Dressed-quark anomalous magnetic moments

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In Dirac’s relativistic quantum mechanics, a fermion with charge $q$ and mass $m$, interacting with an electromagnetic field, has a magnetic moment\(^1\) $\mu = q/[2m]$. This prediction held true for the electron until improvements in experimental techniques enabled the discovery of a small deviation\([1]\), with the moment increased by a multiplicative factor: $1.00119 \pm 0.00005$. This correction was explained by the first systematic computation using renormalized quantum electrodynamics (QED)\([2]\):

$$\frac{q}{2m} \rightarrow \left(1 + \frac{\alpha}{2\pi} \right) \frac{q}{2m}, \quad (1)$$

where $\alpha$ is QED’s fine structure constant. The agreement with experiment established quantum electrodynamics as a valid tool. The correction defines the electron’s anomalous magnetic moment, which is now known with extraordinary precision and agrees with theory at $O(\alpha)\([3]\)\).

The fermion-photon coupling in QED is described by:

$$\int d^4x \, i q \, \bar{\psi}(x) \gamma_\mu \psi(x) \, A_\mu(x), \quad (2)$$

This interaction generates the following electromagnetic current for an on-shell Dirac fermion ($k = p_f - p_i$),

$$iq \, \bar{u}(p_f) \left[ \gamma_\mu F_1(k^2) + \frac{1}{2m} \sigma_{\mu\nu} k_\nu F_2(k^2) \right] u(p_i), \quad (3)$$

where: $F_1(k^2)$, $F_2(k^2)$ are form factors; and $\bar{u}(p)$, $u(p)$ are spinors, the free particle forms of which satisfy

$$\bar{u}(p_f)(i\gamma \cdot p_f + m) = 0, \quad (i\gamma \cdot p_i + m)u(p_i) = 0. \quad (4)$$

A Gordon-identity can be obtained from these “Dirac” equations; viz., with $2\ell_p = p_f + p_i$,

$$2m \, \bar{u}(p_f)i\gamma_\mu u(p_i) = \bar{u}(p_f)[2\ell_\mu + i\sigma_{\mu\nu} k_\nu] \, u(p_i). \quad (5)$$

It follows that a point-particle in the absence of radiative corrections has $F_1 = 1$ and $F_2 = 0$, and hence Dirac’s value for the magnetic moment. The anomalous magnetic moment in Eq. (1) corresponds to $F_2(0) = \alpha/2\pi$.

An anomalous contribution to the moment can therefore be associated with an additional interaction term:

$$\int d^4x \, \frac{1}{2} q \, \bar{\psi}(x) \sigma_{\mu\nu} \psi(x) F_{\mu\nu}(x), \quad (6)$$

where $F_{\mu\nu}(x)$ is the gauge-boson field strength tensor. This term is invariant under local $U(1)$ gauge transformations but is not generated by minimal substitution in the action for a free Dirac field.

Consider the effect of the global chiral transformation $\psi(x) \rightarrow \exp(i \theta \gamma_5) \psi(x)$. The term in Eq. (2) is invariant. However, the interaction of Eq. (6) is not. These observations facilitate the understanding of a general result: $F_2 = 0$ for a massless fermion in a quantum field theory with chiral symmetry realized in the Wigner mode; i.e., when the symmetry is not dynamically broken. A firmer conclusion can be drawn. For $m = 0$ it follows from Eq. (5) that Eq. (2) does not mix with the helicity-flipping interaction of Eq. (6) and hence a massless fermion does not possess a measurable magnetic moment.

A reconsideration of Ref.\([2]\) reveals no manifest conflict with these facts. The perturbative expression for $F_2(0)$ contains a multiplicative numerator factor of $m$ and the usual analysis of the denominator involves steps that are only valid for $m \neq 0$. Fundamentally, there is no conundrum because QED is not an asymptotically free theory and hence does not possess a well-defined chiral limit.

On the other hand, in quantum chromodynamics (QCD) the chiral limit is rigorously defined nonperturbatively\([4, 5]\). This non-Abelian quantum field theory describes quarks interacting via the exchange of gluons, which themselves self-interact. Apart from the inclusion of a matrix to represent the color degree of freedom, the quark-gluon interaction is described by Eq. (2). The analogue of Schwinger’s one-loop calculation can be carried out to find an anomalous chromo-magnetic moment for

\(\footnote{We use natural units, $\hbar = 1 = c$, and a Euclidean metric: $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}; \gamma_5^4 = \gamma_5; \sigma_{\mu\nu} = (i/2)[\gamma_\mu, \gamma_\nu]; a \cdot b = \sum_{i=1}^n a_i b_i$; and $P_\mu$ timelike $\Rightarrow P^2 < 0.$}
the quark. There are two diagrams in this case: one similar in form to that in QED; and another owing to the gluon self-interaction. One reads from Ref. [6] that the perturbative result vanishes in the chiral limit. However, nonperturbative studies of QCD’s gap equation [7] and numerical simulations of the lattice-regularized theory [8] have revealed that chiral symmetry is dynamically broken in QCD. Does this affect the chromomagnetic moment?

Dynamical chiral symmetry breaking (DCSB) is a remarkably effective mass generating mechanism, which can be explained via the dressed-quark propagator

$$S(p; \zeta) = 1/(i\gamma \cdot pA(p^2; \zeta) + B(p^2; \zeta)),$$  

(7)

where \(\zeta\) is the renormalization mass-scale and the dressed-quark mass function \(M(p^2) = B(p^2; \zeta)/A(p^2; \zeta)\) is renormalization point invariant. In the chiral limit, \(M(p^2)\) is identically zero at any finite order in perturbation theory. However, DCSB generates mass from nothing. Thus, in chiral-QCD, \(M(p^2)\) is strongly momentum-dependent and \(M(p^2) \approx 0.5\text{ GeV}\). DCSB is the origin of constituent-quark masses and intimately connected with confinement in QCD [10].

QCD dynamics, and DCSB in particular, also have a material effect on the quark-gluon vertex:

$$\Gamma^a_{\mu}(p_f, p_i; k) = \frac{\lambda^a}{2} \Gamma_{\mu}(p_f, p_i; k),$$  

(8)

where \(\{\lambda^a|a = 1, \ldots, 8\}\) are the color Gell-Mann matrices. \(\Gamma_{\mu}(p_f, p_i)\) can be expressed via twelve independent Dirac-matrix-valued tensors, each multiplied by a scalar function. It has long been known via Dyson-Schwinger equation (DSE) studies [9] that at least three of the tensors are materially modified from their perturbative forms in strongly interacting theories; namely, \(\lambda_{1,2,3}\) in

$$i\Gamma_{\mu}(p_f, p_i; k) = \lambda_1(p_f, p_i; k)\lambda_\mu + 2\lambda_2(\lambda_\mu, k) + \lambda_3(p_f, p_i; k) + [\ldots].$$  

(9)

These terms constitute the so-called longitudinal vertex and are constrained by the Slavnov-Taylor identity, a non-Abelian form of the Ward-Takahashi identity.

Contemporary simulations of lattice-regularized QCD [11] and DSE studies [12] agree that

$$\lambda_3(p, p; 0) \approx \frac{d}{dp^2} B(p^2, \zeta)$$  

(10)

and also on the form of \(\lambda_1(p, p; 0),\) which is functionally similar to \(A(p^2, \zeta).\) However, owing to non-orthogonality of the tensors accompanying \(\lambda_1\) and \(\lambda_2,\) it is difficult to obtain a lattice form for \(\lambda_2.\) We therefore consider the DSE prediction in Ref. [12] more reliable.

Perturbative massless-QCD conserves helicity so the quark-gluon vertex cannot perturbatively have a term with the helicity-flipping characteristics of \(\lambda_3.\) Equation (10) is thus remarkable, showing that the dressed-quark-gluon vertex contains at least one chirally-asymmetric component whose origin and size owes solely to DCSB. A recent advance in understanding the Bethe-Salpeter equation (BSE) has enabled practitioners to establish that \(\lambda_3\) has a big impact on the hadron spectrum [13]; e.g., it generates a very strong spin-orbit interaction.

We will take this reasoning further. As explained above, massless fermions in gauge field theories cannot possess an anomalous chromo/electro-magnetic moment because the term that describes it couples left- and right-handed fermions. However, if chiral symmetry is strongly broken dynamically, then the fermions should also possess large anomalous magnetic moments. Such an effect is expressed in the dressed-quark-gluon vertex via a term

$$\Gamma^\text{acm}_{\mu}(p_f, p_i; k) = \sigma_{\mu
u}k_\nu \tau_5(p_f, p_i, k).$$  

(11)

That QCD generates a strongly momentum-dependent chromomagnetic form factor in the quark-gluon vertex, \(\tau_5,\) with a large DCSB-component, is confirmed in Ref. [11]. Only a particular kinematic arrangement was readily accessible in that lattice simulation but this is enough to learn that, at the current-quark mass considered: \(\tau_5\) is roughly two orders-of-magnitude larger than the perturbative form; and

$$\forall p^2 > 0: |\tau_5(p, -p; 2p)| \gtrsim |\lambda_3(p, p; 0)|.$$

(12)

The magnitude of the lattice result is consistent with instanton-liquid model estimates [14, 15].

This large chromomagnetic moment is likely to have a broad impact on the properties of light-quark systems [15, 16]. In particular, it can probably explain the long-standing puzzle of the mass splitting between the \(\alpha_1\)- and \(\rho\)-mesons in the hadron spectrum [10]. Herein, however, we will elucidate another novel effect; viz., the manner in which the quark’s chromomagnetic moment generates a quark anomalous electromagnetic moment. The method of Ref. [13] makes this possible for the first time.

Following Ref. [13], one need only specify the gap equation’s kernel because the quark-photon vertex BSE is completely defined therefrom. The gap equation is

$$S(p)^{-1} = Z_2(i\gamma \cdot p + m_{\text{bm}}^\mu) + Z_1 \int_k^\Lambda g^2 D_{\mu\nu}(p - k) \times \gamma^a \lambda^a_{\mu} S_f(q) \frac{\lambda^a_{\nu}}{2} \Gamma_{\nu}(k, p),$$  

(13)

where: \(D_{\mu\nu}\) is the gluon propagator; \(\Gamma_{\nu}\) is the quark-gluon vertex, Eq. (8); \(\int_k^\Lambda d^4k/(2\pi)^4\) is a Poincaré invariant regularization of the integral, with \(\Lambda\) the regularization mass-scale; \(m_{\text{bm}}(\Lambda)\) is the Lagrangian bare mass; and \(Z_{1,2}(\Lambda^2, \Lambda^2)\) are renormalization constants.

The kernel can be rendered tractable by writing [5]

$$Z_1 g^2 D_{\rho\sigma}(t) \Gamma_{\sigma}(q, q + t) = G(t^2) D_{\rho\sigma}^\text{free}(t) \tilde{\Gamma}_{\sigma}(q, q + t),$$  

(14)

wherein \(D_{\rho\sigma}^\text{free}\) is the Landau-gauge free-gauge-boson propagator, \(G\) is an interaction model and \(\tilde{\Gamma}_{\sigma}\) is an Ansatz
for the quark-gluon vertex. For the interaction, we use
\[
G(\ell^2) = \frac{4\pi^2}{\omega^6} D \ell^4 e^{-\ell^2/\omega^2},
\]
a simplified form of that in Ref. [5]. This enables us to avoid renormalization, which is straightforward but not germane to an analysis of vertex contributions that are power-law suppressed in the ultraviolet.

In order to explain the vertex Ansatz to be used, we return to perturbation theory. One can determine from Ref. [6] that at leading-order in the coupling, \(\alpha_s\), the three-gluon vertex does not contribute to the QCD analogue of Eq. (1) and the Abelian-like diagram produces the finite and negative correction \((-\alpha_s/[12\pi])\). The complete cancelation of ultraviolet divergences occurs only because of the dynamical generation of another term in the transverse part of the quark-gluon vertex; namely,
\[
\Gamma_{\mu}(p_f, p_i) = i\hat{\epsilon}_\mu \cdot k + i\gamma_\mu \sigma_{\nu\rho} k_\nu k_\rho \Gamma_{4}\,,
\]  
with \(T_{\mu\nu} = \delta_{\mu\nu} - k_\mu k_\nu/k^2\), \(a^2_\mu := T_{\mu\nu} a_{\nu}\).

Cognisant of this, we use a simple Ansatz to express the dynamical generation of an anomalous chromomagnetic moment via the dressed-quark gluon vertex; viz.,
\[
\hat{\Gamma}_\mu(p_f, p_i) = \Gamma_{\mu}^{BC}(p_f, p_i) + \Gamma_{\mu}^{\text{mix}}(p_f, p_i),
\]
\[
i\hat{\Gamma}_\mu^{BC}(p_f, p_i) = i\Sigma_m(p_f^2, p_i^2) (\gamma_\mu + 2\epsilon_\mu [\gamma_\nu \cdot \Delta_m(p_f^2, p_i^2) + \Delta_B(p_f^2, p_i^2)])
\]
where \(\Sigma_m(p_f^2, p_i^2) = [\phi(p_f^2) + \phi(p_i^2)]/2\), \(\Delta_m(p_f^2, p_i^2) = [\phi(p_f^2) - \phi(p_i^2)]/[p_f^2 - p_i^2]\), and
\[
\Gamma_{\mu}^{\text{mix}}(p_f, p_i) = \Gamma_{\mu}^{\text{mix}}(p_f, p_i) + \Gamma_{\mu}^{\text{mix}}(p_f, p_i),
\]
with \(\tau_5(p_f, p_i) = \eta \Delta_B(p_f^2, p_i^2)\), as discussed above, and
\[
\tau_4(p_f, p_i) = \mathcal{F}(z) \left[\frac{1 - 2\eta}{M_E^2} \Delta_B(p_f^2, p_i^2) - \Delta_A(p_f^2, p_i^2)\right].
\]
The denominating factor \(\mathcal{F}(z) = (1 - \exp(-z))/z\), \(z = (p_f^2 + p_i^2 - 2M_E^2)/\Lambda_x^2\), \(\Lambda_x = 1\) GeV, simplifies numerical analysis; and \(M_E = \{s|s > 0, s = M^2(s)\}\) is the Euclidean constituent-quark mass. A realistic description of the light-quark meson spectrum is obtained with \(\omega = 0.5\) GeV, \(D = (0.72\) GeV\(^2\), \(m = 5\) MeV, \(\eta = -7/4\).

A confined quark does not possess a mass-shell [9, 18]. Hence, one cannot unambiguously assign a single value to its anomalous magnetic moment. One can nonetheless compute a magnetic moment distribution. At each value of \(p^2\), we define spinors to satisfy Eqs. (4) with \(m \to M(p^2)\) := \(\varsigma\), and use
\[
\bar{u}(p_f; \varsigma) \Gamma_\mu(p_f, p_i; k) u(p_i; \varsigma)
\]
\[
= \bar{u}(p_f) [F_1(k^2) \gamma_\mu + \frac{1}{2\varsigma} \sigma_{\mu\nu} k_\nu F_2(k^2)] u(p_i),
\]
Now, from Eqs. (17) – (20), one finds
\[
\kappa^{\text{acm}}(\varsigma) = \frac{-2\gamma_5 \delta^\varsigma}{\sigma_A - 2\varsigma^2 \delta_A + 2\varsigma \delta_B},
\]
where \(\sigma_A = \Sigma_A(\varsigma, \varsigma), \delta_A = \Delta_A(\varsigma, \varsigma),\) etc. The numerator’s simplicity owes to a premeditated cancelation between \(\tau_4\) and \(\tau_5\) which replicates the one at leading-order in perturbation theory. Where a comparison of terms is possible, our vertex Ansatz is semi-quantitatively in agreement with Refs. [11, 12]. However, the presence and understanding of the role of \(\Gamma_{\mu}^{\text{mix}}\) is novel. (NB. It is apparent from Eq. (22) that \(\kappa^{\text{acm}} \propto m^2\) in the absence of DCSB, so that \(\kappa^{\text{acm}}/[2m] \rightarrow 0\) in the chiral limit.)

We can now write the BSE for the quark-photon vertex following the method of Ref. [13]. This is nontrivial but details will be reported elsewhere. Since the method guarantees preservation of the Ward-Takahashi identities, the general form of the solution is
\[
\Gamma_\mu(p_f, p_i) = \Gamma_{\mu}^{BC}(p_f, p_i) + \Gamma_{\mu}^{T}(p_f, p_i),
\]
\[
\Gamma_{\mu}^{T}(p_f, p_i) = \gamma_\mu F_1 + \sigma_{\mu\nu} k_\nu F_2 + T_{\mu\nu} \sigma_{\rho\nu} \ell \cdot k F_3
\]
\[+ |(\ell_\mu \gamma_\cdot k + i\gamma_\mu \sigma_{\nu\rho} k_\rho)| F_4 - i\ell_\mu F_5
\]
\[+ \ell_\mu \gamma_\cdot k \cdot \ell \cdot k F_6 - \ell_\mu \gamma_\cdot \ell F_7
\]
\[\ell_\mu \gamma_\cdot k \cdot \ell \cdot k F_8 \]
where \(\{F_i|i = 1, \ldots, 8\}\) are scalar functions. The Ward-Takahashi identity is plainly satisfied; viz., \(k_\mu i\Gamma_\mu(p_f, p_i) = 1/S(p_f) - 1/S(p_i)\).

We have solved for the vertex and computed the quark’s anomalous electromagnetic moment form factor
\[
f_\gamma(p) := \lim_{p_f \to p} -\frac{1}{12\pi} \frac{1}{2k^2} \text{tr} \sigma_{\mu\nu} k_\mu \Gamma_{\nu}(p_f, p) = F_2 + \frac{1}{3} p^2 F_3.
\]
The result is sizable, Fig. 1. We reiterate that \(f_\gamma\) is completely nonperturbative: in the chiral limit, at any finite order in perturbation theory, \(f_\gamma = 0\), both in our model and in QCD. For contrast we also plot the result obtained in the rainbow-ladder truncation of QCD’s DSEs. As the leading-order in a systematic but stepwise symmetry-preserving scheme [19], this truncation only partially expresses DCSB: it is exhibited by the dressed-quark propagator but not present in the quark-gluon vertex. In this case \(f_\gamma\) is nonzero but small. These are artefacts of the truncation that will not be remedied at any finite order of the procedure in Ref. [19] or a kindred scheme.

Employing Eq. (21), one can write an expression for the quark’s anomalous electromagnetic moment distribution
\[
\kappa(\varsigma) = \frac{2\varsigma \hat{F}_2 + 2\varsigma \hat{F}_4 + \Lambda_\kappa(\varsigma)}{\sigma_A + \hat{F}_1 - \Lambda_\kappa(\varsigma)},
\]
where: \(\Lambda_\kappa(\varsigma) = 2\varsigma^2 \delta^\varsigma + 2\varsigma \delta^\varsigma - \varsigma \hat{F}_5 - \varsigma^2 \hat{F}_7\); and the \(\hat{F}_i\) are evaluated at \(p_f^2 = p_i^2 = M(p_f^2)^2 := \varsigma^2, k^2 = 0\).
Plainly, $\kappa(\varsigma) \equiv 0$ in the chiral limit when chiral symmetry is not dynamically broken. Moreover, as a consequence of asymptotic freedom, $\kappa(\varsigma) \rightarrow 0$ rapidly with increasing momentum. Our computed distribution is depicted in Fig. 1. It yields Euclidean mass-shell values: $M_{\text{full}}^E = 0.44$ GeV, $\kappa_{\text{acm}} / M_{\text{RL}} = 0.35$ GeV, $\kappa_{\text{RLL}} = 0.45$ MeV, and $\kappa_{\text{RL}} = 0.22$. Our results should stimulate and provide the basic input for a reanalysis of the hadron spectrum and hadron elastic and transition electromagnetic form factors with these novel effects taken into account. Furthermore, given the magnitude of the muon "$g_{\mu} - 2$ anomaly" and its assumed importance as an harbinger of physics beyond the Standard Model [20], it might also be worthwhile to make a quantitative estimate of the contribution to $g_{\mu} - 2$ from the quark’s DCSB-induced anomalous moments. These contributions appear in the hadronic component of the photon polarization tensor.

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FIG. 1. Upper panel – $f_\gamma$, (GeV$^{-1}$) in Eq. $(25)$ of $\eta\Delta_B(p^2, p^2)$, both computed using Eqs. $(15)$, $(17)$. Lower panel – Anomalous chromo- and electro-magnetic moment distributions for a dressed-quark, computed using Eq. $(26)$. The dashed-curve in both panels is the rainbow-ladder (RL) truncation result.

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