Vizing’s conjecture for cographs

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ABSTRACT. We show that if $G$ is a cograph, that is $P_4$-free, then for any graph $H$, $\gamma(G \Box H) \geq \gamma(G)\gamma(H)$. By the characterization of cographs as a finite sequence of unions and joins of $K_1$, this result easily follows from that of Bartsalkin and German. However, the techniques used are new and may be useful to prove other results.

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1. Introduction

Vizing’s conjecture [12], now open for fifty-three years, states that for any two graphs $G$ and $H$,

$$
\gamma(G \Box H) \geq \gamma(G)\gamma(H)
$$

(1.1)

where $\gamma(G)$ is the domination number of $G$.

The survey [4] discusses many results and approaches to the problem. For more recent partial results see [11], [10], [3], [6], [8], and [9].

A predominant approach to the conjecture has been to show it true for some large class of graphs. For example, in their seminal result, Bartsalkin and German [2] showed the conjecture for decomposable graphs. More recently, Aharoni and Szabó [1] showed the conjecture for chordal graphs and Brešar [3] gave a new proof of the conjecture for graphs $G$ with domination number 3.

We say that a bound is of Vizing-type if $\gamma(G \Box H) \geq c\gamma(G)\gamma(H)$ for some constant $c$, which may depend on $G$ or $H$. It is known [11] that all graphs satisfy the Vizing-type bound,

$$
\gamma(G \Box H) \geq \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G),\gamma(H)\}.
$$

Restricting the graphs, but as a generalization of Bartsalkin and German’s class of decomposable graphs, Contractor and Krop [6] showed

$$
\gamma(G \Box H) \geq \left(\gamma(G) - \sqrt{\gamma(G)}\right) \gamma(H)
$$

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where \( G \) belongs to \( \mathcal{A}_1 \), the class of graphs which are spanning subgraphs of domination critical graphs \( G' \), so that \( G \) and \( G' \) have the same domination number and the clique partition number of \( G' \) is one more than its domination number.

Krop \cite{8} showed that any claw-free graph \( G \) satisfies the Vizing-type bound

\[
\gamma(G \Box H) \geq \frac{2}{3} \gamma(G) \gamma(H)
\]

In this paper we show that the class of induced \( P_4 \)-free graphs, or cographs, satisfies Vizing’s conjecture.

1.1. Notation. All graphs \( G(V, E) \) are finite, simple, connected, undirected graphs with vertex set \( V \) and edge set \( E \). We may refer to the vertex set and edge set of \( G \) as \( V(G) \) and \( E(G) \), respectively. For more on basic graph theoretic notation and definitions we refer to Diestel \cite{7}.

For any graph \( G = (V, E) \), a subset \( S \subseteq V \) dominates \( G \) if \( N[S] = G \). The minimum cardinality of \( S \subseteq V \), so that \( S \) dominates \( G \) is called the domination number of \( G \) and is denoted \( \gamma(G) \). We call a dominating set that realizes the domination number a \( \gamma \)-set.

The Cartesian product of two graphs \( G_1(V_1, E_1) \) and \( G_2(V_2, E_2) \), denoted by \( G_1 \square G_2 \), is a graph with vertex set \( V_1 \times V_2 \) and edge set \( E(G_1 \square G_2) = \{((u_1, v_1), (u_2, v_2)) : v_1 = v_2 \text{ and } (u_1, u_2) \in E_1, \text{ or } u_1 = u_2 \text{ and } (v_1, v_2) \in E_2\} \).

A graph \( G \) is a cograph or \( P_4 \)-free if it contains no induced \( P_4 \) subgraph.

Let \( G \) be any graph and \( S \) a subset of its vertices. Chellali et al. \cite{5} defined \( S \) to be a \([j, k]\)-set if for every vertex \( v \in V - S, j \leq |N(v) \cap S| \leq k \). Clearly, a \([j, k]\)-set is a dominating set. For \( k \geq 1 \), the \([1, k]\)-domination number of \( G \), written \( \gamma_{[1,k]}(G) \), is the minimum cardinality of a \([1, k]\)-set in \( G \). A \([1, k]\)-set with cardinality \( \gamma_{[1,k]}(G) \) is called a \( \gamma_{[1,k]}(G) \)-set.

If \( \Gamma = \{v_1, \ldots, v_k\} \) is a minimum dominating set of \( G \), then for any \( i \in [k] \), define the set of private neighbors for \( v_i \), \( P_i = \{v \in V(G) - \Gamma : N(v) \cap \Gamma = \{v_i\}\} \). For \( S \subseteq [k] \), \( |S| \geq 2 \), we define the shared neighbors of \( \{v_i : i \in S\} \), \( P_S = \{v \in V(G) - \Gamma : N(v) \cap \Gamma = \{v_i : i \in S\}\} \).

For any \( S \subseteq [k] \), say \( S = \{i_1, \ldots, i_s\} \) where \( s \geq 2 \). We may write \( P_S \) as \( P_{\{i_1, \ldots, i_s\}} \) or \( P_{i_1, \ldots, i_s} \) interchangeably.

For \( i \in [k] \), let \( Q_i = \{v_i\} \cup P_i \). We call \( Q = \{Q_1, \ldots, Q_k\} \) the cells of \( G \). For any \( I \subseteq [k] \), we write \( Q_I = \bigcup_{i \in I} Q_i \) and call \( C(Q_I) = \bigcup_{i \in I} Q_i \cup \bigcup_{S \subseteq I} P_S \) the chamber of \( Q_I \). We may write this as \( C_I \).

For a vertex \( h \in V(H) \), the \( G \)-fiber, \( G^h \), is the subgraph of \( G \Box H \) induced by \( \{(g, h) : g \in V(G)\} \).

For a minimum dominating set \( D \) of \( G \Box H \), we define \( D^h = D \cap G^h \). Likewise, for any set \( S \subseteq [k] \), \( P_S^h = P_S \times \{h\} \), and for \( i \in [k] \), \( Q_i^h = Q_i \times \{h\} \).

By \( v_i^h \) we mean the vertex \( (v_i, h) \). For any \( I^h \subseteq [k] \), where \( I^h \) represents the indices of some cells in \( G \)-fiber \( G^h \), we write \( C_{\{I^h\}} \) to mean the chamber of \( Q_{I^h}^h \), that is, the set \( \bigcup_{i \in I^h} Q_i \cup \bigcup_{S \subseteq I^h} P_S^h \).

We may write \( \{v_i : i \in I^h\} \) for \( \{v_i^h : i \in I^h\} \) when it is clear from context that we are talking about vertices of \( G \Box H \) and not vertices of \( G \).
For clarity, assume that our representation of $G \square H$ is with $G$ on the $x$-axis and $H$ on the $y$-axis.

Any vertex $v \in V(G) \times V(H)$ is \textit{vertically dominated} if $(\{v\} \times N_H[h]) \cap D \neq \emptyset$ and \textit{vertically undominated}, otherwise. For $i \in [k]$ and $h \in V(H)$, we say that the cell $Q^i_h$ is \textit{vertically dominated} if $(Q^i \times N_H[h]) \cap D \neq \emptyset$. A cell which is not vertically dominated is \textit{vertically undominated}.

In our argument, we label vertices of a minimum dominating set $D$ of $G \square H$, by labels from $[k]$ so that for any $i \in [k]$, projecting the vertices labeled by $i$ onto $H$ produces a dominating set of $H$. We call a vertex $(x, h) \in D^h$ with the single label $i$, \textit{free}, if there exists another vertex $(y, h) \in D^h$, which is given the label $i$.

2. Cographs

\textbf{Theorem 2.1.} For any cograph $G$ and any graph $H$, $\gamma(G \square H) \geq \gamma(G)\gamma(H)$.

\textbf{Proof.} Let $\Gamma = \{v_1, \ldots, v_k\}$ be a minimum $[1, 2]$ dominating set of $G$ and let $D$ be a minimum dominating set of $G \square H$. By the result of Chellali et al. \cite{5} (Theorem 8), $\gamma(G) = k$. Suppose $u \in V(G) - \{\Gamma\}$ is adjacent to two vertices of $\Gamma$, say $v_1$ and $v_2$. Notice that if neither $v_1$ nor $v_2$ have private neighbors with respect to $\Gamma$, then we could replace $v_1$ and $v_2$ by $u$ in $\Gamma$ and produce a smaller dominating set, which is a contradiction. Hence, at least one of $P_1$ or $P_2$ is nonempty.

\textbf{Claim 2.2.} There exists a vertex in $P_1 \cup P_2$ which is independent from both $u$ and $V(G) - \{Q_1 \cup Q_2\}$.

\textbf{Proof.} Case 1: Suppose $P_1 \neq \emptyset$ and $P_2 = \emptyset$. Note that by the minimality of $\Gamma$ no vertex of $\Gamma - \{v_2\}$ can be adjacent to $v_2$. If $w_1 \in P_1$, then by definition of private neighbors, no vertex of $\Gamma - \{v_1\}$ is adjacent to $w_1$. If $u$ is not adjacent to $w_1$, then we produce $P_1 : w_1v_1uv_2$ which contradicts the definition of $G$. However, if $u$ is adjacent to every vertex of $P_1$, then we could replace $v_1$ and $v_2$ by $u$ in $\Gamma$ which would produce a smaller dominating set, which is impossible.

Case 2: Suppose $P_1, P_2 \neq \emptyset$. By minimality of $\Gamma$, some vertex of $P_1 \cup P_2$ is not adjacent to $u$. Suppose such a vertex is $w_2 \in P_2$. We may assume $v_1$ is adjacent to $v_2$, else we would produce $P_1 : w_2v_2uv_1$. For any vertex $w_1 \in P_1$, we may assume $w_1$ is adjacent to $w_2$, else we would produce $P_1 : w_1v_1w_2w_2$. Notice $u$ is adjacent to $w_1$ to avoid $P_1 : w_2w_1w_1u$. Suppose $u' \in V(G) - \{Q_1 \cup Q_2\}$ is adjacent to $w_2$ and suppose without loss of generality that $u'$ is adjacent to $v_3$.

![Figure 1](image-url)
Thus, we are left with the situation illustrated in Figure 2 where the drawn edges have been shown to exist.

\[
\begin{array}{c}
\text{Figure 2.}
\end{array}
\]

Since \( u' \) may adjacent to at most two vertices of \( \Gamma \) we argue that \( u' \) is adjacent to either \( v_1 \) or \( v_2 \), since otherwise we have \( P_4 : u'w_2v_2v_1 \).

Subcase (i): If \( u' \) is adjacent to \( v_2 \), then \( u' \) is also adjacent to \( w_1 \) to avoid \( P_4 : v_3u'w_2w_1 \). Furthermore, if \( v_3 \) is not adjacent to \( v_1 \) or \( v_1 \), then we produce \( P_4 : v_3u'w_2v_1 \). If \( v_3 \) is adjacent to \( v_1 \), then we produce \( P_4 : w_2v_2v_1v_3 \). Thus, \( v_2 \) is adjacent to \( v_3 \). However, now we have \( P_4 : v_3v_2w_1 \) which is impossible.

Subcase (ii): If \( u' \) is adjacent to \( v_1 \), then \( u' \) is also adjacent to \( w_1 \) to avoid \( P_4 : v_3u'w_2v_1 \). Furthermore, \( v_3 \) is adjacent to \( v_2 \), else we have \( P_4 : v_3u'w_2v_2 \).

For any \( h \in V(H) \), suppose the fiber \( G^h \) contains \( \ell_h(= \ell) \) vertically un-dominated cells \( U^h = \{ Q^h_{i_1}, \ldots, Q^h_{i_k} \} \) for \( 0 \leq \ell \leq k \). We set \( I^h = \{ i_1, \ldots, i_k \} \). Notice that for \( j_1, j_2 \in [k] - I^h \), no vertex of \( P^h_{j_1,j_2} \) may dominate any of \( v_{i_1}, \ldots, v_{i_k} \). Thus, \( \{ v_i : i \in I^h \} \) must be dominated horizontally in \( G^h \) either by shared neighbors of \( \{ v_i : i \in I^h \} \) or by vertices of \( \{ v_i : 1 \leq i \leq k, i \notin I^h \} \). Furthermore, the private neighbors \( \{ P^h_i : i \in I^h \} \) must be dominated horizontally in \( G^h \) either by shared neighbors of \( \{ v_i : i \in I^h \} \) or by vertices of \( \{ P^h_i : 1 \leq i \leq k, i \notin I^h \} \).

We label the vertices of \( D \) by the following Provisional Labeling. If a vertex of \( D^h \) for any \( h \in H \), is \( Q_i^h \) for \( 1 \leq i \leq k \), then we label that vertex by \( i \). If \( v \) is a shared neighbor of some subset of \( \{ v_i : i \in I^h \} \), then it is a member of \( P^h_{i,j} \) for some \( i, j \in I^h \), and we label \( v \) by the pair of labels \( (i, j) \). If \( v \) is a member of \( P^h_{i,j} \) for \( i \in I^h \) and \( j \in [k] - I^h \), then we label \( v \) by \( i \). If \( v \) is a member of \( P^h_{i,j} \) for \( i, j \in [k] - I^h \), then we label \( v \) by either \( i \) or \( j \) arbitrarily.

After the labels are placed, all vertices of \( D \) have a single label or a pair of labels.

Next, we apply a relabeling to some of the vertices of \( D \) which we call the First Refinement. For a fixed \( h \in H \), suppose \( v \) is some shared neighbor of two vertices of \( \{ v_i : i \in I^h \} \) in the chamber of \( Q^h_{j_1} \), which is vertically dominated, say by \( y \in D^{h'} \) for some \( h' \in H \), \( h \neq h' \). In other words, we suppose \( v \in P^h_{j_1,j_2} \) for some \( j_1, j_2 \in I^h \) which implies that \( y \in P^{h'}_{j_1,j_2} \).

The vertex \( y \) may be labeled by one or two labels, regardless of whether the First Refinement had been performed on \( D^{h'} \).
Suppose that $y$ is labeled by one label, say $j_1$. If $D^h$ contains a vertex $x \in P^h_{j_1,j_2}$, then we remove the pair of labels $(j_1, j_2)$ from $x$ and relabel $x$ by $j_2$.

Suppose $y$ is labeled by the pair of labels, $(j_1, j_2)$. If $D^h$ contains a vertex $x \in P^h_{j_1,j_2}$, then we remove the pair of labels $(j_1, j_2)$ from $x$ and then relabel $x$ arbitrarily by one of the single labels $j_1$ or $j_2$.

After the labeling, a vertex $v$ of $D$ may have a pair of labels $(i, j)$ if $v \in P^h_{i,j}$ and for any $h' \in N_H(h)$, $D^{h'} \cap P^h_{i,j} = \emptyset$.

Finally, we relabel some of the vertices of $D$ by the Second Refinement. For every $h \in H$, if $D^h$ contains vertices $x$ and $y$ with pairs of labels $(j_1, j_2), (j_2, j_3)$ respectively, for some integers $j_1, j_2$, and $j_3$, then we relabel $y$ by the label $j_3$. If $x$ and $y$ are labeled $j_1$ and $(j_1, j_2)$ respectively, for some integers $j_1, j_2$, we relabel $y$ by $j_2$. We apply this relabeling to pairs of vertices of $D^h$, sequentially, in any order.

**Claim 2.3.** After the Second Refinement every label on a vertex of $D$ is a single label.

**Proof.** For any $h \in V(H)$, suppose $v \in D^h$ has a pair of labels $(i, j)$. The Provisional Labeling prescribes that $i, j \in I^h$ which means that $Q_i$ and $Q_j$ are vertically undominated cells. If there exists $w \in D^h \cap P^h_{j,m}$ for any $1 \leq m \leq k$, or $x \in D^h \cap P^h_j$, then $v$ would have a single label after the Second Refinement which is not the case. By Claim 2.2 some vertex $x$ in $P^h_i \cup P^h_j$ is independent from $v$ and independent from $V(G) - \{Q_i \cup Q_j\}$. However, this means that $x$ is undominated, which contradicts the fact that $D$ is a dominating set. □

Suppose that for some $h \in V(H)$, $G^h$ contains a cell, $Q^h_i$, which is vertically undominated and the vertices of $D^h$ dominating $Q^h_i$ are not labeled $i$. In this case, $v^h_i$ can only be dominated by other members of $\{v^h_j : j \in [k], j \neq i\}$, so suppose for some $j_1 \neq i, j_1 \in [k]$, there exists $v^h_{j_1} \in D^h$ so that $v_i$ is adjacent to $v_{j_1}$. To avoid a contradiction to the minimality of $\Gamma$, we see that $P_i \neq \emptyset$ and say $u \in P^h_i$. Notice that if $u$ is dominated by some $u' \in P^h_{j_2} \cap D^h$ for some $j_2 \neq i, j_1$, then we produce $P_4 : v_{j_1}v_{j_2}uu'$ in $G^h$ and thus in $G$. Furthermore, if $v \in P^h_{j_1} \cap D^h$ dominates $u$, then $v$ is a free vertex labeled $j_1$ and we may relabel $v$ by $i$ without changing the vertically dominated status of cells $Q^h_{j_1}$ for any $h' \in V(H)$. Finally, suppose $u$ is dominated by some shared neighbor $w \in P^h_{j_1, j_2} \cap D^h$. Notice if $x \in P^h_{j_1}$, then we produce $P_4 : xv_{j_1}v_{j_2}$ and if $y \in P^h_{j_2}$, then we produce $P_4 : yv_{j_2}v_{j_1}$ which cannot occur. Thus, $P^h_{j_1} = P^h_{j_2} = \emptyset$. If for every $u' \in P^h_{j_1, j_2}$, $w$ is adjacent to $w'$, we have a contradiction to the minimality of $\Gamma$, since now we can replace $v_{j_1}$ and $v_{j_2}$ by the projection of $w$ onto $V(G)$ and form a smaller dominating set of $G$. We are left to suppose there exists $w' \in P^h_{j_1, j_2}$ so that $w'$ is not adjacent to $w$. To avoid $P_4 : w'v_{j_1}v_{j_2}$, we must also have $v_{j_2}$ adjacent to $v_i$. At this point, notice that $\{v_{j_1}, v_{j_2}, P^h_{j_1, j_2}\}$ is dominated by $v_i$ and $u$, which is a contradiction to the
minimality of $\Gamma$, since now we can replace $v_{j_1}$ and $v_{j_2}$ in $\Gamma$ by the projection of $u$ onto $V(G)$ and produce a smaller dominating set of $G$.

Notice that for any $i \in [k]$, projecting all vertices with a label $i$ to $H$ produces a dominating set of $H$. Summing over all $i$, we count at least $\gamma(G)\gamma(H)$ vertices in $D$.

□

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