Event-Triggered Boundary Control of 2 × 2 Semilinear Hyperbolic Systems

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Abstract—In this article, we present an event-triggered boundary control scheme for hyperbolic systems. The trigger condition is based on predictions of the state on determinate sets, and the control input is updated only when the predictions deviate from the reference by a given margin. Nominal closed-loop stability, the absence of Zeno behavior, and robustness to uncertainty and disturbances, are all established analytically. For the special case of linear systems, the trigger condition can be expressed in closed-form as an $L_2$-scalar product of kernels with the distributed state. The presented controller can also be combined with existing observers to solve the event-triggered output-feedback control problem. A numerical simulation demonstrates the effectiveness of the proposed approach.

Index Terms—Boundary control, distributed-parameter systems, event-triggered control, semilinear hyperbolic systems.

I. INTRODUCTION

We consider 2 × 2 semilinear hyperbolic systems with one boundary actuator of the form

$$u_t(x,t) = -\lambda_u(x) u_x(x,t) + f^u(u(x,t), x),$$
$$v_t(x,t) = \lambda_v(x) v_x(x,t) + f^v(u(x,t), x),$$
$$u(0,t) = g^u(v(0,t), t),$$
$$v(1,t) = U(t),$$
$$w(\cdot,0) = w_0$$

where $x \in [0,1]$, $t \geq 0$, and $w(x,t) = (u(x,t), v(x,t))$, $\lambda_u$, $\lambda_v$ denote partial derivatives, $U(t)$ is the control input, and $w_0$ the initial condition.

Systems of form (1) to (5) model a range of 1-D transport processes including gas or fluid flow through pipelines, open channel flow, traffic flow, electrical transmission lines, and blood flow in arteries [1]. Consequently, the control and observer design for such systems has received much attention.

Static controllers that achieve asymptotic convergence are designed using dissipative boundary conditions in [2] and using control Lyapunov functions in [3]. The exact-finite-time controllability of such systems is analyzed in [4] and [5]. For the special case of linear systems, backstepping has become a popular method for designing feedback controllers that achieve this kind of performance. See, e.g., [6], [7], [8], [9], [10], [11], [12], [13] for a range of results using different configurations and stabilization or tracking as the objective. For semilinear and quasi-linear systems, the same control performance has been achieved using a predictive approach [14], [15], [16], [17], [18].

Recently, there has been interest in the event-triggered boundary control of (linear) hyperbolic systems [19], [20], [21], [22], [23], [24]. The related topic of sampled-data control is analyzed in [25]. In event-triggered control, the control inputs are held piecewise constant and are updated only when needed, instead of continuously in time or in a periodic fashion. One of the main benefits of such an approach is that it reduces wear and tear on the physical actuators, which is of interest in many applications. See also [26] for event-triggered control of ordinary differential equations (ODE).

The contributions of this article pertain to event-triggered implementations of the predictive feedback control approach from [14], including a novel robustness result. For clarity of presentation, we focus on the stabilization of semilinear hyperbolic systems using state-feedback control, although the approach can be adapted to tracking problems and output-feedback control. Similarly, extensions to $n + 1$ systems (that is, vector-valued $u$ as considered in [27] for linear systems), and to time-varying $f^v$ and $f^u$ are possible.

The rest of this article is organized as follows. Section II contains preliminaries, including declaration of the model assumptions, sufficient conditions for stability and well-posedness, and the continuous-in-time controller implementation. Different variants of event-triggered controllers are then presented in Section III, including a closed-form implementation [i.e., without the need to solve partial differential equations (PDEs) online] for the special case of linear systems in Section III-C. Simulation examples are presented in Section IV. Finally, Section V concludes this article.

II. PRELIMINARIES

A. Assumptions

We consider broad solutions to system (1)-(5) [28, Ch. 3.4], for which well-posedness can be shown under the following assumptions: The speeds $\lambda_u, \lambda_v \in L^\infty([0,1])$ are bounded from below and above by the finite positive values

$$k_u = \text{ess sup}_{x \in [0,1]} \{\min\{\lambda_u(x), \lambda_v(x)\}\}$$
$$k_v = \text{ess sup}_{x \in [0,1]} \{\max\{\lambda_u(x), \lambda_v(x)\}\}$$

and the model data are Lipschitz-continuous with

$$|f^{u/v}(w_1, x_1) - f^{u/v}(w_2, x_2)| \leq l_{f_v} \|w_1 - w_2\|_{\infty}$$
$$|g^{u/v}(v_1, t) - g^{u/v}(v_2, t)| \leq l_{g_v} \|v_1 - v_2\|_{\infty}$$

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Fig. 1. Characteristic lines for system (1)–(5) going upwards (\(a\)) and downwards (\(c\)). The area shaded in green indicates the determinate sets \(A^0(t)\) (including the line \((x, \tau^\ast(x))\)) and \(A^1(t)\) (excluding the line \((x, \tau^\ast(x))\)).

for all \(x_1, x_2 \in [0, 1]\), \(w_1, w_2 \in \mathbb{R}^2\), \(v_1, v_2 \in \mathbb{R}\), and \(t \in \mathbb{R}\), where the notation \(u/v\) in the superscript is a placeholder for either \(u\) or \(v\). We assume further that the origin is an equilibrium for \(U = 0\), i.e.,

\[
\begin{align*}
&f^u(0, x) = f^v(0, x) = 0 & \text{for all } x \in [0, 1] \\
g^u(0, t) = 0 & \text{for all } t \geq 0.
\end{align*}
\]

The initial condition \(w_0 \in L^\infty([0, 1], \mathbb{R}^2)\).

Remark 1: The global Lipschitz conditions are restrictive, but made here to simplify some of the proofs. For locally Lipschitz-continuous data, similar local results can be shown. See also [5] for local controllability results.

B. Characteristic Lines and Determinate Sets

The effect of the control input \(U(t)\) propagates through the domain \(x \in [0, 1]\) with finite speed \(\lambda^+\) (see Fig. 1). More precisely, the control input \(U(t)\) entering at the boundary \(x = 1\) at time \(t\) has an effect on the state at some location \(x\) only after a delay of \(\tau^+(x)\), where

\[
\tau^+(x) = \int_0^x \frac{1}{\lambda^+(\xi)} \, d\xi, \quad \tau^-(x) = \int_x^1 \frac{1}{\lambda^-(\xi)} \, d\xi.
\]

Consequently, the state can be predicted based on the current state \(w(\cdot, t)\) alone, over the determinate sets

\[
A^0(t) = \{(x, s) : x \in [0, 1], s \in [t, t + \tau^+(x)]\}
\]

\[
A^1(t) = \{(x, s) : x \in [0, 1], s \in [t, t + \tau^-(x)]\}.
\]

Lemma 2: For any \(t \geq 0\), the Cauchy problem consisting of (1)–(3) with \(w(\cdot, t)\) as the initial condition at time \(t\), has a unique solution \(u(x, s)\) for \((x, s) \in A^0(t)\) and \(v(x, s)\) for \((x, s) \in A^1(t)\). Moreover, there exists a constant \(c_1 \geq 1\) (depending on the model data) such that

\[
\begin{align*}
&\text{ess sup}_{(x, s) \in A^0(t)} |u(x, s)| \leq c_1 \|w(\cdot, t)\|_\infty \\
&\text{ess sup}_{(x, s) \in A^1(t)} |v(x, s)| \leq c_1 \|w(\cdot, t)\|_\infty.
\end{align*}
\]

Proof: The proof is given in [14, Appendix]. See also [28], [29] for a general discussion of determinate sets.

Remark 3: One convenient implementation for determining the solution of \(u(x, s)\) on \(A^0(t)\), including the solution on the line \((x, t + \tau^+(x))\), \(x \in [0, 1]\), is to solve the system consisting of (1)–(4) over the larger rectangular domain \([0, 1] \times [t, t + \tau^+(0)]\) using \(w(\cdot, t)\) as initial condition and an arbitrary input \(U\), e.g., \(U \equiv \lim_{s \to -1} v(x, t)\). The solution of \(u\) on \(A^0(t)\) can then be selected from this larger solution where, as seen in Lemma 2, it is independent of the choice of \(U\).

C. Sufficient Condition for Convergence to the Origin

Following [14], [17], we exploit that convergence to the origin is easier to characterize via conditions on the uncontrolled boundary value \(v(0, \cdot)\). In particular, we introduce the virtual input \(U^*(t)\), which is the target value for \(v(0, t)\). We then reverse the roles of \(t\) and \(x\) in (1) and (2) to formulate the system as a PDE in the positive \(x\)-direction for \(x \in [0, 1]\)

\[
\begin{align*}
u_x(x, t) &= -\frac{1}{\lambda^+(x, t)} u_t(x, t) + f^u(w(x, t), x) \lambda^+(x) \\
u_x(x, t) &= \frac{1}{\lambda^-(x, t)} v_t(x, t) - f^v(w(x, t), x) \lambda^-(x)
\end{align*}
\]

\[
v(0, t) = U^*(t) \quad \text{(21)}
\]

\[
u(0, t) = g^0(U^*(t), t). \quad \text{(22)}
\]

See Fig. 2 for the characteristic lines of the transformed system (19)–(22). Using the same methodology underlying Lemma 2, it is possible to show the following.

Lemma 4: For any \(T \geq \tau^\ast(0)\), with the domain of \(U^*(t)\) restricted to \(t \geq T\), the unique solution of the transformed system (19)–(22) on

\[
B(T) = \{(x, s) : x \in [0, 1], s \geq T + \tau^+(x)\}
\]

is independent of the initial condition \(w_0\). Moreover, there exists a constant \(c_2 \geq 1\) such that

\[
\text{ess sup}_{(x, s) \in B(T)} \|w(x, s)\|_\infty \leq c_2 \text{ ess sup}_{s \geq T} \|U^*(s)\|_\infty.
\]

D. Continuous-in-Time Control Design

The key idea in [14] is to design \(U(t)\) such that the future boundary value \(v(0, t + \tau^\ast(0))\) becomes equal to the virtual input \(U^*(t + \tau^\ast(0))\). By (24), if \(v(0, t + \tau^\ast(0)) = U^*(t + \tau^\ast(0)) = 0\) for all \(t \geq T\), the state reaches the origin by the time \(T + \tau^\ast(0)\). Defining the future state on the characteristic line

\[
\bar{w}(x, t) = \bar{u}(x, t), \bar{v}(x, t) = w(x, t + \tau^\ast(x))
\]

the following relationship between \(U\) and \(v(0, t + \tau^\ast(0), t)\) is derived in [14, Th. 2].

Lemma 5: For given \(\bar{u}\), the state \(\bar{v}\) satisfies the ODE

\[
\bar{v}_x(x, t) = \frac{-f^\prime((\bar{u}(x, t), \bar{v}(x, t)), x) \lambda^-(x)}{\lambda^+(x)}
\]

\[
\bar{v}(1, t) = U(t).
\]
Crucially, (26) is an ODE in the $x$-direction with no time dynamics. Consequently, it can be solved in either $x$-direction. In (26)–(27), it is solved in the negative $x$ direction, so that $\bar{v}(0, t)$ is a function of $U(t)$. As originally explored in [14], the alternative is to start with the desired $U^*(t + \tau^*(0))$ and solve a copy of (26) in the positive $x$-direction, i.e., backwards relative to how the actual input $U$ propagates through the domain. By setting $U^*(t) \equiv 0$, the closed-loop system converges to the origin in finite (minimum) time.

**Theorem 6:** Consider the system consisting of (1)–(5) in closed loop with $U(t)$ set according to the following algorithm at each time $t \geq 0$.

1. Predict $\bar{u}(\cdot)$ by solving (1)–(3) with $w(\cdot, t)$ as initial condition over the domain $A^u(t)$.
2. Solve the target system
   \[
   \bar{v}^y_\nu(x, t) = -\frac{f^n((\bar{u}(x, t), \bar{v}(x, t)), x)}{\lambda^\nu(x)} \quad (28)
   \]
   \[
   \bar{v}(0, t) = 0. \quad (29)
   \]
3. Set $\tilde{U}(t) = \bar{v}^y(1, t)$.

Then, $w(\cdot, t) = 0$ for all $t \geq \tau^*(0) + \tau^*(1)$.

**Proof:** With $\bar{u}(\cdot)$ determined uniquely by the prediction in step 1, (26) and (28) are equivalent ODEs in $\bar{v}(\cdot, t)$ and $\bar{v}^y(\cdot, t)$, respectively. Since these ODEs have unique solutions under the given assumptions, $\bar{v}(0, t) = \bar{v}^y(0, t) = 0$ if and only if $\bar{U}(t) = \bar{v}(1, t) = \bar{v}^y(1, t)$. That is, the construction in steps (1)–(3) ensures $\bar{v}(v, t) = 0$ for all $t \geq \tau^*(1)$. Convergence to the origin then follows from (24).

### III. EVENT-TRIGGERED CONTROL

#### A. Nominal Case

In this section, we develop an event-triggered implementation of the control strategy from Section II-D. That is, the control input is restricted to be piecewise constant with

\[
U(t) = U(t_k) \quad \text{for all } t \in [t_k, t_{k+1}) \quad (30)
\]

where $t_k, k \in \mathbb{N}$, are update times.

For the continuous-in-time implementation of the feedback controller, the updates occur continuously so that the uncontrolled boundary value is kept at $v(0, t) = U^*(t) = 0$ for $t \geq \tau^*(0)$. In the absence of any disturbances or prediction errors due to uncertainty, this ensures that the state is kept exactly at the origin. While such idealized assumptions are never exactly achievable in practice, it is usually also acceptable if the state remains sufficiently “close” to the origin. By the bound in (24), one can ensure that $|w(\cdot, t)| \leq \omega$ for given $\omega > 0$ and sufficiently large $t$ by keeping $|v(0, t)| \leq \frac{\omega}{2}$. With this in mind, the idea of the proposed event-triggered control law is to compute the control inputs in the same way as in Theorem 6, but to only update the control input at times $t_k$ where the prediction of $|v(0, t_k + \tau^*(0))|$ under the previous control input exceeds some threshold $\varepsilon > 0$. More specifically, at time $t > 0$, given the state $w(\cdot, t)$ and threshold $\varepsilon$, the control input $U(t)$ is set according to the following algorithm, which is initialized with $\tilde{U} = U(0)$ and $t_0 = 0$. The algorithm also produces the sequence of update times $t_k$, which is Zeno-free as established in Theorem 7.

**Theorem 7:** The system consisting of (1)–(5) in closed loop with $U(t)$ set according to Algorithm 1 at each time $t \geq 0$, satisfies $|w(0, s)| \leq \varepsilon$ for all $s \geq \tau^*(0)$ and $|w(0, s)| \leq c_2 \varepsilon$ for all $s \geq \tau^*(0) + \tau^*(1)$. Moreover, there exists $\Delta > 0$, which depends on $\varepsilon$, $\|w_0\|_\infty$, and the model data, such that $t_{k+1} - t_k \geq \Delta$ for all $k \in \mathbb{N}$.

**Proof:** The construction is such that if the prediction $|\bar{v}(0, t)| > \varepsilon$, then $U(t)$ is updated to a value such that $\bar{v}(0, t)$ becomes zero. That is, the design ensures that $|\bar{v}(0, t)| = |v(0, t + \tau^*)| \leq \varepsilon$ for all $t \geq 0$.

### Algorithm 1: Event-Triggered Control Law

1. **Predict** $\bar{u}(\cdot)$ by solving (1)–(3) with state-measurement $w(\cdot, t)$ as initial condition over the domain $A^u(t)$;
2. **Compute** $\bar{v}(0, t)$, which is the prediction of $v(0, t + \tau^*(0))$ with the current input value, by solving
   \[
   \bar{v}^y(x, t) = -\frac{f^n((\bar{u}(x, t), \bar{v}(x, t)), x)}{\lambda^\nu(x)} \quad (31)
   \]
   \[
   \bar{v}(1, t) = \bar{U}; \quad (32)
   \]
3. **if** $|\bar{v}(0, t)| > \varepsilon$ **then**
4. **Solve target dynamics** (28)–(29);
5. **Set** $\tilde{U} = \bar{v}(1, t)$ and append update times by $t_{k+1} = t$;
6. **end if**
7. **Set** $U(t) = \tilde{U}$.

Thus, the bound on $\|w(\cdot, s)\|_\infty$ for $s \geq \tau^*(0) + \tau^*(1)$ follows directly from (24).

It remains to be shown that there exists $\Delta > 0$ such that $t_{k+1} - t_k \geq \Delta$ for all $k \in \mathbb{N}$. The control input $\tilde{U}$ is first initialized at time $t_0 = 0$. Assuming $\tilde{U}$ was last updated at time $t_k$, we show by induction that at least some fixed time $\Delta > 0$ elapses before the update condition in line 3 of Algorithm 1 is triggered. The idea of the proof is to use the continuous-dependence results given in the Appendix to show that for short $\Delta$, as long as $\tilde{U}$ is kept constant, the solution to (31)–(32) at $x = 0$ is continuous in time. For this, rewrite (31)–(32) as the integral equation

\[
\bar{v}^y(x, t) = \bar{U} + \int_{t_k}^t f^n((\bar{u}(\xi, t), \bar{v}(\xi, t)), \xi) \lambda^\nu(\xi) d\xi. \quad (33)
\]

Note that in (33), $\bar{U}$ and $\bar{u}$ play the role of parameters.

We first establish an a priori bound on $\|w\|_\infty$. With

\[
W = \max\{c_1 \|w_0\|_\infty, \varepsilon\}. \quad (34)
\]

Equation (18) implies $\|v(0, \cdot)\|_\infty \leq W$. As in the derivation of (24), by amending (19)–(22) with the left boundary condition $u(\cdot, 0) = u_0(x)$, using $v(0, \cdot)$ instead of $U^*$, and solving in the positive $x$-direction, we obtain the bound $\|w\|_\infty \leq c_2 W$.

The characteristic line of $v$ passing through the point $(x, s_0)$ can be parameterized via the solution of

\[
\frac{ds}{d\zeta} = \lambda^\nu(z(x, s_0; s); z(x, s_0; s) = x. \quad (35)
\]

As shown in [14, Sec. A.3], $u$ is continuous in $s$ along the characteristic line $(z(x, s_0; s), s)$ on the interval where $z(x, s_0; s) \in [0, 1]$. In fact, $u$ satisfies

\[
\frac{ds}{d\zeta} = f^n(u(z(x, s_0; s); s), z(x, s_0; s)) \quad (36)
\]

The right-hand side can be bounded using the Lipschitz assumption in Section II-A and the a priori bound on $\|w\|_\infty$. In particular, in view of (12), we have

\[
\left|\frac{ds}{d\zeta} \right| \leq l_{f_n} c_2 W \quad (37)
\]

and hence

\[
|u(z(x, s_0; s) - u(x, s_0)| \leq l_{f_n} c_2 W |s - s_0| \quad (38)
\]

Now, the difference between the integral term in (33) at given $t_k$ and $t > t_k$ is bounded. Define $y_1$ as the $x$-coordinate of the intersection
of the line \( z(0, t_k + \tau^*(0); s, s) \) parameterized by \( s \) and the line \( (x, t + \tau^*(x)) \) parameterized by \( x \). Define \( y_2 \) as the \( x \)-coordinate of the intersection of the line \( z(1, t; s, s) \), \( s \leq 0 \), and the line \( (x, t_k + \tau^*(x)), x \in [0, 1] \) (see Fig. 3). Note that \( |y_1| \leq k_0(t - t_k) \) and \( |1 - y_2| \leq k_0(t - t_k) \). Then, the integral over \( x \in [0, 1] \) at time \( t_k \) is split into the parts \([y_2, 1]\) and \([0, y_2]\), and the integral at time \( t \) is split into the parts \([y_1, 1]\) and \([0, y_1]\). Bounding the right-hand side of (33) for \( x = y_2 \) via the a-priori bound on \( \|w\|_\infty \), the bound on \(|1 - y_2|\), and the basic assumptions from Section II-A, we get

\[
\left| \bar{v}(y_2, t_k) - \bar{U} \right| \leq l_{f,w} k_{i-1} c_1 c_2 \|u_0\|_\infty (1 - y_2)
\]

Similarly

\[
\left| \bar{v}(y_1, t) - \bar{v}(0, t) \right| \leq l_{f,w} k_{i-1} c_1 c_2 W(t - t_k).
\]  

Next, for \( x \in [0, y_2] \), define \( \bar{z}(x) \) as the \( x \)-coordinate of the intersection of the line \((z(x), t + \tau^*(x); s, s), s \geq 0\), and the line \((\zeta, t + \tau^*(\zeta)), \zeta \in [0, 1]\). Thus, for any \( \zeta \in [y_1, 1] \) there exists \( x \in [0, y_2] \) such that \( \bar{z}(x) = \bar{z}(x) \). We also have that \( |z(x) - x| \leq k_0(t - t_k) \) and \( \frac{d\bar{z}(x)}{dx} - 1 \leq k_0(t - t_k) \). Thus, the coordinate in the integral (33) for time \( t \) can be changed from \( x \) to \( z \) as in

\[
\bar{v}(y_1, t) = \bar{U} + \int_{y_1}^{y_2} \frac{f^*(\bar{u}(\bar{z}(x), t), \bar{v}(\bar{z}(x), t), \bar{z}(x)) \bar{z}(x) dx}{\bar{\lambda}^*(\bar{z}(x))} + \int_{y_2}^{y_2} \frac{f^*(\bar{u}(\bar{z}(x), t), \bar{v}(\bar{z}(x), t), \bar{z}(x)) \bar{z}(x) dx}{\bar{\lambda}^*(\bar{z}(x))}
\]

\[
\bar{v}(y_1, t) = \bar{U} + I_1 + I_2
\]  

where \( I_1 \) and \( I_2 \) are abbreviations for the integral terms in the third and fourth line of (41), respectively.

The integral equation for \( \bar{v}(0, t_k) \) can be rewritten as

\[
\bar{v}(y_2, t_k) + \int_{y_2}^{0} \frac{f^*(\bar{u}(\bar{z}(x), t), \bar{v}(\bar{z}(x), t), \bar{z}(x)) \bar{z}(x) dx}{\bar{\lambda}^*(\bar{z}(x))}.
\]  

Also note that the integral equation

\[
\bar{v}(y_1, t) = \bar{U} + \int_{y_2}^{y_2} \frac{f^*(\bar{u}(\bar{z}(x), t), \bar{v}(\bar{z}(x), t), \bar{z}(x)) \bar{z}(x) dx}{\bar{\lambda}^*(\bar{z}(x))}
\]  

for \( \tilde{v}(y_1, t) \) only differs from (41) for \( \bar{v}(y_1, t) \) by the term \( I_2 \). Then, expressing the bound (38) in terms of the transformation (25) and using the definition of \( \bar{z} \) gives

\[
|\bar{u}(\bar{z}(x), t) - \bar{u}(\bar{z}(x), t_k)| \leq l_{f,w} c_2 W(t - t_k)
\]  

for all \( t > t_k \) satisfying \( z(x, t_k; t) \leq 1 \). With this, noting that the integral equations (42) and (43) are now of the form (68) in the Appendix for \( \bar{x} = y_2 \), the difference between them can be bounded by use of (69). Moreover, \( I_2 \) can be bounded using the a priori bound on \( \|w\|_\infty \), the bound on \( \|\frac{d\bar{z}(x)}{dx}\| - 1 \), and the basic assumptions from Section II-A. Then, in view of (41) and (43), a result similar to Lemma 12 can be used to bound |\( \bar{v}(y_1, t) - \bar{v}(y_1, t) \) | by a term that is proportional to \( I_2 \). In short, we can obtain a bound of the form

\[
|\bar{v}(y_1, t) - \bar{v}(0, t_k)| \leq c_3 (|U - \bar{v}(y_2, t_k)| + \|\bar{u}\|_\infty \|\bar{w}\|_\infty (1 - y_2) + \|\bar{u}\|_\infty (t - t_k))
\]

\[
\leq c_4 W(t - t_k)
\]  

where \( c_3 \) and \( c_4 \) are constants and the bounds (39), (44) and the bound on \( |\bar{z}(x) - x| \) are used in the last step. Finally, by using (40) we have that

\[
|\bar{v}(0, t) - \bar{v}(0, t_k)| \leq |\bar{v}(y_1, t) - \bar{v}(0, t_k)| + |\bar{v}(y_1, t) - \bar{v}(0, t)| \leq c_5 W(t - t_k)
\]  

for some constant \( c_5 \). Therefore, since \( \bar{v}(0, t_k) = 0 \), with

\[
\Delta = \frac{e}{c_5 W}
\]

and \( W \) as defined in (34), the condition in line 3 of Algorithm 1 is not triggered before time \( t_k + \Delta \).

In Theorem 7 the trigger-parameter \( e \) is constant. As a consequence, the minimum intertrigger time \( \Delta \) decreases with increasing state norm, which can lead to frequent sampling. In some situations, it would likely be acceptable if the control specifications were relaxed while the state is far from the origin, so that the state is first brought “closer” to the equilibrium before the control is tightened. By updating \( e \) periodically, depending on the state norm, exponential convergence of the closed-loop system can be achieved.

**Theorem 8:** Given \( \gamma \in (0, 1) \) and \( T = \tau^*(0) + \tau^*(1) \), the system consisting of (1)–(5) in closed loop with \( U(t) \) set according to Algorithm 1, with \( e = \frac{\gamma}{\epsilon} \), for each \( t > 0 \) satisfies

\[
\|w(t, s)\| \leq c_1 c_2 \gamma^{1/(s/T)} \|w_0\|_\infty
\]

for all \( s > 0 \). Moreover, the bound on the intertrigger time becomes \( \Delta = \frac{e}{c_5 W} \), independently of \( \|w_0\|_\infty \).

**Proof:** We apply the approach from Theorem 7 recursively on each interval \( t \in [i T, (i + 1) T], i \in N \). On each such interval, with the given choice of \( e \), and since both \( c_1 \geq 1 \) and \( c_2 \geq 1 \), \( W \) as introduced in (34) becomes \( W = c_1 \|w_i\|_\infty T \). As such, the a priori bound on the state takes the form ess sup \( w_{i, T} \|w_i\|_\infty \leq c_1 c_2 \gamma^{1/(s/T)} \|w_0\|_\infty \) for all \( s > 0 \). Inserting this \( W \) and \( e \) into (47) gives the lower bound \( \Delta = \frac{e}{c_5 W} \), on the intertrigger time. Finally, Theorem 7 provides the recursive bound \( \|w(t, (i + 1) T)\|_\infty \leq \gamma \|w(t, i T)\|_\infty \), from which exponential convergence follows.

**Remark 9:** The developments above focus on stabilization of an equilibrium. Alternatively, tracking objectives of the form \( \varepsilon(0, t) = g_{\text{ref}}(t) \), which includes many objectives of the form \( g_{\text{ref}}(w(0, t), t) = 0 \).
Algorithm 2: Sampled-Data Event-Triggered Control Law.

1: if $\exists (i, j) \in \mathbb{N} \times \{0, \ldots, \left\lceil \frac{T}{\tau} \right\rceil\}$ then
2: Set $\varepsilon = \frac{\sum |u|}{\tau}$; $\gamma \in (0, 1)$; \end{1}
3: end if
4: if $\exists (i, j) \in \mathbb{N} \times \{0, \ldots, \left\lceil \frac{T}{\tau} \right\rceil\}$ then
5: Predict $\bar{u}(t, \cdot)$ and $\bar{v}(t, \cdot)$ as in steps 1 and 2 of Algorithm 1, using measurement $w_{ij}$ as initial condition;
6: if $|\bar{v}(t, \cdot)| > 0.5 \varepsilon$ then
7: Update $\bar{U}$ as in steps 4–5 of Algorithm 1;
8: end if
end if
9: end if
10: Set $U(t) = \bar{U}$.

(substituting $u(0, t)$ by (3) and solving for $v(0, t)$), can be solved by replacing the boundary condition of the target system, (29), by $\bar{v}(0, \cdot) = g_{\text{del}}(t + \tau^0(0))$, and setting the trigger condition in line 3 of Algorithm 1 to $|\bar{v}(0, \cdot) - g_{\text{del}}(t + \tau^0(0))| > \varepsilon$.

B. Robustness to Uncertainty, Disturbances, and Sampling

The analysis in Section III-A makes idealized assumptions in that the update condition is evaluated continuously in time and that the effect of model uncertainty and disturbances on the predictions is neglected. In this section, a periodically sampled event-triggered control algorithm is considered with sampling at times $t_{ij} = t_i + j \theta$, with sampling period $\theta > 0$, $t_i = iT$, $T = \tau^0(0) + \tau^+(1)$, $i \in \mathbb{N}$, and $j \in \{0, \ldots, \left\lceil \frac{T}{\tau} \right\rceil\}$. The actual, uncertain dynamics are governed by

$$u^\delta(x, t) = -\lambda^v(x)u^\delta_x(x, t) + f^v\delta(u^\delta(x, t), x) + d^v(x, t)$$

$$v^\delta(x, t) = -\lambda^v(x)v^\delta_x(x, t) + f^v\delta(u^\delta(x, t), x) + d^v(x, t)$$

$$u^\delta(t, 0) = g^v\delta(u^\delta(0, t), t) + d^v(t)$$

$$v^\delta(1, t) = U^\delta(U(t)) + g^v\delta(u^\delta(1, t), t) + d^v(t)$$

with bounded disturbances $d^v$ and uncertain sampled measurements $w_{ij} = (u_{ij}, v_{ij})^T$ of the actual state $w^\delta(t, t_{ij})$. Assume the uncertainties satisfy the bounds

$$|f^v\delta(w, x) - f^v(w, x)| \leq \delta_f \|w\|_\infty$$

$$|f^v\delta(w, x) - f^v(w, x)| \leq \delta_f \|w\|_\infty$$

$$|g^v\delta(v, t) - g^v(v, t)| \leq \delta_g \|v\|$$

$$|g^v\delta(u, t) - g^v(u, t)| \leq \delta_g \|u\|$$

$$\|w_{ij} - w^\delta(t, t_{ij})\| \leq \delta_w \|w^\delta(t, t_{ij})\|_\infty$$

$$\|\bar{U} - U\| \leq \delta_U \|U\|$$

for all $x \in \mathbb{R}^2$, $u, v, t, \in \mathbb{R}$, $j \in \mathbb{N}$, $i \in \mathbb{N}$, $j \in \{0, \ldots, \left\lceil \frac{T}{\tau} \right\rceil\}$, $x \in [0, 1]$, $U \in \mathbb{R}$, and satisfy the same assumptions of boundedness and Lipschitz continuity as the nominal parameters in Section II-A with $l_\rho^n l_\rho < 1$, where $l_\rho^n$ is a uniform bound on the Lipschitz constant of $g^v\delta$. Let $\delta = \delta_f + \delta_g + \delta_w + \delta_U$.

**Theorem 10:** For all $\gamma \in (0, 1)$, there exist $\theta^* > 0$ and $\theta^* > 0$ such that for all $0 < \theta^* \leq \theta^*$, $\delta \leq \delta^*$, and $\delta^* \leq 0.5$, the system consisting of (48)–(51) in closed loop with $U(t)$ set according to Algorithm 2 for each $t \geq 0$, satisfies

$$\|w^\delta(t, s)\|_\infty \leq c_1 c_2 T |s/T| |w_{0\infty}| + c_0 D \frac{T}{1 - \gamma}$$

for all $s \geq 0$, some constant $c_0 \geq 1$, and $D = \|d^v\|_\infty + \|d^v\|_\infty + \|d^v\|_\infty$.

**Proof:** The idea of the proof is to show the bound

$$|\bar{v}(0, t)| = |\bar{v}(0, t + \tau^0(0))| \leq \frac{\gamma}{c_2} \|w^\delta(t, t_i)\|_\infty + c_0 D \tag{59}$$

for all $t \in [t_i, t_i + \tau^+(1)]$, $i \in \mathbb{N}$. Due to (24), (59) implies the recursive bound

$$\|w^\delta(t, t_{i+1})\|_\infty \leq \gamma \|w^\delta(t, t_i)\|_\infty + c_0 D \tag{60}$$

which leads to (58). For any $i \in \mathbb{N}$, $j = 0, \ldots, \left\lceil \frac{T}{\tau} \right\rceil - 1$, and $t \in [t_i, t_{ij+1}]$, we have that

$$|\bar{v}^\delta(0, t)| \leq |\bar{v}(0, t_{ij})| + |\bar{v}(0, t) - \bar{v}(0, t_{ij})|$$

$$+ |\bar{v}(0, t) - \bar{v}(0, t_{ij})|$$

where the three terms of the right-hand side can be bounded as follows. By design of the trigger condition in lines 2 and 6 of Algorithm 2, and (56), $|\bar{v}(0, t_{ij})| \leq 0.5 \varepsilon \leq \frac{\gamma}{c_2} \|w^\delta(t, t_i)\|_\infty$. Using (46), where as in the proof of Theorem 8, $w$ on each interval $[t_i, t_{ij+1}]$ can be bounded by $c_1 \|w^\delta(t, t_i)\|_\infty$, we get that $|\bar{v}(0, t) - \bar{v}(0, t_{ij})| \leq c_1 c_2 (t - t_{ij})$. Lemma 13 can be used to derive a bound of the form $|\bar{v}(0, t) - \bar{v}^\delta(0, t)| \leq c_2 \delta \|w^\delta(t, t_i)\|_\infty + c_0 D e^{c_0 \theta t}$. Hence, the bound (61) is satisfied.

C. Closed-Form Expression for Linear Systems

Evaluating the trigger condition requires solving a nonlinear PDE online when predicting $\bar{u}(t, \cdot)$ and then solving an ODE to obtain $\bar{v}(0, t)$. As is the case for the continuous-in-time state-feedback controller [14, Sec. 3.4], it is possible to express the trigger condition as a simple integral of kernels (which can be precomputed) weighting the state.

Consider linear systems of the form

$$u_i = -\varepsilon_1 (x) u_x + c_1 (x) v, \quad u(0, t) = q v(0, t)$$

$$v_i = c_2 (x) v_x + c_2 (x) u, \quad v(1, t) = U(t). \tag{63}$$

The derivations in [14, Sec. 3.4] can be modified slightly to show that the trigger condition can be evaluated via

$$\bar{v}(0, t) = \bar{U} - \int_0^1 K^u(1, \xi) u(\xi, t) + K^v(1, \xi) v(\xi, t) d\xi \tag{64}$$

where the kernels $K^u$ and $K^v$ are the solution of the PDE system given in [6, eqs. (24)–(31)]. At each trigger instance, the control inputs can be updated as

$$\bar{U} = \int_0^1 K^u(1, \xi) u(\xi, t_k) + K^v(1, \xi) v(\xi, t_k) d\xi \tag{65}$$

In terms of the original backstepping transformation, this amounts to only updating the control input when the boundary value of the target system (see [6, eqs. (5)–(8)]) would otherwise exceed the threshold $|\delta(1, t)| \leq \varepsilon$. 

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The performance of the controller is illustrated below within the context of numerical simulations of a system with $\lambda^u(x) = 0.2$ for $x < 0.5$ and $\lambda^w(x) = 2 - x$ for $x \geq 0.5$. The simulation example is shown in Fig. 4. Overall, the closed-loop trajectories converge to the origin as expected from theory. Convergence is slightly slower when the event-triggered controller is used, as compared to the continuous-in-time implementation where the control input is updated at every time step of the solver. One can also see the correlation that there are more trigger times when the continuous control input changes quickly, whereas at other times the event-triggered version is constant for long periods. Moreover, for $t < 5$, the steps in the input are larger than around $t = 12$ due to the changes in $\|w(\cdot, t)\|_{\infty}$ and the corresponding values of $e$. If desired, the very minor updates of $U(t)$ after $t = 12$ could be avoided by choosing a trigger-parameter of the form $\varepsilon = \max \{5\|w(\cdot, t_i)\|_{\infty}, \varepsilon\}$, so that the state exponentially converges to a small ball around zero with a uniform lower bound on the intertrigger times.

IV. NUMERICAL EXAMPLES

The simulated trajectories for the nominal system are shown in Fig. 4. The parameters are such that in open loop the origin is an unstable equilibrium. Algorithm 2 is implemented with time-varying trigger parameter $\varepsilon = 0.2\|w(\cdot, t_i)\|_{\infty}$ and sampling period $\theta = 0.1(\tau^w(0) + \tau^u(1)) \approx 0.37$. The simulation and prediction operators are implemented by the method of lines, i.e., semidiscretizing the PDEs in space using first-order finite differences with 50 spatial elements. The resulting high-order ODEs on each interval $[t_i, t_{i+1}]$ are solved in MATLAB using ode45.

The performance of the controller is illustrated below within the context of numerical simulations of a system with $\lambda^u(x) = 0.2$ for $x < 0.5$ and $\lambda^w(x) = 2 - x$ for $x \geq 0.5$. The simulation example is shown in Fig. 4. Overall, the closed-loop trajectories converge to the origin as expected from theory. Convergence is slightly slower when the event-triggered controller is used, as compared to the continuous-in-time implementation where the control input is updated at every time step of the solver. One can also see the correlation that there are more trigger times when the continuous control input changes quickly, whereas at other times the event-triggered version is constant for long periods. Moreover, for $t < 5$, the steps in the input are larger than around $t = 12$ due to the changes in $\|w(\cdot, t)\|_{\infty}$ and the corresponding values of $e$. If desired, the very minor updates of $U(t)$ after $t = 12$ could be avoided by choosing a trigger-parameter of the form $\varepsilon = \max \{5\|w(\cdot, t_i)\|_{\infty}, \varepsilon\}$, so that the state exponentially converges to a small ball around zero with a uniform lower bound on the intertrigger times.

Trajectories for an uncertain system as in (48)–(51) are shown in Fig. 5. Each uncertain term takes one of the values $f_0^A = \{0.9, 1, 1, 1\} f^u$, $f_0^b = \{0.9, 1, 1, 1\} f^w$, $w_{ij} = \{0.9, 1, 1, 1\} w(\cdot, t_{ij})$, and $U^A = \{0.9, 1, 1, 1\} U$, resulting in 81 possible combinations which are all shown in Fig. 5. Moreover, disturbances are included with $d^0(t) = 0.1$ for $t \in [0, 40]$ and $d^0(t) = 0$ else, and $d^i(t) = 0.1$ for $t \in [60, 80]$ and $d^0(t) = 0$ else.

V. CONCLUSION

In this article, we proposed an event-triggered implementation of predictive boundary controllers for semilinear hyperbolic systems. A simple trigger condition is used, which can be tuned to achieve either minimum-time convergence to a ball around the origin or exponential convergence. The same trigger-mechanism can also be applied to backstepping control of linear systems, where a less computationally expensive implementation is possible. Compared to other approaches to event-triggered control of hyperbolic PDEs in the literature, no Lyapunov functions are required, which avoids some conservativeness. The approach is also generalizable. For instance, by replacing the state measurement in our event-triggered control scheme with the state estimate obtained via the finite-time convergent boundary observers from [14], one solves the event-triggered output feedback control problem. Estimation errors can be handled within the robustness framework from Section III-B. In quasi-linear systems, Lipschitz-continuity of the control inputs is necessary for well-posedness of broad solutions. Therefore, when extending the proposed event-triggered control scheme to quasi-linear systems, the transition between different values of $U$ needs to be smooth instead of by a jump. Once that transition is completed, the control input can remain constant until the update condition is triggered again. The extension to nonscalar systems as shown in [30] is also possible but the analysis is even more technical. In the notation of [30], any deviation between $\tilde{v}_1$ (which is associated with the fastest speed $\lambda^1$) and the value that would be achieved with continuous-in-time control affects the prediction operators $\Phi_j$. These prediction errors can be analyzed using the techniques from Section III-B and appropriate update conditions designed so that the closed-loop system still converges.

APPENDIX

EXISTENCE AND CONTINUOUS DEPENDENCE OF SOLUTIONS

The classical Banach fixed point theorem as given, e.g., in [28, Th. 2.7] will form the basis for proving continuous dependence of the predictions and solutions.

Lemma 11: Let $P$ be a metric space, $X$ a Banach space with norm $\| \cdot \|_{\infty}$, and $\Omega : P \times X \mapsto X$ a contractive mapping, i.e.,

$$\|\Omega(p, x_1) - \Omega(p, x_2)\|_{\infty} \leq \kappa \|x_1 - x_2\|_{\infty},$$

(66)

for some $\kappa < 1$ and all $x_1, x_2 \in X$ and $p \in P$. Then, for every $p \in P$ there exists a unique $x(p) \in X$ such that $x(p) = \Omega(p, x)$. Moreover, for all $p_1, p_2 \in P$

$$\|x(p_1) - x(p_2)\|_{\infty} \leq \frac{1}{1 - \kappa} \|\Omega(p_1, x(p_1)) - \Omega(p_2, x(p_1))\|_{\infty},$$

(67)

The following lemma will be used to prove continuity of the boundary value $\tilde{v}^0(0, \cdot)$.

Lemma 12: Let $h : \mathbb{R}^2 \times [0, 1] \rightarrow \mathbb{R}$, $h(u, v, x) = f^u((u, \cdot), x)^{(u, v, x)}$, and let $\tilde{v} \in [0, 1]$, $U \in \mathbb{R}$, $\tilde{u} : [0, \tilde{v}] \rightarrow \mathbb{R}$ and $\tilde{x} : [0, \tilde{x}] \rightarrow [0, 1]$. Then, the
integral equation
\[ \dot{v}(\hat{U}, \hat{u}, \hat{z})(x) = \hat{U} + \int_{\mathbb{R}} h(\hat{u}(\xi), \hat{v}(\xi), \hat{z}(\xi))d\xi, \quad x \in [0, \hat{z}] \] (68)

has a unique solution and there exists a constant \( \hat{c}_1 \) such that
\[ \|v(\hat{U}_1, \hat{u}_1, \hat{z}_1) - v(\hat{U}_2, \hat{u}_2, \hat{z}_2)\| \leq \hat{c}_1 \|\hat{U}_1 - \hat{U}_2\| \]
\[ + \|\hat{u}_1 - \hat{u}_2\| + \|\hat{z}_1 - \hat{z}_2\| \] (69)

for all \( \hat{U}_1, \hat{U}_2, \hat{u}_1, \hat{u}_2, \hat{z}_1, \hat{z}_2 \).

Proof: The proof of existence and uniqueness of the solution uses Lemma 11 and follows [28, Th. 3.1]. In particular, one can write (68) in the form \( \tilde{v}(\cdot) = \Omega[p, \tilde{v}(\cdot)] \) with \( p = [\hat{U}, \hat{u}, \hat{z}]^T \) and \( \Omega[p, \tilde{v}(\cdot)](x) = \hat{U} + \int_{\mathbb{R}} h(\tilde{u}(\xi), \tilde{v}(\xi), \tilde{z}(\xi))d\xi \). The assumptions from Section II-A imply that \( h \) is Lipschitz-continuous in all three arguments, with \( |h(u_1, v_1, x) - h(u_2, v_2, x)| \leq l_h \|u_1 - u_2\| + |v_1 - v_2| \) and \( |h(u, v, x_1) - h(u, v, x_2)| \leq \|u\| \|v\| \|x_1 - x_2\| \) for some constants \( l_h \) and \( l_v \). Then, as in the proof of [28, Th. 3.1], one can show that \( \Omega \) is a contraction with respect to the norm \( \|v(\cdot)\|_\Omega = \text{ess sup}_{x \in [0, \hat{z}]} e^{-\hat{c}_1(x-\hat{z})} \tilde{v}(x) \).

To prove (69), for all \( x \in [0, \hat{z}] \) the Lipschitz assumptions imply
\[ \|\Omega[p_1, \tilde{v}_1(\cdot)](x) - \Omega[p_2, \tilde{v}_2(\cdot)](x)\| = \|\hat{U}_1 - \hat{U}_2\| \]
\[ + \int_{\mathbb{R}} \|\tilde{u}(\xi, \tilde{v}_1(\xi), \tilde{z}_1(\xi)) - \tilde{u}(\xi, \tilde{v}_2(\xi))\|d\xi \]
\[ \leq \|\hat{U}_1 - \hat{U}_2\| + \int_{\mathbb{R}} |h_\hat{u}(\xi, \tilde{v}_1(\xi), \tilde{z}_1(\xi)) - h_\hat{u}(\xi, \tilde{v}_2(\xi))|d\xi \]
\[ + \|\hat{u}_1 - \hat{u}_2\| + \|\hat{z}_1 - \hat{z}_2\| \] (70)

After rescaling by \( e^{-\hat{c}_1(x-\hat{z})} \), (69) follows from (67).

Next, a lemma on continuous dependence on the model data, measurement, control input and disturbance is developed for use in the proof of robustness in Theorem 10. For this, the differential equation (48)–(51) on the domain
\[ C_j = \{(x, t) : x \in [0, 1], t \in [t_j, t_{j+1} + \tau^u(x)]\} \] (71)

at sampling instance \( t_j, j \in \mathbb{N} \), are converted into integral equations. The uncertain terms are treated as a parameter
\[ p = [w_1, w, U, \omega^1, \omega^2, d^1, d^2, d^3, d^4, f^1, f^2, g^1, g^2]^T \] (72)

The parameter-dependent integral equations written in operator-form are
\[ w[p](x, t) = \Omega[p, w[p]](x, t) \] (73)

where the operator \( \Omega \) is given by
\[ \Omega[p, w](x, t) = \left[ P^w[p](x, t) + Q^w[p, w](x, t) + R^w[p, w](x, t) \right] \]
\[ + Q^w[p, w](x, t) + R^w[p, w](x, t) \] (74)

with (omitting arguments on left-hand side to save space)
\[ P^w = p^w(x, t) + \int_{t_{j}}^{t} d^0(z^0(x, t, s), s) ds \] (75)
\[ p^w = \begin{cases} u_j(z^0(x, t, t_j)) & \text{if } t \leq t_j + \tau^u(x) \\ d^0(t - \tau^u(x)) & \text{else} \end{cases} \] (76)

and similar for the other terms. Thus, the bound of form (83) is obtained via (67), where the term \( e^{\tilde{c}_2(x-\hat{z})} \) comes from the rescaling according to \( \|z\|_\Omega \), (note that \( t - t_j \leq \tau^u(0) + \theta \) for all \( x, t \in C_j \), and \( e^{z^0(x)} \) is included in \( \hat{c}_2 \)).
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