The uniqueness and iterative properties of solutions for a general Hadamard-type singular fractional turbulent flow model*

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Abstract. In this paper, we consider the iterative properties of positive solutions for a general Hadamard-type singular fractional turbulent flow model involving a nonlinear operator. By developing a double monotone iterative technique we firstly establish the uniqueness of positive solutions for the corresponding model. Then we carry out the iterative analysis for the unique solution including the iterative schemes converging to the unique solution, error estimates, convergence rate and entire asymptotic behavior. In addition, we also give an example to illuminate our results.

Keywords: unique solutions, turbulent flow, nonlinear operator, iterative analysis.

1 Introduction

In recent years, many researchers were interested in the study of turbulent flow in which the fluid undergoes irregular fluctuations or mixing. For example, Leibenson [17] introduced a $p$-Laplacian differential equation \((\varphi_p(u'(t)))' = f(t, u(t)), t \in (0, 1),\) to model
turbulent flow in a porous medium, where \( \varphi_p(s) = |s|^{p-2}s, p > 1 \). However, the transport of solute in highly heterogeneous porous media often exhibits anomalous diffusion phenomenon [14], and laboratory data and numerical experiments [2, 8] have indicated that solutes moving through a highly heterogeneous porous media violate the basic Fick’s first law of Brownian motion. Thus the works [2, 8] indicated that the fractional differential equation is more suitable for describing the convection–dispersion process of solutes in porous media.

On the other hand, since fractional-order derivative possesses a nonlocal characteristics, so it can provide a possibility to represent the memory occurring in viscoelastic dynamical process [4, 10, 13], blood flow [3], quantum mechanics [7], advection–dispersion process in anomalous diffusion [22, 23, 25] and bioprocesses with genetic attribute [5, 20, 26]. As a powerful tool of modeling the above many abnormal phenomena, in the last few decades, the fractional calculus theory has been enriched, and several different derivatives and integrals such as Caputo, Atangana, Riemann–Liouville, Hadamard, Caputo–Fabrizio, Hilfer, Riesz derivative and so on have been developed.

Thus, in this paper, we choose a general Hadamard-type fractional differential equation involving a nonlinear operator from turbulent flow to study, more specifically, we consider the iterative properties of positive solutions for the equation

\[
-D_t^\alpha (F( -D_t^\beta u(t))(-D_t^\beta u(t))) = f(t,u(t)), \quad 1 < t < e,
\]

\[
u(1) = S(u(1)) = S(u(e)) = 0,
\]

\[
R(-D_t^\beta u(1)) = S(R(-D_t^\beta u(1))) = S(R(-D_t^\beta u(e))) = 0,
\]

where \( \alpha, \beta \in (2, 3) \), \( D_t^\alpha, D_t^\beta \) are \( \alpha \)- and \( \beta \)-order Hadamard fractional derivatives, \( S \) is a differential operator denoted by \( t(d/dt) \), that is, \( Su(t) = t(d/dt)u(t), f \in ((1,e) \times (0, +\infty), [0, +\infty)) \) is a continuous function with singularities at \( t = 0,1 \) and \( u = 0 \). \( R(s) = sF(s) \), and \( F \) is a nonlinear operator satisfying

\[
\chi = \{ F \in C^2([0, +\infty), [0, +\infty)) : \text{there exists a constant } p > 0 \text{ such that} \]

\[
\text{for any } 0 < k < 1, \quad F(kx) \leq k^p F(x) \},
\]

which possesses the following properties:

**Proposition 1.** (See [24].) If \( F \in \chi \), then

- (i) \( R(s) \) has a nonnegative increasing inverse mapping \( R^{-1}(s) \);
- (ii) For \( 0 < l < 1 \), \( R^{-1}(ls) \geq l^{1/(p+1)} R^{-1}(s) \);
- (iii) For \( l \geq 1 \), \( R^{-1}(ls) \leq l^{1/(p+1)} R^{-1}(s) \).

Equation (1) involves a nonlinear operator \( F \), which implies that Eq. (1) covers many interesting and important cases, in particular, if \( F(x) \equiv 1 \) and then \( R(x) = xF(x) = x \).
in this case, Eq. (1) reduces to the following form:

\[ D_t^\alpha \left( D_t^\beta u(t) \right) = f(t, u(t)), \quad 1 < t < e, \]
\[ u(1) = S(u(1)) = S(u(e)) = 0, \]
\[ \varphi_p \left( D_t^\beta u(1) \right) = S \left( \varphi_p \left( D_t^\beta u(1) \right) \right) = 0, \]

where \( \alpha, \beta \in (2, 3), D_t^\alpha, D_t^\beta \) are the \( \alpha \)- and \( \beta \)-order Hadamard fractional derivatives, \( S \) is a differential operator denoted by \( t(d/dt) \), that is, \( Su(t) = t(d/dt)u(t) \). By using the fixed point index and the properties of nonnegative matrices Ding et al. [9] considered the existence of positive solutions for a system of the above Hadamard-type fractional differential equations with semipositone nonlinearities. If \( F(x) = |x|^{p-2}, p > 1 \), Eq. (1) takes the form

\[ D_t^\alpha \left( \varphi_p \left( D_t^\beta u(t) \right) \right) = f(t, u(t)), \quad 1 < t < e, \]
\[ u(1) = S(u(1)) = S(u(e)) = 0, \]
\[ \varphi_p \left( D_t^\beta u(1) \right) = S \left( \varphi_p \left( D_t^\beta u(1) \right) \right) = 0, \]

which is a \( p \)-Poisson turbulent flow equation in highly heterogeneous porous media. By using the fixed point theorem of the mixed monotone operator Zhang et al. [22] studied the uniqueness of positive solution for a fractional-order model of turbulent flow in a porous medium. In addition, on fractional differential equations, some significant work by using fixed theorems has been made by Karapinar and his collaborators [1, 11, 12]. Thus Eq. (1) is a more generalized \( p \)-Poisson turbulent flow equation in highly heterogeneous porous media (3). To the best of our knowledge, no results have been reported on the iterative analysis of positive solutions for Eq. (1) involving a nonlinear operator under singular case.

In addition, a fluid in highly heterogeneous porous media may push the transmission process from a phase into another different phase or state. At absolute zero, this change always leads to transformation process losing continuity and further forms some singular points or singular domains. Normally, near singular points and domains, the “bad” properties such as blow-up properties [24], impulse interference [19], chaotic influence [6] obstruct people’s apperceiving for the essence of related natural phenomena. Thus it is important and interesting to explore the properties of dynamic process governed by singular differential equations, which can deepen people’s comprehension for the natural law of dynamic system.

Thus, to overcome the difficult associated with singular logarithmic kernel and singular nonlinearity of Eq. (1) and follow the work Nieto [18] and Ladde, Lakshmikantham and Vatsala [16], a new double monotone iterative technique will be developed, and more new estimates are given. This paper is organized as follows. In Section 2, we firstly give the definition of Hadamard fractional integral and differential operators and then claim the properties of Green function. The main results are summarized in Section 3.

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2 Preliminaries and lemmas

In this section, we firstly review the definitions of the Hadamard-type fractional integrals and derivatives; for details, see [15].

Let \( \alpha \in \mathbb{C}, \Re \alpha > 0, n = \lfloor \Re \alpha \rfloor \), and let \((a, b)\) be a finite or infinite interval of \( \mathbb{R}^+ \). The \( \alpha \)-order left Hadamard fractional integral is defined by

\[
(I_{a}^{\alpha} x)(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \left( \ln \frac{t}{s} \right)^{\alpha - 1} \frac{x(s)}{s} \, ds, \quad t \in (a, b),
\]

and the \( \alpha \)-order left Hadamard fractional derivative is defined by

\[
(D_{t}^{\alpha} x)(t) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{t}{d} \right)_{t}^{n} \int_{a}^{t} \left( \ln \frac{t}{s} \right)^{n - \alpha - 1} \frac{x(s)}{s} \, ds, \quad t \in (a, b).
\]

In the following, we firstly consider the linear Hadamard fractional equation

\[
-D_{t}^{\alpha} v(t) = h(t), \quad t \in (1, e),
\]

\[v(1) = Sv(1) = Sv(e) = 0.\] (4)

Equation (4) is equivalent to the following Hammerstein-type integral equation [9]:

\[v(t) = \int_{1}^{e} G(t, s)h(s) \frac{ds}{s}, \quad t \in [1, e],\] (5)

and

\[
G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} 
(ln t)^{\alpha - 1}(1 - \ln s)^{\alpha - 2} - (ln t - \ln s)^{\alpha - 1}, & 1 \leq s \leq t \leq e, \\
(ln t)^{\alpha - 1}(1 - \ln s)^{\alpha - 2}, & 1 \leq t \leq s \leq e.
\end{cases}
\]

is the Green’s function of Eq. (4).

Let

\[
H(t, s) = \frac{1}{\Gamma(\beta)} \begin{cases} 
(ln t)^{\beta - 1}(1 - \ln s)^{\beta - 2} - (ln t - \ln s)^{\beta - 1}, & 1 \leq s \leq t \leq e, \\
(ln t)^{\beta - 1}(1 - \ln s)^{\beta - 2}, & 1 \leq t \leq s \leq e,
\end{cases}
\]

for \( h \in L^{1}[0, 1] \), and then we consider the associated linear boundary value problem

\[-D_{t}^{\alpha} (F(-D_{t}^{\beta} u(t))(-D_{t}^{\beta} u(t))) = h(t), \quad 1 < t < e,\]

\[u(1) = S(u(1)) = S(u(e)) = 0,\]

\[R(-D_{t}^{\beta} u(1)) = S(R(-D_{t}^{\beta} u(1))) = S(R(-D_{t}^{\beta} u(e))) = 0.\] (6)

We have the following lemma.
Lemma 1. The associated linear boundary value problem (6) has a unique solution if and only if it solves the Hammerstein-type integral equation

\[ u(t) = \int_1^e H(t, s) R^{-1} \left( \int_1^e G(s, \tau) h(\tau) \frac{d\tau}{\tau} \right) \frac{ds}{s}. \]

Proof. Let \( y(t) = -D_t^\beta u(t), \) \( v(t) = F(y(t))y(t), \) then the associated linear boundary value problem (6) reduces to

\[
\begin{align*}
-D_t^\alpha v(t) &= h(t), \quad t \in (1, e), \\
v(1) &= Sv(1) = Sv(e) = 0.
\end{align*}
\]

By (4) it is equivalent to the Hammerstein-type integral equation

\[ v(t) = \int_1^e G(t, s) h(s) \frac{ds}{s}, \quad t \in [1, e]. \]

Noting that \(-D_t^\beta u(t) = y, y = R^{-1}(v)\) as well as (5) and Proposition 1, the associated linear boundary value problem (6) is equivalent to the following equation:

\[
\begin{align*}
-D_t^\beta u(t) &= R^{-1} \left( \int_1^e G(t, s) h(s) \frac{ds}{s} \right), \quad t \in (1, e), \\
R^{-1}(u(1)) &= R^{-1}(u(e)) = 0,
\end{align*}
\]

and it follows from [9] that if and only if it solves the following Hammerstein-type integral equation

\[ u(t) = \int_1^e H(t, s) R^{-1} \left( \int_1^e G(s, \tau) h(\tau) \frac{d\tau}{\tau} \right) \frac{ds}{s}, \quad t \in [1, e], \]

the proof is completed. \(\square\)

Lemma 2. (See [9].) Let \( \psi_i(t) = \ln t(1 - \ln t)^{i-2}, \) \( i = \alpha, \beta, \) \( t \in [1, e]. \) Then the Green’s functions \( H \) and \( G \) has the following properties:

(i) \( H, G \in C([1, e] \times [1, e], \mathbb{R}^+). \)

(ii) For all \( t, s \in [1, e], \) the following inequalities hold:

\[
\begin{align*}
\frac{1}{\Gamma(\beta)} (\ln t)^{\beta-1} \psi_\beta(s) &\leq H(t, s) \leq \frac{1}{\Gamma(\beta)} (\ln t)^{\beta-1} (1 - \ln s)^{\beta-2}, \\
\frac{1}{\Gamma(\alpha)} (\ln t)^{\alpha-1} \psi_\alpha(s) &\leq G(t, s) \leq \frac{1}{\Gamma(\alpha)} (\ln t)^{\alpha-1} (1 - \ln s)^{\alpha-2}.
\end{align*}
\]
Our work space of this paper is Banach space $E = C[1, e]$ equipped with the norm $\|u\| = \max_{t \in [1, e]} |u(t)|$. Let $P = \{u \in E: u(t) \geq 0, \ t \in [1, e]\}$, then $P$ is a normal cone of $E$ with normality constant 1. Now let us define a subset of $P$ by

$$\Lambda = \{u \in P: \text{there exists a positive number } 0 < l_u < 1 \text{ such that } l_u (\ln t)^{\beta - 1} \leq u(t) \leq l_u^{-1} (\ln t)^{\beta - 1}, \ \text{for all } t \in [1, e]\}$$

and a nonlinear operator $T : E \to E$:

$$(Tu)(x) = \int_1^e H(t, s) R^{-1} \left( \int_1^e G(s, \tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}, \ \ x \in [1, e].$$

It follows from Lemma 1 that $u \in C[1, e]$ is a solution of the Hadamard-type fractional turbulent flow model (1) if and only if $u \in C[1, e]$ is a fixed point of the nonlinear operator $T$.

3 Main results

In order to carry out the iterative analysis of positive solution for the singular Hadamard-type fractional turbulent flow equation (1), we need the following assumption:

(H1) $f : (1, e) \times (0, +\infty) \to [0, +\infty)$ is continuous and increasing with respect to second variable, and for any $c \in (0, 1)$, there exists a constant $0 < \mu < p + 1$, where $p$ is defined by (2), such that for all $(t, u) \in (1, e) \times (0, +\infty)$,

$$f(t, cu) \leq c^{-\mu} f(t, u).$$

Remark 1. Assumption (7) allows $f$ to have a singularity at $t = 1$, $e$, and $u = 0$. For example, $f(t, u) = (t - 1)^{-1}(e - t)^{-1/3}u^{-p}$ satisfies assumption (H1).

Remark 2. Suppose that condition (H1) holds. It is easy to prove that for any $c \geq 1$ and for all $(t, u) \in (1, e) \times (0, +\infty)$, the following formula is also valid:

$$f(t, cu) \geq c^{-\mu} f(t, u).$$

Now we give our main results as follows:

Theorem 1. Suppose that (H1) holds, and the following condition is satisfied:

$$0 < R^{-1} \left( \int_1^e (1 - \ln \tau)^{\alpha - 2} f(\tau, (\ln \tau)^{\beta - 1}) \frac{d\tau}{\tau} \right) < +\infty.$$

Then

(i) The singular Hadamard-type fractional turbulent flow equation (1) has a unique positive solution $u^* \in \Lambda$;
(ii) For any initial value \( v_0 \in \Lambda \), we construct the iterative sequence

\[
v_n = \int_1^e H(t, s) R^{-1} \left( \int_1^e G(s, \tau) f(\tau, v_{n-1}(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}, \quad n = 1, 2, \ldots,
\]

then \( \{v_n\}_{n \geq 1} \) converges uniformly to the unique positive solution \( u^* \) of Eq. (1) on \([1, e]\);

(iii) The error of the iterative value \( v_n \) and the exact solution \( u^* \) can be estimated by the following formula:

\[
\|v_n - u^*\| \leq (1 - \zeta^2(\mu/(p+1)))^{2n} \zeta^{-1}
\]

with an exact convergence rate

\[
\|v_n - u^*\| = o(1 - \zeta^2(\mu/(p+1)))^n)
\]

where \( 0 < \zeta < 1 \) is a positive constant, which is determined by \( u_0 = \rho^\kappa \ln^{\beta-1}(t) \) and some \( \rho \in (0, 1) \);

(iv) The exact solution \( u^* \) of Eq. (1) has an entire asymptotic estimate

\[
l(\ln t)^{\beta-1} \leq u^*(t) \leq l^{-1}(\ln t)^{\beta-1}, \quad t \in [1, e], \quad 0 < l < 1.
\]

Proof. Firstly, we show that \( T: \Lambda \rightarrow \Lambda \) is a compact operator.

To do this, for any \( u \in \Lambda \), by the definition of the set \( \Lambda \) there exists a constant \( 0 < l_u < 1 \) such that

\[
l_u(\ln t)^{\beta-1} \leq u(t) \leq l_u^{-1}(\ln t)^{\beta-1}, \quad t \in [1, e]. \tag{9}
\]

Notice that \( f(t, x) \) is decreasing in \( x \), it follows from Lemma 2, (7), (9) and Proposition 1 that

\[
T(u)(t) = \int_1^e H(t, s) R^{-1} \left( \int_1^e G(s, \tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}
\leq \int_1^e \frac{(\ln t)^{\beta-1}}{\Gamma(\beta)} R^{-1} \left( \frac{1}{\Gamma(\alpha)} \int_1^e (1 - \ln \tau)^{\alpha-2} f(\tau, l_u(\ln \tau)^{\beta-1}) \frac{d\tau}{\tau} \right) \frac{ds}{s}
\leq \frac{l_u^{-\mu/(p+1)}}{\Gamma(\beta)} \left( \frac{1}{\Gamma(\alpha)} + 1 \right)^{1/(p+1)} R^{-1} \left( \int_1^e (1 - \ln \tau)^{\alpha-2} f(\tau, (\ln \tau)^{\beta-1}) \frac{d\tau}{\tau} \right)
\leq +\infty.
\]

So \( T \) is well defined and uniformly bounded.
On the other hand, notice that $H(s, t)$ is uniformly continuous on $[1, e] \times [1, e]$, and let $1 \leq t_1 < t_2 \leq e$ for all $u \in \Lambda$. Then we have

$$
|(Tu)(t_1) - (Tu)(t_2)|
\leq \int_0^1 |H(t_1, s) - H(t_2, s)| R^{-1} \left( \int_1^e G(s, \tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}
\leq \int_0^1 |H(t_1, s) - H(t_2, s)| R^{-1} \left( \frac{1}{\Gamma(\alpha)} \int_1^e (1 - \ln \tau)^{\alpha-2} f(\tau, l_u(\ln \tau)^{\beta-1}) \frac{d\tau}{\tau} \right) \frac{ds}{s}
\leq l_u^{-\mu/(p+1)} \left( \frac{1}{\Gamma(\alpha)} + 1 \right) ^{1/(p+1)} R^{-1} \left( \int_1^e (1 - \ln \tau)^{\alpha-2} f(\tau, (\ln \tau)^{\beta-1}) \frac{d\tau}{\tau} \right)
\times \int_0^1 |H(t_1, s) - H(t_2, s)| \frac{ds}{s},
$$

that is, $T(\Lambda)$ is equicontinuous, and then $T$ is a compact operator in $\Lambda$.

Now we show that $T(\Lambda) \subset \Lambda$. In fact, similar to (10), for any $u \in \Lambda$, one has

$$
T(u)(t) = \int_1^e H(t, s) R^{-1} \left( \int_1^e G(s, \tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}
\leq \int_1^e \frac{(\ln t)^{\beta-1}}{\Gamma(\beta)} R^{-1} \left( \frac{1}{\Gamma(\alpha)} \int_1^e (1 - \ln \tau)^{\alpha-2} f(\tau, l_u(\ln \tau)^{\beta-1}) \frac{d\tau}{\tau} \right) \frac{ds}{s}
\leq l_u^{-\mu/(p+1)} \left( \frac{1}{\Gamma(\beta)} \right) \left( \frac{1}{\Gamma(\alpha)} + 1 \right) ^{1/(p+1)}
\times R^{-1} \left( \int_1^e (1 - \ln \tau)^{\alpha-2} f(\tau, (\ln \tau)^{\beta-1}) \frac{d\tau}{\tau} \right) (\ln t)^{\beta-1}
\leq \tilde{T}_u^{-1} (\ln t)^{\beta-1}
$$

and

$$
(Tu)(t) = \int_1^e H(t, s) R^{-1} \left( \int_1^e G(s, \tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}
\geq (\ln t)^{\beta-1} \int_1^e \frac{1}{\Gamma(\beta)} \psi_\beta(s) R^{-1} \left( \int_1^e G(s, \tau) f(\tau, l_u^{-1}(\ln \tau)^{\beta-1}) \frac{d\tau}{\tau} \right) \frac{ds}{s}
$$

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\[
\frac{l_u^{-\mu/(p+1)}}{\Gamma(\beta)} \int_1^e \psi_\beta(s) R^{-1} \left( \int_1^e G(s, \tau) f(\tau, (\ln \tau)^{\beta-1}) \frac{d\tau}{\tau} \right) ds (\ln t)^{\beta-1} \\
\geq \frac{(\ln t)^{\beta-1} l_u^{-\mu/(p+1)}}{\Gamma(\beta)} \left( \frac{1}{\Gamma(\alpha)} + 1 \right)^{1/(p+1)} \int_1^e \psi_\beta(s) (\ln s)^{\alpha-1}/(p+1) ds \\
\times R^{-1} \left( \int_1^e \psi_\alpha(\tau) f(\tau, (\ln \tau)^{\beta-1}) \frac{d\tau}{\tau} \right) \\
\geq \frac{(\ln t)^{\beta-1} l_u^{-\mu/(p+1)} \Gamma(\frac{\alpha-1}{p+1} + 2)}{\Gamma(\beta + \frac{\alpha-1}{p+1} + 2)} \left( \frac{1}{\Gamma(\alpha)} \right)^{1/(p+1)} \\
\times R^{-1} \left( \int_1^e \psi_\alpha(\tau) f(\tau, (\ln \tau)^{\beta-1}) \frac{d\tau}{\tau} \right),
\]

where \( \tilde{T}_u \) satisfies

\[
0 < \tilde{T}_u \\
< \min \left\{ \frac{1}{2}, \left\{ \frac{l_u^{-\mu/(p+1)}}{\Gamma(\beta)} \left( \frac{1}{\Gamma(\alpha)} + 1 \right)^{1/(p+1)} \right. \right.
\times R^{-1} \left. \left. \left( \int_1^e (1 - \ln \tau)^{\alpha-2} f(\tau, (\ln \tau)^{\beta-1}) \frac{d\tau}{\tau} \right) \right\}^{-1}, \\
\frac{l_u^{-\mu/(p+1)} \Gamma(\frac{\alpha-1}{p+1} + 2)}{\Gamma(\beta + \frac{\alpha-1}{p+1} + 2)} \left( \frac{1}{\Gamma(\alpha)} \right)^{1/(p+1)} \times R^{-1} \left( \int_1^e \psi_\alpha(\tau) f(\tau, (\ln \tau)^{\beta-1}) \frac{d\tau}{\tau} \right) \right\}. \tag{13}
\]

Hence we have \( T(\Lambda) \subset \Lambda \) from (11) and (12).

Next, by developing the double monotone iterative technique we will show that Eq. (1) has a unique positive solution \( u^* \) in \( \Lambda \).

Taking \( \omega(t) = (\ln t)^{\beta-1} \), we have \( \omega \in \Lambda \), and then \( T \omega \in \Lambda \), which implies that there exists a constant \( 0 < l_{T_\omega} < 1 \) such that

\[
l_{T_\omega} \omega(t) \leq T \omega(t) \leq l_{T_\omega}^{-1} \omega(t), \tag{14}
\]

where \( l_{T_\omega} \) is chosen according to (13). Notice that \( 0 < \mu/(p+1) < 1 \) for some \( \rho \in (0, 1) \) and take a sufficiently large positive constant \( \kappa \) such that

\[
\left[ \rho^{1-\mu/(p+1)} \right]^{\kappa} \leq l_{T_\omega}. \tag{15}
\]
Now we construct an iterative sequence with initial value \( u_0(t) = \rho^{\kappa} \omega(t) \)

\[
 u_n(t) = T u_{n-1}(t), \quad t \in [1, e], \quad n = 1, 2, \ldots .
\]  

(16)

Then we assert

\[
 u_0 \leq u_2 \leq \cdots \leq u_{2n} \leq \cdots \leq u_{2n+1} \leq \cdots \leq u_3 \leq u_1.
\]  

(17)

In fact, noticing that \( T \) is a decreasing operator in \( u \), by (14)–(16) we have

\[
 u_0(t) = \rho^{\kappa} \omega(t) \leq \omega(t),
 u_1(t) = T u_0(t) \geq T \omega(t) \geq l_{T \omega} \omega(t) \geq (\rho^{-\mu/(p+1)+1})^{\kappa} \omega(t)
\]  

(18)

and then

\[
 u_2(t) = T u_1(t) \leq T u_0(t) = u_1(t).
\]  

(19)

On the other hand, from (7) and (14) we have the following estimates:

\[
 u_1(t) = T u_0(t) = \int_1^e H(t, s) R^{-1} \left( \int_1^e G(s, \tau) f(\tau, \rho^{\kappa} \omega(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}
 \leq \rho^{-\mu/(p+1)} \int_1^e H(t, s) R^{-1} \left( \int_1^e G(s, \tau) f(\tau, \omega(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}
 = \rho^{-\mu/(p+1)} T \omega \leq \rho^{-\mu/(p+1)} l_{T \omega}^{-1} \omega(t) \leq \rho^{-\kappa} \omega(t).
\]  

(20)

By using (8), (14), (20) and the monotonicity of \( T \) we get

\[
 u_2(t) = T u_1(t) \geq T (\rho^{-\kappa} \omega(t)) = \int_1^e H(t, s) R^{-1} \left( \int_1^e G(s, \tau) f(\tau, \rho^{-\kappa} \omega(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}
 \geq \rho^{\mu/(p+1)} T \omega(t) \geq \rho^{\kappa} \omega(t) = u_0(t).
\]  

(21)

Thus it follows from (18), (19) and (21) that

\[
 u_0(t) \leq u_2(t) \leq u_1(t).
\]

So by mathematical induction the conclusion (17) holds.

On the other hand, for any \( c \in (0, 1) \), it follows from Proposition 1, (7) and (8) that

\[
 T^2 (cu) \geq c^{(\mu/(p+1))^2} T^2 u.
\]  

(22)
Since $T^2$ is increasing operator with respect to $u$, thus let us apply (22) repeatedly, one gets
\[
u_{2n}(t) = T\nu_{2n-1}(t) = T^{2n}\nu_0(t) = T^{2n}(\rho^{2\kappa}\omega(t)) = T^{2n}(\rho^{2\kappa}\rho^{-\kappa}\omega(t)) \\
\geq T^{2n-2}(T^2(\rho^{2\kappa}\nu_1(t))) \geq T^{2n-2}(\rho^{2\kappa}(\mu/(p+1))^2T^2\nu_1(t)) \\
= T^{2n-4}T^2((\rho^{2\kappa}(\mu/(p+1))^2T^2\nu_1(t)) \geq T^{2n-4}((\rho^{2\kappa}(\mu/(p+1))^4T^4\nu_1(t)) \\
\geq \cdots \geq (\rho^{2\kappa}(\mu/(p+1)))^{2n}T^{2n}\nu_1(u) = (\rho^{2\kappa}(\mu/(p+1)))^{2n}T^{2n+1}\nu_0(t) \\
= (\rho^{2\kappa}(\mu/(p+1)))^{2n}\nu_{2n+1}(t),
\]
that is,
\[
(\rho^{2\kappa}(\mu/(p+1)))^{2n}\nu_{2n+1}(t) \leq \nu_{2n}(t) \leq \nu_{2n+1}(t).
\]
So for any natural numbers $n$ and $m$, we have
\[
0 \leq \nu_{2(n+m)}(t) - \nu_{2n}(t) \leq \nu_{2n+1}(t) - \nu_{2n}(t) \\
\leq (1 - (\rho^{2\kappa}(\mu/(p+1))))^{2n}\nu_{2n+1} \leq (1 - (\rho^{2\kappa}(\mu/(p+1))))^{2n}\nu_1(t) \\
\leq (1 - (\rho^{2\kappa}(\mu/(p+1)))^{2n})\rho^{-\kappa}\omega(t) \tag{23}
\]
and
\[
0 \leq \nu_{2n+1}(t) - \nu_{2(n+m)+1}(t) \leq \nu_{2n+1}(t) - \nu_{2n}(t) \\
\leq (1 - (\rho^{2\kappa}(\mu/(p+1)))^{2n})\rho^{-\kappa}\omega(t). \tag{24}
\]
Notice that $P$ is a normal cone with normality constant $1$, thus it follows from (23), (24) that
\[
\|u_{n+m} - u_n\| \leq (1 - (\rho^{2\kappa}(\mu/(p+1)))^{2n})\rho^{-\kappa} \rightarrow 0, \quad n \rightarrow +\infty,
\]
which implies that $\{u_n\}$ is a Cauchy sequence of compact set. Since $\{u_n\} \in \Lambda$ and $T(\Lambda) \subset \Lambda$ is compact, consequently, $\{u_n\}$ converges to some $u^* \in \Lambda$ with
\[
u_{2n} \leq u^* \leq \nu_{2n+1},
\]
and then
\[
u_{2n+2} = T\nu_{2n+1} \leq Tu^* \leq Tu_{2n} = \nu_{2n+1}. \tag{25}
\]
Let $n \rightarrow \infty$ in (25), one has $u^*(t) = Tu^*(t)$, that is, $u^*$ is a positive solution of Eq. (1). In the end, we show the uniqueness of $u^*$ in $\Lambda$ and the asymptotic properties of solution for Eq. (1). Suppose that $\tilde{u}$ is another positive solution of $u^*$. Let $r_1 = \sup \{r > 0: \tilde{u} \geq ru^*\}$. Clearly, $0 < r_1 < +\infty$. We claim that $r_1 \geq 1$. Otherwise, we have $0 < r_1 < 1$, which as well as (22) implies
\[
\tilde{u} = T\tilde{u} = T^2\tilde{u} \geq T^2(r_1u^*) \geq r_1^{(\mu/(p+1))^2}T^2u^* = r_1^{(\mu/(p+1))^2}u^*.
\]
https://www.journals.vu.lt/nonlinear-analysis
Since \( r_1^{\mu/(p+1)} > r_1 \), this contradicts with the definition of \( r_1 \). So we have \( r_1 \geq 1 \), and then \( \tilde{u} \geq u^* \). In the same way, we have \( \tilde{u} \leq u^* \). Consequently, \( \tilde{u} = u^* \), so the positive solution of Eq. (1) in \( \Lambda \) is unique.

On the other hand, for any initial value \( v_0 \in \Lambda \), there exists a constant \( l_{v_0} \in (0, 1) \) such that

\[
l_{v_0} \omega(t) \leq v_0(t) \leq l_{v_0}^{-1} \omega(t), \quad t \in [0, 1].
\]

Let

\[
v_n = \int_1^e H(t, s) R^{-1} \left( \int_1^e G(s, \tau) f(\tau, v_n(t)) \frac{d\tau}{\tau} \right) \frac{ds}{s}, \quad n = 1, 2, \ldots.
\]

Since \( T(\Lambda) \subset \Lambda \), there still exists a constant \( l_{v_1} \in (0, 1) \) such that

\[
l_{v_1} \omega(t) \leq v_1 = T v_0 \leq l_{v_1}^{-1} \omega(t), \quad t \in [0, 1].
\]

As (15), let us choose a sufficiently large \( \kappa > 0 \) such that

\[
\left[ \rho^{(1-\mu/(p+1))} \right]^{\kappa} \leq \min\{l_{v_0}, l_{v_1}, l_T\},
\]

where \( \rho \in (0, 1) \). Thus

\[
\begin{align*}
    u_0 &= \rho^\kappa \omega(t) \leq \left[ \rho^{1-\mu/(p+1)} \right]^{\kappa} \omega(t) \leq l_{v_0} \omega(t) \leq v_0, \\
    u_0 &= \rho^\kappa \omega(t) \leq \left[ \rho^{1-\mu/(p+1)} \right]^{\kappa} \omega(t) \leq l_{v_1} \omega(t) \leq v_1,
\end{align*}
\]

which implies that \( v_1 = T v_0 \leq T u_0 = u_1 \), and then

\[
u_0 \leq v_1 \leq u_1, \quad u_2 \leq u_2 \leq u_1. \quad (26)
\]

According to the fact that \( T^2 \) is increasing operator with respect to \( u \) and (26), we have

\[
\begin{align*}
    u_{2n}(t) \leq v_{2n+1}(t) \leq u_{2n+1}(t), \quad u_{2n+2}(t) \leq v_{2n+2}(t) \leq u_{2n+1}(t). \quad (27)
\end{align*}
\]

Letting \( n \to \infty \) in (27), we get that \( v_n \) uniformly converges to the unique positive solution \( u^* \) of Eq. (1).

Moreover it follows from (24) and (27) that the unique positive solution \( u^* \) satisfies the following estimate of error:

\[
\|v_n - u^*\| \leq \left( 1 - \left( \rho^{2\kappa} \right)^{(\mu/(p+1))^{2n}} \right) \rho^{-\kappa} = \left( 1 - \zeta^{2(\mu/(p+1))^{2n}} \right) \zeta^{-1}
\]

with a exact convergence rate

\[
\|v_n - u^*\| = o\left( 1 - \zeta^{2(\mu/(p+1))^{2n}} \right),
\]

where \( 0 < \zeta = \rho^\kappa < 1 \) is a positive constant, which is determined by \( u_0 = \rho^\kappa \omega(t) \).

Finally, since \( u^* \in \Lambda \), we have the following entire asymptotic analysis, that is, there exists a constant \( 0 < l < 1 \) such that

\[
l(\ln t)^{\beta-1} \leq u^*(t) \leq l^{-1}(\ln t)^{\beta-1}.
\]

The proof is completed.
4 Example

The following example shows that conditions of Theorem 1 are easily verified.
Consider the following singular Hadamard-type fractional turbulent flow model:

\[-D_t^{5/2}(F(-D_t^{9/4}u(t))(-D_t^{9/4}u(t))) = \frac{(1 - \ln t)^{-3/4}}{u^{1/2}(t)} + \ln t,\]

\[1 < t < e,\]

\[u(1) = S(u(1)) = S(u(e)) = 0,\]

\[R(-D_t^{9/4}u(1)) = S(R(-D_t^{9/4}u(1))) = S(R(-D_t^{9/4}u(e))) = 0\]

with a nonlinear operator

\[F(x) = x^{1/2}, \quad R(x) = x^{3/2}\]

Then we have the following conclusions:

(i) The singular Hadamard-type fractional turbulent flow equation (28) has a unique positive solution \(u^* \in \Lambda\);

(ii) For any initial value \(v_0 \in \Lambda\), we construct the iterative sequence

\[v_n(t) = \int_1^e H(t,s) \left( \int_1^e G(s,\tau) \left[ (1 - \ln \tau)^{-3/4}v_{n-1}^{1/2}(\tau) + \ln \tau \right] \frac{d\tau}{\tau} \right)^{3/2} \frac{ds}{s},\]

\[t \in [1, e], \quad n = 1, 2, \ldots,\]

then \(\{v_n\}_{n \geq 1}\) converges uniformly to the unique positive solution \(u^*\) of Eq. (28) on \([1, e]\);

(iii) The error of the iterative value \(v_n\) and the exact solution \(u^*\) can be estimated by the following formula:

\[\|v_n - u^*\| \leq (1 - \zeta^{2(1/2)^{2n}})\zeta^{-1},\]

and the convergence rate is

\[\|v_n - u^*\| = o(1 - \zeta^{2(1/2)^{2n}}),\]

where \(0 < \zeta < 1\) is a positive constant;

(iv) The exact solution \(u^*\) of Eq. (28) has an entire asymptotic estimate

\[l(\ln t)^{5/4} \leq u^*(t) \leq l^{-1}(\ln t)^{5/4}, \quad t \in [1, e], \quad 0 < l < 1.\]

First, we define a set as

\[\Lambda = \{u(x) \in P: \text{there exists a positive number } 0 < l_u < 1 \text{ such that} \]

\[l_u(\ln t)^{5/4} \leq u(x) \leq l_u^{-1}(\ln t)^{5/4}, \quad x \in [1, e] \};\]

Let \(\alpha = 5/2, \beta = 9/4, p = 1/2, \mu = 5/4,\)

\[f(t, u) = (1 - \ln t)^{-3/4} u^{-1/2},\]
then we have
\[
G(t, s) = \frac{1}{\Gamma\left(\frac{5}{2}\right)} \left\{ (\ln t)^{3/2}(1 - \ln s)^{1/2} - (\ln t - \ln s)^{3/2}, \quad 1 \leq s \leq t \leq e, \right.
\]
\[
\left. \quad \text{and} \quad (\ln t)^{3/2}(1 - \ln s)^{1/2}, \quad 1 \leq t \leq s \leq e, \right. \]
and
\[
H(t, s) = \frac{1}{\Gamma\left(\frac{9}{4}\right)} \left\{ (\ln t)^{5/4}(1 - \ln s)^{1/4} - (\ln t - \ln s)^{5/4}, \quad 1 \leq s \leq t \leq e, \right.
\]
\[
\left. \quad \text{and} \quad (\ln t)^{5/4}(1 - \ln s)^{1/4}, \quad 1 \leq t \leq s \leq e. \right. \]

Clearly, \( f : (1, e) \times (0, +\infty) \to [0, +\infty) \) is continuous and decreasing with respect to variable \( u \), and for any \( c \in (0, 1) \), there exists a constant \( 0 < \mu < p + 1 \) such that for all \( (t, u) \in (1, e) \times (0, +\infty) \),
\[
f(t, cu) = (1 - \ln t)^{-3/4}(cu)^{-1/2} + \ln t \leq c^{-1/2}[(1 - \ln t)^{-3/4}u^{-1/2} + \ln t] \leq c^{-3/4}f(t, u).\]
Moreover, we also have
\[
0 < \int_1^e (1 - \ln \tau)^{\alpha - 2}f(\tau, (\ln \tau)^{\beta - 1}) \frac{d\tau}{\tau}
\]
\[
= \int_1^e (1 - \ln \tau)^{-1/4}[(\ln \tau)^{-5/8} + \ln \tau] \frac{d\tau}{\tau}
\]
\[
= \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{8}\right)} + \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{11}{4}\right)} = 3.8469 < +\infty,
\]
and then
\[
0 < R^{-1} \left( \int_1^e (1 - \ln \tau)^{\alpha - 2}f(\tau, (\ln \tau)^{\beta - 1}) \frac{d\tau}{\tau} \right)^{2/3}
\]
\[
= \left( \int_1^e (1 - \ln \tau)^{-1/4}(\ln \tau)^{-5/8} \frac{d\tau}{\tau} \right)^{2/3} < +\infty.
\]

Thus all conditions of Theorem 1 are satisfied. From Theorem 1 it follows that all of the conclusions hold.

Now we give the graphical illustration of the iterative process to show the effectiveness of approximate solution converging to the exact solution.

Figure 1 shows that the convergence speed of iterative sequence is fairly fast, especially when \( n = 4 \), the iterative solution has almost approximated exact solution of Eq. (28). Tables 1 and 2 give a numerical approximation to exact solution, which also shows that iterative process is very effective and the iterative convergence speed is robust.
these difficulties, we introduce a new double monotone iterative technique and give some

\[
\begin{align*}
&n = 1 \quad 0.0085 \\ &n = 2 \quad 0.0585 \\ &n = 3 \quad 0.0205 \\ &n = 4 \quad 0.0273 \\ &n = 5 \quad 0.0263 \\ &n = 6 \quad 0.0263 \\ &n = 7 \quad 0.0263 \\ &n = 8 \quad 0.0263 \\ &n = 9 \quad 0.0263 \\ &n = 10 \quad 0.0263
\end{align*}
\]

Figure 1. The sequence series \( v_n(t) \) converges when \( n \geq 4 \).

Table 1. The numerical approximation of the solution for Eq. (28) in \([1.0859, 1.8591]\).

| \( n \) | \( t \) |
|---|---|
| 1.0859 | 1.1718 | 1.2577 | 1.3437 | 1.4296 | 1.5155 | 1.6014 | 1.6873 | 1.7732 | 1.8591 |
| \( n = 1 \) | 0.0085 | 0.0190 | 0.0297 | 0.0399 | 0.0495 | 0.0585 | 0.0657 | 0.0723 | 0.0781 | 0.0829 |
| \( n = 2 \) | 0.0585 | 0.1300 | 0.2023 | 0.2669 | 0.3277 | 0.3748 | 0.4191 | 0.4477 | 0.4750 | 0.4848 |
| \( n = 3 \) | 0.0205 | 0.0464 | 0.0731 | 0.0998 | 0.1249 | 0.1495 | 0.1714 | 0.1931 | 0.2112 | 0.2299 |
| \( n = 4 \) | 0.0273 | 0.0611 | 0.0956 | 0.1279 | 0.1583 | 0.1849 | 0.2092 | 0.2293 | 0.2474 | 0.2617 |
| \( n = 5 \) | 0.0263 | 0.0589 | 0.0924 | 0.1242 | 0.1542 | 0.1811 | 0.2054 | 0.2265 | 0.2451 | 0.2606 |
| \( n = 6 \) | 0.0263 | 0.0589 | 0.0923 | 0.1241 | 0.1539 | 0.1807 | 0.2049 | 0.2259 | 0.2443 | 0.2598 |
| \( n = 7 \) | 0.0263 | 0.0590 | 0.0924 | 0.1242 | 0.1541 | 0.1809 | 0.2051 | 0.2261 | 0.2445 | 0.2599 |
| \( n = 8 \) | 0.0263 | 0.0589 | 0.0924 | 0.1242 | 0.1541 | 0.1809 | 0.2051 | 0.2261 | 0.2445 | 0.2599 |
| \( n = 9 \) | 0.0263 | 0.0589 | 0.0924 | 0.1242 | 0.1541 | 0.1809 | 0.2051 | 0.2261 | 0.2445 | 0.2599 |
| \( n = 10 \) | 0.0263 | 0.0589 | 0.0924 | 0.1242 | 0.1541 | 0.1809 | 0.2051 | 0.2261 | 0.2445 | 0.2599 |

Table 2. The numerical approximation of the solution for Eq. (28) in \([1.9451, 2.7183]\).

| \( m \) | \( t \) |
|---|---|
| 1.9451 | 2.0311 | 2.1169 | 2.2028 | 2.2887 | 2.3746 | 2.4605 | 2.5465 | 2.6324 | 2.7183 |
| \( m = 1 \) | 0.0870 | 0.0904 | 0.0931 | 0.0952 | 0.0968 | 0.0980 | 0.0988 | 0.0993 | 0.0995 | 0.0996 |
| \( m = 2 \) | 0.4984 | 0.4923 | 0.4937 | 0.4738 | 0.4652 | 0.4324 | 0.4147 | 0.3663 | 0.3367 | 0.2465 |
| \( m = 3 \) | 0.2444 | 0.2603 | 0.2715 | 0.2848 | 0.2932 | 0.3041 | 0.3101 | 0.3187 | 0.3219 | 0.3247 |
| \( m = 4 \) | 0.2744 | 0.2838 | 0.2921 | 0.2978 | 0.3027 | 0.3059 | 0.3084 | 0.3102 | 0.3115 | 0.3136 |
| \( m = 5 \) | 0.2738 | 0.2845 | 0.2933 | 0.3052 | 0.3088 | 0.3114 | 0.3127 | 0.3134 | 0.3133 | 0.3133 |
| \( m = 6 \) | 0.273 | 0.2836 | 0.2924 | 0.2992 | 0.3045 | 0.3083 | 0.3109 | 0.3126 | 0.3134 | 0.3136 |
| \( m = 7 \) | 0.2731 | 0.2838 | 0.2925 | 0.2993 | 0.3045 | 0.3083 | 0.3109 | 0.3126 | 0.3134 | 0.3136 |
| \( m = 8 \) | 0.2731 | 0.2838 | 0.2925 | 0.2993 | 0.3045 | 0.3083 | 0.3109 | 0.3125 | 0.3134 | 0.3136 |
| \( m = 9 \) | 0.2731 | 0.2838 | 0.2925 | 0.2993 | 0.3045 | 0.3083 | 0.3109 | 0.3125 | 0.3134 | 0.3136 |
| \( m = 10 \) | 0.2731 | 0.2838 | 0.2925 | 0.2993 | 0.3045 | 0.3083 | 0.3109 | 0.3125 | 0.3134 | 0.3136 |

5 Conclusion

In this paper, we study the uniqueness and iterative properties of solutions for a general Hadamard-type singular fractional turbulent flow model involving a nonlinear operator. The singularities and nonlinear operator lead to some difficulties of study. To overcome these difficulties, we introduce a new double monotone iterative technique and give some useful estimations. Then we establish some new results of positive solutions including the uniqueness, the iterative sequence converging to the unique solution, error estimates, convergence rate and entire asymptotic behavior. These properties can describe nicely the dynamic behaviour of the turbulent flow. The example also indicates that the conditions of theorem are easy to be verified, the graphic and numerical approximation show that the convergence speed of iterative sequence is robust and effective.
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