Quantum mechanics for military officers

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Abstract
We present a trivial probabilistic illustration for representation of quantum mechanics as an algorithm for approximative calculation of averages.

1 Introduction
In a series of authors papers [1]–[4] there was developed an asymptotic approach to the problem of hidden variables in quantum mechanics. It was shown that, in spite of all “NO-GO” theorems (e.g., von Neumann [5], Cohen-Specker, Bell [6]), it is possible to construct a classical physical model, Pre-quantum Classical Statistical Field Theory – PCSFT, such that quantum mechanics can be considered as an approximation of PCSFT with respect to a small parameter $\alpha$. This parameter is given by the dispersion of fluctuations of classical random field which is represented in quantum formalism by a von Neumann density operator. In PCSFT the role of hidden variables is played by classical fields. This is a model of classical statistical mechanics with the infinite dimensional phase space. The infinite dimension induces a number of rather technical mathematical problems, in particular, using theory of measures on Hilbert spaces. Such purely mathematical difficulties embarrass understanding of PCSFT. In this paper there will be presented a simplified version of the approximation algorithm which induces the quantum rule for calculation of averages – the von Neumann trace-formula [5].
2 Method of Taylor approximation for functions of random variables

Here we follow chapter 11 of the book [7] of Elena Ventzel. This book was the basic book for teaching probability theory in Soviet military colleges.\footnote{I am thankful to my farther-in-law, Alexander Choustov (marine artillery officer) who pointed out to this chapter.} Elena Ventzel wrote her book in the form of precise instructions what student should follow to solve a problem:

“In practice we have very often situations in that, although investigated function of random arguments is not strictly linear, but it differs practically so negligibly from a linear function that it can be approximately considered as linear. This is a consequence of the fact that in many problems fluctuations of random variables play the role of small deviations from the basic law. Since such deviations are relatively small, functions which are not linear in the whole range of variation of their arguments are almost linear in a restricted range of their random changes,” [7], p. 238.

Let \( y = f(x) \). Here in general \( f \) is not linear, but it does not differ so much from linear on some interval \([m_x - \delta, m_x + \delta]\), where \( x = x(\omega) \) is a random variable and

\[
m_x \equiv E_x = \int x(\omega) \, dP(\omega)
\]

is its average. Here \( \delta > 0 \) is sufficiently small. Student of a military college should approximate \( f \) by using the first order Taylor expansion at the point \( m_x \):

\[
y(\omega) \approx f(m_x) + f'(m_x)(x(\omega) - m_x).
\]

By taking the average of both sides he obtains:

\[
m_y \approx f(m_x).
\]

The crucial point is that the linear term \( f'(m_x)(x(\omega) - m_x) \) does not give any contribution! Further Elena Ventzel pointed out [7], p. 245: “For some problems the above linearization procedure may be unjustified, because the method of linearization may be not produce a sufficiently good approximation. In such cases to test the applicability of the linearization method and
to improve results there can be applied the method which is based on preserving not only the linear term in the expansion of function, but also some terms of higher orders.

Let \( y = f(x) \). Student now should preserve the first three terms in the expansion of \( f \) into the Taylor series at the point \( m_x \):

\[
y(\omega) \approx f(m_x) + f'(m_x)(x(\omega) - m_x) + \frac{1}{2} f''(m_x)(x(\omega) - m_x)^2. \tag{3}
\]

Hence

\[
m_y \approx f(m_x) + \frac{\sigma_x^2}{2} f''(m_x), \tag{4}
\]

where

\[
\sigma_x^2 = E (x - m_x)^2 = \int (x(\omega) - m_x)^2 \, dP(\omega)
\]

is the dispersion of the random variable \( x \).

Let us now consider the special case of symmetric fluctuations:

\[
m_x = 0
\]

and let us restrict considerations to functions \( f \) such that

\[
f(0) = 0.
\]

Then we obtain the following special form of (4):

\[
m_y \approx \frac{\sigma_x^2}{2} f''(0). \tag{5}
\]

We emphasize again that the first derivative does not give any contribution into the average.

Thus at the same level of approximation we can calculate averages not by using the Lebesgue integral (as we do in classical probability theory), but by finding the second derivative. Such a “calculus of probability” would match well with experiment. I hope that reader has already found analogy with the quantum calculus of probabilities. But for a better expression of this analogy we shall consider the multi-dimensional case. Let now

\[
x = (x_1, ..., x_n),
\]

so we consider a system of \( n \) random variables. We consider the vector average:

\[
m_x = (m_{x_1}, ..., m_{x_n})
\]
and the covariance matrix:

$$B_x = (B_x^{ij}), \quad B_x^{ij} = E (x_i - m_x_i) (x_j - m_x_j).$$

We now consider the random variable $y(\omega) = f(x_1(\omega), ..., x_n(\omega))$. By using the Taylor expansion we would like to obtain an algorithm for approximation of the average $m_y$. We start directly from the second order Taylor expansion:

$$y(\omega) \approx f(m_{x_1}, ..., m_{x_n}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(m_{x_1}, ..., m_{x_n})(x_i(\omega) - m_{x_i})$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(m_{x_1}, ..., m_{x_n})(x_i(\omega) - m_{x_i})(x_j(\omega) - m_{x_j}),$$  \hspace{1cm} (6)

and hence:

$$m_y \approx f(m_{x_1}, ..., m_{x_n}) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(m_{x_1}, ..., m_{x_n})B_x^{ij}. \hspace{1cm} (7)$$

By using the vector notations we can rewrite the previous formulas as:

$$y(\omega) \approx f(m_x) + (f'(m_x), x(\omega) - m_x) + \frac{1}{2}(f''(m_x)(x(\omega) - m_x), x(\omega) - m_x).$$ \hspace{1cm} (8)

and

$$m_y \approx f(m_x) + \frac{1}{2} \text{Tr} B_x f''(m_x). \hspace{1cm} (9)$$

Let us again consider the special case: $m_x = 0$ and $f(0) = 0$. We have:

$$m_y \approx \frac{1}{2} \text{Tr} B_x f''(0). \hspace{1cm} (10)$$

We now remark that the Hessian $f''(0)$ is always a symmetric operator. Let us now represent $f$ by its second derivative at zero:

$$f \rightarrow A = \frac{1}{2} f''(0).$$

Then we see that, at some level of approximation, instead of operation with Lebesgue integrals, one can use linear algebra:

$$m_y \approx \text{Tr} B_x A \hspace{1cm} (11)$$
References

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