MULTIPLE SOFT PION PRODUCTION WITHIN NONLINEAR CHIRAL SIGMA MODEL

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Abstract

Multiple soft pion production in the baryon scattering reactions is considered in the framework of chiral nonlinear sigma model neglecting the pion mass. Treating baryons in the eikonal approximation as classical sources, a set of analytical solutions for the pion field is found. A tree $S$-matrix is constructed on the basis of these solutions describing the emission (or absorption) of any number of soft pions. Then the contribution of soft virtual pions is taken into account in a closed form. It is shown that the loop corrections strongly suppress the pion radiation, and for the two limiting cases of non-relativistic and ultra-relativistic baryon scatterings there is no pion emission from the "ends". Thus, the mechanism similar to the soft photon bremsstrahlung in the quantum electrodynamics seems to be unable to create a state with a large number of the soft pions.

1. The problem of possible production of the classical pion field in the high energy hadron-hadron or ion-ion collisions has recently drawn much attention [1]. Very interesting example of the classical pion field is a disoriented chiral condensate (DCC) introduced by Bjorken and co-authors [2]. Possible experimental signatures of DCC are of special importance since one has to distinguish between this phenomenon and the effects arising due to the independent emission of a large number of soft pions. It is well-known that these effects can be described in terms of a coherent state resembling bremsstrahlung photon in the electrodynamics [3, 4] (see also [5]). Neglecting the pion interaction, such an approach leads to the predictions for the charge distribution of pions in the multiple production processes, which are really almost identical to what could be expected from DCC [3, 7]. That is why it is of great interest to study how the pion interaction modifies these results.

The multiple soft pion production in the hadron-hadon collisions is treated here within the nonlinear sigma-model. This model includes three isovector fields $\pi_i$, $i = 1, 2, 3$, and an auxiliary scalar field $\sigma$ obeying the constraint

$$\sigma^2 + \vec{\pi}^2 = f^2, \quad f = 93 \text{ MeV}.$$
The lagrangian is
\[ L = L_\pi + L_f, \]  
where \( L_\pi \) is a pure pionic part,
\[ L_\pi = \frac{1}{2}[(\partial_\mu \sigma)^2 + (\partial_\mu \vec{\pi})^2], \]
and \( L_f \) describes the pion interaction with fermions,
\[ L_f = \bar{q} i \gamma^\mu \partial_\mu q - g\bar{q}(\sigma + i\vec{\pi}\vec{\tau}\gamma_5)q. \]
The vacuum of the model corresponds to the expectation values \( \langle \sigma \rangle = f_\pi \), \( \langle \pi_i \rangle = 0 \), which result into the fermion mass \( M = gf_\pi \). The vacuum is degenerate, there is a set of vacuum configurations
\[ \langle \sigma \rangle = f_\pi \cos \theta, \quad \langle \pi_i \rangle = f_\pi n_i \sin \theta \]
for arbitrary parameter \( \theta \) and unit vector \( n_i \).

The model (1) provides a simple but qualitatively satisfactory description for soft pion-nucleon physics, \( g_{\pi NN} \) and \( M \) being pion-nucleon interaction constant and nucleon mass, respectively.

The sigma model can be also regarded as an effective chiral theory describing the low energy limit of QCD. It correctly reflects the important property of QCD, namely, spontaneous chiral symmetry breaking due to the quark condensate \( \langle \bar{q}q \rangle \neq 0 \) and the goldstone nature of pions. In this case \( M = M_q \) is the mass of the constituent quark \( q \).

The pion field represents the chiral phase of the quark condensate, that is why the parametrization in the form of a unitary matrix is quite natural in this approach. Constructing the matrix
\[ U = \frac{1}{f_\pi}(\sigma + i\vec{\pi}\vec{\tau}), \quad U^+U = 1, \]
the lagrangian (2) takes the form
\[ L_\pi = \frac{1}{4}f_\pi^2 Tr(\partial_\mu U^+\partial_\mu U). \]
It is obviously symmetric under the chiral transformation
\[ U \rightarrow SUS \]
and the isotopic rotation
\[ U \rightarrow V^{-1}UV \]
for any unitary matrices \( S \) and \( V \).

In this paper the mass \( M \) is supposed to be much larger than the pion momenta. This is the case of a soft pion bremsstrahlung when pions are
emitted from the ends of the diagram. Each external line acquires, due to
the pion emission, the factor
\[ \exp \left\{ \frac{i}{2M} \int_0^\infty d\tau A(x_\tau) \right\} \]  
(3)
for outgoing particles and
\[ \exp \left\{ \frac{i}{2M} \int_{-\infty}^0 d\tau A(x_\tau) \right\} \]  
(4)
for incoming ones. Here
\[ A(x) = g(\sigma(x) - f_\pi) \bar{N}(p) N(p) + i\pi(x) \bar{N}(p) \tau_5 N(p) = 2M g(\sigma(x) - f_\pi) \]
and the integration goes along a worldline of an emitting particle \( x_\tau = \tau p / M \).
These expressions are clearly equivalent to the effect of a pointlike external
source moving with a constant velocity \( \vec{v} = \vec{p} / M \), with a lagrangian
\[ L_s = g(\sigma(x) - f_\pi) \theta(\pm t) \delta^3_v(\vec{x} - \vec{v}t) \]
in which
\[ \delta^3_v(\vec{x} - \vec{v}t) = \delta\left(\frac{x_3 - vt}{\sqrt{1 - v^2}}\right) \delta^2(x_\perp) \].
In matrix notations it is
\[ L_s = \frac{1}{4} g_f \pi T r(U + U^+ - 2) \delta^3_v(\vec{x} - \vec{v}t) \]  
(5)
Detailed structure of colliding baryons is not important for the soft emission
which depends on the interaction constant \( g \) only, which has a meaning of
the baryon form factor. It can be expressed through ”scalar density” of the
baryon state \( |N\rangle \)
\[ \rho(x) = \langle N | \bar{\pi}(x) q(x) | N \rangle \]
as
\[ g = \int d^3x \rho(x) \]  
(6)
Here the pseudoscalar density \( \rho_{ps} = \langle N | \bar{\pi}(x) \tau_5 q(x) | N \rangle \) is supposed to be
zero. From this point of view both interpretations of the sigma model men-
tioned above give the same answer. However, this is not true for the central
part of the hard process occurring at small distances for \( t = 0 \).
2. To begin with, consider the classical pion field produced by a single point-
like source moving with constant velocity \( \vec{v} \) along the \( x_3 \) direction. Upon
varying the lagrangian with respect to the pion matrix \( U \), the equation of
motion yields
\[ \partial_\mu [U^+ \partial_\mu U] = \frac{g}{2f_\pi} \delta^3_v(\vec{x} - \vec{v}t) [U^+ - U] \]  
(7)
In what follows the matrix \( U(x) \) is sought among exact solutions found in refs. [8, 9]:

\[
U(x) = V^{-1}e^{i\tau_3 f(x)} V,
\]  

(8)

where \( V \) is an arbitrary but constant unitary matrix. The function \( f(x) \) obeys the equation

\[
\partial^2 f(t, \vec{x}) = \frac{g}{f_\pi} \delta^3(\vec{x} - \vec{\nu} t) \sin f(t, \vec{\nu} t).
\]

The solution decreasing at the infinity has the form of "moving Coulomb potential":

\[
f(t, \vec{x}) = -\frac{g}{4\pi f_\pi} \frac{1}{|x_v|} \sin f(\tau_v, \vec{\nu} \tau_v),
\]

(9)

\[
|x_v| = \left[ \left( \frac{x_3 - vt}{\sqrt{1 - v^2}} \right)^2 + x_\perp^2 \right]^{\frac{1}{2}},
\]

\[
\tau_v = \frac{1}{1 - v^2} \left[ t - vx_3 - \sqrt{1 - v^2} |x_v| \right]
\]

(10)

which is defined through unknown function \( f(\tau_v, \vec{\nu} \tau_v) \). To get the closed equation for it one has to take the l.h.s. of (9) at the source worldline \((t, \vec{v}t)\). However \( f(t, \vec{v}t) \) is divergent since \( |x_v| \) turns into zero. As the divergency is a clear consequence of the point-like structure of the source, it is straightforward to cure it by spreading the delta-function in eq. (7) over a small but finite space volume:

\[
\partial_\mu \left[ U^+ \partial_\mu U \right] = \frac{g}{2f_\pi} \rho_v(\vec{x} - \vec{\nu} t) [U^+ - U],
\]

(11)

where the density

\[
\rho_v(\vec{x} - \vec{\nu} t) = \rho(v(\frac{x_3 - vt}{\sqrt{1 - v^2}}, x_\perp))
\]

is normalized according to (8).

The function \( f(t, \vec{x}) \) is generally related to the density through the Green function,

\[
f(t, \vec{x}) = -\frac{1}{4\pi f_\pi} \int d\tau d^3 y \theta(t - \tau) \delta[(t - \tau)^2 - (\vec{x} - \vec{y})^2] \rho_v(\vec{y} - \vec{\nu} \tau) \sin f(\tau, \vec{y}).
\]

(12)

The retarded Green function chosen here ensures \( f(t, \vec{x}) \to 0 \) for \( t \to \pm \infty \), that is the absence of the emission or absorption of real particles. If the distance \( x \) is larger than the size of the source, one can replace \( \vec{y} \) in the delta-function by \( \vec{\nu} \tau \), and the integral (12) results into eq. (9).

For the points on the trajectory the integral is

\[
f(t, \vec{v}t) = \frac{1}{4\pi f_\pi} \int d\tau d^3 y \theta(\tau) \delta[\tau^2 - (\vec{\nu} \tau - \vec{y})^2] \rho_v(\vec{y}) \sin f(t - \tau, \vec{y} + \vec{\nu}(t - \tau)),
\]
or, after integrating over $\tau$,

$$f(t, \bar{v}t) = \frac{1}{4\pi f_\pi} \int d^3y \frac{1}{2\sqrt{(1 - v^2)}} \frac{1}{|y_v|} \rho_v(\bar{y}) \sin f(t - \tau, \bar{y} + \bar{v}(t - \tau)).$$

Keeping here the most singular terms at $y \to 0$, the equation for $f(t, \bar{v}t)$ takes the form

$$f(t, \bar{v}t) = -\frac{1}{a} \sin f(t, \bar{v}t), \quad (13)$$

where the parameter $a$,

$$\frac{1}{a} = \frac{1}{8\pi f_\pi} \int d^3y \frac{\rho(\bar{y})}{|y|},$$

is proportional to the radius of the source.

Since $f(t, \bar{v}t)$ is supposed to be a smooth function, the solutions of eq. (13) are time-independent constants $f(t, \bar{v}t) = f_n$. When $a \to 0$,

$$\sin f_n \simeq \pi an, \quad n = 0, \pm 1, \pm 2, \ldots, \quad |n| \leq \frac{1}{\pi a}.$$

Finally, there is a set of solutions which at large distances are

$$f(t, \bar{x}) = n \frac{r}{|x_v|}, \quad n = 0, \pm 1, \pm 2, \ldots, \quad |n| \leq \frac{g}{4\pi f_\pi r},$$

where $r = g/4f_\pi a$ is the effective radius of the source.

Notice that in contrast to the pure pionic case the solution (8) contains only one arbitrary unitary matrix. This loss of generality has an obvious explanation. While the interaction (5) allows for the independent isotopic rotation of the pion field irrespective of the source, the chiral transformation has to be accompanied by an appropriate transformation of the source densities. The ”chiral phase” of the solution (8) is fixed by the conditions $\langle \bar{q}q \rangle \neq 0, \langle \bar{q}\gamma^5\bar{q} \rangle = 0$ imposed on the source. After chiral transformation it becomes a solution of the pion field equation with chirally transformed density.

3. Using exact solutions described above, consider the $S$-matrix for the soft pion production. Generally, the $S$-matrix is a functional depending on the pion fields:

$$S[\pi^0] = \sum_{n \geq 0} \int dx_1 \cdots dx_n S^{(n)}_{i_1, \ldots, i_n}(x_1, \ldots, x_n) \pi_{i_1}^0(x_1) \cdots \pi_{i_n}^0(x_n). \quad (14)$$

The $\pi^0$ fields are free,

$$\partial^2 \pi_i^0(x) = 0,$$
and expressed through the pion creation and annihilation operators

\[
\pi_i^0(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2k_0} \left[ e^{ikx} a_i^+(k) + e^{-ikx} a_i(k) \right],
\]

(15)

these operators being normally ordered in eq. (14).

Cross section for \( N \) emitted pions which carry the momenta \( k_1, \ldots, k_N \) is given by the matrix element

\[
d\sigma_{i_1, \ldots, i_N}^{(N)}(k_1, \ldots, k_N) = |\langle 0 | a_{i_1}(k_1) \cdots a_{i_N}(k_N) S[\pi^0] | 0 \rangle|^2 d\tau_N.
\]

Here \(|0\rangle\) is the vacuum state and \(d\tau_N\) is the phase volume. The distribution of pions can be evaluated as

\[
d\bar{N}_{i_1, \ldots, i_N}(k_1, \ldots, k_N) = \sum_{N=1}^{\infty} \frac{N!}{\sigma_{tot}} d\sigma_{i_1, \ldots, i_N}^{(N)}(k_1, \ldots, k_N),
\]

(16)

where the total cross section is

\[
\sigma_{tot} = \sum_N \sum_{i_1, \ldots, i_N} \int d\sigma_{i_1, \ldots, i_N}^{(N)}(k_1, \ldots, k_N).
\]

According to general rules for the \( S \)-matrix construction in the tree approximation, one has to find the solution \( \pi_i(x) \) of the classical equation, which is a free field for \( t \to \pm \infty \). The positive-frequency part of the asymptotics for \( t = +\infty \) and the negative one for \( t = -\infty \) have to be equal, respectively, to the positive and negative parts of the field \( \pi^0 \) (15). Then the tree \( S \)-matrix is evaluated through the classical action calculated with this solution:

\[
S[\pi^0] = \mathcal{N} \exp \left\{ i \int d^4x L(\pi) \right\}.
\]

\( \mathcal{N} \) is a normalization constant. Here the action should be regularized by subtracting the asymptotic \( \pi_i^0 \) from the field \( \pi_i \) in the quadratic terms:

\[
\frac{1}{2} \partial_\mu \pi_i \partial_\mu \pi_i \to \frac{1}{2} \partial_\mu (\pi_i - \pi_i^0) \partial_\mu (\pi_i - \pi_i^0).
\]

This way of proceeding follows from the saddle-point approximation for the functional \( S \)-matrix integral [10].

For the soft pion momenta, the terms containing the field derivatives can be omitted, and the tree \( S \)-matrix is expected to be of the local form,

\[
S[\pi^0] = \exp \left\{ \sum_{n \geq 0} \int dx S_{i_1, \ldots, i_n}^{(n)}(x) \pi_{i_1}^0(x) \cdots \pi_{i_n}^0(x) \right\},
\]

(17)

or of the sum of exponents of this type, if there are several solutions of the classical field equations. The isotopic invariance implies the independence
of the $S$-matrix of the direction of $\pi_0$ in the isotopic space, $S = S(|\pi_0(x)|)$. This property enables us to recover the $S$-matrix at the tree level through classical solutions (8), with the constant isotopic orientation.

4. For definiteness the annihilation of two nucleons into pions is taken as an example. The colliding baryons are described in the center-of-mass system as the two sources moving towards each other from the space-time infinity. They give rise to the classical pion field obeying the equation

$$\partial_{\mu} (U + \partial_{\mu} U) = \frac{g}{2 f_\pi} \theta(-t) [\delta_\nu^3(\vec{x} - \vec{v}t) - \delta_\nu^3(\vec{x} + \vec{v}t)] (U^+ - U),$$

or, in terms of the function $f$ in the parametrization (8)

$$\partial^2 f = \frac{g}{f_\pi} \theta(-t) [\delta_\nu^3(\vec{x} - \vec{v}t) - \delta_\nu^3(\vec{x} + \vec{v}t)] \sin f.$$  

Similarly to a single source, the general solution of eq. (18) can be written as a sum of two “moving Coulomb potentials”:

$$f(t, x) = \varphi_0(t, x) - \frac{g}{4\pi f_\pi} \theta(-\nu_0) \frac{1}{|x_\nu|} \sin f(\nu_\nu, \vec{\nu}_\nu) +$$

$$+ \frac{g}{4\pi f_\pi} \theta(-\nu_{-\nu}) \frac{1}{|x_{-\nu}|} \sin f(\nu_{-\nu}, \vec{\nu}_{-\nu}),$$

with so far unknown functions $f(\nu_{\pm\nu}, \vec{\nu}_{\pm\nu})$ of the proper time (10) and the function $\varphi_0$ satisfying the free equation, $\partial^2 \varphi_0 = 0$. For $t \to \pm \infty$ (and fixed $\vec{x}$) $f(t, x) \to \nu_0(t, x)$, so it is the function $\varphi_0$ which defines the asymptotic pion field, $\pi_0(t, x) = f_\pi \varphi_0(t, x)$.

To obtain the closed equations for $f$, one has to take into account the finite sizes of the sources, that is to replace the delta-functions in eq. (18) by more smooth densities $\rho_\nu(x \pm vt)$ localized in a small volume. This leads to the following equations for the function $f$ at the wordlines of the colliding particles

$$f(t, \vec{v}t) - \varphi_0(t, \vec{v}t) =$$

$$= -\left[ \sin f(t, \vec{v}t) - \frac{g}{4\pi f_\pi} \sqrt{1 - \nu^2} \frac{a}{2v|t|} \sin f(\frac{1 - \nu \nu}{1 + \nu \nu}, -\vec{v} \frac{1 - \nu }{1 + \nu }) \right]$$

$$f(t, -\vec{v}t) - \varphi_0(t, -\vec{v}t) =$$

$$= -\left[ \sin f(t, -\vec{v}t) - \frac{g}{4\pi f_\pi} \sqrt{1 - \nu^2} \frac{a}{2v|t|} \sin f(\frac{1 - \nu \nu}{1 + \nu \nu}, \vec{v} \frac{1 - \nu }{1 + \nu }) \right].$$

Second terms in the r.h.s. of (19) are negligible at large distances, and they become of the same order as the first ones only when the distance between the sources is comparable with their size, $v|t| \sim r$. However the region, where
the colliding particles overlap, corresponds to the central part of the process related to the hard stage of the reaction. Here the dynamics is governed by the detailed structure of baryons, which is not incorporated in the model of soft pion emission by classical sources the present approach is based on. This is the reason why the second terms are to be dropped in eq. (19), and the equation determining the soft pion bremsstrahlung amplitude is

\[ f(t, \pm \vec{v}t) - \varphi_0(t, \pm \vec{v}t) = \mp \frac{1}{a} \sin f(t, \pm \vec{v}t). \]

For small \( a \), one gets the two sets of solutions similar to those obtained for a single source:

\[
\begin{align*}
\sin f_n(t, vt) &= -a(\pi n + \varphi_0(t, vt)), \\
n &= 0, \pm 1, \pm 2, \ldots, \quad |\pi n - \varphi_0| \leq \frac{1}{a},
\end{align*}
\]

\[
\sin f_m(t, -vt) = a(\pi m + \varphi_0(t, -vt)), \quad m = 0, \pm 1, \pm 2, \ldots, \quad |\pi m - \varphi_0| \leq \frac{1}{a}.
\]

(20)

Now one can calculate the tree \( S \)-matrix substituting this solution by the classical action. As was mentioned above, one has to regularize the piece of the action which gives rise to the free pion propagator. Separating this term explicitly, the modified action density can be written as

\[
\tilde{L} = \frac{1}{2} \partial_\mu(\pi_i - \pi_i^0)\partial_\mu(\pi_i - \pi_i^0) + \frac{1}{4} f_\pi^2 \text{Tr}(\partial_\mu U^+ \partial_\mu U) - \frac{1}{2} \partial_\mu \pi_i \partial_\mu \pi_i + \frac{1}{4} g f_\pi \left[ \delta^3(\vec{x} - \vec{v}t) - \delta^3(\vec{x} + \vec{v}t) \right] \text{Tr}(U + U^+ - 2),
\]

\[
U = e^{i\pi \vec{v}}.
\]

Then the tree \( S \)-matrix is given by the sum of the terms calculated with this action for each solution of the pion field equation (20)

\[
S = \mathcal{N} \sum_{n,m} \exp i g f_\pi \left\{ \int_{-\infty}^0 dt \left[ \frac{1}{2a} \sin^2 f_n + \cos f_n \right] - \int_{-\infty}^0 dt \left[ \frac{1}{2a} \sin^2 f_m + \cos f_m \right] \right\}.
\]

Using the explicit expression (20), consider the contribution to this sum coming from the first particle:

\[
S_1 = \mathcal{N} \sum_n \exp \left\{ ig f_\pi \int_{-\infty}^0 dt \left[ \frac{1}{2} a(\pi n - \varphi_0)^2 \right] - \right\}
\]

\[
\left. \quad - (-1)^n \sqrt{1 - a^2(\pi n - \varphi_0)^2} \right\}.
\]

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For small $a$ the sum can be converted into two integrals over the variable $z = \pi an$, the odd and even $n$ being taken separately. Keeping only the terms nonvanishing at $a \to 0$, these integrals take the form

$$S_1 = \mathcal{N} \int_{-1}^{1} dz \exp \left\{ ig f_\pi \int_{-\infty}^{0} dt \left[ \frac{1}{2a} z^2 - z \varphi_0 + \sqrt{1 - z^2} \right] \right\} +$$

$$\mathcal{N} \int_{-1}^{1} dz \exp \left\{ ig f_\pi \int_{-\infty}^{0} dt \left[ \frac{1}{2a} z^2 - z \varphi_0 - \sqrt{1 - z^2} \right] \right\}.$$  

The first and the last terms in the exponents are divergent due to the time integrals. Regularizing them by the long-distance cutoff $T$, the first integral

$$\mathcal{N} \int_{-1}^{1} dz \exp ig f_\pi T \left[ \frac{1}{2a} z^2 + \sqrt{1 - z^2} \right] \exp \left\{ -ig f_\pi z \int_{-\infty}^{0} dt \varphi_0 \right\}$$

can be calculated for $T \to \infty$ through the stationary point which is at $\sqrt{1 - z^2} = a$. Since there is no stationary point on the real axis of the complex plane $Z$ for the second integral, it vanishes for large $T$. Collecting all the factors, which do not contain the field $\varphi_0$, into the overall constant $\mathcal{N}$, one gets a simple result:

$$S_1 = \mathcal{N} \sum_{\kappa = \pm} \exp \left\{ ig f_\pi \kappa \int_{-\infty}^{0} dt \varphi_0(t, vt) \right\}. \quad (21)$$

Recall now that due to the isotopic invariance the $S$-matrix actually depends only upon the absolute value of the isotopic vector $\vec{\pi}_0(x)$ and that eq. (21) is obtained for a particular direction of this vector. So one can rewrite it as

$$S_{in} = \mathcal{N} \sum_{\kappa = \pm} \exp \left\{ ig \kappa \int_{-\infty}^{0} dt \sqrt{\vec{\pi}_0^2} \right\}. \quad (22)$$

The generalization for any number of colliding particles is straightforward – each incoming line has to be supplemented with the factor (22), while the outgoing one acquires the factor

$$S_{out} = \mathcal{N} \sum_{\kappa = \pm} \exp \left\{ ig \kappa \int_{0}^{\infty} dt \sqrt{\vec{\pi}_0^2} \right\}, \quad (23)$$

where the integrals go along the particle trajectories.

Notice that the square roots here are due to the nonlinear character of pion field. In the absence of the pion self-interaction the expressions (3), (4) would produce the factors

$$S = \mathcal{N} \exp \left\{ \pm ig \int dt \sqrt{f_\pi^2 - \vec{\pi}_0^2} \right\}$$

\footnote{There is another solution $z = 0$ for both these integrals. However, it results in the constant independent of the field $\varphi_0$.}
(due to the property $\langle \tau_i \gamma_5 \rangle = 0$ only even powers of the pion field contribute). Thus, at least for a weak field, nonlinear effects enhance the amplitude at the tree level.

5. The functions (22), (23) give the $S$-matrix in the tree approximation. A natural question is to what extent the loop corrections change this result. As far as the effective low energy theory is treated, the loop momenta have to be cut off from above by a typical scale of the order of nucleon mass. To find the contribution from soft virtual pions one can use an approach similar to the equivalent photon method of the electrodynamics. Assuming the vertices for the soft virtual and real pion emission to be the same, one can yield the soft loops connecting the incoming and outgoing lines of the tree $S$-matrix by the pion propagator $1/\vec{k}^2$. Then the $S$-matrix accounting for all the loops can be written as the functional averaging of the tree $S$-matrix over virtual pion field $\pi_i(x)$

$$S[\pi^0] = \mathcal{N} \prod_{\kappa_i} \int D\pi_i(x) \exp i\left\{ \int d^4x \left[ \frac{1}{2} (\partial_\mu \vec{\pi}_i)^2 + g\kappa_i \theta(\pm t) \delta^3(\vec{x_i}) \right] \right\},$$

where the product is taken according to eqs. (22), (23) over the external lines of the basic diagram. Actually, eq. (24) is a special case of the general recipe for the $S$-matrix expressed through a functional integral used above. The integral (24) effectively re-sums the quantum fluctuations around the classical solution.

With the tree $S$-matrix linear in the pion field and without a square root, $S \sim \exp \int J_i(x)\pi^0_i(x)dx$ ($J_i$'s are external sources), this integral would be of the form

$$S[\pi^0] = \mathcal{N} \int D\pi_i(x) \exp i\left\{ \int d^4x \left[ \frac{1}{2} (\partial_\mu \pi_i)^2 + J_i(\pi_i + \pi^0_i) \right] \right\} = \mathcal{N}' \exp \left\{ \int d^4x J_i\pi^0_i \right\},$$

and the loop corrections would be completely absorbed by the normalization $\mathcal{N}'$. This is the case of the quantum electrodynamics where the soft virtual photons change the cross section but do not affect the normalized distribution of the type of (16).

To get rid of the square root in eq. (24) one can represent it through the functional integral over auxiliary fields $\vec{n}_i(t)$ and $\alpha(t)$

$$e^{ig\kappa \int_{-\infty}^{\infty} \sqrt{\vec{n}_i^2} dt} = \mathcal{N} \int DnD\alpha \exp i\left\{ \int_{-\infty}^{\infty} dt [g\kappa \vec{n}_i \vec{\alpha} - \frac{1}{2} \alpha \kappa (\vec{n}_i^2 - 1)] \right\},$$

the variable $\alpha(t)$ taking positive values only, $\alpha(t) > 0$. This formula is easily derived by discretizing the time interval in the l.h.s. and making the Fourier
transform for each moment of time. Here the functional measure \( DnD\alpha \) denotes the product of discrete integrals:

\[
DnD\alpha = \prod_t \int_{-\infty}^{\infty} d^3n_t \int_0^\infty d\alpha_t e^{-\alpha_t \delta},
\]

(25)

where \( \delta \to +0 \) ensures the convergency of the \( \alpha_t \)–integrals in the upper limit.

Now the integrals over virtual pions turn out to be of a gaussian type and can be calculated. Returning again to the case of the two colliding baryons, the result is

\[
S[\pi_0] = \mathcal{N} \int Dn_i D\alpha_i e^{-i \int_{-\infty}^0 \frac{1}{2} \alpha_i \kappa_i \langle \bar{n}_i^2 - 1 \rangle - g\bar{n}_i^2} e\Phi,
\]

(26)

where \( \Phi \) can be symbolically written as

\[
\Phi = \frac{1}{2} g^2 \left[ \bar{n}_1(t) \delta^3_v(x-vt) + \bar{n}_2(t) \delta^3_v(x+vt) \right] \frac{1}{\partial^2} \frac{1}{2} \left[ \bar{n}_1(t) \delta^3_v(x-vt) + \bar{n}_2(t) \delta^3_v(x+vt) \right]
\]

and \( 1/\partial^2 \) stands for the pion propagator. Substituting its expression in the coordinate space as follows

\[
\langle x | \frac{1}{\partial^2} | y \rangle = - \frac{i}{4\pi^2} \frac{1}{(x-y)^2 - i\delta},
\]

(27)

one arrives to the explicit form of \( \Phi \):

\[
\Phi = \frac{g^2}{8\pi^2} \int_{-\infty}^0 dt_1 dt_2 \left\{ \sum_{i=1}^2 \frac{1}{(1-v^2)(t_1-t_2)^2 - i\delta} \bar{n}_i(t_2) + \right. \\
+ \left. 2 \bar{n}_1(t_1) \frac{1}{(t_1-t_2)^2 - v^2(t_1+t_2)^2 - i\delta} \bar{n}_2(t_2) \right\}.
\]

(28)

Starting from this point, the two limits will be taken separately: the nonrelativistic limit, \( v \approx 0 \), and the ultra-relativistic one, \( v \approx 1 \).

For \( v \approx 0 \) the function \( \Phi \) takes the form

\[
\Phi_{NR} = \frac{g^2}{8\pi^2} \int_{-\infty}^0 dt_1 dt_2 \sum_{i,k=1}^2 \frac{\bar{n}_i(t_1) \bar{n}_k(t_2)}{(t_1-t_2)^2 - i\delta}.
\]

The integrals in this expression are singular for \( t_1 = t_2 \). They can be regularized by the parameter \( r \) which has a meaning of the ultraviolet cutoff, \( (t_1-t_2)^2 \to (t_1-t_2)^2 + r^2 \). In terms of the variable \( \tau = (t_1-t_2)/2r \), the regularized integral is

\[
\Phi_{NR} = \frac{g^2}{4\pi^2} \frac{1}{r} \int_{-\infty}^0 dt \int_{\tau/r}^0 d\tau \sum_{i,k=1}^2 \frac{\bar{n}_i(t) \bar{n}_k(t-r\tau)}{\tau^2 + 1},
\]

(29)
or, for small \( r \),

\[
\Phi_{NR} = \frac{g^2}{8\pi r} \int_{-\infty}^{0} dt \sum_{i,k=1}^{2} \bar{n}_i(t) \bar{n}_k(t) + \bar{n} K \bar{n}
\]

(29)

where all the terms, but the most singular ones, are collected in the operator \( K \) in such a way that \( r K \to 0 \) for \( r \to 0 \). The functional integral over the auxiliary fields \( \bar{n}_i(t) \) is of the gaussian type, with the quadratic exponential part

\[
\frac{1}{2} \int dt_1 dt_2 \sum_{i,j} \bar{n}_i(t_1) F_{ij}(t_1, t_2) \bar{n}_j(t_2)
\]

given by the \( 2 \times 2 \) block matrix

\[
F(t_1, t_2) = \begin{pmatrix}
\left( \frac{a}{r} - i\kappa_1 \alpha_1 \right) \delta_{t_1, t_2} + K(t_1, t_2) & \frac{a}{r} \delta_{t_1, t_2} + K(t_1, t_2) \\
\frac{a}{r} \delta_{t_1, t_2} + K(t_1, t_2) & \left( \frac{a}{r} - i\kappa_2 \alpha_2 \right) \delta_{t_1, t_2} + K(t_1, t_2)
\end{pmatrix},
\]

\[
a = \frac{g^2}{8\pi}, \quad \delta_{t_1, t_2} \equiv \delta(t_1 - t_2),
\]

in terms of which the integral yields

\[
\text{Det } F^{-\frac{3}{2}} \exp \int dt_1 dt_2 \frac{1}{2} g^2 \sum_{i,j} \bar{\pi}_i^0(t_1) F_{ij}^{-1}(t_1, t_2) \bar{\pi}_j^0(t_2).
\]

(31)

The inverse matrix \( F^{-1} \) and its determinant are expressed through the operator \( K \) (29) as

\[
(F^{-1})_{ik} = i \delta_{ik} \kappa_k \alpha_k^{-1} + \kappa_i \kappa_k \alpha_k^{-1} \left( \frac{a}{r} + K \right) [I + i \sum_{j=1}^{2} \kappa_j \alpha_j^{-1} \left( \frac{a}{r} + K \right)]^{-1} \alpha_k^{-1},
\]

\[
\text{Det } F = \prod_{j=1}^{2} \text{Det } \alpha_j \cdot \text{Det } [I + i \sum_{j=1}^{2} \kappa_j \alpha_j^{-1} \left( \frac{a}{r} + K \right)].
\]

Here \( \alpha_j = \alpha_j(t) \) is understood as a multiplication operator. As is seen from the first line, the matrix \( F^{-1} \) has a finite limit for \( r \to 0 \):

\[
F^{-1} \to \frac{1}{r \to 0} \frac{1}{\kappa_1 \alpha_1 + \kappa_2 \alpha_2} \begin{pmatrix}
I & -I \\
-I & I
\end{pmatrix}.
\]

(32)

In this limit

\[
\text{Det } F = c \text{ Det } |\kappa_1 \alpha_1 + \kappa_2 \alpha_2|,
\]

where all \( \alpha \)-independent factors are included in the constant \( c \).
Thus, the $S$-matrix turns out to be finite for $r \to 0$:

$$S[\pi_0] = N \sum_k \int \prod_i D\alpha_i \text{Det}|\kappa_1 \alpha_1 + \kappa_2 \alpha_2|^{-\frac{3}{2}} \times$$

$$\times \exp i \left\{ \frac{1}{2} \int_{-\infty}^{0} dt \left[ \kappa_1 \alpha_1 + \kappa_2 \alpha_2 + g^2 \frac{\mathbf{p}_1^0 - \mathbf{p}_2^0}{\kappa_1 \alpha_1 + \kappa_2 \alpha_2} \right] \right\}.$$

$$\pi_1^0 = \pi_0(t, vt), \quad \pi_2^0 = \pi_0(t, -vt).$$

Crucial point here is that the matrix $F(t_1, t_2)$ becomes degenerate for a vanishing regulator. Really the result (33) does not depend on the particular choice of the cutoff. Instead of $r$ one can take, for instance, a finite value for the parameter $\delta$ in the propagator (27).

The integral (33) allows for a straightforward calculation being a time product of integrals of the same type, without derivatives in the exponent. Its discretized version reads

$$S[\pi_0] = N \prod_i \int_0^\infty d\alpha_1 d\alpha_2 e^{-\delta(\alpha_1 + \alpha_2)} |\kappa_1 \alpha_1 + \kappa_2 \alpha_2|^{-\frac{3}{2}} \times$$

$$\times \left\{ 1 + \frac{1}{2} i \kappa_1 \alpha_1 + \kappa_2 \alpha_2 + g^2 \frac{(\pi_1^0 - \pi_2^0)^2}{\kappa_1 \alpha_1 + \kappa_2 \alpha_2} \right\} \Delta t \right\}.$$

Or, after assigning a finite value to the parameter $\delta$ in the measure (25), one gets

$$S[\pi_0] = N \prod_i \sqrt{\pi \delta^{-\frac{3}{2}}} \left\{ 1 + i \kappa_1 \alpha_1 + \kappa_2 \alpha_2 + g^2 \frac{(\pi_1^0 - \pi_2^0)^2}{\kappa_1 \alpha_1 + \kappa_2 \alpha_2} \right\} \Delta t \right\}.$$

for the same signs, $\kappa_1 = \kappa_2 \equiv \kappa$, and, when the signs are different, $\kappa_1 = -\kappa_2$,

$$S[\pi_0] = N \prod_i 2 \sqrt{2 \pi \delta^{-\frac{3}{2}}}.$$

The other terms in the last line vanish due to the antisymmetry of the integrand (here the principal value is taken for the term $1/(\alpha_1 - \alpha_2)$).

In the continuous limit the first line turns into the function

$$S[\pi_0] = N' \exp \left\{ i \kappa \int_{-\infty}^{0} dt \left[ \frac{1}{4} \delta^{-1} - g^2 \delta(\pi_1^0 - \pi_2^0)^2 \right] \right\}.$$

Only the first term survives when $\delta \to 0$. However, this term though strongly divergent is included into the constant $N'$. The second one is of the order of $v^2$ and should be also dropped by this reason, since the terms $v^2$ are neglected in eq. (28). Thus, all the pieces of the $S$-matrix result into the constants. Since the pion multiplicity (16) does not depend on the normalization, one can conclude that there is no soft pion emission from the "ends" in the nonrelativistic limit. It is completely suppressed by the loop corrections.
In the ultra-relativistic limit, \( v_1 \sim 1, v_2 \sim -1 \) the function \( \Phi \) takes the form

\[
\Phi_{UR} = g^2 \frac{8\pi^2}{\pi} \int_{-\infty}^{0} dt_1 dt_2 \left\{ \sum_{i=1}^{2} \bar{n}_i(t_1) \frac{1}{(1-v_i^2)(1-t_2)^2 - i\delta} \bar{n}_i(t_2) + \bar{n}_1(t_1) \frac{1}{1-v_1 v_2} \frac{1}{t_1 t_2 - i\delta} \bar{n}_2(t_2) \right\}.
\]

Rewriting again the exponential part of the integral (26) as

\[
\frac{1}{2} \int dt_1 dt_2 \sum_{i,k} \bar{n}_i(t_1) F_{v,ij}(t_1, t_2) \bar{n}_j(t_2),
\]

where

\[
F_v(t_1, t_2) = \left( \begin{array}{c} \frac{a}{\beta_1 r} + i\kappa_1 \alpha_1 \delta_{t_1, t_2} + \frac{1}{\beta_1} K(t_1, t_2) \\ \frac{a}{2\pi} \frac{1}{1-v_1 v_2} \frac{1}{t_1 t_2 - i\delta} \left( \frac{a}{\beta_2 r} + i\kappa_2 \alpha_2 \delta_{t_1, t_2} + \frac{1}{\beta_2} K(t_1, t_2) \right) \end{array} \right),
\]

\[\beta_i = 1 - v_i^2, \quad a = \frac{g^2}{8\pi},\]

the integral over auxiliary fields yields

\[
S[\pi_0] = \mathcal{N} \int \prod_{i=1}^{2} D\alpha_i e^{\frac{i}{\pi} \int_{-\infty}^{0} dt_i \kappa_i \alpha_i(t)} \text{Det} F_v^{-\frac{1}{2}} \times \exp\left\{ \frac{1}{2} g^2 \int dt_1 dt_2 \sum_{i,k} \bar{\pi}_i^0(t_1) F_{v,ij}(t_1, t_2) \pi_j^0(t_2) \right\}.
\]

The limit \( \beta_i \to 0 \) (or \( r \to 0 \)) essentially simplifies this answer, and the \( S \)-matrix is

\[
S[\pi_0] = \mathcal{N}' \exp\left\{ \frac{1}{2} g^2 \sum_{i=1}^{2} \beta_i \int dt_1 dt_2 \bar{\pi}_i^0(t_1) \left[ \frac{a}{r} + K \right]^{-1}(t_1, t_2) \pi_j^0(t_2) \right\},
\]

or, for small \( r \),

\[
S[\pi_0] = \mathcal{N}' \exp\left\{ 4\pi r \sum_{i=1}^{2} \beta_i \int_{-\infty}^{0} dt \bar{\pi}_i^0(t) \pi_i^0(t) \right\}.
\]

Here the terms independent of \( \pi_0 \), in particular the integrals over \( \alpha_i \), are absorbed by the coefficient \( \mathcal{N}' \). This expression shows that the \( S \)-matrix turns into a constant at \( r = 0 \). Thus, like in the nonrelativistic case, the loop corrections suppress the soft pion emission. The suppression is resulted both from the ultraviolet cutoff parameter \( r \) and kinematical factors \( \beta_i \).

The above result remains valid for a more general case. First, consider the nonrelativistic \( 2 \to 2 \) baryon scattering. Taking the factors (4) and (3)
for the incoming and outgoing baryons, respectively, the pion \( S \)-matrix for this process is

\[
S[\pi_0] = N \int \prod_{i=1}^{2} Dn_i D\alpha_i e^{-i \int_{-\infty}^{0} dt_i \left[ \frac{1}{2} \alpha_i \kappa_i (n_i^2 - 1) - g n_i \bar{\pi}_i^0 \right]} \times \int \prod_{k=3}^{4} Dn_k D\alpha_k e^{-i \int_{0}^{\infty} dt_k \left[ \frac{1}{2} \alpha_k \kappa_k (n_k^2 - 1) - g n_k \bar{\pi}_k^0 \right]} e^{\Phi},
\]

where

\[
\Phi = \frac{1}{2} g^2 \mathcal{J} \frac{1}{\partial^2} \mathcal{J},
\]

\[
\mathcal{J} = \theta(-t) \sum_{i=1}^{2} \bar{n}_i(t) \delta^3(x - v_i t) + \theta(t) \sum_{f=3}^{4} \bar{n}_f(t) \delta^3(x - v_f t).
\]

Denoting \( \bar{n}_i^+(t) = \theta(t) \bar{n}_i(t) \) and \( \bar{n}_f^-(t) = \theta(-t) \bar{n}_f(t) \), the part of an action, which is quadratic in \( \bar{n}(t) \), can be written as

\[
\frac{1}{2} \int_{0}^{\infty} dt_1 dt_2 \sum_{i,j} \bar{n}_i^\sigma(t_1) F_{ij}^{\sigma\sigma'}(t_1, t_2) \bar{n}_j^{\sigma'}(t_2), \quad \sigma, \sigma' = +, -,
\]

where the set of the functions \( F_{ij}^{\sigma\sigma'} \) is again represented by the \( 2 \times 2 \) block matrix in the indices \( \sigma, \sigma' \):

\[
F = \begin{pmatrix}
F_{11} & F_{12} \\
F_{21} & F_{22}
\end{pmatrix},
\]

but now each block is itself the \( 2 \times 2 \) block matrix in the indices \( i, j \). The blocks \( F_{11} \) and \( F_{22} \) are given by eq. (30) (the variables \( \alpha_{1,2} \) and \( \kappa_{1,2} \) in the block \( F_{11} \) for the incoming particles have to be replaced in the block \( F_{22} \) by the variables \( \alpha_{3,4} \) and \( \kappa_{3,4} \) for outgoing ones).

The non-diagonal blocks are

\[
F_{12} = F_{21} = \frac{g^2}{8\pi^2} \frac{1}{(t_1 + t_2)^2 + i\delta} \left( \begin{array}{cc}
I & I \\
I & I
\end{array} \right),
\]

where the baryon velocities squared are neglected in the nonrelativistic limit. This expression becomes singular at \( t_1 = t_2 = 0 \) only, namely at the central region of the reaction that is beyond the approximation used here and should be excluded. Therefore, only the diagonal blocks are singular, when the regulator \( r \to 0 \).

The gaussian integral over the fields \( n_i^\sigma(t) \) results again in the formula (31), where the sum runs over the indices \( i, \sigma \). Since the matrix \( F^{-1} \) can be expressed as

\[
F^{-1} = \begin{pmatrix}
F_{11}^{-1} & 0 \\
0 & F_{22}^{-1}
\end{pmatrix} \begin{pmatrix}
I & F_{12} F_{22}^{-1} \\
F_{21} F_{11}^{-1} & I
\end{pmatrix}^{-1}
\]

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and due to the property $F_{12}F_{22}^{-1} = F_{21}F_{11}^{-1} = 0$, which follows from eq. (22) for inverse blocks, the functional integral over $\alpha$ in eq. (33) decays at $r \to 0$ into the product of the two independent integrals for the incoming and outgoing baryons. Both of them result in the constants, so the $S$-matrix turns out to be constant too.

Since in the ultra-relativistic limit only the diagonal blocks, $F_{11}, F_{22}$, are singular for the small $r$ in the matrix $F$ and they have the form given by eq. (34), the integral (35) results into the $S$-matrix

$$S[\pi_0] = \mathcal{N} \exp \left\{ 4\pi r \left[ \sum_i \beta_i \int_{-\infty}^{0} dt \bar{\pi}_i^0(t) \bar{\pi}_i^0(t) + \sum_f \beta_f \int_{0}^{\infty} dt \bar{\pi}_f^0(t) \bar{\pi}_f^0(t) \right] \right\},$$

(36)

which turns into a constant for $r \to 0$ as well.

6. As the two limiting cases lead to the same answer, it is natural to assume the pion emission to be suppressed for all kinematics. This conclusion agrees with the old result [3] and exhibits a general property of loop corrections to reduce tree amplitudes. The reason for the suppression lies in the essentially nonlinear character of the pion field which is actually a phase of the chiral condensate, so the strong field associated with a large number of pions does not really contribute.

The suppression occurs when the ultraviolet cutoff $r \to 0$, that is when the loop momenta are formally allowed to go to infinity. However, an effective low-energy theory can originally imply some cutoff. From this viewpoint the expression (36) can be used for the small finite values of $r$. Notice that the exponential factors in the $S$-matrix include the quadratic combination of pion fields instead of the linear one peculiar to a usual coherent state like in electrodynamics. An opposite limit, $r \to \infty$, kills the loops because of the absence of infrared singularities for soft pions, and the $S$-matrix is given by the tree formulae (22), (23).

One has to stress the point, at last, that the suppression occurs only for pions emitted off the "ends", that is from the most peripheral parts of the process. However, the pion radiation treatment at the hard stage of the reaction is beyond the present approach, since it needs a detailed knowledge of the baryon structure at small distances.

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