EQUIVARIANT COMPARISON OF QUANTUM HOMOGENEOUS SPACES

MAKOTO YAMASHITA

Abstract. We prove that homogeneous spaces over the $q$-deformation of simply connected simple compact Lie groups with respect to Poisson–Lie quantum subgroups are equivariantly KK-equivalent to the classical one, extending the nonequivariant case of Neshveyev–Tuset. As applications of this equivariant equivalence we prove the ring isomorphism of the $K$-group of $G_q$ with respect to the equivariant Kasparov product for the dual discrete quantum group, and the Borsuk–Ulam theorem for quantum spheres.

1. Introduction

After seminal works of Woronowicz [Wor87] and Podleś [Pod87], comodule algebras over quantum groups became known to be a rich class of ‘noncommutative’ spaces in the study of operator algebras and noncommutative geometry. One geometrically interesting direction to generalize their results is given by the $q$-deformation $G_q$ of simply connected simple compact Lie groups and their homogeneous spaces due to Reshetikhin–Takhtadzhyan–Faddeev [RTF89] and Vaksman–Soibelman [VS90].

A $C^*$-algebra obtained this way can be thought as a continuous deformation of the commutative algebra of continuous functions on a homogeneous space of $G$, which is an ordinary compact Riemannian manifold. Standing on this viewpoint, Nagy [Nag98] took a $KK/E$-categorical approach to compute the $K$-theory of quantum groups $G_q$ by utilizing continuous fields of $C^*$-algebras over the parameter space with fiber $C(G_q)$. His method was recently generalized to homogeneous spaces over Poisson–Lie quantum subgroups $K_{S,L}^S$ by Neshveyev–Tuset [NT11b]. Their method proceeds by reducing the comparison of $K$-groups to the case of deformation quantization of open discs which appear as the symplectic leaves in the homogeneous space $K_{S,L}$. The categorical structure of comodule algebras over such quantum groups is also interesting in connection with the Baum–Connes problem of quantum groups due to Meyer–Nest [MN06, MN07]. The analogue of the strong Baum–Connes conjecture for the dual of Hodgkin groups allows us to cast a new light on early results of Hodgkin [Hod75], McLeod [McL79], and Snaith [Sna72] concerning equivariant $K$-groups of $G$-spaces. It is natural to expect that quantum homogeneous spaces are crucial in understanding of more general comodule algebras over $G_q$. Indeed, the equivariant $KK$-theory for the Podleś sphere plays a central role in the proof of the strong Baum–Connes conjecture for $SU_q(2)$ by Voigt [Voi09].

The main result of this paper is that the algebra $C(G_q/K_{S,L}^S)$ of the quantum homogeneous space is equivariantly $KK$-equivalent to the classical one $C(G/K_{S,L}^S)$ with respect to the translation actions of the maximal torus (Theorem 1). This extends the result of [NT11b] to the equivariant setting. The proof occupies Section 3 where we adapt the argument of [NT11b] to the equivariant case using the

2010 Mathematics Subject Classification. Primary 46L80; Secondary 20G42.

Key words and phrases. quantum group, quantum flag variety, KK-theory.
equivariant Universal Coefficient Theorem \cite{RSS6} and the triangulated structure of the equivariant KK-category \cite{MN06}.

We observe two applications of our main theorem in Section \ref{sect:applications} both of which require the equivariant form of the KK-equivalence. The first one concerns the ring structure of $K_*(C(G_q))$ with respect to the Kasparov product on the $G_q$-equivariant category (Theorem \ref{thm:main}). Following the induction-restriction duality trick with respect to the maximal torus of \cite{MN07}, we prove that $K_*(C(G_q))$ is isomorphic to $K^*(G)$ as a ring. This can be interpreted as a quantum version of Hodgkin’s theorem which describes the ring structure of $K^*(G)$.

The second application is an analogue of the Borsuk–Ulam theorem for the quantum spheres (Theorem \ref{thm:Borsuk-Ulam}), conjectured by Baum and Hajac \cite{BH}. The theorem states that there is no equivariant homomorphism from $C(S^n_q)$ to $C(S^{n+1}_q)$ with respect to the antipodal action of $C_2 = \mathbb{Z}/2\mathbb{Z}$. We prove it using the equivariant KK-theory for $C_2$, which is modelled on the Lefschetz number argument of Casson–Gottlieb \cite{CG77}. The crucial point is that the antipodal action on an odd-dimensional sphere is given by a restriction of a $U(1)$-action, which comes from the homogeneous space description of $S^n_q$. As a byproduct, we obtain that the $n$-dimensional spheres with the antipodal action of $C_2$ are mutually inequivalent.

Acknowledgment. A major part of this research was carried out under the support of the Marie Curie Research Training Network MRTN-CT-2006-031962 in Noncommutative Geometry, EU-NCG. The author thanks Sergey Neshveyev for fruitful exchanges which were crucial to the early development of the research. This paper was written during the author’s stay at the Mathematical Sciences Institute, Australian National University. He would like to thank them, particularly Alan Carey and Adam Rennie, for their hospitality. He is also grateful to Piotr M. Hajac, from whom he learned the Borsuk–Ulam problems during his lectures at ANU in August 2011.

2. Preliminaries

Throughout the paper $G$ denotes a simply connected simple compact Lie group and $\mathfrak{g}$ its Lie algebra. We fix a maximal torus $T$ of $G$, and let $W$ denote the associated Weyl group. We also fix a positive root system $P$ of $(G,T)$ and let $\Pi = \{\alpha_i\}$ denote the corresponding set of the simple roots. By abuse of the language we sometimes consider $\Pi$ as the index set $\{i\}$ for the $\alpha_i$. The associated length function on $W$ is denoted by $l(w)$, and the longest element by $w_0$, whose length is denoted by $m_0$.

We briefly review the definition of the quantum group $G_q$ for $0 < q < 1$ and related constructs. The closed unit interval $[0,1]$ is denoted by $I$. We adopt the convention of Neshveyev–Tuset \cite{NT07, NT11}, \cite{NT11b} for the $q$-deformation of $G$, and that of Nest–Voigt \cite{NV10} for the braided tensor products. When $A$ is a continuous field of $C^*$-algebras over $I$ (or $(0,1]$) whose fiber at $q \in I$ is $A_q$, we let $\Gamma_X(A_q)$ denote the $C^*$-algebra of the continuous sections over $X \subset I$.

We put $\mathbb{N} = \{0,1,\ldots\}$ and $\mathbb{N}_+ = \{1,2,\ldots\}$. For each subset $X$ of $\mathbb{N}$, we let $\chi_X$ denote the orthogonal projection onto the subspace $\ell^2 X$ of $\ell^2 \mathbb{N}$.

2.1. Algebras representing $q$-deformations. Let $(a_{i,j})_{i,j \in \Pi}$ be the Cartan matrix $a_{i,j} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ of $\mathfrak{g}$. Let $U_q(\mathfrak{g})$ be the algebra generated by the generators $E_i, F_i, K_i^{\pm 1}$ for $i \in \Pi$ subject to the relations

\begin{equation}
[K_i, K_j] = 0, \quad K_i E_j K_i^{-1} = q_i^{a_{i,j}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{i,j}} F_j
\end{equation}

for $q_i = q^{(\alpha_i, \alpha_i)/2}$ and

\[ [E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \]

where $\delta_{i,j}$ is the Kronecker delta.
together with $q$-analogues of the Serre relations.

The space $\mathbb{C}[G_q]$ of the matrix coefficients of finite dimensional unitary representations of $\mathcal{U}_q(\mathfrak{g})$ is a $\ast$-Hopf algebra with a unique faithful Haar state. We let $(C(G_q), \Delta)$ denote its $C^*$-algebraic closure.

2.2. Braided tensor products. Let $W$ be the multiplicative unitary associated to $G_q$. It is a unitary element in the multiplier algebra $M(C(G_q) \otimes C^*(G_q))$ and satisfying the pentagonal equation $W_{12}W_{13}W_{23} = W_{23}W_{12}$ and

$$C(G_q) = \{ t \otimes \omega \mid \omega \in B(H)_+ \}, \quad C^*(G_q) = \{ \omega \otimes t \mid \omega \in B(H)_+ \}.$$  

Let $D(G_q)$ be the Drinfeld double of $G_q$. It is a locally compact quantum group given by the algebra $C_0(D(G_q)) = C_0(G_q) \otimes C^*(G_q)$ endowed with the coproduct

$$\Delta_{D(G_q)} = (t \otimes (\Sigma \circ \text{ad}_W) \otimes t) \circ (\Delta \otimes \Delta).$$  

Given $D(G_q)$-C$^*$-algebras $A$ and $B$, we obtain the braided tensor product $A \boxtimes_{G_q} B$ as a $D(G_q)$-C$^*$-algebra [Vae95,NY10].

The $C^*$-algebras $C(G_q)$ and $C(G_q/T)$ are $D(G_q)$-algebras [NY10]. The coaction $\delta$ of $C(G_q)$ on $C(G_q)$ is the left translation, and the one $\lambda$ of $C^*(G_q)$ is the left adjoint coaction. From this it follows that there is an isomorphism of $C^*$-algebras

(2) $C(G_q) \boxtimes_{G_q} C(G_q) \rightarrow C(G_q) \otimes C(G_q), \quad \lambda(a)_{12}\delta(b)_{13} \mapsto (a \otimes 1)\Delta(b).$

The above coactions restrict to the (function algebra of) quantum homogeneous space $G_q/H$ for any closed quantum subgroup $H$ of $G_q$. When $H$ is actually a group such as $T$, what is more is true.

Proposition 1. Let $H$ be a closed subgroup of $G_q$. The coaction $\lambda$ commute with the right translation action by $H$.

Proof. Let $p^C_{G_q}_H$ be the restriction homomorphism $C(G_q) \rightarrow C(H)$. Then the structure map $\rho : C(G_q) \rightarrow C(G_q) \otimes C(H)$ of the right translation coaction by $C(H)$ is given by $t \otimes p^C_{G_q}_H \circ \lambda$. We need to show the equality

$$t \otimes \rho \circ \lambda = \lambda \otimes t \circ \rho.$$  

We may instead prove the corresponding equality for the action of $C(G_q)$ on $C(G_q)$ associated to $\lambda$. This action is given by $h.a = h(\xi)\alpha S(h(\zeta))$ for the antipode $S$. The above equality is equivalent to

$$h_{(1)}a_{(1)} S(h_{(22)}) \otimes p^C_{G_q}_H (h_{(12)}a_{(2)} S(h_{(21)})) = h_{(1)}a_{(1)} S(h_{(21)}) \otimes p^C_{G_q}_H (a_{(2)}).$$  

for any $h, a$ in $C(G_q)$. This is indeed the case by the commutativity of $C(H)$. □

By the proposition above we obtain the right translation action of $T$ on $C(G_q) \boxtimes_{G_q} C(G_q) \approx \hat{G}_q \ltimes C(G_q)$. The fixed point algebra is identified to $C(G_q/T) \boxtimes_{G_q} C(G_q)$.

In the rest of the paper we shall be mainly concerned with these $D(G_q)$-C$^*$-algebras.

2.3. Poisson–Lie subgroups. We follow the convention of [NT11b]. Let $S$ be a subset of II, and $\mathfrak{g}_S$ be the subalgebra of $\mathfrak{g}$ generated by $E_i, F_i$ for $\alpha_i \in S$. We let $\hat{K}^S$ denote the closed subgroup of $G$ characterized by $\text{Lie}(\hat{K}^S) = \mathfrak{g}_S$. Let $P^c(S)$ be the subgroup of the weight lattice spanned by the fundamental weights $w_i$ for $\alpha_i \notin S$. When $L$ is a subgroup of $P^c(S)$, we let $T_L$ denote the annihilator subgroup of $T$ for $L$. The Poisson–Lie subgroup $K^{S,L}$ is the closed subgroup of $G$ generated by $\hat{K}^S$ and $T_L$. When $L = P^c(S)$, we have $K^{S,L} = \hat{K}^S$.

We have a subalgebra $\mathcal{U}(K^{S,L}_q)$ of $\mathcal{U}_q(\mathfrak{g})$ spanned by $T_L$ and $E_i, F_i$ for $\alpha_i \in S$. The image of $\mathbb{C}[G_q]$ under the transpose map $\mathcal{U}_q(\mathfrak{g}) \rightarrow \mathcal{U}(K^{S,L}_q)^*$ is a C$^*$-algebra denoted by $\mathbb{C}[K^{S,L}_q]$. 


3. Equivariant comparison of quantum manifolds

Our goal in this section is to show that the $T \times (T/T_L)$-equivariant $KK$-equivalence class of $C(G_q/K_q^{S,L})$ does not depend on $q$.

3.1. Comparison family of quantum discs. For $0 < q < 1$, let $C(\mathbb{D}_q)$ be the noncommutative disc algebra generated by an element $Z_q$ satisfying

$$1 - Z_q^* Z_q = q^2 (1 - Z_q Z_q^*).$$

Then $C(\mathbb{D}_q)$ can be faithfully represented on $\ell^2(\mathbb{N})$ by setting

$$Z_q e_n = \sqrt{1 - q^{2(n+1)}} e_{n+1} \quad (n \in \mathbb{N}).$$

The set of operators generated by this representation is equal to the Toeplitz algebra $T$ on $\ell^2(\mathbb{N})$ generated by the shift operator $S: e_n \mapsto e_{n+1}$. The homomorphism $C(\mathbb{D}_q) \to C(S^1)$ given by $Z_q \mapsto z$ is identified with the standard homomorphism $T \to C(S^1)$, $S \mapsto z$.

Let $C_0(\mathbb{D}_q)$ be the kernel of the above homomorphism $C(\mathbb{D}_q) \to C(S^1)$. It is isomorphic to the algebra $K$ of the compact operators.

Now, let $\alpha$ and $\gamma$ be the standard generators of $C(SU_q(2))$, satisfying

$$\alpha^* \alpha + \gamma^* \gamma = 1, \quad \alpha \alpha^* + q^2 \gamma \gamma^* = 1, \quad \gamma^* \gamma = \gamma \gamma^*, \quad \alpha \gamma = q \gamma \alpha, \quad \alpha \gamma^* = q^* \gamma^* \alpha.$$

Then one has a homomorphism from $C(SU_q(2))$ onto $C(\mathbb{D}_q)$ given by

$$\alpha \mapsto Z_q^*, \quad \gamma \mapsto -(1 - Z_q Z_q^*)^{1/2}.$$

By composing this with the representation $\mathcal{R}$, we obtain a representation $\rho_q$ of $C(SU_q(2))$ on $\ell^2(\mathbb{N})$.

The homomorphism $\mathcal{R}$ restricts to an isomorphism between $C(SU_q(2)/U(1))$ and $C_0(\mathbb{D}_q)^*$. We record the image of the generators of $C(SU_q(2)/U(1))$ under this map:

$$\alpha \alpha^* \mapsto Z_q^* Z_q, \quad \alpha \gamma^* \mapsto -Z_q^* (1 - Z_q Z_q^*)^{1/2}, \quad \gamma \gamma^* \mapsto (1 - Z_q Z_q^*).$$

Let $\tau_\alpha$ be the action of $U(1)$ on $C(SU_q(2))$ defined by

$$\tau_\alpha (\alpha) = e^{2\pi i t} \alpha, \quad \tau_\alpha(\gamma) = \gamma.$$

The conjugation action of $U(1)$ is given by the composition of $\tau$ and the homomorphism $U(1) \to U(1)$ of mapping degree 2.
Let $U_t$ be the unitary representation of $T$ on $\ell^2\mathbb{N}$ defined by
\begin{equation}
U_t e_n = e^{2\pi in t} e_n \quad (n \in \mathbb{N}).
\end{equation}
We consider the adjoint action $\text{ad}_{U_t}$ on $T$. By [3], this is identified to the rotation action $Z_0 \mapsto e^{2\pi i t} Z_0$ on $C(\mathbb{D}_0)$. By [4], its restriction to $C_0(\mathbb{D}_q)$ is identified to the action $\tau$ on $C(SU_q(2)/U(1))$. Note also that the restriction of the conjugation action of $T$ to $C(G_q/T)$ is equal to the left translation action.

For $q = 1$, we formally put $C(\mathbb{D}_1) = C(\mathbb{D})$ and $C_0(\mathbb{D}_1) = C_0(\mathbb{D})$ where $\mathbb{D} = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$ and $\mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \}$.

The algebras above form continuous fields of $C^\ast$-algebras $(C_0(\mathbb{D}_q))_{q \in [0,1]}$ and $(C(\mathbb{D}_q))_{q \in (0,1]}$. The algebra $\Gamma_1(\mathbb{D}_q)$ of the continuous sections is defined as the universal algebra $C^\ast(Q,Z)$ generated by a positive contraction $Q$ and another element $Z$ satisfying
\begin{equation}
QZ = ZQ, \quad 1 - Z^*Z = Q^2 (1 - ZZ^*).
\end{equation}
The inclusion of $C(I) \simeq C^\ast(Q)$ into $C^\ast(Q,Z)$ defines the structure of a $C(I)$-algebra. The algebra $(C_0(\mathbb{D}_q))_{q \in I}$ is defined as the kernel of the homomorphism onto $C(I) \otimes C(S^1)$ given by $Z \mapsto x \in C(S^1)$.

**Lemma 4.** Let $k \in \mathbb{N}$ and $i_1, \ldots, i_k$ be nonnegative integers. For each $q \in (0,1]$, the embedding
\begin{equation}
\iota : C \to C(\mathbb{D}_{q^{i_1}}) \otimes \cdots \otimes C(\mathbb{D}_{q^{i_k}})
\end{equation}
induces a $KK^{U(1)}$-equivalence.

**Proof.** Suppose that the assertion is verified for $k = 1$, with the inverse $\beta$ of $\iota$ in $KK^{U(1)}(C(\mathbb{D}_q), C)$. Then the morphism $\beta^q k$ will be the inverse of $[\iota]$. Hence it is enough to prove the assertion for $k = 1$.

First suppose that $q = 1$. Then the evaluation at the origin $ev_o$ defines a $U(1)$-equivariant extension $C_0(\mathbb{D} \setminus 0) \to C(\mathbb{D}) \to C$. Since the kernel $C_0(\mathbb{D} \setminus 0)$ is equivariantly contractible, $ev_o$ induces a $KK^{T}$-equivalence, which is obviously inverse to $\iota$.

Next we consider the case $q < 1$. Then we may use the following $U(1)$-equivariant graded $C^\ast$-$T$-$C$-module $\beta$, which is already considered by Pimsner in the non-equivariant context [Pim97, Definition 4.3 and Theorem 4.4]. Pimsner’s argument applies to our setting without change, but we briefly review it for the reader’s convenience.

The graded Hilbert space for $\beta$ is given by $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$, endowed with the action of $U(1)$ defined by [5] on each copy of $\ell^2(\mathbb{N})$. We consider the natural action $\pi$ of $T$ on the first copy of $\ell^2(\mathbb{N})$. On the second copy, its action is given by $\chi_N, \pi \chi_N$. The odd operator for $\beta$ is given by the transposition of the direct summands.

On one hand, the composition $\iota \otimes \beta$ equals the identity morphism of $KK^{T}(C, C)$. On the other hand, the morphism $\beta \otimes 1 - \text{Id}_T$ is represented by the following $C^\ast$-$T$-bimodule. The graded space is $(\ell^2(\mathbb{N}) \otimes_C T) \oplus (\ell^2(\mathbb{N}) \otimes_C T)$ with the diagonal action $(U_t \otimes \text{Ad}_{C_1}) \otimes 2$ of $U(1)$. The right action of $T$ is given by the right multiplication action on the second tensor component. The left action of $T$ on the first copy of $\ell^2(\mathbb{N}) \otimes_C T$ is given by $\pi \otimes \text{Id}_T$, and on the second copy by
\begin{equation}
\chi_{(0)} \otimes \pi_t + (\chi_N, \pi \chi_N) \otimes \text{Id}_T,
\end{equation}
where $\pi_t$ is the left multiplication action of $T$ on itself.

Now, we construct a continuous homotopy between $[\iota]$ and $\pi \otimes \text{Id}_T$. For each $t \in [0,1]$, consider the operator
\begin{equation}
S_t \delta_0 \otimes x = \cos(\frac{\pi}{2} t) \delta_1 \otimes x + \sin(\frac{\pi}{2} t) \delta_0 \otimes Sx, \quad S_t \delta_k \otimes x = \delta_{k+1} \otimes x \quad (x \in \mathcal{T}, k \in \mathbb{N}_+).
\end{equation}
Then at each $t$ one has the homomorphism $\pi_t: \mathcal{T} \to \mathcal{L}(E_T)$ by $\pi_t(S) = S_t$. They satisfy $\pi_t - \pi_t' \in \mathcal{K}(E_T)$, and from the choice of $S_0$ above it follows that the connecting morphism $\pi_1$ remain $U(1)$-equivariant. The morphism $\pi_1$ agrees with $[\mathfrak{S}]$, while the one $\pi_0$ equals $\pi \otimes \text{Id}_\mathcal{T}$. Hence we have $\beta \otimes \mathfrak{s} = [\text{Id}_\mathcal{T}]$ in $KK^{U(1)}(\mathcal{T}, \mathcal{T})$. \hfill \Box

The above construction can be formulated in the framework of $U(1)$-$C(I)$-algebras with respect to the trivial action of $U(1)$ on $I$. The action of $U(1)$ on $\Gamma_I(C(\mathbb{D}_0))$ given by

$$\pi_t(Q) = Q, \quad \pi_t(Z) = e^{2\pi i t} Z$$

defines a structure of the $U(1)$-$C(I)$-algebra.

In the following proposition we observe that the evaluation map at an $y$ point induces an equivariant $KK$-equivalence. Note that the $KK$-equivalence without the $U(1)$-action was proved in [NT11b] Lemma 6.3.

**Proposition 5.** Let $k$ be any positive integer. For any $q_0 \in I$, the evaluation map $ev_{q_0}$ for the $C(I)$-algebra

$$A_k = \Gamma_I(C_0(\mathbb{D}_0)) \otimes_{C(I)} \cdots \otimes_{C(I)} \Gamma_I(C_0(\mathbb{D}_0))$$

is a $KK^{U(1)^k}$-equivalence.

**Proof.** The assertion is equivalent to that the kernel of $ev_{q_0}$ is $KK^{U(1)^k}$-contractible. We have the decomposition

$$\ker(ev_{q_0}) = \ker(ev_{q_0} |_{\Gamma_I(\mathbb{D}_0)}) \otimes \ker(ev_{q_0} |_{\Gamma_{[q_0, 1]}(\mathbb{D}_0)})$$

The field $(C_0(\mathbb{D}_0))_{q_0(0, 1)}$ is a trivial field over $[0, 1]$ with the fiber $\mathcal{T}$. Hence the first summand of the right hand side is equivariantly contractible.

To handle the remaining cases, it is enough to consider the case $q_0 = 0$. For $0 \leq m \leq k$, let $A_{k,m}$ denote the $C(I)$-algebra

$$\Gamma_I(C_0(\mathbb{D}_0)) \otimes_{C(I)} \cdots \otimes_{C(I)} \Gamma_I(C_0(\mathbb{D}_0))$$

whose fiber is denoted by

$$A^{(q)}_{k,m} = C_0(\mathbb{D}_0) \otimes \cdots \otimes C_0(\mathbb{D}_0) \otimes C(\mathbb{D}_0) \otimes \cdots \otimes C(\mathbb{D}_0).$$

The kernel of the evaluation map $A_{k,m} \to A^{(1)}_{k,m}$ at 1 is $KK^{U(1)^k}$-contractible, being the trivial field with fiber $A^{(q)}_{k,m}$. We argue by induction on $m$ that $KK^{U(1)^k}(A^{(q)}_{k,m}) \simeq R(U(1)^k)$ and the evaluation map $ev_0$ induces an isomorphism

$$KK^{U(1)^k}(A^{(1)}_{k,m}) \simeq R(U(1)^k) \xrightarrow{\text{Id}} R(U(1)^k) \simeq KK^{U(1)^k}(A^{(0)}_{k,m}).$$

First, let us consider the case for $m = 0$. By Lemma 4 the $K_0^{U(1)}$-groups of both $\Gamma_I(C(\mathbb{D}_0))$ and $C(\mathbb{D}_0)$ are isomorphic to $R(U(1))$ with the basis $[1]$. The evaluation map makes correspondence between these basis.

Now, the $U(1)$-algebra $\mathcal{T}$ is in the equivariant bootstrap class by Lemma 4. It follows that $A_{k,0}$ is also in the $U(1)^k$-equivariant bootstrap class. Since it has an $R(\mathcal{T})$-projective $K^T\mathcal{T}$-group, we can apply the equivariant Universal Coefficient Theorem [RS88, Theorem 10.1] to conclude that $ev_0$ is a $KK^{U(1)^k}$-equivalence when $m = 0$. 

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Next, suppose that the assertion was proved for the continuous fields $A_{k',m'}$ satisfying $k' < k$ and $A_{k,m'}$ for $m' \leq m$. Let us first verify that the map $ev_a$ induces an isomorphism between the $U(1)^k$-equivariant $K_0$-groups of $A_{k,m}$ and $A_{k,m'}^{(0)}$. We have the extension

$$
\begin{array}{ccc}
A_{k,m+1} & \longrightarrow & A_{k,m} & \longrightarrow & B \\
evo & | & | & | & | & | \\
A_{k,m+1}^{(0)} & \longrightarrow & A_{k,m}^{(0)} & \longrightarrow & B^{(0)}
\end{array}
$$

where $B$ is the $C(I)$-algebra defined by

$$
B = \Gamma I(C_0(\mathbb{D}_q) \otimes C(I)) \oplus \cdots \oplus \Gamma I(C_0(\mathbb{D}_q)) \otimes C(S^1 \times I)
$$

Now, the evaluation map for $B$ is the tensor product of the maps $ev_0$ for the $C(I)$-algebras $A_{k,m+1}$ and $C(S^1) \otimes C(I)$. By the induction assumption, the former is a $KK^{U(1)^k}$-isomorphism. The latter is trivially an $KK^{U(1)^k}$-isomorphism. Hence $ev_0$ for $B$ is an $KK^{U(1)^k}$-isomorphism. Next, the evaluation for $A_{k,m}$ is again an $KK^{U(1)^k}$-isomorphism by the induction assumption.

The rows in the diagram (10) are (parts of) mapping cone triangles. Since the category $KK^{U(1)^k}$ is an triangulated category with the mapping cone triangles as its exact triangles [MINO06 Proposition 2.1], the evaluation map for $A_{k,m+1}$ is also an $KK^{U(1)^k}$-isomorphism. Thus we obtain the assertion for $A_{k,m+1}$. □

3.2. Equivariant family of quantum homogeneous spaces. For each simple root $\alpha_i$, let $\sigma_i$ be the homomorphism $U_q(sl_2) \to U_q(g)$ defined by

$$
E \mapsto E_i, \quad F \mapsto F_i, \quad K \mapsto K_i.
$$

Its transpose extends to a $*$-homomorphism $\sigma_i : C(G_q) \to C(SU_q(2))$. Then $\pi_i = \rho_q \sigma_i$ is a representation of $C(G_q)$ on $\ell^2(\mathbb{N})$.

For each element $w$ of $W$, we shall choose and fix an reduced presentation

$$
w = s_{i_1}(w) \cdots s_{i_k}(w).
$$

Then we obtain an irreducible representation $\pi_w$ of $C(G_q)$ on $\ell^2(\mathbb{N}) \otimes^k$ defined by

$$
\pi_w = (\pi_{i_1}(w) \otimes \cdots \otimes \pi_{i_k}(w)) \Delta_q^{(k-1)}.
$$

For the neutral element $e \in W$, the representation $\pi_e$ is understood to be the counit map $C(G_q) \to \mathbb{C}$. Finally, we put $\pi_{w,t} = \pi_w \circ \rho_t$ for each $t \in T$, where $\rho_t$ is the automorphism of $C(G_q)$ given as the right translation by $t$. The representations $(\pi_{w,t})_{w \in W, t \in T}$ give a complete parametrization of the irreducible representations of $C(G_q)$ [KRS95].

By construction the operators generated by the torus $U(1)$ in $U_q(sl_2)$ is mapped to the subalgebra generated by $K_i$ in $U_q(g)$.

Let $a_i$ denote the $i$-th column vector $(a_{j,i})_{j \in \Pi}$ of the Cartan matrix of $g$. It defines a homomorphism $a_i : T \to U(1)$. By the relations [1] and $q^{a_{i,j}} = q^{a_{j,i}}$, the homomorphism $\sigma_i$ intertwines the adjoint action of $T$ on $C(G_q)$ and the one $\tau_{a_i} : T$ on $C(SU_q(2))$ by

$$
\pi_w(Ad_{a_i}f) = \big((\tau_{a_{i_1}(w)} \circ \pi_{i_1}(w)) \otimes \cdots \otimes (\tau_{a_{i_k}(w)} \circ \pi_{i_k}(w))\big) \Delta_q^{(k-1)}(f).
$$
As in Section 2.3, let $S$ be a subset of $\Pi$ and $L$ be a subgroup of $P^c(S)$. Let $W_S$ denote the set of elements $w \in W$ satisfying $w(\alpha_i) \in P$ for all $\alpha_i \in S$. For each $0 \leq m \leq m_0$, put
\[ J_m^q = \bigcap \{ \ker \pi_w \cap C(G_q/K^{S,L}) \mid w \in W^S, l(w) = m, t \in T \}. \]
It is a $T \times (T/T_L)$-invariant bilateral ideal of $C(G_q/K^{S,L})$. By [NTT11b, Theorem 3.1], we obtain a composition sequence
\begin{equation}
0 = J^q_{m_0} \subset J^q_{m_0-1} \subset \cdots \subset J^q_0 \subset J^q_{-1} = C(G_q/K^{S,L}),
\end{equation}
satisfying
\begin{equation}
J^q_{m-1}/J^q_m \cong \bigoplus_{w \in W^S, \ell(w) = m} C(T/T_L) \otimes C_0(D_{q_{1_{t(w)}}}) \otimes \cdots \otimes C_0(D_{q_{m(w)}}),
\end{equation}
given by the surjective map $a \mapsto (\pi_{w,t}(a))_w$ from $J^q_{m-1}$ to the right hand side.

**Remark 6.** By the definitions of $\pi_{w,t}$ and the isomorphism (13), the action of $T$ on $J_{m-1}/J_m$ induced by the adjoint action corresponds to the product of the trivial action on $C(T/T_L)$ and (11) on the rest of the tensor components. The right translation action of $T/T_L$ induces the translation action on $C(T/T_L)$ and the trivial action on the rest.

At $q = 1$, we have the embeddings $\gamma_i : SU(2) \to G$ associated to $i \in \Pi$. It induces the embeddings of discs
\[ \gamma_w : D^k \to G, (g_1, \ldots, g_k) \mapsto \gamma_{i_1(w)}(g_1) \cdots \gamma_{i_k(w)}(g_k). \]
The adjoint action of $T$ on $\text{Img}(\gamma_w)$ is given by the following action determined by the maps $a_{i_1(w)} \cdots a_{i_k(w)}$ [Lm99, Proposition 5] which is linear with respect to the coordinate given by $\gamma_w$:
\begin{equation}
\gamma_w((a_{i_1(w)} g_1, \ldots, a_{i_k(w)} g_k) = t_{\gamma_w}(g_1, \ldots, g_k).
\end{equation}
Thus, we may interpret (11) as an analogue of this action in the quantum setting.

For each $w \in W$, we consider a continuous $T$-$C(I)$-algebra
\[ A_w = \Gamma_I(C_0(D_{q_{i_1(w)}})) \otimes \cdots \otimes \Gamma_I(C_0(D_{q_{i_k(w)}})) \]
whose fiber is given by
\[ A_w^{(q)} = C_0(D_{q_{i_1(w)}}) \otimes \cdots \otimes C_0(D_{q_{i_k(w)}}) = \text{Img}(\pi_w), \]
endowed with the actions of $T$ given by (11) for $q < 1$ and by (14) at $q = 1$. Note that $A_w$ is isomorphic to $A_{t(w)}$.

**Proposition 7.** Let $w \in W$ and $0 < a \leq q_0 \leq b \leq 1$. Then the evaluation map $A_w \to A_w^{(q)}$ induces a $KK^T$-equivalence.

**Proof.** The $T$-$C(I)$-algebra $A_w$ is given by (9) with respect to the integer sequence
\[ \left( \frac{\alpha_{i_1(w)}, \alpha_{i_1(w)}}{2}, \ldots, \frac{\alpha_{i_k(w)}, \alpha_{i_k(w)}}{2} \right). \]
The action of $T$ is induced by the homomorphism $T \to U(1)^k$ determined by the vectors $\alpha_{i_j(w)}$. Hence we obtain the assertion by Proposition 5. \qed

Next we consider the continuous field of $C^*$-algebras $(C(G_q/K^{S,L}))_{q \in (0,1]}$ with fiber $C(G_q/K^{S,L})$ at $q$, constructed in [NTT11b]. This is a field of $T \times (T/T_L)$-algebras coming from the fiberwise left and right translations of $T$.

Regarding this field, we obtain the following $T \times (T/T_L)$-equivariant $KK$-equivalence for the evaluation maps. The equivalence without the action of $T \times (T/T_L)$ was already proved in [NTT11b Theorem 6.1].
Theorem 1. Let $0 < a \leq q_0 \leq b \leq 1$. Then the evaluation map
\[ \text{ev}_{q_0} : \Gamma_{[a,b]}(C(G_q/K_q^{S,L})) \to C(G_{q_0}/K_{q_0}^{S,L}) \]
induces a $KK^{T \times (T/T_L)}$-equivalence.

Proof. The ideals $J_m^{(q)}$ for $q \in (0,1]$ form a continuous field of $T \times (T/T_L)$-algebras. We argue by induction that the evaluation map $\Gamma_{[a,b]}(J_m^{(q)}) \to J_m^{(q_0)}$ is a $KK^{T \times (T/T_L)}$-equivalence. We also note that the $T \times (T/T_L)$-equivalence for the left and right translations is the same as the equivariance for the adjoint action and the right translation.

The case for $m = m_0$ trivially holds as $J_m^{(q_0)} = 0$. Now, suppose that ev$_{q_0}$ is a $KK^T$-equivalence for the bundle $(J_m^{(q)})_{q \in [a,b]}$. Then one has the equivariant exact sequence
\[ 0 \to \Gamma_{[a,b]}(J_m^{(q)}) \to \Gamma_{[a,b]}(J_m^{(q_0)}) \to \bigoplus_{w \in H^2(L/T)} C(T/T_L) \otimes \Gamma_{[a,b]}(A_w^{(q)}) \to 0 \]
by (13). This extension is compatible with the evaluation homomorphism ev$_{q_0}$ of each field.

By Proposition 4 and Remark 6, ev$_{q_0}$ is a $KK^{T \times (T/T_L)}$-equivalence for the quotient of (15). By the induction hypothesis for the middle column and the fact that $KK^{T \times (T/T_L)}$ is a triangulated category implies that the evaluation map for the bundle $(J_m^{(q)})_{q \in [a,b]}$ is also a $KK^{T \times (T/T_L)}$-equivalence. □

Remark. S. Neshveyev pointed out the following partial strengthening of the above result. Let $H$ be a locally compact group. We say that a continuous field of $H$-algebras $A = (A_q)_{q \in [a,b]}$ on an interval $[a,b]$ is an $\mathcal{R}KK^H$-fibration [ENOO09] if $A$ is $\mathcal{R}KK^H$-equivalent to the constant field with fiber $A_q$ for any $q$. Then the field $(C(G_q/K_q^{S,L}))_q$ is an $\mathcal{R}KK^{T/T_L}$-fibration. Indeed, Proposition 4 and [ENOO09] Corollary 1.6 implies that $\Gamma_{[a,b]}(A_w^{(q)})$ is an $\mathcal{R}KK$-fibration. Hence the quotient field in (15) is an $\mathcal{R}KK^{T/T_L}$-fibration. Since the equivariant $\mathcal{R}KK$-category is triangulated, we may use the induction on the ideals $\Gamma_{[a,b]}(J_m^{(q)})$.

4. Applications

4.1. Ring structure of $K_*(C(G_q))$. The group $K_*(C(G_q))$ is naturally isomorphic to $KK^{G_q}(\mathbb{C}, \mathbb{C})$ by the Green–Julg theorem [Ver02 Théorème 5.10]. Hence the Kasparov product on $KK^{G_q}(\mathbb{C}, \mathbb{C})$ defines a structure of ring on $K_*(C(G_q))$. In this section we shall analyze this ring.

Proposition 9. The algebras $C(G_q) \otimes_{G_q} C(G_q)$ for $0 < q \leq 1$ are mutually $KK^{T^4}$-equivalent.

Proof. Via the isomorphism (2), the right translation of $T$ on the first copy of $C(G_q)$ of the braided tensor product becomes $\rho \otimes \lambda$ on $C(G_q) \otimes C(G_q)$. Similarly, the left translation on the first copy and the adjoint action on the second of the braided tensor product becomes the direct product action $\lambda \otimes \rho$ on the usual tensor product.

Now, fix $0 < a < q_0 < b \leq 1$ as before. The tensor product $\Gamma_{[a,b]}(C(G_q)) \otimes \Gamma_{[a,b]}(C(G_q))$ is a continuous field of $C^*$-algebras over the square $[a,b]^2$. By Theorem 1 the evaluation map of this field at $(q_0,q_0)$ is a $KK^{T^4}$-equivalence. □

Corollary 10. The algebras $C(G_q/T) \otimes_{G_q} C(G_q)$ for $0 < q \leq 1$ are mutually $KK^{T^3}$-equivalent.
Proof. We consider the crossed product of $C(G_q) \rtimes C(G_q)$ by $T$ acting on the first copy of $C(G_q)$ by the right translation. By Proposition 3, we have the $KK$-equivalence between the crossed products which is in addition equivariant with respect to the other actions of $T$.

It remains to show that the crossed product is strongly Morita equivalent to the fixed point algebra in an equivariant way. It is enough to show that the right translation action is saturated [Phi87, Chapter 7].

Since the action of $T$ can be identified with $\lambda \otimes \rho$ on $C(G_q) \otimes C(G_q)$, it is enough to prove that the ideals $J_{m}^{(q)}$ are saturated with respect to both $\lambda$ and $\rho$. Since this property is closed under extension [Phi87, Proposition 7.1.10], the problem is reduced to the algebras of the form [13].

The right action is indeed saturated as it can be identified with the translation action on $C(T)$ and the trivial action on $C_0(\mathbb{R})^\mathbb{R}$. The left action is the product of the translation on $C(T)$ and the one on $C_0(\mathbb{R})^\mathbb{R}$ (see Remark 4). By [Phi87, Proposition 7.2.6], this one is also saturated.

**Theorem 2.** Let $R_q = R(G_q)$ be the ring $K_*(C(G_q))$ whose product is given by the Kasparov product on $KK^{G_q}(\mathbb{C}, \mathbb{C})$. Then $R_q$ is isomorphic to $R_1$ for any $q \in (0, 1]$.

Proof. By the Baaj–Skandalis duality [BS93], we have the natural isomorphism

$$KK^G_q(\mathbb{C}, \mathbb{C}) \simeq KK^{G_q}(C(G_q), C(G_q)).$$

Note that we have the adjunction

$$KK^G_q(A, Ind^G_q B) \simeq KK^T_q(Res^G_q A, B)$$

between the restriction functor $Res^G_q: G_q\text{-}alg \to T\text{-}alg$ and the induction functor $Ind^G_q: T\text{-}alg \to G_q\text{-}alg$. Combining this with the identification $C(G_q) = Ind^G_q C(T)$, we arrive at the isomorphism

$$K_*(C(G_q)) \simeq KK^T_q(C(G_q), C(T)).$$

The Kasparov product on $KK^G_q(\mathbb{C}, \mathbb{C})$ can be translated into the following operation on the right hand side of the above formula. The composition of $Res^G_q$ and $Ind^G_q$ gives a map

$$KK^T_q(C(G_q), C(T)) \to KK^T_q(Ind^G_q Res^G_q C(G_q), C(G_q)).$$

In order to make the notation lighter we omit $Res^G_q$ in the rest of the proof. Using the Kasparov product

$$KK^T_q(Ind^G_q C(G_q), C(G_q)) \times KK^T_q(C(G_q), C(T)) \to KK^T_q(Ind^G_q C(G_q), C(T)),$$

we obtain a map

$$KK^T_q(C(G_q), C(T)) \times KK^T_q(C(G_q), C(T)) \to KK^T_q(Ind^G_q C(G_q), C(T)).$$

The homomorphism

$$(16) \quad C(G_q) \to Ind^G_q C(G_q)$$

for the adjunction of $Ind^G_q$ and $Res^G_q$ induces a map

$$KK^T_q(Ind^G_q C(G_q), C(T)) \to KK^T_q(C(G_q), C(T)).$$

From this we obtain the desired composition map

$$KK^T_q(C(G_q), C(T)) \times KK^T_q(C(G_q), C(T)) \to KK^T_q(C(G_q), C(T)).$$

Under the identification

$$Ind^G_q C(G_q) \simeq C(G_q) \rtimes_{G_q} C(G_q),$$
the homomorphism \([\mathbb{Z}]\) can be identified to the embedding of \(C(G_q)\) into the second component of \(C(G_q/T)\#C(G_q)\).

By Theorem 1 the algebras \(C(G_q)\) for different values of \(q\) are mutually \(KK^T\)-equivalent. Corollary 10 implies the analogous equivalence for \(C(G_q/T)\#C(G_q)\). It follows that the structure maps used to define the product on \(KK^T(C(G_q), C(T))\) can be identified for the different values of \(q\).

\(\square\)

Remark 11. The ring \(R_1\) is the classical topological \(K\)-group of the compact topological space \(G\) endowed with the ring structure induced by the tensor product of vector bundles. It follows that \(K_*(C(G_q))\) is isomorphic to the exterior algebra on generated by the classes of \(K_1(C(G_q))\) given by the corepresentations for the fundamental weights \([\text{Hod67}, \text{Ati65}]\).

Remark 12. The algebras \(C(G_q)\) for different values of \(0 < q \leq 1\) have been already known to be \(KK\)-equivalent to each other [NT11a]. Nevertheless, the \(KK\)-equivalence need not respect the ring structure of the \(K\)-groups for locally compact spaces. For example, the 2-sphere \(S^2\) and the two-point set \(\{a, b\}\) are \(KK\)-equivalent but their \(K\)-groups have the different ring structures.

4.2. The Borsuk–Ulam theorem for the quantum spheres. Let \(C_2\) denote the cyclic group of order 2. We consider the \(n\)-sphere \(S^n\) as a \(C_2\)-space with respect to the antipodal map \(x \mapsto -x\), where we consider \(S^n\) as the unit sphere of \(\mathbb{R}^{n+1}\).

The Borsuk–Ulam theorem states that there is no \(C_2\)-equivariant continuous map from \(S^{n+1}\) to \(S^n\). Since Borsuk’s initial proof, there have been numerous alternative proofs and generalizations of the problem. See [Ste85, Mat03] for an overview.

In this section we show that the odd-dimensional quantum spheres \(S^{2n+1}_q\) of Example 3 and the even-dimensional ones \(S^{2n}_q\) introduced by Hong–Szymański [HS02] satisfy an analogous statement, as conjectured by Baum–Hajac [BH].

We briefly review the quantum spheres. The algebra \(C(S^{2n-1}_q)\) is generated by elements \(z_1, \ldots, z_n\) satisfying

\[
\begin{align*}
  z_j z_i &= q z_i z_j \quad (i < j), \\
  z_j^* z_i &= q z_i z_j^* \quad (i \neq j), \\
  z_i^* z_i &= z_i z_i^* + (1 - q^2) \sum_{j \neq i} z_j z_j^* \quad \sum_{i=1}^n z_i z_i^* &= 1.
\end{align*}
\]

The algebra \(C(S^{2n}_q)\) of the even dimensional quantum sphere is defined as the quotient of \(C(S^{2n+1}_q)\) by the ideal generated by \(z_{n+1} - z_n\) [HS02, Section 5].

There is also a surjective homomorphism from \(C(S^{2n}_q)\) to \(C(S^{2n+1}_q)\). These surjections \(C(S^{2n+1}_q) \to C(S^{2n}_q)\), denoted by \(\iota^\#\), correspond to the embedding of \(S^n\) as an equator in \(S^{2n+1}\).

The map \(z_i \mapsto -z_i\) for \(i = 1, \ldots, n+1\) defines an action of \(C_2\) on \(C(S^{2n+1}_q)\), which descends to \(C(S^{2n}_q)\). The homomorphisms \(\iota^\#\) are \(C_2\)-equivariant.

The \(K\)-groups of the quantum spheres are computed as \([\text{VS90}, \text{HS02}]\)

\[
K_*(C(S^{2n-1}_q)) = \mathbb{Z} \oplus \mathbb{Z}, \quad K_*(C(S^{2n}_q)) = \mathbb{Z}^2 \oplus 0.
\]

Thus, these invariants are isomorphic to the classical case.

The action of \(T \times (T/T_L) \cong U(1)^n\) on \(C(S^{2n-1}_q)\) is given by

\[
(\lambda_1, \ldots, \lambda_n)(z_1, \ldots, z_n) = (\lambda_n \lambda_1 z_1, \lambda_n \lambda_1 \lambda_2 z_2, \lambda_n \lambda_2 \lambda_3 z_3, \ldots, \lambda_n \lambda_{n-1} z_n).
\]

Hence it is separate gauge action up to a finite cover.

**Proposition 13.** Let \(n\) be any integer greater than 1. The algebra \(C(S^{2n-1}_q)\) of odd quantum sphere is \(KK^{U(1)^n}\)-equivalent to \(C(S^{2n-1})\).
Proof. This follows from Theorem [1] applied to the Poisson–Lie subgroup SU(n−1) of SU(n) (see Example [3]). □

Definition 14. Let A be a C∗-algebra whose K-groups have finite rank over \( \mathbb{Z} \). The Lefschetz number of an element \( x \) in \( KK_0(A, A) \) is the alternating trace of the map induced by \( x \) on the graded vector space \( K_*(A) \otimes \mathbb{Q} \).

Proposition 15. Let \( x \) be any element of \( KK^C_0(C(S^{2n-1}), C(S^{2n-1})) \). Then the Lefschetz number of \( x \) is an even integer.

Proof. By Kasparov’s Poincaré duality, one has a natural isomorphism

\[
KK^C_0(C(S^{2n-1}), C(S^{2n-1})) \cong K^1((S^{2n-1} \times S^{2n-1})/C_2).
\]

The technique is nothing new, but we include the argument for the reader’s convenience.

When \( X \) is a Riemannian manifold, let \( \Gamma_0\text{Cl}(X) \) denote the \( \mathbb{Z}/2\mathbb{Z} \)-graded C∗-algebra of the continuous sections of the Clifford bundle over \( X \). When \( X \) is compact we shall write \( \Gamma\text{Cl}(X) \) instead. The minimal graded tensor product of graded algebras \( A \) and \( B \) are denoted by \( A \widehat{\otimes} B \). As usual, ordinary C∗-algebras are regarded as graded algebras with the trivial grading and the graded tensor product reduces to the (usual) minimal tensor product for such algebras.

By the Poincaré duality \([Kas88, Theorem 4.10]\), we have

\[
KK^C_0(C(S^{2n-1}), C(S^{2n-1})) \cong K^1((S^{2n-1} \times S^{2n-1})/C_2).
\]

Consider \( \Gamma_0\text{Cl}(\mathbb{R}) \) as a \( C_2 \)-algebra with the trivial action. Then, by the Bott periodicity \([Kas80, Theorem 7]\), we have a \( KK^C_2 \)-equivalence of graded C∗-algebras

\[
\Gamma\text{Cl}(S^{2n-1}) \cong \Gamma\text{Cl}(S^{2n-1}) \otimes \Gamma_0\text{Cl}(\mathbb{R}).
\]

Let \( U \) denote \( \mathbb{R}^{2n} \setminus \{0\} \) endowed with the action \( x \mapsto -x \) of \( C_2 \). The algebra of the right hand side in the above formula can be identified with \( \Gamma_0\text{Cl}(U) \). Since the tangent bundle of \( U \) is trivial, this algebra can be written as \( C_0(U) \otimes \text{Cl}(\mathbb{R}^{2n}) \) with respect to the Clifford algebra \( \text{Cl}(\mathbb{R}^{2n}) \cong M_{2^n}(\mathbb{C}) \) of the Euclidean vector space \( \mathbb{R}^{2n} \). The action of \( C_2 \) on \( \text{Cl}(\mathbb{R}^{2n}) \otimes \text{Cl}(\mathbb{R}^{2n}) \) is cocycle conjugate to the product of the antipodal action on \( C_0(U) \) and the trivial one on \( \text{Cl}(\mathbb{R}^{2n}) \). By the \( KK \)-equivalence \( C \cong \text{Cl}(\mathbb{R}^{2n}) \) as graded C∗-algebras, we obtain

\[
K^C_0(C(S^{2n-1}) \otimes \Gamma_0\text{Cl}(U)) \cong K^C_0(C(S^{2n-1}) \otimes C_0(U)).
\]

Since the action of \( C_2 \) is free, the right hand side is indeed isomorphic to \( K^1((S^{2n-1} \times S^{2n-1})/C_2) \).

Arguing as above, we obtain the Poincaré duality isomorphism

\[
KK_0(C(S^{2n-1}), C(S^{2n-1})) \cong K^1(S^{2n-1} \times S^{2n-1})
\]

for the nonequivariant case. Then we may identify the forgetful homomorphism

\[
KK^C_0(C(S^{2n-1}), C(S^{2n-1})) \to KK_0(C(S^{2n-1}), C(S^{2n-1}))
\]

with the homomorphism of the topological K-groups

\[
K^1((S^{2n-1} \times S^{2n-1})/C_2) \to K^1(S^{2n-1} \times S^{2n-1})
\]

induced by the natural projection map \( S^{2n-1} \times S^{2n-1} \to (S^{2n-1} \times S^{2n-1})/C_2 \).

The space \( (S^{2n-1} \times S^{2n-1})/C_2 \) is the total space of a sphere bundle over \( \mathbb{R}P^{2n-1} \) with fiber \( S^{2n-1} \). The diagonal map induces a section of this bundle. The complement of the image of this section is homeomorphic to \( \mathbb{R}^{2n-1} \times \mathbb{R}P^{2n-1} \). Hence we
obtain the associated split (six-term) exact sequence

\[ K^1(\mathbb{R}^{2n-1} \times \mathbb{R}P^{2n-1}) \xrightarrow{\iota_1} K^1((S^{2n-1} \times S^{2n-1})/C_2) \xrightarrow{\pi_2^*} K^1(\mathbb{R}P^{2n-1}). \]

Since \( K^1(S^{2n-1} \times S^{2n-1}) \) is torsion-free, we only need to consider the image of the (dual) fundamental class \( \pi_2^* [\mathbb{R}P^{2n-1}] \) of the base and the one \( \iota_1([S^{2n-1}] \otimes [1]) \) of the fiber.

Under the map \((17)\), the class \( \pi_2^* [\mathbb{R}P^{2n-1}] \) is mapped to \([2][1] \otimes [S^{2n-1}]\) while \( \iota_1([S^{2n-1}] \otimes [1]) \) is to \([S^{2n-1}] \otimes [1] + [1] \otimes [S^{2n-1}]\). Since \([S^{2n-1}] \otimes [1]\) (resp. \([1] \otimes [S^{2n-1}]\)) corresponds to the projection onto \( K^0(S^{2n+1}) \) (resp. the projection onto \( K^1(S^{2n+1})\)), we obtain the assertion. \(\square\)

**Theorem 3.** Let \( n \) be any positive integer. For any value of \( 0 < q \leq 1 \), there is no \( KK^{C_2} \)-morphism from \( C(S^q_n) \) to \( C(S^q_{n+1}) \) which induces a unital homomorphism on the \( K_0 \)-groups.

**Proof.** The antipodal map on \( C(S^q_{2k-1}) \) can be realized as the action of \(-1 \in U(1)\) with respect to the right translation action on \( C(SU_q(k)/SU_q(k-1))\). By Proposition \(13\) we have a \( KK^{C_2} \)-equivalence between \( C(S^q_{2k-1}) \) and \( C(S^{2k-1})\). By Proposition \(15\) the Lefschetz number of any \( x \in KK^{C_2}(C(S^q_{2k-1}), C(S^{2k-1})) \) must be an even integer.

Now, suppose that \( n \) is odd, and that \( \phi \) is a \( C_2 \)-equivariant \( KK^{C_2} \)-morphism from \( C(S^q_n) \) to \( C(S^q_{n+1}) \) which is unital on \( K_0 \). Then the composition \( \phi \circ [i^{\#}] \) is a \( C_2 \)-equivariant morphism from \( C(S^q_n) \) to itself. The map on \( K_*(C(S^q_n)) \) induced by \( \phi \circ [i^{\#}] \) factors through \( K_*(C(S^q_{n+1})) \), which is trivial in the odd part. Hence the graded trace of \( \phi \circ [i^{\#}] \) on \( K_*(C(S^q_n)) \) becomes equal to 1, which is a contradiction.

Next, suppose that \( n \) is even, and let \( \phi \) be a \( C_2 \)-equivariant \( KK^{C_2} \)-morphism from \( C(S^q_n) \) to \( C(S^q_{n+1}) \) which is unital on \( K_0 \). Then the composition \( [i^{\#}] \otimes \phi \) is a \( C_2 \)-equivariant morphism on \( C(S^q_{n+1}) \). Analogously to the above argument, we obtain a contradiction in this case as well. \(\square\)

**Corollary 16.** Let \( m \) and \( n \) be distinct positive integers. Then \( C(S^m_q) \) and \( C(S^n_q) \) are not \( KK^{C_2} \)-equivalent.

**Remark 17.** Proposition \(15\) and the subsequent proof of the Borsuk–Ulam theorem was motivated by a result of Casson and Gottlieb [CG77, Theorem 4]. We ignored the problem of proving analogous statements for other compact topological spaces with free action of \( C_2 \). Actually, their theorem implies that there is no \( C_2 \)-equivariant continuous map from the suspension \( SX \) to \( X \) when \( X \) is a compact free \( C_2 \)-space of finite covering dimension. It would be interesting to know to which extent Proposition \(15\) can be generalized.

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