Free Vibration Analysis of Smart Laminated Functionally Graded CNT Reinforced Composite Plates via New Four-Variable Refined Plate Theory

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Abstract: This paper presents a new four-variable refined plate theory for free vibration analysis of laminated piezoelectric functionally graded carbon nanotube-reinforced composite plates (PFG-CNTRC). The present theory includes a parabolic distribution of transverse shear strain through the thickness and satisfies zero traction boundary conditions at both free surfaces of the plates. Thus, no shear correction factor is required. The distribution of carbon nanotubes across the thickness of each FG-CNT layer can be functionally graded or uniformly distributed. Additionally, the electric potential in piezoelectric layers is assumed to be quadratically distributed across the thickness. Equations of motion for PFG-CNTRC rectangular plates are derived using both Maxwell’s equation and Hamilton’s principle. Using the Navier technique, natural frequencies of the simply supported hybrid plate with closed circuit and open circuit of electrical boundary conditions are calculated. New parametric studies regarding the effect of the volume fraction, the CNTs distribution, the number of layers, CNT fiber orientation and thickness of the piezoelectric layer on the free vibration response of hybrid plates are performed.

Keywords: free vibration; four-variable refined plate theory; piezoelectric material; FG-CNTRC; laminated composite

1. Introduction

A novel class of functionally graded materials (FGM) titled functionally graded carbon nanotube-reinforced composite (FG-CNTRC) plates was first introduced by Shen [1]. Shen’s study revealed that the distribution of CNT had a remarkable influence on the mechanical behaviors of the FG-CNTRC plates. Since then, static, dynamic, and buckling behaviors of FG-CNTRC structures have been studied and reported in the literature. Alibeigloo and Liew [2] studied the bending response of simply supported FG-CNTRC rectangular plate under thermo–mechanical loads by using the 3D theory of elasticity. Zhu et al. [3] presented a finite element model to study bending and free vibration responses of thin-to-moderately thick FG-CNTRC plates using the first shear deformation plate theory (FSDT). Lei et al. [4] gave the solution for static analysis of laminated FG-CNTRC plates using the element-free kp-Ritz method. Huang et al. [5] analyzed the bending and free vibration characteristics of antisymmetrically laminated FG-CNTRC plates using the FSDT and simple four-variable theory. The static, vibration and buckling responses of FG-CNTRC resting on elastic foundation were investigated by Wattanasakulpong [6] and Nguyen et al. [7]. Additionally, Shen et al. [8] analyzed the buckling and post-buckling behaviors of symmetrically distributed CNT-reinforced composite plate, including thermal effects. Next, Shen et al. [9] examined the buckling loads and post-buckling equilibrium paths of the CNTRC plates assuming properties of CNTs were temperature-dependent. Using a higher-order shear deformation plate theory (HSDT), the nonlinear
free vibration behaviors of the FG-CNTRC plates with an elastic foundation in the thermal environment was investigated by Wang and Shen [10]. That study used the perturbation technique to solve the nonlinear equations of motion. Mehar et al. [11] investigated the static response of the FG-CNTRC doubly curved shell panel, in which the geometric nonlinear and thermal dependent properties of the individual constituents were considered. Using FSDT and piston theory in determining the aerodynamic pressure, Asadi et al. [12] analyzed aeroelastic flutter of FG-CNTRC beams under axial compression and supersonic airflow. These authors continue to study the aero- thermoelastic behaviors of supersonic FG-CNTRC plates taking to account thermal effects in [13].

There have been a limited number of studies related to electromechanical coupling analysis of laminated FG-CNTRC plates with surface-embedded or bonded piezoelectric layers. Using the 3D-theory, Alibeigloo investigated the bending behaviors of the piezoelectric FG-CNTRC (PFG-CNTRC) plates under the mechanical uniform load [14], thermal load, and electric field [2]. Rafiee et al. [15] investigated initial geometrical imperfections in the large amplitude dynamic stability of PFG-CNTRC plates under the simultaneous effect of thermal and electrical loadings. Setoodeh et al. [16] studied the free vibration characteristic of PFG-CNTRC spherical panels by differential quadrature method based on the HSDT. Using the Ritz method with Chebyshev polynomials, Kiani [17] analyzed the free vibration of the PFG-CNTRC plates with opened and closed circuits electrical boundary conditions. In Kiani’s research, the electric potential in the piezoelectric layers was assumed to be linearly distributed through the thickness of the plate. Wu et al. [18] presented a buckling analysis of an arbitrarily thick PFG-CNTRC plate subjected to in-plane compressive loads using unified formulation. Nguyen et al. [19] used the extended isogeometric method with non-uniform rational B-spline and the HSDT to investigate the dynamic response of PFG-CNTRC plates. In the study of Selim et al. [20], an element-free IMLS-Ritz model based on Reddy’s HSDT for the active vibration control of PFG-CNTRC plates was presented. Song et al. [21] used velocity feedback and linear quadratic regulator LQR methods to study active vibration control of PFG-CNTRC cylindrical shells with bonded piezoelectric patches. Zhang et al. [22] used a genetic algorithm to study shape control of FG-CNTRC rectangular plates bonded with piezoelectric patches acting as actuators and sensors.

HSDT [23–28] is often desirable for the design of composite structures since it yields more accurate results than the CPT (classical plate theory) and the FSDT. However, these HSDTs have computational costs because the equations of motions based on these HSDT are more. Therefore, simple HSDT must be developed. Recently, based on HSDT, Shimpi [29] developed a new plate theory that has only two unknown displacements, in which the transverse shear stress variation across the thickness is parabolic and equals zero on free surfaces. After that, several researchers introduced a class of four-variable refined plate theory by adding two in-plane displacements and separating the transverse displacements into the bending component and shear component. Meiche et al. [30] presented a new four-variable refined plate theory with hyperbolic shape function for buckling and vibration analysis of FGM sandwich plates. Thai and Vo [31] developed a new sinusoidal shear deformation theory to analyze static and dynamic behaviors of FG plates. Then, another sinusoidal shear deformation theory was also presented by Thai and Kim [32] to investigate the bending and free vibration response of FG plates. Daoudji et al. [33] presented the static analysis of FG plates using a new higher-order shear deformation model.

In present work, a new plate theory with four unknown displacements is presented for free vibration analysis of FG-CNTR plates with two piezoelectric layers bonded at the free surfaces. The electric potential in piezoelectric layers is assumed to be quadratic through the thickness. Navier solution is applied to solve the governing equation of simply supported rectangular plates to obtain the frequencies of the smart FG-CNTRC plates with closed and open circuit electrical conditions. The accuracy of the proposed plate theory is indicated by comparing the obtained natural frequencies with existing results in the literature. Several examples are carried out to show the effects of volume fraction and distribution type of CNTs, the number of layers, CNT fiber orientation, and thickness of piezoelectric layers on the natural frequencies of hybrid plates.
2. Laminated PFG-CNTRC Plates

A hybrid laminated FG-CNTRC plate with integrated piezoelectric lamina at top and bottom surfaces is depicted in Figure 1. Width, length, core thickness, and thickness of each piezoelectric layer of the plate are denoted by $a$ and $b$, $h$ and $h_p$. Four types of CNT distribution across the thickness of each FG-CNT layer namely UD, FG-V, FG-O, and FG-X are also indicated in Figure 1.

![Figure 1](image)

**Figure 1.** Configuration of the laminated piezoelectric functionally graded carbon nanotube-reinforced composite plates (PFG-CNTRC).

The CNT volume fractions for each FG-CNTRC lamina are assumed as follows [3]:

$$
V_{\text{CNT}} = V_{\text{CNT}}^* \quad \text{(UD)}
$$

$$
V_{\text{CNT}}(z) = \left(1 - \frac{2z}{h}\right) V_{\text{CNT}}^* \quad \text{(FG-O)}
$$

$$
V_{\text{CNT}}(z) = \frac{4z}{h} V_{\text{CNT}}^* \quad \text{(FG-X)}
$$

$$
V_{\text{CNT}}(z) = \left(1 + \frac{2z}{h}\right) V_{\text{CNT}}^* \quad \text{(FG-V)}
$$

(1)

where:

$$
V_{\text{CNT}}^* = \frac{w_{\text{CNT}}}{w_{\text{CNT}} + \left(\rho_{\text{CNT}} / \rho_m\right) - \left(\rho_{\text{CNT}} / \rho_m\right) w_{\text{CNT}}}
$$

(2)

The effective elastic properties of each FG-CNTRC lamina can be written as follows [3]:

$$
E_{11} = \eta_1 V_{\text{CNT}} E_{11}^{\text{CNT}} + V_m E_m; \quad \eta_2 E_{22} = \frac{V_{\text{CNT}} E_{22}^{\text{CNT}} + V_m E_m}{E_m};
$$

$$
\frac{\eta_3}{G_{12}} = \frac{V_{\text{CNT}}}{G_m} \quad \frac{V_m}{G_m}; \quad v_{12} = V_{\text{CNT}} v_{12}^{\text{CNT}} + V_m v_m; \quad \rho = V_{\text{CNT}} \rho_{\text{CNT}} + V_m \rho_m
$$

(3)

where $E_{11}^{\text{CNT}}, E_{22}^{\text{CNT}}, G_{12}^{\text{CNT}}$ and $E_m$, $G_m$ are Young's moduli and shear modulus of CNT and isotropic matrix, respectively; $\eta_1, \eta_2,$ and $\eta_3$ are called efficiency parameters. $V_{\text{CNT}}$ and $V_m$ are the volume
fractions of CNT and of matrix, respectively; the Poisson ratio and mass density of CNT/matrix are denoted as $\nu_{12}^{CNT}$, $\rho^{CNT}$ and $\nu_{12}^{m}$, $\rho^{m}$, respectively.

The linear constitutive relations for the FG-CNTRC core can be expressed as

\[
\begin{bmatrix}
\sigma_{xx}^k \\
\sigma_{yy}^k \\
\tau_{xy}^k \\
\tau_{xz}^k
\end{bmatrix} =
\begin{bmatrix}
Q_{11}^k & Q_{12}^k & 0 & 0 \\
Q_{12}^k & Q_{22}^k & 0 & 0 \\
0 & 0 & Q_{56}^k & 0 \\
0 & 0 & 0 & Q_{44}^k
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy} \\
\gamma_{xz}
\end{bmatrix}
\]

(4)

where \(Q_i^k\) are the transformed elastic coefficients related to elastic coefficients in material coordinates \(Q_i\) [34]:

\[
Q_{11}^k = Q_{11}^m \cos^2 \theta^k + 2(Q_{12}^m + 2Q_{16}^m) \sin^2 \theta^k \cos^2 \theta^k + Q_{22}^m \sin^4 \theta^k \\
Q_{12}^k = (Q_{11}^m + Q_{22}^m - 4Q_{16}^m) \sin^2 \theta^k \cos^2 \theta^k + Q_{12}^m (\sin^4 \theta^k + \cos^4 \theta^k) \\
Q_{22}^k = Q_{11}^m \sin^4 \theta^k + 2(Q_{12}^m + 2Q_{16}^m) \sin^2 \theta^k \cos^2 \theta^k + Q_{22}^m \cos^4 \theta^k \\
Q_{44}^k = Q_{44}^m \cos^2 \theta^k + Q_{55}^m \sin^2 \theta^k \\
Q_{55}^k = Q_{44}^m \sin^2 \theta^k + Q_{55}^m \cos^2 \theta^k
\]

(5)

For each the CNT layer:

\[
Q_{11}^k = \frac{E_{II}}{1-\nu_{12}^m \nu_{11}^m}; \quad Q_{12}^k = \frac{v_{12}^m E_{22}}{1-\nu_{12}^m \nu_{11}^m}; \quad Q_{22}^k = \frac{E_{22}}{1-\nu_{12}^m \nu_{11}^m}; \\
Q_{44}^k = C_{33}^m; \quad Q_{55}^k = G_{11}^m; \quad Q_{66}^k = G_{12}^m
\]

(6)

The constitutive relations for a piezoelectric material can be expressed as [35]

\[
\begin{bmatrix}
\sigma_{xx}^k \\
\sigma_{yy}^k \\
\tau_{xy}^k \\
\tau_{xz}^k
\end{bmatrix} =
\begin{bmatrix}
C_{11}^k & C_{12}^k & 0 & 0 \\
C_{12}^k & C_{22}^k & 0 & 0 \\
0 & 0 & \frac{1}{2}(C_{33}^k - C_{11}^k) & 0 \\
0 & 0 & 0 & C_{55}^k
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy} \\
\gamma_{xz}
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
0 & 0 & e_{15}^k \\
0 & 0 & 0 & e_{15}^k \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
E_x^k \\
E_y^k \\
E_{15}^k \\
E_{33}^k
\end{bmatrix}
\]

(7)

\[
\begin{bmatrix}
D_{xx}^k \\
D_{yy}^k \\
D_{xz}^k \\
D_{yz}^k
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & e_{15}^k \\
0 & 0 & 0 & 0 \\
\varepsilon_{31}^k & \varepsilon_{31}^k & 0 & 0 \\
\bar{\varepsilon}_{31}^k & \bar{\varepsilon}_{31}^k & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy} \\
\gamma_{xz}
\end{bmatrix}
+ \begin{bmatrix}
p_{11}^k \\
0 \\
0 & 0 & p_{13}^k \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
E_x^k \\
E_y^k \\
E_{15}^k \\
E_{33}^k
\end{bmatrix}
\]

(8)

The elastic constants for the piezoelectric layer:
\[
\begin{align*}
\overline{C}_{11}^{k} &= C_{11}^{k} - \left( \frac{C_{11}^{k}}{C_{33}^{k}} \right)^2 ; & \overline{C}_{12}^{k} &= C_{12}^{k} - \left( \frac{C_{11}^{k}}{C_{33}^{k}} \right)^2 ; & \overline{C}_{33}^{k} &= C_{33}^{k} ; \\
\overline{C}_{31}^{k} &= C_{31}^{k} - \left( \frac{C_{11}^{k}}{C_{33}^{k}} \right)^2 ; & p_{33}^{k} &= p_{33}^{k} + \frac{C_{33}^{k}}{C_{33}^{k}}
\end{align*}
\]

(9)

where \([C^{k}]\) is the elastic constants matrix of the piezoelectric lamina, \([p_{ji}^{k}]\) is the dielectric permittivity matrix, \([e_{ji}^{k}]\) is the electromechanical coupling matrix, \([D^{k}]\) is the electrical displacement, and \([E^{k}]\) is the electric field in the piezoelectric lamina.

3. Kinematic Equations

According to the four-variable refined plate theory \([30–33]\), the displacement components at an arbitrary point in the hybrid panel can be expressed as follows:

\[
\begin{align*}
\pi (x, y, z, t) &= u(x, y, t) - z \frac{\partial u}{\partial x} - f(z) \frac{\partial u}{\partial x} ; \\
\tau (x, y, z, t) &= v(x, y, t) - z \frac{\partial v}{\partial y} - f(z) \frac{\partial v}{\partial y} ; \\
w(x, y, z, t) &= w_{r}(x, y, t) + w_{s}(x, y, t)
\end{align*}
\]

(10)

where \(u, v\) are the displacements of the corresponding point on the reference surface of the plate along \(x\) and \(y\) axis, respectively; \(w_{r}\) and \(w_{s}\) are the bending and shear components of the transverse displacement, respectively; the shape function \(f(z)\) represents the distribution of the transverse shear stresses and strains along the thickness.

By supposing the shape function \(f(z)\) satisfies the free transverse shear stress conditions on the free surfaces of the plates, a class of refined plate theory was developed by various researchers as shown in Table 1:

| Researcher | Shape Function |
|------------|----------------|
| Shimpi [29] | \(f(z) = z \left[ \frac{-1}{4} + \frac{5}{3} \left( \frac{z}{h} \right)^2 \right]\) |
| N. E Meiche et al. [30] | \(f(z) = \frac{(h / \pi) \sinh \left( \frac{\pi z}{h} \right) - z \cosh (\pi / 2) - 1}{\cosh (\pi / 2) - 1}\) |
| Huu-Tai Thai and Thuc P. Vo [31] | \(f(z) = z \frac{h}{\pi} \sin \frac{\pi z}{h}\) |
| Huu-Tai Thai and Seung-Eock Kim [32] | \(f(z) = \frac{4z^2}{3h^2}\) |
| Daouadji et al. [33] | \(f(z) = z \left[ z \sec h \left( \frac{\pi z}{h^2} \right) - z \sec h \left( \frac{\pi}{4} \left( \frac{1}{2} \tanh \left( \frac{\pi}{4} \right) \right) \right) \right]\) |

In this study, a new shape function \(f(z)\) is supposed as follows:

\[
f(z) = z \left[ \frac{-1}{8} + \frac{3}{2} \left( \frac{z}{h} \right)^2 \right]
\]

(11)

The linear strain-displacement relations are written as:
\[
\begin{align*}
\varepsilon_x &= \frac{\partial u}{\partial x} - z \frac{\partial^2 w_1}{\partial x^2} - f(z) \frac{\partial^2 w_2}{\partial x^2}; \\
\varepsilon_y &= \frac{\partial v}{\partial y} - z \frac{\partial^2 w_1}{\partial y^2} - f(z) \frac{\partial^2 w_2}{\partial y^2}; \\
\gamma_{xy} &= \frac{\partial^2 w_4}{\partial x \partial y} - 2z \frac{\partial^2 w_3}{\partial x \partial y} - 2 f(z) \frac{\partial^2 w_5}{\partial x \partial y}; \\
\gamma_{yy} &= (1 - f'(z)) \frac{\partial w_6}{\partial y}; \\
\gamma_{xx} &= (1 - f'(z)) \frac{\partial w_7}{\partial x}; \\
\end{align*}
\]

The variation of electric potential through the thickness of the piezoelectric lamina was proposed by Wu et al. [36]:

\[
\Phi(x, y, z, t) = \phi(x, y, t) \left[ 1 - \left( \frac{z - h/2 - h_p/2}{h_p/2} \right)^2 \right] + f_1(x, y, t) z + f_2(x, y, t) \quad h/2 \leq z \leq h/2 + h_p
\]

\[
\Phi(x, y, z, t) = \phi(x, y, t) \left[ 1 - \left( \frac{-z - h/2 - h_p/2}{h_p/2} \right)^2 \right] + f_1(x, y, t) z + f_2(x, y, t) \quad -h/2 - h_p \leq z \leq -h/2
\]

where the unknowns \( f_1, f_2, f_3 \) and \( f_4 \) can be obtained by satisfying the specific electrical boundary condition. In this study, two cases of electrical boundary conditions are considered. For the closed circuit condition, both major surfaces of the piezoelectric lamina are circuited:

\[
\Phi(z = \pm \frac{h}{2}) = 0; \quad \Phi(z = \pm \frac{h}{2} + h_p) = 0
\]

On the other hand, when one surface is kept at zero voltage and the other is electrically insulated, for the open circuit condition, the electrical boundary conditions are

\[
\Phi(z = \pm \frac{h}{2}) = 0; \quad \Phi(z = \pm \frac{h}{2} + h_p) = 0
\]

In addition, from electric potential function, the electric field can be derived as

\[
\vec{E} = -\hat{\nabla} \Phi
\]

Substituting the expressions in Equations (13) and (8) into Equations (14) and (15) yields the electrical potential distribution for the closed circuit (C-circuit) as

\[
\Phi'(x, y, z, t) = \phi'(x, y, t) \left[ 1 - \left( \frac{z - h/2 - h_p/2}{h_p/2} \right)^2 \right] \quad h/2 \leq z \leq h/2 + h_p
\]

\[
\Phi'(x, y, z, t) = \phi'(x, y, t) \left[ 1 - \left( \frac{-z - h/2 - h_p/2}{h_p/2} \right)^2 \right] \quad -h/2 - h_p \leq z \leq -h/2
\]

and for open circuit (O-circuit) as
\[
\Phi'(x,y,z,t) = \Phi'(x,y,t) \left[ 1 - \left( \frac{z - h/2 - h_p/2}{h_p/2} \right)^{2} + \frac{4(z - h/2)}{h_p} \right] + \frac{\tau_{zz}}{\rho} \left[ u_x + v_y + \right.
\]
\[
(h/2 + h_p) \left( w_{x,z} + w_{v,yy} \right) + f(z) \left( w_{x,z} + w_{v,yy} \right) \left( z - h/2 \right) \leq z \leq h/2 + h_p
\]
\[
\Phi'(x,y,z,t) = \Phi'(x,y,t) \left[ 1 - \left( -\frac{z - h/2 - h_p/2}{h_p/2} \right)^{2} + \frac{4(z + h/2)}{h_p} \right] + \frac{\tau_{zz}}{\rho} \left[ u_x + v_y - \right.
\]
\[
(h/2 + h_p) \left( w_{x,z} + w_{v,yy} \right) + f(z) \left( w_{x,z} + w_{v,yy} \right) \left( z + h/2 \right) - h/2 - h_p \leq z \leq -h/2
\]

4. Equations of Motion

Hamilton’s principle is used herein to derive the governing differential equations of motion for the free vibration problem. Without external forces, the principle can be stated as [37]
\[
\int_{t_1}^{t_2} \left( \delta U - \delta K \right) dt = 0
\]

in which \( \delta U \) is the variation of the strain energy of the plate and may be expressed as
\[
\delta U = \int_{A} \left( \sum_{k=1}^{N/2} \sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + \tau_{xy} \delta \gamma_{xy} + \tau_{xz} \delta \gamma_{xz} + \tau_{yz} \delta \gamma_{yz} \right) dA = \]
\[
= \int_{A} \left\{ N_x \frac{\partial \delta \varepsilon_x}{\partial x} - M_y^x \frac{\partial^2 \delta w_x}{\partial x^2} - M_y^x \frac{\partial^2 \delta w_y}{\partial x \partial y} + N_y \frac{\partial \delta \varepsilon_y}{\partial y} - M_x^y \frac{\partial^2 \delta w_y}{\partial y^2} - M_x^y \frac{\partial^2 \delta w_x}{\partial x \partial y} - M_y^x \frac{\partial^2 \delta w_x}{\partial x \partial y} - M_y^y \frac{\partial^2 \delta w_y}{\partial x \partial y} - \right. \]
\[
\left. \right \} dA
\]
where \( N_x, M_y, M_y \) are stress resultants and defined by
\[
\begin{bmatrix} N_x, & N_y, & N_y \end{bmatrix} = \sum_{k=1}^{N/2} \int_{h_0}^{h_z} \begin{bmatrix} \sigma_x^k, \sigma_y^k, \tau_{xy}^k \end{bmatrix} \begin{bmatrix} 1 \\ z \\ f(z) \end{bmatrix} dz;
\]
\[
\begin{bmatrix} Q_{xz}, & Q_{yz} \end{bmatrix} = \sum_{k=1}^{N/2} \int_{h_0}^{h_z} \begin{bmatrix} \tau_{xz}^k, \tau_{yz}^k \end{bmatrix} g(z) dz
\]
and \( \delta K \) is the variation of the kinetic energy of the plate and can be written as follows:
\[
\delta K = \int_{A} \left( \rho \left( \dot{u} \ddot{u} + \dot{v} \ddot{v} + \dot{w} \ddot{w} \right) \right) dA dz
\]
\[
= \int_{A} \left\{ \begin{bmatrix} T_x \left( \dot{u} \ddot{w} + \dot{v} \ddot{w} + \dot{w} \ddot{w} \right) \right. \]
\[
- T_x \left( \frac{\partial \delta \dot{w}_x}{\partial x} + \frac{\partial \delta \dot{w}_y}{\partial x} \frac{\partial \delta \dot{w}_x}{\partial y} + \frac{\partial \delta \dot{w}_y}{\partial x} \right)
\]
\[
+ T_x \left( \frac{\partial \delta \dot{w}_x}{\partial x} + \frac{\partial \delta \dot{w}_y}{\partial x} \frac{\partial \delta \dot{w}_x}{\partial y} + \frac{\partial \delta \dot{w}_y}{\partial x} \right) \]
\[
\left. \right \} dA
\]
where mass moments \( \{ I_0, I_1, I_2, I_3, I_4, I_5 \} \) are defined by
\[
(I_0, I_1, I_2, I_3, I_4, I_5) = \sum_{i=1}^{N_h} (1, z^2, f(z), z f(z), f^2(z)) \rho i dz
\]  
(24)

Substituting Equation (12) into Equation (7), then the obtained results into Equation (21), and combine with the relations in Equation (16), the stress resultants are obtained as follows:
\[
\begin{bmatrix}
N \\
M_x \\
M_y \\
M_z \\
N^p \\
M^p_x \\
M^p_y \\
M^p_z \\
Q^p
\end{bmatrix} =\begin{bmatrix}
A \\
B \\
D \\
F \\
N_x \\
M^p_x \\
M^p_y \\
M^p_z \\
Q^p
\end{bmatrix} + \begin{bmatrix}
\varepsilon \\
\kappa^p_x \\
\kappa^p_y \\
\kappa^p_z \\
\gamma
\end{bmatrix} 
\]  
(25)

where
\[
N = \begin{bmatrix}
N_x \\
N_y \\
N_z
\end{bmatrix}; \quad M^p = \begin{bmatrix}
M^p_x \\
M^p_y \\
M^p_z
\end{bmatrix}; \quad M^p = \begin{bmatrix}
M^p_x \\
M^p_y \\
M^p_z
\end{bmatrix}; \quad Q = \begin{bmatrix}
Q_{1x} \\
Q_{1y}
\end{bmatrix}
\]  
(26)

\[
N^p = \begin{bmatrix}
N^p_x \\
N^p_y \\
N^p_z
\end{bmatrix}; \quad M^p = \begin{bmatrix}
M^p_x \\
M^p_y \\
M^p_z
\end{bmatrix}; \quad M^p = \begin{bmatrix}
M^p_x \\
M^p_y \\
M^p_z
\end{bmatrix}; \quad Q^p = \begin{bmatrix}
Q^p_{1x} \\
Q^p_{1y}
\end{bmatrix} #
\]  
(27)

\[
\varepsilon = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\
\frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}
\end{bmatrix}^T; \quad \kappa^p = \begin{bmatrix}
\frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial y^2} & \frac{\partial^2 w}{\partial z^2} & -2 \frac{\partial^2 w}{\partial x \partial y}
\end{bmatrix}^T; \\
\gamma = \begin{bmatrix}
\frac{\partial w}{\partial x} & \frac{\partial w}{\partial y}
\end{bmatrix}^T #
\]  
(28)

\[
A = \begin{bmatrix}
A_{11} & A_{12} & A_{16} \\
A_{12} & A_{22} & A_{26} \\
A_{16} & A_{26} & A_{66}
\end{bmatrix}; \quad B = \begin{bmatrix}
B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66}
\end{bmatrix}; \quad D = \begin{bmatrix}
D_{11} & D_{12} & D_{16} \\
D_{12} & D_{22} & D_{26} \\
D_{16} & D_{26} & D_{66}
\end{bmatrix}
\]  
(29)

\[
B^p = \begin{bmatrix}
B^p_{11} & B^p_{12} & B^p_{16} \\
B^p_{12} & B^p_{22} & B^p_{26} \\
B^p_{16} & B^p_{26} & B^p_{66}
\end{bmatrix}; \quad D^p = \begin{bmatrix}
D^p_{11} & D^p_{12} & D^p_{16} \\
D^p_{12} & D^p_{22} & D^p_{26} \\
D^p_{16} & D^p_{26} & D^p_{66}
\end{bmatrix}; \quad H^p = \begin{bmatrix}
H^p_{11} & H^p_{12} & H^p_{16} \\
H^p_{12} & H^p_{22} & H^p_{26} \\
H^p_{16} & H^p_{26} & H^p_{66}
\end{bmatrix};
\]  
(30)

\[
A^p = \begin{bmatrix}
A^p_{44} & A^p_{45} \\
A^p_{45} & A^p_{55}
\end{bmatrix} #
\]  

in which
\[
(A_i, B_i, D_i, B^p_i, D^p_i, H^p_i) = \sum_{i=1}^{N_h} (1, z^2, f(z), z f(z), f^2(z))(Q_i) dz \quad (i, j = 1, 2, 6)
\]  
(31)

\[
A^p_i = \sum_{i=1}^{N_h} \int_{Q_i} \left[1 - f^2(z)\right] (Q_i) dz \quad (i, j = 4, 5) #
\]  

And
\[
\begin{align*}
\left\{ \begin{array}{c}
N_i^n \\
N_j^n \\
N_{i\nu}^n \\
N_{j\nu}^n
\end{array} \right\} - \frac{1}{h} \sum_{k=1}^{h} \left\{ \begin{array}{c}
\sigma_i^n \\
\sigma_j^n \\
\sigma_{i\nu}^n \\
\sigma_{j\nu}^n
\end{array} \right\} \left( 1, z, f(z) \right) dz
\end{align*}
\]

(32)

\[
\left\{ \begin{array}{c}
Q_{ix}^p \\
Q_{iy}^p
\end{array} \right\} = \frac{1}{h} \sum_{k=1}^{h} \left\{ \begin{array}{c}
\tau_{ix}^p \\
\tau_{iy}^p
\end{array} \right\} \left[ 1 - f'(z) \right] dz;
\]

Substituting the expressions of \( \delta U \) and \( \delta K \) from Equations (21)–(26) into Equation (20) and after some mathematical manipulations, we obtain the equations of motion of the plate as follow:

\[
\begin{align*}
\delta u : & \frac{\partial^2 \nu}{\partial x^2} + 2 \frac{\partial^2 \nu}{\partial x \partial y} + \frac{\partial^2 \nu}{\partial y^2} = T_1 (\ddot{w}_b + \ddot{w}_s) + T_1 (\ddot{w}_b + \ddot{w}_s) \\
\delta v : & \frac{\partial^2 \nu}{\partial y^2} = T_1 (\ddot{w}_b + \ddot{w}_s) \\
\delta w : & \frac{\partial^2 M_b}{\partial x^2} + 2 \frac{\partial^2 M_b}{\partial x \partial y} + \frac{\partial^2 M_s}{\partial y^2} = T_1 (\ddot{w}_b + \ddot{w}_s) \\
& - T_1 V^2 \ddot{w}_b - T_1 V^2 \ddot{w}_s \\
& - T_1 V^2 \ddot{w}_b - T_1 V^2 \ddot{w}_s
\end{align*}
\]

(33)

In addition, the electric potential in piezoelectric lamina must satisfy Maxwell’s equation:

\[
\int_{-b}^{b} \nabla \cdot D \, dz + \int_{-a}^{a} \nabla \cdot D \, dz = \sum_{k=1}^{h} \int_{0}^{h_k} \left( \frac{\partial D_{ix}^k}{\partial x} + \frac{\partial D_{iy}^k}{\partial y} + \frac{\partial D_{iz}^k}{\partial z} \right) dz = 0
\]

(34)

5. Solution Procedures

In this study, two sets of simply supported boundary conditions (SSSS) are used to develop the Navier solutions for rectangular laminated plates and are shown in Table 2.

| Edges | Boundary Conditions |
|-------|---------------------|
| \( x = 0 \) and \( x = a \) | \( u = w_x = w_s = 0; N_s = M_s = M_i = 0 \) |
| \( y = 0 \) and \( y = b \) | \( u = w_y = w_s = 0; N_y = M_y = M_i = 0 \) |

To satisfy the above boundary conditions, the following expansion displacements \( (u, v, w_x, w_s) \) are chosen as in Table 3:
Table 3. The expansion displacements \( \{u, v, w_v, w_w\} \).

| Displacements \( u(x,y,t) \) | Boundary Conditions \( \text{SS-1} \) | Boundary Conditions \( \text{SS-2} \) |
|-----------------------------|----------------|----------------|
| \( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn} c(\alpha x) s(\beta y); \) | \( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn} s(\alpha x) c(\beta y); \) |
| \( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} v_{mn} c(\alpha x) c(\beta y); \) | \( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} v_{mn} s(\alpha x) s(\beta y); \) |
| \( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn}^v c(\alpha x) s(\beta y); \) | \( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn}^v s(\alpha x) s(\beta y); \) |
| \( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn}^w c(\alpha x) s(\beta y); \) | \( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn}^w s(\alpha x) s(\beta y); \) |

where \( u_{mn}, v_{mn}, w_{mn}^v, w_{mn}^w \) are unknown coefficients to be determined, \( c = \cos, s = \sin \), \( \alpha = m \pi / a \), \( \beta = n \pi / b \).

In addition, the electrostatic potential can be expanded as follows:

\[
\phi(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn} e^{i \omega t} s(\alpha x) z(\beta y)
\]  

Substituting Equation (35) and the displacements in Table 3 into the equations of motion Equations (33) and (34), one obtains the analytical solution in the following matrix form:

\[
\begin{bmatrix}
X_{11} & X_{12} & X_{13} & X_{14} & X_{15} \\
X_{12} & X_{22} & X_{23} & X_{24} & X_{25} \\
X_{13} & X_{23} & X_{33} & X_{34} & X_{35} \\
X_{14} & X_{24} & X_{34} & X_{44} & X_{45} \\
X_{15} & X_{25} & X_{35} & X_{45} & X_{55}
\end{bmatrix}
\begin{bmatrix}
\psi_{11} \\
\psi_{12} \\
\psi_{13} \\
\psi_{14} \\
\psi_{15}
\end{bmatrix}
- \omega^2
\begin{bmatrix}
\psi_{11} \\
\psi_{12} \\
\psi_{13} \\
\psi_{14} \\
\psi_{15}
\end{bmatrix}
= 0
\]

\[
\begin{bmatrix}
\omega_{mn}^u \\
\omega_{mn}^v \\
\omega_{mn}^w
\end{bmatrix}
= 0
\]

\[
\begin{bmatrix}
\phi_{mn}
\end{bmatrix}
= 0
\]

where the matrix elements of Equation (36) are given in the Appendix.

6. Results and Discussions

In this section, several numerical results are carried out and discussed to verify the accuracy and efficiency of the proposed theory in free vibration analysis of simply supported laminated piezoelectric rectangular plates. Furthermore, the influence of volume fraction of CNTs, distribution of CNTs, number of the lamina, CNT fiber orientation, and thickness of piezoelectric lamina on the natural frequencies of laminated plates are also investigated in detail.

6.1. Comparison Studies

6.1.1. Example 1

The non-dimensional natural frequencies \( \bar{\omega} = \omega_{mn} h \sqrt{\rho / G} \) of simply supported isotropic square plate were performed and compared with the existing results in Table 4:
Table 4. Non-dimensional natural frequencies $\bar{\omega}$ of simply supported boundary conditions (SSSS) isotropic square plate: $a/h = 10$; $b = a$.

| Mode | $m$ | $n$ | EXACT [38] | FSDT [38] | CPT [38] | Shimpi [29] | Present |
|------|-----|-----|------------|------------|-----------|-------------|---------|
| 1    | 1   | 1   | 0.0932     | 0.0930     | 0.0955    | 0.0930      | 0.0932  |
| 1    | 2   | 0.2226 | 0.2219     | 0.2360     | 0.2219    | 0.2232      |
| 2    | 2   | 0.3421 | 0.3406     | 0.3732     | 0.3406    | 0.3435      |
| 1    | 3   | 0.4171 | 0.4149     | 0.4629     | 0.4149    | 0.4192      |
| 2    | 3   | 0.5239 | 0.5206     | 0.5951     | 0.5206    | 0.5271      |
| 1    | 4   | -    | 0.6520     | 0.7668     | 0.6520    | 0.6618      |
| 3    | 3   | 0.6889 | 0.6834     | 0.8090     | 0.6834    | 0.6941      |
| 2    | 4   | 0.7511 | 0.7446     | 0.8926     | 0.7447    | 0.7572      |
| 3    | 4   | -    | 0.8896     | 1.0965     | 0.8897    | 0.9069      |
| 1    | 5   | 0.9268 | 0.9174     | 1.1365     | 0.9174    | 0.9356      |

It is worth noting that the results obtained by Srinivas et al. [38] used CPT, FSDT, and exact solutions, whereas the work of Shimpi et al. [29] was implemented using a new FSDT. It is seen that all obtained frequencies are in good agreement with available results.

6.1.2. Example 2

The second comparison study as follows:

The fundamental frequency of a square laminated PFG-CNTRC with piezoelectric lamina was calculated and compared with the results of K. Nguyen-Quang et al. [19] using an isogeometric approach. The plate had length $a = b = 0.4$ m, thickness $h = 0.05a$. Two continuous piezoelectric (PZT-5A) lamina of thickness $h_P = 0.1h$ were bonded to the top and bottom surfaces of the host. The material elastic properties for the matrix, CNT, and piezoelectric are listed in Table 5.

Table 5. Values of material parameters.

| Core | Plate | Matrix | Piezoelectric Layer |
|------|-------|--------|---------------------|
| $E_{11}^{CNT} = 5.64$ TPa | $E_0 = (3.52-0.0034T)$ (GPa) | $E = 63$ GPa; $G = 23.3$ GPa; $\nu = 0.35$ |
| $E_{22}^{CNT} = 7.0800$ TPa | $\nu_0 = 0.34$ | $\rho = 7750$ kg/m$^3$ |
| $G_{12}^{CNT} = 1.9455$ TPa | $\rho_0 = 1150$ kg/m$^3$ | $e_{31} = -7.209$ C/m$^2$, $e_{32} = e_{33}$ |
| $\rho^{CNT} = 0.175$ | | |
| $G_{12}^{CNT}$ | | $e_{33} = 15.118$ C/m$^2$ |
| $G_{33}^{CNT} = 1.2 G_{12}^{CNT}$ | | $p_{11} = p_{22} = 1.53 \times 10^{-8}$ F/m |

The CNT efficiency parameters are shown in Table 6.

Table 6. Carbon nanotube (CNT) efficiency parameters with respect to various volume fractions.

| $V^{CNT}$ | $\eta$ | $\eta_1$ | $\eta_2$ |
|-----------|-------|---------|---------|
| 0.12      | 0.137 | 1.022   | 0.7 $\eta_2$ |
| 0.17      | 0.142 | 1.626   | 0.7 $\eta_2$ |
| 0.28      | 0.141 | 1.585   | 0.7 $\eta_2$ |
The comparison results are listed in Table 7.

Table 7. The fundamental natural frequency (Hz) of the SSSS square piezoelectric laminated piezoelectric functionally graded carbon nanotube-reinforced composite plates (PFG-CNTRC) ($a = b = 0.4$ m; $h_t = 0.1h$; $a/h = 20$).

| $V$ | Type | Electrical Condition | Laminate Configurations | Present | Ref. [19] | Present | Ref. [19] | Present | Ref. [19] |
|-----|------|----------------------|------------------------|---------|-----------|---------|-----------|---------|-----------|
|     | UD   | C-circuit            | (p/p/p/p)              | 587.099 | 583.199   | 587.099 | 583.510   | 662.579 | 656.538   |
|     |      | O-circuit            |                        | 621.839 | 627.416   | 621.839 | 627.716   | 692.687 | 695.085   |
| 0.12| FG-X | C-circuit            |                        | 626.536 | 622.009   | 592.695 | 588.372   | 666.224 | 658.696   |
|     |      | O-circuit            |                        | 658.751 | 662.982   | 627.080 | 632.184   | 696.144 | 697.103   |
|     | FG-V | C-circuit            |                        | 563.624 | 560.042   | 585.314 | 581.714   | 661.328 | 655.606   |
|     |      | O-circuit            |                        | 600.128 | 606.518   | 620.273 | 626.205   | 691.506 | 694.272   |
|     | FG-O | C-circuit            |                        | 544.131 | 540.558   | 581.557 | 578.737   | 659.024 | 654.510   |
|     |      | O-circuit            |                        | 581.965 | 588.764   | 616.659 | 623.343   | 689.323 | 693.196   |
|     | UD   | C-circuit            | (p/p/0/0/p)            | 628.449 | 623.946   | 628.449 | 624.543   | 727.603 | 720.800   |
|     |      | O-circuit            |                        | 660.700 | 665.032   | 660.700 | 665.615   | 754.615 | 755.388   |
| 0.17| FG-X | C-circuit            |                        | 681.622 | 675.814   | 636.195 | 631.317   | 732.516 | 723.781   |
|     |      | O-circuit            |                        | 710.906 | 713.079   | 668.022 | 671.913   | 759.323 | 758.217   |
|     | FG-V | C-circuit            |                        | 595.013 | 591.216   | 625.837 | 621.914   | 726.077 | 719.594   |
|     |      | O-circuit            |                        | 629.510 | 635.182   | 658.370 | 663.359   | 753.169 | 754.324   |
|     | FG-O | C-circuit            |                        | 569.202 | 565.533   | 620.976 | 618.126   | 723.043 | 718.247   |
|     |      | O-circuit            |                        | 605.304 | 611.599   | 653.664 | 659.687   | 750.267 | 752.995   |
|     | UD   | C-circuit            | (p/0/90/0/p)           | 692.023 | 685.587   | 692.023 | 686.852   | 828.991 | 821.713   |
|     |      | O-circuit            |                        | 720.549 | 721.919   | 720.549 | 723.150   | 851.606 | 850.524   |
| 0.28| FG-X | C-circuit            |                        | 767.318 | 757.950   | 703.736 | 697.260   | 836.338 | 826.415   |
|     |      | O-circuit            |                        | 792.364 | 789.814   | 731.760 | 732.991   | 858.755 | 855.093   |
|     | FG-V | C-circuit            |                        | 642.030 | 637.353   | 688.175 | 682.974   | 827.574 | 820.463   |
|     |      | O-circuit            |                        | 673.463 | 677.399   | 717.082 | 719.788   | 850.294 | 849.465   |
|     | FG-O | C-circuit            |                        | 605.283 | 601.032   | 681.060 | 677.986   | 823.309 | 818.750   |
|     |      | O-circuit            |                        | 638.738 | 643.745   | 710.669 | 714.904   | 846.167 | 847.767   |

It can be seen that the present results agree well with those acquired by the isogeometric approach [19] for different volume fractions of CNTs, distribution of CNTs, number of layers, CNT fiber orientation, and electrical condition, which indicates the accuracy and correctness of the present formulation and solution method.

6.2. Parametric Studies

After showing the accuracy of the present model, the following new results for free vibration of laminated FG-CNTRC plates integrated with piezoelectric layers were investigated. The material elastic properties for the matrix, CNT, and piezoelectric material are shown in Tables 5 and 6.

6.2.1. Effect of FG-CNT Parameters

Natural frequencies of anti-symmetric cross-ply and angle-ply laminated PFG-CNTRC ($a = b = 0.4$m; $a/h = 20$) are shown in Table 8 and Tables 9–11, respectively. It is observed from these tables that the FG-X plates had the highest value of frequency, whereas the FG-O plates had the lowest one. Therefore, it can be concluded that the type of CNT distribution has a remarkable influence on the stiffness of the plate. In detail, the CNTs distributed close to the upper and lower surfaces of each FG-CNTRC layer were more efficient than those distributed near the mid-plane of each FG-CNTRC layer in increasing the stiffness of the laminated PFG-CNTRC. Table 8 reveals that with the increase in the
CNT volume fraction, the natural frequencies of the plates increased accordingly; these results are presented in more detail in Figure 2. Table 8 also shows that at the fixed value of the thickness ratio, the stiffness of the plate increased as the number layer of CNT increased. The effects of the width-to-

thickness ratio on the natural frequencies of angle-ply laminated PFG-CNTRC plates are also presented in Table 9. As expected, the frequencies decrease with the increment of $a/h$. This is because the plates become thinner with the increment of $a/h$, and as the results, the stiffness of the plate decreased.

Figure 2 shows the fundamental frequencies of anti-symmetric angle-ply $[p/(\theta/-\theta)/h/p]$ laminated PFG-CNTRC plates versus the lamination angle ($a = b = 0.4m$; $a/h = 20$). It can be seen that the fundamental frequency increased with the increase in lamination angle $\theta$ from 0 to 45, and decreased with $\theta$ values from 45 to 90 for all four CNT distribution types and three CNT volume fractions. This is compatible with conclusions in previous studies in the literature. The previous conclusions regarding the CNT distribution type are confirmed. Noted that the plate with FG-X distribution type had the highest frequency, while with FG-O type had the lowest one.

| $V_{\text{cnt}}$ | Type | Electrical Condition | Configuration |
|-----------------|------|----------------------|---------------|
|                 | UD   | C-circuit            | $[p/(0/90)/p]$ | $[p/(0/90)/p]$ | $[p/(0/90)/p]$ | $[p/(0/90)/p]$ |
| 0.12            |      |                      | 535.019       | 574.472       | 581.514       | 585.093 |
| FG-X            |      | C-circuit            | 546.627       | 577.979       | 583.618       | 586.488 |
| FG-V            |      | C-circuit            | 530.609       | 573.512       | 581.131       | 584.998 |
| FG-O            |      | C-circuit            | 523.307       | 571.070       | 579.521       | 583.809 |
| 0.17            |      |                      | 554.285       | 610.732       | 620.631       | 625.644 |
| FG-X            |      | C-circuit            | 570.880       | 615.687       | 623.633       | 627.665 |
| FG-V            |      | C-circuit            | 547.378       | 609.367       | 620.155       | 625.610 |
| FG-O            |      | C-circuit            | 537.761       | 606.167       | 618.011       | 623.996 |
| 0.28            |      |                      | 575.055       | 664.853       | 680.081       | 687.747 |
| FG-X            |      | C-circuit            | 601.487       | 672.685       | 684.983       | 691.192 |
| FG-V            |      | C-circuit            | 635.165       | 702.232       | 713.903       | 719.805 |
| FG-O            |      | C-circuit            | 563.213       | 663.041       | 679.760       | 688.157 |
| 0.12            |      |                      | 599.863       | 693.172       | 708.977       | 716.931 |
| FG-O            |      | C-circuit            | 549.709       | 658.607       | 676.710       | 685.794 |
| FG-O            |      | C-circuit            | 586.877       | 688.865       | 706.029       | 714.662 |
Table 9. The fundamental natural frequencies $\omega$(Hz) of anti-symmetric angle-ply \[p/(\theta/\theta)/p\]
laminated PFG-CNTRC plate ($a = b = 0.4m$; $h_p = 0.1h$; $V^\text{CNT} = 0.12$).

| Layers \[p/(\beta/\beta)/p\] | Type | Electrical Condition | a/h  | 10   | 20   | 50   | 100  |
|-------------------------------|------|----------------------|------|------|------|------|------|
| UD                            |      | C-circuit            | 1083.983 | 589.563 | 242.289 | 121.632 |
| -                             |      | O-circuit            | 1141.716 | 624.136 | 256.978 | 129.043 |
| FG-X                          |      | C-circuit            | 1087.691 | 591.770 | 243.224 | 122.104 |
| -                             |      | O-circuit            | 1145.139 | 626.204 | 257.859 | 129.488 |
| FG-V                          |      | C-circuit            | 1082.852 | 588.833 | 241.972 | 121.472 |
| -                             |      | O-circuit            | 1140.697 | 623.457 | 256.680 | 128.892 |
| FG-O                          |      | C-circuit            | 1080.548 | 587.458 | 241.389 | 121.177 |
| -                             |      | O-circuit            | 1138.576 | 622.171 | 256.131 | 128.615 |
| [p/(\beta/\beta)/p]          |      |                      |      |      |      |      |      |
| UD                            |      | C-circuit            | 1166.673 | 644.809 | 266.550 | 133.932 |
| -                             |      | O-circuit            | 1217.212 | 675.925 | 279.922 | 140.691 |
| FG-X                          |      | C-circuit            | 1172.045 | 648.119 | 267.971 | 134.651 |
| -                             |      | O-circuit            | 1222.218 | 679.056 | 281.274 | 141.375 |
| FG-V                          |      | C-circuit            | 1164.971 | 643.678 | 266.053 | 133.680 |
| -                             |      | O-circuit            | 1215.655 | 674.862 | 279.450 | 140.452 |
| FG-O                          |      | C-circuit            | 1161.583 | 641.592 | 265.157 | 133.227 |
| -                             |      | O-circuit            | 1212.503 | 672.890 | 278.598 | 140.021 |
| [p/(\beta/\beta)/p]          |      |                      |      |      |      |      |      |
| UD                            |      | C-circuit            | 1192.400 | 662.579 | 274.449 | 137.944 |
| -                             |      | O-circuit            | 1240.859 | 692.687 | 287.437 | 144.514 |
| FG-X                          |      | C-circuit            | 1198.251 | 666.224 | 276.021 | 138.740 |
| -                             |      | O-circuit            | 1246.327 | 696.144 | 288.937 | 145.273 |
| FG-V                          |      | C-circuit            | 1190.536 | 661.328 | 273.896 | 137.664 |
| -                             |      | O-circuit            | 1239.148 | 691.506 | 286.911 | 144.246 |
| FG-O                          |      | C-circuit            | 1186.834 | 659.024 | 272.903 | 137.162 |
| -                             |      | O-circuit            | 1235.694 | 689.323 | 285.965 | 143.767 |
**Table 10.** The fundamental natural frequencies $\omega$(Hz) of anti-symmetric angle-ply $[p/(−0/θ)/p]$ laminated PFG-CNTRC plate ($a = b = 0.4m; h_p = 0.1h; V_{CNT} = 0.17$).

| Layers | Type   | Electrical Condition | a/h     |
|--------|--------|----------------------|---------|
|        |        |                      | 10      | 20      | 50      | 100     |
| $[p/(−5/5)/p]$ | UD     | C-circuit            | 1155.117| 631.748 | 260.141 | 130.634 |
|        |        | O-circuit            | 1208.292| 663.803 | 273.800 | 137.529 |
|        | FG-X   | C-circuit            | 1160.475| 634.869 | 261.455 | 131.296 |
|        |        | O-circuit            | 1213.327| 666.758 | 275.047 | 138.157 |
|        | FG-V   | C-circuit            | 1153.930| 630.866 | 259.743 | 130.431 |
|        |        | O-circuit            | 1207.254| 662.982 | 273.423 | 137.336 |
|        | FG-O   | C-circuit            | 1150.769| 628.983 | 258.944 | 130.028 |
|        |        | O-circuit            | 1204.308| 661.204 | 272.665 | 136.954 |
| $[p/(−30/30)/p]$ | UD     | C-circuit            | 1261.941| 704.587 | 292.376 | 146.997 |
|        |        | O-circuit            | 1306.976| 732.703 | 304.536 | 153.149 |
|        | FG-X   | C-circuit            | 1269.356| 709.089 | 294.304 | 147.971 |
|        |        | O-circuit            | 1314.002| 737.008 | 306.385 | 154.084 |
|        | FG-V   | C-circuit            | 1260.134| 703.205 | 291.744 | 146.674 |
|        |        | O-circuit            | 1305.348| 731.399 | 303.931 | 152.840 |
|        | FG-O   | C-circuit            | 1255.663| 700.434 | 290.549 | 146.070 |
|        |        | O-circuit            | 1301.138| 728.756 | 302.786 | 152.261 |
| $[p/(−45/45)/p]$ | UD     | C-circuit            | 1294.290| 727.603 | 302.726 | 152.263 |
|        |        | O-circuit            | 1337.071| 754.615 | 314.466 | 158.209 |
|        | FG-X   | C-circuit            | 1302.277| 732.516 | 304.840 | 153.332 |
|        |        | O-circuit            | 1344.658| 759.323 | 316.500 | 159.237 |
|        | FG-V   | C-circuit            | 1292.321| 726.077 | 302.022 | 151.904 |
|        |        | O-circuit            | 1335.288| 753.169 | 313.791 | 157.863 |
|        | FG-O   | C-circuit            | 1287.487| 723.043 | 300.707 | 151.238 |
|        |        | O-circuit            | 1330.721| 750.267 | 312.527 | 157.223 |
Table 11. The fundamental natural frequencies $\omega$(Hz) of anti-symmetric angle-ply [p/(−0/0)/p] laminated PFG-CNTRC plate ($a = b = 0.4$ m; $h_p = 0.1$ m; $V_{CNT} = 0.28$).

| Configuration | Type       | Electrical Condition | a/h  |
|---------------|------------|----------------------|------|
|               |            |                      | 10   | 20  | 50  | 100 |
| UD            | C-circuit  | 1252.438             | 696.727 | 288.711 | 145.122 |
|               | O-circuit  | 1298.062             | 725.018 | 300.910 | 151.292 |
| FG-X          | C-circuit  | 1261.724             | 701.745 | 290.768 | 146.154 |
|               | O-circuit  | 1307.040             | 729.846 | 302.884 | 152.282 |
| FG-V          | C-circuit  | 1252.602             | 695.985 | 288.272 | 144.891 |
|               | O-circuit  | 1298.480             | 724.357 | 300.493 | 151.071 |
| FG-O          | C-circuit  | 1247.846             | 693.164 | 287.074 | 144.287 |
|               | O-circuit  | 1293.968             | 721.661 | 299.345 | 150.492 |
| [p/(−5/5)/p]  |            |                      | 100   |     |     |     |
| UD            | C-circuit  | 1391.451             | 797.939 | 334.643 | 168.530 |
|               | O-circuit  | 1428.003             | 821.744 | 345.144 | 173.861 |
| FG-X          | C-circuit  | 1403.564             | 804.759 | 337.484 | 169.959 |
|               | O-circuit  | 1439.786             | 828.365 | 347.899 | 175.247 |
| FG-V          | C-circuit  | 1391.475             | 796.657 | 333.881 | 168.128 |
|               | O-circuit  | 1428.307             | 820.563 | 344.410 | 173.472 |
| FG-O          | C-circuit  | 1385.151             | 792.713 | 332.170 | 167.262 |
|               | O-circuit  | 1422.249             | 816.756 | 342.754 | 172.633 |
| [p/(−30/30)/p]|            |                      | 100   |     |     |     |
| UD            | C-circuit  | 1431.534             | 828.991 | 349.087 | 175.920 |
|               | O-circuit  | 1465.742             | 851.606 | 359.140 | 181.030 |
| FG-X          | C-circuit  | 1444.428             | 836.338 | 352.162 | 177.468 |
|               | O-circuit  | 1478.305             | 858.755 | 362.130 | 182.535 |
| FG-V          | C-circuit  | 1431.592             | 827.574 | 348.231 | 175.467 |
|               | O-circuit  | 1466.081             | 850.294 | 358.314 | 180.591 |
| FG-O          | C-circuit  | 1424.854             | 823.309 | 346.367 | 174.523 |
|               | O-circuit  | 1459.610             | 846.167 | 356.505 | 179.673 |
| [p/(−45/45)/p]|            |                      | 100   |     |     |     |

Figure 2. Effect of lamination angle on the natural frequency of laminated functionally graded carbon nanotube-reinforced composite plates (FG-CNTRC) plate coupled with O-circuit piezoelectric layer: (a) for different carbon nanotube (CNT) distribution types; (b) for different CNT volume fractions.

6.2.2. Effect of Electrical Condition

The natural frequency of laminated cross-ply FG-CNTRC plates ($a = b = 0.4$ m; $a/h = 20$; $V_{CNT} = 0.28$) coupled with closed and open piezoelectric layers are shown in Tables from 8 to 11 with
different inlet parameters: CNT volume fraction, CNT distribution type, number of layers, lamination angle, and width-to-thickness ratio. It is seen from these tables that the frequencies of the plates increased as the electrical boundary conditions changed from the closed circuit to the open circuit. Figure 3, once again, indicates that the FG-CNTRC plates coupled with the open circuit of piezoelectric layers had a greater stiffness than the FG-CNTRC plates coupled with the closed circuit of the piezoelectric layers. This may be because the open circuit converts electric potential to mechanical energy while the closed circuit does not.

![Graphs showing the effect of lamination angle on natural frequency of laminated FG-CNTRC plates](image)

**Figure 3.** Effect of lamination angle on natural frequency of laminated FG-CNTRC plates [p/(-0/0)/p] with electrical condition (a = b = 0.4m; w/h = 20; V_CNT = 0.28); (a) UD; (b) FG-X; (c) FG-V; (d) FG-O.

### 6.2.3. Effect of Piezoelectric Layer Thickness

The effect of piezoelectric layer thickness on the natural frequency of hybrid plates (a = b = 0.4 m; FG-X; [p/(-45/45)/p]) for different CNT volume fraction and width-to-thickness was examined. For this purpose, the natural frequency increment \( \delta \) between O-circuit and C-circuit electrical conditions is defined as:

\[
\delta = \frac{\omega_{O-circuit} - \omega_{C-circuit}}{\omega_{C-circuit}} \times 100\%
\]

(37)

In Figure 4a,b, the effects of piezoelectric layer thickness on the natural frequency increment \( \delta \) for different CNT volume fractions and different \( a/h \) ratio are depicted, respectively. It is found that the natural frequency increment \( \delta \) had a higher value with a lower volume fraction of CNT and a larger \( a/h \) ratio. Furthermore, it can be seen when the \( h_p/h \) ratio increased, the natural frequency increment \( \delta \) increased. Accordingly, piezoelectric layer thickness had a greater effect on the natural frequency of an O-circuit piezoelectric coupled plate than that of a C-circuit.
Figure 4. Variation of the natural frequency increment $\delta$ between O-circuit and C-circuit electrical conditions versus the $h_p/h$ ratio for a square piezoelectric functionally graded carbon nanotube-reinforced composite plates (PFG-CNTRC) plate ($a = b = 0.4$ m; FG-X; $[p/(\pm 45/45)/p]$): (a) for different CNT volume fractions; (b) for different $a/h$ ratio.

Furthermore, the variations of the frequency parameter $\omega$ (Hz) are plotted in Figure 5a,b for the open circuit condition with different CNT volume fractions and different width-to-thickness ratios, respectively. These figures indicate that the natural frequency of the hybrid plate decreased by increasing the thickness of the piezoelectric layer from zeros to a specific value. After this value, the natural frequencies were increased by the incrementing of the piezoelectric layer in the cases of moderately thick plates but seem to be unchanged in cases of thin plates. It can be concluded that the piezoelectric effect is more effective in the case of thick plates rather than thin ones.

Figure 5. Effect of $h_p/h$ ratio on frequency parameter $\omega$ (Hz) of a square FG-CNTRC plate coupled with open circuit piezoelectric layer and different width-to-thickness ratio ($a = b = 0.4$m; $V_{\text{CNT}} = 0.28$; $[p/(\pm 45/45)/p]$): (a) for different CNT volume fractions; (b) for different $a/h$ ratio.

7. Conclusions

In summary, this paper shows our contribution to the development of a new four-variable refined plate theory for free vibration analysis of laminated PFG-CNTRC plates. The comparison studies show that the present theory is not only accurate but also efficient in predicting the free vibration responses of the plates.
Our insight indicates that the natural frequency of the hybrid plates is strongly affected by the volume fraction of CNT and the distribution type of CNT in the matrix. FG-X CNTRC plate had the highest frequency, while the FG-O CNTRC plate had the smallest frequency regarding all inlet studied parameters. In addition, the lamination angles of CNT fiber and number of CNT lamina have a significant effect on the stiffness of the hybrid plate. Numerical results also revealed that the piezoelectric effect was more prominent in plates bonded with O-circuit piezoelectric lamina because, during vibration, the O-circuit converts electric potential to mechanical energy.

The present theory is accurate and efficient in solving free vibration behaviors of laminated FG-CNT reinforced composite plates with the piezoelectric layer and may be useful in the study of similar composite structures.

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**Appendix A**

In case of closed circuit:

\[
\chi_{11} = -2C_{11}h_p\alpha^2 - (C_{11} - C_{12})h_p\beta^2 - A_{11}\alpha^2 - A_{14}\beta^2; \quad \chi_{12} = -(C_{11} + C_{12})\alpha\beta h_p - (A_{12} + A_{14})\alpha\beta; \\
\chi_{15} = 0; \quad \chi_{22} = -(C_{11} - C_{12})\alpha^2 h_p - 2C_{11}\beta^2 h_p - A_{66}\alpha^2 - A_{22}\beta^2; \quad \chi_{25} = 0; \\
\chi_{33} = -D_{11}\alpha^4 - D_{22}\beta^4 - (2D_{13} + 4D_{66})\alpha^2\beta^2 - (2/3)(h_p)^3 + h_p(h_p)(h_p)(h_p)^2 + 1/3(h_p)^3 + 2h_p(h_p)(h_p)(h_p)^2 + (h_p)(h_p)(h_p), \\
\chi_{34} = -D_{11}\alpha^4 - (2D_{13} + 4D_{66})\alpha^2\beta^2 - D_{22}\beta^4; \quad \chi_{35} = 4 / 3\alpha h_p \beta^2 + 4 / 3\alpha h_p \beta^2; \\
\chi_{44} = -2C_{11}\alpha^4 \left(f(z)\right)^2 h_p - 4C_{11}\alpha^4 f(z)^2 h_p - 2C_{11}\alpha^4 (f(z))^2 h_p - 2C_{11}\alpha^4 g(z) h_p - 2C_{11}\alpha^4 g(z) h_p - 2C_{11}\alpha^4 g(z) h_p - 2C_{11}\alpha^4 g(z) h_p; \\
\chi_{45} = 0; \quad \chi_{51} = \chi_{12}; \quad \chi_{35} = \chi_{13}; \quad \chi_{25} = \chi_{23}; \quad \chi_{45} = \chi_{43}; \quad \chi_{35} = \chi_{34}; \quad \chi_{51} = \chi_{15}; \quad \chi_{52} = \chi_{25}; \quad \chi_{53} = \chi_{35}; \quad \chi_{54} = \chi_{45}; \\
\chi_{55} = 4 / 3h_p p_{11} \alpha^2 + 4 / 3h_p p_{22} \beta^2 + 16 / 3p_{15} \alpha^2; \\
\psi_{11} = \psi_{22} = T_{11} h_p; \quad \psi_{33} = \psi_{15} = T_{33} h_p; \quad \psi_{33} = \psi_{25} = T_{25} h_p + T_{21} h_p; \\
\psi_{34} = T_{10} + T_{21} h_p; \quad \psi_{44} = \psi_{14} = \psi_{24} = \psi_{43} = \psi_{14}; \quad \psi_{44} = T_{10} + T_{21} h_p; \quad \psi_{31} = \psi_{33} = 0; \quad \psi_{25} = \psi_{35} = \psi_{45} = \psi_{44} = 0. \\

For cross-ply laminates:

\[
\chi_{13} = B_{11}\alpha^2 + (B_{12} + 2B_{66})\alpha\beta^2; \quad \chi_{14} = 2C_{11}\alpha^3 f(z) h_p + 2C_{11}\alpha^2 f(z) h_p + B_{11}\alpha^2 + (B_{12} + 2B_{66})\alpha\beta^2; \\
\chi_{23} = (B_{12} + 2B_{66})\alpha^2 \beta^2; \quad \chi_{24} = 2C_{11}\alpha^3 f(z) h_p + 2C_{11}\alpha^2 f(z) h_p + B_{22}\beta^2 + (B_{12} + 2B_{66})\alpha\beta^2; \\
\psi_{13} = -T_{10} h_p; \quad \psi_{14} = -T_{10} h_p; \quad \psi_{23} = -T_{10} h_p; \quad \psi_{24} = -T_{10} h_p. \\

For angle-ply laminates:

\[
\chi_{13} = 3B_{16}\alpha^2 \beta^2 + B_{26} \beta^2; \quad \chi_{14} = 3B_{16}\alpha^3 \beta + B_{26}\alpha^2 \beta^2; \quad \chi_{23} = B_{16}\alpha^3 + 3B_{26}\alpha \beta^2; \quad \chi_{24} = B_{16}\alpha^3 + 3B_{26}\alpha \beta^2; \\
\psi_{13} = \psi_{14} = \psi_{23} = \psi_{24} = 0; \\
\]

In case of Open circuit:

\[
\chi_{11} = h_p(C_{12} - C_{11})\beta^2 - 2h_p(C_{11} + (C_{12}^2)^2) \alpha^2 - A_{66}\beta^2 - A_{11}\alpha^2; \\
\chi_{12} = -(A_{12} + A_{14})\alpha\beta - (C_{11} + C_{12} - 2(C_{12}^2)^2) \alpha\beta h_p; \\
\]
\[ \chi_{22} = (C_{12} - C_{11})\alpha^2 h_p - \left( C_{11} + \frac{\beta^2 (\tau_{31})^2}{\bar{P}_{33}} \right)2h_p\beta^2 - A_{66}\alpha^2 - A_{22}\beta^2; \]

\[ \chi_{33} = -D_{11}\alpha^4 - 2(D_{12} + 2D_{66})\alpha^2\beta^2 - D_{22}\beta^4 - \left( 2/3 C_{11}(h_p)^3 + \frac{(\tau_{31})^2 (h_p)^3}{\bar{P}_{33}} + 3/2 C_{31}(h_p)^3 + 1/2 C_{11}(h_p)^3 h_p \right) \]

\[ + \frac{3}{2} \frac{(\tau_{31})^2 h_p (h_p)^3}{\bar{P}_{33}} + 1/2 \frac{(\tau_{31})^2 (h_p)^3 h_p}{\bar{P}_{33}} \right) \alpha^4 - 4 \frac{3 C_{11}(h_p)^3}{\bar{P}_{33}} + 2 \frac{(\tau_{31})^2 (h_p)^3}{\bar{P}_{33}} + 2 C_{11}(h_p)^3 h_p \]

\[ + \frac{3}{2} \frac{(\tau_{31})^2 h_p (h_p)^3}{\bar{P}_{33}} + 1/2 \frac{(\tau_{31})^2 (h_p)^3 h_p}{\bar{P}_{33}} \right) \beta^4; \]

\[ \chi_{35} = -(\tau_{31} h_p + 4/3 \tau_{31} h_p)4\alpha^2 - (\tau_{31} h_p + 4/3 \tau_{31} h_p)4\beta^2; \]

\[ \chi_{44} = -H_{11}\alpha^4 - H_{22}\beta^4 - (H_{11}' + 2\mu_{66})2\alpha^2\beta^2 - A_{66}^\alpha\alpha^2 + A_{66}^\beta\beta^2 - \left( C_{11} + \frac{(\tau_{31})^2}{\bar{P}_{33}} \right) \left( 2(f(z))^2 h_p \alpha^4 \right) \]

\[ - \left( C_{11} + \frac{(\tau_{31})^2}{\bar{P}_{33}} \right) \left( 4 h_p (f(z))^2 \alpha^2 \beta^2 - 2 C_{35}(h_p)^2 \alpha^2 \beta^2 - 2 C_{35}(h_p)^2 \beta^2; \right) \]

\[ \chi_{35} = 16/3 h_p p_{11} \alpha^2 + 16/3 h_p p_{22} \beta^2 + 16 \frac{\bar{P}_{33}}{h_p}; \]

For cross-ply laminates:

\[ \chi_{13} = B_{13} \alpha^3 + (B_{12} + 2B_{66})\alpha^2\beta; \]

\[ \chi_{14} = B_{14} \alpha^3 + B_{14}^\alpha \alpha^2 \beta + 2B_{14}^\beta \alpha^2 \beta^2 + 2C_{11} \alpha^3 f(z)h_p + 2 \frac{\alpha^2 (\tau_{31})^2 f(z)h_p}{\bar{P}_{33}} + 2C_{11} \alpha^2 f(z)h_p + 2 \frac{\alpha^2 (\tau_{31})^2 f(z)h_p}{\bar{P}_{33}} \]

\[ \chi_{23} = B_{23} \beta^3 + (B_{12} + 2B_{66})\alpha^2 \beta; \]

\[ \chi_{24} = B_{24} \beta^3 + (B_{12} + 2B_{66})\alpha^2 \beta + 2(\tau_{31})^2 f(z)h_p + 2(\tau_{31})^2 f(z)h_p + 2(\tau_{31})^2 f(z)h_p; \]

For angle-ply laminates:

\[ \chi_{13} = 3B_{16} \alpha^2 \beta + B_{26} \beta^3; \]

\[ \chi_{14} = 3B_{16} \alpha^2 \beta + B_{26} \beta^3; \]

\[ \psi_{13} = \psi_{14} = \psi_{23} = \psi_{24} = 0; \]
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