Domination Polynomials of Graph Products

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Abstract

The domination polynomials of binary graph operations, aside from union, join and corona, have not been widely studied. We compute and prove recurrence formulae and properties of the domination polynomials of families of graphs obtained by various products, ranging from explicit formulae and recurrences for specific families to more general results. As an application, we show the domination polynomial is computationally hard to evaluate.

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1 Introduction and Definitions

This paper discusses simple undirected graphs $G = (V, E)$. A vertex subset $W \subseteq V$ of $G$ is a dominating set in $G$, if for each vertex $v \in V$ of $G$ either $v$ itself or an adjacent vertex is in $W$.

Definition 1.1. Let $G = (V, E)$ be a graph. The domination polynomial $D(G, x)$ is given by

$$D(G, x) = \sum_{i=0}^{\left|V\right|} d_i(G)x^i,$$

where $d_i(G)$ is the number of dominating sets of size $i$ in $G$.

The domination numbers of graph products have been extensively studied in the literature, see e.g [1, 2, 3, 7, 14, 17, 18, 19, 21, 26, 27, 30]. In particular, a large number of papers have addressed the domination number of Cartesian products, inspired by the famous conjecture by V. G. Vizing [31] (see [11] for a recent survey.) In contrast, although the domination polynomial has been actively studied in recent years, almost no attention has been given to the domination polynomials of graph products.

In [24] we showed that there exist recurrence relations for the domination polynomial which allow for efficient schemes to compute the polynomial for some types of graphs. A recurrence for the domination polynomial of the path graph with $n$ vertices $(P_n)$ was shown in [5] to be

$$D(P_{n+1}, x) = x(D(P_n, x) + D(P_{n-1}, x) + D(P_{n-2}, x))$$

(1)

where $D(P_0, x) = 1$, $D(P_1, x) = x$ and $D(P_2, x) = x^2 + 2x$. Note that the complete graphs $K_j \cong P_j$ for $0 \leq j \leq 2$ and that $D(K_r, x) = (x+1)^r - 1$.

Given any two graphs $G$ and $H$ we define the Cartesian product, denoted $G \square H$, to be the graph with vertex set $V(G) \times V(H)$ and edges between two vertices $(u_1, v_1)$ and $(u_2, v_2)$ if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $u_1u_2 \in E(G)$ and $v_1 = v_2$. For $S \subseteq V(G)$ and $T \subseteq V(H)$ let $(S, T)$ be the subgraph of $G \square H$ containing all vertices $(u, v)$ such that $u \in S$ and $v \in T$.

This product is well known to be commutative and, if $G \cong G_1 \cup G_2$ is a disconnected graph, then $G \square H = (G_1 \square H) \cup (G_2 \square H)$, so that

$$D(G \square H, x) = D(G_1 \square H, x)D(G_2 \square H, x).$$

Despite these properties, it is difficult to determine much about this product, even in such simple cases as the grid graphs $P_n \square P_m$, especially in the case of dominating sets. The strong product $(G \boxtimes H)$ is the graph which is formed by taking the graph $G \square H$ and then additionally adding edges between vertices $(u_1, v_1)$ and $(u_2, v_2)$ if both $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$.

The closed neighborhood $N_G[W]$ of a vertex set $W$ in $G$ contains $W$ and all vertices adjacent to vertices in $W$. When $W = \{v\}$ we will write $N_G[v]$ or just $N[v]$ if the graph we are working in is obvious. We define $N_G(W)$ as the open neighbourhood which includes all neighbours of $W$ that are not in $W$, so that $N_G(W) := N_G[W] \setminus W$. 

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If $S$ is a set of vertices from $G$ we use $G - S$ to mean the graph resulting from the deletion of all vertices in $S$ from $G$, and let $G - v$ be $G - \{v\}$. The vertex contraction $G/v$ denotes the graph obtained from $G$ by the removal of $v$ and the addition of edges between any pair of non-adjacent neighbors of $v$.

An outline of the paper is as follows. In section 2 a decomposition formulae for an arbitrary graph’s Cartesian product with $K_2$ is given and also its strong product with $K_r$ for any $r \geq 2$. In Section 3 we give exact formulae for families of Cartesian products of complete graphs. Section 4 gives a recurrence relation for any graph family which contains $P_n \square K_2$ that uses only six smaller graphs. A generalisation of the result in section 4 is given in section 5, where we give a recurrence for $P_n \square K_r$. In section 6, we discuss why recurrence relations can be deduced to exist for many graph products and show implications of their existence to properties of sequences of coefficients of the domination polynomial. Finally, we use a result from the paper to show the Turing hardness of the domination polynomial.

2 Domination polynomials of products with Complete Graphs

Let us suppose that $V(K_2) := \{v_1, v_2\}$ in the product $G \square K_2$ and let $G$ be any non-null graph. We will concentrate first on the vertices in $(G, \{v_1\})$: every vertex subset $W$ of $(G, \{v_1\})$ can be a subset of some dominating set $S$ in $G \square K_2$ so long as some vertices in $(G, \{v_2\})$ are included in $S$ as described below. Let $W \subseteq V(G)$, so, by definition, $(W, \{v_1\})$ dominates the vertices in $(N_G[W], \{v_1\})$ and those in $(W, \{v_2\})$. For $S$ to dominate, all other vertices $(y, v_1)$ must then be dominated by $(y, v_2)$, their only neighbor outside of $(G, \{v_1\})$.

**Theorem 2.1.** The domination polynomial for $D(G \square K_2, x)$ is equal to:

$$
\frac{x^{|V(G)|}}{x+1} \times \sum_{W \subseteq V(G)} \frac{(D(J_W/v, x) + D(J_W - N_J_W[v], x) + D(J_W, x) - D(J_W - v, x))}{x^{|N_G(W)|}}
$$

where $J_W$ is the subgraph of $G$ induced by $N_G[W]$ with a new vertex $z$ joined to the union of $W$ and $N(V(G) - N_G[W])$.

**Proof.** Suppose that $W \subseteq V(G)$, so that, as above, we know that in any dominating set for $G \square K_2$ if the only vertices from $(G, \{v_1\})$ are $W$ then we must also include $(V(G) - N_G[W], v_2)$. In this way all vertices in $(G, \{v_1\})$ are dominated by these vertices, giving the powers of $x$ as in the theorem.

It now remains to ensure that all of the vertices in $(G, \{v_2\})$ are dominated. Using the vertices forced to dominate $(G, \{v_1\})$ we see that, in $(G, \{v_2\})$, every vertex in either $W$ or in $N[V(G) - N[W]]$ is dominated. The only vertices not dominated are therefore those which are not in $W$ but have no neighbors outside of $N[W]$. Let us call this set $T_W$.  

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We now introduce the graph $J_W$ which is formed by taking the subgraph of $G$ induced by $N[W]$ and adding a new vertex $z$ which is adjacent to every vertex either in $W$ or $N(V(G) - N[W])$. The vertices which $z$ is joined to are exactly those not in $T_W$. Thus we want to count all sets of vertices in $J_W \setminus \{z\}$ such that $T_W$ is dominated.

From [24], $p_z(J_W, x)$ counts the dominating sets for $J_W - N[z]$ which additionally dominate the vertices of $N(z)$. Each of these sets when combined with $z$ is a dominating set for $J_W$ in which $T_W$ is dominated and $z$ is only dominated by itself.

Similarly, $D(J_W, x) - D(J_W - z, x)$ counts all of the dominating sets for $J_W$ which contain $z$ and at least one of its neighbours.

Thus the generating function counting all sets of vertices in $J_W \setminus \{z\}$ such that $T_W$ is dominated satisfies the following relation, using Theorem 2.1 of [24]:

$$\frac{p_z(J_W, x) + D(J_W, x) - D(J_W - z, x)}{x} = \frac{xD(J_W/z, x) + xD(J_W - N_{J_W}[z], x) + D(J_W - z, x) - D(J_W, x)}{x(x + 1)}$$

$$= \frac{xD(J_W/z, x) + D(J_W - N_{J_W}[z], x) + D(J_W, x) - D(J_W - z, x)}{x + 1}$$

Putting this together with our first observation finishes the proof.

The following result is also proven in [12] as Lemma 3:

**Theorem 2.2.** For any graph $G$

$$D(G \boxtimes K_r, x) = D(G, (x + 1)^r - 1)$$

**Proof.** Let $u$ be a vertex of $G$ and $v \in V(K_r)$; the closed neighborhood of the vertex $(u, v)$ is $(N_G[u], K_r)$. For any $X \subseteq V(G)$, let $\{A_x \mid x \in X\}$ be a family of arbitrary non-empty subsets of $V(K_r)$. We then have that such a set $X$ is a dominating set of $G$ if and only if

$$\bigcup_{x \in X} \{(x, v) \mid v \in A_x\}$$

is a dominating set of $G \boxtimes K_r$. Consequently, each vertex of a dominating set of $G$ corresponds to all non-empty subsets of $G \boxtimes K_r$, which are counted by the generating function $(x + 1)^r - 1$.

Theorem 2.2 can be used to generalise recurrence relations for the domination polynomial of any families of graphs, such as for $H_{n,r} := P_n \boxtimes K_r$ as follows:

**Corollary 2.3.** For any integers $n \geq 3$ and $r \geq 1$ we have

$$D(H_{n+1,r}, x) = ((x + 1)^r - 1) (D(H_{n,r}, x) + D(H_{n-1,r}, x) + D(H_{n-2,r}, x))$$
Proof. From equation (1) and using Theorem 2.2 we have

\[ D(H_{n+1}, r, x) = D(P_{n+1} \boxtimes K_r, x) = (x+1)^r - 1 \]

\[ = ((x+1)^r - 1)(D(P_n, (x+1)^r - 1) + D(P_{n-1}, (x+1)^r - 1) + D(P_{n-2}, (x+1)^r - 1)) \]

\[ = ((x+1)^r - 1)(D(H_n, r, x) + D(H_{n-1}, r, x) + D(H_{n-2}, r, x)) \]

as required. \(\square\)

Note that, as shown in [5], the same recurrence as equation (1) holds for the cycle graphs \(C_n\) hence there will be an identical generalisation for the domination polynomial of \(C_n \boxtimes K_r\).

Corollary 2.4. For any integers \(n > 3\) and \(r \geq 1\) we have

\[ D(C_{n+1} \boxtimes K_r, x) = ((x+1)^r - 1)(D(C_n \boxtimes K_r, x) + D(C_{n-1} \boxtimes K_r, x) + D(C_{n-2} \boxtimes K_r, x)) \]

3 The Domination polynomial of \(K_r \boxtimes K_s\)

Theorem 3.1. The domination polynomial for \(K_r \boxtimes K_2\) is

\[ (x+1)^2r - 2(x+1)^r + 2x^r + 1 = ((x+1)^r - 1)^2 + 2x^r \]

Proof. Let us again suppose that \(V(K_2) := \{v_1, v_2\}\) in the product \(K_r \boxtimes K_2\). All sets which contain only vertices from some \((K_r, \{v_j\})\) are counted by \(2((x+1)^r - 1) + 1\) where the first term counts non-empty sets and the last is the empty set. Amongst these sets, only those which contain all \(r\) vertices from \((K_r, v_j)\) (for \(j = 1\) or 2) are dominating in \(K_r \boxtimes K_2\), giving the term \(2x^r\). All vertex subsets of \(K_r \boxtimes K_2\) are counted by \((x+1)^{2r}\), so the domination polynomial for it is

\[ (x+1)^{2r} - 2(x+1)^r + 2x^r + 1 = ((x+1)^r - 1)^2 + 2x^r \]

\[ \square \]

Note that Theorem 3.1 can also be deduced from Theorem 2.1, although it is a more involved calculation, even after using the symmetry inherent when \(G = K_r\).

Theorem 3.2. The domination polynomial for \(K_r \boxtimes K_s\) is

\[ D(K_r \boxtimes K_s, x) = ((x+1)^r - 1)^s - \sum_{k=1}^{s-1} \binom{s}{k} (-1)^k ((x+1)^{s-k} - 1)^r \]

Proof. We can imagine the vertices of \(K_r \boxtimes K_s\) as elements of an \(r \times s\) matrix; for a dominating set in this graph we need to have at least one element in every row and column. The simplest way this can be achieved is to have at least one vertex in every column and the ordinary generating function that counts such sets is \(((x+1)^r - 1)^s\). However, it is also possible to have empty sets in some columns, so long as each row contains at least one element.
Table 1: The domination polynomials for the graphs $P_n \square K_2$

| $n$ | $D(P_n \square K_2, x)$ |
|-----|------------------------|
| 1   | $x^2 + 2x$             |
| 2   | $x^4 + 4x^3 + 6x^2$    |
| 3   | $x^6 + 6x^5 + 15x^4 + 16x^3 + 3x^2$ |
| 4   | $x^8 + 8x^7 + 28x^6 + 52x^5 + 48x^4 + 12x^3$ |
| 5   | $x^{10} + 10x^9 + 45x^8 + 116x^7 + 178x^6 + 148x^5 + 47x^4 + 2x^3$ |
| 6   | $x^{12} + 12x^{11} + 66x^{10} + 216x^9 + 453x^8 + 604x^7 + 470x^6 + 168x^5 + 17x^4$ |

There are $s$ choices for the case of one empty column and, given that choice, the generating function counting non-empty rows of $s-1$ elements is $((x+1)^{s-1} - 1)^r$. However, some of the sets counted in this way will have more than one empty column; by the principle of inclusion-exclusion, we now need to subtract the $\binom{s}{2}$ ways to choose a pair of columns to be empty.

The polynomial counting dominating sets with at least two columns empty is $((x+1)^{s-2} - 1)^r$ but this then includes sets with more than two empty columns and so the inclusion-exclusion process will continue. The final case will be when we have all but one column empty, in which case the only possible dominating set contains all $r$ vertices from one column. The term counting all such sets will be $sx^r = \binom{s}{s-1}((x+1) - 1)^r$, which matches the term in the sum in the theorem when $k = s - 1$. Combining all of these cases together completes the proof.

**Corollary 3.3.** The domination polynomial for $K_r \square K_3$ is

$$(x+1)^r - 1)^3 + 3x^r((x+2)^r - 1)$$

**Proof.** Substituting $s = 3$ into Theorem 3.2 we get

$$D(K_r \square K_3, x) = ((x+1)^r - 1)^3 - \sum_{k=1}^{2} \binom{3}{k}(-1)^k ((x+1)^{3-k} - 1)^r$$

$$= ((x+1)^r - 1)^3 + 3(((x+1)^{2} - 1)^r - ((x+1) - 1)^r)$$

$$= ((x+1)^r - 1)^3 + 3(((x+1)^{2})^r - x^r)$$

$$= ((x+1)^r - 1)^3 + 3x^r((x+2)^r - 1)$$

\[\square\]

4 The Domination Polynomial for $P_n \square K_2$

Let $L_n$ be the graph $P_n \square K_2$ and label the vertices of the two copies of $P_n$ as $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$. Note that the graph $L_{n-1}$ is formed from $L_n$ by deletion of $u_n$ and $v_n$. The domination polynomials of the first six graphs in the family are given in Table 1.
We first prove a small result which will be used in the main theorem of this section.

**Lemma 4.1.** The polynomial $A_n(x)$ counting the dominating sets of $L_n$ such that both $u_n$ and $v_n$ are included satisfies the following:

$$A_n(x) := x^2 (D(L_{n-1}, x) + D(L_{n-2}, x) - A_{n-2}(x))$$

**Proof.** Every dominating set for either $L_{n-1}$ or $L_{n-2}$ will be a dominating set for $L_n$ when combined with $u_n$ and $v_n$ since these two vertices dominate themselves and their neighbours. Any set $S$ which is a dominating set in both $L_{n-1}$ and $L_{n-2}$ cannot contain either $u_{n-1}$ or $v_{n-1}$ since they are not in $L_{n-2}$ and hence $S$ must contain both $u_{n-2}$ and $v_{n-2}$ in order for the former pair of vertices to be dominated. Thus exactly $x^2 A_{n-2}(x)$ sets are counted twice and this is subtracted to give our result.

**Theorem 4.2.** The dominating polynomial for $L_n$ satisfies the recurrence:

$$D(L_n, x) = x(x + 2)D(L_{n-1}, x) + x(x + 1)D(L_{n-2}, x) + x^2(x + 1)D(L_{n-3}, x) - x^3D(L_{n-4}, x) - x^3D(L_{n-5}, x)$$

**Proof.** Let $T$ be a dominating set for $L_n$ and set $T_1 := T/\{u_n, v_n\}$. If $T_1 = T$ then (in order to have $u_n$ and $v_n$ dominated) we can conclude that $|T \cap \{u_{n-1}, v_{n-1}\}| = 2$ and the polynomial counting such sets will be $A_{n-1}(x)$ as in Lemma 4.1. This gives us the contribution $x^2 (D(L_{n-2}, x) + D(L_{n-3}, x) - A_{n-3}(x))$ for our summation.

Now suppose $|T \cap \{u_n, v_n\}| \geq 1$; if $T_1$ is a dominating set for $L_{n-1}$ then $T$ will be a dominating set for $L_n$. Thus we get the term $x(x + 2)D(L_{n-1}, x)$, the $x(x + 2)$ coming from that we can use $u_n$ and/or $v_n$ with $T_1$ to form a dominating set.

However, there are circumstances under which $T_1$ does not have to be a dominating set for $L_{n-1}$, since $u_{n-1}$ and $v_{n-1}$ in $L_{n-1}$ might be only dominated by $u_n$ or $v_n$ in $T$. Let us now consider the ways that exist such that $u_{n-1}$ and $v_{n-1}$ can be not dominated in $T_1$ but dominated in $T$.

If both $u_{n-1}$ and $v_{n-1}$ are undominated by $T_1$ then we must have $|T \cap \{u_n, v_n\}| = 2$ to dominate those vertices and also $|T \cap \{u_{n-3}, v_{n-3}\}| = 2$ to dominate $u_{n-2}$ and $v_{n-2}$, giving the term $x^2 A_{n-3}(x)$ which will cancel that term introduced at the start of the proof.

We are now left to count just the dominating sets for $L_{n-2}$ which include only one of $u_{n-2}$ and $v_{n-2}$. These sets will make a previously uncounted dominating set for $L_n$ when combined with $v_n$ and/or $u_n$ respectively. These are the four different possibilities, defining $S := T \cap \{u_n, v_n, u_{n-1}, v_{n-1}, u_{n-2}, v_{n-2}\}$:

(i) $S = \{u_n, v_n, v_{n-2}\}$
(ii) $S = \{u_n, v_{n-2}\}$
(iii) $S = \{v_n, u_{n-2}\}$
(iv) $S = \{u_n, v_n, u_{n-2}\}$
To count these possibilities we can now consider the different ways that exactly one of \( u_n \) or \( v_n \) can be combined with a dominating set for \( L_{n-2} \) which will lead to the contribution of the term \( xD(L_{n-2}, x) \) to our sum. Suppose \( Q \) is a dominating set for \( L_{n-2} \); we will split into subcases depending on \( r := |Q \cap \{u_{n-2}, v_{n-2}\}| \) as follows:

Every set \( Q \) satisfying \( r = 2 \) can be converted into a set of the type of possibility (i) (by adding \( u_n \) and switching \( v_n \) for \( u_{n-2} \), but this new set will not be a dominating set for \( L_n \) when \( u_{n-3} \) is solely dominated by \( u_{n-2} \) in \( Q \); that is when \( Q \cap \{u_{n-3}, v_{n-3}, u_{n-4}\} = \emptyset \). Let the sets of this form which have \( v_{n-4} \in Q \) be counted by the polynomial \( J(x) \) and such sets which also do not include \( v_{n-4} \) are necessarily \( x^2A_{n-5}(x) \) as in Lemma 4.1.

When \( r = 1 \) we can add \( u_n \) or \( v_n \) as appropriate and have possibilities (ii) and (iii) for \( S \). In the case when \( r = 0 \), \( Q \) must include both \( u_{n-3} \) and \( v_{n-3} \) to be dominating. No such set can be combined with just one more vertex to make a dominating set for \( L_n \), and we can count the sets with \( r = 0 \) (and one additional unspecified vertex) using the polynomial \( xA_{n-3}(x) \). Putting these terms together, we see that possibilities (i),(ii) and (iii) are counted by

\[
x(D(L_{n-2}, x) - J(x) - x^2A_{n-5} - A_{n-3}(x)).
\]

Finally, we can count the dominating sets for \( L_n \) with \( S \) as in possibility (iv) by using \( x^3D(L_{n-3}, x) + xJ(x) \). We make a slight adjustment in the same way as in the subcase when \( r = 0 \) since a set in which only \( u_{n-3} \) is not dominated in \( L_{n-3} \) will still be a dominating set in \( L_n \) when combined with this \( S \), and the polynomial counting such sets exactly matches the definition of \( xJ(x) \).

Using Lemma 4.1 again, we get that

\[
x^3A_{n-5}(x) + xA_{n-3}(x) = x^3(D(L_{n-4}, x) + D(L_{n-5}, x)).
\]

and so, summing all of our terms together, we can count all possible dominating sets \( T \) for \( L_n \) by using the polynomial in the statement in the theorem. \( \Box \)

Note that at no point did we either concern ourselves with the structure beyond \( u_{n-5} \) and \( v_{n-5} \) or utilise the symmetry of \( P_n \square K_2 \), and hence this same recurrence also holds for any family of graphs with \( P_6 \square K_2 \) as a pendant subgraph.

We can again use Theorem 2.2 as in Corollary 2.4 to find the domination polynomial for the strong product \( Z_{n,r} := L_n \square K_r \):

**Corollary 4.3.** For any integers \( n \geq 6 \) and \( r \geq 1 \) we have

\[
D(Z_{n,r}, x) = \left( (x + 1)^{2r} - 1 \right) D(Z_{n-1,r}, x)
+ ((x + 1)^r - 1)(x + 1)^r D(Z_{n-2,r}, x)
+ ((x + 1)^r - 1)^2(x + 1)^r D(Z_{n-3,r}, x)
- ((x + 1)^r - 1)^3(D(Z_{n-4,r}, x) + D(Z_{n-5,r}, x))
\]
Proof. Let us substitute \( y := (x + 1)^r - 1 \) to simplify calculations:

\[
D(Z_{n,r}, x) = D(L_n, (x + 1)^r - 1) = D(L_n, y) = y(y + 2)L_{n-1, y} + y(y + 1)L_{n-2, y} + y^2(y + 1)L_{n-3, y} - y^2D(L_{n-4, y}) - y^3D(L_{n-5, y})
\]

\[
= y(y + 2)L_{n-1, r} + y(y + 1)L_{n-2, r} + y^2(y + 1)L_{n-3, r} - y^2D(L_{n-4, r}) - y^3D(L_{n-5, r})
\]

Utilising now that \( y + 1 := (x + 1)^r \) we get the desired result. \( \square \)

5 The Domination Polynomial of \( P_n \square K_r \)

We denote by \( M_{n,r} := P_n \square K_r \) the Cartesian product of the path \( P_n \) and the complete graph \( K_r \), where \( n \) and \( r \) are non-negative integers. We will utilise the linear structure of \( P_n \) and refer to the copy of \( K_r \) corresponding to a vertex of degree one in \( P_n \) as at the left end and the copy of \( K_r \) adjacent to it as the second one. Let \( m_{n,r}^t(x) \) be the domination polynomial of \( M_{n,r} \) under the condition that exactly \( t \) of the \( r \) vertices of the left end \( K_r \) do not necessarily need to be dominated.

Let \( \delta_{t,r} := [t = r] \) denote the Kronecker delta function. For \( n = 0 \) and \( n = 1 \) the graph \( M_{n,r} \) is the null graph and \( K_r \) respectively and so only the case of the empty dominating set needs to be considered carefully. For \( n = 2 \) the case \( t = 0 \) and \( r > 0 \) corresponds to Theorem 3.1 and that proof can be generalised to give the result here.

\[
\begin{align*}
m_{0,r}^t(x) &= 1 \\
m_{1,r}^t(x) &= (x + 1)^r - 1 + \delta_{t,r} \\
m_{2,r}^t(x) &= (x + 1)^r - 2(x + 1)^r + x^r + 1 + x^{r-t}(x + 1)^t - \delta_{t,r}
\end{align*}
\]

From these equations we can establish the following recurrence relations for \( m_{n,r}^t \) in general

**Theorem 5.1.** The dominating polynomial for \( P_n \square K_r \) satisfies

\[
D(P_n \square K_r, x) = D(M_{n,r}, x) = m_{n,r}^0(x) = \sum_{t=0}^{r} \binom{r}{t} x^t m_{n-1,r}^t(x)
\]

where \( m_{n-1,r}^t(x) \) can be evaluated recursively using recurrence relations.

**Proof.** Suppose we have a set \( S \) from \( M_{n-1,r} \) which dominates all vertices in \( M_{n-2,r} \) and \( t \) vertices in the left end. We first suppose that \( t < r \) so that at least one vertex is undominated.

In order to form a dominating set for \( M_{n,r} \), a non-empty vertex subset of the left end \( K_r \) must be added to \( S \). There are \( \binom{r}{t} \) ways to choose \( t \) vertices and the contribution to the domination polynomial is a factor of \( x^t \) in each case. These \( t \)
vertices will then cover \( t \) vertices of the second \( K_r \), so we get \( m_{n-1,r}^t \) as the polynomial counting the possibilities for \( S \).

It is also possible that all vertices of the left end are dominated by a set which doesn’t include any vertices from its \( K_r \). In this case all \( r - t \) vertices in the second \( K_r \) which are adjacent to the undominated vertices of the left end must be added to \( S \), giving a factor of \( x^{r-t} \) for the domination polynomial. This then allows the choice of an arbitrary subset of the remaining \( t \) vertices of the second \( K_r \) to make up \( S \) which means that some vertices in the third \( K_r \) don’t need to be dominated. Putting these choices together with that from the first paragraph gives us, for \( 0 \leq t < r \):

\[
m_{n,r}^t(x) = \sum_{i=1}^{r} \binom{r}{i} x^i m_{n-1,r}^i(x) + x^{r-t} \sum_{i=0}^{t} \binom{t}{i} x^i m_{n-2,r}^{r-t+i}(x)
\]

When \( t = r \) we have all vertices of the left end already dominated, and so we are free to select an arbitrary subset of it to create a dominating set, giving:

\[
m_{n,r}^r(x) = \sum_{i=0}^{r} \binom{r}{i} x^i m_{n-1,r}^i(x)
\]

These equations when used recursively with equations (2) and (3), produce our result for any \( n \) and \( r \) since \( t = 0 \) is the case when \( M_{n,r} \) is dominated.

\[\square\]

6 Recursively constructible graph sequences

In [28], M. Noy and A. Ribó considered the Tutte polynomials of recursively constructible sequences of graphs. A sequence \( G_1, G_2, \ldots, G_n, \ldots \) is recursively constructible if it can be obtained from an initial graph by repeated application of fixed elementary operations involving addition of vertices and edges and deletion of edges. Some familiar recursively constructible graph sequences are paths, cycles, stars and wheels. Noy and Ribó proved that for every such sequence, the Tutte polynomials \( T(G_n, x, y) \) of the graphs in the sequence satisfy a linear recurrence relation with coefficients in the polynomial ring \( \mathbb{Z}[x,y] \). In a recent paper [10], the authors disprove a conjecture of Noy and Ribó regarding the Tutte polynomials of recursively constructible families.

Noy and Ribó considered graph sequences obtained from various graph products. They proved that if \( G_1, G_2, \ldots, G_n, \ldots \) is a recursively constructible sequence of graphs and \( H \) is a fixed graph, then the sequences of graphs obtained by applying the Cartesian product \( G_n \Box H \), the strong product \( G_n \boxtimes H \), the rooted product, the tensor product \( G_n \times H \) and the join \( G_n + H \) are all recursively constructible, as are variants of them. This implies, for example, that several well known families are recursively constructible, such as cyclic ladders \( (C_n \Box K_2) \) and, for any fixed \( t \), grids \( (P_n \Box P_t) \) and complete bipartite graphs \( (K_{n,t}) \).

E. Fischer and J. A. Makowsky generalized this result to a wide family of graph polynomials, including in particular the domination polynomial in [15]:
Theorem 6.1. (Corollary of [15]) Let \( G_1, G_2, \ldots, G_n, \ldots \) be a recursively constructible sequence of graphs \(^1\). The sequences of domination polynomials of \( G_n \) satisfy a linear recurrence relation with coefficients in \( \mathbb{Z}[x] \).

Fischer and Makowsky’s result is not constructive, in the sense that their proof does not directly emit the desired recurrence relation. Rather, it only attests to the existence of a recurrence relation.

From [15] it follows that, for every fixed \( i \), the sequence \( d_{|V(G_n)|-i}(G_n) \) of dominating sets of size \( |V(G_n)| - i \), satisfies a linear recurrence relation over \( \mathbb{Z} \). Note \( |V(G_n)| = cn + r \) for \( c, r \in \mathbb{N} \) which do not depend on \( n \). It is natural to consider the number of dominating sets of size \( d \) e.g. \( d_{|V(G_n)|-i}(G_n) \), \( p, q \in \mathbb{Q} \). Such dominating sets arise as the sizes of minimum dominating sets of various graphs. For example, M. S. Jacobson and L. F. Kinch showed in [21] the domination number of the ladder \( L_n \) is \( \left\lfloor \frac{x+1}{2} \right\rfloor \). T. Y. Chang and W. E. Clark showed in [13] that the domination number of a \( 5 \times n \) grids is \( \lfloor 6n+8 \rfloor \) for large enough \( n \). In fact, [21] and [13] show that the dominating numbers of grids \( t \times n \) for \( 1 \leq t \leq 6 \) are roundings of numbers of the form \( qn + p \). S. Alikhani and Y. Peng considered the domination polynomials of paths in [4]. In particular, they computed sequences of coefficients such as \( d_n(P_3n) \) and \( d_{n+1}(P_{3n+2}) \). Similar sequences of coefficients were considered by Alikhani and Peng for cycles in [5].

We consider the number of linear sized dominating sets. We prove:

**Theorem 6.2.** Let \( G_1, G_2, \ldots, G_n, \ldots \) be a recursively constructible sequence of graphs. Then the generating functions of

1. \( d_{|V(G_n)|-i}(G_n) \)
2. \( d_{|V(G_n)|-i}(G_n) \)

are algebraic for every \( q, p \in \mathbb{Q} \).

In particular, this implies that \( d_{|V(G_n)|-i}(G_n) \) and \( d_{|V(G_n)|-i}(G_n) \) are P-recursive or holonomic, i.e. that they satisfy linear recurrence relations with coefficients which are polynomials in \( n \).

**Example 6.3.** As a simple example of Theorem 6.2, consider the sequence \( d_n(K_n \sqcap K_2) \).

Theorem 3.1 gives the following explicit expression for \( D(K_n \sqcap K_2, x) \):

\[
(x + 1)^{2n} - 2(x + 1)^n + 2x^n + 1
\]

From this expression we can easily extract \( d_n(K_n \sqcap K_2) \), which is the coefficient of \( x^n \). In \( (x + 1)^{2n} \), the coefficient of \( x^n \) is \( \binom{2n}{n} \) using the binomial expansion of \( (x + 1)^{2n} \). The coefficient of \( x^n \) in \( -2(x + 1)^n + 2x^n + 1 \) is 0. So,

\[
d_n(K_n \sqcap K_2) = \binom{2n}{n}
\]

is the central binomial coefficient, which is well-known to have an algebraic generating function. Similar expressions can be obtained for any fixed \( r \) from Theorem 3.2.

\(^1\)Fischer and Makowsky actually use a wider definition, namely *iteratively constructible sequences* of graphs, which also covers e.g. cliques and complete bipartite graphs \( K_{n,n} \).
Theorem 6.2 can be applied to show that the number of “linearly small” (or "linearly large") dominating sets is P-recursive.

**Theorem 6.4.** Let $G_1, G_2, \ldots, G_n, \ldots$ be a recursively constructible sequence of graphs and let $q \in \mathbb{Q}$. Then the generating functions of the number of dominating sets of size at most (at least) $q|V(G_n)|$ is algebraic.

### 6.1 Proofs

We assume the reader is familiar with the basics of rational and algebraic generating functions, as described e.g. in [29]. The following lemma is folklore:

**Lemma 6.5.** If $F(x)$ and $G(x)$ are rational generating functions, so is their Hadamard product $F \ast G(x)$.

We use the following theorem:

**Theorem 6.6** (H. Furstenburg [16]). If $F(s, t)$ is a rational generating function in $s$ and $t$, then $\text{diag } F$ is algebraic.

We can now prove Theorem 6.2:

**Proof of Theorem 6.2.** We prove case 1. Case 2 can be proven analogously. Consider the generating function

$$F(x, t) = \sum_{n,m=0}^{\infty} d_n(G_m)x^{n+m-\lfloor qm+p \rfloor}t^{m}$$

We will prove $F(x, t)$ is a rational function in $x$ and $t$. The diagonal of $F$ is

$$\text{diag } F(y) = \sum_{m=0}^{\infty} d_{\lfloor qm+p \rfloor}(G_m)y^{m}$$

and by Theorem 6.6, $\text{diag } F(y)$ is algebraic.

It remains to prove that $F(x, t)$ is rational. Since the sequence $D(G_m, x)$ satisfies a linear recurrence relation with coefficients in $\mathbb{Z}[x]$, the power series

$$H(t) = \sum_{m=0}^{\infty} D(G_m, x)t^{m}$$

is rational. Using the definition of $D(G_m, x)$, $H(t)$ can be rewritten as a power series over $\mathbb{Q}$ with indeterminates $x$ and $t$:

$$H(x, t) = \sum_{n,m=0}^{\infty} d_n(G_m)x^{n}t^{m}$$

and note that $H(x, t)$ is rational. Substituting $t$ with $xt$ we get a new rational power series:

$$H_1(x, t) = \sum_{n,m=0}^{\infty} d_n(G_m)x^{n+m}t^{m}.$$
Let

\[ R(t) = \sum_{m=0}^{\infty} x^{-[qm+p]} t^m. \]

We have \( F(x, t) = H_1(t) * R(t) \). Using again the closure property of the Hadamard product, \( F(x, t) \) is rational if \( R(t) := \sum_{m=0}^{\infty} r(m, x)t^m \) is. Let \( q = \frac{a}{b} \) with \( a, b \in \mathbb{Z} \) and \( b > 0 \) so that \( r(m, x) \) satisfies \( r(m, x) = x^{-a}r(m - b, x) \), implying that \( R(t) \) is rational. \( \square \)

We restate Theorem 6.4 as follows:

**Theorem 6.7.** *(Number of small dominating sets)* Let \( G_1, G_2, \ldots, G_n, \ldots \) be a recursively constructible sequence of graphs. Then the generating function of

\[ \sum_{i=0}^{|V(G_n)|+p'} d_i(G_n) \]

is algebraic for every \( q', p' \in \mathbb{Q} \).

**Proof.** Let \( c \) and \( r \) be the natural numbers guaranteed such that \( |V(G_n)| = cn + r \). Let \( q = q'c \) and \( p = p' + rq' \). Then we need to consider the generating function of

\[ \sum_{i=0}^{[qn+p]} d_i(G_n) \]

In the proof of Theorem 6.2 we proved \( F(x, t) \) is rational. Consider

\[ F_1(x, t) = F(x, t) \cdot \frac{1}{1-x}. \]

\( F_1 \) is given by

\[ F_1(x, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} d_n(G_m)x^{i+n+m-[qm+p]} t^m. \]

We have that \( i + n + m - [qm+p] = m \) iff \( i + n = [qm+p] \). Hence,

\[ \text{diag} F(y) = \sum_{m=0}^{\infty} \sum_{i,n \in \mathbb{N}; \ i+n=[qm+p]} d_n(G_m)y^m = \sum_{m=0}^{\infty} \sum_{n=0}^{[qm+p]} d_n(G_m)y^m. \]

Again case (2) is similar. \( \square \)

## 7 An application to complexity

In this section we show an application of Theorem 2.2 to the Turing complexity of the domination polynomial. We assume the reader is familiar with the basics of counting complexity theory as described e.g. in [6, Chapter 17].

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Many graph polynomials studied in the literature have been shown to be \#P-hard to compute with respect to Turing reductions, and the domination polynomial is no exception. This follows from the \#P-completeness of the number of dominating sets. This remains true even on restricted graph classes, see e.g. [23].

A common further step towards understanding the complexity of a particular graph polynomial is to assess the hardness of computation of its evaluations. A classic result of this kind can be found in [22], where it is shown that the Tutte polynomial is \#P-hard to compute for any rational evaluation, except those in a semi-algebraic set of low dimension which are polynomial-time computable. Similar dichotomy theorems have been shown for the cover polynomial [8], the interlace polynomial [9] and the edge elimination polynomial and its specializations [20].

For any fixed \(\gamma \in \mathbb{Q}\), we denote by \(D(-, \gamma)\) the problem of computing for an input graph \(G\) the evaluation \(D(G, \gamma)\) of the domination polynomial. We limit ourselves to rational evaluations and remain within the realm of Turing complexity in order to avoid a discussion of appropriate computation models which is not central to this paper.

**Theorem 7.1.** Given \(\gamma \in \mathbb{Q}\setminus\{0, -1, -2\}\), the graph parameter \(D(-, \gamma)\) is \#P-hard to compute.

**Proof.** Let \(\gamma \in \mathbb{Q}\setminus\{0, -1, -2\}\). We show an algorithm which on an input graph \(G\) with \(n\) vertices computes \(D(G, x)\) in polynomial time in \(n\) using an oracle to \(D(-, \gamma)\). Since \(D(G, x)\) is \#P-hard, so is \(D(-, \gamma)\). The algorithm is as follows:

1. For every \(r \in \{1, \ldots, n+1\}\), compute \(D(G \boxtimes K_r, \gamma) = D(G, (1 + \gamma)^r - 1)\). \(D(G \boxtimes K_r, \gamma)\) is computed using the oracle to \(D(-, \gamma)\) (and \(D(G, (1 + \gamma)^r - 1)\) is obtained by Theorem 2.2).

2. Interpolate \(D(G, x)\) from the values

\[
(x_0, D(G, x_0)) = ((1 + \gamma)^i - 1, D(G, (1 + \gamma)^i - 1)),
\]

\(i = 1, \ldots, n+1\).

\(D(G, x)\) can be interpolated from the computed values since the values \((1+\gamma)^r - 1\) are pairwise distinct (because \(\gamma \neq 0, -1, -2\)) and \(D(G, x)\) has degree \(n\).

\(\square\)

8 Conclusion

In this paper we studied the domination polynomials of families of graphs given by products. While our results cover some important families of graphs obtained by products, there remain some open problems which we believe deserve attention.

Our work can be extended by finding analogous formulae to more families:

**Problem 8.1.**

1. How can Theorem 2.1 be extended to deal with basic Cartesian product families such as \(G \boxtimes K_s\), \(G \boxtimes P_s\), \(G \boxtimes C_s\), etc.?
2. Can analogs of Theorem 2.2 be found for $G \boxtimes P_s$, $G \boxtimes C_s$, etc.?

3. What other families of graphs obtained using graph products have simple explicit formulae in the spirit of Theorem 3.2?

We have not considered families such as square grids $P_n \square P_n$:

**Problem 8.2.** What can be proven about families $G_n \oplus H_n$ for some graph product $\oplus$ of recursively constructible families of graphs, both in general and for important special cases such as grids?

Theorem 6.1 leaves open the complexity of two special evaluations of the domination polynomial, which are not solved using a product-based reduction. $D(−, 0)$ is clearly polynomial-time computable, but the following remains open:

**Problem 8.3.** How hard to compute are the graph parameters $D(−, −1)$ and $D(−, −2)$?

$D(G, −1)$ has been studied from a combinatorial point of view in [25]. To our knowledge $D(G, −2)$ has not received attention in the literature.

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