The $AdS_3 \times S^3 \times S^3 \times S^1$ Hernández–López phases: a semiclassical derivation

Michael C Abbott

Department of Mathematics, University of Cape Town, South Africa

E-mail: michael.abbott@uct.ac.za

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Abstract
This note calculates the Hernández–López phases for strings in $AdS_3 \times S^3 \times S^3 \times S^1$ by semiclassical methods using the $d(2, 1; \alpha)$ algebraic curve. By working at general $\alpha$ we include modes absent from previous semiclassical calculations of this phase in $AdS_5 \times S^5$, and in particular can study the scattering of particles of different mass. By carefully re-deriving the semiclassical formula we clarify some issues of antisymmetrization, cutoffs and surface terms which could safely be ignored in $AdS_5 \times S^5$, and some issues about the terms $c_{1, s}$ which were absent there. As a result we see agreement with the recently calculated all-loop dressing phase in the $AdS_3 \times S^3 \times S^3 \times T^4$ case, and exactly $1/2$ this in the general case $AdS_3 \times S^3 \times S^3 \times S^1$, for any $\alpha$ and any (light) polarizations.

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1. Introduction

Our understanding of integrability in the AdS/CFT correspondence between $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM is largely contained in the exact $S$-matrix and its associated Bethe equations [1]. Much of this is fixed by symmetry of the spin-chain, but some remaining freedom is filled in by the celebrated Beisert–Eden–Staudacher dressing phase [2]. The dressing phase was first introduced by Arutyunov, Frolov and Staudacher (AFS) [3] for classical strings (large $\lambda$), and the extension of this to one loop by Hernández and López [4] (see also [5, 6]) was an important step towards the all-$\lambda$ phase, conventionally $(\sigma_{\text{BES}})^2$. This phase was later observed at small $\lambda$ (four loops) by [7].

The first example of AdS/CFT with less supersymmetry to be well-explored, $AdS_4 \times CP^3$ and ABJM theory [8], turns out to have an almost identical dressing phase: one power of $\sigma_{\text{BES}}$ [9, 10]. The subject of this note is IIB strings in the $R–R$ background $AdS_3 \times S^3 \times S^3 \times S^1$, which along with the related background $AdS_3 \times S^3 \times T^4$ has been the subject of much recent work in integrability [11–29] ¹. It has been discovered that while the AFS phase is modified in

¹ There is related work on deformations [30], quotients [31], $S$-matrices [32, 33], and mixed NS–NS backgrounds [34].
a quite simple way, the HL phase is different to that seen before. This was seen in comparisons of one-loop energy corrections [23], in direct semiclassical calculations of the phase by [14, 24] (for the $T^4$ case), and in near-BMN and near-flat-space calculations [27].

There is now a proposal for an all-loop dressing phase for the $T^4$ case [29]. When this is expanded in the coupling, the coefficients $c_{ir}$ in the HL terms are slightly different to those written down by [24].

The goal of this note is to provide a clear semiclassical derivation of the phases for the $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ case, following the method of [35] but being careful to keep all surface terms dropped there. We find coefficients exactly $1/2$ those found by [29], independent of $\alpha$, and unaffected by whether the scattering involves light particles of the same or different mass. It is easy to go to the $T^4$ case by omitting the modes corresponding to one $S^3$ factor, in which case the agreement with [29] is perfect.

Antisymmetry of the phase under the exchange of the two physical particles emerges naturally. In fact there is both a symmetric and an antisymmetric term in the general result (7), and in the magnon regime $p \sim 1$ the former cancels against surface terms. If however we consider the regime $p \ll 1$, then the instead the symmetric and antisymmetric terms cancel against each other to remove the terms linear in $p$ (or in $x$).

Outline
Section 2 looks at one-loop semiclassical phases in general, and section 3 applies this to magnons in $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ (including left–right section 3.1 and unequal mass section 3.2). Section 4 concludes. The appendix sets up the derivation of the classical phase shifts which are the input for this calculation.

2. Deriving the one-loop phase
The same classical data used to find the one-loop correction to the energy [36] can also be used to find the one-loop correction to the scattering phase. This idea, originally from [37], was used by [35] to recover the Hernández–López phase in $\text{AdS}_5 \times S^5$. (A similar idea was also developed by the slightly earlier paper [38].)

The one-loop correction to the energy is given by a sum of frequencies $\omega_n$ over all mode numbers $n$, and polarizations $r$. This can often be written as an integral over momentum $k$, and the necessary Jacobians for this are derivatives of the classical phase shifts $\delta_r(k)$ experienced by modes of the soliton:

$$\Delta E = \sum_r (-1)^F_r \left[ \sum_n \frac{1}{2} \omega'_n - \sum_n \frac{1}{2} \omega'_n \text{vac} \right]$$

$$= \sum_r (-1)^F_r \frac{1}{4\pi} \int dk \frac{\partial \delta_r(k)}{\partial k} \omega'(k).$$

Rather than working in momentum space we will work in the spectral plane, in which case we obtain a similar integral with $\delta_r(z) \Omega'(z)$. For $\text{AdS}_3$ this kind of energy correction has been discussed by [14, 16, 23, 24, 26], starting from a variety of classical systems.

The two $S^3$ factors are of radius $R/cos \phi$ and $R/sin \phi$, where $R$ is the $\text{AdS}_3$ radius and $\phi$ (or often $\alpha = \cos^2 \phi$) is a free parameter of the theory. As a result of this there are modes of mass $1$, $\cos^2 \phi$, $\sin^2 \phi$ and $0$, approximately corresponding to the factors $\text{AdS}_3 \times S^3 \times S^3$. We are mostly concerned with the light modes, $m = \cos^2 \phi$, $\sin^2 \phi$, as these are fundamental in the description of [22, 25]. In the limit $\phi \rightarrow 0$ the second sphere decompactifies, taking us to $\text{AdS}_3 \times S^3 \times T^4$ at least in terms of the bosonic geometry.
The one-loop correction to the scattering phase is given by [35] in terms of the phase shifts $\delta_i(k, p)$ (for a mode $k$ in the background of a single magnon $p$) as follows:

$$\Delta \Theta(p_1, p_2) = \frac{1}{2\pi} \int dk \sum_i (-1)^i \left( \frac{\partial \delta_i(k, p_1)}{\partial k} \delta_i(k, p_2) \right).$$  \hspace{1cm} (2)

However naïvely applying this formula to the present case leads to a correction which is not antisymmetric in $p_1 \leftrightarrow p_2$. This was cured in [14] by explicitly antisymmetrizing, noting that this amounts to keeping one of the surface terms dropped by [35]. Various approaches to this issue were explored in [24].

To derive (2), Chen et al [35] studied a classical scattering state $\varphi_{1,2}(x, t)$ of two magnons in a box of size $L$, and calculated $1/L$ corrections to the one-loop energy correction (1). Their notation is that quantum corrections $1/g$ are written

$$E(p) = gE_{c1}(p) + \Delta E(p) + O\left( \frac{1}{g} \right)$$

and corrections in $1/L$ are written

$$p = p^{(0)} + \frac{1}{L} p^{(1)} + O\left( \frac{1}{L^2} \right).$$

Consider two physical particles with mode numbers $n_1$ and $n_2$, and for short write $p_{n_1} = p_1$ etc. They have polarizations $r_1, r_2$, for which we similarly write $\Theta_{r_1 r_2} = g\Theta_{c1,12} + \Delta \Theta_{c1,12} + \cdots$. To find the one-loop phase $\Delta \Theta_{c1,12}$, we calculate the one-loop order-$1/L$ correction $\Delta E^{(1)}$ in two different ways.

- The first calculation is about the quantization conditions for the momenta. Demanding periodicity of the solution in the box $L$, we obtain

$$p_1 L = 2\pi n_1 + \Theta_{c1,2}(p_1, p_2), \quad p_2 L = 2\pi n_2 + \Theta_{c1,2}(p_2, p_1).$$

The $1/L$ term is

$$p_1^{(1)} = \Theta_{c1,2}(p_1^{(0)}, p_2^{(0)}) + g\Theta_{c1,12}(p_1^{(0)}, p_2^{(0)}) + \Delta \Theta_{c1,2}(p_1^{(0)}, p_2^{(0)}) + O\left( \frac{1}{L} \right)$$

and similarly for $p_2$. Since scattering is elastic, the energy of a two-particle state is $E = E_1(p_1) + E_2(p_2)$. These functions receive $1/g$ corrections but not $1/L$ corrections, so the only such correction comes from the shifts of the momenta:

$$E^{(1)} = \sum_{i=1}^2 \frac{\Delta E_i(p_i^{(0)})}{d p_i} p_i^{(1)} = \sum_i \left[ gE'_{c1,1}(p_i^{(0)}) p_i^{(1)} + \Delta E'_i(p_i^{(0)}) p_i^{(1)} \right] + O\left( \frac{1}{g} \right).$$

Using $p_i^{(1)}$ above, the one-loop part of this is

$$\Delta E^{(1)} = gE'_{c1,1}(p_1^{(0)}) \Delta \Theta_{c1,2}(p_1^{(0)}, p_2^{(0)}) + gE'_{c1,2}(p_2^{(0)}) \Delta \Theta_{c1,2}(p_2^{(0)}, p_1^{(0)})$$

$$+ \Delta E'_1(p_1^{(0)}) g\Theta_{c1,12}(p_1^{(0)}, p_2^{(0)}) + \Delta E'_2(p_2^{(0)}) g\Theta_{c1,12}(p_2^{(0)}, p_1^{(0)}).$$  \hspace{1cm} (4)

The second line is absent in [35]. (However for magnons in AdS$_5$ we have $\Delta E(p) = 0$.)

- The second calculation is a zero-point-energy mode sum, like (1). The frequencies here are those of a third classical particle (mode $n$, momentum $k_n$) on top of the same two physical particles. We must include in $\omega_n$ the shifts in the energies of the other two particles due to the extra phase shift, which we write as $g\Theta_{c1,1 r_1} = \delta_{r_1} = -\delta_{1 r}$. Summing the $1/L$ term of $\omega_n$ then gives

$$\Delta E^{(1)} = \frac{g}{2} \sum_n \left[ E'_{c1,1}(p_1) \left[ \delta_{1}(k_n, p_1) - \delta_{1 r}(k_n^{(0)}, p_1) \right] + E'_{c1,2}(p_2) \left[ \delta_{r 2}(k_n, p_2) - \delta_{2 r}(k_n^{(0)}, p_2) \right] \right]$$

as in [35]. The subscript $r$ is to indicate that we still need to sum over all polarizations.

\[ \text{J. Phys. A: Math. Theor. 46 (2013) 445401 M C Abbott} \]
These can be written in terms of the spectral parameter as

\[ \Delta \Theta_{12}(p_1, p_2) = \sum_r (-1)^r \sum_n \left\{ \frac{E_{cl1}'(p_1)}{E_{cl1}'(p_1) - E_{cl2}'(p_2)} \left[ \delta_{r1}(k_n, p_1) - \delta_{r1}(k_n^0, p_1) \right] \right. \\
+ \left. \frac{E_{cl2}'(p_2)}{E_{cl1}'(p_1) - E_{cl2}'(p_2)} \left[ \delta_{r2}(k_n, p_2) - \delta_{r2}(k_n^0, p_2) \right] \right\} \\
- \frac{\Delta E_{cl1}'(p_1) - \Delta E_{cl2}'(p_2)}{gE_{cl1}'(p_1) - gE_{cl2}'(p_2)} g\Theta_{12}(p_1, p_2). \]

(6)

To convert the sums over \( n \) into integrals, the relevant Jacobians are

\[ 2\pi \frac{dn}{dk} = L + \frac{\partial \delta_{r1}(k, p_1)}{\partial k} + \frac{\partial \delta_{r2}(k, p_2)}{\partial k}, \quad 2\pi \frac{dn}{dk^{(0)}} = L \]

leading to integrals of the form

\[ I_{XY} = \frac{1}{8\pi} \int dk \frac{\partial \delta_{r1}(k, px)}{\partial k} \delta_{r2}(k, py), \quad I_X = I_{XX} \]

where we now write \( p_1 = px \) and \( p_2 = py \). Clearly \( I_X \) is a surface term. It will be convenient to organize the terms like this:

\[ \Delta \Theta_{12,r}(px, py) = (I_{XX} - I_{XY}) + (I_X - I_Y) + \frac{E_{cl1}'(px) + E_{cl2}'(py)}{E_{cl1}'(px) - E_{cl2}'(py)} [(I_{XX} + I_{XY}) + (I_X + I_Y)]. \]

(7)

The total is then \( \Delta \Theta_{12,\tau} = \sum_r (-1)^r \Delta \Theta_{12,r} + ( \text{term } \Delta E_{\tau} \Theta_{12} ) \). We discuss the last term here (which is the last line of (6)) in section 3.3 below, and until then focus on (7) only.

3. \( AdS_3 \) magnon scattering

The phase shifts we need are those for a mode in the background of a single giant magnon. These can be written in terms of the spectral parameter as

\[ \delta_{r}(z, X^\pm) = 2\pi n_r(z)_{\text{mag}}(X^\pm) - 2\pi n_r(z)_{\text{vac}}. \]

(8)

Here \( n_r(z) \) is the mode number of a pole located at \( z \). The locations for which this is an integer will change slightly when we turn on the soliton; this is why (1) is nonzero. These phase shifts were calculated in [23] for the purpose of finding energy corrections, and they are shown in table 1. (See appendix for formulae needed to calculate these, and [23] for more details.)

Note that all of the phase shifts are independent of \( \phi \), and that all are proportional to either \( G(z) = G(z, X^+) \) or \( G(\frac{z}{2}) \). Let us define expansion coefficients of this function to be \( \tilde{Q}_n \) as follows:

\[ G(z, X^\pm) = -i \log \frac{z - X^+}{z - X^-} - \frac{p}{2} = \sum_{n=0}^{\infty} -\tilde{Q}_{n+1} z^n. \]

(9)

Note however that these coefficients are not quite the standard ones \( Q_n \), which are defined for the resolvent \(-i \log \frac{z - X^+}{z - X^-}\) alone. The term \(-p/2\) is a twist added to the quasimomenta to allow for the fact that one giant magnon is not a closed string [40, 41]. Thus

\[ \tilde{Q}_1 = p, \quad \tilde{Q}_1 = \frac{p}{2} \]

(10)

where \( p = -i \log X^+/X^- \), while for all higher charges

\[ Q_{n+1}(X^\pm) = \tilde{Q}_{n+1}(X^\pm) = -i n \left( \frac{1}{X^{+n}} - \frac{1}{X^{-n}} \right), \quad n \geq 1. \]
See for instance [23] for a diagram of the contours.

At strong coupling these are

$$\tilde{Q}_{n+1}(X^\pm) = \frac{2}{n} \sin \left( np \frac{x}{2} \right) + O \left( \frac{1}{n} \right).$$

This form can be used to obtain the following identity

$$\sum_{r,s=1, r+s \text{ odd}}^{\infty} \tilde{Q}_r(p_1) \tilde{Q}_s(p_2) = \frac{\pi^2}{2} \text{sign}(p_1 p_2) + O \left( \frac{1}{n} \right)$$

(11)

derived using these sums:

$$\sum_{n,m=1, n+m \text{ odd}}^{\infty} \frac{\sin \left( np \frac{x}{2} \right)}{n} \frac{\sin \left( mp \frac{x}{2} \right)}{m} = \frac{\pi}{16} (2\pi - |p_1| - |p_2|) \text{sign}(p_1 p_2).$$

$$\sum_{n \geq 1 \text{ odd}}^{\infty} \frac{\sin \left( np \frac{x}{2} \right)}{n} = \frac{\pi}{4} \text{sign}(p_2).$$

Consider first the case of scattering two physical magnons of the same polarization ‘3’, and the contribution of a mode for which $\delta_r \equiv \pm G(z)$. The integrals we need to do arise from sums over $n$ using the method of [42], which uses the poles of $\cot(\pi n)$ to write a contour integral. This is then converted to an integral in $z$ with (8), and after deforming the contour to the unit circle $U$, we may approximate $\cot(\pi n r(z)) = \pm i$ on the upper/lower half plane. The result is this:

$$I_{XY} = \sum_{r}^{\pm} \frac{1}{16\pi} \left[ \int_{U_k} \mp \int_{C_k} \right] dz \frac{\partial G(z, X^\pm)}{\partial z} G(z, Y^\pm).$$

(12)

Here the contours $C_{\pm}$ are components surrounding the poles and cuts outside the unit circle.

- For the integral $I_{XY}$, the antisymmetric term in (7) is

$$I_{XY} - I_{YX} = \frac{1}{8\pi} \sum_{r,s \geq 1, r+s \text{ odd}} \tilde{Q}_r(X^\pm) \tilde{Q}_s(Y^\pm)$$

See for instance [23] for a diagram of the contours.
where \( c_{r,s}^{BLMT} \) are precisely the antisymmetrized coefficients of [24]:

\[
c_{r,s}^{BLMT} = 2 - \frac{s - r}{r + s - 2}, \quad r + s \text{ odd}, \ r, s \geq 1. \tag{13}
\]

For the symmetric term we can use the identity (11) above to get this:

\[
I_{XY} + I_{YX} = \frac{1}{4\pi} \sum_{r,s \geq 1, r+s \text{ odd}} \tilde{Q}_r \tilde{Q}_s = \frac{\pi}{8} \text{sign}(p_1 p_2).
\]

• The \( I_X \) integrals are surface terms. Evaluating these we find

\[
I_X = \frac{1}{4\pi} \sum_{r,s \geq 1, r+s \text{ odd}} \tilde{Q}_r (X^\pm) \tilde{Q}_s (Y^\pm) = \frac{\pi}{16}, \quad I_Y = \frac{\pi}{16}
\]

again using identity (11). Then in the total \( \Delta \Theta \), (7), the symmetric term \( I_{XY} + I_{YX} \) will cancel with the surface terms \( I_X + I_Y \) provided the signs of \( p_1 \) and \( p_2 \) are different.

• Finally, the contributions from \( C_{\pm} \) (namely poles at \( X^\pm \) and log cuts ending at \( Y^\pm \)) all cancel within each integral.

Then we must consider modes with \( \delta_r = G(1/z) \), but in fact these gives exactly the same results.

It remains only to count the number of modes, using table 1, allowing for pre-factors and counting fermions with \( -1 \). Define \( \eta \) by

\[
\eta = \frac{-2 + 8 - 2 - 2}{\sin^2 \phi, \cos^2 \phi, 1} = \begin{cases} 2, & \text{AdS}_3 \times S^3 \times S^3 \times S^1 \\ 4, & \text{AdS}_3 \times S^3 \times T^4 \end{cases}
\]

(14)

where we omit modes of mass \( \sin^2 \phi \) in the \( T^4 \) case. This is the only point of difference between the \( T^4 \) and \( S^3 \) cases. Then the final result \( \Delta \Theta_{33} \) in the \( S^3 \) case, or \( \Delta \Theta_{(i\bar{4})_(i\bar{4})} \) in the \( T^4 \) case, is

\[
\Delta \Theta (X^\pm, Y^\pm) = \eta \sum_{r,s \geq 1, r+s \text{ odd}} c_{r,s}^{BLMT} \tilde{Q}_r (X^\pm) \tilde{Q}_s (Y^\pm)
\]

\[
= \eta \sum_{r,s \geq 1, r+s \text{ odd}} c_{r,s}^{BOST} Q_r (X^\pm) Q_s (Y^\pm). \tag{15}
\]

On the second line we use (10) to express the correction in terms of \( Q_n \) rather than \( \tilde{Q}_n \), and absorb a factor \( \frac{1}{4\pi} \). The resulting coefficients are

\[
c_{r,s}^{BOST} = \frac{1}{4\pi} \left[ \frac{2 - \frac{s - r}{r + s - 2} - \delta_{r,1} + \delta_{s,1}}{r + s \text{ odd}, \ r, s \geq 1} \right] 
\]

\[
= \begin{cases} \frac{1}{8\pi} c_{r,s}^{BLMT}, & r = 1 \text{ or } s = 1 \\ \frac{1}{4\pi} c_{r,s}^{BLMT}, & r, s \geq 2. \end{cases} \tag{16}
\]

This is the correction to the phase of the \( S \)-matrix, which is conventionally written as \( S = \hat{S} e^{2i\sigma} \) with the dressing phase \( \sigma = e^{i\theta} \). Thus \( \Delta \Theta = 2\theta_{HL} \), with which we see perfect agreement with [29] for the \( T^4 \) case.
3.1. The left–right phase

In the above derivation we can equally well start from a classical scattering state of two different polarizations. When we scatter one left particle and one right particle (for instance by taking \( r_1 = 3, r_2 = 3 \)) then the correction is different, as was observed in the \( T^4 \) case by [24] 4.

The only integral we need consider is the case when \( \delta_X(z) = G(z, X^\pm) \) and \( \delta_Y(z) = -G(1/z, Y^\pm) \). For this,

\[
I_{XY} - I_{YX} = \frac{1}{8\pi} \sum_{s \geq 1, r + s \text{ odd}} c_{r,s}^{BLMT} \tilde{Q}_r(X^\pm)\tilde{Q}_s(Y^\pm)
\]

where

\[
c_{r,s}^{BLMT} = -2 \frac{r + s - 2}{s - r}, \quad r + s, \quad r, s \geq 1.
\]

Then adding up all polarizations we get the same factor \( \eta \) (14), and writing the result in terms of the \( Q_n \) we again see agreement with [29] in the \( T^4 \) case 5:

\[
\Delta \Theta_{33}(X^\pm, Y^\pm) = \frac{\eta}{2} \sum_{r + s \text{ odd}} \tilde{c}_{r,s} \tilde{Q}_r(X^\pm)\tilde{Q}_s(Y^\pm)
\]

(17)

where

\[
\tilde{c}_{r,s}^{BOSST} = \frac{1}{4\pi} \left[ -2 \frac{r + s - 2}{s - r} + \delta_{r,1} \delta_{s,1} \right], \quad r + s, \quad r, s \geq 1.
\]

(18)

The relation between these two phases and the original \( AdS_5 \) HL phase is this:

\[
c_{r,s} + \tilde{c}_{r,s}^{BOSST} = \frac{1}{4\pi} \left[ c_{r,s}^{BLMT} + \tilde{c}_{r,s}^{BLMT} \right] = \frac{1}{4\pi} \left[ -8 \frac{(r - 1)(s - 1)}{(r + s - 2)(s - r)} \right].
\]

(19)

The factor in square brackets is precisely \( c_{r,s} \) defined in [4]. Note that the terms with \( r = 1 \) or \( s = 1 \) cancel here, and that the modifications of [29] likewise cancel.

3.2. Particles of different mass 6

In the \( S^4 \) case there is, in addition to a division into left- and right-sector particles, a distinction between light particles of mass \( \sin^2 \phi \), mass \( \cos^2 \phi \), and heavy particles of mass 1. The one-loop scattering phase for two light particles of different masses is the same, while the heavy modes behave as composites of two light particles.

To show this we must compute \( \Delta \Theta_{ab} \) for all \( a, b \), and it is sufficient to show that the following factor does not vary:

\[
\sum_r (-1)^r \partial_z \delta_{ta}(z) \delta_{tb}(z) = 2 \left[ G'(z) G(z) - \partial_z G \left( \frac{1}{z} \right) \right] G \left( \frac{1}{z} \right).
\]

(20)

The phase shifts \( \delta_{ta} \) needed for light bosons are trivially related to \( \delta_{t3} \) as used above; they are shown in table 2.

The one-loop phase is also the same if one of the particles is a fermion: \( \Delta \Theta_{3,1f} = \Delta \Theta_{33} \). In this case the classical soliton needed for \( \delta_{t1f}(z, X^\pm) \) is no longer a cromulent giant magnon,

4 See also [32] for an earlier appearance of two independent dressing phases.

5 Strictly this is \( \Delta \Theta_{(1,3)} \) in the \( T^4 \) case.

6 I thank the authors of [29] for suggesting this clarification, expanded from a single sentence in the initial preprint.
but proceeding all the same by turning on $G_1 = G_2 = \frac{1}{\sin^2 \phi} G(z)$ (i.e. making a giant $1f$ mode, see table A2) we get the following quasimomenta:

\[
\begin{align*}
p_1(x) &= \frac{1}{2 \sin^2 \phi} G(x, X^\pm) \\
p_2(x) &= \frac{\Delta}{2g} x^2 - 1 + \frac{1}{2 \sin^2 \phi} G(x, X^\pm) \\
p_3(x) &= 0.
\end{align*}
\]

(21)

This and the similar giant $3f$ mode with $G_1 = G_2 = \frac{1}{\cos^2 \phi} G(z)$ lead to the phase shifts on the right of table 2. In this case they are not independent of $\phi$, but nevertheless the total (20) is the same, and hence $\Delta \Theta$ is too.

Finally we can also consider scattering with one of the heavy modes. These are not fundamental particles in the Bethe equations, but in the worldsheet theory they are simply less strongly curved directions in space. The heavy bosons are modes in $AdS_3$ directions, and thus not giant magnons. From the integrable point of view they are a sum of two fermions, $4 = 1f + 3f$, and hence the relevant phase shifts are

$$\delta_{r,4}(z, X^\pm) = \delta_{r,1f}(z, X^\pm) + \delta_{r,3f}(z, X^\pm).$$

Then the result for heavy-light scattering is clearly $\Delta \Theta_{34} = 2 \Delta \Theta_{33}$, and for heavy–heavy $\Delta \Theta_{44} = 4 \Delta \Theta_{33}$.

Note again that this discussion is limited to the $S^1$ case. In the $T^4$ case, the scattering of two $AdS$ modes $(1, 4)$ has exactly the same $\Delta \Theta$ as two sphere modes $(1, 4)$. If we are to regard the $T^4$ case as the limit $\phi \to 0$, then this is an additional discontinuity, and in the opposite direction to (14) above. These two discontinuities can be summarized as follows:

\[
\begin{align*}
sphere: \quad &\Delta \Theta_{(1\bar{4}), (1\bar{4})} = 2 \Delta \Theta_{3,3} \\
AdS: \quad &\Delta \Theta_{(1\bar{4}), (1\bar{4})} = 2 \Delta \Theta_{3,3} = \frac{1}{2} \Delta \Theta_{4,4}.
\end{align*}
\]

(22)

3.3. Dependence on cutoff prescription\(^7\)

So far we have ignored the last term in (6), with $\Delta E' \Theta_{\xi}$. While there is no explicit integral, this term will depend on the cutoff prescription used through $\Delta E$, the one-loop term in the dispersion relation, which was calculated using in (1) in [23]. This is

$$\Delta E = \frac{2c \sin \frac{p}{2}}{2}$$

\(^7\) This section was added in an update. The original preprint mistakenly claimed that this term cancelled out.
where \( c = 0 \) when using a cutoff in the spectral plane, and is given by (A.8) when using a
cutoff on the physical energy. Then the term becomes

\[
- \frac{\Delta E'_1(p_1) - \Delta E'_2(p_2)}{gE'_{cl1}(p_1) - gE'_{cl2}(p_2)} g\Theta_{cl12}(p_1, p_2) = -\frac{c}{2} \Theta_{cl12} + \mathcal{O}\left(\frac{1}{g}\right).
\]  

(23)

This constant \( c \) is the one-loop term in the relation between the Bethe coupling and the
string tension: \( h = 2g + c + \mathcal{O}(1/g) \). The former is the coupling appearing in the integrable
structure, while the latter appears in any pure string calculation. Based on what we learned in
\( AdS_3 \times CP^3 \) [43–45] we might expect the last term in (6) to arise as a result of expanding in \( g \)
some exact result depending on \( h \) but not on \( c \). Comparing the expansion in \( g \) from (3) above
to one in \( h \), we do indeed find such a term:

\[
\Theta = g\Theta_{cl} + \Delta\Theta + \mathcal{O}\left(\frac{1}{g}\right) = \frac{h}{2} \Theta_{cl} + \left[ \Delta\Theta - \frac{c}{2} \Theta_{cl} \right] + \mathcal{O}\left(\frac{1}{h}\right).
\]  

(24)

However the sign of (23) is not right for this explanation—this term is part of \( \Delta\Theta \) and so
doubles the \( -\frac{c}{2} \Theta_{cl} \) in square brackets, rather than cancelling it. Thus using a different cutoff
(and hence a different \( c \)) will lead us to a different scattering phase \( \Theta(h) \).

Something conceptually similar was seen in [23], where changing the cutoff prescription
led to a change in the one-loop term in the cusp anomalous dimension \( f(h) \). It was argued there
that this meant the cutoff was not a matter of indifference, and that there must be a preferred
choice. And further that if we demand a smooth \( \phi \to 0 \) limit, then the preferred choice was
the physical cutoff. Here we saw that a smooth connection to the \( T^4 \) case seems to be ruled
out, (22), undercutting this argument. So the question of what prescription is preferred may
still be open.

3.4. A very small \( p \) limit

The results above assume \( p \sim 1 \) and strong coupling, particularly in (11). This section looks
very briefly at the result of expanding instead at small \( p \), small enough that we may approximate

\[
Q_n(X^\pm) = (pX)^n\left(\frac{h}{2m}\right)^{n-1} + \mathcal{O}(pX)^{n+2}h^{n-1}(1 + h^2).
\]  

(25)

It is a novel feature of the \( AdS_3 \) dressing phase that \( \Delta\Theta \) has terms \( c_1Q_1Q_2 \) which can
contribute at linear order in this limit. In terms of the integrals above, the antisymmetric term gives

\[
I_{XY} - I_{YX} = \frac{1}{4\pi} \sum_{s \geq 2 \text{ even}} Q_s(Y^\pm)p_X + \mathcal{O}(p_X)^2.
\]  

(26)

However, notice that the symmetric term gives exactly the same result:

\[
I_{XY} + I_{YX} = \frac{1}{4\pi} \sum_{s \geq 2 \text{ even}} Q_s(Y^\pm)p_X + \mathcal{O}(p_X)^2.
\]

This is no longer cancelled by the surface terms, as the first term in \( I_X \) is \( Q_1Q_2 \sim p_X^3 \), and
clearly \( I_Y \) is free of \( p_X \). But it comes with a pre-factor

\[
\frac{E'_{cl}(p_X) + E'_{cl}(p_Y)}{E'_{cl}(p_X) - E'_{cl}(p_Y)} = -1 + \mathcal{O}(p_X)
\]

and thus cancels the antisymmetric term.
4. Summary and conclusions

This note gives the first direct semiclassical calculation of the one-loop dressing phase for strings in $AdS_5 \times S^5 \times S^5 \times S^5$. The result, when expanded in the charges $Q_a$, is precisely one half the coefficients found by BOSST [29] from their all-loop dressing phase for $AdS_5 \times S^5 \times T^4$. If we sum over only those virtual modes which remain massive in the $T^4$ limit, then this matches [29] exactly.

Both the $S^4$ and $T^4$ cases feature a division of particles into left and right sectors, and the HL phase for the scattering of two particles of the same persuasion is different to that for the scattering of one of each. In the $S^4$ case there is also a division between particles of mass $\sin^2 \phi$ and those of mass $\cos^2 \phi$, but the HL phase is indifferent to this $^8$.

Some comments on the calculation:

- Antisymmetry is built into this way of calculating the phase: if we do not assume that $\Delta \Theta_{21} = -\Delta \Theta_{12}$ in (4) then equating it to (5) tells us nothing. To get something like (2) we must drop some surface terms which vanish in $AdS_5$ but not here.
- In the analogous $AdS_5$ and $AdS_4$ calculations [9, 35, 38], all modes with classical phase shift $G(z)$ or $G(1/z)$ cancel out, and the entire phase is given by modes $\delta(z) = G(z) - G(1/z)$. (See table A1.) This clearly vanishes at $z = \pm 1$, which is what removes the surface terms.
- The integral implementing the infinite sum over all modes $n$ is finite for each polarization considered (and the integrand regular at $z = \pm 1$) so should not be sensitive to the cutoff used. This is in contrast to the case of energy corrections (1), where the contribution of one boson or fermion diverges quadratically $^9$. However one term in $\Delta \Theta$ arises from the magnons' one-loop energy corrections $\Delta E_{\alpha}$, and through this depends on the cutoff-dependent subleading term in $h = 2g + c + O(1/g)$. The one-loop term when expanding in $h$ is not independent of $c$. This is discussed in section 3.3, and elsewhere assumed to vanish.
- The phase for the $S^4$ case is independent of $\phi$ (or $\alpha$). However to see agreement with the $T^4$ case we must omit those virtual modes which become massless in this limit, (14). Thus if the $T^4$ case is to be thought of as a limit $\phi \to 0$ then it is a discontinuous one. Some related discontinuities were seen in worldsheet calculations in [24]. Note also that the scattering of two heavy modes has four times the $\Delta \Theta$, but in the $T^4$ case this distinction is lost. This is a second discontinuity, and in the opposite direction, see (22).
- When working in the $T^4$ case (and apart from the issue of antisymmetrization) the difference between this calculation and that of [24] is equation (10), arising from the twists needed in the quasimomenta [40, 41]. These were treated correctly in [14]'s calculation of the HL phase, although this for the left–left case only, and the result was not expanded in terms of the charges $Q_a$.
- Finally, and more speculatively, note that for the linear term when expanding $\Delta \Theta$ at small $p$, which is proportional to $c_{1,1}$ (and thus absent in $AdS_4$), there is a cancellation between the antisymmetrized term and the symmetric term $I_{XY} + I_{YX}$ (which in the magnon regime instead cancels surface terms).

Some comments on relations to other work.

The near-flat-space limit calculation of [27] found agreement with the phase given by [24] $^10$. The lowest terms $c_{e,1}$ and $c_{1,4}$ are not visible in this limit (the sum of all such is $h^{-1/2}$)

---

$^8$ The full theory contains also modes of mass 1 (which are composite in the Bethe equation description) and massless modes (absent both here and in the Bethe equations).

$^9$ For discussion of this see [43–45] in $AdS_4$ and [18, 23, 24] in $AdS_5$.

$^{10}$ The near-BMN theory derived there is for the $S^4$ case, with the limit $\phi \to 0$ taken before comparing with [24]. It is not entirely clear whether a factor of 2 discrepancy should be seen.
compared to other terms at $h_0$) and thus nothing changes when using the phase of [29]. In the near-BMN limit all terms contribute at the same order, and thus it would be very interesting if a way to avoid the divergences seen by [27] could be found.

Nothing in the derivation above makes any reference to the Bethe equations. Of course in $AdS_5 \times S^5$ the dressing phase was invented to match the string’s Bethe equations to those for the SYM spin chain [3, 46]. And the goal of [4, 5] was to reproduce the energy of certain strings at one loop.

Some comparisons between Bethe equations and $AdS_3$ string calculations were explored in [24, 26], using the equations proposed in 2011 [15] which reduce (in the sectors considered) to exactly the same $su(2)$ and $sl(2)$ Bethe equations used for $AdS_5 \times S^5$. When calculating the effect of the HL phase on the energy of a given string, the only change needed in [4] is to start the sums at $r, s = 1$. In general the agreement was good but the interpretation of some extra terms is not entirely clear.

These comparisons will certainly be changed by the use of the coefficients of [29], and they should also be updated to test the Bethe equations proposed in [28]. In the $S^3$ case we should use the coefficients derived here, and the relevant Bethe equations are those of [25]. I hope to return to this topic in the near future.

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Appendix. Algebraic curve setup

Here for completeness we recall the parts of [23] we will need; this in turn was largely following [12, 13]. Afterwards we will modify this to use the mixed grading suggested by [25] but the resulting system is equivalent.

The Cartan matrix for $d(2, 1; \alpha)^2$ is

$$A = \begin{bmatrix} 4\sin^2 \phi & -2\sin^2 \phi & 0 \\ -2\sin^2 \phi & 0 & -2\cos^2 \phi \\ 0 & -2\cos^2 \phi & 4\cos^2 \phi \end{bmatrix} \otimes 1_{2 \times 2}. \quad (A.1)$$

For each Cartan generator $A_\ell$, there is a quasimomentum $p_\ell(x)$, where $\ell = 1, 2, 3, \bar{1}, \bar{2}, \bar{3}$. In addition to $A$, we also need to know the matrix $S$ which gives the inversion symmetry:

$$p_\ell \left( \frac{1}{x} \right) = S \sum \sum (x), \quad S = 1_{3 \times 3} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (A.2)$$

The vacuum algebraic curve has poles at $x = \pm 1$, controlled by a vector $\kappa_\ell$:

$$p_\ell(x) = \frac{\kappa_\ell x}{x^2 - 1}, \quad \kappa = \frac{\Delta}{2g} (0, 1, 0, 0, -1, 0). \quad (A.3)$$

Solutions above this vacuum constructed by introducing various cuts. The crucial equation is that when crossing a cut $C$ (of mode number $n$) in sheet $\ell$, the change in $p_\ell$ is given by

$$p_\ell \to p_\ell - A_{\ell m} p_m + 2\pi n. \quad (A.4)$$

By continuity this change must be zero at the end of the cut, and this gives us an equation relating the position of the branch point to the cut’s mode number. The various polarizations (including heavy modes) are summarized in table A2.
For similar Lüscher F-term corrections in $\psi\bar{\psi}\phi$ perturbation of the quasimomenta nearby is given by $m$ magnons in [40] where they play an important role in finite-size corrections. It is needed to take account of the fact that a giant magnon is not a closed string; alternatively, it is a result in working in some $n$-torus. In fact $1, 3$ and $(2, 4)$ alone are sufficient to derive the HL phase, the others cancel. These are precisely the bosons absent from $AdS_3 \times S^3$ (table 1).

\begin{table}[h]
\centering
\caption{Phase shifts in $AdS_3 \times S^3$, for a background giant magnon of $(2, 3)$ polarization. In fact $(1, 3)$ and $(2, 4)$ alone are sufficient to derive the HL phase, the others cancel. These are precisely the bosons absent from $AdS_3 \times S^3$ (table 1).}
\begin{tabular}{ccc}
\hline
$r$ & $\delta_r(z)$ & $r$ \\
\hline
$3, 3$ & $G(z) - G(z)$ & $3, 3$ & $G(z) - G(z)$ \\
$1, 1$ & $G(z) - G(z)$ & $1, 1$ & $G(z) - G(z)$ \\
$0$ & $G(z) - G(z)$ & $0$ & $G(z) - G(z)$ \\
\hline
\end{tabular}
\end{table}

Vibrational modes are very short cuts which may be treated as poles. The resulting perturbation of the quasimomenta nearby is given by

$$\delta p_i(x) = \frac{k_i \alpha(y)}{x - y} + \mathcal{O}(x - y)^0, \quad \alpha(y) = \frac{1}{2 g y^2 - 1}.$$ 

The energy of this perturbation can be read off from $\delta p_i(x)$ near to $x = \infty$, and gives the off-shell frequency $\Omega(y) = \delta \Delta$. The frequency of mode number $n$ is then given by evaluating this at the point fixed by (A.4).

The giant magnon is described by

$$G(x, X^\pm) = -i \log \frac{x - X^+}{x - X^-} + \frac{i}{2} \log \frac{X^+}{X^-}. \tag{A.5}$$

The second term here is a twist of the type first discussed in [41], and employed for giant magnons in [40] where they play an important role in finite-size corrections. It is needed to take account of the fact that a giant magnon is not a closed string; alternatively, it is a result in working in some $Z_n$ orbifold [49] in which the string is closed. (See [50] for earlier work on $Z_n$ symmetries.) The same twists were used for the construction of BMN modes in [23].

Choosing to consider a giant ‘3’ mode we are interested in the curve

$$p_1(x) = 0 \quad p_2(x) = \frac{\Delta}{2 g x^2 - 1} \quad p_3(x) = \frac{1}{2 \cos^2 \phi} G(x, X^\pm). \tag{A.6}$$

This has momentum $p = -i \hbar \frac{\Delta}{2 \cos^2 \phi} (X^+ + 1/X^- + c.c.) = 1$ for an elementary magnon, and dispersion relation

$$E(p) = \Delta - J' = -i \frac{\hbar}{2} \left( X^+ - \frac{1}{X^+} - X^- + \frac{1}{X^-} \right) = \sqrt{Q^2 \cos^2 \phi + 4 \hbar^2 \sin^2 \phi \frac{p}{2}}. \tag{A.7}$$

\footnote{For similar Lüscher F-term corrections in $AdS_4 \times CP^3$, it was necessary to introduce similar twists in [47] compared to the ansatz of [48].}
The Bethe coupling\textsuperscript{12} $h$ is related to $g = R^2/4\pi\alpha'^2 = \sqrt{\lambda}/4\pi$ (half the effective string tension) by\textsuperscript{13}

$$h = 2g + \frac{\sin^2\phi \log(\sin^2\phi) + \cos^2\phi \log(\cos^2\phi)}{2\pi} + \mathcal{O}\left(\frac{1}{g}\right). \quad (A.8)$$

The subleading term was first written down by [14] for $\phi = 0$ and by [18, 23, 24] for general $\phi$. This assumes we are using the ‘physical’ prescription, i.e. using a cutoff on energy. Using a cutoff in the spectral plane, the subleading term vanishes [23].

\subsection*{A.1. Mixed grading}

According to [25] we should use a $d(2, 1; \alpha)$ Dynkin diagram with different grading for modes on the right. Drawing $\bigoplus$ for a fermionic node (and with labels to indicate momentum-carrying roots) the left and right diagrams are:

\[ \begin{array}{c}
\bigoplus \\
1 \\
\bigoplus
\end{array} \quad \begin{array}{c}
\bigoplus \\
1 \\
\bigoplus
\end{array} \]

Above we used the left-hand diagram for both left and right modes, as in [23] and [13, 15]. In this appendix we explicitly change the set-up to use the mixed grading, and confirm that (as expected) this does not affect anything.

To do this we now use the following Cartan matrix:

$$A' = \begin{bmatrix}
4\sin^2\phi & -2\sin^2\phi & 0 \\
-2\sin^2\phi & 0 & -2\cos^2\phi \\
0 & -2\cos^2\phi & 4\cos^2\phi
\end{bmatrix} \oplus \begin{bmatrix}
0 & 2\sin^2\phi & -2 \\
2\sin^2\phi & 0 & 2\cos^2\phi \\
-2 & 2\cos^2\phi & 0
\end{bmatrix}. \quad (A.9)$$

With this we need

$$S = \begin{bmatrix}
1 & 0 & 0 \\
1 & -1 & 1 \\
0 & 0 & 1
\end{bmatrix} \otimes \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}. \quad (A.10)$$

and vacuum

$$\kappa = -\frac{\Delta}{2g}(0, 1, 0, 0, 1, 0). \quad (A.11)$$

The mode corresponding to ‘1’\bigoplus via inversion symmetry is now\bigoplus, colouring in two fermionic nodes to make a boson. All the others are listed in table A.3.

The giant magnon in the ‘3’ polarization now has

$$p_1(x) = 0$$

$$p_2(x) = \frac{\Delta}{2g}x^2 - 1 + \frac{1}{2\cos^2\phi}G(\frac{1}{2}, X^+) \quad (A.12)$$

$$p_3(x) = \frac{1}{2\cos^2\phi}G(\frac{1}{2}, X^+).$$

It is easy to check that when using this the phases $\delta_i$ in table 1 are unchanged.

\textsuperscript{12} This coupling is identified with $\sqrt{\lambda}$ in $\text{AdS}_5$, but not in $\text{AdS}_4$ [51] where it was further investigated at strong coupling by [43–45] and at weak coupling by [52]. Perturbative checks at weak coupling have now been done up to six loops [53], and eight loops [10] at which the first influence of the dressing phase appears.

\textsuperscript{13} Note that [24, 29] define $h$ differently, and [18] defines $g$ differently. My conventions follow [14, 15, 25].
Table A3. Right-hand modes when using the mixed grading Cartan matrix $A'$, (A.9). Filled nodes for $\ell = \bar{1}, \bar{2}, \bar{3}$ here indicate $k_{\ell r} = 1$. 

| $r$ | $m_r$ | $2\pi n_r$ |
|-----|-------|------------|
| $\bar{1}$ | $\sin^2 \phi$ | $-(A'_{1\bar{1}} + A'_{3\bar{1}}) p_\ell$ |
| $\bar{1}f$ | $\sin^2 \phi$ | $-A'_{1\bar{1}} p_\ell$ |
| $\bar{3}$ | $\cos^2 \phi$ | $-(A'_{1\bar{3}} + A'_{3\bar{3}}) p_\ell$ |
| $\bar{3}f$ | $\cos^2 \phi$ | $-A'_{1\bar{3}} p_\ell$ |
| $\bar{4}$ | $1$ | $(A'_{1\bar{4}} - A'_{1\bar{2}}) p_\ell$ |
| $\bar{4}f$ | $1$ | $(A'_{1\bar{4}} - A'_{1\bar{2}} - A'_{1\bar{3}}) p_\ell$ |

References

[1] Beisert N et al 2012 Review of AdS/CFT integrability: an overview Lett. Math. Phys. 99 3–32 (arXiv:1012.3982)
[2] Beisert N, Eden B and Staudacher M 2007 Transcendentality and crossing J. Stat. Mech. 01 P021 (arXiv:hep-th/060251)
[3] Arutyunov G, Frolov S and Staudacher M 2004 Bethe ansatz for quantum strings J. High Energy Phys. JHEP10(2004)016 (arXiv:hep-th/0610251)
[4] Hernández R and López E 2006 Quantum corrections to the string Bethe ansatz J. High Energy Phys. JHEP07(2006)004 (arXiv:hep-th/0603204)
[5] Beisert N and Tseytlin A A 2005 On quantum corrections to spinning strings and Bethe equations Phys. Lett. B 629 102–10 (arXiv:hep-th/0509084)
[6] Freyhult L and Kristjansen C 2006 A universality test of the quantum string Bethe ansatz Phys. Lett. B 638 258–64 (arXiv:hep-th/0604069)
[7] Beisert N, Hernández R and López E 2006 A crossing-symmetric phase for $AdS_5 \times S^5$ strings J. High Energy Phys. JHEP11(2006)070 (arXiv:hep-th/0604044)
[8] Beisert N, McLoughlin T and Roiban R 2007 The four-loop dressing phase of $\mathcal{N} = 4$ SYM Phys. Rev. D 76 046002 (arXiv:0705.0321)
[9] Aharony O, Bergman O, Jafferis D L and Maldacena J M 2008 $\mathcal{N} = 6$ superconformal Chern–Simons-matter theories, M2-branes and their gravity duals J. High Energy Phys. JHEP10(2008)091 (arXiv:0806.1218)
[10] Klose T 2012 Review of AdS/CFT integrability, chapter iv.3: $\mathcal{N} = 6$ Chern–Simons and strings on $AdS_5 \times CP^3$ Lett. Math. Phys. 99 401–23 (arXiv:1012.3999v5)
[11] Gromov N and Vieira P 2009 The all loop $AdS_4/CFT_3$ Bethe ansatz J. High Energy Phys. JHEP01(2009)016 (arXiv:0807.0777)
[12] Mauri A, Santambrogio A and Scoleri S 2013 The leading order dressing phase in ABJM theory J. High Energy Phys. JHEP04(2013)146 (arXiv:1301.7732)
[13] David J R and Sahoo B 2008 Giant magnons in the D1-D5 system J. High Energy Phys. JHEP07(2008)033 (arXiv:0804.3267)
[14] Babichenko A, Stefanski B Jr and Zarembo K 2010 Integrability and the $AdS_3/CFT_2$ correspondence J. High Energy Phys. JHEP03(2010)058 (arXiv:0912.1723)
[15] Zarembo K 2011 Algebraic curves for integrable string backgrounds Proc. Steklov Inst. Math. 272 275–87 (arXiv:1005.1342v2)
[16] David J R and Sahoo B 2010 $S$-matrix for magnons in the D1–D5 system J. High Energy Phys. JHEP10(2010)112 (arXiv:1005.0501)
[17] Ohlsson Sax O and Stefanski B Jr 2011 Integrability, spin-chains and the $AdS_3/CFT_2$ correspondence J. High Energy Phys. JHEP06(2011)029 (arXiv:1106.2558)
[18] Formi V, Pulitetti V G M and Ohlsson Sax O 2012 Generalized cusp in $AdS_3 \times CP^3$ and more one-loop results from semiclassical strings J. Phys. A: Math. Theor. 46 115402 (arXiv:1204.3302)
[19] Rughoonauth N, Sundin P and Wulff L 2012 Near-BMN dynamics of the $AdS_3 \times S^3 \times S^3 \times S^1$ superstring J. High Energy Phys. JHEP07(2012)159 (arXiv:1204.4742)
[39] Gromov N and Vieira P 2008 The $AdS_5 \times S^5$ superstring quantum spectrum from the algebraic curve Nucl. Phys. B 789 175–208 (arXiv:hep-th/0703191)

[40] Gromov N, Schäfer-Nameki S and Vieira P 2008 Quantum wrapped giant magnon Phys. Rev. D 78 026006 (arXiv:0801.3671)

[41] Gromov N and Vieira P 2008 Complete 1-loop test of AdS/CFT J. High Energy Phys. JHEP04(2008)046 (arXiv:0709.3487)

[42] Schäfer-Nameki S 2006 Exact expressions for quantum corrections to spinning strings Phys. Lett. B 639 571–8 (arXiv:hep-th/0602224)

[43] McLoughlin T, Roiban R and Tseytlin A A 2008 Quantum spinning strings in $AdS_4 \times \mathbb{C}P^3$: testing the Bethe ansatz proposal J. High Energy Phys. JHEP11(2008)069 (arXiv:0809.4038)

[44] Gromov N and Mikhaylov V 2009 Comment on the scaling function in $AdS_4 \times \mathbb{C}P^3$ J. High Energy Phys. JHEP04(2009)083 (arXiv:0807.4897)

[45] Abbott M C, Aniceto I and Bombardelli D 2010 Quantum strings and the $AdS_4/CFT_3$ interpolating function J. High Energy Phys. JHEP12(2010)040 (arXiv:1006.2174)

[46] Abbott M C and Sundin P 2012 The near-flat-space and BMN limits for strings in $AdS_4 \times \mathbb{C}P^3$ at one loop J. Phys. A: Math. Theor. 45 025401 (arXiv:1106.0737)

[47] Abbott M C, Aniceto I and Bombardelli D 2012 Real and virtual bound states in Lüscher corrections for $\mathbb{C}P^3$ magnons J. Phys. A: Math. Theor. 45 335401 (arXiv:1111.2839)

[48] Ikeda K 2004 Semiclassical strings on $AdS_5 \times S^5$ and operators in orbifold field theories J. High Energy Phys. JHEP03(2004)013 (arXiv:hep-th/0312029)

[49] Minahan J A, Ohlsson Sax O and Sieg C 2010 Magnon dispersion to four loops in the ABJM and ABJ models Phys. Rev. D 81 126004 (arXiv:0911.0689)