On inverse problem of determination of the coefficient in strongly degenerate parabolic equation

T I Bukharova\textsuperscript{1}, V L Kamynin\textsuperscript{1} and A P Tonkikh\textsuperscript{2}

\textsuperscript{1}National Research Nuclear University MEPhI, Kashirskoe shosse, 31,115409 Moscow, Russia
\textsuperscript{2}Bryansk State Academician I.G.Petrovsky University, Bezhitskaya Street, 14, 241036, Bryansk, Russia

E-mail: bukharova_t@mail.ru, vlkamynin2008@yandex.ru, a_tonkih@mail.ru

Abstract. We prove the existence and uniqueness theorems for inverse problem of determination of the lower coefficient in the Black-Scholes type equation with additional condition of integral observation. These results are based on the investigation of unique solvability of corresponding direct problem which is of independent interest. We give the example of the inverse problem for which the conditions of the theorems proved are fulfilled.

1. Introduction
In present work we study the questions of unique solvability of the inverse problem of determination of a pair of functions \(\{u(t,x),\gamma(t)\}\) satisfying in the rectangle \(Q \equiv [0,T] \times [0,l]\) the parabolic equation

\[
u_t - a(t,x)u_{xx} + b(t,x)u_x + \gamma(t)u = f(t,x),
\]

the initial and boundary conditions

\[
u(0,x) = u_0(x), \quad x \in [0,l]; \quad u(t,l) = 0, \quad t \in [0,T];
\]

and the additional condition of integral observation

\[
\int_0^l u(t,x)\omega(x) \, dx = \varphi(t), \quad t \in [0,T] .
\]

In our investigation we suppose that the leading coefficient \(a(t,x)\) in the equation (1) is strongly degenerate for \(x=0\), namely, we assume that \(a(t,x) - a_0x^2, \quad x \to 0, \quad a_0 = \text{const} > 0\).

In what follows we use Lebesgue, Sobolev and Hölder spaces with corresponding norms in usual sense (see, for example, [1]). The derivatives are understood in generalized sense, the equalities and inequalities are satisfied almost everywhere.

We introduce notation

\[
Q^\delta = [0,T] \times [\delta,l], \quad 0 \leq \delta < l, \quad 0 \leq \tau \leq T; \quad Q^0 = Q, \quad Q^\delta = Q^\delta, \quad Q_\tau = Q; \quad \|\cdot\|_{L_q([0,l])} = \|\cdot\|_q, \quad q \in [0,\infty].
\]

We assume that the functions occurring in the input data of the problem (1)–(3) are measurable and satisfy the following conditions:
\[ x^2 a_1 \leq a(t,x) \leq a_2, \quad (t,x) \in Q; \quad a_1, a_2 = \text{const} > 0; \quad a_s(t,x) \frac{a^2_s(t,x)}{a(t,x)} \in L_\alpha(Q); \quad (A) \]
\[ a(t,0) = 0, \quad t \in [0,T]; \quad \left\| \frac{a^2_s}{a(t,x)} \right\|_{L_\alpha(Q)} \leq K_s; \]
\[ \frac{b^2(t,x)}{a(t,x)} \in L_\alpha(Q), \quad \left\| \frac{b^2}{a(t,x)} \right\|_{L_\alpha(Q)} \leq K_{b,a}; \quad \frac{f^2(t,x)}{a(t,x)} \in L_\alpha(Q), \quad \left\| \frac{f^2}{a(t,x)} \right\|_{L_\alpha(Q)} \leq K_{f,a}(T); \quad (B) \]
\[ u_0(x) \in W^1_2(0,L), \quad \left\| u_0 \right\| \leq M_1; \]
\[ \omega(x) \in W^1_2(0,L), \quad \omega(l) = 0, \quad \left\| \omega \right\| \leq K_\omega, \quad \left\| (a\omega) \right\|_{L_2([0,T];L_2(0,L))} \leq K_{\omega,a}; \quad (C) \]
\[ \phi(t) \in W^1_2(0,T), \quad \left\| \phi \right\|_{L_2([0,T])} \leq K_{\phi}, \quad \left\| \phi \right\|_{L_2([0,T])} \leq K_{\phi}^*; \quad (D) \]
\[ \phi(0) = \int_0^t u_0(x) \omega(x) \, dx, \quad \left\| \phi \right\| \geq \phi_0 > 0, \quad t \in [0,T]. \quad (E) \]

Here \( a_1, a_2, K_s, K_{b,a}, K_{f,a}, \phi_0, M_1, \text{const} > 0, \quad K_{b,a}, K_{f,a}(T), \quad K_{\phi}^*, \text{const} \geq 0. \)

**Definition 1.** By generalized solution of the inverse problem (1)–(3) we mean a pair of functions \((u(t,x), \gamma(t))\),
\[ u(t,x) \in L^\infty_{\alpha}(0,T;W^1_2(0,L)) \cap C^{0,\beta}(Q), \quad \beta \in (0,1), \quad u_s(t,x) \in L^2_\alpha(Q), \]
\[ au^2_s \in L^1(0,T;W^1_2(0,L)) \cap C^{0,\beta}(Q), \quad \beta \in (0,1), \quad u(t,x) \in L^2_\alpha(Q), \quad \forall \delta > 0, \quad \gamma(t) \in L^\infty_\alpha(0,T), \]
satisfying equation (1) almost everywhere in \( Q \) and such that the function \( u(t,x) \) satisfies the conditions (2), (3) in classical sense.

**Remark 1.** Conditions (A) and (B) are satisfied for the well-known Black-Scholes equation (see [2,3]). Hence, this equation can be considered in our investigation.

**Remark 2.** Equation (1) degenerates on the boundary of \( Q \) for \( x = 0 \). From the well-known Fichera’s theory it follows that whether or not boundary conditions should be given at the particular part of the boundary where the equation degenerates, depends on the sign of so-called Fichera function on that part of the boundary (see [4], [5, p. 20]). We impose such conditions on the input data that it is not necessary to specify the values of \( u(t,x) \) for \( x = 0 \) (these values cannot be arbitrary). But in our existence theorem we prove that the function \( u(t,x) \) obtained in this theorem automatically satisfies the condition \( u(t,0) = 0 \).

Inverse problems for degenerate parabolic equations of the type (1) are of great interest, for example, in financial mathematics (see Remark 1). In various settings but different from those considered in present paper they were investigated (including numerical calculations) in [6,7,8,9,10,11], etc.

Method of proving the existence and uniqueness theorems for inverse problem (1)–(3) in our paper is based on the study of the unique solvability of the direct problem (1)–(2) (when the function \( \gamma(t) \) is assumed to be known) and on the estimates of its solution. We note that the investigation of the direct problem (1),(2) is of independent interest and the results obtained here for it are also new.

### 2. Unique solvability and estimates for the direct problem
Suppose that \( \gamma(t) \in L^\infty(0,T) \) is a known function and consider the direct problem (1),(2) with this function \( \gamma(t) \) in the equation (1). A generalized solution \( u(t,x) \) of this problem will be understood in a sense of Definition 1.
Using the technique similar to [11] and the existence theorem for uniformly parabolic equations with two independent variables from [1] we prove the following existence and uniqueness theorems.

**Theorem 1.** Let the conditions (A)–(D) hold, \( \gamma(t) \in L_\infty(0,T) \). Then the generalized solution of the direct problem (1),(2) is unique.

**Theorem 2.** Let the conditions (A)–(D) hold, \( \|v\|_{L_2(0,T)} \leq K_\gamma \). Then there exists a generalized solution \( u(t,x) \) of the direct problem (1),(2). For this solution we have \( u(t,0)=0, \ t \in [0,T] \), and it satisfies the estimates

\[
\sup_{0 \leq t \leq T} \left[ a_u \frac{d}{dt} u(t,x) + |u(t,x)|^2 \right] \leq e^{\lambda T} \left[ M_1^2 + 3K_{f,a} \right],
\]

where \( \lambda = 3K_{b,a} + K_\gamma \).

\[
\|u\|_{L_2(0,T)} \leq \frac{e^{\lambda T}}{\delta^2} \left[ M_1^2 + 3K_{f,a}(T) \right], \quad \forall \delta \in (0,1),
\]

\[
\|u\|_{L_2(0,T)} \leq c_1 \left[ M_1^2 + 3K_{f,a}(T) \right],
\]

where \( c_1 \) is a constant depending only on \( l,T,a,b,K_{b,a},K_{f,a}(T),M_1 \) and \( K_\gamma \).

### 3. Investigation of the inverse problem

Now we consider the inverse problem (1)–(3). Denote

\[
F(t) = \int_0^t f(t,x) \omega(x) \, dx.
\]

**Theorem 3.** Let the conditions (A)–(G) hold. Then there are no two different generalized solutions of the problem (1)–(3).

**Sketch of the proof.** Assume that there are two different solutions \( \{u^{(1)}(t,x), \gamma^{(1)}(t)\} \) and \( \{u^{(2)}(t,x), \gamma^{(2)}(t)\} \) of inverse problem (1)–(3). We put \( \nu(t,x) = u^{(1)}(t,x) - u^{(2)}(t,x) \), \( \eta(t) = \gamma^{(1)}(t) - \gamma^{(2)}(t) \).

Taking into account condition (3) and assumptions (A)–(G), after a series of calculations analogous to those carried out in the proof of the corresponding uniqueness theorem in [12] we obtain that the function \( \nu(t,x) \) satisfies in \( Q \) the integro-differential equation

\[
v_t - a(t,x)v_{xx} + b(t,x)v_x + \gamma^{(1)}(t)v = \frac{1}{\varphi(t)} \left[ (a\omega)_x - (b\omega)_x \right] v \, dy \cdot \cdot \cdot (t,x) \tag{8}
\]

and the function \( \eta(t) \) satisfies the relation

\[
\eta(t) = \frac{1}{\varphi(t)} \left[ (a\omega)_x - (b\omega)_x \right] v(t,x) \, dx. \tag{9}
\]

Using the well-known technique of energy estimates applied to the relation (8) we prove that \( \nu(t,x)=0 \) in \( Q \). Then from (9) we obtain that \( \eta(t)=0 \) in \( [0,T] \). Theorem 3 is proved.

In order to prove the existence of generalized solution of the inverse problem (1)–(3) we in addition to conditions (A)–(G) assume that

\[
f(t,x) \in L_\infty([0,T]; L_2(0;l)), \quad (a\omega)_x, (b\omega)_x \in L_\infty([0,T]; L_2(0;l)).
\]
and
\[
\|f\|_{L_2(0,T;L_2(0,1))} \leq K_f, \quad (10)
\]
\[
\|a\omega\|_{L_2(0,T;L_2(0,1))} \leq K^*_{a,d}, \quad \|b\omega\|_{L_2(0,T;L_2(0,1))} \leq K^*_b. \quad (11)
\]

We derive the operator equation for the unknown function \(\gamma(t) \in L_\infty(0,T)\).

Let pair of functions \(\{u(t,x), \gamma(t)\} \) be a generalized solution of the inverse problem (1)–(3). We multiply equation (1) by \(\omega(x)\) and integrate the result over the segment \([0,l]\). Taking into account conditions (2), (3) and also (A)–(G) we after integration by parts obtain the relation
\[
\gamma(t) = \frac{1}{\varphi(t)} \left\{ F(t) - \varphi'(t) + \int_0^T \left[ (a\omega)_{x,x} + (b\omega)_{x,x} \right] u \, dx \right\}. \quad (12)
\]

In view of this relation let us introduce the operator \(A:L_\infty(0,T) \to L_\infty(0,T)\) by the formula
\[
A(\gamma) = \frac{1}{\varphi(t)} \left\{ F(t) - \varphi'(t) + \int_0^T \left[ (a\omega)_{x,x} + (b\omega)_{x,x} \right] u \, dx \right\}, \quad (13)
\]
where \(\gamma(t)\) is an arbitrary function in \(L_\infty(0,T)\) and \(u \equiv u(t,x,\gamma)\) is a solution of the direct problem (1), (2) with given coefficient \(\gamma(t)\) in the equation (1). Then the relation (12) can be written as
\[
\gamma = A(\gamma). \quad (14)
\]

**Remark 3.** By virtue of Theorems 1 and 2 and assumptions (A)–(G) the operator \(A\) is well defined.

**Lemma 1.** Let conditions (A)–(G) hold. Then the operator equation (14) is equivalent to the inverse problem (1)–(3) in the following sense. If pair \(\{u(t,x), \gamma(t)\} \) is a generalized solution of the inverse problem, then \(\gamma(t)\) satisfies (14). Conversely, if \(\gamma(t) \in L_\infty(0,T)\) is a solution of operator equation (14), and \(u \equiv u(t,x,\gamma)\) is a solution of direct problem (1), (2) with this \(\gamma\) in the equation (1), then the pair \(\{u(t,x,\gamma), \gamma(t)\}\) is a generalized solution of inverse problem (1)–(3).

The proof of this Lemma is standard (see, for example [11]).

**Theorem 4.** Let conditions (A)–(G), (10), (11) hold. Suppose that the value \(T_0\) satisfies the inequality
\[
\frac{1}{\varphi} \left\{ K_f K_f + K^*_d + 1 + e^{-3K_f e^{2/3}} \left[ M_1 + 3K_f (T_0) \right]^{1/2} \right\} \leq \frac{2}{T_0}. \quad (15)
\]

Then in the rectangle \(Q_{x_0}\) there exists a generalized solution \(\{u(t,x), \gamma(t)\}\) of the inverse problem (1)–(3) wherein \(u(t,0) = 0, t \in [0,T]\). Moreover, we have the estimate
\[
\|\gamma\|_{L_\infty(0,T_0)} \leq \frac{2}{T_0} \quad (16)
\]
and the estimates (4)–(7) for \(T=T_0\).

**Sketch of the proof.** We put \(R_0 = 2/T_0\). Then we prove that the operator \(A\) is a completely continuous operator mapping the bounded convex closed set \(B_{R_0} \equiv \{\gamma(t) \in L_\infty(0,T_0): \|\gamma\|_{L_\infty(0,T_0)} \leq R_0\}\) into itself. By the Schauder fixed point theorem (e.g. see [13, p. 193]) in this case the equation (14) has a solution \(\gamma(t)\) from \(B_{R_0}\), so the estimate (16) holds. Then the assertion of the theorem follows from Lemma 1.
Let us show that the conditions of our uniqueness and existence Theorems 3 and 4 are valid for the inverse problem for Black-Scholes equation.

**Example.** In the rectangle $Q_0$ we consider the following inverse problem:

$$u_t - \sigma x^2 u_{xx} + \mu x u_x + \gamma(t) u = 0,$$

$$u(0,x) = u_0(x) \equiv x(l-x), \quad u(t,l) = 0,$$

$$\int_0^l u(t,x)(l-x) \, dx = \frac{l^4}{12}.$$

Here $\sigma, \mu > 0$ are some given constants.

It is easy to check that the conditions (A)-(G), (10), (11) are fulfilled. By simple calculations we obtain that the condition (15) for the problem (17)–(19) can be written in the form

$$12 \left( 2\sigma + \frac{\mu}{\sqrt{3}} \right) \frac{1}{\sqrt{2}} \exp \left( 1 + \frac{3\mu^2}{2\sigma} T_0 \right) \leq \frac{2}{T_0}.$$

Obviously the condition (20) is fulfilled for small $T_0$. In this case Theorems 3 and 4 are valid for the problem (17)–(19) and so the inverse problem (17)–(19) has a solution and this solution is unique.

The first and second authors were partially supported by the Program of competitiveness increase of the National Research Nuclear University MEPhI (Moscow Engineering Physics Institute); contract No.02.a03.21.0005, 27.08.2013.

**References**

[1] Kruzhkov S N 1979 Quasilinear parabolic equations and systems with two independent variables Trudy Sem. im. Petrovskogo I G 5 217–272

[2] Black F and Scholes M 1973 The pricing of options and corporate liabilities. J. Political Economy 81 637–659.

[3] Hull J 2005 Options, futures and other derivatives ( NJ: Prentice Hall, upper Saddle River)

[4] Fichera G 1956 Sulle equazioni differenziali lineari ellitico-paraboliche del secondo ordine Atti Accad. Nazionale dei Lincei. Mem. Cl. Sci. Fis. Mat. Natur. Ser. II(8) 5 1–30

[5] Oleinik O A and Radkevič E A 1973 Second Order Differential Equations with Nonnegative Characteristic Form (New York: AMS. Rhode Island and Plenum Press)

[6] Deng Z C and Yang L 2011 An inverse problem of identifying the coefficient of first-order in a degenerate parabolic equation J. Computational Applied Math. 235 4404–4417

[7] Deng Z C and Yang L 2014 An inverse problem of identifying the radiative coefficient in a degenerate parabolic equation Chinese Annals of Math. Ser. B 35B N3 355–382

[8] Bouchouev I and Isakov V 1999 Uniqueness, stability and numerical methods for the inverse problem that arises in financial markets Inverse problems 15 N3 95–116

[9] Jiang Lishang and Tao Yourshan 2001 Identifying the volatility of underlying assets from option prices Inverse problems 17 N1 137–155

[10] Jiang Lishang, Chen Qihong, Wang Lijun and Zhang J E 2003 A new well-posed algorithm to recover implied local volatility Quantitative Finance 3 N6 451–457

[11] Prilepko A I, Kamynin V L and Kostin A B 2018 Inverse source problem for parabolic equation with the condition of integral observation in time Journal of Inverse and Ill-posed Problems 26 N4 523–539

[12] Bukharova T I and Kamynin V L 2015 Inverse problem of determining the absorption coefficient in the multidimensional heat equation with unlimited minor coefficients Computational Mathematics and Mathematical Physics 55 N7 1183–1195

[13] Lysternik L A and Sobolev V I 1982 Kratkii kurs Functional’nogo Analiza (Moskva: Vyshh. Shkola)