The BKL scenario, infrared renormalization, and quantum cosmology

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Received October 3, 2018
Accepted January 2, 2019
Published January 14, 2019

Abstract. A discussion of inhomogeneity is indispensable to understand quantum cosmology, even if one uses the dynamics of homogeneous geometries as a first approximation. While a full quantization of inhomogeneous gravity is not available, a broad framework of effective field theory provides important ingredients for quantum cosmology. Such a setting also allows one to take into account lessons from the Belinski-Khalatnikov-Lifshitz (BKL) scenario. Based on several new ingredients, this article presents conditions on various parameters and mathematical constructions that appear in minisuperspace models. Examples from different approaches demonstrate their restrictive nature.

Keywords: quantum cosmology, cosmic singularity

ArXiv ePrint: 1810.00238
1 Introduction

Quantum gravity is a quantum theory of many interacting degrees of freedom. Such a theory, in general, requires approximations and assumptions in order to derive reliable predictions of physical phenomena. Given the complexity of such a theory, it is hard to find good candidates without observational assistance. Condensed-matter physics, for instance, provides a wealth of examples in which this program has been followed through.

In quantum gravity, by contrast, no clear observations will be available for the foreseeable future. One could draw the lesson that one should postpone any quantum-gravity phenomenology until observations can indicate a good starting point for such an analysis. However, we also need phenomenology to suggest promising experiments. The main conundrum of quantum gravity is therefore a chicken-and-egg problem — what should come first, good phenomenology that can suggest experiments, or observations that indicate how to do reliable phenomenology? Effective field theory can provide a solution: by parameterizing a large class of potential outcomes, promising effects can be highlighted.

One realm in which important quantum-gravity effects are expected is cosmology. Quantum cosmology has traditionally been performed by various mathematical studies of simple (toy, minisuperspace) models and different kinds of perturbations around them, but no systematic effective field theory is available. The purpose of this paper is to point out several ingredients of quantum cosmology suggested by effective arguments, and to show that some existing models are at odds with these properties. The main contributions to this program, which in individual form are not new but appear here in a novel and fruitful combination, are: (i) A minisuperspace approximation (as opposed to truncation) [1], (ii) the Belinskii-Khalatnikov-Lifshitz (BKL) scenario [2], and (iii) the infrared behavior of gravity [3].

The main ingredient missing in existing models of quantum cosmology is infrared renormalization, introduced here in the context of cosmological models. Three approaches will be analyzed in this new picture. While the effective framework modifies the interpretations of
all these examples, one of them is seen to require a major revision: loop quantum cosmol-
yogy, in its commonly practiced form, does not obey the conditions extracted from effective
field theory.

2 Minisuperspace approximation

We introduce the first major deviation from standard quantum cosmology by way of a brief
review of the results of [1]. As a well-controlled setting, we consider scalar field theory on
Minkowski space-time with a potential $W(\phi)$ and Lagrangian

$$L = \int d^3 x \left( \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} |\nabla \phi|^2 - W(\phi) \right).$$

(2.1)

A minisuperspace truncation of this theory is obtained by assuming that $\phi$ is spatially con-
stant, and then integrating over some fixed spatial region with finite volume $V_0 = \int d^3 x$. The
resulting minisuperspace Lagrangian is

$$L_{\text{mini}} = V_0 \left( \frac{1}{2} \dot{\phi}^2 - W(\phi) \right).$$

(2.2)

It implies the minisuperspace momentum

$$p = V_0 \dot{\phi}.$$  

(2.3)

The minisuperspace Hamiltonian

$$H_{\text{mini}} = \frac{1}{2} \frac{p^2}{V_0} + V_0 W(\phi),$$

(2.4)

is straightforwardly quantized to

$$\hat{H}_{\text{mini}} = \frac{1}{2} \frac{\hat{p}^2}{V_0} + V_0 W(\hat{\phi}).$$

(2.5)

2.1 Quantum theory of regions

Without doing a detailed analysis, it can easily be seen that quantum corrections, unlike
classical effects, depend on the averaging volume $V_0$. For instance, for a quadratic potential,
changing $V_0$ would have the same effects on quantum corrections as changing the mass has
for the standard harmonic oscillator. This minisuperspace result is conceptually related to
physical effects in quantum-field theory, such as the Casimir force between two plates which
depends on the size of an enclosed region. Our first lesson, which will play an important
role in the effective field theory to be outlined in what follows, is therefore that quantum
cosmology is a quantum theory of regions (or patches in the description given in [4]). It is
not a quantum theory of the metric or scale factor, as it is often presented. In particular,
quantum effects in homogeneous models depend on the apparently arbitrary size $V_0$ of an
averaging region.

A variety of methods can be used to derive the effective potential

$$W_{\text{eff}}^{\text{mini}}(\phi) = W(\phi) + \frac{1}{2V_0} \hbar \sqrt{W''(\phi)}.$$  

(2.6)
in the minisuperspace model to first order in $\hbar$, clearly showing the $V_0$-dependence of quantum corrections. For instance, the leading term of the low-energy effective action applied to quantum mechanics as a 0 + 1-dimensional quantum-field theory is of this form [5], and an independent derivation can be done using canonical effective methods [6, 7].

The canonical derivation illustrates the role of quantum fluctuations: if we take the expectation value of the Hamiltonian $\hat{H}$ in a semiclassical state with fluctuations $\langle \Delta \phi \rangle^2 \sim \hbar$ and $(\Delta p)^2 \sim \hbar$, to first order in $\hbar$ we can write

$$
\langle \hat{H} \rangle = \frac{\langle \hat{p}^2 \rangle}{2V_0} + V_0 \langle \hat{W}(\hat{\phi}) \rangle 
$$

$$
= \frac{\langle \hat{p}^2 \rangle}{2V_0} + \frac{\langle \Delta p \rangle^2}{2V_0} + V_0 W(\hat{\phi}) + \frac{1}{2} V_0 W''(\hat{\phi}) (\Delta \phi)^2 + \cdots
$$

(2.7)

Heisenberg’s equations of motion can be used to derive the following time derivatives of fluctuations, coupled to the covariance $\Delta(\phi p) = \frac{1}{2} \langle \hat{\phi} \hat{p} + \hat{p} \hat{\phi} \rangle - \langle \hat{\phi} \rangle \langle \hat{p} \rangle$:

$$
\frac{d(\Delta \phi)^2}{dt} = \frac{2}{V_0} \Delta(\phi p) + \cdots
$$

(2.8)

$$
\frac{d\Delta(\phi p)}{dt} = \frac{1}{V_0} (\Delta p)^2 - V_0 W''(\hat{\phi}) (\Delta \phi)^2 + \cdots
$$

(2.9)

$$
\frac{d(\Delta p)^2}{dt} = -2V_0 W''(\hat{\phi}) \Delta(\phi p) + \cdots
$$

(2.10)

again to first order in $\hbar$. For an expansion around the stationary ground state, the moments are (almost) constant in time. For simplicity, we set the time derivatives exactly equal to zero. Small variations in time can be included by a systematic adiabatic expansion [6–8]. Therefore,

$$
\Delta(\phi p) = 0
$$

(2.11)

from (2.8) or (2.10) and

$$
(\Delta p)^2 = V_0^2 W''(\hat{\phi}) (\Delta \phi)^2
$$

(2.12)

from (2.9). We then minimize the contribution

$$
H_\Delta = \frac{(\Delta p)^2}{2V_0} + \frac{1}{2} V_0 W''(\hat{\phi}) (\Delta \phi)^2
$$

(2.13)

of fluctuations to the Hamiltonian (2.7), again to be close to the ground state, respecting the uncertainty relation

$$
(\Delta \phi)^2 (\Delta p)^2 - \Delta(\phi p)^2 \geq \frac{\hbar^2}{4}.
$$

(2.14)

Since $H_\Delta$ is linear in fluctuations, the minimum is realized at the boundary implied by the inequality (2.14), or for fluctuations saturating the uncertainty relation. Combined with (2.12), we obtain

$$
(\Delta p)^2 = \frac{\hbar^2}{4(\Delta \phi)^2} = V_0^2 W''(\hat{\phi}) (\Delta \phi)^2
$$

(2.15)

or

$$
(\Delta \phi)^2 = \frac{\hbar}{2V_0 \sqrt{W''(\hat{\phi})}}, \quad (\Delta p)^2 = \frac{1}{2} V_0 \hbar \sqrt{W''(\hat{\phi})}.
$$

(2.16)
With these values, $H_{\Delta}/V_0$ equals the correction term in (2.6). As shown by this derivation, the $V_0$-dependence of quantum corrections is determined by two universal properties: the symplectic structure or canonical relationships used to derive Heisenberg’s equations of motion (2.8)–(2.10), together with the uncertainty relation. In the next section, we will confirm the same qualitative behavior in quantum cosmology. But first we have to find a valid interpretation of $V_0$ within effective field theory.

2.2 Infrared contributions

The full theory (2.1), from which we derived the minisuperspace model, also has an effective potential: the Coleman-Weinberg potential [9]

$$W_{\text{eff}}(\phi) = W(\phi) + \frac{1}{2}\hbar \int \frac{d^4k}{(2\pi)^4} \log \left( 1 + \frac{W''(\phi)}{|k|^2} \right).$$  (2.17)

It looks very different from the minisuperspace effective potential, but, as noticed in [10], performing the $k^0$-integration in closed form reveals their similarity: the Coleman-Weinberg potential then equals

$$W_{\text{eff}}(\phi) = W(\phi) + \frac{1}{2}\hbar \int \frac{d^3k}{(2\pi)^3} \left( \sqrt{|\vec{k}|^2 + W''(\phi)} - |\vec{k}| \right).$$  (2.18)

The minisuperspace potential is related to the field-theory effective potential through the infrared contribution

$$W_{\text{IR}}(\phi) = \frac{1}{2}\hbar \int_{|\vec{k}| \leq k_{\text{max}}} \frac{d^3k}{(2\pi)^3} \left( \sqrt{|\vec{k}|^2 + W''(\phi)} - |\vec{k}| \right).$$  (2.19)

of the latter: if we use quantum-field theory to describe the modes with wave length greater than $1/V_0^{1/3}$, which remain inhomogeneous after averaging over a region with volume $V_0$, they result in an infrared contribution with $k_{\text{max}} = 2\pi/V_0^{1/3}$. For large $V_0$ and therefore small $k_{\text{max}}$, we can replace the integrand in (2.19) with the integrand at $\vec{k} = 0$ times the $k$-integration volume:

$$W_{\text{IR}}(\phi) \approx W(\phi) + \frac{\hbar}{12\pi^2} k_{\text{max}}^3 \sqrt{W''(\phi)} = W(\phi) + \frac{2\pi}{3V_0} \hbar \sqrt{W''(\phi)}$$  (2.20)

in agreement with $W_{\text{eff}}^{\text{mini}}$ up to a numerical factor. The different numerical factor can be related to the separation of modes, which is more clear in models with a discrete spectrum of $\vec{k}$. Such models, studied in more detail in [1], can lead to complete agreement between minisuperspace potentials and infrared contributions of field-theory potentials. Here, we are mainly interested in seeing the common $V_0$-dependence.

3 Infrared behavior

A more complicated infrared behavior is obtained in theories with massless excitations, such as gravity. In the scalar model, we have implicitly assumed that $W''(\phi)$ is sufficiently large, such that $\sqrt{|\vec{k}|^2 + W''(\phi)}$ can be replaced in (2.20) by its Taylor expansion with respect to $|\vec{k}|$ in the entire integration region up to $k_{\text{max}}$. If $W''(\phi) \ll |\vec{k}|^2$, however,

$$\sqrt{|\vec{k}|^2 + W''(\phi)} \approx |\vec{k}| + \frac{1}{2} \frac{W''(\phi)}{|\vec{k}|}.$$  (3.1)
is different from the previous expansion. Evaluated at \( k_{\text{max}} \) (since (3.1) is singular at \(|\vec{k}| = 1\) and multiplied with the \( k \)-integration volume in (2.19), we have

\[
W_{\text{IR}}(\phi) \approx W(\phi) + \frac{\hbar}{12\pi^2} \left( k_{\text{max}}^4 + \frac{1}{2} W''(\phi) k_{\text{max}}^2 \right). \tag{3.2}
\]

In particular,

\[
W_{\text{IR}}(\phi) \approx W(\phi) + \frac{4}{3} \pi^2 \hbar V_0^{-4/3} + \frac{1}{6} \hbar W''(\phi) V_0^{-2/3}. \tag{3.3}
\]

contains terms with a different powers of \( V_0 \), compared with (2.20), which can be important for large averaging volumes. If (3.1) is pushed to higher orders in \( 1/|\vec{k}| \), positive powers of \( V_0 \) even appear in \( W_{\text{IR}}(\phi) \). However, the entire infrared expansion of a massless theory requires a more careful derivation.

The problem of infrared contributions to massless theories is that both \( W''(\phi) \) and \( |\vec{k}| \) are small, and no obvious expansion can be done. Although the naive expansion (3.1) could then be misleading, it turns out that it does indicate the correct qualitative properties in an application to gravity: the detailed analysis given in [3] shows that the gravitational effective potential \( W \) has an infrared fixed point where it has a small-\( |\vec{k}| \) expansion of the form

\[
W = \frac{c_1}{16\pi G} |\vec{k}|^2 - \frac{\pi G c_2}{2} |\vec{k}|^3 \tag{3.4}
\]

(slightly adapted to our notation) with \( k \)-independent \( c_1 \) and \( c_2 \), which depend on background values through a parameterization of the infrared flow. The effective potential contributes a term

\[
V_0 W = \frac{8\pi^3}{|\vec{k}|^3} W = \frac{\pi^2}{2G} c_1 - 8\pi^4 G c_2 |\vec{k}|^3 \tag{3.5}
\]

to the Hamiltonian constraint, which receives a term proportional to \( |\vec{k}|^3 \propto 1/V_0 \) from (3.4) just as in (2.20), but also a term proportional to an inverse power \( |\vec{k}|^{-1} \). In particular, the infrared contribution to the Hamiltonian constraint is not a Taylor series in \( |\vec{k}| \), in agreement with (3.1).

### 3.1 Canonical quantum cosmology

We can see the same feature independently in a generic analysis of quantum-cosmological models. Quantum corrections with inverse powers of \( |\vec{k}| \) should then appear as positive powers of \( V_0 \), unlike what is seen in a minisuperspace effective potential such as (2.6). At the same time, we can make a connection with the main lesson from the canonical derivation of the effective potential (2.6): the interplay of canonical relationships with the uncertainty relation.

The relevant canonical relationships are determined by the ADM formulation of general relativity [11].\(^1\) The spatial metric \( h_{ab} \) has momenta given by

\[
p^{ab} = \frac{\sqrt{\text{det} h}}{16\pi G} \left( K^{ab} - K^c_{\ c} h^{ab} \right) \tag{3.6}
\]

in terms of extrinsic curvature

\[
K_{ab} = \frac{1}{2N} \left( \dot{h}_{ab} - D_a N_b - D_b N_a \right) \tag{3.7}
\]

\(^1\)Reprinted in [12].
For isotropic cosmological models, the shift vector \( N^a = 0 \) and its spatial covariant derivatives \( D_a N_b \) are zero, and we can assume \( N = 1 \) for proper time. The spatial part \( h_{ab} = a^2 \delta_{ab} \) of the Friedmann-Robertson-Walker metric then implies that (3.6) simplifies to
\[
p^{ab} = \frac{\dot{a}}{8\pi G} \delta^{ab}.
\] (3.8)
The symplectic potential \( \int d^3p \delta h_{ab} \) in a Lagrangian, integrated over the averaging volume of an isotropic model, is therefore reduced to
\[
\int d^3p \delta h_{ab} = -\frac{3V_0}{8\pi G} \dot{a} \delta a^2
\] (3.9)
from which we read off that the momentum canonically conjugate to \( V_0^{2/3} a^2 \) is equal to \(- (3/8\pi G)V_0^{1/3} \dot{a} \). Here, we have distributed \( V_0 \) such that any \( a \) is accompanied by a factor of \( V_0^{1/3} \), ensuring that the canonical variables are invariant under rescaling spatial coordinates.

For quantum cosmology, we use a general parameterization of basic canonical variables. In the presence of ambiguities, we need to describe quantization choices related to the representation of the scale factor \( a \) and its momentum as operators. We begin with the canonical pair just derived, and apply a 1-parameter family of canonical transformations such that the momentum remains linear in \( \dot{a} \) but \( a \) appears in different power laws:
\[
Q = \frac{3}{8\pi G} (V_0^{1/3} a)^{2(1-x)} \frac{\dot{V}_0^{1/3} a}{1-x}, \quad P = -(V_0^{1/3} a)^{2x} V_0^{1/3} \dot{a}
\] (3.10)
with a real parameter \( x \), such that \( \{Q, P\} = 1 \) for any \( x \). The scale factor as basic variable is obtained from (3.10) for \( x = 1/2 \), while the volume corresponds to \( x = -1/2 \). The only common choice not strictly included in this parameterization is the logarithmic variable \( \log a \), canonically conjugate, up to a multiplicative constant, to \( a^2 \dot{a} \). Formally, this choice can be obtained from (3.10) in the limit \( x \to 1 \).

We can already see that these canonical relationships together with the uncertainty relation restrict possible \( V_0 \)-dependences: we have \( QP \propto V_0 \) for any \( x \), while \( \Delta Q \Delta P \geq \hbar/2 \) is bounded from below by a \( V_0 \)-independent constant. The “semiclassicality” parameters \( Q^{-1} \Delta Q \) and \( P^{-1} \Delta P \) must therefore depend on \( V_0 \), and so do quantum corrections to an effective Hamiltonian.

In a canonical effective theory \([6, 7]\), the basic variables \( Q \) and \( P \) correspond to expectation values of basic operators, while fluctuations are independent quantum variables characterizing a state. These variables, as well as higher moments, are responsible for quantum corrections as in (2.6). Their precise form, such as a dependence on \( a \) for solutions of the theory, would require a detailed dynamical analysis, replacing information provided in standard systems by the no longer existing ground state. For our purposes it is sufficient to continue with parameterized equations. In particular, we assume that
\[
(\Delta Q)^2 \propto (V_0^{1/3} a)^{4y}
\] (3.11)
with a new, generically non-zero constant \( y \). We need not make assumptions about \( \Delta P \) because it is related to \( \Delta Q \) by the uncertainty relation \( (\Delta Q)^2 (\Delta P)^2 \geq \hbar^2/4 \). If the state is nearly semiclassical (in a broad sense, that is, not necessarily Gaussian), it is close to saturating the uncertainty relation. For \( y \neq 0 \), \( \Delta Q \) decreases in forward or backward evolution of an expanding universe, and we will quickly violate the uncertainty relation unless \( \Delta P \) changes suitably. The saturation limit is respected if
\[
(\Delta P)^2 \propto (V_0^{1/3} a)^{-4y},
\] (3.12)
such that \((\Delta Q)^2(\Delta P)^2\) is constant. Importantly, the uncertainty relation, imposing a \(V_0\)-independent lower bound, \(\hbar^2/4\), implies that the product \(\Delta Q\Delta P\) has a dependence on \(V_0\) different from the product \(QP \propto V_0 a^2 \dot{a}\). This general fact is responsible for the \(V_0\)-dependence of quantum corrections.

Combining the preceding equations, we can derive the parameterized scaling of quantum corrections in the Hamiltonian constraint of a quantum cosmological model. The classical constraint \(H(Q, P)\), like (2.4), is such that every term scales like \(V_0\). For dimensional reasons, the leading quantum corrections linear in \((\Delta Q)^2\) and \((\Delta P)^2\) are of the form \((\Delta Q)^2 \partial^2 H/\partial Q^2\) and \((\Delta P)^2 \partial^2 H/\partial P^2\). Since \(H\) scales like \(V_0\), the scaling behavior of the corrections can be read off from \(V_0(\Delta Q)^2/Q^2\) and \(V_0(\Delta P)^2/P^2\). In our parameterization,

\[
V_0\frac{(\Delta Q)^2}{Q^2} \propto V_0^{4(x+y-1/4)/3} \propto k_{\text{max}}^{4(x+y)-1}, \quad V_0\frac{(\Delta P)^2}{P^2} \propto V_0^{-4(x+y-1/4)/3} \propto k_{\text{max}}^{-4(x+y)}.
\]

As a consequence of how we have chosen our parameterization, together with the uncertainty relation, these quantities depend on a single parameter, \(x + y\). It is now easy to see that, unless \(x + y = 1/4\), the leading quantum corrections in the Hamiltonian constraint of quantum cosmology contain an inverse power of \(k_{\text{max}}\), from \(V_0(\Delta Q)^2/Q^2\) if \(x + y > 1/4\) and from \(V_0(\Delta P)^2/P^2\) if \(x + y < 1/4\). This statement agrees with (3.5) derived from the detailed analysis of [3]. The \(V_0\)-behavior of quantum corrections depends on quantization choices (through \(x\)) and quantum dynamics via the behavior of fluctuations (through \(y\)).

### 3.2 Infrared renormalization

Having established a close relationship between quantum corrections in minisuperspace models and infrared contributions to quantum-field theories, we return to the minisuperspace approximation. Quantum corrections in minisuperspace models, implicitly, give an approximate description of interactions of those modes of the full quantum-field theory that have not been averaged out, and therefore have wavelengths greater than the averaging volume \(V_0\) of the minisuperspace model. This conclusion is based on the two main steps in our derivation of this relationship: to obtain (2.20) from (2.18), we (i) include only modes with \(|\vec{k}| \leq k_{\text{max}} = 2\pi/V_0^{1/3}\), and (ii) replace the remaining mode integral by its integrand evaluated at small \(|\vec{k}|\), multiplied by a small \(k\)-volume. The minisuperspace approximation can therefore be expected to be reliable in regimes with

(i) significant inhomogeneity only on scales greater than \(V_0\), provided that

(ii) \(V_0\) is large.

Both conditions are fulfilled in late-universe cosmology, averaging over a Hubble region, but in quantum cosmology we are usually more ambitious and aim to apply quantum theory to the early universe, or even to understand the big-bang singularity.

As we approach a spacelike singularity, for instance in backward evolution from our present nearly homogeneous state, distance scales of structure, and therefore any region which is approximately homogeneous, shrink with a decreasing \(a\). However, inhomogeneity grows even within a comoving region. Existing structure is not only brought to smaller distances by a shrinking scale factor, it is also enhanced by gravitational collapse that forms new structure. If we try to describe the classical dynamics underlying a minisuperspace model, therefore, inhomogeneity grows within any region of constant \(V_0\).
In order to maintain the minisuperspace description, we should then gradually shrink $V_0$ as we evolve toward a spacelike singularity. Since we have associated the averaging volume with an infrared scale, adjusting $V_0$ amounts to infrared renormalization. It is important to understand how this process affects the approach to a singularity, as seen in a homogeneous model. At this stage, the BKL scenario enters the picture. When we get close to a spacelike singularity, the BKL scenario [2] sets in as an asymptotic statement. It tells us that we can assume a homogeneous geometry right up to the spacelike singularity, a conclusion which is often cited as a justification of minisuperspace truncations; see for instance [13] as a recent example. However, as an asymptotic statement, the BKL scenario does not place any lower limit on $V_0$, not even the Planck volume. Since $V_0$ has to decrease on approach to the singularity in order to maintain the minisuperspace assumption, generically we should therefore describe a geometry close to a spacelike singularity using small $V_0$: we obtain a local homogeneous geometry, such as Bianchi IX, that describes how the metric changes at a given point, but not a full Bianchi IX model which includes the topological space on which it is formulated. There is an important difference between these two applications of homogeneous solutions because quantum cosmology, as we have learned, is a quantum theory of regions. It therefore matters whether we can assume a dynamical behavior only locally or for a global space.

In addition, the shrinking $V_0$ implies that the minisuperspace approximation becomes less and less reliable near a spacelike singularity: condition (ii) at the beginning of this subsection is then violated. Even though the BKL scenario allows us to use the classical dynamics of homogeneous models to understand space-time near a spacelike singularity, it is a poor justification of minisuperspace models in quantum cosmology. Using a minisuperspace model to evolve from a nearly homogeneous geometry at late times to a BKL-like geometry at early times means that we begin with a well-justified, approximate infrared contribution of the full theory, but then push the infrared scale all the way into the ultraviolet.

As a technical note, we should expect mixed states in a quantum-mechanical implementation of reducing the averaging volume, while most studies in quantum cosmology are based on pure states. Moreover, there is no unitary transformation that could be used to change $V_0$ in quantum cosmology, since even a classical change of $V_0$ would not be a canonical transformation; see the $V_0$-dependence in (3.10). Therefore, quantum cosmology in a minisuperspace approximation cannot be based on a single equivalence class of Hilbert-space representations. Both features — the appearance of mixed states and the impossibility of using a single Hilbert space — indicate that effective field theory is required for a proper analysis.

4 Examples

Our general discussion can be applied to various approaches to quantum cosmology, often with an important change in viewpoint.

4.1 Bohmian quantum cosmology

As reviewed for instance in [14] Bohmian quantum mechanics [15, 16] applied to a cosmological model with Hamiltonian

$$H = V_0 \left( \frac{1}{2} f^{ab}(q) p_a p_b + U(q) \right)$$

(4.1)
implies a quantum potential
\[ W_Q = -\frac{V_0}{2\sqrt{f}|\psi|} \frac{\partial}{\partial q^a} \left( f^{ab} \sqrt{f} \frac{\partial}{\partial q^b} |\psi| \right) \] (4.2)
derived from the wave function \( \psi \). In this version of quantum cosmology, a logarithmic basic variable is used, which in terms of \( V_0 \) behaves similarly to the scalar example we used to motivate the minisuperspace approximation. Using the \( V_0 \)-dependence of the momentum (2.3) and the Hamiltonian (2.4), we therefore have
\[ f^{ab} \propto V_0^{-2}, \] (4.3)
such that
\[ W_Q \propto V_0^{-1} \] (4.4)
as in (2.6). Any effects based on the quantum potential in quantum cosmology are therefore enhanced by infrared renormalization as we evolve to smaller volumes, closer toward a spacelike singularity. Bounce models based on Bohmian quantum cosmology, such as [17, 18], are then more secure in the effective picture developed here — provided the minisuperspace truncation is reliable.

4.2 Affine quantization
In [19], affine quantization [20, 21] has been applied to derive an effective Friedmann equation
\[ \ddot{a}^2 + \frac{k}{a^2} + \frac{k_2}{\ell^2 a^6} = \frac{8\pi G}{3} \rho(a) \] (4.5)
with an effective energy density
\[ \rho(a) = \frac{k_3 h(N + 1)}{V_0 a^4} \] (4.6)
from harmonic anisotropies, using \( \ell = V_0/\ell_P^2 \). In [19], \( V_0 \) has been assumed to equal the coordinate volume of a Bianchi IX space, but we can easily adapt the equations to a running \( V_0 \) according to infrared renormalization.

The behavior of \( \rho(a) \) agrees with our \( \hbar \sqrt{W''}/V_0 \) in (2.6). In addition to this effective matter term, there is a repulsive
\[ k_2 \ell^{-2} a^{-6} \propto \frac{\hbar^2}{V_0^2}, \] (4.7)
which, for \( k_2 < 0 \), is able to cause a bounce because it dominates all other terms for small \( a \). This domination is enhanced for small \( V_0 \) if the model is combined with infrared renormalization. Also here, the bounce is more secure in the effective picture. (However, at present it does not seem clear whether higher-order (\( \hbar/V_0 \))-corrections in \( \rho(a) \) might compete with the repulsive term.)

4.3 Loop quantum cosmology
An effective Friedmann equation [22]
\[ \ddot{a}^2 = \frac{8\pi G}{3} \rho \left( 1 - \frac{\rho}{\rho_{QG}} \right) \] (4.8)
with
\[ \rho_{\text{QC}}(a) = \frac{3}{8\pi G\delta^2(V_0^{1/3}a)^{2(1+2x)}} \]

\hspace{1cm} (4.9)
can, under certain assumptions about the matter ingredients and properties of a state, be derived from loop quantum cosmology [4]. The parameter \( \delta \), with units of length to the power \(-2x\), characterizes the strength of spatially non-local effects in the theory implied by using holonomies. Following [23], many studies of this and related equations have been published, but usually assuming a macroscopic value of \( V_0 \). The authors of [23] have argued that late-time homogeneity justifies such a choice. But as we have seen here, using this postulate throughout long-term evolution up to high curvature is not compatible with the BKL scenario and an effective description which, through infrared renormalization, requires that \( V_0 \) be adjusted to smaller and smaller values as gravitational collapse proceeds.

The only reliable information we have about inhomogeneity in cosmology, relevant for quantum-cosmological models, is late-time near-homogeneity and the asymptotic statement of the BKL scenario. A bounce somewhere near Planckian curvature does not fall into either if these two regimes, but reaching it from well-understood late times certainly requires long evolution through dense phases with significant gravitational collapse. The assumption that large \( V_0 \) can still be used close to Planckian curvature is therefore very restrictive. There may be cosmological solutions which can be approximated by a minisuperspace model with constant and large \( V_0 \) all the way to the Planck density, but insisting on this assumption amounts to a high degree of fine-tuning of initial data that describe the late-time geometry.

While it is possible to begin evolving with large \( V_0 \) at late times, this parameter must take on smaller and smaller values in order to maintain the minisuperspace assumption as one approaches high curvature. For \( x > -1/2 \), the correction term \( \rho/\rho_{\text{QC}} \propto V_0^{2(1+2x)/3} \) decreases for smaller \( V_0 \). In contrast to the first two examples, effects that may lead to a bounce in loop quantum cosmology therefore become weaker as a consequence of infrared renormalization.

The borderline case \( x = -1/2 \) implies corrections in (4.8) independent of \( V_0 \), but the dynamics remains sensitive to quantum fluctuations which are not included in (4.8): the general effective Friedmann equation derived in [24, 25] shows that quantum fluctuations and correlations contribute to (4.8) by modifying the parenthesis \((1 - \rho/\rho_{\text{QC}})\) such that
\[ \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \rho \left( 1 - \frac{\rho}{\rho_{\text{QC}}} + \sigma \right) \]
\hspace{1cm} (4.10)

where
\[ \sigma = \frac{(\Delta Q)^2 - C + \delta^2 a^2/4}{(Q + \delta h/2)^2} \]
\hspace{1cm} (4.11)
with a correlation parameter \( C \). To be specific, we may assume that \( \delta \sim \ell_p^{-2x} \) is related to the Planck length \( \ell_P \), as usually done in models of loop quantum cosmology. It is now important to remember that \( Q \) is proportional to a positive power of \( V_0 \) for values of \( x \) usually considered in loop quantum cosmology, in particular for \( x = -1/2 \). For large \( V_0 \) and a semiclassical state with \( C \sim \Delta Q \sim \delta h, \sigma \propto V_0^{2(1-x)} \ll 1 \) is negligible. However, when \( V_0 \) has reached a small value after infrared renormalization such that \( GQ \ll \delta \hbar \sim \ell_p^{2(1-x)} \), we have \( \sigma \sim 1 \) even if the state remains semiclassical. For states at small \( V_0 \) that are not semiclassical, as may be expected in a high-curvature phase, \( \sigma \) can be significantly greater than one. Fluctuations at small \( V_0 \) therefore significantly alter the effective Friedmann equation for densities close
to $\rho_{QG}$, where semiclassical large-$V_0$ solutions would provide a bounce. In particular, non-bouncing solutions do exist when small $V_0$ are considered [26], even in simple models in which one can show that all large-$V_0$ solutions bounce. Effective field theory therefore suggests a significant revision of the conclusions drawn in loop quantum cosmology following [23], based on the assumption that large $V_0$ can be used throughout the entire evolution.

5 Conclusions

A possible effective field theory of quantum cosmology combines the BKL scenario with effects from quantum field theory and infrared renormalization. These considerations mainly apply to models in which the early universe is treated as a transition phase, in particular to bounce models. Models which treat the early universe as an initial stage, such as the tunneling [27–29] or the no-boundary proposal [30] as well as recent applications of loop quantum cosmology to such scenarios [31, 32], behave differently: at the initial stage, the scale factor reaches the value $a = 0$ such that the entire space, of any size $V_0$, uniformly collapses to zero size. In these models, $V_0$ is not required to take on small values.

As an important application to quantum cosmology, our effective description shows qualitative differences between various approaches that have led to bounce models. In particular, it strengthens quantum corrections based on fluctuations, but also reveals spurious effects, in particular in loop quantum cosmology where large $V_0$, or large-scale homogeneity within a comoving volume, is often assumed even for early-universe models.

At the same time, the effective description highlights the main problem of minisuperspace models: how do we reconcile the necessity of small homogeneous regions in the asymptotic regime of BKL with the infrared truncation of quantum field theory implied by a minisuperspace approximation?

Acknowledgments

This work was supported in part by NSF grant PHY-1607414.

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