OPERADS AND JET MODULES

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Abstract. Let $A$ be an algebra over an operad in a cocomplete closed symmetric monoidal category. We study the category of $A$-modules. We define certain symmetric product functors of such modules generalising the tensor product of modules over commutative algebras, which we use to define the notion of a jet module. This in turn generalises the notion of a jet module over a module over a classical commutative algebra. We are able to define Atiyah classes (i.e. obstructions to the existence of connections) in this generalised context. We use certain model structures on the category of $A$-modules to study the properties of these Atiyah classes.

The purpose of the paper is not to present any really deep theorem. It is more about the right concepts when dealing with modules over an algebra that is defined over an arbitrary operad, i.e. the aim is to show how to generalise various classical constructions, including modules of jets, the Atiyah class and the curvature, to the operadic context.

For convenience of the reader and for the purpose of defining the notations, the basic definitions of the theory of operads and model categories are included.


date: March 29, 2022.

2000 Mathematics Subject Classification. 18D50; 18G55; 13N15; 14F10.
Let $X$ be a complex manifold. We denote its $\mathcal{O}_X$-module of Kähler differentials by $\Omega_X$. Let $\mathcal{E}$ be another locally free $\mathcal{O}_X$-module. A (global) holomorphic connection on $\mathcal{E}$ is a $\mathbb{C}$-linear morphism $\nabla : \mathcal{E} \to \Omega_X \otimes \mathcal{E}$ of sheaves on $X$ such that

$$\nabla(fs) = df \otimes s + f \nabla s$$

for a local section $f$ of $\mathcal{O}_X$ and a local section $s$ of $\mathcal{E}$. In general, no global holomorphic connection does exist. The obstruction to the existence is given by the so-called Atiyah class $\alpha_E \in \text{Ext}^1_X(\mathcal{E}, \Omega_X \otimes \mathcal{E})$ of $\mathcal{E}$ ([1]). The Atiyah class gives us thus cohomological invariants of the module $\mathcal{E}$. In fact, the Chern classes of $\mathcal{E}$ can be calculated from its Atiyah class. For details, we refer the reader to the first section of [9] or to the first chapter of our book [14].

Let us denote by $\mathcal{A}^{p,q}_X$ the $C^\infty$-sheaf of the $(p,q)$-differential forms on $X$. A $C^\infty$-connection on $\mathcal{E}$ is a $\mathbb{C}$-linear morphism $\nabla : \mathcal{E} \to \mathcal{A}^{1,0}_X \otimes \mathcal{E}$ of sheaves on $X$ such that (1) holds. Using a partition of unity, one can show that such a connection always exists for $\mathcal{E}$. As shown in, e.g., [9], the Atiyah class of $\mathcal{E}$ is then given by extension class represented by $-\bar{\partial} \nabla$, which is in fact $\mathcal{O}_X$-linear (the occurrence of the sign depends on the identification of the Dolbeault cohomology
groups $H^*_0(X, \Omega_X \otimes \mathcal{E}^\vee \otimes \mathcal{E})$ of $\Omega_X \otimes \mathcal{E}^\vee \otimes \mathcal{E}$ with the Ext-groups $\text{Ext}_X^1(\mathcal{E}, \Omega_X \otimes \mathcal{E})$ of $\mathcal{E}$ by $\Omega_X \otimes \mathcal{E}$, see [4].

In this article, we want to generalise these constructions considerably. First of all, the notion of a connection does not truly depend on the universal derivation $d: \mathcal{O}_X \to \Omega_X$. In fact, one may substitute $d: \mathcal{O}_X \to \Omega_X$ by any derivation $d: \mathcal{O}_X \to \mathcal{M}$ where $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module.

In [9] another way besides the Dolbeault method to calculate the Atiyah is presented: one may use Čech resolutions instead of Dolbeault resolution, while substituting the Dolbeault differential with the Čech differential. Formally, both methods are very similar. The idea to handle not only these two resolutions uniformly is to think of these resolutions as fibrant replacements each with respect to a certain model structures on the category of cochain complexes of $\mathcal{O}_X$-modules, where the weak equivalences are the quasi-isomorphisms. The assumption that $\mathcal{M}$ and $\mathcal{E}$ are locally free translate into the assumption that both are cofibrant modules. Thus the theory of model categories enters at this point and generalises, e.g., the Čech construction mentioned above.

The next generalisation is to replace the category of $\mathcal{O}_X$-modules by any closed symmetric monoidal abelian category whose category of cochain complexes $\mathcal{C}^\ast$ possesses a suitable model structure. Given a commutative algebra $A$ in this category $\mathcal{C}^\ast$, we can talk about $A$-modules. Thus we are lead to the notion of an Atiyah class for (cofibrant) $A$-modules and can try to translate everything else from the theory on complex manifolds to this context.

The last step in generalising these constructions and notions is the main reason why this article has been written. Instead of saying that $A$ is a commutative algebra in $\mathcal{C}^\ast$, we may say that $A$ is an algebra over the operad $\text{Com}$ in $\mathcal{C}^\ast$ (see [13]). Replacing the operad $\text{Com}$ by any other operad, we arrive at other types of algebras, say associative or Lie algebras. One can again talk of $A$-modules and we show what to do to get the notion of connections and the Atiyah class also in this operadic context.

There are still ways to generalise the results in this paper even more. Firstly, we only consider the case of cofibrant objects when dealing with their Atiyah class. Similar, the module of differentials is assumed to be cofibrant. One may try to extend the theory to the non-cofibrant case. Secondly, the theory developed here works for model categories of cochain complexes, i.e. for model categories being modules over the category of cochain complexes of abelian groups. To deal with the non-abelian case, one may work with model categories that are modules over the category of simplicial sets. To deal with this case one has to extend the notion of a derivation and a connection to the simplicial case, and one can hope to get the notion of an Atiyah class as well in this case. We haven’t pursued this way in this article.

0.1. Organisation. The article is organised in sections (numbered by 1., 2., etc.), which themselves are divided into subsections (numbered by 1.1., 1.2., etc.). The outline of the sections is as follows:

The first section deals with the notion of a symmetric closed monoidal category, i.e. with a categorical generalisation of the presence of the tensor product in the category of modules over a commutative ring with a unit. We do this so that we can talk about monoids (associative algebras) in these categories and (left) modules over them. We shall apply these notions to categories of cochain complexes.

The next section, which is on operadic notions, brings together all the results of the theory of operads needed here. The first subsections contain material well-known to people working on operads. The other subsections deal with the generalisation of the theory of connections of modules over a commutative algebra $A$.
to modules over arbitrary algebras. A main point here is to generalise the tensor product over $A$ between modules. The answer we have come with is given by what we call the \textit{lax product}. It gives back the tensor product in the commutative case, which gives the category of $A$-algebras a structure of a symmetric monoidal category. In general however, our lax product only gives a structure of a so-called lax symmetric monoidal category (3). Thus the name given to our product.

Having the application to the category of cochain complexes of $O_X$-modules over a complex manifold $X$ in mind, we develop the necessary portions of the model category theory to deal with suitable model structures on the categories of modules over arbitrary algebras in the third and forth section. These sections are mainly for the convenience of the reader and to fix the needed notions. For any further study of model categories, we highly recommend the book [7] and the references therein.

In the last section, we finally deal with the Atiyah class in the operadic case. The Atiyah class becomes a morphism in a homotopy category (which is nothing else than a derived category as our weak equivalences are always exactly the quasi-isomorphisms) and we show how it can be calculated by using fibrant resolutions. In [3], cohomological Bianchi identities for the Atiyah class are proven. We prove these identities in the more general operadic case. We also talk about the notion of (higher) curvature classes. Again we have been inspired by the methods and results in [9]. Our final application is as follows: Starting with a deformation of a free algebra $A$, we consider its category of modules and show that the Atiyah class of its module $M$ of Kähler differentials on $A$ defines a deformation of the free $A$-algebra over $M$. This gives a map compatible with gauge equivalence from the solution set of one Maurer–Cartan equation to the solution set of another Maurer–Cartan equation. In particular, every Lie algebra $\mathfrak{g}$ (in fact every strong homotopy Lie algebra) gives rise to a deformed version of the free commutative algebra over the vector space $\mathfrak{g}$, over which we can study our geometrical notions like jet modules and curvature forms. (This has been inspired by [10].)

The article includes a short appendix in which we give a proof for the result that every object in a Grothendieck category $\mathcal{A}$ is small. This is needed to use Quillen’s small object argument (see [7]) in order to put a model structure on the category of cochain complexes over $\mathcal{A}$, which we want to do. The result itself is well-known and one of the different types of proofs in the literature can be found in [3]. However, all published proofs we know use deep theorems about Grothendieck categories (like the embedding theorem). Therefore we feel that it is time to give a simple proof using just the basic properties of Grothendieck categories.

We hope that our developed notions will prove their usefulness in applications of the theory of modules over arbitrary algebras.

0.2. \textbf{A few remarks.} A lot of material in this article is included to round up the whole exposition but is otherwise well-known to people working in the particular fields. In these cases, we often give references to the literature. However, these are often not the original works where the material originally comes from but textbooks which may be better accessible to the average reader.

A lot of the propositions in this article just end with the “proof end symbol”, meaning that no proof is included. This usually means that the proof is straightforward and follows directly from the definitions. Most of the morphisms used in this paper are defined by some diagrams in some categories. In order to keep the diagrams simple, a lot of arrows between two objects, say $X$ and $Y$, are not annotated. This means that the arrow stands for the “most natural” morphism between the two objects. For example, an arrow $R \otimes_k M \to M$ where $R$ is a $k$-algebra, $k$ a field, and $M$ is an $R$-module over $k$ stands for the operation of $R$ on $M$, i.e. the scalar multiplication.
1. Symmetric closed monoidal categories

For the convenience of the reader, we recall in this section the notion of a symmetric closed monoidal category, on which the whole theory of operads is based.

1.1. Tensor products. Let $C$ be a category. By a tensor product on $C$, we understand a bifunctor $\cdot \otimes \cdot : C \times C \to C$ that is suitably associative, i.e. there are fixed isomorphisms, called associators, $X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$ natural in $X$, $Y$ and $Z$ for which the coherence diagrams

\[
\begin{array}{ccc}
X \otimes (Y \otimes (Z \otimes W)) & \to & (X \otimes Y) \otimes (Z \otimes W) \\
X \otimes ((Y \otimes Z)) \otimes W & \to & ((X \otimes Y) \otimes Z) \otimes W
\end{array}
\]

that are built up from the associators and are natural in $X$, $Y$, $Z$ and $W$, commute.

Remark 1. By Mac Lane’s coherence theorem, the commutativity of the pentagons above suffices to show that in fact all natural diagrams built up from the associators do commute.

1.2. Units. Let $\otimes$ be a tensor product on $C$. By a unit for $\otimes$ we understand an object $1$ of $C$ with fixed isomorphisms, called the left and the right unit law, $1 \otimes X \to X$ and $X \otimes 1 \to X$, natural in $X$, such that the coherence diagrams

\[
\begin{array}{ccc}
X \otimes (1 \otimes Y) & \to & (X \otimes 1) \otimes Y \\
X \otimes 1, & \to & X \otimes Y,
\end{array}
\]

that are built up from the associators and the unit laws and are natural in $X$ and $Y$, commute.

Remark 2. Again, the commutativity of the triangles above (together with the commutativity of the pentagons above) suffices to make all natural diagrams built up from the associators and unit laws commute.

Definition 1. A category together with a tensor product and a unit as defined above is a monoidal category.

1.3. Symmetric tensor products. Let $C$ be a monoidal category. The tensor product $\otimes$ is called symmetric if there are fixed isomorphisms, called (symmetric) braidings, $\gamma : X \otimes Y \to Y \otimes X$ natural in $X$ and $Y$ for which the coherence diagrams

\[
\begin{array}{ccc}
X \otimes Y & \to & Y \otimes X \\
X \otimes Y, & \to & X \otimes Y
\end{array}
\]

that are built up from the associators and braidings and are natural in $X$ and $Y$, commute.
and

\[
\begin{array}{c}
X \otimes (Y \otimes Z) \\

(X \otimes Y) \otimes Z & X \otimes (Z \otimes Y) \\

Z \otimes (X \otimes Y) & (X \otimes Z) \otimes Y \\

(Z \otimes X) \otimes Y
\end{array}
\]

that are built from the associators and the braidings and are natural in \(X, Y\) and \(Z\), commute.

Remark 3. The commutativity of all coherence diagrams suffices to make all natural diagrams built up from the associators, unit laws and braidings commute.

Thus, a specific bracketing or ordering in iterated tensor products does not matter up to a uniquely defined natural isomorphism.

Definition 2. A monoidal category with a symmetric tensor product as defined above is a symmetric monoidal category.

1.4. Symmetric monoidal functors. Later we shall use the notion of a symmetric monoidal functor, which is more or less a functor between symmetric monoidal categories that respects the symmetric monoidal structures. We follow [2]. The authors of this article in turn refer to [12].

Let \(\mathcal{C}\) and \(\mathcal{D}\) be two symmetric monoidal categories. Let \(F : \mathcal{C} \to \mathcal{D}\) be a functor. Assume \(F\) comes equipped with a morphism \(1 \to F(1)\) in \(\mathcal{D}\) and morphisms \(F(X) \otimes F(Y) \to F(X \otimes Y)\) in \(\mathcal{D}\) that are natural in \(X\) and \(Y\).

The functor \(F\) is (left) unital if the diagrams

\[
\begin{array}{c}
1 \otimes F(X) \longrightarrow F(X) \\

F(X) \otimes F(X) \longrightarrow F(1 \otimes X),
\end{array}
\]

natural in \(X\), commute.

The functor \(F\) is symmetric if the diagrams

\[
\begin{array}{c}
F(X) \otimes F(Y) \longrightarrow F(Y) \otimes F(X) \\

F(X \otimes Y) \longrightarrow F(Y \otimes X),
\end{array}
\]

natural in \(X\) and \(Y\), commute.
Finally, the functor $F$ is associative if the diagrams

\[
\begin{array}{ccc}
F(X) \otimes (F(Y) \otimes F(Z)) & \xrightarrow{\sim} & F(X \otimes F(Y) \otimes F(Z)) \\
(F(X) \otimes F(Y)) \otimes F(Z) & \xrightarrow{\sim} & F(X \otimes Y) \otimes F(Z) \\
F(X \otimes Y) \otimes F(Z) & \xrightarrow{\sim} & F((X \otimes Y) \otimes Z)
\end{array}
\]

are natural in $X$, $Y$, and $Z$, commute.

**Definition 3.** A functor $F$ as above that comes equipped with the morphism $1 \to F(1)$ and the morphisms $F(X) \otimes F(Y) \to F(X \otimes Y)$, natural in $X$ and $Y$, which is unital, symmetric, and associative is a symmetric monoidal functor.

1.5. **Inner hom’s.** Let $\mathcal{C}$ be a symmetric monoidal category. An inner hom for $\mathcal{C}$ is a bifunctor $\hom : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ such that there are bijections $\mathcal{C}(X \otimes Y, Z) \to \mathcal{C}(X, \hom(Y, Z))$ natural in $X$, $Y$ and $Z$. In other words, $\hom(Y, \cdot) : \mathcal{C} \to \mathcal{C}$ is right adjoint to $\cdot \otimes Y : \mathcal{C} \to \mathcal{C}$ for each object $Y \in \mathcal{C}$.

**Definition 4.** A closed symmetric monoidal category is a symmetric monoidal category that possesses an inner hom as defined above.

**Example 1.** One of the most prominent example of such a category is the category of $k$-modules for a commutative ring with unit $k$. The tensor product is the tensor product of $k$-modules over $k$, the unit is the object $k$. The inner hom of two objects is the set of $k$-linear maps between these objects considered as a $k$-module. Finally, the associators, unit laws, symmetric braidings and adjunctions are given by the obvious isomorphisms.

**Remark 4.** Let us remark that in a closed symmetric monoidal category the functor $\cdot \otimes Y : \mathcal{C} \to \mathcal{C}$ does commute with every colimit for each object $Y$. This relies solely on the fact that the functor possesses a right adjoint.

Let us end this paragraph with a definition concerning the previously defined structures in the case that the underlying category is additive.

**Definition 5.** A closed symmetric monoidal additive category is a symmetric monoidal category that is at the same time an additive category and such that the symmetric monoidal category structures are compatible with the group laws on the hom-sets.

1.6. **Cochain complexes.** Let $\mathcal{C}$ be an additive category. By a cochain complex over $\mathcal{C}$ we understand a family $(X^n)_{n \in \mathbb{Z}}$ of objects $X^n \in \mathcal{C}$ together with morphisms $\partial^n : X^n \to X^{n+1}$ with $\partial^{n+1} \circ \partial^n = 0$ for all $n \in \mathbb{Z}$. Such a family is usually abbreviated by $X^*$ or simply by $X$. The morphisms $\partial^n$ are the differentials of $X^*$. A morphism between cochain complexes $X^*$ and $Y^*$ over $\mathcal{C}$ is a family $(f^n : X^n \to Y^n)_{n \in \mathbb{Z}}$ of morphisms in $\mathcal{C}$ such that $\partial^n \circ f^n - f^{n+1} \circ \partial^n = 0$ for all $n \in \mathbb{Z}$. One can concatenate morphisms of cochain complexes over $\mathcal{C}$ componentwise in the obvious way. This leads to the following definition.

**Definition 6.** The category of cochain complexes over $\mathcal{C}$ is the category whose objects are the cochain complexes over $\mathcal{C}$ and whose morphisms are the morphisms between cochain complexes defined above. It is denoted by $\mathcal{C}^*$. 
For each cochain complex $X^*$ and each $n \in \mathbb{Z}$ we define the $n$-th cohomology group $H^n(X^*) := \ker \bar{\delta}^n / \text{im} \bar{\delta}^{n-1}$. Note that each morphism $f : X^* \to Y^*$ of cochain complexes induces as usual group homomorphisms between the cohomology groups of $X^*$ and $Y^*$. We need this for the following (well-known) definition:

**Definition 7.** The morphism $f : X^* \to Y^*$ is a **quasiisomorphism** if $f^* : H^*(X^*) \to H^*(Y^*)$ is an isomorphism on each cohomology group.

We shall also make use of the notion of a **free morphism between cochain complexes over** $\mathcal{C}$. A free morphism between two cochain complexes $X^*$ and $Y^*$ is just a family $(f^n : X^n \to Y^n)_{n \in \mathbb{Z}}$ of morphisms in $\mathcal{C}$ with no compatibility condition with the differentials whatsoever. Being a free morphism is denoted by $f : X \to_Y Y$.

Assume that $\mathcal{C}$ is a bicocomplete (i.e. arbitrary small limits and colimits do exist) closed symmetric monoidal additive category. In what follows we describe how to put a structure of a closed symmetric monoidal category on the category of cochain complexes over $\mathcal{C}$.

Let $X^*$ and $Y^*$ be two cochain complexes over $\mathcal{C}$. Let $X^* \otimes Y^*$ be the cochain complex with $(X^* \otimes Y^*)_n = \bigoplus_{p+q=n} X^p \otimes Y^q$ and such that the differential

$$d^n : \bigoplus_{p+q=n} X^p \otimes Y^q \to \bigoplus_{p+q=n+1} X^p \otimes Y^q$$

is given componentwise by

$$\bar{\delta}^n \otimes \text{id}_{Y^q} + (-1)^p \text{id}_{X^p} \otimes \bar{\delta}^q : X^p \otimes Y^q \to X^{p+1} \otimes Y^q \oplus X^p \otimes Y^{q+1}.$$ 

On the defines tensor products of morphisms between cochain complexes over $\mathcal{C}$ in the obvious way. By defining associators in the most obvious way, this makes $\cdot \otimes \cdot : \mathcal{C}^* \times \mathcal{C}^* \to \mathcal{C}^*$ a tensor product in the category of cochain complexes over $\mathcal{C}$.

A unit for this tensor product is given by the cochain complex $1^*$ with $1^0 = 0$ for $n \neq 0$ and $1^0 = \mathbf{1}$. The unit laws are defined in the most obvious way.

Let $\gamma^* : X^* \otimes Y^* \to Y^* \otimes X^*$ be the isomorphism natural in $X^*$ and $Y^*$ that is given componentwise by $(-1)^{pq} \gamma : X^p \otimes Y^q \to Y^q \otimes X^p$ where $\gamma : X^p \otimes Y^q \to Y^q \otimes X^p$ is given by the braiding of $\mathcal{C}$. This gives a symmetric braiding for the category $\mathcal{C}^*$.

Finally, an inner hom $\text{hom}^*(Y^*, Z^*)$ for the tensor product on $\mathcal{C}^*$ is defined by $\text{hom}^*(Y^*, Z^*) = \prod_{p+q=n} \text{hom}(X^p, Y^q)$ and such that the differential

$$\bar{\delta}^n : \prod_{p+q=n} \text{hom}(X^p, Y^q) \to \prod_{p+q+1=n} \text{hom}(X^p, Y^q)$$

is given componentwise by

$$\text{hom}(X^p, \bar{\delta}^q) - (-1)^n \text{hom}(\bar{\delta}^{p-1}, Y^q) : \text{hom}(X^p, Y^q) \to \text{hom}(X^p, Y^{q+1}) \oplus \text{hom}(X^{p-1}, Y^q).$$

The adjunction morphisms are defined in the most obvious way.

**Proposition 1.** With the so defined structures, the category of cochain complexes over a closed symmetric monoidal additive category becomes naturally a closed symmetric monoidal additive category.

\[\square\]

1.7. **Monoids.** Let $\mathcal{C}$ be a monoidal category.

Let $A$ be an object of $\mathcal{C}$. A **multiplication law for** $A$ is a morphism

$$\mu : A \otimes A \to A.$$
It is associative if the canonical diagram

\[
\begin{array}{c}
A \otimes A \otimes A \\
\mu \otimes \text{id} \\
A \otimes A \\
\mu \\
A
\end{array}
\]

commutes.

A morphism \( \eta : 1 \to A \) is a unit for the multiplication law if the canonical diagrams

\[
\begin{array}{c}
A \\
\text{id} \otimes \eta \\
A \otimes A \\
\mu \\
A
\end{array}
\]

and

\[
\begin{array}{c}
A \\
\eta \otimes \text{id} \\
A \otimes A \\
\mu \\
A
\end{array}
\]

commute.

**Definition 8.** A monoid in \( C \) or a unital associative algebra in \( C \) is an object in \( C \) together with an associative multiplication law and a unit for this law.

Assume that \( C \) is symmetric as a monoidal category. Let \( \gamma : A \otimes A \to A \otimes A \) be the canonical morphism that interchanges the two factors \( A \).

The multiplication law \( \mu \) is commutative if the canonical diagram

\[
\begin{array}{c}
A \otimes A \\
\gamma \\
A \otimes A \\
\mu \\
A
\end{array}
\]

commutes.

**Definition 9.** A commutative monoid in \( C \) or a unital commutative algebra in \( C \) is an object in \( C \) together with an associative commutative multiplication law and a unit for this law.

Let \( R \) be a unital (commutative) ring.

**Example 2.** A unital (commutative) algebra in the category of left \( R \)-modules is nothing else than an ordinary (commutative) \( R \)-algebra.

**Remark 5.** We leave the obvious definition of a morphism of (commutative) monoids in \( C \) and the composition law for these morphisms to the reader. In particular, one can define the category of (commutative) monoids in \( C \).

1.8. **Modules over monoids.** Let \( C \) be a monoidal category and let \( A \) be a monoid in \( C \) with the multiplication law \( \mu : A \otimes A \to A \).

Let \( M \) be any object. An \( A \)-operation on \( M \) is given by a morphism

\[
\nu : A \otimes M \to M.
\]
This operation is associative if the canonical diagram

\[
\begin{array}{c}
A \otimes A \otimes M \xrightarrow{id \otimes \nu} A \otimes M \\
\mu \otimes id \\
A \otimes M \xrightarrow{\nu} M
\end{array}
\]

commutes.

It is unital if the canonical diagram

\[
\begin{array}{c}
M \\
\xrightarrow{\nu} A \otimes M \\
\xleftarrow{\mu} M
\end{array}
\]

commutes.

**Definition 10.** A (left) \(A\)-module or a (left) module over \(A\) is an object of \(C\) together with a unital and associative \(A\)-operation.

Let \(R\) be a unital ring and let \(A\) be an \(R\)-algebra, which we also view as a unital associative algebra in the category of left \(R\)-modules.

**Example 3.** A left module over the monoid \(A\) is nothing else than an ordinary left \(A\)-module.

**Remark 6.** We leave the obvious definition of a morphism of left modules over a monoid and the composition law for these morphisms to the reader. In particular, one can define the category of left modules over an algebra.

## 2. Operads, Algebras and Modules

### 2.1. Permutations

We use the latter \(\mathcal{S}\) to denote the permutation groups:

**Definition 11.** The permutation group in \(n\) letters is the group of bijections of the set \(\{1, \ldots, n\}\) into itself and denoted by \(\mathcal{S}_n\).

Let \(m_1, \ldots, m_n\) be non-negative integers and set \(m := \sum_{i=1}^{n} m_i\). Let \(\tau_i \in \mathcal{S}_{m_i}\) be permutations. These induce a permutation \(\tau \in \mathcal{S}_m\) which is given by

\[
\tau \left( \sum_{i=1}^{j-1} m_i + k \right) = \tau_j (k) + \sum_{i=1}^{j-1} m_j
\]

for \(1 \leq k \leq m_j\).

**Definition 12.** The permutation \(\tau\) defined above is the sum of the permutations \(\tau_1, \ldots, \tau_n\) and denoted by \(\tau_1 + \cdots + \tau_n\).

Let \(\sigma \in \mathcal{S}_n\) be a permutation. It induces a permutation \(\hat{\sigma} \in \mathcal{S}_m\) which is given by

\[
\hat{\sigma} \left( \sum_{i=1}^{j-1} m_j + k \right) = \sum_{\sigma(i) < \sigma(j)} m_i + k
\]

for \(1 \leq k \leq m_j\).

**Definition 13.** The permutation \(\hat{\sigma}\) defined above is the block permutation induced by \(\sigma\) for the blocks given by \(m_1, \ldots, m_n\) and denoted by \(\sigma_{m_1, \ldots, m_n}\).
2.2. Operads. Operads describe certain types of algebras. There is, e.g., the associative operad that governs associative algebras. Another example is the commutative operad governing commutative algebras.

For what follows, let \( C \) be a closed symmetric monoidal category, e.g. the category of \( k \)-modules for a commutative ring with unit \( k \).

**Definition 14.** An \( \mathfrak{S} \)-module \( M \) is a sequence \((M(n))_{n \in \mathbb{N}_0}\) of objects \( M(n) \) in \( C \) together with a right action of \( \mathfrak{S}_n \) on each \( M(n) \).

Let \( M \) be an \( \mathfrak{S} \)-module. An operadic composition law on \( M \) is given by a family of morphisms

\[
\gamma : M(n) \otimes M(m_1) \otimes \cdots \otimes M(m_n) \to M(m)
\]

with \( m := \sum_{i=1}^n m_i \).

Such an operadic composition law is called associative if the canonical diagrams

\[
\begin{array}{ccc}
M(n) \otimes \bigotimes_{i=1}^n \left( M(m_i) \otimes \bigotimes_{j=1}^{m_i} M(l_{i,j}) \right) & \to & M(n) \otimes \bigotimes_{i=1}^n M(l_i) \\
\downarrow & & \downarrow \\
M(m) \otimes \bigotimes_{i=1}^n \bigotimes_{j=1}^{m_i} M(l_{i,j}) & \to & M(l)
\end{array}
\]

with \( l_i := \sum_{j=1}^{m_i} l_{i,j} \) and \( l := \sum_{i=1}^n l_i \) built up from the composition law commute.

The operadic composition law is called equivariant if the canonical diagrams

\[
\begin{array}{ccc}
M(n) \otimes (M(m_1) \otimes \cdots \otimes M(m_n)) & \to & M(n) \otimes (M(m_{\sigma(1)}) \otimes \cdots \otimes M(m_{\sigma(n)})) \\
\downarrow & & \downarrow \\
M(m) & \sigma_{m_{\sigma(1)} \cdots m_{\sigma(n)}} & \to M(m)
\end{array}
\]

for all \( \sigma \in \mathfrak{S}_n \) and

\[
\begin{array}{ccc}
M(n) \otimes (M(m_1) \otimes \cdots \otimes M(m_n)) & \to & M(n) \otimes (M(m_1) \otimes \cdots \otimes M(m_n)) \\
\downarrow & & \downarrow \\
M(m) & \to M(m)
\end{array}
\]

for all \( \tau_i \in \mathfrak{S}_{m_i} \) commute.

A morphism \( \eta : 1 \to M(1) \) is a unit for the operadic composition law if the canonical diagrams

\[
\begin{array}{ccc}
M(n) & \to & M(n) \otimes M(1)^{\otimes n} \\
\downarrow & & \downarrow \\
M(n) & & 
\end{array}
\]

and

\[
\begin{array}{ccc}
M(m) & \to & M(1) \otimes M(m) \\
\downarrow & & \downarrow \\
M(n) & & 
\end{array}
\]

commute.
We give the definition of an operad in a second. Before that, however, we want to note that the reader who wants to learn more about operads should consult the monograph and the references therein.

**Definition 15.** An operad is an $\mathfrak{S}$-module together with an associative and equivariant operadic composition law and a unit for this law.

The prototype of an operad is the so called endomorphism operad $\mathcal{E}\text{nd}_V$ over an object $V$ in $\mathcal{C}$: Its underlying $\mathfrak{S}$-module is given by $\mathcal{E}\text{nd}_V(n) = \text{hom}(V^\otimes n, V)$ with the canonical operations of each $\mathfrak{S}_n$ on the right. There is an equivariant operadic composition law on $\mathcal{E}\text{nd}_V$ given by the ordinary composition morphisms

$$\mathcal{E}\text{nd}_V(n) \otimes \mathcal{E}\text{nd}_V(m_1) \otimes \cdots \otimes \mathcal{E}\text{nd}_V(m_n) \to \mathcal{E}\text{nd}_V(m)$$

with $m := \sum_{i=1}^n m_i$.

A canonical unit for the operadic composition law on $\mathcal{E}\text{nd}_V$ is given by the morphism $1 \to \mathcal{E}\text{nd}_V(1) = \text{hom}(V, V)$ which corresponds to the identity morphism on $V$.

*Example 4.* The operad $\mathcal{E}\text{nd}_V$ given by the $\mathfrak{S}$-module and the operadic composition law defined above is the endomorphism operad over $V$.

Consider the $\mathfrak{S}$-module $\mathcal{A}\text{ss}$ which is given by $\mathcal{A}\text{ss}(n) = 1^{1 \mathfrak{S}_n}$ with the canonical operations of each $\mathfrak{S}_n$ on the right. The natural map from $1 \to \mathcal{A}\text{ss}(n)$ corresponding to the inclusion of the summand indexed by a permutation $\sigma \in \mathfrak{S}_n$ is denoted by $\cdot \sigma : 1 \to \mathcal{A}\text{ss}(n)$.

There is a unique equivariant operadic composition law $\gamma$ on $\mathcal{A}\text{ss}$ such that the canonical diagrams commute.

A canonical unit for the operadic composition law on $\mathcal{A}\text{ss}$ is given by $\cdot \text{id} : 1 \to \mathcal{A}\text{ss}(1)$.

*Example 5.* The operad $\mathcal{A}\text{ss}$ given by the $\mathfrak{S}$-module and the operadic composition law defined above is the associative operad.

Consider the $\mathfrak{S}_n$-module $\mathcal{C}\text{om}$ which is given by $\mathcal{C}\text{om}(n) = 1$ with the trivial operation of each $\mathfrak{S}_n$ on the right. A canonical operadic composition law is given by the unit laws of the underlying symmetric monoidal category. This operadic composition law is trivially equivariant and has a canonical unit.

*Example 6.* The operad $\mathcal{C}\text{om}$ given by the $\mathfrak{S}$-module and the operadic composition law defined above is the commutative operad.

*Remark 7.* We leave the obvious definition of a morphism of operads and the composition law for these morphisms to the reader. In particular, one can define the category of operads in $\mathcal{C}$.

2.3. **Algebras.** In this subsection, we define the notion of an algebra over an operad.

Again let $\mathcal{C}$ denote a closed symmetric monoidal category. Let $\mathcal{O}$ be an operad in $\mathcal{C}$.

Let $A$ be an object in $\mathcal{C}$. An $\mathcal{O}$-multiplication law on $A$ is a family of morphisms

$$\mathcal{O}(n) \otimes A^\otimes n \to A.$$
Such a multiplication law is called \textit{associative} if the canonical diagrams
\[
\begin{array}{ccc}
\mathcal{O}(n) \otimes \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n) \otimes A^\otimes m & \rightarrow & \mathcal{O}(n) \otimes A^\otimes n \\
\downarrow & & \downarrow \\
\mathcal{O}(m) \otimes A^\otimes m & \rightarrow & A
\end{array}
\]
built up from the operadic composition law and the \(\mathcal{O}\)-multiplication law with \(m := \sum_{i=1}^{n} m_i\) commute.

The multiplication law is called \textit{equivariant} if the canonical diagrams
\[
\begin{array}{ccc}
\mathcal{O}(n) \otimes A \otimes n & \rightarrow & \mathcal{O}(n) \otimes A^\otimes n \\
\downarrow & & \downarrow \\
\mathcal{O}(m) \otimes A & \rightarrow & A
\end{array}
\]
for all \(\sigma \in \mathfrak{S}_n\) commute.

The multiplication law is called \textit{unital} if the canonical diagram
\[
\begin{array}{ccc}
A & \rightarrow & \mathcal{O}(1) \otimes A \\
\downarrow & & \downarrow \\
A & \rightarrow & A
\end{array}
\]
commutes.

\textbf{Definition 16.} An \textit{algebra over} \(\mathcal{O}\) is an object of \(\mathcal{C}\) together with a unital, equivariant, and associative \(\mathcal{O}\)-multiplication law.

(For more on algebras over operads, one may consult \cite{13}.)

\textbf{Remark 8.} In this monograph, one can also find the one-to-one correspondence between algebra structures over \(\mathcal{O}\) on an object \(A\) on the one hand and morphisms \(\mathcal{O} \rightarrow \text{End}_A\) of operads in \(\mathcal{C}\) on the other hand.

Let \(A\) be a unital, associative algebra in \(\mathcal{C}\). There is a unique equivariant \(\text{Ass}\)-multiplication law \(\gamma\) on \(A\) such that the canonical diagrams
\[
\begin{array}{ccc}
A^\otimes n & \rightarrow & \text{Ass}(n) \otimes A^\otimes n \\
\downarrow \gamma & & \downarrow \\
A
\end{array}
\]
commute. This law is associative and unital. Thus we get the following:

\textbf{Example 7.} Each unital, associative algebra in \(\mathcal{C}\) defines canonically an algebra over \(\text{Ass}\).

(In fact, this defines a one-to-one correspondence between unital, associative algebras and algebras over \(\text{Ass}\).)

Let \(C\) be a unital, commutative algebra in \(\mathcal{C}\). There exists a unique \(\text{Com}\)-multiplication law \(\gamma\) on \(C\) such that the canonical diagrams
\[
\begin{array}{ccc}
C^\otimes n & \rightarrow & \text{Com}(n) \otimes C^\otimes n \\
\downarrow \gamma & & \downarrow \\
C
\end{array}
\]
commute. This law is associative, equivariant, and unital. Thus we get the following:
Each unital, commutative algebra in $C$ defines canonically an algebra over $\text{Com}$. (In fact, this defines a one-to-one correspondence between unital, commutative algebras and algebras over $\text{Com}$.)

Remark 9. Again, we leave the obvious definition of a morphism of algebras over an operad and the composition law for these morphisms to the reader. In particular, one can define the category of algebras over an operad.

2.4. The free algebra. There is a “free construction” in the category of algebras over a fixed operad $O$ in a cocomplete closed symmetric monoidal category $C$.

Let $V$ be an object in $C$. Each $\sigma \in S_n$ induces a morphism
$$(\cdot, \sigma) \otimes (\cdot^{-1}) : O(n) \otimes V^{\otimes n} \to O(n) \otimes V^{\otimes n}.$$ This defines a canonical operation of $S_n$ on $O(n) \otimes V^{\otimes n}$. The corresponding object of coinvariants is denoted by $O(n) \otimes_{S_n} V^{\otimes n}$.

We set
$$F_O V := \bigoplus_{n=0}^{\infty} O(n) \otimes_{S_n} V^{\otimes n}.$$ There is a unique $O$-multiplication law on $F_O V$ which is induced by the morphisms given componentwise by
$$O(n) \otimes (O(m_1) \otimes V^{\otimes m_1}) \otimes \cdots \otimes (O(m_n) \otimes V^{\otimes m_n}) \to O(m) \otimes V^{\otimes m}$$ with $m := \sum_{i=1}^{n+1} m_i$ that are induced by the operadic composition law on $O$. This multiplication law is associative, equivariant, and unital.

Definition 17. The free $O$-algebra over $V$ is the $O$-algebra $F_O V$ given by the $O$-multiplication law defined above.

(More on this free construction can be found in [13].) There is a natural morphism $V \to F_O V$ that is induced by the unit $1 \to O(1)$ of the operad $O$.

Proposition 2. The free algebra over $V$ is in fact free, i.e., it is the initial object in the comma category $(V, \#)$ where $\#$ is the forgetful functor from the category of $O$-algebras to $C$.

Example 9. The free $\text{Ass}$-algebra over an object is naturally isomorphic to the tensor algebra over that object.

Example 10. The free $\text{Com}$-algebra over an object is naturally isomorphic to the symmetric algebra over that object.

2.5. Modules. Let $O$ be an operad over a closed symmetric monoidal category $C$. Let $A$ be an $O$-algebra.

Let $M$ be an object in $C$. An $A$-operation on $M$ is a family of morphisms
$$O(n + 1) \otimes A^{\otimes n} \otimes M \to M.$$ Such an operation is associative if the canonical diagrams
$$O(n + 1) \otimes O(m_1) \otimes \cdots \otimes O(m_{n+1}) \otimes A^{\otimes m} \otimes M \to O(n + 1) \otimes A^{\otimes n} \otimes M$$
$$O(m + 1) \otimes A^{\otimes m} \otimes M \to M$$ with $m := \sum_{i=1}^{n+1} m_i$ commute.
The operation is *equivariant* if the canonical diagrams
\[
O(n + 1) \otimes A^\otimes n \otimes M \xrightarrow{\cdot (\sigma + \text{id}) \otimes (\sigma^{-1}) \otimes \text{id}} O(n + 1) \otimes A^\otimes n \otimes M
\]
for all \(\sigma \in \mathfrak{S}_n\) commute.

The operation is *unital* if the canonical diagram
\[
M \xrightarrow{} O(1) \otimes M \xrightarrow{} M
\]
commutes.

**Definition 18.** A *module over A* (or an *A-module*) is an object of \(\mathcal{C}\) together with a unital, equivariant, and associative \(A\)-operation on it.

**Example 11.** The algebra \(A\) is naturally a module over itself.

Let \(A\) be a unital, associative algebra in \(\mathcal{C}\), which we also view as an \(\text{Ass}\)-algebra, and let \(M\) be an \(A\)-bimodule. There is a unique equivariant \(A\)-operation \(\gamma\) on \(M\) such that the canonical diagrams
\[
A^\otimes n \otimes M \xrightarrow{(\cdot \text{id}) \otimes \text{id} \otimes \text{id}} \text{Ass}(n + 1) \otimes A^\otimes n \otimes M
\]
commute. This law is associative and unital. This can be summarised as follows:

**Example 12.** Each bimodule over \(A\) is canonically an \(A\)-module when \(A\) is viewed as an \(\text{Ass}\)-algebra.

**Example 13.** Similarly, one can show that \(\mathcal{C}\)-modules for a unital, commutative algebra \(C\) are canonically modules over \(C\) when \(C\) is viewed as a \(\text{Com}\)-algebra.

**Remark 10.** Again, we leave the obvious definition of a morphism of modules over an algebra and the composition law for these morphisms to the reader. In particular, one can define the category of modules over an algebra.

**Remark 11.** Assume that \(\mathcal{C}\) is a closed symmetric monoidal additive category. This defines a canonical structure of an additive category on the category of modules over an algebra over an operad in \(\mathcal{C}\). In case \(\mathcal{C}\) is abelian, this structure is also abelian. One way to show this is to show that the category of modules is isomorphic to a category of left modules over a certain associative algebra \(U\) (which is defined in the next subsection).

2.6. **The universal enveloping algebra.** In this subsection, we recall the notion of the universal enveloping algebra, which is an ordinary unital, associative algebra, of an algebra over some operad. For further reading, we suggest [5] and the references therein. Let \(\mathcal{C}\) be a cocomplete closed symmetric monoidal category and \(\mathcal{O}\) an operad in \(\mathcal{C}\). Let \(A\) be an \(\mathcal{O}\)-algebra.

Set
\[
T(A) := \prod_{n=0}^{\infty} \mathcal{O}(n + 1) \otimes_{\mathfrak{S}_n} A^\otimes n,
\]
which is called the tensor algebra over \( A \).

Let \( U(A) \) be the maximal quotient of \( T(A) \) such that the canonical diagrams
\[
\begin{array}{c}
(\mathcal{O}(n+1) \otimes \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_{n+1})) \otimes_{\Sigma_m} A^\otimes m \\
\downarrow \\
\mathcal{O}(m+1) \otimes_{\Sigma_m} A^\otimes m \\
\downarrow \\
U(A)
\end{array}
\]
with \( m := \sum_{i=1}^{n+1} m_i - 1 \) commute.

There is a unique structure of an associative algebra on \( T(A) \) which is induced by the canonical morphisms given componentwise by
\[
(\mathcal{O}(n+1) \otimes A^\otimes n) \otimes (\mathcal{O}(m+1) \otimes A^\otimes m) \rightarrow \mathcal{O}(n+1) \otimes \mathcal{O}(1) \otimes \mathcal{O}(m+1) \otimes A^\otimes (n+m) \\
\rightarrow \mathcal{O}(n + m + 1) \otimes A^\otimes (n+m) \otimes W.
\]

This \( A \)-operation is associative and unital.

**Definition 19.** The unital associative algebra \( U(A) \) in \( C \) as defined above is the universal enveloping algebra of \( A \).

**Example 14.** Let \( A \) be a classical unital associative algebra in \( C \), which we also view as an \( \text{Ass} \)-algebra. The universal enveloping algebra over \( A \) is given by \( A \otimes A^\circ \).

**Example 15.** Let \( A \) be a classical unital commutative algebra in \( C \), which we also view as an \( \text{Com} \)-algebra. The universal envelopping algebra over \( A \) is given by \( A \) itself.

**Remark 12.** As already stated in the previous subsection, the category of modules over the associative algebra \( U(A) \) is isomorphic to the category of \( A \)-modules. In fact, an \( A \)-module \( E \) is the same as a morphism \( U(A) \rightarrow \text{hom}(E, E) \) of associative algebras in \( C \).

Let \( W \) be an object in \( C \). Set
\[
F_A(W) := U(A) \otimes W.
\]

There is a unique equivariant \( A \)-operation on \( F_A(W) \) which is induced by the natural morphisms given componentwise by
\[
\begin{array}{c}
\mathcal{O}(n+1) \otimes A^\otimes n \otimes (\mathcal{O}(m+1) \otimes A^\otimes m \otimes W) \\
\rightarrow \mathcal{O}(n+1) \otimes \mathcal{O}(1) \otimes \mathcal{O}(m+1) \otimes A^\otimes (n+m) \otimes W \\
\rightarrow \mathcal{O}(n + m + 1) \otimes A^\otimes (n+m) \otimes W.
\end{array}
\]

This \( A \)-operation is associative and unital.

**Definition 20.** The free \( A \)-module over \( W \) is the \( A \)-algebra \( F_A(W) \) given by the \( A \)-operation defined above.

(See also [5].)

There is a natural morphism \( W \rightarrow F_A(W) \) induced by the canonical morphism \( 1 \rightarrow U(A) \) (the unit) induced by the unit \( 1 \rightarrow \mathcal{O}(1) \) of the operad \( \mathcal{O} \).

**Proposition 3.** The free \( A \)-module over \( W \) is in fact free, i.e. it is the initial object in the comma category \( (W, \#) \), where \( \# \) is the forgetful functor from the category of \( A \)-modules to \( C \).
2.7. Derivations. Let \( C \) be a closed symmetric monoidal additive category and \( O \) an operad in \( C \). Let \( A \) be an \( O \)-algebra.

Let \( M \) be an \( A \)-module. A morphism \( d : A \to M \) is derivative if the canonical diagrams

\[
\begin{array}{ccc}
\mathcal{O}(n) \otimes A^n & \xrightarrow{id \otimes \sum_{p+q=n} \text{id} \otimes d \otimes \text{id}} & \mathcal{O}(n) \otimes A^n \\
A & \xrightarrow{d} & M
\end{array}
\]

commute.

**Definition 21.** A derivation of \( A \) (into the module \( M \)) is a derivative morphism as defined above.

**Example 16.** Let \( A \) be a unital associative algebra in \( C \), which we also view as an \( \text{Ass} \)-algebra and let \( M \) be an \( A \)-module. Every derivation \( d : A \to M \) in the classical sense is a derivation in the above sense.

**Example 17.** An analogous result holds true in the category of commutative algebras in \( C \), i.e. in the category of algebras over \( \text{Com} \).

Let \( V \) be an object in \( C \) and \( M \) an \( A \)-module. Let \( \phi : V \to M \) be a morphism in \( C \). There is exactly one derivation \( d_\phi : F_O(V) \to M \) such that the following canonical diagram

\[
\begin{array}{ccc}
V & \xrightarrow{\phi} & F_O(V) \\
\downarrow & & \downarrow \\
M & \xrightarrow{d_\phi} & M
\end{array}
\]

commutes.

**Proposition 4.** The mapping \( \phi \mapsto d_\phi \) sets up a natural bijection between the morphisms in \( C \) from \( V \) to \( M \) and the derivations from \( F_O(V) \) into \( M \).

**Example 18.** Let \( V \) and \( W \) be two objects in \( C \). Set \( A := F_O(V) \). Every morphism from \( V \) to \( W \) in \( C \) induces a natural derivation \( A \to F_A(W) \).

Set \( \bar{d} := \eta \otimes \text{id} : A \to U(A) \otimes A \), where \( \eta : 1 \to U(A) \) is the canonical unit of the universal enveloping algebra. Let \( \Omega_A \) be the maximal \( A \)-module quotient of \( F_A(A) = U(A) \otimes A \) such that the canonical diagrams

\[
\begin{array}{ccc}
\mathcal{O}(n) \otimes A^{\otimes n} & \xrightarrow{id \otimes \sum_{p+q=n} \text{id} \otimes \bar{d} \otimes \text{id}} & \mathcal{O}(n) \otimes A^{\otimes (n-1)} \otimes (U(A) \otimes A) \\
A & \xrightarrow{\bar{d}} & U(A) \otimes A \\
\downarrow & & \downarrow \\
U(A) \otimes A & \xrightarrow{\Omega_A} & U(A) \otimes A
\end{array}
\]

commute.

Let \( d : A \to \Omega_A \) be the composition of \( \bar{d} \) and the quotient map \( U(A) \otimes A \to \Omega_A \). The morphism \( d \) is derivative.
**Definition 22.** The derivation \( d : A \to \Omega_A \) is the Kähler (or universal) derivation of \( A \).

**Proposition 5.** The universal derivation \( d : A \to \Omega_A \) is in fact universal, i.e. it is the initial object in the category of derivations of \( A \).

**Example 19.** Let \( A \) be a unital, associative algebra in \( \mathcal{C} \), which we also view as an \( \text{Ass} \)-algebra. The universal derivation is the classical one: set

\[
\tilde{d} := \eta \otimes \eta^\circ \otimes \text{id} : A \to A \otimes A^\circ \otimes A,
\]

where \( \eta \) and \( \eta^\circ \) are the units of \( A \) and \( A^\circ \), respectively. Then \( \Omega_A \) is the maximal quotient of \( A \otimes A^\circ \otimes A \) such that the canonical diagram

commutes. The universal derivation \( d : A \to \Omega_A \) is given by the composition of \( \tilde{d} \) and the quotient map \( A \otimes A^\circ \otimes A \to \Omega_A \).

**Example 20.** In the case of unital commutative algebras, the classical and the operadic notion of the Kähler derivation coincide as well.

**Example 21.** Let \( V \) be an object of \( \mathcal{C} \). Let \( A := F_\mathcal{O}(V) \) be the free \( \mathcal{O} \)-algebra over \( V \). The natural morphism \( d : A \to U(A) \otimes V \) is the universal derivation of \( A \).

### 2.8. Lax products.

Given modules over an operad, one may consider their tensor product in the underlying closed symmetric monoidal category. In the case of the, say, commutative operad, this product, however, does not coincide with the tensor product as modules. Thus we propose here a new product of modules over a general algebra that is closer to the given module structure than the tensor product in \( \mathcal{C} \).

Let \( \mathcal{C} \) be a cocomplete closed symmetric monoidal category and let \( \mathcal{O} \) be an operad in \( \mathcal{C} \). Let \( A \) be an \( \mathcal{O} \)-algebra.

Let \( M_1, \ldots, M_m \) be modules over \( A \). Let \( P_A(M_1, \ldots, M_m) \) be the maximal quotient of \( \prod_{n=0}^\infty \mathcal{O}(n+m) \otimes_{\mathcal{E}_n} A^\otimes n \otimes M_1 \otimes \cdots \otimes M_m \) such that the canonical diagrams

\[
\begin{align*}
(\mathcal{O}(n+m) \otimes \mathcal{O}(k_1) \otimes \cdots \otimes \mathcal{O}(k_{n+m})) & \otimes_{\mathcal{E}_n} A^\otimes k \otimes M_1 \otimes \cdots \otimes M_m & \mathcal{O}(n+m) \otimes_{\mathcal{E}_n} A^\otimes n & \otimes M_1 \otimes \cdots \otimes M_m \\
& \mathcal{O}(k+m) \otimes_{\mathcal{E}_k} A^\otimes k \otimes M_1 \otimes \cdots \otimes M_m & P_A(M_1, \ldots, M_m) \\
\end{align*}
\]

with \( k := \sum_{i=1}^{n+m} k_i - m \) commute.
There is a natural (unital, associative) $A$-operation on $P_A(M_1, \ldots, M_n)$ which is induced by the canonical morphisms given componentwise by

$$O(k+1) \otimes A^\otimes k \otimes (O(n+m) \otimes A^\otimes n \otimes M_1 \otimes \cdots \otimes M_m) \rightarrow O(k+1) \otimes (O(1))^\otimes k \otimes O(n+m) \otimes A^\otimes (k+n) \otimes M_1 \otimes \cdots \otimes M_m$$

$$\rightarrow O(k+n+m) \otimes A^\otimes (k+n) \otimes M_1 \otimes \cdots \otimes M_m.$$  

**Definition 23.** The $A$-module $P_A(M_1, \ldots, M_m)$ as defined above is the **lax product of the $A$-modules** $M_1, \ldots, M_m$.  

**Example 23.** Let $A$ be a unital, associative algebra in $\mathcal{C}$, which we also view as an $\text{Ass}$-algebra. Let $M_1, \ldots, M_m$ be modules over $A$. Their lax product is given by

$$P_A(M_1, \ldots, M_m) = \prod_{\sigma \in \mathfrak{S}_n} M_{\sigma(1)} \otimes_A \cdots \otimes_A M_{\sigma(n)}$$

with the obvious structure of a module over $A$.  

**Example 24.** Let $C$ be a unital, associative algebra in $\mathcal{C}$, which we also view as a $\text{Com}$-algebra. Let $M_1, \ldots, M_m$ be modules over $C$. Their lax product is given by

$$P_C(M_1, \ldots, M_m) = M_1 \otimes_C \cdots \otimes_C M_m$$

with the obvious structure of a module over $C$.  

**Example 25.** Let $W_1, \ldots, W_m$ be objects in $\mathcal{C}$. The lax tensor product of the free $A$-modules $M_i := F_A(W_i) = U(A) \otimes W_i$ is given by

$$P_A(M_1, \ldots, M_m) = U(A) \otimes W_1 \otimes \cdots \otimes W_m = F_A(W_1 \otimes \cdots \otimes W_m).$$

**2.9. Lax inner hom’s.** In this subsection, we want to construct right adjoints to the various lax products defined in the previous subsection.  

Let $\mathcal{C}$ be a bicomplete closed symmetric monoidal category, and let $O$ be an operad in $\mathcal{C}$. Let $A$ be an $O$-algebra.  

Let $M_2, \ldots, M_m$ and $N$ be modules over $A$. Let $H_A(M_2, \ldots, M_m; N)$ be the maximal subobject of $\prod_{n \geq 0} \text{hom}(O(n+m) \otimes \mathfrak{S}_n A^\otimes n \otimes M_2 \otimes \cdots \otimes M_m, N)$ such
that the canonical diagrams
\[
\begin{array}{c}
H_A(M_2, \ldots, M_m; N) \\ \longrightarrow
\end{array}
\begin{array}{c}
\text{hom}(O(n + m) \otimes A \otimes \cdots \otimes M_m, N) \\ \downarrow
\end{array}
\begin{array}{c}
\text{hom}(O(k + m) \otimes A \otimes \cdots \otimes M_m, N) \\ \text{hom}(O(n + m) \otimes O(k_1) \otimes \cdots O(k_{n+m}))
\end{array}
\end{array}
\]
\[
\text{hom}(M_2 \otimes \cdots \otimes M_m, N) \\ \longrightarrow
\text{hom}(O(n + m) \otimes A \otimes \cdots \otimes M_m, N)
\]
with \( k := \sum_{k=1}^{m+n} k_i - m \) commute. There is a natural \( A \)-operation on the object
\( H_A(M_2, \ldots, M_m; N) \) which is induced by the canonical morphisms given componentwise by
\[
\begin{align*}
O(k + 1) \otimes A^\otimes k \otimes \text{hom}(O(n + m) \otimes A^\otimes n \otimes M_2 \otimes \cdots \otimes M_m, N) \\
\longrightarrow \text{hom}(O(n + m) \otimes A^\otimes n \otimes M_2 \otimes \cdots \otimes M_m, O(k + 1) \otimes A^\otimes k \otimes N) \\
\longrightarrow \text{hom}(O(n + m) \otimes A^\otimes n \otimes M_2 \otimes \cdots \otimes M_m, N)
\end{align*}
\]

**Definition 25.** The \( A \)-module \( H_A(M_2, \ldots, M_m; N) \) as defined above is the lax inner hom from the \( A \)-modules \( M_2, \ldots, M_m \) to the \( A \)-module \( N \).

**Example 26.** It is \( H_A(\cdot; N) = N \) as \( A \)-modules.

Let \( M_1 \) be another \( A \)-module. The definition of \( H_A(M_2, \ldots, M_n; N) \) has been chosen so that the following holds:

**Proposition 6.** The set of morphisms \( P_A(M_1, \ldots, M_m) \rightarrow N \) of \( A \)-modules is naturally in one-to-one correspondence with the set of morphisms
\[
M_1 \rightarrow H_A(M_2, \ldots, M_m; N),
\]

i.e. the functor \( H_A(M_2, \ldots, M_m; \cdot) \) is right adjoint to \( P_A(\cdot, M_2, \ldots, M_m) \). The adjunction morphism is induced by the adjunction morphism of the inner hom of \( C \).

**Example 27.** Let \( A \) be a unital, associative algebra in \( C \), which we also view as an \( \text{Ass} \)-algebra. Let \( M_2, \ldots, M_n \) and \( N \) be modules over \( A \). Their lax inner hom is given by
\[
H_A(M_2, \ldots, M_n; N) = \prod_{\sigma \in S_n} \text{hom}_A(M_{\sigma(1)} \otimes_A \cdots \otimes_A M_{\sigma(n)}; N).
\]
where \( M_1 := A \) and where the right hand side carries the obvious structure of a module over \( A \).

**Example 28.** Let \( C \) be a unital, commutative algebra in \( C \), which we also view as a \( \text{Com} \)-algebra. Let \( M_2, \ldots, M_n \) and \( N \) be modules over \( A \). Their lax inner hom is given by
\[
H_A(M_2, \ldots, M_n; N) = \text{hom}_C(M_2 \otimes_C \cdots \otimes_C M_n; N)
\]
with the obvious structure of a module over \( C \).

### 2.10 Algebras over algebras

Let \( C \) be a cocomplete closed symmetric monoidal category and let \( O \) be an operad in \( C \). Let \( A \) be an \( O \)-algebra.

**Definition 26.** An \( A \)-algebra is a morphism \( A \rightarrow B \) of \( O \)-algebras. By abuse of notation, we often write \( B \) instead of \( A \rightarrow B \). The category of \( A \)-algebras is the comma category \((A, \text{id})\) where \( \text{id} \) is the identity functor on the category of \( O \)-algebras.
Remark 14. Let $A \to B$ be an $A$-algebra. Then $B$ is naturally an $A$-module where the $A$-operation is given by the morphisms given componentwise by the canonical maps
\[
\mathcal{O}(n+1) \otimes A^{\otimes n} \otimes B \to \mathcal{O}(n+1) \otimes B^{\otimes (n+1)} \to B.
\]

Let $M$ be an $A$-module. Set
\[
S_n^A(M) := \prod_{n=0}^\infty S_n^A(M).
\]

There is a unique equivariant $\mathcal{O}$-multiplication law on $S_n^A(M)$ that is induced by the morphisms given componentwise by
\[
\mathcal{O}(n) \otimes (\mathcal{O}(m_1 + k_1) \otimes A^{\otimes k_1} \otimes M^{\otimes m_1}) \otimes \cdots \otimes (\mathcal{O}(m_n + k_n) \otimes A^{\otimes k_n} \otimes M^{\otimes m_n}) \to \mathcal{O}(m + k) \otimes A^{\otimes k} \otimes M^{\otimes m}
\]
with $k := \sum_{i=1}^n k_i$ and $m := \sum_{i=1}^n m_i$. It is unital, and associative. The canonical morphism $A \to S_n^A(M)$ given by $A = S_0^A(M)$ is a morphism of $\mathcal{O}$-algebras.

Definition 27. The free $A$-algebra over $M$ is the $A$-algebra given by the natural morphism $A \to S_n^A(M)$ as defined above.

There is a natural morphism of $A$-modules $M \to S_n^A(M)$ induced by the canonical isomorphism $M \cong S_1^A(M)$.

Proposition 7. The free $A$-algebra over $M$ is in fact free, i.e. it is the initial object in the comma category $(M, \#)$, where $\#$ is the forgetful functor from the category of $A$-algebras to the category of $A$-modules.

Example 29. Let $A$ be a unital, associative algebra, which we also view as an $A_{ss}$-algebra. Let $M$ be an $A$-module. It is
\[
S_n^A(M) = \prod_{n=0}^\infty M^{\otimes n},
\]
i.e. the classical tensor algebra.

Example 30. Let $C$ be a unital, commutative algebra, which we also view as a $Com$-algebra. Let $M$ be a $C$-module. It is
\[
S_n^A(M) = \prod_{n=0}^\infty M^{\otimes n},
\]
i.e. the classical symmetric algebra.

Example 31. Let $W$ be an object in $C$. The free $A$-algebra over $M := F_A(W)$ is given by
\[
S_n^A(M) = U(A) \otimes \prod_{n=0}^\infty W^{\otimes n},
\]
i.e. the tensor product of the universal enveloping algebra and the symmetric algebra over $W$ in $C$.

Let $A \to B$ be an $A$-algebra. Note that every $B$-module is canonically an $A$-module via the morphism $A \to B$.

Let $E_1, \ldots, E_r$ be $A$-modules. Set
\[
S_n^A(M, E_1, \ldots, E_r) := \prod_{n=0}^\infty S_n^A(M, E_1, \ldots, E_r).
\]
There is a unique equivariant $S^*_A(M)$-operation on $S^*_A(M, E_1, \ldots, E_r)$ that is induced by the morphisms given componentwise by

$$O(n + 1) \otimes (O(m_1 + k_1) \otimes A^{\otimes k_1} \otimes M^{\otimes m_1}) \otimes \cdots \otimes (O(m_n + k_n) \otimes A^{\otimes k_n} \otimes M^{\otimes m_n})$$

$$\otimes (O(m_{n+1} + k_{n+1} + r) \otimes A^{\otimes k_{n+1}} \otimes M^{\otimes m_{n+1}} \otimes E_1 \otimes \cdots \otimes E_r)$$

$$\rightarrow O(m + k + r) \otimes A^{\otimes k} \otimes M^{\otimes m} \otimes E_1 \otimes \cdots \otimes E_r$$

with $k := \sum_{i=1}^{n+1} k_i$ and $m := \sum_{i=1}^{n+1} m_i$. It is unital, equivariant and associative.

Let $E$ be an $A$-module. The canonical morphism $E \rightarrow S^*_A(M, E)$ induced by the isomorphism $E = S^1(M, E)$ is a morphism of $A$-modules.

**Proposition 8.** The $S^*_A(M)$-module $S^*_A(M, E)$ as defined above is a free object, i.e. it is the initial object in the comma category $(E, \#)$, where $\#$ is the canonical forgetful functor from the category of $S^*_A(M)$-modules to the category of $A$-modules.

**2.11. Connections.** Now we shall propose an extension of the notion of a connection of a module over a commutative algebra to modules over an algebra of any operadic type. For this, one has to make use of the lax product as defined above.

Let $C$ be a cocomplete closed symmetric monoidal additive category and let $O$ be an operad in $C$. Let $A$ be an $O$-algebra and let $d : A \rightarrow M$ be a derivation of $A$.

Let $E$ be an $A$-module. A morphism $\nabla : E \rightarrow P_A(M, E)$ is a $d$-derivative if the canonical diagrams

$$\begin{array}{ccc}
O(n + 1) \otimes A^{\otimes n} \otimes E & \xrightarrow{id \otimes (\sum_{p=1}^{n+1} \cdot \cdot \cdot ) \otimes \id \otimes \id \otimes \nabla} & O(n + 1) \otimes A^{\otimes (n-1)} \otimes M \otimes E \\
E & \xrightarrow{\nabla} & P_A(M, E)
\end{array}$$

commute.

**Definition 26.** A $d$-connection on $E$ is a $d$-derivative morphism $E \rightarrow P_A(M, E)$.

**Example 31.** In this example, we view $A$ as an module over itself. The morphism $d : A \rightarrow M$ composed with the natural morphism $M \rightarrow P_A(M, A)$ is a $d$-connection on $A$.

**Example 33.** Let $A$ be a unital, associative algebra, which we also view as an $Ass$-algebra. Let $d : A \rightarrow M$ be an $A$-derivative and $E$ an $A$-module. A $d$-connection $\nabla : E \rightarrow P_A(M, E) = M \otimes_A E \oplus E \otimes_A M$ is a morphism such that the canonical diagrams

$$\begin{array}{ccc}
A \otimes E & \xrightarrow{d \otimes \id + \id \otimes \nabla} & M \otimes E \oplus A \otimes (M \otimes_A E \oplus E \otimes_A M) \\
E & \xrightarrow{\nabla} & M \otimes_A E \oplus E \otimes_A M
\end{array}$$

and

$$\begin{array}{ccc}
E \otimes A & \xrightarrow{\id \otimes d + \nabla \otimes \id} & E \otimes M \oplus (M \otimes_A E \oplus E \otimes_A M) \otimes A \\
E & \xrightarrow{\nabla} & M \otimes_A E \oplus E \otimes_A M
\end{array}$$

commute.
Example 34. Let $C$ be a unital, commutative algebra, which we also view as a $Com$-algebra. Let $d : A \to M$ be a derivation of $A$ and $E$ an $A$-module. A $d$-connection $\nabla : E \to P_A(M, E) = M \otimes_A E$ is just a $d$-connection in the classical sense.

Example 35. Let $W$ be an object in $C$. There is a canonical $d$-connection on $F_A(W) = U(A) \otimes W$ which is induced by the morphisms given componentwise by

\[
\begin{align*}
\mathcal{O}(n+1) \otimes A^\otimes n \otimes W &\xrightarrow{id \otimes (\sum_{p+1+q=n} id^\otimes p \otimes d^\otimes q) \otimes id} \mathcal{O}(n+1) \otimes A^\otimes (n-1) \otimes M \otimes W \\
&\downarrow \\
&\mathcal{P}_A(M, F_A(W))
\end{align*}
\]

2.12. Jet modules. Let $C$ be a cocomplete closed symmetric monoidal abelian category, and let $O$ be an operad in $C$. Let $A$ be an $O$-algebra and let $d : A \to M$ be a derivation of $A$. Let $E$ be an $A$-module and let $\gamma$ be the multiplication map of $A$ on $E$. Set $J_dE := E \oplus P_A(M, E)$.

There is a unique equivariant $A$-operation on $J_dE$, which is induced by the canonical morphisms given componentwise by

\[
\begin{align*}
\mathcal{O}(n+1) \otimes A^\otimes n \otimes E &\xrightarrow{id \otimes (\sum_{p+1+q=n} id^\otimes p \otimes d^\otimes q) \otimes id} E \\
&\downarrow \\
E \oplus P_A(M, E)
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{O}(n+1) \otimes A^\otimes n \otimes P_A(M, E) &\xrightarrow{\gamma} P_A(M, E).
\end{align*}
\]

This operation is unital, equivariant, and associative.

Definition 29. The $A$-module $J_dE$ as defined above is the (one-) $d$-jet module of $E$.

The $A$-operation on $J_dE$ has been chosen so that the following proposition holds:

Proposition 9. A morphism $\nabla : E \to P_A(M, E)$ is a $d$-connection if and only if $id + \nabla = (id, \nabla) : E \to E \oplus P_A(M, E)$ is a morphism of $A$-modules.

There is a canonical complex

\[
\begin{array}{cccc}
0 & \rightarrow & P_A(M, E) & \rightarrow & J_dE & \rightarrow & E & \rightarrow & 0,
\end{array}
\]

of $A$-modules where the second morphism is the canonical inclusion and the third morphism is the canonical projection. This sequence is exact as the underlying sequence of objects in $C$ is exact.

Definition 30. The short exact sequence

\[
\begin{array}{cccc}
0 & \rightarrow & P_A(M, E) & \rightarrow & J_dE & \rightarrow & E & \rightarrow & 0
\end{array}
\]

of $A$-modules is the (first) $d$-jet module sequence of $E$.

For the classical (i.e. non-operadic case) in the geometrical setting this sequence is discussed in [9].
3. Model categories

The rest of the article uses the language of model categories to do the necessary homological algebra. In order to fix the notation, we repeat the main notions.

3.1. Retracts. Let $\mathcal{C}$ be a category. Let $f$ and $g$ be two morphisms in $\mathcal{C}$ such that a commutative diagram of the form

\[
\begin{array}{ccc}
A & \xrightarrow{f} & C \\
\downarrow & & \downarrow \\
B & \xrightarrow{g} & D
\end{array}
\]

exists.

**Definition 31.** In this situation, $f$ is a **retract** of $g$.

Let $\mathcal{W}$ a subcategory of $\mathcal{C}$.

**Definition 32.** The subcategory $\mathcal{W}$ is **closed under retracts** if for any two morphisms $f$ and $g$ such that $f$ is a retract of $g$ and $g$ is in $\mathcal{W}$, $f$ is as well.

3.2. Lifting properties. Let $\mathcal{C}$ be a category. Consider commutative diagrams of the form

\[
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
B & \xrightarrow{g} & Y
\end{array}
\]

in $\mathcal{C}$.

**Definition 33.** A morphism $i$ in $\mathcal{C}$ has the **left lifting property with respect to a morphism** $p$ in $\mathcal{C}$ (and $p$ the **right lifting property with respect to $i$**) if for all morphism $f$ and $g$ in $\mathcal{C}$ such that the solid square in the above diagram commutes, there exists a morphism $h$ making the whole diagram commutative.

3.3. Two-out-of-three axiom. Let $\mathcal{C}$ be a category and $\mathcal{W}$ a subcategory of $\mathcal{C}$.

**Definition 34.** The subcategory $\mathcal{W}$ satisfies the two-out-of-three axiom if for any three morphisms $f$, $g$, and $h$ of $\mathcal{C}$ with $h = gf$ such that two of them are in $\mathcal{W}$, the third is as well.

3.4. Model categories. Let $\mathcal{C}$ be a category. Let us assume that there are three distinguished subcategories of $\mathcal{C}$ whose morphisms are called **weak equivalences**, **cofibrations**, and **fibrations**, respectively.

Morphisms that are weak equivalences and cofibrations at the same time are **acyclic (or trivial) cofibrations**. Morphisms that are weak equivalences and fibrations at the same time are **acyclic (or trivial) fibrations**.

**Definition 35.** The three distinguished subcategories define a **model structure on $\mathcal{C}$** if

- the subcategory of the weak equivalences fulfills the two-out-of-three axiom,
- each of the three subcategories of weak equivalences, cofibrations, and fibrations is closed under retracts,
- the trivial cofibrations have the left lifting property with respect to the fibrations, the trivial fibrations have the right lifting property with respect to the cofibration, and
- each morphisms $f$ in $\mathcal{C}$ has two functorial factorisations $f = p \circ i = q \circ j$ such that $p$ is a fibration, $i$ is a acyclic cofibration, $q$ is a acyclic fibration and $j$ is a cofibration.
Definition 36. A bicomplete category $C$ together with a model structure on it is a model category.

Definition 37. An object in a model category is cofibrant if the unique morphism from the initial object to the object is a cofibration. An object is fibrant if the unique morphism from the object to the terminal object is a fibration.

For more on model categories, we refer the reader to the monograph [7] and the references therein.

3.5. Monoidal model categories. Let $C$ be a bicomplete closed (symmetric) monoidal category. Let $f : A \to B$ and $g : X \to Y$ be two morphisms in $C$. Let $f \square g : A \otimes Y \amalg B \otimes X \to B \otimes Y$ be the natural morphism induced by $f$ and $g$.

Definition 38. The morphism $f \square g$ as defined above is the pushout tensor product of $f$ and $g$.

Assume that $C$ is endowed with a model structure. Let $0 \to q \to 1$ be the functorial factorisation of $0 \to 1$ into a cofibration followed by an acyclic fibration.

Definition 39. The category $C$ is a symmetric monoidal model category if for any two cofibrations $f$ and $g$, their pushout tensor product $f \square g$ is a cofibration which is acyclic if $f$ or $g$ is acyclic, and if the natural morphisms $q \square X : Q1 \otimes X \to 1 \otimes X$ are weak equivalences for all cofibrant objects $X$.

See also [6] and [16].

4. Cofibrantly generated model categories

Generally it is not an easy task to construct model structures on categories “by hand”. There is, however, a general method to construct certain model structures, the so-called cofibrantly generated model structures. This method is based on Quillen’s “small object argument”, see [15] and [7]. We repeat the main notions here.

4.1. Smallness. A limit ordinal is an ordinal that is not the direct successor of an ordinal.

Definition 40. Let $\lambda$ be a limit ordinal. The cofinality $\text{cofin} \lambda$ of $\lambda$ is the least cardinal $\kappa$ such that there exists a subset $T$ of $\lambda$ with $|T| = \kappa$ and $\text{sup} T = \lambda$.

Example 36. It is $\text{cofin} \kappa = \kappa$ for each cardinal $\kappa$.

Let $C$ be a cocomplete category, $A$ an object in $C$ and $\kappa$ a cardinal. Let $D$ be a subcategory of $C$.

Definition 41. The object $A$ is $\kappa$-small (relative to $D$) if, for every ordinal $\lambda$ with $\text{cofin} \lambda > \kappa$ and every colimit-preserving functor $X : \lambda \to C$ ($X : \lambda \to D$), the natural map $\lim_{\mu < \lambda} \text{hom}(A, X_\mu) \to \text{hom}(A, \lim_{\mu < \lambda} X_\mu)$ is an isomorphism.

Example 37. If $A$ is $\kappa$-small for a cardinal $\kappa$, it is also $\kappa'$-small for each cardinal $\kappa' \geq \kappa$.

Definition 42. The object $A$ is small if it is $\kappa$-small for some cardinal $\kappa$. 
Theorem 1. Let $\mathcal{C}$ be a Grothendieck category (i.e. a bicomplete abelian category with a generator and exact filtered colimits). Any object of $\mathcal{C}$ is small, i.e. small relative to the whole category $\mathcal{C}$.

Proof. Usually, the proof is given by using an embedding theorem for Grothendieck categories or by using that any Grothendieck category is locally presentable. For one these proofs, we refer to [8].

In the appendix, we give a proof that does not use any deep theorem and relies solely on the basic properties of a Grothendieck category. □

4.2. Cells. Let $\mathcal{C}$ be a cocomplete category. Let $I$ be a class of morphisms in $\mathcal{C}$.

A morphism in $\mathcal{C}$ is $I$-injective if it has the right lifting property with respect to every morphism in $I$.

Definition 43. The class of all $I$-injective morphisms is denoted by $I$-inj.

For example, an object $Y$ in $\mathcal{C}$ is injective (in the sense of injective modules) if $0 \to Y$ is in $I$-inj, where $I$ is the class of monomorphisms in $\mathcal{C}$.

A morphism in $\mathcal{C}$ is an $I$-cofibration if it has the left lifting property with respect to every morphism in $I$-inj.

Definition 44. The class of all $I$-cofibrations is denoted by $I$-cof.

In particular, it is $I$ a subset of $I$-cof.

A morphism in $\mathcal{C}$ is a relative $I$-cell complex if it is a transfinite composition of pushouts of morphisms in $I$. Here, a transfinite composition is just a colimit over a colimit-preserving functor $X : \lambda \to \mathcal{C}$, where $\lambda$ is a limit ordinal. The monograph [7] is also a good reference for this notion.

Definition 45. The class of all relative $I$-cell complexes is denoted by $I$-cell.

For example, relative CW-complexes in topology are exactly the relative $I$-cell complexes when $I$ is the set of all morphisms $\partial D^n \to D^n$, where $D^n$ denotes the $n$-disk.

Remark 15. It is $I$-cell a subclass of $I$-cof.

4.3. Cofibrantly generated model categories. Our main reference for this subsection is again [7].

Let $\mathcal{C}$ be a model category. Let $I$ and $J$ be two sets of morphisms in $\mathcal{C}$.

Definition 46. The model category $\mathcal{C}$ is cofibrantly generated if the domains of the morphisms in $I$ are small relative to $I$-cell, the domains of the morphisms in $J$ are small relative to $J$-cell, the class of fibrations is $J$-inj, and the class of trivial fibrations is $I$-inj. $I$ is the set of generating cofibrations and $J$ the set of generating acyclic cofibrations.

Let $R$ be a unital commutative ring (in the ordinary sense).

Example 38. The category of cochain complexes of $R$-modules is a cofibrantly generated model category whose weak equivalences are the quasi-isomorphisms and whose fibrations are all degree-wise surjective morphisms. For the proof and the generating (acyclic) cofibrations see, e.g., [7].

In fact, the category is a symmetric monoidal model category.

Let $(X, O_X)$ be a ringed space with finite hereditary global dimension (see [8] in this context), e.g. $X$ is a finite-dimensional noetherian topological space or a finite-dimensional locally compact Hausdorff space that is countably at infinity.
Example 39. This example is one of the main results in [5]. The category of cochain complexes of $O_X$-modules is a cofibrantly generated model category whose weak equivalences are the quasiisomorphisms and whose fibrations are the degree-wise surjections with degree-wise flabby kernel.

In fact, the category is a symmetric monoidal model category. This example subsumes the previous one (for $X = \{\ast\}$, the one-pointed space).

4.4. Categories of modules. This subsection builds on the ideas and results in [16] and [6].

Let $C$ be a cofibrantly generated symmetric monoidal model category. Let $U$ be a monoid (i.e. a unital, associative algebra) in $C$.

Let $I$ be the set of generating cofibrations and $J$ be the set of generating trivial cofibrations.

Assume that the following conditions hold: The domains of the morphisms in $I$ are small relative to $(U \otimes I)$-cell. The domains of the morphisms in $J$ are small relative to $(U \otimes J)$-cell. Every morphism in $(U \otimes J)$-cell is a weak equivalence. (If these conditions hold, we say the cofibrantly generated monoidal model category $C$ fulfills the monoid axiom for $U$.)

The following theorem is proved in [6].

Theorem 2. There is a cofibrantly generated model structure on the category of left $U$-modules, where a morphism of left $U$-modules is a weak equivalence (respectively fibration) if and only if it is a weak equivalence (respectively fibration) in $C$. The set of generating cofibrations is $U \otimes I$, the set of generating acyclic cofibrations is $U \otimes J$.

We should remark that Hinich provides us in [5] with a different proof for the existence of a model structure on a category of modules over an algebra.

Definition 47. In the situation of the previous theorem, we say that the category of left $U$-modules admits a model structure over $C$.

Let $C$ be a cofibrantly generated symmetric monoidal abelian model category that is a Grothendieck category. Assume that $C$ fulfills the monoid axiom. Let $O$ be an operad in $C$ and let $A$ be an algebra over $O$.

Example 40. The category of modules over $A$ admits a model structure over $C$ as it is isomorphic to the category of left $U(A)$-modules.

4.5. The category of modules over an algebra. In this subsection, we apply the previous theorem to the case of modules over an algebra over an operad.

Let $C$ be a cofibrantly generated closed symmetric model category. Let $I$ be the set of generating cofibrations and $J$ be the set of generating acyclic cofibrations of $C$. Let $O$ be an operad in $C$. Let $A$ be an $O$-algebra.

Consider the monoid $U(A)$ in $C$. Assume that $C$ satisfies the monoid axiom for $U(A)$. Recall that the free module functor $F_A$ is just tensoring by $U(A)$.

Example 41. The category of $A$-modules becomes cofibrantly generated model category, where a morphism of $A$-modules is a weak equivalence (respectively a fibration) if and only if it is a weak equivalence (respectively a fibration) in $C$. The set of generating cofibrations is $F_A(I)$, the set of generating acyclic cofibrations is $F_A(J)$.

4.6. The category of modules over a ringed space as an example. Let $(X, O_X)$ be a ringed space with finite hereditary global dimension. The tensor product over $O_X$ makes the category of cochain complexes of $O_X$-modules a closed
symmetric monoidal category, which is by a result of Hovey (see [8]) in fact a closed symmetric monoidal model category.

**Proposition 10.** The category of $O_X$-modules satisfies the monoid axiom for any monoid.

*Proof.* Let $J$ be the set of generating acyclic cofibrations of the category $X^*$ of cochain complexes of $O_X$-modules. We have to show that transfinite compositions of pushouts of morphisms in $U \otimes J$ are quasiisomorphism for any object $U$ in $X^*$. As cohomology commutes with filtered colimits, we just have to consider pushouts of morphisms in $U \otimes J$. That these are quasiisomorphisms can be verified stalk-wise. However, stalk-wise, the morphisms in $J$ (which are given in [8]) are by definition either isomorphisms of exact complexes of free modules or inclusions of the zero complex into exact complexes of free modules. Thus stalk-wise, the morphisms in $U \otimes J$ are either isomorphisms or inclusions of the zero complex into exact complexes. Pushouts of such morphisms are quasiisomorphisms. □

Let $U$ be any monoid in the category of $O_X$-modules. As any object in the Grothendieck category of cochain complexes of $O_X$-modules is small and this category satisfies the monoid axiom, we arrive at the following example due to the theorem in the previous subsection:

**Example 42.** The category of (left) $U$-modules is a cofibrantly generated model category, where a morphism of left $U$-modules is a weak equivalence if and only if it is a quasiisomorphism, and where a morphism of left $U$-modules is a fibration if and only if it is a degree-wise surjection with degree-wise flabby kernel.

4.7. **Fibrant replacements.** Let $\mathcal{C}$ be a model category with terminal object $\ast$.

By the axioms of a model structure, every morphism $f : X \to \ast$ in $\mathcal{C}$ can be decomposed functorially as $f = pi$ where $i : X \to RX$ is an acyclic cofibration and $p : RX \to \ast$ is a fibration. We call $RX$ a (the) fibrant replacement for $X$. This leads to the following definition:

**Definition 48.** A fibrant replacement functor $R : \mathcal{C} \to \mathcal{C}$ is part of a natural transformation $\eta : \text{id} \Rightarrow R$ such that each natural morphism $X \to RX$ is a weak equivalence and each $RX$ is a fibrant object for all objects $X$ in $\mathcal{C}$

**Example 43.** As we have seen above, each model category comes with a canonical fibrant replacement functor.

The main reason why one considers other fibrant replacement functors is the presence of a tensor product and the question of compatibility.

Assume that $\mathcal{C}$ is a closed symmetric monoidal model category with a fibrant replacement functor $R$.

The following definition is from [2].

**Definition 49.** The fibrant replacement functor $R$ is symmetric monoidal if it is symmetric monoidal as a functor of symmetric monoidal categories and the natural diagrams

\[
\begin{array}{ccc}
X \otimes Y & \rightarrow & RX \otimes RY \\
\downarrow & & \downarrow \Rightarrow \\
RX \otimes SY & \rightarrow & R(X \otimes Y),
\end{array}
\]

natural in $X$ and $Y$, commute.

**Remark 16.** By adjunction, every symmetric monoidal fibrant replacement functor $R$ in $\mathcal{C}$ defines morphisms $\text{hom}(X, X) \to \text{hom}(RX, RX)$, natural in $X$. 

The following proposition yields plenty of examples of closed symmetric monoidal categories with a fibrant replacement functor $R$.

**Proposition 11.** Let $C$ be cofibrantly generated with $J$ being the set of generating acyclic cofibrations. Assume that $J \Box J$ is a subset of $J$-cell. Then $C$ admits a symmetric monoidal fibrant replacement functor.

**Proof.** The proof makes use of Quillen’s small object argument. In fact, we start with the standard proof of constructing a functorial factorisation into a acyclic cofibration and a fibration.

Let $J$ be the set of generating acyclic cofibrations. Let $\kappa$ be a cardinal such that every domain of the morphisms in $J$ is $\kappa$-small. Let $\lambda$ be a limit ordinal with $\text{cofin} \lambda > \kappa$.

For each object $X$ in $C$, we shall naturally define a functor $(X_\beta)_{\beta < \lambda} : \lambda \to C$ inductively as follows: We set $X_0 := X$. For each ordinal $\beta < \lambda$ let $X_{\beta + 1}$ be the simultaneous pushout of all diagrams of the form

$$
\begin{array}{ccc}
A & \to & X_{\beta} \\
\downarrow & & \downarrow \\
B & \to & X_{\beta + 1},
\end{array}
$$

where the vertical arrow runs through all morphisms in $J$ and the horizontal morphism through all morphisms into $X_{\beta}$. For a limit ordinal $\beta < \lambda$, we set $X_\beta := \varinjlim_{\beta' < \beta} X_{\beta'}$. Finally set $RX := \varinjlim_{\beta < \lambda} X_{\beta}$.

The natural morphism $X \to RX$ is in $J$-cell and thus a weak equivalence by the axioms of a cofibrantly generated model category. We have to show that the morphism $RX \to *$ has the right lifting property with respect to morphisms in $J$-cell, which means that $RX \to *$ is a fibration. (In fact, we shall construct natural lifts.) To show this, consider a pushout diagram

$$
\begin{array}{ccc}
A & \to & Z \\
j & \downarrow & \downarrow \\
B & \to & Z',
\end{array}
$$

with $j \in J$. Assume that there is given a morphism $Z \to RX$. We want to extend this map to $Z'$. The composition $A \to Z \to RX$ factors by the $\kappa$-smallness of $A$ through some $X_\beta$. By construction, this morphism $A \to X_\beta$ extends naturally to a morphism $B \to X_{\beta + 1}$. Passing to the colimit shows that the morphism $A \to RX$ extends naturally to a morphism $B \to RX$. By definition of the pushout, this defines an extension $Z' \to RX$ of $Z \to RX$. Now, every morphism in $J$-cell is a filtered colimit of such pushouts $Z \to Z'$. Thus $RX \to *$ has the right lifting property with regard to morphisms in $J$-cell.

It remains to construct natural morphisms $RX \otimes RY \to R(X \otimes Y)$. By construction and the fact that pushout products of maps in $J$ are in $J$-cell, the morphism $X \otimes Y \to RX \otimes RY$ is in $J$-cell, so by the above considerations, there exists a natural lift $RX \otimes RY \to R(X \otimes Y)$ making the diagram

$$
\begin{array}{ccc}
X \otimes Y & \to & R(X \otimes Y) \\
\downarrow & & \downarrow \\
RX \otimes RY & \to & R(X \otimes Y)
\end{array}
$$

commutative. □
Remark 17. Assume that pushouts and filtered colimits of monomorphisms are monomorphisms. Then the symmetric monoidal fibrant replacement functor constructed in the proof above maps monomorphisms to monomorphisms.

For example, using this proposition, the symmetric monoidal model category of cochain complexes of $\mathcal{O}_X$-modules over a topological space of finite hereditary global dimension admits a symmetric monoidal fibrant replacement functor.

The remark holds for example true in the category of cochain complexes $\mathcal{O}_X$-modules considered in the previous paragraph.

4.8. The homotopy category. Let $\mathcal{C}$ be a model category. By $\text{Ho}\mathcal{C}$ we denote the "category" which is the localisation of $\mathcal{C}$ by all weak equivalences, i.e. we formally invert all weak equivalences in $\mathcal{C}$. The word "category" is printed in quotes as it is a priori not clear if $\text{Ho}\mathcal{C}$ is really a (locally small) category, i.e. if each class of morphisms between two objects is in fact a set. In this section, we give one argument for the well-known fact that this is true.

Let $s : X' \rightarrow X$ be a weak equivalence and $f : X' \rightarrow Y$ be any morphism in $\mathcal{C}$. This defines a morphism $fs^{-1} : X \rightarrow Y$ in $\text{Ho}\mathcal{C}$.

Let $QX \rightarrow X$ be a cofibrant replacement for $X$, i.e. $QX$ is a cofibrant object and $QX \rightarrow X$ is an acyclic fibration. Let $Y \rightarrow RY$ be a fibrant replacement for $Y$, i.e. $RY$ is a fibrant object and $Y \rightarrow RY$ is an acyclic cofibration. Further let us decompose the morphism $s : X' \rightarrow X$ as $s = p \circ i$ where $i : X' \rightarrow \tilde{X}$ is an acyclic cofibration and $p : \tilde{X} \rightarrow X$ is an acyclic fibration. By functoriality of the fibrant replacement functor, the morphism $f : X' \rightarrow Y$ induces a morphism $Rf : RX' \rightarrow RY$ where $RX'$ is the fibrant replacement of $X'$. By two lifting properties due to the axioms of a model structure there exist dashed arrows making the diagram

\[
\begin{array}{ccc}
RX' & \overset{Rf}{\rightarrow} & RY \\
\nearrow & & \nearrow \\
X' & \overset{f}{\rightarrow} & Y \\
\downarrow & & \downarrow \\
X & \overset{s}{\rightarrow} & X
\end{array}
\]

commutative. By this construction, the morphism $fs^{-1}$ can be written as the composition of the formal inverse of $Y \rightarrow RY$, a morphism $QX \rightarrow RY$ in $\mathcal{C}$ (!), and the formal inverse of $QX \rightarrow X$. We call the result of this construction a standard representation of the morphism $fs^{-1}$ in $\text{Ho}\mathcal{C}$. Generally, we shall call a standard representation of $fs^{-1}$ a representation of $fs^{-1}$ of the form

\[
X \rightarrow Q^M X \rightarrow R^N Y \rightarrow Y,
\]

where the first arrow and third arrow are given by the inverses (in $\text{Ho}\mathcal{C}$!) of the arrows given by the cofibrant and fibrant replacement and the second arrow is a morphism in $\mathcal{C}$.

Lemma 1. Let $t : Y \rightarrow Y'$ be a weak equivalence and $g : X \rightarrow Y'$ any morphism in $\mathcal{C}$. Assume that $Y$ is fibrant. Then there exists a weak equivalence $s : \tilde{X} \rightarrow X$ and a morphism $f : \tilde{X} \rightarrow Y$ such that $t^{-1}g = fs^{-1}$ in $\text{Ho}\mathcal{C}$.
Proof. Consider the pullback diagram

\[ \begin{array}{ccc}
X' & \downarrow s & Y \\
\downarrow \tilde{X} & & \downarrow \tilde{Y} \\
X & \downarrow t & Y' \\
\end{array} \]

made up from the outer solid arrows, where \( X' \to \tilde{X} \) is a cofibration, \( Y \to \tilde{Y} \) is an acyclic cofibration, \( \tilde{X} \to X \) and \( \tilde{Y} \to Y' \) are acyclic fibrations and where the composition \( Y \to Y' \) is the given morphism \( t \). The dashed line making the diagram commutative exists due to the left lifting property of cofibrations with respect to acyclic fibrations. Let \( s \) be the acyclic fibration given by \( \tilde{X} \to X \). As \( Y \) is fibrant by assumption, the left lifting property of acyclic cofibrations with respect to fibrations yields an left inverse \( j : \tilde{Y} \to Y \) of the acyclic cofibration \( \tilde{Y} \to Y \). Finally let \( f : \tilde{X} \to Y \) be the composition of \( \tilde{X} \to \tilde{Y} \) with this inverse \( j \).

\[ \square \]

Lemma 2. Let \( QX \to X \) be a cofibrant replacement of an object \( X \) and \( Y \to RY \) be a fibrant replacement of an object \( Y \) in \( C \). Let \( X \to Y \) be a morphism in \( \text{Ho}C \). Then it can be written as a composition of the formal inverse of \( QX \to X \), a proper morphism \( QX \to RY \) in \( C \), and the formal inverse of \( Y \to RY \).

Proof. Any morphism \( QX \to RY \) in \( \text{Ho}C \) can be written as a composition \( f_n \circ s_n^{-1} \circ \cdots \circ f_1 \circ s_1^{-1} \) where the \( f_i \) are proper morphisms in \( C \) and the \( s_i \) are weak equivalences in \( C \). We may assume that all intermediate objects are fibrant. By the previous lemma, this morphism can also be written as a composition \( f \circ s^{-1} \) where \( f \) is any morphism in \( C \) and \( s \) is a weak equivalence in \( C \). Finally by the construction above, the composition \( f \circ s^{-1} \) is given by a single morphism \( \tilde{f} : QX \to RY \) in \( C \).

Thus \( \text{Ho}C \) is in fact a category.

Definition 50. The category \( \text{Ho}C \) is the homotopy category of \( C \).

5. The operadic Atiyah class

5.1. Cofibrations. Let \( C \) be a cofibrantly generated closed symmetric monoidal model category that satisfied the monoid axiom for any monoid. Let \( \mathcal{O} \) be an operad in \( C \). Let \( A \) be an \( \mathcal{O} \)-algebra.

Let \( f_1 : M_1 \to N_1, \ldots, f_n : M_n \to N_n \) be morphisms of \( A \)-modules. Let

\[
\begin{align*}
P_{A, \Box}(f_1, \ldots, f_n) : P_A(N_1, M_2, \ldots, M_n) \coprod_{P_A(M_1, \ldots, M_n)} P_A(M_1, N_2, \ldots, N_n) & \to P_A(N_1, \ldots, N_n)
\end{align*}
\]

be the induced morphism of \( A \)-modules.

Let \( g : X \to Y \) be another morphism of \( A \)-modules. Let

\[
\begin{align*}
H_{A, \Box}(f_2, \ldots, f_n, g) : H_A(N_2, \ldots, N_n; R) & \to H_A(M_2, \ldots, M_n; R) \times_{H(M_2, \ldots, M_n; S)} H_A(N_2, \ldots, N_n; S)
\end{align*}
\]
be the canonical morphism of $A$-modules. By adjunction there is a one-to-one correspondence between diagrams of $A$-modules of the form

$$
\begin{array}{ccc}
P_A(M_1, N_2, \ldots, N_n) \amalg_{P_A(M_1, \ldots, M_n)} P_A(N_1, M_2, \ldots, M_n) & \to & R \\
\downarrow & & \downarrow \\
P_A(N_1, \ldots, N_n) & \to & S
\end{array}
$$

and

$$
\begin{array}{ccc}
M_1 & \to & H_A(N_2, \ldots, N_m; R) \\
\downarrow & \Downarrow & \downarrow \\
N_1 & \to & H_A(M_2, \ldots, M_n; R) \times H_A(M_2, \ldots, M_n; S) H_A(N_2, \ldots, N_n; S).
\end{array}
$$

The following lemma has been inspired by [8]. The proof is based on an idea from that article.

**Lemma 3.** Let $I_1, \ldots, I_n$ and $K$ classes of morphisms of $A$-modules. Assume that for all $i \in I_i$ it is $P_{A, □}(i_1, \ldots, i_n)$ a morphism in $K$. Then $P_{A, □}(f_1, \ldots, f_n)$ is a morphism in $K$-cof whenever each $f_i$ is a morphism in $I_i$-cof.

**Proof.** By assumption and adjointness, the morphisms in $I_1$ have the left lifting property with respects to morphisms of the form $H_{A, □}(i_2, \ldots, i_n; k)$ with $i_j \in J_j$ where $k$ is in $K$-inj. Thus every morphism $f_1$ in $I_1$-cof has the left lifting property with respects to morphisms of the form $H_{A, □}(i_2, \ldots, i_n; k)$ where $k$ is in $K$-inj. Applying adjointness again, we see that morphisms of the form $P_{A, □}(f_1, i_2, \ldots, i_n)$ are morphisms in $K$-cof. Then one repeats analogous arguments to replace successively each $i_j$ by an $f_j$ in $I_j$-cof, $j \geq 2$. □

Recall the free module functor $F_A$.

**Lemma 4.** Let $I$ be the set of generating cofibrations of $C$ and $J$ be the set generating acyclic cofibrations of $C$. Without loss of generality, we may assume that $J \subset I$. Then $P_{A, □}(F_A(i_1), \ldots, F_A(i_n))$ is a cofibration of $A$-modules for all $i_j \in I$ which is acyclic if there is a $k \in \{1, \ldots, n\}$ with $i_k \in J$.

**Proof.** It is $P_{A, □}(F_A(i_1), \ldots, F_A(i_n)) = F_A(i_1 □ (i_2 \otimes \cdots \otimes i_n))$. Now use that $C$ is a monoidal model category that satisfies the monoid axiom. □

Recall the generating (acyclic) cofibrations of the model structure on the category of $A$-modules.

**Proposition 12.** Let $f_1, \ldots, f_n$ be cofibrations of $A$-modules. Then $P_{A, □}(f_1, \ldots, f_n)$ is a cofibration which is acyclic if one of the $f_i$ is acyclic.

**Proof.** This follows directly from putting the previous two lemmata together. □

### 5.2. Fibrations

Let $C$ be a cofibrantly generated closed symmetric model category that satisfies the monoid axiom and has a symmetric monoidal fibrant replacement functor $R : C \to C$. Let $O$ be an operad in $C$. Let $A$ be an $O$-algebra.

Let $M$ be a module over $A$. The $A$-operation on $M$ is given by a morphism $U(A) \to \text{hom}(M, M)$. By composition with the natural morphism $\text{hom}(M, M) \to \text{hom}(RM, RM)$, this defines an $A$-operation on $RM$. This makes $RM$ naturally an $A$-module, i.e. the functor $R$ on $C$ induces an endofunctor $R$ on the category of $A$-modules, also equipped with a natural transformation from the identity endofunctor to itself.
Proposition 13. The so-defined functor $R$ from the category of $A$-modules into itself is a fibrant replacement functor on the model category of $A$-modules.

Proof. This follows at once from the fact that the forgetful functor from the category of $A$-modules to $C$ reflects fibrations and weak equivalences. \qed

In a certain sense, this fibrant replacement functor is compatible with the lax tensor products: Let $M_1, \ldots, M_n$ be $A$-modules.

Remark 18. As the fibrant replacement functor $R : C \to C$ is symmetric monoidal, there exist natural commutative diagrams of the form

$$
P_A(M_1, \ldots, M_n) \overset{\cong}{\longrightarrow} P_A(RM_1, \ldots, RM_n) \longrightarrow RP_A(M_1, \ldots, M_n)
$$

of $A$-modules. Moreover, these morphisms are compatible with the coherence diagrams involving the functors $P_A$ that have been mentioned in the previous section.

5.3. Symmetric monoidal model categories of cochain complexes. Let $C$ be a bicomplete closed symmetric monoidal Grothendieck category. Let the category of cochain complexes $C^*$ over $C$ be endowed with the structure of a monoidal model category. We make the following further assumptions:

- The weak equivalences in $C^*$ are exactly the quasiisomorphisms.
- The monoidal model structure on $C^*$ satisfies the monoid axiom for any monoid.
- The property of a morphism between two cochain complexes being a fibration can be tested degree-wise.
- There exists a symmetric monoidal fibrant replacement functor $R : C^* \to C^*$.
- The translation functor on the category of cochain complexes $C^*$ respects all of the structure.

Definition 51. Under these assumptions, we call the category $C^*$ a symmetric monoidal model category of cochain complexes for short.

As our own interest lies in examples coming from geometry, the following is important for us:

Example 44. Let $(X, \mathcal{O}_X)$ be a ringed space with finite hereditary dimension. The category cochain complexes of $\mathcal{O}_X$-modules is canonically a symmetric monoidal category of cochain complexes.

Remark 19. The axioms imposed on the symmetric monoidal model category of cochain complexes on $C^*$ imply in particular that there is a symmetric monoidal model structure on the category of modules over an algebra over an operad in $C^*$ that possesses a fibrant replacement functor that is compatible with the lax tensor products.

5.4. Extension classes. Let $C$ be a symmetric monoidal model category of cochain complexes. Let

$0 \longrightarrow E' \overset{f}{\longrightarrow} E \longrightarrow E'' \longrightarrow 0$

be a short exact sequence of objects in $C^*$. Recall the definition of the cone cone $f$ of $f$: It is $(\text{cone } f)^n = E'^{n+1} \oplus E''^n$ and the differential on $(\text{cone } f)^n$ is given by $-d_{E'}^{n+1} + f + d_E^n$. The morphism $E \to E''$ induces (via the projection $\text{cone } f \to E$,
which is itself not a map of complexes) a quasiisomorphism \(s : \text{cone } f \to E''\). There is further a canonical projection morphism \(p : \text{cone } f \to E'[1]\).

Let \(R : \mathcal{C} \to \mathcal{C}\) be a fibrant replacement functor. Then \(\text{cone}(Rf)\) is a fibrant object.

**Definition 52.** The extension class associated to the short exact sequence above is the morphism \(ps^{-1} : E'' \to E'[1]\) in \(\text{Ho } \mathcal{C}\).

Recall the standard representation of the morphism \(ps^{-1} : E'' \to E''\): Let \(QE'' \to E''\) be a cofibrant replacement of \(E''\) and \(E' \to RE'\) be a fibrant replacement of \(E'\) (which makes \(E'[1] \to RE'[1]\) a fibrant replacement of \(E'[1]\)). Then dashed arrows exist making the diagram

\[
\begin{array}{cccccc}
\text{E''} & \longrightarrow & \text{E''} \\
\text{\text{cone } f} & \leftarrow & \text{\text{cone } f} \\
\text{\text{cone}(Rf)} & \longrightarrow & \text{RE'[1]} \\
\end{array}
\]

commutative, where the \(\text{cone } f \to \text{cone } f\) is a cofibration and \(\text{cone } f \to E''\) is an acyclic fibration. The extension class associated to the short exact sequence above is given by the composition \(\hat{\alpha}\) of the formal inverse of \(QE'' \to E''\), the composition \(QE'' \to RE'[1]\) of the dashed arrows with the morphism \(\text{cone}(Rf) \to RE'[1]\) and the formal inverse of \(E'[1] \to RE'[1]\). By abuse of notion we often call the morphism \(\alpha : QE'' \to RE'[1]\) the extension class associated to the short exact sequence above.

5.5. **Definition of the Atiyah class.** Let \(\mathcal{C}\) be a symmetric monoidal model category of cochain complexes.

Let \(\mathcal{O}\) be an operad in \(\mathcal{C}\) and let \(A\) be an \(\mathcal{O}\)-algebra. Let \(d : \mathcal{O} \to M\) be a derivation where \(M\) is supposed to be a cofibrant \(A\)-module. We make this assumption (which can be seen as a kind of “smoothness” assumption), to simplify things in what follows.

Let \(E\) be any cofibrant \(A\)-module. Recall that there is a short exact sequence

\[
\begin{array}{cccccc}
0 & \longrightarrow & P_A(M, E) & \longrightarrow & J_dE & \longrightarrow & E & \longrightarrow 0 \\
\end{array}
\]

of \(A\)-modules, the \(d\)-jet module sequence of \(E\).

**Definition 53.** The extension class \(\alpha_E : E \to P_A(M, E)[1]\) of the \(d\)-jet module sequence of \(E\) is the (operadic) \(d\)-Atiyah class of \(E\).

**Remark 20.** By the considerations of the previous subsection, the extension class is represented by a morphism \(E \to RP_A(M, E)[1]\) in \(\mathcal{C}\) composed with the formal inverse of the natural morphism \(P_A(M, E)[1] \to RP_A(M, E)[1]\).

**Example 45.** Let \(A\) be a unital, commutative algebra in \(\mathcal{C}\), which we also view as \(\text{Com}\)-algebra. Then the definition of the operadic Atiyah class above corresponds to the usual definition of the Atiyah class of a cofibrant module over a unital commutative algebra.

This example is considered in [9] where \(\mathcal{C}\) is the category of \(\mathcal{O}_X\)-modules over a smooth complex manifold \(X\).
5.6. A stabilised category. Let $C$ be a symmetric monoidal model category of cochain complexes. Let $R$ be the symmetric monoidal fibrant replacement functor of $C^*$.

Let $X$ and $Y$ be two objects in $C^*$. Thanks to the natural transformations $R^nY \to R^{n+1}Y$, there is a natural sequence

$$
\hom(X, Y) \to \hom(X, RY) \to \hom(X, R^2Y) \to \ldots
$$

of sets. In particular, we can form the colimit $\lim_{\to n \in \mathbb{N}_0} \hom(X, R^nY)$.

As the morphisms $R^nY \to R^{n+1}Y$ are weak equivalences, i.e. induce isomorphisms in the homotopy category, there is a well-defined natural map

$$
\lim_{\to n \in \mathbb{N}_0} \hom(X, R^nY) \to \text{Ho}\mathcal{C}(X, Y).
$$

**Definition 54.** An element in the colimit $\lim_{\to n \in \mathbb{N}_0} \hom(X, R^nY)$ is a **stable morphism** from $X$ to $Y$. Stable morphisms are denoted by arrows of the form $X \leftarrow Y$.

**Remark 21.** There is a canonical way to compose two stable morphisms $X \leftarrow Y$ and $Y \leftarrow Z$ to a stable morphism $X \leftarrow Z$. Thus, one may form the category of stable morphisms of $C^*$ that has the same objects as $C^*$.

Let $O$ be an operad in $C^*$, and let $A$ be an $O$-algebra. The induced fibrant replacement functor on the model category of $A$-modules is also denoted by $R$.

**Remark 22.** Using this functor, one can also define stable morphisms of $A$-modules and the category of stable morphisms of $A$-modules. There is a faithful forgetful functor from the category of stable morphisms of $A$-modules to the category of stable morphisms of $C^*$.

5.7. Free connections. Let $C$ be a symmetric monoidal model category of cochain complexes.

Let $O$ be an operad in $C$ and let $A$ be an $O$-algebra. Let $d : O \to M$ be a derivation where $M$ is supposed to be a cofibrant $A$-module.

Let $E$ be any cofibrant $A$-module. A stable free morphism $\nabla : E \to P_A(M, E)$ of cochain complexes over $C$ is called **freely $d$-derivative** if the canonical diagrams

$$
\begin{array}{ccc}
\mathcal{O}(n+1) \otimes A^n \otimes E & \xrightarrow{id \otimes (\sum_{p+q=n} \text{id} \otimes d \circ \text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \circ \text{id})} & \mathcal{O}(n+1) \otimes A^{\otimes(n-1)} \otimes M \otimes E \\
E & \xrightarrow{\nabla} & 0 \oplus \mathcal{O}(n+1) \otimes A^{\otimes n} \otimes P_A(M, E)
\end{array}
$$

commute.

**Definition 55.** A **free $d$-connection on** $E$ is a freely $d$-derivative stable morphism $\nabla : E \to P_A(M, E)$.

**Remark 23.** Note that a free $d$-connection is in spite of its name not a special case of a $d$-connection.

However, every $d$-connection induces naturally a free $d$-connection as follows:

**Example 46.** Any $d$-connection $\nabla_0 : E \to P_A(M, E)$ on $E$ composed with the natural stable morphism $P_A(M, E) \to P_A(M, A)$ gives a free $d$-connection on $E$. 
Let $f : P_A(M, E) \to J_d E$ be the canonical morphism. Consider the diagram

$$\begin{array}{c}
\text{cone } Rf \\
\text{cone } f,
\end{array}$$

which is a small part of the big diagram in the previous subsection about general extension classes. The dashed arrow $\tilde{\alpha}$ making the diagram commutative exists as $E$ is by assumption already cofibrant.

Recall that up to the differential (!)

$$\text{cone } Rf = RP_A(E, A)[1] \oplus RE \oplus RP_A(E, A).$$

Thus the morphism $\tilde{\alpha}$ given above, can be decomposed as $\tilde{\alpha} = (\beta, i, \nabla) : E \to RP_A(E, A)[1] \oplus RE \oplus RP_A(E, A)$. In particular, $\beta$ is the extension class associated to the $d$-jet module sequence of $E$, and $i : E \to RE$ is the natural acyclic cofibration. As $\tilde{\alpha}$ is a morphism of $A$-modules, it follows that the component $\nabla$ defines a free $d$-connection on $E$.

Thus we have proven the following proposition:

**Proposition 14.** There exists a free $d$-connection on $E$. 

\[\square\]

This proposition has a kind of inverse.

**Proposition 15.** Let $\nabla : E \to P_A(M, E)$ be a free $d$-connection. Then

$$-[\bar{\partial}, \nabla] := \nabla \circ \bar{\partial} - \bar{\partial} \circ \nabla : E \to P_A(E)[1]$$

is a standard representation of the $d$-Atiyah class of $E$.

**Proof.** Assume that the stable morphism $\nabla$ is induced by a morphism $\nabla : E \to R^N P_A(M, E)$ for some $N \gg 0$. Set $\beta := -[\bar{\partial}, \nabla]$. The morphism $\tilde{\alpha} = (\beta, i, \nabla) : E \to \text{cone } R^N f$, where $i : E \to R^N E$ is the natural acyclic cofibration, is a morphism of $A$-modules and makes the diagram

$$\begin{array}{c}
\text{cone } R^N f \\
\text{cone } f
\end{array}$$

commutative. By definition of the extension class of the $d$-jet module sequence, the proposition follows. \[\square\]

5.8. **Derivations of morphisms.** Let $C$ be a symmetric monoidal model category of cochain complexes.

Let $O$ be an operad in $C$ and let $A$ be an $O$-algebra. Let $d : A \to M$ be a derivation where $M$ is supposed to be a cofibrant $A$-module.

Let $E_1, \ldots, E_m$ be cofibrant $A$-modules. Let $\nabla_i : E_i \to P_A(M, E_i)$ be free $d$-derivations. There is a free stable morphism

$$\nabla : P_A(E_1, \ldots, E_m) \to P_A(M, E_1, \ldots, E_m)$$
of cochain complexes over $C$ which is induced by the canonical morphisms

$$\mathcal{O}(n + m) \otimes A^{\otimes n} \otimes E_1 \otimes \cdots \otimes E_m$$

$$\oplus_{j=1}^{m} P_A(E_1, \ldots, E_{j-1}, P_A(M, E_j), E_{j+1}, \ldots, E_m)$$

$$P_A(M, E_1, \ldots, E_m)$$

**Definition 56.** The free stable morphism

$$\nabla : P_A(E_1, \ldots, E_m) \rightarrow P_A(M, E_1, \ldots, E_m)$$

of cochain complexes is denoted by $(\nabla_1, \ldots, \nabla_m)$.

**Remark 24.** It is easy to see that $\nabla$ as defined above, behaves similar to a free $d$-connection, i.e. it makes the natural diagram

$$\mathcal{O}(n + 1) \otimes A^n \otimes P_A(E_1, \ldots, E_m) \xrightarrow{\nabla} P_A(M, E_1, \ldots, E_m)$$

commutative.

Let $E$ be another $A$-module and $\nabla : E \rightarrow P_A(M, E)$ a free $d$-connection.

Let $f : E \rightarrow P(A, E_1, \ldots, E_m)$ be a stable morphism of $A$-modules. Note that $f$ induces naturally a stable morphism $P_A(id, f) : P_A(M, E) \rightarrow P_A(M, E_1, \ldots, E_m)$ of $A$-modules.

Define the free stable morphism

$$\nabla f := [\nabla, f] := (\nabla_1, \ldots, \nabla_m) \circ f - (id, f) \circ \nabla : E \rightarrow P_A(M, E_1, \ldots, E_m).$$

**Definition 57.** The morphism $\nabla f$ is the derivative of $f$ (with respect to $\nabla$ and $\nabla_1, \ldots, \nabla_m$).

### 5.9. The free $A$-algebra

Now we are ready to come to the central subsections. The main object of our studies will be the free algebra over a module and derivations on it.

Let $\mathcal{C}$ be a symmetric monoidal model category of cochain complexes. Let $\mathcal{O}$ be an operad in $\mathcal{C}$ and let $A$ be an $\mathcal{O}$-algebra.

Let $d : A \rightarrow M$ be a derivation where $M$ is supposed to be a cofibrant $A$-module.

Let $\nabla : M \rightarrow P_A(M, M)$ be a free $d$-derivation. As detailed in the previous subsection, it induces free stable morphisms $\nabla : P^n_A(M) \rightarrow P^{n+1}_A(M), n \geq 0$.

It is easy to see that there exist unique free stable morphisms $\nabla : S^n_A(M) \rightarrow S^{n+1}_A(M)$ such that the canonical diagrams

$$P^n_A(M) \xrightarrow{\nabla} P^{n+1}_A(M)$$

$$S^n_A(M) \xrightarrow{\nabla} S^{n+1}_A(M)$$
commute. Here, the vertical arrows denote the canonical projections onto the coinvariants.

A free stable morphism \( Q : S^*_A(M) \to 0 S^*_A(M) \) is freely derivative over \( d \) if \( Q \) is a free stable derivation and the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{d} & M \\
\downarrow & & \downarrow \\
S^*_A(M) & \xrightarrow{Q} & S^*_A(M)
\end{array}
\]

commutes.

**Definition 58.** A free stable morphism \( Q : S^*_A(M) \to 0 S^*_A(M) \) that is freely derivative over \( d \) is a free derivation over \( d \).

An example of such a free derivation over \( d \) is given as follows. First note that the free stable morphisms \( \nabla : S^n_A(M) \to 0 S^{n+1}_A(M) \) as constructed above altogether induce a free stable morphism \( Q_\nabla : S^*_A(M) \to 0 S^*_A(M) \). It easily follows that:

**Proposition 16.** The morphism \( Q_\nabla : S^*_A(M) \to 0 S^*_A(M) \) is a free derivation over \( d \). \( \square \)

Let \( E \) be another cofibrant \( A \)-module. Let \( \nabla_E : E \to 0 P_A(M, E) \) be a free \( d \)-connection.

By analogous considerations as above, there exist unique free stable morphisms \( \nabla_E : S^n_A(M, E) \to 0 S^{n+1}_A(M, E) \) such that the canonical diagrams

\[
\begin{array}{ccc}
P^n_A(M, E) & \xrightarrow{\nabla_E} & P^{n+1}_A(M, E) \\
\downarrow & & \downarrow \\
S^n_A(M, E) & \xrightarrow{\nabla_E} & S^{n+1}_A(M, E)
\end{array}
\]

commute. Again, the vertical arrows denote the canonical projections onto the coinvariants.

Let \( Q : S^*_A(M) \to 0 S^*_A(M) \) be a free derivation over \( d \). A free stable morphism \( D : S^*_A(M, E) \to 0 S^*_A(M, E) \) is freely \( Q \)-derivative over \( \nabla_E \) if \( D \) is a free \( Q \)-connection and the canonical diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\nabla_E} & P_A(M, E) \\
\downarrow & & \downarrow \\
S^*_A(M, E) & \xrightarrow{D} & S^*_A(M, E)
\end{array}
\]

commutes.

**Definition 59.** A free stable morphism \( D : S^*_A(M, E) \to 0 S^*_A(M, E) \) that is freely \( Q \)-derivative over \( \nabla_E \) is a free \( Q \)-derivation over \( \nabla_E \).

Again we can easily construct an example: By \( D_\nabla \) we denote the free stable morphism \( S^*_A(M, E) \to 0 S^*_A(M, E) \) induced by the morphisms \( \nabla_E : S^n(M, E) \to 0 S^{n+1}(M, E) \).

**Proposition 17.** The free stable morphism \( D_\nabla : S^*_A(M, E) \to 0 S^*_A(M, E) \) is a free \( Q_\nabla \)-connection over \( \nabla_E : E \to 0 P_A(M, E) \).
5.10. The total curvature form. Let $C$ be a symmetric monoidal model category of cochain complexes. We assume that $C$ is $\mathbb{Q}$-linear, i.e. that the hom-sets are $\mathbb{Q}$-vector spaces.

Let $O$ be an operad in $C$ and let $A$ be an $O$-algebra. Let $d : O \to M$ be a derivation where $M$ is supposed to be a cofibrant $A$-module.

Let $\nabla : M \to P_A(M,M)$ be a free stable $d$-connection. Set

$$R_{\nabla} := \exp(ad Q_{\nabla}) \bar{\partial} = \sum_{n=0}^{\infty} \frac{1}{n!}(ad Q_{\nabla})^n(\bar{\partial}) : S^*_A(M) \to S^*_A(M)[1],$$

where, as always, $\bar{\partial} : S^*_A(M) \to S^*_A(M)[1]$ denotes the differential of $S^*_A(M)$. We note that $R_{\nabla}$ is a stable morphism and a stable derivation of $S^*_A(M)$ of degree one over $d$.

**Definition 60.** The stable morphism $R_{\nabla} : S^*_A(M) \to S^*_A(M)[1]$ is the total curvature form of $M$ associated to $\nabla$.

In fact, $R_{\nabla}$ is a differential:

**Proposition 18.** One has $R_{\nabla} \circ R_{\nabla} = \frac{1}{2}[R_{\nabla}, R_{\nabla}] = 0$.

**Proof.** This follows from $[R_{\nabla}, R_{\nabla}] = [\exp(ad Q_{\nabla}) \bar{\partial}, \exp(ad Q_{\nabla}) \bar{\partial}] = \exp(ad Q_{\nabla})[\bar{\partial}, \bar{\partial}] = 0$. \hfill $\square$

Denote by $R^{(n)}_{\nabla} : M \to S^*_A(M)[1], n \geq 0$ be those stable morphisms such that the diagrams

$$\begin{array}{ccc}
M & \xrightarrow{R^{(n)}_{\nabla}} & S^*_A(M) \\
\downarrow & & \downarrow \\
S^*_A(M) & \xrightarrow{R_{\nabla}} & S^*_A(M)
\end{array}$$

commute.

**Example 47.** We have $R^{(0)}_{\nabla} = 0$ and $R^{(1)}_{\nabla} = \bar{\partial}$, the differential of the complex $M$. Furthermore, we have

$$R^{(2)}_{\nabla} = -[\bar{\partial}, \nabla] : M \to S^*_A(M),$$

i.e. $R^{(2)}_{\nabla}$ represents the symmetrisation of the Atiyah class of $M$ in the homotopy category of $A$-modules.

Let $E$ be a cofibrant $A$-module and $\nabla_E : E \to P_A(M,E)$ be a free stable $d$-connection. Set

$$T_{\nabla} := \exp(ad D_{\nabla}) \bar{\partial} = \sum_{n=0}^{\infty} \frac{1}{n!}(ad D_{\nabla})^n(\bar{\partial}) : S^*_A(M,E) \to S^*_A(M,E)[1],$$

where $\partial : S^*_A(M,E) \to S^*_A(M,E)[1]$ denotes the differential of $S^*_A(M)$. We note that $T_{\nabla}$ is a stable derivation of $S^*_A(M,E)$ of degree one over $d$.

**Definition 61.** The morphism $T_{\nabla} : S^*_A(M,E) \to S^*_A(M,E)[1]$ is the total curvature form of $E$ associated to $\nabla$ and $\nabla_E$.

In fact, $T_{\nabla}$ is a differential. The proof goes as in the case of $R_{\nabla}$.

**Proposition 19.** One has $T_{\nabla} \circ T_{\nabla} = \frac{1}{2}[T_{\nabla}, T_{\nabla}] = 0$.
Denote by $T^{(n)}_\nabla : M \to S^n_A(M,E)[1]$, $n \geq 0$ those stable morphisms such that the diagrams
\[
\begin{array}{ccc}
E & \xrightarrow{T^{(n)}_\nabla} & S^n_A(M,E) \\
\downarrow & & \downarrow \\
S^*_A(M,E) & \xrightarrow{T_\nabla} & S^*_A(M,E)
\end{array}
\]
commute.

**Example 48.** We have $T^{(0)}_\nabla = \bar{\partial}$, the differential of the complex $M$. Furthermore, we have $T^{(1)}_\nabla = -[\bar{\partial}, \nabla] : M \to P_A(M,E)$, i.e. $T^{(1)}_\nabla$ represents the Atiyah class of $E$ in the homotopy category of $A$-modules.

We owe the idea to define the total curvature classes $R_\nabla$ and $T_\nabla$ the article [9] by Kapranov, where this is done in the classical geometrical context.

5.11. **Bianchi identity.** Let $\mathcal{C}$ be a $\mathcal{Q}$-linear symmetric monoidal model category of cochain complexes.

Let $\mathcal{O}$ be an operad in $\mathcal{C}$ and let $A$ be an $\mathcal{O}$-algebra. Let $d : \mathcal{O} \to M$ be a derivation where $M$ is supposed to be a cofibrant $A$-module.

Let $\nabla : M \to P_A(M,M)$ be a free stable $d$-connection. Let $\alpha : M \to S^2_A(M)[1]$ be the (symmetrised) Atiyah class of $M$ (i.e. it is a morphism in the homotopy category). It induces a morphism $\hat{\alpha} : S^2_A(M) \to S^3_A(M)[1]$ such that the canonical diagram (in the homotopy category)
\[
\begin{array}{ccc}
P_A(M,M) & \xrightarrow{P_A(\alpha, \text{id}) + P_A(\text{id}, \alpha)} & P_A(M,M)[1] \\
\downarrow & & \downarrow \\
S^2_A(M) & \xrightarrow{\hat{\alpha}} & S^3_A(M)[1]
\end{array}
\]
commutes.

The following proposition is known under the name “homological Bianchi identity” in the classical, i.e. non-operadic, case. See [9].

**Proposition 20.** It is
\[
\hat{\alpha} \circ \alpha : M \to S^3_A(M)[2]
\]
the zero morphism in the homotopy category.

This is called the Bianchi identity of the Atiyah class.

**Proof.** A representative of $\alpha$ is the free stable morphism $R^{(2)}_\nabla : M \to S^2_A(M)$. Let $\tilde{R}^{(2)}_\nabla : S^2(M) \to S^3_A(M)$ be the free stable morphism such that the canonical diagram
\[
\begin{array}{ccc}
P_A(M,M) & \xrightarrow{P_A(\tilde{R}^{(2)}_\nabla, \text{id}) + P_A(\text{id}, \tilde{R}^{(2)}_\nabla)} & P_A(S^2_A(M),M) \\
\downarrow & & \downarrow \\
S^2_A(M) & \xrightarrow{\tilde{R}^{(2)}_\nabla} & S^3_A(M)[0]
\end{array}
\]
commutes, i.e. $\tilde{R}^{(2)}_\nabla$ is a representative of $\hat{\alpha}$. 


From $[R_{\nabla}, R_{\nabla}] = 0$ we deduce $\hat{R}_{\nabla}^{(2)} \circ R_{\nabla}^{(2)} = [\partial, R_{\nabla}^{(3)}]$ by looking at the terms of low degree. Thus the left hand side, which gives $\hat{\alpha} \circ \alpha$ in the homotopy category, is null-homotopic, which means that it is the zero morphism in the homotopy category.

We can do the same for the Atiyah class of a cofibrant $A$-module $E$. Let $\nabla_E : M \to P(M, E)$ be a free stable $d$-connection. Let $\alpha_E : M \to P_A(M, E)$ be the Atiyah class of $E$. Together with (the non-symmetrised version of) $\alpha_E$ it is induces a morphism $\hat{\alpha}_E : S^1(M, E) \to S^2_A(M, E)[1]$ such that the canonical diagram

\[
\begin{array}{ccc}
P_A(M, E) & \xrightarrow{P_A(\alpha, \text{id}) + P_A(\text{id}, \alpha_E)} & P_A(M, M, E)[1] \\
\downarrow & & \downarrow \\
S^1_A(M, E) & \xrightarrow{\hat{\alpha}_E} & S^2_A(M, E)[1]
\end{array}
\]

commutes.

Again we have a cohomological Bianchi identity:

**Proposition 21.** It is $\hat{\alpha}_E \circ \alpha_E : M \to S^2_A(M, E)[2]$ the zero morphism in the homotopy category.

5.12. **An example.** Let $\mathcal{C}$ be a $\mathbb{Q}$-linear symmetric monoidal model category of cochain complexes.

Let $\mathcal{O}$ be an operad in $\mathcal{C}$ and let $V$ be a cofibrant object in $\mathcal{C}$. Consider the free $\mathcal{O}$-algebra $A := F_{\mathcal{O}}(V)$.

Recall that any free morphism $g : V \to [1]$ induces a free derivation $\hat{g} : A \to A[1]$.

**Definition 62.** The morphism $g$ is a solution of the Maurer–Cartan equation for $A$ if and only if

$$ [\bar{\partial} + \hat{g}, \bar{\partial} + \hat{g}] = 0, $$

where $\bar{\partial}$ is the differential on $A$.

In other words, $g$ is a solution if and only if $\bar{\partial} + \hat{g}$ defines a new differential on $A$. In that case, we denote by $A(g)$ the $\mathcal{O}$-algebra we get when we substitute its differential $\bar{\partial}$ by $\bar{\partial} + \hat{g}$. We consider $A(g)$ as a deformation of $A$. It is not a free algebra anymore.

An example of such a solution is given by a (strong homotopy) Lie coalgebra structure on $V[1]$, see, e.g., [10].

**Definition 63.** Two solutions $g_0$ and $g_1$ of the Maurer–Cartan equation for $A$ are gauge equivalent if and only if they appear simultaneously in a family $g(t)$ of solutions with

$$ \hat{g}'(t) = [\hat{\xi}(t), \bar{\partial} + \hat{g}(t)] $$

for a family $\xi(t)$ of morphisms $V \to A$ (that induces a family $\hat{\xi}(t)$ of derivations $A \to A$).

The free $A$-module $M := F_A(V)$ is a cofibrant $\mathcal{O}$-module by definition of the model structure on the category of $A$-modules as $V$ is cofibrant.

Denote by $d : A \to M$ the derivation of $A$ into $M$ that is induced by the identity map $V \to V$. If $g$ is a solution of the Maurer–Cartan equation for $A$, we denote by $M(g)$ the object $M$ with the unique differential $\delta(g)$ such that it becomes an $A(g)$-module and $d$ a morphism from $A(g)$ to $M(g)$. 
Recall further that we have a canonical $d$-connection $\nabla : M \to P_A(M, M)$. This is a free $d$-connection of $A(g)$-modules when viewed as a morphism $\nabla : M(g) \to P_A(M(g), M(g))$.

Thus we can consider its total curvature form $R(g) : M(g) \to S_A(M(g))$. Let $R(g) : M(g) \to S_A(M(g))$ be the restriction of $\bar{R}(g)$ to $M(g)$ considered as a free morphism of $A(g)$-modules. As $[R(g), R(g)] = 0$, we have the following proposition:

**Proposition 22.** The morphism $R(g) : M(g) \to S_A(M(g))$ viewed a free morphism $R(g) : M \to S_A(M)$ is a solution of the Maurer–Cartan equation for $S_A(M)$, i.e. $[\bar{\partial}_0 + \bar{R}(g), \bar{\partial}_0 + \bar{R}(g)] = 0$.

In fact, we have the following:

**Proposition 23.** Let $g_1$ and $g_2$ be gauge equivalent solutions of the Maurer–Cartan equation for $A$. Then $R(g_1)$ and $R(g_2)$ are gauge equivalent.

**Proof.** Let $g(t)$ be a family of solutions connecting $g_1$ and $g_2$ with $\dot{g}(t) = [\xi(t), \bar{\partial} + \hat{g}(t)]$. Then $\bar{R}(g(t)) = [\exp(\nabla \xi(t), \bar{\partial}_0 + \bar{R}(g(t)))].$ □

**Remark 25.** Thus, what we have defined is a canonical map from the set of solutions of the Maurer–Cartan equation for $A$ up to gauge equivalence to set of solutions of the Maurer–Cartan equation for $S_A(M)$ up to gauge equivalence.

**Appendix A. Proofs**

**A.1. Smallness in Grothendieck categories.** In this appendix we give a simple proof for the following theorem:

**Theorem 3.** Any object in a Grothendieck category is small.

**Proof.** Let $A$ be an object in a Grothendieck category $C$. Let $G$ be a generator of $C$. Let $\mathfrak{A}$ be the set of subobjects of $A$ and set $\hat{G} := \bigoplus_{A' \in \mathfrak{A}} G$. Let $\kappa$ be the sum of the cardinality of the set $\mathfrak{A}$ of subobjects of $A$ and the cardinality of the set of subobjects of $G$. We show that $A$ is $\kappa$-small. Let $\lambda$ be an ordinal with cofinal $\lambda > \kappa$ and let $X : \lambda \to C$ be a colimit-preserving functor. We have to show that the natural map $\varinjlim_{\mu < \lambda} \text{hom}(A, X_\mu) \to \text{hom}(A, \varinjlim_{\mu < \lambda} X_\mu)$ is an isomorphism.

First, we show the surjectivity. Let $f : A \to X_\lambda := \varinjlim_{\mu < \lambda} X_\mu$ be any morphism in $C$. We have to show that $A$ factors through some $X_\beta$ for $\beta < \lambda$. By dividing out the kernel if necessary, we may assume that $f$ is a monomorphism. For each subobject $A'$ of $A$ there is a $\beta < \lambda$ such that the map from the fibre product $X_\beta \times_{X_\lambda} A \to X_\lambda \to X_\lambda/f(A')$ is not the zero map if $A'$ is a proper subobject of $A$. As the cofinality of $\lambda$ is greater than the cardinality of the set of subobjects of $A$, there is a $\beta < \lambda$ such that $X_\beta \times_{X_\lambda} A \to X_\lambda \to X_\lambda/f(A')$ is not the zero map for each proper subobject $A'$ of $A$. For each subobject $A'$ there is a morphism $G \to A$ and a morphism $G \to X_\gamma$ such that $G \to A \to X_\lambda$ equals $G \to X_\gamma \to X_\lambda$ and such that $G \to A \to X_\lambda \to X_\lambda/f(A')$ is non-zero if $A'$ is a proper subobject of $A$. These morphisms induce by the universal property of the direct sum $\hat{G}$ a morphism $\hat{G} \to A$ and a morphism $\hat{G} \to X_\beta$ such that $\hat{G} \to A \to X_\lambda$ equals $\hat{G} \to X_\beta \to X_\lambda$. Let $A'$ be the image of the morphism $\hat{G} \to A$. Suppose that $A'$ is a proper subobject of $A$. Then $A \to X_\lambda \to X_\lambda/f(A')$ is not the zero map. In particular, $\hat{G} \to A \to X_\lambda \to X_\lambda/f(A')$ is not the zero map which contradicts the fact that the image of $\hat{G} \to A$ is $A'$. Thus $A'$ cannot be a proper subobject of $A$, thus $A' = A$, i.e. $\hat{G}$ is an epimorphism. Let us denote the kernel of $\hat{G} \to A$ be $K$. For each ordinal $\gamma$ with $\beta \leq \gamma < \lambda$ denote the kernel of $\hat{G} \to X_\beta \to X_\gamma$ by $K_\gamma.$
As $A \to X_\lambda$ is a monomorphism, the kernel $K_\gamma$ is a subobject of $K$ for all ordinals $\gamma$ with $\beta \leq \gamma < \lambda$. Consider the filtered colimit over $\lambda$ of the left exact sequence $0 \to K_\gamma \to \hat{G} \to X_\gamma$. As $\hat{G}$ is a Grothendieck category, this yields the exact sequence $0 \to \lim_{\gamma < \lambda} K_\gamma \to \hat{G} \to X_\gamma$, which shows that $\lim_{\gamma < \lambda} K_\gamma = K$. As the cofinality of $\lambda$ is greater than the cardinality of the set of subobjects of $\hat{G}$ the sequence $(K_\gamma)_{\gamma < \lambda}$ stabilises, i.e. there is a $\gamma < \lambda$ with $K_{\gamma} = \lim_{\gamma < \lambda} K_\gamma = K$. Thus the morphism $\hat{G} \to X_\gamma$ induces a morphism $\hat{G}/K \to X_\gamma$ and this morphism composed with the inverse of the isomorphism $\hat{G}/K \to A$ yields a morphism $A \to X_\gamma$ such that $A \to X_\gamma \to X_\lambda$ equals $f$.

We now show the injectivity. Let $A \to X_{\beta_1}$ and $A \to X_{\beta_2}$ with $\beta_1, \beta_2 < \lambda$ be two morphisms such that $A \to X_{\beta_1} \to X_\lambda$ and $A \to X_{\beta_2} \to X_\lambda$ are equal. We have to show that there is an ordinal $\gamma$ with $\beta_1, \beta_2 \leq \gamma < \lambda$ such that $A \to X_{\beta_1} \to X_\gamma$ and $A \to X_{\beta_2} \to X_\gamma$ are equal. For this, let $K_\gamma$ be the equaliser of $A \to X_{\beta_1} \to X_\gamma$ and $A \to X_{\beta_2} \to X_\gamma$ for any ordinal $\gamma$ with $\beta_1, \beta_2 \leq \gamma < \lambda$. Consider the filtered colimit over the exact sequence $0 \to K_\gamma \to A \to X_\gamma$. As $\hat{G}$ is a Grothendieck category, this yields the exact sequence $0 \to \lim_{\gamma < \lambda} K_\gamma \to A \to X_\gamma$, which shows that $\lim_{\gamma < \lambda} K_\gamma = 0$. As the cofinality of $\lambda$ is greater than the set of subobjects of $A$, the sequence $(K_\gamma)_{\gamma < \lambda}$ stabilises. In particular, there is a $\gamma < \lambda$ with $K_{\gamma} = \lim_{\delta < \lambda} K_\delta = 0$. Thus $A \to X_{\beta_1} \to X_\gamma$ and $A \to X_{\beta_2} \to X_\gamma$ are equal. 

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