Tractable vs. Intractable Cases of Matching Dependencies for Query Answering under Entity Resolution

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ABSTRACT

Matching Dependencies (MDs) are a relatively recent proposal for declarative entity resolution. They are rules that specify, on the basis of similarities satisfied by values in a database, what values should be considered duplicates, and have to be matched. On the basis of a chase-like procedure for MD enforcement, we can obtain clean (duplicate-free) instances; actually possibly several of them. The resolved answers to queries are those that are invariant under the resulting class of resolved instances. Previous work identified certain classes of queries and sets of MDs for which resolved query answering is tractable. Special emphasis was placed on cyclic sets of MDs. In this work we further investigate the complexity of this problem, identifying intractable cases, and exploring the frontier between tractability and intractability. We concentrate mostly on acyclic sets of MDs. For a special case we obtain a dichotomy result relative to \( NP \)-hardness.

Keywords: data cleaning, entity resolution, matching dependencies, query answering, data complexity

1. INTRODUCTION

A database may contain several representations of the same external entity. In this sense it contains “duplicates”, which is in general considered to be undesirable; and the database has to be cleaned. More precisely, the problem of duplicate-or entity-resolution (ER) is about (a) detecting duplicates, and (b) merging duplicate representations into single representations. This is a classic and complex problem in data management, and in data cleaning in particular [11, 13, 4]. In this work we concentrate on the merging part of the problem, in a relational context.

A generic way to approach the problem consists in specifying what attribute values have to be matched (made identical) under what conditions. A declarative language with a precise semantics could be used for this purpose. In this direction, matching dependencies (MDs) have been recently introduced [14, 15]. They represent rules for resolving pairs of duplicate representations (considering two tuples at a time). Actually, when certain similarity relationships between attribute values hold, an MD indicates what attribute values have to be made the same (matched).

Example 1. The similarities of phone and address indicate that the tuples refer to the same person, and the names should be matched. Here, 723-9583 \( \approx \) (750) 723-9583 and 10-43 Oak St. \( \approx \) 43 Oak St. Ap. 10.

| People | Name    | Phone       | Address       |
|--------|---------|-------------|---------------|
| John Smith | 723-9583 | (750) 723-9583 | 43 Oak St. Ap. 10 |
| J. Smith  | (750) 723-9583 | 10-43 Oak St. |               |

The following MD captures this resolution policy: (with \( P \) standing for predicate \( \text{People} \))

\[
P[\text{Phone}] \approx P[\text{Phone}] \land P[\text{Address}] \approx P[\text{Address}] \rightarrow P[\text{Name}] = P[\text{Name}].
\]

This MD involves only one database predicate, but in general, an MD may involve two different relations. We can also have several MDs on the database schema.

The framework for MD-based entity resolution used in this paper was introduced in [16], where a precise semantics for MDs involving a chase procedure for cleaning the database instance was introduced. This semantics made precise the rather intuitive semantics for MDs originally introduced in [15].

Also in [16], the problem of resolved query answering was introduced. For a fixed set of MDs, and a fixed query, this is the problem of deciding, given an unresolved database instance, and a candidate query answer \( \bar{a} \), whether \( \bar{a} \) is an answer to the query under all admissible ways of resolving the duplicates as dictated by the MDs. It was shown that this problem is generally intractable by giving an \( NP \)-hard case of the problem involving a pair of MDs. By identifying the elements of this set of MDs that lead to intractability, tractability of resolved query answering was obtained for other pairs of MDs.

The resolved query answering problem was studied further in [18, 17]. Specifically, a class of tractable cases of the problem was identified [18], for which a method for retrieving the resolved answers based on query rewriting into stratified Datalog with aggregation was developed [17].
In those tractable cases, we find conjunctive queries with certain restrictions on joins, and sets of MDs that depend cyclically on each other, in the sense that modifications produced by one MD may affect the application of the next MD in the enforcement cycle. These are the (cyclic) HSC sets identified in [18]. It was shown that, in general, cyclic dependencies on MDs make the problem tractable, because the requirement of chase termination implies a relatively simple structure for the clean database instances [18].

In this work we concentrate on acyclic sets of MDs. This completely changes the picture w.r.t. previous work. As just mentioned, for HSC sets, tractability of resolved query answering holds [18]. This is the case, for example, for the cyclic \( M = \{ R[A] \approx R[A] \rightarrow R[B] \approx R[B], R[B] \approx R[B] \rightarrow R[A] \approx R[A] \} \). However, as we will see later on, for the following acyclic, somehow syntactically similar example, \( M' = \{ R[A] \approx R[A] \rightarrow R[B] \approx R[B], R[B] \approx R[B] \rightarrow R[C] \approx R[C] \} \), resolved query answering can be tractable. This example, and our general results, show that, possibly contrary to intuition, the presence of cycles in sets of MDs tends to make resolved query answering easier.

In this work, we further explore the complexity of resolved query answering. Rather than considering isolated tractable cases as in previous work, here we take a more systematic approach. We develop a set of syntactic criteria on sets of two MDs that, when satisfied by a given pair of MDs, implies intractability of the resolved query answer problem.

We also show, under an additional assumption about the nature of the similarity operator, that resolved query answering is tractable for sets of MDs not satisfying these criteria, leading to a dichotomy result. We extend these results also considering tractability/intractability of sets of more than two MDs.

All these results apply to acyclic sets of MDs, and thus are complementary to those of [18, 17], providing a broader view of the complexity landscape of query answering under matching dependencies.

Summarizing, this paper, we undertake a systematic investigation of the data complexity of the problems of deciding and computing resolved answers to conjunctive queries under MDs. This complexity analysis sheds some light on the intrinsic computational limitations of retrieving, from a database with unresolved duplicates, the information that is invariant under the entity resolution process as captured by MDs. The main contributions of this paper are as follows:

1. We identify a class of conjunctive queries that are relevant for the investigation of tractability vs. intractability of resolved query answering. Intuitively, these queries return data that can be modified by application of the MDs. We call them changeable attribute queries.

2. Having investigated in [17, 18] cases of cyclic sets of MDs, we complement these results by studying the complexity of resolved query answering for sets of MDs that do not have cycles.

3. For certain sets of two MDs that satisfy a syntactic condition, we establish an intractability result, proving that deciding resolved answers to changeable attribute queries is \( \textit{NP} \)-hard in data.

4. For similarity relations that are transitive (a rare case), we establish that the conditions for hardness mentioned in the previous item, lead to a dichotomy result: pairs of MDs that satisfy them are always \textit{hard}, otherwise they are always \textit{easy} (for resolved query answering). This shows, in particular, that the result mentioned in item 3. cannot be extended to a wider class of MDs for arbitrary similarity relations.

We also prove that the dichotomy result does not hold when the hypothesis on similarity is not satisfied.

5. Relying on the results for pairs of MDs, we consider acyclic sets of MDs of arbitrary size. In particular, we prove intractability of the resolved query answering problem for certain acyclic sets of MDs that have the syntactic property of \textit{non-inclusiveness}.

The structure of the paper is as follows. Section 2 introduces notation and terminology used in the paper, and reviews necessary results from previous work. Section 3 identifies classes of MDs, queries and assumptions that are relevant for this research. Sections 3.1 and 4 investigate the complexity of the problem of computing resolved answers for sets of two MDs. Section 5 extends those results to sets of MDs of arbitrary size. In Section 6 we summarize results, including a table of known complexity results (obtained in this and previous work). We also draw some final conclusions, and we point to open problems. Full proofs of our results can be found in the appendix.\(^1\)

## 2. PRELIMINARIES

In this work we consider relational database schemas and instances. Schemas are usually denoted with \( S \), and contain relational predicates. Instances are usually denoted with \( D \). Matching dependencies (MDs) are symbolic rules of the form:

\[
\bigwedge_{i,j} R[A_i] \approx_{ij} S[B_j] \rightarrow \bigwedge_{k,l} R[A_k] \approx_{k,l} S[B_l],
\]

where \( R, S \) are relational predicates in \( S \), and the \( A_1, \ldots \) are attributes for them. The LHS captures similarity conditions on a pair of tuples belonging to the extensions of \( R \) and \( S \) in an instance \( D \). We abbreviate this formula as: \( R[A] \approx S[B] \rightarrow R[C] \approx S[E] \).

The similarity predicates (or operators) \( \approx \) (there may be more than one in an MD depending on the attributes involved) are domain-dependent and treated as built-ins. \(^1\)

\(^1\)An extended abstract containing in preliminary form some of results in this paper is [9].
ferent attribute domains may have different similarity predicates. We assume they are symmetric and reflexive. Transitivity is not assumed (and in many applications it may not hold).

MDs have a dynamic interpretation requiring that those values on the RHS should be updated to some (unspecified) common value. Those attributes on a RHS of an MD are called changeable attributes. MDs are expected to be “applied” iteratively until duplicates are solved.

In order to keep track of the changes and comparing tuples and instances, we use global tuple identifiers, a non-changeable surrogate key for each database predicate that has changeable attributes. The auxiliary, extra attribute (when shown) appears as the first attribute in a relation, e.g. \( t \) is the identifier in \( R(t, x) \). A position is a pair \( (t, A) \) with \( t \) a tuple id, and \( A \) an attribute (of the relation where \( t \) is an id). The position’s value, \( t[A] \), is the value for \( A \) in tuple (with id) \( t \).

2.1 MD semantics

A semantics for MDs acting on database instances was proposed in [16]. It is based on a chase procedure that is iteratively applied to the original instance \( D \). A resolved instance \( D' \) is obtained from a finitely terminating sequence of database instances, say

\[
D := D_0 \Rightarrow D_1 \Rightarrow D_2 \Rightarrow \cdots \Rightarrow D_n =: D'.
\]  
\( \text{(2)} \)

\( D' \) satisfies the MDs as equality generating dependencies [1], i.e. replacing \( \doteq \) by equality.

The semantics specifies the one-step transitions or updates allowed to go from \( D_{i-1} \) to \( D_i \), i.e. “\( \Rightarrow \)” in (2). Only modifiable positions within the instance are allowed to change their values in such a step, and as forced by the MDs. Actually, the modifiable positions syntactically depend on a whole set \( M \) of MDs and instance at hand; and can be recursively defined (see [16, 17] for the details). Intuitively, a position \( (t, A) \) is modifiable iff: (a) There is a \( t' \) such that \( t \) and \( t' \) satisfy the similarity condition of an MD with \( A \) on the RHS; or (b) \( t[A] \) has not already been resolved (it is different from one of its other duplicates).

Example 2. For the schema \( R(A, B) \), consider the MD \( R[A] = R[B] \Rightarrow R[B] \doteq R[B] \), and the instance \( R(D) \) below. The positions of the underlined values in \( D \) are modifiable, because their values are unresolved (wrt the MD and instance \( R(D) \)).

| \( R(D) \) | \( A \) | \( B \) |
|---|---|---|
| \( t_1 \) | \( a \) | \( b \) |
| \( t_2 \) | \( a \) | \( c \) |

\( D' \) is a resolved instance since it satisfies the MD interpreted as the FD \( R : A \Rightarrow B \). Here, the update value \( d \) is arbitrary.

\( D' \) has no modifiable positions with unresolved values: the values for \( B \) are already the same, so there is no reason to change them.

More formally, the single step semantics (\( \Rightarrow \) in (2)) is as follows. Each pair \( D_i, D_{i+1} \) in an update sequence (2), i.e. a chase step, must satisfy the set \( M \) of MDs modulo unmodifiability, denoted \( (D_i, D_{i+1}) \models_{um} M \), which holds iff: (a) For every MD in \( M \), say \( R[A] \approx S[B] \Rightarrow R[C] \doteq S[D] \), and pair of tuples \( t_R \) and \( t_S \), if \( t_R[A] \approx t_S[B] \) in \( D_i \), then \( t_R[C] = t_S[D] \) in \( D_{i+1} \); and (b) The value of a position can only differ between \( D_i \) and \( D_{i+1} \) if it is modifiable wrt \( D_i \). Accordingly, in (2) we also require that \( (D_i, D_{i+1}) \not\models_{um} M \), for \( i < n \), and \( (D_n, D_n) \models_{um} M \) (the stability condition).

This semantics stays as close as possible to the spirit of the MDs as originally introduced [15], and also uncommitted in the sense that the MDs do not specify how the matchings have to be realized (c.f. Section 6 for a discussion).

Example 3. Consider the following instance and set of MDs. Here, attribute \( R(C) \) is changeable. Position \( (t_2, C) \) is not modifiable wrt. \( M \) and \( D \). There is no justification to change its value in one step on the basis of an MD and \( D \). However, position \( (t_1, C) \) is modifiable. \( D \) has two resolved instances, \( D_1 \) and \( D_2 \).

\[
\begin{array}{|c|c|c|c|}
\hline
R(D) & A & B & C \\
\hline
\hline
\text{\( t_1 \)} & a & b & d \\
\text{\( t_2 \)} & a & c & e \\
\text{\( t_3 \)} & a & b & e \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
R(D_1) & A & B & C \\
\hline
\text{\( t_1 \)} & a & b & d \\
\text{\( t_2 \)} & a & b & d \\
\text{\( t_3 \)} & a & b & d \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
R(D_2) & A & B & C \\
\hline
\text{\( t_4 \)} & b & e & d \\
\text{\( t_5 \)} & b & e & d \\
\hline
\end{array}
\]

For arbitrary sets of MDs, some (admissible) chase sequences may not terminate. However, it can be proved that there are always terminating chase sequences. As a consequence, for some sets of MDs, there are both terminating and non-terminating chase sequences. In any case, the class of resolved instances is always well-defined.

Example 4. Consider relation \( R(A, B) \), equality as the similarity relation, and the MDs and instance below:

\[
\begin{array}{|c|c|}
\hline
R(D) & A & B \\
\hline
\text{\( t_1 \)} & a & c \\
\text{\( t_2 \)} & b & c \\
\text{\( t_3 \)} & b & d \\
\text{\( t_4 \)} & a & d \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
m_1 : R[A] = R[B] \Rightarrow R[B] \doteq R[B] \\
m_2 : R[B] = R[B] \Rightarrow R[A] \doteq R[A] \\
\hline
\end{array}
\]

The chase may not terminate, which happens when the values oscillate, as in the following update sequence:

\[
\begin{array}{|c|c|}
\hline
R(D) & A & B \\
\hline
\text{\( t_1 \)} & a & c \\
\text{\( t_2 \)} & b & c \\
\text{\( t_3 \)} & b & d \\
\text{\( t_4 \)} & a & d \\
\hline
\end{array}
\quad
\Rightarrow
\quad
\begin{array}{|c|c|}
\hline
R(D) & A & B \\
\hline
\text{\( t_1 \)} & a & c \\
\text{\( t_2 \)} & a & d \\
\text{\( t_3 \)} & b & d \\
\text{\( t_4 \)} & b & c \\
\hline
\end{array}
\]

\( ^2 \)The case \( D' = D_0 \) occurs only when \( D \) is already resolved.
However, there are non-trivial terminating chase sequences:

| $R(D)$ | $A$ | $B$ |
|--------|-----|-----|
| $t_1$  | $a$ | $c$ |
| $t_2$  | $b$ | $c$ |
| $t_3$  | $b$ | $d$ |
| $t_4$  | $a$ | $d$ |

with $e$, $f$, $g$, and $h$ arbitrary. After this, a stable instance can be obtained by updating all values in the $A$ and $B$ to the same value.

We prefer resolved instances that are the closest to the original instance. A minimally resolved instance (MRI) of $D$ is a resolved instance $D'$ whose the number of changes of attribute values wrt. $D$ is a minimum. In Example 3, instance $D_2$ is an MRI, but not $D_1$ (2 vs. 3 changes). We denote with $\text{Res}(D, M)$ and $\text{MinRes}(D, M)$ the classes of resolved, resp. minimally resolved, instances of $D$ wrt $M$.

Infinite chase sequences may occur when the MDs cyclically depend on each other, in which case updated instances in such a sequence may alternate between two or more states [18, Example 6] (see also Example 4). However, for the chase sequences that do terminate in a minimally resolved instance, the chase imposes a relatively easily characterizable structure [18, 17], allowing us to obtain a query rewriting methodology. So, cycles help us achieve tractability for some classes of queries [17] (cf. Section 2.2).

On the other side, it has been shown that if a set of MDs satisfies a certain acyclicity property, then all chase sequences terminate after a number of iterations that depends only on the set of MDs and not on the instance [16, Lemma 1] (cf. Theorem 1 below). But the number of resolved instances may still be “very large”. Sets of MDs considered in this work are acyclic.

### 2.2 Resolved query answers

Given a conjunctive query $Q$, a set of MDs $M$, and an instance $D$, the resolved answers to $Q$ from $D$ are invariant under the entity resolution process, i.e. they are answers to $Q$ that are true in all MRIs of $D$:

$$\text{ResAns}_M(Q, D) := \{ \bar{a} \mid D' \models Q[\bar{a}], \text{ for every } D' \in \text{MinRes}(D, M) \}. \quad (3)$$

The corresponding decision problem is $\text{RA}(Q, M) := \{(D, \bar{a}) \mid \bar{a} \in \text{ResAns}_M(Q, D)\}$.

In [17, 18], a query rewriting methodology for resolved query answering (RQA) under MDs (i.e. computing resolved answers to queries) was presented. In this case, the rewritten queries turn out to be Datalog queries with counting, and can be obtained for two main kinds of sets of MDs: (a) MDs do not depend on each other, i.e. non-interacting sets of MDs [16]; (b) MDs that depend cyclically on each other, e.g. as in the set containing $R[A] \approx R[A] \rightarrow R[B] \equiv R[B]$ and $R[B] \approx R[B] \rightarrow R[A] \equiv R[A]$ (or relationships like this by transitivity).

For these sets of MDs just mentioned, a conjunctive query can be rewritten to retrieve, in polynomial time in data, the resolved answers, provided the queries have no joins on existentially quantified variables corresponding to changeable attributes. The latter form the class of unchangeable attribute join conjunctive (UJCQ) queries [18].

For example, for the MD $R[A] = R[A] \rightarrow R[B, C] \equiv R[B, C]$ on schema $R[A, B, C]$, $Q : \exists x \exists y \exists z (R(x, y, c) \land R(z, y, d))$ is not UJCQ; whereas $Q' : \exists x \exists z (R(x, y, z) \land R(x, y', z'))$ is UJCQ. For queries outside UJCQ, the resolved answer problem can be intractable even for one MD [18].

The case of a set of MDs consisting of both

$$R[A] \approx R[A] \rightarrow R[B] \equiv R[B], \quad (4)$$

$$R[B] \approx R[B] \rightarrow R[C] \equiv R[C],$$

which is neither non-interacting nor cyclic, is not covered by the positive cases for Datalog rewriting above. Actually, for this set RQA becomes intractable for very simple queries, like $Q(x, z) : \exists y R(x, y, z)$, that is UJCQ [16]. Sets of MDs like (4) are the main focus of this work.

### 3. INTRACTABILITY OF RQA

In the previous section we briefly described classes of queries and MDs for which RQA can be done in polynomial time in data (via the Datalog rewriting). We also showed that there are intractable cases, by pointing to a specific query and set of MDs. Natural questions that we start to address in this section are the following: (a) What happens outside the Datalog rewritable cases in terms of complexity of RQA? (b) Do the exhibited query and MDs correspond to a more general pattern for which intractability holds?

For all sets $M$ of MDs we consider below, we assume that at most two relational predicates, say $R, S$, appear in $M$, e.g. as in $M = \{R[A] \approx S[B] \rightarrow R[C] \equiv S[E]\}$. In same cases we assume that there are exactly two predicates. The purpose of this restriction is to simplify the presentation. All results can be generalized to sets of MDs with more than two predicates. To do this, definitions and conditions concerning the two relations in the MDs can be extended to cover the additional relations as well.

At the other extreme, when a single predicate occurs in $M$, say $R$, as in Example 3, the results for at most two predicates can be reformulated and applied by replacing $S$ with $R'$. Although $R$ and $R'$ are the same relation in this case, the prime is used to distinguish between the two tuples to which the MD refers.

All the sets of MDs considered below are both interacting (non-interaction does not bring complications) and acyclic. Both notions and others can be captured in terms of the MD graph, $\text{MDG}(M)$, of $M$. It is a directed graph, such that,
for $m_1, m_2 \in M$, there is an edge from $m_1$ to $m_2$ if there is an overlap between $RHS(m_1)$ and $LHS(m_2)$ (the right- and left-hand sides of the arrows as sets of attributes) [16]. Accordingly, $M$ is acyclic when $MDG(M)$ is acyclic.

**Definition 1.** [16] 1. Let $M$ be a set of MDs on schema $\mathcal{S}$. (a) The symmetric binary relation $\equiv_r$ relates attributes $R[A]$ and $S[B]$ of $\mathcal{S}$ whenever there is $m \in M$ in which $R[A] \equiv_r S[B]$ occurs. (b) The attribute closure of $M$ is the reflexive and transitive closure of $\equiv_r$. (c) $E_{R[A]}$ denotes the equivalence class of attribute $R[A]$ in the attribute closure of $M$.

2. The **augmented MD-graph** of $M$, denoted $AMDG(M)$, is a directed graph with a vertex labeled with $m$ for each $m \in M$, and with an edge from $m$ to $m'$ if there is an attribute, say $R[A]$, with $R[A] \in RHS(m)$ and $E_{R[A]} \cap LHS(m') \neq \emptyset$.

3. $M$ is strongly acyclic if $AMDG(M)$ has no cycles. □

Because $R[A] \in E_{R[A]}$, for any set $M$ of MDs, all edges in $MDG(M)$ are also edges in $AMDG(M)$. Therefore, strong acyclicity implies acyclicity. However, the converse is not true, as shown in the next example.

**Example 5.** The set $M$ of MDs

$m_1 : R[F] \equiv S[G] \rightarrow R[A] \equiv S[H],$

$m_2 : R[A] \equiv S[B] \rightarrow R[C] \equiv S[E],$

$m_3 : R[C] \equiv S[E] \rightarrow R[I] \equiv S[H],$

is acyclic but not strongly acyclic. $MDG(M)$ has three vertices, $m_1, m_2, m_3$, and edges $(m_1, m_2)$ and $(m_2, m_3)$. $AMDG(M)$ has the additional edge $(m_3, m_2)$, because $E_{R[I]} = \{R[I], S[H], R[A]\} \cap LHS(m_2) = \{R[A]\}$. □

In this work, we consider strongly acyclic sets of MDs. In particular, two interesting and common kinds that form large sets of MDs: **linear pairs**, which consist of two MDs such that $MDG(M)$ contains a single edge from one to the other (c.f. Definition 5); and **acyclic sets** that are **pair-preserving** (c.f. Definition 7). From the definitions of these two kinds of sets of MDs it will follow that they are strongly acyclic.

**Theorem 1.** [16] Let $M$ be a strongly acyclic set of MDs on schema $\mathcal{S}$, and $D$ an instance for $\mathcal{S}$. Every sequence of $M$-based updates to $D$ as in (2) terminates with a resolved instance after at most $d + 1$ steps, where $d$ is the maximum length of a path in $AMDG(M)$. □

As mentioned previously, the chase can be infinite if the set is not acyclic. Theorem 1 only tells us about the chase termination and lengths, but it does not involve the data. So, it does not guarantee tractability for RQA, leaving room, in principle, for both tractable and intractable cases. Actually, it can still be the case that there are exponentially many minimally resolved instances. A reason for this is that the application of an MD to an instance may produce new similarities among the values of attributes in $RHS(m_1)$ that are not strictly required by the chase, but result from a particular choice of update values. Such “accidental similarities” affect subsequent updates, resulting in exponentially many possible update sequences. This is illustrated in the next example.

**Example 6.** Consider the strongly acyclic set $M$:

$R[A] \approx R[A] \rightarrow R[B] \equiv R[B],$

$R[B] \approx R[B] \rightarrow R[C] \equiv R[C].$

When the instance

| $R(D_1)$ | $A$ | $B$ | $C$ |
|----------|-----|-----|-----|
| $t_1$    | $a$ | $m$ | $e$ |
| $t_2$    | $a$ | $d$ | $f$ |
| $t_3$    | $b$ | $c$ | $g$ |
| $t_4$    | $b$ | $k$ | $h$ |

is updated according to $M$, the sets of value positions $\{t_1[B], t_2[B]\}$ and $\{t_3[B], t_4[B]\}$ must be merged. One possible update is

| $R(D_2)$ | $A$ | $B$ | $C$ |
|----------|-----|-----|-----|
| $t_1$    | $a$ | $m$ | $e$ |
| $t_2$    | $a$ | $m$ | $f$ |
| $t_3$    | $b$ | $m$ | $g$ |
| $t_4$    | $b$ | $m$ | $h$ |

The similarities between the attribute $B$ values of the top and bottom pairs of tuples are accidental, because they result from the choice of update values. In the absence of accidental similarities, there is only one possible set of sets of values that are merged in the second update, namely $\{\{t_1[C], t_2[C]\}, \{t_3[C], t_4[C]\}\}$.

Accidental similarities increase the complexity of query answering over the instance by adding another possible set of sets of merged values, $\{\{t_1[C], t_2[C], t_3[C], t_4[C]\}\}$. More generally, for an instance with $n$ sets of merged value positions in the $B$ column, the number of possible sets of sets of value positions in the $C$ column that are merged in the second update is $\Omega(2^{n^2})$. □

We want to investigate the frontier between tractability and intractability. For this reason, we make the assumption that, for each similarity relation, $\approx$, there is an infinite set of mutually dissimilar values. Actually, without this assumption, the resolved answer problem becomes immediately tractable for certain similarity operators (e.g. transitive similarity operators). This is because, for these operators, the whole class of minimal resolved instances of an instance can be computed in polynomial time.

**Proposition 1.** For strongly acyclic sets of MDs, if the similarity predicates are transitive and there is no infinite set of mutually dissimilar values, then the set of minimal resolved
instances for a given instance $D$ can be computed in polynomial time in the size of $D$.

Proof (sketch): By Theorem 1, the chase terminates after a number of updates that is constant in the size of the instance. We claim that, at each step of the chase, the number of updates that could be made that would lead to a minimal resolved instance is polynomial in the size of the instance. Thus, all minimal resolved instances can be computed in polynomial time by exhaustively going through all possible choices of updates at each step.

To prove the claim, we consider, for a given MD $m$, a conjunct $R[A] \approx S[B]$ appearing to the left of the arrow in $m$. Consider a set of tuples in $R$ and $S$ whose values are merged by the application of $m$. All tuples in this set must be in the same equivalence class of the transitive closure of the binary relation expressed by $R[A] \approx S[B]$. By transitivity of $\approx$, this means that the values that these tuples take on $R[A]$ or $S[B]$ must similar. Thus, there are at most $b$ sets of tuples whose values are merged, with $b$ the maximum number of mutually distinct values in a set.

A minimal resolved instance is obtained by choosing, for each set of values that are merged by application of $m$, an update value from a set of values consisting of the union of the set of all the values to be merged and a maximal set of mutually dissimilar values. If the update value $v$ is not one of the values to be merged, then the values are updated to $v^*$, which represents any value from the equivalence class of $\approx$ to which $v$ belongs. There are a constant number of sets of merged values, and $O(n)$ possible update values for each set, with $n$ the size of the instance. This proves the claim.

Our next results require some terms and notation that we now introduce.

Definition 2. Let $M$ be a set of MDs with predicates $R$ and $S$. A changeable attribute query $Q$ is a (conjunctive) query in UJCQ, containing a conjunct of the form $R(\bar{x})$ or $S(\bar{y})$ such that all variables in the conjunct are free and none occur in another conjunct of the form $R(\bar{x})$ or $S(\bar{y})$. Such a conjunct is called a join-restricted free occurrence of the predicate $R$ or $S$.

By definition, the class of changeable attribute queries (CHAQ) is a subclass of UJCQ. Both classes depend on the set of MDs at hand. For example, for the MDs in (4), $\exists \bar{y}\exists \bar{z} R(x, y, z) \in \text{UJCQ} \setminus \text{CHAQ}$, but $\exists \bar{w}\exists \bar{t} (R(x, y, z) \land S(x, w, t)) \in \text{CHAQ}$. We confine our attention to UJCQ and subsets of it, because, as mentioned in the previous section, intractability limits the applicability of the duplicate resolution method for queries outside UJCQ.

The requirement that the query contains a join-restricted free occurrence of $R$ or $S$ eliminates from consideration certain queries in UJCQ for which the resolved answer problem is immediately tractable. For example, for the MDs in (4), the query $\exists \bar{y}\exists \bar{z} R(x, y, z)$ is not CHAQ, and is tractable simply because it does not return the values of a changeable attribute (the resolved answers are the answers in the usual sense). The restriction on joins simplifies the analysis while still including many useful queries.

In order to eliminate queries like $\exists \bar{y}\exists \bar{z} R(x, y, z)$ wrt. $M$ in (4), CHAQ imposes a strong condition. Actually, the condition can be weakened, requiring to have at least one of the variables satisfying the condition in the definition for CHAQ. Weakening the condition makes the presentation much more complex since a finer interaction with the MDs has to be brought into the picture. (We leave this issue for an extended version.)

Definition 3. A set $M$ of MDs is hard if for every CHAQ Q, $RA(Q, M)$ is NP-hard. $M$ is easy if for every CHAQ Q, $RA(Q, M)$ is in PTIME.

Of course, a set of MDs may not be hard or easy. For the resolved answer problem, membership of NP is an open problem. However, for strongly acyclic sets, the bound on the length of the chase implies an upper bound of $\Pi^p_2$ [16, Theorem 5].

In the following we give some syntactic conditions that guarantee hardness for classes of MDs. To state them we need to introduce some useful notions first.

Definition 4. Let $m$ be an MD. The symmetric binary relation $LRel(m)$ ($RRel(m)$) relates each pair of attributes $R[A]$ and $R[B]$ such that an atom $R[A] \approx S[B]$ (resp. $R[A] \approx S[B]$) appears in $m$. An L-component (R-component) of $m$ is an equivalence class of the reflexive and transitive closure $LRel(m) = +$ (resp. $RRel(m) = +$) of $LRel(m)$ (resp. $RRel(m)$).

Example 7. For $m : R[A] \approx S[B] \land R[A] \approx S[C] \rightarrow R[E] = S[F] \land R[G] = S[H]$, there is only one L-component: \{ $(R[A], S[B], S[C])$ \}; and two R-components: \{ $(R[E], S[F])$ \} and \{ $(R[G], S[H])$ \}.

3.1 Hardness of linear pairs of MDs

Most of the results that follow already hold for pairs of MDs, we concentrate on this case first.

Definition 5. A set $M = \{m_1, m_2\}$ of MDs is a linear pair, denoted by $(m_1, m_2)$, if its graph $MDG(M)$ consists of the vertices $m_1$ and $m_2$ with only an edge from $m_1$ to $m_2$.

First, notice that if $(m_1, m_2)$ is a generic linear pair, with say

\begin{align*}
m_1 : & R[\bar{A}] \approx_1 S[\bar{B}] \rightarrow R[\bar{C}] = S[\bar{E}], \\
m_2 : & R[\bar{F}] \approx_2 S[\bar{G}] \rightarrow R[\bar{H}] = S[\bar{I}],
\end{align*}

then, from the definition of the MD graph, it follows that $(R[\bar{C}] \cup S[\bar{E}]) \cap (R[\bar{F}] \cup S[\bar{G}]) \neq \emptyset$, whereas $(R[\bar{H}] \cup S[\bar{I}]) \cap (R[\bar{A}] \cup S[\bar{B}]) = \emptyset$. In the following we have to
analyze other different forms of (non-)interaction between the attributes in linear pairs.

**Definition 6.** Let \((m_1, m_2)\) be a linear pair as in (5). (a) \(B_R\) is a binary (reflexive and symmetric) relation on attributes of \(R\): \((R[U_1], R[U_2]) \in B_R\) iff \(R[U_1] \equiv R[U_2]\) and \(R[U_2] \equiv R[U_1]\) are in the same \(R\)-component of \(m_1\) or the same \(L\)-component of \(m_2\). Similarly for \(B_S\).

(b) An \(R\)-equivalent set \((R-ES)\) of attributes of \((m_1, m_2)\) is an equivalence class of \(\text{TC}(B_R)\), the transitive closure of \(B_R\), with at least one attribute in the equivalence class belonging to \(\text{LHS}(m_2)\). The definition of an \(S\)-equivalent set \((S-ES)\) is similar, with \(R\) replaced by \(S\).

(c) An \((R\text{ or }S)\)-ES \(E\) of \((m_1, m_2)\) is bounded if \(E \cap \text{LHS}(m_1)\) is non-empty.

**Example 8.** Consider the schema \(R[A, C, F, H, I, M]\), \(S[B, D, E, G, N]\), and the linear pair \((m_1, m_2)\) with:

\[
m_1 : R[A] \approx S[B] \rightarrow R[C] \approx S[D] \wedge R[F] \approx S[E] \wedge R[H] \approx S[G],
\]

\[
m_2 : R[F] \approx S[E] \wedge R[I] \approx S[E] \wedge R[A] \approx S[E] \wedge R[M] \approx S[N].
\]

It holds:

(a) \(B_R(R[F], R[H])\) due to the occurrence of \(R[F] \equiv S[G]\), \(R[H] \equiv S[G]\).

(b) \(B_R(R[F], R[I])\) due to \(R[F] \equiv S[E], R[I] \equiv S[E]\).

(c) \(B_R(R[I], R[A])\) due to \(R[I] \equiv S[E], R[A] \equiv S[E]\).

(d) \(\{R[A], R[F], R[I], R[H]\}\) is an \(R\)-ES, and since \(\{R[A], R[F], R[I], R[H]\} \cap \text{LHS}(m_1) = \{R[A]\} \neq \emptyset\), it is also bounded.

**Theorem 2.** Let \((m_1, m_2)\) be a linear pair, with relational predicates \(R\) and \(S\). Let \(E_R, E_S\) be the sets of \(R\)-ESs and \(S\)-ESs, resp. The pair \((m_1, m_2)\) is hard if \(RHS(m_1) \cap RHS(m_2) = \emptyset\), and at least one of (a) and (b) below holds:

(a) All of the following hold:

(i) \(\text{Attr}(R) \cap (\text{RHS}(m_1) \cap \text{LHS}(m_2)) \neq \emptyset\).

(ii) There are unbounded ESs in \(E_R\).

(iii) For some \(L\)-component \(L\) of \(m_1\), \(\text{Attr}(R) \cap (L \cap \text{LHS}(m_2)) = \emptyset\).

(b) Same as (a), but with \(R\) replaced by \(S\).

Theorem 2 says that a linear pair of MDs is hard unless the syntactic form of the MDs is such that there is a certain association between changeable attributes in \(\text{LHS}(m_2)\) and attributes in \(\text{LHS}(m_1)\) as specified by conditions (ii) and (iii).

For pairs of MDs satisfying the negation of (a)(ii) or that of (a)(iii) (or the negation of (b)(ii) or that of (b)(iii)) in Theorem 2, the similarities resulting from applying \(m_2\) are restricted to a subset of those that are already present among the values of attributes in \(\text{LHS}(m_1)\), making the problem tractable. However, when condition (ii) or (iii) is satisfied, accidental similarities among the values of attributes in \(\text{RHS}(m_1)\) cannot be passed on to values of attributes in \(\text{RHS}(m_2)\).

**Example 9.** The linear pair \((m_1, m_2)\) with

\[
m_1 : R[A] \approx S[B] \rightarrow R[C] \approx S[D]\n\]

\[
m_2 : R[C] \approx S[D] \rightarrow R[E] \approx S[F]
\]

is hard. In fact, first: \(\text{RHS}(m_1) \cap \text{RHS}(m_2) = \emptyset\).

Now, it satisfies condition (a): Condition (a)(i) holds, because \(R[C] \not\in \text{RHS}(m_1) \cap \text{LHS}(m_2)\). Conditions (a)(ii) and (a)(iii) are trivially satisfied, because there are no attributes of \(\text{LHS}(m_1)\) in \(\text{LHS}(m_2)\).

As mentioned above, Theorem 2 generalizes to the case of more or fewer than two database predicates. It is easy to verify, for the former case, that if there are more than two predicates in a linear pair, then there must be exactly three of them, one of which appears in both MDs. In this case, hardness is implied by condition (a) in Theorem 2 alone, with \(R\) the predicate in common.

**Example 10.** The linear pair with three predicates:

\[
m_1 : R[A] \approx S[B] \rightarrow R[C] \approx S[E],
\]

\[
m_2 : R[C] \approx P[B] \rightarrow R[F] \approx P[G].
\]

is hard if it satisfies condition (a) in Theorem 2. It does satisfy it:

(i) \(\text{Attr}(R) \cap (\text{RHS}(m_1) \cap \text{LHS}(m_2)) = \{R[C]\}\).

(ii) The ES \(\{R[C]\}\) is unbounded.

(iii) Part (iii) holds with \(L = \{R[A], S[B]\}\).

For the case with only one predicate \(R\) in the linear pair, in order to apply Theorem 2, we need to derive from it a special result, Corollary 1 below. It is obtained by first labeling the different occurrences of the (same) predicate in \(M\), and then generating conditions (four of them, analogous to (a) and (b) in Theorem 2) for the labeled version, \(M'\). When \(M'\) satisfies those conditions, the original set \(M\) is hard. The algorithm **Conditions** in Table 1 does both the labeling and the condition generation to be checked on \(M'\). Notice that, after the labeling, there is still only one predicate in \(M'\). The labeling simply provides a convenient way to refer to different sets of attributes. Example 11 demonstrates the use of the algorithm and the application of the corollary.

**Corollary 1.** A linear pair containing one predicate is hard if it satisfies \(\text{RHS}(m_1) \cap \text{RHS}(m_2) = \emptyset\) and at least one of the four sets of three conditions (i)-(iii) generated by Algorithm **Conditions**. 

\[\square\]
Input: A linear pair $M = (m_1, m_2)$ with a single predicate $R$
Output: Two-MD labeled set $M' = \{m'_1, m'_2\}$ and four conditions for $M'$

1. Subscript all occurrences of $R$ in $m_1$ ($m_2$) with $1$ ($2$)
2. Superscript all occurrences of $R$ to the left (right) of $\approx$ with $1$ ($2$)
3. For each choice of $X, Y \in \{1, 2\}$ generate the following conditions (i),(ii),(iii):
   (i) $Attr(R^X_l) \cap Attr(R^2_l) \cap (RHS(m'_1) \cap LHS(m'_2)) \neq \emptyset$.
   (ii) There are $R^1_l$-equivalent sets that do not contain attributes in $Attr(R^X_l) \cap LHS(m'_1)$.
   (iii) For some $L$-component $L$ of $m'_1$, $Attr(R^1_l) \cap (L \cap LHS(m'_2)) = \emptyset$.

Table 1: Algorithm Conditions

Example 11. Consider the linear pair $M$:

\[
\begin{align*}
m_1 & : R[A] \approx R[B] \land R[C] \approx R[E] \rightarrow R[F] \approx R[G] \\
& \land R[B] \approx R[G], \\
m_2 & : R[G] \approx R[H] \land R[B] \approx R[I] \land R[L] \approx R[I] \rightarrow R[J] \approx R[K].
\end{align*}
\]

Algorithm Conditions produces the following labeling:

\[
\begin{align*}
m'_1 & : R^1_1[A] \approx R^2_1[B] \land R^1_1[C] \approx R^2_1[E] \\
& \iff R^1_2[F] \approx R^2_2[B] \land R^1_2[B] \approx R^2_2[G], \\
m'_2 & : R^2_2[G] \approx R^2_2[H] \land R^2_2[B] \approx R^2_2[I] \land \\
& \iff R^1_2[L] \approx R^2_2[I] \rightarrow R^1_2[J] \approx R^2_2[K].
\end{align*}
\]

With the above labeling, $R^1_l$ ($R^2_l$)-equivalent sets can be defined analogously to $R(S)$-equivalent sets in the two relation case, except that they generally include attributes from two "relations", $R^1_1$ and $R^1_1$ ($R^1_2$ and $R^2_2$), instead of one. For example, in $\{m'_1, m'_2\}$, one $R^1$-ES is $\{R^1_1[F], R^1_1[B], R^1_2[B], R^1_2[L]\}$.

The conditions output by Conditions for the combination $X = 1, Y = 2$ is the following: (i) $Attr(R^1_1) \cap Attr(R^2_1) \cap (RHS(m'_1) \cap LHS(m'_2)) \neq \emptyset$, (ii) There are $R^1_l$-equivalent sets that do not contain attributes in $Attr(R^2_1) \cap LHS(m'_1)$, and (iii) For some $L$-component $L$ of $m_1$, $Attr(R^1_1) \cap Attr(R^1_2) \cap (L \cap LHS(m'_2)) = \emptyset$.

These conditions are satisfied by $M'$. In fact, for (i) this set is $\{R^2_1[G]\}$; for (ii) the $R^1$-ES $\{R^1_1[G]\}$ satisfies the condition; and for (iii) $L = \{R^1_1[C], R^1_1[E]\}$ satisfies the condition. Thus, by Corollary 1, $M$ is hard. \hfill \Box

Example 12. (example 6 cont.) This set $M$ is hard by Corollary 1. In fact, Algorithm Conditions produces the following labeled set $M'$:

\[
\begin{align*}
R^1_1[A] & \approx R^2_1[A] \rightarrow R^1_2[B] \approx R^2_1[B], \\
R^2_2[B] & \approx R^2_2[B] \rightarrow R^2_1[C] \approx R^1_2[C],
\end{align*}
\]

which satisfies the conditions (i)-(iii) for the choice $X = 1, Y = 2$: for (i) this set is $\{R[B]\}$; for (ii) the $R^1$-ES $\{R^1_1[B], R^1_2[B]\}$ satisfies the property; and for (iii) we use $L = \{R^1_1[A], R^1_2[A]\}$.

As mentioned in Section 2.2, for the given $M$ and the query $Q(x, z) : \exists y R(x, y, z)$, RQA is intractable [16]. This query is in UJCQ \& CHAQ. Now, we have just obtained that RQA, for that $M$, is also intractable for all CHAQ queries. \hfill \Box

Example 13. Consider $M$ consisting of

\[
\begin{align*}
m_1 & : R[A] \approx R[A] \rightarrow R[B] \approx R[B], \\
m_2 & : R[A] \approx R[A] \land R[B] \approx R[B] \rightarrow R[C] \approx R[C].
\end{align*}
\]

It does not satisfy the conditions of Theorem 2 (actually, Corollary 1). The sole $L$-component of $m_1$ is $\{R[A]\}$, and all attributes of this set occur in $LHS(m_2)$. Actually, the set is easy, because the non-interacting set

\[
\begin{align*}
R[A] & \approx R[A] \rightarrow R[B] \approx R[B], \\
R[A] & \approx R[A] \rightarrow R[C] \approx R[C],
\end{align*}
\]

is equivalent to it in the sense that, for any instance, the MRIs are the same for either set. This is because applying $m_1$ to the tuples of $R$ and $S$ results in an instance such that all pairs of tuples satisfying the first conjunct to the left of the arrow in $m_2$ satisfy the entire similarity condition. \hfill \Box

Theorem 2 gives a syntactic condition for hardness. It is an important result, because it applies to simple sets of MDs such as that in Example 6 that we expect to be commonly encountered in practice. Moreover, in Section 5, we use Theorem 2 to show that similar sets involving more than two MDs are also hard.

The conditions for hardness in Theorem 2 are not necessary conditions. Actually, the set of MDs in Example 14 below is hard, but does not satisfy the conditions this theorem.

4. A DICHOTOMY RESULT

All syntactic conditions/constructs on attributes above, in particular, the transitive closures on attributes, are "orthogonal" to semantic properties of the similarity relations. When similarity predicates are transitive, every linear pair not satisfying the hardness criteria of Theorem 2 is easy.

Theorem 3. Let $\{m_1, m_2\}$ be a linear pair with $RHS(m_1) \cap RHS(m_2) = \emptyset$. If the similarity operators are transitive, then $\{m_1, m_2\}$ is either easy or hard. More precisely, if the conditions of Theorem 2 hold, $M$ is hard. Otherwise, $M$ is easy.

Theorem 3 does not hold in general when similarity is not transitive (c.f. Proposition 2 below). The possibilities for accidental similarities are reduced by disallowing that two dissimilar values are similar to a same value. Actually, the complexity of the problem is reduced to the point where the resolved answer problem becomes tractable.
Proof (sketch): As discussed in Section 3.1, intractability occurs as a result of the effect of particular choices of update value on subsequent updates. Obviously, if condition (a)(i) ((b)(i)) of Theorem 2 does not hold, then changes to values in \( R(S) \) in the first update cannot affect subsequent updates. If operators are transitive and (ii) or (iii) hold, then the effect is sufficiently restricted that the set of MDs becomes easy.

To illustrate, we will consider updates when condition (a)(iii) does not hold. Let \( m_1 \) and \( m_2 \) be as in Theorem 2. Let \( A \) be the set of sets of tuples in \( R \) whose values are merged as a result of applying \( m_1 \). Let \( B \) be the set of sets of tuples in \( R \) whose values are merged as a result of applying \( m_2 \) in the second update. We claim that, for any \( B_1 \in B \), there is at most one \( A_1 \in A \) such that \( A_1 \cap B_1 \neq \emptyset \). This implies that no accidental similarity between updated values can affect subsequent updates as in Example 6, from which it follows that the set of MDs is easy.

Let \( L \) be an L-component of \( m_1 \). To prove the claim, we first prove that, for any attribute \( E \in L \), if a pair of tuples \( t_1 \) and \( t_2 \) in \( R \) whose values are modified by application of \( m_1 \) satisfies \( t_1[E] \approx t_2[E] \), then \( t_1[E] \approx t_2[E] \) for all \( E \in L \). Suppose \( t_1[E] \approx t_2[E] \). Since \( t_1 \) and \( t_2 \) are modified by \( m_1 \), there must be tuples \( t_3 \) and \( t_4 \) in \( S \) such that the pair \( t_1, t_4 \) and the pair \( t_2, t_4 \) satisfy the similarity condition of \( m_1 \). Let \( F \) be an attribute of \( S \) such that \( R[E] \approx S[F] \) is a conjunct of \( m_1 \). By \( t_1[E] \approx t_2[E] \) and transitivity of \( \approx \), \( t_3[F] \approx t_4[F] \) holds. More generally, for any pair of attributes \( R[E'] \) and \( S[F'] \) such that \( R[E'] \approx S[F'] \) is a conjunct of \( m_1 \), \( t_1[E'] \approx t_2[E'] \) if \( t_3[F'] \approx t_4[F'] \). It then follows from the definition of L-component that \( t_1[E] \approx t_2[E] \) for all \( E \in L \).

Suppose that the values of a pair of tuples \( t_1 \) and \( t_2 \) in \( R \) are merged by application of \( m_2 \) in the second update. By an argument similar to the preceding, this means that \( t_1[A] \approx t_2[A] \) for any attribute \( A \) of \( R \) to the left of the arrow in \( m_2 \). Since (a)(iii) does not hold, by the result of the preceding paragraph, \( t_1 \) and \( t_2 \) satisfy the similarity condition of \( m_1 \). This proves the claim.

Example 14. The linear pair \( M \) consisting of

\[
\begin{align*}
m_1 &: R[A] \approx S[B] \land R[I] \approx S[J] \rightarrow R[E] \approx S[F], \\
m_2 &: R[E] \approx S[F] \land R[A] \approx S[J] \land R[I] \approx S[B] \rightarrow R[G] \approx S[H].
\end{align*}
\]

does not satisfy the conditions of Theorem 2, because \( m_1 \) has two L-components, \( \{ R[A], S[B] \} \) and \( \{ R[I], S[J] \} \). Since LHS(\( m_2 \)) includes one attribute of \( R \) and \( S \) from each of these L-components, conditions (a)(iii) and (b)(iii) are not satisfied. Then, by Theorem 3, \( M \) is easy when \( \approx \) is transitive.

In Example 13, we showed that a pair of MDs is easy for arbitrary \( \approx \) by exhibiting an equivalent non-interacting set. This method cannot be applied in Example 14, because the similarity condition of \( m_1 \) is not included in that of \( m_2 \).

Actually, the set of MDs in Example 14 can be hard for non-transitive similarity relations, as the following proposition shows.

Proposition 2. There exist (non-transitive) similarity operators \( \approx \) for which the set of MDs in Example 14 is hard. \( \square \)

5. HARDNESS OF ACYCLIC SETS OF MDs

We consider now acyclic sets of MDs of arbitrary finite size, concentrating on a class of them that is common in practice.

Definition 7. A set \( M \) of MDs is pair-preserving if for every attribute appearing in \( M \), say \( R[A] \), there is exactly one attribute appearing in \( M \), say \( S[B] \), such that \( R[A] \approx S[B] \) or \( R[A] \approx S[B] \) (or the other way around) occurs in \( M \). \( \square \)

It is easy to verify that pair-preserving, acyclic sets of MDs are strongly acyclic.

Example 15. \( M \) in Example 13 is pair-preserving. However, the set of MDs in Example 14 is not pair-preserving, because \( S[B] \) is paired with both \( R[A] \) and \( R[C] \) in \( m_1 \). It is also possible for cyclic sets of MDs to be pair-preserving. For example, the set

\[
\begin{align*}
R[A] & \approx R[A] \rightarrow R[B] \approx R[B], \\
R[B] & \approx R[B] \rightarrow R[A] \approx R[A],
\end{align*}
\]

is pair-preserving. \( \square \)

Pair-preservation typically holds in entity resolution, because the values of pairs of attributes are normally compared only if they hold the same kind of information (e.g. both addresses or both names).

Now, recall from the previous section that syntactic conditions on linear pairs \( (m_1, m_2) \) imply hardness. One of the requirements is the absence of certain attributes in LHS(\( m_1 \)) from LHS(\( m_2 \)) (c.f. conditions (a)(iii) or (b)(iii)). The condition of non-inclusiveness wrt subsets of \( M \) is a syntactic condition on acyclic, pair-preserving sets \( M \) of MDs that generalizes the conditions that ensure hardness for linear pairs.

Definition 8. Let \( M \) be acyclic and pair-preserving, \( B \) an attribute in \( M \), and \( M' \subseteq M \). \( B \) is non-inclusive wrt. \( M' \) if, for every \( m \in M \setminus M' \) with \( B \in \text{RHS}(m) \), there is an attribute \( C \) such that: (a) \( C \in \text{LHS}(m) \), (b) \( C \not\in \bigcup_{m'\in M'} \text{LHS}(m') \), and (c) \( C \) is non-inclusive wrt. \( M' \). \( \square \)

This is a recursive definition of non-inclusiveness. The base case occurs when \( C \) is not in \( \text{RHS}(m) \) for any \( m \), and so must be inclusive (i.e. not non-inclusive). Because \( C \in \text{LHS}(m) \) in the definition, for any \( m_1 \) such that \( C \in \text{RHS}(m_1) \), there is an edge from \( m_1 \) to \( m \). Therefore, we are traversing an edge backwards with each recursive step, and the recursion terminates by the acyclicity assumption.
Example 16. In the set acyclic and pair-preserving set of
MDs containing

\[
\begin{align*}
\{m_1\} & : R[I] \equiv S[J] \rightarrow R[A] \equiv S[E], \\
\{m_2\} & : R[A] \equiv S[E] \rightarrow R[C] \equiv S[B], \\
\{m_3\} & : R[G] \equiv S[H] \rightarrow R[I] \equiv S[J], \\
\end{align*}
\]

\(R[A]\) is non-inclusive wrt. \(\{m_2\}\) because \(R[A] \in \text{RHS}(m_1)\) and there is an attribute, \(R[I]\), in \(LHS(m_1)\) that satisfies conditions (a), (b), and (c) of Definition 8. Conditions (a) and (b) are obviously satisfied. Condition (c) is satisfied, because \(R[G]\) is non-inclusive wrt. \(\{m_1\}\). This is trivially true, since \(R[G] \notin \text{RHS}(m_1) \cup \text{RHS}(m_3)\).

Non-inclusiveness is a generalization of conditions (a) (iii) and (b) (iii) in Theorem 2 to a set of arbitrarily many MDs. It expresses a condition of inclusion of attributes in the left-hand side of one MD in the left-hand side of another. In particular, suppose \(M = \{m_1, m_2\}\) is a pair-preserving linear pair, and take \(M' = \{m_2\}\). It is easy to verify that the requirement that there is an attribute in \(\text{RHS}(m_1)\) that is non-inclusive wrt. \(M'\) is equivalent to conditions (a)(iii) and (b)(iii) of Theorem 2.

Theorem 4 tells us that a set of MDs that is non-inclusive in this sense is hard.

Theorem 4. Let \(M\) be acyclic and pair-preserving. Assume there is \(\{m_1, m_2\} \subseteq M\), and attributes \(C \in \text{RHS}(m_2), B \in \text{RHS}(m_1) \cap \text{LHS}(m_2)\) with: (a) \(C\) is non-inclusive wrt \(m_1, m_2\), and (b) \(B\) is non-inclusive wrt \(m_2\). Then, \(M\) is hard.

Example 17. (example 16 cont.) The set of MDs is hard. This follows from Theorem 4, with \(m_1, m_2\) in the theorem being the \(m_1, m_2\) in the example. \(C, B\) in the theorem are \(R[C], R[A]\) in the example, resp. Part (b) of the theorem was shown in the first part of this example. Part (a) holds trivially, since \(R[C] \notin \text{RHS}(m_3)\).

Example 18. Consider \(M = \{m_1, m_2, m_3\}\) with

\[
\begin{align*}
\{m_1\} : & \quad R[G] \equiv S[H] \rightarrow R[I] \equiv S[J], \\
\{m_2\} : & \quad R[G] \equiv S[H] \wedge R[I] \equiv S[J] \rightarrow R[A] \equiv S[E], \\
\{m_3\} : & \quad R[G] \equiv S[H] \wedge R[A] \equiv S[E] \rightarrow R[C] \equiv S[B]. \\
\end{align*}
\]

It does not satisfy the condition of Theorem 4. The only candidates for \(m_1\) and \(m_2\) in the theorem are \(m_1, m_2\), respectively, and \(m_2\) and \(m_3\), respectively, because of the requirement that \(\text{RHS}(m_1) \cap \text{LHS}(m_2) \neq \emptyset\). In the first case, \(B\) in the theorem is \(R[I]\) (or \(S[J]\)), which does not satisfy (b) because \(\text{LHS}(m_1) \cap \text{LHS}(m_2) = \emptyset\). In the second case, \(B\) in the theorem is \(R[A]\) (or \(S[E]\)). Because \(R[G]\) and \(S[H]\) are in \(\text{LHS}(m_3)\), \(R[A]\) can only satisfy (b) if \(R[I]\) does. \(R[I]\) does not satisfy (b), since \(\text{LHS}(m_1) \setminus \text{LHS}(m_3) = \emptyset\).

Actually, \(M\) is easy, because it is equivalent to the non-interacting set

\[
\begin{align*}
\{m'_1\} : & \quad R[G] \equiv S[H] \rightarrow R[I] \equiv S[J], \\
\{m'_2\} : & \quad R[G] \equiv S[H] \rightarrow R[A] \equiv S[E], \\
\{m'_3\} : & \quad R[G] \equiv S[H] \rightarrow R[C] \equiv S[B], \\
\end{align*}
\]

which can be shown with the same argument as in Example 13 to \(m_1, m_2\), and then to \(m_2\) and \(m_3\).

Our dichotomy result applies to linear pairs (and transitive similarities). However, tractability can be obtained in some cases of larger sets of MDs for which hardness cannot be obtained via Theorem 4 (because the conditions do not hold). The following is a general result concerning sets such as \(M\) in Example 18.

Theorem 5. Let \(M\) be an acyclic, pair-preserving set of MDs. If for all \(m \in M\), all changeable attributes \(A\) such that \(A \in \text{LHS}(m)\) are inclusive wrt \(\{m\}\), then \(M\) is easy.

Proof: Consider the MD graph \(MDG(M)\) of \(M\). We transform \(M\) to an equivalent set of MDs \(M'\) as follows. For each MD \(m\) such that its corresponding vertex \(v(m)\) in \(MDG(M)\) has an incoming edge from a vertex \(v'(m')\) that has no incoming edges incident on it, we delete from \(LHS(m)\) all attributes in \(RHS(m')\). If it is readily verified that the maximum length of a path in \(MDG(M')\) is one less than the maximum length of a path in \(MDG(M)\), and that \(M'\) satisfies the conditions of the theorem. Therefore, the transformation can be applied repeatedly until an equivalent non-interacting set of MDs is obtained.

Example 19. (example 18 cont.) As expected, the set \(M\) of MDs \(\{m_1, m_2, m_3\}\) satisfies the requirement of Theorem 5.

To show this, the only attributes to be tested for inclusiveness wrt an MD are \(R[A]\) and \(R[I]\). Specifically, it must be determined whether \(R[I]\) is inclusive wrt \(m_2\) and whether \(R[A]\) is inclusive wrt \(m_3\). \(R[I]\) is inclusive wrt \(m_2\), because all attributes in \(\text{LHS}(m_1)\) are in \(\text{LHS}(m_2)\). \(R[A]\) is inclusive wrt \(m_3\), since \(R[G] \in \text{LHS}(m_3)\) and \(R[I]\) is inclusive wrt \(m_3\).

Example 20. (example 17 cont.) The set \(\{m_1, m_2, m_3\}\) in Example 16 was shown to be hard in Example 17.

As expected, it does not satisfy the requirement of Theorem 5. This is because \(R[A]\) is changeable, \(R[A] \in \text{LHS}(m_2)\), and \(R[B]\) is non-inclusive wrt \(m_2\) since \(R[I] \in \text{LHS}(m_1)\), \(R[I] \notin \text{LHS}(m_2)\), and \(R[I]\) is non-inclusive wrt \(m_2\).

As expected, the conditions of Theorems 4 and 5 are mutually exclusive. In fact, \(B\) in Theorem 4 is changeable (since \(B \in \text{RHS}(m_1)\)), \(B \in \text{LHS}(m_2)\), and \(B\) is non-inclusive wrt \(m_2\). However, together they do not provide a dichotomy result, as the following example shows.
Example 21. The set
\[m_1 : R[E] \approx R[E] \rightarrow R[B] = R[B]\]
\[m_2 : R[B] \approx R[B] \rightarrow R[C] = R[C]\]
\[m_3 : R[E] \approx R[E] \rightarrow R[C] = R[C]\]
does not satisfy the conditions of Theorems 4 or 5. It does not satisfy the condition of Theorem 5 because \(R[B]\) is changeable and non-inclusive wrt \(m_2\). It does not satisfy condition (a) of Theorem 4, because \(C\) is inclusive wrt \(m_1, m_2\) \((R[E] \in LHS(m_1))\).

Although tractability of this case cannot be determined through the theorems above, it can be shown that the set is easy. The reason is that, for any update sequence that leads to an MR, each set of merged duplicates must be updated to a value in the set (to satisfy minimality of change). It is easily verified that, with this restriction, the second update to the values of \(R[C]\) is subsumed by the first, and therefore this update has no effect on the instance. Thus, sets of duplicates can be computed in the same way as with non-interacting sets.

Notice that the condition of Theorem 2 that there exists an ES that is not bounded does not appear in Theorem 4. This is because, for pair-preserving, acyclic sets of MDs, this condition is always satisfied by any subset of the set that is a linear pair. Indeed, consider such a subset \((m_1, m_2)\). If all ESs are bounded for this pair, then by the pair-preserving requirement, \(LHS(m_2) \subseteq LHS(m_1)\). Since \((m_1, m_2)\) is a linear pair, \(LHS(m_2) \cap RHS(m_1) \neq \emptyset\). This implies \(LHS(m_1) \cap RHS(m_1) \neq \emptyset\), contradicting the acyclicity assumption.

For linear pairs, Theorem 4 becomes Theorem 2. For such pairs, condition (a) of Theorem 4 is always satisfied. If the (acyclic) linear pair is also a pair-preserving, as required by Theorem 4, the conditions of Theorem 2 reduce to conditions (a)(iii) and (b)(iii), which, as noted previously, are equivalent to condition (b) of Theorem 4.

6. DISCUSSION AND CONCLUSIONS

In this paper we have shown that resolved query answering is typically intractable when the MDs have a non-cyclic dependence on each other.

The results in this paper shed additional light on the complexity landscape of resolved query answering under MDs, complementing previously known results. Actually, Table 2 summarizes the current state of knowledge of the complexity of the resolved answer problem.

The definition of resolved answer is reminiscent of that of consistent query answer (CQA) in databases that may not satisfy given integrity constraints (ICs) [2, 5]. Much research in CQA has been about developing (polynomial-time) query rewriting methodologies. The idea is to rewrite a query, say conjunctive, into a new query such that the new query on the inconsistent database returns as usual answers the consistent answers to the original query.

In all the cases identified in the literature on CQA (see [6, 26] for recent surveys) depending on the class of conjunctive query and ICs involved, the rewritings that produce polynomial time CQA have been first-order. For MDs, the exhibited rewritings that can be evaluated in polynomial time are in Datalog [17].

Resolved query answering under MDs brings many new challenges in comparison to CQA, and results for the latter cannot be applied (at least not in an obvious manner): (a) MDs contain the usually non-transitive similarity relations. (b) Enforcing consistency of updates requires computing the transitive closure of such relations. (c) The minimality of value changes that is not always used in CQA or considered for consistent rewritings. Actually, tuple-based repairs are usually considered in CQA [6]. (d) The semantics of resolved query answering for MD-based entity resolution is given, in the end, in terms of a chase procedure. However, the semantics of CQA is model-theoretic, given in terms repairs that are not operationally defined, but arise from set-theoretic conditions.

In this paper we have presented the first dichotomy result for the complexity of resolved query answering. The cases for this dichotomy depend on the set of MDs, for a fixed class of queries. In CQA with functional dependencies, dichotomy results have been obtained for limited classes of conjunctive queries [20, 22, 25, 21]. However, in CQA the cases depend mainly on the queries, as opposed to the FDs.

Some open problems that are subject to ongoing research are about: (a) Obtaining tighter upper-bounds on the complexity of resolved query answering. (b) Extending the class of CHAQ queries, considering additional projections, and also boolean queries. (c) Since a condition for easiness was presented for linear pairs with transitive similarity, deriving similar results for other commonly used similarity relations, e.g. edit distance. (d) Deriving a dichotomy result for acyclic, pair-preserving sets analogous to the one for linear pairs. (e) Since, functional dependencies (and other equality generating dependencies) can be expressed as MDs, with equality as a transitive symmetry relation, applying the dichotomy result in Theorem 3 to CQA under FDs (EGDs) under a value-based repair semantics [6].

The results in this paper depend on the chase-based semantics for clean instances that was introduced in Section 2.1. Alternative semantics for clean instances in relation to the chase sequence in (2) can be investigated [19]. A couple of them are essentially as follows:

(a) Apply a chase that, instead of applying all the MDs,
In case (b) above, the same rewriting techniques of [17] apply, but now also to some sets of MDs with non-cyclic dependencies.

Still in case (b) and acyclic pairs of MDs, we may obtain a different behavior wrt the semantics used in this work. For example, the resolved query answer problem for $M$ consisting of

\begin{align*}
  m_1 & : R[A] \approx R[A] \rightarrow R[B] \doteq R[B], \\
  m_2 & : R[B] \approx R[B] \rightarrow R[C] \doteq R[C],
\end{align*}

was established as hard in Example 12. However, under the semantics in (b), it becomes tractable for every UJCCQ [19]. The reason is that, while accidental similarities can arise among values of $R[B]$ in the update process, these similarities cannot affect subsequent updates to values of $R[C]$ (c.f. Example 6). If a pair of tuples must have their $R[C]$ attribute values merged in the second update as a result of an accidental similarity between their $R[B]$ values, these values would have to be merged anyway, to preserve similarities generated in the $R[C]$ column by the first update.

In a different direction, even with the semantics used in this work (as in Section 2.1), we could consider an alternative definition of resolved answer to the one given in (3), namely those that are true in all, not necessarily minimal, resolved instances, i.e. in the instances in $Res(D, M)$ (as opposed to $MinRes(D, M)$), obtaining a subset of the original resolved answers. For some sets of MDs, like the one in (6) above, the different possible sets of merged positions in resolved instances (not directly the resolved instances though) can be specified in (extensions of) Datalog.7 These rules can be combined with a query to produce a new query that retrieves the resolved answers under this alternative query answer semantics [19].

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7This does not extend to minimally resolved instances since the sets of merged positions may not coincide with those for general resolved instances.

| Kind of MD Set | Data Complexity |
|----------------|-----------------|
| linear pair    | does not satisfy condition of Thm. 2 | transitive $\approx$ | easy (Thm. 3) |
|                | satisfies condition of Thm. 2 | non-transitive $\approx$ | hard for some $\approx$ ** |
| acyclic, pair-preserving | does not satisfy condition (b) of Thm. 4 | easy |
| HSC (cyclic)   | satisfies condition (a) of Thm. 4 | does not satisfy | can be easy *** |
|                | does not satisfy condition (b) of Thm. 4 | can be easy *** |

Table 2: Complexity of RQA for Sets of MDs (CHAQ queries)
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APPENDIX

A. PROOFS OF RESULTS

For the proofs below, we need some auxiliary definitions and results.

Definition 9. Let m be an MD. Consider the binary relation that relates pairs of tuples that satisfy the similarity condition of m. We denote the transitive closure of this relation by Tm.

The relation Tm is an equivalence relation, since reflexivity and symmetry are satisfied by the relation of which it is the transitive closure.

Lemma 1. Let D be an instance and let m be an MD. An instance D′ satisfies (D, D′) |= m iff for each equivalence class of Tm, for tuples t1 and t2 in the equivalence class, and for attributes A and B in the same R-component of m, it holds that t1′[A] = t2′[B], where t1′ (t2′) is the tuple in D′ with the same identifier as t1 (t2).

Proof: Suppose (D, D′) |= m, and let t1, t2, A, and B be as in the statement of the theorem. The tuples t1′[A] and t2′[B] are equivalent under the equivalence relation obtained by taking the transitive closure of the equality relation. But the equality relation is its own transitive closure. Therefore, t1′[A] = t2′[B]. The converse is trivial.

Definition 10. Let S be a set and let S1, S2,...,Sn be subsets of S whose union is S. A cover subset is a subset Si, 1 ≤ i ≤ n, that is in a smallest subset of {S1, S2,...,Sn} whose union is S. The problem Cover Subset (CS) is the problem of deciding, given a set S, a set of subsets {S1, S2,...,Sn} of S, and an subset Si, 1 ≤ i ≤ n, whether or not Si is a cover subset.

Lemma 2. CS and its complement are NP-hard.

Proof: The proof is by Turing reduction from the minimum set cover problem, which is NP-complete. Let O be an oracle for CS. Given an instance of minimum set cover consisting of set S, subsets S1, S2,...,Sn of S, and integer k, the following algorithm determines whether or not there exists a cover of S of size k or less. The algorithm queries O on (S, {S1,...Sn}, Si) until a subset Si is found for which O answers yes. The algorithm then invokes itself recursively on the instance consisting of set S\Si, subsets {S1,...Si−1, Si+1,...Sn}, and integer k − 1. If the input set in a recursive call is empty, the algorithm halts and returns yes, and if the input integer is zero but the set is nonempty, the algorithm halts and returns no. It can be shown using induction on k that this algorithm returns the correct answer. This shows that CS is NP-hard. The complement of CS is hard by a similar proof, with the oracle for CS replaced by an oracle for the complement of CS.

Proof of Theorem 2: For simplicity of the presentation, we make the assumption that, for relations R and S, the domain of all attributes that occur in m1 and m2 is the same. If this assumption does not hold, the general form of the instance produced by the reduction would be the same, but it would have different sets of values for attributes with different domains. All pairs of distinct values in an instance are dissimilar. Unless otherwise noted, when we refer to the equivalence classes of Tm1 or Tm2, we mean the non-singleton equivalence classes of these relations.

Wlog, we will assume that part (a) of Theorem 2 does not hold. A symmetric argument proves the theorem for the case in which (b) does not hold. Let E and L denote an ES and an L-component that violate part (a) of Theorem 2. We prove the theorem separately for the following three cases: (1) There exists such an E that contains only attributes of m1, (2) there exists such an E that contains both attributes not in m1 and attributes in m1, and (3) (1) and (2) don’t hold (so there exists such an E that contains only attributes not in m1). Case (1) is divided into two subcases: (1)(a) Only one R-component of m1 contains attributes of E and (1)(b) more than one R-component contains attributes of E.

In addition to the constants that are introduced for each case, we introduce a constant cd from each attribute domain d. R (S) contains a tuple that takes the value cd on each attribute of R (S), where d is the domain of the attribute. For all other tuples besides this one, the values of attributes not in m1 or m2 are arbitrary for the instance produced by the reduction.

For any relation W other than R and S, the tuples that are contained in W are specified in terms of those contained in R and S as follows. Let X be the set of attributes of W whose domain is the same as that of the attributes in m1 and m2, and let Y be the set of all other attributes of W. W contains the set of all tuples such that, for each attribute in Y, the attribute takes the value cd, where d is the domain of the attribute, and for each attribute in X, the attribute takes a value of an attribute in m1 or m2.

Case (1)(a): We reduce an instance of the compliment of CS (c.f. Definition 10) to this case, which is NP-hard by lemma 2. Let F be an instance of CS with set of elements U = {e1, e2,...en} and set of subsets V = {f1, f2,...fm}. Wlog, we assume in all cases that each element is contained in at least two sets. With each subset in V we associate a value in the set K = {k1, k2,...km}. With each element in U we associate a value in the set P = {v1, v2,...vp}. We also define a set of values J = {vij | 1 ≤ i ≤ n, 1 ≤ j ≤ p}, where p is one greater than the number of attributes in some R-component Z of m2. The instance will also contain a value b.

Relation R (S) contains a set SijR (SijS) of tuples for each value vij in J. Specifically, there is a tuple in SijR and SijS for each set to which ej belongs. On attributes in L, all tuples in SijR and SijS take the value vij. There is a tuple in SijR and a tuple in SijS for each value in K corresponding to a set to
which $e_i$ belongs that has that value as the value of all attributes in the R-component of $m_1$ that contains an attribute in $E$. On all other attributes, all tuples in all $S_{i,j}^R$ and $S_{i,j}^S$ take the value $b$.

Relation $S$ also contains a set $G_1$ of $m$ other tuples. For each value in $K$, there is a tuple in $G_1$ that takes this value on all attributes $A$ such that there is an attribute $B \in E$ such that $B \approx A$ occurs in $m_2$. This tuple also takes this value on all attributes of $S$ in $Z$. For all other attributes, all tuples in $G_1$ take the value $b$.

Relation $R$ also contains a set $G_2$ of $m$ other tuples. For each value in $K$, there is a tuple in $G_1$ that takes this value on all attributes in $E$ and all attributes of $R$ in $Z$. Tuples in $G_2$ take the value $a$ on all attributes in $L$. For all other attributes, all tuples in $G_1$ take the value $b$.

A resolved instance is obtained in two updates. We first describe a sequence of updates that will lead to an MRI, which we call our candidate update process. It is easy to verify that the equivalence classes of $T_{m_2}$ are the sets $S_{i,j}^R \cup S_{i,j}^S$. In the first update, the effect of applying $m_1$ is to update to a common value all modifiable positions of attributes in $RHS(m_1)$ for each equivalence class (c.f. Lemma 1). For some minimum cover set $C$, we choose as the update value for $S_{i,j}^R \cup S_{i,j}^S$ for all $j$ a value $k$ in $K$ that is associated with a set in $C$ containing $e_i$.

Before the first update, there is one equivalence class of $T_{m_2}$ for each value in $K$. Let $E_k$ be the equivalence class for the value $k \in K$. Each $k$ contains all the tuples in $R$ with $k$ as the value for the attributes in $E$, as well as a tuple in $G_1$ with $k$ as the value for the attributes in $Z$. We choose $k$ as the update value for the modifiable positions of attributes in $RHS(m_2)$ for $E_k$.

After the first update, applying $m_1$ has no effect, since none of the positions of attributes in $RHS(m_1)$ are modifiable. For each update value that was chosen for the modifiable attributes of $RHS(m_1)$ in the first update there is an equivalence class of $T_{m_2}$ that contains the union over all sets $S_{i,j}^R$ whose tuples’ $RHS(m_1)$ attributes were updated to that value as well as the tuple of $G_1$ containing the given value. Given the choices of update values in the previous update, it is easy to see that the positions of attributes in $RHS(m_2)$ that were modifiable before the first update are modifiable after the first update. Thus, the first update is “overwritten” by the second. We choose $b$ as the update value for the equivalence classes of $T_{m_2}$ in the second update.

We now show that (i) our sequence of updates leads to an MRI, and (ii) in an MRI, none of the positions of attributes of $S$ in $Z$ for tuples in $G_1$ can have their values differ from the original value, unless the value corresponds to a cover set. (i) and (ii) together imply that a value of an attribute of $S$ in $Z$ for a tuple in $G_1$ is changed in some MRI iff the value corresponds to a cover set.

Consider an arbitrary sequence of two updates. When $m_1$ is applied to the instance during the first update, the set of modifiable positions of attributes in $RHS(m_1)$ for each set $S_{i,j}^R \cup S_{i,j}^S$ of tuples is updated to a common value. Our update sequence satisfies the two conditions that (a) in the update resulting from applying $m_1$, the update value chosen for all $S_{i,j}^R \cup S_{i,j}^S$ is the value in $K$ of a subset to which $e_i$ belongs and (b) after the second update, all tuples in all $S_{i,j}^R \cup S_{i,j}^S$ have the value $b$ for all attributes in $RHS(m_2)$. In an arbitrary update sequence, these conditions will generally be satisfied only for some pairs $(i,j)$ of indices. Let $I$ be the set of all pairs that satisfy (a) and (b). It is easy to verify that, in the resulting resolved instance, for all $(i,j)$ not in $I$, the number of changes to positions of tuples in $S_{i,j}^R \cup S_{i,j}^S$ is at least one greater than in our candidate update process.

First, we show that, for an MRI, $I$ must include all pairs $(i,j)$. To prove this, we first show that, for any fixed value $i^*$, either all $(i^*,j)$ are in $I$ or none of them are. Suppose only some of the $(i^*,j)$ are in $I$. Suppose the update sequence is modified so that for all $(i^*,j)$ not in $I$, the tuples in $S_{i,j}^R \cup S_{i,j}^S$ are instead updated the same way as $S_{i,j}^R \cup S_{i,j}^S$, for some $(i^*,j^*) \in I$. Then the number of changes to tuples in $S_{i,j}^R \cup S_{i,j}^S$ for each $(i^*,j) \notin I$ decreases by at least one, while the number of changes to other tuples is unchanged.

Suppose that there exists an MRI $M$ such that there is an $(i^*,j^*) \notin I$. By the preceding paragraph, $(i^*,j^*) \notin I$ for all $j$. Consider a modification of the update sequence used to obtain $M$ that updates the tuples in $S_{i,j}^R \cup S_{i,j}^S$, $1 \leq j \leq p$, according to our candidate update process, while leaving all other updates the same. This new update process will make at least $p$ fewer changes to the tuples in $S_{i,j}^R \cup S_{i,j}^S$, $1 \leq j \leq p$ (at least one fewer for the tuples in each $S_{i,j}^R \cup S_{i,j}^S$). Furthermore, it can make at most $p-1$ additional changes to positions in other tuples. This is because the only other tuples that are updated as a result of having their values merged with those of the tuples in $S_{i,j}^R \cup S_{i,j}^S$ are the tuples in $G_1$ and $G_2$ containing the value in $K$ that was used to update the tuples in $S_{i,j}^R \cup S_{i,j}^S$ according to $m_1$. The values modified in these tuples are those of attributes in the R-component $Z$, of which there are $p-1$. Thus, the number of changes decreases as a result of changing the update process, contradicting the statement that $M$ is an MRI.

For an MRI $M$, let $H$ be the set of update values used when applying $m_1$ to the tuples in $S_{i,j}^R \cup S_{i,j}^S$. For each value in $H$, the positions of all attributes in $Z$ are modified in $M$ for the tuple in $G_1$ that takes this value on all attributes in $Z$. Since these are the only positions that are modified by applying $m_2$, there are no more than $|H| \cdot (p-1)$ changes to the value positions of attributes in $RHS(m_2)$ in the second update used to produce $M$. Therefore, $|H|$ must be as small as possible, implying that $H$ corresponds to a minimum set cover. Furthermore, no other positions can be updated besides the ones updated during the second update. This proves (i) and (ii).

Let $Q$ be a query as in the statement of the theorem. Let $k$ be the value in $K$ corresponding to the candidate cover set in the CS instance. We construct an assignment to the free variables of $Q$ as follows. For some join-restricted free oc-
instance from case (1)(a), and define all sets of values as be-
take the same values on all such attributes as on
the tuples in
the preceding paragraph, at least one more change is made to
$S$

component of
set. The converse is obvious.

Case (1)(b): This case uses the same set of values as
(1)(a). The instance is the same, except that the tuples in
$S'_{ij}$ and $S''_{ij}$ that took a certain value on attributes in the R-
component of $m_1$ that contains an attribute in $E$ now take
that value on all such R-components. The update sequence
that we specify for obtaining an MRI is also the same, but
we add the requirement that for a given equivalence class of
$T_{m_1}$, the update value must be the same for all R-components
of $m_1$.

The difference between this case and (1)(a) is that differ-
ent update values can be chosen for different R-components
of $m_1$ for the same equivalence class of $T_{m_1}$. It is easy to
verify that if different values are chosen, all tuples in the
equivalence class would be in singleton equivalence classes
of $T_{m_1}$ after the first update. Therefore, any changes made to
positions of attributes in $RHS(m_2)$ for tuples in the equiva-
ience class in the first update cannot be undone in the second
update.

Let $X$ denote the set of all $(i, j)$ such that there are two
R-components of $m_1$ that are updated to different values for
tuples in $S'_{ij} \cup S''_{ij}$. For $(i, j) \notin X$, we use the same criteria as
in part (1)(a) to classify $(i, j)$ as being in $I$ or not. For some
$(i, j) \in X$, consider the update values chosen for tuples in
$S'_{ij} \cup S''_{ij}$ when $m_2$ is applied during the first update. If any
of the update values are not $b$, then, by the last sentence of
the preceding paragraph, at least one more change is made to
the tuples in $S'_{ij} \cup S''_{ij}$ than in our candidate update process.
In this case, we say $(i, j)$ is not in $I$. Otherwise, $(i, j)$ is in
$I$. The remainder of the proof is the same as in part (1)(a),
except that $H$ also contains, for each $(i, j) \in X \cap I$, all the
values from $K$ in tuples in $S'_{ij}$.

Case (2): For simplicity of the presentation, we will as-
sume that there exists only one attribute $A$ in $E$ not in $m_1$.
If there is more than one such attribute, then all tuples will
take the same values on all such attributes as on $A$ in the in-
stance produced by the reduction. Let $F$ be the min set cover
instance from case (1)(a), and define all sets of values as be-
fore. We also have value $a$. In addition, we define a set $Y$ of
$2m^2np^2$ values, which we denote by $y_{ij}$, $1 \leq i \leq 2mnp^2$,
$1 \leq j \leq m$. We also define a set $X$ of $2mnp^2$ values.

Relations $R$ and $S$ contain sets $S'_{ij}$ and $S''_{ij}$ for each $e_i$, $1 \leq i \leq n$, as before. However, $S'_{ij}$ and $S''_{ij}$ now contain
two tuples for each set to which $e_i$ belongs. On attributes
in $L$, tuples in each $S'_{ij}$ and $S''_{ij}$ take the same value as in
case (1)(a). Let $K' = \{k'_1, k'_2, ..., k'_{|S'_{ij}|/2}\}$ and $K'' = \{k''_1, k''_2, ..., k''_{|S''_{ij}|/2}\}$ be lists of all the values in $K$
corresponding to sets to which $e_i$ belongs such that $k'_i = k''_i \mod \lfloor |S'_{ij}|/2 \rfloor + 1$.

For each value $k'_i \in K'$, there are two tuples in $S'_{ij}$ and
two in $S''_{ij}$ that take this value on all attributes in all R-
components of $m_1$ containing an attribute of $E$. On the attri-
bute $A$, one of the two tuples in $S'_{ij}$ takes the value $k'_i$ and
the other takes the value $k''_i$. (This ensures that the tuples that
take the value $k''_i$ will be in singleton equivalence classes of
$T_{m_2}$ before the first update.) On all other attributes, all tuples
in all $S'_{ij}$ and $S''_{ij}$ take the value $b$.

Relation $S$ ($R$) also contains a set $G_1$ ($G_2$) of $m$ tuples, which is the same as the set $G_1$ ($G_2$) from case (1)(a).

Relation $R$ also contains a set $G_3$ of $2m^2np^2 \cdot (2mnp^2 + 1)$ other tuples. For each value $y_{ij} \in Y$, there is a set $Y_{ij}$ of
$2mnp^2 + 1$ tuples that have this value as the value of all attributes of $R$ in $L$. For each value in the set $X$, there is a tuple in $Y_{ij}$ that takes this value on attribute $A$. On
all other attributes in $E$, these tuples take the value $a$. On
attributes in $Z$, they take the value $k_j$ from the set $K$. On
all other attributes they take the value $b$. There is also a tuple in
$Y_{ij}$ that takes the value $k_j$ on all attributes in $E$ and on all attributes in $Z$. On all other attributes, this tuple takes the value $b$.

Relation $S$ also contains a set $G_4$ of $2m^2np^2$ tuples. For
each value in $Y$, there is a tuple in $G_4$ that takes this value on
all attributes of $S$ in $L$. On attributes in R-components of
$m_1$ that contain an attribute in $E$, all tuples in $G_4$ take the
value $a$. On all other attributes, they take the value $b$.

We now describe an update sequence that leads to the value $b$,
which we call our candidate update process. In this sequence,
there are equivalence classes of $T_{m_2}$ that are the sets $S'_{ij} \cup S''_{ij}$, as in cases (1)(a) and (1)(b), and we choose the update
values for these equivalence classes in the same way as in
those cases. There are also equivalence classes of $T_{m_2}$ that
involve tuples in $G_3$ and $G_4$. Each of these consists of one of
the $Y_{ij}$ sets and the tuple in $G_4$ containing $y_{ij}$. We use $a$
as the update value for these equivalence classes. This results
in all tuples in $G_3$ being in singleton equivalence classes of
$T_{m_2}$ after the first update.

Before the first update, there is one equivalence class of
$T_{m_2}$ for each value in $K$. Let $E_{k_j}$ be the equivalence class
for the value $k_j \in K$. $E_{k_j}$ contains all the tuples in $S'_{ij}$ and
$G_3$ with $k_j$ as the value for all of the attributes in $E$
(including $A$), as well as the tuple in $G_1$ with $k_j$ as the value
for attributes in $Z$. We choose $k_j$ as the update value for the
modifiable positions of attributes in $RHS(m_2)$ for $E_{k_j}$.
After the first update, the equivalence classes of $T_{m_2}$ include the two tuples in $S_{ij}^R$ that have the value to which their RHS($m_1$) attributes were updated in the first update as the value of $A$. We choose $b$ as the update value for these equivalence classes. Note that the fact that one of the two tuples was in a singleton equivalence class of $T_{m_2}$ before the first update guarantees that all their positions that were modified by application of $m_2$ during the first update are modifiable during the second update.

We claim that for an MRI, (i) the update values chosen in the first update for equivalence classes of $T_{m_2}$ must be the same as in our candidate update process, and (ii) the update values chosen for the equivalence classes of $T_{m_2}$ containing tuples of $G_3$ must be the same as in our candidate update process. Statement (ii) follows from the fact that, if any value other than $a$ is chosen as the update value for such an equivalence class, it would result in at least $2mn^2 + 1$ more changes to the positions of attributes in RHS($m_2$) than in our update sequence. Since our candidate update process makes no more than $2mn^2$ changes to the positions of attributes in RHS($m_2$), no such alternative update sequence could produce an MRI. Similarly, (i) follows from the fact that, if for any $E_{kj}$, any value other than $k_j$ is chosen as the update value, there would be at least $2mn^2$ changes to positions of attributes in RHS($m_2$) for tuples in $G_3 \cap E_{kj}$ during the first update. If that is the case, then some of these positions must be restored to their original values in the second update. However, this would require some of the tuples in $G_3 \cap E_{kj}$ to be in non-singleton equivalence classes of $T_{m_2}$ after the first update, which by (ii) is not possible.

Consider an arbitrary sequence of two updates. When $m_1$ is applied to the instance during the first update, the set of modifiable positions of attributes in RHS($m_1$) for each set $S_{ij}^R \cup S_{ij}^S$ of tuples is updated to a common value. Our candidate update process satisfies the three conditions that (a) in the update resulting from applying $m_1$, the update value chosen for each set $S_{ij}^R \cup S_{ij}^S$ is the same for all R-components, (b) this update value is the value in $K$ of a subset to which $e_i$ belongs and (c) in the second update, the update value chosen for modifiable positions in tuples in each set $S_{ij}^R$ is $b$. In an arbitrary update sequence, these conditions will generally be satisfied only for some pairs $(i, j)$ of indices. Clearly, for pairs of indices not satisfying (b), there will be at least one more change to the values of tuples in $S_{ij}^R$ than in our candidate update process. Given (i) above, this is also true for pairs of indices not satisfying (c). For pairs of indices not satisfying (a), all tuples in $S_{ij}^R$ are in singleton equivalence classes of $T_{m_2}$, and therefore (c) cannot be satisfied. Therefore, failing to satisfy any of (a), (b), and (c) results in at least one more change to the values of tuples in $S_{ij}^R$ than in our candidate update process. Let $I$ be the set of all pairs that satisfy (a), (b), and (c). We now use exactly the same argument involving the set $I$ as in part (1)(a) to prove the result.

Case (3): Let $F$ be the CS instance from case (1)(a), and define sets of values $K$ and $P$ as before. Let $E'$ be an ES containing attributes of $m_1$. Since the MDs are interacting, there must be at least one such ES, and by assumption, it must contain an attribute of RHS($m_1$). Let $C_1$ denote some R-component of $m_1$ that contains an attribute of $E'$, and let $p$ denote the number of attributes in $C_1$. Let $C_2$ denote some R-component of $m_2$. Let $q_R$ and $q_S$ be the number of attributes of $R$ and $S$ in $C_2$, respectively. Let $d_i$ be the number of elements in the set $f_i$. We define a set $W_j$ of values of size $4q_Rp^2$ for each $j$ such that $e_j \in f_i$. We also define sets $Y_{ij}$ and $Z_{ij}$ of values each that contains value $e_j$ in $4q_Rp^2$ values each, respectively, for all pairs of indices $i, j$ such that $e_j \in f_i$. We also define set $X$ containing $nq_S$ values, and values $a$ and $b$.

Relation $R(S)$ contains a set $S_{ij}^R(S_{ij}^S)$ for each set $f_i$, $1 \leq i \leq m$, in $V$. For each element $e_j \in f_i$, $S_{ij}^R(S_{ij}^S)$ contains a set $S_{ij}^R(S_{ij}^S)$ of $4q_Rp^2$ tuples each. On all attributes of $L$, all tuples in $S_{ij}^R$ and $S_{ij}^S$ take the value $k_i$ in $K$ corresponding to $f_i$. For each $S_{ij}^R$ and $S_{ij}^S$, there is a set of $4q_Rp^2$ tuples $S_{ij}^R$ in $S_{ij}^R$ and a set of $4q_Sp^2$ tuples $S_{ij}^S$ in $S_{ij}^S$. For each $i$, for all $j$ such that $e_j \in f_i$ except one, each value in $W_j$ occurs once as the value of an attribute of either $R$ or $S$ in $C_1$ for a tuple in $S_{ij}^R$ (or $S_{ij}^S$). For the remaining $j$, all but two of the values in $W_j$ occur once as the value of an attribute of either $R$ or $S$ in $C_1$ for a tuple in $S_{ij}^R$ (or $S_{ij}^S$). This leaves the values of two positions of tuples in $S_{ij}^R$ and $S_{ij}^S$ for attributes in $C_1$ undefined, and two values in the set $W_j$ unassigned. These positions take the same value, which is one of the unassigned values from $W_j$. We call this value $s_i$. All other tuples in $S_{ij}^R$ and $S_{ij}^S$ take the value $a$ on all attributes in $C_1$.

For each value in $Y_{ij}$, there are $4q_S$ tuples in $S_{ij}^R$ that take the value on all attributes in $C_2$. For each value in $Z_{ij}$, there is a tuple in $S_{ij}^R$ not in $S_{ij}^R$ that takes the value on all attributes in $C_2$. On all attributes of $E$, each tuple in $S_{ij}^R$, $1 \leq i \leq m$, takes the value $v_j$ in $P$ that is associated with $e_j$, and all other tuples in $S_{ij}^R$ take the value $b$. On all other attributes, all tuples in $S_{ij}^R$ and $S_{ij}^S$ take the value $a$.

Relation $S$ also contains a set of $n$ tuples $G_1$. For each value in $X$, there is a tuple in $G_1$ that takes the value on an attribute of $C_2$. For each value in $P$, there is a tuple in $G_1$ that takes this value on all attributes $B$ such that there is an attribute $A$ of $R$ in $E$ such that $A \approx B$ is a conjunct of $m_2$. On all other attributes, tuples in $G_1$ take the value $a$.

A resolved instance is obtained in two updates. Before the first update, the equivalence classes of $T_{m_2}$ are all singletons. The equivalence classes of $T_{m_2}$ are the sets $S_{ij}^R \cup S_{ij}^S$, $1 \leq i \leq n$. The effect of applying $m_1$ is to change all positions of all attributes in $C_1$ for tuples in $S_{ij}^R \cup S_{ij}^S$ to a common value. It is easy to verify that if the update value is not $a$, then all tuples in $S_{ij}^R$ will be in singleton equivalence classes of $T_{m_2}$ after the update. Thus, the equivalence classes of $T_{m_2}$ after the update are $\bigcup_{i \leq j} S_{ij}^R \cup x_j$, $1 \leq j \leq n$, where $I \equiv \{i \mid a$ was chosen as the update value for $S_{ij}^R \cup S_{ij}^S\}$ and $x_j$ is the tuple in $G_1$ containing the value $v_j$. If the update
value \( a \) is chosen for set \( S^R_i \cup S^S_i \) for some \( i \), we say that \( S^R_i \) is \textit{unblocked}. Otherwise, it is \textit{blocked}.

For a given \( i \), we will consider the number of changes resulting from different choices of update values for tuples in \( S^R_i \cup S^S_i \). These changes include all changes that are affected by the choice of update value for tuples in \( S^R_i \cup S^S_i \).

Consider a blocked \( S^R_i \). In the first update, the minimum number of changes to positions of attributes in \( RHS(m_1) \) for tuples in \( S^R_i \cup S^S_i \) is \( 4qspd_i(p + qR) - 2 \), where \( d_i \) is the number of elements in \( f_i \). All tuples in \( S^R_i \) are in singleton equivalence classes of \( T_{m_2} \) before and after the first update.

Therefore, the number of changes resulting from this choice of update value is \( 4qspd_i(p + qR) - 2 \).

For an unblocked \( S^R_i \), the minimum number of changes to values for attributes in \( RHS(m_1) \) for tuples in \( S^R_i \cup S^S_i \) is \( 4qsp^2d_i \). A set \( S^R_{ij} \) is \textit{good} if, in the second update, all positions in the set of positions of attributes in \( C_2 \) for tuples in \( S^R_{ij} \) are modified to the value of a position in the set. The set \( S^R_i \) is \textit{good} if it contains a good \( S^R_{ij} \). Sets \( S^R_{ij} \) and \( S^R_i \) that are not good are \textit{bad}. The total number of changes to positions of attributes in \( RHS(m_2) \) for tuples in a bad unblocked \( S^R_i \) is \( 4qspqpd_i \), and for a good unblocked \( S^R_i \) it is \( 4qspqpd_i - 4qspqg_i \), where \( g_i \) is the number of good \( S^R_{ij} \) in \( S^R_i \). Thus, the total number of changes for the bad case is \( 4qspd_i(p + qR) \) and for the good case it is \( 4qspd_i(p + qR) - 4qspqg_i \).

The number of changes to tuples in a bad unblocked \( S^R_i \) is larger than that in tuples in a blocked \( S^R_i \). Since tuples in a blocked \( S^R_i \) are in singleton equivalence classes of \( T_{m_2} \), both before and after the first update, choosing a \( S^R_i \) to be blocked also minimizes the number of changes to tuples not in \( S^R_i \). Therefore, in an MRI, all unblocked \( S^R_i \) are good. Let \( U \) (\( B \)) be the set of \( i \) for which \( S^R_i \) is unblocked (blocked). For an MRI, the total number of changes for all \( S^R_i \) is

\[
\sum_{i \in U} [4pqsd_i(p + qR) - 4qsgqR] + \sum_{i \in B} [4pqsd_i(p + qR) - 2]
\]

plus the number of changes to tuples in \( G_1 \). To compute the latter, we note that values in a tuple in \( G_1 \) can change if the tuple contains \( v_j \), where \( e_j \in f_i \) for some \( i \in G \). In this case, there must be some \( i^* \in U \) such that \( S_{ij} \) is good. Indeed, if this were not the case, then the tuple in \( G_1 \) containing \( v_j \) would always be in a singleton equivalence class of \( T_{m_2} \). Therefore the number of tuples in \( G_1 \) that change is \( \sum_{i \in G} g_i \), and the total number of changes is

\[
\sum_{i \in U} [4pqsd_i(p + qR) - 4qsgqR] + \sum_{i \in B} [4pqsd_i(p + qR) - 2] = 4pqspqR + \sum_{1 \leq i \leq m} d_i (4qsgqR - qS).
\]

The first term in (7) depends only on the database instance and not on the choice of update values. Therefore, the number of changes is minimized by choosing the update values so as to maximize the magnitude of the last two terms.

The sum over \( g_i \) in the second term in (7) is bounded above by \( n \). This can be shown as follows. After the first update, there is one equivalence class for each value of \( j \), containing the set of all \( S^R_{ij} \) such that \( S^R_i \) is unblocked. Furthermore, the sets of values of modifiable positions of attributes in \( RHS(m_2) \) for tuples in a given \( S^R_{ij} \) do not overlap with those of any other \( S^R_{ij} \). Therefore, at most one \( S^R_{ij} \) can be good for any value of \( j \).

If the sum over \( g_i \) equals \( n \), then the set of subsets corresponding to the set of \( i \) for which \( S^R_i \) is good is a set cover. If it is a min set cover, then \( |B| \) is maximized for this value of the sum.

We claim that the magnitude of the last two terms in (7) is maximized by choosing the set of good \( S^R_i \) so that \( \{ e_i \mid S^R_i \text{ is good} \} \) is a min set cover, from which it follows that this choice is required for the resolved instance to be an MRI. Suppose for a contradiction that there is an MRI \( M \) for which the sum over \( g_i \) in (7) is \( n - c \) for some \( 1 \leq c \leq n \). This implies that, for \( M \), there is a set \( J \) of \( c \) values of \( j \) such that there is no \( i \) such that \( S^R_{ij} \) is good. Consequently, for \( j^* \in J \), for any \( i \) such that \( S^R_{ij^*} \) exists, \( S^R_i \) must be blocked. This is because if there were an unblocked \( S^R_i \), then the second update could be changed so that \( S^R_i \) is good, reducing the number of changes.

We modify the update sequence used to obtain \( M \) in the following way. For each \( j \in J \), choose an \( i \) such that \( S^R_{ij} \) exists. For each such \( i \), change the first update so that \( S^R_i \) is unblocked, and change the second update so that \( S^R_i \) is good. This will increase the magnitude of the second term in (7) by \( (4qspqR - qS)c \geq 3c \) and decrease the magnitude of the third term by at most \( 2c \). Therefore, the number of changes decreases as a result of this modification to the update sequence, contradicting the assumption that \( M \) is an MRI.

The value \( s_i \) is the value of an attribute in \( C_1 \) for a tuple in \( R \) or \( S \) in \( RHS(m_1) \) occurs in \( LHS(m_2) \). The other cases are...
similar. For each L-component of $m_1$, there is an attribute of $R$ and an attribute of $S$ from that L-component in $LHS(m_2)$. Let $t_1 \in R$ be a tuple not in a singleton equivalence class of $T_{m_1}$. Suppose there exist two conjuncts in $LHS(m_1)$ of the form $A \approx B$ and $C \approx B$. Then it must hold that there exists $t_2 \in S$ such that $t_1[A] \approx t_2[B]$ and $t_1[C] \approx t_2[B]$ and by transitivity, $t_1[A] \approx t_1[C]$. More generally, it follows from induction that $t_1[A] \approx t_1[E]$ for any pair of attributes $A$ and $E$ of $R$ in the same L-component of $m_1$.

We now prove that for any pair of tuples $t_1, t_2 \in R$ satisfying $T_{m_1}(t_1, t_2)$ such that each of $t_1$ and $t_2$ is in a non-singleton equivalence class of $T_{m_1}$, for any instance $D$ it holds that $T_{m_1}(t_1, t_2)$. By symmetry, the same result holds with $R$ replaced with $S$. Suppose for a contradiction that $T_{m_1}(t_1, t_2)$ but $\neg T_{m_1}(t_1, t_2)$ in $D$. Then it must be true that $t_1[A] \neq t_2[A]$, since, by assumption, there exists a $t_3 \in S$ such that $t_1[A] \approx t_3[B]$, which together with $t_1[A] \approx t_2[A]$ would imply $T_{m_1}(t_1, t_2)$. Therefore, there must be an attribute $A' \in A$ such that $t_1[A'] \neq t_2[A']$, and by the previous paragraph and transitivity, $t_1[A'] \neq t_2[A']$ for all $A'$ in the same L-component of $m_1$ as $A'$. By transitivity of $\approx$, this implies $\neg T_{m_2}(t_1, t_2)$, a contradiction.

A resolved instance is obtained in two updates. Let $T_{m_2}^0$ and $T_{m_2}^1$ denote $T_{m_2}$ before and after the first update, respectively. The first update involves setting the attributes in $RHS(m_1)$ to a common value for each non-singleton equivalence class of $T_{m_1}$. The relation $T_{m_2}^0$ will depend on these common values, because of accidental similarities. However, because of the property proved in the previous paragraph, this dependency is restricted. Specifically, for each equivalence class $E$ of $T_{m_2}^1$, there is at most one non-singleton equivalence class $E_1$ of $T_{m_1}$, such that $E$ contains tuples of $E_1 \cap R$ and at most one non-singleton equivalence class $E_2$ of $T_{m_1}$ such that $E$ contains tuples of $E_1 \cap S$. A given choice of update values for the first update will result in a set of sets of tuples from non-singleton equivalence classes of $T_{m_1}$ (ns tuples) that are equivalent under $T_{m_2}^0$. Let $K$ be the set of all such sets of ESs. Clearly, $|K| \in O(n^2)$, where $n$ is the size of the instance.

Generally, when the instance is updated according to $m_1$, there will be more than one set of choices of update values that will lead to the ns tuples being partitioned according to a given $k \in K$. This is because an equivalence class of $T_{m_2}^1$ will also contain tuples in singleton equivalence classes of $T_{m_1}$ (s tuples), and the set of such tuples contained in the equivalence class will depend on the update values chosen for the modifiable attribute values in the ns tuples in the equivalence class. For a set $E \in k$, let $E'$ denote the union over all sets of update values for $E$ of the equivalence classes of $T_{m_2}^1$ that contain $E$ that result from choosing that set of update values. By transitivity and the result of the second paragraph, these $E'$ cannot overlap for different $E \in k$. Therefore, minimization of the change produced by the two updates can be accomplished by minimizing the change for each $E'$ separately. Specifically, for each equivalence class $E$, consider the possible sets of update values for the attributes in $RHS(m_1)$ for tuples in $E$. Call two such sets of values equivalent if they result in the same equivalence class $E_1$ of $T_{m_2}^1$. Clearly, there are at most $O(n^c)$ such sets of ESs of values, where $c$ is the number of R-components of $m_1$. Let $V$ be a set consisting of one set of values $v$ from each set of sets of equivalent values. For each set of values $v \in V$, the minimum number of changes produced by that choice of value can be determined as follows. The second application of $m_1$ and $m_2$ updates to a common value each element in a set $S_2$ of sets of value positions that can be determined using lemma 1. The update values that result in minimal change are easy to determine. Let $S_1$ denote the corresponding set of sets of value positions for the first update. Since the second update “overwrites” the first, the net effect of the first update is to change to a common value the value positions in each set in $\{S_1 | S_1 = S \setminus \bigcup_{S' \in S_2} S', S \in S_1\}$. It is straightforward to determine the update values that yield minimal change for each of these sets. This yields the minimum number of changes for this choice of $v$. Choosing $v$ for each $E$ so as to minimize the number of changes allows the minimum number of changes for resolved instances in which the ns tuples are partitioned according to $k$ to be determined in $O(n^{c+2})$ time. Repeating this process for all other $k \in K$ allows the determination of the update values that yield an MRI in $O(n^{c+2})$ time. Since the values to which each value in the instance can change in an MRI can be determined in polynomial time, the result follows.

Proof of Theorem 4: For simplicity, we prove the theorem for the special case in which $M$ is defined on a single relation $R$ and both attributes in each conjunct are the same. The same argument can be used for arbitrary sets of pair-preserving MDs by adding the additional restriction that the set $I$ defined below contains only instances for which the set of values taken by pairs of attributes occurring in the same conjunct are the same.

The proof is by reduction from the resolved answer problem for a set of MDs that is hard by Theorem 2. Specifically, we will construct a set $I$ of database instances. We then give a polynomial time reduction from (a) the resolved answer problem for a specific pair of MDs to (b) the current problem, where for both (a) and (b), the input to the problem is restricted to having instances in $I$. We will show that (a) remains intractable when instances are restricted to $I$. Since (b) restricted to $I$ can obviously be reduced to the current problem in polynomial time, this proves the theorem.

We define a set $S_1$ of attributes recursively according to Definition 8. An attribute $A$ is in $S_1$ if (a) $A \in LHS(m)$ for some $m$ such that $C \in RHS(m)$, (b) $A \notin LHS(m_1) \cup LHS(m_2)$ and (c) $A$ is non-inclusive wrt $\{m_1, m_2\}$, or if $A$ satisfies (a), (b), and (c) with $C$ replaced by an attribute in $S_1$. For all attributes $A \in S_1$, all values in the $A$ column for instances in $I$ are dissimilar to each other.

The set $S_2$ of attributes is defined similarly. An attribute
A is in $S_2$ if (a) $A \in LHS(m)$ for some $m$ such that $B \in RHS(m)$, (b) $A \notin LHS(m_2)$ (c) $A$ is non-inclusive wrt \{m_2\}, and (d) $A \notin S_1$, or $A$ satisfies (a), (b), (c), and (d) with $B$ replaced by an attribute in $S_2$. The second requirement for an instance to be in $I$ is that, for any pair of tuples in the instance, the tuples are either equal on all attributes in $S_2$ or dissimilar on all attributes in $S_2$.

For all attributes not in $S_1$ or $S_2$ besides $B$ and $C$, all tuples in instances in $I$ have the same value for the attribute.

Consider the set $M'$ of MDs

\[
\begin{align*}
m'_1 : & \ R[E] \approx R[E] \rightarrow R[B] \equiv R[B] \\
m'_2 : & \ R[B] \approx R[B] \rightarrow R[C] \equiv R[C]
\end{align*}
\]

where $E \in LHS(m_1) \cap S_2$ (there must be such an $E$ by assumptions (a) and (b) of the theorem). By Theorem 2, $M'$ is hard. We claim that (1) $RA_Q,M'$ for a changeable attribute query $Q$ remains intractable when input instances are restricted to $I$, and (2) $RA_Q,M'$ for any $Q$ reduces in polynomial time to $RA_Q,M$ when input instances are restricted to $I$. These two claims imply the theorem.

Claim (1) is true because the reduction in the proof of Theorem 2 can be made to always produce an instance in $I$ by making a specific choice of the values in the instance that were allowed to be arbitrary in that proof. Specifically, since $R[B]$ and $R[C]$ are not in $S_1 \cup S_2$, the values that tuples in instances in $I$ can take on attributes $R[E]$, $R[B]$, and $R[C]$ are unrestricted. Given the values that tuples in an instance in $I$ take on $R[E]$, $R[B]$, and $R[C]$, the values that the tuples can take on attributes not in $m_1$ and $m_2$ are restricted.

However, in the proof of Theorem 2, the values for these attributes in the instance produced by the reduction were (mostly) left unspecified, and it is easily verified that they can always be chosen so that this instance is in $I$.

To prove claim (2), we show that the set of all updates that can be made under $M'$ is the same as that under $M$, for any instance in $I$. Thus, the reduction is simply the identity transformation.

First, we show that, for any MD $m$ other than $m_1$ and $m_2$, applying $m$ has no effect. Such MDs can therefore be ignored when updating the instance. If $RHS(m)$ consists of an attribute not in $S_1 \cup S_2$ and is not $B$ or $C$, then applying $m$ cannot change the values of the attribute, because these values are already the same. If $RHS(m)$ is an attribute of $S_1$, then by definition of these sets, $LHS(m)$ contains an attribute of $S_1$ or $S_2).$ Therefore, any pair of tuples satisfying the similarity condition of $m$ must already have equal values for the attribute in $RHS(m)$, and applying $m$ has no effect. If $C$ is the attribute in $RHS(m)$, then there must be an attribute of $S_1$ or $S_2$. Since all values for this attribute are mutually dissimilar and are never updated, no pair of tuples satisfies the similarity condition of $m$, so applying $m$ has no effect. Lastly, if $B$ is the attribute in $RHS(m)$, we claim that updates resulting from $m$ are subsumed by those resulting from $m_1$. Indeed, by definition of $S_1$, there are no attributes of $S_1$ in $LHS(m_1)$, and by the acyclic property, neither $B$ nor $C$ are in $LHS(m_1)$. Given this and the fact that there is an attribute of $S_2$ in $LHS(m)$ (by definition of $S_2$), it is easy to verify that if a pair of tuples satisfies the similarity condition of $m$, it must satisfy the similarity condition of $m_1$.

We now show that the effect of applying \{m_1, m_2\} to an instance in $I$ is the same as that of applying $M'$ to the instance, thus proving the theorem. All attributes in $LHS(m_1)$ are either in $S_2$ or have the same value for all tuples. Thus, all tuples satisfying the conjunct $R[E] \approx R[E]$ also satisfy all other conjuncts to the left of the arrow in $m_1$. By definition, $LHS(m_2)$ contains no attributes of $S_1$ and $S_2$, and by the acyclic property, it does not contain $C$. Therefore, all attributes besides $B$ in $LHS(m_2)$ have the same value for all tuples. This implies that all pairs of tuples satisfying $R[B] \approx R[B]$ satisfy the similarity condition of $m_2$.

**Proof of Proposition 2:** We take finite strings of bits as the domain of all attributes. We number each bit within a string consecutively from left to right starting at 1. Two strings are similar if they both have a 1 bit with the same number. Otherwise, they are dissimilar. For example, 011 and 001 are similar, but 010 and 100 are dissimilar. It is readily verified that this satisfies the properties of a similarity operator.

As in the proof of Theorem 2, we reduce the complement of CS to the resolved answer problem for the given MDs. Let $F$ be an instance of CS as in Case (1)(a) of the proof of Theorem 2, and define $U$, $V$, $K$, and $P$ as before. We take $v_i$ to be a string with $n + 1$ bits, all of which are 0 except the $i^{th}$ bit. We take $k_i$ to be a string with $m$ bits, with all bits 0 except the $i^{th}$ bit. The instance will also contain strings $a$, $b$, and $c$ of length $n + 1$. String $a$ is all zeros, $b$ is all zeros except the $(n + 1)^{th}$ bit, and $c$ is all ones.

There are $n$ sets of tuples $S_i$, $1 \leq i \leq n$, which contain \{f_i\} tuples of $R$ and \{f_i\} tuples of $S$. On attributes $R[A]$ and $S[B]$, the tuples in $S_i$ take the value $v_i$. On attributes $R[I]$ and $S[J]$, all tuples in all $S_i$ take the value $e$. On attributes $R[G]$ and $S[H]$, all tuples in all $S_i$ take the value $a$. In each $S_i$, for each set $f_i$ to which $e_i$ belongs, there is one tuple in $R$ and one tuple in $S$ that has $k_i$ as the value of $R[E]$ (or $S[F]$).

There is also a set $G_1$ of $m$ tuples in $S$. On $S[B]$, all tuples in $G_1$ take the value $b$. On $S[J]$, all tuples in $G_1$ take the value $a$. For each value in $K$, there is a tuple in $G_1$ that takes the value on $S[F]$ and $S[H]$.

The result is now proved analogously to part (1)(a) of the proof of Theorem 2.