High-accuracy scaling exponents in the local potential approximation

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We test equivalences between different realisations of Wilson’s renormalisation group by computing the leading, subleading, and anti-symmetric corrections-to-scaling exponents, and the full fixed point potential for the Ising universality class to leading order in a derivative expansion. We discuss our methods with a special emphasis on accuracy and reliability. We establish numerical equivalence of Wilson-Polchinski flows and optimised renormalisation group flows with an unprecedented accuracy in the scaling exponents. Our results are contrasted with high-accuracy findings from Dyson’s hierarchical model, where a tiny but systematic difference in all scaling exponents is established. Further applications for our numerical methods are briefly indicated.

I. INTRODUCTION

Renormalisation group (RG) methods provide powerful tools in the computation of universal scaling exponents or anomalous dimensions in quantum field theories \cite{1}. Modern functional RG approaches are based on Wilson’s idea to integrate-out momentum modes within a path-integral representation of the theory \cite{2}. A particular strength of these methods is their flexibility when it comes to approximations. Wilsonian flows can be implemented in many different ways, which helps adapting the technique to the problem at hand \cite{3}.

A useful systematic expansion scheme is the derivative expansion, which, to leading order, is known as the local potential approximation (LPA). For the study of critical phenomena, the derivative expansion is expected to provide good results as long as quantum corrections to propagators and anomalous dimensions remain small \cite{4}. Important results in LPA have been accumulated over the years based on different implementations of the RG including the Wilson-Polchinski RG \cite{5,6}, the functional RG \cite{2} for the effective average action \cite{7} and optimised versions thereof \cite{8}, and Dyson’s hierarchical RG \cite{9,10}. Interestingly, and despite qualitative differences in the respective formalisms, these approaches are closely related in LPA. In fact, it has been established by Felder \cite{11} that the Wilson-Polchinski RG and the hierarchical RG are equivalent. More recently, following a conjecture first stated in \cite{12,13}, the equivalence between the Wilson-Polchinski flow and an optimised functional RG has been proven by Morris \cite{14}. Consequently, universal scaling exponents from either of the three approaches should be identical.

Quantitatively, these equivalences have been put to test for 3d critical scalar theories \cite{13,15}. While scaling exponents from either approach agree at the order $10^{-4}$, an unexpected disagreement starting at the order $10^{-5}$ was found. Presently, however, this last observation solely relies on results from the hierarchical RG and the optimised RG, where exponents have been obtained with sufficiently high accuracy. Therefore it becomes mandatory to obtain results from the Wilson-Polchinski RG with a higher precision, in order to confirm or refute the above-mentioned discrepancy.

In this paper, we close this gap in the literature and study both the Wilson-Polchinski RG and the optimised RG for the 3d Ising universality class with an unprecedented accuracy. To that end, we introduce several methods to solve non-linear eigenvalue problems with special emphasis on the numerical reliability. We expect that these techniques are equally useful for other non-linear problems in mathematical physics. The outline of the paper is as follows. We introduce the relevant differential equations (Sec. \textsuperscript{III}) and discuss our numerical methods (Sec. \textsuperscript{IV}) and the error control (Sec. \textsuperscript{IV}). Results for the fixed point solution and scaling exponents are discussed (Sec. \textsuperscript{V}) and compared with the hierarchical model (Sec. \textsuperscript{VI}). We close with a discussion and conclusions (Sec. \textsuperscript{VII}).

II. LOCAL POTENTIAL APPROXIMATION

We restrict ourselves to a real scalar field theory in three dimensions – the Ising universality class – and to the leading order in the derivative expansion, meaning that the scale-dependent effective action is approximated by $\Gamma_k = \int d^3x \left[ \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + V_k(\phi) \right]$. Here, $k$ denotes the scale of the Wilsonian momentum cutoff. We provide the flows in terms of the dimensionless potentials $u(\rho) = V_k(\phi)/k^3$ with
\[ \rho = \frac{1}{2} \phi^2 / k, \text{ and } v(\phi) = V_\phi(\phi)/k^2 \text{ with } \phi = \phi/\sqrt{k}. \]

Within an effective average action approach, the flow equation for \( u(\rho) \) depends on the infrared momentum cutoff. For an optimised choice of the latter, and neglecting an irrelevant (field-independent) term, it is given by

\[ \partial_t u = -3u + \rho u' + \frac{1}{1 + u' + 2\rho u''}. \tag{1} \]

Here, \( t = \ln k \) denotes the logarithmic scale parameter. At a fixed point \( \partial_t u' = 0 \), the potential obeys

\[ \frac{d u''}{d \rho} = \left( \frac{u''}{2} - \frac{u'}{\rho} \right) (1 + u' + 2\rho u'')^2 - \frac{3u''}{2\rho}. \tag{2} \]

There exists a unique non-trivial and well-defined solution \( u(\rho) \) to \( (1) \), and in this case, we write

\[ \partial_t v = -3v + \frac{1}{2} \varphi v' + \frac{1}{1 + v' + v''}, \tag{3} \]

\[ \frac{d v''}{d \varphi} = \frac{1}{2} (v'' - 5v') (1 + v'')^2. \tag{4} \]

For the Wilson-Polchinski RG, we follow an analogous ansatz for the effective action. In terms of the potential \( u(\rho) \) and with \( t = \ln k \) we find

\[ \partial_t u = -3u + \rho u' - u' - 2\rho u'' + 2\rho (u')^2 \tag{5}, \]

\[ \frac{d u''}{d \rho} = \frac{u''}{2} - \frac{u'}{\rho} + \frac{(u')^2}{\rho} + 2u'' u'' - \frac{3u''}{2\rho}. \tag{6} \]

The corresponding equations for \( v(\varphi) \) are

\[ \partial_t v = -3v + \frac{1}{2} \varphi v' - v'' + (v')^2, \tag{7} \]

\[ \frac{d v''}{d \varphi} = \frac{1}{2} (v'' - 5v') + 2v' v''. \tag{8} \]

Numerical solutions for the fixed point potentials and their derivatives \( (2), (4) \) and \( (8) \) are displayed in Figs. 1, 2, 3, 6 and 7 below.

The potential \( v(\varphi) \) in \( (3) \) is related to the Wilson-Polchinski potential \( v_{WP}(\varphi_{WP}) \) in \( (4) \) by a Legendre transform \( (1) \),

\[ v = v_{WP} + \frac{1}{2} (\varphi_{WP} - \varphi)^2, \tag{9} \]

\[ \varphi = \varphi_{WP} - \varphi_{WP}', \tag{10} \]

up to an irrelevant constant (the absolute value of the potentials at vanishing field). An immediate consequence of \( (10) \) is that the two potentials take their absolute minimum at the same field value \( \varphi_{WP,0} = \varphi_0 \). It follows that the barrier heights

\[ v(0) - v(\varphi_0) = v_{WP}(0) - v_{WP}(\varphi_0) \tag{11} \]

are identical. Further similarities and differences between \( (1) - (8) \) have been studied in \( (12) - (14) \).

### III. Numerical Methods

In this section, we summarise the numerical techniques used to solve \( (11) - (8) \) and similar to high accuracy at a fixed point, and to deduce the leading and subleading scaling exponents. These include tools for local polynomial expansions with various degrees of sophistication (a) - (c), tools for initial-value problems (d), and tools for solving (two-point) boundary-value problems (e). Analytical methods for solving \( (11) \) can be found in \( (16) - (17) \), and are not further elaborated here.

#### A. Local behaviour

For small fields, we use polynomial expansions of the effective potential \( (12) \). Their convergence has been addressed in \( (16) - (19) \). We implement the expansion in different ways (a) - (c). At a fixed point, the potential is \( Z_2 \)-symmetric under \( \varphi \rightarrow -\varphi \). Therefore, we expand the effective potential in \( \rho \) rather than in \( \varphi \).

(a) We Taylor-expand the potential \( u(\rho) \) in field monomials about vanishing field \( \sim \rho^n \) up to the maximum order \( m \) in the truncation,

\[ u(\rho) = \sum_{n=1}^{m} \frac{1}{n!} \lambda_n \rho^n. \tag{12} \]

Inserting \( (12) \) into \( (11) \) and solving for \( \partial_t u' = 0 \) leads to coupled ordinary differential equations \( \partial_t \lambda_n = \beta_n(\{\lambda_i\}) \) for the couplings \( \lambda_i \), which are solved numerically. We note that the \( m \) differential equations \( \partial_t \lambda_n \) depend, in general, on \( m + 2 \) couplings up to order \( \lambda_{m+2} \). Therefore, we have to provide boundary conditions for the couplings \( \lambda_{m+1} \) and \( \lambda_{m+2} \), as their values are undetermined by the truncated flow. For \( (12) \), we employ \( \lambda_{m+1} = 0 \) as a boundary condition (in an expansion about vanishing field, the dependence on \( \lambda_{m+2} \) drops out), which shows good convergence with increasing \( m \); see Tab. 1 for numerical results for the couplings at the fixed point of \( (11) \).

(b) The convergence is further enhanced by expanding \( u(\rho) \) about non-vanishing field \( (16) - (19) \). In this case, we write

\[ u(\rho) = \sum_{n=2}^{m} \frac{1}{n!} \lambda_n (\rho - \lambda_1)^n, \tag{13} \]

where the expansion point \( \rho = \lambda_1 \) is defined implicitly by \( u'(\lambda_1) = c \) with \( c \) a free parameter. From \( (11) \) we conclude that \( c \) can take values between \(-1\).
where the potential is normalised to $\rho_0$. The absolute minimum is at $\rho = 1.814898403687 \cdots$, where the potential is normalised to $u(\rho_0) = 0$. The axes are rescaled as $x \to \frac{\rho}{\rho_0}$ with $x = \rho/\rho_0$, and $u \to \frac{u}{2\xi u}$. Numerically, it reads $u'(0) = -0.186064249470 \cdots$ at vanishing argument. $u'$ is a monotonously increasing function of the field. Same rescaling as in Fig. 1 (see text).

and $\infty$. This is confirmed by the explicit result, see Fig. 2. A natural choice is $c = 0$, in which case $\lambda_1$ denotes the unique potential minimum, $u'(\lambda_1) = 0$. The numerical convergence of the expansion is best for small $c$. In a given truncation to order $m$, the boundary condition $\lambda_{m+1} = 0 = \lambda_{m+2}$ is used. The expansion displays very good convergence properties with increasing $m$ [16]; see Tab. 1 for numerical results for the couplings at the fixed point of 1.

(c) The simple boundary conditions $\lambda_{m+1} = 0$ or $\lambda_{m+1} = 0 = \lambda_{m+2}$ used in (a) or (b), respectively, fail to provide convergent solutions for the Wilson-Polchinski flow [5]. The reason behind this is that the Wilson-Polchinski flow at small fields is more strongly sensitive to the couplings at large fields [13]. Identifying a stable solution is then more demanding and boils down to providing better-adapted boundary conditions for those higher order couplings, which are undetermined in a given truncation. Equations which fix the higher order couplings can be derived from the asymptotic behaviour of the scaling solution. An algebraic procedure which determines the highest order couplings from the asymptotic behaviour of [2], i.e. from $u'' \to 0$ for $\rho \to \infty$ (and similarly for [6]) has been given in [21]. We have adopted this procedure for [6] (for technical details, see [22]). As a result, we find that the polynomial expansion [12] for the Wilson-Polchinski flow [4] with appropriate boundary condition converges nicely; see Tab. 2 for numerical results for the couplings at the fixed point of [5]. Although it is more demanding to stabilise the small field expansion based on [5] as opposed to [1], we note that the Wilson-Polchinski potential for small fields differs only mildly from the optimised fixed point potential (see Fig. 6 below).

Polynomial expansions like [12] and [14] have, in general, a finite radius of convergence in field space, restricted to field values $\rho_{\text{min}} \leq \rho \leq \rho_{\text{max}}$. Here, we find that $\rho_{\text{min}} < 0$ and $\rho_{\text{max}} > \rho_0$.

In either of these cases (a) – (c), the flows [11] and [15] transform into a set of $m$ coupled ordinary differential equations $\partial_{\lambda} \lambda_n = \beta_n(\{\lambda_i\})$ for the coefficient functions $\lambda_n$. The fixed point equations $\beta_n(\{\lambda_i\}) = 0$ can be solved to very high accuracy with standard methods. The scaling exponent $\nu$ and subleading corrections-to-scaling exponents are deduced from the eigenvalues of the stability matrix $M$ at criticality, $M_{ij} = \partial_{\lambda_i} \beta_j |_{\lambda_i u''=0}$. Asymmetric corrections-to-scaling, i.e. eigenperturbations not symmetric under $\varphi \to -\varphi$, can be obtained as well [23].
\[
\begin{array}{|c|c|c|}
\hline
\text{coupling} & \rho = 0 & u'(\rho = \lambda_1) = 0 \\
\hline
\lambda_1 & -0.186 \, 064 \, 249 \, 470 & 1.814 \, 898 \, 403 \, 687 \\
\lambda_2 & 0.082 \, 177 \, 310 \, 824 & 0.126 \, 164 \, 421 \, 218 \\
\lambda_3 & 0.018 \, 980 \, 341 \, 099 & 0.029 \, 814 \, 964 \, 767 \\
\lambda_4 & 0.005 \, 252 \, 082 \, 509 & 0.006 \, 262 \, 816 \, 384 \\
\lambda_5 & 0.001 \, 103 \, 954 \, 106 & -0.000 \, 275 \, 905 \, 516 \\
\hline
\end{array}
\]

Table 1: The first five couplings at the fixed point of the optimised flow \([1]\) at vanishing field (first column), and at the potential minimum (second column).

\[
\begin{array}{|c|c|c|}
\hline
\text{coupling} & \rho = 0 & u'(\rho = \lambda_1) = 0 \\
\hline
\lambda_{1WP} & -0.228 \, 598 \, 202 \, 437 & 1.814 \, 898 \, 403 \, 687 \\
\lambda_{2WP} & 0.187 \, 236 \, 893 \, 730 & 0.086 \, 535 \, 420 \, 434 \\
\lambda_{3WP} & -0.105 \, 930 \, 606 \, 484 & -0.028 \, 253 \, 169 \, 622 \\
\lambda_{4WP} & 0.101 \, 611 \, 201 \, 027 & 0.015 \, 928 \, 269 \, 983 \\
\lambda_{5WP} & -0.135 \, 786 \, 295 \, 049 & -0.012 \, 666 \, 298 \, 430 \\
\hline
\end{array}
\]

Table 2: The first five couplings at the fixed point of the Wilson-Polchinski flow \([3]\) at vanishing field (first column), and at the potential minimum (second column).

B. Global behaviour

Next we discuss two methods (d) and (e) which integrate \((2)\) or \((6)\) directly, without using polynomial approximations.

(d) Initial value problem. The differential equations \((2)\) or \((6)\) are studied as initial value problems with boundary conditions given at some starting point \(\rho_1\). The necessary boundary conditions can be obtained within the polynomial expansion (a) and (b) in their domain of validity, see Tab. 1 for the couplings at vanishing field and at the potential minimum. The boundary conditions are

\[
\begin{align*}
&u'(0) = \lambda_1, \quad u''(0) = \lambda_2 = -\frac{3}{2} \lambda_1(1 + \lambda_1)^2, \\
&u'(0) = \lambda_{1WP}, \quad u''(0) = \lambda_{2WP} = -\frac{3}{2} \lambda_{1WP}(1 + \lambda_{1WP})
\end{align*}
\]

at \(\rho_1 = 0\) for the optimised RG flow \([16]\) and the Wilson-Polchinski flow, respectively. Identifying the well-defined fixed point solution \(u'(\rho)\) which extends over all fields \(\rho\) requires a high degree of fine-tuning in the boundary condition \(\lambda_1 \approx 0\). Integrating \((2)\) towards larger fields, starting at some non-vanishing field \(|\rho| \sim \mathcal{O}(1)\) with boundary conditions from (a) or (b) is numerically more stable. Following this strategy, we have computed in Figs. 1, 2 and 3 the fixed point potential \(u\), its first derivative \(u'\), and the field-dependent mass term \(u' + 2\rho u''\), respectively, for all \(\rho \in [-\infty, \infty]\).

We note that \(u\) displays only one global minimum. Both \(u'\) and \(u' + 2\rho u''\) are monotonously increasing functions of the field variable. The numerical solution fits the expected analytical behaviour for asymptotically large fields.

(e) Finally, we discuss solvers for two-point boundary value problems. The shooting method \([24]\) requires boundary conditions at two points \(\rho_1\) and \(\rho_2\) in field space. Initial conditions at \(\rho_1\) are varied until the boundary condition at \(\rho_2\) is matched. The procedure is iterated until the desired accuracy in the solution is achieved \([24]\). If one is not constrained by the \(Z_2\) symmetry \(\varphi \to -\varphi\) and because of the potentially singular behaviour of the right-hand sides of \((2)\) and \((6)\) for \(\rho \to 0\), it is preferable to implement \((1)\) and \((8)\). Then the differential equations at the origin \(\varphi = 0\) are better under control and one may shoot from the origin \(\varphi_1 = 0\) (with the required symmetry conditions imposed) to large fields \(\varphi_2 \approx \varphi_{\text{bound}}\), where the asymptotic large-field behaviour is imposed. Shooting in the reverse direction may provide a better accuracy. The large-field behaviour of \((2)\), \((4)\) and \((1)\), \((8)\) has previously been determined in the literature, e.g. \([13]\). The RG eigenvalues are deduced from \((1)\) or \((8)\) in the vicinity of the fixed point solution. The implementation of \((5)\), \((1)\) and \((7)\), \((8)\) in terms of \(\varphi\) allows a direct computation of asymmetric correction-to-scaling exponents.
IV. ERROR CONTROL

Within the numerical approaches (a) – (e), there are several sources for numerical errors, the control of which is discussed in this section.

(i) Within the polynomial expansions (a) – (c), the fixed point is determined by seeking simultaneous zeros of all $\beta$-functions. Solutions for $\partial_t \lambda_n = 0$ are found to very high precision. To ensure that the fixed point is numerically a good approximation to the full fixed point solution, we have computed $|\partial_t u'(\rho)|$ for all $\rho$ within the domain of validity of the polynomial approximation. This serves as a measure for the quality of a polynomial expansion to order $m$. We define the accuracy $N_{\text{acc}}$ of the fixed point solution to order $m$ as

$$10^{-N_{\text{acc}}} = \max_{\rho \in [0, \rho_{\text{max}}]} |\partial_t u'(\rho)|.$$  \hfill (14)

Here, $[0, \rho_{\text{max}}]$ denotes a compact neighbourhood of the expansion point $\lambda_1$, which needs not to coincide with the potential minimum $\rho_0$. We take $\rho_{\text{max}} > \rho_0$ from $u'(\rho_{\text{max}}) + 2\rho_{\text{max}} u''(\rho_{\text{max}}) = 1$. This notion of accuracy is a good measure for how well a local Taylor expansion to order $m$ approximates the full fixed point potential on $[0, \rho_{\text{max}}]$. We define the accuracy $N_{\text{acc}}$ of the fixed point solution to order $m$ as

$$10^{-N_{\text{acc}}} = \max_{\rho \in [0, \rho_{\text{max}}]} |\partial_t u'(\rho)|.$$  \hfill (14)

Here, $[0, \rho_{\text{max}}]$ denotes a compact neighbourhood of the expansion point $\lambda_1$, which needs not to coincide with the potential minimum $\rho_0$. We take $\rho_{\text{max}} > \rho_0$ from $u'(\rho_{\text{max}}) + 2\rho_{\text{max}} u''(\rho_{\text{max}}) = 1$. This notion of accuracy is a good measure for how well a local Taylor expansion to order $m$ approximates the full fixed point potential on $[0, \rho_{\text{max}}]$. Using (b) with $c = 0$, we find that (14) achieves its extremum typically at $\rho = 0$. In Fig. 4 we display $N_{\text{acc}}$ as a function of $m$. Full red (open blue) dots indicate that $\max_{\rho} \partial_t u'$ is positive (negative). The slope is approximately 1/2, indicating that an increase in the truncation by $\Delta m \approx 2$ increases the accuracy in the fixed point solution roughly by a decimal place. This serves as an indicator for the reliability in the scaling exponents [see (ii)].

(ii) Within the polynomial expansions (a) – (c), we study the numerical convergence of both the fixed point values and the scaling exponents with increasing degree of truncation $m$ \hfill (15).

In analogy to (14), we define the number of significant figures $N_X$ in a fixed point coupling or a critical index $X_m$ at order $m$ in the expansion as

$$10^{-N_X} = \left| 1 - \frac{X_m}{X} \right|,$$  \hfill (15)

where $X$ denotes the full (asymptotic) result. For the leading scaling exponent $X = \nu$, we display $N_\nu$ as a function of $m$ in Fig. 5 based on the expansion (13) with $c = 0$. An open blue (full red) dot indicates that $\nu(m)$ is smaller (larger) than the asymptotic value. The expansion converges roughly in the pattern $+ + - -$. Note that the slope, in comparison with Fig. 4, slightly decreases towards larger $m$. This part of the analysis is conveniently performed with Mathematica. The accuracy of the matrix inversion (leading to the scaling exponents) has been checked independently with standard routines from Matlab.

(iii) Within the polynomial expansions (a) and (b), we study the numerical stability of the result by varying the expansion point. The radius of convergence of polynomial expansions depends on the latter. This check serves as an indicator for a possible break-down of the expansion at the highest orders. We have confirmed that only the rate of
convergence depends on the expansion point, but not the asymptotic result.

(iv) The numerical approach (d) is checked in several ways. First, starting at intermediate fields, the numerical accuracy in the integration can be made large. Second, the domain of validity for (d) and (a) – (c) overlap. This allows for a quantitative cross-check. Third, for large fields $|\rho| \to \infty$, the results from (d) match the expected asymptotic behaviour. Fourth, we have cross-checked the result with a two-point boundary value routine.

(v) The shooting method (e) is controlled in several ways. First, the numerical precision for the solution on $[0, \rho_{\text{bound}}]$ is only limited by the machine precision. We use standard routines under Fortran f77 with double precision. The boundary condition at large fields has not to be known to very high accuracy to achieve a reliable result. Stability in the result is confirmed by varying $\rho_{\text{bound}}$ as well as the boundary condition at $\rho_{\text{bound}}$. These procedures are applied for both (2) and (6). The shooting method is checked independently using a different solver under Maple, based on a different boundary value integration algorithm.

V. RESULTS

Our results for the scaling exponents from (1) and (7) are given in Tab. 3 and 4. All digits are significant, except for possible rounding effects. Earlier findings with a lower level of accuracy [16, 25, 26] are fully consistent with Tab. 3. Specifically, in Tab. 3, the exponents from the optimised RG flows have been obtained using (a), (b), and (e), whereas those for the Wilson-Polchinski flow follow from (c) and (e). The method (b) has been used up to the order $m = 32$, leading to roughly 11 significant digits in the result for $\nu$, and roughly one significant digit less with increasing order for the subleading exponents, see [16, 23]. Method (e) has been pushed to an accuracy of 14 digits for all scaling exponents. The asymmetric corrections-to-scaling exponents in Tab. 4 have been computed using (e), again to an accuracy of 14 digits. Again, all earlier results to lower order in the accuracy [23, 27] are consistent with Tab. 4.

In Figs. 1 – 3, we have plotted our results for the fixed point potential $u(\rho)$ from (2) and its derivatives for all $\rho \in [-\infty, \infty]$. It is noteworthy that the solution extends to all negative $\rho$, which in terms of the field $\varphi$ corresponds to the analytical...
continuation to purely imaginary field values. For \( \rho \to -\infty \), the solution approaches the convexity bound where \( u' + 2\rho u'' \to -1 \). In the flow equation (1), this limit is potentially singular as it corresponds to a pole. However, the explicit solution for \( \partial_t u' = 0 \) shows that the pole is suppressed, leading to a well-behaved solution even in this limit. In the physical domain where \( \rho \geq 0 \), the fixed point solution stays far away from the pole, since \( \min_{\rho \geq 0} (u' + 2\rho u'') = u'(0) > -1 \). General (non-fixed point) solutions to (1) approach the pole only in a phase with spontaneous symmetry breaking where \( V''(\phi) \) is negative. Then the pole ensures convexity of the physical potential \( V_{k=0}(\phi) \) in the infrared limit where \( V''_{k=0}(\phi) \geq 0 \) [23].

In Figs. 6 and 7 we compare the fixed point potentials \( v(\phi) \) from the optimised RG and the Wilson-Polchinski RG for small and large fields. We have checked that \( u(\rho) \) in Fig. 1 matches \( v(\phi) \) in Fig. 2 as it should. Both potentials have their absolute minimum at the same field value \( \rho_0^{\text{WP}} = \rho_0 = 1.814 898 403 687 \); see Tab. 1 and 2. It is also confirmed that the potentials display the same barrier height \( v(0) - v(\varphi_0) \) as expected from (11). For large fields, the potentials scale differently with the fields [13].

We have checked numerically that the fixed point solutions to (2) and (6) are related by a Legendre transform, see (9), (10). Consequently, at vanishing field, the couplings from Tab. 1 are related to the Wilson-Polchinski couplings in Tab. 2 by

\[
\lambda_1 = \frac{\lambda_1^{\text{WP}}}{1 - \lambda_1^{\text{WP}}}, \quad \lambda_2 = \frac{\lambda_2^{\text{WP}}}{(1 - \lambda_1^{\text{WP}})^2}, \quad (16)
\]

\[
\lambda_1^{\text{WP}} = \frac{\lambda_1}{1 + \lambda_1}, \quad \lambda_2^{\text{WP}} = \frac{\lambda_2}{(1 + \lambda_1)^2}. \quad (17)
\]

Similar expressions are found for the higher order couplings. The numerical results in Tab. 1 and 2 confirm these relations within the present accuracy.

Table 3: The leading scaling exponent \( \nu \) and the first six subleading eigenvalues for the Ising universality class in three dimensions.

| exponent | optimised RG | Wilson–Polchinski RG |
|-----------|-------------|---------------------|
| \( \nu \) | 0.649 561 773 880 65 | 0.649 561 773 880 65 |
| \( \omega \) | 0.655 745 939 193 35 | 0.655 745 939 193 35 |
| \( \omega_2 \) | 3.180 006 512 059 2 | 3.180 006 512 059 2 |
| \( \omega_3 \) | 5.912 230 612 747 7 | 5.912 230 612 747 7 |
| \( \omega_4 \) | 8.796 092 825 414 | 8.796 092 825 414 |
| \( \omega_5 \) | 11.798 087 658 337 | 11.798 087 658 336 9 |
| \( \omega_6 \) | 14.896 053 175 688 | 14.896 053 175 688 |

Table 4: The first four asymmetric correction-to-scaling exponents for the Ising universality class in three dimensions.

| exponent | optimised RG | Wilson–Polchinski RG |
|-----------|-------------|---------------------|
| \( \varpi \) | 1.886 703 838 091 4 | 1.886 703 838 091 4 |
| \( \varpi_2 \) | 4.524 390 733 670 8 | 4.524 390 733 670 8 |
| \( \varpi_3 \) | 7.337 650 643 354 4 | 7.337 650 643 354 4 |
| \( \varpi_4 \) | 10.283 900 724 026 | 10.283 900 724 026 |

Our value \( \lambda_1^{\text{WP}} = -0.228 598 202 437 02 \) for the dimensionless mass term squared at vanishing field deviates at the order \( 10^{-5} \) from the corresponding value \(-0.228 601 293 102 \) given in [28].

In summary, we have confirmed the equivalence of an optimised RG and the Wilson-Polchinski RG to the order \( 10^{-14} \) in the universal indices, and in the Legendre-transformed fixed point solutions.

VI. COMPARISON

Next we compare our results with those based on Dyson’s hierarchical model [2]. To leading order in the derivative expansion, it has been proven by Felder [11] that the hierarchical RG is equivalent to the Wilson-Polchinski RG, and hence to the optimised RG. Then scaling exponents should come out identical. With high-accuracy numerical data at hand, we can test this assertion quantitatively.

In hierarchical models, the block-spin transformations are characterised by a decimation parameter \( \ell \geq 1 \), where \( \ell \to 1 \) refers to Felder’s limit of continuous transformations. Numerical studies with \( \ell = 2^{1/3} \approx 1.26 \) have been performed in [15, 30, 31, 32, 33, 34, 35]. In Tab. 3 we compare our results with previous computations of \( \nu \) from the Wilson-Polchinski RG [28, 29], the optimised RG [12, 16], and the hierarchical RG [13, 34, 31, 32, 33, 34, 35]. The most advanced hierarchical RG (HRG) and functional RG (FRG) studies agree amongst themselves at least to the order \( 10^{-6} \) and \( 10^{-12} \), respectively. We emphasize that the functional RG results clearly deviate from the hierarchical RG results, although the difference \((\nu_{\text{FRG}} - \nu_{\text{HRG}})/\nu_{\text{FRG}} \approx -1.3 \times 10^{-5} \) is quantitatively small.

In Tab. 4 we compare the best results from the functional RG (i.e. both the Wilson-Polchinski RG and the optimised RG) with those from the hierarchical RG for various other indices, some of which have been obtained.
with up to 12 significant figures \( \nu \) (see also \( \nu \)). The variations \( (X_{\text{FRG}} - X_{\text{HRG}})/X_{\text{FRG}} \) in the exponents \( X = \nu, \omega, \gamma, \Delta, \alpha \) read \((-1.3, 1.5, -1.3, 0.15, 5.0) \times 10^{-5} \), respectively. Hence, we confirm that the findings disagree, beginning at the order \( 10^{-5} \). Given Felder’s proof of equivalence for the limit \( \ell \to 1 \) \( \ell \), we conclude that the hierarchical model displays a dependence on the decimation parameter \( \ell \), with a tiny slope of the order of \( 10^{-6} \) for variations in the exponents.

### VII. DISCUSSION AND CONCLUSIONS

Equivalences between non-linear differential or difference equations, and more generally functional relationships between different implementations of the renormalisation group, often allow for a deeper understanding of both the underlying physics and the adequacy and efficiency of the methods at hand. It has been proven previously that three different implementations of Wilson’s renormalisation group — the Wilson-Polchinski RG, an optimised version of the effective average action RG, and Dyson’s hierarchical RG in the limit of continuous block-spin transformations — are equivalent in the local potential approximation. This implies that the corresponding non-linear RG equations carry identical universal content, \( e.g. \) identical scaling exponents.

Quantitatively, these equivalences have been confirmed up to the order \( 10^{-4} \) in the literature, but a discrepancy at the order \( 10^{-5} \) has recently been pointed out. This last observation relies on studies within an optimised RG and Dyson’s hierarchical RG with decimation parameter \( \ell = 2^{1/3} \), where results with a sufficiently high accuracy are available, while previous results from the Wilson-Polchinski RG were in agreement with either of them. Here, we have closed this gap in the literature by computing scaling exponents for the Ising universality class from the Wilson-Polchinski RG and the optimised RG with an unprecedented accuracy of the order \( 10^{-14} \). We confirm their equivalence, first conjectured in \( \ell \), based on the leading, subleading and asymmetric correction-to-scaling exponents. Equally important, our central numerical results are obtained in several ways, and furthermore independently from both the Wilson-Polchinski and the optimised RG flow. We conclude that their equivalence is rock solid for all technical purposes, and in full agreement with the explicit proof given by Morris \[14\].

In contrast to this, we now have clear indications for a non-equivalence of our results with those from Dyson’s hierarchical model for discrete block-spin transformations. In the 3d Ising universality class, all scaling exponents from the functional RG, \( e.g. \) from both the Wilson-Polchinski and the optimised RG, differ systematically at the order \( 10^{-5} \) from high accuracy studies based on Dyson’s hierarchical RG. Given the high degree of accuracy in the critical indices from the present and earlier studies, and the fact that the scaling exponents in all three approaches have been obtained from several independent numerical implementations and collaborations, it is unlikely that the discrepancy originates from numerical insufficiencies. Furthermore, these differences are absent in the limit of continuous block-spin transformations \( 11 \). We conclude that the tacit assumption of \( \ell \)-independence in the scaling exponents of the hierarchical model \( 15 \) cannot be maintained. Rather, the hierarchical model carries an inherent \( \ell \)-dependence, similar to a dependence previously observed in \( \ell \) (see also \( \ell \)). In functional RG approaches, scheme-dependences related to the underlying Wilsonian momentum cutoff can arise in truncations \( \ell \). Here, powerful control and optimisation mechanisms are available which decrease truncational artefacts and increase convergence.

| Wilson – Polchinski | optimised RG | hierarchical RG |
|---------------------|--------------|-----------------|
| \( 0.649^e \)       | \( 0.6496^g \) | \( 0.6496^a \)  |
| \( 0.6496^d \)       | \( 0.64956^h \) | \( 0.64957^h \) |
| \( 0.649561773881 \) | \( 0.649561773881 \) | \( 0.649570^c \) |

Table 5: Comparison of the scaling exponent \( \nu \) for the Ising universality class in three dimensions (see text). Data from this work, and from a) \( [31] \), b) \( [31] \), c) \( [28] \), d) \( [26] \), e) \( [22] \), f) \( [33] \), g) \( [12] \), h) \( [16] \), i) \( [15] \).

| exponent | functional RG | hierarchical RG |
|----------|---------------|-----------------|
| \( \nu \) | \( 0.649561773881 \) | \( 0.649570^c \) |
| \( \omega \) | \( 0.655745939193 \) | \( 0.655736^d \) |
| \( \gamma \) | \( 1.299123547761 \) | \( 1.29914073015^c \) |
| \( \Delta \) | \( 0.425947495477 \) | \( 0.425946859881^c \) |
| \( \alpha \) | \( 0.05131467835805 \) | \( 0.051289^i \) |

Table 6: Comparison of universal indices \( \nu, \omega, \gamma = 2 \nu, \Delta = \omega \nu \) and \( \alpha = 2 - 3 \nu \) (for \( \eta = 0 \)) from the functional RG (this work) and the hierarchical RG (same referencing as in Tab. 5).
towards the physical theory \[3, 8, 38\]. It will be interesting to see if the \(\ell\)-dependence of hierarchical models or its extensions can be understood along similar lines.

For our numerical work, we have developed methods to solve non-linear eigenvalue problems with high precision. The combined use of local polynomial expansions techniques with global integration methods allows for an efficient determination of the full fixed point solutions with reasonable numerical efforts. We have also put some emphasis on a reliable error control, in view of the high accuracy aimed for in the universal eigenvalues. We expect that this combination of techniques will be equally useful for other non-linear problems in mathematical physics, including e.g. the derivative expansion to higher order.

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