Krajewski diagrams and the Standard Model

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Abstract

This paper provides a complete list of Krajewski diagrams representing the standard model of particle physics. We will give the possible representations of the algebra and the anomaly free lifts which provide the representation of the standard model gauge group on the fermionic Hilbert space. The algebra representations following from the Krajewski diagrams are not complete in the sense that the corresponding spectral triples do not necessarily obey to the axiom of Poincaré duality. This defect may be repaired by adding new particles to the model, i.e. by building models beyond the standard model.

The aim of this list of finite spectral triples (up to Poincaré duality) is therefore to provide a basis for model building beyond the standard model.

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1 Introduction

The aim of this paper is to provide a geometrical basis to enlarge the standard model of particle physics within the setting of noncommutative geometry [1,2]. Krajewski diagrams [3] are a particularly useful tool to classify the finite part of spectral triples [3,4]. Especially the minimal Krajewski diagrams, for finite spectral triples in $KO$-dimension zero as well as in $KO$-dimension six, already allowed to reduce the number of interesting finite geometries significantly [5] and showed the singular role of the standard model within the class of almost-commutative geometries.

We will therefore give in this paper a complete list of minimal Krajewski diagrams which represent the finite part of spectral triples that allow to recover the first generation of the standard model in $KO$-dimension zero as well as in the more promising $KO$-dimension six. By recovering the standard model we mean that one is able to reconstruct from the geometric data the fermionic Hilbert space of the standard model with its well known gauge group $G_{SM} = U(1)_Y \times SU(2)_w \times SU(3)_c$ in the correct representation and the correct charge assignment.

Only the first generation of the standard model without right-handed neutrinos will be taken into account. The reason for this limitation is the fact that noncommutative geometry does not explain why there are three generations of Fermions. Furthermore right-handed neutrinos usually appeared as a reducible extension of the pure standard model [5–8] and they can always be added later to each realisation of the standard model shown below.

Depending whether one works in $KO$-dimension zero or six there are several mass mechanisms for the neutrinos available. In $KO$-dimension zero they are usually based on the Higgs mechanism [6,7] while $KO$-dimension six allows also for Majorana masses [8–10], although at the expense of the orientability axiom [11]. Recently A. Sitarz proposed a third possibility [12] which builds on a modification of the spectral action principle, resulting in a radiative generation of neutrino masses. This mechanism is also compatible with all the models presented below.

The minimal Krajewski diagrams will in general respect all axioms for spectral triples [1], save the axiom of the Poincaré duality. For the first generation of the standard model, restricted to the suitable finite matrix algebra [2,5], the axiom holds of course. But if the finite algebra is enlarged, new fermions are in general needed [13,14] to satisfy the Poincaré duality.

Therefore the minimal Krajewski diagrams presented here that do not necessarily satisfy the Poincaré duality, should serve as basic building blocks to construct models beyond the standard model within the noncommutative framework. They may also allow to push the classification begun in [5] further by enlarging the minimal diagrams in all possible ways. This has the advantage that the standard model will always appear as a sub-model and thus ensure the correct “low energy limit” of such particle models.

The paper is organised as follows: Starting with the basic definitions of spectral triples and Krajewski diagrams we fix the physical requirements coming from the standard model and the resulting geometric data. To obtain the correct hyper-charge assignment we will use the central extension approach [15]. The central charges will then be fixed by the
requirement of being free of harmful anomalies, or equivalently by the requirement of producing the standard model hyper-charge assignment.

Then we will construct the corresponding minimal Krajewski diagrams in \(KO\)-dimension six and zero. We will start with the more restrictive case of \(KO\)-dimension six and give in a second step the remaining diagrams in \(KO\)-dimension zero.

These basic minimal Krajewski diagrams can then be used as building blocks for more sophisticated particle models beyond the standard model.

2 Basic Definitions

In this section we will give the necessary basic definitions for finite noncommutative geometries [1]. We will use the classical axioms and not the modified versions of orientability and Poincaré duality as in [10]. We restrict ourselves to real, finite spectral triples \((\mathcal{A}, \mathcal{H}, \mathcal{D}, J, \chi)\). The algebra \(\mathcal{A}\) is a finite sum of matrix algebras \(\mathcal{A} = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{K}_i)\) with \(\mathbb{K}_i = \mathbb{R}, \mathbb{C}, \mathbb{H}\) where \(\mathbb{H}\) denotes the quaternions. A faithful representation \(\rho\) of \(\mathcal{A}\) is given on the finite dimensional Hilbert space \(\mathcal{H}\). The Dirac operator \(\mathcal{D}\) is a selfadjoint operator on \(\mathcal{H}\) and plays the role of the fermionic mass matrix. \(J\) is an antiunitary involution, \(J^2 = 1\), and is interpreted as the charge conjugation operator of particle physics. The chirality \(\chi\) is a unitary involution, \(\chi^2 = 1\), whose eigenstates with eigenvalue +1 are interpreted as right-handed particle states and left-handed antiparticle states, whereas the eigenstates with eigenvalue −1 represent the left-handed particle states and right-handed antiparticle states. These operators are required to fulfill Connes’ axioms for spectral triples:

- \([J, \mathcal{D}] = [J, \chi] = 0\), \(\mathcal{D} \chi = -\chi \mathcal{D}\), \([\chi, \rho(a)]= [\rho(a), J \rho(a')J^{-1}] = [[\mathcal{D}, \rho(a)], J \rho(a')J^{-1}] = 0\), \(\forall a, a' \in \mathcal{A}\),

where \([J, \chi] = 0\) is the commutator in \(KO\)-dimension zero and the anticommutator \([J, \chi] = 0\) in \(KO\)-dimension six.

- The intersection form \(\cap_{ij} := \text{tr}(\chi \rho(p_i)J \rho(p_j)J^{-1})\) is non-degenerate, \(\det \cap \neq 0\). The \(p_i\) are minimal rank projections in \(\mathcal{A}\). This condition is called Poincaré duality. Demanding the Poincaré duality to hold requires in \(KO\)-dimension six an even number of summands in the matrix algebra [5, 8].

- The chirality can be written as a finite sum \(\chi = \sum_i \rho(a_i)J \rho(a'_i)J^{-1}\), which is a 0-dim Hochschild cycle. This condition is called orientability.

The representation \(\rho(a)\) takes the general form

\[
\rho(a) = (\bigoplus_{i,j=1}^{N} \rho(a_i, a_j)) \oplus (\bigoplus_{i,j=1}^{N} \rho^c(a_i, a_j))
\]

where \(\rho(\cdot)\) and \(\rho^c(\cdot)\) are the representation on the particle and anti-particle Hilbert subspace. Without restricting generality they can be taken to be

\[
\rho(a_i, a_j) := a_i \otimes 1_{(m_{ij})} \otimes 1_{(n_j)} \quad \rho^c(a_i, a_j) := 1_{(n_i)} \otimes 1_{(m_{ij})} \otimes a_j.
\]
The multiplicities \( (m_{ij}) \) are non-negative integers. Here \( n = n \) for \( K = \mathbb{R}, \mathbb{C} \) and \( n = 2n \) for \( K = \mathbb{H} \). We denote by \( I(n) \) the \( (n) \times (n) \) identity matrix and set by convention \( I_0 := 0 \). Algebra elements \( a_i \) are taken to be from the \( i \)-th summand \( M_{n_i}(K_i) \) of the algebra \( A = \bigoplus_{i=1}^N M_{n_i}(K_i) \).

We will now present the basics of Krajewski diagrams, but only treat the easy case, \( K = \mathbb{R}, \mathbb{H} \) in all components. For further details on the complex case and on multiple arrows we refer to [5].

We define the multiplicity matrix \( \mu \in M_N(\mathbb{Z}) \), \( N \) being the number of summands in \( A \), such that \( \mu_{ij} := \chi_{ij} m_{ij} \), with \( m_{ij} \) being the multiplicities of the representation (2.1) and \( \chi_{ij} \) the signs of the chirality. There are \( N \) minimal projectors in \( A \), each of the form 
\[
p_i = 0 \oplus \cdots \oplus 0 \oplus \text{diag}(1_{(1)}, 0, \ldots, 0) \oplus 0 \oplus \cdots \oplus 0.
\]
With respect to the basis \( p_i \), the matrix of the intersection form is \( \cap = \mu \pm \mu^T \), the relative plus (minus) sign has its origin in the (anti-)commutation relation of the real structure \( J \) and the chirality \( \chi \).

If both entries \( \mu_{ij} \) and \( \mu_{ji} \) of the multiplicity matrix are non-zero, then they must have the same (opposite) sign in \( KO \)-dimension zero (six).

- **Poincaré duality:** The last condition to be satisfied by the multiplicity matrix reflects the Poincaré duality and requires the multiplicity matrix to obey \( \det(\cap = \mu \pm \mu^T) \neq 0 \). Since the intersection form is an anti-symmetric matrix in \( KO \)-dimension six, this case restricts to an even number of summands in the matrix algebra.

- **The Dirac operator:** The components of the (internal) Dirac operator are represented by horizontal or vertical lines connecting two nonvanishing entries of opposite signs in the multiplicity matrix \( \mu \) and we will orient them from plus to minus. Each arrow represents a nonvanishing, complex submatrix in the Dirac operator: For instance \( \mu_{ij} \) can be linked to \( \mu_{ik} \) by

\[
\begin{array}{ccc}
\mu_{ij} & \mu_{ik} \\
\end{array}
\]

and this arrow represents respectively submatrices of \( M \) in \( D \) of type \( m \otimes 1_{(n_i)} \) with \( m \) a complex \( (n_j) \times (n_k) \) matrix.

Every arrow comes with three algebras: Two algebras that localize its end points, let us call them **right and left algebras** and a third algebra that localizes the arrow, let us call it **colour algebra**. For the arrow presented above the left algebra is \( A_j \), the right algebra is \( A_k \) and the colour algebra is \( A_i \).

We deduced however in [11] that if \( i = j \) or \( k = j \) the corresponding spectral triple does not satisfy the axiom of orientability, so the colour algebra must not coincide with the left of the right algebra. Translated into the language of Krajewski diagrams this means that the arrow must not touch the diagonal of the diagram.

We will restrict ourselves to minimal Krajewski diagrams. A minimal Krajewski diagram is defined in detail in [16], in short it means that it is not possible to remove an arrow from the diagram without changing the multiplicity matrix.

- **Convention for the diagrams:** Usually arrows always point from right chirality for particles and antiparticles, to left chirality for particles and antiparticles. But since we will only consider the general structure of the particle model and therefore left-handedness and right-handedness are purely conventional, we will not draw the arrowheads. As a
further convention the horizontal arrows will encode particles and its vertical copies encode antiparticles. This choice is of course also arbitrary. We will only draw the horizontal arrows in the Krajewski diagrams below to keep them as uncluttered as possible.

3 General requirements for the standard model

To fix the geometrical data that will lead us to the minimal Krajewski diagrams, we assume as a physical input only the first generation of the standard model without right-handed neutrinos.

For the geometrical realisation there is a choice in the so called $KO$-dimension of the spectral triple. In physicists terms the $KO$-dimension can be thought of as the signature of the metric of the internal space modulo 8. In this sense $KO$-dimension six has the signature $-2$, corresponding to the Minkowski version of the finite spectral triple [8]. In the rather general construction presented below the $KO$-dimension is of little importance. For $KO$-dimension six it only results in two extra constraints: From the axiom of Poincaré duality follows that the number of summands in the matrix algebra has to be even. Also the representation of the algebra is not allowed to represent the same summand on a the same left- and right-handed particle species and anti-particle species [11].

3.1 The physical constraints

As physical constraints we assume the following:

- All standard model fermions, i.e. quarks and leptons, share for their Dirac masses the same mass generating mechanism. This is the standard Higgs mechanism emerging from the spectral action [2].

- We require the group of unitaries lifted to the Hilbert space of the standard model fermions, to be the standard model gauge group $G_{SM} = U(1)_Y \times SU(2)_w \times SU(3)_c$

- We also require the models to be free of harmful anomalies, i.e. the hyper-charge assignment is the one of the standard model.

- For simplicity we will assume only one $U(1)$-subgroup in the standard model gauge group, the hypercharge gauge group. It was shown in [15] that, due to the central extension, each additional $U(1)$-subgroup results in an unphysical, completely decoupled extra photon.

3.2 The algebra and its representation

Let us now construct the matrix algebra, its representation and the internal Dirac operator which contains the Yukawa couplings. Here we have to take care of the physical constraints specified in the previous section as well as the axioms from noncommutative geometry.

From the standard model we know that the gauge group of any extension of the standard model has to contain $G_{SM} = U(1)_Y \times SU(2)_w \times SU(3)_c$ as a sub-group. In
noncommutative geometry the non-abelian part of the gauge group emerges as the group of unitary elements of the matrix algebra. This unitary group is then lifted to the particle Hilbert space; we will cover the details of the lift in the next section. For simplicity we choose as noncommutative subalgebra $\mathbb{H} \oplus M_3(\mathbb{C})$ which has as unitary group $\text{Aut}(\mathbb{H} \oplus M_3(\mathbb{C})) = SU(2) \times U(3)$. But $M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ will lead to similar results with an extra $U(1)$ subgroup since $\text{Aut}M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) = U(2) \times U(3)$. This subtlety has no effect on the Krajewski diagrams, we will therefore ignore it.

The abelian part of the gauge group emerges from a central extension of the lift, using the $U(1)$ subgroup of the $U(3)$ subgroup of the unitary group. To obtain the correct $U(1)$ hyper-charge assignment of the standard model the lift needs at least one abelian subalgebra $\mathbb{C}$ of the matrix algebra as a receptacle for the $U(1)$ group [15]. This leads directly to the minimal matrix algebra $\mathcal{A} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ which is the valid candidate for the case of $KO$-dimension zero [2]. In $KO$-dimension six an even number of summands is needed and one has to add a second copy of the complex numbers, i.e. $\mathcal{A} = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ [5], if assuming the classical axioms.

For the most general matrix algebra of a finite spectral triple containing the standard model we find therefore

$$\mathcal{A} = M_3(\mathbb{C}) \oplus \mathbb{H} \oplus \bigoplus_{i=1}^{N-2} M_i(\mathbb{K}) \ni (a, b, x_1, ..., x_{N-2}), \quad (3.1)$$

with at least one summand being the complex numbers. We also assume a finite number of summands with $N \geq 3$.

What is now the maximal number of summands equal to the complex numbers which can affect the standard model particles? To determine this, we take the standard model with an algebra of four summands. Its Krajewski diagram is [5]:

Here we have included the arrowheads in their standard form and we have $\det(\cap = \mu \pm \mu^t) \neq 0$ so the Poincaré duality is fulfilled. The algebra of the model is

$$\mathcal{A}_{SM} = M_3(\mathbb{C}) \oplus \mathbb{H} \oplus \mathbb{C} \oplus \mathbb{C} \ni (a, b, x_1, x_2), \quad (3.2)$$
and its representation

\[ \rho_{SM,L}(b) = \begin{pmatrix} b \otimes 1_3 & 0 \\ 0 & b \end{pmatrix}, \quad \rho_{SM,R}(x_1) = \begin{pmatrix} x_1 1_3 & 0 & 0 \\ 0 & \bar{x}_1 1_3 & 0 \\ 0 & 0 & \bar{x}_1 1_2 \end{pmatrix}, \]

\[ \rho_{SM,L}^{c}(a, x_2) = \begin{pmatrix} 1_2 \otimes a & 0 \\ 0 & x_2 1_2 \end{pmatrix}, \quad \rho_{SM,R}^{c}(a, x_2) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & x_2 1_2 \end{pmatrix} \]

\[ \rho_{SM}(a, b, x_1, x_2) = \rho_{SM,L}(b) \oplus \rho_{SM,R}(x_1) \oplus \rho_{SM,L}^{c}(a, x_2) \oplus \rho_{SM,R}^{c}(a, x_2). \]

(3.3)

The Dirac operator takes the form

\[ D = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}, \]

(3.6)

with the sub-matrices

\[ \Delta = \begin{pmatrix} 0 & 0 & M_d \otimes 1_3 & M_u \otimes 1_3 & 0 \\ 0 & 0 & 0 & 0 & M_e \\ M_d^* \otimes 1_3 & 0 & 0 & 0 & 0 \\ M_u^* \otimes 1_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & M_e^* & 0 & 0 & 0 \end{pmatrix} \]

(3.7)

where \( M_d, M_u \) and \( M_e \) are in \( M_{2 \times 1}(\mathbb{C}) \) and contain the Yukawa couplings of the down-quark, the up-quark and the electron.

The axioms of noncommutative geometry require now that \([\rho(a), J \rho(a') J^{-1}]) = [[D, \rho(a)], J \rho(a') J^{-1}]) = 0\) which results in the following constraint: While the complex numbers \( x_2 \) in the anti-particle representations \( \rho_{SM,L}^{c}(a, x_2) \) and \( \rho_{SM,R}^{c}(a, x_2) \) must have their origin in the same summand of the matrix algebra this cannot be said for three copies \( x_1 \) of the complex numbers in the particle representations \( \rho_{SM,R}(x_1) \). They can, in principle, come from three different summands of complex numbers.

We conclude that we can accommodate at most four summands of complex numbers in the matrix algebra which are represented on the standard model fermions. Now the Krajewski diagram for this model is
where the dots indicate the more possible summands in the matrix algebra. If only the standard model fermions are included we find $\det(\cap = \mu \pm \mu^t) = 0$ so the axiom of the Poincaré duality is not fulfilled. For a viable spectral triple more fermions, i.e. more arrows have to be included.

Ignoring the Poincaré duality for now, the matrix algebra has then the maximal form

$$A_{\text{max}} = M_3(\mathbb{C}) \oplus \mathbb{H} \oplus \mathbb{C}_1 \oplus \mathbb{C}_2 \oplus \mathbb{C}_3 \oplus \mathbb{C}_4 \oplus \bigoplus_{i=5}^{N-2} M_i(\mathbb{K}) \ni (a, b, x_1, x_2, x_3, x_4, \ldots, x_{N-2}) \quad (3.8)$$

where the first six summands are represented on the standard model Hilbert subspace in the following way:

$$\rho_L(b) = \begin{pmatrix} b \otimes 1_3 & 0 \\ 0 & b \end{pmatrix}, \quad \rho_R(x_1, x_2, x_3) = \begin{pmatrix} x_1 1_3 & 0 & 0 \\ 0 & x_2 1_3 & 0 \\ 0 & 0 & x_3 1_2 \end{pmatrix},$$

$$\rho_L^c(a, x_4) = \begin{pmatrix} 1_2 \otimes a & 0 \\ 0 & x_4 1_2 \end{pmatrix}, \quad \rho_R^c(a, x_4) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & x_4 1_2 \end{pmatrix}. \quad (3.9)$$

The Dirac operator for the standard model does not change. The remaining part of the algebra $\bigoplus_{i=5}^{N-2} M_i(\mathbb{K})$, its representation and corresponding part of the Dirac operator, belong then to the “beyond the standard model” part and have to be determined separately.

Smaller algebras with less summands represented on the standard model Hilbert space are readily obtained by identifying two or more of the complex number summands. This leads then, due to the necessary compatibility of the corresponding representations, also to a representation of the standard model.
### 3.3 The lift and the Standard Model charges

Let us now turn to the lift of the group of inner unitary group of the matrix algebra $A_{\text{max}}$. We will restrict ourselves to the first six summands and their representation on the Hilbert subspace of the standard model. The group of unitaries is $SU(2) \times U(3)$ and contains just a single $U(1)$ subgroup which is represented via a central extension.

The lift of the unitaries of $A_{\text{max}}$ to the Hilbert space is in general given by

$$L = \rho J \rho J^{-1}.$$ (3.10)

For the particle part of the standard model it takes the form

$$L^P \left( (\det u)^q u, v, (\det u)^{p_1}, (\det u)^{p_2}, (\det u)^{p_3}, (\det u)^{p_4}, \ldots \right) |_{SM}$$

(3.11)

where $v \in SU(2)$ and $u \in U(3)$. The central charges $p_i$ have to be chosen to match the standard model representation of the gauge group $G_{SM} = U(1)_Y \times SU(2)_W \times SU(3)_C$.

Comparing to the well known lift of the standard model [15]

$$L^P_{SM} \left( (\det u)^q u, v, (\det u)^p, (\det u)^{-p} \right)$$

(3.11)

$$= \text{diag}[(\det u)^q v \otimes u, (\det u)^{q-p} v, (\det u)^{q+p} u, (\det u)^{q-p} u, (\det u)^{-p}]$$

with the relation

$$q = \frac{p - 1}{3},$$

(3.12)

we find the following identifications that allow to recover the standard model hyper-charge assignment:

$$p = p_1 = -p_2 = -p_3 = -p_4.$$ (3.13)

It is now immediately clear why the $C$-summands in $A_{\text{max}}$ may be identified (if the axioms allow it): They all contribute the same central charge, modulo a sign which can be obtained by taking the complex conjugate in the respective representation.

### 4 Implementing the constraints into the Krajewski diagrams

We will now implement the physical constraints as well as the constraints coming from the axioms into the Krajewski diagrams. To keep the diagrams uncluttered we will only draw the arrows representing the particles of the model. The anti-particle arrows are obtained by reflecting the particle arrows at the main diagonal. All of the following constraints are therefore valid for the particle arrows only but the anti-particles behave automatically in the correct manner.

We choose the first line and column of the diagram to represent the $M_3(\mathbb{C})$ summand of the matrix algebra and the second line and column the $\mathbb{H}$ summand. This already
Figure 1: Diagram for $KO$-dimension zero with a quark double arrow drawn in. The dashed column represents the $\mathbb{H}$-line to which $SU(2)$ doublets have to connect. The continuous lines and columns are prohibited for the lepton arrow.

fixes the double arrow of the quarks to lie on the first line with its connection point at the second algebra, i.e. at the crossing of the first line and the second column. Reading off the representation this would correspond to the particle part $\rho_L(b) = b \otimes 1_3$ and the anti-particle part $\rho_L^c(a) = 1_2 \otimes a$ (left-handedness and right-handedness are again purely conventional).

Since the colour of the quarks coming from the unitaries of $M_3(\mathbb{C})$ is not broken by the standard model fermions, no particles may connect to the first column [5]. So arrows can only connect on the second column and on columns further to the right in the diagram. The choice of a specific line to represent the “colour algebra” of the quarks is of course also purely conventional.

Also neither quarks nor leptons couple vectorially to the $SU(2)$ subgroup and therefore no arrows can lie on the second line of the diagram. But both, leptons and quarks, couple with their right- or left-handed doublets chirally to the $SU(2)$ subgroup and therefore have to connect to the second column.

What left-handed or right-handed means is also conventional and this choice is usually indicated by the direction of the arrow head. To keep the diagrams here as general as possible, we will drop the arrowheads.

The last physical constraint is that the leptons are neutral to the colour group. As a consequence the lepton arrow cannot lie on the $M_3(\mathbb{C})$-line, that is in our case the first line.

Putting these physical constraints together, we find the diagram depicted in figure 1, where a quark double arrow has been drawn to fix the $M_3(\mathbb{C})$-line as well as the $\mathbb{H}$-line. Each line/column represents a summand in the algebra $\mathcal{A} \ni (a, b, c, d, e, ...)$ going from left to right. The connected end of the quark arrow represents the $SU(2)$ doublet.
Figure 2: Diagram for $KO$-dim six with a quark double arrow drawn in. The dashed column represents the $\mathbb{H}$-column to which $SU(2)$ doublets have to connect. The continuous lines and columns are prohibited for the lepton arrow. No arrow is allowed to connect to the continuously drawn diagonal.

and the two ends the $U(1)$ singlets. Continuous lines represent the physical constraints specified above. The lepton arrow will be added next has to connect to the second column, accentuated by the dashed line, and must not connect to or lie on any of the continuous lines.

In the case of $KO$-dimension six we have an additional constraint from the orientability axiom [11]. It translates into the requirement that no arrow, including the quark arrow, may connect to the main diagonal. We depict this by another continuous line in figure 2.

4.1 The Krajewski diagrams of the Standard Model

To construct the full list of Krajewski diagrams of the standard model (up to Poincaré duality) we start as before with the quarks to fix the first three or four summands of the matrix algebra.

We note that the spectral triples are invariant under simultaneous permutations of lines and columns of the respective Krajewski diagrams. These permutations result only in a reshuffle of the algebra’s summands, its representation, Hilbert space and corresponding Dirac operator. But they do not alter the physical content of the theory [16]. Therefore Krajewski diagrams which can be obtained by permutations are equivalent.

Figure 3 shows the two possible ways to put the quarks into a Krajewski diagram. The algebra truncated to three summands of the left diagram is $\mathcal{A} = M_3(\mathbb{C}) \oplus \mathbb{H} \oplus \mathbb{C} \ni (a,b,c)$ with the representation $\rho_L(b) = b \otimes 1_3$, $\rho_R(c) = \text{diag}(c1_3,c1_3)$, $\rho_L^c(a) = 1_2 \otimes a$ and $\rho_R^c(a) = \text{diag}(a,a)$. For the left diagram in figure 3 we have $\mathcal{A} = M_3(\mathbb{C}) \oplus \mathbb{H} \oplus \mathbb{C} \oplus \mathbb{C} \ni$
Figure 3: The two representation of the standard model quarks in a Krajewski diagram. All other possibilities can be obtained by simultaneous permutation of the lines and the columns.

\[(a, b, c, d)\] with the representation \(\rho_L(b) = b \otimes 1_3\), \(\rho_R(c, d) = \text{diag}(c_{13}, d_{13})\), \(\rho^c_L(a) = 1_2 \otimes a\) and \(\rho^c_R(a) = \text{diag}(a, a)\).

Let us now add the lepton arrow according to the physical and geometrical constraints depicted in figure 1 and figure 2. We will begin with the more restrictive case of a finite spectral triple in \(KO\)-dimension six as shown in figure 2.

Building on the left diagram of figure 3 we add a lepton arrow on the third line, the first allowed line. Connecting it according to the rules to the second column, the closest end point is at the third column. The whole diagram is shown in figure 4 together with a possible permutation obtained by interchanging the fourth and fifth line/column (\(d \leftrightarrow e\)). The algebra and its representation truncated to the standard model are

\[\mathcal{A} = M_3(\mathbb{C}) \oplus \mathbb{H} \oplus \mathbb{C} \oplus \mathbb{C} \ni (a, b, c, d)\]

with

\[
\begin{align*}
\rho_L(b) &= \begin{pmatrix} b \otimes 1_3 & 0 \\ 0 & b \end{pmatrix}, & \rho_R(c, d) &= \begin{pmatrix} c_{13} & 0 & 0 \\ 0 & c_{13} & 0 \\ 0 & 0 & d_{13} \end{pmatrix}, \\
\rho^c_L(a, c) &= \begin{pmatrix} 1_2 \otimes a & 0 \\ 0 & c_{12} \end{pmatrix}, & \rho^c_R(a, c) &= \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & c_{12} \end{pmatrix}.
\end{align*}
\]

The Dirac operator and the other operators appearing in the spectral triple remain the same for all realisations of the standard model.

The next in-equivalent diagram is shown in figure 5 together with an equivalent diagram obtained by permuting the fourth and fifth line/column (\(d \leftrightarrow e\)). Note that the left diagram in figure 5, when truncated to the first four summands in the algebra, cor-
responds exactly to the Krajewski diagram of the minimal standard model found in the classification [5]. Its algebra and representation are given by (3.2) and (3.3) with the identification \( c = x_1 \) and \( d = x_2 \). For the following diagrams we will not give the details of the algebra and its representation.

We proceed in this spirit (only depicting one representative for each equivalence class of Krajewski diagrams) and find five more diagrams which concur with the physical and geometrical constraints for finite spectral triples with \( KO \)-dimension six. These five diagrams are shown in figures 6, 7 and 8.

In \( KO \)-dimension zero, the conditions on the Krajewski diagrams are more relaxed since the lepton arrow may touch the diagonal, see figure 1. The previous seven inequivalent diagrams shown in figures 3-8 are also admissible in \( KO \)-dimension zero but we find four more diagrams, see figure 9 and figure 10. Note again that the left diagram in figure 9, if truncated to the first three summands in the algebra, is the Krajewski diagram [3] which represents the classical version of the noncommutative standard model by A. Chamseddine and A. Connes [2].

5 Conclusions

In this paper we have presented all inequivalent Krajewski diagrams which represent spectral triples constituting the first family of the standard model of particle physics without right-handed neutrinos. We have ignored for the moment the axiom of Poincaré duality [1], which is of course respected for suitable truncations leading to the well known Krajewski diagrams of the standard model, i.e. figure 5 and figure 9 (left) truncated at four summands or three summands.

We find eleven inequivalent diagrams for spectral triples with \( KO \)-dimension zero. Of these eleven diagrams the first seven, figures 4-8 are also compatible with the more restrictive conditions for spectral triples with \( KO \)-dim six.

The eleven Krajewski diagrams may now be used as basic building blocks for models beyond the standard model. Only a few models beyond the standard model are known within noncommutative geometry [13, 14, 17] and these extensions have been found by trial and error methods. Now it appears to be possible to explore the realm beyond the standard model in a more organised way by starting with one of the standard model diagrams presented here and extending it by enlarging the number of summands in the algebra and its particle content. Thereby one is always sure to obtain the standard model as a sub-model.

This procedure will still be extremely restricted, not only by the axiom of Poincaré duality that should be obeyed by the final model. But also the spectral action principle poses extra constraints on the physical models [2, 10] which result for example in restrictions on the masses and gauge couplings of the new particles as it is the case for the \( \theta \)-particle model, [14]. As an example let us provide its Krajewski diagram (with the arrowheads put into place) which consists of an extension of diagram 5:
Since we know that extensions of the standard model within the noncommutative framework lead to models of physical interest like the AC-model [13], which even provides an interesting dark matter candidate [18], this endeavour to seek for new physics seems very promising.

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Figure 4: Krajewski diagram with one quark double arrow (left diagram in figure 3) and one lepton arrow according to the restrictions for $KO$-dimension six. The two diagrams show two possible permutation, i.e. $d \leftrightarrow e$, giving equivalent diagrams.

Figure 5: Krajewski diagram of the standard model with constraints compatible with $KO$-dimension six. This diagram is in-equivalent to the Krajewski diagram shown in figure 4. The permutation $d \leftrightarrow e$ leads to the equivalent diagram on the right.
Figure 6: In-equivalent Krajewski diagrams compatible with KO-dimension six.

Figure 7: In-equivalent Krajewski diagrams compatible with KO-dimension six.
Figure 8: In-equivalent Krajewski diagram compatible with $KO$-dimension six.

Figure 9: In-equivalent Krajewski diagram compatible with $KO$-dimension zero.
Figure 10: In-equivalent Krajewski diagram compatible with $KO$-dimension zero.