LENGTH FUNCTION COMPATIBILITY FOR GROUP ACTIONS 
ON REAL TREES

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Abstract. Let \( G \) be a finitely generated group. Given two length functions \( \ell \) and \( m \) of irreducible \( G \) actions on real trees \( A \) and \( B \), when is the point-wise sum \( \ell + m \) again the length function of an irreducible \( G \) action on a real tree? Guirardel and Levitt showed that additivity is equivalent to the existence of a common refinement of \( A \) and \( B \), this equivalence is established using Guirardel’s core. Moreover, in this case the sum \( \ell + m \) is the length function of the common refinement of \( A \) and \( B \) given explicitly by the Guirardel core. The core can be difficult to compute in general. Behrstock, Bestvina, and Clay give an algorithm for computing the core for free group actions on simplicial trees. In this article we give a geometric characterization of existence of a common refinement that generalizes the criterion underlying Behrstock, Bestvina, and Clay’s algorithm, as well as two equivalent characterizations in terms of the associated translation length functions.

1. Introduction

Suppose \( \lambda \) and \( \mu \) are sets of disjoint simple closed geodesics on a closed hyperbolic surface \( \Sigma \). Further suppose the geometric intersection number \( i(\lambda, \mu) = 0 \) so that the union \( \lambda \cup \mu \) is again a set of disjoint simple closed geodesics. The lifts of \( \lambda \) to the universal cover, \( \tilde{\lambda} \), describe a simplicial tree \( A \) with \( \pi_1(\Sigma) \) action. The vertices of \( A \) are the connected components of \( \mathbb{H}^2 \setminus \tilde{\lambda} \), and vertices \( X \) and \( Y \) are joined by an edge if \( \tilde{X} \cap \tilde{Y} = \gamma \) is a geodesic in \( \tilde{\lambda} \). Similarly the lifts of \( \mu \) describe a tree \( B \) with \( \pi_1(\Sigma) \) action, and \( \lambda \cup \mu \) a tree \( T \). Since \( \lambda \) and \( \mu \) are disjoint, \( T \) comes with equivariant surjections \( T \to A \) and \( T \to B \). From the construction, these surjections have the property that every segment \( [a,b] \) is sent to the segment \( [f(a), f(b)] \), we say that these surjections preserve alignment. Further, each tree has a metric induced by assigning each edge length one, which allows us to define translation length functions \( \ell_A, \ell_B, \ell_T : \pi_1(\Sigma) \to \mathbb{R}^+ \). From the construction of \( T \) it follows that the surjections to \( A \) and \( B \) are Lipschitz; and we can calculate \( \ell_T = \ell_A + \ell_B \).

This kind of compatibility generalizes to the setting of a finitely generated group \( G \) acting on real trees \( A \) and \( B \) (hereafter \( G \)-trees).

Definition 1.1. A \( G \)-tree \( T \) is a common refinement of \( G \)-trees \( A \) and \( B \) if there are equivariant Lipschitz surjections \( T \to A \) and \( T \to B \) that preserve alignment.

Guirardel [3] introduces a convex core (and a notion of intersection number) for a pair of \( G \)-trees, and proves [3, Theorem 6.1] that this convex core is one-dimensional if and only if there exists a common refinement of the two \( G \)-trees.

In the opening example, the compatibility of two sets of simple closed curves also entailed a compatibility for the resulting length functions on \( \pi_1(\Sigma) \). One family of actions, known as irreducible actions (Definition 3.8), are completely characterized
by their length functions \([4, 9]\). In the more general setting of \(G\)-trees, Guirardel and Levitt [6, Appendix A] prove that two minimal, irreducible \(G\)-trees have a common refinement if and only if the sum of their length functions is again the length function of a minimal, irreducible \(G\)-tree [6, Appendix A]. Moreover, in this case there is a common refinement with length function equal to the sum.

This article introduces three new equivalent characterizations of the existence of a common refinement of \(G\)-trees. Each is motivated from the surface setting, we defer formal definitions to Section 5 after the necessary background is introduced.

Generalizing slightly from the opening example, consider two measured geodesic laminations \(\lambda\) and \(\mu\) on a closed hyperbolic surface \(\Sigma\). There are naturally associated \(\pi_1(\Sigma)\)-trees \(A\) and \(B\) dual to \(\lambda\) and \(\mu\) [8]. To define incompatibility, suppose \(\lambda\) and \(\mu\) have leaves that intersect transversely. This intersection produces arcs \(a \subseteq A\) and \(b \subseteq B\) in the dual trees to \(\lambda\) and \(\mu\) such that the Gromov boundaries of particular complementary components of \(a\) and \(b\), all intersect, as in Figure 1. Definition 5.1 re-interprets this boundary intersection as an intersection condition for four particular subsets of \(G\) associated to arcs in the \(G\)-trees under consideration, and two \(G\)-trees have incompatible horizons if such an intersection occurs.

The intersection of the laminations \(\lambda\) and \(\mu\) is also detected by certain pairs of elements of \(\pi_1(\Sigma)\). The hyperbolic structure on \(\Sigma\) gives an action of \(\pi_1(\Sigma)\) on the hyperbolic plane \(\mathbb{H}^2\), and elements of \(\pi_1(\Sigma)\) act hyperbolically. Given two elements \(x, y \in \pi_1(\Sigma)\), if the axes of \(x\) and \(y\) are separated by a set of leaves of positive measure of the lift \(\tilde{\lambda} \subseteq \mathbb{H}^2\) then the axes of \(x\) and \(y\) in the dual tree \(A\) will be disjoint. On the other hand, if the axes of \(x\) and \(y\) cross a common set of leaves of \(\tilde{\mu}\) (with positive measure) then their axes in the dual tree \(B\) will overlap in an arc. The presence of a pair of such elements detects the intersection of \(\lambda\) and \(\mu\), as illustrated in Figure 2. Generalizing to \(G\)-trees, we say (Definition 5.7) two \(G\)-trees have compatible combinatorics if there is no pair of elements \(x, y \in G\) that have disjoint axes in one tree and overlapping axes in the other. This definition is stated synthetically, entirely in terms of length function inequalities.
Another situation in which a pair of fundamental group elements \( x, y \in \pi_1(\Sigma) \) detect the intersection of \( \lambda \) and \( \mu \) occurs when the axes of \( x \) and \( y \) cross a common set of leaves of positive measure in both \( \tilde{\lambda} \) and \( \tilde{\mu} \). In this case, if the axes of \( x \) and \( y \) cross their common leaves of \( \tilde{\lambda} \) with differing orientations and their common leaves of \( \tilde{\mu} \) with the same orientation, then \( \tilde{\lambda} \) and \( \tilde{\mu} \) must intersect, as in Figure 3.

In the general setting of \( G \)-trees, Definition 5.9 captures compatible orientations purely in terms of length functions.

The main contribution of this article is that each of these conditions is a characterization of compatibility.
Theorem 1.2. Suppose $A$ and $B$ are irreducible $G$-trees with length functions $\ell$ and $m$. The following are equivalent:

1. The trees $A$ and $B$ have compatible horizons.
2. The length functions $\ell$ and $m$ have compatible combinatorics.
3. The length functions $\ell$ and $m$ are coherently oriented.
4. The sum $\ell + m$ is a length function for an irreducible $G$-tree.
5. The two trees $A$ and $B$ have a common refinement.

Proof. The equivalence of items 4 and 5 is given by Guirardel and Levitt [6, Theorem A.10]. We prove the equivalence of items 1–3 in Lemma 5.12, and show the equivalence of 1–3 to 4 in Theorem 6.1.

These characterizations have been used in work of the author both to certify incompatibility in the setting of $F_r$-trees and to extract geometric information from incompatible trees [2].

The article is organized as follows. Sections 2 and 3 recall the theory of $G$-trees and their length functions. The interplay between the boundary of a $G$-tree and its length function is elaborated on in Section 4. This is a warm-up for Section 5 which introduces the three new criteria and proves that they are equivalent to one another. Finally, Section 6 proves the equivalence of the three new criteria to compatibility.

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2. Real trees
An arc $e$ in a metric space $X$ is the image of an embedding of an interval $\gamma_e : [a, b] \to X$. In a uniquely geodesic metric space $X$, let $[p, q]$ denote the geodesic from $p$ to $q$. If $p, q, r \in X$ and $r \in [p, q]$, we will use the notation $[p, r, q]$ for the geodesic path, for emphasis.

Definition 2.1. A real tree or $\mathbb{R}$-tree $T$ is a connected uniquely geodesic metric space such that for any pair of points $p, q \in T$ the geodesic $[p, q]$ from $p$ to $q$ is the unique arc from $p$ to $q$. A subtree of a real tree is a connected subset $S \subseteq T$.

Throughout this article, when $e \subseteq T$ is an arc in a real tree we will assume this arc is oriented, that is we have a fixed isometry $\gamma : [0, \text{length}_T(e)] \to T$ whose image is $e$. We will use the notation $o(e) = \gamma(0)$ and $t(e) = \gamma(\text{length}_T(e))$ for the origin and terminus of the arc, and $\bar{e}$ for the reversed orientation. To keep the notation uncluttered we will not refer to the isometry $\gamma$ unless it is desperately necessary for clarity. We will always specify an orientation when specifying an arc (or it will inherit one from context, by being a sub-arc of an oriented arc).

Definition 2.2. Let $T$ be a real tree. A point $p \in T$ is a branch point if $T \setminus \{p\}$ has more than two connected components. The order of a branch point is the number of connected components of $T \setminus \{p\}$. A direction based at $p$, $\delta_p \subseteq T$, is a connected component of $T \setminus \{p\}$.

Definition 2.3. The visual boundary of a real tree $T$ based at $p \in T$ is the set $\partial_p T = \{\rho \subseteq T | \rho \text{ is a geodesic ray based at } p \}$.

The boundary is topologized by the basis of open sets $V(\rho, r) = \{\gamma \in \partial_p T | B(p, r) \cap \gamma = B(p, r) \cap \rho\}$.
for \( r > 0 \) and \( \rho \in \partial_p T \).

Different base points \( p \) give different identifications of the same boundary [3 Proposition II.8.8].

We will write \( \partial T \) when the choice of basepoint is not important and \( \omega_T(S) \subseteq \partial T \) for the subset of the boundary determined by the geodesic rays contained in a subtree \( S \). If \( S \) is a bounded subtree, \( \omega(S) = \emptyset \).

3. LENGTHS AND ACTIONS

Definition 3.1. Let \( G \) be a group and \( \rho : G \to Isom(T) \) be an injection, with \( T \) a real tree, so that \( G \) acts on \( T \) on the right. The triple \((G, \rho, T)\) is a \( G \)-tree.

The action of \( G \) will be clear from context and \( G \) will be fixed, so we suppress the notation and refer to a tree \( T \) as a \( G \)-tree. The restriction to actions where \( \rho : G \to Isom(T) \) is injective is not standard in the literature, some authors allow group actions with kernel; these authors call actions without kernel effective.

We study the geometry of \( G \)-tree actions via their translation length functions. The elements of \( G \)-tree geometry reviewed here are for the most part based on the exposition given by Culler and Morgan [4], with other developments cited as relevant.

Definition 3.2. The translation length function of a \( G \)-tree \( T \), denoted \( \ell_T : G \to \mathbb{R} \) is defined by 
\[
\ell_T(g) = \inf_{p \in T} d_T(p, p \cdot g).
\]

Any \( G \)-tree \( T \) divides the elements \( g \in G \) into hyperbolic elements, when \( \ell_T(g) > 0 \) and elliptic elements, when \( \ell_T(g) = 0 \). When an element \( g \in G \) is elliptic, \( Fix(g) \) will denote the set of fixed points of \( g \).

3.1. A taxonomy.

Definition 3.3. A \( G \)-tree \( T \) is minimal if there is no proper \( G \) invariant subtree \( T' \subsetneq T \).

Definition 3.4. A \( G \)-tree \( T \) where for all \( g \in G \), \( Fix(g) \neq \emptyset \) is trivial.

For finitely generated groups this is equivalent to the condition that \( G \) has a global fixed point, but this is not true for infinitely generated groups [7][10].

Definition 3.5. A \( G \)-tree \( T \) is lineal if there is a \( G \) invariant subtree isometric to the line.

Definition 3.6. A \( G \)-tree \( T \) is reducible if it is minimal and neither trivial, lineal, nor reducible.

Lineal and reducible actions are uninteresting from the perspective of translation length functions.

Theorem 3.7 ([4 Theorem 2.4.2.5]). If \( T \) is a lineal or reducible \( G \)-tree, then there is a homomorphism \( \rho : G \to Isom(\mathbb{R}) \) such that \( \ell_T(g) = N(\rho(g)) \), where \( N \) is the translation length function of the induced action on \( \mathbb{R} \).

The \( G \)-trees of interest for this article are the ones whose study is not an indirect study of subgroups of \( Isom(\mathbb{R}) \).

Definition 3.8. A \( G \)-tree \( T \) is irreducible if it is minimal and neither trivial, lineal, nor reducible.

The translation length function is an isometry invariant of irreducible \( G \) trees.

Theorem 3.9 ([4 Theorem 3.7]). Suppose \( A \) and \( B \) are two irreducible \( G \) trees and \( \ell_A = \ell_B \). Then there is an equivariant isometry from \( A \) to \( B \).
3.2. Axes.

**Definition 3.10.** The characteristic set of some \( g \in G \) in a \( G \)-tree \( T \) is the set

\[
C^T_g = \{ p \in T | d(p, p \cdot g) = \ell_T(g) \}
\]

of points achieving the translation length. When \( T \) is clear from context we write \( C_g \).

**Lemma 3.11 ([H Lemma 1.3]).** For any \( G \)-tree \( T \) and \( g \in G \), the characteristic set \( C^T_g \) is a closed non-empty subtree of \( T \) invariant under \( g \). Moreover,

- If \( \ell_T(g) = 0 \) then \( C_g = \text{Fix}(g) \).
- If \( \ell_T(g) > 0 \) then \( C_g \) is isometric to the real line and the action of \( g \) on \( C_g \) is translation by \( \ell_T(g) \). In this case we call \( C_g \) the axis of \( g \).
- For any \( p \in T \), \( d(p, p \cdot g) = \ell_T(g) + d(p, C_g) \).

When \( g \) is a hyperbolic element of a \( G \)-tree \( T \), \( \omega_T(C_g) \) is a pair of boundary points fixed by \( g \). The action of \( g \) on \( C_g \) gives \( C_g \) a natural orientation and we always consider an axis oriented by the element specifying it, so that \( C_{g^{-1}} \) is the same set as \( C_g \) but with the opposite orientation. The point of \( \partial T \) in the equivalence class of a positive ray along \( C_g \) with the \( g \) orientation will be denoted \( \omega_T(g) \). If \( g \) is elliptic and \( T \) minimal, \( \omega_T(C_g) = \emptyset \), and \( \omega_T(g) \) is undefined.

**Definition 3.12.** Let \( T \) be a \( G \)-tree. The \( T \)-boundary of \( g \in G \), \( \partial_T g \) is the empty set if \( g \) is elliptic, and the set \( \{ \omega_T(g), \omega_T(g^{-1}) \} \) if \( g \) is hyperbolic.

The intersection of characteristic sets is detected by the translation length function.

**Lemma 3.13 ([H Lemma 1.5]).** Let \( T \) be a \( G \)-tree. For any \( g, h \in G \) such that \( C_g \cap C_h = \emptyset \), we have

\[
\ell(gh) = \ell(gh^{-1}) = \ell(g) + \ell(h) + 2d(C_g, C_h)
\]

This lemma is also used in its contrapositive formulation, if \( \ell(gh) \leq \ell(g) + \ell(h) \), then \( C_g \cap C_h \neq \emptyset \). For hyperbolic isometries there is a more precise relationship between the intersection of characteristic sets and the length function.

**Lemma 3.14 ([H Lemma 1.8]).** Suppose \( g \) and \( h \) are hyperbolic in a \( G \)-tree \( T \). Then \( C_g \cap C_h \neq \emptyset \) if and only if

\[
\max\{\ell_T(gh), \ell_T(gh^{-1})\} = \ell_T(g) + \ell_T(h).
\]

Moreover \( \ell(gh) > \ell(gh^{-1}) \) if and only if \( C_g \cap C_h \) contains an arc and the orientations of \( C_g \) and \( C_h \) agree on \( C_g \cap C_h \).

These two lemmas are proved by the construction of explicit fundamental domains. These fundamental domains are sufficiently useful that we detail them here. That these domains have the claimed properties is a consequence of the proofs of the previous two lemmas.

**Definition 3.15.** Let \( T \) be a \( G \)-tree and suppose \( g \) and \( h \) are such that \( C_g \cap C_h \neq \emptyset \). Let \( \alpha = [p, q] \) be the geodesic joining \( C_g \) to \( C_h \). The Culler-Morgan fundamental domain for the action of \( gh \) on \( C_{gh} \) is the geodesic

\[
[p \cdot g^{-1}, p, q, q : h, p : h] = [p, q, h, p : h, q : h, p : h].
\]

**Definition 3.16.** Let \( T \) be a \( G \)-tree and suppose \( g \) and \( h \) are such that \( C_g \cap C_h \neq \emptyset \), at least one of \( g \) and \( h \) is hyperbolic, and that if both \( g \) and \( h \) are hyperbolic the orientations agree. Let \( \alpha = [p, q] \) be the possibly degenerate \((p = q)\) common arc of intersection with the induced orientation. The Culler-Morgan fundamental domain for the action of \( gh \) on \( C_{gh} \) is the geodesic
If $gh^{-1}$ is also hyperbolic, then the Culler-Morgan fundamental domain for the action of $gh^{-1}$ on $C_{gh^{-1}}$ is the geodesic $[q \cdot g^{-1}, q \cdot h]$.

The axes of a minimal $G$-tree provide complete information about the $G$-tree.

**Proposition 3.17** ([4 Proposition 3.1]). A minimal non-trivial $G$-tree $T$ is equal to the union of the axes of the hyperbolic elements.

### 3.3. Axioms

For irreducible $G$-trees, length functions provide a complete invariant, as noted above. Culler and Morgan characterized these length functions in terms of a list of useful properties; Parry showed that any length function satisfying these axioms comes from an irreducible $G$-tree [9].

**Definition 3.18.** An axiomatic length function (or just length function) is a function $\ell : G \to \mathbb{R}_{\geq 0}$ satisfying the following six axioms.

1. $\ell(id) = 0$.
2. For all $g \in G$, $\ell(g) = \ell(g^{-1})$.
3. For all $g, h \in G$, $\ell(g) = \ell(hgh^{-1})$.
4. For all $g, h \in G$, either $\ell(gh) = \ell(gh^{-1})$ or $\max\{\ell(gh), \ell(gh^{-1})\} \leq \ell(g) + \ell(h)$.
5. For all $g, h \in G$ such that $\ell(g) > 0$ and $\ell(h) > 0$, either $\ell(gh) = \ell(gh^{-1}) > \ell(g) + \ell(h)$ or $\max\{\ell(gh), \ell(gh^{-1})\} = \ell(g) + \ell(h)$.
6. There exists a pair $g, h \in G$ such that $0 < \ell(g) + \ell(h) - \ell(gh^{-1}) < 2\min\{\ell(g), \ell(h)\}$.

**Proposition 3.19** ([4]). If $\ell_T$ is the translation length function of an irreducible $G$-tree then $\ell_T$ is an axiomatic length function.

**Theorem 3.20** ([9]). If $\ell$ is an axiomatic length function on a group $G$ then there is an irreducible $G$-tree $T$ such that $\ell = \ell_T$.

In the wider literature, axiom VI is omitted, including the consideration of all $G$-trees, instead of only irreducible $G$ trees. Without this axiom, length functions are no longer a complete isometry invariant. A pair of elements witnessing Axiom VI for a given length function $\ell$ is called a good pair for $\ell$.

### 3.4. Good pairs

Culler and Morgan used good pairs in the proof of their uniqueness statement for $G$-trees coming from a given length function. They give a geometric definition.

**Definition 3.21.** Let $T$ be a $G$-tree. A pair of elements $g, h \in G$ is a good pair for $T$ if

- the elements $g$ and $h$ are hyperbolic;
- the axes $C_g$ and $C_h$ meet in an arc of positive length;
- the orientations of $C_g$ and $C_h$ agree on the intersection;
- $\text{length}(C_g \cap C_h) < \min\{\ell(g), \ell(h)\}$. 

Proposition 3.22 ([4, Lemma 3.6]). A pair of elements $g, h \in G$ is a good pair for a $G$-tree $T$ if and only if $g$ and $h$ witness Axiom VI for $\ell_T$.

Lemma 3.23. Suppose $g, h \in G$ is a pair of hyperbolic elements of a $G$-tree $T$ whose axes intersect in an arc of finite length and the induced orientations agree. Then there are integers $A, B > 0$ so that for all $a \geq A$ and $b \geq B$, $g^a, h^b$ is a good pair.

Proof. By hypothesis $g$ and $h$ satisfy the first three points of the geometric definition of a good pair. Let $N = \text{length}(C_g \cap C_h)$. It is immediate that $A = \left\lceil N/\ell_T(g) \right\rceil$ and $B = \left\lceil N/\ell_T(h) \right\rceil$ are the desired integers. □

The axes of a pair of group elements satisfying the hypotheses of Lemma 3.23 have distinct boundary points; this is a form of independence seen by the tree, and closely related to the algebraic independence of group elements in the subgroup generated by a good pair [4, Lemma 2.6]. In the sequel we only need to reference this boundary independence.

Definition 3.24. Let $T$ be a $G$-tree. Two hyperbolic elements $g, h \in G$ are $T$-independent when

$$\partial_T g \cap \partial_T h = \emptyset.$$
direction $\delta_p^e$ based at $p$ such that $t(e) \in \delta_p^e$. The subset of the boundary of the tree corresponding to $e$ is then
\[ \bigcap_{p \in e^o} \omega_T(\delta_p^e). \]
In the sequel we will be more concerned with describing this directly from the group.

**Definition 4.1.** The *group ends* of a direction $\delta \subseteq T$ is the set of group elements
\[ \delta(G) = \{ g \in G \mid \omega_T(g) \in \omega_T(\delta) \}. \]

**Definition 4.2.** The *asymptotic horizon* of an oriented arc $e \subseteq T$ of a $G$-tree is
\[ [e] = \bigcap_{p \in e^o} \delta_p^e(G), \]
where $\delta_p^e$ is the unique direction based at $p$ such that $t(e) \in \delta_p^e$.

**Remark 4.3.** In some figures $[e]$ will be used to indicate the set $\{ \omega(g) \mid g \in [e] \} \subseteq \partial X$ where $X$ is hyperbolic. This abuse of notation is used only in illustrative figures, and the set of group elements will play the important role in the text.

The asymptotic horizon of an oriented arc $e$ is all hyperbolic group elements whose axes have an endpoint visible from $e$, when looking in the forward direction specified by the orientation. The visibility of group ends is sufficient to find group elements whose axes either contain $e$ or are disjoint from $e$, exercises in the calculus of axes that are recorded in the next two lemmas.

To fix notation, for an oriented arc $e \subseteq T$ in a $G$-tree, let $R_e^-$ be the connected component of $T \setminus e^o$ containing $o(e)$ and $R_e^+$ the component containing $t(e)$.

**Lemma 4.4.** Suppose $e \subseteq T$ is an oriented arc in a $G$-tree $T$. Suppose $g \in [e]$ and $h \in [e]$. Then there is an $N > 0$ such that for all $n \geq N$, $f = h^{-n}g^n$ is hyperbolic and $e \subseteq C_f$. Moreover the orientation of $e$ agrees with the orientation on $C_f$ induced by $f$.

**Proof.** Consider the intersection $C_g \cap C_h$. There are three cases.

**Case 1:** $C_g \cap C_h = \emptyset$. Let $a$ be the unique shortest oriented arc joining $C_g$ to $C_h$ with $t(a) \in C_g$. Take
\[ N > \frac{d_T(e, a) + \text{length}(e)}{\min\{\ell_T(g), \ell_T(h)\}} \]
and suppose $n \geq N$. Consider the Culler-Morgan fundamental domain for the action of $f = h^{-n}g^n$ on its axis: the geodesic path $b$ in $T$ passing through the points
\[ [o(a) \cdot h^n, o(a), t(a), t(a) \cdot g^n, o(a) \cdot g^n]. \]
By hypothesis, the axis $C_h$ meets $R_e^-$ in at least a positive ray and $b R_e^- \subseteq R_e^-$. If $o(a) \in T \setminus R_e^-$, then the ray of $C_h$ based at $o(a)$ directed at $\omega_T(h)$ must pass through $o(a)$. By the choice of $N$, $o(a) \cdot h^n \in R_e^-$. Similarly, $t(a) \cdot g^n \in R_e^-$. The arc $e$ is the unique geodesic in $T$ joining $R_e^-$ to $R_e^+$, hence $e \subseteq b$. Moreover, the action of $f$ takes $o(b) = o(a) \cdot h^n$ to $t(b) = o(a) \cdot g^n$, so the orientations of $e$ and $b$ agree, as required.

**Case 2:** $C_g \cap C_h = a \neq \emptyset$, $a$ a point or arc. Orient $a$ according to the orientation of $g$. (When $a$ is a point, orientation does not matter; we use the convention $o(a) = a = t(a).$) Take
\[ N > \frac{d_T(e, a) + \text{length}(e) + \text{length}(a)}{\min\{\ell_T(g), \ell_T(h)\}} \]
and suppose $n \geq N$. Again consider the Culler-Morgan fundamental domain for the action of $f = h^{-n}g^n$ on its axis. It contains (regardless of the agreement
between the orientations of $h$ and $a$) the geodesic path $b$ in $T$ passing through the points

$[t(a) \cdot h^n, t(a), t(a) \cdot g^n]$.

As in the previous case, we find $t(a) \cdot h^n \in R^+_e$ and $t(a) \cdot g^n \in R^+_e$. We conclude $e \subseteq b$ and the orientations agree.

Case 3: $C_g \cap C_h$ contains a ray. If $C_g = C_h$ then $e \subseteq C_{h^{-1}g} = C_g = C_h$ and $N = 1$ suffices. So suppose $C_g \neq C_h$. Let $p \in C_g \cap C_h$ be the basepoint of the common ray. Take

$N > d_T(p, e) + \text{length}(e)$

and suppose $n \geq N$. Once more, a fundamental domain for the action of $f = h^{-n}g^n$ on its axis can be described. It contains the geodesic path $b$ in $T$ passing through the points

$[p \cdot h^n, p, p \cdot g^n]$. 

By the choice of $n$, we find $p \cdot h^n \in R^+_e$ and $p \cdot g^n \in R^+_e$. We conclude $e \subseteq b$ and the orientations agree. □

Lemma 4.5. Suppose $e \subseteq T$ is an oriented arc in a $G$-tree $T$. Suppose $g, h \in \llbracket e \rrbracket$ and $\omega_T(g) \neq \omega_T(h)$. Then there is an $N > 0$ such that for all $n \geq N$, $f = h^{-n}g^n$ is hyperbolic and $C_f \subseteq R^+_e$.

Proof. As in the proof of the previous lemma, there are three cases depending on $C_g \cap C_h$.

Case 1: $C_g \cap C_h = \emptyset$. Let $a$ be the oriented geodesic from $C_h$ to $C_g$, so that $t(a) \in C_g$. Let $C^+_g$ and $C^+_h$ be the positive rays of $C_g$ and $C_h$ based at $t(a)$ and $o(a)$ respectively. The infinite geodesic $C^+_h \cup a \cup C^+_g$ has both endpoints in $\partial R^+_e$, so must be contained in $R^+_e$, therefore $a \subseteq R^+_e$. At this point it is tempting to take $N = 1$, however we must exercise care to ensure that the axis of the product is contained in $R^+_e$, as this axis is not the infinite geodesic previously mentioned.

Since $g, h \in \llbracket e \rrbracket$, there is an integer $N_1 > 0$ such that for all $n \geq N_1$ we have

$d(t(a) \cdot g^n, t(e)) > d(t(a), t(e))$

and

$d(o(a) \cdot h^n, t(e)) > d(o(a), t(e))$.

Let $\alpha_g$ and $\alpha_h$ be the geodesics from $t(e)$ to $C_g$ and $C_h$ respectively, oriented such that $o(\alpha_g) = o(\alpha_h) = o(e)$. Since $g$ acts by translation on its axis in the direction of $\omega_T(g)$, there is an $N_2$ such that for all $n \geq N_2$, $t(a) \cdot g^n > t(\alpha_g)$ (in the orientation on $C_g$ induced by the action of $g$). Similarly there is an $N_3$ such that for all $n \geq N_3$, $o(a) \cdot h^n > t(\alpha_h)$. Take $N = \max\{N_1, N_2, N_3\}$.

Suppose $n \geq N$. As in the previous lemma, we use the Culler-Morgan fundamental domain for the action of $f = h^{-n}g^n$ on $C_f$: the geodesic $b$ passing through the points

$[o(a) \cdot h^n, o(a), t(a), t(a) \cdot g^n]$. 

By construction, $b \subseteq R^+_e$. Further, the geodesic from $t(\alpha_g)$ to $t(\beta_h)$ is a proper subarc of $b$. Therefore, the center $u$ of the geodesic triangle $t(\alpha_g), t(\alpha_h), t(e)$ is in the interior of $b$. This point is, by construction, the unique closest point of $b$ to $o(e)$. Since $u$ is in the interior of $b$, $u$ is also the unique closest point of $C_f$ to $o(e)$, whence $e \not\subseteq C_f$ and so $C_f \subseteq R^+_e$ as required.

Case 2: $C_g \cap C_h = a \neq \emptyset$, $a$ an arc or point. Orient $a$ so that it agrees with the orientation of $C_g$ induced by the action of $g$ (again with the convention that if $a$ is a point, $o(a) = a = t(a)$). If the orientations of $C_g$ and $C_h$ disagree on $a$, then with $C^+_g$ and $C^+_h$ defined as in the previous case, the previous argument applies. If the orientations of $C_g$ and $C_h$ agree on $a$, let $C^+_g$ be as before and instead take
5. Synthetic compatibility conditions

Recall the motivating examples of the introduction. The intersection of boundary sets is naturally captured \textit{in the group} by the asymptotic horizons, and Figure [1] gives the geometric motivation for the following definition.

**Definition 5.1.** Two \(G\)-trees \(A\) and \(B\) have \textit{incompatible horizons} if there are oriented arcs \(a \subseteq A\) and \(b \subseteq B\) such that the four sets

\[
[a] \cap [b], \quad [a] \cap [b], \quad [a] \cap [b], \quad [a] \cap [b]
\]

are non empty.

**Remark 5.2.** Behrstock, Bestvina, and Clay [1] consider a similar collection of sets when giving a criterion for the presence of a rectangle in the Guiraud core of two free simplicial \(F_r\) trees.

Pairs of group elements with either overlapping or disjoint axes for a given action capture the situations in Figure [2] and Figure [3]. Let \(P(G) = G \times G \setminus \Delta\) be the set of all distinct pairs of elements in our group.

**Definition 5.3.** For a \(G\)-tree \(T\) the \textit{overlap set}, \(O^T \subseteq P(G)\), is all pairs \((g, h) \in P(G)\) such that \(g\) and \(h\) are hyperbolic and \(C_g \cap C_h\) contains an arc.

The \textit{disjoint set}, \(D^T \subseteq P(G)\), is all pairs \((g, h) \in P(G)\) such that \(C_g \cap C_h = \emptyset\).

This definition can also be stated for length functions.

**Definition 5.4.** For a length function \(\ell : G \to \mathbb{R}_{\geq 0}\) the \textit{overlap set}, \(O^\ell \subseteq P(G)\) is all pairs \((g, h) \in P(G)\) such that

\[
\ell(gh) \neq \ell(gh^{-1})
\]

The \textit{disjoint set}, \(D^\ell \subseteq P(G)\) is all pairs \((g, h) \in P(G)\) such that

\[
\ell(gh) = \ell(gh^{-1}) > \ell(g) + \ell(h).
\]

In the definition for a tree, the hyperbolicity requirement for membership in \(O^T\) is necessary, but the length function requirement implies that \(O^\ell\) consists of pairs of hyperbolic elements.

**Lemma 5.5.** Suppose \(\ell\) is a length function on \(G\). If \((g, h) \notin D^\ell\) satisfies \(\ell(g) = 0\), then

\[
\ell(gh) = \ell(gh^{-1}) = \ell(h).
\]

In particular all pairs in \(O^\ell\) are pairs of hyperbolic elements.

**Proof.** First, suppose \(\ell(h) = 0\) also. Since \((g, h) \notin D^\ell\), length function axiom IV implies

\[
\max\{\ell(gh), \ell(gh^{-1})\} \leq \ell(g) + \ell(h) = 0
\]

and we are done. So suppose \(\ell(h) > 0\). Let \(T\) be the irreducible tree realizing \(\ell\). It must be the case that \(C_g \cap C_h\) is non-empty, by Lemma [5.13]. Consider \(p \in T\) and \(\alpha\) the shortest arc from \(p\) to \(C_g \cap C_h\). Let \(q\) be the endpoint of \(\alpha\) in \(C_g \cap C_h\).
Since $g$ is elliptic, $\alpha \cdot g \cap C_g \cap C_h$ contains $q$, as does $\alpha \cap \alpha \cdot g \cap C_h$. The element $h$ is hyperbolic, therefore
\[d_T(p, p \cdot gh) \geq d_T(q, q \cdot gh) = d_T(q, q \cdot h) = \ell(h)\]
\[d_T(p, p \cdot gh^{-1}) \geq d_T(q, q \cdot h^{-1}) = \ell(h),\]
and we conclude $\ell(gh) = \ell(gh^{-1}) = \ell(h)$ as required. \qed

**Proposition 5.6.** Suppose $T$ is an irreducible $G$-tree with length function $\ell$. Then $O^T = O^\ell$ and $D^T = D^\ell$, that is, definitions 5.3 and 5.4 are equivalent.

**Proof.** It is immediate from the definitions that $O^T \subseteq O^\ell$ and similarly $D^T \subseteq D^\ell$.

To demonstrate the reverse inclusions, suppose $(g, h) \in O^\ell$. By Lemma 5.5, $g$ and $h$ are hyperbolic. If, for a contradiction, $(g, h) \notin O^T$ then either $C_g \cap C_h = \emptyset$ or $C_g \cap C_h = \{\ast\}$. In either case we have
\[\ell(gh) = \ell(gh^{-1}) = \ell(g) + \ell(h) + d_T(C_g, C_h),\]
a contradiction.

If $(g, h) \in D^\ell$ but $(g, h) \notin D^T$ then $C_g \cap C_h$ is non-empty, and so
\[\max\{\ell(gh), \ell(gh^{-1})\} \leq \ell(g) + \ell(h),\]
a contradiction. \qed

Note that the definitions of $O^\ell$ and $D^\ell$ depend only on the projective class of $\ell$; the axis overlap condition is a topological property of a tree, so this is expected. Also be aware that $O^\ell \cup D^\ell \neq \mathcal{P}(G)$; pairs such that $\ell(gh) = \ell(gh^{-1}) = \ell(g) + \ell(h)$ exist.

The interaction of overlap and disjoint sets captures the situations pictured in Figure 2 and Figure 3. We state the definitions in terms of length functions.

**Definition 5.7.** Two length functions $\ell$ and $m$ on a group $G$ have compatible combinatorics if
\[O^\ell \cap D^m = D^\ell \cap O^m = \emptyset.\]

**Remark 5.8.** The equivalent definition for trees is vacuous for lineal actions. For an lineal action the tree is a line, and the disjoint set is empty, hence all lineal actions have compatible combinatorics.

**Definition 5.9.** Two length functions $\ell$ and $m$ on a group $G$ are coherently oriented if for all $(g, h) \in O^\ell \cap O^m$
\[\ell(gh^{-1}) \leq \ell(gh) \Leftrightarrow m(gh^{-1}) \leq m(gh).\]

The figures in the motivating discussion strongly suggest that these three compatibility definitions are equivalent, at least for irreducible $G$-trees. Further motivation is provided by the following lemma, which produces pairs of group elements with distinct axes, mirroring the pictures.

**Lemma 5.10.** Suppose $A$ and $B$ are irreducible $G$-trees that are have incompatible horizons. Let $a \subseteq A$ and $b \subseteq B$ be arcs witnessing this fact. Then there exist group elements $g \in \llbracket a \rrbracket \cap \llbracket b \rrbracket$ and $\alpha \in \llbracket a \rrbracket \cap \llbracket b \rrbracket$ such that $C^A_g \cap C^A_\alpha$ is bounded; and elements $h \in \llbracket a \rrbracket \cap \llbracket b \rrbracket$ and $\beta \in \llbracket a \rrbracket \cap \llbracket b \rrbracket$ such that $C^B_h \cap C^B_\beta$ is bounded.

**Proof.** The argument is symmetric so we give the construction of $g$ and $\alpha$. Since $A$ and $B$ are incompatible the relevant sets are non-empty. Take any $g \in \llbracket a \rrbracket \cap \llbracket b \rrbracket$ and $\alpha \in \llbracket a \rrbracket \cap \llbracket b \rrbracket$. If $C^A_g \cap C^A_\alpha$ is bounded we are done. Suppose $C^A_g \cap C^A_\alpha$ contains a ray. Let $s \in G$ be any $A$-hyperbolic element such that $C^A_s \cap C^A_\alpha$ is bounded. Such an element exists since $A$ is irreducible (see Proposition 3.17). If $s$ is elliptic in $B$ then $\alpha s$ is hyperbolic in both $A$ and $B$ and $C^A_{\alpha s} \cap C^A_\alpha$ is bounded, so we
may suppose that \( s \) is hyperbolic in both \( A \) and \( B \). Since \( g \in [a] \cap [b] \) there is some \( N > 0 \) such that \( g^N s g^{-N} \in [a] \cap [b] \). Take \( g' = g^N s g^{-N} \). By construction \( C^A_g \cap C^A_h \) is bounded, so \( g', \alpha \) is the desired pair.

\[ \square \]

**Corollary 5.11.** The group elements \( g \) and \( \alpha \) are \( A \)-independent, and the group elements \( h \) and \( \beta \) are \( B \)-independent.

For irreducible \( G \)-trees, the three definitions of compatibility are equivalent. The strategy suggested by the pictures is to use boundary points to pick suitable elements of \( G \). This philosophy guides the proof below.

**Lemma 5.12.** Suppose \( \ell \) and \( m \) are length functions on \( G \) corresponding to the irreducible \( G \)-trees \( A \) and \( B \) respectively. The following are equivalent.

1. The length functions \( \ell \) and \( m \) do not have compatible combinatorics.
2. The length functions \( \ell \) and \( m \) are not coherently oriented.
3. The trees \( A \) and \( B \) have incompatible horizons.

**Proof.** We will show \( 1 \iff 3 \) and \( 2 \iff 3 \).

(1 \( \Rightarrow \) 3.) Suppose, without loss of generality, \( (g, h) \in D^f \cap O^m \). In \( A \), by definition \( C^A_g \cap C^A_h = \emptyset \); let \( a \subseteq A \) be the geodesic joining \( C^A_g \) and \( C^A_h \), oriented so that \( t(a) \in C^A_g \). We have \( g^a \in [a] \) and \( h^a \in [a] \). In \( B \), again by definition there is an arc \( b = C^B_g \cap C^B_h \). Without loss of generality we assume \( g \) and \( h \) induce the same orientation on \( b \), and use this orientation. Then \( g, h \in [b] \) and \( g^{-1}, h^{-1} \in [b] \). We conclude the four sets

\[
[a] \cap [b] \quad [a] \cap [b] \quad [a] \cap [b] \quad [a] \cap [b],
\]

are all non-empty. (See Figure 5 for an illustration.)

(2 \( \Rightarrow \) 3.) Let \( g, h \in G \) witness the incoherent orientation of \( \ell \) and \( m \), so that \( \ell(gh^{-1}) < \ell(gh) \) but \( m(gh^{-1}) > m(gh) \). Let \( a = C^A_g \cap C^A_h \) and \( b = C^B_g \cap C^B_h \). Since \( (g, h) \in O^f \cap O^m \), both \( a \) and \( b \) are arcs. Orient \( a \) according to the orientation induced by \( g \) on \( C^A_g \), and similarly orient \( b \). The inequality implies that the orientation on \( a \) induced by \( h \) agrees with the orientation on \( a \); thus \( g, h \in [a] \) and \( g^{-1}, h^{-1} \in [a] \). Similarly, the inequality \( m(gh^{-1}) > m(gh) \) implies \( g, h \in [b] \) and \( g^{-1}, h \in [b] \). We conclude the four sets

\[
[a] \cap [b] \quad [a] \cap [b] \quad [a] \cap [b] \quad [a] \cap [b],
\]

are all non-empty. (See Figure 6 for an illustration.)

(3 \( \Rightarrow \) 1 and 2.) Let \( a \subseteq A \) and \( b \subseteq B \) be arcs witnessing the incompatibility of \( A \) and \( B \). Fix group elements \( g \in [a] \cap [b], h \in [a] \cap [b], \alpha \in [a] \cap [b], \) and

![Figure 5. Incompatible combinatorics implies incompatible trees.](image-url)
Suppose Theorem 6.1.

Let $N_B$ be the integer guaranteed by Lemma 4.4 applied to $g$ and $\alpha$ in $B$, and $N_A$ be the integer supplied by Lemma 4.3 applied to $g$ and $\alpha$ in $A$. (Note that the hypothesis of Lemma 4.4 on the ends of $g$ and $\alpha$ is satisfied.) Set $N = \max\{N_A, N_B\}$ and consider $\rho = \alpha^{-N}g^N$. Lemma 4.3 implies $b \subseteq C^B_\rho$, and Lemma 4.5 implies $C^{A}_\rho \subseteq R^+_g$. Choose $M$ by a similar process applied to $h$ and $\beta$, so that $\sigma = \beta^{-M}h^M$ satisfies $b \subseteq C^B_\sigma$ and $C^A_\sigma \subseteq R^-_g$. By construction $C^B_\rho \cap C^B_\sigma \supseteq b$, so $(\rho, \sigma) \in O^m$; and $C^A_\rho \cap C^A_\sigma = \emptyset$, so $(\rho, \sigma) \in D^f$. Hence $D^f \cap O^m \neq \emptyset$ and $\ell$ and $m$ do not have compatible combinatorics, as required.

Continuing the theme, let $J_a$ be the integer given by Lemma 4.3 applied to $g, h$ and $a \subseteq A$, $J_b$ be the integer given by the application to $g, h$ and $b \subseteq B$, and $J = \max\{J_a, J_b\}$. Similarly, let $K_a$ be the integer given by Lemma 4.3 applied to $\alpha, \beta$ and $a$, $K_b$ be the integer given by the application to $\alpha, \beta$ and $b \subseteq B$ (note the reversed orientation), and $K = \max\{K_a, K_b\}$. Consider $c = h^{-1}g^J$ and $\gamma = \beta^{-K}\alpha^K$. By Lemma 4.3 $a \subseteq C^A_\rho \cap C^A_\sigma$ and all three orientations agree; however $b \subseteq C^B_\rho \cap C^B_\sigma$, but the orientation of $C^B_\rho$ induced by $c$ agrees with $b$, while that of $C^B_\sigma$ induced by $\gamma$ agrees with $b$. Translating this to the length functions $\ell$ and $m$ we find $(c, \gamma) \in O^f \cap O^m$ and $\ell(c\gamma^{-1}) < \ell(\gamma)$ but $m(c\gamma^{-1}) > m(\gamma)$, hence $\ell$ and $m$ are not coherently oriented, as required.

In light of this lemma a single definition of compatible will be used throughout the remainder of this article.

**Definition 5.13.** Two irreducible $G$-trees $A$ and $B$ with length functions $\ell$ and $m$ are **synthetically compatible** if, equivalently
- The length functions $\ell$ and $m$ have compatible combinatorics.
- The length functions $\ell$ and $m$ are coherently oriented.
- The trees $A$ and $B$ have compatible horizons.

Note that this definition applies equally well to projective classes of trees. The first two points depend only on the projective class, so if $\ell$ and $m$ are synthetically compatible then so are $s\ell$ and $tm$ for all $s, t \in \mathbb{R}_{>0}$.

6. **Synthetic compatibility is equivalent to a common refinement**

**Theorem 6.1.** Suppose $\ell$ and $m$ are length functions on a group $G$. The sum $\ell + m$ is a length function on $G$ if and only if $\ell$ and $m$ are synthetically compatible.
Proof. First observe that $\ell + m$ always satisfies length function axioms I–III. We will focus on IV–VI.

For the forward implication, suppose $\ell + m$ is a length function on $G$. For a contradiction suppose that $\ell$ and $m$ do not have coherent orientation, and there is some pair $(g, h) \in O^\ell \cap O^m$ such that $\ell(gh^{-1}) < \ell(gh)$ and $m(gh) < m(gh^{-1})$. By Lemma 5.5, $g$ and $h$ are hyperbolic with respect to both $\ell$ and $m$, so both $g$ and $h$ must be hyperbolic in $\ell + m$. Length function axiom V implies that for $\ell$ and $m$ respectively,

$$
\ell(gh) = \ell(g) + \ell(h)
$$

and

$$
m(gh^{-1}) = m(g) + m(h).
$$

Taking a sum we have

$$
\ell(gh) + m(gh^{-1}) = \ell(g) + m(g) + \ell(h) + m(h).
$$

By hypothesis, both

$$
\ell(gh) + m(gh) < \ell(gh) + m(gh^{-1})
$$

and

$$
\ell(gh^{-1}) + m(gh^{-1}) < \ell(gh) + m(gh^{-1}).
$$

We conclude that

$$
\max\{(\ell + m)(gh), (\ell + m)(gh^{-1})\} < \ell(gh) + m(gh^{-1})
$$

$$
= (\ell + m)(g) + (\ell + m)(h).
$$

This is a contradiction, since $\ell + m$ satisfies length function axiom V, which implies the above strict inequality must be equality. We conclude that $\ell$ and $m$ are compatible.

For the converse, suppose $\ell$ and $m$ are compatible. As remarked previously, $\ell + m$ satisfies length function axioms I–III. We will show $\ell + m$ satisfies the remaining axioms.

Axiom IV. Suppose $g, h \in G$. We will proceed through the following cases:

- $(g, h) \in O^\ell$,
- $(g, h) \in O^m$,
- $(g, h) \in P(G) \setminus (O^\ell \cup O^m)$.

Case $(g, h) \in O^\ell$. Since $\ell$ satisfies axiom IV,

$$
\max\{\ell(gh), \ell(gh^{-1})\} \leq \ell(g) + \ell(h).
$$

Since $\ell$ and $m$ have compatible combinatorics, $(g, h) \in P(G) \setminus D^m$, which implies that

$$
\max\{m(gh), m(gh^{-1})\} \leq m(g) + m(h).
$$

Hence we may calculate

$$
\max\{\ell(gh) + m(gh), \ell(gh^{-1}) + m(gh^{-1})\} \leq \max\{\ell(gh), \ell(gh^{-1})\}
$$

$$
+ \max\{m(gh), m(gh^{-1})\}
$$

$$
\leq \ell(g) + \ell(h) + m(g) + m(h)
$$

and conclude that in this case $\ell + m$ satisfies axiom IV.

Case $(g, h) \in O^m$. The proof is symmetric with the previous case.

Case $(g, h) \in P(G) \setminus (O^\ell \cup O^m)$. In this case, by hypothesis both

$$
\ell(gh) = \ell(gh^{-1})
$$

and

$$
m(gh) = m(gh^{-1}).
$$
Adding, we conclude
\[ \ell(gh) + m(gh) = \ell(gh^{-1}) + m(gh^{-1}) \]
as required.

**Axiom V.** Suppose \( g, h \in G \) satisfy \( \ell(g) + m(g) > 0 \) and \( \ell(h) + m(h) > 0 \). This implies that \( g \) and \( h \) are both hyperbolic in at least one of \( \ell \) and \( m \). We proceed through the same cases.

- \((g, h) \in O^f\),
- \((g, h) \in O^m\),
- \((g, h) \in P(G) \setminus (O^f \cup O^m)\).

**Case** \((g, h) \in O^f\). In this case, since \( \ell \) and \( m \) are compatible, \((g, h) \notin D^m\) and we argue by subcases.

- \( m(g) > 0 \) and \( m(h) > 0 \),
- \( m(g) = 0 \) and \( m(h) \geq 0 \),
- \( m(g) \geq 0 \) and \( m(h) = 0 \).

**Subcase** \( m(g) > 0 \) and \( m(h) > 0 \). Since \( \ell \) and \( m \) are coherently oriented we have, without loss of generality,

\[ \ell(gh^{-1}) < \ell(gh) \]
and
\[ m(gh^{-1}) \leq m(gh) \]

Appealing to axiom V for \( \ell \) and \( m \) we have,

\[ \ell(gh) = \ell(g) + \ell(h) \]
and
\[ m(gh) = m(g) + m(h). \]

Summing, we conclude
\[ \ell(gh^{-1}) + m(gh^{-1}) \leq \ell(gh) + m(gh) \]
\[ = \ell(g) + m(g) + \ell(h) + m(h). \]

Therefore in this subcase \( \ell + m \) satisfies axiom V.

**Subcase** \( m(g) = 0 \) and \( m(h) \geq 0 \). By Lemma 5.3, \( m(gh^{-1}) = m(gh) = m(h) \), so axiom V for \( \ell + m \) follows immediately from axiom V for \( \ell \).

**Subcase** \( m(g) \geq 0 \) and \( m(h) = 0 \). This subcase is symmetric with the previous one.

**Case** \((g, h) \in O^m\). This case is symmetric with the previous case.

**Case** \((g, h) \in P(G) \setminus (O^f \cup O^m)\). In this case we have

\[ \ell(gh) = \ell(gh^{-1}) = \ell(g) + \ell(h) + \Delta_\ell \]
\[ m(gh) = m(gh^{-1}) = m(g) + m(h) + \Delta_m \]
for real numbers \( \Delta_\ell, \Delta_m \geq 0 \). Immediately we have that

\[ \ell(gh) + m(gh) = \ell(gh^{-1}) + m(gh^{-1}) \]

and from
\[ \ell(gh) + m(gh) = \ell(g) + m(g) + \ell(h) + m(h) + \Delta_\ell + \Delta_m \]
we conclude that axiom V is satisfied by \( \ell + m \).

**Axiom VI.** Finally we confirm that \( \ell + m \) has a good pair of elements. Let \((g, h)\) be a good pair of elements for \( \ell \), so that

\[ 0 < \ell(g) + \ell(h) - \ell(gh^{-1}) < 2 \min \{ \ell(g), \ell(h) \}. \]

We check the following cases

- \((g, h) \in O^m\),
• \((g, h) \not\in O^m\).

Case \((g, h) \in O^m\). In this case, since \(\ell\) and \(m\) are coherently oriented, 
\[
m(gh^{-1}) < m(gh).
\]
By Lemma 5.5, \(g\) and \(h\) are hyperbolic in \(m\). Therefore, by Lemma 5.23, there are positive integers \(a\) and \(b\) so that \((g^a, h^b)\) is a good pair for \(m\). Further by Lemma 5.23, the property of being a good pair is preserved under taking positive powers, so \((g^a, h^b)\) is a good pair for \(\ell\) also. Adding the good pair inequalities, we calculate 
\[
0 < \ell(g^a) + m(g^a) + \ell(h^b) + m(h^b) - \ell(g^ah^{-b}) - m(g^ah^{-b}) \\
< 2\min\{\ell(g^a), \ell(h^b)\} + \min\{m(g^a), m(h^b)\} \\
\leq 2\min\{\ell(g^a) + m(g^a), \ell(h^b) + m(h^b)\}.
\]
Hence \((g^a, h^b)\) is a good pair for \(\ell + m\).

Case \((g, h) \not\in O^m\). In this case, since \(\ell\) and \(m\) have compatible combinatorics, \((g, h) \not\in D^m\), and we have 
\[
m(gh^{-1}) = m(gh) = m(g) + m(h).
\]
Adding this to the \(\ell\) good pair inequality for \((g, h)\), we have 
\[
0 < \ell(g) + \ell(h) - \ell(gh^{-1}) = \ell(g) + m(g) + \ell(h) + m(h) - \ell(gh^{-1}) - m(gh^{-1})
\]
Since 
\[
2\min\{\ell(g), \ell(h)\} \leq 2\min\{\ell(g) + m(g), \ell(h) + m(h)\},
\]
we conclude \((g, h)\) is again a good pair for \(\ell + m\).

This concludes the case analysis. We have verified axioms IV–VI for \(\ell + m\), and conclude that \(\ell + m\) is a length function, as required. \(\square\)

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