A general necessary and sufficient optimality conditions for singular control problems

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Abstract

We consider a stochastic control problem where the set of strict (classical) controls is not necessarily convex and the the variable control has two components, the first being absolutely continuous and the second singular. The system is governed by a nonlinear stochastic differential equation, in which the absolutely continuous component of the control enters both the drift and the diffusion coefficients. By introducing a new approach, we establish necessary and sufficient optimality conditions for two models. The first concerns the relaxed-singular controls, who are a pair of processes whose first component is a measure-valued processes. The second is a particular case of the first and relates to strict-singular control problems. These results are given in the form of global stochastic maximum principle by using only the first order expansion and the associated adjoint equation. This improves and generalizes all the previous works on the maximum principle of controlled stochastic differential equations.

Keywords. Stochastic differential equation, Strict-singular control, Relaxed-singular control, Maximum principle, Adjoint process, Variational inequality.

AMS subject classification. 93Exx.

1 Introduction

We study a stochastic control problem where the system is governed by a nonlinear stochastic differential equation (SDE for short) of the type

\[
\begin{align*}
   & dx^{(v, \eta)}_t = b\left(t, x^{(v, \eta)}_t, v_t\right) dt + \sigma\left(t, x^{(v, \eta)}_t, v_t\right) dW_t + G_t d\eta_t, \\
   & x^{(v, \eta)}_0 = x,
\end{align*}
\]

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where $b, \sigma$ and $G$ are given functions, $x$ is the initial data and $W = (W_t)_{t \geq 0}$ is a standard Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathcal{P})$, satisfying the usual conditions.

The control variable, called strict-singular control, is a suitable process $(v, \eta)$ where $v : [0, T] \times \Omega \to U_1 \subset \mathbb{R}^k$, $\eta : [0, T] \times \Omega \to U_2 = ([0, \infty))^m$ are $B [0, T] \otimes \mathcal{F}$-measurable, $(\mathcal{F}_t)$- adapted, and $\eta$ is an increasing process (componentwise), continuous on the left with limits on the right with $\eta_0 = 0$. We denote by $\mathcal{U}$ the class of all strict-singular controls.

The criteria to be minimized, over the set $\mathcal{U}$, has the form

$$J(v, \eta) = E \left[ g \left( x_{T}^{(v, \eta)} \right) + \int_{0}^{T} h \left( t, x_{t}^{(v, \eta)}, v_{t} \right) dt + \int_{0}^{T} k_{t} d\eta_{t} \right],$$

where, $g$, $h$ and $k$ are given maps and $x_{t}^{(v, \eta)}$ is the trajectory of the system controlled by $(v, \eta)$.

A control $(u, \xi) \in \mathcal{U}$ is called optimal if it satisfies

$$J(u, \xi) = \inf_{(v, \eta) \in \mathcal{U}} J(v, \eta).$$

This kind of stochastic control problems have been studied by many authors, both by the dynamic programming approach and by the Pontryagin stochastic maximum principle. The first approach was studied by Benêš, Shepp and Witsenhausen [6], Chow, Menaldi and Robin [10], Karatzas and Shreve [21], Davis and Norman [11], Haussmann and Soo [17, 18, 19]. See [17] for a complete list of references on the subject. It was shown in particular that the value function is solution of a variational inequality, and the optimal state is a reflected diffusion at the free boundary. Note that in [17], the authors apply the compactification method to show existence of an optimal singular control.

In this paper, we are concerned with the second approach, whose objective is to establish necessary (as well as sufficient) conditions for optimality of controls. The first version of the stochastic maximum principle that covers singular control problems was obtained by Cadenillas and Haussmann [8], in which they consider linear dynamics, convex cost criterion and convex state constraints. The method used in [8] is based on the known principle of convex analysis, related to the minimization of convex, continuous and Gâteaux-differentiable functional defined on a convex closed set. Necessary conditions of optimality for non linear SDEs with convex control domain, where the coefficients depend explicitly on the absolutely part of the control, was derived by Bahlali and Chala [1] by applying a convex perturbation on the pair of controls. The result in then obtained in weak form. Bahlali and Mezerdi [2] generalize the work of [1] to the case of nonconvex control domain, and derive necessary optimality conditions by using a strong perturbation (spike variation) on the absolutely continuous component of the control and a convex perturbation on the singular one. The Peng stochastic maximum principle is then used and te result is given with two adjoint equations and a variational inequality of the second order. Version
of stochastic maximum principle for relaxed-singular controls was established by Bahlali, Djehiche and Mezerdi [4] in the case of uncontrolled diffusion, by using the previous works on strict-singular controls, Ekeland’s variational principle and some stability properties of the trajectories and adjoint processes with respect to the control variable.

In a recent work, Bahlali [5] generalizes and improves all the previous results on stochastic maximum principle for controlled SDEs, by introducing a new approach and establish necessary and sufficient optimality conditions for both relaxed and strict controls, by using only the first order expansion and the associated adjoint equation. The main idea of [5], is to use the property of convexity of the set of relaxed controls and treat the problem with the convex perturbation on relaxed controls (instead of the spike variation on strict one).

Our aim in this paper, is to follow the method used by [5] and derive necessary as well as sufficient conditions of optimality in the form of global stochastic maximum principle, for both relaxed-singular and strict-singular controls, without using the second order expansion. We introduce then a bigger new class $R$ of processes by replacing the $U_1$-valued process $(v_t)$ by a $P(U_1)$-valued process $(q_t)$, where $P(U_1)$ is the space of probability measures on $U_1$ equipped with the topology of stable convergence. This new class of processes is called relaxed-singular controls and have a richer structure, for which the control problem becomes solvable.

In the relaxed-singular model, the system is governed by the SDE

$$
\begin{align*}
\left\{ \begin{array}{l}
\text{dx}^{(q,\eta)}_t = \int_{U_1} b\left(t, x^{(q,\eta)}_t, a\right) q_t(da) \, dt + \int_{U_1} \sigma\left(t, x^{(q,\eta)}_t, a\right) q_t(da) \, dW_t + G_t d\eta,
\end{array} \right.
\end{align*}
$$

$$
\begin{align*}
x^{(q,\eta)}_0 = x.
\end{align*}
$$

The functional cost to be minimized, over the class $R$ of relaxed-singular controls, is defined by

$$
\begin{align*}
\mathcal{J}(q, \eta) &= \mathbb{E}\left[ g\left(x^{(q,\eta)}_T\right) + \int_0^T \int_{U_1} h\left(t, x^{(q,\eta)}_t, a\right) q_t(da) \, dt + \int_0^T k_t \, d\eta \right],
\end{align*}
$$

A relaxed-singular control $(\mu, \xi)$ is called optimal if it solves

$$
\mathcal{J}(\mu, \xi) = \inf_{(q,\eta) \in R} \mathcal{J}(q, \eta).
$$

The relaxed-singular control problem finds its interest in two essential points. The first is that an optimal solution exists. Haussmann and Suo [17] have proved that the relaxed-singular control problem admits an optimal solution under general conditions on the coefficients. Indeed, by using a compactification method and under some mild continuity hypotheses on the data, it is shown by purely probabilistic arguments that an optimal solution for the problem exists. Moreover, the value function is shown to be Borel measurable. The second interest is that it is a generalization of the strict-singular control problem. Indeed, if $q_t(da) = \delta_{v_t}(da)$ is a Dirac measure concentrated at a single point $v_t$
of $U_1$, then we get a strict-singular control problem as a particular case of the relaxed one.

To achieve the objective of this paper and establish necessary and sufficient optimality conditions for these two models, we proceed as follows.

Firstly, we give the optimality conditions for relaxed controls. The main idea is to use the fact that the set of relaxed controls is convex. Then, we establish necessary optimality conditions by using the classical way of the convex perturbation method. More precisely, if we denote by $(\mu, \xi)$ an optimal relaxed control and $(q, \eta)$ is an arbitrary element of $\mathcal{R}$, then with a sufficiently small $\theta > 0$ and for each $t \in [0, T]$, we can define a perturbed control as follows

$$(\mu^\theta_t, \xi^\theta_t) = (\mu_t, \xi_t) + \theta [(q_t, \eta_t) - (\mu_t, \xi_t)].$$

We derive the variational equation from the state equation, and the variational inequality from the inequality

$$0 \leq \mathcal{J}(\mu^\theta, \xi^\theta) - \mathcal{J}(\mu, \xi).$$

By using the fact that the drift, the diffusion and the running cost coefficients are linear with respect to the relaxed control variable, necessary optimality conditions are obtained directly in the global form. The result is given by using only the first-order expansion and the associated adjoint equations.

To enclose this part of the paper, we prove under minimal additional hypotheses, that these necessary optimality conditions for relaxed-singular controls are also sufficient.

The second main result in the paper characterizes the optimality for strict-singular control processes. It is directly derived from the above results by restricting from relaxed to strict-singular controls. The main idea is to replace the relaxed controls by a Dirac measures charging a strict controls. Thus, we reduce the set $\mathcal{R}$ of relaxed-singular controls and we minimize the cost $\mathcal{J}$ over the subset $\delta(\mathcal{U}_1) \times \mathcal{U}_2 = \{(q, \eta) \in \mathcal{R} / q = \delta_v ; v \in \mathcal{U}_1\}$. Then, we derive necessary optimality conditions by using only the first order expansion and the associated adjoint equation. We don’t need anymore the second order expansion. Moreover, we show that these necessary optimality conditions for strict-singular controls are also sufficient, without imposing neither the convexity of $U_1$ nor that of the Hamiltonian $H$ in $v$.

The results of this paper are an important improvement of those of Bahlali and Mezerdi [2] and an extension of the works by Bahlali [5] to the class of singular controls.

The paper is organized as follows. In Section 2, we formulate the strict-singular and relaxed-singular control problems and give the various assumptions used throughout the paper. Section 3 is devoted to study the relaxed-singular control problems and we establish necessary as well as sufficient conditions of optimality for relaxed-singular controls. In the last section, we derive directly from the results of Section 3, the optimality conditions for strict-singular controls.
Along with this paper, we denote by $C$ some positive constant and for simplicity, we need the following matrix notations. We denote by $\mathcal{M}_{n \times d}(\mathbb{R})$ the space of $n \times d$ real matrix and $\mathcal{M}_{d \times n}(\mathbb{R})$ the linear space of vectors $M = (M_1, \ldots, M_d)$ where $M_i \in \mathcal{M}_{n \times n}(\mathbb{R})$. For any $M, N \in \mathcal{M}_{d \times n}(\mathbb{R})$, $L, S \in \mathcal{M}_{n \times d}(\mathbb{R})$, $Q \in \mathcal{M}_{n \times n}(\mathbb{R})$, we use the following notations
\[
\alpha \beta = \sum_{i=1}^{n} \alpha_i \beta_i \in \mathbb{R}^n;
\]
\[
LS = \sum_{i=1}^{d} L_i S_i \in \mathbb{R}, \text{ where } L_i \text{ and } S_i \text{ are the } i^{th} \text{ columns of } L \text{ and } S;
\]
\[
ML = \sum_{i=1}^{d} M_i L_i \in \mathbb{R}^n;
\]
\[
M \alpha \gamma = \sum_{i=1}^{d} (M_i \alpha) \gamma_i \in \mathbb{R}^n;
\]
\[
MN = \sum_{i=1}^{d} M_i N_i \in \mathcal{M}_{n \times n}(\mathbb{R});
\]
\[
MQN = \sum_{i=1}^{d} M_i Q N_i \in \mathcal{M}_{n \times n}(\mathbb{R});
\]
\[
MQ \gamma = \sum_{i=1}^{d} M_i Q \gamma_i \in \mathcal{M}_{n \times n}(\mathbb{R}).
\]

We denote by $L^*$ the transpose of the matrix $L$ and $M^* = (M_1^*, \ldots, M_d^*)$.

2 Formulation of the problem

Let $\left( \Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P} \right)$ be a filtered probability space satisfying the usual conditions, on which a $d$-dimensional Brownian motion $W = (W_t)_{t \geq 0}$ is defined. We assume that $(\mathcal{F}_t)$ is the $\mathbb{P}$- augmentation of the natural filtration of $(W_t)_{t \geq 0}$.

Let $T$ be a strictly positive real number and consider the following sets

$U_1$ is a non empty subset of $\mathbb{R}^k$,

$U_2 = ([0, \infty))^m$,

$U_1$ is the class of measurable, adapted processes $v : [0, T] \times \Omega \rightarrow U_1$ such that
\[
E \left[ \sup_{t \in [0,T]} |v_t|^2 \right] < \infty.
\]

$U_2$ is the class of measurable, adapted processes $\eta : [0, T] \times \Omega \rightarrow U_2$ such that $\eta$ is nondecreasing (componentwise), left-continuous with right limits, $\eta_0 = 0$, and
\[
E \left[ |\eta_T|^2 \right] < \infty.
\]
2.1 The strict-singular control problem

Definition 1 A strict-singular control is a pair of processes \((v, \eta) \in U_1 \times U_2\).

We denote by \(U = U_1 \times U_2\) the set of all strict-singular controls.

For any \((v, \eta) \in U\), we consider the following SDE

\[
\begin{align*}
\frac{dx_t^{(v, \eta)}}{dt} &= b\left(t, x_t^{(v, \eta)}, v_t\right) dt + \sigma\left(t, x_t^{(v, \eta)}, v_t\right) dW_t + G_t d\eta_t, \\
x_0^{(v, \eta)} &= x,
\end{align*}
\]

where

\[
\begin{align*}
b &: [0, T] \times \mathbb{R}^n \times U_1 \longrightarrow \mathbb{R}^n, \\
\sigma &: [0, T] \times \mathbb{R}^n \times U_1 \longrightarrow M_{n \times d}(\mathbb{R}), \\
G &: [0, T] \longrightarrow M_{n \times m}(\mathbb{R}).
\end{align*}
\]

The criteria to be minimized is defined from \(U\) into \(\mathbb{R}\) by

\[
J(v, \eta) = E\left[g\left(x_T^{(v, \eta)}\right) + \int_0^T h\left(t, x_t^{(v, \eta)}, v_t\right) dt + \int_0^T k_t d\eta_t\right],
\]

Where

\[
\begin{align*}
g &: \mathbb{R}^d \longrightarrow \mathbb{R}, \\
h &: [0, T] \times \mathbb{R}^d \times U_1 \longrightarrow \mathbb{R}, \\
k &: [0, T] \longrightarrow ([0, \infty))^d.
\end{align*}
\]

A strict-singular control \((v, \eta)\) is called optimal if it satisfies

\[
J(u, \xi) = \inf_{(v, \eta) \in U} J(v, \eta).
\]

We assume that

\[
\begin{align*}
b, \sigma, g \text{ and } h &\text{ are continuously differentiable with respect to } x. \\
\text{The derivatives } b_x, \sigma_x, g_x \text{ and } h_x, &\text{ are continuous in } (x, v) \text{ and uniformly bounded.} \\
b \text{ and } \sigma &\text{ are bounded by } C (1 + |x| + |v|). \\
G \text{ and } k &\text{ are continuous and } G \text{ is bounded.}
\end{align*}
\]

Under the above assumptions, for every \((v, \eta) \in U\), equation (1) has an unique strong solution and the functional cost \(J\) is well defined from \(U\) into \(\mathbb{R}\).
2.2 The relaxed-singular model

The strict-singular control problem \{(1), (2), (3)\} formulated in the last sub-section may fail to have an optimal solution. Let us begin by a deterministic examples which shows that even in simple cases, existence of a strict optimal control is not ensured (see Fleming [16] and Yong and Zhou [28] for other examples).

**Example 1.** The problem is to minimize, over the set \( U \) of measurable functions \( v : [0, T] \to \{-1, 1\} \), the following functional cost

\[
J(v) = \int_0^T (x_v^t)^2 \, dt,
\]

where \( x_v^t \) denotes the solution of

\[
\begin{align*}
\frac{dx_v^t}{dt} &= v_t \, dt, \\
x_v^0 &= 0.
\end{align*}
\]

We have

\[
\inf_{v \in U} J(v) = 0.
\]

Indeed, consider the following sequence of controls

\[
v^n_t = (-1)^k \quad \text{if} \quad \frac{k}{n} T \leq t \leq \frac{k+1}{n} T, \quad 0 \leq k \leq n - 1.
\]

Then clearly

\[
\left| x_v^n \right| \leq \frac{T}{n}, \\
|J(v^n)| \leq \frac{T^3}{n^2}.
\]

Which implies that

\[
\inf_{v \in U} J(v) = 0.
\]

There is however no control \( v \) such that \( J(v) = 0 \). If this would have been the case, then for every \( t \), \( x_v^t = 0 \). This in turn would imply that \( v_t = 0 \), which is impossible. The problem is that the sequence \( (v^n) \) has no limit in the space of strict controls. This limit if it exists, will be the natural candidate for optimality.

If we identify \( v^n_t \) with the Dirac measure \( \delta_{v^n_t} (da) \) and set \( q_n(dt, dv) = \delta_{v^n_t} (dv) \, dt \), we get a measure on \([0, 1] \times U\). Then, the sequence \( (q_n(dt, dv))_n \) converges weakly to \( \frac{1}{2} dt. [\delta_{-1} + \delta_{1}] (da) \).

**Example 2.** Consider the control problem where the system is governed by the SDE

\[
\begin{align*}
\frac{dx_t}{dt} &= v_t \, dt + dW_t, \\
x_0 &= 0.
\end{align*}
\]
The functional cost to be minimized is given by

\[ J(v) = \mathbb{E} \int_0^T \left[ x_t^2 + (1 - v_t^2)^2 \right] dt. \]

\( U = [-1, 1] \) and \( x, v, W \) are one dimensional. The control \( v \) (open loop) is a measurable function from \([0, T]\) into \( U \).

The separation principle applies to this example, the optimal control minimizes

\[ \int_0^T \left[ \bar{x}_t^2 + (1 - v_t^2)^2 \right] dt, \]

where \( \bar{x}_t = \mathbb{E}[x_t] \) satisfies

\[
\begin{align*}
\frac{d\bar{x}_t}{dt} &= v_t dt, \\
\bar{x}_0 &= 0.
\end{align*}
\]

This problem has no optimal strict control. A relaxed solution is to let

\[ \mu_t = \frac{1}{2} \delta_1 + \frac{1}{2} \delta_{-1}, \]

where \( \delta_a \) is an Dirac measure concentrated at a single point \( a \).

This suggests that the set of strict controls is too narrow and should be embedded into a wider class with a richer topological structure for which the control problem becomes solvable. The idea of relaxed-singular control is to replace the absolutely continuous part \( v_t \) of the strict-singular control by a \( \mathbb{P}(U_1) \)-valued process \( q_t \), where \( \mathbb{P}(U_1) \) is the space of probability measures on \( U_1 \) equipped with the topology of stable convergence of measures.

**Definition 2** A relaxed-singular control is a pair \( (q, \eta) \) of processes such that

i) \( q \) is a \( \mathbb{P}(U_1) \)-valued process progressively measurable with respect to \( (\mathcal{F}_t) \) and such that for each \( t, 1_{[0,t]} q \) is \( \mathcal{F}_t \)-measurable.

ii) \( \eta \in U_2 \).

We denote by \( \mathcal{R} = R_1 \times U_2 \) the set of relaxed-singular controls.

For more details on relaxed controls, see [3], [4], [5], [15], [16], [24], [25] and [26].

For any \( (q, \eta) \in \mathcal{R} \), we consider the following relaxed-singular SDE

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{d}{dt} x_t^{q,\eta} = \int_{U_1} b(t, x_t^{q,\eta}, a) q_t(da) dt + \int_{U_1} \sigma(t, x_t^{q,\eta}, a) q_t(da) dW_t + G_t \, \eta_t \\
x_0^{q,\eta} = x.
\end{array} \right.
\end{align*}
\]

(5)
The expected cost to be minimized, in the relaxed-singular model, is defined from \( \mathbb{R} \) into \( \mathbb{R} \) by

\[
J(q, \eta) = E \left[ g \left( x_T^{(q, \eta)} \right) + \int_0^T \int_{\mathbb{U}_1} h \left( t, x_t^{(q, \eta)}, a \right) q_t(da) dt + \int_0^T k_t d\eta_t \right].
\] (6)

A relaxed-singular control \((\mu, \xi)\) is called optimal if it solves

\[
J(\mu, \xi) = \inf_{(q, \eta) \in \mathcal{R}} J(q, \eta).
\] (7)

Haussmann and Suo [17] have proved that the relaxed-singular control problem admits an optimal solution under general conditions on the coefficients. Indeed, by using a compactification method and under some mild continuity hypotheses on the data, it is shown by purely probabilistic arguments that an optimal solution for the problem exists. Moreover, the value function is shown to be Borel measurable. See Haussmann and Suo [17], Section 3, page 925 to page 934 and essentially Theorem 3.8, page 933.

**Remark 3** If we put for any \((q, \eta) \in \mathcal{R}\)

\[
\begin{align*}
\mathbb{b} \left( t, x_t^{(q, \eta)}, q_t \right) &= b \left( t, x_t^{(q, \eta)}, a \right) q_t(da), \\
\mathbb{\sigma} \left( t, x_t^{(q, \eta)}, q_t \right) &= \sigma \left( t, x_t^{(q, \eta)}, a \right) q_t(da), \\
\mathbb{h} \left( t, x_t^{(q, \eta)}, q_t \right) &= h \left( t, x_t^{(q, \eta)}, a \right) q_t(da).
\end{align*}
\]

Then, equation (5) becomes

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{dx_t^{(q, \eta)}}{dt} = \mathbb{b} \left( t, x_t^{(q, \eta)}, q_t \right) dt + \mathbb{\sigma} \left( t, x_t^{(q, \eta)}, q_t \right) dW_t + G_t d\eta_t, \\
x_T^{(q, \eta)} = x.
\end{array} \right.
\]

With a functional cost given by

\[
J(q, \eta) = E \left[ g \left( x_T^{(q, \eta)} \right) + \int_0^T \mathbb{h} \left( t, x_t^{(q, \eta)}, q_t \right) dt + \int_0^T k_t d\eta_t \right].
\]

Hence, by introducing relaxed-singular controls, we have replaced \( \mathbb{U}_1 \) by a larger space \( \mathbb{P}(\mathbb{U}_1) \). We have gained the advantage that \( \mathbb{P}(\mathbb{U}_1) \) is convex. Furthermore, the new coefficients of equation (5) and the running cost are linear with respect to the relaxed control variable.

**Remark 4** The coefficients \( \mathbb{b} \) and \( \mathbb{\sigma} \) (defined in the above remark) check respectively the same assumptions as \( b \) and \( \sigma \). Then, under assumptions (4), for every \((q, \eta) \in \mathcal{R}\), equation (5) has an unique strong solution.

On the other hand, It is easy to see that \( \mathbb{h} \) checks the same assumptions as \( h \). Then, the functional cost \( J \) is well defined from \( \mathcal{R} \) into \( \mathbb{R} \).
**Remark 5** If $q_t = \delta_{v_t}$ is an atomic measure concentrated at a single point $v_t \in \mathcal{P}(U_1)$, then for each $t \in [0, T]$ we have

$$x^{(q, \eta)} = x^{(v, \eta)},$$
$$J(q, \eta) = J(v, \eta),$$

and we get a strict-singular control problem. So the problem of strict-singular controls \{(1), (2), (3)\} is a particular case of relaxed-singular control problem \{(5), (6), (7)\}.

### 3 Optimality conditions for relaxed-singular controls

In this section, we study the problem \{(5), (6), (7)\} and we establish necessary as well as sufficient conditions of optimality for relaxed-singular controls.

#### 3.1 Preliminary results

Since the set of relaxed-singular controls $\mathcal{R}$ is convex, a classical way of treating such a problem is to use the convex perturbation method. More precisely, let $(\mu, \xi)$ be an optimal relaxed-singular control and $x_t^{(\mu, \xi)}$ the solution of (5) controlled by $(\mu, \xi)$. Then, for each $t \in [0, T]$, we can define a perturbed relaxed-singular control as follows

$$(\mu_t^\theta, \xi_t^\theta) = (\mu_t, \xi_t) + \theta [(q_t, \eta_t) - (\mu_t, \xi_t)],$$

where, $\theta > 0$ is sufficiently small and $(q_t, \eta_t)$ is an arbitrary element of $\mathcal{R}$.

Denote by $x_t^{(\mu_t^\theta, \xi_t^\theta)}$ the solution of (5) associated with $(\mu_t^\theta, \xi_t^\theta)$.

From optimality of $(\mu, \xi)$, the variational inequality will be derived from the fact that

$$0 \leq J(\mu_t^\theta, \xi_t^\theta) - J(\mu, \xi).$$

For this end, we need the following classical lemmas.

**Lemma 6** Under assumptions (4), we have

$$\lim_{\theta \to 0} \left[ \sup_{t \in [0, T]} \mathbb{E} \left[ \left| x_t^{(\mu_t^\theta, \xi_t^\theta)} - x_t^{(\mu, \xi)} \right|^2 \right] \right] = 0. \quad (8)$$
Proof. We have

\[ x_t^{(\mu, \xi)} - x_t^{(\mu, \xi)} = \int_0^t \left[ \int_{U_1} \left( b \left( s, x_s^{(\mu, \xi)}, a \right), \mu_s^\theta (da) - \int_{U_1} b \left( s, x_s^{(\mu, \xi)}, a \right), \mu_s^\theta (da) \right) ds 
+ \int_0^t \left[ \int_{U_1} \left( b \left( s, x_s^{(\mu, \xi)}, a \right), \mu_s^\theta (da) - \int_{U_1} b \left( s, x_s^{(\mu, \xi)}, a \right), \mu_s (da) \right) ds 
+ \int_0^t \left[ \int_{U_1} \left( \sigma \left( s, x_s^{(\mu, \xi)}, a \right), \mu_s^\theta (da) - \int_{U_1} \sigma \left( s, x_s^{(\mu, \xi)}, a \right), \mu_s (da) \right) dW_s 
+ \int_0^t \left[ \int_{U_1} \left( \sigma \left( s, x_s^{(\mu, \xi)}, a \right), \mu_s^\theta (da) - \int_{U_1} \sigma \left( s, x_s^{(\mu, \xi)}, a \right), \mu_s (da) \right) dW_s 
+ \int_0^t G_t d \left( \xi_t^\theta - \xi_t \right) \right] \right] \right). \]

By using the definition of \((\mu, \xi)\) and taking expectation, we have

\[ E \left| x_t^{(\mu, \xi)} - x_t^{(\mu, \xi)} \right|^2 \leq CE \int_0^t \left[ \int_{U_1} \left( b \left( s, x_s^{(\mu, \xi)}, a \right), \mu_s (da) \right) ds 
+ C \theta^2 \int_0^t \left[ \int_{U_1} \left( b \left( s, x_s^{(\mu, \xi)}, a \right), q_s (da) - \int_{U_1} b \left( s, x_s^{(\mu, \xi)}, a \right), \mu_s (da) \right) ds 
+ C \int_0^t \left[ \int_{U_1} \left( b \left( s, x_s^{(\mu, \xi)}, a \right), \mu_s (da) \right) ds 
+ C \int_0^t \left[ \int_{U_1} \left( b \left( s, x_s^{(\mu, \xi)}, a \right), \mu_s (da) \right) ds 
+ C \theta \int_0^t \left[ \int_{U_1} \left( b \left( s, x_s^{(\mu, \xi)}, a \right), \mu_s (da) \right) ds 
+ C \theta \left[ \int_0^t G_t d (\eta_t - \xi_t) \right]^2 \right] \right] \right] \right]. \]

Since \(b\) and \(\sigma\) are uniformly Lipschitz with respect to \(x\), and \(G\) is bounded, then

\[ E \left| x_t^{(\mu, \xi)} - x_t^{(\mu, \xi)} \right|^2 \leq CE \int_0^t \left| x_s^{(\mu, \xi)} - x_s^{(\mu, \xi)} \right|^2 ds + C \theta E |\eta_T - \xi_T|^2 + C \theta^2. \]

By using Gronwall’s lemma and Buckholder-Davis-Gundy inequality, we obtain the desired result. \( \blacksquare \)
Lemma 7 Let \( z \) be the solution of the linear SDE (called variational equation)

\[
\begin{align*}
\frac{dz}{dt} &= \int_{U_1} b \left( t, x_{t_{i_{1}}}^{(\mu, \xi)}, a \right) \mu_t (da) z_t dt \\
&+ \int_{U_1} \sigma \left( t, x_{t_{i_{1}}}^{(\mu, \xi)}, a \right) \mu_t (da) z_t dW_t \\
&+ \int_{U_1} \sigma \left( t, x_{t_{i_{1}}}^{(\mu, \xi)}, a \right) \mu_t (da) - \int_{U_1} \sigma \left( t, x_{t_{i_{1}}}^{(\mu, \xi)}, a \right) q_t (da) dW_t \\
&+ G_t d(\eta_t - \xi_t),
\end{align*}
\]

\( z_0 = 0 \).

Then, we have

\[
\lim_{\theta \to 0} \mathbb{E} \left[ \left\| \frac{x_{t_{i_{1}}}^{(\mu, \xi)}}{\theta} - x_{t_{i_{1}}}^{(\mu, \xi)} - z_t \right\|^2 \right] = 0. \tag{10}
\]

**Proof.** It is easy to see that

\[
\frac{x_{t_{i_{1}}}^{(\mu, \xi)}}{\theta} - x_{t_{i_{1}}}^{(\mu, \xi)} - z_t,
\]

does not depend on the singular part. Then the result follows immediately by the same method that in [5, Lemma 10, page 2086-2088].

Lemma 8 Let \((\mu, \xi)\) be an optimal relaxed-singular control minimizing the cost \( J \) over \( \mathcal{R} \) and \( x_{t_{i_{1}}}^{(\mu, \xi)} \) the associated optimal trajectory. Then, for any \((q, \eta) \in \mathcal{R}\), we have

\[
0 \leq \mathbb{E} \left[ g_x \left( x_{t_{i_{1}}}^{(\mu, \xi)} \right) z_T \right] + \mathbb{E} \int_0^T \int_{U_1} h_x \left( t, x_{t_{i_{1}}}^{(\mu, \xi)}, a \right) \mu_t (da) z_t dt \\
+ \mathbb{E} \int_0^T \int_{U_1} h \left( t, x_{t_{i_{1}}}^{(\mu, \xi)}, a \right) q_t (da) - \int_{U_1} h \left( t, x_{t_{i_{1}}}^{(\mu, \xi)}, a \right) \mu_t (da) dt \\
+ E \int_0^T k_t d(\eta_t - \xi_t). \tag{11}
\]

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Proof. Since \((\mu, \xi)\) minimizes the cost \(J\) over \(R\), then

\[
0 \leq J(\mu^0, \xi^0) - J(\mu, \xi)
\]

\[
\leq E \left[ g \left( x_T^{(\mu^0, \xi^0)} \right) - g \left( x_T^{(\mu, \xi)} \right) \right] + \mathbb{E} \int_0^T \int_{U_1} h \left( t, x_t^{(\mu^0, \xi^0)}, a \right) \mu_t^0 \left( da \right) dt - \mathbb{E} \int_0^T \int_{U_1} h \left( t, x_t^{(\mu, \xi)}, a \right) \mu_t^0 \left( da \right) dt + \mathbb{E} \int_0^T \int_{U_1} h \left( t, x_t^{(\mu, \xi)}, a \right) \mu_t \left( da \right) dt - \mathbb{E} \int_0^T \int_{U_1} h \left( t, x_t^{(\mu, \xi)}, a \right) \mu_t \left( da \right) dt + \mathbb{E} \int_0^T k_t d(\xi_t - \xi_t).
\]

By using the definition of \((\mu^0, \xi^0)\), we get

\[
0 \leq E \left[ g \left( x_T^{(\mu^0, \xi^0)} \right) - g \left( x_T^{(\mu, \xi)} \right) \right] + E \int_0^T \int_{U_1} h \left( t, x_t^{(\mu^0, \xi^0)}, a \right) \mu_t \left( da \right) dt - E \int_0^T \int_{U_1} h \left( t, x_t^{(\mu, \xi)}, a \right) \mu_t \left( da \right) dt + \theta E \int_0^T \int_{U_1} h \left( t, x_t^{(\mu^0, \xi^0)}, a \right) q_t \left( da \right) - E \int_0^T \int_{U_1} h \left( t, x_t^{(\mu, \xi)}, a \right) \mu_t \left( da \right) dt + \theta E \int_0^T k_t d(\eta_t - \xi_t).
\]

Hence,

\[
0 \leq E \int_0^1 g_x \left( x_T^{(\mu, \xi)} + \lambda \theta \left( X^\theta_T + z_T \right) \right) z_T d\lambda (12) + E \int_0^T \int_{U_1} h_x \left( t, x_t^{(\mu, \xi)} + \lambda \theta \left( X^\theta_t + z_t \right), a \right) \mu_t \left( da \right) z_t d\lambda dt + E \int_0^T \int_{U_1} h \left( t, x_t^{(\mu, \xi)}, a \right) q_t \left( da \right) - E \int_0^T \int_{U_1} h \left( t, x_t^{(\mu, \xi)}, a \right) \mu_t \left( da \right) dt + E \int_0^T k_t d(\eta_t - \xi_t) + \rho_t^0,
\]

where

\[
X^\theta_t = \frac{x_t^{(\mu^0, \xi^0)} - x_t^{(\mu, \xi)}}{\theta} - z_t,
\]

and \(\rho_t^0\) is given by

\[
\rho_t^0 = E \int_0^1 g_x \left( x_T^{(\mu, \xi)} + \lambda \theta \left( X^\theta_T + z_T \right) \right) X^\theta_T d\lambda + E \int_0^T \int_{U_1} h_x \left( t, x_t^{(\mu, \xi)} + \lambda \theta \left( X^\theta_t + z_t \right), a \right) \mu_t \left( da \right) X^\theta_t d\lambda dt.
\]
By (10), we have
\[ \lim_{\theta \to 0} \mathbb{E} \left| X_t^{\theta} \right|^2 = 0. \]

Since \( g_x \) and \( h_x \) are continuous and bounded, then by using the Cauchy-Schwartz inequality we get
\[ \lim_{\theta \to 0} \rho_t^\theta = 0, \]
and by letting \( \theta \) go to 0 in (12), the proof is completed.

### 3.2 Variational inequality and adjoint equation

In this subsection, we introduce the adjoint process. With this process, we derive the variational inequality from (11). The linear terms in (11) may be treated in the following way. Let \( \Phi \) be the fundamental solution of the linear SDE
\[
\begin{cases}
    d\Phi_t = \int_{U_1} b_x (t, x^{(\mu, \xi)}_t, a) \mu_t (da) \Phi_t dt + \int_{U_1} \sigma_x (t, x^{(\mu, \xi)}_t, a) \mu_t (da) \Phi_t dW_t, \\
    \Phi_0 = I_d.
\end{cases}
\]

This equation is linear with bounded coefficients. Hence, it admits an unique strong solution which is invertible, and its inverse \( \Psi_t \) is the unique solution of
\[
\begin{cases}
    d\Psi_t = \left[ \int_{U_1} \sigma_x (t, x^{(\mu, \xi)}_t, a) \mu_t (da) \Psi_t + \int_{U_1} \sigma^*_x (s, x^{(\mu, \xi)}_s, a) \mu_s (da) \Psi_t \right] dt \\
    \Psi_0 = I_d.
\end{cases}
\]

Moreover, \( \Phi \) and \( \Psi \) satisfy
\[ \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \Phi_t \right|^2 \right] + \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \Psi_t \right|^2 \right] < \infty. \quad (13) \]

We introduce the following processes
\[ \alpha_t = \Psi_t z_t, \quad (14) \]
\[ X = \Phi_T g_x (x_T^{(\mu, \xi)}) + \int_0^T \int_{U_1} \Phi_t h_x (t, x_t^{(\mu, \xi)}, a) \mu_t (da) dt, \quad (15) \]
\[ Y_t = \mathbb{E} [X / \mathcal{F}_t] - \int_0^t \int_{U_1} \Phi_s h_x (s, x_s^{(\mu, \xi)}, a) \mu_s (da) ds. \quad (16) \]

We remark from (14), (15) and (16) that
\[ \mathbb{E} [\alpha_T Y_T] = \mathbb{E} \left[ g_x (x_T^{(\mu, \xi)}) z_T \right]. \quad (17) \]
Since \( g_x \) and \( h_x \) are bounded, then by (13), \( X \) is square integrable. Hence, the process \( \mathbb{E}[X / \mathcal{F}_t] \) is a square integrable martingale with respect to the natural filtration of the Brownian motion \( W \). Then, by Itô’s representation theorem we have

\[
Y_t = \mathbb{E}[X] + \int_0^t Q_s dW_s - \int_0^t \int_{U_1} \Phi_s h_x \left( s, x_s^{(\mu, \xi)}, a \right) \mu_s \left( da \right) ds,
\]

where, \( Q \) is an adapted process such that \( \mathbb{E} \int_0^T |Q_s|^2 ds < \infty \).

By applying Itô’s formula to \( \alpha_t \) then with \( \alpha_t Y_t \) and using (17), the variational inequality (11) becomes

\[
0 \leq \mathbb{E} \left[ \int_0^T \left( \mathcal{H} \left( t, x_t^{(\mu, \xi)}, q_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)} \right) - \mathcal{H} \left( t, x_t^{(\mu, \xi)}, \mu_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)} \right) \right) dt 
+ \mathbb{E} \int_0^T (k_t + G_t^* p_t) d(\eta_t - \xi_t) \right],
\]

where, the Hamiltonian \( \mathcal{H} \) is defined from \([0, T] \times \mathbb{R}^n \times \mathbb{P} (U_1) \times \mathbb{R}^n \times \mathcal{M}_{n \times d} (\mathbb{R})\) into \( \mathbb{R} \) by

\[
\mathcal{H} (t, x, q, p, P) = \int_{U_1} h \left( t, x, a \right) q \left( da \right) + \int_{U_1} b \left( t, x, a \right) q \left( da \right) p + \int_{U_1} \sigma \left( t, x, a \right) q \left( da \right) P,
\]

\( (p^{(\mu, \xi)}, P^{(\mu, \xi)}) \) is a pair of adapted processes given by

\[
p_t^{(\mu, \xi)} = \Psi_t^* Y_t, \quad p_t^{(\mu, \xi)} \in \mathcal{L}^2 \left( [0, T]; \mathbb{R}^n \right)
\]

\[
P_t^{(\mu, \xi)} = \Psi_t^* Q_t - \int_{U_1} \sigma_t^* \left( t, x_t^{(\mu, \xi)}, a \right) \mu_t \left( da \right) \mu_t^{(\mu, \xi)}, \quad P^{(\mu, \xi)} \in \mathcal{L}^2 \left( [0, T]; \mathbb{R}^n \times \mathbb{R}^n \right),
\]

and the process \( Q \) satisfies

\[
\int_0^t Q_s dW_s = \mathbb{E} \left[ \Phi_t^* g_x \left( x_t^{(\mu, \xi)} \right) + \int_0^T \Phi_t^* \int_{U_1} h_x \left( t, x_t^{(\mu, \xi)}, a \right) \mu_t \left( da \right) dt / \mathcal{F}_t \right]
- \mathbb{E} \left[ \Phi_t^* g_x \left( x_t^{(\mu, \xi)} \right) + \int_0^T \Phi_t^* \int_{U_1} h_x \left( t, x_t^{(\mu, \xi)}, a \right) \mu_t \left( da \right) dt \right].
\]

The process \( p^{(\mu, \xi)} \) is called the adjoint process and from (15), (16) and (19), it is given explicitly by

\[
p_t^{(\mu, \xi)} = \mathbb{E} \left[ \Psi_t^* \Phi_t^* g_x \left( x_t^{(\mu, \xi)} \right) + \Psi_t^* \int_0^T \Phi_s^* h_x \left( s, x_s^{(\mu, \xi)}, a \right) \mu_s \left( da \right) ds / \mathcal{F}_t \right].
\]
By applying Itô’s formula to the adjoint processes \( p^{(\mu, \xi)} \) in (19), we obtain the adjoint equation, which is a linear backward SDE, given by

\[
\begin{align*}
\frac{dp_t^{(\mu, \xi)}}{dt} &= -\mathcal{H}(t, x_t^{(\mu, \xi)}, \mu_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)}) + P_t^{(\mu, \xi)} dW_t, \\
p_T^{(\mu, \xi)} &= g(x_T^{(\mu, \xi)}).
\end{align*}
\] (21)

### 3.3 Necessary optimality conditions for relaxed-singular controls

Starting from the variational inequality (18), we can now state the necessary optimality conditions, for the relaxed-singular control problem \{(5), (6), (7)\}, in integral form.

**Theorem 9** (Necessary optimality conditions for relaxed-singular controls in integral form). Let \((\mu, \xi)\) be an optimal relaxed-singular control minimizing the cost \( J \) over \( \mathcal{R} \) and \( x^{(\mu, \xi)} \) denotes the corresponding optimal trajectory. Then, there exists a unique pair of adapted processes

\[
\left( p^{(\mu, \xi)}, P^{(\mu, \xi)} \right) \in \mathcal{L}^2([0, T]; \mathbb{R}^n) \times \mathcal{L}^2([0, T]; \mathbb{R}^{n \times d}),
\]

solution of the backward SDE (21) such that, for every \((q, \eta) \in \mathcal{R}\)

\[
0 \leq \mathbb{E} \int_0^T \left[ \mathcal{H}(t, x_t^{(\mu, \xi)}, q_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)}) - \mathcal{H}(t, x_t^{(\mu, \xi)}, \mu_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)}) \right] dt
\]
\[
+ \mathbb{E} \int_0^T \left( k_t + G^*_t p_t^{(\mu, \xi)} \right) d(\eta_t - \xi_t),
\] (22)

**Proof.** The result follows immediately from (18). \( \blacksquare \)

We are ready now state necessary optimality conditions for the relaxed-singular control problem \{(5), (6), (7)\}, in global form.

**Theorem 10** (Necessary optimality conditions for relaxed-singular controls in global form). Let \((\mu, \xi)\) be an optimal relaxed-singular control minimizing the cost \( J \) over \( \mathcal{R} \) and \( x^{(\mu, \xi)} \) denotes the corresponding optimal trajectory. Then, there exists a unique pair of adapted processes

\[
\left( p^{(\mu, \xi)}, P^{(\mu, \xi)} \right) \in \mathcal{L}^2([0, T]; \mathbb{R}^n) \times \mathcal{L}^2([0, T]; \mathbb{R}^{n \times d}),
\]

solution of the backward SDE (21) such that

\[
\mathcal{H}(t, x_t^{(\mu, \xi)}, \mu_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)}) = \inf_{q_t \in \mathbb{P}(U_1)} \mathcal{H}(t, x_t^{(\mu, \xi)}, q_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)}), \text{ a.e., a.s.,}
\]
\[
\mathcal{P} \left\{ \forall t \in [0, T], \forall i : \left( k_i(t) + G^*_t(t) p_t^{(\mu, \xi)} \right) \geq 0 \right\} = 1,
\] (23) (24)
\[
\mathcal{P} \left\{ \sum_{i=1}^{d} 1 \{ k_i(t) + G_i(t)p_i(t) \geq 0 \} d\xi^i_t = 0 \right\} = 1.
\]

**Proof.** Let \((\mu, \xi)\) be an optimal solution of problem \{(5), (6), (7)\}. The necessary optimality conditions in integral form (22) is valid for every \((q, \eta) \in \mathcal{R}\). If we put in (22) \(\eta = \xi\), then (23) becomes immediately. On the other hand, if we choose in (22) \(q = \mu\), then we can show (24) and (25), by the same method that in [8, Theorem 4.2] or [4, Theorem 3.7]. \(\blacksquare\)

### 3.4 Sufficient optimality conditions for relaxed-singular controls

**Theorem 11 (Sufficient optimality conditions for relaxed-singular controls).** Assume that the functions \(g\) and \(x \mapsto \mathcal{H}(t, x, q, P)\) are convex. Then, \((\mu, \xi)\) is an optimal solution of problem \{(5), (6), (7)\}, if it satisfies (23), (24) and (25).

**Proof.** We know that the set of relaxed-singular controls \(\mathcal{R}\) is convex and the Hamiltonian \(\mathcal{H}\) is linear in \(q\).

Let \((\mu, \xi)\) be an arbitrary element of \(\mathcal{R}\) (candidate to be optimal). For any \((q, \eta) \in \mathcal{R}\), we have

\[
\mathcal{J}(\mu, \xi) - \mathcal{J}(q, \eta) = \mathbb{E} \left[ g \left( x_T^{(\mu,\xi)} \right) - g \left( x_T^{(q,\eta)} \right) \right] \\
+ \mathbb{E} \int_0^T \left[ \int_{\mathcal{U}_1} h \left( t, x_t^{(\mu,\xi)}, a \right) \mu_t(da) - \int_{\mathcal{U}_1} h \left( t, x_t^{(q,\eta)}, a \right) q_t(da) \right] dt \\
+ \mathbb{E} \int_0^T k_t d(\xi_t - \eta_t).
\]

Since \(g\) is convex, we get

\[
g \left( x_T^{(q,\eta)} \right) - g \left( x_T^{(\mu,\xi)} \right) \geq g_x \left( x_T^{(\mu,\xi)} \right) \left( x_T^{(\mu,\xi)} - x_T^{(q,\eta)} \right).
\]

Thus,

\[
g \left( x_T^{(\mu,\xi)} \right) - g \left( x_T^{(q,\eta)} \right) \leq g_x \left( x_T^{(\mu,\xi)} \right) \left( x_T^{(\mu,\xi)} - x_T^{(q,\eta)} \right).
\]

Hence,

\[
\mathcal{J}(\mu, \xi) - \mathcal{J}(q, \eta) \leq \mathbb{E} \left[ g_x \left( x_T^{(\mu,\xi)} \right) \left( x_T^{(\mu,\xi)} - x_T^{(q,\eta)} \right) \right] \\
+ \mathbb{E} \int_0^T \left[ \int_{\mathcal{U}_1} h \left( t, x_t^{(\mu,\xi)}, a \right) \mu_t(da) - \int_{\mathcal{U}_1} h \left( t, x_t^{(q,\eta)}, a \right) q_t(da) \right] dt \\
+ \mathbb{E} \int_0^T k_t d(\xi_t - \eta_t).
\]
We remark that $p_T^{(\mu, \xi)} = g_x \left( x_T^{(\mu, \xi)} \right)$, then we have

\[
\mathcal{J}(\mu, \xi) - \mathcal{J}(q, \eta) \leq \mathbb{E} \left[ p_T^{(\mu, \xi)} \left( x_T^{(\mu, \xi)} - x_T^{(q, \eta)} \right) \right]
\]

\[
+ \mathbb{E} \int_0^T \left[ \int_{U_1} h \left( t, x_t^{(\mu, \xi)}, a \right) \mu_t(da) - \int_{U_1} h \left( t, x_t^{(q, \eta)}, a \right) q_t(da) \right] dt
\]

\[
+ \mathbb{E} \int_0^T k_t d(\xi_t - \eta_t).
\]

Applying Itô’s formula to $p_t^{(\mu, \xi)} \left( x_t^{(\mu, \xi)} - x_t^{(q, \eta)} \right)$ and taking expectation, we obtain

\[
\mathcal{J}(\mu, \xi) - \mathcal{J}(q, \eta)
\]

\[
\leq \mathbb{E} \int_0^T \left[ \mathcal{H} \left( t, x_t^{(\mu, \xi)}, \mu_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)} \right) - \mathcal{H} \left( t, x_t^{(q, \eta)}, q_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)} \right) \right] dt
\]

\[
- \mathbb{E} \int_0^T \mathcal{H}_x \left( t, x_t^{(\mu, \xi)}, \mu_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)} \right) \left( x_t^{(\mu, \xi)} - x_t^{(q, \eta)} \right) dt
\]

\[
+ \mathbb{E} \int_0^T \left( k_t + G_t^* p_t^{(\mu, \xi)} \right) d(\xi_t - \eta_t).
\]

Since $\mathcal{H}$ is convex in $x$ and linear in $\mu$, then by using the Clarke generalized gradient of $\mathcal{H}$ evaluated at $(x_t, \mu_t)$ and (23), it follows that

\[
\mathcal{H} \left( t, x_t^{(\mu, \xi)}, \mu_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)} \right) - \mathcal{H} \left( t, x_t^{(\mu, \xi)}, \mu_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)} \right)
\]

\[
\geq \mathcal{H}_x \left( t, x_t^{(\mu, \xi)}, \mu_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)} \right) \left( x_t^{(\mu, \xi)} - x_t^{(q, \eta)} \right).
\]

Or equivalently

\[
0 \geq \mathcal{H} \left( t, x_t^{(\mu, \xi)}, \mu_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)} \right) - \mathcal{H} \left( t, x_t^{(q, \eta)}, q_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)} \right)
\]

\[
- \mathcal{H}_x \left( t, x_t^{(\mu, \xi)}, \mu_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)} \right) \left( x_t^{(\mu, \xi)} - x_t^{(q, \eta)} \right).
\]

By the above inequality and (26), we have

\[
\mathcal{J}(\mu, \xi) - \mathcal{J}(q, \eta) \leq \mathbb{E} \int_0^T \left( k_t + G_t^* p_t^{(\mu, \xi)} \right) d(\xi_t - \eta_t).
\]

By (24) and (25), we show that

\[
\mathbb{E} \int_0^T \left( k_t + G_t^* p_t^{(\mu, \xi)} \right) d(\eta_t - \xi_t) \geq 0.
\]

Then, we get

\[
\mathcal{J}(\mu, \xi) - \mathcal{J}(q, \eta) \leq 0.
\]

The theorem is proved. ■
4 Optimality conditions for strict-singular controls

In this section, we study the strict-singular control problem \{ (1), (2), (3) \} and from the results of section 3, we derive the optimality conditions for strict-singular controls.

Throughout this section and in addition to the assumptions (4), we suppose that

\[ U_1 \text{ is compact,} \]
\[ b, \sigma \text{ and } h \text{ are bounded.} \tag{27} \]

Consider the following subset of \( \mathbb{R} \delta (U_1) \times U_2 \)
\[ \delta (U_1) \times U_2 = \{(q, \eta) \in \mathbb{R} \mid q = \delta_v; \quad v \in U_1\}. \]

Denote by \( \delta (U_1) \times U_2 \) the action set of all relaxed-singular controls in \( \delta (U_1) \times U_2 \).

If \( (q, \eta) \in \delta (U_1) \times U_2 \), then \( (q, \eta) = (\delta_v, \eta) \) with \( v \in U_1 \). In this case we have for each \( t \), \( (q_t, \eta_t) = (\delta_{v_t}, \eta_t) \in \delta (U_1) \times U_2 \).

We equipped \( \mathbb{P} (U_1) \) with the topology of stable convergence. Since \( U_1 \) is compact, then with this topology \( \mathbb{P} (U_1) \) is a compact metrizable space. The stable convergence is required for bounded measurable functions \( f (t, a) \) such that for each fixed \( t \in [0, T] \), \( f (t, \cdot) \) is continuous (Instead of functions bounded and continuous with respect to the pair \( (t, a) \) for the weak topology). The space \( \mathbb{P} (U_1) \) is equipped with its Borel \( \sigma \)-field, which is the smallest \( \sigma \)-field such that the mapping \( q \mapsto \int f (s, a) q (ds, da) \) are measurable for any bounded measurable function \( f \), continuous with respect to \( a \). For more details, see Jacod and Memin [20] and El Karoui et al [15].

This allows us to summarize some of lemmas that we will be used in the sequel.

**Lemma 12** (Chattering Lemma). Let \( q \) be a predictable process with values in the space of probability measures on \( U_1 \). Then there exists a sequence of predictable processes \( (u^n) \) with values in \( U_1 \) such that

\[ d t q^n_t (da) = d t \delta_{u^n_t} (da) \quad \rightarrow \quad d t q_t (da) \text{ stably, } \mathbb{P} - a.s. \quad (29) \]

where \( \delta_{u^n_t} \) is the Dirac measure concentrated at a single point \( u^n_t \) of \( U_1 \).

**Proof.** See El Karoui et al [15].
Lemma 13  Let $q \in \mathcal{R}_1$ and $(u^n)_n \subset \mathcal{U}_1$ such that (29) holds. Then for any bounded measurable function $f : [0, T] \times U_1 \to \mathbb{R}$, such that for each fixed $t \in [0, T]$, $f(t, \cdot)$ is continuous, we have

$$
\int_{U_1} f(t, a) \delta_{u^n_t}(da) \to_{n \to \infty} \int_{U_1} f(t, a) q_t(da) ; \ dt - a.e.
$$

(30)

**Proof.**  By (29), and the definition of the stable convergence (see Jacod and Memin [20, definition 1.1, page 529], we have

$$
\int_0^T \int_{U_1} f(t, a) \delta_{u^n_t}(da) dt \to_{n \to \infty} \int_0^T \int_{U_1} f(t, a) q_t(da) dt.
$$

Put

$$
g(s, a) = 1_{[0, s]}(s) f(s, a).
$$

It’s clear that $g$ is bounded, measurable and continuous with respect to $a$. Then, by (29) we get

$$
\int_0^T \int_{U_1} g(s, a) \delta_{u^n_t}(da) ds \to_{n \to \infty} \int_0^T \int_{U_1} g(s, a) q_s(da) ds.
$$

By replacing $g(s, a)$ by its value, we have

$$
\int_0^t \int_{U_1} f(s, a) \delta_{u^n_s}(da) ds \to_{n \to \infty} \int_0^t \int_{U_1} f(s, a) q_s(da) ds.
$$

The set $\{(s, t) : 0 \leq s \leq t \leq T\}$ generate $\mathcal{B}_{[0, T]}$. Then for every $B \in \mathcal{B}_{[0, T]}$, we have

$$
\int_B \int_{U_1} f(s, a) \delta_{u^n_s}(da) ds \to_{n \to \infty} \int_B \int_{U_1} f(s, a) q_s(da) ds.
$$

This implies that

$$
\int_{U_1} f(s, a) \delta_{u^n_s}(da) \to_{n \to \infty} \int_{U_1} f(s, a) q_s(da) , \ dt - a.e.
$$

The lemma is proved. ■

The next lemma gives the stability of the controlled SDE with respect to the control variable.

Lemma 14  Let $(q, \eta) \in \mathcal{R}$ be a relaxed-singular control and $x^{(q, \eta)}$ the corresponding trajectory. Then there exists a sequence $(u^n, \eta)_n \subset \mathcal{U}$ such that

$$
\lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \in [0, T]} \left| x_t^{(u^n, \eta)} - x_t^{(q, \eta)} \right|^2 \right] = 0,
$$

(31)

$$
\lim_{n \to \infty} J(u^n, \eta) = J(q, \eta).
$$

(32)

where $x^{(u^n, \eta)}$ denotes the solution of equation (1) associated with $(u^n, \eta)$. 

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Proof. i) Proof of (31). We have

\[
\mathbb{E}\left|x_t^{(u_n, \eta)} - x_t^{(q, \eta)}\right|^2 \leq C \int_0^t \mathbb{E}\left|b\left(s, x_s^{(u_n, \eta)}, u_s^n\right) - b\left(s, x_s^{(q, \eta)}, u_s^q\right)\right|^2 ds
\]

\[
+ C \int_0^t \mathbb{E}\left|b\left(s, x_s^{(q, \eta)}, u_s^q\right) - \int_{U_1} b\left(s, x_s^{(q, \eta)}, a\right) q_s(da)\right|^2 ds
\]

\[
+ C \int_0^t \mathbb{E}\left|\sigma\left(s, x_s^{(u_n, \eta)}, u_s^n\right) - \sigma\left(s, x_s^{(q, \eta)}, u_s^q\right)\right|^2 ds
\]

\[
+ C \int_0^t \mathbb{E}\left|\sigma\left(s, x_s^{(q, \eta)}, u_s^q\right) - \int_{U_1} \sigma\left(s, x_s^{(q, \eta)}, a\right) q_s(da)\right|^2 ds
\]

Since \(b\) and \(\sigma\) are uniformly Lipschitz with respect to \(x\), then

\[
\mathbb{E}\left|x_t^{(u_n, \eta)} - x_t^{(q, \eta)}\right|^2 \leq C \int_0^t \mathbb{E}\left|x_s^{(u_n, \eta)} - x_s^{(q, \eta)}\right|^2 ds
\]

\[
+ C \int_0^t \mathbb{E}\left|\int_{U_1} b\left(s, x_s^{(q, \eta)}, a\right) \delta_{u_s^n} (da) - \int_{U_1} b\left(s, x_s^{(q, \eta)}, a\right) q_s(da)\right|^2 ds
\]

\[
+ C \int_0^t \mathbb{E}\left|\int_{U_1} \sigma\left(s, x_s^{(q, \eta)}, a\right) \delta_{u_s^n} (da) - \int_{U_1} \sigma\left(s, x_s^{(q, \eta)}, a\right) q_s(da)\right|^2 ds
\]

Since \(b\) and \(\sigma\) are bounded, measurable and continuous with respect to \(a\), then by (30) and the dominated convergence theorem, the second and third terms in the right hand side of the above inequality tend to zero as \(n\) tends to infinity. We conclude then by using Gronwall’s lemma and Bukholder-Davis-Gundy inequality.

ii) Proof of (32). By using the Cauchy-Schwartz inequality and the fact that \(g\) and \(h\) are uniformly Lipschitz with respect to \(x\), we get

\[
|J (q^n, \eta) - J (q, \eta)|
\]

\[
\leq C \left(\mathbb{E}\left|x_T^{(u_n, \eta)} - x_T^{(q, \eta)}\right|^2\right)^{1/2} + C \left(\int_0^T \mathbb{E}\left|x_t^{(u_n, \eta)} - x_t^{(q, \eta)}\right|^2 dt\right)^{1/2}
\]

\[
+ \left(\mathbb{E}\int_0^T \left|\int_{U_1} h\left(s, x_s^{(q, \eta)}, a\right) \delta_{u_s^n} (da) dt - \int_{U_1} h\left(t, x_t^{(q, \eta)}, a\right) q_t (da)\right|^2 dt\right)^{1/2}.
\]

By (31), the first and second terms in the right hand side converge to zero. Moreover, since \(h\) is bounded, measurable and continuous in \(a\), then by (30) and the dominated convergence theorem, the third term in the right hand side tends to zero as \(n\) tends to infinity. This prove (32). ■
Lemma 15 As a consequence of (32), the strict-singular and the relaxed-singular control problems have the same value functions. That is
\[
\inf_{(v,\eta) \in \mathcal{U}} J(v, \eta) = \inf_{(q,\eta) \in \mathcal{R}} J(q, \eta).
\] (34)

Proof. Let \((u, \xi) \in \mathcal{U}\) and \((\mu, \xi) \in \mathcal{R}\) be respectively a strict-singular and relaxed-singular controls such that
\[
J(u, \xi) = \inf_{(v,\eta) \in \mathcal{U}} J(v, \eta),
\]
\[
J(\mu, \xi) = \inf_{(q,\eta) \in \mathcal{R}} J(q, \eta).
\] (35) (36)

By (36), we have
\[
J(\mu, \xi) \leq J(q, \eta), \quad \forall (q, \eta) \in \mathcal{R}.
\]

Since \(\delta(\mathcal{U}_1) \times \mathcal{U}_2 \subset \mathcal{R}\), then
\[
J(\mu, \xi) \leq J(q, \eta), \quad \forall (q, \eta) \in \delta(\mathcal{U}_1) \times \mathcal{U}_2.
\]

Since \((q, \eta) \in \delta(\mathcal{U}_1) \times \mathcal{U}_2\), then \((q, \eta) = (\delta_v, \eta)\), where \(v \in \mathcal{U}_1\). Then we get
\[
\begin{align*}
x^{(q, \eta)} &= x^{(v, \eta)}, \\
J(q, \eta) &= J(v, \eta).
\end{align*}
\]

Hence
\[
J(\mu, \xi) \leq J(v, \eta), \quad \forall (v, \eta) \in \mathcal{U}.
\]

The control \((u, \xi)\) becomes an element of \(\mathcal{U}\), then we get
\[
J(\mu, \xi) \leq J(u, \xi).
\] (37)

On the other hand, by (35) we have
\[
J(u, \xi) \leq J(v, \eta), \quad \forall (v, \eta) \in \mathcal{U}.
\] (38)

The process \(\mu\) becomes an element of \(\mathcal{R}_1\), then by the Chattering lemma (Lemma 12), there exists a sequence \((u^n)\) such that
\[
\frac{dt\mu^n_t}{(da)} = \delta_{u^n_t}(da) \longrightarrow \frac{dt\mu_t}{(da)} \text{ stably, } \mathcal{P} - \text{a.s.}
\]

Relation (38) holds for every \((v, \eta) \in \mathcal{U}\). This is true for \((u^n, \xi) \in \mathcal{U}, \forall n \in \mathbb{N}\). We get then
\[
J(u, \xi) \leq J(u^n, \xi), \quad \forall n \in \mathbb{N},
\]

By using (32) and letting \(n\) go to infinity in the above inequality, we get
\[
J(u, \xi) \leq J(\mu, \xi).
\] (39)
Finally, by (37) and (39), we have

\[ J(u, \xi) = J(\mu, \xi). \]

The lemma is proved. ■

To establish necessary optimality conditions for strict-singular controls, we need the following lemma

**Lemma 16** The strict-singular control \((u, \xi)\) minimizes \(J\) over \(U\) if and only if the relaxed-singular control \((\mu, \xi) = (\delta_u, \xi)\) minimizes \(J\) over \(R\).

**Proof.** Suppose that \((u, \xi)\) minimizes the cost \(J\) over \(U\), then

\[ J(u, \xi) = \inf_{(v, \eta) \in U} J(v, \eta). \]

By using (34), we get

\[ J(u, \xi) = \inf_{(q, \eta) \in R} J(q, \eta). \]

Since \((\mu, \xi) = (\delta_u, \xi)\), then

\[
\begin{cases}
  x^{(\mu, \xi)} = x^{(u, \xi)}, \\
  J(\mu, \xi) = J(u, \xi),
\end{cases}
\]

This implies that

\[ J(\mu, \xi) = \inf_{(q, \eta) \in R} J(q, \eta). \]

Conversely, if \((\mu, \xi) = (\delta_u, \xi)\) minimize \(J\) over \(R\), then

\[ J(\mu, \xi) = \inf_{(q, \eta) \in R} J(q, \eta). \]

From (34), we get

\[ J(\mu, \xi) = \inf_{(v, \eta) \in U} J(v, \eta). \]

Since \((\mu, \xi) = (\delta_u, \xi)\), then relations (40) hold, and we obtain

\[ J(u, \xi) = \inf_{(v, \eta) \in U} J(v, \eta). \]

The proof is completed. ■

The following lemma, who will be used to establish sufficient optimality conditions for strict-singular controls, shows that we get the results of the above lemma if we replace \(R\) by \(\delta(U_1) \times U_2\).

**Lemma 17** The strict-singular control \((u, \xi)\) minimizes \(J\) over \(U\) if and only if the relaxed control \((\mu, \xi) = (\delta_u, \xi)\) minimizes \(J\) over \(\delta(U_1) \times U_2\).
Proof. Let \((\mu, \xi) = (\delta_u, \xi)\) be an optimal relaxed-singular control minimizing the cost \(J\) over \(\delta(U_1) \times U_2\), we have then
\[
J(\mu, \xi) \leq J(q, \eta), \quad \forall (q, \eta) \in \delta(U_1) \times U_2.
\] (41)

Since \(q \in \delta(U_1)\), then there exists \(v \in U_1\) such that \(q = \delta_v\). Hence, \((\delta_v, \eta) = (q, \eta)\), and since \((\mu, \xi) = (\delta_u, \xi)\), it is easy to see that
\[
\begin{align*}
&x^{(\mu, \xi)} = x^{(u, \xi)}, \\
x^{(q, \eta)} = x^{(v, \eta)}, \\
&J(\mu, \xi) = J(u, \xi), \\
&J(q, \eta) = J(v, \eta).
\end{align*}
\] (42)

By (41) and (42), we get then
\[
J(u, \xi) \leq J(v, \eta), \quad \forall (v, \eta) \in U.
\]

Conversely, let \((u, \xi)\) be a strict-singular control minimizing the cost \(J\) over \(U\). Then
\[
J(u, \xi) \leq J(v, \eta), \quad \forall (v, \eta) \in U.
\]

Since the controls \(u, v \in U_1\), then there exist \(\mu, q \in \delta(U_1)\) such that \(\mu = \delta_u\) and \(q = \delta_v\). Then
\[
(\mu, \xi) = (\delta_u, \xi), \\
(q, \eta) = (\delta_v, \eta).
\]

This implies that relations (42) hold. Consequently, we get
\[
J(\mu, \xi) \leq J(q, \eta), \quad \forall (q, \eta) \in \delta(U_1) \times U_2.
\]

The proof is completed. \(\blacksquare\)

4.1 Necessary optimality conditions for strict-singular controls

Define the Hamiltonian in the strict case from \([0, T] \times \mathbb{R}^n \times U_1 \times \mathbb{R}^n \times \mathcal{M}_{n \times d} (\mathbb{R})\) into \(\mathbb{R}\) by
\[
H(t, x, v, p, P) = h(t, x, v) + b(t, x, v) p + \sigma(t, x, v) P.
\]

Theorem 18 (Necessary optimality conditions for strict-singular controls in global form). Suppose that \((u, \xi)\) is an optimal strict-singular control minimizing the cost \(J\) over \(U\) and \(x^{(u, \xi)}\) denotes the solution of (1) controlled by \((u, \xi)\). Then, there exists an unique pair of adapted processes
\[
\left( p^{(u, \xi)}, P^{(u, \xi)} \right) \in L^2([0, T]; \mathbb{R}^n) \times L^2([0, T]; \mathbb{R}^{n \times d}),
\]
solution of the backward SDE

\[
\begin{align*}
    dp_t^{(u, \xi)} &= -H_x \left( t, x_t^{(u, \xi)}, u_t, p_t^{(u, \xi)}, P_t^{(u, \xi)} \right) dt + P_t^{(u, \xi)} dW_t, \\
p_t^{(u, \xi)} &= g_x(x_t^{(u, \xi)}),
\end{align*}
\]

such that

\[
H \left( t, x_t^{(u, \xi)}, u_t, p_t^{(u, \xi)}, P_t^{(u, \xi)} \right) = \inf_{v_t \in \mathcal{U}_t} H \left( t, x_t^{(u, \xi)}, v_t, p_t^{(u, \xi)}, P_t^{(u, \xi)} \right), \text{ a.e., a.s.}
\]

\[
\mathcal{P} \left\{ \forall t \in [0, T], \forall i : \left( k_i(t) + G_i^* (t).p_t^{(u, \xi)} \right) \geq 0 \right\} = 1, \quad (44)
\]

\[
\mathcal{P} \left\{ \sum_{i=1}^{d} 1 \left\{ k_i(t) + G_i^* (t).p_t^{(u, \xi)} \geq 0 \right\} d\xi_t = 0 \right\} = 1. \quad (46)
\]

**Proof.** The optimal strict-singular control \((u, \xi)\) is an element of \(\mathcal{U}\), then there exists \((\mu, \xi) \in \delta (\mathcal{U}_1) \times \mathcal{U}_2\) such that

\[(\mu, \xi) = (\delta_u, \xi).\]

Since \((u, \xi)\) minimizes the cost \(J\) over \(\mathcal{U}\), then by lemma 16, \((\mu, \xi)\) minimizes \(J\) over \(\mathcal{R}\). Hence, by the necessary optimality conditions for relaxed-singular controls (Theorem 10), there exists an unique pair of adapted processes \((p^{(\mu, \xi)}, P^{(\mu, \xi)})\), solution of (21), such that relations (23), (24) and (25) hold.

Since \(\delta (\mathcal{U}_1) \subset \mathbb{P}(\mathcal{U}_1)\), then by (23), we get

\[
\mathcal{H} \left( t, x_t^{(\mu, \xi)}, \mu_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)} \right) \leq \mathcal{H} \left( t, x_t^{(\mu, \xi)}, q_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)} \right), \quad \forall q_t \in \delta (\mathcal{U}_1), \text{ a.e., a.s.} \quad (47)
\]

Since \(q \in \delta (\mathcal{U}_1)\), then there exists \(v \in \mathcal{U}_1\) such that \(q = \delta_u\).

We note that \(v\) is an arbitrary element of \(\mathcal{U}_1\) since \(q\) is arbitrary.

Now, since \((\mu, \xi) = (\delta_u, \xi)\) and \((q, \xi) = (\delta_v, \xi)\), we can easily see that

\[
\begin{align*}
    x^{(\mu, \xi)} &= x^{(u, \xi)}, \\
x^{(q, \xi)} &= x^{(v, \xi)}, \\
\mathcal{H} \left( t, x_t^{(\mu, \xi)}, \mu_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)} \right) &= \mathcal{H} \left( t, x_t^{(u, \xi)}, u_t, p_t^{(u, \xi)}, P_t^{(u, \xi)} \right), \\
\mathcal{H} \left( t, x_t^{(\mu, \xi)}, q_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)} \right) &= \mathcal{H} \left( t, x_t^{(u, \xi)}, v_t, p_t^{(u, \xi)}, P_t^{(u, \xi)} \right),
\end{align*}
\]

where \((p^{(u, \xi)}, P^{(u, \xi)})\) is the unique solution of (43).

By using (47) and (48), we deduce (44). Relations (45) and (46) follows immediately from (24), (25) and (48). The proof is completed. \(\blacksquare\)

**Remark 19** Bahlali and Mezerdi [2], established necessary optimality conditions for strict-singular controls of the second-order with two adjoint processes. The result of the above theorem improves that of [2], in the sense where, we consider the same strict-singular control problem, with nonconvex control domain and a general state equation in which the control variable enters both the drift and the diffusion coefficients, and we establish necessary optimality conditions of the first-order with only one adjoint process.
4.2 Sufficient optimality conditions for strict-singular controls

Theorem 20 (Sufficient optimality conditions for strict-singular controls). Assume that the functions $g$ and $x \mapsto H(t, x, q, p, P)$ are convex. Then, $(u, \xi)$ is an optimal solution of problem $(1), (2), (3)$ if it satisfies $(44), (45)$ and $(46)$.

Proof. Let $(u, \xi) \in U$ be a strict-singular control (candidate to be optimal) and $(v, \eta)$ an arbitrary element of $U$.

The controls $u, v$ are elements of $U_1$, then there exist $\mu, q \in \delta(U_1)$ such that $\mu = \delta_u$ and $q = \delta_v$. Hence,

$$(\mu, \xi) = (\delta_u, \xi),$$

$$(q, \eta) = (\delta_v, \eta).$$

This implies that relations $(48)$ hold. Then, by $(44), (45)$ and $(46)$, we deduce respectively the relaxed relations

$$H \left( t, x_t^{(\mu, \xi)}, \mu_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)} \right) = \inf_{q_t \in \delta(U_1)} H \left( t, x_t^{(\mu, \xi)}, q_t, p_t^{(\mu, \xi)}, P_t^{(\mu, \xi)} \right), \text{ a.e, a.s,}$$

$$P \left\{ \forall t \in [0, T], \forall i : \left( k_i(t) + G_i^*(t)p_t^{(\mu, \xi)} \right) \geq 0 \right\} = 1, \quad \text{(50)}$$

$$P \left\{ \sum_{i=1}^{d} 1 \{ k_i(t) + G_i^*(t)p_t^{(\mu, \xi)} \geq 0 \} d\xi_t^i = 0 \right\} = 1. \quad \text{(51)}$$

We remark that the infimum in $(49)$ is taken over $\delta(U_1)$.

Now, since $H$ is convex in $x$, it is easy to see that $H$ is convex in $x$, and since $g$ is convex, then by using $(49), (50)$ and $(51)$, and by the same proof that in theorem 11, we show that $(\mu, \xi)$ minimizes the cost $J$ over $\delta(U_1) \times U_2$. Then, by Lemma 17, we deduce that $(u, \xi)$ minimizes the cost $J$ over $U$. The theorem is proved. \qed

Remark 21 The sufficient optimality conditions for strict-singular controls are proved without assuming neither the convexity of $U_1$ nor that of the Hamiltonian $H$ in $v$.

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