On the condition for the central caustic degeneracy of the planetary microlensing

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ABSTRACT

It is shown that the linear approximation of the central caustic for the planetary \( (q \ll 1) \) microlensing is valid if \(|1 - s| \gg q^{1/3}\) (where \( q \) is the mass ratio and \( s \) is the projected separation in the unit of the Einstein ring radius of the primary). The condition is also consistent with the requirement that the binary separation is far from those in the resonant binary regime resulting in a single six-cusp caustic. Given that the linear approximation of the caustic is invariant under \( s \leftrightarrow s^{-1} \), the close/wide binary degeneracy observed under the same condition may be understood via the linear approximation of the central caustic. Finally it is argued that the local degeneracies of lensing features associated with caustic crossings can still persist in the planetary events even when \(|1 - s| \sim q^{1/3} \) although the overall caustic shape may not be degenerate at all.

Key words: gravitational lensing: micro–planetary systems

1 INTRODUCTION

Recently, Yee et al. (2021) have reported a planetary \( (q \sim 10^{-5}) \) microlensing event with two degenerate solutions resembling the classical \( s \leftrightarrow s^{-1} \) degeneracy (Griest & Safizadeh 1998; Dominik 1999) but \( s \sim 1 \). As the close/wide binary degeneracy is originally derived from a certain symmetry present in the lens equation when the projected separation is far from the Einstein ring radius, they have questioned whether the observed degeneracy can be considered as a particular manifestation of the well-known degeneracy. In fact, if the mass ratio is sufficiently small so that the system can be approximated as a point mass lens under a planetary perturbation, the central asteroid caustic is also invariant under the same \( s \leftrightarrow s^{-1} \) transformation of the separation up to the linear order of the perturbation (Bozza 1999; An 2005). Whilst this approximation is known to fail as \( s \to 1 \), the condition under which the approximation is valid is only vaguely mentioned in the literature and so it is difficult to judge if the observed degeneracy can be properly understood as an example of this classical approximation approach. This short paper attempts to rectify the situation and address more precise condition for the central caustic degeneracy of the planetary microlensing.

2 THE CAUSTICS OF THE BINARY LENS SYSTEM

Let us consider the gravitational lensing system described by the complexified lens equation (Witt 1990) given by

\[
\zeta = \frac{z - \alpha(z)}{z} = \frac{m_1}{z} + \frac{m_2}{z - \ell},
\]

where \( \zeta \) and \( z \) are the complexified source and image positions and the overbar notation represents the complex conjugation. The system corresponds to the binary lens system with the primary of the mass \( m_1 \) at the origin and the secondary of the mass \( m_2 \) at the location \( \ell \). The Jacobian of the lens mapping is given by

\[
J = 1 - \left| \frac{\partial \zeta}{\partial z} \frac{\partial \bar{\zeta}}{\partial \bar{z}} \right| = 1 - \left| \frac{dz}{d\zeta} \right|^2, \tag{2}
\]

and the critical curve is defined to be the lens plane locus of \( J = 0 \), whereas the caustic is the source plane image of the critical curve under the mapping in equation (1).

The critical curve \( \zeta_{cc}(\phi) \) may also be seen as the parametric curve satisfying \( f(\zeta_{cc}) = e^{-\beta \phi} \) (Witt 1990) where

\[
f(z) = \frac{dz}{d\zeta} = \frac{m_1}{z^2} + \frac{m_2}{(z - \ell)^2}, \tag{3}
\]

and the caustic is similarly parametrized through \( \zeta(\phi) = \zeta_{cc} - \alpha(\zeta_{cc}) \). Unless \( f'(\zeta_{cc}) = 0 \), the parametric critical curve \( \zeta_{cc}(\phi) \) is differentiable such that \( f'(\zeta)\zeta''_{cc}(\phi) = -2ie^{-2\beta \phi} \), and it follows that

\[
\zeta''_{cc}(\phi) = \zeta''_{cc}(\phi) + f'(\zeta_{cc})\zeta'_{cc}(\phi) = 2\bar{f}'e^{2\beta \phi} \tag{4}
\]

Hence if \( \bar{f}' = f'e^{2\beta \phi} \) or equivalently \( f'^2 e^{2\beta \phi} = f'^2 f^2 f^3 = |f'|^2 \in \mathbb{R}^+ \) is positive real at a point on the critical curve, then \( \zeta''_{cc}(\phi) = 0 \) at the corresponding point on the caustic (Daněk & Heyrovský 2015); that is, the parametrized caustic is locally stationary and so it develops a cusp there. If \( f'(\zeta_{cc}) = 0 \) on the other hand, it can be shown that \( f(\zeta_{cc}) = e^{-2\beta \phi} \) possesses a degenerate solution. This implies that the critical curve (and the caustic as well) bifurcates at the corresponding point and globally it becomes self-intersecting.

For the binary lens system, Schneider & Weiss (1986) have shown that there exist three possible topologies for the caustics depending on the projected separation between two lens components: that is, one asteroid and two deltoids for “close” binaries, two asteroids for “wide” binaries, and a single simply-connected curve with six cusps for intermediate (“resonant”) cases. At the transition between
these intersegregates and so the exact values of the separations dividing these three regimes correspond to those permitting simultaneous solutions for \( f(z) = e^{-2b} \) and \( f'(z) = 0 \). For \( f(z) \) in equation (3), the equation \( f'(z) = 0 \) reduces to \((1 - ℓ/z)^3 = m_2/m_1 \) and so its solution is \( z_0 = f(1 + \omega q^{1/3} |\ell|) \), where \( \omega = \{\frac{1}{3} e^{2/3} (1 - \sqrt[3]{2}) \} \) is a cube root of unity and \( q = m_2/m_1 \). In order for \( z_0 \) to be a solution of \( f(z) = e^{-2b} \), we then must have

\[
\ell e^{-2a} = m_1(1 + \omega q^{1/3} |\ell|) = (m_1^{1/3} + \omega m_2^{1/3})^3.
\]

Hence the separations dividing these three regimes are

\[
|\ell_0| = (m_1^{1/3} + m_2^{1/3})^2,
\]

and

\[
|\ell_c| = \left| \frac{2m_1^{1/3} - m_2^{1/3} \pm \sqrt{3}m_2^{1/3}}{2} \right| = (m_2^{1/3} - m_1^{1/3} + m_2^{1/3})^2.
\]

Here \( |\ell_0| \geq (m_1 + m_2)^{1/3} \geq |\ell_c| \) (equal only if \( m_1 = m_2 = 0 \) and so \( |\ell_c| \) is the minimum separation for the wide binaries (two caustics) whilst \( |\ell_0| \) is the maximum separation for the close binaries (three caustics). We note that the value for \( \ell_0 \) is first derived by Erdl & Schneider (1993, eq. 18) and also reproduced in Dominik (1999). However, both authors only provide with an implicit equation for \( \ell_0 \). In fact we find \((m_1 + m_2)^2 - |\ell|^2 = 3(m_1m_2)^{1/3}|\ell|^{8/3} > 0 \) for \(|\ell_c| \) in equation (7), which is equivalent to Erdl & Schneider (1993, eq. 17) if \( m_1 + m_2 = 1 \) and to Dominik (1999, eq. 57) if \( m_1^{1/2} d_0 = |\ell_c| \) and \( q = m_1/m_2 \).

## 3 THE LINEAR APPROXIMATION FOR THE CENTRAL CAUSTIC IN THE PLANETARY LENSING

For a point lensing, the critical curve is basically identical to the Einstein ring: that is, \( z = m_1^{1/2} e^\phi \) is the solution to equation (3) if \( m_2 = 0 \). If we reparameterize the critical curve as a deviation from the Einstein ring, namely \( z_a = m_1^{1/2} (1 + e^\phi) \), equation (3) is then reducible to the equation for the fractional deviation \( \epsilon \) (which is complex):

\[
\frac{1}{(1 + e^\phi)} + \frac{q}{(1 - s - e^{-2\phi} + e^\phi)} = 1,
\]

where \( q = m_1/m_2 \) and \( s = m_1^{1/2}/(1 + e^\phi) \). Under the assumptions that \(|\epsilon| \ll 1 \) and \(|\epsilon| \ll |1 - s|\), expanding the left-hand side of equation (8) in a power series of \( \epsilon \) and equating up to its linear term results in

\[
\epsilon \approx \frac{q}{2(1 - s - e^{-2\phi} + e^\phi)} \left[ 1 - \frac{q}{1 - s - e^{-2\phi}} \right]^{-1}.
\]

It follows that, if \( q \ll 1 \), the critical curve in the linear approximation of \( q \) is given by

\[
z_a \approx m_1^{1/2} e^\phi \left[ 1 + \frac{q}{2(1 - s - e^{-2\phi} + e^\phi)} + O(q^2) \right].
\]

Here \(|\epsilon| \ll |1 - s - e^{-4\phi}|^{-2} \) and so the original assumptions (i.e. \(|\epsilon| \ll 1 \) and \(|\epsilon| \ll |1 - s| \)) also hold if \( q \ll |1 - |s|| \). Note that equations (10) with \( \phi \neq \phi + \pi \) are both approximate solutions to the same equation \( f(z) = e^{-2a} = e^{-2b(\ln m_1)} \), and so the perturbative solution in equation (10) accounts for two of four solutions of \( f(z) = e^{-2b} \).

The preceding derivation is somewhat heuristic but it makes the conditions for its validity (viz. \( q \ll 1 \)) rather more clear. Alternatively if the (uniformly) convergent power series for \( \epsilon = \sum_{i=1}^{\infty} \epsilon_i i q^i \)

\[
(1 - |s|)^2 \leq |1 - s - e^{-4\phi}|^{-2} \leq (1 + |s|)^2
\]

exist, the coefficients \( \epsilon_i \) can be determined by inserting the expression back to equation (8) or the equivalent (quartic on \( e \)) polynomial equation, \((1 - s - e^{-4\phi} + e^\phi)q(1 + e^\phi) = (1 + e^\phi)^2(1 - s - e^{-4\phi} + e^\phi) \) and assembling terms with the same power on \( q \) which results in

\[
\epsilon = \frac{q}{2(1 - s - e^{-4\phi} + e^\phi)} \left( \frac{1 + 3 s e^{-4\phi}}{8(1 - s - e^{-4\phi})^2} q^2 + \frac{1}{16(1 - s - e^{-4\phi})} q^3 + \cdots \right),
\]

the linear term of which is indeed consistent with equation (10). Without knowing the general expression for the arbitrary coefficient \( \epsilon_a \), it is difficult to assess if the power series in equation (11) actually converges. Nevertheless equation (11) suggests that the ratio between the subsequent terms behaves like \( \epsilon_a / (\epsilon_a + 1) \), and so one expects the convergence criterion to exist in a form of \( q[1 - |s|] < C \) with some constant \( C \).

As for the caustics, with \( z_c = m_1^{1/2} (1 + e^\phi) \) and \( z_\ast = m_1^{1/2} (1 + \bar{\epsilon}) \), equation (1) leads to

\[
z_a = m_1^{1/2} \left[ 1 + \epsilon - \frac{1}{1 + \bar{\epsilon}} - \frac{q}{1 - s - e^{-2\bar{\epsilon}} + e^\phi} \right].
\]

Here if \( \epsilon \) is given by equation (11) and \( \bar{\epsilon} \) is its complex conjugate, expanding equation (12) in a power series on \( q \) then results in

\[
z_a \approx m_1^{1/2} e^\phi \left[ \frac{q}{2(1 - s - e^{-2\phi} + e^\phi)} \left( \frac{1}{1 - s - e^{-2\phi}} - 1 \right) \right] = \frac{q}{8} \left[ \frac{1 + 3 s e^{-4\phi}}{1 - s - e^{-4\phi}} - \frac{5 - 4 - s e^{-4\phi}}{(1 - s - e^{-4\phi})^2} \right] + \cdots.
\]

Note that the linear term is invariant under the transform \( s \rightarrow -s \), but this invariance does not hold for the higher order term. Formally, if we consider the relative departure from the invariance, namely \( \delta = (z_\ast - z_a)/(z_\ast + z_a) \) where \( z_\ast \) is equation (13) under the transformation of \( s \rightarrow \bar{s} \), we find that \( |\delta| \approx q|\bar{s}|^{-1} \) as \( \chi \rightarrow 0 \) where \( \chi = 1 - s - e^{-4\phi} \). In other words, the central caustic invariance under \( s \rightarrow \bar{s} \) is again applicable (globally) when \( |1 - |s|| \gg |q|^{1/3} \).

### 3.1 Relation to the non-resonance condition

The linear-\( q \) approximation of the caustic in equation (13) results in an (distorted) astroid-shaped curve with four cusps, which corresponds to the “central caustics” of either close or wide planetary binaries. For the true binary lens system however, such caustics only exist if the separation is not in the resonant region; that is,

\[
|\epsilon| > m_1^{1/3} \left| \frac{d_\epsilon}{d_\epsilon} \right| = (1 + q^{1/3})^2 \approx 1 + \frac{3}{2} q^{1/3} + \frac{3}{8} q^{2/3} + \cdots
\]

or

\[
|\epsilon| < m_1^{-1/3} \left| \frac{d_\epsilon}{d_\epsilon} \right| = (1 - q^{1/3} + q^{2/3})^2 \approx \frac{3}{4} + \left( \frac{\pi}{2} - q^{1/3} \right)^2 + \cdots
\]

where \( d_\epsilon \) and \( d_\epsilon \) are the transition value derived in equations (6) and (7). With a sufficiently small \( q \), this condition is approximately 2

2 The relation is not actually valid for \( |s| < 1 \) if \( q > 0 \). Rather the actual precise statement would be \( |s| - 4/3 \approx 4/3 \) with \( s_c = m_1^{1/3} \left| \frac{d_\epsilon}{d_\epsilon} \right| \). Alternatively, the relevant zeroth order model for the close binary is the point mass with the total mass of the system and \( q^{1/3} \) times \( s_c = m_1^{1/3} \left| \frac{d_\epsilon}{d_\epsilon} \right| \), albeit an increasing function of \( q \), is still bounded for all \( q \in [0, 1] \).

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3.2 Global vs local degeneracy

Technically the preceding linear approximations of the critical curves and the caustics start to break down as $\phi \rightarrow 1$ and not in fact $|s| \rightarrow 1$. In other words, even if $|1-s| \lesssim q^{1/3}$, the approximation may be locally valid for the portion of the caustic corresponding to the values of $\phi$ with which $\phi^{5/3}$ is sufficiently far from the unity. For example, Figures 1 and 2 show the portions of caustics for the planetary $(q = 10^{-3})$ and $10^{-4}$ respectively) lensing models with $q^{1/3}/|1-s| \approx 1$. Whilst the approximation in equation (13) is invalid globally for these models and the overall caustic shapes are completely different between two models related by $s \leftrightarrow s^{-1}$, the caustic shown in the region of Figures 1 and 2 actually passably resembles each other and also the linear approximation. In fact, we expect that slight adjustments of $(q, s)$ separately for the close and wide binary might result in a pair of locally degenerate models although they would not be in the exact $s \leftrightarrow s^{-1}$ correspondence.

Whilst the true observed degeneracy of the microlensing lightcurve should be properly analyzed in the magnification map and not just with the shape of the caustics, most noticeable detailed structures of the microlensing events are dominated by the caustic crossings and the cusp approaches. Since the behaviour of the magnifications near the caustics is known to be fairly generic (e.g., Keeton, Gaudi, & Petters 2003, 2005), the resemblance of the caustics may be a good starting point to understand the degeneracy in the lightcurve modelling, especially given the fact that the lightcurve actually samples the magnification map only along a 1-d slice.