$K_{\ell 3}$ and $\pi_{\ell 3}$ transition form factors

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Abstract

$K_{\ell 3}$ and $\pi_{\ell 3}$ transition form factors are calculated as an application of Dyson-Schwinger equations. The role of nonanalytic contributions to the quark–W-boson vertex is elucidated. A one-parameter model for this vertex provides a uniformly good description of these transitions, including the value of the scalar form factor of the kaon at the Callan-Treiman point. The $K_{\ell 3}$ form factors, $f_{K^\pm}$, are approximately linear on $t \in [m_e^2, m_\mu^2]$ and have approximately the same slope. $f_{K^+}(0)$ is a measure of the Euclidean constituent-quark mass ratio: $M_E^S/M_E^U$. In the isospin symmetric limit: $-f_{\pi^+}^+(0) = F_{\pi^+}(t)$, the electromagnetic pion form factor, and $f_{\pi^-}^+(t) \equiv 0$.

Keywords: Electroweak interactions; Semileptonic decays, $f_{K_{\ell 3}}(t)$, $f_{\pi_{\ell 3}}(t)$; Dyson-Schwinger equations; Confinement; Nonperturbative QCD; Quark models.

1. Introduction.

The semileptonic transitions $K^+ \rightarrow \pi^0 \ell \nu_\ell$ [$K_{\ell 3}^+$], $K^0 \rightarrow \pi^- \ell \nu_\ell$ [$K_{\ell 3}^0$] and $\pi^+ \rightarrow \pi^0 e \nu_e$ [$\pi_{\ell 3}$] proceed via the flavour-changing, vector piece of the $V - A$ electroweak interaction, in particular $j_{\mu}^{su}$ and $j_{\mu}^{du}$. The axial-vector component does not contribute because, in every case, the two mesons involved have the same parity. Neither $j_{\mu}^{sa}$ nor $j_{\mu}^{da}$ is conserved; in each case the symmetry breaking term is a measure of the current-quark mass difference, $m_s - m_u$ or $m_d - m_u$, and its enhancement due to nonperturbative effects. Therefore these processes can be employed to probe $SU_f(3)$ flavour symmetry violation.

The current status of experimental analyses of $K_{\ell 3}^+$ and $K_{\ell 3}^0$ transitions is summarised in Ref. [1]. It is not completely satisfactory, with the value of two of the four observables showing a surprising sensitivity to the initial state (see
Table 2. Contemporary theoretical analyses fall into two classes: those employing quark-gluon degrees of freedom; e.g., Refs. [2,3], and those using meson degrees of freedom; e.g., Refs. [4,5]. While there is agreement between these approaches for some observables, in Sect. 3 we identify qualitatively important quantitative disagreements whose resolution requires an appreciation of the origin and role of nonanalytic contributions to the quark-W-boson vertex.

Our analysis of the transition amplitudes, and their form factors, is a phenomenological application of Dyson-Schwinger equations [DSEs]. It is an extension of the calculation of the electromagnetic form factors: \( F_{K^0}(t) \) \( F_{K^0}(t) \) [6] and \( F_\pi(t) \) [7]. As therein, primary elements of this calculation are the dressed-quark propagator (2-point Schwinger function) for the \( u- \), \( d- \) and \( s- \)quarks \( u- \), \( d- \) and \( s- \)quarks and the Bethe-Salpeter amplitudes for the \( \pi- \) and \( K- \)mesons, the behaviour of which follows from extensive nonperturbative, model-DSE studies [8].

A significant new feature of this study is the dressed-vertex (3-point Schwinger function) describing the quark-W-boson coupling. A qualitative understanding of this is crucial in resolving the discrepancies between quark- and meson-based analyses of the transition form factors; and is an important precursor to the study of other weak-interaction processes such as those involving baryons.

2. Transition Form Factors.

The matrix elements for the \( K_{e3} \) and \( \pi_{e3} \) transitions are

\[
J_{\mu}^{K^+}(K,Q) \equiv \langle \pi^0(p)|\bar{d}_\mu s|K^+(k)\rangle \equiv \frac{1}{\sqrt{2}} \left( f_{K^+}^+(t) K_\mu + f_{K^+}^-(t) Q_\mu \right) \\
J_{\mu}^{K^0}(K,Q) \equiv \langle \pi^+(p)|\bar{u}_\mu s|K^0(k)\rangle \equiv f_{K^0}^+(t) K_\mu + f_{K^0}^-(t) Q_\mu \\
J_{\mu}^{\pi^+}(K,Q) \equiv \langle \pi^0(p)|\bar{u}_\mu d|\pi^+(k)\rangle \equiv \frac{1}{\sqrt{2}} \left( f_{\pi^+}^+(t) K_\mu + f_{\pi^+}^-(t) Q_\mu \right),
\]

where \( K = k + p \), \( Q = k - p \) and the squared-momentum transfer \( t = -Q^2 \). [1]

In the isospin-symmetric case, \( m_u = m_d \), \( j^{dn}_\mu \) is conserved and: 1) \( f_{K^0}^\pm \equiv f_{K^+}^\pm \); 2) \( f_{\pi}^\pm(t) = -F_\pi(t) \); and 3) \( f_\pi^\pm \equiv 0 \), while in the case of \( SU_f(3) \) symmetry, \( j^{su}_\mu \) is also conserved, \( f_{K^0,K^+,\pi}^\pm(t) \equiv -F_\pi(t) \) and \( f_{K^0,K^+,\pi}^\pm \equiv 0 \). Away from the symmetry limits, the Ademollo-Gatto theorem [10] entails \( f_\pi^2(0) \approx 1 \); i.e., in the vector form factor, \( SU_f(3) \) symmetry breaking effects are suppressed at \( t = 0 \). However, one expects \( f_\pi(0) \) to be sensitive to the nonperturbative enhancement of the \( SU_f(3) \) symmetry breaking current-quark mass differences, since \( f_\pi(0) \) depends on the \( s:u \) ratio of constituent-quark masses in Ref. [11].

\[1\] We employ a Euclidean space formulation with \( \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \gamma_\mu^\dagger = \gamma_\mu \) and \( a \cdot b = \sum_{i=1}^{4} a_i b_i \). A timelike vector, \( Q_\mu \), has \( Q^2 < 0 \).
The scalar form factor

\[ f_0^K(t) \equiv f_+^K(t) + \frac{t}{m_K^2 - m_\pi^2} f_0^K(t) \]  

measures the divergence of the currents in Eqs. (1) and (2), which makes it a particularly useful observable. Current algebra predicts the value of \( f_0^K \) at the Callan-Treiman point, \( t = m_K^2 - m_\pi^2 \equiv \Delta \): \( f_0^K(\Delta) = -f_K/f_\pi \), the ratio of the weak-decay constants [13]. The Callan-Treiman point is not accessible experimentally, however, the robust nature of the derivation of this result makes it a useful tool in constraining and improving a given theoretical framework.

In impulse approximation

\[ J_\mu^+ (K, Q) = \sqrt{2N_c} \int \frac{d^4 \ell}{(2\pi)^4} \text{tr}_D \left[ \bar{\Gamma}_{\pi 0} \left( \ell + \frac{1}{2} [K + Q]; -p \right) \times S_u(\ell + \frac{1}{2} K) \Gamma_{K^+} \left( \ell + \frac{1}{2} [K - Q]; k \right) S_u(\ell - \frac{1}{2} Q) iV_\mu^s(-Q) S_u(\ell + \frac{1}{2} Q) \right], \]  

with obvious modifications for \( K_{\ell 3} \) and \( \pi_{\ell 3} \). In Eq. (3): \( S_f \) is the dressed-propagator for quark flavour \( f \); \( \Gamma_M(q; P) \) is the Bethe-Salpeter amplitude for a meson \( M \), with \( q \) the relative quark-antiquark momentum and \( P \) the total momentum; and \( V_\mu^s(q; P) \) is the dressed \( s-u-W \)-vertex. Herein we work in the isospin symmetric limit; i.e., we do not distinguish between \( u- \) and \( d-\)quarks, and hence \( S_u \equiv S_d \).

Extensive studies of the DSE for the dressed-quark propagator [8] lead to the following algebraic approximating form, used successfully in Refs. [6,9]:

\[ S_f(p) = -i\gamma \cdot p \bar{\sigma}_V(p) + \sigma_\Delta(p), \]

where \( x = p^2/(2D) \) and:

\[ \bar{\sigma}_V(x) = 2D \sigma_\Delta(p^2); \quad \bar{\sigma}_S(x) = \sqrt{2D} \sigma_\Delta(p^2); \quad \bar{m}_f = m_f/\sqrt{2D}, \]  

with \( D \) a mass scale. We write the inverse of the propagator as

\[ S_f^{-1}(p) = i\gamma \cdot p A_f(p^2) + B_f(p^2). \]

The Bethe-Salpeter equation [BSE] for the pseudoscalar mesons has also been studied extensively [8]. The realisation of Goldstone’s theorem entails that, in the chiral limit, \( m_f = 0 \), the pseudoscalar Bethe-Salpeter amplitude is
completely determined by the dressed-quark propagator \[ [13] \]. As a consequence one finds that

\[
\Gamma_\pi(p; P^2 = -m_\pi^2) \approx i\gamma_5 \frac{1}{f_\pi} B^u_{m_u=0}(p^2),
\]

\[
\Gamma_K(p; P^2 = -m_K^2) \approx i\gamma_5 \frac{1}{f_K} B^s_{m_s=0}(p^2),
\]

with \( f_\pi \approx 92 \text{ MeV} \) and \( f_K \approx 113 \text{ MeV} \) the calculated pion and kaon normalisation constants, respectively, are good pointwise approximations. These results follow from BSE studies and have proven phenomenologically efficacious \[ [6,9] \].

The parameters \( C^f(\bar{m}_f), \bar{m}_f, b^f_{1,3} \) in Eqs. \((6), (7)\) are determined in a \( \chi^2 \)-fit to a range of hadronic observables. This is described in Ref. \[ [6] \] and leads to the values in Table 1. \( C^f(\bar{m}_f) \) is a function of \( \bar{m}_f \); it is only non-zero when evaluating the Bethe-Salpeter amplitudes using Eqs. \((8), (9)\), which allows the algebraic approximation to represent those differences between \( B_{m_u=0}(0) \) and \( B_{m_s=0}(0) \) observed in numerical studies of the quark DSE. In the fit, the difference between the \( u \)- and \( d \)-quarks was neglected and only that minimal difference between \( u \)- and \( s \)-quarks allowed that was necessary to ensure: \( \langle \bar{s}s \rangle < \langle \bar{u}u \rangle \); and \( m_s/m_u \gg 1 \). (\( \Lambda = 10^{-4} \) is included in Eqs. \((8), (9)\) only for the purpose of separating the small- and intermediate-\( p^2 \) behaviour of the algebraic form, characterised by \( b_0 \) and \( b_2 \); a separation in magnitude observed in numerical studies.)

Table 1

| \( C^f(\bar{m}_f = 0) \) | \( \bar{m}_f \) | \( b^f_0 \) | \( b^f_1 \) | \( b^f_2 \) | \( b^f_3 \) |
|-----------------|----------------|----------------|----------------|----------------|----------------|
| \( u \)        | 0.121          | 0.00897        | 0.131          | 2.90           | 0.603          | 0.185          |
| \( s \)        | 1.69           | 0.224          | 0.105          | 2.90           | 0.740          | 0.185          |

The algebraic approximation represented by Eqs. \((6)-(9)\) provides a good description of pion and kaon observables \[ [6] \] and has recently been employed successfully in the prediction of a wide range of vector-meson electroproduction observables \[ [9] \]; a process far outside the domain on which it was constrained.

2.1 Dressed Quark W-Boson Vertex. \( V_{su}^{\mu}(p; P) \) in Eq. \((4)\) is the new element in this study. It satisfies a DSE, from which one can derive the following Ward-Takahashi identity

\[
Q_\mu iV_{su}^{\mu}(p; Q) = S^{1}_{s}(p_+) - S^{1}_{u}(p_-) - (m_s - m_u) \Gamma^{su}_{I}(p; Q)
\]

where \( p_\pm \equiv p \pm Q/2 \) and \( \Gamma^{su}_{I}(p; Q) \) is the flavour-dependent scalar vertex (neglecting interactions, \( \Gamma^{su}_{I}(p; Q) = I_D \)). This entails that the flavour-changing
vector current is not conserved and that the symmetry breaking term measures the current-quark mass difference. Nonperturbative contributions to $\Gamma_{I}^{s}(p; Q)$ enhance this effect.

Herein we use what is known from extensive studies of the flavour-dependent, electromagnetic quark-photon vertex, $\Gamma_{I}^{f}(p; Q)$, to construct an Ansatz for $V_{I}^{su}(p; Q)$. In studies of pion and kaon electromagnetic form factors [6,7] and vector meson electroproduction [9] the Ball-Chiu Ansatz [14]

$$i\Gamma_{I}^{f}(p; Q) = i\gamma_{\mu} f_{1}^{f}(p; Q) + i\gamma_{\cdot} p p_{\mu} f_{2}^{f}(p; Q) + p_{\mu} f_{3}^{f}(p; Q),$$

(11)

has proven successful. As described in Ref. [7], this Ansatz is successful because it is the minimal solution of the Abelian Ward-Identity that is completely determined by the dressed-quark propagator and: A) has a well defined limit as $Q^{2} \to 0$; B) transforms correctly under $C, P, T$ and Lorentz transformations; and C) reduces to the bare vertex in the manner prescribed by perturbation theory. It therefore satisfies four of the six physical constraints proposed in Ref. [15] and explored in detail in Ref. [16]. Consequently, a first, minimal Ansatz is

$$V_{I}^{su}(p; Q) = \frac{1}{2} \left( \Gamma_{I}^{s}(p; Q) + \Gamma_{I}^{u}(p; Q) \right),$$

(13)

which satisfies Eq. (11), with a particular form of the symmetry breaking term, and also A)-C) above.

A property that our study, when using Eq. (13), shares with other quark-based studies; for example, Refs. [2,3,11], is that the $s-u-W$-vertex is analytic on the real-$t$ axis. An Ansatz with this property excludes resonance contributions, such as $K-\pi$ loops, which although unimportant for $t < 0$, may provide significant contributions for $t \in [m_{e}^{2}, m_{\mu}^{2}]$. We now address this issue.

3. Results and Discussion.

Equations (5)-(9) and (13) yield the results in the first column of Table 2. They are in quantitative agreement with the results of Refs. [2,3,11] and, with some qualitatively important exceptions that we discuss below, with Refs. [4,5].

Before addressing these exceptions we discuss other points of interest. The result for $f_{+}(0)$ is consistent with the Ademollo-Gatto theorem. Further, the
calculated value of $f_-(t_m)$ can be compared with Ref. \[11\]. Using our dressed-quark propagators we define a Euclidean constituent-quark mass, $M^E_f$, which provides a single, indicative and quantitative measure of the importance of nonperturbative dressing of the quarks in the infrared: $(M^E_f)^2$ is the solution of $p^2[A^f(p^2)]^2 - [B^f(p^2)]^2 = 0$. In the present case we find $M^E_{u,d} = 330$ MeV, $M^E_s = 490$ MeV. From Fig. 1 of Ref. \[11\] these values lead to $f_-(t_m) \sim 0.28$, consistent with our result. (There is a relative “−” sign between our definitions of $f_\pm$ and those of Ref. \[11\].) This emphasises that $f_-(0)$ is a probe of the nonperturbative enhancement of $SU_f(3)$ breaking.

In the experimental analysis of $K_{\ell 3}$ decays it is often assumed that, on $t \in [m^2_{\pi}, m^2_{\mu}]$, $f_+(t)$ is linear; i.e., $f_+(t) = f_+(0)[1 + \lambda_+ t/m^2_{\pi}]$, and $f_-(t) = f_-(0)[1 + \lambda_- t/m^2_{\pi}]$, with $\lambda_- = 0$; i.e., $f_-(t) = \text{constant}$. Using the definition of $\lambda$ in Table 2 we find that the difference $\lambda^e_+ - \lambda^u_+$ is small and hence that a linear approximation to $f_+$ is reasonable. However, in our study we find $\lambda_+ \sim \lambda_-$, consistent with Refs.\[2,3\] where $\lambda^e_+ = \lambda^\mu_+$, but inconsistent with the assumption that $f_-(t) = \text{constant}$.

In addressing the exceptions, we note from Table 2 that, in contrast to Ref. \[3\], neither our calculation using Eq. (13) nor Refs.\[2,3\] reproduce the anticipated value of the scalar form factor at the Callan-Treiman point, $f_0(\Delta) = -f_K/f_\pi$. Correlated with this are the results for $\lambda_0$: in comparison with the experimental analysis of $K^0_{\mu 3}$ decays, which have the smallest error, $\lambda_0$ is of the wrong sign or, if of the correct sign, it is an order-of-magnitude too small. These results are systematic effects due to the magnitude of $f_\pm(0)$ and the fact that $\lambda_+ \simeq \lambda_-$. To establish this we employed many variants of Eq. (13), the simplest of which replaces $[\Gamma^e_\mu + \Gamma^u_\mu]/2$ by $\alpha \Gamma^e_\mu + (1 - \alpha) \Gamma^u_\mu$, $0 < \alpha < 1$. The various Ansätze could introduce quantitative changes of $\sim 5\%$ in $f_0(\Delta)$ and $\sim 50\%$ in $\lambda_0$, however, such changes are patently inadequate.

3.1 Nonanalytic Contributions to the Vertex. This discrepancy is a defect of the vertex Ansatz, Eq. (13); a defect that is implicit in Refs.\[2,3\] and similar studies. That this is the case is signalled by the small value of $r_{\pi K}$ in the first column of Table 2 Nonanalytic, resonance contributions to the quark-photon vertex or, analogously, $\pi-\pi$ rescattering, provide $\lesssim 10\%$ of $r_{\pi}$ \[18\], which measures $F_1(t = 0)$. Such nonanalytic contributions rapidly becomes insignificant as $(-t)$ increases because resonances do not contribute to the vertex at spacelike momentum transfer \[19\]. However, their importance increases in the timelike region and, in calculating $f_0(t = \Delta \approx 0.56(m_K + m_\pi)^2)$, one is nearing the $K-\pi$ production threshold where such contributions dominate.

In a careful study of the DSE for the $s-u-W$-vertex, these contributions are manifest in the solution at $t > 0$. They should therefore be included in constructing a realistic Ansatz for $V^s_{\mu u}$ and hence, following Ref. \[18\], we consider
a heuristic minimal modification of Eq. (13), redefining $f_1$ as follows:

$$f_1(k; Q) \rightarrow f_1(k; Q) \mathcal{M}(t), \quad \mathcal{M}(t) \equiv \left[ 1 + c_t e^{t/t_p} \mathcal{L}(t) \right],$$

with $t_p = (m_K + m_\pi)^2$, and where

$$\mathcal{L}(t) \equiv 2 + \ln \left[ \frac{m^2_\pi}{m^2_K} \left( \frac{\Delta}{t} - \frac{\Sigma}{\Delta} \right) - \frac{\nu(t)}{t} \ln \left[ \frac{(t + \nu(t))^2 - \Delta^2}{(t - \nu(t))^2 - \Delta^2} \right] \right],$$

with $\Sigma = m^2_K + m^2_\pi$ and $\nu(t)^2 = (t - t_p)(t - t_m)$, expresses the essence of the nonanalytic structure of the $K-\pi$ loop \[5\]. In Eq. (14), $c_t$ parametrises the

| $-f_+(t_m)$ | 1.11 | 1.24 | 1.04$^a$ |
| $f_-(t_m)$ | 0.27 | 0.30 | 0.29$^a$ |
| $-f_+(0)$ | 0.98 | 0.98 | 0.93$^a$, 0.98$^c$ |
| $f_-(0)$ | 0.24 | 0.24 | 0.26$^a$, 0.15$^d$ |
| $-\xi(0)$ | 0.25 | 0.25 | 0.28$^a$, 0.28$^b$, 0.15$^d$ | 0.35 ± 0.15 |
| $\lambda^e_+$ | 0.018 | 0.030 | 0.019$^a$, 0.028$^b$, 0.030$^d$ | 0.0286 ± 0.0022 |
| $\lambda^\mu_+$ | 0.018 | 0.031 | 0.019$^a$, 0.028$^b$, 0.030$^d$ | 0.033 ± 0.008 |
| $\lambda^e_-$ | 0.012 | 0.024 | 0.019$^a$, 0.028$^b$, |
| $\lambda^\mu_-$ | 0.012 | 0.025 | 0.019$^a$, 0.028$^b$, |
| $-f_0(\Delta)$ | 0.95 | 1.22 | 0.88$^a$, 1.22$^d$ |
| $\lambda^e_0$ | −0.0024 | 0.0082 | −0.005$^a$, 0.0026$^b$, 0.017$^d$ |
| $\lambda^\mu_0$ | −0.0024 | 0.0089 | −0.005$^a$, 0.0026$^b$, 0.017$^d$ | 0.004 ± 0.007 |
| $r_{\pi K}$ (fm) | 0.47 | 0.60 | 0.48$^a$, 0.60$^d$ | 0.025 ± 0.006 |

Table 2
$t_m = (m_K - m_\pi)^2$ is the largest, physically accessible value of the squared momentum-transfer, $t$: $\xi(t) \equiv f_-(t)/f_+(t)$; $\lambda^\alpha_\alpha \equiv m^2_\alpha f'_\alpha(m^2_\alpha)/f_\alpha(0)$, $\alpha \in \{+, -, 0\}$; and $r^2_{\pi K} = 6 f'_+(0)/f_+(0)$. Experimental results are taken from Ref. \[1\]. $m^2_\mu/m^2_\pi \ll 1$ means that $\lambda^\alpha_\alpha$ is not accessible experimentally. References to theoretical comparisons are labelled: $a = \text{Ref. } \[2\], b = \text{Ref. } \[3\], c = \text{Ref. } \[4\], d = \text{Ref. } \[5\]$; where necessary, results have been calculated using the information provided.
relative weight of the analytic and nonanalytic terms at \( t = \Delta \). Its value is
determined by the requirement that this Ansatz yield \( f_0(\Delta) = -f_K/f_\pi \). This
Ansatz, which herein is only sampled on the domain \( t \in (-\infty, \Delta] \), is only a
useful heuristic tool if the required value of \( c_t \) is small, for then it satisfies our
physical requirements: it leaves the vertex unmodified for spacelike-\( t \); and it
has a logarithmic branch point at the \( K-\pi \) production threshold.

The results obtained using Eq. (14), with \( c_t = 0.17 \), are presented in the second
column of Table 2. With this value of \( c_t \), \( 0.94 < M(t < 0) \leq 1 \); i.e., for \( t < 0 \),
the modification of the vertex is small. The value of the slope parameters, \( \lambda_{+0} \),
are in agreement with experiment and Ref. [5], as is \( r_{\pi K} \). As expected, \( f_\pm(0) \)
and \( \xi(0) \) are unchanged. This outcome is qualitatively insensitive to the exact
form of \( L \); for example, using \( L(t) = -\{1 + (t_p/t) \ln[1 - t/t_p]\} \) with \( c_t = 0.335 \),
leads only to a \( \sim 3\% \) reduction in \( f_\pm(t_m) \) and a \( \sim 27\% \) reduction in \( \lambda_0 \).

![Graph](image)

**Fig. 1.** \( f_+^K(Q^2) \): solid line, Eq. (14); dashed line, Eq. (13). \( f_+^K \) is approximately linear
for \( -t_m < Q^2 < 0 \).

In Figs. 1 and 2, to further illustrate the effect of nonanalytic contributions, we
plot the \( K_{\ell_3} \) transition form factors. \( K_{\ell_3}^+ \) is not distinguished from \( K_{\ell_3}^0 \) because
hitherto we have made no attempt to fine-tune isospin breaking effects.

In Fig. 3 we compare \( f_+^K(Q^2) \) with \( f_+^\pi(Q^2) \). Since we work in the isospin
symmetric limit, \( f_+^\pi(Q^2) = -F_\pi(Q^2) \). The results are qualitatively similar to
those obtained in Ref. [3], although, as in Ref. [3], our results for \( f_+^{K\pi}(Q^2) \) are
uniformly smaller in magnitude.

4. **Summary and Conclusions.**

We have analysed the \( K_{\ell_3} \) and \( \pi_{\ell_3} \) transition form factors. An important new
element of this study is the development of an understanding of the form of the
dressed \( s-u-W \)-vertex, \( V_{\mu}^{uu}(k; Q) \), on the domain \(- (m_K + m_\pi)^2 < Q^2 < \infty \).
A simple heuristic Ansatz for \( V_{\mu}^{uu} \), with one adjustable parameter, yields a
Fig. 2. $f_K^{-}(Q^2)$: solid line, Eq. (14); dashed line, Eq. (13). $f_K^{-}$ is approximately linear for $-t_m < Q^2 < 0$ and it varies almost as rapidly as $f_K^{+}$.

Fig. 3. $-Q^2 f_K^{+}(Q^2)$: solid line; $-Q^2 f_K^{-}(Q^2) = Q^2 F_{\pi}(Q^2)$: dot-dashed line. The peak in $Q^2 f_{\pi}(Q^2)$, most pronounced for the kaon, is a characteristic signal of quark-antiquark recombination into the meson final state in exclusive processes.

We identified a systematic discrepancy between the studies of Refs. [2,3] and Refs. [4,5] arising because, in Refs. [2,3], nonanalytic contributions to $V_{\mu}^{\text{snn}}$ in the timelike region are overlooked. These contributions are important in the domain $0 \lesssim t \lesssim (m_K^2 - m_{\pi}^2)$ and allow for a correct description of the scalar form factor, $f_0^K$, in this region; for example, without these contributions it is not possible to reproduce the current-algebra prediction for $f_0^K(t)$ at the
Callan-Treiman point: \( t = m_K^2 - m_\pi^2 \). The difference between the results in columns one and two of Table 2 is a measure of the relative importance of these contributions to a given quantity.

Our results are consistent with the Ademollo-Gatto theorem, \( f_2^2(0) \approx 1 \). They also support the expectation [11] that \( f_K^- (0) \) is a quantitative measure of the nonperturbative enhancement of the current-quark mass difference \( m_s - m_u \); i.e., it is a measure of the Euclidean constituent-quark mass-ratio: \( M_s^E / M_u^E \).

Figures 1 and 2 illustrate that it is a good approximation to consider \( f_+(t) \) to be linear functions on the domain \( 0 < t < (m_K - m_\pi)^2 \). In common with other studies that employ quark-gluon degrees-of-freedom, we find \( \lambda_- \approx \lambda_+ \). This suggests that the assumption, employed in experimental analyses, that \( f_-(t) \) is a constant on this domain, should be reconsidered.

A strength of the DSE framework is that it allows the study of the \( t \)-dependence of the transition form factors on the entire domain \( -\infty < t < (m_K - m_\pi)^2 \). In the isospin symmetric limit, \( m_d = m_u \), \( f_\pi^+(t) = -F_\pi(t) \), the electromagnetic pion form factor and \( f_\pi^-(t) \equiv 0 \). One also has \( f_K^+(t) = f_K^0(t) \), however, \( f_K^+(t) \neq F_K(t) \), the electromagnetic kaon form factor. There are qualitative similarities; for example, \( -t f_K^+(t) \) exhibits the peak characteristic of quark-antiquark recombination into the meson final state in this exclusive process, however, \( |f_K^+(t)| \) falls-off less rapidly than \( |F_K(t)| \) at large-\( -t \) and this is a measure of the difference between the pion and kaon Bethe-Salpeter amplitudes; i.e., of nonperturbative effects in QCD.

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References

[1] Particle Data Group, Phys. Rev. D 50 (1994) 1173.
[2] D. Scora and N. Isgur, Phys. Rev. D 52 (1992) 2783.
[3] A. Afanasev and W. W. Buck, *Unified description of kaon electroweak form factors*, archive: [hep-ph/9606296](http://arxiv.org/abs/hep-ph/9606296).
[4] H. Leutwyler and M. Roos, Z. Phys. C 25 (1984) 91.
[5] J. Gasser and H. Leutwyler, Nucl. Phys. B 250 (1985) 517.
[6] C. J. Burden, C. D. Roberts and M. J. Thomson, Phys. Lett. B 371 (1996) 163.

[7] C. D. Roberts, Nucl. Phys. A 605 (1996) 475.

[8] C. D. Roberts and A. G. Williams, Prog. Part. Nucl. Phys. 33 (1994) 477.

[9] M. A. Pichowsky and T. S. -H. Lee, Phys. Lett. B 379, 1 (1996); Exclusive, diffractive processes and the quark substructure of mesons, preprint ANL-PHY-8529-TH-96.

[10] N. Ademollo and R. Gatto, Phys. Rev. Lett. 13 (1964) 264.

[11] N. Isgur, Phys. Rev. D 12 (1975) 3666.

[12] C. G. Callan and S. B. Treiman, Phys. Rev. Lett. 16 (1966) 153.

[13] A. Bender, C. D. Roberts and L. v. Smekal, Phys. Lett. B 380 (1996) 7; C. D. Roberts, Confinement, Diquarks and Goldstone’s theorem, to appear in the proceedings of Quark Confinement and the Hadron Spectrum, II (World Scientific, Singapore), archive: nucl-th/9609039.

[14] J. S. Ball and T.-W. Chiu, Phys. Rev. D 22 (1980) 2542.

[15] C. J. Burden and C. D. Roberts, Phys. Rev. D 47 (1993) 5581.

[16] A. Bashir and M. R. Pennington, Phys. Rev. D 50 (1994) 7679.

[17] M. R. Pennington and D. Walsh, Phys. Lett. B 253 (1991) 246.

[18] R. Alkofer, A. Bender and C. D. Roberts, Int J. Mod. Phys. A 10 (1995) 3319.

[19] K. L. Mitchell and P.C. Tandy, Pion loop contribution to ρ-ω mixing and mass splitting, archive: nucl-th/9607025.