Constraints on Cosmographic Functions of Cosmic Chronometers Data Using Gaussian Processes

A. M. Velasquez-Toribio¹ · Júlio C. Fabris¹,²

Received: 15 October 2021 / Accepted: 11 April 2022 / Published online: 3 May 2022
© The Author(s) under exclusive licence to Sociedade Brasileira de Física 2022

Abstract
We study observational constraints on the cosmographic functions up to the fourth derivative of the scale factor with respect to cosmic time, i.e., the so-called snap function, using the nonparametric method of Gaussian processes. As observational data, we use the Hubble parameter data. Also we use mock data sets to estimate the future forecast and study the performance of this type of data to constrain cosmographic functions. The combination between a nonparametric method and the Hubble parameter data is investigated as a strategy to reconstruct cosmographic functions. In addition, our results are quite general because they are not restricted to a specific type of functional dependency of the Hubble parameter. We investigate some advantages of using cosmographic functions instead of cosmographic series, since the former are general definitions free of approximations. In general, our results do not deviate significantly from ΛCDM. We determine a transition redshift $z_{tr} = 0.637^{+0.165}_{-0.175}$ and $H_0 = 69.45 \pm 4.34$. Also assuming priors for the Hubble constant, we obtain $z_{tr} = 0.670^{+0.121}_{-0.120}$ with $H_0 = 67.44$ (Planck) and $z_{tr} = 0.710^{+0.159}_{-0.111}$ with $H_0 = 74.03$ (SH0ES). Our main results are summarized in Table 2.

Keywords Cosmography function · Jerk · Snap · Gaussian processes

1 Introduction

The accelerated expansion of the universe [1, 2] is currently one of the main problems of cosmology. The simplest explanation is given by the ΛCDM model making use of cold dark matter and a cosmological constant which matches well with different observational data types: supernovae Ia, cosmic microwave background (CMB), baryons acoustic oscillations (BAO), etc., see, e.g., [3–5].

The ΛCDM model includes a component of dark energy which is responsible for the accelerating expansion of the universe. However, from a theoretical point of view, one of the big problems that affect this model is the difference between the measured value of the cosmological constant and that required by quantum theory (depending on the energy scale), which can reach a part in $10^{120}$. This gives rise to the cosmological constant fine-tuning problem. Other theoretical problem is the cosmological coincidence problem, i.e., why does the universe start to accelerate when the structures reach the nonlinear regime? Or why the present values of the densities of dark energy and dark matter are of the same order of magnitude? For a review of these and another theoretical problems on the ΛCDM model, see [6–8].

On the other hand, there are many theoretical models that try to explain the accelerating expansion of the universe, for example, among the most popular we have: the scalar field models (quintessence) [9], k-essence models [10], models including dissipation [11], models inspired by the renormalization group [12], modified gravity models and the Horndeski’s model that have the most general scalar–tensor action yielding up to second-order equations of motion ([13]) and so forth. For a review, see [14].

In addition, another way to understand the accelerated expansion is the phenomenological approach, which consists in assuming a specific functional form for the equation of state and studying its implications. The prototype of this approach is the parameterization CPL [15, 16]. Other examples are: the $q(z)$ parametrization [17], the Wetterich parameterizations [18], among others, e.g., [19].
However, all the mentioned models to explain the accelerated expansion are dependent on the theory of gravitation. In this regard, there is an approach that avoids it by assuming only the metric structure, without considering the theory of gravitation and is called the model-independent approach. We assumed this approach in our investigation. It is also common to call this approach as the cosmographic approach and derive it from a Taylor series of the scale factor or, equivalently, of the Hubble parameter, e.g., see references [20–24].

It is worth noting that the Taylor series expansion implies fundamental difficulties with the convergence and the truncation of the series. Thus, we cannot use observational data for redshift greater than one, \( z > 1 \). For a detailed discussion of this question, see reference [25]. Consequently, in the present paper, to avoid this problem we calculate the cosmographic functions instead of the cosmographic parameters derived from the Taylor series. The cosmographic functions are assumed as kinematic definitions on the scale factor or, equivalently, the Hubble parameter. In this case, the cosmographic parameters correspond to the cosmographic functions with \( z = 0 \), i.e., currently evaluated.

In addition, a statistical method complementary to the model-independent approach is the method of Gaussian processes (GP), which is a nonparametric statistical method. This means that the reconstruction of the Hubble function and its derivatives are performed without assuming a specific theoretical model. Also as observational data, we used the Hubble parameter data, since from the theoretical point of view the cosmographic functions are directly dependent on the Hubble parameter and its derivatives. On the other hand, a reconstruction that uses, for example, measurements of the luminosity distance of supernovae Ia involves the computation of the derivative of the supernovae Ia distance data to determine the Hubble parameter and its derivatives. Thus, this process adds a source of error propagation. This is the main reason because we only use Hubble parameter data in our research.

Our main objective is to determine observational constraints in model-independent form using a nonparametric statistical technique, such as Gaussian processes and observational data from the Hubble parameter. In this way, we avoid using the Taylor series expansion. We also discuss how to use these results to derive observational constraints on specific models by constructing a simple correspondence between the parameters of a given model and the cosmographic parameters. We discuss the implications of this methodology.

The present paper is organized as follows. We present the basic definitions in Sect. 2. The observational data and the simulated data are discussed in Sect. 3. We summarize the method of Gaussian processes in Sect. 4. We present the results of the reconstruction of our cosmographic functions in Sect. 5 and in Sect. 6 we give our conclusions.

### 2 Cosmographic Functions

We assume that our universe is homogenous and isotropic and therefore can be described by the Friedmann–Lemaître–Robertson–Walker (FLRW) metric,

\[
ds^2 = -c^2 dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right],
\]

where \( a(t) \) is the scale factor and the \( k \) is a constant that can have the value = +1, 0, −1 and represent the three-dimensional curvature. Using this metric, it is possible to introduce the cosmographic functions defined directly from the scale factor,

\[
\begin{align*}
H(t) &= \frac{1}{a} \frac{da}{dt}, \\
q(t) &= -\frac{1}{a} \frac{d^2 a}{dt^2} \frac{1}{H^2}, \\
j(t) &= \frac{1}{a} \frac{d^3 a}{dt^3} \frac{1}{H^3}, \\
s(t) &= \frac{1}{a} \frac{d^4 a}{dt^4} \frac{1}{H^4}, \\
&\quad \cdots = \cdots \\
m_n &= \frac{1}{a} \frac{d^n a}{dt^n} \frac{1}{H^n}.
\end{align*}
\]

If we evaluate these functions at the current time (\( z = 0 \)), then they are called the cosmographic parameters and are usually referred as the Hubble constant \((H_0)\), the deceleration parameter \((q_0)\), the jerk parameter \((j_0)\), the snap parameter \((s_0)\), etc.\(^1\)

In the literature, the traditional way to introduce cosmographic parameters is by using the Taylor expansion for the Hubble parameter see, for example, [20, 22]. However, the use of observational data and the Taylor series leads to two main inconsistencies. The first arises when we use observational data with \( z > 1 \), since from the mathematical point of view the series has a convergence radius at the most \( |z| = 1 \), thus it is only justified to use data with redshift \( z << 1 \). Otherwise the errors would increase and the series would not be useful to represent the original function. The other inconsistency arises when a new redshift variable is introduced to transform the domain of redshift values and to be able to use data with \( z > 1 \), the problem is that there is no one-way to do this. We briefly review these two inconsistencies:

\(^1\) Curiously in the literature, there are proposals of names for the fifth and sixth derivatives as crackle and pop ([26]).
Taylor series and their error in cosmological models: the Hubble’s function can be expressed in Taylor series as a function of the redshift using the relation $1 + z = \frac{a_t}{a_0}$ [20],

$$H(z) = H_0 + \frac{dH}{dz} \bigg|_0 \left( z + \frac{1}{2!} \frac{d^2H}{dz^2} \right) z^2 + \frac{1}{3!} \frac{d^3H}{dz^3} \right) z^3 + O(z^4).$$

(3)

The series is written around the current redshift, $z = 0$, and mathematically represents the approximate Hubble’s law for an arbitrary order.\(^2\) In this context, we consider the truncation error. To briefly illustrate this question we have used three cosmological models with Hubble’s law exact, so that we can easily determine its Taylor series. We consider the flat $ΛCDM$ model, the CPL model and a $q(z)$ parametrization. In all case, we consider the flat universe. Thus, for the case of the flat cosmological constant model the Hubble parameter can be written,

$$H(z) = H_0 [\Omega_m(1 + z)^3 + (1 - \Omega_m)]^{1/2},$$

(4)

where $\Omega_m$ is the matter density parameter. On the other hand, for the CPL (Chevallier–Polarski–Linder) parametrization [15, 16] we have,

$$H(z) = H_0[\Omega_m(1 + z)^3 + (1 - \Omega_m)(1 + z)^{3w_0+w_1}e^{3w_0(z-1)^{1/2}},$$

(5)

where $w_0$ and $w_1$ are the parameters of the model and although it does not behave very well for high redshift it allows reproduce the Hubble law and the distances of different models of dark energy with very good precision and in the literature it has been intensively used, see for example [29, 30] and references therein. This model represents dynamic dark energy. We are also consider another model of dynamic dark energy with four free parameters [17] for which have

$$H(z) = H_0 \left[ (1 + z)^{(1+q_t)} \left( \frac{q_t}{(1+q_t)} \right)^{1/r} - q_t \left( \frac{1+q_t}{1+z} \right)^{2(1+q_t)z} - q_t \right]^r(q_t-q_i)^s,.$$

(6)

where $q_t$ and $q_f$ are the initial and final deceleration parameter, respectively, and $z_t$ is the transition redshift from a decelerated universe to an accelerated universe and $r$ is the width of the transition. This model is quite generic and allows to study in greater detail the transition for an accelerated universe. It is also interesting to know that this parametrization includes several other models of dark energy, for details and observational constraints of the parameters see [31]. All these models have an analytical expression for the Hubble parameter and have been constrained observationally. We calculate the Taylor series for each of these models up to the third power in the redshift variable. In Fig. 1, we show the exact Hubble function versus the third-order approximation. We can observe that up around $z = 0.5$ the values between the exact function and the third-order approximation are more or less equivalent. To quantify explicitly this difference, we plot the magnitude of the error defined as:

$$Error = \frac{H(z)_{\text{exact}} - H(z)_{\text{approximate}}}{H(z)_{\text{exact}}}. $$

(7)

Also in Fig. 1, right column, we can see that the error between the exact function and the third-order function for the range of redshift, $0 < z < 1$, is relatively small, but for higher values of the redshift the error increases considerably. Therefore, the use of the Taylor series to compute observational constraints for data with $z > 1$ leads to considerable truncation errors. One way to circumvent this problem could be to select observational data with $z < 1$, this would give consistent results within the truncation error of the Taylor series.

• New redshift variable: a widely used proposal in the literature on the truncation problem is to use a new variable for the redshift. The initial and best motivated proposal is given by reference [25] and establishes the following relationship

$$y_1 = \frac{z}{1 + z},$$

(8)

where $y_1$ is the new redshift variable. This new redshift variable allows mapping the redshift values with $z > 1$ to a region with $y_1 < 1$ and can be defined as the change of the emission wavelength divided by the observed wavelength,

$$1 - y_1 = \frac{\lambda_e}{\lambda_0} = \frac{1}{\bar{z}}.$$

(9)

The correspondence between the values of redshift $z$ and the values of the variable $y_1$ are: in the past $z \in (0, \infty)$ and corresponding the values $y_1 \in (0, 1)$; in the future $z \in (-1, 0)$ and corresponding to the values $y_1(\infty, 0)$. However, this redshift variable transform is not unique in the literature there are other proposals such as [32],

$$y_2 = \arctan(z),$$

(10)

$$y_3 = \arctan\left(\frac{z}{z + 1}\right),$$

(11)

\(^2\) On the generality of the equation above, it is important to mention that for gravitation theories of high derivative one cannot reconstruct his equation of state without making use of an external hypothesis, because these theories include new degrees of freedom. For details, see references [27] and [28].
In the top left, the Hubble function for $\Lambda$CDM, in the middle left for the CPL model and in the bottom left for the $q(z)$ model and in the right side the quantification of the error, respectively, for each model. In all case, we use $\Omega_m = 0.32$ and we consider two values for the today Hubble parameter: the highest value corresponds to the SH0ES Collaboration and the lowest value corresponds to the Planck Collaboration.

$$y_i = \frac{z}{1 + z^2}. \quad (12)$$

It is important to note that the new redshift variable ($y_i$) maintains the model independence of the cosmographic approximation. However, there is no a priori reason to consider the variable $y_i$ better justified than the variable $y_j$ for example. In this regard, it is interesting to note that a possible strategy to justify the use of a variable $y$ with respect to another variable $y$ can be through the use of statistical selection criteria. For example, we can assume LCDM as reference (this means for a given set of observational data estimate the cosmographic parameters ($q_0, j_0, s_0, ...$) and use these values as base for comparison) and then estimate the same cosmographic parameters, but using the different variables $y_1, y_2, y_3$, etc. and then we determine the selection criteria AIC, BIC and the Bayesian factor. This would be a possibility to catalog the different functions $y_i$. More in general, we should expect that for a given redshift variable the estimated cosmographic parameters will be different depending on the new redshift variable used. But this is beyond the scope of our work and we simply mention this inconsistency in the use of Taylor Series. However, it is instructive to consider the expressions for the luminosity distances in the variables $z$ and $y_i$ [20, 25],

$$d_L(z) = \frac{c}{H_0} z[1 + \frac{1}{2}(1 - q_0)z + \frac{1}{6}(1 - q_0 - 3q_0^2 + j_0)z^2 + O(z^3)], \quad (13)$$

$$d_L(y_i) = \frac{c}{H_0} y_i[1 + \frac{1}{2}(3 - q_0)y_i + \frac{1}{6}(11 - 5q_0 - j_0 + q_0^2)y_i^2 + O(y_i^3)]. \quad (14)$$

$$d_L(y_2) = \frac{c}{H_0} y_2[1 + \frac{1}{2}(1 - q_0)y_2 + \frac{1}{6}(1 - q_0 - 3q_0^2 + j_0)y_2^2 + O(y_2^3)]. \quad (15)$$

$$d_L(y_3) = \frac{c}{H_0} y_3[1 + \frac{1}{2}(3 - q_0)y_3 + \frac{1}{6}(13 - j_0 - 5q_0^2 + q_0^2)y_3^2 + O(y_3^3)]. \quad (16)$$

$$d_L(y_4) = \frac{c}{H_0} y_4[1 + \frac{1}{2}(1 - q_0)y_4 + \frac{1}{6}(5 - j_0 + 3q_0^2 + q_0^2)y_4^2 + O(y_4^3)]. \quad (17)$$

A simple way to see that different redshift variables will determine different estimates of the cosmographic parameters results from comparing the modulus distance, $\mu(z)$, used as the observable to determine observational constraints of SNIa data, defined as:

$$\mu = 5 \log_{10}(d_L) + 25. \quad (18)$$

This simple comparison is presented in Fig. 2 for all luminosity distance definitions. We fix the values of $q_0 = -0.550$ and $j_0 = 1.100$ for all cases. As we can see, all the redshift variables coincide for values of $z < 0.2$, but for higher values, a divergence between the curves can be observed. Therefore, the estimated results on the cosmographic parameters: ($q_0, j_0, s_0, ...$) are expected to be different. For example, in reference [33] the authors use data from Supernovae Ia, the Hubble parameter and the shift parameter of the CMB and estimate values for the cosmographic parameters for different Taylor series. Their results, shown in Tables 1 and 2 of reference [33], reveal that the final parameter estimations can be quite different.

Therefore, to avoid the problems mentioned about the Taylor series we propose to use directly the definitions of cosmographic functions as given above. Thus, for our analysis it is convenient to rewrite the cosmographic functions explicitly as a function of the Hubble parameter and its derivatives with respect to the redshift,

$$q(z) = (1 + z) \frac{H'}{H} - 1 \quad (19)$$

$$j(z) = \frac{H''}{H}(1 + z)^2 + \left(\frac{H'}{H}\right)^2 (1 + z)^2 - 2 \frac{H'}{H} (1 + z) + 1 \quad (20)$$

$$s(z) = -\frac{1}{2} (1 + z)^2 \left[\frac{H(z)}{H(z)}\right]'' + \frac{1}{2} (1 + z)^2 \left[\frac{H(z)}{H(z)}\right]'' \frac{H(z)}{H(z)} + \frac{1}{2} \frac{H(z)}{H(z)} \frac{1}{H(z)} + 1, \quad (21)$$

where the prime represents derivative with respect to redshift. Furthermore, using the snap parameter definition we have $\ddot{a} = saH^4$, from the jerk parameter we have $\dddot{a} = jaH^3$ and of the deceleration parameter we have $\dddot{a} = -qaH^2$, thus we can write a useful relationship between these cosmographic functions given by the expression [34],

$$s(z) = -(1 + z) \frac{d(j(z))}{dz} - j(z)(2 + 3q(z)). \quad (22)$$

It is also useful to write the cosmographic parameters for the flat $\Lambda$CDM model,

$$q_0 = -1 + \frac{3}{2} \Omega_m^0 \quad (23)$$

$$j_0 = 1 \quad (24)$$
As we can see for the flat ΛCDM model these cosmographic parameters are defined by the value of the matter parameter today. These results are useful to make a comparison with our results. We use the set of Eqs. (16)–(18) to reconstruct the cosmographic functions, i.e., \( q(z) \), \( j(z) \) and \( s(z) \) using observational data from the Hubble parameter and simulated data.

\[
s_0 = 1 - \frac{9}{2} \Omega_m. \tag{25}\]

As we can see for the flat ΛCDM model these cosmographic parameters are defined by the value of the matter parameter today. These results are useful to make a comparison with our results. We use the set of Eqs. (16)–(18) to reconstruct the cosmographic functions, i.e., \( q(z) \), \( j(z) \) and \( s(z) \) using observational data from the Hubble parameter and simulated data.

### 3 Data

As we have observed in the previous section, the cosmographic functions can be written directly as a function of the Hubble parameter and its derivatives. This allows us to discern that the most appropriate data to minimize error propagation are the Hubble data. Therefore, we focus on this class of data.

#### 3.1 Observational Data

There are two efficient and widely used forms to obtain Hubble parameter measurements.

##### 3.1.1 Cosmic Chronometers (CC)

This method is based on the expression of the differential age of the universe as a function of redshift,

\[
H(z) = -\frac{1}{1+z} \frac{dz}{dt}. \tag{26}\]

This method was proposed by [35] and consists in directly measuring the amount \( dz/dt \) and, consequently, the Hubble parameter. The most used data to measure this amount have been passively evolving galaxies with high-resolution spectroscopic data along with synthetic catalogs to limit the age of the oldest stars in the galaxy. A complete description of this methodology can be reviewed for the SDSS and GDS data in reference [36].

##### 3.1.2 The Radial BAO Size Method

The method is based on measurements of the acoustic scale of BAO, which is more accurate with respect at the CC method [37]. This accuracy is understandable because BAO mainly depends on a spatial measurement compared to the first method where a time measurement is required which increases systematic errors. However, this method of BAO requires assuming a prior in the radius of the sound horizon. Hence, we have

\[
H(z) = -\frac{r_{bao}}{r_{emb}} H_{fiducial}. \tag{27}\]

This method depends on the fiducial model which is assumed to be the flat ΛCDM model.

On the other hand, in the literature there are different compilations of samples of the Hubble parameters data. We use the sample presented by [38] what includes data of both CC and BAO, but we are not including the points \( z = 0.4497 \) and \( z = 2.34 \). The first point is excluded because it overlaps with another data and has a negligible
effect on the results. The other point has a strong influence on the estimation of parameters and because it is very restrictive we have excluded it. Therefore our sample has 34 points. This sample has been used in [39] showing results compatible with the literature.

In Fig. 3, we show the data used. We can see that the errors of the CC data can reach up to a value of \( \Delta H \approx 62Km \, s^{-1} \, Mpc^{-1} \) and the errors of the BAO data have a mean of about \( 7Km \, s^{-1} \, Mpc^{-1} \).

Before using these data, it is important to verify their Gaussian nature [40]. To verify this, we determine the factor \( N \) which must be small to be consistent with the Gaussianity hypothesis. Briefly, the methodology used is shown in Appendix. The results are presented in Table 1 and we can see that the data are compatible with the Gaussianity hypothesis. Therefore, we have a suitable sample to use the Gaussian process method.

3.2 Simulation Data

Our simulation follows the prescriptions established in [41]: we choose as background cosmology the flat \( \Lambda CDM \) model using the values \( \Omega_m = 0.315 \pm 0.001 \) of the [4] and \( H_0 = 69.45 \pm 4.34 \) (best fit of \( H(z) \) observational data). Then, with these values we determine the \( H_{fiducial} \) using equation (5). We assume that the deviation of the simulated value with respect to fiducial value is calculated as: \( \Delta H = H_{sim} - H_{fiducial} \), where the \( \Delta H \) can be derived from a Gaussian distribution of the form \( N(0, \tilde{\sigma}) \), where the \( \tilde{\sigma} \) is a random number which also can be drawn from the following Gaussian distribution \( N(\sigma_0, \epsilon(z)) \). In this Gaussian distribution, we introduce the information about the error as follows: we define the \( \epsilon(z) \) as \( \epsilon(z) = \frac{\sigma_+ - \sigma_-}{4} \) and the parameter \( \tilde{\sigma} \) is chosen so that the probability of \( \tilde{\sigma} \) falling within the strip of 95.4%. We also define the \( \sigma_0 = (\sigma_+ + \sigma_-)/2 \). We assume a linear function for the \( \sigma \) that allows us to model the simulated errors. This can be deduced from Fig. 3 where we can see that, except for some special cases, the error increases with the redshift. Thus, in general we define \( \sigma_\pm = a_\pm + b_\pm \) where \( a \) and \( b \) are chosen according to the errors of the Hubble data and the + sign indicates the upper line and the – sign the lower line as shown in the lower part of Fig. 3. For example, for the case of CC we assume the straight lines (orange in Fig. 3): \( \sigma_+ = 19.999z + 11.105 \) and \( \sigma_- = 3.650z + 1.65 \) and in the case of BAO the blue lines. We simulated 100 simulated data using the CC prescription and 100 data using the BAO prescription.

4 Gaussian Processes

The Gaussian processes (GP) are a nonparametric statistical method allowing the reconstruction of a given function directly from the manipulation of data. They can be thought as an extension to a multivariate normal distribution, that is, a GP is an infinity collection of random variables, but where every finite set have a normal distribution. A GP is a distribution over functions, but in the practice a GP allows us to derive a posterior distributions by simply considering a finite set of points that are associated with the observational data. For details, see [42]. A GP can be written as:

\[
 f(x) \sim GP(\mu(x), k(x, \tilde{x})),
\]

where the value of \( f \) when evaluated at a point \( x \) is a Gaussian random variable with mean \( \mu(x) \). Additionally, the value of the function \( f \) at the point \( x \) is not independent of the value of the function \( f \) at some other point nearby \( \tilde{x} \), but is related by the covariance function \( k(x, \tilde{x}) \). Considering observational data \( (x_i, y_i) \), assuming that the errors are Gaussian and that \( y_i = f(x_i) + \sigma_i \) where \( \sigma_i \) are the 1\( \sigma \) error bars and \( i = 1, ..., N \), we can to reconstruct the function \( f \) at chosen points. This
function is denoted by \( f^* \). This reconstruction can be done through the joint distribution between \( f^* \) and \( y_i \). For details of the exact expressions, see reference [46]. In general, this reconstructed function has a reconstructed mean given by,

\[
\tilde{f}^*(x) = \sum_{i,j=1}^{N} k(x, x_i)(M^{-1})_{ij}(f(x_i) - \mu(z))
\]

(29)

in our reconstruction we chosen as prior mean function \( \mu(z) \) a constant value and for the standard deviation:

\[
\sigma(x) = k(x, x) - \sum_{i,j=1}^{N} k(x, x_i)(M^{-1})_{ij}k(x_j, x)
\]

(30)

where \( M_{ij} = k(x_i, x_j) + c_{ij} \) and \( c_{ij} \) represents the covariance of the input data.

For our calculations, we assume the exponential function as a covariance function which is given by

\[
k(x, \bar{x}) = \sigma_f^2 \exp \left( -\frac{(x - \bar{x})^2}{2\sigma^2} \right),
\]

(31)

where \( \sigma_f \) and \( \sigma \) are called hyperparameters which are determined by maximizing the log marginal likelihood, see [42],

\[
\ln L = -\frac{1}{2} \sum_i \left[ f(x_i) - \mu(x_i) \right] (M^{-1})_{ij} \left[ f(x_j) - \mu(x_j) \right] \\
-\frac{1}{2} \ln |M| - \frac{1}{2} N \ln 2\pi,
\]

(32)

where \( M \) represents the determinant of \( M_{ij} \). It is important to note that instead of optimizing these hyperparameters, they can be marginalized by using, for example, the MCMC method. However, a simple method of testing the validity of the optimization process is to vary the initial value of the hyperparameters and see if the values obtained change significantly. We have done this process and we determine that our hyperparameters do not change significantly. Therefore, we are confident in using the optimization process. On the other hand, the MCMC method should be used to obtain high precision restrictions associated with high quality data.

The question of how to choose a suitable covariance function is an important problem for GP. In the present investigation, we show some evidence to consider the Gaussian nature of the data, see Table 1. However, we consider that this evidence is not conclusive. Some papers as [43, 44] have used other covariance functions to estimate \( H_0 \) and have shown that within 1\( \sigma \) different choices are consistent. In the paper [45] the authors use three different covariance functions (exponential, Cauchy and Matérn) and also use a database for the Hubble parameter similar to our data. They find that within 1\( \sigma \) the three covariance functions lead to consistent results. Therefore, in the present paper we only use the squared exponential covariance function and we leave the specific study of the effects of choosing different covariance functions for further investigation.

Additionally, the GP allows to reconstruct the derivative of the data. To implement this method, we use the public package GaPP [46]. For applications of GP in cosmology consider references [47, 48] and for another GP methodology to see [49–52].

## 5 Results

In Fig. 4, we present the reconstruction of the Hubble parameter using the GaPP code. In the top, we present the model-independent reconstruction of \( H(z) \), where we have used the \( CC + BAO \) data to determine the Hubble constant \( (H_0 = 69.45 \pm 4.34) \). In the other two figures, we reconstructed \( H(z) \) using as prior for the Hubble constant the values of the Planck and SH0ES Collaborations. In all cases, the reconstruction is done with 1\( \sigma \) of uncertainty. In Fig. 5, we can observe the reconstruction of the cosmographic functions \( q(z) \), \( j(z) \) and \( s(z) \) without using a prior value for the Hubble constant. In particular the deceleration parameter shows that the transition redshift, \( z_{tr} \), from a decelerated universe to an accelerated universe, defined as \( q(z_{tr}) = 0 \), is into the region \( z < 1 \) with 2\( \sigma \).

We obtain the values; \( z_{tr} = 0.637^{+0.105}_{-0.175} \) (model-independent) and \( z_{tr} = 0.670^{+0.210}_{-0.120} \) with \( H_0 = 67.44 \) (Planck) and \( z_{tr} = 0.710^{+0.159}_{-0.111} \) with \( H_0 = 74.03 \) (SH0ES). In accordance with these results, we can conclude that \( z_{tr} \in [0.550, 0.870] \). These values for, \( z_{tr} \), are in accordance with other results determined in the literature, for example, see [53–58].

On the other hand, for values \( z > 1 \) the function \( q(z) \) shows a trend to remain in the region \( q > 0 \) associated with a phase dominated by matter. However, for high redshift values we see an abrupt drop which can be associated with the few observational data for this region (see Fig. 3).

In Fig. 6, we show the estimated \( H_0 \) value and compare our value with the different estimates published in the literature using other cosmological data and methods. We can observe that due to the high dispersion of the data from the Hubble parameter, our estimate has a considerable error, but it is in agreement with recent measurements.

In Fig. 7, we can see that the reconstructed function \( j(z) \) does not exclude the flat \( \Lambda CDM \) model with 2\( \sigma \), however the average value is different from the flat \( \Lambda CDM \) model for all the reconstruction. The figure shows a trend for \( j < 0 \) when \( z > 1 \), this is due to the propagation of errors for high redshift of the derivatives of the Hubble parameter. For the case of the Hubble constant of the Planck collaboration the value of \( j_0 \) is close to the value of the flat \( \Lambda CDM \) model.
In the top, the reconstruction of the Hubble parameter using observational data of Hubble parameter in a model-independent form. In the middle, the reconstruction of the Hubble parameter with a prior $H_0 = 74.03$ (SH0ES collaboration). In the bottom, the reconstruction using a prior $H_0 = 67.44$ (Planck’s collaboration). The red dashed line represents the Taylor series for the Hubble parameter up to the third order (includes $q_0$ and $f_0$). We can observe significant deviations for $z > 0.5$ in all cases. We used the values from Table 2. It is important to note that the Taylor series is only feasible for $z < 1$, however, in the figure we show the region with $z > 1$ only to illustrate the incompatibility with the data.
With respect to the snap parameter if we consider the value of the matter density parameter measured by the Planck collaboration, $\Omega_{m0} = 0.315$, then for the flat $\Lambda CDM$ model implies a value of $s_0 = -0.4175$. Our results with 1$\sigma$ are marginally in agreement with this value (see values in Table 2). Secondly when $z > 1$ the error increases remarkably because reconstructing the parameter $s(z)$ implies reconstructing third-order derivatives for the Hubble parameter.

In Fig. 8, we show the same results as Fig. 7 but for simulated data. On the left side, we use simulated data with the error prescription from the CC technique and on the right side we use the simulated data compatible with the error from the BAO technique. In both cases, we observe that the reconstructed functions behave similarly to the observational data, but with less propagation of errors.

6 Discussion and Conclusions

In this paper, we presented observational constraints on some of the cosmographic functions, specially the deceleration parameter, the jerk parameter and the snap parameter using a model-independent approximation. The statistical method used has been the Gaussian processes through the GaPP public package. The data used for this study are observational and simulated data from the Hubble parameter. We directly estimate the value of the Hubble constant from the data used, $H_0 = 69.45 \pm 4.34$, and use this value to perform our simulations. However, in order to observe the effect of priors values of the Hubble constant on the functions $q(z)$, $j(z)$ and $s(z)$ we use the values from the Planck collaboration and from the SH0ES collaboration [59].

On the other hand, even though it is not our objective to discuss the Hubble tension, it is interesting to note that our result marginally includes both the Planck result and the SH0ES result, see Fig. 6. However, this is primarily due to the large dispersion of the CC measures. But what is interesting is to observe that following the same methodology presented here and using data from future observational projects such as the LSST the CC measurements can become excellent discriminators for the Hubble tension problem.

We propose to use the cosmographic functions given by equations (15-17) instead of the cosmographic parameters derived from Taylor’s expansion commonly used in the literature. By using cosmographic functions, we avoid the Taylor’s expansion truncation problem and consequently we can use data with $z > 1$ without restrictions. Also if we use cosmographic functions, it is not necessary for a new redshift variable which can lead to inconsistent results.

In general, we can observe that the current observational data of the Hubble parameter do not allow to exclude the model $\Lambda CDM$ which still fits the reconstruction of the cosmographic functions. On the other hand, the simulated data, in particular, the Hubble parameter data obtained using the BAO technique seems promising to distinguish between alternative models to the $\Lambda CDM$ model. However, data from the Hubble parameter derived from the BAO technique have the disadvantage of being dependent on the fiducial model, but a compromise between the choice of different fiducial models can cover a wide spectrum of data allowing to determine robust constraints.

Fig. 5 Observational constraints on the cosmography functions using the observational data of the Hubble parameter in a model-independent form
An interesting application of our reconstruction is the study of specific dark energy models by writing the parameters of a given dark energy model as a function of the cosmographic parameters.

To implement this methodology, we can proceed as follows: given a specific Hubble function we can use our system of equations (15-17) to determine a system of algebraic equations between the cosmographic parameters \((q_0, j_0, s_0)\) and the parameters of a given model \(\theta_i\). Then, we can solve the model parameters as a function of the cosmographic parameters: \(\theta_i = \theta_i(q_0, j_0, s_0)\).

As an instructive example in the present work, we are going to apply this methodology to a two-parameter cosmological model, specifically the \(\text{CPL}\) model. In this case, the Hubble function is given by equation (5) so that the parameters of the model are \(\theta_i = (w_0, w_1, \Omega_{m0})\), where initially we assume the parameter \(\Omega_{m0}\) as known. This means that to determine the parameters \((w_0, w_1)\) we only need equations (15) and (16). Thus, we get an algebraic system of equations that provide us with the following solutions:

---

![Observational constraints on the Hubble constant, \(H_0\), where we show our result in comparison with the main results published in the literature, for example, compiled in [65]. The orange vertical band corresponds to the SH0ES team and the blue vertical band corresponds to the Planck Collaboration. The figure was made using the public program available online at github.com/lucavisinelli/H0TensionRealm. [65]](image-url)
Fig. 7 Observational constraints on the cosmography functions using the observational data of Hubble parameter for the two values of the Hubble constant. In left side for $H_0 = 74.03$ and in right side for $H_0 = 67.44$.
Now we can use this equation to derive the value of \( \Omega_{m0} \) directly from the cosmographic parameters and then we can use this value to determine the parameters \( w_0, w_1 \) using equations (29-30). If we do this we get: \( w_0 = -1.132, w_1 = -0.461 \).

Additionally, if we consider the equation for the slope parameter, equation (17), we can also determine the parameter \( \Omega_{m0} \) as a function explicitly of the cosmographic parameters after some manipulations we obtain,

\[
\Omega_{m0} = \frac{-1 + 9 j_0 + 6 q_0 + 10 j_0 q_0 - 10 q_0^2 + 2 q_0^3}{2(-2 q_0 + (3 + q_0)(2 j_0 + q_0^2) + s_0)}
\]

\[
+ \frac{2(-2 q_0 + (3 + q_0)(2 j_0 + q_0^2) + s_0)}{1 - 2 q_0}
\]

\[
+ \frac{2(-2 q_0 + (3 + q_0)(2 j_0 + q_0^2) + s_0)}{1 - 2 q_0 + (3 + q_0)(2 j_0 + q_0^2) + s_0}.
\]

Now we can use this equation to determine the value of \( \Omega_{m0} \) directly from the cosmographic parameters and then we can use this value to determine the parameters \( w_0, w_1 \) using equations (29-30). If we do this we get: \( w_0 = -1.036, w_1 = -0.119, \Omega_{m0} = 0.247 \).

In principle, we can use this methodology for other cosmological models. If we add more cosmographic functions, then we can include models with more free parameters. In a subsequent investigation, we intend to extend this methodology to other models of dynamic dark energy. Therefore, it is important to note that in this way we can infer the values of the free parameters of a given model using the cosmographic approximation without using the Taylor expansion.

It is important to mention that recently reference [60] has proposed a similar method but in another context. They determine the cosmographic parameters as function of the parameters of different dark energy models and use observational data to estimate the values of the parameters of dark energy and, with these results, the cosmographic parameters. Also recently in the paper [61], the authors argue that the reconstruction of \( j_0 \) using the GP method is inconsistent with respect to the \( \Lambda \)CDM model using data from H(z) for low redshift and high redshift. On the other hand, this inconsistency is with respect to the LCDM cosmology. We reconstruct the functions but do not assume an LCDM model for comparison. Also our Hubble parameter data samples are relatively different with respect to reference [61]. This may perhaps explain why the authors estimate values of \( H_0 > 70 \). It is important to note that the number and quality of Hubble data can give noticeable effects on observational constraints.

As we mentioned, our methodology is very useful to avoid using the Taylor expansion, but in compensation it depends on the data of the Hubble parameter, which are still a small sample when compared to supernovae Ia, for example [62–64]. Furthermore, we need to reconstruct derivatives of observational data. The fact that a small noise in the measurement data can cause a large error in the derivatives is a difficult problem to analyze. For this reason, we hope to investigate in a later work the use of different statistical methods to reconstruct the derivatives as well as to discuss the importance of data quality.

### Appendix: The Gaussianity of the data

An important test of Gaussianity is to determine the parameter \( \chi^2 \) as shown in Table 1. Then, the weighted mean for the Hubble parameter is given by [40],

\[
\bar{H} = \frac{\sum_{i=1}^{N} H_i(z_i)/\sigma_i^2}{\sum_{i=1}^{N} 1/\sigma_i^2}.
\]

We also compute the weighted bin redshift, it is given by,

\[
\bar{z} = \frac{\sum_{i=1}^{N} z_i/\sigma_i^2}{\sum_{i=1}^{N} 1/\sigma_i^2}.
\]

The associated weighted error is given by
Fig. 8 In the column on the left, we used the simulation that uses the CC prescription and in the right column we have used the BAO prescription. In both cases, we have used the value of the Hubble parameter estimated directly from the observational data. The value used is in Table 2.
\[ \bar{\sigma} = \left( \sum_{i=1}^{N} 1/\sigma_i^2 \right)^{-1/2}. \]  
\[ \chi^2 = \frac{1}{N-1} \sum \frac{H - \bar{H}}{\sigma_i^2}. \]

And thus the number of standard deviations with respect from unity is defined as

\[ N_s = |\chi - 1| \sqrt{2(N-1)}. \]

Acknowledgements  A.M.V.T. acknowledges the computational facilities of the UNESP to develop the present investigation and also the authors thank Célia Escamilla-Rivera for helpful discussions on the interpretation of cosmographic parameters. J. C. Fabris thanks Fundação de Amparo Pesquisa e Inovação do Espírito Santo (FAPES, Project Number 80598935/17) and Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq, Grant Number 304521/2015-9) for partial support.

Declarations

Conflicts of interest  The authors declare that they have no conflict of interest.

References

1. A.G. Riess et al., (Supernova Search Team), Astron. J. 116, 1009 (1998). astro-ph/9805201
2. S. Perlmutter et al., (Supernova Cosmology Project), Nature 391, 51 (1998). astro-ph/9712212
3. D.M. Scolnic et al., Astrophys. J. 859(2), 101 (2018)
4. P. Collaboration et al., 1807.06209 (2018) arXiv e-prints arXiv:1807.06209
5. S. Alam et al., Mon. Not. Roy. Astron. Soc. 470, 2617 (2017)
6. P.J.E. Peebles, B. Rastra, Rev. Mod. Phys. 75, 559 (2003). [arXiv:astro-ph/0207347]
7. L. Perivolaropoulos, arxiv:0811.4684 (2008)
8. J.S. Bullock, M. Boylan-Kolchin, Annu. Rev. Astron. Astrophys. 55, 343–387 (2017)
9. R.R. Caldwell, R. Dave, P.J. Steinhardt, Phys. Rev. Lett. 80(8), 959 (1998) astro-ph/9708069
10. P. Almendarez, V. Mukanov, P. Steinhardt, Phys. Rev. D 63, 103510(2001)
11. W. Zimdahl, J. Triginer, D. Pavon, Phys. Rev. D 54, 6101–6110 (1996)
12. I.L. Shapiro, T. Sola, J. Phys. A 40, 6583–6593 (2007)
13. G.W. Horndeski, Int. J. Theor. Phys. 10, 363 (1974)
14. D. Weinberg et al., Phys. Rep. 530(2), 87–255 (2013)
15. M. Chevallier, D. Polarski, Int. J. Mod. Phys. D 10, 213 (2001)
16. E.V. Linder, Phys. Rev. Lett. 90, 091301(2003)
17. E.E.O. Ishida, R.R.R. Reis, A.V. Toribio, I. Waga, Astropart. Phys. 28, 7 (2007)
18. C. Wetterich, Phys. Lett. B 594, 17 (2004)
19. E.J. Copeland, M. Sami, S. Tsujikawa, Int. J. Mod. Phys. D 15, 1753 (2006). [arXiv:hep-th/0603057]
20. S. Weinberg, Gravitation and cosmology: Principles and applications of the general theory of relativity (Wiley, New York, 1972)
21. T. Chiba, T. Nakamura, Prog. Theor. Phys. 100, 1077 (1998). [arXiv:astro-ph/9808022]
22. M. Visser, Class. Quant. Grav. 21, 2603 (2004). [arXiv:gr-qc/0309109]
23. M. Visser, Gen. Rel. Grav. 37, 1541 (2005). [arXiv:gr-qc/0411131]
24. R. Caldwell, M. Kamionkowski, JCAP 09, 009 (2004) e-Print: astro-ph/0403003
25. C. Cattoni, M. Visser, Class. Quant. Grav. 24, 5985–5998 (2007)
26. M. Dunajski, G. Gibbons, Classical Quantum Gravity 25, 235012 (2008)
27. A. de la Cruz-Dombriz et al., JCAP 1612(12), 042 (2016)
28. C. Busti et al., Phys. Rev. D 92(12), 123512 (2015)
29. T. Padmanabhan, T.R. Choudhury, Mon. Not. Roy. Astron. Soc. 344, 823 (2003)
30. R.J. Scherrer, Phys. Rev. D 92, 043001 (2015)
31. V.M. dos santos, R.R.R Reis, I. Waga, JCAP 02, 066 (2016)
32. A. Aviles, C. Gruber, O. Luongo, H. Quevedo, Phys. Rev. D 86, 123516 (2012)
33. S.R. Capozziello, D. Agostino, O. Luongo, Mon. Not. Roy. Astron. Soc. 494(2), 2576 (2020)
34. N.J. Poplawski, Class. Quant. Grav. 24, 3013–3020 (2007)
35. R. Jimenez, A. Loeb, ApJ 573, 37 (2002)
36. R. Jimenez, L. Verde, T. Treu, D. Stern, ApJ 593, 622 (2003)
37. E. Gaztanaga, R. Miquel, E. Sanchez, Phys. Rev. Lett. 103, 091302 (2009)
38. X. Zheng, X. Ding, M. Biesiada, S. Cao, Z. Zhu, Astrophys. J. 825, 17 (2016). [arXiv:1604.07910]
39. A.M. Velasquez-Toribio, A. Magnano, EPJC 80, 562 (2020)
40. S. Podariu, T. Souradeep, J.R. Gott III, B. Ratra, M.S. Vogele, ApJ 559, 9 (2001)
41. C. Ma, T.-J. Zhang, Astrophys. J. 730, 74 (2011)
42. C.E. Rasmussen, and Williams (C. K. I, Gaussian Process for Machine Learning, 2006)
43. V.C. Busti, C. Clarkson, M. Seikel, MNRAS 441(1), 11 (2014)
44. H. Yu, B. Ratra, F.Y. Wang, ApJ 856, 3 (2018)
45. A. Gómez-Valent, L. Amendola, JCAP 1804, 051 (2018)
46. M. Seikel, C. Clarkson, M. Smith, JCAP 6, 36 (2012)
47. M. Seikel, C. Clarkson, Phys. Rev. D 86, 083001 (2012)
48. A.M. Velasquez-Toribio, M.M. Machado, J. Fabris, EPJC 79, 1010 (2019)
49. T. Holsclaw et al., Phys. Rev. Lett. 105, 241302 (2010). [arXiv:1011.3079]
50. T. Holsclaw et al., Phys. Rev. D 82, 103502 (2010). [arXiv:1009.5443]
51. T. Holsclaw et al., Phys. Rev. D 84, 083501 (2011). [arXiv:1104.2041]
52. A. Shafieloo, A.G. Kim, E.V. Linder, Phys. Rev. D 85, 123530 (2012)
53. O. Farooq, B. Ratra, ApJ 766, L7 (2013). arXiv:1301.5243
54. O. Farooq, S. Crandall, B. Ratra, Phys. Lett. B 726, 72 (2013)
55. O. Farooq et al., Astrophys. J. 835(1), 26 (2017)
56. H. Yu, B. Ratra, F.-Y. Wang, ApJ 856(1), 3 (2018). arXiv:1711.03437
57. A.A. Mamon, Mod. Phys. Lett. A 33, 1830056 (2018)
58. A.M. Velasquez-Toribio, M.L. Bedran, Braz. J. Phys. 41, 59 (2011)
59. SHOES Collaboration, ApJ. 855, 136 (2018)
60. C. Z. Muniz, C. Escamilla-Rivera, JCAP 12, 007 (2020)
61. A. Mehrabi, M. Razea, ApJ 923, 274 (2021). [arXiv:2110.14950]
62. R. Tripp, A&A 331, 815 (1998)
63. A. Conley et al., ApJS 192, 1 (2011)
64. A. Arjona, A. Cardona, S. Nesseris, Phys. Rev. D 99, 043516 (2019). [arXiv:1811.02469]
65. E. Di Valentino et al., (2021). ArXiv:2103.01183