NONCONSTANT POSITIVE SOLUTIONS TO THE RATIO-DEPENDENT PREDATOR-PREY SYSTEM WITH PREY-TAXIS IN ONE DIMENSION

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Abstract. Resorting to M.G. Crandall and P.H. Rabinowitz’s well-known bifurcation theory we first obtain the local structure of steady states concerning the ratio–dependent predator–prey system with prey-taxis in spatial one dimension, which bifurcate from the homogeneous coexistence steady states when treating the prey–tactic coefficient as a bifurcation parameter. Based on this, then the global structure of positive solution is established. Moreover, through asymptotic analysis and eigenvalue perturbation we find the stability criterion of such bifurcating steady states. Finally, several numerical simulations are performed to show the pattern formation.

1. Introduction. Prey–taxis describes the movement of predators towards the area with higher density of prey, playing a key role in biological control and in ecological balance such as regulating prey (pest) population or incipient outbreaks of prey or forming large-scale aggregation for survival (cf. [10, 22, 26]). A diffusive predator–prey model with prey–taxis, in general, can be formulated as (details cf. [16, p.249])

\[
\begin{cases}
    u_t = \nabla \cdot (d_1 \nabla u - uq(u,v)\nabla v) + \beta uF(u,v) - g(u), & x \in \Omega, \ t > 0, \\
    v_t = d_2 \Delta v + vf(v) - uF(u,v), & x \in \Omega, \ t > 0,
\end{cases}
\]

(1)

where \( u = u(x,t) \) and \( v = v(x,t) \) represent the predator density and the prey density at position \( x \) in an open domain \( \Omega \subset \mathbb{R}^N \) (\( N \geq 1 \)) and time \( t > 0 \) respectively;

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coefficients $d_1$ and $d_2$ are diffusion rates of predators and prey severally. The term $-\nabla \cdot (uq(u,v)\nabla v)$ accounts for the prey–taxis with a sensitivity coefficient $q(u,v)$ measuring the strength of prey-taxis; $F(u,v)$ refers to predation rate and often is called a functional response function; $\beta > 0$ is the conversion rate; $g(u)$ is the predator’s death typically defined as:

(linear) $g(u) = \theta u$, (quadratic) $g(u) = \theta u + lu^2$, $\theta > 0$, $l \geq 0$;

$f(v)$ delegates the per capita population growth rate of prey at density $v$. Note that $vf(v)$ is usually assumed to be negative for large $v > 0$ due to the limitation of resources (or crowding effect) and its typical forms (cf. [31] and [12]) include

$$vf(v) = rv\left(1 - \frac{v}{K}\right) \text{ (Logistic)}, \quad vf(v) = rv\left(1 - \frac{v}{K}\right)\left(\frac{v}{a_0} - 1\right) \text{ (Strong Allee)},$$

$$vf(v) = rv\left(1 - \frac{v}{K}\right) - \frac{av}{v+b}, \quad a < br \text{ (Weak Allee)},$$

where $r > 0$ is the intrinsic growth rate of prey and $K > 0$ is called the carrying capacity, and $0 < a_0, \frac{b^2}{d} < K$. There are also several classical forms on the functional response $F(u,v)$, one of which is prey–dependent predator functional response function if it depends only on the prey population, i.e., $F(u,v) = F(v)$. For instance (cf. [31, 14] and references therein)

$$F(v) = v \text{ (Holling I)}, \quad F(v) = \frac{p_1v}{m + p_2v} \text{ (Holling II)},$$

$$F(v) = \frac{v^\alpha}{m^\alpha + v^\alpha} \text{ (Holling III)}, \quad F(v) = c[1 - \exp(-\alpha v)] \text{ (Ivlev)},$$

where constants $c, p_1, p_2, m > 0$ and $\alpha > 1$.

Denoting by $\vec{\nu}$ the outward normal vector of the smooth boundary $\partial\Omega$ with $\partial_{\nu} := \frac{\partial}{\partial \nu}$, then system (1) is usually imposed by the zero (homogeneous) Neumann boundary condition as

$$\partial_{\nu} u = \partial_{\nu} v = 0, \quad x \in \partial\Omega, \quad t > 0$$

(2)

to explicate area-restricted or no flux boundary conditions and initial values are given by

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega.$$  

(3)

Below we may briefly review some interesting results in connection with (1), (2), and (3).

- If $d_2 > 0$ is a constant, $d_1 = d(v), g(u,v) = \chi(v)$ and $g(u)$ is quadratic, then system (1) becomes the main model in [15] where Jin and Wang studied its global boundedness, asymptotic stability and pattern formation in a two-dimensional bounded domain.

- If $d_1, d_2 > 0$ are constants, $g(u,v) = \chi(1 - u)$ with the constant $\chi > 0$, $\beta uF(u,v) - g(u) \text{ is logistic growth of cell and } vf(v) - uF(u,v)$ is changed into $\alpha u - \beta v$ in order to describe chemical having a linear production and degradation, then Ma et al. considered the existence and stability of nonconstant steady states in [20].

- The numerical solutions of (1) coupled with the predator’s velocity equation in the interval $[-1,1]$ were examined in [6], illustrating that both the initial conditions and the forms of $F(u,v)$ have a great impact on the pattern formation.

- If $g(u,v) = \frac{1}{1+e^u} (\gamma = 1, 2)$, Lee et al. [17] investigated traveling wave solutions of system (1) with $x \in \mathbb{R}$, deriving that prey-taxis virtually reduces the likelihood of effective biocontrol on prey spread.
If \( q(u,v) \) is a non-negative non-increasing function of prey density \( v \) such as \( \chi(\geq 0) \) or \( \chi/v \), Lee et al. [18] established necessary conditions for pattern formation of prey–taxis system (1) concerning a variety of \( F(u,v) \) and \( f(v) \) in a bounded interval. Wu et al. [31] showed that the solution is globally bounded if \( q(u,v) = \chi \) is small. Subsequently in [14], Jin and Wang showed that the two-dimensional solution is globally bounded for any \( \chi > 0 \) and found the condition on the global asymptotical stability of the homogeneous steady states. For any constant \( \chi \), the work in [29] revealed the one-dimensional nonconstant positive steady states with general population kinetics.

Moreover, Rosenzweig–MacArthur predator–prey dynamics was considered in [25], namely,

\[
F(v) = \frac{p_1 v}{m + p_2 v}, \quad p_1, p_2, m > 0, \quad f(v) = r \left(1 - \frac{v}{K}\right), \quad K > 0, \quad (4)
\]

and further if the prey–tactic coefficient \( q(u,v) = q(u) \) is truncated at some number \( u_m > 0 \) (i.e. \( q(u_m) = 0, q(u) > 0 \) for \( 0 \leq u < u_m \)), then there are some other results as below.

- B. Ainseba et al. [23] obtained the global weak solutions of (1) through the Schauder fixed point theorem and duality technique. Tao [28] extended it to the global classical solutions for \( n \leq 3 \) via \( L^p \)-estimates and Schauder estimates, where the solution bound depends on time. He and Zheng [11] obtained the global existence of solution with uniform-in-time bound. The existence of non–constant steady states of (1) was studied in [19, 30] via Hopf bifurcation theorem, index degree theory and bifurcation theory.

2. Main model and main results.

2.1. Main model and known results. As stated in [3, 1, 7], when predators have to forage (leading to sharing or competition for food), a more suitable general predator–prey theory may be built on a sort of ratio–dependent functional response, i.e. \( F(u,v) \) should be a function of the ratio of prey to predator density. In Arditi and Ginzburg [2], for example,

\[
F(u,v) = \frac{v}{mu + v} = \frac{v/u}{m + v/u} =: F(v/u). \quad (5)
\]

Therefore, by taking \( q(u,v) = \chi > 0, d_1 = 1, d_2 = d > 0 \) and making \( g(u) \) linear in (1) coupled with (2), (3) and (5), one can obtain the following ratio-dependent predator-prey system with prey-taxis

\[
\begin{cases}
  u_t = \Delta u - \nabla \cdot (\chi u \nabla v) + \frac{\partial uv}{mu + v} - \theta u, & x \in \Omega, \; t > 0, \\
  v_t = d\Delta v - \frac{uv}{mu + v} + vf(v), & x \in \Omega, \; t > 0, \\
  \partial_n u = \partial_n v = 0, & x \in \partial \Omega, \; t > 0, \\
  u(x,0) = u_0(x), \; v(x,0) = v_0(x), & x \in \Omega,
\end{cases} \quad (6)
\]

which has been investigated in [5] and references therein. In the sequel, we shall maintain the hypothesis of [5] as:

(H1) The function \( f : [0, \infty) \rightarrow \mathbb{R} \) is continuously differentiable satisfying \( f'(v) < -\delta^* < 0 \) for all \( v \geq 0 \) and some constant \( \delta^* > 0 \), and there exist two constants \( r, K > 0 \) such that \( f(v) \leq r \) for any \( v \geq 0 \), \( f(K) = 0 \) and \( f(v) < 0 \) for all \( v > K \);

and add
(H2) \(0 < \frac{\beta - \theta}{m\beta} < r\) (so \(\theta < \beta\)).

It is easy to see that \(f\) satisfies (H1) if \(vf(v)\) is Logistic. Furthermore, denote by \(I\) an interval in \(\mathbb{R}\) and by \(W^{1,\infty}(\Omega)\) the standard Sobolev space \(W^{k,p}(\Omega)\) with \(k = 1, p = \infty\). Set

\[
C^{k,l}(\Omega \times I) = \left\{ h(x,t) \in C(\Omega \times I) : \partial_x^k h(x,t) \in C(\Omega \times I) \text{ and } \partial_t^l h(x,t) \in C(\Omega \times I) \right\}.
\]

To the current system (6), Cai et al. [5] have obtained the following existence of global classical solution:

**Proposition 2.1** (Global existence). Let \(\Omega\) be a bounded domain in \(\mathbb{R}^N (N \geq 1)\) with smooth boundary and hypothesis (H1) hold. Assume \((u_0, v_0) \in [W^{1,\infty}(\Omega)]^2\) with \(u_0, v_0 \geq 0(\neq 0)\). Then system (6) has a unique global classical solution \((u, v) \in [C(\Omega \times [0, \infty)) \cap C^{2,1}(\Omega \times (0, \infty))]^2\) satisfying \(u, v \geq 0\) for all \(t > 0\) and

\[
||u(\cdot, t)||_{L^\infty(\Omega)} + ||v(\cdot, t)||_{W^{1,\infty}(\Omega)} \leq C,
\]

where \(C > 0\) is a constant independent of \(t\). In particular, \(0 < v(x, t) \leq \max\{||v_0||_{L^\infty}, K\}\).

In many population systems it is of importance to determine whether the population can reach either the coexistent steady state or competitive exclusion and extinction over time. This amounts to exploring the global stability of steady states of (6).

To system (6) the corresponding steady state problem in one dimension is

\[
\begin{align*}
-\nabla \cdot (\nabla u - \chi u \nabla v) &= \frac{\beta uv}{mu + v} - \theta u, & x \in \Omega, \\
-d\Delta v &= -\frac{\nu v}{mu + v} + vf(v), & x \in \Omega, \\
\partial_v u &= \partial_v v = 0, & x \in \partial \Omega,
\end{align*}
\]

where \(\chi, \theta, \beta > 0\), and \(\Omega = (0, \ell)\).

In the present paper our main focuses are on (linear) stability and existence of nonconstant positive solution of (7) (i.e., nonconstant nonnegative steady states of the system (6)). Since \((0, 0)\) is not well-defined herein in fact, supposing that \((u_*, v_*)\) is a nonzero constant solution of (7) (i.e., a homogeneous steady state of (6)), then

\[
\frac{\beta u_* v_*}{mu_* + v_*} = \theta u_*, \quad \frac{u_* v_*}{mu_* + v_*} = v_* f(v_*).
\]

One may easily follow from \(f(K) = 0\) in (H1) that \((0, K)\) is a nonnegative solution of (7) (i.e., a homogenous prey-only steady state of (6)), which also has a positive constant solution \((u_*, v_*)\) (a homogenous coexistent steady state of (6)) as follows:

\[
(u_*, v_*) = \left(\frac{\beta - \theta}{m\beta} v_*, \frac{f^{-1}((\beta - \theta)/m\beta)}{v_*}\right)
\]

guaranteed by \(0 < \frac{\beta - \theta}{m\beta} < r\) (i.e. (H2)). We shall take \(w_* := \frac{u_*}{v_*}\) throughout this paper.

Cai et al. [5] have shown the global asymptotical stability and linear stability of constant steady state \((u_*, v_*)\) of the system (6). Here we record the latter one for our purpose.

**Proposition 2.2.** (i) For \(\beta < \theta\), the constant equilibrium solution \((0, K)\) is linearly stable, which implies that system (6) has no spatial pattern.

(ii) Given \(0 < \frac{\beta - \theta}{m\beta} < r\), let \(m\beta \geq 1\) or \(m\beta < 1\) and \(-v_* f'(v_*) > \frac{\theta(\beta - \theta)(1 - m\beta)}{m\beta^2}\). Then the following statements hold.
2.2. Our main results. Before showing our results, some notations are claimed at first.

(A) To the following Neumann eigenvalue problem

\[
\begin{cases}
-\Delta \Phi = \lambda \Phi, & x \in \Omega, \\
\partial_n \Phi = 0, & x \in \partial \Omega,
\end{cases}
\]

it is well-known that the Laplacian has a discrete spectrum of infinitely many nonnegative eigenvalues which form a strictly increasing sequence

\[0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots,\]

where \(\lambda_j\) has finite multiplicities. In particular, in one dimension \(\Omega = (0, \ell)\) with \(\ell > 0\), the sequence of simple eigenvalues and eigenfunctions associated with are explicitly given by

\[
\lambda_j = (\pi j/\ell)^2, j = 0, 1, 2, \ldots, \quad \Phi_j(x) = \begin{cases} 
\frac{1}{\sqrt{\ell}}, & j = 0, \\
\sqrt{\frac{2}{\ell}} \cos(\pi jx/\ell), & j = 1, 2, \ldots.
\end{cases}
\]

Clearly, this set of eigenfunctions constitutes an orthonormal basis of \(L^2(0, \ell)\).

(B) Set

\[
\chi_j = \frac{v_* f'(v_*) + \frac{F^2(w_*)}{w_*} (1 - d\beta m) - d\lambda_j + \beta w_* v_* f'(v_*) F'(w_*) \lambda_j^{-1}}{u_* F^2(w_*)} > 0; \quad \text{(See also (27))}
\]

(C) Suppose that for \(i, j \in \mathbb{N}_+\) if \(i \neq j\) then

\[
d\lambda_i \lambda_j \neq -\beta w_* v_* f'(v_*) F'(w_*), \quad \text{(See also (39))},
\]

which indicates the simple bifurcation.
(D) One may deduce from (9) that \( \chi_0^j \leq \chi_c \) (see Remark 3). We denote by \( j_0 \) the number satisfying \( \chi_0^{j_0} = \chi_c \) or \( |\chi_c - \chi_0^{j_0}| = \min_j |\chi_c - \chi_0^j| \). Hence when \( \Omega = (0, \ell) \subset \mathbb{R} \) and \( \chi_0^{j_0} = \chi_c \), simple calculations give

\[
j_0 = \left( -\frac{\beta w_\ast v_\ast f'(w_\ast)F'(w_\ast)}{d} \right)^{\frac{1}{2}} \ell \frac{\ell}{\pi}. \tag{14}
\]

Our main results based on the hypothesis \((H1)\) and \((H2)\) consist of the following two parts. First, the existence of nonconstant positive solutions to system (7) is acquired.

**Theorem 2.3.** (Global bifurcation) Suppose that the conditions (2) and (10) in Proposition 2.2 (ii) and (C) hold. If \( \chi \in (0, \chi_{\max}) \) and \( \chi \neq \chi_0^j \) for any integer \( i > 0 \), then the system (7) possesses at least one nonconstant positive solution where \( \chi_{\max} = \max_{j \in \mathbb{N}_+} \{ \chi_0^j \} \).

Note that all these positive solutions bifurcate from the positive constant solution \((u_\ast, v_\ast, \chi_0^j)\) and locally can be parameterized as \((u_j(x, \varepsilon), v_j(x, \varepsilon), \chi_j(\varepsilon))\) where \( |\varepsilon| < \delta \) is small enough (see Lemma 3.4) and

\[
\chi_j(\varepsilon) = \chi_0^j + \varepsilon \chi_1 + \varepsilon^2 \chi_2 + o(\varepsilon^2). \tag{See also (40)}
\]

Furthermore, we can ascertain the stability of such bifurcating solutions.

**Theorem 2.4.** (Stability criterion) Suppose that the conditions in (2) of Proposition 2.2 (ii), (10) and (C) hold. Then

- for \( \varepsilon \in (-\delta, \delta), \varepsilon \neq 0 \), the solution \((u_{j_0}(\varepsilon, x), v_{j_0}(\varepsilon, x))\) of system (7) is asymptotically stable if \( \chi_2 < 0 \) and it is unstable if \( \chi_2 > 0 \);
- for each \( j \in \mathbb{N}_+ \setminus \{ j_0 \} \), the bifurcating solutions \((u_j(\varepsilon, x), v_j(\varepsilon, x))\) are unstable when \( \varepsilon \in (-\delta, \delta) \).

Considering that nonconstant solutions indicates likelihood of patterns, thus some numerical simulations of pattern formation will be shown in the final section.

3. **Proof of main results.** In this section we will establish the existence of the nonconstant positive solutions of system (7) by adopting M.G. Crandall and P.H. Rabinowitz’s canonical bifurcation theory and demonstrate their stability by means of asymptotic analysis and eigenvalue perturbation.

We start with an explanation of notations used latter. Let

\[
X := \left\{ (u, v) : u, v \in C^2(\Omega), \partial_\nu u = \partial_\nu v = 0 \text{ on } \partial \Omega \right\}, \tag{15}
\]

which is a Banach space endowed with standard \( C^2 \) norm \( \| \phi \|_{C^2(\Omega)} := \max_{0 \leq |\nu| \leq 2} \sup_{x \in \Omega} |D^\nu \phi(x)| \) with \( \alpha \) a multi-index). A Hilbert space \( Y := L^2(\Omega) \times L^2(\Omega) \) is equipped with the inner product

\[
(U_1, U_2)_Y = (u_1, u_2)_{L^2(\Omega)} + (v_1, v_2)_{L^2(\Omega)}
\]

for \( U_1 = (u_1, v_1) \in Y, U_2 = (u_2, v_2) \in Y \).

First of all, the Moser’s iteration allows us to have the following a priori estimate for any nonnegative solution of system (7), which will be applied in subsequent global bifurcation analysis.
Lemma 3.1. Suppose the assumptions in (H1) hold. Let \((u(x), v(x))\) be a nonnegative (classical) solution to the boundary value problem of elliptic model (7). Then \((u(x), v(x))\) satisfies

\[0 < v(x) \leq K, \quad 0 < ||u(x)||_{\infty} \leq K_2, \quad \text{for all } x \in \Omega \subset \mathbb{R}^N (N \geq 1),\]

where \(K_2\) is independent of \(u, v\). Moreover, if \(r > \frac{1}{m}\), then \(v(x) \geq \tilde{v}\) where \(\tilde{v}\) is a positive constant such that \(f(\tilde{v}) = \frac{1}{m}\) and \(f(v) > \frac{1}{m}\) when \(0 \leq v < \tilde{v}\).

Proof. From the Hopf’s Lemma and strong maximum principle, we have \(u(x) > 0, 0 < v(x) \leq K\) for all \(x \in \Omega\). Define differential operator \(\mathcal{A} : \mathbb{R} \to \mathbb{R}\) as \(\mathcal{A}v = -d\Delta v \) and function \(g(u, v) = vF(v/u)\). Then the second equation of model (7) can be written as \(\mathcal{A}v = g(u, v)\) for \(x \in \Omega\) with boundary condition \(\partial_n v = 0\). If \(r > \frac{1}{m}\), then

\[0 = \mathcal{A}\tilde{v} = \tilde{v}
left(f(\tilde{v}) - \frac{1}{m}\right) \leq g(u, \tilde{v}) = \tilde{v}
left(f(\tilde{v}) - \frac{F(\tilde{v}/u)}{\tilde{v}/u}\right), \quad x \in \Omega,
\]

which, together with boundary condition \(\partial_n v = 0\) for \(x \in \partial\Omega\), indicates that \(\tilde{v}\) is a low-solution of the equation \(\mathcal{A}v = g(u, v)\) for all \(x \in \Omega\).

Now we estimate \(u(x)\). Integrating model (7) over \(\Omega\), using integration by parts, and summing the obtained equations, we have

\[
\theta \int_{\Omega} u = \int_{\Omega} vf(v) + (\beta - 1) \int_{\Omega} v \cdot \frac{F(v/u)}{v/u} \leq \int_{\Omega} vf(v) + \frac{1}{m}(|\beta - 1| \int_{\Omega} v).
\]

By \(f(v) \leq r\) for \(0 \leq v(x) \leq K\), we can obtain from (17) that there exists a positive constant \(K'\) such that

\[||u||_1 \leq K'.\]

For any \(k > 0\), multiplying the equation for \(u\) by \(u^k\), and using integration by parts and the boundary conditions for \(u\) and \(v\), we have

\[
\int_{\Omega} ku^{k-1}|\nabla u|^2 = \int_{\Omega} k\chi u^k \nabla u \cdot \nabla v + \int_{\Omega} u^{k+1}(\beta F(v/u) - \theta) \leq \int_{\Omega} k\chi u^k \nabla u \cdot \nabla v + C_0 \int_{\Omega} u^{k+1},
\]

where the boundedness of \(\beta F(v/u) - \theta\) has been used.

We now turn to the estimate of \(\int_{\Omega} k\chi u^k \nabla u \cdot \nabla v\). For any \(k > 0\) multiplying the equation for \(v\) with \(\chi u^{k+1}\) and integrating by parts, we deduce that

\[
\chi(k + 1) \int_{\Omega} u^{k+1} \nabla u \cdot \nabla v = \frac{1}{\theta} \left( \int_{\Omega} \chi u^{k+1}vf(v) - \int_{\Omega} \chi u^{k+1}vF(v/u)/v/u \right) \leq C_1 \int_{\Omega} \chi u^{k+1},
\]

where \(C_1 = \frac{K'}{\theta}\). Then

\[
\int_{\Omega} \chi ku^k \nabla u \cdot \nabla v \leq \frac{\chi k C_1}{k + 1} \int_{\Omega} u^{k+1}.
\]

We then deduce from (19) and (20) that

\[
\frac{k}{(k+1)^2} \int_{\Omega} |\nabla u^{k+1/2}|^2 \leq \frac{k}{4} \int_{\Omega} u^{k-1} |\nabla u|^2 \leq \int_{\Omega} k\chi u^k \nabla u \cdot \nabla v + C_0 \int_{\Omega} u^{k+1} \leq \left( \frac{\chi k C_1}{k + 1} + C_0 \right) \int_{\Omega} u^{k+1}.
\]
Thus we get

\[ \int_{\Omega} |\nabla u + \frac{k}{x^2}|^2 \leq C_2(k)(k + 1)^2 \int_{\Omega} u^{k+1} \]  

(21)

where \( C_2(k) = \frac{C_1 \chi}{k^2} + \frac{C_0}{k^2} \leq C_1 \chi + C_0(= \frac{\chi}{k^2} + C_0 \text{ if } k \geq 1) \).

On the other hand, for sufficiently smooth domain \( \Omega \) there exists the classical Sobolev inequality \( \|w\|_q^2 \leq C_3 \left( \|w\|_2^2 + \|\nabla w\|_2^2 \right) \) where if \( n > 2 \) then \( 1 \leq 2q \leq \frac{2n}{n-2} \) and \( C_3 = C_3(n, \Omega) \); if \( n = 2 \) then \( 1 \leq 2q < +\infty \) and \( C_3 = C_3(q, \Omega) \) but in this case one can arbitrarily choose \( 1 \leq q < +\infty \) and fix it; and if \( n = 1 \) then \( C_3 = C_3(\Omega) \) and \( 1 \leq 2q \leq +\infty \). Thus letting \( w = u^{k+1} \) together with (21) one has

\[ \left( \int_{\Omega} u^{(k+1)q} \right)^{\frac{1}{q}} \leq C_3 \int_{\Omega} \left( |\nabla u + \frac{k}{x^2}|^2 + u^{k+1} \right) \leq C_4(k)(k + 1)^2 \int_{\Omega} u^{k+1}, \]

where \( C_3 \) is independent of \( k \). Then we have for any \( k > 0, \)

\[ \|u\|_{k+1, q} \leq (C_4(k + 1)^2)^{\frac{1}{k+1}} \|u\|_{k+1}. \]  

(22)

where \( C_4(k) = C_3 C_2(k) + \frac{C_0}{k^{1/q}} \leq (\frac{\chi}{k^2} + C_0) C_3 =: C_5 \text{ as } k \geq 1 \). Let \( \gamma > 1 \) be fixed. Select appropriate \( q > 1 \) and fix it. Then by assuming \( k \geq 1 \) one can update (22) by \( k + 1 := \gamma q^i \) as follow:

\[ \|u\|_{\gamma q^i + 1} \leq (C_5 \gamma^2 q^{2i})^{\frac{1}{\gamma q^i}} \|u\|_{\gamma q^i}, \quad i = 0, 1, 2, 3, \ldots. \]  

(23)

Iterating (23) repeatedly and using the interpolation inequality (for \( L^p \) norms), lead to

\[ \|u\|_{\infty} \leq C_6 \mu q^r \|u\|_{\gamma} \leq C_6 \mu q^r \|u\|_{\infty} \leq (C_5 \gamma^2 q^{2i})^{\frac{1}{\gamma q^i}} \|u\|_{\gamma q^i}, \]  

(24)

where \( \mu = \sum_{i=0}^{2i} \frac{1}{\gamma q^i} = \frac{q}{\gamma(q-1)}, \nu = 2 \sum_{i=1}^{2i} \frac{i}{\gamma q^i} = \frac{2q}{\gamma(q-1)^2} \), and \( C_6 = \gamma^2 C_5 \). It follows from (18), (23) and (24) that \( \|u\|_{\infty} \leq C_6 \mu q^r \gamma \nu K := K_2 \). This competes the proof.

\[ \square \]

Remark 1. Suppose that the conditions in Lemma 3.1 hold. If \( \chi = 0 \) and \( r > \frac{1}{m} \), then \((u(x), v(x))\) satisfies

\[ \dot{v} \leq v(x) \leq K, \quad 0 < u(x) \leq \frac{\beta K}{m \theta} \quad \text{for all } x \in \Omega. \]

From Proposition 2.2 (\( ii \)), any potential nonconstant positive solutions must bifurcate from \((u^*, v^*)\) under the conditions in (2) of Proposition 2.2 (\( ii \)) and condition (10). Hence, we shall treat prey–tactic coefficient \( \chi > 0 \) as a bifurcation parameter and fix others.

3.1. A standard local bifurcation (Lemma 3.4). Now we define a mapping \( F : X \times \mathbb{R}^+ \to Y \)

\[ F(u, v, \chi) = \left( \begin{array}{c} \nabla \cdot (\nabla u - \chi u \nabla v) + \beta u F(v/u) - \theta u \\ d\Delta v - F(v/u)u + vf(v) \end{array} \right). \]  

(25)

Then \( F(u^*, v^*, \chi) = 0 \) and system (7) is equivalent to

\[ F(u, v, \chi) = 0. \]

The Fréchet derivative of \( F(u, v, \chi) \) with respect to \((u, v)\) at \((u^*, v^*)\) is given by

\[ L(\chi) := D_{(u,v)}F(u^*, v^*, \chi) = \left( \begin{array}{cc} \Delta - \beta F'(w^*)w^* & -\chi u^* \Delta + \beta F'(w^*) \\ -F^2(w^*) & d\Delta + \rho(u^*, v^*) \end{array} \right). \]  

(26)
where \( w_* = \frac{u_*}{v_*} \), \( \rho(u_*, v_*) := -F'(w_*) + f(v_*) + v_*f'(v_*) \) and \( F(z) = \frac{z^+}{m+z} \). Hence \( F(z) - zF'(z) = F(z)^2 \) and \( w_* f(v_*) = F(w_*) \) under (8). By the method in [27], it is easy to obtain that \( L(\chi) \) is a Fredholm operator with index zero. To make the kernel space of the operator \( L(\chi_0^j) \) nonempty, we can set

\[
\chi_j^0 = \frac{v_*f'(v_*) + \frac{F^2(w_*)(1-d\beta m)}{w_*} - d\lambda_j + \beta w_* v_* f'(v_*) F'(w_*) \chi_j^{-1}}{u_* F^2(w_*)} > 0. 
\]

(27)

Since we mainly consider the case \( \chi > 0 \), where \( d > 0 \), \( j \) satisfying (27) is finite as \( \lambda_j \) is strictly increasing with respect to \( j \) (see (13)). Then we can let

\[
\chi_{\text{max}} := \max_{j \in \mathbb{N}_+} \left\{ \chi_j^0 \right\} =: \chi_{j_0}^0 \text{ for some } j_0 \in \mathbb{N}_+.
\]

By the way (27) indicates that

\[
\chi_j^0 u_* F^2(w_*) = -d\lambda_j + \rho(u_*, v_*) - d\beta w_* F'(w_*) + \beta w_* v_* f'(v_*) F'(w_*) \chi_j^{-1} > 0,
\]

(28)

which means that if \( d\lambda_j \chi_j^0 = -\beta w_* v_* f'(v_*) F'(w_*) \), \( i, j \in \mathbb{N}_+ \), then \( \chi_i^j = \chi_j^j \) if and only if \( i = j \).

From [8, Theorem 1.7], \( (u_*, v_*, \chi_0^j) \) is a bifurcation point provided that:

(a) \( D_\chi F, D_{(u,v)} F \) and \( D_{\chi(u,v)} F \) exist and are continuous;

(b) \( \dim \text{Ker}(\chi_0^j) = \text{codim Range} L(\chi_0^j) = 1 \);

(c) Let \( \text{Ker} L(\chi_0^j) = \text{span} \{\Psi\} \). Then \( D_{\chi(u,v)} F(u_*, v_*, \chi_0^j) \Psi \notin \text{Range} (D_{(u,v)} F(u_*, v_*, \chi_0^j)) \),

where co-dimension (abbr. codim) of \( \text{Range} L(\chi_0^j) \) is dimension (abbr. dim) of \( \mathcal{Y}/\text{Range} L(\chi_0^j) \).

A mapping \( L \) from a Banach space \( \mathcal{X} \) to another Banach space \( \mathcal{Y} \) is Fredholm if the dimension of \( \text{Ker}(L) \) and the co-dimension of \( \text{Range}(L) \) are both finite. Moreover, index of Fredholm mapping \( L \) is defined by \( \text{index} [L] = \dim \text{Ker}(L) - \text{codim Range}(L) \). Note that if condition (b) holds then the operator \( L(\chi_0^j) \) is Fredholm with index 0.

One can easily verify that the linear operators \( D_\chi F, D_{(u,v)} F \) and \( D_{\chi(u,v)} F \) are continuous, thus (a) holds. We proceed to verify (b) and (c). The linear operator \( L(\chi_0^j) \) is given by letting \( \chi = \chi_0^j \) in (26) for \( j \in \mathbb{N}_+ \) where \( \chi_0^j \) is from (27).

**Lemma 3.2.** *Condition (b) holds, i.e.,*

\[
\dim \text{Ker} L(\chi_0^j) = \text{codim Range} L(\chi_0^j) = 1 \text{ for } j \in \mathbb{N}_+.
\]

(29)

**Proof.** First of all, we prove

\[
\text{Ker} L(\chi_0^j) \neq \{ 0 \}.
\]

(30)

The null space of \( L(\chi) \) consists of solution to the following system

\[
\begin{aligned}
\Delta u - \chi u \Delta v - \beta F'(w_*) w_* u + \beta F'(w_*) v &= 0, & x \in \Omega, \\
\partial_\nu u = \partial_\nu v &= 0, & x \in \partial \Omega.
\end{aligned}
\]

(31)

We expand \( u \) and \( v \) into their eigenexpansions

\[
u = \sum_{j=0}^{\infty} T_j \Phi_j, \quad v = \sum_{j=0}^{\infty} S_j \Phi_j.
\]

(32)
where $T_j$ and $S_j$ are constants to be determined. Then (30) is equivalent to that there exists at least one $j \in \mathbb{N}$ such that $(T_j, S_j)$ is nontrivial. Substituting (32) into (31), we obtain

$$
\sum_{j=0}^{\infty} \begin{pmatrix} -\lambda_j - \beta F'(w_*) w_* & \lambda_j \chi u_* + \beta F'(w_*) \\ -F^2(w_*) & -d\lambda_j + \rho(u_*, v_*) \end{pmatrix} \begin{pmatrix} T_j \\ S_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

(33)

$j = 0$ can be excluded because $\lambda_0 = 0$ and $-\beta w_* F'(w_*) F'(w_*) > 0$ in light of hypothesis (H1). For each $j \in \mathbb{N}_+$, we can solve (33) directly and get non-constant solutions

$$
(\bar{u}_j, \bar{v}_j) := (Q_j, 1) \Phi_j, \quad Q_j := \frac{\rho(u_*, v_*) - d\lambda_j}{F^2(w_*)} > 0 \quad \text{(by (28))}
$$

(34)

as long as $\chi = \chi_0^j$, $j \in \mathbb{N}_+$. Hence, we have

$$
\text{Ker } \mathcal{L}(\chi_0^j) = \text{span} \{ \Psi_j \}, \quad \Psi_j = \begin{pmatrix} \bar{u}_j \\ \bar{v}_j \end{pmatrix} = \begin{pmatrix} Q_j \\ 1 \end{pmatrix} \Phi_j
$$

(35)

and dim Ker $\mathcal{L}(\chi_0^j) = 1$ for $j \in \mathbb{N}_+$. The adjoint operator is defined by

$$
\mathcal{L}^*(\chi_0^j) = \begin{pmatrix} \Delta - \beta F'(w_*) w_* & -F^2(w_*) \\ -\chi_0^j u_* \Delta + \beta F'(w_*) & d\Delta + \rho(u_*, v_*) \end{pmatrix}.
$$

Similarly,

$$
\text{Ker } \mathcal{L}^*(\chi_0^j) = \text{span} \{ \Psi_j^\ast \}, \quad \Psi_j^\ast = \begin{pmatrix} 1 \\ Q_j^\ast \end{pmatrix} \Phi_j,
$$

where $Q_j^\ast := -\frac{\lambda_j + \beta F'(w_*) w_*}{F^2(w_*)} < 0$. Since Range (L) = Ker $(\mathcal{L}^*)^\perp$, it is easy to derive codim Range $\mathcal{L}(\chi_0^j) = \dim \text{Ker } \mathcal{L}^*(\chi_0^j) = 1$. Therefore, we obtain (29).

Lemma 3.3. Condition (c) holds, i.e.,

$$
D_{\chi(u,v)} \mathcal{F}(u_*, v_*, \chi_0^j) \Psi_j \notin \text{Range } \mathcal{L}(\chi_0^j) \text{ for } j \in \mathbb{N}_+.
$$

(36)

Proof. By direct calculations, we get

$$
D_{\chi(u,v)} \mathcal{F}(u_*, v_*, \chi_0^j) \Psi_j = \begin{pmatrix} u_* \lambda_j \Phi_j \\ 0 \end{pmatrix}.
$$

To prove (36), we shall argue by a contradiction. If (36) is false, then there exists a nontrivial pair $(\bar{u}, \bar{v})$ such that

$$
\begin{cases}
\Delta \hat{u} - \chi_0^j u_* \Delta \hat{v} - \beta F'(w_*) w_* \hat{u} + \beta F'(w_*) \hat{v} = u_* \lambda_j \Phi_j, & x \in \Omega, \\
\partial_{\nu} \hat{u} - \hat{v} = 0, & x \in \partial \Omega.
\end{cases}
$$

(37)

Let

$$
\hat{u} = \sum_{j=0}^{\infty} \bar{T}_j \Phi_j, \quad \hat{v} = \sum_{j=0}^{\infty} \bar{S}_j \Phi_j,
$$

where $\bar{T}_j$ and $\bar{S}_j$ are constants to be determined. Substitute this into (37) and we obtain

$$
\sum_{j=0}^{\infty} \begin{pmatrix} -\lambda_j - \beta F'(w_*) w_* & \lambda_j \chi_0^j u_* + \beta F'(w_*) \\ -F^2(w_*) & -d\lambda_j + \rho(u_*, v_*) \end{pmatrix} \begin{pmatrix} \bar{T}_j \\ \bar{S}_j \end{pmatrix} = \begin{pmatrix} u_* \lambda_j \\ 0 \end{pmatrix}.
$$

From the definition of $\lambda_0^j$ in (27) it is easy to see that this linear system has no solutions. Then we have completed the proof of (36).
Let $Z$ be a closed complement of Ker $(D_{(u,v)}F(u_*,v_*,\chi_0))$ in $X$. Without loss of generality, denote
\[
Z = \left\{(u,v) \in X \left| \int_\Omega (u\bar{u}_j + v\bar{v}_j)dx = 0 \right. \right\}, \tag{38}
\]
where $(\bar{u}_j, \bar{v}_j)$ is from (34). We have the following standard local bifurcation result.

**Lemma 3.4.** *(Local bifurcation)* Suppose that the conditions in (2) of Proposition 2.2 (ii) and (10) hold. Let $\Omega = (0,\ell) \subset \mathbb{R}^1$. If
\[
d\lambda_i \lambda_j \neq -\beta w_* v_* f'(v_*) F'(w_*), \quad i \neq j, i, j \in \mathbb{N}_+,
\]
then for $j \in \mathbb{N}_+$ there exists a constant $\delta > 0$ (small enough) such that the solutions of system (7) around $(u_*, v_*, \chi_0)$ consist precisely of the continuous curve
\[
\Gamma_j(\varepsilon) = \left\{ (u_j(x,\varepsilon), v_j(x,\varepsilon), \chi_j(\varepsilon)) : \varepsilon \in (-\delta, \delta), x \in \Omega \right\}
\]
with
\[
\begin{align*}
(u_j(x,\varepsilon), v_j(x,\varepsilon)) &= (u_*, v_*) + \varepsilon (Q_j, 1) \cos(\pi jx/\ell) + \varepsilon (\xi_j(x,\varepsilon), \zeta_j(x,\varepsilon)), \\
\chi_j(\varepsilon) &= \chi_0^1 + \varepsilon \chi_1 + \varepsilon^2 \chi_2 + o(\varepsilon^2),
\end{align*}
\]
where $\chi_0^1$ is defined by (27), $\chi_1$ and $\chi_2$ are constants, $Q_j$ is from (34), $(\xi_j(x,\varepsilon), \zeta_j(x,\varepsilon))$ is an element in $Z$ defined in (38) with $(\xi_j(x,0), \zeta_j(x,0)) = (0,0)$ and eigenvalues $\lambda_j$ are defined in (13).

**Proof.** Utilizing M.G. Crandall and P.H. Rabinowitz [8, Theorem 1.7], it follows from Lemma 3.2 and 3.3 that local bifurcation does occur at $(u_*, v_*, \chi_0^j)$ for each $j \in \mathbb{N}_+$ satisfying (27). One can further give the expression of $(u_j, v_j)$ in (40) with the help of eigenexpansions available in $L^2(\Omega)$. This completes the proof. \( \square \)

**Remark 2.** The condition (39) implies that $\chi_0^i \neq \chi_0^j$, $i \neq j$, in order that $(u_*, v_*, \chi_0^j)$ is a bifurcation point from simple eigenvalue.

**Remark 3.** It follows from the definition $\chi_e$ in (9) and $\chi_0^j$ below (27) that
\[
\chi_0^j \leq \chi_e
\]
where “$\leq$” holds if and only if $\lambda_j = \left( -\beta u_* v_* f'(v_*) F'(w_*) \right)^{\frac{1}{2}}$, in particular, in one dimension $j_0 = \left( -\beta u_* v_* f'(v_*) F'(w_*) \right)^{\frac{1}{2}} \frac{\ell}{\pi}$ and is an integer. Thus the conclusion (2) of Proposition 2.2 (ii) implies that when $0 < \chi < \chi_0^j$, $(u_*, v_*)$ is always unstable.

**3.2. Proof of Theorem 2.3 (Global bifurcation).** According to the local bifurcation analysis, we have established nonconstant positive local solutions of the system (7) which are small perturbations of $(u_*, v_*)$. We proceed to extend the local curves $\Gamma_j(\varepsilon)$ by the global bifurcation theory.

**Proof of Theorem 2.3** Similar to [13, 21], we first rewrite the system (7) in a form that the standard global bifurcation theory can be applied more conveniently:
\[
\begin{align*}
-\Delta u &= -\chi \nabla u \cdot \nabla v + \frac{\lambda_0}{\ell} \left( vf(v) - F(v/u)u \right) + \beta u F(v/u) - \theta u, \quad x \in \Omega, \\
-\Delta v &= \frac{1}{\ell} \left( vf(v) - F(v/u)u \right), \quad x \in \Omega, \\
\partial_\nu u &= \partial_\nu v = 0, \quad x \in \partial \Omega.
\end{align*}
\]
\[
\begin{align*}
-\Delta u &= -\chi \nabla u \cdot \nabla v + \frac{\lambda_0}{\ell} \left( vf(v) - F(v/u)u \right) + \beta u F(v/u) - \theta u, \\
-\Delta v &= \frac{1}{\ell} \left( vf(v) - F(v/u)u \right), \\
\partial_\nu u &= \partial_\nu v = 0.
\end{align*}
\]
\[
\begin{align*}
-\Delta u &= -\chi \nabla u \cdot \nabla v + \frac{\lambda_0}{\ell} \left( vf(v) - F(v/u)u \right) + \beta u F(v/u) - \theta u, \\
-\Delta v &= \frac{1}{\ell} \left( vf(v) - F(v/u)u \right), \\
\partial_\nu u &= \partial_\nu v = 0.
\end{align*}
\]
Then letting \( \hat{u} = u - u_* \), \( \hat{v} = v - v_* \), (41) is transformed into

\[
\begin{align*}
-\Delta \hat{u} &= -f_0 \hat{u} + f_1 \hat{v} + f_2(\hat{u}, \hat{v}), \quad x \in \Omega, \\
-\Delta \hat{v} &= -g_0 \hat{u} + g_1 \hat{v} + g_2(\hat{u}, \hat{v}), \quad x \in \Omega, \\
\partial_\nu \hat{u} = \partial_\nu \hat{v} &= 0, \quad x \in \partial \Omega,
\end{align*}
\]

(42)

where \( f_2, g_2 \) are higher-order terms of \( \hat{u} \) and \( \hat{v} \) respectively, and

\[
g_0 = \frac{F'(w_*)}{d}, \quad g_1 = \frac{\rho(u_*, v_*)}{d}, \quad f_0 = \chi_* u_* g_0 + \beta F'(w_*) w_*, \quad f_1 = \chi_* u_* g_1 + \beta F'(w_*)
\]

with \( \rho(u_*, v_*) \) from (26). The constant solution \((u_*, v_*)\) of system (7) is transformed to the zero solution \(O = (0, 0)\) of (42). Denote by \( G_\chi \) and \( G \) the inverse of the operators \( f_0 - \Delta \) and \( g_1 - \Delta \) with homogeneous Neumann boundary condition, respectively. Set

\[
U = (\hat{u}, \hat{v}), \quad \mathcal{K}(\chi)U = (f_1 G_\chi(\hat{v}), -g_0 G(\hat{u}) + 2g_1 G(\hat{v}))
\]

Then the boundary value problem (42) is equivalent to the equation

\[
U = \mathcal{K}(\chi)U + \mathcal{W}(\chi, U) := T(\chi, U), \quad \mathcal{K}(\chi) = \begin{pmatrix} 0 & f_1 G_\chi \\ -g_0 G & 2g_1 G \end{pmatrix} \quad \text{in } X.
\]

(43)

Here \( \mathcal{K}(\chi) \) is a compact linear operator on \( X \) for any given \( \chi > 0 \), and \( \mathcal{W}(\chi, U) = o(||U||) \) is also a compact operator on \( X \) both for \( U \) near zero uniformly and for \( \chi \) on closed sub-intervals of \((0, \infty)\).

Next as shown in [13, Theorem 4], to apply the argument in the proof (rather than directly using the conclusion) of Rabinowitz’s global bifurcation theorem ([24, Theorem 1.3]), two preparations are in need as follows.

1. From the first equality of (43), we should verify that 1 is an eigenvalue of \( \mathcal{K}(\chi_0^*) \) of algebraic multiplicity one (hence odd).

Indeed by (35), \( \mathcal{L}(\chi_0^*) \Psi_j = 0 \Leftrightarrow (\mathcal{K}(\chi_0^*) - \mathcal{I}) \Psi_j = 0. \) Thus we know that 1 is an eigenvalue of \( \mathcal{K}(\chi_0^*) \) with the unique eigenfunction \( \Psi_j = \begin{pmatrix} Q_j \\ 1 \end{pmatrix} \Phi_j \) which indicates

\[
\ker (\mathcal{K}(\chi_0^*) - \mathcal{I}) = \ker \mathcal{L}(\chi_0) = \text{span} \{ \Psi_j \} \quad \text{and dim } \ker (\mathcal{K}(\chi_0^*) - \mathcal{I}) = 1 \quad \text{where } \mathcal{I} \text{ is the identity operator.}
\]

We proceed to show the eigenvalue 1 is simple. According to [9, Definition 1.2] and [4], 1 is an algebraically simple eigenvalue of \( \mathcal{K}(\chi_0^*) \) if and only if

\[
\ker (\mathcal{K}(\chi_0^*) - \mathcal{I}) \cap \text{range}(\mathcal{K}(\chi_0^*) - \mathcal{I}) = \{ 0 \}.
\]

Denote \( \mathcal{K}(\chi_0^*) \) by \( \mathcal{K} \). Let \( \mathcal{K}^* \) be the adjoint of \( \mathcal{K} \). We now compute \( \ker (\mathcal{K}^* - \mathcal{I}) \). Let \( (\varphi, \psi) \in \ker (\mathcal{K}^* - \mathcal{I}) \), we have from (43) that

\[
\begin{align*}
-g_0 G(\psi) &= \varphi, \\
f_1 G(\varphi) + 2g_1 G(\psi) &= \psi,
\end{align*}
\]

(44)

where \( f_1 = \chi_0^* u_* g_0 + \beta F'(w_*) w_* \), \( f_1 = \chi_0^* u_* g_1 + \beta F'(w_*) \). According to the definition of \( G_\chi \) and \( G \), (44) can be rewritten as

\[
\begin{align*}
-\Delta \varphi &= -g_1 \varphi - g_0 \psi, \\
-g_0 \Delta \psi &= g_\varphi \varphi + g_\psi \psi,
\end{align*}
\]

(45)
Hence, by (39) or Remark 2, \( \det L \) where

\[
\sum_{i=0}^{\infty} \Phi_i = 0, \quad \sum_{i=0}^{\infty} b_i \Phi_i. \]

By (45), we have

\[
\sum_{i=0}^{\infty} \mathcal{L}_i^* \left( \begin{array}{c} a_i \\ b_i \end{array} \right) = 0, \quad \mathcal{L}_i^* = \left( \begin{array}{cc} -g_1 - \lambda_i & -g_0 \\ g_\psi - g_0 \lambda_i \end{array} \right).
\]

A straightforward calculation gives

\[
\det \mathcal{L}_i^* = \frac{d^2(w_+)(\lambda_i - \lambda_j)}{d^2\lambda_j} (d\lambda_i \lambda_j + \beta w_+ f'(v_+) F'(w_+)).
\]

Hence, by (39) or Remark 2, \( \det \mathcal{L}_i^* = 0 \) if and only if \( i = j \) and

\[
\text{Ker} (K^* - I) = \text{span} \{ \Psi_j \}, \quad \Psi_j = \left( \begin{array}{c} -g_0 \\ g_1 + \lambda_j \end{array} \right) \Phi_j.
\]

In addition, we can check \( \int_0^L \Psi_j \Psi_j' = 2\lambda_j \), which implies that

\[
\Psi_j \notin \text{Ker} (K^* - I)^\perp = \text{Range} (K - I),
\]

and \( \text{Ker} (K - I) \cap \text{Range} (K - I) = \{0\} \). Therefore the algebraic multiplicity of the eigenvalue 1 is one.

Moreover, we can claim that 1 is an eigenvalue of \( K(\chi) \), if and only if \( \chi = \chi_0^j \) (see analysis below (48)). Therefore, if \( 0 < \chi \neq \chi_0^j \) and \( \chi \) lies in a small neighborhood of \( \chi_0^j \), then the linear operator \( I - K(\chi) : X \to X \) is a bijection and thus \( O \) is an isolated solution of (43) for this fixed \( \chi \).

Hence, the index of this isolated zero fixed point of \( I - T(\chi, U) \) is given by

\[
\text{index} (I - T(\chi, U), (\chi, O)) = \text{deg} (I - K(\chi), B, O) = (-1)^\gamma, \quad (46)
\]

where \( B \) is a sufficiently small ball centered at point \( O \) (indicating that \( O \notin \partial B \) and thus Leray–Schauder degree here is well–defined), and \( \gamma \) is the sum of the algebraic multiplicities of the eigenvalues of \( K(\chi) \) greater than 1 (cf. [4]).

Based upon (46), we will obtain that this index changes as \( \chi \) crosses \( \chi_0^j \), i.e.,

\[
\text{index} (I - T(\chi_0^j - \epsilon), (\chi_0^j - \epsilon, O)) \neq \text{index} (I - T(\chi_0^j + \epsilon), (\chi_0^j + \epsilon, O)) \quad (47)
\]

for \( \epsilon > 0 \) sufficiently small.

Indeed if \( \tau \) is an eigenvalue of \( K(\chi) \) with eigenfunction \( (\varphi, \psi) \), then we have

\[
\begin{cases}
-\tau \varphi'' = -\tau f_0 \varphi + f_1 \psi, \\
-\tau \psi'' = -g_0 \varphi + g_1 (2 - \tau) \psi.
\end{cases}
\]

Using the Fourier cosine series \( \varphi = \sum_{i=0}^{\infty} a_i \Phi_i, \quad \psi = \sum_{i=0}^{\infty} b_i \Phi_i \) leads to

\[
\sum_{i=0}^{\infty} \left( \begin{array}{c}
-\tau (f_0 + \lambda_i) \\
-g_0 \\
\end{array} \right)
\left( \begin{array}{c}
f_1 \\
g_1 (2 - \tau) - \tau \lambda_i \\
\end{array} \right)
\left( \begin{array}{c} a_i \\ b_i \end{array} \right) \Phi_i = 0.
\]

Thus the set of eigenvalues of \( K(\chi) \) is composed of all \( \tau \) that solve the characteristic equation

\[
(f_0 + \lambda_i) (g_1 + \lambda_i) \tau^2 - 2 g_1 (f_0 + \lambda_i) \tau + f_1 g_0 = 0, \quad i = 0, 1, 2, \cdots. \quad (48)
\]
Taking \( \chi = \chi_0^j \), if \( \tau = 1 \) is a root of (48), then a simple calculation gives \( \chi_0^j = \chi_0^j \), so \( i = j \) by the assumption (see. Remark 2). Further from (48) if 1 is an eigenvalue of \( K(\chi) \), then \( \chi = \chi_0^j \).

Therefore, *without* counting the eigenvalues corresponding to \( i = j \) in (48), the eigenvalues of \( K(\chi) \) have the *same* number of eigenvalues > 1 for all \( \chi \) close to \( \chi_0^j \) by homotopy invariance of Leray–Schauder degree, and they have the same multiplicities.

On the other hand, for \( i = j \), (48) can be rewritten as
\[
(\tau - 1) (\beta w, v_*) f'(w_*) F'(w_*) + \rho(u_*, v_*) \rho(u_*, v_*) \tau - \rho(u_*, v_*) = 0
\]
where \( \rho(u_*, v_*) \) is from (26), which has the following two roots:
\[
\tau_1(\chi_0^j) = 1, \quad \tau_2(\chi_0^j) = \frac{\rho(u_*, v_*)}{\rho(u_*, v_*) + d\lambda_j} < 1.
\]

Now for \( \chi \) close to \( \chi_0^j \), \( \tau_2(\chi) < 1 \). So when \( \chi \) passes through \( \chi_0^j \), the change of \( \tau_1(\chi) \) with the variable \( \chi \) plays a critical role. Differentiating (48) with respect to \( \chi \), we obtain
\[
\tau'_1(\chi) = -\frac{u_*, g_0 (g_1 + \lambda_i) \tau_1^2 - 2 g_1 \tau_1 + g_1}{2(f_0 + \lambda_i)(g_1 + \lambda_i) \tau_1^2 - g_1} = -\frac{u_*, g_0 (g_1 + \lambda_i) - f_1 g_0}{2(f_0 + \lambda_i)^2((g_1 + \lambda_i) \tau_1^2 - g_1)}
\]
whose sign is determined by
\[
g_1 (f_0 + \lambda_i) - f_1 g_0 = g_1 (\chi u_*, \lambda_i + \beta F'(w_*) w_*) - g_0 (\chi u_*, g_1 + \beta F'(w_*)) + g_1 \lambda_i
\]
\[
= \frac{\beta F'(w_*) w_*}{d} \left( \rho(u_*, v_*) - \frac{F'(w_*)}{w_*} \right) + \frac{\rho(u_*, v_*) \lambda_i}{d}
\]
\[
= \frac{\lambda_i}{d} \left( \rho(u_*, v_*) + \beta w_*, v_*, F'(w_*) f'(v_*) \lambda_j^{-1} \right) > 0 \quad (\text{due to (28)}).
\]
It follows that \( \tau_1(\chi) \) is a decreasing function of \( \chi \) and thus
\[
\tau_1(\chi_0^j + \epsilon) < 1, \quad \text{and} \quad \tau_1(\chi_0^j - \epsilon) > 1.
\]
Consequently, \( K(\chi_0^j + \epsilon) \) has exactly one more eigenvalues (larger than 1) than \( K(\chi_0^j - \epsilon) \) does, and by a similar argument above we can show that this eigenvalue has algebraic multiplicity one. This verifies (47).

Denote the solution set of (7) by
\[
S = \left\{ (u, v, \chi) \in X \times \mathbb{R}^+ : \mathcal{F}(u, v, \chi) = 0, (u, v) \neq (u_*, v_*), u, v \geq 0 \right\}.
\]
From (40) and for small \( |\epsilon| \), there exists an \( \Gamma_j \in S \) for each \( j \) satisfying \( \chi_0^j > 0 \), and thus \( S \) is not empty. In addition, Lemma 3.1 indicates that \( (u, v) \) is bounded and the bound is dependent on \( \chi \). Taking \( 0 < \chi \leq \chi_{\text{max}} \) into consideration indicates that \( S \) is bounded in \( X \times \mathbb{R}^+ \). Denote this bounded domain by \( \mathcal{M} \subset X \times \mathbb{R}^+ \). Let \( \mathcal{C} \) be the maximal connected subset of \( S \) (the closure of \( S \)) to which \( (u_*, v_*, \chi_0^j) \) belongs. For this \( j \) based on (1) and (2), and applying the argument in the proof of [24, Theorem 1.3] (its extended version in [27, Theorem 4.4 and Corollary 1.12]), one can claim that \( \mathcal{C} \) satisfies one of the following alternatives:
(A1) \( \mathcal{C} \) approaches \( \partial \mathcal{M} \) or
(A2) \( \mathcal{C} \) contains another \( (u_j, v_j, \chi_0^k) \) for some \( k \neq j, \chi_0^k > 0 \).

Below we shall show that assertion (A1) holds by a contradiction through a reflective and periodic extension method exactly the same as [13, Theorem 4]. As a matter of fact, if assertion (A2) holds, then we can assume that \( \mathcal{C} \) meets bifurcation
point \((u_*, v_*, \chi^k_0)\) but not \((u_*, v_*, \chi^i_0)\) for any \(i > k\). We reconsider the main system (7) on \((0, I/k)\) as follow
\[
\begin{cases}
-u''(x) = -u'(x)\chi(x) + u(x)\nu(x) + \frac{\beta u(x)v(x)}{m\nu(x)+v(x)} - \theta u(x), & x \in (0, I/k), \\
-dv''(x) = -u'(x)\chi(x) + v(x)f(v(x)), & x \in (0, I/k), \\
u'(x) = 0 = v'(x), & x = 0, I/k.
\end{cases}
\]
(49)

Note that if \(U(x) := (u(x), v(x))\) solves (49), then one can construct a solution to (7) by a reflective and periodic extension: Define \(x_n = nI/k, n = 0, 1, 2, \ldots, k\) and
\[
\tilde{U} = \begin{cases}
U(x - x_{2n}), & \text{if } x_{2n} \leq x \leq x_{2n+1}; \\
U(x_{2n+2} - x), & \text{if } x_{2n+1} \leq x \leq x_{2n+2}.
\end{cases}
\]
(50)

From lemma 3.4, this operation does not change the bifurcation structure of solutions to (7), and then \((u_*, v_*, \chi^k_0)\) is also a bifurcation point of problem (49). We denote by \(C_k\) this bifurcating branch \(C \cup (u_*, v_*, \chi^k_0)\) of this new problem.

For this new problem, we still have that \(C_k\) approaches \(\partial M\) or meets another \((u_*, v_*, \chi^{k'}_0)\) for some \(k' > k\) (note that if \(k' < k\), then using the same method as above, after finite steps, we can only obtain (A1) since the number of bifurcation point \((u_*, v_*, \chi^k_0)\) with \(k' < k\) is finite). If the latter holds, by the same discussion as above we can construct a new system through the reflective and periodic extension and it contains \(C_k\) as its one of bifurcating branches. However, this contradicts that \(C_k\) does not meet \((u_*, v_*, \chi^i_0)\) for any \(i > k\). Thus system (49) only has (A1).

Now we know that for system (7), \(C\) at most meets finite bifurcation points like \((u_*, v_*, \chi^k_0)\) in (A2) but after that, the connected component including \(C \cup (u_*, v_*, \chi^k_0)\) meets \(\partial M\). In particular, the projection of the branch \(C_k\) on \(\chi\)-axis contains \((0, \chi_{\text{max}})\). According to \(S\), this branch is a nonnegative nonconstant solution to (7). Again from the \(a\ priori\) estimate in Lemma 3.1, any nonnegative solutions to (7) are positive. Thus such a nonnegative nonconstant solution to (7) is positive. Then Theorem 2.3 is proved.

\[\square\]

3.3. Proof of Theorem 2.4. In this subsection, we investigate the stability of the spatially inhomogeneous solution \((u_j(x, \varepsilon), v_j(x, \varepsilon))\) that bifurcates from \((u_*, v_*)\) at \(\chi = \chi^i_0\) in one dimensional domain \(\Omega = (0, I)\), based on the results of Crandall and Rabinowitz in [9].

Since \(F\) defined in (25) is \(C^4\)-smooth, according to [8, Theorem 1.18], \((u_j(x, \varepsilon), v_j(x, \varepsilon), \chi_j(\varepsilon))\) are \(C^3\)-smooth functions of \(\varepsilon\) and we can have the following expansions:
\[
\begin{align*}
&u_j(x, \varepsilon) = u_* + \varepsilon Q_j \Phi_j(x) + \varepsilon^2 \varphi_1(x) + \varepsilon^3 \varphi_2(x) + o(\varepsilon^3), \\
v_j(x, \varepsilon) = v_* + \varepsilon \Phi_j(x) + \varepsilon^2 \psi_1(x) + \varepsilon^3 \psi_2(x) + o(\varepsilon^3), \\
&\chi_j(\varepsilon) = \chi^i_0 + \varepsilon \chi_1 + \varepsilon^2 \chi_2 + o(\varepsilon^2),
\end{align*}
\]
(51)
where \((\varphi_i, \psi_i) \in Z\) for \(i = 1, 2\) and \(\chi_1, \chi_2\) are constants. Note that the \(o(\varepsilon^3)\) terms in \(u_j(x, \varepsilon)\) and \(v_j(x, \varepsilon)\) are measured in the normal \(H^2\)-norms.

Before discussing the stability of such bifurcating solutions, we begin with determination of the coefficients in the expansion of \(\chi_j(\varepsilon)\).
Substituting (51) into system (7) and equating the $\varepsilon^2$-term, we have

$$
\begin{cases}
\Delta \varphi_1 - \chi_0 U_s \Delta \psi_1 - \beta F'(w_\ast) w_\ast \varphi_1 + \beta F'(w_\ast) \psi_1 = F_1, & x \in \Omega, \\
d\Delta \psi_1 - F^2(w_\ast) \varphi_1 + \rho(u_\ast, v_\ast) \psi_1 = F_2, & x \in \Omega, \\
\partial_n \varphi_1 = \partial_n \psi_1 = 0, & x \in \partial \Omega,
\end{cases}
$$

(52)

where

$$
F_1 = Q_j \chi_0^{1/2} \nabla (\Phi_j \nabla \Phi_j) + \chi_1 U_s \Delta \Phi_j - \frac{\beta F''(w_\ast)(w_\ast Q_j - 1)^2 \Phi_j^2}{2u_\ast},
$$

$$
F_2 = \frac{1}{2u_\ast} \left( F''(w_\ast) w_\ast Q_j (w_\ast Q_j - 2) - \left( f''(v_\ast) w_\ast u_\ast^2 + 2 f'(v_\ast) u_\ast - F''(w_\ast) \right) \Phi_j^2 \right),
$$

Multiplying the second equation in (52) by $\Phi_j$ and integrating it over $\Omega$ lead us to

$$
-F^2(w_\ast) \int_\Omega \varphi_1 \Phi_j dx + (-d\lambda_j + \rho) \int_\Omega \psi_1 \Phi_j dx = 0,
$$

(53)

where we have used (11), (38) and $\int_\Omega \Phi_j dx = 0$ ($j > 0$).

Since $(\varphi_1, \psi_1) \in Z$ is defined in (38), we have $\int_\Omega (\varphi_1 \bar{u}_j + \psi_1 \bar{v}_j) dx = 0$, i.e.,

$$
(d\lambda_j - \rho) \int_\Omega \varphi_1 \Phi_j dx - F^2(w_\ast) \int_\Omega \psi_1 \Phi_j dx = 0.
$$

(54)

From (53) and (54), we arrive at the following system

$$
M \begin{pmatrix} \int_\Omega \varphi_1 \Phi_j dx \\ \int_\Omega \psi_1 \Phi_j dx \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
$$

where

$$
M = \begin{pmatrix} -F^2(w_\ast) & -d\lambda_j + \rho(u_\ast, v_\ast) \\ d\lambda_j - \rho(u_\ast, v_\ast) & -F^2(w_\ast) \end{pmatrix}.
$$

Obviously,

$$
\det(M) = F^4(w_\ast) + (d\lambda_j - \rho(u_\ast, v_\ast))^2 \neq 0,
$$

and then

$$
\int_\Omega \varphi_1 \Phi_j dx = \int_\Omega \psi_1 \Phi_j dx = 0.
$$

(55)

Multiplying the first equation in (52) by $\Phi_j$ and integrating it over $\Omega$ lead us to

$$
(-\lambda_j - \beta F'(w_\ast) u_\ast) \int_\Omega \varphi_1 \Phi_j dx + (\lambda_j \chi_0^{1/2} u_\ast + \beta F'(w_\ast)) \int_\Omega \psi_1 \Phi_j dx
$$

$$
= \int_\Omega \left( Q_j \chi_0^{1/2} \nabla (\Phi_j \nabla \Phi_j) + \chi_1 U_s \Delta \Phi_j \right) \Phi_j dx = -\chi_1 U_s \lambda_j \int_\Omega \Phi_j^2 dx.
$$

(56)

It follows from (55) that $\chi_1 = 0$. As an auxiliary result, above we have proved the following observation.

**Lemma 3.5.** (Bifurcation of pitchfork type) If $\chi_i \neq \chi_j$ for any positive integer $i \neq j$, then $\chi_1 = 0$ and the local bifurcation curve $\Gamma_j(\varepsilon)$ of system (7) at $(u_\ast, v_\ast, \chi_0)$ is of pitchfork type if $\chi_2 \neq 0$. 

show that the each eigenvalue according to the principle of linearized stability (see [9, Theorem 1.16]), it remains to part:

Case 1: \( \Gamma \)

Now we turn to proving the stability of bifurcation solutions.

**Proof of Theorem 2.4.**

To consider the stability of \((u_j(\varepsilon, x), v_j(\varepsilon, x), \chi_j(\varepsilon))\) around \((u_*, v_*, \chi_0^*)\), we linearize system (7) around this bifurcating solution. According to the principle of linearized stability (see [9, Theorem 1.16]), it remains to show that the each eigenvalue \( \sigma \) of the following elliptic problem has negative real part:

\[
D_{(u,v)} F(u_j(\varepsilon, x), v_j(\varepsilon, x), \chi_j(\varepsilon))(u, v) = \sigma \cdot (u, v), \quad (u, v) \in X,
\]

which is equivalent to

\[
\begin{align*}
\nabla \cdot (\nabla u - \chi_j(\varepsilon)(u\nabla v_j(\varepsilon, x) + u_j(\varepsilon, x)\nabla v)) &- \beta F'(z_j(\varepsilon, x))z_j(\varepsilon, x)u \\
+ \beta F'(z_j(\varepsilon, x))v + \beta F(z_j(\varepsilon, x))u - \theta u &\quad = \sigma u, \quad x \in \Omega, \\
\quad \quad d\Delta v - F^2(z_j(\varepsilon, x))u + (v_j(\varepsilon, x)f'(v_j(\varepsilon, x)) + f(v_j(\varepsilon, x))) \\
- F'(z_j(\varepsilon, x))v &\quad = \sigma v, \quad x \in \Omega, \\
\partial_\nu u = \partial_\nu v &\quad = 0, \quad x \in \partial \Omega,
\end{align*}
\]

where \( u_j(\varepsilon, x), v_j(\varepsilon, x) \) and \( \chi_j(\varepsilon) \) are defined in Lemma 3.4 and \( z_j(\varepsilon, x) = \frac{v_j(\varepsilon, x)}{u_j(\varepsilon, x)} \).

At this moment we will discuss the stability of bifurcating solutions \( \Gamma_j \) in the following two aspects.

**Case 1:** \( \Gamma_j(\varepsilon) \) with \( j \in \mathbb{N}_+ \setminus \{j_0\} \) for \(|\varepsilon|\) being small, where \( j_0 \) is from Remark 3.

In this case, by taking \( \varepsilon = 0 \), we see that 0 is a simple eigenvalue of \( D_{(u,v)} F(u_*, v_*, \chi_0^*) \)

with an eigenspace equal to \( \text{span}(\tilde{u}_j, \tilde{v}_j) \). As \( j \neq j_0 \), we already know from (2) of Proposition 2.2 (ii) and Remark 3 that when \( \varepsilon = 0 \), (57) has an eigenvalue with positive real part. Then by the standard eigenvalue perturbation theory (e.g. in [32]), it always has a positive root for small \( \varepsilon \). This implies that the bifurcation branch \( \Gamma_j(\varepsilon) \) around \((u_*, v_*, \chi_0^*)\) is unstable for each \( j \in \mathbb{N}_+ \setminus \{j_0\} \).

**Case 2:** \( \Gamma_{j_0}(\varepsilon) \) for \(|\varepsilon|\) being small.

In such a case, it follows from [9, Corollary 1.13] that there exist an interval \( I \) containing \( \chi_0^{j_0} \) and two continuously differentiable functions \( \chi \in I \rightarrow \mu(\chi), \varepsilon \in (-\delta, \delta) \rightarrow \sigma(\varepsilon) \) with \( \sigma(0) = 0 \) and \( \mu(\chi_0^{j_0}) = 0 \) such that \( \sigma(\varepsilon) \) is an eigenvalue of (57) and \( \mu(\chi) \) is an eigenvalue of the following eigenvalue problem

\[
D_{(u,v)} F(u_*, v_*, \chi)(u, v) = \mu \cdot (u, v), \quad (u, v) \in X.
\]

Moreover, \( \sigma(\varepsilon) \) is the only eigenvalue of (57) in any fixed neighborhood of the origin in the complex plane (the same assertion can be made on \( \mu(\chi) \)).

We also know from [9] that the eigenfunctions of (58) can be represented by \((u(\chi, x), v(\chi, x))\) which smoothly depend on \( \chi \) and are uniquely determined through \((u(\chi_0^{j_0}, x), v(\chi_0^{j_0}, x)) = (Q_{j_0}\Phi_{j_0}, \Phi_{j_0})\) together with \((u(\chi_0^{j_0}, x) - Q_{j_0}\Phi_{j_0}, v(\chi_0^{j_0}, x) - \Phi_{j_0}) \in Z \).
arrive at the following system
\[
\begin{cases}
\Delta(\dot{u} - u_\epsilon \Phi_{j_0} - \chi^{j_0}_0 u_\epsilon \dot{v}) - \beta F'(w_*) w_* \dot{u} + \beta F'(w_*) \dot{v} = \dot{\mu}(\chi^{j_0}_0) Q_{j_0} \Phi_{j_0}, & x \in \Omega \\
d \Delta \dot{v} - F^2(w_*) \dot{u} + \rho(u_*, v_*) \dot{v} = \dot{\mu}(\chi^{j_0}_0) \Phi_{j_0}, & x \in \Omega \\
\partial_{\nu} \dot{u} = \partial_{\nu} \dot{v} = 0 & x \in \partial \Omega, \quad (59)
\end{cases}
\]
where \( \dot{u} = \frac{\partial u(x, \epsilon)}{\partial \chi} \bigg|_{\chi=\chi^{j_0}_0}, \dot{v} = \frac{\partial v(x, \epsilon)}{\partial \chi} \bigg|_{\chi=\chi^{j_0}_0}, \) and \( \dot{\mu}(\chi^{j_0}_0) = \frac{d\mu(\chi)}{d\chi} \bigg|_{\chi=\chi^{j_0}_0}. \) Multiplying both equations of (59) by \( \Phi_{j_0} \) and then integrating over \( \Omega \) by parts, we obtain that
\[
\begin{pmatrix}
-\lambda_{j_0} - \beta F'(w_*) w_* \\
-\beta F'(w_*)
\end{pmatrix}
\begin{pmatrix}
\lambda_{j_0} \chi^{j_0}_0 u_* + \beta F'(w_*) \\
- \lambda_{j_0} + \rho(u_*, v_*)
\end{pmatrix}
= \begin{pmatrix}
\int_\Omega \dot{\mu}(\chi^{j_0}_0) Q_{j_0} - \lambda_{j_0} u_* \\
\int_\Omega \dot{\mu}(\chi^{j_0}_0)
\end{pmatrix}.
\]
With the coefficient matrix being singular and to make the system above solvable, one can set
\[
\frac{\lambda_{j_0} + \beta F'(w_*) w_*}{F^2(w_*)} = \frac{\dot{\mu}(\chi^{j_0}_0) Q_{j_0} - \lambda_{j_0} u_*}{\dot{\mu}(\chi^{j_0}_0)}
\]
which means
\[
\dot{\mu}(\chi^{j_0}_0) = \frac{\lambda_{j_0} u_* F^2(w_*)}{Q_{j_0} F^2(w_*) - (\lambda_{j_0} + \beta F'(w_*) w_*)} = \frac{\rho(u_*, v_*) - \beta F'(w_*) w_* - (d + 1) \lambda_{j_0}}{\lambda_{j_0} u_* F^2(w_*) - (d + 1) \lambda_{j_0}} < 0.
\]
The final inequality holds due to \((d + 1) \lambda_{j_0} > 0\) and
\[
\rho(u_*, v_*) - \beta F'(w_*) w_* - (d + 1) \lambda_{j_0} < \rho(u_*, v_*) - \beta F'(w_*) w_* \leq 0.
\]
In fact from Proposition 2.2 (ii),

• if \( m\beta > 1 \) then owing to \( v_* f'(v_*) < 0 \) and by (8)
\[
\rho(u_*, v_*) - \beta F'(w_*) w_* = -F'(w_*) + f(v_*) + v_* f'(v_*) - \beta w_* F'(w_*) \\
\leq - F'(w_*) + f(v_*) - \beta w_* F'(w_*) \\
= \frac{w_* (1 - m\beta)}{(m + w_*)^2} < 0;
\]

• if \( m\beta < 1 \) and \( v_* f'(v_*) + \frac{\theta (\beta - \theta)(1 - m\beta)}{m\beta^2} < 0 \) then
\[
\rho(u_*, v_*) - \beta F'(w_*) w_* = -F'(w_*) + f(v_*) + v_* f'(v_*) - \beta w_* F'(w_*) \\
\leq -F'(w_*) + f(v_*) - \beta w_* F'(w_*) - \frac{\theta (\beta - \theta)(1 - m\beta)}{m\beta^2} = 0.
\]

By [9, Theorem 1.16], for \( \varepsilon \in (-\delta, \delta) \), the functions \( \sigma(\varepsilon) \) and \(-\varepsilon \chi^{j_0}_0(\varepsilon) \dot{\mu}(\chi^{j_0}_0)\) have the same zeros and the same signs. Moreover
\[
\lim_{\varepsilon \to 0, \sigma(\varepsilon) \neq 0} \frac{-\varepsilon \chi^{j_0}_0(\varepsilon) \dot{\mu}(\chi^{j_0}_0)}{\sigma(\varepsilon)} = 1.
\]
Now it follows from \( \chi_1 = 0 \) that
\[
\lim_{\varepsilon \to 0, \sigma(\varepsilon) \neq 0} \frac{2\varepsilon^2 \chi_2 \dot{\mu}(\chi^{j_0}_0)}{\sigma(\varepsilon)} = -1
\]
and L'Hospital's rule gives \( \sigma(\varepsilon) = \text{sign}(\chi_2) \) for \( \varepsilon \in (-\delta, \delta) \). Thus Theorem 2.4 is proved.

4. **Numerical simulation of spatial patterns.** For completeness and comparison with [5], in this section we shall perform numerical simulations in an interval \( \Omega = [0, \ell] \) to illustrate the possible pattern formation of the system (6) through the MATLAB pde solver based upon the finite difference scheme. Without loss of generality, we select the initial value \((u_0, v_0)\) of system (6) to be a small perturbation on its homogeneous steady state \((u_\ast, v_\ast)\), namely,

\[
\begin{align*}
  u_0 &= u_\ast + 0.001 \cdot \sin(\pi x / n), \\
  v_0 &= v_\ast + 0.001 \cdot \sin(\pi x / n),
\end{align*}
\]

where \( n \) is a real number flexibly chosen to make the patterns more abundant. We below explore the numerical solutions of the system (6) with logistic growth of prey density, i.e.

\[
f(v) = r \left( 1 - \frac{v}{K} \right),
\]

and with prey-taxi coefficient \( \chi \) taking different values.

To obtain fruitful patterns, all parameters chosen in system (6) ought to satisfy the conditions in (2) and (10) of Proposition 2.2 (ii) and (C). There is a particular example as follow if we choose corresponding parameters to be

\[
n = 0.001, \quad d = 0.01, \quad \beta = 0.03, \quad \theta = 0.01, \quad r = 0.8, \quad m = 0.95, \quad K = 3, \quad \ell = 100,
\]

then according to (9) and (27),

\[
(u_\ast, v_\ast) = (0.775623, 0.368421), \quad j_0 = 16, \quad \chi_{\text{max}} = \chi_j^{\text{th}} = 1.5141146155 < \chi_c = 1.5141195026.
\]

As is shown in figure 1, we can see how prey-taxis rate \( \chi \) affects pattern formations in the system (6) with logistic type and summarize them as follow:

- The prey \( v(x, t) \) is always more concentrated (or denser) than predator \( u(x, t) \).
- When \( \chi \leq \chi_{\text{max}} \) and \( \chi \) approaches to 0, the patterns are generated. As \( \chi \) tends to \( \chi_{\text{max}} \), the patterns eventually disappear when time \( t \) becomes large enough.
- From the spatial distribution of \((u, v)\) at \( t = 1000 \) (the third column in the figure 1) we observe that smaller \( \chi \) supports more aggregates than larger \( \chi \) does.

As indicated in Proposition 2.2 (ii), we can only make sure that when \( \chi > \chi_c \), the patterns will disappear as time becomes large enough, but cannot ascertain the exact bounds of \( \chi(\leq \chi_c) \) which allow us to predict the starting point of the disappearance and appearance of pattern when \( \chi \in (0, \chi_c) \). For example, in figure 1, this bounded interval should be included in \((0, 0.6) \subset (0, \chi_c) \) since we only consider the case \( \chi > 0 \) in this paper.

**Appendix A. Calculation of \( \chi_2 \).** In this appendix, we calculate the sign of \( \chi_2 \) in Lemma 3.6. To this end, we need to collect the \( \varepsilon^3 \)-term in system (7).

Since \( \chi_1 = 0 \), we readily see that collecting the \( \varepsilon^3 \)-term in system (7) leads us to

\[
\begin{align*}
\Delta \varphi_2 - \chi_0 u_\ast \Delta \psi_2 - \beta F'(w_\ast)w_\ast \varphi_2 + \beta F'(w_\ast)\psi_2 &= F_3, & x \in \Omega, \\
d \Delta \psi_2 - F'(w_\ast)\psi_2 + \rho(u_\ast, v_\ast)\psi_2 &= F_4, & x \in \Omega, \\
\partial_n \varphi_2 = \partial_n \psi_2 &= 0, & x \in \partial \Omega,
\end{align*}
\]

where
Figure 1. Pattern formation of system (6) where $x \in (0, 100)$ with grid number 1000 and $t \in (0, 8000)$ with grid number 1000.
where
\[ F_3 = Q_j \chi_0 \nabla (\Phi_j \nabla \psi_j) + \chi_0 \nabla (\phi_j \nabla \Phi_j) + \chi_2 u_\omega \Delta \Phi_j - \frac{\beta F''(w_\omega) w_\omega (w_\omega Q_j - 1)}{w_\omega} \nabla \phi_j \]
+ \frac{\beta F''(w_\omega) (w_\omega Q_j - 1)}{w_\omega} \psi_1 \Phi_j + \frac{\beta (w_\omega Q_j - 1)^2 (F''(w_\omega) w_\omega Q_j - F'''(w_\omega) + 3 F''(w_\omega) Q_j)}{6 u_\omega^2} \phi_j^3
\]
:= Q_j \chi_0 \nabla (\Phi_j \nabla \psi_j) + \chi_0 \nabla (\phi_j \nabla \Phi_j) + \chi_2 u_\omega \Delta \Phi_j - F_{31} \phi_1 \Phi_j + F_{32} \phi_2 \Phi_j + F_{33} \phi_j^3.

Multiplying (52) by \( \Phi_j \), we have
\[
J \left( \int_\Omega \frac{\phi_1 dx}{\psi_1 dx} \right) = \begin{pmatrix} -F_{10} \\ F_{20} \end{pmatrix},
\]
where \( J = \begin{pmatrix} -\beta F''(w_\omega) w_\omega & \beta F'(w_\omega) \\ -F''(w_\omega) & \rho(u, v) \end{pmatrix} \). Solving (62) gives us
\[
\int_\Omega \phi_1 dx = D_1, \quad \int_\Omega \psi_1 dx = D_2,
\]
where \( D_1 = \frac{\rho(u_\omega, v_\omega) F_{10} + \beta F'(w_\omega) F_{20}}{-\det(J)} \), \( D_2 = \frac{(F''(w_\omega) w_\omega - F'(w_\omega)) F_{10} - \beta F'(w_\omega) w_\omega F_{20}}{\det(J)} \).

Multiplying (52) by \( \Phi_j(2x) \) and integrating it over \( \Omega \) by part, we have
\[
\Pi \left( \int_\Omega \frac{\phi_1 \Phi_j(2x) dx}{\psi_1 \Phi_j(2x) dx} \right) = \left( \frac{\sqrt{2}}{7} \right)^3 \begin{pmatrix} -\frac{4}{3} F_{10} - \frac{\beta^2 x^2}{2} Q_j \chi_0^2 \\ \frac{2}{3} F_{20} \end{pmatrix},
\]
where \( \Pi = \begin{pmatrix} -4 \lambda_j - \beta F'(w_\omega) w_\omega & 4 \lambda_j \chi_0^2 u_\omega + \beta F'(w_\omega) \\ -F^2(w_\omega) & -4d \lambda_j + \rho \end{pmatrix} \). Since \( \chi_0^2 \neq \chi_j^2 \) for any positive integer \( i \neq j \), we have \( \det(\Pi) \neq 0 \). Solving (64) gives us
\[
\int_\Omega \phi_1 \Phi_j(2x) dx = E_1, \quad \int_\Omega \psi_1 \Phi_j(2x) dx = E_2,
\]
where
\[
E_1 = -\left( \frac{\sqrt{2}}{7} \right)^3 \frac{\beta^2 F_{10}(-4d \lambda_j + \rho) + \beta F_{20} (4 \lambda_j \chi_0^2 u_\omega + \beta F'(w_\omega)) + 2 \pi^2 \lambda_j^2 Q_j \chi_0^2 (-4d \lambda_j + \rho)}{4 \det(\Pi)},
\]
\[
E_2 = -\left( \frac{\sqrt{2}}{7} \right)^3 \frac{F^2(w_\omega) F_{10} + \beta F_{20} (4 \lambda_j \chi_0^2 u_\omega + \beta F'(w_\omega)) + 2 \pi^2 \lambda_j^2 Q_j \chi_0^2 F^2(w_\omega)}{4 \det(\Pi)}.
\]

Multiplying the second equation in (61) by \( \Phi_j(x) \) and integrating it over \( \Omega = (0, \ell) \) lead us to
\[
- F^2(w_\omega) \int_\Omega \phi_2 \Phi_j(x) dx + (-d \lambda_j + \rho) \int_\Omega \psi_2 \Phi_j(x) dx = A_0,
\]
(66)
where
\[
A_0 = F_{41} \int_\Omega \varphi_2 \Phi_1^2(x)dx - F_{42} \int_\Omega \psi_1 \Phi_1^2(x)dx - \frac{F_{43}}{6} \int_\Omega \Phi_1^4(x)dx
\]
\[
= \frac{F_{41}}{\sqrt{2}} \int_\Omega \varphi_1 \Phi_1(x)dx - \frac{F_{42}}{\sqrt{2T}} \int_\Omega \psi_1 \Phi_1(x)dx + \frac{F_{43}}{T} \int_\Omega \varphi_1 dx - \frac{F_{42}}{T} \int_\Omega \psi_1 dx - \frac{F_{43}}{4T} = A^*.
\]
Since \((\varphi_2, \psi_2) \in Z\) as defined in (38), we have
\[
(d\lambda_j - \rho) \int_\Omega \varphi_2 \Phi_j(x)dx - F^2(w_*) \int_\Omega \psi_2 \Phi_j(x)dx = 0.
\]
From (66) and (67), we arrive at the following system
\[
M \begin{pmatrix} 
\int_\Omega \varphi_2 \Phi_j(x)dx \\
\int_\Omega \psi_2 \Phi_j(x)dx
\end{pmatrix} = \begin{pmatrix} 
A^* \\
0
\end{pmatrix},
\]
where \(M\) is defined in section 3.3. Then, solving (68) gives
\[
\int_\Omega \varphi_2 \Phi_j(x)dx = B_1, \quad \int_\Omega \psi_2 \Phi_j(x)dx = B_2,
\]
where \(B_1 = -\frac{A^*F^2(w_*)}{\det(M)}, B_2 = -\frac{A^*}{\det(M)}(d\lambda_j - \rho).
\]
Multiplying the first equation in (61) by \(\Phi_j(x)\) and integrating it over \(\Omega\) lead us to
\[
(-\lambda_j - \beta F'(w_*)u_*) \int_\Omega \varphi_2 \Phi_j(x)dx + (\lambda_j \chi_0^0 u_* + \beta F'(w_*)) \int_\Omega \psi_2 \Phi_j(x)dx
\]
\[
= Q_j \lambda_0^0 \int_\Omega \nabla(\Phi_j \nabla \psi_1) \Phi_j(x)dx + \lambda_j^0 \int_\Omega \nabla(\varphi_1 \nabla \Phi_j) \Phi_j(x)dx - \chi_2 \lambda_j \int_\Omega \Phi_j^2(x)dx
\]
\[
- \frac{F_{41}^2}{\sqrt{2}} \int_\Omega \varphi_1 dx - \frac{F_{41}}{\sqrt{2T}} \int_\Omega \varphi_1 \Phi_1^2(x)dx + \frac{F_{42^2}}{T} \int_\Omega \psi_1 dx + \frac{F_{42}}{\sqrt{2T}} \int_\Omega \psi_1 \Phi_1^2(x)dx + \frac{3}{2T} F_{33}. \tag{70}
\]
Since
\[
\int_\Omega \nabla(\Phi_j(x) \nabla \psi_1) \Phi_j(x)dx = \int_\Omega \psi_1(\Phi_j^2(x) + \Phi_j \Phi_j^\prime(x))dx = -\lambda_j \sqrt{T^2} \int_\Omega \psi_1 \Phi_j^2(x)dx = -\lambda_j \sqrt{T^2} E_2,
\]
and
\[
\int_\Omega \nabla(\varphi_1 \nabla \Phi_j) \Phi_j(x)dx = - \int_\Omega \varphi_1 \Phi_j^2(x)dx = \lambda_j^0 \int_\Omega \varphi_1(\sqrt{T^2} \Phi_j^2(x) - 1)dx = \lambda_j^0 \left( \sqrt{T^2} E_1 - D_1 \right),
\]
we have
\[
\lambda_j u_* \chi_2 = (\lambda_j + \beta F'(w_*)u_*) \int_\Omega \varphi_2 \Phi_1(x)dx - (\lambda_j \chi_0^0 u_* + \beta F'(w_*)) \int_\Omega \psi_2 \Phi_j(x)dx
\]
\[
- \lambda_j \sqrt{T^2} Q_j \lambda_0^0 \int_\Omega \psi_1 \Phi_1(x)dx + \lambda_j \sqrt{T^2} \int_\Omega \varphi_1 \Phi_1^2(x)dx - \frac{\lambda_j \chi_0^0}{T} \int_\Omega \varphi_1 dx
\]
\[
- \frac{F_{41}^2}{\sqrt{2}} \int_\Omega \varphi_1 dx - \frac{F_{41}}{\sqrt{2T}} \int_\Omega \varphi_1 \Phi_1^2(x)dx + \frac{F_{42^2}}{T} \int_\Omega \psi_1 dx + \frac{F_{42}}{\sqrt{2T}} \int_\Omega \psi_1 \Phi_1^2(x)dx + \frac{3}{2T} F_{33}
\]
\[
= (\lambda_j + \beta F'(w_*)u_*) \int_\Omega \varphi_2 \Phi_j(x)dx - (\lambda_j \chi_0^0 u_* + \beta F'(w_*)) \int_\Omega \psi_2 \Phi_j(x)dx
\]
\[
+ \left( \frac{F_{42^2}}{\sqrt{2T}} - \lambda_j \sqrt{T^2} Q_j \lambda_0^0 \right) \int_\Omega \psi_1 \Phi_1(2x)dx + \left( \lambda_j \chi_0^0 - \frac{F_{41}^2}{\sqrt{2T}} \right) \int_\Omega \psi_1 \Phi_1^2(x)dx
\]
\[
- \lambda_j \sqrt{T^2} Q_j \lambda_0^0 E_2 - \left( \frac{\lambda_j \chi_0^0}{T} + \frac{F_{41}}{T} \right) D_1 + \frac{F_{42}}{T} D_2 + \frac{3}{2T} F_{33}, \tag{71}
\]
where $B_1 = \int_{\Omega} \varphi_2 \Phi_j(x) dx$ and $B_2 = \int_{\Omega} \psi_2 \Phi_j(x) dx$.

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