ON VALUE DISTRIBUTIONS FOR QUASIMEROMORPHIC MAPPINGS
ON $\mathbb{H}$-TYPE CARNOT GROUPS

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Abstract. In the present paper we define quasimeromorphic mappings on homogeneous
groups and study their properties. We prove an analogue of results of L. Ahlfors, R. Nevan-
linna and S. Rickman, concerning the value distribution for quasimeromorphic mappings
on $\mathbb{H}$-type Carnot groups for parabolic and hyperbolic type domains.

Introduction

The classical value distribution theory for analytic functions $w(z)$ studies the system of
sets $z_a$ of a domain $G$, where the function $w(z)$ takes the value $w = a$ for an arbitrary $a$.
A central result in the distribution theory is the Picard theorem, stating that a meromorphic
function in the punctured plane assumes all except for at most two values $a_1, a_2$, $a_1 \neq a_2$
ininitely often. In an equivalent way, we can say that an analytic function $w(z) : \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{a_1, a_2\}$ must be constant if $a_1$ and $a_2$ are distinct points in $\mathbb{R}^2$. We mention results
of J. Hadamard, E. Borel, G. Julia, A. Beurling, L. V. Ahlfors and others [1, 2, 7, 28] in
general value distribution theory, going far beyond Picard-type theorems. Nevertheless, in
those extensions and deep generalizations the nature of conformal mappings was not actively
involved. New ideas of the function theory and potential theory point of view were incoming
by R. Nevanlinna [44, 45]. The most important achievements of the Nevanlinna theory were
not only analytic deep results, but its geometric aspects and relations with Riemannian
surfaces of analytic functions. Such principal notions, as a characteristic function, a defect
function, a branching index connect the asymptotic behavior of an analytic function $w(z)$
with properties of the Riemannian surface which is the conformal image of the domain of
$w(z)$.

A natural generalization of an analytic function of one complex variable to the Euclidean
space of the dimension $n > 2$ was firstly introduced and studied by Yu. G. Reshetnyak in
1966—1968 [50, 51, 52]. In some sense this is a quasiconformal mapping admitting branch
points. Such mappings were called in Russian school the mappings with bounded distortion.
The main contribution of Yu. G. Reshetnyak to the foundation of this theory is a discovery of
a connection between mappings with bounded distortion and non-linear partial differential
equations. Yu. G. Reshetnyak has proved also that an analytic definition of mappings
with bounded distortion implies the topological properties: the continuity, the openness,
and the discreteness. Later these mappings, under the name quasiregular mappings, were
investigated intensively by O. Martio, S. Rickman, J. Väisälä, F. W. Gehring, M. Vuorinen,
B. Bojarski, T. Iwaniec and others [5, 6, 21, 41, 42, 58, 60, 77]. Briefly a quasiregular

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mapping can be defined as an appropriate Sobolev mapping with nonnegative Jacobian and such that an infinitesimal ball is transformed into infinitesimal ellipsoid with bounded ratio of the largest and the smallest semi-exes.

The Picard theorem is true for quasiregular mappings in \( \mathbb{R}^2 \). In fact, an arbitrary quasiregular mapping \( f \) in \( \mathbb{R}^2 \) has a representation \( f = g \circ h \) where \( h : \mathbb{R}^2 \to \mathbb{R}^2 \) is a quasiconformal mapping and \( g \) is an analytic function of \( \mathbb{R}^2 \) omitting two points \[36\]. In 1967 V. A. Zorich \[73\] asked whether a Picard-type theorem exists for quasiregular mappings in higher dimensions. S. Rickman has given a complete answer to the question and developed the value distribution theory for quasimeromorphic mappings in \( \mathbb{R}^n \), \( n > 2 \), based on the potential theory, metric and topological properties of quasiregular mappings \[54, 55, 56, 57, 58\]. Quasimeromorphic mappings \( f : \mathbb{R}^n \to \mathbb{R}^n \) generalize quasiregular mappings in the same way as meromorphic mappings do the analytic functions.

A stratified nilpotent group (of which \( \mathbb{R}^n \) and the Heisenberg group are the simplest examples) is a Lie group equipped with an appropriate family of dilations. Thus, this group forms a natural habitat for extensions of many of the objects studied in the Euclidean space. The fundamental role of such groups in analysis was noted by E. M. Stein \[61, 62\]. There has been since a wide development in the analysis of the so-called stratified nilpotent Lie groups, nowadays, also known as Carnot groups. The theory of quasiconformal and quasiregular mappings on Carnot groups is presented in the works \[14, 25, 27, 34, 37, 59, 63, 69, 72\].

In the present paper we define quasimeromorphic mappings on Carnot groups and study their properties. The main difference with the Euclidean definition of a quasimeromorphic mapping is an absence of inversions on general Carnot groups. Therefore, we are not able to use neither a stereographic projection nor conformal metric on Carnot groups. Nevertheless, the definition of quasimeromorphic mappings on Carnot groups we give, allows us not only to obtain the analogues of their Euclidean properties but to adopt also the ideas and methods of the value distribution theory for quasimeromorphic mappings in the Euclidean spaces, developed by S. Rickman. The main difference with respect to Rickman’s approach is that we do not use the inversion as a conformal mapping defined on the one-point compactification of a Carnot group. We present some results concerning the value distribution of \( K \)-quasimeromorphic mappings on \( \mathbb{H}(eisenberg) \)-type Carnot groups in a domain with one boundary point and for \( K \)-quasimeromorphic mappings defined on the unit ball.

The paper is organized as follows. In Section 1 we give the necessary definitions. Section 2 is devoted to the properties of quasimeromorphic mappings and capacity estimates on an arbitrary Carnot group. In Section 3 we consider module inequalities playing fundamental role in the proofs of the main theorems. Section 4 is dedicated to the relationships between the module of a family of curves, a counting function, and averages of the counting function over spheres. In Section 5 we state the first main theorem and prove auxiliary lemmas. Sections 6 and 7 are devoted to proofs of the first and the second principal theorems respectively. The theorems are stated and showed for the \( \mathbb{H} \)-type Carnot groups.

It is well-known, that S. Rickman employed a special family of curves in order to find a method for estimating their modules. A suitable counterpart of such families on Carnot groups exists in frame of ”polar coordinates” in the \( \mathbb{H} \)-type Carnot groups. The key property is that the radial curves have the finite length in Carnot-Carathéodory metric. It allows us to involve the classical module methods \[33, 34, 67\]. In \[4\], ”polarizable” Carnot groups were introduced. These groups admit the analogue of polar coordinates. Unfortunately, nowadays, it is unknown an example of polarizable Carnot group, which is not of \( \mathbb{H} \)-type.
By the way, there are no non-trivial examples of quasiregular mappings on an arbitrary Carnot groups. Nevertheless, if the theory of quasiregular mappings is not degenerate on the polarizable Carnot groups, then our results are true also for this setting.

Now we state the principal result of our work.

**Theorem 0.1.** Let $G$ be a $\mathbb{H}$-type Carnot group, $f : G \to \overline{G}$ be a nonconstant $K$-quasimeromorphic mapping. Then there exists a set $E \subset [1, \infty[$ and a constant $C(Q,K) < \infty$ such that

$$\limsup_{r \to \infty} \sup_{r \notin E} \sum_{j=0}^{q} \left(1 - \frac{n(r,a_j)}{\nu(r,1)}\right) \leq C(Q,K) \quad \text{with} \quad \int_{E} \frac{dr}{r} < \infty,$$

whenever $a_0, a_1, \ldots, a_q$ are distinct points in $\overline{G}$.

The definitions of the counting function $n(r,a_j)$ and the average $\nu(r,1)$ see in Section 4.

We would like to call the attention on the difference between our assertion and Rickman’s one (see for instance [58, p. 80]). S. Rickman employed a version of (0.1) which is conformally invariant and used essentially this property in his proof. R. Nevanlinna pointed out that averages of the counting function with respect to distinct measures can find different applications and physical-geometrical meaning (see also O. Frostman [18, 19]). For this reason, possessing only limited geometrical and analytical tools, we deal with expression (0.1) which is not conformally invariant but still carries an information sufficient to effectively control the distribution of values of a quasimeromorphic mapping. As a corollary of our main result we get the Picard theorem.

**Theorem 0.2.** Let $G$ be a $\mathbb{H}$-type Carnot group. For each $K \geq 1$, there exists a constant $q(G,K)$ such that every $K$-quasiregular mapping $f : G \to G \setminus \{a_1, \ldots, a_q\}$, where $q \geq q(G,K)$ and $a_1, \ldots, a_q$ are distinct, is constant.

Another way of proving this assertion can be found in [72].

The next theorem is stated for $K$-quasimeromorphic mappings in the unit ball $B(0,1)$. The proof of the statement essentially uses the method developed for Theorem 0.1.

**Theorem 0.3.** Let $G$ be a $\mathbb{H}$-type Carnot group, $f : B(0,1) \to \overline{G}$ be a nonconstant $K$-quasimeromorphic mapping such that

$$\limsup_{r \to 1} (1 - r) A(r)^{\frac{1}{Q-1}} = \infty.$$

Then there exists a set $E \subset (0,1)$ satisfying

$$\liminf_{r \to 1} \frac{\text{mes}_1(E \cap [r,1))}{(1-r)} = 0,$$

and a constant $C(Q,K) < \infty$ such that

$$\limsup_{r \to 1} \sup_{r \notin E} \sum_{j=0}^{q} \left(1 - \frac{n(r,a_j)}{\nu(r,1)}\right) \leq C(Q,K),$$

whenever $a_0, a_1, \ldots, a_q$ are distinct points in $\overline{G}$. 
1. Notations and definitions

The Carnot group is a connected and simply connected nilpotent Lie group $G$ whose Lie algebra $\mathcal{G}$ decomposes into the direct sum of vector subspaces $V_1 \oplus V_2 \oplus \ldots \oplus V_m$ satisfying the following relations:

$$[V_1, V_k] = V_{k+1}, \quad 1 \leq k < m, \quad [V_1, V_m] = \{0\}.$$ 

We identify the Lie algebra $\mathcal{G}$ with a space of left-invariant vector fields. Let $X_{11}, \ldots, X_{1n_1}$ be a basis of $V_1$, $n_1 = \dim V_1$, and $\langle \cdot, \cdot \rangle_0$ be a left-invariant Riemannian metric on $V_1$ such that

$$\langle X_{1i}, X_{1j} \rangle_0 = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Then, $V_1$ determines a subbundle $HT$ of the tangent bundle $TG$ with fibers

$$HT_q = \operatorname{span}\{X_{11}(q), \ldots, X_{1n_1}(q)\}, \quad q \in G.$$ 

We call $HT$ the horizontal tangent bundle of $G$ with $HT_q$ as the horizontal tangent space at $q \in G$. Respectively, the vector fields $X_{1j}$, $j = 1, \ldots, n_1$, are said to be horizontal vector fields.

Next, we extend $X_{11}, \ldots, X_{1n_1}$ to a basis

$$X_{11}, \ldots, X_{1n_1}, X_{21}, \ldots, X_{2n_2}, \ldots, X_{m1}, \ldots, X_{mn_m}$$

of $\mathcal{G}$. Here, each vector field $X_{ij}$, $2 \leq i \leq m$, $1 \leq j \leq n_i = \dim V_i$, is a commutator

$$X_{ij} = [\ldots [[X_{1k_1}, X_{1k_2}], X_{1k_3}], \ldots, X_{1k_i}]$$

of the length $i - 1$ of basic vector fields of the chosen basis of $V_1$.

It is known (see, for instance, [16]) that the exponential map $\exp : \mathcal{G} \to G$ from the Lie algebra $\mathcal{G}$ into the Lie group $G$ is a global diffeomorphism. We can identify the points $q \in G$ with the points $x \in \mathbb{R}^N$, $N = \sum_{i=1}^{m} \dim V_i$, by means of the mapping $q = \exp(\sum_{i,j} x_{ij} X_{ij})$. The collection $\{x_{ij}\}$ is called the normal coordinates of $q \in G$. The number $N = \sum_{i=1}^{m} \dim V_i$ is the topological dimension of the Carnot group. The bi-invariant Haar measure on $G$ is denoted by $dx$; this is the push-forward of the Lebesgue measure in $\mathbb{R}^N$ under the exponential map. The family of dilations $\{\delta_\lambda(x) : \lambda > 0\}$ on the Carnot group is defined as

$$\delta_\lambda x = \delta_\lambda(x_{ij}) = (\lambda x_1, \lambda^2 x_2, \ldots, \lambda^m x_m),$$

where $x_i = (x_{i1}, \ldots, x_{in_i}) \in V_i$. Moreover, $d(\delta_\lambda x) = \lambda^Q dx$ and the quantity $Q = \sum_{i=1}^{m} i \dim V_i$ is called the homogeneous dimension of $G$.

Example 1. The Euclidean space $\mathbb{R}^n$ with the standard structure exemplifies an Abelian group: the exponential map is the identical mapping and the vector fields $X_i = \frac{\partial}{\partial x_i}$, $i = 1, \ldots, n$, have trivial commutators only and constitute a basis for the corresponding Lie algebra.

Example 2. The simplest example of a non-abelian Carnot group is the Heisenberg group $\mathbb{H}^n$. The non-commutative multiplication is defined as

$$pq = (x, y, t)(x', y', t') = (x + x', y + y', t + t' - 2xy' + 2yx'),$$
where \( x, x', y, y' \in \mathbb{R}^n, \ t, t' \in \mathbb{R}. \) Left translation \( L_p(\cdot) \) is defined as \( L_p(q) = pq. \) The left-invariant vector fields
\[
X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad i = 1, \ldots, n, \quad T = \frac{\partial}{\partial t},
\]
constitute the basis of the Lie algebra of the Heisenberg group. All non-trivial relations are only of the form \([X_i, Y_i] = -4T, i = 1, \ldots, n,\) and all other commutators vanish. Thus, the Heisenberg algebra has the dimension \( 2n + 1 \) and splits into the direct sum \( \mathcal{G} = V_1 \oplus V_2. \) The vector space \( V_1 \) is generated by the vector fields \( X_i, Y_i, i = 1, \ldots, n,\) and the space \( V_2 \) is the one-dimensional center which is spanned by the vector field \( T. \) More information see [31][32].

**Example 3.** A Carnot group is said to be of \( \mathbb{H} \)-type if the Lie algebra \( \mathcal{G} = V_1 \oplus V_2 \) is two-step and if the inner product \( \langle \cdot, \cdot \rangle_0 \) in \( V_1 \) can be extended to an inner product \( \langle \cdot, \cdot \rangle \) in all of \( \mathcal{G} \) so that the linear map \( J : V_2 \to \text{End}(V_1) \) defined by \( \langle J_Z U, V \rangle = \langle Z, [U, V] \rangle \) satisfies \( J_Z^2 = -\langle Z, Z \rangle \text{Id} \) for all \( Z \in V_2. \) For the moment we introduce the notation \( \|Z\|^2 = \langle Z, Z \rangle. \) Then \( \|J_Z V\| = \|Z\| \cdot \|V\| \) and \( \langle V, J_Z V \rangle = 0 \) for all \( V \in V_1 \) and \( Z \in V_2. \) More details and information see in [12][13][29][30][40].

A homogeneous norm on \( \mathcal{G} \) is, by definition, a continuous function \( |\cdot| \) on \( \mathcal{G} \) which is smooth on \( \mathcal{G} \setminus \{0\} \) and such that \( |x| = |x^{-1}|, \ |\delta_t(x)| = \lambda|x|, \) and \( |x| = 0 \) if and only if \( x = 0. \) All homogeneous norms are equivalent. We choose one of them that admits an analogue of polar coordinates on \( \mathbb{H} \)-type Carnot groups (see [3]). This norm \( |\cdot| = u_2^{1/(1-Q)} \) is associated to Folland’s singular solution \( u_2 \) for the sub-Laplacian \( \Delta_0 = \sum_{j=1}^{n_1} X_{ij}^2 \) at \( 0 \in \mathcal{G}. \) Another advantage of this norm is that it gives the exact value for the \( Q \)-capacity of spherical ring domains. The norm \( |\cdot| \) defines a pseudo-distance: \( d(x, y) = |x^{-1}y| \) satisfying the generalized triangle inequality \( d(x, y) \leq \omega (d(x, z) + d(z, y)) \) with a positive constant \( \omega. \) By \( B(x, r) \) we denote an open ball of radius \( r > 0 \) centered at \( x \) in the metric \( d. \) Note that \( B(x, r) = \{ y \in \mathcal{G} : d(x, y) < r \} \) is the left translation of the ball \( B(0, r) \) by \( x, \) which is the image of the “unit ball” \( B(0, 1) \) under \( \delta_x. \) By mes(\( E \)) we denote the measure of the set \( E. \) Our normalizing condition is such that the balls of radius one have measure one: \( \text{mes}(B(0, 1)) = \int_{B(0, 1)} dx = 1. \) We have \( \text{mes}(B(0, r)) = r^Q \) because the Jacobian of the dilation \( \delta_r \) is \( r^Q. \)

A continuous map \( \gamma : I \to \mathcal{G} \) is called a curve. Here \( I \) is a (possibly unbounded) interval in \( \mathbb{R}. \) If \( I = [a, b] \) then we say that \( \gamma : [a, b] \to \mathcal{G} \) is a closed curve. A closed curve \( \gamma : [a, b] \to \mathcal{G} \) is rectifiable if
\[
\sup\left\{ \sum_{k=1}^{p-1} d(\gamma(t_k), \gamma(t_{k+1})) \right\} < \infty,
\]
where the supremum ranges over all partitions \( a = t_1 < t_2 < \ldots < t_p = b \) of the segment \( [a, b]. \) P. Pansu proved in [40] that any rectifiable curve is differentiable almost everywhere in \( [a, b] \) in the Riemannian sense and there exist measurable functions \( a_j(s), s \in [a, b], \) such that
\[
\dot{\gamma}(s) = \sum_{j=1}^{n_1} a_j(s)X_{1j}(\gamma(s)) \quad \text{and} \quad d\left( \gamma(s + \tau), \gamma(s) \exp(\dot{\gamma}(s)\tau) \right) = o(\tau) \text{ as } \tau \to 0
\]
for almost all \( s \in (a, b). \)

An absolutely continuous curve is a continuous map \( \gamma : I \to \mathcal{G} \) satisfying the following property: for an arbitrary number \( \varepsilon > 0, \) there exists \( \delta > 0 \) such that for arbitrary disjoint
collections of segments \((\alpha_i, \beta_i) \subset I\) with \(\sum_i (\beta_i - \alpha_i) \leq \delta\) we have \(\sum_i d(\gamma(\beta_i), \gamma(\alpha_i)) \leq \varepsilon\).

A closed absolutely continuous curve \(\gamma : [a, b] \to G\) is always rectifiable. Its length \(l(\gamma)\) can be calculated by the formula

\[
l(\gamma) = \int_a^b \langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle_0^{1/2} ds = \int_a^b \left( \sum_{j=1}^{n_1} |a_j(s)|^2 \right)^{1/2} ds
\]

where \(\langle \cdot, \cdot \rangle_0\) is the left invariant Riemannian metric on \(V_1\).

A result of [11] implies that one can connect two arbitrary points \(x, y \in G\) by a rectifiable curve. The Carnot-Carathéodory distance \(d_c(x, y)\) is the infimum of the lengths over all rectifiable curves with endpoints \(x\) and \(y \in G\). Since \(\langle \cdot, \cdot \rangle_0\) is left-invariant, the Carnot-Carathéodory metric is also left-invariant. The metric \(d_c(x, y)\) is finite since the points \(x, y \in G\) can be joined by a rectifiable curve with endpoints \(x, y\). The Hausdorff dimension of the metric space \((G, d_c)\) coincides with the homogeneous dimension \(Q\) of the group \(G\). More information see in [13] [14] [16].

The Sobolev space \(W^1_p(\Omega)\) (\(L^1_p(\Omega)\)), \(1 \leq p < \infty\), consists of locally summable functions \(u : \Omega \to \mathbb{R}, \Omega \subset G\), having distributional derivatives \(X_{1j}u\) along the vector fields \(X_{1j}\):

\[
\int_{\Omega} X_{1j}u \varphi \, dx = - \int_{\Omega} u X_{1j} \varphi \, dx, \quad j = 1, \ldots, n_1,
\]

for any test function \(\varphi \in C_0^\infty\), and the finite norm

\[
\|u\|_{W^1_p(\Omega)} = \left( \int_{\Omega} |u|^p \, dx \right)^{1/p} + \left( \int_{\Omega} |\nabla_0 u|^p_0 \, dx \right)^{1/p}
\]

(semi-norm

\[
\|u\|_{L^1_p(\Omega)} = \left( \int_{\Omega} |\nabla_0 u|^p_0 \, dx \right)^{1/p}.
\]

Here \(\nabla_0 u = (X_{11}u, \ldots, X_{1n_1}u)\) is the subgradient of \(u\) and \(|\nabla_0 u|_0 = \langle \nabla_0 u, \nabla_0 u \rangle_0\). We say, that \(u\) belongs to \(W^1_{p,loc}(\Omega)\) if for an arbitrary bounded domain \(U, \overline{U} \subset \Omega\), the function \(u\) belongs to \(W^1_p(U)\). Henceforth, for a bounded domain \(U \subset \Omega\) whose closure \(\overline{U}\) belongs to \(\Omega\), we write \(\overline{U} \Subset \Omega\) and say that \(U\) is a compact domain in \(\Omega\).

**Definition 1.1** ([19] [70] [71]). Suppose that \((\mathbb{X}, r)\) is a complete metric space, \(r\) is a metric on \(\mathbb{X}\), and \(\Omega\) is a domain on a Carnot group \(G\). We say that a mapping \(f : \Omega \to \mathbb{X}\) belongs to Sobolev class \(W^1_{p,loc}(\Omega; \mathbb{X})\) if the following conditions hold.

(A) For each \(x \in \mathbb{X}\), the function \([f]_x : x \in \Omega \mapsto r(f(x), z)\) belongs to the class \(W^1_{p,loc}(\Omega)\).

(B) The family of functions \((\nabla_0 [f]_x)_{x \in \mathbb{X}}\) has a dominant belonging to \(L^1_{p,loc}(\Omega)\), i.e., there is a function \(g \in L^1_{p,loc}(\Omega)\) independent of \(z\) and such that \(|\nabla_0 [f]_z(x)|_0 \leq g(x)\) for almost all \(x \in \Omega\).

**Definition 1.2.** A function \(u : \Omega \to \mathbb{R}, \Omega \subset G\), is said to be absolutely continuous on lines \((u \in ACL(\Omega))\) if for any domain \(U \Subset \Omega\), and any fibration \(X_j\) defined by the left-invariant vector fields \(X_{1j}, j = 1, \ldots, n_1\), the function \(u\) is absolutely continuous on \(\gamma \cap U\) with respect to the \(H^1\)-Hausdorff measure for \(d\gamma\)-almost all curves \(\gamma \in X_j\). (Recall that the measure \(d\gamma\) on \(X_j\) equals the inner product \(i(X_j)\) of the vector field \(X_j\) by the bi-invariant volume form \(dx\)).
For a function \( u \in \text{ACL}(\Omega) \), the derivatives \( X_{1j}u \) along the vector fields \( X_{1j}, j = 1, \ldots, n_1 \), exist almost everywhere in \( \Omega \). It is known that a function \( u : \Omega \to \mathbb{R} \) belongs to \( W^1_p(\Omega) \) \( (L^p_p(\Omega)) \), \( 1 \leq p < \infty \), if and only if it can be modified on a set of measure zero by such a way that \( u \in L^p(\Omega) \) (\( u \) is locally \( p \)-summable), \( u \in \text{ACL}(\Omega) \), and \( X_{1j}u \in L^p(\Omega) \) hold, \( j = 1, \ldots, n_1 \). The reader can find more information on ACL-functions in [34, 65, 70].

**Proposition 1.3** ([70, 71]). A mapping \( f : \Omega \to \mathbb{G} \), \( \Omega \subset \mathbb{G} \), belongs to the Sobolev class \( W^1_{p,\text{loc}}(\Omega) \), \( 1 \leq p < \infty \), if and only if it can be modified on a set of measure zero by such a way that

1. \( |f(x)| \in L^p_{\text{loc}}(\Omega) \);
2. the coordinate functions \( f_{ij} \) belong to \( \text{ACL}(\Omega) \) for all \( i \) and \( j \);
3. \( f_{ij} \in W^1_{p,\text{loc}}(\Omega) \) for \( 1 \leq j \leq n_1 \);
4. the vector

\[
X_{1k}(f(x)) = \sum_{1 \leq l \leq m, 1 \leq \omega \leq n_1} X_{1k}(f_{l\omega}(x)) \frac{\partial}{\partial x_{l\omega}}
\]

belongs to \( HT_{f(x)} \) for almost all \( x \in \Omega \) and all \( k = 1, \ldots, n_1 \).

In [22, 70, 72], one can find various definitions of the Sobolev space on Carnot groups and their correlations. The matrix \( X_{1k}f = (X_{1k}f_{ij})_{k,j=1,\ldots,n_1} \) defines a linear operator \( D_Hf : V_1 \to V_1 \) which is called a **formal horizontal differential**. A norm of the operator \( D_Hf \) is defined by

\[
|D_Hf(x)| = \sup_{\xi \in V_1, |\xi|_0 = 1} |D_Hf(x)(\xi)|_0.
\]

The norm \(|D_Hf|\) is equivalent to \(|\nabla_{0}f|_0 = \left( \sum_{i=1}^{n_1} |X_{1i}f|^2_0 \right)^{\frac{1}{2}} \). It has been proved in [65, 70] that the formal horizontal differential \( D_Hf \) generates a homomorphism \( Df : \mathbb{G} \to \mathbb{G} \) of Lie algebras which is called a **formal differential**. The determinant of the matrix \( Df(x) \) is denoted by \( J(x,f) \) and called a **(formal) Jacobian**.

A continuous mapping \( f : \Omega \to \mathbb{G} \), \( \Omega \subset \mathbb{G} \), is **open** if the image of an open set is open and **discrete** if the pre-image \( f^{-1}(y) \) of each point \( y \in f(\Omega) \) consists of isolated points. We say that \( f \) is sense-preserving if a topological degree \( \mu(y,f,U) \) is strictly positive for all domains \( U \subset \Omega \) and \( y \in f(U) \setminus f(\partial U) \). The precise definition of the topological degree see in Subsection 2.4.

**Definition 1.4.** Let \( \Omega \) be a domain on the group \( \mathbb{G} \). A mapping \( f : \Omega \to \mathbb{G} \) is said to be a **quasiregular mapping** if

1. \( f \) is continuous open discrete and sense-preserving ;
2. \( f \) belongs to \( W^1_{p,\text{loc}}(\Omega) \);
3. the formal horizontal differential \( D_Hf \) satisfies the condition

\[
(1.1) \quad \max_{|\xi|_0 = 1} |D_Hf(x)(\xi)|_0 \leq K \min_{|\xi|_0 = 1} |D_Hf(x)(\xi)|_0
\]

for almost all \( x \in \Omega \).

It is known [70] that the pointwise inequality \((1.1)\) is equivalent to the following one: the formal horizontal differential \( D_Hf \) satisfies the condition

\[
(1.2) \quad |D_Hf(x)|^Q \leq K'J(x,f)
\]
for almost all \( x \in \Omega \) where \( K' \) depends on \( K \). The smallest constant \( K' \) in inequality (1.2) is called the outer distortion and denoted by \( K_O(f) \). It is not hard to see that for a quasiregular mapping the inequality

\[
0 \leq J(x, f) \leq K'' \min_{|\xi|=1, \xi \in V_1} |D_H f(x)(\xi)|_0^Q
\]

also holds for almost all \( x \in \Omega \) where \( K'' \) depends on \( K \). The smallest constant \( K'' \) in inequality (1.3) is called the inner distortion and denoted by \( K_I(f) \).

It is established in [74, 75] that the conditions 2 and 3 of Definition 1.4 provide for a non-constant mapping on a two-step Carnot group to be continuous open discrete and sense-preserving if there exists a singular solution \( w \in W^{1,\infty}_\text{loc}(\mathbb{R} \setminus 0) \) to the equation

\[
\text{div}(|\nabla w|^2 \nabla w) = 0.
\]

In [14], the same result is proved under stronger assumption that the solution \( w \) belongs to \( C^1 \). Such a singular solution exists on the \( \mathbb{R} \)-type Carnot groups [25]. By another words, on the \( \mathbb{R} \)-type Carnot groups a mapping with bounded distortion (that is a mapping satisfying conditions 2 and 3 of Definition 1.4) is also a quasiregular one. As soon as on Carnot groups, there is no a complete counterpart of the Euclidean theory of mappings with bounded distortion we will distinguish mappings with bounded distortion and quasiregular mappings.

**Definition 1.5.** A continuous mapping \( f : \Omega \to \mathbb{G} \) is \( P \)-differentiable at \( x \in \Omega \) if the family of maps \( f_t = \delta_{1/t}(f(x)^{-1} f(x \delta_t y)) \) converges locally uniformly to an automorphism of \( \mathbb{G} \) as \( t \to 0 \).

In the following theorem we formulate analytic properties of quasiregular mappings [69, 70, 72, 73]. In the statement of the theorem we use notions of a topological degree \( \mu(y, f, D) \) of the mapping \( f \) and a multiplicity function \( N(y, f, A) = \text{card}\{x \in f^{-1}(y) \cap A\} \) (see the precise definitions in Subsection 2.4).

**Theorem 1.1.** Let \( f : \Omega \to \mathbb{G}, \Omega \subset \mathbb{G}, \) be a quasiregular mapping. Then it possesses the following properties:

1) \( f \) is \( P \)-differentiable almost everywhere in \( \Omega \);
2) \( N \)-property: if \( \text{mes}(A) = 0 \) then \( \text{mes}(f(A)) = 0 \);
3) \( N^{-1} \)-property: if \( \text{mes}(A) = 0 \) then \( \text{mes}(f^{-1}(A)) = 0 \);
4) \( \text{mes}(B_f) = \text{mes}(f(B_f)) = 0 \);
5) \( J(x, f) > 0 \) almost everywhere in \( \Omega \);
6) for every compact domain \( D \subset \Omega \) such that \( \text{mes}(f(\partial D)) = 0 \) (every measurable set \( A \subset \Omega \)) and every measurable function \( u \), the function \( y \mapsto u(y)\mu(y, f, D) \) (\( y \mapsto u(y)N(y, f, D) \)) is integrable in \( \mathbb{G} \) if and only if the function \( (u \circ f)(x)J(x, f) \) is integrable on \( D \). Moreover, the following change of variable formulas hold:

\[
\int_D (u \circ f)(x)J(x, f) \, dx = \int_{\mathbb{G}} u(y)\mu(y, f, D) \, dy,
\]

\[
\int_A u(x)J(x, f) \, dx = \int_{\mathbb{G}} \sum_{x \in f^{-1}(y) \cap A} u(x) \, dy,
\]

\[
\int_A (u \circ f)(x)J(x, f) \, dx = \int_{\mathbb{G}} u(y)N(y, f, A) \, dy.
\]
We use the notation $\Omega = \mathbb{G} \cup \{ \infty \}$ for the one-point compactification of the Carnot group $\mathbb{G}$. The system of neighborhoods for $\{ \infty \}$ are generated by the complement to homogeneous closed balls. It is evident that $\Omega$ is topologically equivalent to the unit Euclidean sphere $S^N$ in the Euclidean space $\mathbb{R}^{N+1}$. Later on, we use the symbol $\Omega$ to denote a domain (open connected set) on the Carnot group $\mathbb{G}$. It is not excluded that $\Omega$ coincides with $\mathbb{G}$.

**Definition 1.6.** A continuous mapping $f : \Omega \rightarrow \Omega$ is said to be a *quasimeromorphic mapping* if

1) $f : \Omega \setminus f^{-1}(\infty) \rightarrow \mathbb{G}$ is a quasiregular mapping;
2) for any domain $\omega \in \Omega$, the multiplicity function $N(y, f, \omega)$ is essentially bounded:

$$N(f, \omega) = \operatorname{ess} \sup_{y \in \mathbb{G}} N(y, f, \omega) = \operatorname{ess} \sup_{y \in \mathbb{G}} \operatorname{card}\{f^{-1}(y) \cap \omega\} < \infty.$$ 

An ordered triplet $(F_0, F_1; \Omega)$ of nonempty sets, where $\Omega$ is open in $\Omega$, $F_0$ and $F_1$ are compact subsets of $\Omega$, is said to be a *condenser* on $\Omega$. We define the $p$-capacity, $1 \leq p < \infty$, of the condenser $E = (F_0, F_1; \Omega)$ as

$$\operatorname{cap}_p(E) = \operatorname{cap}_p(F_0, F_1; \Omega) = \inf \int_{\Omega \setminus \{\infty\}} |\nabla v|^p_0 dx,$$

where the infimum is taken over all nonnegative functions $v \in C(\Omega \cup F_0 \cup F_1) \cap L^1_p(\Omega \setminus \{\infty\})$ such that $v = 0$ in a neighborhood of $F_0 \cap \Omega$ and $v \geq 1$ in a neighborhood of $F_1 \cap \Omega$. Functions taking part in the definition of the $p$-capacity of a condenser are said to be *admissible* for this condenser. If the set of admissible functions is empty then the $p$-capacity of a condenser equals infinity, by definition.

If $\Omega \subset \Omega$ is an open set and $C$ is a compact set in $\Omega$ then, for brevity, we denote the condenser $E = (C, \partial C; \Omega)$ by $E = (C, \Omega)$, and we shall write $\operatorname{cap}_p(E) = \operatorname{cap}_p(C, \Omega)$ instead of $\operatorname{cap}_p(C, \partial C; \Omega)$. The notion of the $p$-capacity $\operatorname{cap}_p(C, \Omega)$ is extended to an arbitrary set $E \subset \Omega$ by the usual way (see, for instance [10, 26] in the case $\mathbb{G} = \mathbb{R}^n$ and [8, 9] in the geometry of vector fields satisfying the Hörmander hypoellipticity condition).

We say that a compact $C \subset \Omega$ has the $p$-capacity zero and write $\operatorname{cap}_p C = 0$, if $\operatorname{cap}_p(C, U) = 0$ for some open set $U \subset \Omega$ such that $\operatorname{cap}_p(B, U) > 0$ for some ball $B \subset U$. One can prove

1) if $\operatorname{cap}_p(B_1, U) > 0$ for some ball $B_1 \subset U$ then $\operatorname{cap}_p(B_2, U) > 0$ for an arbitrary other ball $B_2 \subset U$;
2) if $\operatorname{cap}_p(C, U) = 0$ then $\operatorname{cap}_p(C, V) = 0$ whenever $V$ is any bounded open set containing $C$.

An arbitrary Borel set $E$ has the $p$-capacity zero, if the same holds for any compact subset of $E$, otherwise $\operatorname{cap}_p E > 0$.

The chosen homogeneous norm gives the following exact value for the $p$-capacity of spherical rings $(B(x, r), B(x, R))$, $0 < r < R < \infty$, [4]:

$$\operatorname{cap}_p(B(x, r), B(x, R)) = \begin{cases} \kappa(\mathbb{G}, p) \left( \frac{p-Q}{p-1} \right)^{p-1} \left( R^{p-Q} - r^{p-Q} \right)^{1-p}, & p \neq Q, \\ \kappa(\mathbb{G}, Q) \left( \ln \frac{R}{r} \right)^{1-Q}, & p = Q, \end{cases}$$

where $\kappa(\mathbb{G}, p)$ is a positive constant whose an exact value was obtained in [4] (we give it in Section 4).
2. Properties of quasimeromorphic mappings

Lemma 2.1 ([72]). Let \( f : \Omega \to \mathbb{C} \) be a quasimeromorphic mapping. For any open set \( U \subset \Omega \) such that \( N(f, U) < \infty \) the operator \( f^* : L^1_Q(f(U) \setminus \{\infty\}) \to L^1_Q(U \setminus f^{-1}(\infty)) \) where \( f^*(u) = u \circ f \), is bounded:

\[
\|f^*(u)\|_{L^1_Q(U \setminus f^{-1}(\infty))} \leq (K_0(f), N(f, U))^{1/q} \|u\|_{L^1_Q(f(U) \setminus \{\infty\})},
\]

and the chain rule works: \( \nabla f^*(u)(x) = D_H f(x)^T \nabla u(f(x)) \) almost everywhere in \( U \).

Proof. The set \( U \setminus f^{-1}(\infty) \) is an open set in \( \Omega \). Consider an arbitrary function

\[
u \in C^1(f(U) \setminus \{\infty\}) \cap L^1_Q(f(U) \setminus \{\infty\}).
\]

Then \( v = u \circ f \in \text{ACL}(U \setminus f^{-1}(\infty)) \) (since the function \( u \) is locally Lipschitz) and \( \nabla v(x) = D_H f(x)^T \nabla u(f(x)) \) almost everywhere in \( U \setminus f^{-1}(\infty) \) (since the mapping \( f \) is \( P \)-differentiated a. e.). Using (1.6) and the property \( N(f, U \setminus f^{-1}(\infty)) \leq N(f, U) < \infty \), we obtain

\[
\int_{U \setminus f^{-1}(\infty)} |\nabla v_0(u \circ f)|_0^Q(x) dx \leq K_0(f) \int_{U \setminus f^{-1}(\infty)} |\nabla u_0^Q(f(x))| D_H f^Q(x) dx
\]

\[
\leq K_0(f) \int_{U \setminus f^{-1}(\infty)} |\nabla u_0^Q(f(x))| J(x, f) dx
\]

\[
\leq K_0(f) N(f, U) \int_{f(U) \setminus \{\infty\}} |\nabla u_0^Q(y)| dy.
\]

Since the composition operator \( f^* : C^1(f(U) \setminus \{\infty\}) \cap L^1_Q(f(U) \setminus \{\infty\}) \to L^1_Q(U \setminus f^{-1}(\infty)) \) is bounded, this operator can be continuously extended to \( L^1_Q(f(U) \setminus \{\infty\}) \) making use of arguments of [66], and the extended operator will be also the composition operator.

\[\square\]

Lemma 2.2. If \( f : \Omega \to \mathbb{C} \), \( \Omega \subset \mathbb{C} \), is a quasimeromorphic mapping and \( S = f^{-1}(\infty) \), then

\[\text{cap}_Q(S) = 0.\]

Proof. According to the definition of the quasimeromorphic mapping, we have \( S \neq \Omega \).

Therefore, there exists a ball \( B_0 \) such that \( \overline{B}_0 \subset \Omega \setminus S \). Let \( \omega \) be a domain satisfying \( \omega \subset \Omega \), \( B_0 \subset \omega \), and \( \omega \cap S \neq \emptyset \). We shall prove that \( \text{cap}_Q(\overline{B}_0, S \cap \overline{B}; \omega) = 0 \) for an arbitrary ball \( B \subset \omega \) such that \( \overline{B}_0 \cap \overline{B} = \emptyset \) and \( S \cap \overline{B} \neq \emptyset \).

Fix some domain \( \omega \subset \Omega \), \( \omega \cap S \neq \emptyset \), a ball \( B \subset \omega \) with \( S \cap \overline{B} \neq \emptyset \), and a point \( y \in f(\omega \setminus \overline{B}) \setminus \{\infty\} \). For any \( R > 1 \) we consider a condenser \( E_R = (CB(y, R), C\overline{B}(y, r)) \) where \( r < 1 \) is small enough to provide \( B(y, r) \subset f(\omega \setminus \overline{B}) \setminus \{\infty\} \) and \( f^{-1}(B(y, r)) \subset \Omega \) contains some open ball \( B_0 \subset \omega \) satisfying \( \overline{B}_0 \cap S \cap \overline{B} = \emptyset \). Notice that since \( \overline{B}(y, r) \subset B(y, R) \) we may choose as an admissible function for the \( Q \)-capacity of the condenser \( E_R \) a function \( \varphi_R \) such that \( \varphi_R|_{\overline{B}(y, r)} = 0 \) and \( \varphi_R|_{\partial B(y, R)} = 1 \). By this we define

\[
\varphi_R(z) = \begin{cases} 
0 & \text{if } z \in \overline{B}(y, r), \\
\frac{\ln |z - y| + 1}{\ln R} & \text{if } z \in B(y, R) \setminus \overline{B}(y, r), \\
1 & \text{if } z \notin B(y, R) \setminus \{\infty\}.
\end{cases}
\]
Then \( \text{cap}_Q(E_R) \leq \int_{B(y,R) \setminus B(y,r)} |\nabla \varphi_R(z)|_0^Q \, dz \leq C(\ln \frac{R}{r})^{1-Q} \), where \( C \) is the Lipschitz constant of the function \( z \mapsto |y^{-1}z| \). We denote by \( F_R \) the set \( \{ x \in \Omega : \varphi_R(f(x)) = 1 \} \). Then \( S \subset F_R \) for any real \( R > 1 \) and therefore \( S \subset \bigcap_{R \geq 2} F_R \). It is clear also that \( \omega \subset F_R \cap \overline{B} \) and \( f^*(\varphi_R) = \varphi_R \circ f \) is an admissible function for the \( Q \)-capacity of the condenser \( (\overline{B}_0, S \cap \overline{B}; \omega) \) for all \( R \geq k_0 \) where \( k_0 \) is some number greater than one. Now, we use Lemma 2.1 to derive

\[
\text{cap}_Q(\overline{B}_0, S \cap \overline{B}; \omega) \leq \| f^*(\varphi_R) \|_{L^1_Q(\omega)} \| L^1_Q(\omega \setminus S) \|^{Q} \leq K_0(f)N(f, \omega)\| \varphi_R \|_{L^1_Q(f(\omega) \setminus \{ \infty \})}^{Q} \leq K_0(f)N(f, \omega)C\left( \ln \frac{R}{r} \right)^{1-Q}.
\]

The right-hand side of this inequality goes to 0 as \( R \to \infty \). Therefore,

\[
\text{cap}_Q(\overline{B}_0, S \cap \overline{B}; \omega) = 0
\]

and the lemma is proved. \( \square \)

As a consequence of Lemma 2.2 we have the following property [68]: if \( f : \Omega \to \overline{\mathbb{G}}, \Omega \subseteq \mathbb{G} \), is a quasimeromorphic mapping, then \( S(x, t) \cap f^{-1}(\infty) = \emptyset \) for an arbitrary point \( x \in \Omega \) and for almost all \( t \) such that the sphere \( S(x, t) \) belongs to \( \Omega \).

We say that a mapping \( f \) is light if \( f^{-1}(y) \) is totally disconnected for all \( y \). Thus, from the previous considerations we have the following statement.

**Corollary 2.3.** A quasimeromorphic mapping is light.

### 2.4. Topological degree

Recall that we identify the Carnot group \( \mathbb{G} \) with its Lie algebra \( \mathbb{G} \) and thus with \( \mathbb{R}^N, N = \sum_{i=1}^{m} \dim V_i \). Moreover, the one-point compactification of \( \overline{\mathbb{G}} \) is topologically equivalent to the unit sphere \( S^N \) centered at 0 in \( \mathbb{R}^{N+1} \). Therefore the topological degree \( \mu(y, f, D) \) of a continuous mapping \( f : \Omega \to \overline{\mathbb{G}} \) where \( D \Subset \Omega \) is a compact domain, can be treated as the topological degree of the continuous mapping \( f : \Omega \to S^N \) with the standard orientation in \( \Omega \subseteq \mathbb{R}^N \) and \( S^N \). The topological degree \( \mu(y, f, D) \) of the continuous mapping \( f : \Omega \to \overline{\mathbb{G}} \) at \( y \) is well-defined whenever \( D \) is a compact domain in \( \Omega \) and \( y \in \overline{\mathbb{G}} \setminus f(\partial D) \). The degree is integer-valued function and has the following properties:

1. the function \( y \mapsto \mu(y, f, D) \) is a constant in every connected component of \( \overline{\mathbb{G}} \setminus f(\partial D) \) and \( \mu(y, f, D) = 0 \) if \( y \notin f(\partial D) \);
2. if \( U \) is a connected component of \( \overline{\mathbb{G}} \setminus f(\partial D) \) such that \( \mu(y, f, D) \neq 0 \) for some point \( y \in U \) then for any \( z \in U \) there exists \( x \) such that \( f(x) = z \);
3. if \( y \in f(D) \setminus f(\partial D) \) and the restriction of \( f \) to \( \overline{D} \) is one-to-one then \( |\mu(y, f, D)| = 1 \);
4. if \( D_1, \ldots, D_k \in \Omega \) are disjoint open sets and if \( D \cap f^{-1}(y) \subset \bigcup_{i=1}^{k} D_i \subset D \Subset \Omega \), then

\[
\mu(y, f, D) = \sum_{i=1}^{k} \mu(y, f, D_i), \quad y \notin f(\partial D), \text{ and } y \notin f(\partial D_i), \quad i = 1, \ldots, k.
\]

Other properties of the mapping degree can be found in [61, 62].

**Lemma 2.5.** Let \( f : \Omega \to \overline{\mathbb{G}}, \Omega \subseteq \mathbb{G} \), be a quasimeromorphic mapping. If \( f(x_0) = \infty \) then the image of any neighborhood of \( x_0 \) is a neighborhood of \( \{ \infty \} \).
Proof. Let \( x_0 \) be a point such that \( f(x_0) = \infty \). Since \( f \) is light we can find a sphere \( S(x_0, r) \in \Omega \) such that \( \{ \infty \} \notin f(S(x_0, r)) \). We choose an open connected component \( U_\infty \in \mathbb{C} \setminus f(S(x_0, r)) \) containing \( \{ \infty \} \). There exists a point \( z \in U_\infty \) such that \( z = f(x) \) for some point \( x \in B(x_0, r) \). According to the properties of quasiregular mappings, the image \( W = f(B(x_0, r) \setminus f^{-1}(\infty)) \) is an open neighborhood of \( z \). Then the following properties hold (see \([70, 72, 73]\)):

a) for all \( y \in W \), the pre-image \( f^{-1}(y) \cap B(x_0, r) \) contains finitely many points;

b) for almost all points \( y \in W \), the \( \mathcal{P} \)-differential exists in all points \( x \in f^{-1}(y) \) and \( J(x, f) \) does not vanish;

c) for almost all points \( y \in W \),

\[
\mu(y, f, B(x_0, r)) = \sum_{x \in f^{-1}(y)} \text{sign } J(x, f) > 0.
\]

By properties of the topological degree, the last expression implies that the degree \( \mu(y, f, B(x_0, r)) \) does not vanish at all points \( y \in U_\infty \). Thus, for any \( y \in U_\infty \) there exists \( x \in B(x_0, r) \) such that \( f(x) = y \).

Corollary 2.6. A quasimeromorphic mapping is open and discrete.

Proof. The openness follows from Lemma 2.5 and the definition of quasimeromorphic mappings. If a map is open and light, then it is discrete. The complete proof can be found in \([52, 58, 63]\). \( \square \)

Lemma 2.7. Let \( f : \Omega \to \mathbb{C} \) be a quasimeromorphic mapping and \( U \subset \Omega \) be a domain such that \( N(f, U) < \infty \). Then the condenser \( E = (F_0, F_1; U) \) meets the inequality

\[
\text{cap}_Q(F_0, F_1; U) \leq K_O(f) N(f, U) \text{cap}_Q(f(F_0), f(F_1); f(U)).
\]

Proof. We have to consider two cases: \( F_1 \cap f^{-1}(\infty) = \emptyset \) and \( F_1 \cap f^{-1}(\infty) \neq \emptyset \). The first case is well known (see, for instance, \([72]\)). The second one is more interesting for us. We note that since a quasimeromorphic mapping is open, the triplet \((f(F_0), f(F_1); f(U))\) is a condenser. Let \( u \) be an admissible function for \((f(F_0), f(F_1); f(U))\). Then, in view of Lemma 2.4, the function \( u \circ f \) is admissible for the condenser \((F_0, F_1; U)\) and \( u \circ f = 1 \) in some neighborhood of \( f^{-1}(\infty) \). Therefore, at the same neighborhood we have \( \nabla_0 (u \circ f) = 0 \). Applying estimate (2.1), we obtain

\[
\text{cap}_Q(F_0, F_1; U) \leq \int_U |\nabla_0(u \circ f)|_0^Q(x) \, dx = \int_{U \setminus f^{-1}(\infty)} |\nabla_0(u \circ f)|_0^Q(x) \, dx
\]

\[
\leq K_O(f) N(f, U) \int_{f(U) \setminus \{ \infty \}} |\nabla_0 u|_0^Q(z) \, dz.
\]

(2.2)

Since \( u \) is an arbitrary admissible function, the lemma is proved. \( \square \)

We need the following \( Q \)-capacity estimate.

Theorem 2.1 (\([40, 72]\)). Let \( f : \Omega \to \mathbb{C} \) be a non-constant quasiregular mapping and \( E = (C, U) \) be a condenser such that \( C \) is a compact in \( U \) and \( U \subset \Omega \). Then \( f(E) = (f(C), f(U)) \) is also a condenser and

\[
\text{cap}_Q(f(C), f(U)) \leq K_I(f) \text{cap}_Q(C, U).
\]

(2.3)
Proof. In the case $\mathbb{G} = \mathbb{R}^n$ the estimate (2.3) is proved in [41]. The proof in our case is based on the following construction. Since $U$ is compact then $N(f, U) < \infty$. We define the pushforward of a non-negative function $u \in C_0(U)$ to be the function $v = f_\sharp u : f(\Omega) \to \mathbb{R}$, given by
\[
v(y) := \begin{cases} 
\sup \{u(x) : f(x) = y\} & \text{if } y \in f(U), \\
0 & \text{otherwise}. 
\end{cases}
\]

By the same way as in [41, Lemma 7.6], one can prove that if $f$ is continuous discrete and open, and the non-negative function $u : U \to \mathbb{R}$ is continuous with compact support, then the function $v = f_\sharp u : f(\Omega) \to \mathbb{R}$ is also continuous and supp $v \subset f(\text{supp } u)$. Moreover, if additionally $u \in C^1_0(U)$ and the mapping $f$ is quasiregular then $v = f_\sharp u$ belongs to $W^1_Q(f(\Omega))$. Below the precise statement follows [40, 72].

Let $f : \Omega \to \mathbb{G}$ be a non-constant quasiregular mapping. Then the operator $f_\sharp$ possesses the following properties:

1) $f_\sharp : C^1_0(U)^+ \to W^1_Q(f(\Omega)) \cap C_0(f(\Omega))$ where the symbol $C^1_0(U)^+$ denotes all non-negative functions of $C^1_0(U)$,

2) $\int_{f(\Omega)} |\nabla_0 f_\sharp(u)(x)|^Q dx \leq K_1 \int_U |\nabla_0 u|^Q dx$ for any $u \in C^1_0(U)$,

3) if the function $u$ is admissible for the condenser $E = (U, C)$ then $f_\sharp u$ is admissible for the condenser $f(E) = (f(U), f(C))$. To prove the proposition one needs to check that $f_\sharp u \in \text{ACL}(f(\Omega))$ (see details in [72] where ACL-property is verified for a function of similar nature).

From the last two properties everyone can deduce the inequality (2.3). \hfill $\square$

We use the estimate (2.3) to prove the removability property of quasimeromorphic mappings. Before to formulate it we prove some auxiliary assertions.

Let $E \subset \mathbb{G}$ be a closed set of positive $Q$-capacity. We say that the set $E$ has the essentially positive $Q$-capacity at a point $x \in E$, $x \neq \infty$, if
\[
cap_Q(E \cap \overline{B(x, r)}, B(x, 2r)) > 0
\]
for any positive $r$. One is able to check that

1) it is sufficient to verify (2.4) for $r \in (0, r_0)$, where $r_0$ is a positive number;

2) the set $\overline{E} = \{x \in E : \text{the set } E \text{ has the essentially positive } Q\text{-capacity at } x\}$ is not empty and closed.

Then there exists a point $x_0$ such that $|x_0| = \inf \{|x| : x \in \overline{E}\}$. Let us denote the intersection $E \cap B(x_0, 1)$ by the symbol $E_0$. By definition, we have cap$_Q(E_0, B(x_0, 2))$ is positive.

Lemma 2.8 ([40]). Let $E$ be a closed subset of $\overline{\mathbb{G}}$ with cap$_Q(E) > 0$. Then for every $a > 0$ and $d > 0$ there exists $\delta > 0$ such that cap$_Q(C, CE) \geq \delta$ whenever $C \subset CE$ is a continuum such that $\text{diam}(C) \geq a > 0$ and $\text{dist}(C, E_0) \leq d$.

Proof. It is enough to prove the assertion under assumption that $E$ is a non-empty bounded set.

We use the rule of contraries. Then there exist $a > 0$ and $d > 0$ such that for any $\delta_n = \frac{1}{n}$, $n \in \mathbb{N}$, we can find a continuum $C_n$ with the diameter $\text{diam}(C_n) \geq a > 0$ and $\text{dist}(C_n, E_0) \leq d$ but cap$_Q(C_n, CE) \leq \delta_n$. By these assumptions we derive existence of a real
number $R$, $R \geq d > 0$, such that some connected part of the intersection $\gamma_n = C_n \cap \overline{B(x_0, R)}$ has the diameter $\text{diam}(\gamma_n) \geq a/2 > 0$ and

$$\text{cap}_Q(E_0 \cap \overline{B(x_0, R)}, B(x_0, 2R)) > 0. \tag{2.5}$$

Since

$$\text{cap}_Q(C_n, CE) \geq \text{cap}_Q(\gamma_n, CE) \geq \text{cap}_Q(\gamma_n, E_0 \cap \overline{B(x_0, R)}) \geq \text{cap}_Q(\gamma_n, E_0 \cap \overline{B(x_0, R)}; B(x_0, 2R))$$

we can choose admissible functions $\phi_n(x) \in C(\overline{B(x_0, 2R)} \cap L^1_Q(B(x_0, 2R)))$ for condensers $(\gamma_n, E_0 \cap \overline{B(x_0, R)}; B(x_0, 2R))$ such that $\phi_n(x) \in (0, 1)$ when $x \in B(x_0, 2R)$,

$$\phi_n(x) = \begin{cases} 0 & \text{if } x \in \gamma_n, \\ 1 & \text{if } x \in E_0 \cap \overline{B(x_0, R)}, \end{cases}$$

and

$$\int_{B(x_0, 2R)} |\nabla_0 \phi_n|^Q dx \to 0 \quad \text{as } n \to \infty.$$ 

Using Poincar’e inequality we can extract a subsequence (that we denote by the same symbol) such that $\phi_n(x) \to \alpha \in [0, 1]$ almost everywhere in $B(x_0, 2R)$ as $n \to \infty$. Additionally, we can also assume that $|\nabla_0 \phi_n(x)|_0 \to 0$ almost everywhere in $B(x_0, 2R)$ as $n \to \infty$.

Let $\psi \in C_0(B(x_0, 2R)) \cap L^1_Q(B(x_0, 2R))$ be a function such that $\psi(x) = 1$ if $x \in (E_0 \cap \overline{B(x_0, R)}) \cup \gamma_n$ and $\psi(x) \in [0, 1]$.

1st CASE: $\alpha < 1$. The product $g_n = (1 - \alpha)^{-1}(\phi_n - \alpha)\psi$ is an admissible function for the condenser $(E_0 \cap \overline{B(x_0, R)}, B(x_0, 2R))$. Since $|\nabla_0 \phi_n|_0 \to 0$ and $|\phi_n - \alpha| \to 0$ almost everywhere as $n \to 0$ we derive

$$\int_{B(0,2R)} |\nabla_0 g_n|^Q dx \leq \frac{2^{Q-1}}{(1-\alpha)^Q} \left( \int_{B(0,2R)} |\psi \nabla_0 \phi_n|^Q dx \right) + \int_{B(0,2R)} |(\phi_n - \alpha) \nabla_0 \psi|^Q dx \to 0$$

as $n \to \infty$ by the Lebesgue dominated theorem. This contradicts to (2.5). 

2nd CASE: $\alpha = 1$. In this case the product $g_n = \psi(1 - \phi_n)$ is an admissible function for a condenser $(\gamma_n, B(x_0, 2R))$, $n \in \mathbb{N}$. According to the above estimates $\int_{B(0,2R)} |\nabla_0 g_n|^Q dx \to 0$ as $n \to \infty$. Results of [23] (see also [9, 35]) imply that $\text{diam}(\gamma_n) \to 0$ as $n \to \infty$ that contradicts to the choice of $\gamma_n$. \hfill \Box

**Theorem 2.2 ([30]).** Let $\Omega$ be a domain in $\mathbb{G}$, $E \subset \Omega$ be a closed set with $\text{cap}_Q(E) = 0$. If $f : \Omega \setminus E \to \overline{\mathbb{G}}$ is a quasimeromorphic mapping and $\text{cap}_Q(\mathbb{C}f(\Omega \setminus E))$ is positive, then $f$ can be extended to a continuous mapping $f^* : \Omega \to \overline{\mathbb{G}}$. Moreover, if the domain $\Omega$ is unbounded then there exists also a limit

$$\lim_{x \to \infty, x \in \Omega} f^*(x) \in \overline{\mathbb{G}}. \tag{2.6}$$
Proof. We may assume that $E$ contains the set $f^{-1}(\infty)$. Since the $Q$-capacity of $E$ is zero, the set $\Omega \setminus E$ is connected. To show that $f$ has a limit at a point $b \in E$ we choose a sphere $S(b, R) \subset \Omega \setminus E$ and two different sequences $\{x_j\} \in \Omega \cap B(b, R/4)$, $\{x_j'\} \in \Omega \cap B(b, R/4)$ going to $b$ as $j \to \infty$. Let $r_j = 2 \max\{|b^{-1} x_j|, |b^{-1} x_j'|\}$. By $C_j \subset B(b, r_j) \setminus E$, we denote a rectifiable curve with endpoints $x_j$ and $x_j'$. In view of $\text{cap}_Q(E) = 0$ the set $E$ is removable and we have

\[
\text{cap}_Q(C_j, B(b, R) \setminus E) = \text{cap}_Q(C_j, B(b, R)) \leq \text{cap}_Q(\overline{B}(b, r_j), B(b, R)) = \kappa(G, Q)\left(\ln \frac{R}{r_j}\right)^{1-Q}. \tag{2.7}
\]

Since the right-hand side of this relation tends to $0$ as $j \to \infty$ then the left-hand side of it does the same.

Suppose that $f : \Omega \setminus E \to \overline{G}$ has no limit in $b \in E$. It follows that for some subsequences (that we denote by the same symbols $\{x_j\}$ and $\{x_j'\}$) we have simultaneously

a) at least one of the sequences $f(x_j)$ and $f(x_j')$ is bounded in $f(\Omega \setminus E)$,

b) $\text{diam}(f(C_j)) \geq \alpha > 0$ for some constant $\alpha > 0$ and for all $j \in \mathbb{N}$.

Applying Lemma 2.8, we obtain the inequality

\[
\text{cap}_Q(f(C_j), f(\Omega \setminus E)) \geq \delta > 0 \tag{2.8}
\]

for some $\delta$ and all $j \in \mathbb{N}$.

On the other hand, for fixed $j$, we can exhaust a domain $B(b, R) \setminus E$ by compact domains $\omega_k$ such that $C_j \subset \omega_1 \subset \ldots \subset \omega_k \subset \ldots \subset \Omega \cap B(b, R) \setminus E$, $\bigcup_k \omega_k = B(b, R) \setminus E$. By Theorem 2.1 and properties of capacity, we deduce

\[
\text{cap}_Q(f(C_j), f(\Omega \setminus E)) \leq \text{cap}_Q(f(C_j), f(\omega_k)) \leq K_I(f) \text{cap}_Q(C_j, \omega_k). \tag{2.9}
\]

Letting $k \to \infty$ in the right-hand side of (2.9), we obtain

\[
\text{cap}_Q(f(C_j), f(\Omega \setminus E)) \leq K_I(f) \text{cap}_Q(C_j, B(b, R) \setminus E) \tag{2.10}
\]

by properties of the capacity. The inequalities (2.8) and (2.10) imply

\[
0 < \delta \leq \text{cap}_Q(f(C_j), f(\Omega \setminus E)) \leq K_I(f) \text{cap}_Q(C_j, B(b, R) \setminus E). \tag{2.11}
\]

We have a contradiction, since, by (2.7), the right-hand side of (2.11) goes to $0$ as $j \to 0$.

It remains to show 2.11. Since the set $\Omega \setminus E$ is open we can find a ball $B(x_0, R) \subset \Omega \setminus E$ and two different sequences $\{x_j\} \in (\Omega \setminus E) \cap \overline{C}(x_0, 4R)$, $\{x_j'\} \in (\Omega \setminus E) \cap \overline{C}(x_0, 4R)$ going to $\infty$ as $j \to \infty$. Put $r_j = \frac{1}{2} \min\{|x_0^{-1} x_j|, |x_0^{-1} x_j'|\}$. We denote by $C_j$ a rectifiable curve connecting points $x_j$ and $x_j'$ in $(\Omega \setminus E) \cap \overline{C}(x_0, r_j)$ if the points $x_j$ and $x_j'$ belong to the same connected component of $(\Omega \setminus E) \cap \overline{C}(x_0, r_j)$. In the case when $x_j$ and $x_j'$ are in the different components of $(\Omega \setminus E) \cap \overline{C}(x_0, r_j)$, then $C_j$ will denote the union of rectifiable curves joining $x_j$ and $x_j'$ with the sphere $S(x_0, r_j)$. Then

\[
\text{cap}_Q(C_j, (\Omega \setminus E) \cap \overline{C}(x_0, R)) = \text{cap}_Q(C_j, \Omega \cap \overline{C}(x_0, R)) \leq \text{cap}_Q(S(x_0, r_j) \cap \Omega, S(x_0, R); \Omega) \tag{2.12}
\]

\[
\leq \text{cap}_Q(\overline{B}(x_0, R), B(x_0, r_j)) = \kappa(G, Q)\left(\ln \frac{r_j}{R}\right)^{1-Q}.
\]
If (2.6) does not exist then the following limit
\[ \lim_{x \to \infty, x \in \Omega \setminus B} f(x) \]
does the same and the properties a), b) hold. Then, by Lemma 2.8 we obtain the inequality (2.8) for some positive \( \delta \) and all \( j \in \mathbb{N} \). Arguing like above, we get an analogue of (2.11):
\[
0 < \delta \leq \text{cap}_Q(f(C_j), f(\Omega \setminus E)) \leq \text{cap}_Q(f(C_j), f((\Omega \setminus E) \cap \overline{C}(x_0, R))) \\
\leq K_I(f) \text{cap}_Q(C_j, (\Omega \setminus E) \cap \overline{C}(x_0, R)).
\]
We come to a contradiction, since the right-hand side goes to 0 as \( j \to \infty \) by (2.10).

**Definition 2.9.** If \( f : \Omega \to \overline{\mathbb{G}} \) is a quasimeromorphic mapping, and if \( b \) is an isolated point of \( \partial \Omega \) such that \( f \) has no limits at \( b \), then we call \( b \) the (isolated) essential singularity of \( f \).

**Corollary 2.10.** Let \( b \) be an isolated essential singularity of a quasimeromorphic mapping \( f : \Omega \to \overline{\mathbb{G}} \). Then \( \text{cap}_Q\mathcal{C}(U \setminus \{b\}) = 0 \) for an arbitrary neighborhood \( U \subset \Omega \cup \{b\} \) of the point \( b \).

**Proof.** If we suppose that there exists a neighborhood \( U \subset \Omega \cup \{b\} \) such that \( \text{cap}_Q\mathcal{C}(U \setminus \{b\}) > 0 \) then \( f|_{U \setminus \{b\}} \) is extended to the point \( b \) by Theorem 2.2. This contradicts to the assumption that \( b \) is the essential singularity of \( f \). \( \square \)

**Lemma 2.11 (10).** Let \( b \) be an isolated essential singularity of a quasimeromorphic mapping \( f : \Omega \to \overline{\mathbb{G}} \). Then there exists an \( F_a \)-set \( E \subset \overline{\mathbb{G}} \) with \( \text{cap}_Q(E) = 0 \), such that \( N(y, f, U \setminus \{b\}) = \infty \) for every \( y \in \overline{\mathbb{G}} \setminus E \) and all neighborhoods \( U \subset \Omega \cup \{b\} \) of the point \( b \).

**Proof.** We consider two possibilities. If \( b \neq \infty \) then we can assume that \( B(b, 1) \subset \Omega \cup \{b\} \) and denote the set \( B(b, \frac{1}{k}) \) by \( V_k \), \( k = 1, 2, \ldots \). In the case \( b = \infty \), we will use the notation \( V_k = \mathbb{C}(0, k) \cap \Omega \), \( k = 1, 2, \ldots \). Set \( E = \bigcup_{k=1}^{\infty} \mathcal{C}(V_k) \). Then \( E \) contains \( f(U \setminus \{b\}) \) for any neighborhood \( U \subset \Omega \cup \{b\} \) of the point \( b \). The properties of the capacity imply that \( \text{cap}_Q(E) = 0 \) because of \( \text{cap}_Q(\mathcal{C}(U \setminus \{b\})) = 0 \) by Corollary 2.10. For an arbitrary neighborhood \( U \subset \Omega \cup \{b\} \) and given \( y \in \overline{\mathbb{G}} \setminus E = \bigcup_{k=1}^{\infty} f(V_k) \), we can find a sequence \( \{x_j\} \) with pairwise disjoint elements such that \( x_j \in V_{k_j}, f(x_j) = y \). The lemma is proved. \( \square \)

### 3. Main inequalities for modulus

Here and subsequently \( \langle a, b \rangle \) stands for an interval of one of the following type \( (a, b), [a, b), (a, b], \) and \( [a, b] \). We say that a curve \( \gamma : \langle a, b \rangle \to \overline{\mathbb{G}} \) is locally rectifiable if \( \gamma \) is locally rectifiable on \( \langle a, b \rangle \setminus \gamma^{-1}(\infty) \). A restriction \( \gamma' = \gamma|_{[a, \beta]} \), \( [a, \beta] \subset \langle a, b \rangle \setminus \gamma^{-1}(\infty) \), is said to be a closed part of \( \gamma \). The closed part \( \gamma' \) is rectifiable and we may use the length arc parameter \( s \) on \( \gamma' \). The linear integral is defined by
\[
\int_{\gamma} \rho \, ds = \sup_{\gamma'} \int_{0}^{l(\gamma')} \rho(\gamma'(s)) \, ds,
\]
where the supremum is taken over all closed parts \( \gamma' \) of \( \gamma \) and \( l(\gamma') \) is the length of \( \gamma' \). Let \( \Gamma \) be a family of curves in \( \overline{\mathbb{G}} \). Denote by \( \mathcal{F}(\Gamma) \) the set of Borel functions \( \rho : \overline{\mathbb{G}} \to [0; \infty] \) such
that the inequality
\[ \int_{\gamma} \rho \, ds \geq 1 \]
holds for a locally rectifiable curve \( \gamma \in \Gamma \). Otherwise we put \( \int_{\gamma} \rho \, ds = \infty \). An element of the family \( \mathcal{F}(\Gamma) \) is called an admissible density for \( \Gamma \).

**Definition 3.1.** Let \( \Gamma \) be a family of curves in \( \mathbb{G} \) and \( p \in (1, \infty) \). The quantity
\[ M_p(\Gamma) = \inf_{\rho} \int_{\mathbb{G}} \rho^p \, dx \]
is called the \( p \)-module of the family of curves \( \Gamma \). The infimum is taken over all admissible densities \( \rho \in \mathcal{F}(\Gamma) \).

It is known that the \( p \)-module of a family of non-rectifiable curves vanishes \( [20] \). If some property fails to hold for a family of curves whose \( p \)-module vanishes, then we say that the property holds \( p \)-almost everywhere.

Let \( F_0, F_1 \) be disjoint compacts in \( \mathbb{G} \). We say that a curve \( \gamma : (a, b) \rightarrow \Omega \) connects \( F_0 \) and \( F_1 \) in \( \Omega \) (starts on \( F_0 \) in \( \Omega \)) if
\begin{enumerate}
  \item \( \gamma((a, b)) \cap F_1 \neq \emptyset \), \( i = 0, 1 \), \( \gamma((a, b)) \cap F_0 \neq \emptyset \),
  \item \( \gamma(t) \in \Omega \) for all \( t \in (a, b) \).
\end{enumerate}

A family of curves connecting \( F_0 \) and \( F_1 \) (starting at \( F_0 \) in \( \Omega \)) is denoted by \( \Gamma(F_0, F_1; \Omega) \) \( (\Gamma(F_0; \Omega)) \). In the next theorem the relation between the \( p \)-capacity of the condenser \( (F_0, F_1; \Omega) \) and the \( p \)-module of the family \( \Gamma(F_0, F_1; \Omega) \) is given.

**Theorem 3.1.** \( [37] \) Let \( \Omega \) be a bounded domain in the Carnot group \( G \). Suppose that \( K_0 \) and \( K_1 \) are disjoint non-empty compact sets in the closure of \( \Omega \). Then
\[ M_p(\Gamma(F_0, F_1; \Omega)) = \operatorname{cap}_p(F_0, F_1; \Omega), \quad p \in (1, \infty) \).

**Remark 3.2.** Let \( f : \Omega \rightarrow \mathbb{G} \) be a quasimeromorphic mapping and \( \Gamma \) be a family of curves in \( \Omega \). We correlate the parametrization of the curves in \( \Gamma \subset \Omega \) and in \( \Gamma^* = f(\Gamma) \subset f(\Omega) \). Let \( \gamma^* \in \Gamma^* \) be a rectifiable curve. We introduce the length arc parameter \( s^* \) in the curve \( \gamma^* \in \Gamma^* \). Thus \( s^* \in I^* = [0, l(\gamma^*)] \) where \( l(\gamma^*) \) is the length of the curve \( \gamma^* \). If \( t \) is any other parameter on \( \gamma^* \): \( \gamma^*(t) = f(\gamma(t)) \), then the function \( s^*(t) \) is strictly monotone and continuous, so the same holds for its inverse function \( t(s^*) \). For the curve \( \gamma(t) \in \Gamma \) such that \( f(\gamma(t)) = \gamma^* \) the parameter \( s^* \) can be introduced by the following way
\[ f(\gamma(t(s^*))) = f(\gamma(s^*)) = \gamma^*(s^*), \quad s^* \in I^*. \]

We note that if we take the length arc parameter \( s \) on \( \gamma \), \( s \in I = [0, l(\gamma)] \), then
\[ 1 = \left| \frac{d\gamma(s)}{ds} \right|_0 = \left| \frac{d\gamma(s^*)}{ds^*} \right|_0 \cdot \left| \frac{ds^*}{ds} \right|. \]

 luego, from now on, we use the letters \( s \) and \( s^* \) to denote the length arc parameters on curves \( \gamma \) and \( \gamma^* = f(\gamma) \). The corresponding domains of \( s \) and \( s^* \) are denoted by \( I = [0, l(\gamma)] \) and \( I^* = [0, l(\gamma^*)] \), respectively.

**Theorem 3.3.** Let \( f : \Omega \rightarrow \mathbb{G} \) be a nonconstant quasimeromorphic mapping. Then, for a Borel set \( A \), \( A \subset \Omega \), with \( N(f, A) < \infty \) and a family of curves \( \Gamma \) in \( A \), we have
\[ M_Q(\Gamma) \leq K_0(f)N(f, A)M_Q(f(\Gamma)). \]
Then \( Q \) that since mes(\( \rho \)) is absolutely continuous. A result of B. Fuglede [20] implies that for rectifiable curves such that there exists an image \( f(\gamma') \) of a closed part \( \gamma' \) of \( \gamma \) that is not absolutely continuous. A result of B. Fuglede [20] implies that \( M_Q(\Gamma_0) = 0 \). Notice also, that since mes(\( E \)) = 0 the family of curves \( \Gamma_1 \subset \Gamma \) where \( \int_{\gamma} \chi_E ds > 0 \), \( \gamma \in \Gamma_1 \), has the \( Q \)-module zero. In fact, define \( \tilde{\rho} : A \to \mathbb{R}^1 \) as

\[
\tilde{\rho}(x) = \begin{cases} 
\infty & \text{for } x \in E, \\
0 & \text{for } x \notin E.
\end{cases}
\]

Then \( \int_{\gamma} \tilde{\rho} ds = \infty \) for \( \gamma \in \Gamma_1 \). Hence, \( \tilde{\rho} \) is admissible for \( \Gamma_1 \) and

\[
M_Q(\Gamma_1) \leq \int_E \tilde{\rho}^Q dx = 0.
\]

Suppose that the closed parts \( \gamma' \) and \( f(\gamma') \) of curves \( \gamma \in \Gamma \setminus (\Gamma_0 \cup \Gamma_1) \) and \( f(\gamma) \in f(\Gamma \setminus (\Gamma_0 \cup \Gamma_1)) \) are parameterized as in Remark 5.2. We have

\[
(3.2) \quad 1 = \left| \frac{df(\gamma'(s^*))}{ds^*} \right|_0 \leq \left| D_H f(\gamma') \cdot \frac{d\gamma'(s^*)}{ds^*} \right|_0 = \left| D_H f(\gamma') \right| \left| \frac{ds}{ds^*} \right|,
\]

where \( s \) is the length arc parameter on \( \gamma' \). By [5.2], we deduce

\[
\int_{\gamma} \rho ds \geq \int_{\gamma'} \rho ds \geq \int_{\gamma'} \rho^* \left| D_H f(\gamma'(s)) \right| ds \geq \int_{f(\gamma')} \rho^* ds^* \geq \int_{f(\gamma')} \rho^* ds^*
\]

for any closed part \( \gamma' \) of \( \gamma \in \Gamma \setminus (\Gamma_0 \cup \Gamma_1) \). Taking supremum over all closed parts of \( f(\gamma) \) we see \( \int_{\gamma} \rho ds \geq \int_{f(\gamma')} \rho^* ds^* \geq 1 \). We conclude that \( \rho \) is admissible for the family \( \Gamma \setminus (\Gamma_0 \cup \Gamma_1) \).

Since mes(\( E \)) = 0 we write

\[
M_Q(\Gamma) = M_Q(\Gamma \setminus (\Gamma_0 \cup \Gamma_1)) \leq \int_{\Gamma} \rho^Q dx = \int_A \rho^*(f(x))^Q |D_H f(x)|^Q dx \leq K_O(f) \int_A \rho^*(f(x))^Q J(x, f) dx = K_O(f) \int_G \rho^*(y)^Q N(y, f, A) dy \leq K_O(f) N(f, A) \int_G \rho^*(y)^Q dy
\]

by [1.2] and [1.6]. Taking infimum over all \( \rho^* \in F(f(\Gamma)) \) we end the proof. \( \square \)
Remark 3.4. Within the proof of Theorem 3.3 we have derived the estimate

\begin{equation}
M_Q(\Gamma) \leq K_O(f) \int_{\Omega} \rho(y)^Q N(y, f, A) \, dy
\end{equation}

which will be used below. Inequality (3.4) holds for any function \( \rho \in \mathcal{F}(f(\Gamma)) \).

We state here Poleskii type lemma. Its complete proof can be found in [38, 39].

Lemma 3.2. Let \( f : \Omega \to \mathbb{G} \) be a non-constant quasiregular mapping and \( U \subset \Omega \) be a domain, such that \( \overline{U} \subset \Omega \). Assume \( \Gamma \) to be a family of curves in \( U \) such that \( \gamma^*(s^*) = f(\gamma(s^*)) \) is locally rectifiable and there exists a closed part \( \gamma'(s^*) \) of \( \gamma(s^*) \) that is not absolutely continuous (the parameterization of \( \Gamma \) and \( f(\Gamma) \) is correlated as in Remark 3.2). Then, \( M_Q(f(\Gamma)) = 0 \).

Theorem 3.5. Let \( f : \Omega \to \mathbb{G} \) be a nonconstant quasimeromorphic mapping and \( \Gamma \) be a family of curves in \( \Omega \). Then

\begin{equation}
M_Q(f(\Gamma)) \leq K_I(f) M_Q(\Gamma).
\end{equation}

Proof. Let \( \rho \) be an admissible function for a family \( \Gamma \). We can assume that \( \int_{\Omega} \rho \, dx < \infty \). We take a sequence \( \Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega \) of subdomains that exhausts \( \Omega \). Then \( \int_{\Omega \setminus \Omega_i} \rho \, dx \to 0 \) as \( i \to \infty \). Now we define an admissible function for a family \( f(\Gamma) \). If \( x \in \Omega_i \setminus B_f \) then \( f \) is \( \mathcal{P} \)-differentiable almost everywhere with strictly positive Jacobian \( J(x, f) \). By \( E \), we denote a Borel set of \( \text{mes}(F) = 0 \) containing all points \( x \in \Omega_i \setminus B_f \) where \( f \) is not \( \mathcal{P} \)-differentiable. Notice that \( \text{mes}(F \cup B_f) = \text{mes}(F \cup B_f) = 0 \). Let \( E \subset \mathbb{G} \) be a Borel set of zero measure such that \( f(F \cup B_f) \cup \{ \infty \} \subset E \). We set \( \lambda_f(x) = \min_{|\xi_0 = 1, \xi \in V_1} \frac{1}{|D_H f(x)|} \) for \( x \in \Omega_i \setminus (F \cup B_f) \) and define

\[
\rho^*_i(y) = \begin{cases} 
\max_{x \in f^{-1}(y) \cap \Omega_i} \rho(x) \lambda_f(x) & \text{for } y \in f(\Omega_i) \setminus E, \\
\infty & \text{for } y \in E, \\
0 & \text{for } y \notin f(\Omega_i).
\end{cases}
\]

The function \( \rho^*_i(y) \) is nonnegative and Borel.

Since the \( \mathcal{Q} \)-module of a family of non-rectifiable curves vanishes we consider only rectifiable curves. We correlate the parametrization of the curves in \( \Gamma \) and \( \Gamma^* = f(\Gamma) \) as in Remark 3.2. Denote by \( I_i \) the maximal sub-interval of \([0, l(\gamma^*))] \) such that \( \gamma^*|_{I_i} \subset \Omega_i \). Let \( \Gamma_i \) be the family of curves \( \gamma_i = \gamma|_{I_i^*} \) and \( \Gamma^*_i = f(\Gamma_i) \). Lemma 3.2 states that the \( \mathcal{Q} \)-module of a family \( \Gamma^*_{i} \) of curves \( \gamma^*_i \in \Gamma^*_{i} \), for which \( \gamma_i \) is not absolutely continuous, vanishes. As in Theorem 3.3 it can be shown that the family of curves \( \Gamma^*_1 \subset \Gamma^* \) where \( \int \chi_E \, ds > 0 \), \( E = f(F \cup B_f) \cup \{ \infty \} \), \( \gamma^* \in \Gamma^*_1 \), has the \( \mathcal{Q} \)-module zero. So, we restrict our attention to the family \( \Gamma^*_1 \setminus (\Gamma^*_{0,1} \cup \Gamma^*_1) \).

Notice that

\begin{equation}
1 = \left| \frac{df(\gamma(s^*))}{ds^*} \right|_0 = \left| \frac{1}{\lambda_f(\gamma_i)} \cdot \frac{d\gamma_i(s^*)}{ds^*} \right|_0 = \left| \frac{1}{\lambda_f(\gamma_i)} \cdot \frac{ds}{ds^*} \right|
\end{equation}
by (3.1). The curves \( \gamma_i \) are absolutely continuous. It implies

\[
\int_{f(\gamma)} \rho^*_i \, ds^* \geq \int_{f(\gamma)} \rho^*_j \, ds^* \geq \int_{\Omega_i} \rho(\gamma_i(s^*)) \lambda f(\gamma_i(s^*)) \, ds^* \\
\geq \int_0^{h(\gamma_i)} \rho(\gamma_i(s)) \lambda f(\gamma_i(s)) \left| \frac{ds^*}{ds} \right| \, ds \geq \int_{\gamma_i} \rho \, ds
\]

(3.7)

by (3.6). If we show that \( \lim_{i \to \infty} \int_{f(\gamma)} \rho^*_i \, ds^* \) exists, then (3.7) and \( \lim_{i \to \infty} \int \rho \, ds \geq 1 \) will imply that the limit function \( \rho^* \) belongs to \( F(f(\Gamma)) \). We argue as follows

\[
\int_{f(\Omega)} (\rho^*_i)^Q \, dy = \int_{f(\Omega)} \max_{x \in f^{-1}(y) \cap \Omega_i} (\rho(x) \lambda f(x))^Q \, dy \\
\leq \int_{f(\Omega)} \sum_{x \in f^{-1}(y) \cap \Omega_i} (\rho(x) \lambda f(x))^Q \, dy \\
\leq K_I(f) \int_{f(\Omega)} \left( \sum_{x \in f^{-1}(y) \cap \Omega_i} \rho(x)^Q J^{-1}(x, f) \right) \, dy = K_I(f) \int_{\Omega_i} \rho^Q \, dx,
\]

(3.8)

by (1.3) and (1.5). Since for \( i \geq k > 0 \)

\[
\int_{f(\Omega)} |\rho^*_i - \rho^*_k|^Q \, dy \leq K_I(f) \int_{\Omega_i \setminus \Omega_k} \rho^Q \, dx \to 0 \quad \text{as} \quad k \to \infty,
\]

we deduce that there exists a function \( \rho^* \in L_Q(f(\Omega)) \) such that

\[
\lim_{i \to \infty} \int_{f(\Omega)} |\rho^*_i - \rho^*|^Q \, dy = 0.
\]

A result of [20] implies that there is a subsequence of \( \rho^*_i \) (for the simplicity we use the same symbol \( \rho^*_i \)) with \( \lim_{i \to \infty} |\rho^*_i - \rho^*| \, ds^* = 0 \) for \( \gamma^* \in \Gamma^* \setminus \Gamma^0 \) where \( M_0(\Gamma^0) = 0 \). From here it follows that \( \int \rho^* \, ds^* \geq 1 \) for \( \gamma^* \in \Gamma^* \setminus (\Gamma^0 \cup \Gamma^1 \cup (\bigcup_i \Gamma^0_{0,i})) \). Moreover, the inequality

\[
\int_{f(\Omega)} (\rho^*)^Q \, dy \leq K_I \int_{f(\Omega)} \rho^Q \, dx
\]

holds. Finally, we conclude

\[
M_Q(f(\Gamma)) = M_Q(f(\Gamma) \setminus (\Gamma^0 \cup \Gamma^1 \cup (\bigcup_i \Gamma^0_{0,i}))) \leq K_I \int_{f(\Omega)} \rho^Q \, dx.
\]

We obtain (3.5) taking infimum over all admissible functions \( \rho \) for \( \Gamma \).

3.3. **Lifting of curves.** Let \( f : \Omega \to \overline{\mathcal{G}} \) be continuous discrete and open mapping of a domain \( \Omega \in \mathcal{G} \). Let \( \beta : [a, b] \in \overline{\mathcal{G}} \) be a curve and let \( x \in f^{-1}(\beta(a)) \). A curve \( \alpha : [a, c] \to \Omega \) is called an \( f \)-lifting of \( \beta \) starting at point \( x \) if

1. \( \alpha(a) = x \),
2. \( f \circ \alpha = \beta|_{[a, c]} \).

□
We say that a curve $\alpha : [a, c] \to \Omega$ is a maximal $f$-lifting of $\beta$ starting at point $x$ if both 1), 2) and the following property hold:

3) if $c < c' < b$ then there does not exist a curve $\alpha' : [a, c'] \to \Omega$ such that $\alpha = \alpha'|_{[a, c]}$ and $f \circ \alpha' = \beta|_{[a, c']}$. Let $f^{-1}(\beta(a)) = \{x_1, \ldots, x_k\}$ and $m = \sum_{j=1}^{k} i(x_j, f)$. We say that $\alpha_1, \ldots, \alpha_m$ is a maximal essentially separate sequence of $f$-liftings of $\beta$ starting at the points $x_1, \ldots, x_k$ if

1) each $\alpha_j$ is a maximal lifting of $f$,
2) $\text{card}\{j : \alpha_j(a) = x_l\} = i(x_l, f)$, $1 \leq l \leq k$,
3) $\text{card}\{j : \alpha_j(t) = x\} \leq i(x, f)$ for all $x \in \Omega$ and all $t$.

Similarly, we define a maximal sequence of $f$-liftings terminating at $x_1, \ldots, x_k$ if $f : [b, a] \to \mathbb{G}$.

**Theorem 3.6.** Let $f : \Omega \to \mathbb{G}$ be a quasimeromorphic mapping, $\beta : [a, b] \to \mathbb{G}$, and let $x_1, \ldots, x_k$ be distinct points in $f^{-1}(\beta(a))$. Then $\beta$ has a maximal sequence of $f$-liftings starting (terminating) at $x_1, \ldots, x_k$.

**Proof.** The theorem is formulated for quasimeromorphic mappings but actually this is a topological assertion. For the proof we refer to [58] where it is shown the local existence of $f$-liftings. Since topological properties of $S = f^{-1}(\infty)$ coincides with topological properties of the pre-image of a finite point of $\mathbb{G}$, we can apply the arguments of [58] almost verbatim. Then, we can show that the local existence of maximal $f$-liftings implies the global existence.

□

In the next statement we present a generalization of the inequality of J. Väisälä. The Väisälä inequality is an essential tool on the study of value distribution of quasimeromorphic mappings.

**Theorem 3.7.** Let $f : \Omega \to \mathbb{G}$ be a nonconstant quasimeromorphic mapping, $\Gamma$ be a family of curves in $\Omega$, $\Gamma^*$ be a family in $\mathbb{G}$ and $m$ be a positive integer such that the following is true. For every locally rectifiable curve $\beta : (a, b) \to \mathbb{G}$ in $\Gamma^*$ there exist curves $\alpha_1, \ldots, \alpha_m$ in $\Gamma$ such that

1) $(f \circ \alpha_j) \subset \beta$ for all $j = 1, \ldots, m$,
2) $\text{card}\{j : \alpha_j(t) = x\} \leq i(x, f)$ for all $x \in \Omega$ and for all $t \in (a, b)$.

Then

$$M_Q(\Gamma^*) \leq \frac{K(f)}{m} M_Q(\Gamma).$$

**Proof.** Let $\rho$ be an admissible function for a family $\Gamma$. If $x \in \Omega \setminus B_f$ then $f$ is $\mathcal{P}$-differentiable almost everywhere with strictly positive Jacobian $J(x, f)$. By $\bar{F}$, we denote a Borel set of measure zero containing all points $x \in \Omega \setminus B_f$ where $f$ is not $\mathcal{P}$-differentiable. Since $\text{mes}(F \cup B_f) = \text{mes} f(F \cup B_f) = 0$ we find a Borel set $E$ such that $f(F \cup B_f) \cup \{\infty\} \subset E \subset \mathbb{G}$ and $\text{mes}(E) = 0$. We define a function $\rho^*(y)$ on $f(\Omega)$ by the following way:

$$\rho^*(y) = \begin{cases} \frac{1}{m} \sum_{x \in f^{-1}(y) \cap \Omega} \rho(x) \lambda_f(x) & \text{if } y \in f(\Omega) \setminus E, \\ \infty & \text{if } y \in E, \\ 0 & \text{if } y \notin f(\Omega), \end{cases}$$
where \( \lambda_f(x) = \left( \min_{|\xi|=1, \xi \in V_1} |D_H f(x)|_{\xi} \right)^{-1} \). The function \( \rho^*(y) \) is a Borel nonnegative function. We show that \( \rho^* \in \mathcal{F}(\Gamma^*) \).

Let \( \gamma^* \in \Gamma^* \) and \( \alpha_1, \ldots, \alpha_m \) be as in conditions of the theorem. Lemma 3.2 implies that the family \( \Gamma_0^* \) of curves \( \gamma^* \) for which \( \alpha_1, \ldots, \alpha_m \) are not absolutely continuous, vanishes: \( \mathcal{M}_Q(\Gamma_0^*) = 0 \). The family \( \Gamma_1^* \subset \Gamma^* \) where \( \int_{\gamma^*} \chi_E ds > 0 \), \( E = f(F \cup B_f) \cup \{\infty\} \), \( \gamma^* \in \Gamma_1^* \), has also the \( Q \)-module zero (it can be shown as in Theorem 3.3). Throughout the proof we restrict our attention to the family \( \Gamma^* \setminus (\Gamma_0^* \cup \Gamma_1^*) \).

Let, for the moment, suppose that \( \gamma^* \in \Gamma^* \setminus (\Gamma_0^* \cup \Gamma_1^*) \) be a closed curve: \( \gamma^* : [a, b] \to G \).

We correlate the parameterization for \( \gamma^* \) and \( \alpha_1, \ldots, \alpha_m \) as in Remark 3.2. We follow the notations: \( s^* \) is the length arc parameter on \( \gamma^* \), \( s^* \in I^* = [0, l(\gamma^*)) \); for each \( \alpha_k \) the interval \( I_k^* \) is such that \( f(\alpha_k(s^*)) \subset \gamma^* \) when \( s^* \in I_k^* \); \( s \) is the length arc parameter on \( \alpha_k \), \( k = 1, \ldots, m \), and \( s \in I_k^* \). Then by (3.6) we see

\[
1 \leq \int_{\alpha_k} \rho \, ds = \int_{I_k^*} \rho(\alpha_k(s)) \, ds = \int_{I_k^*} \rho(\alpha_k(s^*)) \frac{ds}{ds^*} \, ds^* \leq \int_{I_k^*} \rho(\alpha_k(s^*)) \lambda_f(\alpha_k(s^*)) \, ds^* \tag{3.9}
\]

for each curve \( \alpha_k \), \( k = 1, \ldots, m \).

Set \( K_{s^*} = \{k : s^* \in I_k^*\} \). Then for almost all \( s^* \in I^* \) the points \( \alpha_k(s^*) \), \( k \in K_{s^*} \), are distinct points in \( f^{-1}(\gamma^*(s^*))) \). Therefore,

\[
\rho^*(\gamma^*(s^*)) \geq \frac{1}{m} \sum_{k=1}^{m} \rho(\alpha_k(s^*)) \lambda_f(\alpha_k(s^*)) \chi_{I_k^*}, \tag{3.10}
\]

where \( \chi_{I_k^*} \) is the characteristic function of \( I_k^* \). We conclude from (3.9), (3.10) and

\[
1 \leq \frac{1}{m} \sum_{k=1}^{m} \int_{I_k^*} \rho \, ds \leq \frac{1}{m} \sum_{k=1}^{m} \int_{I_k^*} \rho(\alpha_k(s^*)) \lambda_f(\alpha_k(s^*)) \, ds^* \leq \int_{\gamma^*} \rho^* \, ds^*.
\]

that \( \rho^* \in \mathcal{F}(\Gamma^* \setminus (\Gamma_0^* \cup \Gamma_1^*)) \). If the curve \( \gamma^* \) is not closed we obtain the same result taking supremum over all closed parts of \( \gamma^* \).
Now we estimate $M_Q(\Gamma^*)$. We choose an exhaustion of $\Omega$ by measurable sets $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega$, $\cup_i \Omega_i = \Omega$. Since $\text{mes}(E) = 0$, we obtain

$$
\int_G \rho^*(y)^Q \chi_{\Omega_i} dy \leq \int_G \left( \frac{1}{m} \sum_{x \in f^{-1}(y) \cap \Omega} \rho(x)^Q \chi_{\Omega_i}(x) \right)^Q dy
$$

$$
\leq \frac{1}{m} \int_G \sum_{x \in f^{-1}(y) \cap \Omega} \rho(x)^Q \chi_{\Omega_i}(x) dy
$$

$$
\leq \frac{K_I(f)}{m} \int_G \sum_{x \in f^{-1}(y) \cap \Omega} \rho(x)^Q \chi_{\Omega_i} J^{-1}(x, f) dy
$$

$$
\leq \frac{K_I(f)}{m} \int_\Omega \rho(x)^Q dx \leq \frac{K_I(f)}{m} \int_\Omega \rho(x)^Q dx,
$$

by (1.3) and (1.5). Letting $i \to \infty$, we deduce

$$
M_Q(\Gamma^*) = M_Q(\Gamma^* \setminus (\Gamma_0^* \cup \Gamma_1^*)) \leq \int_\Omega \rho^*(y)^Q dy \leq \frac{K_I(f)}{m} \int_\Omega \rho(x)^Q dx,
$$

and the proof is complete. \hfill \Box

Corollary 3.4. Let $f$ be as in Theorem 3.7, $U$ be a normal domain for $f$ with $m = N(f, U)$, $\Gamma^*$ be a family of curves in $f(U)$, $\Gamma$ be a family of curves in $U$ such that $f \circ \alpha \in \Gamma^*$ for any $\alpha \in \Gamma$. Then

$$
M_Q(\Gamma^*) \leq \frac{K_I(f)}{m} M_Q(\Gamma).
$$

Proof. Let $\beta : [a, b) \to f(U)$ be a curve from $\Gamma^*$ and $\{x_1, \ldots, x_k\} = U \cap f^{-1}(\beta(a))$. Since $U$ is a normal domain, we have $\sum_{i=1}^k i(x_i, f) = m$. Theorem 3.6 implies that there exists a maximal sequence $\alpha_1, \ldots, \alpha_m$ of $f$-liftings starting at $x_1, \ldots, x_k$ defining on $[a, b)$. We have $\sum_{x \in f^{-1}(y) \cap U} i(x, f) = m$ for every $y \in f(U)$. Therefore, the condition 2) of Theorem 3.7 also holds.

If $\beta$ is defined on an arbitrary interval $(a, b)$, then a slight change in the proof give the same conclusion. \hfill \Box

4. Relations between module and counting functions

From now on, we restrict our considerations on $\mathbb{H}$-type Carnot groups, for which the Heisenberg group is the simplest example. For the definition of $\mathbb{H}$-type Carnot groups see the example 3 and references therein. As was mentioned in the introduction, the presented value distribution theory can be extended to "polarizable" Carnot group introduced in the work [4]. We give the necessary definitions.

Let $\Omega \subset G$ be a domain. A function $u \in W^1_2(\Omega)$ is said to be harmonic if it is a weak solution to the equation

$$
\Delta_0 u = \text{div}(\nabla_0 u) = \sum_{j=1}^{n_1} X_{1j}^2 u = 0.
$$

(4.1)
The norm $|\cdot| = u^{1/(2-Q)}$, that we use below, is associated to the fundamental solution of (4.1). The fundamental solution exists and it is unique by a result of Folland [15, Theorem 2.1]. Denote a set of characteristic points by $Z = \{0\} \cup \{x \in \mathbb{G} \setminus \{0\} : \nabla_0 |x| = 0\}$.

**Theorem 4.1.** [4] Let $S = S(0,1) = \{x \in \mathbb{G} : |x| = 1\}$. There exists a unique Radon measure $\sigma^*$ on $S \setminus Z$ such that for all $u \in L^1(\mathbb{G})$

$$\int_{\mathbb{G}} u(x) \, dx = \int_{S \setminus Z} \int_0^\infty u(\varphi(s,y)) s^{Q-1} \, ds \, d\sigma^*(y),$$

where $dx$ denotes the Haar measure on $\mathbb{G}$.

Here $\varphi : (0,\infty) \times \mathbb{G} \setminus Z \to \mathbb{G} \setminus Z$ is a flow of radial rectifiable curves. The flow $\varphi$ satisfies the following properties:

(i) $|\varphi(s,x)| = s|x|$ for $s > 0$ and $x \in \mathbb{G} \setminus Z$;

(ii) $\left| \frac{\partial \varphi(s,x)}{\partial s} \right|_0 = \frac{|x|}{\nabla_0 |x|}_0$ and, in particular, is independent of $s$;

(iii) $J(x,\varphi(s,x)) = s^Q$ for $s > 0$ and $x \in \mathbb{G} \setminus Z$.

We present the value of the constant $\kappa(\mathbb{G},p)$ from (1.8) (see [4]). If we use the notation $v(x) = \frac{|\nabla_0 |x||_0}{|x|}$, $x \in \mathbb{G} \setminus Z$, then

$$\kappa(\mathbb{G},p) = \int_{S \setminus Z} v(y)^p \, d\sigma^*(y).$$

The set $Z \cap S$ has Hausdorff dimension at most $N - 2$ [17], where $N$ is the topological dimension of the group.

The polar coordinates can be introduced on any Carnot group for any homogeneous norm $|\cdot|'$. The integration formula for $u \in L^1(\mathbb{G})$ is of the form

$$\int_{\mathbb{G}} u(x) \, dx = \int_{S'} \int_0^\infty u(\delta_s y) s^{Q-1} \, ds \, d\sigma'(y),$$

where $d\sigma'$ is a Radon measure on $S' = S'(0,1) = \{x \in \mathbb{G} : |x'| = 1\}$ and $\delta_s$ is the dilation that was introduced in Section 1 (more details see in [16]). The formula (4.2) differs from (4.1) in one important respect: the curves $\varphi(s,y)$ have the finite Carnot-Carathéodory length. This is not the case for the curves $\gamma(s,y) = \delta_s y$ in the most situations. The pointed out difference allows us to employ the standard family curves arguments. On the Heisenberg group the polar coordinates with a rectifiable radial flow were described in [34].

Let $f : \Omega \to \overline{\mathbb{G}}$, $\Omega \subset \mathbb{G}$, be a quasimeromorphic mapping. For a point $y \in \overline{\mathbb{G}}$ and for a Borel set $E \subset \Omega$ such that $\overline{E}$ is a compact in $\Omega$ we set

$$n(E,y) = \sum_{x \in f^{-1}(y) \cap E} i(x, f).$$

In the case $E = B(0,r)$ we use the notation $n(r,y)$.

**Lemma 4.1.** The function $y \mapsto n(E,y)$ is upper semicontinuous.
Proof. Since a quasimeromorphic mapping is discrete, any point $x \in E$ has a normal neighborhood. If $U(x)$ is a normal neighborhood of $x \in E$, we have
\[ i(x, f) = \mu(f(x), f, U(x)) = \mu(f(z), f, U(x)) \geq i(z, f) \]
for any $z \in U(x)$. It shows that the function $x \mapsto i(x, f)$ is upper semicontinuous. \qed

If $S(z, s)$ is a sphere in $\mathbb{G}$ we denote by $\nu(E, S(z, s))$ the average of $n(E, y)$ over the sphere $S(z, s)$ with respect to a measure $\sigma = \nu Q \sigma^*$, where $\nu(x)$ is a function from $\mathbb{G}$ and $\sigma^*$ is the Radon measure on a sphere defined in Theorem 4.1. The measure $\sigma$ is absolutely continuous with respect to the measure $\sigma^*$. In particular, we denote by $\nu(r, s)$ the average of $n(r, y)$ over $S(0, s)$. Hence
\[ \nu(r, s) = \frac{1}{\sigma(S \setminus \mathcal{Z})} \int_{S \setminus \mathcal{Z}} n(r, \varphi(s, y)) d\sigma(y), \]
where $\sigma(S \setminus \mathcal{Z}) = \sigma(S(0, 1) \setminus \mathcal{Z})$ is the measure of the unit sphere coinciding with the constant $\kappa(\mathbb{G}, Q)$. We observe that value of $\nu(x)$ is invariant under the left translation. In fact, since $v(x) = \left| \frac{\partial \varphi_{s,x}}{\partial s} \right|_0^{-1} = |\dot{\varphi}(s, x)|^{-1}_0$ we need to show
\[ |\dot{\varphi}(s, x)|_0 = |\dot{w}\varphi(s, x)|_0, \]
where $w\varphi(s, x)$ is the image of $\varphi(s, x)$ under the left translation by the element $w$. We have $(w\varphi)_{1j}(s, x) = \varphi_{1j}(s, x) + w_{1j}$, $j = 1, \ldots, n_1$. Thus $w\varphi_{1j}(s, x) = \dot{\varphi}_{1j}(s, x)$. The curves $\varphi(s, x), w\varphi(s, x)$ are rectifiable. Then
\[ \dot{\varphi}(s, x) = \sum_{j=1}^{n_1} \dot{\varphi}_{1j}(s, x)X_{1j}(\varphi(s, x)), \]
\[ w\varphi(s, x) = \sum_{j=1}^{n_1} w\varphi_{1j}(s, x)X_{1j}(w\varphi(s, x)) = \sum_{j=1}^{n_1} \dot{\varphi}_{1j}(s, x)X_{1j}(w\varphi(s, x)). \]
Since the left invariant basis is orthonormal at an arbitrary point of $\mathbb{G}$ we deduce the necessary result. Therefore, $\kappa(\mathbb{G}, Q) = \sigma(S \setminus \mathcal{Z}) = \sigma(S(0, 1) \setminus \mathcal{Z}), w \in \mathbb{G}$.

We use below Definition 2.9 of an isolated essential singularity of $f$.

Lemma 4.2. Let $f : \mathbb{G} \rightarrow \mathbb{C}$ be a quasimeromorphic mapping with $\{ \infty \}$ as an essential singularity and $Y = S(w, s)$ is a sphere in $\overline{\mathbb{G}}$, then
\[ \lim_{r \rightarrow \infty} \nu(r, Y) = \infty. \]

Proof. By Lemma 2.11 there is a $\mathcal{F}_0$-set $E \in \mathbb{C}$ of the $Q$-capacity zero such that
\[ N(y, f, \mathbb{C}(0, r) \setminus \{ \infty \}) = \infty \quad \text{for all} \quad y \in \mathbb{C} \setminus E \quad \text{and for all} \quad r > 0. \]
Let $F_k(r) = \{ y \in Y \setminus \mathcal{Z} : n(r, y) \geq k \}$ and $E_0 = (Y \setminus \mathcal{Z}) \setminus E$. We claim that $\sigma(E_0) = 0$ if $\operatorname{cap}_Q(E_0) = 0$. Results of [9, 35] imply that $\mathcal{H}_1^\alpha(E_0) = 0$ for any $\alpha > 0$. It follows $\mathcal{H}_{Q-1}^1(E_0) = 0$. Here $\mathcal{H}_{Q-1}^1$ is an $\alpha$-dimensional Hausdorff measure with respect to the homogeneous norm $|\cdot|$. If we denote the Riemannian area element on $Y \setminus \mathcal{Z}$ by $dA$, then the connection between the Riemannian area element and the Hausdorff measure is expressed by the formula $dA = |\nabla x| \cdot |\nabla y| d\mathcal{H}_{Q-1}^1$ (see, for instance, [24, Proposition 4.9]). It follows that
A(E_0) = 0. The Radon measure \( \sigma^* \) is absolutely continuous with respect to \( dA \), that gives \( \sigma^*(E_0) = \sigma(E_0) = 0 \).

We continue to estimate \( \lim_{r \to \infty} \nu(r, Y) \).

\[
\lim_{r \to \infty} \nu(r, Y) \geq \frac{1}{\sigma(Y \setminus Z)} \lim_{r \to \infty} \int_{F_k(r)} k \, d\sigma(y) = \frac{k}{\sigma(Y \setminus Z)} \lim_{r \to \infty} \sigma(F_k(r)) \geq \frac{k}{\sigma(Y \setminus Z)} \sigma((Y \setminus Z) \setminus E_0) = k
\]

for every \( k > 0 \). The lemma is proved. \( \square \)

From now on, we use the symbol \( S \) to denote the unit sphere \( S(0,1) \) centered at the identity of the group. Let us choose a parametrization of radial curves \( \varphi(s, x) \) in such a way that \( |\varphi(1, y)| = |y| = 1 \) for \( y \in S \setminus Z \). We fix the notation \( \varphi_s(y), y \in S \setminus Z \) for such curve. Then any \( x \in G \setminus Z \) can be obtained as an image of \( y \in S \setminus Z \) under the map \( y \mapsto \varphi_s(y) \) for some \( s > 0 \). Therefore \( |x| = |\varphi_s(y)| = s \).

**Lemma 4.3.** Let \( E \) be a Borel set on \( S \setminus Z \) and let \( C \) be the cone \( \{ x \in G : x/|x| \in E \} \). Set \( \Gamma_E \) be the family of all curves \( \varphi_s(y) : [a, b] \to G, y \in E \). Then

\[
\sigma(E) \left( \ln \frac{b}{a} \right)^{1-Q} = M_Q(\Gamma_E).
\]

**Proof.** Let \( \rho \in F(\Gamma_E) \) and \( \varphi_s(y) : [a, b] \to G \) be a radial curve, introduced before Lemma 4.3. Recall that \( |\frac{\partial \varphi_s(y)}{\partial s}|_0 = \frac{|\nabla y|}{|y|} = v(y)^{-1} \). The Hölder inequality implies

\[
1 \leq \left( \int_{\varphi_s(y)} \rho \, dt \right)^Q = \left( \int_a^b \rho(\varphi_s(y)) \left| \frac{\partial \varphi_s(y)}{\partial s} \right|_0 \, ds \right)^Q 
\]

\[
\leq \int_a^b \rho(\varphi_s(y))^Q v(y)^{-Q} s^{-Q} \, ds \left( \int_a^b \frac{ds}{s} \right)^{Q-1} 
\]

\[
= \left( \ln \frac{b}{a} \right)^{Q-1} \int_a^b \rho(\varphi_s(y))^Q v(y)^{-Q} s^{-Q} \, ds.
\]

Integrating over \( y \in E \) with respect to the measure \( \sigma \) yields

\[
\sigma(E) \leq \left( \ln \frac{b}{a} \right)^{Q-1} \int_a^b \rho(\varphi_s(y))^Q s^{-Q} \, ds \, d\sigma^*(y) = \left( \ln \frac{b}{a} \right)^{Q-1} \int_C \rho(x) \, dx.
\]

Taking the infimum over all \( \rho \in F(\Gamma_E) \) we obtain the inequality \( \sigma(E) \left( \ln \frac{b}{a} \right)^{1-Q} \leq M_Q(\Gamma_E) \).

To deduce the reverse inequality, we take the function

\[
\rho(x) = \begin{cases} 
\frac{v(x)}{|x| \ln b/a} & \text{if } x \in C, \\
0 & \text{otherwise}.
\end{cases}
\]
We claim that \( \rho \) is admissible for \( \Gamma_E \). Indeed, since
\[
(4.6) \quad v(\varphi_s(y))^{-1} = \frac{\partial \varphi_s(\varphi_s(y))}{\partial s}\bigg|_0 = \frac{\nabla_0 \varphi_s(y)}{|\varphi_s(y)|} = \frac{|\nabla_0 s| |y|}{s|y|} = v(y)^{-1},
\]
we estimate
\[
\int \rho(s) \, ds = \int_a^b \rho((\varphi_s(y))) \frac{\partial \varphi_s(y)}{\partial s}\bigg|_0 \, ds
= \left( \ln \frac{b}{a} \right)^{-1} \int_a^b v(\varphi_s(y)) v^{-1}(y) \, ds = \left( \ln \frac{b}{a} \right)^{-1} \int_a^b \frac{ds}{s} = 1.
\]
Thus, by (4.6) we see
\[
M_Q(\Gamma_E) \leq \int_C \rho(x)^Q \, dx = \left( \ln \frac{b}{a} \right)^{-Q} \int_E \int_a^b \frac{v(\varphi_s(y))^Q}{|\varphi_s(y)|^Q} s^{Q-1} \, ds \, d\sigma^*
= \left( \ln \frac{b}{a} \right)^{-Q} \int_E \int_a^b \frac{ds}{s} \, d\sigma = \sigma(E) \left( \ln \frac{b}{a} \right)^{1-Q}.
\]
The proof is completed. \( \square \)

We would like to compare averages of \( n(r, y) \) over two distinct concentric spheres on the Carnot group.

**Proposition 4.4.** Let \( m : S \setminus Z \to Z \) be a nonnegative integer valued Borel function, \( t, s > 0 \), and, for \( y \in S \setminus Z \), \( \beta_y(\tau) \) be a radial curve \( \omega \varphi_s(y) \) connecting the spheres \( S(\omega, s) \) and \( S(\omega, t) \). Suppose \( \Gamma \) is a family consisting of \( m(y) \) essentially separate partial \( f \)-liftings of each \( \beta_y \) when \( y \) runs over \( S \setminus Z \). Then
\[
(4.7) \quad \int_{S \setminus Z} m(y) \, d\sigma(y) \leq K_I(f) \left| \ln \frac{t}{s} \right|^{Q-1} M_Q(\Gamma).
\]

**Proof.** We use the following notations: \( E_k = \{ y \in S \setminus Z : m(y) = k \} \), \( k = 0, 1, 2, \ldots \), \( \Gamma^*_k = \{ \beta_y : y \in E_k \} \), \( \Gamma_k = \{ f \text{-liftings of } \beta_y \in \Gamma^*_k \} \). Theorem 3.7 and Lemma 4.3 imply
\[
(4.8) \quad k \sigma(E) \left| \ln \frac{t}{s} \right|^{1-Q} \leq kM_Q(\Gamma^*_k) \leq K_I(f)M_Q(\Gamma_k).
\]
Since the curves in \( \Gamma_k \) are separate, we deduce \( (4.7) \) summing \( (4.8) \) over \( k \). \( \square \)

**Theorem 4.2.** Let \( f : \Omega \to \mathbb{C} \) be a quasimeromorphic mapping, \( \varrho > r > 0 \), and \( t, s > 0 \). If \( \overline{B}(0, \varrho) \subset \Omega \), then
\[
(4.9) \quad \nu(\varrho, S(\omega, t)) \geq \nu(r, S(\omega, s)) - K_I(f) \left| \ln \frac{t}{s} \right|^{1-Q} \left( \ln \frac{\varrho}{r} \right)^{1-Q}
\]
for any \( \omega \in \mathbb{G} \).

**Proof.** We may assume that \( t > s > 0 \) and we set \( m(y) = \max \{ 0, n(r, \omega \varphi_s(y)) - n(\varrho, \omega \varphi_t(y)) \} \) for a point \( y \in S \setminus Z \). If \( m(y) > 0 \), then there exist at least \( m(y) \) maximal \( f \)-liftings of \( \beta_y = \omega \varphi_s(y) \) starting in \( \overline{B}(0, r) \) and terminating on \( \partial B(0, \varrho) \) which are essentially separate.
We denote by $\Gamma$ the set of these $f$-liftings and, making use of Proposition 4.1 and (1.8), deduce

$$
\int_{S \setminus Z} m(y) \, d\sigma(y) \leq K_I(f) \left( \frac{\ln t}{s} \right)^{Q-1} M_Q(\Gamma)
$$

$$
\leq K_I(f) \kappa(\mathbb{G}, Q) \left( \frac{\ln \frac{t}{s}}{r} \right)^{1-Q}.
$$

If $E = \{y \in S \setminus Z : m(y) > 0\}$, then

$$
\int_{S \setminus Z} n(q, \omega \varphi(t)(y)) \, d\sigma(y) = \int_{(S \setminus Z) \setminus E} n(q, \omega \varphi(t)(y)) \, d\sigma(y) + \int_E n(q, \omega \varphi(t)(y)) \, d\sigma(y)
$$

$$
\geq \int_{(S \setminus Z) \setminus E} n(r, \omega \varphi_s(y)) \, d\sigma(y)
$$

$$
+ \int_E \left( n(r, \omega \varphi_s(y)) - m(y) \right) \, d\sigma(y)
$$

$$
\geq \int_{S \setminus Z} n(r, \omega \varphi_s(y)) \, d\sigma(y) - \int_{S \setminus Z} m(y) \, d\sigma(y).
$$

Dividing by $\kappa(\mathbb{G}, Q)$ and using (4.10), we finish the proof. $\square$

**Lemma 4.5.** Let $\varrho > r > 0$, $\theta > 1$, and $f : \Omega \to \mathbb{G}$ be a quasimeromorphic mapping of a domain $\Omega$ with $\overline{B}(0, \varrho) \subset \Omega$. Let $a_1, \ldots, a_q$, $q \geq 2$, be distinct finite points in $\mathbb{G}$. Set $\sigma_m = \frac{1}{4} \min_{i \neq j} \text{dist}(a_i, a_j)$ and $0 < s < t \leq \sigma_m$. Assume that $F_1, \ldots, F_\lambda$, $2 \leq \lambda \leq q$, are disjoint compact sets in $\overline{B}(0, \varrho)$ such that $f(F_j) \subset \overline{B}(a_j, s)$ for each $j \leq \lambda$, and $F_1, \ldots, F_\lambda$ intersect spheres $S(0, \tau)$ for almost all $\tau \in [r, \rho]$. Then there are positive constants $b_1$ and $b_2$ depending on $Q, \vartheta$ only such that

$$
(M_Q(\Gamma_j) - b_1 K_O(f) K_I(f)) \left( \frac{\ln \frac{t}{s}}{r} \right)^{Q-1} \leq b_2 K_O(f) \nu(\varrho, S(a_j, t))
$$

for all $j$. Here $\Gamma_j$ is a family of locally rectifiable curves connecting $F_j$ with $\bigcup_{i \neq j} F_i$ in $B(0, \varrho) \setminus \overline{B}(0, r)$.

**Proof.** Fix $j \leq \lambda$ and assume that $a_j = 0$. We put

$$
\rho(z) = \begin{cases} 
\frac{\nu(z)}{|z| \ln t/s} & \text{if } z \in B(0, t) \setminus B(0, s), \\
0 & \text{elsewhere}.
\end{cases}
$$

The proof of Lemma 4.3 shows that the function $\rho(z)$ is admissible for the family $f(\Gamma_j)$. Then

$$
M_Q(\Gamma_j) \leq K_O(f) \int_{\mathbb{G}} \rho(z) Q n(q, z) \, dz = K_O(f) \kappa(\mathbb{G}, Q) \left( \frac{\ln \frac{t}{s}}{r} \right)^{-Q} \int_s^t \nu(\varrho, \tau) \frac{d\tau}{\tau}
$$

$$
\leq K_O(f) \kappa(\mathbb{G}, Q) \nu(\varrho, t) \left( \frac{\ln \frac{t}{s}}{r} \right)^{1-Q} + K_O(f) K_I(f) \kappa(\mathbb{G}, Q) (\ln \theta)^{1-Q}
$$
Lemma 4.7. Let the function (4.12) for pact \( Q \) be used with constants \( b_1 = \kappa(G, Q)(\ln \theta)^{-1} \) and \( b_2 = \kappa(G, Q) \).

Corollary 4.6. Under the assumptions of Lemma 4.5, there exist positive constants \( b_1 \) and \( b_2 \) depending on \( Q, \theta \) only such that

\[
(\ln \frac{r}{\theta} - b_1 K_O(f) K_I(f)) \left(\ln \frac{t}{s}\right)^{-1} \leq b_2 K_O(f) \nu(\theta^Q, S(a_j, t))
\]

for arbitrary \( j = 1, \ldots, \lambda \).

Proof. Let \( \Gamma_j \) be as in the condition of Lemma 4.5. Then the following estimate \( M_Q(\Gamma_j) \geq c(Q) \ln \frac{r}{\theta} \) is true for arbitrary \( j = 1, \ldots, \lambda \). We deduce the desired result applying this inequality to the left-hand side of (4.11). Here \( b_1 = c^{-1}(Q) \kappa(G, Q)(\ln \theta)^{-1} \) and \( b_2 = c^{-1}(Q) \kappa(G, Q) \).

We would like to obtain an analogue of Lemma 4.5 when one of the points \( a_j \) is \( \{\infty\} \).

Lemma 4.7. Let \( \theta > r > 0, \theta > 1 \), and \( f : \Omega \rightarrow \mathbb{G} \) be a quasimeromorphic mapping of a domain \( \Omega \) with \( B(0, \theta) \subset \Omega \). Let \( a_0 = \infty \) and \( a_1, \ldots, a_q \) be distinct finite points in \( \mathbb{G} \). Set \( \sigma_M = 4 \max \{\{a_j\}, 1\} \) and \( \sigma_m = \frac{1}{4} \min \text{dist}(a_i, a_j) \). Assume that \( F_0, \ldots, F_\lambda \), \( 1 \leq \lambda \leq q \), are disjoint compact sets in \( \overline{B}(0, \rho) \) such that \( f(F_j) \subset \mathbb{G}(0, t) \) for \( t > s > \sigma_M, f(F_j) \subset \mathbb{G}(0, t) \) for \( e \) each \( 1 \leq j \leq \lambda \) and \( 0 < s < t < \sigma_m \). We suppose that \( F_0, F_1, \ldots, F_\lambda \) intersect spheres \( S(0, \tau) \) for almost all \( \tau \in [r, \rho] \). Then there are positive constants \( b_1 \) and \( b_2 \) depending only on \( Q, \theta \) such that

\[
(M_Q(\Gamma_0) - b_1 K_O(f) K_I(f)) \left(\ln \frac{t}{s}\right)^{-1} \leq b_2 K_O(f) \nu(\theta^Q, S(0, s)).
\]

Here \( \Gamma_0 \) is the family of locally rectifiable curves in \( B(0, \rho) \) that connect the compact \( F_i \) to \( \bigcup_{1 \leq i \leq \lambda} F_i \).

Proof. The function (4.12) for \( s, t \) such that \( \sigma_M < s < t < \infty \) is admissible for the family \( f(\Gamma_0) \). We have

\[
\nu(\theta, \tau) \leq \nu(\theta^Q, s) + K_I(f) \left(\ln \frac{t}{s}\right)^{-1} \leq \nu(\theta^Q, s) + \frac{K_I(f)}{(\ln \theta)^{-1}} \left(\ln \frac{t}{s}\right)^{-1}
\]

for all \( \tau \in (s, t) \) by Theorem 4.2. Then, using the same estimates as in (4.13), we finish the proof as in Lemma 4.5 with \( b_1 = \kappa(G, Q)(\ln \theta)^{-1} \) and \( b_2 = \kappa(G, Q) \).

Corollary 4.8. Under the assumptions of Lemma 4.7, there exist positive constants \( b_1 \) and \( b_2 \) depending on \( Q, \theta \) only such that

\[
(\ln \frac{r}{\theta} - b_1 K_O(f) K_I(f)) \left(\ln \frac{t}{s}\right)^{-1} \leq b_2 K_O(f) \nu(\theta^Q, S(0, t)),
\]

Proof. We argue as in Corollary 4.6.

The average \( \nu(r, s) \) over a sphere of radius \( s \) is a discontinuous function in the variable \( r \). We need an auxiliary continuous function \( A(r) \) related to a quasimeromorphic mapping \( f : \Omega \rightarrow \mathbb{G} \). In the classical value distribution theory for analytic functions, it is considered the integral \( A(r) = \int r \eta(r, y) \, d\mu(y) \) with respect to a nonnegative measure distributed over a closed set \( E \) in a target plane of an analytic function. Then \( A(r) \) has a geometric-physical significance: it equals the total mass distributed over the Riemann surface of
the analytic function onto which this function maps the disk $B(0, r)$. In the Euclidean space $\mathbb{R}^n$, S. Rickman employed the $n$-dimensional normalized spherical measure $\frac{c(n) \, dy}{(1+|y|^2)^n}$ as a measure $\mu(y)$ which is invariant under conformal mappings. In this case, $A(r)$ has a geometric meaning also. In the arguments below, our version of $A(r)$ is used as an auxiliary tool only.

We define $A(r)$ as

$$
A(r) = \frac{2Q}{\pi \kappa(\mathbb{G}, Q)} \int_{\mathbb{G}} n(r, y)v(y)^Q \, dy = \frac{2Q}{\pi \kappa(\mathbb{G}, Q)} \int_{B(0, r)} J(x, f)v(f(x))^Q \, dx.
$$

In the definition of $A(r)$ we have taken into account that $\text{mes}(B_f) = \text{mes}(f(B_f)) = 0$.

**Lemma 4.9.** If $\theta > 1$, $r > 0$, $\overline{B}(0, \theta r) \subset \Omega$, and $Y = S(\omega, t)$ is a sphere with radius $t > 0$, then

$$
\nu\left(\frac{r}{\theta}, Y\right) - \frac{K_I(f)c_1}{(\ln \theta)^{Q-1}} (|\ln t|^{Q-1} + c_0) \leq A(r)
$$

$$
\leq \nu(\theta r, Y) + \frac{K_I(f)c_1}{(\ln \theta)^{Q-1}} (|\ln t|^{Q-1} + c_0),
$$

where $c_0$ and $c_1$ some positive constants depending on $Q$ only.

**Proof.** Writing (4.9) in the form

$$
\nu(\theta r, t) \geq \nu(r, s) - \frac{K_I(f)|\ln \frac{r}{s}|^{Q-1}}{(\ln \theta)^{Q-1}}
$$

multiplying both sides of (4.17) by $\frac{s^{Q-1}}{1+s^{2Q}}$, and integrating from 0 to $\infty$, we obtain

$$
\nu(\theta r, t) \int_0^\infty \frac{s^{Q-1}}{1+s^{2Q}} \, ds \geq \int_0^\infty \frac{\nu(r, s)s^{Q-1}}{1+s^{2Q}} \, ds - \frac{K_I(f)}{(\ln \theta)^{Q-1}} \int_0^\infty \frac{|\ln \frac{r}{s}|^{Q-1}s^{Q-1}}{1+s^{2Q}} \, ds.
$$

The integral in the left hand side of (4.18) equals $\frac{\pi}{2Q}$. The first integral in the right hand side gives

$$
\int_0^\infty \frac{\nu(r, s)s^{Q-1}}{1+s^{2Q}} \, ds = \frac{1}{\kappa(\mathbb{G}, Q)} \int_{S^2} \int_0^\infty \frac{n(r, \varphi_s(y))v(\varphi_s(y))Qs^{Q-1}}{1-|\varphi_s(y)|^{2Q}} \, ds \, d\sigma^+(y)
$$

$$
= \frac{1}{\kappa(\mathbb{G}, Q)} \int_{\mathbb{G}} \frac{n(r, z)v(z)^Q}{1+|z|^{2Q}} \, dz = \frac{\pi}{2Q} A(r).
$$

To estimate the second one we use the inequality

$$
|\ln \frac{t}{s}|^{Q-1} \leq 2Q-2(|\ln t|^{Q-1} + |\ln s|^{Q-1})
$$

and the finiteness of the integral $\int_0^\infty \frac{|\ln s|^{Q-1}s^{Q-1}}{1+s^{2Q}} \, ds$. To simplify notation, we write $\frac{\pi}{2Q} c_0$ for the last integral. Thus

$$
\frac{K_I(f)}{(\ln \theta)^{Q-1}} \int_0^\infty \frac{|\ln \frac{r}{s}|^{Q-1}s^{Q-1}}{1+s^{2Q}} \, ds \leq \frac{K_I(f)2Q-2}{(\ln \theta)^{Q-1}} \left(\frac{\pi}{2Q}|\ln t|^{Q-1} + \frac{\pi}{2Q} c_0\right).
$$
Joining the estimates of all integrals, we get the right hand side of (4.16) with $c_1 = 2^{Q-2}$.

To obtain the left hand side of (4.16) we multiply

$$\nu(r,t) \geq \nu(r/\theta,s) - \frac{K_1(f)\ln v^{Q-1}}{(\ln \theta)^{Q-1}}$$

by $\frac{Q-1}{1+2\nu^2}$, and integrate the result from 0 to $\infty$. Arguing as above, we prove the lemma. □

**Corollary 4.10.** Let $f : \mathbb{C} \to \mathbb{C}$ be a quasimeromorphic mapping with $\{\infty\}$ as an essential singularity and let $A(r)$ be as in (4.15). Then $\lim_{r \to \infty} A(r) = \infty$.

**Proof.** The corollary follows from Lemmas 4.12 and 4.19 □

We need an inequality that will permit us to compare the average over spheres with different centers and radii.

**Lemma 4.11.** If $\theta > 1$, $r > 0$, $\overline{B}(0, \theta r) \subset \Omega$, and if $Z = S(\omega, u)$ and $Y = S(\omega, v)$ are spheres in $\mathbb{C}$ with radii $u$ and $v$ then

$$\nu(\theta r, Z) \geq \nu(r, Y) - \frac{c_1 K_1(f)}{(\ln \theta)^{Q-1}} (|\ln u|^{Q-1} + |\ln v|^{Q-1} + 2c_0)$$

where $c_1 = 2^{Q-3}$.  

**Proof.** To show Lemma 4.11 we replace $\theta$ by $\theta^{1/2}$ and $r$ by $r \theta^{1/2}$ in the inequality (4.16). □

5. **Auxiliary statements for the proof of the main theorem**

Recall the statement of Theorem 0.1. In the formulation we use the notation $u_+ = \max\{0, u\}$.

Let $f : \mathbb{C} \to \mathbb{C}$ be a nonconstant $K$-quasimeromorphic mapping. Then there exists a set $E \subset [1, \infty)$ and a constant $C(Q, K) < \infty$ such that

$$\lim_{r \to \infty} \sup_{r \notin E} \sum_{j=0}^q \left(1 - \frac{n(r, a_j)}{\nu(r, 1)}\right) \leq C(Q, K)$$

with $\int_E \frac{dr}{r} < \infty$,

whenever $a_0, a_1, \ldots, a_q$ are distinct points in $\mathbb{C}$.

To prove our main result we need to realize the following steps.

1. To set up the exceptional set $E \subset [1, \infty)$ mentioned in Theorem 0.1.
2. To construct a special decomposition of the ball $B(0, s)$ to some smaller sets $U_i, i = 1, 2, \ldots, p$. The sets $U_i$ constitute a finite-to-one covering: there is $M > 0$ such that $\sum_i \chi_{U_i} \leq M$ for all $x \in B(0, s)$. Here $\chi_{U_i}$ is the characteristic function of $U_i$.
3. To apply Lemmas 4.5 and 4.7. For realizing this, we consider a chain of increasing balls $U_i \subset V_i \subset W_i \subset X_i \subset Y_i \subset Z_i \subset \overline{B}(0, s')$ having finite-to one covering property. Then we calculate the number of $f$-liftings of radial curves connecting points $z \in S(0, \sigma_M)$ with $a_j \in B(0, \sigma_M/2), j = 1, \ldots, q$, or $a_0 = \infty$.
4. To estimate $Q$-module of families related to these liftings in the spherical ring type domains $V_i \setminus U_i$ and $W_i \setminus V_i$.
5. To sum obtained results over all $U_i$ obtaining by this way a bound from above for the sum $\sum_{j \in J \cup \{0\}} \Delta_j$ (the definition of $\Delta_j$ see at the end of Subsection 5.4).
5.1. Construction of the exceptional set. In the present section we construct the exceptional set $E \in [1, \infty)$ mentioned in Theorem 5.1.

**Theorem 5.1.** There exists a set $E \in [1, \infty)$ such that $\int_{E} \frac{dr}{r} < \infty$ and the following is true: if $0 < \varepsilon_0 < 1/5$ and we write

$$ s' = s + \frac{s}{\varepsilon_0(A(s))^{q-1}} \quad \text{for} \quad s > 0, $$

then this is an increasing function $\omega : [0, \infty) \to [D_{\varepsilon_0}, \infty]$ such that

$$ \nu(s,Y) \left\| \nu(s',Y) - 1 \right\| < \varepsilon_0 \quad (5.1) $$

and

$$ \nu(s,Y) \nu(s',Y) > 1 - \varepsilon_0 \quad (5.2) $$

hold whenever $Y = S(w,t)$ is a sphere of radius $t$, and $s' \in [\omega(\ln t), \infty] \setminus E$. Here $D_{\varepsilon_0} > 0$, $A(D_{\varepsilon_0}) > \varepsilon_0^{1-\frac{1}{Q}}$.

**Proof.** First, we construct a set $E$. Set $\phi(r) = m^{-2}A(r)^{\frac{1}{q-1}}$ for each integer $m \geq 2$. We can choose $r''_0 = r''_0(m) \geq 1$ such that $\phi(r''_0) \geq 1$, by Corollary 4.10. Let $F_m = \{ r > r''_m : A(r + \frac{m}{r''_m}) > \frac{m}{m-1}A(r) \}$ and assume that $F_m \neq \emptyset$. We define inductively a sequence

$$ r''_0 \leq r_1 < r''_1 \leq r_2 < r''_2 \leq \ldots $$

by

$$ r_k = \inf\{ r > r''_{k-1} : r \in F_m \} \quad \text{and} \quad r''_k = r_k + \frac{2r_k}{\phi(r_k)}. $$

Then $F_m \subset \bigcup_{k \geq 1} [r_k, r''_k]$. Also we define $\rho_k = r''_k + \frac{2r_k}{\phi(r_k)}$, and put $H_m = \bigcup_{k \geq 1} [r_k, \rho_k]$. We estimate the logarithmic measure of $H_m$ as follows:

$$ \int_{H_m} \frac{dr}{r} \leq \sum_{k \geq 1} \frac{\rho_k - r_k}{r_k} = \sum_{k \geq 1} \frac{1}{r_k} \left( r''_k + \frac{2r''_k}{\phi(r_k)} - r_k \right) $$

$$ = \sum_{k \geq 1} \frac{1}{r_k} \left( r_k + \frac{2r_k}{\phi(r_k)} + \frac{2r_k}{\phi(r_k)} - r_k \right) $$

$$ \leq \sum_{k \geq 1} \frac{8}{\phi(r_k)} = \sum_{k \geq 1} \frac{8m^2}{(A(r_k))^{q-1}}. $$

Since the function $A(r)$ increases, we have $A(r_{k+1}) \geq A(r''_k) > \frac{m}{m-1}A(r_k)$ and

$$ \int_{H_m} \frac{dr}{r} \leq \frac{8m^2}{(A(r_1))^{q-1}} \sum_{k \geq 1} \left( \frac{m-1}{m} \right)^{\frac{k}{m}} < \infty. $$

If $F_m = \emptyset$, then we set $H_m = \emptyset$.

Further, we choose a sequence $d_2 < d_3 < \ldots$ of numbers such that

$$ d_m \geq 3r''_0(m) \quad \text{and} \quad \int_{E_m} \frac{dr}{r} < \frac{1}{2m}. $$
where \( E_m = H_m \cap [d_m, \infty) \). If we denote the union \( \bigcup E_m \) by \( E \) then we get \( \int_E \frac{dr}{r} < \infty \). Let \( \varepsilon_0 \in (0, 1/5) \) and let \( Y = S(w, t) \) be a sphere with radius \( t \). We choose the least integer \( m \) satisfying

(i) \( m \geq 4 \),
(ii) \( \frac{m^2}{(m-1)^2} < 1 + \frac{2}{\varepsilon} \), and
(iii) \( c_1K_1(f)(|\ln t|^{Q-1} + c_0) \leq \frac{m}{2m-1} \).

In this case the inequality \( \frac{2}{m} < \varepsilon_0 \) also holds. We take \( s' \geq d_m \) and \( s' \notin E \). Then there is \( r \geq r_0'' \) such that

\[
s' = s + \frac{s}{\varepsilon_0(A(s))^{\frac{1}{Q-1}}} \quad \text{with} \quad s = r + \frac{r}{\phi(r)}.
\]

We claim that \( r \notin F_m \). Indeed, if we suppose that \( r \in [r_k, r''_k) \) for some \( k \), then we obtain

\[
r_k < r < s' = r + \frac{r}{\phi(r)} + \frac{r + \frac{r}{\phi(r)}}{\varepsilon_0(A(r + \frac{r}{\phi(r)})^{\frac{1}{Q-1}})} \leq r + \frac{r}{\phi(r)} + \frac{r + \frac{r}{\phi(r)}}{2\phi(r)} \leq r + 2r/\phi(r) = r''_k
\]

from the inequalities \( A(r + 1/\phi(r)) > A(r), \phi(r) > 1, \) and \( \varepsilon_0 > 2/m^2 \). The estimates \( r_k < s' < \rho_k \) imply that \( s' \notin H_m \), but it contradicts to choice of \( s' \notin E \).

Now, we apply Lemma [19] with \( \theta = 1 + 1/\phi(r) \). Using the condition (iii), the estimates \( \ln \theta \geq 1/2/\phi(r), s' < r + 2r/\phi(r) \), and the definition of \( F_m \), we get

\[
\nu(s, Y) = \nu(r\theta, Y) \geq A(r) - \frac{c_1K_1(f)}{(\ln \theta)^{Q-1}}(|\ln t|^{Q-1} + c_0) \geq A(r) - m\phi(r)^{Q-1}
\]

\[
= A(r) - \frac{m}{m^{2Q-2}}A(r) \geq \frac{m-1}{m}A(r) \geq \frac{(m-1)^2}{m^2}A\left(r + \frac{2r}{\phi(r)}\right)
\]

\[
\geq \frac{(m-1)^2}{m^2}A(s').
\]

This and the condition (ii) imply

\[
\frac{\nu(s, Y)}{A(s')} \geq \left(\frac{m-1}{m}\right)^2 > 1 - \frac{\varepsilon_0}{2}.
\]

Notice, that since \( s' \notin E \) and \( s' \geq d_m \), we have \( s' \notin H_m \). Therefore, \( s' \notin F_m \). Now, we apply Lemma [19] with \( \theta = 1 + 1/\phi(s') \). Arguing as above we deduce

\[
\nu(s', Y) \leq \nu\left(s' + \frac{s'}{\phi(s')}\right) + m\phi(s')^{Q-1} \leq \left(\frac{m}{m-1} + \frac{1}{m}\right)A(s') \leq \left(\frac{m}{m-1}\right)^2A(s').
\]

Finally,

\[
\frac{\nu(s', Y)}{A(s')} \leq \left(\frac{m}{m-1}\right)^2 < 1 + \frac{\varepsilon_0}{2}.
\]

The inequalities \([5.4]\) and \([5.5]\) imply \([5.1]\). Moreover, we have

\[
\nu(s, Y) > \frac{2 - \varepsilon_0}{2}A(s') > \frac{2 - \varepsilon_0}{2 + \varepsilon_0}\nu(s', Y) \geq (1 - \varepsilon_0)\nu(s', Y),
\]
that prove (5.2).

By $m_0 \geq 4$, we denote the least integer satisfying the condition (iii) and $\frac{2}{m} < \varepsilon_0$. Then since $s' \geq d_{m_0}$ we can put $D_{\varepsilon_0} = d_{m_0}$. In this case

$$1 \leq \frac{(A(d_{m_0}))^{-1}}{m_0^2} < \varepsilon_0(A(D_{\varepsilon_0}))^{-1}$$

that gives the last statement of the theorem. \qed

Let $E \subset [1, \infty)$ be the exceptional set constructed in the previous theorem. The points $a_1, \ldots, a_q$ belong to the ball $B(0, \sigma_M/2)$, $\sigma_M = 4 \max_{1 \leq j \leq q} \{1, |a_j|\}$, and $a_0 = \infty$. To apply Theorem 5.1 we fix a positive $\varepsilon_0 \leq \min(\frac{1}{4}, \frac{1}{8q+9})$. Then (5.2) implies that

$$\frac{\nu(s', \sigma_M)}{\nu(s, \sigma_M)} < 1 + \frac{\varepsilon_0}{1 - \varepsilon_0} < \frac{3}{2}$$

and $\frac{\nu(s', 1)}{\nu(s, 1)} < \frac{1}{1 - \varepsilon_0}$. Moreover, we have $\frac{\nu(s', 1)}{\nu(s, \sigma_M)} < 1 + \varepsilon_0$ and $\frac{A(s')}{\nu(s, \sigma_M)} < \frac{1}{1 - \varepsilon_0}$ from (5.1). Three last inequalities yield

$$\frac{\nu(s', 1)}{\nu(s, \sigma_M)} < \frac{1 + \varepsilon_0}{(1 - \varepsilon_0)^2} \leq \left(1 - \frac{2\varepsilon_0}{1 - \varepsilon_0}\right)^2 \leq 1 + \frac{8\varepsilon_0}{1 - \varepsilon_0} < 1 + \frac{1}{q + 1}.$$ Estimations (5.6) and (5.7) hold whenever $s > 0$ is such that $s' \in [\hat{k}, \infty) \setminus E$ with $\hat{k} > \max\{\omega(\ln 2\sigma_M), \omega(0)\}$. The numbers $\varepsilon_0$ will be specified later. Set $J = \{1, \ldots, q\}$. To prove Theorem 0.1 it suffices to show

$$\sum_{j \in J \cup \{0\}} \left(1 - \frac{n(s', a_j)}{\nu(s', 1)}\right) + \leq C(Q, K) < \infty.$$ We may assume that $\frac{n(s', a_j)}{\nu(s', 1)} < 1$ for all $j \in J \cup \{0\}$. Introduce the auxiliary value

$$\Delta_j = 1 - \frac{n(s', a_j)}{\nu(s, \sigma_M)}.$$

Then

$$\sum_{j \in J \cup \{0\}} \left(1 - \frac{n(s', a_j)}{\nu(s', 1)}\right) = \sum_{j \in J \cup \{0\}} \Delta_j + \sum_{j \in J \cup \{0\}} \frac{n(s', a_j)}{\nu(s', 1)} \left(\frac{\nu(s', 1)}{\nu(s, \sigma_M)} - 1\right)$$

$$\leq \sum_{j \in J \cup \{0\}} \Delta_j + \frac{1}{q + 1} \sum_{j \in J \cup \{0\}} \frac{n(s', a_j)}{\nu(s', 1)}$$

$$\leq \sum_{j \in J \cup \{0\}} \Delta_j + 1.$$ 5.2. Decomposition of the ball $B(0, s)$. Here we construct the precise decomposition of the ball $B(0, s)$ into finitely overlapping sets. Let $d = s' - s$. We start from the ball $B(0, s)$ and, inside of it, we construct rings of increasing diameters. The diameters increase when we move from the sphere $\partial B(0, s)$ to the center of $B(0, s)$.

$$R_0 = \overline{B}(0, s) \setminus B(0, s - d), R_1 = \overline{B}(0, s - d) \setminus B(0, s - 2d), \ldots$$

$$R_n = \overline{B}(0, s - 2^{n-1} d) \setminus B(0, s - 2^nd), \ldots.$$ We continue this process up to the step $L$ when $B(0, s - 2^L d) \subset B(0, 2^L d)$. 


Every ring $R_n$ has a diameter $2^{n-1}d$, $n = 1, \ldots, L$. Then we use Wiener lemma (see for instance [15], p. 53) and cover the ring $R_n$ by balls $B(x_i, \frac{2^{n-2}d}{100\kappa x})$, $x_i \in R_n$, such that every point of $R_n$ belongs to at most $M$ balls. Here $\kappa$ is the constant from the generalized triangle inequality and $x > 0$ is a number specifying later. Also we can suppose that $\kappa \geq 1$.

We cover the ring $R_0$ by balls $B(x, \frac{d}{100\kappa x})$ and the ball $B(0, 2d)$ by $B(x, \frac{2d-1}{100\kappa x})$. Notice, that the balls $B(x, \frac{2^{n-2}d}{100\kappa x})$ intersect only three rings $R_{n-1}$, $R_n$, and $R_{n+1}$.

Estimate the quantity of balls covering $B(0, s)$. The volume $V(R_n)$ of one ring $R_n$ is estimated by

$$V(R_n) = (s - 2^{n-1}d)^Q - (s - 2^n d)^Q \leq C_1(Q)s^{Q-1}d^{n-1}.$$  

Since the volume of a ball $B(x, \frac{2^{n-2}d}{100\kappa x}) \cap R_n \neq \emptyset$ is $\left(\frac{2^{n-2}d}{100\kappa x}\right)^Q$, we deduce the upper bound for the number $p_n$ of balls that have nonempty intersection with $R_{n+1}$, $R_n$ or $R_{n-1}$:

$$p_n \leq \frac{V(R_{n+1}) + V(R_n) + V(R_{n-1})}{\left(\frac{2^{n-2}d}{100\kappa x}\right)^Q} \leq \frac{C_2(Q, M, \kappa, x)^Q}{(2^{-Q}(n-1))} \left(\frac{s}{d}\right)^{Q-1}.$$  

Summing over $n$, we get the estimate for number $p$ of balls covering $B(0, s)$:

$$p \leq \sum_{n=0}^{L} p_n \leq C_3(Q, M, \kappa, x) \left(\frac{s}{d}\right)^{Q-1} = C_3\kappa^{Q-1}A(s) \leq 2C_3\kappa^{Q-1}A(s, \sigma M).$$

Now, we denote by $B(x_i, r_i)$ the balls constructed in the decomposition of $B(0, s)$. Then we write

$$U_i = B(x_i, r_i) \cap B(0, s), \quad V_i = B(x_i, 2r_i), \quad W_i = B(x_i, 4r_i),$$  

$$X_i = B(x_i, 6r_i), \quad Y_i = B(x_i, 2r_i) \quad Z_i = B(x_i, 4r_i).$$

We obtain that $Z_i \in \overline{B}(0, s')$. Let us estimate the multiplicity of overlapping of balls $Z_i$. It is sufficient to estimate the multiplicity in one ring, for instance, in $R_n$. Firstly, we observe that $Z_k = B(x_k, \frac{2^{n-2}d}{25\kappa x})$, $x_k \in R_n$, can intersect only the balls with the same radius and center from $R_n$, either the balls $Z_i = B(x_i, \frac{2^{n-1}d}{25\kappa x})$, $x_i \in R_{n+1}$, or $Z_j = B(x_j, \frac{2^{n-3}d}{25\kappa x})$, $x_j \in R_{n-1}$. Indeed, if $y \in B(x_m, \frac{2^{n-2}d}{25\kappa x}) \cap B(x_k, \frac{2^{n-2}d}{25\kappa x})$, $x_m \in R_{n+2}$, $x_k \in R_n$, then

$$2^n d \leq d(x_m, x_k) \leq \kappa \left(\frac{2^n d}{25\kappa x} + \frac{2^{n-2}d}{25\kappa x}\right) \leq \kappa \frac{2^n d}{5}.$$  

We get a contradiction. If $y \in B(x_i, \frac{2^{n-4}d}{25\kappa x}) \cap B(x_k, \frac{2^{n-2}d}{25\kappa x})$, $x_i \in R_{n-2}$, $x_k \in R_n$, then

$$2^{n-2} d \leq d(x_i, x_k) \leq \kappa \left(\frac{2^{n-4}d}{25\kappa x} + \frac{2^{n-2}d}{25\kappa x}\right) \leq \kappa \frac{2^{n-4}d}{5},$$

which is a contradiction again. We note also, that if $y \notin B(x_i, \frac{2^{n-2}d}{25\kappa x})$, $x_i \in R_n$, then the ball $B(x_i, \frac{2^{n-2}d}{25\kappa x})$ does not meet the balls $B(y, \frac{2^{n-3}d}{25\kappa x})$, $y \in R_{n+1}$, $B(y, \frac{2^{n-2}d}{25\kappa x})$, $y \in R_n$, and $B(y, \frac{2^{n-3}d}{25\kappa x})$, $y \in R_{n-1}$. It follows that the multiplicity $M$ of overlapping of $Z_i = B(x_i, \frac{2^{n-2}d}{25\kappa x})$ cannot exceed

$$\tilde{M} \leq M \left(\frac{2^{n-2}d}{5}\right)^Q \left(\frac{2^{n-7}d}{100\kappa x}\right)^Q = (40\kappa x)^Q M.$$  

Now we can choose $\kappa$. Setting

$$\theta = 2 \exp \left(\frac{(K_0(Q)\kappa K_0(f)K_1(f))^{\frac{1}{\theta}}}{c(Q)\ln 65}\right).$$
we put \( \varkappa = 3\theta \). Here the constant \( c(Q) \) is from the proof of Corollary 4.6.

**Remark 5.2.** Notice, that by the choice of \( \theta \) we have

\[
\ln \theta > \left( \frac{(G, Q)K_O(f)K_I(f)}{c(Q)\ln 6/5} \right)^{\frac{1}{q-1}}
\]

and

\[
\ln \frac{3}{2} - \frac{(G, Q)K_O(f)K_I(f)}{c(Q)(\ln \theta)^{q-1}} > \ln \frac{5}{4}.
\]

**5.3. Estimates for partial liftings.** We remember that the points \( a_1, \ldots, a_q \) belong to \( B(0, \sigma_M/2) \) and \( a_0 = \infty \). Now we describe a rectifiable curve \( \gamma_y^j(t) \) that connect the points \( a_j, j = 0, \ldots, q \) with \( z \in S(0, \sigma_M) \setminus \mathcal{Z} \).

Let \( G_y(t) : [0, \sigma_M] \to [0, t_y] \) be an affine mapping with \( G_y(0) = t_y, G_y(\sigma_M) = 0 \), and such that \( a_j \varphi_{G_y(\sigma_M)}(y) = a_j \varphi_0(y) = a_j, a_j \varphi_{G_y(\sigma_M)}(y) = a_j \varphi_y(t_y) = y \) where \( z \in S(0, \sigma_M) \), \( j = 1, \ldots, q \), and \( \varphi_y \) is a radial curve for \( y \in S \setminus \mathcal{Z} \). We put \( \gamma_y^j(t) = a_j \varphi_{G_y(t)}(y) \), \( j = 1, \ldots, q \). Then the rectifiable curve \( \gamma_y^j(t) : [0, \sigma_M] \to \mathcal{B}(0, \sigma_M) \) connect \( a_j \) with points \( z \in S(0, \sigma_M) \setminus \mathcal{Z} \) such that \( a_j = \gamma_y^j(\sigma_M), z = \gamma_y^j(0) \). In view of \( t_y \in [\sigma_M/2, 2\sigma_M] \) and [4.5], we deduce

\[
|\varphi_y(t)|_0 = |\varphi_{G_y(t)}(y)|_0 = v^{-1}(y)|G_y(t)| \leq 2v^{-1}(y).
\]

Also, we consider locally rectifiable curves \( \gamma_y^0(t) : [\sigma_M, \infty) \to \mathbb{C}B(0, \sigma_M), \gamma_y^0(t) = \varphi_t(y), y \in S \setminus \mathcal{Z} \). These curves joint the point \( a_0 = \{\infty\} \) with \( z \in S(0, \sigma_M) \setminus \mathcal{Z}, a_0 = \lim_{t \to \infty} \gamma_y^0(t), z = \gamma_y^0(\sigma_M). \)

> From now on, up to the end of the paper, we suppose that the points \( y \in S \setminus \mathcal{Z} \) correspond to the points \( z \in S(0, \sigma_M) \setminus \mathcal{Z} \) as it was described in the preceding paragraph.

Let \( f_0 = f|_{B(0,2\sigma_M)} \). We fix \( i \in I = \{1, \ldots, p\} \) and \( j \in J \cup \{0\} = \{0, 1, \ldots, q\} \). Put \( \{x_1, \ldots, x_k\} = f^{-1}(z) \cap U_i, z \in S(0, \sigma_M) \setminus \mathcal{Z} \), \( z = \gamma_y^j(0) \) for \( j \in J \) or \( z = \gamma_y^0(\sigma_M) \). Let us consider the maximal sequence of essentially separate \( f_0 \)-liftings \( \beta_1, \ldots, \beta_m, m = n(U_i, z) \), of a curve \( \gamma^j \) starting at \( \{x_1, \ldots, x_k\} \). In this section we will work only with those curves of \( \beta_\mu \) whose locus is not contained in \( \mathcal{B}(0, s') \). We denote them by \( \alpha_1, \ldots, \alpha_m \) for each \( y \in S \setminus \mathcal{Z} \). We correlate the parametrization of \( \gamma^j, j \in J, \) and \( \alpha_\mu \) as in Remark 3.2.

Then \( \alpha_\mu : [0, \sigma_M] \to \mathbb{G} \). The curve \( \gamma_y^0(t)|_{[\sigma_M, R]}, \sigma_M < R < \infty, \) is rectifiable. Making use of correlation of Remark 3.2 we may assume that \( f_0 \)-liftings of \( \gamma_y^0(t)|_{[\sigma_M, R]} \) are curves \( \alpha_\mu : [\sigma_M, R] \to \mathbb{G}, \sigma_M < R < \infty. \)

Taking the supremum over all closed parts of \( \gamma_y^0 \) we get that \( f_0 \)-liftings of \( \gamma_y^0 \) are curves \( \alpha_\mu : [\sigma_M, \infty) \to \mathbb{G} \). We fix values of parameters \( 0 \leq u_{\mu,y} < v_{\mu,y} < w_{\mu,y} < \sigma_M \) if \( j \in J \) and \( \sigma_M \leq u_{\mu,y} < v_{\mu,y} < w_{\mu,y} < \infty \) if \( j = 0 \) for each \( \alpha_\mu \) such that

\[
\alpha_\mu(u_{\mu,y}) \in \partial U_i, \quad \alpha_\mu(v_{\mu,y}) \in \partial V_i, \quad \alpha_\mu(w_{\mu,y}) \in \partial W_i.
\]

We use the notation

\[
\alpha_\mu^{(1)} = \alpha_\mu|_{u_{\mu,y}}, \quad \alpha_\mu^{(2)} = \alpha_\mu|_{v_{\mu,y}^{-}, w_{\mu,y}^{+}}.
\]

We want to estimate the number of \( f_0 \)-liftings of different parts of \( \gamma^j \), moving from \( z = \gamma_y^j(0) \) to \( a_j \) in the case when \( j \in J \) or advancing from \( z = \gamma_y^0(\sigma_M) \) to \( \infty \) for \( j = 0 \).
Set \( s_0 = \frac{1}{16} \min_{1 \leq j \neq k \leq q} |a_j^{-1}a_k| \). We introduce

\[
L_j(y) = \{ \mu \in \{1, \ldots, \mu_y\} : \frac{\sigma_M - u_{\mu,y}}{\sigma_M - v_{\mu,y}} \leq \frac{\sigma_M}{s_0} \} \quad \text{for} \quad j = 1, \ldots, q
\]

and

\[
L_0(y) = \{ \mu \in \{1, \ldots, \mu_y\} : \frac{v_{\mu,y}}{u_{\mu,y}} \leq \frac{\sigma_M}{s_0} \}.
\]

Since we can not say anything about measurability of \( \text{card} \ L_j(y) \) we need to estimate \( \text{card} \ L_j(y) \) by some measurable function.

**Lemma 5.4.** There exists a nonnegative measurable function \( l_j(y) : S \setminus Z \to \mathbb{R}^1 \) such that

\[
\text{card} \ L_j(y) \leq l_j(y) \quad \text{for any} \quad y \in S \setminus Z, \quad j \in J \cup \{0\},
\]

\[
\int_{S \setminus Z} l_j(y) \, d\sigma(y) \leq c_5 K_1(f) \left( \ln \frac{\sigma_M}{s_0} \right)^{Q-1}.
\]

**Proof.** The properties of a quasimeromorphic mapping imply that there is a Borel set \( F \subset \mathbb{C} \) with \( \text{mes}(F) = 0 \) containing the set of points, where \( f \) is not \( P \)-differentiable. Then \( \text{mes}(f(F)) = 0 \). In this case the one dimensional Hausdorff measure of intersection \( \gamma_j(t) \cap f(F) \) vanishes for almost all points \( z \in S(0, \sigma_M) \setminus Z \) with respect to the \( \sigma \)-measure. Then, making use of the Fubini theorem and that \( \sigma \) is absolutely continuous with respect to \( \sigma^* \), we get a contradiction with \( \text{mes}(f(F)) = 0 \). Let \( \Gamma_0 \) be a family of locally rectifiable curves \( \gamma_j(t) \setminus a_j, \ y \in S \setminus Z \), that have closed parts with the \( f_0 \)-liftings \( \alpha_{\mu,y} \) of which are not absolutely continuous. Lemma 3.2 states that the \( Q \)-module of \( \Gamma_0 \) vanishes. We conclude that for almost all points \( z \in S(0, \sigma_M) \setminus Z \) curves \( \alpha_{\mu,y}, \ mu = 1, \ldots, \mu_y \), are absolutely continuous for all \( j = 0, 1, \ldots, q \) by Lemma 4.3. We denote by \( E_2 \) this exceptional set of \( S(0, \sigma_M) \). Put \( E \subset S(0, \sigma_M) \) be a Borel set such that \( E_1 \cup E_2 \subset E \) and \( \sigma(E) = 0 \).

We start from the case when \( j = 1, \ldots, q \). Let \( \rho \) be an admissible function for the family of curves connecting \( U_i \) with \( \partial V_i \) in \( V_i \) such that \( \rho|_{G \setminus (V_i \setminus V) = 0} \) and \( \int_{V_i} \rho^2 \, dx \leq 2M_Q(U_i, V_i) \).

We fix \( z \in S(0, \sigma_M) \setminus (E \cup Z) \). For the corresponding \( y \in S \) and \( \mu = 1, \ldots, \mu_y \) we define \( \rho^*_\mu \) on \( f(\alpha_\mu) \) by

\[
\rho^*_\mu(z) = \rho^*_\mu(f(x)) = \begin{cases} \rho(x)\lambda_f(x), & x \in \alpha_\mu \cap CF, \\ 0, & \text{otherwise}, \end{cases}
\]
where $1/\lambda_f(x) = \min_{|\xi|=1, \xi \in V_1} |D_H f(x) \xi|_0$. Then by (5.10)

$$
2 \int_{u_{\mu,y}} \rho^*_\mu(\gamma^j_y(t)) v(y)^{-1} dt \geq \int_{u_{\mu,y}} \rho^*_\mu(\gamma^j_y(t)) |\dot{\gamma}^j_y(t)|_0 dt
$$

$$
= \int_{u_{\mu,y}} \rho(\alpha^1_\mu(t)) \lambda_f(\alpha^1_\mu) |\dot{\alpha}^1_\mu(t)|_0 dt
$$

(5.15)

$$
\geq \int_{u_{\mu,y}} \rho(\alpha^1_\mu(t)) |\dot{\alpha}^1_\mu(t)|_0 dt = \int_{\alpha^1_\mu} \rho dt \geq 1
$$

for each point $y \in S$ such that the corresponding point $z$ belongs to $S(0, \sigma_M) \setminus (E \cup Z)$. Applying the Hölder inequality we obtain

$$
1 \leq \left(2^Q \int_{u_{\mu,y}} (\rho^*_\mu(\gamma^j_y(t)))^Q v(y)^{-Q}(\sigma_M - t)^{Q-1} dt \right)^{\frac{1}{Q}} \left( \int_{u_{\mu,y}} \frac{dt}{\sigma_M - t} \right)^{1-Q}.
$$

We get the next estimate for the last integral

$$
\left( \ln \frac{\sigma_M - u_{\mu,y}}{\sigma_M - v_{\mu,y}} \right)^{1-Q} \leq 2^Q \int_{u_{\mu,y}} (\rho^*_\mu(\gamma^j_y(t)))^Q v(y)^{-Q}(\sigma_M - t)^{Q-1} dt.
$$

Summing over $L^1_i(y)$ yields

$$
\text{card } L^1_i(y) \left( \ln \frac{\sigma_M}{s_0} \right)^{1-Q} \leq \sum_{\mu=1}^{\mu_y} \left( \ln \frac{\sigma_M - u_{\mu,y}}{\sigma_M - v_{\mu,y}} \right)^{1-Q}
\leq 2^Q \sum_{\mu=1}^{\mu_y} \int_{u_{\mu,y}} (\rho^*_\mu(\gamma^j_y(t)))^Q v(y)^{-Q}(\sigma_M - t)^{Q-1} dt
\leq 2^Q \int_{\sigma_M}^\sigma \sum_{x \in f^{-1}(\gamma^j_y(t))} (\rho(x) \lambda_f(x))^Q v^{-Q}(\sigma_M - t)^{Q-1} dt.
$$

Now, we define the measurable function $\tilde{l}_i(y) : S \setminus Z \to \mathbb{R}$ satisfying (5.12), by

$$
\tilde{l}_i(y) = 2^Q \left( \ln \frac{\sigma_M}{s_0} \right)^{Q-1} \int_{\sigma_M}^\sigma \sum_{x \in f^{-1}(\gamma^j_y(t))} (\rho(x) \lambda_f(x))^Q v(y)^{-Q}(\sigma_M - t)^{Q-1} dt
$$

if $z \in S(0, \sigma_M) \setminus (E \cup Z)$, and $\tilde{l}_i(y) = n(U_i, z)$ if $z \in E$. Let us show that $\tilde{l}_i(y)$ satisfies (5.13). We denote by $\phi(w)$ an auxiliary map from $B(0, \sigma_M)$ to itself such that $\phi(w) = \phi(\varphi_{(\sigma_M - t)}(y)) = \gamma^j_y(t)$ for $t \in [0, \sigma_M], y \in S \setminus Z$. Then integrating over $S \setminus Z$
with respect to the $\sigma$-measure we deduce

$$
\int_{S \setminus \mathcal{Z}} l_i^1(y) \, d\sigma(y) = 2^Q \left( \ln \frac{\sigma_M}{\sigma_0} \right)^{Q-1} \int_{S \setminus \mathcal{Z}} d\sigma^*(y) \int_0^{\sigma_M} \left( \rho(x) \lambda_f(x) \right)^Q \, dt \, dt^-1 \, dt
\leq c_4 \left( \ln \frac{\sigma_M}{\sigma_0} \right)^{Q-1} \int_{B(0,\sigma_M)} \sum_{x \in f^{-1}(z)} \left( \rho(x) \lambda_f(x) \right)^Q \, dz.
$$

For any point $z \in B(0,\sigma_M) \setminus f(B_f)$, there exists some neighborhood $W$ where the inverse mapping $f^{-1}$ exists and homeomorphic. We write $V_1, \ldots, V_k$ for the components of $f^{-1}(W) \cap \overline{B(s')}$. The mappings $f_j = f|_{V_j} : V_j \to W$ are quasiconformal. Then

$$
\int_{B(0,\sigma_M) \setminus f(B_f)} \sum_{x \in f^{-1}(z)} \left( \rho(x) \lambda_f(x) \right)^Q \, dz = \sum_{j=1}^k \int_{V_j} \left( \rho(x) \lambda_f(x) \right)^Q J(x, f) \, dx
\leq K_1(f) \int_{f^{-1}(W)} \rho(x)^Q \, dx.
$$

The set $B(0,\sigma_M) \setminus f(B_f)$ can be covered up to the a set of measure zero by disjoint neighborhoods of this kind. It follows

$$
\int_{S \setminus \mathcal{Z}} l_i^1(y) \, d\sigma(y) \leq c_4 K_1(f) \left( \ln \frac{\sigma_M}{\sigma_0} \right)^{Q-1} \int_G \rho(x)^Q \, dx.
$$

We have [5.13] with $\zeta' = c_4 2M_Q(\overline{l_1}, V_i)$.

Now we consider a locally rectifiable curve $\gamma^0_y(t) : [\sigma_M, \infty) \to \mathbb{C}(0, \sigma_M)$ joining $z \in S(0, \sigma_M) \setminus \mathcal{Z}$ with $\infty$, such that $\gamma^0_y(\sigma_M) = z$ and $\lim_{t \to \infty} \gamma^0_y(t) = \infty$. We will define a measurable function $l_i^0(y) : S \setminus \mathcal{Z} \to \mathbb{R}$ satisfying [5.12] and [5.13]. As at the beginning of the proof, for the functions $\rho$ and $\rho^*$, we obtain [5.15] for any $y \in S \setminus \mathcal{Z}$, $j = 0$. The Hölder inequality implies

$$
1 \leq \left( \int_{u_{\mu,y}} \left( \rho^*_u(\gamma^0_y(t)) \right)^Q \, v(y)^{-Q} \, dt \right)^{-1} \left( \int_{u_{\mu,y}} \frac{dt}{t} \right)^{Q-1}.
$$

Making use of the estimate

$$
\left( \ln \frac{\sigma_M}{\sigma_0} \right)^{1-Q} \leq \int_{u_{\mu,y}} \left( \rho^*_u(\gamma^0_y(t)) \right)^Q \, v(y)^{-Q} \, dt
$$

and summing over $\mu \in L_i^0(y)$, we get

$$
\frac{\text{card } L_i^0(y)}{(\ln(\sigma_M/\sigma_0))^{Q-1}} \leq \sum_{\mu=1}^{M_y} \left( \ln \frac{\sigma_M}{\sigma_0} \right)^{1-Q} \leq \sum_{\mu=1}^{M_y} \int_{u_{\mu,y}} \left( \rho^*_u(\gamma^0_y(t)) \right)^Q \, v(y)^{-Q} \, dt
\leq \int_{\sigma_M}^{\infty} \sum_{x \in f^{-1}(\gamma^0_y(t))} \left( \rho(x) \lambda_f(x) \right)^Q \, v(y)^{-Q} \, Q \, dt.
$$
We define the function \( l_i^0(y) \) by
\[
l_i^0(y) = \left( \frac{\ln \frac{\sigma_M}{s_0}}{\ln \frac{\sigma_M}{s_0}} \right)^{Q-1} \int_{\sigma_M}^{\infty} \sum_{x \in f^{-1}(\gamma_i^0(t))} (\rho(x)\lambda_f(x)) Q v(y)^{-Q} t^{Q-1} dt
\]
if \( z \in S(0, \sigma_M) \setminus (Z \cup E) \), and \( l_i^0(y) = n(U_i, z) \) if \( z \in E \). Obviously, that \( l_i^0(y) \) satisfies (5.12).

To show (5.13) we argue as in the previous case, integrate over \( S \setminus Z \), and obtain
\[
\int_{S \setminus Z} l_i^0(y) \, d\sigma(y) = \left( \frac{\ln \frac{\sigma_M}{s_0}}{\ln \frac{\sigma_M}{s_0}} \right)^{Q-1} \int_{\sigma_M}^{\infty} \sum_{x \in f^{-1}(z)} (\rho(x)\lambda_f(x)) Q d\nu(x).
\]
Covering \( \mathbb{C}B(0, \sigma_M) \) by disjoint neighborhoods, where the mapping \( f^{-1} \) is homeomorphic, we deduce
\[
\int_{S \setminus Z} l_i^0(y) \, d\sigma(y) \leq K_i(f) \left( \frac{\ln \frac{\sigma_M}{s_0}}{\ln \frac{\sigma_M}{s_0}} \right)^{Q-1} \int_{\mathbb{C}B(0,\sigma_M)} \rho^Q \, dx \leq c_5' K_i(f) \left( \frac{\ln \frac{\sigma_M}{s_0}}{\ln \frac{\sigma_M}{s_0}} \right)^{Q-1}.
\]
We have (5.13) with \( c_5'' = 2 M_Q(U_i, V_i) \). Setting \( c_5 = \max\{c_5', c_5''\} \) we end the proof. □

Now we estimate the cardinality of \( \alpha_{\mu}^{(2)} \). For each \( i \in I \), we let \( \sigma_i \in [0, s_0] \) be a number defined by
\[
\left( \frac{\ln \frac{s_0}{\sigma_i}}{\ln \frac{s_0}{\sigma_i}} \right)^{Q-1} = A_i \nu(U_i, \sigma_M),
\]
where \( A_i \) will be given later. Set
\[
M_i^0(y) = \left\{ \mu \in \{1, \ldots, \mu_y\} : \frac{\sigma_M - \nu_{\mu,y}}{\sigma_M - \nu_{\mu,y}} \leq \frac{s_0}{\sigma_i} \right\} \quad \text{for} \quad j \in J
\]
and
\[
M_i^0(y) = \left\{ \mu \in \{1, \ldots, \mu_y\} : \frac{\nu_{\mu,y}}{\nu_{\mu,y}} \leq \frac{s_0}{\sigma_i} \right\},
\]
where index 0 corresponds to the curve \( \gamma_i^0 \) joining \( \infty \) with \( z \in S(0, \sigma_M) \setminus Z \).

**Lemma 5.5.** There exists a nonnegative measurable function \( m_i^j(y) : S \setminus Z \to \mathbb{R} \) such that
\[
\text{card} \left( M_i^j(y) \setminus L_i^j(y) \right) \leq m_i^j(y) \quad \text{for any} \ y \in S \setminus Z, j \in J \cup \{0\}, \ \text{and}
\]
\[
\sum_{j \in I \cup \{0\}} \int_{S \setminus Z} m_i^j(y) \, d\sigma(y) \leq c_6 K_i \left( \frac{\ln \frac{s_0}{\sigma_i}}{\ln \frac{s_0}{\sigma_i}} \right)^{Q-1}.
\]

**Proof.** Let \( E \subset S(0, \sigma_M) \) be the exceptional set that we defined in the previous lemma. Let \( \rho \) be an admissible function for \( \Gamma(V_i, W_i) \) such that \( \rho|_{\mathbb{C}\setminus(W_i \cup V_i)} = 0 \).

Fix \( z \in S(0, \sigma_M) \setminus (Z \cup E) \) and put the function \( \rho^\mu_\mu^* \) as in (5.14). If \( \mu \in M_i^j(y) \setminus L_i^j(y) \) for \( j \in J \) then \( \sigma_M - \nu_{\mu,y} < s_0 \). We estimate
\[
\text{card} \left( M_i^j(y) \setminus L_i^j(y) \right) \leq 2^Q \left( \frac{\ln \frac{s_0}{\sigma_i}}{\ln \frac{s_0}{\sigma_i}} \right)^{Q-1} \sum_{\mu \in M_i^j \setminus L_i^j} \int_{\nu_{\mu,y}}^{w_{\mu,y}} (\rho^\mu_\mu^*(\gamma_i^0(t))) Q v^{-Q}(\sigma_M - t)^{Q-1} dt
\]
\[ \leq 2^Q \left( \frac{\ln \frac{S_0}{\sigma_i}}{Q} \right)^{Q-1} \int_{\sigma_M - \omega_0}^{\sigma_M} \sum_{x \in f^{-1}(\gamma_i^0(t))} \left( \rho(x) \lambda_f(x) \right)^Q v(y)^{-Q} \chi(\sigma_M - t)^{Q-1} dt. \]

Define the measurable function \( m_0^j(y) : S \setminus Z \to \mathbb{R} \) by

\[ m_0^j(y) = 2^Q \left( \frac{\ln \frac{S_0}{\sigma_i}}{Q} \right)^{Q-1} \int_{\sigma_M - \omega_0}^{\sigma_M} \sum_{x \in f^{-1}(\gamma_i^0(t))} \left( \rho(x) \lambda_f(x) \right)^Q v(y)^{-Q} \chi(\sigma_M - t)^{Q-1} dt \]

if \( z \in S(0, \sigma_M) \setminus (Z \cup E) \) and \( m_0^j(y) = n(U_i, z) \) if \( z \in E \). The property (5.17) is evident. To show (5.18) we integrate over \( S \setminus Z \) and obtain

\[ \int_{S \setminus Z} m_0^j(y) d\sigma(y) \leq \hat{c}\left( \ln \frac{S_0}{\sigma_i} \right)^{Q-1} \int_{B(a_j, 3\omega_0/2)} \sum_{x \in f^{-1}(w)} \left( \rho(x) \lambda_f(x) \right)^Q dw. \]

Here we used an auxiliary map \( \phi(w) = \phi(\varphi(\sigma_{M-t}(y)) = \gamma_i^0(t) \) as in the previous lemma. Since the balls \( B(a_j, 3\omega_0/2) \) are disjoint, summing over \( j \in J \), we deduce

\[ (5.19) \sum_{j \in J} \int_{S \setminus Z} m_0^j(y) d\sigma(y) \leq \hat{c}\left( \ln \frac{S_0}{\sigma_i} \right)^{Q-1} \int_{B(0, \sigma_M)} \sum_{x \in f^{-1}(w)} \left( \rho(x) \lambda_f(x) \right)^Q dw. \]

As in the proof of Lemma 5.4 the estimate (5.19) gives (5.18) with the constant \( \hat{c}_i^0 = \hat{c}_i 2M_Q(V_i, W_i) \).

Now, we show (5.17) and (5.18) for \( j = 0 \). Arguing as in Lemma 5.4 we obtain

\[ \text{card}(M_0^i(y) \setminus L_0^i(y)) \leq \left( \ln \frac{S_0}{\sigma_i} \right)^{Q-1} \sum_{x \in f^{-1}(\gamma_i^0(t))} \left( \rho(x) \lambda_f(x) \right)^Q v(y)^{-Q} Q^Q dt \]

\[ \leq \left( \ln \frac{S_0}{\sigma_i} \right)^{Q-1} \int_{\sigma_M}^\infty \sum_{x \in f^{-1}(\gamma_i^0(t))} \left( \rho(x) \lambda_f(x) \right)^Q v(y)^{-Q} Q^Q dt. \]

We define \( m_0^0(y) \) by

\[ m_0^0(y) = \left( \ln \frac{S_0}{\sigma_i} \right)^{Q-1} \int_{\sigma_M}^\infty \sum_{x \in f^{-1}(\gamma_i^0(t))} \left( \rho(x) \lambda_f(x) \right)^Q v(y)^{-Q} Q^Q dt \]

if \( z \in S(0, \sigma_M) \setminus (Z \cup E) \) and \( m_0^0(y) = n(U_i, z) \) if \( z \in E \). The function \( m_0^0(y) : S \setminus Z \to \mathbb{R} \) is measurable and satisfies (5.17). Integrating over \( S \setminus Z \) and arguing as in Lemma 5.4 we have

\[ \int_{S \setminus Z} m_0^0(y) d\sigma(y) = \left( \ln \frac{S_0}{\sigma_i} \right)^{Q-1} \int_{B(0, \sigma_M)} \sum_{x \in f^{-1}(w)} \left( \rho(x) \lambda_f(x) \right)^Q dw \]

\[ \leq K_I(f) \left( \ln \frac{S_0}{\sigma_i} \right)^{Q-1} \int_{E} \rho(x) Q^Q dx \]

\[ \leq K_I(f) 2M_Q(V_i, W_i) \left( \ln \frac{S_0}{\sigma_i} \right)^{Q-1} \]

\[ = c_i^0 K_I(f) \left( \ln \frac{S_0}{\sigma_i} \right)^{Q-1}. \]
We obtain (5.18) from (5.19) and (5.20) with the constant $c_0 = \max\{c'_0, c''_0\}$. □

5.6. **Estimates for extremal maximal sequences of liftings.** In the previous section we estimated the number of different parts of $f_0$-liftings of the curve $\gamma_d^j$. But we do not know precisely how to chose these parts. In the present section we give a rule how to choose the $f_0$-liftings.

Fix $j \in J \cup \{0\}$ for a moment and consider the curves $\gamma_d^j$. For each point $y \in S \setminus Z$ we consider maximal essentially separate sequences $\Lambda = (\lambda_1, \ldots, \lambda_l)$ of $f_0$-liftings of $\gamma_d^j$ starting at the points $(x_1, \ldots, x_l) = f^{-1}(y) \cap B(0,s)$. Here $z = \gamma_d^j(0)$ for $j \in J$ or $z = \gamma_d^0(\sigma_M)$.

In this case $g = n(s,z) = \sum_{k=1}^l i(x_k,f)$. Let $\Omega_y$ denote the set of all these sequences. We introduce measurable functions $\psi_d^j(y) = \psi_i(y,a_j)$, $\psi_d(y) : S \setminus Z \to \mathbb{R}$, $i = 1, \ldots, p$, that help us to calculate the number of $f_0$-liftings of $\gamma_d^j$, starting from $U_i$, locus of which belongs to $B(0,s')$. We recall that $p$ is the number of sets $U_i$ in the decomposition of $B(0,s)$.

For $\Lambda \in \Omega_y$ we define

$$N(\Lambda) = \text{card}\{\nu : \lambda_\nu \subset \overline{B}(0,s')\}, \quad \psi_d(y) = \sup\{N(\Lambda) : \Lambda \in \Omega_y\}.$$ 

By $\Omega(y)$, we denote the set of sequences $\Lambda \in \Omega_y$ for which $N(\Lambda) = \psi_d(y)$. Set

$$U = B(0,s) \setminus \left( \bigcup_{i=1}^p \partial U_i \right).$$

**Lemma 5.7.** Under the above-mentioned notations, the function $\psi_d(y)$ is upper semicontinuous at points $y \in S \setminus (Z \cup f(\partial U))$.

**Proof.** Let $y_0 \in S \setminus (Z \cup f(\partial U))$ and $\{y_h\}_{h=1}^\infty$ be some sequence of points in $S \setminus (Z \cup f(\partial U))$ that converges to $y_0$. We want to show

$$\lim_{h \to \infty} \sup \psi_d^j(y_h) \leq \psi_d^j(y_0).$$

Firstly, we prove that if $y_h \to y_0$, then the maximal sequences of $f_0$-liftings of $\gamma_d^j$ converge to some sequence of $f_0$-liftings of $\gamma_d^{y_0}$. Then we show that the obtained sequence is essentially separate and deduce the inequality (5.21).

Passing to a subsequence of $\{y_h\}$ (that we denote by the same symbol $\{y_h\}$) we can assume that, for some integer $m$ such that $\psi_d^j(y_h) = m$ and for all $h \geq 1$, the following properties hold.

(i) There exist normal neighborhoods $V_1, \ldots, V_l \subset U$ of points $(x_1, \ldots, x_l) = f^{-1}(z_0) \cap U$ such that $z_h \in f(V_1) \cap \ldots \cap f(V_l)$ for all $h \geq 1$. Here $z_0 = \gamma_d^{y_0}(0)$, $z_h = \gamma_d^j(0)$ for $j \in J$ or $z_0 = \gamma_d^0(\sigma_M)$, $z_h = \gamma_d^{y_0}(\sigma_M)$.

(ii) There is a maximal essentially separate sequence $\Lambda_h = (\lambda_{h,1}, \ldots, \lambda_{h,M}) \in \Omega(y_h)$, $g = n(s,z_0)$, such that each $\lambda_{h,\nu}$ starts at a point in $f^{-1}(z_h) \cap V_\mu$ for some $\mu$ that depends on $\nu$ and is independent of $h$. Moreover, the locus of each curve $\lambda_{h,\nu}$ belongs to $\overline{B}(0,s')$ for $1 \leq \nu \leq m$.

(iii) The bound $v(y_h)^{-1} \leq K$ holds for all $h = 0, 1, \ldots$

This choice is possible because of properties of the local index and the topological degree.

Next, we consider two cases: 1. $1 \leq \nu \leq m$ and 2. $m + 1 \leq \nu \leq g$.

**Case 1.** We claim that the family $\{\lambda_{h,\nu}\}_{h=1}^\infty$ is equicontinuous for any $\nu$. Here we need to separate the consideration for $j \in J$ and $j = 0$. We start from the indexes $j \in J$. Since
\( \lambda_{h,\nu} \subset \overline{B}(0, s') \) for \( \nu = 1, \ldots, m \), each curve \( \lambda_{h,\nu} \) is defined on \([0, \sigma_M]\). Fix \( \varepsilon > 0 \). For each \( t \in [0, \sigma_M] \) there is \( \delta(t) > 0 \) such that for all \( \rho \in (0, \delta(t)] \), \( U(\xi, \rho) = f^{-1}(B(f(\xi), \rho)) \) is a normal neighborhood of \( \xi \in f^{-1}(\gamma_{0}(t)) \cap \overline{B}(0, s') \) with \( \text{diam}(U(\xi, \rho)) < \varepsilon \). We cover the curve \( \gamma_{0}(t) \) by finitely many balls \( B(\gamma_{0}(t), \delta(t)/2\varepsilon) \), which we denote by \( B(\eta_{u}, \rho_{u}) \), \( u = 1, \ldots, v \). Here \( \varepsilon \) is the constant from the generalized triangle inequality. We may suppose that the curves \( \gamma_{h_{j}} \), \( h \geq 1 \), belong to the tube \( \bigcup_{u=1}^{v} B(\eta_{u}, \rho_{u}) \). For any \( t \in [0, \sigma_M] \), there is \( u \) such that \( \gamma_{h_{j}}(t) \in B(\eta_{u}, \rho_{u}) \). We recall that the distances \( d(\cdot, \cdot) \) and \( d_{c}(\cdot, \cdot) \) are equivalent with a constant \( \tilde{c} > 0: \tilde{c}^{-1}d(\cdot, \cdot) \leq d_{c}(\cdot, \cdot) \leq \tilde{c}d(\cdot, \cdot) \). We have, by (5.10),

\[
d(\gamma_{h_{j}}(t), \gamma_{h_{j}}(t')) \leq \tilde{c}d_{c}(\gamma_{h_{j}}(t), \gamma_{h_{j}}(t')) \leq \tilde{c} \int_{t}^{t'} |\dot{\gamma}_{h_{j}}(s)|_{0} ds = 2\tilde{c}v^{-1}(y_{h})|t - t'| \leq 2\tilde{c}K|t - t'|.
\]

Thus, if \( |t - t'| \leq \frac{\rho_{u}}{2\tilde{c}K} \), then

\[
d(\gamma_{h_{j}}(t), \gamma_{h_{j}}(t')) \leq \varepsilon \left( d(\gamma_{h_{j}}(t), \gamma_{h_{j}}(t)) + d(\gamma_{h_{j}}(t), \gamma_{h_{j}}(t')) \right) \leq 2\varepsilon \rho_{u}
\]

and \( \gamma_{h_{j}}(t') \in B(\eta_{u}, 2\varepsilon \rho_{u}) \) for all \( h \geq 1 \). Then there exists \( \xi \in f^{-1}(\eta_{u}) \cap \overline{B}(0, s') \) such that if \( |t - t'| < \frac{\rho_{u}}{2\tilde{c}K} \), then \( \lambda_{h,\nu}(t') \in U(\xi, 2\varepsilon \rho_{u}) \) for \( h \geq 1 \). This means that the considered family \( \{\lambda_{h,\nu}\}_{h=1}^{\infty} \) is equicontinuous. Since the families \( \{\lambda_{h,\nu}\}_{h=1}^{\infty}, \nu = 1, \ldots, m \), are also uniformly bounded, we may apply the Ascoli theorem and find a subsequence \( \{\lambda_{h_{j},\nu}\}_{j=1}^{\infty} \) that converges uniformly to the curve \( \lambda_{\nu} : [0, \sigma_M] \rightarrow \overline{B}(0, s') \). The curve \( \lambda_{\nu} \) is a maximal \( f_{0}\)-lifting of \( \gamma_{0} \) starting in \( \overline{B}(0, s) \).

We continue now, studying the \( f_{0}\)-liftings of the curve \( \gamma_{0}(t) : [\sigma_{M}, \infty] \rightarrow \overline{B}(0, \sigma_M) \) joining \( z_{0} \) with \( \infty \), provided \( \gamma_{0}(\infty) = \infty \). We show that the family \( \{\lambda_{h,\nu}\}_{h=1}^{\infty} \) is equicontinuous for each \( \nu = 1, \ldots, m \). Here \( \lambda_{h,\nu} \) are the \( f_{0}\)-liftings of \( \gamma_{0} \) as described above. Each of the curves \( \lambda_{h,\nu} \) is defined in \( [\sigma_{M}, \infty) \). Fix \( \varepsilon > 0 \). For every \( t \in [\sigma_{M}, \infty) \), there exists \( \delta(t) > 0 \) such that, for all \( \rho \in (0, \delta(t)] \), \( U(\xi, \rho) = f^{-1}(B(f(\xi), \rho)) \) is a normal neighborhood of \( \xi \in f^{-1}(\gamma_{0}(t)) \cap \overline{B}(0, s') \) with \( \text{diam}(U(\xi, \rho)) < \varepsilon \). We find a ball \( B(0, R) \subset G, R > \sigma_{M} \), such that \( f^{-1}(\overline{B}(0, R)) = W_{1} \cup W_{2} \cup \ldots \cup W_{s} \), \( W_{j} \) are disjoint normal domains in \( \overline{B}(0, s') \) with \( \text{diam}(W_{j}) < \varepsilon, j = 1, \ldots, s \). We cover the intersection \( \gamma_{0} \cap \overline{B}(0, R) \) by finitely many balls \( B(\gamma_{0}(t), \delta(t)/2\varepsilon) \), that we still denote by \( B(\eta_{u}, \rho_{u}), u = 1, \ldots, v \). We may suppose that the curves \( \gamma_{h_{j}}, h \geq 1 \), belong to \( \bigcup_{u=1}^{v} B(\eta_{u}, \rho_{u}) \). For every \( t \in [\sigma_{M}, R] \), there exists \( u \) such that \( \gamma_{h_{j}}(t) \in B(\eta_{u}, \rho_{u}) \). We argue as in the previous case and deduce that if \( |t - t'| < \frac{\rho_{u}}{2\tilde{c}K}, t, t' \in [\sigma_{M}, R] \), then \( \lambda_{h,\nu}(t') \in U(\xi, 2\varepsilon \rho_{u}) \) for \( h \geq 1 \). This means that the considered family is equicontinuous in \( [\sigma_{M}, R] \). For \( t > R \), the curves \( \gamma_{h_{j}}(t) \) belong to \( \overline{B}(0, R) \). Then for each \( h > 1 \) one can find \( W_{j} \), \( \text{diam}(W_{j}) < \varepsilon, j = 1, \ldots, s \), such that \( \lambda_{h,\nu}(t) \in W_{j} \). This proves that the family under consideration is equicontinuous for \( t \in [\sigma_{M}, \infty] \).

Applying the Ascoli theorem we conclude that the limit curve \( \lambda_{\nu} \) is a maximal \( f_{0}\)-lifting of \( \gamma_{0} \) starting in \( \overline{B}(0, s) \).

**Case 2.** \( m + 1 \leq \nu \leq g \). Fix \( j \in J \). If \( \lambda_{h,\nu} \) is half opened, it extends to the closed curve in \( \overline{B}(0, s' + 1) \). Let \( \tilde{\lambda}_{h,\nu} : [0, t_{h}] \rightarrow \overline{B}(0, s' + 1) \) be extended curves. We may assume (if it necessary passing to the subsequence) that \( t_{h} \rightarrow t_{0} \in (0, 1] \). Let \( G_{h} : [0, t_{0}] \rightarrow [0, t_{h}] \) be an affine mapping with \( G(0) = 0 \). Arguing as above we deduce that the sequence \( \{\tilde{\lambda}_{h_{j},\nu}\}_{j=1}^{\infty} \) converges uniformly to \( \tilde{\lambda}_{\nu} : [0, t_{0}] \rightarrow \overline{B}(0, s' + 1) \) which is lifting of \( \gamma_{0} \) on \( [0, t_{0}] \). If \( \Delta \subset [0, t_{0}] \) is
the largest interval such that \(0 \in \Delta\) and \(\overline{\lambda_\nu|_\Delta} \subset B(0, s'+1)\), then \(\lambda_\nu = \overline{\lambda_\nu|_\Delta}\) is a maximal \(f_0\)-lifting of \(\gamma_{y_0}^o\) starting in \(\overline{B}(0, s)\).

It remains to consider the \(f_0\)-liftings \(\lambda_{h, \nu}\) of \(\gamma_{y_0}^o\) for \(\nu = m + 1, \ldots, g\). Fix \(\nu\). We may assume that the locus of \(\lambda_{h, \nu}\) is not contained in \(\overline{B}(0, s'+2)\). (If it is contained in \(\overline{B}(0, s'+2)\), then we argue as above for \(\lambda_{h, \nu}\), \(\nu \leq m\), and find a maximal \(f_0\)-lifting of \(\gamma_{y_0}^o\) starting in \(\overline{B}(0, s)\).) By \(t_h\), for every \(h\), we denote the first value of parameter when \(\lambda_{h, \nu}(t)\) intersects \(\partial B(0, s'+1)\). Then \([\sigma_M, t_h] \subset [\sigma_M, L]\) for some \(\sigma_M < L < \infty\) and for all \(h > 1\). We extend each \(\lambda_{h, \nu}(t)\) to the closed curve \(\overline{\lambda_{h, \nu}}\) in \(\overline{B}(0, s'+1)\). Assuming that \(t_h \to t_0 \in (\sigma_M, L]\), we argue as in the previous paragraph and find the limit sequence \(\lambda_\nu = \overline{\lambda_\nu|_\Delta}\) that is a maximal \(f_0\)-lifting of \(\gamma_{y_0}^o\) starting in \(\overline{B}(0, s)\). Here \(\Delta \subset [\sigma_M, t_0]\) is the largest interval such that \(\sigma_M \in \Delta\) and \(\overline{\lambda_\nu|_\Delta} \subset B(0, s'+1)\).

If we show that the constructed limit sequence \(\Lambda_0 = (\lambda_1, \ldots, \lambda_g)\) is essentially separate, then we conclude that \(\Lambda_0 \in \Omega_{y_0}\). Fix a point \(x \in B(0, s')\). Let \(A = \{\nu: \lambda_\nu(t) = x\} \neq \emptyset\) and \(U(x, r)\) be a normal neighborhood of \(x\). Fix \(h_0\) such that \(\lambda_{h, \nu} \cap U(x, r) \neq \emptyset\) for all \(h \geq h_0\) and \(\nu \in A\). We choose \(h \geq h_0\) and, then, find a point \(\eta = \gamma_{y_0}^o(t')\) in \(\bigcap_{\nu \in A} f(\lambda_{h, \nu} \cap U(x, r))\).

Let \(\xi_1, \ldots, \xi_w\) be the points in \(\{\lambda_{h, \nu}(t'): \nu \in A\} \subset f^{-1}(\eta) \cap U(x, r)\). Since the curves \(\lambda_{h_1, 1}, \ldots, \lambda_{h, g}\) are essentially separate, we have

\[
\theta_u = \text{card}\{\nu: \lambda_{h, \nu}(t') = \xi_u\} \leq i(\xi_u, f), \quad u = 1, \ldots, w.
\]

Hence

\[
\text{card}\ A = \sum_{u=1}^w \theta_u \leq \sum_{u=1}^w i(\xi_u, f) \leq i(x, f).
\]

The claim is proved.

In the limit sequence \(\Lambda_0 = (\lambda_1, \ldots, \lambda_g)\) we have \(\lambda_\nu \subset \overline{B}(0, s')\) for \(1 \leq \nu \leq m\) with guarantee. By the limit process, it can happened that \(\lambda_\nu \subset \overline{B}(0, s')\) for some other \(\nu > m\). Thus, we have \(\psi_i(\gamma_{y_0}^o) \geq N(\Lambda_0) \geq m\). The lemma is proved. \(\Box\)

Since \(\text{mes}(\partial U) = 0\) and a quasimeromorphic mapping possesses the Luzin property, we have \(\text{mes}(f(\partial U)) = 0\). Then \(\sigma(S(0, t) \cap f(\partial U)) \neq 0\) for almost all \(t > 0\). If it holds for \(t = \sigma_M\), then \(\psi_i^0(y)\) is a measurable function. If it is not so, then we choose \(t\) sufficiently close to \(\sigma_M\) such that Theorem 5.1 still holds. Thus, from now on we can think, that \(\sigma(S(0, \sigma_M) \cap f(\partial U)) = 0\) and \(\psi_i^0(y)\) is a measurable function.

We are interested in estimating of the quantity of \(f_0\)-liftings starting from different sets \(U_i\). For this we define the functions \(\psi_i^0(y)\) which will calculate the number of \(f_0\)-liftings of \(\gamma_{y}^o\) starting in \(B(0, s) \setminus U_i\). For sequences \(\Lambda \in \Omega(y)\), we let

\[
N(i, \Lambda) = \text{card}\ \left\{\nu: \lambda_\nu \subset \overline{B}(0, s'), \ \lambda_\nu \text{ starts in } B(0, s) \setminus U_i \right\}
\]

and

\[
\psi_i^0(y) = \sup\{N(i, \Lambda): \Lambda \in \Omega(y)\}.
\]

By \(\Omega(i, y)\), we denote the set of sequences \(\Lambda \in \Omega(y)\) for which \(\psi_i^0 = N(i, \Lambda)\).

**Lemma 5.8.** The functions \(\psi_i^0: S \setminus Z \to \mathbb{N}, i = 1, \ldots, p,\) are measurable.
Proof. We prove that \( \psi^j \) is measurable. Fix \( j \in J \cup \{0\} \) and \( i \in I \). Since we know that \( \psi^j \) is measurable, it is enough to show that the restriction of \( \psi^j \) to each set

\[
A_m = \{ y \in S \setminus \mathcal{Z} : \psi^j(y) = m \}, \quad m = 0, \ldots, m_{\max},
\]
is measurable. Here \( m_{\max} = \max\{n(s,z) : z \in S(0,\sigma_M) \setminus \mathcal{Z}\} \). Fix \( m \) and verify that \( \psi^j \) is upper semicontinuous in \( B_m = A_m \setminus f(\partial U) \). Let \( y_0 \in B_m \) and \( \{y_h\} \in B_m \) be a sequence that converges to \( y_0 \). We may assume that for some integer \( m_1 \leq m \) we have \( \psi^j = m_1 \) for all \( h \geq 1 \) and the following properties hold.

(i) There exist normal neighborhoods \( V_1, \ldots, V_l \subset U \) of points \( (x_1, \ldots, x_l) = f^{-1}(z_0) \cap U \) such that \( z_i \in f(V_i) \cap \ldots \cap f(V_l) \) for all \( h \geq 1 \). Here \( z_0 = \gamma_{y_0}(0) \), \( z_h = \gamma_{y_h}(0) \) for \( j \in J \) or \( z_0 = \gamma_{y_0}^h(\sigma_M) \), \( z_h = \gamma_{y_h}^h(\sigma_M) \).

(ii) There is a maximal essentially separate sequence \( \Lambda_0 = (\lambda_{h,1}, \ldots, \lambda_{h,q}) \in \Omega(i,z_h) \), \( g = n(s,z_0) \), such that every \( \lambda_{h,v} \) starts at a point in \( f^{-1}(z_h) \cap V_{\mu} \) for some \( \mu \) dependent on \( v \) and independent of \( h \). Moreover, the locus of every curve \( \lambda_{h,v} \) belongs to \( \overline{B}(0,s') \) for \( 1 \leq v \leq m_1 \).

(iii) The bound \( v(y_h)^{-1} \leq K \) holds for all \( h = 0, 1, \ldots \).

As in the proof of Lemma 5.7 we find a maximal essentially separate sequence \( \Lambda_0 \in \Omega_{y_0} \) such that \( N(\Lambda_0) \geq m \). Since \( y_0 \in A_m \), we have \( N(\Lambda_0) \leq \psi^j(y_0) = m \). Thus \( N(\Lambda_0) = m \) and therefore \( \Lambda_0 \in \Omega_{y_0} \). The construction of \( \Lambda_0 \) implies that \( \psi^j(y_0) \geq N(i,\Lambda_0) \geq m_1 \). It means the upper semicontinuity of \( \psi^j \).

5.9. Estimates for \( \Delta_j \). If the sum \( \sum_{j \in J \cup \{0\}} \Delta_j = \sum_{j \in J \cup \{0\}} 1 - \frac{n(s,a_j)}{\nu(s,\sigma_M)} \) is bounded, then it is nothing to prove. In this section we show that if the sum \( \sum_{j} \Delta_j \) is large we come to the situation when Lemma 4.5 can be applied. We may assume that \( \sum_{j} \Delta_j > 20 M \).

Let \( i \in I = \{1, \ldots, p\} \) and \( j \in J \cup \{0\} = \{0, 1, \ldots, q\} \). For each \( y \in S \setminus \mathcal{Z} \) and \( U_i \in B(0,s) \) we choose a maximal essentially separate \( f_0 \)-lifts \( \Lambda_y = (\lambda_{y,1}, \ldots, \lambda_{y,q}) \), \( g = n(s,z) \), \( \Lambda_y \in \Omega(i,y) \). Here \( z = \gamma_{y}^{j}(0) \) for \( j \in J \) or \( z = \gamma_{y}^{0}(\sigma_M) \). Those curves \( \lambda_{y,v} \) in \( \Lambda_y \) that start in \( U_i \) form a maximal sequence \( (\beta_1, \ldots, \beta_m) \), \( m = n(U_i, z) \), of essentially separate \( f_0 \)-lifts of \( \gamma_{y}^{j} \). For this sequence we consider the sequence \( \alpha_1, \ldots, \alpha_{\mu_{y}} \) of those liftings of \( \beta_{\mu} \) whose locus \( |\beta_{\mu}| \) is not contained in \( \overline{B}(0,s') \). From now on, we denote the quantity of such \( \alpha_{\mu} \) by \( n_{i}^{j}(y) \). The number of curves \( \beta_{\mu} \), starting on \( U_i \) and the locus of which is contained in \( \overline{B}(0,s') \), equals the difference \( \psi^{j}(y) - \psi^{j}(y) \). We have

\[
n(U_i, z) = n_{i}^{j}(y) + (\psi^{j}(y) - \psi^{j}(y)).
\]

Since the functions \( \psi^{j}, \psi^{j}_{i} \) are measurable by Lemmas 5.7, 5.8 and \( n(U_i, z) \) are upper semicontinuous, we get that \( n_{i}^{j}(y) \) are measurable for arbitrary \( i \) and \( j \).

For \( i \in I \), set

\[
J_i = \left\{ j : \frac{1}{\kappa(G, Q)} \int_{S \setminus \mathcal{Z}} n_{i}^{j}(y) \, d\sigma > \frac{1}{2M} \nu(U_i, \sigma_M) \Delta_j \right\},
\]

(5.22)
where $M$ is the constant of multiplicity in the decomposition of $B(0, s)$. Observe, that the overlapping of the decomposition with the multiplicity $M$ imply the following inequality

\[(5.23) \quad \frac{1}{M} \sum_{i=1}^{p} \nu(U_i, \sigma_M) \leq \nu(s, \sigma_M) \leq \sum_{i=1}^{p} \nu(U_i, \sigma_M).\]

We start to estimate the sum $\sum \Delta_j$ with the next lemma.

**Lemma 5.10.** The function $n^j_i(y)$ satisfy

\[
\sum_{i \in I} \sum_{j \in J_i \setminus S \setminus Z} \int n^j_i(y) \, d\sigma \geq \frac{\kappa(G, Q)}{2} \nu(s, \sigma_M) \sum_{j \in J \cup \{0\}} \Delta_j.
\]

**Proof.** Summing the functions $n^j_i$ over $i$, we obtain

\[
\sum_{i \in I} n^j_i(y) \geq \sum_{i \in I} n(U_i, z) - \psi^j(y) \geq n(s, z) - n(s', a_j).
\]

Integrating $n^j_i(y)$ over $S \setminus Z$ we have

\[
\frac{1}{\kappa(G, Q)} \int_{S \setminus Z} \sum_{i \in I} n^j_i(y) \, d\sigma \geq \nu(s, \sigma_M) - n(s', a_j) = \nu(s, \sigma_M) \Delta_j.
\]

From the last estimate, (5.22), and (5.23) we deduce

\[
\frac{\kappa(G, Q)}{2} \nu(s, \sigma_M) \sum_{j \in J \cup \{0\}} \Delta_j \geq \frac{\kappa(G, Q)}{2M} \sum_{i \in I} \sum_{j \in J \cup \{0\}} \nu(U_i, \sigma_M) \Delta_j
\]

\[
\geq \frac{\kappa(G, Q)}{2M} \sum_{i \in I} \sum_{j \in (J \cup \{0\}) \setminus J_i} \nu(U_i, \sigma_M) \Delta_j \geq \sum_{i \in I} \sum_{j \in (J \cup \{0\}) \setminus J_i} \int n^j_i(y) \, d\sigma
\]

\[
= \sum_{i \in I} \sum_{j \in J \cup \{0\} \setminus S \setminus Z} \int n^j_i(y) \, d\sigma - \sum_{i \in I} \sum_{j \in J \setminus S \setminus Z} \int n^j_i(y) \, d\sigma
\]

\[
\geq \kappa(G, Q) \nu(s, \sigma_M) \sum_{j \in J \cup \{0\}} \Delta_j - \sum_{i \in I} \sum_{j \in J \setminus S \setminus Z} \int n^j_i(y) \, d\sigma.
\]

\[\square\]

Now, we try to estimate the average number over $S(0, \sigma_M)$ of $f_0$-liftings of parts of $\gamma^j_M$ such as $\alpha^{(1)}_\nu$ and $\alpha^{(2)}_\nu$. This helps us to refine the upper bound for $\sum_{j \in J \cup \{0\}} \Delta_j$. We set

\[
J^i = \left\{ j \in J_i : \int_{S \setminus Z} n^i_j(y) \, d\sigma \leq 3 \int_{S \setminus Z} \nu^i_j(y) \, d\sigma \quad \text{or} \quad \int_{S \setminus Z} n^i_j(y) \, d\sigma \leq 3 \int_{S \setminus Z} m^i_j(y) \, d\sigma \right\},
\]
Derive the bound on average number over $S(0, \sigma_M)$ for indexes $J^i$ from estimates for curves $\alpha^{(1)}_\nu$ and $\alpha^{(2)}_\nu$. By Lemmas 5.11 and 5.12, we deduce
\[
\sum_{j \in J^i S \setminus Z} \int n_i^j(y) \, d\sigma \leq 3 \sum_{j \in J^i S \setminus Z} \int (l_i^j(y) + m_i^j(y)) \, d\sigma
\]
(5.24)
\[
\leq 3c_5 K_I(f) q \left( \ln \frac{\sigma_M}{\sigma_0} \right)^{Q-1} + 3c_6 K_I(f) A_i \nu(U_i, \sigma_M),
\]
where $A_i$ are constants from (5.16).

We continue to estimate, making use of Lemma 5.10, the inequality (5.24) and upper bound for $p: p \leq 2C_3 \varepsilon_0^{Q-1} \nu(s, \sigma_M)$,
\[
\sum_{i \in I} \sum_{j \in J^i \setminus J^i S \setminus Z} \int n_i^j(y) \, d\sigma = \sum_{i \in I} \sum_{j \in J^i S \setminus Z} \int n_i^j(y) \, d\sigma - \sum_{i \in I} \sum_{j \in J^i S \setminus Z} \int n_i^j(y) \, d\sigma
\geq \frac{\kappa(G, Q)}{2} \nu(s, \sigma_M) \sum_{j \in J^i \setminus \{0\}} \Delta_j
\]
(5.25)
\[
- 6C_3c_5 K_I(f) q \left( \ln \frac{\sigma_M}{\sigma_0} \right)^{Q-1} \varepsilon_0^{Q-1} \nu(s, \sigma_M)
- 3c_6 K_I(f) \sum_{i \in I} A_i \nu(U_i, \sigma_M).
\]

We need to choose the constants $A_i$ to obtain an effective lower bound for the sum $\sum_{i \in I} \sum_{j \in J^i \setminus J^i S \setminus Z} \int n_i^j(y) \, d\sigma$. We introduce the value $\lambda_i = \text{card}(J_i \setminus J^i)$. The set $J_i \setminus J^i$ contains the indexes $j \in J$ for which the following inequalities hold:
\[
\int_{S \setminus Z} n_i^j(y) \, d\sigma > \frac{\kappa(G, Q)}{2M} \nu(U_i, \sigma_M) \Delta_j,
\]
(5.26)
\[
\int_{S \setminus Z} n_i^j(y) \, d\sigma > 3 \int_{S \setminus Z} l_i^j(y) \, d\sigma,
\]
\[
\int_{S \setminus Z} n_i^j(y) \, d\sigma > 3 \int_{S \setminus Z} m_i^j(y) \, d\sigma.
\]

We recall the definition of $A_i$: $A_i \nu(U_i, \sigma_M) = \left( \ln \frac{\sigma_i}{\sigma_0} \right)^{Q-1}$, where $\sigma_i \in (0, \sigma_0]$. When $A_i$ increases in the range $[0, \infty)$ the number $\lambda_i$ decreases from $\lambda_i^0$ to some value $\lambda_i^\infty$. From (5.26) we have
\[
\lambda_i^0 = \text{card}\left\{ j \in J_i : \int_{S \setminus Z} n_i^j(y) \, d\sigma > 3 \int_{S \setminus Z} l_i^j(y) \, d\sigma \right\}.
\]
We may assume that the jumps at the discontinuities of the function $A_i \mapsto \lambda_i$ equal 1. If it is not so, we can make a small variation of function $m_i^j$ changing $\sigma_i$ for different $j$'s. Finally, we conclude that we can choose $A_i \geq 0$ with
\[
\lambda_i - 1 \leq 9\kappa^{-1}(G, Q) c_6 K_I(f) A_i \leq \lambda_i.
\]
(5.27)
We use the value of the constants $A_i$ to terminate the estimation of (5.25). From (5.25) and (5.27) we obtain
\[
\sum_{i \in I} \lambda_i \nu(U_i, \sigma_M) = \sum_{i \in I} \sum_{j \in J \setminus J_i} \nu(U_i, \sigma_M) \geq \sum_{i \in I} \sum_{j \in J \setminus J_i} \frac{1}{\kappa(G, Q)} \int_{S \setminus Z} n_i^j(y) \, d\sigma
\]
\[
\geq \frac{\nu(s, \sigma_M)}{2} \sum_{j \in J \setminus \{0\}} \Delta_j
\]
\[
- \frac{6C_3c_5K_I(f)q_{e_0}^Q}{\kappa(G, Q)} \left( \ln \frac{\sigma_M}{\sigma_0} \right)^{Q-1} \nu(s, \sigma_M)
\]
\[
- \frac{1}{3} \sum_{i \in I} \lambda_i \nu(U_i, \sigma_M).
\]
(5.28)

We are free in the choice of $\varepsilon_0$. Take $\varepsilon_0$ such that
\[
(5.29) \quad 6C_3c_5K_I(f)q_{e_0}^Q \left( \ln \frac{\sigma_M}{\sigma_0} \right)^{Q-1} < M.
\]

Then (5.28) implies
\[
(5.30) \quad \sum_{i \in I} \lambda_i \nu(U_i, \sigma_M) \geq \frac{\nu(s, \sigma_M)}{4} \sum_{j \in J \cup \{0\}} \Delta_j,
\]

Now we exclude some set of indexes from $I$ where $\sum \lambda_i \nu(U_i, \sigma_M)$ can be bound from above by $\sum_{j \in J \cup \{0\}} \Delta_j$. Namely, let
\[
I_1 = \left\{ i \in I; \lambda_i \leq \frac{\sum_{j \in J \cup \{0\}} \Delta_j}{10M} \quad \text{or} \quad \nu(U_i, \sigma_M) \leq P \right\}.
\]
Here $P = \frac{c_1K_I(f)}{(\ln 2)^{Q-1}} \left( |\ln \frac{\sigma_m^2}{\sigma_0}|^{Q-1} + |\ln \sigma_M^{Q-1} + 2c_0 | \right)$ with $\sigma_m$ defined in Lemma 4.5. We need the first choice for using Lemmas 4.5 and 4.7. The second one serves for applying of Lemma 4.11.

**Lemma 5.11.** Under the previous notations we have
\[
(5.31) \quad \sum_{i \in I \setminus I_1} \lambda_i \nu(U_i, \sigma_M) \geq \frac{\nu(s, \sigma_M)}{8} \sum_{j \in J \cup \{0\}} \Delta_j.
\]

**Proof.** Summing over $I_1$ we get
\[
\sum_{i \in I_1} \lambda_i \nu(U_i, \sigma_M) \leq \frac{1}{10} \nu(s, \sigma_M) \sum_{j \in J \cup \{0\}} \Delta_j + Pq_p
\]
\[
\leq \frac{1}{10} \nu(s, \sigma_M) \sum_{j \in J \cup \{0\}} \Delta_j + Pq2C_3q_{e_0}^Q \nu(s, \sigma_M)
\]
from (5.28). Let us add one more restriction on $\varepsilon_0$. We choose $\varepsilon_0$ such that
\[
(5.32) \quad 2PqC_3q_{e_0}^Q < M/2.
\]

Then
\[
(5.33) \quad \sum_{i \in I_1} \lambda_i \nu(U_i, \sigma_M) \leq \frac{\nu(s, \sigma_M)}{8} \sum_{j \in J \cup \{0\}} \Delta_j.
\]
Joining the estimates (5.30) and (5.33) we deduce (5.31). □

**Lemma 5.12.** If \( \sum_{j \in J \setminus \{0\}} \Delta_j > 20M \), then for \( i \in I \setminus I_1 \), the following inequality

\[
\lambda_i \nu(U_i, \sigma_M) \leq c_7 K_0(f) K_1(f) \nu(Z_i, \sigma_M).
\]

holds.

**Proof.** Let \( i \in I \setminus I_1 \). The inequalities (5.26) imply for \( j \in J \setminus J' \) that

\[
\int_{S \setminus Z} (n_i^j - l_i^j - m_i^j)(y) \, d\sigma \geq \frac{1}{3} \int_{S \setminus Z} n_i^j(y) \, d\sigma > \frac{\kappa(G,Q)}{6M} \nu(U_i, \sigma_M) \Delta_j > 0.
\]

We conclude that the function \( n_i^j(y) - l_i^j(y) - m_i^j(y) \) is positive for some \( y \in S \setminus Z \). Therefore

\[
0 < n_i^j(y) - l_i^j(y) - m_i^j(y) < n_i^j(y) - \text{card}(M_i^j(y) \cup L_i^j(y))
\]

by Lemmas 5.3 and 5.5. It means that for these indexes \( i \) and \( j \in (J \setminus J') \setminus \{0\} \) there are indexes \( \nu \in \{1, \ldots, n_i^j(y)\} \) such that we have \( \frac{\sigma_M - w_{y,\nu}}{\sigma_M - w_{y,\nu}} > \frac{s_0}{2} \) and \( \frac{\sigma_M - w_{y,\nu}}{\sigma_M - w_{y,\nu}} > \frac{s_0}{2} \). We also can say that the corresponding curves has left \( W_i \) and reached \( \partial B(0,s') \). Here the values \( w_{y,\nu}, w_{y,\nu}, w_{y,\nu} \) were defined in (5.11). The two last inequalities imply that \( \sigma_M - w_{y,\nu} \leq \sigma_i \) and for the corresponding \( f_0 \)-lifting \( \alpha_\nu \) of \( \gamma_0 \) the following properties hold

(i) the restriction \( \alpha_i^j = \nu|_{[w_{y,\nu},t]} \) is a curve in \( X_i \) connecting \( \partial W_i \) and \( \partial X_i \) for some \( t \leq \sigma_M \);

(ii) \( f(\alpha_i^j) \subset B(a_j, 3\sigma_i/2) \).

If \( j \in (J \setminus J') \setminus \{0\} \), then we apply Lemma 4.15 to the mapping \( f \) and the set \( F_j = \alpha_i^j \). We consider the balls \( W_i \subset X_i \subset Y_i \) with radii \( r = 4r_i \), \( \rho = 6r_i \), and \( \theta \rho \), where \( \theta \) was defined in the decomposition of \( B(0,s) \). We obtain

\[
(M_Q(G_j) - \frac{\kappa(G,Q) K_0(f) K_1(f)}{(\ln \theta)^{Q-1}}) \left( \ln \frac{2\sigma_m}{3\sigma_i} \right)^{Q-1} \leq \kappa(G,Q) K_0(f) \nu(6\theta r_i, S(a_j, \sigma_m)).
\]

Note, that we use the relations \( \sigma_i \leq s_0 < \frac{2}{3} \sigma_m \). Thanks to our choice of \( r \) and \( \rho \) we have

\[
M_Q(G_j) \geq c(Q) \ln \frac{2}{3} = c(Q) \ln \frac{2}{3}.
\]

By Remark 5.2 we get

\[
c(Q) \ln \frac{5}{4} \left( \ln \frac{2\sigma_m}{3\sigma_i} \right)^{Q-1} \leq \left( \ln \frac{3}{2} - \frac{\kappa(G,Q) K_0(f) K_1(f)}{c(Q) \ln \theta} \right) \left( \ln \frac{2\sigma_m}{3\sigma_i} \right)^{Q-1}.
\]

Making use of Lemma 4.11 with \( Z = S(0, \sigma_M), Y = S(a_j, \sigma_m), r = 6\theta r_i \) and \( \theta = 2 \) we deduce

\[
\nu(6\theta r_i, S(a_j, \sigma_m)) \leq \nu(12\theta r_i, S(0, \sigma_M))
\]

\[
+ \frac{c_1 K_1(f)}{(\ln 2)^{Q-1}} \left( |\ln \sigma_m|^{Q-1} + |\ln \sigma_M|^{Q-1} + 2c_0 \right)
\]

\[
\leq \nu(Z_i, \sigma_M) + P = 2\nu(Z_i, \sigma_M).
\]

The last inequality was possible because of the choice of \( P < \nu(U_i, \sigma_M) \leq \nu(Z_i, \sigma_M) \) for \( i \in I \setminus I_1 \) and \( \theta = \kappa/3 \). We conclude

\[
\left( \ln \frac{2\sigma_m}{3\sigma_i} \right)^{Q-1} \leq \frac{2\kappa(G,Q)}{c(Q) \ln 5/4} K_0(f) \nu(Z_i, \sigma_M)
\]
from (5.35), (5.36), and (5.37). Now, we use the definition (5.27) of $A_i$. We can assume also that $\lambda_i \geq 2$. We have
\begin{equation}
(5.39) \quad \frac{\kappa(G, Q)\lambda_i \nu(U_i, \sigma_M)}{18c_9 K_i(f)} \leq A_i \nu(U_i, \sigma_M) = \left( \ln \frac{s_0}{\sigma_i} \right)^{Q-1} < \left( \ln \frac{2\sigma_m}{3\sigma_i} \right)^{Q-1}.
\end{equation}

Finally, (5.38) and (5.39) imply (5.34).

It remains to consider the case $j = 0$. We should slightly change the arguments. If $0 < n_1^0(y) - \text{card}(M_1^0(y) \cup L_1^0(y))$ then there are indexes $\nu \in \{1, \ldots, n_1^0(y)\}$ such that $\frac{w_{y,\nu}}{v_{y,\nu}} > \frac{2m}{s_0}$ and $\frac{w_{y,\nu}}{v_{y,\nu}} > \frac{s_0}{\sigma_i}$, where $\sigma_M \leq u_{y,\nu} < v_{y,\nu} < w_{y,\nu} < \infty$. We deduce
\[ w_{y,\nu} > v_{y,\nu} \frac{s_0}{\sigma_i} > u_{y,\nu} \frac{\sigma_m}{\sigma_i} > \frac{2\sigma_m^2}{3\sigma_i} > \frac{\sigma_M^2}{\sigma_m} > \sigma_M. \]

For the corresponding $f_0$-lifting $\alpha_{\nu}$ the following properties hold for some $t > w_{y,\nu}$

\begin{enumerate}[\normalfont (i)]
    \item the restriction $\alpha^0_{\nu} = \alpha_{\nu}|_{[w_{y,\nu}, t]}$ is a curve in $X_i$ connecting $\partial W_i$ and $\partial X_i$;
    \item $f(\alpha^0_{\nu}) \subset \mathbb{C}B(0, \frac{2\sigma^2_{\nu}}{3\sigma_i})$.
\end{enumerate}

Instead of Lemma 4.13 we apply Lemma 4.14 to the sets $F_0 = \alpha^0_i$, $F_j = \alpha^i_j$ in the balls $W_i \subset X_i \subset Y_i$, $t = \frac{2\sigma^2_{\nu}}{3\sigma_i}$, $s = \frac{\sigma^2_{\nu}}{\sigma_m}$. We get
\begin{equation}
(5.40) \quad \left( M_Q(\Gamma_0) - \frac{\kappa(G, Q)K_O(f)K_I(f)}{(\ln \theta)^{Q-1}} \right) \left( \ln \frac{2\sigma_m}{3\sigma_i} \right)^{Q-1} \leq \kappa(G, Q)K_O(f)\nu\left( 6\theta r_i, S\left( 0, \frac{\sigma_M^2}{\sigma_m} \right) \right).
\end{equation}

Then we estimate $M_Q(\Gamma_0)$ from below $M_Q(\Gamma_0) \geq c(Q)\ln 3/2$ and deduce (5.36). Applying Lemma 4.11 with $Z = S(0, \sigma_M)$, $Y = S(0, \frac{\sigma^2_{\nu}}{\sigma_m})$, $r = 6\theta r_i$ and $\theta = 2$ we have
\begin{equation}
\nu(6\theta r_i, S(0, \frac{\sigma^2_{\nu}}{\sigma_m})) \leq \nu(12\theta r_i, S(0, \sigma_M))
\end{equation}
\begin{equation}
\begin{align*}
&+ \frac{c_1 K_I(f)}{(\ln 2)^{Q-1}} \left( |\ln \sigma_m|^{Q-1} + |\ln \frac{\sigma^2_{\nu}}{\sigma_m}|^{Q-1} + 2c_0 \right) \\
&\leq 2 \nu(Z, \sigma_M).
\end{align*}
\end{equation}

Combining (5.40), (5.36), and (5.41) we obtain (5.38) and continue as in the previous case.$\square$

6. PROOF OF THEOREM 0.1

We recall that at the end of Subsection 5.1 we observed that for the proof of the main theorem it is sufficient to show the finiteness of the sum $\sum_{j \in J \cup \{0\}} \Delta_j$. If we assume that
\[ \sum_{j \in J \cup \{0\}} \Delta_j > 20 M, \]
then we can apply Lemma 5.12. We get
\begin{align*}
\frac{\nu(s, \sigma_M)}{8} \sum_{j \in J \cup \{0\}} \Delta_j &\leq \sum_{i \in I \setminus I_i} \lambda_i \nu(U_i, \sigma_M) \leq c_7 K_O(f) K_I(f) \sum_{i \in I} \nu(Z_i, \sigma_M) \\
&= c_7 M K_O(f) K_I(f) \nu(s', \sigma_M) < \frac{3}{2} c_8 K_O(f) K_I(f) \nu(s, \sigma_M)
\end{align*}
by Lemma 5.11, the inequality (5.6), the estimate (5.9), and the decomposition of $B(0, s')$.
The final conclusion is that
\[ \sum_{j \in J \cup \{0\}} \Delta_j \leq C(Q, K_0(f), K_I(f)). \]
We proved Theorem 0.1.

7. Proof of Theorem 0.3

Let \( \mathbb{G} \) be a \( \mathbb{H} \)-type Carnot group and let \( B(0, 1) \) be the unit ball in the group \( \mathbb{G} \). We recall the statement of Theorem 0.3: Let \( f : B(0, 1) \to \mathbb{G} \) be a nonconstant \( K \)-quasimeromorphic mapping such that
\[ \limsup_{r \to 1} (1 - r) A(r) = \infty. \]
Then there exists a set \( E \subset (0, 1) \) satisfying
\[ \liminf_{r \to 1} \frac{\text{mes}_1(E \cap [r, 1))}{1 - r} = 0. \]
and a constant \( C(Q, K) < \infty \) such that
\[ \lim_{r \to 1} \sup_{r \notin E} \left( \sum_{j=0}^{q} \left( 1 - \frac{\nu(r, a_j)}{\nu(r, 1)} \right) \right) \leq C(Q, K), \]
whenever \( a_0, a_1, \ldots, a_q \) are distinct points in \( \mathbb{G} \).

To prove Theorem 0.3 we need an analogue of Theorem 5.1 and a construction of the exceptional set \( E \). In spite of the different definition of the function \( A(r) \), the proof of the next lemma repeats the proof from [58] almost verbatim, because we have used the continuity of \( A(r) \) only. We present the proof for the completeness.

**Lemma 7.1.** Let \( B(0, 1) \in \mathbb{G} \) and \( f : B(0, 1) \to \mathbb{G} \) be a quasimeromorphic mapping with the property that
\[ (7.1) \quad \limsup_{r \to 1} (1 - r) A(r) = \infty. \]
Then there exists a set \( E \subset [0, 1) \) such that
\[ (7.2) \quad \liminf_{s \to 1} \frac{\text{mes}_1(E \cap [s, 1))}{1 - s} = 0 \]
and such that the following is true: If \( \epsilon_0 \in (0, 1/5) \) and if for \( s \in (0, 1) \) we write
\[ s' = s + \frac{s}{\epsilon_0 A(s)^{1/(Q-1)}}, \]
then there exists an increasing function \( \omega : [0, \infty) \to [D_{\epsilon_0}, 1) \) such that for any sphere \( Y = S(w, t) \) in \( \mathbb{G} \) and any \( s' \in [\omega(|\log t|), 1) \setminus E \) there is an \( s \in (0, 1) \), for which inequality (5.1):
\[ \left| \frac{\nu(s, Y)}{A(s')} - 1 \right| < \epsilon_0 \]
and inequality (5.2):
\[ \frac{\nu(s, Y)}{\nu(s', Y)} > 1 - \epsilon_0. \]
hold. Moreover \( A(D_{\epsilon_0}) > \epsilon^{2-2Q} \).
Proof. We inductively define an increasing sequence \( t_3, t_4, t_5, \ldots \) in the interval \((0, 1)\) tending to 1 such that \( 1 - t_m < \frac{1 - t_{m-1}}{m} \). At the same time we define a sequence \( E_3, E_4, \ldots \) of subsets of \((0, 1)\). Set \( t_3 = 3/4 \) and \( E_3 = [3/4, 1) \). Suppose that \( m \geq 4 \) and \( t_{m-1} \) is defined. Set 
\[
\phi(r) = \frac{A(r)^{1/(Q-1)}}{m^2}.
\]
There exists \( t_m \) such that
\[
1 - t_m < \frac{1 - t_{m-1}}{m} \quad \text{and} \quad (1 - t_m)A(t_m)^{1/(Q-1)} > \frac{8m^3}{1 - (m-1)/m^{1/(Q-1)}}
\]
by (7.3). Set \( t_m^* = t_m + \frac{1 - t_m}{m} \), \( \overline{t}_m = 1 - \frac{1 - t_m}{m} \), and
\[
F_m = \{ r \in (t_m, 1) : A\left( r + \frac{2r}{\phi(r)} \right) > \frac{m}{m-1}A(r) \quad \text{or} \quad r + \frac{2r}{\phi(r)} > 1 \}.
\]
Let us assume that \( F_m \cap (t_m, t_m^*] \neq \emptyset \) and put \( t_m = r''_m \). Now we define inductively the sequence \( r''_0 \leq r_1 < r''_1 \leq r_2 < r''_2 \ldots \leq r_h < r''_h \) of numbers in \((t_m, 1)\) by
\[
r_k = \inf \{ r \in (r''_{k-1}, 1) : r \in F_m \}, \quad r''_k = r_k + \frac{2r_k}{\phi(r_k)}.
\]
Here the number \( h \) is the last index \( k \) for which \( F_m \cap (r''_{k-1}, 1) \neq \emptyset \) and \( r_k \leq t_m^* \). If we denote \( \rho_k = r_k'' + \frac{2r_k}{\phi(r_k)} \), then we obtain \( \rho_k < 1 \) for \( k = 1, \ldots, h \). In this case, we put \( E_m^1 = \bigcup_{k=1}^{h} (r_k, \rho_k) \).

If \( F_m \cap (t_m, t_m^*] = \emptyset \), then we set \( E_m^1 = \emptyset \). To estimate the 1-measure of \( E_m^1 \) we use (7.3) and the definition of \( F_m \). Thus
\[
\text{mes}_1(E_m^1) < \sum_{k=1}^{h} (\rho_k - r_k) < \sum_{k=1}^{h} \frac{4}{\phi(r_k)} = \sum_{k=1}^{h} \frac{4m^2}{A(r_k)^{1/(Q-1)}} \leq \frac{4m^2}{A(t_m)^{1/(Q-1)}} \left( 1 - \frac{m-1}{m^{1/(Q-1)}} \right) \leq \frac{1}{2m}(1 - t_m).
\]
Set \( E_m = [t_m, t_m^*] \cup [\overline{t}_m, 1) \cup E_m^1 \). Then \( \text{mes}_1(E_m) < \frac{3(1 - t_m)}{m} \). Making use of definitions of sequences \( \{t_m\} \) and \( \{E_m\} \), we set
\[
E = \bigcup_{m \geq 3} (E_m \cap [t_m, t_{m+1}]) .
\]
Then, clearly
\[
\lim_{m \to \infty} \frac{\text{mes}_1(E \cap [t_m, 1])}{1 - t_m} = 0 .
\]

Let \( \varepsilon \in (0, 1/5) \), let \( S(w, t) \) be a sphere in \( G \), and let \( m_1 \) be the integer \( m \) satisfying the conditions (i) – (iii) in the proof of Theorem 5.1. Suppose \( s' \in [t_{m_1}, 1) \setminus E \). Then \( s' \) belongs to some interval \([t_m, t_{m+1}]\) with \( m > m_1 \). By the definition of \( E \), we have \( s' \in (t_m, \overline{t}_m) \setminus E_m^1 \).

We claim that there exists \( r \in (t_m, \overline{t}_m) \) such that
\[
s' = s + \frac{s}{\varepsilon_0(A(s))^{1/(Q-1)}} \quad \text{with} \quad s = r + \frac{r}{\phi(r)} .
\]
and \( r \notin F_m \). In fact, if we suppose that \( r \leq t_m \) from (7.3) we get
\[
s' = r + \frac{r}{\phi(r)} + \frac{r + t_m}{\varepsilon_0 A(r + \frac{r}{\phi(r)})^{1/(Q-1)}} \leq t_m + \frac{t_m + \frac{t_m}{\phi(t_m)}}{\varepsilon_0 A(t_m)^{1/(Q-1)}}
\]
\[
\leq t_m + \frac{1 - t_m}{8m} + \frac{1 + (1 - t_m)/8m}{16m}(1 - t_m) \leq t_m + \frac{1 - t_m}{4m} < t_m^*.
\]
It follows that \( r > t_m \). If we assume that \( r > \overline{t}_m \), then \( \overline{t}_m < r < s < s' \) and we get a contradiction with \( s' \in (t_m^*, \overline{t}_m) \). To show that \( r \notin F_m \) we argue as in (5.3). Since, in addition, \( F_m \subset E^1_m \subset E_m \) we obtain inequalities (5.1) and (5.2) as in the proof of Theorem 5.1. If \( m_0 \) is the least positive integer with \( \frac{2}{m_0} < \varepsilon \), we can put \( D(\varepsilon_0) = t_{m_0} \).

From (7.3) it follows that
\[
A(D(\varepsilon_0)) > \left( \frac{8m^3_0}{1 - D(\varepsilon_0)} \right)^{Q-1} > (8m^4_0)^{Q-1} > \varepsilon_0^{2-2Q}.
\]
The lemma is proved.

We are at the point to show Theorem 0.3.

Proof of Theorem 0.3. Let \( E \subset (0,1) \) be the set constructed in Lemma 7.1. We fix \( \varepsilon_0 \in \left(0, \min\{1, \frac{1}{5}, \frac{1}{8q+9}\}\right) \) such that, in addition, \( \varepsilon_0 \) satisfies (5.29) and (5.32). By Lemma 7.1 we find \( \kappa \in (0, 1) \) such that, for every \( s' \in (\kappa, 1) \setminus E \), there exists \( s \) with \( s' = s + \frac{s}{\phi(s)} \) and such that the estimates (5.6), (5.7) hold. Fix such an \( s \). Then we denote by \( f_0 \) the restriction \( f|_{B(0,s'+(1-s')/2)} \). Now, for proving of Theorem 0.3 we repeat the arguments used in the proof of Theorem 0.1.

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