Remarks on compositions of some random integral mappings

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Abstract. The random integral mappings (some type of functionals of Lévy processes) are continuous homomorphisms between convolution subsemigroups of the semigroup of all infinitely divisible measures. Compositions of those random integrals (mappings) can be always expressed as another single random integral mapping. That fact is illustrated by some old and new examples.

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For the last few decades random integrals were used to describe some classes of limiting distributions. For example, Lévy class L (selfdecomposable) distributions or s-selfdecomposable distributions (the class U). In those and other situations one had to identify an integrand, an interval (or a half-line) over each one integrates, a class of integrators (Lévy processes) and a time change in the process; cf. Jurek (2011) for a review of the research in that area and appropriate references; cf. Jurek (2014) for examples of some specific mappings.

Here we discuss a quite general set-up for such random integral mappings and prove, among others, that they are closed under compositions. It is illustrated by some explicit examples.

For an interval \((a, b]\) in the positive half-line, two deterministic functions \(h\) and \(r\), and a Lévy process \(Y_\nu(t), t \geq 0\), where \(\nu\) is the law of random variable \(Y_\nu(1)\), we consider the following mapping

\[
D_{(a,b]}^{h,r} \ni \nu \mapsto I_{(a,b]}^{h,r}(\nu) := L\left( \int_{(a,b]} h(t) \, dY_\nu(r(t)) \right), \quad (*)
\]

where \(L\) denotes the probability distribution of the random integral and \(D_{(a,b]}^{h,r}\) is the domain of the mapping \(I_{(a,b]}^{h,r}\) in (*).

Our results and proofs are given for \(\mathbb{R}^d\) variables. An infinite dimensional case is discussed in the Concluding Remarks.

1. Main results.

1 a). Compositions of the random integral mappings.

Let time changes \(r(t), a < t \leq b\), be either \(\rho\{s : s > t\}\) or \(\rho\{s : s \leq t\}\) for some positive, possibly infinite, measure \(\rho\) on \([0, \infty)\). For functions \(h_1, ..., h_m\) on the intervals \((a_1, b_1], ..., (a_m, b_m]\), respectively, and positive measures \(\rho_1, ..., \rho_m\), let us define

\[
\mathbf{h} := h_1 \otimes ... \otimes h_m, \quad \text{(the tensor product of functions)}
\]

i. e. \(h(t_1, t_2, ..., t_m) := h_1(t_1) \cdot h_2(t_2) \cdot ... \cdot h_m(t_m)\), where \(a_i < t_i \leq b_i\);

\[
(a,b] := (a_1, b_1] \times ... \times (a_m, b_m], \quad \mathbf{\rho} := \rho_1 \times ... \times \rho_m, \quad \text{(product measure)}. \quad (1)
\]

**THEOREM 1.** Let functions \(h_i\), measures \(\rho_i\) (given by increments of functions \(r_i\)) and intervals \((a_i, b_i]\), for \(i = 1, 2, ..., m\), be as above.

If the image \(\mathbf{h}((a,b]) = (c,d] \subset \mathbb{R}^+\) and \(\nu \in ID(\mathbb{R}^d)\) is from an appropriate domain then we have

\[
I_{(a,b]}^{h_1,\rho_1}(I_{(a_2,b_2]}^{h_2,\rho_2}(...(I_{(a_m,b_m]}^{h_m,\rho_m}((\nu))...)) = I_{(c,d]}^{h,\mathbf{\rho}}(\nu),
\]

\[
(2)
\]
where $h\rho$ is the image of the product measure $\rho = \rho_1 \times \cdots \times \rho_m$ under the mapping $h := h_1 \otimes \cdots \otimes h_m$.

Random integrals $I^{h_i,\rho_i}_{(a_i,b_i)}$, $i = 1, 2, \ldots, m$, commute on the domain $D^{t,h\rho}_{(c,d)}$ of the mapping $I^{t,h\rho}_{(c,d)}$.

Examples illustrating Theorem 1 are given in Section 3 below.

**COROLLARY 1.** If $Z_1, Z_2, \ldots, Z_m$ are stochastically independent variables with their probability distributions $\rho_i$ concentrated on intervals $(a_i, b_i)$, respectively then

$$r(t) := h\rho(s \leq t) = P[h_1(Z_1) \cdots h_m(Z_m) \leq t]$$

is the time change in the corresponding composition of random mappings $I^{h_i,r_i}_{(a_i,b_i)}$ where $r_i(t) = \rho_i\{s : s \leq t\}$.

If a random mapping is a composition of other mappings we may infer some inclusions of their ranges. Namely we have

**COROLLARY 2.** If an equality $I^{h,r}_{(a,b)} = I^{h_1,r_1}_{(a_1,b_1)} \circ I^{h_2,r_2}_{(a_2,b_2)}$ (a composition) holds on the domain $D^{h,r}_{(a,b)}$, then we have

$$R^{h,r}_{(a,b)} = I^{h,r}_{(a,b)}(D^{h,r}_{(a,b)} \subset I^{h_1,r_1}_{(a_1,b_1)}(D^{h_1,r_1}_{(a_1,b_1)} \cap I^{h_2,r_2}_{(a_2,b_2)}(D^{h_2,r_2}_{(a_2,b_2)} = R^{h_1,r_1}_{(a_1,b_1)} \cap R^{h_2,r_2}_{(a_2,b_2)}.$$  

1 b). Properties of the random integral mappings.

**THEOREM 2.** (a) Assume that $h(a) := h(a+), r(a) := r(a+)$ exist in $\mathbb{R}$. Then the mapping

$$D^{h,r}_{(a,b)} \ni \nu \to I^{h,r}_{(a,b)}(\nu) \in ID$$

is a continuous homomorphism between the corresponding measure convolution semigroups.

(b) For given $s > 0$, we have that $\nu \in D^{h,r}_{(a,b)}$ if and only if $\nu^{**} \in D^{h,r}_{(a,b)}$, and

$$I^{h,r}_{(a,b)}(\nu^{**}) = (I^{h,r}_{(a,b)}(\nu))^{**} = (I^{h,r}_{(a,b)}(\nu)).$$

(c) For $u > 0$ and the dilation operator $T_u : \mathbb{R}^d \to \mathbb{R}^d$ defined as $T_u(x) = u x$, we have that $\nu \in D^{h,r}_{(a,b)}$ if and only if $T_u \nu \in D^{h,r}_{(a,b)}$, and

$$T_u(I^{h,r}_{(a,b)}(\nu)) = I^{h,r}_{(a,b)}(T_u \nu) = I^{uh,r}_{(a,b)}(\nu).$$

(d) For bounded linear operator $A$ on $\mathbb{R}^d$ and $\nu \in D^{h,r}_{(a,b)}$ we have that $A\nu \in D^{h,r}_{(a,b)}$ and $A(I^{h,r}_{(a,b)}(\nu)) = I^{h,r}_{(a,b)}(A\nu)$.  

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2. Proofs.

First we will recall some basic definitions and facts.

2 a). Lévy-Khintchine representations.

Here $ID \equiv ID(\mathbb{R}^d)$ stands for the class of all infinitely divisible probability measures $\nu$ on $\mathbb{R}^d$. Thus their characteristic functions (Fourier transforms) are of the form (the famous Lévy-Khintchine formula)

$$\hat{\nu}(y) = e^{\Phi(y)}, \quad \Phi(y) \equiv \log \hat{\nu}(y) = i < y, z > - \frac{1}{2} < y, Ry >$$

$$+ \int_{\mathbb{R}^d \setminus \{0\}} \left[ e^{i<y,x>} - 1 - i < y, x > 1_{\{|x| \leq 1\}}(x) \right] M(dx), \quad y \in \mathbb{R}^d,$$  \(4\)

where $< \cdot, \cdot >$ denotes the scalar product and the triple: a vector $z \in \mathbb{R}^d$, a covariance operator $R$ of a Gaussian part of $\nu$ and a Lévy (spectral) measure $M$ (of Poissonian part), is uniquely determined by $\nu$. In short, we write: $\nu = [z, R, M]$ and $\Phi$ is referred to as the Lévy exponent of $\nu \in ID$; cf. Meerschaert and Scheffler (2001). [For more general case than $\mathbb{R}^d$, cf. Araujo-Gine (1980) or Parthasarathy (1967).]

2 b). Definition of path-wise random integrals.

For an interval $(a, b]$ in a positive half-line, a real-valued continuous of bounded variation function $h$ on $(a, b]$, a positive non-decreasing right-continuous (or non-increasing left-continuous) time change function $r$ on $(a, b]$ and a càdlàg Lévy stochastic processes $(Y_{\nu}(t), 0 \leq t < \infty)$, let us define, via the formal integration by parts formula, the following random integral

$$\int_{(a,b]} h(t)dY_{\nu}(r(t)) := h(b)Y_{\nu}(r(b)) - h(a)Y_{\nu}(r(a)) - \int_{(a,b]} Y_{\nu}(r(t) - )dh(t) \in \mathbb{R}^d,$$  \(5\)

and the corresponding random integral mapping

$$D^{h,r}_{(a,b]} \ni \nu \rightarrow I^{h,r}_{(a,b]}(\nu) := \mathcal{L} \left( \int_{(a,b]} h(t)dY_{\nu}(r(t)) \right) \in ID,$$  \(6\)

where $D^{h,r}_{(a,b]}$ is the domain of the mapping $I^{h,r}_{(a,b]}$, that is a subset of the class $ID$ consisting of those measures $\nu$ for which the integral (5) is well defined.

From properties of infinitely divisible measures (distributions) one concludes that the law of the random integral (5) is infinitely divisible one; cf. Jurek-Verervaat (1983), Lemma 1.1 (or for a particular case, cf. Jurek-Mason (1993), Section 3.6)).
Remark 1. (a) Since Lévy processes are semi-martingales the random integral (5) can be defined as an Ito stochastic integral. However, for our purposes we do not need that generality of stochastic calculus. [Term: random integral emphasizes that h in (5) is a deterministic function (not a stochastic process).]

(b) Integrals over intervals (a,b) or (a,∞) or [a,b] and others are defined as weak limits of integrals over intervals (a,b] in (5). Thus, the random integral \( \int_{(a,\infty)} h(t)dY_\nu(r(t)) \) is well-defined if and only if the function

\[
\mathbb{R}^d \ni y \rightarrow \int_{(a,\infty)} \Phi(h(t)y)dr(t) \in \mathbb{C}
\]

is a Lévy exponent (a functional of the form (3), above).

2 c). Lévy exponents.

If \( \nu \in D_{(a,b]}^{h,r} \) and \( I_{h,r}^{h} \) have the Lévy exponents \( \Phi \) and \( \Phi_{h,r}^{h,r} \), respectively, then, from already mentioned in Lemma 1.1 in Jurek-Vervaat (1983), we get

\[
\Phi_{h,r}^{h,r}(a,b] = \int_{(a,b]} \Phi(h(t)y)dr(t), \quad y \in \mathbb{R}^d
\]

for non-decreasing \( r \). Similarly we have that

\[
\Phi_{h,r}^{h,r}(a,b] = \int_{(a,b]} \Phi(-h(t)y)|dr(t)|, \quad y \in \mathbb{R}^d
\]

for non-increasing \( r \), because for \( 0 < u < w \), we have \( L(Y_\nu(u) - Y_\nu(w)) = (\nu^-)^{u-w} \) where \( \nu^- := L(-Y_\nu(1)) \). Consequently we have distributional equality of two processes: \( (-Y_\nu(t), t \geq 0) \overset{d}{=} (Y_{\nu^+}(t), t \geq 0) \).

2 d). Proofs of Theorem 1, Corollaries 1 and 2.

For \( \nu \in D_{(a,b]}^{h,r} \) and its Lévy exponent \( \Phi \) let us define the script mapping \( T_{h,r}^{h,r} \) as follows

\[
T_{(a,b]}(\Phi)(y) := \Phi_{(a,b]}^{h,r} = \int_{(a,b]} \Phi(\pm h(s)y)d(\pm)r(s),
\]

where the sign minus is in the case of decreasing time change \( r \). Then to justify (2) it is enough to notice that

\[
\int_{(a,b]} T_{(a_1,b_1]}^{h_1} T_{(a_2,b_2]}^{h_2} \cdots T_{(a_m,b_m]}^{h_m}(\Phi)(y)) = \int_{(c,d]} \Phi(t y)(\mathbf{1}((h \rho)(dt)),
\]

(11)
which follows from the Fubini and the image measure theorems.

To conclude the second part of Theorem 1 (commutativity) one needs to note that

\[ h_1 \otimes \cdots \otimes h_m (\rho_1 \times \rho_2 \times \cdots \times \rho_m) = h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(m)} (\rho_{\sigma(1)} \times \rho_{\sigma(2)} \times \cdots \times \rho_{\sigma(m)}), \]

for any permutation \( \sigma \) of 1, 2, ..., \( m \).

Corollary 1 follows from the definition of tensor product of functions.

**Remark 2.** Corollary 1, for \( h_i(t) = |t| \) and standard normal random variables \( Z_i \), was investigated by Aoyama (2009) via polar decomposition of Lévy spectral measures.

For a proof of Corollary 2 note that the equality

\[ I_{(a,b)}^{h,r} (D_{(a,b)}^{h,r}) = I_{(a_1,b_1)}^{h_1,r_1} (I_{(a_2,b_2)}^{h_2,r_2} (D_{(a,b)}^{h,r})) \text{ implies } I_{(a_2,b_2)}^{h_2,r_2} (D_{(a,b)}^{h,r}) \subset D_{(a_1,b_1)}^{h_1,r_1}. \]

Hence \( I_{(a,b)}^{h,r} (D_{(a,b)}^{h,r}) \subset I_{(a_1,b_1)}^{h_1,r_1} (D_{(a_1,b_1)}^{h_1,r_1}) \). By the commutative property we also get \( I_{(a,b)}^{h,r} (D_{(a,b)}^{h,r}) \subset I_{(a_2,b_2)}^{h_2,r_2} (D_{(a_2,b_2)}^{h_2,r_2}) \), which completes a proof of Corollary 2.

2 c). **Proof of Theorem 2.**

Part (a). The homomorphism property of \( I_{(a,b)}^{h,r} \), that is, the equality

\[ I_{(a,b)}^{h,r} (\nu_1 * \nu_2) = I_{(a,b)}^{h,r} (\nu_1) * I_{(a,b)}^{h,r} (\nu_2), \]

in terms of the corresponding Lévy exponents, follows from (8) or (9).

For the continuity, let us note that \( 0 \leq |r(b) - r(a)| < \infty \) and the cadlag property imply that functions \( t \to Y(r(t)) \) are bounded and with at most countable many discontinuities; cf. Billingsley (1968), Chapter 3, Lemma 1. Furthermore, the mapping

\[ D_{R^d}[a,b] \ni y \to \int_{(a,b)} h(t)dy(r(t)) := h(b)y(r(b)) - h(a)y(r(a)) - \int_{(a,b)} y(r(t)) dh(t) \in R^d, \quad (12) \]

is continuous in Skorohod topology (for details see Billingsley (1968), p. 121.). Furthermore, if \( \nu_n \Rightarrow \nu \) then \( (Y_{\nu_n}(t), a \leq t \leq b) \Rightarrow (Y_{\nu}(t), 0 \leq t \leq b) \) in Skorohod space \( D_{R^d}[a,b] \) of cadlag functions. Consequently, we have

\[ \mathcal{L} \left( \int_{(a,b)} h(t)dy_{\nu_n}(r(t)) \right) \Rightarrow \mathcal{L} \left( \int_{(a,b)} h(t)dy_{\nu}(r(t)) \right), \]
which proves the continuity of mappings $I_{(a,b)}^{h,r}$ and completes the proof of part (a) of Theorem 2.

Equality in parts (b), (c) and (d) follow from (8) and (9).

3. Applications and illustrations of Theorem 1.

We begin with the following auxiliary fact.

**Lemma 1.** Let $h_1(t) := e^{-t}$, $r_1(t) := t$, $h_2(s) := s$ and $r_2(s) := 1 - e^{-s}$, $0 < s, t < \infty$. Then the corresponding measures are: $d\rho_1(t) = dt$, $d\rho_2(s) = e^{-s}ds$ and $d\rho(t, s) = d(\rho_1 \times \rho_2)(t, s) = e^{-s}dt ds$. Finally, for the image measure $h \rho(dw) = (h_1 \otimes h_2)(\rho_1 \times \rho_2)(dw) = \frac{e^{-w}}{w} dw$.

**Proof.** For Borel measurable, bounded and non-negative functions $g$ we have

\[
\int_0^\infty g(u)(h_1 \otimes h_2)(\rho_1 \times \rho_2)(du) = \int_0^\infty \int_0^\infty g((h_1 \otimes h_2)(t, s))\rho_1(dt)\rho_2(ds) = \int_0^\infty \int_0^\infty g(e^{-s}t) dt e^{-s} ds = \int_0^\infty g(s) e^{-s} ds,
\]

which completes the proof of Lemma 1.

From Theorem 1 and Lemma 1 we conclude the following.

**Example 1.** For $\nu \in ID_{\log}$ we have

\[
I_{(0,\infty)}^{1, e^{-t}}(I_{(0,\infty)}^{e^{-s}, s}(\nu)) = I_{(0,\infty)}^{e^{-s}, s}(I_{(0,\infty)}^{1, e^{-t}}(\nu)) = I_{(0,\infty)}^{-w, \Gamma(0; w)}(\nu) = I_{(0,\infty)}^{w, \Gamma(0; w)}(\nu^-).
\]

Moreover, $\Gamma(0; w) = (h_1 \otimes h_2)(\rho_1 \times \rho_2)(\{x : x > w\}) = \int_w^\infty \frac{e^{-s}}{s} ds$, for $w > 0$.

**Remark 3.** (a) For the Euler constant $C$ we have

\[
-\Gamma(0; w) = Ei(-w) = C + \ln w + \int_0^w \frac{e^{-t} - 1}{t} dt, \text{ for } w > 0,
\]

where $Ei$ is the special exponential-integral function; cf. Gradshteyn-Ryzhik (1994), formulas 8.211 and 8.212.

(b) Recall that the class $I_{(0,\infty)}^{1, e^{-t}}(ID) \equiv E$ was introduced in Jurek (2007), where the mapping $I_{(0,\infty)}^{1, e^{-t}}$ was denoted by $K(e)$; (here $(e)$ stands for the exponential distribution).

More importantly, the class $E$ was related to the class of Voiculescu free-infinitely divisible measures; cf. Corollary 6 in Jurek (2007). Note also that $I_{(0,\infty)}^{1, e^{-t}} = I_{(0,1)}^{\log, s,s}$ and thus it coincides with the upsilon mapping $\Upsilon$ studied in Barndorff-Nielsen, Maejima and Sato (2006).
(c) Similarly $I_{(0,\infty)}^{e^{-s}, s}(ID_{\log}) \equiv L$ coincides with the Lévy class of selfdecomposable probability measures; cf. Jurek-Vervaat (1983), Theorem 3.2 or Jurek-Mason (1993), Theorem 3.6.6.

(d) Finally we get identity $I_{(0,\infty)}^{e^{-s}, s}(I_{(0,\infty)}^{1-e^{-t}}(ID_{\log})) \equiv T$, which is the Tho-$\text{rin}$ class; cf. Grigelionis (2007), Maejima and Sato (2009) or Jurek (2011).

From Corollary 2, Remark 3 (c) and (d) we infer the following inclusion.

**COROLLARY 3.** For the three classes: Thorin class $T$, Lévy class $L$ (selfdecomposable measures) and $E$ of probability measures on $\mathbb{R}^d$, we have that $T \subset L \cap E$.

This inclusion was first noticed in Barndorff-Nielsen, Maejima and Sato (2006) and also in Remark 2.3 in Maejima-Sato (2009) but by using different methods. See note (c) in Concluding Remarks.

Finally, we give additionally three examples of compositions of random integral mappings.

**Example 2.** For $\beta > 0$ we have

$$I_{[0,1]}^{t^{1/\beta}, t} \circ I_{[0,1]}^{s^{1/2\beta}, s} = I_{[0,1]}^{w, 2w^{\beta}(1-(1/2)w^{\beta})} = I_{[0,1]}^{(1-\sqrt{\beta})^{1/\beta}, t}.$$ (13)

Or equivalently, for Lebesgue measure $l_1$ on the unit interval and $0 < w \leq 1$ we get

$$\left(t^{1/\beta} \otimes s^{1/(2\beta)}\right)(l_1 \times l_1)(dw) = \int dt^{2(\beta^{2-1})} \times 2\beta s^{2\beta-1} dt)(dw) = 2\beta w^{\beta-1}(1-w^{\beta}) dw.$$

**Proof.** As in Example 1, it simply follows from Theorem 1 and identity (8), because all time change functions are strictly increasing on the unit interval.

**Example 3.** For $\beta > 0$

$$I_{[0,1]}^{t^{1/\beta}, t} \circ I_{(0,\infty)}^{e^{-s}, s} = I_{(0,\infty)}^{e^{-s}, s} + \beta^{-1} e^{-\beta s} - \beta^{-1} = I_{[0,1]}^{-w, \beta^{-1} w^{\beta} - \log w - \beta^{-1}}.$$ 

Or equivalently, for $0 < w \leq 1$

$$(t^{1/\beta} \otimes e^{-s})(l_1 \times l)(dw) = (\beta^{-1} w^{\beta} - \log w - \beta^{-1}) dw.$$

This is a consequence of Theorem 1. Also cf. Czyżewska-Jankowska and Jurek (2011), Proposition 2.
Example 4. For $\alpha \in \mathbb{R}$, let $\Gamma(\alpha; x) := \int_x^\infty t^{\alpha-1} e^{-t} dt, x > 0$ be the incomplete Euler function. Then we have

$$I_{(0,\infty)}^t \Gamma(\alpha; t) \circ I_{(0,\infty)}^{e^{-x}, s} = I_{(0,\infty)}^t \int_{(0,\infty)} s^{-1} \Gamma(\alpha; s) ds,$$

which follows from Theorem 1.

4. Concluding Remarks.

(a) Let $E$ be a real separable Banach space with the dual $E'$ and a bilinear form $\langle \cdot, \cdot \rangle: E' \times E \to \mathbb{R}$, (a scalar product when $E$ is a Hilbert space). Then we define a convolution $\ast$ of measures $\mu$, a characteristic functional (Fourier transform) $\hat{\mu}$ and, in particular, the notion of infinite divisibility of (Borel) probability measures in $E$ and the convolution semigroup $ID(E)$. The Lévy-Khintchine representation (4) holds true with $\mathbb{R}^d$ replaced by $E$. However, for infinite dimensional Banach spaces, the condition $\int_E \min(||x||^2, 1) M(dx) < \infty$ is neither sufficient or necessary for $M$ to be a Lévy (spectral) measure. Moreover, in general case of $E$, convergence of characteristic functionals $\hat{\mu}_n(y) \to \hat{\mu}(y), y \in E'$, does not imply the weak convergence $\mu_n \Rightarrow \mu$. cf. Araujo-Gine (1980) or Parthasarathy (1967).

(b) Because in this note we did not use any of those two exceptions our results are valid for measures on real separable Banach spaces $E$; (and $E$-valued Lévy processes $Y$). Note that in the proof of continuity of random integral mappings (Theorem 2) we did not use characteristic functional arguments.

(c) Corollary 3, inclusions of three ranges of a given three random integral mappings, is valid for measures on Banach spaces. However, we do not define Thorin class $T$ via properties of Lévy measures $M$; cf. for instance Jurek (2011) or Grigelionis (2007).

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