Cohomology of Deligne-Lusztig varieties for groups of type $A$

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Abstract

We study the cohomology of parabolic Deligne-Lusztig varieties associated to unipotent blocks of $GL_n(q)$. We show that the geometric version of Broué’s conjecture over $\mathbb{Q}_\ell$, together with Craven’s formula, holds for any unipotent block whenever it holds for the principal $\Phi_1$-block.

Introduction

Let $G$ be a connected reductive algebraic group over $F = \mathbb{F}_p$ with an $\mathbb{F}_q$-structure associated to a Frobenius endomorphism $F$. Let $\ell$ be a prime number different from $p$ and $b$ be a unipotent $\ell$-block of $G^F = G(\mathbb{F}_q)$. When $\ell$ is large, the defect group of $b$ is abelian, and the geometric version of Broué’s conjecture predicts that the cohomology of some Deligne-Lusztig variety should induce a derived equivalence between $b$ and its Brauer correspondent $b^\alpha$.

When the centraliser of the defect group of $b$ is a torus, then in [4] Broué and Michel identified which specific class of Deligne-Lusztig varieties should be considered. They correspond to good $d$-regular elements or equivalently to $d$-roots of $\pi = w_0^2$ in the Braid monoid. In a recent work [9], Digne and Michel introduced the notion of $d$-periodic element to generalise this to the parabolic setting. If $b$ is a unipotent $\Phi_d$-block, then it is to be expected that there exists a $d$-periodic element $(I, w)$ such that the corresponding parabolic Deligne-Lusztig variety $\tilde{X}(I, w^F)$ is a good candidate for inducing the derived equivalence predicted by Broué’s conjecture. Furthermore, Chuang and Rouquier conjectured in [6] that this equivalence is perverse, with a perversity function that has recently been conjectured by Craven in [7]. Surprisingly, it can be expressed by a function $C_d$ depending only on the generic degrees of the corresponding characters.

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If we restrict our attention to the characteristic zero then we obtain a conjectural explicit description of the unipotent part of the cohomology of $\widetilde{X}(\mathcal{I}, wF)$. The fundamental property that we derive from Broué's conjecture is that the cohomology groups of $X(\mathcal{I}, wF)$ are mutually disjoint. More precisely, it can be formulated as follows:

**Conjecture 1.** Any $d$-cuspidal pair is conjugate to a pair $(L w, \chi)$ where $(\mathcal{I}, w)$ is a $d$-periodic element. Moreover, if $F^\chi$ is the corresponding $\mathbb{Q}_\ell$-local system on $X(\mathcal{I}, wF)$, then $(\mathcal{I}, w)$ can be chosen such that

(i) The $\mathbb{Q}_\ell G^F$-modules $H^i(X(\mathcal{I}, wF), F^\chi)$ are mutually disjoint.

(ii) If $\rho$ is a irreducible unipotent constituent of $H^i(X(\mathcal{I}, wF), F^\chi)$ then $\rho$ lies in the $\Phi_d$-block associated to $\chi$ and $i = C_d(\deg \rho / \deg \chi)$.

In addition, the endomorphism algebra $\text{End}_{G^F}(H'(X(\mathcal{I}, wF), F^\chi))$ should be endowed with a natural structure of $d$-cyclotomic Hecke algebra. Let us note the following important consequence of this property: the eigenvalue of any sufficiently divisible power $F^m$ of $F$ on $\rho$ should be $q^{m(n(\alpha_\rho + A_\rho - \alpha_\chi - A_\chi)/d)}$.

The choice of a specific $d$-periodic element in this conjecture is not very relevant: it is conjectured that any other $d$-periodic element $(\mathcal{I}, w')$ can be obtained from $(\mathcal{I}, w)$ by cyclic shifts, so that the cohomology of the corresponding varieties are isomorphic. This has already been proven when $\mathcal{I} = \emptyset$ and $F$ acts trivially on $W$ (see [9, Remark 7.4]).

When $d = 1$, the unipotent blocks correspond to the usual Harish-Chandra series. In particular, when $(G, F)$ has type $A$, there is a unique unipotent block and it contains all the unipotent characters. The purpose of this paper is to show that from the cohomology of $X(\pi)$ one can actually deduce all the other interesting cases (see Corollary 3.2):

**Theorem.** For groups of type $A$, Conjecture 1 holds whenever it holds for $d = 1$, that is for $X(\pi)$.

Let us emphasize that Conjecture 1 is known to be true only in a very small number of cases, namely when $d = h$ is the Coxeter number by Lusztig [12], for groups of rank 2 by Digne, Michel and Rouquier [10] and when $d = n$ for $A_n$ and $d = 4$ for $D_4$ by Digne and Michel [8]. Therefore this theorem represents a very important step towards a proof of the geometric version of Broué's conjecture.

Even though this result depends on the conjectural description of $X(\pi)$, one can give an effective proof of Conjecture 1 for principal $\Phi_d$-blocks when $d > (n + 1)/2$. In that case the defect group is cyclic, and the modular representation theory of the block is fully understood. We will address this problem in a subsequent paper, where we will compute the cohomology of $\mathcal{Z}_\ell$ of the corresponding Deligne-Lusztig variety.

To give a flavour of the proof of the main theorem, recall that for groups of type $A$, we have by [3] a combinatorial description of the Deligne-Lusztig
induction associated to $\tilde{X}(I, wF)$. In terms of partitions, it corresponds to adding a certain number of $d$-hooks. By transitivity, one can decompose $\tilde{X}(I, wF)$ by means of simpler varieties $\tilde{X}_{n,d}$, each of which corresponds to adding a single $d$-hook to a partition. Now, using the methods developed in [11] one can compute the cohomology of (some quotient of) $\tilde{X}_{n,d}$ in terms of $\tilde{X}_{n-1,d}$ and $\tilde{X}_{n-1,d-1}$ (see Theorem 2.1), providing an inductive argument to tackle Conjecture 1.

Note finally that Theorem 2.1 can be generalised to many other situations in type $B$, $C$ and $D$. However, two main problems arise: firstly, the limit case is either $X(w_0)$ or $X(\pi)$ and does not contain all the unipotent characters. Secondly, the methods in [11] work obviously for non-cuspidal unipotent characters only. We can obtain partial results on the principal series in that situation, which we believe are too coarse to be mentioned in this paper.

1 Parabolic varieties in type $A$

Throughout this paper, $G$ will denote any connected reductive algebraic group of type $A_n$ over $\mathbb{F} = \overline{\mathbb{F}}_p$. We will consider a Frobenius endomorphism $F: G \rightarrow G$ defining a standard $F_q$-structure on $G$. Since we will be interested in unipotent characters only, we will not make any specific choice for $(G, F)$ in its isogeny class. If $H$ is any $F$-stable subgroup of $G$ we will denote by $H = H^F$ the associated finite group.

The Weyl group $W$ of $G$ is the symmetric group $S_n$ and its Braid monoid $B^+$ is the usual Artin monoid. It is generated by a set $S = \{s_1, \ldots, s_n\}$ corresponding to simple reflections $s_1, \ldots, s_n$ of $W$. Following [9], we define for $1 \leq d \leq n + 1$

$$v_d = s_1 s_2 \cdots s_{n-\lfloor \frac{d}{2} \rfloor} s_n s_{n-1} \cdots s_{\lfloor \frac{d+1}{2} \rfloor}$$

and

$$J_d = \{s_i \mid \lfloor \frac{d+1}{2} \rfloor + 1 \leq i \leq n - \lfloor \frac{d}{2} \rfloor \} \subset S.$$  

We are interested in computing the cohomology of the variety

$$X_{n,d} = X(J_d, v_d F)$$

with coefficients in any unipotent local system. Note that for $d > 1$, the element $v_d$ is reduced so that we can work with the variety $X(J_d, v_d F)$. By [9, Lemma 11.7 and 11.8], the pair $(J_d, v_d)$ is $d$-periodic so that it makes sense to study the cohomology of $X_{n,d}$. Recall from [9] that a $d$-periodic element is any pair $(I, b)$ with $I \subset S$ and $b \in B^+$ such that $bF(b) \cdots F^{d-1}(b) = \pi/\pi_I$ where $\pi = w_0^0$ is the generator of the pure Braid group. It has been shown in [9] that this forces $bF$ to normalise $I$. Note that when $d \leq (n+1)/2$, $v_d$ is not maximal in the sense that it is not extendable to a $d$th root of $\pi/\pi_I$ for a proper subset $I$ of $J_d$. However, it can still be used to associate to any unipotent block a "good" parabolic Deligne-Lusztig variety. Before making any precise statement, we shall briefly recall the combinatorial objects that we will use.

1.1. $\Phi_d$-blocks of $G$. The unipotent characters of $G$ are labeled by the partitions of $n+1$. If $\lambda$ is such a partition, we will denote by $\chi_\lambda$ the corresponding character,
with the convention that $\chi_{(1, 1, \ldots, 1)} = \text{St}_G$ is the Steinberg character of $G$. We shall also fix a representation $V_\lambda$ over $\overline{Q}_\ell$ of character $\chi_\lambda$. For $1 \leq d \leq n + 1$, the pair $(L_{J_d}, \nu_d F)$ represents a $d$-Levi subgroup of $G$. From [3], we know how to express the $d$-Harish-Chandra induction in terms of combinatorics of partitions. To fix the notation, let $\mu$ be a partition of $n + 1 - d$ and $X = \{x_1 < x_2 < \cdots < x_s\}$ be a $\beta$-set associated to $\mu$. We may and will assume that $X$ is big enough, so that it contains $[0, 1, \cdots, d - 1]$. Let $X'$ be the subset of $X$ defined by $X' = \{x \in X | x + d \notin X\}$. It represents the possible $d$-hooks that can be added to $\mu$. For $x \in X'$ we will denote by $\mu * x$ the partition of $n + 1$ which has $(X \setminus \{x\}) \cup \{x + d\}$ as a $\beta$-set.

We fix an $F$-stable Tits homomorphism $t : B^+ \longrightarrow N_G(T)$. By [9] the variety $X_{n, d}$ has an étale covering $\tilde{X}_{n, d} = \tilde{X}(J_d, \nu_d F)$ with Galois group $L_{J_d}^{t(\nu_d)F}$. Since $(L_{J_d}, t(\nu_d)F)$ is a split group of type $A_{n-d}$, the partition $\mu$ defines a unipotent local system $\mathcal{F}_\mu$ on $X_{n, d}$ such that $H_1^c(X_{n, d}, \mathcal{F}_\mu)$ and $H_1^c(\tilde{X}_{n, d}, \overline{\nu}_\ell)_{|_{\mu}}$ are isomorphic. Then we deduce from [3, Section 3.4] that there exist signs $\varepsilon_x = \pm 1$ such that the $d$-Harish-Chandra induction of $\chi_\mu$ is given by

$$R_{L_{J_d}}^G(\chi_\mu) = \sum (-1)^i H_1^c(X_{n, d}, \mathcal{F}_\mu) = \sum_{x \in X'} \varepsilon_x \chi_\mu * x.$$ 

In particular, the $d$-Harish-Chandra restriction of $\chi_\lambda \in \text{Irr} G$ is non-zero until we reach the $d$-core $\nu$ of $\lambda$, which corresponds to a $d$-cuspidal character $\chi_\nu$. The unipotent characters in the $\Phi_d$-block of $G$ containing $\chi_\lambda$ are all the characters that can be obtained by successive $d$- inductions from $\chi_\nu$. They correspond to partitions of $n + 1$ that have $\nu$ as a $d$-core.

### 1.2. A parabolic variety associated to a $\Phi_d$-block.

The cohomology of the variety $X_{n, d}$ induces to a minimal $d$-induction since there is no $d$-split Levi between $(L_{J_d}, t(\nu_d)F)$ and $(G, F)$. By transitivity, one can form a Deligne-Lusztig variety $X(I, w)$ associated to the $d$- cuspidal character $\chi_\nu$. Let $n + 1 - ad$ be the size of $\nu$ and consider for $i = 1, \ldots, a$ the pairs $(J_d^i, \nu_d^i)$ where $(J_d^{(0)}_d, \nu_d^{(0)}) = (J_d, \nu_d)$ and $(J_d^{(i+1)}_d, \nu_d^{(i+1)})$ is the analogue of the pair $(J_d, \nu_d)$ for the split group $(L_{J_d^i}, t(\nu_d^{i}) \cdots \nu_d^{(1)})$ of type $A_{n-id}$. Then one can readily check that the pair $(I, w) = (J_d^a, \nu_d^{(a)} \cdots \nu_d^{(1)})$ is $d$-periodic.

By [9, Proposition 8.26] the associated Deligne-Lusztig variety $\tilde{X}(I, w F)$ is isomorphic to the following amalgamated product

$$\tilde{X}(J_d^{(1)}, \nu_d^{(1)}) \times_L L_{J_d^{(1)}}, \nu_d^{(1)} \cdots \times L_{J_d^{(a-1)}}, \nu_d^{(a-1)} \tilde{X}_{n, d}^{(a-1)}(J_d^{(a)}, \nu_d^{(a)}) \cdot (\nu_d^{(a-1)} \cdots \nu_d^{(1)}) F).$$

Now each variety in this decomposition corresponds to a variety $\tilde{X}_{n, -id, d}$ for some $i = 0, \ldots, a - 1$. Since the cohomology of the latter with coefficients in a unipotent local system depends only on the isogeny class of the group (here, the split type $A_{n-id}$) we obtain

$$R_{\text{c}}(X(I, w F), F_\nu) = R_{\text{c}}(\tilde{X}_{n, d}, \overline{\nu}_\ell) \otimes_{A_{n-d}} \cdots \otimes_{A_{n-(a-1)d}} R_{\text{c}}(X_{n-(a-1)d, d}, F_\nu).$$

(1.1)
Consequently, if we believe that the vanishing property in Conjecture 1 holds for the cohomology of $X(I, w)$, then for any partition $μ$ of $n + 1 − d$, the graded $G$-module $H^i_c(X_{n,d}, F_μ)$ should be multiplicity-free.

1.3. Craven’s formula in type $A$. Conjecturally, the unipotent character $χ_λ$ is a constituent of only one cohomology group of $RΓ(X(I, wF), F_v)$. Craven proposed in [7] a formula which gives the degree of this cohomology group in terms of $d$ and the generic degree of $χ_λ$ and $χ_μ$. More precisely, he considers a function $C_d$ on some set of enhanced cyclotomic polynomials and conjectured that

$$\langle χ_λ; H^i(X(I, w), F_v)\rangle_G \neq 0 \iff i = C_d(\deg χ_λ) - C_d(\deg χ_μ).$$

Let us recall the definition of $C_d$: assume that $P \subset Q[x]$ is a polynomial such that the non-zero roots $z_1, \ldots, z_m$ (written with multiplicity) of $P$ are all roots of unity. Let us denote by $d^c(P)$ the degree of $P$ and by $v(P)$ its valuation, that is the degree of $x^d(P)x^{-1}).$. Then Craven’s function $C_d$ is defined by

$$C_d(P) = \frac{1}{d} \left(d^c(P) + v(P)\right) + \# \{i = 1, \ldots, m | \text{Arg} z_i < 2\pi/d\} - \frac{1}{2} \# \{i = 1, \ldots, m | z_i = 1\}.$$

Here, the argument $\text{Arg} z$ of a non-zero complex number $z$ is taken in $[0; 2\pi)$. More generally, if $\zeta = \exp(2\pi k/d)$ is a primitive $d$-root of unity, one can define a function $C_\zeta$ by replacing $d$ by $dk$ and 1.2 should hold for $dth$ roots of $(\pi/\pi)^k$. Note also that Craven’s function is additive: it satisfies $C_\zeta(PQ) = C_\zeta(P) + C_\zeta(Q)$.

For groups of type $A$, the degree $\deg χ_λ$ of the unipotent character $χ_λ$ is explicitly known (see for example [5, Section 13]). It is a polynomial in $q$ of degree $A_λ$ and valuation $a_λ$ and no factors of the form $(q - 1)$ can appear. In particular, Craven’s function can be written

$$C_\zeta(\deg χ_λ) = \frac{2\pi}{\text{Arg} \zeta} (a_λ + A_λ) + \# \{i = 1, \ldots, m | \text{Arg} z_i < \text{Arg} \zeta\}$$

where $z_1, \ldots, z_m$ are the roots with multiplicity of the polynomial $\deg χ_λ$. Note that with this description it is already not obvious that the rational number on the right-hand side of 1.2 is actually an integer.

Since $C_d$ is additive, formula 1.2 together with the quasi-isomorphism 1.1 suggests that the partition $ν$ should not be necessarily a $d$-core. In the case of an elementary $d$-induction (when $a = 1$) we can write everything explicitly using [7, Proposition 9.1]; the second equality follows from an easy calculation:

**Lemma 1.3.** Let $μ$ be a partition with corresponding $β$-set $X$ that we assume to be large enough. For $x \in X'$, we have

$$C_d(\deg χ_{μ+x}) - C_d(\deg χ_μ) = 2(n + 1 - d - x + \# \{y \in X | y < x\}) + \# \{y \in X | x < y < x + d\}$$

and

$$a_{μ+x} + A_{μ+x} - a_μ - A_μ = d(n - d - s - x).$$

These integers give conjecturally the degree of the cohomology group of $X_{n,d}$ in which $χ_{μ+x}$ will appear, as well as the corresponding eigenvalue of the Frobenius. Since we will work with the cohomology with compact support, we shall rather work with the integers.
\[ \pi_d(X, x) = 2(n + x - \#(y \in X \mid y < x)) - \#(y \in X \mid x < y < x + d) \]

and
\[ \gamma_d(X, x) = n + 1 + x - s. \]

They are readily deduced from the previous ones by taking into account the dimension of \(X_{n,d}\) (which is equal to \(\ell(v_d) = 2n + 1 - d\)). Now Conjecture 1 can be deduced from the following:

**Conjecture 1.4.** Let \(n \geq 1\) be a positive integer and \(1 \leq d \leq n + 1\). Let \(\mu\) be a partition of \(n - d + 1\) and \(X\) be its \(\beta\)-set, assumed to be large enough. Then
\[ R\tilde{\Gamma}_c(X_{n,d}, F_\mu) \cong \bigoplus_{x \in X'} V_{\mu+x}(-\pi_d(X, x)) \otimes \overline{Q}_\ell(\gamma_d(X, x)) \]
as a complex of \(G \times \langle F \rangle\)-modules.

The main result of this section gives a geometric interpretation of this phenomenon.

**Theorem 2.1.** Assume that \(d \geq 2\). Let \(I = \{s_j \mid 1 \leq j \leq n - 1\}\). Let \(\mu\) be a partition of \(n - d + 1\) and \(\{\mu^{(j)}\}\) be the set of partitions of \(n - d\) obtained by restricting \(\mu\). Then there is a distinguished triangle in \(D^b(\overline{Q}_\ell L_I \times \langle F \rangle\cdot\text{-mod})\)
\[ R\tilde{\Gamma}_c(G_m \times X_{n-1,d-1}, \overline{Q}_\ell \otimes F_\mu) \rightarrow R\tilde{\Gamma}_c(X_{n,d}, F_\mu)^{U_I} \rightarrow R\tilde{\Gamma}_c(X_{n-1,d-1}, \bigoplus F_{\mu^{(j)}})[-2](1) \rightarrow \]

**Remark 2.2.** From [1, Proposition 1.1] we can deduce that the cohomology of a Deligne-Lusztig variety with coefficients in a unipotent local system depends only on the type of \((G, F)\). Therefore there is no ambiguity in the statement of the theorem.
We will use the results in [11] to compute the quotient of $\tilde{X}(J_d, \partial_d F)$ by the finite group $U_I$. Recall that $X_{n,d} = X(J_d, \partial_d F)$ can be decomposed into locally closed $P_J$-subvarieties $X_x$, where $x$ is a $I$-reduced-$J_d$ element of $W$. In our situation, at most two pieces will appear:

**Lemma 2.3.** Assume that $2 \leq d \leq n$. The variety $X_x$ is non-empty if and only if $x$ is one of the following two elements:

1. $x_0 = s_n s_{n-1} \cdots s_1$
2. $x_1 = s_n s_{n-1} \cdots s_{n-\lfloor \frac{d}{2} \rfloor + 1}$

**Proof.** For simplicity, we shall denote $a = \lfloor \frac{d+1}{2} \rfloor$ and $b = n - \lfloor \frac{d}{2} \rfloor$ so that $w = v_d = s_1 \cdots s_b s_n \cdots s_a$ and $J = J_d = \{a+1, \ldots, b\}$. If $x$ is a $I$-reduced-$J$ element, then $x = s_n s_{n-1} \cdots s_i$ with $b + 1 \leq i \leq n + 1$ or $1 \leq i \leq a$. Recall from [11] that the variety $X_x$ is non-empty if and only if there exists $y = y_1 \cdots y_r \in W_J$ and an $x$-distinguished subexpression $\gamma$ of $y w$ such that the products of the elements of $\gamma$ lies in $(W_I)^x$. We first observe that for $i \notin \{1, n + 1\}$ we have

$$(W_I)^x = \langle s_1, \ldots, s_{i-2}, s_i s_{i-1} s_i, s_{i+1}, \ldots, s_n \rangle.$$

Now, since $x$ is reduced-$J$, the subexpression $\gamma$ is the concatenation of $(y_1, \ldots, y_r)$ and an $xy$-distinguished subexpression $\tilde{\gamma}$ of $w$. If $i > b + 1$ or $2 \leq i \leq a$, the group $W_J$ is included in $(W_I)^x$. Therefore the product of the elements of $\tilde{\gamma}$ must lie in $(W_I)^x$. We shall distinguish two cases:

**Case (1).** We assume that $i > b + 1$. In that case $x$ commutes with any element of $W_J$, so that $\tilde{\gamma}$ is an $xy$-distinguished subexpression of $w$. Then

- if $x$ is trivial (that is if $i = n + 1$), then any $y$-distinguished subexpression of $w$ contains necessarily $s_n$ and hence cannot produce any element of $(W_I)^x = W_I$;
- if $x$ is non-trivial then $i \leq n$, and a subexpression of $w$ lies in $(W_I)^x$ if and only if it does not contain $s_i$ or $s_{i-1}$. However, such a subexpression will never be $yx$-distinguished since for all $v \in W_I$ we have $v x s_{i-1} > v x$.

We deduce that the variety $X_x$ is empty in this case.

**Case (2).** We assume that $2 \leq i \leq a$. The subexpression $\tilde{\gamma}$ is $x y x$-distinguished.

Since $i \leq a$, we have $x W_J = W_a, \ldots, b-1$. For $j < i - 1$, we have $x s_j = s_j x$; moreover, $x s_{i-1}$ is $I$-reduced, so that $\tilde{\gamma}$ should start with $(s_1, \ldots, s_{i-1})$. In that case, the product of the elements of $\tilde{\gamma}$ cannot belong to $(W_I)^x$. Indeed, a subexpression of $s_{i-1} s_i \cdots s_b s_n \cdots s_a$ starting with $s_{i-1}$ will never give an element of $(W_I)^x$, the only non-trivial situation being the case $a = i$:

- with the notation in [10, Section 2.1.2] we have $s_{a-1} s_{a+1} \cdots s_b s_n \cdots s_a = s_{a+1} \cdots s_b s_n \cdots s_{a-1} s_{a-2}$ and neither $s_{a-1} s_a$ nor $s_{a-1} s_a$ belongs to $(W_I)^x$;
- $s_{a-1} s_{a+1} \cdots s_b s_n \cdots s_a = (s_{a-1} s_{a+1}) s_{a-1} s_{a+1} \cdots s_b s_n \cdots s_a$ and we are back to the previous case.

This forces the variety $X_x$ to be empty.  

\[\square\]
Proof of the Theorem. From the previous lemma we deduce that \( \tilde{X}(J_d, v_d F) \) decomposes as a disjoint union \( \tilde{X}(J_d, v_d F) = \tilde{X}_{x_0} \cup \tilde{X}_{x_1} \) with \( \tilde{X}_{x_0} \) being open. Using [11] we shall now determine the cohomology of the quotient of each of these varieties by \( U_I \). Throughout the proof, we will denote \( \bar{I} = \{ s_2, \ldots, s_n \} \subset S \) the conjugate of \( I \) by \( w_0 \).

When \( x = x_0 = w_1 w_0 \) and \( d > 2 \) we are in the situation of [11, Proposition 3.4]. Indeed, \( v_d = s_1 w' \) with \( w' \in W_{2,\ldots,n} \) and \( s_1 \) commutes with \( W_d \subset W_{3,\ldots,n-1} \) so that we obtain

\[
\text{R} \Gamma_c(U_I \setminus \tilde{X}_{x_0}/N, \overline{Q}_f) \approx \text{R} \Gamma_c(G_m \times \tilde{X}_{U_d}(K_{x_0}, \bar{v} F)/\tilde{N}' \overline{Q}_f)
\]

with \( v = w_0 w' \) and \( K_{x_0} = I \cap w_0 \Phi J_d = w_0 J_d \). For simplicity, we shall rather consider the conjugate by \( x_0 \) of the right-hand side.

Recall that \( N \) and \( N' \) are normal subgroup of \( L_d \) and are both contained in \( T \). In particular, any unipotent character of \( L_d \) is trivial on \( N \) (resp. \( N' \)). Consequently, for any unipotent character \( \chi \) of \( L_d \) we obtain the following quasi-isomorphism of complexes of \( L_I \times \langle F \rangle \)-modules:

\[
\text{R} \Gamma_c(U_I \setminus \tilde{X}_{x_0}/\tilde{Q}_f) \approx \text{R} \Gamma_c(G_m \times \tilde{X}_{U_d}(w_0 J_d, \bar{v} F), \tilde{Q}_f)^{w_0 \chi}.
\]

Finally, we observe that the varieties \( X_{U_d}(w_0 J_d, \bar{v} F) \) and \( X_{n-1, d-1} \) have the same cohomology with coefficients in any unipotent local system. Indeed, if we denote \( (s_1, \ldots, s_{n-1}) \) by \( (t_1, \ldots, t_{n-1}) \) if \( d \) is odd or by \( (t_{n-1}, \ldots, t_1) \) if \( d \) is even, then we have

\[
v = t_1 t_2 \cdots t_{n-1} (d-1)/t_{n-1} t_{n-2} \cdots t_0.
\]

which corresponds to the element \( v_{d-1} \) in the Weyl group \( W_I = \langle t_1, \ldots, t_{n-1} \rangle \) of type \( A_{n-1} \).

When \( x = x_0 \) and \( d = 2 \), we can write \( v_2 = w w' \) with \( w = s_n s_{n-1} \cdots s_2 \) and \( w' = s_1 s_2 \cdots s_n = s_1 w'' \) so that \( X_{n, 2} = X(\{ s_2, \ldots, s_{n-1} \}, \bar{w} w' F) \). Moreover, via this isomorphism we have

\[
X_{x_0} \simeq \bigcup_{y \in W} X_{(x_0, y)}.
\]

We claim that \( X_{(x_0, y)} \) is empty unless \( y \in W_I w_0 W_{J'_2} \) where \( J'_2 = J_d^w = \{ s_3, s_4, \ldots, s_n \} \). The piece \( X_{(x_0, y)} \) consists of pairs \( (p x_0 P_{J_d}, p' y P_{J'_2}) \) with \( p, p' \in P_I \) such that \( p^{-1} p' \in x_0 P_{J_d} \bar{w} P_{J'_2} y^{-1} \) and \( p'^{-1} p \in y P_{J'_2} \bar{w} P_{J_d} x_0^{-1} \). In particular, if \( X_{(x_0, y)} \) is non-empty then the double coset \( P_I y P_{J'_2} \) has a non-trivial intersection with \( x_0 B w \). But \( x_0 = w_1 w_0 \) is reduced-\( \bar{I} \) so that \( \ell(x_0 w) = \ell(x_0) + \ell(w) \) and \( x_0 B w \subset P_I w_0 P_{J_d} \). This forces \( y \) to lie in \( W_I w_0 W_{J'_2} \). Note that \( w_0 (J'_2) \subset I \) so that the minimal element in this coset is \( x_0 \) and we have \( X_{x_0} = X_{(x_0, x_0)} \).

Now \( s_1 \) commutes with \( J'_2 \) and we can apply [11, Proposition 3.4] to obtain, after conjugation by \( x_0 \):

\[
\text{R} \Gamma_c(U_I \setminus \tilde{X}_{x_0}/N, \overline{Q}_f) \approx \text{R} \Gamma_c(G_m \times \tilde{X}_{U_d}(K_{x_0}, \bar{v} v' F)/\tilde{x}_0 N' \overline{Q}_f).
\]
with \( K_{x_0} = x_0 J_2 \), \( v = x_0 w \) and \( v' = x_0 w' \). If we denote \((s_1, \ldots, s_{n-1})\) by \((t_{n-1}, \ldots, t_1)\) we obtain \( x_0 J_2 = (t_2, \ldots, t_{n-1}) \) and \( vv' = t_1 \cdots t_{n-1} t_{n-1} \cdots t_1 \) so that the pair \((K_{x_0}, vv')\) corresponds to \((J_1, v_1)\) in the Weyl group \( W_l \) of type \( A_{n-1} \). As before, \( N \) and \( N' \) do not play any role if we consider the unipotent part of the previous quasi-isomorphism.

When \( x = x_1 \) we use [11, Proposition 3.2]: the conjugate of \( v_d \) by \( x_1 \) is

\[
v = x_1 v_d x_1^{-1} = s_1 s_2 \cdots s_{n-1-d} s_{n-1} s_{n-2} \cdots s_{\lfloor d/2 \rfloor}
\]

which corresponds exactly to the element \( v_d \) in \( W_l \). We can therefore identify the cohomology of the varieties \( X_{L_d}(K_{x_1}, v_F) \) and \( X_{n-1,d} \) with coefficients in any unipotent local system (see Remark 2.2). The group \( P_l \cap x_1 L_{d_d} \) is a \( vF \)-stable parabolic subgroup of \( x_1 L_{d_d} \) and \( L_{K_{x_1}} = L_{x_1} \cap x_1 L_{d_d} \) is a stable Levi complement. Therefore it makes sense to consider the Harish-Chandra restriction \( ^*\rho^J_K \chi \) of any unipotent character \( \chi \) of \( L_{d_d}^{\nu_d F} \simeq (x_1 L_{d_d})^{\nu_d F} \) to \( L_{K_{x_1}}^{\nu_d F} \). From [11, Proposition 3.2] (see also [11, Remark 3.12]) we deduce the following quasi-isomorphism

\[
\Gamma_c(U_l \backslash \tilde{X}_{x_1}, \overline{Q}_l)_{\chi_{\mu}}[1] \simeq \Gamma_c(\tilde{X}_{L_d}(K_{x_1}, v_F), \overline{Q}_l). \overline{R}_{\nu_d}^J \chi.
\]

Let \( \mu \) be a partition of \( n - d + 1 \). The cohomology of the variety \( X_{n,d} \) with coefficients in the local system \( \mathcal{F}_\mu \) is given by

\[
\Gamma_c(X_{n,d}, \mathcal{F}_\mu) \simeq \Gamma_c(\tilde{X}_{L_d}(K_{x_1}, v_d F), \overline{Q}_l)_{\chi_{\mu}}
\]

where \( \chi_{\mu} \) is the unipotent character of \( L_{d_d}^{\nu_d F} \) corresponding to the partition \( \mu \). Since \( (L_{K_{x_1}}, v_F) \) is a split group of type \( A_{n-d-1} \), the Harish-Chandra restriction of \( \chi_{\mu} \) from \( L_{d_d}^{\nu_d F} \simeq (x_1 L_{d_d})^{\nu_d F} \) to \( L_{K_{x_1}}^{\nu_d F} \) is the sum of the \( \chi_{\mu_i} \)'s where the \( \mu_i \)'s are the partitions of \( n - d \) obtained by restricting \( \mu \). With with description, we get the following isomorphisms in \( D^b(\overline{Q}_l L_l \times \langle F \rangle \text{-mod}) \)

\[
\Gamma_c(U_l \backslash \tilde{X}_{x_0}, \overline{Q}_l)_{\chi_\mu} \simeq \Gamma_c(G_m \times X_{n-1,d-1}, \overline{Q}_l \otimes \mathcal{F}_\mu)
\]

and

\[
\Gamma_c(U_l \backslash \tilde{X}_{x_1}, \overline{Q}_l)_{\chi_\mu} \simeq \Gamma_c(X_{n-1,d}, \bigoplus \mathcal{F}_{\mu_i})[-2](1).
\]

We conclude using the distinguished triangle associated to the decomposition \( \tilde{X}_{n,d} = \tilde{X}_{x_0} \cup \tilde{X}_{x_1} \) in which \( \tilde{X}_{x_0} \) is open. \( \Box \)

## 3 Cohomology over \( \overline{Q}_l \)

We have just seen how to relate the Harish-Chandra restriction of the cohomology of \( X_{n,d} \) to the cohomology of smaller parabolic Deligne-Lusztig varieties. We shall now explain how this strategy provides an inductive method for a thorough determination of the cohomology of \( X_{n,d} \) with coefficients in any unipotent local system. The main result in this section gives an inductive strategy towards a proof of Conjecture 1:
Theorem 3.1. Let \( n \geq 1 \) and \( 2 \leq d \leq n \). If Conjecture 1.4 holds for \((n,d+1)\), \((n-1,d-1)\) and \((n-1,d)\) then it holds for \((n,d)\).

Note that we already know from [12] that Conjecture 1.4 holds in the Coxeter case, corresponding to \((n,n+1)\). Therefore \( d = 1 \) is the only limit case. But \( \pi = w_0^2 \) is a maximal 1-periodic element in the sense of [9] and in that specific case, a general conjecture for the cohomology has been already formulated in [4]: a unipotent character \( \chi_\lambda \) can appear in \( H^i_c(X(\pi)) \) for \( i = 4v_\lambda - 2A_\lambda \) only, where \( v_\lambda \) is the number of positive roots. An important consequence of Theorem 3.1 is that knowing the cohomology of \( X(\pi) \) is sufficient for determining all the other interesting cases:

Corollary 3.2. For groups of type \( A \), Conjecture 1 holds for any \( d \geq 1 \) as soon as it holds for \( d = 1 \).

Proof. Assume that Conjecture 1 holds for \( d = 1 \), that is for the variety \( X(\pi) \). Let \( I = J = \{ s_2, \ldots, s_n \} \) and \( b = v_1 = s_1 \cdots s_n \cdots s_1 \). By [9, Proposition 8.26] we have

\[
\Gamma_c[X(\pi),\bar{\ell}^\ell] = \Gamma_c[X(I,bF),\bar{\ell}^\ell] \otimes_{\bar{\ell}^\ell \Gamma_c F} \Gamma_c[X(\pi_I),\bar{\ell}^\ell].
\]

Since the cohomology of \( X(\pi_I) \) contains all the unipotent characters of \( L^F_I \), we deduce that for any partition \( \mu \) of \( n \), the groups \( H^i_c(X_{n,1},F_\mu) \) are submodules of the cohomology groups of \( X(\pi) \). Consequently, they are disjoint as soon as Conjecture 1 holds for \( X(\pi) \). Since we have assumed that it holds also for \( X(\pi_I) \) we have actually

\[
H^i_c(X(\pi),\bar{\ell}^\ell) \cong \bigoplus_{\mu \neq 0} H^i_c(X_{n,1},F_\mu)(2\nu-L-a_\mu-A_\mu) \tag{3.3}
\]

as a \( G \times (F) \)-module. Now, the alternating sum of the cohomology groups of \( X_{n,1} \) represents the Deligne-Lusztig induction from \( L^F_I \) to \( G \). Therefore a character \( \chi_\lambda \) appear in \( H^i_c(X_{n,1},F_\mu) \) if and only if \( \mu \) is the restriction of \( \lambda \), or equivalently, if \( \lambda \) is obtained from \( \mu \) by adding a 1-hook. This, together with 3.3 and Lemma 1.3 proves that Conjecture 1.4 holds for \( X_{n,1} \), and therefore for any variety \( X_{n,d} \) by 3.1. We use 1.1 to conclude.

Proof of the Theorem. Let \( X \) be a \( \beta \)-set associated the partition \( \mu \) of \( n-d+1 \). We can always assume that it contains \( \{0,1,\ldots,d-1\} \). The partitions \( \mu^{(b)} \) of \( n-d \) which are obtained by restricting \( \mu \) can be associated to the following \( \beta \)-set:

\[
X^{(b)} = \{ x_1^{(b)} < \cdots < x_s^{(b)} \} \quad \text{with} \quad x_i^{(b)} = \begin{cases} x_j-1 & \text{if } i = j; \\ x_i & \text{otherwise.} \end{cases}
\]

Let \( I = \{ s_1, \ldots, s_{n-1} \} \). By Theorem 2.1, the Harish-Chandra restriction of the cohomology of \( X_{n,d} \) can be fitted into the following distinguished triangle:

\[
\Gamma_c(G_m \times X_{n-1,d-1},\bar{\ell}^\ell \otimes F_\mu) \longrightarrow \Gamma_c(X_{n,d},F_\mu)^{U_I} \longrightarrow \Gamma_c(X_{n-1,d},\bigoplus F_{\mu^{(b)}})(-2)(1) \sim
\]

Now, if we assume that Conjecture 1.4 holds for both \((n-1,d-1)\) and \((n-1,d)\), the complexes on the left and right-hand side are completely determined. Let us examine the different eigenvalues of \( F \) that can appear:
(a) on $\mathcal{C} = \text{RG}_c(G_m \times X_{n-1,d-1}, \overline{Q}_\ell \otimes \mathcal{F}_\mu)$, the eigenvalues of $F$ are $q^{n+x-s}$ and $q^{n+1+x-s}$ with $x \in X$ such that $x + d - 1 \notin X$. The character of the corresponding eigenspace is $\chi$ where $\lambda$ is the partition of $n$ obtained by adding to $\mu$ a $(d-1)$-hook represented by $x$;

(b) on $\mathcal{D}^{(j)} = \text{RG}_c(X_{n-1,d}, \mathcal{F}_{d^{(j)}})(-2)(1)$, the eigenvalues of $F$ are $q^{n+1+x-s}$ where $x \in X^{(j)}$ is such that $x + d \notin X^{(j)}$. The character of the corresponding eigenspace is $\chi$ where $\lambda$ is the partition of $n$ obtained by adding to $\mu^{(j)}$ a $d$-hook represented by $x$.

We shall now determine $H^*_c(X_{n,d}, \mathcal{F}_\mu)^{U_1}$ by studying each eigenspace of $F$ separately. For $x$ a positive integer, we can separate the following cases:

**Case (1).** Assume first that $x \in X$ and $x + d \notin X$. Let $\lambda = \mu \ast x$ be the partition of $n + 1$ obtained by adding to $\mu$ a $d$-hook from $x$. We want to prove that the $q^{n+1+s-x}$-eigenspace of $F$ on $H^*_c(X_{n,d}, \mathcal{F}_\mu)^{U_1}$ is non-zero in degree $\pi_d(X,x)$ only and that its character is the Harish-Chandra restriction of $\chi$.

By (a), the $q^{n+1+x-s}$-eigenspace of $F$ on $\mathcal{C}$ will produce non-zero representations in the following two cases:

- if $x + d - 1 \notin X$, then one obtains a character associated to the $\beta$-set $(X \setminus \{x\}) \cup \{x + d - 1\}$ and it is concentrated in degree
  \[
  2 + \pi_{d-1}(X,x) = 2 + 2\{n - 1 + x - \#(y \in X | y < x)\} - \#(y \in X | x < y < x + d - 1) \\
  = 2\{n + x - \#(y \in X | y < x)\} - \#(y \in X | x < y < x + d - 1) \\
  = \pi_d(X,x).
  \]

- if $x + 1 \in X$, then the corresponding $\beta$-set is $(X \setminus \{x + 1\}) \cup \{x + d\}$ and the associated character appears in degree $1 + \pi_{d-1}(X,x) + 1$ only. But we have
  \[
  \pi_{d-1}(X,x + 1) = 2\{n + x - \#(y \in X | y < x + 1)\} - \#(y \in X | x + 1 < y < x + d) \\
  = 2\{n + x - 1 - \#(y \in X | y < x)\} - \#(y \in X | x < y < x + d - 1) \\
  = \pi_d(X,x) - 1.
  \]

On the other hand, the $q^{n+1+x-s}$-eigenspace of $F$ on $\mathcal{D}^{(j)}$ is non-zero if and only if $x \in X^{(j)}$ and $x + d \notin X^{(j)}$. This happens if and only if $x$ and $x + d + 1$ are different from $x_j$. In that case, the $\beta$-set corresponding to the character of the eigenspace will be $(X^{(j)} \setminus \{x\}) \cup \{x + d\}$. Furthermore, the degree in which this character will appear is $2 + \pi_d(X^{(j)},x)$, which is clearly equal to $\pi_d(X,x)$ in that case.

Now, the $\beta$-set $Y = (X \setminus \{x\}) \cup \{x + d\}$ is associated to the partition $\lambda = \mu \ast x$. As mentioned in the beginning of Section 2, the restriction of $\lambda$ is obtained by restricting the hook (usually in two different ways) or by restricting $\mu$. In the framework of $\beta$-sets, it corresponds to decreasing specific elements of $Y$:

- if $x + d - 1 \notin X$, one can replace $x + d$ by $x + d - 1$ in $Y$ and we obtain the $\beta$-set $(X \setminus \{x\}) \cup \{x + d - 1\}$;
• if $x + 1 \in X$, one can replace $x + 1$ by $x$ in $Y$ and we obtain the $\beta$-set $(X \setminus \{x + 1\}) \cup \{x + d\}$;

• if $x_j \in X$ is different from $x$ or from $x + d + 1$, and if $x_j - 1 \notin X$, then one can replace $x_j$ by $x_j - 1$ in $Y$ to obtain the $\beta$-set $(X^{(j)} \setminus \{x\}) \cup \{x + d\}$.

This shows that the character of the $q^{n+1+x-s}$-eigenspace of $F$ on $H^*_c(X_{n,d}, F_\mu)_{U I}$ is the Harish-Chandra restriction of $\chi_{\mu \ast x}$.

**Lemma 3.4.** Let $\lambda$ be a partition of $n+1$, with $n \geq 3$ and let $\chi$ be a (non-necessarily irreducible) unipotent character of $G$. Then the Harish-Chandra restriction of $\chi$ and $\chi_\lambda$ are equal if and only if $\chi = \chi_\lambda$.

**Proof of the Lemma.** Assume that there exists a partition $\nu = (\nu_1 \leq \nu_2 \leq \cdots \leq \nu_r)$ of $n + 1$ with $\nu_1 \neq 0$ such that the difference between the Harish-Chandra restriction of $\chi_\lambda$ and $\chi_\nu$ is still a unipotent character. This means that in the Young diagram of $\nu$, any box that can be removed can be replaced to form the Young diagram of $\lambda$. If $\nu \neq \lambda$, this is possible only if $\nu_1 = \nu_2 = \cdots = \nu_r$.

Let $\chi$ be a unipotent character of $G$ which has the same Harish-Chandra restriction as $\chi_\lambda$. If $\chi \neq \chi_\lambda$, we deduce from the previous argument that all the irreducible constituents of $\chi$ are of the form $\chi_\nu$ with $\nu = (a, a, \ldots, a)$. This can happen if and only if $n = 2$ and $\lambda = (1, 2)$.

When $n \geq 3$, we deduce that the $q^{n+1+x-s}$-eigenspace of $F$ on $H^*_c(X_{n,d}, F_\mu)$ is actually $\chi_{\mu \ast x}$. If $n = 2$, then the only ambiguity concerns $\chi_{\mu \ast x}$ when $\mu \ast x = (1, 2)$. In that case, the $q^{n+1+x-s}$-eigenspace can be either $\chi_{\mu \ast x}$ or $1_G + St_G$. But by [9, Corollary 8.28], the trivial character and the Steinberg character cannot occur in the same cohomology group of $X_{n,d}$ as soon as the dimension of this variety is non-zero.

**Case (2).** Assume now that $x \notin X$. The $q^{n+1+x-s}$-eigenspace of $F$ on $G^{(j)}$ is non-zero if and only if $x \in X^{(j)}$ and $x + d \notin X^{(j)}$. Since $x \notin X$, this forces $x = x^{(j)}_j = x_j - 1$ and $x_j + d - 1 \notin X$. In that case, its character corresponds to a partition with $\beta$-set $(X \setminus (x + 1)) \cup \{x + d\}$ and it appears in degree $2 + \pi_d(X^{(j)}, x)$ only. On the other hand, the $q^{n+1+x-s}$-eigenspace of $F$ on $G$ is non-zero if and only if $x + 1 \in X$ and $x + 1 + d - 1 = x + d \notin X$. By (a), the character of this eigenspace corresponds to a partition with $\beta$-set $(X \setminus (x + 1)) \cup \{x + d\}$. Furthermore, it appears in degree $1 + \pi_{d-1}(X, x+1)$ only. Note that in that case we have

$$\pi_{d-1}(X, x+1) = 2(n + x - \#\{y \in X | y < x + 1\}) - \#\{y \in X | x + 1 < y < x + d\}$$

$$= 2(n + x - \#\{y \in X | y < x\}) - (\#\{y \in X | x < y < x + d\} - 1)$$

$$= \pi_d(X, x) + 1$$

and

$$\pi_d(X^{(j)}, x) = 2(n - 1 + x - \#\{y \in X^{(j)} | y < x\}) - \#\{y \in X^{(j)} | x < y < x + d\}$$

$$= 2(n - 1 + x - \#\{y \in X | y < x\}) - (\#\{y \in X | x < y < x + d\} - 1)$$

$$= \pi_d(X, x) - 1.$$
We deduce that the $q^{n+1+x-s}$-eigenspace of $F$ on $H^*(X_{\mu,d}, F_\mu)^{U_I}$ is isotypic and concentrated in two consecutive degrees. However, there are only a few unipotent characters that can have an isotypic Harish-Chandra restriction: they correspond to partitions of the form $(a, a, \ldots, a)$. Among them we can find the Steinberg character $St_G$ (with $a = 1$) and the trivial character $1_G$ (with $a = n + 1$).

But by [9, Corollary 8.28] they have respective eigenvalues $1$ and $q^{2n+1-d}$. Let us write the $\beta$-set of $\mu$ as $X = \{0, 1, \ldots, k-1, \mu_1 + k, \mu_2 + k + 1, \ldots, \mu_r + s - 1\}$ with $k \geq d$. Since $x \not\in X$, one must have $k - 1 < x < \mu_r + s - 1$ and hence

$$d - 1 \leq n + k - s < n + 1 + x - s < n + 1 + \mu_r - 1 \leq 2n + 1 - d.$$  

We deduce that the $q^{n+1+x-s}$-eigenspace of $F$ on $H^*(X_{n,d}, F_\mu)$ is either zero, or consists of two copies of the character $\chi_\lambda$ in two consecutive degrees, where $\lambda = (a, a, \ldots, a)$ with $1 < a < n + 1$. We shall actually prove that it is always zero, but before that we need to study the last case.

**Remark 3.6.** The Harish-Chandra restriction of $\chi_\lambda$ corresponds to the partition $(a-1, a, \ldots, a)$. Therefore if the associated character appears in the cohomology of $G$ then the $\beta$-set $(X \setminus (x + 1)) \cup \{x + d\}$ must correspond to the partition $(a - 1, a, \ldots, a)$. This gives a rather strong condition on $X$: we will have either

$$X = \{0, 1, \ldots, k-1, x+1, b, b+2, b+3, \ldots, \overline{x+d}, \ldots, b+r\}$$

with $b + 2 \leq x + d \leq b + r$, or

$$X = \{0, 1, \ldots, k-1, x+1, x+d+2, x+d+3, \ldots, x+d+r\}.$$

**Case (3).** Finally, assume that $x \in X$ and $x + d \in X$. The $q^{n+1+x-s}$-eigenspace of $F$ on $G^{(j)}$ is non-zero if and only if $x \in X^{(j)}$ and $x + d \not\in X^{(j)}$. Since $x + d \in X$, this forces $x + d = x_j$ (and therefore $x_j - 1 \not\in X$). In that case, the character of the eigenspace corresponds to a partition with $\beta$-set $(X^{(j)} \setminus \{x\}) \cup \{x + d\} = (X \setminus \{x\}) \cup \{x + d - 1\}$. On $G$, the Frobenius has a non-zero $q^{n+1+x-s}$-eigenspace if and only if $x + d - 1 \not\in X$ and its character is again associated to the $\beta$-set $(X \setminus \{x\}) \cup \{x + d - 1\}$. This ensures that the $q^{n+1+x-s}$-eigenspace of $F$ on $H^*(X_{n,d}, F_\mu)^{U_I}$ is isotypic. Using $x + d - 1$ instead of $x$ in the inequalities 3.5 yields $0 < n + 1 + x - s < 2n + 2 - 2d$ and therefore the previous argument applies. We deduce that the $q^{n+1+x-s}$-eigenspace of $F$ on $H^*(X_{n,d}, F_\mu)$ is again either zero or consists of two copies of the character $\chi_\lambda$ in two consecutive degrees, namely $\pi_d(X, x) - 1$ and $\pi_d(X, x)$, where $\lambda = (a, a, \ldots, a)$ and $1 < a < n + 1$.

To conclude, we need to prove that the $q^{n+1+x-s}$-eigenspaces of $F$ are actually zero whenever $x \not\in X$ or $x + d \in X$. Let us first summarize what we have proven so far:

1. If $x \in X$ and $x + d \not\in X$ then the $q^{n+1+x-s}$-eigenspace of $F$ on $H^*(X_{n,d}, F_\mu)$ is $\chi_{\mu+x}$ and it appears in degree $\pi_d(X, x)$ only;

2. If $x \not\in X$, the $q^{n+1+x-s}$-eigenspace of $F$ is zero unless $x + 1 \in X$ and $x + d \not\in X$.

In that case, it may consist of two copies of $\chi_\lambda$, one in degree $\pi_d(X, x) + 1$ and one in degree $\pi_d(X, x) + 2$, where $\lambda = (a, a, \ldots, a)$ with $1 < a < n + 1$. 


Moreover, the $\beta$-set $(X \setminus \{x + 1\}) \cup \{x + d\}$ must correspond to the partition $(a - 1, a, \ldots, a)$ (see Remark 3.6).

(3) if $x \in X$ and $x + d \in X$, the $q^{n+1+x-s}$-eigenspace of $F$ is zero unless $x + d - 1 \notin X$. In that case, it can only be $\chi_\lambda$-isotypic with $\lambda = (a, a, \ldots, a)$ and $1 < a < n + 1$. Moreover, it is non-zero in degrees $\pi_d(X, x) - 1$ and $\pi_d(X, x)$ only, and $(X \setminus \{x\}) \cup \{x + d - 1\}$ must be a $\beta$-set of the partition $(a - 1, a, \ldots, a)$.

Now, if we assume that Conjecture 1.4 holds for the variety $X_{n,d+1}$, then we can use the distinguished triangle

$$\text{RI}_c(G_m \times X_{n,d}, \overline{Q}_\ell \otimes F_\mu) \rightarrow \text{RI}_c(X_{n,d+1}, F_\mu)^{U_1} \rightarrow \text{RI}_c(X_{n,d+1}, \bigoplus F_{\mu(j)})(-2)(1) \rightarrow$$

from Theorem 2.1 to prove that the eigenspaces of $F$ on $H^*_c(X_{n,d}, F_\mu)$ in cases (2) and (3) are indeed zero.

Assume that $x \notin X$ and that there exists $1 < a < n + 1$ such that the character $\chi_\lambda = \chi(a, a, \ldots, a)$ appears twice in the $q^{n+1+x-s}$-eigenspace of $F$ on the cohomology of $X_{n,d}$ — that is in degrees $\pi_d(X, x) + 1$ and $\pi_d(X, x) + 2$. Then,

- if $x - 1 \notin X$, the $q^{n+1+x-s}$-eigenspace of $F$ on $H^*_c(G_m \times X_{n,d}, \overline{Q}_\ell \otimes F_\mu)$ is $\chi_\lambda$-isotypic by (2) (we have $x - 1 + 1 \notin X$). Moreover, the $q^{n+1+x-s}$-eigenspace of $F$ on $H^*_c(X_{n,d+1}, \bigoplus F_{\mu(j)})(-2)(1)$ is zero since Conjecture 1.4 holds for the variety $X_{n,d+1}$. We deduce that the eigenspace on $H^*_c(X_{n,d+1}, F_\mu)^{U_1}$ is $\chi_\lambda$-isotypic, which is impossible since no unipotent character can have $\chi_\lambda$ as a Harish-Chandra restriction when $1 < a < n + 1$.

- if $x - 1 \in X$, then since $x - 1 + d + 1 \notin X$, the $q^{n+1+x-s}$-eigenspace of $F$ on $H^*_c(G_m \times X_{n,d}, \overline{Q}_\ell \otimes F_\mu)$ and $H^*_c(X_{n,d+1}, \bigoplus F_{\mu(j)})(-2)(1)$ can be determined as in case (1). It corresponds to the Harish-Chandra restriction of the partition $\mu * (x - 1)$ obtained from $\mu$ by adding a $(d + 1)$-hook from $x - 1$. Furthermore, they will appear in degree $\pi_{d+1}(X, x - 1)$ only, which is equal to

$$\pi_{d+1}(X, x - 1) = 2(n + x - \#\{y \in X \mid y < x - 1\}) - \#\{y \in X \mid x - 1 < y < x + d\}$$

$$= 2(n + x + 1 - \#\{y \in X \mid y < x\}) - \#\{y \in X \mid x < y < x + d\}$$

$$= \pi_d(X, x) + 2.$$

To these characters we have to add the contribution of $\chi_\lambda$ and possibly of an other character $\chi_{\lambda'}$ corresponding to $\lambda = (a', a', \ldots, a')$ (when the case (3) applies to $x - 1$). Now, we claim that neither $\chi_\lambda$ nor $\chi_{\lambda'}$ can appear in the $q^{n+1+x-s}$-eigenspace of $F$ on $H^*_c(X_{n,d+1}, \bigoplus F_{\mu(j)})(-2)(1)$. Indeed the assumptions on $x$ force $X$ on $F_\mu$ (see Remark 3.6) to be either

$$X = \{0, 1, \ldots, k - 1, k + 1, b, b + 2, b + 3, \ldots, k + d, \ldots, b + r\}$$

with $b + 2 \leq k + d \leq b + r$, or

$$X = \{0, 1, \ldots, k - 1, k + 1, k + d + 2, k + d + 3, \ldots, k + d + r\}.$$

Therefore a $\beta$-set corresponding to the partition $\mu * (x - 1)$ of $n + 2$ is either
with $b + 2 \leq k + d \leq b + r$, or

$$\{0, 1, \ldots, k - 2, k + 1, k + d, k + d + 2, k + d + 3, \ldots, k + d + r\}.$$ 

We deduce that the restriction of $\mu^*(x-1)$ will never produce $\lambda$ or $\lambda'$ unless $r = 2, b = k + 2$ and $d = 4$ in the first case, or $r = 2$ and $d = 2$ in the second case. In these very specific cases, we have either $X = \{0, \ldots, k - 1, k + 1, k + 2\}$, which corresponds to the partition $\mu = (1, 1)$ or $X = \{0, \ldots, k - 1, k + 1, k + 4\}$, which corresponds to $\mu = (1, 3)$. In this situation, we get $\lambda = (3, 3)$ and $\mu^*(x-1) = (2, 2, 3)$. But $(3, 3)$ cannot be obtain by restricting $\mu^*(x-1)$. This proves that the $q^{a+1+x-s}$-eigenspace of $F$ on $H^c_{\mu}(X, x)^{s}(X_{p+1,d+1},F_{\mu})^{U_1}$ is just $\chi_{\lambda}$ (plus possibly $\chi_{\lambda'}$), which is impossible by the properties of the Harish-Chandra restriction.

The same argument can be adapted to deal with the case (3), if we rather look at the $q^{a+2+x-s}$-eigenspace and distinguish whether $x + 1 + d$ is an element of $X$ or not. The details are left to the reader.

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\section*{References}

[1] C. Bonnafé and R. Rouquier. Coxeter orbits and modular representations. Nagoya Math. J., 183:1–34, 2006.

[2] M. Broué and G. Malle. Zyklotomische Heckealgebren. Astérisque, (212):119–189, 1993. Représentations unipotentes génériques et blocs des groupes réductifs finis.

[3] M. Broué, G. Malle, and J. Michel. Generic blocks of finite reductive groups. Astérisque, (212):7–92, 1993. Représentations unipotentes génériques et blocs des groupes réductifs finis.

[4] M. Broué and J. Michel. Sur certains éléments réguliers des groupes de Weyl et les variétés de Deligne-Lusztig associées. In Finite reductive groups (Luminy, 1994), volume 141 of Progr. Math., pages 73–139.

[5] R. W. Carter. Finite groups of Lie type. Wiley Classics Library. John Wiley & Sons Ltd., Chichester, 1993. Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication.
[6] J. Chuang and R. Rouquier. Calabi-Yau algebras and perverse Morita equivalences. *In preparation.*

[7] D. Craven. On the cohomology of Deligne-Lusztig varieties, arXiv:math/1107.1871. *Preprint*, 2011.

[8] F. Digne and J. Michel. Endomorphisms of Deligne-Lusztig varieties. *Nagoya Math. J.*, 183:35–103, 2006.

[9] F. Digne and J. Michel. Parabolic Deligne-Lusztig varieties, arXiv:math/1110.4863. *preprint*, 2011.

[10] F. Digne, J. Michel, and R. Rouquier. Cohomologie des variétés de Deligne-Lusztig. *Adv. Math.*, 209(2):749–822, 2007.

[11] O. Dudas. Quotient of Deligne-Lusztig varieties, arXiv:math/1112.4942. *preprint*, 2011.

[12] G. Lusztig. Coxeter orbits and eigenspaces of Frobenius. *Invent. Math.*, 38(2):101–159, 1976/77.