Integrability of geodesic flows for metrics on suborbits of the adjoint orbits of compact groups

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Abstract
Let $G/K$ be an orbit of the adjoint representation of a compact connected Lie group $G$, $\sigma$ be an involutive automorphism of $G$ and $\tilde{G}$ be the Lie group of fixed points of $\sigma$. We find a sufficient condition for the complete integrability of the geodesic flow of the Riemannian metric on $\tilde{G}/(\tilde{G}\cap K)$, which is induced by the bi-invariant Riemannian metric on $\tilde{G}$. The integrals constructed here are real analytic functions, polynomial in momenta. It is checked that this sufficient condition holds when $G$ is the unitary group $U(n)$ and $\sigma$ is its automorphism defined by the complex conjugation.

Introduction

Let $G/K$ be a homogeneous space of a compact Lie group $G$. We consider the problem of the complete integrability of the geodesic flow of the Riemannian metric on $G/K$, which is induced by a bi-invariant Riemannian metric on $G$. This problem was solved for some types of homogeneous manifolds, including symmetric spaces, spherical spaces, Stiefel manifolds, flag manifolds, orbits of the adjoint actions and others (see [Mi], [GS], [My2], [MS], [BJ1], [BJ3], [MP]). Here we consider a new family of homogeneous manifolds – suborbits of orbits of the adjoint actions.

This paper is motivated by the paper [DGJ], where were constructed integrable geodesic flows of $\tilde{G}$-invariant metrics on the homogeneous space $\tilde{G}/\tilde{K} = SO(n)/(SO(k_1) \times SO(k_2) \times \cdots \times SO(k_r))$, $k_1 + k_2 + \cdots + k_r \leq n$. The method of the proof in [DGJ] is based on investigations of bi-Poisson structures on the Lie algebras $\mathfrak{u}(n)$ and $\mathfrak{so}(n)$ associated with Lie algebra deformations. We consider the Lie-algebraic aspects of the integrability problem for such homogeneous spaces. Our approach is based on the following observation: the space $\tilde{G}/\tilde{K}$ is a $G$-suborbit of the adjoint orbit $G/K = U(n)/(U(k_1) \times U(k_2) \times \cdots \times U(k_r))$ of the Lie algebra $\mathfrak{u}(n)$ of the unitary group, i.e. $\tilde{G}/\tilde{K} = \text{Ad}(\tilde{G})(a)$, where $a \in \mathfrak{u}(n)$.

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and \( G/K = \text{Ad}(G)(a) \). Moreover, \( \tilde{G} \) is the group of fixed points of the involutive automorphism \( \sigma \) of \( U(n) \) induced by the complex conjugation. In other words, the space \( G/K \) is defined uniquely by the pair \((G/K, \sigma)\), where \( G/K = \text{Ad}(G)(a) \) is an arbitrary adjoint orbit of the Lie group \( G \) in its Lie algebra \( g \) with \( a \in (1 - \sigma_\ast)g \), \( \sigma_\ast \) is the tangent automorphism of the Lie algebra \( g \).

Let \( G \) be an arbitrary compact connected Lie group \( G \) with an involutive automorphism \( \sigma : G \to G \) and let \( \tilde{G} \) be the set of fixed points of \( \sigma \). In the article we investigate the integrability of the geodesic flow on the cotangent bundle \( T^\ast(\tilde{G}/\tilde{K}) \) defined by a \( \tilde{G} \)-invariant metric on \( \tilde{G}/\tilde{K} \), which is induced by a bi-invariant Riemannian metric on \( \tilde{G} \). As a homogeneous space \( \tilde{G}/\tilde{K} \) we consider the homogeneous space associated with the adjoint orbit \( G/K = \text{Ad}(G)(a) \) of arbitrary point \( a \in (1 - \sigma_\ast)g \), i.e. \( \tilde{K} = \tilde{G} \cap K \). We found a sufficient purely algebraic condition for the integrability of this geodesic flow on the symplectic manifold \( T^\ast(\tilde{G}/\tilde{K}) \) (Theorem 2.17). We prove that this sufficient condition holds when \( G \) is the unitary group \( U(n) \) and \( \sigma \) is its automorphism defined by the complex conjugation (Theorem 2.18). Our approach is based on the fact that \( G/K \subset G/K \) is a totally real (Lagrangian) submanifold of the homogeneous Kähler manifold \((G/K, \sigma)\) (the compact orbit) \( G/K \) and \( T(G/K) \subset T(G/K) \) is a totally real submanifold of \( T(G/K) \). But to simplify calculations we reformulate this fact in some algebraic terms (not explicitly, since explicit reformulation is very complicated from the point of view of calculations on \( T(T(G/K)) \)).

One calls a Hamiltonian system on \( T^\ast M \) (completely) integrable if it admits a maximal number of independent integrals in involution, i.e. \( \dim M \) functions commuting with respect to the Poisson bracket on \( T^\ast M \) whose differentials are independent in an open dense subset of \( T^\ast M \). By Liouville's theorem the integral curves of an integrable Hamiltonian system under a certain additional compactness hypothesis are quasiperiodic (are the orbits of a constant vector field on an invariant torus).

Let \( A^G \) be the set of all \( G \)-invariant real analytic functions on the cotangent bundle \( T^\ast M \) of \( M = G/K \). This space is an algebra with respect to the canonical Poisson bracket on the symplectic manifold \( T^\ast M \). The natural extension of the action of \( G \) on \( M \) to an action on the symplectic manifold \( T^\ast M \) is Hamiltonian with the moment mapping \( \mu^\text{can} : T^\ast M \to g^\ast \). The functions of type \( h \circ \mu^\text{can} : g^\ast \to \mathbb{R} \), are integrals for any \( G \)-invariant Hamiltonian flow on \( T^\ast M \), in particular, for the geodesic flow corresponding to any \( G \)-invariant Riemannian metric on \( M \). In general, a maximal involutive subset of \( \{ h \circ \mu^\text{can}, h : g^\ast \to \mathbb{R} \} \) is not a maximal involutive subset of the algebra \( C^\infty(T^\ast M) \). But for the compact Lie group \( G \) the problem of constructing of a maximal commutative set of real analytic functions on \( T^\ast(M/K) \) is reduced to the problem of a finding of a maximal commutative set of real analytic functions from the set \( A^G \) (see [My], [BJ], Lemma 3, [Pa]). This is true also for the group \( \tilde{G} \) and the corresponding algebra \( \tilde{A}^G \subset C^\infty(T^\ast(\tilde{G}/\tilde{K})) \).

The algebra of functions \( A^G \) on \( T^\ast(G/K) \), where, recall, \( G/K \) is an adjoint orbit, contains some maximal involutive subset \( F \) of \( A^G \) consisting of independent functions (MP Theorem 3.10), see also [BJ], [BJ1]. The homogeneous space \( \tilde{G}/\tilde{K} \), as we remarked above, is a submanifold of \( G/K \) and therefore \( T(\tilde{G}/\tilde{K}) \) is a submanifold of \( T(G/K) \). Moreover, \( T(\tilde{G}/\tilde{K}) \) is a symplectic submanifold of \( T(G/K) \), where the symplectic structures on these spaces are defined via iso-
morphisms $T(G/K) \simeq T^*(G/K)$ and $T(G/K) \simeq T^*(G/K)$ using a standard $G$-invariant metric on $G/K$ and its restriction to $G/K$ (see Proposition 1.14). The set $\tilde{\mathcal{F}} = \{ f \mid T(\tilde{G}/\tilde{K}), f \in \mathcal{F} \}$ of restrictions is an involutive subset of the algebra $A^\tilde{G}$.

This involutiveness of the functions from $\tilde{\mathcal{F}}$ is a consequence of the fact that $G/\tilde{G}$ is a symmetric space and follows easily from results published in [MF], [TF]. The following observation is crucial in our approach:

if the functions from the set $\mathcal{F}$ are independent at some point of the symplectic submanifold $T(\tilde{G}/\tilde{K}) \subset T(G/K)$, then the set $\tilde{\mathcal{F}}$ is a maximal involutive subset of the algebra $A^\tilde{G}$.

Therefore we describe explicitly some open dense subset $O$ of $T(G/K)$, where all functions from the set $\mathcal{F}$ are independent (Theorem 2.13) in the paper [MP] only the existence of such a set was proved). We prove that $O \cap T(\tilde{G}/\tilde{K}) \neq \emptyset$ if $G$ is the unitary group $U(n)$ and $\sigma$ is its automorphism defined by the complex conjugation (Theorem 2.18).

1 G-invariant bi-Poisson structures and moment maps

1.1 Some definitions, conventions, and notations

All objects in this paper are real analytic, $X$ stands for a connected manifold, $\mathcal{E}(X)$ for the space of real analytic functions on $X$.

We will say that some functions from the set $\mathcal{E}(X)$ are independent if their differentials are independent at each point of some open dense subset in $X$. For any subset $\mathcal{F} \subset \mathcal{E}(X)$ denote by $\dim_x \mathcal{F}$ the maximal number of independent functions from the set $\mathcal{F}$ at a point $x \in X$. Put $\dim_x \mathcal{F} \equiv \max_{x \in X} \dim_x \mathcal{F}$.

Let $\eta$ be a Poisson bi-vector on $X$ and let $\mathcal{A} \subset \mathcal{E}(X)$ be a Poisson subalgebra of $(\mathcal{E}(X), \eta)$, i.e. $\mathcal{A}$ is a vector space closed under the Poisson bracket $\{ , \}$ : $(f_1, f_2) \mapsto \eta(df_1, df_2)$ on $X$. Put $(DA)_x \equiv \{ df_x : f \in \mathcal{A} \} \subset T_x^* X$ for any $x \in X$. Let $B_x$ denote the restriction of $\eta_x$ to this subspace $(DA)_x$. We say that a subset $\mathcal{F} \subset \mathcal{A}$ is a maximal involutive subset of the algebra $(\mathcal{A}, \eta)$ if at each point $x$ of some open dense subset in $X$ the subspace $V_x = \{ df_x : f \in \mathcal{F} \} \subset (DA)_x$ is a maximal isotropic with respect to the form $B_x$, i.e. $B_x(V_x, V_x) = 0$ and $B_x(v, V_x) = 0$ for $v \in (DA)_x$ implies $v \in V_x$. In particular, any two functions $f_1, f_2 \in \mathcal{F}$ are in involution on $X$, i.e. $\{ f_1, f_2 \} = 0$.

**Definition 1.1.** A pair $(\eta_1, \eta_2)$ of linearly independent bi-vector fields (bi-vectors for short) on a manifold $X$ is called Poisson if $\eta^t \equiv t_1 \eta_1 + t_2 \eta_2$ is a Poisson bi-vector for any $t = (t_1, t_2) \in \mathbb{R}^2$, i.e. each bi-vector $\eta^t$ determines on $X$ a Poisson structure with the Poisson bracket $\{ , \}^t : (f_1, f_2) \mapsto \eta^t(df_1, df_2)$; the whole family of Poisson bi-vectors $\{ \eta^t \}_{t \in \mathbb{R}^2}$ is called a bi-Poisson structure.

A bi-Poisson structure $\{ \eta^t \}$ (we shall often skip the parameter space) can be viewed as a two-dimensional vector space of Poisson bi-vectors, the Poisson pair $(\eta_1, \eta_2)$ as a basis in this space.
The set $X_A$ of all points $x \in X$ for which $\dim_x A = \dim A$ is an open dense subset in $X$. But sometimes the exact description of this set is impossible or not constructive. Therefore we will consider some greater open subset $R_A \subset X$ containing $X_A$ and such that there exists a smooth subbundle $\mathbb{C}_x \subset T^*_x X$, $x \in R_A$ of dimension $\dim \mathbb{C}_x$ of the cotangent bundle $T^*(R_A)$, where $D_x = (DA)_x$ if $x \in X_A$.

Suppose that a linear subspace $A \subset E$ is a Poisson subalgebra of $(E(X), \eta)$ for each $t \in \mathbb{R}^2 \setminus \{0\}$. Let $B^*_x$ denote the restriction of $\eta_x$ to the subspace $D_x$, $x \in R_A$.

**Definition 1.2.** We say that the pair $(A, \{\eta^t\})$ is Kronecker at a point $x \in R_A \subset X$ if the linear space $\{B^*_x, t \in \mathbb{C}^2\}$ is two dimensional and $\text{rank}_{\mathbb{C}} B^*_x$ is constant with respect to $(t_1, t_2) \in \mathbb{C}^2 \setminus \{0\}$. We mean $B^*_t, t \in \mathbb{C}^2$, as the complex bilinear form $t_1 B^*_x(\cdot, 1) + t_2 B^*_x(\cdot, 1)$ on the complexification $D^*_x \subset (T^*x)^{\mathbb{C}}$. We say that $(A, \{\eta^t\})$ is micro-Kronecker if it is Kronecker at any point of some open dense subset in $X$.

It is evident that this (micro)definition is independent of the choice of this greater open dense subset $R_A \subset X$.

The definitions above are motivated by the following assertion of Bolsinov which is fundamental for our considerations.

**Proposition 1.3.** [35] Let $B_1$ and $B_2$ be two linearly independent skew-symmetric bilinear forms on a vector space $V$. Suppose that the kernel of each form $B^t = t_1 B_1 + t_2 B_2, t \in \mathbb{R}^2$, is non-trivial, i.e. $0 < r = \min_{t \in \mathbb{R}^2} \dim \ker B^t$. Put $T = \{t \in \mathbb{R}^2 : \dim \ker B^t = r\}$.

Then

1. the subspace $L \defeq \sum_{t \in T} \ker B^t$ is isotropic with respect to any form $B^t$, $t \in \mathbb{R}^2$, i.e. $B^t(L, L) = 0$;
2. the space $L$ is maximal isotropic with respect to any form $B^t$, $t \in T$, i.e.

$$
\dim L = \frac{1}{2}(r + \dim V) \text{ and } B^t(v, L) = 0, v \in V \implies v \in L,
$$

iff $\dim_{\mathbb{C}} \ker B^t = r$ for all $t \in \mathbb{C}^2 \setminus \{0\}$.

Suppose that the Poisson bi-vector $\eta$ on $X$ is non-degenerate. Then there exists a unique symplectic form $\omega$ on $X$ such that $\eta(df_1, df_2) = -\omega(\xi_{f_1}, \xi_{f_2})$, where $\xi_{f_i}, i = 1, 2$ are the Hamiltonian vector fields of the functions $f_i$ ($df_i = -\omega(\xi_{f_1}, \cdot)$). In other words, $\eta(\cdot, \cdot) = -\omega(\omega^{-1}(\cdot), \omega^{-1}(\cdot))$, where $\omega : TX \to T^*X$ is the natural isomorphism given by the contraction with the 2-form $\omega$ on the second index. Such a Poisson bi-vector $\eta$ will be denoted by $\omega^{-1}$.

Let $G$ be a connected Lie group acting on a symplectic manifold $(X, \omega)$ and preserving its symplectic structure $\omega$. Let $\mathfrak{g}$ be the Lie algebra of $G$. For each vector $\xi \in \mathfrak{g}$ denote by $\xi$ the fundamental vector field on $X$ generated by the one-parameter diffeomorphism group $\exp(t\xi) \subset G$. The group $G$ acts on the symplectic manifold $(X, \omega)$ in a Hamiltonian fashion if there is a $G$-equivariant map $\mu : X \to \mathfrak{g}^*$ such that for each $\xi \in \mathfrak{g}$ the field $\xi$ is the Hamiltonian vector field with the Hamiltonian function $f_\xi : X \to \mathbb{R}, x \mapsto \mu(x)(\xi)$, i.e. $df_\xi = -\omega(\xi, \cdot)$. The equivariance property $\mu(g^{-1}x)(\xi) = \mu(x)(\text{Ad}(g)\xi)$, where $g \in G, x \in X$, of the moment map $\mu$ implies the identity $\{f_\xi, f_\eta\} = f_{\{\xi, \eta\}}$, where $\xi, \eta \in \mathfrak{g}$ and $\{,\}
is the Poisson bracket associated with \( \omega^{-1} \). In other words, the mapping \( \mu \) is canonical with respect to the Poisson structure \( \omega^{-1} \) on \( X \) and the standard linear Poisson structure on \( \mathfrak{g}^* \). Moreover, by definition \( \{ f, h \circ \mu \} = 0 \) for any \( G \)-invariant function \( f \) on \( X \) and \( h \in \mathcal{E}(\mathfrak{g}^*) \).

Consider a connected Riemannian manifold \((M, g)\) and its connected Riemannian submanifold \((\tilde{M}, \tilde{g})\), where \( \tilde{g} = g|\tilde{M} \). The cotangent bundles \( T^* M \) and \( T^* \tilde{M} \) are symplectic manifolds with canonical symplectic structures \( \Omega \) and \( \tilde{\Omega} \) respectively. Using the metric \( g \) (resp. \( \tilde{g} \)) we can identify \( T^* M \) with \( T^* \tilde{M} \) (resp. \( T^* \tilde{M} \)) with \( T^* \tilde{M} \) ) the corresponding diffeomorphism. Let \( p : TM \to M \) (resp. \( \tilde{p} : T\tilde{M} \to \tilde{M} \)) be the natural projection and let \( \theta \) (resp. \( \tilde{\theta} \)) be the canonical 1-form on \( T^* M \) (resp. on \( T^* \tilde{M} \)).

**Proposition 1.4.** The symplectic manifold \((T\tilde{M}, \tilde{\varphi}^*\tilde{\Omega})\) is a symplectic submanifold of \((TM, \varphi^*\Omega)\), i.e. \( \tilde{\varphi}^*\Omega = \varphi^*\Omega|T\tilde{M} \). Moreover, \( \tilde{\varphi}^*\tilde{\theta} = \varphi^*\theta|T\tilde{M} \).

**Proof.** By definition, \( \theta_{x'}(Z') = x'((\pi_{x'}Z')) \), where \( \pi : T^* M \to M \) is the natural projection, \( x' \in T^*_q M \), \( q = \pi(x') \). Putting \( x' = \varphi(x) \in T^*_q M \) (to simplify notation) and taking into account that \( \pi \circ \varphi = p \) we obtain that for any \( Z \in T_q T M \)

\[
(\varphi^*\theta)x(Z) \overset{\text{def}}{=} \theta_{\varphi(x)}(\varphi_\ast xZ) \overset{\text{def}}{=} x'(\pi_{x'}(\varphi_\ast xZ)) = x'((\pi \circ \varphi)_\ast xZ) = x'(p_\ast xZ) = g_q(x, p_\ast xZ).
\]

Similarly, we obtain that

\[
(\tilde{\varphi}^*\tilde{\theta})\tilde{x}(\tilde{Z}) = \tilde{g}_{\tilde{q}}(\tilde{x}, \tilde{p}_\ast \tilde{Z}) \quad \text{for any} \quad \tilde{q} \in \tilde{M}, \; \tilde{x} \in T\tilde{q} \tilde{M}, \; \tilde{Z} \in T_{\tilde{x}} T\tilde{M}.
\]

In other words \( \tilde{\varphi}^*\tilde{\theta} = \varphi^*\theta|T\tilde{M} \), because \( \tilde{p}|T\tilde{M} = \tilde{p} \) and \( g|\tilde{M} = \tilde{g} \). Now to complete the proof it is sufficient to note that \( \Omega = d\theta \) and \( \tilde{\Omega} = d\tilde{\theta} \).

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**1.2 Hamiltonian actions on cotangent bundles and maximal involutive sets of functions**

Let \( G \) be a compact connected Lie group with a closed subgroup \( K \). Denote by \( \mathfrak{g} \) and \( \mathfrak{t} \) the Lie algebras of the Lie groups \( G \) and \( K \). Let \( \Omega \) be the canonical symplectic form on the cotangent bundle \( X = T^* M \), where \( M = G/K \). The natural action of \( G \) on \((X, \Omega)\) is Hamiltonian with the moment map \( \mu^{\mathrm{can}} : X \to \mathfrak{g}^* \). This equivariant moment map \( \mu^{\mathrm{can}} \) has the form \( \mu^{\mathrm{can}}(x)(\xi) = \theta(\xi_X)(x) \), where \( \theta \) is the canonical 1-form on \( X = T^*(G/K) \).

Since the canonical form \( \Omega \) is \( G \)-invariant, the set \( A^G_X \) of \( G \)-invariant function on \( X \) is a subalgebra of \((\mathcal{E}(X), \eta^{\mathrm{can}})\), \( \eta^{\mathrm{can}} = \Omega^{-1} \). As we remarked above, the moment map \( \mu^{\mathrm{can}} \) is a Poisson map and therefore the set \( \{ h \circ \mu^{\mathrm{can}}, h \in \mathcal{E}(\mathfrak{g}^*) \} \) is also a subalgebra of \((\mathcal{E}(X), \eta^{\mathrm{can}})\). The following assertion is known [My3, §2], [BJJ] Lemma 3], [Pa], but here we formulate it in terms of maximal involutive subsets of Poisson algebras:

**Proposition 1.5.** Suppose that there exist a set of functions \( F \subset A^G_X \) which is a maximal involutive subset of the algebra \((A^G_X, \eta^{\mathrm{can}})\). Then there is a set \( \mathcal{H} \) of
polynomial function on $g^*$ such that the set $\mathcal{F} \cup \{ h \circ \mu^\text{can}, h \in \mathcal{H} \}$ form maximal involutive set of independent functions on $T^*(G/K)$.

2 The integrability of geodesic flows

In this section for any compact Lie algebra $a$ by $z(a)$ we will denote its center and by $a_s$ its maximal semisimple ideal, i.e. $a = z(a) \oplus a_s$; for any real vector space or Lie algebra $a$ by $a^\mathbb{C}$ we will denote its complexification.

2.1 Commutator on $A^K_m$ induced by canonical Poisson structure on $T^*(G/K)$

Let $M = G/K$ be a homogeneous space of a compact connected Lie group $G$ with the Lie algebra $g$. There exists a faithful representation $\chi$ of $g$ such that its associated bilinear form $\Phi^\chi$ is negative-degenerate on $g$ (if $g$ is semi-simple we can take the Killing form associated with the adjoint representation of $g$). Let $m = k^\perp$ be the orthogonal complement to $k$ with respect to $\Phi^\chi$. Then $g = k \oplus m$, $[k, m] \subset m$.

The form $\langle \cdot, \cdot \rangle = -\Phi^\chi$ defines a $G$-invariant metric on $G/K$. This metric identifies the cotangent bundle $T^*(G/K)$ and the tangent bundle $T(G/K)$. Thus we can also talk about the canonical 2-form $\Omega$ on the manifold $T(G/K)$ (extension of the action of $G$ on $G/K$).

We can identify the tangent space $T_o(G/K)$ at the point $o = p(e)$ with the space $m$ by means of the canonical projection $p : G \to G/K$. Let $A^G$ (resp. $A^K_m$) be the set of all $G$-invariant (resp. $\text{Ad}(K)$-invariant) functions on $T(G/K)$ (resp. on $m$). There is a one-to-one correspondence between $G$-orbits in $T(G/K)$ and $\text{Ad}(K)$-orbits in $m$. Thus we can identify naturally the spaces of functions $A^G$ and $A^K_m$. For any smooth function $f$ on $m$ write $\text{grad}_m f$ for the vector field on $m$ such that

$$df(y) = \langle \text{grad}_m f(x), y \rangle \quad \text{for all} \quad y \in m.$$

The Poisson bracket of two functions $f_1, f_2$ from the set $A^K_m = A^G$ with respect to the canonical Poisson structure $\eta^\text{can}$ (determined by the canonical 2-form $\Omega$) has the form [MP] Lemma 3.1:

$$\{f_1, f_2\}^\text{can}(x) = -\langle x, [\text{grad}_m f_1(x), \text{grad}_m f_2(x)] \rangle, \quad x \in m. \quad (2.2)$$

Now, let us consider an important for our considerations subset of $m$. For any $x \in m$ define the subspace $m(x) \subset m$ putting

$$m(x) \overset{\text{def}}{=} \{ y \in m : [x, y] \in m \} = \{ y \in m : \langle y, \text{ad} x(t) \rangle = 0 \}, \quad (2.3)$$

in particular,

$$\text{ad} x(m(x)) \subset m \quad \text{and} \quad m(x) \oplus \text{ad} x(t) = m. \quad (2.4)$$
For any element $x \in \mathfrak{g}$ denote by $g^x$ its centralizer in $\mathfrak{g}$, i.e. the set of all $z \in \mathfrak{g}$ satisfying $[x, z] = 0$. Put $\mathfrak{k} \subseteq \mathfrak{g}^z \cap \mathfrak{k}$. Consider in $\mathfrak{m}$ a nonempty Zariski open subset:

$$R(\mathfrak{m}) = \{x \in \mathfrak{m} : \dim \mathfrak{g}^x = q(\mathfrak{m}), \dim \mathfrak{k} = p(\mathfrak{m})\},$$

where $q(\mathfrak{m})$ (resp. $p(\mathfrak{m})$) is the minimum of dimensions of the spaces $\mathfrak{g}^y$ (resp. $\mathfrak{t}^y$) over all $y \in \mathfrak{m}$. Put $r(\mathfrak{m}) = q(\mathfrak{m}) - p(\mathfrak{m})$. Remark that the number $p(\mathfrak{m})$ is determined only by Ad-representation of $\mathfrak{k}$ in $\mathfrak{m}$.

Let $(\cdot)_m$ be the projection in $\mathfrak{g}$ into $\mathfrak{m}$ along $\mathfrak{k}$. For each $x \in R(\mathfrak{m})$ the spaces $\mathfrak{m}(x)$ and $(\mathfrak{g}^x)_m \subset \mathfrak{m}(x)$ have the same dimensions

$$\dim \mathfrak{m}(x) = \dim \mathfrak{m} - (\dim \mathfrak{k} - p(\mathfrak{m})) \quad \text{and} \quad \dim((\mathfrak{g}^x)_m) = \dim \mathfrak{g}^x - \dim \mathfrak{k} = r(\mathfrak{m}).$$

Moreover, for each $x \in R(\mathfrak{m})$ the maximal semi-simple ideal $\mathfrak{g}_x^x = [\mathfrak{g}^x, \mathfrak{g}^x]$ of $\mathfrak{g}^x$ is contained in the algebra $\mathfrak{k}^x$, i.e.

$$\mathfrak{g}_x^x = [\mathfrak{g}^x, \mathfrak{g}^x] = [\mathfrak{k}^x, \mathfrak{k}^x] = \mathfrak{k}^x,$$

if $x \in R(\mathfrak{m})$, (2.6) (see [My3, Prop.10] or [Mi]). Therefore, $\dim(\mathfrak{g}^x/\mathfrak{k}^x) = \mathfrak{g} - \mathfrak{k}^x$, i.e.

$$\dim \mathfrak{g}^x = \mathfrak{g} - \mathfrak{k}^x + (\dim \mathfrak{k}^x - \dim \mathfrak{k}^x), \quad \text{if} \ x \in R(\mathfrak{m}).$$

It is clear that for any $f \in \mathcal{A}^K_m$ grad$_m f(x) \in \mathfrak{m}(x)$. Moreover, since the Lie group $K$ is compact, for each $x$ from some nonempty Zariski open subset of $\mathfrak{m}$ the space $\mathfrak{m}(x)$ is generated by vectors grad$_m f(x)$, $f \in \mathcal{A}^K_m$. Taking into account relation (2.2) and that $\dim V^0_x = r(\mathfrak{m})$ for $x \in R(\mathfrak{m})$, where

$$V^0_x \overset{\text{def}}{=} \{y \in \mathfrak{m}(x) : \langle y, [y, \mathfrak{m}(x)] \rangle = 0 \} = \{y \in \mathfrak{m}(x) : \langle [x, y], \mathfrak{m}(x) \rangle = 0 \}$$

$$= \{y \in \mathfrak{m}(x) : [x, y] \in \mathfrak{ad} x(\mathfrak{k}) \} = (\mathfrak{g}^x)_m \cap \mathfrak{m}(x) = (\mathfrak{g}^x)_m,$$

we obtain that the number

$$\frac{1}{2} (r(\mathfrak{m}) + \dim \mathfrak{m}(x)) = \frac{1}{2} (r(\mathfrak{m}) + \dim \mathfrak{m} - \dim \mathfrak{k} + p(\mathfrak{m}))$$

is a maximal number of functions in involution from the set $\mathcal{A}^K_m$ functionally independent at the point $x$.

For arbitrary element $x \in R(\mathfrak{m})$ we have [My3, Prop.9]

$$[\mathfrak{m}(x), \mathfrak{k}^x] = 0. \quad (2.8)$$

Remark 2.1. The subspace $\mathfrak{m}(x) \subset \mathfrak{m}$ in some sense characterizes the Ad-action of $K$ on $\mathfrak{m}$ because $\mathfrak{ad} K(\mathfrak{m}(x)) = \mathfrak{m}$ (for any $y \in \mathfrak{m}$ the function $f_y : k \mapsto \langle \mathfrak{ad} k(y), x \rangle$ on the compact Lie group $K$ takes its maximum value at some point $k_1 \in K$ and therefore $\mathfrak{ad} k_1(y) \perp \mathfrak{ad} x(\mathfrak{t})(\text{see [My2, Lemma 2.1]}))$. By the dimension arguments (see (2.7) from (2.8) and definition (2.5) we get that for $x \in R(\mathfrak{m})$ and each element $y \in \mathfrak{m}(x) \cap R(\mathfrak{m})$: $\mathfrak{t}^y = \mathfrak{k}^x$.

The compact Lie algebra $\mathfrak{k}$ is a direct sum $\mathfrak{k} = \mathfrak{z}(\mathfrak{k}) \oplus \mathfrak{k}$, of the center and of the semisimple ideal. The center $\mathfrak{z}(\mathfrak{k})$ of $\mathfrak{k}$ we will denote simply by $\mathfrak{z}$ for short. Then we have the following orthogonal splittings with respect to the invariant form $\langle \cdot, \cdot \rangle$

$$\mathfrak{t} = \mathfrak{z} \oplus \mathfrak{t}_s, \quad \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{z} \oplus \mathfrak{t}_s, \quad \mathfrak{m}_3 = \mathfrak{z} \oplus \mathfrak{m}, \quad \mathfrak{g} = \mathfrak{m}_3 \oplus \mathfrak{t}_s,$$
which serve as definition for $m_3$ (if $j = 0$ then $m_3 = m$ and $t_s = t$).

Consider the set $R(m_3)$ determined by (2.8) for the pair $(g, t_s)$. Then for any $z + x \in m_3$ such that $z \in m_3, x \in R(m)$ and $z + x \in R(m_3)$ we have $t_{s+x} = t_{s+x}^*$ because $[3, t] = 0$. But $t_s$ is the maximal semi-simple ideal of the compact Lie algebra (centralizer) $t^x$, and therefore $(\dim t^x - \text{rank } t^x) = (\dim t_{s+x}^* - \text{rank } t_{s+x}^*)$.

Thus by (2.7)

$$q(m) = \dim g^x = \text{rank } g + (\dim t^x - \text{rank } t^x) = \text{rank } g + (\dim t_{s+x}^* - \text{rank } t_{s+x}^*)$$

$$= \text{rank } g + (\dim t_{s+x}^* - \text{rank } t_{s+x}^*) = 0.$$ 

In other words, the following lemma is proved:

**Lemma 2.2.** The subset $R(m) \cap R(3 \oplus m) \subset m$ is a nonempty Zariski open subset of the vector space $m$ and this set coincides with the set

$$\{x \in m : \dim g^x = q(m), \ \dim t^x = \min_{y \in m} \dim t^y, \ \dim t_{s+x}^* = \min_{y \in m} \dim t_{s+x}^y\}. \quad (2.9)$$

Remark that in the last condition in relation (2.9) describing the set $R(m) \cap R(3 \oplus m)$ we choose $y \in m$.

The following proposition is Proposition 2.3 from [MP] adapted to the case of compact Lie algebras.

**Proposition 2.3.** [MP] Assume that $x_0 \in R(m)$. Let $g_0$ and $t_0$ be the centralizers of $t^{x_0}$ in $g$ and $t$ respectively. Let $m_0 = \{y \in g_0 : (y, t_0) = 0\}$. Then

1. $m_0 = g_0 \cap m$ (this set contains $x_0$ by definition);
2. for any $x \in m_0 \cap R(m)$ (this set contains $x_0$) we have $m_0(x) = m(x)$;
3. for any $x \in m_0 \cap R(m)$ we have $t^x = t^{x_0}$ and the centralizer $t_{s+x}^0$ is contained in the center $z(g_0)$ of the compact Lie algebra $g_0$.
4. any element $x \in m_0 \cap R(m)$ is a regular element of the compact Lie algebra $g_0$ and $x \in R(m_0)$ (i.e. $(m_0 \cap R(m)) \subset R(m_0)$).

**Remark 2.4.** Recall that a subalgebra $a \subset g$ is regular if its normalizer $N(a)$ in $g$ has maximal rank, i.e. rank $N(a) = \text{rank } a$. It is well known that the centralizer $g^x$ of the element $x \in R(m)$ is a regular subalgebra of $g$ of maximal rank (containing some Cartan subalgebra of $a$), and, consequently, rank $g = \dim g^{x_0} + \text{rank } g_{s+x_0}$, where recall $g^{x_0}$ is the center of $g^x$, $g_{s+x_0} = [g^x, g^x]$ is its maximal semisimple ideal and $g^x = g^{x_0} \oplus g_{s+x_0}$. In particular, the semisimple Lie algebra $g_{s+x_0}$ is a regular subalgebra of $g$.

**Corollary 2.5.** The Lie algebra $g_0$ is a regular subalgebra of $g$ and $g^{x_0} = g^{x_0}_0 \oplus t_{s+x_0}$ for any element $x$ from the nonempty Zariski open subset $m_0 \cap R(m) \subset m_0$. In particular, rank $g_0 = \text{rank } g - \text{rank } t^{x_0}$ and $r(m) = \text{rank } g_0 - \dim t_{s+x_0}^y$.

**Proof.** As we remarked above, $g^{x_0} = 3(g^x) \oplus g_{s+x_0}^x$ and $t^{x_0} = 3(t^x) \oplus t_{s+x_0}^x$. But by (2.8) $g_{s+x_0}^x$ coincides with the maximal semisimple ideal $t_{s+x_0}^x$ of $t^x$. Therefore $3(t^x) \subset 3(g^x)$ and, consequently, $[3(g^x), t^x] = 0$. But by item (3) above $t^x = t^{x_0}$. Thus $[3(g^x), t^{x_0}] = 0$.

In other words, $3(g^x) \subset g_0 \cap t^{x_0} = g_0$. The intersection $g_0 \cap t^{x_0} = 0$ vanishes because the Lie algebra $t_{s+x_0}^x$ is semisimple. Since

$$\text{rank } g_0^x = \text{rank } g_0 \leq (\text{rank } g - \text{rank } t^{x_0}) = (\text{rank } g - \text{rank } g_{s+x_0}^x) = \dim 3(g^x),$$
we obtain that $\mathfrak{z}(\mathfrak{g}^{\ast}) = \mathfrak{g}^{\ast}_{0}$ and rank $\mathfrak{g}_{0} = \text{rank } \mathfrak{g} - \text{rank } \mathfrak{g}^{\ast}_{0}$. The last equality means that $\mathfrak{g}_{0}$ is a regular subalgebra of $\mathfrak{g}$. By dimension arguments $\mathfrak{k}_{0} = \mathfrak{z}(\mathfrak{t}^{\ast})$. So that $\mathfrak{t}^{\ast}_{0} = \mathfrak{z}(\mathfrak{t}^{\ast})$ because $\mathfrak{t}^{\ast} = \mathfrak{t}^{\ast}_{0}$. 

2.2 The bi-Poisson structure $\{\eta^{i}(\omega_{O})\}$: exact formulas and involutive sets of functions

Consider the adjoint action Ad of $G$ on the Lie algebra $\mathfrak{g}$. Suppose now in addition that the Lie subgroup $K$ is an isotropy group of some element $a \in \mathfrak{g}$, i.e. $K = \{g \in G : \text{Ad } g(a) = a\}$ and $\mathfrak{t} = \mathfrak{g}^{\ast}$. Moreover, now by invariance of the form $\langle \cdot, \cdot \rangle$

$$\mathfrak{t} = \mathfrak{z} \oplus \mathfrak{t}_{a}, \quad a \in \mathfrak{z}, \quad \text{and } \text{ad } a(\mathfrak{m}) \subset \mathfrak{m}.$$ 

Using the invariant form $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g}$, we identify the dual space $\mathfrak{g}^{\ast}$ and $\mathfrak{g}$. So the orbit $O = G/K$ is a symplectic manifold with the Kirillov-Kostant-Souriau symplectic structure $\omega_{O}$. By definition the form $\omega_{O}$ is $G$-invariant and at the point $a \in O$ we have

$$\omega_{O}(a)([a, \xi_{1}], [a, \xi_{2}]) = -\langle a, [\xi_{1}, \xi_{2}] \rangle, \quad \forall \xi_{1}, \xi_{2} \in \mathfrak{g},$$

where we consider the vectors $[a, \xi_{1}], [a, \xi_{2}] \in T\_a \mathfrak{g} = \mathfrak{g}$ as tangent vectors to the orbit $O \subset \mathfrak{g}$ at the point $a \in O$. Let $\tau : TO \rightarrow O$ be the natural projection. Using the closed 2-form $\tau^{\ast} \omega_{O}$ on $TO$ (the lift of $\omega_{O}$) we construct a bi-Poisson structure on $TO$.

Consider on $TO$ two symplectic forms: $\omega_{1} = \Omega$ and $\omega_{2} = \Omega + \tau^{\ast} \alpha$. Write $\eta_{1} = \omega_{1}^{-1}, \eta_{2} = \omega_{2}^{-1}$ for the inverse Poisson bi-vectors. Then the family $\{\eta^{i}(\omega_{O}) = \eta^{i} = t_{1}\eta_{1} + t_{2}\eta_{2}, t_{1}, t_{2} \in \mathbb{R}\}$, is a bi-Poisson structure [MP Prop.1.6]. Putting $t_{2} = \lambda, t_{1} = 1 - \lambda, \lambda \in \mathbb{R}$ or $t_{1} = -1, t_{2} = 1$ we exclude a considering of proportional bi-vectors. The corresponding bi-vectors we denote by $\eta^{\lambda}, \lambda \in \mathbb{R}$ and $\eta^{a}$ (the singular bi-vector). The Poisson bracket of two functions $f_{1}, f_{2}$ from the set $A^{K}_{\mathfrak{m}} = A^{G} \subset \mathcal{E}(TO)$ with respect to the Poisson structure $\eta^{\lambda}, \lambda \in \mathbb{R}$ or $\eta^{a}$ has the form [MP Lemma 3.1]:

$$\{f_{1}, f_{2}\}^{\lambda}(x) = -\langle x + \lambda a, [\text{grad}_{\mathfrak{m}} f_{1}(x), \text{grad}_{\mathfrak{m}} f_{2}(x)]\rangle,$$

$$\{f_{1}, f_{2}\}^{a}(x) = -\langle a, [\text{grad}_{\mathfrak{m}} f_{1}(x), \text{grad}_{\mathfrak{m}} f_{2}(x)]\rangle. \quad (2.10)$$

Remark that the structure $\eta^{0} (\lambda = 0)$ is the canonical Poisson structure (see [22]). Since the set $R(\mathfrak{m}) \subset \mathfrak{m} = T_{o}(G/K)$ is Ad $K$-invariant, the set of G-orbits $G \cdot R(\mathfrak{m})$ in $X = T(G/K)$ is an open dense subset of $X$ such that its intersection with $\mathfrak{m} = T_{o}(G/K)$ is equal $R(\mathfrak{m})$. Let $X_{A^{G}}$ be the set of all points $x \in X = T(G/K)$ for which $\text{ddim} \_x A^{G} = \text{ddim} A^{G}$. It is clear that for any $f \in A^{K}_{\mathfrak{m}} \quad \text{grad}_{\mathfrak{m}} f(x) \in \mathfrak{m}(x)$. Moreover, since the Lie group $K$ is compact, for each $x$ from some nonempty Zariski open subset of $\mathfrak{m}$ the space $\mathfrak{m}(x)$ is generated by vectors $\text{grad}_{\mathfrak{m}} f(x), f \in A^{K}_{\mathfrak{m}}$, i.e. $\text{ddim} A^{G} = \text{dim } \mathfrak{m}(y), y \in R(\mathfrak{m})$. Thus we can choose as an open dense subset $R_{A^{G}} \subset X$ containing $X_{A^{G}}$ the set $G \cdot R(\mathfrak{m})$ (see definition in subsection 1.1). We will investigate points in $R(\mathfrak{m})$, where the pair $(A^{K}_{\mathfrak{m}}, \eta^{i})$ is Kronecker.

Let $x$ be an element of $R(\mathfrak{m}) = R_{A^{G}} \cap T_{o}(G/K)$, i.e an element of $\mathfrak{m}$ for which

$$\text{dim } \mathfrak{g}^{\ast} = q(\mathfrak{m}), \quad \text{dim } \mathfrak{t}^{\ast} = p(\mathfrak{m}), \quad \text{dim } \mathfrak{m}(x) = \text{dim } \mathfrak{m} - (\text{dim } \mathfrak{t} - p(\mathfrak{m})). \quad (2.11)$$
The bi-Poisson structure \( \{ \eta^t = \eta^t(\omega_G) \} \) determines at this point \( x \in m = T_o(G/K) \) the bilinear forms \( B^t_x : D_x \times D_x \to \mathbb{R} \), where recall that \( B^t_x \) is the restriction \( \eta^t|_{D_x} \) \( (D_x = (DA^t)^x) \) if \( x \in X_{AG} \subset R_{AG} \subset X \), see subsection [1.1]. Since we identified the spaces \( A^G \) and \( A^G_m \), \( B^t_x \) defines the following complex-valued bilinear forms (which we denote also by \( B^t_x, B^\lambda_x \) and \( B^{ei}_x \) for short) on \( m(x) \times m(x) \) [MP (3.11)]:

\[
\begin{align*}
B^t_x : (y_1, y_2) & \mapsto -((t_1 + t_2)x + t_2a, [y_1, y_2]), \ t_1, t_2 \in \mathbb{C}, \\
B^\lambda_x : (y_1, y_2) & \mapsto -(x + \lambda a, [y_1, y_2]), \ \lambda \in \mathbb{C}, \\
B^{ei}_x : (y_1, y_2) & \mapsto -a, [y_1, y_2]).
\end{align*}
\]

(2.12)

Let \( m^C(x) \) be the complexification of the space \( m(x) \), \( x \in R(m) \). It is easy to see that the kernel of the form \( B^\lambda_x \) in \( m^C(x) \) is the subspace of \( m^C(x) \) given by

\[
ker B^\lambda_x = \{ y \in m^C(x) : [x + \lambda a, y] \in \text{ad} x(t^C) \}
\]

\[
= \{ y \in m^C(x) : [x + \lambda a, y] \in \text{ad}(x + \lambda a)(t^C) \}
\]

because \( \text{ad} x(t^C) = (m^C(x))^\perp \) in \( m^C(x) \) and \([a, t^C] = 0 \). Thus

\[
ker B^\lambda_x = ((g^C)^{x+\lambda a})_{m^C} \cap m^C(x),
\]

where \((\cdot)_{m^C}\) denotes the projection onto \( m^C \) along \( t^C \). But \(((g^C)^{x+\lambda a})_{m^C} \subset m^C(x)\) because \([a, m^C] \subset m^C, [a, t^C] = 0, \text{ad} x(t^C) \subset m^C \) and by [2.3] \( y \in m^C \) is an element of \( m^C(x) \) iff \([x, y] \in m^C \). Thus

\[
ker B^\lambda_x = ((g^C)^{x+\lambda a})_{m^C}, \ \lambda \in \mathbb{C}.
\]

(2.13)

In particular, for \( \lambda = 0 \) (for the canonical Poisson structure on \( T(G/K) \)),

\[
ker B^0_x = ((g^C)^x)_{m^C} = ((g^+)^C).
\]

Since \( x \in R(m) \), a (real) dimension of the space \( (g^+)_m \) is equal to the constant \( r(m) = q(m) - p(m) \). Therefore a maximal isotropic subspace of the space \( m(x) \) with respect to the form \( B^0_x \) has dimension \( \frac{1}{2}(r(m) + \dim m(x)) \). It is clear that

\[
ker B^{ei}_x = \{ y \in m^C(x) : [a, y] \in \text{ad} x(t^C) \} = m^C(x) \cap (\text{ad}_a^{-1}\text{ad} x(t^C)),
\]

(2.14)

where \( \text{ad}_a^{-1} \) \( \overset{\text{def}}{=} \text{ad} a|_{m^C}^{-1} \). As an immediate consequence of Proposition [1.3] we obtain

**Proposition 2.6.** The pair \( (A^G, \eta^t(\omega_G)) \) is Kronecker at the point \( x \in R(m) \) iff

1. \( \dim_{\mathbb{C}}((g^C)^{x+\lambda a})_{m^C} = r(m) \) for each \( \lambda \in \mathbb{C} \) and
2. \( \dim_{\mathbb{C}} ker B^{ei}_x = r(m) \).

Denote by \( I(g) \) the space of all \( \text{Ad}(G) \)-invariant polynomials on \( g \). If \( h \in I(g) \) then it is clear that the function \( h^\lambda : g \to \mathbb{R} \), \( h^\lambda(y) = h(y + \lambda a), \lambda \in \mathbb{R} \), is \( \text{Ad}(K) \)-invariant on \( g \). Therefore the set \( \mathcal{F} = \{ h^\lambda | m, h \in I(g), \lambda \in \mathbb{R} \} \) is a subset of \( A^K_m = A^G \) (of \( G \)-invariant function on \( T(G/K) \)). The following assertion was proved in [MP] (see Proposition 3.6, Lemma 3.3 and Theorem 3.9).
Theorem 2.7. Let $\mathcal{F}$ be a maximal involutive subset of the Poisson algebra $(A^G, \eta^G)$ and in this set $\mathcal{F}$ there are $\frac{1}{2}(r(m) + \dim m - \dim \mathfrak{k} + p(m))$ functions functionally independent at each point of some nonempty Zariski open subset $O^F$ of $m = T_o(G/K)$.

The pair $(A^G, \eta^G(\omega_G))$ is micro-Kronecker, in particular, it is Kronecker at each point of some nonempty Zariski open subset $O^K_r$ of $m = T_o(G/K)$.

But in our paper [MP] the subsets $O^F$ and $O^K_r$ of $m$ are not described explicitly. Nevertheless, using Theorem 2.7 we prove that each of these sets contains the subset $R(m) \cap R(\mathfrak{z} \oplus m) \cap Q_\alpha(m)$ of $m$, where

$$Q_\alpha(m) = \{x \in m : \dim_{\mathbb{C}} \ker B^\alpha_x = r(m)\}$$

$$= \{x \in m : \dim m(x) \cap (\text{ad}_a^{-1} \text{ad}_x(t)) = r(m)\}. \quad (2.15)$$

By relations (2.14) and by Proposition 2.6 the set $Q_\alpha(m)$ contains the open subset $O^K_r \subset m$, in particular,

$$r(m) = \min_{x \in m} \dim \left(m(x) \cap (\text{ad}_a^{-1} \text{ad}_x(t))\right). \quad (2.16)$$

Since $m(x) = (\text{ad}_x(t))^\perp$ in $m$, the set $Q_\alpha(m)$ is a nonempty Zariski open subset of $m$.

Theorem 2.8. The pair $(A^G, \eta^G(\omega_G))$ is Kronecker at each point of the nonempty Zariski open subset $R(m) \cap R(\mathfrak{z} \oplus m) \cap Q_\alpha(m)$ of $m = T_o(G/K)$.

Proof. Let us consider in the complex spaces $(\mathfrak{z} \oplus m)^C = m_3^C$ and $m^C$ nonempty Zariski open subsets $R(m_3^C)$ and $R(m^C)$ defined as $R(m_3)$ and $R(m)$ in the real case (see (2.5)). For example,

$$R((\mathfrak{z} \oplus m)^C) = \{y \in (\mathfrak{z} \oplus m)^C : \dim_{\mathbb{C}}(g_{\mathbb{C}})^y = q(m_3^C), \dim_{\mathbb{C}}((t_\mathfrak{z})^C)^y = p(m_3^C)\},$$

where $q(m_3^C)$ (resp. $p(m_3^C)$) is the minimum of (complex) dimensions of the spaces $(g_{\mathbb{C}})^y$ (resp. $(t_\mathfrak{z})^C)^y$ over all $y \in m_3^C$. It is clear that $q(m_3^C) = q(m_3)$ and $p(m_3^C) = p(m_3)$, $q(m^C) = q(m)$ and $p(m^C) = p(m)$ (a complex polynomial function which vanishes on a real form of a complex space vanishes identically on this complex space).

Lemma 2.9. Let $y \in R(m) \cap R(\mathfrak{z} \oplus m)$. Then $\dim_{\mathbb{C}}((g_{\mathbb{C}})^{y + \lambda a})_{m^C} = r(m)$ for each $\lambda \in \mathbb{C}$ iff the complex affine line $l(y; a) = \{y + \lambda a, \lambda \in \mathbb{C}\}$ is a subset of the set $R((\mathfrak{z} \oplus m)^C)$.

Proof. It is clear that $(t_\mathfrak{z})^{y + \lambda a} = (t_{\mathbb{C}})^y$ and $((t_\mathfrak{z})^C)^{y + \lambda a} = ((t_{\mathbb{C}})^C)^y$ because $[a, t] = [a, \mathfrak{z}] = 0$. Therefore

$$\dim_{\mathbb{C}}((t_{\mathbb{C}})^{y + \lambda a} = p(m) \quad \text{and} \quad \dim_{\mathbb{C}}((t_{\mathbb{C}})^C)^{y + \lambda a} = p(m_3) \quad \text{for all} \quad \lambda \in \mathbb{C}$$

by definitions of the sets $R(m)$ and $R(m_3)$ containing $y$. Then $\dim_{\mathbb{C}}((g_{\mathbb{C}})^{y + \lambda a})_{m^C} = r(m) = q(m) - p(m)$ iff $\dim_{\mathbb{C}}((g_{\mathbb{C}})^{y + \lambda a})_{m^C} = q(m)$. But $q(m) = q(m_3)$ by (2.4). Now taking into account that $p(m_3^C) = p(m_3)$ and $q(m^C) = q(m_3)$, we complete the proof. □
Fix some element $x \in R(m) \cap R(\mathfrak{j} \oplus m) \cap Q_\alpha(m)$. If $x \in O^{Kr}$, the assertion of the theorem is evident. Suppose that $x \notin O^{Kr}$ and choose some point $x_0 \in O^{Kr} = R(m) \cap R(\mathfrak{j} \oplus m) \cap O^{Kr}$. Since $O^{Kr}$ is a Zariski open subset of $m$ and $x_0 \in O^{Kr}$, the whole real affine line $\{y_t = x_0 + t(x - x_0), \; t \in \mathbb{R}\}$ with the exception of a finite set of points with $t \in T_N = \{t_1, \ldots, t_N\}$ belongs to $O^{Kr} \subset O^{Kr}$. In other words, at each point $y_t, \; t \in \mathbb{R} \setminus T_N$ the pair $(A^G, \eta^F(\omega_0))$ is Kronecker and therefore by Proposition 2.6, $\dim C((\mathbb{C}^{\infty})^{\mu} \setminus \lambda_0)_{m^C} = r(m)$ for all $\lambda \in \mathbb{C}$. Since each such $y_t \in O^{Kr}$ is an element of $R(m) \cap R(\mathfrak{j} \oplus m)$, by Lemma 2.9 the set $R((\mathfrak{j} \oplus m)^C)$ contains each complex affine line $l(y_t; a), \; t \in \mathbb{R} \setminus T_N$.

Consider the complex affine plane $\pi(x_0; x; a) = \{x_0 + \lambda a + \mu(x - x_0), \; \lambda, \mu \in \mathbb{C}\}$ in $(\mathfrak{j} \oplus m)^C$ containing the points $x_0$ and $x$. But $R((\mathfrak{j} \oplus m)^C)$ is a Zariski open subset, i.e. is defined by a finite family $\{P_1, \ldots, P_k\}$ of complex polynomial functions. The restriction $p_j \overset{\text{def}}{=} P_j|\pi(x_0; x; a), \; j = 1, \ldots, k$ is a polynomial function of the two variables $\lambda, \mu \in \mathbb{C}$. Since $l(y_t; a) \subset R((\mathfrak{j} \oplus m)^C), \; t \in \mathbb{R} \setminus T_N$, then each polynomial $p_j$ is constant on such a line $l(y_t; a)$. In other words, $p_j(\lambda, \mu)_{\mu \neq t} = c_j(t)$ for all $t \in \mathbb{R} \setminus T_N$, where $c_j(t) \in \mathbb{C}$. Since the set $t \in \mathbb{R} \setminus T_N$ is infinite, the polynomial $p_j, \; j = 1, \ldots, l$ is a function of only one variable $\mu$, i.e. $p_j(\lambda, \mu) = c_j(\mu)$, where $c_j(\mu)$ is a polynomial.

Suppose that the complex affine line $l(x; a) = l(y_t; a) \subset \pi(x_0; x; a)$ is not a subset of the set $R((\mathfrak{j} \oplus m)^C)$. Then all polynomials $P_j, \; j = 1, \ldots, k$ vanish in some point $x + \lambda_0 a$ of this line, and, consequently, vanish identically on this line: $0 = p_j(\lambda_0, 1) = c_j(1)$ for all $j = 1, \ldots, k$. But $x \in R(\mathfrak{j} \oplus m) \subset R((\mathfrak{j} \oplus m)^C)$, i.e. $P_j(x) = p_j(0, 1) = c_j(1) \neq 0$ for some $1 \leq j \leq k$, the contradiction. Thus the line $l(x; a)$ is a subset of the set $R((\mathfrak{j} \oplus m)^C)$. But $x \in Q_\alpha(m)$, i.e. $\dim C \ker B_x^0 = r(m)$. Now the assertion of theorem follows immediately from Lemma 2.9 and Proposition 2.6.

**Theorem 2.10.** The set $\mathcal{F}$ is a maximal involutive subset of the Poisson algebra $(A^G, \eta^F)$. For each point $x$ from the nonempty Zariski open subset $R(m) \cap R(\mathfrak{j} \oplus m) \cap Q_\alpha(m)$ of $m$ there are $\frac{1}{2}(r(m) + \dim m(x))$ functions from the set $\mathcal{F}$ functionally independent at $x$.

**Proof.** Our proof of the theorem is based on the proof of Proposition 3.6. in [MP]. Let $x \in R(m) \cap R(\mathfrak{j} \oplus m) \cap Q_\alpha(m)$. By Theorem 2.8 the pair $(A^G, \eta^F(\omega_0))$ is Kronecker at $x$. Then by Proposition 1.3 the space $L_x = \sum_{t \in \mathbb{R}^2 \setminus \{0\}} V_x^t$, where $V_x^t = \ker B_x^t \subset m(x)$, is a maximal isotropic subspace of $m(x)$ with respect to the form $B_x^{1,0} = B_x^0$ (of maximal rank) corresponding to the canonical Poisson structure $\eta^F$. Here for $t \in \mathbb{R}^2$ we consider $B_x^t$ as a real form on $m(x)$ with $\mathbb{C}$-linear extension described by relations (2.12).

But the space $L_x$ is generated by a finite subset of spaces from the set $\{V_x^t\}$. Since by the first relation in (2.12) the family $V_x^t$ depends smoothly on the parameter $t \in \mathbb{R}^2 \setminus \{0\}$, we can suppose that this finite subset of spaces does not contain the kernel of the singular form $B_x^0$. In other words, $L_x = \bigoplus_{j=1}^N V_x^j$, where each space $V_x^i = \ker B_x^i$ is defined by (2.13) with $\lambda_j \in \mathbb{R}, \; j = 1, N$. Moreover, since $x \in R(\mathfrak{j} \oplus m)$ and $R(\mathfrak{j} \oplus m)$ is a Zariski open subset of $\mathfrak{j} \oplus m$, we can choose these numbers $\{\lambda_j\}$ such that each $x + \lambda_j a \in R(\mathfrak{j} \oplus m)$.

Let $h \in I(g)$ and $y \in g$. Then $[y, \text{grad}_g b(h)] = 0$ by invariance of the form $\langle , \rangle$, i.e. $\text{grad}_g b(h) \in g^\theta$. But since $y$ is a semisimple element of the reductive Lie
sections on where ˜
\[\langle \eta \rangle \]

Let \( \sigma \) through this element \( a \). Denote by ˜
\[G \]

g, respect of an arbitrary automorphism of \( \{ \text{the vectors} \} \)

For any point \( x \in R(m) \cap R(\mathfrak{z} \oplus m) \cap Q_e(m) \) the subspace \( L_x = \{ \text{grad}_m f(x), f \in \mathcal{F} \} \subset m(x) \) is a maximal isotropic subspace of \( m(x) \), i.e. \( \langle x, y, L_x \rangle = 0 \) and \( y \in m(x) \) implies \( y \in L_x \).

2.3 Adjoint orbits and involutive automorphisms

Let \( \sigma \) be an involutive automorphism of \( \mathfrak{g} \) and let \( \mathfrak{g} = \tilde{\mathfrak{g}} \oplus \mathfrak{g}' \) be the decomposition of \( \mathfrak{g} \) into the eigenspaces of \( \sigma \) for the eigenvalues \(+1\) and \(-1\) respectively:

\[ [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] \subset \tilde{\mathfrak{g}}, \quad [\mathfrak{g}', \mathfrak{g}'] \subset \mathfrak{g}', \quad [\tilde{\mathfrak{g}}, \mathfrak{g}'] \subset \mathfrak{g}'. \] (2.17)

Denote by \( \tilde{G} \) the closed connected subgroup of \( G \) with the Lie algebra \( \tilde{\mathfrak{g}} \). Fix some element \( a \in \mathfrak{g}' \) (\( \sigma(a) = -a \)) and consider the orbit \( \tilde{O} = \text{Ad}(\tilde{G})(a) = \tilde{G}/\tilde{K} \) in \( \mathfrak{g} \) through this element \( a \). It is clear that \( \tilde{O} \) is a submanifold (\( \tilde{G} \)-suborbit) of the \( G \)-orbit \( O = G/K \) of \( a \) and \( \tilde{K} = \tilde{G} \cap K \).

Since \( \sigma(a) = -a \), the algebra \( \mathfrak{k} = \mathfrak{g}'' \) is \( \sigma \)-invariant. Suppose that the form \( \langle , \rangle = -\Phi_\chi \) is also \( \sigma \)-invariant (if \( \mathfrak{g} \) is semi-simple its Killing form is invariant with respect of an arbitrary automorphism of \( \mathfrak{g} \)). Then \( \sigma(m) = m \) and we have in addition to \( (2.17) \) the following orthogonal decompositions of algebras \( \mathfrak{g}, \tilde{\mathfrak{g}}, \mathfrak{k} \) with respect to the form \( \langle , \rangle \)

\[ \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^\perp \oplus \tilde{m} \oplus m', \quad \tilde{\mathfrak{g}} = \mathfrak{k} \oplus \tilde{m}, \quad [\mathfrak{k}, \tilde{m}] \subset \tilde{m}, \quad [\mathfrak{k}, m'] \subset m', \quad \mathfrak{k} = \mathfrak{k} \oplus \mathfrak{k}', \] (2.18)

where \( \mathfrak{k}, \tilde{m} \) are subspaces of \( \mathfrak{g}, \mathfrak{k}', \mathfrak{m}' \) are subspaces of \( \mathfrak{g}' \). In particular, \( (\mathfrak{k}, \mathfrak{k}') \) is a symmetric pair of compact Lie algebras, i.e. \( \mathfrak{k} \) is the fixed point set of the involutive automorphism \( \sigma|\mathfrak{k} \).

Since \( \ker \text{ad} a = \mathfrak{k} \) and \( m = \mathfrak{k}^\perp \) in \( \mathfrak{g} \), then \( \text{ad} a(m) = m \) and the operator \( \text{ad} a|m : m \to m \) is invertible. Moreover, for \( m' \subset m \) and \( \tilde{m} \subset m \) we have

\[ \text{ad} a(m') \subset [\mathfrak{g}', m'] \cap m \subset \tilde{\mathfrak{g}} \cap m \subset m, \quad \text{ad} a(\tilde{m}) \subset [\mathfrak{g}', \tilde{m}] \cap m \subset \mathfrak{g}' \cap m \subset m', \]

and therefore \( \dim \tilde{m} = \dim m' \). Since \( \ker(\text{ad} a|m) = 0 \), we have

\[ \text{ad} a(m') = \tilde{m}, \quad \text{ad} a(\tilde{m}) = m'. \] (2.19)

Let \( A^\tilde{G} \) (resp. \( A^\tilde{K} \)) be the set of all \( \tilde{G} \)-invariant (resp. \( \text{Ad}(\tilde{K}) \)-invariant) functions on \( T(\tilde{G}/\tilde{K}) \) (resp. on \( \tilde{m} \)). The Poisson bracket of two functions \( \tilde{f}_1, \tilde{f}_2 \in A^\tilde{K} \) with respect to the canonical Poisson structure \( \tilde{\mathfrak{g}}^\text{can} \) (determined by the canonical 2-form \( \Omega \) on \( T\tilde{O} \)) has the form (see \( (2.2) \)):

\[ \{ \tilde{f}_1, \tilde{f}_2 \}^\text{can}(x) = -\langle x, [\text{grad}_m \tilde{f}_1(x), \text{grad}_m \tilde{f}_2(x)] \rangle, \quad x \in \tilde{m}. \] (2.20)
Since \( \sigma(a) = -a \) the center \( \mathfrak{z} \) of the reductive Lie algebra \( \mathfrak{k} = \mathfrak{g}^0 \) is \( \sigma \)-invariant, i.e. \( \mathfrak{z} = \mathfrak{z} + \mathfrak{g}' \), where \( \mathfrak{z} = \mathfrak{g} \cap \mathfrak{g}' \). It is clear that \( a \in \mathfrak{z}' \). Then for each element \( b \in \mathfrak{z}' \) we can consider the endomorphism \( \varphi_{a,b} : \mathfrak{g} \to \mathfrak{g} \) on \( \mathfrak{g} \) putting \( \varphi_{a,b}(x) = \text{ad}_a^{-1}(\{b, x\}) \) for \( x \in \mathfrak{m} \) and \( \varphi_{a,b}(z) = z \) for \( z \in \mathfrak{k} \), where, recall, \( \text{ad}_a^{-1} \overset{\text{def}}{=} (\text{ad} |_{\mathfrak{m}})^{-1} \). Remark that \( \varphi_{a,b}(\mathfrak{m}) \subset \mathfrak{m} \) because \( \{b, \mathfrak{k}\} = 0 \). Moreover, \( \varphi_{a,b}(\mathfrak{m}) \subset \mathfrak{m} \) and \( \varphi_{a,b}(\mathfrak{m}') \subset \mathfrak{m}' \) because \( a, b \in \mathfrak{g}' \) (see also (2.19)).

It is clear that the endomorphism \( \varphi_{a,b} \) is symmetric and the group \( \text{Ad}(\mathbf{K}) \) commutes elementwise with \( \varphi_{a,b} \) on \( \mathfrak{m} \). Therefore the operator \( \varphi_{a,b}(\mathfrak{m}) \) is also symmetric and the group \( \text{Ad}(\mathbf{K}) \) commutes elementwise with \( \varphi_{a,b} \) on \( \mathfrak{m} \). So the function \( \tilde{H}_{a,b}(x) = \frac{1}{2}\langle x, \varphi_{a,b}(x) \rangle \), \( x \in \mathfrak{m} \) is \( \text{Ad}(\mathbf{K}) \)-invariant. Then \( \tilde{H}_{a,b} \) (as a function on \( T(G/\mathbf{K}) \) from the set \( A^{G} = A^{\mathbf{K}}_{\mathfrak{m}} \) is a Hamiltonian function of the geodesic flow of some pseudo-Riemannian metric on \( 
abla G/\mathbf{K} \) if \( \varphi_{a,b}|_{\mathfrak{m}} \) is non-degenerate.

Consider the space \( I(\mathfrak{g}) \) of all \( \text{Ad}(G) \)-invariant polynomials on \( \mathfrak{g} \). As we remarked in the previous subsection for each \( h \in I(\mathfrak{g}) \) the function \( h^{\lambda} : \mathfrak{g} \to \mathbb{R} \), \( h^{\lambda}(y) = h(y + \lambda a) \), \( \lambda \in \mathbb{R} \), is an \( \text{Ad}(\mathbf{K}) \)-invariant function on \( \mathfrak{g} \). Therefore the set \( F = \{ h^{\lambda} |_{\mathfrak{m}}, h \in I(\mathfrak{g}), \lambda \in \mathbb{R} \} \) is a subset of \( A^{K}_{\mathfrak{m}} = A^{G} \) (of \( G \)-invariant function on \( T(G/\mathbf{K}) \)) and the set \( \tilde{F} = \{ \tilde{h}^{\lambda} |_{\mathfrak{m}}, h \in I(\mathfrak{g}), \lambda \in \mathbb{R} \} \) is a subset of \( A^{G}_{\mathfrak{m}} = A^{G} \) (of \( G \)-invariant function on \( T(G/\mathbf{K}) \)). Put \( H^{\lambda} = h^{\lambda}|_{\mathfrak{m}} \) and \( \tilde{H}^{\lambda} = \tilde{h}^{\lambda}|_{\mathfrak{m}} \). The following lemma follows easily from the results of \cite{MP} (see also [TF] Ch.6.16.Lemma) or [DGJ] sec.3).



\begin{lemma}[DGJ] \begin{enumerate}
\item For any functions \( h_1, h_2, h \in I(\mathfrak{g}) \) and arbitrary parameters \( \lambda_1, \lambda_2, \lambda \in \mathbb{R} \) we have \( \{ H_1^{\lambda_1}, H_2^{\lambda_2} \}^{\text{can}} = 0 \) and \( \{ H^{\lambda}, H_{a,b} \}^{\text{can}} = 0 \).
\end{enumerate}
\end{lemma}

\begin{proof}
Mainly to fix notations we shall prove this lemma here. Since \( \sigma \) is an automorphism of \( \mathfrak{g} \) and \( \text{Ad}(G) \) is a normal subgroup of \( \text{Aut}(\mathfrak{g}) \), we have \( f = h \circ \sigma \in I(\mathfrak{g}) \) if \( h \in I(\mathfrak{g}) \). But

\[ 2 \text{grad}_{\mathfrak{g}} h(x + \lambda a) = \text{grad}_{\mathfrak{g}} h(x + \lambda a) + \text{grad}_{\mathfrak{g}} f(x - \lambda a) \quad \text{for any} \quad x \in \mathfrak{g}, \quad (2.21) \]

because \( \sigma(a) = -a \) and \( \sigma(x) = x \). The five functions \( h_{1}^{\lambda_1}, h_{2}^{\lambda_2}, f_{1}^{-\lambda_1}, f_{2}^{-\lambda_2}, h_{a,b} \) commute pairwise on \( \mathfrak{g} \simeq \mathfrak{g}' \) with respect to the standard (linear) Lie-Poisson bracket on \( \mathfrak{g} \) \cite{MP}. This means that for any pair of functions \( F_1, F_2 \) from this set we have

\[ \langle x, [\text{grad}_{\mathfrak{g}} F_1(x), \text{grad}_{\mathfrak{g}} F_2(x)] \rangle = 0 \quad \text{for all} \quad x \in \mathfrak{m} \subset \mathfrak{g}. \]

Then by (2.21)

\[ \langle x, [\text{grad}_{\mathfrak{g}} h_{1}^{\lambda_1}(x), \text{grad}_{\mathfrak{g}} h_{2}^{\lambda_2}(x)] \rangle = 0 \quad \text{for all} \quad x \in \mathfrak{m} \subset \mathfrak{g}. \]

Now taking into account that \( \text{grad}_{\mathfrak{m}} H_1^{\lambda_1}(x) \in \mathfrak{m}(x), [x, \mathfrak{m}(x)] \subset \mathfrak{m} \) for \( x \in \mathfrak{m} \) and \( \mathfrak{m} \perp \mathfrak{k} \), we obtain that

\[ \langle x, [(\text{grad}_{\mathfrak{g}} h_{1}^{\lambda_1}(x))_{\mathfrak{m}}, (\text{grad}_{\mathfrak{g}} h_{2}^{\lambda_2}(x))_{\mathfrak{m}}] \rangle = 0, \]

i.e. \( \{ \tilde{H}_1^{\lambda_1}, \tilde{H}_2^{\lambda_2} \}^{\text{can}}(x) = 0 \). Similarly we can show that \( \{ \tilde{H}^{\lambda}, \tilde{H}_{a,b} \}^{\text{can}} = 0 \). \( \square \) \( \square \)
As follows from the lemma above the set $\tilde{F}$ is an involutive subset of $(A_m^K, \tilde{\eta}^\text{can})$. Put
\[ \hat{O} = R(m) \cap R(j \oplus m) \cap Q_a(m) \cap \tilde{m} \subset \tilde{m}, \] (2.22)
Let us define the numbers $q(\tilde{m})$, $p(\tilde{m})$, $r(\tilde{m})$ and the subset $R(\tilde{m}) \subset \tilde{m}$ similarly to the numbers $q(m)$, $p(m)$, $r(m)$ and the subset $R(m) \subset m$ but for the pair of algebras $(\tilde{m}, \tilde{\eta})$ (see (2.3)).

**Theorem 2.13.** Suppose that the Zariski open subset $\hat{O}$ of $\tilde{m}$ is nonempty. Then the set $\tilde{F}$ is a maximal involutive subset of the algebra $(A_m^K, \tilde{\eta}^\text{can})$. For each point $x$ from the nonempty Zariski open subset $\hat{O} \cap R(\tilde{m}) \subset \tilde{m}$ there are $\frac{1}{2}(r(\tilde{m}) + \dim \tilde{m}(x))$ functions from the set $\tilde{F}$ functionally independent at $x$.

**Proof.** Fix some point $x \in \hat{O} \cap R(\tilde{m}) \subset \tilde{m}$. Put
\[ L_x = \{ \text{grad}_m f(x), f \in F \} \subset m(x) \subset m, \quad \tilde{L}_x = \{ \text{grad}_{\tilde{m}} \tilde{f}(x), \tilde{f} \in \tilde{F} \} \subset \tilde{m}(x) \subset \tilde{m}. \]
It is evident that $\tilde{L}_x = (L_x)_{\tilde{m}}$, where $(\cdot)_{\tilde{m}}$ denotes the projection onto $\tilde{m}$ along $m'$ in $m = \tilde{m} \oplus m'$. Moreover, since $x \in \tilde{m}$ and $\tilde{\eta} = \tilde{\eta} \oplus \tilde{\eta}'$, we have $\text{ad} \ x(\tilde{\eta}) = \text{ad} x(\tilde{\eta}) \oplus \text{ad} x(\tilde{\eta}')$, where $\text{ad} x(\tilde{\eta}) \subset \tilde{m}$ and $\text{ad} x(\tilde{\eta}') \subset m'$, and therefore
\[ m(x) = \tilde{m}(x) \oplus (m(x) \cap m'). \] (2.23)

By Lemma 2.12 the space $\tilde{L}(x)$ is an isotropic subspace of $\tilde{m}(x)$ with respect to the form $\tilde{B}_x : (y_1, y_2) \rightarrow \langle x, [y_1, y_2] \rangle$ on $\tilde{m}(x)$ associated with Poisson bracket (2.20). So that to prove the theorem it is sufficient to show that this subspace is maximal isotropic.

To this end suppose that $\langle x, [y, \tilde{L}_x] \rangle = 0$ for some $y \in \tilde{m}(x)$. Then $\langle [x, y], \tilde{L}_x \rangle = 0$ by invariance of the form $\langle , \rangle$. Taking into account that $[x, \tilde{m}(x)] \subset \tilde{m}$ by definition (2.23) and $\tilde{m} \perp m'$, we obtain that
\[ 0 = \langle [x, y], \tilde{L}_x \rangle = \langle [x, y], L_x \rangle = \langle x, [y, L_x] \rangle. \]
But $y \in m(x)$ by (2.23). Also by Corollary 2.11 the space $L_x$ is a maximal isotropic in $m(x)$ and, consequently, $y \in L_x$. Then $y \in \tilde{L}_x$, because $y \in \tilde{m}$ and $L_x \cap \tilde{m} \subset (L_x)_{\tilde{m}} = \tilde{L}_x$. In other words, $L_x$ is a maximal isotropic subspace in $\tilde{m}(x)$ with respect to the form $\tilde{B}_x$. $\square$

As follows from Theorem 2.13 we have to establish when the set $\hat{O}$ (2.22) is nonempty. It is clear that $\hat{O}$ is nonempty iff $R(m) \cap R(j \oplus m) \cap \tilde{m} \neq \emptyset$ and $Q_a(m) \cap \tilde{m} \neq \emptyset$. Therefore we consider these two Zariski open subsets in more detail.

Suppose that the set $R(m) \cap R(j \oplus m) \cap \tilde{m}$ is nonempty and that $x_0$ is a common element of this set and the set $R(\tilde{m})$. Let $\mathfrak{t}_0$ be the centralizer of the Lie algebra $\mathfrak{t}^0$ in $\mathfrak{t}$, put $\mathfrak{k}_0 = \mathfrak{t}_0 \cap \mathfrak{k}$. Since $\sigma(x_0) = x_0$, then $\sigma(\mathfrak{t}^0) = \mathfrak{t}^0$ and, consequently, $\sigma(\mathfrak{k}_0) = \mathfrak{k}_0$. We have the following splitting of $\mathfrak{k}_0 = \mathfrak{t}_0 \oplus \mathfrak{k}_0$, $[\mathfrak{t}_0, \mathfrak{t}_0] \subset \mathfrak{k}_0$, associated with $\sigma$. Let $K_0$ and $\tilde{K}_0$ be the connected Lie subgroups of $K$ with the Lie algebras $\mathfrak{t}_0$ and $\tilde{\mathfrak{t}}_0$ respectively. These subgroups are closed in $K$ and $K_0/\tilde{K}_0$ is a symmetric space. Let $V$ be a some vector subspace of the space $m$. Put
\[ m_a(V) = \min_{x \in V} \dim \left( m(x) \cap (\text{ad}^{-1} \text{ad} x(\mathfrak{t})) \right), \] (2.24)
where, recall, \( \text{ad}^{-1}_a \overset{\text{def}}{=} (\text{ad} \circ \langle \cdot, \cdot \rangle)^{-1} \). By (2.16) \( m_a(m) = r(m) \) and \( m_a(V) \geq r(m) \).

**Proposition 2.14.** Suppose that \( R(m) \cap R(j \oplus m) \cap \tilde{m} \neq \emptyset \) and choose arbitrary point \( x_0 \in R(m) \cap R(j \oplus m) \cap R(\tilde{m}) \). Then

\[
m_a(\tilde{m}) = \min_{\alpha \in t_0} \dim f^\alpha - \dim f^\alpha_0 = \min_{\alpha \in t_0} \dim f^\alpha_0 - \dim f^\alpha.
\]

(2.25)

**Proof.** By Remark 2.1 \( \text{Ad}(K)(\langle m(x_0) \rangle) = m \). Since \( \text{Ad} k(a) = a \) for all \( k \in K \), we obtain that \( m_a(m) = m_a(m(x_0)) = r(m) \). Similarly, since \( \text{Ad}(K)(\langle \tilde{m}(x_0) \rangle) = \tilde{m} \) and \( \tilde{K} = K \cap \tilde{G} \subset K \), then

\[
m_a(\tilde{m}) = m_a(\tilde{m}(x_0)).
\]

(2.26)

We will use the moment map theory to calculate the number \( m_a(\tilde{m}) \) (this method was proposed in [Pa]). For our aim it is convenient to use the moment map constructed in our previous paper [MP]. So here we briefly describe main properties of this moment map [MP, Remark 3.2].

Consider on the vector space \( m \) the non-degenerate bilinear form

\[
\beta(y_1, y_2) = \langle y_1, \text{ad}_a^{-1}(y_2) \rangle, \quad y_1, y_2 \in m.
\]

Since the endomorphism \( \text{ad} a| m : m \to m \) is skew-symmetric (with respect to the form \( \langle \cdot, \cdot \rangle \)), the form \( \beta \) is also skew-symmetric. Identifying the tangent space \( T_x m \) with \( m \) for each \( x \in m \), we can consider \( \beta \) as a symplectic form on \( m \). Since the \( \text{Ad} \)-action of \( K \) on \( m \) preserves the form \( \beta \), this action of \( K \) is Hamiltonian with the \( K \)-equivariant moment map \( \mu^\beta : m \to \mathfrak{t}^* \), \( \mu^\beta(x)(\zeta) = -\frac{1}{2} \langle \text{ad}_a^{-1}(x), [\zeta, x] \rangle, \forall \zeta \in \mathfrak{t} \) (see [MP, Remark 3.2]). In particular, for each \( \zeta \in \mathfrak{t} \) the vector field \( \zeta X(x) = [\zeta, x] \in T_x m \) is the Hamiltonian vector field of the function \( f_\beta(x) = \mu^\beta(x)(\zeta) \) on the manifold \( X = m \).

Let \( x \in m \), \( W_x \subset T_x m \) be the tangent space to the \( K \)-orbit in \( m \) and let \( W_x^\beta \) be the (skew)orthogonal complement to \( W_x \) in \( T_x m \) with respect to the form \( \beta \). It is easy to see that \( W_x = \text{ad} x(\mathfrak{t}) \) and \( W_x^\beta = \text{ad} a(m(x)) \), i.e.

\[
\dim(W_x \cap W_x^\beta) = \dim(\text{ad} a(\langle m(x) \rangle)) \cap \text{ad} x(\mathfrak{t})).
\]

But by the \( K \)-equivariance of the moment map \( \mu^\beta \), \( \zeta X(x) \in W_x \cap W_x^\beta \) iff \( \text{ad}^{*} \zeta(\alpha) = 0 \), where \( \alpha = \mu^\beta(x) \) [CS]. In other words, \( \dim(W_x \cap W_x^\beta) + \dim\{\zeta \in \mathfrak{t} : \zeta X(x) = 0\} \) equals to codimension of the orbit \( \text{Ad}^{*}(K) \cdot \alpha \) in \( \mathfrak{t}^* \).

Identifying the space \( \mathfrak{t} \) with its dual \( \mathfrak{t}^* \) using the form \( \langle , \rangle \), we obtain that

\[
\mu^\beta : m \to \mathfrak{t}, \quad \mu^\beta(x) = \frac{1}{2} \langle x, \text{ad}_a^{-1}(x) \rangle \mathfrak{t}
\]

(2.27)

and \( \dim(W_x \cap W_x^\beta) = \dim f^\alpha - \dim f^\alpha_0 \), because \( \zeta X(x) = 0 \) iff \( \zeta \in \mathfrak{t}^\alpha \). Thus

\[
m_a(V) = \min_{x \in \mathfrak{t}}(\dim f^\alpha(x) - \dim f^\alpha), \quad \text{where} \quad \alpha(x) = \mu^\beta(x) \in \mathfrak{t}.
\]

(2.28)

Fix some element \( x_0 \in R(m) \cap R(j \oplus m) \cap R(\tilde{m}) \). Since \( x_0 \in \tilde{m} \subset m \), we have that \( \tilde{m}(x_0) \subset m(x_0) \) (see definition (2.23) or proof of (2.23)). Let \( g_0 \) and \( t_0 \)
be the centralizers of $\mathfrak{t}^{x_0}$ in $\mathfrak{g}$ and $\mathfrak{t}$ respectively. Let $\mathfrak{m}_0 = \mathfrak{t}_0 \subset \mathfrak{g}_0$. Then by Proposition 2.3

$$\mathfrak{m}_0 = \mathfrak{g}_0 \cap \mathfrak{m}, \quad \tilde{\mathfrak{m}}(x_0) \subset \mathfrak{m}(x_0) \subset \mathfrak{m}_0 \subset \mathfrak{g}_0, \quad x_0 \in \mathfrak{m}_0 \cap R(\mathfrak{m}) \subset R(\mathfrak{m}_0),$$

(2.29)

the centralizer $\mathfrak{t}_0^{x_0}$ of $x_0$ in $\mathfrak{t}_0$ is the center $\mathfrak{z}(\mathfrak{t}^{x_0})$ of the Lie algebra $\mathfrak{t}^{x_0}$ and a subalgebra of the center $\mathfrak{z}(\mathfrak{g}_0)$ of the Lie algebra $\mathfrak{g}_0$:

$$\mathfrak{t}_0^{x_0} \overset{\text{def}}{=} (\mathfrak{g}_0 \cap \mathfrak{t}) \cap \mathfrak{t}^{x_0} = \mathfrak{g}_0 \cap \mathfrak{t}^{x_0} = \mathfrak{z}(\mathfrak{t}^{x_0}) \subset \mathfrak{z}(\mathfrak{g}_0), \quad \mathfrak{t}_0 \cap \mathfrak{t}^{x_0} = \mathfrak{z}(\mathfrak{t}) = \mathfrak{t}_0^{x_0}. \quad (2.30)$$

But as we remarked above, $\sigma(\mathfrak{t}^{x_0}) = \mathfrak{t}^{x_0}$ because $\sigma(x_0) = x_0$ and $\sigma(\mathfrak{t}) = \mathfrak{t}$. It is clear also that the spaces $\mathfrak{g}_0$, $\mathfrak{t}_0$ and $\mathfrak{m}_0$ are $\sigma$-invariant, i.e.

$$\mathfrak{g}_0 = \mathfrak{g}_0 \cap \mathfrak{t}^{x_0}, \quad \mathfrak{g}_0 = \mathfrak{g}_0 \cap \mathfrak{t}^{x_0}, \quad [\mathfrak{g}_0, \mathfrak{m}_0] \subset \mathfrak{m}_0, \quad [\mathfrak{g}_0, \mathfrak{m}_0'] \subset \mathfrak{m}_0', \quad \mathfrak{t}_0 = \mathfrak{t}_0 \cap \mathfrak{t}^{x_0},$$

where $\tilde{\mathfrak{g}}_0$, $\tilde{\mathfrak{t}}_0$, $\tilde{\mathfrak{m}}_0$ are subspaces of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}_0'$, $\tilde{\mathfrak{t}}_0'$, $\tilde{\mathfrak{m}}_0'$ are subspaces of $\mathfrak{g}'$. Also by (2.20)

$$\sigma(\mathfrak{t}^{x_0}) = \mathfrak{t}^{x_0}. \quad \text{As } \tilde{\mathfrak{m}}(x_0) \subset \mathfrak{m}_0 \cap \tilde{\mathfrak{g}} = \tilde{\mathfrak{m}}_0 \text{ by (2.29) and } \mathfrak{m}_0 \subset \tilde{\mathfrak{m}} \text{ by definition, it follows from (2.20) that}$$

$$m_\alpha(\tilde{\mathfrak{m}}) = m_\alpha(\tilde{\mathfrak{m}}(x_0)) = m_\alpha(\tilde{\mathfrak{m}}_0). \quad (2.31)$$

It is evident that $a \in \mathfrak{g}_0$ and $\ker(\operatorname{ad} a|\mathfrak{g}_0) = \mathfrak{t}_0$, $\operatorname{ad}(a|\mathfrak{m}_0) = \mathfrak{m}_0$ and the operator $\operatorname{ad} a|\mathfrak{m}_0$ is invertible. Then by (2.27) $\mu^\beta(\mathfrak{m}_0) \subset (\mathfrak{g}_0)\mathfrak{t} = \mathfrak{t}_0$. We can prove also that

$$\operatorname{ad} a|\mathfrak{m}_0 = \mathfrak{m}_0', \quad \operatorname{ad} a|\mathfrak{m}_0' = \tilde{\mathfrak{m}}_0 \quad (2.32)$$

using the same arguments as in the proof of relations (2.19).

Taking into account that $\dim \mathfrak{t}^{x_0}$ is the minimum of dimensions of centralizers $\mathfrak{t}^y$, $y \in \mathfrak{m}$ and $x_0 \in \tilde{\mathfrak{m}}(x_0) \subset \mathfrak{m}_0 \subset \tilde{\mathfrak{m}}$, using (2.28) and (2.31), we conclude that

$$m_\alpha(\mathfrak{m}) + \dim \mathfrak{t}^{x_0} = \min_{x \in \mathfrak{m}} \dim \mathfrak{t}^{x_0}(x) = \min_{x \in \mathfrak{m}_0} \dim \mathfrak{t}^{x_0}(x), \quad \text{where } \alpha(x) = \mu^\beta(x) \in \mathfrak{t}.$$

As we remarked above $\mu^\beta(\mathfrak{m}_0) \subset \mathfrak{t}_0$ and the operator $\operatorname{ad} a|\mathfrak{m}_0$ is invertible. Therefore it is naturally to consider on the vector space $\mathfrak{m}_0$ the non-degenerate skew-symmetric form $\beta_0(y_1, y_2) = (y_1, \operatorname{ad}_{a^{-1}}(y_2))$, where $y_1, y_2 \in \mathfrak{m}_0$. It is clear that the pair $(\mathfrak{m}_0, \beta_0)$ is a symplectic submanifold of the symplectic manifold $(\mathfrak{m}, \beta)$. Since the Ad-action of $K_0$ on $\mathfrak{m}_0$ preserves the form $\beta_0$, this action of $K_0$ is Hamiltonian with the $K_0$-equivariant moment map $\mu^\beta_0 : \mathfrak{m}_0 \to \mathfrak{t}_0$, $\mu^\beta_0(x) = \frac{1}{2}[x, \operatorname{ad}_{a^{-1}}(x)]|_{\mathfrak{t}_0}$, i.e. $\mu^\beta_0 = \mu^\beta|\mathfrak{m}_0$.

But the Lie algebra $\mathfrak{t}_0^{x_0}$ is a subalgebra of the center $\mathfrak{z}(\mathfrak{g}_0)$ of the Lie algebra $\mathfrak{g}_0$ and is $\sigma$-invariant (see (2.30)). In particular, $\mathfrak{t}_0^{x_0} = \mathfrak{z}(\mathfrak{t}^{x_0})$ is subalgebra of the center $\mathfrak{z}(\mathfrak{g}_0)$ of the Lie algebra $\mathfrak{g}_0$. Therefore, the orthogonal complement to $\mathfrak{t}_0^{x_0}$ in $\mathfrak{t}_0$ is a compact $\sigma$-invariant Lie subalgebra $\mathfrak{t}_c$ of $\mathfrak{t}_0$ and $\mathfrak{t}_0 = \mathfrak{t}_c \perp \mathfrak{t}_0^{x_0}$. Let us prove that $\mu^\beta_0(\mathfrak{m}_0)$ is subset of the Lie algebra $\mathfrak{t}_c$ ("effective part"). Indeed, as we remarked above $[\mathfrak{t}_0^{x_0}, \mathfrak{g}_0] = 0$ and $[\mathfrak{m}_0, \operatorname{ad}_{a^{-1}}(\mathfrak{m}_0)] \subset \mathfrak{g}_0$. Then by invariance of the scalar product $\langle [\mathfrak{m}_0, \operatorname{ad}_{a^{-1}}(\mathfrak{m}_0)]|_{\mathfrak{t}_c}, \mathfrak{t}_0^{x_0} \rangle = \langle [\mathfrak{m}_0, \operatorname{ad}_{a^{-1}}(\mathfrak{m}_0)], \mathfrak{t}_0^{x_0} \rangle = 0$.

To determine the number $\min_{x \in \mathfrak{m}_0} \dim \mathfrak{t}^{x_0}(x)$ we will show that the image $\mu^\beta_0(\tilde{\mathfrak{m}}_0)$ contains an open subset in the space $\mathfrak{t}_c' \overset{\text{def}}{=} (1 - \sigma)\mathfrak{t}_c = \mathfrak{t}_c \cap \mathfrak{t}_0'$. It is easy to calculate that for any tangent vector $y \in \mathfrak{m}_0 = T_{x_0}\mathfrak{m}_0$

$$D_{x_0}(y) \overset{\text{def}}{=} \mu^\beta_0(x_0)(y) = \frac{1}{2}[y, \operatorname{ad}_{a^{-1}}x_0]|_{\mathfrak{t}_0} + \frac{1}{2}[x_0, \operatorname{ad}_{a^{-1}}y]|_{\mathfrak{t}_0}.$$
Taking into account relations (2.32) and (2.18) and the inclusion $x_0 \in \tilde{m}_0 \subset \tilde{g}$, we obtain that
\[ D_{x_0}(m_0') \subset ([m'_0, ad_{a_0}^{-1} m_0] + [\tilde{m}_0, ad_{a_0}^{-1} m_0'])_{t_0} \subset [m'_0, m_0]_{t_0} \subset (\tilde{g})_{t_0} \subset \tilde{t}_0. \]
and, similarly, $D_{x_0}(\tilde{m}_0) \subset (\tilde{g}')_{t_0} \subset \tilde{t}_0$. In other words, $D_{x_0}(m_0) = D_{x_0}(m_0') \oplus D_{x_0}(\tilde{m}_0)$.

The image $\mu^{\beta_0}(T_{x_0}m_0) \subset t_0$ of the tangent map of the moment map $\mu^{\beta_0}$ at $x_0$ coincides with the annihilator in $t_0^\ast \simeq t_0$ of the Lie algebra $t_0^\ast$ of the isotropy group $\{ k \in K_0 : Ad k(x_0) = x_0 \}$ of $x_0 \in m_0$ [GS]. Since this annihilator coincides with $t_\sigma$, then $t_\sigma = D_{x_0}(m_0)$. Since the Lie algebra $t_\sigma$ is $\sigma$-invariant, then $t_\sigma \overset{\text{def}}{=} t_\sigma \cap t_0' = D_{x_0}(m_0) \cap t_0 = D_{x_0}(\tilde{m}_0)$. Thus $\mu^{\beta_0}(x_0)(\tilde{m}_0) = t_\sigma'$ and, consequently, the set $\mu^{\beta_0}(\tilde{m}_0) = \mu^{\beta_0}(\tilde{m}_0)$ contains some open subset in $t_\sigma' = t_\sigma \cap t_0'$. Therefore
\[ m_\alpha(\tilde{m}_0) = \min_{\alpha \in t_\sigma'} \dim \mathfrak{t}^\alpha - \dim \mathfrak{t}^{t_0}. \] (2.33)

It is not evident that $\min_{\alpha \in t_\sigma'} \dim \mathfrak{t}^\alpha = \min_{\alpha \in t_0} \dim \mathfrak{t}^\alpha$, where, recall, $t_0 = t_\sigma \oplus t_0^\ast$. We will prove this fact, using the moment map $\mu^{\beta_0}$. To this end first of all remark that since $\tilde{m}_0 \cap R(m)$ is nonempty Zariski open subset of $\tilde{m}_0$ (containing $x_0$)
\[ \min \dim(\text{ad } a(m(x)) \cap \text{ad } x(\mathfrak{t})) = \min \dim(\text{ad } a(m(x)) \cap \text{ad } x(\mathfrak{t})). \]
Choose an arbitrary element $x \in m_0 \cap R(m)$. By item (2) of Proposition 2.3 $m(x) = m_0(x) \subset m_0$ and by (2.4) $m_0 = m(x) \oplus \text{ad } x(t_0)$ and $m = m(x) \oplus \text{ad } x(t)$. Therefore $\text{ad } a(m(x)) \subset m_0$ and $\text{ad } x(\mathfrak{t}) \cap m_0 = \text{ad } x(t_0)$. In other words,
\[ m_\alpha(\tilde{m}_0) \overset{\text{def}}{=} \min_{x \in m_0 \cap R(m)} \dim(\text{ad } a(m(x)) \cap \text{ad } x(\mathfrak{t})). \]
Let $W_x \subset T_x m_0$ be the tangent space to the $K_0$-orbit in $m_0$ and let $W_x^{\beta_0 \perp}$ be the (skew)orthogonal complement to $W_x$ in $T_x m_0$ with respect to the form $\beta_0$. It is easy to see that $W_x = \{ [\zeta, x], \zeta \in t_0 \} = \text{ad } x(t_0)$ and $W_x^{\beta_0 \perp} = \text{ad } a(m(x))$ because $m_0(x) = m(x)$, i.e.
\[ \dim(W_x \cap W_x^{\beta_0 \perp}) = \dim(\text{ad } a(m(x)) \cap \text{ad } x(t_0)). \]
Now we can apply the method used above to prove expression (2.28) changing the moment map $\mu^\beta$ by $\mu^{\beta_0}$. By the $K_0$-equivariance of the moment map $\mu^{\beta_0}$, $[\zeta, x] \in W_x \cap W_x^{\beta_0 \perp}$ iff $\text{ad } \zeta(\alpha) = 0$, where $\alpha = \mu^{\beta_0}(x)$. In other words, $\dim(W_x \cap W_x^{\beta_0 \perp}) = \dim([\zeta, x] = 0)$ equals to codimension of the orbit $\text{Ad}(K_0) \cdot \alpha$ in $t_0$. In other words, $\dim(W_x \cap W_x^{\beta_0 \perp}) = \dim(t_0^\ast - \dim t_0^\ast$ and thus
\[ m_\alpha(\tilde{m}_0) = \min_{x \in m_0 \cap R(m)} (\dim t_0^\ast(\alpha(x) = \alpha(\mu^{\beta_0}(x)) \in t_0). \] (2.34)
Since $\dim t_0^\ast \leq \dim t_0^\ast$ for all $y \in m_0$ and the image $\mu^{\beta_0}(\tilde{m}_0) \subset t_\sigma'$ contains an open subset of $t_\sigma'$, we can rewrite (2.34) as
\[ m_\alpha(\tilde{m}_0) = \min_{\alpha \in t_\sigma'} \dim t_0^\ast - \dim t_0^\ast. \] (2.35)
Let us compare expressions (2.33) and (2.35) (by (2.31) \( m_\gamma(m_0) = m_\alpha(m) \)). The algebra \( t^{x_0} \) is the center \( z(t^{x_0}) \) of \( t^{x_0} \), and 
\[
t^{x_0} = t^{x_0}_0 \oplus t^{x_0}_e, \quad t_0 = t_e \oplus t^{x_0}_e, \quad [t_0, t^{x_0}] = 0, \quad t_0 \cap t^{x_0} = t^{x_0}_0.
\]

Since for any element \( \alpha \in t'_e \subset t_0 \) its centralizer \( t^\alpha \) contains the algebra \( t^\alpha \oplus t^{x_0}_0 \oplus t^{x_0}_e \), we obtain, comparing the dimensions in (2.33) and (2.35), that \( t^\alpha = t^\alpha_0 \oplus t^{x_0}_0 \oplus t^{x_0}_e \subset t \) for almost all \( \alpha \in t'_e \). Taking into account that \( t^{x_0}_0 \) is a subalgebra of the center of \( t^e \oplus t^{x_0}_0 \oplus t^{x_0}_e \), we can replace the space \( t^\alpha_e \) in expressions (2.33) and (2.35) by the space \( t^{x_0}_0 \oplus t^{x_0}_e \). As a result we obtain (2.25). Remark also that the algebra \( t^e \oplus t^{x_0}_0 \oplus t^{x_0}_e \) is the normalizer \( N(t^{x_0}) \) of \( t^{x_0} \) in \( t \).

As an immediate consequence of the proof we have

**Corollary 2.15.** For all elements \( \alpha \) from some nonempty Zariski open subset of the space \( t'_0 \subset t_0 \) the centralizer \( t^\alpha \) belongs to the subalgebra \( t_0 + t^{x_0} = N(t^{x_0}) \subset t \).

**Corollary 2.16.** Suppose that \( R(m) \cap R(1 \oplus m) \cap \tilde{m} \neq \emptyset \) and choose arbitrary point \( x_0 \in R(m) \cap R(1 \oplus m) \cap R(\tilde{m}) \). The set \( Q_\alpha(m) \cap \tilde{m} \) is nonempty, i.e. \( m_\alpha(\tilde{m}) = r(m) \), iff one of the following equivalent conditions holds:

1. for some element \( \alpha \in t'_0 \) \( \dim t^\alpha = \dim g^{x_0} \);
2. the subspace \( t^{x_0}_0 \subset t^e \) contains regular elements of the Lie algebra \( t_0 \).

**Proof.** To prove item (1) it is sufficient to remark that by definition \( m_\alpha(\tilde{m}) \geq m_\alpha(m) = r(m) \), \( m(\tilde{m}) = \dim g^{x_0} - \dim t^{x_0} \), and by (2.25) for all \( \gamma \) from some nonempty Zariski open subset of \( t'_0 \) we have

\[
\dim t^\gamma = m_\alpha(\tilde{m}) + \dim t^{x_0} = (m_\alpha(\tilde{m}) - r(m)) + \dim g^{x_0}.
\]

By Corollary 2.16 \( r(m) = \rank g_0 - \dim t^\alpha_0 \) and \( x_0 \) is a regular element of the Lie algebra \( g_0 \), i.e. \( g^{x_0}_0 \) is a Cartan subalgebra of \( g_0 \). As above for all \( \gamma \) from some nonempty Zariski open subset of \( t'_0 \) we have

\[
\dim t^{x_0}_0 = m_\alpha(\tilde{m}) + \dim t^{x_0}_0 = (m_\alpha(\tilde{m}) - r(m)) + \dim g^{x_0}_0.
\]

By definition the algebra \( g_0 \) contains the element \( a \in g' (\sigma(a) = -a) \) and \( t = g^a \). Therefore \( t_0 = t \cap g_0 = g^a_0 \), i.e. the Lie algebra \( t_0 \) is a subalgebra of \( g_0 \) of maximal rank: \( \rank t_0 = \rank g_0 \). Therefore, \( \dim t^{x_0}_0 = \dim g^{x_0}_0 \) for some \( \alpha \in t'_e \) iff \( t^{x_0}_0 \) is a Cartan subalgebra of \( t_0 \).

### 2.4 Integrable geodesic flows

Here we will use notations of Subsections 2.2 and 2.3. Consider the suborbit \( \tilde{O} = \Ad(G) \cdot a \simeq \hat{G}/K \) of the adjoint orbit \( O = \Ad(G) \cdot a \simeq G/K \) in the compact Lie algebra \( g \).

**Theorem 2.17.** Suppose that the Zariski open subset \( O = R(m) \cap R(1 \oplus m) \cap \tilde{m} \) of \( \tilde{m} \) is nonempty and the subspace \( t'_e \) contains regular elements of the Lie algebra \( t_0 \). Here \( t_0 \) is the centralizer of \( t^{x_0} \) for arbitrary \( x_0 \in O \), \( t_0 = (1 - \sigma)t_0 \).

Then there exists a maximal involutive set of independent real analytic functions on \( (T(\hat{G}/K), \Omega) \). These functions are integrals for 1) the geodesic flow determined by the Riemannian metric \( \langle , \rangle \) on \( \hat{G}/K \); 2) the Hamiltonian flow with the Hamiltonian function \( H_{a,b} \) on \( T(\hat{G}/K) \).

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Proof. By Corollary 2.16 and by Theorem 2.13 there exists $m = \frac{1}{2}(r(\tilde{m}) + \text{ddim} A^{\tilde{G}})$ independent involutive functions from the set $A^{\tilde{G}}$. These functions form a maximal involutive subset of independent functions in the algebra $A^{\tilde{G}} = A_m^{\tilde{G}}$ with respect to the canonical Poisson structure on $T(\tilde{G}/\tilde{K})$. Moreover, by Lemma 2.12 these functions are integrals of 1) the geodesic flow on $T(\tilde{G}/\tilde{K})$ determined by the form $\langle , \rangle$; 2) the Hamiltonian flow with the Hamiltonian function $\tilde{H}_{ab}$ on $T(\tilde{G}/\tilde{K})$. Now the assertion of the theorem follows immediately from Proposition 1.5.

2.5 Integrable geodesic flows on $SO(n)/(SO(n_1) \times \cdots \times SO(n_p))$

In this subsection we show that the conditions of Theorem 2.17 hold for the homogeneous space $SO(n)/(SO(n_1) \times \cdots \times SO(n_p))$.

Consider the symmetric space $G/K = U(n)/SO(n)$, where $n \leq 4$, with the involution $\sigma$ on the Lie algebra of skew-hermitian matrices $\mathfrak{g} = u(n)$ defined by the complex conjugation. Then the Lie algebra $\mathfrak{g} = (1+\sigma)\mathfrak{g}$ is the Lie algebra $\mathfrak{so}(n)$ of all real skew-symmetric $n \times n$ matrices. The space $\mathfrak{g}' = (1-\sigma)\mathfrak{g}$ coincides with the set $i\text{sym}(n)$, where $\text{sym}(n)$ is the space of all real symmetric $n \times n$ matrices.

Fix some element $a \in \mathfrak{g}'$, $a = \text{diag}(i\lambda_1, \ldots, i\lambda_1, i\lambda_2, \ldots, i\lambda_2, \ldots, i\lambda_p, \ldots, i\lambda_p)$, where all real numbers $\lambda_1, \ldots, \lambda_p$ are pairwise different and the multiplicity of each $i\lambda_j$ is equal to $n_j \geq 1$, $n_1 + \cdots + n_p = n$. Without loss of generality (to simplify calculations) we may assume that $1 \leq n_1 \leq n_2 \leq \cdots \leq n_p < n$.

It is clear that the Lie algebra $\mathfrak{k} = \mathfrak{g}^*$ is the Lie algebra $u(n_1) \oplus \cdots \oplus u(n_p)$ (with the standard block-diagonal embedding) and $\mathfrak{k}$ is the real part of this Lie algebra, i.e. $\mathfrak{k}$ coincides with $\mathfrak{so}(n_1) \oplus \cdots \oplus \mathfrak{so}(n_p)$ ($\mathfrak{so}(1) = 0$). In this case $G/K = SO(n)/(SO(n_1) \times \cdots \times SO(n_p))$.

Putting $(X, Y) = -\text{Tr} XY$ (using the trace-form) we define an invariant scalar product on $\mathfrak{g}$. To describe the space $\mathfrak{m} = \mathfrak{k}^\perp$ consider any matrix $X \in \mathfrak{g}$ as a block-matrix consisting of rectangle elements $X^{k,l}$, which are rectangle complex $n_k \times n_l$ matrices, $1 \leq k, l \leq p$. It is clear that $(X^{k,l})^* = -X^{l,k}$ and therefore any element of $u(n)$ is defined by its blocks $X^{k,l}$ with $k \leq l$. As a space the Lie algebra $\mathfrak{g} = u(n)$ is a direct sum of its block-type subspaces $V^{k,l}$, $1 \leq k \leq l \leq p$. In this notation the Lie subalgebra $\mathfrak{k}$ is the direct sum $\bigoplus_{j=1}^p V^{j,j}$ and $\mathfrak{m} = \bigoplus_{1<k<l \leq p} V^{k,l}$. We will denote the corresponding to $X^{k,l}$ element of the space $V^{k,l}$ by $\varphi(X^{k,l})$. Each subspace $V^{k,l}$ is $\mathfrak{k}$-module, i.e. $[V^{k,l}, \mathfrak{k}] \subset V^{k,l}$.

First of all consider the simplest case when $p = 2$. In this case $G/K = U(n_1 + n_2)/(U(n_1) \times U(n_2))$ is a Hermitian symmetric space. Therefore there exists a Cartan subspace $\mathfrak{a}$ in $V^{1,2} = \mathfrak{m}$ (a maximal commutative subspace in $V^{1,2}$) consisting of real matrices (belonging to $\mathfrak{so}(n)$) [He, Ch.X, sec.2.1]. This $n_1$-dimensional Cartan subspace $\mathfrak{a}$ can be described by the "diagonal" matrices $X^{1,2}$, $j = 1, \ldots, n_1 \leq n_2$, in which all entries vanish with the exception of $n_1$ entries $X^{1,2}_{j,j}$, $j = 1, \ldots, n_1 \leq n_2$, which are arbitrary real numbers. Then the centralizer $\mathfrak{g}^\mathfrak{a}$ of a regular element $x_0 = \varphi(X^{1,2}) \in \mathfrak{m} \subset V^{1,2}$ in $u(n_1) \oplus u(n_2) = V^{1,1} \oplus V^{2,2}$ is a direct sum $\mathfrak{h}_* \oplus \mathfrak{b}_*$, where $\mathfrak{b}_* \simeq u(n_2 - n_1)$ and $\mathfrak{h}_*$ is a commutative algebra of dimension $n_1$, consisting of diagonal matrices $\text{diag}(ix_1, \ldots, ix_{n_1}, ix_{n_1}, \ldots, ix_{n_1}, 0, \ldots, 0)$, $\forall x_j \in \mathbb{R}$. Remark that the maximal semisimple ideal of $\mathfrak{h}_*$ coincides with the maximal semisimple ideal of the centralizer of $\mathfrak{h}_*$ in $u(n_1) \oplus u(n_2)$. It is easy to check that the (real) regular element $x_0 = \varphi(X^{1,2})$ belongs to the set $R(\mathfrak{m}) \cap R(\mathfrak{z} \oplus \mathfrak{m})$ ($\mathfrak{z}$ is a two-dimensional center.
of $\mathfrak{k}$. The centralizer $\mathfrak{z}_0$ of $\mathfrak{k}^{\circ_0}$ in $\mathfrak{k}$ is the commutative Lie algebra of dimension $2n_1$ consisting of diagonal matrices $\text{diag}(i\lambda_1,\ldots,i\lambda_{n_1},0,\ldots,0)$. Since $\mathfrak{z}_0$ is commutative, each element from $\mathfrak{z}_0 = \mathfrak{k} \subset \mathfrak{g}'$ is regular in $\mathfrak{k}_0$. Thus the conditions of Theorem 2.17 hold.

Suppose now that $p \geq 3$ and $n_p \leq n_1 + \ldots + n_{p-1}$. We claim that for some element $x_0 \in \mathfrak{m}$ its centralizer $\mathfrak{z}_0$ is the one-dimensional center $\mathfrak{j}(\mathfrak{g})$ of $\mathfrak{g} = \mathfrak{u}(n)$. To simplify our calculations remark that each space $V^{k,k} \oplus V^{l,l} \oplus V^{k,l}$, $k < l$ is a Lie subalgebra of $\mathfrak{u}(n)$ isomorphic to $\mathfrak{u}(n_k + n_l) \oplus \mathfrak{u}(n_l)$ and $[V^{k,l}, V^{k,l}] \subset V^{k,k} \oplus V^{l,l}$. But $U(n_k + n_l)/(U(n_k) \times U(n_l))$ is a Hermitian symmetric space and therefore we can use our calculation for the case $p = 2$. Since each subspace $V^{k,l}$ is $\mathfrak{k}$-module, we will construct the element $x_0$ selecting step by step its $V^{k,l}$-entries. For our aim it is enough to consider the submodule $\sum_{j=1}^{p-1} V^{j,j+1} \oplus \sum_{j=1}^{p-2} V^{j,j} \oplus V^{j,p}$ of $\mathfrak{m}$. Choosing in each $k$-module $V^{j,j+1}$ of the first component the "diagonal" element $\varphi(X^{j,j+1})$ as above, we obtain that their common isotropy algebra is a direct sum $\mathfrak{h}_s \oplus \mathfrak{b}_s$, where $\mathfrak{h}_s \simeq \mathfrak{u}(n_p - n_{p-1})$ and $\mathfrak{h}_s$ is of commutative algebra of dimension $n_{p-1}$, consisting of diagonal matrices $\text{diag}(i\lambda_1,\ldots,i\lambda_{n_1},0,\ldots,0)$, $\lambda_1,\ldots,\lambda_{n_1} \in \mathbb{R}$. Remark that the maximal semisimple ideal of $\mathfrak{b}_s$ coincides with the maximal semisimple ideal of the centralizer of $\mathfrak{h}_s$ in $\mathfrak{k}$. Now we consider $V = \sum_{j=1}^{p-2} V^{j,j} \oplus V^{j,p}$ as a $\mathfrak{h}_s \oplus \mathfrak{b}_s$-module (not as $\mathfrak{k}$-module) of complex dimension $N \times n_p$. Then $V$ is direct sum of $\mathfrak{h}_s \oplus \mathfrak{b}_s$-modules $V^{(1)} \oplus V^{(2)}$, $V^{(1)} \perp V^{(2)}$, where $V^{(1)}$ (of complex dimension $N \times n_{p-1}$) is trivial $\mathfrak{b}_s$-module, and therefore the isotropy subalgebra $\mathfrak{h}_s$ in the diagonal subalgebra $\mathfrak{h}_s$ of a real generic point from $V^{(1)}$ is one-dimensional (consisting of elements of $\mathfrak{h}_s$ with $x_1 = x_2 = \ldots = x_{n_{p-1}} = \lambda$). Considering the module $V^{(2)}$ as the space of complex $N \times (n_p - n_{p-1})$ matrices with elements $B$, the ad-representation of $\mathfrak{h}_s \oplus \mathfrak{b}_s$ in $V^{(2)}$ is described as follows: $(i\lambda,A) \cdot B = i\lambda B - BA$, $A \in \mathfrak{u}(n_p - n_{p-1})$. Since the number of rows $N$ in $B$ is greater then $n_p - n_{p-1}$ by our assumption, then for any real $B$ of maximal rank $i\lambda B - BA = 0$ iff $A = i\lambda$ (is a scalar matrix). Therefore $\mathfrak{k}_0^{(0)} = \mathfrak{j}(\mathfrak{g})$ for some real matrix $x_0 \in \mathfrak{m}$ and $\mathfrak{k}_0 = \mathfrak{k}$. This element belong to the set $R(\mathfrak{m}) \cap R(\mathfrak{j} \oplus \mathfrak{m})$. Since the space $\mathfrak{k}_0 = \mathfrak{k}_0 = \mathfrak{sym}(n_1) \oplus \ldots \oplus \mathfrak{sym}(n_p)$ contains a regular elements of $\mathfrak{u}(n_1) \oplus \ldots \oplus \mathfrak{u}(n_p)$, the conditions of Theorem 2.17 hold.

Suppose now that $p \geq 3$ and $n_p > n_1 + \ldots + n_{p-1}$. In this case, since the last component $\mathfrak{g}_p \simeq \mathfrak{u}(n_p)$ of $\mathfrak{g}$ is dominant in $\mathfrak{k}$, the calculation problem can be reduced to the previous case with $n_p = n_1 + \ldots + n_{p-1}$. To this end we consider the representation of the Lie group $K_p \subset K$ with the Lie algebra $\mathfrak{g}_p \simeq \mathfrak{u}(n_p)$ in the $\mathfrak{k}$-submodule $V = \sum_{j=1}^{p-1} V^{j,p}$ of $\mathfrak{m}$. Identifying $V$ with the space of complex $N \times n_p$, $N = n_1 + \ldots + n_{p-1}$ matrices with elements $B$, the Ad-action of $K_p$ in $V$ is described as follows: $k \cdot B = Bk^{-1}$, $k \in K_p = U(n_p)$. Since the number of rows $N$ in $B$ is less then its number of columns $n_p$, then the Ad($K_p$)-orbit of $B$ in $V$ contains an matrix in which last $n_p - N$ columns vanish. In other words, each element of $\mathfrak{m}$ is Ad($K$) conjugated to some element of the first component $\mathfrak{g}_2$ of the Lie algebra $\mathfrak{u}(2N) \oplus \mathfrak{u}(n_p - N) \subset \mathfrak{u}(n)$. Taking into account that the pair $\mathfrak{g}_2$ and $\mathfrak{g}_2 = \mathfrak{g}_2 \cap \mathfrak{g}'$ has the properties considered above, the space $\mathfrak{m} \cap \mathfrak{g}_2$ contains a real matrix $x_0$ with $\mathfrak{k}_0^{(0)} \simeq \mathfrak{g}_2$ because $x_0$ is regular element of $\mathfrak{g}_2$ by property 2.17 and therefore $\mathfrak{k}_0^{(0)} \simeq \mathfrak{g}_2 \cap \mathfrak{g}_2 \simeq \mathfrak{g}_2 \cap \mathfrak{g}_2$. It can be checked easily that the conditions of Theorem 2.17 hold. The following
theorem is proved:

**Theorem 2.18.** There exists a maximal involutive set of independent real analytic functions on \( (T(\tilde{G}/\tilde{K}), \Omega) \), where \( \tilde{G} = SO(n) \) and \( \tilde{K} = (SO(n_1) \times \cdots \times SO(n_p)) \) with the standard embedding of \( \tilde{K} \subset \tilde{G} \). These functions are integrals for 1) the geodesic flow determined by the Riemannian metric \( \langle \cdot, \cdot \rangle \) on \( \tilde{G}/\tilde{K} \); 2) the Hamiltonian flow with the Hamiltonian function \( \tilde{H}_{a,b} \) on \( T(\tilde{G}/\tilde{K}) \).

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