THE BRYANT-SALAMON $G_2$-MANIFOLDS AND HYPERSURFACE GEOMETRY

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Abstract. We show that two of the Bryant-Salamon $G_2$-manifolds have a simple topology, $S^7 \setminus S^3$ or $S^7 \setminus CP^2$. In this connection, we show there exists a complete Ricci-flat (non-flat) metric on $S^n \setminus S^m$ for some $n-1 > m$. We also give many examples of special Lagrangian submanifolds of $T^*S^n$ with the Stenzel metric. Hypersurface geometry is essential for these arguments.

1. Introduction

Physicists as well as mathematicians are interested in Ricci-flat (Kähler) manifolds as a special case of Einstein manifolds. Ricci-flat metrics are often constructed on vector bundles over Riemannian manifolds where some group action of cohomogeneity one is effectively used. In this case, the base manifold is regarded as a degenerate orbit, while the principal orbits are of codimension one in the total space, i.e., the sphere bundles.

This reminds us of the theory of isoparametric hypersurfaces. In fact, isoparametric hypersurfaces in the sphere $S^n$ exist in one parameter families, which laminate $S^n$ with two other degenerate submanifolds, called the focal submanifolds. If we delete from $S^n$ one of the focal submanifolds, then the remaining part is a disk bundle over another focal submanifold $[23]$. Two of the Bryant-Salamon $G_2$-manifolds fit this theory exactly. Namely, the spin bundle $S$ over $S^3$ is associated to the isoparametric hypersurfaces $S^3 \times S^3$ in $S^7$, and the anti-self-dual bundle $\Lambda^2(\mathbb{C}P^2)$ over $\mathbb{C}P^2$ is to the so-called Cartan hypersurfaces in $S^7$. Using these, we prove:

Theorem 1.1. There exist homeomorphisms $S \cong S^7 \setminus S^3$ and $\Lambda^2(\mathbb{C}P^2) \cong S^7 \setminus \mathbb{C}P^2$, where $S^3$ and $\mathbb{C}P^2$ are embedded in $S^7$ in the standard way. In other words, a compactification of $S$ and $\Lambda^2(\mathbb{C}P^2)$ is given by $S^7$.

On the topology of $S$ and $\Lambda^2(M)$, some details are described in [4], but our viewpoint from the theory of isoparametric hypersurfaces is new. Moreover, this theorem reminds us of the Calabi-Yau problem on open manifolds, [7], [29], which asks if there exists a complete Ricci-flat Kähler metric on the complement of some divisor $D$ of a compact Kähler manifold $\overline{M}$ with positive Ricci curvature. It seems natural to pose a real version of this problem:

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Problem. When can we construct a complete Ricci-flat, non-flat metric on the complement of some subset $D$ of a compact irreducible Riemannian manifold $\overline{M}$ with positive Ricci curvature?

In the case $\overline{M} = S^n$, we give a partial answer (§3).

Proposition 1.2. A complete non-flat Ricci flat metric exists on

(i) $S^7 \setminus \mathbb{C}P^2$, $S^7 \setminus S^3$: the Bryant-Salamon metric
(ii) $S^{m+2n+2} \setminus S^{2n+1}$, $n, m \geq 1$: the Lü-Page-Pop metric
(iii) $S^6 \setminus S^2$, $S^{14} \setminus S^6$: the Stenzel metric

Another aspect of Ricci flat manifolds is a relation with special geometry. With respect to the Stenzel metric, $T^*S^n$ becomes a Calabi-Yau manifold with calibrations $\mathcal{R}(e^{i\theta}\Omega)$, where $\Omega$ is the global holomorphic $n$-form (see §5). Then what are special Lagrangian submanifolds? Here again the theory of isoparametric hypersurfaces works well. Harvey and Lawson’s result [14], generalized by Karigiannis and Min-Oo recently [19], tells us that the conormal bundle over an austere submanifold in $S^n$ is a special Lagrangian submanifold of $T^*S^n$. We have many examples of austere submanifolds of $S^n$ in [16], hence we obtain

Theorem 1.3. The conormal bundles of the focal submanifolds $W_{\pm}$ of any isoparametric hypersurfaces are special Lagrangian submanifold of $T^*S^n$ equipped with the Stenzel metric. Infinitely many non-homogeneous examples are included among them. Moreover, the conormal bundle of the following minimal isoparametric hypersurfaces in $S^n$ are special Lagrangian submanifolds of $T^*S^n$,

$$W = S^n_{d-1}$$
$$W^{2d} = S^d \times S^d, \quad n = 2d + 1$$
$$W^{3d}, \quad n = 3d + 1 \quad d = 1, 2, 4, 8$$
$$W^{4d}, \quad n = 4d + 1 \quad d = 1, 2$$
$$W^{6d}, \quad n = 6d + 1, \quad d = 1, 2$$

where $W$ has, respectively, 1, 2, 3, 4, 6 principal curvatures. The phase $e^{i\theta}$ is determined by the dimension of $W_\pm$ or $W$.

Remark 1.4: In [16], we prove that the conormal bundles of the cones in $\mathbb{R}^{n+1}$ over all above $W_\pm$ or $W$ are special Lagrangian cones in $\mathbb{C}^{n+1}$.

2. The Bryant-Salamon $G_2$-manifolds

By a $G_2$-manifold, we mean a Riemannian manifold with the holonomy group $G_2$. The metric is called a $G_2$-metric.

Denoting the exterior product of three vectors of an orthonormal coframe $e^1, \ldots, e^7$ of $\mathbb{R}^7$ by $e^{ijk} = e^i \wedge e^j \wedge e^k$, define a 3-form $\phi$ on $\mathbb{R}^7$ by

\begin{equation}
\phi = e^{125} - e^{426} + e^{136} - e^{427} + e^{147} - e^{237} + e^{567}.
\end{equation}

The automorphism group $G_2$ of the Cayley numbers can be defined also as the subgroup of $GL(7, \mathbb{R})$ preserving $\phi$ [5]. A $G_2$-structure on a 7-dimensional manifold $X$ is a reduction of the structure group of the linear frame bundle to $G_2$. Let $\mathcal{O}$ be the $GL(7, \mathbb{R})$-orbit of $\phi (\cong GL(7, \mathbb{R})/G_2)$, then a $G_2$-structure is equivalent to the existence of a global 3-form $\phi_x$ on $X$ satisfying $\phi_x \in \mathcal{O}_x$. Since $G_2 \subset SO(7)$, a $G_2$-structure induces a Riemannian metric $g$ on $X$. The holonomy group is contained
in $G_2$ if and only if $d\phi = d*\phi = 0$, and is equal to $G_2$ if and only if there are no non-trivial parallel 1-forms on $X$, provided that $X$ is simply connected and connected. Note that a $G_2$-metric is Ricci flat $[5]$. The first examples of $G_2$-manifolds were given by Bryant $[5]$. Later on, complete $G_2$ metrics were constructed by Bryant and Salamon $[8]$ on the spin bundle $S$ over $S^3$ and on the anti-self dual bundle $\Lambda_2^+ (M)$ of the self-dual Einstein manifolds $M = S^4$ and $\mathbb{C}P^2$. Compact Riemannian manifolds with holonomy $G_2$ were constructed by Joyce $[17]$ via a generalization of the Kummer construction, and by Kovalev $[18]$ via a twisted gluing method. In both cases, Ricci flat metrics on non-compact manifolds are essential tools. Indeed, in the former case, $G_2$-metrics which are asymptotically locally Euclidean were used to connect two non-compact parts, and in the latter, $SU(3)$-metrics obtained by solving the open Calabi-Yau problem $[29]$ played an important role. However, the latter two constructions do not give metrics explicitly.

The Bryant-Salamon metrics are explicit. Indeed, let $X = S, \Lambda_2^+ (S^4)$ or $\Lambda_2^+ (\mathbb{C}P^2)$. Let $g_b$ and $g_f$ be the standard metrics on the base and the fiber space, respectively, normalized appropriately by constant multiples. Consider a hypersurface $N_r$ of $X$ consisting of fiber vectors of length $r$. Now, on $X = \cup_{r \geq 0} N_r$, they seek a metric so that the 3-form which depends on the metric satisfies the non-linear partial differential equations $d\phi = d*\phi = 0$. Restricting the metric to the warped product form $g = u(r) f_b + v(r) g_f$, where $u(r)$ and $v(r)$ are functions of $r$, they reduce the equations to ODE’s, and obtain $G_2$-metrics

\[ g = (\lambda + r^2)^{2/3} g_b + (\lambda + r^2)^{-1/3} g_f \quad \text{on} \quad S, \]
\[ g = (\lambda + r^2)^{1/2} g_b + (\lambda + r^2)^{-1/2} g_f \quad \text{on} \quad \Lambda_2^+ (M^4). \]

Actually, when $\lambda > 0$, these metrics extend to complete ones on $X = \cup_{r \geq 0} N_r$. The non-existence of non-trivial parallel 1-forms is then proved, which establishes Hol($g$) = $G_2$. It turns out that a homogeneous metric is used on $N_r$, where $N_r \cong S^3 \times S^3$ for $X = S$, $N_r \cong \mathbb{C}P^3$ for $X = \Lambda_2^+ (S^4)$, and $N_r \cong SU(3)/\mathbb{T}^2$ ($\mathbb{T}^2$ is a maximal torus) for $X = \Lambda_2^+(\mathbb{C}P^2)$. We note that, however, the metric is different from the standard one. Indeed when $X = \Lambda_2^+ (S^4)$, the metric on $N_r \cong \mathbb{C}P^3$ is not the Fubini-Study metric, but a non-Kähler Einstein metric. The one for $N_r$ in the case $X = \Lambda_2^+ (\mathbb{C}P^2)$ is also non-Kähler Einstein.

We notice here that $S^3 \times S^3$ and the flag manifold $SU(3)/\mathbb{T}^2$ are homeomorphic to typical isoparametric hypersurfaces in $S^7$. In particular, the latter is called the Cartan hypersurface, on which the induced metric is Kähler Einstein.

3. Homogeneous and isoparametric hypersurfaces in the sphere

Now, we give a brief review of homogeneous and isoparametric hypersurfaces in the sphere.

By isoparametric hypersurfaces, we mean hypersurfaces with constant principal curvatures (see $[28]$). These are given as level sets $W_t = f^{-1}(t)$ of the so-called Cartan-Münzner function $f : S^n \to [-1,1]$, for $t \in (-1,1)$. The level sets $W_{\pm} = f^{-1}(\pm 1)$ have lower dimension and are called the focal submanifolds. The function $f$ is extended to $F : \mathbb{R}^{n+1} \to \mathbb{R}$ so that $f = F|_{S^n}$, where $F$ is a homogeneous polynomial of degree equal to the number of the principal curvatures, satisfying two PDE’s, see $[24]$. Note that $W_{\pm}$ and $W_0$ are minimal submanifolds. The most
important fact in our argument is that the ambient sphere is stratified as
\[ S^n = \bigcup_{t \in [-1,1]} W_t, \]
by hypersurfaces \( W_t, t \in (-1,1) \) and the two focal submanifolds \( W_{\pm} \).

Typical examples of isoparametric hypersurfaces are given by homogeneous hypersurfaces. Let \( G/K \) be an \((n+1)\)-dimensional rank two symmetric space of compact type, and let \( \mathfrak{g} = \mathfrak{k} + \mathfrak{m} \) be the associated decomposition, where \( \mathfrak{g} \) and \( \mathfrak{k} \) are Lie algebras of \( G \) and \( K \), respectively. At \( o = \mathfrak{g}/K \), the tangent space \( T_o(G/K) \) is identified with \( \mathfrak{m} \cong \mathbb{R}^{n+1} \), equipped with the invariant metric induced from the one on \( \mathfrak{g} \). Then \( K \) acts on \( \mathfrak{m} \) as an isometry by the adjoint action, which we call the isotropy action. Let \( S^n \) be the unit sphere of \( \mathfrak{m} \). Then the principal orbit of \( K \) through \( x \in S^n \) is a hypersurface \( N = (\text{Ad}K)x \) of \( S^n \), because the rank of \( G/K \) is two. Note that we obtain a one parameter family of such hypersurfaces, and the two singular orbits called the focal submanifolds. Conversely, every homogeneous hypersurface in a sphere is obtained in this way \cite{15}, and all such hypersurfaces are classified.

On the other hand, there exist infinitely many non-homogeneous isoparametric hypersurfaces with four principal curvatures. These were constructed by Ozeki and Takeuchi \cite{24} by using the representation of Clifford algebras. Later on, Ferus, Karcher and Minguzzi generalized the method and obtained systematically the so-called isoparametric hypersurfaces of FKM type \cite{13}. The only known examples of non-homogeneous isoparametric hypersurfaces are of this type.

**Remark 3.1 :** Recently, isoparametric hypersurfaces with four principal curvatures are classified by Cecil, Chi and Jensen \cite{10} except for 10 cases, and they are either of FKM type or homogeneous.

The number \( g \) of principal curvatures is limited to 1, 2, 3, 4, or 6, and of particular interest is in the case of 3 and 6, where orbits of a subgroup of \( G_2 \) appear \cite{22}. Recall that those with three principal curvatures are known as Cartan hypersurfaces, which are tubes over standard embedded Veronese surfaces \( \mathbb{F}P^2 \) where \( \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{C} \) in \( S^4, S^7, S^{13}, S^{25} \) (\( \mathbb{C} \) is the Cayley algebra) \cite{9}. Veronese surfaces are related to the so-called Severi varieties \cite{2}.

We are concerned with the case \( S^7 \). All isoparametric hypersurfaces in \( S^7 \) are homogeneous and so are classified. We denote a \( k \)-dimensional sphere with radius \( a \) by \( S^k(a) \). Let \( \mathbf{x} = (x_1, \ldots, x_8) \) be the coordinate of \( \mathbb{R}^8 \).\n
\begin{itemize}
  \item[(a)] \( g = 1 : F(\mathbf{x}) = x_8, f^{-1}(t) = S^6(\sqrt{1-t^2}), t \in (-1,1). \) \( W_{\pm} \) = north and south poles.
  \item[(b)] \( g = 2 : F(\mathbf{x}) = \sum_{i=1}^{k+1} x_i^2 - \sum_{j=k+2}^8 x_j^2, f^{-1}(t) = S^k(\sqrt{1-t^2}) \times S^{6-k}(\sqrt{1-t^2}), t \in (-1,1), 1 \leq k \leq 5 \) : generalized Clifford torus. \( W_{\pm} = S^k(1) \) and \( S^{6-k}(1) \).
  \item[(c)] \( g = 3 : F(\mathbf{x}) = u^3 - 3uv^2 + \frac{3}{2}u(|x|^2 + |y|^2 - 2|z|^2) + \frac{3\sqrt{3}}{2}v(|x|^2 - |y|^2) + \frac{3\sqrt{3}}{2}(xyz + \bar{x}\bar{y}\bar{z}), \mathbf{x} = (u, v, x, y, z) \in \mathbb{R}^2 \times \mathbb{C}^3 = \mathbb{R}^8. f^{-1}(t) \cong SU(3)/T^2 \) : Cartan hypersurface, isotropy orbits of \( SU(3) \times S(3)/SU(3) \). \( W_{\pm} \) = two copies of \( \mathbb{C}P^2 \).
  \item[(d)] \( g = 4 : F(\mathbf{x}) \) is a polynomial of degree 4 (see \cite{24}). \( f^{-1}(t) = \) isotropy orbits of \( SO(6)/SO(2) \times SO(4) \).
\end{itemize}
Proof of Theorem 1.1 and of Proposition 1.2 (1)
Consider a hypersurface \( N_r \) of \( S \) consisting of the fiber vectors of length \( 0 < r < \infty \). Then \( N_r \) is homeomorphic to \( S^3 \times S^3(r) \), where \( S^3 \) denotes the base manifold. Let \( W_t \) be as in (b) where \( k = 3 \). Now we define
\[
\phi : S \rightarrow S^7 \setminus S^3
\]
by identifying \( S^3 \) with \( W_- \cong S^3 \), and then \( N_r, r \in (0, \infty) \) with \( W_t = S^3(\sqrt{\frac{1-t}{2}}) \times S^3(\sqrt{\frac{1-t}{2}}), t \in (-1, 1), \) by
\[
t = \frac{r - 1}{r + 1},
\]
where we keep the correspondence of \( S^3 \) with the first \( S^3(\sqrt{\frac{1-t}{2}}) \) by homothety. The continuity at \( r = 0 \) follows because \( X \) is a tube over \( S^3 \). This evidently implies that \( S \cong \cup_{r \in (0, \infty)} S^3 \times S^3(r) \) is homeomorphic to \( \cup_{t \in [-1, 1]} W_t = S^7 \setminus S^3 \), since \( r = \infty \) corresponds to \( W_+ \cong S^3 \).

The hypersurface \( N_r \) of \( \Lambda^2_+ (\mathbb{C}P^2) \) is homeomorphic to \( SU(3)/S^2 \cong W_t, t \in (-1, 1), \) where \( W_t = f^{-1}(t) \) in (c). Now we define similarly a map,
\[
\phi : \Lambda^2_+ (\mathbb{C}P^2) \rightarrow S^7 \setminus \mathbb{C}P^2
\]
by identifying the base manifold \( \mathbb{C}P^2 \) with \( W_- \cong \mathbb{C}P^2 \), and \( N_r, r \in (0, \infty) \) with \( W_t, t \in (-1, 1), \) by (3). The continuity when \( r \rightarrow 0 \) is guaranteed since \( W_t \) is a tube over \( \mathbb{C}P^2 \). Since \( r = \infty \) corresponds to \( \mathbb{C}P^2 \cong W_+ \), we obtain Theorem 1.1 and Proposition 1.2 (1). \( \square \)

Remark 3.2 : For any isoparametric family, Münzner shows that the ambient sphere \( S^n \) can be decomposed into two disk bundles \( D_\pm \) over the focal submanifolds \( W_\pm \) so that \( S^n = D_+ \cup D_- \), and \( D_+ \cap D_- \) is an isoparametric hypersurface, say, \( W_0 \) [Mii]. If we know this and the stratification [2], the description \( X_1 \cup_Y X_2 \) (in fact = \( S^7 \)) given in (5.63) of [4] becomes clearer.

Remark 3.3 : In the case \( M = S^4 \), the hypersurface \( N_r \) of \( \Lambda^2_+ (S^4) \) is diffeomorphic to \( \mathbb{C}P^3 \). By the result of Cleyton and Swann, [11], Theorem 9.3, \( \Lambda^2_+ (S^4) \cong \mathbb{C}P^3 \times \mathbb{R}^1 \), hence a compactification is given by \( \mathbb{C}P^3 \times S^1 \).

4. Complete Ricci flat metric on \( S^n \setminus D \)

Lü, Page and Pope constructed complete Ricci flat metrics on \( S^m \times \mathbb{R}^{2n+2} \) \((m, n \geq 1)\), modifying the construction of non-homogeneous Einstein metrics on compact manifolds [21]. Since \( S^m \times \mathbb{R}^{2n+2} \cong \cup_{r \geq 0} S^m \times S^{2n+1}(r) \), using the isoparametric embedding of \( S^m(\sqrt{\frac{1-t}{2}}) \times S^{2n+1}(\sqrt{\frac{1-t}{2}}) \) into \( S^{m+2n+2} \), we see that \( S^{m+2n+2} \setminus S^{2n+1} \) is a disk bundle over \( S^m \), i.e., \( S^m \times \mathbb{R}^{2n+2} \). Thus we obtain (2) of Proposition 1.2 (part (3) will be proved in the next section):

Note that \( S^N \setminus S^n \cong \mathbb{R}^N \setminus \mathbb{R}^n \). As a less topologically trivial case, we may ask the following question which we will discuss on another occasion.

**Proposition 4.1.** For each isoparametric family \( \{W_t\} \) in \( S^n \), does there exist a complete Ricci-flat, non-flat metric on \( S^n \setminus W_\pm \)?
5. Stenzel metric and calibrated geometry

We give a brief introduction to the Stenzel metric on $T^*S^n$. Identify $T^*S^n$ with $\mathcal{Q}^n = \{ z \in \mathbb{C}^{n+1} | z_0^2 + \cdots + z_n^2 = 1 \}$ by $T^*S^n \ni (x, \xi) \mapsto x \cosh |\xi| + i \xi / |\xi| \sinh |\xi|$, and induce a complex structure on $T^*S^n$ from $\mathcal{Q}^n$. Then consider a holomorphic $(n,0)$ form $\Omega$ given by

$$\Omega(T) := (dz_0 \wedge \cdots \wedge dz_n)(Z,T), \quad Z = z_0 \frac{\partial}{\partial z_0} + \cdots + z_n \frac{\partial}{\partial z_n} \in \mathbb{C}^{n+1}$$

The Kähler form of the Stenzel metric is given by

$$\omega_{St} = \frac{i}{2} \sum_{j,k=1}^{n} a_{jk} dz_j \wedge d\bar{z}_k$$

where

$$a_{jk} = (\delta_{jk} + \frac{z_j \bar{z}_k}{|z_0|^2})u' + 2\Re(z_j \bar{z}_k - \frac{z_0}{z_0} z_j \bar{z}_k)u'',$$

and $u$ is a function of $r = |z|$, of which details we need not here. This is a highly generalized Eguchi-Hanson metric, first constructed explicitly in 26. Note that the Stenzel metric restricted to $S^n$ is the standard metric on $S^n$.

**Proposition 5.1.** There exists a non-flat complete Ricci-flat Kähler metric on $S^6 \setminus S^2$ and $S^{14} \setminus S^6$.

**Proof:** Because $S^3$ and $S^7$ are parallelizable, it follows that $T^*S^n \cong S^n \times \mathbb{R}^n \cong \cup_{r>0} S^n \times S^{n-1}(r) \cong S^{2n} \setminus S^{n-1}$. \hfill $\square$

**Remark 5.2:** Stenzel also constructs Ricci-flat Kähler metrics on the cotangent bundles of rank one symmetric spaces 26.

In the calibrated geometry developed by Harvey and Lawson 14, one way to obtain special Lagrangian submanifolds in $\mathbb{C}^{n+1}$ is to take the conormal bundle of the so-called austere submanifold in $\mathbb{R}^{n+1}$. A submanifold $N$ in $\mathbb{R}^{n+1}$ or in $S^n$ is austere if any shape operator has eigenvalues in pairs $\{ \pm \lambda_j \}$, and if the multiplicities of $\pm \lambda_j$ coincide, where $\lambda_j = 0$ is admissible. The cone over an austere submanifold of $S^n$ is austere in $\mathbb{R}^{n+1}$. Austere surfaces are nothing but minimal surfaces. In some cases, austere submanifolds are classified 6, 12.

In 10, we found a large class of compact austere submanifolds in $S^n$:

**Theorem 5.3.** [16] The focal submanifolds of any isoparametric hypersurfaces in $S^n$ are austere. Minimal isoparametric hypersurfaces in $S^n$, whose principal curvatures have the same multiplicity, are austere.

In fact, non-zero principal curvatures of minimal isoparametric hypersurfaces appear in $\pm$ pairs, and when $m_j$ is the multiplicity of $\lambda_j$, where $\lambda_1 < \cdots < \lambda_n$, it is known 23 that

(a) If $g = 2$, then $1 \leq m_1 \leq m_2 = n - m_1 - 1 < n - 1$.
(b) If $g = 3$, then $m = m_j \in \{1, 2, 4, 8\}$ does not depends on $j$ [9].
(c) If $g = 4$, then $m_3 = m_1$, $m_4 = m_2$, where the pair $(m_1, m_2)$ is restricted to those of homogeneous ones or of FKM type 27.
(d) If $g = 6$, then $m = 1$ or $2$ and $m_j$ does not depend on $j$ [1].

The shape operators of the focal submanifolds have the following eigenvalues:

(a) If $g = 2$, then 0.
(b) If \( g = 3 \), then \( \pm 1/\sqrt{3} \).
(c) If \( g = 4 \), then \( 0, \pm 1/\sqrt{3}, \pm 1 \).
(d) If \( g = 6 \), then \( 0, \pm 1/\sqrt{3}, \pm \sqrt{3} \).

where the \( \pm \) pair of eigenvalues have the same multiplicity.

On the other hand, Karigiannis and Min-Oo proved:

**Theorem 5.4.** \([19]\) The conormal bundle of a submanifold \( N \) in \( S^n \) is a special Lagrangian submanifold of \( T^* S^n \) with the Stenzel metric if and only if \( N \) is austere.

From Theorem 5.3 and 5.4, we obtain Theorem 1.3.

**Remark 5.5.** The conormal bundle is given by
\[
\Psi(x(s), \sum t_k \nu^k) = x(s) \cosh |t| + i \hat{\nu}(s, t) \sinh |t|,
\]
where \( \hat{\nu} = \sum t_k \nu^k/|t| \), for \( |t|^2 = t_{p+1}^2 + \cdots + t_n^2 \), and an orthonormal frame \( \nu^{p+1}, \cdots, \nu^n \) of the conormal space.

**Remark 5.6.** When \( n = 3 \), any orientable compact (topological) surface can be minimally immersed in \( S^3 \) \([20]\), hence could be austere. Thus the conormal bundles of such surfaces are special Lagrangian submanifolds of \( T^* S^3 \).

**Remark 5.7.** B. Palmer shows that the Gauss map of an isoparametric hypersurface \( M \) in \( S^n \) given by \( x \wedge n \), where \( x \in M \) and \( n \) is the unit normal, defines a Lagrangian submanifold of \( \text{Gr}^+_2(n+1, \mathbb{R}) \) \([25]\). Here the oriented 2-plane Grassmannian is identified with the complex quadric \( Q^{n-1} = \{ [z] \in CP^n | z_0^2 + \cdots + z_n^2 = 0 \} \) equipped with the metric induced from the Fubini-Study metric on \( CP^n \). The induced metric has positive Ricci curvature. Palmer shows that the Lagrangian submanifold obtained in this way is Hamiltonian stable if and only if \( M \) is a hypersphere.

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