K-Core based Temporal Graph Convolutional Network for Dynamic Graphs

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Abstract—Graph representation learning is a fundamental task of various applications, aiming to learn low-dimensional embeddings for nodes which can preserve graph topology information. However, many existing methods focus on static graphs while ignoring graph evolving patterns. Inspired by the success of graph convolutional networks (GCNs) in static graph embedding, we propose a novel k-core based temporal graph convolutional network, namely CTGCN, to learn node representations for dynamic graphs. In contrast to previous dynamic graph embedding methods, CTGCN can preserve both local connective proximity and global structural similarity in a unified framework while simultaneously capturing graph dynamics. In the proposed framework, the traditional graph convolution operation is generalized into two parts: feature transformation and feature aggregation, which gives CTGCN more flexibility and enables CTGCN to learn connective and structural information under the same framework. Experimental results on 7 real-world graphs demonstrate CTGCN outperforms existing state-of-the-art graph embedding methods in several tasks, such as link prediction and structural role classification. The source code of this work can be obtained from https://github.com/jhljx/CTGCN.

Index Terms—Dynamic graph embedding, k-core, structural similarity, graph convolutional network

1 INTRODUCTION

Graph embedding is a method mapping nodes into a dense and low-dimensional embedding space, aiming to preserve topological structure, node content and other side information [1]. Recently, graph embedding has aroused considerable interests and has been proven as an effective way to capture and preserve graph properties in a wide variety of applications, such as node classification [2], link prediction [3] and anomaly detection [4].

Learning graph representations faces two main challenges. The first challenge is how to sufficiently capture the graph topology. Most existing graph embedding methods focus on either connective proximity [5] or structural similarity [6]. However, these methods may omit some potentially important information in graphs. As illustrated in Figure 1, both connective proximity and structural similarity are important in mining potential financial frauds.

In other words, connective proximity is helpful to reveal financial fraud gangs in the graph, such as people in black dashed rectangles, while structural similarity can be utilized to identify financial frauds with similar anomalous social behaviors, such as people highlighted by red dashed circles. Although SNS [7] has already taken both connective proximity and structural similarity into account, it is still a static graph embedding method and is less able to capture highly non-linear graph structure.

Another challenge for learning graph representation is many real-world graphs are dynamic with the addition, deletion and changing of nodes and edges, as shown in Figure 2. Various attempts have been made in recent years to overcome this challenge. However, some approaches are highly dependable on the assumption of temporal smoothness [8], [9], thereby failing to capture sharp changes during the graph evolving process. Furthermore, although some approaches [3] can capture temporal evolving patterns, they have limited capacity to preserve rich structural information in graphs. Hence, how to effectively capture the graph dynamics and preserve structural information is of vital importance and still unresolved.

Recently, there has been a surge of interest in graph convolutional network (GCN) due to its promising performance in both static and dynamic graph embedding. However, GCN only preserves local connective proximity and suffers
from over-smoothing problem when graph convolution layers increase. Accordingly, GCN is less capable to preserve global structural similarity in graphs. For instance, as shown in Figure 1, GCN will fail to capture the connections between those financial frauds who share similar structural roles or positions. Although DEMO-Net has attempted to retain graph structure information for GCN, such as degree-specific information \[10\], it relies on too many learnable degree-specific weight matrices, which limits its efficiency of solving large-scale graph embedding problems.

To address the above issues, we propose a novel k-core based temporal GCN, namely CTGCN, to preserve both connective proximity and structural similarity under a unified framework. Meanwhile, the proposed method is efficient to learn node representations in dynamic graphs. As k-core number(Degeneracy) \[11\] is an important graph centrality and k-core decomposition \[12\] naturally extracts subgraphs at multiple different scales, we design a novel graph convolution layer to capture topological and hierarchical properties of graphs within these nested k-core subgraphs. Most importantly, we generalize the traditional graph convolution operation into feature transformation and feature aggregation operations. In practice, we leverage k-core based graph convolution layers as the feature aggregation module. Then by utilizing different feature transformation functions, CTGCN can preserve connective and structural information under the same framework. Empirically, extensive experiments are conducted on 7 real-world graphs for several tasks. The results show CTGCN achieves gains over baselines on these graphs and demonstrate CTGCN can learn node representations efficiently in large-scale dynamic graphs.

Overall, the main contributions of this paper are as follows:

- We present a novel temporal GCN which can preserve both connective proximity and structural similarity in dynamic graphs under a unified framework.
- We generalize traditional graph convolution into feature transformation and feature aggregation, which provides GCN more flexibility and efficiency.
- We propose a novel k-core based graph convolution layer(CGCL) which can uncover topological and hierarchical properties of graphs.
- We evaluate the effectiveness and efficiency of CTGCN on several real-world graphs in both connective proximity related applications (i.e. link prediction) and structural similarity related applications (i.e. structural role classification).

2 RELATED WORK

2.1 Graph Convolutional Networks

Graph convolutional network has been widely studied in recent years. It can be categorized into two types. One is to apply convolution operation directly in the spatial domain \[13, 14\], which first rearranges the vertices into certain grid forms and then process them by normal convolutional operations. The other is to apply convolution in the spectral domain based on graph Fourier transforms \[15, 16\]. We refer to the graph convolutional network proposed by \[16\] as the vanilla GCN. After the vanilla GCN, many GCN variants have been proposed, such as graph attention network(GAT) \[17\], GraphSAGE \[18\] and FastGCN \[19\]. Furthermore, GCN variants have been utilized to solve spatial-temporal problems. Most of these temporal GCN combine GCN variants and recurrent architecture, such as AddGraph \[20\] and EvolveGCN \[21\].

2.2 Static Graph Embedding

Most existing static graph embedding approaches focus on mining connectivity patterns in graphs. The most commonly preserved connectivity patterns are connective proximities which define neighbourhood information in a graph from different levels of scope, such as first-order, second-order proximity \[22\] and high-order proximity \[23\]. Previous studies preserving connective proximities include random walk based algorithms such as DeepWalk \[24\] and node2vec \[5\], matrix factorization based algorithms such as TADW \[25\], and deep learning methods such as SDNE \[26\]. Furthermore, a few other static approaches take structural similarity into account. For instance, struct2vec \[6\] is designed for learning node representations which preserves structural identity of nodes.

2.3 Dynamic Graph Embedding

Dynamic graph embedding can be generally divided into two categories, structure preserving embedding and property preserving embedding \[1\]. Structure preserving embedding methods for dynamic graphs aim to preserve important structural information when graphs evolve. DynGEM \[27\] proposes a deep learning method to learn dynamic graph embeddings while preserving first-order and second-order proximities. HOPE \[9\] preserves high-order proximity and updates node embeddings through an acceleration technique. Matrix Factorization based methods, such as TIMERS \[28\], learn dynamic graph embeddings with global structure preserved. In addition, property preserving embedding methods for dynamic graphs consider graph properties, such as transitivity, to better model the graph evolving process. For example, DynamicTriad \[8\] introduces triad closure and models the triad closure process to capture important graph dynamics. However, most aforementioned approaches consider dynamic graphs as a sequence of static graph snapshots. Hence, some methods consider temporal graph embedding for dynamic graphs by modeling graph evolving process into a continuous time space, such as HTNE \[29\].

3 PRELIMINARIES

3.1 Problem Definition

A dynamic graph at time step \(t\) is an undirected graph which is defined as \(G^{(t)} = (V, E^{(t)})\), where \(V = \{v_1, \ldots, v_N\}\) denotes a set of \(N\) nodes, \(E^{(t)} = \{e_{ij}^{(t)}\}\) is a set of edges between nodes in \(V\). Each edge \(e_{ij}^{(t)} \in E^{(t)}\) is associated with a weight \(\Lambda_{ij}^{(t)} > 0\) in the adjacent matrix \(\Lambda^{(t)}\). Negative weighted edges are not considered in this paper.
Graph convolutional networks have achieved promising performance in various tasks. Formally, a single graph convolution layer in the vanilla GCN is defined as

\[
H = \sigma \left( D^{-\frac{1}{2}} \hat{A} D^{-\frac{1}{2}} X W \right) \tag{1}
\]

where \(X\) is the input feature matrix, \(\hat{A} = A + I\) is an adjacent matrix with self-loop, \(D\) is a diagonal matrix with \(D_{ii} = \sum_{j=1}^{N} A_{ij}\), \(W\) is a learnable weight matrix, \(\sigma(\cdot)\) is a non-linear activation function and \(H\) is the output embedding matrix.

Obviously, the graph convolutional layer is similar to the full connected layer, except for the diffusion matrix \(\hat{A} = D^{-\frac{1}{2}} \hat{A} D^{-\frac{1}{2}}\) which propagates the features of a node’s neighbors to this node. In essence, graph convolution is a special form of Laplacian smoothing which computes the new features of a node as the weighted average of itself and its neighbors [20]. A one-layer GCN can preserve first-order proximity and multi-layer GCN is often utilized in practice as it can capture high-order proximity information. However, due to the intrinsic Laplacian smoothing property, GCN suffers from over-smoothing and over-fitting when graph convolution layers increase. These problems reflect the inability of GCN in exploring global graph structure. Hence, GCN normally includes at most 3 graph convolution layers in practice.

4 THE PROPOSED METHOD
4.1 Generalized GCN Framework

Inspired by the graph convolution operation in Equation 1, we generalize it into two parts: feature transformation and feature aggregation, which are given by

\[
Z = f(X) \tag{2}
\]
\[
H = h(g(A), Z) \tag{3}
\]

where \(X\) is the input feature matrix, \(f\) is the feature transformation function which maps \(X\) into a latent space, \(A\) is the adjacent matrix, \(g\) is a function of \(A\) which defines the propagation rule, \(h\) is the feature aggregation function which aggregates features for each node based on the propagation rule \(g\), \(H\) is the embedding matrix which integrates the information from feature transformation and graph structure.

Many previous works can be included in this framework, as summarized in Table 1. We note that most GCNs define a linear transformation as \(f\), while PPNP method utilizes a multi-layer perceptron (MLP) to model a non-linear \(f\). Although these methods have different \(f\) and \(g\) combinations, they all choose the outputs of \(h\) as the final embeddings in practice. We argue that both the outputs of \(f\) and \(h\) can be utilized as node representations, which will be discussed later. By generalizing the traditional graph convolution operation into feature transformation and feature aggregation, we provide more flexibility to design different kinds of GCNs.

4.2 Core-based Temporal GCN

We propose a novel k-core based temporal GCN, namely CTGCN, to preserve both connective proximity and structural similarity in dynamic graphs. As shown in Figure 2, CTGCN includes two modules: core-based GCN(CGCN)

![Diagram of the k-core based temporal graph convolutional network (CTGCN).](image)

Fig. 2. The architecture of the k-core based temporal graph convolutional network (CTGCN).
module and graph evolving module. Note that it is also an instance of the generalized GCN framework. Although CTGCN is a temporal GCN method, we can regard $h$ as a complex spatio-temporal feature aggregation function.

### 4.2.1 Core-based GCN Module

In order to capture rich graph structure information, we leverage k-core subgraphs to design core-based GCN. We first give the definition of k-core [34].

**Definition 1 (k-core).** Consider a graph $G$ and a subgraph $G'$ of the graph $G$. $G'$ is a k-core of $G$ if $G'$ is a maximal subgraph of $G$ in which all nodes have degree at least $k$.

From the definition above, we can induce some interesting properties of k-core. One property is that k-core numbers of nodes have strong positive correlation with node degrees in real-world graphs [35]. Another property is that k-cores form nested structure. Formally, let $C = \{C_0, C_1, \cdots, C_{k_{\text{max}}} \}$ be the k-core set of a graph $G$, then all k-cores in $C$ form a nested chain, which is given by

$$C_{k_{\text{max}}} \subseteq \cdots \subseteq C_1 \subseteq C_0 = G$$

where $k_{\text{max}}$ is the max k-core number of $G$. All k-cores can be extracted by k-core decomposition algorithms, which have a linear time complexity with the edge number [36] and are efficient to process large-scale graphs.

### 4.2.2 Graph Evolving Module

To effectively capture graph dynamics during the graph evolving process, we utilize RNN to model the temporal dependency among node representations of CGCNs at different timestamps. In this module, we simply utilize either GRU or LSTM to capture temporal features. Note that this module can also be augmented by other RNN architectures if more complex temporal features are needed. Formally, the temporal GCN is defined as follows:

$$H^t = \text{RNN}(\text{ReLU}(A'(Z), H^{t-1}), \ l = 1, \cdots, k_{\text{max}})$$

where $H^0 = 0$ is a zero matrix, and $A' = A_{k_{\text{max}}-l+1}$ is the normalized adjacent matrix of the $(k_{\text{max}} - l + 1)$-th k-core. Note that $Z$ is shared across all RNN units, which reduces the number of learnable weight matrices. As 1-core is a subgraph without single nodes in 0-core and single points cannot aggregate features from their neighbors, the adjacent matrix of 0-core is not utilized in CGCL. Like the traditional graph convolution layer, CGCL can only preserve first-order proximity. Hence, multiple CGCLs are helpful to capture connective information in a larger receptive field.

### 4.3 Connective Proximity Preserving CTGCN

In order to preserve connective proximity in dynamic graphs, we follow previous GCNs to utilize the outputs of $h$ function as final embeddings. In connective proximity preserving CTGCN(CTGCN-C), $f$ is denoted as a linear mapping, that is $Z = WX$. For each $CGCN^{(t)}$, we employ two GCCLs to capture the connective information within 2-hop neighbors. To this end, the overall objective of CTGCN-C is summarized below:

$$\mathcal{L}_c = \sum_{t=1}^{T} \sum_{u \in V} L_u^t$$

$$L_u^t = \sum_{v \in N^t_u(u)} \log (F_p) - Q \cdot \sum_{v' \in F^t_n(u)} \log (1 - F_n)$$

where $F_p = \sigma(\langle h_u, h_v \rangle)$, $F_n = \sigma(\langle h_u, h_v \rangle)$, $N^t_u(u)$ is the set of nodes that co-occur with $u$ on fixed-length random walks, $P^t_n(u)$ is a negative sampling distribution which is usually a function of node degrees, and $Q$ is a hyper-parameter to balance the positive and negative samples.
Through this unsupervised loss function, nodes co-occurring in fixed-length random walks are encouraged to have similar representations. Hence, the learned node representations can preserve local connective proximity in dynamic graphs.

4.4 Structural Similarity Preserving CTGCN

In previous works, structural similarity preserving graph embedding methods can be summarized into two categories, one is to uncover structural similarity based on the structural similarity graph which is derived from the original graph(i.e. struc2vec), the other is to encode the neighboring structure into embeddings(i.e. DEMO-Net). We focus on the second kind of methods in this work. Formally, we first give some more accurate definitions to describe structural similarity [39].

Definition 2 (structural equivalence). Consider two nodes \( u \) and \( v \) in a graph, \( u \) and \( v \) are structurally equivalent if and only if \( N_u = N_v \).

Definition 3 (regular equivalence). Consider two nodes \( u \) and \( v \) in a graph, \( u \) and \( v \) are regularly equivalent if and only if \( N_u \) and \( N_v \) are regular equivalent.

From the definitions above, we note that SDNE [28] utilizes a deep autoencoder to encode each row of an adjacent matrix into a latent embedding space, which preserves structural equivalence in graphs. Furthermore, both DEMO-Net [10] and DRNE [39] encodes the neighboring information to make nodes with similar neighbors have similar embeddings, which preserves regular equivalence in graphs.

In structural similarity preserving CTGCN(CTGCN-S), \( f \) is denoted as a non-linear mapping, that is \( Z = MLP(X) \). CTGCN-S have the same CGCN architecture with CTGCN-C. However, the biggest difference is that CGCLs in CTGCN-S are regarded as a guide to make \( Z \) encode neighboring structural information of each node. And \( Z \) is utilized as the final embedding matrix. Therefore, the overall objective of CTGCN-S is summarized below:

\[
\mathcal{L}_s = \sum_{t=1}^{T} \sum_{i \in V} \|Z_u^{(t)} - H_u^{(t)}\|_F^2
\]

where \( H^{(t)} \) is the \( t \)-th output embedding matrix of the graph evolving module. Similar to DRNE, the embedding matrix \( Z \) is also optimized recursively to preserve structural similarity in graphs. Then nodes with similar neighboring structure are encouraged to have similar node representations.

4.5 Complexity Analysis

In order to obtain all k-cores in dynamic graphs, we utilize a k-core decomposition algorithm which is linear to the edge number [36]. Let \( |E|_{max} \) be the maximal edge number across all dynamic graphs, then the time complexity of extracting k-cores is \( O(T \cdot |E|_{max}) \). If parallel computing techniques are utilized, the time complexity can be reduced to \( O(|E|_{max}) \).

As for CGCN module, the time complexity of feature transformation is linear to the feature dimension of \( X \). If \( X \) is a one-hot sparse matrix, then the total time complexity is \( O(TNc') \) where \( c' \) is a constant independent with \( N \). Note that matrix multiplications in CGCLs are time-consuming operations. If adjacent matrices are stored as sparse matrices, then the time complexity of feature aggregations is \( O(Tlk_{max}|E|_{max}) \) where \( l \) is the layer number of CGCLs. Here the time complexity of GRUs(or LSTMs) in CGCLs are not considered, as their time complexity is lower than that of matrix multiplications. Therefore, the time complexity of feature aggregation in CGCNs is linear to the edge number.

For the graph evolving module, the time complexity is equal to that of GRU(or LSTM). As \( CGCN^{(l)}(A^{(l)}, X^{(l)}) \) and \( H^{(l-1)} \) are inputs of the RNN model, assume they are \( N \times d \) and \( N \times d' \) matrices respectively. Then the time complexity is \( O(k'N(d+d')) \) where \( k' \) is a constant independent with \( N \). Overall, the time complexity of CTGCN is linear to the node number, which indicates our proposed method can be applied into large-scale graph settings.

5 Experiments

In this section, we evaluate the proposed method in the dynamic setting from four aspects: the performance on the connective proximity related tasks, the structural similarity related tasks, the embedding stability and the time cost. Experimental analyses are presented as follows.

5.1 Experiment Setup

Baselines We employ the following graph embedding methods as baselines:

- DeepWalk [24]. It is a static graph embedding model combining truncated random walk with SkipGram model [40] to learn node representations.
- node2vec [3]. It designs a biased random walk procedure to explore diverse neighbourhoods and utilizes SkipGram model to learn static node embeddings.
- struc2vec [6]. It constructs a hierarchical graph to measure structural similarity at different scales and leverages SkipGram model to learn structural representations for nodes in a static graph.
- GCN [16]. It encodes the graph structure directly using a neural network model based on a simple and well-behaved layer-wise propagation rule.
- GAT [17]. It is a variant of GCN, which leverages masked self-attentional layers to aggregate neighbourhoods’ features with different weights.
- DynGEM [27]. It extends SDNE into dynamic graphs and leverages deep autoencoders to generate dynamic graph embedding incrementally.
- TIMERS [28]. It utilizes singular value decomposition(SVD) to incrementally generate dynamic graph embedding and automatically restart the SVD when accumulated error exceeds the preset error threshold.
- EvolveGCN [21]. It is a dynamic variant of GCN, which leverages RNN to evolve GCN parameters. There are two versions of EvolveGCN method: EvolveGCN-O and EvolveGCN-H.

Parameter Settings We set window size as 10, walk length as 40 and walks per node as 20 for DeepWalk. We set p as 1 and q as 2 in node2vec to discover more neighborhood
information, other parameter settings are the same as DeepWalk. For GCN, we set the graph convolution layer number as 2 and the dropout probability as 0.5. We also utilize 2 one-head graph attention layers for GAT, and set $\alpha$ as 0.2 and the dropout probability as 0.6. As for DynGEM, we set $\alpha$ as $10^{-5}$, $\beta$ as 10, $\gamma_1$ as $10^{-4}$, $\gamma_2$ as $10^{-4}$. We set error threshold as 0.17 in TIMERS. We utilize 2 evolve graph convolution layers in EvolveGCN and report the EvolveGCN version with better performance in each task.

As for the proposed method, CTGCN-C has a linear layer and 2 CGCLs in each CGCN, while CTGCN-S has a 3-layer MLP and a CGCL in each CGCN. We adopt Adam optimizer \cite{kingma2014adam} with learning rate as 0.001 and weight decay as $5 \times 10^{-4}$ in this work. The epoch number is set as 50 for CTGCN-C and 20 for CTGCN-S. The hyper-parameter $Q$ is set as 10. Note that we also evaluate the performance of static versions of CTGCN. CGCN-C and CGCN-S have same feature transformation and feature aggregation layers with CTGCN-C and CTGCN-S, respectively. We use one-hot node feature matrix for CGCN-C and CTGCN-C, and use degree-based node feature matrix for CGCN-S and CTGCN-S. For fair comparison, the dimensionality of embeddings $d$ is set to 128 for all compared methods.

### 5.2 Link Prediction

Given a series of dynamic graphs, the link prediction is to predict the existence of an edge in the next time step $t + 1$ based on the information of embeddings in the current time step $t$. To obtain a labeled edge set from graph $G^{(t)}$, we generate random sampled edges in time step $t$ as positive examples, and generate negative samples by sampling node pairs which are not connected by any edges. Then we follow the methodology in \cite{DBLP:conf/kdd/KipfW16} and compute edge feature vectors by utilizing the hadamard operation between embedding vectors of node pairs in the labeled edge set. We train a logistic regression(LR) classifier to discriminate positive and negative edge samples. Note that link prediction is performed on 20 randomly split edge sets for each timestamp. The area under curve(AUC) is utilized as the evaluation metric and the averaged AUC score of each timestamp is formed on 20 randomly split edge sets for each timestamp. Negative edge samples. Note that link prediction is performed on 20 randomly split edge sets for each timestamp. The area under curve(AUC) is utilized as the evaluation metric and the averaged AUC score of each timestamp is reported to reduce the impact of different edge sets on the link prediction results.

We employ three data sets in this experiment, including a social network of Facebook Corporation\footnote{1. http://networkrepository.com/fb-wosn-friends.php} an E-mail contact network of Enron Corporation\footnote{2. http://networkrepository.com/ia-enron-email-dynamic.php} and a user interaction network of the stack exchange website Math Overflow\footnote{3. http://snap.stanford.edu/data/ssx-mathoverflow.html}. In practice, we split the data sets by month and remove incomplete data. Then the complete data of continue months are retained for link prediction task. Detailed statistics of these data sets are shown in Table \ref{table:dataset}.

We report the link prediction results on these real-world graphs where the best results are indicated in bold, as illustrated in Table \ref{table:link预测}. It can be observed that the proposed CTGCN-C method significantly outperforms other embedding methods across all three dynamic graphs. It indicates that CTGCN-C can capture hierarchical graph structure information in dynamic graphs, which might be helpful for its node representations to predict link formation and deletion. Furthermore, We observe that CTGCN-S is also comparative with CTGCN-C in link prediction task. It demonstrates that CTGCN-S is also capable of preserving local connective proximity in dynamic graphs, and feature transformation function $f$ can produce expressive node representations under the guide of feature aggregation function $h$, as shown in Equation \ref{eq:aggregation}. Especially, the static versions(CGCN-C, CGCN-S) of the proposed method outperform other static graph embedding methods, which suggests that CGCLs are helpful to the performance improvements in link prediction.

![Fig. 4. AUC of link prediction on dynamic graphs.](image)

To further evaluate the stability of all methods over

| Method      | Facebook | Enron | Math |
|-------------|----------|-------|------|
| DeepWalk    | 0.6769   | 0.8038| 0.7742 |
| node2vec    | 0.6819   | 0.8090| 0.7802 |
| GCN         | 0.6942   | 0.7879| 0.7986 |
| GAT         | 0.6991   | 0.8344| 0.8331 |
| CGCN-C      | 0.6954   | 0.8801| 0.8946 |
| CGCN-S      | 0.8333   | 0.9446| 0.9451 |
| DynGEM      | 0.8019   | 0.8926| 0.9025 |
| TIMERS      | 0.6129   | 0.7707| 0.7177 |
| EvolveGCN   | 0.7194   | 0.8332| 0.8810 |
| CTGCN-C     | 0.8744   | 0.9650| 0.9854 |
| CTGCN-S     | 0.8324   | 0.9321| 0.9266 |

Table 2: Average AUC scores of all timestamps for link prediction.
timestamps, we record the AUC scores of all compared methods on each timestamp, as shown in Figure 4. We see that AUC values of most approaches fluctuate sharply when the graph evolves, while CGCN-C is more stable than other methods on most timestamps. Although CGCN-C performs worse than DynGEM on the last several timestamps of Facebook data set, its stability on other data sets is more remarkable than all baselines. Hence, the results further demonstrate that CTGCN can learn stable and expressive node representations on different timestamps.

5.3 Regular Equivalence Prediction
To evaluate the structural similarity preserving performance of all compared methods, we set up the regular equivalence prediction on dynamic graphs. As centrality measures have been utilized to characterize the structural role and the importance of nodes, we choose four popular centralities: closeness centrality [42], betweenness centrality [43], eigenvector centrality [44] and k-core centrality [45] as ground truth in this task. Our aim is to use node representations of all compared methods to train a ridge regression model and predict the above four centrality scores. Mean square error (MSE) is utilized as the evaluation metric.

We employ two data sets in this task, including a collaboration network of Jazz musicians [46] and a social network from BlogCatalog website [47], which is one of the largest blog directories available on the Internet. As both data sets are static graphs, we randomly choose 10% percent of edges as the initial edges and add 10% percent of edges gradually to build 10 dynamic graph snapshots. Detailed statistics of both data sets are shown in Table 4.

| Dataset    | #Nodes | #Edges | $D_{\text{max}}$ | $k_{\text{max}}$ | #Snapshots |
|------------|--------|--------|------------------|------------------|------------|
| Jazz       | 198    | 2742   | 100              | 29               | 10         |
| BlogCatalog| 10312  | 333983 | 3992             | 114              | 10         |

We report the regular equivalence prediction results of each data set in Table 5 and Table 6, respectively. We observe that CGGCN-S and CTGCN-S outperform baselines in prediction accuracy of closeness, betweenness and k-core centrality, while CGCN-C and CTGCN-C receive higher MSE errors in this task, just like other GCN baselines. It indicates that by optimizing the reconstruction loss in Equation 9 in a recursive way, CGCN-S and CTGCN-S can capture structural role related information in dynamic graphs. Therefore, the results demonstrate that CTGCN is capable of preserving structural similarity between nodes in a global sense. We also note that in contrast to structure preserving baselines, such as struct2vec, DynGEM and TIMERS, the proposed CGGCN-S and CTGCN-S perform more accurately in predicting most of the four centrality scores, especially on large dynamic graphs. However, other baselines can only predict one of the four centralities accurately on both data sets.

| Method     | closeness | betweenness | eigenvector | k-core |
|------------|-----------|-------------|-------------|--------|
| DeepWalk   | 1.3330    | 2.8911      | 2.98695     | 139.5198 |
| node2vec   | 1.0464    | 2.7627      | 2.7213      | 129.0802 |
| struct2vec | 0.3274    | 1.8019      | 0.8664      | 19.4234  |
| GCN        | 1.2369    | 2.6445      | 3.1017      | 130.4015 |
| GAT        | 1.7526    | 2.6593      | 3.9321      | 150.2643 |
| CGCN-C     | 2.3221    | 2.9775      | 5.5942      | 229.5273 |
| CGCN-S     | 0.3239    | 1.1865      | 1.0163      | 20.8869  |
| DynGEM     | 0.4363    | 1.9467      | 0.4605      | 28.9027  |
| TIMERS     | 0.9527    | 1.9805      | 0.0060      | 71.4703  |
| EvolveGCN  | 0.8352    | 2.5945      | 3.9347      | 139.0959 |
| CTGCN-C    | 1.8488    | 2.6580      | 4.3826      | 140.7794 |
| CTGCN-S    | 0.3276    | 1.1606      | 1.0283      | 21.4351  |

5.4 Structural Role Classification
We use dynamic structural role classification task to further evaluate the structure preserving ability of all compared methods. In this task, the learned node representations are utilized to predict the structure role related labels of nodes in each dynamic graph. The classifier we used is logistic regression with sklearn package, and the evaluation metric is Accuracy. We evaluate the performance of all methods on two data sets, one is a European air-traffic network, the other is an American air-traffic network [6]. Both data sets are also static graphs, and we use the same technique as shown in Section 5.3 to build dynamic graphs. Detailed statistics of both data sets are shown in Table 7.

| Method     | closeness | betweenness | eigenvector | k-core |
|------------|-----------|-------------|-------------|--------|
| DeepWalk   | 0.6822    | 1.2666      | 1.1358      | 1285.6192 |
| node2vec   | 0.4872    | 1.2977      | 1.1970      | 1408.2082 |
| struct2vec | 0.1514    | 0.6340      | 0.2620      | 237.3233 |
| GCN        | 0.4525    | 1.3028      | 1.2263      | 1538.0220 |
| GAT        | 1.8572    | 1.3127      | 1.3242      | 1801.8154 |
| CGCN-C     | 1.8914    | 1.3137      | 1.3278      | 1827.9269 |
| CGCN-S     | 0.3056    | 0.0531      | 0.0526      | 21.8264  |
| DynGEM     | 1.0827    | 0.0744      | 0.0155      | 68.6522  |
| TIMERS     | 1.2086    | 0.2394      | 1.3590e-5   | 252.1401 |
| EvolveGCN  | 0.2360    | 1.3018      | 1.1771      | 1423.8467 |
| CTGCN-C    | 1.8908    | 1.3148      | 1.3399      | 1838.7318 |
| CTGCN-S    | 0.2895    | 0.0985      | 0.0542      | 22.1570  |

The node classification results are shown in Figure 5. We observe that our proposed CTGCN-S method outperforms other baselines on both graphs. It indicates CTGCN-S can preserve structural similarity between nodes in dynamic graphs. Furthermore, we note that struct2vec, DynGEM and TIMERS also achieve good classification performance on both data sets. It suggests that building node-node structural similarity relationship, encoding the neighboring information and extracting eigenvectors of adjacent matrices are all effective ways to preserve graph structure information. The proposed method is an instance of encoding the neighboring information through CGCLs. Hence, the results further demonstrate the effectiveness of the proposed
5.5 Parameter Sensitivity

The proposed CTGCN methods involve a number of parameters that may affect its performance. We first examine how different choices of embedding dimension $d$ and the hyper-parameter $Q$ in Equation 7 affect the link prediction performance of CTGCN methods on Facebook data set. To save the computation time, we only test the link prediction performance on the last 5 snapshots.

![Fig. 5. Accuracy of structural role classification on dynamic graphs.](image)

![Fig. 6. Average AUC scores with respect to hyper-parameter $Q$ and embedding dimension $d$ on Facebook data set.](image)

From the link prediction results shown in Figure [6], we can see that the performance of CTGCN-C increases when embedding dimension $d$ increases from 50 to 100, but tends to saturate when $d$ is greater than 100. Similarly, the performance of CTGCN-S also becomes stable when embedding dimension $d$ reaches 100, but CTGCN-S is more insensitive than CTGCN-C with respect to the changes of $d$. We also observe that CTGCN-C is sensitive with respect to the hyper-parameter $Q$. The performance of CTGCN-C increases and becomes stable when $Q$ reaches a certain value greater than 0, but drops when $Q$ is too large. It indicates choosing appropriate $Q$ value is helpful to the performance improvement of CTGCN-C.

Then we analyze the effect of transformation layer number and CGCL number on the link prediction performance of CTGCN-C and CTGCN-S. The experiments are conducted on the last 5 snapshots of Facebook data set. As illustrated in Figure [7], we can see that CTGCN-S can learn robust node representations in dynamic graphs.

![Fig. 7. Average AUC scores with respect to the number of transformation layers and CGCLs on Facebook data set.](image)

suggests that CTGCN-C also encounters the over-smoothing problem when stacking too many graph convolution layers, just like GCN and GAT. In addition, CTGCN-C is only compatible with 1-layer linear feature transformation. Utilizing either multi-layer linear transformation or multi-layer non-linear transformation will result in the decrease of its performance.

We also observe that CTGCN-S is more robust with respect to the changes of layer numbers. When CGCL number increases, the performance of CTGCN-S is almost stable. When the transformation layer numbers reach 3, the performance of CTGCN-S also becomes stable. For transformation layers, CTGCN-S is more compatible with non-linear transformation layers, as non-linear transformation can yield more expressive node representations. It’s worth noting that CTGCN-S with either linear transformation layers or non-linear transformation layers are comparable in link prediction task, which suggests that CTGCN-S can learn robust node representations in dynamic graphs.

5.6 Efficiency Analysis

In this section, we evaluate the efficiency of the proposed method from two aspects. First, we compare the running time of CGCN-C and CGCN-S with those of the vanilla GCN and GAT. Second, we evaluate the scalability of CTGCN-C and CTGCN-S with respect to the node number and the timestamp number.

5.6.1 Running Time

To make fair comparison, we utilize 2-layer GCN, GAT and CGCN-C with the same hidden dimension in this task. GAT has one head mask attention in each layer. As CGCN-S has a different architecture, we use the same CGCN-S settings as those in Section 5.1. Then we retrain all GCNs on all snapshots of Facebook and Enron data set. All GCNs are trained separately to reduce interference.

Then we count the average training time of all timestamps for each method. As illustrated in Figure [8], we can see that GCN is highly efficient on both large-scale data sets. CGCN-C and CGCN-S are less efficient than GCN, but are faster than GAT. The possible reason is that the attention operation is time-consuming on large graphs, while the proposed CGCN-C and CGCN-S share the feature transformation matrix across all k-core adjacent matrices,
which highly reduces the time complexity. Therefore, the results demonstrate that CGCN-C and CGCN-S are efficient on large-scale dynamic graphs.

5.6.2 Scalability
We count the actual running time per epoch of CTGCN-C and CTGCN-S on Facebook data set to test their scalability. We sample subgraphs with different node numbers varying from 100 to 60730 on a graph to test the scalability with respect to the node number, and train models on different timestamp ranges to test the scalability with respect to the timestamp number. The experiments are run in a machine with 48 processors Intel Xeon Gold 2.30GHz CPU and 256G RAM. The settings of CTGCN-C and CTGCN-S are same as those in Section 5.1.

As shown in Figure 9, we can observe that the training time of CTGCN-C scales linearly with the number of nodes and the slope of the curve is close to 1. The training time of CTGCN-S also scales linearly with the node number, and CTGCN-S runs faster than CTGCN-C. Furthermore, we can see that both CTGCN-C and CTGCN-S scale linearly with the timestamp number. The results are consistent with the complexity analysis in Section 4.5 and demonstrate our methods are scalable on large-scale dynamic graphs.

6 Conclusion
In this paper, we propose a novel k-core based temporal GCN(CTGCN) for dynamic graph embedding. It can preserve both connective proximity and structural similarity in a unified generalized GCN framework. In CTGCN, we present a novel core-based graph convolution layer(CGCL) which can capture hierarchical structure information in nested k-core subgraphs. Experimental results on several real-world graphs demonstrate the effectiveness and efficiency of the proposed method. In contrast to previous GCNs, CTGCN is essentially an instance of our proposed generalized GCN framework which regard graph convolution as feature transformation and feature aggregation operations. Our future direction is to develop novel GCN models under the proposed generalized GCN framework.

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