Cosmic Acceleration without dark energy

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Abstract. In this work, we investigate the global dynamics of the universe within the framework of the Interacting Dark Matter (IDM) scenario. Considering that the dark matter obeys the collisional Boltzmann equation, we can obtain analytical solutions of the global density evolution, which can accommodate an accelerated expansion, equivalent to either the quintessence or the standard Λ models. This is possible if there is a disequilibrium between the DM particle creation and annihilation processes with the former process dominating, which creates an effective source term with negative pressure. We also find realistic solutions in which the present time is located after the inflection point.

1. Introduction

The detailed analysis of the available high quality cosmological observations ([1, 2, 3, 4] and references therein) have converged during the last decade towards a cosmic expansion history that involves a spatial flat geometry and a recent accelerating expansion of the universe. This expansion has been attributed to an energy component (the so called dark energy) with negative pressure which dominates the universe at late times and causes the observed accelerating expansion. The nature of the dark energy is still a mystery and it is one of the most fundamental current problems in physics and cosmology. Indeed, due to the absence of a physically well-motivated fundamental theory, there have been many theoretical speculations regarding the nature of the above exotic dark energy (DE) among which a cosmological constant, scalar or vector fields (see [5, 6, 7, 8, 9, 10, 11] and references therein).

Most of the recent papers in this kind of studies are based on the assumption that the DE evolves independently of the dark matter (DM). The unknown nature of both DM and DE implies that we can not preclude future surprises regarding the interactions in the dark sector. This is very important because interactions between the DM and quintessence could provide possible solutions to the cosmological coincidence problem. Recently, several papers have been published in this area [12, 13] proposing that the DE and DM could be coupled, assuming also that there is only one type of non-interacting DM.

However, there are other possibilities. (a) It is plausible that the dark matter is self-interacting (IDM) [14], a possibility that has been proposed to solve discrepancies between theoretical predictions and astrophysical observations, among which the gamma-ray and microwave emission from the center of our galaxy (e.g. [15, 16], [17, 18] and references therein). It has also been shown that some dark matter interactions could provide an accelerated expansion phase of the Universe [19, 20, 21]. (b) The DM could potentially contain more than one particle species, for
example a mixture of cold and warm or hot dark matter [22, 23], with or without inter-component interactions.

In this work we are not concerned with the viability of the different such possibilities, nor with the properties of interacting DM models. The aim of this work is to investigate only whether there are repercussions of DM self-interactions or interactions between different species of a two-component DM, for the global dynamics of the universe and specifically whether such models can yield an accelerated phase of the cosmic expansion, without the need of the dark energy. Note that we do not “design” the fluid interactions to produce the desired accelerated cosmic evolution, as in some previous works (e.g. [20]), but rather we investigate whether the analytical solution space of the collisional Boltzmann equation, in the expanding Universe, allows for a late accelerated phase of the Universe.

2. Collisional Boltzmann Equation in the Expanding Universe

As we described in the introduction the DM is usually considered to contain only one type of particles that are stable and neutral. In this work, we would like to investigate the cosmological potential of a different scenario, in which the dominant ”cosmic” fluid is not in equilibrium and thus the time evolution of its overall density is described by the collisional Boltzmann equation (e.g. [24]):

$$\frac{d\rho}{dt} + 3H(t)\rho + \kappa \rho^2 - \Psi = 0$$

(1)

where $H(t) \equiv \dot{\alpha}/\alpha$ is the Hubble function, $\kappa$ is a constant related to the cross-section for annihilation and the mean particle velocity and $\Psi$ is the rate of creation of the corresponding DM particle pairs. It is obvious that if we impose in the current analysis an effective pressure term such as

$$P_{\text{eff}} = \frac{(\kappa \rho^2 - \Psi)}{3H}$$

(2)

then the collisional Boltzmann equation reduces to the usual fluid equation: $\dot{\rho} + 3H(\rho + P_{\text{eff}}) = 0$. Now if the DM is collisionless or the collisional annihilation and pair creation processes are in equilibrium (i.e., $P_{\text{eff}} = 0$), the corresponding solution of the above differential equation is $\rho \propto \alpha^{-3}$, as it should (where $\alpha$ is the scale factor of the universe). In contrast, for the case of a non-perfect fluid (i.e., having a disequilibrium between the annihilation and particle pair creation processes) it becomes clear that we can either have a positive or a negative effective pressure term.

The possibility of a negative such effective pressure term, i.e., the case for which the DM particle creation term is larger than the annihilation term ($\kappa \rho^2 - \Psi < 0$), is of particular interest for its repercussions on the global dynamics of the Universe (see appendix A). Note, that in the literature, there is a wealth of papers discussing the DM particle creation processes in the context of thermodynamics (for example, e.g. [25]):

We now proceed in an attempt to analytically solve Eq. (1). To this end we change variables from $t$ to $\alpha$ and thus Eq. (1) can be written:

$$\frac{d\rho}{d\alpha} = f(\alpha)\rho^2 + g(\alpha)\rho + R(\alpha)$$

(3)

where

$$f(\alpha) = -\frac{\kappa}{\alpha H(\alpha)} \quad g(\alpha) = -\frac{3}{\alpha} \quad R(\alpha) = \frac{\Psi(\alpha)}{\alpha H(\alpha)}$$

(4)

In general it is not an easy task to solve analytically Eq. (3), which is a Riccati equation, due to the fact that it is a non-linear differential equation. However, Eq. (3) could be fully solvable if (and only if) a particular solution is known. Indeed, we find that for some special cases regarding the functional form of the interactive term, such as $\Psi = \Psi(\alpha, H)$, we can derive
analytical solutions (see appendix B). We have phenomenologically identified two functional forms for which we can solve the previous differential equation analytically, only one of which is of interest since it provides \( \alpha^{-3} \) dependence of the scale factor. This is:

\[
\Psi(\alpha) = \alpha H(\alpha) R(\alpha) = C_1 (n + 3) \alpha^n H(\alpha) + \kappa C_1^2 \alpha^{2n},
\]

(5)

The general solution of equation (3) using Eq. (5) is:

\[
\rho(\alpha) = C_1 \alpha^n + \frac{\Phi(\alpha)}{[C_2 - \int_1^\alpha f(x) \Phi(x) dx]},
\]

(6)

where the kernel function \( \Phi(\alpha) \) has the form:

\[
\Phi(\alpha) = \alpha^{-3} \exp \left[ -2 \kappa C_1 \int_1^\alpha \frac{x^{n-1}}{H(x)} dx \right].
\]

(7)

Note that \( n, C_1 \) and \( C_2 \) are the corresponding constants of the problem.

3. Mimicking the Dark Energy

In this section we investigate the solution of our phenomenological model for \( n = -3 \) (see Eq. 6). The basic kernel Eq. (7) of the problem is given by: \( \Phi(\alpha) = \alpha^{-3} F(\alpha) \) where

\[
F(\alpha) = \exp \left[ -2 \kappa C_1 \int_1^\alpha \frac{1}{x^4 H(x)} dx \right].
\]

(8)

Note that at the present time we have \( F(1) = 1 \). Therefore, using the above relation and the functional form of \( f(\alpha) \) (see Eq. 4), the corresponding integral in equation (6) takes the form

\[
\int_1^\alpha f(x) \Phi(x) dx = - \int_1^\alpha \frac{\kappa}{x^4 H(x)} F(x) dx.
\]

(9)

Considering now that the function \( F(\alpha) \) is exponential, we have the following useful formula:

\[
\frac{dF(\alpha)}{d\alpha} = - \frac{2 \kappa C_1 F(\alpha)}{\alpha^4 H(\alpha)}
\]

(10)

and indeed, it is routine to perform the integration in the exponential part of Eq. (9) to obtain:

\[
\int_1^\alpha f(x) \Phi(x) dx = \frac{F(\alpha) - 1}{2 C_1}.
\]

(11)

Based on this analysis it is straightforward to derive the density evolution from Eq. (6):

\[
\rho(\alpha) = \alpha^{-3} \left[ C_1 + \frac{2 C_1 F(\alpha)}{D - F(\alpha)} \right],
\]

(12)

with \( D = 2 C_1 C_2 + 1 \). It is obvious from Eq. (12) that when \( \kappa \to 0 \), our cosmological model tends to the usual \( \rho \propto \alpha^{-3} \) case, as it should. Evaluating Eq. (12) at the present time \( (\alpha = 1) \) we obtain the present-time total effective cosmic density, which is: \( \rho_0 = C_1 + 1/C_2 \).

It is interesting to note that although Einstein’s theory requires local energy conservation, which appears to be incompatible with creation and annihilation processes, the reinterpretation of the creation/annihilation terms as an effective pressure of the medium (Eq. 2) lifts the apparent inconsistency. Below, we investigate the conditions under which Eq. 12 could provide accelerating solutions, similar to the usual dark energy case. Notice, that the current analysis is based on the matter era and thus, the radiation component does not play an important role in the global dynamics, since it evolves as \( \propto \alpha^{-4} \).
3.1. Conditions to have an inflection point
Using the Hubble function: \( H^2(\alpha) = (\dot{\alpha}/\alpha)^2 = 8\pi G \rho(\alpha)/3 \), we derive the second derivative of the scale factor:
\[
\ddot{\alpha} = \frac{4\pi G}{3} \left( 2\alpha \dot{\rho} + \alpha^2 \frac{d\rho}{d\alpha} \right).
\] (13)
Differentiating Eq. (12) with respect to the scale factor and using Eq. (10) we obtain:
\[
\frac{d\rho}{d\alpha} = -\frac{3C_1}{\alpha^4} \left[ \frac{D + F(\alpha)}{D - F(\alpha)} \right] - \frac{4D^2C_1^2 \kappa F(\alpha)}{\alpha^6[H(\alpha)[D - F(\alpha)]^2}. \] (14)
From Eqs. (13) and (14) we have:
\[
\ddot{\alpha} = \frac{4\pi G C_1 Z(\alpha)}{3H(\alpha) \alpha^6[D - F(\alpha)]^2}, \] (15)
where \( Z(\alpha) = \alpha^3 H(\alpha)[D + F(D - F) + 4C_1 \kappa F] \). In order to have an inflection point (\( \ddot{\alpha} = 0 \)), the function \( Z(\alpha) \) should contain roots which are real and such that \( \alpha \in (0, 1) \). For this to be the case we must have:
\[
Z(0)Z(1) < 0 \Rightarrow \lim_{\alpha \to 0} Z(\alpha) \lim_{\alpha \to 1} Z(\alpha) < 0, \] (16)
from which we obtain, for \( C_1 > 0 \): \( \kappa D(D - \lambda_1)(D - \lambda_2) < 0 \) with roots: \( \lambda_{2,1} = -\tau \pm \sqrt{\tau^2 + 1} \) with \( \tau = 2kC_1/H_0 \). Evidently, for the \( \kappa \) > 0 case we have that \( \lambda_1 < -1 \) and \( 0 < \lambda_2 \leq 1 \), and the range of values for which an inflection point is present in the evolution of the scale factor, \( \alpha \), is: \( D \in (-\infty, \lambda_1) \cup (0, \lambda_2) \). Furthermore, we have an additional constrain arising from the fact that \( H_0^2 \) should be positive: \( H^2_0 = 8\pi G C_1 (D + 1)(D - 1)^{-1}/3 > 0 \), implying \( D \in (-\infty, -1) \cup (1, +\infty) \). Joining the above restrictions we finally obtain that \( D \in (-\infty, \lambda_1) \). Considering the possibility of \( \kappa < 0 \) we have that \( -1 < \lambda_1 < 0 \) and \( \lambda_2 \geq 1 \), and a similar analysis, as for the \( \kappa > 0 \) case, also provides a range for which an inflection point exists: \( D \in (\lambda_2, +\infty) \). Although \( \kappa < 0 \) appears artificial, one may envision the case of a two particle type gas, and specifically a Lorentz gas with elastic scattering, in which the scattering velocity of the low-mass particles in the rest-frame of the large-mass particles could be negative, i.e., bunching back with \( \delta \theta = 180^\circ \). However, we do not investigate further such a possibility.

Since the avenue by which the IDM model provides a cosmic acceleration may appear slightly involved, we present in appendix C the correspondence between our model and the usual \( \Lambda \) or dark energy models.

| Table 1. General solutions for \( C_1 > 0, \kappa > 0 \) and \( w_0 < 0 \). |
|-----------------|-----------------|-----------------|-----------------|
| Solution \((\bar{C}_1 - \rho_{m,0})\) | \( D \) | \( C_2 \) | Accel. |
| \( I \) | \(< 0\) | \( D > 1 - \frac{2k_1}{C_1 - \rho_{m,0}} \) | \( > 1 \) | \( C_2 > -\frac{1}{C_1 - \rho_{m,0}} \) | NO |
| \( II \) | \( > 0\) | \( D < 1 - \frac{2k_1}{C_1 - \rho_{m,0}} \) | \( < -1 \) | \( C_2 < -\frac{1}{C_1 - \rho_{m,0}} \) | YES |

3.2. Relation to Quintessence Dark Energy
We know that for homogeneous and isotropic flat cosmologies \( (\Omega_m + \Omega_Q = 1) \), driven by non relativistic DM and a DE with a time varying equation of state, the density evolution of this cosmic fluid can be written as:
\[
\rho(\alpha) = \alpha^{-3} \left[ \rho_{m,0} + \rho_{Q,0} \exp \left( -3 \int_1^\alpha \frac{w(x)}{x} dx \right) \right], \] (17)
where $\rho_{m,0}$ and $\rho_{Q,0}$ are the present-day DM and DE densities, respectively, while $w(\alpha)$ is the equation of state parameter. Now, equating Eq. (12) and Eq. (17) we can investigate the circumstances under which DM self-interactions can act as the usual quintessence. At the present time ($\alpha = 1$) we see that our effective global density, $\rho_0$, can be seen as the sum: $\rho_0 = \rho_{m,0} + \rho_{Q,0}$. In the general case we have, after some algebra and a differentiation, that:

$$w(\alpha) = \frac{4\kappa C_1^2 F}{3\alpha^3 H(\alpha)} \left[ \frac{D}{D - F} \right] \frac{1}{\left[ (C_1 - \rho_{m,0})(D - F) + 2FC_1 \right]},$$

(18)

which evaluated at the present time we obtain:

$$w_0 = \frac{4\kappa C_1^2}{3H_0} \frac{D}{(D - 1)} \frac{1}{\left[ (C_1 - \rho_{m,0})(D - 1) + 2C_1 \right]}.$$

(19)

We present in Table 1 the combination of values of the constants for which $w_0 < 0$. Now we join the analysis of section 3.1 to check which of the solutions shown in Table 1 are viable solutions for our Universe, i.e., for which of these solution Eq. (16) is satisfied (i.e., the inflection point is located before the present time). This constrain excludes solution I, leaving however the solution II as a viable quintessence dark-energy look-alike, as far as the global dynamics is concerned.

4. Mimicking the $\Lambda$ Cosmology

We will show that for $n = 0$ the global dynamics, provided by our model (see Eq. 6), is equivalent to that of the traditional $\Lambda$ cosmology. To this end we use $dt = d\alpha/(\alpha H)$ and the basic kernel (Eq. 7) for this case becomes:

$$\Phi(\alpha) = \alpha^{-3} \exp \left[ -2\kappa C_1 \int_1^\alpha \frac{1}{xH(x)} dx \right] = \alpha^{-3} e^{-2\kappa C_1 (t-t_0)},$$

(20)
where $t_0$ is the present age of the universe. In addition, the integral in equation (6) takes now the following form: $\int_0^\alpha f(x)\Phi(x)dx = -\kappa F(t)$ with $F(t) = \int_0^t \Phi(u)du$. Note that at the present time we have $F(t_0) = 0$. Therefore, using the above formula, the global density evolution (Eq. 6) can be written:

$$\rho(\alpha) = C_1 + \alpha^{-3} e^{-2\kappa C_1(t-t_0)}/[C_2 + \kappa F(t)].$$ \hspace{1cm} (21)

As expected, at early enough times ($t \rightarrow 0$) the overall density scales according to: $\rho(\alpha) \propto a^{-3}$, while close to the present epoch the density evolves according to:

$$\rho(\alpha) \simeq C_1 + \alpha^{-3}/C_2,$$ \hspace{1cm} (22)

which is approximately the corresponding evolution in the $\Lambda$ cosmology, in which the term $C_1$ acts as the constant-vacuum term ($\rho_\Lambda$) and the $1/C_2$ term acts like matter ($\rho_m$).

Note that the effective pressure term (Eq. 2), for $\kappa \rightarrow -0$, becomes: $\Psi \sim 3C_1 H$, which implies that: $P \sim -\Psi/3H = -C_1$. Therefore, this case relates to the traditional $\Lambda$ cosmology, since $C_1$ corresponds to $\rho_\Lambda$ (see Eq. 22). We now investigate in detail the dynamics of the $n = 0$ model.

4.1. Conditions to have an inflection point

Using the Hubble function we derive the second derivative of the scale factor, which is:

$$\ddot{\alpha}/\alpha = -\frac{4\pi G Z(\alpha)}{3\alpha^6 H(\alpha)[C_2 + \kappa F]^2},$$ \hspace{1cm} (23)

where

$$Z(\alpha) = -2C_1\alpha^3 H(\alpha)[C_2 + \kappa F]^2 + \kappa e^{-\kappa C_1(t-t_0)} + \alpha^3 (C_2 + \kappa F)(2\kappa C_1 + H)e^{-\kappa C_1(t-t_0)}.$$ \hspace{1cm} (24)

As already discussed for the $n = -3$ case, in order to have an inflection point ($\ddot{\alpha} = 0$), the function $Z(\alpha)$ should contain roots which are real and such that Eq. (16) holds. For this to be the case we must have: $-C_1\kappa (C_2 - \lambda_1)(C_2 - \lambda_2) < 0$ with roots: $\lambda_{1,2} = (\tau \pm \sqrt{\tau^2 + 2\kappa c})/c$ and $\tau = 2\kappa C_1 + H_0$, $c = 4H_0 C_1$. For $\kappa C_1 > 0$ we have $C_2 \in (-\infty, \lambda_1) \cup (\lambda_2, +\infty)$, while for $\kappa C_1 < 0$ we have $C_2 \in (\lambda_1, \lambda_2)$. Therefore independent of the value of $C_1$ there is always a range of solutions which accommodate an inflection point and therefore an accelerated phase of the scale factor.

4.2. Relation to the Standard $\Lambda$ Cosmology

From Eq. (21), using the usual unit-less $\Omega$-like parameterization, we have after some algebra that:

$$\left(\frac{H}{H_0}\right)^2 = \Omega_{\gamma,0} + \frac{\Omega_{\gamma,0}\Omega_{\delta,0}\alpha^{-3} e^{-2\kappa C_1(t-t_0)}}{\Omega_{\gamma,0} + \kappa C_1 \Omega_{\delta,0} F(t)},$$ \hspace{1cm} (25)

with $\Omega_{\gamma,0} = 8\pi G C_1/3H_0^2$ and $\Omega_{\delta,0} = 8\pi G/3H_0^2 C_2$, which in the usual $\Lambda$ cosmology they correspond to $\Omega_\Lambda$ and $\Omega_m$, respectively. We can now attempt to compare the Hubble function of Eq. (25) to that corresponding to the usual $\Lambda$ model. To this end, we use a $\chi^2$ minimization between our model (Eq. 25) and the Hubble relation derived directly from early type galaxies at high redshifts (Simon et al. 2005). This procedure is performed by using $\Omega_{\delta,0} = 0.26$, $H_0 = 72$ km/sec/Mpc and $t_0 = 13.6$ Gyrs (the age of the universe of the corresponding $\Lambda$ cosmology). The resulting minimization provides $\kappa C_1 = 0.0017$, $\chi^2/\text{d.f.} = 1.18$ and a Hubble relation, seen in figure 1 as the dashed line, which closely resembles the corresponding one of the traditional $\Lambda$ model.
5. Conclusions
In this work we investigate analytically the evolution of the global density of the universe in the framework of an interacting DM scenario by solving analytically the collisional Boltzmann equation in an expanding Universe. The possible disequilibrium between the DM particle creation and annihilation processes creates an effective source term with negative pressure which, acting as dark energy, provides an accelerated expansion phase of the scale factor. Furthermore, we also find a realistic solution for which the present time is after the inflection point.

Appendix A. The main assumptions of our model
The reason why a cosmological constant or a component mimicking it leads to a cosmic acceleration is because it introduces in Friedmann’s equation, which governs the global dynamics, a component which has an equation of state with negative pressure. Our model creates exactly the equivalent to the above, i.e., an effective source term which has negative pressure. The avenue through which this is accomplished is via the collisional Boltzmann equation in an expanding universe (e.g. [24]) in which the disequilibrium between the annihilation (\(\kappa \rho^2\)) and particle creation (\(\Psi\)) processes provides the effective pressure: 
\[
P_{\text{eff}} = \left(\kappa \rho^2 - \Psi\right)/3H,
\]
which when negative acts exactly as a repulsive force, and therefore provides a cosmic acceleration (depending on the combination of parameters, as indicated in our Table 1). The basic assumption of this work is that the DM fluid is non-perfect, i.e., there is a disequilibrium between the particle pair annihilation and creation processes. Note that in our general solution (Eq. 6) we do not specify the direction of this disequilibrium. Only in the detailed analysis we seek solutions which provide an accelerated phase of the universe (for which of course one condition is \(P < 0\)), which we indeed find.

Appendix B. Solutions of the Riccati equation
With the aid of the differential equation theory we present solutions that are relevant to our Eq. (3). In general a Riccati differential equation is given by
\[
y' = f(x)y^2 + g(x)y + R(x)
\]
and it is fully solvable only when a particular solution is known. Below we present two case in which analytical solutions are possible.

- **Case 1:** For the case where:
  \[
  R(x) = C_1 nx^{n-1} - C_2^2 x^{2n} f(x) - C_1 x^n g(x)
  \]
  the particular solution is \(x^n\) and thus the corresponding general solution can be written as:
  \[
  y(x) = C_1 x^n + \Phi(x) \left[ C_2 - \int_1^x f(u) \Phi(u) du \right]^{-1},
  \]
  where
  \[
  \Phi(x) = \exp \left[ \int_1^x \left( 2C_1 u^n f(u) + g(u) \right) du \right]
  \]
  and \(C_1,C_2\) are the integration constants. Using now Eq. (4) we get \(\Psi(x) = xH(x)R(x) = C_1(n + 3)x^n H(x) + \kappa C_2^2 x^{2n}\).

- **Case 2:** For the case where:
  \[
  R(x) = h'(x) \quad \text{with} \quad g(x) = -f(x)h(x)
  \]
the particular solution is $h(x)$ [in our case we have $h(x) = -3\kappa^{-1}H(x)$]. The general solution now becomes:

$$y(x) = h(x) + \Phi(x) \left[ C_2 - \int_1^x f(u)\Phi(u)du \right]^{-1}, \quad (B.6)$$

where

$$\Phi(x) = \exp \left[ \int_1^x f(u)h(u) \right]. \quad (B.7)$$

In this framework, using Eq. (4) we finally get $\Psi(x) = xH(x)R(x) = -3\kappa^{-1}xH(x)H'(x)$. Note that the solution of Case 1 (for $n = -3$ and $n = 0$) is the only one providing a $\propto \alpha^{-3}$ dependence of the scale factor (see Eqs. 12, 21 and 22).

Appendix C. Correspondence between our model and the usual DE models

The necessary criteria to have cosmic acceleration and an inflection point in our past ($t_i < t_0$), are: (a) $P < 0$ and (b) $\ddot{\alpha} = 0$, which implies, from Eq. (13), that $(2\rho + \alpha d\rho/d\alpha) = \alpha Z(\alpha) = 0$ should have a real root within $\alpha \in (0, 1)$. Simple mathematics show that these conditions are respected both in the DE models and in our case, when:

- Dark Energy models: $P = \kappa \rho^2 - \Psi < 0$ and $Z(0)Z(1) < 0$.
- IDM models: $P = \kappa \rho^2 - \Psi < 0$ and $Z(0)Z(1) < 0$.

For the benefit of the reader and in order to see the equivalence of the corresponding criteria ($w < -1/3$ and $Z(0)Z(1) < 0$) we briefly apply our mathematical methodology (for the inflection point) in the case of the usual dark energy model ($w$ constant). Using the Friedmann equation, the second derivative of the scale factor becomes:

$$\ddot{\alpha} = \frac{4\pi G}{3} \alpha Z(\alpha) \quad \text{where} \quad Z(\alpha) = \left(2\alpha^{-1} \rho + \frac{d\rho}{d\alpha}\right). \quad (C.1)$$

The general condition to have an inflection point is $\ddot{\alpha} = 0$ or equivalently the equation $Z(\alpha) = 0$ should contain a real root before the present time, i.e., $\alpha \in (0, 1)$. This reads mathematically, as follows:

$$Z(0)Z(1) < 0 \Rightarrow \lim_{\alpha \to 0} Z(\alpha) \lim_{\alpha \to 1} Z(\alpha) < 0. \quad (C.2)$$

For the simple case of the dark energy models with constant $w(< 0)$, the above restriction simply leads to $w < -1/3$. Indeed in this case $Z(\alpha) = (3w + 1)\alpha^{-3w}\rho_{Q,0} + \rho_{m,0}$ and from $\lim_{\alpha \to 0} Z(\alpha) \lim_{\alpha \to 1} Z(\alpha) < 0$, it can be proved easily that in order to reach an inflection point we need $w < -1/3$. In other words from a mathematical point of view the condition $\lim_{\alpha \to 0} Z(\alpha) \lim_{\alpha \to 1} Z(\alpha) < 0$ is equivalent to $w < -1/3$. For a time-varying equation of state parameter the situation becomes more complicated.

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