Distribution of amplitude in an extended phase space of a generic quantum system

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Abstract

The action reaction principle is violated by the projection of state in some simple quantum measurements. A formulation of Quantum Mechanics in an extended phase space is proposed in order to restore the action reaction principle. All predictions of the standard theory are reproduced. Observable effects of an accompanying de Broglie wave are also predicted.

Keywords: quantum mechanics, quantum measurement, amplitude of probability, hidden variables, correlation, entanglement.

1 Introduction

The action reaction principle (ARP) is as firmly established as the conservation of energy. Theoretical models where one or the other is not fulfilled have most probably incomplete phase spaces, and additional variables in an extended phase space will restore them. The axiom of measurement in Quantum Mechanics (QM) violates the ARP in simple examples. This paradox would not appear if the phase space of standard QM were incomplete. The wave function collapse, projection of state, generates many paradoxes in the interpretation of QM, and a possible incompleteness of the theory was already considered in the seminal paper [7].

Alternative formulations and interpretations incessantly appear although the scientific success of QM is overwhelming; see e.g. [6] for a detailed bibliographic list up to 2004. No–go mathematical results [2] [9] rule out alternative formulations with hidden variables fulfilling the hypothesis of the theorems. On the other hand, alternatives as Bohmian mechanics [3] are explicit examples of consistent models with hidden variables, obviously not fulfilling the hypothesis of those no–go theorems.

With regards to Bell’s theorems, there has been a wide research activity around which of the hypothesis is (or are) not fulfilled by the laws of Physics. Logic, axioms of mathematical probability, even realism, have each open more or less fruitful lines of study in different fields, although none seems to be relevant in fundamental Physics.

Non locality is acknowledged as the most probable property of the real world that explains the violation of Bell’s inequalities in experiments [11]. But that seems to be incompatible with relativity. The non local interaction between
measurements of an entangled pair is very different from known fundamental interactions (electromagnetic, gravitational, ...). It acts between events either spatially or causally separated, it does not decay with distance and it is specifically directed towards the entangled pair. No theoretical model of its mechanisms (as, e.g., gauge theories) has been proposed, neither an isolated effect of an hypothetical signal over an apparatus has ever been detected.

Standard Quantum Mechanics is non local through the projection of state, the same axiom that crashes with the ARP, unitary evolution, etc. It seems reasonable to look at the established theory in order to find some other ingredient of the mathematical formalism that could be the cause of violation of Bell’s inequalities, instead of a non local projection of state.

Born’s rule, a distribution of probability obtained from a distribution of amplitude, is not considered in the hypothesis of Bell’s theorems; e.g., typical interference phenomena as in the two slit experiment can not be reproduced with a distribution of probability. A window is open to assign the violation of Bell’s inequalities in Nature to the description of physical states through distributions of amplitude, instead of non locality of the wave function collapse.

Violation of the ARP in the projection of state suggest to extend the phase space. An extended Hilbert space will necessarily contain elementary states with joint precise values of non commuting magnitudes. It seems that such states would violate the uncertainty principle. The point is if the uncertainty principle applies to the description of states or it just determines a fundamental limit of accuracy to the knowledge (through measurement) of conjugate variables.

Commutation relations apply to evolution in Hamiltonian dynamics. In Classical Mechanics, functions on the phase space (e.g., conjugate variables) commute under the usual product, and do not generically commute under the Poisson bracket operation. The existence of quantum states with joint precise (hidden) values of non commuting magnitudes could be consistent with an uncertainty principle for joint measurement of these magnitudes.

In [10], I presented a mathematical framework with an extended phase space for spin variables, consistent with the standard theory. The model is inspired in the path integral formalism, where I consider the formal hypothesis that there are families of virtual paths characterized by joint precise final values of non commuting physical magnitudes, as position and momentum \((x, p)\), or spin in two or more directions \((s_1, s_2, \ldots)\). Sum of \(\exp(iS/\hbar)\), \(S\) the path integral, for all paths in the family determines a distribution of amplitude \(\Psi(x, p)\) or \(\Psi(s_1, s_2, \ldots)\) in an extended phase space. The orthodox state in the standard phase space is then obtained by addition of amplitudes, i.e., computation of marginal amplitudes, for all families with common value of the considered magnitude, e.g., \(\Psi(x) = \int dp \, \Psi(x, p)\). Bell’s type inequalities do not apply; the fundamental ingredient is, as in the usual formulation, a distribution of am

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1In a minimum of the diffraction pattern there is no way to get a near vanishing total probability by addition of two positive independent ones, which is a kind of trivial Bell’s type inequality.

2This fact necessarily limits accuracy of classical joint measurements, although there is no Plank’s constant as fundamental lower bound.
plitude (here in an extended phase space with hidden variables), and not a distribution of probability,

The main physical hypothesis of the proposed theory, introduced to restate the ARP, are the existence of an accompanying subsystem, the de Broglie wave \[5\], and additional variables for the corpuscular (sub)system. Elementary states of a quantum system are characterized by a family of precise joint values of all physical magnitudes representing the particle (corpuscle) behavior, and a distribution of amplitude (ray in a Hilbert space) representing its wave like behavior. Bell’s experiments with spin variables on a sample of entangled pairs \[2\] (and Aspect’s for polarization variables \[1\]) appear in this theory as local interference phenomena, analogous to the two slit experiment.

The aim of this paper is to apply the former hypothesis to a generic quantum system and arbitrary family of magnitudes. Section 2 reviews the physical arguments, the need to extend the phase space in order to preserve the ARP. In Section 3 the mathematical framework of the theory is presented, and its correspondence with standard QM established. Section 4 develops the description of composite systems and entanglement. Interference terms in the marginal amplitudes give account of the apparently non local correlation between measurements. Particular cases, the two slit experiment and the phase space of spin variables, are presented in Section 5.

2 Quantum measurement and the action reaction principle

The ARP represents, in its most generic terms, a basic hypothesis of dynamical theories. An isolated composite system evolves, with regards to some coarse–grained variables, as a free elementary system. Internal variables must be correlated: departure from the free dynamics of a subsystem is accompanied by a corresponding departure of the other. Third Newton’s law is necessary for the consistency of Classical Mechanics; otherwise, the first law would not have been established, because elementary classical systems are composite at a microscopic level. The ARP is independent of the particular type of interaction between subsystems; it belongs to the foundations of Physics and its generic framework (ideal isolated systems, free dynamics, . . . ).

Friction forces apparently violate the conservation of energy; obviously, there are additional (microscopic) variables giving account of the lost energy in the balance, and all known fundamental interactions are conservative. In Brownian motion, the pollen grain receives impulses from unobservable systems, the fluid molecules; once incorporated the corresponding variables in an extended phase space, an opposite impulse balances the momentum equation.

The ARP and the axiom of quantum measurement are contradictory. One or the other must be rejected in its present formulation. Let \[|a_1\rangle\] be the state of a quantum system, eigenstate of a physical magnitude \(A\). When \(A\) is measured, the result of measurement is \(a_1\) with certainty, and the final state of the system
is again $|a_1>$, with trivial projection of state. The pointer of the measurement apparatus has changed of state, from a neutral position to the result “$a_1$”. ARP is violated.

In a theory with additional variables in an extended phase space, elementary states of the quantum system should be described by joint precise values of non commuting magnitudes, because maximal families of compatible magnitudes are already considered in the standard formulation. In the orthodox Hilbert space $\mathcal{H}$ there are not common eigenstates for incompatible magnitudes; an extended Hilbert space $\mathcal{H}_{ext}$, with different representation of physical magnitudes, will be the first ingredient of the proposed framework.

Let $c_1|a_1> + c_2|a_2>$ be the initial state of a quantum system, an elementary particle, and suppose wave packets of the two components are spatially separated (e.g., with a Stern Gerlach apparatus). A particle detector is located in one of the “virtual” paths, $|a_1>$, and the result of measurement is negative $3$. The new, projected state of the system is $|a_2>$. The detector has not, apparently, changed of state, and the ARP is violated in this indirect measurement.

Obviously, the detector is designed to show an observable response when interacting with a particle located in the surrounding spatial region. Perhaps some type of non local interaction between particle at $|a_2>$ region and detector at $|a_1>$ happens, an interaction that does not generate an observable response in the detector.

I will consider an alternative hypothesis: the detector interacts with a system spatially located in its neighborhood. This hypothetical system is not the particle, which is certainly located away. An hypothetical wave like system, which will be denoted the de Broglie wave, does not generate the same reaction than a corpuscular system, and another type of detector should be designed to observe a response. An accompanying wave of an isolated particle is a wave in vacuum, the vacuum becoming a relevant physical ingredient in non relativistic QM, as it is relevant in the opposite length scales of Quantum Field Theory (e.g., Casimir energy) and Cosmology (dark energy).

Let us suppose that an elementary particle, and in general a quantum system, is a composite of a corpuscular subsystem and a wave like subsystem. The corresponding phase space of the composite system must describe states of both subsystems. If the standard representation $c_1|a_1> + c_2|a_2>$ in $\mathcal{H}$, or the corresponding vector in an extended Hilbert space $\mathcal{H}_{ext}$, is associated to the de Broglie wave (as seems to suggest interaction of the detector with the $|a_1>$ wave component and not with the particle), additional variables of state for the corpuscular component (known to be located at the spatial region of the other wave packet $|a_2>$) must be considered for a total phase space $\mathcal{H}_{ext}^{wave} \times \mathcal{P}_{corp}$.

### 3 Extended phase space

We must confront the standard phase space of a quantum system, the Hilbert space $\mathcal{H}$, with an extended phase space $\mathcal{H}_{ext}^{wave} \times \mathcal{P}_{corp}$ describing both com-

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$3$ It will happen with relative frequency $|c_2|^2$. 
ponents, wave like and corpuscular subsystems. First, a formal path integral formalism is considered. Then, the definition of an extended phase space for a generic quantum system $S$ and arbitrary family of physical magnitudes $F$ is given. A correspondence between states in both formalism determines equivalence of the theories, with regards to phase space description.

### 3.1 Abstract path integral formalism

$S$ denotes a quantum system, $F = \{A, B, C, \ldots\}$ a generic family of $N$ physical magnitudes (observables) of $S$, $M_A = \{a_i\}$ the set of possible values of magnitude $A$, $M_B = \{b_j\}$ of $B$, etc. Additional magnitudes can be incorporated if needed; in many cases some magnitudes are ignored, when they are not correlated to those of our interest.

An element $\lambda \in M_F \equiv M_A \times M_B \times M_C \cdots$, $\lambda = (a_i, b_j, c_k, \ldots)$, is a $N$–tuple of values of all magnitudes in $F$.

Let us consider an abstract, formal set of virtual paths, such that we can assign to each of them an action integral $S(path)$ and a final value $\lambda$. $[path](\lambda)$ denotes the subset of paths with final value $\lambda = (a_i, b_j, \ldots)$. We formally define the amplitude

$$Z(\lambda) = \sum_{[path](\lambda)} e^{iS(path)}$$

(1)

It is possible to fix the values of a subset of magnitudes $F_1 \subset F$, and define $Z(\lambda_1)$, $\lambda_1 \in M_{F_1}$, by addition of elementary terms $exp(iS/\hbar)$ for all paths with common values $\lambda_1$. The amplitude $Z$ associated to a family of paths union of disjoint subfamilies with corresponding amplitudes $Z_1$, $Z_2$, $\ldots$, is $Z = Z_1 + Z_2 + \cdots$. For example, $Z(\lambda_1) = \sum_{\pi_1(\lambda)=\lambda_1} Z(\lambda)$ in the previous case $\lambda_1 \in M_{F_1}$, with $\pi_1$ the natural projection $\pi_1 : M_F \to M_{F_1}$.

In the standard treatment, the family $F_1$ considered is one made of compatible magnitudes, with vanishing commutation relations, e.g. final position coordinates. A quantum state of $S$ is completely determined through a distribution of amplitude $Z(\lambda_1)$, which defines a vector $|S> = \sum_{\lambda_1} Z(\lambda_1)|\lambda_1>$ in the corresponding Hilbert space $\mathcal{H}$, generated by elementary states $|\lambda_1> = |(a_i, b_j, \ldots)>$, common eigenvectors of the family of self adjoint operators representing the magnitudes in $F_1$.

Another family of magnitudes $F_2$ (compatible among themselves, but not with all of $F_1$) has its corresponding basis of eigenvectors, and the relations between bases of eigenvectors $|\lambda_1>$ and $|\lambda_2>$ is consistent with the commutation relations (quantization rules) of operators. Given the change of bases the new distribution $Z(\lambda_2)$ is obtained from $Z(\lambda_1)$, $Z(\lambda_2) = \sum_{\lambda_1} <\lambda_2|\lambda_1> Z(\lambda_1)$.

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4 Magnitudes in $F$ are not necessarily functionally independent, as would be a coordinate description of classical phase space. Angular momentum, for example, is a vector but components in three independent directions do not exhaust the quantum information, because operators in additional directions do not commute with the selected three.

5 A specific description of the set of paths and rules of assignment should be given if no alternative way to determine states is available.
In the abstract path integral formalism, we can express the amplitude $Z(\lambda_2)$ as

$$Z(\lambda_2) = \sum_{\lambda_1} \sum_{[\text{path}] (\lambda_1) \cap [\text{path}] (\lambda_2)} e^{i\hat{S}(\text{path})}$$

where the set of paths $[\text{path}] (\lambda_2)$ has been decomposed into disjoint subsets $[\text{path}] (\lambda_1) \cap [\text{path}] (\lambda_2)$. Denoting $Z(\lambda_1, \lambda_2)$ the amplitude associated to the family of paths $[\text{path}] (\lambda_1) \cap [\text{path}] (\lambda_2)$,

$$Z(\lambda_1, \lambda_2) = \sum_{[\text{path}] (\lambda_1) \cap [\text{path}] (\lambda_2)} e^{i\hat{S}(\text{path})}$$

we can obtain both $Z(\lambda_1)$ and $Z(\lambda_2)$ as marginal amplitudes

$$Z(\lambda_1) = \sum_{\lambda_2} Z(\lambda_1, \lambda_2) \quad Z(\lambda_2) = \sum_{\lambda_1} Z(\lambda_1, \lambda_2)$$

In this way, we can get any orthodox representation of a state (associated to a family of compatible magnitudes) from a representation in an extended phase space (associated to a larger family of non-commuting magnitudes).

We consider next the formulation of a non-relativistic Quantum Mechanics inspired in the former abstract path integral formalism, making the hypothesis that a distribution of amplitude defining a vector (ray) in an extended Hilbert space represents a state of the quantum system. Being inspired in the path integral formalism, a relativistic quantum theory in an extended phase space could be formulated when a Lorentz invariant action were considered. Notice that virtual paths arriving to a space time event are contained inside its past light cone, i.e., acausal interactions do not appear through the amplitudes obtained in this way.

### 3.2 Extended phase space

If the phase space of the quantum system is to be extended, the family of magnitudes considered in $\mathcal{F}$ must contain non-commuting operators. In the former abstract path integral, a distribution of amplitude $Z(\lambda)$ determines the quantum state, where $\lambda \in \mathcal{M}_F$ is a $N$-tuple of values of magnitudes in $\mathcal{F}$.

We define a Hilbert space $\mathcal{H}_{ext}$, generated by vectors $|\lambda> = (a_i, b_j, c_k, \ldots >$, orthonormal vectors representing elementary states of $\mathcal{S}$ with joint precise values of all magnitudes in $\mathcal{F}$. Operators representing the magnitudes (denoted with the same symbols) $A, B, \text{etc.}$, have the basis $|\lambda>$ of common eigenvectors, with eigenvalues $a_i, b_j, \ldots$. $|S> = \sum_\lambda Z(\lambda)|\lambda>$ is a generic vector of state in $\mathcal{H}_{ext}$.

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6Families $\mathcal{F}_1$ and $\mathcal{F}_2$ can contain a common subfamily $\mathcal{F}_c = \mathcal{F}_1 \cap \mathcal{F}_2$ of magnitudes; the set of paths characterized by joint parameters $\lambda_1$ and $\lambda_2$ which do not share the same values in $\mathcal{F}_c$ is void. Both $\lambda_1$ and $\lambda_2$ must project onto a common $\lambda_c$. 
By construction, $A$, $B$, etc., commute. This represents the fact that elementary states of the system have joint precise values of all considered magnitudes. Together with operators $A$, $B$, etc. introduced in the definition of the phase space $\mathcal{H}_{\text{ext}}$, we will need additional operators $A^d$, $B^d$, ..., acting on $\mathcal{H}_{\text{ext}}$ to represent the dynamical role of magnitudes, e.g., when they appear in the Hamiltonian. The “phase space” representation $A$, $B$, etc., preserves the commutation property of usual multiplication of functions, while “dynamical” representation $A^d$, $B^d$, ..., would fulfill the usual quantization rules. Dynamics is not considered in this article, only an alternative phase space representation of quantum states is analyzed.

Together with the extended Hilbert space $\mathcal{H}_{\text{ext}}$, we introduce a label $\lambda_0 \in M_F$ characterizing the physical state of the corpuscular component of $S$. Both subsystems, de Broglie wave and particle, are described by ($\|S\rangle, \lambda_0$) in a complete phase space $\mathcal{H}_{\text{ext}} \times M_F$. $\lambda_0$ determines the value of an arbitrary measurement on state ($\|S\rangle, \lambda_0$): the result of an $A$ measurement is $\pi_A(\lambda_0)$ (some $a_i$), where $\pi_A : M_F \rightarrow M_A$ is the natural projection.

$\lambda_0$ is hidden, we can at most know precise values for a maximal family of compatible magnitudes, a subfamily of $\mathcal{F}$. The observable relative frequencies for an arbitrary measurement on an ensemble $\|S\rangle$ of states ($\|S\rangle, \lambda_0$), i.e., with common component $\|S\rangle$ and all allowed values of $\lambda_0$, is obtained from the distribution of amplitude through Born’s rule.

Born’s rule [4] at $\mathcal{H}_{\text{ext}}$ is defined in two steps as follows. If we want to obtain a (perhaps formal) distribution of probability $P(\lambda_1)$ for a subfamily $\mathcal{F}_1 \subset \mathcal{F}$, $\mathcal{F}_1 = \{D, E, G, \ldots\}$, $\lambda_1 \in M_{\mathcal{F}_1}$, we first project $\|S\rangle$ onto the Hilbert space generated by vectors $\|\lambda_1\rangle$, $\tau_1 : \mathcal{H}_{\text{ext}} \rightarrow \mathcal{H}_1$, $\tau_1(\|\lambda\rangle) = \|\pi_1(\lambda)\rangle = \|\lambda_1\rangle$, i.e.,

$$\tau_1(\|a_i, b_j, c_k, d_l, e_m, g_n, \ldots\rangle) = \|d_l, e_m, g_n, \ldots\rangle > \quad (5)$$

with $\lambda = (a_i, b_j, c_k, d_l, e_m, g_n, \ldots) \in M_F$ and $\lambda_1 = (d_l, e_m, g_n, \ldots) \in M_{\mathcal{F}_1}$.

For $\|S\rangle = \sum_\lambda Z(\lambda) \|\lambda\rangle$, the projection $\|S_1\rangle$ is

$$\|S_1\rangle = \tau_1(\|S\rangle) = \sum_\lambda \left( \sum_{\pi_1(\lambda) = \lambda_1} Z(\lambda) \right) \|\lambda_1\rangle \quad (6)$$

The coefficient $Z(\lambda_1)$ of an elementary state $\|\lambda_1\rangle$ in $\mathcal{H}_1$ is the marginal amplitude obtained from $Z(\lambda)$,

$$Z(\lambda_1) = \sum_{\pi_1(\lambda) = \lambda_1} Z(\lambda) \quad (7)$$

In the abstract path integral, this is the union of disjoint families of virtual paths characterized by different $\lambda$ in a larger family characterized by a common $\lambda_1$.

The second step is the standard $P(\lambda_1) = |Z(\lambda_1)|^2$. Obviously, in general $|Z(\lambda_1)|^2 \neq \sum_{\pi_1(\lambda) = \lambda_1} |Z(\lambda)|^2$, that is, the probability distribution $P(\lambda_1)$ does
not match the marginal probability distribution $P'(\lambda_1) = \sum_{\pi_1(\lambda) = \lambda_1} P(\lambda)$ obtained from the formal probability distribution $P(\lambda) = |Z(\lambda)|^2$ associated to the amplitude distribution $Z(\lambda)$.

The former Born’s rule can be interpreted as a representation of the correlation (or interaction) between the corpuscular and wave like subsystems. Contextuality of QM is in this formulation a consequence of interference, with different interference results for different marginal amplitudes. The interference that appears in marginal amplitudes can not be generically reproduced with marginals of a distribution of probability in $M_F$, as Bell’s theorems show.

The formal $P(\lambda)$ is not observable, no joint measurements of all magnitudes can be consistently performed. The unobservable $P(\lambda)$ does not reproduce observable probabilities through marginal probabilities. For example, in the two slit experiment the diffraction pattern is not obtained from a sum of probability distributions for each individual slit; interference, or sum of wave like degrees of freedom, is a more adequate analogy. Generically, an observable $P(\lambda_1)$ distribution is calculated through the sum of wave like degrees of freedom in the marginal amplitude and Born’s rule. According to the path integral point of view, all amplitudes are calculated in a strictly causal way, through integral along virtual paths inside the past light cone.

### 3.3 Correspondence

For a quantum system $S$, let $\mathcal{F}$ be a family of physical magnitudes, and $\mathcal{H}_{ext}$ the associated extended Hilbert space, generated by the basis $\{|\lambda >\}$. $\mathcal{H}_{QM}$ will denote the standard Hilbert space, orthodox quantum phase space of $S$. For each orthonormal basis $\{|\lambda_1 >\}$ in $\mathcal{H}_{QM}$ of common eigenvectors of a maximal family of compatible operators in a subfamily $\mathcal{F}_1 \subset \mathcal{F}$, we describe next the mathematical conditions for a correspondence between vectors $|S >\in \mathcal{H}_{ext}$ and $|S >\in \mathcal{H}_{QM}$ representing the same physical state (or ensemble) of $S$.

The most direct (and strongest) conditions are simply the equations ($\pi_1(\lambda) = \lambda_1$)

$$\tau_1(|\lambda >) \equiv |\lambda_1 > = |\lambda_1 > \quad \tau_1(|S >) = |S > \quad (8)$$

Marginal amplitudes of $Z(\lambda)$ will match $z(\lambda_1)$, for $|S > = \sum_{\lambda} Z(\lambda)|\lambda >$, $|S > = \sum_{\lambda} z(\lambda_1)|\lambda_1 >$, i.e., $\sum_{\pi_1(\lambda) = \lambda_1} Z(\lambda) = z(\lambda_1)$.

However, the observable properties of the physical state are determined by the ray $|S > = \{e|S >\}$ (c arbitrary complex numbers), so that it is enough that rays $[\tau_1(|S >)]$ and $|S >$ coincide. The corresponding equations will be projective ones, which I will denote with the symbol $::$. Maintaining the identification $|\lambda_1 > = |\lambda_1 >$ we have

$$\sum_{\pi_1(\lambda) = \lambda_1} Z(\lambda) :: z(\lambda_1) \quad (9)$$

The mildest way of the correspondence takes into account the freedom in phase that exists in the definition of a basis of eigenvectors, even after}
malisation. We could state, for each subfamily $\mathcal{F}_1$ and each one dimensional eigenspace $|\lambda_1\rangle$, a relation

$$\parallel \lambda_1 \rangle = e^{i\theta(\lambda_1)} |\lambda_1\rangle$$

which determines the projective equation

$$|\sum_{\pi_1(\lambda) = \lambda_1} Z(\lambda) | : |z(\lambda_1)|$$

These equations should be solved for the $Z(\lambda)$ (defining $\parallel S \rangle$), given the $z(\lambda_1)$, $z(\lambda_2)$, $z(\lambda_3)$, . . . of $|S\rangle$ for all maximal subfamilies $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots \subset \mathcal{F}$ of compatible magnitudes. The abstract path integral suggest there could be solution in a generic physical case; I do not have proof of that. Equations are linear (projective), and obviously no positivity condition appears, amplitudes are not even real numbers.

A correspondence for the dynamical operators $A^d$ would also be needed. In particular, an extended Hamiltonian in $\mathcal{H}_{ext}$ should determine a dynamics $\parallel S \rangle (t)$ compatible with the standard $|S\rangle (t)$ through the previous correspondence of states.

There is not a correspondence for the corpuscular variables $\lambda_0$, which do not exist in the orthodox theory. We could incorporate hidden $\pi_1(\lambda_0)$ labels (for each set $\mathcal{F}_1$ of compatible magnitudes) to the interpretation of the standard theory, without observable consequences.

An interesting property of this formulation is the classical limit. Classical limit for an elementary particle has no physical relevance, Plank’s constant can not be driven to 0, and quantum effects from the wave component are unavoidably relevant. The relevant classical limit applies to complex systems, with many components. For them, only coarse-grained, global, macroscopic variables are observable. Classical additive state variables of the corpuscular component survive in the classical limit, while wave variables become irrelevant and we can apply $\hbar \rightarrow 0$. In particular, only corpuscular variables are observable in the interaction with a macroscopic system as a measurement apparatus.

The quantum Hamiltonian $H_0$ for $\lambda_0(t)$ evolution can be rewritten as $H_0 = H_{\text{class}} + V_{QM}$, with $H_{\text{class}}$ the classical Hamiltonian. $V_{QM}$ is the (possibly stochastic) interaction term (quantum potential in Bohm’s formulation) between corpuscular and wave subsystems, and also disappears in the classical limit.

Quantum interaction between particle and vacuum wave is out of reach of observation, we can not analyse its deterministic or genuinely probabilistic character. Therefore, the observable evolution must be described in a probabilistic

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7Equivalently, given $z(\lambda_1)$ and the change of bases of eigenvectors in $\mathcal{H}_{QM}$ for all maximal compatible subfamilies $\mathcal{F}_2, \mathcal{F}_3, \ldots$ of the considered physical magnitudes $\mathcal{F}$.

8Wigner’s quasiprobability distribution in the phase space of a point particle is an example of partial solution for the analogous problem of existence of a distribution of probability in an extended phase space, with hidden variables. The positivity requirement for probabilities can not be generically fulfilled.
formulation, that incorporates wave like effects through amplitudes and Born’s rule. In the classical limit, once these effects become negligible for global state variables, we get a deterministic theory.

An observable ingredient that does not exist in the orthodox theory is the de Broglie wave subsystem; recall it was introduced in order to preserve the ARP in indirect measurements, when particle and detector in different spatial regions can not interact, if we maintain the hypothesis of locality.

4 Composite system

Let \( S \) be a composite quantum system, with components (subsystems) \( S_I \) and \( S_{II} \), \( \mathcal{F} \) a family of physical magnitudes of \( S \), \( \mathcal{F}_I \) magnitudes specific of subsystem \( S_I \), \( \mathcal{F}_{II} \) of \( S_{II} \), and \( \mathcal{F}_{int} \) magnitudes, as a potential of interaction, defined on the composite. As before, a Hilbert space \( \mathcal{H}_{ext} \) is defined through an orthonormal basis of vectors \( \| \lambda > = \| \lambda_I > \| \lambda_{II} > \) for \( \lambda_I \in M_{\mathcal{F}_I}, \lambda_{II} \in M_{\mathcal{F}_{II}} \).

For example, two independent systems with vectors of state

\[
\| S_I > = \sum_{\lambda_I} Z_I(\lambda_I)\| \lambda_I > \quad \| S_{II} > = \sum_{\lambda_{II}} Z_{II}(\lambda_{II})\| \lambda_{II} >
\]

(12)

define a state of the composite

\[
\| S > = \sum_{\lambda=(\lambda_I,\lambda_{II})} Z(\lambda_I) Z_{II}(\lambda_{II})\| \lambda_I > \| \lambda_{II} >
\]

(13)

In general, when both subsystems interact the state of the composite will not be a direct product, but some

\[
\| S > = \sum_{\lambda} Z(\lambda)\| \lambda >
\]

(14)

in which additional magnitudes of interaction could be taken into account, and appear in \( \lambda \).

If \( A_I \) and \( B_{II} \) are a maximal family of compatible operators in \( \mathcal{F}_I \) and \( \mathcal{F}_{II} \) respectively (magnitudes of a system trivially commute with magnitudes of the other), the projection of \( \| S > \) onto \( \mathcal{H}_{A_I} \times \mathcal{H}_{B_{II}} \) is

\[
\tau_{A_I B_{II}}(\| S >) = \sum_{i,j} \left( \sum_{\pi A_I(\lambda)=a_I, \pi B_{II}(\lambda)=b_{II}} \| a_I > \| b_{II} > \right.
\]

\[
\left. \sum_{i,j} Z(a_{II},b_{III})\| a_I > \| b_{III} > \right)
\]

(15)

(16)

with \( Z(a_{II},b_{III}) \) the corresponding marginal amplitudes.

Correspondence with the standard formalism will be fulfilled if the previous state is an equivalent representation of the orthodox \( |S > = \sum_{i,j} c_{ij} | a_I > \)
A state is said to be entangled if \( c_{ij} \) does not factorize, or in the extended formalism, if \( Z(\lambda) \) does not factorize.

Relative frequencies are obtained as usual, \( P(a_{II}, b_{IIj}) = |c_{ij}|^2 \) in the standard formulation when \(|S\rangle\) is normalized, and \( P(a_{II}, b_{IIj}) \propto |Z(a_{II}, b_{IIj})|^2 \) in the alternative formulation. Obviously, \( P(a_{II}) \) is obtained as a marginal probability \( P(a_{II}) = \sum_j P(a_{II}, b_{IIj}) \), and it is independent of the chosen magnitude in the second system, e.g. \( P(a_{II}) = \sum_k P(a_{II}, c_{IIk}) \) for a magnitude (more precisely, maximal family) \( C_{II} \).

The (extended) state of system \( S \) is determined by some vector \(|S\rangle\) and label \( \lambda_0 = (\lambda_I, \lambda_{II}) \), if we ignore for simplicity magnitudes \( F_{\text{int}} \), for example because after interaction both subsystems are far apart. \( \lambda_0 \) determines the result of an arbitrary measurement, or pair of independent measurements on each subsystem. When measuring system \( S_I \) its new label \( \lambda'_{I0} \) will change (magnitudes non commuting with the one measured will evolve along the measurement interaction), but this does not modify the label \( \lambda_{II0} \), because all magnitudes in \( F_{II} \) commute with the measured magnitude in \( F_I \). With regards to the relative frequencies, as we said \( P(b_{IIj}) \) is independent of magnitude \( A_I \) measured, although there will be in general a correlation in \( P(a_{II}, b_{IIj}) \).

If we understand the amplitudes \( Z(\lambda) \) as result of a relativistic path integral calculation, virtual paths characterized by \( \lambda = (\lambda_I, \lambda_{II}) \) are contained in the past light cones of events corresponding to \( \lambda_I \) and \( \lambda_{II} \), and in the shared past region of both there will be contributions of interaction terms in the action integral. This is the origin of correlation. After both subsystems separate, do not interact, the additional contributions along each individual path will not modify the established correlation, although obviously they can generate individual evolution of each subsystem. That is, all correlation information in \( Z(\lambda) \) has a causal origin in the path integral formalism.

Given a magnitude \( A_I \) of \( S_I \) and two non commuting magnitudes \( B_{II} \) and \( C_{II} \) of \( S_{II} \) we can calculate the marginal \( Z(a_{II}, b_{IIj}, c_{IIk}) \), and a formal \( P(a_{II}, b_{IIj}, c_{IIk}) \). Again, the marginal \( P'(a_{II}, b_{IIj}) = \sum_k P(a_{II}, b_{IIj}, c_{IIk}) \) will not match generically the observable \( P(a_{II}, b_{IIj}) \), because of interference in the marginal amplitude \( Z(a_{II}, b_{IIj}) = \sum_k Z(a_{II}, b_{IIj}, c_{IIk}) \).

Let us consider a correlated magnitude \( C_T = C_I + C_{II} \), with value \( c_T \), as a consequence of a past interaction. The label \( \lambda_0 \) (corpuscular degrees of freedom) will fulfill the equation

\[
\pi_{CT}(\lambda_0) = c_T = \pi_{C_I}(\lambda_0) + \pi_{C_{II}}(\lambda_0) = c_{Ii} + c_{IIk}
\]  \hspace{1cm} (17)

so that perfect correlation appears in a measurement of magnitudes \( C_I \) and \( C_{II} \) on the entangled pair. A subset \( M_{\text{corr}} \subset M_T \) determines the correlated values.

Similarly, the vector of state (wave degrees of freedom) fulfills \( C_T \parallel |S\rangle = c_T |S\rangle \). If \( \{C_I, C_{II}\} \) is a maximal family of compatible magnitudes, the projected state
\[ \tau_{CI} \left( |S > \right) = \sum_{k,l} \left( \sum_{\pi} \pi_{CI}(\lambda) = c_{Ik} \right) \pi_{CII}(\lambda) = c_{IIl} \|c_{Ik} > \|c_{IIl} > (18) \]
defines, as usual, the marginal amplitudes \( Z(c_{Ik}, c_{IIl}) \), which vanish of \( c_{Ik} + c_{IIl} \neq c_{T} \).

However, not all terms \( Z(\lambda) \) with \( \lambda \in M_{F}/M_{corr} \) necessarily vanish, it is enough that interference in the marginal \( Z(c_{Ik}, c_{IIl}) \) is destructive. That is, \( \lambda_{0} \in M_{corr} \), but parameter \( \lambda \) in the sum defining \( |S > \) belongs to \( M_{F} \). As an example, in the two slit experiment wave amplitudes coming from both slits do not vanish in a zero of the diffraction pattern, and a destructive interference of both determines the null probability density there. A correspondence with the standard state \( |S > = \sum_{k} z_{k} |c_{Ik} > |c_{IIk} > (c_{Ik} + c_{IIk} = c_{T}) \) can now be established, \( |z_{k}| :: |Z(c_{Ik}, c_{IIk})| \).

With another correlated magnitude \( D_{T} = D_{l} + D_{II} \), with value \( d_{T} \), we can define the marginals

\[ Z(c_{Ik}, c_{IIl}, d_{Im}, d_{IIm}) \equiv Z_{klmn} (19) \]
as usual, and formal, unobservable, probability distributions \( P(c_{Ik}, c_{IIl}, d_{Im}, d_{IIm}) \) whose marginal probabilities do not match those obtained through Born’s rule.

As pointed out before, \( Z_{klmn} \) does not vanish generically for \( c_{Ik} + c_{IIl} \neq c_{T} \) or \( d_{Im} + d_{IIm} \neq d_{T} \). Only the marginals, used in Born’s rule to obtain observable relative frequencies, \( Z(c_{Ik}, c_{IIl}) = Z_{kl} = \sum_{mn} Z_{klmn} \) will vanish for \( c_{Ik} + c_{IIl} \neq c_{T} \). Therefore, in a marginal \( Z(c_{Ik}, d_{IIm}) = \sum_{lm} Z_{klmn} \) there will be contributions of terms \( Z_{klmn} \) where \( c_{Ik} + c_{IIl} \neq c_{T} \). We can compare this generic case with the two slit experiment, corpuscular variables \( \lambda_{0} \) have a definite, but hidden, value (e.g., left or right slit) while wave degrees of freedom, the amplitudes, have generically all possible components, although they can interfere destructively in some marginals. In next section a particularly relevant case, pairs of spin \( 1/2 \) entangled particles with total null spin, is analysed.

The double role of amplitudes, as source of wave like effects and in Born’s rule for observable relative frequencies, is behind the quantumness of QM in the proposed formulation, instead of a non local wave function collapse. There is no essential distinction between entanglement and the two slit experiment.

5 Applications

5.1 Two slit experiment

The relevant variables in the two slit experiment are the slit variable \( S \), with value \( L \) or \( R \), and the position at the final screen \( R \), with values \( r_{s} \). Slit and
final position do not commute. Paths can be grouped in families\(^9\) of \([\text{path}] (L, r_i)\) and \([\text{path}] (R, r_i)\). The extended Hilbert space is generated by elementary states \(|L, r_i\rangle\) and \(|R, r_i\rangle\). The vector of state is

\[
|S\rangle = \sum_i Z(L, r_i) |L, r_i\rangle + \sum_i Z(R, r_i) |R, r_i\rangle
\]  

with projections

\[
\tau_S\langle S | = \frac{1}{\sqrt{2}} (|L\rangle + |R\rangle)
\]  

and

\[
\tau_R\langle S | = \sum_i (Z(L, r_i) + Z(R, r_i)) |r_i\rangle
\]

This one is also the orthodox vector of state in the position coordinates representation at the final screen.

The formal distribution of probability

\[
P(L, r_i) = |Z(L, r_i)|^2 \quad P(R, r_i) = |Z(R, r_i)|^2
\]

is not observable. The marginal

\[
Z(r_i) = Z(L, r_i) + Z(R, r_i)
\]

determines the observable diffraction pattern \(P(r_i) = |Z(r_i)|^2\).

Let us suppose that an external system interacts with our system of interest (particle plus de Broglie wave) at the R slit. For simplicity, let us consider that the new amplitude \(Z'(R, r_i)\) is similar in modulus to the unperturbed \(Z(R, r_i)\). However, there will be some phase shift \(Z'(R, r_i, \phi) \simeq e^{i\phi} Z(R, r_i)\). If we use, for example, a beam of coherent photons, and the additional system is a phase plate of known phase shift, we will obtain a displaced diffraction pattern.

If the additional system is a (more or less complex) measurement apparatus, the phase shift will be unknown. A statistical average on the phase (e.g., with uniform distribution) determines a total distribution of relative frequencies

\[
P'(r_i) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi |Z'(R, r_i, \phi) + Z(L, r_i)|^2 \simeq |Z(R, r_i)|^2 + |Z(L, r_i)|^2
\]

and the diffraction pattern disappears. We can observe now, by correlation with the positive/negative result of measurement in slit \(R\), the marginal distributions \(P'(R, r_i)\) and \(P'(L, r_i)\). In practice, we can apply the projection rule according to the result of measurement at \(R\) slit. A stochastic phase shift at measurement interactions determines the projection of state as a practical rule.

\(^9\)Magnitudes associated to a path are not necessarily values at its final point. Virtual paths going through \(L\) and \(R\) slits are disjoint families.
5.2 Spin variables

Let us consider a spin $1/2$ particle. The most general family of spin variables will contain spin (up or down) in an arbitrary number of directions $\mathbf{n}_j$, unit vectors in space. $\mathcal{F} = \{S_1, S_2, \ldots\}$, with $M_{S_j} = \{+, -\}$ and $M_{\mathcal{F}} = \{(s_1, s_2, \ldots)\}$: $s_j$ is $+$ for spin up and $-$ for spin down in direction $\mathbf{n}_j$.

The extended Hilbert space $\mathcal{H}_{ext}$ is generated by elementary states $|\lambda\rangle$, $\lambda = (s_1, s_2, \ldots) \in M_{\mathcal{F}}$. Orthodox Hilbert space is a two dimensional one $\mathcal{H}_{QM}$, where we can use each $\{|\pm_j\rangle, |\mp_j\rangle\}$ basis of spin up/down states in direction $\mathbf{n}_j$.

Let us associate a fix “path” amplitude to each spin value, $s_j e^{i\theta_j}$ for coplanar directions or $s_j N_j$ for directions in space, with $N_j$ a quaternion number (with null real part) $N_j = n_j^x I + n_j^y J + n_j^z K$ associated to the vector $\mathbf{n}_j = n_j^x \mathbf{i} + n_j^y \mathbf{j} + n_j^z \mathbf{k}$ \[^{[10]}\]. The amplitude associated to an elementary state $|\lambda\rangle$ will be

$$Z(\lambda) = \sum_j s_j N_j$$

A states with known spin $s_1$ in direction $\mathbf{n}_1$ is

$$|s_1\rangle = \sum_{\pi_1(\lambda) = s_1} Z(\lambda) |\lambda\rangle$$

Projection over the two dimensional Hilbert space for operator $S_1$, when using the basis $\{|\pm_1\rangle, |\mp_1\rangle\}$, trivially reproduces the orthodox $|s_1\rangle$: in the marginal amplitude, terms $s_j N_j$ and $-s_j N_j$ for $j > 1$ cancel out, and a global factor can be ignored.

Projection over the Hilbert space for operator $S_2$, using the basis $\{|\pm_2\rangle, |\mp_2\rangle\}$, gives

$$\tau_2(|s_1\rangle) = \mathcal{N} ((s_1 N_1 + N_2) |\pm_2\rangle + (s_1 N_1 - N_2) |\mp_2\rangle)$$

with $\mathcal{N}$ a global factor. There is a correspondence, when appropriate phases are introduced in the correspondence of vectors, between $\tau_2(|s_1\rangle)$ in the two dimensional quaternion Hilbert space $\mathcal{H}_{S_2} = \langle |\pm_2\rangle, |\mp_2\rangle \rangle$ and the standard expression in the orthodox complex Hilbert space.

All orthodox spin states of an individual particle are eigenstates of the spin operator in some spatial direction. The former correspondence allows to represent them in the extended phase space. On the other hand, the label $\lambda_0$ in the state ($|s_1\rangle, \lambda_0$) will project onto a label $s_1$ (known for state $|s_1\rangle$) or $s_2$ (unknown at the ensemble state $(s_1 N_1 + N_2) |\pm_2\rangle + (s_1 N_1 - N_2) |\mp_2\rangle$). The (hidden) label determines the value of spin measurement in arbitrary direction $\mathbf{n}_j$, $s_j = \pi_j(\lambda_0)$. Born’s rule in two steps for the extended formalism reproduces the standard results. Operators $S_j$ commute. For dynamical purposes, additional operators $S^d_j$ fulfilling the standard $[S^d_x, S^d_y] = i\hbar S^d_z$ should be considered.

\[^{[10]}\text{Direction } -\mathbf{n}_j \text{ is redundant, because } S_{-\mathbf{n}_j} = -S_{\mathbf{n}_j}.\]
Let us consider a composite system of two spin 1/2 particles $S^a$ and $S^b$ in a total null spin state. The label $\lambda_0$ will fulfill $\pi_j(\lambda_{00}) + \pi_j(\lambda_{00}) = 0$, i.e., perfect correlation $s^a_j + s^b_j = 0$, determining a subset $M_{corr} \subset M_F$; the total family of operators is $F = F^a \cup F^b$, $F^a = \{S^a_1, S^a_2, \ldots\}$ and $F^b = \{S^b_1, S^b_2, \ldots\}$.

The vector of state for the composite will have the form

$$\| S \| = \sum_{\lambda \in M_F} Z_T(\lambda) \| \lambda^a > \| \lambda^b >$$

(29)

Notice that $\lambda$ is not restricted to $M_{corr}$, all we have to impose is that marginals $Z(s^a_1, s^b_1) = 0$, through destructive interference. A solution for the amplitude distribution is

$$Z_T(\lambda) = Z_T(\lambda^a, \lambda^b) = Z(\lambda^a) - Z(\lambda^b)$$

(30)

where $Z(\lambda)$ is as before $Z(\lambda) = \sum_s n_s \lambda_j$.

Let us consider two directions $n_1$ and $n_2$, with variables $(s^a_1, s^a_2, s^b_1, s^b_2)$. The marginal amplitude becomes

$$Z_{(1,2)^a,(1,2)^b}(s^a_1, s^a_2, s^b_1, s^b_2) = (s^a_1 - s^b_1)N_1 + (s^a_2 - s^b_2)N_2$$

(31)

up to a total factor; all coefficients of $N_j$ for $j > 2$ vanish. Therefore, $Z_{(1,2)^a,(1,2)^b}(s^a_1, s^a_2, s^b_1, s^b_2) = (s^a_1 - s^a_2)N_1 \neq 0$. However, the observable amplitudes of measurement in direction $n_1$ for both particles is

$$Z_{1^a,1^b}(s^a_1, s^b_1) = (s^a_1 - s^b_1)N_1$$

(32)

We get $Z_{1^a,1^b}(s^a_1, s^b_1) = 0$, $Z_{1^a,1^b}(s^a_1, s^b_1) = 2s^a_1N_1$, and a probability distribution $P(s_1, s_1) = 0$, $P(s_1, -s_1) = 1/2$.

Similarly, the observable amplitude $Z_{1^a,2^b}$ for measurement $S_1$ of particle $a$ and $S_2$ of particle $b$ is

$$Z_{1^a,2^b}(s^a_1, s^b_2) = s^a_1N_1 - s^b_2N_2$$

(33)

with the quantum associated probability distribution $P(s^a_1, s^b_2) :: |s^a_1N_1 - s^b_2N_2|^2$

$$= 2(1 - s^a_1 s^b_2 n_1 \cdot n_2)$$

If we calculate the marginal amplitude for direction $S_1$ of particle $a$ and directions $S_2$ and $S_3$ of particle $b$

$$Z_{1^a,(2,3)^b}(s^a_1, s^b_2, s^b_3) = s^a_1N_1 - (s^b_2N_2 + s^b_3N_3)$$

(34)

we can define a formal distribution of probability

$$P(s^a_1, (s^b_2, s^b_3)) = |Z_{1^a,(2,3)^b}(s^a_1, (s^b_2, s^b_3))|^2$$

(35)

\(^{11}N_1^* = -N_1\) for quaternions without real part, and $N_1^*N_2 = -n_1 \cdot n_2 - n_1 \times n_2$; in the former expression, the dot product is the real part and the cross product the imaginary part of a quaternion.
The marginal probability \( P'(s_1, s_2) = P(s_1^a, (s_2, +3)^b) + P(s_1^a, (s_2, -3)^b) \) does not match the former one, because of interference when applying Born’s rule in two steps,

\[
Z_{1^a, 2^b}(s_1^a, s_2^b) = Z_{1^a, (2, 3)^b}(s_1^a, (s_2, +3)^b) + Z_{1^a, (2, 3)^b}(s_1^a, (s_2, -3)^b) 
\]  
(36)

and

\[
P(s_1^a, s_2^b) :: \left| Z_{1^a, (2, 3)^b}(s_1^a, (s_2, +3)^b) + Z_{1^a, (2, 3)^b}(s_1^a, (s_2, -3)^b) \right|^2 
\]  
(37)

differs, even projectively, from

\[
\left| Z_{1^a, (2, 3)^b}(s_1^a, (s_2, +3)^b) \right|^2 + \left| Z_{1^a, (2, 3)^b}(s_1^a, (s_2, -3)^b) \right|^2 
\]  
(38)

There is a full analogy with interference in the two slit experiment, for example if we identify the third magnitude \( s_3 \in \{+, -\} \) with the slit \( \{L, R\} \). Bell’s inequalities prove that there is no distribution of probability in a space with hidden variables whose marginal probabilities reproduce the quantum result. What we have here is a distribution of amplitude in the extended space of hidden variables; through the defined correspondence (marginals, interference, Born’s rule) it reproduces the orthodox quantum phase space representation.

Corpuscular variables \( \lambda_0 \) fulfil the perfect correlation condition, and determine the result of arbitrary measurements in a correlated pair of particles (\( \|S\rangle \), \( \{(\lambda_{a_0}, \lambda_{b_0})\} \)). The distribution of probability for observable magnitudes depends on the wave degrees of freedom \( \|S\rangle \) (common for an ensemble), the distribution of amplitudes. Born’s rule in two steps gives way to interference, wave like phenomenon that does not appear in a marginal probability. The distribution \( Z(\lambda) \) of the entangled system, as well as the corpuscular variable \( \lambda_0 \), are fixed from the generation event of the two entangled particles. No non local phenomenon is invoked in the former mathematical description.

6 Summary and Outlook

A formalism of non relativistic Quantum Mechanics in which elementary states have joint precise values of non commuting magnitudes has been developed in order to restore the action reaction principle, which is contradictory with the projection of state.

An abstract path integral formalism, with families of virtual paths more restrictive than the standard one, suggest that a correspondence with all predictions of the orthodox theory could exist. The two slit experiment and the phase space of spin variables have been formulated in the extended phase space.

Contextuality and non locality of Quantum Mechanics appear here as interference properties of wave like degrees of freedom. The mysterious double role of amplitudes, at interference and determining probabilities, is the source of these non classical properties of quantum systems.
An accompanying, wave like subsystem is predicted when particle and detector are spatially separated in indirect measurements, in order to preserve locality of interactions and the action reaction principle. An appropriately designed detector could show observable reactions; they would be assigned either to a non local interaction between the corpuscular component and the detector or to a local interaction between a de Broglie wave component and detector.

7 Acknowledgements

Financial support from research project MAT2011-22719 is acknowledged. I also kindly acknowledge helpful comments from members of the audience in both seminars at Zaragoza and Valladolid Universities, where I presented some conclusions of this research in April and June 2015 respectively.

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