Polar Duality Between Pairs of Transversal Lagrangian Planes; Applications to Uncertainty Principles

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Abstract

We extend the notion of polar duality to pairs \((\ell, \ell')\) of transversal Lagrangian planes in the standard symplectic space \((\mathbb{R}^{2n}, \omega)\). This allows us to show that polar duality has a natural interpretation in terms of symplectic geometry. We show that the orthogonal projections \(\Omega_\ell\) and \(\Omega_{\ell'}\) of a centrally symmetric convex body \(\Omega\) satisfying \(c_{\min}(\Omega) \geq \pi\) satisfy the duality relation \(\Omega_\ell \subset \Omega_{\ell'}\).

Keywords: polar duality; Lagrangian plane; polar duality; symplectic capacity; John and Löwner ellipsoids; uncertainty principle

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1 Introduction

Polar duality, in its usual acceptation, associates to a centrally symmetric convex body \(X\) in the Euclidean space \(\mathbb{R}^n\) another convex body \(X^o\) in the dual space \((\mathbb{R}^n)^*\) which is, for all practical purposes, identified with \(\mathbb{R}^n\) itself: \(p' \in X^o\) if and only if \(p' \cdot x \leq 1\) for all \(x \in X\). The present paper is based on the observation that if one identifies \(\mathbb{R}^n\) and is dual vector space \((\mathbb{R}^n)^*\) with the coordinate Lagrangian planes \(\ell_X = \mathbb{R}^n \times 0\) and \(\ell_P = 0 \times \mathbb{R}^n\),

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respectively (for the canonical symplectic structure $\omega$ on $T^*\mathbb{R}^n$), then the defining relation $p' \cdot x \leq 1$ can be rewritten as $\omega(z, z') \leq 1$ where $z = (x, 0) \in \ell_X$ and $z' = (0, p') \in \ell_P$. Thus, polar duality can be reformulated in terms of the symplectic structure on $T^*\mathbb{R}^n$, and motivates a more general definition: let $\ell$ and $\ell'$ be two transverse Lagrangian planes in $(T^*\mathbb{R}^n, \omega)$, and $X_\ell \subset \ell$ a centrally symmetric convex body; the polar dual $X_{\ell'}^o \subset \ell'$ is then defined as being the set of all $z' \in \ell'$ such that $\omega(z, z') \leq 1$ for all $z \in \ell$. This highlights the fact that polar duality is essentially a property of symplectic geometry. In the present work apply this extended version of polar duality to the study of the projections of convex sets containing a symplectic ball on pairs of transverse Lagrangian planes. More precisely:

- In Section 2 we review the geometric tools we will need (symplectic capacities, standard polar duality, Lagrangian planes), and define the new concept of Lagrangian polar duality which is central to this work; we establish the main properties of this duality relate it to symplectic capacities;

- In Section 3 we state and prove the main geometric results of this paper (Proposition 11, Theorems 13 and 15). In Proposition 11 we determine the John ellipsoid of the product $X \times X^o$ and in Theorem 13 we show that if $(\ell, \ell')$ is a pair of transverse Lagrangian planes then the projections $\Pi_\ell : \mathbb{R}^{2n} \rightarrow \ell$ and $\Pi_{\ell'} : \mathbb{R}^{2n} \rightarrow \ell$ in the directions $\ell'$ and $\ell$ of a symmetric convex body $\Omega \subset \mathbb{R}^{2n}$ containing a symplectic ball $S(B^{2n}(1))$ form a Lagrangian polar dual pair. Theorem 15 is a partial converse to Theorem 13 highlighting the role played by the John ellipsoid of a Lagrangian polar dual pair.

- In Section 4 we discuss from a new point of view the indeterminacy principle of quantum mechanics. We justify in Proposition 20 the following claim: a quantum system localized in a Lagrangian plane $\ell$ cannot be localized in a transverse Lagrangian plane $\ell'$ in a set smaller than its Lagrangian polar dual. This is a geometric restatement of the uncertainty principle, which has the advantage of being independent of the choice of any statistical measure.

- In Section 5 we apply Lagrangian polar duality to a reformulation of the multi-dimensional Hardy uncertainty principle about the simultaneous decrease at infinity of a function and its Fourier transform, thus generalizing previous results of ours.
Notation. We denote by $\omega$ the standard symplectic form on $T^*\mathbb{R}^n \equiv \mathbb{R}^{2n}$: $\omega = \sum_{j=1}^{n} dp_j \wedge dx_j$ (we use the splitting $z = (x, p)$). A diffeomorphism $f \in \text{Diff}(n)$ of $\mathbb{R}^{2n}$ is called a symplectomorphism of $(\mathbb{R}^{2n}, \omega)$ if $f^* \omega = \omega$. Symplectomorphisms form a subgroup $\text{Symp}(n)$ of $\text{Diff}(n)$ equipped with the usual composition operation \[34\]. Linear symplectomorphisms form a subgroup $\text{Sp}(n)$ of $\text{Symp}(n)$. We denote by $\text{Mp}(n)$ the unitary representation (metaplectic group) of the double cover of $\text{Sp}(n)$. Every $S \in \text{Sp}(n)$ is the projection of two operators $\pm \hat{S} \in \text{Mp}(n)$.

2 Prerequisites

We will use the following notation for closed balls and symplectic cylinders in $\mathbb{R}^{2n}$:

$$B^{2n}(R) = \{ z \in \mathbb{R}^{2n} : |z|^2 \leq R^2 \}$$

$$Z_j^{2n}(R) = \{ z \in \mathbb{R}^{2n} : |z_j|^2 \leq R^2 \}$$

where $z_j = (x_j, p_j), 1 \leq j \leq n$. Similarly, we denote by $B^{2n}(z_0, R)$ (resp. $Z_j^{2n}(z_0, R)$) the ball (resp. cylinder) centered at $z_0 \in \mathbb{R}^{2n}$. Recall that in view of Gromov’s \[22\] symplectic non-squeezing theorem there exists $f \in \text{Symp}(n)$ such that $f(B^{2n}(R)) \subset Z_j^{2n}(r)$ if and only if $R \leq r$. We will call “symplectic ball” (with radius $R$) the image of $B^{2n}(R)$ by $S \in \text{Sp}(n)$.

2.1 Symplectic capacities on the standard symplectic space

A (normalized) symplectic capacity on $(\mathbb{R}^{2n}, \omega)$ associates to every subset $\Omega \subset \mathbb{R}^{2n}$ a number $c(\Omega) \in [0, +\infty]$ such that the following properties hold \[3\] \[9\]:

- **Monotonicity**: If $\Omega \subset \Omega'$ then $c(\Omega) \leq c(\Omega')$;
- **Conformality**: For every $\lambda \in \mathbb{R}$ we have $c(\lambda \Omega) = \lambda^2 c(\Omega)$;
- **Symplectic invariance**: $c(f(\Omega)) = c(\Omega)$ for every $f \in \text{Symp}(n)$;
- **Normalization**: We have, for $1 \leq j \leq n$,

$$c(B^{2n}(R)) = \pi R^2 = c(Z_j^{2n}(R)). \quad (1)$$

The number $\pi R^2$ is sometimes called the symplectic radius of $B^{2n}(R)$ (or $Z_j^{2n}(R)$). We will avoid this terminology because it is confusing. Note
that the conformality and normalization properties show that for \( n > 1 \) symplectic capacities are not related to volume; they have the dimension of an area. This is exemplified by the Hofer–Zehnder capacity \([26]\) which has the property that when \( \Omega \) is a compact convex set in \((\mathbb{R}^{2n}, \omega)\) with smooth boundary \( \partial \Omega \) then

\[
c_{\text{HZ}}(\Omega) = \int_{\gamma_{\text{min}}} p_1 dx_1 + \cdots + p_n dx_n \tag{2}
\]

where \( \gamma_{\text{min}} \) is the shortest (positively oriented) Hamiltonian periodic orbit carried by \( \partial \Omega \).

It follows from Gromov’s symplectic non-squeezing theorem \([22]\) that the formulas

\[
c_{\text{min}}(\Omega) = \sup_{f \in \text{Symp}(n)} \{ \pi R^2 : f(B^{2n}(R)) \subset \Omega \} \tag{3a}
\]

\[
c_{\text{max}}(\Omega) = \inf_{f \in \text{Symp}(n)} \{ \pi R^2 : f(\Omega) \subset Z_j^{2n}(R) \} \tag{3b}
\]

define two symplectic capacities (\( c_{\text{min}} \) is sometimes called “Gromov’s width” while \( c_{\text{max}} \) is referred to as the “cylindrical capacity”). The notation is motivated by the fact that for every symplectic capacity on \((\mathbb{R}^{2n}, \omega)\) we have \( c_{\text{min}} \leq c \leq c_{\text{max}} \).

Linear symplectic capacities \( c_{\text{lin}} \) are defined as above, replacing everywhere the group \( \text{Symp}(n) \) of symplectomorphisms with the affine symplectic group \( \text{ASp}(n) \) generated by the affine symplectic transformations \( ST(z_0) = T(Sz_0)S \) where \( z_0 \in \mathbb{R}^{2n} \) and \( S \in \text{Sp}(n) \) \( (T(z_0) \) is the translation \( z \mapsto z + z_0 \). One defines the minimum and maximum linear symplectic capacities by

\[
c_{\text{lin}}_{\text{min}}(\Omega) = \sup_{S \in \text{Sp}(n) \atop z_0 \in \mathbb{R}^{2n}} \{ \pi R^2 : S(B^{2n}(z_0, R)) \subset \Omega \} \tag{4a}
\]

\[
c_{\text{lin}}_{\text{max}}(\Omega) = \inf_{S \in \text{Sp}(n) \atop z_0 \in \mathbb{R}^{2n}} \{ \pi R^2 : ST\Omega) \subset Z_j^{2n}(z_0, R) \} \tag{4b}
\]

and one has \( c_{\text{lin}}_{\text{min}} \leq c_{\text{lin}} \leq c_{\text{lin}}_{\text{max}} \) for every linear symplectic capacity \( c_{\text{lin}} \).

Of particular interest is the symplectic capacity of an ellipsoid. In what follows we assume that \( \Omega \) is an ellipsoid in \( \mathbb{R}^{2n} \) given by

\[
\Omega = \{ z \in \mathbb{R}^{2n} : Mz \cdot z \leq 1 \}
\]
where $M$ is a positive definite (symmetric) matrix of order $2n$. It is well-known that for every symplectic capacity $c$ (resp. linear symplectic capacity $c^{\text{lin}}$) we have
\[ c(\Omega) = c^{\text{lin}}(\Omega) = \frac{\pi}{\lambda^\omega_{\text{max}}} \] (5)
where $\lambda^\omega_{\text{max}}$ is the largest symplectic eigenvalue of the matrix $M$ (the symplectic eigenvalues of $M$ are the numbers $\lambda^\omega_j > 0$ ($1 \leq j \leq n$) such that the $\pm i\lambda^\omega_j$ are the eigenvalues of the antisymmetric matrix $M^{1/2}JM^{1/2}$).

## 2.2 Polar duality

Let $X$ be a closed convex body in $\mathbb{R}^n_x$ (a convex body in an Euclidean space is a compact convex set with non-empty interior). We assume in addition that $X$ contains 0 in its interior. This is the case if, for instance, $X$ is centrally symmetric: $X = -X$. By definition The polar dual of $X$ is the subset
\[ X^o = \{ p \in \mathbb{R}^n : px \leq 1 \text{ for all } x \in X \} \] (6)
of the dual space $\mathbb{R}^n_p \equiv (\mathbb{R}^n)^*$ of $\mathbb{R}^n_x \equiv \mathbb{R}^n$ (with this notation the phase space $\mathbb{R}^{2n}$ is identified with the product $T^*(\mathbb{R}^n) \equiv \mathbb{R}^n_x \times \mathbb{R}^n_p$).

It follows from the definition (6) that $X^o$ is convex. The following properties of the polar dual are obvious:

\textbf{Biduality:} $(X^o)^o = X$ ; \hspace{1cm} (7)
\textbf{Anti-monotonicity:} $X \subset Y \implies Y^o \subset X^o$ ; \hspace{1cm} (8)
\textbf{Scaling:} $L \in GL(n, \mathbb{R}) \implies (LX)^o = (L^T)^{-1}X^o$ . \hspace{1cm} (9)

We also list the following properties of the polar duals of balls and ellipsoids:

Let $B^n_X(R)$ (resp. $B^n_p(R)$) be the ball $\{ x : |x| \leq R \}$ in $\mathbb{R}^n_x$ (resp. $\{ p : |p| \leq R \}$ in $\mathbb{R}^n_p$). We have
\[ B^n_X(R)^o = B^n_p(R^{-1}) . \] (10)

In particular $B^n_X(1)^o = B^n_p(1)$. When the context is clear we will write $B^n_X(R) = B^n_p(R) = B^n(R)$, etc.

Let $L \in GL(n, \mathbb{R})$. In view of (9) we have, for $L \in GL(n, \mathbb{R})$,
\[ (L(B^n(R)))^o = (L^T)^{-1}B^n(R^{-1}) \] (11)
hence, if $A = A^T \in GL(n, \mathbb{R})$
\[ \{ x \in \mathbb{R}^n_x : Ax \cdot x \leq 1 \}^o = \{ p \in \mathbb{R}^n_p : A^{-1}p \cdot p \leq 1 \} \] (12)
since the left-hand side is $(A^{-1/2}(B^n(1)))^o = A^{1/2}(B^n(1))$. 

5
Definition 1 A pair $(X, P)$ of centrally symmetric convex bodies $X \subset \mathbb{R}_x^n$ and $P \subset \mathbb{R}_p^n$ is called a “dual pair” if we have $X^o \subset P$ (or, equivalently, $P^o \subset X$). If $X^o = P$ we say that $(X, P)$ is an exact dual pair.

The following elementary result is straightforward:

Lemma 2 Let $(X, P)$ be a dual pair and $Y, Q$ be symmetric convex bodies such that $X \subset Y$ and $P \subset Q$. Then $(Y, Q)$ is also a dual pair.

Proof. It follows from the chain of inclusions $Y^o \subset X^o \subset P$ using the anti-monotonicity of the passage to the dual. ■

Proposition 3 The ellipsoids $X = \{ x : Ax^2 \leq 1 \}$ and $P = \{ p : Bp^2 \leq 1 \}$ $(A, B > 0)$ form a dual pair if and only if $A \leq B^{-1}$ (or, equivalently, $AB \leq I_{n \times n}$), and $(X, P)$ is an exact dual pair if and only if $AB = I_{n \times n}$.

Proof. We have $X = A^{-1/2}(B^n(1))$ and $P = B^{-1/2}(B^n(1))$; in view of (11) the relation $X^o \subset P$ is thus equivalent to $A^{1/2} \leq B^{-1/2}$ (with equality if and only if $X^o = P$). ■

In [3], Remark 4.2, Artstein-Avidan et al prove that if $(X, P)$ is an arbitrary pair of centrally symmetric convex bodies $X \subset \mathbb{R}_x^n$ and $P \subset \mathbb{R}_p^n$ then we have

$$c_{HZ}(X \times P) = c_{\max}(X \times P) = 4 \sup \{ \lambda > 0 : \lambda X^o \subset P \}. \quad (13)$$

It follows that if $(X, P)$ is a dual pair, then

$$c_{HZ}(X \times P) = c_{\max}(X \times P) \geq 4 \quad (14)$$

and, in particular,

$$c_{HZ}(X \times P) = c_{\max}(X \times P) = 4. \quad (15)$$

2.3 Lagrangian polar duality

Let $(\mathbb{R}^{2n}, \omega)$ be the standard symplectic space. We denote by $\Lambda(n)$ the Lagrangian Grassmannian of $(\mathbb{R}^{2n}, \omega)$: its elements (the “Lagrangian planes”) are the $n$-dimensional subspaces $\ell$ of $\mathbb{R}^{2n}$ on which $\omega$ vanishes identically. We denote by $\text{Sp}(n)$ the standard symplectic group. It consists of all automorphisms of $\mathbb{R}^{2n}$ preserving the symplectic form $\omega$. It follows that $\text{Sp}(n)$ is a closed subgroup of $\text{GL}(2n, \mathbb{R})$ and hence a classical Lie group. The symplectic group acts transitively on the Lagrangian Grassmannian: for every
pair \((\ell, \ell') \in \Lambda(n)^2\) there exists \(S \in \text{Sp}(n)\) such that \(\ell' = S\ell\). This is most easily seen choosing two bases \(\{e_1, \ldots, e_n\}\) and \(\{e'_1, \ldots, e'_n\}\) of \(\ell\) and \(\ell'\), respectively and completing these bases to symplectic bases \(\{e_1, \ldots, e_n; f_1, \ldots, f_n\}\) and \(\{e'_1, \ldots, e'_n; f'_1, \ldots, f'_n\}\) of \((\mathbb{R}^{2n}, \omega)\). The automorphism \(S\) of \(\mathbb{R}^{2n}\) taking the first basis to the second is in \(\text{Sp}(n)\) and we have \(\ell' = S\ell\) by construction.

Let \(U(n)\) be the subgroup of \(\text{Sp}(n)\) consisting of symplectic rotations:

\[
U(n) = \text{Sp}(n) \cap O(2n, \mathbb{R})
\]

The monomorphism \(A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}\) identifies the unitary group \(U(n, \mathbb{C})\) with \(U(n)\). An argument similar to the one above shows that the action \(U(n) \times \Lambda(n) \to \Lambda(n)\) is transitive as well; it allows to identify \(\Lambda(n)\) topologically with the homogeneous space \(U(n)/O(n)\) where \(O(n)\) is the subgroup of \(U(n)\) consisting of the mappings \((x, p) \mapsto (Ax, Ap)\), \(A \in O(n, \mathbb{R})\). It implies that:

**Lemma 4** Let \(\ell\) be a \(n\)-dimensional subspace of \(\mathbb{R}^{2n}\). It is a Lagrangian plane in \((\mathbb{R}^{2n}, \omega)\) if and only if there exists real \(n \times n\) matrices \(A\) and \(B\) satisfying \(A^T B = B^T A\) and \(A^T A + B^T B = I_{n \times n}\) (or \(AB^T = BA^T\) and \(AA^T + BB^T = I_{n \times n}\)) such that \((x, p) \in \ell\) if and only \(Ax + Bp = 0\).

**Proof.** It follows from the transitivity of the action of \(U(n)\) on \(\Lambda(n)\) which allows to parametrize \(\ell\) by \(x = B^T u\) and \(p = -A^T u\), \(u \in \mathbb{R}^n\) (observe that \(AA^T + BB^T = I_{n \times n}\) implies that \(\text{rank}(A, B) = n\)).

The following classical result from symplectic geometry \([12]\) will be used several times. We will say that two Lagrangian planes \(\ell\) and \(\ell'\) are transverse if \(\ell \cap \ell' = 0\); equivalently \(\ell \oplus \ell' = \mathbb{R}^{2n}\).

**Lemma 5** The symplectic group \(\text{Sp}(n)\) acts transitively on the set \(\Lambda_0(n)^2\) of all transverse Lagrangian planes in \((\mathbb{R}^{2n}, \omega)\): if \((\ell_1, \ell'_1) \in \Lambda(n)^2\) and \((\ell_2, \ell'_2) \in \Lambda(n)^2\) are such that \(\ell_1 \cap \ell'_1 = \ell_2 \cap \ell'_2 = 0\) then there exists \(S \in \text{Sp}(n)\) such that \((\ell_1, \ell'_1) = S(\ell_2, \ell'_2)\).

**Proof.** Choose a basis \(\{e_{11}, \ldots, e_{1n}\}\) of \(\ell_1\) and a basis \(\{f_{11}, \ldots, f_{1n}\}\) of \(\ell'_1\) such that \(\{e_{1i}, f_{1j}\}_{1 \leq i, j \leq n}\) is a symplectic basis of \((\mathbb{R}^{2n}_2, \sigma)\). Similarly choose bases of \(\ell_2\) and \(\ell'_2\) whose union \(\{e_{2i}, f_{2j}\}_{1 \leq i, j \leq n}\) is also a symplectic basis. The linear mapping \(S : \mathbb{R}^{2n} \to \mathbb{R}^{2n}\) defined by \(S(e_{1i}) = e_{2i}\) and \(S(f_{1i}) = f_{2i}\) for \(1 \leq i \leq n\) is in \(\text{Sp}(n)\) and \((\ell_2, \ell'_2) = (S\ell_1, S\ell'_1)\).

We define Lagrangian polar duality:
**Definition 6** Let $\ell$ and $\ell'$ be two transversal Lagrangian planes, and $X_\ell$ a centrally symmetric convex body in $\ell$. The Lagrangian polar dual $X^o_{\ell'}$ of $X_\ell$ in $\ell'$ is the subset of $\ell'$ consisting of all $z' \in \ell'$ such that

$$\omega(z, z') \leq 1 \quad \text{for all} \quad z' \in X.$$  \hfill (16)

The Lagrangian polar dual $X^o_{\ell'}$ is also centrally symmetric. In the particular case $\ell = \ell_X = \mathbb{R}^n_x \times 0$ and $\ell' = \ell_P = 0 \times \mathbb{R}^n_p$ this reduces to ordinary polarity as studied above: for $z = (x, p) \in X_\ell$ and $z' = (x', p') \in X^o_{\ell'}$ we have indeed

$$\omega(z, z') = \omega((0, p; x', 0) = px'.

Using the transitivity of the action $\text{Sp}(n) \times \Lambda(n) \to \Lambda(n)$ one has the following symplectic covariance of which relates Lagrangian polar duality to ordinary polar duality:

**Proposition 7** Let $S \in \text{Sp}(n)$ be such that $\ell = S(\ell_X, \ell_P)$. The Lagrangian polar dual $X^o_{\ell'} \subset \ell'$ of $X_\ell \subset \ell$ is given by $X^o_{\ell'} = S(X^o)$, that is

$$(X_\ell, X^o_{\ell'}) = S(X, X^o)$$ \hfill (17)

where $X^o$ is the polar dual of $X = S^{-1}(X_\ell) \subset \ell_X$.

**Proof.** Let us define $P \subset 0 \times \mathbb{R}^n_p$ by $(X_\ell, X^o_{\ell'}) = S(X, P)$. It suffices to prove that $P = X^o$. But this readily follows from the definition (16) and the symplectic invariance of the symplectic form $\omega$. 

This leads us to the following definition:

**Definition 8** Let $X_\ell \subset \ell$ and $Y_{\ell'} \subset \ell'$ be centrally symmetric convex bodies. We will say that $(X_\ell, Y_{\ell'})$ is a polar dual Lagrangian pair if $X^o_{\ell'} \subset Y_{\ell'}$. If the equality $X^o_{\ell'} = Y_{\ell'}$ holds we will say that $(X_\ell, Y_{\ell'})$ is an exact Lagrangian polar dual pair.

Let $S \in \text{Sp}(n)$ be such that $\ell = S(\ell_X, \ell_P)$; in view of Proposition 7 $(X_\ell, Y_{\ell'})$ is a dual polar Lagrangian polar pair if and only $(X, Y) = S^{-1}(X_\ell, Y_{\ell'})$ is a polar dual pair in the usual sense.

The notion of Mahler volume generalizes without difficulty to the Lagrangian context. Recall that the Mahler volume of a centrally symmetric convex body $X \subset \mathbb{R}^n_x$ is defined by

$$v(X) = \text{Vol}(X \times X^o)$$
where Vol is the standard volume in $\mathbb{R}^{2n}$. We define accordingly

$$v(X_\ell) = \text{Vol}(X_\ell \times X_\ell^o).$$

Since symplectomorphisms are volume preserving it follows that $v(X_\ell) = v(X)$ if $X_\ell \times X_\ell^o = S(X, X^o)$ for $S \in \text{Sp}(n)$.

### 2.4 L"owner–John ellipsoids

Let $\Omega$ be a convex body in $\mathbb{R}^n$. The L"owner and John ellipsoids $\Omega_{\text{L"owner}}$ and $\Omega_{\text{John}}$ are defined as follows [4, 5]:

- $\Omega_{\text{L"owner}}$ is the unique ellipsoid in $\mathbb{R}^n$ with minimum volume containing $\Omega$.
- $\Omega_{\text{John}}$ is the unique ellipsoid in $\mathbb{R}^n$ with maximum volume contained in $\Omega$.

The L"owner and John ellipsoids are linearly covariant: if $L \in GL(n, \mathbb{R})$ then

$$\left( L(\Omega) \right)_{\text{L"owner}} = L(\Omega_{\text{L"owner}}), \quad \left( L(\Omega) \right)_{\text{John}} = L(\Omega_{\text{John}})$$

and $\Omega_{\text{L"owner}}$ and $\Omega_{\text{John}}$ are transformed into each other by polar duality:

$$\Omega_{\text{L"owner}} = (\Omega^o)_{\text{John}}, \quad \Omega_{\text{John}} = (\Omega^o)_{\text{L"owner}}.$$  

We will say that $L \in GL(n, \mathbb{R})$ brings $\Omega$ in L"owner position (resp. John position) if $\left( L(\Omega) \right)_{\text{L"owner}} = B^n(1)$ (resp. $\left( L(\Omega) \right)_{\text{John}} = B^n(1)$).

Replacing $n$ with $2n$ we have the following basic example:

**Lemma 9** The John ellipsoid of $B_X^n(1) \times B_P^n(1)$ is $B^{2n}(1)$.

**Proof.** The inclusion

$$B^{2n}(1) \subset B_X^n(1) \times B_P^n(1)$$

is obvious, and we cannot have $B^{2n}(R) \subset B_X^n(1) \times B_P^n(1)$ if $R > 1$. Assume now that the John ellipsoid $\Omega_{\text{John}}$ of $\Omega = B_X^n(1) \times B_P^n(1)$ is defined by

$$Ax \cdot x + Bx \cdot p + Cp \cdot p \leq 1$$

where $A, C > 0$ and $B = B^T$ are real $n \times n$ matrices. Since $\Omega$ is invariant by the transformation $(x, p) \mapsto (p, x)$ so is $\Omega_{\text{John}}$ and we must thus have $A = C$ and $B = B^T$. Similarly, $\Omega$ being invariant by the reflection $(x, p) \mapsto (-x, p)$
we get $B = 0$ so $\Omega_{\text{John}}$ is defined by $Ax \cdot x + Ap \cdot p \leq 1$. The next step is to observe that $\Omega$ and hence $\Omega_{\text{John}}$ is invariant under all transformations $(x,p) \mapsto (Hx,Hp)$ where $H \in O(n,\mathbb{R})$ so we must have $AH = HA$ for all $H \in O(n,\mathbb{R})$, but this is only possible if $A = \lambda I_{n \times n}$ for some $\lambda \in \mathbb{R}$. The John ellipsoid is thus of the type $B^{2n}(\lambda^{-1/2})$ for some $\lambda \geq 1$ and this concludes the proof in view of (20) since $\lambda > 1$ is excluded. 

The considerations above allow us to determine the John ellipsoid of $X_\ell \times X^o_\ell$. Our proof relies on the following straightforward result:

**Lemma 10** Let $\Omega \subset \mathbb{R}^{2n}$ be a centrally symmetric body. We have

$$c_{\text{min}}(\Omega) = \sup_{S \in \text{Sp}(n)} \{ \pi R^2 : S(B^{2n}(R)) \subset \Omega \}.$$  \hspace{1cm} (21)

**Proof.** Since $\Omega$ is centrally symmetric we have $S(B^{2n}(z_0,R)) \subset \Omega$ if and only if $S(B^{2n}(-z_0,R)) \subset \Omega$. The ellipsoid $S(B^{2n}(R))$ is interpolated between $S(B^{2n}(z_0,R))$ and $S(B^{2n}(-z_0,R))$ using the mapping $t \mapsto z(t) = z - 2tz_0$ where $z \in S(B^{2n}(z_0,R))$, and is hence contained in $\Omega$ by convexity. 

**Proposition 11** Let $(\ell,\ell')$ be a pair of transverse Lagrangian planes and $X_\ell \subset \ell$ a centered ellipsoid. (i) There exists $S \in \text{Sp}(n)$ such that the John ellipsoid of $X_\ell \times X^o_\ell$ is

$$(X_\ell \times X^o_\ell)_{\text{John}} = S(B^{2n}(1)).$$

(ii) We have

$$c_{\text{min}}(X_\ell \times X^o_\ell) = \pi.$$  \hspace{1cm} (22)

**Proof.** (i) In view of Proposition [7] there exists $S' \in \text{Sp}(n)$ such that

$$(X_\ell, X^o_\ell) = S'(X, X^o)$$  \hspace{1cm} (23)

where $X \subset \ell_X$ and $X^o \subset \ell_P$. Using formula [18] with $L = S$ we have

$$(X_\ell \times X^o_\ell)_{\text{John}} = (S'(X, X^o))_{\text{John}} = S'(X, X^o)_{\text{John}}$$

so we may assume that there exists $A > 0$ such that $X_\ell = X$ and $X^o_\ell = X^o$ with

$$X = \{ x \in \mathbb{R}^n : Ax \cdot x \leq 1 \} = A^{-1/2}(B^n_X(1))$$  \hspace{1cm} (24)

$$X^o = \{ p \in \mathbb{R}^n : A^{-1}p \cdot p \leq 1 \} = A^{1/2}(B^n_P(1)).$$  \hspace{1cm} (25)
This can be rewritten as

\[(X, X^o) = S^\prime\prime(B^n_X(1) \times B^n_P(1))\]

where

\[S^\prime\prime = \begin{pmatrix} A^{-1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} \in \text{Sp}(n)\]

and hence

\[(X_\ell \times X^o_\ell)_{\text{John}} = S' S''((B^n_X(1) \times B^n_P(1))_{\text{John}}).\]

The claim now follows from Lemma (9) taking \(S = S' S''\). (ii) It is sufficient to assume \(X_\ell = X \subset \ell_X\) and \(X^o_\ell = X^o \subset \ell_P\). In view of formula (21) we have to show that

\[c_{\min}(X_\ell \times X^o_\ell) = \sup_{S \in \text{Sp}(n)} \{ \pi R^2 : S(B^{2n}(R)) \subset X \times X^o \} = 4.\]

In view of (i) the largest ellipsoid contained in \(X \times X^o\) is \(S(B^{2n}(1))\) for some \(S \in \text{Sp}(n)\) hence

\[c_{\min}(X \times X^o) = c(B^{2n}(1)) = \pi.\]

\[\Box\]

3 The Main Result: Statement and Proof

We begin by recollecting some well-known facts from linear algebra; see for instance Zhang [37].

3.1 Orthogonal projections of ellipsoids

Consider a non-degenerate ellipsoid

\[\Omega = \{ z \in \mathbb{R}^{2n} : M z \cdot z \leq 1 \}. \tag{26}\]

where \(M\) is a real positive definite symmetric \(2n \times 2n\) matrix (written for short as \(M > 0\)). We will use the block matrix representation

\[M = \begin{pmatrix} M_{XX} & M_{XP} \\ M_{PX} & M_{PP} \end{pmatrix} \tag{27}\]

where the blocks are \(n \times n\) matrices, the condition \(M > 0\) ensures us that \(M_{XX} > 0, M_{PP} > 0,\) and \(M_{PX} = M_{XP}^T\). The \(n \times n\) matrices

\[M/M_{PP} = M_{XX} - M_{XP} M_{PP}^{-1} M_{PX} \tag{28}\]

\[M/M_{XX} = M_{PP} - M_{PX} M_{XX}^{-1} M_{XP} \tag{29}\]
are the Schur complements in $M$ of $M_{PP}$ and $M_{XX}$, respectively, and we have $M/M_{PP} > 0$, $M/M_{XX} > 0$ (see [37]).

**Lemma 12** Let $M > 0$ and consider the ellipsoid

$$ \Omega = \{ z \in \mathbb{R}^{2n} : Mz \cdot z \leq 1 \}. $$

The orthogonal projections

$$ \Omega_X = \Pi_X \Omega \ , \ \Omega_P = \Pi_P \Omega \ . $$

(30)

of $\Omega$ onto $\ell_X = \mathbb{R}^n_x \oplus 0$ and $\ell_P = 0 \oplus \mathbb{R}^n_p$, respectively, are the ellipsoids

$$ \Omega_X = \{ x \in \mathbb{R}^n_x : (M/M_{PP})x^2 \leq 1 \} $$

(31)

$$ \Omega_P = \{ p \in \mathbb{R}^n_p : (M/M_{XX})p^2 \leq 1 \} \ . $$

(32)

**Proof.** Let us set $Q(z) = Mz^2 - 1$; the boundary $\partial \Omega$ of the hypersurface $Q(z) = 0$ is defined by

$$ M_{XX}x^2 + 2M_{PX}x \cdot p + M_{PP}p^2 = 1 \ . $$

(33)

We have $x \in \partial \Omega_X$ (the boundary of $\Omega_X$) if and only if the normal vector to $\partial \Omega$ at the point $z = (x, p)$ is parallel to $\mathbb{R}^n_x \times 0$ hence we get the constraint $\partial_z Q(z) = 2Mz \in \mathbb{R}^n_x \times 0$; this is equivalent to saying that $M_{PX}x + M_{PP}p = 0$, that is to $p = -M_{PP}^{-1}M_{PX}x$. Inserting this value of $p$ in the equation (33) shows that $\partial \Omega_X$ is the set of all $x$ such that $(M/M_{PP})x^2 = 1$, which yields (31). Formula (32) is proven in a similar way. ■

### 3.2 The projection theorem

Let us state and prove our first main result. Recall that we use the notation $\ell_X = \mathbb{R}^n_x \oplus 0 \ , \ \ell_P = 0 \oplus \mathbb{R}^n_p$ for the coordinate Lagrangian planes.

**Theorem 13** Let $\Omega$ be a symmetric convex body in $(\mathbb{R}^{2n}, \omega)$ containing a symplectic ball $S(B^{2n}(1))$. Let $(\ell, \ell')$ be a pair of transverse Lagrangian planes and denote by $\Pi_\ell : \mathbb{R}^{2n} \rightarrow \ell$ and $\Pi_{\ell'} : \mathbb{R}^{2n} \rightarrow \ell'$ the oblique projections in the directions $\ell'$ and $\ell$, respectively. Then $(\Pi_\ell \Omega, \Pi_{\ell'} \Omega)$ is a Lagrangian dual pair

$$ (\Pi_\ell \Omega)_{\ell'} \subset \Pi_{\ell'} \Omega \quad (34) $$

with equality if and only if $\Omega = S(B^{2n}(1))$. 

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Proof. First case: we choose $\ell = \ell_X$ and $\ell' = \ell_P$. Let $S \in \text{Sp}(n)$ be such that $S(B^{2n}(1)) \subset \Omega$ (such an $S$ is generally not unique). The ellipsoid $S(B^{2n}(1))$ is the set of all $z \in \mathbb{R}^{2n}$ such that $Mz^2 \leq 1$ where $M = (S^T)^{-1}S^{-1}$ is a positive definite symplectic matrix; writing $M$ in block form

$$M = \begin{pmatrix} M_{XX} & M_{XP} \\ M_{PX} & M_{PP} \end{pmatrix}$$

we have $M_{XX} > 0$, $M_{PP} > 0$, $M_{PX} = M_{XP}^T$. In view of the projection Lemma [12]

$$\Omega_X = \Pi_X \Omega = \{ x \in \mathbb{R}^n_x : (M/M_{PP})x \cdot x \leq 1 \} \quad (35)$$
$$\Omega_P = \Pi_P \Omega = \{ p \in \mathbb{R}^n_p : (M/M_{XX})p \cdot p \leq 1 \} \quad (36)$$

where

$$M/M_{PP} = M_{XX} - M_{XP}M_{PP}^{-1}M_{PX} \quad (37)$$
$$M/M_{XX} = M_{PP} - M_{PX}M_{XX}^{-1}M_{XP} \quad (38)$$

In view of formula (37) we have

$$\Omega_X^o = \{ p : (M/M_{PP})^{-1}p^2 \leq 1 \}.$$ 

Let us prove that $\Omega_X^o \subset \Omega_P$; the result will then follow by Lemma [2]. The condition $\Omega_X^o \subset \Omega_P$ is equivalent to

$$(M/M_{PP})^{-1} \geq M/M_{XX} ; \quad (39)$$

let us show that this is the case here. The conditions $M \in \text{Sp}(n)$ and $M = M^T$ being equivalent to $MJM = J$ we have the following relations between the blocks:

$$M_{XX}M_{PX} = M_{XP}M_{XX} \, , \, M_{PX}M_{PP} = M_{PP}M_{XP} \quad (40)$$
$$M_{XX}M_{PP} - M_{XX}^2 = I_{n \times n} . \quad (41)$$

These equalities imply that

$$M/M_{PP} = (M_{XX}M_{PP} - M_{XX}^2)M_{PP}^{-1} = M_{PP}^{-1} \quad (42)$$
$$M/M_{XX} = M_{XX}^{-1}(M_{XX}M_{PP} - M_{XX}^2) = M_{XX}^{-1} \quad (43)$$

and hence the inequality (39) holds if and only if $M_{XX}M_{PP} \geq I_{n \times n}$. Now $M_{XX}M_{PP} = I_{n \times n} + M_{XX}^2$ in view of (41) hence we will have $M_{XX}M_{PP} \geq I_{n \times n}$. Now
$I_{n \times n}$ if $M_{XP}^2 \geq 0$. To prove that $M_{XP}^2 \geq 0$ it suffices to show that the eigenvalues of $M_{XP}$ are real. In view of the first formula (40) we have $M_{XX}^{1/2} M_{PP} M_{XX}^{1/2} = M_{XX}^{1/2} M_{XP} M_{XX}^{1/2}$ hence $M_{XX}^{1/2} M_{XP} M_{XX}^{1/2}$ is symmetric and its eigenvalues are real as claimed. **General case:** Choose $S \in \text{Sp}(n)$ such that $(\ell, \ell') = S(\ell_X, \ell_P)$, and consider the centrally symmetric convex body $S^{-1}(\Omega)$. In view of Proposition 7 the sets $\Pi_X(S^{-1}(\Omega))$ and $\Pi_P(S^{-1}(\Omega))$ form a Lagrangian dual pair. Now $\Pi_\ell = S \Pi_X S^{-1}$ satisfies $\Pi_\ell^2 = \Pi_\ell$ and $\ker(\Pi_\ell) = S \ell_P = \ell'$ and similarly $\Pi_{\ell'}^2 = \Pi_{\ell'}$ and $\ker(\Pi_{\ell'}) = S \ell_X = \ell$ hence $\Pi_\ell$ and $\Pi_{\ell'}$ are indeed the projections described in the statement of the theorem. There remains to study the case $(\Pi_\ell \Omega)_{\ell'} = \Pi_{\ell'} \Omega$. The statement follows from Proposition 11. 

The following consequence is easy:

**Corollary 14** Let $\Omega$ be a centrally symmetric convex body in $(\mathbb{R}^{2n}, \omega)$ such that $c_{\text{min}}(\Omega) \geq \pi$. Then $(\Pi_\ell \Omega, \Pi_{\ell'} \Omega)$ is a Lagrangian dual pair and the formulas (34) hold.

**Proof.** The result follows from Theorem 13 in view of formula (21). 

### 3.3 A partial converse

Let us address the following problem: given two transverse Lagrangian planes $\ell$ and $\ell'$ in $(\mathbb{R}^{2n}, \omega)$ and a Lagrangian polar dual pair $(X_\ell, Y_{\ell'}) (X_\ell \subset \ell$ and $X_{\ell'} \subset Y_{\ell'} \subset \ell')$, can we find a (centrally symmetric) convex body $\Omega \subset \mathbb{R}^{2n}$ such that (with the notation of Theorem 13) $X_\ell = \Pi_\ell \Omega$ and $Y_{\ell'} = \Pi_{\ell'} \Omega$? It is intuitively clear that the problem has in general infinitely many solutions. To make things more visible, consider the case $n = 1$ where we choose $\ell = \ell_X$ (the “x-axis”) and $\ell = \ell_P$ (the “p-axis”). Choose $X = [-a, a]$ so that $X^o = [-1/a, 1/a]$. Any centered ellipse $\Omega$ inscribed in the rectangle $X \times X^o$ (which has area 4) will have orthogonal projections $X$ and $X^o$ on the $x$ and $p$ axes. However, If we require the area of this ellipse to be $c(\Omega) = \pi$ then the solution is unique, and $\Omega$ is the ellipse defined by $x^2/a^2 + a^2 p^2 \leq 1$, which is the John ellipse of $X \times X^o$. This discussion can be extended to the case of arbitrary dimension $n$:

**Theorem 15** Let $(\ell, \ell')$ be a pair of transverse Lagrangian planes in $(\mathbb{R}^{2n}, \omega)$ and $X_\ell \subset \ell$ a centered ellipsoid. Let $X_{\ell'}^o$ be the Lagrangian dual of $X_\ell$. There exists a unique symplectic ball $\Omega = S(B^{2n}(1))$ ($S \in \text{Sp}(n)$) having projections $\Pi_\ell \Omega = X_\ell$ and $\Pi_{\ell'} \Omega = X_{\ell'}^o$. 

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**Proof. First case:** Assume that \( \ell \) and \( \ell' \) are the coordinate planes \( \ell_X \) and \( \ell_P \) and that \( X_\ell \) and \( X_\ell' \) are the ellipsoids

\[
X = \{ x \in \mathbb{R}^n : Ax \cdot x \leq 1 \} \tag{44}
\]

\[
X^o = \{ p \in \mathbb{R}^n : A^{-1}p \cdot p \leq 1 \}. \tag{45}
\]

Let us show that there exists a symplectic ball \( \Omega = S(B^{2n}(1)) \) having orthogonal projections \( X \) and \( X^o \). Setting \( M = (SS^T)^{-1} \) the ellipsoid \( S(B^{2n}(1)) \) is the set

\[
\Omega = \{ z \in \mathbb{R}^{2n} : Mz \cdot z \leq 1 \}.
\]

Let us write \( M \) in block-matrix form, as in the proof of Theorem 13:

\[
M = \begin{pmatrix} M_{XX} & M_{XP} \\ M_{PX} & M_{PP} \end{pmatrix}.
\]

In view of Lemma 12 the orthogonal projections \( \Pi_X \Omega \) and \( \Pi_P \Omega \) of \( \Omega \) on \( \ell_X \) and \( \ell_P \) are the ellipsoids

\[
\Omega_X = \{ x \in \mathbb{R}^n : (M/M_{PP})x \cdot x \leq 1 \} \tag{46}
\]

\[
\Omega_P = \{ p \in \mathbb{R}^n : (M/M_{XX})p \cdot p \leq 1 \}. \tag{47}
\]

The conditions \( M \in \text{Sp}(n) \), \( M = M^T \) imply that we have \( MJM = J \) hence the following set of relations must be satisfied by the block-entries of \( M \):

\[
M_{XX}M_{PP} - M_{XP}^2 = I_{n \times n} \tag{48}
\]

\[
M_{XX}M_{PX} = M_{XP}M_{XX} \tag{49}
\]

\[
M_{PX}M_{PP} = M_{PP}M_{XP}. \tag{50}
\]

Using the identities (49) and (50) we get

\[
M/M_{PP} = M_{XX} - M_{PP}^{-1}M_{PX}^2 = M_{PP}^{-1}(M_{PP}M_{XX} - M_{PX}^2) = M_{PP}^{-1}
\]

the last equality being obtained by transposition of the identity (48). A similar calculation leads to the formula

\[
M/M_{XX} = M_{PP} - M_{XX}^{-1}M_{XP}^2 = M_{XX}^{-1}
\]

and comparing with (44) and (45) we must thus have \( A = M_{XX} = M_{PP}^{-1} \).

From the identity (48) follows that \( M_{XX}^2 = 0 \); multiplying both sides of
the identity \((49)\) on the left by \(M_{XP}\) then implies that 
\[ M_{PX}^T M_{XX} M_{PX} = M_{XP} M_{XX} M_{PX} = 0 \] 
and hence \(M_{XP} = 0\) since \(M_{XX}\) is invertible. The symplectic matrix \(M\) must thus be the block diagonal matrix
\[
M = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}
\] (51)
and \(\Omega\) is hence the ellipsoid
\[
\Omega = \{(x,p) \in \mathbb{R}^n \times \mathbb{R}^n : Ax \cdot x + A^{-1} p \cdot p \leq 1\}.
\] (52)
The latter is the John ellipsoid of \((X,X^o)\) in view of Lemma \([9]\) **Second case**: It is similar to the proof given at the end of Theorem \([13]\) choose \(S' \in \text{Sp}(n)\) such that \((\ell,\ell') = S'(\ell_X,\ell_P)\) and consider the ellipsoid
\[
\Omega' = S' \Omega = S' \mathcal{S}(B^{2n}(1))
\]
\((S' \text{ and } \Omega \text{ as above})\). The projections \(\Pi_{\ell'}\) and \(\Pi_{\ell'}\) are given by \(\Pi_{\ell} = S' \Pi_X S'^{-1}\) and \(\Pi_{\ell'} = S' \Pi_P S'^{-1}\) hence
\[
(\Pi_{\ell} \Omega', \Pi_{\ell'} \Omega') = S'(\Pi_X \Omega, \Pi_P \Omega)
\]
is a Lagrangian dual pair. The uniqueness of \(\Omega'\) follows using the linear covariance \([18]\) of the John ellipsoid. Formula \((??)\) is obvious by definition of \(c_{\text{lin}}\) and the fact that if \(S(B^{2n}(R)) \subset (X \times X^o)\) for some \(S \in \text{Sp}(n)\) then we must have \(R \leq 1\) by definition of the John ellipsoid. ■

**Remark 16** The fact that \(X\) and \(X^o\) are the orthogonal projections on the coordinate Lagrangian planes of the ellipsoid \((??)\) is obvious and might lead to think that the rather long argument developed in the proof is superfluous; what is not obvious in the multi-dimensional case is the uniqueness of an ellipsoid having this property.

4 The Uncertainty Principle of Quantum Mechanics

4.1 Uncertainties

The term “uncertainty principle” commonly hints in mathematics at a constellation of related results based on the observation of that there is a kind of “trade-off” between a function \(\psi\) and its Fourier transform \(F\psi\) which prevents them to be simultaneously too sharply located. Perhaps the most
emblematic of these results (and one of the oldest) is Hardy’s uncertainty principle which says that if \( \psi(x) = O(e^{-ax^2/2}) \) and \( F\psi(p) = O(e^{-ap^2/2}) \) as \( x, p \to \infty \) then we must have \( ab \leq 1 \), with equality if and only if \( \psi(x) = Ce^{-ax^2/2} \) for some constant \( C \). In practice, the best known uncertainty principles are those arising from quantum mechanics, and are variations on the theme of the Heisenberg principle of indeterminacy \( \sigma_{xx}\sigma_{pp} \geq \frac{1}{4}\hbar^2 \). The most used of these results is, with no doubt, the Robertson–Schrödinger inequality

\[
\sigma_{xx}\sigma_{pp} \geq \sigma_{xp}^2 + \frac{1}{4}\hbar^2
\]

where \( \sigma_{xx} \) and \( \sigma_{pp} \) are variances (also called standard deviations) and \( \sigma_{xp} \) is the covariance corresponding to joint position and momentum measurements. As we already observed in [18] the rub with these types of inequalities, generalizing Heisenberg’s principle, is that they crucially depend on the choice of a privileged (and arbitrary) way of measuring uncertainties, in his case the usual (co-)variances from standard statistics. In fact, Hilgevoord [24] already noticed some time ago (also see the follow-up [25] with Uffink), that variances and covariances give good measurements of the spread only for Gaussian (or almost Gaussian) quantum states, this being due to the fact they usually fail to take into account the “tails” of \( \psi \) which can be quite large and influence the variance and covariance.

We are going to see that the notion of dual polarity is a very intuitive and general tool for expressing uncertainties, which avoids the pitfalls due to the use of standard statistical tools. We will thereafter show that Hardy’s uncertainty principle can also be interpreted in a convincing way using polar duality.

We will from now on use units in which \( \hbar = 1 \).

4.2 The Robertson–Schrödinger inequalities

Let \( \mathcal{B} = \mathcal{B}(L^2(\mathbb{R}^n)) \) be the algebra of bounded operators on \( L^2(\mathbb{R}^n) \). We denote by \( \mathcal{L}_1 = \mathcal{L}_1(L^2(\mathbb{R}^n)) \) the two-sided ideal of \( \mathcal{B} \) consisting of all trace-class operators on \( L^2(\mathbb{R}^n) \). An operator \( \hat{\varrho} \in \mathcal{B}(L^2(\mathbb{R}^n)) \) is called a density operator if it is positive semi-definite: \( \hat{\varrho} \geq 0 \) (and hence self-adjoint) and if it has trace \( \text{Tr}(\hat{\varrho}) = 1 \). It follows from the spectral theorem that there exists an orthonormal sequence of vectors \( (\psi_j)_{j \in \mathbb{N}} \) in \( L^2(\mathbb{R}^n) \) and a corresponding sequence \( (\lambda_j)_{j \in \mathbb{N}} \) of non-negative real numbers summing up to one such that for every \( \psi \in L^2(\mathbb{R}^n) \)

\[
\hat{\varrho}\psi = \sum_{j \in \mathbb{N}} \lambda_j (\psi|\psi_j)_{L^2} \psi_j.
\]
One proves [15] that the Weyl symbol of $\widehat{\rho}$ is the function ("Wigner distribution")

$$\rho = (2\pi)^n \sum_{j \in \mathbb{N}} \lambda_j W\psi_j$$  \hspace{1cm} (54)

where $W\psi_j$ is the Wigner transform of $\psi_j$; recall [15, 16] that for $\psi \in L^2(\mathbb{R}^n)$ the function $W\psi$ is given by the absolutely convergent integral

$$W\psi(x, p) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-ip \cdot y} \psi(x + \frac{1}{2}y)\overline{\psi(x - \frac{1}{2}y)} dy. \hspace{1cm} (55)$$

In order to define the covariance matrix of $\widehat{\rho}$ we need to consider density operators whose Wigner distribution decreases sufficiently well at infinity: [15, 11, 27, 21]:

**Definition 17** We will say that $\widehat{\rho}$ has the Feichtinger property for $s \geq 0$ if the functions $\psi_j$ in the spectral decomposition (53) of $A_s$ satisfy

$$W(\psi_j, \phi) \in L^1_s(\mathbb{R}^{2n}) \hspace{1cm} , \hspace{1cm} j \in \mathbb{N}$$  \hspace{1cm} (56)

for one (and hence every) every $\phi \in \mathcal{S}(\mathbb{R}^n)$.

In formula (56) $W(\psi, \phi)$ is the cross-Wigner transform, defined by the integral

$$W(\psi, \phi)(z) = (\frac{1}{2\pi})^n \int_{\mathbb{R}^n} e^{-ip \cdot y} \psi(x + \frac{1}{2}y)\overline{\phi(x - \frac{1}{2}y)} dy \hspace{1cm} (57)$$

and $L^1_s(\mathbb{R}^{2n})$ is the weighted $L^1$-space

$$L^1_s(\mathbb{R}^{2n}) = \{ a : \mathbb{R}^{2n} \rightarrow \mathbb{C} \hspace{1cm} , \hspace{1cm} \langle z \rangle^s a \in L^1(\mathbb{R}^{2n}) \} \hspace{1cm} (58)$$

where $\langle z \rangle = (1 + |z|^2)^{1/2}$. It is easy to show [19] that if $\rho$ has the Feichtinger property for some $s \geq 2$ then $\rho \in L^1(\mathbb{R}^{2n})$ and the symmetric $2n \times 2n$ matrix

$$\Sigma = \int_{\mathbb{R}^{2n}} (z - \bar{z})(z - \bar{z})^T a(z) dz \hspace{1cm} (59)$$

where $\bar{z} = \int_{\mathbb{R}^{2n}} |z| zp(z) dz$. is defined (both integrals being absolutely convergent). This matrix $\Sigma$ is called the covariance matrix of the density operator $\widehat{\rho}$, and it is well-known [31, 15] that the positive semi-definiteness of $\widehat{\rho}$ implies the condition

$$\Sigma + \frac{i}{2} J \geq 0. \hspace{1cm} (60)$$
This property, which is related to the so-called KLM conditions \([29, 30]\), has the following converse: let \(\Sigma\) be a real symmetric \(2n \times 2n\) matrix satisfying (60). Then, for every \(z \in \mathbb{R}^{2n}\), the Gaussian probability distribution
\[
\rho(z) = \frac{1}{(2\pi)^n \sqrt{\det \Sigma}} e^{-\frac{1}{2}(z-z)^t \Sigma^{-1} (z-z)^t}
\]
is the Wigner distribution of a density operator \(\hat{\varrho}\) (and \(\hat{\varrho}\) has trivially the Feichtinger property for every \(s \geq 0\)). Writing the covariance matrix in block form
\[
\Sigma = \begin{pmatrix} \Sigma_{XX} & \Sigma_{XP} \\ \Sigma_{PX} & \Sigma_{PP} \end{pmatrix}, \quad \Sigma_{PX} = \Sigma_{XP}^t
\]
with \(\Sigma_{XX} = (\sigma_{x,x})_{1 \leq j,k \leq n}, \Sigma_{PP} = (\sigma_{p,p})_{1 \leq j,k \leq n}, \) and \(\Sigma_{XP} = (\sigma_{x,p})_{1 \leq j,k \leq n}\) it follows from Sylvester’s criterion for the leading principal minors of a positive matrix that the condition (60) is equivalent to the Robertson–Schrödinger (RS) inequalities
\[
\sigma_{x,x_j} \sigma_{p,p_j} \geq \sigma_{x,p_j}^2 + \frac{1}{4}, \quad 1 \leq j \leq n,
\]
for position and momentum measurements \([13, 15]\).

As discussed in previous section, this formulation of the uncertainty principle is heavily dependent on the way uncertainties are measured (in this case the variances and covariances of certain stochastic variables). We are going to show that polar duality allows to formulate a more general principle.

### 4.3 Polar duality and a conjecture

Suppose that a large number of position and momentum measurements are made on some physical system, identified with a point in \(T^* \mathbb{R}^n = \mathbb{R}^n_x \times \mathbb{R}^n_p \equiv \mathbb{R}^{2n}\). These measurements yield “clouds” of points in \(\mathbb{R}^n_x\) and \(\mathbb{R}^n_p\), respectively. Using standard methods from statistics and optimization theory \([28,35]\) we can model these subsets by “best fit” ellipsoids \(\Omega_X \subset \mathbb{R}^n_x\) and \(\Omega_P \subset \mathbb{R}^n_p\) (which we assume, for simplicity, both centered at the origin). For instance, a standard and robust statistical method is to use Van Aelst and Rousseeuw’s \([36]\) Minimum Volume Ellipsoid method (MVE), of which we have given a short description in \([13]\); also see our paper \([14]\) where the MVE method is related to the notion of symplectic capacity. Now, according to the principle of quantum indeterminacy \([32]\), it does not make sense to attribute a precise value to the position or the momentum, so the sets of measurements, and hence the ellipsoids \(\Omega_X\) and \(\Omega_P\), cannot be simultaneously arbitrarily small. This leads us to make the following physical conjecture, which could be (in principle) justified experimentally:
**Conjecture 18** The ellipsoids $\Omega_X \subset \mathbb{R}^n_x$ and $\Omega_P \subset \mathbb{R}^n_p$ form a dual pair: $\Omega_X^o \subset \Omega_P$.

We are going to make the above conjecture plausible by showing that it implies (but is more general then) the RS inequalities (63). We begin by defining two real symmetric $n \times n$ ellipsoids by identifying the ellipsoids $\Omega_X$ and $\Omega_P$ with the sets

\[
\Omega_X = \{x \in \mathbb{R}^n_x : \frac{1}{2} \Sigma_{XX}^{-1} x \cdot x \leq 1\}
\]

\[
\Omega_P = \{p \in \mathbb{R}^n_p : \frac{1}{2} \Sigma_{PP}^{-1} p \cdot p \leq 1\}.
\]

Let us assume that $(\Omega_X, \Omega_P)$ is an exact polar dual pair, that is $\Omega_X^o = \Omega_P$. In view of the duality property (12) for ellipsoids we must then have $\Sigma_{XX} \Sigma_{PP} = \frac{1}{4} I_{n \times n}$ (which reduces to the equality $\sigma_{xx} \sigma_{pp} = \frac{1}{4}$ for $n = 1$). It follows that we have

\[
\frac{1}{2} \Sigma^{-1} = \begin{pmatrix}
\frac{1}{2} \Sigma_{XX}^{-1} & 0 \\
0 & \frac{1}{2} \Sigma_{PP}^{-1}
\end{pmatrix} \in \text{Sp}(n)
\]

hence $\Sigma$ automatically satisfies the condition (60) so the RS inequalities (63) are satisfied, in fact we have $\sigma_{x_j x_j} \sigma_{p_j p_j} \geq \frac{1}{4}$ for $1 \leq j \leq n$. Notice that the Gaussian distribution (61) is here the Wigner transform of the function

\[
\psi(x) = \left(\frac{1}{2\pi}\right)^{n/4} \left(\det \Sigma_{XX}\right)^{-1/4} e^{-\frac{1}{4} \Sigma_{XX}^{-1} x \cdot x}. 
\]

(64)

This discussion generalizes to arbitrary dual polar pairs; the key to this generalization is the following result:

**Lemma 19** Let $\Sigma > 0$. The condition $\Sigma + \frac{1}{2} iJ \geq 0$ is equivalent to the existence of $S \in \text{Sp}(n)$ such that $S(B^{2n}(1)) \subset \Omega$ (that is to the condition $c(\Omega) \geq \pi$) where $\Omega$ is the ellipsoid

\[
\Omega = \{z \in \mathbb{R}^{2n} : \frac{1}{2} \Sigma^{-1} z \cdot z \leq 1\}.
\]

**Proof.** See [13, 15] for detailed proofs. ■

As we have discussed in [13] the condition $c(\Omega) \geq \pi$ thus implies the RS inequalities (63), and can therefore be viewed as a symplectically invariant expression of the uncertainty principle of quantum mechanics.

**Proposition 20** Let $(\ell, \ell')$ be a pair of transverse Lagrangian planes and $\Omega_X \subset \ell$, $\Omega_P \subset \ell'$ two centered ellipsoids. (i) If $(\Omega_X, \Omega_P)$ is a Lagrangian dual pair, there exists ellipsoids $\Omega$ such that $\Pi_\ell \Omega = \Omega_X$ and $\Pi_{\ell'} \Omega = \Omega_P$ and
\(\Omega\) has symplectic capacity \(c(\Omega) \geq \pi\); (ii) Equivalently, the matrix \(\Sigma\) defined by

\[\Omega = \{ z \in \mathbb{R}^{2n} : \frac{1}{2}\Sigma^{-1}z \cdot z \leq 1 \}\]

is the covariance matrix of a density operator with Gaussian Wigner distribution

\[\rho(z) = \frac{1}{(2\pi)^{n}\sqrt{\det\Sigma}}e^{-\frac{1}{2}\Sigma^{-1}(z-x)^2}.\]  

(65)

(iii) When \((\Omega_X, \Omega_P)\) is an exact pair, i.e. when \(\Omega_X = \Omega_P\), then \(\rho(z) = \psi(\pi)\)

where \(\psi\) is a Gaussian

\[\psi(x) = \left(\frac{1}{4\pi}\right)^{n/4}(\det A)^{1/4}e^{-\frac{1}{2}(A+iB)x^2}\]  

where \(A\) and \(B\) are symmetric real \(n \times n\) matrices and \(A > 0\).

**Proof.** Suppose first that \(\Omega_X = \Omega_P\). In view of Theorem 15(i) there exists a unique symplectic ball \(\Omega = S(B^{2n}(1)) (S \in \text{Sp}(n))\) having projections \(\Pi_\ell \Omega = X_\ell\) and \(\Pi_{\ell'} \Omega = X_{\ell'}\) and this ellipsoid is the John ellipsoid of \(X_\ell \times X_{\ell'}\).

\[\Box\]

**Remark 21** The result above is closely related to “Pauli’s reconstruction problem” \([33, 11]\) where one wants to recover a function \(\psi\) knowing its modulus and the modulus of its Fourier transform; see our discussion of Pauli’s problem in \([18]\).

5 **Hardy’s Uncertainty Principle**

We briefly discussed Hardy’s uncertainty principle in the beginning of Section 4. Let us study the multi-dimensional variant of it from the point of view of polar duality; we will extend previous results in \([17]\). The Fourier transform of \(\psi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\) is defined by

\[F\psi(p) = \left(\frac{1}{2\pi}\right)^{n/2}\int_{\mathbb{R}^n} e^{-ipx}\psi(x)dx\]  

(67)

and extends into a unitary automorphism \(L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)\) also denoted by \(F\). In \([20]\) we have proven the following generalization Hardy’s result: assume that \(\psi \in L^2(\mathbb{R}^n)\), \(||\psi|| \neq 0\), satisfies the sub-Gaussian estimates

\[|\psi(x)| \leq Ce^{-\frac{1}{2}A_{x}} \quad \text{and} \quad |F\psi(p)| \leq Ce^{-\frac{1}{2}B_{p}}\]  

(68)

where \(A\) and \(B\) are two real positive definite symmetric matrices and \(C > 0\). Then:
Proposition 22  (i) The eigenvalues \( \lambda_j, j = 1, \ldots, n \), of \( AB \) are \( \leq 1 \); (ii) If \( \lambda_j = 1 \) for all \( j \), then \( \psi(x) = Ce^{-\frac{1}{2}Ax^2} \) for some complex constant \( C \).

The proof of this result is based on the following diagonalization result which is a refinement of Williamson’s symplectic diagonalization theorem in the block-diagonal case:

Lemma 23  Let \( A \) and \( B \) be two real positive-definite symmetric \( n \times n \) matrices. There exists \( L \in GL(n, \mathbb{R}) \) such that

\[
\begin{pmatrix}
L^T & 0 \\
0 & L^{-1}
\end{pmatrix}
\begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix}
\begin{pmatrix}
L & 0 \\
0 & (L^T)^{-1}
\end{pmatrix}
= \begin{pmatrix}
\Lambda & 0 \\
0 & \Lambda
\end{pmatrix}
\]

(69)

where \( \Lambda = \text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}) \) is the diagonal matrix whose eigenvalues are the square roots of the eigenvalues \( \lambda_1, \ldots, \lambda_n \) of \( AB \).

Proposition 22 implies the following geometric version of Hardy’s uncertainty principle:

Corollary 24  Let \( X_A = \{ x \in \mathbb{R}^n : Ax \cdot x \leq 1 \} \) and \( P_B = \{ p \in \mathbb{R}^n : Bp \cdot p \leq 1 \} \). (i) There exists a \( \psi \in L^2(\mathbb{R}^n) \) with \( ||\psi|| \neq 0 \) satisfying the multi-dimensional Hardy inequalities (68) if and only if \((X_A, P_B)\) is a dual pair, (ii) we have \( X_A^o = P_B \) if and only if \( \psi(x) = Ce^{-\frac{1}{2}Ax^2} \) for some constant \( C > 0 \).

Proof. (i) immediately follows from Proposition 22 using Proposition 3. (ii) We have \( X_A^o = P_B \) if and only if all the eigenvalues of \( AB \) are all equal to one hence the result by Proposition 22.

5.1 Sub-Gaussian estimates for the Wigner transform

Assuming that \( \psi \) and its Fourier transform \( F\psi \) are in \( L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) the marginal conditions

\[
\int_{\mathbb{R}^n} W\psi(z)dp = |\psi(x)|^2
\]

(70)

\[
\int_{\mathbb{R}^n} W\psi(z)dx = |F\psi(p)|^2
\]

(71)

hold [12, 16]. Since the knowledge of \( W\psi \) determines \( \psi \) up to a constant unimodular factor, Hardy’s uncertainty principle can be reformulated in terms of the Wigner transform \( W\psi \).
We apply the multi-dimensional Hardy uncertainty principle to the study of estimates of the type
\[ W_\psi(z) \leq Ce^{-Mz\cdot z} \] (72)
where \( M > 0 \) (i.e. \( M \) is real positive-definite and symmetric), \( C > 0 \) and \( \psi \in L^2(\mathbb{R}^n) \). We will in fact assume that, in addition, \( \psi \) and its Fourier transform \( F\psi \) are in \( L^2(\mathbb{R}^n) \) so that the marginal conditions (70) and (71) hold.

We have the following result which can be viewed as an extension of Corollary 24:

**Proposition 25** Assume that there exists a non-zero \( \psi \in L^2(\mathbb{R}^n) \) such that the inequality
\[ W_\psi(z) \leq Ce^{-Mz\cdot z}, \quad z \in \mathbb{R}^{2n} \]
holds for some \( C > 0 \). Let \( \Omega = \{z \in \mathbb{R}^{2n} : Mz \cdot z \leq 1\} \).

(i) For every pair \((\ell, \ell')\) of transversal Lagrangian planes \((\Pi_\ell \Omega, \Pi_{\ell'} \Omega)\) is a dual Lagrangian dual pair. (ii) If \((\Pi_\ell \Omega)_{\tilde{\nu}} = \Pi_{\ell'} \Omega\) then \( W_\psi \) is a Gaussian \( Ce^{-Mz\cdot z} \).

**Proof.** (i) In view of Theorem 13 it is again sufficient to prove that \( c(\Omega) \geq \pi \) for any symplectic capacity \( c \). In view of Williamson’s symplectic diagonalization theorem [12, 26] there exists \( S \in \text{Sp}(n) \) such that
\[ M = S^TDS, \quad D = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} \]
where \( \Lambda \) is the diagonal matrix with non-zero entries the symplectic eigenvalues of \( M \). Using the symplectic covariance of the Wigner distribution [12] it follows that we have
\[ W(\hat{S}\psi)(z) = W_\psi(S^{-1}z) \leq Ce^{-Dz\cdot z} \] (73)
where \( \hat{S} \in \text{Mp}(n) \) is any of the two metaplectic operators covering \( S \). Setting \( \psi' = \hat{S}\psi \) and integrating this inequality with respect to \( p \) and then \( x \) we get, using the marginal conditions (70) and (71),
\[ |\psi'(x)|^2 \leq C'e^{-Dx\cdot x}, \quad |F\psi'(p)|^2 \leq C'e^{-Dp\cdot p} \] (74)
where \( C' > 0 \) is a constant depending only on \( C \) and the symplectic eigenvalues \( \lambda_j^\omega \) of \( M \). Applying the multidimensional Hardy uncertainty principle, we must have \( \lambda_j^\omega \leq 1 \) for \( 1 \leq j \leq n \) and hence \( c(\Omega) \geq \pi \) in view of formula (5). The result now follows from Theorem 13. \[ \blacksquare \]

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References

[1] S. Artstein-Avidan, V. D. Milman, and Y. Ostrover. The M-ellipsoid, Symplectic Capacities and Volume. *Comment. Math. Helv.* 83(2), 359–369 (2008)

[2] S. Artstein-Avidan and Y. Ostrover. Bounds for Minkowski billiard trajectories in convex bodies. *Intern. Math. Res. Not.* (IMRM) (2012)

[3] S. Artstein-Avidan, R. Karasev, and Y. Ostrover. From Symplectic Measurements to the Mahler Conjecture. *Duke Math. J.* 163(11), 2003–2022 (2014)

[4] G. Aubrun and S. J. Szarek, *Alice and Bob meet Banach*. Vol. 223. American Mathematical Soc., 2017

[5] K. M. Ball. Ellipsoids of maximal volume in convex bodies. *Geom. Dedicate.* 41(2), 241–250 (1992)

[6] M. J. Bastiaans. Wigner distribution function and its application to first-order optics, *J. Opt. Soc. Am.* 69, 1710 (1979)

[7] E. Cordero, M. de Gosson, and F. Nicola. On the positivity of trace class operators. *Adv. Theor. Math. Phys.* 23(8), 2061–2091 (2019)

[8] I. Ekeland and H. Hofer. Symplectic topology and Hamiltonian dynamics, *Math. Z.* 200(3), 355–378 (1989)

[9] I. Ekeland and H. Hofer. Symplectic topology and Hamiltonian dynamics, *Math. Z.* 203, 553–567 (1990)

[10] H. G. Feichtinger, Modulation Spaces: Looking Back and Ahead Sampl, *Theory Signal Image Process.* 5(2), 109–140 (2006)

[11] G. Esposito, G. Marmo, G. Miele, and G. Sudarshan. *Advanced Concepts in Quantum Mechanics*, Cambridge University Press, 2015

[12] M. de Gosson, *Symplectic geometry and quantum mechanics*. Vol. 166. Springer Science & Business Media, 2006

[13] M. de Gosson. The Symplectic Camel and the Uncertainty Principle: The Tip of an Iceberg? *Found. Phys.* 99, 194 (2009)
[14] M. de Gosson, On the Use of Minimum Volume Ellipsoids and Symplectic Capacities for Studying Classical Uncertainties for Joint Position–Momentum Measurements. *J. Stat. Mech.: Theory and Experiment* 11, P11005 (2010)

[15] M. de Gosson. *Symplectic Methods in Harmonic Analysis and in Mathematical Physics*. Birkhäuser, Basel, 2011

[16] M. de Gosson. *The Wigner Transform*, World Scientific, Series: Advanced Texts in mathematics, 2017

[17] M. de Gosson. Two Geometric Interpretations of the Multidimensional Hardy Uncertainty Principle. *Appl. Comput. Harmon. Anal.* 42(1), 143–153 (2017)

[18] M. de Gosson. Quantum Polar Duality and the Symplectic Camel: a New Geometric Approach to Quantization. *Found. Phys.* 51, Article number: 60 (2021)

[19] C. de Gosson and M. de Gosson. On the Non-Uniqueness of Statistical Ensembles Defining a Density Operator and a Class of Mixed Quantum States with Integrable Wigner Distribution. *Quantum Rep.*, 3, 473–481 (2021)

[20] M. de Gosson and F. Luef. Quantum States and Hardy’s Formulation of the Uncertainty Principle: a Symplectic Approach, *Lett. Math. Phys.* 80, 69–82 (2007)

[21] K. Gröchenig. *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, 2000

[22] M. Gromov. Pseudoholomorphic curves in symplectic manifolds. *Inv. Math.* 82(2), 307–347 (1985)

[23] G. H. Hardy. A theorem concerning Fourier transforms. *J. London. Math. Soc.* 8, 227–231 (1933)

[24] J. Hilgevoord. The standard deviation is not an adequate measure of quantum uncertainty. *Am. J. Phys.* 70(10), 983 (2002)

[25] J. Hilgevoord and J. B. M. Uffink. Uncertainty Principle and Uncertainty Relations. *Found. Phys.* 15(9) 925 (1985)
[26] H. Hofer and E. Zehnder. *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser Advanced Texts (Basler Lehrbücher), Birkhäuser Verlag, 1994.

[27] M. S. Jakobsen. On a (no longer) New Segal Algebra: a review of the Feichtinger algebra, *J. Fourier Anal. Appl.* 24(6), 1579–1660 (2018).

[28] Q. Li and J. G. Griffiths. Least squares ellipsoid specific fitting. In *Geometric modeling and processing*, 2004. Proceedings (pp. 335-340). IEEE, 2004.

[29] G. Loupias and S. Miracle-Sole. $C^*$-Algèbres des systèmes canoniques, I, *Commun. math. Phys.* 2, 31–48 (1966).

[30] G. Loupias and S. Miracle-Sole. $C^*$-Algèbres des systèmes canoniques, II, *Ann. Inst. Henri Poincaré* 6(1), 39–58 (1967).

[31] F. J. Narcowich. Conditions for the convolution of two Wigner functions to be itself a Wigner function, *J. Math. Phys.* 30(11), 2036–2041 (1988).

[32] J. von Neumann. *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, 1955 [Original German edition in 1932].

[33] W. Pauli. *General principles of quantum mechanics*, Springer Science & Business Media, 2012 [original title: *Prinzipien der Quantentheorie*, publ. in : Handbuch der Physik, v.5.1, 1958].

[34] L. Polterovich. *The geometry of the group of symplectic diffeomorphisms*. Birkhäuser, 2012.

[35] D. A. Turner, I. J. Anderson, J. C. Mason, and M. G. Cox. An algorithm for fitting an ellipsoid to data. *National Physical Laboratory*, UK (1999).

[36] S. Van Aelst and P. Rousseeuw, Minimum volume ellipsoid. Wiley Interdisciplinary Reviews: *Computational Statistics*, 1(1), 71–82 (2009).

[37] F. Zhang. *The Schur Complement and its Applications*, Springer, Berlin, 2005.