Convergence Properties for the Physarum Solver

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Abstract

The Physarum solver [31, 18] is an intuitive mechanism for solving optimisation problems. The solver is based on the idea of a current reinforced electrical network, whereby the conductivity \( \sigma(t) \in \mathbb{R}_+^E \) is reinforced by the current or flow, \( \phi(t) \in \mathbb{R}_+^E \). In this paper, we show how the Physarum solver obtains the solution to the linear transshipment problem on a digraph \( G = (V, E) \). We prove that the current \( \phi(t) \) and \( \sigma(t) \) converge with an exponential rate to a positive flow minimising \( \ell(\phi) = \sum_{ij} \ell_{ij}\phi_{ij} \). The limit flow has full support on the optimal set \( \hat{H} \). If we assume that \( \hat{H} \) is a connected subgraph, the electrical potential vector \( p(t) \in \mathbb{R}^V \) converges to a solution \( p^* \) of the dual problem which is a discrete \( \infty \)-harmonic function ([14, 25]) defined on the vertices of a subgraph \( H^* \) which in many cases is a spanning subgraph, i.e. \( V(H^*) = V(G) \).

1 Introduction

Many biological systems solve problems in a decentralized manner. As a leading example, path-finding by *Physarum polycephalum* (a giant amoeba of true slime mold) is well studied. Physarum can find the shortest path in a maze and the risk-minimum path in the inhomogeneous field of risk [20, 21, 22]. On the basis of experimental work, a differential equation model of such path minimisation has been proposed [23, 19]. The model is based on the idea of a current reinforced electrical network. As current or flow increases through a medium the conductivity is increased.
The Physarum solver has been primarily studied by numerical simulation. Indeed, the algorithm was extended in order to make it applicable to a wider range of problems concerning the optimal design of networks, e.g. the shortest network problem \[29\] and multi-objective optimisation \[30, 32\]. Numerical simulation shows a good performance in problem solving and a correspondence to some interesting characteristics of the biological system \[24\]. Onishi et al. have published a few papers on the rigorous analysis of the Physarum solver \[16, 18, 17\]. The work is pioneering and just at the beginning.

A mathematical description of the Physarum solver is as follows. We consider a weighted directed graph \(G = (V, E, \ell)\), where the positive weights \(\ell_{ij} > 0\) are interpreted as lengths of the arcs \(ij \in E\). For the physarum solver the candidate flow \(\phi(t), t > 0\), is obtained as an electric current in an electrical resistor network obtained from an adapting positive conductivity vector \(\sigma(t) = (\sigma_{ij}(t)), t \geq 0\).

We apply fixed external current sources, described by a fixed source vector \(b = (b_i)\) which defines the flow requirements, and we obtain from \(\sigma(t)\) the current \(\phi(t) = (\phi_{ij}(t))\) and potential \(p(t) = (p_i(t))\) using Kirchhoff’s laws. The physarum solver is then completely specified by updating the conductivities according to the recipe

\[
\frac{d}{dt}\sigma_{ij}(t) + \sigma_{ij}(t) = \phi_{ij}(t). \tag{1}
\]

We can arbitrarily choose the initial state \(\sigma_{ij}(0) > 0\), for all \(ij \in E\). We show that \(\phi(t)\) converges at an exponential rate to a positive flow \(\hat{\phi}\) that minimises the cost \(\ell(\phi) := \sum_{ij} \ell_{ij} \phi_{ij}\). In \[5\] and \[15\] minimum cost flow problems are solved using electrical networks with non-linear resistors. In contrast, the Physarum Solver works by modifying networks with ordinary linear resistors.

In this report, we state a couple of theorems and proofs regarding the algorithm by means of mathematical analysis. Using the above model, we obtain a more general result than that obtained in \[18\] which gives the convergence for certain shortest-path problems in planar graphs. As stated above, we show convergence to a solution of the transshipment problem \[28\] for the given weighted digraph. One consequence is that the shortest path in a maze is certainly obtained and the analysis indicates additional utility and performance of the algorithm.

We also show that \(\hat{p} = \lim p(t)\) converge to a canonical dual solution in the form of an \(\infty\)-harmonic function, which, generally, is defined outside the node support of the minimum flows. Discrete infinity-harmonic functions have been introduced in \[14, 25\] as value functions for certain type of games. (Our definition of discrete \(\infty\)-harmonic functions is slightly different for directed problems and take into consideration the direction of the arcs in another way.) The discrete infinity-laplacian is highly non-linear, but here we obtain the harmonic solution as a limit using solutions of the ordinary linear laplacians \(L(\sigma(t))\).

What perhaps makes the Physarum solver stand out as an algorithm is that its implementation is physically immediate; the computation relies on physical quantities like conductivity, potential and flow, which are present for many natural systems; in particular, they can be derived from underlying diffusion
1.1 Preliminaries

A vector \( \phi \in \mathbb{R}^A \) is a real-valued functions on the finite index-set \( A \). We write \( x = (x_\alpha) \), \( \alpha \in A \). The support \( \text{supp} \, x \) is the set of \( \alpha \) where \( x_\alpha \neq 0 \). Arithmetic operations and relations between vectors should be interpreted component-wise, for example \( xy^2/\ell \leq z \) should mean the vector \( (x_\alpha y^2_\alpha/\ell_\alpha \leq z_\alpha) \). Scalars are interpreted as constant vectors of appropriate dimensions when needed. We restrict the domain of vector \( x \) to \( B \subset A \) by writing \( x_B \) or \( x|_B \). The characteristic function for a set \( S \subset A \) is denoted \( 1_S \). When using matrix-algebra, all vectors are column vectors and the corresponding row-vector is denoted by \( x^T \). For a vector \( x = (x_\alpha) \), we use \( |x| \) to denote the vector \((|x_\alpha|)\). We use \( x^+ \) and \( x^- \) to denote the positive and negative parts, respectively, so that \( x = x^+ - x^- \) and \( |x| = x^+ + x^- \). The norm \( \|x\| \) is used to denote the \( \ell_1 \)-norm of \( x \), i.e. \( \|x\| = \sum_\alpha |x_\alpha| \), and the norm \( \|x\|_\infty \) is the maximum norm \( \|x\|_\infty = \max_\alpha |x_\alpha| \).

The language from graph theory is hopefully standard. However, we consider mainly directed graphs \( G = (V,E) \) without loops and multiple edges. The term graph will usually mean such digraphs. Elements of \( V \) are called nodes (or vertices) and are denoted \( i, j, k, l \) etc.; the elements of \( E \subset V \times V \) are called directed arcs (edges) and are denoted \( ij, kl \), etc. For the arc \( ij \), \( i \) is the tail and \( j \) is the head of the arc. We sometimes write \( |G| \) for the number of edges \( |E(G)| \). The underlying undirected graph of a digraph \( G \) is denoted by \( U(G) \). An oriented subgraph \( H \) of \( G \) is a digraph \( H \) such that \( U(H) \) is a subgraph of \( U(G) \). An oriented graph corresponds with a vector \( \xi_H \in \{0,1,-1\}^E \), where \( \xi_H(ij) = 1 \) if \( ij \in H \), \(-1 \) if \( ji \in H \) and zero otherwise. The oriented subgraph \( H \) is thus a directed subgraph if \( \xi_H = 1_{E(H)} \). As oriented subgraphs goes, we will mainly deal with oriented paths, oriented cycles and oriented cuts (cut-sets) of \( G \), and if the corresponding subgraph is a sub-digraph of \( G \) we talk of a directed cycle, path or cut in \( G \).

We use standard asymptotic notation. Thus \( g = \Omega(f) \) means \( \lim \inf \|g/|f| > 0 \), \( g = O(f) \) that \( \lim \sup \|g/|f| < \infty \) and \( g = o(f) \) that \( \lim \sup \|g/|f| = 0 \), as the relevant limit is taken.

1.2 The transshipment problem

In a discrete setting the transshipment problem has the formulation of a minimum cost flow problem on a directed connected graph \( G = (V,E) \) without upper capacities and positive linear costs. A flow (or a b-flow) is an arc-vector \( \phi = (\phi_{ij})_{ij \in E} \in \mathbb{R}^E \) that satisfies, for all nodes \( i \in V \), Kirchhoff’s Current Law

\[
\sum_{ij \in E} \phi_{ij} - \sum_{ki \in E} \phi_{ki} = b_i. \tag{2}
\]
The source vector is the node vector $b \in \mathbb{R}^V$ obtained as the prescribed net out-flow at nodes. We assume that $\sum_i b_i = 0$. If $b_i > 0$ we say $i$ is a supply node (or source) and if $b_i < 0$ a demand node (or sink). A flow is positive if $\phi_{ij} \geq 0$, meaning that the flow goes in the direction of the arcs. The cost is a positive linear function on flows

$$\ell(\phi) := \ell^T \phi = \sum_{ij \in E} \ell_{ij} \phi_{ij}, \quad \ell_{ij} \geq 0.$$ 

The cost coefficients $\ell_{ij} > 0$ will be referred to as lengths prescribed to the arcs.

The transshipment problem is to minimise the cost among all positive flows. It can be shown (see e.g. [28, 33]) to be equivalent to the more general minimum cost flow problem and subsumes among others the shortest path problem and the minimum cost assignment problem. An undirected problem is a problem without the positive flow constraint, with cost function $\ell(|\phi|)$. It can be mapped to an equivalent transshipment problem if we replace each arc with a pair of arcs in both directions.

For the given directed graph $G = (V, E)$, let $B = B_G \in \mathbb{R}^{V \times E}$ denote the boundary operator to $G$, i.e. the matrix given by

$$B_{ie} := \begin{cases} -1 & \text{if } i \text{ is the head of arc } e \\ +1 & \text{if } i \text{ is the tail of arc } e \\ 0 & \text{if } i \text{ is not incident with } e. \end{cases}$$

The linear transshipment problem can be stated quite effectively using the matrix $B$, i.e.

$$\text{minimise } z = \ell^T \phi, \text{ where } B\phi = b \text{ and } \phi \geq 0, \quad (3)$$

For the given source vector $b \in \mathbb{R}^V$, we denote the affine space of corresponding flows by $\Phi = \Phi(b, G) \subset \mathbb{R}^G$. The convex polyhedron of positive flows is denoted by $\Phi^+ = \Phi^+(b, G)$, i.e. $\Phi^+ = \{ \phi \in \mathbb{R}^G : B\phi = b, \phi \geq 0 \}$. The target for the Physarum Solver is to reach the set of flows $\phi \in \Phi^+$ where the cost $\ell(\phi)$ is minimum. Since $\ell > 0$, this makes up a convex polytope (i.e. a bounded polyhedron) $\hat{\Phi} = \hat{\Phi}(b, G, \ell)$.

A cut is a partition $(S, V \setminus S)$ of $V$ in two parts and the corresponding oriented cut is given by the vector $\kappa^S := B^T 1_S \in \{-1, +1, 0\}^E$. Given a flow $\phi \in \Phi(b, G)$, it holds that for any cut $\kappa = \kappa^S$ the net flow $\kappa^T \phi$ equals

$$b(S) := b^T 1_S = \sum_{i \in S} b_i = - \sum_{i \in V \setminus S} b_i. \quad (4)$$

The transshipment problem (3) is feasible, i.e. $\Phi^+ \neq \emptyset$, if and only if, for every $S \subset V$ with $b(S) > 0$, there is some arc from $S$ to $V \setminus S$.

A negative cost cycle is an oriented cycle $\gamma \in \{0, 1, -1\}^E$ such that the cost $\ell(\gamma) < 0$. That a flow $\phi \in \Phi^+$ admits an augmenting cycle $\gamma$ means that the flow $x + \delta \gamma$ is feasible, i.e. positive, for some scalar $\delta > 0$. It is well known (e.g. [12]) that a positive flow is of minimum cost if and only if it admits no
augmenting negative cost cycle. Since the problem (3) does not have any upper capacities, a positive flow \( \hat{\phi} \in \Phi^+ \) belongs to \( \hat{\Phi} \) if and only if for every negative cost cycle \( \gamma \) there is some arc \( ij \) such that \( \gamma_{ij} = -1 \) and \( \hat{\phi}_{ij} = 0 \).

From linear programming theory \cite{12}, we obtain that the dual problem corresponding to (3) is

\[
\text{maximise } w = p^T b, \text{ where } B^T p \leq \ell
\]

for the dual variables \( p = (p_i) \in \mathbb{R}^V \). In the physarum solver, the dual variables \( p_i, i \in V \), are interpreted as potential values in the electric network. It is the cocycle of potential differences, \( B^T p = (p_i - p_j)_{ij \in E} \), that carry all information and \( p \in \mathbb{R}^V \) will only be determined up to a constant. The cocycle space is the range of \( B^T : \mathbb{R}^V \to \mathbb{R}^E \) and constitutes the orthogonal complement to the cycle space \( \Phi(0,G) \) with respect to the standard inner product \( (x,y) \to x^T y \) on \( \mathbb{R}^E \). Elements of the cycle space are also referred to as circulations.

Instead of working with cocycles, we mainly use the corresponding fields (“electrical field strengths”), or slopes, by which we mean vectors \( \psi \in \mathbb{R}^G \) such that \( \ell \psi \) is a cocycle on \( G \). In other words, fields are vectors of the form

\[
\psi = \Psi(p) := (B^T p)/\ell = \left( \frac{p_i - p_j}{\ell_{ij}} \right)_{ij \in E}.
\]

Notice that, the constraint in (5) can be written \( \psi \leq 1 \). Fields and cocycles are for our purposes equivalent entities and the potential can be recovered, as \( p_i = p_k + \ell_{kj} \psi\pi \), where \( \pi \) is any oriented path from the (fixed) vertex \( k \) to vertex \( i \).

### 1.3 The non-symmetric physarum solver

Given a transshipment problem specified by \((G, b, \ell)\) as above, an electrical network is specified by the positive conductivity \( \sigma = (\sigma_{ij}) \in \mathbb{R}^E_+ \). The conductance vector is then \( \sigma/\ell \) and the resistance vector is given by \( \ell/\sigma \). Kirchhoff’s equations can be stated using the weighted graph laplacian

\[
L(\sigma) := BGB^T,
\]

where \( G = \text{diag}(\sigma/\ell) \in \mathbb{R}^{E \times E} \) is the diagonal matrix with the conductance vector \( \sigma/\ell \) along the diagonal. In our setting, Kirchhoff’s equations amounts to finding a solution \( p \in \mathbb{R}^V \) to the discrete Neumann problem

\[
L(\sigma)p = b
\]

A solution \( p \) gives the flow (the current) \( \phi \) via Ohm’s law

\[
\phi_{ij} = \sigma_{ij} \frac{p_i - p_j}{\ell_{ij}} = \sigma_{ij} \psi,
\]

where

\[
\psi = \Psi(p) = B^T p/\ell.
\]
In the Physarum Solver, we consider an electrical network which evolves through “time” \( t \in [0, \infty) \), where the state is specified by the corresponding time-varying conductivity vector \( \sigma(t) \in \mathbb{R}_+^E \). We let \( \phi(t) \in \mathbb{R}^E \) and \( p(t) \in \mathbb{R}^V \) denote the corresponding current and potential, which are derived from \( G, b, \ell \) and \( \sigma(t) \) by solving Kirchhoff’s equations. The current \( \phi(t) \) will at all times constitute a flow in \( \Phi(b, G) \), but not necessarily a positive flow, nor will the potential vector \( p(t) \) automatically be feasible for the dual problem. Since \( \text{supp} \sigma(t) = G \) which is connected by assumption, we can make the vector \( p(t) \) unique by stipulating that its lowest value is zero.

In this paper we use a version of the Physarum Solver, where the conductivity vector \( \sigma(t) = (\sigma_{ij}(t))_{ij \in E} \) is updated according to the non-linear equation

\[
\frac{d}{dt} \sigma_{ij}(t) + \sigma_{ij}(t) = \phi_{ij}(t). \tag{8}
\]

We can arbitrarily choose the initial condition as long as \( \sigma_{ij}(0) > 0 \), for all \( ij \in G \). For fixed \( \sigma(0) > 0, \ell \) and \( b \), in this paper, we refer to the electrical network obtained by letting \( \sigma(t) \), evolve, for \( t \geq 0 \), according to (9), as the Physarum Solver. Ohm’s law (7) immediately gives the following alternative form of (8)

\[
\frac{d}{dt} \log \sigma(t) = \psi(t) - 1, \tag{9}
\]

where \( \psi(t) := \Psi(p(t)) \).

The “non-symmetric” physarum solver defined by (8), or equivalently by (9), is different from the previous symmetric solver, presented in [31] and analysed in e.g. [18], where \( |\psi(t)| - 1 \) and \( |\phi_{ij}(t)| \) was used on the right hand sides of (9) and (8), respectively. The use of a non-symmetric conductivity vector \( \sigma_{ij}(t) \) is perhaps somewhat surprising; in an electrical network, conductivity and conductance works symmetrically at a fundamental level. If we have a double (undirected) arc \( \{ij, ji\} \), the electrical network will have an effective conductivity \( \sigma_{ij} + \sigma_{ji} \) across the corresponding arc and the flow across the two arcs will be proportional to these conductances. The two terms will not evolve identically; if the flow is from \( i \) to \( j \) consistently then the conductivity \( \sigma_{ji} \) in the opposite direction will be of order \( O(e^{-t}) \) as \( t \to \infty \).

1.4 Duality analysis

The convergence of the physarum solver to an optimal flow can be discussed in the context of primal-dual methods (see [12], [3]) to solve mathematical programming problems. By the Theorem of Complementarity of Slackness ([12, Theorem 13.4]), a primal-dual pair of feasible solutions \((p, \phi)\), corresponds to optimal solutions of the dual and primal problems, respectively, precisely when

\[
(\psi_{ij} - 1)\phi_{ij} = 0, \quad \forall ij \in E, \tag{10}
\]

which means that, for each arc \( ij \in E \), the points \((\psi_{ij}, \phi_{ij})\) lie on the broken “in-kilter line” \( \{0\} \times (-\infty, 1] \cup [0, \infty) \times \{1\} \) depicted in figure 1. This illustration
of duality is used in e.g. [5] [15], where it is showed that a non-linear resistor
network having the kilter-line as a “characteristic curve” for the resistor along
edge $ij$ will produce a current that solves the corresponding minimum cost flow
problem.

For the linear resistor networks used in the Physarum Solver, Ohm’s law,
(7) ensures that the point $(\phi_{ij}, \psi_{ij})$ is on the line going through the origin and
the point $(\sigma_{ij}, 1)$. The optimality criterion (10) can therefore be reformulated
as
$$\psi \leq 1 \text{ and } \phi = \sigma.$$  \hspace{1cm}  \text{(11)}

The dynamics of the physarum solver increase the logarithm of conductivity if
$\psi_{ij} > 1$ and make it decrease if $\psi_{ij} < 1$. From this, we obtain the following
observation.

**Proposition 1.** Any attractive fixed-point for the Physarum Solver must cor-
respond to a primal-dual pair $(\phi, \psi)$ where, for all $ij \in E$, $(\phi_{ij}, \psi_{ij})$ lie on the
kilter line: $\psi_{ij} = 1$, if $\phi_{ij} > 0$, and $\psi_{ij} \leq 1$ if $\phi_{ij} = 0$. It will thus correspond
to an optimal solution of the primal problem (3) and the dual problem (5).

### 1.5 The primal convergence theorem

However, it remains to show that $\sigma_{ij}(t)$ actually converges to some attractive
fixed point. The optimal set is the subgraph $\hat{H}$ of $G$ induced by arcs supporting
some minimum cost flows, i.e.

$$\hat{H} = G[\cup\{\text{supp } f : f \in \tilde{\Phi}(b, G)\}].$$

By the characterisation of optimal flows and the convexity of $\tilde{\Phi}$, $\hat{H}$ only support
zero-cost cycles and it follows that a positive flow is optimal precisely if it’s
Theorem 1. Assume that the stated problem (3) is feasible. Then the current $\phi(t)$ converges exponentially fast to some optimal flow $\tilde{\phi} \in \hat{\Phi}$, i.e. $\phi(t) = \tilde{\phi} + e^{-|\Omega(t)|}$. The limit flow $\tilde{\phi}$ has moreover full support, i.e. $\text{supp} \tilde{\phi} = \hat{H}$.

Note that, the fact that the Physarum Solver converges to a flow with full support on the optimal set puts it on par with interior point methods (see e.g. [26]) in linear optimisation. Although there are some similarities, the Physarum Solver seems to be distinct from interior point algorithms: For instance, neither the current $\phi(t)$ nor the potential $p(t)$ are supposed to be feasible solutions to (3) and (5) and are not interior points in this sense. In section 2, we briefly discuss possible extensions of the Physarum Solver to more general linear and non-linear optimisation problems.

1.6 Infinity harmonic functions and the dual convergence theorem

Given a directed graph $G = (V,E)$ and a subset $S \subset V$, we say that a potential $p \in \mathbb{R}^V$ with corresponding field $\psi = \Psi(p)$, is a discrete $\infty$-harmonic function on $G \setminus S$, if for all $i \in V(G) \setminus S$, we have

$$\max_{k:ki \in E} \psi_{ki} = \max_{j:i j \in E} \psi_{ij} \geq 0$$

We can restate this condition as follows: Consider the level sets $F_r := \psi^{-1}(r)$, $r \geq 0$, of the field $\psi = \Psi(p)$ and define

$$H_r := \cup_{s > r} F_s = \psi^{-1}((r, \infty)) .$$

Then $p$ is $\infty$-harmonic on $F_0 \cup H_0 \setminus S$ if and only if, for all $r$, $F_r$ is a union of directed paths with endpoints contained in $S \cup V(H_r)$.

Discrete functions satisfying (12) have been studied [14, 13, 25] as values for stochastic two-person games which are symmetric in a certain sense. The results in [25] concern the more general concept of length spaces. Our definition above is a little bit different, since we take the maximum over in-going and outgoing arcs in (12), separately. For the undirected problems, i.e. digraphs $G = (V,E)$ where $ji \in E$ whenever $ij \in E$, the definitions are equivalent.

Since all oriented cycles in $\hat{H}$ have zero cost, it follows that $\hat{\psi} = \xi_{\hat{H}}$ is a field on $\hat{H}$. The following lemma is an adaptation of Theorem 12 in [13]. For $A > 0$, let

Lemma 1. Assume that $\hat{H}$ is connected and that $\hat{p} \in \mathbb{R}^V(\hat{H})$ is the potential on $\hat{H}$ given by $\Psi(\hat{p}) = \xi_{\hat{H}}$ and $\min \hat{p} = 0$. Then there is a unique extension $p^* \in \mathbb{R}^V(H^*)$ of $\hat{p}$, where $\hat{H} \subset H^* \subset G$, such that $p^*_|_{V(H^*)}$ is $\infty$-harmonic on $H^* \setminus \text{supp} \hat{b}$. It is the unique such extension that maximises

$$S(p^*, H^*, A) := \sum_{r \in \mathbb{R}} |\Psi(p)^{-1}(r)| A^r .$$
for all sufficiently large $A$. Moreover, $p^*$ is dually feasible for (5), i.e. $\Psi(p^*) \leq 1$ for all arcs $ij \in G$.

We defer the precise construction to section 3.3. However, it should be noted that, if $G$ is connected and comes from an undirected problem, i.e. if all arcs comes in pairs of 2-cycles, then $\hat{H}$ is connected and $V(H^*) = V(G)$. Notice also that $\hat{H}$ is connected if, for all $S \subset V$ such that the symmetric difference $S \triangle (\text{supp } b) \neq \emptyset$, we have $b(S) \neq 0$; in particular, this holds for a generic source vector $b$.

The following theorem states that $p(t)$ converges to a $\infty$-harmonic function defined on $H^*$.

**Theorem 2.** Assume that $\hat{H}$ is connected and (3) is feasible. Then

$$\|p(t)|_{V(H^*)} - p^*|_{V(H^*)}\|_{\infty} = e^{-|\Omega(t)|},$$

where $p^*|_{H^*}$ is the unique $\infty$-harmonic extension of $\hat{p}$ obtained in Lemma 1.

The connection between discrete $\infty$-harmonic functions and dual solutions of the transshipment problem has, as far as we know, not been noted elsewhere. For the shortest path problem between a source $s$ and a sink $t$, Dijkstra’s algorithm construct a canonical limiting dual solution given by the distance functions to $s$ (or $t$). The $\infty$-harmonic dual solution $\hat{p}$ obtained is, in contrast to these solutions, symmetric under the symmetry of “time-reversal”, i.e. if we change the sign of $B$, $h$, we obtain a new solution $-\hat{p} + \text{const}$. In continuous theory, the relation between transport problems and the Neumann problem for the $\infty$-laplacian has been noted e.g. in [9], but, as far as we understand, then only defined on the transport set of optimal flows. The physarum solver, if it can be extended to the continuous setting, would perhaps be interesting for such problems.

### 1.7 Feasibility detection

The physarum solver will detect an infeasible problem fairly quickly. We state the following proposition without proof.

**Proposition 2.** If the problem (3) is infeasible then $\|p(t)\|_{\infty} \to +\infty$ before time $t_0$ for some constant $t_0 = t_0(b, \ell, \sigma(0))$. Otherwise, if the problem (3) is feasible then there is a constant $p_{\max} > 0$ such that $\|p(t)\|_{\infty} < p_{\max}$.

## 2 Discussion

### 2.1 General costs and general linear programs

The duality analysis stated in Proposition 1 remains valid for the more general class of linear programs of the form (3) given by more general coefficient matrices $B = A$ and linear cost $\ell(\phi) = \ell^T \phi$ where $\ell > 0$. That is, provided the laplacian
matrix $L(\sigma) = BGB^T$ is well behaved and provides a solution, any stable fixed-point to (8) should correspond to an optimal solution and vice versa. Although the proof in the next section uses arguments specific to graphs, one can hope to extend the applicability of the Physarum Solver to a larger class of linear programming problems.

Proposition 1, can also be generalised to more general convex costs, i.e., flows where the cost $C(\phi)$ is a convex increasing function of the flow $\phi \geq 0$. (See e.g. [15] or [8].) In the argument one should then substitute $\ell$ with the gradient $\nabla_\phi C(\phi) \geq 0$. However, as is well known, arguments based on duality breaks down for concave costs; these problems are in general NP-hard [10] and contains, e.g. the famous Steiner tree problem on graphs. The actual models [30] of the physarum organism use non-linear functions $|\phi_{ij}|^{1+\gamma}$, $\gamma > 0$, on the right hand side of (8), to fit empirical data. It may be taken as an heuristic method for the minimisation with increasing concave costs. The Physarum organism seems to do quite well solving simple Steiner problem [29], which partly can be explained by such concave cost minimising.

### 2.2 Efficient implementation of the physarum model

In this paper, we have not analysed the physarum solver as a computer algorithm for transport problems. Thus we make no assertions about the time or space complexity of an eventual computer implementation. However, some observations regarding its eventual place among existing algorithms can be made. It is clear that the physarum solver does not rely on any centralised synchronous computation and, assuming that the map that takes the conductiv ity vector to the corresponding current vector and vector of potential differences is provided by the environment, it is readily implemented from local dynamics with simple rules.

The one important algorithmic complexity in the physarum solver as defined here is the cost of solving Kirchhoff’s equations for a given conductivity vector. There are certainly efficient and localised solvers for Kirchhoff’s equations for general weighted laplacians (see e.g. [27]). However, it may be more elegant and perhaps ultimately more efficient to side-step this issue by using the well-known ([7] or [11]) relations between random walks on weighted graphs and electrical networks. As such, the physarum solver could be implemented in an entirely decentralised manner as a reinforced random walk. There are algorithms for the minimum cost flow problem, like the various auctions algorithm by Bertsekas et. al. (see [2]), that allows for being implemented asynchronously and in parallel.

The physarum solver seems quite similar to the variant of Baum’s algorithm [1], the “knee-jerk algorithm”, used in [4] to compute resistive inverses in electrical networks — if the obtained voltage over an edge is too large or small, we increase or decrease the conductivity, quite oblivious of global considerations. It would be nice to know if it is possible to find a similar interpretation for the physarum solver. This could possibly lead to faster implementations allowing for optimal steps in the basic iterations.

Another motivation for studying a randomised physarum solver is to model
biological systems. A randomised physarum solver can take many forms, for example as an “ant algorithm” with positive reinforcement of walks. It should be noted that, unlike the prototypical edge-reinforced ant algorithm \cite{6}, the randomised distributed physarum solver should update conductivities according to the net transport rather than total transport across arcs. In a forthcoming paper, we plan to investigate the possibilities of different distributed implementations of the physarum solver using the random walk connection. Indeed, the physarum solver can be used as a unifying model for several biological transport systems among them foraging ants and, of course, the physarum organism.

3 The proofs

In this section we prove Theorem 1 and Theorem 2 and is divided in several subsections. We start stating a couple of lemmas, including Lemma 5 and Lemma 4, concerning the continuity of Kirchhoff’s equations and the existence of solutions having constant field-strength locally. The construction of the infinity harmonic dual solution is made in section 3.3. Finally, we prove Theorem 1 and later we prove Theorem 2 by induction, using the result from Theorem 1 as the base case.

3.1 Some facts and lemmas

Recall that, for $S \subset V$, $b(S)$ is the required net flow across the cut $E(S,V \setminus S)$. Let $b^*_{\text{max}} = \max_S |b(S)|$ and let $b^*_{\text{min}} = \min\{|b(S)| : b(S) \neq 0\}$. The following elementary lemma states that if the total flow across a set of arcs is small enough we can find a nearby flow avoiding those arcs.

Lemma 2. Let $\phi \in \Phi$ be a given flow and let $E' \subset E$ be a set of arcs. If

$$w := \sum_{ij \in E'} |\phi_{ij}| \leq b^*_{\text{min}}$$

then there is a flow $\phi' \in \Phi$ with support $\text{supp } \phi' \subset \text{supp } \phi \setminus E'$ and such that

$$||\phi - \phi'||_{\infty} \leq w.$$  

Moreover, $\phi_{ij}\phi'_{ij} \geq 0$ for all $ij \in E$, so the flow $\phi'$ is in the same direction as that given by $\phi$.

Proof of Lemma 2. It is enough to show this for the case $|E'| = |\{ab\}| = 1$, the general case follows easily by induction. We can assume that the flow $\phi$ is positive, since we can reorient the graph $G$ without changing the statement of the lemma.

Let therefore $f = \phi - \phi_{ab} \cdot 1_{ab}$ where $\phi_{ab} = w > 0$. We intend to repair $f$ by constructing a $ab$-flow $\delta$ of value $w$ such that $f = f + \delta$ does not change the orientation of any arc. By the Max-Flow-Min-Cut Theorem and Theorem 4.3 in \cite{12}, this flow problem can be solved with $||\delta||_{\infty} \leq w$ unless there is an cut
$(S, V \setminus S)$ in $G$ separating $a \in S$ and $b \in V \setminus S$ with the following properties: Firstly, no arc in $G$, except $ab$ goes from $S$ to $V \setminus S$. Secondly, the capacity of the cut, which is given by $c := \sum_{i \in V \setminus S, j \in S} \phi_{ij}$, satisfies $c < w$.

Hence, for the original flow $\phi$, the flow goes from $V \setminus S$ to $S$ for all arcs in the cut except for the arc $ab$. Since the net flow of $\phi$ across the cut equals $c - w < b_{min}^*$, we can conclude that $b(S) = 0$. But then $c = w$ contradicting the existence of the cut.

The set $T$ of basic feasible solutions (bfs) are positive flows with support on directed graphs without oriented cycles, i.e. the underlying undirected graphs are forests. For a $\tau \in T$, removing an edge $ij$ from the support of $\tau$ results in a graph with two new components and $\tau_{ij} = b(S) > 0$, where $S$ is the component containing $i$. Hence, we have

$$b_{min}^* \leq \tau_{ij} \leq b_{max}^*, \ \forall \ ij \in \text{supp}\tau. \ \ (13)$$

A positive flow $\phi \in \Phi^+$ is circulation free if it is supported on a subgraph without a directed cycle. A circulation free positive flow $\phi$ with minimal edge support is a bfs; if $\gamma \in \{-1, +1, 0\}$ is an oriented cycle with $\text{supp}\gamma \subset \text{supp}\phi$ then $\phi' = \phi - \alpha \gamma$ is a positive circulation free flow with $\text{supp}\phi' \subset \text{supp}\phi$ for $\alpha = \min\{\phi_{ij}/\gamma_{ij}^- : \gamma_{ij}^- > 0\}$. It follows that, for any circulation free positive flow $\phi$, we can find a $\tau \in T$ and a $c > 0$ such that $c\tau_{ij} \leq \phi_{ij}$, where equality holds for at least one edge. It is then easy to deduce that $\phi$ can be written as a convex combination $\phi = \sum_{\tau} c_{\tau}\tau$, where at most $|\text{supp}\phi|$ coefficients $c_{\tau}$ are non-zero. We also have the following lemma.

**Lemma 3.** Let $\phi$ be a circulation free positive flow and $ab$ an edge in the support of $\phi$. There is a $\tau \in T$ with the property that $\tau_{ab} > 0$ and such that

$$\min \phi(\text{supp}\tau) \geq \frac{b_{min}^*}{b_{max}^*} \cdot \frac{\phi_{ab}}{|\text{supp}\phi|}. \ \ (14)$$

**Proof.** Write $\phi$ as a convex combination $\phi = \sum_{\tau} c_{\tau}\tau$ of bfs where at most $|\text{supp}\phi|$ of the $c_{\tau}$s are non-zero. Hence,

$$\max\{c_{\tau}\tau_{ab} : \tau_{ab} > 0\} \geq \frac{\phi_{ab}}{|\text{supp}\phi|} \implies c_{\tau} \geq \frac{1}{b_{max}^*} \cdot \frac{\phi_{ab}}{|\text{supp}\phi|}.$$  

The stated inequality then follows, since positiveness implies that $\phi \geq c_{\tau}\tau$, for all $\tau$, and thus

$$\min \phi(\text{supp}\tau) \geq c_{\tau} \cdot \min \tau(\text{supp}\tau) \geq c_{\tau} b_{min}^*$$

by (13). \qed

### 3.2 Kirchhoff’s equations

Given a weighted graph $G = (V, E, \ell)$ and a conductivity vector $\sigma = (\sigma_{ij}) > 0$, we will consider Kirchhoff’s equation with flow requirements (a Neumann problem)

$$L(\sigma)p = b, \ \ b \in \mathbb{R}^V \ \ (14)$$
and also with voltage prescriptions (a Dirichlet problem)

$$(L(\sigma)p)\big|_{V \setminus S} = 0, \quad p|_S = q|_S. \quad (15)$$

The orientation of the edges of $G$ are inessential. The equation (15) has a unique solution as long as $S$ intersect each component of $G$. The equation (14) has a solution as long as $b(W) = 0$ for every component vertex-set $W$ of $G$ and is unique up to linear combinations of the indicators of component vertex-sets of $G$. In particular, the corresponding cocycles and fields are unique. We say that $p \in \mathbb{R}^V$ is harmonic outside $S$ if $L(\sigma) = b$ and $\text{supp} b \subset S$.

The set of solutions to (15) are denoted $D(\sigma|_G, q|_S)$. The solution space to problem (14) are similarly written $N(\sigma|_G, b|_S)$, where $\text{supp} b \subset S$. For definiteness, we assume the disambiguation rule that the lowest value for $p$ in each component of $G$ is zero and we write $p = D(\sigma|_G, q|_S)$ and $p = N(\sigma|_G, b|_S)$ to stress that the solutions are assumed to exist uniquely. The vector $\sigma$ should have domain containing $G$ and $q$ should be defined on a subset of $V(G)$. For the problem (14) we implicitly extend $b$ to $V(G)$ by setting $b$ to zero outside the singular set $S$ if needed.

A conductivity vector which at the same time is a minimum cost flow allow solutions to (15) and (14) where the corresponding field is constant.

**Lemma 4.** Assume $\sigma \in \mathbb{R}_+^H$ is a conductivity vector supported on a graph $H$ where all oriented cycles satisfies $\ell(\gamma) = 0$. Assume further that $S$ is a subset $S \subset V(H)$ and that $b \in \mathbb{R}^V$ is an admissible source vector such that $\text{supp} b \subset S$.

If $\sigma$ is a positive flow with $B\sigma = b$ then

(a) the solution $p = N(\sigma|_H, b|_S)$ is given by $\Psi(p) \equiv 1$.

(b) if $\Psi(p) \equiv r$, $r > 0$, then $p = D(\sigma|_H, p|_S)$.

**Proof of Lemma 4.** By assumption, $\ell$ is orthogonal to all oriented cycles and is thus a cocycle. Hence, there is a vector $p \in \mathbb{R}^{V(H)}$ such that $\Psi(p) \equiv 1$. The first statement in the lemma then follows since

$$L(\sigma)p = B \cdot \text{diag}(\sigma) \cdot \Psi(p) = B\sigma = b.$$ 

The potential $p$ is essentially unique since $\text{supp} b$ must intersect all components of $H = \text{supp} \sigma$.

The second statement follows by the same reasoning since $\text{supp} b \subset S$ and the obtained $p$ is thus harmonic outside $S$. \qed

The following elementary, but crucial, lemma expresses the continuity of the solutions in Kirchhoff’s equations. Assume $\sigma$ and $\hat{\sigma}$ are two conductivity vectors on the connected digraph $G = (V, E)$ where we assume that $\text{supp} \sigma = E$ and where $H \subset G$ is the graph induced by $\text{supp} \hat{\sigma}$. Consider two cases

**(D)** $p = D(\sigma|_G, q|_S)$ and $\hat{p} = D(\hat{\sigma}|_H, q|_S), \ S \subset V(H)$.

**(N)** $p = N(\sigma|_G, b)$ and $\hat{p} \in N(\hat{\sigma}|_H, b_{V(H)}), \ S := \text{supp} b \subset V(H)$.

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Lemma 5. For the solutions $p$ and $\hat{p}$ in both cases above, define

$$\tilde{p} := D(\sigma|G, \hat{p}|V(H)).$$

Let

$$\delta(\sigma, \hat{\sigma}) := \frac{\|\sigma - \hat{\sigma}\|_{\infty}}{\min_+ \sigma},$$

where $\min_+(x_\alpha) := \min\{x_\alpha : x_\alpha > 0\}$. Then

$$\|p - \tilde{p}\|_{\infty} = O(\delta \|\hat{p}\|_{\infty})$$

as $\delta \to 0$.

3.2.1 Proof of Lemma 5

Let $\psi = \Psi(p)$, $\check{\psi} = \Psi(\check{p})$ and $\hat{\psi} = \Psi(\hat{p})$. Notice that $\check{\psi}_{ij} = \hat{\psi}_{ij}$, for $i, j \in H$. Let $r := p - \hat{p}$, and $\rho := \Psi(r) = \psi - \check{\psi}$.

We expand $\sigma \check{\psi}^2 = \sigma (\psi - \rho)^2$ and obtain

$$\ell(\sigma \check{\psi}^2) = \ell(\sigma \psi^2) + \ell(\sigma \rho^2) - 2 \ell(\rho \sigma \psi).$$

(16)

Similarly, since $\psi = \check{\psi} + \rho$, we obtain

$$\ell(\hat{\sigma} \check{\psi}^2) = \ell(\hat{\sigma} \check{\psi}^2) + \ell(\hat{\sigma} \rho^2) + 2 \ell(\rho \hat{\sigma} \check{\psi}),$$

(17)

Let $\epsilon = \hat{\sigma} - \sigma$. Adding (16) to (17), and rearranging the terms gives

$$\ell \left( \epsilon (\psi^2 - \check{\psi}^2) \right) = \ell \left( (2 \hat{\sigma} - \epsilon) \rho^2 \right) + 2 \ell \left( \rho (\check{\phi} - \hat{\phi}) \right)$$

(18)

where $\phi = \sigma \psi$ and $\hat{\phi} = \hat{\sigma} \check{\psi}$. Both $\phi$ and $\hat{\phi}$ are currents with sources contained in $S$ and the last term in (18)

$$\ell \left( \rho (\check{\phi} - \hat{\phi}) \right) = r^T B (\check{\phi} - \hat{\phi}) = 0$$

vanishes. To see this, note that the Dirichlet case (D) implies that $r = p - \hat{p}$ is zero on supp $B(\check{\phi} - \hat{\phi}) \subset S$. In the Neumann problem (N), $B(\phi - \hat{\phi}) = b - b = 0$, since $\phi$ and $\hat{\phi}$ are flows with source vector $b$, by assumption.

Hence, after some more rearranging and using that $\psi^2 - \check{\psi}^2 = \rho (\check{\psi} + \psi)$ we obtain from (18) the equality

$$\ell(\check{\sigma} \rho^2) = \frac{1}{2} \ell \left( \rho c (\check{\psi} + \psi + \rho) \right) = \ell(\rho c \psi)$$

(19)

Since $r$ equals $p - \hat{p}$ on $V(H)$ and since $r$ is harmonic outside $V(H)$, we have $\|r\|_{\infty} = \|r|_{V(H)}\|_{\infty}$ by the maximum principle (e.g. [7]). Moreover, since $\rho = \Psi(r)$ and since, by assumption, $r_k = 0$ for some vertex $k$ in each component of $H$, we have, for any $i \in V(H)$ that

$$r_i = r_i - r_k = \ell(\pi_i \rho),$$

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where \( \pi_i \) is a vector representing some oriented path connecting \( i \) with the grounded vertex \( k \). The Cauchy-Schwarz inequality then implies that

\[
\|r\|_\infty^2 = \max_i (\ell(\rho_{\pi_i}))^2 \leq \text{diam } H \cdot \ell \left( \rho_{\pi_H}^2 \right),
\]

where \( \text{diam } H \) denotes the length of a longest shortest path in \( H \).

It follows from (20) that the left hand side in (19)

\[
\ell(\rho^2) \geq (\min_+ \hat{\sigma}) \cdot \ell \left( \rho_{H}^2 \right) \geq \frac{\min_+ \hat{\sigma}}{\text{diam } H} \cdot \|r\|_\infty^2.
\]

By Cauchy-Schwarz, the right hand side of (19)

\[
\ell(\rho \psi) \leq \frac{4|G|}{\min \ell} \cdot \|r\|_\infty \cdot \|\rho\|_\infty \cdot \|\psi\|_\infty,
\]

which gives

\[
\|r\|_\infty \leq \frac{4|G| \cdot \text{diam } H}{\min \ell} \cdot \delta \cdot \|\rho\|_\infty.
\]

where \( \delta = \|\epsilon\|_\infty / \min_+ \hat{\sigma} \).

It just remains to verify that \( \|\rho\|_\infty = O(\|\hat{\rho}\|_\infty) \): In case (D) it follows immediately that \( \|\rho\|_\infty = \|\hat{\rho}\|_\infty \) by the maximum principle. In case (N), we have \( \sigma \geq (1 - \delta)\hat{\sigma} \). The solution to the Neumann problem with conductivity \( (1 - \delta)\hat{\sigma} \) is given by \( \hat{\rho} / (1 - \delta) \) and we deduce from Rayleigh’s monotonicity principle ([7]) that \( \|\rho\|_\infty \leq \|\hat{\rho}\|_\infty / (1 - \delta) \).

### 3.3 The construction of \( (H^*, p^*) \) in Lemma 1

Given a pair \((H, p)\), where \( H \) is a subgraph of the weighted digraph \( G = (V, E, \ell) \) and \( p \in \mathbb{R}^V \) is a potential on \( H \), we say that a directed path \( \pi \subset G \) with endpoints \( \{a, b\} \subset V(H) \) has \((p-) slope \( r(p; \pi) = (p_a - p_b)/\ell(\pi) \). Note that, the slope along a directed path in \( H \) is a weighted average of the slopes of arcs along the path, i.e.

\[
r(p; \pi) = M_\pi(\psi) := \frac{1}{\ell(\pi)} \ell(\pi \psi)
\]

where \( \psi = \Psi(p) \) is the field on \( H \) corresponding to \( p \). We use \( \pi \) to both denote the directed path and the corresponding vector \( \xi_\pi \geq 0 \) in \( \mathbb{R}^E \).

The \( p \)-slope is constant on the path \( \pi \subset H \) if \( \psi_{ij} = r(p; \pi) \) for all arcs \( ij \in \pi \). A directed \( kl \)-path \( \pi \subset G \) is a directed trajectory to a connected subgraph \( H \subset G \) if both endpoints, \( k \) and \( l \), of \( \pi \) belongs to \( V(H) \) and all internal nodes are in \( V(G) \setminus V(H) \).
To construct the graph $H^*$ and $p^*$, we consider a given pair $(H, p)$ where $H \subset G$ is a connected subgraph and $p \in \mathbb{R}^{V(H)}$ is a potential on $H$. Initially, we have $(H, p) = (H, \hat{p})$ and we iterate the elementary extension $(H', p')$ of $(H, p)$ defined as follows: Let $\pi$ denote a directed trajectory to $H$ having maximum positive slope $r = r(p, \pi) > 0$. Let $H' = H \cup \pi$ and extend $p$ to $p'$ by requiring that $p'$ has constant slope on $\pi$. In the case we have many paths of the same maximum positive slope, we pick the first path in some lexicographic order. If no trajectory of positive slope exists then we stop and set $H^* = H$ and $p^{|_{V(H')}} = p$.

The construction ensures that the slope of extension paths will decrease in value: If $(H', p')$ has a trajectory $\pi'$ such that the $p'$-slope exceeds $r$ then $\pi$ cannot be a trajectory of maximum slope to $(H, p)$. To see this, let $\pi''$ be the unique trajectory to $H$ that contains $\pi'$ and note that the $p$-slope of $p''$ equals it $p'$-slope: which is obtained as an average of the $p'$-slope along sub-paths of $\pi$ and $\pi'$ and would thus exceed the $p$-slope of $\pi$, contradicting the choice of $\pi$ as a trajectory of maximum slope.

Furthermore, the $\hat{p}$-slope along any $\hat{H}$-trajectory, $\pi$, with endpoints $a, b \in V(\hat{H})$, must be strictly less than 1. Otherwise $\pi''$ can be extended to an oriented cycle $\gamma$, with $\pi$ as a positively oriented subsegment and $\gamma \setminus \pi$ an oriented $ba$-path in $\hat{H}$. Thus, $\ell(\gamma) = \ell(\pi) + \ell(\gamma \setminus \pi) = \ell(\pi) - (\hat{p}_a - \hat{p}_b)$.

If $r(\hat{\phi}; \pi) \geq 1$ then $\ell(\gamma) \leq 0$ and $\gamma$ is an augmenting negative- or zero-cost cycle contradicting the choice of $\hat{H}$.

The constructed extension clearly maximises $S(p^*, H^*, A)$ for large $A$, since it is easy to see that $F_r = (p^*)^{-1}(r)$ contains all edges contained in some $H_r = (p^*)^{-1}(r, \infty)$-trajectory of slope $r$: If we inductively assume that the extension to $H_r$ is maximising — which is trivial when $H_r = \hat{H}$ — then, so, is the extension to $H_r \cup F_r$, since $r$ and then $|F_r|$ are maximal.

### 3.4 The proof of Theorem 1

#### 3.4.1 Some relations and identities

We first state some more equivalent formulations of the adaptation rule given by (8) and (9): By multiplying (8) with the integrating factor $e^t$ and then integrate, we obtain

$$\sigma(t) = (1 - e^{-t})\tilde{\phi}(t) + e^{-t}\sigma(0),$$

where $\tilde{\phi}(t) \in \Phi$ is the discounted time-averaged flow

$$\tilde{\phi}(t) := \frac{1}{1 - e^{-t}} \cdot \int_0^t \phi(s) e^{-(t-s)} \, ds.$$  

That $\tilde{\phi}(t) \in \mathbb{R}^E$ is a flow follows from the convexity of $\Phi(b, G)$.

If we integrate (9), we derive

$$\sigma(t) = \sigma(0) e^{-(\hat{\phi} - 1)t}.$$  

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where
\[ \bar{p}(t) := \frac{1}{t} \int_0^t p(s) \, ds, \quad (26) \]
and \( \bar{\psi}(t) = \Psi(\bar{p}(t)) \) is the time average of \( \psi(t) \).

For a flow obtained as a current in an electrical network, the flow is always from higher potential to lower potential. Thus, if \( \phi \) is a current then each arc \( ij \) is contained in the cut given by \( S = \{ k : p_k \geq \max\{p_i, p_j\} \} \) and the flow across this cut is from \( S \) to \( V \setminus S \) only. Hence,
\[ |\phi_{ij}| \leq b_{\text{max}}^* \quad (27) \]
Since \( \hat{\phi} \) in (23) is an average of the currents \( \phi(s), s \leq t \), we deduce that conductivities stay bounded
\[ \sigma(t) = (1 - e^{-t})\hat{\phi}(t) + e^{-t}\sigma(0) \leq K_0 := b_{\text{max}}^* + \max \sigma(0). \quad (28) \]
(We will label constants \( K_1, K_2, \ldots \) and \( c_1, c_2, \ldots \).)

Since \( \sigma_{ij}(t) > 0 \), we also obtain that
\[ \min_{ij} \hat{\phi}_{ij}(t) \geq -\sigma(0) \cdot \frac{e^{-t}}{1 - e^{-t}}, \quad (29) \]
which means that the flow \( \hat{\phi}_{ij}(t) \) is “almost positive” for large \( t \). Let therefore \( f(t) \) be the non-circulatory positive flow \( f(t) \in \Phi^+ \) obtained from \( \hat{\phi}(t) \) by the application of Lemma 2 with respect to arc-set
\[ E := \{ ij : \hat{\phi}_{ij}(t) \leq 0 \} \cup \{ ij : \bar{\psi}_{ij}(t) \leq 0 \}. \]
The flow \( f(t) \) is non-circulatory, since \( \bar{\psi}_{ij}(t) < 0 \) for at least one \( ij \) along any directed cycle. Lemma 2 in conjunction with (29) and (25) gives that \( f(t) = \hat{\phi}(t) + O(e^{-t}) \) and we deduce from (28) that
\[ \sigma(t) = f(t) + O(e^{-t}). \quad (30) \]
Hence, there is a constant \( K_1 \), such that
\[ \frac{1}{2} f_{ij}(t) \leq \sigma_{ij}(t) \leq 2 f_{ij}(t) \quad \text{whenever } \sigma_{ij}(t) \geq K_1 e^{-t}. \quad (31) \]

The observation (34) below is in some sense the key to the proof of Theorem 1 and the statement is essentially the same as that given in by Onishi et al. in [18]. Let
\[ \xi(t) := \log \left( \frac{\sigma(t)}{\sigma(0)} \right). \quad (32) \]
Let \( \gamma = \gamma^+ - \gamma^- \in \mathbb{R}^E \) be fixed. Applying \( \ell(\gamma) \) on both sides of (25) gives
\[ \ell(\xi(t)\gamma) = \left( \ell(\bar{\psi}(t)\gamma) - \ell(\gamma) \right) t - \ell(\gamma) t. \quad (33) \]
After dividing by $-\frac{1}{\ell(\gamma)}$ and rearranging, we obtain

$$M_\gamma - (\xi(t)) = r \cdot M_\gamma(\xi(t)) + (r - 1) t - t \frac{\ell(\tilde{\psi}(t)\gamma)}{\ell(\gamma)}, \quad (34)$$

where $r(\gamma) := \frac{\ell(\gamma^+)}{\ell(\gamma^+)}$ and $M_\gamma(x)$, $\gamma \geq 0$, is the weighted mean

$$M_\gamma(x) := \frac{1}{\ell(\gamma)} \ell(\gamma x).$$

If $\gamma \in \Phi(0, G)$ is a circulation, then $\ell(\bar{\psi}(t)\gamma) = 0$ and we obtain

$$M_\gamma - (\xi(t)) = r \cdot M_\gamma + (\xi(t)) + (r - 1) t \quad (35)$$

and if $\gamma = -\pi$, where $\pi$ is a directed $ab$-path, we obtain

$$M_\pi(\xi(t)) = t(M_\pi(\bar{\psi}(t)) - 1) = t \left( \frac{\bar{p}_a - \bar{p}_b}{\ell(\pi)} - 1 \right) \quad (36)$$

3.4.2 The proof of Theorem 1

Let $kl \in G \setminus \hat{H}$. Using (35), we show below, for some $a_1 > 0$, that

$$\sigma_{kl}(t) \leq O(e^{-a_1 t}), \quad \forall kl \in G \setminus \hat{H} \quad (37)$$

It follows from (37) and Lemma 2 that we can write the circulation free positive flow $f(t)$ as $f(t) = \hat{f}(t) + O(e^{-a_1 t})$, where $\hat{f}(t)$ is an optimal flow supported on $\hat{H}$. (Recall that any such flow is optimal.)

Hence we have from (30) the decomposition

$$\sigma(t) = \hat{f}(t) + O(e^{-a_1 t}).$$

We also show below that

$$\hat{f}_{ij}(t) = \Omega(1), \quad \forall ij \in \hat{H}. \quad (38)$$

The conditions of Lemma 5 are now fulfilled for $\sigma = \sigma(t)$ and $\hat{\sigma} = \hat{f}(t)$. Moreover, Lemma 4 implies that the solution $\tilde{p} = N(\hat{f}(t)|\hat{H}, b|\hat{H})$ is constant and given by $\Psi_{\hat{H}}(\tilde{p})_{ij} = \psi_{*ij}^{\hat{H}} = 1$, $ij \in \hat{H}$. The estimates (37) and (38) shows that we may take $\delta(\hat{f}(t), \sigma(t)) = O(e^{-a_1 t})$ in Lemma 5. Thus, we obtain from Lemma 5, that $\forall i \in V(\hat{H})$, $p_i(t) = \tilde{p}_i + O(e^{-a_1 t})$ or, equivalently that

$$\psi_{ij}(t) = \psi_{*ij} + O(e^{-a_1 t}), \quad \forall ij \in \hat{H}; \quad (39)$$

where $\psi_{*ij} = 1$ for $ij \in \hat{H}$.

Write

$$\psi_{ij}(t) = \psi_{*ij} + \epsilon_{ij}(t).$$

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As a direct consequence of (25), we have
\[ \sigma_{ij}(t) = (1 - \tilde{\epsilon}_{ij}(t)) \cdot \sigma^* \cdot e^{(\psi^*_{ij} - 1)t} \] (40)
where \( \sigma^* > 0 \) is the constant vector
\[ \sigma^* := \sigma(0) \exp \left( \int_0^\infty \epsilon(s) \, ds \right). \]
and
\[ \tilde{\epsilon}(t) = \exp \left\{ \int_t^\infty \epsilon(s) \, ds \right\} - 1, \]
since the integrals are convergent by (39). We obtain from (40), for \( ij \in \hat{H} \), that
\[ \lim_{t \to \infty} \sigma(t) = \lim_{t \to \infty} \hat{f}(t) = \hat{\phi} \in \hat{\Phi}. \]
and that \( |\sigma(t) - \hat{\phi}| = O(e^{-a_1 t}) \).

For \( ij \in \hat{H} \), the exponential convergence of
\[ \lim_{t \to \infty} \phi_{ij}(t) = \lim_{t \to \infty} (\sigma_{ij}(t) \psi_{ij}(t)) = (\lim_{t \to \infty} \sigma_{ij}(t)) \cdot 1 = \hat{\phi}_{ij}, \]
follows. For \( ij \notin \hat{H} \), we have on account of Ohm’s law and (37) that
\[ |\phi_{ij}(t)| \leq K_3(t) \cdot \sigma_{ij}(t) = O(e^{-a_1 t}), \]
where, due to the maximum principle, we can take
\[ K_3(t) := \max_{i,j \in \text{supp} \tau} \left| p_i(t) - p_j(t) \right| \]
\[ \min \ell. \]
The numerator is stays bounded and is for large \( t \) at most \( \text{diam} \hat{H} \) since \( \psi(t) \) converge to 1 on \( \hat{H} \).

Proof of (37). Using Lemma 3 on the circulation free flow \( f(t) \) and taking the
(30) into account, we deduce the following: There is a constant \( c_3 \) such that,
for any given edge \( ij \), either \( \sigma_{ij} < K_1 e^{-t} \) or else there is a \( \tau = \tau(t) \in \mathcal{T} \) such
that \( \tau_{ij} > 0 \)
\[ \min \sigma(\text{supp} \tau) \geq c_3 \sigma_{ij}. \] (41)
(We suppress the time variable for readability.)

As before, let \( \xi(t) = \log \left( \sigma(t)/\sigma(0) \right) \). With \( kl \in G \setminus \hat{H} \) as in (37), either
\( \sigma_{kl}(t) \leq K_1 e^{-t} \) in which case we are through. Otherwise, there is a bfs \( \tau_1 \) such that
\[ M_{\tau_1}(\xi(t)) \geq \min \xi(\text{supp} \tau_1) \geq \log \sigma_{kl}(t) + \log K_4, \] (42)
where \( K_4 = c_3/\max \sigma(0) \), with \( c_3 \) as in (41).

Let \( \mathcal{T} \) denote the set of basic optimal solutions, i.e. feasible solutions sup-
ported on forests contained in \( \hat{H} \). If \( \tau_2 \in \mathcal{T} \) then we can apply (35) on the
circulation \( \gamma = \tau_2 - \tau_1 \) and deduce from (42) that
\[ \log \sigma_{kl}(t) \leq r M_{\tau_2}(\xi(t)) - (1 - r)t - \log K_4, \] (43)
where
\[ r = \frac{\ell(\tau_2)}{\ell(\tau_1)} \leq \max_{\tau \in \calT} \frac{\ell(\tau_2)}{\ell(\tau)} =: 1 - a_1 < 1. \]

Moreover, from (28), we know that \( M_{\tau_2}(\xi(t)) \leq \log K_0 \), whence
\[ \log \sigma_{kl}(t) \leq (1 - a_1) \log K_0 - a_1 t - \log K_4, \]
which proves (37).

\[ \square \]

Proof of (38). Clearly, we have for all \( t > 0 \) that \( \max_{ij} f_{ij}(t) \geq b_{max}^*/|E| \), and hence, by (41), there is, for all \( t \), some \( \tau_0 = \tau_0(t) \in \calT \) such that
\[ M_{\tau_0(t)}(\xi(t)) \geq \log \left( b_{max}^*/|E| \right) + \log c_3 - \max \log \sigma(0) =: \log c_0. \]

Choose \( \tau_1 \in \hat{\calT} \) arbitrary. If we apply (35) on the circulation \( \gamma = \gamma(t) = \tau_0(t) - \tau_1 \) we get
\[ M_{\tau_1}(\xi(t)) \geq r \cdot \log c_0 + (r - 1)t \geq \log c_6. \]

since, in this case \( r \geq 1 \) on account of \( \tau_1 \) being optimal. Since \( \xi_{ij}(t) \leq \log K_0 \), for all \( ij \), by (28), it follows that
\[ \min \xi(\text{supp} \tau_1) \geq \frac{\ell(\tau_1) \cdot (\log c_6 - \log K_0)}{\min \ell}. \]

This proves (38), since \( \tau_1 \) was chosen arbitrary among basic solutions with support in \( \hat{H} \).

\[ \square \]

3.5 The proof of Theorem 2

Since \( \hat{H} \) is connected, the construction of \( p^* \) as in section 3.3 goes through. Let
\[ \psi^*(H^*) = \{r_0, r_1, \ldots, r_M\} \subset (0, 1], \tag{44} \]
denote the slopes obtained by \( p^* \) on \( H^* \), where \( 1 = r_0 > r_1 > \cdots > r_M > 0 \).

Let further \( r_{M+1} = 0 \). For \( r = r_k > 0 \), let \( F_r = (\psi^*)^{-1}(r) \), \( H_r := \bigcup_{s > r} F_s \) and \( F_0 := G \setminus H^* \) and \( H_1 \) the empty graph vertex-set \( \text{supp} \theta \). Note that \( F_1 = \hat{H} \) and that \( H_r \) is connected for \( r < 1 \) and that \( H_0 = H^* \).

Assume that \( r = r_k \in \psi^*(H^*) \) is fixed. We simplify the notation by putting \( H = H_r, F = F_r \). We let \( r' = r_{k+1} < r \) and put \( H' = H_{r'} = H \cup F \). We prove Theorem 2 with an induction argument over \( k = 1, 2, \ldots \) using the induction hypothesis that, for some constants \( a_k > 0 \) and \( C_k > 0 \),
\[ |p_i(t) - p_i^*| \leq C_k \cdot e^{-a_k t}, \quad \forall i \in V(H_r). \tag{45} \]
This is shown to hold for \( r = r_1 (k = 1) \) on account of (39) in the proof of Theorem 1. The induction step comprise of showing that, for \( r > 0 \), we can extend (45) to hold for \( i \in V(F_r) \) for some constants \( a_{k+1} > 0 \) and \( C_{k+1} > 0 \).
The following two relations, (46) and (47), take us a good way through the induction step. Let $\xi(t) = \log(\sigma(t)/\sigma(0))$. If $r > 0$ then

$$\xi_{ij}(t) = (r - 1) t + O(1)$$

(46)

for all $ij \in F$ and

$$\xi_{ij}(t) \leq (r' - 1) t + O(1),$$

(47)

for $ij \in G \setminus H'$. We defer the proof of these two claims to the end.

We use Lemma 5 together with (46) and (47) to conclude the proof. We want to show that $p(t) = \mathcal{D}(\sigma(t)|_G, p(t)|_{V(H')})$, satisfies

$$|p_i(t) - p_i^*| = O(e^{-a_{k+1}t})$$

for $i \in V(H')$, by the maximum principle and the induction hypothesis (45). It is therefore enough to show that

$$|q_i(t) - p_i^*| = O(e^{-a_{k+1}t})$$

(50)

By (47), (30) and Lemma 2,

$$\|f'(t) - \sigma(t)\|_{\infty} = O(e^{(r-1)t}).$$

(51)

and by (46), (30) and Lemma 2,

$$f'_{ij}(t) = \Theta(e^{(r-1)t}) + O(e^{(r'-1)t}) = \Theta(e^{(r'-1)t}),$$

(52)

for $ij \in H'$. It follows that the conditions of Lemma 5 are fulfilled for the solutions in (49) and (48) with $\sigma = \sigma(t)$ and $\hat{\sigma} = f'(t)$ and

$$\delta(\sigma(t)|_G, f'_{\hat{\sigma}}|_G) = O(exp\{- (r - r')t\}),$$

and thus we can conclude from Lemma 5 that (50) holds with

$$a_{k+1} = \min\{a_k, r - r'\}.$$
Proof of (46) and (47). Integration gives that if \(|p_i(t) - p_i^*| = O(e^{-a_k t})\) then

\[|\bar{p}_i(t) - p_i^*| = O(1/t).\]  \hspace{1cm} (53)

We first show that

\[\xi_{ij}(t) \leq (r - 1) t + O(1), \quad \text{for all } ij \in G \setminus H.\]  \hspace{1cm} (54)

Given \(kl \in G \setminus H\), we obtain either that \(\xi_{kl}(t) \leq -t + O(1)\), or else, by (41), a basic solution \(\tau = \tau(t) \in T\) such that \(\tau_{kl} > 0\) and

\[\min \{\xi(\text{supp } \tau) \} \geq \xi_{kl}(t) + O(1).\]

The forest \(\tau\) contains a unique directed \(ab\)-path \(\pi\) containing \(kl\) with endpoints \(a,b \in H\) and internal vertices in \(G \setminus H\). Then (36) and (53) gives that

\[\xi_{kl}(t) \leq M_\pi(\xi) + O(1) = t(\bar{p}_a - \bar{p}_b) - 1) + O(1)
\]
\[= t(\bar{p}_a^* - \bar{p}_b^*) - 1) + tO(1/t) + O(1)
\]
\[\leq t(r - 1) + O(1),\]

since, by the construction of \(H = H_r\), we know that \(\pi\) has \(p^*\)-slope at most \(r = r_k\). This shows (54).

Furthermore, it follows by the construction of \(F = F_r\) that, if \(ij \in F\), then \(ij\) is the endpoint of an \(ab\)-path \(\pi\), \(a,b \in V(H)\), which has \(p^*\)-slope exactly equal to \(r\). Thus the \(\bar{p}(t)\)-slope of \(\pi\) equals \(r + O(1/t)\), by (53). But, by (25) and (54),

\[\bar{\psi}_{ij} = \xi_{ij}(t)/t + 1 \leq r + O(1/t),\]

and since the \(\bar{p}\)-slope of \(\pi\) equals the average

\[M_\pi(\bar{p}) = \frac{1}{\ell(\pi)} \sum_{ij \in \pi} \ell_{ij} \bar{\psi},\]

we obtain

\[\bar{\psi}_{ij} \geq r + \frac{\ell(\pi)}{\ell_{ij}} O(1/t) = r + O(1/t), \quad \forall ij \in \pi.\]

Together with (25) and (54) this shows (46).

It also follows that \(\bar{\psi}_{ij}(t) = r + O(1/t)\) for \(ij \in F\). Hence,

\[\bar{p}_i(t) = p_i^* + O(1/t) \quad \text{for } i \in V(H').\]  \hspace{1cm} (55)

Hence, we can use the same argument as for (54) to deduce (47), since that argument only used (53) which can be replaced by (55).

\[\square\]

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