EMERGENCE FOR DIFFEOMORPHISMS WITH NONZERO LYAPUNOV EXponents

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Abstract. We consider the set of points with high pointwise emergence for $C^{1+\alpha}$ diffeomorphisms preserving a hyperbolic measure. We find a lower bound on the Hausdorff dimension of this set in terms of unstable Hausdorff dimension of the hyperbolic measure. If the measure is an SRB, we prove that the set of points with high emergence has full Hausdorff dimension.

1. Introduction

The notion of (metric) emergence has been first introduced by Berger in [6] as a tool to 'evaluate the complexity to approximate a system by statistics'. Metric emergence quantifies such phenomena as the Newhouse phenomenon or KAM phenomenon. The list of research works on metric emergence includes [7, 9, 8, 37, 14]. Following similar ideas, Nakano, Kiriki, and Soma [21] introduced a concept of pointwise emergence (see (2.1) for the definition) to measure complexity of irregular orbits. We recall that a point $x$ is said to be irregular, if the sequence

$$\delta^n_x = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}, \quad n \geq 1,$$

does not converge (here $\delta_x$ denotes the atomic measure at the point $x$).

By the Birkhoff Ergodic Theorem, the set of irregular points has zero measure with respect to any invariant measure. On the other hand, for many dynamical systems this set is known to be large from different points of view - to have full Hausdorff dimension and full entropy. Pesin and Pitskel’ [32] obtained the first result of this type, proving that the set of irregular points for the full shift has full entropy and full Hausdorff dimension. This result was later extended to topologically mixing subshifts of finite type in [5], to graph directed Markov systems in [10], to continuous maps with specification property in [13] (see also [38]), and to continuous maps with almost specification property in [39]. Other works related to the study of the set of irregular points include [18, 34, 36, 1, 11, 12, 40, 10, 22, 2, 4, 41].

In this paper we consider irregular points with high complexity, that is points for which (1.1) oscillates between infinitely many ergodic measures. This corresponds to the pointwise emergence being super-polynomial and we will also refer to it as high (for the definition see (2.2) in the next section).

As pointed out in [6], there is a consensus among computer scientists that super-polynomial algorithms are impractical. From that perspective dynamical systems with high metric emergence are not feasible to be studied numerically. The set of
points with high (pointwise) emergence can be then considered as statistically very complex (see also [7, 9] for other motivations to study high emergence).

It was proved by the author of this note and Nakano in [28] that the set of points with high emergence for topologically mixing subshifts of finite type has full entropy, full Hausdorff dimension, as well as the full pressure for any Hölder continuous function. The aim of this paper is to find a lower bound for the Hausdorff dimension of the set of points with high emergence for $C^{1+\alpha}$ diffeomorphisms preserving a hyperbolic measure (see Theorem A). This is done by finding an appropriate approximation by Horseshoes (Theorem B) and then applying the result from [28]. The first construction of a horseshoe approximating the support of a hyperbolic measure was presented by Katok in [19] and since then there have been many works establishing different variants of this classical result. Examples include [3, 15, 17, 24, 26, 27, 29, 33, 35, 42] but this list is far from complete. For our purposes we need a horseshoe maximizing the unstable dimension. This was done in [26] for diffeomorphisms on surfaces preserving an ergodic SRB and in [35] this result has been extended to higher dimensions. Theorem B of this paper is a generalization of [26] and [35] to general hyperbolic measures and as such, we believe that it may be of independent interest.

1.1. Structure of the paper. In Section 2 we introduce the setting and state the main results - Theorem A and Theorem B. We prove Theorem B in Section 3. The proof of Theorem A occupies Sections 4 and 5.

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2. The setting and the main results

Consider a $C^{1+\alpha}$ diffeomorphism $f$ of a smooth compact Riemannian manifold $M$ without boundary and $\mu$ - an $f$-invariant hyperbolic probability measure. Let $0 < \chi_1 < \chi_2 < \ldots < \chi_l$ denote distinct positive Lyapunov exponents of $f$ with multiplicities $n_1, n_2, \ldots, n_l$. The following statement was proved by Ledrappier and Young in [23] (see also [31, Theorem 14.1.18]).

**Theorem 2.1.** There are numbers $d_1, d_2, \ldots, d_l$ such that:

1. $0 \leq d_i \leq n_i$,
2. $\sum_{i=1}^{l} d_i = d_{\mu}$, and
3. $h_{\mu}(f) = \sum_{i=1}^{l} d_i \chi_i$.

Here $d_{\mu}$ denotes the unstable dimension of the measure and can be obtained [31, Theorem 14.1.6] as the limit

$$\lim_{r \to 0} \frac{\log \mu^u(x)(B^u(x, r))}{\log r}$$

for $\mu$- almost every $x \in M$,

where $\mu^u(x)$ is the conditional measure on the local unstable manifold $V^u(x)$ and $B^u(x, r) := B(x, r) \cap V^u(x)$.

For the rest of this paper we assume that:
A1. there exists a number $0 < d \leq 1$ such that
\[ d = \frac{d_1}{n_1} = \frac{d_2}{n_2} = \ldots = \frac{d_l}{n_l}. \]

A2. \[ \frac{\chi_l}{\chi_1} (1 - d) < 1. \]

In particular our results apply if $\mu$ is an SRB measure, in which case $d = 1$. Another example is $\mu$ being $u$-conformal, meaning that there is only one positive Lyapunov exponent $\chi^+$ of multiplicity $n^+ \geq 1$. In the latter case the statement of Theorem 2.1 reduces to the formula $h_\mu(f) = d^u \chi^+$.

The pointwise emergence $E_{f,x}(\epsilon)$ at scale $\epsilon > 0$ at $x \in M$ (with respect to $f$) is defined as

(2.1) \[ E_{f,x}(\epsilon) = \min \left\{ N \in \mathbb{N} \mid \text{there exists } \{\mu_j\}_{j=1}^N \subset \mathcal{P}(M) \text{ such that} \right. \]
\[ \limsup_{n \to \infty} \min_{1 \leq j \leq N} d_W \left( \sum_{k=0}^{n-1} \delta_{f^k(x), \mu_j} \right) \leq \epsilon, \]

where:
- $\mathcal{P}(M)$ denotes the set of Borel probability measures on $M$,
- $d_W$ is the first Wasserstein metric:
  For $j = 1, 2$, let $\pi_j : M \times M \to M$ be the canonical projection to the $j$-th coordinate. Let $\Pi(\mu, \nu)$ be the set of probability measures $P$ on $M \times M$ such that $P \circ \pi_1^{-1} = \mu$ and $P \circ \pi_2^{-1} = \nu$. The first Wasserstein metric $d_W$ is defined as
\[ d_W(\mu, \nu) = \inf_{P \in \Pi(\mu, \nu)} \int_{M \times M} d(x, y) dP(x, y) \text{ for } \mu, \nu \in \mathcal{P}(M), \]
- $\delta_x$ denotes the atomic measure at the point $x$.

The pointwise emergence at $x \in M$ is called super-polynomial (or high) if

(2.2) \[ \limsup_{\epsilon \to 0} \frac{\log E_{f,x}(\epsilon)}{-\log \epsilon} = \infty. \]

The following was shown in [28, Theorem 1.1].

**Theorem 2.2.** Let $X$ be a topologically mixing subshift of finite type of $\{1, 2, \ldots, \kappa\}^\mathbb{N}$ with $\kappa \geq 2$. Let $\sigma : X \to X$ be the left-shift operation on $X$. Let $\mathcal{E}_\sigma$ be the set of points $x \in X$ satisfying
\[ \lim_{\epsilon \to 0} \frac{\log E_{\sigma,x}(\epsilon)}{-\log \epsilon} = \infty. \]

Then,
\[ h_{\text{top}}(\sigma|_{\mathcal{E}_\sigma}) = h_{\text{top}}(\sigma|_X) \text{ and } \dim_H(\mathcal{E}_\sigma) = \dim_H(X). \]
In addition, for any Hölder continuous function $\varphi$, we have that $P(\sigma|_{\mathcal{E}_\sigma}, \varphi) = P(\sigma|_X, \varphi)$. That is, the set of points with high emergence carries full topological pressure.

The aim of this note is to use the above result to obtain the following.
**Theorem A.** Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a smooth compact Riemannian manifold $M$ without boundary and $\mu$ - an $f$-invariant hyperbolic probability measure satisfying Conditions A1. and A2.

Let $E_f$ be the set of points $x \in M$ satisfying

$$\lim_{\epsilon \to 0} \frac{\log \delta_{E_f}(x)}{-\log \epsilon} = \infty.$$ 

Then,

$$\dim_H(E_f) \geq \dim(E^s) + \left(1 - (1 - d)\frac{\chi_1}{\chi_1}\right) \dim E^u,$$

where $E^s$ and $E^u$ denote the stable and unstable subbundles respectively.

In the two special cases we obtain the following corollaries.

**Corollary 2.3.** Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a smooth compact Riemannian manifold $M$ without boundary preserving a hyperbolic SRB measure. Then,

$$\dim_H(E_f) = \dim(M).$$

**Corollary 2.4.** Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a smooth compact Riemannian manifold $M$ without boundary and $\mu$ - an $f$-invariant hyperbolic probability measure with one positive Lyapunov exponent $\chi^+$ of multiplicity $n^+ \geq 1$. Then,

$$\dim_H(E_f) \geq \dim(E^s) + d^u_\mu.$$

We will prove Theorem A by constructing an appropriate horseshoe in a neighborhood of the support of $\mu$. We prove the following result in Section 3 by combining arguments from [3] and [26].

**Theorem B.** Consider a $C^{1+\alpha}$ diffeomorphism $f : M \to M$ preserving a hyperbolic probability measure $\mu$ satisfying Conditions A1 and A2. For any $\delta > 0$ and $0 < \delta_1 \ll \delta$ there exists a compact $f$-invariant set $\Lambda \subset M$ such that:

1. there exists a dominated splitting on $\Lambda$:

$$T\Lambda\Lambda = E^s \oplus E_1 \oplus E_2 \oplus \ldots \oplus E_l,$$

   with $\dim(E_i) = n_i$;

2. there exists $n_0 \geq 1$ such that for each $i = 1, \ldots, l$, each $x \in \Lambda$, each unit vector $v \in E_i(x)$ and all $n \geq n_0$,

   $$\exp((\chi_i - \delta_1)n) \leq \|Df^n_x(v)\| \leq \exp((\chi_i + \delta_1)n);$$

3. denote $E^n = E_1 \oplus E_2 \oplus \ldots \oplus E_l$;

4. the function $\log |\det Df_{|E^n}|$ is Hölder continuous on $\Lambda$;

5. there exists an ergodic, $f$-invariant measure $\nu$ supported on $\Lambda$ and such that

$$d^u_\nu \geq \dim E^n \left(1 - \frac{\chi_1}{\chi_1}(1 - d)\right) - \delta > 0.$$

**Remark 2.5.** A construction of a hyperbolic horseshoe satisfying (1), (2), and with topological entropy close to $h_\mu(f)$ was presented in [3, Theorem 3.3]. Even though it was not stated there, Statement (3) is an immediate consequence of statements (1) and (2), see for example [31] Theorem 5.3.2 or [20] Theorem 19.1.6 and Corollary
19.1.13]. For our purposes we need a horseshoe with unstable Hausdorff dimension close to \(d_\mu^\ast\). This can be obtained by replacing in \(\Xi\) quantities corresponding to entropy with analogous quantities corresponding to pressure of an appropriate potential. We will then show that (4) implies (5).

3. Proof of Theorem B

We need the following Oseledec-Pesin reduction theorem for the derivative cocycle \(\Xi\) Theorem 5.6.1 and Theorem 3.5.5].

**Theorem 3.1.** Let \(f : M \to M\) be a \(C^{1+\alpha}\) diffeomorphism preserving an ergodic probability measure \(\mu\) with Lyapunov exponents \(\lambda_1 > \ldots > \lambda_s\) of multiplicities \(k_1, \ldots, k_s\). Then there exists an invariant set \(Z \subset M\) with \(\mu(Z) = 1\) such that for any \(\eta > 0\) the following holds on \(Z\):

1. There exists a measurable family of invertible linear maps \(C(x) : T_x M \to \mathbb{R}^{\dim M}\) and \(A_i(x) \in \text{GL}(k_i, \mathbb{R}), i = 1, \ldots, s\), such that
   
   \[
   D f(x) = C^{-1}(f(x)) \text{diag}[A_1(x), \ldots, A_s(x)] C(x),
   \]

2. \(e^{\lambda_i - \eta} < \|A_i^{-1}\| \leq \|A_i\| < e^{\lambda_i + \eta}\),
3. \(C(x)\) is tempered, that is for all \(x \in Z\)
   
   \[
   \lim_{m \to \pm \infty} \frac{1}{m} \log \|C(f^m(x))\| = \lim_{m \to \pm \infty} \frac{1}{m} \log \|C^{-1}(f^m(x))\| = 0;
   \]

4. there exist measurable functions \(r, K : Z \to (0, 1]\), and a collection of embeddings \(\psi_x : B(0, r(x)) \to M\) such that:
   
   \(\psi_x(0) = x\) and \(\psi_x = \exp_x \circ C(x)\),
   
   the maps \(f_x = \psi_{f(x)} \circ f \circ \psi_x^{-1}\) satisfy \(d_{C(x)}(f_x, D f(x)) < \eta\),
   
   there exists a constant \(Q > 0\) such that for any \(y, y' \in \psi_x^{-1}(B(0, r(x)))\),
   
   \[
   \frac{1}{K(x)} d(y, y') \leq \|\psi_x(y) - \psi_x(y')\| \leq \frac{1}{Q} d(y, y'),
   \]

   \[
   e^{-\eta} < \frac{r(f(x))}{r(x)} < e^\eta \text{ and } e^{-\eta} < \frac{K(f(x))}{K(x)} < e^\eta.
   \]

Statement \(\Xi\) guarantees that choosing \(m_0 \in \mathbb{N}\) large enough, we can find a set \(X_0 \subset Z\) of measure arbitrarily close to one on which \(\|C \circ f^m\|, \|C^{-1} \circ f^m\| \in (e^{-\eta m}, e^{\eta m})\) for all \(m \geq m_0\). This combined with Lusin’s theorem gives that for any \(\eta > 0\) there exists a compact set \(X \subset X_0 \subset Z\) of measure arbitrarily close to one on which the functions \(r, K, C, C^{-1}\) are continuous. Following the terminology used in \(\Xi\), we will refer to any such set as a “uniformity block of tolerance \(\eta\). It is worth pointing out that those are closely related to Pesin’s regular sets.

3.0.1. Thermodynamic quantities. For \(n \geq 1\) we define the dynamical metric \(d_n\) on \(M\) as

\[
d_n(x, y) := \max_{0 \leq k \leq n} d(f^k(x), f^k(y)).
\]

For \(x \in M\) and \(\rho > 0\) we denote by \(B_n(x, \rho)\) the open ball with respect to the metric \(d_n\) centered at \(x\) of radius \(\rho\). We say that a set \(E \subset M\) is

- \(\mu - (n, \rho, \beta)\)- spanning if \(\mu(\bigcup_{x \in E} B_n(x, \rho)) > 1 - \beta\).
• \((n, \rho)\)-separated if for any two elements \(x, x' \in E\) one has \(d_n(x, x') > \rho\).

In addition, for any subset \(A \subset M\) and an \((n, \rho)\)-separated set \(E \subset A\) we say that

• \(E\) is a maximal \((n, \rho)\)-separated set in \(A\), if there is no other \((n, \rho)\)-separated set in \(A\) containing \(E\).

We observe that for any \(A \subset M\) one has that if \(E\) is a maximal \((n, \rho)\)-separated set in \(A\), then \(E\) is \(\mu - (n, 2\rho, \mu(A))\)-spanning. We denote by \(C(n, \rho, \beta)\) the minimal cardinality of a \(\mu - (n, \rho, \beta)\)-spanning set. The following was shown in [19, Theorem 1.1].

**Theorem 3.2.** For any \(\beta > 0\),

\[
\mu(f) = \lim_{\rho \to 0} \lim_{n \to \infty} \frac{1}{n} \log C(n, \rho, \beta).
\]

We now consider the function

\[
\phi(x) := -d \log |\det Df^{\mu}(x)| \text{ and } S_n \phi(x) = \sum_{k=1}^{n-1} \phi(f^k(x)).
\]

Observe that \(\phi\) is only measurable, however it is continuous on every uniformity block. If \(X \subset M\) is a uniformity block, we denote by

\[
Q(n, \rho, \beta, X) = \inf \left\{ \sum_{x \in E} \exp(S_n \phi(x)) | E \subset X \text{ is } \mu - (n, \rho, \beta)\text{-spanning} \right\}.
\]

Restricting to a uniformity block is an essential modification of the quantity studied in [20]. Because \(\phi\) is not continuous, Theorem 1.1 in [20] does not apply. Instead, we can prove the following.

**Lemma 3.3.** If \(\eta > 0\) and \(X \subset Z\) is a uniformity block of tolerance \(\eta\), then there exists \(m_0 \in \mathbb{N}\) such that:

1. for all \(m \geq m_0\) and any \(x \in X\) one has

\[
\exp S_m \phi(x) = |\det Df^m(x)|^{\dim E^u} e \pm 3\eta d \dim E^u \prod_{i=1}^{l} e^{\chi_i d n_i m};
\]

2. for all \(m \geq m_0\) and any finite set \(A \subset X\),

\[
\frac{1}{m} \log \sum_{x \in A} \exp(S_m \phi(x)) > -3\eta d \dim(E^u) + \left[ \frac{1}{m} \log \# A - h_\mu(f) \right];
\]

3. for all \(m \geq m_0\) and for any \(\beta \in (0, 1)\) one has

\[
\frac{1}{m} \log Q(m, \rho, \beta, X) > -3\eta d \dim(E^u) + \left[ \frac{1}{m} \log C(m, \rho, \beta) - h_\mu(f) \right],
\]

4. and in particular

\[
\lim_{\rho \to 0} \liminf_{m \to \infty} \frac{1}{m} \log Q(m, \rho, \beta, X) > -3\eta d \dim(E^u).
\]

\footnote{We use the notation \(A = \pm B\) to mean \(-B \leq A \leq B\).}
Proof. By Theorem 3.1 for any \( x \in Z \) we have that
\[
Df(x)|_{E^s(x)} = C^{-1}(f(x)) diag[A_1(x), \ldots, A_l(x)] C(x),
\]
consequently, since \( Z \) is \( f \)-invariant,
\[
Df^m(x)|_{E^s(x)} = Df^{m-1}(f(x))|_{E^s(f(x))} \circ Df(x)|_{E^s(x)} = \ldots = Df(f^{m-1}(x))|_{E^s(f^{m-1}(x))} \circ \ldots \circ Df(f(x))|_{E^s(f(x))} \circ Df(x)|_{E^s(x)} = C^{-1}(f^m(x)) diag[\prod_{k=1}^{m-1} A_1(f^k(x)), \ldots, \prod_{k=1}^{m-1} A_l(f^k(x))] C(x).
\]

We then have that
\[
|\det Df^m(x)|_{E^s(x)} = |\det C^{-1}(f^m(x))| \cdot \prod_{k=1}^{m-1} |\det A_1(f^k(x))| \cdot \ldots \cdot |\det A_l(f^k(x))| \cdot |\det C(x)|.
\]

We estimate all the terms in the product above. By Statement (1c) in Theorem 3.1 and because \( X \) is a uniformity block, there exists \( \tilde{m}_0 \in \mathbb{N} \) such that
\[
||C(f^m(x))||, ||C^{-1}(f^m)|| = e^{\pm \eta m} \text{ for all } m \geq \tilde{m}_0.
\]
Consequently, by Hadamard’s determinant inequality,
\[
|\det C^{-1}(f^m(x))| = e^{\pm \eta m \dim E^u} \text{ for all } m \geq \tilde{m}_0.
\]
By continuity of \( C \) and \( C^{-1} \) on \( X \), there exists a constant \( Q > 0 \) such that
\[
|\det C(x)| = Q^{\pm 1}.\]
In addition, for \( i = 1, \ldots, l \) and any \( k \in \mathbb{N} \), Statement (1b) of Theorem 3.1 implies that
\[
|\det A_i(f^k(x))| = e^{(\chi_i \pm \eta)n_i}.
\]
Together we obtain that
\[
|\det Df^m(x)|_{E^s(x)} = e^{\pm \eta m \dim E^u} \prod_{i=1}^{l} e^{(\chi_i \pm \eta)n_i, m} Q^{\pm 1} = e^{\pm 2\eta m \dim E^u} Q^{\pm 1} \prod_{i=1}^{l} e^{\chi_i, n_i, m}
\]
for every \( x \in X \) and \( m \geq \tilde{m}_0 \). We set \( m_0 = \tilde{m}_0 \) so that \( Q < e^{\eta \dim E^u m_0} \). Then for any \( m \geq m_0 \) one has
\[
\exp S_m \phi(x) = |\det Df^m(x)|_{E^s(x)}^{-d} = e^{\pm 3\eta m \dim E^u} \prod_{i=1}^{l} e^{-\chi_i, d, m}
\]
so we proved the first statement of the lemma. If now \( A \subset X \) is any finite set, then for all \( m \geq m_0 \) we can estimate
\[
\sum_{x \in A} \exp S_m \phi(x) = \sum_{x \in A} |\det Df^m(x)|_{E^s(x)}^{-d} \geq e^{-3\eta m \dim E^u} \prod_{i=1}^{l} e^{-\chi_i, d, m} \#A.
\]
From this we obtain
\[
\frac{1}{m} \log \sum_{x \in A} \exp S_m \phi(x) \geq -3\eta d \dim E^u - \sum_{i=1}^{l} \chi_i d, + \frac{1}{m} \log \# A
\]
Let $L$ be a finite cover of $X$ such that there exists $\tilde{\eta}(1)$.

$\lim \inf_{m \to \infty} \frac{1}{m} \log Q(m, \rho, \beta, X) \geq \lim \inf_{\rho \to 0} \lim \inf_{m \to \infty} \left[ -3\eta d \dim E^u - \sum_{i=1}^{l} \chi_{d^n} + \frac{1}{m} \log C(m, \rho, \beta) \right]$

$= -3\eta d \dim E^u - \sum_{i=1}^{l} \chi_{d^n} + h_\mu(f) = -3\eta d \dim E^u.$

The following is the analogue of Lemma 8.6 in [3].

**Lemma 3.4.** Let $X \subset Z$ be a uniformity block of tolerance $\eta > 0$. For any $\delta_0 > 0$ there exists $\rho > 0$ such that the following holds. For any $\epsilon > 0$ and $0 < \rho < \tilde{\rho}$ there exists $L \in \mathbb{N}$ (arbitrarily large), an open set $B \subset M$ of diameter less than $\epsilon$, and a set $\tilde{X} \subset B \cap X$ such that:

1. $\tilde{X}$ is $(L, \rho)$-separated,
2. $f^L(\tilde{X}) \subset B \cap X$,
3. $\sum_{x \in \tilde{X}} \exp(S_{\phi}(x)) \geq \exp((-3\eta d \dim(E^u) - \delta_0)L]$.

**Proof.** Let $\epsilon, \delta_0 > 0$ be fixed. We then consider $\xi := \frac{\delta_0}{h_\mu(f) + \frac{1}{2}}$. Let $\alpha = \alpha_1, \ldots, \alpha_t$ be a finite cover of $X$ by open sets of diameter less than $\epsilon$. For $m \in \mathbb{N}$ define $X_m := \{ x \in X \mid x \text{ and } f^n(x) \text{ are in the same element of } \alpha \text{ for some } n \in [m, (1+\xi)m] \}$. The Birkhoff Ergodic Theorem implies that $m(X_m) \to \mu(X)$ as $m \to \infty$. Let $M_0$ be such that $m(X_m) > m(X)/2$ for all $m \geq M_0$. Let $\tilde{\rho} > 0$ be small enough so that

$$\lim \inf_{m \to \infty} \frac{1}{m} \log C(m, \rho, \mu(X)/2) \geq h_\mu(f) - \frac{\xi}{2} \text{ for every } 0 < \rho \leq 2\tilde{\rho}.$$

Let $M_1 > m_0$ (where $m_0$ is the constant in Lemma 3.3) be big enough so that

$$\frac{1}{m} \log C(m, 2\tilde{\rho}, \mu(X)/2) \geq h_\mu(f) - \xi \text{ for all } m \geq M_1.$$

We now fix $m > \max(M_0, M_1, \xi^{-1} \log t)$. Chose any $\rho \in (0, \tilde{\rho})$. Let $E_m$ be a maximal $(m, \rho)$-separated set in $X_m$. Then $E_m$ is $\mu - (m, 2\rho, \mu(X)/2)$-spanning, consequently

$$\#E_m \geq \exp((h_\mu(f) - \xi)m).$$

For $n \in [m, (1+\xi)m]$ consider the set $V_n := \{ x \in E_m \mid x \text{ and } f^n(x) \text{ are in the same element of } \alpha \}$. Let $L$ be a value of $n$ maximizing $\#V_n$. Then

$$\#V_L \geq \frac{\#E_m}{\xi m} \geq \frac{\exp((h_\mu(f) - \xi)m)}{\xi m} \geq \exp((h_\mu(f) - 2\xi)m).$$
Consequently,

\[
\frac{1}{L} \log \#V_L - h_\mu(f) = \left( h_\mu(f) - 2\xi \right) - \frac{\xi}{1 + \xi} - h_\mu(f) \\
= \frac{h_\mu(f) - 2\xi - h_\mu(f) - \xi h_\mu(f)}{1 + \xi} = -\frac{2\xi - \xi h_\mu(f)}{1 + \xi} \geq -2\xi - \xi h_\mu(f).
\]

By Statement (2) in Lemma 3.3 we obtain that

\[
\sum_{x \in V_L} \exp(S_L(x)) \geq \exp(-3\eta d \dim(E^n) - 2\xi - \xi h_\mu(f))L).
\]

We now choose an element \( \bar{\alpha} \) of \( \alpha \) which maximizes the sum \( \sum_{x \in V_L \cap \bar{\alpha}} \exp(S_L(x)) \). We set \( \bar{X} := V_L \cap \bar{\alpha} \). Observe that \( \bar{X} \subset V_L \subset E_n \) is \((m, \rho)\)-separated, then it is also \((L, \rho)\)-separated since \( L \geq m \). In addition, we have that

\[
\sum_{x \in \bar{X}} \exp(S_L(x)) \geq \exp(-3\eta d \dim(E^n) - 2\xi - \xi h_\mu(f))L)
\]

Finally, since \( \xi = \frac{\delta_0}{\pi_\mu(f, e+4)} \), we conclude that

\[
\sum_{x \in \bar{N}} \sum_{x \in \bar{X}} \exp(S_L(x)) \geq \exp(-3\eta d \dim(E^n) - \delta_0) L]
\]

If now \( \bar{X} \) is a set constructed in Lemma 3.3, one can define the shift space \( \bar{X} = \bar{X}^\mathbb{Z} \) over the alphabet \( \bar{X} \). To each \( (x_n) \in \bar{X} \) one can associate a pseudo-orbit

\[
o(x_n) := \ldots, x_0, \ldots, f^{L-1}(x_0), x_1, \ldots, f^{L-1}(x_1), \ldots
\]

Observe that for each \( i \in \mathbb{Z} \) we have that \( x_i \in \bar{X} \) and \( d(f^L(x_i), x_{i+1}) < \epsilon \), where \( \epsilon > 0 \) was chosen in Lemma 3.3 as an upper bound for the diameter of the set \( B \) and can be made arbitrarily small. The following was proved in [3] Theorem 8.3.

**Theorem 3.5.** There exists \( \kappa > 0 \) such that if \( \eta > 0 \) is sufficiently small, then there are constants \( C_0, \epsilon_0 > 0 \) such that if \( \epsilon \in (0, \epsilon_0) \), then:

1. For each sequence \( (x_n) \in \mathcal{X} \) there is a unique point \( y \in M \) whose orbit \( C_0 \epsilon \)-shadows \( o(x_n) \).
2. \( y \) belongs to the regular neighborhood \( \psi_{x_0}^{-1}(B(0, r(x_0)/2)) \).
3. If \( y, y' \) shadow two pseudo-orbits \( o(x_n), o(x'_n) \) such that \( x_n = x'_n \) for \( |n| \leq N \), then \( d(y, y') \leq C_0 e^{-\kappa N} \).

We consider \( \Lambda_0 \) - the set of all the orbits that shadow the elements of \( \mathcal{X} \). We claim that the set

\[
\Lambda = \Lambda_0 \cup f(\Lambda_0) \cup \ldots \cup f^{L-1}(\Lambda_0)
\]

satisfies the assertion of Theorem B. Let \( y \in \Lambda_0 \) be the unique point that shadows \( o(x_n) \). We define cones \( C_i(f^m(y)) \) along the trajectory of \( y \) in the following way. For \( i = 1, \ldots, l, n \in \mathbb{N}, \) and \( k \in [0, L - 1] \), we define a cone \( C_i(f^m(y)) \) in \( T_{f^L(y)} M \) as the parallel transport
of the cone $\mathcal{C}(E^i(f^kx_n), \pi/4)$ - the cone at $T^*_f(x_n)M$ of angle $\pi/4$ around the subspace $E^i(f^kx_n)$.

Let $\delta_1$ be a small positive number. It was shown in [3] that by choosing $\eta > 0$ sufficiently small, one can guarantee that such defined cones are preserved under $Df$ along the orbit of $y$. In addition, there exist subspaces $E^i(f^m(y)) \subset \mathcal{C}(f^m(y))$ such that:

- $\dim(E^i) = n_i$,
- $Df^m(y) E^i(f^m(y)) = E^i(f^{m+1}(y))$,
- there exists $m_1$ such that for all $m \geq m_1$ and for all $n \in \mathbb{Z}$, and a vector $v \in E^i(f^{mL}(y))$ one has that
  \[
  \exp((\chi_i - \delta_1/2)m) \leq \|Df^m f^{mL}(v)\| \leq \exp((\chi_i + \delta_1/2)m)
  \]
  and then for large enough $m$, all $n \in \mathbb{Z}$, $k \in [0, L-1]$, and a vector $v \in E^i(f^{mL}(y))$ one has that
  \[
  \exp((\chi_i - \delta_1)\eta) \leq \|Df^m f^{mL+\eta}(v)\| \leq \exp((\chi_i + \delta_1/\eta)m).
  \]

Then the function $\phi$ is defined along the trajectory of $y$. We then conclude that $\phi$ is defined on the compact invariant set $\Lambda = \Lambda_0 \cup f(\Lambda_0) \cup \ldots \cup f^{L-1}(\Lambda_0)$. Theorem 5.3.2 in [31] gives that the subspaces $E^i(y)$ vary Hölder continuously with the point $y \in \Lambda$ and so $\phi$ is Hölder continuous on $\Lambda$.

Using Hadamard’s determinant inequality we can estimate

\[
\exp S_m \phi(y) = \prod_{i=1}^l e^{-\chi_i d_{mL} e^{\pm \delta_1 d \dim E^m} \eta} \text{ for all } y \in \Lambda.
\]

If $y \in \Lambda_0$ then by Statement (1) in Lemma 3.3 we can write

\[
\exp S_m L \phi(y) = \prod_{k=0}^{m-1} \exp S_L \phi(x_k) e^{\pm (\delta_1 + 3\eta) d \dim E^m Lm}.
\]

Now consider the collection $\mathcal{I}_m$ of all distinct $m$-tuples $(x_0, \ldots, x_{m-1}) \in X^m$. To each element in $\mathcal{I}_m$ we can assign a point $y \in \Lambda_0$ that shadows it. Denote the corresponding set of $y$'s by $Y_m$. Since the set $X$ is $(L, \rho)$-separated, choosing $\epsilon > 0$ small enough so that $C_\rho \epsilon < \rho/3$ we can guarantee that if $y, y'$ are two points shadowing two distinct $m$-tuples: $(x_0, \ldots, x_{m-1})$ and $(x'_0, \ldots, x'_{m-1})$, then $y$ and $y'$ are $(mL, \rho/3)$-separated. That means that the set $Y_m$ is $(mL, \rho/3)$-separated. In addition, we have that

\[
\sum_{y \in Y_m} \exp S_m L \phi(y) = \sum_{(x_0, \ldots, x_{m-1}) \in \mathcal{I}_m} \prod_{k=0}^{m-1} \exp S_L \phi(x_k) e^{\pm (\delta_1 + 3\eta) d \dim E^m Lm} = e^{\pm (\delta_1 + 3\eta) d \dim E^m Lm} \left( \sum_{x \in X} \exp S_L \phi(x_k) \right)^m.
\]

By Statement (3) in Lemma 3.4 we conclude

\[
\sum_{y \in Y_m} \exp S_m L \phi(y) \geq \exp \left[ \left( (L - \delta_1 - 6\eta) d \dim E^m - \delta_0 \right) Lm \right]
\]
and
\[
P(f_{\Lambda}, -d \log |\det Df_{E^0}|) = P(f_{\Lambda_0}, -d \log |\det Df_{E^0}|)
\]
\[
> (-\delta_1 - 6\eta)d \dim E^n - \delta_0.
\]

Observe that $\Lambda$ is a compact, invariant hyperbolic set for $f$ and therefore there exists a unique equilibrium measure $\nu$ for $\phi$ with respect to $f_{\Lambda}$. This measure is supported on $\Lambda$ and in particular it is hyperbolic. Because $\nu$ is an equilibrium measure, choosing sufficiently large $m > 0$, we can write that that

(3.2) \( (-\delta_1 - 6\eta)d^u_\mu - \delta_0 < P_\nu(f, \phi) = h_\nu(f) - \int d \log |\det Df_{E^0}| d\nu \)

\[
= h_\nu(f) - \frac{1}{m} \int d \log |Df_{E^0}| d\nu = h_\nu(f) - \left( \sum_{i=1}^l \chi_i d\nu_i \right) \pm \delta_1 d^u_\mu.
\]

On the other hand, since $\nu$ is hyperbolic, by Theorem 2.1 for each $i = 1, \ldots, l$ there are:

- numbers $1 \leq k_i \leq n_i$ such that for $j = 1, \ldots, k_i$ there are
- $1 \leq n_j^i \leq n_i$ with $\sum_{j=1}^{k_i} n_j^i = n_i$,
- $\chi_j^i = \chi_i \pm \delta_1$,
- $0 < d_j^i \leq n_j^i$ with $\sum_{j=1}^{k_i} \sum_{j=1}^{k_i} d_j^i = d^u_\nu$, and such that

(3.3) \[
\sum_{i=1}^l \sum_{j=1}^{k_i} d_j^i \chi_j^i = 0.
\]

Comparing the equations (3.2) and (3.3) we obtain that

\[
\sum_{i=1}^l \sum_{j=1}^{k_i} d_j^i \chi_j^i = h_\nu(f) = \left( \sum_{i=1}^l \chi_i d\nu_i \right) \pm (2\delta_1 + 6\eta)d^u_\mu \pm \delta_0.
\]

Substituting $\chi_j^i = \chi_j \pm \delta_1$, we can write

\[
\sum_{i=1}^l \left( \sum_{j=1}^{k_i} d_j^i \right) \chi_i \pm d^u_\mu \delta_1 = \left( \sum_{i=1}^l \chi_i d\nu_i \right) \pm (2\delta_1 + 6\eta)d^u_\mu \pm \delta_0.
\]

Combining like terms we obtain that

\[
\sum_{i=1}^l \chi_i \left( \sum_{j=1}^{k_i} d_j^i \right) = \pm (3\delta_1 + 6\eta)d^u_\mu \pm \delta_0.
\]

Denote $s := (3\delta_1 + 6\eta)d^u_\mu + \delta_0$ and $\gamma_i := \sum_{j=1}^{k_i} d_j^i$. We estimate

\[
-s < \sum_{i=1}^l \chi_i (\gamma_i - d\nu_i) = \sum_{\{i|\gamma_i > d\nu_i\}} \chi_i (\gamma_i - d\nu_i) + \sum_{\{i|\gamma_i \leq d\nu_i\}} \chi_i (\gamma_i - d\nu_i)
\]

\[
\leq \sum_{\{i|\gamma_i > d\nu_i\}} \chi_i (\gamma_i - d\nu_i) + \sum_{\{i|\gamma_i \leq d\nu_i\}} \chi_i (\gamma_i - d\nu_i).
\]

It follows that
\[
\sum_{\{i | \gamma_i \leq d_n\}} (\gamma_i - d_n) > -\frac{s}{\chi_1} - \frac{\chi_1}{\chi_1} \sum_{\{i | \gamma_i > d_n\}} (\gamma_i - d_n).
\]

Consequently,
\[
d^u_{\nu} - d^u_{\mu} = \sum_{i=1}^{l} (\gamma_i - d_n) = \sum_{\{i | \gamma_i > d_n\}} (\gamma_i - d_n) + \sum_{\{i | \gamma_i \leq d_n\}} (\gamma_i - d_n)
\[
> \sum_{\{i | \gamma_i > d_n\}} (\gamma_i - d_n) - \frac{s}{\chi_1} - \frac{\chi_1}{\chi_1} \sum_{\{i | \gamma_i > d_n\}} (\gamma_i - d_n).
\]
\[
\geq -\frac{s}{\chi_1} - \left(\frac{\chi_1}{\chi_1} - 1\right) (1 - d) \dim E^u.
\]

Where we estimated
\[
\sum_{\{i | \gamma_i > d_n\}} (\gamma_i - d_n) \leq \dim E^u - d \dim E^u.
\]

Finally, this gives that
\[
d^u_{\nu} \geq -\frac{s}{\chi_1} + \dim E^u \left(1 + (d - 1) \frac{\chi_1}{\chi_1}\right).
\]

To finish the proof of Theorem B it is enough to choose \(\delta_0, \delta_1\) and \(\eta\) small enough so that
\[
\frac{(3\delta_1 + 6\eta)d \dim E^u + \delta_0}{\chi_1} < \delta.
\]

4. Emergence

In this section we exploit the symbolic representation of the set \(\Lambda_0\) constructed in the previous section. Recall that \(\tilde{X}\) is the set constructed in Lemma 3.4 and we consider the shift space \(X = \tilde{X}^\mathbb{Z}\) over the alphabet \(\tilde{X}\). For \(x, x' \in \tilde{X}\) set \(\delta(x, x') = 1\) if \(x \neq x'\) and \(\delta(x, x) = 0\). We can then define the metric \(d_x\) on \(X\) as follows,
\[
d_x((x_n), (x'_n)) := \sum_{n \in \mathbb{Z}} \delta(x_n, x'_n).
\]
As before, to each \((x_n) \in X\) we associate a pseudo-orbit
\[
o(x_n) := \ldots, x_0, \ldots, f^{L-1}(x_0), x_1, \ldots, f^{L-1}(x_1), \ldots\]
Finally, we consider \(\Lambda_0\) - the set of all the orbits that shadow the elements of \(X\). We have the following.

Lemma 4.1. If \(\eta, \epsilon > 0\) are sufficiently small, then:

1. \(f^L_{\Lambda_0}\) is topologically conjugate to the full shift in \(\# \tilde{X}\) symbols,
2. the conjugacy \(\pi : X \to \Lambda_0\) is H"older continuous with respect to the metric \(d_x\) on \(X\).

Proof. Let \(\epsilon > 0\) be small enough so that \(C_0 \epsilon < \rho/3\). Since the set \(\tilde{X}\) is \((L, \rho)\)-separated, if \(y, y'\) are two points shadowing two \(\epsilon\)-pseudo orbits \(o(x_n), o(x'_n)\) with \(x_m \neq x'_m\), then \(y\) and \(y'\) are \((mL, \rho/3)\)-separated and in particular different. That means that the map \(\pi : X \to \Lambda_0\) defined by assigning to each sequence \((x_n)\) a
EMERGENCE FOR DIFFEOMORPHISMS 13

unique orbit $y \in \Lambda_0$ that shadows $o(x_n)$ is one-to-one. The fact that $\pi$ is Hölder continuous follows from Statement (3) in Theorem 3.5. □

Choose $\tilde{y} \in \Lambda_0$ and consider the set $Y := V^u(\tilde{y}) \cap \Lambda_0$, where $V^u(\tilde{y})$ denotes the local unstable manifold at $\tilde{y}$. We consider the map $F : Y \to Y$ defined by $F(y) = \theta(f^L(y))$, where for $z \in \Lambda_0$ we denote $\theta(z) = V^s(z) \cap V^u(\tilde{y}) \in Y$. Observe that:

- if a point $y \in Y$ has high pointwise emergence with respect to $F$, then it also has high pointwise emergence with respect to $f$,
- if $y \in Y$ has high pointwise emergence with respect to $f$, then so does every point $z$ on its stable manifold,
- by Lemma 4.2, $F$ is topologically conjugate to the left shift $(\sigma^+, \mathcal{X}^+)$, where $\mathcal{X}^+ = \bar{X}^N$,
- the conjugacy $\pi^+ : \mathcal{X}^+ \to Y$ is Hölder continuous.

In addition to the above observations, we prove the following.

Lemma 4.2. If $\omega^+ \in \mathcal{X}^+$ has high pointwise emergence with respect to $\sigma^+$, then $\pi^+(\omega^+)$ has high pointwise emergence with respect to $F$.

We conclude that

\begin{equation}
\dim_H(\mathcal{E}_f) \geq \dim E^s + \dim_H(\pi^+(E^+)),
\end{equation}

where $E^+$ denotes the set of points with high emergence for $\sigma^+$. We estimate $\dim_H(\pi^+(E^+))$ and conclude Theorem A in the next section.

Proof of Lemma 4.2. We consider the metric $d^+((x_n), (x_n')) := \sum_{j=0}^{\infty} \delta_{x_m,x'_m}$ on $\mathcal{X}^+$. If $m$ is the smallest natural number for which $x_m \neq x'_m$, then $d^+((x_n),(x_n')) \leq \frac{C}{2^m}$ for some uniform constant $C > 0$. At the same time, for $y = \pi^+((x_n))$, $y' = \pi^+((x'_n))$ we must have that $d(f^k(y), f^k(y')) > \rho/3$ for some $(m-1)L \leq k \leq mL$. Consequently,

\[
d(y,y') > \frac{\rho}{3} \max_{x \in \Lambda} ||Df_{E^u(x)}||^{-mL} \geq \frac{\rho}{3C} d^+((x_n),(x_n'))^K,
\]

where $K = L \log_2 \max_{x \in \Lambda} ||Df_{E^u(x)}||$.

Let now $\nu$ and $\mu$ be two Borel probability measures on $Y$ and let $\mathbb{P}$ be a Borel probability measure on $Y \times Y$ with the property that $\mathbb{P} \circ \pi_1^{-1} = \nu$ and $\mathbb{P} \circ \pi_2^{-1} = \mu$. Here $\pi_i$ is the canonical projection on the $i$'th coordinate. We consider the map $\pi^+_n : \mathcal{X}^+ \times \mathcal{X}^+ \to Y \times Y$ given by $\pi^+_n((x_n),(x'_n)) = (\pi^+(x_n), \pi^+(x'_n))$. We denote $\mathbb{P}_\pi = \mathbb{P} \circ \pi_1^{-1} = \mu \circ \pi^+$ and $\mathbb{P}_\pi = \mathbb{P} \circ \pi_2^{-1} = \nu \circ \pi^+$. Consequently,

\[
\int_{Y \times Y} d(y,y') \mathbb{P} = \int_{\mathcal{X}^+ \times \mathcal{X}^+} d(\pi^+(x_n), \pi^+(x'_n)) \mathbb{P}_\pi \geq \int_{\mathcal{X}^+ \times \mathcal{X}^+} \frac{\rho}{3C} d^+((x_n),(x'_n))^K \mathbb{P}_\pi,
\]

where the infimum is taken over all Borel probability measures with the property that $\mathbb{P}_\pi \circ \pi_1^{-1} = \mu \circ \pi^+$ and $\mathbb{P}_\pi \circ \pi_2^{-1} = \nu \circ \pi^+$. Using Jensen’s inequality, we obtain that

\[
\int_{Y \times Y} d(y,y') \mathbb{P} \geq \frac{\rho}{3C} \left( \inf \mathbb{P}_\pi \int_{\mathcal{X}^+ \times \mathcal{X}^+} d^+((x_n),(x'_n)) \mathbb{P}_\pi \right)^K.
\]
and finally,
\[ d_W(\nu, \mu) \geq \frac{P}{3C} d_W(\nu \circ \pi^+, \mu \circ \pi^+) K. \]
Then for any \( \epsilon > 0 \) and \( y \in Y \) we have that
\[ \mathcal{E}_{F,y}(\frac{P}{3C} \epsilon^K) \geq \mathcal{E}_{\sigma^+,\pi^+}^{-1}(y)(\epsilon). \]
Consequently, if \( \omega^+ \in \mathcal{X}^+ \) has high pointwise emergence with respect to \( \sigma^+ \), then for \( y = \pi^+ (\omega^+) \) we have that
\[ \limsup_{\epsilon \to 0} \frac{\mathcal{E}_{F,y}(\frac{P}{3C} \epsilon^K)}{-\log \frac{P}{3C} \epsilon^K} \geq \limsup_{\epsilon \to 0} \frac{\mathcal{E}_{\sigma^+,\omega^+}^{-1}(y)(\epsilon)}{-\log \frac{P}{3C} \epsilon^K} = \frac{1}{\dim(E^u)} \]
\[ = \frac{1}{\dim(E^u)} \dim(E^u) \]

5. **HAUSDORFF DIMENSION**

In this section we show that \( \dim_H(\pi^+(E^+)) \geq (d - d') \dim E^u \), for any
\[ d' = \frac{\delta + 6\delta_1 d \dim E^u + (1 - d) \sum_{i=1}^l \chi_i \nu_i}{\chi_1 \dim E^u}. \]
Before we start on the proof of this claim, let us see how it implies Theorem A.

**Proof of Theorem A.** We observe that by Theorem B, \( \tilde{r} = \delta + 6\delta_1 d \dim E^u \) can be made arbitrarily small and that ultimately \( d' \) can be chosen arbitrarily close to \( \frac{(1 - d) \sum_{i=1}^l \chi_i}{\chi_1 \dim E^u} \). Then conclude that
\[ \dim_H(E^u) \geq \dim(E^u) + \left( d - \frac{(1 - d) \sum_{i=1}^l \chi_i}{\chi_1 \dim E^u} \right) \]
\[ \geq \dim(E^u) + \left( d - \frac{(1 - d) \chi_1}{\chi_1} \right) \dim E^u \]
\[ = \dim(E^u) + \left( 1 - d - \frac{\chi_1}{\chi_1} \right) \dim E^u. \]

We now prove the claim. By Lemma A2.1 in [30], there exists a Hölder continuous function \( \phi_+^\pi \) on \( \mathcal{X} \), which is cohomologous to \( S_L \phi \circ \pi \) and only depends on the positive side of the sequence. By Proposition A2.2 in [30], denoting by \( \phi^+ \) the restriction of \( \phi_+^\pi \) to \( \mathcal{X}^+ \), one has that \( P(\sigma, S_L \phi \circ \pi) = P(\sigma^+, \phi^+). \) Let \( E^+ \) denote the set of points with high emergence for \( \sigma^+ \). By Theorem 1.1 in [28], \( P(\sigma^+_{|E^+}, \phi^+) = P(\sigma^+, \phi^+). \) We will use this to find a lower bound for \( \dim_H(\pi^+(E^+)) \).

Let \( d \) be as in Condition A1. Recall that
\[ P(f_{|\Lambda_0}, -d \log |Df_{|E^u}|) > -\delta, \]
and let \( \tilde{r} = \delta + 6\delta_1 d \dim E^u. \) We choose \( d' < d \) such that
\[ d' > \frac{\tilde{r} + (1 - d) \sum_{i=1}^l \chi_i}{\chi_1 \dim E^u}. \]
We then have \( d = d' + d'' \) for some \( d'' > 0. \) Since \( \phi_+^\pi \) is cohomologous to \( S_L \phi \circ \pi \), there exists a Hölder continuous function \( u : Y \to \mathbb{R} \) such that
\[ \phi^+ \circ (\pi^+)^{-1}(x) = S_L \phi(x) + u(F(x)) - u(x) \]
for all $x \in Y$. Since $P(f|\Lambda_0, -d \log |\det Df|_{Ec}) > -\delta$, then also $P(F_{\pi^+(E^+)} \circ \phi^+ \circ (\pi^+)^{-1}) > -\delta L$ and we have that

$$\infty = \liminf_{N \to \infty} \sum_{(x,m) \in I_F} \exp \left[ S_{F,m} \phi^+ (\pi^+)^{-1} (x) + \delta mL \right]$$

$$= \liminf_{N \to \infty} \sum_{(x,m) \in I_F} \exp \left[ S_{mL} \phi(x) + u(f^{mL}(x)) - u(x) + \delta mL \right],$$

where the infimum is taken over all collections of pairs $(x, m)$ with $x \in Y, m \geq N$ such that $\pi^+(E^+) \subset \bigcup_{(x,m) \in I_F} B_{F,m}(x,s)$. The notation $S_{F,m}, B_{F,m}$ indicates that the summation and Bowen balls are with respect to $F$. Recall that $S_m, B_m$ denote the summation and Bowen ball with respect to $f$. By \[14\], we continue

$$\leq \liminf_{N \to \infty} \sum_{(x,m) \in I} \exp \left[ \sum_{i=1}^{l} (\chi_i dn_i m) + \delta_i md \dim E^u + U + \delta mL \right],$$

where $U := 2 \max_{x \in Y} |u(x)|$ and the infimum is taken over all collections of pairs $(x, m)$ with $x \in Y, m \geq NL$ is a multiple of $L$, and such that $\pi^+(E^+) \subset \bigcup_{(x,m) \in I} B_{m}(x,s)$. For a fixed $N \in \mathbb{N}$ denote

$$\zeta_N := \inf_I \sum_{(x,m) \in I} \exp \left[ \sum_{i=1}^{l} (\chi_i dn_i m) + \delta_i md \dim E^u + U + \delta mL \right].$$

We have the following foliated structure on $Y$ induced by ”intermediate” unstable manifolds $V^1(x), \ldots, V^l(x)$ corresponding to distinct Lyapunov exponents:

$$V^u(x) = V^1(x) \supset V^2(x) \supset \ldots \supset V^l(x),$$

with

$$V^i(x) := \{ y \in B(x, r(x)) \mid \limsup_{n \to \infty} \frac{1}{n} \log d(f^{-n}(x), f^{-n}(y)) < -\chi_i \}.$$ 

By slightly abusing the notation, we will consider sets

$$B_{m_1}^1(x,s) \times \ldots \times B_{m_l}^l(x,s), \text{ where } B_{m_i}^i(x,s) := B_{m_i}(x,s) \cap V^i(x)$$

and $B_{m_1}^1(x,s) \times \ldots \times B_{m_l}^l(x,s)$ is defined as follows.

We first denote

$$T_l := B_{m_l}^l(x,s).$$

Then we define $T_{l-1} := \bigcup_{x^{(l)} \in T_l} B_{m_{l-1}}^{l-1}(x^{(l)}, s),$ 

$$T_{l-2} := \bigcup_{x^{(l-2)} \in T_{l-1}} B_{m_{l-2}}^{l-2}(x^{(l-1)}, s).$$

Continuing in this manner we finally define $B_{m_1}^1(x,s) \times \ldots \times B_{m_l}^l(x,s)$ as

$$T_1 := \bigcup_{x^{(1)} \in T_2} B_{m_1}^1(x^{(1)}, s).$$

Observe that there exist $s_1, s_2 > 0$ such that for every $m \in \mathbb{N}$

$$B_m(x,s_1) \subset B_m^1(x,s) \times \ldots \times B_m^l(x,s) \subset B_m(x,s_2).$$
Given $m \in \mathbb{N}$, and $x \in Y$, we set $m_i = m_i(m, x, s)$ for $i = 2, \ldots, l$ such that
\begin{equation}
1 \leq \frac{\text{diam}(B_m^i(x, s))}{\text{diam}(B_{m_1}^i(x, s))} \leq \max_{x \in \Lambda} \| Df_{E^u(x)} \|.
\end{equation}

We then consider collections $\mathcal{I}^E$ of tuples $(x, x^2, \ldots, x^l, m)$ such that
\[ x^l \in B_{m_1}^l(x, s), \ x^l-1 \in B_{m_{l-1}}^{l-1}(x_1, s), \ldots, x^2 \in B_{m_2}^2(x_3, s), \text{ and} \]
\[ \pi^+(E^+) \subseteq \bigcup_{(x,x^2,\ldots,x^l,m) \in \mathcal{I}^E} (B_m^1(x^2, s) \times B_m^2(x^3, s) \times \cdots \times B_m^l(x^l, s)), \]
and for every given $x \in Y$ and $m \in \mathbb{N}$ the corresponding subcollection $\mathcal{I}_{x,m} := \{(x, x^2, \ldots, x^l, m) \} \subseteq \mathcal{I}^E$ (if not empty) satisfies
\[ B_{m_1}^1(x, s) \times \cdots \times B_{m_1}^l(x, s) \subseteq \bigcup_{(x,x^2,\ldots,x^l,m) \in \mathcal{I}_{x,m}} B_m^1(x^2, s) \times B_m^2(x^3, s) \times \cdots \times B_m^l(x^l, s). \]

Given $x \in Y$ and $m \in \mathbb{N}$ one can construct $\mathcal{I}_{x,m}$ inductively as follows. We first choose $\{x_{j_1}^l\} \subset B_{m_1}^l(x, s)$ such that
\[ \bigcup_{j_1} B_{m_1}^l(x_{j_1}^l, s) \supseteq B_{m_1}^l(x, s). \]

The elements $\{x_{j_1}^l\}$ can be chosen in such a way that their number does not exceed
\[ \left( \frac{\text{diam}B_{m_1}^l(x, s)}{1/3 \min_j \text{diam}B_{m_1}^l(x_{j_1}^l, s)} \right)^{n_1} \leq \left( \frac{\text{diam}B_m^1(x, s) \max_{x \in \Lambda} \| Df_{E^u(x)} \|}{1/3 \min_j \text{diam}B_{m_1}^l(x_{j_1}^l, s)} \right)^{n_1} \leq \left( \frac{\max_{x \in \Lambda} \| Df_{E^u(x)} \|}{e(-\chi_1 + \delta_1)m} \right)^{n_1} \leq e^{(\chi_1 - \chi_1 + 3\delta_1)mn_1}. \]

Next, for each $x_{j_1}^l$ in the previous step, we find a collection $\{x_{j_1-1}^{l-1}\} \subset B_{m_{l-1}}^{l-1}(x_{j_1}^l, s)$ such that
\[ \bigcup_{j_1} B_{m_{l-1}}^{l-1}(x_{j_1-1}^{l-1}, s) \supseteq T_{l-1}. \]

The number of elements produced in this step (for each $x_{j_1}^l$) can be estimated from above by
\[ \left( \frac{\text{diam}B_{m_{l-1}}^{l-1}(x, s)}{1/3 \min_j \text{diam}B_{m_{l-1}}^{l-1}(x_{j_1-1}^{l-1}, s)} \right)^{n_{l-1}} \leq e^{(\chi_{l-1} - \chi_1 + 3\delta_1)mn_{l-1}}. \]

Continuing the construction in this manner we finally obtain that the set $B_{m_1}^1(x, s) \times \cdots \times B_{m_1}^l(x, s)$ is covered by a union of no more than
\[ \prod_i e^{(\chi_1 - \chi_1 + 3\delta_1)mn_i} \]

elements of the form $B_m^1(x^2, s) \times B_m^2(x^3, s) \times \cdots \times B_m^l(x^l, s)$. \]
We then continue with the estimate of $\zeta_N$, 

$$
\zeta_N \leq \inf_{\mathcal{I}^S} \sum_{(x, m, m_2, \ldots, m_l) \in \mathcal{I}^S} \exp \left[ \sum_{i=1}^{l} (-\chi_i d_i n_i m) + \delta_1 m d \dim E^u + U + \delta m \right] 
$$

$$
\leq \inf_{\mathcal{I}^S} \sum_{(x, m, m_2, \ldots, m_l) \in \mathcal{I}^S} \exp \left[ \sum_{i=1}^{l} (-\chi_i d_i n_i m) + \tau_1 m + \sum_{i=1}^{l} (\chi_i - \chi_1) mn_i \right]. 
$$

where we denoted $\tau_1 := \delta + 5 \delta_1 d \dim(E^u)$ and the last infimum is taken over all collections of tuples $(x, m, m_2, \ldots, m_l)$ with $x \in Y, m \geq N$ being a multiple of $N$, $m_i = m_i(m, x, s)$ such that

$$
\pi^+(E^+) \subset \bigcup_{(x, m, m_2, \ldots, m_l) \in \mathcal{I}^S} B_m^1(x, s) \times B_m^2(x, s) \times \cdots \times B_m^l(x, s).
$$

Writing $d = d' + d''$ we continue,

$$
\zeta_N \leq \inf_{\mathcal{I}^S} \sum_{(x, m, m_2, \ldots, m_l) \in \mathcal{I}^S} \exp \left[ -\chi_1 d'' \dim E^u m - \chi_1 d' \dim E^u m + \tau_1 m + (1 - d)m \sum_{i=1}^{l} (\chi_i - \chi_1) n_i \right] 
$$

$$
\leq \inf_{\mathcal{I}^S} \sum_{(x, m, m_2, \ldots, m_l) \in \mathcal{I}^S} \diam B_m^l(x, s) \sum_{(x, m, m_2, \ldots, m_l) \in \mathcal{I}^S} \exp \left[ -\chi_1 d' \dim E^u m + \tilde{\tau} m + (1 - d)m \sum_{i=1}^{l} (\chi_i - \chi_1) n_i \right]. 
$$

In addition, by the choice of $d'$, we have that

$$
\exp \left[ -\chi_1 d' \dim E^u m + \tilde{\tau} m + (1 - d)m \sum_{i=1}^{l} (\chi_i - \chi_1) n_i \right] < 1. 
$$

We observe that there exist uniform constants $c_1, c_2$ such that for every $(x, m, m_2, \ldots, m_l) \in \mathcal{I}^S$ one has that

$$
B(x, c_1 \diam B_m^l(x, s)) \subset B_m^1(x, s) \times B_m^2(x, s) \times \cdots \times B_m^l(x, s) \subset B(x, c_2 \diam B_m^l(x, s)).
$$

Consequently,

$$
\inf_{(B(x, \epsilon))} \sum \diam (B(x, \epsilon)) \cdot \sum \diam (E^u) \geq \zeta_N,
$$

where the infimum is taken over all collections of open balls $B(x, \epsilon)$ with $\epsilon < \bar{\epsilon}$ covering $\pi^+(E^+)$ and $N = N(\bar{\epsilon})$ is large enough. By the definition of the Hausdorff dimension we conclude that $\dim_H(\pi^+(E^+)) \geq d'' \dim E^u$.

**References**

[1] F. Abdenur, C. Bonatti, S. Crovisier, *Nonuniform hyperbolicity for C^1-generic diffeomorphisms*, Israel Journal of Mathematics 183 (2011), 1–60.

[2] V. Araújo, V. Pinheiro, *Abundance of wild historic behavior*, Bulletin of the Brazilian Mathematical Society, New Series (2019), 1–36.

[3] A. Avila, S. Crovisier, and A. Wilkinson, *C^1 density of stable ergodicity*, Advances in Mathematics, (2021) 379:107496.

[4] P. Barrientos, S. Kiriki, Y. Nakano, A. Raibekas, T. Soma, *Historic behavior in non-hyperbolic homoclinic classes*, Proceedings of the American Mathematical Society 148 (2020), 1195–1206.

[5] L. Barreira, J. Schmeling, *Sets of “non-typical” points have full topological entropy and full Hausdorff dimension*, Israel Journal of Mathematics 116 (2000), 29–70.

[6] P. Berger, *Emergence and non-typicality of the finiteness of the attractors in many topologies*, Proceedings of the Steklov Institute of Mathematics 297 (2017), 1–27.
[7] P. Berger, Complexities of differentiable dynamical systems, Journal of Mathematical Physics (2020).
[8] P. Berger, S. Biebler, Emergence of wandering stable components, arXiv preprint arXiv:2001.08649 (2020).
[9] P. Berger, J. Bochi, On Emergence and Complexity of Ergodic Decompositions, Advances in Mathematics. 2021 ; Vol. 390.
[10] T. Bomfim, P. Varandas, Multifractal analysis for weak Gibbs measures: from large deviations to irregular sets, Ergodic Theory and Dynamical Systems 37 (2017), 79–102.
[11] Y. Cao, L. Zhang, Y. Zhao, The asymptotically additive topological pressure on the irregular set for asymptotically additive potentials, Nonlinear Analysis: Theory, Methods & Applications 74 (2011), 5015–5022.
[12] T. Bomfim, P. Varandas, Multifractal analysis for weak deviations to irregular sets, Ergodic Theory and Dynamical Systems 37 (2017), 79–102.
[13] E. Catsigeras, X. Tian, E. Vargas, Topological entropy on points without physical-like behaviour, Mathematical Zeitschrift (2015), 1–13.
[14] E. Chen, T. Küpper, L. Shu, Topological entropy for divergence points, Ergodic Theory and Dynamical Systems (2005), 1173–1208.
[15] E. Chen, Y. Ji, X. Zhou, Entropy and Emergence of Topological Dynamical Systems, arXiv preprint arXiv:2005.01548 (2020).
[16] E. Chen, T. Küpper, L. Shu, Topological entropy for divergence points, Ergodic Theory and Dynamical Systems (2005), 1173–1208.
[17] K. Gelfert, Horseshoes for diffeomorphisms preserving hyperbolic measures, Math. Z., 283:685–701, 2016.
[18] F. Hofbauer, G. Keller, Quadratic maps without asymptotic measure, Communications in mathematical physics 127 (1990), 319–337.
[19] A. Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, Publications Mathématique de l’IHEs 51 (1980), 137–173.
[20] A. Katok, B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, 54, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1995.
[21] S. Kiriki, Y. Nakano, T. Soma, Emergence via non-existence of averages, arXiv preprint arXiv:1904.03421 (2019).
[22] S. Kiriki, T. Soma, Tokens’ last problem and existence of non-trivial wandering domains, Advances in Mathematics 306 (2017), 524–588.
[23] F. Ledrappier, L.-S. Young, The Metric Entropy of Diffeomorphisms: Part II: Relations between Entropy, Exponents and Dimension, Annals of Mathematics 122, no. 3 (1985): 540–74.
[24] S. Luzatto, F. J. Sánchez-Salas, Uniform hyperbolic approximation of measures with non-zero Lyapunov exponents, Proc. Amer. Math. Soc. 141 (2013), 3157–3169.
[25] McCluskey, H., Manning, A. Hausdorff dimension for horseshoes, Ergodic Theory and Dynamical Systems, 3(2), 251-260, (1983).
[26] L. Mendoza, Ergodic attractors for diffeomorphisms of surfaces, J. London Math. Soc. (2) 37 (1988) 362-374.
[27] M. Misiurewicz and W. Szlenk, Entropy of piecewise monotone mappings, Studia Math. 67 (1980), 1, 45–63.
[28] Y. Nakano, A. Zelerowicz Highly irregular orbits for subshifts of finite type: large intersections and emergence, 2021 Nonlinearity 34 7609.
[29] T. Persson and J. Schmeling, Dyadic diophantine approximation and Katok’s horseshoe approximation, Acta Arith. 132 (2008), 205–230.
[30] Ya. Pesin, Dimension theory in dynamical systems: Contemporary Views and Applications, University of Chicago Press, 1997.
[31] L. Barreira, and Y. Pesin, Nonuniform Hyperbolicity: Dynamics of Systems with Nonzero Lyapunov Exponents (Encyclopedia of Mathematics and its Applications). (2007). Cambridge: Cambridge University Press.
[32] Y. Pesin, B. Pitskel’, Topological pressure and the variational principle for noncompact sets, Functional Analysis and its Applications 18 (1984), 307–318.
[33] F. Przytycki and M. Urbański, Conformal Fractals – Ergodic Theory Methods, Cambridge University Press, 2010.
[34] D. Ruelle, Historical behaviour in smooth dynamical systems, Global Analysis of Dynamical Systems (eds. H. W. Broer et al), Institute of Physics Publishing (2001), 63–66.
[35] F. Sánchez-Salas, Ergodic attractors as limits of hyperbolic horseshoes, (2002) Ergodic Theory and Dynamical Systems, 22(2), 571-589.

[36] F. Takens, Orbits with historic behaviour, or non-existence of averages, Nonlinearity 21 (2008), 33–36.

[37] A. Talebi, Non-statistical rational maps, arXiv preprint arXiv:2003.02185 (2020).

[38] D. Thompson, The irregular set for maps with the specification property has full topological pressure, Dynamical Systems 25 (2010), 25–51.

[39] D. Thompson, Irregular sets, the $\beta$-transformation and the almost specification property, Transactions of the American Mathematical Society 364 (2012), 5395–5414.

[40] X. Tian, Topological pressure for the completely irregular set of Birkhoff averages, Discrete & Continuous Dynamical Systems-A 37 (2017), 2745–2763.

[41] D. Yang, On the historical behavior of singular hyperbolic attractors, Proceedings of the American Mathematical Society 148 (2020), 1641–1644.

[42] Y. Yang, Horseshoes for $C^{1+\alpha}$ mappings with hyperbolic measures, Disc. Contin. Dynam. Sys., 35: 5133–5152, 2015.

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