A Kind of Extension of The Famous Young Inequality

Luo Xianqiang

Department of Mathematics, Shanghai University, Shanghai 200444, China
Mathematics College, Wuyi University, Guangdong 529020, China

Abstract: Young Inequality was extended in the references [2] and [3], which has extensive use and great effort in analysis mathematics. By the kind of extended Young Inequality, we can get the famous Holder Inequality and Minkowski Inequality. But until now we have not found its strict analysis proof. In the references [2] and [3], only the probable pattern description was found. In this paper, we will get the strict analysis proof of a kind of Extension of Young Inequality with the approximation method.

Key words: Young Inequality, N-function, Strictly convex function

MR(2000) Subject Classification: 46B20, 46B02, 46A22

§1 Introduction

The original Young Inequality [1] has been proposed in an integral form by W. H. Young in 1912: Suppose $f(x)$ is strictly increasing and continuous function defined in $[0,c]$, $f^{-1}(x)$ is the inverse function of $f(x)$, $f(0) = 0$, $a \in [0,c]$, $b \in [f(0), f(c)]$. Then

$$\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \geq ab,$$

where the equality holds if and only if $b = f(a)$.

Young Inequality has extensive use and great effort in analysis mathematics. Now Young Inequality was extended as follows:

Let $M(u)$, and $N(v)$ are complementary N-function each other (see Definition 2.1 and Definition 2.2), then the kind of Young Inequality: $uv \leq M(u) + N(v)$ holds, and the equality holds if and only if $u = q(|v|)sign v$ or $v = p(|u|)sign u$, for all $u, v \in (-\infty, +\infty)$.

This research was partially supported by the National Natural Science Foundation of China(No:11271245), and the Natural Science Foundation Guangdong Province of China(2012KJCX0101).

Corresponding author:luoxq1978@126.com
By the kind of Young Inequality, we can get the famous Holder Inequality and Minkowski Inequality (see references [2] and [3]). But until now we have not found its strict analysis proof. In the references [2] and [3], only the probable pattern description was found.

In this paper, we will get its strict analysis proof with the approximation method.

§2 Preliminaries

Definition 2.1[2] The mapping $M : (-\infty, +\infty) \rightarrow (-\infty, +\infty)$ is called an N-function if it has the following properties:

(i) $M(u)$ is even, continuous, convex and $M(0) = 0$.
(ii) $M(u) > 0$ for all $u \neq 0$.
(iii) $\lim_{u \to 0} \frac{M(u)}{u} = 0$ and $\lim_{u \to \infty} \frac{M(u)}{u} = \infty$.

Lemma 2.1[2] $M(u)$ is an N-function if and only if there exists $p(u) : [0, +\infty) \rightarrow [0, +\infty)$ with the following properties:

(i) $p(u)$ is right-continuous and nondecreasing;
(ii) $p(u) > 0$ whenever $u \neq 0$;
(iii) $p(0) = 0$ and $p(\infty) = \infty$, $M(u) = \int_{0}^{[u]} p(t) dt$.

Record 2.1[2] $p(u)$ is the right-derivative of N-function $M(u)$.

Lemma 2.2 Let $p_{-}(u)$ is the left-derivative of N-function $M(u)$, then $p_{-}(u) = \lim_{h \to 0^{+}} p(u-h)$, and $\int_{0}^{[u]} p_{-}(t) dt = M(u)$.

Proof From the proof process of Theorem 1.4 in reference[2], we know $p_{-}(u)$ is left continuous, and for all $0 < u < v$, $p(u) \leq p_{-}(v) \leq p(v)$.

Hence, for $h > 0$, we have $p(v-h) \leq p_{-}(v)$.

Therefore,

$$\lim_{h \to 0^{+}} p(v-h) \leq p_{-}(v).$$

On the other hand, since $p_{-}(v) \leq p(v)$ and $p_{-}(v)$ is left continuous, we get

$$p_{-}(v) = \lim_{h \to 0^{+}} p_{-}(v-h) \leq \lim_{h \to 0^{+}} p(v-h).$$

Therefore, we have

$$p_{-}(v) = \lim_{h \to 0^{+}} p(v-h).$$

Since for all $h > 0$, $p(v-h) \leq p_{-}(v) \leq p(v)$, then we have

$$\int_{0}^{[u]} p(t-h) dt \leq \int_{0}^{[u]} p_{-}(t) dt \leq \int_{0}^{[u]} p(t) dt = M(u).$$

That is,

$$M(|u| - h) - M(-h) \leq \int_{0}^{[u]} p_{-}(t) dt \leq M(u).$$
Let $h \to 0$, by the property (i) of $M(u)$ in Definition 2.1, we have
\[
\int_0^{[u]} p_-(t)dt = M(u).
\]

**Definition 2.2** [2] Suppose $M(u)$ is an N-function. Let $p(t)$ is the right derivative of $M(u)$. Let $q(s) = \sup t = \inf t$, called the right-inverse function of $p(t)$. By Theorem 1.5 in reference [2], we know $q(s)$ also satisfies the three properties of Lemma 2.0, and $N(s) = \int_0^{|v|} q(s)ds$ is called the complementary N-function of $M(u)$. It is obvious, the left derivative $q_-(s)$ of $N(v)$ satisfies $q_-(s) = \sup t = \inf t$.

**Lemma 2.3** [2] $q(p(t)) \geq t, p(q(s)) \geq s; q(p(t) - \varepsilon) \leq t, p(q(s) - \varepsilon) \leq s$.

**Lemma 2.4** [2] $M(u)$ is strictly convex if and only if $p(t)$ is strictly increasing, that is, $q(s)$ is continuous.

**Lemma 2.5** [2] For any N-function $M(u)$ and $\varepsilon > 0$, there exists a strictly convex N-function $M_1(u)$, such that

\[
(1 - \varepsilon)p(t) \leq p_1(t) \leq (1 + \varepsilon)p(t), (1 - \varepsilon)M(u) \leq M_1(u) \leq (1 + \varepsilon)M(u),
\]

where $p(t)$ and $p_1(t)$ is the right derivative of $M(u)$ and $M_1(u)$ respectively.

**Record 2.2** Lemma 2.5 is Theorem 1.10 in reference [2], but it reverses the old conclusion “$M(u) \leq M_1(u) \leq (1 + \varepsilon)M(u)$”, for the new conclusion “$(1 - \varepsilon)M(u) \leq M_1(u) \leq (1 + \varepsilon)M(u)$”.

From the construction process of $p_1(t)$ in the proof in reference [3], we know if $p(t)$ is continuous, then $p_1(t)$ is also continuous.

**Lemma 2.6** Suppose $u \geq 0$ and $v \geq 0$, then $u = q(v)$ or $v = p(u)$ if and only if $u \in [q_-(v), q(v)]$. By the symmetry we get another necessary and sufficient condition is $v \in [p_-(u), p(u)]$.

**Proof** Sufficiency.
Suppose $u \in [q_-(v), q(v)]$.
(i) If $q_-(v) = q(v)$, it is clear that $u = q_-(v) = q(v)$.
(ii) If $q_-(v) \neq q(v)$, then $q_-(v) < q(v)$. If $u = q(v)$, then the conclusion holds.
If $q_-(v) \leq u < q(v)$, we need only prove $p(u) = v$.
From Definition 2.2, we have
\[
q_-(v) = \sup_{p(t) < v} t = \inf_{p(t) \geq v} t.
\]

Since $q_-(v) \leq u \Rightarrow \sup_{p(t) < v} t \leq u$, then for any $\frac{1}{n}$, we get
\[
p(u + \frac{1}{n}) \geq v.
\]

3
Let $n \to \infty$, since $p(t)$ is right continuous, then we have

\[ p(u) \geq v. \]

On the other hand, from $u < q(v) = \sup_{p(t) \leq v} t = \inf_{p(t) \geq v} t$, we get

\[ p(u) = v. \]

So, we have

\[ p(u) = v. \]

Necessity.

If $u = q(v)$, it is clearly established.

If $p(u) = v$, then from

\[ q_-(v) = \inf_{p(t) \geq v} t \leq u, \]

and

\[ q(v) = \sup_{p(t) \leq v} t \geq u, \]

We have

\[ u \in [q_-(v), q(v)]. \]

The next two lemmas are about change of variable of integral and distribute integral.

**Lemma 2.7** Suppose $f(x)$ and $g(x)$ are defined on the interval $[a, b]$, and the Stieltjes integral of $f(x)$ about $g(x)$ exists. Suppose $x(t)$ is a strictly increasing and continuous function on the interval $[\alpha, \beta]$, and $x(\alpha) = a$ and $x(\beta) = b$, then

\[ \int_{\alpha}^{\beta} f(x(t))dg(x(t)) = \int_{a}^{b} f(x)dg(x). \]

**Lemma 2.8** Suppose $f(x)$ and $g(x)$ are defined on the interval $[a, b]$, and the Stieltjes integral of $f(x)$ about $g(x)$ exists, then

\[ \int_{a}^{b} f(x)dg(x) + \int_{a}^{b} g(x)df(x) = f(b)g(b) - f(a)g(a). \]

§3 Main Result

**Theorem 3.1** Suppose $M(u)$ be an N-function and $N(v)$ is the complementary N-function of $M(u)$, then Young Inequality $uv \leq M(u) + N(v)$ holds, and $uv = M(u) + N(v)$ holds if and only if $u = q(|v|) \text{sign } v$ or $v = p(|u|) \text{sign } u$. 


Proof. Suppose $u \geq 0$ and $v \geq 0$.

Firstly, we will prove the necessity of the equality.

Suppose there exist $u_0 \geq 0$ and $v_0 \geq 0$ satisfying

$$M(u_0) + N(v_0) = u_0v_0.$$ 

Let

$$F(u, v) = M(u) + N(v) - uv.$$ 

From Young Inequality we have known that for all $u$ and $v$, $F(u, v) \geq 0$.

And from $M(u_0) + N(v_0) = u_0v_0$, we have $F(u, v_0) = M(u) + N(v_0) - uv_0$ can get the minimum 0 in $u_0$.

If $u_0 = 0$, from $M(u_0) + N(v_0) = u_0v_0$, we get that $v_0 = 0$,
then $u_0 = q(v_0) = 0$ or $v_0 = p(u_0) = 0$, that is, the necessity of the equality holds.

If $u_0 \neq 0$, then $F(u_0, v_0)$ is the minimum of the $F(u, v_0)$ on the interval $(0, +\infty)$.

Therefore, the left derivative of $F(u, v_0)$ is less than or equal to zero on the point $u_0$, and the right derivative of $F(u, v_0)$ is more than or equal to zero on the point $u_0$.

That is,

$$p_-(u_0) - v_0 \leq 0, p(u_0) - v_0 \geq 0.$$ 

Then

$$v_0 \in [p_-(u_0), p(u_0)].$$ 

From Lemma 2.6 we get $u_0 = q(v_0)$ or $v_0 = p(u_0)$.

That is, the necessity of the equality holds.

Secondly, we will get the proof of Young Inequality and the sufficiency of the equality in three steps:

Step I. Suppose $M(u)$ and $N(v)$ are all strictly convex. From Lemma 2.4, the right derivative $p(t)$ and $q(s)$ are all strictly increasing, continuous, and are the right inverse-function each other. From the reference [4], we have that the Stieltjes integral $\int_0^q(v) tdp(t)$ exists.

From Lemma 2.7 and Lemma 2.8, we have

$$M(u) + N(v)$$

$$= \int_0^u p(t)dt + \int_0^v q(s)ds$$

$$= \int_0^u p(t)dt + \int_0^{q(v)} tdp(t)$$

$$= \int_0^u p(t)dt + vq(v) - \int_0^{q(v)} p(t)dt$$

$$= vq(v) + \int_{q(v)}^u p(t)dt.$$ (1)
(i) If \( u > q(v) \), then \( p(u) > v \).

Hence, by the expression (1), we have

\[
M(u) + N(v) = \int_{q(v)}^{u} p(t) dt + vq(v) \\
= \int_{q(v)}^{u} p(t) dt + uv - v(u - q(v)) \\
\geq p(q(v))(u - q(v)) + uv - v(u - q(v)) \\
= uv.
\]

(ii) If \( u < q(v) \), then \( p(u) \leq v \).

Hence, by the expression (1), we have

\[
M(u) + N(v) = vq(v) - \int_{q(v)}^{u} p(t) dt \\
= uv + v(q(v) - u) - \int_{q(v)}^{u} p(t) dt \\
\geq uv + v(q(v) - u) - p(q(v))(q(v) - u) \\
= uv.
\]

(iii) If \( u = q(v) \), then \( v = p(u) \).

From the expression (1), we have \( uv = M(u) + N(v) \).

That is, the sufficiency of the equality holds.

Step II. Suppose \( M(u) \) is strictly convex, then from Lemma 2.4, the right derivative \( p(t) \)

is strictly increasing, and the right-inverse function \( q(s) \) is continuous and nondecreasing.

From Lemma 2.5 and Record 2.2, \( \forall 0 < \varepsilon < \frac{1}{2} \), we can construct a function strictly
increasing and continuous \( q_1(s) \),

such that

\[
(1 - \varepsilon)q(s) \leq q_1(s) \leq (1 + \varepsilon)q(s).
\]

Hence,

\[
\frac{1}{1 + \varepsilon} q_1(s) \leq q(s) \leq \frac{1}{1 - \varepsilon} q_1(s).
\]

Let \( p_1(t) \) is the right-inverse function of \( q_1(s) \), then \( p_1(t) \) is strictly increasing and continuous.

In the following we will get the relation of \( p_1(t) \) and \( p(t) \).
we have
\[ \frac{1}{1+\varepsilon} q_1(p_1(t)) \leq q(p_1(t)) \leq \frac{1}{1-\varepsilon} q_1(p_1(t)). \]

That is,
\[ \frac{t}{1+\varepsilon} \leq q(p_1(t)) \leq \frac{t}{1-\varepsilon}. \]

(3)

From Lemma 2.3 and the expression (3), we get
\[ q(p(t)) \geq \int_0^u p(t)dt + \int_0^v q_1(s)ds. \]

(4)

Since \( q(s) \) is nondecreasing, by the expression (4), we get
\[ p\left(\frac{t}{1-2\varepsilon}\right) > p_1(t). \]

(5)

From the result in step I, we get
\[ uv \leq M_1(u) + N_1(v) = \int_0^u p_1(t)dt + \int_0^v q_1(s)ds. \]

Therefore,
\[ M(u) + N(v) = \int_0^u p(t)dt + \int_0^v q(s)ds \]
\[ \geq \int_0^u p(t)dt + \int_0^v \frac{1}{1+\varepsilon} q_1(s)ds \quad (by \ (2)) \]
\[ \geq \int_0^u p(t)dt + \frac{1}{1+\varepsilon} (uv - \int_0^u p_1(t)dt) \]
\[ \geq \int_0^u p(t)dt - \frac{1}{1+\varepsilon} \int_0^u p_1(t)dt \quad (by \ (5)) \]
\[ \geq \int_0^u p(t)dt - \frac{1}{1+\varepsilon} \int_0^u p(t)dt. \]

Let \( \varepsilon \to 0 \), we have
\[ M(u) + N(v) \geq uv. \]

In the following, we will prove the sufficiency of the equality.
If \( v = p(u) \), from Lemma 2.3 and the expression (3), for the above \( 0 < \varepsilon < \frac{1}{2} \), we have
\[ q(p(\frac{t}{1+2\varepsilon}) - \varepsilon) \leq \frac{t}{1+2\varepsilon} < \frac{t}{1+\varepsilon} \leq q(p_1(t)) \Rightarrow p(\frac{t}{1+2\varepsilon}) - \varepsilon < p_1(t). \]

(6)

In the expression (6), let \( \varepsilon \to 0 \), by Lemma 2.2,
Therefore, we have
\[ p_-(t) \leq \lim_{\varepsilon \to 0} p_1(t). \] (7)

On the other hand, in the expression (5), let \( \varepsilon \to 0 \), we get
\[ \lim_{\varepsilon \to 0} p_1(t) \leq p(t). \]

Therefore,
\[ p_-(t) \leq \lim_{\varepsilon \to 0} p_1(t) \leq p(t). \] (8)

By Lemma 2.2, we get
\[ \int_0^u p_-(t) dt = M(u) = \int_0^u \lim_{\varepsilon \to 0} p_1(t) dt = \int_0^u p(t) dt \] (9)

Now we need prove
\[ \int_{p_1(u)}^{p(u)} q(s) ds = u(p(u) - p_1(u)). \] (10)

In fact, if \( s = p(u) \), from Definition 2.2, since \( p(u) \) is strictly increasing, then we have \( q(s) = \sup_{p(t) \leq s} t = \sup_{p(t) \leq p(u)} t = u \). If \( s \in [p_-(u), p(u)] \), from Lemma 2.6 we get \( q(s) = u \).

Therefore, we have \( \int_{p_1(u)}^{p(u)} q(s) ds = \int_{p_1(u)}^{p(u)} u ds = u(p(u) - p_1(u)) \).

By the result in Step I, we have
\[ up_1(u) = M_1(u) + N_1(p_1(u)) = \int_0^u p_1(t) dt + \int_{p_1(u)}^{p(u)} q_1(s) ds. \] (11)

From the expressions (9) and (11), we get
\[
\begin{align*}
M(u) + N(p(u)) & = \int_0^u p(t) dt + \int_0^{p(u)} q(s) ds \\
& = \int_0^u p(t) dt + \int_0^{p_1(u)} q(s) ds + \int_{p_1(u)}^{p(u)} q(s) ds \\
& \leq \int_0^u p(t) dt + \int_{p_1(u)}^{p_1(u)} \frac{1}{1 - \varepsilon} q_1(s) ds + \int_{p_1(u)}^{p(u)} q(s) ds \quad \text{(by (2))} \\
& = \int_0^u p(t) dt + \frac{1}{1 - \varepsilon} (up_1(u) - \int_0^u p_1(t) dt) + \int_{p_1(u)}^{p(u)} q(s) ds \quad \text{(by (11))} \\
& = \int_0^u p(t) dt - \frac{1}{1 - \varepsilon} \int_0^u p_1(t) dt + \frac{1}{1 - \varepsilon} up_1(u) + u(p(u) - p_1(u)) \quad \text{(by (10))} \\
& = M(u) - \frac{1}{1 - \varepsilon} \int_0^u p_1(t) dt + \left( \frac{u}{1 - \varepsilon} - u \right) p_1(u) + up(u)
\end{align*}
\]
Let \( \varepsilon \to 0 \), we have
\[
M(u) + N(p(u)) \leq up(u).
\] (12)

On the other hand, we have got the inequality \( uv \leq M(u) + N(v) \).
Let \( v = p(u) \), we have
\[
up(u) \leq M(u) + N(p(u)).
\]
Therefore, together with the expression (12), we have
\[
M(u) + N(p(u)) = up(u).
\]

That is, the sufficiency of the equality holds.

Step III, for any N-function \( M(u) \), suppose its complementary N-function is \( N(v) \), \( p(t) \) is the right-inverse function of \( M(u) \), and \( q(s) \) is the right-inverse function of \( N(v) \). From Lemma 2.5, for the above \( 0 < \varepsilon < \frac{1}{2} \), we can find a strictly convex N-function \( M_1(u) \) and its right-derivative \( p_1(t) \),

such that
\[
(1 - \varepsilon)p(t) \leq p_1(t) \leq (1 + \varepsilon)p(t), (1 - \varepsilon)M(u) \leq M_1(u) \leq (1 + \varepsilon)M(u)
\] (13)

Suppose \( N_1(v) \) is the complementary N-function of \( M_1(u) \), \( q_1(s) \) is the right derivative of \( N_1(v) \).

In the following we will get the relation of \( q_1(t) \) and \( q(t) \).
for the above \( 0 < \varepsilon < \frac{1}{2} \). In the expression (13), let \( t = q_1(s) - \varepsilon \),
we have
\[
(1 - \varepsilon)p(q_1(s) - \varepsilon) \leq p(q_1(s) - \varepsilon) \leq (1 + \varepsilon)p(q_1(s) - \varepsilon).
\] (14)

From Lemma 2.3, we have that
\[
p_1(q_1(s) - \varepsilon) \leq s, \quad p_1(q_1(s)) \geq s.
\] (15)

Therefore, by the expressions (14) and (15), we have
\[
p(q_1(s) - \varepsilon) \leq \frac{s}{1 - \varepsilon}, \quad p(q_1(s)) \geq \frac{s}{1 + \varepsilon}.
\] (16)

And then, by Lemma 2.3, together with the expression (16), we have
\[
p(q_1(s) - \varepsilon) \leq \frac{s}{1 - \varepsilon} < \frac{s}{1 - 2\varepsilon} \leq p(q(s - \varepsilon)),
\]
\[
p(q_1(s)) \geq \frac{s}{1 + \varepsilon} > \frac{s}{1 + 2\varepsilon} \geq p(q(s + 2\varepsilon)) - \varepsilon).
\] (17)
Since \( p(t) \) is nondecreasing, then by the expression (17), we get
\[
q\left(\frac{s}{1 + 2\varepsilon}\right) - \varepsilon < q_1(s) < q\left(\frac{s}{1 - 2\varepsilon}\right) + \varepsilon.
\] (18)

From the result in Step II, we get
\[
uv \leq M_1(u) + N_1(v) \\
\leq (1 + \varepsilon)M(u) + \int_0^v q_1(s)\,ds \\
< (1 + \varepsilon)M(u) + \int_0^v (q\left(\frac{s}{1 - 2\varepsilon}\right) + \varepsilon)\,ds \quad \text{(by (18))} \\
= (1 + \varepsilon)M(u) + \varepsilon v + (1 - 2\varepsilon)\int_0^\frac{v}{1 - 2\varepsilon} q(s)\,ds \\
= (1 + \varepsilon)M(u) + \varepsilon v + (1 - 2\varepsilon)N\left(\frac{v}{1 - 2\varepsilon}\right)
\]

Let \( \varepsilon \to 0 \), we have
\[
uv \leq M(u) + N(v).
\]

In the following we will prove sufficiency of the equality.

By the result in Step II, we have
\[
M_1(u) + N_1(p_1(u)) = up_1(u).
\] (19)

Therefore,
\[
M(u) + N(p(u)) \\
\leq \frac{1}{1 - \varepsilon}M_1(u) + N(p(u)) \\
= \left(\frac{1}{1 - \varepsilon} - 1\right)M_1(u) + M_1(u) + N_1(p_1(u)) + N(p(u)) - N_1(p_1(u)) \\
= \left(\frac{1}{1 - \varepsilon} - 1\right)M_1(u) + up_1(u) + N(p(u)) - N_1(p_1(u)) \quad \text{(by (19))} \\
\leq \left(\frac{1}{1 - \varepsilon} - 1\right)M_1(u) + up_1(u) + N(p(u)) - \int_0^{p_1(u)} (q\left(\frac{s}{1 + 2\varepsilon}\right) - \varepsilon)\,ds \quad \text{(by (18))} \\
= \left(\frac{1}{1 - \varepsilon} - 1\right)M_1(u) + up_1(u) + \int_0^{p(u)} q(s)\,ds - (1 + 2\varepsilon)\int_0^{p_1(u)} q(s)\,ds + \varepsilon p_1(u) \\
= \left(\frac{1}{1 - \varepsilon} - 1\right)M_1(u) + up_1(u) + N(p(u)) - (1 + 2\varepsilon)N\left(\frac{p_1(u)}{1 + 2\varepsilon}\right) + \varepsilon p_1(u).
\]

Let \( \varepsilon \to 0 \), together with the expression (13), we get \( p_1(u) \to p(u), M_1(u) \to M(u) \), and \( N\left(\frac{p_1(u)}{1 + 2\varepsilon}\right) \to N(p(u)) \) since \( N(v) \) is continuous.

Therefore,
\[
M(u) + N(p(u)) \leq up(u).
\] (20)
On the other hand, we have got the inequality $uv \leq M(u) + N(v)$.

Let $v = p(u)$, we have

$$up(u) \leq M(u) + N(p(u)).$$

Therefore, together with the expression (20),

we have

$$M(u) + N(p(u)) = up(u).$$

That is, the sufficiency of the equality holds.

References:

[1] Pei L W: Typical problems and methods in mathematical analysis (in chinese). G: Higher education press, Beijing (2000)

[2] Wu C X, Wang T Q, Chen S T, Wang Y W: Geometry of Orlicz spaces (in chinese). G: Harbin Institute of Technology press, Harbin (1986)

[3] Chen S T: Geometry of Orlicz spaces. G: Dissertations Mathematicae Warszawa, Warszawa (1996)

[4] Jang Z J: Theory of functions of a real variable (in chinese). G: Higher education press, Beijing (1994)

[5] Chen S, He X, Hudzik H, Kamińska A: Monotonicity and best approximation in Orlicz-Sobolev spaces with the Luxemburg norm. J. Math. Anal. Appl. 344, 687-698 (2008)

[6] Gong W, Shi Z: Drop proper ties and approximative compactness in Orlicz-Bochner function spaces. J. Math. Anal. Appl. 344, 748-756 (2008)

[7] Shi Z, Gong W: Monotone points in Orlicz-Bochner function spaces. MATHEMATICA APPLICATA. 23(2), 376-383 (2010)

[8] Liu C Y, Shi Z R: U properties in Orlicz spaces (in chinese). Journal of Mathematical Physics. 31(2), 328-334 (2011)

[9] Shi Z R, Liu C Y: Noncreasy and uniformly noncreasy Orlicz-Bochner function spaces. Nonlinear Analysis. 74, 6153-6161 (2011)