ON THE DIOPHANTINE EQUATION $x^2 + 2^a \cdot 3^b \cdot 11^c = y^n$

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Abstract. In this note, we find all the solutions of the Diophantine equation $x^2 + 2^a \cdot 3^b \cdot 11^c = y^n$, in nonnegative integers $a, b, c, x, y, n \geq 3$ with $x$ and $y$ coprime.

1. Introduction

The history of the Diophantine equation

$$x^2 + C = y^n, \quad n \geq 3 \tag{1.1}$$

in positive integers $x$ and $y$ goes back to 1850’s. In 1850, Lebesque [19] proved that the equation (1.1) has no solutions when $C = 1$. This equation is a particular case of the Diophantine equation $ay^2 + by + c = dx^n$, where $a \neq 0$, $b, c$ and $d \neq 0$ are integers with $b^2 - 4ac \neq 0$. This equation has at most finitely many integer solutions $x, y, n \geq 3$. This was proved to be so by Landau and Ostrowski (see [17]) for a fixed $n \geq 3$ in 1920. The fact that $n$ itself is also bounded was proved only 63 years later by Stewart and Shorey in [28]. For the best theoretical upper bounds available today on the exponent $n$, we refer to [7] and [16]. However, these estimates are based on Baker’s theory of lower bounds for linear forms in logarithms of algebraic numbers, so they are quite impractical.

We next survey some results concerning the actual resolution of the Diophantine equation (1.1) for various values of $C$. In 1993, Cohn [14] studied the Diophantine equation (1.1) and found all its integer solutions $(x, y, n)$ for most values of $C$ in the interval $[1, 100]$. In [25], Mignotte and de Weger dealt with the cases $C = 74$ and 86, which had not been covered by Cohn. In both these cases, the only interesting value of the exponent $n$ is $n = 5$. The remaining cases were finally dealt with by Bugeaud, Mignotte and Siksek in [11].

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Variations of the Diophantine equation (1.1) have also been intensively studied. For example, if we replace $y^n$ from the right hand side of equation (1.1) with $2y^n$, but keep the conditions that $n \geq 3$ and the coprimality condition on $x$ and $y$, we get an equation whose solutions were found [29] for all values of $C$ which are squares of an odd integer $B \in \{3, 5, 7, \ldots, 501\}$.

Recently, several authors studied the case when $C$ is a positive integer which is an $S$-unit, where $S$ is some small set of primes. Recall that if $S := \{p_1, \ldots, p_k\}$ is some finite set of primes, then an $S$-unit is an integer all whose prime factors are in $S$. When $S = \{2\}$, we obtain the equation $x^2 + 2^k = y^n$ which was first studied by Cohn in 1992 (see [13]) who found all the integer solutions $(x, y, k, n)$ with $n \geq 3$ when $k$ is odd, even without the coprimality condition on $x$ and $y$. The case when $k$ is even in the above Diophantine equation generated a few papers before it was finally completely settled in [4], without the coprimality condition on $x$ and $y$, and independently but one year later by Le [18], under the coprimality condition on $x$ and $y$. The two proofs used different tools. The recorded work on the Diophantine equation $x^2 + 3^k = y^n$ is more entertaining.

In 1998, Abu Muriefah and Arif [2], found its solutions with $k$ odd and two years later, in 2000, Luca [21], found all the solutions with $k$ even. Unaware of this work, the above results were rediscovered in 2008 by Tao Liqun [20]. The case when $S = \{5\}$ was dealt with in [3] and [5]. Partial results on the case when $S = \{7\}$ appear in [23]. Recently, Bérczes and Pink [8], found all the solutions of the Diophantine equation (1.1) when $C = p^k$ and $k$ is even, where $p$ is any prime in the interval $[2, 100]$.

The paper [22] is the first recorded instance in which all the solutions of the Diophantine equation (1.1) were found when $C$ is some positive $S$-unit for a set $S$ containing more than one prime. In that instance, the set $S$ was $\{2, 3\}$. Since then, all solutions of the same Diophantine equation (1.1) when $C > 0$ is some $S$-unit were found in [24] for $S = \{2, 5\}$, in [6] for $S = \{5, 13\}$, in [12] for $S = \{2, 11\}$, and in [15] for $S = \{2, 5, 13\}$. In [27] Pink has obtained some results for $S = \{2, 3, 5, 7\}$.

Here, we add to the literature on the topic and study the case when $C > 0$ is an $S$-unit, where $S = \{2, 3, 11\}$. More precisely, we study the Diophantine equation

$$x^2 + 2^a \cdot 3^b \cdot 11^c = y^n, \quad (x, y) = 1 \quad \text{and} \quad n \geq 3. \quad (1.2)$$

Our result is the following.
Theorem 1.1. The only solutions of the Diophantine equation (1.2) are:

\[ n = 3 : \quad \text{the solutions given in Table 1 and Table 2;} \]
\[ n = 4 : \quad \text{the solutions given in Table 3;} \]
\[ n = 5 : \quad (x, y, a, b, c) = (1, 3, 1, 0, 2), \ (241, 9, 3, 0, 2); \]
\[ n = 6 : \quad (x, y, a, b, c) \in \{(5, 3, 6, 0, 1), (37, 5, 4, 4, 1), (117, 5, 4, 0, 2)\}; \]
\[ n = 10 : \quad (x, y, a, b, c) = (241, 3, 3, 0, 2); \]

A few words about the proofs.

We start by treating the cases \( n = 3 \) and \( n = 4 \). This is achieved in Section 2 and Section 3, respectively. As a method, we transform equation (1.2) into several elliptic equations written in cubic and quartic models, respectively, for which we need to determine all their \( \{2, 3, 11\}\)-integral points. As a byproduct of our results, we also read easily that the only exponents \( n \geq 3 \) whose prime factors are in the set \( \{2, 3\} \) and for which equation (1.2) has a solution \((x, y, a, b, c, n)\) are \( n = 3, 4, 6 \). In Section 4, we assume that \( n \geq 5 \) and study the equation (1.2) under this assumption. The method here uses the properties of the Primitive Divisors of Lucas sequences. All the computations are done with MAGMA [10] and with Cremona’s program mwrank.

Before digging into the proofs, we note that since \( n \geq 3 \), it follows that \( n \) is either a multiple of 4, or \( n \) is a multiple of an odd prime \( p \). Furthermore, if \( d \mid n \) is such that \( d \in \{4, p\} \) with \( p \) an odd prime and \((x, y, a, b, c, n)\) is a solution of our equation (1.2), then \((x, y^{n/d}, a, b, c, d)\) is also a solution of our equation (1.2) satisfying the same restrictions. Thus, we may replace \( n \) by \( d \) and \( y \) by \( y^{n/d} \), and from now on assume that \( n \in \{4, p\} \). Furthermore, note that when \( c = 0 \) our equation becomes \( x^2 + 2^a3^b = y^n \) all solutions of which are already known from what we have said earlier (see [22]), while when \( b = 0 \) our equation becomes \( x^2 + 2^a11^c = y^n \) all solutions of which have been found in [12]. Thus, we shall assume that \( bc > 0 \). Since \( 3^b11^c \equiv 1, 3 \pmod{8} \) according to whether \( b + c \) is even or odd, it follows by considerations modulo 8 that either \( a > 0 \), or that \( x \) is even. This observation will be useful later on.
Table 1. Solutions for \( n = 3 \).

| \( \alpha \) | \( \beta \) | \( \gamma \) | \( z \) | \( a \) | \( b \) | \( c \) | \( x \) | \( y \) |
|-------------|-------------|-------------|--------|--------|--------|--------|--------|--------|
| 0           | 0           | 1           | 1      | 0      | 0      | 1      | 4      | 3      |
| 0           | 0           | 1           | 1      | 0      | 0      | 1      | 58     | 15     |
| 0           | 0           | 1           | 2      | 0      | 6      | 1      | 5066   | 295    |
| 0           | 0           | 1           | 2      | 6      | 0      | 1      | 5      | 9      |
| 0           | 0           | 1           | 8      | 18     | 0      | 1      | 6179   | 345    |
| 0           | 0           | 1           | 6      | 6      | 6      | 1      | 5491   | 313    |
| 0           | 0           | 2           | 1      | 0      | 0      | 2      | 2      | 5      |
| 0           | 0           | 2           | 2      | 6      | 0      | 2      | 835    | 89     |
| 0           | 0           | 2           | 4      | 12     | 0      | 2      | 404003 | 5465   |
| 0           | 0           | 2           | 3      | 0      | 6      | 2      | 908    | 97     |
| 0           | 0           | 3           | 1      | 0      | 0      | 3      | 9324   | 443    |
| 0           | 0           | 4           | 6      | 6      | 6      | 4      | 589229 | 7033   |
| 0           | 0           | 1           | 2      | 3      | 0      | 7      | 2      | 910    | 103    |
| 0           | 0           | 1           | 4      | 9      | 0      | 13     | 4      | 14259970 | 58807 |
| 0           | 0           | 2           | 2      | 33     | 0      | 8      | 8      | 1549034 | 15613 |
| 0           | 0           | 3           | 2      | 6      | 6      | 9      | 2      | 4085   | 553    |
| 0           | 0           | 4           | 0      | 1      | 0      | 4      | 0      | 46     | 13     |
| 0           | 0           | 4           | 1      | 1      | 0      | 4      | 1      | 170    | 31     |
| 0           | 0           | 4           | 3      | 4      | 12     | 4      | 3      | 239363 | 3865   |
| 0           | 0           | 5           | 0      | 1      | 0      | 5      | 0      | 10     | 7      |
| 0           | 0           | 5           | 1      | 1      | 0      | 5      | 1      | 7910   | 397    |
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Table 2. Solutions for $n = 3$.

| $\alpha$ | $\beta$ | $\gamma$ | $z$ | $a$ | $b$ | $c$ | $x$   | $y$   |
|----------|---------|----------|-----|-----|-----|-----|-------|-------|
| 1        | 0       | 0        | 1   | 0   | 0   | 5   | 3     |       |
| 1        | 0       | 2        | 1   | 1   | 0   | 2   | 5805  | 323   |
| 1        | 3       | 0        | 1   | 1   | 3   | 0   | 17    | 7     |
| 1        | 3       | 1        | 3   | 1   | 9   | 1   | 1043  | 115   |
| 1        | 4       | 1        | 2   | 7   | 4   | 1   | 2196415 | 16897 |
| 1        | 5       | 1        | 1   | 1   | 5   | 1   | 865   | 91    |
| 1        | 5       | 1        | 9   | 1   | 17  | 1   | 94517 | 2275  |
| 2        | 0       | 0        | 1   | 2   | 0   | 0   | 11    | 5     |
| 2        | 0       | 0        | 22  | 8   | 0   | 6   | 5497  | 785   |
| 2        | 0       | 1        | 1   | 2   | 0   | 1   | 9     | 5     |
| 2        | 0       | 1        | 6   | 8   | 6   | 1   | 4069  | 265   |
| 2        | 2       | 2        | 3   | 2   | 8   | 2   | 241397 | 3877  |
| 2        | 4       | 1        | 1   | 2   | 4   | 1   | 217   | 37    |
| 2        | 4       | 2        | 1   | 2   | 4   | 2   | 107   | 37    |
| 2        | 4       | 5        | 2   | 8   | 4   | 5   | 335802455 | 483121 |
| 2        | 5       | 0        | 1   | 2   | 5   | 0   | 35    | 13    |
| 2        | 5       | 2        | 1   | 2   | 5   | 2   | 1495  | 133   |
| 2        | 5       | 2        | 2   | 8   | 5   | 2   | 34255 | 1057  |
| 3        | 2       | 2        | 3   | 3   | 8   | 2   | 912668635 | 940897 |
| 3        | 4       | 0        | 1   | 3   | 4   | 0   | 955   | 97    |
| 3        | 5       | 1        | 3   | 3   | 11  | 1   | 8099  | 433   |
| 4        | 0       | 2        | 1   | 4   | 0   | 2   | 117   | 25    |
| 4        | 0       | 3        | 2   | 10  | 0   | 3   | 9959  | 465   |
| 4        | 1       | 0        | 3   | 4   | 7   | 0   | 595   | 73    |
| 4        | 2       | 1        | 12  | 16  | 8   | 1   | 73225 | 2161  |
| 4        | 4       | 0        | 1   | 4   | 4   | 0   | 2681  | 193   |
| 4        | 4       | 1        | 1   | 4   | 4   | 1   | 37    | 25    |
| 4        | 4       | 1        | 1   | 4   | 4   | 1   | 97129 | 2113  |
| 4        | 4       | 1        | 2   | 10  | 4   | 1   | 17    | 97    |
| 4        | 4       | 4        | 1   | 4   | 4   | 4   | 3419  | 313   |
| 4        | 5       | 2        | 1   | 4   | 5   | 2   | 665   | 97    |
| 5        | 1       | 1        | 3   | 5   | 7   | 1   | 53333 | 1417  |
| 5        | 5       | 0        | 1   | 5   | 5   | 0   | 39151 | 1153  |
Table 3. Solutions for $n = 4$.

| $\alpha$ | $\beta$ | $\gamma$ | $z$ | $a$ | $b$ | $c$ | $x$ | $y$ |
|----------|---------|----------|-----|-----|-----|-----|-----|-----|
| 0        | 3       | 1        | 4   | 8   | 3   | 1   | 233 | 19  |
| 0        | 3       | 2        | 4   | 8   | 3   | 2   | 1607| 43  |
| 1        | 0       | 0        | 2   | 5   | 0   | 0   | 7   | 3   |
| 1        | 0       | 1        | 6   | 5   | 4   | 1   | 7   | 13  |
| 1        | 1       | 0        | 2   | 5   | 1   | 0   | 23  | 5   |
| 1        | 1       | 1        | 4   | 9   | 1   | 1   | 4223| 65  |
| 1        | 1       | 2        | 6   | 5   | 5   | 2   | 235223| 485 |
| 2        | 1       | 0        | 2   | 6   | 1   | 0   | 47  | 7   |
| 2        | 1       | 1        | 2   | 6   | 1   | 1   | 17  | 7   |
| 2        | 1       | 1        | 2   | 6   | 1   | 1   | 527 | 23  |
| 2        | 1       | 1        | 4   | 10  | 1   | 1   | 223 | 17  |
| 2        | 1       | 2        | 2   | 6   | 1   | 2   | 73  | 13  |
| 2        | 2       | 0        | 2   | 6   | 2   | 0   | 7   | 5   |
| 3        | 1       | 1        | 1   | 3   | 1   | 1   | 19  | 5   |
| 3        | 2       | 0        | 2   | 7   | 2   | 0   | 287 | 17  |
| 3        | 2       | 1        | 2   | 7   | 2   | 1   | 343 | 19  |
| 3        | 3       | 1        | 1   | 3   | 3   | 1   | 5   | 7   |
| 3        | 3       | 1        | 3   | 3   | 7   | 1   | 2165| 47  |
2. The case when \( n = 3 \)

**Lemma 2.1.** All solutions with \( n = 3 \) and \( bc > 0 \) of the Diophantine equation (1.2) are given in Tables 4 and 5:

**Table 4. Solutions for \( n = 3 \).**

| \( \alpha \) | \( \beta \) | \( \gamma \) | \( \delta \) | \( a \) | \( b \) | \( c \) | \( x \) | \( y \) |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 0 | 6 | 1 | 5066 | 295 |
| 0 | 0 | 1 | 6 | 6 | 6 | 1 | 5491 | 313 |
| 0 | 0 | 2 | 3 | 0 | 6 | 2 | 908 | 97 |
| 0 | 0 | 4 | 6 | 6 | 6 | 4 | 589229 | 7033 |
| 0 | 1 | 2 | 3 | 0 | 7 | 2 | 910 | 103 |
| 0 | 1 | 4 | 9 | 0 | 13 | 4 | 14259970 | 58807 |
| 0 | 2 | 2 | 33 | 0 | 8 | 8 | 1549034 | 15613 |
| 0 | 3 | 2 | 6 | 6 | 9 | 2 | 4085 | 553 |
| 0 | 4 | 3 | 4 | 12 | 4 | 3 | 239363 | 3865 |
| 0 | 4 | 1 | 1 | 0 | 4 | 1 | 170 | 31 |
| 0 | 5 | 1 | 1 | 0 | 5 | 1 | 7910 | 397 |
| 1 | 3 | 1 | 3 | 1 | 9 | 1 | 1043 | 115 |
| 1 | 4 | 1 | 2 | 7 | 4 | 1 | 2196415 | 16897 |
| 1 | 5 | 1 | 1 | 1 | 5 | 1 | 865 | 91 |
| 1 | 5 | 1 | 9 | 1 | 17 | 1 | 94517 | 2275 |
In particular, if \( n \geq 3 \) is a multiple of 3 and the Diophantine equation \((1.2)\) has an integer solution \((x, y, a, b, c, n)\), then \( n = 6 \). Furthermore, when \( n = 6 \), the only solution \((x, y, a, b, c)\) is \((37, 5, 4, 4, 1)\).

**Proof.** Equation \((1.2)\) can be rewritten as

\[
\left( \frac{x}{z^2} \right)^2 + A = \left( \frac{y}{z^3} \right)^3,
\]

where \( A \) is cubefree and defined implicitly by the relation \( 2^a \cdot 3^b \cdot 11^c = Az^6 \). One can see that \( A = 2^a \cdot 3^\beta \cdot 11^\gamma \) with some exponents \( \alpha, \beta, \gamma \in \{0, 1, 2, 3, 4, 5\} \). We thus get

\[
V^2 = U^3 - 2^\alpha \cdot 3^\beta \cdot 11^\gamma,
\]

where \( U = y/z^2 \), \( V = x/z^3 \), \( \alpha, \beta, \gamma \in \{0, 1, 2, 3, 4, 5\} \) and all prime factors of \( z \) are in \( \{2, 3, 11\} \). Thus, we need to determine all the \( \{2, 3, 11\} \)-integral points on the totality of the 216 elliptic curves above. Recall that if \( S \) is a finite set of prime numbers, then an \( S \)-integer is rational number \( a/b \) where \( a \) and \( b \) are coprime integers and \( b \) is an \( S \)-unit. We use MAGMA \[10\] to determine all the \( \{2, 3, 11\} \)-integral points on the above elliptic curves from which one can reconstruct easily the solutions \((x, y, a, b, c)\) listed in Tables 4 and 5.

When \( n = 6 \), we replace \( n \) by 3 and \( y \) by \( y^2 \) and get a solution of the same equation \((1.2)\) with \( n = 3 \) and the value of \( y \) being a perfect square. Looking in Tables 4 and 5 we get only the possibility \((37, 25, 4, 4, 1)\) for \((x, y, a, b, c)\).
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Therefore, the only solution to equation (1.2) having $n = 6$ is $(37, 5, 4, 4, 1)$. This completes the proof of this lemma.

3. The case $n = 4$

Lemma 3.1. The only solutions with $n = 4$ and $bc > 0$ of the Diophantine equation (1.2) are given Table 6.

Table 6. Solutions for $n = 4$.

| $\alpha$ | $\beta$ | $\gamma$ | $z$ | $a$ | $b$ | $c$ | $x$ | $y$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 3 | 1 | 4 | 8 | 3 | 1 | 233 | 19 |
| 0 | 3 | 2 | 4 | 8 | 3 | 2 | 1607 | 43 |
| 1 | 0 | 1 | 6 | 5 | 4 | 1 | 7 | 13 |
| 1 | 1 | 1 | 4 | 9 | 1 | 1 | 4223 | 65 |
| 1 | 1 | 2 | 6 | 5 | 5 | 2 | 235223 | 485 |
| 2 | 1 | 1 | 2 | 6 | 1 | 1 | 17 | 7 |
| 2 | 1 | 1 | 2 | 6 | 1 | 1 | 527 | 23 |
| 2 | 1 | 1 | 4 | 10 | 1 | 1 | 223 | 17 |
| 2 | 1 | 2 | 2 | 6 | 1 | 2 | 73 | 13 |
| 3 | 1 | 1 | 1 | 3 | 1 | 1 | 19 | 5 |
| 3 | 2 | 1 | 2 | 7 | 2 | 1 | 343 | 19 |
| 3 | 3 | 1 | 1 | 3 | 3 | 1 | 5 | 7 |
| 3 | 3 | 1 | 3 | 3 | 7 | 1 | 2165 | 47 |

Proof. We use a similar method as in the case $n = 3$ except that we write equation (1.2) as

$$U^2 + A = V^4,$$

(3.1)

where now $U = x/z^2$, $V = y/z$, and $A$ is fourth powerfree and defined implicitly by the relation $2^a \cdot 3^b \cdot 11^c = Az^4$. Thus, $A = 2^a \cdot 3^b \cdot 11^c$ holds with some exponents $\alpha, \beta, \gamma \in \{0, 1, 2, 3\}$. Observe that all prime factors of $z$ are in the set $\{2, 3, 11\}$. Hence, we reduced the problem to determining all the $\{2, 3, 11\}$-integral points the totality of the 64 elliptic curves above. We used again MAGMA to determine these points from which we easily determined all the corresponding solutions $(x, y, a, b, c)$ listed in Table 6. 

□

4. The case when $n \geq 5$ is prime

Lemma 4.1. The Diophantine equation (1.2) has no solutions with $n \geq 5$ prime and $bc > 0$. 

Proof. We change $n$ to $p$ to emphasize that $p$ is a prime number. We rewrite the Diophantine equation (1.2) as $x^2 + dz^2 = y^p$, where
\begin{equation}
\label{eq:1.1}
d \in \{1, 2, 3, 6, 11, 22, 33, 66\}
\end{equation}
according to the parities of the exponents $a$, $b$ and $c$. Here, $z = 2^{\alpha_1} \cdot 3^{\beta_1} \cdot 11^{\gamma_1}$ for some nonnegative exponents $\alpha_1$, $\beta_1$, and $\gamma_1$. Write $\mathbb{K} := \mathbb{Q}(i\sqrt{d})$. Observe that since $bc > 0$ and either $a > 0$ or $x$ is even (see the end of Section 1), it follows that $y$ is always odd. A standard argument tells us now that in $\mathbb{K}$ we have
\begin{equation}
\label{eq:1.2}
(x + i\sqrt{d}z)(x - i\sqrt{d}z) = y^n,
\end{equation}
where the ideals generated by $x + iz\sqrt{d}$ and $x - iz\sqrt{d}$ are coprime in $\mathbb{K}$. Hence, the ideal $x + iz\sqrt{d}$ is a $p$th power of some ideal $O_\mathbb{K}$. The class number of $\mathbb{K}$ belongs to $\{1, 2, 4, 8\}$. In particular, it is coprime to $p$. Thus, by a standard argument, it follows that $x + iz\sqrt{d}$ is associated to a $p$th power in $O_\mathbb{K}$. The cardinality of the group of units of $O_\mathbb{K}$ is 2, 4, or 6, all coprime to $p$. Furthermore, $\{1, i\sqrt{d}\}$ is always an integral base for $O_\mathbb{K}$ except for when $d = 3$, and $d = 11$, in which cases an integral basis for $O_\mathbb{K}$ is $\{1, (1 + i\sqrt{d})/2\}$. Thus, we may assume that the relation
\begin{equation}
\label{eq:1.3}
x + i\sqrt{d}z = \eta^p
\end{equation}
holds with some algebraic integer $\eta \in O_\mathbb{K}$. We write $\eta = u + i\sqrt{dv}$, where either both $u$ and $v$ are integers, or both $2u$ and $2v$ are odd integers, the last case occurring only when $d = 3$ or $d = 11$. Conjugating equation (1.3) and subtracting the two relations, we get
\begin{equation}
\label{eq:1.4}
2i\sqrt{d} \cdot 2^{\alpha_1} \cdot 3^{\beta_1} \cdot 11^{\gamma_1} = \eta^p - \overline{\eta}^p.
\end{equation}
The right hand side of the above equation is an integer multiple of $2i\sqrt{d}v = \eta - \overline{\eta}$. We deduce that $v \mid 2^{\alpha_1} \cdot 3^{\beta_1} \cdot 11^{\gamma_1}$, and that
\begin{equation}
\label{eq:1.5}
\frac{2^{\alpha_1} \cdot 3^{\beta_1} \cdot 11^{\gamma_1}}{v} = \frac{\eta^p - \overline{\eta}^p}{\eta - \overline{\eta}} \in \mathbb{Z}.
\end{equation}
Now let $\{L_m\}_{m \geq 0}$ be the sequence of general term $L_m = (\eta^m - \overline{\eta}^m)/(\eta - \overline{\eta})$ for all $m \geq 0$. This is a Lucas sequence and it consists of integers. Its discriminant is $(\eta - \overline{\eta})^2 = -4dv^2$. For nonzero integer $k$, let $P(k)$ be the largest prime factor of $k$ with the convention that $P(\pm 1) = 1$. Equation (1.5) now leads to the conclusion that
\begin{equation}
\label{eq:1.6}
P(L_p) = P\left(\frac{2^{\alpha_1} \cdot 3^{\beta_1} \cdot 11^{\gamma_1}}{v}\right) \leq 11.
\end{equation}
Recall that a prime factor $q$ of $L_m$ is called primitive if $q \nmid L_k$ holds for any $0 < k < m$ and also $q \nmid (\eta - \overline{\eta})^2$. When $q$ exists, it satisfies the congruence $q \equiv \pm 1 \pmod{m}$ where the sign coincides with $(\frac{-4dv^2}{q}) = (\frac{-d}{q})$. Here, and in what follows, $(\frac{a}{q})$ stands for the Legendre symbol of the integer $a$ with respect to the odd prime $q$. Recall that a particular instance of the Primitive Divisor
Theorem for the Lucas sequences implies that if \( p \geq 5 \), then \( L_p \) always has a primitive prime factor except for finitely many pairs \((\eta, \eta)\) all of which appear in Table 1 in [9] (see also [1]). These exceptional Lucas numbers are called defective.

Let us first assume that we are dealing with a number \( L_p \) without primitive divisors. Then a quick look at Table 1 in [9] reveals that the only defective Lucas numbers whose roots are in \( K = \mathbb{Q}(i\sqrt{d}) \) with \( d \) appearing in the list (4.1) is \((\eta, \eta) = ((1 + i\sqrt{11})/2, (1 - i\sqrt{11})/2)\) for which \( L_5 = 1 \) and \( y = 3 \). However, this is not convenient since for us \( b > 0 \) and \( x \) and \( y \) are coprime so \( y \) cannot be a multiple of 3.

Now let us look at the possibility when the Lucas number \( L_p \) appearing in the right hand side of equation (4.5) has a primitive divisor. Since \( p \geq 5 \), it follows that 11 is primitive for \( L_p \). Thus, 11 \( \equiv \pm 1 \pmod{p} \). We now see that the only possibility is \( p = 5 \) and since 11 \( \equiv 1 \pmod{5} \), we get that \((\eta, \eta) = (\pm 1)\) and \( d \in \{1, 2, 3, 6, 11, 22, 33, 66\} \), we get that \( d = 2 \) and \( d = 6 \). In particular, \( u \) and \( v \) integers.

In the remaining of this section, we shall treat each one of these two cases separately.

4.1. The case \( d = 2 \). Since \( P(L_n) = 11 \) is coprime to \(-4du^2 = -8v^2\), we get the possibilities

\[ v = \pm 2^{\alpha_1}, \quad v = \pm 3^{\beta_1}, \quad v = \pm 2^{\alpha_1}3^{\beta_1}. \]

Since \( y = u^2 + 2v^2 \), we get that \( u \) is odd.

**Case 1:** \( v = \pm 2^{\alpha_1} \).

In this case, equation (4.5) becomes

\[ \pm 3^{\beta_1}11^{\gamma_1} = 5u^4 - 20u^2v^2 + 4v^4. \]

Since \( u \) is odd, it follows that the right hand side of the last equation above is congruent to 5 (mod 8). So \( \pm 3^{\beta_1}11^{\gamma_1} \equiv 5 \pmod{8} \), showing that the sign on the left hand side is negative and \( \beta_1 + \gamma_1 \) is odd.

Assume first that \( \beta_1 = 2\beta_0 + 1 \) be odd. Then \( \gamma_1 = 2\gamma_0 \). We get

\[ -3V^2 = 5U^4 - 20U^2 + 4, \]

where \((U, V) := (u/v, 3^{\beta_0}11^{\gamma_0}/v^2)\) is a \( \{2\} \)-integral point on the above elliptic curve. Using MAGMA we get no solution.

Assume now that \( \beta_1 = 2\beta_0 \) is even. Then \( \gamma_1 = 2\gamma_0 + 1 \) is odd and we get that

\[ -11V^2 = 5U^4 - 20U^2 + 4, \]

where \((U, V) := (u/v, 3^{\beta_0}11^{\gamma_0}/v^2)\) is a \( \{2\} \)-integral point on the above elliptic curve. With MAGMA, we get that the only such points on the above curve are \((U, V) = (\pm 1, \pm 1)\) and \((\pm 1/2, \pm 1/4)\), leading to \((u, v) = (\pm 1, \pm 1)\) and \((\pm 1, \pm 2)\), respectively. These give the solution (1, 3, 1, 0, 2) and (241, 9, 3, 0, 2) respectively.
The second solution is also a solution for \( n = 10 \) for which \( y = 3 \). However, as \( b = 0 \), we ignore such solutions.

**Case 2**: \( v = \pm 3^{\beta_1} \).

In this case, the equation \((4.5)\) becomes
\[
\pm 2^{\alpha_1}1^{11^{\gamma_1}} = 5u^4 - 20u^2v^2 + 4v^4.
\]
Since \( u \) is odd, the right hand side is congruent to 5 (mod 8). So we get the congruence \( \pm 2^{\alpha_1}11^{\gamma_1} \equiv 5 \) (mod 8). When \( \alpha_1 > 0 \), there is no solution, while when \( \alpha_1 = 0 \), we have \( -11^{\gamma_1} \equiv 5 \) (mod 8). Thus, \( \gamma_1 \) is odd and we get that
\[
-11v^2 = 5U^4 - 20U^2 + 4,
\]
where \((U, V) := (u/v, 11^{\gamma_0}/v^2)\) is a \(\{2,3\}\)-integral point on the above elliptic curve. With MAGMA, we get a few points which lead to two solutions of the original equation \((1.2)\) which are not convenient for us since they have \( b = 0 \).

**Case 3**: \( v = \pm 2^{\alpha_1}3^{\beta_1} \).

In this case, the equation \((4.5)\) becomes
\[
\pm 3^{\beta_1}11^{\gamma_1} = 5u^4 - 20u^2v^2 + 4v^4.
\]
Similar to the above cases, there is no solution with \( b > 0 \).

**4.2. The case \( d = 6 \).** In this case, for the equation \((4.5)\) we get the possibilities
\[
v = \pm 2^{\alpha_1}, \quad v = \pm 3^{\beta_1}, \quad v = \pm 2^{\alpha_1}3^{\beta_1}.
\]
Since \( y = u^2 + 6v^2 \), we get that \( u \) is odd.

**Case 1**: \( v = \pm 2^{\alpha_1} \).

Here, the equation \((4.5)\) becomes
\[
\pm 3^{\beta_1}11^{\gamma_1} = 5u^4 - 60u^2v^2 + 36v^4.
\]
Since \( u \) and \( v \) are coprime, we get
\[
\pm 3^{\beta_1}11^{\gamma_1} \equiv 5 \pmod{8}.
\]
So \( \pm 3^{\beta_1}11^{\gamma_1} \equiv 5 \) (mod 8), showing that the sign on the left hand side is negative and that \( \beta_1 + \gamma_1 \) is odd.

First let \( \beta_1 = 2\beta_0 + 1 \) and \( \gamma_1 = 2\gamma_0 \). Then we get that
\[
-3V^2 = 5U^4 - 20U^2 + 4,
\]
where \((U, V) := (u/v, 3^{\gamma_0}11^{\gamma_0}/v^2)\) is a \(\{2\}\)-integral point on the above elliptic curve. Using MAGMA, we get no solution.

Assume next that \( \beta_1 = 2\beta_0 \) be even. Then \( \gamma_1 = 2\gamma_0 + 1 \), and we have
\[
-11V^2 = 5U^4 - 60U^2 + 36,
\]
where \((U, V) := (u/v, 3^{\gamma_0}11^{\gamma_0}/v^2)\) is a \(\{2\}\)-integral point on the above elliptic curve. With MAGMA, we obtain that the only solutions as \((U, V) = (\pm 3, \pm 3)\) and \((\pm 9/4, \pm 57/16)\), leading to \((u, v) = (\pm 3, \pm 1)\). This leads to the solution

\[\ldots\]
(837, 15, 1, 5, 2) of the initial equation \([1.2]\), but as \(\operatorname{gcd}(837, 15) = 3 \neq 1\), this is not a convenient solution.

**Case 2:** \(v = \pm 3^{2\alpha_1}\).

In this case, the equation \([4.5]\) becomes
\[
\pm 2^{\alpha_1} 11^{\gamma_1} = 5u^4 - 60u^2v^2 + 36v^4.
\]
This leads to
\[
\pm 2^{\alpha_1} 11^{\gamma_1} \equiv 5 \pmod{8},
\]
and when \(\alpha_1 > 0\), we get no solution, while when \(\alpha_1 = 0\), we get that
\[
-11^{\gamma_1} \equiv 5 \pmod{8}.
\]
Therefore \(\gamma_1\) is odd, and we have
\[
-11V^2 = 5U^4 - 60U^2 + 36,
\]
where \((U, V) := (u/v, 2^{\alpha_1}11^{\gamma_1}/v^2)\) is a \(\{3\}\)-integral point on the above elliptic curve. With MAGMA we obtain the only solutions as \((U, V) = (\pm 3, \pm 3)\) and \((\pm 9/4, \pm 57/16)\). They do not lead to solutions of our original equation.

**Case 3:** \(v = \pm 2^{\alpha_1} 3^{2\beta_1}\).

In this case, the equation \([4.5]\) becomes
\[
\pm 11^{\gamma_1} = 5u^4 - 60u^2v^2 + 36v^4.
\]
We now get that \(\pm 11^{\gamma_1} \equiv 5 \pmod{8}\). This last congruence is possible only when the left hand side has minus sign. Therefore \(\gamma_1\) must be odd. We are led to computing the \(\{2, 3\}\)-integer points on a certain elliptic curve and similarly as in the previous cases, we get no solution.

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ON THE DIOPHANTINE EQUATION $x^2 + 2^a \cdot 3^b \cdot 11^c = y^n$

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