The Hitchin-Witten Connection and Complex Quantum Chern-Simons Theory

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Abstract

We give a direct calculation of the curvature of the Hitchin connection, in geometric quantization on a symplectic manifold, using only differential geometric techniques. In particular, we establish that the curvature acts as a first-order operator on the quantum spaces. Projective flatness follows if the Kähler structures do not admit holomorphic vector fields. Following Witten, we define a complex variant of the Hitchin connection on the bundle of prequantum spaces. The curvature is essentially unchanged, so projective flatness holds in the same cases. Finally, the results are applied to quantum Chern-Simons theory, both for compact and complex gauge groups.

1 Introduction

Since their introduction by Atiyah [Ati], Segal [Seg] and Witten [Wit1, Wit2], topological quantum field theories (TQFTs) have been studied intensely using a wide range of techniques. The first construction in 2 + 1 dimensions was given by Reshetikhin and Turaev [RT2, RT1, Tur] using representation theory of quantum groups at roots of unity to construct link invariants and in turn derive invariants of 3-manifolds through surgery and Kirby calculus. Shortly thereafter, a combinatorial construction was given by Blanchet, Habegger, Masbaum and Vogel [BHMV1, BHMV2] in the language of skein theory.

A geometric realization was proposed by Witten [Wit1], suggesting the use of quantum Chern-Simons theory or conformal field theory to construct the 2-dimensional part. The gauge theoretic approach was studied independently by Axelrod, Della Pietra and Witten [ADPW] and Hitchin [Hit3], proving that the quantum spaces arising from geometric quantization of Chern-Simons theory for compact gauge group are indeed independent of the conformal structure on the surface, in the sense that they are identified by parallel transport of a projectively flat connection over the Teichmüller space of the surface. These constructions have been expressed and generalized in purely differential geometric terms in [And5] and [AGL], and we shall be mainly concerned with this description in

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the present paper. The other construction proposed by Witten, through conformal field theory, was provided by Tsuchiya, Ueno and Yamada [TUY], and the link to the gauge theoretic construction was established by Laszlo [Las].

Only recently has the relation to Reshetikhin and Turaev been fully demonstrated. In a series of papers [AU1, AU2, AU3, AU4], the first author of this paper and Ueno obtain a modular functor, from a twist of the conformal field theory construction, and identify it with the modular functor constructed from skein theory, and hence with the original construction of Reshetikhin and Turaev. This has paved the way for studying the TQFT through geometric quantization of moduli space and in particular the application of Toeplitz operator theory, see e.g. [And6, And1, And4, AB1, And5, And2, And3, AG, AH, AHJ].

For non-compact gauge group, the situation is very different. In the paper [Wit3], Witten initiated the study of Chern-Simons topological quantum field theory for complex gauge groups from a physical point of view. He proposed that the smooth sections of the Chern-Simons line bundle over the moduli space of flat connections in the corresponding compact real form of the group should be the appropriate pre-Hilbert space of this theory. He reduced the description to this model space by considering a real polarization, which we review in section 5, on the space of connections with values in the complex gauge group. Although this model space itself does not depend on a choice of complex structure on the surface, the polarization, and hence the interpretation as the quantum space, does. Witten argued that the needed infinitesimal change of polarization, under infinitesimal change of the complex structure on the surface, can be encoded as a connection in the trivial bundle with the fixed model space as fiber. Furthermore, Witten provided infinite-dimensional gauge theory arguments for the projective flatness of this connection.

In this paper, we review the general differential geometric construction of the Hitchin connection for a rigid family of complex structures on a symplectic manifold with vanishing first Betti number and first Chern class represented essentially by the symplectic form. Inspired by Witten’s considerations, we also consider a certain one-parameter family of connections on the prequantum spaces. For any value of the parameter, the resulting connection will be called the Hitchin-Witten connection. The main feature of both connections is projective flatness, at least when the Kähler structures have few symmetries, and we shall establish this fact by direct curvature calculations.

Let us briefly introduce the setting and state the main results. The basic assumptions and their implications will be explored in greater detail in the following sections. Consider a symplectic manifold \((M, \omega)\), with vanishing first Betti number and first Chern class given by \(c_1(M, \omega) = \lambda \left[ \frac{\omega}{2\pi} \right] \), for some integer \(\lambda\). Furthermore, let \(\mathcal{T}\) be a complex manifold parametrizing a holomorphic family \(J: \mathcal{T} \rightarrow C^\infty(M, \text{End}(TM))\) of integrable almost complex structures on \((M, \omega)\), none of which admit non-constant holomorphic functions on \(M\). The variation of the Kähler structure, along the holomorphic part of a vector field on \(\mathcal{T}\), is encoded by a section \(G(V) \in C^\infty(M, S^2(TM))\), defined by \(V'[J] = G(V) \cdot \omega\), of the second symmetric power of the holomorphic tangent bundle, and we will assume that the family \(J\) is rigid in the sense that the bivector field \(G(V)\) defines a holomorphic section \(G(V) \in H^0(J(M, S^2(TM)))\).

By the assumption on the Chern class, the symplectic manifold \((M, \omega)\) admits a Hermitian line bundle \(\mathcal{L}\) with a compatible connection of curvature \(F_{\mathcal{L}} = -i\omega\). For any \(\sigma \in \mathcal{T}\),
the space $H_2^{(k)} = H^0_2(M, L^k)$ of holomorphic sections is the quantum space, at level $k \in \mathbb{N}$, arising from geometric quantization using the complex structure $J_{\sigma}$. These spaces sit inside the prequantum space $\mathcal{H}^{(k)} = C^\infty(M, L^k)$ of smooth sections, and as $\sigma \in \mathcal{T}$ varies, they form a sub-bundle of the trivial bundle $\hat{\mathcal{H}}^{(k)} = \mathcal{T} \times \mathcal{H}^{(k)}$. As proved in [And5], this sub-bundle is preserved by the explicitly given connection,

$$\nabla_V = \nabla_T + \frac{1}{4k + 2n}(\Delta_{G(V)} + 2\nabla G(V) \cdot dF - 2\lambda V'[F]) + V'[F],$$

where $\Delta_{G(V)}$ is a second-order operator with symbol $G(V)$, and $F \in C^\infty(\mathcal{T} \times M)$ is the Ricci potential, expressing the relation between the Ricci form $\rho = \lambda \omega + 2i\partial \bar{\partial}F$. This connection will be called the Hitchin connection, and generalizes the connections studied by [Hit3] and [ADPW] in the setting of quantum Chern-Simons theory for compact gauge group.

Inspired by the work of Witten [Wit3], we may also use the family of Kähler structures to define another connection on $\hat{\mathcal{H}}^{(k)}$ by the expression,

$$\tilde{\nabla}_V = \nabla_T + \frac{1}{2t}(\Delta_{\bar{G}(V)} + 2\nabla \bar{G}(V) \cdot dF - 2\lambda V''[F]) - \frac{1}{2t}(\Delta_{\bar{G}(V)} + 2\nabla \bar{G}(V) \cdot dF - 2\lambda V''[F]) + V'[F],$$

for any complex number $t \in \mathbb{C}$ with real part equal to $k$. We shall refer to this as the Hitchin-Witten connection at level $k$. Of course it also depends on the imaginary part of $t$, so for each level $k$ we get family of theories, parametrized by one real parameter.

Unlike the Hitchin connection, the Hitchin-Witten connection will not preserve the sub-bundle of quantum spaces. Both connections do, however, share another meritorious feature, captured by the main theorem of the paper.

**Theorem 1.1.** If all complex structures in the family have zero-dimensional symmetry group, then both the Hitchin and Hitchin-Witten connections are projectively flat.

Since the moduli spaces have no holomorphic vector fields, which is equivalent to the condition stated in the theorem, we get the following immediate corollary

**Theorem 1.2.** The Hitchin and Hitchin-Witten connections for the moduli spaces of flat connections on a closed oriented surface are projectively flat.

This result on the Hitchin connection, for the moduli spaces of flat $\text{SU}(n)$ connections, is due to Hitchin, but his proof uses algebraic geometric properties of these moduli spaces. As mentioned above, Witten gave an infinite-dimensional gauge-theoretic argument for this result on the Hitchin-Witten connection, for the same moduli spaces. In this paper, we provide a purely differential geometric finite-dimensional argument for projective flatness, which applies in the more general setting we have described.

As explained above, the quantum representation of the mapping class groups from the Reshetikhin-Turaev TQFT for $\text{SU}(n)$ coincides with the representation obtained from the Hitchin connection. We expect that the same will hold for the Hitchin-Witten connection and quantum Chern-Simons theory for the complex gauge group $\text{SL}(n, \mathbb{C})$. In the paper [And7], the first author has computed explicitly the resulting representation of the mapping class group for genus 1, the gauge group $\text{SL}(2, \mathbb{C})$ and all integer levels $k$. 

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Only recently has the whole TQFT been rigorously constructed by the first author of this paper and Kashaev in [AK1] and [AK2] for the case of PSL(2, C) and level \(k = 1\), using quantum Teichmüller theory and the Faddeev quantum dilogarithm. In the paper [AK3], the same authors have constructed quantum Chern-Simons theory for PSL(2, C) and all non-negative integer levels \(k\) and further understood how it relates to the geometric quantization of the PSL(2, C)-moduli spaces. In fact, they have proposed a very general scheme which just requires a Pontryagin self-dual locally compact group, which we expect will lead to the construction of the SL(\(n, C\)) for all non-negative integer levels \(k\). This should be seen in parallel to the developments on indeces [Gar, DG1], which should be related to the level \(k = 0\) theory. In the physics literature, the complex quantum Chern-Simons theory has been discussed from a path integral point of view in a number of papers [Dim1, DGG, DGLZ, DG2, GM, Guk, Hik1, Hik2, Wit4, BNW] and latest by Dimofte [Dim2] using the more advanced 3d-3d correspondence.

Outline

Let us briefly outline the organization of the paper. Section 2 discusses general aspects of families of Kähler structures on a fixed symplectic manifold. In particular, a rather serious holomorphicity condition, called rigidity (Definition 2.4), on the infinitesimal deformation of the Kähler structure will be discussed. We derive a symmetry result (Proposition 2.6) for certain tensor fields associated with such families, which will prove crucial in the calculation of the curvature of the Hitchin connection.

In section 3, we briefly recall the basics of geometric quantization, described as a two stage process where the prequantum space \(H^{(k)}\) is first constructed as sections of a line bundle, and the quantum space \(H^{(k)}\) is then defined as the subspace of polarized sections with respect to some choice of auxiliary Kähler polarization. One way of understanding the influence of this choice on quantization is by studying the infinitesimal behaviour of a family of such polarizations, and this is exactly the approach employed with the Hitchin connection, which relates the quantum spaces through parallel transport. Following the initial discussion, we calculate the commutators of general second-order differential operators acting on the prequantum spaces. These results will be useful when calculating the curvature of the Hitchin connection.

The Hitchin connection is the subject of section 4, which contains the main results. After briefly reviewing the differential geometric construction of the Hitchin connection, we move on to the straightforward but rather lengthy calculation of its curvature, culminating with the first major result in Theorem 4.8. The fact that the curvature acts as a differential operator of order at most one is a crucial point. Building on these computations and inspiration from Witten’s work on quantum Chern-Simons theory for complex gauge group [Wit3], we then consider the Hitchin-Witten connection defined in (44) as well as above. The second major result is the calculation of its curvature in Theorem 4.10. The expressions turn out to be essentially equal to the curvature of the Hitchin connection in Theorem 4.8, and in particular it acts as a differential operator of order at most one. This property, shared by both connections, entails projective flatness if, for instance, the family of complex structures does not admit holomorphic vector fields. This is the content of Theorem 4.11, leading ultimately to Theorem 1.1.
The final section applies the results to quantum Chern-Simons theory, both for compact and complex gauge groups. The Hitchin connection was originally studied in this setting, with compact gauge group $\text{SU}(n)$, by Hitchin [Hit3] and Axelrod, Della Pietra and Witten [ADPW]. Our results provide another proof of projective flatness, which still relies on the absence of holomorphic vector fields, but uses general properties of rigid families of Kähler structures to prove the vanishing of higher-order symbols. As mentioned, the case of complex gauge group was studied by Witten in [Wit3], where he used a real polarization to reduce the quantum space from complex to unitary connections. The real polarization, and hence the reduction, depends on the conformal structure on the surface, and Witten derived a formula for an analogue of the Hitchin connection in this model, arriving at exactly the expression (44). We recall the necessary theory and connect it with the results of previous sections, providing a differential geometric and purely finite-dimensional construction of a projectively flat connection, the Hitchin-Witten connection, in Witten’s model of quantum Chern-Simons theory with complex gauge group.

2 Families of Kähler Structures

Before we recall the construction of the Hitchin connection and calculate its curvature, we will explore the properties of families of Kähler structures on a symplectic manifold. Such families are central to the notion of a Hitchin connection, and the results obtained will play a fundamental role in subsequent parts. The section serves to introduce notation, establish conventions and provide a number of basic results for later reference. The result is somewhat lengthy and can be read swiftly on first reading.

Let $(M, \omega)$ be a symplectic manifold. If $T$ is a manifold, we say that a smooth map,

$$J: T \rightarrow C^\infty(M, \text{End}(TM)),$$

is a family of Kähler structures on $(M, \omega)$ if it defines an integrable and $\omega$-compatible almost complex structure for every point $\sigma \in T$. Smoothness of $J$ means that it defines a smooth section of the pullback bundle $\pi_M^*\text{End}(TM)$ over $T \times M$, where $\pi_M: T \times M \rightarrow M$ denotes the projection.

For any point $\sigma \in T$, the almost complex structure $J_\sigma$ induces a splitting,

$$TM_\sigma = T'M_\sigma \oplus T''M_\sigma,$$

of the complexified tangent bundle of $M$ into the two eigenspaces of $J_\sigma$, with associated subspace projections $\pi^{1,0}_\sigma: TM_\sigma \rightarrow T'M_\sigma$ and $\pi^{0,1}_\sigma: TM_\sigma \rightarrow T''M_\sigma$, explicitly given by

$$\pi^{1,0}_\sigma = \frac{1}{2}(\text{Id} - iJ_\sigma) \quad \text{and} \quad \pi^{0,1}_\sigma = \frac{1}{2}(\text{Id} + iJ_\sigma).$$

We denote by $X = X'_\sigma + X''_\sigma$ the associated splitting of a vector field $X$ on $M_\sigma$. In general, the subscript $\sigma$ indicates dependence on the complex structure, but we shall typically omit it when the dependence is obvious and the formula is valid for any point in $T$.

The Kähler metric associated with the complex structure $J$ is given by

$$g = \omega \cdot J,$$
where the dot denotes contraction of tensors. The inverses of $g$ and $\omega$ are denoted by $\tilde{g}$ and $\tilde{\omega}$, respectively, and they are the unique symmetric, respectively anti-symmetric, bivector fields satisfying

$$g \cdot \tilde{g} = \tilde{g} \cdot g = \text{Id} \quad \text{and} \quad \omega \cdot \tilde{\omega} = \tilde{\omega} \cdot \omega = \text{Id}.$$  

As above, a dot will be used to denote contraction of tensors, and the placement of the tensors relative to each other indicates which entries to contract. We will, however, also encounter more complicated expressions, where the entries to be contracted cannot be indicated by simply placing the tensors next to each other. In such cases, we will use abstract indices to denote the entries of each tensor. The indices only name the entries of a tensor, and do not represent a choice of local coordinates. As usual, subscript indices refer to covariant entries of a tensor, whereas superscript indices refer to contravariant entries, and following the Einstein convention, repeated indices will indicate contraction. If the two contracted indices are both either subscript or superscript, the Kähler metric is used for contraction. With these conventions, the above identities become

$$g_{ab} = \omega_{au} J^u_b \quad \text{and} \quad g_{ab} \tilde{g}^{ab} = \omega_{au} \tilde{\omega}^{ub} = \text{Id}^b_a.$$  

In a few places, we will also need to apply the projections $\pi^{1,0}$ and $\pi^{0,1}$ to the entries of the tensor. In the index notation, composition with $\pi^{1,0}$ will be indicated by a prime on the index, whereas composition with $\pi^{0,1}$ will be indicated by two primes. As an example we can write

$$\omega_{ab} = \omega_{a'b'^{r'}} + \omega_{a''b''}$$  

for the symplectic form, which is of type (1,1) with respect to a compatible complex structure.

Associated with the Kähler metric $g$, we have the Levi-Civita connection $\nabla^g$, and as usual its curvature is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$  

for any vector fields $X, Y$ and $Z$ on $M$. In abstract index notation, the curvature writes $R^d_{abc}$, and we can use the metric to lower the upper index and get the curvature tensor

$$R_{abcd} = R^r_{abc} g_{rd}.$$  

The Ricci curvature is the symmetric $J$-invariant tensor $r$ defined by

$$r(X,Y) = \text{Tr}(Z \mapsto R(Z,X)Y),$$  

and its corresponding skew-symmetric two-form is the Ricci form $\rho = J \cdot r$, which would correspond to

$$r_{ab} = R^u_{uab} = R_{uabu} = R_{auub} \quad \text{and} \quad \rho_{ab} = J^u_a r_{ub} = \frac{1}{2} R_{abuv} \tilde{\omega}^{uv}.$$  

in index notation. Finally, the scalar curvature $s$ is the metric trace of the Ricci curvature

$$s = r_{uu} = r_{uv} \tilde{g}^{uv} = \rho_{uv} \tilde{\omega}^{vu}.$$  

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If $E$ is a vector bundle over $M$ and $D \in \mathcal{D}(M,E)$ is a differential operator of order at most $n$, we can assign the principal symbol $\sigma_p(D) \in C^\infty(M,S^n(TM))$, which is a symmetric section of the $n$'th tensor power of the tangent bundle. If the principal symbol vanishes, then $D$ is of order at most $n - 1$. In general, there is no good notion of lower order symbols of differential operators, but a connection on $E$ and can be combined with the Levi-Civita connection on $M$ to define symbols of all orders. For vector fields $X_1, \ldots, X_n$ on $M$, we consider the inductively defined differential operator on sections of $E$,

$$\nabla^n_{X_1, \ldots, X_n} s = \nabla X_1 \nabla^{n-1}_{X_2, \ldots, X_n} s - \sum_j \nabla^{n-1}_{X_1, \ldots, \nabla X_j, \ldots, X_n} s,$$

with the obvious induction start given by the covariant derivative. It is easily verified that this expression is tensorial in the vector fields, so we get a map

$$\nabla^n : C^\infty(M,TM^n) \to \mathcal{D}(M,E).$$

For any tensor field $T_n \in C^\infty(M,TM^n)$, the symbol of $\nabla^n_{T_n}$ is given by the symmetrization $S(T_n) \in C^\infty(M,S^n(TM))$ of $T_n$. If $D \in \mathcal{D}(M,E)$ is an operator, of order at most $n$, with principal symbol $\sigma_p(D) = S_n \in C^\infty(M,S^n(TM))$, then the operator $D - \nabla^n_{S_n}$ is of order at most $n - 1$, since its principal symbol vanishes. Inductively, it follows that the operator $D$ can be written uniquely in the form

$$D = \nabla^n_{S_n} + \nabla^{n-1}_{S_{n-1}} + \cdots + \nabla S_1 + S_0,$$

where $S_d \in C^\infty(M,S^d(TM))$ is called the symbol of order $d$ and gives rise to a map

$$\sigma_d : \mathcal{D}(M,E) \to C^\infty(M,S^d(TM)).$$

Any finite order differential operator on $E$ is uniquely determined by the values of these symbol maps. In fact, through the expression (2), a choice of symbols specifies a differential operator on any vector bundle with connection, and in particular on functions.

We will also need the notion of divergence of vector fields and more general contravariant tensors. Recall that the divergence $\delta X$ of a vector field $X$ on $M$ is defined in terms of the Lie derivative and volume form by the equation $\mathcal{L}_X \omega^m = (\delta X) \omega^m$. Although the divergence of a vector field only depends on the symplectic volume, and not on the Kähler metric itself, a simple computation reveals that the divergence can be calculated using the Levi-Civita connection by the formula

$$\delta X = \text{Tr} \nabla X = \nabla_a X^a,$$

in which the independence of the Kähler structure is perhaps not so evident. The Laplace-de Rham operator on functions can be expressed in terms of the divergence by

$$\Delta f = -2i \delta X'_f,$$

where $X'_f = \bar{\partial} f \omega$ denotes the $(1,0)$-part of the Hamiltonian vector field associated with the function $f \in C^\infty(M)$.

The formula (3) generalizes to tensors of higher degree. For vector fields $X_1, \ldots, X_n$ on $M$, we define

$$\delta(X_1 \cdots \cdot X_n) = \delta(X_1)X_2 \cdots \cdot X_n + \sum_j X_2 \cdots \cdot \nabla X_j X_j \cdots \cdot X_n.$$
This defines a map $\delta: C^\infty(M, TM^n) \to C^\infty(M, TM^{n-1})$, also called the divergence, which does depend on the Kähler structure.

The generalization of divergence to sections of the endomorphism bundle of the tangent bundle will also be convenient. If $\alpha \in \Omega^1(M)$ is a one-form and $X$ is a vector field, we define

$$\delta(X \otimes \alpha) = \delta(X)\alpha + \nabla_X \alpha,$$

which gives a map $\delta: C^\infty(M, \text{End}(TM)) \to \Omega^1(M)$.

Finally, for any bivector field $B \in C^\infty(M, T M^2)$, we introduce the second-order differential operator,

$$\Delta_B = \nabla^2_B + \nabla_\delta B,$$

which will appear repeatedly throughout the paper.

### Infinitesimal Deformations

For a smooth family of Kähler structures, we can take its derivative along a vector field $V$ on $\mathcal{T}$ to obtain a map

$$V[J]: \mathcal{T} \to C^\infty(M, \text{End}(TM)).$$

Differentiating the identity $J^2 = -\text{Id}$, we see that $V[J]$ and $J$ anti-commute,

$$V[J]J + JV[J] = 0,$$

so $V[J]_\sigma$ interchanges types on the Kähler manifold $M_\sigma$. Therefore, it splits as

$$V[J] = V[J]_\sigma + V[J]_\pi,$$

where $V[J]_\sigma \in C^\infty(M, TM_\sigma \otimes T^*M^*_\sigma)$ and $V[J]_\pi \in C^\infty(M, T^*M_\sigma \otimes TM^*_\sigma)$ is its conjugate.

Notice that this splitting occurs for any infinitesimal deformation of an almost complex structure on $M$ and defines an almost complex structure on the space of almost complex structures on $M$.

Differentiating the integrability condition on $J$, expressed through the vanishing of the Nijenhuis tensor, reveals that $V[J]' \in \Omega^{0,1}(M, TM)$ satisfies the holomorphicity condition $\bar{\partial}V[J]' = 0$, and the associated cohomology class in $H^1(M, TM)$ is the Kodaira-Spencer class of the deformation (see [Kod]).

Define a bivector field $\tilde{G}(V) \in C^\infty(M, T M^G \otimes T M^G)$ by the relation

$$V[J] = \tilde{G}(V) \cdot \omega,$$

for any vector field $V$ on $\mathcal{T}$. Differentiating the identity $\tilde{g} = -J \cdot \omega$ along $V$, we get

$$V[\tilde{g}] = -V[J] \cdot \omega = -\tilde{G}(V),$$

and since $\tilde{g}$ is symmetric, this implies that $\tilde{G}(V)$ is a symmetric bivector field. Furthermore, the combined types of $V[J]$ and $\omega$ yield a decomposition,

$$\tilde{G}(V) = G(V) + \tilde{G}(V),$$
where \( G(V) \in C^\infty(M, S^2(TM)) \) and \( \check{G}(V) \in C^\infty(M, S^2(T^*M)) \). In other words, the real symmetric bivector field \( \check{G}(V) \) has no \((1,1)\)-part. The variation of the Kähler metric is obtained by differentiating the identity \( g = \omega \cdot J \), which yields

\[
V[g] = \omega \cdot V[J] = \omega \cdot \check{G}(V) \cdot \omega = g \cdot \check{G}(V) \cdot g.
\]

We shall also need the variation of the Levi-Civita connection, which is the tensor field \( V[\nabla g] \in C^\infty(M, S^2(T^*M) \otimes TM) \) given by (see [Bes] Theorem 1.174)

\[
2g(V[\nabla g], X, Y) = \nabla_X (V[g])(Y, Z) + \nabla_Y (V[g])(X, Z) - \nabla_Z (V[g])(X, Y),
\]

for any vector fields \( X, Y \) and \( Z \) on \( M \). In index notation, this translates to

\[
2V[\nabla g]_{ab} = \nabla_a \check{G}(V)_{cu} g_{ub} + g_{au} \nabla_b \check{G}(V)^{uc} - g_{au} \check{G}(V)^{uv} \nabla_w \check{G}(V)_{uv} g_{wb},
\]

and we remark that the trace \( V[\nabla g]_{ab} \) of this tensor vanishes. Indeed, we get that

\[
V[\nabla g]_{ab} = \nabla_x \check{G}(V)^{cu} g_{ub} + g_{xu} \nabla_b \check{G}(V)^{ux} - g_{xu} \check{G}(V)^{uv} \nabla_w \check{G}(V)_{uv} g_{wb} = 0,
\]

where the first and last term cancel, and the middle term vanishes because \( \check{G}(V) \) has no part of type \((1,1)\), which is the type of the metric.

We will also need to know the variation of the Ricci curvature, which will result from the Bianchi identity of a certain line bundle associated with the family of complex structures.

**The Canonical Line Bundle of a Family**

For a family \( J \) of Kähler structures, we can consider the vector bundle,

\[
\check{T}M \to \mathcal{T} \times M,
\]

with fibers \( \check{T}M_{(\sigma, p)} = T'_p M_{\sigma} \) given by the holomorphic tangent spaces of \( M \). Throughout the paper, we shall generally use a hat in the notation to indicate that we are working over the product \( \mathcal{T} \times M \). Following this convention, the exterior differential on \( \mathcal{T} \times M \) is denoted by \( d \), whereas the differential on \( \mathcal{T} \) is denoted by \( d_{\tau} \) and by \( d \) on \( M \).

The Kähler metric induces a Hermitian structure \( \check{h}^{TM} \) on \( \check{T}M \), and the Levi-Civita connection gives a compatible partial connection along the directions of \( M \). We can extend this partial connection to a full connection \( \check{\nabla}^{TM} \) on \( \check{T}M \) in the following way. If \( Z \in C^\infty(\mathcal{T} \times M, \check{T}M) \) is a smooth family of sections of the holomorphic tangent bundle, and \( V \) is a vector field on \( \mathcal{T} \), then we define

\[
\check{\nabla}_V Z = \pi^{1,0} V[Z].
\]

In other words, we regard \( Z \) as a smooth family of sections of the complexified tangent bundle \( TM_{\mathbb{C}} \), and then we simply differentiate \( Z \) along \( V \) in this bundle, which does not depend on the point in \( \mathcal{T} \), and project the result back onto the holomorphic tangent bundle.
Clearly, the connection $\hat{\nabla}^T M$ preserves the Hermitian structure in the directions of $M$, since it is induced by the Levi-Civita connection. Moreover, if $V$ is a vector field on $T$, and $X$ and $Y$ are sections of $T^M$, we get that

$$V[\hat{h}^T M(X, Y)] = V[g(X, \bar{Y})] = V[g](X, \bar{Y}) + g(V[X], \bar{Y}) + g(X, \bar{V}[Y])$$

$$= h(\nabla_V X, Y) + h(X, \nabla_Y Y),$$

since the $(1, 1)$-part of $V[g]$ vanishes. It follows that $\hat{\nabla}^T M$ preserves the Hermitian structure on $\hat{T} M$.

Now consider the line bundle

$$\hat{K} = \mathcal{A}^n \hat{T}^* M \rightarrow T \times M,$$

which will be referred to as the canonical line bundle of the family of Kähler structures. As usual, the Hermitian structure and connection on $\hat{T}^* M$ induce a Hermitian structure $\hat{h}^K$ and a compatible connection $\hat{\nabla}^K$ on $\hat{K}$. The curvature of $\hat{\nabla}^K$ was calculated in [AGL] and will be recalled below, but before stating it, we introduce the following important notation.

For any vector fields $V$ and $W$ on $T$, we define $\Theta \in \Omega^2(T, S^2(TM))$ by

$$\Theta(V, W) = S(\tilde{G}(V) \cdot \omega \cdot \tilde{G}(W)),$$

where $S$ denotes symmetrization. We also give a name to the metric trace of the symmetric bivector field $\Theta(V, W)$ and define

$$\theta(V, W) = -\frac{1}{4} g(\Theta(V, W)) = \frac{1}{4} g_{uv} \Theta(V, W)^{uv}. \quad (10)$$

Clearly, this defines a real two-form $\theta \in \Omega^2(T, C^\infty(M))$ on $T$ with values in smooth functions on $M$.

We note that the two-form $\Theta$ over $T$ is exact. To see this, we take the variation of $G(V) = \pi_2^0(\tilde{G}(V)) = (\pi_1^0 \otimes \pi_1^0)\tilde{G}(V)$ along $W$ to get

$$2W[G(V)] = i G(V) \cdot \omega \cdot \tilde{G}(W) - i \tilde{G}(W) \cdot \omega \cdot G(V) - \pi_2^0(WV[\bar{g}])$$

$$= 2i S(G(V) \cdot \omega \cdot G(W)) - \pi_2^0(WV[\bar{g}]), \quad (11)$$

which in turn shows that

$$V[G(W)] - W[G(V)] = -i S(\tilde{G}(V) \cdot \omega \cdot G(W)) - i S(G(V) \cdot \omega \cdot \tilde{G}(W)) = -i \Theta(V, W), \quad (12)$$

for commuting vector fields $V$ and $W$ on $T$. This can be rephrased as

$$d_t G = -i \Theta, \quad (13)$$

where $G$ is viewed as a one-form in $\Omega^1(T, S^2(TM))$.

The following proposition gives the curvature of the canonical line bundle of a family of Kähler structures and is proved in [AGL].

**Proposition 2.1.** The curvature of $\hat{\nabla}^K$ is given by

$$F_{\hat{\nabla}^K}(X, Y) = i \rho(X, Y), \quad F_{\hat{\nabla}^K}(V, X) = \frac{i}{2} \delta \tilde{G}(V) \cdot \omega \cdot X, \quad F_{\hat{\nabla}^K}(V, W) = i \theta(V, W),$$

for any vector fields $X, Y$ on $M$ and $V, W$ on $T$. 
By applying the Bianchi identity to the connection $\hat{\nabla} K$, and using the formulas of Proposition 2.1, we get three useful results. The first is the fact that the two-form $\theta \in \Omega^2(T, C^\infty(M))$ is closed, which is a trivial reformulation of the Bianchi identity for three vector fields on $T$. By applying the Bianchi identity to two vector fields $V$ and $W$ on $T$, and one vector field on $M$, we get

$$d\theta(V, W) = \frac{1}{2} W[\delta \tilde{G}(V)] \omega - \frac{1}{2} V[\delta \tilde{G}(W)] \omega.$$  \hspace{1cm} (14)

Finally, the Bianchi identity for two vector fields on $M$ and one on $T$ gives following important formula for the variation of the Ricci form,

$$V[\rho] = \frac{1}{2} d(\delta \tilde{G}(V) \cdot \omega),$$  \hspace{1cm} (15)

for any vector field $V$ on $T$. As an immediate consequence of (15), we get the following simple formula for the variation of the scalar curvature

$$V[s] = V[\rho] \cdot \omega^\nu = \frac{1}{2} \nabla_u (\delta \tilde{G}(V))^v \omega^\nu_{uv} \cdot \omega^\nu - \frac{1}{2} \nabla_v (\delta \tilde{G}(V))^u \omega^\nu_{uv} \cdot \omega^\nu = \delta \delta \tilde{G}(V),$$  \hspace{1cm} (16)

for any vector field $V$ on $T$.

**Holomorphic Families of Kähler Structures**

In case the manifold $T$ is itself a complex manifold, we can require the family $J$ to be a holomorphic map from $T$ to the space of complex structures. This is made precise by the following definition, which uses the splitting (6) of $V[J]$.

**Definition 2.2.** Suppose that $T$ is a complex manifold, and that $J$ is a family of complex structures on $M$, parametrized by $T$. Then $J$ is holomorphic if

$$V'[J] = V[J'] \quad \text{and} \quad V''[J] = V[J''].$$

for any vector field $V$ on $T$.

If $I$ denotes the integrable almost complex structure on $T$ induced by its complex structure, then we get an almost complex structure $\hat{J}$ on $T \times M$ defined by

$$\hat{J}(V \oplus X) = IV \oplus J_\sigma X, \quad V \oplus X \in T(\sigma, \sigma)(T \times M).$$

The following proposition gives another characterization of holomorphic families [AGL].

**Proposition 2.3.** The family $J$ is holomorphic if and only if $\hat{J}$ is integrable.

By this proposition, a holomorphic family induces a complex structure on the product manifold $T \times M$. Clearly, the projection $\pi_T: T \times M \to T$ is a holomorphic map, and its differential is the projection $d\pi_T: T' T \oplus T' M \to T' T$, where $T' T$ is the pullback of $T' T$ by $\pi_T$. Since the bundle $T' M$ over $T \times M$ is the kernel of this map, it has the structure of a holomorphic vector bundle, and it is easily verified that the connection $\hat{\nabla}^{T M}$ is compatible with this holomorphic structure. Since the connection also preserves the Hermitian structure, it must be the Chern connection.
Holomorphicity has several useful consequences. First of all, it implies that
\[
\tilde{G}(V') = V'[J] \tilde{\omega} = V'[J] \tilde{\omega} = G(V),
\]
and similarly \(\tilde{G}(V'') = \tilde{G}(V)\). This means that \(G\), viewed as a one-form over the complex manifold \(T\), has type (1,0) and that \(\Theta\) and \(\theta\) have type (1,1). These important facts will be used without reference going forward.

Finally, for commuting vector fields \(V'\) and \(W''\), the identity (12) reduces to
\[
W''[G(V)] = \frac{i}{2} G(V) \cdot G(W) - \frac{i}{2} \tilde{G}(W) \cdot G(V) = i \Theta(V', W'').
\] (17)
This expression for the second-order variation of the complex structure will prove very useful in later calculations, but we emphasize the fact that it only holds for commuting vector fields.

**Rigid Families of Kähler Structures**

The following rather serious assumption on a family of Kähler structures turns out to be crucial to the construction of the Hitchin connection as well as the calculation of its curvature.

**Definition 2.4.** A family of Kähler structures is called rigid if
\[
\nabla_X G(V) = 0,
\] (18)
for all vector fields \(V\) on \(T\) and \(X\) on \(M\).

In other words, the family \(J\) is rigid if \(G(V)\) is a holomorphic section of \(S^2(T'M)\), for any vector field \(V\) on \(T\). For examples of rigid families in a basic setting, we refer to [AGL]. By differentiating the rigidity condition (18) along \(T\), we get the following crucial result.

**Proposition 2.5.** Any rigid family of Kähler structures satisfies the symmetry property
\[
S(G(V) \cdot \nabla G(W)) = S(G(W) \cdot \nabla G(V)),
\] (19)
for any vector fields \(V\) and \(W\) on \(T\).

**Proof.** Throughout the proof, let \(V\) and \(W\) be commuting vector fields on \(T\). By differentiating the holomorphicity condition on the bivector field \(G(V)\) along \(W\), we obtain
\[
0 = W[\nabla_a G(V)]^{bc}
= W[\tilde{G}(V)]_a^{bc} \omega \omega_{ab} G(V) + \nabla_a W[G(V)]^{bc}
+ W[\nabla]_a^{bc} \omega G(V) + W[\nabla]^{bc} \omega G(V) + W[\omega]_a^{bc} G(V).
\] (20)
Let us work out each of these terms individually. Using rigidity, the first term reduces to
\[
2W[\tilde{G}(V)]_a^{bc} = i \tilde{G}(W) \omega \omega_{ab} \nabla a G(V)^{bc} = -g_{av} G(W) \omega \omega_a G(V)^{bc}.
\]
For the second term of (20), we simply apply (11) and rigidity to obtain

\[ 2
\n\]for any vector fields \( V \) and \( W \) on \( \mathcal{T} \). Then Proposition 2.5 ensures that \( \Gamma_3 \) actually defines a symmetric two-form on \( \mathcal{T} \).

Clearly, the symmetry of the two-form over \( \mathcal{T} \) is not affected by taking the divergence of the tri-vector field part of \( \Gamma_3 \) over \( M \). In other words, \( \delta \Gamma_3 \) defines a symmetric two-form on \( \mathcal{T} \) with values in the second symmetric power of the holomorphic tangent bundle on \( M \). It is given by

\[ 3 \delta \Gamma_3(V, W) = \Delta_{G(V)} G(W) + 2 S(\nabla G(V) \cdot \nabla \delta G(W)) + 2 S(\nabla_u G(V)^{au} \nabla_v G(W)^{ub}), \]

for any vector fields \( V \) and \( W \) on \( \mathcal{T} \). In contrast to the first two terms of this expression, the last term is obviously symmetric in \( V \) and \( W \). This leads us to define \( \Gamma_2 \in \Omega^2(\mathcal{T}, C^\infty(M, S^2(\mathcal{T}^* M))) \) by

\[ \Gamma_2(V, W) = \Delta_{G(V)} G(W) + 2 S(\nabla G(V) \cdot \nabla \delta G(W)), \]

which is also a symmetric two-form on \( \mathcal{T} \) and encodes the interesting part of the symmetry statement for \( \delta \Gamma_3 \).
Repeatedly taking the divergence, and removing obviously symmetric parts, we get the following important proposition.

**Proposition 2.6.** The four two-forms $\Gamma_j$ on $T$, with values in symmetric contravariant tensors on $M$, defined by

$$
\begin{align*}
\Gamma_3(V, W) &= S(G(V) \cdot \nabla G(W)) \\
\Gamma_2(V, W) &= 2S(G(V) \cdot \delta G(W)) + \Delta G(V)G(W) \\
\Gamma_1(V, W) &= 2\Delta G(V) \delta G(W) + \nabla \omega(G(V)^{uw})\nabla^2_{uv}(G(W)^w) \\
\Gamma_0(V, W) &= \Delta G(V) \delta \delta G(W) + \nabla \omega(G(V)^{uw})\nabla^2_{uv} \delta G(W)^w,
\end{align*}
$$

are all symmetric in the vector fields $V$ and $W$ on $T$.

**Families of Ricci Potentials**

Suppose that the first Chern class of $(M, \omega)$ is represented by the symplectic form, that is

$$c_1(M, \omega) = \lambda \left[ \frac{\omega}{2\pi} \right],$$

(21)

for some integer $\lambda \in \mathbb{Z}$. Now the first Chern class is also represented by the Ricci form $\rho$, so the difference between the forms $\rho$ and $\lambda \omega$ is exact. A smooth real function $F \in C^\infty(T \times M)$, which can be viewed as a smooth map $F: T \to C^\infty(M)$, is called a family of Ricci potentials if it satisfies

$$\rho_\sigma = \lambda \omega + 2i\partial_\sigma \bar{\partial}_\sigma F_\sigma,$$

(22)

for every point $\sigma \in T$.

Any two Ricci potentials differ by a global pluriharmonic real function. If we assume that $H^1(M, \mathbb{R})$ vanishes, such a function is globally the real part of a holomorphic function, so a family of Ricci potential is then uniquely determined up to a function on $T$ if the family of Kähler structures does not allow any non-constant holomorphic functions on $M$.

The existence of a Ricci potential is clearly a global issue over $M$, as the local $\partial \bar{\partial}$-lemma ensures local existence around any point on $M$ when (21) holds. If the manifold $M$ is compact, the global $\partial \bar{\partial}$-lemma from Hodge theory ensures the existence of a Ricci potential, using the fact that the Kähler form $\omega$ is harmonic. In this case, the Ricci potential is certainly unique up to a function on $T$, and by imposing zero average over $M$,

$$\int_M F \omega^m = 0,$$

(23)

we can fix it uniquely.

The following proposition gives an important identity, involving the variation of a family of Ricci potentials.

**Proposition 2.7.** Suppose that $M$ is a symplectic manifold with $H^1(M, \mathbb{R}) = 0$ and $c_1(M, \omega) = \lambda \left[ \frac{\omega}{2\pi} \right]$, and let $J$ be a holomorphic family of Kähler structures on $M$, none of which admit non-constant holomorphic functions on $M$. Then

$$4i\partial \bar{\partial}^m[F] = \delta G(V) \cdot \omega + 2dF \cdot G(V) \cdot \omega,$$

(24)

for any family of Ricci potentials $F$ and any vector field $V$ on $T$. 

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Proof. By differentiating the identity (22) in the direction of $V'$, we get

$$V'[ho] = -d(dF \cdot G(V) \cdot \omega) + 2i\partial\bar{\partial}V'[F],$$

and by using (15) on the left-hand side, this yields

$$d(\delta G(V) \cdot \omega) + 2d(dF \cdot G(V) \cdot \omega) - 4id\bar{\partial}V'[F] = 0.$$

On one hand, it follows that the one-form

$$\delta G(V) \cdot \omega + 2dF \cdot G(V) \cdot \omega - 4i\bar{\partial}V'[F]$$

is closed, and hence exact by the assumption $H^1(M, \mathbb{R}) = 0$. On the other hand, it is of type $(0,1)$, so it cannot be exact unless it is zero, because we assumed that none of the Kähler structures admit non-constant holomorphic functions. This proves the lemma.

Another equivalent form of (24) is the following,

$$4iV'\bar{\partial}F = \delta G(V) \cdot \omega,$$

and if we combine this with (14), for commuting vector fields $V'$ and $W''$ on $\mathcal{M}$, we get

$$2d\theta(V', W'') = W''[\delta G(V) \cdot \omega] - V'[\delta \bar{G}(W) \cdot \omega]$$

$$= 4iW''V'\bar{\partial}F - 4iV'W''\bar{\partial}F$$

$$= 4i\partial_r \bar{\partial}_r F(V', W'').$$

Since the family is holomorphic, the form $\theta$ has type $(1,1)$ on $\mathcal{T}$, so we have shown

**Proposition 2.8.** In the setting of Proposition 2.7, any family of Ricci potentials satisfies

$$\theta - 2i\partial_r \bar{\partial}_r F \in \Omega^{1,1}(\mathcal{T}).$$

In other words, the form takes values in constant functions on $M$.

This ends the general discussion of families of Kähler structures. In the next section, we discuss general aspects of geometric quantization, and in particular the need for a choice of auxiliary polarization in the construction. Understanding the effects of this choice naturally leads us to consider families of Kähler structures, and ultimately to the Hitchin connection relating the different choices. The results from this section will play a fundamental role in the discussion.

### 3 Geometric Quantization

In very broad terms, geometric quantization concerns the passage from classical mechanics to quantum mechanics. It aims to produce a Hilbert space of quantum states from a classical phase space, in the form of a symplectic manifold, and a self-adjointed operator from a classical observable, in the form of a function on the classical phase space.
In the following, we shall briefly review the basic notions from geometric quantization relevant to us. A key role is played by an auxiliary choice of polarization, which is often chosen to be Kähler. For a broader treatment of geometric quantization, the reader is referred to [Woo] and [AE].

After reviewing the elements of geometric quantization, we calculate the commutators of certain differential operators acting on the prequantum spaces. These will be relevant in later discussions of the Hitchin connection and its curvature.

**Prequantization**

In geometric quantization, the Hilbert space of quantum states arises as sections of a certain Hermitian line bundle over the classical phase space. As a model for this classical phase space, we consider a symplectic manifold \((M, \omega)\) of dimension \(2m\). A prequantum line bundle over the \(M\) is a complex line bundle \(L\) endowed with a Hermitian metric \(h\) and a compatible connection \(\nabla\) of curvature

\[
F_\nabla = -i\omega.
\]

A symplectic manifold admitting a prequantum line bundle is called prequantizable. Evidently, this is not the case for every symplectic manifold. Indeed, the real first Chern class of a prequantum line bundle is given by

\[
\frac{i\omega}{2\pi} \in \operatorname{Im} \left( H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R}) \right).
\]

This is, in fact, also sufficient to ensure the existence of a prequantum line bundle, and the inequivalent prequantum line bundles over \(M\) are parametrized by \(H^1(M, \mathbb{U}(1))\).

For any natural number \(k\), called the **level**, we consider the **prequantum space**

\[
\mathcal{H}^{(k)} = C^\infty(M, \mathcal{L}^k),
\]

of smooth sections of the \(k\)’th tensor power of the line bundle \(\mathcal{L}\). These sections play the role of wave functions in the quantum theory. If \(f \in C^\infty(M)\) is a function on \(M\), the corresponding **prequantum operator**, acting on \(\mathcal{H}^{(k)}\), is defined by

\[
P_k(f) = \frac{i}{k} \nabla X_f + f,
\]

where \(X_f\) is the Hamiltonian vector field of the function \(f\). The virtue of (26) is that the prequantum operators satisfy the correspondence principle

\[
[P_k(f), P_k(g)] = \frac{i}{k} P_k(\{f, g\}),
\]

which is one of the distinctive features of a viable quantization.

From a physical perspective, the wave functions in \(\mathcal{H}^{(k)}\) depend on twice the number of variables they should. A standard way to remedy this is to pick an auxiliary polarization on \(M\) and consider the space of polarized sections of the line bundle. The polarization can be by real or complex Lagrangian subspaces. In the following, we will focus on the case of complex Kähler polarizations.
Kähler Quantization

From now on, we assume that the symplectic manifold admits a Kähler structure, in the form of an integrable almost complex structure $J$ on $M$ which is compatible with the symplectic structure. Since the Kähler form $\omega$ has type $(1,1)$, it follows that the $(0,1)$-part of the connection on the prequantum line bundle $L$ defines a holomorphic structure. Therefore, we can define the quantum space to be the space of holomorphic (or polarized) sections,

$$H^0_J = H^0(M, J^k) = \{ s \in \mathcal{H}^k \mid \nabla_Z s = 0, \forall Z \in T''M_J \},$$

which is a subspace of the prequantum space $\mathcal{H}^k$ of smooth sections. If the manifold $M$ is compact, then $H^0_J$ is a finite-dimensional space by standard theory of elliptic operators.

Unfortunately, the prequantum operators do not in general preserve the space of holomorphic sections. A function $f \in \mathcal{C}_\infty(M)$ is polarized if the $(1,0)$-part of its corresponding Hamiltonian vector field is holomorphic. In other words, the space of polarized functions is given by

$$C^\infty_J(M) = \{ f \in \mathcal{C}_\infty(M) \mid \nabla_Z X_f = 0, \forall Z \in T''M_J \}.$$

Now, the operator $P_k(f)$ preserves $H^0_J$ if and only if $f$ is a polarized function. In fact, it is easily verified that a first-order differential operator of the form $i\frac{\partial}{\partial Z} + f$ on $\mathcal{H}^k$ preserves the subspace $H^0_J$ if and only if $f$ is a polarized function and $X' = X_f$. This gives another justification for the choice of prequantum operators. In fact, the $(0,1)$-part of the vector field makes no difference to the action of $P_k(f)$ on $H^0_J$, so if we define a polarized variation of the prequantum operators by

$$P'_k(f) = i\frac{\partial}{\partial Z} X_f + f,$$

then these are essentially the only first-order operators with a chance of preserving the subspace $H^0_J$ of $\mathcal{H}^k$. For a real polarized function $f \in C^\infty_J(M)$, the Hamiltonian vector field $X_f$ must be Killing for the Kähler metric, effectively reducing the quantizable observables to an at most finite-dimensional, and often trivial, space (see [Woo]).

To get more quantizable observables, their quantization is modified in the following way. The space $H^0_J$ is in fact a closed subspace of $\mathcal{H}^k$, and therefore we have the orthogonal projection $\pi^k_J : \mathcal{H}^k \to H^0_J(k)$. For $f \in C^\infty(M)$, we then define the corresponding quantum operator by

$$Q_k(f)_J = \pi^k_J \circ P_k(f).$$

These operators do not form an algebra, but they satisfy a weaker form of (27) (at least if $M$ is compact) in the sense that

$$\left\| [Q_k(f), Q_k(g)] - \frac{i}{k} Q_k(\{f, g\}) \right\| = O(k^{-2}) \quad \text{as} \quad k \to \infty,$$

with respect to the operator norm on $H^0_J(k)$. The proof of (29) relies on the fact that these operators are Toeplitz operators (see [BMS]).
Although this quantization scheme gives a Hilbert space of the right size, it still fails to produce the right answers on basic examples from quantum mechanics. In the end, what really matters is the spectrum of the operators, and if the above procedure is applied to the one-dimensional harmonic oscillator, the quantization yields a spectrum which differs from the correct one by a shift. To deal with this problem, the so-called metaplectic correction can be introduced. We shall not pursue this direction further in the present paper, but refer the interested reader to [AGL], where the construction of the Hitchin connection in the metaplectic setting is discussed.

**Commutators of Differential Operators**

The description of the Hitchin connection, and the calculation of its curvature in particular, requires the calculation of commutators of a number of differential operators on sections on the prequantum line bundle $\mathcal{L}$ and its tensor powers. Since the commutators of even second-order operators are quite complicated, it will be convenient to encode the operators through their symbols, as in (2), and have general but explicit descriptions of the symbols of the commutators. The followings lemmas give exactly such descriptions.

The very definition of the curvature of the line bundle $\mathcal{L}^k$ implies the basic relation

$$[\nabla_X, \nabla_Y]s = \nabla_{[X,Y]}s - ik\omega(X,Y)s$$

for any vector fields $X, Y$ on $M$ and any smooth section $s \in H^{(k)}$. Things become a little more complicated when second-order operators are introduced.

**Lemma 3.1.** For any Kähler structure on $M$, any vector field $X$ on $M$, and any symmetric bivector field $B \in C^\infty(M, S^2(TM))$, we have the symbols

$$\sigma_2[\nabla_B^2, \nabla_X] = 2S(B \cdot \nabla X) - \nabla_X B$$
$$\sigma_1[\nabla_B^2, \nabla_X] = \nabla_B^2 X - 2ikB \omega X + B^{uv}R^a_{uwv}X^w$$
$$\sigma_0[\nabla_B^2, \nabla_X] = -ik\omega(B \cdot \nabla X)$$

for the commutator of the operators $\nabla_X$ and $\nabla_B^2$ acting on $H^{(k)}$.

**Proof.** By straightforward calculation, we get

$$B^{uv}\nabla^{2}_{uv} X^x \nabla_x s = B^{uv} X^x \nabla^{3}_{uxs} s + 2B^{uv} \nabla_v (X^x) \nabla^{2}_{uxs} s + B^{uv} \nabla^2_{uv} (X^x) \nabla_x s$$
$$= B^{uv} X^x \nabla^{3}_{uxv} s - 2ikB^{uv} \omega_{ux} X^x \nabla_u s - B^{uv} X^x R^r_{uxv} \nabla_r s$$
$$+ B^{uv} \nabla_v (X^x) \nabla^{2}_{uxv} s + B^{uv} \nabla_v (X^x) \nabla^{2}_{uxs} s - ikB^{uv} \omega_{uxv} \nabla_v (X^x)s$$
$$+ B^{uv} \nabla^{2}_{uv} (X^x) \nabla_x s,$$

where we used the fact that the symplectic form $\omega$ is parallel with respect to the Levi-Civita connection. To get an expression for the desired commutator, we subtract

$$X^x \nabla_x B^{uv} \nabla^2_{uv} s = X^x \nabla_x (B^{uv}) \nabla^{2}_{uv} s + B^{uv} X^x \nabla^{3}_{xuv} s,$$

and the stated symbols can easily be extracted from the result. \qed
Naturally, things get even more complicated for two second-order operators.

**Lemma 3.2.** For any Kähler structure on $M$ and any symmetric bivector fields $A, B \in C^\infty(M, S^2(TM))$, we have the symbols

$$\sigma_3[\nabla_A^2, \nabla_B^2] = 2\mathcal{S}(A \nabla B) - 2\mathcal{S}(B \nabla A)$$
$$\sigma_2[\nabla_A^2, \nabla_B^2] = \nabla_A^2 B - \nabla_B^2 A - 4ik\mathcal{S}(A \omega - B) + 2S(A^{xy}R^u_{axy}B^v) - 2S(B^{uv}R^a_{xuv}A^{xy})$$
$$\sigma_1[\nabla_A^2, \nabla_B^2] = -2ikA^{xy}\omega_{yu}\nabla_x(B^{uv}) + 2ikB^{uv}\omega_{uv}\nabla_u(A^{xy})$$
$$- A^{xy}\nabla_x(R^a_{yuv})B^{uv} + B^{uv}\nabla_v(R^a_{xy})A^{xy}$$
$$- \frac{4}{3}A^{xy}R^a_{xuv}\nabla_yB^{uv} + \frac{4}{3}B^{uv}R^a_{xyv}\nabla_yA^{xy}$$
$$\sigma_0[\nabla_A^2, \nabla_B^2] = \frac{ik}{2}A^{xy}J^i_yR^xuvB^{uv} - \frac{ik}{2}B^{uv}J^x_iR^xyuvA^{xy}$$

for the commutator of the operators $\nabla_A^2$ and $\nabla_B^2$ acting on $\mathcal{H}^{(k)}$.

**Proof.** Once again, the proof proceeds by straightforward calculation. We get

$$A^{xy}\nabla^2_{xy}B^{uv}\nabla^2_{uv}s = A^{xy}B^{uv}\nabla^3_{xuv}s + 2A^{xy}\nabla_y(B^{uv})\nabla^3_{xuv}s + A^{xy}\nabla^2_{xy}(B^{uv})\nabla^2_{uv}s. \quad \text{(30)}$$

Focusing on the first term, we commute the indices $x$ and $y$ past $u$ and $v$.

$$A^{xy}B^{uv}\nabla^4_{xuv}s$$
$$= A^{xy}B^{uv}\left(\nabla_x^4 - \nabla_x^2 R^x_{xuv}\nabla_r - i\omega_{xy}\nabla_{xy}\right)s$$
$$= A^{xy}B^{uv}\left(\nabla^4 - i\omega_{xy}\nabla^2 - \nabla_x^2 R^x_{yuv}\nabla_r - i\omega_{yu}\nabla_{xy}\right)s$$
$$= A^{xy}B^{uv}\left(\nabla^4 - \nabla_x^2 R^x_{yuv}\nabla_r - \nabla_x^2 R^y_{xuv}\nabla_r - i\omega_{xy}\nabla_{xy}\right)s$$
$$- i\omega_{xy}\nabla^2_{xy}$$
$$= A^{xy}B^{uv}\left(\nabla^4 - \nabla_x^2 R^x_{yuv}\nabla_r - \nabla_x^2 R^y_{xuv}\nabla_r - i\omega_{xy}\nabla_{xy}\right)s$$
$$- R^x_{xuv}\nabla^2_r - R^y_{xuv}\nabla^2_r - i\omega_{xy}\nabla^2_{xy} - i\omega_{xy}\nabla^2_{xy} - \nabla_x^2 R^x_{yuv}\nabla_r - i\omega_{xy}\nabla_{xy}\right)s,$$

where we used the fact that $\omega$ is parallel, but otherwise just added the curvature terms. Expanding by the Leibniz rule, using symmetries, and collecting terms, this can be rewritten as

$$A^{xy}B^{uv}\nabla^4_{xuv}s$$
$$= A^{xy}B^{uv}\left(\nabla^4 - \nabla_x^2 R^x_{yuv}\nabla_r - R^x_{xuv}\nabla^2_r - R^y_{xuv}\nabla^2_r + R^x_{uxy}\nabla^2_r + R^y_{uxy}\nabla^2_r\right)$$
$$- 2i\omega_{xy}\nabla^2_{xy} - 2i\omega_{xy}\nabla^2_{xy} + \nabla_u(R^x_{uxy})\nabla_r - \nabla_x(R^y_{yuv})\nabla_r - i\omega_{xy}\nabla^2_{xy},$$

where the order of differentiation was interchanged for the term $A^{xy}B^{uv}R^x_{xuv}\nabla^2_r$. The second term of (30) can be rewritten as

$$2A^{xy}\nabla_y(B^{uv})\nabla^3_{xuv}s$$
$$= \frac{2}{3}A^{xy}\nabla_y(B^{uv})\left(\nabla^3_{xuv} + 2\nabla^3_{uuv} - 2R^x_{xuv}\nabla_r - i2\omega_{xy}\nabla_{xy}\right)s$$
$$= \frac{2}{3}A^{xy}\nabla_y(B^{uv})\left(\nabla^3_{xuv} + \nabla^3_{uuv} + \nabla^3_{uuv} - 2R^x_{xuv}\nabla_r - i3\omega_{xy}\nabla_{xy}\right)s.$$
Analagous to (30), the other term of the commutator $[\nabla^2, \nabla^2]$ yields
\[B^{uv}\nabla^2_{uv}A^{xy}\nabla^2_{xy}s = A^{xy}B^{uv}\nabla^4_{uvxy}s + 2B^{uv}\nabla_v(A^{xy})\nabla^3_{uxy}s + B^{uv}\nabla^2_v(A^{xy})\nabla^2_{xy}s, \tag{33}\]
where the second term can be rewritten as
\[2B^{uv}\nabla_v(A^{xy})\nabla^3_{uxy}s = \frac{2}{3}B^{uv}\nabla_v(A^{xy})(\nabla^3_{uxy} + 2\nabla^3_{xuy} - 2R^r_{uxy}\nabla_r - i2k\omega_{ux}\nabla_y)s \tag{34}\]
By subtracting (33) from (30), substituting (32) and (34), and collecting terms by order of covariant differentiation, one verifies the claimed symbols.

4 The Hitchin Connection

In this section, we study the Hitchin connection and calculate its curvature. We start by recalling the differential geometric construction of the Hitchin connection in geometric quantization. The results concerning this construction are all proved in [And5], to which the reader is referred for further details.

Consider a symplectic manifold $(M, \omega)$, equipped with a prequantum line bundle $L$, and assume that $H_1(M, \mathbb{C}) = 0$ and that the real first Chern class of $(M, \omega)$ is given by
\[c_1(M, \omega) = \lambda \frac{[\omega]}{2\pi}, \tag{35}\]
for some integer $\lambda \in \mathbb{Z}$. Further, assume that $M$ is of Kähler type, and let $J$ be a rigid and holomorphic family of Kähler structures on $(M, \omega)$, parametrized by some complex manifold $T$. Finally, assume that the family of Kähler structures admits a family of Ricci potentials $F$, and that it does not admit any non-constant holomorphic functions on $M$.

The prequantum space $\mathcal{H}^{(k)} = C^\infty(M, L^k)$ forms the fiber of a trivial, infinite-rank vector bundle over $T$,
\[\mathcal{H}^{(k)} = T \times \mathcal{H}^{(k)}.\]
If $\nabla^T$ denotes the trivial connection on $\mathcal{H}^{(k)}$, we consider a connection of the form
\[\nabla = \nabla^T + a, \tag{36}\]
where $a \in \Omega^1(T, \mathcal{D}(M, L^k))$ is a one-form on $T$ with values in the space of differential operators on sections of $L^k$, and we seek an $a$ for which the connection $\nabla$ preserves the quantum subspaces $H^{(k)}_\sigma = H^0(M_\sigma, L^k)$ of holomorphic sections inside each fiber of $\mathcal{H}^{(k)}$.

**Definition 4.1.** A Hitchin connection on the bundle $\mathcal{H}^{(k)}$ is a connection of the form (36) which preserves the fiberwise subspaces $H^{(k)}$.  

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It turns out that, with the assumptions made above, an explicit construction of a Hitchin connection can be given. For any vector field $V$ on $\mathcal{T}$, the operator $a(V)$ is of order two, with principal symbol $G(V)$ and lower order symbols given in terms of $G(V)$ and the Ricci potential. The precise statement is contained in the following theorem from [And5].

**Theorem 4.2.** Let $(M, \omega)$ be a prequantizable symplectic manifold with $H^1(M, \mathbb{R}) = 0$ and $c_1(M, \omega) = \lambda \left[ \frac{\omega}{2\pi} \right]$. Further, let $J$ be a rigid, holomorphic family of Kähler structures on $M$, parametrized by a complex manifold $\mathcal{T}$, admitting a family of Ricci potentials $F$ but no non-constant holomorphic functions on $M$. Then the expression

$$\nabla_V = \nabla_V^T + \frac{1}{4k + 2\lambda} (\Delta_{G(V)} + 2\nabla_{G(V)} dF + 4kV'[F])$$

defines a Hitchin connection in the bundle $\hat{H}^{(k)}$ over $\mathcal{T}$.

The characterizing feature of the operator-valued one-form $a$ is the fact that it satisfies

$$[\nabla^{0,1}, a(V)] s = -\frac{i}{2} \omega \cdot G(V) \cdot \nabla s,$$

for any section $s$ of $\hat{H}^{(k)}$. In fact, this property, and the fact that the Hitchin connection preserves the quantum subspaces $H^{(k)}$ inside $\mathcal{H}^{(k)}$, implies that these subspaces form a bundle $\hat{H}^{(k)}$ over $\mathcal{T}$, and this is part of the statement in Theorem 4.2.

Having reviewed the explicit differential geometric construction of a Hitchin connection in geometric quantization, we turn to the calculation of its curvature.

**Curvature of the Hitchin Connection**

Suppose the assumptions of Theorem 4.2 are satisfied, ensuring the existence of the Hitchin connection, and let us calculate its curvature. For this calculation, it will be convenient to rewrite the Hitchin connection slightly as

$$\nabla_V = \nabla_V^T + \frac{1}{4k + 2\lambda} b(V) + V'[F] \quad \text{with} \quad b(V) = \Delta_{G(V)} + 2\nabla_{G(V)} dF - 2\lambda V'[F],$$

essentially splitting the operator $a(V)$ into orders of $k$. In particular, the one-form $b$ does not involve the level $k$.

We shall divide the calculation of the curvature into a number of propositions. The first relies on Lemma 3.1 and Lemma 3.2, in combination with Proposition 2.6, to compute a commutator of fundamental importance to the curvature calculation.

**Proposition 4.3.** For any rigid family of Kähler structures, the commutator of $\Delta_{\tilde{G}(V)}$ and $\Delta_{\tilde{G}(W)}$, acting on $\mathcal{H}^{(k)}$, has the symbols

$$\sigma_3[\Delta_{\tilde{G}(V)}, \Delta_{\tilde{G}(W)}] = 0$$

$$\sigma_2[\Delta_{\tilde{G}(V)}, \Delta_{\tilde{G}(W)}] = -4ik \Theta(V, W)$$

$$\sigma_1[\Delta_{\tilde{G}(V)}, \Delta_{\tilde{G}(W)}] = \Delta_{G(V)} \delta G(W) - \Delta_{G(W)} \delta G(V) + \Delta_{\tilde{G}(V)} \delta \tilde{G}(W) - \Delta_{\tilde{G}(W)} \delta \tilde{G}(V)$$

$$- 4ik \delta (\Theta(V, W)) - \delta (\tilde{G}(V) - r) \tilde{G}(W) + \delta (\tilde{G}(W) - r) \tilde{G}(V)$$

$$\sigma_0[\Delta_{\tilde{G}(V)}, \Delta_{\tilde{G}(W)}] = -ik \delta \Theta(V, W) + ikr(\Theta(V, W)),$$

for any vector fields $V$ and $W$ on $\mathcal{T}$. 21
Proof. We verify that the third-order symbol vanishes. Using Proposition 2.6 and its conjugated version, which rely heavily on rigidity on the family, we get

$$
\sigma_3[\Delta_{\tilde{G}(V)}, \Delta_{\tilde{G}(W)}] = \sigma_3[\nabla^2_{\tilde{G}(V)}, \nabla^2_{\tilde{G}(W)}] \\
= 2S(\tilde{G}(V)\cdot \nabla \tilde{G}(W)) - 2S(\tilde{G}(W)\cdot \nabla \tilde{G}(V)) \\
= 2S(G(V)\cdot \nabla G(W)) - 2S(G(W)\cdot \nabla G(V)) \\
+ 2S(\tilde{G}(V)\cdot \nabla \tilde{G}(W)) - 2S(\tilde{G}(W)\cdot \nabla \tilde{G}(V)) \\
= 2\Gamma_3(V, W) - 2\bar{\Gamma}_3(W, V) + 2\Gamma_3(V, W) - 2\bar{\Gamma}_3(W, V) \\
= 0.
$$

To calculate the second-order symbol, we first notice that

$$
\sigma_2[\Delta_{G(V)}, \Delta_{G(W)}] = \sigma_2[\nabla^2_{G(V)}, \nabla^2_{G(W)}] + \sigma_2[\nabla^2_{\delta G(V)}, \nabla^2_{\delta G(W)}] - \sigma_2[\nabla^2_{\tilde{G}(W)}, \nabla^2_{\tilde{G}(V)}] \\
= \nabla^2_{G(V)}G(W) + \nabla^2_{\delta G(V)}G(W) + 2S(G(V)\cdot \nabla \delta G(W)) \\
- \nabla^2_{\tilde{G}(W)}G(V) - \nabla^2_{\tilde{G}(W)}G(V) - 2S(G(W)\cdot \nabla \delta G(V)) \\
= \Gamma_2(V, W) - \Gamma_2(W, V) \\
= 0.
$$

For the mixed-type terms of the second-order symbol, we first observe that

$$
G(V)\cdot \nabla \delta \tilde{G}(W) = G(V)^{xy}\nabla_y \tilde{G}(W)^{uv} \\
= G(V)^{xy}R_{yuv}^a \tilde{G}(W)^{uv} + G(V)^{xy}R_{yuv}^a \tilde{G}(W)^{uv} \\
= -G(V)^{xy}\tilde{G}(W) + G(V)^{xy}R_{yuv}^a \tilde{G}(W)^{uv}.
$$

(39)

Using this, we calculate that

$$
\sigma_2[\Delta_{G(V)}, \Delta_{G(W)}] \\
= \sigma_2[\nabla^2_{G(V)}, \nabla^2_{G(W)}] + \sigma_2[\nabla^2_{\delta G(V)}, \nabla^2_{\delta G(W)}] - \sigma_2[\nabla^2_{\tilde{G}(W)}, \nabla^2_{\tilde{G}(V)}] \\
= -4ikS(G(V)\cdot \omega \cdot \tilde{G}(W)) + 2S(G(V)^{xy}R_{yuv}^a \tilde{G}(W)^{uv} - 2S(\tilde{G}(W)^{uv}R_{xuv}^a G(V)^{xb}) \\
+ 2S(G(V)\cdot \nabla \delta \tilde{G}(W)) - 2S(\tilde{G}(W)\cdot \nabla \delta G(V)) \\
= -4ikS(G(V)\cdot \omega \cdot \tilde{G}(W)),
$$

where the last equality follows by inserting (39) and its conjugate. In total, this means that the second-order symbol is given by

$$
\sigma_2[\Delta_{\tilde{G}(V)}, \Delta_{\tilde{G}(W)}] = -4ikS(G(V)\cdot \omega \cdot \tilde{G}(W)) - 4ikS(\tilde{G}(V)\cdot \omega \cdot G(W)) \\
= -4ikS(\tilde{G}(V)\cdot \omega \cdot \tilde{G}(W)) \\
= -4ik \Theta(V, W).
$$

Next, we find the first-order symbol. First of all, we get

$$
\sigma_1[\Delta_{G(V)}, \Delta_{G(W)}] = \Delta_{G(V)}\delta G(W) - \Delta_{G(W)}\delta G(V).
$$
For the terms of mixed type on \( M \), we get the following computation,

\[
\sigma_1 [\Delta G(V), \Delta \tilde{G}(W)] \\
= \sigma_1 [\nabla^2 G(V), \nabla^2 \tilde{G}(W)] + \sigma_1 [\nabla \delta G(V), \nabla \delta \tilde{G}(W)] + \sigma_1 [\nabla G(V), \nabla \delta \tilde{G}(W)] - \sigma_1 [\nabla^2 \tilde{G}(W), \nabla \delta G(V)] \\
= -2ikG(V)^{xy} R_{xuy} \nabla_x (\tilde{G}(W)^{yu}) + 2ik \tilde{G}(W)^{yu} \nabla_x G(V)^{xy} \\
- G(V)^{xy} \nabla_x (R_{yuv} \tilde{G}(W)^{uv}) + \tilde{G}(W)^{uv} \nabla_v (R_{xuvy} G(V)^{xy}) \\
- \frac{1}{4} G(V)^{xy} \nabla_x (R_{yuv} \tilde{G}(W)^{uv}) + \tilde{G}(W)^{uv} \nabla_x G(V)^{xy} \\
= -\delta V G(V) \cdot \tilde{G}(W) - \delta \tilde{G}(W) \cdot G(V) + 2ik G(V) \cdot \tilde{G}(W) - \tilde{G}(W)^{uv} R_{xuvy} G(V)^{xy} \\
- \delta \tilde{G}(W) \cdot G(V) + 2ik \tilde{G}(W) \cdot G(V) - \tilde{G}(W)^{uv} R_{xuvy} G(V)^{xy} \\
= -\delta G(V) \cdot \tilde{G}(W) - \delta \tilde{G}(W) \cdot G(V) \\
- \delta G(V) \cdot \tilde{G}(W) + 2ik \tilde{G}(W) \cdot G(V) \\
+ \tilde{G}(W)^{uv} \nabla_u (R_{xuvy} G(V)^{xy}) + 2ik \tilde{G}(W) \cdot G(V) \\
= -\delta G(V) \cdot \tilde{G}(W) + 2ik \tilde{G}(W) \cdot G(V) \\
= -2i \delta (\Theta(V, W)) - \delta \tilde{G}(V) \cdot \tilde{G}(W) + \delta \tilde{G}(W) \cdot \tilde{G}(V) \\
+ \Delta G(V) \delta G(W) - \Delta \tilde{G}(W) \delta \tilde{G}(V) + \Delta G(V) \delta G(W) - \Delta \tilde{G}(W) \delta \tilde{G}(V),
\]

where we have indicated cancelling terms. Analogously, we compute

\[
\sigma_1 [\Delta \tilde{G}(V), \Delta G(W)] = \delta (\tilde{G}(V) \cdot r \tilde{G}(W)) - \delta (\tilde{G}(W) \cdot \tilde{G}(V)) - 4i \delta \delta (\tilde{G}(V) \cdot \tilde{G}(W)),
\]

so that finally

\[
\sigma_1 [\Delta \tilde{G}(V), \Delta \tilde{G}(W)] = -4i \delta (\Theta(V, W)) - \delta (\tilde{G}(V) \cdot r \tilde{G}(W)) + \delta (\tilde{G}(W) \cdot r \tilde{G}(V)) \\
+ \Delta G(V) \delta G(W) - \Delta \tilde{G}(W) \delta \tilde{G}(V) + \Delta G(V) \delta G(W) - \Delta \tilde{G}(W) \delta \tilde{G}(V).
\]

For the symbol of order zero, we first observe that

\[
\sigma_0 [\Delta G(V), \Delta \tilde{G}(W)] = 0.
\]

For the mixed terms, we get

\[
\sigma_0 [\Delta G(V), \Delta \tilde{G}(W)] \\
= \sigma_0 [\nabla G(V), \nabla \tilde{G}(W)] + \sigma_0 [\nabla \delta G(V), \nabla \delta \tilde{G}(W)] + \sigma_0 [\nabla G(V), \nabla \delta \tilde{G}(W)] - \sigma_0 [\nabla \tilde{G}(W), \nabla \delta G(V)] \\
= -k G(V)^{xy} R_{xuy} \tilde{G}(W)^{yu} - ik \delta G(V) \cdot \omega \delta \tilde{G}(W) \\
- i k \omega (G(V) \cdot \nabla \delta \tilde{G}(W)) + i k \omega (\tilde{G}(W) \cdot \nabla G(V)) \\
= -ik \delta G(V) \cdot \omega \tilde{G}(W) - ik \tilde{G}(W) \cdot \delta G(V) - ik \omega (G(V) \cdot \nabla \delta \tilde{G}(W)) \\
= -ik \delta G(V) \cdot \omega \tilde{G}(W) + i k r (G(V) \cdot \omega \tilde{G}(W)),
\]

the desired result.
where (39) was applied for the last equation, and this finally gives
\[ \sigma_0[\Delta_{G(V)}, \Delta_{G(W)}] = -ik\delta\Theta(V, W) + ikr(\Theta(V, W)). \]
This proves the proposition. \[\square\]

Before stating the next proposition, we introduce a one-form \(c \in \Omega^{1,0}(\mathcal{T}, C^\infty(M))\), with values in smooth functions on \(M\), which will play a central role in the calculations. It is defined by the expression,
\[ c(V) = -\Delta_{G(V)}F - dF \cdot G(V) \cdot dF - 2\lambda V'[F]. \tag{40} \]
This one-form serves, in fact, as the zero-order part of the Hitchin connection in metaplectic quantization, studied in [AGL], but as we will see, it also appears in the curvature of the Hitchin connection in the setting considered here. In the metaplectic case, the crucial property satisfied by this one-form is the following relation,
\[ \bar{\partial}c(V) = i\frac{\delta(G(V) \cdot \rho)}{2}, \tag{41} \]
which will also be useful to us. This is verified through the calculation
\[
2\bar{\partial}\Delta_{G(V)}F = 2\bar{\partial}\delta(G(V) \cdot dF)
= -2i\rho G(V) \cdot dF + 2\delta(G(V) \cdot \partial\bar{\partial}F)
= -2i\lambda \omega \cdot G(V) \cdot dF + 4\bar{\partial}\partial\bar{\partial}F \cdot G(V) \cdot dF - i\delta(G(V) \cdot \rho) + i\lambda \delta G(V) \cdot \omega
= -4\lambda \partial V'[F] - 2\bar{\partial}(dF \cdot G(V) \cdot dF) - i\delta(G(V) \cdot \rho),
\]
where we applied (22) twice for the third equality and (24) for the last equality.

As hinted above, the exterior derivative of the one-form \(c\) over \(\mathcal{T}\) appears in the curvature of the Hitchin connection. To see how, we must be able to recognize this derivative.

**Proposition 4.4.** The exterior derivative of the one-form \(c\) defined by (40) is given by
\[
\partial_r c(V, W) = b(V)W'[F] - b(W)V'[F]
\]
\[
\bar{\partial}c(V, W) = i\frac{\delta\Theta(V, W)}{4} - i\frac{r(\Theta(V, W)) - i\lambda \theta(V, W)}{4} - 2\lambda \partial_r \bar{\partial}c(F, V, W)
\]
for any vector fields \(V\) and \(W\) on \(\mathcal{T}\).

**Proof.** For the first statement, choose \(V\) and \(W\) so that \(V'\) and \(W'\) commute. Using the identity \(\Delta_{G(V)}F = \delta(G(V) \cdot dF)\) and the fact that the divergence operator on vector fields does not depend on the Kähler structure, we get
\[
\partial_r c(V, W) = V'[c(W)] - W'[c(V)]
= -\Delta_{V'[G(W)]}F - \Delta_{G(W)}V'[F] - 2dF \cdot G(W) \cdot dV'[F]
+ \Delta_{W'[G(V)]}F + \Delta_{G(V)}W'[F] + 2dF \cdot G(V) \cdot dW'[F]
= b(V)W'[F] - b(W)V'[F],
\]
where we used the the fact that \(W'[G(V)] = -W'V'[\bar{g}] = V'[G(W)]\), since the vector fields were chosen to commute.
Finally, the identities (potential satisfies the equation $\rho$ where rigidity of the family of Kähler structures was used for the last equality. The Ricci

As usual, the Hamiltonian vector field in the statement is determined by

Proof. To calculate the right-hand side of this, we use the fact that

Combining the identities above, we get

where rigidity of the family of Kähler structures was used for the last equality. The Ricci potential satisfies the equation $\rho = \lambda \omega + 2i \partial \bar{\partial} F$, so we get that

Finally, the identities (17) and (24) can be used to verify that

Combining the identities above, we get

For the first term of the last equality, we used the fact that repeated application of the divergence operator to a bivector field only depends on its symmetric part. This finishes the proof of the proposition.\[\Box\]

We will also need the following

**Proposition 4.5.** The one-form $c$ defined in (40) satisfies

for any vector fields $V$ and $W$ on $\mathcal{T}$.

Proof. As usual, the Hamiltonian vector field in the statement is determined by

To calculate the right-hand side of this, we use the fact that $c$ satisfies (41) to get

\[\frac{i}{2} W'\{\delta(G(V)\cdot dF)\} = W'\{\delta c(V)\} = \delta W'\{c(V)\} - \frac{i}{2} \omega \cdot G(W)\cdot dc(W).\]
On the other hand, we calculate

\[ W'[\delta(G(V), \rho)] = W'[\nabla_x(G(V) \cdot \rho_x)] \]
\[ = W'[\nabla^y \nabla^p G(V) \cdot \rho_{x_a} - W'[\nabla^y \nabla^p G(V) \cdot \rho_{y_z} + \delta(W'[G(V), \rho])] \]
\[ = \delta(W'[G(V)], \rho) + \delta(G(V), W'[\rho]), \]

where the last equality uses (8) and (9) and type considerations. Applying (15), we get

\[ 2\delta(G(V), W'[\rho]) = \delta(G(V), d(\delta G(W) \cdot \omega)) = \delta(G(V), \nabla \delta G(W) \cdot \omega) = \Delta_{G(V)} \delta G(W) \cdot \omega. \]

Combining the previous three identities, we find

\[ \partial_{\partial \tau} c(V, W) = \partial W'[c(V)] - \partial_{\partial \tau} W'[c(V)] \]
\[ = \frac{i}{2} \omega G(V) \cdot d c(W) - \frac{i}{2} \omega G(W) \cdot d c(V) + \frac{i}{4} \Delta_{G(W)} \delta G(V) \cdot \omega - \frac{i}{4} \Delta_{G(V)} \delta G(W) \cdot \omega, \]

for commuting vector fields \( V' \) and \( W'' \). Raising the index with \( \omega \) ends the proof. \( \square \)

The curvature calculation for the Hitchin connection proceeds with expressions for the one-form \( b \) defined in (38).

**Proposition 4.6.** The commutator of the operators \( b(V) \) and \( b(W) \), acting on sections of \( \mathcal{H}^{(k)} \), is a first-order operator with symbols given by

\[ \sigma_1 [b(V), b(W)] = 4iX_{\partial \tau} c(V, W) \]
\[ \sigma_0 [b(V), b(W)] = -2\lambda \partial_{\partial \tau} c(V, W), \]

for any vector fields \( V \) and \( W \) on \( T \).

**Proof.** The vanishing of the third-order symbol is essentially Proposition 4.3,

\[ \sigma_3 [b(V), b(W)] = \sigma_3 [\Delta_{G(V)}, \Delta_{G(W)}] = -4i k \Theta(V', W') = 0. \]

Vanishing of the second-order symbol is seen through the following calculation using Lemma 3.1 and Lemma 3.2,

\[ \sigma_2 [b(V), b(W)] \]
\[ = \sigma_2 [\Delta_{G(V)}, \Delta_{G(W)}] + 2\sigma_2 [\Delta_{G(V)}, \nabla_{G(W)} d F] - 2\sigma_2 [\Delta_{G(W)}, \nabla_{G(V)} d F] \]
\[ = 2\sigma_2 [\nabla_{G(V)}^2, \nabla_{G(W)} d F] - 2\sigma_2 [\nabla_{G(W)}^2, \nabla_{G(V)} d F] \]
\[ = 4 \mathcal{S}(G(V) \cdot \nabla(G(W)) d F) - 4 \mathcal{S}(G(W) \cdot \nabla(G(V)) d F) \]
\[ - 2d F \cdot G(W) \cdot \nabla G(V) + 2d F \cdot G(V) \cdot \nabla G(W) \]
\[ = 6 \mathcal{S}(G(V) \cdot \nabla G(W)) d F - 6 \mathcal{S}(G(W) \cdot \nabla G(V)) d F \]
\[ = 6 \Gamma_3 (V, W) d F - 6 \Gamma_3 (V, W) d F \]
\[ = 0. \]

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The first-order symbol is the first non-vanishing part. We split its calculation in two by first calculating

\[
\sigma_1[\Delta_{G(V)}, \nabla_{G(W)}] = \sigma_1[\Delta_{G(W)}, \nabla_{G(V)}] = \sigma_1[\Delta_{G(V)} + \sigma_1[\nabla_{G(W)} - \nabla_{G(V)}]
\]

This finishes the proof of the proposition.
Finally, to calculate the curvature of the Hitchin connection, we need the exterior derivative of the one-form \( b \) over \( T \). This is calculated in the following proposition.

**Proposition 4.7.** The two-form \( d_r b \in \Omega^2(T, D(M, L^k)) \) is given by

\[
d_r b(V, W) = -i \Delta_{\Theta(V, W)} - 2i \nabla_{\Theta(V, W)} df - 2\nabla_{G(V)} df - 2\nabla_{G(W)} df - 2\lambda \partial_r \bar{\partial}_r F(V, W)
\]

on sections of \( \hat{H}^{(k)} \), and by

\[
d_r b(V, W) = 2\nabla_{G(W)} W' df - 2\nabla_{G(V)} W' df - 2\lambda \partial_r \bar{\partial}_r F(V, W)
\]

when restricted to sections of \( \hat{H}^{(k)} \).

**Proof.** For any section \( s \) of \( \hat{H}^{(k)} \) over \( T \), we have the identity

\[
W[\Delta_{G(V)}]s = W[\nabla_x G(V) z y \nabla y s] = W[\nabla_x G(V) z y \nabla y s + \nabla_x W[G(V)] z y \nabla y s] = \Delta_{W[G(V)]} s,
\]

where the last equality follows from (9). In particular, for commuting \( V \) and \( W \), we get

\[
V[\Delta_{G(W)}] - W[\Delta_{G(V)}] = \Delta_{V[G(W)]} - \Delta_{W[G(V)]} = \Delta_{d_r G(V, W)} = -i \Delta_{\Theta(V, W)},
\]

where (13) was used for the last equation. Similarly, we calculate

\[
W[\nabla_{G(V)} df] s = \nabla_{G(V)} W' df + \nabla_{W[G(V)]} df s,
\]

so that finally

\[
d_r b(V, W) = V[b(W)] - W[b(V)] = -i \Delta_{\Theta(V, W)} + 2i \nabla_{\Theta(V, W)} df - 2\nabla_{G(V)} df - 2\nabla_{G(W)} df + 2\lambda \partial_r \bar{\partial}_r F(V, W).
\]

This proves the first statement of the proposition.

For the second statement, observe that the stated formulas on sections of \( \hat{H}^{(k)} \) and \( \hat{H}^{(k)} \) agree for \( \partial_r b \). For the case of \( \partial_r b \), suppose that \( s \) is a section of \( \hat{H}^{(k)} \) and observe that

\[
2\nabla^2 \Theta(V', W'') s = -\nabla^2_{G(W)} \omega(-G(V)) s = ik\omega(G(W) \omega G(V)) s = 4k\omega(V', W'') s.
\]

Using the conjugated version of (24), we also get that

\[
2\nabla \omega(V', W') df + 4\nabla \omega(V', W') df = \nabla G(V) \omega s - \nabla G(W) s + 2\nabla G(V) \omega - \nabla G(W) \omega \omega df s = 2i \nabla_{G(W)} df s.
\]

Finally, plugging these two expressions into (43), we get

\[
d_r b(V', W'') s = -2\nabla_{G(V)} W' df - 2i \lambda \omega(V', W'') s + 2\lambda \partial_r \bar{\partial}_r F(V', W'') s,
\]

which finishes the proof of the proposition. \( \square \)
With the above propositions at hand, calculating the curvature of the Hitchin connection is a straightforward matter.

**Theorem 4.8.** The curvature of the Hitchin connection acts by

\[ F^{2,0}_{\nabla} = \frac{k}{(2k + \lambda)^2} P_k(\partial_r c) \quad F^{1,1}_{\nabla} = -\frac{ik}{2k + \lambda}(\theta - 2i(\partial_r \bar{\partial}_r F)) \quad F^{0,2}_{\nabla} = 0, \]

on sections of the bundle \( \hat{H}^{(k)} \).

**Proof.** For commuting vector fields \( V \) and \( W \), we get

\[ F_{\nabla}(V, W) = \left[ \nabla_V, \nabla_W \right] = \frac{[b(V), b(W)]}{(4k + 2\lambda)^2} + \frac{d_r b(V, W) + [b(V), W'[F]] - [b(W), V'[F]]}{4k + 2\lambda} - \partial_r \bar{\partial}_r F(V, W). \]

Now Proposition 4.4 can be applied to see that

\[ [b(V), W'[F]] - [b(W), V'[F]] = 2\nabla_{G(V), W'[F]} - 2\nabla_{G(W), V'[F]} + \partial_r c(V, W), \]

and combining the above with Proposition 4.6 and Proposition 4.7 yields the following expression for the \( (2,0) \)-part of the curvature

\[ F^{2,0}_{\nabla}(V, W) = \frac{[b(V), b(W)]}{(4k + 2\lambda)^2} + \frac{\partial_r c(V, W)}{4k + 2\lambda} = \frac{k}{(2k + \lambda)^2} P_k(\partial_r c(V, W)), \]

where \( P_k \) denotes the prequantum operator defined in (26). For the \( (1,1) \)-part of the curvature, Proposition 4.7 yields

\[ F^{1,1}_{\nabla}(V, W) = \frac{\bar{\partial}_r b(V, W)}{4k + 2\lambda} - \partial_r \bar{\partial}_r F(V, W) = -\frac{ik}{2k + \lambda}(\theta - 2i(\partial_r \bar{\partial}_r F)). \]

Finally, the \( (0,2) \)-part of the curvature clearly vanishes, and the theorem is proved. \( \square \)

The fact that the \( (1,1) \)-part of the curvature is a zeroth-order operator, combined with the fact that the Hitchin connection preserves the subbundle of quantum spaces \( \hat{H}^{(k)} \) implies that the \( (1,1) \)-part must take values in holomorphic and hence constant functions on \( M \). This was, however, already known from Proposition 2.8 for the particular expression we found in Theorem 4.8. Also, being a first-order operator, it is no surprise that the \( (2,0) \)-part of the curvature acts by prequantum operator. After all, prequantum operators of the form (28) are the only first-order operators which preserve the holomorphic sections.

**The Hitchin-Witten Connection**

In the previous section, we saw how the higher-order symbols of the curvature of the Hitchin connection vanished and left a first-order operator with a relatively simple expression. For the \( (2,0) \)-part, the higher order symbols vanished for general reasons, related to rigidity of the family of Kähler structures, but for the \( (1,1) \)-part, the curvature reduced to a zero-order operator only when restricted the subbundle \( \hat{H}^{(k)} \) of quantum spaces.
It turns out that, with a slight modification of the formula, we can achieve a cancellation of higher-order terms for the (1,1)-part of the connection defined on the whole bundle $\hat{\mathcal{H}}^{(k)}$ of prequantum spaces over $\mathcal{T}$, but still maintain the vanishing of higher-order symbols for the $(2,0)$ and $(0,2)$-parts.

In this section, we let $t \in \mathbb{C}$ be any complex number with integer real part, $k = \text{Re}(t) \in \mathbb{Z}$, and we consider the connection on $\hat{\mathcal{H}}^{(k)}$ given by

$$\bar{\nabla}_V = \nabla^c_V + \frac{1}{2t}b(V) - \frac{1}{2t}\bar{b}(V) + V[F],$$

where $\bar{b}(V)$ has the conjugated symbols of $b(V)$ so that

$$b(V) = \Delta_{G(V)} + 2\nabla_{G(V)}dF - 2\lambda V'[F] \quad \text{and} \quad \bar{b}(V) = \Delta_{\bar{G}(V)} + 2\nabla_{\bar{G}(V)}dF - 2\lambda V''[F].$$

We will refer to $\bar{\nabla}$ as the Hitchin-Witten connection. It is a generalization of the connection for quantum Chern-Simons theory with complex gauge group $\text{SL}(n, \mathbb{C})$ discussed by Witten in [Wit3], where he arrives at exactly the formula (44). This relation will be further explored in the final section of the paper.

We will prove that the curvature of $\bar{\nabla}_V$, acting on sections of $\hat{\mathcal{H}}^{(k)}$, has essentially the same expression as the curvature of $\nabla$, acting on sections of $\hat{\mathcal{H}}^{(k)}$. We start with the following.

**Proposition 4.9.** The commutator of the operators $b(V)$ and $\bar{b}(W)$, acting on sections of $\hat{\mathcal{H}}^{(k)}$, is a second-order operator with symbols given by

$$\sigma_2[b(V), \bar{b}(W)] = -4ik\Theta(V', W'')$$
$$\sigma_1[b(V), \bar{b}(W)] = -4ik\delta\Theta(V', W'') - 8ik\Theta(V', W'')dF$$
$$\sigma_0[b(V), \bar{b}(W)] = -\frac{ik}{4}\delta\Theta(V', W'') - \frac{ik}{4}r(\Theta(V', W'')) - 2ik\lambda\theta(V', W'')$$

and furthermore

$$\sigma_0[b(V), \bar{b}(W)] = 2\bar{b}(V)W''[F] + 2t\bar{b}(W)V'[F] - 4ik\lambda\theta(V', W''),$$

for any vector fields $V$ and $W$ on $\mathcal{T}$.

**Proof.** The third-order symbol vanishes, as Proposition 4.3 readily gives

$$\sigma_3[b(V), \bar{b}(W)] = \sigma_3[\Delta_{G(V)}, \Delta_{\bar{G}(W)}] = 0.$$

The calculation of the second-order symbol is also straightforward,

$$\begin{align*}
\sigma_2[b(V), \bar{b}(W)] &= \sigma_2[\Delta_{G(V)}, \Delta_{\bar{G}(W)}] + 2\sigma_2[\nabla^2_{G(V)}, \nabla_{\bar{G}(W)}dF] - 2\sigma_2[\nabla^2_{\bar{G}(W)}, \Delta_{G(V)}dF] \\
&= -4ik\Theta(V', W'') + 4\mathcal{S}(G(V)\cdot \partial\bar{\partial}\bar{F}\cdot G(W)) - 4\mathcal{S}(\bar{G}(W)\cdot \bar{\partial}\partial\bar{F}\cdot G(V)) \\
&= -4ik\Theta(V', W'').
\end{align*}$$
For the first-order symbol, we first calculate $\sigma_1 [\Delta_{G(V)}, \nabla_{\bar{G}(W)} - dF]$. We get

$$2 \sigma_1 [\nabla_{\bar{G}(V)}, \nabla_{\bar{G}(W)} - dF] = 2 \nabla_{\bar{G}(V)} (\bar{G}(W) - dF) - 4i k G(V) \cdot \omega \cdot \bar{G}(W) - dF + 2 G(V)^{xy} R_{uv}^{a} G(W)^{uv} dF_v$$

and

$$2 \sigma_1 [\nabla_{\delta G(V)}, \nabla_{\bar{G}(W)} - dF] = 2 \nabla_{\delta G(V)} (\bar{G}(W) - dF) - 2 dF \cdot \bar{G}(W) \cdot \nabla \delta G(V)$$

$$= 2 \nabla_{\delta G(V)} (\bar{G}(W) - dF) + 2i dF \cdot \bar{G}(W) \cdot \rho \cdot G(V) - 2d F \cdot \bar{G}(W)^{uv} R_{uv}^{a} G(V)^{xy},$$

and when combining these, we get

$$2 \sigma_1 [\Delta_{G(V)}, \nabla_{\bar{G}(W)} - dF] = 2 \Delta_{G(V)} (\bar{G}(W) - dF) - 4i k G(V) \cdot \omega \cdot \bar{G}(W) - dF - 2i G(V) \cdot \rho \cdot \bar{G}(W) - dF.$$

The first term of this expression can be rewritten as

$$2 \Delta_{G(V)} (\bar{G}(W) - dF) = 2 \delta (G(V) \cdot \nabla (\bar{G}(W) - dF)) = 2 \delta (G(V) \cdot \bar{\delta F} \cdot \bar{G}(W))$$

$$= -i \delta (G(V) \cdot \rho \cdot \bar{G}(W)) + i \lambda \delta (G(V) \cdot \omega \cdot \bar{G}(W)).$$

For the first-order symbol, we must also understand $\sigma_1 [\nabla_{\bar{G}(V)}, W''[F]]$, which yields

$$2 \sigma_1 [\nabla_{\bar{G}(V)}, W''[F]] = 4 G(V) \cdot d W''[F] = -i \delta (G(V) \cdot \omega \cdot \bar{G}(W) - 2i G(V) \cdot \omega \cdot \bar{G}(W) - dF.$$

Altogether, the previous identities yield

$$2 \sigma_1 [\Delta_{G(V)}, \nabla_{\bar{G}(W)} - dF] = 2 \lambda \sigma_1 [\nabla_{\bar{G}(V)}, W''[F]]$$

$$= 2 \Delta_{G(V)} (\bar{G}(W) - dF) - 4i k G(V) \cdot \omega \cdot \bar{G}(W) - dF - 2i G(V) \cdot \rho \cdot \bar{G}(W) - dF$$

$$- i \delta (G(W) \cdot \omega \cdot \bar{G}(W)) + 2i \lambda G(V) \cdot \omega \cdot \bar{G}(W) - dF$$

$$= -i \delta (G(V) \cdot \rho \cdot \bar{G}(W)) - 4i k G(V) \cdot \omega \cdot \bar{G}(W) - dF + 4 G(V) \cdot \bar{\delta F} \cdot \bar{G}(W) - dF$$

$$+ 2i \lambda \delta (V', W'')$$

$$= \delta (G(V) \cdot \rho \cdot \bar{G}(W)) - 4i k G(V) \cdot \omega \cdot \bar{G}(W) - dF + 4d F \cdot \bar{G}(W) \cdot \nabla (G(V) - dF)$$

$$+ 2i \lambda \delta (V', W'').$$

A completely analogous computation shows

$$2 \sigma_1 [\Delta_{\bar{G}(W)}, \nabla_{\bar{G}(V)} - dF] = 2 \lambda \sigma_1 [\nabla_{\bar{G}(W)}, V''[F]]$$

$$= \delta (G(W) \cdot \rho \cdot G(V)) - 4i k G(W) \cdot \omega \cdot G(V) - dF + 4d F \cdot G(V) \cdot \nabla (G(W) - dF)$$

$$+ 2i \lambda \delta (V', W'').$$
Finally, combining all of the above with Proposition 4.3, we get

\[
\sigma_1 [b(V), b(W)] = \sigma_1 [\Delta_{G(V)}, \Delta_{G(W)}] + 4 [G(V) \cdot dF, \tilde{G}(W) \cdot dF] \\
+ 2\sigma_1 [\Delta_{G(V)}, \nabla_{G(W)} \cdot dF] - 2\lambda \sigma_1 [\nabla^2_{G(V)}, W'' [F]] \\
- 2\sigma_1 [\Delta_{G(W)}, \nabla_{G(V)} \cdot dF] + 2\lambda \sigma_1 [\nabla^2_{G(W)}, V' [F]] \\
= \sigma_1 [\Delta_{G(V)}, \Delta_{G(W)}] - 8ik \Theta(V', W'') \cdot dF \\
+ \delta(G(V) \cdot r) \cdot \tilde{G}(W) - \delta(G(W) \cdot r) \cdot G(V) \\
= -4ik \delta(\Theta(V', W'') - 8ik \Theta(V', W'') \cdot dF
\]

It only remains to calculate the zero-order symbol. First we compute

\[
2\sigma_0 [\nabla^2_{G(V)}, \nabla_{G(W)} \cdot dF] = -2ik \omega(G(V) \cdot \nabla(dF, \tilde{G}(W))) \\
= -2ik \omega(G(V) \cdot \tilde{\omega}F \cdot \tilde{G}(W)) \\
= -2ik \partial \tilde{\omega}F(G(V) \cdot \omega \cdot \tilde{G}(W)) \\
= -\kappa \rho(G(V) \cdot \omega \cdot \tilde{G}(W)) + \kappa \lambda \omega(G(V) \cdot \omega \cdot \tilde{G}(W)) \\
= -ik r(\Theta(V', W'')) - 4ik \lambda \theta(V', W''),
\]

and similarly, we have

\[
2\sigma_0 [\nabla^2_{G(W)}, \nabla_{G(V)} \cdot dF] = ik r(\Theta(V', W'')) + 4ik \lambda \theta(V', W''),
\]

Furthermore, we calculate

\[
2\sigma_0 [\nabla_{\delta G(V)}, \nabla_{G(W)} \cdot dF] - 2\sigma_0 [\nabla_{\delta G(W)}, \nabla_{G(V)} \cdot dF] \\
= -2ik \delta G(V) \cdot \omega \cdot \tilde{G}(W) \cdot dF + 2ik \delta G(W) \cdot \omega \cdot G(V) \cdot dF \\
= -4ik \delta(\Theta(V', W'')) \cdot dF,
\]

and finally

\[
4\sigma_0 [\nabla_{G(V)} \cdot dF, \nabla_{G(W)} \cdot dF] = -4ik \delta F \cdot \Theta(V', W'') \cdot dF.
\]

All of the above contribute to the zero-order symbol, but we also need to calculate \(b(V)W'' [F]\) and \(b(W)V' [F]\). First we compute

\[
4\Delta_{G(V)} W'' [F] \\
= 4\delta(G(V) \cdot dW'') [F] \\
= -i\delta(G(V) \cdot \omega \cdot \delta \tilde{G}(W)) - 2i\delta(G(V) \cdot \omega \cdot \tilde{G}(W)) \cdot dF \\
= i\delta \delta(G(W) \cdot \omega \cdot G(V)) - 2i\delta \tilde{\omega}F(G(V) \cdot \omega \cdot \tilde{G}(W)) - 2i\delta G(V) \cdot \omega \cdot \tilde{G}(W) \cdot dF \\
= -i\delta(\Theta(V', W'')) - ir(\Theta(V', W'')) - 4i\lambda \theta(V', W'') - 2i\delta G(V) \cdot \omega \cdot \tilde{G}(W) \cdot dF,
\]

where the last equality used the fact that the double divergence of a bivector field only depends on its symmetric part. Similarly, we find

\[
4\Delta_{G(W)} V' [F] = i\delta \delta(\Theta(W'', V')) + ir(\Theta(W'', V')) + 4i\lambda \theta(W'', V') + 2i\delta \tilde{G}(W) \cdot \omega \cdot G(V) \cdot dF.
\]
This proves the proposition.

The curvature of the Hitchin-Witten connection

\[ 4\nabla_{G(V)} df W''[F] = -idF \cdot G(V) \cdot \omega \cdot \delta \tilde{G}(W) - 2idF \cdot \Theta(V', W'') \cdot dF \]

\[ 4\nabla_{\tilde{G}(W)} df V'[F] = +idF \cdot \tilde{G}(W) \cdot \omega \cdot \delta \tilde{G}(V) + 2idF \cdot \Theta(W'', V') \cdot dF, \]

so finally, we conclude that

\[ 4b(V)W''[F] = -i\delta (\Theta(V', W'')) - i r(\Theta(V', W'')) - 4i\lambda \theta(V', W'') \]

\[ - 4i\delta \Theta(V', W'') \cdot dF - 2idF \cdot \Theta(V', W'') \cdot dF - 8\lambda V'[F]W''[F]. \]

Using this identity, conjugation yields

\[ \tilde{b}(W)V'[F] = \overline{b(W)V'[F]} = b(V)W''[F], \]

so the contributions of \( \tilde{b}(W)V'[F] \) and \( b(V)W''[F] \) to the zero-order symbol of the commutator cancel. This finally gives

\[ \sigma_0[b(V), \tilde{b}(W)] = \sigma_0[\Delta G(V), \Delta G(W)] + 2\sigma_0[\Delta G(V), \nabla_{G(W)} df] - 2\sigma_0[\Delta G(W), \nabla_{G(V)} df] \]

\[ + 4\sigma_0[\nabla_{G(V)} df, \nabla_{G(W)} df] - 2\lambda \tilde{b}(V)W''[F] + 2\lambda \tilde{b}(W)V'[F] \]

\[ = -i\delta \Theta(V', W'') - i r(\Theta(V', W'')) - 8i\lambda \theta(V', W'') \]

\[ - 4i dF \cdot \Theta(V', W'') \cdot dF - 4i \delta \Theta(V', W'') \cdot dF, \]

as claimed in the proposition. To prove the final statement of the proposition, we note from the above that

\[ 2\tilde{b}(V)W''[F] + 2t \tilde{b}(W)V'[F] \]

\[ = -i\delta \Theta(V', W'') - i r(\Theta(V', W'')) - 4i\lambda \theta(V', W'') \]

\[ - 4i dF \cdot \Theta(V', W'') \cdot dF - 4i \delta \Theta(V', W'') \cdot dF - 4\lambda V'[F]W''[F], \]

which gives

\[ \sigma_0[b(V), \tilde{b}(W)] = 2\tilde{b}(V)W''[F] + 2t \tilde{b}(W)V'[F] - 4i\lambda \theta(V', W'') + 4\lambda V'[F]W''[F] \]

\[ = 2\sigma_0[b(V), W''[F]] + 2t \sigma_0[\tilde{b}(W), V'[F]] - 4i \lambda \theta(V', W''). \]

This proves the proposition.

With the bulk of computations encoded in Proposition 4.9, we can calculate the curvature of the Hitchin-Witten connection. This is the content of the following theorem.

**Theorem 4.10.** The curvature of the Hitchin-Witten connection \( \nabla \) acts as a first-order operator with symbols

\[ \sigma_{1F_{\nabla}}^{2,0} = \frac{i}{t^2} \chi' \partial_{\bar{r}} \partial_{\bar{c}} \quad \sigma_{1F_{\nabla}}^{1,1} = 0 \quad \sigma_{1F_{\nabla}}^{0,2} = -i \frac{\partial'}{t^2} \chi'' \partial_{\bar{r}} \partial_{\bar{c}} \]

\[ \sigma_{0F_{\nabla}}^{2,0} = \frac{t - \lambda}{2t^2} \partial_{\bar{r}} \partial_{\bar{c}} \quad \sigma_{0F_{\nabla}}^{1,1} = \frac{i k \lambda}{t l} (\theta - 2i \partial_{\bar{r}} \partial_{\bar{c}} F) \quad \sigma_{0F_{\nabla}}^{0,2} = -\frac{\bar{t} + \lambda}{2t^2} \partial_{\bar{r}} \partial_{\bar{c}} \]

on sections of the bundle \( \hat{H}^{(k)} \) over \( \mathcal{T} \).
Proof. The calculation of the (2,0)-part proceeds as in the proof of Theorem 4.8 and yields a first-order operator with symbols given by

\[ \sigma_1(F^{2,0}_\Theta(V, W)) = \frac{i}{t^2} X^\prime_{\partial \tau c}(V, W) \quad \text{and} \quad \sigma_0(F^{2,0}_\Theta(V, W)) = \frac{t - \lambda}{2t^2} \partial \tau c(V, W), \]
as claimed in the theorem.

The calculation of the (0,2)-part is completely analogous. We have the conjugate statements to Proposition 4.6,

\[ \sigma_1(\bar{b}(V), \bar{b}(W)) = -4iX''_{\partial \tau \bar{c}(V, W)} \quad \text{and} \quad \sigma_0(\bar{b}(V), \bar{b}(W)) = -2\lambda \partial \tau \bar{c}(V, W), \]

and Proposition 4.7,

\[ d_\tau \bar{b}(V, W) = i\Delta_\Theta(V, W) + 2i\nabla_\Theta(V, W) \cdot dF - 2\nabla \tilde{G}(V) \cdot dV[F] - 2\lambda \partial \tau \tilde{\partial}_\tau F(V, W), \]
and these can be combined with the conjugate of Proposition 4.4 to compute

\[ \sigma_1(F^{0,2}_\Theta(V, W)) = -\frac{i}{t^2} X''_{\partial \tau \bar{c}(V, W)} \quad \text{and} \quad \sigma_0(F^{2,0}_\Theta(V, W)) = -\frac{t + \lambda}{2t^2} \partial \tau c(V, W). \]

It only remains to compute the (1,1)-part, and we get

\[ 4t \tilde{i} F^{1,1}_\Theta(V', W'') = -[\bar{b}(V), \bar{b}(W)] + 2t d_\tau b(V', W'') + 2t d_\tau b(V', W'') \]
\[ + 2\tilde{i} \bar{b}(V'), W''[F] + 2\tilde{i} \bar{b}(W), V'[F]. \]

By Proposition 4.9 and the above expression for \( d_\tau \bar{b} \), the second-order symbol vanishes,

\[ 4t \tilde{i} \sigma_2(F^{1,1}_\Theta(V', W'')) = -\sigma_2(\bar{b}(V), \bar{b}(W)] + 2\tilde{i} \sigma_2(d_\tau b(V', W'')) - 2t \sigma_2(d_\tau \bar{b}(V', W'')) \]
\[ = 4ik \Theta(V', W'') - 2t(t + \tilde{i}) \Theta(V', W'') = 0. \]

The same is true for the first-order symbol

\[ 4t \tilde{i} \sigma_1(F^{1,1}_\Theta(V', W'')) = -\sigma_1(\bar{b}(V), \bar{b}(W)] + 2\tilde{i} \sigma_1(d_\tau b(V', W'')) - 2t \sigma_1(d_\tau \bar{b}(V', W'')) \]
\[ + 2\tilde{i} \sigma_1(\bar{b}(V), W''[F]) + 2t \sigma_1(\bar{b}(W), V'[F]) \]
\[ = 4ik \delta \Theta(V', W'') + 8ik \Theta(V', W'') \cdot dF \]
\[ - 2i(t + \tilde{i}) \delta \Theta(V', W'') - 4i(t + \tilde{i}) \Theta(V', W'') \cdot dF \]
\[ - 4ik G(V) \cdot W''[F] - 4ik G(W) \cdot dV'[F] \]
\[ + 4ik G(V) \cdot W''[F] + 4ik G(W) \cdot V'[F] \]
\[ = 0. \]

Finally, the last statement of Proposition 4.9 gives the zeroth-order part

\[ 4t \tilde{i} \sigma_0(F^{1,1}_\Theta(V', W'')) = 4ik \lambda \Theta(V', W'') + 4\lambda(t + \tilde{i}) \partial \tau \tilde{\partial}_\tau F(V', W'') \]
\[ = 4ik \lambda \Theta(V', W'') - 2i \partial \tau \tilde{\partial}_\tau F(V', W''). \]

This completes the proof of the theorem. \( \square \)
Projective Flatness

In case the Hitchin connection $\nabla$ in the bundle $\hat{H}^{(k)}$ over $\mathcal{T}$ is projectively flat, the parallel translation maps along homotopic curves are equal up to scale. Thus, if the parameter space $\mathcal{T}$ is simply connected, the Hitchin connection gives a canonical identification of the projectivized quantum spaces associated with different complex structures. In this sense, the quantization is independent of the complex structure.

The following theorem is an immediate consequence of the explicit curvature calculations of Theorem 4.8 and Theorem 4.10.

**Theorem 4.11.** The connections $\nabla$ and $\tilde{\nabla}$, acting on $\hat{H}^{(k)}$ and $\hat{H}^{(k)}$, respectively, are both projectively flat if and only if the holomorphic vector field on $\mathcal{M}$ given by

$$X'_{\partial Tc(V,W)} = i \Delta G(W)\delta G(V) - i \Delta G(V)\delta G(W) + i G(W)\cdot dc(V) - i G(V)\cdot dc(W)$$

vanishes, for all vector fields $V$ on $\mathcal{T}$.

**Proof.** Projective flatness amounts to the curvature being a two-form on $\mathcal{T}$ with values in constant functions on $\mathcal{M}$. In particular, it takes values in zeroth-order differential operators. Being proportional to the first-order symbol of the $(2,0)$-part of the curvature of both $\nabla$ and $\tilde{\nabla}$, the vector field $X'_{\partial Tc(V,W)}$ will of course vanish if either of these are projectively flat.

Conversely, suppose that $X'_{\partial Tc(V,W)}$ vanishes, which means that the curvature of $\nabla$ is a zeroth-order operator, and since it preserves the subbundle $\hat{H}^{(k)}$ of holomorphic sections of $\mathcal{L}^k$ it takes values in holomorphic functions on $\mathcal{M}$. By assumption, such functions are constants, which proves that $\nabla$ is projectively flat. In particular, the form $\partial Tc$ takes values in constant functions on $\mathcal{M}$, which is also the case for $\theta - 2i\partial_r\bar{\partial}_r F$, as already noted in Proposition 2.8. By Theorem 4.10, this proves that the Hitchin-Witten connection $\tilde{\nabla}$ is projectively flat.

We observe that our main Theorem 1.1 follows as an immediate corollary of the theorem above. This ends the general discussion of the Hitchin and the Hitchin-Witten connection. In the final section, we will apply the results to the setting originally motivating the study of these connections.

5 Quantum Chern-Simons Theory

In this section, we apply the results of previous sections to Chern-Simons theory with either compact or complex gauge group. To be specific, let $G$ denote the real Lie group $\text{SU}(n)$, sitting as the maximal compact subgroup of its complexification $G_{\mathbb{C}}$, which can be identified with $\text{SL}(n, \mathbb{C})$. Indeed, if $\mathfrak{g}$ denotes the Lie algebra of $G$, consisting of skew-Hermitian traceless matrices, then its complexification $\mathfrak{g}_{\mathbb{C}}$ can be identified with the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ of traceless matrices through the unique splitting of any complex matrix into Hermitian and skew-Hermitian parts. Furthermore, let $\langle \cdot, \cdot \rangle$ be an invariant inner product on $\mathfrak{g}_{\mathbb{C}}$, such as $-\frac{1}{16\pi^2}\text{Tr}$, which is normalized so that $\frac{1}{8}\langle \theta \wedge [\theta \wedge \theta] \rangle$ represents an integral generator of $H^3(G, \mathbb{R})$, where $\theta$ is the Maurer-Cartan form on $G$. 

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Let \( \Sigma \) be a closed surface of genus \( g \geq 2 \), and let \( \Sigma_p \) be the surface obtained by puncturing \( \Sigma \) at a point \( p \in \Sigma \). Fix an element \( d \in \mathbb{Z}/n\mathbb{Z} \) and a small loop \( \gamma \) around the puncture, and consider the moduli spaces

\[
M = \text{Hom}_d(\pi_1(\Sigma_p), G)/G \\
M_c = \text{Hom}_d^+(\pi_1(\Sigma_p), G_c)/G_c, 
\]

of representations of the fundamental group mapping \( \gamma \) to the element \( e^{2\pi id/n}I \) in the common center of \( G \) and \( G_c \). For \( M_c \), we restrict to reductive representations, which is indicated by a plus in the notation. The subspaces \( M^s \) and \( M^s_c \) corresponding to irreducible connections are of particular relevance to us. For these spaces, we have an embedding \( M^s \subset M^s_c \), since any irreducible unitary representation can only be conjugate to another such by a unitary transformation.

The spaces \( M \) and \( M_c \) also have a gauge theoretic realization as moduli spaces of flat connections. Let \( P \) denote the trivial principal bundle over \( \Sigma_p \) with structure group \( G \). Fix an element \( a_d \in g \) such that \( \exp(2\pi a_d) = e^{2\pi id/n}I \), and denote by \( F \) the space of flat connections on \( P \) which are all equal to \( a_d d\theta \) for some fixed polar coordinate system, with angular coordinate \( \theta \), in a disc around the puncture. These connections are said to be in temporal gauge near the puncture and clearly have holonomy \( e^{2\pi it/n}I \) around \( \gamma \). If \( \mathcal{G} \) denotes the space of gauge transformations equal to the identity in a neighbourhood of the puncture, then the holonomy representation then gives an identification

\[
M \cong F/\mathcal{G}. 
\]

Similarly, let \( F^+_c \) be the space of flat reductive connections on the trivial \( G_c \)-bundle \( P_c \) over \( \Sigma_p \), and let \( \mathcal{G}_c \) be the space of gauge transformations, both spaces with restrictions similar to the above around the puncture. The holonomy representation then gives an identification

\[
M_c \cong F^+_c/\mathcal{G}_c. 
\]

By a more careful selection of connections and gauge transformations, using exponential decay in weighted Sobolev norms around the puncture [DW], the subsets \( M^s \) and \( M^s_c \) corresponding to irreducible connections can be endowed with the structure of smooth manifolds. The tangent space \( T_{[A]} M^s_c \) at a connection \( A \), is given by the compactly supported first cohomology \( H^1_A(\Sigma_p, g_c) \), with values in the adjoint bundle and exterior derivative \( d_A \) induced by \( A \). We can then define a complex symplectic form \( \omega_c \) by

\[
\omega_c([\alpha], [\beta]) = -4\pi \int_\Sigma \langle \alpha \wedge \beta \rangle, 
\]

In complete analogy, the tangent space \( T_{[A]} M^s \) is given by the cohomology \( H^1_A(\Sigma_p, g) \). The formula (46) defines in this case a real symplectic form on \( M^s \subset M^s_c \), which is of course just the restriction of \( \omega_c \).

If \( t \in \mathbb{C} \) is a complex number with integer real part, \( k = \text{Re } t \in \mathbb{Z} \), we wish to quantize the space \( M^s_c \) with respect to the real symplectic form

\[
\omega_t = \frac{1}{2} (t \omega_c + t^* \omega_c),
\]

as well as the space \( M^s \) equipped with the restriction of \( \omega_t \), which is given by \( k \omega \).
Suppose that $A \in \mathcal{F}_c^+$ is a connection, and that $g: \Sigma_p \to G_c$ is a gauge transformation in $G_c$. Since $G_c$ is simply connected and hence 2-connected, we can choose a homotopy \( \tilde{g}: \Sigma_p \times [0,1] \to G_c \) from the trivial gauge transformation to $g$, keeping identity values fixed. If $\pi: \Sigma_p \times [0,1] \to \Sigma_p$ denotes projection onto the first factor, we can pull back $A$ to a connection $\tilde{A} = \pi^* A$ on $\Sigma_p \times [0,1]$ and consider the Chern-Simons form

$$\alpha_c(\tilde{A}) = (\tilde{A} \wedge F_{\tilde{A}}) - \frac{1}{6}(\tilde{A} \wedge [\tilde{A} \wedge \tilde{A}]),$$

which will of course be complex-valued in general. Using the complex number $t \in \mathbb{C}$ from before, we shall consider the real form

$$\alpha_t(\tilde{A}) = \frac{1}{2}(t\alpha_c(\tilde{A}) + \overline{t\alpha_c(\tilde{A})}),$$

and the Chern-Simons cocycle given by

$$\Theta_t(A, g) = \exp \left( 2\pi i \int_{\Sigma_p \times [0,1]} \alpha_t(\tilde{A}^g) \right) \in U(1),$$

where $\tilde{A}^g$ denotes the gauge-transformed connection. Since $A$ is flat and in temporal gauge, the Chern-Simons form vanishes in a neighbourhood of $\{p\} \times [0,1]$ and the integral converges. Furthermore, since the real part of $t$ is an integer, the expression is independent of the choice of homotopy $\tilde{g}$ and defines a map $\Theta: \mathcal{F}_c \times G_c \to U(1)$, which can be shown to satisfy the cocycle relation

$$\Theta_t(A^g, h) \Theta_t(A, g) = \Theta_t(A, gh).$$

This cocycle can be used to lift the action by the gauge group $\mathcal{G}_c$ on $\mathcal{F}_c^+$ to the trivial line bundle $\mathcal{F}_c^+ \times \mathbb{C}$ by

$$(A, z) \cdot g = (A^g, \Theta_t(A, g)z),$$

and the quotient defines a Hermitian line bundle,

$$\mathcal{L}_t^c \to M_c^s,$$

over the smooth part of the moduli space. Furthermore, this bundle comes with a unitary connection $\nabla$, given on the trivial bundle $\mathcal{F}_c^+ \times \mathbb{C}$ by the one-form

$$B_t(\alpha) = 2\pi i \int_{\Sigma_p} \langle tA \wedge \alpha + \overline{tA} \wedge \alpha \rangle,$$

for a tangent vector $\alpha \in T_{A\mathcal{F}} \cong \Omega^1(\Sigma, \mathfrak{g}_c)$ at $A \in \mathcal{F}_c^+$. It is easily verified that the curvature of this connection is given by

$$F_\nabla = -i\omega_t,$$

so that $\mathcal{L}_t^c$ defines a prequantum line bundle over $M_c^s$, with symplectic form $\omega_t$, and clearly this restricts to a prequantum line bundle $\mathcal{L}^k$ over $M_s$, equipped with $k\omega$ as its symplectic structure.
To perform geometric quantization, we must introduce a polarization. This will come from a choice of Riemann surface structure on $\Sigma$, which amounts to a Hodge star-operator $\ast: \Omega^1(\Sigma) \to \Omega^1(\Sigma)$, satisfying $\ast^2 = -\text{Id}$ for dimensional reasons. Using Hodge theory to identify $H^1_A(\Sigma, g)$ with the space of harmonic forms, we define an almost complex structure $J$ on $M^s_k$ by the expression

$$J\alpha = -\ast\omega$$

(47)
on harmonic representatives, the space of which is preserved by $\ast$. It is easily checked that $J$ is compatible with $\omega$. There is another obvious almost complex structure $I$ on $M^s_k$, which is simply given by $I\alpha = i\alpha$, and clearly this anti-commutes with $J$. Both of these almost complex structures are in fact integrable, as seen through the correspondence with Higgs bundles discussed below, giving $M^s_k$ the structure of a hyperkahler manifold.

The Kähler structure $J$ only depends on the Riemann surface structure up to isotopy, so in fact we get a family $J: \mathcal{T}_\Sigma \to C^\infty(M^s_k, \text{End}(TM^s_k))$ of Kähler structures parametrized by the Teichmüller space $\mathcal{T}_\Sigma$ of the surface. At a unitary connection $A$, the family $J$ clearly preserves the real subspace $H^1_A(\Sigma, g)$ tangent to $M^s$, so $J$ also defines a family of Kähler structures on $M^s$. This is indeed the polarization we shall use for quantizing $M^s$, but for $M^s_k$ we shall instead rely on a certain real polarization, considered by Witten in [Wit3], which also depends on the Riemann surface structure of $\Sigma$. Let us first recall how the quantization of the moduli space $M^s$ for compact gauge group proceeds.

The moduli space $M^s$ is simply connected, so in particular $H^1(M^s, \mathbb{R})$ vanishes, and furthermore $H^2(M^s, \mathbb{Z}) = \mathbb{Z}$ (see [AB2, DW, AHJ]). For a given point $\sigma \in \mathcal{T}_\Sigma$ in Teichmüller space, the properties of $M^s_k$ as a complex manifold can be understood through the classical work of Narasimhan and Seshadri [NS], which identifies $M$ with the moduli space of $S$-equivalence classes of semi-stable holomorphic vector bundles, over the Riemann surface $\Sigma$, of rank $n$, degree $d$ and fixed determinant. This space has the structure of an normal projective algebraic variety which is typically singular but contains the moduli space of stable bundles as an open smooth subvariety, corresponding exactly to the manifold $M^s$ of irreducible connections. In this picture, the holomorphic tangent space $T_{[\sigma]}M^s$ at a stable holomorphic bundle $E$ is given by the cohomology $H^1(\Sigma, \text{End}_0E)$, with values in the traceless endomorphisms. As we saw above, the moduli space $M^s$ admits a prequantum line bundle $\mathcal{L}$, which is just the determinant line bundle in this picture. It generates the Picard group, so in particular $[\omega_{1}]$ is a generator of $H^2(M^s, \mathbb{Z})$, and $c_1(M^s) = \lambda[\omega_{1}]$ with $\lambda = 2\text{gcd}(r, d)$ as proved in [DN].

If the rank $n$ and the degree $d$ are coprime, there are no strictly semi-stable bundles, so $M = M^s$ is a compact Kähler manifold. In this case, clearly $H^0(M^s, \mathcal{O}) = \mathbb{C}$, but as observed by Hitchin [Hit3], this holds even for non-coprime $n$ and $d$ by the Hartogs theorem, since the complement of $M^s$ in $M$ has codimension at least 2. Hitchin also notes that the family $J$ of Kähler structures on $M^s$ parametrized by Teichmüller space, which is itself in a canonical way a contractible complex manifold, is holomorphic and rigid in the sense of Definition 2.2 and Definition 2.4. Finally, the Kähler metric on the moduli space of stable bundles was studied by Zograf and Takhtajan in [ZT], where they give a Ricci potential in terms of the determinant of the Laplacian on the endomorphism bundle of a stable bundle.

Altogether, the discussion above demonstrates that the moduli space $M^s$ satisfies all the conditions of Theorem 4.2 to ensure the existence of a Hitchin connection. Furthermore, in
the case of coprime \( n \) and \( d \), Narasimhan and Ramanan \([NR]\) have shown that \( M^* \) does not admit any holomorphic vector fields, so Theorem \( 1.1 \) implies that the Hitchin connection must be projectively flat.

Projective flatness also holds in the non-coprime case, which similarly does not admit holomorphic vector fields. Hitchin \([Hit3]\) proves this by regarding a holomorphic vector field on \( M^* \) as a holomorphic function on the cotangent bundle \( T^* M^* \), which sits inside the moduli space of semi-stable Higgs bundles \( \mathcal{M} \), as discussed below. Once again appealing to the Hartogs theorem, the function, which is homogeneous of degree 1 in the action of \( \mathbb{C}^* \) and constant along fibers of the Hitchin fibration, can be extended to \( \mathcal{M} \), which is quite easily seen not to support such functions. The proof does not apply to the special situation when the genus \( g \) and the rank \( n \) are both equal to 2, and the holonomy around \( \gamma \) is trivial, because the moduli space \( M_{c} \) is isomorphic to \( \mathbb{C}P^3 \) and clearly has holomorphic vector fields.

In summary, we have proved projective flatness of the Hitchin connection for the moduli spaces as claimed in Theorem \( 1.2 \). Our proof of projective flatness follows the original proof by Hitchin \([Hit3]\) from the point where the curvature is known to be at most order one, but vanishing of the higher-order terms was established by Hitchin through particular properties of the Hitchin integrable system \([Hit1]\), whereas we derive it from rigidity of the family of Kähler structures.

We now turn to the problem of quantizing the moduli space \( M_{c}^* \) of irreducible \( G_{c} \)-connections. The theory of Higgs bundles naturally enters in this discussion as well. Suppose again that \( \Sigma \) is endowed with a Riemann surface structure and recall that a Higgs bundle is a pair \( (E, \Phi) \), where \( E \to \Sigma \) is a holomorphic vector bundle and the Higgs field \( \Phi \in H^0(\Sigma, \text{End}_0 E \otimes K) \) is a holomorphic one-form with values in the traceless endomorphisms of \( E \). Stability for Higgs bundles is defined by imposing the usual stability condition on the slope of subbundles \( E' \subset E \), but only the invariant ones satisfying \( \Phi(E') \subset E' \otimes K \). This leads to the moduli spaces \( \mathcal{M} \) and \( \mathcal{M}^* \) of semi-stable, respectively stable, Higgs bundles of rank \( n \), degree \( d \) and fixed determinant. Nitsure \([Nit]\) proves that \( \mathcal{M} \) is a quasi-projective algebraic variety, which contains \( \mathcal{M}^* \) as an open smooth subvariety. Through the work of Hitchin, Simpson, Donaldson and Corlette \([Hit2, Sim1, Sim2, Don, Cor]\), the moduli space \( \mathcal{M} \) can be identified, via the Hitchin equations and non-abelian Hodge theory, with the moduli space \( \mathcal{M}_{c}^* \) of flat reductive \( G_{c} \)-connections, with stable bundles corresponding to irreducible connections.

If \( E \) is itself already a stable bundle, then obviously stability as a Higgs bundle is implied for any Higgs field \( \Phi \). In particular, by simply taking the Higgs field to be zero, the moduli space \( M^* \) sits canonically inside \( \mathcal{M}^* \cong M_{c}^* \), as we discussed already at the level of representations. But in fact, the entire cotangent bundle of \( M^* \) can be embedded in \( M_{c}^* \), although the embedding depends crucially on the Riemann surface structure of \( \Sigma \). Indeed, we get the following identification by Serre duality,

\[
H^0(\Sigma, \text{End}_0 E \otimes K) \cong H^1(\Sigma, \text{End}_0 E)^* ,
\]

where we recognize the right-hand side as the holomorphic cotangent space of the moduli space \( M^* \). In other words, the Higgs fields on stable bundles can be viewed as cotangent vectors to \( M^* \) through Serre duality. The Higgs bundle model \( \mathcal{M}^* \) of \( M_{c}^* \) illuminates the hyperkähler structure but it also carries a natural action of \( \mathbb{C}^* \) by scaling the Higgs field.
To quantize the moduli space $M_s$, we follow Witten [Wit3] and define a family of real polarizations, parametrized by the Teichmüller space of $\Sigma$, in the following way. For any Riemann surface structure on $\Sigma$, in the form of a star-operator $*$ as usual, we can use Hodge theory to split the space $H^1_A(\Sigma, g_c)$ into types,

$$H^1_A(\Sigma, g_c) = H^1_{A,0}(\Sigma, g_c) \oplus H^0_{A,1}(\Sigma, g_c).$$

Although this uses the complex structure on $H^1_A(\Sigma, g_c)$, it defines a splitting of the underlying real space, which is a model for the real tangent space $T[\lambda]M_s$. Since the symplectic form $\omega$ is invariant under the Hodge-star, the same of course applies to $\omega_1$, so that each of the summands are Lagrangian.

We shall take the subspaces $H^1_{A,0}(\Sigma, g_c)$ as a polarization, and define the quantum space to be the polarized sections of $L^k$, which are covariantly constant along its leaves. Notice that the polarization at a unitary connection $A$ is transverse to the tangent space $T[\lambda]M_s = H^1_A(\Sigma, g)$, simply because the only real form of type $(1,0)$ is the zero form. This means that a polarized section is determined by its values on the space $M_s$, or in other words, that the quantum space is identified with the prequantum space $C^\infty(M_s, L^k)$ of smooth sections over $M_s$, although the identification depends on the complex structure on $\Sigma$. Surely this identification requires covariantly constant sections to exist on the leaves of the foliation, which would be guaranteed for instance if they were simply connected, but we shall not deal with this question here. Instead, the discussion can be taken as motivation for using $C^\infty(M_s, L^k)$ as a model for the quantum space of $M_s$.

To understand how the identification between the quantum space of $M_s$ and the prequantum space $C^\infty(M_s, L^k)$ depends on the Riemann surface structure, we consider the latter as the fiber of a trivial bundle over Teichmüller space,

$$T_{\Sigma} \times C^\infty(M_s, L^k) \rightarrow T_{\Sigma}.$$

Then the expression (44) defines a connection $\nabla_V$ on this bundle, which is projectively flat by Theorem 1.1, once again due to the fact that the moduli space $M_s$ does not admit any holomorphic vector fields for any of the Kähler structures in the family $J$ parametrized by Teichmüller space. This establishes the second statement of Theorem 1.2 on the Hitchin-Witten connection. We also expand the statement in the following theorem.

**Theorem 5.1.** For any $t \in \mathbb{C}$ with $k = \text{Re}(t) \in \mathbb{Z}$, the trivial bundle $T_{\Sigma} \times C^\infty(M_s, L^k)$ over Teichmüller space, with fiber given by the smooth sections of the Chern-Simons line bundle over the moduli space of flat SU($n$) connections, has a projectively flat connection given by

$$\nabla_V = \nabla^T + \frac{1}{2t}(\Delta_G(V) + 2\nabla_G(V) \cdot dF - 2\lambda V'[F]) - \frac{1}{2t}(\Delta_G(V) + 2\nabla_G(V) \cdot dF - 2\lambda V''[F]) + V[F],$$

for any vector field $V$ on Teichmüller space.

In this way, the quantum spaces arising from different real polarizations as above are projectively identified through the parallel transport of the Hitchin-Witten connection.
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