Inside $s$-inner product sets and Euclidean designs

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Abstract

Let $X$ be a finite subset of Euclidean space $\mathbb{R}^d$. We define for each $x \in X$, $B(x) := \{(x, y) \mid y \in X, x \neq y, (x, x) \geq (y, y)\}$ where $(,)$ denotes the standard inner product. $X$ is called an inside $s$-inner product set if $|B(x)| \leq s$ for all $x \in X$. In this paper, we prove that the cardinalities of inside $s$-inner product sets have the Fisher type upper bound. An inside $s$-inner product set is said to be tight if its cardinality attains the Fisher type upper bound. Tight inside $s$-inner product sets are closely related to tight Euclidean designs. Indeed, many tight Euclidean designs are tight inside $s$-inner product sets. Generally, a tight inside $s$-inner product set becomes a Euclidean $s$-design with a weight function which may not be positive. In the last part of this paper, we prove the non-existence of tight 2- or 3-inner product sets supported by a union of two concentric spheres. In order to prove this result, we give a new upper bound for the finite subset of $\mathbb{R}^d$ which does not have positive inner products.

1 Introduction

Delsarte, Goethals, and Seidel [15] studied a finite subset $X$ of the unit sphere $S^{d-1}$. In their article, the two concepts, spherical $t$-designs and $s$-distance sets, play important roles. The cardinalities of spherical $t$-designs have the Fisher type lower bound. A spherical $t$-design is said to be tight if its cardinality attain the Fisher type lower bound. Similarly, the cardinalities of $s$-distance sets have the Fisher type upper bound, and the tightness for $s$-distance sets is defined. There is strong relationship between these concepts, that is, a tight spherical design is a tight distance set, and the converse also holds. Moreover, this result has been generalized for the relationship between weighted spherical designs and locally distance sets [25]. When we consider this kind of relationship, the key is the Fisher type lower bound and upper bound. In this paper, a bound is said to be Fisher type if the bound is given as the dimension of the linear space of certain functions by the method of Koornwinder [19]. A design and a distance set are said to be tight if their cardinalities attain the Fisher type lower bound and upper bound, respectively.

We want to generalize this theory to Euclidean space $\mathbb{R}^d$. However, this problem is very difficult, and we do not have an effective idea. Euclidean designs are known as a generalization of weighted spherical designs, and have the Fisher type lower bound [16] [24]. A Euclidean $t$-design is a finite weighted set in $\mathbb{R}^d$ with a certain approximation property. The concept of $s$-distance sets in $\mathbb{R}^d$ is naturally defined. Furthermore, Bannai, Kawasaki, Nitamizu and Sato proved that the cardinalities of $s$-distance sets in $\mathbb{R}^d$ have the Fisher type upper bound [14]. However, known tight Euclidean designs are hardly related to tight $s$-distance sets in $\mathbb{R}^d$. Indeed, the only known tight $s$-distance set supported by at least two concentric spheres is Lisoněk’s 2-distance set supported by a union of two concentric spheres [20], which is a Euclidean 3-design and not tight (see [14]). Therefore, it seems that a new concept is needed instead of $s$-distance sets or Euclidean designs.

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In this paper, we introduce a new concept, inside $s$-inner product sets in $\mathbb{R}^d$. The concept of inside $s$-inner product sets is a generalization of locally $s$-distance sets on the unit sphere. Furthermore, inside $s$-inner product sets are essential for the Fisher type upper bound for Euclidean space because the approach of Koornwinder is naturally applied to a proof of this upper bound. Moreover, we can find many examples of tight inside $s$-inner product sets, which are not $s$-inner product set. In fact, many known examples of tight inside $s$-inner product sets are tight Euclidean designs. Generally, a tight inside $s$-inner product set become a Euclidean $s$-design with a weight function which is not necessarily positive.

The classification of tight inside $s$-inner product sets is also an interesting problem. First, we consider the problem of the classification of tight $s$-inner product sets. Note that an $s$-inner product set is also an inside $s$-inner product set. In Section 5, we prove the non-existence of tight 2- or 3-inner product sets supported by a union of two concentric spheres for any dimension at least 2. In order to prove this result, we give a new upper bound for the finite subset of $\mathbb{R}^d$ which does not have positive inner products. This upper bound is a generalization of the Rankin bound in [20].

2 Definitions and the Fisher type upper bound

Let $\mathbb{R}^d$ denote the $d$-dimensional Euclidean space. For $x, y \in \mathbb{R}^d$, we denote their standard inner product by $\langle x, y \rangle$. Let $X$ be a finite subset of $\mathbb{R}^d$. We define $A(X) := \{(x, y) \mid x, y \in X, x \neq y\}$. For a fixed $x \in X$, we define $A(x) := \{(x, y) \mid y \in X, x \neq y\}$ and $B(x) := \{(x, y) \mid y \in X, x \neq y, ||x|| = ||y||\}$ where $||x|| := \sqrt{\langle x, x \rangle}$ is the norm of $x$. Let $|*|$ denote the cardinality.

**Definition 2.1.** Let $X$ be a finite subset of $\mathbb{R}^d$.

1. $X$ is called an $s$-inner product set if $|A(X)| = s$.
2. $X$ is called a locally $s$-inner product set if $|A(x)| \leq s$ for all $x \in X$.
3. $X$ is called an inside $s$-inner product set if $|B(x)| \leq s$ for all $x \in X$.

Note that an $s$-inner product set is a locally $s$-inner product set, and a locally $s$-inner product set is an inside $s$-inner product set. Let $d(x, y)$ denote the Euclidean distance between $x$ and $y$. We define $A_{dis}(X) := \{d(x, y) \mid x, y \in X, x \neq y\}$. For a fixed $x \in X$, we define $A_{dis}(x) := \{d(x, y) \mid y \in X, x \neq y\}$.

**Definition 2.2.** Let $X$ be a finite subset of $\mathbb{R}^d$.

1. $X$ is called an $s$-distance set if $|A_{dis}(X)| = s$.
2. $X$ is called a locally $s$-distance set if $|A_{dis}(x)| \leq s$ for all $x \in X$.

Let $S^{d-1} \subset \mathbb{R}^d$ denote the unit sphere whose center is the origin. We note that an $s$-inner product set on $S^{d-1}$ is an $s$-distance set on $S^{d-1}$.

We prepare some notation. Let $S := S_1 \cup S_2 \cup \cdots \cup S_p$ be a union of $p$ concentric spheres, where $S_i$ is a sphere whose center is the origin and whose radius is $r_i$. We assume $0 \leq r_1 < r_2 < \cdots < r_p$. If $r_1$ is equal to zero, then $S_1$ is the origin and regarded as a special sphere. If $S$ contains the origin, then $\varepsilon_S := 1$, and if $S$ does not contain the origin, then $\varepsilon_S := 0$. Let $P_l(\mathbb{R}^d)$ be the linear space of all real polynomials of degree at most $l$, with $d$ variables. Let $\text{Hom}_l(\mathbb{R}^d)$ be the linear space of all real homogeneous polynomials of degree $l$. Let $\text{Harm}_l(\mathbb{R}^d)$ be the linear space of all real harmonic polynomials of degree $l$, i.e., $\text{Harm}_l(\mathbb{R}^d) := \{f \in \text{Hom}_l(\mathbb{R}^d) \mid \Delta f = 0\}$, where $\Delta := \sum_{i=1}^d \partial^2/\partial x_i^2$. We define $P^*_l(\mathbb{R}^d) := \sum_{a\leq l/2}^{|a|} \text{Hom}_{l-2a}(\mathbb{R}^d)$, where $[a]$ denotes the maximum integer in all integers at most $a$ for a real number $a$. Let $P_l(S)$, $\text{Hom}_l(S)$, $\text{Harm}_l(S)$ and $P^*_l(S)$ be the linear space of all functions which are the restrictions of the corresponding polynomials to $S$. For example, $P_l(S) := \{f|_S \mid f \in P_l(\mathbb{R}^d)\}$.

The dimensions of these linear space are well defined. Define $p' = p - \varepsilon_S$.

**Theorem 2.3** ([3] [16] [18]).

1. $\dim P_l(S) = \begin{cases} 
\varepsilon_S + \sum_{i=0}^{2p'-1} \binom{d+i-1}{d-1} & \text{if } l \geq 2p', \\
(\varepsilon_S + \sum_{i=0}^{2p'-1} \binom{d+i-1}{d-1}) & \text{if } l \geq 2p' - 1.
\end{cases}$

2. $\dim P^*_l(S) = \begin{cases} 
\varepsilon_S + \sum_{i=0}^{p'-1} \binom{d+i-2l-1}{d-1} & \text{if } l \text{ is even and } l \geq 2p', \\
\varepsilon_S + \sum_{i=0}^{p'-1} \binom{d+i-2l-1}{d-1} & \text{if } l \text{ is odd and } l \geq 2p', \\
\dim P^*_l(\mathbb{R}^d) = \sum_{i=0}^{2p'} \binom{d+i-2l-1}{d-1} & \text{if } l \leq 2p' - 1.
\end{cases}$
Let \( X \subseteq S \). \( X \) is called antipodal if \(-x\) is an element of \( X \) for all \( x \in X \). We define \( X_i := X \cap S_i \). \( X \) is said to be supported by \( S \), if \( X_i \) is not empty for \( i = 1, 2, \ldots, p \). We introduce the known Fisher type upper bounds for \( s \)-distance sets and \( s \)-inner product sets.

**Theorem 2.4** ([11, 14, 15, 17]). (1) Let \( X \) be a locally \( s \)-distance set (locally \( s \)-inner product set) on \( S^{d-1} \). Then, \(|X| \leq (d+s-1)/(s-1) = \dim P_s(S^{d-1})\).

(2) Let \( X \) be an antipodal locally \( s \)-distance set (locally \( s \)-inner product set) on \( S^{d-1} \). Then, \(|X| \leq 2(d+s-2)/(s-1) = \dim P_{s-1}(S^{d-1})\).

(3) Let \( X \) be a locally \( s \)-inner product set in \( \mathbb{R}^d \). Then, \(|X| \leq (d+s)/(s) = \dim P_s(\mathbb{R}^d)\).

(4) Let \( X \) be an \( s \)-distance set in \( \mathbb{R}^d \). Then, \(|X| \leq (d+s)/(s) = \dim P_s(\mathbb{R}^d)\).

(5) Let \( X \) be an \( s \)-distance set supported by \( S \). Assume \( 0 \notin X \). Then, \(|X| \leq \dim P_s(S)\).

(6) Let \( X \) be an antipodal \( s \)-distance set supported by \( S \). Assume \( 0 \notin X \). Then, \(|X| \leq 2 \dim P_{s-1}(S)\).

It is written in [17] as the referee’s comment that Theorem 2.4 (3) can be deduced also using the approach of Koornwinder [19]. However, a proof by this method has not been published. In this paper, by the approach of Koornwinder, we prove the Fisher type inequality for inside \( s \)-inner product sets as a generalization of Theorem 2.4 (1), (2) and (3). If \( X \) is an antipodal inside \( s \)-inner product set, and \( 0 \in X \), then there exists odd \( k \), such that \( k \leq s \) and \( X \) is an inside \( k \)-inner product set.

**Theorem 2.5.** Let \( S \subseteq \mathbb{R}^d \) be a union of \( p \) concentric spheres centered at the origin.

(1) Let \( X \) be an inside \( s \)-inner product set supported by \( S \). Then,

\[
|X| \leq \dim P_s(S).
\]

(2) Let \( X \) be an antipodal inside \( s \)-inner product set supported by \( S \). Then,

\[
|X| \leq \begin{cases} 2 \dim P_{s-1}(S) + \varepsilon_S & \text{if } s \text{ is even,} \\ 2 \dim P_{s-1}(S) & \text{if } s \text{ is odd and } 0 \notin X. \end{cases}
\]

**Proof.** Let \( X \) be an inside \( s \)-inner product set supported by \( S \subseteq \mathbb{R}^d \). Note that \(-||x||^2 \leq \alpha < ||x||^2\) for any \( \alpha \in B(x) \). For each \( x \in X \), we define the polynomial \( f_x(\xi) \) in the variables \( \xi = (\xi_1, \xi_2, \ldots, \xi_d)\):

\[
f_x(\xi) = \begin{cases} \prod_{\alpha \in B(x)} \frac{\xi - \alpha}{x - \alpha} & \text{if } B(x) \neq \emptyset, \\ 1 \text{ (constant)} & \text{otherwise}. \end{cases}
\]

Then, \( f_x(\xi) \) is a polynomial of degree at most \( s \). \( B(x) \) is an empty set if and only if \( x \in X_1 \) and \( |X_1| = 1 \). Hence, the number of \( x \) such that \( f_x(\xi) = 1 \) (constant) is at most \( 1 \). By the definition of \( f_x, f_x(x) = 1, \) and \( f_x(y) = 0 \) for \( x \neq y \in X \) and \( ||y|| \leq ||x|| \). We order the elements of \( X \) as \( X = \{x_1, x_2, \ldots, x_n\} \) in such a way that \( ||x_i|| \leq ||x_{i+1}|| \), where \( |X| = n \). Let \( M \) be the \( n \times n \) matrix whose \((i, j)\)-entry is \( f_x(x_j) \).

Then, we get

\[
M = \begin{bmatrix} I_{|X_1|} & * & \cdots & * \\ 0 & I_{|X_2|} & * & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & * & \vdots \\ 0 & 0 & \cdots & 0 & I_{|X_n|} \end{bmatrix}, \tag{2.1}
\]

where \( I_m \) denotes the \( m \times m \) identity matrix. Therefore, \( M \) is a nonsingular matrix, and hence \( \{f_x\}_{i=1,2,\ldots,n} \) are linearly independent. This proves (1).

Let \( X \) be an antipodal inside \( s \)-inner product set supported by \( S \). Then there exists a subset \( Y' \) such that \( X \setminus \{0\} = Y' \cup (-Y') \). We define \( Y := Y' \cup \{0\} \) or \( Y' \), according to \( 0 \in X \) or not. Note that \(|X| = 2|Y| - \varepsilon_S\). For each \( y \in Y \), we define

\[
B^2(y) := \{\alpha^2 \mid \alpha \in B(y), \alpha \neq 0, \alpha \neq -(y, y)\}.
\]
Then, \( |B^2(y)| \leq [(s - 1)/2] \). For any \( \alpha^2 \in B^2(y) \), we have \( 0 < \alpha^2 < (y, y)^2 \). For each \( y \in Y \), we define the polynomial \( f_y(\xi) \):

\[
f_y(\xi) := \begin{cases} 1 \text{ (constant)} & \text{if } y = 0, \\ \left( \frac{(y, \xi)}{(y, y)} \right)^{(s-1)/2} \prod_{\alpha^2 \in B^2(y)} \frac{(y, \xi)^2 - \alpha^2}{(y, y)^2 - \alpha^2} & \text{otherwise}, \end{cases}
\]

If \( 0 \in Y \), and \( s \) is even, then \( 1 \not\in P_{s-1}(\mathbb{R}^d) \). Therefore, \( f_y(\xi) \in P_{s-1}(\mathbb{R}^d) + \varepsilon \text{Hom}(\mathbb{R}^d) \). Note that \( f_y(y) = 1 \), and \( f_y(z) = 0 \) for \( y \neq z \in Y \) and \( ||z|| \leq ||y|| \). By an argument similar to that in the proof of Theorem 2.5 (1), \( \{f_y \}_{y \in Y} \) are linearly independent as elements of \( P_{s-1}(\mathbb{S}) + \varepsilon S \text{Hom}(\mathbb{S}) \). Hence, \( |Y| \leq \dim(P_{s-1}(\mathbb{S}) + \varepsilon \text{Hom}(\mathbb{S})) \)

\[
= \begin{cases} \dim P_{s-1}(\mathbb{S}) + \varepsilon \text{ if } s \text{ is even,} \\ \dim P_{s-1}(\mathbb{S}) \text{ if } s \text{ is odd.} \end{cases}
\]

Since \( |X| = 2|Y| - \varepsilon_S \), this proves (2).

\[\square\]

Of course, Theorem 2.3 is applicable for \( s \)-inner product sets and locally \( s \)-inner product sets. If \( p = 1 \) and \( r_1 \neq 0 \), then the upper bounds in Theorem 2.3 (1) and (2) coincide with those in Theorem 2.3 (1) and (2), respectively. If \( s \leq 2p' - 1 \), then the upper bound in Theorem 2.3 (1) coincides with that in Theorem 2.3 (3). Moreover, if \( s \geq 2p' \), then the upper bound in Theorem 2.3 (1) is better than that in Theorem 2.3 (3).

An (antipodal) inside \( s \)-inner product set (resp. locally \( s \)-inner product set, or resp. \( s \)-inner product set) is said to be tight, if the equality in Theorem 2.3 holds. We introduce examples of tight inside \( s \)-inner product sets in Section 4. Many examples of tight inside inner product sets are tight Euclidean designs.

It has not proved that locally \( s \)-distance sets supported by \( \mathcal{S} \) have the same upper bound as in Theorem 2.3.

### 3 Euclidean designs

Euclidean designs were introduced by Neumaier and Seidel in 1988 [23]. In 2008, Ei. Bannai, Et. Bannai, Hirao, and Sawa generalized the concept of Euclidean designs, namely, \( X \) may contain the origin 0. We consider the Haar measure \( \sigma_t \) on each \( S_t \). For \( S_t \neq \{0\} \), we assume \( |S_t| = \int_{S_t} d\sigma_t(x) \) where \( |S_t| \) is the volume of \( S_t \). If \( S_t = \{0\} \), then we define \( \frac{1}{|S_t|} \int_{S_t} f(x) d\sigma_t(x) = f(0) \).

**Definition 3.1 (Euclidean design).** Let \( X \) be a finite set supported by \( \mathcal{S} \subset \mathbb{R}^d \). Let \( w(x) : X \to \mathbb{R}_{>0} \) be a positive weight function. \((X, w)\) is called a Euclidean \( t \)-design if the following equality holds for all \( f \in P_t(\mathbb{R}^d) \):

\[
\sum_{i=1}^p \frac{w(X_i)}{|S_t|} \int_{S_t} f(x) d\sigma_t(x) = \sum_{x \in X} w(x) f(x),
\]

where \( w(X_i) := \sum_{x \in X_i} w(x) \).

The largest value of \( t \) for which \((X, w)\) is a Euclidean \( t \)-design is called the maximum strength of the design. If \( p = 1 \), then Euclidean \( t \)-designs are called weighted spherical \( t \)-designs, and if \( p = 1 \) and \( w \) is a constant function, then Euclidean \( t \)-designs are called spherical \( t \)-designs. We can check the strength \( t \) by the following theorem.

**Theorem 3.2.** The following are equivalent:

1. \((X, w)\) is a Euclidean \( t \)-design.
2. \( \sum_{x \in X} w(x)||x||^2 \varphi(x) = 0 \), for any polynomial \( \varphi \in \text{Har}_l(\mathbb{R}^d) \) with \( 1 \leq l \leq t \) and \( 0 \leq j \leq \lfloor \frac{l}{2} \rfloor \).
3. \( \sum_{x \in X} \sum_{y \in X} w(x) w(y)||x||^{2+l}||y||^{2+l} G_{l}(d) \left( \left( \frac{|x|}{||x||}, \frac{|y|}{||y||} \right) \right) = 0 \), for Gegenbauer polynomial \( G_l(d) \) with \( 1 \leq l \leq t \) and \( 0 \leq j \leq \lfloor \frac{l}{2} \rfloor \). Here, \( G_l(d)(1) = \dim \text{Har}_l(\mathbb{R}^d) \).
The equation (3.1) implies the equivalence of the second and third statements. Let \( \{\varphi_{i,k}\}_{1 \leq k \leq b_i} \) be an orthonormal basis of \( \operatorname{Harm}(S^{d-1}) \) with respect to the inner product \( (f, g) = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(x) g(x) d\sigma(x) \). By the addition formula,

\[
\sum_{k=1}^{b_i} \left( \sum_{x \in X} w(x) ||x||^2 \varphi_{i,k}(x) \right)^2 = \sum_{k=1}^{b_i} \left( \sum_{x \in X} w(x) ||x||^{2j+4} \varphi_{i,k} \left( \frac{x}{||x||} \right) \right)^2 = \sum_{(x,y) \in X \times X} w(x) w(y) ||x||^{2j+4} ||y||^{2j+4} \sum_{k=1}^{b_i} \varphi_{i,k} \left( \frac{x}{||x||} \right) \varphi_{i,k} \left( \frac{y}{||y||} \right) \tag{3.1}
\]

The equation (3.1) implies the equivalence of the second and third statements. 

Proof. The equivalence of the first and second statements is well known \[24\]. We prove the equivalence of the second and third statements. Let \( \{\varphi_{i,k}\}_{1 \leq k \leq b_i} \) be an orthonormal basis of \( \operatorname{Harm}(S^{d-1}) \) with respect to the inner product \( (f, g) = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(x) g(x) d\sigma(x) \). By the addition formula,

\[
\sum_{k=1}^{b_i} \left( \sum_{x \in X} w(x) ||x||^2 \varphi_{i,k}(x) \right)^2 \]

\[
= \sum_{k=1}^{b_i} \left( \sum_{x \in X} w(x) ||x||^{2j+4} \varphi_{i,k} \left( \frac{x}{||x||} \right) \right)^2 = \sum_{(x,y) \in X \times X} w(x) w(y) ||x||^{2j+4} ||y||^{2j+4} \sum_{k=1}^{b_i} \varphi_{i,k} \left( \frac{x}{||x||} \right) \varphi_{i,k} \left( \frac{y}{||y||} \right) \tag{3.1}
\]

Theorem 3.3. Let \( X \) be a Euclidean \( 2e \)-design supported by \( S \). Then,

\[
|X| \geq \dim \mathcal{P}_e(S).
\]

(2) Let \( X \) be a Euclidean \( (2e - 1) \)-design supported by \( S \). Then,

\[
|X| \geq \begin{cases} 
2 \dim \mathcal{P}_{e-1}(S) - 1 & \text{if } e \text{ is odd and } 0 \in X, \\
2 \dim \mathcal{P}_{e-1}(S) & \text{otherwise.}
\end{cases}
\]

A Euclidean design is said to be tight if its cardinality attains the equality in Theorem 3.3. We note that these lower bounds coincide with the previous upper bounds in Theorem \[25\] when \( s = e \), except when \( s = 1 \) and \( e \) (or \( s \)) is even. If \( s = 1 \) and \( e \) is even, then we have some examples of tight antipodal inside \( s \)-inner product sets (Examples \[1,1 \] and \[4,2 \] in section \[4 \]). Some examples of tight spherical designs are given by Delsarte, Goethals, and Seidel \[15 \]. Moreover, the classification of tight spherical \( t \)-designs is complete, except for \( t = 4, 5, 7 \) \[9,10,11 \]. Some examples of tight Euclidean designs are given by Bajnok, Ei. Bannai, Et. Bannai, Suprijanto, Hirao, and Sawa \[11,12,4,6,8 \].

Clearly, inside \( s \)-inner product sets on \( S^{d-1} \) are locally \( s \)-inner product sets. We note that tight locally inner product sets on \( S^{d-1} \) are strongly related to tight weighted spherical designs. If \( X \) is a tight (resp. antipodal) locally \( s \)-inner product set on \( S^{d-1} \), then for some positive weight function \( w \), \( (X, w) \) is a tight weighted spherical \( 2s \)-design (resp. \( (2s - 1) \)-design) \[25 \]. Conversely, if \( (X, w) \) is a tight weighted spherical \( 2s \)-design (resp. \( (2s - 1) \)-design), \( X \) is a tight spherical \( 2s \)-design (resp. \( (2s - 1) \)-design) and tight (resp. antipodal) \( s \)-inner product set. Therefore, tight (resp. antipodal) locally \( s \)-inner product sets on the unit sphere are tight (resp. antipodal) \( s \)-inner product sets \[25 \].

We can expect that tight inside inner product sets are Euclidean \( t \)-designs with a high value of \( t \). This theorem implies that a tight inside \( s \)-inner product set becomes a Euclidean \( s \)-design if we allow that a weight function is not necessarily positive.

Theorem 3.4. Let \( X \) be a tight inside \( s \)-inner product set supported by a union of \( p \) concentric spheres \( S \subset \mathbb{R}^d \). Let \( a_i \) be any positive real numbers for \( i = 1, 2, \ldots, p \). Then, for some weight function \( w : X \to \mathbb{R} \), the following condition holds:

\[
\sum_{i=1}^{p} \frac{a_i}{|S_i|} \int_{S_i} f(x) d\sigma_i(x) = \sum_{x \in X} w(x) f(x)
\]

for any polynomial \( f(x) \in \mathcal{P}_s(S) \).
Proof. Let \( X := \{x_1, x_2, \ldots, x_n\} \) be a tight inside \( s \)-inner product set supported by \( S \subset \mathbb{R}^d \), where \( n = \dim \mathcal{P}_s(S) \) and \( \|x_i\| \leq \|x_{i+1}\| \). For each \( x_i \in X \), we define \( g_{x_i}(\xi) \):

\[
\begin{bmatrix}
g_{x_1}(\xi) \\
g_{x_2}(\xi) \\
\vdots \\
g_{x_n}(\xi) 
\end{bmatrix} := M^{-1} \begin{bmatrix}
f_{x_1}(\xi) \\
f_{x_2}(\xi) \\
\vdots \\
f_{x_n}(\xi)
\end{bmatrix},
\]

(3.2)

where the polynomials \( f_{x_i} \) and the matrix \( M \) are defined in the proof of Theorem 2.5. Then, \( g_{x_i}(\xi) \) is a \( d \)-variable polynomial of degree at most \( s \) and

\[
g_{x_i}(x_j) = \begin{cases} 
1 & \text{if } i = j, \\
0 & \text{if } i \neq j.
\end{cases}
\]

Since \( |X| = \dim \mathcal{P}_s(S) \), \( \{g_{x_i}(\xi)\}_{i=1,2,\ldots,n} \) is a basis of \( \mathcal{P}_s(S) \). Therefore, for any \( f(\xi) \in \mathcal{P}_s(S) \), we can write \( f(\xi) = \sum_{i=1}^c c_ig_{x_i}(\xi) \) where \( c_i \) are real. Defining the weight function \( w(x_j) := \sum_{i=1}^p \frac{a_i}{|S_i|} \int_{S_i} g_{x_i}(x)d\sigma_i(x) \),

\[
\sum_{i=1}^p \frac{a_i}{|S_i|} \int_{S_i} f(x)d\sigma_i(x) = \sum_{i=1}^p \frac{a_i}{|S_i|} \int_{S_i} \sum_{j=1}^n c_j g_{x_j}(\xi)d\sigma_i(x)
= \sum_{j=1}^n c_j w(x_j) = \sum_{j=1}^n w(x_j) \sum_{k=1}^n c_k g_{x_k}(x_j)
= \sum_{j=1}^n w(x_j)f(x_j)
\]

for any \( f(\xi) \in \mathcal{P}_s(S) \).

Corollary 3.5. Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a tight inside \( s \)-inner product set supported by \( S \subset \mathbb{R}^d \). Let \( \{g_{x_i}\}_{i=1,2,\ldots,n} \) be polynomials as defined above. If there exist positive real numbers \( \{a_i\}_{i=1,2,\ldots,p} \) such that

\[
w(x_j) := \sum_{i=1}^p \frac{a_i}{|S_i|} \int_{S_i} g_{x_i}(x)d\sigma_i(x) > 0
\]

for any \( x_j \in X \). Then, \( (X, w) \) is a Euclidean \( s \)-design.

Proof. By Theorem 3.4

\[
\sum_{i=1}^n w(x_i)||x_i||^{2j} \varphi(x_i) = \sum_{i=1}^p \frac{a_i}{|S_i|} \int_{S_i} ||x||^{2j} \varphi(x)d\sigma_i(x) = 0
\]

for any polynomial \( \varphi \in \text{Harm}_l(\mathbb{R}^d) \) with \( 1 \leq l \leq t \) and \( 0 \leq j \leq \left\lfloor \frac{d}{2} \right\rfloor \).

Remark 3.6. Let \( X \) be a tight inside \( s \)-inner product set and a Euclidean \( t \)-design with a positive weight function \( w \). If \( t \geq s \), then \( a_i := w(X_i) \). Because

\[
\sum_{i=1}^p \frac{w(X_i)}{|S_i|} \int_{S_i} g_{x_i}(x)d\sigma_i(x) = \sum_{x \in X} w(x)g_{x_i}(x) = w(x_j).
\]

4 Examples of tight inside \( s \)-inner product sets

In this section, we introduce some examples of tight inside inner product sets, and their maximum achievable strengths as Euclidean designs whose weight functions are constant on each \( X_i \).

If \( p = 1 \), then tight inside inner product sets are tight inner product sets and tight spherical designs. Clearly, tight inside 1-inner product sets are tight 1-inner product sets. We easily construct tight 1-inner
product sets (for an example, see [8]). We note that \( X \) is a tight 1-inner product set with a negative inner product if and only if \( X \) is a tight Euclidean 2-design [8]. A tight 1-inner product set with a non-negative inner product is not even a Euclidean 1-design. Tight antipodal inside 2-inner product sets are easily classified. Namely, tight antipodal inside 2-inner product sets are the following:

\[
X = \{0, \pm a_1 e_1, \pm a_2 e_2, \ldots, \pm a_d e_d\}, \quad \text{if } S \text{ contains the origin,}
\]

\[
X = \{\pm a_1 e_1, \pm a_2 e_2, \ldots, \pm a_d e_d\}, \quad \text{if } S \text{ does not contain the origin,}
\]

where \( a_i \) are positive numbers and \( \{e_i\}_{i \leq d} \) is an orthonormal basis of \( \mathbb{R}^d \). Note that tight antipodal inside 2-inner product sets are tight antipodal locally 2-inner product sets. Therefore, a tight antipodal locally 2-inner product set which does not contain the origin (resp. contains the origin) is a tight (resp. an almost tight) Euclidean 3-design [13], and the converse also holds. An almost tight design was defined in [6], and that is a union of a tight design satisfying \( 0 \not\in X \) and the origin.

We can find examples of tight (antipodal) inside \( s \)-inner product sets in tight Euclidean designs (see Table 1).

\[
p = 2, \ r_1 = 1, \ w_1 = 1
\]

| \( t \) | \( s \) | \( d \) | \( |X| \) | \( |X_1| \) | \( |X_2| \) | \( B(X_1) \) | \( B(X_2) \) | \( r_2 \) | \( w_2 \) | \( \text{ref.} \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 4   | 2   | 2   | 6   | 3   | 3   | \(-\frac{1}{2}\) | -2, 1 | 2   | \( \frac{1}{2} \) | 12   |
| 2   | 4   | 15  | 10  | 5   | \(-\frac{1}{2}, \frac{1}{6}\) | \(-\frac{1}{2}, 1\) | \( \sqrt{6}\) | \( \frac{1}{27} \) | 12   |
| 2   | 5   | 21  | 6   | 15  | \(-\frac{1}{5}\) | -\(\frac{4}{5}, \frac{2}{5}\) | \( \frac{\sqrt{5}}{2}\) | \( \frac{1}{27} \) | 12   |
| 2   | 4   | 15  | 6   | 9   | \(-\frac{1}{4}, 0\) | -1, \(\frac{1}{4}\) | \( \sqrt{2}\) | \( \frac{1}{12} \) | 12   |
| 5   | 3   | 14  | 6   | 8   | \(-1, 0\) | -3, 1, \(\pm 1\) | \( \sqrt{3}\) | \( \frac{1}{3} \) | 12   |
| 3   | 5   | 32  | 12  | 20  | \(-1, \pm \frac{1}{5}\) | \(-\frac{9}{5}, \pm \frac{3}{5}\) | \( \frac{3}{5}\) | \( \frac{1}{3} \) | 13   |
| 3   | 5   | 32  | 12  | 12  | \(-1, \pm \frac{1}{3}\) | \(-5, 1, \pm 1\) | \( \sqrt{5}\) | \( \frac{1}{3} \) | 13   |
| 3   | 6   | 44  | 12  | 32  | \(-1, 0\) | \(-\frac{4}{2}, \pm \frac{1}{2}\) | \( \sqrt{2}\) | \( \frac{1}{12} \) | 13   |
| 7   | 4   | 48  | 24  | 24  | \(-1, 0, \pm \frac{1}{2}\) | \(-2, 0, \pm 1\) | \( \sqrt{2}\) | \( \frac{1}{6} \) | 5     |
| 4   | 7   | 182 | 56  | 126 | \(-1, \pm \frac{1}{4}\) | \(-\frac{1}{2}, 0, \pm \frac{1}{4}\) | \( \sqrt{2}\) | \( \frac{1}{6} \) | 5     |
| 4   | 7   | 182 | 126 | 56  | \(-1, 0, \pm \frac{1}{2}\) | \(-3, 0, \pm 1\) | \( \sqrt{3}\) | \( \frac{1}{3} \) | 5     |

\[
p = 3, \ r_1 = 1, \ w_1 = 1
\]

| \( t \) | \( s \) | \( d \) | \( |X| \) | \( |X_1| \) | \( |X_2| \) | \( |X_3| \) | \( B(X_1) \) | \( B(X_2) \) | \( B(X_3) \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 7   | 4   | 26  | 12  | 6   | 8   | -1, 0, \(\pm \frac{1}{2}\) | -2, 0, \(\pm 1\) | -6, 0, \(\pm 2\) | 4     |

Weight function \( w(x) = w_i \) for \( x \in X_i \). \( B(x) = B(X_i) \) for \( x \in X_i \).

Table 1

These examples in Table 1 are not locally \( s \)-inner product sets except \( (s, d, |X|) = (3, 3, 14) \) which is a tight locally \( s \)-inner product sets (and is not an \( s \)-inner product set). We give some examples that are tight inside \( s \)-inner product sets but not tight Euclidean designs.

**Example 4.1** (antipodal, \( s = 4, \ p = 2, \ d = 8, 23 \)). Let \( X_2 \) be a tight spherical 7-design. Let \( X_1 \) be the set whose element is only the origin. Then, \( X = X_1 \cup X_2 \) is a tight antipodal 4-inner product set supported by a union of two concentric spheres. Note that \( X \) is an almost tight Euclidean 7-design (the maximum strength is 7). Tight spherical 7-designs are known only for \( d = 8, 23 \)—the \( E_8 \) root system and the kissing configuration of the minimum vectors of the Leech lattice [15].

**Example 4.2** (antipodal, \( s = 6, \ p = 2, \ d = 24 \)). Let \( X_2 \) be the minimum vectors of the Leech lattice. Let \( X_1 \) be the set whose element is only the origin. Then, \( X = X_1 \cup X_2 \) is a tight 6-inner product set supported by a union of two concentric spheres in \( \mathbb{R}^{24} \). \( X \) is an almost tight Euclidean 11-design (the maximum strength is 11).
Example 4.3 $(s = 2, p = 2, d \geq 2)$. Let $X_2$ be the $d$-dimensional regular simplex in $\mathbb{R}^d$ and

$$X_1 := \left\{ \frac{1}{2}(x_i + x_j) \left| x_i, x_j \in X_2, i \neq j \right. \right\}.$$ 

Then, $r_1 = \sqrt{(1 - 1/d)/2}$, $r_2 = 1$,

$$B(x) = \left\{ -\frac{1}{d^2}, \frac{1}{4} \left( 1 - \frac{3}{d} \right) \right\} \text{ for } x \in X_1,$$

and

$$B(x) = \left\{ -\frac{1}{d}, \frac{1}{2} \left( 1 - \frac{1}{d} \right) \right\} \text{ for } x \in X_2.$$

$X = X_1 \cup X_2$ is a tight inside 2-inner product set supported by a union of two concentric spheres in $\mathbb{R}^d$. If $d = 2$, then $X$ is a tight Euclidean 4-design in Table 1. If $d \neq 2$, then $(X, w)$ is a Euclidean 2-design (the maximum strength is 2) where $w(x) = 1$ for any $x \in X$.

Example 4.4 $(s = 2, p = 2, d \geq 4)$. Assume $d \geq 4$. Let $X_2$ be the $d$-dimensional regular simplex in $\mathbb{R}^d$ and

$$X_1 := \left\{ -\frac{x_i + x_j}{d - 1} \left| x_i, x_j \in X_2, i \neq j \right. \right\}.$$ 

Then, $r_1 = \sqrt{2/(d(d-1))}$, $r_2 = 1$,

$$B(x) = \left\{ -\frac{4}{d(d-1)^2}, \frac{d - 3}{d(d-1)^2} \right\} \text{ for } x \in X_1,$$

and

$$B(x) = \left\{ -\frac{1}{d}, \frac{2}{d(d-1)} \right\} \text{ for } x \in X_2.$$ 

$X = X_1 \cup X_2$ is a tight inside 2-inner product set supported by a union of two concentric spheres in $\mathbb{R}^d$. If $d = 4$, then $X$ is a tight Euclidean 4-design in Table 1. If $d \geq 5$, then $(X, w)$ is a Euclidean 2-design (the maximum strength is 3) where

$$w(x) = \begin{cases} 1 \\ \frac{d - 3}{(d-1)^2} \end{cases} \text{ for } x \in X_1,$$ 

and

$$w(x) = \frac{d - 3}{(d-1)^2} \text{ for } x \in X_2.$$ 

Let $a = -\frac{1 + (d-1)/\sqrt{d}}{d \sqrt{d}}$ and $b = -\frac{1 + (d-1)/\sqrt{d}}{d \sqrt{d}}$. We define $u_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,n})$ for $i = 1, 2, \ldots, d$ by

$$u_{i,j} = \begin{cases} a & \text{for } j = i, \\ b & \text{for } j \neq i, \end{cases}$$ 

and $u_{d+1} := (1/\sqrt{d}, 1/\sqrt{d}, \ldots, 1/\sqrt{d})$. Then, $U_d = \{u_1, u_2, \ldots, u_{d+1}\}$ is a regular simplex on the unit sphere. We define

$$L := \frac{1 + b \sqrt{d}}{a - b} I + \frac{b \sqrt{d}}{a - b} (J - I),$$ 

where $I$ is the identity matrix, and $J$ is the matrix whose entries are all 1. The following two examples are constructed from tight Euclidean designs.

Example 4.5 $(s = 2, p = 2, d = 6)$. Let $X_1$ be the regular simplex $U_6$ and

$$X_2 = \left\{ \frac{9}{2} \sum_{i=1}^{7} \epsilon_i \right\} \epsilon_i \in \{ -1, 1 \}, |\{ i \mid \epsilon_i = -1 \}| = 1 \text{ or } 2 \right\}.$$
where \( u_7 = (1/\sqrt{6}, 1/\sqrt{6}, 1/\sqrt{6}, 1/\sqrt{6}, 1/\sqrt{6}, 1/\sqrt{6}, 1/\sqrt{6}) \). The elements of \( X_2 \) are on the sphere of radius \( \sqrt{15} \). Let \( X' = X_1 \cup X_2 \). We define the weight function \( w \) on \( X' \) by

\[
w(x) = \begin{cases} 
1 & \text{for } x \in X_1, \\
\frac{1}{3} & \text{for } x \in X_2.
\end{cases}
\]

Then, \( X' \) is a tight Euclidean 4-design on a union of two concentric spheres in \( \mathbb{R}^6 \).

Let \( X_3 = 1/15 X_2 \) and \( X = X_1 \cup X_3 \). Then the elements of \( X_3 \) are on the sphere of radius \( 1/\sqrt{15} \). We can easily check that

\[
B(x) = \left\{ \frac{1}{6}, \frac{1}{15} \right\} \quad \text{for } x \in X_1,
\]

\[
B(x) = \left\{ \frac{2}{75}, \frac{1}{50} \right\} \quad \text{for } x \in X_3.
\]

Therefore, \( X \) is a tight inside 2-inner product set supported by a union of two concentric spheres. We can check that \( X \) is a Euclidean 3-design (the maximum strength is 3) with the weight function

\[
w(x) = \begin{cases} 
\frac{125}{3} & \text{for } x \in X_1, \\
\frac{1}{81} & \text{for } x \in X_2.
\end{cases}
\]

**Example 4.6** \((s = 2, p = 2, d = 22)\). Let \( X_1 \) be the regular simplex \( U_{22} \) and

\[
X_2 = \left\{ \frac{27}{4} u_{23} + 69/88 (f_1, f_2, \ldots, f_{22}) \mid (f_1, f_2, \ldots, f_{22}) \in Y \right\},
\]

where \( u_{23} = 1/\sqrt{22}(1, \ldots, 1) \in \mathbb{R}^{22} \) and \( Y \) is a 253-point set in \( \{(f_1, f_2, \ldots, f_{22}) \mid f_i \in \{-1, 1\}, |\{i \mid f_i = -1\}| = 7\} \), satisfying \( |\{i \mid f_i = f'_i = -1\}| = 1 \) or 3 for any distinct \( (f_1, \ldots, f_{22}), (f'_1, \ldots, f'_{22}) \) in \( Y \). \( Y \) is combinatorial tight 4-design in \( J(23, 7) \), i.e., tight 4-(23, 7, 1) design. The elements of \( X_2 \) are on the sphere of radius \( \sqrt{126/11} \). Let \( X' = X_1 \cup X_2 \). We define the weight function \( w \) on \( X' \) by

\[
w(x) = \begin{cases} 
\frac{1}{81} & \text{for } x \in X_1, \\
\frac{1}{81} & \text{for } x \in X_2.
\end{cases}
\]

Then, \( X' \) is a tight Euclidean 4-design on a union of two concentric spheres in \( \mathbb{R}^{22} \).

Let \( X_3 = 1/24 X_2 \) and \( X = X_1 \cup X_3 \). Then the elements of \( X_3 \) are on the sphere of radius \( 1/4\sqrt{7/22} \). We can easily check that

\[
B(x) = \left\{ \frac{1}{22}, \frac{7}{352} \right\} \quad \text{for } x \in X_1,
\]

\[
B(x) = \left\{ \frac{13}{2816}, \frac{5}{1408} \right\} \quad \text{for } x \in X_3.
\]

Therefore, \( X \) is a tight inside 2-inner product set supported by a union of two concentric spheres. We can check that \( X \) is a Euclidean 3-design (the maximum strength is 3) with the weight function

\[
w(x) = \begin{cases} 
\frac{312}{3} & \text{for } x \in X_1, \\
\frac{312}{3} & \text{for } x \in X_3.
\end{cases}
\]

**Example 4.7** \((s = 2, p = 4, d = 2)\). We define \( X_1 := \{(0, 0)\}, X_2 := \{(1, 0), (0, 1)\}, X_3 := \{(1, 1)\} \) and \( X_4 := \{(2, 0), (0, 2)\} \). Then, \( r_1 = 0, r_2 = 1, r_3 = \sqrt{2}, r_4 = 2, B(x) \) is empty for \( x \in X_1, B(x) = \{0\} \) for \( x \in X_2, B(x) = \{0, 1\} \) for \( x \in X_3 \) and \( B(x) = \{0, 2\} \) for \( x \in X_4 \). Therefore, \( X := X_1 \cup X_2 \cup X_3 \cup X_4 \) is a tight inside 2-inner product set supported by a union of four concentric spheres in \( \mathbb{R}^2 \). However, \( X \) is not even a Euclidean 1-design.
5 The non-existence of tight 2- or 3-inner product sets

In this section, we prove the non-existence of tight 2- or 3-inner product sets supported by a union of two concentric spheres. We collect some results to prove this result.

**Theorem 5.1** (Rankin bound \[26\]). Let \( X \) be a finite set in \( S^{d-1} \).

1. If \( \alpha < 0 \) for all \( \alpha \in A(X) \), then \( |X| \leq d + 1 \).
2. If \( \alpha \leq 0 \) for all \( \alpha \in A(X) \), then \( |X| \leq 2d \).

It is known that there exist many examples that attain the upper bound in Theorem 5.1 (1), such as a \( d \)-dimensional regular simplex. Examples that attain the upper bound in Theorem 5.1 (2) are only the cross polytopes \( \{ \pm e_1, \ldots, \pm e_d \} \) where \( \{ e_i \}_{1 \leq i \leq d} \) is an orthonormal basis of \( \mathbb{R}^d \).

We prove the following theorem as a generalization of the Rankin bound from \( S^{d-1} \) to \( \mathbb{R}^d \).

**Theorem 5.2** (Rankin bound for \( \mathbb{R}^d \)). Let \( X \) be a finite set in \( \mathbb{R}^d \).

1. If \( \alpha < 0 \) for all \( \alpha \in A(X) \), then \( |X| \leq d + 1 \).
2. If \( \alpha \leq 0 \) for all \( \alpha \in A(X) \), then \( |X| \leq 2d + 1 \).

**Proof.** We prove the first statement. Let \( X \) be a finite subset of \( \mathbb{R}^d \) with \( \alpha < 0 \) for \( \alpha \in A(X) \). Note that \( 0 \not\in X \). For any distinct \( x, y \in X \), we have \( x/||x||, y/||y||| < 0 \), and hence \( x/||x|| \neq y/||y|| \). We define \( X' := \{ x/||x|| \mid x \in X, x \neq 0 \} \subseteq S^{d-1} \). Since \( (x, y) < 0 \) for any distinct \( x, y \in X' \), \( |X| = |X'| \leq d + 1 \) by the Rankin bound in Theorem 5.1.

We prove the second statement. Let \( X \) be a finite subset of \( \mathbb{R}^d \) with \( \alpha \leq 0 \) for \( \alpha \in A(X) \). Define \( \varepsilon_X = 1 \) if \( 0 \in X \), and \( \varepsilon_X = 0 \) if \( 0 \notin X \). For any distinct non-zero elements \( x, y \in X \), we have \( x/||x||, y/||y||| \leq 0 \), and hence \( x/||x|| \neq y/||y|| \). We define \( X' = \{ x/||x|| \mid x \in X, x \neq 0 \} \subseteq S^{d-1} \). Since \( (x, y) \leq 0 \) for any distinct \( x, y \in X' \), \( |X| = |X'| + \varepsilon_X \leq 2d + 1 \) by the Rankin bound in Theorem 5.1.

There exist many examples that attain the upper bound in Theorem 5.2 (1), such as a \( d \)-dimensional regular simplex. Since we have classified the sets that attain the upper bound in Theorem 5.1 (2), it is easy to see that examples that attain the upper bound in Theorem 5.2 (2) are only the following sets:

\[
X = \{0, a_1 e_1, a_2 e_2, \ldots, a_d e_d, -a_{d+1} e_1, -a_{d+2} e_2, \ldots, -a_{2d} e_d \}
\]

where \( a_i \) are positive real numbers.

The following upper bound is useful for classifying \( s \)-inner product sets on \( S^{d-1} \).

**Theorem 5.3** \([25]\). Let \( X \) be an \( s \)-inner product sets on \( S^{d-1} \). Let \( h_l \) be the dimension of \( \text{Harm}_l(\mathbb{R}^d) \).

Let \( G_l^{(d)}(t) \) be the \( d \)-dimensional Gegenbauer polynomial of degree \( l \), normalized by \( G_l^{(d)}(1) = h_l \). We define the polynomial \( F_X(t) \) of degree \( s \):

\[
F_X(t) := \prod_{\alpha \in A(X)} (t - \alpha) = \sum_{i=0}^s f_i G_i^{(d)}(t),
\]

where \( f_i \) are real numbers. Then,

\[
|X| \leq \sum_{i \text{ with } f_i > 0} h_i.
\]

If \( f_i > 0 \) for all \( i \), then this upper bound coincides with the Fisher type upper bound in Theorem 2.4 (1).

**Corollary 5.4.** Let \( X \) be a 2-inner product set and \( A(X) = \{\alpha, \beta\} \). Then, \( F_X(t) := (t - \alpha)(t - \beta) = \sum_{i=0}^2 f_i G_i^{(d)}(t) \) where \( f_0 = \alpha \beta + 1/d, f_1 = -\alpha \beta/d \) and \( f_2 = 2/(d(d-2)) \). If \( \alpha + \beta \geq 0 \), then

\[
|X| \leq \binom{d + 1}{2}.
\]
Lemma 5.5. Let \( X = \{x_1, x_2, \ldots, x_n\} \) be an (resp. antipodal) inside s-inner product set supported by \( S \subset \mathbb{R}^d \). Let \( \{\varphi_i\}_{1 \leq i \leq v} \) be a basis of \( P_s(S) \) (resp. \( P^*_s(S) \)), and \( v \) be the dimension of \( P_s(S) \) (resp. \( P^*_s(S) \)). Let \( M \) be the \( n \times v \) matrix whose \((i, j)\)-entry is \( \varphi_j(x_i) \) (resp. \( \varphi_j(y_i) \)). Then, \( n \leq v \) and the rank of \( M \) is \( n \). In particular, if \( n = v \), then \( M \) is a nonsingular matrix.

Proof. We define the \( d \)-variable polynomial \( f_{x_k}(\xi) \) as in the proof of Theorem \[23\]. Since \( f_{x_k}(\xi) \) is of degree at most \( s \), we can write \( f_{x_k}(\xi) = \sum_{i=1}^v a_i(x_k) \varphi_i(\xi) \) for some \( a_i(x_k) \) are real numbers. \( N \) is defined by the \( n \times v \) matrix, whose \((i, j)\)-entry is \( a_j^{x_k} \). Then, \( N M^T \) is an upper triangular matrix whose diagonal entries are all 1, and hence is of rank \( n \). Therefore, \( M \) is of rank \( n \).

Lemma 5.6. Let \( X \) be an s-inner product set in \( \mathbb{R}^d \), and \( F_X(t) = \sum_{\alpha \in A(X)} \frac{(t-\alpha)}{t} = \sum_{i=0}^s f_i t^i \), where \( f_i \) are real numbers. If \( |X| > 2d + 1 \), then there exists \( i \) such that \( f_i < 0 \).

Proof. Let \( X \) be an s-inner product set in \( \mathbb{R}^d \). Assume \( |X| > 2d + 1 \). By Theorem \[5.2\] there exists \( \alpha \in A(X) \) such that \( \alpha > 0 \). If we assume \( f_i \geq 0 \) for all \( 0 \leq i \leq s \), then \( F_X(t) \) is monotonically increasing for \( t > 0 \). Since \( f_0 \geq 0 \), this is a contradiction for \( \alpha > 0 \).

Let \( x = (x_1, x_2, \ldots, x_d) \) in \( \mathbb{R}^d \), \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{Z}_{\geq 0}^d \), and \( |\lambda| = \sum_{i=1}^d \lambda_i \). We define \( x^\lambda := x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_d^{\lambda_d} \) and \( (\lambda_1, \lambda_2, \ldots, \lambda_d) := \frac{|\lambda|!}{\lambda_1! \lambda_2! \cdots \lambda_d!} \). Then, for \( x, y \in \mathbb{R}^d \)

\[
(x, y)^k = \sum_{|\lambda|=k} \left( \frac{|\lambda|!}{\lambda_1! \lambda_2! \cdots \lambda_d!} \right) x^\lambda y^\lambda = \sum_{|\lambda|=k} \varphi_\lambda(x) \varphi_\lambda(y)
\]

(5.2)

where \( \varphi_\lambda(x) := \sqrt{\frac{|\lambda|!}{\lambda_1! \lambda_2! \cdots \lambda_d!}} x^\lambda \). Let \( H_1 \) be the matrix whose rows and columns indexed by \( X \) and \( \{\varphi_\lambda | |\lambda| = l\} \) respectively, whose \((x, \lambda)\) entry is \( \varphi_\lambda(x) \). For any polynomial \( F(t) = \sum_{i=0}^s f_i t^i \) of degree \( k \) where \( f_i \) are real numbers, define

\[
M(F) = \begin{bmatrix}
0 & f_1 H_1^T & \cdots & f_k H_k^T \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}
\]

where \( I_m \) is the identity matrix of degree \( m \) and \( h_i^l = \dim \text{Hom}_i(\mathbb{R}^d) \). By the equality \[5.2\], we have

\[
[H_0 \ H_1 \ \cdots \ H_k] M(F) = \left[ \begin{array}{c}
H_0^T \\
H_1^T \\
\vdots \\
H_k^T
\end{array} \right] = \sum_{i=0}^k f_i H_i H_i^T
\]

(5.3)

\[
= \sum_{i=0}^k f_i (x, y)^i \bigg|_{(x, y) \in X \times X} = [F((x, y))]_{(x, y) \in X \times X}.
\]

In particular, if \( X \) is an s-inner product set and \( F(t) := \prod_{\alpha \in A(X)} (t - \alpha) \), then \([F((x, y))]_{(x, y) \in X \times X} \) is a diagonal matrix.

Lemma 5.7. Let \( X \) be a tight 2-inner product set, which is supported by a union of two concentric spheres, does not contain the origin, and is not antipodal. Let \( A(X) = \{\alpha, \beta\} \) where \( \alpha > \beta \). Then, \((|X_1|, |X_2|) = (1, d + (d+1)) \), \((d, 1 + (d+1)) \), or \((d + 1, 1 + (d+1)) \). Furthermore, \((|X_1|, |X_2|) = (1, d + (d+1)) \) if and only if \( \alpha > 0 \), \( \beta < 0 \) and \( \alpha + \beta < 0 \). \((|X_1|, |X_2|) = (d + 1, 1 + (d+1)) \) if and only if \( \alpha > 0 \), \( \beta > 0 \). \((|X_1|, |X_2|) = (d + 1, 1 + (d+1)) \) if and only if \( \alpha > 0 \), \( \beta < 0 \) and \( \alpha + \beta > 0 \).
Proof. Let \( X \) be a tight 2-inner product set, which is supported by a union of two concentric spheres, does not contain the origin, and is not antipodal, and \( A(X) = \{\alpha, \beta\} \). Then, \(|X| = \dim P_d(\mathbb{R}^d) = \left(\frac{d+2}{2}\right) = 1 + d + \left(\frac{d+1}{2}\right)\). We define \( F(t) := (t - \alpha)(t - \beta) = \sum_{i=0}^{2} f_i t^i \) where \( f_0 = \alpha \beta, f_1 = -(\alpha + \beta), \) and \( f_2 = 1 \). By the equality (5.3), we may write

\[
[H_0 \ H_1 \ H_2] M(F) \begin{bmatrix} H_0^T \\ H_1^T \\ H_2^T \end{bmatrix} = N
\]

where

\[
N = \begin{bmatrix} (r_1^2 - \alpha)(r_2^2 - \beta)I_{|X_1|} & 0 \\ 0 & (r_1^2 - \alpha)(r_2^2 - \beta)I_{|X_2|} \end{bmatrix}.
\]

By Lemma 5.5, \([H_0 \ H_1 \ H_2] \) is a nonsingular matrix. Since the signature as quadratic form is independent under congruence transformations by the Sylvester’s law of inertia, the numbers of positive, negative and zero eigenvalues of \( M(F) \) are equal to those of \( N \) respectively. By Theorem 5.2 \( \alpha > 0 \). Hence, by Lemma 5.6, some \( f_i \) are negative. Furthermore, \((r_2^2 - \alpha)(r_2^2 - \beta) > 0 \). Therefore,

\[
|X_1| = \sum_{i \text{ with } f_i < 0} b_i', \quad |X_2| = \sum_{i \text{ with } f_i > 0} b_i'.
\]

Since \( f_2 > 0 \), we can determine \(|X_1|, |X_2|) = (d + (\frac{d+1}{2})), (d + 1 + (\frac{d+1}{2})) \), or \((d + 1, (\frac{d+1}{2})) \). Therefore, the statement of this Lemma is immediate. \( \square \)

Remark that if \( X \) is a tight 2-inner product set on a union of two concentric spheres, then \( X_1 \) is a 1-inner product set or one point set, because \((r_1^2 - \alpha)(r_1^2 - \beta) \) is negative.

**Lemma 5.8.** Let \( X \) be a tight 3-inner product set, which is supported by a union of two concentric spheres, does not contain the origin, and is not antipodal. Let \( A(X) = \{\alpha, \beta, \gamma\} \) where \( \alpha > \beta > \gamma \). Then, \(|X_1|, |X_2|) = (d + (\frac{d+1}{2})), (d + 1 + (\frac{d+1}{2})) \), or \((d + 1, (\frac{d+1}{2})) \). Furthermore, \(|X_1|, |X_2|) = (d + (\frac{d+1}{2})), (d + (\frac{d+2}{3})) \) if and only if \( \alpha > 0, \beta > 0, \gamma < 0, \alpha + \beta + \gamma > 0, \) and \( \alpha \beta + \beta \gamma + \gamma \alpha < 0 \). \(|X_1|, |X_2|) = (1 + (\frac{d+1}{2})), (d + (\frac{d+2}{3})) \) if and only if \( \alpha > 0, \beta < 0, \gamma < 0, \alpha + \beta + \gamma > 0, \) and \( \alpha \beta + \beta \gamma + \gamma \alpha > 0 \). \(|X_1|, |X_2|) = (d + 1, (\frac{d+1}{2})), (d + (\frac{d+2}{3})) \) if and only if \( \alpha > 0, \beta < 0, \gamma < 0, \alpha + \beta + \gamma < 0, \) and \( \alpha \beta + \beta \gamma + \gamma \alpha > 0 \).

Proof. Let \( X \) be a tight 3-inner product set, which is supported by a union of two concentric spheres, does not contain the origin, and is not antipodal. Let \( A(X) = \{\alpha, \beta, \gamma\} \). Define \( F(X) = (t - \alpha)(t - \beta)(t - \gamma) = \sum_{i=0}^{3} f_i t^i \) where \( f_0 = -\alpha \beta \gamma, f_1 = \alpha \beta + \beta \gamma + \gamma \alpha, f_2 = -(\alpha + \beta + \gamma) + 1 \). Since \(|X| = \dim P_d(\mathbb{R}^d) \) we can use the same method as in the proof of Lemma 5.6. Therefore, we can get \(|X_1|, |X_2|) = (1 + (\frac{d+1}{2})), (d + (\frac{d+2}{3})) \), \((d + (\frac{d+2}{3})), (1 + (\frac{d+1}{2}), d + (\frac{d+2}{3})) \), \((1 + d, (\frac{d+1}{2})), (d + (\frac{d+2}{3})) \). \((1, d + (\frac{d+1}{2})), (d + 1 + (\frac{d+1}{2})) \), \((d + (\frac{d+2}{3})), (1 + (\frac{d+1}{2})), d + (\frac{d+2}{3})) \). Since \((r_1^2 - \alpha)(r_1^2 - \beta)(r_1^2 - \beta) \) is negative, \( X_1 \) is at most 2-inner product set on a sphere. Therefore, \(|X_1| \leq (\frac{d+1}{2}) + d \). \( X_2 \) is at most 3-inner product set on a sphere, and hence \(|X_2| \leq (\frac{d+1}{2}) + (\frac{d+2}{3}) \). This proves the statement. \( \square \)

The following are the main theorems in this section.

**Theorem 5.9.** There does not exist a tight 2-inner product set, that is supported by a union of two concentric spheres and is not antipodal, for any \( d \geq 2 \).

Proof. Let \( X \) be a tight 2-inner product set supported by a union of two concentric spheres, and \( A(X) = \{\alpha, \beta\} \) where \( \alpha > \beta \). We have \(|X| = \dim P_d(\mathbb{R}^d) = \left(\frac{d+2}{2}\right) \). If \( X \) contains the origin, then \( X_1 \) is the origin and \( X_2 \) is a tight spherical 4-design. Since \( X_2 \) is a tight spherical 4-design, \( A(X_2) \) does not contain zero. Hence, \( X \) cannot contain the origin. By Lemma 5.7, we can know \(|X_1|, |X_2| \), and the properties of \( \alpha \) and \( \beta \).
Assume $|X_1| = 1$ and $|X_2| = d + (d+1)$. Then $\alpha > 0$, $\beta < 0$, $\alpha + \beta < 0$ and $X_2$ is a tight spherical 4-design. Since $X_2$ is a tight spherical 4-design, for $x_1 \in X_1$, we have
\[
\sum_{x \in X_2} (x, x_1) = n_\alpha \alpha + n_\beta \beta = 0,
\]
\[
\sum_{x \in X_2} (x, x_1)^3 = n_\alpha \alpha^3 + n_\beta \beta^3 = 0,
\]
where $n_k = |\{x \in X_2 \mid (x, x_1) = k\}|$ (see (11)). This implies that $n_\alpha = 0$ and $n_\beta = 0$, a contradiction.

If $|X_1| = d$ and $|X_2| = 1 + (d+1)$, then $\alpha + \beta > 0$. By Corollary 5.3 we have $|X_2| \leq (d+1)$, a contradiction.

Assume $|X_1| = d + 1$ and $|X_2| = (d+1)/2$. Then, $\alpha > 0$, $\beta < 0$, and $\alpha + \beta < 0$. Since $X_1$ is a 1-inner product set and $|X_1| = d + 1$, $X_1$ is a tight spherical 2-design. Without loss of generality, we assume $r_1 = 1$. Since $X_1$ is a tight spherical 2-design, for $x_2 \in X_2$,
\[
\sum_{x \in X_1} (x, x_2)^0 = n_\alpha + n_\beta = d + 1,
\]
\[
\sum_{x \in X_1} (x, x_2) = n_\alpha \alpha + n_\beta \left(-\frac{1}{d}\right) = 0,
\]
\[
\sum_{x \in X_1} (x, x_2)^2 = n_\alpha \alpha^2 + n_\beta \left(-\frac{1}{d}\right)^2 = r_2^2 d + 1,
\]
where $n_k = |\{x \in X_1 \mid (x, x_2) = k\}|$. This implies $\alpha = r_2^2$, a contradiction.

**Theorem 5.10.** There does not exist a tight 3-inner product set, that is supported by a union of two concentric spheres and is not antipodal, for any $d \geq 2$.

**Proof.** Let $X$ be a tight 3-inner product set supported by a union of two concentric spheres, and $A(X) := \{\alpha, \beta, \gamma\}$ where $\alpha > \beta > \gamma$. If $X$ contains the origin, then $X_2$ is a tight spherical 6-design. A tight spherical 6-design does not exist, except for the regular heptagon on $S^3$. Thus, no tight 3-inner product set exists on $\{0\} \cup S_2$. Assume that $X$ does not contain the origin. Then, $|X| = \dim P_3(\mathbb{R}^d) = (d+3)/3$. By Lemma 5.8 we can know $|X_1|$, $|X_2|$, and the properties of $\alpha$, $\beta$ and $\gamma$.

Assume $|X_1| = d + (d+1)/2$ and $|X_2| = 1 + (d+3)/2$. Since $X_1$ is a 2-distance set and $|X_1| = d + (d+1)/2$, $X_1$ is a tight spherical 4-design. Without loss of generality, we assume $r_1 = 1$. Then $A(X_1) = \{-1/2, 1/2\}$. For a fixed $x \in X_2$ and $k \in A(X_2)$, we define $n_k = |\{y \in X_1 \mid (y, x) = k\}|$. Since $X_1$ is a tight spherical 4-design, we have the following equations:
\[
n_\alpha + n_\beta + n_\gamma = |X_1|, \tag{5.4}
\]
\[
\alpha n_\alpha + \beta n_\beta + \gamma n_\gamma = 0, \tag{5.5}
\]
\[
\alpha^3 n_\alpha + \beta^3 n_\beta + \gamma^3 n_\gamma = 0. \tag{5.6}
\]
Since $\beta = -1/2d+3/2$ and $\gamma = -1/2d+3/2$, we can determine
\[
n_\alpha = \frac{d(d+3)}{2 + \alpha^3(2 + d)^2 - \alpha(d+6)}.
\]
Since $(1 - \alpha)(1 - \beta)(1 - \gamma) < 0$ and hence $\alpha > 1$, we have $n_\alpha < 1$, a contradiction.

Assume $|X_1| = 1 + (d+1)/2$ and $|X_2| = d + (d+2)/3$. Then, $\alpha > 0$, $\beta < 0$, $\gamma < 0$, $\alpha + \beta + \gamma > 0$, and $\alpha \beta + \beta \gamma + \gamma \alpha = 0$. We have $\alpha \beta + \beta \gamma + \gamma \alpha < -(\beta + \gamma)^2 + \beta \gamma = -\beta^2 - \beta \gamma - \gamma^2 < 0$, a contradiction.

Assume $|X_1| = 1 + d$ and $|X_2| = (d+1)/2 + (d+2)/3$. Then, $X_2$ is a tight spherical 6-design. It is well known that the only tight spherical 6-design on $S^{d-1}$ is the regular heptagon on $S^3$, a contradiction.

Assume $|X_1| = (d+1)/2$ and $|X_2| = 1 + d + (d+2)/3$. Then, $\alpha + \beta + \gamma > 0$. We have the Gegenbauer expansion $(t - \alpha)(t - \beta)(t - \gamma) = \sum_{k=0}^{3} a_k G_k^{(d)}(t)$ where $a_2 = -2(\alpha + \beta + \gamma)/d(d+2)$. By Theorem 3.3 $|X_2| \leq (d+2)/3 + 1$, a contradiction.

\[
\square
\]
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