We study the stability of standing shock waves in advection-dominated accretion flows into a Schwarzschild black hole using two-dimensional general relativistic hydrodynamic simulations, as well as linear analysis, in the equatorial plane. We demonstrate that the accretion shock is stable against axisymmetric perturbations but becomes unstable to nonaxisymmetric perturbations. The results of the dynamical simulations show good agreement with the linear analysis on the stability and the oscillation and growth timescales. A comparison of different wave-travel times with the growth timescales of the instability suggests that it is likely to be of the Papaloizou-Pringle type, induced by the repeated propagations of acoustic waves. However, the wavelengths of the perturbations are too long to allow a clear definition of the reflection point. By analyzing the nonlinear phase in the dynamical simulations, we show that quadratic mode couplings precede the nonlinear saturation. It is also found that not only short-term random fluctuations due to turbulent motion, but also quasi-periodic oscillations on longer timescales, take place in the nonlinear phase. We give some possible implications of the instability for black hole quasi-periodic oscillations and the central engine in gamma-ray bursts.

Subject headings: accretion, accretion disks — black hole physics — hydrodynamics — instabilities — shock waves

1. INTRODUCTION

Accretion flows with an embedded shock wave have attracted much attention from researchers. Hydrodynamic instabilities of shocked accretion flows may explain the time variability of the emission from many black hole candidates, since a shock wave is a promising mechanism to transform gravitational energy into radiation. The possibility that shocks might exist in black hole accretion disks was first suggested by Hawley et al. (1984a, 1984b), although they did not make clear the essential conditions for such an occurrence. The possible structure of shocked accretion flows in the equatorial plane was described by Fukue (1987), who suggested the importance of the multiplicity of critical points for the existence of a standing shock. Multiple critical points exist only for appropriate values of the injection parameters, such as the specific angular momentum and Bernoulli constant. There are generally two possible shock locations, which are referred to as the inner and outer shocks.

The stability of a standing shock wave in an accretion flow has also been investigated by many authors, both analytically and numerically. Nakayama (1994, 1996) showed by linear analysis in the equatorial plane that if the postshock matter is accelerated, the flow is unstable against radial perturbations, which is true in both Newtonian and general relativistic dynamics. As a result, we know for black hole accretion that the inner shock is generally unstable against radial perturbations and that a nonrotating, steady accretion flow into a black hole cannot have a stable standing shock wave in it. These features were also observed in one-dimensional axisymmetric simulations for a pseudo-Newtonian potential (Chakrabarti & Molteni 1993; Nobuta & Hanawa 1994).

More recently, Foglizzo (2001, 2002) pointed out from linear analysis that the outer shock wave, which is stable against radial perturbations, is in fact unstable to nonradial axisymmetric perturbations. He argued that the advective-acoustic cycle could be responsible for the instability. In this mechanism, the velocity and entropy fluctuations initially generated at the shock are advected inward, producing pressure perturbations, which then propagate outward, reach the shock, and generate entropy and velocity fluctuations there, thus causing the cycle to repeat with increased amplitude. The same instability appears to be at work in accretion flows onto the nascent proto–neutron star in a supernova core, in which large-scale oscillations of the shock wave of $\ell = 1$ mode are observed (Blondin et al. 2003), where $\ell$ denotes the index in the spherical harmonic functions $Y_{\ell m}$. Although this mechanism remains controversial (Ohnishi et al. 2006; Foglizzo et al. 2006; Blondin & Mezzacappa 2006; Laming 2007), the so-called standing accretion shock instability or SASI is currently attracting much attention as a promising explanation for the asymmetric explosions of supernovae, as well as for young pulsars’ proper motions (Scheck et al. 2004, 2006) and spins (Blondin & Mezzacappa 2007; W. Iwakami et al. 2008, in preparation).

As for the stability of the shock wave against nonaxisymmetric perturbations, Molteni et al. (1999) carried out two-dimensional simulations of an adiabatic, shocked accretion flow using a pseudo-Newtonian potential and found a nonaxisymmetric instability. They showed that the shock instability saturates at a low level, at which point a new, quasi-steady asymmetric configuration is realized. To investigate the mechanism of this instability, Gu & Foglizzo (2003) and Gu & Lu (2006) performed a linear analysis for both isothermal and adiabatic flows. They concluded that the instability seems to be of the Papaloizou-Pringle type and that the repeated propagations of acoustic waves between the corotation radius and the shock surface are the driving mechanism. These conclusions are based on the WKB approximation and a comparison between the growth rate and the period of the acoustic cycle.

We also note that Yamasaki & Foglizzo (2008) investigated the linear stability of a shocked accretion flow onto a proto–neutron star against nonaxisymmetric perturbations in the equatorial plane. They demonstrated that the counterrotating spiral modes are significantly
damped, whereas the growth rate of the corotating modes is increased by rotation. They also claim that the instability is not of the Papaloizou-Pringle type, since the stability is not affected by the presence or absence of the corotation point. Instead, they suggest the advective-acoustic cycle, based on WKB analysis. The purely acoustic cycle was found to be stable.

As described, although many efforts have been made to clarify the nonradial instability, a complete understanding of its mechanism remains elusive. It is known that the unperturbed accretion flows should be treated carefully, since they strongly affect the instability. In black hole accretion, gravitation is one of the main factors determining the flow features, such as the sonic points and shock locations. So far, however, the shock stability in accretion flows into black holes has been investigated under the Newtonian or pseudo-Newtonian approximation, and there has been no full general relativistic treatment. As we show below, the range of injection parameters that admit an existence of a standing shock wave is changed substantially when general relativity (GR) is fully taken into account.

In this paper, we carry out a fully general relativistic investigation of the shock stability in an advection-dominated accretion flow into a Schwarzschild black hole by using both linear analysis and nonlinear dynamical simulations. In so doing, we consider only the equatorial plane, assuming that the \( \theta \)-component of the 4-velocity (\( u^\theta \)) and all the \( \theta \)-derivatives are vanishing. We use spherical coordinates \((r, \theta, \phi)\). The evolution of the metric is not taken into account, which is justified so long as the mass of the accretion flow is much smaller than that of the black hole. We show that such shocks are indeed unstable against nonaxisymmetric perturbations and that a spiral arm structure is formed as the instability grows. We discuss the instability mechanisms, comparing various timescales. Finally, we mention possible implications of our findings for gamma-ray bursts (GRBs) and black hole quasi-periodic oscillations (QPOs).

The paper is organized as follows: In \( \S \) 2, we describe the steady axisymmetric accretion flows with an embedded shock. In \( \S \) 3, we formulate the linear analysis. The numerical method for the dynamical simulations is explained in \( \S \) 4. The main numerical results and their analysis are provided in \( \S \) 5. The implications for GRBs and black hole QPOs are visited in \( \S \) 6. Finally, we summarize and conclude in \( \S \) 7.

2. AXISYMMETRIC STEADY ACCRETION FLOWS WITH A SHOCK

2.1. The Multiplicity of Sonic Points

One of the key features of an accretion flow into a black hole is that the inflow velocity is supersonic at the event horizon. This immediately means that there should be at least two sonic points if a steady shock wave is to exist in the accretion flow, since both the pre- and postshock flows are transonic. This is in sharp contrast to accretion onto a neutron star, in which the postshock flow is subsonic.

One of the consequences of this fact is that spherical, adiabatic accretions into a Schwarzschild black hole cannot harbor a steady shock wave, since they have only a single sonic point. For a rotating accretion flow in the equatorial plane, the locations of the sonic points are determined by the adiabatic index and injection parameters, such as the Bernoulli constant and specific angular momentum. (Note that the accretion rate is irrelevant for the locations of the sonic points and standing shocks.)

The basic equations are the relativistic continuity equation and the equation of energy-momentum conservation:

\[
(\rho u^\mu)_{,\mu} = 0, \quad (T^{\mu\nu})_{,\nu} = 0,
\]

where the Greek indices represent the spacetime components. As already mentioned, we consider accretion only in the equatorial plane, and assume that the \( \theta \)-component of the velocity (\( u^\theta \)) and all \( \theta \)-derivatives vanish. In this case, the basic equations reduce to ordinary differential equations with respect to the radial coordinate:

\[
\partial_r p + \rho_0 u^r \partial_r (hu_r) = \frac{1}{2} \rho_0 h \left[ (\partial_r g_{rr})(u^r)^2 + (\partial_r g_{\theta\theta})(u^\theta)^2 + (\partial_r g_{\phi\phi})(u^\phi)^2 \right],
\]

\[
\partial_r (r^2 \rho_0 u^r) = 0, \quad \partial_r (hu_r) = 0, \quad \partial_r (hu_\theta) = 0.
\]
This treatment is slightly different from those of Molteni et al. (1999), Gu & Foglizzo (2003), and Gu & Lu (2006), who employed cylindrical coordinates and integrated out the vertical structure, thus considering the accretion flow only in the equatorial plane. We instead use spherical coordinates in this paper if general relativity were taken into account. Because of this difference, however, the accretion flows obtained with these previous formulations would still be different from those considered in this paper if general relativity were taken into account.

Figure 1 (left) shows the locations of the sonic points around a Schwarzschild black hole as a function of the Bernoulli constant and specific angular momentum (see also Lu 1985). As can be seen, there are indeed two or three sonic points for some combinations of these quantities. The maximum and minimum specific angular momenta are $\lambda_{\text{max}} \sim 4.0 M_\bullet$ and $\lambda_{\text{min}} \sim 3.2 M_\bullet$, respectively, for a Bernoulli constant $E = 1.004$, for example, where $M_\bullet$ denotes the black hole mass. It should be noted that one of the most important differences between the full GR and the pseudo-Newtonian treatments is the range of injection parameters that allow the existence of multiple sonic points. As the Bernoulli constant becomes smaller, the maximum specific angular momentum increases without limit in the pseudo-Newtonian case, whereas it is bounded for the full GR case. For Bernoulli constants that may be typical of massive stellar collapse, for example, $E = 1.003$, the maximum specific angular momentum is larger by about 60% under the pseudo-Newtonian approximation than in the full GR treatment. The reason for this difference is that the gravity near the black hole is too strong in the pseudo-Newtonian approximation. It should also be pointed out that the range of injection parameters that allow the existence of multiple sonic points is not very wide and that this difference may be important when considering the implications for astrophysical phenomena.

### 2.2. The Locations of Standing Shock Waves

The sonic points discussed in the previous subsection correspond to the so-called critical points of a dynamical system. It is known that the innermost and outermost critical points are of the saddle type, while the middle critical point is of center type. Hence, transonic accretion flows can be constructed only for the first two. These two transonic flows have the same Bernoulli constant and angular momentum but different entropies, and they can be connected by a standing shock wave, where the Rankine-Hugoniot relations hold:

$$ [\rho u^\mu]_{\mu} = 0, \quad [T^{\mu\nu}]_{\nu} = 0. \quad (4) $$

Here $l_\mu$ is a four-dimensional vector normal to the shocked surface and is set to $l_\mu = (0, 1, 0, 0)$ in the axisymmetric steady flow (see also eq. [13] for nonaxisymmetric perturbations). We use the notation $[Q] \equiv Q_+ - Q_- $, where the plus subscript represents a postshock quantity and minus represents a preshock quantity. The Bernoulli constant and specific angular momentum are defined respectively as

$$ E \equiv -hu_t, \quad \lambda \equiv -u_0/u_t. \quad (5) $$

Since the Bernoulli constant, specific angular momentum, and mass flux remain unchanged across the shock, we only need to consider the radial component of energy-momentum conservation across the shock. Then the Rankine-Hugoniot relation for an axisymmetric steady flow can be written as

$$ [\rho hu_t u^\mu + p]_{\mu} = 0. \quad (6) $$

### TABLE 1

| Model | Adiabatic Index $\Gamma$ | Bernoulli Constant $E$ | Specific Angular Momentum $\lambda$ ($M_\odot$) | Inner Sonic Point $r_{\text{iso}}$ ($M_\odot$) | Shock Point $r_{\text{sh}}$ ($M_\odot$) | Mach Number | Initial Perturbation Mode | Initial Perturbation Amplitude ($\%$) |
|-------|--------------------------|------------------------|---------------------------------------------|---------------------------------------------|---------------------------------------------|-------------|--------------------------|---------------------------------------------|
| M1    | 4/3                      | 1.004                  | 3.43                                        | 5.3                                         | 16.1                                        | 2.4         | 1                        | 1                                           |
| M2    | 4/3                      | 1.004                  | 3.46                                        | 5.2                                         | 23.2                                        | 2.3         | 1                        | 1                                           |
| M3    | 4/3                      | 1.004                  | 3.50                                        | 5.0                                         | 34.8                                        | 2.1         | 1                        | 1                                           |
| M4    | 4/3                      | 1.004                  | 3.56                                        | 4.8                                         | 78.4                                        | 1.5         | 1                        | 1                                           |
| M5    | 4/3                      | 1.001                  | 3.50                                        | 5.1                                         | 16.9                                        | 4.1         | 1                        | 1                                           |
| M6    | 4/3                      | 1.005                  | 3.50                                        | 5.0                                         | 30.2                                        | 1.6         | 1                        | 1                                           |
| M7    | 1.033                    | 1.13                   | 3.80                                        | 4.4                                         | 38.7                                        | 2.2         | 1                        | 1                                           |
| M8    | 1.167                    | 1.02                   | 3.70                                        | 4.6                                         | 64.2                                        | 1.4         | 1                        | 1                                           |
| M9    | 1.167                    | 1.02                   | 3.60                                        | 5.0                                         | 14.0                                        | 2.7         | 1                        | 1                                           |
| M10   | 1.167                    | 1.03                   | 3.60                                        | 5.0                                         | 32.4                                        | 1.5         | 1                        | 1                                           |
| M11   | 1.433                    | 1.001                  | 3.35                                        | 5.2                                         | 40.6                                        | 2.3         | 1                        | 1                                           |
| M12   | 1.433                    | 1.004                  | 3.15                                        | 6.0                                         | 36.5                                        | 1.3         | 1                        | 1                                           |
| M1m2  | 4/3                      | 1.004                  | 3.43                                        | 5.5                                         | 16.1                                        | 2.4         | 2                        | 1                                           |
| M1m3  | 4/3                      | 1.004                  | 3.43                                        | 5.3                                         | 16.1                                        | 2.4         | 3                        | 1                                           |
| M1a10 | 1.433                    | 1.004                  | 3.43                                        | 5.3                                         | 16.1                                        | 2.4         | 10                       | 1                                           |
| M1a100| 4/3                      | 1.004                  | 3.43                                        | 5.3                                         | 16.1                                        | 2.4         | 10                       | 1                                           |

Notes.—The locations of the inner sonic point and the shock surface are determined by the adiabatic index, Bernoulli constant, and specific angular momentum. The Mach number is calculated in the corotating observer’s frame, and $M_\odot$ is the mass of the central black hole.
In general, there are two possible shock locations, which we refer to as the inner and outer shocks. This is apparent from the right panel of Figure 1, where we show the parameter region that allows multiple shock locations. It is well known, however, that the inner shock is unstable against axisymmetric perturbations (Nakayama 1996), which we have confirmed with our own linear analysis. We have also performed numerical simulations with a single grid point in the azimuthal direction, thus suppressing nonaxisymmetric modes, and observed that the inner shock is either swallowed by the black hole or moves outward to become the outer shock after radial perturbations are imposed. On the other hand, we have seen that the outer shock is stable against radial perturbations without requiring that the perturbation amplitude be small. In the following, we consider only the outer shock.

We constructed several axisymmetric steady accretion flows with an outer shock for different combinations of the adiabatic index and injection parameters. These are summarized in Table 1.

### 3. LINEAR ANALYSIS OF NONAXISYMMETRIC SHOCK INSTABILITY

Here we give the basic equations and boundary conditions for the linear analysis. The obtained eigenvalues are later compared with the numerical simulations, and the eigenstates are employed to impose the initial perturbations.

The basic equations are the linearized relativistic continuity and energy-momentum tensor conservation equations (see eq. [1]). We again neglect the \( \theta \)-component of the velocity and all \( \theta \)-derivatives and consider only the equatorial plane. Under these assumptions, the system of equations can be written as follows:

\[
\begin{align*}
\partial_r f &= \frac{i}{\rho_0(0)u^r(0)} \left[ \rho_0(1)u^l(0) \sigma + \rho_0(0) (\omega u^r(1) - mu^\phi(1)) \right], \\
\partial_r q &= \frac{i}{\rho_0(0)u^r(0)\rho_0(0)u^\theta(0)} (\omega p_1 + \rho_0(0)u^r(0)\rho_1 u^\phi(0) \sigma q), \\
\partial_r V(1) &= i \frac{u^l(0)}{u^\theta(0)} \sigma V(1), \\
\partial_r S(1) &= i \frac{u^l(0)}{u^\theta(0)} \sigma S(1),
\end{align*}
\] (7)

where \( S \) is the entropy and the following definitions are used:

\[
\begin{align*}
f &= \rho_0(1) + \frac{u^r(1)}{u^\theta(0)}, & q &= \frac{h_1(1)}{h_0} + u^r(1) u^\phi(1), & \sigma &= \omega - m \frac{u^\phi(0)}{u^\theta(0)}, \\
V(1) &= \omega \left( h_1(1) u^\phi(0) + h_0 u^\phi(1) \right) + m \left( h_1(1) u^\phi(0) + h_0 u^\phi(1) \right).
\end{align*}
\] (9)

Following the standard procedure for linear stability analysis, the perturbed quantities are assumed to be proportional to \( e^{-i \omega t + im \phi} \). All perturbed quantities are calculated from \( f, q, V(1), \) and \( S(1) \). The latter two can be integrated analytically as

\[
V(1) = V(1)|_R \exp \left( i \frac{u^l(0)}{u^\theta(0)} \sigma \right), \quad S(1) = S(1)|_R \exp \left( i \frac{u^l(0)}{u^\theta(0)} \sigma \right),
\] (11)

where the subscript \( R \) denotes a quantity evaluated at \( r = R \). Thus, we need to integrate numerically only the two formulae of equation (7).

These linearized equations can be solved with appropriately set boundary conditions, which are imposed at the shock surface and inner sonic point. In this study, we assume that the perturbations are confined to the postshock region and that the preshock region remains unperturbed. As a result, the outer boundary condition is set at the shock surface. We express the shock radius as follows:

\[
R_{sh} = R_{sh(0)} + \eta \exp (-i \omega t + im \phi),
\] (12)

where \( \eta \) denotes the initial amplitude of the shock displacement. Defined as the four-dimensional vector normal to the shock surface, \( l_\nu \) can be written as

\[
l_\nu = (i \omega \eta e^{-i \omega t + im \phi}, 1, 0, -im \eta e^{-i \omega t + im \phi}).
\] (13)

Using these formulae in the Rankine-Hugoniot relations expressed in general as \( [Q^l]_{l\nu} = 0 \) (eq. [4]), we can write

\[
[Q^{(0)}] l \omega \eta - [Q^{(0)}] im \eta + [Q^{(1)}] = 0,
\] (14)

where the following notation is employed:

\[
Q^{(0)} \equiv Q^{(0)}|_{R_{sh(0)}}, \quad Q^{(1)} \equiv Q^{(1)}|_{R_{sh(0)}} + \eta (dQ^{(0)}/dr)|_{R_{sh(0)}},
\] (15)
As mentioned above, we assume that the preshock quantities are unperturbed, that is, \((Q^{\alpha}(1)|_{r=0}) = 0\). The explicit forms of these equations are

\[
(\rho_0 u^\alpha u^\beta)_{,\alpha} + \rho_0 u^\beta = 0,
\]

\[
\left[ \frac{\rho_0 u^\alpha u^\beta}{\rho_0} \left( \frac{h_0}{h_0} + \frac{u^\alpha}{u^\beta} \right) + \frac{u^\beta}{u^\alpha} \right]_{,\alpha} + B = 0,
\]

\[
\left\{ [\rho_0 h u^\beta]_{,\alpha} \left( \frac{\rho_0}{\rho_0} \left( \frac{h_0}{h_0} + 2 \frac{u^\alpha}{u^\beta} \right) + p_0 g^\beta_{\gamma\delta} \right) \right\}_{,\alpha} + C = 0,
\]

\[
\left[ \frac{\rho_0 u^\alpha u^\beta}{\rho_0} \left( \frac{h_0}{h_0} + \frac{u^\alpha}{u^\beta} \right) + \frac{u^\beta}{u^\alpha} \right]_{,\alpha} + D = 0,
\]

with the definitions

\[
A \equiv \left[ \left( \frac{d}{dr} \rho_0 u^\alpha u^\beta \right)_{,\alpha} - \left( \frac{d}{dr} \rho_0 u^\beta \right) \right] - \eta
\]

\[
+ i \omega \left( \rho_0 u^\alpha u^\beta \right)_{,\alpha} - \left( \rho_0 u^\alpha u^\beta \right)_{,\alpha}
\]

\[
B \equiv \left[ \left( \frac{d}{dr} \rho_0 h_0 u^\alpha u^\beta \right)_{,\alpha} - \left( \frac{d}{dr} \rho_0 h_0 u^\beta \right) \right] - \eta
\]

\[
+ i \omega \left( \rho_0 h_0 u^\alpha u^\beta \right)_{,\alpha} - \left( \rho_0 h_0 u^\alpha u^\beta \right)_{,\alpha}
\]

\[
C \equiv \left[ \left( \frac{d}{dr} \rho_0 h_0 (u^\alpha u^\beta)^2 + p_0 g^\beta_{,\gamma\delta} \right)_{,\alpha} - \left( \frac{d}{dr} \rho_0 h_0 u^\alpha u^\beta \right) \right] - \eta
\]

\[
+ i \omega \left( \rho_0 h_0 u^\alpha u^\beta \right)_{,\alpha} - \left( \rho_0 h_0 u^\alpha u^\beta \right)_{,\alpha}
\]

\[
D \equiv \left[ \left( \frac{d}{dr} \rho_0 h_0 u^\alpha u^\beta \right)_{,\alpha} - \left( \frac{d}{dr} \rho_0 h_0 u^\beta \right) \right] - \eta
\]

\[
+ i \omega \left( \rho_0 h_0 u^\alpha u^\beta \right)_{,\alpha} - \left( \rho_0 h_0 u^\alpha u^\beta \right)_{,\alpha}
\]

For the inner boundary condition, on the other hand, a regularity condition is imposed at the sonic point. By combining equations (7)–(10), we obtain a differential equation for \(u_r\), which is written generally as

\[
F(u_r) = G.
\]

The linearized form of this equation becomes

\[
(u_r)_{,\alpha} = \frac{G(1) - F(1) u_r}{F(0)} - \frac{F(1)}{F(0)} u_r,
\]

where the explicit forms of \(F(0), F(1), \) and \(G(1)\) are

\[
F(0) \equiv (2 \rho_0 h_0 g_{,\gamma\delta} (u^\alpha u^\beta)_{,\alpha} - (b_{,\gamma\delta} (1 + u^\alpha u^\beta))
\]

\[
F(1) \equiv \rho_0 h_0 (u^\alpha u^\beta)^2 + \rho_0 h_0 \rho_0 (u^\alpha u^\beta)^2 + 2 \rho_0 h_0 (u^\alpha u^\beta)^2 - \rho_0 h_0 (u^\alpha u^\beta)^2 - \rho_0 h_0 (u^\alpha u^\beta)^2 + 2 \rho_0 (u^\alpha u^\beta)^2 (u^\alpha u^\beta)^2
\]

\[
G(1) \equiv \Gamma (g_{,\gamma\delta} + 2 e^{-1}) \left[ \rho_0 h_0 (u^\alpha u^\beta)^2 + \rho_0 h_0 (u^\alpha u^\beta)^2 + \rho_0 h_0 (u^\alpha u^\beta)^2 + \rho_0 h_0 (u^\alpha u^\beta)^2 \right]
\]

\[
+ \frac{1}{2} \left[ \rho_0 h_0 (u^\alpha u^\beta)^2 + \rho_0 h_0 (u^\alpha u^\beta)^2 \right] + \rho_0 h_0 (u^\alpha u^\beta)^2 + \rho_0 h_0 (u^\alpha u^\beta)^2 + \rho_0 h_0 (u^\alpha u^\beta)^2
\]

\[
+ \rho_0 h_0 (u^\alpha u^\beta)^2 + \rho_0 h_0 (u^\alpha u^\beta)^2 + \rho_0 h_0 (u^\alpha u^\beta)^2 + \rho_0 h_0 (u^\alpha u^\beta)^2
\]

\[
+ i \rho_0 h_0 (u^\alpha u^\beta)_{,\alpha} u_r - \Delta \rho_0 (1 + u^\alpha u^\beta)(-\omega u^\alpha + mu^\beta) - i u^\alpha p_{\gamma\delta}.
\]

Here \(b_{,\gamma\delta}\) and \(\Gamma\) are the unperturbed sound velocity in the comoving frame and the adiabatic index, respectively. Since \(F(0)\) vanishes at the sonic point, we impose the regularity condition there:

\[
G(1) - F(1) u_r = 0.
\]

4. NUMERICAL METHOD AND MODELS

The growth of the initial perturbations was computed with a multidimensional general relativistic hydrodynamics (GRHD) code based on a modern technique, the so-called high-resolution central scheme (Kurganov & Tadmor 2000). The details of the numerical scheme are given in the Appendix.
We use Kerr-Schild coordinates with the Kerr parameter set to zero, since they have no coordinate singularity at the event horizon and we can place the inner boundary inside the event horizon. This is very advantageous for numerical simulations. We employ a Γ-law equation of state (EOS), \( p = (\Gamma - 1)\rho \epsilon \), where \( p \), \( \rho \), and \( \epsilon \) are the pressure, rest-mass density, and specific internal energy, respectively.

The computational domain is that part of the equatorial plane with \( 1.5M \leq r \leq 200M \), and the computation times are \(~6 \times 10^4M\) sec, where \( M \) again denotes the black hole mass and units with \( c = G = 1 \) are used. We employ \( 600 \times 60 \times 60 \) grid points. The radial grid width is nonuniform, with the grid being smallest (\( \Delta r = 0.1M \)) at the inner boundary and increasing geometrically by 0.34% per zone toward the outer boundary.

![Fig. 2.—Time evolution of velocity for model M1. The color scale indicates the magnitude of the radial velocity. The arrows represent the velocity at their respective positions. The central region in dark blue corresponds to the black hole.](image)
The initial perturbation modes are summarized in Table 1. For models M1 to M12 we employ the $m = 1$ mode, where $m$ stands for the azimuthal mode number in $e^{im\phi}$. For models M1m2 and M1m3, on the other hand, the $m = 2$ and $m = 3$ modes are initially imposed, respectively. These models are meant for the study of the initial-mode dependence. In order to investigate the dependence on initial amplitude, we also ran models M1a10 and M1a100, whose initial amplitudes are 10\% and 100\%, respectively. The unperturbed flows for models M1m2, M1m3, M1a10, and M1a100 the same as those in the other models.

The radial distributions of these modes are obtained from the linear analysis. This is important in order to compare the linear growth of purely single modes between the linear analysis and numerical simulations and is one of the differences between this work and that of
Molteni et al. (1999). We choose the most unstable mode for each azimuthal wavenumber except for model M1a100, in which the following perturbation is employed to avoid a superluminal velocity:

\[ v' = v_{\text{sta}}' (1 + \sin \phi), \]  

(30)

where \( v_{\text{sta}}' \) is the unperturbed radial velocity. The initial perturbations are added to the whole of the postshock region, again except for model M1a100, in which the perturbation is imposed in the region between the inner sonic point and the shock to prevent the flow velocity from being greater than the velocity of light.

In the following, we set the black hole mass to \( M_M = 3 M_\odot \) where \( M_\odot \) is the solar mass. We have in mind here some applications to astrophysical phenomena such as GRBs and QPOs. Since the formulation is dimensionless, the scaling is quite obvious.

5. NUMERICAL RESULTS

In this section, we describe the time evolution of the nonaxisymmetric instability obtained from our GRHD simulations. In the following analysis, we frequently employ mode decomposition of the shock surface according to the Fourier transform

\[ a_m(t) = \int_0^{2\pi} R_{\phi t}(\phi, t)e^{i\phi} d\phi, \]  

(31)

where \( R_{\phi t}(\phi, t) \) is the radius of the shock wave as a function of \( \phi \) and \( t \) and where \( a_m(t) \) is the amplitude of mode \( m \) as a function of \( t \).

5.1. Basic Features

Figures 2 and 3 show the temporal evolution of velocity and entropy for the baseline model M1, respectively. The perturbation grows exponentially at the beginning and the shock wave is deformed according to the imposed \( m = 1 \) mode, for which the deformed shock surface rotates in a prograde fashion, that is, in the direction of the unperturbed flow. Then the shock radius, or the \( m = 0 \) mode, starts to grow. After several revolutions, a spiral arm develops and the instability becomes saturated with more complex structures. In this nonlinear regime, several shocks are generated and collide with each other. As a result of these interactions, the original shock oscillates radially.

As listed in Table 2, the saturation levels of the shock radius, or the \( m = 0 \) mode, differ widely among the models. As an example, Figure 4 (left) illustrates the time evolution of the \( m = 0 \) mode for model M1. Both large- and small-amplitude oscillations can be seen, with periods of \( \sim 100 \) and \( \sim 20 \) ms, respectively. These two kinds of axisymmetric oscillations are also found in the other models. Their characteristics are summarized in Table 2. The growth of the \( m = 0 \) mode in the nonlinear phase is strongly dependent on the Mach number (see Tables 1 and 2); that is, a stronger shock tends to become more unstable to nonaxisymmetric perturbations. Although there is no entry for the maximum amplitude of the \( m = 0 \) mode in Table 2 for model M5, this is not an exception. In fact, the shock is so unstable in this model that it leaves the computational domain soon after the addition of nonaxisymmetric perturbations.

On the other hand, the oscillation periods of the nonaxisymmetric modes are much shorter in general than those of the axisymmetric ones. Indeed, the right panel of Figure 4 shows the time evolution of the \( m = 1 \)–3 for model M1. The typical periods range from several to a few dozen milliseconds.

| Model   | Dominant Mode in Nonlinear Phase | Maximum Amplitude of \( m = 0 \) Mode | \( \tau_{la} \) (ms) | \( \tau_{sa} \) (ms) |
|---------|---------------------------------|--------------------------------------|---------------------|---------------------|
| M1      | \( m = 1 \)                      | 3.9                                  | \( \approx 100 \)   | \( \approx 20 \)    |
| M2      | \( m = 1 \)                      | 5.1                                  | \( \approx 120 \)   | \( \approx 20 \)    |
| M3      | \( m = 1 \)                      | 3.2                                  | \( \approx 200 \)   | \( \approx 20 \)    |
| M4      | \( m = 1 \) or \( m = 2 \)      | 1.3                                  | ...                 | \( \approx 50 \)    |
| M5      | \( m = 1 \)                      | ...                                  | ...                 | ...                 |
| M6      | \( m = 1 \)                      | 1.5                                  | \( \approx 210 \)   | \( \approx 30 \)    |
| M7      | \( m = 1 \)                      | 4.5                                  | ...                 | \( \approx 50 \)    |
| M8      | \( m = 1 \)                      | 1.4                                  | \( \approx 300 \)   | \( \approx 20 \)    |
| M9      | \( m = 1 \)                      | 3.5                                  | \( \approx 80 \)    | \( \approx 20 \)    |
| M10     | \( m = 1 \)                      | 1.3                                  | ...                 | \( \approx 10 \)    |
| M11     | \( m = 1 \)                      | 4.2                                  | \( \approx 300 \)   | \( \approx 60 \)    |
| M12     | ...                              | ...                                  | ...                 | ...                 |
| M1m2    | \( m = 1 \)                      | 3.1                                  | \( \approx 100 \)   | \( \approx 20 \)    |
| M1m3    | \( m = 1 \)                      | 2.9                                  | \( \approx 100 \)   | \( \approx 20 \)    |
| M1a10   | \( m = 1 \)                      | 3.5                                  | \( \approx 100 \)   | \( \approx 20 \)    |
| M1a100  | \( m = 1 \)                      | 4.0                                  | \( \approx 100 \)   | \( \approx 20 \)    |

Notes.—Here \( \tau_{la} \) and \( \tau_{sa} \) are the large- and small-amplitude oscillation periods, respectively. Blank entries signify the lack of an identification.

| Model   | Maximum Amplitude of \( m = 0 \) Mode | \( \tau_{la} \) (ms) | \( \tau_{sa} \) (ms) |
|---------|--------------------------------------|---------------------|---------------------|
| M1      | 3.9                                  | \( \approx 100 \)   | \( \approx 20 \)    |
| M2      | 5.1                                  | \( \approx 120 \)   | \( \approx 20 \)    |
| M3      | 3.2                                  | \( \approx 200 \)   | \( \approx 20 \)    |
| M4      | 1.3                                  | ...                 | \( \approx 50 \)    |
| M5      | ...                                  | ...                 | ...                 |
| M6      | 1.5                                  | \( \approx 210 \)   | \( \approx 30 \)    |
| M7      | 4.5                                  | ...                 | \( \approx 50 \)    |
| M8      | 1.4                                  | \( \approx 300 \)   | \( \approx 20 \)    |
| M9      | 3.5                                  | \( \approx 80 \)    | \( \approx 20 \)    |
| M10     | 1.3                                  | ...                 | \( \approx 10 \)    |
| M11     | 4.2                                  | \( \approx 300 \)   | \( \approx 60 \)    |
| M12     | ...                                  | ...                 | ...                 |
| M1m2    | 3.1                                  | \( \approx 100 \)   | \( \approx 20 \)    |
| M1m3    | 2.9                                  | \( \approx 100 \)   | \( \approx 20 \)    |
| M1a10   | 3.5                                  | \( \approx 100 \)   | \( \approx 20 \)    |
| M1a100  | 4.0                                  | \( \approx 100 \)   | \( \approx 20 \)    |
In the nonlinear phase, the dominant mode is $m = 1$ for almost all the models. Remarkably, although the initially added perturbations are not the $m = 1$ mode in models M1m2 and M1m3, the nonlinear mode couplings eventually lead to the dominance of this mode (see Fig. 9, top).

5.2. Comparison with the Linear Analysis

Equations (7)–(8) with the boundary conditions at the shock surface and sonic point were solved numerically to find eigenmodes. Figure 5 shows the real and imaginary parts of the eigenfrequencies for some of the $m = 1$–3 modes for model M1. They are all unstable nonaxisymmetric modes. We find, on the other hand, that the axisymmetric perturbations are stable, which is consistent both with the dynamical simulations presented here and with previous work (Nakayama 1996). The oscillation periods, which correspond to $\omega_r$, are 1.5–3.7 ms for the most unstable mode in each $m$-sequence for model M1, whereas the growth times, which are obtained from $\omega_i$, are 2.6–3.2 ms.

Figure 6 compares the amplitudes obtained from the linear analysis and from the dynamical simulation. As can clearly be seen, the oscillation period and growth time are in good agreement for the first 10 ms, which is the linear growth phase. After this the dynamical simulation starts to deviate from the linear analysis, which indicates the beginning of the nonlinear phase. The amplitude saturates and the oscillation period becomes slightly longer. Figure 7 shows the evolution of the axisymmetric $m = 0$ mode; it starts to grow at $t \sim 10$ ms, leading to an increase in the average shock radius. This is the reason why the oscillation period becomes longer in the nonlinear phase.

As mentioned in the previous subsection and summarized in Table 2, the dominant modes in the nonlinear phase are prograde; that is, the deformation pattern rotates in the same direction as the unperturbed flow. This is also true for all the linearly unstable modes. In fact, the linear analysis shows that there is no unstable mode for $m < 0$. In contrast, Yamasaki & Foglizzo (2008) demonstrated by linear analysis that for accretion onto a neutron star, the prograde modes are enhanced and retrograde ones are suppressed by rotation.

As the specific angular momentum of the unperturbed flow becomes larger, the distance between the shock wave and the inner sonic point increases (compare models M1 and M4 in Table 1). According to the linear analysis, the number of unstable modes also increases, whereas the growth time of the most unstable modes becomes longer. This suggests that the larger distance tends to stabilize the non-axisymmetric instability, although it is not the only factor in the shock instability. In fact, we also find that the growth rate of unstable modes is affected by the shock strength; that is, stronger shocks tend to be more unstable.

Fig. 4.—Time evolution of the $m = 0$ mode (left) and the $m = 1$–3 modes (right).

Fig. 5.—Real and imaginary parts of the eigenfrequencies for some of the $m = 1$–3 modes for model M1.

Fig. 6.—Comparison of the amplitudes obtained from the linear analysis and from the dynamical simulation.
Finally, we show in Figure 8 the most unstable modes for different values of $m$ in model M1. As is clear, the most unstable of all is $m = 4$, and the modes with $m > 12$ are stable for this model. It is also interesting to note that the real part of the eigenfrequency for the most unstable modes becomes larger as the mode number $m$ increases (Fig. 8, left), while the pattern frequency $\omega_r/m$ becomes smaller (right).

5.3. Dependence on Initial Perturbations

Comparing models M1, M1a10, and M1a100, we find that the qualitative features of the dynamics are almost the same. In the linear phase, the growth of the mode amplitudes is unaffected by the initial conditions. Only the duration of the linear phase is affected, becoming shorter as the initial amplitude is increased, as expected. The saturation levels do not differ very much among the three models. In fact, the nonlinear phase is rather chaotic and forgets the differences in the initial conditions. It is also important to point out that despite the large initial amplitude for model M1a100, the shock continues to exist. This implies that if the injection parameters are appropriate, a standing shock will exist oscillating violently in an accretion flow into a black hole.

Next we show what happens if we impose initially the $m = 2$ or $m = 3$ mode instead of $m = 1$, comparing models M1, M1m2, and M1m3. In the top panels of Figure 9, we show the time evolution of the amplitudes for various modes in models M1m2 (left) and M1m3 (right).

We first examine the evolution up to $\sim 150$ ms, where the differences are most evident. Note that the linear phase lasts only for $\sim 10$ ms; the nonlinear phase thereafter is the focus here. In the left panel, we see the growth and saturation of the $m = 4$ mode in addition to the original $m = 2$ mode. On the other hand, the $m = 6$ mode forms and grows to saturation in the right panel. Note that the $m = 0$ mode is also generated in these models (discussed below). These modes are produced by the nonlinear mode couplings, which are quadratic in nature, and other modes, with odd $m$ for model M1m2, for example, are not generated.

After $\sim 150$ ms, however, the other modes also emerge and grow to saturation. These are probably generated by numerical noise, as they have much smaller amplitudes initially and spend a longer time in the linear phase. After the saturation of the modes, the dynamics is almost identical for the three models.
In the bottom panel of Figure 9, we show the time evolution of the amplitude of the $m = 0$ mode in the three models. As mentioned above, the modes are produced by quadratic couplings of the modes that are initially imposed. Up to $\sim 150$ ms, the evolution is quite different among the three models. For model M1, the amplitude grows to 4 times the initial value within $\sim 50$ ms and oscillates violently thereafter. On the other hand, the maximum amplitudes are much smaller for models M1m2 and M1m3, and the subsequent oscillations have much smaller amplitude. In fact, for model M1m3 the amplitude becomes nearly constant after $\sim 50$ ms. The saturation level is much lower than that of model M1m2. It is interesting that even though the $m = 3$ mode is unstable in the linear phase (see Fig. 5), the average shock radius, or the $m = 0$ mode, is most strongly affected by the $m = 1$ mode. After $\sim 150$ ms, all modes saturate and the behavior of the $m = 0$ mode becomes almost identical among the different models.

5.4. Dependence on the Adiabatic Index

In this paper, we employ a simple $\Gamma$-law EOS, and so far we have discussed only cases with $\Gamma = 4/3$. In reality, the EOS will not be so simple and the adiabatic index may not be constant. In order to infer the differences that the EOS may make, in this subsection we vary the adiabatic index in the $\Gamma$-law EOS and observe the changes.

In the bottom panel of Figure 9, we show the time evolution of the amplitude of the $m = 0$ mode in the three models. As mentioned above, the modes are produced by quadratic couplings of the modes that are initially imposed. Up to $\sim 150$ ms, the evolution is quite different among the three models. For model M1, the amplitude grows to 4 times the initial value within $\sim 50$ ms and oscillates violently thereafter. On the other hand, the maximum amplitudes are much smaller for models M1m2 and M1m3, and the subsequent oscillations have much smaller amplitude. In fact, for model M1m3 the amplitude becomes nearly constant after $\sim 50$ ms. The saturation level is much lower than that of model M1m2. It is interesting that even though the $m = 3$ mode is unstable in the linear phase (see Fig. 5), the average shock radius, or the $m = 0$ mode, is most strongly affected by the $m = 1$ mode. After $\sim 150$ ms, all modes saturate and the behavior of the $m = 0$ mode becomes almost identical among the different models.

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It is the unperturbed accretion flows that are most affected by a change of the adiabatic index. As the index becomes larger, both the specific angular momentum and Bernoulli constant that allow the existence of a standing shock wave get smaller. This can be understood as follows: The structure of an unperturbed accretion flow and, hence, the existence of a shock are determined by balance between the attractive gravity and the repulsive centrifugal force and pressure. As the EOS becomes harder, the pressure increases, and as a result, the centrifugal force can be reduced. For the same specific angular momentum, on the other hand, the Bernoulli constant can be smaller for a harder EOS. (Note that the Bernoulli constant is a measure of the matter temperature at infinity.)

The instability itself is also affected by a change in the adiabatic index, since it depends on the structure of the unperturbed accretion flow and, hence, the existence of a shock are determined by balance between the attractive gravity and the repulsive centrifugal force and pressure. As the EOS becomes harder, the pressure increases, and as a result, the centrifugal force can be reduced. For the same specific angular momentum, on the other hand, the Bernoulli constant can be smaller for a harder EOS. (Note that the Bernoulli constant is a measure of the matter temperature at infinity.)

5.5. Instability Mechanisms

To gain some insight into the instability mechanism, we calculate some timescales as follows:

\[
\tau_{c\rightarrow(c)} = \int_{r_{\text{cor}}}^{r_m} \left( \frac{1}{|\nu'|} + \frac{1}{C_{s}^+} \right) dr, \quad \tau_{c\rightarrow(c)} = \int_{r_{\text{cor}}}^{r_m} \left( \frac{1}{C_s^-} + \frac{1}{C_{s}^-} \right) dr, \tag{32}
\]

\[
\tau_{d\rightarrow(c)} = \int_{r_{\text{cor}}}^{r_{\text{max}}} \left( \frac{1}{|\nu'|} + \frac{1}{C_{s}^+} \right) dr, \quad \tau_{c\rightarrow(c)} = \int_{r_{\text{cor}}}^{r_{\text{max}}} \left( \frac{1}{C_s^-} + \frac{1}{C_{s}^-} \right) dr, \tag{33}
\]

where \( r_{\text{max}} \) is the radius of inner sonic point and the \( C_{s}^\pm \) are the outgoing (+) and ingoing (−) sound velocities, respectively, given in the observer’s frame for the Schwarzschild geometry as

\[
C_{s}^\pm = \left| \frac{(1 - b_s^2)u' u' \pm [(1 - b_s^2)u' u']^2 - [(1 - b_s^2)(u')^2 - b_s^2 g'''][(1 - b_s^2)(u')^2 - b_s^2 g''']^{1/2}}{(1 - b_s^2)(u')^2 - b_s^2 g'''} \right|. \tag{34}
\]

Here \( b_s \) denotes the sound velocity in the comoving frame. The corotation point of the perturbation is defined as

\[
\omega_r - m \frac{\nu'(r_{\text{cor}})}{u'(r_{\text{cor}})} = 0, \tag{35}
\]

where its radius is expressed as \( r_{\text{cor}} \). This investigation is inspired by the previous work (Gu & Foglizzo 2003; Gu & Lu 2006) mentioned in § 1. These timescales and the radius of the corotation point for all the models are summarized in Table 3. For comparison, we list the oscillation and growth timescales, which were obtained by linear analysis for the most unstable mode.
Figure 10 compares the growth times with the cycle periods given by equations (32) and (33) for all the models except M1a10 and M1a100. It is found that the periods of the acoustic-acoustic cycle are closer to the growth times than those of the advective-acoustic cycle, which appears to support the claim by Gu & Foglizzo (2003) and Gu & Lu (2006) that the instability is of the Papaloizou-Pringle type. It is important, however, to point out that we cannot identify the reflection point clearly (see Fig. 10, in which the left panel adopts the corotation point as the inner reflection point and the right panel adopts the inner sonic point). This is mainly because the wavelengths of the perturbations are rather long. In fact, we estimate the wavelength of the acoustic perturbation as

$$\lambda \approx \min \left( \frac{2\pi b_s}{\omega}, \frac{2\pi B_s}{\omega} \right),$$  

which should be much smaller than the scale height of the unperturbed flow for the WKB approximation to be justified. The wavelength of the dominant unstable mode for each model is also given in Table 3. According to this estimate, the wavelengths are comparable to or longer than the scale height. Hence the WKB approximation is not justified, at least for these models. Indeed, the “reflection point of the waves” loses its meaning. Incidentally, the WKB approximation may be applicable to higher harmonics. There are sequences of unstable modes up to $m = 12$ for model M1, for example, and the wavelengths of their high harmonics are shorter than the scale height. It should be noted however that they have smaller growth rates and are subdominant in driving the instability. Note also that the above analysis neither proves nor disproves a particular mechanism in a mathematically rigorous sense. Further investigation is definitely needed.

6. IMPLICATIONS FOR ASTROPHYSICAL PHENOMENA

In the previous sections, we found that the standing shock wave in an accretion flow into a Schwarzschild black hole is generally unstable to nonaxisymmetric perturbations and will oscillate with large amplitude in the nonlinear regime. Here we consider the astrophysical implications of the shock instability, picking black hole QPOs and GRBs as examples.

6.1. Black Hole QPOs

As mentioned already, quasi-periodic oscillations have been observed from a couple of black hole candidates, and they are attributed to some activity of the accretion disk around the black hole. The shock oscillation model for black hole QPOs has been investigated by many authors (Das et al. 2003a, 2003b; Chakrabarti et al. 2004; Aoki et al. 2004; Okuda et al. 2007). Recently, for example, Okuda et al. (2007) performed two-dimensional pseudo-Newtonian numerical simulations of the shock oscillation in the meridian section, assuming axisymmetry and taking into account the cooling and heating of gas and radiative transport. They demonstrated that a quasi-periodically oscillating shock wave forms around the black hole. They compared their numerical results with observations of GRS 1915+105 and suggest that the intermediate-frequency QPO in this source might be due to the shock instability.

Our models are different from those of Okuda et al. (2007). Whereas they considered axisymmetric oscillations, we investigate the nonaxisymmetric variety. We take general relativity fully into account. In addition, we calculate the energy density spectra for our models. In so doing, we employ the data from the nonlinear regime, that is, 100 ms after the beginning of the computations. Note that the dynamics in the nonlinear phase is almost identical in all the models, including those in which the $m = 2$ or $m = 3$ mode is initially imposed instead of $m = 1$.

Figure 11 shows the power spectra for the $m = 0$–3 modes in model M1. The $m = 0$ mode has a quasi-periodic feature around 8 Hz, which corresponds to the period of large oscillations observed for this mode. Although there are some hints of other QPOs, they are much less remarkable. This axisymmetric quasi-periodic oscillation is similar to those found by Okuda et al. (2007). The most important point here, however, is that the quasi periodicity of the $m = 0$ mode is induced by the nonaxisymmetric instability through the quadratic mode coupling.

Similar QPOs are found in the other models. Their frequencies depend on the unperturbed flow and, hence, on the Bernoulli constant and specific angular momentum. A quantitative comparison with observations is beyond the scope of this paper, since we have neglected radiative processes and viscosity and, among other things, considered only the equatorial plane. It can be mentioned, however,
that the nonaxisymmetric shock instability is a good candidate for the source of QPOs and that further investigations are certainly needed.

6.2. Fluctuations in GRB Jets

Long gamma-ray bursts are currently thought to be associated with massive stellar collapses and the subsequent formation of black holes. Although the central engine remains a mystery, it is widely believed that a highly relativistic jet is somehow produced near the black hole and that its kinetic energy is later dissipated in internal shocks at greater distances, emitting gamma rays (see Mészáros 2006 for a recent review). In the so-called patchy-shell model, the jet is due to mass shells that have slightly different velocities and collide with each other, generating internal shock waves. Although the timescale of the velocity fluctuations is thought to be set by the dynamical timescale of the black hole, the exact physical processes that produce the velocity variations are unknown at present.

During the collapse of a massive star giving rise to a GRB, a large amount of matter will accrete, on a timescale of seconds, onto a proto-neutron star at first and into a black hole later. If the progenitor is rotating rapidly prior to the collapse (Yoon et al. 2008), the accreting matter will form a disk around the compact object at the center. The accretion disk is expected to be advection dominated (Popham et al. 1999). We are thus interested in the stability of accretion flows into a black hole, especially in the accretion-dominated regime. Here we consider accretion flows with a standing shock wave in them, since the core bounce produces a shock wave that soon after becomes a standing shock in the core and will continue to exist during the subsequent accretion onto the proto-neutron star, the phase just preceding black hole formation. Even if the bounce shock were not to survive, there are many chances for shock formation as long as the standing shock is robust, since in reality the velocity and pressure of the accreting matter will experience fluctuations.

According to the patchy-shell model, gamma rays are emitted when the kinetic energy of the ultrarelativistic jet is dissipated in the internal shock waves originating from the inhomogeneity of the jet. Although the mechanism of the jet formation remains unknown, one would suppose the black hole to be involved. The source of the inhomogeneity is also an unsolved problem. If a standing shock wave exists in the accretion flow—for example, as a relic of the shock wave produced at core bounce—we would speculate that the intrinsic instability of the system against nonradial perturbations will be a natural source of fluctuations in GRB jets if it is formed from some interaction between the accretion disk and the black hole, which is not unlikely (Blandford & Znajek 1977). We mention in passing that recent progenitor models, which can produce GRBs (MacFadyen & Woosley 1999; Heger et al. 2005), predict injection parameters that are appropriate for the existence of a standing accretion shock. For example, Heger et al. (2005) calculated the evolution of massive stars, taking into account magnetic fields, and obtained a specific angular momentum of several times $10^{16}$ cm$^2$ s$^{-1}$ and a temperature of $\lesssim10^{10}$ K for the matter that will later form an accretion disk. These numbers are just adequate for the existence of a standing shock wave in the accretion disk around a black hole of several solar masses.

Owing to the nonaxisymmetric instability, the mass flux fluctuates very much indeed in our models. In this context, it is noteworthy that the QPO with period much longer than the dynamical timescale found in the previous sections may somehow leave an imprint in the prompt gamma-ray emission or early X-ray afterglows. It is certainly necessary, however, to study the possible effects of cooling (Popham et al. 1999) on the instability. The disk thickness, which we have ignored in this paper, is also a concern for future work. We finally mention that gravitational radiation due to the nonradial shock instability may also have interesting implications.

7. SUMMARY AND CONCLUSIONS

We have investigated the nonaxisymmetric shock instability in an accretion disk around a Schwarzschild black hole, employing fully general relativistic hydrodynamic simulations as well as linear analysis. Both the linear and nonlinear phases have been analyzed in detail. We have also given some possible implications for astrophysically interesting phenomena such as black hole QPOs and GRBs.

The main findings in the present work are as follows:

1. The standing shock is generally unstable against nonaxisymmetric perturbations, and a spiral arm structure forms as a result of the growth of the instability. It is typically one-armed, implying that the dominant mode in the nonlinear phase is the $m = 1$ mode.
2. In the linear phase, the dynamical simulations are in good agreement with the linear analysis regarding such features as stability and the oscillation and growth timescales. The prograde modes, in which the deformed shock pattern rotates in the same direction as the unperturbed flow, are unstable, and the retrograde modes are stable. This is consistent with previous works.

3. In the nonlinear phase, various modes are produced by nonlinear couplings, which are mainly quadratic in nature, and the amplitudes saturate. The axisymmetric mode is also induced by the nonaxisymmetric instability, and the shock radius oscillates with large amplitude. The oscillation periods become slightly longer than in the linear analysis because of the larger shock radii.

4. Even if strong perturbations are added initially, the shock continues to exist. The disk-plus-shock system is quite robust in this sense.

5. Comparison of the various cycle timescales with the linear growth times seems to support the claim that the instability is induced by the acoustic-acoustic cycle, although the inner reflection point is not identified unambiguously. It is important to note in this respect that the wavelength of the perturbations is longer than the scale height, which makes the WKB approximation inapplicable.

6. The black hole SASI found by Molteni et al. (1999) may be a promising candidate as the source of black hole QPOs and the fluctuations in GRB jets.

In this study, we also found that the nonaxisymmetric instability is sensitive to the structure of the unperturbed steady flow. General relativity is important in this respect. It should be stressed that the injection parameters that allow the existence of a standing shock wave are different between the GR and pseudo-Newtonian treatments. In fact, we have found by direct comparison that the maximum specific angular momentum for the existence of multiple sonic points is different by more than 60% for a Bernoulli constant $E \leq 1.003$.

Note also that general relativity is indispensable in discussing accretion into a Kerr black hole, since frame dragging will play an important role. This is currently under study (H. Nagakura & S. Yamada 2008, in preparation). For a more detailed comparison with observations, it will be necessary to include cooling and heating for the GRB case and the magnetic field and viscosity for black hole QPOs. Last but not least, comparable simulations including the polar dimension are inevitable.

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APPENDIX

GENERAL RELATIVISTIC HYDRODYNAMIC CODE

Here we describe the GRHD code used in this paper. As mentioned already, it is based on the so-called central scheme, which guarantees good accuracy even if the flows include strong shocks or high Lorentz factors. Magnetic fields can also be included (Del Zanna & Bucciantini 2002; Shibata & Sekiguchi 2005; Duez et al. 2005).

Although we do not take into account the evolution of the gravitational field, the so-called 3+1 formalism is suitable for hydrodynamics as well. Following Duez et al. (2005), we write the metric in the form

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),$$  

(A1)

where $\alpha$, $\beta^i$, and $\gamma_{ij}$ are the lapse, shift vector, and spatial metric, respectively. The basic equations for fluid dynamics in 3+1 form are expressed as

$$\partial_t \rho + \partial_j (\rho u^j) = 0,$n \partial_j S^i + \partial_i (\alpha \sqrt{\gamma} T^i_j) = \frac{1}{2} \alpha \sqrt{\gamma} T^\alpha_\beta g_{\alpha\beta},$$ \n$$\partial_i \tau + \partial_j (\alpha^2 \sqrt{\gamma} T^0_0 - \rho u^i) = s,$$  

(A2)

where the different variables are defined as follows:

$$v^j \equiv u^j/u^t,$$ \n$$\rho_s \equiv \alpha \sqrt{\gamma} \rho u^t,$$ \n$$S_j \equiv \alpha \sqrt{\gamma} T^0_j = \rho_s h u^j,$$ \n$$\tau \equiv \alpha^2 \sqrt{\gamma} T^{00} - \rho_s = \rho_s hu^t - \sqrt{\gamma} p - \rho_s,$$ \n$$s \equiv \alpha \sqrt{\gamma}(T^{00} \beta^j + T^{0j} \beta^0 + T^{ij} K_{ij} - (T^{00} \beta^j + T^{0j} \beta^0) \partial_\alpha).$$  

(A3) \n(A4) \n(A5) \n(A6) \n(A7)

In the above equations, $\gamma$ and $K_{ij}$ are the determinants of the 3-metric and extrinsic curvature, respectively. We refer to $\rho_s$, $S_j$, and $\tau$ as “conserved variables,” whereas $\rho_0$, $p$, and $v^j$ are called “primitive variables” (collectively denoted by $P$).

The conserved variables can be calculated directly from the primitive variables via equations (A4), (A5), and (A6). There is no analytical expression for the primitive variables as a function of the conserved variables, on the other hand. Since the code updates the conserved variables rather than the primitive variables, we must solve for the latter numerically at each time step because they are necessary for the calculation of the characteristic wave speed at each cell interface, as shown below. If we use a $\Gamma$-law EOS, the inversion
can be performed easily, as done by Duez et al. (2005). The same method cannot be applied to a general EOS, however. Hence we adopt a different procedure based on the Newton-Raphson method, as explained below.

We first write down a useful relation between \( u' \) and \( u_j \):

\[
    u' = \alpha^{-1}(1 + \gamma_j u_j)^{1/2}.
\]  

We define two more quantities as

\[
    f_1 = \rho_0^2 \gamma (\rho^2 v^2 + \gamma \rho S S_j) - \rho_0^2 h^2, \quad f_2 = \tau + \rho_s - \rho_s \alpha h u' + \sqrt{\gamma} p.
\]  

We then search iteratively for the primitive variables that satisfy \( f_1 = f_2 = 0 \). We first guess the thermodynamic quantities \( \rho_0 \) and \( p \). Then the other thermodynamic quantities can be obtained from the EOS. Next we obtain \( u \) from equation (A5) using \( S_j, \rho_s, \) and \( h; u' \) is determined by \( u_j \) from equation (A8). Thus the right-hand sides of the equations for \( f_1 \) and \( f_2 \) (eq. [A9]) are expressed as functions of only two thermodynamic quantities. We solve them with the Newton-Raphson method. The initial guess is obtained from the values in the previous step.

The net flux at the cell interface is given by an approximate solution to the Riemann problem. Our code adopts the HLL (Harten–Lax–van Leer) flux, which does not require complete knowledge of the solutions to the Riemann problem, although the maximum wave speed in each direction is needed (Harten et al. 1983). The first step in calculating the flux is to obtain \( P_R \) and \( P_L \), which are the values of the primitive variables interpolated to the right- and left-hand side of each cell interface. We implemented the MUSCL method (Hirsch 1990) for this purpose. From \( P_R \) and \( P_L \), the maximum wave speeds at each side of the cell interface, \( c_{\pm, R} \) and \( c_{\pm, L} \), can be calculated as in Duez et al. (2005).

The HLL flux is then expressed in terms of the maximum wave speeds, defined by \( c_{\text{max}} \equiv \max \{0, c_{+, R}, c_{+, L}\} \) and \( c_{\text{min}} \equiv \max \{0, c_{-, R}, c_{-, L}\} \), as

\[
    f_{\text{int}} = \frac{f_{\text{max}} - f_{\text{min}}}{c_{\text{max}} + c_{\text{min}}},
\]

where \( f_{\text{max}} \) and \( f_{\text{min}} \) are the fluxes calculated with \( P_R \) and \( P_L \), respectively. Note that if we define \( c_{\text{max}} = c_{\text{min}} = \max \{0, c_{+, R}, c_{+, L}, c_{-, R}, c_{-, L}\} \), then \( f_{\text{int}} \) becomes the local Lax-Friedrichs flux.

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