Existence and Stability of Static Spherical Fluid Shells in a
Schwarzschild-Rindler-anti-de Sitter Metric

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We demonstrate the existence of static stable spherical fluid shells in the Schwarzschild-Rindler-anti-de Sitter (SRAdS) spacetime where $ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2(d\theta^2 + \sin^2 \theta d\phi^2)$ with $f(r) = 1 - \frac{2\rho}{r} + 2br - \frac{\Lambda}{3}r^2$. This is an alternative to the well known gravastar geometry where the stability emerges due to the combination of the repulsive forces of the interior de Sitter space with the attractive forces of the exterior Schwarzschild spacetime. In the SRAdS spacetime the repulsion that leads to stability of the shell comes from a negative Rindler term while the Schwarzschild and anti-de Sitter terms are attractive. We demonstrate the existence of such stable spherical shells for three shell fluid equations of state: vacuum shell ($p = -\sigma$), stiff matter shell ($p = \sigma$) and dust shell ($p = 0$) where $p$ is the shell pressure and $\sigma$ is the shell surface density. We also identify the metric parameter conditions that need to be satisfied for shell stability in each case. The vacuum stable shell solution in the SRAdS spacetime is consistent with previous studies by two of the authors that demonstrated the existence of stable spherical scalar field domain walls in the SRAdS spacetime.

I. INTRODUCTION

Thin spherical shells in GR are 2+1 boundary hypersurfaces with energy momentum tensor $S^{ij} = f_R^{-1} T^i_j dr = \text{diag}(-\sigma, p, p)$, where $R$ is the shell radius, $r$ is the radial coordinate of the 3+1 dimensional spacetime, $\sigma$ is the surface energy density and $p$ is the surface pressure on the shell hypersurface with equation of state $p = p(\sigma)$. The thin shell interpolates between an interior and an exterior spherically symmetric metric. The exterior metric is related to the interior metric in the context of the Israel junction conditions [1–3].

A well known spherical static stable thin shell configuration corresponds to the gravastar that interpolates between an interior de Sitter metric and an exterior Schwarzschild metric and constitutes an extension of the Schwarzschild metric with eliminated singularity [4–8].

An alternative thin shell solution obtained using spherically symmetric scalar field dynamical equations in a non-trivial background geometry has been obtained in Ref. [9]. It was demonstrated that static metastable solutions can exist in the presence of a Schwarzschild-anti-deSitter curved spacetime [9, 10] supplemented with the Rindler acceleration term. Thus the total metric is a Schwarzschild-Rindler-anti-de Sitter (SRAdS) metric [11],

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

(1.1)

where $b$ is the Rindler acceleration parameter and $\Lambda$ is the cosmological constant.

The metric (1.1) has been constrained by solar system observations, indicating that $|b| < 3 \text{mm/}sec^2$ [12, 13] and it has been shown that it can lead to the production of flat rotation curves as well as contribute to the explanation [14, 15] of the Pioneer anomaly [16, 17] for $b > 0$. It also emerges generically [14] as a vacuum solution of spherically symmetric scalar-tensor theories, in GR when one considers the presence of a spherically symmetric fluid with an equation of state $\rho = -p_r = -\rho_0 = -b_0$ [9] and a black hole in the center. It also appears as a vacuum solution in conformal (Weyl) gravity [18].

In view of these interesting features of the SRAdS metric it is intriguing to identify possible additional properties of this metric with possible observational effects. In particular, the property of the metric to support metastable spherical domain walls motivates the search of additional stable shell solutions described as general fluid thin shells as opposed to scalar field vacuum energy shells (domain walls). Such an analysis would be based on the Israel junction conditions formalism as opposed to the solution of dynamical scalar field equations. The following questions therefore emerge:

- Are there static, stable fluid shell solutions in a SRAdS background geometry?
- If yes, what are the conditions for their stability given the equation of state of the fluid shell?
- What is the metric parameter range for shell stability and how does the stability radius change as a function of these parameters?

These questions will be addressed in the present analysis. We implement the Israel junction conditions in the context of a fixed equation of state of the fluid shell and a SRAdS background metric with a discontinuous value of $m$ across the shell and fixed values of $b$ and $\Lambda$ with no discontinuity as the shell is crossed. We thus derive the stability conditions and identify the range of metric
parameters $b, \Lambda$, that satisfy these conditions for given values of the shell coordinate radius $R$, shell surface density $\sigma$ and mass parameters inside and outside the shell $(m_-, m_+)$. The conditions that need to be satisfied for stability by the shell density and shell radius are also determined.

The structure of this paper is the following: In the next section we develop the general formalism for the derivation of stability conditions by implementing the Israel junction conditions on the SRAAdS metric for a shell with a general fluid equation of state. In section III we consider three specific applications of the method for corresponding shell fluid equations of state: vacuum shell, stiff matter shell and matter shell and find the particular stability conditions and parameter regions in each case. Finally in section IV we conclude summarize and discuss possible extensions of this analysis. In what follows we set $G = c = 1$. In most cases we will also set the interior mass parameter $m_-$ to 1.

II. THIN SHELLS: EXISTENCE AND STABILITY

Consider a thin spherical shell with coordinate radius $R$ interpolating between an interior (-) and an exterior (+). Let the interior and exterior metrics be of the form $[4, 5, 19],\,$

$$ds^2 = f_\pm(r_\pm)dt^2 - \frac{dr_\pm^2}{f_\pm(r_\pm)} - r_\pm^2(d\theta^2 + \sin^2\theta d\phi^2)$$

(2.1)

where,

$$f_\pm(r_\pm) = 1 - \frac{2m_\pm(r_\pm)}{r_\pm}$$

(2.2)

and

$$m_\pm(r_\pm) = m_\pm - br_\pm^2 + \frac{\Lambda}{6}r_\pm^3.$$  

(2.3)

We now impose the following conditions:

1. Continuity of the metric on the shell $(r_- = r_+ = R)$. This implies

$$f_+(r_+)dt_+^2 - \frac{dr_+^2}{f_+(r_+)} = f_-(r_-)dt_-^2 - \frac{dr_-^2}{f_-(r_-)},$$

(2.4)

which leads to

$$t_- = f_+(R) t_+$$

(2.5)

$$\frac{dr_-}{dr_+} = f_-(R).$$

(2.6)

2. The Israel junction conditions [1] expressed through a discontinuity of the extrinsic curvature on the shell hypersurface $\Sigma$. The extrinsic curvature (second fundamental form) at either side of the three-dimensional $(2+1)$ hypersurface $\Sigma$ swept by a spherically symmetric shell, embedded in the four-dimensional spacetime are:

$$K^\pm_{ij} = \left( h_{\mu\nu}^\pm \frac{dx^\mu}{dx^i} \frac{dx^\nu}{dx^j} \right)_{\Sigma},$$

(2.7)

where $x^i$ are coordinates on $\Sigma$, $h_{\mu\nu}^\pm = g_{\mu\nu} - n_\mu n_\nu$, and $(:)$ denote covariant derivative with respect to $g_{\mu\nu}^\pm$. The unit 4-normals to $\Sigma$ in the four-dimensional spacetime are given by [3],

$$n_\alpha = \pm \left( g^{\beta\gamma} \frac{\partial g}{\partial x^\beta} \frac{\partial g}{\partial x^\gamma} \right)^{-1/2} \frac{\partial g}{\partial x^\alpha},$$

(2.8)

where the parametric equation for $\Sigma$ is of the form

$$g(x^\alpha(x^i)) = 0.$$  

(2.9)

We assume $n_\alpha \neq 0$ and label $\Sigma$ as timelike for $n_\alpha n^\alpha = 1$ (a spacelike normal). The Israel junction conditions are expressed as discontinuities of the extrinsic curvature of the shell

$$[[K_{ij}]] = -8\pi \left[ S_{ij} - \frac{1}{2} Sh_{ij} \right]$$

(2.10)

where $[[X]] \equiv [X_+]-[X_-]$ denotes the discontinuity of the quantity $X$ as the shell is crossed. In the case of a static shell in the SRAAdS metric (1.1) the extrinsic curvature tensor takes the form

$$K_{ij} = \sqrt{f_\pm(r_\pm)} \text{diag} \left( \frac{f_\pm(r_\pm)}{r_\pm}, \frac{1}{r_\pm}, \frac{1}{r_\pm} \right).$$

(2.11)

and the Israel junction conditions for a dynamic shell are of the form [5]

$$\sigma = -\frac{1}{4\pi R} \left[ \sqrt{1 - 2m_\pm(R)/R + \hat{R}^2} \right]$$

(2.12)

$$p = \frac{1}{8\pi R} \left[ \frac{1 - m_\pm(R)/R - m_\pm(R)' + \hat{R}^2 + R\hat{R}}{\sqrt{1 - 2m_\pm(R)/R + \hat{R}^2}} \right]$$

(2.13)

where $(\cdot)'$ denotes derivative of $m_\pm(r)$ with respect to $r$ at $r = R$ and dot denotes derivative with respect to the proper time of the shell defined as
\[ d\tau^2 = \left[ 1 - \frac{2m_\pm(R)}{R} \right] dt^2 - \frac{1}{1 - 2m_\pm(R)/R} \left[ \frac{dR}{dt} \right]^2 dt^2 \quad (2.14) \]

These equations lead also to the energy conservation equation on the shell,

\[ \frac{d}{d\tau}(\sigma R^2) + p \frac{d}{d\tau} R^2 = 0 \quad (2.15) \]

and \( E = 0 \).

Clearly, eq. (2.15) is identical with the energy conservation equation of a particle moving in one dimension with coordinate \( R(\tau) \) and zero energy. Thus, the conditions for the existence of a static, stable shell may be written as,

\[ V(R) = 0 \]

\[ V'(R) = 0 \]

\[ V''(R) > 0 \quad (2.18) \]

\[ V(R) = 1 - \frac{m_\pm(R)}{R} + 2bR - \frac{\Lambda R^2}{3} \frac{m_\pm(R) + m_\mp(R)}{4\pi R^2} \quad (2.19) \]

In the context of a constant shell fluid equation of state we have \( p = w\sigma \) and it is easy to show that energy conservation (2.15) leads to

\[ \sigma = \sigma_0' \left( \frac{R}{R_0} \right)^{-2(w+1)} \quad (2.20) \]

where \( \sigma_0' \) is the surface density of a shell of radius \( R_0 \). In what follows we define \( \sigma_0 \equiv \sigma_0' R_0^{2(w+1)} \). For example in the special case of a pressureless matter shell \( (w = 0) \) we obtain the expected result \( \sigma(R) \sim R^{-2} \) while for a vacuum shell we have \( \sigma(R) = \sigma_0 = const \).

In the special case when \( \sigma \) is independent of \( R \) discussed in the next section (vacuum shell) it is straightforward to show that a minimum of the potential (2.19) exists for \( b < 0 \) and \( \Lambda < 0 \) due to the attractive nature of the linear potential term \( 2bR \) which dominates at large \( R \) competing with the repulsive effects of the quadratic potential term \( -\Lambda R^2/3 \) which dominates at even larger \( R \).

The eq. (2.12) may also be expressed as,

\[ \frac{1}{2} \dot{R}^2 + V(R) = E \quad (2.16) \]

where,

\[ V(R) \equiv 1 + \frac{4m_+(R) - m_-(R)}{16\pi^2\sigma^2 R^4} - \frac{4\pi\sigma R^2}{2R} + \frac{m_+(R) + m_-(R)}{4\pi\sigma R^2} \quad (2.17) \]

These conditions, along with the equation of state \( p(\sigma) \) and the energy conservation eq. (2.15) may be used to identify constraints on the metric parameters appearing in the expressions of \( m_-(R) \) and \( m_+(R) \) required for the existence of a stable spherical shell with given radius \( R \). In the present analysis we consider the particular forms of \( m_\pm(r) \) given by eq. (2.3) corresponding to the SRAdS metric. In this case the potential of eq. (2.17) takes the form,

\[ V(R) = 1 - \frac{m_\pm(R)}{R} + 2bR - \frac{\Lambda R^2}{3} \frac{m_\pm(R) + m_\mp(R)}{4\pi^2 R^2} - \frac{(m_\pm(R) - m_\mp(R))^2}{16\pi^2 R^4 \sigma(R)^2} \quad (2.19) \]

For more general metrics or fluid equations of state than the one considered here it is clearly possible to have several minima for the potential corresponding to configurations of more than one stable concentric shells.

In the next section we identify the metric parameter ranges of the \( b \) and \( \Lambda \) that allow for stable shells in the spatial cases of three shell fluid equations of state. We then proceed by finding the ranges of the parameters \( b, \Lambda \) which allow for stable spherical shell solutions, via the implementation of these conditions for different cases of interior and exterior equations of state.

III. SPECIAL CASES

III.1. Vacuum fluid shell \( (w = -1) \)

The simplest case of a stable spherical shell is obtained assuming a vacuum fluid equation of state,
This case is similar to the case of a stable domain wall in the SRAdS metric discussed in [9] using theoretical methods. It was shown that such metastable topological field configurations may indeed exist for $b < 0$, $\Lambda < 0$ due to the competing attractive-repulsive effects of the linear and quadratic terms of the metric functions. In the vacuum fluid case we have from eq. (2.20)

$$\sigma(R) = \sigma_0 = \text{const.}$$  \hspace{1cm} (3.2)

In this case the system (2.18) becomes,

$$V(R) = 1 - \frac{m_- + m_+}{R} + 2bR - \frac{\Lambda R^2}{3} - \frac{(m_- - m_+)^2}{16\pi^2 R^4 \sigma_0^2} - 4\pi^2 \sigma_0^2 R^2 = 0$$ \hspace{1cm} (3.3)

$$\left. \frac{\partial V}{\partial r} \right|_{r=R} = 2b + \frac{m_- + m_+}{R^2} - 2\Lambda R - \frac{(m_- - m_+)^2}{4\pi^2 R^6 \sigma_0^2} - 8\pi^2 \sigma_0^2 R = 0$$ \hspace{1cm} (3.4)

$$\left. \frac{\partial^2 V}{\partial r^2} \right|_{r=R} = -\frac{2\Lambda}{3} - \frac{2(m_- + m_+)}{R^3} - \frac{5(m_- - m_+)^2}{4\pi^2 R^6 \sigma_0^2} - 8\pi^2 \sigma_0^2 R > 0.$$ \hspace{1cm} (3.5)

The solution of the system (3.3 - 3.5) may be written as:

$$p = -\sigma$$ \hspace{1cm} (3.1)

FIG. 1. The shell stability region (grey region) in the $b - \Lambda$ parameter space for two pairs of $m_+ - m_-$. The colored curves correspond to fixed value of surface density in the stability range $\sigma_0 \equiv \sigma_{0\text{min}} + \Delta \sigma > \sigma_{0\text{min}}$ while $R$ varies such that $R > R_{\text{min}}$. 

$p = -\sigma$
The existence of lower limits on the values of \( R \) and \( \sigma_0 \) allows the analytical derivation of the boundaries in the \( b, \Lambda \) parameter space of the region that permits a stable shell solution. In particular when the shell radius takes its lower limit value \( R = R_{\text{min}} \), we have \( \sigma_{0_{\text{min}}} = \infty \) and \( \Lambda \to -\infty \) which implies the existence of a low bound on \( b \) for large \( |\Lambda| \) as

\[
\Lambda \to -\infty \implies b \to -\frac{1}{6(m_+ + m_-)}.
\]  

(3.10)

Similarly for large shell radius \( (R \to \infty) \) we have,

\[
\sigma_{0_{\text{min}}} \to 0 \text{ and } \Lambda \to -12\pi^2 \sigma_0^2 > 0.
\]  

(3.11)

From eq. (3.7) implies the existence of an upper bound for the parameter \( b \),

\[
b < 0 \text{ with } \Lambda < \Lambda_{\text{max}} = -12\pi^2 \sigma_0^2.
\]  

(3.12)

These analytically derived boundaries of the stability parameter region may be displayed by showing contours in the \( (b, \Lambda) \) parameter space that show the shell stability regions in the context of the constraints (3.6 - 3.9) for fixed values of \( m_+, m_- \). Clearly, the boundaries expressed by eqs (3.10 - 3.12) are respected by these re-

![Image](image_url)
regions as demonstrated in Fig. 1. In Fig. 2 we show the form of the potential (2.17) for three sets of parameters \((R, \sigma, b, \Lambda)\) inside and outside the stability region of Fig. 1. As expected the potential develops a minimum with \(V(R) = 0\) only for the parameters inside the stability region while the parameter values in the instability region correspond only to a local maximum of the potential at the corresponding value of \(R\).

In order to illustrate the validity of the stability boundaries shown in Fig. 1 we show a random set of stability parameter points in Fig. 3 which is constructed as follows:

1. We fix \(m_- = 1, m_+ = 1.5\) and construct the stability boundary as the set of points with \(b = b(R, \sigma_{\text{omin}}(R))\), \(\Lambda = \Lambda(R, \sigma_{\text{omin}}(R))\), where \(R > R_{\text{min}}\) (see eq. (3.8)), \(\sigma_{\text{omin}}(R)\) is obtained from eq. (3.9).
2. We construct a random selection of shell radius values \(R_i\) respecting the stability constraint (3.8). For each value of \(R = R_i\) we consider a random value for \(\sigma_i\) such that \(\sigma_i > \sigma_{\text{omin}}(R_i)\) (see eq. (3.9)). For the given random pair \((R_i, \sigma_i)\) we obtain the stability parameters \((\Lambda, b)\) and plot the corresponding point in Fig. 3.

3. We repeat this process for \(i = 1, \ldots, N \ (N = 5 \times 10^4)\) thus constructing Fig. 3.

Clearly all the points corresponding to stable shell parameter values are within the stable region thus testing the validity of this region and the consistency of Fig. 1.

### III.2. Stiff matter fluid shell \((w = 1)\)

A stiff matter shell has equation of state

\[
p = \sigma. \quad (3.13)
\]

From the eq. (2.20) with \(w = 1\) we obtain

\[
\sigma(R) = \sigma_0 R^{-4} \quad (3.14)
\]

For this equation of state the potential (2.19) takes the form,

\[
V(R) = 1 + 2bR - \frac{\Lambda R^2}{3} - \frac{(m_- - m_+)^2 R^4}{16\pi^2 \sigma_0^2} - \frac{m_- + m_+}{R} - \frac{4\pi^2 \sigma_0^2}{R^6} \quad (3.15)
\]

The system of stability conditions (2.18) in this case takes the form,

\[
V(R) = 1 - \frac{m_- + m_+}{R} + 2bR - \frac{\Lambda R^2}{3} - \frac{(m_- - m_+)^2 R^4}{16\pi^2 \sigma_0^2} - \frac{4\pi^2 \sigma_0^2}{R^6} = 0 \quad (3.16)
\]

\[
\left. \frac{\partial V}{\partial r} \right|_{r=R} = 2b + \frac{m_- + m_+}{R^2} - \frac{2\Lambda R}{3} - \frac{(m_- - m_+)^2 R^3}{4\pi^2 \sigma_0^2} + \frac{24\pi^2 \sigma_0^2}{R^7} = 0 \quad (3.17)
\]

\[
\left. \frac{\partial^2 V}{\partial r^2} \right|_{r=R} = -\frac{2\Lambda}{3} - \frac{2(m_- + m_+)}{R^3} - \frac{3(m_- - m_+)^2 R^2}{4\pi^2 \sigma_0^2} - \frac{168\pi^2 \sigma_0^2}{R^8} > 0 \quad (3.18)
\]

with solution for existence of shell solution

\[
\Lambda(R, \sigma_0) = -\frac{9(m_- - m_+)^2 R^2}{16\pi^2 \sigma_0^2} + \frac{6(m_+ + m_-) - 3R}{R^3} + \frac{84\pi^2 \sigma_0^2}{R^8} \quad (3.19)
\]

\[
b(R, \sigma_0) = -\frac{(m_- - m_+)^2 R^3}{16\pi^2 \sigma_0^2} + \frac{3(m_+ + m_-) - 2R}{2R^2} + \frac{16\pi^2 \sigma_0^2}{R^5} \quad (3.20)
\]

The stability condition (3.18) leads to the constraints

\[
-4\pi \sqrt{\sigma_0^2 (R_6 - 6m_+ R^5 - 100\pi^2 \sigma_0^2)} < \frac{3[(m_- - m_+)^2 R^5 + 8\pi^2 \sigma_0^2]}{\sqrt{3}} < 4\pi \sqrt{\sigma_0^2 (R_6 - 6m_+ R^5 - 100\pi^2 \sigma_0^2)}, \quad (3.21)
\]

\[
R^6 > 6m_+ R^5 + 100\pi^2 \sigma_0^2. \quad (3.22)
\]

which must be met simultaneously in order for a stability region to exist.
FIG. 5. A Monte-Carlo map of the dust matter shell stability parameter region \((b, \Lambda)\) for \(m_- = 1, m_+ = 1.5\).

Using again the Monte-Carlo method of Fig. 3 with random values of \(R\) and \(\sigma_0\) in the region allowed by eqs. (3.21)-(3.22), we obtain the corresponding stability values of \(\Lambda\) and \(b\) which map the stability region shown in Fig. 4.

The range of the \((b, \Lambda)\) parameters for which we have stable solutions for the stiff matter case appears to be significantly narrower than the corresponding one for the case of vacuum shell. The reduction of the stability region in this case is due to the repulsive term of the potential of eq. (3.15) proportional to \(R^4\) which is not present in the vacuum shell case and spoils the attractive effects of the anti-deSitter term \(\sim \Lambda R^2 (\Lambda < 0)\) needed for the formation of a potential minimum at large \(R\).

### III.3. Pressureless dust fluid shell \((w = 0)\)

For a pressureless dust fluid shell we have \(p = 0\) and eq. (2.15) leads to a surface energy density of the form

\[
\sigma(R) = \sigma_0 R^{-2}
\]

In this case the potential takes the form,

\[
V(r) = 1 + 2bR - \frac{\Lambda R^2}{3} - \frac{(m_- - m_+)^2}{16\pi^2\sigma_0^2} - \frac{m_- + m_+}{R} - \frac{4\pi^2\sigma_0^2}{R^2}
\]

Solving the system (2.18) for this potential yields the following forms for \(\Lambda\) and \(b\) (existence conditions)

\[
\Lambda(R, \sigma_0) = \frac{3(m_- - m_+)^2}{16\pi^2 R^2 \sigma_0^2} + \frac{6(m_- + m_+)}{R^3} - 3R - \frac{36\pi^2\sigma_0^2}{R^4}
\]

\[
b(R, \sigma_0) = \frac{(m_- - m_+)^2}{16\pi^2 R \sigma_0^2} + \frac{3(m_- + m_+)}{2R^2} - \frac{8\pi^2\sigma_0^2}{R^3}
\]

While stability of the shell implies that

\[
\|24\pi^2\sigma_0^2 + (m_- - m_+ R)\| < 4\pi \sqrt{\sigma_0^2(12\pi^2\sigma_0^2 - 6m_+ R + R^2)},
\]

\[
0 < 12\pi^2\sigma_0^2 - 6m_+ R + R^2.
\]

Via the same Monte-Carlo process as in the former cases we show a map the stability parameter region in the \((b, \Lambda)\) space (Fig. 5).

### IV. CONCLUSION - OUTLOOK

We have demonstrated the existence of static, stable spherically symmetric thin fluid shells in a Schwarzschild-Rindler-anti-de Sitter (SRAdS) metric. We have found analytically the conditions for stability and the corresponding range of values of metric parameters that admit stable fluid shells for different forms of fluid equation of state. These structures have similarities with the well known gravastar shell structures [4, 5, 20–22]. In our shell structures the interior de Sitter term of the gravastars is replaced by a combination of Rindler-anti-de Sitter terms present in a continuous form (same values both in the interior and in the exterior of the shell) allowing for the existence of a minimum of the stability effective potential.

Interesting extensions of this analysis include the following:

- The investigation of alternative forms of metrics
that may admit stable shell solutions. For example an interesting alternative simple metric would be one with a Rindler term inside the shell and a Schwarzschild term outside. Such a metric would be free of singularities and would differ from a gravastar in the replacement of the de Sitter interior by a Rindler interior. Other types of metrics could accept multiple concentric shell structures if the corresponding stability potential has multiple minima at different radii $R$.

- The investigation of observational effects of such shell structures. For example signatures of such SRAdS shell structures in typical lensing patterns could be identified and compared to observed lensing patterns around black holes [23–28]. Signatures of SRAdS shells in such optical images could be specified and compared with predicted signatures of other similar exotic objects like gravastars [29].

- The investigation of non-spherical junctions and shells. An interesting problem would be the study of joining rotating spacetimes in the presence of the cosmological constant.

- The consideration of more general fluid shell equations of state. In the case of phantom shells it may be possible to have stable shells in a pure Schwarzschild background due to the tendency of such shells to expand rather than contract (negative tension). This is easily shown using the energy conservation equation (2.15) with $w < −1$ which leads to a surface density $\sigma(R) = \sigma_0 R^{-2(w+1)}$ which increases with $R$. The positive value of the exponent for $w < −1$ indicates that it is energetically favourable for such phantom shell to expand rather than contract leading to a negative tension (pressure) that would tend to stabilize the shell even in a pure Schwarzschild background.

- The investigation of the dynamical evolution of the shell in the context of spherical symmetry and beyond. Non-spherical dynamical excitations of the shell could also lead to interesting gravitational wave signatures.

**Numerical Analysis Files**: The numerical files for the reproduction of the figures can be found in [30].

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