Abstract
Multi-path becomes a main source of positioning errors in new-generation global navigation satellite systems (GNSS). Code multi-correlator discriminators (MCD), represented by narrow early-minus-late (NEMIL) and double delta (DD) discriminators which are designed for binary phase offset keying (BPSK) signals, are one of the main multi-path mitigation techniques. However, because of the complicated auto-correlation function and intricate relationships among multiple correlators, it is hard to implement a trial and error approach, which is the conventional design method of discriminator structures, to specifically design code MCD structures for new GNSS signals which have complicated spreading waveforms. In this article, instead of a trial and error approach, a general design methodology for designing code MCD structures based on heuristic optimisation for multi-path mitigation is proposed. The proposed method can specifically design code MCD structures for signals with various modulations and receivers with different bandwidths, indicating the adaptability of the proposed method. Multi-path mitigation performances of such designed structures are better than those of traditional code discriminator structures with strong tracking robustness and slight degradation of thermal noise performance. Designed code MCD structures are also insensitive to multi-path relative amplitudes and bandwidths of receivers, ensuring the practicality of the designed code MCD structures.

1 | INTRODUCTION

Global navigation satellite systems (GNSS) provide positioning, navigation and timing service. To promote the accuracy of positioning, on one hand, several new signals are employed in the new generation GNSS. Compared with the binary phase shift keying (BPSK) signal, which was first used in GNSS, binary offset carrier (BOC) [1] signals and multiplexed BOC (MBOC) [2] signals have more complicated auto-correlation functions (ACFs) and greater potential to improve the accuracy of positioning. On the other hand, with the development of various GNSS receiving and processing techniques, such as high precision differential techniques, performance of positioning is significantly improved. Nevertheless, there are still several non-negligible factors that reduce positioning accuracy, such as multi-path, which has become one of the main sources of positioning errors.

To alleviate the effects of multi-path, various multi-path mitigation techniques have been proposed that can be divided into three classes. The first class is the antenna-based multi-path cancellation technique, such as the choke ring [3, 4], multi-path-limiting antenna (MLA) [5, 6] and phased-array antennas [7], which mitigate multi-path signals before they enter the receiver. The second class is the multi-path elimination technique based on multi-path parameter estimation, such as multi-path estimation delay lock loop (MEDLL) [8, 9]. MEDLL can effectively mitigate multi-path by estimating multi-path parameters, such as the number of multi-paths, relative delay and amplitudes. However, it is hard to mitigate multi-path in real time owing to the large calculation burden, especially when the number of multi-path is unknown. In addition to MEDLL, there is also a Bayesian method implemented by particle filters to estimate multi-path parameters [10]. The third class is the multi-path suppression technique based on discriminator
structure design. Several types of techniques within this class include: the early-minus-late (EML) structure, the narrow early-minus-late (NEML) structure and the enhanced traditional code discriminator structure with better multi-path mitigation performance, such as double delta (DD) correlator structures and strobe correlator structures, which can be collectively termed as high resolution correlator (HRC) structures. HRC structures have better multi-path mitigation performance than any other traditional code discriminator structures.

The code multi-correlator discriminator (MCD) structure in a delay lock loop (DLL) is a general discriminator structure, of which traditional code discriminator structures are specific implementations. Since the code MCD structure has great design flexibility to fulfill diverse design requirements and potential to reach better multi-path mitigation performance, it is widely used in various applications. A specific designed code MCD structure can track the high-order BOC signal without ambiguity. The code MCD structures designed by the method based on quadratic programming have better characteristics of the discriminator’s output, which means that the tracking loop is less susceptible to noise and interference. Furthermore, MCD structures can also be used for spoofing signal suppression. According to these applications in various fields, code MCD structures have a very broad development prospect, implying that code MCD structures have great potential to improve the multi-path mitigation performance. Therefore, this article mainly focusses on the design of code MCD structures.

In the field of multi-path mitigation, the conventional method of designing code discriminator structures is the trial and error approach and traditional code discriminator structures, such as NEMl structures and DQ structures, are both specifically designed for BPSK signals by this method. On one hand, since new signals such as BOC signals and MBOC signals, have much more complicated ACFs than BPSK signals, it is difficult for the trial and error approach to specifically design code discriminator structures, and the existing traditional code discriminator structures are obviously suboptimal for these signals, causing that the positioning accuracy of new signals can be significantly influenced by multi-path. On the other hand, although code MCD structures have great potential to mitigate multi-path, the intricate relationships among multiple correlators make it harder to design code MCD structures by trial and error.

Corresponding to the above issues in designing code MCD structures, this article proposes a general design methodology of code MCD structures for multi-path mitigation based on heuristic optimisation. The proposed method can specifically design code MCD structures for signals with various modulations and receivers with different bandwidths, which illustrates the adaptability of the proposed method. Such code MCD structures have better multi-path mitigation performance than traditional code discriminator structures with strong tracking robustness and slight sacrifice of thermal noise performance. The proposed method is also insensitive to multi-path relative amplitudes and bandwidths of receivers. On the other hand, the code MCD structure designed with one specific multi-path relative amplitude can greatly mitigate multi-path signals with other multi-path relative amplitudes. On the other hand, the code MCD structure designed with one specific bandwidth still has better anti-multi-path performance than that of traditional code discriminator structures when applied to receivers with other bandwidths.

This article is organised as follows. Section 2 formalises the satellite baseband spreading signals and analyses the ACF and power spectral density (PSD) characteristics of signals. Then the code MCD model is established to show the relationship among the multiple correlators. Section 3 introduces the multi-path error optimisation model whose goal is to minimise the mean multi-path errors and to propose the general design methodology of code MCD structures for multi-path mitigation. In Section 4, the adaptability, insensitivity, thermal noise and tracking robustness analysis of the proposed design method are represented by case studies. Finally, conclusions are discussed in Section 5.

2 | SATELLITE SIGNALS AND CODE MCD MODEL

This section first models satellite baseband spreading signals and presents the ACF and PSD characteristics of signals. Then the code multi-correlator discriminator model is established to illustrate relationships between multiple correlators in code MCD structures.

2.1 | Satellite baseband spreading Signal

GNSS signals adopt direct sequence spread spectrum (DSSS) technology. DSSS signal, which can also be called as spreading signal, is utilised to modulate navigation data signals. Since the spreading sequence, alternatively termed as pseudo random noise (PRN) code, has much higher code rate than navigation data sequence, frequent phase reversal of spreading sequence can be used for precise ranging. According to the great orthogonal cross-correlation characteristics of spreading sequence, which is the basis of code division multiple access (CDMA) system, satellites can transmit multiple signals on the same carrier frequency, and GNSS receivers can separate different satellite signals by utilising the cross-correlation characteristic of spreading sequence.

A general form of the baseband spreading signal $g(t)$ can be expressed as

$$g(t) = \sum_{i=-\infty}^{\infty} c_i p(t - iT_c)$$

where, $\{c_i\}$ is a periodic spreading sequence with the code rate $f_c = n \times f_0$, where $f_0 = 1.023$ MHz is the reference frequency in GNSS, $T_c = 1/f_c$ is the length of one chip, and $p(t)$ is a spreading chip waveform, which takes non-zero values for
$0 \leq t < T_c$ and takes zero values elsewhere. It can be further expressed as

$$p(t) = \sum_{j=0}^{K-1} m_j \psi(t - jT_s)$$

(2)

where, $m = \{m_j\}$ is the modulation sequence of the spreading chip waveform with rate $f_s = m \times f_0$, $\psi(t)$ is the waveform with the rectangular pulse which takes the value one for $0 \leq t < T_s = 1/(2f_0)$, and zero elsewhere and $K$ is the length of the modulation sequence. To align the spreading sequence with the modulation sequence, the ratio $K = T_c/T_s$ should be a positive integer. The modulation sequence of BPSK(1) signals, sine-phased BOC(1,1) signals and cosine-phased BOC(1,1) signals are

$$m_{\text{BPSK}(1)} = [1]$$
$$m_{\text{BOC}_{\text{s}}(1,1)} = [1, -1]$$
$$m_{\text{BOC}_{\text{c}}(1,1)} = [1, -1, -1, 1]$$

(3)

respectively. The ACF of baseband spreading signals $g(t)$ can be calculated as

$$R(\tau) = \frac{1}{T} \int_0^T g(t)g(t - \tau)dt$$

(4)

where, $T$ is the integral length.

Since signals received by front-end antennas are multiplied with the local carrier and then filtered by the front-end low-pass filter (LPF), the obtained baseband spreading signals are always bandlimited. In the finite bandwidth condition, with the impulse response function of LPF $h_{\text{BW}}(t)$, filtered baseband spreading signals $g_{\text{BW}}(t)$ should be written as

$$g_{\text{BW}}(t) = g(t) \otimes h_{\text{BW}}(t)$$

(5)

where $\otimes$ is the convolution operator. The single-side bandwidth is denoted as $BW$. The ACF of bandlimited signals, denoted as $R_{\text{BW}}(\tau)$, can be calculated by substituting $g(t)$ for $g_{\text{BW}}(t)$ in Equation (4). Figure 1 shows the ACFs of four kinds of modulated signals, which are BPSK(1), BOC(1,1), BOC$_{\text{s}}$(15,2.5) and CBOC(6,1,1/11,+), in infinite and finite bandwidth.

Except for ACF of signals, the normalised PSD of signals is another critical characteristic, which can be calculated as

$$G_s(f) = \frac{1}{T_c} Q(f) Q^*(f)$$

(6)

where, $G_s(f)$ is the normalised PSD of modulated signals, $Q(f)$ is the Fourier Transform (FT) of baseband spreading signals. The normalised PSDs of BPSK(1), BOC($kn$, $n$) and BOC$_{\text{c}}$($kn$, $n$) signals can be written as

$$G_s^{\text{BPSK}(1)}(f) = T_c \left( \frac{\sin(\pi f T_c)}{\pi f T_c} \right)^2$$
$$G_s^{\text{BOC}_{\text{s}}(kn,n)}(f) = \frac{1}{nT_s} \left( \frac{\sin(\pi f n T_s)\sin(\pi f T_s)}{\pi f \cos(\pi f T_s)} \right)^2$$
$$G_s^{\text{BOC}_{\text{c}}(kn,n)}(f) = \frac{1}{nT_s} \left( \frac{2\sin(\pi f n T_s)\sin^2(\pi f T_s)}{\pi f \cos(\pi f T_s)} \right)^2$$

(7)

respectively, where $n = 2T_c/T_s$. According to the definition of CBOC signals, the PSD of CBOC(6,1,1/11,+ signals is
where $Q_{BOC(1,1)}(f)$ and $Q_{BOC(6,1)}(f)$ are the FT of BOC(1,1) and BOC(6,1) signals, respectively. Figure 2 shows the normalised PSDs of BPSK(1), BOC(1,1) and BOC(15,2.5) and CBOC(6,1,1/11,+) signals, which will be utilised in thermal noise analysis in Section 4.

### 2.2 Code MCD model

Code multi-correlator discriminators can be divided into two classes: the non-coherent discriminator and the coherent discriminator. Compared with coherent discriminators, non-coherent discriminators have greater mean code tracking errors when the carrier of signals is steadily tracked by a phase lock loop (PLL) [18, 20, 21]. This article focuses on the design of coherent discriminator structures to fully utilise their advantages. Although coherent discriminators are invalid when the carrier phase is out of lock, receivers can take the strategy that non-coherent discriminators are first used to ensure locking of the carrier phase, then coherent discriminators are applied to reach better performance in the estimation of code phase and multi-path mitigation. Figure 3 is the DLL structures with multiple correlators discriminators. The output of a general coherent code MCD can be written as

$$S(\tau|w,d) = \sum_{i=1}^{N} w_i R(\tau + d_i)$$

(9)

where, $N$ is the number of correlators, $w = [w_1, \ldots , w_N]$ is the weights vector of correlators, $d = [d_1, \ldots , d_N]$ is the code delay vector of correlators, which can also be termed as the location vector, $w_i$ and $d_i$ are the weight and location of the $i$th correlator, respectively.

Traditional discriminator structures, such as EML, NEML and DD structures, are special cases of code MCD structures. EML and NEML structures in DLL usually utilise two correlators: an early correlator and a late correlator. The parameters of traditional code discriminators are $w = [1,-1]$ and $d = [-\Delta T/2, \Delta T/2]$, where $\Delta T$ is the early-to-late space. DD structures have four correlators: a very-early correlator, an early correlator, a late correlator and a very-late correlator. The parameters of DD structures are $w = [-0.5, 1.0, -1.0, 0.5]$ and $d = [-\Delta T, -\Delta T/2, \Delta T/2, \Delta T]$.

Figure 4 (a) and (b) show the traditional code discriminator’s output of EML, NEML and DD structures in finite and infinite bandwidths conditions for BPSK(1) signals. The output of code MCD discriminators, denoted as $S$-curve, can be affected by the Gibbs effect in finite bandwidth condition. The presence of multi-path signals, as Figure 4 (c) and (d) show, can cause the zero-crossing point of the $S$-curve tracked by DLL, which is the closest to zero chip, to deviate from the zero-chip point by contaminating the ACF of signals. Shifted distance between the zero-chip point and the zero-crossing point is the code phase error caused by multi-path signals, which cannot be eliminated by DLL. Compared with EML structures, the code multi-path errors of NEML and DD structures are smaller, illustrating the better anti-multi-path performance.

### 3 DESIGN METHODOLOGY

This section establishes the multi-path error optimisation model. The design methodology of code MCD structures for multi-path mitigation will then be introduced.

#### 3.1 Multi-path error optimisation model

Signals received by antennas consist of two parts: the direct path signal and the multi-path signals. Compared with the direct path signal, multi-path signals have a longer travel path, decay in signal power and relative shifted phase by reflecting and diffracting. With the presence of multi-path signals, $S$-curve of code MCD structures can be written as

$$S_{MP}(\tau|w,d,\tau_m,\phi,a) = \sum_{i=1}^{N} w_i \{R(\tau + d_i) + aR(\tau + d_i - \tau_m)\cos\phi\}$$

(10)

where, $a$ is the multi-path relative amplitude, $\phi$ is the multi-path relative phase and $\tau_m$ is the multi-path relative delay. Since the received signal is contaminated by multi-path signals, the symmetry of the signal’s ACF is broken and the code phase
**Figure 3** Delay lock loop structures with multi-correlator discriminator

**Figure 4** S-curves of discriminators with EML, NEML and DD structures for BPSK signals with path relative amplitude $\alpha = 0.5$ in (a) infinite bandwidth and $\tau_m = 0$ chips; (b) one-side bandwidth = $5f_c$ and $\tau_m = 0.4$ chips; (c) infinite bandwidth and $\tau_m = 0.4$ chips; (d) one-side bandwidth = $\tau_m = 0.4$ chips conditions. BPSK, binary phase shift keying; DD, double delta; EML, early-minus-late; NEML, narrow early-minus-late
tracked by DLL has multi-path errors, which are denoted as \( \tau_d \) and can be expressed as
\[
\tau_d = \arg \min_{\tau} \{ S_{\text{MP}}(\tau | w, d, \tau_m, \phi, \alpha) = 0 \}
\]

(11)

Multi-path error \( \tau_d \) is determined by multi-path relative delay \( \tau_m \) and multi-path relative phase \( \phi \) with the given multi-path relative amplitude \( \alpha \) and code MCD structure parameters \( w \) and \( d \). Multi-path errors under different multi-path relative delays constitute the multi-path error curve. When absolute amplitudes of multi-path signals have maximal values at \( \phi = 0^\circ \) or \( 180^\circ \), the multi-path error curve reaches its envelope. For any \( \phi \), multi-path error curves lie in the range of the multi-path error envelope (MEE).

Figure 5 exhibits the MEE of traditional code discrimination structures with EML, NEML, and DD structures for BPSK(1) signals. As Figure 5 shows, DD structures have the smallest MEE of these three traditional code discriminator structures, which indicates the best multi-path mitigation performance. The area enclosed by multi-path error envelopes can be defined as the mean multi-path error (MME), and can be written as
\[
\text{MME}(w, d | \alpha) = \int_{0}^{\infty} \left[ | \text{MEE}_+(\tau_m | w, d, \alpha) | + | \text{MEE}_-(\tau_m | w, d, \alpha) | \right] d\tau_m
\]

(12)

where \( \text{MEE}_+ \) and \( \text{MEE}_- \) are multi-path error curves with \( \phi = 0^\circ \) and \( \phi = 180^\circ \), respectively.

Since weights vector \( w \) and location vector \( d \) are continuous bounded parameters, a minimal value can be derived from the range of MME values, which indicates that the optimisation goal of this model is to minimise mean multi-path errors. More specifically, to obtain optimal code MCD structure which has the best multi-path mitigation performance, code MCD structure parameters \( w \) and \( d \) should be optimised. According to the symmetry of the signal's ACF, \( w \) and \( d \) have the inner relationship: \( w_i = -w_{N-i}, d_i = -d_{N-i}, i = 1, 2, \ldots, N \). Therefore, a general multi-path error optimisation model can be concluded as
\[
\min_{w,d} \text{MME}(w, d | \alpha) \quad \text{s.t.} \quad \begin{cases} w_i + w_{N-i} = 0, & i = 1, \ldots, N \quad \text{(13)} \\ d_i + d_{N-i} = 0, & i = 1, \ldots, N \end{cases}
\]

where the multi-path error optimisation model is optimised in one specific multi-path relative amplitude condition. Although multi-path signals with one specific multi-path relative amplitude cannot represent the complicated multi-path signals received from the real-world environment, studies derived from it can provide useful diagnostic insights. The solutions of \( w \) and \( d \) obtained by optimising the multi-path error optimisation model can construct code MCD structures that have the best anti-multi-path performance.
The multi-path error optimisation model with the goal to minimise mean multi-path errors can be effectively solved by optimisation methods. Considering that ACFs of signals are segmented, which can be derived from Figure 1, outputs of discriminators, which are the linear combination of shifted ACFs, should have complicated segmented mathematical expressions. Owing to the segmented characteristics of S-curves, zero-crossing points of S-curves should be piecewise-solved, causing their mathematical expressions to be much more complicated [22]. Therefore, it is difficult to apply derivative-based optimisation methods to the design of code MCD structures, such as gradient descent and Newton’s method.

Except for derivative-based optimisation methods, the heuristic algorithm, which is inspired by natural creatures, can effectively solve optimisation problems in a swarm intelligence method, such as ant colony optimisation [23], simulated annealing [24, 25], genetic algorithm (GA) [26, 27] and differential evolution (DE) algorithms [28, 29]. Genetic algorithms and differential evolution algorithms imitate the evolution of biological populations by natural rules such as genetic hybridisation, genetic mutation and survival of the fittest. Since binary encoding for genes of populations adopted by GA introduces the ‘Hamming cliff’ problem [30], which includes the instability of the algorithm; the DE algorithm directly mutates and crosses populations’ genes in the real number domain. Because of the DE algorithm’s ability to reach global optimal solutions and the strong algorithm robustness, the proposed method adopts it to assist the design of code MCD structures.

Algorithm 1 is the proposed design method of code MCD structures for multi-path mitigation. To implement this code MCD structure design method, hyperparameters, which can be divided into three classes: structure parameters, environment parameters and DE parameters, should be set in advance. The structure parameters, such as bandwidths of receivers and number of correlators (N) in code MCD structures, determine the direction of the optimisation. The environment parameters, such as multi-path relative amplitudes, simulate the multi-path environment in the optimisation. The DE parameters, such as sizes of weights, location populations (Nw, Nd), generation sizes (Gw, Gd), crossover probability (Pc) and iteration size (G), control the convergence speed and the optimal multi-path mitigation performance.

In this algorithm, w_i is the i-th generation of weights population in i-th iteration, which have Nw populations, w_i^{G_w, opt} is the population which owns better multi-path mitigation performance than any other weights population in w_i, X_i is the fitness of w_i, w_{m,j} is weights populations obtained by mutating w_i, and w_{c,d} is weights populations obtained by crossing w_{m,j}. The meanings of symbols are the same for the location vector d. Fit(·) is the function of calculating fitness, which is the inverse value of mean multi-path errors in this algorithm. M(·) is the process of mutation, which can generate new values in populations. For each population in X_i, where X = {w, d}, a mutant population is obtained by

\[ X_i^{c,k} = X_i^{c,n} + F_m(X_i^{c,r2} - X_i^{c,r1}), k = 1, ..., N_x \]  

where r1, r2 and r3 are randomly selected in the range of 1 to N_x, X_i^{c,r1}, X_i^{c,r2} and X_i^{c,r3} are r1th, r2th and r3th population in X_i. C(·) is the process of crossover, which simulates the reproduction of populations. A crossover population is generated by

\[ X_i^{c,b} = \begin{cases} X_{m,i}^{c,b}, & \text{if } R_b < P_c \text{ or } D_b = b \\ X_i^{c,b}, & \text{else} \end{cases} \]

where X_i^{c,b}, X_{m,i}^{c,b} and X_i^{c,b} are bth vector element of X_i, X_{m,i}^{c,b} and X_i^{c,b} population, respectively, R_b is a random value in [0,1] and D_b is a random integer in \{1, ..., N\}. S(·) is the process of selection, which selects populations which have higher fitness in X_i^{c,b} and X_i to constitute next generation of populations.

Since the two parameters, weights vector w and location vector d should be optimised, the proposed design method optimises w and d in turn. In the i-th iteration, the proposed method first optimises w_i with the (i-1)th optimal location population \(d_i^{G_w,\text{opt}}\), of which the 0th optimal location population \(d_0^{G_w,\text{opt}}\) is randomly selected in \(d_0\). In the jth generation of w_i, the optimisation method calculates the fitness of each weights population in w_i and updates the jth optimal weights population \(w_i^{G_w,\text{opt}}\). Then the proposed method applies mutation and crossover operations on \(w_i^{G_w,\text{opt}}\) to get \(w_i^{c,i}\). Populations which have better fitness in \(w_i^{c,i}\) and \(w_i^{G_w,\text{opt}}\) constitute the next generation of weights populations \(w_i^{G_w,\text{opt}}\). After the optimisation of w_i in i-th iteration, weights populations w_{i+1} and i-th optimal weights population \(w_i^{G_w,\text{opt}}\) are obtained. Then the proposed method optimises location populations \(d_i\) with \(w_i^{G_w,\text{opt}}\). Location populations in the next iteration \(d_{i+1}\) and i-th optimal location population \(d_i^{G_w,\text{opt}}\) will be obtained by the same optimisation procedure as \(w_i\). Having finished the optimisation of these two parameters w and d, the obtained optimal weights population \(w_i^{G_w,\text{opt}}\) and location population \(d_i^{G_w,\text{opt}}\).
can construct the code MCD structure which has the optimal multi-path mitigation performance.

### 3.3 Implementation of the proposed method

From the perspective of algorithm implementation, the time complexity of the proposed method is $O(G \cdot G_X \cdot N_X \cdot D)$, which means that the time consumption of the proposed design method is positively correlated with iteration size, generation size, population size and correlator number. Table 1 exhibits the time consumption of the algorithm under different hyperparameter settings, where the computing equipment is Intel(R) Core(TM) i7-4770K CPU @ 3.50 GHz. To reach better multi-path mitigation performance of code MCD structures designed by the proposed method, generation size $G_X$ and population size $N_X$ should be large enough to ensure that the algorithm can approach its optimal solutions. Under the trade-off between time cost of the proposed design method and the anti-multi-path performance of designed code MCD structures, hyperparameters adopted in case studies in Section 4 are $G = 2$, $G_X = 50$, $N_X = 150$, $D = 4$.

When utilising the proposed method to design code MCD structures, hyperparameters of algorithm such as the modulation and bandwidth of the signal should be set in advance, which means that the proposed method can design one code MCD structure for one signal with certain modulation and bandwidth. According to the modulation of the signal and bandwidth of the receiver, the ACF of the signal, which can be calculated by Equation (4) and Equation (5), is input into the design method. The outputs of the proposed method should be optimal weights and location vectors of the code MCD structure which is specifically designed for the signal with given modulation and bandwidth of the receiver. Although receivers equipped with designed code MCD structures have more correlators compared with traditional receivers, the implementation methods of early-late correlators in both traditional receivers and receivers with code MCD structures are consistent. [18, 19] Since the type and bandwidth of signals cannot vary with time when receiving GNSS signals, the designed code MCD structure can be solidified in the corresponding receiving channel and utilised all the time. Therefore, the design time cost of several hours consumed by the proposed method is acceptable.

### 4 CASE STUDIES

This section illustrates the characteristics of the proposed method by case studies, which will be represented in four aspects: adaptability, sensitivity, thermal noise and tracking robustness analysis.

To analyse these characteristics of the proposed method, several representative GNSS signals and bandwidths of receivers are selected in the following case studies. BPSK(1) signals are widely applied in GNSS, such as global positioning system (GPS) L1 C/A and L2C. BOC(1,1) signals are used in GPS L1C and the BeiDou navigation satellite system (BDS) B1C, as the baseline components, and BOC($\cos(15,2,5)$) signals are high order BOC signals which are adopted by Galileo E1-A. Composite BOC (CBOC) signals belong to MBOC signals and the CBOC(6,1,1/11,+) signal is a specific implementation which is used in GALILEO E1-B. Three single-side bandwidths of receivers are also adopted in the case studies: 10$\omega_c$, 30$\omega_c$, and $\infty$. The $\infty$ represents the infinite bandwidth.

Since the DD structure has great multi-path mitigation performances in traditional code discriminator structures, it is adopted as a benchmark for multi-path mitigation performance analysis of code MCD structures.

Table 2 lists the hyperparameter settings of case studies in this section and Table 3 is the value range of weights vector $w$ and location vector $d$. Since $w_i$ and $d_i$ have the opposite values with $w_{N-1}$ and $d_{N-1}$, the first half designed parameters of $w$ and $d$ are shown and the remaining half designed parameters can be calculated according to that relationship.

| Parameters          | Value  |
|---------------------|--------|
| Correlator number ($N$) | 8      |
| Multi-path relative amplitude ($\alpha$) | 0.5    |
| Population size ($N_D, N_W$) | 150    |
| Generation size ($G_D, G_W$) | 50     |
| Mutation coefficient ($\mu_W$) | 0.4/0.2 |
| Crossover probability ($P_c$) | 0.7    |
| Iteration size ($G$) | 2      |

### 4.1 Adaptability analysis

This section discusses the adaptability of the proposed method. As long as modulation types of the signals and bandwidths of the receivers can be determined, the proposed method can specifically design code MCD structures which will have better multi-path mitigation performance than DD structures in every case.

Case studies for adaptability analysis can be divided into two classes. The first class is code MCD structures designed for...
various signals with one specific bandwidth of the receiver. The second class is code MCD structures designed for one specific signal with various bandwidths of receivers. Table 4 lists the designed code MCD structures for signals with four modulation types and receivers with three bandwidths, respectively.

### 4.1.1 Code MCD structures designed for various signals with one specific bandwidth of the receiver

In this class of cases, the proposed method designs code MCD structures for four signals with one specific bandwidth of the receiver. As Figure 6 (a) shows, when the bandwidth of the receiver is fixed, the mean multi-path error of the designed code MCD structures for each signal is smaller than that of the DD structures, which indicates that the designed code MCD structures have greater capability to mitigate multi-path. Compared with the mean multi-path error of DD structures, it can be seen that the mean multi-path errors relative degradation of code MCD structures designed for BOCcos(15,2,5) signals and CBOC(6,1,1/11,+) signals, is larger than that of code MCD structures designed for BPSK(1) signals and BOC(1,1) signals, indicating that the DD structures cannot effectively degrade mean multi-path errors for signals with complicated ACF. Since the DD structure is designed for BPSK signals by trial and error, it is no longer suitable for signals with complicated ACF, while the proposed design method can systematically design code MCD structures by the optimisation method. These structures have great anti-multi-path performance, especially for signals with complicated ACFs, which indicates the proposed method's adaptability to various signals.

Figures 7–9 show the multi-path error envelopes of the designed code MCD structures and DD structures for different modulation types and receiver bandwidths. For most cases, the multi-path error envelopes of code MCD structures are smaller than that of the DD structures, which illustrates a smaller mean multi-path error and better multi-path mitigation performance.

| Signals | Bandwidths | Parameter types | Value 1 | Value 2 | Value 3 | Value 4 |
|---------|------------|----------------|--------|--------|--------|--------|
| BPSK(1) | 10f<sub>c</sub> | w              | 0.0418 | -0.5582 | 0.9000 | -0.4465 |
|         |            | d              | -0.1260 | -0.1012 | -0.0904 | -0.0675 |
|         | 30f<sub>c</sub> | w              | -0.0583 | 0.0510 | -0.5815 | 0.8328 |
|         |            | d              | -0.1250 | -0.1250 | -0.0751 | -0.0535 |
|         | ∞f<sub>c</sub> | w              | -0.0139 | 0.0902 | 0.5013 | -0.8578 |
|         |            | d              | -0.1387 | -0.1000 | -0.075 | -0.0521 |
| BOC(1,1) | 10f<sub>c</sub> | w              | 0.1137 | 0.1771 | -0.6390 | 0.4819 |
|         |            | d              | -0.1250 | -0.1000 | -0.0921 | -0.0561 |
|         | 30f<sub>c</sub> | w              | -0.0013 | 0.0065 | 0.3723 | -0.5131 |
|         |            | d              | -0.1360 | -0.1201 | -0.0750 | -0.0556 |
|         | ∞f<sub>c</sub> | w              | -0.0033 | -0.0614 | 0.5409 | -0.6237 |
|         |            | d              | -0.1274 | -0.1115 | -0.0750 | -0.0534 |
| BOCcos(15,2,5) | 10f<sub>c</sub> | w              | 0.0461 | 0.2265 | -0.2194 | 0.2628 |
|         |            | d              | -0.1499 | 0.1032 | -0.0750 | -0.0500 |
|         | 30f<sub>c</sub> | w              | 0.4030 | -0.2104 | -0.2825 | 0.6914 |
|         |            | d              | -0.1398 | -0.1171 | -0.0841 | -0.0732 |
|         | ∞f<sub>c</sub> | w              | -0.2691 | 0.0336 | -0.1875 | -0.2255 |
|         |            | d              | -0.1333 | -0.1090 | -0.0819 | -0.0509 |
| CBOC(6,1,1/11,+) | 10f<sub>c</sub> | w              | -0.0365 | 0.2083 | -0.3931 | 0.2420 |
|         |            | d              | -0.1459 | -0.1061 | -0.0839 | -0.0666 |
|         | 30f<sub>c</sub> | w              | 0.1923 | -0.6319 | 0.6696 | -0.1927 |
|         |            | d              | -0.1250 | -0.1000 | -0.0796 | -0.0689 |
|         | ∞f<sub>c</sub> | w              | 0.0000 | -0.3463 | 0.4438 | -0.1287 |
|         |            | d              | -0.1251 | -0.1001 | -0.0931 | -0.0567 |

Abbreviations: BPSK, binary phase shift keying; BOC, binary offset carrier; MDC, multi-correlator discriminator.
In Figure 7 (c), multi-path error envelopes of code MCD structures have an obvious multi-path error peak at $r_m = 1$ chip, which greatly exceeds the multi-path error envelopes of the DD structures. Since $BOC_{\cos}(15,2.5)$ signals are high-order BOC signals, the Gibbs effect of $BOC_{\cos}(15,2.5)$ signals is much more severe than that of any other signals when the single-sideband bandwidth is 10$f_c$. As the bandwidth of the receiver grows wider, the Gibbs effect is attenuated and the multi-path error peak of multi-path error envelopes at $r_m = 1$ chip will disappear.

### 4.2.1 Sensitivity to different mMulti-path relative amplitudes

Corresponding to the multi-path error optimisation model, the proposed method optimises code MCD structure parameters with only one specific multi-path relative amplitude, which is the hyperparameter set before the optimisation. In the real-world complex environment, since multi-path signals received by antennas have various multi-path relative amplitudes, the code MCD structure designed with one specific multi-path relative amplitude is expected to greatly degrade code errors caused by multi-path signals with other multi-path relative amplitudes.

As Figure 10 shows, mean multi-path errors of the proposed design method designs code MCD structures and DD structures are both positively correlated with multi-path relative amplitudes, which means that when the multi-path relative amplitude increases, multi-path errors will become larger. In every bandwidth condition, mean multi-path errors of code MCD structures designed with $\alpha = 0.5$ are smaller than that of the DD structures when applied to multi-path signals with other multi-path relative amplitudes. This shows that the proposed design method is insensitive to multi-path relative amplitudes.

### 4.2.2 Sensitivity to bandwidths of receivers

To put the proposed design method into practice, the bandwidth of the receiver, which is the hyperparameter of the multi-path error optimisation model, should be set before the optimisation, that is the proposed method specifically designs one code MCD structure with only one specific bandwidth of the receiver. Since receivers have different bandwidths according
to their various applications, it will be complicated to design various code MCD structures for these receivers by the proposed method. Therefore, the code MCD structure designed with one specific bandwidth is expected to be able to greatly mitigate multi-path when applied to receivers with other bandwidths.

In case studies shown in this section, the proposed method designs code MCD structures for receivers with three bandwidths, respectively, and mean multi-path errors of these code MCD structures applied to receivers with other bandwidths are represented in Figure 11 (a), (b) and (c).

As Figure 11 (c) shows, the code MCD structure designed with infinite bandwidth has smaller mean multi-path errors than DD structures when applied to receivers with other bandwidths. This illustrates the insensitivity of the proposed method to bandwidths of receivers. When the bandwidth used in the design becomes narrower, as Figure 11 (a) shows, the code MCD structure designed with single-side bandwidth = 10f_c has worse multi-path mitigation performance than the DD structures in several cases. Since the ACFs of signals suffer severe Gibbs effect when the bandwidth is 10f_c, the proposed design method is strongly
influenced by it and the designed code MCD structures have poor multi-path mitigation performance when applied to receivers with other bandwidths. Therefore, to migrate a designed code MCD structure to receivers with other bandwidths, the proposed method can design code MCD structures for a wider bandwidth condition.
4.3 Thermal noise analysis

According to adaptive and sensitive analysis discussed above, designed code MCD structures have great anti-multi-path performance and abilities to adapt to various multi-path environments and signals with different bandwidths. Thermal noise performance, which is one of the important standards to evaluate the performance of discriminators, should also be analysed by observing the tracking jitter of DLL with multi-correlator discriminators. Since designed discriminators adopt multi-correlator structures, the two-correlator coherent thermal noise performance formula given by [31] should be extended, which can be written as

\[
\sigma^2 = \frac{B_L (1 - 0.5B_L T) \int_{-BW}^{+BW} G_i(f) \left( \sum_{i=1}^{N/2} a_i \sin(\pi f \delta) \right)^2 df}{\pi C/N_0} \]

where, \( \sigma^2 \) is the code delay estimation variance of DLL, which represents the thermal noise performance, \( B_L \) is the single-side loop bandwidth, \( C/N_0 \) is the signal-to-noise ratios (SNR), \( \delta \) is the early-minus-late spacing for \( i \)th correlator, which equals to \( 2 |a_i| \). A detailed derivation of Equation (16) is displayed in Appendix.

Figure 12 exhibits tracking jitters of discriminators with NEML, DD and code MCD structures for four types of signal modulations, respectively, with single-side bandwidth 10\( f_c \), single-side loop bandwidth 1 Hz and integration time 1 ms. As Figure 12 (a), (b) and (d) show, for BPSK(1), BOC(1,1) and CBOC(6,1,1/11,+), the DLL tracking error variance of code MCD structures is slightly degraded when compared with DD structures. As Figure 12 (c) shows, for BOC\(_{\text{cos}}\)(15,2.5) signals which are high-order BOC signals, designed code MCD structures have the same tracking error variance as DD structures without any degradation in thermal noise performance.

Since the multi-path mitigation performance of code MCD structures is greatly enhanced, taking both the influence of thermal noise and multi-path into account, the code phase estimation errors of code MCD structures are much smaller than that of the DD structures, especially for high-order BOC signals, illustrating the superiority of the proposed method when designing for signals with complicated spreading waveforms.

4.4 Tracking robustness analysis

Tracking robustness can be characterised by the linear tracking region of S-curves, which is located around the zero-code phase. When inputs of discriminators fall in this region, discriminators can react accordingly and their outputs can modify the code phase, so that the code phase error can gradually converge to zero code phase error. Therefore, a wider linear tracking region can handle a wider range of discriminators’ inputs, which means that the tracking loop has better robustness.

Figure 13 exhibits the normalised S-curves of DD and code MCD structures for four distinct signals with \( BW = 10f_c \). As Figure 13 shows, the range of linear tracking regions of code MCD structures are the same as that of DD structures, which means that code MCD structures have the similar tracking robustness with DD structures. In Figure 13 (c), for BOC\(_{\text{cos}}\)(15,2.5) signals, there are many side peaks in the S-curve of DD structures, which may cause that tracking loop to deviate from the linear tracking region around zero code phase and lock on to the linear region of side peaks. However, code MCD structures can effectively restrain side peaks introduced by high-order BOC signals and the probability of false lock will be then reduced. Designed code MCD structures can not only outperform DD structures in multi-path mitigation performance, but also have the similar tracking robustness with DD structures, which indicates the superiority of the proposed method.

4.5 Summary

Corresponding to case studies in this section, the proposed method can specifically design code MCD structures for various signals and bandwidths of receivers, which illustrates the adaptability of the proposed design method. The multi-
path mitigation performance of the designed code MCD structures is always better than that of the DD structures, especially for signals with complicated ACFs. Although the proposed design method can only design one code MCD structure with one specific multi-path relative amplitude and one specific bandwidth of the receiver, the designed code MCD structures have greater multi-path mitigation performance than DD structures when applied to other multi-path relative amplitudes and bandwidths of receivers. This indicates the insensitivity of the proposed method to these effects. Although thermal noise performance of designed code MCD structures is slightly degraded, the multi-path mitigation performance of code MCD structures is greatly improved and the code tracking loop also has strong robustness.

5 | CONCLUSION

This article proposes a general methodology to design code MCD structures for multi-path mitigation with the goal to minimise the mean multi-path error. To conquer the problem that the multi-path error optimisation model cannot be effectively solved by derivative-based method, the heuristic algorithm DE is adopted to assist the design of code MCD structures without the problem of derivations.

In the presented case studies, some significant characteristics of the proposed method are developed. First, the proposed method can specifically design code MCD structures for signals with various modulations and receivers with different bandwidths. Compared with traditional code discriminator structures, the designed code MCD structures always have better multi-path mitigation performance, especially for signals with complicated ACFs. This illustrates the adaptability of the proposed method.

Second, the multi-path signal relative amplitude and the bandwidth of the receiver which are the two hyperparameters of the proposed method should be set in advance of optimisation. Code MCD structures designed with one specific multi-path relative amplitude can effectively mitigate multi-path signals with other multi-path relative amplitudes, and code MCD structures designed for the receiver with one specific bandwidth can be migrated effectively to receivers with other bandwidths. Therefore, the proposed method is insensitive to multi-path relative amplitudes and bandwidths of receivers, indicating the great practicality of the proposed method.

Third, although thermal noise performance of designed code MCD structures is slightly degraded, multi-path mitigation performance of code MCD structures is promoted with strong tracking robustness, so that the accuracy of positioning can be eventually improved.

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ORCID

Yunhan Qi  https://orcid.org/0000-0003-3599-698X
Zheng Yao  https://orcid.org/0000-0002-7657-644X

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APPENDIX

This appendix provides the derivation of thermal noise performance of discriminators with multiple correlators.

Taking the influence of noise into account, the relationship between output of discriminators and code estimated errors can be modelled as

\[ e_c = K(\tau) + N_c \quad (17) \]

where \( K(\cdot) \) is the relation function, \( N_c \) is the complex noise introduced by code discriminators. Equation (17) can be linearised around \( \tau = 0 \):

\[ e_c = k_c \tau + N_c \quad (18) \]

where \( k_c \) is the slope of discriminator at \( \tau = 0 \). The variance of estimated code phase \( \hat{\tau} \) can be written as

\[ \text{var}\{\hat{\tau}\} = E\{\hat{\tau}^2\} = \frac{E\{\Re\{N_c^2\}\}}{k_c^2} \quad (19) \]

where, \( \Re\{\cdot\} \) takes the real part. According to the closed-loop delay estimation formulas given in [31], the closed-loop code phase estimation variance can be written as

\[ \sigma_c^2 = \text{var}\{\hat{\tau}\} = 2B_L T (1 - 0.5B_L T) \quad (20) \]

Considering the presence of noise, baseband signals with finite bandwidth can be modelled as:

\[ u(t) = A \cdot g_{BW}(t) e^{-j\Delta \varphi} + n(t) \quad (21) \]

where, \( A \) is the magnitude of baseband signals, \( \Delta \varphi \) is the carrier phase error, and \( n(t) \) is the zero-mean Gaussian white noise with power spectrum density \( N_0 \). The multiple correlators discriminator function is

\[ e_c(\tau) = \Re\{\int_0^T e^{j\Delta \varphi} u(t) \sum_{i=1}^N g(t - \tau + d_i) dt\} \]

\[ = A \sum_{i=1}^N R_{BW}(\tau + d_i) + \Re\{N_c\} \quad (22) \]

\[ = A \sum_{i=1}^N R_{BW}(\tau + d_i) + \Re\{\sum_{i=1}^N N_c\} \]

\[ = A \sum_{i=1}^N R_{BW}(\tau + d_i) + \Re\{\sum_{i=1}^N N_c\} \]

where

\[ N_{ci} = \frac{e^{j2\varphi}}{T} \int_0^T n(t) g(t + d_i) dt \quad (23) \]

Therefore, \( E\{\Re\{N_c^2\}\} \) in Equation (19) can be expanded to

\[ E\{\Re\{N_c^2\}\} = \frac{1}{2T^2} E\left\{\left| \int_0^T n(t) \sum_{i=1}^N w_i g(t + d_i) dt \right|^2 \right\} \]

\[ = \frac{1}{2T^2} E\left\{ \int_0^T \int_0^T R_n(t - u) \sum_{i=1}^N w_i g(t + d_i) dt du \right\} \]

\[ \times \left[ \sum_{i=1}^N w_i g(u + d_i) \right]^* \quad (24) \]

where \( R_n(t - u) \) is the ACF of noise, which can be further expressed as

\[ R_n(t - u) = \int_{-BW}^{+BW} G_n(f) e^{j2\pi f(t-u)} df \]

\[ = \int_{-BW}^{+BW} N_0 e^{j2\pi f(t-u)} df \quad (25) \]

where \( G_n(f) \) is the PSD of noise. Therefore,

\[ E\{\Re\{N_c^2\}\} = \frac{N_0}{2T} \int_{-BW}^{+BW} \left[ \int_0^T \int_0^T \sum_{i=1}^N w_i g(t + d_i) e^{j2\pi f} dt \right] \]

\[ \times \left[ \sum_{i=1}^N w_i g(u + d_i) e^{j2\pi f} du \right]^* df \quad (26) \]

\[ = \frac{N_0}{2T} \int_{-BW}^{+BW} G_n(f) Q^*(f) \left( \sum_{i=1}^N w_i e^{j2\pi f d_i} \right)^2 df \]

\[ = \frac{N_0}{2T} \int_{-BW}^{+BW} G_n(f) \left( \sum_{i=1}^{N/2} w_i \sin(\pi f d_i) \right)^2 df \]

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When stably tracking, multiple correlators discriminator function can be rewritten as

$$
\varepsilon_c(\zeta) = 9\left\{ A N \sum_{i=1}^{N} \int_{0}^{T} g_{BW}(t)g(t - \zeta + d_i)dt \right\}
$$

$$
= 9\left\{ A N \sum_{i=1}^{N} \int_{-BW}^{+BW} Q(f) \int_{0}^{T} g(t - \zeta + d_i)e^{2\pi j f t}dt df \right\}
$$

$$
= 9\left\{ A N \sum_{i=1}^{N} \int_{-BW}^{+BW} G_i(f)Q(f)e^{2\pi j f (\zeta - d_i)}df \right\}
$$

$$
\cong 2A \int_{-BW}^{+BW} G_i(f) \sum_{i=1}^{N/2} \sin(\pi f \delta_i)\sin(2\pi f \zeta)df
$$

(27)

where \( \zeta \) is near the zero value. Therefore, the slope of discriminators is

$$
k_c = 4\pi A \int_{-BW}^{+BW} f G_i(f) \sum_{i=1}^{N/2} \sin(\pi f \delta_i)df
$$

(28)

From Equation (19), Equation (20), Equation (26) and Equation (28), code phase estimation variance is

$$
\sigma^2_c = B_L(1 - 0.5B_L T) \int_{-BW}^{+BW} G_i(f) \left( \sum_{i=1}^{N/2} w_i \sin(\pi f \delta_i) \right)^2 df
$$

$$
= \frac{C}{N_0} \left( 2\pi \int_{-BW}^{+BW} f G_i(f) \sum_{i=1}^{N/2} w_i \sin(\pi f \delta_i)df \right)^2
$$

(29)

where \( C/N_0 = A^2/N_0 \) is the SNR of signals.