The Semiclassical Instability of de Sitter Space

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The effect of the spontaneous nucleation of black holes in de Sitter space is reviewed and the main steps of the calculation in Nucl.Phys. B 582, 313, 2000 of the one-loop amplitude of this process are summarized. The existence of such an effect suggests that de Sitter space is not a ground state of quantum gravity with a positive cosmological constant.

The recent observational evidence for a positive cosmological constant has renewed interest in the dynamics of de Sitter space. In particular, quantum gravity in de Sitter space is attracting new attention (see for example [2] and references therein). In this connection I would like to review in this lecture the effect of semiclassical instability of de Sitter space with respect to spontaneous nucleation of black holes. Such an instability suggests that de Sitter space is not a ground state of quantum gravity with a positive cosmological constant \( \Lambda > 0 \).

The existence of the semiclassical instability in quantum gravity at finite temperature has been known for a long time. This is a non-perturbative effect mediated by certain gravitational instantons. In the \( \Lambda = 0 \) case the effect was described in detail by Gross, Perry and Yaffy [14], while the case with \( \Lambda > 0 \) was considered by Ginsparg and Perry [13]. The simplest way to understand this effect is to consider the semiclassical partition function for Euclidean quantum gravity with \( \Lambda > 0 \) as the sum over gravitational instantons. It turns out that in this sum there is one term, determined by the \( S^2 \times S^2 \) solution of the Euclidean Einstein equations, whose contribution is purely imaginary, because this (and only this) instanton is not a local minimum of the action. This renders the partition function and free energy

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complex, and the existence of an imaginary part of the free energy signals that the system is metastable. Since the $S^2 \times S^2$ instanton is the analytic continuation of the extreme Schwarzschild-de Sitter solution, one can argue that this instability results in the nucleation of black holes in de Sitter space. The energy for this process comes from the de Sitter heat-bath.

This effect is inherently present in quantum gravity with $\Lambda > 0$, since the value of the de Sitter temperature is fixed by $\Lambda$, and this gives rise to the thermal instability. For comparison, in the $\Lambda = 0$ case the situation is different, since the temperature is then a free parameter and one can take the zero temperature limit. In this limit Minkowski space is stable [14].

Summarizing, the $S^2 \times S^2$ instanton is analogues to the “bounce” solution [5], and so it is responsible for the formation of bubbles of a new phase. Its physical negative mode living in the physical TT-sector should not be confused with the negative modes from the conformal sector [10]. The latter exist for any background, and it seems that they do not imply any physical instabilities but emerge merely as a result of a bad choice of variables in the path integral. It seems likely that if one was able to start from the path integral over the phase space and then make it covariant in some way, these modes would be absent [18]. Negative modes from the TT-sector are on the other hand physically significant. In fact, having factorized the conformal sector, it is only these remaining modes that render the partition function divergent. This reflects the breakdown of the canonical ensemble for gravity due to black holes, whose degeneracy factor grows too fast with the energy. After complex-rotating the physical negative modes, the partition function becomes finite but complex, thus again indicating the breakdown of the canonical description and the existence of metastability.

A very nice feature of the $S^2 \times S^2$ instanton is its high symmetry. This allows one to carry out all the calculations exactly (in one loop), which is an exceptional situation in quantum gravity for non-trivial backgrounds. Such a calculation had not been done though until our recent paper with Andreas Wipf [19], where a complete derivation of the amplitude of the process was presented. In brief, we succeeded to explicitly determine the spectra of all the relevant fluctuation operators and to exactly compute their determinants within the $\zeta$-function scheme. Since calculations of this type are rather involved (and also rare), I would like to briefly summarize below the essential steps of our procedure, referring to [19] for more details and references.
1 Qualitative description

Let us consider the partition function for the gravitational field

\[ Z = \int D[g_{\mu\nu}] e^{-I}, \]  

(1)

where the integral is taken over Riemannian metrics with the action

\[ I[g_{\mu\nu}] = -\frac{1}{16\pi G} \int_{\mathcal{M}} (R - 2\Lambda) \sqrt{g} d^4x, \]  

(2)

with \( \Lambda > 0 \). The extrema of this action are solutions \((\mathcal{M}, g_{\mu\nu})\) of the Euclidean Einstein equations

\[ R_{\mu\nu} = \Lambda g_{\mu\nu}. \]  

(3)

Such solutions are called instantons; all of them are compact. In the semiclassical approximation, for \( G\Lambda \ll 1 \), the path integral (1) is given by the sum over all instantons

\[ Z \approx \sum_l Z[l] \equiv \sum_l \frac{\exp(-I[l])}{\sqrt{\text{Det}\Delta[l]}}. \]  

(4)

Here \( I[l] \) is the action of the \( l \)-th instanton and the prefactor comes from the Gaussian integration over small fluctuations around this background, \( \Delta[l] \) being the corresponding fluctuation operator.

The leading contribution to this sum is given by those instantons whose action is minimal. One can expect that these will be the instantons whose symmetry is maximal. Among all solutions to equations (3) there is one which is a maximal symmetry space – this is the four-sphere with radius \( \sqrt{3/\Lambda} \) and the standard metric which can be written as

\[ ds^2 = \left(1 - \frac{\Lambda}{3} r^2\right) dt^2 + \frac{dr^2}{1 - \frac{\Lambda}{3} r^2} + r^2(d\vartheta^2 + \sin^2 \vartheta d\phi^2), \]  

(5)

with \( r \leq \sqrt{3/\Lambda} \). The action of this \( S^4 \) instanton is \( I = -3\pi/\Lambda G \). Hence, for \( G\Lambda \ll 1 \), the path integral (1) approximately is

\[ Z \approx Z[S^4] = \frac{\exp(3\pi/\Lambda G)}{\sqrt{\text{Det}\Delta[S^4]}}. \]  

(6)
One can rewrite this expression as the partition function of some thermal system, \( Z = e^{-F/T} \). Here the inverse temperature \( 1/T = 2\pi \sqrt{3}/\Lambda \) is determined by the proper length of geodesics on \( S^4 \) (all of them are periodic), and up to subleading terms the free energy is \( F = -\frac{\sqrt{3}}{2G\Lambda} \).

One can wonder to which physical system refer these temperature and free energy? In fact, they relate to the de Sitter space \([8]\). Upon analytic continuation \( t \to it \) the metric (5) becomes the de Sitter metric restricted to the region inside the cosmological horizon, \( r < r_c = \sqrt{3}/\Lambda \). Let us call this region the Hubble region, or the causal diamond. De Sitter space has the temperature \( T \) determined by the surface gravity of the horizon, the entropy \( S = \pi r_c^2/G \) determined by the area of the horizon, and the free energy \( F = -TS \) contained inside the horizon \([8, 9]\). The values of \( T \) and \( F \) exactly coincide with those obtained above from the geometry of the \( S^4 \) instanton.

Summarizing, the partition function of quantum gravity with \( \Lambda > 0 \) and \( G\Lambda \ll 1 \) is determined by the thermodynamic parameters of de Sitter space, \( Z \approx Z[S^4] = e^{-F/T} \). (7)

Let us now consider corrections to this formula due to other gravitational instantons contributing to (4):

\[
Z \approx e^{-F/T} \left( 1 + \sum_{l \neq S^4} \frac{Z[l]}{Z[S^4]} \right). \tag{8}
\]

For \( \Lambda G \ll 1 \) all terms in this sum are exponentially small and can be neglected, if only they are real. If there are complex terms, their contribution will be physically important despite their smallness. Such complex terms can arise due to those instantons which are not local minima but saddle points of the action. If the number of their negative modes is odd, the determinants of the corresponding fluctuation operators will be negative, and the prefactors \( \sqrt{\text{Det} \Delta_{[l]} } \) in (8) will therefore be complex.

The question therefore arises: if among solutions of Eqs. (3) there are those which are not local minima of the action? A theorem quoted by Gibbons in Ref.[4] states that for \( \Lambda > 0 \) there exists only one such solution, which is the geometrical direct product of \( S^2 \times S^2 \) with the standard metric:

\[
ds^2 = \frac{1}{\Lambda} \left( d\vartheta_1^2 + \sin^2 \vartheta_1 d\varphi_1^2 + d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right). \tag{9}\n\]
Its action is \( I = -2\pi / GA \). As we will see, the solution has exactly one negative mode. In view of this, the partition function can actually be approximated by the semiclassical contributions of only two instantons, \( S^4 \) and \( S^2 \times S^2 \):

\[
Z \approx e^{-F/T} \left( 1 + \frac{Z[S^2 \times S^2]}{Z[S^4]} \right) \approx \exp \left\{ -\frac{1}{T} \left( F - T \frac{Z[S^2 \times S^2]}{Z[S^4]} \right) \right\},
\]

where \( Z[S^2 \times S^2] \) is purely imaginary. Rewriting this as \( Z \approx e^{-F/T} \), the free energy \( F \) here will have the real part coinciding with the free energy \( F \) in the Hubble region, and also the exponentially small imaginary part

\[
\Im(F) = -T \frac{Z[S^2 \times S^2]}{Z[S^4]}.
\]

The existence of the imaginary part of the free energy indicates that the system is metastable \([16, 1]\). The conclusion therefore is that de Sitter space, which is classically stable, becomes unstable semiclassically when the non-perturbative effects are taken into account.

The qualitative description of this instability was first given by Ginsparg and Perry \([13]\), who argued that the quantum decay of de Sitter space will lead to the spontaneous nucleation of black holes. The logic is that, as the decay is the tunneling transition mediated by the \( S^2 \times S^2 \) instanton, the structure of the configuration created during such a transition will be inherited from that of this instanton. The basic observation is then that the \( S^2 \times S^2 \) instanton can be obtained via the analytic continuation of the Lorentzian Schwarzschild-de Sitter solution \([12, 13, 4]\)

\[
\text{ds}^2 = -N \text{dt}^2 + \frac{dr^2}{N} + r^2 \left( d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right).
\]

Here \( N = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \), and for \( 9M^2 \Lambda < 1 \) this function has roots at \( r = r_+ > 0 \) (black hole horizon) and at \( r = r_{++} > r_+ \) (cosmological horizon). If one analytically continues via \( t \rightarrow it \), the metric becomes Euclidean, but then one has to restrict the range of \( r \) to \( r_+ < r < r_{++} \), since \( N \) is negative otherwise. In addition, one has to identify \( t \) with a suitable period, since otherwise the geometry will have conical singularities at \( r = r_+ \) and \( r = r_{++} \). In general one cannot remove both singularities at the same time, since the period of the identification of \( t \) is determined by the surface gravity, which
is different for the two
horizon. However, in the limit $r_+ \to r_{++} \to \frac{1}{\sqrt{\Lambda}}$
the surface gravities will be the same and both conical singularities can be
removed at the same time. Although one might think that nothing will
remain of the solution in this limit, this is not so. The limit $r_+ \to r_{++}$
implies that $9M^2\Lambda = 1 - 3\epsilon^2$ with $\epsilon \to 0$. One can introduce new coordinates
$\vartheta_1$ and $\varphi_1$ via $\cos \vartheta_1 = (\sqrt{\Lambda}r - 1)/\epsilon + \epsilon/6$ and $\varphi_1 = \sqrt{\Lambda} \epsilon \tau$. Passing to the
new coordinates and taking the limit $\epsilon \to 0$, the result is exactly the metric (1).

The conclusion is that a tunneling transition via the $S^2 \times S^2$ instanton
will create an extreme Schwarzschild-de Sitter black hole. The radius of such
a black hole is equal to the radius of the cosmological horizon, and so it will
fill completely the Hubble region. However, the total volume of de Sitter
space is infinite and it contains infinitely many Hubble regions. The black
holes will emerge in some of these regions, but most of the regions will remain
empty. The number of the filled regions divided by the number of the empty
ones is the probability for the black hole nucleation in one region. This is
proportional to $\Im(\mathcal{F})$ in (11). As a result, black holes will nucleate with a
certain probability all over the space, like the bubbles in boiling water.

The temperature of the nucleated black holes can be read off from the
$S^2 \times S^2$ metric as the inverse length of the equator of any of the two spheres:
$T_{\text{BH}} = \frac{\sqrt{\Lambda}}{2\pi}$ (the same value can be obtained from the Lorentzian solution (12) [4]). This is different from the temperature of the de Sitter heat bath.

The origin of this discrepancy can be traced to the causal structure of de
Sitter space: the fluctuations cannot absorb energy from and emit energy
into the whole of de Sitter space, but can only exchange energy with the
Hubble region. Thus the energy exchange is restricted. As a result, the local
temperature in the vicinity of a created defect may be different from that of
the heat bath.

Using finally the classical formula of Langer and Affleck [16, 1], the qua-
si-classical decay rate of de Sitter space is

$$\Gamma = 2|\Im(\mathcal{F})| = 2T \frac{|Z_{[S^2 \times S^2]}|}{Z_{[S^4]}},$$

where $T = \frac{1}{2\pi} \sqrt{\frac{\Lambda}{3}}$ is the temperature of the de Sitter heat bath. This formula
gives the probability of a black hole nucleation per Hubble volume and
per unit time of a freely falling observer [19].
2 Calculation of the path integral

Let us now briefly describe the one-loop calculation of the partition functions $Z_{[S^2 \times S^2]}$ and $Z_{[S^4]}$ entering formula (13). We use the standard Faddeev-Popov procedure and complex-rotate the conformal and TT-negative modes. Only the ghost operator has zero modes in its spectrum, which modes are associated with the background isometries. We integrate over these modes non-perturbatively, with the result (Eq.(24)) being proportional to the volume of the isometry group. The main steps of the procedure are as follows.

For small fluctuations $h_{\mu\nu}$ around an instanton $(\mathcal{M}, g_{\mu\nu})$ the action expands as $I[g_{\mu\nu} + h_{\mu\nu}] = I[g_{\mu\nu}] + \delta^2 I + \ldots$. Since $\delta^2 I$ has many zero modes associated with diffeomorphisms, one has to add a gauge fixing term, which is chosen to correspond to the covariant background gauge [11]:

\[ \delta^2 I_g = \gamma \left\langle \nabla_\sigma h^\sigma_\rho - \frac{\gamma + 1}{4\gamma} \nabla_\rho h, \nabla^\alpha h^\rho_\alpha - \frac{\gamma + 1}{4\gamma} \nabla^\rho h \right\rangle. \]  

(14)

Here $\gamma$ is a real parameter and the scalar product is defined as

\[ \langle h_{\mu\nu}, h'^{\mu\nu} \rangle = \frac{1}{32\pi G} \int_{\mathcal{M}} h_{\mu\nu} h'^{\mu\nu} \sqrt{g} \, d^4 x; \]

similarly for vectors and scalars. The one-loop partition function then reads

\[ Z = e^{-I} \int D[h_{\mu\nu}] \, D_{FP} \exp \left( -\delta^2 I - \delta^2 I_g \right), \]  

(15)

where the Faddeev-Popov factor is given by the integral over all diffeomorphisms

\[ (D_{FP})^{-1} = \int D[\xi_\mu] \exp \left( -\delta^2 I_g \right). \]  

(16)

It is convenient to use the Hodge decomposition for fluctuations,

\[ h_{\mu\nu} = \phi_{\mu\nu} + \frac{1}{4} h g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - \frac{1}{2} g_{\mu\nu} \nabla^\sigma \xi^\sigma, \]  

(17)

where $\phi_{\mu\nu}$ is the transverse tracefree (TT) part, $\nabla_\mu \phi^\mu_\nu = \phi^\mu_\mu = 0$, and $h$ is the trace. The longitudinal vector part also decomposes as

\[ \xi_\mu = \eta_\mu + \nabla_\mu \chi + \xi^H_\mu, \]  

(18)
\[ \nabla_\mu \eta^\mu = 0 \] and the harmonic piece \( \xi^\mu \) vanishes for simply-connected manifolds. With these decompositions the gauge fixing term and the gauge-fixed action \( \delta^2 I_{gf} = \delta^2 I + \delta^2 I_g \) are diagonal:

\[ \delta^2 I_g = \gamma \langle \eta_\mu, \Delta_1^2 \eta^\mu \rangle + \frac{1}{16 \gamma} \langle (\tilde{h} + 2 \tilde{\Delta}_0 \chi), \Delta_0 (\tilde{h} + 2 \tilde{\Delta}_0 \chi) \rangle. \tag{19} \]

\[ \delta^2 I_{gf} = \frac{1}{2} \langle \phi^{\mu\nu}, \Delta_2 \phi^{\mu\nu} \rangle + \gamma \langle \eta_\mu, \Delta_1^2 \eta^\mu \rangle + \frac{1}{4} \langle \chi, \Delta_0 \tilde{\Delta}_0 \chi \rangle - \frac{1}{16 \gamma} \langle h, \tilde{\Delta}_0 h \rangle. \]

Here the tensor fluctuation operator is

\[ \Delta_2 \phi^{\mu\nu} = -\nabla_\sigma \nabla^\sigma \phi^{\mu\nu} - 2R^{\mu\nu;\beta} \phi^{\alpha\beta}, \tag{20} \]

the vector operator and scalar operators are

\[ \Delta_1 = -\nabla_\sigma \nabla^\sigma - \Lambda, \quad \Delta_0 = -\nabla_\sigma \nabla^\sigma, \tag{21} \]

while \( \tilde{\Delta}_0 = 3\Delta_0 - 4\Lambda \) and \( \tilde{\Delta}_0 = \gamma \Delta_0 - \Delta_0 \) also \( \tilde{h} = h - 2\nabla_\mu \xi^\mu \).

To compute the path integrals in (15),(16), all fields are expanded with respect to the bases associated with complete sets of eigenstates of the operators \( \Delta_2 \), \( \Delta_1 \) and \( \Delta_0 \) with some Fourier coefficients \( C \). For example, \( h = \sum_n C_n h_n \) where \( \Delta_0 h_n = \lambda_n h_n \). The quadratic actions (19) then reduces to quadratic forms in the coefficients \( C \), and the path-integration in (15),(16) is performed by integrating over all \( dC \). The perturbative path integration measure is defined as the square root of the determinant of the metric on the function space of fluctuations:

\[ D[h_{\mu\nu}] \sim \sqrt{\text{Det}(\langle dh_{\mu\nu}, dh_{\mu\nu} \rangle)}, \quad D[\xi_\mu] \sim \sqrt{\text{Det}(\langle d\xi_\mu, d\xi_\mu \rangle)}, \tag{22} \]

where the differentials refer to the Fourier coefficients \( C \) of the expansions of \( h_{\mu\nu} \) and \( \xi_\mu \). For example, \( dh = \sum_n dC_n h_n \). The normalization is chosen such that \( \int D[h_{\mu\nu}] \exp \left( -\mu_o^2 \langle h_{\mu\nu}, h_{\mu\nu} \rangle \right) = 1 \) and \( \int D[\xi_\mu] \exp \left( -\mu_o^4 \langle \xi_\mu, \xi_\mu \rangle \right) = 1 \), where \( \mu_o \) is an arbitrary renormalization parameter with the dimension of an inverse length.

Using (17),(18) the metrics on the space of fluctuations are

\[ \langle dh_{\mu\nu}, dh_{\mu\nu} \rangle = \langle d\phi^{\mu\nu}, d\phi^{\mu\nu} \rangle + 2\langle d\eta_\mu, \Delta_1 d\eta^\mu \rangle + \langle d\chi, \Delta_0 \tilde{\Delta}_0 d\chi \rangle + \frac{1}{4} \langle dh, dh \rangle, \]

\[ \langle d\xi_\mu, d\xi_\mu \rangle = \langle d\eta_\mu, d\eta^\mu \rangle + \langle d\chi, \Delta_0 d\chi \rangle. \tag{23} \]
The path integrals in (15),(16) reduce then to infinite products of ordinary integrals over the Fourier coefficients $C$. Most of these integrals are Gaussian, and their computation gives products of eigenvalues $\sigma_s$ for coexact vectors and eigenvalues $\epsilon_k$ for TT-tensors in the resulting formula for $Z$ in Eq.(23) below. There are, however, also non-Gaussian integrals, which can be of three different types.

- First, for the $S^2 \times S^2$ instanton there exists a TT tensor mode with a negative eigenvalue $\epsilon_- < 0$. The integration over this mode is carried out with the complex contour rotation [5] leading to the complex factor $\Omega_{neg} = \mu_o/(2i\sqrt{|\epsilon_-|})$ in (26). For any other instanton $\Omega_{neg} = 1$.

- Next, for any instanton there are infinitely many conformal negative modes associated with the 'wrong' sign term $-\frac{1}{16\gamma}\langle h, \hat{\Delta} h \rangle$ in (19). This is the manifestation of the well-known problem of conformal sector in Euclidean quantum gravity [10]. Although its real understanding is lacking, it seems that this problem is essentially an artifact of the bad choice of variables in the path integral [13]. The prescription is then to perform the complex rotation $h \to ih$ in the space spanned by positive eigenstates of the operator $\hat{\Delta}_0$ [19]. After this the quadratic forms become positive-definite and the integrals can be computed. Remarkably, the resulting effect of conformal modes is then exactly cancelled by the contribution of the exact parts $\chi$ of the vectors. There is only one scalar mode which contributes to the final answer: the constant conformal mode present for any background and giving the factor $\Omega_{conf} = \frac{\mu_o}{\sqrt{3}}$ to (25). Only for the $S^4$ instanton there are additional 5 conformal Killing scalars which also contribute, and so the answer in the $S^4$ case is $\Omega_{conf} = \frac{\mu_o}{\sqrt{3}} \left(\frac{1}{\sqrt{3\mu_o}}\right)^5$.

- Finally, after fixing the gauge, there remain a finite number of zero modes of the vector operator $\Delta_1$. These are associated with the background isometries. Such modes are not contained in the fluctuation measure $D[h_{\mu\nu}]$, since they do not contribute to the metric $\langle dh_{\mu\nu}, dh^{\mu\nu} \rangle$ in (23), but they are contained in the ghost measure $D[\xi_\mu]$ and should therefore be taken into account. In fact, the effect of these modes turns out to be very important. In treating these modes, we follow the approach of t’Hooft [15] and Osborn [17], whose idea is to carry out the integration over these modes non-perturbatively, which amounts to integrating over the isometry group $\mathcal{H}$. The corresponding integration measure must be proportional to the Haar measure. If $\mathcal{H}$ acts on $\mathcal{M}$ via $x^\mu \to x^\mu (C_j)$, where $j = 1, \ldots \dim \mathcal{H}$, the vector
zero modes are the Killing vectors $K_j = \frac{\partial}{\partial C_j} \equiv \frac{\partial \mu^\nu}{\partial x^\nu} \frac{\partial}{\partial x^\nu}$, and the integration over these modes gives $\Omega_{\text{Killing}} = \int (\prod_j \mu^\nu \sqrt{\sigma_s \mu}) d\mu(C)$. Here $\frac{\partial}{\partial C_j}$ is computed at $C_j = 0$ and the normalization of the Haar measure $d\mu(C)$ of $\mathcal{H}$ is fixed by the condition that at the unity, $C_j = 0$, the perturbative measure is reproduced: $d\mu(C) \to \prod_j dC_j$ as $C_j \to 0$. For the $S^2 \times S^2$ instanton the isometry group is $\mathcal{H} = SO(3) \times SO(3)$, while in the $S^4$ case $\mathcal{H} = SO(5)$.

This gives, respectively

\begin{equation}
\Omega_{\text{Killing}} = \frac{64\pi^4 (\mu_o)^{12}}{27(AG)^3}, \quad \Omega_{\text{Killing}} = \left( \frac{9}{10} \right)^5 \frac{128\pi^6}{3} (\mu_o)^{20} (AG)^5.
\end{equation}

All this finally leads to the following expression for the one-loop partition function of small fluctuations around a given instanton background:

\begin{equation}
Z = \frac{\Omega_{\text{neg}}}{\Omega_{\text{Killing}}} \left( \prod_{\sigma_s>0} \frac{\sqrt{\sigma_s}}{\mu_o} \right) \left( \prod_{\epsilon_k>0} \frac{\mu_o}{\sqrt{\epsilon_k}} \right) e^{-I},
\end{equation}

where all dependence on the gauge parameter $\gamma$ has disappeared.

In order to use this formula, we need to explicitly know the spectra of the tensor fluctuation operator (20) on the space of the TT-tensors subject to $\nabla_\mu \phi^\mu_\nu = \phi^\mu_\nu = 0$, and also those for the vector operator (21) on the space of coexact vectors $\nabla_\mu \eta^\mu = 0$. For the $S^2 \times S^2$ instanton one finds [19]

| operator | eigenvalue | degeneracy |
|----------|------------|------------|
| $\Delta_2$ | $-2\Lambda$ | 2 \(\Lambda\) |
| | $(j(j+1) - 2)\Lambda$ | \(j(j+1)\) |
| | $(j_1(j_1+1) + j_2(j_2+1) - 2)\Lambda$ | \(j_1, j_2 \geq 2\) |
| $\Delta_1$ | $(j(j+1) - 2)\Lambda$ | 2 \(2j+1\) |
| | $(j_1(j_1+1) + j_2(j_2+1) - 2)\Lambda$ | \(2j_1 + 1, 2j_2 + 1\) |
| $\Delta_0$ | $(j_1(j_1+1) + j_2(j_2+1))\Lambda$ | \(j_1 + 1, 2j_2 + 1\) |

These spectra contain one negative tensor mode and six vector zero modes associated with the $SO(3) \times SO(3)$ isometry group; all other modes are positive. For the $S^4$ instanton one has
In this case all eigenvalues are positive, apart from ten vector zero modes which are generators of $SO(5)$.

The next step is to compute the infinite products of these eigenvalues in Eq. (25). This can be done with the use of the $\zeta$-function regularization, in which scheme the product of an infinite discrete set of numbers $\sigma_s$ by definition is

$$\prod_s \sigma_s = \exp \{-\zeta'(0) - \zeta(0) \ln \mu\}, \quad (26)$$

where the $\zeta$-function

$$\zeta(z) = \sum_s (\sigma_s)^{-s}. \quad (27)$$

The spectra in the tables above lead to the $\zeta$-functions of the following three basic types

$$Z(k, \nu | z) = \sum_{n=k}^{\infty} \sum_{m=k}^{n} \frac{(2n+1)(2m+1)}{(2n+1)^2 + (2m+1)^2 + \nu^2} \quad (28)$$

$$\zeta(k, \nu | z) = \sum_{n=k}^{\infty} \frac{(2n+1)}{(2n+1)^2 + \nu^2} \quad ,$$

$$Q(k, \nu, c | z) = \sum_{j=k}^{\infty} \frac{(2j+3)(j(j+3)+c)}{(j(j+3)+\nu)^2} \quad (29)$$

where $\Re(z)$ must be large enough to ensure convergence. The analytic continuation of these expressions to arbitrary values of $z$ has been carried out in [19]. This gives for $z = 0$

$$Z(k, \nu | 0) = \frac{1}{32} \nu^2 - \frac{1}{24} \nu + \frac{1}{2} k^2 \nu + 2 k^4 - \frac{2}{3} k^2 + \frac{13}{360},$$

$$\zeta(k, \nu | 0) = \frac{1}{12} - \frac{1}{4} \nu - k^2, \quad (30)$$

$$Q(k, \nu, c | 0) = -\frac{k^4}{2} - 2k^3 - (2c+1)\frac{k^2}{2} + (3-2c)k + \frac{\nu^2}{2} + (4-3\nu)\frac{c}{3} - \frac{11}{15},$$

which quantities define the ‘regularized numbers of eigenvalues’ of the operators and determine the anomalous scaling behavior of the determinants.
The values of the determinants themselves are given by the derivatives of the \( \zeta \)-functions at zero, which can be expressed in quadratures and computed numerically. The values needed in Eq.(25) are:

\[
Q'(2, 0, -4|0) = 3.72344, \quad Q'(2, -4, 0|0) = 6.65246, \quad Z'(2, -10|0) = -18.3118 \]

Collecting everything together, the one-loop partition functions in the \( \zeta \)-function regularization scheme are:

\[
Z_{[S^2 \times S^2]} = -i 0.3667 \times (\Lambda G)^3 \mu_o^{-\frac{98}{45}} \exp \left( \frac{2\pi}{\Lambda G} \right) \quad \text{(31)}
\]

and

\[
Z_{[S^4]} = 0.0047 \times (\Lambda G)^5 \mu_o^{-\frac{571}{45}} \exp \left( \frac{3\pi}{\Lambda G} \right). \quad \text{(32)}
\]

Up to my knowledge, such closed expressions were obtained for the first time in [19]. Here the numerical prefactors are determined by the \( \zeta \)-regularized determinants, the factors of \( \Lambda G \) come from the background isometries (see Eq.(24)). The powers of \( \mu_o \) are the anomalous dimensions, they receive contributions from all modes, including the isometries. As is seen from Eq.(24), the anomalous effect of the isometries is considerable, and had it been neglected the result would be very different. The anomalous dimension \(-\frac{571}{45}\) for \( S^4 \) in (32) agrees with the analysis of [6, 20].

The last step is to insert into Eq.(13) to find

\[
\Gamma = 14.338 \sqrt{\Lambda} (G \Lambda)^{-2} (\mu_o \Lambda)^{-\frac{473}{45}} \exp \left( -\frac{\pi}{\Lambda G} \right). \quad \text{(33)}
\]

This gives the probability of the spontaneous nucleation of a black hole inside the finite volume enclosed by the de Sitter cosmological horizon per unit time of a freely falling observer. The formula applies for \( \Lambda G \ll 1 \). Due to non-renormalizability of gravity, the cutoff parameter \( \mu_o \) remains undetermined; for estimates one can for example set \( \mu_o = G \). The subsequent real time evolution of the created black holes is an issue requiring special study [3].
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