Abstract. In this paper, we present a review of the canonical structure of field theories defined on manifolds with time-like boundaries. The notion of differentiable generator is shown to be a requirement coming from the consistency of the symplectic structure. We show how this structure can be applied to classify the possible boundary conditions of a general gauge theory. We then review the definition and properties of surface charges. We show how the notion of differentiable generators allows the direct computation of the phase-space of boundary gauge degrees of freedom.
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1 Introduction

The usual way one develops the lagrangian and hamiltonian formulation for field theories is by taking the continuous limit from a discrete system [1]. This allows for a local definition of field theories. For most applications we can neglect boundary contributions and this structure is sufficient: either we work on manifolds without boundary or we impose asymptotic conditions strong enough to put all boundary contributions to zero. Unfortunately, those setups are too restrictive for a lot of physically relevant problems. In general, we need to relax the asymptotic behavior and deal with the boundary terms.

The situation is easily solved in the Lagrangian framework: one chooses boundary conditions and adds a corresponding boundary term to the Lagrangian in order to make it well-defined. A well-defined Lagrangian $L$ is such that, for any allowed variation of the fields $\delta \phi^a$, the variation of $L$ does not produce any boundary term:

$$\delta \int_M d^{n+1}x L = \int_M d^{n+1}x \frac{\delta L}{\delta \phi^a} \delta \phi^a.$$

This well-defined action is the one needed in the path integral [2].

In the hamiltonian picture, it seems that we don’t need any modification as the fundamental object, the poisson bracket, is independent of total derivatives. In the same way that hamiltonian generators are defined up to a constant in discrete classical mechanics, they are usually defined up to a boundary term in field theories. A famous problem in that context is the definition of mass in general relativity. The main issue is that the hamiltonian density is given by a sum of constraints and is zero on all solutions of the equations of motion. The answer is to add to the Hamiltonian a well chosen boundary term [3]: this doesn’t change the equations of motion and gives the expected value for the energy. However, this construction is ad-hoc and is not a solution to the problem as the formalism does not constrain this boundary term.

This phenomenon is general and appears whenever one wants to define conserved quantities related to gauge symmetries. The generators of gauge symmetries are constraints and always give zero when evaluated on solutions. The conserved quantities are then given by specific boundary terms known as surface charges: one example is the electric charge given by the flux of the electric field through the boundary. As for gravity, these boundary terms are ad-hoc and not constrained by the formalism.

The final solution was proposed by Regge-Teitelboim in [4] and was later refined in [5, 6, 7, 8]. Using the definition of well-defined action explained above and applying it to the hamiltonian action fixes the form of the boundary term in the definition of the Hamiltonian. This leads to the notion of differentiable generator. Restricting the set of functionals to the set of differentiable functionals fixes the boundary term for any
hamiltonian generator. In particular it fixes the boundary term for gauge symmetries and gives a systematic definition for the algebra of asymptotic symmetries and their associated surface charges.

In the last 15 years, this technique and its lagrangian counterpart [9, 10, 11] have been very useful in the study of holography and of the AdS/CFT conjecture. The conjecture relates a bulk gravity theory to a field theory without gravity living in one dimension lower. The two theories being equivalent, they share the same symmetry algebra. In particular, the asymptotic symmetries of the bulk gravity theory are global symmetries of the dual theory. The most famous exemple in that context is maybe the original computation by Brown-Henneaux in [7]. They showed that the asymptotic symmetry algebra of the asymptotically AdS3 space-times is given by two copies of the Virasoro algebra with central charges given by $c^\pm = \frac{3}{2\ell}$. This proves that this theory is described by a conformal field theory in two dimensions. Recently, it has played a major role in the study of higher spin theories in 3 dimensions and their holographic duals. In [12, 13, 14], it was shown that higher spin theories have asymptotic symmetry algebras given by $W$-algebras. This provides good indications in favor of the conjecture that higher spin theories on AdS3 are duals to certains minimal cosets models (see [15] for a review).

In this paper, we present a constructive introduction to the notion of differentiable functional. We show how this structure can be applied to classify the possible boundary conditions of a general gauge theory. We then review the definition and properties of surface charges associated to asymptotic gauge symmetries. In the last part of the paper, we show how the same notion of differentiable generator allows the direct computation of the reduced phase-space of some topological theories. The appendix contains the definitions and conventions used to describe the differential structure of the phase-space of field theories.

The plan of the paper is the following:

- In section 2 we present the canonical structure of field theories defined on a manifold with boundary. Requiring field theories to behave like discrete mechanical systems naturally introduces the notion of differentiable functional. We then show how this structure is related to the notion of well-defined action and we end with a review of Noether’s theorem.

- Section 3 is devoted to gauge theories. We start by describing the set-up and the requirements for consistent boundary conditions. We then introduce the notion of differentiable gauge generator and use them to classify the possible boundary conditions on the lagrange multipliers. The last part contains a review of the definition of the asymptotic symmetry algebra and the associated surface charges.
In section 4 we show how the notion of differentiable gauge generator allows the computation of the phase-space of boundary gauge degrees of freedom of some topological theories without the need to solve the constraints. We also present how one can make a complete classification of the possible boundary conditions. The technique is presented using Chern-Simons in three dimensions and BF theory in four dimensions as examples.
2 Canonical Structure for Field Theories

We explained in the introduction that one has to be careful with the boundary terms when studying field theories on manifolds with boundaries. The problem is even deeper: in presence of time-like boundaries, the usual Poisson bracket does not satisfy the Jacobi identity. It means that the canonical structure is not well-defined.

Let’s consider a simple example: 3D Chern-Simons theory on a cylinder $\mathbb{R} \times D$ with standard coordinates $x^\mu = (t, r, \phi)$ the time-like boundary being given by $r = R$. The action is given by

$$S[A_\mu^a] = -\frac{\kappa}{2\pi} \int d^2 x \frac{1}{2} \epsilon^{ij} g_{ab} \left( A_i^a \dot{A}_j^b - A_0^a F_{ij}^b \right), \quad (2.1)$$

where $\epsilon^{12} = 1$ and the metric $g_{ab}$ is a symmetric non-degenerate invariant tensor on the Lie algebra $g$. We use the field strength $F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + f_{bc}^a A_b^i A_c^j$ with $f_{bc}^a$ being the structure constants of $g$. If we impose the boundary condition $A_0^a|_{\partial D} = 0$, the action is well-defined and the lagrangian picture makes sense [16, 17]. Let’s now compute the Jacobi identity for the following gauge generators:

$$I = -\frac{\kappa}{4\pi} \int_D d^2 x \rho A_r^a \epsilon^{ij} g_{ab} F_{ij}^b, \quad J = -\frac{\kappa}{4\pi} \int_D d^2 x \eta^a \epsilon^{ij} g_{ab} F_{ij}^b, \quad (2.2)$$

$$K = -\frac{\kappa}{4\pi} \int_D d^2 x \xi^a \epsilon^{ij} g_{ab} F_{ij}^b, \quad (2.3)$$

where $\eta^a, \xi^a, \rho$ are independent of the dynamical fields and $\rho = 0$ in a neighborhood of the origin. For simplicity, let’s assume that $\eta^a|_{\partial D} = 0$ and $\rho = 1$ on a neighborhood of $\partial D$. A straightforward computation gives:

$$\{I, \{J, K\}\} + \{J, \{K, I\}\} + \{K, \{I, J\}\} \approx \frac{\kappa}{2\pi} \int_{\partial D} d\phi \partial_r \eta^a g_{ab} D_\phi \xi^b. \quad (2.4)$$

which is non-zero in general. The covariant derivative is defined by $D_i \xi^a = \partial_i \xi^a + f_{bc}^a A_b^i \xi^c$ and we used the symbol $\approx$ to denote equality on the constraints surface.

The notion of differentiable functional introduced by Regge and Teitelboim in [4] solves this problem and allows for a good definition of the canonical structure. The idea is to restrict the set of allowed functionals to the set of differentiable functionals. In the above example, the functional $I$ is not differentiable and should not be allowed in the Poisson bracket. More general definitions of the canonical structure in presence of a boundary have been developed in [18, 19, 20], but they add non-trivial dynamics on the boundary and will not be needed for our description.

In this section, we will present a constructive introduction to the Regge-Teitelboim idea and its link to the Lagrangian framework. We will start with the description of the
symplectic structure for field theories and introduce the idea of differentiable functionals. We will then make the link with the lagrangian notion of well-defined action. In the last part, we will show that Noether’s theorem associates a differentiable generator to any symmetry of the action.

The main point of this construction is that, using the notion of differentiable generators, the hamiltonian structure of field theories behaves exactly as the one of discrete mechanical systems.

2.1 Symplectic Structure ans Poisson Bracket

The notion of symplectic manifold can be taken as the starting point of the hamiltonian theory [21]:

**Definition 2.1.** Let $\mathcal{M}$ be an even-dimensional differentiable manifold. A symplectic structure on $\mathcal{M}$ is a closed non-degenerate differential 2-form $\omega$ on $\mathcal{M}$:

$$d\omega = 0 \quad \text{and} \quad \forall \xi \neq 0, \exists \eta : \omega(\xi, \eta) \neq 0, \xi, \eta \in T_x\mathcal{M}. \quad (2.5)$$

The pair $(\mathcal{M}, \omega)$ is called a symplectic manifold.

If we have Darboux coordinates on $\mathcal{M}$, the symplectic structure takes the form:

$$\omega = dp_i dq^i \quad (2.6)$$

where the $q^i$ describe the position of the system and the $p_i$ are the associated momenta.

For field theories, the equivalent of $\mathcal{M}$ is the set of allowed configurations of the fields $z^A$ that we will denote $\mathcal{F}(\Sigma) = \{z^A(x), x^i \in \Sigma; \chi^\mu(z)|_{\partial \Sigma} = 0\}$. The manifold $\Sigma$ describes constant time slices of the space-time under consideration. The conditions $\chi^\mu(z)|_{\partial \Sigma} = 0$ are the set of boundary condition. We will assume in the following that they are imposed in all equalities.

A differential 2-form on $\mathcal{F}(\Sigma)$ is a functional 2-form (see appendix A). We will restrict ourselves to the most simple case where we have Darboux coordinates for the fields and we will assume that the symplectic structure is given by:

$$\Omega = \int_\Sigma \frac{1}{2} \sigma_{AB} \delta z^A \delta z^B d^n x, \quad (2.7)$$

where $\sigma_{AB}$ is a non degenerate constant antisymmetric matrix whose inverse will be denoted $\sigma^{AB}$. The results we will present in this paper apply only to this case.

**Example 2.1.**
• Electromagnetism: the phase-space can be parametrized by

\[ z^A = (A_i, E^i) \]  

(2.8)

where \( A_i \) is the potential vector and \( E^i \) is the electric field. The symplectic structure is then given by

\[ \Omega = \int \Sigma -\delta E^i \delta A_i \, d^n x \]  

(2.9)

• Gravity: in this case, we can use the spacial metric \( g_{ij} \) and its conjugate momentum \( \pi^{ij} \) to describe the phase-space:

\[ z^A = (g_{ij}, \pi^{ij}) \]  

(2.10)

and

\[ \Omega = \int \Sigma \delta \pi^{ij} \delta g_{ij} \, d^n x. \]  

(2.11)

The symplectic structure of a manifold defines an isomorphism between 1-forms and vector fields. In the field-theoretic case, the 1-forms are functional 1-forms and the vector fields are evolutionary vector fields preserving the boundary conditions. From an allowed evolutionary vector field \( Q^A \frac{\partial}{\partial z^A} \) we can build a functional 1-form \( \Theta_Q \) using the symplectic structure (2.7):

\[ \Theta_Q = \iota_Q \Omega = \int \sigma_{AB} Q^A \delta z^B \, d^n x. \]  

(2.12)

Due to the particular form of \( \Theta_Q \) and the restrictions on \( Q^A \), the image of this application is never the full set of functional 1-forms.

**Definition 2.2.** A differential 1-form is a functional 1-form of the form

\[ \Theta = \int \Sigma \theta_A \delta z^A \, d^n x, \]  

(2.13)

such that the evolutionary vector field \( \sigma^{AB} \theta_B \frac{\partial}{\partial z^A} \) preserves the boundary conditions.

Any functional 1-form can be put into this form up to boundary terms. The key point is that these boundary terms must be zero using the boundary conditions. The application (2.12) defines an isomorphism between the differential 1-forms and the evolutionary vector fields preserving the boundary conditions. We will denote by \( J \) the inverse of this isomorphism.

The fact that we needed to restrict the set of functional 1-forms in order to have an isomorphism with the evolutionary vector fields means that we will not be able to associate a hamiltonian vector field to all functionals. Only functionals for which the differential \( \delta \) gives a differential 1-form will generate a hamiltonian transformation. This leads to the following definition:
Definition 2.3. A functional $G = \int_{\Sigma} g \, d^n x$ is called differentiable if its differential $\delta G$ is a differential 1-form:

$$\delta G = \int_{\Sigma} \frac{\delta g}{\delta z^A} \delta z^A \, d^n x \quad \Leftrightarrow \quad \oint_{\partial \Sigma} I^n(g \, d^n x) = 0 \quad (2.14)$$

and the evolutionary vector field $\sigma^{AB} \frac{\delta g}{\delta z^A} \frac{\partial}{\partial z^B}$ preserves the boundary conditions.

The property (2.14) can also be written in term of evolutionary vector fields by asking that for all variations $\delta Q$ preserving the boundary conditions, we have

$$\delta_Q G = \int_{\Sigma} \frac{\delta g}{\delta z^A} Q^A \, d^n x \quad \Leftrightarrow \quad \oint_{\partial \Sigma} I^n_Q(g \, d^n x) = 0. \quad (2.15)$$

This definition of differentiable functional is the one introduced in [4], but we see that it comes naturally from the analysis of the symplectic structure.

Definition 2.4. An evolutionary vector field $Q^A \frac{\partial}{\partial z^A}$ is called hamiltonian if there exists a differentiable functional $G = \int_{\Sigma} g \, d^n x$ such that

$$Q^A \frac{\partial}{\partial z^A} = J G \quad \Leftrightarrow \quad Q^A = \sigma^{AB} \frac{\delta g}{\delta z^B}. \quad (2.16)$$

The functional $G$ is the generator of $Q^A \frac{\partial}{\partial z^A}$.

Using these definitions, field theories behave in exactly the same way as standard mechanical systems. We will now derive some of the most important hamiltonian results that we will need later.

Proposition 2.5. Let the phase-space $\mathcal{F}(\Sigma)$ be path-connected. If two differentiable functionals $G_1$ and $G_2$ generate the same hamiltonian vector field, then they differ only by a constant.

Proof. We have

$$\frac{\delta g_1}{\delta z^A} = \frac{\delta g_2}{\delta z^A}. \quad (2.17)$$

Because $G_1$ and $G_2$ are both differentiable, it imposes $\delta(G_1 - G_2) = 0$. Due to the path-connectedness of the phase-space, the functional $G_1 - G_2$ is a constant. \qed

This property relies heavily on the notion of differentiable functional. If we drop the differentiability condition and use $Q^A = \sigma^{AB} \frac{\delta g}{\delta z^B}$ as the definition of the evolutionary vector field associated to $G$ then a generator would be defined only up to a boundary term.

The following definition and properties describe the Poisson bracket induced on the differentiable functionals by the symplectic structure $\Omega$. 

**Definition 2.6.** The bracket of two differentiable functionals $F$ and $G$ is the functional given by

$$ \{F, G\} = \iota_{F^A} \iota_{G} \Omega = \Omega(G^A, F^A) $$

(2.18)

where $F^A$ and $G^A$ are the characteristics of the hamiltonian vector fields associated to $F$ and $G$:

$$ F^A = \sigma^{AB} \frac{\delta f}{\delta z^B}, \quad G^A = \sigma^{AB} \frac{\delta g}{\delta z^B}. $$

(2.19)

The bracket takes the simple form:

$$ \{F, G\} = \int_{\Sigma} \frac{\delta f}{\delta z^A} G^A \frac{\delta g}{\delta z^B} d^n x. $$

(2.20)

**Proposition 2.7.** The variation of a differentiable functional $F$ along the hamiltonian vector field generated by $G$ is given by

$$ \delta_G F = \{F, G\}. $$

(2.21)

**Proof.** We have

$$ \delta_G F = \int_{\Sigma} \frac{\delta f}{\delta z^A} G^A d^n x + \int_{\partial \Sigma} I^n_G (f d^n x) $$

$$ = \int_{\Sigma} \frac{\delta f}{\delta z^A} G^A d^n x = \int_{\Sigma} \frac{\delta f}{\delta z^A} \sigma^{AB} \frac{\delta g}{\delta z^B} d^n x. $$

(2.22)

The boundary term is zero because the vector field $G^A \frac{\partial}{\partial z^A}$ preserves the boundary conditions and $F$ satisfies (2.15).

**Proposition 2.8.** The bracket $\{F, G\}$ defines a Poisson bracket on the set of differentiable functionals:

- $\{F, G\}$ is a differentiable functional
- $\{F, G\} = - \{G, F\}$
- $\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0$

where $F$, $G$ and $H$ are differentiable functionals.

**Proof.** We will only prove the first condition as the other two can be proved easily following the discrete case. This proof has first been done in [8]. We have to prove that $d \nu \{F, G\}$ does not contain boundary terms and that its associated vector field preserves the boundary conditions.

One can prove that

$$ \delta \frac{\delta f}{\delta z^A} G^A d^n x = \delta z^A \delta_G \frac{\delta f}{\delta z^A} d^n x + d (\delta I^n_G (f d^n x) - \delta_G I^n (f d^n x)). $$

(2.23)
Integrating over $\Sigma$, we obtain
\[ \int_{\Sigma} \delta \frac{\delta f}{\delta z^A} G^A d^n x = \int_{\Sigma} \delta z^A \delta G \frac{\delta f}{\delta z^A} d^n x. \] (2.24)

The boundary terms disappear because $F$ is a differentiable functional and $G^A \frac{\partial}{\partial z^A}$ preserves the boundary conditions (see appendix A.2). The differential of $\{F, G\}$ is then easily computed
\[ \delta \{F, G\} = \int_{\Sigma} \left( \delta \frac{\delta f}{\delta z^A} G^A - \delta \frac{\delta g}{\delta z^A} F^A \right) d^n x \]
\[ = \int_{\Sigma} \delta z^A \left( \delta G \frac{\delta f}{\delta z^A} - \delta F \frac{\delta g}{\delta z^A} \right) d^n x, \] (2.25)

which does not contain any boundary term. The characteristics of the hamiltonian vector field generated by $\{F, G\}$ can be read off from the above equation:
\[ \{F, G\}^A = \sigma^{AB} \left( \delta G \frac{\delta f}{\delta z^B} - \delta F \frac{\delta g}{\delta z^B} \right) \]
\[ = \delta_G F^A - \delta_F G^A \]
\[ = - [F, G]^A, \] (2.26)

where the last expression is the Lie bracket of the two hamiltonian vector fields $F^A \frac{\partial}{\partial z^A}$ and $G^A \frac{\partial}{\partial z^A}$. The Lie bracket of two evolutionary vector fields preserving the boundary conditions preserves the boundary conditions which implies that $\{F, G\}$ is a differentiable functional.

**Corollary 2.9.** *The application sending a differentiable generator onto its associated hamiltonian vector field is a homomorphism of Lie algebras.*

It is important to keep in mind that only differentiable functionals can enter the Poisson bracket. For all purposes, functionals that are not differentiable don’t exist in the hamiltonian framework. This fact has a lot of consequences. It is, for instance, the property used to solve the problem of charges in gauge theories. We will also use it in section 4 to build functionals in order to probe the reduced phase-space of theories with no local degrees of freedom.

### 2.2 Well defined Actions

We saw in the previous section that the notion of differentiable generator is a key point of the canonical formalism for field theories defined on a manifold with boundary. This condition can be reinterpreted as follows: if $G$ is a differential functional then the hamiltonian action generating the evolution along the associated hamiltonian vector field is well defined
\[ S_G[z^A] = \int ds \left( \int_{\Sigma} d^n x \frac{1}{2} \sigma_{AB} z^A \partial_s z^B - G[z^A] \right). \] (2.27)
We use “well defined” in the sense that the variation of the action $S_G$ will not generate any boundary term on $\partial \Sigma$. The condition that $G^A \frac{\partial}{\partial x}$ preserves the boundary conditions is equivalent to the requirement that the evolution along the parameter $s$ stays inside the allowed configurations. Those are of course important properties in the case where the differentiable functional is the Hamiltonian of the theory: $H[z^A]$.

The canonical structure and Hamiltonian of a theory are usually deduced from the Lagrangian description of the theory. One expects that a well defined Lagrangian action will lead to a differentiable Hamiltonian. This is indeed the case.

Let’s assume that we have a set of boundary conditions for the dynamical fields $\phi^a$ and a well-defined Lagrangian $L$ on $\mathcal{M} = \mathbb{R} \times \Sigma$. As we saw in the introduction, the differentiation of a well-defined Lagrangian does not create any boundary term:

$$\delta S[\phi] = \int dt \int_\Sigma d^n x \left( \frac{\delta L}{\delta \phi^a} \delta \phi^a + \frac{\delta L}{\delta \dot{\phi}^a} \delta \dot{\phi}^a \right). \quad (2.28)$$

We have also assumed that $L$ does not depend on second or higher time derivatives. The Euler-Lagrange derivatives are only defined on $\Sigma$, they don’t take into account the derivatives with respect to $t$. We will restrict our analysis to boundary conditions on $\partial \Sigma$ that are independent of time. If this is not the case, the phase-space is time-dependent and the canonical structure developed in the previous section needs to be improved.

The momenta are defined as

$$\pi_a \equiv \frac{\delta L}{\delta \dot{\phi}^a}. \quad (2.29)$$

If this relation can be inverted, we can express $\dot{\phi}^a$ as local functions of $\pi_a$ and $\phi^a$. The boundary conditions on $\phi^a$ imply boundary conditions on $\pi_a$. The Hamiltonian is then defined as

$$H[\phi^a, \pi_a] = \int_\Sigma d^n x \left( \pi_a \dot{\phi}^a - L \right)_{\phi^a=\phi^a(\pi,\phi)}. \quad (2.30)$$

The variation of $H$ can be easily computed:

$$\delta H = \int_\Sigma d^n x \left( \delta \pi_a \dot{\phi}^a + \pi_a \delta \dot{\phi}^a - \delta L \right)_{\phi^a=\phi^a(\pi,\phi)}$$

$$= \int_\Sigma d^n x \left( \delta \pi_a \dot{\phi}^a + \pi_a \delta \dot{\phi}^a - \frac{\delta L}{\delta \phi^a} \delta \phi^a - \frac{\delta L}{\delta \dot{\phi}^a} \delta \dot{\phi}^a \right)_{\phi^a=\phi^a(\pi,\phi)}$$

$$= \int_\Sigma d^n x \left( \delta \pi_a \dot{\phi}^a - \frac{\delta L}{\delta \phi^a} \delta \phi^a \right)_{\phi^a=\phi^a(\pi,\phi)} \quad (2.31)$$

which does not contain any boundary term. The hamiltonian vector field associated to $H$ is the time evolution. A consistent choice of boundary conditions for $L$ requires these boundary conditions to be preserved by the time evolution. The Hamiltonian $H$ is a differentiable generator by construction.
If (2.29) is not invertible, we have to add primary constraints $\psi_\alpha = 0$ (see [22] for the details). The remarkable property of $H$ is that it depends only on $\phi^a$ and $\pi_a$ even when the relation (2.29) is not invertible. The analysis above is still valid and $H[\pi, \phi]$ is again a differentiable functional. The constraints $\psi_\alpha$ as such are not differentiable functionals and can not enter the Poisson bracket. The solution is to build the smeared quantities:

$$\Gamma_\lambda[\phi^a, \pi_a] = \int_\Sigma d^n x \lambda^\alpha \psi_\alpha,$$

where the different possible functions $\lambda^\alpha(x)$ play the role of labels. The differentiability of $\Gamma_\lambda$ will impose boundary conditions on $\lambda^\alpha(x)$. The local constraints $\psi_\alpha = 0$ are equivalent to the requirement that $\Gamma_\lambda = 0$ for any allowed function $\lambda^\alpha$. The hamiltonian action is then given by:

$$S[\phi^a, \pi_a, \lambda^\alpha] = \int dt \left( \int_\Sigma d^n x \pi_a \dot{\phi}^a - H[\phi^a, \pi_a] - \Gamma_\lambda[\phi^a, \pi_a] \right).$$

The functions $\lambda^\alpha$ play the role of Lagrange multipliers enforcing the constraints $\psi_\alpha = 0$. Both $H$ and $\Gamma_\lambda$ are differentiable generators as expected.

The action (2.33) is not the end of the story and one can continue the Dirac algorithm to build the full set of constraints of the theory. The secondary constraints are obtained by requiring the preservation in time of the primary constraints: $\frac{d}{dt} \Gamma_\rho = 0$ for all allowed $\rho$. This gives:

$$0 = \frac{d}{dt} \Gamma_\rho = \left\{ \int_\Sigma d^n x \rho^\alpha \psi_\alpha, H[\phi^a, \pi_a] + \Gamma_\lambda[\phi^a, \pi_a] \right\}.$$

This gives conditions on the Lagrange multipliers $\lambda^\alpha$ or new constraints. If there are new constraints, this procedure has to be continued to check their preservation in time by constructing the associated differentiable smeared quantities.

**Example 2.2. Electromagnetism in 4D.**

We will work with $\Sigma$ a ball of finite radius in $\mathbb{R}^3$. The metric on $\mathbb{R} \times \Sigma$ is the flat metric $\eta_{\mu\nu}$. The action is then given by:

$$S[A_\mu] = \int_{\mathbb{R} \times \Sigma} d^4 x \frac{-1}{4} F_{\mu\nu} F^{\mu\nu}.$$  

(2.35)

It is well defined if $A_0, A_I$ are fixed on the boundary where $x^I = (\theta, \phi)$ are coordinates on the sphere $\partial \Sigma$ and $(A_0, A_I)$ are the components of the pull-back of $A_\mu$ on the boundary $\mathbb{R} \times \partial \Sigma$. We will take these boundary values to be time independent. The momenta are given by:

$$\pi^0 = 0, \quad \pi^i = F^{i0}, \quad \pi_i = \partial_0 A_i - \partial_i A_0.$$  

(2.36)
We have one primary constraint $\psi = \pi^0$. The smeared quantity (2.32) becomes $\Gamma_\lambda = \int_\Sigma d^3x \lambda \pi^0$. Its associated hamiltonian vector field will preserve the boundary conditions ($A_0$ fixed) only if $\lambda = 0$ on the boundary. The hamiltonian action is then

$$S[\pi^\mu, A_\mu, \lambda] = \int dt \int_\Sigma d^3x \left( \pi^\mu \dot{A}_\mu - h - \lambda \pi^0 \right),$$

$$h = \frac{1}{2} \pi^i \pi_i + \frac{1}{4} F_{ij} F^{ij} + \pi^i \partial_i A_0.$$ (2.37)

The boundary conditions are $\lambda = 0$ and $A_0, A_I$ fixed on the boundary. One can easily check that the Hamiltonian is a differentiable generator.

The preservation of $\psi$ will imply a secondary constraint:

$$0 = \left\{ \int_\Sigma d^3x \rho \pi^0, H + \Gamma_\lambda \right\}$$

$$= \int_\Sigma d^3x \rho \partial_i \pi^i.$$ (2.38)

This must be zero for all $\rho$ (with $\rho = 0$ on the boundary) which gives us the secondary constraint $\partial_i \pi^i = 0$. In this case, the Dirac algorithm stops here and the total action can be written

$$S[\pi^\mu, A_\mu, \lambda_1, \lambda_2] = \int dt \int_\Sigma d^3x \left( \pi^\mu \dot{A}_\mu - h - \lambda_1 \pi^0 - \lambda_2 \partial_i \pi^i \right),$$

with $\lambda_1 = 0 = \lambda_2$ on the boundary. This is not yet the usual hamiltonian action. It is obtained by solving the constraint $\pi^0 = 0$ and introducing the electric field $E^i = -\pi^i$:

$$S[E^i, A_\mu, \lambda_2] = \int dt \int_\Sigma d^3x \left[ -E^i \dot{A}_i - \left( \frac{1}{2} E^i E_i + \frac{1}{4} F_{ij} F^{ij} \right) + E^i \partial_i (A_0 - \lambda_2) \right].$$ (2.41)

We can absorb $\lambda_2$ in $A_0$ which then takes the role of lagrange multiplier for the Gauss constraint $\partial_i E^i = 0$. However, only the bulk part of $A_0$ is a lagrange multiplier, its boundary value is still fixed and contributes to the hamiltonian through the boundary term:

$$H[z^A] \approx \int_\Sigma d^3x \left( \frac{1}{2} E^i E_i + \frac{1}{4} F_{ij} F^{ij} \right) - \oint_{\partial \Sigma} d\Omega^2 A_0 E^r,$$ (2.42)

where we used spherical coordinates $(r, \theta, \phi)$ on $\Sigma$ with $d\Omega^2$ the usual measure on the 2-sphere. The sign $\approx$ denotes equality on the surface of the constraints.

In a similar way, in gravity, the lapse and shift become lagrange multipliers after solving the primary constraints, but only their bulk part are really arbitrary. Their boundary values contribute in a non-trivial way to the Hamiltonian of the theory (see section 3.7).
2.3 Global Symmetries

In the hamiltonian description of discrete systems, Noether’s theorem relates symmetries of the Hamiltonian action and generators that commute with the Hamiltonian of the theory. In this section, we will study the same problem in field theories and its interaction with the notion of differentiable generator. We will do this analysis without constraints but it can be extended to gauge field theories (see exercise 3.24 of [22]).

Let’s consider the following hamiltonian action:

\[ S[z^A] = \int dt \left( \int d^n x \frac{1}{2} \sigma_{AB} z^A \dot{z}^B - H[z^A] \right) \]  \hspace{1cm} (2.44)

with a differentiable Hamiltonian. Let’s assume that we have a variation \( \delta Q \) preserving the boundary conditions, with \( Q^A \) local functions of \( z^A \) and not of their time derivatives, such that it also preserves the action:

\[ \delta Q S = \int dt \int d^n x \frac{d}{dt} k, \quad \delta Q z^A = Q^A. \]  \hspace{1cm} (2.45)

We only allowed conservation up to a total time derivative as we expect the theory to behave exactly as a discrete mechanical system. We will see later that this conditions is necessary (see theorem 2.11). Expanding the left hand side, we obtain:

\[ \delta Q S = \int dt \int d^n x \left( \frac{1}{2} \sigma_{AB} Q^A \dot{z}^B + \frac{1}{2} \sigma_{AB} z^A \frac{d}{dt} Q^B - \delta h \delta z^A Q^A \right). \]  \hspace{1cm} (2.46)

There is no boundary term coming from the variation \( \delta Q H \) as \( H \) is a differential functional. Let’s introduce the functional:

\[ G[z^A] = \int d^n x \left( \frac{1}{2} \sigma_{AB} z^A Q^B - k \right), \quad g = \frac{1}{2} \sigma_{AB} z^A Q^B - k. \]  \hspace{1cm} (2.47)

We can rewrite condition (2.45) as

\[ \frac{d}{dt} G = \int d^n x \left( -\sigma_{AB} Q^A \dot{z}^B + \frac{\delta h}{\delta z^A} Q^A \right). \]  \hspace{1cm} (2.48)

This equality is valid for any values of the fields \( z^A \) and any values of their time derivative \( \dot{z}^A \). We can expand the left hand side as

\[ \frac{d}{dt} G = \int d^n x \left( \frac{\partial g}{\partial t} + \frac{\delta g}{\delta z^A} \dot{z}^A \right) + \oint_{\partial \Sigma} I^n_i (g d^n x) \]  \hspace{1cm} (2.49)

where \( \frac{\partial}{\partial t} \) is the partial derivative with respect to \( t \) and \( I^n_i \) is the homotopy operator \( (A.9) \) associated to \( \delta z^A = \dot{z}^A \). Because \( \dot{z}^A \) is an arbitrary variation of \( z^A \) and there is no boundary term involving it in the right hand side of (2.48), it implies that the boundary
term in the above expression is zero. We recognize part of the differentiability of the functional $G$:

$$\delta G = \int_\Sigma d^nx \frac{\delta g}{\delta z^A} \delta z^A, \quad \forall \delta z^A.$$  \hspace{1cm} (2.50)

Locally, equation (2.48) gives also

$$\frac{\delta g}{\delta z^A} = \sigma_{AB} Q^B,$$  \hspace{1cm} (2.51)

$$\frac{\partial g}{\partial t} = \frac{\delta h}{\delta z^A} Q^A.$$  \hspace{1cm} (2.52)

The first equation implies that $Q^A \frac{\partial}{\partial z^A}$ is a hamiltonian vector field with generator $G$ and that $G$ is a differentiable functional as $\delta Q$ preserves the boundary conditions. The second equation gives the conservation of $G$:

$$\frac{d}{dt} G = \frac{\partial G}{\partial t} + \{ G, H \} = 0.$$  \hspace{1cm} (2.53)

**Theorem 2.10.** If we have a variation $\delta Q$ preserving the boundary conditions and preserving the action in the sense (2.45), then there exists a differentiable generator $G$ such that $Q^A \frac{\partial}{\partial z^A}$ is the associated hamiltonian vector field and satisfying

$$\frac{\partial G}{\partial t} + \{ G, H \} = 0.$$  \hspace{1cm} (2.54)

The converse is also true. Let’s assume that we have a conserved differentiable functional $G$, i.e. satisfying equation (2.54). Then the variation of the action along the associated hamiltonian vector field $G^A \frac{\partial}{\partial z^A}$ is given by:

$$\delta_G S = \int dt \left[ \int d^n x \left( \frac{1}{2} \sigma_{AB} G^A \dot{z}^B + \frac{1}{2} \sigma_{AB} \frac{d}{dt} G^B \right) - \delta_G H \right]$$

$$= \int dt \left[ \int d^n x \left( \sigma_{AB} G^A \dot{z}^B + \frac{1}{2} \sigma_{AB} \frac{d}{dt} (z^A G^B) \right) - \{ H, G \} \right]$$

$$= \int dt \left[ \int d^n x \left( - \frac{\delta g}{\delta z^A} \dot{z}^A - \frac{\partial g}{\partial t} \right) + \frac{d}{dt} \int d^n x \frac{1}{2} \sigma_{AB} z^A G^B \right]$$

$$= \int dt \frac{d}{dt} \left[ G + \int d^n x \frac{1}{2} \sigma_{AB} z^A G^B \right],$$  \hspace{1cm} (2.55)

where we have used the fact that $\dot{z}^A$ is a variation of the fields $z^A$ preserving the boundary conditions to obtain the last line.

**Theorem 2.11.** If $G$ is a differentiable functional such that

$$\frac{\partial G}{\partial t} + \{ G, H \} = 0,$$  \hspace{1cm} (2.56)

then the variation generated by $G$ preserves the action in the sense of (2.45).
These results are a direct application of Noether’s theorem but they are free of the problems we presented in the introduction. The conserved charges that we built are defined up to a constant only and not up to a boundary term. Applying Noether’s theorem to a gauge symmetry will construct the associated generator with the right boundary term. The source of the problem of gauge theories has been handled by a careful treatment of the boundary conditions. We will see in the next section how to apply these ideas to the computation of surface charges.
3 Applications to Gauge Theories

As we saw in the introduction, surface charges are the conserved quantities associated to gauge-like transformations. For instance, in electromagnetism, we have the electric charge or, in gravity, the energy and the angular momentum of the system. In both cases, these conserved quantities are associated with transformations that look like gauge transformations. As they are generated by constraints, we expect those charges to give zero on the constraints surface. We will see in this section how the notion of differentiable generator introduced in the previous section solves the problem and associates non-zero charges to a certain class of gauge-like transformations.

Usually when we want to compute the surface charges of a theory, we have a set of solutions for which we want to compute those charges. This means that we only know the local form of the action of the theory and not all the boundary terms necessary to make it well defined. In that spirit, we will consider theories of the form

\[ S[z^A, \lambda^\alpha] = \int dt \int d^n x \left( \frac{1}{2} \sigma_{AB} z^A \dot{z}^B - h - \lambda^\alpha \gamma_\alpha \right), \tag{3.1} \]

where \( h \) is the first class hamiltonian density and \( \gamma_\alpha \) are the full set of first-class constraints. The weak equality sign \( \approx \) will be used for the equality on the constraints surface \( \gamma_\alpha \approx 0 \). We will denote \( h_T = h + \lambda^\alpha \gamma_\alpha \) the total hamiltonian density. Such an action is in general well defined only for very restrictive boundary conditions that usually don’t contain the solutions we are interested in.

The first step in the analysis is to define a set of boundary conditions containing the solutions of interest and add the right boundary term to (3.1) in order to make everything well defined.

3.1 Boundary Conditions and Total Hamiltonian

The choice of boundary conditions is a very tricky one and there are a lot of different possibilities. The only necessary restriction imposed by the consistency of the theory is that the total Hamiltonian be a differential functional. When dealing with boundaries at infinity, we will also require finiteness of the total Hamiltonian.

If we want to compute the charges of some particular solutions associated to specific symmetries, we need boundary conditions that both contain the solutions and are preserved by the symmetries under consideration.

Let’s assume that a set of boundary conditions for both the dynamical fields \( z^A \) and for the lagrange multipliers \( \lambda^\alpha \) has been selected. As in section 2.2, we need the boundary
conditions on the dynamical fields $z^A$ to be time independent. The differentiability of $H_T$ implies two conditions. The first one is that its associated evolutionary vector field $\sigma^{AB} \frac{\delta h_T}{\delta z^B}$ preserves the boundary conditions on $z^A$. The second condition is that there exist a $(n-1,0)$-form $b(z, \lambda)$ such that the total Hamiltonian defined by

$$H_T = \int_{\Sigma} h_T d^n x + \oint_{\partial \Sigma} b,$$

is a differentiable functional. The boundary term must satisfy

$$-\oint_{\partial \Sigma} \delta b = \oint_{\partial \Sigma} I^n (h_T d^n x),$$

where the right hand side is the boundary term produced by the variation of the bulk term $\int_{\Sigma} h_T d^n x$. As we saw in the previous section (proposition 2.5), this boundary term is defined only up to a constant with respect to $z^A$: it is defined up to a function of the lagrange multipliers $\lambda^\alpha$.

The candidate action is then given by

$$S[z^A, \lambda^\alpha] = \int dt \left[ \int d^n x \left( \frac{1}{2} \sigma^{AB} z^A \dot{z}^B - h - \lambda^\alpha \gamma_\alpha \right) - \oint_{\partial \Sigma} b \right],$$

$$= \int dt \left[ \int d^n x \frac{1}{2} \sigma^{AB} z^A \dot{z}^B - H_T[z^A, \lambda^\alpha] \right].$$

The above considerations make this action well defined with respect to variations of the dynamical fields $z^A$. With this, the hamiltonian structure of the theory is well defined and we can continue the analysis.

In general, a variation with respect to the lagrange multipliers $\lambda^\alpha$ will still produce boundary term in the action which lead to extra constraints on the boundary. It will be particularly useful in section 4.2 to encode part of the boundary conditions as extra constraints. However, in most cases, those boundary constraints are not welcome. To remove them, one needs to select the total Hamiltonian so that the action is also well defined with respect to variations of the lagrange multipliers:

$$\delta^\lambda S[z^A, \lambda^\alpha] = \int dt \left[ \int d^n x \delta^\lambda \gamma_\alpha + \oint_{\partial \Sigma} \delta^\lambda b \right],$$

where $\delta^\lambda$ only hits the lagrange multipliers. It means that $b$ must be independent of $\lambda^\alpha$. If such a $b$ exists, this last requirement fixes the boundary term up to a constant.

For certain sets of boundary conditions, it might not be possible to find a boundary term satisfying all the above conditions. This is referred to as the integrability problem. In that case, the selected boundary conditions are too relaxed and it is not possible to write an associated well defined theory. The only solution is to restrict the boundary conditions.
In the following, we will assume that a set of boundary conditions for both $z^A$ and $\lambda^\alpha$ has been selected such that the total Hamiltonian $H_T$ is a differentiable generator. In general, the boundary conditions of $\lambda^\alpha$ can depend on $z^A$. It will be useful to decompose the lagrange multipliers $\lambda^\alpha$ as $\lambda^\alpha = \bar{\lambda}^\alpha + \mu^\alpha$ such that $\bar{\lambda}^\alpha$ are fixed in term of $z^A$ in order to encode this dependence. The quantities $\mu^\alpha$ are left to vary freely up to boundary conditions independent of $z^A$ with:

$$H_T[z^A, \mu^\alpha] \approx 0 \Rightarrow H_T[z^A, \lambda^\alpha] \approx H_T[z^A, \bar{\lambda}^\alpha(z^A)].$$

(3.6)

The fields $\mu^\alpha$ are the real lagrange multipliers and $\bar{\lambda}^\alpha(z^A)$ encode the contribution of $\lambda^\alpha$ to $H_T$ through the boundary term.

Example 3.1. Gravity.

The local action for gravity in $n + 1$ dimensions is given by [3, 4]:

$$S[\pi^{ij}, g_{ij}, N, N^i] = \int dt \int_\Sigma d^n x \left( \pi^{ij} \dot{g}_{ij} - N^r \mathcal{R} - N^i \mathcal{R}_i \right),$$

(3.7)

$$\mathcal{R} = -\sqrt{g} R - \frac{1}{\sqrt{g}} \left( \frac{1}{n-1} \pi^{ij} \pi^{ij} - \pi^{ij} \pi^{ij} \right),$$

(3.8)

$$\mathcal{R}_i = -2 \nabla_j \pi^j_i,$$

(3.9)

where indices are lowered and raised using the metric $g_{ij}$ and its inverse $g^{ij}$. The derivative $\nabla_i$ is the covariant derivative associated to $g_{ij}$ and $R$ is the corresponding Ricci scalar. The tensor $\pi^{ij}$ is treated as a density. The lapse $N$ and shift $N^i$ are coming from the $3 + 1$ decomposition of the 4 dimensional metric. We will assume that $\Sigma$ is a finite manifold with a boundary $\partial \Sigma$. The boundary term adapted to the Dirichlet boundary conditions in the Hamiltonian formalism can be found in [23]. The boundary conditions are given by:

$$N^r|_{\partial \Sigma} = 0, \quad N|_{\partial \Sigma} = \bar{N},$$

(3.10)

$$N^A|_{\partial \Sigma} = \bar{N}^A, \quad g_{AB}|_{\partial \Sigma} = \gamma_{AB},$$

(3.11)

where the coordinates are given by $x^i = r, x^A$ and the boundary $\partial \Sigma$ is a surface at constant $r$. Fixing $N, N^A, g_{AB}$ is equivalent to fixing $g_{00}, g_{0A}, g_{AB}$ when $N^r = 0$. The surface term needed to make the total Hamiltonian well defined is then easily computed. The variation of the smeared constraints gives

$$- \oint_{\partial \Sigma} \Gamma^n \left( N^r \mathcal{R} + N^i \mathcal{R}_i \right) = \oint_{\partial \Sigma} d^{n-1} x \left\{ 2N^A \delta \pi_A^r + \sqrt{\gamma} N \left( \delta K + \gamma^{AB} \delta K_{AB} \right) \right\},$$

(3.12)

where $K$ is the trace of the extrinsic curvature of the boundary. The total Hamiltonian is then defined as:

$$H_T = \int_\Sigma d^n x \left( N^r \mathcal{R} + N^i \mathcal{R}_i \right) + 2 \oint_{\partial \Sigma} d^{n-1} x \left\{ N^A \pi_A^r + N \sqrt{\gamma} K \right\}.$$

(3.13)
When evaluated on a solution, the total Hamiltonian gives the energy of the system. In this case, the only non-zero contribution comes from the boundary term and we have

$$H_T \approx 2 \oint_{\partial \Sigma} d^{n-1} x \left\{ \bar{N}^A \pi^r_A + \bar{N} \sqrt{\gamma} K \right\}.$$  (3.14)

This value of the Hamiltonian is tied to the boundary conditions we imposed on $N$ and $N^i$. This reflects the fact that those boundary values are not behaving as Lagrange multipliers but carry information about the dynamics of the system.

### 3.2 Differentiable Gauge Transformations

Gauge transformations are transformations generated by the first-class constraints of the theory through the Poisson bracket. In field theories, we call gauge transformation any transformation of the form

$$\delta_\eta z^A = \sigma^{AB} \frac{\delta}{\delta z^B} (\eta^\alpha \gamma_\alpha),$$  (3.15)

where the gauge parameters $\eta^\alpha$ can be functions of the dynamical fields $z^A$. The algebra of these transformations closes:

$$[\delta_\eta, \delta_\rho] = \delta_\epsilon,$$  (3.16)

$$[\eta, \rho]_g^\alpha \equiv \epsilon^\alpha$$

where $\delta_\epsilon$ is a gauge transformation and $[\eta, \rho]_g^\alpha$ is the bracket induced on the gauge parameters. The transformations $3.15$ can be extended to the Lagrange multipliers $\lambda^\alpha$ in order to leave the action invariant up to a boundary term $[22]$. The resulting variation is:

$$\delta_\eta \lambda^\alpha = \frac{\partial}{\partial t} \eta^\alpha + [\lambda, \eta]_g^\alpha - V(\eta)^\alpha.$$  (3.17)

The quantity $V(\eta)^\alpha$ is defined from the first class Hamiltonian $h$ by

$$\delta_\eta h = V(\eta)^\alpha \gamma_\alpha,$$  (3.18)

where the equality is up to boundary terms. In general, gauge transformations describe the redundancy of the description but, as we saw in the introduction, in the presence of a spatial boundary the story is different. In order to avoid confusion, we will call the transformations $3.15$-(3.17) gauge-like transformations.

The above considerations are only local and don’t take into account the boundary structure of the theory. As we saw in section $2.1$ only differentiable functionals are allowed in the Poisson bracket, however there is no guarantee that gauge-like transformations are generated by differentiable functionals.

For this to happen, we saw in section $2.1$ that one needs two requirements to be satisfied. The first is the preservation of the boundary conditions of $z^A$: they must transform
allowed configurations into allowed configurations of the fields. Requiring that the transformations \( (3.15) \) preserve the boundary conditions of \( z^A \) will impose boundary conditions on the gauge parameters \( \eta^\alpha \).

An important observation here is that we don’t need to require the preservation of the boundary conditions of the Lagrange multipliers in order to build differentiable generators. We will see in section 3.4 that this additional restriction selects transformations that are also symmetries of the theory. This confirms the fact that the boundary conditions of \( \Lambda^\alpha \) contains dynamical information.

The second condition comes from requiring the existence of a suitable boundary term to complete the generator. Its bulk part is given by:

\[
\Gamma_{\eta}[z^A] = \int_{\Sigma} d^n x \eta^\alpha \gamma_\alpha.
\]  

(3.19)

We need a \((n - 1, 0)\)-form \( k_\eta \) such that:

\[
\oint_{\partial \Sigma} \delta k_\eta = -\oint_{\partial \Sigma} I^n (\eta^\alpha \gamma_\alpha d^n x).
\]  

(3.20)

If we can find such form \( k_\eta \), then the generator defined by

\[
\Gamma_{\eta}[z^A] = \tilde{\Gamma}_{\eta}[z^A] + \int_{\partial \Sigma} k_\eta
\]  

(3.21)

is differentiable. It is not always possible to find such a boundary term and the integrability problem might also appear here. In that case, only a subset of the allowed gauge-like transformations will have an associate differentiable generator. We will call this subset of gauge-like transformations differentiable gauge transformations.

As the two requirements we added for differentiable gauge transformations are preserved by the bracket of evolutionary vector fields, the differentiable gauge transformations form a subalgebra of the algebra of gauge-like transformations \( (3.16) \). The algebra of the associated generators forms a representation of this subalgebra:

**Theorem 3.1.** Let the phase-space \( \mathcal{F}(\Sigma) \) be path-connected. If \( \eta^\alpha \) and \( \rho^\alpha \) are two differentiable gauge transformations then:

\[
\{ \Gamma_{\eta}[z^A], \Gamma_{\rho}[z^A] \} = \Gamma_{[\rho,\eta]}[z^A] + K_{\eta,\rho}
\]  

(3.22)

where \( K_{\eta,\rho} \) is a possible central extension. It satisfies

\[
K_{\eta,\rho} = -K_{\rho,\eta},
\]  

(3.23)

\[
K_{[\eta,\rho],\epsilon} + K_{[\rho,\epsilon],\eta} + K_{[\epsilon,\eta]_{\rho}} = 0,
\]  

(3.24)

for all differentiable gauge transformations \( \eta^\alpha, \rho^\alpha \) and \( \epsilon^\alpha \).
Proof. The proof of this theorem is direct. We know that the algebra of the hamiltonian transformations is homomorphic to the algebra of the Hamiltonian generators. The commutator of two differentiable gauge transformations is a gauge-like transformation $[\eta, \rho]_g^\alpha$, the Poisson bracket of the associated generators will be a differentiable generator associated to its resulting gauge-like transformation $[\eta, \rho]_g^\alpha$. The possibility of a central extension comes from the fact that hamiltonian generators are defined up to a constant [21].

If one drops the hypothesis of path-connectedness of the phase-space, the extension of the algebra can become field dependent: it could take different values on different path-connected components of the phase-space.

Differentiable gauge transformations split into two categories, the proper and improper gauge transformations:

- proper gauge transformations $\delta_\eta$ are defined by $\Gamma_\eta[z^A] \approx 0$ for all configurations on the constraints surface. These transformations are generated by constraints of the theory.

- improper gauge symmetries $\delta_\eta$ are those for which there exist field configurations on the constraints surface such that $\Gamma_\eta[z^A] \neq 0$.

There is a common misconception that improper gauge transformations are still generated by constraints, first or second class. This is not the case since, for improper gauge transformations, the integral $\int_\Sigma d^n x \eta^\alpha \gamma_\alpha$ is not a differentiable functional: it can not enter the Poisson bracket.

The proper gauge transformations form an ideal subalgebra of the differentiable gauge transformations: if $\eta^\alpha$ is a proper gauge transformation and $\rho^\alpha$ is a differential gauge transformation, the commutator of the two gauge transformations $[\eta, \rho]_g$ is generated by

$$\{ \Gamma_\rho[z^A], \Gamma_\eta[z^A] \} = -\delta_\rho \Gamma_\eta[z] \approx 0. \quad (3.25)$$

This is zero on the constraint surface because $\Gamma_\eta[z^A] \approx 0$ and $\delta_\rho$ preserves the constraints.

Up until now, we have been very careful of not talking about symmetries. The differentiable gauge transformations are in general not symmetries of the theory: they don’t satisfy

$$\frac{\partial}{\partial t} \Gamma_\eta + \{ \Gamma_\eta, H_T \} \approx 0. \quad (3.26)$$

The weak equality here denotes equality on the constraints surface and is the conservation condition [22] for gauge theories.
However, the proper gauge transformations are symmetries of the system. It comes from the fact that the evolutionary vector field associated to $H_T$ transforms allowed configurations into allowed configurations and preserves the constraints which implies

$$ \{ \Gamma_\eta, H_T \} \approx 0 $$

(3.27)

for proper gauge transformations $\delta_\eta$. Combining it with $\Gamma_\eta \approx 0 \Rightarrow \frac{\partial}{\partial t} \Gamma_\eta \approx 0$ we obtain the conservation condition (3.26).

The proper gauge transformations are the real gauge symmetries of the system. They represent the redundancy of the description. On the other hand, part of the improper gauge transformations will satisfy (3.26) and also be symmetries of the system. Those transformations are global symmetries of the theory, they change the state of the system. They are called asymptotic symmetries and will be the subject of section 3.4.

The fact that proper gauge transformations form an ideal of the differentiable gauge transformations leads to a very interesting result

**Theorem 3.2.** The differentiable gauge transformations are first-class quantities.

First-class functionals in gauge theories are the building blocks. They are gauge invariant and, as such, are the observables of the theory. In most theories, the set of first-class differentiable functionals that one can build in this way is rather small and does not bring a lot of information, see example below. However, in theories with no local degrees of freedom, this set is a lot bigger and we will see in a few examples in section 4 that, up to topological information, it can describe completely the reduced phase-space of the theory.

**Example 3.2. Electromagnetism in 4D**

As we saw in example 2.2, the action is written:

$$ S[E^i, A_i, A_0] = \int dt \int_\Sigma d^3x \left[ -E^i \dot{A}_i - \left( \frac{1}{2} E^i E_i + \frac{1}{4} F_{ij} F^{ij} \right) + E^i \partial_i A_0 \right], $$

(3.28)

where the values of $A_0$ and $A_I$ are fixed on the boundary $\partial \Sigma$. The gauge-like transformations are given by:

$$ \delta_\eta A_j = \frac{\delta (\eta \partial_i E^i)}{\delta E^j} \approx \partial_j \eta. $$

(3.29)

Preservation of the boundary conditions for $A_i$ imposes that $\partial_i A_\eta \approx 0$ on the boundary $\partial \Sigma$. The gauge parameter must tend to a constant $\eta_R$ on the boundary when the constraints are satisfied. Those transformations are generated by:

$$ \Gamma_\eta = \int_\Sigma d^3x \eta \partial_i E^i - \oint_{\partial \Sigma} d\Omega^2 \eta E^r \approx -\eta_R Q, $$

(3.30)

where $Q$ is the electric charge $Q = \oint_{\partial \Sigma} d\Omega^2 E^r$. 

"
3.3 Classification of Boundary Conditions

As we saw in section 2, only the boundary conditions for \( z^A \) are part of the definition of the canonical structure of the theory. The boundary conditions on the Lagrange multipliers \( \lambda^\alpha \) appear for the definition of the total Hamiltonian only: they contain dynamical information.

Let’s assume that, for fixed boundary conditions on \( z^A \), we have two different sets of boundary conditions for the Lagrange multipliers, \( \lambda^\alpha = \bar{\lambda}^\alpha + \mu^\alpha \) and \( \lambda^\alpha = \bar{\lambda}^\alpha + \mu^\alpha \), leading to two differentiable total Hamiltonians, \( H_{T1} \) and \( H_{T2} \). The boundary conditions for \( \mu^\alpha \) are the same as it will span the set of proper gauge transformations which is independent of \( H_T \). The difference \( H_{T1} - H_{T2} \) is a differentiable generator and its bulk term is given by a sum of constraints: it is the generator of a differentiable gauge transformation.

Conversely, let’s assume that we have boundary conditions for \( \lambda^\alpha \), their associated Hamiltonian \( H_T \) and a differentiable generator \( \Gamma_\eta \). We can build new boundary conditions for \( \lambda^\alpha \) such that the new Hamiltonian is given by \( H_T + \Gamma_\eta \). The answer is obviously:

\[
\bar{\lambda}_2^\alpha = \bar{\lambda}^\alpha + \eta^\alpha, \quad \lambda^\alpha = \bar{\lambda}^\alpha + \eta^\alpha + \mu^\alpha, \tag{3.31}
\]

leading to

\[
H_{T2} = H_T + \Gamma_\eta. \tag{3.32}
\]

The new Hamiltonian being the sum of two differentiable generators is differentiable. If \( \Gamma_\eta \) is the generator of a proper gauge transformation, the two Hamiltonians are equivalent: \( H_{T2} \approx H_T \).

**Theorem 3.3.** For gauge theories of the form \( (3.1) \), once boundary conditions for the dynamical variables \( z^A \) have been selected, the possible boundary conditions for the Lagrange multipliers \( \lambda^\alpha \) are in one to one correspondence with the differentiable gauge transformations modulo proper gauge transformations.

The various theories obtained that way share the same local form of the action \( (3.1) \). However, the value of their total Hamiltonian will be different due to a different boundary term. The theories obtained are different.

3.4 Asymptotic Symmetries

We saw in section 3.2 that proper gauge transformations are symmetries of the theory. As they are generated by functionals that are zero on the constraint surface they are the set of true gauge transformations of the theory. We will now study the set of asymptotic symmetries which are the global symmetries of the theory hidden inside the set of improper gauge transformations.
The necessary and sufficient condition for an improper gauge transformation $\delta\eta$ to be a symmetry is that its generator $\Gamma_\eta[z]$ satisfies equation (2.56):

$$\frac{\partial}{\partial t} \Gamma_\eta + \{\Gamma_\eta, H_T\} \approx 0. \tag{3.33}$$

The only explicit time dependence of $\Gamma_\eta[z^A]$ is in the gauge parameter $\eta^\alpha$:

$$\frac{\partial}{\partial t} \Gamma_\eta[z^A] = \Gamma_\eta \dot{\eta}[z^A] \tag{3.34}$$

and, because $\Gamma_\eta[z^A]$ is differentiable, $\Gamma_\eta[z^A]$ is also differentiable. It implies that the functional $F$ appearing in (3.33) is differentiable:

$$F[z^A, \eta^\alpha, \lambda^\alpha] \equiv \Gamma_\eta[z^A] + \{\Gamma_\eta[z^A], H_T[z^A, \lambda^\alpha]\}. \tag{3.35}$$

By definition of the variation of the Lagrange multipliers under a gauge transformation (3.17), we know that the bulk term of $F$ is given by $\int_\Sigma \delta_\eta \lambda^\alpha \gamma_\alpha$ $d^a x$: it is the generator of the differentiable gauge transformation with parameter $\delta_\eta \lambda$:

$$F[z^A, \eta^\alpha, \lambda^\alpha] = \Gamma_{\delta_\eta \lambda}[z^A]. \tag{3.36}$$

We have proved the following result:

**Theorem 3.4.** An improper gauge transformation $\delta\eta$ is a symmetry of the theory if and only if the differentiable gauge transformation generated by $\Gamma_{\delta_\eta \lambda}[z^A]$ is a proper gauge transformation.

The set of differentiable gauge symmetries forms a subalgebra of the differentiable gauge transformations and the proper gauge transformations form an ideal of this subalgebra. This leads to the standard definition:

**Definition 3.5.** The algebra obtained by taking the quotient of the algebra of the differentiable gauge symmetries by the proper gauge symmetries is called the asymptotic symmetry algebra.

As we said earlier, it is the algebra of global symmetries of the theory hidden inside the set of gauge-like transformations. Their differentiable generators $\Gamma_\eta$ are constants of motion given by boundary terms when evaluated on the constraints

$$\Gamma_\eta[z^A] \approx \int_{\partial \Sigma} k_\eta. \tag{3.37}$$

Those quantities are called surface charges. They form a representation of the asymptotic symmetry algebra through the Poisson bracket:
Theorem 3.6. Let the phase-space $\mathcal{F}(\Sigma)$ be path-connected. If $\eta^\alpha$ and $\rho^\alpha$ are two differentiable gauge symmetries then:

$$\{ \Gamma_{\eta}[z^A], \Gamma_{\rho}[z^A] \} = \Gamma_{[\rho,\eta]}[z^A] + \mathcal{K}_{\eta,\rho}$$

(3.38)

where $\mathcal{K}_{\eta,\rho}$ is a possible central extension. It satisfies

$$\mathcal{K}_{\eta,\rho} = -\mathcal{K}_{\rho,\eta},$$

(3.39)

$$\mathcal{K}_{[\eta,\rho]} + \mathcal{K}_{[\rho,\rho]} + \mathcal{K}_{[\epsilon,\eta]} = 0,$$

(3.40)

for all differentiable gauge symmetries $\eta^\alpha$, $\rho^\alpha$ and $\epsilon^\alpha$.

As for theorem[3.1], if one drops the hypothesis of path-connectedness of the phase-space, the extension of the algebra can become field dependent.

A subset of those symmetries can be computed easily. Let’s assume that $\delta_{\eta}\lambda^\alpha$ preserve the boundary conditions of $\lambda^\alpha$. This is a new condition; it was not required for the differentiability of the generator $\Gamma_{\eta}[z^A]$. This means that $\delta_{\eta}\lambda^\alpha$ satisfies the same boundary conditions as the true lagrange multipliers $\mu^\alpha$ and that

$$H_T[z^A, \delta_{\eta}\lambda^\alpha] \approx 0.$$  

(3.41)

By construction, $H_T[z^A, \mu^\alpha]$ is a differentiable generator when $\mu^\alpha$ is independent of $z^A$. However, $\delta_{\eta}\lambda^\alpha$ might depend on $z^A$ and we have no guarantee that the functional $H_T[z^A, \delta_{\eta}\lambda^\alpha]$ is differentiable. The only thing we have is:

$$\oint_{\partial\Sigma} I^n H_T[z^A, \delta_{\eta}\lambda^\alpha] \approx 0$$

(3.42)

where the variation $\delta z^A$ does not necessarily satisfy the linearized constraints. The generator $\Gamma_{\delta_{\eta}\lambda}$ and $H_T[z^A, \delta_{\eta}\lambda^\alpha]$ have the same bulk term, it implies that

$$\delta \left( \Gamma_{\delta_{\eta}\lambda} - H_T[z^A, \delta_{\eta}\lambda^\alpha] \right) = -\oint_{\partial\Sigma} I^n H_T[z^A, \delta_{\eta}\lambda^\alpha] \approx 0.$$  

(3.43)

Up to an irrelevant constant, we then have

$$\Gamma_{\delta_{\eta}\lambda} \approx H_T[z^A, \delta_{\eta}\lambda^\alpha] \approx 0,$$

(3.44)

which is the symmetry condition.

We see that requiring the preservation of the boundary conditions of the lagrange multipliers guarantees that differentiable gauge transformations become symmetries. This subset of the asymptotic symmetries is the one usually computed. In practice, one computes the set of gauge-like transformations preserving both the boundary conditions of
This computation can be done in the lagrangian formalism where it is easier because we now treat both dynamical field and lagrange multipliers in the same way. One then restricts to the subset of those transformations generated by a differentiable generator. This technique has been applied in many different cases to compute both the charges and the algebra under the Poisson bracket.

Among the most famous examples, we find the original computation of the Poincaré charges for asymptotically flat space times in 4D by Regge and Teitelboim [4], where they recover the ADM definition of the mass [3], and the computation of the asymptotic symmetry algebra by Brown and Henneaux for asymptotically $AdS_3$ space-times [7]. An interesting non-trivial integrability problem appeared in the study of gravity coupled to scalar fields [24, 25].

### 3.5 Brown-York Quasi-local Charges

We will now, as an example, apply the technique presented in the previous section to the computation of the Brown-York quasi-local charges [23].

The boundary conditions have been introduced in section 3.1. They are given by:

\[ N^r |_{\partial \Sigma} = 0, \quad N |_{\partial \Sigma} = \tilde{N}, \]  
\[ N^A |_{\partial \Sigma} = \tilde{N}^A, \quad g_{AB} |_{\partial \Sigma} = \gamma_{AB}, \]  

with the associated differentiable Hamiltonian:

\[ S[g_{ij}, \pi^{ij}, N, N^i] = \int dt \left\{ \int_{\Sigma} d^n x \, \pi^{ij} \partial_t g_{ij} - H_T \right\}, \]  
\[ H_T = \int_{\Sigma} d^n x \left( N R + N^i R_i \right) + 2 \oint_{\partial \Sigma} d^{n-1} x \left( N^A \pi^r_A + N \sqrt{\gamma} K \right). \]  

Following the result of section 3.4, all differentiable gauge transformations preserving the boundary conditions on the lagrange multipliers $N, N^i$ give rise to conserved quantities. The gauge-like transformations take the form:

\[ \delta_\xi N = \partial_t \xi - [N, \xi]^t_{\partial \Sigma}, \]  
\[ \delta_\xi N^i = \partial_t \xi^i - [N, \xi]_{\partial \Sigma}^i, \]  
\[ \delta_\xi g_{ij} = 2 \frac{\xi}{\sqrt{g}} \left( \pi_{ij} - \frac{\pi}{n-1} g_{ij} \right) + \nabla_i \xi_j + \nabla_j \xi_i. \]

For simplicity, we will only consider the transformations with $\xi^r |_{\partial \Sigma} = 0$. Preservation of
the boundary conditions imposes:
\[
0 = \partial_t \xi - \bar{N}^A \partial_A \xi + \xi^A \partial_A \bar{N}, \\
0 = \partial_t \xi^A - \bar{N}^B \partial_B \xi^A + \xi^B \partial_B \bar{N}^A - g^{A j} \left( \bar{N} \partial_j \xi - \xi \partial_j \bar{N} \right), \\
0 = -g^{r j} \left( \bar{N} \partial_j \xi - \xi \partial_j \bar{N} \right), \\
0 = \frac{\xi}{\bar{N}} \left( \partial_t \gamma_{AB} - \bar{N}^C \partial_C g_{AB} - \partial_A \bar{N}^C g_{CB} - \partial_B \bar{N}^C g_{CA} \right) \\
+ \xi^C \partial_C g_{AB} + \partial_A \xi^C g_{CB} + \partial_B \xi^C g_{CA},
\]
where all the equalities hold on the boundary. In term of the induced metric \( \gamma_{\alpha\beta} \) on the space-time boundary \( \partial \Sigma \times \mathbb{R} \) and the vector \( \eta^\alpha = \left( \frac{\xi}{\bar{N}}, \xi^A - \frac{\bar{N}^A \xi}{\bar{N}} \right) \), they become
\[
\gamma_{\alpha\beta} = \begin{pmatrix} -\bar{N}^2 + \bar{N}_c \bar{N}^c & \bar{N}_B \\
\bar{N}_A & \gamma_{AB} \end{pmatrix}, \\
\eta^\alpha \partial_\alpha \gamma_{\beta\delta} + \partial_\beta \eta^\alpha \gamma_{\alpha\delta} + \partial_\delta \eta^\alpha \gamma_{\alpha\beta} = 0, \\
g^{rr} \left( \bar{N} \partial_r \xi - \xi \partial_r \bar{N} \right) = g^{rA} \left( \bar{N} \partial_A \xi - \xi \partial_A \bar{N} \right)
\]
where \( x^\alpha = (t, x^A) \). Equation (3.57) is the Killing equation for the metric \( \gamma_{\alpha\beta} \). Once we have selected a Killing vector \( \eta^\alpha \), the last equation (3.58) gives \( \partial_t \xi \) in term of the other quantities.

The next step is to select the allowed transformation for which we can build a differentiable generator. This computation is similar to the one we did in order to compute the total Hamiltonian in section 3.1. As the boundary values of \( \xi, \xi^i \) are independent of the dynamical fields, we see easily that they are all differentiable with a generator given by:
\[
\Gamma_\xi = \int_\Sigma d^n x \left( \xi R + \xi^i R_i \right) + 2 \oint_{\partial \Sigma} d^{n-1} x \left( \xi^A \pi^r_A + \xi \sqrt{\gamma} K \right).
\]

**Theorem 3.7.** To each Killing vector \( \eta^\alpha \) of the induced metric \( \gamma_{\alpha\beta} \) on the boundary \( \partial \Sigma \times \mathbb{R} \), we can associate a differentiable generator \( \Gamma_\xi \) given in (3.59) where the boundary values of \( \xi \) satisfy:
\[
\left. \xi \right|_{\partial \Sigma} = \bar{N} \eta^0, \quad \left. \xi^A \right|_{\partial \Sigma} = \eta^A + \bar{N}^A \eta^0, \quad \left. \xi^r \right|_{\partial \Sigma} = 0,
\]
and \( \partial_t \xi \) is given by (3.58). Evaluated on the constraints surface, these generators are conserved quantities:
\[
\Gamma_\xi [g_{ij}, \pi^{ij}] \approx \oint_{\partial \Sigma} d^{n-1} x \sqrt{\gamma} \left\{ \eta^A \sigma_A + \eta^0 \epsilon \right\}
\]
where the energy and momentum density are defined by:
\[
\epsilon = 2 \bar{N} K + \frac{2}{\sqrt{\gamma}} \bar{N}^A \pi^r_A, \quad \sigma_A = \frac{2}{\sqrt{\gamma}} \pi^r_A.
\]
This is exactly the result obtained in [23].
4 Boundary Gauge Degrees of Freedom and Holography

We saw in section 3 how the notion of differentiable functional solves the problem of surface charges by selecting the right generator for the gauge symmetries. We will show in this section how we can use the same notion to extract information about the reduced phase-space of some theories with no local degrees of freedom without solving the constraints. In those cases, the phase-space of boundary gauge degrees of freedom is described by dynamical fields living on the boundary of the spatial manifold $\Sigma$. This can be interpreted as direct examples of the holography mechanism. We will also obtain a complete classification of the possible boundary conditions for the bulk theory and the dictionary with the corresponding Hamiltonians of the boundary theory.

In this section, we will consider field theories of the form:

$$S[z^A, \lambda^\alpha] = \int dt \left[ \int_\Sigma d^n x \left( \frac{1}{2} \sigma_{AB} z^A z^B - \lambda^\alpha \gamma_\alpha \right) + \oint_{\partial \Sigma} b \right],$$

where $\gamma_\alpha$ are first-class constraints. We will assume that a set of boundary conditions for $\lambda^\alpha$ and $z^A$ as been selected such that the total Hamiltonian $H_T$ is a differentiable generator:

$$H_T = \int d^n x \lambda^\alpha \gamma_\alpha + \oint_{\partial \Sigma} b.$$

Using the results of section 3.3, we see that once boundary conditions have been selected for $z^A$, the possible total Hamiltonians are given by the set of differentiable gauge generators. Computing the set of differentiable gauge transformations $\delta_\eta$ gives the set of possible boundary conditions on $\lambda^\alpha$.

We will restrict our analysis to theories with no local degrees of freedom, i.e. where the number of independent first-class constraints is equal to the number of canonical pairs. The reason is that if we have local degrees of freedom, we need boundary conditions on the dynamical fields $z^A$, see example 3.2. This would considerably restrict the set of possible improper gauge transformations. Some examples of theories with no local degrees of freedom are Poisson sigma models in 2D, Chern-Simons theories, pure gravity in 3 dimensions, BF theories, ...

In the following, we will use Chern-Simons theories in 3D and BF theory in 4D to present our technique. For simplicity, we will restrict our analysis to finite manifold $\Sigma$. If $\Sigma$ has a boundary at infinity, the analysis is similar with the additional requirement that all differentiable generators have to be finite. The analysis of pure gravity in 3D will the subject of a following work.

We will start by studying 3D Chern-Simons without imposing any boundary conditions on the dynamical variables $A^a_i$. We will compute the Dirac bracket using differen-
tiable gauge generators and make the link with the Wess-Zumino-Witten description. We will then impose boundary conditions on the dynamical variables $A_i^a$ and show how those new boundary conditions can be interpreted as boundary constraints on the theory. This allows us to completely classify the possible boundary conditions for Chern-Simons in 3D. In the last part, we will apply the same analysis to the 4D BF theory. In particular, we will show that the reduced phase-space of 4D BF is related to the phase space of a 3D Chern-Simons theory defined on the boundary.

4.1 3D Chern-Simons

Chern-Simons theory in 3 dimensions is a good toy model for us. The constraints can be solved exactly and one can show that the reduced phase-space theory is given by a WZW model on the boundary [16, 17]. In this section, we will recover the same result using differentiable gauge transformations without having to solve the constraints.

The hamiltonian bulk action is:

$$S[A_i^a, A_0^a] = -\frac{\kappa}{2\pi} \int dt \int_\Sigma d^2x \frac{1}{2} \epsilon^{ij} g_{ab} \left( A_i^a \dot{A}_j^b - A_0^a F_{ij}^b \right).$$

(4.3)

We use $\epsilon^{12} = 1$ and the metric $g_{ab}$ is a symmetric non-degenerate invariant tensor on the Lie algebra $\mathfrak{g}$ of the Lie group $\mathfrak{g}$. The fields $A_i^a$ are the dynamical variables and $A_0^a$ plays the role of a lagrange multiplier. They are all valued in the algebra $\mathfrak{g}$: $A_i^a \equiv A_i^0 T_a \in \mathfrak{g}$ and $A_0^a \equiv A_0^a T_a \in \mathfrak{g}$ where $T_a$ are the generators of $\mathfrak{g}$. We also use the usual field strength in 2 dimensions $F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + f_{abc} A_b^i A_c^j$. The poisson bracket and the constraints can be easily read from the action to be:

$$\{F, G\} = \frac{2\pi}{\kappa} \int_\Sigma d^2x \frac{\delta F}{\delta A_i^a} \epsilon^{ij} g_{ab} \frac{\delta G}{\delta A_j^b},$$

$$\Phi_a = -\frac{\kappa}{4\pi} g_{ab} \epsilon^{ij} F_{ij}^b \approx 0.$$

(4.4)

(4.5)

The constraints satisfy the following closed algebra

$$\{\Phi_a(x), \Phi_b(y)\} = -f_{ab}^c \Phi_c(x) \delta^2(x - y).$$

(4.6)

Because the hamiltonian is a combination of the primary constraints and those constraints satisfy to a closed algebra, we have the full set of constraints and they are all first-class.

For simplicity, we will consider $\Sigma$ to be a disk of finite radius $R$ and we will use adapted coordinates $r, \phi$. There are multiple choices for the boundary conditions, the most common being $A_0|_{\partial \Sigma} = 0$ or $A_0 - A_\phi|_{\partial \Sigma} = 0$. Those are boundary conditions on the lagrange multipliers only: they don’t impose anything on the dynamical fields. We can study the canonical structure and the set of differentiable gauge transformations without imposing any boundary conditions on $A_i^a$. 
The gauge-like transformations are given by:

$$\delta_\eta A^a_i = -\frac{1}{2} g^{ab} \varepsilon_{ij} \frac{\delta}{\delta A^b_j} \left( \eta^c g_{cd} \epsilon_{kl} F^d_{kl} \right).$$  \hspace{1cm} (4.7)

The proper gauge transformations are those for which the gauge parameters are zero on the boundary \( \eta|_{\partial\Sigma} = 0 \). On the constraints surface, the finite gauge transformations of \( A_r \) are given by:

$$A'_r = h^{-1} A_r h + h^{-1} \partial_r h,$$  \hspace{1cm} (4.8)

where \( h \) is valued in the group \( \mathfrak{g} \). The subset of finite proper gauge transformations is the set of transformations for which \( h \) is the identity on the boundary. Using a finite proper gauge transformation, we can put \( A^a_i \) to zero in a neighborhood of the boundary but this uses all the gauge freedom that we have in that neighborhood.

The suitable group element is given by:

$$h = \mathcal{P} \exp \left( \int_{r}^R dr' A_r(r', \theta) \right),$$  \hspace{1cm} (4.9)

where \( \mathcal{P} \) is the path ordering symbol. With \( A_r = 0 \), the constraints take the form

$$0 \approx \Phi_a = -\frac{\kappa}{4\pi} g_{ab} \left( \partial_r A^b_r - \partial_\phi A^b_\phi + f^b_{cd} A^c_r A^d_\phi \right) = -\frac{\kappa}{4\pi} g_{ab} \partial_r A^b_\phi.$$  \hspace{1cm} (4.10)

The value of \( A^a_\phi \) is completely characterized by its boundary value \( A^a_\phi|_{\partial\Sigma} \). The gauge transformation of \( A^a_\phi \) is

$$\delta_\eta A^a_\phi = \partial_\phi \eta^a + f^a_{bc} A^b_\phi \eta^c \equiv D_\phi \eta^a.$$  \hspace{1cm} (4.11)

Evaluated on the boundary for a proper gauge transformation, we obtain \( \delta_\eta A^a_\phi|_{\partial\Sigma} = 0 \): this boundary value is a gauge invariant quantity. It means that, up to topological issues, the reduced phase-space of the theory is parametrized by the boundary value of \( A^a_\phi \). We will now use the differentiable gauge generators and their algebra to compute the induced bracket.

We have no boundary conditions on the dynamical fields \( A^a_i \): the only restrictions from differentiability are coming from the existence of the boundary term for the generator. The bulk generator is \( \bar{\Gamma}_\eta[A^a_i] = \int_{\Sigma} d^2x \eta^a \Phi_a \) and its variation is given by:

$$\delta \bar{\Gamma}_\eta [A^a_i] = \int_{\Sigma} d^2x \delta \eta^a \Phi_a + \frac{\kappa}{2\pi} \int_{\Sigma} d^2x D_\phi \eta^a g_{ab} \epsilon^{ij} \delta A^b_j
- \frac{\kappa}{2\pi} \int_{\partial\Sigma} (d^1x)_i \epsilon^{ij} \eta^a g_{ab} \delta A^b_j.$$  \hspace{1cm} (4.12)

There might be boundary terms coming from \( \delta \eta^a \) but they will be proportional to \( \Phi_a \). When evaluated on the constraints surface, the boundary term is given by the last term only. We are looking for a \((n - 1, 0)\)-form \( k_\eta \) such that:

$$\delta \int_{\partial\Sigma} k_\eta^{n-1} \approx \frac{\kappa}{2\pi} \int_{\partial\Sigma} d\phi \eta^a g_{ab} \delta A^b_\phi.$$  \hspace{1cm} (4.13)
The only gauge parameters for which we can build a boundary term are those such that there exists \( f_\eta \) a local function of \( \phi \) and \( A_\phi \) defined on the boundary \( \partial \Sigma \) with:

\[
\eta^a|_{\partial \Sigma} = \frac{2\pi}{\kappa} g^{ab} \sum_{k=0}^{\infty} (-\partial_\phi)^k \frac{\partial^S f_\eta}{\partial \phi^k A^b_\phi} \bigg|_{\partial \Sigma} \equiv \frac{2\pi}{\kappa} g^{ab} \frac{\delta f_\eta}{\delta A^b_\phi} \bigg|_{\partial \Sigma}. \tag{4.14}
\]

We have defined \( \frac{\delta}{\delta \Sigma} \) as the Euler-Lagrange derivative on the circle \( \partial \Sigma \). The differentiable generators of the gauge transformations are then given by:

\[
\Gamma_\eta[A^a_1] = \int_\Sigma d^2 x \, \eta^a \Phi_a + \oint_{\partial \Sigma} d\phi \, f_\eta \approx \oint_{\partial \Sigma} d\phi \, f_\eta. \tag{4.15}
\]

This is the complete set of differentiable generators of gauge transformations even if we only have done the computation on the constraints surface. We could add to the boundary value of \( \eta^a \) a contribution depending on \( \Phi_a \) but this would be zero on the constraints surface and irrelevant.

We showed that to any functional \( F[A_\phi] = \int_{\partial \Sigma} d\phi \) on the circle \( \partial \Sigma \), we can associate a differentiable gauge generator \( \Gamma_F[A^a_1] \) by choosing a gauge parameter \( \eta \) satisfying (4.14). There are multiple possible generators but they are all equal on the constraint surface:

\[
\Gamma_F[A^a_1] \approx F. \tag{4.16}
\]

As differentiable gauge generators are first-class quantities, we can compute their Dirac bracket by computing their Poisson bracket and evaluate the result on the constraints surface. If one considers two functionals of \( A_\phi \) on the circle \( \partial \Sigma \), \( F_1 \) and \( F_2 \), the Poisson bracket of the associated differentiable gauge generators \( \Gamma_{F_1} \) and \( \Gamma_{F_2} \) is easily computed. We obtain:

\[
\{ \Gamma_{F_1}, \Gamma_{F_2} \} \approx \frac{2\pi}{\kappa} \int_\Sigma d^2 x \, \left( \frac{\kappa}{2\pi} \epsilon^{ijk} g_{ac} D_k \eta_1 \right) \epsilon_{ij} g^{ab} \left( \frac{\kappa}{2\pi} \epsilon^{ijkl} g_{bd} D_l \eta_2 \right) \\
\approx \frac{\kappa}{2\pi} \oint_{\partial \Sigma} (d^1 x)_i \epsilon^{ij} \left( g_{ab} \eta_1^a \partial_j \eta_2^b - A^a_2 g_{ab} f^{b}_{cd} \eta_1^c \eta_2^d \right). \tag{4.17}
\]

We can then read the Dirac bracket induced on the functionals \( F_1 \) and \( F_2 \):

\[
\{ F_1, F_2 \}^{*} \approx \frac{2\pi}{\kappa} \oint_{\partial \Sigma} d\phi \left( g^{ab} \frac{\delta f_1}{\delta A^a_\phi} \partial_\phi \frac{\delta f_2}{\delta A^b_\phi} - A^a_\phi g_{ab} f^{b}_{cd} \frac{\delta f_1}{\delta A^c_\phi} \frac{\delta f_2}{\delta A^d_\phi} \right). \tag{4.18}
\]

This can also be written in term of \( A^a_\phi|_{\partial \Sigma} \):

\[
\{ A^a_\phi|_{\partial \Sigma}(\phi), A^b_\phi|_{\partial \Sigma}(\phi') \}^{*} \approx \frac{2\pi}{\kappa} \left( g^{ab} \partial_\phi - g_{cd} f^{dab} A^c_\phi|_{\partial \Sigma} \right) \delta(\phi - \phi'). \tag{4.19}
\]

The fields \( A^a_\phi|_{\partial \Sigma} \) parametrize the reduced phase-space of the theory. Their Dirac bracket is given by (4.18) and (4.19). The only thing we are missing to have the full description is the Hamiltonian.
The total Hamiltonian $H_T$ is the differentiable gauge generator associated to the lagrange multipliers $A_0$. For this generator to exists, $A_0$ must satisfy boundary conditions of the form (4.14) for a particular functional $\oint_{\partial \Sigma} d\phi h_B$ of $A_0^a$:

$$A_0^a|_{\partial \Sigma} = \frac{2\pi}{\kappa} g^{ab} \frac{\delta h_B}{\delta A_b^a} |_{\partial \Sigma}. \tag{4.20}$$

The total Hamiltonian is then given by:

$$H_T = \int_{\Sigma} A_0^a \Phi_a + \oint_{\partial \Sigma} d\phi h_B. \tag{4.21}$$

The Hamiltonian on the reduced phase-space is the value of $H_T$ evaluated on the constraints surface:

$$H_T \approx \oint_{\partial \Sigma} d\phi h_B. \tag{4.22}$$

As we said earlier, the two most common boundary conditions for $A_0^a$ are $A_0^a = 0$ or $A_0^a = A_0^a|_{\partial \Sigma}$. Respectively, they correspond to $h_B = 0$ and $h_B = \frac{\kappa}{4\pi} g_{ab} A_0^a A_0^b$.

The global picture is the following: up to topological issues, the reduced phase-space is parametrized by the value of $A_0^a|_{\partial \Sigma}$ on the boundary and the Dirac bracket is given by:

$$\{ A_0^a|_{\partial \Sigma}(\phi), A_0^b|_{\partial \Sigma}(\phi') \}^* = \frac{2\pi}{\kappa} \left( g^{ab} \partial_\phi - g_{cd} f^{dab} A_0^c|_{\partial \Sigma} \right) \delta(\phi - \phi'). \tag{4.23}$$

This is the current algebra associated to the algebra $\mathfrak{g}$ and we recovered the result obtained in [17]. The Hamiltonian is given by a functional of $A_0^a$:

$$H_T[A_0^a|_{\partial \Sigma}] \approx \oint_{\partial \Sigma} d\phi h_B(A_0^a) \tag{4.24}$$

which is determined by the boundary conditions on the lagrange multipliers $A_0^a$ through:

$$A_0^a|_{\partial \Sigma} = \frac{2\pi}{\kappa} g^{ab} \frac{\delta h_B}{\delta A_b^a} |_{\partial \Sigma}. \tag{4.25}$$

If we consider the problem from the other direction, we see that only the reduced phase-space structure is controlled by the bulk action. The Hamiltonian (4.24) is determined by the boundary conditions of $A_0^a$ and by tuning them, we can build any local function $h_B$. It has important consequences. For instance, it implies that any theory in two dimension with the phase-space structure given by (4.23) is equivalent to a 3D Chern-Simons theory with specific boundary conditions. It also means that two different choices for the boundary conditions of $A_0^a$ will lead to different Hamiltonians on the reduced phase-space: they describe two completely different theories.

The analysis has been done strictly at the level of the boundary terms without taking into account the topological structure of $\Sigma$. This structure will in general restrict the set
of available values for $A_\phi$ on the boundary. In the case where $\Sigma$ is a disk, the reduced phase-space is exactly parametrized by the value of $A_\phi$ on the boundary such that the holonomy around the $\phi$-circle is the identity of the group $\mathfrak{G}$:

$$W \equiv \mathcal{P} \exp \left( - \int_0^{2\pi} d\phi \, A_\phi|_{\partial\Sigma} \right) = 1 \in \mathfrak{G}. \quad (4.26)$$

### 4.2 Boundary conditions on the phase-space

In the previous section, we considered only boundary conditions on the Lagrange multipliers. However, one can construct sets of boundary conditions that also include boundary conditions on the canonical variables $A^a_\phi$. For instance, boundary conditions on $A^a_\phi$ are present in the study of Chern-Simons gravity [26, 27]. Those additional boundary conditions are responsible for the second step of the reduction from Chern-Simons to Liouville in the Brown-Henneaux boundary conditions case: first, one goes to WZW on the boundary and then, there is a second reduction to Liouville.

Let’s assume that we want additional boundary conditions on the canonical variables:

$$\chi^\alpha(A^0_\phi)|_{\partial\Sigma} = 0 \quad (4.27)$$

where $\chi^\alpha$ are local functions of $A^0_\phi$ and their derivatives $\partial_k^b A^a_\phi$. Any boundary condition on $A^a_\phi$ is irrelevant because $A^a_\phi$ is pure gauge even on the boundary. The conservation of $\chi^\alpha = 0$ in time will impose restrictions on the possible boundary conditions for $A^0_\phi$:

$$\sum_k \frac{\partial \chi^\alpha}{\partial \partial^k A^0_\phi} \partial_k (D_\phi A^0_\phi) \bigg|_{\partial\Sigma} = 0. \quad (4.28)$$

By adding test functions $\nu_\alpha$ on the circle, we can rewrite this as an integral:

$$\oint_{\partial\Sigma} d\phi \, \bar{\delta}(\nu_\alpha \chi^\alpha) \frac{\delta}{\delta A^0_\phi} D_\phi A^0_\phi = 0 \quad \forall \nu_\alpha. \quad (4.29)$$

Let’s assume that we have a set of boundary conditions on $A^0_\phi$ satisfying (4.29) such that the differentiable total Hamiltonian $H_T$ can be constructed. Remark that the differentiability condition here is different than the one used in section 4.1 as we now have boundary conditions on the dynamical variables.

We want to relate the two problems and describe the case with the additional conditions $\chi^\alpha$ in term of the canonical structure of section 4.1. The idea will be to treat those additional conditions as additional constraints. Imposing everything, $F_{ij} = 0$ and $\chi^\alpha|_{\partial\Sigma} = 0$, the total hamiltonian $H_T$ becomes a functional of $A^0_\phi|_{\partial\Sigma}$:

$$H_T|_{\chi^\alpha = 0} \approx \oint_{\partial\Sigma} d\phi \, h_B(A^0_\phi). \quad (4.30)$$
We saw in the previous section that there exist a differentiable generator $\tilde{H}_T$ for the canonical structure without imposing $\chi^\alpha$ such that

$$\tilde{H}_T \approx \int_{\partial \Sigma} d\phi \, h_B(A^a_\phi). \tag{4.31}$$

By construction, if we impose the additional boundary conditions (4.27), we have:

$$H_T|_{\chi^\alpha=0} = \tilde{H}_T|_{\chi^\alpha=0}, \tag{4.32}$$

which implies that $\tilde{H}_T$ is differentiable using either of the two canonical structures. From here on, we will work with the general canonical structure of section 4.1 and the Hamiltonian will be taken as $\tilde{H}_T$. For any function $\nu_\alpha$ on the circle, we can build the associated differentiable gauge generator $\Gamma_\nu$ such that:

$$\Gamma_\nu[A^a_\phi] \approx \int_{\partial \Sigma} d\phi \, \nu_\alpha \chi^\alpha. \tag{4.33}$$

The well-defined action for the total system can then be written as:

$$S[A^a_t, A^a_0 = \bar{A}^a_0 + \mu^a, \nu_\alpha] = \frac{-\kappa}{2\pi} \int dt \left\{ \int_{\Sigma} d^2x \frac{1}{2} \epsilon^{ij} g_{ab} A^a_i \dot{A}^b_j - \tilde{H}_T - \Gamma_\nu \right\},$$

where $\mu^a$ with $\mu^a|_{\partial \Sigma} = 0$ are the true lagrange multipliers enforcing $\Phi_a$. The action is written without imposing a priori the conditions $\chi^\alpha|_{\partial \Sigma} = 0$; they will be enforced by the boundary term coming from the variation of $\nu$ in $\Gamma_\mu$. Solving the constraints $\chi^\alpha = 0$ reduces the action to:

$$S[A^a_t, A^a_0 = \bar{A}^a_0 + \mu^a] = \frac{-\kappa}{2\pi} \int dt \left\{ \int_{\Sigma} d^2x \frac{1}{2} \epsilon^{ij} g_{ab} A^a_i \dot{A}^b_j - H_T \right\}. \tag{4.34}$$

This is the expected action when $\chi^\alpha|_{\partial \Sigma}$ hold. On the reduced phase-space, we obtain:

$$\tilde{H}_T + \Gamma_\nu \approx \int_{\partial \Sigma} d\phi \, (h_B + \nu_\alpha \chi^\alpha), \tag{4.35}$$

which is the Hamiltonian of a constrained system. The condition (4.29) can be rewritten

$$\left\{ \int_{\partial \Sigma} d\phi \, h_B, \int_{\partial \Sigma} d\phi \, \nu_\alpha \chi^\alpha \right\} = 0 \quad \forall \nu_\alpha. \tag{4.36}$$

As expected, this condition is the conservation of the constraints $\chi^\alpha$ under time evolution.

We see explicitly that boundary conditions on the canonical variables of the Chern-Simons theory will produce a constrained Wess-Zumino-Witten model on the boundary. In the case of gravity, solving those additional constraints and going to the fully reduced phase-space is what gives Liouville theory.
The other side of the same coin is that, playing with the boundary conditions of the Lagrange multipliers $A_0^a$, we can build any Hamiltonian on the reduced phase-space. In particular, we can build a Hamiltonian containing some additional constraints on $A_0^a|_{\partial \Sigma}$.

Let’s assume that we want a Hamiltonian containing constraints on our boundary. We select $h_B$ to be of the form:

$$h_B = \tilde{h}_B + \nu_\alpha \chi^\alpha, \quad (4.37)$$

where $\nu_\alpha$ are test function of the boundary that can vary arbitrarily. The associated boundary conditions for $A_0^a$ are given by:

$$A_0^a|_{\partial \Sigma} = \frac{2\pi}{\kappa} g^{ab} \frac{\delta}{\delta A^b_0} \left( \tilde{h}_B + \nu_\alpha \chi^\alpha \right). \quad (4.38)$$

Because $\nu_\alpha$ is not fixed, part of the $A_0^a$ are not fixed on the boundary but are in reality playing the role of the Lagrange multipliers enforcing the additional boundary constraints $\chi^\alpha$. This is particularly obvious if the boundary constraints $\chi^\alpha$ are simple boundary conditions like:

$$A^a_0|_{\partial \Sigma} = 0, \quad (4.39)$$

where $\bar{a}$ is fixed. In that case, the combination $g_{\bar{a}b} A^b_0$ plays the role of the Lagrange multiplier enforcing (4.39).

As the analysis we did on one boundary can be reproduced independently on all boundaries, we obtained a complete classification of the possible boundary conditions for Chern-Simons theory in 3 dimensions:

**Theorem 4.1.** For Chern-Simons theory on $\Sigma \times \mathbb{R}$, the possible boundary conditions for the fields $A_\mu^a$ on each connected component $C_n$ of $\partial \Sigma$ are in one to one correspondence with the functionals of $a_n^a$ defined on the circle $C_n$ where the fields $a_n^a$ are the pullback of $A_1^a$ on $C_n$.

### 4.3 4D BF Theory

In this section, we will study BF theory in 4 dimensions with a cosmological term [28, 29, 30]. The bulk term of the hamiltonian action can be written as:

$$S[A^a_1, B^a_{ij}, A_0^a, B^a_{0i}] = \int dt \int_{\Sigma} d^3x \left\{ B^a_{ij} \epsilon^{ijk} g_{ab} \partial_t A^a_k - A_0^a \Phi_a - B^a_{0i} \Psi^i \right\}, \quad (4.40)$$

$$\Phi_a = -g_{ab} \epsilon^{ijk} D_i B^b_{jk}, \quad (4.41)$$

$$\Psi^i_a = -g_{ab} \epsilon^{ijk} \left( F^b_{jk} + \frac{\Lambda}{6} D^b_{jk} \right). \quad (4.42)$$
All fields are valued in the algebra \( g \) and we use the same convention as in section 4.1. The Poisson bracket is given by:

\[
\{I, J\} = \frac{1}{2} \int_{\Sigma} d^3x \left( \frac{\delta I}{\delta A^a_i} g^{ab} \epsilon_{ijk} \frac{\delta J}{\delta B^b_{jk}} - I \leftrightarrow J \right). \tag{4.43}
\]

The constraints are first-class and satisfy the following algebra:

\[
\{\Phi_a(x), \Phi_b(y)\} = -f^{c}_{ab} \Phi_c \delta^3(x - y), \tag{4.44}
\]

\[
\{\Phi_a(x), \Psi^i_b(y)\} = -f^{c}_{ab} \Psi^i_c \delta^3(x - y), \tag{4.45}
\]

\[
\{\Psi^{i a}_a(x), \Psi^{j b}_b(y)\} = 0. \tag{4.46}
\]

We have \( 3N \) canonical pairs and \( 4N \) first-class constraints. The naive counting leads to \(-N\) local degree of freedom. However, locally, the constraints are not independent:

\[
D_i \Psi^i_a = \frac{\Lambda}{6} \Phi_a. \tag{4.47}
\]

There are only \( 3N \) locally independent constraints and, as expected, zero local degrees of freedom. On the boundary theory however, the story will be different.

The setup is very similar to the 3D Chern-Simons theory described in the previous sections and we don’t need any boundary conditions on the canonical variables to make the action well-defined. The only boundary conditions that we need are on the Lagrange multipliers and will give the Hamiltonian of the reduced theory. Let’s first compute the reduced phase-space. As before, we will focus on one boundary and ignore possible topological obstructions.

The smeared constraints will be denoted

\[
\bar{\Gamma}_{\epsilon, \eta} = \int_{\Sigma} d^3x \left( \epsilon^a \Phi_a + \eta^a_i \Psi^i_a \right). \tag{4.48}
\]

They are associated to the following gauge-like transformations:

\[
\delta_{\epsilon, \eta} A^a_i = D_i \epsilon^a - \frac{\Lambda}{6} \eta^a_i, \tag{4.49}
\]

\[
\delta_{\epsilon, \eta} B^a_{ij} = D_i \eta^a_j - D_j \eta^a_i + f^{a}_{bc} B^b_{ij} \epsilon^c. \tag{4.50}
\]

The boundary term in the variation of \( \bar{\Gamma}_{\epsilon, \eta} \) is

\[
\oint_{\partial \Sigma} I^n(\bar{\Gamma}_{\epsilon, \eta}) = -\oint_{\partial \Sigma} (d^2x)_i \epsilon^{ijk} g_{ab} \left( \epsilon^a \delta B^b_{jk} - 2\eta^a_i \delta A^b_k \right), \tag{4.51}
\]

\[
(d^2x)_i = \frac{1}{2} \epsilon^{ijk} dx^j dx^k. \tag{4.52}
\]

As before, let’s use coordinates adapted to the boundary \( x^i = (r, x^A) \) with the boundary under consideration given by \( r \) constant. In that case, the gauge-like transformations
with \( \epsilon^a = 0 \) and \( \eta^a_A = 0 \) on the boundary are proper gauge transformations. On the constraints surface, the finite gauge transformations are generated by the two following transformations:

\[
\begin{align*}
A'_i &= h^{-1}A_i h + h^{-1}\partial_i h, \\
B'_{ij} &= h^{-1}B_{ij} h,
\end{align*}
\tag{4.53}
\]

and

\[
\begin{align*}
A'_i &= A_i - \frac{\Lambda}{6}\eta_i, \\
B'_{ij} &= B_{ij} + D_i\eta_j - D_j\eta_i + \frac{\Lambda}{6}[\eta_i, \eta_j],
\end{align*}
\tag{4.54}
\]

where \( B_{ij} = B_{ij}^a T_a \). The finite proper gauge transformations are those generated by transformations with \( h \) equals to the identity and \( \eta_i \) equals to zero on the boundary. Using a proper transformation of the form (4.53), we can put \( A_r = 0 \) in a neighborhood of the boundary (see section 4.1). We can then use a transformation of the form (4.54), with \( \eta_r = 0 \) and \( \eta_A \) solution to

\[
\partial_r \eta_A = -B_r A, \quad \eta_A|_{\partial \Sigma} = 0,
\tag{4.55}
\]

to also put \( B_r A = 0 \) in a neighborhood of the boundary. This fixes the gauge close to the boundary and the reduce-phase space is then completely parametrized by the boundary value of \( A^a_A \) and \( B^a \equiv \epsilon^{AB} B^a_{AB} \) with \( \epsilon^{AB} = \epsilon_r^{AB} \). However, they are not independent:

\[
\Psi^r_a = g_{ab} \left( \frac{\Lambda}{6} B^b + \epsilon^{AB} F^b_{AB} \right) \approx 0.
\tag{4.56}
\]

The 4 sets of constraints are dependent in the bulk but we see that on the boundary it is not the case: imposing \( \Phi_a \approx 0 \) and \( \Psi^A_a \approx 0 \) imply \( D_r \Psi^r_a \approx 0 \) but we still need to impose \( \Psi^r_a \approx 0 \) on the boundary for it to be valid everywhere.

The Dirac bracket is easily computed using the differentiable functionals \( \Gamma_F \)

\[
F[A^a_A|_{\partial \Sigma}, B^a|_{\partial \Sigma}] = \oint_{\partial \Sigma} d^2 x f(A^a_A|_{\partial \Sigma}, B^a|_{\partial \Sigma}),
\tag{4.57}
\]

\[
\Gamma_F = \Gamma_{\epsilon_F, \eta_F} + F[A^a_A|_{\partial \Sigma}, B^a|_{\partial \Sigma}],
\tag{4.58}
\]

\[
\eta^r_F|_{\partial \Sigma} = -\epsilon_{AB} g^{ab} \frac{\delta f}{\delta A^b_A}, \quad \epsilon^a_F|_{\partial \Sigma} = g^{ab} \frac{\delta f}{\delta B^b},
\tag{4.59}
\]

where the Euler-Lagrange derivatives \( \frac{\delta}{\delta} \) are defined on the boundary coordinates only. For two arbitrary functionals \( I[A^a_A|_{\partial \Sigma}, B^a|_{\partial \Sigma}] \) and \( J[A^a_A|_{\partial \Sigma}, B^a|_{\partial \Sigma}] \), a direct computation
leads to:

\[ \{ I, J \}^* \approx \{ \Gamma_I, \Gamma_J \} \]

\[ \approx \oint_{\partial \Sigma} d^2 x \, g_{ab} \left( \left( -B^a f^{cb} \Gamma_I \Gamma_J - 2\epsilon^{AB} D_A \epsilon^b_{I J} \right) 
\right.

\[ + 2\epsilon^{AB} D_A \epsilon^b_{I J} + \epsilon^{AB} \frac{\Lambda}{3} \eta^a_{I A} \eta^b_{J B} \left. \right) \]

\[ \approx \oint_{\partial \Sigma} d^2 x \left( -B^a g_{ab} f^{bcd} \frac{\delta I}{\delta B^c} \frac{\delta J}{\delta B^d} - g^{ab} D_A \frac{\delta I}{\delta B^a} \frac{\delta J}{\delta A^b} \right.
\]

\[ + g^{ab} D_A \frac{\delta J}{\delta B^a} \frac{\delta I}{\delta A^b} + \frac{\Lambda}{12} \frac{\delta I}{\delta A^a} g^{ab} \epsilon_{AB} \frac{\delta J}{\delta A^b} \left. \right) \].

The residual constraints on the boundary \( \Psi^r_a|_{\partial \Sigma} \) are Casimir functions of the Dirac bracket:

\[ \{ \Psi^r_a|_{\partial \Sigma}, J \}^* \approx 0, \]

for all functional \( J \).

The Hamiltonian \( H_T \) will be the differentiable gauge generator associated to the Lagrange multipliers:

\[ H_T = \Gamma_{A_0, B_0}. \]

As in the case of Chern-Simons, by tuning the boundary conditions on the Lagrange multipliers, we can build \( H_T \) to be any functional of the reduced phase-space.

**Theorem 4.2.** For BF theory in 4D on \( \Sigma \times \mathbb{R} \), the possible boundary conditions for the fields \( A^a_{\mu}, B^a_{\mu\nu} \) on each connected component \( C_n \) of \( \partial \Sigma \) are in one to one correspondance with the functionals of \( a^a_{nA} \) and \( b^a_{nAB} \) defined on the circle \( C_n \). The fields \( a^a_{nA} \) and \( b^a_{nAB} \) are respectively the pullback of \( A^a_i \) and \( B^a_{ij} \) on \( C_n \) and satisfy the constraint implied by the pullback of \( \epsilon_{ijk} \Psi^k_a \approx 0 \).

If \( \Lambda \) is different than zero, we can solve \( \Psi^r_a|_{\partial \Sigma} \approx 0 \) exactly with

\[ B^b|_{\partial \Sigma} = -\frac{6}{\Lambda} \epsilon^{AB} F^b_{AB}|_{\partial \Sigma}, \]

and describe the reduced phase-space in term of \( A^a_A|_{\partial \Sigma} \). The Dirac bracket becomes:

\[ \{ I[A^a_A|_{\partial \Sigma}], J[A^a_A|_{\partial \Sigma}] \}^* \approx \frac{\Lambda}{12} \oint_{\partial \Sigma} d^2 x \left( \frac{\delta I}{\delta A^a_A} g^{ab} \epsilon_{AB} \frac{\delta J}{\delta A^b_B} \right. \]

\[ \left. \right). \]

This is exactly the Poisson bracket \( \{4,4\} \) of the Chern-Simons theory in 3 dimensions. In this case, the reduced phase-space of the 4D B-F theory is the phase-space of Chern-Simons theory in 3 dimensions. However, the Hamiltonians will in general be different.

If we can build any Hamiltonian, in principle we should be able to reproduce the one of Chern-Simons. The easiest way of constructing it is to add a boundary condition on
the canonical variable $B_{AB}^a$: $B_{AB}^a|_{\partial \Sigma} = 0$. Following the arguments of the previous section, this can be done by relaxing the boundary condition on the corresponding Lagrange multiplier: $A_0^a$. If we put the other two relevant Lagrange multipliers, $B_{0A}^a$, to zero on the boundary, the associated differentiable total Hamiltonian is given by:

$$B_{0A}^a|_{\partial \Sigma} = 0,$$

$$H_T = \int_{\Sigma} d^3x \left( A_0^a \Phi_a + B_{0i}^a \Psi_i^a \right) + \oint_{\partial \Sigma} d^2x g_{ab} \epsilon^{AB} A_0^a B_{AB}^b. \quad (4.65)$$

On the constraint surface $\Phi_a \approx 0$ and $\Psi_i^a \approx 0$, we obtain:

$$H_T \approx -\frac{6}{\Lambda} \oint_{\partial \Sigma} d^2x A_0^a g_{ab} \epsilon^{AB} F_{AB}^b, \quad (4.67)$$

which is the Hamiltonian of the 3D Chern-Simons theory (4.3).
5 Conclusions

Starting with the notion of symplectic structure and requiring field theories to behave like discrete mechanical system naturally introduces the notion of differentiable generators. Restricting the set of functionals to the subset of differentiable one is then mandatory for the definition of a Poisson bracket. With these definitions, the hamiltonian structure of field theories behaves exactly like the one of discrete mechanical systems.

In the context of gauge theories, we showed that boundary conditions split into two categories: the boundary conditions on the dynamical variables are part of the definition of the canonical structure whereas the boundary conditions on the lagrange multipliers are part of the choice of the Hamiltonian of the theory. The canonical structure leads to the definition of differentiable gauge transformations as the gauge-like transformations generated by differentiable functionals. We gave a complete classification of the possible boundary conditions on the lagrange multipliers in term of these differentiable gauge transformations. In theories with local degrees of freedom, we need boundary conditions on the dynamical variables in order to control the flux of radiation. This restricts the set of differentiable generators a lot and consequently the set of possible boundary conditions on the lagrange multipliers. We also showed how the restriction to the differentiable gauge transformations preserving the boundary conditions on the lagrange multipliers leads to the usual notion of surface charges.

In theories with no local degrees of freedom, one can often remove the boundary conditions on the dynamical variable which makes the set of differentiable gauge transformations a lot bigger. This leads to two interesting consequences. Firstly, using differentiable gauge generators, we can probe the reduced phase-space and compute the Dirac bracket without solving the constraints. Secondly, by tuning the boundary conditions on the lagrange multipliers, we can construct any hamiltonian on the reduced phase-space. We used 3D Chern-Simons and 4D BF theory as example. In particular, we showed derived the complete set of possible boundary conditions for these theories when defined on manifold with time-like boundaries located at a finite distance.

The reduced phase-space of topological theories like Chern-Simons contains boundary gauge degrees of freedom. However, they are not a feature of topological theories and we expect them to exist in any gauge theory. In this paper, we used the differentiable generator of gauge transformations to describe them. Unfortunately, this technique is not generalizable to theories with local degrees of freedom like gravity in 4 dimensions. As we saw, the problem comes from the necessity of boundary conditions on the dynamical variables. In the future, it would be interesting to generalize the notion of canonical structure presented in this paper in order to relax these kind of boundary conditions. The hope
would then be that the boundary gauge degrees of freedom would again be described by
gauge generators. This would give some new insight on the notion of holography for
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A The Phase-Space

We will consider the space manifold $\Sigma$ to be of dimension $n$ and described by coordinates $x^i$. Its exterior derivative will be denoted $d$ and will be treated as a Grassmann odd quantity: $dx^i dx^j = -dx^j dx^i$.

The dynamical fields of the theory will be denoted $z^A$. The phase-space of the theory is the set of allowed configurations:

$$\mathcal{F} = \{ z^A(x), x^i \in \Sigma; \chi^\mu(z)|_{\partial\Sigma} = 0 \}.$$  \hfill (A.1)

The conditions $\chi^\mu|_{\partial\Sigma} = 0$ are the set of boundary conditions imposed on the fields; this set may be empty. If the boundary of $\Sigma$ is at infinity, boundary conditions are replaced by asymptotic conditions. We will assume that the boundary conditions are imposed on all equalities.

We will now describe the differential structure of $\mathcal{F}$. We will start by ignoring the boundary conditions and then describe the implications they have on the general structure.

A.1 Differential Structure

The exterior differential associated to the infinite dimensional manifold $\mathcal{F}$ will be denoted $\delta$. We will also treat it as a Grassmann odd quantity, $\delta z^A \delta z^B = -\delta z^B \delta z^A$, $\delta x^i = 0$, and assume that it commutes with the base manifold differential: $[d, \delta] = 0$. A general form will have components in both directions. A $(p, q)$-form will be a $p$-form over $\Sigma$ and a $q$-form over $\mathcal{F}$.

A vector field $Q^A \frac{\partial}{\partial z^A}$ over $\mathcal{F}$ is called an evolutionary vector field with characteristic $Q^A$. It represents a variation of the fields by an amount $Q^A(z, x)$. The operator measuring this variation is Grassmann even and denoted $\delta Q$. It satisfies to $[\delta Q, \delta] = 0$, $[\delta Q, d] = 0$ and $[\delta_q, \partial_i] = 0$ where $\partial_i$ is the total derivative with respect to $x^i$. The algebra of evolutionary vector fields is given by:

$$[\delta Q_1, \delta Q_2] = \delta [Q_1, Q_2] \quad \text{with} \quad [Q_1, Q_2]^A = \delta Q_1 Q_2^A - \delta Q_2 Q_1^A.$$  \hfill (A.2)

The interior product $\iota_Q$ between an evolutionary vector field $Q^A \frac{\partial}{\partial z^A}$ and a general $(p, q)$-form $\theta^{p,q}$ is given by:

$$\iota_Q \theta^{p,q}(\delta z^{A_1}, ..., \delta z^{A_q}, dx^{i_1}, ..., dx^{i_p}) =$$

$$\sum_{k=1}^q (-1)^k \theta^{p,q}(\delta z^{A_1}, ..., \delta Q_k z^{A_k}, ..., \delta z^{A_q}, dx^{i_1}, ..., dx^{i_p}).$$  \hfill (A.3)

We have

$$\iota_Q \delta + \delta \iota_Q = \delta Q, \quad \iota_{Q_1} \delta Q_2 + \delta Q_2 \iota_{Q_1} = \iota_{[Q_1, Q_2]}.$$  \hfill (A.4)
In our analysis, we will work with functionals and their differentials under $\delta$. A functional $F$ is defined as the integral of a $(n, 0)$-form:

$$F[z] = \int_{\Sigma} f^m x.$$  \hspace{1cm} (A.5)

We will use lowercase letters for the integrant and uppercase letters for integrated quantities. If the integrant is a $(n, s)$-form $\theta^{m,s}$, the resulting integrated quantity $\Theta^s$ will be a functional $s$-form on the space of configurations $\mathcal{F}$:

$$\Theta^s[z] = \int_{\Sigma} \theta^{m,s}.$$  \hspace{1cm} (A.6)

When $\delta$ acts on a functional $F$, we obtain:

$$\delta F[z] = \int_{\Sigma} \delta f^m x = \int_{\Sigma} \delta z^A \frac{\delta f}{\delta z^A} d^m x + \oint_{\partial \Sigma} I^n (f d^m x),$$  \hspace{1cm} (A.7)

where $\frac{\delta}{\delta z^A}$ is the Euler-Lagrange derivative and $I^n (f d^m x)$ denotes the $(n-1, 1)$-form obtained by integration by parts. In a similar way, we have

$$\delta_Q F[z] = \int_{\Sigma} Q^A \frac{\delta f}{\delta z^A} d^m x + \oint_{\partial \Sigma} I^n_Q (f d^m x),$$  \hspace{1cm} (A.8)

where $I^n_Q (f d^m x)$ is a $(n-1, 0)$-form given by

$$I^n_Q (f d^m x) = \iota_Q I^n (f d^m x).$$  \hspace{1cm} (A.9)

### A.2 Boundary Conditions

In the description of the differential structure we did not take into account the boundary conditions. They will impose restrictions on both $\delta$ and the allowed evolutionary vector fields.

The boundary conditions $\chi^\mu |_{\partial \Sigma} = 0$ are valid for all allowed field configurations: they must be preserved by $\delta$. The exterior derivative $\delta$ satisfies

$$\delta \chi^\mu |_{\partial \Sigma} = 0.$$  \hspace{1cm} (A.10)

In a similar way, an allowed evolutionary vector field must transform allowed configurations into allowed configurations:

$$\delta_Q \chi^\mu |_{\partial \Sigma} = 0.$$  \hspace{1cm} (A.11)

We then have the two following important results:
Theorem A.1. For any \((p, q)\)-form \(\theta^{n,s}\) such that we have

\[ \theta^{p,q}|_{\partial \Sigma} = 0 \]  

(A.12)

for all allowed values of \(z^A\) and \(\delta z^A\), we have

\[ \delta \theta^{p,q}|_{\partial \Sigma} = 0, \quad \iota_Q \theta^{p,q}|_{\partial \Sigma} = 0, \]

\[ \delta_Q \theta^{p,q}|_{\partial \Sigma} = 0, \]

(A.13)

(A.14)

for all allowed evolutionary vector fields \(Q^A \frac{\partial}{\partial z^A}\).

Corollary A.2. The set of evolutionary vector fields preserving the boundary conditions forms an algebra.

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