FUNK FUNCTIONS AND PROJECTIVE DEFORMATIONS OF SPRAYS AND FINSLER SPACES OF SCALAR FLAG CURVATURE

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Abstract. In this paper, we address the following question of Zhongmin Shen, [11, page 184]. Is it possible to have two projectively related Finsler metrics with the same Riemann curvature? In Theorem 3.1, we show that the answer is negative, when referring to Finsler metrics of scalar flag curvature. In other words, we show that for an isotropic spray, its projective class contains at most one Finsler metrizable spray having the same Riemann curvature as the original spray. Moreover, in Proposition 3.3, we show how to construct a spray whose projective class does not contain any Finsler metrizable spray with the same Riemann curvature.

1. Introduction

A system of second order ordinary differential equations (SODE), whose coefficients functions are positively two-homogeneous, can be identified with a second order vector field, which is called a spray. If such a system represents the variational (Euler-Lagrange) equations of the energy of a Finsler metric, the system is said to be Finsler metrizable, the corresponding spray represents the geodesic spray of the Finsler metric. In such a case, the system comes with a fixed parameterization, which is given by the arc-length of the Finsler metric.

An orientation preserving reparameterization of a homogeneous SODE can change substantially the geometry of the given system. Two sprays that are obtained by such reparametrization are called projectively related. It has been shown in [3] that the property of being Finsler metrizable is very unstable to reparameterization and hence to projective deformations.

Within the geometric setting one can associate to a spray, important information are encoded in the Riemann curvature tensor ($R$-curvature or Jacobi endomorphism). Projective deformations that preserve the Riemann curvature are called Funk functions. In this paper we are interested in the following question, which is due to Zhongmin Shen, [11, page 184]. Can we projectively deform a Finsler metric, by a Funk function, and obtain a new Finsler metric? In other words, can we have, within the same projective class, two Finsler metrics with the same Jacobi endomorphism? We prove, in Theorem 3.1, that projective deformations by Funk functions of non-$R$-flat geodesic sprays, of scalar flag curvature, do not preserve the property of being Finsler metrizable. As a consequence we obtain that for an isotropic spray, its projective class cannot have more then one geodesic spray with the same Riemann curvature as the given spray. We also show, in Proposition 3.3, that there are sprays whose projective class do not contain any geodesic sprays with the same $R$-curvature tensor as the original one.

The negative answer to Shen’s Question is somehow surprising and heavily relies on the fact that the original geodesic spray is not $R$-flat. It is known that in the case of an $R$-flat spray, any
A projective deformation by a Funk function leads to a Finsler metrizable spray, see [9, Theorem 7.1], [11, Theorem 10.3.5].

2. A geometric framework for sprays and Finsler spaces

In this section, we provide a geometric framework that we will use to study, in the next sections, some problems related to projective deformations of sprays and Finsler spaces by Funk functions. The main references that we use for providing this framework are [2, 8, 11, 12].

2.1. A geometric framework for sprays. In this work, we consider a manifold $M$ of $n$-dimensional smooth and connected, and $(TM, \pi, M)$ its tangent bundle. Local coordinates on $M$ are denoted by $(x^i)$, while induced coordinates on $TM$ are denoted by $(x^i, y^i)$. Most of the geometric structures in our work will be defined not on the tangent space $TM$, but on the slit tangent space $T_0M = TM \setminus \{0\}$, which is the tangent space with the zero section removed. Standard notations will be used in this paper, $C^\infty(M)$ represents the set of smooth functions on $M$, $\mathfrak{X}(M)$ is the set of vector fields on $M$, and $\Lambda^k(M)$ is the set of $k$-forms on $M$. The geometric framework that we will use in this work is based on the Frölicher-Nijenhuis formalism, [7, 9]. There are two important derivations in this formalism. For a vector valued form $L$ on $M$, consider $i_L$ and $d_L$ the corresponding derivations of degree $(\ell - 1)$ and $\ell$, respectively. The two derivations are connected by the following formula

$$d_L = i_L \circ d - (-1)^{\ell-1} d \circ i_L.$$  

If $K$ and $L$ are two vector valued forms on $M$, of degrees $k$ and $\ell$, then the Frölicher-Nijenhuis bracket $[K, L]$ is the vector valued $(k + \ell)$-form, uniquely determined by

$$d[K, L] = d_K \circ d_L - (-1)^{k\ell} d_L \circ d_K.$$  

In this work, we will use various commutation formulæ for these derivations and the Frölicher-Nijenhuis bracket, following Grifone and Muzsnay [9, Appendix A].

There are two canonical structures on $TM$, one is the Liouville (dilation) vector field $\mathbb{C}$ and the other one is the tangent structure (vertical endomorphism) $J$. Locally, these two structures are given by

$$\mathbb{C} = y^i \frac{\partial}{\partial y^i}, \quad J = \frac{\partial}{\partial y^i} \otimes dx^i.$$  

A system of second order ordinary differential equations (SODE), in normal form,

$$(2.1) \quad \frac{d^2 x^i}{dt^2} + 2G^i \left( x, \frac{dx}{dt} \right) = 0,$$

can be identified with a special vector field $S \in \mathfrak{X}(TM)$, which is called a semispray and satisfies the condition $JS = \mathbb{C}$. In this work, a special attention will be paid to those SODE that are positively homogeneous of order two, with respect to the fiber coordinates. To address the most general cases, the corresponding vector field $S$ has to be defined on $T_0M$. The homogeneity condition reads $[\mathbb{C}, S] = S$ and the vector field $S$ is called a spray. Locally, a spray $S \in \mathfrak{X}(T_0M)$ is given by

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}.$$  

The functions $G^i$, locally defined on $T_0M$, are 2-homogeneous with respect to the fiber coordinates. A curve $c(t) = (x^i(t))$ is called a geodesic of a spray $S$ if $S \circ \dot{c}(t) = \dot{c}(t)$, which means that it satisfies the system (2.1).
An orientation-preserving reparameterization of the second-order system \(2.1\) leads to a new second order system and therefore gives rise to a new spray \(\tilde{S} = S - 2PC\), \([11\), Chapter 12]. The two sprays \(S\) and \(\tilde{S}\) are said to be projectively related. The 1-homogeneous function \(P\) is called the projective deformation of the spray \(S\).

For discussing various problems for a given SODE \(2.1\) one can associate a geometric setting to the corresponding spray. This geometric setting uses the Frölicher-Nijenhuis bracket of the given spray \(S\) and the tangent structure \(J\). The first ingredient to introduce this geometric setting is the horizontal projector associated to the spray \(S\), and it is given by \([8\]
\[
h = \frac{1}{2} (\text{Id} - [S, J]).
\]

The next geometric structure carries curvature information about the given spray \(S\) and it is called the Jacobi endomorphism, \([12, \text{Section 3.6}]\), or the Riemann curvature, \([11, \text{Definition 8.1.2}]\).

A spray \(S\) is said to be isotropic if its Jacobi endomorphism takes the form
\[
\Phi = \rho J - \alpha \otimes C.
\]

The function \(\rho\) is called the Ricci scalar and it is given by \((n - 1)\rho = \text{Tr}(\Phi)\). The semi-basic 1-form \(\alpha\) is related to the Ricci scalar by \(i_S \alpha = \rho\).

In this work, we study when projective deformations preserve or not some properties of the original spray. Therefore, we recall first the relations between the geometric structures induced by two projectively related sprays. For two such sprays \(S_0\) and \(S = S_0 - 2PC\), the corresponding horizontal projectors and Jacobi endomorphisms are related by \([3, (4.8)\],
\[
\Phi = \Phi_0 + (P^2 - S_0(P)) J - (dJ(S_0(P) - P^2) + 3(PdJ - dh_0 P)) \otimes C.
\]

As one can see from the two formulae \(2.4\) and \(2.5\), projective deformations preserve the isotropy condition. In this work, we will pay special attention to those projective deformations that preserve the Jacobi endomorphism. Such a projective deformation is called a Funk function for the original spray. From formula \(2.4\), we can see that a 1-homogeneous function \(P\) is a Funk function for the spray \(S_0\), if and only if it satisfies
\[
dh_0 P = PdJ P.
\]

See also \([11, \text{Prop. 12.1.3}]\) for alternative expressions of formulae \(2.5\) and \(2.6\) in local coordinates.

### 2.2. A geometric framework for Finsler spaces of scalar flag curvature.

We recall now the notion of a Finsler space, and pay special attention to those Finsler spaces of scalar flag curvature.

**Definition 2.1.** A Finsler function is a continuous non-negative function \(F : T_0M \rightarrow \mathbb{R}\) that satisfies the following conditions:

1. \(F\) is smooth on \(T_0M\) and \(F(x, y) = 0\) if and only if \(y = 0\);
2. \(F\) is positively homogeneous of order 1 in the fiber coordinates;
3. the 2-form \(ddJ F^2\) is a symplectic form on \(T_0M\).

There are cases when the conditions of the above definition can be relaxed. One can allow for the function \(F\) to be defined on some open cone \(A \subset T_0M\), in which case talk about a conic-pseudo Finsler function. We can also allow for the function \(F\) not to satisfy the condition iii) of Definition 2.1 in which case we will say that \(F\) is a degenerate Finsler function, \([1]\).
A spray $S \in \mathfrak{X}(T_0 M)$ is said to be **Finsler metrizable** if there exists a Finsler function $F$ that satisfies
\begin{equation}
  i_S dd J F^2 = -d F^2.
\end{equation}
In such a case, the spray $S$ is called the **geodesic spray** of the Finsler function $F$. Using the homogeneity properties, it can be shown that a spray $S$ is Finsler metrizable if and only if
\begin{equation}
  d_h F^2 = 0.
\end{equation}
The Finsler metrizability problem, has been studied intensively, using various techniques in [4, 5, 6, 9, 10, 12].

**Definition 2.2.** Consider a Finsler function $F$ and let $S$ be its geodesic spray. The Finsler function $F$ is said to be of **scalar flag curvature** if there exists a function $\kappa \in C^\infty (T_0 M)$ such that the Jacobi endomorphism of the geodesic spray $S$ is given by
\begin{equation}
  \Phi = \kappa (F^2 J - F d J F) \otimes \mathbb{C}.
\end{equation}

By comparing the two formulae (2.4) and (2.10) we observe that for Finsler functions of scalar flag curvature, the geodesic spray is isotropic. The converse of this statement is true in the following sense. If an isotropic spray is Finsler metrizable, then the corresponding Finsler function has scalar flag curvature, [11, Lemma 8.2.2].

3. **Projective deformations by Funk functions**

In [11, page 184], Zhongmin Shen asks the following question: given a Funk function $P$ on a Finsler space $(M, F_0)$, decide whether or not there exists a Finsler metric $F$ that is projectively related to $F_0$, with the projective factor $P$. Since Funk functions preserve the Jacobi endomorphism under projective deformations, one can reformulate the question as follows. Decide whether or not there exists a Finsler function $F$, projectively related to $F_0$, having the same Jacobi endomorphism with $F_0$. When the Finsler function $F_0$ is $R$-flat, the answer is known, every projective deformation by a Funk function leads to a Finsler metrizable spray, [9, Theorem 7.1], [11, Theorem 10.3.5].

In the next theorem, we prove that the answer to Shen’s Question is negative, for the case when the Finsler function that we start with has non-vanishing scalar flag curvature.

**Theorem 3.1.** Let $F_0$ be a Finsler function of scalar flag curvature $\kappa_0 \neq 0$ and having the geodesic spray $S_0$. Then, there is no projective deformation of $S_0$, by a Funk function $P$, that will lead to a Finsler metrizable spray $S = S_0 - 2PC$.

**Proof.** Consider $F_0$, a Finsler function of non-vanishing scalar flag curvature $\kappa_0$, and let $S_0$ be its geodesic spray. All geometric structures associated with the Finsler space $(M, F_0)$ will be denoted with the subscript $0$. The Jacobi endomorphism of the spray $S_0$ is given by
\begin{equation}
  \Phi_0 = \kappa_0 F_0^2 J - \kappa_0 d J F_0 \otimes \mathbb{C}.
\end{equation}

We will prove the theorem by contradiction. Therefore, we assume that there exists a non-vanishing Funk function $P$ for the Finsler function $F_0$, such that the projectively related spray $S = S_0 - 2PC$ is Finsler metrizable by a Finsler function $F$. Since, $P$ is a Funk function, it follows that the Jacobi endomorphism $\Phi$ of the spray $S$ is given by $\Phi = \Phi_0$. From formula (3.1), it follows that $\Phi = \Phi_0$ is isotropic and using the fact that $S$ is metrizable, we obtain that $S$ has scalar flag curvature $\kappa$. Consequently, its Jacobi endomorphism is given by formula (2.10). By comparing the two formulae (3.1) and (2.10) and using the fact that $\Phi_0 = \Phi$, we obtain
\begin{align*}
  \kappa_0 F_0^2 &= \kappa F^2, \\
  \kappa_0 d J F_0 &= \kappa d J F.
\end{align*}
From the above two formulae, and using the fact that $\kappa_0 \neq 0$, we obtain

$$\frac{d_j F}{F} = \frac{d_j F_0}{F_0},$$

which implies $d_j (\ln F) = d_j (\ln F_0)$ on $T_0 M$. Therefore, there exists a basic function $a$, locally defined on $M$, such that

$$F(x, y) = e^{2a(x)} F_0(x, y), \forall (x, y) \in T_0 M.$$

Now, we use the fact that $S$ is the geodesic spray of the Finsler function $F$, which, using formula (2.9), implies that $S(F) = 0$. $S$ is projectively related to $S_0$, which means $S = S_0 - 2PC$ and hence $S_0(F) = 2PC(F)$. Last formula fixes the projective deformation factor $P$, which in view of formula (3.2) and the fact that $S_0(F_0) = 0$, is given by

$$P = \frac{S_0(F)}{2F} = \frac{S_0(e^{2a} F_0)}{2 e^{2a} F_0} = \frac{S_0(e^{2a}) F_0}{2 e^{2a} F_0} = \frac{S_0(e^{2a})}{2} = S_0(a) = a^c.$$

In the above formula $a^c$ is the complete lift of the function $a$. Since we assumed that the projective factor $P$ is non-vanishing, it follows that $a^c$ has the same property. Again, from the fact that $S$ is the geodesic spray of the Finsler function $F$, it follows that $d_h F = 0$. We use now formula (2.9) that relates the horizontal projectors $h$ and $h_0$ of the two projectively related sprays $S$ and $S_0$. It follows

$$d_{h_0} F - P d_j F - d_j PC(F) = 0.$$

We use the above formula, as well as formula (3.2), to obtain

$$2e^{2a} F_0 da - a^c e^{2a} d_j F_0 - e^{2a} F_0 da = 0.$$

To obtain the above formula we did use also that $a$ is a basic function and therefore $d_{h_0} a = da$ and $d_j a^c = da$. In view of these remarks, we can write formula (3.1) as follows

$$F_0 d_j a^c - a^c d_j F_0 = 0.$$

Using the fact that $a^c \neq 0$, we can write above equation as

$$d_j \left( \frac{F_0}{a^c} \right) = 0.$$

Last formula implies that $F_0 / a^c = b$ is a basic function and therefore $F_0(x, y) = b(x) \frac{\partial a}{\partial x^i}(x) y^i$, $\forall (x, y) \in T_0 M$, which is not possible due to the regularity condition that the Finsler function $F_0$ has to satisfy.

One can give an alternative proof of Theorem 3.1 by using the scalar flag curvature (SFC) test provided by [5] Theorem 3.1. Within the same hypothesis of Theorem 3.1, it can be shown that the projective deformation $S = S_0 - 2PC$, by a Funk function, is not Finsler metrizable since one condition of the SFC test is not satisfied. We presented here a direct proof, to make the paper self contained.

We can reformulate the result of Theorem 3.1 as follows. Let $F_0$ be a Finsler function of scalar flag curvature $\kappa_0 \neq 0$ and let $S_0$ be its geodesic spray with the Jacobi endomorphism $\Phi_0$. Then, within the projective class of $S_0$, there is exactly one geodesic spray, and that one is exactly $S_0$ that has $\Phi_0$ as the Jacobi endomorphism. We point out here the importance of the condition $\kappa_0 \neq 0$. The proof of Theorem 3.1 is based on formula (3.2) which is not true, in view of the previous two formulae, in the case $\kappa_0 = 0$. For the alternative proof of the Theorem 3.1 using [5] Theorem 3.1, we mention that the SFC test is valid only if the Ricci scalar does not vanish.
In the case $\kappa_0 = 0$, which means that the spray $S_0$ is $R$-flat, it is known that any deformation of the geodesic spray $S_0$ by a Funk function leads to a spray that is Finsler metrizable, \cite[Theorem 7.1]{11}, \cite[Theorem 10.3.5]{11}.

The following corollary is a consequence of Theorem 5.1 and of the above discussion.

**Corollary 3.2.** Let $S_0$ be an isotropic spray, with Jacobi endomorphism $\Phi_0$ and non-vanishing Ricci scalar. Then, the projective class of $S_0$ contains at most one Finsler metrizable spray that has $\Phi_0$ as Jacobi endomorphism.

The statement in the above corollary gives rise to a new question: is there any case when we have none? In the next proposition, we will show that the answer to this question is affirmative if the dimension of the configuration manifold is greater than two.

**Proposition 3.3.** We assume that $\dim M \geq 3$. Then, there exists a spray $S_0$ with the Jacobi endomorphism $\Phi_0$ such that the projective class of $S_0$ does not contain any Finsler metrizable spray having the same Jacobi endomorphism $\Phi_0$.

**Proof.** We consider $\tilde{S}$ the geodesic spray of a Finsler function $\tilde{F}$ of constant flag curvature $\tilde{\kappa}$. According to \cite[Theorem 5.1]{3}, the spray

\begin{equation}
S_0 = \tilde{S} - 2\lambda \tilde{F} \mathcal{C}
\end{equation}

is not Finsler metrizable for any real value of $\lambda$ such that $\tilde{\kappa} + \lambda^2 \neq 0$ and $\lambda \neq 0$. We fix such $\lambda$ and the spray $S_0$. Using formula (2.10), it follows that the Jacobi endomorphism $\Phi_0$ of the spray $S_0$ is given by

\begin{equation}
\Phi_0 = \left(\tilde{\kappa} + \lambda^2\right) \left(\tilde{F}^2 - \tilde{F}dJF \otimes \mathcal{C}\right).
\end{equation}

We will prove by contradiction that the projective class of $S_0$ does not contain any Finsler metrizable spray, whose Jacobi endomorphism is given by formula (3.6). Accordingly, we assume that there is a Funk function $P$ for the spray $S_0$ such that the spray $S = S_0 - 2P\mathcal{C}$ is metrizable by a Finsler function $F$. Since $P$ is a Funk function, it follows that $S_0$ and $S$ have the same Jacobi endomorphism, $\Phi_0 = \Phi$. A first consequence is that the spray $S$ is isotropic and being Finsler metrizable, it follows that it is of scalar flag curvature $\kappa$. Therefore, the Jacobi endomorphism $\Phi$ is given by formula (2.10).

By comparing the two formulae (3.6) and (2.10) and using the fact that $\Phi_0 = \Phi$, we obtain that the two Ricci scalars, as well as the two semi-basic 1-forms coincide

\begin{equation}
\rho_0 = \left(\tilde{\kappa} + \lambda^2\right) \tilde{F}^2 = \rho = \kappa F^2,
\end{equation}

\begin{equation}
\alpha_0 = \left(\tilde{\kappa} + \lambda^2\right) \tilde{F}dJF = \alpha = \kappa FdJF.
\end{equation}

From the above formulae we have that $dJ \rho_0 = 2 \alpha_0$ and therefore $dJ \rho = 2 \alpha$. Last formula implies $F^2dJ\kappa + 2\kappa FdJF = 2\kappa FdJF$, which means $dJ\kappa = 0$. At this moment we have that $\kappa$ is a function which does not depend on the fibre coordinates. With this argument, using the assumption that $\dim M \geq 3$ and the Finslerian version of Schur's Lemma \cite[Lemma 3.10.2]{2} we obtain that the scalar flag curvature $\kappa$ is a constant.

We express now the spray $S$ in terms of the original spray $\tilde{S}$ that we started with,

\begin{equation}
S = \tilde{S} - 2 \left(\lambda \tilde{F} + P\right) \mathcal{C}.
\end{equation}

Since $S$ is the geodesic spray of the Finsler function $F$ and $\tilde{S}$ is the geodesic spray of the Finsler function $\tilde{F}$ it follows that $S(F) = 0$ and $\tilde{S}(\tilde{F}) = 0$. From first formula (3.7) we have $(\tilde{\kappa} + \lambda^2) \tilde{F}^2 = \kappa F^2$. We apply to both sides of this formula the spray $S$ given by (3.8) and obtain $\lambda \tilde{F} + P = 0$.\]
Therefore, the projective factor is given by $P = -\lambda \tilde{F}$. However, we will show that this projective factor $P$ does not satisfy the equation (2.7) and therefore it is not a Funk function for the spray $S_0$. The projectively related sprays $S_0$ and $\tilde{S}$ are related by formula (3.5). Using the form of the projective factor $P = -\lambda \tilde{F}$, as well as the formula (2.5), we obtain that the corresponding horizontal projectors $h_0$ and $\tilde{h}$ are related by:

$$h_0 = \tilde{h} + \lambda \tilde{F}J + \lambda d_J \tilde{F} \otimes C.$$ 

We evaluate now the two sides of the equation (2.7) for the projective factor $P = -\lambda \tilde{F}$. For the right hand side we have

$$d_{h_0} P = -\lambda d_{\tilde{h}} \tilde{F} - \lambda^2 \tilde{F} d_J \tilde{F} - \lambda^2 \tilde{F} d_J \tilde{F} = -2\lambda^2 \tilde{F} d_J \tilde{F}.$$ 

In the above calculations we used the fact that $\tilde{S}$ is the geodesic spray of $\tilde{F}$ and hence $d_{\tilde{h}} \tilde{F} = 0$. For the right hand side of the equation (2.7) we have

$$P d_J P = \lambda^2 \tilde{F} d_J \tilde{F}.$$ 

It follows that the projective factor $P$ is not a Funk function for the spray $S_0$.

Therefore, we can conclude that for the spray $S_0$, given by formula (3.5) and that is not Finsler metrizable, there is no projective deformation by a Funk function that will lead to a Finsler metrizable spray. $\square$

We can provide an alternative proof of Proposition 3.3 using the constant flag curvature (CFC) test from [4, Theorem 4.1]. More exactly, we can show that the spray $S$ given by formula (3.8) is not metrizable by a Finsler function of constant flag curvature, and hence not Finsler metrizable, see also [4, Theorem 4.2].

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