PROPRIATION DYNAMICS IN A DIFFUSIVE SIQR MODEL FOR CHILDHOOD DISEASES

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Abstract. This paper is concerned with the propagation dynamics in a diffusive susceptible-infective nonisolated-isolated-removed model that describes the recurrent outbreaks of childhood diseases. To model the spatial-temporal modes on disease spreading, we study the traveling wave solutions and the initial value problem with special decay condition. When the basic reproduction ratio of the corresponding kinetic system is larger than one, we define a threshold that is the minimal wave speed of traveling wave solutions as well as the spreading speed of some components. From the viewpoint of mathematical epidemiology, the threshold is monotone decreasing in the rate at which individuals leave the infective and enter the isolated classes.

1. Introduction. Many differential systems have been established/studied to understand the disease spreading since the work of Kermack and McKendrick [10]. To obtain a mathematical model, the total population are often split into different classes including the susceptible, the infected and the removed. For some childhood diseases including measles, infected children who become infective at the end of the latency period get severe symptoms at the end of the incubation period that causes them to stay at home, and their infectious impact is greatly reduced because they are almost kept from making contacts outside their families. Therefore, it is reasonable to consider the class of isolated infected individuals based on the classical SIR models, so Feng and Thieme [5] split the total population into the susceptible to the disease $S$, infective nonisolated individuals $I$, isolated individuals $Q$, recovered or immune individuals $R$, and they studied the following mathematical model

\[
\begin{align*}
\frac{dS(t)}{dt} &= \Lambda - \mu S(t) - \frac{\sigma S(t)I(t)}{S(t)+I(t)+R(t)}, \\
\frac{dI(t)}{dt} &= \sigma S(t)I(t) - (\mu + \gamma)I(t), \\
\frac{dQ(t)}{dt} &= \gamma I(t) - (\mu + \xi)Q(t), \\
\frac{dR(t)}{dt} &= \xi Q(t) - \mu R(t),
\end{align*}
\]

in which $t > 0$, all the parameters are positive, $\Lambda$ is the birth rate and all newborns are assumed to be susceptible, $\mu$ denotes the mortality rate, $\sigma$ is the per capita infection rate of an average susceptible individual provided that everybody else is infected, $\gamma$ is the rate at which individuals leave the infected and enter the isolated,
\( \xi \) formulates the rate at which individuals leave the isolated and enter the removed. From [5], we see that the success or failure of disease spreading depends on the sign of \( \frac{\sigma}{\mu + \gamma} - 1 \), in which \( \frac{\sigma}{\mu + \gamma} \) is the so-called basic reproduction ratio. To further know the effect of isolation period \( 1/\xi \), we may refer to Wu and Feng [26].

The purpose of this paper is to further describe the spatio-temporal modes in the corresponding reaction-diffusion system by considering the movement of individuals, which focuses on the spatial expansion thresholds when \( \sigma > \mu + \gamma \). Therefore, we study the following reaction-diffusion system

\[
\begin{align*}
\frac{\partial S(x,t)}{\partial t} &= d_1 \Delta S(x,t) + \Lambda - \mu S(x,t) - \frac{\sigma S(x,t)I(x,t)}{S(x,t) + I(x,t) + R(x,t)}, \\
\frac{\partial I(x,t)}{\partial t} &= d_2 \Delta I(x,t) + \frac{\sigma S(x,t)I(x,t)}{S(x,t) + I(x,t) + R(x,t)} - (\mu + \gamma)I(x,t), \\
\frac{\partial Q(x,t)}{\partial t} &= d_3 \Delta Q(x,t) + \gamma I(x,t) - (\mu + \xi)Q(x,t), \\
\frac{\partial R(x,t)}{\partial t} &= d_4 \Delta R(x,t) + \xi Q(x,t) - \mu R(x,t),
\end{align*}
\]

in which \( x \in \mathbb{R}, d_1 > 0, d_2 > 0, d_3 \geq 0, d_4 > 0 \) are diffusive parameters of the corresponding classes, \( \Delta = \frac{\partial^2}{\partial x^2} \) is the Laplacian operator that models the random movement of individuals and has been utilized in many mathematical models [17, 18], and other parameters are the same as those in (1). In particular, \( d_3 > 0 \) is admissible because of the possible movement of the isolated. We shall try to formulate the spatial spreading process of the infected when the infected individuals were introduced into the habitat of the susceptible.

In different parabolic systems, traveling wave solutions and asymptotic spreading have been widely studied to model the spatial spreading of individuals since [1, 6, 11], and we refer to some related pioneer work in mathematical epidemiology by [2, 22] and a survey paper [20]. For coupled non-cooperative systems modeling disease spreading, we refer to [3, 14] for asymptotic spreading and [4, 9, 12, 21, 23, 24, 28] for traveling wave solutions. In what follows, we first study the corresponding initial value problem of (2) with a kind of initial condition. Evidently, this system does not generate monotone semiflows, and the results for monotone semiflows (see [13, 16, 25]) do not work directly. In this paper, by constructing some auxiliary equations, we finally obtain some scalar equations of the infected, which are monotone. By the comparison principle, we can prove that \( 2\sqrt{d_2}[\sigma - (\mu + \gamma)] \) is the rough spreading speed of \( I, Q, R \) when the initial condition satisfies proper decay behavior.

Besides the asymptotic spreading, we also study the traveling wave solutions of (2). Due to the special form of the equations of \( Q, R \), we obtain a coupled system of \( S, I \). However, this system of \( S, I \) does not satisfy the classical comparison principle of predator-prey systems due to the nonlocal delays [7]. Applying the recipe of generalized upper and lower solutions of delayed systems [15], we obtain the existence of nonconstant traveling wave solutions. Moreover, the nonexistence and asymptotic behavior of traveling wave solutions are studied with the help of asymptotic spreading and some auxiliary monotone scalar equations. We confirm that \( 2\sqrt{d_2}[\sigma - (\mu + \gamma)] \) is also the threshold determining the existence and nonexistence of nontrivial traveling wave solutions modeling the disease spreading.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries and present the main results. The initial value problem is studied in Section 3 to estimate the upper and lower bounds of spreading speed. The existence and nonexistence of traveling wave solutions are investigated in Section 4. In Section 5, two numerical examples are given to illustrate our conclusions. Finally, we give a brief discussion on our results and further investigation.
2. Main results. In this paper, we use the standard partial ordering in $\mathbb{R}^4$. Moreover, $C$ is the set of all uniformly continuous and bounded functions from $\mathbb{R}$ to $\mathbb{R}$, and $u \in C^k$ implies $u, u', \cdots, u^{(k)} \in C, k \in \mathbb{N}$. If $b > a$, then

$$C_{[a,b]} = \{ u \in C : a \leq u(x) \leq b, x \in \mathbb{R} \},$$

and $C^+$ is defined by

$$C^+ = \{ u \in C : u(x) \geq 0, x \in \mathbb{R} \}.$$

If $\sigma > \mu + \gamma$, we define

$$\hat{S} = \frac{\Lambda}{\mu} \hat{t} = \frac{1}{\mu} \left[ \frac{\sigma \Lambda}{\mu + \gamma} - \Lambda \right], \hat{Q} = \frac{\gamma \hat{t}}{\mu + \xi}, \hat{R} = \frac{\xi \gamma \hat{t}}{\mu (\mu + \xi)}$$

and $S_0 > 0$ such that

$$\Lambda - \mu S_0 - \frac{\sigma S_0 \hat{t}}{S_0 + \hat{t} + \hat{R}} = 0.$$

Moreover, if $d_3 = 0$, then

$$e^{-(\mu + \xi)t} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4\pi d_4 t}} Q(y) dy := e^{-(\mu + \xi)t} Q(x)$$

for $t > 0, Q \in C$.

Firstly, we consider the initial value problem

$$\begin{cases}
\frac{\partial S(x,t)}{\partial t} = d_1 \Delta S(x,t) + \Lambda - \mu S(x,t) - \frac{\sigma S(x,t) I(x,t)}{S(x,t) + I(x,t) + R(x,t)}, \\
\frac{\partial I(x,t)}{\partial t} = d_2 \Delta I(x,t) + \frac{\sigma S(x,t) I(x,t)}{S(x,t) + I(x,t) + R(x,t)} - (\mu + \gamma) I(x,t), \\
\frac{\partial Q(x,t)}{\partial t} = d_3 \Delta Q(x,t) + \gamma I(x,t) - (\mu + \xi) Q(x,t), \\
\frac{\partial R(x,t)}{\partial t} = d_4 \Delta R(x,t) + \xi Q(x,t) - \mu R(x,t), \\
S(x,0) = S(x), I(x,0) = \mathcal{I}(x), Q(x,0) = Q(x), R(x,0) = \mathcal{R}(x),
\end{cases}$$

in which $S, \mathcal{I}, Q, R \in C^+, S(x) + \mathcal{I}(x) + R(x) > 0, x \in \mathbb{R}$. Due to our question, we require that the initial condition satisfies the following assumption.

(1C): $\mathcal{I}(x)$ has nonempty support such that

$$\limsup_{x \to \pm \infty} \frac{\mathcal{I}(x)}{|x|e^{-a|x|}} < \infty$$

with $a = \sqrt{[\sigma - (\mu + \gamma)]/d_2}$.

For the initial value problem, we use the following threshold to describe its long time behavior.

**Definition 2.1.** A constant $c_u$ is called the spreading speed of a nonnegative function $u(x,t), x \in \mathbb{R}, t > 0$, if

(a): $\limsup_{t \to \infty} \sup_{|x| > (c_u + \varepsilon)t} u(x,t) = 0$ for any given $\varepsilon > 0$;

(b): $\liminf_{t \to \infty} \inf_{|x| < (c_u - \varepsilon)t} u(x,t) > 0$ for any given $\varepsilon \in (0, c_u)$.

Moreover, we also study the traveling wave solutions of (2). Hereafter, a traveling wave solution of (2) is a special solution taking the following form

$$S(x,t) = s(z), I(x,t) = \iota(z), Q(x,t) = q(z), R(x,t) = r(z), z = x + ct,$$
in which $c > 0$ is the wave speed parameter while $s, i, q, r \in C^2(d_3 > 0)$ or $s, i, r \in C^2, q \in C^1(d_3 = 0)$ are wave profiles. By the definition, we have

\[
\begin{aligned}
    d_1 s''(z) - cs'(z) + \Lambda - \mu s(z) - \frac{\sigma s(z) i(z)}{s(z) + i(z) + r(z)} &= 0, \\
    d_2 s''(z) - c'(z) + \frac{\sigma s(z) i(z)}{s(z) + i(z) + r(z)} - (\mu + \gamma) i(z) &= 0, \\
    d_3 q''(z) - cq'(z) + \gamma i(z) - (\mu + \xi) q(z) &= 0, \\
    d_4 r''(z) - cr'(z) + \xi q(z) - \mu r(z) &= 0 \\
\end{aligned}
\]  

(4)

for $z \in \mathbb{R}$. Moreover, to simulate the disease spreading, we also require that

\[
\begin{aligned}
    \lim_{z \to -\infty} s(z) = \frac{A}{\mu}, \\
    \lim_{z \to -\infty} i(z) = \lim_{z \to -\infty} q(z) = \lim_{z \to -\infty} r(z) = 0, \\
    \lim \inf_{z \to -\infty} s(z), \lim \inf_{z \to -\infty} i(z), \lim \inf_{z \to -\infty} q(z), \lim \inf_{z \to -\infty} r(z) > (\theta, \theta, \theta, \theta) \\
\end{aligned}
\]  

(5)

for some constant $\theta > 0$.

Define

\[c^* = 2\sqrt{d_2[\sigma - (\mu + \gamma)]}\]

if $\sigma > \mu + \gamma$, then $c^*$ has the following properties.

**Theorem 2.2.** $c^*$ is the propagation threshold of (2) in the following sense.

1. Assume that $\sigma > \mu + \gamma$ and (IC) hold. Then $c^* = c_I = c_Q = c_R$ with $I, Q, R$ defined by the initial value problem (3).

2. Assume that $\sigma \leq \mu + \gamma$. Then

\[\lim_{t \to \infty} S(x, t) = \tilde{S}, \quad \lim_{t \to \infty} I(x, t) = \lim_{t \to \infty} Q(x, t) = \lim_{t \to \infty} R(x, t) = 0,\]

in which the convergence is uniform. Here, $S, I, Q, R$ are defined by the initial value problem (3).

3. Assume that $\sigma > \mu + \gamma$. Then (4)-(5) has a positive solution if and only if $c \geq c^*$.

4. Assume that $\sigma \leq \mu + \gamma$. Then (4)-(5) does not have a positive solution for any $c > 0$.

3. **Initial value problem.** In this part, we study the initial value problem (3) and prove (1)-(2) of Theorem 2.2 by several lemmas. Firstly, we have the following result on the existence and invariance of (3), we omit the verification here since it is evident by the definition of these constants.

**Lemma 3.1.** Assume that $\sigma > \mu + \gamma$. Then

\[
\begin{aligned}
    0 < S(x, t) \leq \max \left\{ \sup_{x \in \mathbb{R}} S(x), \frac{\sigma \tilde{S}}{\mu + \gamma} - \tilde{s} \right\} := \tilde{S}, \\
    0 < I(x, t) \leq \max \left\{ \sup_{x \in \mathbb{R}} I(x), \frac{\sigma \tilde{I}}{\mu + \gamma} - \tilde{i} \right\} := \tilde{I}, \\
    0 < Q(x, t) \leq \max \left\{ \sup_{x \in \mathbb{R}} Q(x), \frac{\gamma \tilde{I}}{\mu + \xi} - \tilde{q} \right\} := \tilde{Q}, \\
    0 < R(x, t) \leq \max \left\{ \sup_{x \in \mathbb{R}} R(x), \frac{\xi \tilde{Q}}{\mu} \right\} := \tilde{Q}. \\
\end{aligned}
\]

By the uniform boundedness,

\[S_2(x, t), S_3(x, t), I_2(x, t), I_3(x, t), x \in \mathbb{R}, t > 1\]
are uniformly bounded. Moreover, there exist $T > 0, L > 0$ such that
$$\inf_{x \in \mathbb{R}} S(x, t) \geq S_0/2, t \geq T,$$
and
$$\frac{\sigma S(x, t)I(x, t)}{S(x, t) + I(x, t) + R(x, t)} > \sigma I(x, t)[1 - L I(x, t) - LR(x, t)], \ t \geq T.$$ Consider the Fisher type equation
$$\frac{\partial u(x, t)}{\partial t} = d_2 \Delta u(x, t) + \frac{\sigma \Delta u(x, t)}{\Lambda + \bar{\mu} u(x, t)} - (\mu + \gamma) u(x, t), x \in \mathbb{R}, t > 0 \ (6)$$
with $\bar{\mu} > 0$, then the following results can be found in many works, for example, see Ye et al. [27].

**Lemma 3.2.** If $c = c^*$, then (6) has a positive traveling wave solution $u(x, t) = \phi(x + ct)$ such that $\phi(z)$ is strictly increasing and
$$\lim_{z \to -\infty} \phi(z) = 0, \lim_{z \to \infty} \phi(z) = \frac{1}{\bar{\mu}} \left[ \frac{\sigma \Lambda}{\mu + \gamma} - \Lambda \right], \lim_{z \to -\infty} \frac{\phi(z)}{z e^{\sigma z}} = 1.$$ Moreover, if the initial condition $u(x, 0)$ admits nonempty compact support in (6), then for any $\epsilon \in (0, c^*)$, we have
$$\liminf_{t \to \infty} \inf_{|x| < (c^* - \epsilon)t} u(x, t) = \limsup_{t \to \infty} \sup_{|x| < (c^* - \epsilon)t} u(x, t) = \frac{1}{\bar{\mu}} \left[ \frac{\sigma \Lambda}{\mu + \gamma} - \Lambda \right]$$
and
$$\limsup_{t \to \infty} \sup_{|x| > (c^* + \epsilon)t} u(x, t) = 0.$$ The outer spreading property is presented as follows, which shows the upper bounds of spreading speed of unknown functions.

**Lemma 3.3.** If $\sigma > \mu + \gamma$ and (IC) hold, then
$$\limsup_{t \to \infty} \sup_{|x| > c^*} I(x, t) = \limsup_{t \to \infty} \sup_{|x| > c^*} Q(x, t) = \limsup_{t \to \infty} \sup_{|x| > c^*} R(x, t) = 0$$
for any given $c > c^*$.

**Proof.** Let
$$\max \left\{ \sup_{x \in \mathbb{R}} S(x), \hat{S} \right\} = \hat{s}.$$ By Lemma 3.1, we have
$$\begin{align*}
\frac{\partial I(x, t)}{\partial t} & \leq d_2 \Delta I(x, t) + \frac{\sigma \Delta I(x, t)}{S(x, t) + I(x, t) + R(x, t)} - (\mu + \gamma) I(x, t), \\
\frac{\partial Q(x, t)}{\partial t} & = d_3 \Delta Q(x, t) + \gamma I(x, t) - (\mu + \xi) Q(x, t), \\
\frac{\partial R(x, t)}{\partial t} & = d_4 \Delta R(x, t) + \xi Q(x, t) - \mu R(x, t), \\
I(x, 0) & = I(x), Q(x, 0) = Q(x), R(x, 0) = R(x)
\end{align*} \ (7)$$
and so
$$\begin{align*}
\frac{\partial I(x, t)}{\partial t} & \leq d_2 \Delta I(x, t) + \frac{\sigma \Delta I(x, t)}{S(x, t) + I(x, t) + R(x, t)} - (\mu + \gamma) I(x, t), \\
\frac{\partial Q(x, t)}{\partial t} & = d_3 \Delta Q(x, t) + \gamma I(x, t) - (\mu + \xi) Q(x, t), \\
\frac{\partial R(x, t)}{\partial t} & = d_4 \Delta R(x, t) + \xi Q(x, t) - \mu R(x, t), \\
I(x, 0) & = I(x), Q(x, 0) = Q(x), R(x, 0) = R(x)
\end{align*} \ (8)$$
for all \(x \in \mathbb{R}, t > 0\). That is, \(I(x, t)\) satisfies
\[
\begin{cases}
\frac{\partial I(x, t)}{\partial t} \leq d_2 \Delta I(x, t) + \frac{\sigma \bar{I}(x, t)}{\sigma + I(x, t)} - (\mu + \gamma)I(x, t), \\
\bar{I}(x, 0) = I(x), x \in \mathbb{R}, t > 0.
\end{cases}
\] (9)

By letting \(\tilde{\mu} > 0\) in Lemma 3.2 such that
\[
\frac{1}{\tilde{\mu} \left[ \frac{\sigma \Lambda}{\mu + \gamma} - \Lambda \right]} > \max \left\{ \sup_{x \in \mathbb{R}} I(x), \frac{\sigma \hat{\sigma}}{\mu + \gamma} - \hat{\sigma} \right\},
\]
then there exists \(z_0 \in \mathbb{R}\) such that
\[
\bar{I}(x) \leq \min \{\phi(x + z_0), \phi(-x + z_0)\}, x \in \mathbb{R},
\]
in which \(\phi\) is given by Lemma 3.2. Using the comparison principle in (9), we have
\[
\limsup_{t \to \infty} \sup_{|x| > ct} I(x, t) = 0
\]
for any given \(c > c^\ast\).

Note that
\[
Q(x, t) = e^{-\frac{(\mu + \xi)t}{\sqrt{4\pi dt}}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4\pi ds}} Q(y) dy
+ \gamma \int_0^t e^{-\frac{(\mu + \xi)(t-s)}{2\sqrt{4\pi ds}}} I(y, s) dy ds, x \in \mathbb{R}, t > 0,
\]
then \(\lim_{t \to \infty} e^{-\frac{(\mu + \xi)t}{\sqrt{4\pi dt}}} = 0\) implies
\[
\limsup_{t \to \infty} \sup_{|x| > ct} Q(x, t) = 0
\]
for any given \(c > c^\ast\). In a similar way, we can finish the proof.

By (6) and (9), we have the following result, of which the result is evident and the proof is omitted here.

**Lemma 3.4.** If \(\sigma \leq \mu + \gamma\), then (2) of Theorem 2.2 holds.

We now consider the property of inner spreading, which implies the lower bounds of spreading speed of unknown functions.

**Lemma 3.5.** If \(\sigma > \mu + \gamma\) and \(\bar{I}(x)\) has nonempty support, then
\[
\liminf_{t \to \infty} \inf_{|x| < ct} I(x, t) > 0, \liminf_{t \to \infty} \inf_{|x| < ct} Q(x, t) > 0, \liminf_{t \to \infty} \inf_{|x| < ct} R(x, t) > 0
\]
for any given \(c < c^\ast\).

**Proof.** By Lemma 3.1, \(I(x, t)\) satisfies
\[
\begin{cases}
\frac{\partial I(x, t)}{\partial t} = d_2 \Delta I(x, t) + \frac{\sigma S(x, t)I(x, t)}{\sigma I(x, t) + R(x, t)} - (\mu + \gamma)I(x, t) \\
\geq d_2 \Delta I(x, t) + I(x, t) [\sigma - (\mu + \gamma) - \sigma LI(x, t) - \sigma LR(x, t)], \\
I(x, T) > 0, x \in \mathbb{R}, t \geq T.
\end{cases}
\]

From the third equation in (2), we have
\[
Q(y, s) = e^{-\frac{(\mu + \xi)s}{\sqrt{4\pi ds}}} \int_{\mathbb{R}} e^{-\frac{(y-z)^2}{4\pi dz}} Q(z) dz
+ \gamma \int_0^s e^{-\frac{(\mu + \xi)(s-v)}{2\sqrt{4\pi dv}}} I(v, d) dv \int_{\mathbb{R}} e^{-\frac{(z-v)^2}{4\pi d(s-v)}} I(z, v) dz
\]
for $y \in \mathbb{R}, s \geq 0$. Furthermore, the fourth equation implies that
\[
R(x, t) = \frac{e^{-\mu t}}{\sqrt{4\pi d_4 t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4d_4 t}} R(y) dy \\
+ \xi \int_{0}^{t} \frac{e^{-\mu(t-s)}}{\sqrt{4\pi d_4 (t-s)}} ds \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4d_4 (t-s)}} Q(y, s) dy \\
= \frac{e^{-\mu t}}{\sqrt{4\pi d_4 t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4d_4 t}} R(y) dy \\
+ \xi \int_{0}^{t} \frac{e^{-\mu(t-s)}}{\sqrt{4\pi d_4 (t-s)}} ds \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4d_4 (t-s)}} e^{-(\mu + \gamma)s} dy \int_{\mathbb{R}} e^{-\frac{(y-z)^2}{4d_3 t}} Q(z) dz \\
+ \gamma \xi \int_{0}^{t} e^{-\frac{\gamma(t-s)}{4d_4 (t-s)}} ds \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4d_4 (t-s)}} dy \int_{0}^{s} e^{-(\mu + \xi)(s-v)} dv \int_{\mathbb{R}} e^{-\frac{(y-z)^2}{4d_3 (s-v)}} I(z, v) dz
\]
for $x \in \mathbb{R}, t \geq 0$. We now obtain an auxiliary inequality that only depends on $I$, of which the corresponding equality admits nonlocal delay and is not quasimonotone.

Let $\epsilon > 0$ be a constant such that
\[
\epsilon > 2\sqrt{d_2 [\sigma - (\mu + \gamma) - 4\epsilon]},
\]
then we can fix $T_1 > 0, \tau > 0, N > 0$ such that
\[
R(x, t) = \frac{e^{-\mu t}}{\sqrt{4\pi d_4 t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4d_4 t}} R(y) dy \\
+ \xi \int_{0}^{t} \frac{e^{-\mu(t-s)}}{\sqrt{4\pi d_4 (t-s)}} ds \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4d_4 (t-s)}} e^{-(\mu + \xi)s} dy \int_{\mathbb{R}} e^{-\frac{(y-z)^2}{4d_3 t}} Q(z) dz \\
+ \gamma \xi \int_{0}^{t} e^{-\frac{\gamma(t-s)}{4d_4 (t-s)}} ds \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4d_4 (t-s)}} dy \int_{0}^{s} e^{-(\mu + \xi)(s-v)} dv \int_{\mathbb{R}} e^{-\frac{(y-z)^2}{4d_3 (s-v)}} I(z, v) dz
\]
\[
\leq \frac{\epsilon}{\sigma L} + \gamma \xi \int_{0}^{t} \frac{e^{-\mu(t-s)}}{\sqrt{4\pi d_4 (t-s)}} ds \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4d_4 (t-s)}} dy \int_{0}^{s} e^{-(\mu + \xi)(s-v)} dv \int_{\mathbb{R}} e^{-\frac{(y-z)^2}{4d_3 (s-v)}} I(z, v) dz
\]
\[
< \frac{2\epsilon}{\sigma L} + \gamma \xi \int_{0}^{t} \frac{e^{-\mu(t-s)}}{\sqrt{4\pi d_4 (t-s)}} ds \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4d_4 (t-s)}} dy \int_{0}^{s} e^{-(\mu + \xi)(s-v)} dv \int_{\mathbb{R}} e^{-\frac{(y-z)^2}{4d_3 (s-v)}} I(z, v) dz
\]
\[
= : \frac{2\epsilon}{\sigma L} + \tilde{R}(x, t)
\]
for any $x \in \mathbb{R}, t > 2T_1 + 1$, in which $\tilde{R}(x, t)$ only depends on $I(y, s), y \in [x-2N, x+2N], s \in [t-2T_1, t-\tau]$.

In fact, as for a fixed $\epsilon$, we can fix $T_1, \tau$, such
\[
\gamma \xi \int_{0}^{t} \frac{e^{-\mu(t-s)}}{\sqrt{4\pi d_4 (t-s)}} ds \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4d_4 (t-s)}} dy \int_{0}^{s} e^{-(\mu + \xi)(s-v)} dv \int_{\mathbb{R}} e^{-\frac{(y-z)^2}{4d_3 (s-v)}} I(z, v) dz
\]
\[
+ \gamma \xi \int_{0}^{t} \frac{e^{-\mu(t-s)}}{\sqrt{4\pi d_4 (t-s)}} ds \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4d_4 (t-s)}} dy \int_{0}^{s} e^{-(\mu + \xi)(s-v)} dv \int_{\mathbb{R}} e^{-\frac{(y-z)^2}{4d_3 (s-v)}} I(z, v) dz
\]
\[
< \frac{\epsilon}{4\pi L}
\]
by the decay behavior of $e^{-\mu t}$. 
Note that the \( e^{-\mu(t-s)} \), \( e^{-\frac{(x-y)^2}{4\pi(t-s)}} \) are uniformly continuous and uniformly bounded for \( s \in [t - T_1, t - \tau] \), we can fix \( N \) such
\[
\gamma \xi \int_{t-T_1}^{t-\tau} \frac{e^{-\mu(t-s)}}{4\pi d_4(t-s)} \, ds \int_{-\infty}^{\infty} e^{-\frac{(y-z)^2}{4\pi(t-s)}} \, dy \int_{0}^{t} e^{-\frac{(y-s)^2}{4\pi d_3(s-v)}} I(z,v) \, dz \\
+ \gamma \xi \int_{t-T_1}^{t-\tau} \frac{e^{-\mu(t-s)}}{4\pi d_4(t-s)} \, ds \int_{N}^{\infty} e^{-\frac{(y-z)^2}{4\pi(t-s)}} \, dy \int_{0}^{t} e^{-\frac{(y-s)^2}{4\pi d_3(s-v)}} I(z,v) \, dz
\]
\[
< \frac{\epsilon}{4\sigma L}
\]
hold. Repeating the process, we can fix the \( N, T_1, \tau \).

If \( \sigma L \hat{R}(x,t) \leq \epsilon \), then
\[
\frac{\partial I(x,t)}{\partial t} \geq d_2 \Delta I(x,t) + I(x,t) [\sigma - (\mu + \gamma) - \sigma L I(x,t) - 3\epsilon].
\]
If \( \sigma L \hat{R}(x,t) > \epsilon \), then the uniform continuity in Lemma 3.1 implies that there exist \( \eta > 0 \), \( \nu > 0 \) such that
\[
I(y,s) > \eta, y \in [x_0 - \nu, x_0 + \nu] \text{ for some } x_0 \in [x - 2N, x + 2N], s \in [t - 2T_1, t - \tau],
\]
in which \( \eta > 0 \), \( \nu > 0 \) are independent on \( x, t \). Since
\[
I(x,t) \geq e^{-\frac{\omega(t-s)}{4\pi d_2}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4\pi d_3}} I(y,0) \, dy, x \in \mathbb{R}, t > 0,
\]
then there exists \( \epsilon > 0 \) independent on \( x, t \) such that
\[
I(x,t) > \epsilon
\]
once \( \sigma L \hat{R}(x,t) > \epsilon \). By Lemma 3.1, we may further fix a finite constant \( M > 0 \) such that
\[
\begin{cases} 
\frac{\partial I(x,t)}{\partial t} \geq d_2 \Delta I(x,t) + I(x,t) [\sigma - (\mu + \gamma) - 3\epsilon - MI(x,t)], \\
I(x,2T_1+1) > 0, x \in \mathbb{R}, t \geq 2T_1 + 1.
\end{cases}
\]
Applying Lemma 3.2 and comparison principle, \( I \) satisfies
\[
\liminf_{t \to \infty} \inf_{|x| < ct} I(x,t) \geq \frac{\sigma - (\mu + \gamma) - 3\epsilon}{M} > 0.
\]
By the third and fourth equations in (2), we further have
\[
\liminf_{t \to \infty} \inf_{|x| < ct} Q(x,t) > 0, \liminf_{t \to \infty} \inf_{|x| < ct} R(x,t) > 0.
\]
The proof is completed. \( \square \)

4. **Minimal wave speed.** In this part, we consider the traveling wave solutions and prove (3)-(4) of Theorem 2.2 by several lemmas. By the limit behavior in (5), we have
\[
r(z) = \int_{0}^{\infty} e^{-\mu s} \int_{\mathbb{R}} e^{-\frac{y^2}{4\pi d_4 s}} q(z - y - cs) \, dy \, ds, z \in \mathbb{R},
\]
\[
q(z) = \int_{0}^{\infty} e^{-(\mu + \xi) s} \int_{\mathbb{R}} e^{-\frac{y^2}{4\pi d_3 s}} \gamma i(z - y - cs) \, dy \, ds, z \in \mathbb{R},
\]
which further implies that \( r(z) =: F(i)(z), z \in \mathbb{R} \) with \( F(i)(z) \) defined by
\[
\gamma \xi \int_{0}^{\infty} e^{-\frac{\mu z^2}{4\pi d_4 s_2}} \int_{\mathbb{R}} e^{-\frac{y^2}{4\pi d_2 s_2}} dy_2 ds_2 \int_{0}^{\infty} e^{-\frac{(\mu + \xi) s_1}{4\pi d_3 s_1}} \int_{\mathbb{R}} e^{-\frac{y_1^2}{4\pi d_1 s_1}} i(z, c, y_1, y_2, s_1, s_2) dy_1 ds_1
\]
where \(i(z,c,y_1,y_2,s_1,s_2) = i(z - (y_1 + y_2) - c(s_1 + s_2))\).

After such a process, it suffices to consider the existence or nonexistence of the positive solutions in the following system

\[
\begin{cases}
    d_1 s''(z) - cs'(z) + \Lambda - \mu s(z) - \frac{\sigma s(z)i(z)}{s(z) + i(z) + F(i(z))} = 0, \\
    d_2 s''(z) - ci'(z) + \frac{\sigma s(z)i(z)}{s(z) + i(z) + F(i(z))} - (\mu + \gamma)i(z) = 0
\end{cases}
\]  

(11)

with \(s(z) > 0, z \in \mathbb{R}\), and

\[
\begin{cases}
    \lim_{z \to -\infty} s(z) = \frac{\Lambda}{\mu}, \lim_{z \to -\infty} i(z) = 0, \\
    \liminf_{z \to -\infty} s(z) > \theta, \liminf_{z \to -\infty} i(z) > \theta
\end{cases}
\]  

(12)

for some constant \(\theta > 0\).

Assume that \(s \in C_{[S_0, \hat{S}]}, i \in C_{[0, \hat{i}]}\). Let \(A > \mu + \gamma\) be a constant such that

\[f_1(s,i)(z) = As(z) + \Lambda - \mu s(z) - \frac{\sigma s(z)i(z)}{s(z) + i(z) + F(i(z))}\]

is monotone in \(s \in C_{[S_0, \hat{S}]}\) and define

\[f_2(s,i)(z) = Ai(z) + \frac{\sigma s(z)i(z)}{s(z) + i(z) + F(i(z))} - (\mu + \gamma)i(z)\]

Denote

\[d_j y_{i,j}^2 - cy_{i,j} - A = 0, y_{i,1} < 0 < y_{i,2}, i,j = 1,2,\]

and

\[F_j(s,i)(z) = \frac{1}{d_j(y_{j,2} - y_{j,1})} \left[ \int_{-\infty}^{t} e^{y_{j,1}(z-t)} + \int_{z}^{\infty} e^{y_{j,2}(z-t)} \right] f_j(s,i)(t)dt, j = 1,2\]

with \(s,i \in C^+\), then a solution of (11) is a fixed point of \((F_1,F_2)\). Moreover, any bounded fixed point of \((F_1,F_2)\) is uniformly continuous.

Firstly, we show the bounds of \(s,i\) as follows.

**Lemma 4.1.** Assume that \((s,i)\) is a nonconstant positive solution of (11). Then

\[S_0 < s(z) < \hat{S}, 0 < i(z) < \hat{i}, z \in \mathbb{R}\]

**Proof.** We first prove that \(s(z) \leq \hat{S}, z \in \mathbb{R}\). Since

\[s(z) = \frac{1}{d_1(y_{1,2} - y_{1,1})} \left[ \int_{-\infty}^{\infty} e^{y_{1,1}(z-t)} + \int_{z}^{\infty} e^{y_{1,2}(z-t)} \right] f_1(s,i)(t)dt \leq \frac{1}{d_1(y_{1,2} - y_{1,1})} \left[ \int_{-\infty}^{2} e^{y_{1,1}(z-t)} + \int_{z}^{\infty} e^{y_{1,2}(z-t)} \right] [As(t) + \Lambda - \mu s(t)]dt\]

and

\[\frac{1}{d_1(y_{1,2} - y_{1,1})} \left[ \int_{-\infty}^{\infty} e^{y_{1,1}(z-t)} + \int_{z}^{\infty} e^{y_{1,2}(z-t)} \right] dt = \frac{1}{A},\]

we have \(s(z) \leq \hat{S}, z \in \mathbb{R}\). Again by the continuity, if \(\Lambda - \mu s(z_0) = 0\) at some \(z_0 \in \mathbb{R}\), then \(s(z) = \hat{S}, z \in \mathbb{R}\). It is impossible, so \(s(z) < \hat{S}, z \in \mathbb{R}\).
we have

Assume that Lemma 4.2.

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i

which further implies that

then we have

c

A contradiction occurs.

by

and

Let

We first assume that

Proof.

Since

\[
1 = \frac{1}{d_2(y_{2,1} - y_{2,1})} \left[ \int_{-\infty}^{\infty} e^{y_{2,1}(z-t)} + \int_{z}^{\infty} e^{y_{2,2}(z-t)} \right] f_2(s,i)(t)dt
\]

\[
\leq \frac{1}{d_2(y_{2,1} - y_{2,1})} \left[ \int_{-\infty}^{\infty} e^{y_{2,1}(z-t)} + \int_{z}^{\infty} e^{y_{2,2}(z-t)} \right] \left[ Ai(t) - (\mu + \gamma)i(t) + \frac{\sigma s(t)i(t)}{s(t) + i(t)} \right] dt
\]

\[
\leq \frac{1}{d_2(y_{2,1} - y_{2,1})} \left[ \int_{-\infty}^{\infty} e^{y_{2,1}(z-t)} + \int_{z}^{\infty} e^{y_{2,2}(z-t)} \right] \left[ Ai(t) - (\mu + \gamma)i(t) + \frac{\sigma \hat{s}i(t)}{S + i(t)} \right] dt
\]

and

\[
\frac{1}{d_2(y_{2,1} - y_{2,1})} \left[ \int_{-\infty}^{\infty} e^{y_{2,1}(z-t)} + \int_{z}^{\infty} e^{y_{2,2}(z-t)} \right] dt = \frac{1}{A},
\]

we have \( i(z) \leq \hat{I}, z \in \mathbb{R} \). Again by the continuity, if \( i(z_0) = \hat{I} \) at some \( z_0 \in \mathbb{R} \), then

\( i(z) = \hat{I}, z \in \mathbb{R} \). It is impossible, so \( i(z) < \hat{I}, z \in \mathbb{R} \).

In a similar way, we can prove the remainder. The proof is completed. \( Q.E.D. \)

With the above result, we study the nonexistence of traveling wave solutions.

Lemma 4.2. Assume that \( \sigma > \mu + \gamma, c < c^* \) or \( \sigma \leq \mu + \gamma, c > 0 \). Then (11)-(12) does not have a positive solution.

Proof. We first assume that \( \sigma > \mu + \gamma, c < c^* \). If (11)-(12) has a positive solution \((s,i)\), then Lemma 4.1 implies that they are strictly positive such that

\[
d_2 i''(z) - ci'(z) + \frac{\sigma i(z)}{1 + i(z)/S_0 + F(i)(z)/S_0} - (\mu + \gamma)i(z) \leq 0, z \in \mathbb{R}.
\]

Let \( \epsilon > 0 \) such that

\[
d_2 x^2 - cx + \frac{\sigma}{1 + 2\epsilon} - (\mu + \gamma) > 0, x > 0.
\]

By (12), there exists \( z_0 \in \mathbb{R} \) such that \( F(i)(z) < \epsilon, z \leq z_0 \). Moreover, \( \inf_{z \geq z_0} i(z) > 0 \) by (12) and Lemma 4.1, so there exists \( M > 0 \) such that

\[
d_2 i''(z) - ci'(z) + \frac{\sigma i(z)}{1 + \epsilon + Mi(z)} - (\mu + \gamma)i(z) \leq 0, z \in \mathbb{R},
\]

and \( I(x,t) = i(z) \) satisfies

\[
\begin{cases}
\frac{dI(x,t)}{dt} & \geq d_2 \Delta I(x,t) + \frac{\sigma I(x,t)}{1 + \epsilon + MI(x,t)} - (\mu + \gamma)I(x,t), \\
I(x,0) & = i(x), x \in \mathbb{R}, t > 0.
\end{cases}
\]

Select

\[
c_1 = 2 \sqrt{d_2 \left[ \frac{\sigma}{1 + 2\epsilon} - (\mu + \gamma) \right]},
\]

then we have \( c_1 > c \) and

\[
\lim_{t \to \infty} I(\pm c_1 t, t) > 0
\]

which further implies that

\[
\lim_{z \to -\infty} i(z) > 0
\]

by

\[
z = x + ct = -c_1 t + ct \to -\infty, t \to \infty.
\]

A contradiction occurs.
We now suppose that \( \sigma \leq \mu + \gamma, c > 0 \). Since a traveling wave solution is a special entire solution that admits invariant wave profile, then a positive solution of (4) satisfies
\[
i(z) = 0, z \in \mathbb{R}
\]
by (2) of Theorem 2.2. The proof is completed.

We now study the existence of traveling wave solutions. For the purpose, we introduce the following definition of generalized upper and lower solutions.

**Definition 4.3.** Assume that \( s(z), \sigma(z) \in C_{[S_0, \hat{S}]} \), \( i(z), \bar{i}(z) \in C_{[0, T]} \) satisfy the following facts:

1. \( s(z) \leq \sigma(z), i(z) \leq \bar{i}(z), z \in \mathbb{R} \);
2. except in a finite set \( T \subset \mathbb{R} \), they have bounded and continuous second derivatives such that
\[
\begin{align*}
\frac{d}{dz} s''(z) &\leq \frac{d}{dz} \left( \sigma(z) - c \sigma'(z) + \Lambda - \mu \sigma(z) - \frac{\sigma(z) \bar{i}(z) - \sigma(z) \bar{i}(z) - \sigma(z) i(z)}{\bar{i}(z) + \bar{i}(z) + F(z)(z)} \right) \\
\frac{d}{dz} \sigma''(z) &\leq \frac{d}{dz} \left( \sigma(z) - c \sigma'(z) + \Lambda - \mu \sigma(z) - \frac{\sigma(z) \bar{i}(z) - \sigma(z) \bar{i}(z) - \sigma(z) i(z)}{\bar{i}(z) + \bar{i}(z) + F(z)(z)} \right)
\end{align*}
\]
for all \( z \in T \);
3. if \( z \in \mathbb{R} \setminus T \), then
\[
\begin{align*}
\frac{d}{dz} s''(z) &\leq \frac{d}{dz} \left( \sigma(z) - c \sigma'(z) + \Lambda - \mu \sigma(z) - \frac{\sigma(z) \bar{i}(z) - \sigma(z) \bar{i}(z) - \sigma(z) i(z)}{\bar{i}(z) + \bar{i}(z) + F(z)(z)} \right) \\
\frac{d}{dz} \sigma''(z) &\leq \frac{d}{dz} \left( \sigma(z) - c \sigma'(z) + \Lambda - \mu \sigma(z) - \frac{\sigma(z) \bar{i}(z) - \sigma(z) \bar{i}(z) - \sigma(z) i(z)}{\bar{i}(z) + \bar{i}(z) + F(z)(z)} \right)
\end{align*}
\]

Then (\( \sigma(z), \bar{i}(z) \)), (\( s(z), \bar{i}(z) \)) are a pair of generalized upper and lower solutions of (11).

**Remark 1.** Because of the delayed effect, the system (11) does not satisfy the classical monotone conditions of predator-prey systems, we use the generalized upper and lower solutions.

Similar to Lin and Ruan [15], the existence of (11) can be obtained by the existence of a pair of generalized upper and lower solutions.

**Lemma 4.4.** For given \( c > 0 \), if (11) has a pair of generalized upper and lower solutions \((\sigma(z), \bar{i}(z)), (s(z), \bar{i}(z))\), then (11) has a solution such that
\[
(s(z), \bar{i}(z)) \leq (s(z), \bar{i}(z)) \leq (\sigma(z), \bar{i}(z)), z \in \mathbb{R}.
\]

**Lemma 4.5.** For any given \( c > c^*, \sigma > \mu + \gamma \), (11) has a strictly positive solution such that
\[
\lim_{z \rightarrow -\infty} s(z) = \frac{\Lambda}{\mu}, \lim_{z \rightarrow -\infty} \bar{i}(z) = 0.
\]

**Proof.** By Lemmas 4.1 and 4.4, it suffices to construct proper generalized upper and lower solutions such that \( \bar{i}(z) \neq 0, z \in \mathbb{R} \). For the purpose, we introduce some constants by several steps:

1. \( \lambda_{2,1} < \lambda_{2,2} \) be two positive roots of \( d_2 \lambda^2 - c \lambda + \sigma - (\mu + \gamma) = 0 \);
2. fix \( \lambda_1 \in (0, \lambda_{2,1}) \) such that \( -d_1 \lambda_1^2 + c \lambda_1 + \mu > 0 \);
3. select \( K > 0 \) such that \( K = \hat{S} + \left( \frac{\sigma(z) \bar{i}(z) - \sigma(z) \bar{i}(z) - \sigma(z) i(z)}{\bar{i}(z) + \bar{i}(z) + F(z)(z)} \right) \);
4. fix \( \tau \in (0, \lambda_{2,1}) \) such that \( F(I)(0) < \infty, I(z) = e^{\tau z}, z \in \mathbb{R} \), and select \( \Pi_1 > 0, \Pi = \Pi(\Pi_1) > 0 \) such that \( \Pi_1 e^{\tau z} > \min \{T, e^{\lambda_{2,1} z} \}, z \in \mathbb{R} \), \( F(\Pi_1 I)(0) = \Pi \);
5. give \( \rho > 0 \) such that \( \rho < \min \{\lambda_{2,1}, \lambda_{2,2} - \lambda_{2,1}, \sigma \} \).
(S6): let \( L > 0 \) be defined by Lemma 3.1 and
\[
q > \frac{-(L + LI)\sigma}{d_2(\lambda_{2,1} + \rho)^2 - c(\lambda_{2,1} + \rho) + \sigma - (\mu + \gamma) + 1}
\]
such that
\[
\min\{\hat{I}, e^{\lambda_{2,1}z}\} > \max\{0, e^{\lambda_{2,1}z} - qe^{(\lambda_{2,1} + \rho)z}\}, z \in \mathbb{R}.
\]

With these constants, we define
\[
\bar{s}(z) = \hat{S}, \quad \bar{s}(z) = \max\{S_0, \hat{S} - Ke^{\lambda_{1}z}\}
\]
and
\[
\bar{t}(z) = \min\{\hat{I}, e^{\lambda_{2,1}z}\}, \quad \bar{t}(z) = \max\{0, e^{\lambda_{2,1}z} - qe^{(\lambda_{2,1} + \rho)z}\}.
\]

Due to their monotonicity and smoothness, it suffices to verify (3) of Definition 4.3. We now verify this by four steps.

(1) By \( \bar{s}(z) = \hat{S} = \frac{\lambda}{\mu} \), we have
\[
d_1 \bar{s}''(z) - c\bar{s}'(z) + \Lambda - \mu \bar{s}(z) - \frac{\sigma \bar{s}(z) \bar{t}(z)}{\bar{s}(z) + \bar{t}(z) + F(\bar{t})(z)} = -\frac{\sigma \bar{s}(z) \bar{t}(z)}{\bar{s}(z) + \bar{t}(z) + F(\bar{t})(z)} \leq 0, \quad z \in \mathbb{R}.
\]

(2) If \( \bar{t}(z) = \hat{I} < e^{\lambda_{2,1}z} \), then it is differentiable such that
\[
d_2 \bar{t}''(z) - c\bar{t}'(z) + \frac{\sigma \bar{s}(z) \bar{t}(z)}{\bar{s}(z) + \bar{t}(z) + F(\bar{t})(z)} - (\mu + \gamma) \bar{t}(z) = \frac{\sigma \bar{s}(z) \bar{t}(z)}{\bar{s}(z) + \bar{t}(z) + F(\bar{t})(z)} - (\mu + \gamma) \bar{t}(z)
\]
\[
\leq \frac{\sigma \bar{s}(z) \bar{t}(z)}{\bar{s}(z) + \bar{t}(z)} - (\mu + \gamma) \bar{t}(z)
\]
\[
= \frac{\sigma \hat{S} \bar{t}}{\hat{S} + \bar{t}} - (\mu + \gamma) \bar{t} = 0.
\]

If \( \bar{t}(z) = e^{\lambda_{2,1}z} < \hat{I} \), then it is differentiable such that
\[
d_2 \bar{t}''(z) - c\bar{t}'(z) + \frac{\sigma \bar{s}(z) \bar{t}(z)}{\bar{s}(z) + \bar{t}(z) + F(\bar{t})(z)} - (\mu + \gamma) \bar{t}(z) \leq d_2 \bar{t}''(z) - c\bar{t}'(z) + \frac{\sigma \bar{s}(z) \bar{t}(z)}{\bar{s}(z) + \bar{t}(z) + F(\bar{t})(z)} - (\mu + \gamma) \bar{t}(z)
\]
\[
\leq e^{\lambda_{2,1}z}[d_2 \lambda_{2,1}^2 - c\lambda_{2,1} + \sigma - (\mu + \gamma)] = 0.
\]

(3) If \( s(z) = S_0 > \frac{\Lambda}{\mu} - Ke^{\lambda_{1}z} \), then
\[
d_1 s''(z) - c s'(z) + \Lambda - \mu s(z) - \frac{\sigma s(z) \bar{t}(z)}{s(z) + \bar{t}(z) + F(\bar{t})(z)} = \Lambda - \mu s(z) - \frac{\sigma s(z) \bar{t}(z)}{s(z) + \bar{t}(z) + F(\bar{t})(z)}
\]
\[ \geq \lambda - \mu s_i(z) - \frac{\sigma s_i(z)\tilde{t}(z)}{s_i(z) + i(z)} \]
\[ \geq \lambda - \mu s_0 - \frac{\sigma s_0\tilde{t}}{s_0 + \tilde{t}} \]
\[ = 0. \]

When \( s(z) = \frac{\lambda}{\mu} - Ke^{\lambda_1 z} > S_0 \), then \( z < 0 \) by \( K > \tilde{S} \) and it is differentiable such that

\[ d_1 s''(z) - c_1'(z) + \lambda - \mu s_i(z) - \frac{\sigma s_i(z)\tilde{t}(z)}{s_i(z) + i(z) + F(\tilde{t})} \]
\[ \geq d_1 s''(z) - c_1'(z) + \lambda - \mu s_i(z) - \frac{\sigma s_i(z)\tilde{t}(z)}{s_i(z) + i(z)} \]
\[ \geq d_1 s''(z) - c_1'(z) + \lambda - \mu s_i(z) - \sigma \tilde{t}(z) \]
\[ = Ke^{\lambda_1 z}(-d_1 \lambda_1^2 + c\lambda_1 + \mu) - \sigma \tilde{t}(z) \]
\[ \geq Ke^{\lambda_1 z}(-d_1 \lambda_1^2 + c\lambda_1 + \mu) - \sigma e^{\lambda_1 z} \]
\[ \geq 0. \]

(4) When \( \tilde{t}(z) = 0 > e^{\lambda_2,1 z} - qe^{(\lambda_2,1 + \rho)z} \), it is clear that

\[ d_2 s''(z) - c_2'(z) + \frac{\sigma s(z)\tilde{t}(z)}{s(z) + \tilde{t}(z) + F(\tilde{t})} - (\mu + \gamma)\tilde{t}(z) = 0. \]

If \( \tilde{t}(z) = e^{\lambda_2,1 z} - qe^{(\lambda_2,1 + \rho)z} > 0 \), then \( q > 1 \) implies \( z < 0 \) and it is differentiable such that

\[ d_2 s''(z) - c_2'(z) + \frac{\sigma s(z)\tilde{t}(z)}{s(z) + \tilde{t}(z) + F(\tilde{t})} - (\mu + \gamma)\tilde{t}(z) \]
\[ \geq d_2 s''(z) - c_2'(z) + \tilde{t}(z)[\sigma - (\mu + \gamma) - \sigma L\tilde{t}(z) - \sigma LF(\tilde{t})(z)] \]
\[ \geq d_2 s''(z) - c_2'(z) + \tilde{t}(z)[\sigma - (\mu + \gamma)] - \sigma L\tilde{t}(z) - \sigma LF(\tilde{t})(z) \]
\[ = -qe^{(\lambda_2,1 + \rho)z}[d_2(\lambda_2,1 + \rho)^2 - c(\lambda_2,1 + \mu) + \sigma - (\mu + \gamma)] - \sigma Le^{\lambda_2,1 z} - \sigma L\Pi e^{(\lambda_2,1 + \pi)z} \]
\[ \geq -qe^{(\lambda_2,1 + \rho)z}[d_2(\lambda_2,1 + \rho)^2 - c(\lambda_2,1 + \mu) + \sigma - (\mu + \gamma)] - \sigma (L + \Pi)e^{(\lambda_2,1 + \rho)z} \]
\[ \geq 0. \]

The proof is completed. \( \square \)

**Lemma 4.6.** Let \( c = c^* \), then (11) has a strictly positive solution such that

\[ \lim_{z \to -\infty} s(z) = \frac{\Lambda}{\mu}, \lim_{z \to -\infty} i(z) = 0. \]

**Proof.** By Lemma 4.4, we construct proper generalized upper and lower solutions and introduce some constants.

(S1): let \( a = \frac{\mu}{Ks} \) as that in Section 2;

(S2): fix \( \lambda_1 \in (0, a/2) \) such that \(-d_1 \lambda_1^2 + c\lambda_1 + \mu > 0\);

(S3): select \( J > \tilde{I} + 1 \) such that \(-1 + J)e^{-a} > \tilde{I} \), and denote \( z_1 < -1 \) such that \((-z_1 + J)e^{az} = \tilde{I}, (z - z_1)e^{az} < \tilde{I}, z < z_1\);

(S4): fix \( \pi \in (0,a) \) such that \( F(e^{\pi})(0) < \infty \), and select \( \Pi_1 > 0, \Pi > 0 \) such that

\[ \Pi_1 e^{\pi z} > (-z + J)e^{az}, z \leq z_1, F(\Pi_1 e^{\pi})(0) \leq \Pi; \]
(S5): fix \( \lambda_2 \in (a/2, a) \) such that \( \lambda_2 + \pi > a; \)
(S6): choose \( M > 0 \) such that \((-z + J)e^{az} \leq Me^{\lambda_2 z}, \ z \leq -1; \)
(S7): select \( K > 0 \) such that \( K = \hat{S} + \frac{\lambda_2}{d_1\lambda_1 + c\lambda_1 + \mu}; \)
(S8): select \( q > 1 \) such that
\[
\frac{d_2 q}{4\sqrt{(-z)^3}}e^{az} - \sigma LM^2e^{2\lambda_2 z} - \sigma LMI e^{(\lambda_2 + \pi)z} \geq 0, \ z < -1;
\]
and
\[
\min\{\hat{I}, (-z + J)e^{az}\} > \max\{0, -ze^{az} - q\sqrt{(-z)e^{az}}\}, \ z < -q^2.
\]
With these constants, we define
\[
s(z) = \hat{S}, \ s(z) = \max\{S_0, \hat{S} - Ke^{\lambda_1 z}\}
\]
and
\[
\tilde{I}(z) = \begin{cases} \hat{I}, & z \geq z_1, \\ (-z + J)e^{az}, & z \leq z_1, \end{cases}
\]
\[
\tilde{I}(z) = \begin{cases} -ze^{az} - q\sqrt{(-z)e^{az}}, & z \leq -q^2; \\ 0, & z \geq -q^2. \end{cases}
\]

We now verify they are a pair of upper and lower solutions by four steps.

(1) For \( s(z) = \hat{S} = \frac{A}{\mu} \), the proof is the same as that in the proof of Lemma 4.5.

(2) If \( \tilde{I}(z) = \hat{I} \) and it is differentiable, then the proof is the same as that in the proof of Lemma 4.5. If \( \tilde{I}(z) = (-z + J)e^{az} \) is differentiable, then
\[
d_2 \tilde{I}''(z) - c\tilde{I}'(z) + \frac{\sigma s(z)\tilde{I}(z)}{s(z) + \tilde{I}(z) + F(\tilde{I})(z)} - (\mu + \gamma)\tilde{I}(z)
\]
\[
\leq d_2 \tilde{I}''(z) - c\tilde{I}'(z) + \sigma \tilde{I}(z) - (\mu + \gamma)\tilde{I}(z)
\]
\[
= Je^{az}[d_2a^2 - ca + \sigma - (\mu + \gamma)] + e^{az}[-2d_2a + c - z(d_2a^2 - ca + \sigma - \mu - \gamma)]
\]
\[
= 0.
\]

(3) If \( s(z) = S_0 > \frac{A}{\mu} - Ke^{\lambda_1 z}, \) the proof is the same as that in the proof of Lemma 4.5. When \( s(z) = \frac{A}{\mu} - Ke^{\lambda_1 z} > S_0, \) then \( z < 0 \) such that
\[
d_1 s''(z) - c_s'(z) + \Lambda - \mu s(z) - \frac{\sigma s(z)\tilde{I}(z)}{s(z) + \tilde{I}(z) + F(\tilde{I})(z)}
\]
\[
\geq d_1 s''(z) - c_s'(z) + \Lambda - \mu s(z) - \sigma \tilde{I}(z)
\]
\[
= Ke^{\lambda_1 z}(-d_1\lambda_1 + c\lambda_1 + \mu) - \sigma \tilde{I}(z)
\]
\[
\geq Ke^{\lambda_1 z}(-d_1\lambda_1 + c\lambda_1 + \mu) - \sigma Me^{\lambda_2 z}
\]
\[
\geq Ke^{\lambda_1 z}(-d_1\lambda_1 + c\lambda_1 + \mu) - \sigma Me^{\lambda_1 z}
\]
\[
\geq 0.
\]

(4) When \( \tilde{I}(z) = 0 \) is differentiable, the result is clear. If \( \tilde{I}(z) = -ze^{az} - q\sqrt{(-z)e^{az}} > 0, \) then \( z < -q^2 \) implies that \( z < -1 \) and it is differentiable such that
\[
\tilde{I}'(z) = a(-z - q\sqrt{-z})e^{az} - e^{az} + \frac{q}{2\sqrt{-z}}e^{az},
\]
\[
\tilde{I}''(z) = a^2(-z - q\sqrt{-z})e^{az} - 2ae^{az} + \frac{qa}{\sqrt{-z}}e^{az} + \frac{q}{4\sqrt{(-z)^3}}e^{az}.
\]
By direct calculation, we have
\[
d_{2}^{ii}(z) - c_{i}^{ii}(z) + \frac{\sigma s(z)i(z)}{s(z) + i(z) + F(i)(z)} - (\mu + \gamma)i(z)
\]
\[
\geq d_{2}^{ii}(z) - c_{i}^{ii}(z) + \frac{\sigma i(z)}{1 + i(z)/S_{0} + F(i)(z)/S_{0}} - (\mu + \gamma)i(z), \quad z \in \mathbb{R}.
\]
Let \(\beta > 0\) be a constant such that \(\frac{\beta}{1 + 2\gamma} > \mu + \gamma\). If \(F(i)(z)/S_{0} \leq \beta\), then
\[
d_{2}^{ii}(z) - c_{i}^{ii}(z) + \frac{\sigma i(z)}{1 + \beta i(z)/S_{0}} - (\mu + \gamma)i(z) \leq 0.
\]
If \(F(i)(z)/S_{0} > \beta\), then the convergence of \(F\) and the uniform continuity from (F1, F2) imply that there exist \(i_{0} > 0, z_{1} > 0, z_{2} > 0\) independent on \(z\) and \(z_{0} \in [z - z_{1}, z + z_{1}]\) such that
\[
i(z') > i_{0}, z' \in [z_{0} - z_{2}, z_{0} + z_{2}].
\]
Note that a traveling wave solution is a special entire solution, then similar to that in (10), we can select a constant \(i_{1} > 0\) independent on \(z\) such that
\[
i(z) > i_{1}.
\]
Due to the uniform boundedness of \(F(i)(z)/S_{0}\), we can select a constant \(M > 0\) such that
\[
d_{2}^{ii}(z) - c_{i}^{ii}(z) + \frac{\sigma i(z)}{1 + \beta + Mi(z)} - (\mu + \gamma)i(z) \leq 0, \quad z \in \mathbb{R}.
\]
Since \(i(z) = I(x, t)\) also satisfies
\[
\begin{align*}
\frac{\partial I(x,t)}{\partial t} & \geq d_{2}I(x,t) + \frac{\sigma I(x,t)}{1 + \beta + Mi(x,t)} - (\mu + \gamma)I(x,t), \\
I(x, 0) & = i(x), \quad x \in \mathbb{R}, \quad t > 0,
\end{align*}
\]
then Lemma 3.2 implies
\[
\lim_{t \to \infty} \inf I(0,t) \geq i_{2} > 0
\]
with \( \sigma^{1+\beta+\frac{M}{1+M_{t}}} = (\mu + \gamma) \), which further implies that \( \liminf_{z \to \infty} i(z) \geq i_2 \). The proof is completed.

5. **Numerical simulations.** In this section, we give two examples to illustrate our main results and explore further propagation dynamics of this system. Denote the level set by

\[
L_u^u(\alpha) = \inf \{ x \in \mathbb{R} : u(x, t) = \alpha \}, u \in \{ I, Q, R \}.
\]

**Example 1.** We first simulate our main conclusion and consider the following initial value problem

\[
\begin{align*}
\frac{\partial S(x, t)}{\partial t} &= \Delta S(x, t) + 1 - S(x, t) - \frac{2S(x, t)I(x, t)}{S(x, t) + I(x, t) + R(x, t)}, \\
\frac{\partial I(x, t)}{\partial t} &= \Delta I(x, t) + 2S(x, t)I(x, t) - 1.5I(x, t), \\
\frac{\partial Q(x, t)}{\partial t} &= 2\Delta Q(x, t) + 0.5I(x, t) - 1.5Q(x, t), \\
\frac{\partial R(x, t)}{\partial t} &= 2\Delta R(x, t) + 0.5Q(x, t) - R(x, t),
\end{align*}
\]

(15)

when

\[
I(x, 0) = I_0, Q(x, 0) = Q_0, R(x, 0) = R_0, x \in \mathbb{R}, t > 0
\]

\[
S(x, 0) = 1, I(x, 0) = I(x), Q(x, 0) = Q(x), R(x, 0) = R(x), x \in \mathbb{R}, t > 0
\]

By our results, the threshold in (15) is \( c^* = \sqrt{2} \approx 1.414 \). We show the spatial-temporal plots of unknown functions as that in Figure 1, from which we see \( I, Q, R \) invade the habitat almost in a constant speed. To further estimate the invasion speed, we compare some level sets in Figure 2, by which we observe the invasion speed approximating \( \sqrt{2} \) (see Table 1). Moreover, for such a model, the local convergence to the constant steady state is possible.

![Figure 1](image1.png)

**Figure 1.** Spatial-temporal plots of (15).

**Table 1.** Approximate level sets in Figures 2.

| Level sets | \( L^I_t(0.1) \) | \( L^I_t(0.02) \) | \( L^R_t(0.02) \) |
|------------|------------------|------------------|------------------|
| \( t = 80 \) | -105             | -107.2           | -102.2           |
| \( t = 100 \) | -133.8           | -134.6           | -130.2           |
| Averaging moving speed of level sets | 1.42 | 1.37 | 1.40 |
Example 2. We now explore the role of noncooperative properties in this system. Further simulate the following initial value problem

\[
\begin{align*}
\frac{\partial S(x,t)}{\partial t} &= \Delta S(x,t) + 1 - 0.1S(x,t) - \frac{2S(x,t)I(x,t)}{S(x,t)+I(x,t)+R(x,t)}, \\
\frac{\partial I(x,t)}{\partial t} &= \Delta I(x,t) + \frac{2S(x,t)I(x,t)}{S(x,t)+I(x,t)+R(x,t)} - 1.6I(x,t), \\
\frac{\partial Q(x,t)}{\partial t} &= 2\Delta Q(x,t) + 1.5I(x,t) - 1.6Q(x,t), \\
\frac{\partial R(x,t)}{\partial t} &= 2\Delta R(x,t) + 1.5Q(x,t) - 0.1R(x,t), \\
S(x,0) &= 10, I(x,0) = I(x), Q(x,0) = R(x,0) = 0, x \in \mathbb{R}, t > 0
\end{align*}
\]

(17)

when \(I(x)\) is defined by (16).

For this system (17), we do not estimate the spreading speed, and only observe the possible oscillation. We show the spatial-temporal plots of unknown functions in Figure 3, from which we see \(S, I, Q, R\) may vibrate and have complex dynamics.

6. Discussion. For the noncooperative system (1), Feng and Thieme [5] obtained the basic reproduction ratio. In this paper, we consider the reaction-diffusion system (2). Our results imply that the persistence or extinction of the disease is independent on the spatial variable by (9). However, if the disease could spread successfully, then the expansion speed only depends on the diffusive parameter of the infected. Moreover, from our study on initial value problem, we can obtain several different factors that determine the spreading speed. In particular, the diffusive parameters of
$S, Q, R$ do not change the spreading speed, and the threshold is monotone decreasing in the rate at which individuals leave the infective and enter the isolated classes.

In this paper, we only study the simple limit behavior formulating the spreading or vanishing of the disease. Because (2) is not a cooperative system, its dynamics is very plentiful, see [5, 26] for the role of isolation period $1/\xi$, which indicates that increasing the length of the isolation period makes the endemic equilibrium less stable and even leads to its instability with a simultaneous rise of stable periodic oscillations. If the length of the isolation period is further increased, however, the endemic equilibrium gains its stability back at long isolation periods. By our analysis on $F$, we conjecture that if $\frac{\gamma \xi}{\mu (\mu + \xi)} - 1 > 0$ is large, then the oscillation is possible due to the dynamics of the corresponding functional differential equation [8]. The oscillation was observed numerically in Figure 2. The long time behavior is useful to understand the final size of the infected, and it is necessary to further study the convergence results on asymptotic spreading and traveling wave solutions.

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