MOTIVES AND ORIENTED COHOMOLOGY OF A LINEAR ALGEBRAIC GROUP

ALEXANDER NESHITOV

Abstract. For a cellular variety $X$ over a field $k$ of characteristic 0 and an algebraic oriented cohomology theory $h$ of Levine-Morel we construct a filtration on the cohomology ring $h(X)$ such that the associated graded ring is isomorphic to the Chow ring of $X$. Taking $X$ to be the variety of Borel subgroups of a split semisimple linear algebraic group $G$ over $k$ we apply this filtration to relate the oriented cohomology of $G$ to its Chow ring. As an immediate application we compute the algebraic cobordism ring of a group of type $G_2$ and of some other groups of small ranks, hence, extending several results by Yagita.

Using this filtration we also establish the following comparison result between Chow motives and $h$-motives of generically cellular varieties: any irreducible Chow-motivic decomposition of a generically split variety $Y$ gives rise to a $h$-motivic decomposition of $Y$ with the same generating function. Moreover, under some conditions on the coefficient ring of $h$ the obtained $h$-motivic decomposition will be irreducible. We also prove that if Chow motives of two twisted forms of $Y$ coincide, then their $h$-motives coincide as well.

1. Introduction

We work over the base field $k$ of characteristic 0. For an algebraic oriented cohomology theory $h$ of Levine-Morel [11] and a cellular variety $X$ of dimension $N$ we construct a filtration

$$h(X) = h^0(X) \supseteq h^1(X) \supseteq \ldots \supseteq h^N(X) \supseteq 0$$

on the cohomology ring such that the associated graded ring

$$Gr^* h(X) = \bigoplus_{i \geq 0} h^{(i)}(X)/h^{(i+1)}(X)$$

is isomorphic (as a graded ring) to the Chow ring $CH^*(X, \Lambda)$ of algebraic cycles modulo rational equivalence relation with coefficients in a ring $\Lambda$. We exploit this filtration and isomorphism in two different contexts:

First, we consider the (cellular) variety $X = G/B$ of Borel subgroups of a split semisimple linear algebraic group $G$ over $k$. By [7, Prop. 5.1] the cohomology ring $h(G)$ can be identified with a quotient of $h(G/B)$, so there is an induced filtration on $h(G)$. One of our key results (Prop. 4.3) shows that $CH^*(G, \Lambda)$ covers the associated graded ring $Gr^* h(G)$ and describes the kernel of this surjection. As an immediate application for $h = \Omega$ (the algebraic cobordism of Levine-Morel) we compute the cobordism ring for groups $G_2$, $SO_3$, $SO_4$, $Spin_n$ for $n = 3, 4, 5, 6$ and $PGL_n$ for $n \geq 2$, in terms of generators and relations, hence, extending several previously known results by Yagita [19]; as an application for $h = K_0$ (the
Grothendieck $K_0$) we construct certain elements in the difference between topological and the Grothendieck $\gamma$-filtration on $K_0(X)$, hence, extending some of the results by Garibaldi-Zainoulline [6].

The second deals with the study of $h$-motives of generically cellular varieties. The latter is a natural generalization of the notion of the Chow motives to the case of an arbitrary algebraic oriented cohomology theory of Levine-Morel. It was introduced and studied by Nenashev-Zainoulline [13] and Vishik-Yagita in [17].

Let $\Lambda$ denote the coefficient ring of $h$ and let $\Lambda^i$ denote its $i$-th graded component. We prove the following theorem which relates $h$-motives of generically cellular varieties to its Chow motives:

**Theorem A.** Let $X$ be a generically cellular variety over $k$, i.e. cellular over the function field $k(X)$. Assume that the Chow motive of $X$ with coefficients in $\Lambda^0$ splits as

$$M^{CH}(X, \Lambda^0) = \bigoplus_{i \geq 0} R(i)^{\oplus c_i}, \ c_i \geq 0,$$

for some motive $R$ which splits as a direct sum of twisted Tate motives $\overline{R} = \bigoplus_{j \geq 0} \Lambda^0(j)^{\oplus d_j}$ over its splitting field.

Then the $h$-motive of $X$ (with coefficients in $\Lambda$) splits as

$$M^h(X) = \bigoplus_{i \geq 0} R_h(i)^{\oplus c_i}$$

for some motive $R_h$ and over the same splitting field $R_h$ splits as a direct sum of twisted $h$-Tate motives $\overline{R}_h = \bigoplus_{j \geq 0} \Lambda(j)^{\oplus d_j}$.

This result can also be derived from the arguments of [17] where it is proved that sets of isomorphism classes of objects of category of Chow motives and $\Omega$-motives coincide. However, our approach gives more explicit correspondence between idempotents defining the (Chow) motive $R$ and the $h$-motive $R_h$. The latter allows us to prove the following result concerning the indecomposability of the $h$-motive $R_h$:

**Theorem B.** Assume that $\Lambda^1 = \ldots = \Lambda^N = 0$, where $N = \dim X$.

If the Chow motive $R$ is indecomposable (over $\Lambda^0$), then the $h$-motive $R_h$ is indecomposable (over $\Lambda$).

and also the following comparison property:

**Theorem C.** Suppose that $X,Y$ are generically cellular and $Y$ is a twisted form of $X$, i.e. $Y$ becomes isomorphic to $X$ over some splitting field. If $M^{CH}(X, \Lambda^0) \cong M^{CH}(Y, \Lambda^0)$, then $M^h(X) \cong M^h(Y)$.

The paper is organized as follows: In section 2 we recall concepts of an algebraic oriented cohomology theory $h$ of Levine-Morel and the corresponding category of $h$-motives. In section 3 we introduce the filtration on the cohomology ring $h(X)$ of a cellular variety $X$ which plays a central role in the paper. In section 4 we apply the filtration to obtain several comparison results between $CH(G)$ and $h(G)$. In particular, in section 5 we compute the algebraic cobordism $\Omega$ for some groups of small ranks and construct explicit elements in the difference of between the topological and the $\gamma$-filtration on $K_0(G/B)$. In section 6 we apply the filtration to obtain comparison results between $h$-motives and Chow-motives of generically split varieties.
2. Preliminaries

In the present section we recall notions of an algebraic oriented cohomology theory, a formal group law and of a cellular variety. We recall the definition of the category of smooth varieties over Spec $k$, denoted by $\text{Sm}_k$.

**Oriented cohomology theories.** The notion of an algebraic oriented cohomology theory was introduced by Levine-Morel [11] and Panin-Smirnov [14]. Let $\text{Sm}_k$ denote the category of smooth varieties over Spec $k = \text{pt}$. An algebraic oriented cohomology theory $h^*$ is a functor from $\text{Sm}_k^{op}$ to the category of graded rings. We will denote by $f^* : h^*(Y) \to h^*(X)$ the induced morphism $f : X \to Y$ and call it the pullback of $f$. By definition, the functor $h^*$ is equipped with the pushforward map $f_* : h^*(X) \to h^!(\dim X - \dim X + \nu(Y))$ for any projective morphism $f : X \to Y$. These two structures satisfy the axioms of [11, Def. 1.1.2]. We denote its coefficient ring $h_*(pt)$ by $\Lambda^*$. As for the Chow groups, we will also use the lower grading for $h$, i.e. $h_*(X) = h^\dim X^{−i}(X)$ for an irreducible variety $X$.

**Formal group law.** For an oriented cohomology theory $h^*$ there is a notion of the first Chern class of a line bundle. For $X \in \text{Sm}_k$ and a line bundle $L$ over $X$ it is defined as $c_1^L(L) = z^2 c_2(1) \in h^1(X)$ where $z : X \to L$ is a zero section. There is a commutative associiative 1-dimensional formal group law $F$ over $\Lambda^*$ such that for any two line bundles $L_1, L_2$ over $X$ we have $c_1^L(L_1 \otimes L_2) = F(c_1^L(L_1), c_1^L(L_2))$ [11 Lem. 1.1.3]. We will use the notation $x +_F y$ for $F(x, y)$. For any $x$ we will denote by $−_F x$ the unique element such that $x +_F (−_F x) = 0$. For any $n \in \mathbb{Z}$ we will denote by $n \cdot_F x$ the expression $x +_F \ldots +_F x (n$ times) if $n$ is positive, and $(−_F x) +_F \ldots +_F (−_F x) (−n$ times) if $n$ is negative. By [11] there is a natural transformation that commutes with pushforwards:

$$\nu_X : \Omega^*(X) \otimes_{\text{L}*} \Lambda^* \to h^*(X),$$

where $\text{L} = \Omega(pt)$ is the Lazard ring and the map $\text{L}^* \to \Lambda^*$ is obtained by specializing the coefficients of the universal formal group law to the coefficients of the corresponding $F$.

**Cellular and generically cellular varieties.** A variety $Y \in \text{Sm}_k$ is called cellular if there is a filtration of $Y = Y_0 \supseteq Y_1 \supseteq \ldots \supseteq Y_m \supseteq \emptyset$ such that each $Y_i \setminus Y_{i+1}$ is a disjoint union of affine spaces of the same rank $c_i : Y_i \setminus Y_{i+1} \cong \mathbb{A}^{c_i}_k \bigsqcup \ldots \bigsqcup \mathbb{A}^{c_i}_k$.

We call a variety $X$ generically cellular if $X_k(X)$ is a cellular variety over the function field $k(X)$.

2.1. Example. Let $G$ be a split semisimple algebraic group, $B$ its Borel subgroup containing a fixed maximal split torus $T$ and $W$ the corresponding Weyl group. For any $w \in W$ let $l(w)$ denote its length. Let $w_0 \in W$ denote the longest element of $W$ and $N = l(w)$. It is well known that the flag variety $X = G/B$ has the cellular structure given by the Schubert cells $X_w$:

$$X = X_{w_0} \supseteq \bigcup_{l(w) = N-1} X_w \supseteq \bigcup_{l(w) = N-2} X_w \supseteq \ldots \supseteq \bigcup_{l(w) = 1} X_w = pt,$$
where \(X_w\) is the closure of \(BwB/B\) in \(X\).

2.2. Example. Let \(\zeta \in Z^1(k, G)\) be a 1-cocycle with values in \(G\). Then the twisted form \(\zeta(G/B)\) of \(X = G/B\) provides an example of a generically split variety.

\(h\)-motives. The notion of \(h\)-motives for the algebraic oriented cohomology theory \(h\) was studied by Nenashev-Zainoulline in [13], and Vishik-Yagita in [17]. We refer to [17, §2] for definition of the category of effective \(h\)-motives. In the present paper we will deal with the category of \(h\)-motives \(\mathcal{M}_h\) with the inverted Tate object. It is constructed as follows:

Let \(\text{SmProj}_k\) denote the category of smooth projective varieties over \(k\). Following [5] we consider the category \(\text{Corr}_h\) defined as follows: For \(X, Y \in \text{SmProj}_k\) with irreducible \(X\) and \(m \in \mathbb{Z}\) we set

\[
\text{Corr}_m(X, Y) = h_{\dim X + m}(X \times Y).
\]

Objects of \(\text{Corr}_h\) are pairs \((X, i)\) with \(X \in \text{SmProj}_k\) and \(i \in \mathbb{Z}\). For \(X \in \text{SmProj}_k\) with irreducible components \(X_i\) define the morphisms

\[
\text{Hom}_{\text{Corr}}((X, i), (Y, j)) = \prod_i \text{Corr}_{i-j}(X_i, Y).
\]

For \(\alpha \in \text{Hom}((X, i), (Y, j))\) and \(\beta \in \text{Hom}((Y, j), (Z, k))\) the composition is given by the usual correspondence product: \(\alpha \circ \beta = (p_{XZ})_h((p_{YZ})^h(\beta) \cdot (p_{XY})^h(\alpha))\), where \(p_{XY}, p_{YZ}, p_{XZ}\) denote the projections from \(X \times Y \times Z\) onto the corresponding summands.

Taking consecutive additive and idempotent completion of \(\text{Corr}_h\) we obtain the category \(\mathcal{M}_h\) of \(h\)-motives with inverted Tate object. Objects of this category are \((\prod_i (X_i, n_i), p)\) where \(p\) is a matrix with entries \(p_{i,j} \in \text{Corr}_{n_i-n_j}(X_i, X_j)\) such that \(p \circ p = p\). Morphisms between objects are given by the set

\[
\text{Hom}((\prod_i (X_i, n_i), p), (\prod (Y_j, m_j), q)) = q \circ \bigoplus_{i,j} \text{Corr}_{n_i-m_j}(X_i, Y_j) \circ p
\]

considered as a subset of \(\bigoplus_{i,j} \text{Corr}_{n_i-m_j}(X_i, Y_j)\). This is an additive category where each idempotent splits. There is a natural tensor structure inherited from the category \(\text{Corr}_h\):

\[
(\prod_i (X_i, n_i), p) \otimes (\prod (Y_j, m_j), q) = (\prod_i (X_i \times Y_j, n_i + n_j), p \times q)
\]

where \(p \times q\) denotes the projector \(p_{((i_1, j_1), (i_2, j_2))} = p_{i_1, j_1} \times q_{i_2, j_2}: X_{i_1} \times Y_{j_1} \to X_{i_2} \times Y_{j_2}\).

There is a functor \(M^h: \text{SmProj}_k \to \mathcal{M}_h\) that maps a variety \(X\) to the motive \(M^h(X) = ((X, 0), id_X)\) and any morphism \(f: X \to Y\) to the correspondence \((\Gamma_f)_h(1) \in h_{\dim X}(X \times Y) = \text{Corr}_0(X, Y)\), where \(\Gamma_f: X \to X \times Y\) is the graph inclusion. We will denote by \(\Delta: X \to X \times X\) the diagonal embedding. Then \(\Delta_0(1)\) is the identity in \(\text{Corr}_0(X, X)\).

Denote \(M^h(pt)\) by \(\Lambda\) and \(((pt, 1), id_w)\) by \(\Lambda(1)\). We call \(\Lambda(1)\) the \(h\)-Tate motive. We write \(\Lambda(n)\) for \(\Lambda(1)^{\otimes n}\) and \(M^h(X)(n)\) for \(M^h(X) \otimes \Lambda(n)\). The motive \(M^h(X)(n)\) is called the \(n\)-th twist of the motive \(M^h(X)\).

By definition we have

\[
h^i(X) = \text{Hom}_{\mathcal{M}_h}(M^h(X), \Lambda(i))\text{ and } h_i(X) = \text{Hom}_{\mathcal{M}_h}(\Lambda(i), M^h(X)).
\]
2.3. Lemma. For \( X \in \text{SmProj}_k \) with the structure morphism \( \pi: X \to \text{pt} \) any isomorphism \( M^p(X) \cong \bigoplus_i \Lambda(\alpha_i) \) corresponds to a choice of two \( \Lambda \)-basis sets

\[
\{ \tau_i \in h^{\alpha_i}(X) \}_i \text{ and } \{ \zeta_i \in h_{\alpha_i}(X) \}_i,
\]

such that \( \pi_h(\tau_i \zeta_i) = \delta_{i,j} \) in \( \Lambda \) and \( \sum_i \zeta_i \otimes \tau_i = \Delta_h(1) \) in \( h(X \times X) \).

Proof. In the direct sum decomposition \( M^h(X) \cong \bigoplus_i \Lambda(\alpha_i) \) the \( i \)-th projection \( p_i: M^h(X) \to \Lambda(\alpha_i) \) is defined by an element \( \tau_i \in h^{\alpha_i}(X) \) and the \( i \)-th inclusion \( \iota_i: \Lambda(\alpha_i) \to h(X) \) is defined by an element \( \zeta_i \in h_{\alpha_i}(X) \). Then by definition of the direct sum we obtain

\[
\pi_h(\tau_i \zeta) = \delta_{i,j} \text{ and } \sum_i \zeta_i \otimes \tau_i = \Delta_h(1).
\]

Let us check that \( \{ \zeta_i \}_i \) form a basis of \( h(X) \). Indeed, we have \( h(X) = \bigoplus_j h^j(X) = \bigoplus_j \text{Hom}_{M_k}(M^h(X), \Lambda(j)) \cong \bigoplus_j \text{Hom}_{M_k}(\bigoplus_i \Lambda(\alpha_i), \Lambda(j)) = \bigoplus_j \Lambda_{\alpha_i \to \alpha_j} \) and \( \zeta_i \) are the images of standard generators. So \( \{ \zeta_i \}_i \) form a \( \Lambda \)-basis of \( h(X) \). Finally, since \( \{ \tau_i \}_i \) are dual to \( \{ \zeta_i \}_i \), \( \{ \tau_i \}_i \) is also a basis. \( \square \)

2.4. Remark. Observe that any isomorphism \( M^h(X) \cong \bigoplus \Lambda(\alpha_i) \) gives rise (canonically) to an isomorphism \( h^*(X) \cong \bigoplus \Lambda^{*-\alpha_i} \).

3. Filtration on the cohomology ring

In the present section we construct a filtration on the oriented cohomology \( h(X) \) of a cellular variety \( X \) which will play an important role in the sequel.

3.1. Proposition. Assume that \( X \) is a cellular variety over \( k \) with the cellular decomposition \( X = X_0 \supseteq X_1 \supseteq \ldots \supseteq X_n \supseteq \emptyset \) where \( X_i \setminus X_{i+1} = \prod c_i \mathbb{A}^{c_i} \). Then

1. the \( h \)-motive of \( X \) splits as \( M^h(X) = \bigoplus_i \Lambda(\alpha_i)^{\otimes c_i} \);
2. the Künneth formula holds, i.e. the natural map \( h(X) \otimes \Lambda h(X) \to h(X \times X) \) is an isomorphism;
3. the specialization maps \( \nu_X: \Omega(X) \otimes \Lambda \to h(X) \) and \( \nu_{X \times X}: \Omega(X \times X) \otimes \Lambda \to h(X \times X) \) are isomorphisms.

Proof. By \( \text{CH} \) Cor. 66.4] the Chow motive \( M^\text{CH}(X) \) splits, then \( \text{CH} \) Cor. 2.9] implies that the motive \( M^{\Omega}(X) \) splits into a sum of twisted Tate motives \( M^{\Omega}(X) = \bigoplus_{i \in I} L(\alpha_i)^{\otimes c_i} \). By Lemma 2.3 there are elements \( \zeta_{i,j}^\Omega \in h_{\alpha_i}(X) \) and \( \tau_{i,j}^\Omega \in h^{\alpha_i}(X) \), \( j \in \{1, \ldots, c_i\} \) such that \( \pi^{\Omega}(\zeta_{i,j}^\Omega \cdot \tau_{i,j}^\Omega) = \delta_{(i,j), (i', j')} \) and \( \Delta^{\Omega}(1) = \sum_{i,j} \zeta_{i,j}^\Omega \otimes \tau_{i,j}^\Omega \). Denote \( \zeta_{i,j}^h = \nu(\zeta_{i,j}^\Omega \otimes 1) \) and \( \tau_{i,j}^h = \nu(\tau_{i,j}^\Omega \otimes 1) \). Since \( \nu \) commutes with pullbacks and push-forwards, \( \pi_h(\zeta_{i,j}^h \cdot \tau_{i,j}^h) = \delta_{(i,j),(i', j')} \) and \( \Delta_h(1) = \sum_{i,j} \zeta_{i,j}^h \otimes \tau_{i,j}^h \). Then by Lemma 2.3 we have \( M^h(X) = \bigoplus_i \Lambda(\alpha_i)^{\otimes c_i} \), so (1) holds.

The Künneth map fits into the diagram

\[
\begin{array}{ccc}
\nu_X: \Omega(X) \otimes \Lambda & \longrightarrow & h(X) \times h(X) \\
\bigoplus \Lambda^{*-\alpha_i} \otimes \bigoplus \Lambda^{*-\alpha_i} & \longrightarrow & \bigoplus_{i,j} \Lambda^{*-\alpha_i - \alpha_j}
\end{array}
\]

where the bottom arrow is an isomorphism, so the Künneth formula (2) holds.

Note that the natural map \( \nu_X \) can be factored as follows

\[
\nu_X: \Omega(X) \otimes \Lambda = \oplus_m \text{Hom}_{M^{\text{cha}}}(\bigoplus \Lambda(\alpha_i), \Lambda(m)) \to \text{Hom}_{M_k}(\bigoplus \Lambda(\alpha_i), \Lambda(m)) = h(X).
\]
Thus $\nu_X$ is an isomorphism. The same reasoning proves the statement for $\nu_{X \times X}$, hence, (3) holds.

3.2. **Definition.** Let $X$ be a cellular variety. Consider two basis sets $\zeta_i \in h_{a_i}(X)$ and $\tau_i \in h^{a_i}(X)$ provided by Proposition 3.1 and Lemma 2.3. We define the filtration $h^{(l)}(X)$ as the $\Lambda$-linear span

$$h^{(l)}(X) = \bigoplus_{\alpha_i \geq l} \Lambda \zeta_i = \bigoplus_{\alpha_i \geq l} \Lambda \tau_i.$$ 

We denote $h^{(l+1)}(X) = h^{(l)}(X)/h^{(l+1)}(X)$ and $Gr^{h} h(X) = \bigoplus_{l} h^{(l+1)}(X)$ to be the associated graded group. Lemma 3.3 implies that the latter is a graded ring.

3.3. **Remark.** In the case when the theory $h$ is generically constant and satisfies the localization property, the filtration introduced above coincides with the topological filtration on $h(X)$, i.e. with the filtration where the $l$-th term is generated over $\Lambda$ by classes $[Z \to X]$ of projective morphisms $Z \to X$ birational on its image and $\dim X - \dim Z \leq l$. This fact follows from the generalized degree formula [11, Thm. 4.4.7].

3.4. **Lemma.** $h^{(i_1)}(X) \cdot h^{(i_2)}(X) \subseteq h^{(i_1+i_2)}(X)$.

**Proof.** We have $\tau_i \tau_j \Omega = \sum_1 a_i \zeta_j$ in $\Omega(X)$ for some $a_i \in \mathbb{L}$. Then $\alpha_i + \alpha_j = \deg(a_i) + \alpha_i$. Since $\deg(a_i) \leq 0, \alpha_i + \alpha_i \geq \alpha_i + \alpha_j \geq l_1 + l_2$ for any nontrivial $a_i$. Since $\zeta_i = \nu(\zeta_i \otimes 1)$ we have $\zeta_i \zeta_j = \sum_1 (a_i \otimes 1)\zeta_j$ with $\alpha_i \geq \alpha_i + \alpha_j \geq l_1 + l_2$. So $\zeta_i \zeta_j \in h^{(i_1+i_2)}(X)$. □

3.5. **Proposition.** For a cellular $X$ there is a graded ring isomorphism:

$$\Psi: \bigoplus_{l=0}^N h^{(i/i+1)}(X) \to CH(X, \Lambda).$$

**Proof.** By Proposition 3.4 it is sufficient to prove the statement for $h = \Omega$. Observe that $\Omega^{(l+1)}(X)$ is a free $\mathbb{L}$-module with the basis $\tau_i \Omega + h^{(l+1)}(X)$ with $\alpha_i = l$ and $CH(X, \mathbb{L})$ is a free $\mathbb{L}$-module with basis $\tau_i^{CH}$ with $\alpha_i = l$. Thus the $\mathbb{L}$-module homomorphism $\Psi_l$ defined by

$$\Psi_l(\tau_i \Omega + h^{(l+1)}(X)) = \tau_i^{CH}$$

is an isomorphism.

Let us check that $\Psi = \bigoplus \Psi_l$ preserves multiplication. For any $i, j$ we have

$$\tau_i \Omega \tau_j = \sum_m a_m \tau_m^{\Omega}$$

for some $a_m \in \mathbb{L}$. Then for any $m$ we have $\deg(a_m) + \alpha_m = \alpha_i + \alpha_j$. Then in $h^{(\alpha_i+\alpha_j)}(X)$ we have

$$\tau_i \Omega \tau_j = \sum_{\alpha_m = \alpha_i + \alpha_j} a_m \tau_m^{\Omega} \text{ modulo } h^{(\alpha_i+\alpha_j+1)}(X)$$

Observe that $\mathbb{L}^0 = \mathbb{Z}$ and for all $a_m \in \mathbb{L}$ such that $\deg(a_m) < 0$ we have that $a_m \otimes 1 = 0$ in $\mathbb{Z}$. Thus tensoring (*) with $1_\mathbb{L}$ we get

$$\tau_i^{CH} \tau_j^{CH} = \sum_{\alpha_m = 0} (a_m \otimes 1)\tau_m^{CH}.$$
So \( \Psi_{\alpha_i + \alpha_j}(\tau_i^{\Omega} + h^{(\alpha_i + 1)}(X) \cdot \tau_j^{\Omega} + h^{(\alpha_j + 1)}(X)) = \tau_i^{\text{CH}} \cdot \tau_j^{\text{CH}} \). Hence, \( \Psi \) is a graded ring isomorphism.

3.6. Lemma. \( \Psi(\zeta^h + h^{(\alpha_i + 1)}(X)) = \zeta^i \).

Proof. It is sufficient to show the statement for \( h = \Omega^* \). Consider the expansion

\[
\zeta^i = \sum a_j \tau_j^{\Omega} \mod \Omega^{(N - \alpha_i + 1)}(X).
\]

Therefore, \( \Psi(\zeta^i + \Omega^{(N - |w| + 1)}(X)) = \zeta^i \). \( \square \)

4. ORIENTED COHOMOLOGY OF A GROUP

In this section we assume that the associated to \( h \) weak Borel-Moore homology theory satisfies the localization property of [11, Definition 4.4.6]. Examples of such theories include \( h(-) = \Omega(-) \otimes \Lambda \), or any oriented cohomology theory in the sense of Panin-Smirnov [14].

Consider the variety \( X = G/B \) where \( G \) is a split semisimple algebraic group. Let \( \pi_{G/B}: G \to X \) be the quotient map. According to Example [22] \( X \) is cellular. For any \( w \in W \) we fix a minimal decomposition \( w = s_{i_1} \cdots s_{i_m} \) into simple reflections. Denote the corresponding multiindex by \( I_w = (i_1, \ldots, i_m) \) and consider the Bott-Samelson variety \( X_{I_w}/B \) [4, §11]. Then \( p_{I_w}: X_{I_w}/B \to G/B \) is a desingularisation of the Schubert cell \( X_w \). Take

\[
\zeta_w = (p_{I_w})_*(1) \in \mathfrak{h}_{l(w)}(X) = \text{Hom}_{\mathcal{M}_h}(\Lambda(l(w)), M^h(X))
\]

to be the embedding in the direct sum decomposition \( \bigoplus_{w \in W} \Lambda(l(w)) \cong M^h(G/B) \). So, with this choice of isomorphism \( M^h(G/B) \cong \bigoplus_{w \in W} \Lambda(l(w)) \) the basis given by Lemma [2, 3] coincides with the basis \( \zeta_{I_w} \) constructed in [4, §13].

Let \( \Lambda[[T^*]]_F \) be the formal group algebra introduced by Calmès-Petrov-Zainoulline in [4, §2], where \( T^* \) is the character lattice of \( T \) and \( F \) is the formal group law of the theory \( h \). There is the characteristic map \( \zeta_F: \Lambda[[T^*]]_F \to \mathfrak{h}(G/B) \) such that \( \zeta_F(x_\lambda) = \zeta^i(L(\lambda)) \) for a character \( \lambda \). By [4, Prop. 5.1] there is a short exact sequence

\[
0 \to c(I_F) \to \mathfrak{h}(G/B) \xrightarrow{\pi^h_{G/B}} \mathfrak{h}(G) \to 0,
\]

where \( I_F \) denotes the ideal in \( \Lambda[[T^*]]_F \) generated by \( x_\alpha \) for \( \alpha \in T^* \). By [4, Lem. 4.2] there is a graded algebras isomorphism \( \psi: \bigoplus_{m=0}^{\infty} I_F^m/I_F^{m+1} \to S^*_\Lambda(T^*) \) where \( S^*_\Lambda(T^*) \) denotes the symmetric algebra over \( T^* \). Let \( F^\alpha \) denote the additive formal group law. We will need the following
4.1. **Lemma.** The following diagram commutes:
\[
\begin{array}{ccc}
\bigoplus_{m=0}^{\infty} I_F^m/I_F^{m+1} & \xrightarrow{\psi} & \bigoplus_{m=0}^{\infty} h^{(m/m+1)}(G/B) \\
S^i_{\Lambda}(T^*) & \xrightarrow{\psi} & \text{CH}(G/B, \Lambda)
\end{array}
\]

**Proof.** By definition, it is sufficient to prove that \(\Psi(pr_1(c_1^L(\mathcal{L}_\Lambda))) = c_1^{CH}(\mathcal{L}_\Lambda)\) Consider the expansion \(c_1^L(\mathcal{L}_\Lambda) = \sum r_\omega \Omega\) we have that \(r_\omega \in \mathbb{L}^0 = \mathbb{Z}\) for \(|\omega| = 1\). Then \(c_1^L(\mathcal{L}_\Lambda) = \sum (r_\omega \otimes 1_\Lambda)r_\omega^0\) and
\[
\Psi(pr_1(c_1^L(\mathcal{L}_\Lambda))) = \sum_{|\omega| = 1} r_\omega \Omega = c_1^L(\Lambda) \otimes 1_\mathbb{Z} = c_1^{CH}(\mathcal{L}_\Lambda).
\]

4.2. **Lemma.** For the additive group law the induced filtration satisfies
\[
\epsilon(I_{F_z})\text{CH}(X, \Lambda) \cap \text{CH}^i(X, \Lambda) = \sum \epsilon(x_\alpha)\text{CH}^{i-1}(X, \Lambda).
\]

**Proof.** Note that \(\epsilon(I_a)\text{CH}(X, \Lambda)\) is generated by the elements \(\epsilon(x_\alpha) \in \text{CH}^1(X, R)\). For any element \(z = \sum \epsilon(x_\alpha)y_\alpha\) of \(\epsilon(I_a)\text{CH}(X, \Lambda)\) we have that \(z\) lies in the \(\text{CH}^i(X, \Lambda)\) if and only if \(y_\alpha \in \text{CH}^{i-1}(X, \Lambda)\). Therefore \(\epsilon(I_a)\text{CH}(X, \Lambda) \cap \text{CH}^i(X, \Lambda) = \sum \epsilon(x_\alpha)\text{CH}^{i-1}(X, \Lambda)\).

Let \(h^{i}(G)\) denote the image \(\pi_{G/B}^i(h^{i}(G/B))\) and let \(h^{(i,i+1)}(G)\) denote the quotient \(h^i(G)/h^{(i,i+1)}(G)\).

4.3. **Proposition.** For every \(i\) there is an exact sequence:
\[
0 \rightarrow \frac{\epsilon(I)h^{(i-1)}(X)}{h^{(i,i+1)}(X)} \rightarrow \frac{\epsilon(I)h(X)}{h^{(i)}(X)} \rightarrow \text{CH}^i(G, \Lambda) \rightarrow h^{(i,i+1)}(G) \rightarrow 0
\]

**Proof.** By [3] Prop. 2, §2.4] we obtain from [1] the short exact sequence:
\[
0 \rightarrow \frac{\epsilon(I)h(X) \cap h^{(i)}(X)}{h^{(i,i+1)}(X)} \rightarrow \frac{h^{(i)}(X)}{h^{(i,i+1)}(X)} \rightarrow h^{(i,i+1)}(G) \rightarrow 0.
\]

By Lemma [4,2] applied to the case of additive formal group law, the above sequence turns into
\[
0 \rightarrow \sum \epsilon(x_\alpha)\text{CH}^{i-1}(X, R) \rightarrow \text{CH}^i(X, R) \rightarrow \text{CH}^i(G, R) \rightarrow 0.
\]

Observe that for isomorphism \((\Psi^i)^{-1}\) we have
\[
(\Psi^i)^{-1}\left(\sum \epsilon(x_\alpha)\text{CH}^{i-1}(X, \Lambda)\right) = \left(\frac{\epsilon(I)h^{(i-1)}(X)}{h^{(i,i+1)}(X)}\right) \subseteq \left(\frac{\epsilon(I)h(X) \cap h^{(i)}(X)}{h^{(i,i+1)}(X)}\right)
\]

Then we get the following diagram with exact rows:
\[
\begin{array}{cccccc}
0 & \rightarrow & \sum \epsilon(x_\alpha)\text{CH}^{i-1}(X, \Lambda) & \rightarrow & \text{CH}^i(X, \Lambda) & \rightarrow & \text{CH}^i(G, \Lambda) & \rightarrow & 0 \\
& & \downarrow & & \Psi^i & & \downarrow (\Psi^i)^{-1}
\end{array}
\]
\[
\begin{array}{cccccc}
0 & \rightarrow & \frac{\epsilon(I)h(X) \cap h^{(i)}(X)}{h^{(i,i+1)}(X)} & \rightarrow & h^{(i,i+1)}(X) & \rightarrow & h^{(i,i+1)}(G) & \rightarrow & 0
\end{array}
\]

The latter sequence and (1) gives rise to the exact sequence
\[
0 \rightarrow \frac{\epsilon(I)h^{(i-1)}(X)}{h^{(i,i+1)}(X)} \rightarrow \frac{\epsilon(I)h(X) \cap h^{(i)}(X)}{h^{(i,i+1)}(X)} \rightarrow \text{CH}^i(G, \Lambda) \rightarrow h^{(i,i+1)}(G) \rightarrow 0.
\]
4.4. **Corollary.** Assume that pullbacks $p_{G/B}^*(X_w)$ of Schubert cells generate $\text{CH}(G)$ as $\mathbb{Z}$-algebra for some elements $w_1, \ldots, w_m$ in $W$. Then the pullbacks of Schubert cells $p_{G/B}^*(\zeta_w)$ generate $\mathfrak{h}(G)$ as $\Lambda$-algebra.

**Proof.** By classes of $p_{G/B}^*(\zeta_w)$ generate the associated graded ring $\bigoplus_i \mathfrak{h}^{(i+1)}(G)$. Then $p_{G/B}^*(\zeta_w)$ generate $\mathfrak{h}(G)$, since the filtration is finite. □

The following observations will be useful for computations

4.5. **Lemma.** If $\text{CH}^i(G) = 0$ then $c(I) \mathfrak{h}^{(i-1)}(G/B) / \mathfrak{h}^{(i+1)}(G/B)$.

4.6. **Lemma.** Assume that $\text{CH}^1(G) = 0$. Then for $i \geq 2$

$$c(I) \mathfrak{h}(G/B) \cap \mathfrak{h}^{(i)}(G/B) = c(I) \mathfrak{h}^{(1)}(G/B) \cap \mathfrak{h}^{(i)}(G/B)$$

**Proof.** Note that ideal $c(I) \mathfrak{h}(G/B)$ is generated by $c(x_\alpha)$ where $\alpha$ runs over the basis of the character lattice. □

5. **Examples of Computations**

The results of the previous section allow us to obtain some information concerning the ring $\mathfrak{h}(G)$ from $\text{CH}(G)$. Moreover, in some cases it allows us to compute $\mathfrak{h}(G)$.

We follow the notation of the previous section. We denote $\bigoplus \mathfrak{h}^{(i+1)}(G)$ by $\text{Gr}^* \mathfrak{h}(G)$. For $a \in \text{CH}^i(G)$ let $\overline{a} \in \mathfrak{h}^{(i+1)}(G)$ denote its residue class. Let $\alpha$ be the projection $\text{CH}^i(G, \Lambda) \to \text{Gr}^* \mathfrak{h}(G)$.

**Algebraic cobordism of $G_2$.** According to [12] we have

$$\text{CH}^*(G_2, \mathbb{Z}) = \mathbb{Z}[x_3]/(x_3^2, 2x_3)$$

where $x_3 = \pi^\text{CH}(\zeta_{212})$, $\pi : G \to G/B$ is the projection and $\zeta_{212}$ is the Schubert cell corresponding to the word $w = s_2s_1s_2$. Let $y_3$ denote the pullback $\pi^\Omega(\zeta_{212})$ of the corresponding Schubert cell in the ring $\Omega(G/B)$ (see Theorem 13.12 of [3]). Observe that $\alpha^3(x_3) = \overline{y_3}$. Since $\text{CH}(G_2, \mathbb{L})$ is generated by 1 and $x_3$, $\text{Gr}^* \Omega(G_2)$ is generated by 1 and $y_3$. Then by [3] $\Omega(G_2)$ is generated by 1 and $y_3 \in \Omega(3)(G_2)$. Since $2y_3 = 0$, then $2y_3$ so $2y_3 \in \Omega(4)(G_2)$ which is zero since $\text{CH}^4(G_2) = 0$ for $i > 4$. Thus $2y_3 = 0$ and $y_3^2 \in \Omega(6)(G_2) = 0$.

Let us now compute $\Omega^{(3)}(G_2)$. Proposition [4.3] gives us the exact sequence

$$0 \to \frac{c(I)\Omega^2(G_2/B)}{\Omega^3(G_2/B)} \to \frac{c(I)\Omega(G_2/B) \cap \Omega^3(G_2/B)}{\Omega^3(G_2/B)} \to \mathbb{L}/2 \cdot x_3 \to \Omega^3(G_2).$$

Note that since $c(I)\Omega(G/B)$ is generated by $c(x_1)$ and $c(x_2)$. By Lemma [4.5] we have

$$\frac{c(I)\Omega^2(G_2/B)}{\Omega^3(G_2/B)} = \frac{\langle \zeta_{1212}, \zeta_{21212} \rangle \Omega(G_2/B)}{\Omega^3(G_2/B)}$$

Then $c(I) \cap \Omega^3(G_2/B) / \Omega^3(G_2/B)$ equals to the set

$$\{ x = a_1\zeta_{1212} + b\zeta_{21212} \mid x \in \Omega^{(3)} \}.$$

It is enough to consider only $a, b$ in $\Omega^1(G_2/B) \setminus \Omega^2(G_2/B)$ since for $a, b \in \Omega^1(G_2/B) \setminus \Omega^2(G_2/B)$ we have $x \notin \Omega^2(G_2/B)$. So we consider

$$a = r_1\zeta_{1212} + r_2\zeta_{21212}$$

and

$$b = s_1\zeta_{1212} + s_2\zeta_{21212}$$

for $r_1, r_2, s_1, s_2 \in \mathbb{L}$. 

Using the multiplication table for $G_2/B$ from [4] we obtain that $x$ equals

$$(r_2 + s_1 + s_2)ζ_{1212} + (3r_1 + r_2 + s_1)ζ_{2121} + (r_2 + s_1 + 3r_1)a_1ζ_{1212} + (r_2 + s_1)a_1ζ_{2121}$$

modulo $Ω^{(4)}(G_2/B)$. Then $x \in Ω^{(3)}(G_2/B)/Ω^{(4)}(G_2/B)$ iff $r_2 + s_1 + s_2 = 0$ and $3r_1 + r_2 + s_1 = 0$. Therefore, $r_2 + s_1 = -3r_1$ and $s_2 = 3r_1$. So

$$x + Ω^{(4)}(G_2/B) = -3r_1a_1ζ_{2121} + Ω^{(4)}(G_2/B).$$

Hence, the kernel of $L ∙ 2x_3 \rightarrow Ω^{(3)}(G_2)$ is generated by $3a_1x_3$. Then

$$Ω^{(3)}(G_2) = L/(2, 3a_1) ∙ y_3 = L/(2, a_1) ∙ y_3$$

and we obtain that

$$(2) \quad Ω(G_2) = L[y_3]/(y_3^2, 2y_3, a_1y_3).$$

Observe that taking the latter equality modulo 2 we obtain the result established by Yagita in [19].

**Algebraic cobordism of groups SO$_n$, Spin$_m$ for $n = 3, 4$ and $m = 3, 4, 5, 6$.** According to [12]

$$CH(Spin_i) = Z \text{ for } i = 3, 4, 5, 6.$$

Then by Proposition 4.3 we obtain

$$(3) \quad Ω(Spin_i) = L \text{ for } i = 3, 4, 5, 6.$$

We have $CH(SO_3) = Z[x_1]/(2x_1, x_2^2)$ where $x_1 = πCH(ζ_{w_0s_1})$. Since $CH^i(SO_3) = 0$ for $i \geq 2$, $Ω^{(2)}(SO_3) = 0$. For $i = 1$ two left terms of exact sequence of Proposition 4.3 coincide, so there is an isomorphism $CH^1(SO_3) \rightarrow Ω^{(1)}(SO_3)$. Hence, we obtain

$$(4) \quad Ω(SO_3) = L[y_1]/(2y_1, y_1^2), \quad \text{where } y_1 = πΩ(ζ_{w_0s_1}).$$

Since $CH(SO_4) = Z[x_1]/(2x_1, x_2^2)$ the same reasoning proves that

$$(5) \quad Ω(SO_4) = L[y_1]/(2y_1, y_1^2).$$

**Oriented cohomology of PGL$_n$.**

5.1. **Lemma.** For any oriented cohomology theory $h$ with the coefficient ring $Λ$ and the formal group law $F$ we have

$$h(PGL_n) = Λ[x]/(x^n, nx^{n-1}, \ldots, (a^d)x^n, \ldots, nx, n \cdot F x).$$

**Proof.** Consider the variety of complete flags $X = SL_n/B$. Let $F_i$ denote the tautological vector bundle of dimension $i$ over $X$. Let $L_1 = F_1$ and $L_i = F_i/F_{i-1}$ for $i = 2, \ldots, n$. Then, by [9] Thm. 2.6] we have

$$h(X) ≅ Λ[x_1, \ldots, x_n]/S(x_1, \ldots, x_n) \quad (\ast)$$

where $S(x_1, \ldots, x_n)$ denotes the ideal generated by positive degree symmetric polynomials in variables $x_1, \ldots, x_n$, and the isomorphism sends $x_i$ to the Chern class $c_1(F_i)$. The maximal split torus $T ⊆ SL_n$ consists of diagonal matrices with trivial determinant. Let $χ_i ∈ T$ denote the character that sends the diagonal matrix to its $i$-th entry. So, the character lattice equals to $M = Zχ_1 ⊕ \ldots ⊕ Zχ_n/(χ_1 + \ldots + χ_n)$. Observe that $L_i$ coincides with the line bundle $L(χ_i)$, so by definition we have that $x_i = c(χ_i)$, where $c : Λ[[M]]_F → h(X)$ is the characteristic map. Note that the roots of $PGL_n = PSL_n/μ_2$ are equal $nχ_1, χ_2 − χ_1, \ldots, χ_n − χ_1$. 


According to \([7, 5.1]\) we have
\[
h(PGL_n) = h(X) / (c(n\chi_1), c(\chi_2 - \chi_1), \ldots, c(\chi_n - \chi_1)).
\]
Then in the quotient we have
\[
c(\chi_i) = c(\chi_1 + \chi_i - \chi_1) = c(\chi_1) + \gamma c(\chi_i - \chi_1) = c(\chi_1)^{\gamma}.
\]
Taking \(x = c(\chi_1)^{\gamma}\) by (*) we get
\[
h(PGL_n) = \Lambda[x] / (S(x, \ldots, x), n \cdot F \cdot x).
\]
According to \([9]\) \(S(x_1, \ldots, x_n)\) is generated by polynomials \(f_n(x_n), f_{n-1}(x_n, x_{n-1}), \ldots, f_1(x_n, \ldots, x_1)\) where \(f_i(x_n, \ldots, x_i)\) denotes the sum of all degree \(i\) monomials in \(x_n, \ldots, x_i\). Note that \(\binom{n}{d}\) equals to the number of degree \(d\) monomials in \(n - d + 1\) variables. Then substituting \(x_1 = \ldots x_n = x\) we obtain that \(x^n, nx^{n-1}, (\binom{n}{d})x^{n-1}, \ldots, nx\) generate the ideal \(S(x, \ldots, x)\).

### 5.2. Example.
For a prime number \(p\) and \(0 < d < p\) the coefficient \(\binom{p}{d}\) is divisible by \(p\). By \([8, \text{Rem. 5.4.8}]\) over \(\Lambda/\text{pA}\) we have \(p \cdot F \cdot x = p\beta_0(x) + \beta_1(x^p)\). Thus, the ideal \(I = (x^p, px^{p-1}, \ldots, \binom{n}{d}x^d, \ldots, px, p \cdot F \cdot x)\) is generated by \(x^p, px\). So for any prime \(p\) we have
\[
h(PGL_p) = \Lambda[x] / (px, x^p).
\]
In the case \(h = K_0\) this agrees with \([20, 3.6]\).

### Topological and the \(\gamma\)-filtration.
Proposition \([13, 4.3]\) allows to estimate the difference between the topological and the Grothendieck \(\gamma\)-filtration on \(K_0(G/B)\) for a split linear algebraic group \(G\). Namely, consider two filtrations on \(K_0(G/B)\):
\[
\gamma\text{-filtration: } \gamma^i(G/B) = \langle c_1(L(A)) \mid \lambda \in T^* \rangle, \\
\text{topological filtration: } \tau^i(G/B) = \langle [O_V] \mid \text{codim}(V) \geq i \rangle.
\]

### 5.3. Proposition.
Let \(G\) be a split semisimple simply connected linear algebraic group such that \(\text{CH}^i(G) = 0\) for \(1 \leq i \leq n - 1\) and \(\text{CH}^n(G) \neq 0\). Let \(\zeta_w\) be a Schubert cell such that \(\tau^\chi(\zeta)\) is nontrivial in \(\text{CH}^n(G)\).

Then \(\gamma^i(G/B) + \tau^{i+1}(G/B) = \tau^i(G/B)\) for \(i < n\) and the class of \(\zeta_w^{K_0}\) is nontrivial in \(\tau^n(G/B)/\gamma^n(G/B)\).

#### Proof.
As shown in \([15]\) \(K_0(G) = \mathbb{Z}\) for a simply connected group \(G\). Then characteristic map \(\chi\) is surjective \([6, \S 1B]\). We have \(K_0^{(1)}(G/B) = \tau^i = \gamma^1\). Note that \(K_0^{(i)}(G/B) = \tau^i\). Then \(\gamma^1 \tau^0 \cap \tau^i = \tau^i\) and the Proposition \([4, 3]\) gives us a short exact sequence for all \(i \geq 1\):
\[
0 \rightarrow \frac{\tau^{i+1}}{\tau^{i+1}} \rightarrow \frac{\tau^i}{\tau^i} \rightarrow \text{CH}^i(G, \mathbb{Z}[\beta, \beta^{-1}]) \rightarrow 0.
\]
Then for any \(1 \leq i < n\) we have \(\gamma^1 \tau^{i-1} / \tau^{i+1} = \tau^i / \tau^{i+1}\). By induction we get \(\tau^i = \gamma^i + \tau^{i+1}\) for \(i < n\) and for \(i = n\) we get By induction we get \(\tau^i = \gamma^i + \tau^{i+1}\) for \(i < n\) and for \(i = n\) we get
\[
0 \rightarrow \frac{\tau^n}{\tau^{n+1}} \rightarrow \frac{\tau^n}{\tau^{n+1}} \rightarrow \text{CH}^n(G, \mathbb{Z}[\beta, \beta^{-1}]) \rightarrow 0.
\]
So for any nontrivial element of of \(\text{CH}^n(G)\) the class of its preimage is nontrivial in \(\tau^n / \gamma^n\).
6. Applications to $h$-motivic decompositions

Throughout this section we consider a generically cellular variety $X$ of dimension $N$ and an oriented cohomology theory $h^*$ that is generically constant and is associated with weak Borel-Moore homology $h_*$ which satisfies the localization property. These assumptions imply that the generalized degree formula of Levine-Morel [11, Theorem 4.4.7] holds. The aim of this section is to prove theorems A, B and C of the introduction which provide a comparison between the Chow motive $M(X)$ and the $h$-motive $M^h(X)$ of $X$.

Let $L$ be the splitting field of $X$ and $\overline{X} = X \times \text{Spec } k \text{ Spec } L$. Let $p$ denote the projection $p: \overline{X} \times \overline{X} \to X \times X$. Since $\overline{X}$ is cellular, we may consider a filtration on $h^*(\overline{X})$ introduced in 3.2. It gives rise to a filtration on $h^*(\overline{X} \times \overline{X}) = \bigotimes_{i+j=l} h^i(\overline{X}) \otimes \Lambda h^j(\overline{X})$.

Namely, we set $h^l(\overline{X} \times \overline{X}) = \bigoplus_{i+j=l} h^i(\overline{X}) \otimes \Lambda h^j(\overline{X})$.

On $h^*(\overline{X} \times \overline{X})$ we consider the induced filtration $h^l(\overline{X} \times \overline{X}) = (p^h)^{-1}(h^l(\overline{X} \times \overline{X}))$.

Denote the quotient $h^l(\overline{X} \times \overline{X})/h^{l+1}(\overline{X} \times \overline{X})$ by $h^{l+1}(\overline{X} \times \overline{X})$ and denote by $p^l: h^l(\overline{X} \times \overline{X}) \to h^{l+1}(\overline{X} \times \overline{X})$ the usual projection. Denote $h^l_2(\overline{X} \times \overline{X}) = h^l(\overline{X} \times \overline{X}) \cap h^{2N-l}(\overline{X} \times \overline{X})$ and $h^l_2(\overline{X} \times \overline{X}) = h^l(\overline{X} \times \overline{X}) \cap h^{2N-l}(\overline{X} \times \overline{X})$.

6.1. Lemma. There is a graded ring isomorphism

$\Phi: \bigoplus_{i=0}^{2N} h^{i+i+1}(\overline{X} \times \overline{X}) \to \text{CH}^*(\overline{X} \times \overline{X}, \Lambda)$.

Proof. Since $\bigoplus_{i=0}^{2N} h^{i+i+1}(\overline{X} \times \overline{X}) = \bigoplus_{i=0}^{N} h^{i+i+1}(\overline{X}) \otimes \Lambda \bigoplus_{i=0}^{N} h^{i+i+1}(\overline{X})$ and $\text{CH}(\overline{X} \times \overline{X}, \Lambda) = \text{CH}(\overline{X}, \Lambda) \otimes \Lambda \text{CH}(\overline{X}, \Lambda)$ take $\Phi = \Psi \otimes \Psi$, where $\Psi$ is defined in [3.5].

6.2. Remark. The restriction of $\Phi_1$ gives an isomorphism $\Phi_1: h^{i+i+1}_2(\overline{X} \times \overline{X}) \to \text{CH}^*(\overline{X} \times \overline{X}, \Lambda^0)$.

The following lemma provides an $h$-version of the Rost Nilpotence Theorem:

6.3. Lemma. The kernel of the pullback map $p^h: \text{End}(M^h(X)) \to \text{End}(M^h(\overline{X}))$ consists of nilpotents.

Proof. Consider a diagram

\[
\begin{array}{ccc}
\text{End}(M^\Omega(X)) & \xrightarrow{p^\Omega} & \text{End}(M^\Omega(\overline{X})) \\
\Downarrow & & \Downarrow \\
\text{End}(M^{\text{CH}}(X)) & \longrightarrow & \text{End}(M^{\text{CH}}(\overline{X}))
\end{array}
\]
where vertical arrows are ring homomorphisms that arise from the canonical map \( \Omega(-) \to \text{CH}(-) \). By [17, Prop. 2.7] they are surjective with kernels consisting of nilpotents. The kernel of the bottom arrow consists of nilpotents by [13, Prop 3.1]. Then the kernel of the upper arrow consists of nilpotents as well.

Tensoring the upper arrow with \( \Lambda \) we obtain \( \ker(p^H) \otimes \Lambda \to \Omega_N(X \times X) \otimes \Lambda \overset{\rho^H \otimes id}{\longrightarrow} \Omega(X \times X) \otimes \Lambda \), so \( \ker(p^H) \otimes \Lambda \) covers the kernel of \( p^H \otimes id \), thus \( \ker(p^H \otimes id) \) consists of nilpotents. Now the specialization maps fit into the commutative diagram.

\[
\begin{array}{ccc}
\Omega_N(X \times X) \otimes \Lambda & \overset{\rho^H \otimes id}{\longrightarrow} & \Omega(X \times X) \otimes \Lambda \\
\downarrow{\mu_{X \times X}} & & \downarrow{\phi} \\
\h(X \times X) & \overset{p^h}{\longrightarrow} & \h(X \times X)
\end{array}
\]

where the right arrow is an isomorphism by [2.4] and [3.1], and the map \( \mu_{X \times X} \) is surjective. So the kernel of the bottom map consists of nilpotents. \( \square \)

6.4. Lemma. We have \( h^{(N+j)}(X, \bar{X}) \otimes h^{(N+j)}(X, \bar{X}) \subseteq h^{(N+i+j)}(X, \bar{X}) \).

Proof. Consider a generator \( \zeta_n \otimes \tau_n \in h^{(N+j)}(X, \bar{X}) \) where \( N - \alpha_m + \alpha_n \geq N + i \) and \( \zeta_{m'} \otimes \tau_{m'} \in h^{(N+j)}(X, \bar{X}) \) where \( N - \alpha_{m'} + \alpha_{n'} \geq N + j \). The composition

\[(\zeta_n \otimes \tau_n) \circ (\zeta_{m'} \otimes \tau_{m'}) = \deg(\tau_n \zeta_{m'}) (\zeta_m \otimes \tau_n') = \delta_{n, m'} \cdot (\zeta_m \otimes \tau_{n'})
\]

is nonzero if \( n = m' \). In this case \( N - m + n' = (N - m + n) + (N - m' + n') - N \geq N + i + j \). Thus \( \zeta_m \otimes \tau_{n'} \) lies in \( h^{(N+i+j)}(X, \bar{X}) \). \( \square \)

6.5. Remark. Indeed, the lemma implies that \( h^{(N)}(X, \bar{X}) \) is a ring with respect to the composition product, and \( h^{(N+1)}(X, \bar{X}) \) is its two-sided ideal. Since the composition of homogeneous elements is homogeneous, \( h^{(N)}_N(X, \bar{X}) \) is also a ring with respect to the composition.

6.6. Lemma. The isomorphism \( \Phi_N : h^{(N+N+1)}(X, \bar{X}) \to \text{CH}(X, \bar{X}, \Lambda) \) is a ring homomorphism with respect to the composition product.

Proof. This immediately follows from the fact that \( \Phi \) maps residue classes of \( \zeta^h_w \otimes \tau^h_v \) to \( \zeta^\text{CH}_w \otimes \tau^\text{CH}_v \). \( \square \)

6.7. Lemma. Let \( Y \) be a twisted form of \( X \), i.e. \( Y_L \cong X_L = \bar{X} \). For every codimension \( m \) consider the diagram, where \( p : \bar{X} \times \bar{X} \to X \times Y \) denotes the projection.

\[
\begin{array}{ccc}
h^{(m)}_{2N-m}(X \times Y) & \overset{pr_m \circ p^h}{\longrightarrow} & h^{(m/(m+1)}_{2N-m}(X, \bar{X}) \\
& \Phi^m \downarrow & \\
\text{CH}^n(X \times Y, \Lambda^0) & \overset{p^\text{CH}}{\longrightarrow} & \text{CH}^m(X, \bar{X}, \Lambda^0)
\end{array}
\]

Then \( \text{im}(\Phi^m \circ p^\text{CH}) \subseteq \text{im} pr_m \circ p^h \).

Proof. Note that \( \text{CH}^m(X \times Y, \Lambda^0) \) is generated over \( \Lambda^0 \) by classes \( i_{CH}(1) \) where \( i : \bar{Z} \to Z \mapsto X \times Y \), \( Z \) is a closed integral subscheme of codimension \( m \), \( \bar{Z} \in \text{Sm}_k \)
and \( \tilde{Z} \to Z \) is projective birational. Consider the Cartesian diagram

\[
\begin{array}{c}
\tilde{Z} \\
\downarrow^i \\
X \times_k Y
\end{array}
\quad
\begin{array}{c}
\downarrow^j \\
\tilde{Z}_L
\end{array}
\quad
\begin{array}{c}
X \times_L \overline{X}
\end{array}
\]

Since this diagram is transverse, then

\[ j_h \circ \phi^h = \phi^h \circ i_h \quad \text{and} \quad j_{\text{CH}} \circ \epsilon^\text{CH} = p^\text{CH} \circ i_{\text{CH}}. \]

By lemma 6.8 we have \( \Phi^m \circ j_{\text{CH}}(1) = pr_m(j_h(1)) \). Then \( \Phi^m \circ p^\text{CH}(i_{\text{CH}}(1)) = \Phi^m \circ j_{\text{CH}}(1) = pr_m(j_h(1)) = pr_m(p_h \circ i_h(1)) \in \text{im} \, pr_m \circ p_m^h. \]

6.8. Lemma. Consider a morphism \( j : \tilde{Z} \to \overline{X} \times_L \overline{X} \), where \( \tilde{Z} \) is a smooth irreducible scheme and \( j \) is projective of relative dimension \(-m\). It induces two pushforward maps \( j_h : h(\tilde{Z}) \to h(\overline{X} \times \overline{X}) \) and \( j_{\text{CH}} : \text{CH}(\tilde{Z}, \Lambda) \to \text{CH}(\overline{X} \times \overline{X}, \Lambda). \)

Then \( j_h(1) \in h^{(m)}_{2N-m}(\overline{X} \times \overline{X}) \) and \( \Phi^m(j_{\text{CH}}(1)) = pr_m(j_h(1)). \)

Proof. Observe that

\[ j_h(1) = j_{\Omega}(1) \otimes_L 1_{\Lambda} \quad \text{and} \quad j_{\text{CH}}(1) = j_{\Omega}(1) \otimes_L 1_Z. \]

Expanding in the basis we obtain

\[ j_{\Omega}(1) = \sum_{r_{i_1,i_2} \in \mathbb{L}} r_{i_1,i_2} \tau_{i_1}^\Omega \otimes \tau_{i_2}^\Omega \quad \text{for some } r_{i_1,i_2} \in \mathbb{L}. \quad (*) \]

Since \( j_{\Omega}(1) \) is homogeneous of degree \( m \), we have

\[ r_{i_1,i_2} \in \mathbb{L}^{m-\alpha_{i_1}+\alpha_{i_2}}. \quad (***) \]

Then for every nonzero \( r_{i_1,i_2} \) we have \( \alpha_{i_1} + \alpha_{i_2} \geq m \). So each \( \tau_{i_1}^\Omega \otimes \tau_{i_2}^\Omega \in \Omega^m(\overline{X} \times \overline{X}) \)

and, thus, \( j_{\Omega}(1) \in \Omega^m_{2N-m}(\overline{X} \times \overline{X}) \). Taking \((*)\) modulo \( \Omega^{m+1}(\overline{X} \times \overline{X}) \) we obtain

\[ j_{\Omega}(1) + \Omega^{m+1}(\overline{X} \times \overline{X}) = \sum_{\alpha_{i_1} + \alpha_{i_2} = m} r_{i_1,i_2} \tau_{i_1}^\Omega \otimes \tau_{i_2}^\Omega + \Omega^{m+1}(\overline{X} \times \overline{X}). \]

If \( \alpha_{i_1} + \alpha_{i_2} = m \) then \( r_{i_1,i_2} \in \mathbb{L}^0 = \mathbb{Z} \) by \((***)\). Thus taking \( j_h(1) = j_{\Omega}(1) \otimes_L 1_{\Lambda} \) and \( j_{\text{CH}}(1) = j_{\Omega}(1) \otimes_L 1_Z \) we get

\[ pr_m(j_h(1)) = pr_m(\sum_{\alpha_{i_1} + \alpha_{i_2} = m} r_{i_1,i_2} \tau_{i_1}^h \otimes \tau_{i_2}^h) \]

and

\[ j_{\text{CH}}(1) = j_{\Omega}(1) \otimes_L 1_Z \otimes_L 1_{\Lambda} = \sum_{\alpha_{i_1} + \alpha_{i_2} = m} r_{i_1,i_2} \tau_{i_1}^{\text{CH}} \otimes \tau_{i_2}^{\text{CH}}. \]

Then \( \Phi^m(j_{\text{CH}}(1)) = pr_m(j_h(1)) \), since \( \Phi^m(\tau_{i_1}^{\text{CH}} \otimes \tau_{i_2}^{\text{CH}}) = pr_m(\tau_{i_1}^h \otimes \tau_{i_2}^h) \).

6.9. Lemma. The kernel of the composition homomorphism

\[ pr_N \circ p^h : h^{(N)}(X \times X) \to h^{(N)}(\overline{X} \times \overline{X}) \to h^{(N/N+1)}(\overline{X} \times \overline{X}) \]

consists of nilpotents.

Proof. This follows from Rost nilpotence and the fact that \( h^{(N+1)}(\overline{X} \times \overline{X}) \) is nilpotent by Lemma 6.3.
6.10. Lemma. Let C be an additive category, A, B ∈ Ob(C). Let \( f \in \text{Hom}_C(A, B) \) and \( g \in \text{Hom}_C(B, A) \) such that \( f \circ g - \text{id}_B \) is nilpotent in the ring \( \text{End}_C(B) \) and \( g \circ f - \text{id}_A \) is nilpotent in the ring \( \text{End}_C(A) \). Then A is isomorphic to B.

Proof. Denote \( \alpha = \text{id}_A - gf \) and \( \beta = \text{id}_B - fg \). Take natural \( n \) such that \( \alpha^{n+1} = 0 \) and \( \beta^{n+1} = 0 \). Then \( g \cdot f = \alpha \) is invertible and \( (gf)^{-1} = \alpha + \ldots + \alpha^n \). Analogously \( (fg)^{-1} = \beta_B + \beta + \ldots + \beta^n \). So we have

\[
gf(id_A + (id_A - gf) + \ldots + (id_A - gf)^n) = id_A
\]

Since \( (id_A - gf)^n = \sum_{i=0}^n (-1)^i \binom{n}{i} (gf)^i \), we have

\[
g \sum_{m=0}^n \left( \sum_{i=0}^m (-1)^i \binom{m}{i} (fg)^i f \right) = id_A \quad (*)
\]

and

\[
gf(id_B + (id_B - fg) + \ldots + (id_B - fg)^n) = id_B
\]

implies

\[
\sum_{m=0}^n \left( \sum_{i=0}^m (-1)^i \binom{m}{i} f(gf)^i \right) g = id_B. \quad (**)\]

Then take

\[
f_1 = \sum_{m=0}^n \left( \sum_{i=0}^m (-1)^i \binom{m}{i} (fg)^i f \right) = \sum_{m=0}^n \left( \sum_{i=0}^m (-1)^i \binom{m}{i} f(gf)^i \right).
\]

Then \((*)\) implies \(gf_1 = id_A\) and \((**)\) implies \(f_1g = id_B\). So \(f_1\) and \(g\) establish inverse isomorphisms between \(A\) and \(B\). \(\square\)

6.11. Corollary. Suppose \(p_1\) and \(p_2\) are two idempotents in \(\text{End}(M^k(\mathbb{X}))\) such that \(p_1 - p_2\) is nilpotent. Then the motives \(\mathbb{X}, p_1\) and \(\mathbb{X}, p_2\) are isomorphic.

Proof. Take \(f = p_2 \circ p_1 \in \text{Hom}_M((\mathbb{X}, p_1), (\mathbb{X}, p_2))\) and \(g = p_1 \circ p_2 \in \text{Hom}_M((\mathbb{X}, p_2), (\mathbb{X}, p_1))\).

Let us check that \(f \circ g - \text{id}(\mathbb{X}, p_2) = p_2 p_1 p_2 - p_2 = p_2(p_1 - p_2)p_2\) is nilpotent.

It is sufficient to check that \((p_2(p_1 - p_2)p_2)^m = p_2(p_1 - p_2)^m p_2\) for any \(m\). Note that if \(x \in \ker p_2 \cap \text{im} p_1\) then \((p_1 - p_2)(x) = x\). Since \(p_1 - p_2\) is nilpotent, \(x = 0\). Thus, \(\ker p_2 \cap \text{im} p_1 = 0\). Since \(p_2\) is idempotent, \(\text{im} p_2 \cap \ker p_2 = 0\). Then endomorphism \(p_1 - p_2\) of \(M(\mathbb{X}) = \ker p_2 \oplus \text{im} p_2\) can be represented as the matrix

\[
p_1 - p_2 = \begin{pmatrix} E_1 & E_2 \\ 0 & 0 \end{pmatrix}
\]

where \(E_1\) is a homomorphism from \(\text{im} p_2\) to \(\text{im} p_2\) and \(E_2\) is a homomorphism from \(\ker p_2\) to \(\text{im} p_2\). We have

\[
p_2(p_1 - p_2)^m p_2 = p_2 \circ \begin{pmatrix} E_1^m & E_1^{m-1} E_2 \\ 0 & 0 \end{pmatrix} \circ p_2 = \begin{pmatrix} E_1^m & 0 \\ 0 & 0 \end{pmatrix} = (p_2(p_1 - p_2)p_2)^m.
\]

Then \(f \circ g - \text{id}(\mathbb{X}, p_2) = p_2(p_1 - p_2)p_2\) is nilpotent. Symmetrically, \(g \circ f - \text{id}(\mathbb{X}, p_1)\) is nilpotent. So \((\mathbb{X}, p_1)\) and \((\mathbb{X}, p_2)\) are isomorphic by Lemma 6.10. \(\square\)

We are now ready to prove theorems A, B and C of the introduction:
Theorem A. Suppose $X$ is generically cellular. Assume that there is a decomposition of Chow motive with coefficients in $\Lambda^0$

$$M^{\text{CH}}(X, \Lambda^0) = \bigoplus_{i=0}^{n} R(\alpha_i)$$

such that over the splitting field $L$ the motive $R$ equals to the sum of twisted Tate motives: $R = \bigoplus_{j=0}^{m} \Lambda^0(\beta_j)$.

Then there is a $h$-motive $R_h$ such that

$$M^h(X) = \bigoplus_{i=0}^{n} R_h(\alpha_i)$$

such that over the splitting field $R_h$ splits into the $h$-Tate motives $R_h = \bigoplus_{j=0}^{m} \Lambda(\beta_j)$.

Proof. We may assume that $\alpha_0 = 0$ in $(\ast)$. Then each summand $R(\alpha_i)$ equals to $(X, p_i)$ for some idempotent $p_i$ and there are mutually inverse isomorphisms $\phi_i$ and $\psi_i$ of degree $\alpha_i$ between $(X, p_0)$ and $(X, p_i)$. So we have

- idempotents $p_i \in \text{CH}^N(X \times X)$, $\sum p_i = \Delta^X_k(1)$
- isomorphisms $\phi_i \in p_0 \circ \text{CH}^{N+\alpha_i}(X \times X) \circ p_i$ and $\psi_i \in p_i \circ \text{CH}^{N-\alpha_i}(X \times X) \circ p_0$
- such that $\phi_i \circ \psi_i = \psi_0 = p_0$ and $\psi_i \circ \phi_i = p_i$

Consider the diagram of Lemma 6.7:

$$\xymatrix{ h^{(m)}_{2N-m}(X \times Y) \ar[r]^{pr_m \circ p^h} \ar[d]^{\Phi^m} & h^{(m/m+1)}_{2N-m}(X \times X) \ar[d]^{\Phi^m} \cr \text{CH}^m(X \times Y, \Lambda^0) \ar[r]^{p^{\text{CH}}} & \text{CH}^m(X \times X, \Lambda^0) \cr}$$

By Lemma 6.7 the elements $\Phi^N \circ p^{\text{CH}}(p_i)$ and $\Phi^{N+\alpha_i} \circ p^{\text{CH}}(\phi_i)$ and $\Phi^{N-\alpha_i} \circ p^{\text{CH}}(\psi_i)$ lie in $\im pr_N \circ p^h$, $\im pr_{N-\alpha_i} \circ p^h$ and $\im pr_{N+\alpha_i} \circ p^h$ respectively.

By Lemma 6.9 the kernel of $pr_N \circ p^h$, $h^{(N)}_N(X \times X) \rightarrow h^{(N+N+1)}_N(X \times X)$ is nilpotent. Then by [16, Prop. 27.4] there is a decomposition $r_i$ such that $pr_N \circ p^h(r_i) = p_i$.

Let us construct the isomorphisms between $r_i$ and $r_0$. Let $\phi_i'$ and $\psi_i'$ be some preimages of $\Phi^{N+\alpha_i} \circ p^{\text{CH}}(\phi_i)$ and $\Phi^{N-\alpha_i} \circ p^{\text{CH}}(\psi_i)$. Then [16, Lem. 2.5] implies that there are elements $\phi_i'' \in r_0 h^{(N)}_N(X \times X) r_1$ and $\psi_i'' \in r_i h^{(N)}_N(X \times X) r_0$, such that $\phi_i \psi_i \phi_i' = r_0$ and $\psi_i \phi_i \phi_i' = r_i$. So the $h$-motives $(X, r_i)$ and $(X, r_0)(\alpha_i)$ are isomorphic. Taking $R_h = (X, r_0)$ we have

$$M^h(X) = \bigoplus_{i=0}^{n} (X, r_i) = \bigoplus_{i=0}^{n} (X, r_0)(\alpha_i) = \bigoplus_{i=0}^{n} R_h(\alpha_i).$$

Over the splitting field the motive $R_h$ becomes isomorphic to $(X, p^h(r_0))$ and $pr_N \circ p^h(r_0) = \Phi^N(p_0 \circ h(X, p^h(p_0)))$. Since the Chow motive $(X, p^h(p_0))$ splits into $\bigoplus \Lambda^0(\beta_j)$ we have $p^{\text{CH}}(p_0) = \sum f_j \otimes g_j$ with $f_j \in \text{CH}^{\alpha_j}(X), g_j \in \text{CH}_{\alpha_j}(X)$ and $\pi^{\text{CH}}(f_j g_i) = \delta_{i,j}.f$. Take $\varphi_j$ and $\gamma_j$ to be the liftings of $f_j$ and $g_j$ in $h^{(N+1)}_N(X)$ and $h^{(N-\alpha_j)}_N(X)$ respectively.

Note that $\varphi_j \gamma_j + h^{(N+1)}_N(X) = \Psi^N(f_j g_i)$. Since $h^{(N+1)}_N(X) = 0$, we have $\pi^h(\varphi_j \gamma_j) = \pi^{\text{CH}}(f_j g_i) = \delta_{i,j}.f$. Then the element $\sum \varphi_j \otimes \gamma_j$ is an idempotent in $\text{Corr}_0(X \times X)$.
By the degree formula [11, Thm 4.4.7]
Proof.

\[ \text{pushforwards} \]
and in the diagram of Lemma 6.7
we obtain

\[ (\mathcal{X}, p^N(r_0)) \cong (\mathcal{X}, \sum_j \varphi_j \otimes \gamma_j) = \bigoplus \Lambda(\beta_j). \]

6.12. Lemma. Assume that \( \Lambda^1 = \ldots \Lambda^N = 0 \). Then \( h_N(\mathcal{X} \times \mathcal{X}) \subseteq h^{(N)}(\mathcal{X} \times \mathcal{X}) \) and in the diagram of Lemma 6.7

\[
\begin{align*}
h_N^{(N)}(X \times Y) \xrightarrow{pr_N \circ p^h} h_N^{(N/N+1)}(\mathcal{X} \times \mathcal{X}) \\
\xrightarrow{\Phi_N} CH^N(X \times Y, \Lambda^0) \xrightarrow{p^\text{CH}} CH^N(\mathcal{X} \times \mathcal{X}, \Lambda^0)
\end{align*}
\]
the inverse inclusion holds: \( \text{im} \text{pr}_N \circ p^h \subseteq \text{im} \Phi_N \circ p^\text{CH} \).

Proof. By the degree formula [11 Thm 4.4.7] \( h(\mathcal{X} \times \mathcal{X}) \) is generated as \( \Lambda \)-module by pushforwards \( h_i(1) \), where \( i: Z \rightarrow X \times X \) is projective, \( Z \in Sm_k \) and \( i: Z \rightarrow \mathcal{X} \) is birational. Following [11] we will denote such classes by \( [Z \rightarrow X \times X]_h \).
Then \( h_N(X \times X) \) is additively generated by elements \( \lambda[Z \rightarrow X \times X]_h \), where \( \lambda \) is homogeneous such that \( \text{deg} \lambda + \text{codim} Z = N \). Since \( \Lambda^1 = \ldots \Lambda^N = 0 \), we have \( \text{codim} Z \geq N \). Then in \( \Omega(X \times X) \) we have

\[ [Z_L \rightarrow X \times X]_\Omega = \sum \omega_{i,j} \zeta_i \otimes \tau_j \text{ for some } \omega_{i,j} \in \mathbb{L}. \]

Since all elements of the Lazard ring have negative degrees and \( [Z_L \rightarrow X \times X]_\Omega \) has degree \( N \), each \( \zeta_i \otimes \tau_j \) in the expansion is contained in \( \Omega^{(n)}(\mathcal{X} \times \mathcal{X}) \). Then

\[ [Z_L \rightarrow X \times X]_h = \nu_{X \times X} [Z_L \rightarrow X \times X]_\Omega \in h^{(N)}(\mathcal{X} \times \mathcal{X}) \]
and

\[ [Z \rightarrow X \times X]_h \in h^{(N)}(X \times X). \]

By the same reasons \( [Y \rightarrow X \times X] \) belongs to \( h^{(N+1)}(X \times X) \) if \( \text{codim} Y > N \). Then \( \text{im} \text{pr}_N \circ p^h \) is generated over \( \Lambda^0 \) by classes of \( [Z_L \rightarrow X \times X]_h \), where \( Z \rightarrow X \times X \) has codimension \( N \).

By Lemma 6.8 for any \( Z \rightarrow X \times X \) of codimension \( N \) we have

\[ \text{pr}_N \circ p^h([Z \rightarrow X \times X]_h) = \Phi_N \circ p^\text{CH}([Z \rightarrow X \times X]). \]

Then \( \text{im} \text{pr}_N \circ p^h \subseteq \Phi_N \circ p^\text{CH} \) and the theorem is proven. \( \Box \)

Theorem B. Let \( h \) be oriented cohomology theory with coefficient ring \( \Lambda \). Assume that the Chow motive \( \mathcal{R} \) is indecomposable over \( \Lambda^0 \) and \( \Lambda^1 = \ldots = \Lambda^N = 0 \). Then the \( h \)-motive \( \mathcal{R}_h \) from theorem A is indecomposable.

Proof. By definition, \( \mathcal{R}_h = (X, r_0) \) where \( r_0 \) is an idempotent in \( h_N^{(N)}(X \times X) \). If \( \mathcal{R}_h \) is decomposable, then \( r_0 = r_1 + r_2 \) for some idempotents in \( r_1, r_2 \in h_N(X \times X) \). Then by Lemma 6.12 \( r_1, r_2 \in h_N^{(N)}(X \times X) \) and \( p_1 = (\Phi_N)^{-1} \circ \text{pr}_N \circ p^h(r_1) \) and \( p_2 = (\Phi_N)^{-1} \circ \text{pr}_N \circ p^h(r_2) \) are rational idempotents and \( p^\text{CH}(p_0) = p_1 + p_2 \). These idempotents are nontrivial, since \( \text{ker}(\Phi_N)^{-1} \circ \text{pr}_N \circ p^h \) is nilpotent. Hence, the Chow motive \( \mathcal{R} = (X, p_0) \) is decomposable, a contradiction. \( \Box \)
6.13. Example. If \( h = \Omega \) or connective \( K \)-theory, all the elements in the coefficient ring have negative degree. Then Theorems A and B prove that \( h \)-motivic irreducible decomposition coincides with integral Chow-motivic decomposition. This gives another proof of the result by Vishik-Yagita [17 Cor. 2.8].

6.14. Example. Take \( h \) to be Morava \( K \)-theory \( h = K(n)^* \). The coefficient ring is \( \mathbb{F}_p[v_n, v_n^{-1}] \), where \( \deg(v_n) = -2(p^n - 1) \). In the case \( n > \log_p(\frac{n}{n} + 1) \) Theorems A and B prove that \( M^{K(n)}(X) \) has the same irreducible decomposition as Chow motive modulo \( p \).

**Theorem C.** Suppose that \( X, Y \) are generically cellular and \( Y \) is a twisted form of \( X \), i.e. \( Y \cong X \).

If \( M^{\text{CH}}(X, \Lambda^0) \cong M^{\text{CH}}(Y, \Lambda^0) \), then \( M^h(X) \cong M^h(Y) \).

**Proof.** Let \( f \in \text{CH}^N(X \times Y) \) and \( g \in \text{CH}^N(Y \times X) \) be correspondences, that give mutually inverse isomorphisms between \( M^{\text{CH}}(X) \) and \( M^{\text{CH}}(Y) \). Consider the diagram

\[
\begin{array}{ccc}
\text{h}_N(X \times Y) & \xymatrix{ \ar[r]^{pr \circ \text{op}^h_1} } & \text{h}_N(X \times Y) \\
\text{CH}^N(X \times Y, \Lambda^0) & \xymatrix{ \ar[r]^{pr \circ \text{op}^h_1} } & \text{CH}^N(X \times Y, \Lambda^0)
\end{array}
\]

Then by Lemma 6.10 and the theorem is proven.

\( \square \)

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Department of Mathematics and Statistics, University of Ottawa, Canada
E-mail address: anesh094@uottawa.ca