Partial Strong Structural Controllability

Yuan Zhang, Yuanqing Xia
Department of Automatic Control, Beijing Institute of Technology, Beijing, China
Email: {zhangyuan14, xia_yuanqing}@bit.edu.cn

Abstract

This paper introduces a new controllability notion, termed partial strong structural controllability (PSSC), on a structured system whose entries of system matrices are either fixed zero or indeterminate, which naturally extends the conventional strong structural controllability (SSC) and bridges the gap between structural controllability and SSC. Dividing the indeterminate entries into two categories, generic entries and unspecified entries, a system is PSSC, if for almost all values of the generic entries in the parameter space except for a set of measure zero, and any nonzero (complex) values of the unspecified entries, the corresponding system is controllable. We highlight that this notion generalizes the generic property embedded in the conventional structural controllability for single-input systems. We then give algebraic and (bipartite) graph-theoretic necessary and sufficient conditions for single-input systems to be PSSC. Conditions for multi-input systems are subsequently given for a special case. It is shown the established results can induce a new maximum matching based criterion for SSC over the system bipartite graph representations.

Keywords: Structural controllability, strong structural controllability, generic property, maximum matching

1. Introduction

The past decades have witnessed an explosion of research interest into control and observation of complex networks [1, 2, 3]. This is perhaps because many real-world systems could be naturally modeled as complex dynamic networks, such as social networks, biological networks, traffic networks, as well as the internet [4]. Among the related problems, network controllability has been extensively explored both from a qualitative [5, 6, 7] and quantitative perspective [8, 9, 10].

A well-accepted alternative of the classical Kalman controllability is the notion of structural controllability. This notion was first introduced by Lin [10] by inspecting that controllability is a generic property, in the sense that depending on the structure of the state-space matrices, either the corresponding system realization is controllable for almost all values of the indeterminate entries, or there is no controllable system realization. Various criteria for structural controllability were found subsequently [11, 12]. Structural controllability is promising from a practical view, since it does not require the accurate values of system parameters, thus immune to numerical errors. Moreover, criteria for structural controllability usually are directly linked to some mild interconnection conditions of the graph associated with the system structure [13], making it attractive in analyzing large-scale network systems [14, 15, 16].

A more stringent notion than structural controllability is the strong structural controllability (SSC), which requires the system to be controllable for all (nonzero) values of the indeterminate entries [10]. As argued in [10], although the uncontrollable system realizations are atypical if a system is structurally controllable, there do exist scenarios where the existence of an uncontrollable system realization is prohibited, especially in some critical infrastructures where high-level robustness of the system controllability is required. The first criterion for SSC of single-input systems was given in [10], followed by some graph-theoretic characterizations of SSC of multi-input systems in [17, 18, 19] provided a constrained-matching based criterion, while [20] related SSC to the zero-forcing set and graph-coloring. Further, SSC of undirected networks was studied in [21]. For a comprehensive comparison of these criteria, see [22].

It is well-accepted that being SSC requires a system to have a much more restrictive structure than being structurally controllable. In fact, it is shown in [24, 25] that, for structural controllability, the ratio of controllable graphs (networks) to the total number of graphs with $n$ nodes tends to one, as $n \to \infty$. By contrast, [26] demonstrates that for SSC, the same ratio approaches zero, provided all nodes have self-loops and the ratio of control nodes to $n$ converges to zero. This phenomenon indicates that there is an essential gap between structural controllability and SSC. On the other hand, from a generic view, SSC requires that the system realization is controllable subject to all possible interrelations among the (nonzero) indeterminate parameters. In contrast, structural controllability only requires the system realization to be controllable assuming independence among all the indeterminate parameters. Note in practice, it is possible that only a partial subset of indeterminate parameters may be interrelated while the rest are independent of each other. This means, even a system itself is not SSC, the robustness requirement for system controllability might be met. The following example illustrates this:

Example 1 (Motivating example). Consider the following
inverted-pendulum system shown in Fig. 1, which also appears in [23]. Let \( \theta \) and \( x \) be respectively the angle of the pendulum rod and the location of the cart. Define state variables \( x_1, x_2, x_3, \) and \( x_4 \) by \( x_1 = \theta, x_2 = \dot{\theta}, x_3 = x, x_4 = \dot{x} \). According to [23], the linearized state-space equation of this system reads

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & a & b \\
0 & 0 & 0 & 0 \\
-\frac{mg}{l} & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} +
\begin{bmatrix}
0 \\
\frac{1}{l} \\
0 \\
0
\end{bmatrix} u
\]

where \( m \) and \( M \) are the masses of the pendulum bob and the cart, \( l \) is the length of the massless rod, \( u \) is the force imposed on the cart, and \( g \) is the acceleration of gravity. From [27], the zero/nonzero structure of \((A, B)\) is not SSC. However, direct calculation shows that the determinant of the controllability matrix of \((A, B)\) is \( a_{12}^2a_{14}b_{21}(a_{21}b_{42} - a_{41}b_{21})^2 \). This means, if we regard \( a_{21}, a_{14}, b_{41}, \) and \( b_{21} \) as independent parameters, leading to the usual satisfaction of \( a_{21}b_{42} - a_{41}b_{21} \neq 0 \), then whatever nonzero values \( a_{12} \) and \( a_{34} \) may take, the corresponding system will be controllable. In fact, back to this example, \( a_{21}b_{42} - a_{41}b_{21} = \frac{mg}{l} \neq 0 \) is always fulfilled.

Motivated by the above observations, this paper proposes a new controllability notion, named partial strong structural controllability (PSSC), trying to bridge the gap between structural controllability and SSC. In this notion, the indeterminate entries of a (structured) system are divided into two categories, generic entries that are assumed to take independent values, and unspecified entries that can take arbitrary nonzero (complex) values. A system is PSSC, if for almost all values of the generic entries, the corresponding system realization is controllable for all nonzero values of the unspecified entries. The main contributions are detailed as follows:

1) We propose a novel controllability notion of PSSC for linear structured systems. This notion naturally extends the conventional SSC, and for single-input systems, inherits and generalizes the generic property characterized by the conventional structural controllability.

2) We give an algebraic and a graph-theoretic necessary and sufficient conditions for single-input systems to be PSSC, the latter of which can be verified in polynomial time.

3) We extend 2) to the multi-input systems subject to the single unspecified entry constraint.

4) Finally, it is shown the established results can provide a new graph-theoretic criterion for SSC of single-input systems involving maximum matchings over the system bipartite graph representations.

Notably, our work provides a unifying viewpoint towards two seemingly different controllability notions: structural controllability and SSC. It is worth noting that a related notion is the perturbation-tolerant structural controllability proposed in [27]. Compared to that notion, PSSC addressed in this paper is an essentially different concept with many different properties. Particularly, PSSC contains the conventional SSC as a special case, while PTSC does not. Furthermore, from a technical view, due to the nonzero constraint of the unspecified entries, the techniques in [23] are not sufficient to characterize PSSC, and it is nontrivial to extend them to obtain the presented criteria here.

The rest of this paper is organized as follows. Section 2 presents the problem formulation and some preliminaries. Section 3 justifies the generic property involved in PSSC and provides necessary and sufficient conditions for single-input systems to be PSSC. Section 4 extends these results to the multi-input case subject to the single unspecified entry constraint. Section 5 discusses the implications of the established results for the existing SSC theory. All proofs of the technical results are given in the appendix.

Notations: \( \mathbb{N}, \mathbb{R}, \mathbb{C} \) denote the sets of natural, real, and complex numbers, respectively. Let \( \mathbb{R} = \mathbb{R}\setminus \{0\}, \mathbb{C} = \mathbb{C}\setminus \{0\}, \) and \( \mathbb{C}^n \) (resp. \( \mathbb{R}^n \)) be the set of \( n \)-dimensional complex (real) vectors with each of its entries being nonzero. For \( n \in \mathbb{N}, J_n \) stands for \( \{1, ..., n\}\). For an \( n \times m \) matrix \( M, M[i, j] \) denotes the submatrix of \( M \) whose rows are indexed by \( I_1 \subseteq J_n \) and columns by \( I_2 \subseteq J_m \).

2. Problem formulation and preliminaries

2.1. Structured matrix and generic matrix

We often use a structured matrix to denote the sparsity pattern of a numerical matrix [23]. A structured matrix is a matrix whose entries are chosen from the set \{0, *, x\}. The * entries and \( x \) entries are both called indeterminate entries. We use \( \{0, *, x\}^{p \times q} \) to denote set of all \( p \times q \) structured matrices with entries from \{0, *, x\}. Moreover, for \( M \in \{0, *, x\}^{p \times q} \), \( S_M \) is defined as

\[
S_M = \{ M \in \mathbb{C}^{p \times q} : M_{ij} \neq 0 \text{ if } M_{ij} = *, M_{ij} = 0 \text{ if } M_{ij} = 0 \}.
\]

That is, a * entry can take either zero or nonzero, while a \( x \) entry can take only nonzero value. An \( M \in S_M \) is called a realization of \( M \). We also define two structured matrices \( M^{*} \in \{0, *, x\}^{p \times q} \) and \( M^{x} \in \{0, *, x\}^{p \times q} \) associated with \( M \in \{0, *, x\}^{p \times q} \) respectively as

\[
M^{*} = \begin{cases} * & \text{if } M_{ij} = * \\
M_{ij} & \text{otherwise}
\end{cases} \quad M^{x} = \begin{cases} x & \text{if } M_{ij} = * \\
M_{ij} & \text{otherwise}
\end{cases}
\]

That is, \( M^{*} \) (resp. \( M^{x} \)) is obtained from \( M \) by replacing all of its indeterminate entries with * (resp. \( x \)) entries.

The operation ‘+’ between two structured matrices with the same dimension is an entry-wise addition operation so that every indeterminate entry in the sum appears exactly once in one of the addends. We define a generic realization of \( M \) as the realization whose indeterminate entries are assigned with independent parameters, and call such a realization a generic matrix without specifying the corresponding structured matrix. The generic rank of a generic matrix (or structured matrix), denoted by \( \text{rank}(\cdot) \), is the maximum rank it achieves as the function of its indeterminate parameters. For a generic matrix \( M \) and a constant matrix \( N \) with the same dimension, \( M + \lambda N \) defines a generic matrix pencil [28], which is a matrix-valued polynomial of the independent parameters in \( M \) and the variable \( \lambda \).

2.2. Notion of PSSC

Consider a linear time invariant system as

\[
\dot{x}(t) = Ax(t) + Bu(t),
\]

where \( x(t) \in \mathbb{C}^n \) is the state variable, \( u(t) \in \mathbb{C}^m \) is the input, \( A \in \mathbb{C}^{n \times n} \) and \( B \in \mathbb{C}^{n \times m} \) are respectively the state transition matrix and input matrix. For description simplicity, by the pair \((A, B)\), we refer to a system described by (2) that can be either single-input (i.e., \( m = 1 \)) or multi-input (i.e., \( m > 1 \)), while by \((A, B)\) we refer to a single-input system. Note dynamic systems with complex-valued system matrices and state variables are not rare in the literature [23, 30, 31, 51, 52, 53], either for practical utility or for theoretical studies. Following a similar manner as \((A, B)\) in the real field [23, Chap. 9], it turns out the existence
and uniqueness of solutions of system 2 are guaranteed even in the complex domain.

Denote the controllability matrix of system 2 by $\mathcal{C}(A, B)$, i.e.,

$$\mathcal{C}(A, B) = [B, AB, \cdots, A^{n-1}B].$$

It is known from [29, 51, 53] that $(A, B)$ is controllable, if and only if $\mathcal{C}(A, B)$ is of full row rank.

Throughout this paper, given $A \in \{0, *, x\}^{n \times n}$, $B \in \{0, *, x\}^{n \times m}$, $N_x$ denotes the set of positions of all $\times$ entries in $[A, B]$, i.e., $N_x = \{(i, j) : [A, B]_{ij} = x\}$. $N_e = \{(i, j) : [A, B]_{ij} = *\}$ is defined similarly. We use $\alpha_x$, and $\alpha_e$ to denote the numbers of $\times$ entries and $\times$ entries, respectively. Let $\alpha_0$, $\alpha_1$, and $\alpha_\infty$ be respectively the parameters for the $\times$ entries and $\times$ entries (also called indeterminate parameters) in $[A, B]$. Denote $p = (p_1, \ldots, p_{\alpha_0})$ and $\bar{p} = (p_1, \ldots, p_{\alpha_\infty})$. The following notions are natural extensions of the original ones from the real field to the complex field.

**Definition 1 (Structural controllability, [10]).** Given $A \in \{0, *, x\}^{n \times n}$, $B \in \{0, *, x\}^{n \times m}$, $(A, B)$ is structurally controllable, if there is an $(A, B)$ satisfying $[A, B] \in S_{[A, B]}$ that is controllable.

**Definition 2 (SSC, [16]).** Given $A \in \{0, *, x\}^{n \times n}$, $B \in \{0, *, x\}^{n \times m}$, $(A, B)$ is SSC, if $(A, B)$ is controllable for all $[A, B] \in S_{[A, B]}$.

Note that controllability is a generic property in the sense if $(A, \bar{B})$ is structurally controllable, where $A \in \{0, *, x\}^{n \times n}$, $\bar{B} \in \{0, *, x\}^{n \times m}$, then $(A, B)$ is controllable for almost all $[A, B] \in S_{[A, \bar{B}]}$.

Hereafter, ‘almost all’ means ‘all except for a set of zero Lebesgue measure in the corresponding parameter space’ Motivated by Example [1] as argued in Section 1 the PSSC is formally defined as follows.

**Definition 3 (PSSC).** Given $A \in \{0, *, x\}^{n \times n}$, $B \in \{0, *, x\}^{n \times m}$, suppose $(\bar{A}, B)$ is divided into $[A, B] = [A_x, B_x] + [A_\infty, B_\infty]$ in which $A_x \in \{0, *, x\}^{n \times n}$, $B_x \in \{0, *, x\}^{n \times m}$, $A_\infty \in \{0, *, x\}^{n \times \times}$, and $B_\infty \in \{0, *, x\}^{n \times m}$. $(A, B)$ is PSSC, if for almost all $[A_x, B_x] \in S_{[A_x, B_x]}$, $(A + A_\infty, B + B_\infty)$ is controllable for all $[A_x, B_x] \in S_{[A_x, B_x]}$.

Equivalently, $(\bar{A}, B)$ is PSSC, if for all $p \in C^{*n}$ except a set of zero measure in $C^{*n}$ and all $\bar{p} \in C^{*m}$, the corresponding realization of $(\bar{A}, \bar{B})$ is controllable. PSSC of $(\bar{A}, B)$ indicates that, if we randomly generate values for the $\times$ entries (from a continuous interval), then with probability 1, the corresponding system will be controllable for all nonzero values of the $\times$ entries. For this reason, we may also call an $\times$ entry a generic entry, and the $\times$ one an unspecified entry.

To show PSSC is well-defined, we introduce the concept perturbation-tolerant structural controllability (PTSC) from [27], which is defined for a structured system as the property that, for almost all values of its generic entries, there exist no complex values (including zero) for its unspecified entries such that the corresponding system realization is uncontrollable (see Definition 3 of [27] for a precise description). It is proven in [27] that, depending solely on the structure of the structured system, for almost all values of its generic entries, either there exist no values for its unspecified entries that can make the system realization uncontrollable, or there exist such values. Obviously, PTSC is a sufficient condition for the property in Definition 3 to hold. This means, there do exist structured systems satisfying the property in Definition 3.

Further, it is easy to see, if no $\times$ entries exist in $(\bar{A}, \bar{B})$, then PSSC collapses to SSC, which explains the term ‘partial’ in this terminology. On the other hand, if no $\times$ entries exist in $(\bar{A}, \bar{B})$, then PSSC collapses to structural controllability (this also demonstrates that PSSC is well-defined). In this sense, PSSC bridges the gap between structural controllability and SSC. Moreover, it is apparent that for a given $(\bar{A}, \bar{B})$, SSC implies PSSC, and PSSC implies structural controllability while the inverse direction is not necessarily true. The following Example 2 highlights the differences among these concepts.

**Example 2.** Consider a single-input structured system as

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{bmatrix}, \ b = \begin{bmatrix} x \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (3)$$

Let $(\bar{A}, \bar{b})$ be a generic realization of $(\bar{A}, b)$ by assigning $a_{ij}$ (resp. $b_{ij}$) to the $(i, j)$th indeterminate entry of $A$ (resp. $b$). We have

$$\det(\bar{A}, \bar{b}) = a_{21}a_{31}b_1a_{41}a_{44} + a_{21}a_{42} = 0. \quad (4)$$

Hence, $(\bar{A}, \bar{b})$ is structurally controllable. Moreover, in case $a_{41}a_{44} + a_{21}a_{42} = 0$, the obtained system will be uncontrollable, indicating that $(\bar{A}, \bar{b})$ is not SSC. However, for all $(a_{21}, a_{41}, a_{42}, a_{44}) \in \mathbb{C}^4$ except $\{a_{21}, a_{41}, a_{42}, a_{44}\} \in \mathbb{C}^4 : a_{21}a_{42}(a_{41}a_{44} + a_{21}a_{42}) = 0$, whatever nonzero values $a_{21}$ and $b_{11}$ take, the obtained system is controllable. This means $(\bar{A}, \bar{b})$ is PSSC.

**Remark 1 (Complex field and real field).** It is noted that in Definition 3 PSSC is defined for $(A, B)$ in the complex filed. As shown in Section 2.3, this can guarantee that if $(\bar{A}, \bar{b})$ is not SSC, then for almost all values of its $\times$ entries, there exist nonzero values for its $\times$ entries, such that the corresponding realization is uncontrollable. We may also define PSSC in the real field, by restricting that all of the indeterminate parameters should take real values. However, as explained in Remark 3, it then may happen that even for single-input systems, this property will not solely depend on the combinatorial properties of system structures. By Definition 3 and from the property of algebraic independence [34], it is not hard to see, PSSC in the complex field is sufficient for the same property to hold in the real field.

### 2.3. Preliminaries

A (directed) graph $G$ is represented by $G = (V, E)$, where $V$ is the vertex set and $E \subseteq V \times V$ is the edge set. If there is a path (i.e., a sequence of successive edges) from vertex $v_i$ to vertex $v_j$, we say $v_i$ is reachable from $v_j$. A subgraph $G_s = (V_s, E_s)$ of $G$ is a graph such that $V_s \subseteq V$ and $E_s \subseteq E$. And $G_s$ is said to be induced by $V_s$, if $E_s = (V_s \times V_s) \cap E$. For $V_s \subseteq V$, $\bar{G} - V_s$ denotes the graph obtained from $\bar{G}$ by deleting the vertices in $V_s$, together with their incident edges (i.e., $\bar{G} - V_s$ is the subgraph of $\bar{G}$ induced by $V \setminus V_s$).

A bipartite graph is written as $\bar{G} = (V^+, V^-, E)$ with $V^+$ and $V^-$ being the bipartitions and $E$ the edges [28]. For a $v \in V^+ \cup V^-$, $N(v, c)$ denotes the set of neighbors in $G$, i.e., the set of vertices that are connected with $v$ by an edge. A matching of $G$ is a set of its edges, no two of which share a common end vertex. The size (or cardinality) of a matching is the number of
edges it contains. A vertex is said to be matched by a matching if it is an end vertex of edges in this matching. A maximum matching of \(G\) is the largest (smallest) weight maximum matching of the largest (smallest) weight maximum matching is the maximum matching of \(G\) with the weight of a matching is the sum of its edge weights.

**Definition 4 (DM-decomposition, [28]).** Given a bipartite graph \(G = (V^+, V^-, E)\), the Dung-Mendelsohn decomposition (DM-decomposition) of \(G\) is to decompose \(G\) into subgraphs \(G_i = (V_i^+, V_i^-, E_i)\) \(i = \frac{1}{d}, \ldots, \frac{1}{d}, \infty\), each \(G_i\) is called a DM-component satisfying:

1. \(V^+ = \bigcup_{i=0}^\infty V_i^+; V_i^+ \cap V_j^+ = \emptyset\) if \(i \neq j\), with \(* = +, -\);
2. \(E_i = \{(v^+, v^-) \in E : v^+ \in V_i^+, v^- \in V_i^-\}\);
3. \(E_{ij} = \emptyset\) unless \(0 \leq i \leq j \leq \infty\), and \(E_{ij} \neq \emptyset\) only if \(0 \leq i \leq j \leq \infty\), where \(E_{ij} = \{(v^+, v^-) \in E : v^+ \in V_i^+, v^- \in V_j^-\}\);
4. \(M(\subseteq E)\) is a maximum matching of \(G\) if \(M \subseteq \bigcup_{i=0}^\infty E_i\) and \(M \cap E_i = \emptyset\) for \(i = \frac{1}{d}, \ldots, \infty\);
5. \(G\) cannot be decomposed into more components satisfying conditions 1)-4).

In Definition 4 \(G_0\) (if exists) is called the horizontal tail, and \(G_\infty\) the vertical tail. Additionally, if \(G\) contains only one DM-component, then \(G\) is called DM irreducible.

For an \(n_1 \times n_2\) matrix \(M\), the bipartite graph associated with \(M\) is given by \(B(M) = (V^+, V^-, E_M)\), where \(V^+ = \{v_{11}^+ \ldots v_{n_2}^+\}\) (resp. \(V^- = \{v_{11}^- \ldots v_{n_2}^-\}\)) corresponds to the rows (columns) of \(M\), and \(E_M = \{(v_{ij}^+, v_{ij}^-) : M_{ij} \neq 0, v_{ij}^+ = v_{ij}^- \in V^+, v_{ij}^- \in V^-\}\). It is known that, for a generic matrix \(M\), \(\text{rank}(M(B(M))) = [28]\). For \(V_i^+ \subseteq V^+\) and \(V_i^- \subseteq V^-\), we also write \(M(V_i^+, V_i^-)\) as the submatrix of \(M\) whose rows correspond to \(V_i^+\) and columns to \(V_i^-\), where elements in \(V_i^+\) and \(V_i^-\) are ordered. Note such an expression of submatrices is invariant subject to row and column permutations on \(M\), that is, upon letting \(P \in \mathbb{R}^{n_1 \times n_1}\) and \(Q \in \mathbb{R}^{n_2 \times n_2}\) be two permutation matrices and \(M' = PMQ\), \(M'[V_i^+, V_i^-]\) is the same as \(M[V_i^+, V_i^-]\).

For a structured pair \((\bar{A}, \bar{B})\), its associated graph is given by \(G(\bar{A}, \bar{B}) = (X, E_X)\), where \(X = \{x_1, \ldots, x_{n+m}\}\), \(E_X = \{(x_i, x_j) : [\bar{A}, \bar{B}]_{ij} \neq 0\}\). A vertex \(x_i\) in \(X\) is said to be input-reachable, if \(x_i\) is reachable from at least one vertices of \(x_{i+1}, \ldots, x_{n+m}\) in \(G(\bar{A}, \bar{B})\).

**Lemma 1.** [13] Given \(\bar{A} \in \{0, *, \}^{n \times n}, \bar{B} \in \{0, *, \}^{n \times n}\), \((\bar{A}, \bar{B})\) is structurally controllable, if and only if i) every vertex \(x \in X\) is input-reachable, and ii) \(\text{rank}[\bar{A}, \bar{B}] = n\).

**3. Properties and conditions of PSSC for single-input systems.**

In this section, we first present some properties of PSSC of single-input systems. Particularly, we show PSSC generalizes the generic property embedded in the conventional structural controllability. We then give a computationally efficient graph-theoretic criterion for PSSC.

**3.1. Properties.**

We first give an algebraic condition for PSSC, in terms of determinants of the controllability matrix of the generic realization of \((\bar{A}, \bar{B})\).

**Theorem 1.** Given \(\bar{A} \in \{0, *, \}^{n \times n}, \bar{B} \in \{0, *, \}^{n \times 1}\), let \([\bar{A}, \bar{B}]\) be a generic realization of \([\bar{A}, \bar{B}]\), with the parameters for the \(*\) entries being \(p_1, \ldots, p_n\), and for the \(\star\) entries being \(b_1, \ldots, b_n\). \((\bar{A}, \bar{B})\) is PSSC, if and only if det\([\bar{A}, \bar{B}]\) \neq 0 and has the following form

\[
\det([\bar{A}, \bar{B}]) = f(p_1, ..., p_n) \prod_{k=1}^{\infty} p_k^\nu_k,
\]

where \(f(p_1, ..., p_n)\) is a nonzero polynomial (including the constant 1) of \(p_1, ..., p_n\), \(\nu_k \geq 0\).

Although the verification of Theorem 1 is prohibitive for large-scale systems, as computing the determinant of a symbolic matrix has computational complexity increasing exponentially with its dimensions ([25]), it is theoretically significant in proving some properties of PSSC. Particularly, based on Theorem 1 we have the following two properties of PSSC of single-input systems.

**Proposition 1.** Given \(\bar{A} \in \{0, *, \}^{n \times n}, \bar{B} \in \{0, *, \}^{n \times 1}\), let \([\bar{A}, \bar{B}]\) and \([\bar{A}, \bar{B}]\) be defined in Definition 3. Then, depending on \([\bar{A}, \bar{B}]\), either for almost all \([\bar{A}_s, \bar{B}_s] \in S_{\bar{A}_s, \bar{B}_s}\), \((\bar{A}_s + \bar{A}_s + \bar{B}_s)\) is controllable for every \([\bar{A}, \bar{B}]\) or, for almost all \([\bar{A}_s, \bar{B}_s] \in S_{\bar{A}_s, \bar{B}_s}\), there is a \([\bar{A}_s, \bar{B}_s] \in S_{\bar{A}_s, \bar{B}_s}\) such that \((\bar{A}_s + \bar{A}_s + \bar{B}_s)\) is uncontrollable.

The above proposition reveals for a single-input structured system, if it is not PSSC, then for almost all values of the \(*\) entries, there exist nonzero (complex) values for the \(\star\) entries so that the realization is uncontrollable; otherwise, for almost all values of the \(\star\) entries, the corresponding realization is controllable for all nonzero values of the \(*\) entries. This generalizes the generic property embedded in structural controllability. This proposition also explains the motivation behind the definition of PSSC.

**Remark 2 (PSSC in real field).** From Theorem 1, it is easy to see, if \((\bar{A}, \bar{B})\) is PSSC, then for almost all \(p \in \mathbb{R}^\times\) and \(\bar{A} \in \mathbb{R}^{n \times n}\), the corresponding realization is controllable. However, if \((\bar{A}, \bar{B})\) is not PSSC, it is possible that for \(p \in \mathbb{R}^\times\), the realization is uncontrollable; otherwise, in almost all values of the \(*\) entries, the corresponding realization is controllable for all nonzero values of the \(\star\) entries. This generalizes the generic property embedded in structural controllability. This proposition also explains the motivation behind the definition of PSSC.
The above proposition is necessary in deriving conditions for PSSC, as it actually transforms the problem of verifying PSSC of \((A, b)\) to \(n_x\) subproblems of verifying PSSC of \((A^p, b^p)\) for each \(p \in \mathbb{N}_x\).

3.2. Necessary and sufficient conditions

In this subsection, we give testable necessary and sufficient conditions for single-input systems to be PSSC. Owing to Proposition 2 in the following, we first focus on conditions of PSSC for \((A, b),\) i.e., systems that contain only one \(x\) entry and then on the general systems. Recall from the PBH test \([27, 51]\), an uncontrollable mode for system \((A, b)\) is a \(\lambda \in \mathbb{C}\) such that \(\text{rank}(A - \lambda B) < n_x\). Inspired by \([19, 22]\), we shall respectively give the conditions for the nonexistence of zero uncontrollable and nonzero uncontrollable modes. Particularly, the following lemma, relating the ‘structure’ of a vector in the left null space of a given matrix to the full row rank of its submatrices, is fundamental to our subsequent derivations.

Lemma 2 (Lemma 7 of \([36]\)). Given \(M \in \mathbb{C}^{m \times n_x}\), let \(T\) consist of \(r\) linearly independent row vectors that span the left null space of \(M\). Then, for any \(K \subseteq J_{m_t}\), \(T[J_{m_t}, K]\) is of full row rank, if and only if \(M[J_{m_t}, K, J_{m_t}]\) is of full row rank.

Proposition 3. Suppose \((\bar{A}, \bar{b})\) is structurally controllable and \([\bar{A}, \bar{b}]\) contains only one \(x\) entry, with its position being \((i, j)\). Moreover, let \([\bar{A}, \bar{b}]\) be divided into \([A_x, b_x]\) in the way described in Definition 3. For almost all \([A_x, b_x] \in S_{[\bar{A}, \bar{b}]_l}\), there exist no \([A_x, b_x] \in S_{[\bar{A}, \bar{b}]_l}\) and nonzero vector \(q \in \mathbb{C}^n\) that satisfy \(q^{T}[A_x + A, b_x + b_x] = 0\), if and only if one of the following conditions holds:

1) \(\text{rank}([\bar{A}, \bar{b}], J_{m_t} + \{j\}) = n\); 
2) For each \(k \in N_{l}^{2}\), \(\text{rank}([\bar{A}, \bar{b}, J_{m_t} \backslash \{k\}, J_{m_t} + \{j\}) = n - 2\), in which \(N_{l} = \{k \in J_{m_t} \backslash \{i\} : [\bar{A}, \bar{b}]_{kj} = 0\}\); 
3) \(\text{rank}([\bar{A}, \bar{b}, J_{m_t} \backslash \{i\}, J_{m_t} + \{j\}) = n - 2\).

Proposition 4 gives an algebraic condition for the nonexistence of zero uncontrollable modes. The following proposition gives an equivalent bipartite graph form of Proposition 3. Recall the bipartite graph \(B([\bar{A}, \bar{b}]) = (V^+, V^-, E_{[\bar{A}, \bar{b}]})\) associated with \([\bar{A}, \bar{b}]\) is defined in Section 2.3 where \(V^+ = \{v^+_1, ..., v^+_n\}, V^- = \{v^-_1, ..., v^-_{m_t}\}\), and \(E_{[\bar{A}, \bar{b}]}\) is the bipartite graph associated with \(H_x\), defined as follows: \(V^+ = \{v^+_1, ..., v^+_n\}, V^- = \{v^-_1, ..., v^-_{m_t}\}\), and \(E_{[\bar{A}, \bar{b}]_l} = E_{1} \cup E_{[\bar{A}, \bar{b}]_l}\) in which \(E_{1} = \{(v^+_i, v^-_j) : k = 1, ..., n_x\}\) and \(E_{[\bar{A}, \bar{b}]_l} = \{(v^+_i, v^-_j) : [\bar{A}, \bar{b}]_l \neq 0\}\). An edge \(e\) is called an \(\lambda\)-edge if \(e \in E_{1}\), and a self-loop \(e\) if \(e \in E_{1} \cap E_{[\bar{A}, \bar{b}]_l}\). \(B(H_x^\lambda)\) is the subgraph of \(B(H_x)\) induced by \(V^+ \cup V^\lambda\) \(\backslash \{v^+_j\}\).

Let \(G^\lambda_k = (V^+_k, V^-_k, E_k)\) \((k = 0, ..., d)\) be the DM-components of \(B(H_x^\lambda)\). From \([27\text{, Lem 4]}\), if \((\bar{A}, \bar{b})\) is structurally controllable, we have \(\text{mt}(B(H_x^\lambda)) = n = j = 1, ..., n + 1\). Hence, \(G^\lambda_0 = G^\lambda_{d+1} = \emptyset, \forall j \in J_{m_t}\). By the correspondence between a matrix and its associated bipartite graph, the DM-decomposition of \(B(H_x^\lambda)\) corresponds to that the matrix \(H_x^\lambda\) is transformed into the block-triangular form via two \(n \times n\) permutation matrices \(P\) and \(Q\):

\[
PH_x^\lambda Q = \begin{bmatrix}
M^\lambda_1(\lambda) & \cdots & M^\lambda_n(\lambda) \\
0 & \ddots & \vdots \\
0 & 0 & M^\lambda_n(\lambda)
\end{bmatrix} \pm M^\lambda_n, \tag{6}
\]

where the submatrix \(M^\lambda_k(\lambda) = H_x^\lambda[V^+_k, V^-_k]\) corresponds to \(G_k\) \((k = 1, ..., d)\). Accordingly, let \(M^\lambda_0 = P^H_xQ^\lambda\).

For each \(k \in \{1, ..., d\}\), let \(\gamma_{\text{min}}(G^\lambda_k)\) and \(\gamma_{\text{max}}(G^\lambda_k)\) be respectively the minimum and maximum numbers of \(\lambda\)-edges contained in a matching over all maximum matchings of \(G^\lambda_k\). We borrow the boolean function \(\gamma_{\text{max}}(\cdot)\) for \(G^\lambda_k\) from \([27]\), which is defined as

\[
\gamma_{\text{max}}(G^\lambda_k) = \begin{cases}
1 & \text{if } \gamma_{\text{max}}(G^\lambda_k) - \gamma_{\text{min}}(G^\lambda_k) > 0 \\
0 & \text{otherwise.}
\end{cases}
\tag{7}
\]

The following lemma explains the motivation of introducing \(\gamma_{\text{max}}(\cdot)\).

Lemma 3 (Lemma 9 of \([27]\)). With notations as above, \(\text{det}M^\lambda_k(\lambda) (k \in \{1, ..., d\})\) generically has nonzero roots for \(\lambda\), if and only if \(\gamma_{\text{max}}(G^\lambda_k) = 1\)

Next, define the set

\[
\Omega_j = \{k \in \{1, ..., d\} : \gamma_{\text{max}}(G^\lambda_k) = 1\}. \tag{8}
\]

Moreover, for a vertex \(v^+_j \in V^+, \) define a set \(\Omega_j' \subseteq \Omega_j\) as

\[
\Omega_j' = \{k \in \Omega_j : G^\lambda_k \text{ is not covered by at least one maximum matching of } B(H_x^\lambda) = \{v^+_j\}\}. \tag{9}
\]

It is clear \(\Omega_j' = \emptyset\) implies that every maximum matching of \(B(H_x^\lambda) \backslash \{v^+_j\}\) covers \(\bigcup_{k \in \Omega_j} G^\lambda_k\). As we shall see, \(\Omega_j'\) is the set of indices of all DM-components associated with which \(\text{det} M^\lambda_k(\lambda)\) has a nonzero root \(z\), such that \(M^\lambda_k[V^+ \backslash \{v^+_j\}, V^-]\) is of full row rank \((M^\lambda_k)\) is obtained by substituting \(\lambda = z\) into \(M^\lambda_k\).

Hereafter, by saying \(M^\lambda_k\) (or its sub-matrices) satisfies certain properties, we mean these properties are satisfied for almost all values of the corresponding indeterminate parameters (i.e., they are satisfied generically).

Lemma 4. Let \(M^\lambda_k\) be defined in \([0]\). Assume \(\Omega_j' = \emptyset\). The following properties are true:

1) Given a \(v^+_j \in V^+, M^\lambda_k[V^+ \backslash \{v^+_j\}, V^-]\) is generically row rank deficient for all \(z \in \{\lambda \in \mathbb{C}\setminus\{0\} : \text{det} M^\lambda_k = 0\}\), if and only if \(\Omega_j' = \emptyset\). \(^{3}\)
2) Suppose \( \Omega_i \neq \emptyset \). For all nonzero \( z \) making \( M_z^{k}\{V^+\{v_i^+\}, V^-\} \) of full row rank and \( \det M_z^{k} = 0 \) simultaneously, \( M_z^{k}\{V^+\{v_i^+\}, V^-\} \) is not of full row rank for a given \( v_i^+ \in V^+\{v_i^+\} \), if and only if every maximum matching of \( \mathcal{B}(H^{k}_{c}) - \{v_i^+\} \) covers \( \bigcup_{k \in \Omega_{i}} \mathcal{G}_c^{k} \).

Proposition 5. Suppose \( \bar{A}(\bar{b}) \) is structurally controllable and \( [A,b] \) contains only one \( x \times b \) entry, with its position being \((i,j)\), let \([A,b] = \{A_i,b_i\} + \{A_x,b_x\} \) in the way described in Definition 3. For almost all \( [A_x,b_x] \in S[A_i,b_i] \), there exist no \( [A_x,b_x] \in S[A_i,b_i], \) nonzero complex number \( z \) and nonzero vector \( q \in C^n \) that satisfy \( q^{T}\{A_x + A_x - zI_x + b_x + b_x\} = 0 \), if and only if one of the following conditions holds:

\( c1 \) \( \Omega_i = \emptyset \);

\( c2 \) \( \Omega_i \neq \emptyset, i \neq j \), and for each \( v_i^+ \in \mathcal{N}(B(H_x), v_j^-) \{v_i^+\} \), every maximum matching of \( \mathcal{B}(H^{k}_{c}) - \{v_i^+\} \) covers \( \bigcup_{k \in \Omega_{i}} \mathcal{G}_c^{k} \).

Proposition 5 gives the necessary and sufficient condition for the nonexistence of nonzero uncontrollable modes. Combining Propositions 2 and 5 yields a necessary and sufficient condition for general single-input systems to be PSSC.

Theorem 2. Given \( \bar{A}(\bar{b}) \in \{0,*,x\}^{n \times n}, \bar{b} \in \{0,*,x\}^{n \times 1}, \) \( (\bar{A},\bar{b}) \) is PSSC, if and only if \( (\bar{A},\bar{b}) \) is structurally controllable, and for each \( \pi = (i,j) \in N_{x} \), the following two conditions hold for the system \( (\bar{A}^\pi,\bar{b}^\pi) \), recalling \( (\bar{A}^\pi,\bar{b}^\pi) \) is defined in Proposition 3:

1) Condition b1), condition b2), or condition b3) holds;
2) Condition c1) or condition c2) holds.

We present some examples to illustrate Theorem 2.

Example 3. Consider the system \((\bar{A},\bar{b})\) in Example 1. In this system, \( N_x = \{(3,2),(1,5)\} \). For \( (\bar{A},\bar{b}) \), the associated \( \mathcal{B}(\bar{A}^\pi,\bar{b}^\pi) \), \( \mathcal{B}(H_x) \), and \( \mathcal{B}(H^{k}_{c}) \) \( (j = 2) \), as well as its DM-decomposition of \( \mathcal{B}(H^{k}_{c}) \), are given respectively in Figs. 2(a) and 2(c). From Fig. 2(a), condition b2) is fulfilled, as the bipartite graph \( \mathcal{B}(\bar{A}^\pi,\bar{b}^\pi) - \{v_i^+,v_j^+\} \) has a maximum matching with size 2. From Fig. 2(c), \( \Omega_1 = \{2\} \) and \( \Omega_2 = \emptyset \), implying condition c1) is fulfilled. Similarly, for \( (\bar{A},\bar{b}) \), the associated \( \mathcal{B}(\bar{A}^\pi,\bar{b}^\pi) \), and \( \mathcal{B}(H^{k}_{c}) \) \( (j = 5) \), as well as its DM-decomposition, are given respectively in Figs. 2(a) and 2(d). Fig. 2(a) shows \( \mathcal{N}(\mathcal{B}(\bar{A}^\pi,\bar{b}^\pi),v_i^-)\{v_i^+\} = \emptyset \), meaning condition b2) is satisfied. Fig. 2(d) indicates \( \Omega_5 = \{2\} \) and \( \Omega_6 = \{2\} \), meanwhile, \( \mathcal{N}(\mathcal{B}(H_x), v_i^-)\{v_i^+\} = \emptyset \). This means condition c2) is satisfied. As a consequence, \( (\bar{A},\bar{b}) \) is PSSC, which is consistent with the analysis in Example 2. Further, suppose we change \((\bar{A},\bar{b})\) to

\[
\bar{A} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad \bar{b} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}.
\]

Then, \( N_{x} = \{(4,1),(4,4)\} \). For each \( \pi \in N_{x} \), \( \mathcal{B}(\bar{A}^\pi,\bar{b}^\pi) \) and the corresponding \( \mathcal{B}(H_x) \) are of the same form as Figs. 2(a) and 2(b) respectively. For \( (\bar{A},\bar{b}) \), \( \mathcal{B}(\bar{A}^\pi,\bar{b}^\pi) \) satisfies condition b2), with the corresponding \( \mathcal{B}(H^{k}_{c}) \) \( (j = 4) \) and its DM-decomposition given in Fig. 2(e). It turns out that \( \Omega_3 = \Omega_4 = \{3\} \), and meanwhile, \( i \neq j \). Therefore, neither condition c1) nor c2) is fulfilled, meaning \( (\bar{A},\bar{b}) \) in (19) is not PSSC. This can be validated by using Theorem 4. Alternatively, we can obtain the same conclusion by inspecting that, for \( \pi = (4,1) \), the corresponding \( \mathcal{B}(H^{k}_{c}) \) \( (j = 1) \); see Fig. 2(f), does not satisfy condition c1) or c2). See, associated with \( \mathcal{B}(H^{k}_{c}) \), \( \Omega_1 = \{2\} \), and there is a maximum matching of \( \mathcal{B}(H^{k}_{c}) - \{v_i^+\} \) that does not cover \( \mathcal{G}_c^{k} \).
determine the set $\Omega_j^\prime$. Afterwards, if $i \neq j$ and $\Omega_j^\prime \neq 0$, for a
given $v_i^+ \in N(B(H_s), v_j^-) \{v_j^+\}$, to determine whether the third item of condition (c2) is fulfilled, we can adopt a similar manner to the preceding scenario, that is, assigning weight to the edge $e$ of $B(H_s^-) - \{v_i^+\}$ as follows

$$w(e) = \begin{cases} 
1 & \text{if } e \in \left( \bigcup_{k \in \Omega_j^\prime} E_k \right) \setminus \{(v_i^+, v_j^-) : (v_i^+, v_j^-) \in E_{H_s}\} \\
0 & \text{otherwise}.
\end{cases}$$

(13)

Then, similarly, it follows that, every maximum matching of $B(H_s^-) - \{v_i^+\}$ covers $\Omega_j^\prime$, if and only if the unique minimum weight over all maximum matchings of the weighted $B(H_s^-) - \{v_i^+\}$ equals $|\bigcup_{k \in \Omega_j^\prime} V_k^+|$. It is remarkable that determining whether $\Omega_j^\prime = \emptyset$ can be implemented in a similar manner, i.e., by replacing $\Omega_j^\prime$ in (13) with $\Omega_i^\prime$.

Let us figure out the computational complexity of the above procedure. Note that determining the maximum matching of a bipartite graph with $|V|$ vertices and $|E|$ edges incurs $O(\sqrt{|V||E|})$ time via the Hopcroft-Karp algorithm, and there are algorithms computing the maximum weighted matching in $O(|V|^3\frac{\log^2|V|}{\log|E|})$. In addition, DM-decomposition has the same complexity as finding a maximum matching [25]. Verifying whether $(\bar{A}, \bar{B})$ is structurally controllable can invoke the strongly-connected component decomposition and maximum matching algorithms, which incurs $O(n^{2.5})$. Moreover, as analyzed above, for each $(\bar{A}^n, \bar{B}^n)$, conditions (b1), (b2) and (b3) can be verified in $O(n \cdot n^{0.5} \cdot |E_X|) \rightarrow O(n^{3.5})$ time, and conditions (c1) and (c2) can be checked in time complexity at most

$$O(n^2) \quad \text{DM-decomposition} \quad \text{finding } \Omega_j^\prime \quad \text{determining } \Omega_j^\prime \quad \text{checking (c2)}$$

$$\rightarrow O(n^4).$$

To sum up, since there are $|V_X| \times \text{entries}$, the total time complexity of Theorem 2 is most $O(|V_X| n^4)$.

4. A special case for multi-input systems

In this section, we consider PSSC of a special case in multi-input systems, namely, when, for each $(\bar{A}, \bar{B})$, the set $\{\bar{V}_i\} \neq \emptyset$ at almost any position. Thus, the corresponding property does not hold for multi-input systems, for which the general case might need further inspection beyond the scope of this paper.

In the rest of this section, recall $[\bar{A}, \bar{B}]$ is divided into $[\bar{A}, \bar{B}] = [\bar{A}_s, \bar{B}_s] + [\bar{A}_c, \bar{B}_c]$ in the way described in Definition 3. First, the result below indicates the similar generic property in Proposition 3 still holds for multi-input systems with a single $\times$ entry.

**Proposition 6.** For a multi-input system $(\bar{A}, \bar{B})$, assume that there is only one $\times$ entry in $[\bar{A}, \bar{B}]$. Then, either for almost all $[\bar{A}_s, \bar{B}_s] \in \tilde{S}^\prime_{\bar{A}_s, \bar{B}_s}$, $(\bar{A}_s + \bar{A}_c, \bar{B}_s + \bar{B}_c)$ is controllable for each $[\bar{A}_s, \bar{B}_s] \in \tilde{S}^\prime_{\bar{A}_s, \bar{B}_s}$, or for almost all $[\bar{A}_s, \bar{B}_s] \in \tilde{S}^\prime_{\bar{A}_s, \bar{B}_s}$, there is a $[\bar{A}_c, \bar{B}_c] \in \tilde{S}^\prime_{\bar{A}_c, \bar{B}_c}$ such that $(\bar{A}_s + \bar{A}_c, \bar{B}_c + \bar{B}_c)$ is uncontrollable.

The generic property presented above is characterized by PSSC of $(\bar{A}, \bar{B})$. Next, similar to Proposition 3, the following proposition gives the necessary and sufficient condition for the nonexistence of zero uncontrollable modes.

**Proposition 7.** Suppose $(\bar{A}, \bar{B})$ is structurally controllable and $[\bar{A}, \bar{B}]$ contains only one $\times$ in its $(i, j)$th position. For almost all $[\bar{A}_s, \bar{B}_s] \in \tilde{S}^\prime_{\bar{A}_s, \bar{B}_s}$, there exist no $[\bar{A}_c, \bar{B}_c] \in \tilde{S}^\prime_{\bar{A}_c, \bar{B}_c}$ and nonzero vector $q \in \mathbb{C}^n$ that satisfy $q^\dagger (\bar{A}_c + \bar{A}_x, \bar{B}_c + \bar{B}_x) = 0$, if and only if one of the following conditions holds

1) $B([\bar{A}, \bar{B}])$ contains a maximum matching that does not match $v_j^-$;

2) For each $v_i^+ \in N(B([\bar{A}, \bar{B}], v_j^-) \{v_j^+\}$ (if exists), $\text{mt}(B([\bar{A}, \bar{B}]) - \{v_i^+, v_j^-\}) = n - 2$;

3) $\text{mt}(B([\bar{A}, \bar{B}]) - \{v_i^+, v_j^-\}) = n - 2$.

Suppose $(\bar{A}, \bar{B})$ is structurally controllable and $[\bar{A}, \bar{B}]$ contains only one $\times$ in its $(i, j)$th position. Let $[\bar{A}, \bar{B}]$ be a generic realization of $(\bar{A}, \bar{B})$, define a generic matrix pencil $H_s = [\bar{A} - \lambda I, \bar{B}]$, and let $H_s^* = H_s|_{J_0, J_{n+1}\setminus\{j\}}$. Let $B(H_s) = (V^+, V^-, E_{H_s})$ and $B(H_s^*)$ be the bipartite graphs associated with $H_s$ and $H_s^*$, respectively, defined in the same way as in Section 4. Note compared with the single-input case, the essential difference is that $|V^\prime - \{v_j^-\}| = n + m - 1 \geq |V^\prime|$, which results in that there are horizontal or vertical tails in DM-decomposing $B(H_s^*)$. Owing to the structural controllability of $(\bar{A}, \bar{B})$, a trivial extension of Lemma 4 shows that $\text{mt}(B(H_s^*)) = n$. Consequently, by Definition 4 there is only a horizontal tail in the DM-decomposition of $B(H_s^*)$ ($m > 1$). Let $G_k^\prime = (V_k^+, V_k^-, E_k)$ ($k = 0, 1, ..., d$) be the DM-components of $B(H_s^*)$. The following intermediate result is crucial for the subsequent derivations.

**Lemma 5.** If $m > 1$, there is generically no nonzero $\lambda$ that can make $H_s^* \{V_k^+, V_k^-\}$ row rank deficient.

Moreover, associated with $B(H_s^*)$ and $G_k^\prime$ ($k = 0, 1, ..., d$), let $\Omega_i^\prime$ and $\Omega_j^\prime$ ($v_i^+ \in V_k^+$) be defined in the same way as [9] and [9], respectively. Particularly, Lemma 5 implies $H_s^* \{V_k^+, V_k^-\}$ would contribute no nonzero $\lambda$ that can make $H_s^* \{V_k^+, V_k^-\}$ row rank deficient (thus $\emptyset \notin \Omega_j^\prime$). We have the following proposition, providing a necessary and sufficient condition for the nonexistence of nonzero uncontrollable modes.

**Proposition 8.** Suppose $(\bar{A}, \bar{B})$ is structurally controllable and $[\bar{A}, \bar{B}]$ contains only one $\times$ in its $(i, j)$th position. For almost all $[\bar{A}_s, \bar{B}_s] \in \tilde{S}^\prime_{\bar{A}_s, \bar{B}_s}$, there exist no $[\bar{A}_c, \bar{B}_c] \in \tilde{S}^\prime_{\bar{A}_c, \bar{B}_c}$ and nonzero complex number $z$ and nonzero vector $q \in \mathbb{C}^n$ that satisfy $q^\dagger (\bar{A}_c + \bar{A}_x - zI, \bar{B}_c + \bar{B}_x) = 0$, if and only if one of the following conditions holds

1) $\Omega_j^\prime = \emptyset$;

2) $\Omega_j^\prime \neq \emptyset$, $i \neq j$, and for each $v_i^+ \in N(B(H_s), v_j^-) \{v_j^+\}$, every maximum matching of $B(H_s^*) - \{v_i^+\}$ covers $\bigcup_{k \in \Omega_j^\prime} G_k^\prime$.

Combining Propositions 7 and 8 yields a necessary and sufficient condition for PSSC of $(\bar{A}, \bar{B})$ with a single $\times$ entry.

**Theorem 3.** Suppose $(\bar{A}, \bar{B})$ contains only one $\times$ in its $(i, j)$th position. $(\bar{A}, \bar{B})$ is PSSC, if and only if: i) $(\bar{A}, \bar{B})$ is structurally controllable, ii) Condition d1), d2) or d3) holds, and iii) Condition e1) or e2) holds.

Similar to the single-input case, Theorem 3 can be verified in polynomial time mainly via the (weighted) maximum matching computations. Specifically, following a similar manner to Section 4, it can be found the total complexity of Theorem 3 is at most $O(n(n+m)^3)$.
Remark 3. Although presented in an analogous form to the single-input case, results in this section are not simple extensions of the previous section. As shown in our derivations, since $C(A, B)$ is not square, the proof for generarity needs to consider multiple $n \times n$ submatrices of $C(A, B)$. Besides, as $H^A_{\lambda}$ is no longer square, we have to consider the horizontal tail of the DM-components of $B(H^B_{\lambda})$.

While Theorem 3 is devoted to the single-input case, it can provide some necessary conditions for PSSC of more general cases. Specifically, it is easy to see, for $(\bar{A}, \bar{B})$ to be PSSC, by preserving arbitrary one of its $x$ entries and changing the remaining $x$ entries to $\ast$, the obtained structured system should be PSSC, i.e., satisfying the conditions in Theorem 3; otherwise, $(\bar{A}, \bar{B})$ cannot be PSSC by Proposition 4.

Example 5. Consider $(\bar{A}, \bar{B})$ as

$$
\bar{A} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & \ast & \ast & 0 & 0 \\
0 & 0 & 0 & \ast & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & \ast & 0 & 0 & 0
\end{bmatrix}, \quad \bar{B} = \begin{bmatrix}
\ast & 0 \\
0 & \ast \\
0 & 0 \\
0 & \ast \\
0 & \ast
\end{bmatrix},
$$

which is structurally controllable. The associated $B(\bar{A}, \bar{B})$ and $B(H^B_{\lambda})$ (and its DM-decomposition, $j = 5$) are given in Figs. 3(a) and 3(b). From them, we know $B(\bar{A}, \bar{B}) = \{v_3^b, v_7^b\}$ has a maximum matching with size 4, and $\Omega_3 = \emptyset$. This means the conditions of Theorem 3 are satisfied. Hence, $(\bar{A}, \bar{B})$ is PSSC. Further, it can be found that, by replacing arbitrary one of the indeterminate entries of $(\bar{A}, \bar{B})$ with $\ast$ and changing the remaining ones to $\ast$, the obtained structured system is still PSSC. This is consistent with the fact that $(\bar{A}, \bar{B})$ is actually SCC (c.f. [14, Theo 4]).

5. Implications for the existing SSC theory

In this section, we point out the proposed PSSC criteria in the special case can provide new graph-theoretic conditions for SSC, even restricted to the real field. Further, we demonstrate the new conditions can provide a classification of edges (or indeterminate entries) with respect to system controllability.

As mentioned earlier, when there is no $\ast$ entry in $[\bar{A}, \bar{b}]$, Theorem 3 collapses to the criterion for SSC (in the complex field). With this idea, the following corollary provides a new necessary and sufficient condition for SSC in terms of (weighted) maximum matchings over the system bipartite graph representation.

Corollary 1. Given $\bar{A} \in \{0, 1\}^{n \times n}$, $\bar{b} \in \{0, 1\}^{n \times 1}$, $(\bar{A}, \bar{b})$ is SSC in the real field (i.e., $(\bar{A}, \bar{b})$ is controllable for all real-valued $[\bar{A}, \bar{b}] \in \mathbf{S}_{\bar{A}, \bar{b}}$), if and only if $(\bar{A}, \bar{b})$ is structurally controllable, and for each $\pi \in \mathcal{N}_\pi$, $([\bar{A}, \bar{b}]^\pi$) satisfies: i) at least one of conditions b1), b2) or b3) holds; and ii) condition c1) or condition c2) holds.

Although it is not hard to expect that, necessary and sufficient conditions for SSC in the real field and the complex field should have the same form (c.f. [17]), we have provided a self-contained proof for the corollary above from the developed PSSC theory (see the Appendix). Compared to [14, Theo 5], Corollary 1 is not appealing in terms of computational complexity. Nevertheless, since Corollary 1 is an entry-wise criterion, it does provide some deep insight into the role of each edge of $G(\bar{A}, \bar{b})$ in system controllability.

Example 6. Consider $(\bar{A}, \bar{b})$ that is obtained from system 5 by replacing all its indeterminate entries with $\ast$ entries. Corollary 1 yields that $(\bar{A}, \bar{b})$ is not SSC. Further, a byproduct of Corollary 4 on this system is the following classifications for its edges (which can be obtained from Example 3): (a) critical edges: $(x_1, x_2)$, $(x_1, x_4)$, $(x_2, x_4)$ and $(x_3, x_4)$; (b) stable edges: $(x_2, x_3)$, and $(x_5, x_1)$. It is easy to validate the above assertion by noting that the determinant of a generic realization of $(\bar{A}, \bar{b})$ is exactly the right-hand side of 4.

6. Conclusions

In this paper, a new controllability notion, named PSSC, has been proposed for linear systems, with the aim to extend the existing SSC and bridge the gap between structural controllability and SSC. Algebraic and bipartite graph-theoretic necessary and sufficient conditions are given for single-input systems to be SSC, depending on what conditions in Theorem 2 are satisfied for $(\bar{A}^\pi, \bar{b}^\pi)$, the edge $(x_i, x_j)$ can be classified into:

- Critical edge: there is a nonzero (complex) value for $\bar{p}_{i,j}$ making the corresponding realization uncontrollable, for almost all $\bar{p}_{i,j} \in \mathbb{C}^{n \times n}$, if $([\bar{A}, \bar{b}]^\pi$) does not satisfy condition 1) or condition 2) of Theorem 2;
- Stable edge: there is no nonzero value for $\bar{p}_{i,j}$ that can make the corresponding realization uncontrollable, for almost all $\bar{p}_{i,j} \in \mathbb{C}^{n \times n}$, if $([\bar{A}, \bar{b}]^\pi$) satisfies conditions 1) and 2) of Theorem 2.

The above classification rule is immediate from the definitions of PSSC and Theorem 2.

Appendix: Proofs of the technical results

Proof of Theorem 1: Sufficiency: Let $p \neq (p_1, ..., p_n)$, since $f(p)$ is a nonzero polynomial, $p \in \mathbb{C}^{n \times 1}$ making $f(p) =$
0 has zero measure in $\mathbb{C}^{n\times n}$. Therefore, in case $\tilde{p}_i \neq 0$ ($i = 1, \ldots, n_x$) and $f(p) \neq 0$, it follows $f(p) \prod_{i=1}^{n_x} \tilde{p}_i^r \neq 0$, indicating the corresponding realization is always controllable.

Necessity: The necessity of $\det(\tilde{A}, \tilde{b}) \neq 0$ is obvious. If $\det(\tilde{A}, \tilde{b}) \neq 0$ but the remaining condition is not satisfied, then there is a $\tilde{p}_i$ ($1 \leq i \leq n_x$) that exists in two different monomials with different degrees (including zero) for $\tilde{p}_i$ (the degree of $\tilde{p}_i$ is the exponent of $\tilde{p}_i$) in $\det(\tilde{A}, \tilde{b})$. Let $\tilde{p}^r = (\tilde{p}_1, \ldots, \tilde{p}_{i-1}, \tilde{p}_{i+1}, \ldots, \tilde{p}_{n_x})$ and $\tilde{p}^s = (\tilde{p}_1, \ldots, \tilde{p}_{n_x}) \setminus \tilde{p}_i$. In this case, write $\det(\tilde{A}, \tilde{b})$ as a polynomial of $\tilde{p}_i$ as

$$\det(\tilde{A}, \tilde{b}) = f(\tilde{p}_i^r) + \cdots + f(\tilde{p}_i) + f_0 = g(\tilde{p}_i; \tilde{p}_i),$$

in which the coefficients $f_j$ ($j = 0, \ldots, r$) are polynomials of $\tilde{p}_i^r$, and $f_0$ as well as another $f_j$ ($0 \leq j < r$) is not identically zero. Consider the set $P_1 = \{\tilde{p}_1, \ldots, \tilde{p}_{n_x}\} \subset \mathbb{C}^{n\times n}$: $3^\pi \tilde{p}_i \neq 0$ in $\mathbb{C}^{n\times n}$, s.t. $f_0 \neq 0, f_0 \neq 0, \prod_{i=1, i \neq j}^{n_x} \tilde{p}_i \neq 0$. Obviously, the complement of $P_1$ in $\mathbb{C}^{n\times n}$ has zero measure, as $\{\tilde{p}^r \in \mathbb{C}^{n\times n}\}$ is a full-dimensional set in $\mathbb{C}^{n\times n}$. Note in case $f_0 \neq 0$ and $f_0 \neq 0, g(\tilde{p}_i; \tilde{p}_i)$ has at least one nonzero root for $\tilde{p}_i$ (as otherwise $g(\tilde{p}_i^r; \tilde{p}_i) = f(\tilde{p}_i)$). Therefore, for all $(\tilde{p}_1, \ldots, \tilde{p}_{n_x}) \in P_1$, there exists $(\tilde{p}_1, \ldots, \tilde{p}_{n_x}) \in \mathbb{C}^{n\times n}$ such that $g(\tilde{p}_i; \tilde{p}_i) = 0$, making the obtained realization uncontrollable.

**Proof of Proposition 1.** The statement follows directly from the proof of Theorem 4. To be specific, for a given $(\tilde{A}, \tilde{b})$, the first case emerges if $(\tilde{A}, \tilde{b})$ is PSSC, while the second case emerges if $(\tilde{A}, \tilde{b})$ is not PSSC.

**Proof of Proposition 2.** By Theorem 4 if $(\tilde{A}, \tilde{b})$ is PSSC, then for its generic realization $(\tilde{A}, \tilde{b})$, $\det(\tilde{A}, \tilde{b})$ has the form of (4). Let $[\tilde{A}^*, \tilde{b}^*]$ be the generic realization of $[\tilde{A}, \tilde{b}]$. It is easy to see, for every $\pi \in N_\ast$, $\det(\tilde{A}^*, \tilde{b}^*)$ then has the form of (4), which indicates $[\tilde{A}^*, \tilde{b}^*]$ is PSSC.

On the other hand, suppose $(\tilde{A}^*, \tilde{b}^*)$ is PSSC, $\forall \pi \equiv (i, j) \in N_\ast$. Let $\tilde{p}_s$ be the parameter for the $(i, j)$-th entry of $[\tilde{A}, \tilde{b}]$. By Theorem 4 for each $\pi \in N_\ast$, $\det(\tilde{A}^*, \tilde{b}^*)$ has a factor $\tilde{p}_s^{*\pi}$ for some $\pi_\ast \geq 0$, and any other factor containing $\tilde{p}_s$ does not exist. Consequently, $\det(\tilde{A}, \tilde{b})$ must have a form of (5). This means, $(\tilde{A}, \tilde{b})$ is PSSC.

**Proof of Proposition 3.** Sufficiency: The sufficiency of condition a1) is obvious, as in this case for almost all $[A_b, b_1] \in S_{[\tilde{A}_b, b_1]}$: $\text{rank}([A_b, b_1][J_n, J_{n+1} \setminus \{j\}]) = n - 1$. Suppose condition a2) is fulfilled. We only need to consider the case where condition a1) is not fulfilled. In this case, as $(\tilde{A}, \tilde{b})$ is structurally controllable, by Lemma 2 we have $\text{rank}([A_b, b_1][J_n, J_{n+1} \setminus \{j\}]) = n - 1$ must hold. Then, for almost all $[A_b, b_1] \in S_{[\tilde{A}_b, b_1]}$: $\text{rank}([A_b, b_1][J_n, J_{n+1} \setminus \{j\}]) = n - 1$. Next, consider two cases: i) $\text{rank}([A_b, b_1][J_n, J_{n+1} \setminus \{j\}]) = n - 1$ and ii) $\text{rank}([A_b, b_1][J_n, J_{n+1} \setminus \{j\}]) < n - 1$, i.e., $\text{rank}([A_b, b_1][J_n, J_{n+1} \setminus \{j\}]) = n - 2$. Let $\tilde{q} \in \mathbb{C}^{n\times n}$ ($\tilde{q} \neq 0$) be in the left null space of $[A_b, b_1][J_n, J_{n+1} \setminus \{j\}]$. Note $\tilde{q}$ is unique up to scaling. By Lemma 2 we generically have $\tilde{q}_s = 0$ for each $k \in N_\ast$; additionally, $\tilde{q}_s \neq 0$ in case i), and $\tilde{q}_s = 0$ in case ii). Hence, in case i), for almost all $[A_b, b_1] \in S_{[\tilde{A}_b, b_1]}$ and all $[A_b, b_1] \in S_{[\tilde{A}_b, b_1]}$ it holds

$$q^T[A_b + A_x, b_x + b_x][J_n, J_1] = \sum_{k \in N_\ast} q_s[A_b, b_1][J_k] + q_s[A_b, b_1][J_k] = q_s[A_b, b_1][J_k] = 0.$$
common nonzero root for \( A \) with \( M_{e,z}^X(A) \), generically. Therefore, denoting \( V_{d,e}^* = V_{d,e}^* \cup V_{d+1,2}^* \cup \cdots \cup V_{d,n}^* \) where \( e = +, - \), if \( M_{e,z}^X(V_{d,e}^* \setminus \{v_i^X\}, V_{d+1}^w) \) is of full row rank, then \( M_{e,z}^X(V_{d+1}^w \setminus \{v_i^X\}, V_{d+2}^w) \) will be (due to its block-triangular structure). Noting \( \det(M_{e,z}^X(V_{d+1}^w \setminus \{v_i^X\}, V_{d+2}^w \setminus \{v_i^X\})) \) is not identically zero (because of the existence of \( M_z^X \); note \( v_i^X \in V_{d+1}^w \)), as otherwise every maximum matching of \( B(H_{e,z}^X) - \{v_i^X\} \) will cover \( G_{e,z}^X \), any nonzero root of \( \det(M_{e,z}^X(V_{d+1}^w \setminus \{v_i^X\}, V_{d+2}^w \setminus \{v_i^X\})) \) is independent of the parameters in the column of \( M_{e,z}^X(A) \) corresponding to \( v_i^X \). This means, \( M_{e,z}^X(V_{d+1}^w \setminus \{v_i^X\}, V_{d+2}^w) \) is generically of full row rank, so is \( M_{e,z}^X(V_{d+1}^w \setminus \{v_i^X\}, V_{d+2}^w) \).

**Sufficiency:** Suppose condition 1) holds. If \( \Omega_z^X = \emptyset \), by Lemma 3, there is generically no \( z \in C \setminus \{0\} \) making \( M_{e,z}^X \) row rank deficient, due to the block-triangular structure of \( M_{e,z}^X \). Noting \( H_{e,z}^X = P^{-1}M_{e,z}^X Q^{-1} \) from \( \emptyset \), \( H_{e,z}^X \) is generically of full rank for all nonzero \( z \). This means \( \Omega_z^X = \emptyset \) is sufficient. Now consider \( \Omega_z^X \neq \emptyset \) but \( \Omega_z^X = \emptyset \). In this case, consider an arbitrary \( z \in C \setminus \{0\} \) that makes \( H_{e,z}^X \) row rank deficient. As \( (A, \tilde{b}) \) is structurably controllable, \( H_{e,z}^X \) generically has rank \( n - 1 \) (otherwise \( [H_{e,z}^X, H_{e,z}^X] \) will be row rank deficient). Hence, upon letting \( q \) be a nonzero vector in the left null space of \( H_{e,z}^X \), \( q \) is unique up to scaling. From property 1) of Lemma 3, \( H_{e,z}^X(V_{d+1}^w \setminus \{v_i^X\}, V_{d+2}^w) \) is row rank deficient (recalling \( H_{e,z}^X = P^{-1}M_{e,z}^X Q^{-1} \)). By Lemma 2 we have \( q \neq 0 \). Hence, for any nonzero value of \( \tilde{\rho}_j \),

\[
q^1H_{e,z}^X = \sum_{k=1}^{n} q_k[H_{e,z}^X]_{k}^{(a)} \neq 0,
\]

where \( (a) \) is due to the controllability of \( (A, \tilde{b}) \) and that \( q^1H_{e,z}^X \) is independent of \( \tilde{\rho}_j \). Therefore, condition 1) is sufficient.

Suppose condition 2) holds. In this case, for those nonzero roots \( z \) of \( H_{e,z}^X \) that make \( H_{e,z}^X(V_{d+1}^w \setminus \{v_i^X\}, V_{d+2}^w) \) row rank deficient, following a similar argument to the above, we can obtain that there is no \( \tilde{\rho}_j \) and \( q (\tilde{\rho}_j \neq 0 \) and \( q \in C^w \setminus \{0\} \) making \( q^1H_{e,z}^X = 0 \). For the nonzero root \( z \) of \( H_{e,z}^X \) that makes \( H_{e,z}^X(V_{d+1}^w \setminus \{v_i^X\}, V_{d+2}^w) \) row rank deficient, we have \( q^1H_{e,z}^X \) is generically of full space of \( \tilde{\rho}_j \), property 2) of Lemma 3, it turns out that \( q_{k} \neq 0 \) and \( q_{k} \neq 0 \) for all \( k \in \{i : \{H_{k}^X\} = \emptyset\}(i) \). We therefore have

\[
q^1H_{e,z}^X = \sum_{k=1}^{n} q_k[H_{e,z}^X]_{k}^{(a)} \neq 0.
\]

where \( (a) \) is due to \( [H_{e,z}^X] = \emptyset, j \neq i \). Hence, condition 2) is also sufficient.

**Necessity:** Suppose neither condition 1), nor condition 2) holds. Then, either i): \( \Omega_z^X \neq \emptyset \), and the third item of condition 2) does not hold, or ii): \( \Omega_z^X = \emptyset \), \( i = j \), and the third item of condition 2) holds. In case i), suppose the third item of condition 2) does not hold for a vertex \( v_i^X \in N(B(H_{e,z}^X), v_{j}^X \setminus \{v_i^X\}) \). Considering the generic realization \( (A, \tilde{b}) \), by property 2) of Lemma 3 we know there is some nonzero \( z \) making \( M_{e,z}^X \) row rank deficient, while \( M_{e,z}^X(V_{d+1}^w \setminus \{v_i^X\}, V_{d+2}^w) \) and \( M_{e,z}^X(V_{d+1}^w \setminus \{v_i^X\}, V_{d+2}^w) \) are of full row rank. Note it generically holds rank \( M_{e,z}^X(m_{i,j}) = n - 1 \), as otherwise rank \( (M_{e,z}^X, M_{e,z}^X) \) < \( n \), contradicting the structural controllability of \( (A, \tilde{b}) \). Let \( q \) be a nonzero vector spanning the left null space of \( M_{e,z}^X \). Then, by Lemma 2, \( \tilde{\rho}_j \neq 0 \) and \( \tilde{\rho}_j \neq 0 \). Upon letting \( |M_{e,z}^X|_{e,z} = -1/\tilde{\rho}_j \sum_{k=1}^{n} k[M_{e,z}^X]_{k} \), we have

\[
q_{k}M_{e,z}^X = \tilde{\rho}_j[M_{e,z}^X]_{e,z} + \sum_{k=1}^{n} k[M_{e,z}^X]_{k},
\]

which leads to \( q|P[H_{e,z}^X, H_{e,z}^X] = 0 \), by substituting \( M_{e,z}^X = PH_{e,z}^X Q \) and noting \( Q \) is invertible. That is, by assigning

\[
\tilde{\rho}_j = \begin{cases} 0 & \text{if } i = j \\ -1/\tilde{\rho}_j \sum_{k=1}^{n} k[M_{e,z}^X]_{k} \text{ if } i \neq j, \end{cases}
\]

we can make \( q|P(A - zI, \tilde{b}) = 0 \). Note if \( i = j \), \( M_{e,z}^X \) is independent of \( q \) and \( z \) (as \( (M_{e,z}^X, M_{e,z}^X) \) contains a free parameter); if \( i \neq j \), in case \( \sum_{k=1}^{n} k[M_{e,z}^X]_{k} \) contains more than one nonzero items, at least one \( M_{e,z}^X \) is independent of \( q \) and \( z \).
Hence, in all these circumstances, it is assured $\bar{q}$, $p$, $r$, $\bar{p}$, $\bar{q}$, $p$, $r$ different degrees for rectangular matrices. This means the proof can be completed.

Proof of Theorem 2: The result comes immediately from the PBH test and Propositions 2 and 3.

Proof of Proposition 5: The case where $(A, B)$ is not structurally controllable is trivial. Now assume structural controllability of $(A, B)$, and consider its generic realization $(\hat{A}, \hat{B})$. Let $p$ be the parameter for the unique $x$ entry in $[A, B]$, and $p$ the collection of parameters for the remaining indeterminate entries. Let $\Gamma(p, \bar{p})$ be the greatest common divisor among all determinants of the $n \times n$ submatrices of $\Gamma(\hat{A}, \hat{B})$. According to [33, Lem 2], for almost all $p \in C^{n \times n}$ (recalling $n$ is the number of entries), the determinants of all the $n \times n$ submatrices of $\Gamma(\hat{A}, \hat{B})$ share a common zero for $\bar{p}$, if and only if the leading degree for $\bar{p}$ in $\Gamma(p, \bar{p})$ is no less than one. Therefore, if $\Gamma(p, \bar{p}) = p\tilde{p}$, where $r \geq 1$ and $P(p)$ is a nonzero polynomial of $p$, then for almost all $p \in C^{n \times n}$ satisfying $P(p) \neq 0$ and all $\bar{p} \neq 0$, $\Gamma(\hat{A}, \hat{B})$ is of full row rank, indicating that the corresponding realization is controllable. Otherwise, if $\Gamma(p, \bar{p})$ contains two monomials with different degrees for $\bar{p}$, then for almost all $p \in C^{n \times n}$, there exists a nonzero solution $\bar{p}$ satisfying $\Gamma(p, \bar{p}) = 0$ (see the proof of Proposition 4), making the corresponding realization uncontrollable.

Proof of Proposition 7: Note Lemma 2 is applicable to rectangular matrices. This means the proof can be completed in the same manner as in Propositions 3 and 4, which thus is omitted.

Proof of Lemma 5: Since $G^{0}_i$ is the horizontal tail, from [29, Corollary 2.2.23], for each $v_i \in V_0^i$, $\det(G^{0}_i - \{v_i\}) = |V_0^i|$. From [29, Lem 9], for all $v_i \in V_0^i$, every maximum matching of $G^{0}_i$ corresponds to a nonzero term in the determinant of a $|V_0^i| \times |V_0^i|$ submatrix of $H^g_i$ that cannot be cancelled out by other terms. Therefore, suppose there is a nonzero value, denoted by $z$, such that $H^g_i(V_0^i, V_0^i)$ is of low rank, $z$ should depend only on the free parameters in $H^g_i(V_0^i, V_0^i, \{v_i\})$, $\bar{v}_i \in V_0^i$. Applying this across $\bar{v}_i \in V_0^i$, it turns out that $z$ is independent of the free parameters in each column of $H^g_i$, causing a contradiction.

Proof of Proposition 8: From Lemmas 3 and 5, if $z$ is a nonzero root of $det\{H^g_i(V_0^i, V_0^i)\}$ for some $k \in \Omega_i$, then $z$ also makes $H^g_i$ row rank deficient owing to the block-triangular structure of its DM-decomposition; and vice versa. With these results, it can be proved easily that, a nonzero $z$ makes $det\{H^g_i(V_0^i, V_0^i)\}$ row rank deficient for a given $\bar{v}_i \in V_0^i$, if and only if $z$ makes $H^g_i(V_0^i, V_0^i, \{v_i\})$ row rank deficient. Having observed this, it can be found Lemma 4, still holds for the rectangular matrix $H^g_i$ (by changing $det\{M^{0}_i\}$ to the row-rank deficient of $H^g_i$). Hence, the proof can be completed in the similar manner to that for the single-input case, i.e., the proof for Proposition 5.

The details are omitted due to their similarities.

Proof of Corollary 1: Sufficiency: From Theorems 1 and 2, we know det $\{\hat{A}(\bar{p})\}$ has the form $\det(\hat{A}(\bar{p})) = \prod_{i \geq 0} p_i$, $p_i$, $\geq 0$ under the proposed conditions, where $[A, B]$ is a generic realization of $[\bar{A}, \bar{B}]$ with the indeterminate parameters being $p_1, ..., p_{n_x}$. It is then obvious that for all $(\bar{p}_1, ..., \bar{p}_n) \in \mathbb{R}^{n_x}$, the corresponding system realization is controllable.

Necessity: We prove the necessity by contradiction. Let $\bar{p}_c$ be the parameter for the $x$th entry of $[\bar{A}, \bar{B}]$ for a $p \in N_x$, and $\bar{p}_c$ the vector consisting of the parameters for the remaining $x$ entries (except $\bar{p}_c$). The necessity of structural controllability of $(\bar{A}, \bar{B})$ is obvious. Now assume $(\bar{A}, \bar{B})$ is structurally controllable. Suppose $i$ is not fulfilled for a $p \in \{i, j\} \in N_x$. From the proof for necessity of Proposition 3 for almost all $\bar{p}_c \in \mathbb{R}$ there exists nonzero real $\bar{p}_c$ (expressed in (19), where $q$ is chosen to be real), such that the corresponding real system realization is uncontrollable. This means $(\bar{A}, \bar{B})$ is not SSC in the real field.

Furthermore, suppose $i$ is not fulfilled for a $p = \{i, j\} \in N_x$. We first consider the case $\Omega_i \neq \emptyset$ and the third item of condition $c2$ does not hold for some $\{v_i\} \neq N(\hat{B}(\bar{A}), v_i) \backslash \{v_i\}$. Following the proof for necessity of Proposition 5 in this case, to construct a real uncontrollable realization of $(\bar{A}, \bar{B})$, we only need to demonstrate $\bar{q}$ in (15) can be real (thus the vector $\bar{q}$ therein can be real too). To this end, recall in (15), $\bar{z}$ needs to satisfy: requirement 1) $z$ makes $M^{0}_i$ row rank deficient, and requirement 2) $z$ makes $M^{0}_i \backslash \{v_i\}$ row rank deficient. Notice from the proof for necessity of Lemma 3, we need to prove the necessity by contradiction. Let $\bar{p}_c$, $\bar{q}$ be the parameter for the $x$th entry of $[\bar{A}, \bar{B}]$ for a $p \in N_x$, and $\bar{q}$ the vector consisting of the parameters for the remaining $x$ entries (except $\bar{p}_c$). The necessity of structural controllability of $(\bar{A}, \bar{B})$ is obvious. Now assume $(\bar{A}, \bar{B})$ is structurally controllable. Suppose $i$ is not fulfilled for a $p \in \{i, j\} \in N_x$. From the proof for necessity of Proposition 3 for almost all $\bar{p}_c \in \mathbb{R}$ there exists nonzero real $\bar{p}_c$ (expressed in (19), where $q$ is chosen to be real), such that the corresponding real system realization is uncontrollable. This means $(\bar{A}, \bar{B})$ is not SSC in the real field.
$z \in \mathbb{R}$, we can always find $p_{\pi} \in \mathbb{R}^{n \times 1}$ such that requirements 1) and 2) are satisfied. Then, after determining $p_{\pi} \text{ according to Equation (35)}$, where $q$ and $z$ are both real, the corresponding real system realization will be uncontrollable, meaning that $(A, b)$ is not SSC in the real field. For the case where $\Omega_{i} \neq \emptyset, i = j$, and the third item of condition 2) holds, we can adopt a similar argument to construct a real uncontrollable realization of $(A, b)$. This proves the necessity.

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