OPERATOR INEQUALITIES VIA ACCRETIVE AND DISSIPATIVE TRANSFORMS

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Abstract. In this article, we employ certain properties of the transform $\mathcal{C}_{M,m}(A) = (M1_H - A^*)(A - m1_H)$ to obtain new inequalities for the bounded linear operator $A$ on a complex Hilbert space $H$. In particular, we obtain new relations among $|A|, |A^*|, |\mathfrak{R}A|$ and $|\mathfrak{I}A|$. Further numerical radius inequalities that extend some known inequalities will be presented too.

1. Introduction

While studying inequalities of Kantorovich type, Dragomir [8] defined the transform $\mathcal{C}_{M,m} : \mathcal{B}(H) \to \mathcal{B}(H)$ by

$$\mathcal{C}_{M,m}(A) = (M1_H - A^*)(A - m1_H),$$

where $M > m > 0$ are predefined real numbers, $\mathcal{B}(H)$ is the $C^*$-algebra of all bounded linear operators on a complex Hilbert space $H$, $1_H$ is the identity operator in $\mathcal{B}(H)$ and $A^*$ is the adjoint of $A \in \mathcal{B}(H)$.

Basic properties and applications of $\mathcal{C}_{M,m}$ were presented in [8]. Later, Niezgoda [20] used this transform to obtain certain Cassel type inequalities.

Our primary goal in this work is to use the transform $\mathcal{C}_{M,m}$ to obtain new operator inequalities that involve relations among $|A|, |A^*|, \mathfrak{R}A$ and $\mathfrak{I}A$, where the last two quantities refer respectively to the real and imaginary parts of the operator $A$. Then new forms of numerical radius inequalities are found using this transform.

To this end, we need to remind the reader of some terminologies. Recall that an operator $A \in \mathcal{B}(H)$ is said to be positive semi-definite if $\langle Ax, x \rangle \geq 0$ for all vectors $x \in H$ and we write $A \geq 0$, while it is said to be positive (or positive definite) if $\langle Ax, x \rangle > 0$ for all nonzero $x \in H$, and we write $A > 0$. The real and imaginary parts of the operator $A$ are defined respectively by $\mathfrak{R}A = \frac{A + A^*}{2}$ and $\mathfrak{I}A = \frac{A - A^*}{2i}$. When $\mathfrak{R}A \geq 0$, we say that $A$ is accretive, while $A$ is said to be dissipative if $\mathfrak{I}A \geq 0$. If $\mathfrak{R}A, \mathfrak{I}A \geq 0$, then $A$ is said to be accretive-dissipative. Accretive, dissipative, and accretive-dissipative operators have received considerable attention.

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in the literature due to their applicability in operator theory and its inequalities. We refer the reader to [1, 2, 10, 11, 14, 18, 21] as a list of references dealing with such operators.

Our approach here will be to assume accretivity or dissipativity of the transform $C_{M,m}$. For this, we begin by presenting simple properties of this transform for this context.

**Proposition 1.1.** Let $m < M$ be given real numbers and let $C_{M,m} : B(H) \to B(H)$ be the transform $C_{M,m}(A) = (M1_H - A^*)(A - m1_H)$. Then

1. $C_{M,m}(A^*) = C_{M,m}^*(A) \iff A$ is normal.
2. $C_{M,m}(A) = C_{M,m}^*(A) \iff A$ is self-adjoint.
3. $C_{M,m}(|A|)$ is accretive $\iff m1_H \leq |A| \leq M1_H$.
4. $\Re C_{M,m}(iA^*) + \Re C_{M,m}(A) \leq \left(\frac{M - m}{2}\right)^2 1_H$.
5. $\Im C_{M,m}(A) \geq 0$ if and only if $\Im A \geq 0$.
6. If $\Re C_{M,m}(A) \geq 0$, then $\Re A \geq 0$.
7. If $C_{M,m}(A)$ is accretive-dissipative, then $A$ is accretive-dissipative.

**Proof.** The statement (1) follows noting that

$$C_{M,m}^*(A) - C_{M,m}(A^*) = |A^*|^2 - |A|^2,$$

while (2) follows immediately from

$$C_{M,m}(A) - C_{M,m}^*(A) = (M - m)(A - A^*).$$

The third statement follows from the following fact

$$(M1_H - |A|)(|A| - m1_H) \geq 0 \iff m1_H \leq |A| \leq M1_H.$$

It is not hard to check that

$$\Re C_{M,m}(A) + \left| A - \frac{M + m}{2}1_H \right|^2 = \left(\frac{M - m}{2}\right)^2 1_H,$$

which together imply (4). On the other hand, direct calculations show that

$$\Im C_{M,m}(A) = (M - m)\Im A.$$
which implies (5), and (6) follows from the definition of \( C_{M,m} \) likewise. The last statement (7) follows from both (5) and (6).

\[ \square \]

Having established these fundamental properties, we proceed to our main results in the coming sections, where operator inequalities are discussed first, then numerical radius inequalities are presented.

## 2. Operator Inequalities

In this section, we present several operator inequalities using properties of the transform \( C_{M,m} \). In particular, the results of this section will be focused on relations among \(|A|, |A^*|, \mathcal{R}A, \text{ and } \mathcal{I}A\). We should remark that, in general, such relations do not exist. However, by imposing an extra condition on \( C_{M,m} \), we obtain such ties.

We notice the appearance of the constant \( \frac{M + m}{2\sqrt{Mm}} \), which is the ratio between the arithmetic and geometric means of \( M \) and \( m \). This constant is in fact the square root of the well known Kantorovich constant. In the sequel, \( m \) and \( M \) are positive numbers.

**Theorem 2.1.** Let \( A \in \mathcal{B}(\mathcal{H}) \).

(i) If \( C_{M,m}(A) \) is accretive, then

\[
|A| \leq \frac{M + m}{2\sqrt{Mm}} \mathcal{R}A.
\]

(ii) If \( C_{M,m}(iA^*) \) is accretive, then

\[
|A^*| \leq \frac{M + m}{2\sqrt{Mm}} \mathcal{I}A.
\]

(iii) If \( A \) is invertible and \( C_{M,m}(A^{-1}) \) is accretive, then

\[
|A^{-1}| \leq \frac{M + m}{2\sqrt{Mm}} \mathcal{R}A^{-1}.
\]

**Proof.** For the first statement, the assumption implies that \( \mathcal{R}C_{M,m}(A) \geq 0 \). This is equivalent to saying

\[
\frac{(MA - Mm1_\mathcal{H} - |A|^2 + mA^*) + (MA - Mm1_\mathcal{H} - |A|^2 + mA^*)^*}{2} \geq 0.
\]

Namely,

\[
(M + m) \mathcal{R}A \geq Mm1_\mathcal{H} + |A|^2.
\]

Applying the arithmetic-geometric mean inequality, we infer that

\[
Mm1_\mathcal{H} + |A|^2 \geq 2\sqrt{Mm} |A|.
\]

Combining the last two inequalities, we get (2.1).
To prove parts (ii) and (iii), we replace, in part (i), $A$ by $iA^*$, and $A$ by $A^{-1}$, respectively. \[\square\]

We note that the inequality (2.1) in Theorem 2.1 has been given in [20, Proposition 2.4], using a different method.

Theorem 2.1 entails the following reverse of the triangle inequality.

**Theorem 2.2.** Let $S, T \in \mathcal{B}(\mathcal{H})$. If $\mathcal{C}_{M,m}(\begin{bmatrix} 0 & S \\ T^* & 0 \end{bmatrix})$ is accretive, then

$$
\|S\| + \|T\| \leq \frac{M + m}{2\sqrt{Mm}} \|S + T\|.
$$

More precisely,

$$
\|S\| + \|T\| + \|S\| - \|T\| \leq \frac{M + m}{2\sqrt{Mm}} \|S + T\|.
$$

**Proof.** Let $A = \begin{bmatrix} 0 & S \\ T^* & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. If $\mathcal{C}_{M,m}(A)$ is accretive, then

$$
\|S + T\| = \|A + A^*\|
\geq \|\Re(A + A^*)\|
= 2\|\Re A\|
\geq \frac{4\sqrt{Mm}}{M + m} \|A\| \quad \text{(by (2.1))}
= \frac{4\sqrt{Mm}}{M + m} \left\| \begin{bmatrix} 0 & S \\ T^* & 0 \end{bmatrix} \right\|
= \frac{4\sqrt{Mm}}{M + m} \max(\|S\|, \|T\|).
$$

Noting the identity $\max\{a, b\} = \frac{a + b + |a - b|}{2}$, the desired inequality follows. \[\square\]

An upper bound of the difference $|A| - \Re A$ is given next.

**Corollary 2.1.** Let $A \in \mathcal{B}(\mathcal{H})$.

(i) If $\mathcal{C}_{M,m}(A)$ is accretive, then

$$
0 \leq |A| - \Re A \leq \frac{(\sqrt{M} - \sqrt{m})^2}{M + m} \|A\| \mathbf{1}_{\mathcal{H}}.
$$

(ii) If $\mathcal{C}_{M,m}(iA^*)$ is accretive, then

$$
0 \leq |A^*| - \Im A \leq \frac{(\sqrt{M} - \sqrt{m})^2}{M + m} \|A\| \mathbf{1}_{\mathcal{H}}.
$$
(iii) If $A$ is invertible and $\mathcal{C}_{M,m}(A^{-1})$ is accretive, then

$$0 \leq |A^{-1}| - \Re A^{-1} \leq \frac{(\sqrt{M} - \sqrt{m})^2}{M + m} \| A^{-1} \| 1_{\mathcal{H}}.$$

Proof. By Theorem 2.1,

$$|A| - \Re A \leq \left( 1 - \frac{2\sqrt{Mm}}{M + m} \right) |A|$$

$$= \frac{(\sqrt{M} - \sqrt{m})^2}{M + m} |A|$$

$$\leq \frac{(\sqrt{M} - \sqrt{m})^2}{M + m} \| |A| \| 1_{\mathcal{H}}$$

$$= \frac{(\sqrt{M} - \sqrt{m})^2}{M + m} \| A \| 1_{\mathcal{H}}.$$

This completes the proof of part (i).

The other two parts can be proven similarly.

In the following theorem, we will need the fact that the function $f(x) = x^2$ is operator convex on $\mathbb{R}$. This means that when $A$ and $B$ are self adjoint operators in $\mathcal{B}(\mathcal{H})$, we have $((1 - t)A + tB)^2 \leq (1 - t)A^2 + tB^2$ for $0 \leq t \leq 1$. We refer the reader to [3, Example V.1.3] for further information on this topic.

**Theorem 2.3.** Let $A \in \mathcal{B}(\mathcal{H})$. If $\mathcal{C}_{M,m}(A)$ and $\mathcal{C}_{M,m}(iA^*)$ are accretive, then for any $0 \leq t \leq 1$,

$$(1 - t) |A^*| + t |A| \leq \frac{M + m}{2\sqrt{Mm}} ((1 - t) \Re A + t\Re A).$$

Proof. By (i) and (ii) of Theorem 2.1, we have

$$(1 - t) Mm1_{\mathcal{H}} + (1 - t) |A^*|^2 \leq (1 - t) (M + m) \Re A$$

and

$$tMm1_{\mathcal{H}} + t|A|^2 \leq t (M + m) \Re A.$$

Adding these inequalities, we obtain

$$Mm1_{\mathcal{H}} + (1 - t) |A^*|^2 + t|A|^2 \leq (M + m) ((1 - t) \Re A + t\Re A).$$

Now, using the operator convexity of the function $f(t) = t^2$ on $(0, \infty)$ and the arithmetic-geometric mean inequality, we get

$$2\sqrt{Mm} ((1 - t) |A^*| + t |A|) \leq Mm1_{\mathcal{H}} + ((1 - t) |A^*| + t |A|)^2$$

$$\leq Mm1_{\mathcal{H}} + (1 - t) |A^*|^2 + t|A|^2.$$
This completes the proof. □

Squaring operator inequalities are not as straightforward as squaring real inequalities. In other words, if \( a, b \) are positive number such that \( a \leq b \) then \( a^2 \leq b^2 \). Now, if \( A \leq B \), where \( A, B \) are positive operators, then we cannot conclude \( A^2 \leq B^2 \) since the function \( f(x) = x^2 \) is not operator monotone. We refer the reader to [3, Chapter V] to get more insight about this.

The following result shows the inequalities in Theorem 2.1 can be squared.

**Theorem 2.4.** Let \( A \in \mathcal{B}(\mathcal{H}) \).

(i) If \( \mathcal{C}_{M,m}(A) \) is accretive, then

\[
|A|^2 \leq \left( \frac{M + m}{2\sqrt{Mm}} \right)^2 (\Re A)^2.
\]

(ii) If \( \mathcal{C}_{M,m}(iA^*) \) is accretive, then

\[
|A^*|^2 \leq \left( \frac{M + m}{2\sqrt{Mm}} \right)^2 (\Im A)^2.
\]

(iii) If \( A \) is invertible and \( \mathcal{C}_{M,m}(A^{-1}) \) is accretive, then

\[
|A^{-1}|^2 \leq \left( \frac{M + m}{2\sqrt{Mm}} \right)^2 (\Re A^{-1})^2.
\]

**Proof.** By the inequality (2.2),

\[
|A|^2 \leq (M + m) \Re A - Mm1_{\mathcal{H}}.
\]

So, to prove the inequality (2.3), it is enough to show that

\[
(M + m) \Re A - Mm1_{\mathcal{H}} \leq \left( \frac{M + m}{2\sqrt{Mm}} \right)^2 (\Re A)^2,
\]

holds. Define

\[
f(t) = \left( \frac{M + m}{2\sqrt{Mm}} \right)^2 t^2 - (M + m) t + Mm.
\]

Then

\[
f'(t) = \frac{(M + m)^2}{2Mm} t - (M + m),
\]

and

\[
f''(t) = \frac{(M + m)^2}{2Mm} > 0.
\]

Namely, \( f \) is convex. On the other hand, if we put \( f'(t) = 0 \), then we get \( t = 2Mm/(M + m) \), and \( f(2Mm/(M + m)) = 0 \). So \( f(t) \) is positive, i.e.,

\[
(M + m) t - Mm \leq \left( \frac{M + m}{2\sqrt{Mm}} \right)^2 t^2.
\]

We get the desired result by applying functional calculus for the positive operator \( \Re A \).

The other parts can be established similarly, so we omit the details. □
In fact, Theorem 2.1 is a direct consequence of Theorem 2.4, since \( f(t) = t^{1/2} \) is operator monotone on \((0, \infty)\), \([4, \text{Theorem 1.5.9}]\).

Our next target is to investigate commutators of \(|A|\) and \( \mathcal{R}A \). To this end, the following lemma will be needed.

**Lemma 2.1.** \([19, \text{Lemma 2.1}]\) Let \( A, B \in \mathcal{B} (\mathcal{H}) \) be positive operators and let \( \alpha > 0 \). Then

\[
A \leq \alpha B \iff \left\| A^{1/2}B^{-1/2} \right\| \leq \sqrt{\alpha}.
\]

**Corollary 2.2.** Let \( A \in \mathcal{B} (\mathcal{H}) \) be such that both \( \mathcal{R}A \) and \( \mathcal{I}A \) are invertible.

(i) If \( \mathcal{E}_{m,m} (A) \) is accretive, then

\[
\left\| |A| (\mathcal{R}A)^{-1} + (\mathcal{R}A)^{-1} |A| \right\| \leq \frac{M + m}{\sqrt{Mm}} 1_\mathcal{H},
\]

and

\[
|A| (\mathcal{R}A)^{-1} + (\mathcal{R}A)^{-1} |A| \leq \frac{M + m}{\sqrt{Mm}} 1_\mathcal{H}.
\]

(ii) If \( \mathcal{E}_{m,m} (iA^*) \) is accretive, then

\[
\left\| |A^*| (\mathcal{I}A)^{-1} + (\mathcal{I}A)^{-1} |A^*| \right\| \leq \frac{M + m}{\sqrt{Mm}} 1_\mathcal{H},
\]

and

\[
|A^*| (\mathcal{I}A)^{-1} + (\mathcal{I}A)^{-1} |A^*| \leq \frac{M + m}{\sqrt{Mm}} 1_\mathcal{H}.
\]

**Proof.** By Lemma 2.1, the inequality (2.3) is equivalent to

\[
\left\| |A| (\mathcal{R}A)^{-1} \right\| \leq \frac{M + m}{2\sqrt{Mm}}
\]

By \([16, \text{Lemma 3.5.12}]\), we get

\[
\begin{bmatrix}
\frac{M + m}{2\sqrt{Mm}} 1_\mathcal{H} & |A| (\mathcal{R}A)^{-1} \\
(\mathcal{R}A)^{-1} |A| & \frac{M + m}{2\sqrt{Mm}} 1_\mathcal{H}
\end{bmatrix} \geq 0 \quad \text{and} \quad \begin{bmatrix}
\frac{M + m}{2\sqrt{Mm}} 1_\mathcal{H} & (\mathcal{R}A)^{-1} |A| \\
|A| (\mathcal{R}A)^{-1} & \frac{M + m}{2\sqrt{Mm}} 1_\mathcal{H}
\end{bmatrix} \geq 0.
\]

Adding these two operator matrices, we have

\[
\begin{bmatrix}
\frac{M + m}{\sqrt{Mm}} 1_\mathcal{H} & |A| (\mathcal{R}A)^{-1} + (\mathcal{R}A)^{-1} |A| \\
(\mathcal{R}A)^{-1} |A| + |A| (\mathcal{R}A)^{-1} & \frac{M + m}{\sqrt{Mm}} 1_\mathcal{H}
\end{bmatrix} \geq 0.
\]

This completes the proof. \(\square\)

**Remark 2.1.** To show how Corollary 2.2 improves Theorem 2.4, notice that

\[
4 |A| (\mathcal{R}A)^{-2} |A| \leq 4 \left\| |A| (\mathcal{R}A)^{-2} |A| \right\| 1_\mathcal{H} \leq \left\| |A| (\mathcal{R}A)^{-1} + (\mathcal{R}A)^{-1} |A| \right\|^2 1_\mathcal{H} \quad \text{(by \([5]\))}
\]

\[
\leq \left( \frac{M + m}{\sqrt{Mm}} \right)^2 1_\mathcal{H}.
\]
which is equivalent to saying that

\[
(\mathcal{R}A)^{-2} \leq \| |A| (\mathcal{R}A)^{-2} |A| |A|^{-2}
\]
\[
\leq \left\| \frac{|A| (\mathcal{R}A)^{-1} + (\mathcal{R}A)^{-1} |A|}{2} \right\|^2 |A|^{-2}
\]
\[
\leq \left( \frac{M + m}{2\sqrt{Mm}} \right)^2 |A|^{-2}.
\]

Now, by taking the inverse, we get

\[
(\mathcal{R}A)^2 \geq \| |A| (\mathcal{R}A)^{-2} |A| |A|^{-1} |A|^2
\]
\[
\geq \left\| \frac{|A| (\mathcal{R}A)^{-1} + (\mathcal{R}A)^{-1} |A|}{2} \right\|^{-2} |A|^2
\]
\[
\geq \left( \frac{2\sqrt{Mm}}{M + m} \right)^2 |A|^2.
\]

For an arbitrary \( A \in B(\mathcal{H}) \), the inequality

\[(2.4) \quad \Phi^\frac{1}{2}(|A|^2) \geq \Phi(|A|)\]

is well known for the unital positive linear map \( \Phi : B(\mathcal{H}) \to B(\mathcal{H}), \) [6, 7]. In this context, recall that such map is a map that satisfies \( \Phi(A) \geq 0 \) whenever \( A \geq 0 \) and \( \Phi(1_H) = 1_H \). In what follows a reversed version is presented via the transform \( \mathcal{C}_{M,m} \).

**Lemma 2.2.** Let \( A \in B(\mathcal{H}) \) and let \( \Phi \) be a unital positive linear map on \( B(\mathcal{H}) \). If \( \mathcal{C}_{M,m} (|A|) \) is accretive, then

\[
\Phi^\frac{1}{2}(|A|^2) \leq \frac{M + m}{2\sqrt{Mm}} \Phi (|A|).
\]

**Proof.** Since \( \mathcal{C}_{M,m} (|A|) \) is accretive, we have by [13, Theorem 1.32 (iii)],

\[
\Phi (|A|^2) \leq \frac{(M + m)^2}{4Mm} \Phi^2 (|A|).
\]

The result follows by taking into account that the function \( f(t) = t^\frac{1}{2} \) is operator monotone on \((0, \infty)\). \( \square \)

In the next theorem, reverses of the inequalities of Theorem 2.1 are presented.

**Theorem 2.5.** Let \( A \in B(\mathcal{H}) \).

(i) If \( \mathcal{C}_{M,m} (|A|) \) is accretive, then

\[
\mathcal{R}A \leq \frac{M + m}{2\sqrt{Mm}} |A|.
\]
(ii) If $C_{M,m}(|A^*|)$ is accretive, then

$$\Im A \leq \frac{M + m}{2\sqrt{Mm}} |A^*|.$$  

(iii) If $A$ is invertible and $C_{M,m}(|A^{-1}|)$ is accretive, then

$$\Re A^{-1} \leq \frac{M + m}{2\sqrt{Mm}} |A^{-1}|.$$  

Proof. We prove part (i) since the other parts are easy to prove. By the inequality (2.4), we have for any unit vector $x \in H$,

$$|\langle Ax, x \rangle| \leq \langle |A|^2 x, x \rangle^{\frac{1}{2}}.$$  

Applying Lemma 2.2 for $\Phi (T) = \langle Tx, x \rangle 1_H (x \in H, \|x\| = 1)$, implies

$$\langle |A|^2 x, x \rangle^{\frac{1}{2}} \leq \frac{M + m}{2\sqrt{Mm}} \langle |A| x, x \rangle.$$  

Hence,

$$|\langle Ax, x \rangle| \leq \frac{M + m}{2\sqrt{Mm}} \langle |A| x, x \rangle.$$  

Now, by combining this inequality with the fact that $\langle \Re Ax, x \rangle = \Re \langle Ax, x \rangle \leq |\langle Ax, x \rangle|$, we reach the desired result. \hfill \Box

Corollary 2.3. Let $A \in B(H)$.

(i) If $C_{M,m}(|A|)$ is accretive, then

$$\Re A - |A| \leq \frac{(M - m)^2}{2\sqrt{Mm}} \|A\| 1_H.$$  

(ii) If $C_{M,m}(|iA^*|)$ is accretive, then

$$\Im A - |A^*| \leq \frac{(M - m)^2}{2\sqrt{Mm}} \|A\| 1_H.$$  

3. Numerical Radius Inequalities

In this section, we use the properties mentioned above of the transform $C_{M,m}$ and its consequences to obtain some new numerical radius inequalities. In this context, we recall that the numerical radius of an operator $A \in B(H)$ is defined by $\sup_{\|x\|=1} |\langle Ax, x \rangle|$. The numerical radius has a notable recognition in the literature due to its impact in understanding the geometry of the numerical range of the operator. Among the most basic inequalities of the numerical radius, we have

$$\frac{1}{2} \|A\| \leq \omega(A) \leq \|A\|$$
and

\[(3.2) \quad \|\mathcal{R}A\| \leq \omega(A) \text{ and } \|\mathcal{I}A\| \leq \omega(A).\]

We begin with the following observation.

**Remark 3.1.** From Theorem 2.1, we have

\[\|A\| \leq \left\| \frac{M + m}{2\sqrt{Mm}} \mathcal{R}A \right\| = \frac{M + m}{2\sqrt{Mm}} \|\mathcal{R}A\|.
\]

That is

\[\|A\| \leq \frac{M + m}{2\sqrt{Mm}} \|\mathcal{R}A\| \quad \text{and} \quad \|\mathcal{R}A\| - \|A\| \leq \left(\frac{\sqrt{M} - \sqrt{m}}{M + m}\right)^2 \|\mathcal{R}A\|.
\]

Noting that for any operator \(A \in \mathcal{B}(\mathcal{H})\), \(\omega(A) \leq \|A\|\) and \(\omega(A) \geq \mathcal{R}A\), we deduce that when \(\mathcal{C}_{M, m}(A)\) is accretive, one has

\[(3.3) \quad \omega(A) \leq \frac{M + m}{2\sqrt{Mm}} \|\mathcal{R}A\| \quad \text{and} \quad \omega(A) - \|\mathcal{R}A\| \leq \left(\frac{\sqrt{M} - \sqrt{m}}{M + m}\right)^2 \omega(A),
\]

and

\[(3.4) \quad \|A\| \leq \frac{M + m}{2\sqrt{Mm}} \omega(A) \quad \text{and} \quad \|A\| - \omega(A) \leq \left(\frac{\sqrt{M} - \sqrt{m}}{M + m}\right)^2 \|A\|.
\]

The last two inequalities in (3.4) have been proved in [9, Remark 35]. Before proceeding, it is worth mentioning the importance of the above inequalities. We know that for any \(A \in \mathcal{B}(\mathcal{H})\), \(\|\mathcal{R}A\| \leq \omega(A)\). Thus, the first inequality in (3.3) provides a reversed version of this known inequality. Of course, this is valid when \(\mathcal{C}_{M, m}\) is accretive. Further, under this condition, the first inequality in (3.4) provides a refinement of the well known inequality \(\|A\| \leq 2\omega(A)\) in case we have \(\frac{M + m}{2\sqrt{Mm}} < 2\). Notice that this latter ratio is always not less than one.

In the following, we present an inequality that relates the numerical radius of \(A\) with the norms of its real and imaginary parts, as a reversed type of (3.2).

**Theorem 3.1.** Let \(A \in \mathcal{B}(\mathcal{H})\). If \(\mathcal{C}_{M, m}(A)\) and \(\mathcal{C}_{M, m}(iA^*)\) are accretive, then

\[\omega(A) \leq \frac{M + m}{2\sqrt{Mm}} \sqrt{\|\mathcal{R}A\| \|\mathcal{I}A\|}.
\]
Proof. Let \( x \in \mathcal{H} \) be a unit vector. Then by the mixed Schwarz inequality [15, pp. 75-76], and Theorem 2.1, we have
\[
|\langle Ax, x \rangle| \leq \sqrt{\langle |A| x, x \rangle \langle |A^*| x, x \rangle}
\]
\[
\leq \frac{M + m}{2\sqrt{Mm}} \sqrt{\langle |A A^*| x, x \rangle \langle |A^* A| x, x \rangle}
\]
\[
\leq \frac{M + m}{2\sqrt{Mm}} \sqrt{\| |A^*| \| \| A \|}
\]
Thus,
\[
|\langle Ax, x \rangle| \leq \frac{M + m}{2\sqrt{Mm}} \sqrt{\| |A^*| \| \| A \|}.
\]
Now, by taking supremum over all unit vector \( x \in \mathcal{H} \), we get the desired result.

In the following, we present a lower bound of the numerical radius in terms of \( || A^2 + |A^*|^2 || \). The significance of this result is explained in Remark 3.2 below.

**Theorem 3.2.** Let \( A \in \mathcal{B}(\mathcal{H}) \). If \( \mathcal{C}_{M, m}(A) \) is accretive, then
\[
\frac{2Mm}{(M + m)^2} || A^2 + |A^*|^2 || \leq \omega^2 (A).
\]

**Proof.** We know that
\[
|| A ||^2 = || |A| ||^2 = || |A^2| || = || |A^*|^2 || = || |A^*|^2 ||.
\]
This, together with Theorem 2.1, implies that
\[
|| |A^2 + |A^*|^2 || \leq \frac{(M + m)^2}{2Mm}\omega^2 (A),
\]
as desired.

**Remark 3.2.** If
\[
Mm \geq \frac{1}{4}(M - m)^2,
\]
then, Theorem 3.2 improves (see [17, Theorem 1])
\[
\frac{1}{4} || |A^2 + |A^*|^2 || \leq \omega^2 (A).
\]

If we let \( f(x) = x - \frac{(x-1)^2}{4}, \ x \geq 1 \), we can see that \( f \) is increasing on \([1, 3]\) and is decreasing afterwards. Calculating, we find that \( f(x) = 0 \) when \( x = 3 + 2\sqrt{2} \), and that \( f \geq 0 \) on \([3 + 2\sqrt{2}, \infty)\), while it is negative on \([3 + 2\sqrt{2}, \infty)\). Letting \( x = \frac{M}{m} \), this means that the condition \( Mm \geq \frac{1}{4}(M - m)^2 \) when \( 1 < \frac{M}{m} \leq 3 + 2\sqrt{2} \).

On the other hand, a submultiplicative inequality for the numerical radius may be shown as follows.
Corollary 3.1. Let $A, B \in \mathcal{B}(\mathcal{H})$. If $\mathcal{C}_{M,m}(A)$ and $\mathcal{C}_{N,n}(B)$ are accretive, then

$$\omega(AB) \leq \frac{(M + m)(N + n)}{4\sqrt{MNmn}} \omega(A) \omega(B).$$

Proof. We have

$$\omega(AB) \leq \|AB\| \leq \|A\| \|B\| \leq \frac{(M + m)(N + n)}{4\sqrt{MNmn}} \omega(A) \omega(B),$$

where we have used (3.4) to obtain the last inequality. This completes the proof. \qed

Remark 3.3. If

$$\left(\sqrt{MN} - \sqrt{mn}\right)^2 + \left(\sqrt{Mn} - \sqrt{Nm}\right)^2 \leq 12\sqrt{MNmn},$$

then, Corollary 3.1 refines (see [14, Theorem 2.5-2])

$$\omega(AB) \leq 4\omega(A) \omega(B).$$

It is well known that when $A, B \in \mathcal{B}(\mathcal{H})$ then $\omega(AB - BA^*) \leq 2\|A\|\omega(B)$, [12]. In the following, we present a refinement of this inequality when $\mathcal{C}_{M,m}$ is accretive.

Theorem 3.3. Let $A, B \in \mathcal{B}(\mathcal{H})$.

(i) If $\mathcal{C}_{M,m}(A)$ is accretive, then

$$\omega(AB - BA^*) \leq (M - m) \omega(B).$$

(ii) If $\mathcal{C}_{M,m}(iA)$ is accretive, then

$$\omega(AB + B^*A) \leq (M - m) \omega(B).$$

Proof. By the inequality (see [12])

$$\omega(AB - BA^*) \leq 2\|A\|\omega(B),$$

and the relation (1.1), we can write

$$\omega(AB - BA^*) = \omega\left(\left(A - \frac{M + m}{2}1_\mathcal{H}\right)B - B\left(A^* - \frac{M + m}{2}1_\mathcal{H}\right)\right)$$

$$\leq 2\left\|A^* - \frac{M + m}{2}1_\mathcal{H}\right\|\omega(B)$$

$$= 2\left\|A - \frac{M + m}{2}1_\mathcal{H}\right\|\omega(B)$$

$$\leq (M - m) \omega(B),$$

as desired.

The inequality in part (ii) can be shown similarly, so we omit the details. \qed
In the following we give an example to show how Theorem 3.3 improves the inequality \( \omega(AB - BA^*) \leq 2\|A\|\omega(B) \).

**Example 3.1.** Let \( A = \begin{bmatrix} 2 & 0 \\ -1 & 4 \end{bmatrix} \), \( M = 8 \), and \( m = 0.01 \). A simple calculation shows that

\[
\mathcal{K}_{M,m}(A) = \begin{bmatrix}
\frac{551}{200} & -\frac{1601}{200} \\
\frac{1601}{200} & \frac{399}{25}
\end{bmatrix} > 0.
\]

In this case

\[
2\|A\| \approx 8.31
\]

while

\[
M - m = 7.99.
\]

These values imply that Theorem 3.3 improves the inequality

\[
\omega(AB - BA^*) \leq 2\|A\|\omega(B).
\]

We conclude with the following result.

**Corollary 3.2.** Let \( A, B \in B(H) \). If \( \mathcal{C}_{M,m}(A) \) and \( \mathcal{C}_{M,m}(iA) \) are accretive, then

\[
\omega(AB) \leq (M - m)\omega(B).
\]

More precisely,

\[
\omega(AB) + \frac{1}{2}|\omega(AB + B^*A) - \omega(AB - B^*A)| \leq (M - m)\omega(B).
\]

**Proof.** By Theorem 3.3,

\[
\omega(AB \pm B^*A) \leq (M - m)\omega(B).
\]

This implies,

\[
\omega(AB) + \frac{1}{2}|\omega(AB + B^*A) - \omega(AB - B^*A)|
\leq \frac{1}{2}(\omega(AB + B^*A) + \omega(AB - B^*A) + |\omega(AB + B^*A) - \omega(AB - B^*A)|)
= \max\{\omega(AB + B^*A), \omega(AB - B^*A)\}
\leq (M - m)\omega(B),
\]

as desired. \(\square\)
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