An improved method based on block pulse functions for the numerical solution of Volterra type delay integral equations

S. C. Shiralashetti, 1* B. S. Hoogar 2* and Lata Lamani 3

Abstract
In this paper, an improved numerical scheme is proposed based on block pulse functions to solve Volterra type integral equations with time delay. By using the block pulse functions operational matrix, the integral problems with time delay would be reduced into a linear system of algebraic equations which can be solved easily by a suitable tool. The preliminaries of block pulse functions, construction of an operational matrix of integration and their properties are introduced. Numbers of test examples are included to demonstrate the efficiency, validity and wide range of applicability of the proposed method.

Keywords
Block Pulse Functions (BPFs), Volterra Integral Equations with time delay, Operational Matrix.

AMS Subject Classification
65D20, 34A12, 47A56.

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1. Introduction

Recently, delay integral equations have attracted a great deal of attention of researchers and scientists as they occur in a wide range of mathematical formulations of a variety of modelling problems. Delay integral equations are the important class of delay differential equations. These equations originated from various fields like fluid dynamics, biomedical sciences, chemical kinetics, ecology, control systems, aerospace engineering, etc. All of these equations cannot be solved by classical techniques. Therefore the numerical techniques play an important role. In the mathematical description of a physical process, one can easily guess that the performance of the course of action depends only on the present state. However, there are many situations on which the system’s behaviour not only depends on the present state but also on its previous state. These types of conditions are referred to as time-delay in systems and they occur often. Many researchers studied the different computational techniques for solving delay integral equations. Mirzaei et al. used application of Bernoulli wavelet method for estimating a solution of linear stochastic Ito-Volterra integral equations [1]. Chandra et al. Chandra et al. applied single term Walsh series method for numerical solution of the system of nonlinear delay Volterra integro-differential equations describing biological species living together [2]. Tain Y et al. solved some nonlinear delay Volterra-Fredholm type dynamic integral inequalities on time scales [14], S.C. Shiralashetti et al. attained the solution of linear and nonlinear delay differential equations by Hermite wavelet numerical scheme [10]. S.Narayanamoorthy and S.P.Sathiya Priya solved linear and nonlinear fuzzy Volterra integral equations by Ho-
motopy perturbation procedure [11], S. Narayananmoorthy and T.L. Yookesh applied approximate method for solving differential equations with linear fuzzy delay [12], S. Narayananmoorthy and K. Murugan obtained the solution of higher order fuzzy integro-differential equations using method of variational iteration [13], etc. In this paper we have used block pulse functions basis operational matrix to solve Volterra integral equations with time delay of the form:

\[ f(x) = g(x) + \int_{0}^{x} k(x,s) f(s - \tau) ds, \]  

(1.1)

for every \( x \in [0, T] \) the term \( \tau \) is the time delay and \( \tau \in (0, x) \). This article is structured as follows: In section 2, we have discussed basics of Block Pulse Functions (BPFs), construction of operational matrix and their properties. Method of solution of the present numerical scheme is given in section 3. In section 4, to prove the efficiency and applicability of the proposed method, we solved some test problems of the kind Volterra type integral equations with time delay. Conclusions and future scope is drawn in section 5.

\[ \phi_{i}(x) = \begin{cases} 1 & \text{when } (i-1) \frac{T}{m} \leq x \leq i \frac{T}{m}, \\ 0 & \text{otherwise} \end{cases} \]  

(2.1)

with \( x \in [0, T] \), \( i = 1, 2, ..., m \) and \( h = \frac{T}{m} \).

\section{2. Preliminaries}

This section contains the essential notations and basic definitions of Block Pulse Functions (BPFs) that have expressed in [7].

\subsection{2.1 Definition of Block Pulse Functions (BPFs):}

The \( m \)-set of Block Pulse Functions (BPFs) is defined as,

\[ \phi_{i}(x) = \begin{cases} 1 & \text{when } (i-1) \frac{T}{m} \leq x \leq i \frac{T}{m}, \\ 0 & \text{otherwise} \end{cases} \]  

(2.1)

with \( x \in [0, T] \), \( i = 1, 2, ..., m \) and \( h = \frac{T}{m} \).

Properties of BPFs:

(i) Disjointness property: By using the description of a block pulse functions this property can be easily obtained as

\[ \phi_{i}(x)\phi_{j}(x) = \delta_{ij}\phi_{i}(x) \]  

for every \( i, j = 1, 2, ..., m \)  

(ii) Orthogonality Property: This property can be obtained by using property 1 and is

\[ \int_{0}^{T} \phi_{i}(x)\phi_{j}(x)dx = h\delta_{ij} \]  

(2.3)

where \( \delta_{ij} \) is Kronecker \( \delta \) and \( i, j = 1, 2, ..., m \).

(iii) Completeness Property: BPFs set is complete, when for every \( m \rightarrow \infty \) defined in \( f \in L^{2}[0,T] \) the Parseval’s identity holds.

\[ \int_{0}^{T} f^{2}(x)dx = \sum_{i=1}^{\infty} f_{i}^{2} \|\phi_{i}(x)\|^{2} \]  

where \( f_{i} = \frac{1}{h} \int_{0}^{T} f(x)\phi_{i}(x)dx \)  

(2.4)

\subsection{2.2 Functions Approximation}

In order to reduce the mean square error between \( f(x) \) and its approximation, an arbitrary real compact function \( f(x) \) defined in \( x \in [0,T] \) can be extended into a block pulse sequence as:

\[ f(x) \cong f_{m}(x) = \sum_{i=1}^{m} f_{i}\phi_{i}^{m}(x) \]  

(2.5)

where \( f_{i} \) is the block pulse coefficient factor with respect to the \( i \)th BPF \( \phi_{i}^{m} \). We have in form vector as

\[ f(x) \cong f_{m}(x) = F^{T} \phi(x) = \phi^{T}(x)F \]  

(2.6)

where \( F = (f_{1}, f_{2}, ..., f_{m})^{T} \). Let \( k(s,t) \in L^{2}([0,T_{1}] \times [0,T_{2}]) \), then it can expanded as

\[ k(s,x) = \psi^{T}(s)K \phi(x) = \phi^{T}(x)K^{T} \psi(s) \]  

(2.7)

where \( \psi(s) \) and \( \phi(x) \) are \( m_{1} \) and \( m_{2} \) dimensional BPFs vector coefficients respectively and \( K = (k_{ij}), i = 1, 2, ..., m_{1} \) & \( j = 1, 2, ..., m_{2} \) is the \( m_{1} \times m_{2} \) matrix with block pulse coefficient as

\[ k_{ij} = \frac{1}{h_{1}h_{2}} \int_{0}^{T_{1}} \int_{0}^{T_{2}} k(s,x)\psi^{(m_{1})}_{i}(s)\phi^{(m_{2})}_{j}(x)dxds \]  

(2.8)

Where \( h_{1} = \frac{T_{1}}{m_{1}} \) & \( h_{2} = \frac{T_{2}}{m_{2}} \). For convenience, we take it as \( m_{1} = m_{2} = m \).

\subsection{2.3 Integration Operation Matrix}

We find \( \int_{0}^{x} \phi_{i}(s)ds \) as

\[ \int_{0}^{x} \phi_{i}(s)ds = \begin{cases} 0 & 0 \leq x \leq (i-1)h \\ x - (i-1)h & (i-1)h \leq x \leq ih \\ ih \leq x \leq T \end{cases} \]  

(2.9)

Since \( x - (i-1)h = \frac{h}{2} \), at the centre point of \([ (i-1)h, ih ] \) we can approximate \( x - (i-1)h \) for \((i-1)h \leq x \leq ih \) by \( \frac{h}{2} \). Then we express \( \int_{0}^{x} \phi_{i}(s)ds \) as in terms of BPFs as follows:

\[ \int_{0}^{x} \phi_{i}(s)ds \cong \left(0, ..., 0, \frac{h}{2}, h, ..., h\right) \phi(x) \]  

(2.10)

where \( \frac{h}{2} \) is the \( i \)th component of a vector.

From equation ((2.3)) , we have \( \int_{0}^{x} \phi_{i}(s)ds \cong P\phi(x) \), where integration operational matrix is given by:

\[ P = \frac{h}{2} \begin{bmatrix} 1 & 2 & 2 & \ldots & 2 \\ 0 & 1 & 2 & \ldots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}_{m \times m} \]  

(2.11)

So the integral part of every function \( f(x) \) can be approximated as:

\[ \int_{0}^{x} \phi_{i}(s)ds \cong \int_{0}^{x} F^{T} \phi(s)ds \cong F^{T} P \phi(x) \]  

(2.12)
2.4 Functions containing Time Delay $f(x - \tau)$

For approximating time-delay function, we consider a block pulse function with time delay $\tau = (q + \alpha)h$ with $q \geq 0$ and $0 \leq \alpha \leq 1$ that can be introduced as:

$$\phi_i(x - \tau) = \begin{cases} 
\phi_{i+q}(x) + \phi_{\alpha}(x - (i + q)h) - \phi_{\alpha}(x - (i + q - 1)h) & \text{if } i < m - q \\
\phi_{i-q}(x) - \phi_{\alpha}(x - (i - q + 1)h) & \text{if } i = m - q \\
0 & \text{if } i > m - q 
\end{cases}$$

(2.13)

It can also be expressed in the vector form as:

$$\phi_i(x - \tau) = \Delta^T_i H^q \phi(x) - \Delta^T_i H^q \phi_{\alpha}(x) + \Delta^T_i H^{q+1} \phi_{\alpha}(x)$$

(2.14)

To avoid the expression $\phi_{\alpha}(x)$ in the above equation, we expand the function $\phi_i(x - \tau)$ into its block pulse series as:

$$\phi_i(x - \tau) = (k_{i1}, k_{i2}, \ldots, k_{im}) \phi(x)$$

Where the coefficients of pulse $k_{ij}$ $(i, j = 1, 2, \ldots, m)$ are:

$$k_{ij} = \frac{1}{h} \int_0^T \phi_i(x - \tau) \phi_j(x) dx,$$

$$= \frac{1}{h} \int_{(j-1)h}^{jh} \phi_i(x) dx,$$

$$= \frac{1}{h} \Delta_i^T \left( \int_{(j-1)h}^{jh} \phi(x) dx - \int_{(j-1)h}^{jh} \phi_{\alpha}(x) dx \right)$$

$$+ H_j \int_{(j-1)h}^{jh} \phi_{\alpha}(x) dx,$$

$$k_{ij} = \Delta_i^T \left( (1 - \alpha) H^q + \alpha H^{q+1} \right) \Delta_j$$

(2.15)

It is remarked that the expression $\Delta_i^T \left( (1 - \alpha) H^q + \alpha H^{q+1} \right) \Delta_j$ is only single entry which is located in the matrix of $i$th row and $j$th column of the matrix $(1 - \alpha) H^q + \alpha H^{q+1}$, we can expand the entire time delay pulse function vector having $\tau = (q + \alpha)h$ into its block pulse series in a vector form:

$$\phi(x - \tau) = \left( (1 - \alpha) H^q + \alpha H^{q+1} \right) \phi(x)$$

(2.16)

In the equation (2.15), the matrix $(1 - \alpha) H^q + \alpha H^{q+1}$ is generally referred as the time delay operating matrix for block pulse or simply operating matrix for delay and is:

$$\begin{bmatrix}
0 & \cdots & 0 & 1 - \alpha & \alpha & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 - \alpha & \alpha & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 - \alpha & \alpha & 0 & \cdots & 0 \\
: & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & \alpha \\
0 & \cdots & 0 & 0 & 0 & \cdots & 1 - \alpha & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
: & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & \cdots \\
\end{bmatrix}_{m \times m}$$

(2.17)

Therefore it is easy to obtain the block pulse sequence of a function that contains time delay $\tau = (q + \alpha)h$ as follows:

$$f(x - \tau) \equiv F^T \phi(x - \tau) = F^T \left( (1 - \alpha) H^q + \alpha H^{q+1} \right) \phi(x)$$

3. Method of Solution

Consider Volterra integral equation with time delay (constant time delay) as $\tau > 0$ as

$$f(x) = g(x) + \int_0^x k(x, s) f(s - \tau) ds, \text{with } x \in [0, T] \& \tau \in (0, T)$$

(3.1)

In the above equation, the function $f \in L^2 [0, T]$ is the unknown function and the functions $f \in L^2 [0, T] \& k(x, s) \in L^2 ([0, T] \times [0, T])$ are the known functions. We are approximating the functions $f(x), g(x), k(x, s)$ by the equations (2.3) & (2.7) as follows:

$$f(x) \approx G^T \phi(x) = \phi^T (x) G,$$

$$g(x) \approx F^T \phi(x) = \phi^T (x) F,$$

$$k(x, s) \approx \phi^T (x) K \psi(s) = \psi^T (s) K^T \phi^T (x).$$

We are approximating $f(s - \tau)$ by using the equation (2.3) as follows,

$$f(s - \tau) \approx G^T \psi(s - \tau) \approx G^T \left( (1 - \alpha) H^q + \alpha H^{q+1} \right) \psi(s),$$

By taking $Q = (1 - \alpha) H^q + \alpha H^{q+1}$ the above equation can be written as,

$$f(x - \tau) \approx G^T Q \psi(s)$$

(3.2)

With the above approximation and substituting in equation (3.3), we have

$$G^T \phi(x) \approx F^T \phi(x) + \int_0^x G^T Q \psi(s) \psi^T (s) K^T \phi(x) ds,$$

$$G^T \phi(x) \approx F^T \phi(x) + G^T \left( \int_0^x \psi(s) \psi^T (s) ds \right) K^T \phi(x).$$

(3.3)

Let $K_i$ be the $i$th row of the constant matrix $K^T R_i$ be the $i$th row of the operational matrix of integration $P$, and $D_{Ki}$ be a diagonal matrix with $K_i$ as its diagonal entries. By the earlier relations & assuming relations and also assuming $m_1 = m_2$, we have,

$$\left( \int_0^x \psi(s) \psi^T (s) ds \right) K^T \phi(x) = N^T G$$

$$= F \left( \int_0^x \phi(s) \phi^T (s) ds \right) K^T \phi(x)$$

$$= \begin{bmatrix} V_1 \phi(x) & 0 & \cdots & 0 \\
0 & V_2 \phi(x) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & V_m \phi(x) \end{bmatrix} \begin{bmatrix} k_1 \\
k_2 \\
k_3 \\
\vdots \end{bmatrix} \phi(x)$$

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\[ \begin{bmatrix} R_1 \theta(x) k_1 \theta(x) \\ R_2 \theta(x) k_2 \theta(x) \\ \vdots \\ R_m \theta(x) k_m \theta(x) \end{bmatrix} = \begin{bmatrix} R_1 D_{k_1} \\ R_2 D_{k_2} \\ \vdots \\ R_m D_{k_m} \end{bmatrix} \]

So that

\[ U = \frac{h}{2} \begin{bmatrix} k_{11} & 2k_{21} & \cdots & 2k_{m1} \\ 0 & k_{22} & \cdots & 2k_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_{mm} \end{bmatrix}_{m \times m} \] (3.5)

By substituting (3.3) in (3), we have

\[ G^T \phi(x) = F^T \phi(x) + G^T QU \phi(x), \]

Therefore

\[ G^T (I - QU) \cong F^T \] (3.6)

So, by putting \( N = I - QU \) and changing \( \cong \) by \( = \), we get

\[ N^T G = F. \] (3.7)

It is a linear system of equations with lower coefficients in triangular matrix and this gives the unknown function \( f(x) \) approximate block pulse coefficients.

**4. Numerical Test Examples**

In this part, we present a number of test examples to prove the efficiency and performance of the proposed technique.

**Example 1.** Consider the following Volterra integral equation with time delay [6],

\[ f(t) = \sin t + t^2 \cos (t - 1) - t^2 \cos (-1) + \int_0^t t^2 f(s - 1) ds \] (4.1)

with \( f(t) = t - t^3 / 3! \), for all \( t \in [-1, 0] \). The analytical solution of equation (4.1) is \( f(t) = \sin t \), for \( t \in [0, 1] \). We solve this example by using the procedure as described in section 3, we get the numerical solution for different \( m \)-values and \( \tau \) and are given in table 1.

| At \( m=6 \) | At \( m=8 \) | At \( m=9 \) |
|------------|------------|------------|
| \( \tau \) | \( ||E||_\infty \) | \( \tau \) | \( ||E||_\infty \) | \( \tau \) | \( ||E||_\infty \) |
| 0.001 | 3.1813E-03 | 0.001 | 3.1817E-03 | 0.001 | 3.1818E-03 |
| 0.002 | 3.1798E-03 | 0.002 | 3.1806E-03 | 0.002 | 3.1808E-03 |
| 0.004 | 3.1768E-03 | 0.004 | 3.1783E-03 | 0.004 | 3.1788E-03 |
| 0.010 | 3.1738E-03 | 0.010 | 3.1761E-03 | 0.010 | 3.1768E-03 |
| 0.012 | 3.1678E-03 | 0.012 | 3.1693E-03 | 0.012 | 3.1678E-03 |
| 0.014 | 3.1618E-03 | 0.014 | 3.1670E-03 | 0.014 | 3.1688E-03 |
| 0.016 | 3.1588E-03 | 0.016 | 3.1648E-03 | 0.016 | 3.1688E-03 |

Table 1.Numerical solution at different \( m \)-values and \( \tau \) of example 1.
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Figure 2. Numerical solution of present method and exact solution.

5. Conclusion

The Block Pulse Functions (BPFs) used as basis functions to solve the Volterra type integral equations with time delay which is very easy to implement as compared to other existing methods. The efficiency and accuracy of the proposed numerical scheme is examined with some test examples. From the tables and figures, we found that proposed method is giving excellent numerical results as compared to the exact solutions. This exhibits the better advantage of the method, given its simplicity when applying to the problems of complex nature. Therefore, we are optimistic that the usefulness of this approach gives it much broader applicability that should be further explored.

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