DOUBLE EXTENSIONS OF RESTRICTED LIE ALGEBRAS

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Abstract. A double extension (D-extension) of a Lie algebra \( \mathfrak{a} \) with a non-degenerate invariant symmetric bilinear form \( B_\mathfrak{a} \), briefly a NIS Lie algebra, is an enlargement of \( \mathfrak{a} \) by means of a central extension and a derivation; the affine Kac-Moody algebras are the most known examples.

Let \( \mathfrak{a} \) be a restricted Lie algebra equipped with a NIS \( B_\mathfrak{a} \). Suppose \( \mathfrak{a} \) has a restricted derivation \( D \) such that \( B_\mathfrak{a} \) is \( D \)-invariant. We show that the double extension of \( \mathfrak{a} \) caries a \( p \)-mapping constructed by means of \( B_\mathfrak{a} \) and \( D \). We show that, the other way round, any restricted NIS Lie algebra can be obtained as a \( D \)-extension of another restricted NIS Lie algebra of codimension 2 provided that the center is not trivial together with an extra condition pertaining to the central element.

We give examples of \( D \)-extensions of restricted Lie algebras in small characteristic related with Manin triples for the Heisenberg algebra, and with the Hamiltonian and the Jacobson-Witt algebras. These \( D \)-extensions are classified up to an isometry; some of them are new.

Keywords. restricted Lie algebra, \( p \)-mapping, double extension, vectorial Lie algebra

1. Introduction

For basics, in particular, a list of examples and notation, see [BKLS, BKLLS]. Let \( \mathfrak{a} \) be a Lie algebra equipped with a non-degenerate invariant symmetric bilinear form \( B_\mathfrak{a} \), suggestively abbreviated to NIS Lie algebra in [BKLS], defined over a field \( \mathbb{K} \) of positive characteristic \( p \). The notion of double extension of the Lie algebra \( \mathfrak{a} \), called \( D \)-extension in [BeBou], was distinguished by Medina and Revoy, see [MR]. The double extension of \( \mathfrak{a} \), denoted by \( \mathfrak{g} \), simultaneously involves three ingredients:

- a central extension \( \mathfrak{a}_x \) of \( \mathfrak{a} \) with the center spanned by \( x \).
- a derivation \( D \) of \( \mathfrak{a}_x \) such that \( \mathfrak{g} = \mathfrak{a}_x \ltimes \mathbb{K}D \), a semidirect sum.
- a \( D \)-invariant NIS \( B_\mathfrak{g} \) on \( \mathfrak{g} \).

Note that the 2-cocycle needed to construct the central extension we consider is the form \( B_\mathfrak{a}(D(\cdot), \cdot) \). This is not always so, e.g., affine Kac-Moody algebras.

Favre and Santharoubane [FS] introduced an important ingredient in the study of double extensions of NIS Lie algebras: they suggested to consider them up to isometry, i.e., an isomorphism \( \pi : \mathfrak{g} \to \tilde{\mathfrak{g}} \), where \( \mathfrak{g} \) and \( \tilde{\mathfrak{g}} \) are double extensions of the same Lie algebra \( \mathfrak{a} \), such that

\[ B_{\tilde{\mathfrak{g}}}(\pi(f), \pi(g)) = B_\mathfrak{g}(f, g) \text{ for any } f, g \in \mathfrak{g}. \]

The equivalence of double extensions up to isometry turns out to be very important and useful notion, as demonstrated in [BeBou, BE, BDRS] by several new examples.

The double extensions constructed by Median and Revoy were originally carried out for Lie algebras defined over the field \( \mathbb{R} \). But it turns out that the construction holds also true for modular Lie algebras, i.e., defined over any field of characteristic \( p > 0 \). The inductive

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We are thankful to D. Leites for the critical reading of the manuscript. SB was supported by the grant NYUAD-065.
description à la Medina and Revoy, however, becomes the most challenging part, because Lie’s theorem and the Levi decomposition do not hold true if $p > 0$.

Observe that for Lie algebras over fields $\mathbb{K}$ of characteristic $p > 0$, the bilinear form we are studying is not necessarily the Killing form, since the later turns out to be degenerate in some cases and therefore does not help to study the structure of many simple Lie algebras, cf. $[SF, GP, BKLS]$. Unlike the Killing form, other non-degenerate bilinear form do not play an important role in the classification theory of simple Lie algebras of prime characteristic, see $[BGP, S, Fa]$; their significance primarily rests on cohomology theory.

Typical examples of Lie algebras with NIS are:

- $\mathfrak{g}(A)$ with indecomposable Cartan matrix $A$, see $[BKLS]$.
- Some simple $\mathbb{Z}$-graded vectorial Lie algebras of the four series of ‘Cartan type’ in characteristic $p \geq 3$ that have NIS, see $[Dz, F, BKLS]$. For instance, the vectorial Lie algebra $\mathfrak{vect}(n, N)$ for $(n, p) = (1, 3)$ or $(2, 2)$, the divergence free Lie algebra $\mathfrak{svect}(1)(n, N)$ for $n = 3$, the contact Lie algebra $\mathfrak{k}(2n + 1; N)$ for $2n + 2 \equiv -4 \pmod{p}$ and $p > 2$, the Hamiltonian Lie algebras $\mathfrak{h}(2n; N)$ and – the other types – $\mathfrak{h}_\omega(2n, N)$ studied by Skryabin, see $[Sk]$.

As far as we know, the notion of a restricted Lie algebra was introduced by Jacobson $[J]$. Roughly speaking, one requires the existence of a mapping on the modular Lie algebra that enjoys the same basis properties as the Frobenius mapping $x \mapsto x^p$ in the case of associative algebras. Lie algebras associated with algebraic groups turn out to be restricted, and this class has the strongest resemblance with the characteristic 0 case; for instance, the description of Cartan subalgebras can be done by means of maximal tori, see $[SF, S]$.

The question we consider in this paper is the following. Let $(\mathfrak{a}, B_\mathfrak{a})$ be a restricted NIS Lie algebra with a restricted derivation $D$ such that the bilinear form $B_\mathfrak{a}$ is $D$-invariant, namely,

$$B_\mathfrak{a}(D(a), b) + B_\mathfrak{a}(a, D(b)) = 0 \quad \forall a, b \in \mathfrak{a}.$$ 

Under what condition can the $p$-mapping on $\mathfrak{a}$ be extended to a $p$-mapping on the double extension of $(\mathfrak{a}, B_\mathfrak{a})$?

We answer this question for all characteristics provided the restricted derivation satisfies the condition $[N]$ which says that $D^p$ and $D$ have to be cohomolgous in the Hochschild cohomology; for more details, see $[2.3]$ This is rather an unexpected condition that has not been observed before.

It is worth mentioning that there is a large class of derivations of restricted Lie algebras that satisfy the condition $[N]$. For instance, nilpotent restricted Lie algebras necessarily satisfy condition $[N]$, due to a result of $[FSW]$. In addition to nilpotent Lie algebras, the restricted derivations of the simple restricted Lie algebras we provide as examples in $[4]$ also satisfy this condition.

We show that, the other way round, any restricted Lie algebra $\mathfrak{g}$ with NIS can be obtained as a double extension of a restricted NIS Lie algebra $\mathfrak{a}$ of codimension 2 provided the center of $\mathfrak{g}$ is not trivial, and the orthogonal complement of the central element is a $p$-ideal; for more details, see $[3]$.

We introduce the notion of equivalent double extensions that takes the $p$-mapping into account., see Theorem $[3.3.2]$. However, the formula is exhibited only if $p = 2$ and 3. The case where $p > 3$ is still out of reach.

The last section is devoted to some examples in low dimension and small characteristic. These examples are classified up to an isometry using Theorem $[3.3.2]$ We distinguish several new restricted Lie algebras that are double extensions of NIS Lie algebras. The table below summarizes our results in the case of simple Lie algebras.
| The Lie algebra       | Its non-isometric double extensions | \( p \) | Restrictedness |
|----------------------|-------------------------------------|-------|----------------|
| \( \mathfrak{psl}(4) \simeq \mathfrak{h}(1)(4; 1) \) | \( \mathfrak{po}(4; 1), \mathfrak{po}(4; 1) \) | 2     | Yes            |
| \( \mathfrak{psl}(3) \) | \( \mathfrak{gl}(3), \mathfrak{gl}(3) \) | 3     | Yes            |
| \( \mathfrak{sve}(1)(3; 1) \) | \( \mathfrak{sve}(1)(3; 1) \) | 3     | Yes            |
| \( \mathfrak{ve}(1; (2, 2)) \) | \( \mathfrak{ve}(1; (2, 2)) \) | 3     | No             |
| \( \mathfrak{ve}(2; (2, 1)) \) | \( \mathfrak{ve}(2; (2, 1)) \) | 2     | No             |

1.1. **Notation.** The statements proved with the aid of *SuperLie* code \([Gr]\) are called **Claims.**

The 1-cochain \( \tilde{x} \in C^1(\mathfrak{g}) \) denotes the dual of \( x \in \mathfrak{g} \).

2. **Main definitions**

Hereafter, \( \mathbb{K} \) is an arbitrary field of characteristic \( \text{char}(\mathbb{K}) = p \). Throughout this section, \( \mathfrak{a} \) stands for a finite-dimensional modular Lie algebra over \( \mathbb{K} \). For a comprehensive study on modular Lie algebras, see \([S, SF]\).

2.1. **Restricted Lie algebras.** Following \([J, SF]\), a mapping \( [p]_a : \mathfrak{a} \to \mathfrak{a} \), \( a \mapsto a[p]_a \) is called a \( p \)-mapping if

(R1) \( \text{ad}^a_{[p]_a} = (\text{ad}^a)^p \) for all \( a \in \mathfrak{a} \).

(R2) \( (\alpha a)[p]_a = \alpha^p a[p]_a \) for all \( a \in \mathfrak{a} \) and \( \alpha \in \mathbb{K} \).

(R3) \( (a + b)[p]_a = a[p]_a + b[p]_a + \sum_{1 \leq i \leq p-1} s_i(a, b) \), where the coefficients \( s_i(a, b) \) can be obtained from

\[
(ad^a_{\lambda a+b})^{p-1}(a) = \sum_{1 \leq i \leq p-1} i s_i(a, b) \lambda^{i-1}.
\]

The pair \((\mathfrak{a}, [p]_a)\) is referred to as a **restricted** Lie algebra.

The following Theorem, due to Jacobson, is very useful to us.

**Theorem.** \([J]\) Let \((e_j)_{j \in J}\) be a basis of \( \mathfrak{a} \) such that there are \( f_j \in \mathfrak{a} \) satisfying \( (\text{ad}^a_{e_j})^p = \text{ad}^a_{f_j} \).

Then there exists exactly one \( p \)-mapping \([p]_a : \mathfrak{a} \to \mathfrak{a} \) such that

\[
e_j[p]_a = f_j \quad \text{for all } j \in J.
\]

Let \((\mathfrak{a}, [p]_a)\) and \((\tilde{\mathfrak{a}}, [\tilde{p}]_{\tilde{a}})\) be two restricted Lie algebras. A linear map \( \pi : \mathfrak{a} \to \tilde{\mathfrak{a}} \) is called a \( p \)-**homomorphism** if \( \pi \) is a homomorphism of Lie algebras and

\[
\pi(x[p]_a) = (\pi(x))[\tilde{p}]_{\tilde{a}} \quad \text{for all } x \in \mathfrak{a}.
\]

An ideal \( I \) of \( \mathfrak{a} \) is called a \( p \)-**ideal** if \( x[p]_a \in I \) for all \( x \in I \).

For an arbitrary subset \( S \subset \mathfrak{a} \), we denote

\[
S[p]_a := \{ x[p]_a \mid x \in S \},
\]

where the expression \([p]_a \) stands for the composition \([p]_a \circ \cdots \circ [p]_a \) applied \( i \) times.

For an arbitrary ideal \( I \), we denote

\[
I_p := \sum_{i \geq 0} \text{Span}(I[p]_a^i).
\]

One can show that \( I_p \) is a \( p \)-ideal of \((\mathfrak{a}, [p]_a)\) (see, e.g., \([SF]\) Prop. 1.3)). By definition, \( I_p \) is the smallest \( p \)-ideal containing the ideal \( I \). In particular, \( \mathfrak{z}(a)_p = \mathfrak{z}(a) \), where \( \mathfrak{z}(a) \) is the center of \( \mathfrak{a} \), a consequence of the condition (R1).
If $I$ is a $p$-ideal, then the quotient Lie algebra $a/I$ has a $p$-structure defined by 
\[ (a + I)^{[p]/a/I} := a^{[p]/a} + I \quad \text{for any } a \in a, \]
and the natural map $\pi : a \to a/I$ is a $p$-homomorphism.

For each restricted Lie algebra $a$, one can construct its $p$-enveloping algebra 
\[ u(a) := U(a)/I, \]
where $I$ is the ideal generated by the central elements $a^{[p]/a} - a^n \in U(a)$.

For more details, we refer to [J, SF].

An $a$-module $M$ is called restricted if

\[ a(\cdots (a \cdot m) \cdots) = a^{[p]/a} \cdot m \quad \text{for all } a \in a \text{ and any } m \in M. \]

A derivation $D \in \mathfrak{der}(a)$ is called restricted if

\[ D(a^{[p]/a}) = (\operatorname{ad}_a^{p-1})(D(a)) \quad \text{for all } a \in a. \]

Denoted the space of restricted derivations by $\mathfrak{der}_p(a)$. Every inner derivation $\operatorname{ad}_a$, where $a \in a$, is a restricted derivation. Denote by $\mathfrak{out}_p(a) := \mathfrak{der}_p(a)/\operatorname{ad}_a$ the space of restricted outer derivations.

### 2.2. The Hochschild cohomology.

We denote by $H^n(a; M)$ the usual Chevalley-Eilenberg cohomology of the Lie algebra $a$ with coefficient in the $a$-module $M$. Following Hochschild [Ho], the restricted cohomology of a restricted Lie algebra $a$ with coefficients in a restricted module $M$ is given by

\[ H^n_{\text{res}}(a; M) := \operatorname{Ext}^n_{u(a)}(\mathbb{K}, M), \quad \text{where } n \geq 0. \]

In [Ho], Hochschild showed that there is a six-term exact sequence given by

\[ 0 \to H^1_{\text{res}}(a; M) \to H^1(a; M) \to S(a; M^a) \to H^2_{\text{res}}(a; M) \to H^2(a; M) \to S(a; H^1(a; M)), \]

where $S(X, Y)$ is the space of $p$-semi-linear maps $X \to Y$, and $M^a := \{ m \in M \mid a \cdot m = 0 \}$ is the space of $a$-invariants.

An explicit description of the space of cochains $C^k(a; M)$ for $k \leq 3$ was carried out in [EF]. This description was used to classify extensions of restricted modules and infinitesimal deformations of restricted Lie algebras.

The canonical homomorphism

\[ H_{\text{res}}(a; M) \to H(a; M) \]

maps $H^1_{\text{res}}(a; M)$ isomorphically into the subspace of $H^1(a; M)$ whose elements are represented by the 1-cocycles which satisfy the relation

\[ x^{p-1} \cdot f(x) = f(x^{[p]/a}), \]

see [Ho] Theorem 2.1, page 563. In particular, $H^1_{\text{res}}(a; a) \simeq H^1(a; a)$ if $3(a) = 0$.

In [F], there was established a relationship between the ordinary cohomology of finite-dimensional Lie algebras and their finite-dimensional $p$-envelopes.

\[ \text{A map } f : X \to Y \text{ is called } p \text{-semi-linear if } f(x + \lambda y) = f(x) + \lambda^p f(y) \text{ for all } x, y \in X \text{ and for all } \lambda \in \mathbb{K}. \]
2.3. **Restricted outer derivations.** In this paper, our tool is the space of restricted outer derivations (see [Ho, EF]):

\[
\text{out}_p(a) \simeq H^1_{\text{res}}(a; a).
\]

In all the examples we provide in §4, we do have \(\text{out}_p(a) \neq 0\).

Observe that for nilpotent restricted Lie algebras, the space \(\text{out}_p(a) \neq 0\), (see [FSW, Theorem 3.6]).

Now, let us impose one more condition on \(D \in \mathfrak{der}(a)\). Suppose that there exist \(\gamma \in K\) and \(a_0 \in a\) such that the condition below is fulfilled:

\[
\frac{D^p - \gamma D + \text{ad}_{a_0}}{D(a_0)} = 0.
\]

The restricted outer derivations we provide as main examples in §4 do satisfy the condition \(\text{N}\). Moreover, for nilpotent Lie algebras the existence of such derivations can be guaranteed by the following results of \([\text{FSW}]\):

- A derivation of a torus identically vanishes (see \([\text{FSW}], \text{Prop. 3.1}]\));
- Every derivation of \(\mathfrak{hei}(2)\) for \(p = 2\) is inner (see \([\text{FSW}]. \text{Prop. 3.2}]\));
- Apart from a torus and \(\mathfrak{hei}(2)\) for \(p = 2\), every outer restricted derivation \(D\) of any nilpotent restricted Lie algebra satisfies \(D^2 = 0\) (see \([\text{FSW}], \text{Theorem 3.3}]\)).

2.4. **Double extensions of Lie algebras.** Let \(B_a\) be a bilinear form on \(a\). We say that

(A) \(B_a\) is symmetric if \(B_a(a, b) = B_a(b, a)\) for any \(a, b \in a\);

(B) \(B_a\) is invariant if \(B_a([a, b]_a, c) = B_a(a, [b, c]_a)\) for any \(a, b, c \in a\).

We call the Lie algebra \(a\) a \(\text{NIS-Lie algebra}\) (sometimes used to be called quadratic in the literature) if it admits a non-degenerate, invariant and symmetric bilinear form \(B_a\). We denote such an algebra by \((a, B_a)\).

For a list of non-degenerate, invariant and symmetric bilinear forms on a wide class of simple modular Lie algebras (and superalgebras), see \([\text{BKLS}]\).

A NIS-Lie algebra \((a, B_a)\) is said to be reducible if it can be decomposed into a direct sum of ideals, namely \(a = \oplus I_i\), such that the ideals \(I_i\) are mutually orthogonal.

The following Theorem was proved in \([\text{MR}]\) for \(K = \mathbb{R}\). Passing to a field of characteristic \(p \neq 2\), the proof is absolutely the same.

2.4.1. **Theorem.** \([\text{MR}], \text{BeBou}\) Let \((a, B_a)\) be a NIS-Lie algebra in characteristic \(p\). Let \(D \in \mathfrak{der}(a)\) be a derivation satisfying the following conditions:

\[
B_a(D(a), b) + B_a(a, D(b)) = 0 \quad \text{for any } a, b \in a; \tag{1}
\]

If \(p = 2\), we additionally require \(B_a(D(a), a) = 0 \quad \text{for any } a \in a\).

Then there exists a NIS-Lie algebra structure on \(g := \mathcal{K} \oplus a \oplus \mathcal{K}^*\), where \(\mathcal{K} := \text{Span}\{x\}\) given as brackets. The bracket is defined by

\[
[\mathcal{K}, g]_g := 0, \quad [a, b]_g := [a, b]_a + B_a(D(a), b)x, \quad [x^*, a]_g := D(a) \quad \text{for any } a, b \in a. \tag{2}
\]

The non-degenerate symmetric bilinear form \(B_g\) on \(g\) is defined as follows:

\[
B_g|_{a \times a} := B_a, \quad B_g(a, \mathcal{K}) := 0, \quad B_g(x, x^*) := 1, \quad B_g(a, \mathcal{K}^*) := 0,
\]

\[
B_g(x, x) := 0, \quad B_g(x^*, x^*) := \begin{cases} \text{arbitrary,} & \text{if } p = 2 \\ 0, & \text{if } p > 2 \end{cases}
\]

Moreover, the form \(B_g\) is \(g\)-invariant.

\[\text{Recall that the Heisenberg Lie algebra } \mathfrak{hei}(2n) \text{ is spanned by elements } c_i, a_i \text{ for } i = 1, \ldots, n, \text{ and a central element } z \text{ with the only non-zero relation } [c_i, a_i] = z.\]
We call the Lie algebra \((g, B_g)\) constructed in Theorem 2.4.1 a \(D\)-extension of \((a, B_a)\) by means of \(D\).

2.4.2. Remark. If the derivation \(D\) is inner, the double extension is isomorphic to \(a \oplus c\), where \(c\) is a 2-dimensional center, see Theorem 3.3.1.

The converse of Theorem 2.4.1 is given by the following.

2.4.3. Theorem [MR] If \(\mathfrak{z}(g) \neq 0\), then \((g, B_g)\) can be obtained from a NIS-Lie algebra by means of a \(D\)-extension.

2.5. Vectorial Lie algebras. Over any field \(K\) of characteristic \(p > 0\), consider not polynomial coefficients but divided powers in \(n\) indeterminates, whose powers are bounded by the shearing vector \(N = (N_1, \ldots, N_n)\). We get a commutative algebra (here \(p^\infty := \infty\))

\[\mathcal{O}(n; N) := K[u; N] := \text{Span}_K(u^{(r)} \mid 0 \leq r_i < p^{N_i}),\]

where \(u^{(r)} = \prod_{1 \leq i \leq n} u_i^{(r_i)}\). The addition in \(\mathcal{O}(n; N)\) is the natural one; the multiplication is defined by

\[u_i^{(r_i)} \cdot u_i^{(s_i)} = u_i^{(r_i + s_i)}\]

Set \(1 := (1, \ldots, 1)\) and set \(\tau(N) := (p^{N_1} - 1, \ldots, p^{N_n} - 1)\).

Let us introduce distinguished partial derivatives \(\partial_i\) each of them serving as several partial derivatives at once, for each of the generators \(u_i, u_i^{(p)}, u_i^{(p^2)}, \ldots\) (or, in terms of \(y_{i,j} := u_i^{(p^j - 1)}\) : \(\partial_i(u_j^{(k)}) := \delta_{ij}u_j^{(k-1)}\) for all \(k\), i.e., \(\partial_i = \sum_{j \geq 1} (-1)^{j-1} y_{i,1}^{p-1} \cdots y_{i,j-1}^{p-1} \partial_{y_{i,j}}\).

Here we list some \(Z\)-graded vectorial Lie algebras having counterparts over \(\mathbb{C}\). We have adopted the notations of [BKLS], [BGLLS1], which differ from that of [S], [SF] and follow Bourbaki.

The general vectorial Lie algebra, known as the Jacobson-Witt algebra:

\[\text{\texttt{vect}}(n; N) := \{ \sum_i f_i \partial_i \mid f_i \in \mathcal{O}(n, N)\},\]

where the bracket is given by the Lie bracket of vector fields.

The divergence-free Lie algebra:

\[\text{\texttt{svect}}(n; N) := \{ D \in \text{\texttt{vect}}(n; N) \mid \text{div}(D) = 0\}.\]

The Lie algebra \(\text{\texttt{svect}}(n; N)\) is not simple; however its first derived \(\text{\texttt{svect}}^{(1)}(n; N)\) is simple for \(n \geq 3\), see [S], [SF]. Now, if we define the map

\[\mathcal{O}(n, N) \to \text{\texttt{vect}}(n; N) \quad f \mapsto D_{i,j}(f) = \partial_j(f)\partial_i - \partial_i(f)\partial_j,\]

then we can write

\[\text{\texttt{svect}}^{(1)}(n; N) = \text{Span}\{ D_{i,j}(f) \mid f \in \mathcal{O}(n, N), 1 \leq i < j \leq n\}.\]

The Hamiltonian Lie algebra:

\[\mathfrak{h}(2n; N) := \text{Span}\{ H_f \mid f \in \mathcal{O}(n, N)\},\]

where

\[H_f = \sum_{1 \leq i \leq n} \left( \frac{\partial f}{\partial u_i} \frac{\partial}{\partial u_{i+n}} - \frac{\partial f}{\partial u_{i+n}} \frac{\partial}{\partial u_i} \right).\]
The Lie bracket \([H_f, H_g] = H_{(f,g)}\) is given by the Poisson bracket:

\[
\{f, g\} := \sum_{1 \leq i \leq n} \left( \frac{\partial f}{\partial u_i} \frac{\partial g}{\partial u_{i+n}} - \frac{\partial f}{\partial u_{i+n}} \frac{\partial g}{\partial u_i} \right).
\]

Here again, \(\mathfrak{h}(2n; N)\) is not simple but its first derived \(\mathfrak{h}^{(1)}(2n; N)\) is simple.

Each of the deformed hamiltonian Lie algebras \(\mathfrak{h}_d(2n; N)\) classified by Skryabin [Sk] has a NIS, see [BKLS]. There are several more simple Lie (super)algebras, see [BKLS, SF], but we do not consider them in this paper.

For the classification of NISes on simple \(\mathbb{Z}\)-graded vectorial Lie algebras of the above types in characteristic \(p > 0\), see [Dz, SF]. Here is a short summary:

1) The Lie algebra \(\text{s vect}(n; N)\) has a NIS ([SF, Theorems 6.3 and 6.4 in Ch. 4]) if and only if either \(n = 1\) and \(p = 3\) when NIS is

\[
B(u^{(a)} \partial, u^{(b)} \partial) := \int u^{(a)} u^{(b)} du,
\]

or \(n = p = 2\) in which case NIS is

\[
B(u^{(a)} \partial_i, u^{(b)} \partial_j) := (i + j) \int u^{(a)} u^{(b)} du_1 \wedge du_2,
\]

where \(\int f(u) du_1 \wedge \cdots \wedge du_n := \text{coefficient of } u^{(\tau(N))}\) and \(\tau(N) := (p^{N_1} - 1, \ldots, p^{N_n} - 1)\).

2) \(\text{s vect}^{(1)}(n; N)\) has a NIS if and only if \(n = 3\); explicitly

\[
B(\partial_i, D_{jk}(u^{(\tau(N))})) = \text{sign}(i, j, k),
\]

and extending the form to other pairs of elements by invariance and linearity.

3) The Lie algebra \(\mathfrak{h}^{(1)}(2n; N)\) has a NIS given by the analogue of the Berezin integral in the super setting:

\[
B(f, g) := \int fg \text{ vol}(u) \quad (=\text{the coefficient of the monomial } u^{\tau(N)}).
\]

**Remark.** There is no NIS on simple Lie algebras \(\text{s vect}_{exp}(n; N)\) and \(\text{s vect}^{(1)}_{1+a}(n; N)\), see [BKLS, T, W].

### 3. The main results

Let \((a, B_a)\) be a restricted NIS-Lie algebra in characteristic \(p\). We denote by \(\mathfrak{g}\) the \(D\)-extension of \(a\) by means of a restricted derivation \(D\) as in Theorem 2.4.1. Namely,

\[
\mathfrak{g} := \mathcal{K} \oplus a \oplus \mathcal{K}^*, \quad \text{where } \mathcal{K} := \text{Span}\{x\},
\]

and the bracket is given by

\[
[[\mathcal{K}, \mathfrak{g}]_\mathfrak{g} = 0, \quad [a, b]_\mathfrak{g} := [a, b]_a + B_a(D(a), b)x, \quad [x^*, a]_\mathfrak{g} := D(a) \text{ for any } a, b \in a.
\]

We will show how to extend the \(p\)-mapping from \(a\) to \(\mathfrak{g}\).

The following definition is essential to us. Denote by \(\sigma_i(a, b)\) the coefficient that can be obtained from the expansion

\[
B_a(D(\lambda a + b), (ad_{a+b}^a)^{p-2}(a)) = \sum_{1 \leq i \leq p-1} i\sigma_i(a, b)\lambda^{i-1}.
\]

For instance,

- If \(p = 2\), then \(\sigma_1(a, b) = B_a(D(b), a)\).
- If \(p = 3\), then \(\sigma_1(a, b) = B_a(D(b), [b, a])\) and \(\sigma_2(a, b) = 2B_a(D(a), [b, a])\).

We will need the following Lemma.
3.1. Lemma. We have

\[ s_1^0(a, b) = s_1^0(a, b) + \sigma_1(a, b)x \quad \text{for all } a, b \in a. \]

Proof. Indeed,

\[ (ad_{\lambda a + b}^0)^{-1}(a) = (ad_{\lambda a + b}^a)^{-1}(a) + B_a(D(a + b), (ad_{\lambda a + b}^a)^{-2}(a))x. \]

The result follows immediately. \(\square\)

3.2. The \(p\)-mapping on double extensions.

3.2.1. Theorem. Let \((a, B_a)\) be a restricted NIS-Lie algebra. Let \(D \in \mathfrak{Der}_p(a)\) satisfy the conditions \((\text{N})\) and \((\text{I})\). For arbitrary \(m, l \in \mathbb{K}\), the \(p\)-structure on \(a\) can be extended to its \(D\)-extension \(g = \mathcal{X} \oplus a \oplus \mathcal{X}^*\) as follows (for any \(a \in a\), and \(s, t \in \mathbb{K}\)):

\[ a^{[p]}_a := a^{[p]} + Q(a)\], \( (tx^*)^{[p]} := t^p a_0 + t^p l x + t^p \gamma x^* \), \( (sx)^{[p]} := s^p (mx + b_0) \),

where \(b_0 \in \mathfrak{z}(a)\) such that \(D(b_0) = 0\), and \(Q\) is a map satisfying (for any \(a, b \in a\) and any \(\lambda \in \mathbb{K}\)):

\[ Q(\lambda a) = \lambda^p Q(a), \]
\[ Q(a + b) - Q(a) - Q(b) = \sum_{1 \leq i \leq p-1} \sigma_i^a(a, b). \]

Proof. Using Jacobson’s Theorem \([21]\) it suffices to show that

\[ (ad_{a}^g)^p = ad_{a^{[p]} + Q(a)x}, \quad (ad_{x^*}^g)^p = ad_{a_0 + tx^* + \gamma x^*}, \quad (ad_x^g)^p = ad_{mx + b_0}. \]

Indeed, let \(f = ux + b + vx^* \in g\). We have

\[ ad_{mx + b_0}(f) - (ad_x^g)^p(f) = [mx + b_0, f]_g - (ad_x^g)^p(f) = [b_0, b]_a + B_a(D(b_0), b) x - vD(b_0) = 0, \]

since \(x\) is central, \(b_0 \in \mathfrak{z}(a)\) and \(D(b_0) = 0\). Besides, using condition \((\text{N})\) we get

\[ ad_{a_0 + tx + \gamma x^*}(f) - (ad_x^g)^p(f) = [a_0 + lx + \gamma x^*, f]_g - D^p(b) = [a_0, b]_a + B_a(D(a_0), b)x - vD(b_0) + \gamma D(b) - D^p(b) = 0, \]

and

\[ ad_{a^{[p]} + Q(a)x}(f) - (ad_x^g)^p(f) = a^{[p]} + Q(a)x - (ad_x^g)^p(b) + v(ad_x^g)^p-1(D(a)) = [a^{[p]}, b]_a + B_a(D(a^{[p]}), b)x - vD(a^{[p]}) - (ad_x^g)^p(b) + v(ad_x^g)^p-1(D(a)) = [a^{[p]}, b]_a + B_a(D(a^{[p]}), b)x - vD(a^{[p]}) - (ad_x^g)^p(b) - B_a(D(a), (ad_x^g)^p-1(b))x + v(ad_x^g)^p-1(D(a)) + vB_a(D(a), (ad_x^g)^p-2(D(a))x = 0, \]

because

\[ B_a(D(a), (ad_x^g)^p-1(b)) = (-1)^{p-1}B((ad_x^g)^p-1 \circ D(a), b), \]
and also (using again conditions (N) and (I)):

\[
B_a(D(a), (\text{ad}_a^p)_{p-2}(D(a))) = \begin{cases} 
B_a(D(a), (\text{ad}_a^p)_{p-2} \circ \text{ad}_a^p \circ (\text{ad}_a^p)_{p-2}(D(a))) & \text{if } p > 2 \\
B_a(D(a), D(a)) & \text{if } p = 2 \\
(-1)^{\frac{p-3}{2}}B((\text{ad}_a^p)_{p-2} \circ D(a), \text{ad}_a^p(\text{ad}_a^p)_{p-2} \circ D(a)) & \text{if } p > 2 \\
B_a(D(a) + [a_0, a], a) & \text{if } p = 2 \\
= 0. 
\end{cases}
\]

Now, the property (7) of \( q \) comes from conditions (R1) and (R3) as follows. Let \( a, b \in a \).

Using Lemma 3.1, we have

\[
(a + b)^{[\varrho]} = (a + b)^{[\varrho]} + q(a + b)x \\
= a^{[\varrho]} + b^{[\varrho]} + \sum_{1 \leq i \leq p-1} s^a_i(a, b) + q(a + b)x \\
= a^{[\varrho]} + b^{[\varrho]} + \sum_{1 \leq i \leq p-1} s^a_i(a, b) + \left( q(a + b) - q(a) - q(b) - \sum_{1 \leq i \leq p-1} \sigma_i(a, b) \right)x.
\]

The proof is now complete. \( \square \)

The converse of Theorem 3.2.1 is the following.

3.2.2. Theorem. Let \((g, B_g)\) be an irreducible restricted NIS-Lie algebra of dim > 1 such that \( \mathfrak{z}(g) \neq 0 \). Let \( 0 \neq x \in \mathfrak{z}(g) \) be such that \((\mathcal{K})_p = \mathcal{K}^\perp\), where \( \mathcal{K} := \text{Span}\{x\} \). Then \((g, B_g)\) is a \( D \)-extension of a restricted NIS-Lie algebra \((a, B_a)\), where \( D \) is a restricted derivation satisfying condition (N).

Proof. The subspace \( \mathcal{K} := \text{Span}\{x\} \) is an ideal in \((g, B_g)\) because \( x \) is central in \( g \). Moreover, \( \mathcal{K}^\perp \) is also an ideal in \((g, B_g)\), see [BeBou [MR]]. Since \( g \) is irreducible, it follows that \( \mathcal{K} \subset \mathcal{K}^\perp \) and \( \dim(\mathcal{K}^\perp) = \dim(g) - 1 \). Therefore, there exists a non-zero \( x^* \in g \) such that

\[ g = \mathcal{K}^\perp \oplus \mathcal{K}^\ast, \quad \text{where } \mathcal{K}^\ast := \text{Span}\{x^*\}. \]

This \( x^* \) can be normalized so that \( B_g(x, x^*) = 1 \). Besides, \( B_g(x, x) = 0 \) since \( \mathcal{K} \cap \mathcal{K}^\perp = \mathcal{K} \).

Set \( a := (\mathcal{K} + \mathcal{K}^\ast)^\perp \). We then obtain a decomposition \( g = \mathcal{K} \oplus a \oplus \mathcal{K}^\ast \).

There exists a NIS-Lie algebra structure on the vector space \( a \) for which \( g \) is its double extension by Theorem 2.4.3. We denote a NIS on \( a \) by \( B_a \).

It remains to show that there is a \( p \)-mapping on \( a \). Since \( a \subset \mathcal{K}^\perp \), then

\[ a^{[\varrho]} \subset \mathcal{K}^\perp = \mathcal{K}^\perp = \mathcal{K} \oplus a \quad \text{for any } a \in a. \]

It follows that

\[ a^{[\varrho]} = q(a)x + s(a). \]

We will show that the map

\[ s : a \rightarrow a, \quad a \mapsto s(a), \]

is a \( p \)-mapping on \( a \). The fact that \((\lambda a)^{[\varrho]} = \lambda^{p}(a^{[\varrho]})\) implies that \( s(\lambda a) = \lambda^{p}s(a) \) and \( q(\lambda a) = \lambda^{p}q(a) \). Besides,

\[
0 = [a^{[\varrho]}, b]_g - (\text{ad}_a^p)^p(b) \\
= [s(a), b]_a + B_a(D(s(a)), b)x - (\text{ad}_a^p)^p(b) - B_a(D(a), (\text{ad}_a^p)^{p-1}(b))x.
\]

Therefore,

\[ D(s(a)) = (\text{ad}_a^p)^{p-1} \circ (D(a)), \quad [s(a), b]_a = (\text{ad}_a^p)^p(b). \]
On the other hand, denote by $v_i(a, b)$ and $u_i(a, b)$ the coefficients obtained from

\[
\sum_{1 \leq i \leq p-1} iv_i(a, b)\lambda^{i-1} = (\text{ad}^a_{\lambda a + b})^{p-1}(a),
\]

\[ (8) \]

\[
\sum_{1 \leq i \leq p-1} iu_i(a, b)\lambda^{i-1} = B_a(D(\lambda a + b), (\text{ad}^a_{\lambda a + b})^{p-2}(a)).
\]

Now, since

\[
\sum_{1 \leq i \leq p-1} is^i(a, b)\lambda^{i-1} = (\text{ad}^a_{\lambda a + b})^{p-1}(a),
\]

\[ (10) \]

\[
= (\text{ad}^a_{\lambda a + b})^{p-1}(a) + B_a(D(\lambda a + b), (\text{ad}^a_{\lambda a + b})^{p-2}(a))x,
\]

it follows that

\[
s^i(a, b) = v_i(a, b) + u_i(a, b)x.
\]

Moreover,

\[
0 = (a + b)[p]_\theta - a[p]_\theta - b[p]_\theta - \sum_{1 \leq i \leq p-1} s^i(a, b)
\]

\[
= (q(a + b) - q(a) - q(b) - \sum_{1 \leq i \leq p-1} u_i(a, b))x
\]

\[
+ s(a + b) - s(a) - s(b) - \sum_{1 \leq i \leq p-1} v_i(a, b).
\]

Consequently,

\[
s(a + b) = s(a) + s(b) + \sum_{1 \leq i \leq p-1} v_i(a, b),
\]

\[
q(a + b) - q(a) - q(b) = \sum_{1 \leq i \leq p-1} u_i(a, b).
\]

It follows that $s$ defines a $p$-mapping on $a$, that $D$ is a restricted derivation of $a$ (relative to the $p$-mapping $s$), and $q$ is a $p$-form on $a$.

Suppose that

\[ (x^*)[p]_\theta = a_0 + \beta x + \gamma x^*, \quad \text{where } a_0 \in a \text{ and } \beta, \gamma \in \mathbb{K}. \]

For all $a \in a$, we have

\[
0 = [(x^*)[p]_\theta, a]_\theta - (\text{ad}^a_{x^*})^p(a) = [a_0, a] + B_a(D(a_0), a)x + \gamma D(a) - D^p(a).
\]

It follows that $D^p = \gamma D + \text{ad}_a$ and $D(a_0) = 0$ ($B_a$ is non-degenerate).

Suppose now that

\[ x[p]_\theta = b_0 + mx + \delta x^*, \quad \text{where } m, \delta \in \mathbb{K} \text{ and } b_0 \in a. \]

For any $b \in a$, we have

\[
0 = [x[p]_\theta, b]_\theta - (\text{ad}^a_x)^p(b) = [b_0, b]_a + B_a(D(b_0), b)x - \delta D(b).
\]

It follows that $-\delta D(b) + [b_0, b]_a = 0$ and $D(b_0) = 0$ (since $B_a$ is non-degenerate).

The case where $D \not\simeq 0$ in $\text{H}_{\text{res}}^1(a; a)$: It follows that $\delta = 0$ and $b_0 \in z(a)$. Therefore, $a$ can be obtained from the restricted Lie algebra $a$ by means of the restricted derivativation $D$ as in Theorem 3.2.1

The case where $D \simeq 0$ in $\text{H}_{\text{res}}^1(a; a)$: Without loss of generality, we can assume that $D = 0$, cf. Theorem 3.3.1. In this case, $a \simeq a \oplus c$, where $c$ is the center generated by $x$ and $x^*$, and the $p$-structure is given by Eqs. (9), (10), where $a_0, b_0 \in z(a)$, $\gamma, \delta, m$ and $\beta$ are arbitrary.
The proof now is complete. □

3.3. Isometries of NIS Lie algebras. For a given NIS-Lie algebra $\mathfrak{a}$ with a bilinear form $B_{\mathfrak{a}}$, denote by $\mathfrak{g}$ (resp. $\tilde{\mathfrak{g}}$) the double extension of $\mathfrak{a}$ by means of a derivation $D$ (resp. $\tilde{D}$). An isometry between $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ is an isomorphism $\pi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ such that:

$$
\pi([f,g]_{\mathfrak{g}}) = [\pi(f),\pi(g)]_{\tilde{\mathfrak{g}}} \quad \text{for any } f, g \in \mathfrak{g},
$$

$$
B_{\tilde{\mathfrak{g}}}(\pi(f),\pi(g)) = B_{\tilde{\mathfrak{g}}}(f,g) \quad \text{for any } f, g \in \mathfrak{g}.
$$

We will assume further that the isometry satisfies $\pi(\mathcal{X} \oplus \mathfrak{a}) = \tilde{\mathcal{X}} \oplus \mathfrak{a}$, and call it an adapted isometry, see [FS, BeBou]. We will see how the derivations $D$ and $\tilde{D}$ are related with each other when $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are isometric. Hereafter, $\mathcal{X} = \text{Span}\{x\}$, $\mathcal{X}^* = \text{Span}\{x^*\}$, $\tilde{\mathcal{X}} = \text{Span}\{\tilde{x}\}$, and $\tilde{\mathcal{X}}^* = \text{Span}\{\tilde{x}^*\}$.

The following Theorem was proved in [FS] in the case where $K = \mathbb{R}$. The passage to any $p \neq 2$, the proof is absolutely the same.

3.3.1. Theorem. [FS, BeBou] Let $(\mathfrak{a}, B_{\mathfrak{a}})$ be a NIS-Lie algebra, and $D, \tilde{D} \in \text{der}(\mathfrak{a})$ satisfying condition (1). There exists an adapted isometry $\pi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ if and only if there exists an isometry $\pi_0 : \mathfrak{a} \to \mathfrak{a}, \lambda \in K^*$ and $\mathcal{X} \in \mathfrak{a}$ (all unique) such that

\begin{align}
\pi_0^{-1}\tilde{D}\pi_0 &= \lambda D + \text{ad}_\mathcal{X}; \\
\text{(Only when } p = 2) \ B_{\tilde{\mathfrak{g}}}(x^*, x^*) &= \lambda^{-2}(B_{\mathfrak{a}}(\mathcal{X}, \mathcal{X}) + B_{\tilde{\mathfrak{g}}}(\tilde{x}^*, \tilde{x}^*));
\end{align}

\begin{align}
\pi &= \pi_0 + B_{\mathfrak{a}}(\mathcal{X}, \cdot)\tilde{x} \quad \text{on } \mathfrak{a}; \\
\pi(x) &= \lambda\tilde{x}; \\
\pi(x^*) &= \lambda^{-1}(\tilde{x}^* - \pi_0(\mathcal{X}) - \rho\tilde{x}), \quad \text{where }
\rho = \begin{cases} 
\text{arbitrary} & \text{if } p = 2, \\
\frac{1}{2}B_{\mathfrak{a}}(\mathcal{X}, \mathcal{X}) & \text{if } p \neq 2.
\end{cases}
\end{align}

Remark. If $\pi_0 = \text{Id}_\mathfrak{a}$, the condition (11) means that $D \simeq \tilde{D}$ in $H^1(\mathfrak{a}; \mathfrak{a})$. Moreover, if the derivations are restricted, then condition (11) means that $D \simeq \tilde{D}$ in $H^1_{\text{res}}(\mathfrak{a}; \mathfrak{a})$.

Suppose now that $\mathfrak{a}$ is restricted with a $p$-mapping $[p]_{\mathfrak{a}}$. In Theorem 3.2.1 we proved that it is possible to extend the $p$-mapping to any double extension. Let us denote by $[p]_{\mathfrak{g}}$ (resp. $[p]_{\tilde{\mathfrak{g}}}$) the $p$-mapping on $\mathfrak{g}$ (resp. $\tilde{\mathfrak{g}}$) written in terms of $m, l, a_0, b_0, \gamma$ and $q$ (resp. $\tilde{m}, \tilde{l}, \tilde{a}_0, \tilde{b}_0, \tilde{\gamma}$ and $\tilde{q}$). The following Theorem characterizes the equivalence class of $p$-mapping on double extensions, but we prove it only for $p = 2, 3$. The formula for $p > 3$ is still out of reach.

3.3.2. Theorem. The adapted isometry $\pi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ given in Theorem 3.3.1 is a $p$-mapping if and only if

$$
\pi_0(a^{[p]_\mathfrak{a}}) = (\pi_0(a))^{[p]_\mathfrak{a}} + B_{\mathfrak{a}}(\mathcal{X}, a)p\tilde{b}_0,
$$
Moreover, if $z$.

\[ \bar{\gamma} = \lambda^{p-1} \gamma, \]
\[ \bar{\ell} = \lambda^p (B_a(x, a_0) + \lambda l - \lambda^{-1} \gamma \rho) + \rho^p \bar{m} + q(\pi_0(x)) - B_a(\bar{D}^{-1}(\pi_0(x)), \pi_0(x))), \]
\[ \bar{a}_0 = \lambda^p (\pi_0(a_0) - \lambda^{-1} \gamma \pi_0(x)) + (\pi_0(x))^{[p]}_a + \rho^p \bar{b}_0 + \bar{D}^{-1}(\pi_0(x))) \]
\[ - \left\{ \begin{array}{ll}
\lambda \pi_0([D(x), x]_a), & \text{if } p = 3, \\
0, & \text{if } p = 2.
\end{array} \right. \]

Moreover, if $\tilde{z}(a) = 0$, then the isometry $\pi_0$ of $a$ is also a $p$-mapping.

**Proof.** Let us study the $p$-structure. We have
\[ \pi(x)^{[p]}_a - (\pi(x))^{[p]}_a = \pi(mx + b_0) - (\lambda \tilde{x})^{[p]}_a = m \lambda \tilde{x} + \pi_0(x) + B_a(x, b_0) \tilde{x} - \lambda^p (\tilde{m} \tilde{x} + \bar{b}_0). \]
Therefore,
\[ \bar{m} = \lambda^{-p} (\lambda m + B_a(x, b_0)), \]
\[ \bar{b}_0 = \lambda^{-p} \pi_0(b_0), \]
\[ \tilde{q} \circ \pi_0 = \lambda q + B_a(x, (\cdot)^{[p]}_a) - B_a(x, (\cdot)^{p} \bar{m}). \]

Let us study the $p$-structure. We have
\[ \pi(x)^{[p]}_a = \pi((\cdot + \bar{x})^{[p]}_a = \pi_0(a_0) + B_a(x, a_0) \tilde{x} + l \lambda \tilde{x} + \gamma \lambda^{-1}(\tilde{x}^* - \pi_0(x) - \rho \tilde{x}) \]
\[ = (B_a(x, a_0) + \lambda \lambda^{-1} \gamma \rho \tilde{x} - \lambda^{-1} \gamma \pi_0(x) + \pi_0(a_0) + \lambda^{-1} \gamma \tilde{x}^*). \]

On the other hand,
\[ \pi((x^*)^{[p]}_a) = \pi_0((a^* + \gamma x^*)_a) = \pi_0(a_0) + B_a(x, a_0) \tilde{x} + l \lambda \tilde{x} + \gamma \lambda^{-1}(\tilde{x}^* - \pi_0(x) - \rho \tilde{x}) \]
\[ = (B_a(x, a_0) + \lambda \lambda^{-1} \gamma \rho \tilde{x} - \lambda^{-1} \gamma \pi_0(x) + \pi_0(a_0) + \lambda^{-1} \gamma \tilde{x}^*). \]

On the other hand,
\[ \pi((x^*)^{[p]}_a) = \lambda^{-p} (\bar{a}_0 + \gamma \tilde{x}^* - \pi_0(x) - \rho \tilde{x})^{[p]}_a \]
\[ = \lambda^{-p} (\bar{a}_0 + \gamma \tilde{x}^* - \pi_0(x)^{[p]}_a - \rho \bar{b}_0 - \bar{D}^{-1}(\pi_0(x)) + (\bar{I} - \rho^p \bar{m} - \tilde{q}(\pi_0(x))) \tilde{x}) \]
\[ + \lambda^{-p} B_a(\bar{D}^{-1}(\pi_0(x)), \pi_0(x)) \tilde{x} + \left\{ \begin{array}{ll}
\lambda^{-p} [\bar{D}(\pi_0(x)), \pi_0(x)]_a, & \text{if } p = 3, \\
0, & \text{if } p = 2.
\end{array} \right. \]

Therefore,
\[ \lambda^{-1} \gamma = \lambda^{-p} \gamma, \]
\[ B_a(x, a_0) + \lambda \lambda^{-1} \gamma \rho = \lambda^{-p} (\bar{I} - \rho^p \bar{m} - \tilde{q}(\pi_0(x)) + B_a(\bar{D}^{-1}(\pi_0(x)), \pi_0(x))), \]
\[ \pi_0(a_0) - \lambda^{-1} \gamma \pi_0(x) = \lambda^{-p} (\bar{a}_0 - \pi_0(x)^{[p]}_a - \rho \bar{b}_0 - \bar{D}^{-1}(\pi_0(x))) \]
\[ + \left\{ \begin{array}{ll}
\lambda^{-p} [\bar{D}(\pi_0(x)), \pi_0(x)]_a, & \text{if } p = 3, \\
0, & \text{if } p = 2.
\end{array} \right. \]
If \( \mathfrak{z}(\mathfrak{a}) = 0 \), we have \( b_0 = \bar{b}_0 = 0 \) and the isometry \( \pi_0 \) becomes a \( p \)-mapping. The proof is now complete.

\[ \square \]

4. Examples in low characteristic

4.1. Manin triples (\( p = 2 \)). Let \( (\mathfrak{h}, [p]_\mathfrak{h}) \) be a finite-dimensional restricted Lie algebra (not necessarily “NIS”), and let the dual space have the structure of an abelian Lie algebra. A NIS Lie algebra structure on \( \mathfrak{g} := \mathfrak{h} \oplus \mathfrak{h}^* \) is naturally defined as follows. First, the 2-mapping on \( \mathfrak{g} \) is defined as follows (for any \( h \in \mathfrak{h} \) and \( \pi \in \mathfrak{h}^* \), hence for any \( h + \pi \in \mathfrak{g} \)):

\begin{equation}
(h + \pi)[2]_\mathfrak{g} := h[p]_\mathfrak{h} + \pi \circ \text{ad}_h^\mathfrak{h}.
\end{equation}

The bracket of two elements is defined as follows:

\begin{equation}
[h + \pi, h' + \pi']_\mathfrak{g} := [h, h']_\mathfrak{h} + \pi \circ \text{ad}_{h'} + \pi' \circ \text{ad}_h
\end{equation}

It is easy to show that the bracket \([\cdot, \cdot]_\mathfrak{g}\) defined by Eq. (16) satisfies the Jacobi identity. Let us show that the map defined by Eq. (15) is a 2-mapping. Indeed,

\[
[h + \pi, [h + \pi, h' + \pi']_\mathfrak{g}] = [h + \pi, [h, h']_\mathfrak{h} + \pi \circ \text{ad}_{h'} + \pi' \circ \text{ad}_h]_\mathfrak{g}
\]

\[
= [h, [h, h']_\mathfrak{h} + \pi \circ \text{ad}_{h'} + \pi' \circ \text{ad}_h]_\mathfrak{h} - (\pi \circ \text{ad}_{h'} - \pi' \circ \text{ad}_h) \circ \text{ad}_h
\]

\[
= [h^2, h']_\mathfrak{h} + \pi \circ \text{ad}_{h} - \pi \circ \text{ad}_{h'} + \pi' \circ \text{ad}_h + \pi' \circ \text{ad}_h \circ \text{ad}_h
\]

\[
= [h^2, h']_\mathfrak{h} + \pi \circ \text{ad}_h \circ \text{ad}_{h'} + \pi' \circ \text{ad}_h \circ \text{ad}_{h'}
\]

\[
= [(h + \pi)[2], h' + \pi']_\mathfrak{g}.
\]

On the other hand,

\[
(h + \pi + h' + \pi')[2] = (h + h')^2 + (\pi + \pi') \circ \text{ad}_{h+h'}
\]

\[
= h^2 + h'^2 + [h, h'] + \pi \circ \text{ad}_h + \pi' \circ \text{ad}_h + \pi \circ \text{ad}_{h'} + \pi' \circ \text{ad}_{h'} + \pi' \circ \text{ad}_{h'}
\]

\[
= (h + \pi)[2] + (h' + \pi')[2] + [h + \pi, h' + \pi']_\mathfrak{g}.
\]

We define a bilinear form on \( \mathfrak{g} \) as follows:

\begin{equation}
B_\mathfrak{g}(h + \pi, h' + \pi') := \pi(h') + \pi'(h)
\end{equation}

for any \( h + \pi, h' + \pi' \in \mathfrak{g} \).

It is easy to show that the bilinear form \( B_\mathfrak{g} \) is NIS.

4.2. Heisenberg algebra \( \mathfrak{hei}(2) \). Consider the Heisenberg algebra \( \mathfrak{hei}(2) \) spanned by \( p, q \) and \( z \), with the only nonzero bracket: \([p, q] = z\). The 2-structure is given by

\[
p[2] = q[2] = 0, \quad z[2] = z.
\]

We consider the NIS-Lie algebra \( \mathfrak{a} := \mathfrak{hei}(2) \oplus \mathfrak{hei}(2)^* \) constructed as in \( \S 4.1 \). A direct computation using Eq. (15) shows that (for any \( s, w, u, v \in \mathbb{K} \))

\[
(rz + sp + wq + up^* + vq^* + tz^*)[2]_\mathfrak{a} = swz + s(up^* + vq^* + tz^*) \circ \text{ad}_p + w(uq^* + vq^* + tz^*) \circ \text{ad}_q + r(uq^* + vq^* + tz^*) \circ \text{ad}_z
\]

\[
= swz + stq^* + wt*p^*.
\]

A direct computation using Eqs. (15) and (16) shows that the only nonzero brackets are

\[
[p, q]_a = z, \quad [p, z^*]_a = q^*, \quad [q, z^*]_a = p^*.
\]
4.2.1. Claim. The space $H^1_{\text{res}}(\alpha; a)$ is spanned by the (classes of the) following cocycles:

\[ D_1 = q^* \otimes \hat{p}, \quad D_2 = q^* \otimes \hat{q}, \quad D_3 = q^* \otimes z^*, \]
\[ D_4 = p^* \otimes \hat{p}, \quad D_5 = p^* \otimes \hat{q}, \quad D_6 = z \otimes z^*, \]
\[ D_7 = p \otimes \hat{p} + q^* \otimes \hat{q} + z \otimes \hat{z}, \quad D_8 = q \otimes \hat{q} + p^* \otimes \hat{p} + z \otimes \hat{z}, \]
\[ D_9 = q^* \otimes \hat{q} + p^* \otimes \hat{p} + z^* \otimes \hat{z}. \]

Let us fix an ordered basis as follows: $p, q, z, p^*, q^*, z^*$. In this basis, the Gram matrix of the bilinear form $B_\alpha$ in \([17]\) is as follows (here $I_n$ denotes the identity $n \times n$-matrix)

\[
\begin{pmatrix}
0 & I_3 \\
I_3 & 0
\end{pmatrix}.
\]

Any derivation $D$ has the following matrix representation:

\[
D = \begin{pmatrix} A & B \\ C & F \end{pmatrix}.
\]

An easy but boring check shows that $B_\alpha$ is $D$-invariant if and only if $F = A^t, B^t = B$ and $C^t = C$. Let us consider the most general derivation $D = \sum_{1 \leq i < 9} \alpha_i D_i$, where $D_i$ are the cocycles given in Claim 4.2.1. In the same basis $p, q, z, p^*, q^*, z^*$, we have

\[
D = \begin{pmatrix} \alpha_7 E^{1,1} + (\alpha_7 + \alpha_8) E^{3,3} + \alpha_8 E^{2,2} \\ \alpha_4 E^{1,1} + \alpha_1 E^{2,1} + \alpha_2 E^{2,2} \\ \alpha_8 E^{1,1} + \alpha_7 E^{2,2} + \alpha_3 E^{2,3} + \alpha_5 E^{1,3} + \alpha_9 F \end{pmatrix},
\]

where $E^{i,j}$ is the $(i, j)$th $3 \times 3$ matrix unit.

It follows that $B_\alpha$ is $D$-invariant if and only if $\alpha_1 = \alpha_3 = \alpha_5 = 0$ and $\alpha_9 = \alpha_7 + \alpha_8$.

The most general derivation is of the form $\alpha_2 D_2 + \alpha_4 D_4 + \alpha_7 D_7 + \alpha_8 D_8 + (\alpha_7 + \alpha_8) D_9$.

Now, we define two quadratic forms on $\alpha$ as follows:

\[
q_1( rz + sp + wq + tz^* + up^* + vq^*) = rt + su,
\]
\[
q_2( rz + sp + wq + tz^* + up^* + vq^*) = rt + wv.
\]

Let us check the condition (1). Let $a = rz + sp + wq + up^* + vq^* + tz^* \in g$. The fact that

\[ B_\alpha(a, D(a)) = B_\alpha(a, (\mu_2 w + \mu_7 v + (\mu_7 + \mu_8)v)q^* + (\mu_4 s + u\mu_8 + u(\mu_7 + \mu_8))p^* + (\mu_6 t + \mu_7 r + \mu_8 r)z + + \mu_6 sp + \mu_8 wq + (\mu_7 + \mu_8)tz^*) \]

\[ = (\mu_2 w + (\mu_8)v)w + (\mu_4 s + u(\mu_7))s + (\mu_6 t + \mu_7 r + \mu_8 r)t + \mu_7 su + \mu_8 wv + + (\mu_7 + \mu_8)tr \]

\[ = (\mu_2 w)w + (\mu_4 s)s + (\mu_6 t)t = 0, \]

implies that $\mu_2 = \mu_4 = \mu_6 = 0$. Now, let $a = rz + sp + wq + up^* + vq^* + tz^* \in a$.

We have

\[ B_\alpha(a, D(b)) = B_\alpha(a, (\mu_2 w + \mu_8 v)q^* + (\mu_4 s + u\mu_7)p^* + (\mu_6 t + \mu_7 r + \mu_8 r)z + \mu_7 sp + + \mu_8 wq + (\mu_7 + \mu_8)tz^*) \]

\[ = (\mu_2 w + \mu_8 v)\tilde{w} + (\mu_4 s + u\mu_7)\tilde{s} + (\mu_6 t + \mu_7 r + \mu_8 r)\tilde{t} + \mu_7 su + \mu_8 w\tilde{v} + + (\mu_7 + \mu_8)\tilde{t} \]

\[ = (\mu_8 v)\tilde{w} + (\mu_7 r + \mu_8 r)\tilde{t} + \mu_7 su + \mu_8 w\tilde{v} + (\mu_7 + \mu_8)\tilde{t} \]

\[ = \mu_7(su + u\tilde{s}) + (\mu_7 + \mu_8)(\tilde{r}t + \tilde{r}) + \mu_8(wv + t\tilde{v}). \]
It follows that we have two $D$-extensions given by the following data (where $\alpha, \beta \in \mathbb{K}$):

\[
(D = D_7 + D_9, a_0 = \beta q^*, b_0 = \alpha q^*, q_1, m, l, \gamma = 1),
\]

\[
(D = D_8 + D_9, a_0 = \beta p^*, b_0 = \alpha p^*, q_2, \bar{m}, \bar{l}, \bar{\gamma} = 1).
\]

Let us show that these two $D$-extensions are isometric if $m, l, \bar{m}$ and $\bar{l}$ are suitably chosen. Indeed, the isometry is given by (for notation, see Theorem 3.3.1)

\[
\pi_0(z) = z, \quad \pi_0(z^*) = z^*, \quad \pi_0(p) = q,
\]

\[
\pi_0(q) = p, \quad \pi_0(q^*) = p^*, \quad \pi_0(p^*) = q^*,
\]

\[
\varepsilon = 0, \quad \lambda = 1, \quad \rho = 0.
\]

On the other hand, let us show that the $D$-extension by means of $D_7 + D_9$ is not a trivial one; namely, it is not isometric to the one by means of $\text{ad}_T$ for some $T \in \mathfrak{a}$. Suppose there is an isometry, say $\pi$. Let us write

\[
\pi_0(z) = mz, \quad \pi_0(z^*) = m_1z + m_2p + m_3q + m_4p^* + m_5q^* + m^{-1}z^*.
\]

Now, because $q^* = [z^*, p]$, it follows that

\[
\pi_0(q^*) = [m_1z + m_2p + m_3q + m_4p^* + m_5q^* + m^{-1}z^*, \pi_0(p)] = c_1p^* + c_2q^* + c_3z \quad \text{for some } c_1, c_2, c_3 \in \mathbb{K}.
\]

Similarly, since $p^* = [z^*, q]$, it follows that

\[
\pi_0(p^*) = [m^{-1}z^*, \pi_0(q)] = \tilde{c}_1p^* + \tilde{c}_2q^* + \tilde{c}_3z \quad \text{for some } \tilde{c}_1, \tilde{c}_2, \tilde{c}_3 \in \mathbb{K}.
\]

We have (here $T = W_1p + W_2p^* + W_3q + W_4q^* + W_5z + W_6z^*$, where $W_i \in \mathbb{K}$):

\[
((D_7 + D_9) \circ \pi_0 - \pi_0 \circ \text{ad}_T)(z^*) = (D_7 + D_9)(m_1z + m_2p + m_3q + m_4p^* + m_5q^* + m^{-1}z^*)
\]

\[
- \pi_0[T, z^*]
\]

\[
= m^{-1}z^* + m_1z + m_2p + m_4p^* - W_1\pi_0(q^*) - W_3\pi_0(p^*)
\]

\[
= m^{-1}z^* + m_1z + m_2p + m_4p^* - W_1(c_1p^* + c_2q^* + c_3z)
\]

\[
- W_3(\tilde{c}_1p^* + \tilde{c}_2q^* + \tilde{c}_3z).
\]

But this is never zero, hence a contradiction.

4.3. **An exceptional example: $\mathfrak{psl}(4) \cong \mathfrak{h}^{(1)}(4; (1111))$ for $p = 2$.** Consider the Hamiltonian algebra $\mathfrak{h}(4; 1)$ where the bracket is given by Eq. (3), see (2.5). The derived Lie superalgebra $\mathfrak{h}^{(1)}(4; 1)$ admits a NIS given by the Berezin integral

\[
B_{\mathfrak{h}^{(1)}(4; 1)}(f, g) := \int f g \text{vol}(\xi, \eta) \quad (= \text{the coefficient of the monomial } \xi_1 \xi_2 \eta_1 \eta_2).
\]

4.3.1. **Claim.** The space $H^1_{\text{res}}(\mathfrak{h}^{(1)}(4; 1); \mathfrak{h}^{(1)}(4; 1))$ is spanned by the cocycles:

(18)

\[
\deg = -2 : \quad D_1 = \xi_1 \otimes (\xi_1 \xi_2 \eta_2) + \xi_2 \otimes (\xi_1 \xi_2 \eta_2) + \eta_1 \otimes (\xi_2 \eta_1 \eta_2) + \eta_2 \otimes (\xi_1 \eta_1 \eta_2);
\]

\[
\deg = 0 : \quad D_2 = \xi_1 \otimes \eta_1 + (\xi_1 \xi_2) \otimes \xi_2 \eta_1 + (\xi_2 \eta_1) \otimes \eta_1 \xi_2 + (\xi_1 \xi_2) \otimes (\xi_2 \eta_1) ;
\]

\[
D_3 = \xi_2 \otimes \eta_2 + (\xi_1 \xi_2) \otimes \xi_2 \eta_2 + (\xi_2 \eta_1) \otimes \eta_1 \xi_2 + (\xi_1 \xi_2) \otimes (\xi_2 \eta_1) ;
\]

\[
D_4 = \eta_1 \otimes \xi_1 + (\xi_1 \eta_1) \otimes \xi_1 \eta_1 + (\xi_1 \eta_1) \otimes (\xi_1 \eta_1) ;
\]

\[
D_5 = \eta_2 \otimes \xi_2 + (\xi_1 \eta_2) \otimes \xi_2 \eta_2 + (\xi_1 \eta_2) \otimes (\xi_2 \eta_2) ;
\]

\[
D_6 = \xi_2 \otimes \xi_2 + \eta_1 \otimes \eta_2 + (\xi_1 \eta_2) \otimes \xi_2 \eta_2 + (\xi_1 \eta_2) \otimes (\xi_2 \eta_2) ;
\]

\[
\deg = 2 : \quad D_7 = (\xi_1 \eta_1) \otimes (\xi_2 \eta_2) + (\xi_1 \eta_2) \otimes (\xi_2 \eta_1) + (\xi_1 \eta_2) \otimes (\xi_2 \eta_2) + (\xi_1 \eta_2) \otimes (\xi_2 \eta_2).
\]
Fix a lexicographically ordered basis of \( \mathfrak{h}^{(1)}(4; 1) \). In this basis, we identify the bilinear form \( B \) with its Gram matrix antidiag(1, ..., 1). All derivations (18) satisfy
\[
B_{\mathfrak{h}^{(1)}(4; 1)}(D(f), g) = B_{\mathfrak{h}^{(1)}(4; 1)}(f, D(g)) \quad \text{for any } f, g \in \mathfrak{h}^{(1)}(4; 1) \quad \text{and}
\]
\[
B_{\mathfrak{h}^{(1)}(4; 1)}(D(f), f) = 0 \quad \text{for any } f \in \mathfrak{h}^{(1)}(4; 1).
\]

Let us give the proof only for the cocycle \( D_1 \). The proof is identical for the other derivations. The matrix representation of \( D_1 \) in the same basis is \( D_1 \cong E^{1,12} + E^{2,11} + E^{3,14} + E^{4,13} \). Now, the condition \( B^t D_1 = D_1 B \) is easily seen. Besides, a direct computation shows that
\[
B_{\mathfrak{h}^{(1)}(4; 1)}(D_1(f), f) = 0 \quad \text{for any } f \in \mathfrak{h}^{(1)}(4; 1).
\]

Let \( a = \lambda_1 \xi_1 + \ldots + \lambda_8 \xi_2 \eta_1 \eta_2 \). The quadratic forms associated with \( D_1 \) and \( D_7 \) are given, respectively, by:
\[
\alpha_1(a) = \lambda_6 \lambda_8 + \lambda_5 \lambda_7, \quad \alpha_7(a) = \lambda_2 \lambda_4 + \lambda_1 \lambda_3.
\]

Let us show that, up to an isometry, the derivations \( (D_1, \alpha_1) \) and \( (D_7, \alpha_7) \) give the same Lie algebra. Indeed, the isometry is given by (other generators are fixed):
\[
\xi_1 \leftrightarrow \xi_1 \xi_2 \eta_2, \quad \xi_2 \leftrightarrow \xi_1 \xi_2 \eta_1, \quad \eta_1 \leftrightarrow \xi_2 \eta_1 \eta_2, \quad \eta_2 \leftrightarrow \xi_1 \eta_1 \eta_2, \quad \lambda = 1, \quad \nu = 0, \quad t = 0.
\]

On the other hand, the \( D \)-extension of the Lie algebra \( \mathfrak{h}^{(1)}(4; 1) \) by means of \( (D_7, \alpha_7) \) is isomorphic to the Poisson algebra \( \mathfrak{po}(4; 1) \). Indeed the isomorphism is given by
\[
\varphi(x) = 1, \quad \varphi(x^*) = \xi_1 \xi_2 \eta_1 \eta_2, \quad \varphi|_{\mathfrak{h}^{(1)}(4; 1)} = \text{id}.
\]

The \( D_2 \)-extension and \( D_4 \)-extension are isometric where the isometry is given by \( \xi_i \leftrightarrow \eta_i \) for \( i = 1, 2 \); similarly, the same isometry shows that the \( D_3 \)-extension and the \( D_5 \)-extension are isometric. Moreover, all extensions by means of the derivations \( D_2, D_3, D_4 \) and \( D_5 \) are isometric. Here is the isometry relating \( D_2 \) with \( D_3 \):
\[
D_3: \quad \xi_1 \leftrightarrow \xi_2, \quad \xi_2 \leftrightarrow \xi_1, \quad \eta_1 \leftrightarrow \eta_2, \quad \eta_2 \leftrightarrow \eta_1, \quad \lambda = 1, \quad \nu = 0; \quad t = 0,
\]
then extended to monomials in \( \xi \)'s and \( \eta \)'s.

Moreover, the double extension of \( \mathfrak{h}^{(1)}(4; 1) \) by means of \( (D_6, \alpha_6) \), see Table (19), is isometric to \( \mathfrak{gl}(4) \) with the standard NIS given by the trace. The isometry is explicitly given on generators by the following correspondences (other elements are obtained by bracketing)
\[
\eta_1 \leftrightarrow E^{3,2}, \quad \xi_1 \xi_2 \eta_2 \leftrightarrow E^{2,1}, \quad \xi_1 \xi_2 \leftrightarrow E^{3,1}, \quad \xi_1 \xi_2 \eta_2 \leftrightarrow E^{2,3}, \quad \xi_2 \eta_1 \leftrightarrow E^{1,2}, \quad \eta_1 \eta_2 \leftrightarrow E^{3,4}, \quad x \leftrightarrow I, \quad x^* \leftrightarrow E^{2,2}.
\]

Table (19) gives the quadratic form associated with each derivation, and the respective double extension, up to an isometry:

| Derivation | \( \alpha(a) \) | Double extension |
|------------|------------------|-----------------|
| \( D_2 \)  | \( \lambda_3 \lambda_8 \) | \( \mathfrak{po}(4; 1) \) |
| \( D_6 \)  | \( \lambda_2 \lambda_7 + \lambda_3 \lambda_6 \) | \( \mathfrak{gl}(4) \) |
| \( D_7 \)  | \( \lambda_2 \lambda_4 + \lambda_1 \lambda_3 \) | \( \mathfrak{po}(4; 1) \) |

4.3.2. **Claim.** \( \dim \text{out}(\mathfrak{po}(4; 1)) = 3 \), \( \dim \text{out}(\mathfrak{gl}(4)) = 1 \) and \( \dim \text{out}(\mathfrak{po}(4, 1)) = 5 \), hence \( \mathfrak{po}(4, 1) \), \( \mathfrak{gl}(4) \) and \( \mathfrak{po}(4; 1) \) are pairwise not isomorphic.

4.4. **An exceptional example:** \( \mathfrak{psl}(3) \) for \( p = 3 \). Let us fix a basis of \( \mathfrak{psl}(3) \) generated by the root vectors \( x_1, x_2, x_3 = [x_1, x_2] \) (positive) and \( y_1, y_2, y_3 = [y_1, y_2] \) (negative). The Lie algebra \( \mathfrak{psl}(3) \) admits a NIS given in the ordered basis \( e_1 = [x_1, y_1], e_2 = x_1, e_3 = x_2, e_4 = x_3, e_5 = y_1, e_6 = y_2, e_7 = y_3 \) by the Gram matrix
\[
B_{\mathfrak{psl}(3)} = \begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{2,1} & 0 & 0 & 0 & 0 \\
0 & I_{2,1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
where $I_{p,q} = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$ with $p$-many 1s and $q$ many $(-1)$s.

### 4.4.1. Claim

The space $H^1_{\text{res}}(\mathfrak{psl}(3); \mathfrak{psl}(3))$ is spanned by the cocycles:

$$
\deg = -3: \quad D_1 = y_1 dx_3 + y_3 dx_1, \quad D_2 = y_2 dx_3 + y_3 dx_2.
$$

$$
\deg = 0: \quad D_3 = 2 x_2 dx_1 + y_1 dy_2, \quad D_4 = 2 x_1 dx_2 + y_2 dy_1, \quad D_5 = x_1 dx_1 + x_3 dx_3 + 2 y_1 dy_1 + 2 y_3 dy_3
$$

$$
\deg = 3: \quad D_6 = x_1 dy_3 + x_3 dy_1, \quad D_7 = x_2 dy_3 + x_3 dy_2.
$$

Let us consider the most general derivation $D = \sum_{1 \leq i \leq 7} \alpha_i D_i$, where $D_i$ are the cocycles given in Eq. (20). An easy computation shows that $B_\alpha$ is $D$-invariant if and only if $\alpha_1 = \alpha_2 = \alpha_6 = \alpha_7 = 0$.

The Cartan matrix associated with $\mathfrak{psl}(3)$, or rather of $\mathfrak{gl}(3)$, as explained in [BGL], is symmetric, so the isometry

$$
x_1 \leftrightarrow x_2, \quad x_3 \leftrightarrow -x_3, \quad y_1 \leftrightarrow y_2, \quad y_3 \leftrightarrow -y_3,
$$

sends the derivation $D_3$ to $D_4$.

Now, we define two cubic forms on $a$ as follows:

$$
q_4(\sum_{1 \leq i \leq 7} \lambda_i e_i) := \lambda_7 \lambda_5^2 + 2 \lambda_1 \lambda_5 \lambda_3 + \lambda_4 \lambda_5^2,
$$

$$
q_5(\sum_{1 \leq i \leq 7} \lambda_i e_i) := \lambda_1 \lambda_2 \lambda_5 + \lambda_4 \lambda_6 \lambda_5 + 2 \lambda_2 \lambda_3 \lambda_7 + \lambda_1 \lambda_4 \lambda_7.
$$

Now, let $a = \sum_{1 \leq i \leq 7} \lambda_i e_i$, $b = \sum_{1 \leq i \leq 7} \mu_i e_i \in \mathfrak{psl}(3)$. We have

$$
B_{\mathfrak{psl}(3)}(D_5(a-b), [a,b]) = (\lambda_2 - \mu_2)(\lambda_1 \mu_5 + \lambda_7 \mu_3 - 5 \mu_1 - 3 \mu_7)
$$

$$
- (\lambda_4 - \mu_4)(\lambda_7 \mu_1 + 5 \mu_6 - \mu_1 \mu_7 - 6 \mu_5)
$$

$$
- (\lambda_5 - \mu_5)(\lambda_2 \mu_1 + 4 \mu_6 - \mu_1 \mu_2 - 6 \mu_4)
$$

$$
+ (\lambda_7 - \mu_7)(\lambda_2 \mu_3 + \mu_1 \mu_4 - 3 \mu_3 \mu_2 - \mu_4 \mu_1).
$$

Besides,

$$
B_{\mathfrak{psl}(3)}(D_4(a-b), [a,b]) = - (\lambda_3 - \mu_3)(\lambda_1 \mu_5 + \lambda_7 \mu_3 - 5 \mu_1 - 3 \mu_7)
$$

$$
+ (\lambda_5 - \mu_5)(\lambda_3 \mu_1 + 5 \mu_4 - \mu_1 \mu_3 - 4 \mu_5).
$$

It is easy to see that

$$
q_4(a+b) - q_4(a) - q_4(b) = B_{\mathfrak{psl}(3)}(D_4(a-b), [a,b])
$$

$$
q_5(a+b) - q_5(a) - q_5(b) = B_{\mathfrak{psl}(3)}(D_5(a-b), [a,b]).
$$

Let us summarize:

| Derivation | $q(a)$ | $\gamma$ | $\alpha_0$ | Double extension |
|------------|--------|---------|-----------|----------------|
| $D_4$      | $\lambda_7 \lambda_5^2 + 2 \lambda_1 \lambda_5 \lambda_3 + \lambda_4 \lambda_5^2$ | 0       | 0         | $\mathfrak{gl}(3)$ |
| $D_5$      | $\lambda_1 \lambda_2 \lambda_5 + \lambda_4 \lambda_6 \lambda_5 + 2 \lambda_2 \lambda_3 \lambda_7 + \lambda_1 \lambda_4 \lambda_7$ | 1       | 0         | $\mathfrak{gl}(3)$ |

### 4.4.2. Claim

$\dim H^2(\mathfrak{gl}(3)) = 0$ and $\dim H^2(\tilde{\mathfrak{gl}}(3)) = 2$, hence $\mathfrak{gl}(3)$ and $\tilde{\mathfrak{gl}}(3)$ are not isomorphic.

### 4.5. The case: $\mathfrak{vect}(1; \mathcal{N})$ and $p = 3$.

For the notation, see [2.5].

#### 4.5.1. Claim

[SF, Theorem 8.5] We have $H^1(\mathfrak{vect}(1; \mathcal{N}); \mathfrak{vect}(1; \mathcal{N})) = 0$.

It follows that $\mathfrak{vect}(1; \mathcal{N})$ can have only a trivial double extension, given by $\mathfrak{vect}(1; \mathcal{N}) \oplus \mathfrak{c}$, where $\mathfrak{c}$ is a 2-dimensional center. Recall that $\mathfrak{vect}(1; \mathcal{N})$ is restricted if and only if $\mathcal{N} = \mathcal{1}$, see [SF, Theorem 2.4, page 149]. The case $\mathcal{N} = (2, 2)$ gives a non-restricted Lie algebra; however, the double extensions à la Median and Revoy can be applied because of the following:

#### 4.5.2. Claim

$H^1(\mathfrak{vect}(1; \mathcal{N}); \mathfrak{vect}(1; \mathcal{N}))$, where $\mathcal{N} = (2, 2)$, is spanned by the cocycle

$$
D_1 = \partial_1 \otimes (u_1^{(1)} \partial_1) + (u_1 \partial_1) \otimes (u_1^{(4)} \partial_1) + (u_1^{(2)} \partial_1) \otimes (u_1^{(5)} \partial_1) + (u_1^{(3)} \partial_1) \otimes (u_1^{(6)} \partial_1)
$$

$$
+ (u_1^{(4)} \partial_1) \otimes (u_1^{(7)} \partial_1) + (u_1^{(5)} \partial_1) \otimes (u_1^{(8)} \partial_1).
$$
An easy computation shows that the bilinear form (4) is $D_1$-invariant. The double extension of vect$(1; (2, 2))$ is a Lie algebra of dimension 11 that we denote by $\tilde{\text{vect}}(1; \underline{N})$, for $\underline{N} = (2, 2)$.

Observe that vect$(1; \underline{N})$, for $\underline{N} = (2, 2)$, is not restricted.

4.5.3. The case: vect$(2; \underline{N})$ and $p = 2$. We have

4.5.4. Claim. $H^1(\text{vect}(2; 1); \text{vect}(2; 1)) = 0$.

Again, no non-trivial double extensions for vect$(2; 1)$. However,

4.5.5. Claim. $H^1(\text{vect}(2; \underline{N}); \text{vect}(2; \underline{N}))$, for $\underline{N} = (1, 2)$, is spanned by the cocycle

\[
D_1 = \partial_1 \otimes (u_2^{(2)} \partial_1) + \partial_2 \otimes (u_2^{(2)} \partial_2) + (u_1 u_2 \partial_1) \otimes (u_1 u_2^{(3)} \partial_1) + (u_1 u_2 \partial_2) \otimes (u_1 u_2^{(3)} \partial_2) \\
+ (u_1 \partial_1) \otimes (u_1 u_2^{(2)} \partial_1) + (u_1 \partial_2) \otimes (u_1 u_2^{(2)} \partial_2) + (u_2 \partial_1) \otimes (u_2^{(3)} \partial_1) + (u_2 \partial_2) \otimes (u_2^{(3)} \partial_2).
\]

In the ordered basis

$\partial_1, \partial_2, u_2 \partial_1, u_2 \partial_2, u_1 \partial_1, u_1 \partial_2, u_2^{(2)} \partial_1, u_2^{(2)} \partial_2, (u_1 u_2) \partial_1, (u_1 u_2) \partial_2, u_2^{(3)} \partial_1, u_2^{(3)} \partial_2, (u_1 u_2^{(2)}) \partial_1, (u_1 u_2^{(2)}) \partial_2, (u_1 u_2^{(3)}) \partial_1, (u_1 u_2^{(3)}) \partial_2$,

the Gram matrix of the bilinear form in Eq. (4) is of the form $B = \text{antidiag}(1, \ldots, 1)$, and the derivative $D_1$ has a matrix representation given by

\[
E^{1,7} + E^{2,8} + E^{3,11} + E^{4,12} + E^{5,13} + E^{6,14} + E^{9,15} + E^{10,16}.
\]

A direct computation shows that $D_1$ satisfies Eq. (1) for the bilinear form (4). The double extension of vect$(2; (1, 2))$ is a Lie algebra of dimension 18 that we denote by $\tilde{\text{vect}}(1; (1, 2))$.

Observe that vect$(2; (1, 2))$ is not restricted, see [SF, Theorem 2.4, page 149].

4.5.6. The case: svect$(3; \underline{N})$ and $p = 2$. We have the following occasional isomorphisms

\[
\text{svect}^{(1)}(3; 1) \simeq \mathfrak{h}^{(1)}(4; 1) \simeq \mathfrak{psl}(4) \quad \text{for } p = 2 \text{ (as shown in [CK]).}
\]

This case has been studied in \S 4.3

4.5.7. The case: svect$(1)(3; 1)$ and $p = 3$. 
4.5.8. **Claim.** $H^2_{H_0}({\mathfrak{svect}}^{(1)}(3; 1); {\mathfrak{svect}}^{(1)}(3; 1))$ is spanned by the cocycles

\[
\text{deg = 3} : D_2 = 2D_{1,3}(u^{(r)}) \otimes (D_2(\sigma^{(r-c_2-2c_2)})) + D_{1,2}(u^{(r)}) \otimes (D_{1,3}(\sigma^{(r-c_2-2c_2)})) + D_{1,3}(u^{(r-c_1)}) \otimes (D_{1,2}(\sigma^{(r-c_2-2c_2)})) + D_{1,2}(u^{(r-c_1)}) \otimes (D_{1,3}(\sigma^{(r-c_2-2c_2)})),
\]

\[
\text{deg = 0} : D_1 = (D_{1,2}(u^{(r-c_1)})) + 2D_{1,3}(u^{(r-c_1)}) \otimes (D_{1,2}(u^{(r-c_2-2c_2)})) + 2D_{2,3}(u^{(r-c_1)}) \otimes (D_{1,2}(u^{(r-c_2-2c_2)})) + 2D_{2,3}(u^{(r-c_1)}) \otimes (D_{2,3}(u^{(r-c_2-2c_2)})) + D_{1,2}(u^{(r-c_1)}),
\]

A direct computation shows that the bilinear form given in Eq. (5) is not $D_2$-invariant. Therefore, the Lie algebra $\mathfrak{svect}^{(1)}(3; 1)$ cannot be double extended by means of $D_2$. However, a direct computation shows that this bilinear form is $D_1$-invariant and, moreover, $D_1^3 = 0$. Therefore, $a_0 = \gamma = 0$, see (2.3).

Now we define the cubic form

\[
q(\sum_{1 \leq i \leq 52} \lambda_i e_i) = 2\lambda_2 \lambda_3^2 + \lambda_9 \lambda_{12} \lambda_2 + \lambda_{10} \lambda_{17} \lambda_2 + 2\lambda_4 \lambda_2 \lambda_2 + 2\lambda_8 \lambda_{21} \lambda_2 + \lambda_3 \lambda_{37} \lambda_2 + \lambda_7 \lambda_5^2 + \lambda_5 \lambda_9 + \lambda_4 \lambda_8 \lambda_{10} + 2\lambda_4 \lambda_8 \lambda_{11} + 2\lambda_4 \lambda_8 \lambda_{12} + \lambda_3 \lambda_{11} \lambda_{12} + \lambda_3 \lambda_{8} \lambda_{15} + \lambda_4 \lambda_{14} \lambda_{17} + 2\lambda_3 \lambda_{17} + 2\lambda_3 \lambda_{12} + 2\lambda_2 \lambda_{28},
\]

where it suffices to describe the $e_i$ that appear in the expression of $q(a)$:

\[
e_2 = \partial_2, \quad e_3 = \partial_3, \quad e_4 = 2D_{1,2}(u^{(r-c_2-2c_2)}),
\]

\[
e_7 = D_{2,3}(u^{(r-c_2-2c_2)}), \quad e_8 = D_{1,3}(u^{(r-c_2-2c_3)}), \quad e_9 = D_{2,3}(u^{(r-c_2-2c_3)}),
\]

\[
e_{10} = D_{1,2}(u^{(r-c_2-2c_3)}), \quad e_{11} = D_{1,3}(u^{(r-c_2-2c_3)}), \quad e_{12} = D_{2,3}(u^{(r-c_2-2c_3)}),
\]

\[
e_{15} = 2D_{1,2}(u^{(r-c_2-2c_3)}), \quad e_{17} = D_{2,3}(u^{(r-c_2-2c_3)}), \quad e_{20} = D_{1,3}(u^{(r-c_2-2c_3)}),
\]

\[
e_{21} = D_{1,2}(u^{(r-c_2-2c_3)}), \quad e_{22} = D_{1,3}(u^{(r-c_2-2c_3)}), \quad e_{28} = D_{2,3}(u^{(r-c_2-2c_3)}),
\]

\[
e_{32} = D_{2,3}(u^{(r-c_2-2c_3)}), \quad e_{37} = D_{2,3}(u^{(r-c_2-2c_3)}).
\]

A direct computation shows that (here $B$ is the bilinear form in Eq. (5))

\[
q(a + b) - q(a) - q(b) = B(D_1(a - b), [a, b]) \quad \text{for all} \ a, b \in \mathfrak{svect}^{(1)}(3; 1).
\]

Therefore, the $D_1$-extension of $\mathfrak{svect}^{(1)}(3; 1)$ is a Lie algebra of dimension 54 that we denote by $\mathfrak{svect}^{(1)}(3; 1)$. 
Let us summarize:

\[(22)\]

| Derivation | Compatibility with $B$ | $q(a)$ | $\gamma$ | $a_0$ | Double extension |
|------------|-----------------------|--------|----------|-------|-----------------|
| $D_1$      | No                    | $2\lambda_{32} \lambda_{7} + \lambda_{9} \lambda_{12} \lambda_{2} + \lambda_{10} \lambda_{17} \lambda_{3}$ | $+$ | $2\lambda_{4} \lambda_{20} \lambda_{2} + 2\lambda_{8} \lambda_{21} \lambda_{2} + \lambda_{3} \lambda_{37} \lambda_{2}$ | 0 | $\sim \text{svect}^{(1)} (3; 1)$ |
| $D_2$      | Yes                   | $+ \lambda_{7} \lambda_{2}^2 + \lambda_{9} \lambda_{9} + \lambda_{4} \lambda_{8} \lambda_{10}$ | 0      | $+ 2\lambda_{4} \lambda_{8} \lambda_{11} + 2\lambda_{1} \lambda_{8} \lambda_{12} + \lambda_{3} \lambda_{11} \lambda_{12}$ | $\sim \text{svect}^{(1)} (3; 1)$ |
|            |                       | $+ \lambda_{3} \lambda_{8} \lambda_{15} + \lambda_{1} \lambda_{4} \lambda_{17} + 2\lambda_{3} \lambda_{7} \lambda_{17}$ | $\sim \text{svect}^{(1)} (3; 1)$ | $+ 2\lambda_{3} \lambda_{4} \lambda_{22} + 2\lambda_{3}^2 \lambda_{28}$ | |

5. **SUPERIZATION: AN OPEN PROBLEM**

Several superizations of results by Medina and Revoy were carried out by several authors, see [ABB] [ABBQ] [B], [BB], [BBB], and [B2]. Observe that, in super setting, the bilinear form $B_a$ and the derivativation $D$ can be odd, see [ABB] [BeBou] [BKLS]. These constructions remain valid for modular Lie superalgebras defined over a field of characteristic $p \neq 2$.

For the (rather non-trivial) case $p = 2$ in the super setting, the construction had to be modified to take the *squaring* into account, see [BeBou].

**An open problem:** Is it possible to superize Theorem 3.2.1 and Theorem 3.2.2?

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3S. A. Translator’s mistake.