Sum Rate Maximized Resource Allocation in Multiple DF Relays Aided OFDM Transmission
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Abstract—In relay-aided wireless transmission systems, one of the key issues is how to manage the energy resource at the source and each individual relay, to optimize a certain performance metric. This paper addresses the sum rate maximized resource allocation (RA) problem in an orthogonal frequency division modulation (OFDM) transmission system assisted by multiple decode-and-forward (DF) relays, subject to individual sum power constraints of the source and the relays. In particular, the transmission at each subcarrier can be either the direct mode without any relay assisting, or the relay-aided mode with one or several relays assisting. We propose two RA algorithms which optimize the assignment of transmission mode and source power for every subcarrier, as well as the assisting relays and the power allocation to them for every relay-aided subcarrier. First, it is shown that the considered RA problem has zero Lagrangian duality gap when there is a big number of subcarriers. In this case, a duality based algorithm finds a globally optimum RA is developed. Most interestingly, the sensitivity analysis in convex optimization theory is used to derive a closed-form optimum solution to a related convex optimization problem, for which the method based on the Karush-Kuhn-Tucker (KKT) conditions is not applicable. Second, a coordinate-ascent based iterative algorithm, which finds a suboptimum RA but is always applicable regardless of the duality gap of the RA problem, is developed. The effectiveness of these algorithms has been illustrated by numerical experiments.

Index Terms—Orthogonal frequency division modulation, resource allocation, decode and forward, relaying, Lagrangian duality gap, dual decomposition method, energy efficiency.

I. INTRODUCTION

RELAY-AIDED cooperative transmission finds plenty of promising applications when it is difficult to install multiple antennas at the same radio equipment, and therefore has been attracting intensive research interest in both academia and industry lately [1]. Low complexity yet efficient protocols, such as amplify and forward (AF) as well as decode and forward (DF), have been proposed to simplify the implementation with practical devices [2], [3]. Typically, both protocols propose to carry out a relay-aided transmission within two time slots, namely a broadcasting slot and a relaying slot. In [3], the AF/DF which fixes every transmission in the relay-aided mode independently of source-relay channel conditions is referred to as fixed relaying AF/DF. In fact, improved protocols may be built for better performance. For instance, selection relaying AF/DF, which selects either the direct or relay-aided transmission mode depending on channel conditions, has been proposed to improve spectral efficiency [3]. In particular, the direct mode, which refers to the direct source to destination transmission without any relay assisting, is used when the source-destination channel gain is higher than the source-relay channel gain. Most interestingly, not fixed but selection relaying DF can achieve full diversity [3].

We consider in this paper a point to point orthogonal frequency division modulation (OFDM) transmission system aided by multiple DF relays. The motivation behind this is that the OFDM transmission has been widely recognized for current and future wireless systems, thanks to its flexibility to incorporate dynamic resource allocation (RA) for performance improvement [4]. In such a transmission system, one of the key issues is how to decide for every subcarrier the transmission mode and assisting relays if the relay-aided mode is chosen, and the power of the source and every individual relay, to maximize a certain objective related to system performance. Obviously, this RA problem is more complicated compared to those for conventional OFDM systems without relays. Therefore, RA algorithms are solicited for relay-aided OFDM systems.

To date, some related research works have been reported. To name a few, RA algorithms have been proposed in [5]–[8] for OFDM systems aided by AF relays, and in [9]–[14] for multi-user OFDM systems aided by DF relays. As for OFDM systems aided by DF relays, RA algorithms have been proposed in [15] to minimize sum power under rate constraints, and in [16], [17] to maximize sum rate subject to power constraints, when a single relay exists. However, at a subcarrier in the direct mode the source is idle during the relaying slot, which wastes spectrum resource. To address this issue, rate-optimized RA algorithms, which allow for the source to destination transmission during the two slots at the subcarrier in the direct mode, have been proposed in [18]–[21].

So far, the majority of proposed RA algorithms, as the aforementioned ones, restrict that at most one relay can assist the source at every relay-aided subcarrier. In fact, when there are multiple relays available, allowing not just one, but each of them to be eligible for assisting can exploit all degrees of freedom in the system to improve performance. For illustration purposes, example patterns of selecting single or multiple assisting relays are shown in Figure 1. For example, it has been shown in [22] that the sum power can be reduced in a multi-user OFDM system if multiple DF relays assist the transmission at every relay-aided subcarrier.

In this paper, we address the sum rate maximized RA problem in an OFDM system aided by multiple DF relays subject
to the individual (per device, i.e. a source or a relay) sum (over all subcarriers) power constraints\(^1\) of the source and the relays. In particular, one or several relays may cooperate with the source to transmit at every relay-aided subcarrier. When the sum power consumed by the source and that by every relay are fixed, the optimum RA to this problem leads to the maximum energy efficiency for the system under consideration, because the total energy consumed by the source and the relays for transmitting per information bit is minimized. Specifically, our contributions lie in the following aspects:

- when a big number of subcarriers is used, it is shown that the duality gap of the RA problem is equal to zero, based on the same idea first proposed in [23]. Assuming the number of subcarriers is sufficiently large, we develop a duality based algorithm which finds a globally optimum RA for the considered problem. Most interestingly, the sensitivity analysis in convex optimization theory is used to derive a closed-form optimum solution to a related convex optimization problem, for which the method based on the Karush-Kuhn-Tucker (KKT) conditions is not applicable.

- we develop a coordinate-ascent based iterative algorithm, which finds a suboptimum RA but is always applicable regardless of the duality gap of the RA problem. Specifically, this algorithm produces a successive set of RAs with nondecreasing sum rate until convergence.

The remainder of this paper is organized as follows. In the next section, the OFDM system under consideration is described and the RA problem is formulated. Then, the duality based algorithm and the iterative algorithm are developed in Sections III and IV, respectively. In Section V, the effectiveness of the proposed algorithms is illustrated by numerical experiments. Finally, some conclusions wrap up this paper in Section VI.

\(^1\)In this paper, the word “individual” will mean per device and “sum” will refer to a summation over all subcarriers, unless otherwise stated.

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**II. SYSTEM DESCRIPTION AND RA PROBLEM FORMULATION**

**A. System description**

We consider the OFDM transmission from a source to a destination aided by \(N\) DF relays collected in the set \(\Psi = \{r_i| i = 1, \ldots, N\}\). All links are assumed to be frequency selective, and OFDM with properly designed cyclic prefix is used to transform every link into \(K\) parallel channels, each at a different subcarrier facing flat fading. At every subcarrier, the transmission of a symbol is in either the direct mode, or the relay-aided mode spanning across two equal-duration time slots, namely the broadcasting slot and the relaying slot. We assume the destination decodes the signal samples received at each subcarrier separately from those received at any other subcarrier.

We make the following assumptions about the RA in the system. First, the RA is determined by an algorithm running at a central controller, which knows precisely the noise power at each node, as well as the channel coefficients at every subcarrier from the source to every \(r_i\), from the source to the destination, and from every \(r_i\) to the destination, respectively. Second, all channels remain unchanged within a sufficiently long duration, over which RA can be carried out accordingly. Third, the RA information can be reliably disseminated to the source, every relay, and the destination.

Let’s consider the transmission of a unit-variance symbol \(\theta\) at subcarrier \(k\). The coefficient of the channel between any two of the source, \(r_i\), and the destination, are noted according to Table I. We first describe the transmission in the relay-
aided mode. The source first emits in the broadcasting slot the symbol $\sqrt{P_s P_{s,k}} \theta$ as illustrated in Figure 2.a, where $P_s$ and $P_{s,k}$ represent the source sum power and the fraction of that sum power allocated to the transmission at subcarrier $k$, respectively. At the end of this slot, both the destination and the relays receive the source signal. The signal samples received at the destination and $r_i$ can be expressed respectively by

$$y_k = \sqrt{P_s P_{s,k}} h_s(k) \theta + n_k$$

and

$$y_{r_i,k} = \sqrt{P_s P_{s,k}} h_{s,r_i}(k) \theta + n_{r_i,k},$$

where $n_k$ and $n_{r_i,k}$ represent the corruption of the additive white Gaussian noise (AWGN) at the destination and $r_i$, respectively. We assume $\forall i, n_{r_i,k}$ is a zero-mean circularly Gaussian random variable with the variance $\sigma^2$. The signal to noise ratio (SNR) at $r_i$ can be computed as $G_{s,r_i}(k)$, where $G_{s,r_i}(k) = \frac{P_{s} |h_{s,r_i}(k)|^2}{\sigma^2}$ represents the normalized channel power gain from the source to $r_i$.

For subcarrier $k$, we define $\Psi_k$ as the set containing all relays sorted in the increasing order of $G_{s,r_i}(k), r_i(k)$ as the $i$-th relay in $\Psi_k$, and $\Psi_k(i)$ as the set containing relays in $\Psi_k$ with indices from $i$ to $N$. We assume the $r_i$ that has the minimum $G_{s,r_i}(k)$ among all assisting relays for subcarrier $k$ is $r_{b_k}(k)$ where $b_k$ represents the index of that $r_i$ in $\Psi_k$, and all $r_i’s$ in $\Psi_k(b_k)$ decode the received samples to recover $\theta$.

After the recovery of $\theta$, all relays in $\Psi_k(b_k)$ transmit simultaneously to the destination in the relaying slot, which in effect establishes a distributed multiple input and single output (MISO) transmission link as illustrated in Figure 2.b. Specifically, $r_i \in \Psi_k(b_k)$ transmits $w_{r_i} \theta$, where $w_{r_i}$ is the complex weight for transmit beamforming and satisfies $|w_{r_i}|^2 = P_{r_i} P_{s,k}$. $P_{r_i}$ and $P_{s,k}$ represent the sum power of $r_i$ and the fraction of that sum power allocated to subcarrier $k$, respectively. To have the relays’ signals add coherently when received at the destination, $w_{r_i} = \sqrt{P_{r_i} P_{s,k}} e^{-j \arg(h_{r_i,d}(k))}$ is used, where $\arg(h_{r_i,d}(k))$ stands for the phase of $h_{r_i,d}(k)$. It should be noted that this transmission protocol enables a flexible use of all relays opportunistically through a general form of adaptive transmit beamforming, in that $b_k$ and $\{P_{r_i,k} | r_i \in \Psi_k(b_k)\}$ are determined by the RA algorithm depending on channel state information, as will be developed later. At the end of the relaying slot, the signal sample received at the destination is denoted by

$$z_k = \sum_{r_i \in \Psi_k(b_k)} \sqrt{P_{r_i,k}} |h_{r_i,d}(k)|^2 \theta + v_k,$$

(3)

where $v_k$ represents the AWGN corruption at the destination. We assume $n_k$ and $v_k$ are independent zero-mean circularly Gaussian random variables with the same variance $\sigma^2$.

Finally, $y_k$ and $z_k$ are processed at the destination. From (1) and (3), it can be seen that the relay-aided transmission at subcarrier $k$ is in effect over a channel with a single input and two outputs. To achieve the capacity of this channel, the maximum ratio combining (MRC) should be used, i.e., the destination combines $y_k$ and $z_k$ to construct a decision variable

$$c_k = \sqrt{P_{s,k} G_s(k)} y_k + \left( \sum_{r_i \in \Psi_k(b_k)} \sqrt{P_{r_i,k} G_{r_i,d}(k)} \right)^2 z_k,$$

(4)

which is then decoded (please refer to page 179 of [24] for more details). After mathematical arrangements, the SNR for decoding $c_k$ is derived as

$$\eta_k = P_{s,k} G_s(d)(k) + \left( \sum_{r_i \in \Psi_k(b_k)} \sqrt{P_{r_i,k} G_{r_i,d}(k)} \right)^2$$

where $G_{s,d}(k) = \frac{P_{s} |h_{s,d}(k)|^2}{\sigma^2}$ and $G_{r_i,d}(k) = \frac{P_{r_i} |h_{r_i,d}(k)|^2}{\sigma^2}$ represent the normalized channel power gains from the source to the destination and from $r_i$ to the destination, respectively.

To ensure the reliable recovery of $\theta$ at every relay in $\Psi_k(b_k)$, the source transmission rate at subcarrier $k$ should not be higher than $\log_2 (1 + P_{s,k} G_{s,r_b(k)})$ bits/two-slots (bpts). Moreover, the source transmission rate should not be higher than $\log_2 (1 + \eta_k)$ bpts to ensure reliable decoding of $c_k$ at the destination. Therefore, the source transmission rate at subcarrier $k$ in the relay-aided mode should be [3]

$$R_{k,1} = \min \{ \log_2 (1 + P_{s,k} G_{s,r_b(k)}) \}, \log_2 (1 + \eta_k) \}$$

(5)

$$= \log_2 (1 + \min \{ \eta_k, P_{s,k} G_{s,r_b(k)} \})$$

bpts. (6)

When $b_k$ is fixed, $R_{k,1}$ is a concave function of $\{P_{s,k}, P_{r_i,k} | \forall r_i \in \Psi\}$ due to the following reasons. First, $\eta_k$ and $P_{s,k} G_{s,r_b(k)}$ are both concave functions of $\{P_{s,k}, P_{r_i,k} | \forall r_i \in \Psi\}$, and therefore $\min \{ \eta_k, P_{s,k} G_{s,r_b(k)} \}$ is a concave function of $\{P_{s,k}, P_{r_i,k} | \forall r_i \in \Psi\}$ due to the minimum of two concave functions is still a concave function. Second, $\log_2 (1 + x)$ is an increasing and concave function of $x$. Therefore, $R_{k,1}$, as the composition of $\log_2 (1 + x)$ and $x = \min \{ \eta_k, P_{s,k} G_{s,r_b(k)} \}$, is a concave function of $\{P_{s,k}, P_{r_i,k} | \forall r_i \in \Psi\}$ when $b_k$ is fixed, because the composition of an increasing concave function and a concave function is still a concave function [25].

As for the direct transmission mode at subcarrier $k$, we consider a more efficient protocol compared to restricting the source to transmit only in the broadcasting slot as in the related works [3], [15]–[17]. Specifically, the source emits two independent symbols in the two slots, respectively, and only the destination decodes the corresponding two received signal samples. We assume the AWGN corruptions for the
two received samples are independent zero-mean circularly Gaussian distributed with variance $\sigma^2$, and the source uses the power $P_1 P_{s,k}$ in total to transmit the two symbols. The maximum source transmission rate at subcarrier $k$ in the direct mode can be derived as

$$R_{k,2} = 2 \log_2 \left( 1 + \frac{G_{s,d}(k)}{2} P_s P_{s,k} \right) \text{ bpts},$$  \tag{7}$$
and it is achieved when the source uses the power $\frac{P_s P_{s,k}}{2}$ to transmit each symbol.

Note that we assume the same subcarrier is used by the source and the relays for transmitting a symbol in the relay-aided mode. In fact, optimized subcarrier pairing could also be implemented, which would further increase the degrees of freedom for optimization [16]. However, it would be more difficult to solve the RA problem, and that is why in the current work, the RA algorithms are designed under the assumption of nonoptimized subcarrier pairing. We will nevertheless use the insights gained here to guide future work, which will also consider subcarrier pairing.

### B. Formulation of the RA problem

We consider the RA problem to maximize the sum rate, by optimizing the transmission mode and source power allocation for each subcarrier, as well as assisting relays and power allocation to them for every relay-aided subcarrier. To formulate this RA problem, we define a binary variable $t_k$ which indicates the transmis-sion at subcarrier $k$ is in the relay-aided mode (resp. the direct mode) if $t_k = 1$ (resp. $t_k = 0$). Mathematically, the RA problem is formulated as

$$\begin{align*}
\max_{t_k, b_k, P_{s,k}, P_{t_i,k}, \forall i} & \sum_{k=1}^{K} (t_k R_{k,1} + (1-t_k) R_{k,2}) \\
\text{s.t.} & \sum_{k=1}^{K} P_{s,k} \leq 1, \quad \sum_{k=1}^{K} P_{t_i,k} \leq 1, \forall i \\
& P_{s,k} \geq 0, \forall k, \quad P_{t_i,k} \geq 0, \forall k, \forall i \\
& t_k \in \{0, 1\}, \forall k, \quad b_k \in \{1, \ldots, N\}, \forall k,
\end{align*}$$  \tag{8}$$

where $\{t_k, b_k, P_{s,k}, P_{t_i,k}, \forall i \in \Psi, \forall k\}$ are to be optimized by the RA algorithm.

To facilitate analysis in the following sections, (8) can be formulated into the following equivalent form

$$\begin{align*}
\max_{x} & \quad f(x) \\
\text{s.t.} & \quad g(x) \preceq 1,
\end{align*}$$  \tag{9}$$

where $x$ represents the vector stacking all optimization variables, $D_x$ stands for the definition domain of $x$, $f(x)$ denotes the sum rate, $g(x) = [g_1(x), \ldots, g_{N+1}(x)]^T$ where $g_1(x) = \sum_{k=1}^{K} P_{s,k}$ and $g_{i+1}(x) = \sum_{k=1}^{K} P_{t_i,k}$ ($i = 1, \ldots, N$) stacks the sum power of the source and those of the relays, $1$ represents an $(N+1) \times 1$ vector with every entry equal to 1, and $\preceq$ denotes the entrywise “smaller than” inequality.

### III. The Duality Based RA Algorithm

The Lagrangian for (9) is defined as

$$L(x, \mu) = f(x) + \mu^T (1 - g(x)), \tag{10}$$

where $\mu = [\mu_s, \mu_{t_1}, \ldots, \mu_{t_N}]^T$, with $\mu_s$ and $\mu_{t_i}$ representing the dual variables related to the sum power constraints of the source and $t_i$, respectively. The dual function is defined as $d(\mu) = \max_{x \in D_x} L(x, \mu)$, and we denote a $x$ that maximizes $L(x, \mu)$ as $x_{\mu}$. If there exist multiple $x$’s that maximize $L(x, \mu)$, $d(\mu)$ is not differentiable at $x$, and a subgradient of $d(\mu)$ at $x$ is equal to $1 - g(x_{\mu})$, where any $x$ that maximizes $L(x, \mu)$ can be chosen as $x_{\mu}$ [26].

It is important to note that the optimum objective value of (9), denoted by $f^*$, always satisfies $f^* \leq d(\mu)$ for any $\mu \succ 0$, where $0$ represents an $(N+1) \times 1$ vector with every entry equal to 0, and $\succ$ denotes the entrywise “greater than” inequality [25]. The duality gap is defined as $\min_{\mu} d(\mu) - f^*$. Note that $D_x$ is not a convex set, and therefore the duality gap of (9) might not be zero since the Slater constraint qualification is not satisfied [26]. Interestingly, it is shown in Appendix A based on the same idea proposed in [23] that, when $K$ is sufficiently large (9) has zero duality gap.

Assuming sufficiently large $K$, a duality based RA algorithm will be developed in this section to find a globally optimum solution to (9). This algorithm relies on finding the optimum dual variable $\mu^*$ that minimizes $d(\mu)$ based on the subgradient method, i.e., updating $\mu$ with $\mu_{t_i} = [\mu_s - \delta_s (1 - g(x_{\mu}))^+], \mu_{t_i}^+ \text{ represents a vector built from } \mu \text{ by only raising its negative entries to zero, and } \delta_s \text{ represents the step size used in the } g\text{-th iteration, until } g(x_{\mu}) \preceq 1 \text{ and } \mu^T (1 - g(x_{\mu})) = 0 \text{ are fulfilled. Then, } x_{\mu}^* \text{ is a globally optimum solution to } (9) \text{ as explained in Appendix A. If } \delta_s \text{ satisfies the diminishing conditions (i.e., } \lim_{\delta_s \to 0} \delta_s = 0 \text{ and } \sum_{\delta_s = 1}^{\infty} \delta_s = \infty) \text{, } \mu \text{ approaches } \mu^* \text{ as } g \text{ increases} [26]$. In practice, it may take an unaffordably large number of iterations before these optimality conditions are satisfied. To address this issue, the iteration can be terminated when $g(x_{\mu}) \preceq 1$ and $\mu^T (1 - g(x_{\mu})) < \epsilon$, where $\epsilon$ is a prescribed small positive value. In this case, $f(x_{\mu}) \geq f^* - \epsilon$ follows because of

$$f^* - f(x_{\mu}) \leq d(\mu) - f(x_{\mu}) \leq \mu^T (1 - g(x_{\mu})) < \epsilon,$$

which means that $x_{\mu}$ is a good approximation of a globally optimum solution to (9).

The overall duality based RA algorithm is summarized in Algorithm 1. Specifically, $\mu$ is initialized with 1 at the beginning, and $x_{\mu}$ is found as the optimum solution to the Lagrangian maximization problem

$$\begin{align*}
\max_{x} & \quad L(x, \mu) = \sum_{k=1}^{K} (t_k R_{k,1} + (1-t_k) R_{k,2}) + \\
& \mu_s (1 - \sum_{k=1}^{K} P_{s,k}) + \sum_{i=1}^{N} \mu_{t_i} (1 - \sum_{k=1}^{K} P_{t_i,k}), \\
\text{s.t.} & \quad P_{s,k} \geq 0, \forall k, \quad P_{t_i,k} \geq 0, \forall k, \forall i \\
& \quad t_k \in \{0, 1\}, \forall k, \quad b_k \in \{1, \ldots, N\}, \forall k,
\end{align*}$$  \tag{11}$$

where $t_k$ is the transmission mode at subcarrier $k$, $P_{s,k}$ and $P_{t_i,k}$ are the source power and the $i$-th relay power at subcarrier $k$, respectively, $\mu_s$ and $\mu_{t_i}$ are the dual variables related to the sum power constraints of the source and the $i$-th relay, respectively, and $\epsilon$ is a small positive value.
with Algorithm 2 developed in Section III-A. Furthermore, we choose $\delta_q = \frac{1 + Q_1}{Q_1}$ ($Q_1$ is a prescribed positive integer) which satisfies the aforementioned diminishing conditions.

### A. Algorithm to solve the Lagrangian maximization problem

We can see that the optimum solution to (11) can be found on a per subcarrier basis, i.e., \( \{k, b_k, \mu_{s,n}, P_{s,n}, k, P_{t_i, k}\} \forall t_i \in \Psi \) representing the \( \{k, b_k, P_{s,n}, k, P_{t_i, k}\} \forall t_i \in \Psi \) contained in \( x_{\mu} \) for subcarrier \( k \), can be found as an optimum solution to

\[
\text{maximize } L_k = t_k R_{k,1} + (1-t_k) R_{k,2} - \mu_{s,n} P_{s,n,k} - \sum_{i=1}^{N} \mu_{t_i, k} P_{t_i, k}
\]

subject to
\[
P_{s,n,k} \geq 0, \quad P_{t_i, k} \geq 0, \forall t_i \in \Psi, \quad \mu_{s,n,k} \geq 0
\]

When \( t_k \) and \( b_k \) are fixed, (12) is reduced to a convex optimization problem. When \( t_k = 0 \), the maximum \( L_k \) is not influenced by \( b_k \), while this not the case when \( t_k = 1 \) and \( b_k \) is fixed. Based on the above analysis, an exhaustive-search based algorithm is solved to (12). To facilitate algorithm design, let’s denote the maximum \( L_k \) when \( t_k = 0 \) by \( L_{k,0} \), the maximum \( L_k \) and the optimum \( \{P_{s,n,k}, P_{t_i, k}\} \forall t_i \in \Psi \) when \( t_k = 1 \) and \( b_k \) is fixed by \( L_{k,1}(b_k) \) and \( \{P_{s,n,k}, P_{t_i, k}\} \forall t_i \in \Psi \), respectively. Note that \( L_{k,1}(b_k) \) and \( \{P_{s,n,k}, P_{t_i, k}\} \forall t_i \in \Psi \) can be evaluated with Algorithm 3 developed in Section III.B.

It is important to note that \( L_{k,1}(b_k) \leq L_{k,0} \) if \( G_{s,n, t_i}(k) \leq G_{s,d}(k) \), because of

\[
L_{k,1}(b_k) = \log_2(1 + (1 - \mu_{s,n}) G_{s,d}(k)) \geq \log_2(1 + P_{s,n,k} G_{s,d}(k) + X_{b_k}) \leq \log_2\left(1 + P_{s,n,k} G_{s,d}(k) + \frac{2 \log_2 G_{s,d}(k)}{\mu_{s,n}} + X_{b_k}\right) \leq L_{k,0},
\]

where \( X_{b_k} = -\mu_{s,n} P_{s,n,k} - \sum_{i=1}^{N} \mu_{t_i, k} P_{t_i, k} \) and \( \eta_k \) represents the \( \eta_k \) corresponding to \( \Psi_{k}(b_k) \) and \( \{P_{s,n,k}, P_{t_i, k}\} \forall t_i \in \Psi \).

Based on the above analysis, \( t_k(\mu), b_k(\mu), P_{s,n,k}(\mu), \text{ and } \{P_{t_i, k}(\mu)\} \forall t_i \in \Psi \) can be found with one of the following procedures:

1) when \( \max_{s,n} G_{s,n, t_i}(k) \leq G_{s,d}(k), t_k(\mu) = 0 \), because \( L_{k,0} \geq L_{k,1}(b_k) \) holds for every feasible value of \( b_k \) according to (13). In this case, it can be derived according to the KKT conditions that

\[
P_{s,n,k}(\mu) = \frac{2 \log_2(e) - 1}{\mu_{s,n}} G_{s,d}(k) + 1,
\]

\[
P_{t_i, k}(\mu) = 0, \forall t_i \in \Psi,
\]

\[
L_{k,0} = 2 \log_2 \left(1 + G_{s,d}(k) \left[\frac{2 \log_2(e) - 1}{\mu_{s,n}} G_{s,d}(k) + 1\right]\right)
\]

2) when \( \max_{s,n} G_{s,n, t_i}(k) > G_{s,d}(k), t_k(\mu) \) can be determined by an exhaustive-search based method, i.e., if \( \max_{b_k \in \{1, \ldots, N\}} L_{k,1}(b_k) > L_{k,0}, t_k(\mu) = 1, \) otherwise \( t_k(\mu) = 0 \). Let’s denote \( t_k \) as the minimum \( t \) satisfying \( G_{s,n, t_i}(k) > G_{s,d}(k) \). Note that when \( 1 \leq b_k \leq b_k - 1, G_{s,n, t_i}(k) \leq G_{s,d}(k) \), hence \( L_{k,1}(b_k) \leq L_{k,0} \) holds for sure according to (13). This means that the comparison of \( L_{k,0} \) with \( \max_{b_k \in \{1, \ldots, N\}} L_{k,1}(b_k) \) is equivalent to comparing \( L_{k,0} \) with \( \max_{b_k \in B_k} L_{k,1}(b_k) \) where \( B_k = \{1, \ldots, N\} \). Based on this idea, \( L_{k,0} \) is evaluated with (16), and \( L_{k,1}(b_k) \) is computed for all values of \( b_k \in B_k \) with Algorithm 3. If \( L_{k,0} > \max_{b_k \in B_k} L_{k,1}(b_k) \), \( t_k(\mu) = 0 \), and \( P_{s,n,k}(\mu) \) and \( \{P_{t_i, k}(\mu)\} \forall t_i \in \Psi \) are computed with (14) and (15), respectively. Otherwise, \( t_k(\mu) = 1, b_k(\mu) = \arg \max_{b_k \in B_k} L_{k,1}(b_k), \) and the \( \{P_{s,n,k}(b_k), P_{t_i, k}(b_k)\} \forall t_i \in \Psi \) when \( b_k = b_k(\mu) \) is taken as \( \{P_{s,n,k}(b_k), P_{t_i, k}(b_k)\} \forall t_i \in \Psi \).

In summary, the overall algorithm of finding \( x_{\mu} \) is summarized in Algorithm 2. We will proceed with developing Algorithm 3 to find \( L_{k,1}(b_k) \) and \( \{P_{s,n,k}(b_k), P_{t_i, k}(b_k)\} \forall t_i \in \Psi \) in the next section.
B. Algorithm to solve (12) when \( t_k = 1 \) and \( b_k \) is fixed

When \( t_k = 1 \) and \( b_k \) has a fixed value in \( B_k \), (12) is equivalent to

\[
\max L_k = R_{k,1} - \mu_k P_{s,k} - \sum_{i=1}^{N} \mu_{r_i} P_{r_i,k}
\]

\[
= \log_2 (1 + \gamma_k) - \mu_k P_{s,k} - \sum_{i=1}^{N} \mu_{r_i} P_{r_i,k}
\]

\[
s.t. \quad \gamma_k \leq P_{s,k} G_{s,r_k}(k)(k),
\]

\[
\gamma_k \leq P_{s,k} G_{s,d}(k) + \left( \sum_{r_i \in \Psi_k(b_k)} \sqrt{P_{r_i,k} G_{r_i,d}(k)} \right)^2,
\]

\[
\gamma_k \geq 0,
\]

\[
P_{r_i,k} \geq 0, \forall r_i \in \Psi,
\]

where \( \gamma_k \) is an intermediate optimization variable to guarantee the equivalence.

It can readily be shown that (17) is a convex optimization problem. To solve it, one may formulate a set of equations based on the KKT conditions and then solve them for \( \{P_{s,k}(b_k), P_{r_i,k}(b_k)\} \forall r_i \in \Psi \). It is very important to note that this method is effective only when the objective function and all the constraint functions are differentiable at the optimum solution. However, the second term in the right hand side of the second constraint in (17) is not differentiable at \( P_{r_i,k} = 0 \), \( \forall r_i \in \Psi_k(b_k) \). This means that if \( \exists r_i \in \Psi_k(b_k), P_{r_i,k}(b_k) = 0 \), which might happen as shown later, the KKT conditions based method is not capable of finding that optimum solution.

To address this issue, we solve (17) based on the idea that \( P_{s,k}(b_k) \) is the optimum solution to

\[
\max \quad \log_2 (1 + \gamma_k) - \mu_k P_{s,k}
\]

\[
s.t. \quad \gamma_k \leq P_{s,k} G_{s,r_k}(k)(k),
\]

\[
\gamma_k \leq P_{s,k} G_{s,d}(k) + x,
\]

\[
\gamma_k \geq 0,
\]

\[
\{P_{r_i,k}(b_k)\} \forall r_i \in \Psi \}
\]

and \( \{P_{r_i,k}(b_k)\} \forall r_i \in \Psi \} \) is the optimum solution to

\[
\max \quad \sum_{i=1}^{N} (-\mu_{r_i} P_{r_i,k})
\]

\[
s.t. \quad \sum_{r_i \in \Psi_k(b_k)} \sqrt{P_{r_i,k} G_{r_i,d}(k)} = \sqrt{x},
\]

\[
\{P_{r_i,k}(b_k)\} \forall r_i \in \Psi \}
\]

when \( x = x_0 \) with

\[
\gamma_k = \left( \sum_{r_i \in \Psi_k(b_k)} \sqrt{P_{r_i,k}(b_k) G_{r_i,d}(k)} \right)^2.
\]

Specifically, \( x_0 \) is first determined, and then (18) and (19) with \( x = x_0 \) are solved to compute \( \{P_{s,k}(b_k), P_{r_i,k}(b_k)\} \forall r_i \in \Psi \}. At first glance, this method seems confronted with a chicken-and-egg dilemma: though \( \{P_{s,k}(b_k), P_{r_i,k}(b_k)\} \forall r_i \in \Psi \} \) can be computed by solving (18) and (19) once \( x_0 \) is known, it seems that \( \{P_{s,k}(b_k), P_{r_i,k}(b_k)\} \forall r_i \in \Psi \} \) needs to be known first in order to compute \( x_0 \). In fact, this dilemma can be elegantly circumvented by using the sensitivity analysis in convex optimization theory to first determine \( x_0 \) without knowing \( \{P_{s,k}(b_k), P_{r_i,k}(b_k)\} \forall r_i \in \Psi \}, as elaborated in the following.

1) Solutions to (18) and (19) given \( x \): Let’s denote the optimum objective values of (18) and (19) by \( f_1(x) \) and \( f_2(x) \), respectively. Obviously, (18) is a convex optimization problem. Let’s denote the optimum \( \gamma_k \) and the optimum dual variables associated with the first and second constraints of (18) by \( \gamma_k(x), \alpha_k(x), \) and \( \beta_k(x) \), respectively. According to the KKT conditions of (18),

\[
\mu_k = G_{s,r_k}(k)(k) \alpha_k(x) + G_{s,d}(k) \beta_k(x)
\]

and

\[
\gamma_k(x) = \left[ \frac{\log_2 e}{\alpha_k(x)} + \beta_k(x) - 1 \right]^+
\]

should be satisfied.

As for (19), it can readily be derived that

- when \( \forall r_i \in \Psi_k(b_k), \mu_{r_i} > 0, f_2(x) = -\frac{x}{\theta_{b_k}}, \) and the optimum \( P_{r_i,k} \) to (19) is

\[
P_{r_i,k} = \begin{cases} 0 & \text{if } r_i \notin \Psi_k(b_k), \\ \frac{G_{r_i,d}(k) \mu_{r_i}}{\theta_{b_k}} x & \text{if } r_i \in \Psi_k(b_k), \end{cases}
\]

(22)

where \( \theta_{b_k} = \sum_{r_i \in \Psi_k(b_k)} G_{r_i,d}(k) / \mu_{r_i} \).

- when \( \exists r_i \in \Psi_k(b_k), \mu_{r_i} = 0, f_2(x) = 0, \) and the optimum \( P_{r_i,k} \) to (19) is equal to 0 if \( r_i \notin \Psi_k(b_k) \) or if \( r_i \in \Psi_k(b_k) \) with \( \mu_{r_i} > 0 \). The optimum \( \{P_{r_i,k}\} \forall r_i \in \Psi_k(b_k), \mu_{r_i} = 0 \} \) is any set of nonnegative values satisfying

\[
\sum_{r_i \in \Psi_k(b_k), \mu_{r_i} = 0} \sqrt{P_{r_i,k} G_{r_i,d}(k)} = \sqrt{x}.
\]

(23)

Moreover, it can readily be shown based on the Schwartz inequality that, \( \forall r_i : r_i \in \Psi_k(b_k) \) and \( \mu_{r_i} = 0 \), the optimum \( P_{r_i,k} \) satisfying (23) and minimizing the sum power of relays is

\[
P_{r_i,k} = \frac{G_{r_i,d}(k) \mu_{r_i}}{\theta_{b_k} \mu_{r_i} x^2}.
\]

2) Finding \( x_0 \) based on the sensitivity analysis: Let’s denote the \( L_k \) in (17) computed with the optimum \( \{\gamma_k, P_{s,k}\} \) to (18) and the optimum \( \{P_{r_i,k}\} \forall r_i \in \Psi \} \) to (19) by \( L_{k,1}(x, b_k) \) when \( x \geq 0 \). Obviously, \( L_{k,1}(x, b_k) = f_1(x) + f_2(x) \) and \( L_{k,1}(x, b_k) \leq L_{k,1}(b_k) \), since \( L_{k,1}(b_k) \) is defined in Section III.A as the maximum \( L_k \) for (17), while \( L_{k,1}(x, b_k) \) is the \( L_k \) computed with the above mentioned \( \{\gamma_k, P_{s,k}\} \) and \( \{P_{r_i,k}\} \forall r_i \in \Psi \} \) which are feasible for (17). When \( x = x_0 \), the \( P_{s,k} \) and \( \{P_{r_i,k}\} \forall r_i \in \Psi \} \) used for computing \( L_{k,1}(x, b_k) \) are equal to \( P_{s,k}(b_k) \) and \( \{P_{r_i,k}(b_k)\} \forall r_i \in \Psi \}, respectively, and therefore \( L_{k,1}(x_0, b_k) = L_{k,1}(b_k) \). This means that \( x_0 = \arg \max_{x \geq 0} L_{k,1}(x, b_k) \).

To determine \( x_0 \), let’s consider \( L_{k,1}'(x, b_k) \) which represents the first order derivative of \( L_{k,1}(x, b_k) \) with respect to \( x \). According to convex optimization theory, \( \beta_k(x) \) represents the sensitivity of \( f_1(x) \) with respect to \( x \), i.e., \( f_1'(x) = \beta_k(x) \) (please refer to pages 249-253 in [25] for more details).

Therefore,

\[
L_{k,1}'(x, b_k) = \begin{cases} \beta_k(x) - 1/\theta_{b_k} & \text{if } \forall r_i \in \Psi_k(b_k), \mu_{r_i} > 0, \\ \beta_k(x) & \text{if } \exists r_i \in \Psi_k(b_k), \mu_{r_i} = 0. \end{cases}
\]

(24)
Based on the above analysis, the determination of $x_0$, $\beta_k(x_0)$, $\alpha_k(x_0)$ and $P_{s,k}(b_k)$ falls into one of the following cases:

- when $\forall r_i \in \Psi_k(b_k)$, $\mu_{r_i} > 0$ and $\frac{\mu_{r_i}}{G_{s,d}(k)} \leq \frac{1}{G_{s,\tau_{s_k}(k)}}$, $x_0 = 0$.
  This is because $L_{s,1}(x, b_k)$ is a nonincreasing function of $x$ since $\beta_k(x) \in [0, \mu_{s}/G_{s,d}(k)]$ and $L_{s,1}(x, b_k) \leq 0$ for any $x \geq 0$. In this case, the first constraint in (18) when $x = x_0$ is relaxed, whereas the second one is saturated, since $G_{s,d}(k) < G_{s,\tau_{s_k}(k)}(k)$. Therefore, $\alpha_k(x_0) = 0$, $\beta_k(x_0) = \frac{\mu_{s} - G_{s,\tau_{s_k}(k)}(k)}{G_{s,d}(k)}$, and
  \[
P_{s,k}(b_k) = \frac{\gamma_k(x_0)}{G_{s,\tau_{s_k}(k)}(k)} = \left[ \frac{\log_2 e}{\frac{\mu_s}{\Delta_b + 1/\theta_{b_k}} - \frac{\Delta_{b_k}}{G_{s,\tau_{s_k}(k)}(k)}} \right]^+, \tag{25}
\]
  and
  \[
x_0 = P_{s,k}(b_k)\Delta_{b_k} = \left[ \frac{\log_2 e}{\frac{\mu_s}{\Delta_b + 1/\theta_{b_k}} - \frac{\Delta_{b_k}}{G_{s,\tau_{s_k}(k)}(k)}} \right]^+, \tag{27}
\]
  where $\Delta_{b_k} = G_{s,\tau_{s_k}(k)}(k) - G_{s,d}(k)$.

- when $\exists r_i \in \Psi_k(b_k)$, $\mu_{r_i} = 0$, $\beta_k(x_0) = 0$ since $L_{s,1}(x, b_k) = 0$ should be satisfied. In this case, $\alpha_k(x_0) = \frac{\mu_{s} - G_{s,\tau_{s_k}(k)}(k)}{G_{s,d}(k)}$, and $\gamma_k(x_0) = \left[ \frac{G_{s,\tau_{s_k}(k)}(k)}{G_{s,d}(k)} \log_2 e \right]^+ - 1$. This means that the first constraint in (18) when $x = x_0$ is satisfied, whereas the second one is relaxed. Therefore,
  \[
P_{s,k}(b_k) = \frac{\gamma_k(x_0)}{G_{s,\tau_{s_k}(k)}(k)} = \left[ \frac{\log_2 e}{\frac{\mu_s}{\Delta_b + 1/\theta_{b_k}} - \frac{\Delta_{b_k}}{G_{s,\tau_{s_k}(k)}(k)}} \right]^+, \tag{28}
\]
  and $x_0$ can be any value satisfying $x_0 \geq \gamma_k(x_0) - P_{s,k}(b_k)G_{s,d}(k) = x_{th}$ where
  \[
x_{th} = \Delta_{b_k} \left[ \frac{\log_2 e}{\frac{\mu_s}{\Delta_b + 1/\theta_{b_k}} - \frac{\Delta_{b_k}}{G_{s,\tau_{s_k}(k)}(k)}} \right]^+. \tag{29}
\]

After knowing $x_0$, we can find the optimum $\{P_{r_{i,k}}\| \forall r_i \in \Psi\}$ to (19) when $x = x_0$ as $\{P_{r_{i,k}}(b_k)\| \forall r_i \in \Psi\}$. Note that in the third case $x_0$ can be any value no smaller than $x_{th}$, and $\{P_{r_{i,k}}(b_k)\| \forall r_i \in \Psi_k(b_k), \mu_{r_i} = 0\}$ can be any set of nonnegative values satisfying (23) with $x = x_0$. To improve the system energy efficiency, we choose $x_0 = x_{th}$, and $\forall r_i : r_i \in \Psi_k(b_k)$ and $\mu_{r_i} = 0$, $P_{r_{i,k}}(b_k) = G_{s,d}(k)x_{th}^2$ to minimize the sum power of the relays.

The overall algorithm of finding $\{P_{s,k}(b_k), P_{r_{i,k}}(b_k)\| \forall r_i \in \Psi\}$ is summarized in Algorithm 3. Based on the above analysis, it can be seen that the $\eta_i$ corresponding to $\{P_{s,k}(b_k), P_{r_{i,k}}(b_k)\| \forall r_i \in \Psi\}$ and $b_k$ must be equal to or smaller than $P_{s,k}(b_k)G_{s,\tau_{s_k}(k)}(k)$.

IV. THE ITERATIVE RA ALGORITHM

In case (8) has a nonzero duality gap, Algorithm 1 fails to find a globally optimum solution. To address this issue, we develop in this section a coordinate-ascent based iterative algorithm which is suboptimum but always applicable regardless of the duality gap of (8).

A. The coordinate-ascent based iterative RA algorithm

First of all, it should be noted that $R_{k,1} \leq R_{k,2}$ always holds independently of $\{P_{s,k}, P_{r_{i,k}}\| \forall r_i \in \Psi\}$ when $b_k$ satisfies $G_{s,\tau_{s_k}(k)}(k) \leq G_{s,d}(k)$. This means that when $\max_{r_i \in \Psi} G_{s,\tau_{s_k}(k)}(k) \leq G_{s,d}(k)$, $t_k = 0$ is the optimum independently of $b_k$ and $\{P_{s,k}, P_{r_{i,k}}\| \forall r_i \in \Psi\}$, i.e., the direct transmission mode should always be used for subcarrier $k$.

To simplify algorithm design, we assume $t_k$ is fixed as 0 for every $k \in D_a = \{ k \max_{r_i \in \Psi} G_{s,\tau_{s_k}(k)}(k) \leq G_{s,d}(k) \}$. If $k \notin D_a$, the optimum $t_k$ and $b_k$ must lie in the set $\{0, 1\} \times B_k$, where $\times$ represents the Cartesian product. Furthermore, when $\{t_k, b_k\| k \notin D_a\}$ is fixed, (8) is reduced to a convex optimization problem, which has zero duality gap since the Slater constraint qualification is satisfied. Thus a globally optimum $\{P_{s,k}, P_{r_{i,k}}\| \forall r_i \in \Psi, \forall k\}$ and the maximum sum rate when $\{t_k, b_k\| k \notin D_a\}$ is fixed can be found with a duality based algorithm as shown later.

Now the difficulty of solving (8) lies in finding the optimum $\{t_k, b_k\| k \notin D_a\}$. To this end, one may use an exhaustive-search based algorithm. Specifically, this algorithm finds the maximum sum rate for every possible $\{t_k, b_k\| k \notin D_a\}$ in $\prod_{k \notin D_a} \{0, 1\} \times B_k$, then chooses the best one as the optimum $\{t_k, b_k\| k \notin D_a\}$. However, the complexity of exhaustive search might be affordable for practical systems using a big number of subcarriers.
Algorithm 4 The iterative RA algorithm

∀ k ∈ D_a, t_k = 0; ∀ k ∉ D_a, t_k = 1 and b_k = N;
m = 0;
repeat

Find \{P_m^k, P_m^k \forall r_k \in \Psi, \forall k\} as the optimum solution to (8) when \(t_k = t_k^m\) and \(b_k = b_k^m\), ∀ k ∉ D_a with Algorithm 5;

for every k ∉ D_a do

if \(t_k^m = 0\) then

\(t_k^{m+1} = 0;\)

else

Compute \(\eta_k^m(b_k^m), R_k^{m}(b_k^m),\) and \(R_k^{m+1}\);

if \(R_k^{m+1}(b_k^m) < R_k^{m+2}\) then

\(t_k^{m+1} = 0;\)

else

\(t_k^{m+1} = 1;\)

end if

end for

until ∀ k ∉ D_a, \(\{t_k^{m+1} = 0 \forall k \in D_a, t_k^{m+1} = 1 \forall k ∉ D_a\}\) produced in the last iteration are a suboptimality solution.

To address this issue, we develop a coordinate-ascent based iterative algorithm which produces a successive set of RAs with nondecreasing sum rate until convergence. In the following, a superscript \(m\) added to a variable indicates that variable is produced at the \(m\)-th iteration to facilitate description. At the beginning, ∀ k ∉ D_a, \(t_k^0\) and \(b_k^0\) are initialized as 1 and \(N\), respectively, i.e., every subcarrier not in D_a is set in the relay-aided mode with only the relay having the highest source-relay channel gain enabled for assisting. In the \(m\)-th iteration, \(\{P_m^k, P_m^{k} \forall r_k \in \Psi, \forall k\}\), which is the optimum solution to (8) when \(t_k = t_k^m\) and \(b_k = b_k^m\), ∀ k ∉ D_a, is first found with a duality based algorithm, namely Algorithm 5 developed in Section IV-B. This step can be interpreted as finding the optimum source-relay power allocation when \(\{t_k, b_k \forall k \notin D_a\}\) is fixed as \(\{P_m^k, b_k^m \forall k \notin D_a\}\).

Then, \(\{t_k^{m+1}, b_k^{m+1} \forall k \notin D_a\}\) which maximizes the sum rate is found when the power allocation is fixed as \(\{P_m^k, P_m^{k} \forall r_k \in \Psi, \forall k\}\). This step can be interpreted as finding the optimum mode and assisting relays when the source-relay power allocation is prescribed by \(\{P_m^k, P_m^{k} \forall r_k \in \Psi, \forall k\}\). Note that this can be accomplished on a per subcarrier basis, i.e., for every subcarrier k ∉ D_a, \(t_k^{m+1}\) and \(b_k^{m+1}\) are found to maximize the rate when the power allocation is fixed as \(\{P_m^k, P_m^{k} \forall r_k \in \Psi\}\). In this case, the rate is denoted by \(R_{k,1}^{m+2} = 2 \log_2(1 + \frac{P_m^k}{2} G_{s,r,k}(k))\) if the direct mode is used. If the relay-aided mode with a given \(b_k\) is used, the rate is denoted by \(R_{k,1}^{m+1}(b_k) = \log_2(1 + \min(\eta_k^m(b_k), P_m^k G_{s,r,k}(k)))\) where \(\eta_k^m(b_k)\) represents the \(\eta_k\) corresponding to \(\Psi_k(b_k)\) and \(\{P_m^k, P_m^{k} \forall r_k \in \Psi\}\) is evaluated. Suppose the relay-aided mode with any \(b_k^{m+1}\) is now used instead, the rate is reduced independently of \(b_k^{m+1}\) because of

\[
R_{k,1}^{m}(b_k^{m+1}) = \log_2(1 + \min(\eta_k^m(b_k^m), P_m^k G_{s,r,k}(k))) \\
\leq \log_2(1 + \eta_k^m(b_k^{m+1})) = \log_2(1 + P_m^k G_{s,r,k}(k)) \\
\leq \log_2(1 + P_m^k G_{s,r,k}(k)) \\nR_{k,2}^{m},
\]

where the equality in the third line is because of ∀ \(r_k \in \Psi, P_m^k, r_k = 0\) as will be shown in Section IV-B. In order to maximize the rate, \(t_k^{m+1} = 0\), i.e. subcarrier k remains in the direct mode.

Next, we consider the evaluation of \(t_k^{m+1}\) and \(b_k^{m+1}\) when \(t_k^m = 1\), meaning that subcarrier k was set in the relay-aided mode with \(b_k = b_k^m\) when \(\{P_m^k, P_m^{k} \forall r_k \in \Psi, \forall k\}\) was evaluated. As will be shown in Section IV-B, \(\eta_k^m(b_k^m) \leq P_m^k G_{s,r,k}(k)\) always holds and thus \(R_{k,1}^{m}(b_k^m) = \log_2(1 + \eta_k^m(b_k^m))\). It can readily be seen from (5) that

\[
\begin{align*}
\eta_k^m(b_k) & \begin{cases} \
\leq \eta_k^m(b_k^m) & \text{if } b_k > b_k^m; \\
\geq \eta_k^m(b_k^m) & \text{if } b_k < b_k^m,
\end{cases}
\end{align*}
\]

because ∀ \(r_k \notin \Psi_k(b_k^m), P_m^k, r_k = 0\) will be shown in Section IV-B. This means that ∀ \(b_k \in \{1, \ldots, N\}\), \(R_{k,1}^{m}(b_k) = \log_2(1 + \eta_k^m(b_k)) \leq R_{k,1}^{m+1}(b_k^m)\) always follows, i.e., if subcarrier k remains in the relay-aided mode, the rate can not be increased no matter which value is assigned to \(b_k\). In this case, \(t_k^{m+1}\) and \(b_k^{m+1}\) are determined with one of the following procedures:

- if \(R_{k,2}^{m+1} > R_{k,1}^{m+1}(b_k^m)\), \(t_k^{m+1}\) is set as 0 since the rate is increased if the direct mode is used;
- if \(R_{k,2}^{m} \leq R_{k,1}^{m}(b_k^m)\), \(t_k^{m+1}\) is still set as 1. \(b_k^{m+1}\) is assigned as the smallest \(b_k\) satisfying \(R_{k,1}^{m+1}(b_k) = R_{k,1}^{m+1}(b_k^m)\). Obviously, \(b_k^{m+1} \leq b_k^m\), and \(\Psi_k(b_k^{m+1})\) is the biggest set of assisting relays that leads to the same rate as \(\Psi_k(b_k^m)\). The motivation behind this assignment is twofold. One is to guarantee the rate when \(b_k = b_k^m\) is not smaller than when \(b_k = b_k^m\). The other is to increase the degrees of freedom for optimizing the power allocation in the next iteration, since ∀ \(r_k \in \Psi_k(b_k^{m+1})\) and \(r_k \notin \Psi_k(b_k^m)\), \(P_m^k, r_k = 0\) always holds but \(P_m^k, r_k \in \Psi_k(b_k^m)\) is free to be optimized in the \(m\)-th iteration. It can easily be shown that \(b_k^{m+1}\) is actually equal to the minimum \(b_k^m\) satisfying \(R_{k,1}^{m+1}(b_k^m) \leq R_{k,1}^{m+1}(b_k^m)\). Note that \(G_{s,r,k}(k) \geq \frac{P_m^k}{b_k} \geq G_{s,r,k}(k)\), which means that \(t_k^{m+1} \in B_k\).

It can easily be seen that a successive set of RAs with nondecreasing sum rate are produced as the iteration proceeds. After the algorithm converges, the RA produced by the last iteration is output as a suboptimum solution. This RA is at least a locally optimum solution to (8) (please refer to pages 160-162 of [26] for more details). In summary, the iterative algorithm is described in Algorithm 4.
Algorithm 5 The duality based RA algorithm to solve (8) when $t_k = t_k^m$ and $b_k = b_k^m$, $\forall k \notin D_a$, and $t_k = 0$, $\forall k \in D_a$.\[\begin{align*}
& l = 1, \nu = 1; \\
& \text{repeat} \nu_0 = [\nu_0 - \frac{1-iQ_2}{1+iQ_2}(1 - \sum_{k=1}^{K} P_s(k(\nu)))]; \\
& \nu_1 = [\nu_1 - \frac{1+iQ_2}{1+iQ_2}(1 - \sum_{k=1}^{K} P_{t,k}(\nu))], \forall r_i \in \Psi; \\
& l = l + 1; \\
& \text{for } k = 1 \text{ to } N \text{ do} \\
& \text{if } k \notin D_a \text{ or } t_k^m < 0 \text{ then} \\
& P_{s,k}(\nu) \text{ and } P_{t,k}(\nu), \forall r_i \text{ are found with (14) and (15) after replacing } \mu_k \text{ with } \nu_0; \\
& \text{else} \text{ if } k \notin D_a \text{ and } t_k^m = 0 \text{ then} \\
& P_{s,k}(\nu) \text{ and } P_{t,k}(\nu), \forall r_i \text{ are found with Algorithm 3 after replacing } \{b_k, \mu_s, \mu_r\} \forall r_i \in \Psi \text{ with } \{b_k^m, r_0, \nu_0\}; \\
& \text{end if} \\
& \text{end for} \\
& \text{until } P_{s,k} \text{ is feasible for (30) and} \\
& \nu_i(1 - \sum_{k=1}^{K} P_{s,k}(\nu)) + \sum_{k=1}^{K} \nu_i(1 - \sum_{k=1}^{K} P_{t,k}(\nu)) < \epsilon \\
& P_{s,k}(\nu) \text{ and } P_{t,k}(\nu), \forall r_i \in \Psi \text{ produced in the last iteration} \\
& \text{are the optimum solution to (8) when } t_k = t_k^m \text{ and } b_k = b_k^m, \forall k \notin D_a, \text{ and } t_k = 0, \forall k \in D_a. \]

V. NUMERICAL EXPERIMENTS

For illustration purposes, we consider an OFDM transmission system aided by 6 DF relays shown in Figure 3. The source is located at the origin, the destination is located at the coordinates $(0, -15)$, and $r_i, i = 1, \cdots, 6,$ is located at the coordinates $(-6, -7), (-4, -7), (-2, -7), (2, -7), (4, -7),$ and $(6, -7), \text{ respectively.}$ Note that all the aforementioned coordinate-related values have the unit of meter. The parameters are set as $\sigma^2 = -50 \text{ dBW}, \epsilon = 0.1, Q_1 = Q_2 = 50,$ and the duration of one time slot is 1 millisecond. In numerical experiments, we found that the convergence of Algorithm 1 was always observed when $K \geq 256$. Thus we choose $K = 256$.

We generate every channel according to the following assumptions. First, it is modeled as a 6-tap delay line, and the average received power at a distance of $d$ is equal to $\xi d^{-3}$, where $\xi$ represents a log-normal shadowing effect of 1 dB (i.e., $10 \log_{10}(\xi)$ is Gaussian distributed with zero mean and variance equal to 1 dB). Second, we assume the amplitude of the $i$-th tap is a circularly symmetric complex Gaussian random variable with zero mean and variance as $\sigma_i^2 = \sigma_0^2 e^{-3i}$, $i = 0, 1, \cdots, 5$.

In order to illustrate the effectiveness of the proposed algorithms, we consider a heuristic RA algorithm against which
the proposed RA algorithms are compared. This heuristic algorithm allocates the source sum power uniformly to all subcarriers, i.e., \( P_{s,k} = \frac{P_s}{|D_s|} \). Then, for every \( r_i \), \( \Omega_{r_i} = \{ k | k \not\in D_{s}, g_{s_r_i}(k) = \max_{k \in \Psi} g_{s_r_i}(k) \} \) which contains every subcarrier not belonging to \( D_{s} \) and at which \( r_i \) has the maximum source to relay channel gain, is first found. The sum power of \( r_i \) is uniformly allocated to all subcarriers in \( \Omega_{r_i} \) (i.e., if \( k \not\in \Omega_{r_i} \), \( P_{r_i,k} = 0 \), otherwise \( P_{r_i,k} = \frac{P_{r_i}}{|\Omega_{r_i}|} \)) where \( |\Omega_{r_i}| \) denotes the number of subcarriers in \( \Omega_{r_i} \). For every subcarrier \( k \in D_s \), the direct transmission mode is used. For every subcarrier \( k \not\in D_s \), \( R_{k,2} \) is first computed, and then the \( R_{k,1} \) when a single \( r_i \) with the maximum \( g_{s_r_i}(k) \) assists relaying is computed. If \( R_{k,2} \) is equal to or greater than the computed \( R_{k,1} \), the direct mode is used. Otherwise, the relay mode is used, and only the \( r_i \) used for computing \( R_{k,1} \) assists relaying.

We have generated a single random channel realization as shown in Figure 4, and tested the three RA algorithms when \( P_s = P_{r_i}, \forall r_i \in \Psi \), and \( P_s \) varies from 10 to 50 dBW. For the RA evaluated by each algorithm, the sum rate and the total energy of the source and the relays for transmitting per information bit (TETIB) are shown in Figure 5. We can see that for each \( P_s \) either Algorithm 1 or Algorithm 4 produces a better RA with a higher sum rate and a lower TETIB than the heuristic algorithm, and Algorithm 1 produces a better RA than Algorithm 4. Moreover, the RA evaluated by Algorithm 1 or Algorithm 4 results in a much smaller TETIB than that computed by the heuristic algorithm especially when \( P_s \) is low. For the sake of space limitation, the RAs evaluated by the three algorithms for the random channel realization only when \( P_s = P_{r_i} = 20 \) dBW, \( \forall r_i \in \Psi \), are shown in Figure 6, 7, and 8, respectively. When the RA evaluated by Algorithm 1 is used, \( r_3 \) and \( r_4 \) are enabled to assist relaying simultaneously at a few subcarriers, while only \( r_3 \) does when the RA computed by Algorithm 4 or the heuristic algorithm is used.

We have also generated 100 random channel realizations, and tested the three RA algorithms when \( P_s = P_{r_i}, \forall r_i \in \Psi \), and \( P_s \) varies from 10 to 50 dBW. For the RAs evaluated by each algorithm, the average sum rate and the average TETIB are shown in Figure 9. It can be seen that for each \( P_s \), either
Algorithm 4 or Algorithm 1 produces better RAs with a higher average sum rate and a lower average TETIB than the heuristic algorithm, and Algorithm 1 produces better RAs than Algorithm 4. Moreover, the RAs evaluated by Algorithm 1 or Algorithm 4 result in a much smaller average TETIB than those computed by the heuristic algorithm especially when $P_s$ is low.

VI. CONCLUSION

We have considered the sum rate maximized RA problem in a point to point OFDM transmission system assisted by multiple DF relays subject to the individual sum power constraints of the source and relays. In particular, one or several relays may cooperate with the source to transmit at every relay-aided subcarrier. We have proposed two RA algorithms which optimize the assignment of transmission mode and source power for every subcarrier, as well as the optimum assisting relays and the power allocation to them for every relay aided subcarrier. The effectiveness of the two algorithms has been illustrated by numerical experiments.

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APPENDIX A

When the duality gap is equal to zero, the duality based algorithm is one of the state-of-the-art methods to find a globally optimum solution to (9). There exist two important theorems justifying the ability of the duality based algorithm to find a globally optimum solution to (9). The first (Proposition 3.3.4 in [26]) is that $x_{\mu}$ is a globally optimum solution to (9), if it is feasible to (9) and $\mu^T(1-g(x_{\mu})) = 0$. The second (Proposition 5.1.4 in [26]) is that the $\mu$ satisfying the aforementioned conditions, denoted by $\mu^*$, must minimize $d(\mu)$ when $\mu \succeq 0$.

In general, the duality gap of (9) can be studied with a visualization based method proposed in [26]. Specifically, we can plot a cloud of points collected in the set $S = \{(p, w)|p =$...
has the coordinate $S$ function of $w$ where the gray area represents $S$ is equivalent to show that for any $\theta$ and a line passing through $x$ and perpendicular to the vector $(-\mu, 1)$. As shown in Fig. 10, the duality gap is equal to zero for any $\theta$. Thus $f(x')$ follows. Therefore, the duality gap of (9) is equal to zero when $K$ is very large.

![Diagram](image)

**Fig. 10.** The visualization of the duality gap in the hyperplane of $(p, w)$, where the gray area represents $S = \{(p, w) | p = g(x), w = f(x), x \in D_x\}$.

$g(x), w = f(x), x \in D_x$ in the hyperplane of $(p, w)$ shown in Figure 10. Most interestingly, $d(\mu)$ is equal to the $w$-coordinate of the highest intersection between the line $p = 1$ and a line passing through $S$ and perpendicular to the vector $(-\mu, 1)$. As shown in Fig. 10, the duality gap is equal to zero if the $w$-coordinate of the upper border of $S$ is a concave function of $p$. Mathematically, a point on the upper border of $S$ has the coordinate $(p, f(x_p))$ where

$$x_p = \arg \max_{x : x \in D_x, g(x) = p} f(x).$$

(31)

Based on the same idea first proposed in [23], we can show $f(x_p)$ is a concave function of $p$ when $K$ is very large. This is equivalent to show that for any $\theta \in [0, 1]$,

$$f(x_{p1} + (1 - \theta)p_2) \geq \theta f(x_{p1}) + (1 - \theta)f(x_{p2})$$

holds for any $p_1$ and $p_2$ stacking the sum power of the source and relays. Note that the above condition can be interpreted in a very interesting way as follows. Let’s adopt the RAs $x_p$ and $x_p$ in the $\theta$ and $1 - \theta$ portions of the whole transmission duration, respectively, which is called $\theta$ time sharing of $x_p$. and $x_p$ hereafter. In this way, an average sum rate $\theta f(x_{p1}) + (1 - \theta)f(x_{p2})$ can be achieved with an average sum power $\theta p_1 + (1 - \theta)p_2$. This means that showing the validity of (32) for any $\theta \in [0, 1]$, is equivalent to show that the optimum RA for the sum power $\theta p_1 + (1 - \theta)p_2$ provides a higher sum rate than $\theta$ time sharing of $x_p$ and $x_p$.

To this end, we show in the following that when $K$ is very large, a RA $x' \in D_x$, which is of the sum power $\theta p_1 + (1 - \theta)p_2$ and yields a sum rate equal to $\theta$ time sharing of $x_p$, and $x_p$, can be constructed by $\theta$ spectrum sharing of $x_p$ and $x_p$, i.e., taking the entries of $x_p$ at $\theta$ portion of all subcarriers, and the entries of $x_p$ at all the other subcarriers to construct $x'$. Specifically, we first divide all subcarriers into $S$ subbands, each consisting of $N_s$ adjacent subcarriers experiencing the same channel conditions. Suppose $K$ is sufficiently large so that $N_s$ is also very large, and $\forall \theta \in [0, 1]$ there exists an integer $N_s$ to $\{1, \ldots, N_s\}$ satisfying $\theta \approx \frac{N_s}{N}$. Since the subcarriers in each subband experience the same channel conditions, the entries of $x_p$ (or $x_p$) at the subcarriers in the same subband should be the same. Let’s construct $x'$, which in every subband adopts for the first $N_s$ subcarriers the same entries as in $x_p$, and for the remaining $N_s - N_s$ subcarriers the same entries as in $x_p$. We can easily show that the conditions $x' \in D_x$ and

$$g(x') \approx \theta p_1 + (1 - \theta)p_2,$$

$$f(x') \approx \theta f(p_1) + (1 - \theta)f(p_2)$$

are all satisfied. Thus $f(x_{p1} + (1 - \theta)p_2) \geq f(x') \approx \theta f(p_1) + (1 - \theta)f(p_2)$ follows. Therefore, the duality gap of (9) is equal to zero when $K$ is very large.

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