On a question of Hof, Knill and Simon on palindromic substitutive systems

Tero Harju\textsuperscript{a}, Jetro Vesti\textsuperscript{a}, Luca Q. Zamboni\textsuperscript{a,b,1}

\textsuperscript{a}FUNDIM, University of Turku, Finland
\textsuperscript{b}Université de Lyon, Université Lyon 1, CNRS UMR 5208, Institut Camille Jordan, 43 boulevard du 11 novembre 1918, F69622 Villeurbanne Cedex, France

Abstract

In a 1995 paper, Hof, Knill and Simon obtain a sufficient combinatorial criterion on the hull $\Omega$ of the potential of a discrete Schrödinger operator which guarantees purely singular continuous spectrum on a generic subset of $\Omega$. In part, this condition requires the existence of infinitely many palindromic factors. In this same paper, they introduce the class $P$ of morphisms $f : A^* \to B^*$ of the form $a \mapsto pqa$ and ask whether every palindromic subshift generated by a primitive substitution arises from morphisms of class $P$ or by morphisms of the form $a \mapsto q_ap$ where again $p$ and $q_a$ are palindromes. In this paper we give a partial affirmative answer to the question of Hof, Knill and Simon: we show that every rich primitive substitutive subshift is generated by at most two morphisms each of which is conjugate to a morphism of class $P$. More precisely, we show that every rich (or almost rich in the sense of finite defect) primitive morphic word $y \in B^\omega$ is of the form $y = f(x)$ where $f : A^* \to B^*$ is conjugate to a morphism of class $P$, and where $x$ is a rich word fixed by a primitive substitution $g : A^* \to A^*$ conjugate to one in class $P$.

Keywords: Discrete one-dimensional Schrödinger operators, class P conjecture, primitive morphic words, rich words.

2000 MSC: 37B10

Email addresses: harju@utu.fi (Tero Harju), jejove@utu.fi (Jetro Vesti), lupastis@gmail.com (Luca Q. Zamboni)

\textsuperscript{1}Partially supported by a FiDiPro grant (137991) from the Academy of Finland and by ANR grant SUBTILE.
1. Introduction

Let $A$ be a finite non-empty set. Associated to each uniformly recurrent word $w \in A^\omega$, is the subshift $\Omega = \Omega(w)$ of all two-sided infinite words having the same factors as $w$. In many interesting cases (including when $w$ is a Sturmian word, or generated by a primitive substitution), $\Omega(w)$ is a 1-dimensional quasicrystal modeled by a family of aperiodic words which are locally indistinguishable. To each $x \in \Omega$, one associates a discrete one-dimensional Schrödinger operator $H_x$ which acts in the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z})$. If $\phi \in \mathcal{H}$, then $H_x\phi$ is given by

$$(H_x\phi)(n) = \phi(n+1) + \phi(n-1) + f(x_n)\phi(n),$$

where the potential $f : A \to \mathbb{R}$ is any injective mapping. In case $f$ is bounded, $H_x$ becomes a bounded self-adjoint operator. Given an initial state $\phi \in \mathcal{H}$, the Schrödinger time evolution is given by $\phi(t) = \exp(-itH_x)\phi$, where $\exp(-itH_x)$ is given by the spectral theorem. An important question in connection with the conductivity of the given structure is whether $\phi(t)$ spreads out in space, and if so, how fast. In this context, it is natural to consider the spectral measure $\mu_{\phi}$ associated with $\phi$ defined by

$$\langle \phi, (H_x - z)^{-1}\phi \rangle = \int_\mathbb{R} \frac{d\mu_{\phi}(x)}{x-z} \text{ for every } z \text{ with } \text{Im } z > 0.$$

Roughly speaking, the more continuous $\mu_{\phi}$, the faster the spreading of $\phi(t)$; compare, for example, [BGT, Gu, Las].

In physical terms, the spectral properties of $H_x$ determine the "conductivity properties" of the given structure. Roughly, if the spectrum is absolutely continuous, then the structure behaves like a conductor, while in the case of pure point spectrum, it behaves like an insulator. An intermediate spectral type, known as singular continuous spectrum, is expected to give rise to intermediate transport properties. For periodic structures, singular continuous spectra does not occur. However, for one-dimensional quasicrystals, this spectral type is experimentally seen to be rather common. In [HKS], Hof, Knill and Simon give a sufficient combinatorial criterion for purely singular continuous spectrum in terms of a strong palindromicity property of the underlying word. More precisely, a word $x \in A^\mathbb{Z}$ is said to be strongly palindromic if there exist $B > 0$ and a sequence $(u_i)_{i \geq 1}$ of palindromic factors of $x$ centered at $m_i \to +\infty$ such that $e^{Bm_i}/|u_i| \to 0$. They then show that if $x \in A^\mathbb{Z}$ is aperiodic and palindromic, meaning that $x$ contains infinitely many distinct palindromes, then its subshift contains uncountably many strongly palindromic words (see Proposition 2.1 in [HKS]). It follows from a result of Jitomirskaya and Simon in [JS] that if $x$ is strongly palindromic, then the spectrum of $H_x$ is empty. They then deduce that if $\Omega$ is uniquely ergodic and generated by an aperiodic palindromic word $w$, then the operator $H_x$ has purely
singular continuous spectrum for uncountably many $x \in \Omega$. In this same paper they introduce the class $\mathcal{P}$ of (non-erasing) morphisms $f : A^* \to B^*$ of the form $a \mapsto pq_a$ where $p, q_a$ are each palindromes. Morphisms in this class are said to be of class $P$. Actually in [HKS] they consider only primitive substitutions in $\mathcal{P}$. As was observed in [HKS], substitutions in $\mathcal{P}$ generate palindromic subshifts. They also point out that substitutions of the form $a \mapsto q_ap$ with $p$ and $q_a$ palindromes also generate palindromic subshifts. Thus they extend $\mathcal{P}$ to include also substitutions of the form $f(a) = q_ap$ and remark:

Remark 1 (Hof, Knill, Simon, Remark 3 in [HKS]). We do not know whether all palindromic subshifts generated by primitive substitutions arise from substitutions in this extended class.

Over the years this remark has evolved into what is now called the class $P$ conjecture. The first step in the evolution process, which is perhaps non consequential, was to convert this remark into a question. The second step, which in our minds represents a significant alteration, was to replace the entire subshift by a single element within the subshift which is fixed by a primitive substitution. The third was to give a precise interpretation to “arise from” as meaning “fixed by”. The fourth and final step was to call it a conjecture.

If we agree to focus on a single element of the subshift, then it is natural to widen the class of possible morphisms. In fact, if $x \in A^\omega$ is generated by a primitive substitution $f : A \to A^+$ and each of the images $f(a)$ for $a \in A$ begins or ends in a common letter, then one may conjugate each of the images by this common letter to obtain a new primitive substitution which generates the same subshift as $f$. Hence if $f$ is in class $P$, then this new substitution need no longer be in class $\mathcal{P}$ although it, or some power of it, will have a palindromic fixed point. For instance, Blondin Massé proved that the fixed point $x$ of the primitive substitution $a \mapsto abbab, b \mapsto abb$ is palindromic, but that $x$ itself is not fixed by a primitive substitution in $\mathcal{P}$ (see Proposition 3.5 in [Lab1]). However this morphism is conjugate to the class $P$ morphism $a \mapsto bbaba, b \mapsto bba$. Thus it is reasonable to consider the class $\mathcal{P}'$ of all morphisms $f$ which are conjugate to some morphism in $\mathcal{P}$. Let $\mathcal{FP}'$ denote the set of all infinite words which are fixed by some primitive substitution in class $\mathcal{P}'$. Then the original remark of Hof, Knill and Simon was reformulated in terms of the following conjecture, called the class $P$ conjecture:

Conjecture 1 (Blondin Massé, Labbé in [Lab1]). If $x$ is a palindromic word fixed by a primitive substitution, then $x \in \mathcal{FP}'$.

Partial results in support of the conjecture were obtained by Allouche et al. in case $x$ is periodic (see [ABCD]) and by Tan in case $x$ is a binary word (see [Tan]). Tan proves that if $x$ is a palindromic binary word fixed by a primitive substitution $f$, then $f^2 \in \mathcal{P}'$. 

3
Recently Labbé [Lab2] produced a counter-example to the class P conjecture on a ternary alphabet. The counter-example is given by the fixed point

\[ x = acabacabacabacabacabacabacabacabacabacabacabacabacabac \cdots \]

of the primitive substitution:

\[ f : \quad a \mapsto ac, \quad b \mapsto acab, \quad c \mapsto ab. \]

He proves that \( x \) is palindromic but not in \( \mathcal{FP}' \) (see [Lab2]). But let us remark that Labbé’s counter-example to the class P conjecture does not constitute a negative answer to the original question (or remark) of Hof, Knill and Simon. In fact, it is readily verified that the second shift

\[ T^2(x) = abacacabacabacabacabacabacabacabacab \cdots \]

is the fixed point of the class P morphism:

\[ g : \quad a \mapsto ab, \quad b \mapsto acac, \quad c \mapsto ac. \]

So the subshift generated by \( x \) is in fact generated by a morphism in class P. The morphism \( g \) is closely related to the Toeplitz period-doubling word.

What is surprising is that Labbé’s counter-example to the class P conjecture is not only palindromic, but is as rich as possible in palindromes. More precisely, Droubay, Justin and Pirillo observed that any finite word \( u \) has at most \(|u| + 1\) distinct palindromic factors (including the empty word). Accordingly, an infinite word \( x \) is called rich if each factor \( u \) of \( x \) has \(|u| + 1\) many distinct palindromic factors. It turns out that Labbé’s counter-example to the class P conjecture is a rich word. To see this, we observe that \( x \) is obtained from the fixed point \( y \) of the morphism

\[ \tau : \quad b \mapsto ccb, \quad c \mapsto cb \]

by inserting the symbol \( a \) before every occurrence of each of the symbols \( b \) and \( c \). It is readily verified that \( \tau = \tau_3 \tau_2 \tau_1 \) where \( \tau_1 : b \mapsto c, c \mapsto b, \tau_2 : b \mapsto b, c \mapsto cb \) and \( \tau_3 : b \mapsto cb, c \mapsto c \), and hence by Corollary 6.3 and Proposition 6.6 in [GJWZ] it follows that \( y \) is rich. Given that \( y \) is rich, it now follows from Corollary 6.3 in [GJWZ] that \( x \) is rich.

We regard the class P conjecture as an attempt to explain how a fixed point of a primitive substitution can contain infinitely many palindromes. Of course, a typical substitution does not preserve palindromes, hence one would expect that a palindrome generating substitution would have some particular inherent structure. In this paper, we give such an explanation in case \( x \) is rich, or close to being rich. More precisely, the defect of a finite word \( u \), defined by \( D(u) = |u| + 1 - |\text{Pal}(u)| \), is a measure of the extent to which \( u \) fails to be rich. The defect of a
infinite word $x$ is defined by $D(x) = \sup\{D(u) | u \text{ is a prefix of } x\}$. This quantity can be finite or infinite, and an infinite word is rich if and only if its defect is equal to 0. Any infinite word of finite defect is necessarily palindromic, but not conversely as is evidenced, for example, by the Thue-Morse word. We show that if $y$ has finite defect and is generated by a primitive substitution, then there exists a morphism $f \in \mathcal{P}'$ and a rich word $x \in \mathcal{FP}'$ such that $y = f(x)$. Actually, our result is more general as it applies as well to all primitive morphic words (i.e., morphic images of fixed points of primitive substitutions). More precisely:

**Theorem 2.** Let $y$ be a primitive morphic word with finite defect. Then there exists a morphism $f \in \mathcal{P}'$ and a rich word $x \in \mathcal{FP}'$ such that $y = f(x)$.

In this respect, every primitive morphic word $y$ with finite defect is generated by not one, but two morphisms in $\mathcal{P}'$. The first which generates the fixed point $x$ in Theorem 2 and the second which maps $x$ to $y$. Labbé’s counter-example shows that even in the case of pure primitive morphic rich words, one cannot hope to have $y$ itself in $\mathcal{FP}'$. A key ingredient in our proof is Durand’s characterization of primitive morphic words in terms of the finiteness of the set of derived words build from first returns to prefixes (see [Du]). A second involves a result of Balková et al. in [BPS1] linking words having finite defect with rich words.

### 2. Preliminaries

Given a finite non-empty set $A$, we denote by $A^*$ the set of all finite words $u = u_1u_2 \cdots u_n$ with $u_i \in A$. The quantity $n$ is called the length of $u$ and is denoted $|u|$. The empty word, denoted $\varepsilon$, is the unique element in $A^*$ with $|\varepsilon| = 0$. We set $A^+ = A^* - \{\varepsilon\}$. For each word $v \in A^+$, let $|v|$ denote the number of occurrences of $v$ in $u$. We denote by $A^\omega$ the set of all one-sided infinite words $x = x_0x_1x_2 \cdots$ with $x_i \in A$. Given $x \in A^\omega$, let $\text{Fact}^+(x) = \{x_ix_{i+1} \cdots x_{i+j} | i, j \geq 0\}$ denote the set of all (non-empty) factors of $x$. Recall that $x$ is called **recurrent** if each factor $u$ of $x$ occurs an infinite number of times in $x$, and **uniformly recurrent** if for each factor $u$ of $x$ there exists a positive integer $n$ such that $u$ occurs at least once in every factor $v$ of $x$ with $|v| \geq n$. An infinite word $x$ is called **periodic** if $x = u^\omega$ for some $u \in A^+$, and is called **ultimately periodic** if $x = vu^\omega$ for some $v \in A^*$, and $u \in A^+$. The word $x$ is called **aperiodic** if $x$ is not ultimately periodic. Let $x \in A^\omega$ and $u \in \text{Fact}^+(x)$. A factor $v$ of $x$ is called a **first return to $u$ in $x$** if $vu \in \text{Fact}^+(x)$, $vu$ begins and ends in $u$ and $|vu|_u = 2$. If $v$ is a first return to $u$ in $x$, then $vu$ is called a **complete first return to $u$ in $x$**. We note that the two occurrences of $u$ in $vu$ may overlap. We denote by $R_u(x)$ the set of all first returns to $u$ in $x$.

A function $\tau : A \rightarrow A^+$ is called a **substitution**. A substitution $\tau$ extends by concatenation to a morphism from $A^*$ to $A^*$ and to a mapping from $A^\omega$ to $A^\omega$, i.e., $\tau(a_1a_2 \cdots) =$
\(\tau(a_1)\tau(a_2)\cdots\). By abuse of notation we denote each of these extensions also by \(\tau\). A substitution \(\tau : A \to A^+\) is \textit{primitive} if there exists a positive integer \(N\) such that \(|\tau^N(a)|_b > 0\) for all \(a, b \in A\). A word \(x \in A^\omega\) is a called a \textit{fixed point} of a substitution \(\tau\) if \(\tau(x) = x\). We say \(x \in A^\omega\) is \textit{pure primitive morphic} if \(x\) is a fixed point of some primitive substitution \(\tau : A \to A^+\). A word \(y \in B^\omega\) (where \(B\) is a finite non-empty set) is called \textit{primitive morphic} if there exists a morphism \(f : A^* \to B^*\) and a pure primitive morphic word \(x \in A^\omega\) with \(y = f(x)\). It is readily verified that every primitive morphic word is uniformly recurrent.

In [Du], Durand obtains a nice characterization of primitive morphic words in terms of so-called derived words. Let \(x \in A^\omega\) be uniformly recurrent. Then \(#\mathcal{R}_u(x) < +\infty\) for each \(u \in \text{Fact}^+(x)\). Let \(u \in \text{Pref}(x)\) be a non-empty prefix of \(x\). Then \(x\) induces a linear order on \(\mathcal{R}_u(x)\) as follows: given distinct \(v, v' \in \mathcal{R}_u(x)\) we declare \(v < v'\) if the first occurrence of \(v\) in \(x\) occurs prior to that of \(v'\). Let \(A_u(x) = \{0, 1, \ldots, #\mathcal{R}_u(x) - 1\}\), and let \(f_u : A_u(x) \to \mathcal{R}_u(x)\) denote the unique order preserving bijection. We can write \(x\) uniquely as a concatenation of first returns to \(u\), i.e., \(x = u_1u_2u_3\cdots\) with \(u_i \in \mathcal{R}_u(x)\). Following [Du] we define the \textit{derived word} of \(x\) at \(u\), denoted \(\mathcal{D}_u(x)\), as the infinite word with values in \(A_u(x)\) given by

\[
\mathcal{D}_u(x) = f_u^{-1}(u_1)f_u^{-1}(u_2)f_u^{-1}(u_3)\cdots.
\]

For example, let

\[x = 01101001100101101001011001101001\cdots\]

denote the Thue-Morse word fixed by the substitution \(0 \mapsto 01, 1 \mapsto 10\). It is readily verified that \(\mathcal{R}_0(x) = \{011, 01, 0\}\). So \(A_0(x) = \{0, 1, 2\}\) and \(f_0 : A_0(x) \to \mathcal{R}_0(x)\) is given by \(f_0(0) = 011, f_0(1) = 01, f_0(2) = 0\). Writing \(x\) as a concatenation of first returns to 0 we find

\[x = (011)(01)(0)(011)(0)(01)(01)(0)(011)(0)(011)(0)(011)(0)(011)(0\cdots\]

and hence

\[\mathcal{D}_0(x) = 012021012102012\cdots\]

It is readily verified that \(\mathcal{D}_0(x)\) is the well known Hall word.

The following result of Durand gives a characterization of primitive morphic words:

**Theorem 2.1** (F. Durand, Theorem 2.5 in [Du]). A word \(x \in A^\omega\) is primitive morphic if and only if the set \(\{\mathcal{D}_u(x) | u \in \text{Pref}(x)\}\) is finite.

Given a finite or infinite word \(u \in A^*\) we denote by \(\text{Pal}(u)\) the set of all palindromic factors of \(u\) (including the empty word). Droubay, Justin and Pirillo proved that any word \(u \in A^*\) has at most \(|u| + 1\) many distinct palindromic factors including the empty word (see [DJP]). A finite word \(u\) is called \textit{rich} if \(#\text{Pal}(u) = |u| + 1\). For instance \(aababbab\) is rich while
aababb {a} is not. An infinite word is called rich if all of its factors are rich. Rich words were first introduced by Glen et al. in [GJWZ] and have been since studied in various papers.

The defect of a finite word $u$ is defined by $D(u) = |u| + 1 - |\text{Pal}(u)|$. The defect of an infinite word $x$ is defined by $D(x) = \sup\{D(u)|u$ is a prefix of $x\}$. This quantity can be finite or infinite. Thus an infinite word is rich if and only if its defect is equal to 0.

We will make use of the following result from [GJWZ] characterizing rich words according to complete first returns.

**Theorem 2.2 (GJWZ), Theorem 2.14 and Remark 2.15.** An infinite word $x \in A^\omega$ is rich if and only if all complete first returns to any palindromic factor in $x$ are themselves palindromes.

A morphism $f : A^* \rightarrow B^*$ is said to be of class P if there exists a palindrome $p \in B^*$ and palindromes $\{q_a\}_{a \in A} \subset B^*$ such that $f(a) = pg_a$ for each $a \in A$. Let $P$ denote the set of all morphisms of class P. One problem with class P morphisms is that they are not closed under composition. For instance, the morphisms $f : 0 \mapsto 0, 1 \mapsto 01$ and $g : 0 \mapsto 01, 1 \mapsto 011$ are both in class P while the composition $gf : 0 \mapsto 01, 1 \mapsto 01011$ is not.

Two morphisms $f, g : A^* \rightarrow B^*$ are said to be conjugate if there exists $u \in B^*$ such that either $f(a)u = ug(a)$ for all $a \in A$, or $uf(a) = g(a)u$ for all $a \in A$. For example, the morphisms $f : 0 \mapsto 001, 1 \mapsto 0010010010010$ is conjugate to the morphism $g : 0 \mapsto 010, 1 \mapsto 01001001001010$. In fact, taking $u = 001001001010$, it is readily verified that $f(a)u = ug(a)$ for $a \in \{0, 1\}$. Let $P'$ denote the set of all morphisms $f : A^* \rightarrow B^*$ conjugate to some morphism in $P$. We shall now show that the class $P'$ is closed under composition. For this we need four lemmas the first of which is a basic combinatorial result of words:

**Lemma 2.3** (Proposition 1.3.4 in Lothaire [Lot]). Let $w_1u = uw_2$ for words $w_1, w_2, u \in A^*$. Then there are words $x$ and $y$ such that $w_1 = xy$, $w_2 = yx$ and $u = (xy)^kx$ for some non-negative $k \geq 0$.

**Lemma 2.4.** Conjugation of morphisms is an equivalence relation.

**Proof.** Clearly conjugation of morphisms is both reflexive and symmetric by definition. For transitivity, suppose $f, g, h : A^* \rightarrow B^*$ are such that $f$ is conjugate to $g$, and $g$ is conjugate to $h$. We divide the proof to cases according to the two types in the definition of conjugacy.

**Case 1:** Suppose there exist $u$ and $v$ in $B^*$ such that $f(a)u = ug(a)$ and $g(a)v = vh(a)$ for all $a \in A$. Then for all $a \in A$, we have $f(a)uv = ug(a)v = uvh(a)$ and hence $f$ is conjugate to $h$.

**Case 2:** Suppose there exist $u$ and $v$ such that $f(a)u = ug(a)$ and $vg(a) = h(a)v$ for all $a \in A$. 


We show first that $u$ is a suffix of $v$, or $v$ is a suffix of $u$. By Lemma 2.3, $f(a)u = ug(a)$ implies that $f(a) = xy$, $g(a) = yx$ and $u = (xy)^k x$ for some $x, y$ and $k \geq 0$. Similarly, $v g(a) = h(a)v$ implies that $g(a) = rs$, $h(a) = sr$ and $v = (sr)^t s$ for some $r, s$ and $t \geq 0$. Now, $g(a) = yx = rs$, where by symmetry we can assume that $y = rw$ and hence $s = wx$ for $w \in A^*$. Then $u = (xrw)^k x$ and $v = (wrx)^t wx = w(xrw)^t x$ and, indeed one is a suffix of the other.

Without restriction we can assume that $u$ is a suffix of $v$, say $v = wu$. Then we have
\[ wf(a)u = wug(a) = vg(a) = h(a)v = h(a)wu, \]
and hence $wf(a) = h(a)w$ for all $a \in A$. Hence $f$ is conjugate to $h$.

The other two cases (Case 3: $uf(a) = g(au)$ and $vg(a) = h(a)v$ for all $a \in A$; Case 4: $uf(a) = g(a)u$ and $g(a)v = vh(a)$ for all $a \in A$) are obtained from Cases 1 and 2 by interchanging $f$ and $h$ and using the symmetry condition of conjugation. \qed

The following lemma shows that the conjugation of morphisms is compatible with composition.

**Lemma 2.5.** Let $f, f': A^* \to B^*$ and $g, g': B^* \to C^*$ be morphisms such that $f$ is conjugate to $f'$ and $g$ is conjugate to $g'$. Then the compositions $gf$ and $g'f'$ are conjugate.

**Proof.** Again we have cases to consider. Notice first that if $h, h': A^* \to B^*$ are any functions that satisfy $h(a)x = xh'(a)$ for all $a \in A$ and some $x \in B^*$, then $h(w)x = xh'(w)$ for all $w \in A^*$.

**Case 1:** Suppose there exist $u \in B^*$ and $v \in C^*$ such that $f(a)u = uf'(a)$ and $g(a)v = vg'(a)$ for all $a \in A$. Then for all $a \in A$, we have
\[ gf(a) \cdot vg'(u) = vg'(f(a))g'(u) = vg'(f(a)u) = vg'(uf'(a)) = vg'(u) \cdot g'f'(a), \]
and hence $gf$ is conjugate to $g'f'$.

**Case 2:** Suppose there exist $u$ and $v$ such that $f(a)u = uf'(a)$ and $vg(a) = g'(a)v$ for all $a \in A$. By Case 1, we have that $gf'$ is conjugate to $g'f$. Also, $gf$ is conjugate to $gf'$, since $gf(a) \cdot g(u) = g(f(a)u) = g(uf'(a)) = g(u) \cdot g'f'(a)$ and similarly $g'f$ is conjugate to $g'f'$. Hence, by transitivity, $gf$ is conjugate to $g'f'$.

Again, the other two cases follow from the above cases. \qed

**Lemma 2.6.** Let $f: A^* \to B^*$ and $g: B^* \to C^*$ be class $\mathcal{P}$ morphisms. Then the composition $gf$ is in $\mathcal{P}'$. 

8
Proof. Let $f(a) = pu_a$ for all $a \in A$ where $p$ and each $u_a$ is a palindrome, and $g(b) = qv_b$ for all $b \in B$ where $q$ and each $v_a$ is a palindrome. If $p$ is empty, then $u_a$ is nonempty, and $gf(a) = g(u_a) = q \cdot q^{-1}g(u_a)$ where $q$ and $q^{-1}g(u_a)$ are palindromes for all $a$. If $p$ is nonempty, then $gf(a) = g(pu_a) = g(p)g(u_a)$ and $gf$ is conjugate to $h$ defined by $h(a) = q^{-1}g(p) \cdot g(u_a)q$, if $u_a$ is non-empty, and $h(a) = q^{-1}g(p)$ if $u_a$ is empty. The words $q^{-1}g(p)$ and $g(u_a)q$ are palindromes, and hence $h \in \mathcal{P}$, and $gf \in \mathcal{P}'$ as required.

Proposition 2.7. Class $\mathcal{P}'$ is closed under composition.

Proof. Suppose $f : A^* \to B^*$ and $g : B^* \to C^*$ are in $\mathcal{P}'$. Then there exist class $\mathcal{P}$ morphisms $f' : A^* \to B^*$ and $g' : B^* \to C^*$ with $f$ conjugate to $f'$ and $g$ conjugate to $g'$. By Lemma 2.5, the composition $gf$ is conjugate to $g'f'$ and, by Lemma 2.6, the composition $g'f'$ is conjugate to a class $\mathcal{P}$ morphism $h$. By transitivity in Lemma 2.4, $gf$ is conjugate to $h$, and hence by definition of $\mathcal{P}'$, the composition $gf$ is in $\mathcal{P}'$.

In [BPS1], Balková et al. introduce a related class of morphisms, denoted $\mathcal{P}_{ret}$, defined as follows: A morphism $f : A^* \to B^*$ is in $\mathcal{P}_{ret}$ if there exists a palindrome $p$ such that for each $a \in A$ we have that $f(a)p$ is a palindrome, $f(a)p$ begins and ends in $p$, $|f(a)p|_p = 2$, and $f(a) \neq f(b)$ whenever $a, b \in A$ with $a \neq b$. We call the palindrome $p$ the marker. For instance, the morphism $f : 0 \mapsto 0, 1 \mapsto 01$ is in $\mathcal{P} \cap \mathcal{P}_{ret}$. Here the marker is $p = 0$. In contrast, the morphism $f : 0 \mapsto 00, 1 \mapsto 01$ is in $\mathcal{P}$ but not in $\mathcal{P}_{ret}$. While the morphism $f : 0 \mapsto 001, 1 \mapsto 0010$ is in $\mathcal{P}_{ret}$ (with marker $p = 00100$) but not in $\mathcal{P}$. They show that:

Proposition 2.8 (Balková et al., Proposition 5.4 in [BPS1]). $\mathcal{P}_{ret} \subset \mathcal{P}'$.

We note that while the definition of “conjugacy” of two morphisms given in [BPS1] is not the same as ours, their proof of Proposition 5.4 is consistent with our definition while inconsistent with theirs. As it turns out, they intended for their definition to read the same as ours [BPS2].

3. Primitive morphic words of finite defect

Let $\mathcal{FP}'$ denote the set of all infinite words $x$ which are fixed by some primitive substitution $f \in \mathcal{P}'$. We recall that the class $\mathcal{P}$ conjecture states that if $y$ is a palindromic pure primitive morphic word, then $y \in \mathcal{FP}'$. Labbé’s counter-example in [Lab2] shows that the conjecture as stated is false even if $y$ is rich. Instead, we show:

Theorem 3.1. Let $y \in A^*$ be a rich primitive morphic word. Then there exists a morphism $g \in \mathcal{P}'$ and a rich word $x \in \mathcal{FP}'$ such that $y = gx$.
Proof. We can suppose without loss of generality that $A$ contains the symbol 0, and that $y$ begins in 0. Let $\mathcal{R}_0(y)$ denote the set of all first returns to 0 in $y$, $A_0(y) = \{0, 1, \ldots, \#\mathcal{R}_0(y) - 1\}$, and $f_0 : A_0(y) \rightarrow R_0(y)$ be the unique order preserving bijection. Let $f : A_0(y)^* \rightarrow A^*$ be the morphism defined by $f(a) = f_0(a) \in \mathcal{R}_0(y) \subset A^+$ for each $a \in A_0(y)$. Writing $y = u_1u_2u_3 \cdots$ with each $u_i \in \mathcal{R}_0(y)$, let $D_0(y) = f_0^{-1}(u_1)f_0^{-1}(u_2)f_0^{-1}(u_3) \cdots$ denote the derived word of $y$ at the prefix 0. Thus $f(D_0(y)) = y$.

The next three lemmas are stated in terms of the prefix 0 of $y$ since they are needed only in this special case. But in fact they hold for all palindromic prefixes $u$ of $y$.

**Lemma 3.2.** The morphism $f : A_0(y)^* \rightarrow A^*$ is in $\mathcal{P}$ and thus in $\mathcal{P}'$.

*Proof.* Since $y$ is rich, by Theorem [2.2] it follows that for each $v \in \mathcal{R}_0(y)$ there is a palindrome $v' \in A^*$ (possibly empty) such that $v = 0v'$. Thus, for each $a \in A$, there exists a palindrome $u_a$ such that $f(a) = f_0(a) = 0u_a$. Hence $f \in \mathcal{P} \subset \mathcal{P}'$. \hfill \square

**Lemma 3.3.** $D_0(y) \in A_0(y)^\omega$ is rich and begins in 0.

*Proof.* Since $f_0$ is order preserving, $f_0^{-1}(u_1) = 0$, whence $D_0(y)$ begins in 0. Let $z$ be a complete first return in $D_0(y)$ to a palindrome $u \in \text{Fact}^+(D_0(y))$. By Lemma 3.2 we deduce that $f(u)0$ is a palindromic factor of $y$ and $f(z)0$ is a complete first return in $y$ to $f(u)0$. Since $y$ is rich it follows from Theorem 2.2 that $f(z)0$ is a palindrome. By Lemma 3.2 and item 3. in Remark 5.2 in [BPS1] we deduce that $z$ is a palindrome, and hence $D_0(y)$ is rich by Theorem 2.2. \hfill \square

**Lemma 3.4.** $D_0(y)$ is primitive morphic.

*Proof.* In item 5. of Proposition 2.6 of [Du], it is shown that every derived word of $D_0(y)$ is also a derived word of $y$. Since $y$ is primitive morphic, it follows from Theorem 2.1 that $y$ has only finitely many distinct derived words, and hence $D_0(y)$ has finitely many distinct derived words, and hence by Theorem 2.1 $D_0(y)$ is primitive morphic. \hfill \square

Combining the three previous lemmas we deduce that if $y \in A^\omega$ is a rich primitive morphic word beginning in 0, then $D_0(y) \in A_0(y)^\omega$ is a rich primitive morphic word beginning in 0, and if $y = u_1u_2u_3 \cdots$ with $u_i \in \mathcal{R}_0(y)$, then $D_0(y) = f^{-1}(u_1)f^{-1}(u_2)f^{-1}(u_3) \cdots$ where $f : A_0(y)^* \rightarrow A^*$ belongs to $\mathcal{P}'$.

Thus, we can inductively define a sequence of infinite words $(S_n(y))_{n \geq 0}$ with values in finite sets $(A_n)_{n \geq 0}$ by $S_0(y) = y$, and $A_0 = A$, and for $n \geq 0 : S_{n+1}(y) = D_0(S_n(y))$ and $A_n = A_0(S_n(y))$. Moreover, for each $n \geq 1$ there exists a morphism $g_n : A_n^* \rightarrow A_{n-1}^*$ in $\mathcal{P}'$ such that writing $S_{n-1}(y) = u_1u_2u_3 \cdots$ with $u_i \in \mathcal{R}_0(S_{n-1}(y))$, we have $S_n(y) =
In other words, the sequence \((S_n(y))_{n \geq 0}\) is just the sequence of iterated derived words of \(y\) corresponding each time to the prefix 0.

By Theorem 2.1 there exist \(0 \leq m < n\) such that \(S_m(y) = S_n(y)\). Let \(x = S_m(y)\). Let \(h = g_{m+1}g_m \cdots g_n\). Then \(h(x) = x\) and by the proof of Proposition 3.3 in [Du] we deduce that \(h\) is a primitive substitution. By Proposition 2.7 the morphism \(h\) is in \(\mathcal{P}'\). Thus \(x \in \mathcal{FP}'\).

Finally let \(g = g_mg_{m-1} \cdots g_1\). Then \(y = g(x)\) and by Proposition 2.7 we deduce that \(g \in \mathcal{P}'\) as required. This completes the proof of Theorem 3.1.

\[\square\]

**Corollary 3.5.** Let \(z \in A^\omega\) be a primitive morphic word having finite defect. Then there exists a morphism \(g \in \mathcal{P}'\) and a rich word \(x \in \mathcal{FP}'\) such that \(z = g(x)\).

**Proof.** In Theorem 5.5 in [BPS1], the authors show that if \(z \in A^\omega\) is a uniformly recurrent word of finite defect, then there exists a rich word \(y \in B^\omega\) and a morphism \(f : B^* \rightarrow A^*\) in \(\mathcal{P}_{ret}\) such that \(z = f(y)\). In the proof of the theorem, it is revealed that \(y\) is actually a derived word of \(z\). Thus, if \(z\) is primitive morphic, then by Theorem 2.1 so is \(y\), and hence by Theorem 3.1 there exists a rich word \(x \in \mathcal{FP}'\) and a morphism \(h \in \mathcal{P}'\) such that \(y = h(x)\).

Let \(g = fh\). Then \(z = g(x)\) and by Proposition 2.7 and Proposition 2.8 we deduce that \(g \in \mathcal{P}'\).

We end with an illustration applied to Labbé’s example. Let

\[y = acabacacabacacabacacabacacabacacabacacabacacabac \cdots\]

be the fixed point of the morphism \(a \mapsto ac, b \mapsto acab, c \mapsto ab\). Then \(\mathcal{R}_a(y) = \{ac, ab\}\) and the derived word \(\mathcal{D}_a(y) \in \{0, 1\}^\omega\) is the fixed point of the morphism \(0 \mapsto 01, 1 \mapsto 001\) which is clearly in \(\mathcal{P}'\). Thus, setting \(x = \mathcal{D}_a(y)\), we have that \(x \in \mathcal{FP}'\) and \(y = f(x)\) where \(f : 0 \mapsto ac, 1 \mapsto ab\) is in \(\mathcal{P}'\).

**References**

**References**

[ABCD] J.-P. Allouche, M. Baake, J. Cassaigne, D. Damanik, Palindrome complexity. Theoret. Comput. Sci., 292(1) (2003), p. 9–31.

[BPS1] L. Balková, E. Pelantová, Š. Starosta, Infinite words with finite defect, Adv. in Appl. Math., 47 (3), 2011, p. 562–574.

[BPS2] L. Balková, E. Pelantová, Š. Starosta, Private communication, 2013.
[BGT] J. M. Barbaroux, F. Germinet, and S. Tcheremchantsev, Fractal dimensions and the phenomenon of intermittency in quantum dynamics, Duke Math. J., 110 (2001), p. 161–193

[DJP] X. Droubay, J. Justin, G. Pirillo, Episturmian words and some constructions of de Luca and Rauzy, Theoret. Comput. Sci. 255 (1-2), 2001, p. 539–553.

[Du] F. Durand, A characterization of substitutive sequences using return words, Discrete Math. 179, 1998, p. 89–101.

[GJWZ] A. Glen, J. Justin, S. Widmer, L.Q. Zamboni, Palindromic richness, European J. of Combin. 30, 2009, p. 510–531.

[Gu] I. Guarneri, Spectral properties of quantum diffusion on discrete lattices, Europhys. Lett., 10 (1989), p. 95–100

[HKS] A. Hof, O. Knill and B. Simon, Singular continuous spectrum for palindromic Schrödinger operators, Commun. Math. Phys. 174, 1995, p. 149-159.

[JS] S. Jitomirskaya, B. Simon, Operators with singular continuous spectrum: III. Almost periodic Schrödinger operators, Commun. Math. Phys. 165 (1994), p. 201–205.

[Lab2] S. Labbé, A counterexample to a question of Hof, Knill and Simon, arXiv:1307.1589v1 [math.CO], Jul 05 2013.

[Lab1] S. Labbé, Propriétés combinatoires des f-palindromes, Master’s thesis, Université du Québec Montréal, Montréal, 2008, M10615.

[Las] Y. Last, Quantum dynamics and decompositions of singular continuous spectra, J. Funct. Anal., 142 (1996), p. 406–445

[Lot] M. Lothaire, Combinatorics on Words, Encyclopedia of Mathematics Vol. 17, Addison-Wesley, 1983.

[Tan] B. Tan, Mirror substitutions and palindromic sequences, Theoret. Comput. Sci. 389 (1-2), 2007, p. 118–124.