n \), simultaneously, we can choose prime filtrations of \( R/I^n \) such that the set of primes appearing in these filtrations is finite.

1. Introduction

Let \( R \) be a Noetherian ring, let \( I \) be an ideal of \( R \), and let \( M \) be a finitely generated \( R \)-module. In 1979, Brodmann [1] proved that the sets of associated primes of \( M/I^nM \) stabilize for \( n \) sufficiently large. In particular, the union of the associated primes of all \( M/I^nM \) is a finite set. This result furthered results of Ratliff [8] proved in 1976, and has since been used by many authors. In this paper, we prove a result which seems to have been overlooked: there is a finite set of primes such that for all \( n \), \( M/I^nM \) has a prime filtration involving only primes in that finite set. Moreover, in the case in which \( R \) has infinite residue fields, we prove that the set can be chosen to be stable for large \( n \). We recall that a prime filtration of an \( R \)-module \( N \) is a filtration \( 0 = N_0 \subset N_1 \subset \cdots \subset N_n = N \) such that for all \( i \), \( N_i/N_{i-1} \cong R/P_i \) for some prime \( P_i \). By abuse of language, the set of all such \( P_i \) is said to be the primes in the filtration.

On the face of it, our result is a stronger result than that of Brodmann; however, the results are not perfectly comparable. While it is true that all associated primes are always among the primes in a prime filtration (thus our result does prove that the set of associated primes of \( M/I^nM \) is finite as \( n \) varies), it is not clear that, just because the primes in a prime filtration of \( M/I^nM \) stabilize, also the set of associated primes stabilizes.

We also give an estimate on the number of times that a given prime appears in these special filtrations, and we use it to bound the length of the local cohomology modules, reproving a result of Ulrich and Validashti [11].

We were motivated to prove such a result by the second author’s research concerning the upper semi-continuity of the Hilbert–Kunz multiplicity. At one point, it seemed necessary to prove that one could invert a single element to make \( R/I^n \) Cohen–Macaulay for all \( n \). If \( I = P \) is prime, then one can obtain this result by applying generic flatness to the associated graded ring of \( P \); see Remark 10. In general, however, it is not clear. We apply our result to prove results concerning the openness of loci where all \( M/I^nM \) are Cohen–Macaulay, and further generalize to more general types of filtrations. The method we use comes from the theory of superficial elements. We include an appendix which proves some results on superficial elements. These results are essentially folklore, but we could not find a reference for them in the generality we need for this paper.

2. Main results

Definition 1. Let \( R \) be a Noetherian ring and \( I \) be an ideal of \( R \). In this note, we say that a finite \( R \)-module \( M \) is an \( I \)-filtered module if it is endowed with a filtration

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\[ M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq \cdots \text{ such that} \]

(i) \( IM_n \subseteq M_{n+1} \) for all \( n \);

(ii) \( \bigoplus M_n \) is a finitely generated module over the Rees ring, \( \mathcal{R}(I) := \bigoplus I^n \).

**Remark 2.** It is worth remarking that an \( I \)-filtered module \( M \) satisfies the following properties:

(a) \( \text{gr}(M) = \bigoplus_{n \geq 0} M_n/M_{n+1} \) is a finite \( \text{gr}_I(R) \)-module;

(b) the filtrations \( \{M_n\} \) and \( \{I^n M\} \) are cofinal.

The last condition means (under the assumption that \( IM_n \subseteq M_{n+1} \) for all \( n \)) that for any \( n \) there exists \( k_n \) such that \( M_n \subseteq I^{k_n} M \) and \( \lim_{n \to \infty} k_n = \infty \).

**Definition 3.** Let \( R \) be a ring and \( M \) be a module over \( R \), with a prime filtration

\[ 0 = M_0 \subset M_1 \subset \cdots \subset M_N = M. \]

For a given prime ideal \( P \), its multiplicity \( \mu_P \) in the prime filtration \( \{M_k\} \) is the number of quotients \( M_k/M_{k-1} \) isomorphic to \( R/P \). Note that \( \mu_P \) depends on the choice of filtration.

**Remark 4.** Given a short exact sequence

\[ 0 \to L \to M \to N \to 0 \]

and prime filtrations \( \{L_k\} \) and \( \{N_k\} \) of \( L \) and \( N \), respectively, we can build a prime filtration of \( M \) in the following way. Lift submodules \( N_k \) to their preimages \( N_k + L \) in \( M \), then it is easy to check that

\[ 0 = L_0 \subset L_1 \subset \cdots \subset L_N = L = N_0 + L \subset N_1 + L \subset \cdots \subset M \]

is a prime filtration of \( M \). Also, note that for these fixed filtrations, \( \mu_P(M) = \mu_P(N) + \mu_P(L) \).

**Theorem 5.** Let \( R \) be a Noetherian ring, \( I \) be an ideal, and let \( M \) be an \( I \)-filtered module. Then there exists a finite set of prime ideals \( \Lambda \) such that for any \( n \) there exists a prime filtration of \( M/M_n \) that consists only of prime ideals in \( \Lambda \). For these filtrations, we can estimate the number of times that any given prime in \( \Lambda \) appears in the filtration of \( M/M_n \) as \( O(\dim M) \).

Furthermore, if \( R \) has infinite residue fields, then we can choose prime filtrations with stabilizing sets of prime factors, that is, there exists a subset \( \Lambda' \subseteq \Lambda \) such that for all \( n \) sufficiently large \( \Lambda' \) is precisely the set of prime factors of the chosen filtration of \( M/M_n \).

**Proof.** We prove all the claims by contradiction. There are some small differences in the argument in the second case in which \( R \) has infinite residue fields which we point out at the relevant points in the argument. By Noetherian induction, there is a maximal submodule (under inclusion) \( L \) such that the theorem is false in \( M/L \) (for the induced filtration). Note that the theorem is trivially true for the zero module, so \( L \) is a proper submodule. Since quotients of \( I \)-filtered modules are also \( I \)-filtered, without loss of generality, we assume that \( L = 0 \). We reach the situation in which for every nonzero submodule \( M' \) of \( M \), the theorem holds with the induced filtration on \( M/M' \).

By Proposition A.2, there is an integer \( m \geq 1 \) such that \( M \) has a superficial element \( x \) of order \( m \). Moreover, if \( R \) has infinite residue fields, then \( m \) can be taken to be 1. By Proposition A.3, there exists an integer \( N \) such that for any \( n \geq N \) the sequence

\[ 0 \to M/((0 :_M x) + M_{n-m}) \to M/M_n \to M/(M_n + xM) \to 0 \]
is exact. By our assumption, the theorem holds in $M/xM$. Hence, if $R$ has infinite residue fields, then there exists a finite set of prime ideals $\Lambda_1$ and an integer $N_1$, such that for all $n \geq N_1$, $M/(xM + M_n)$ has a prime filtration with the set of factors $\Lambda_1$. Otherwise, set $N_1 = 0$ and let $\Lambda_1$ be a finite set of primes such that, for any $n$, $M/(xM + M_n)$ has a prime filtration with factors from $\Lambda_1$. Also, we can choose these filtrations and a constant $C > 0$ such that the multiplicity for every prime in $\Lambda_1$ in the chosen filtration of $M/(M_n + xM)$ is at most $C_n^{\dim M/xM}$.

If $x$ is a zerodivisor on $M$, then $0 :_M x \neq 0$, hence the assertion is true in $M/(0 :_M x)$. Thus there exists a finite set of prime ideals $\Lambda_2$ and a constant $D > 0$ such that $M/((0 :_M x) + M_{n−m})$ has a prime filtration of the required form for any $n$ (or $n$ sufficiently large for the second part), and the multiplicity of the appearing primes is bounded by $D_n^{\dim M/(0 :_M x)}$. Gluing the filtrations of $M/((0 :_M x) + M_{n−m})$ and $M/(M_n + xM)$, we obtain a prime filtration of $M/M_n$ with all factors in the finite set of primes $\Lambda = \Lambda_1 \cup \Lambda_2$ of multiplicities at most $C_n^{\dim M/xM} + D_n^{\dim M/(0 :_M x)} \leq (C + D)n^{\dim M}$. Also, if the prime factors of the filtrations of $M/((0 :_M x) + M_{n−m})$ and $M/(M_n + xM)$ stabilize, then the glued filtrations will have the same property.

Otherwise, if $0 :_M x = 0$, then choose arbitrary prime filtrations of $M/M_{N+i}$, for $i = 0 \cdots m−1$, and let $\Lambda$ be the union of $\Lambda_1$ and all prime factors appearing in these filtrations. (Here $N$ and $\Lambda_1$ are as in the paragraphs above.) Using the exact sequence

$$0 \longrightarrow M/M_{n−m} \longrightarrow M/M_n \longrightarrow M/(M_n + xM) \longrightarrow 0$$

and induction on $n$, one can easily see that for any $n \geq N$, $M/M_n$ has a prime filtration with the set of factors in $\Lambda$.

If $m = 1$, then we only need to choose an arbitrary prime filtration of $M/M_K$ where $K = \max(N, N_1)$. Then again, by induction on $n$, one obtains prime filtrations of $M/M_n$ consisting exactly of the prime factors of $M/M_K$ and $\Lambda_1$.

For these filtrations, we can count the multiplicity of any fixed prime in $\Lambda$ in the following way. Let $\mu_M(n)$ and $\mu_{M/xM}(n)$ be multiplicities of this prime in the filtrations of $M/M_n$ and $M/(xM + M_n)$ that we just obtained. For $n \geq 0$, let $n − K = dm + i$, where $i < m$ is the remainder, then, by the construction,

$$\mu_M(n) = \mu_M(n − m) + \mu_{M/xM}(n) = \mu_M(n−2m) + \mu_{M/xM}(n−m) + \mu_{M/xM}(n) = \cdots$$

$$= \mu_M(K + i) + \sum_{j=0}^{d} \mu_{M/xM}(n − jm) \leq \mu_M(K + i) + C \sum_{j=0}^{d} (n − jm)^{\dim M/xM}.$$  

Moreover, there exists a constant $C'$ such that $C \sum_{j=0}^{d} (n − jm)^{\dim M/xM} \leq C' n^{\dim M/xM+1}$. But $\dim M/xM \leq \dim M − 1$ since $x$ is a regular element on $M$, so $\mu_M(n)$ has the required asymptotic behavior.

For the first part of the claim, we have showed that for all $n \geq N$, $M/M_n$ has a prime filtration with all factors from $\Lambda$. But then the claim follows, since we can choose arbitrary prime filtrations of $M/M_n$ for $n < N$ and add their prime factors to $\Lambda$.

**Remark 6.** In the general case, the proof above can be used to show that we can choose the filtrations that have the sets of prime factors stabilizing periodically, that is, there are finitely many finite sets $\Lambda_1, \ldots, \Lambda_m \subseteq \text{Spec} R$ such that, for some $N \geq 0$ and all $i \geq 0$, $\Lambda_k$ is exactly the set of prime factors of $M/M_{N+ki}$ where $k = 1, \ldots, m$.

**Remark 7.** To appreciate the theorem better, let us give an example of prime filtrations with an infinite set of prime factors.

Consider $R = k[x, y]$ and $I = (x)$ (or, even, $I = 0$). For every $n > 0$, let $f_n$ be a nonzero element of $k[y]$. Then we can embed $R/(x)$ into $R/(x^n)$ by mapping $1 \mapsto f_nx^{n−1} + (x^n)$ and...
obtain an exact sequence

\[ 0 \longrightarrow R/(x) \longrightarrow R/(x^n) \longrightarrow R/x^n(f_n, x) \longrightarrow 0. \]

Thus any minimal prime of \((f_n, x)\) is an associated prime of \(R/x^n(f, x)\), so we can use it to build the filtration further. Hence, we can choose \(f_n\) to obtain infinitely many distinct minimal primes of \((f_n, x)\) and, thus, infinitely many prime factors.

As a corollary, we recover the celebrated result of Ratliff [8].

**Corollary 8.** Let \(R\) be a Noetherian ring and \(I\) be an ideal of \(R\). Then \(\bigcup_n \text{Ass}(I^n)\) is finite.

**Corollary 9.** Let \(R\) be an excellent ring and \(I\) be an ideal of \(R\). Then there exists an element \(f \notin \sqrt{I}\) such that \(R_f/I^nR_f\) is Cohen–Macaulay for all \(n\).

**Proof.** By the theorem, we can choose prime filtrations of all \(R/I^n\) such that there are only finitely many primes \(P_i, 1 \leq i \leq l\), appearing in those filtrations.

Without loss of generality, let \(P_1\) be a minimal prime of \(I\). Then we can invert an element \(s \in \bigcap_l P_l \setminus P_1\) to make \(P_1\) be the only prime appearing in the prime filtrations. Since \(R/P_1\) is excellent, its Cohen–Macaulay locus is open [4, 7.8.3(iv)], so we can further localize at an element \(t\) outside of \(P_1\) to make it Cohen–Macaulay. We claim that \(R/I^n\) are Cohen–Macaulay in the localization by \(f = st\).

Let \(n\) be arbitrary and let \(0 \subset M_1 \subset \cdots \subset R_f/I^nR_f\) be the prime filtration of \(R_f/I^n R_f\) induced by the original filtration, so that all the quotients are isomorphic to \(R_f/P_1 R_f\). Now, if \(q\) is an arbitrary prime ideal containing \(P_1\), then it is easy to prove by induction that \((M_k)_q\) are Cohen–Macaulay. 

**Remark 10.** If \(I = P\) is a prime ideal, then one can easily deduce the corollary from Generic Freeness [7, 22.A]. Namely, since \(R/P\) is an excellent domain, we can invert an element and assume that it is regular.

Now \(\text{gr}_P(R)\) is a finitely generated \(R/P\)-algebra, so by Generic Freeness we can invert an element of \(R/P\) and make it free over the regular ring \(R/P\). It follows that \(P^n/P^{n+1}\) are projective \(R/P\)-modules for all \(n\). Then, using the sequences

\[ 0 \longrightarrow P^n/P^{n+1} \longrightarrow R/P^{n+1} \longrightarrow R/P^n \longrightarrow 0, \]

we obtain that all residue rings \(R/P^n\) are Cohen–Macaulay in this localization.

For an ideal \(I\), let \(\text{Minh}(I)\) be the set of minimal primes \(P\) of \(I\) such that \(\dim R/P = \dim R/I\).

**Corollary 11.** Let \(R\) be a locally equidimensional excellent ring and \(I\) be an ideal of \(R\). Then there exists an element \(f \notin \bigcup \text{Minh}(I)\) such that \(R_f/I^nR_f\) is Cohen–Macaulay for all \(n\).

**Proof.** By the theorem, we can choose prime filtrations of all \(R/I^n\) such that there are only finitely many primes \(P_i, 1 \leq i \leq l\), appearing in those filtrations. Without loss of generality, let \(\{P_1, \ldots, P_k\} = \text{Minh}(I)\).

By prime avoidance, we can find an element \(t \in \bigcap_{i=1}^l P_i \setminus \bigcup_{i=1}^k P_i\). Then the induced prime filtrations in \(R_t\) contain only \(\text{Minh}(I)\) as prime factors. For \(1 \leq i \leq k\), let \(J_i\) be a preimage in \(R\) of an ideal defining the non-Cohen–Macaulay locus of \(R/P_i\), so \(\text{ht} J_i > \text{ht} P_i = \text{ht} I\).

Let \(J = J_1 \cdots J_k\). We claim that there exists \(s \in J \setminus \bigcup P_i\). If not, then for some \(i, j\) there would be a containment \(J_i \subseteq P_j\). But this is impossible since \(\text{ht} J_i > \text{ht} I = \text{ht} P_j\). Now, we let \(f = st\) and prove that \(R_f/I^nR_f\) is Cohen–Macaulay for all \(n\).
Let \( n \) be arbitrary and let \( 0 \subset M_1 \subset \cdots \subset R_f/I^n R_f \) be the prime filtration of \( R_f/I^n R_f \) induced by the original filtration. Since \( R \) is locally equidimensional, for any prime \( Q \) containing \( I \), \( \text{Minh}(IR_Q) \) consists of the primes in \( \text{Minh}(I) \) contained in \( Q \). So we may localize at \( Q \) and assume that \( R \) is local.

We prove by induction that \( \text{depth} \ M_k = \dim R/I \). The base case of \( M_1 = R/P_i \) is clear. Now, consider the sequence

\[
0 \rightarrow M_k \rightarrow M_{k+1} \rightarrow R/P_i \rightarrow 0
\]

and apply the induction hypothesis.

\[ \square \]

**Corollary 12.** Let \( R \) be an excellent ring and \( I \) be an ideal of \( R \). Then there exists an element \( f \notin \sqrt{I} \) such that the associated graded ring \( \text{gr}_{I_f}(R_f) \) is a Cohen–Macaulay module over \( R_f/I_f \). If \( R \) is locally equidimensional, then we can choose \( f \notin \bigcup \text{Minh}(I) \).

**Proof.** By Corollary 9 (or Corollary 11 if \( R \) is locally equidimensional), we can invert an element \( f \notin \sqrt{I} \) and make all \( R/I^n \) Cohen–Macaulay. Note, that there are exact sequences

\[
0 \rightarrow I^n R_f \rightarrow R_f/I^{n+1} R_f \rightarrow R_f/I^n R_f \rightarrow 0,
\]

so all \( I^n R_f/I^{n+1} R_f \) are Cohen–Macaulay and the assertion follows.

\[ \square \]

**Corollary 13.** Let \( R \) be an analytically unramified ring, \( I \) be an arbitrary ideal of \( R \), and \( M \) be a finite \( R \)-module. Then there is a finite set of prime ideals \( \Lambda \) such that for all \( n \) the module \( M/\overline{I^n} M \) has a prime filtration where all prime factors are in \( \Lambda \).

Furthermore, if \( R \) is a locally equidimensional excellent ring, then there exists an element \( f \notin \bigcup \text{Minh}(I) \) such that \( R_f/\overline{I^n} R_f \) are Cohen–Macaulay for all \( n \).

**Proof.** We will show that \( \overline{I^n} M \) satisfies the conditions of Definition 1.

The first condition holds, since \( I^{n+1} \subset I^n \subset I^{n+1} \). Since \( R \) is analytically unramified, the ring \( \bigoplus_{n \geq 0} \overline{I^n} \) is a finite algebra over the Rees algebra \( R[It] \) (see [6, Corollary 9.2.1]) so the second condition holds.

The second part of the proof is the same as in Corollary 11.

\[ \square \]

In [10, Theorem 1.4 and Lemma 1], Rees showed that \( R \) is analytically unramified if and only if \( \{ \overline{I^n} \} \) is cofinal with \( \{ I^n \} \). Via property (b) of Remark 2, this highlights the necessity of the assumption.

**Remark 14.** We should note that one could modify the proof of Proposition A.2 in our Appendix and get that a superficial element for \( R \) with respect to a filtration \( I_n \) will exist if the Rees ring of the filtration \( \{ \overline{I^n} \} \) is Noetherian. However, this is also equivalent for \( R \) to be analytically unramified, so this will not lead to a generalization of the corollary above. To prove this, we first record a lemma for which it is hard to find a proof in print. However, see [2, 3, 5]. The proof we record here is found in [5].

**Lemma 15.** Let \( \bigoplus_{n \geq 0} I_n \) be a Noetherian nonnegatively graded ring, where \( I_n \) form a decreasing chain of ideals in \( R = I_0 \). Then there exists an integer \( l \) such that for all \( n \geq 1 \), \( (I_l)^n = I_{ln} \).

**Proof.** The ideal generated by all positive degree elements is finitely generated, say with generators up to degree \( k \). Hence for all \( m \geq k \), \( I_m = \sum I_1^{i_1} I_2^{i_2} \cdots I_k^{i_k} \), where the sum ranges over all nonnegative integers \( j_1, \ldots, j_k \) satisfying \( \sum_j ij_j = m \). Set \( l = k \cdot k! \).
We first claim that for $m \geq 1$, $I_m = I_{m-k}I_k!$. For if $\sum_{i} ij_i = m \geq k \cdot k!$, then for some $1 \leq a < k$, $a_j \geq k!$. Note that $q = k!/a$ is an integer. Therefore,

$$I_m = I_1^{i_1} \cdots I_k^{i_k} = I_2^{i_1} \cdots I_a^{i_{a+1}} \cdots I_k^{i_k} \subseteq I_{m-k}I_k!,$$

and the opposite inclusion is obvious.

We finish by proving $I^n_l = I_{ln}$ by induction on $n$. However, $I_{ln} = I_{ln-k}I_k! = I_{ln-2k}I_{2k}! \cdots = I_{(n-1)l}I_l$ by above. So, the induction hypothesis finishes the proof.

The following proposition should be well known; however, we could not find a reference for it.

**Proposition 16.** Let $R$ be a Noetherian ring and $I$ be an ideal. Then the Rees algebra $S = \bigoplus_{n \geq 0} \frac{T^n}{I^n}$ is Noetherian if and only if $R$ is analytically unramified.

**Proof.** If $R$ is analytically unramified, then $S$ is module finite over the Rees algebra $R[I]$, so it is Noetherian.

Now, assume that $S$ is Noetherian. Then by [9, Theorem 2.7 and Corollary 4.5], there exists $k$ such that for all $n \geq 1$, $\overline{I^n_k} = (\overline{I}^k)^n$.

Since $I^k$ is a reduction of $\overline{I}^k$, there exists $n_0$ such that $I^k(\overline{I}^k)^n = (\overline{I}^k)^{n+1}$ for all $n \geq n_0$. Therefore, for any $n \geq n_0$

$$\overline{I^n_k} = I^{k(n-n_0)}(\overline{I}^k)^{n_0} \subseteq I^{k(n-n_0)}.$$

Now, let $m \geq 0$ be arbitrary. We divide $m = nk + r$, where the reminder $r < k$. Hence

$$\overline{I^n_k} \subseteq \overline{I^n_k} \subseteq I^{k(n-n_0)} \subseteq I^{m-k(n_0+1)}.$$

Since $k(n_0 + 1)$ is a fixed number, we have shown that the filtrations $\{\overline{I^n_k}\}$ and $\{I^n\}$ are cofinal, thus, by [10, Lemma 1], $R$ is analytically unramified.

Using the multiplicity estimates of the filtrations constructed by Theorem 5, we give a different proof of the existence of $\epsilon$-multiplicity introduced in [11]. Here $\ell(M)$ denotes the length of $M$.

**Corollary 17.** Let $R$ be a Noetherian ring, $I$ be an ideal in $R$, and $M$ be an $I$-filtered module. Then $\ell(H^0_m(M/M_n)) = O(n^{\dim M})$.

In particular, we have

$$\epsilon(I, M) = \limsup_{n \to \infty} \frac{d\ell(H^0_m(M/I^nM))}{n^{\dim M}} < \infty.$$

**Proof.** Since $H^0_m(-)$ is a semi-additive functor, we obtain that for a prime filtration $0 = L_0 \subset \cdots \subset L_k \subset \cdots \subset L_N = M/M_n$

$$\ell(H^0_m(M/M_n)) = \ell(H^0_m(L_N)) \leq \ell(H^0_m(L_{N-1})) + \ell(H^0_m(R/P_N)) \leq \cdots \leq \sum_{i=1}^{N} \ell(H^0_m(R/P_i)).$$

Thus, if we take the filtrations obtained by Theorem 5, then we obtain

$$\ell(H^0_m(M/M_n)) \leq \sum_{P \in \Lambda} \mu_P(M/M_n)^{\ell(H^0_m(R/P))} \leq \sum_{P \in \Lambda} Cn^{\dim M} \ell(H^0_m(R/P)) = Cn^{\dim M}.$$

\qed
Appendix. Superficial elements for filtered modules

This appendix contains results that are well known, but, surprisingly, we could not find a reference for the generality we need.

**Definition A.1.** Let $R$ be a ring, $I$ be an ideal, and $M$ be an $I$-filtered module. We say that $x \in I^m$ is a superficial element for $M$ of order $m$, if there exists $c \in \mathbb{N}$ such that for all $n \geq c$, $(M_{n+m} :_M x) \cap M_c = M_n$.

**Proposition A.2.** Let $R$ be a Noetherian ring, $I$ be an ideal, and $M$ be an $I$-filtered module. Then $M$ has a superficial element of some order $m$. Furthermore, if $R$ has infinite residue fields, then $m$ can be taken to be 1.

**Proof.** The proof appears in [6, Proposition 8.5.7].

Let $0 = N_1 \cap \cdots \cap N_r$ be a primary decomposition of the zero submodule in $\text{gr}(M)$. For $i = 1, \ldots, r$, let $P_i$ be the associated prime of $\text{gr}(M)/N_i$. Without loss of generality, $P_1, \ldots, P_s$ contain all elements of $\text{gr}(R)$ of positive degree, and $P_{s+1}, \ldots, P_r$ do not. Then there exists an integer $d$ such that $\text{gr}(R)_{\geq d} \subseteq \text{Ann} \text{gr}(M)/N_i$ for all $i = 1, \ldots, s$. Since $\text{gr}(M)$ is a finitely generated $\text{gr}(R)$-module, there exists a constant $n_0$ such that $\text{gr}(M)_{\geq n} \subseteq \text{gr}(R)_{\geq n-n_0} \text{gr}(M)$ for all $n \geq n_0$. Thus there is a constant $c = d + n_0$ such that for all $i = 1, \ldots, s$

$$\text{gr}(M)_{\geq c} \subseteq \text{gr}(R)_{\geq d} \text{gr}(M) \subseteq N_i.$$

By Prime Avoidance, there exists a homogeneous element $h$ of positive degree in $\text{gr}_R(R)$ that is not contained in any $P_i$ for $i > s$. Say $h = x + I^{m+1}$ for some $x \in I^m$. If $R$ has infinite residue fields, then this $m$ can be taken to be 1.

Note that $M_n \subseteq (M_{n+m} :_M x) \cap M_c$ for any $n \geq c$. Suppose $n \geq c$ and there exists $y \in (M_n :_M x) \cap M_c \setminus M_{n-m}$. Let $k$ be the largest integer such that $y \in M_k$. Then $c \leq k < n$. In $\text{gr}(M)$, $(x + I^{m+1}) \cdot (y + M_{k+1}) = 0$. Thus by the choice of $x + I^{m+1}$,

$$y + M_{k+1} \in N_{s+1} \cap \cdots \cap N_r.$$

By the choice of $y$, $y + M_{k+1} \in M_c \cap N_1 \cdots \cap N_s$, hence, by the choice of $c$, $y + M_{k+1} = 0$, a contradiction with the choice of $k$.

**Proposition A.3.** Let $R$ be a Noetherian ring, $I$ be an ideal, and $M$ be an $I$-filtered module. Suppose that $x$ is a superficial element for $M$ of order $m$, then $M_n :_M x = 0 :_M x + M_{n-m}$ for all sufficiently large $n$.

**Proof.** Since the filtration is cofinal with $\{ I^nM \}$, there exists $k_n$ such that $M_n \subseteq I^{k_n}M$. Thus

$$x(M_n :_M x) = M_n \cap xM \subseteq I^{k_n}M \cap xM.$$

Now, by the Artin–Rees Lemma, there exists $e$ such that $I^{k_n}M \cap xM \subseteq I^{k_n-e}(xM)$. Since $\{ M_n \}$ is cofinal to $I^nM$ and $k_n$ grows without a bound as a function of $n$, $xI^{k_n-e}M \subseteq xM_c$ for $n$ sufficiently large. Thus, we obtain that $(M_n :_M x) \subseteq M_c + 0 :_M x$.

Therefore, $(M_n :_M x) = (M_n :_M x) \cap (M_c + 0 :_M x) = (M_n :_M x) \cap M_c + 0 :_M x = M_{n-m} + 0 :_M x$.

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