Gravitational perturbation of traversable worm hole

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In this paper, we study the perturbation problem of the scalar, electromagnetic, and gravitational waves under the traversable Lorentzian worm hole geometry. The unified form of the potential for the Schrodinger type one-dimensional wave equation is found.

I. INTRODUCTION

The worm hole has the structure which is given by two asymptotically flat regions and a bridge connecting two regions. For the Lorentzian worm hole to be traversable, it requires exotic matter which violates the known energy conditions. To find the reasonable models, there had been studying on the generalized models of the worm hole with other matters and/or in various geometries. Among the models, the matter or wave in the worm hole geometry and its effect such as radiation are very interesting to us. The scalar field could be considered in the worm hole geometry as the primary and auxiliary effects. Recently, the solution for the electrically charged case was also found.

Scalar wave solutions in the worm hole geometry were in special worm hole models only and the transmission and reflection coefficients were found. The electromagnetic wave in worm hole geometry is recently discussed along the method of scalar field case. These wave equations in worm hole geometry draws attention to the research on radiation and wave.

Also there is a suggestion that the worm hole would be one of the candidates of the gamma ray bursts. With such suggestions, we can also suggest the worm hole as one of the candidates of the gravitational wave sources. If the gravitational wave detections are achieved in future, the identification of the worm hole might be followed by the unique waveform from the perturbed exotic matter consisting of worm hole.

For the gravitational radiation in any form, the scattering problem to calculate the cross section in more generalized models of worm hole should be considered. Thus the study of scalar, electromagnetic, and gravitational waves under worm hole geometry is necessary to the research on the gravitational radiation.

In this paper, we found the general form of the gravitational perturbation of the traversable worm hole, which will be a key to extend the worm hole physics into the problems similar to those relating to gravitational wave of black holes. The main idea and resultant equation is similar to Regge-Wheeler equation for black hole perturbation. Here we adopt the geometric unit, i.e., $G = c = h = 1$.

II. SCALAR PERTURBATION

The spacetime metric for static uncharged worm hole is given as

$$ds^2 = e^{2\phi(r)}dt^2 + \frac{dr^2}{b(r)r} + r^2(d\theta^2 + \sin^2\theta d\phi^2);$$

(2.1)

where $(\phi)$ is the lapse function and $b(r)$ is the worm hole shape function. They are assumed to be dependent on $r$ only for static case.

The wave equation of the minimally coupled massless scalar field is given by

$$\Box \phi = 0;$$

(2.2)

In spherical symmetric space-time, the scalar field can be separated by variables,

$$\varphi = Y_m(\theta) \frac{U_1(\rho;\theta)}{\rho};$$

(2.3)

where $Y_m(\theta)$ is the spherical harmonics and $l$ is the quantum angular momentum.
If \( l = 0 \) and the scalar field \( \varphi \) depends on \( r \) only, the wave equation simply becomes the following relation [3]:

\[
\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} = \frac{\partial^2 \varphi}{\partial \theta^2} + \frac{1}{r \sin \theta} \frac{\partial \varphi}{\partial \theta} = A = \text{const.} \tag{2.4}
\]

In this relation, the back reaction of the scalar wave on the worm hole geometry is neglected. Thus the static scalar wave without propagation is easily found as the integral form of

\[
Z = A e^{-r^2} \int_0^1 \frac{b}{r} \, dr \tag{2.5}
\]

The scalar wave solution was already given to us for the special case of worm hole in Ref. [3,9].

More generally, if the scalar field \( \varphi \) depends on \( r \) and \( \theta \), the wave equation after the separation of variables \( (\theta, \varphi) \) becomes

\[
\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{\sin \theta} \frac{\partial u}{\partial \theta} = \frac{1}{r^2} \frac{\partial^2 u}{\partial r^2} = V_1 u; \tag{2.6}
\]

where the potential is

\[
V_1(r) = \frac{L^2}{r^2} \frac{e^{2r}}{e^{-2r}} + \frac{1}{e} \frac{b r}{r} + \frac{1}{e} \frac{b_0 r}{r} \tag{2.7}
\]

and the proper distance \( r \) has the following relation to \( r \):

\[
\frac{\partial \theta}{\partial r} = \frac{e}{r^2} \frac{b}{r} \frac{e}{r} \tag{2.8}
\]

Here, \( L^2 = l(l+1) \) is the square of the angular momentum.

The properties of the potential are determined by the shape of \( \varphi \); if only the explicit form of \( V_1 \) and \( b \) are given. If the time dependence of the wave is harmonic as \( u_1(r; t) = A_1(r; \! t) e^{i \omega t} \), the equation becomes

\[
\frac{d^2}{dr^2} + l^2 \quad V_1(r) \quad A_1(r; \! t) = 0 \tag{2.9}
\]

It is just the Schrödinger equation with energy \( \omega^2 \) and potential \( V_1(r) \). When \( e^2 \) is finite, \( V_1 \) approaches zero as \( r \to 0 \), which means that the solution has the form of the plane wave \( A_1 e^{i \omega t} \) asymptotically. The result shows that if a scalar wave passes through the worm hole the solution is changed from \( e^{-i \omega t} \) into \( e^{i \omega t} \), which means that the potential acts on the wave and experience the scattering.

As the simplest example for this problem, we consider the special case \( \theta = 0; b = b_0 = r \) as usual, the potential should be in terms of \( r \) or \( r \) as

\[
V_1 = \frac{l(l+1)}{r^2} + \frac{b_0^2}{r^4} = \frac{l(l+1)}{r^2 + b_0^2} + \frac{b_0^2}{(r^2 + b_0^2)^2} \tag{2.10}
\]

where the proper distance \( r \) is given by

\[
r = Z \int_1^{1/b_0^2} \frac{dr}{r^2 + b_0^2} = \frac{q}{r^2 + b_0^2} \tag{2.11}
\]

There is the hyperbolic relation between \( r \) and \( q \) which is plotted in Fig. 1. The potentials are depicted in Fig. 2.

The potential has the maximum value as

\[
V_1(r)_{\text{max}} = V_1(0) = \frac{l(l+1) + 1}{b_0^2} \tag{2.12}
\]
Figure 1: Plot of the proper distance $r^*$ versus $r$. Here we set $b_0 = 1$. The dotted line is the asymptotic line to the hyperbolic relation, the dashed line.

Figure 2: Plot of the potentials of the scalar wave under the specified wormhole $b = b_0 = r$ in terms of $r$ for $l = 1, 2, 3$. Here we set $b_0 = 1$. To have a positive potential, $l \geq 0$.

### III. ELECTROMAGNETIC WAVE

We just followed the result of Berglid and Gibbend. They used the electromagnetic wave under wormhole geometry of the Morris-Thorne type wormhole like our model. Maxwell's equation in a gravitational field is

$$
H_{,a} = 4 I_{,a}; \quad H_{,a} + H_{,b} + H_{,c} = 0;
$$

where

$$
H = \frac{P}{g F}; \quad I = \frac{P}{g J},
$$

and the electromagnetic field strength tensors are defined by

$$
F = (E; B); \quad H = (D; H).
$$

Defining the vectors

$$
F = E \wedge B; \quad S = D \wedge B;
$$

$$
(3.4)
$$
the Einstein-Maxwell equations are

\[ \mathbf{F} \wedge \mathbf{F} = \frac{\partial \mathbf{S}}{\partial t} = \nabla \mathbf{F} \times \mathbf{F} ; \quad (3.5) \]

\[ \mathbf{F} \cdot \mathbf{S} = 0; \quad (3.6) \]

where \( n \) is the refraction index

\[ n_{ik} = ik = \frac{\rho_{jik}}{\rho_{j00}} n_{ik} \quad (3.7) \]

for a medium characterized by diagonal electric and magnetic permeabilities.

The Morris-Thorne type worm hole metric, Eq. (2.1) can be rewritten as

\[ ds^2 = \frac{e^2 (r) dr^2}{r} - \frac{1}{r^2} dz^2 + \frac{A^2 (r)}{r^2} d\theta^2 ; \quad (3.8) \]

where \( A (\cdot) \) is defined by

\[ n (\cdot) = \frac{A (\cdot)}{e (\cdot)} \quad (3.9) \]

Here we consider the special case \( \frac{b(r)}{r} = 0; b(r) = \frac{b_0}{r} \) like the scalar wave case. The Heaviside vector is decomposed into

\[ \mathbf{F}_J (\cdot t) = \mathbf{X} J \mathbf{F}_J (\cdot t) \quad (3.10) \]

with the generalized spherical harmonics \( Y_J^m (\cdot) \)

\[ Y_J^m (\cdot t) = \frac{X}{J \mathbf{F}_J (\cdot t)} Y_J^m (\cdot t) e^{imt} ; \quad (3.11) \]

The Maxwell equation becomes

\[ \frac{d}{d} (F_J^m) = n! F_J^m \quad (3.12) \]

\[ \frac{d}{d} (F_J^m) \frac{P}{J (J + 1) F_J^m} = n! F_J^m \quad (3.13) \]

and

\[ \frac{1}{J (J + 1) F_J^m} = n! F_J^m \quad (3.14) \]

Let the new coordinate \( x \) be

\[ \frac{dx}{d} = n (\cdot); \quad x = q \frac{z}{r^2 b_0^2} \quad (3.15) \]

and introduce the function

\[ \mathbf{F}_M^m (x; \cdot) = \mathbf{F}_J (x; \cdot) \quad (3.16) \]

Here \( x \) plays the role of the proper distance \( r \). The wave equation is finally

\[ \frac{d^2 F_J^m}{dz^2} + [1 \frac{2}{b_0^2} U_J (z)] F_J^m = 0; \quad (3.17) \]
where $z = x/b_0$ and the potential is

$$U_J(z) = 4J(J + 1) \frac{z^p}{(z^2 + 1) 2 \sqrt{+1 + z^2} + 1} .$$

The potential in our context becomes

$$V(r) = \frac{1(J + 1)}{r^2} .$$

The potentials are depicted in Fig. 3. Here $l = 1$ for the positive potential.

IV. GRAVITATIONAL PERTURBATION

We follow the conventions of Chandrasekhar in Ref. [11] where the gravitational perturbations are derived. Start from the axially symmetric spacetime which is given by

$$ds^2 = e^2 dt^2 + e^2 \left[ q_1 dq_1 + q_2 dq_2 + q_3 dq_3 + e^2 2 \, dr^2 + e^2 3 \, d\Omega^2 \right] .$$

For unperturbed case, the worm hole spacetime is

$$e^2 = e^2 ; \quad e^2 2 = 1 \quad \frac{b}{r} = \frac{r^2}{b} ; \quad e^2 3 = r ; \quad e = r \sin$$

and

$$q_1 = q_2 = q_3 = 0 .$$

Axial perturbations are characterized by the nonvanishing of small $q_1$, $q_2$, and $q_3$. When there are linear perturbations $q_1$, $q_2$, $q_3$, then there are polar perturbations with even parity which will not be considered here. From Einstein's equation

$$(e^3 + 2 \, 2 \, Q_{23})\beta = e^3 + 2 \, Q_{02,0}$$

where $x^2 = x^3$ and $Q_{AB} = q_{k} b_{k} ; Q_{A} = q_{k} 0$ $q_{k} :$ This becomes

$$e \frac{1}{r^3 \sin^3} \frac{\partial Q}{\partial \theta} = (q_{12} + q_{2,0})$$
where \( Q \) is

\[
Q (t; r; \theta) = Q_{23} \sin^3 \theta = (q_{13}, q_{03}) \sin^3 \theta; \tag{4.6}
\]

Another equation

\[
(e^3 + \zeta^2)Q_{23} = e^3 + \zeta^2 Q_{03}; \tag{4.7}
\]

This becomes

\[
\frac{e^p - \frac{1}{r^3 \sin^3 \theta}}{\theta} = (q_{13}, q_{03}) \sin \theta; \tag{4.8}
\]

If the time dependence is \( e^{it} \), then

\[
\frac{e^p - \frac{1}{r^3 \sin^3 \theta}}{\theta} = i! q_{03} + \zeta^2 q_1, \tag{4.9}
\]

\[
\frac{e^p - \frac{1}{r^3 \sin^3 \theta}}{\theta} = + i! q_{13} + \zeta^2 q_0; \tag{4.10}
\]

Let \( Q (t; r) = Q (r)C_n^{-2} (\zeta) \), where Gegenbauer function \( C_n (\zeta) \) satisfy the differential equation

\[
\frac{d}{dr} \sin^2 r \frac{d}{dr} + n (n + 2) \sin^2 r C_n (\zeta) = 0; \tag{4.11}
\]

Then

\[
\frac{d}{dr} \sin^2 r \frac{d}{dr} + \frac{d}{dr} \frac{d}{dr} + 2 \zeta^2 + \zeta^2 Q = 0; \tag{4.12}
\]

where \( \zeta = (1, 1) (l + 2) \). If \( Q = rZ \) and \( \frac{d}{dr} = e^p - \frac{1}{r^3 \sin^3 \theta} \),

\[
\frac{d^2}{dr^2} + \zeta^2 V (r) Z = 0; \tag{4.13}
\]

where the potential is

\[
V (r) = e^{2} \frac{1}{r^2} + \frac{3}{r} \cdot \frac{1}{2}, \tag{4.14}
\]

or

\[
V (r) = e^{2} \left( \frac{1}{r^2} + \frac{3}{r} \cdot \frac{1}{2} \right) \tag{4.15}
\]

in terms of \( b \) and \( s \). The first term is the same as the former two cases, but the signs and coefficients of the second and third terms are different from the scalar and electromagnetic wave cases.

For the simplest special case \( s = 0; b = b_0^0 = r \) like the former cases, the potential is

\[
V (r) = \frac{l(l + 1)}{r^2} + \frac{3b}{r^2}, \tag{4.16}
\]

whose shapes are shown in Fig. 4. By comparing with scalar and electromagnetic cases, the unified general formula is

\[
V (r) = \frac{l(l + 1)}{r^2} + \left( s^2 \right) \frac{b_0^0}{r^2}, \tag{4.17}
\]

where \( s = 0; 1; 2 \) is spin, or

\[
V (r) = \frac{l(l + 1)}{r^2} + \frac{b_0^2}{r^2}, \tag{4.18}
\]

where \( s = 0; 1; 2 \) for scalar, electromagnetic, and gravitational perturbations, respectively. This unified form is similar to the black hole case. The condition of the positive potential is \( l = s \).
FIG. 4: Plot of the potentials of the gravitational wave under the specified worm hole $b = b_0 = r$ for $l = 1; 2; 3$. Here we set $b_0 = 1$. To have a positive potential, $l = 2$.

V. DISCUSSION

We found the Regge-Wheeler type equation for gravitational perturbation. This unified form will give us new ideas and insights in the areas of worm hole physics and gravitational wave. In this paper we only consider the axially perturbation for simplicity. For further problems, Zerilli[11] type equation should be considered in order to see the exotic matter perturbation, and checked whether the potential form is similar to that of our Regge-Wheeler type potential like the black hole case or not.

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