Review of Open Superstring Field Theory

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Abstract

I review the construction of an action for open superstring field theory which does not suffer from the contact term problems of other approaches. This action resembles a Wess-Zumino-Witten action and can be constructed in a manifestly D=4 super-Poincaré covariant manner. This review is based on lectures given at the ICTP Latin-American String School in Mexico City and the Komaba 2000 Workshop in Tokyo.

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1 Problems with Conventional Approach

The construction of a field theory action for the superstring is an important problem since it may lead to information about non-perturbative superstring theory which is unobtainable from the on-shell perturbative S-matrix. This information might be useful for understanding the non-perturbative dualities of the superstring. Although there was much activity ten years ago concerning a field theory action for the bosonic string, there was not much progress on constructing a field theory action for the superstring.

After discussing the problems with conventional approaches to superstring field theory in section 1, a Wess-Zumino-Witten-like action will be constructed for open Neveu-Schwarz string field theory in section 2. In section 3, this action will be generalized to any open string with critical N=2 superconformal invariance, and section 4 will review an open superstring field theory action with manifest four-dimensional super-Poincaré covariance which includes all sectors of the superstring.

The covariant string field theory action for the bosonic string is based on a BRST operator $Q$ and a string field $V$ of +1 ghost-number. In Witten’s approach to open string field theory, the gauge-invariant action is

$$S = \frac{1}{\lambda^2} Tr \left( \frac{1}{2} V Q V + \frac{1}{3} V^3 \right)$$

where string fields are glued together at their midpoint.

In generalizing this approach to superstring field theory, the main difficulty comes from the requirement that the string field carries a definite “picture”. Recall that each physical state of the superstring is represented by an infinite number of BRST-invariant vertex operators in the covariant RNS formalism. To remove this infinite degeneracy, one needs to require that the vertex operator carries a definite picture, identifying which modes of the $(\beta, \gamma)$ ghosts annihilate the vertex operator. For open superstring fields, the most common choice is that all Neveu-Schwarz (NS) string fields carry picture $-1$ and all Ramond (R) string fields carry picture $-\frac{1}{2}$.

Since the total picture must equal $-2$ for open superstrings, the obvious generalization of the action is

$$S = \frac{1}{\lambda^2} Tr \left( \frac{1}{2} V_{NS} Q V_{NS} + \frac{1}{2} V_{R} Q Y V_{R} + \frac{1}{3} Z V_{NS}^3 + \frac{1}{2} V_{NS} V_{R} V_{R} \right)$$

(2)
where the \((\beta, \gamma)\) ghosts are fermionized as
\[
\beta = e^{-\phi} \partial \xi \quad \text{and} \quad \gamma = \eta e^{\phi},
\]
\(Z = \{Q, \xi\}\) is the picture-raising operator of picture +1,
\(Y = \mu \partial \xi e^{2\phi}\) is the picture-lowering operator of picture −1,
and these picture-changing operators are inserted at the midpoint of the interacting strings. However, as shown by Wendt [4], the action of (2) is not gauge-invariant because of the contact-term divergences occurring when two \(Z\)'s collide. One way to make the action gauge-invariant would be to introduce contact terms to cancel the divergences coming from colliding \(Z\)'s. However, the coefficients of these contact terms would have to be infinite in the classical action since the divergences are present already in tree-level amplitudes. Note that infinite contact terms are also expected in light-cone superstring field theory (either in the RNS or Green-Schwarz formalisms) to cancel the divergences when interaction points collide [5].

Although one can choose other pictures for the string field \(V\) which change the relative factors of \(Z\) and \(Y\) [6] [7], there is no choice for which the action is cubic and gauge-invariant [8]. For example, choosing \(V_{NS}\) in the zero picture leads to the action [9]
\begin{equation}
S = \frac{1}{\lambda^2} Tr \left( \frac{1}{2} V_{NS} Y^2 Q V_{NS} + \frac{1}{2} V_{R} Q Y V_{R} + \frac{1}{3} Y^2 V^3_{NS} + \frac{1}{2} Y V_{NS} V_{R} V_{R} \right). \tag{3}
\end{equation}

The kinetic term of (3) implies that \(Y^2 Q V_{NS} = 0\) is the linearized equation of motion for the Neveu-Schwarz field. But since \(Y^2\) has a non-trivial kernel, this equation of motion has additional solutions which are not in the cohomology of \(Q\). Although one could restrict \(V_{NS}\) to only include states not in the kernel of \(Y^2\), such a projection would break gauge-invariance since Witten’s midpoint gluing prescription does not preserve this projection of the string field. Modifying the gluing prescription to preserve the projection would ruin its associativity properties. Note that a similar problem [14] exists for the Ramond kinetic term in the action of (2).

\footnote{This action was recently used to compute the tachyon potential in NS string theory. However, besides the gauge invariance problems mentioned here, there appears to be an error in their computation of the D-brane tension by a factor of \(\sqrt{2}\). When written in terms of the closed string coupling constant, the tension is background-dependent and picks up a factor of \(\sqrt{2}\) if the D-brane is non-BPS. But when written in terms of the open string coupling constant, the tension is background independent and does not pick up a factor of \(\sqrt{2}\) [10] [11] [12] [13].}
2 Open Neveu-Schwarz String Field Theory

In this section, it will be shown how to construct a ten-dimensional Lorentz-covariant action for the Neveu-Schwarz sector of open superstring field theory. It is not yet known how to extend this action to the Ramond sector in a ten-dimensional Lorentz-covariant manner. However, as will be shown in section 4, it can be generalized to a four-dimensional super-Poincaré covariant action which includes both the NS and R sectors.

To construct a field theory action, one first needs to define a NS string field. The first attempts to construct an action used the fermionic string field

$$V = c e^{-\phi} \psi^\mu A_\mu(x) + ...$$

where $V$ was constrained to carry +1 ghost number and −1 picture, and to satisfy $\eta_0 V = 0$, i.e. to be independent of the $\xi_0$ mode. Although quadratic actions were successfully constructed using $V$, the cubic interaction term had problems due to the necessity of introducing the picture-raising operator $Z$.

The solution to this problem is to express $V$ and $ZV$ in terms of a more fundamental NS string field $\Phi$ which is bosonic and carries zero ghost-number and zero picture. If one defines $\Phi = \xi_0 V$, i.e.

$$\Phi = \xi c e^{-\phi} \psi^\mu A_\mu(x) + ...,$$

then $V = \eta_0 \Phi$ and $ZV = Q \Phi$. However, as will be shown below, $\Phi = \xi_0 V$ is a specific gauge choice for $\Phi$, and one needs a more gauge-invariant description for $\Phi$ in order to construct an NS string field theory action without contact term problems.

The quadratic action for $\Phi$ will be defined as

$$S = \frac{1}{2\lambda^2} Tr \langle \Phi Q \eta_0 \Phi \rangle.$$

Because of the $\xi$ zero mode, the non-vanishing inner product will be defined as in the “large” Hilbert space of $\mathbb{Z}_2$, i.e.

$$\langle \xi c \partial c \partial^2 c e^{-2\phi} \rangle = 1.$$

\footnote{The ghost-number operator will be defined as $f(c b + \eta \xi)$ so that $(\eta, \xi)$ carries ghost-number $(+1, -1)$ and $e^{i\phi}$ carries zero ghost number.}
Since \( Q^2 = \eta_0^2 = \{Q, \eta_0\} = 0 \), \( S \) is invariant under the linearized gauge transformation
\[
\delta \Phi = \eta_0 \tilde{\Lambda} + Q \Lambda
\]
where \( \tilde{\Lambda} \) and \( \Lambda \) are independent gauge parameters. Note that \( \Phi \) carries picture 0, \( \eta_0 \) carries picture \(-1\) and \( Q \) carries picture 0, so \( \tilde{\Lambda} \) and \( \Lambda \) must carry picture \(+1\) and 0 respectively.

Since \( \{\eta_0, \xi_0\} = 1 \), the \( \tilde{\Lambda} \) parameter can be used to gauge \( \Phi = \xi_0 V \) for some \( V \) annihilated by \( \eta_0 \). The equation of motion for \( S \) is
\[
Q \eta_0 \Phi = 0,
\]
which in this gauge implies that
\[
Q \eta_0 (\xi_0 V) = QV = 0.
\]
Furthermore, the remaining gauge parameter \( \Lambda \) generates the gauge transformation
\[
\delta V = \eta_0 (Q \Lambda) = Q \Omega
\]
if one chooses \( \Lambda = \xi_0 \Omega \). So one recovers the desired linearized equations of motion and gauge invariances for \( V \).

To include interactions, one needs to find an action which allows a non-linear generalization of the gauge invariances \( \delta \Phi = \eta_0 \tilde{\Lambda} + Q \Lambda \). This can be obtained by drawing an analogy with the two-dimensional Wess-Zumino-Witten (WZW) action [15]
\[
S_{WZW} = \frac{1}{2\lambda^2} Tr \int d^2 z ( (g^{-1} \partial g)(g^{-1} \bar{\partial} g) - \int_0^1 dt (\dot{g}^{-1} \partial_t \dot{g})(\dot{g}^{-1} \bar{\partial} \dot{g} , \dot{g}^{-1} \bar{\partial} \dot{g}) )
\]
where \( g(z, \bar{z}) \) is a group-valued two-dimensional field and \( \dot{g}(t, z, \bar{z}) \) is any continuous group-valued three-dimensional field defined on the three-volume with boundary at \( t = 0 \) and \( t = 1 \) such that \( \dot{g}(1, z, \bar{z}) = g(z, \bar{z}) \) and \( \dot{g}(0, z, \bar{z}) = 1 \). Recall that \( S_{WZW} \) is invariant under the gauge transformation
\[
g(z, \bar{z}) \rightarrow \tilde{\Omega}(\bar{z}) g(z, \bar{z}) + g(z, \bar{z}) \Omega(z)
\]
where \( \partial \tilde{\Omega}(\bar{z}) = \bar{\partial} \Omega(z) = 0 \). If one writes \( g = e^\Phi \) where \( \Phi \) is Lie-algebra valued, the gauge transformation on \( \Phi \) is \( \delta \Phi = \tilde{\Omega}(\bar{z}) + \Omega(z) + ... \) where ... depends on \( \Phi \). Furthermore, the WZW equation of motion \( \partial (g^{-1} \partial g) = 0 \) implies that \( \partial \bar{\partial} \Phi = ... \) where ... is non-linear in \( \Phi \).
This suggests writing the NS string field theory action as

\[ S = \frac{1}{2\lambda^2} Tr \langle (e^{-\Phi} Q e^\Phi)(e^{-\Phi} \eta_0 e^\Phi) - \int_0^1 dt (e^{-\Phi} \partial_t e^\Phi) \{ e^{-\Phi} Q e^\Phi, e^{-\Phi} \eta_0 e^\Phi \} \rangle \]  

(4)

where \( e^\Phi = 1 + \Phi + \frac{1}{2} \Phi^2 + ... \) is defined using the midpoint gluing prescription, \( \Phi \) is the NS string field discussed earlier, \( \hat{\Phi}(t = 0) = 0 \) and \( \hat{\Phi}(t = 1) = \Phi \).

One can show that \( S \) is invariant under the WZW-like gauge invariance

\[ \delta(e^\Phi) = e^\Phi(\eta_0 \Lambda) + (Q \Lambda)e^\Phi, \]

and that on-shell,

\[ \eta_0(e^{-\Phi} Q e^\Phi) = 0, \]

which generalize the linearized gauge invariance and equations of motion of the quadratic action.

To explicitly evaluate the action of (4), one first performs a Taylor expansion in \( \Phi \) to obtain

\[ S = \frac{1}{\lambda^2} Tr \left( \frac{1}{2} \Phi Q \eta_0 \Phi - \frac{1}{6} \Phi \{ Q \Phi, \eta_0 \Phi \} + ... \right) \]

where all string fields are multiplied together using the midpoint interaction. The cubic term can be evaluated in precisely the same manner as in the CS-like action by mapping three half circles for the external states into \( 2\pi/3 \) wedges in the complex plane using the map

\[ f_r^{(3)} = e^{\frac{2\pi i (r-1)}{3}} \left( \frac{1 - iz}{1 + iz} \right)^{\frac{2}{3}} \]

for \( r = 1 \) to 3. To define the order \( N \) term, one uses the functions

\[ f_r^{(N)} = e^{\frac{2\pi i (r-1)}{N}} \left( \frac{1 - iz}{1 + iz} \right)^{\frac{2}{N}} \]

for \( r = 1 \) to \( N \) to map \( N \) half circles into \( 2\pi/N \) wedges in the complex plane. It has been shown by explicit computation [13] that the four-point tree amplitude is correctly reproduced by the action of (4), and there are indirect arguments based on gauge invariance that all \( N \)-point tree amplitudes are correctly reproduced by this action. The action of (4) has also been used to compute the NS tachyon potential [14,15,16,17,18] and kink solutions [14,21] using the level truncation scheme of [22,23], and the results appear to agree with the predictions of Sen coming from \( D \)-brane analysis [10].
3 Open N=2 String Field Theory

Although the action of the previous section is manifestly Lorentz invariant, it is not clear how to generalize it to include the Ramond sector. At the moment, the only action which includes all sectors of the open superstring is based on a hybrid formalism of the superstring with \( \hat{c} = 2 \) N=2 superconformal invariance. In order to understand the relation of this hybrid action with that of the previous section, it will be useful to first recall the relation of the bosonic open string field theory action and Chern-Simons theory.

As shown by Witten in [24], the action for open bosonic string field theory, 
\[
S = \frac{1}{\lambda^2} \text{Tr} \left( \frac{1}{2} V Q V + \frac{1}{3} V^3 \right),
\]
(5)
can also be used to describe the topological string theory version of Chern-Simons. This Chern-Simons string theory is defined by a \( \hat{c} = 3 \) N = 2 superconformal field theory constructed from the worldsheet variables \( [x^j, \bar{x}_j, \psi_j, \bar{\psi}_j] \) for \( j = 1 \) to \( 3 \) with the twisted \( N = 2 \) generators:
\[
T = \partial x^j \partial \bar{x}_j + \bar{\psi}_j \partial \psi^j,
\]
\[
G^+ = \psi^j \partial \bar{x}_j, \quad G^- = \bar{\psi}_j \partial x^j,
\]
\[
J = \psi^j \bar{\psi}_j.
\]
If one identifies \( Q \) with \( \int G^+ \) and the ghost-number with U(1) charge, the action of (3) reproduces the Chern-Simons action
\[
S_{CS} = \frac{1}{\lambda^2} \text{Tr} \int d^3 x \epsilon^{jkl}[\frac{1}{2} A_j \partial_k A_l + \frac{1}{3} A_j A_k A_l].
\]
To obtain the Chern-Simons action from (3), one uses the normalization that \( \langle \psi^j \psi^k \psi^l \rangle = \epsilon^{jkl} \) and expands the ghost-number +1 string field \( V \) as
\[
V = A_j(x) \psi^j + \ldots.
\]
The terms \( \ldots \) involve derivatives on \( \psi \) and \( x \) and describe massive fields which do not propagate. One can easily generalize the above construction to any \( \hat{c} = 3 \) N=2 superconformal field theory and the corresponding action computes topological quantities in the \( N = 2 \) superconformal field theory [25].
The action of (4) in the previous section involves operators $Q$ and $\eta_0$, but $\eta_0$ has no analog in $\hat{c} = 3 \, N = 2$ superconformal field theory. However, as will now be shown, $Q$ and $\eta_0$ do have analogs in $\hat{c} = 2 \, N = 2$ superconformal field theory, i.e. in critical $N = 2$ strings. When $\hat{c} = 2$, the operators $J = \partial H$, $J^{++} = e^{H}$ and $J^{--} = e^{-H}$ generate an SU(2) affine Lie algebra. Commuting these SU(2) generators with the fermionic generators $G^+$ and $G^-$ generate two new fermionic generators defined as

$$\tilde{G}^+ = [\int e^{H}, G^-], \quad \tilde{G}^- = [\int e^{-H}, G^+].$$

These four fermionic generators combine with the SU(2) generators and the stress tensor to form a set of “small” $N=4$ superconformal generators. After twisting by $J$, $(G^+, \tilde{G}^+)$ carry conformal weight +1 and $(G^-, \tilde{G}^-)$ carry conformal weight +2. So if the U(1) charge is identified with ghost-number, one can identify $\int G^+$ with $Q$ and $\int \tilde{G}^+$ with $\eta_0$ [29].

There are three critical $N=2$ superconformal field theories which will be relevant here. The first is the self-dual string which describes self-dual Yang-Mills [27]. The worldsheet variables of the self-dual string consist of $[x^j, \bar{x}_j, \psi^j, \bar{\psi}_j]$ where $j = 1$ to 2. For the self-dual string, the $N=4$ generators after twisting are

$$T = \partial x^j \partial \bar{x}_j + \bar{\psi}_j \partial \psi^j, \quad G^+ = \psi^j \partial \bar{x}_j, \quad \tilde{G}^+ = \epsilon_{jk} \psi^j \partial x^k, \quad (6)$$

$$G^- = \bar{\psi}_j \partial x^j, \quad \tilde{G}^- = \epsilon^{jk} \bar{\psi}_j \partial \bar{x}_k,$$

$$J^{++} = \frac{1}{2} \epsilon_{jk} \psi^j \psi^k, \quad J = \psi^j \bar{\psi}_j, \quad J^{--} = \frac{1}{2} \epsilon^{jk} \bar{\psi}_j \bar{\psi}_k.$$

If one replaces $Q$ with $\int G^+$ and $\eta_0$ with $\int \tilde{G}^+$ of (3), the action of (4) reproduces the Donaldson-Nair-Schiff action [28][29] for self-dual Yang-Mills, in terms of the Yang field $\phi(x)$. To obtain this action from (4), one uses the normalization that $\langle \psi^j \psi^k \rangle = \epsilon^{jk}$ and expands the ghost-number zero string field $\Phi$ as

$$\Phi = \phi(x) + ...$$
where the terms ... involve derivatives on \( \psi \) and \( x \) and correspond to non-
propagating massive fields. Note that the equation of motion

\[
\int \tilde{G}^+ [e^{-\Phi} \left( \int G^+ \right) e^{\Phi}] = 0
\]

implies \( \tilde{\partial}^j (e^{-\phi} \partial_j e^{\phi}) = 0 \) which is Yang’s equation for self-dual Yang-Mills.

A second critical N=2 superconformal field theory is given by the N=2 embedding of the RNS superstring. As shown in [30], any critical N=1 superconformal field theory (such as the ten-dimensional superstring) can be described by a critical N=2 superconformal field theory. The worldsheet fields of this critical N=2 superconformal field theory are the usual RNS worldsheet variables \([x^\mu, \psi^\mu, b, c, \xi, \eta, \phi]\) for \( \mu = 0 \) to 9 and the twisted N=4 generators are defined by [31][30][26]

\[
T = T_{RNS}, \quad G^+ = j_{BRST}, \quad \tilde{G}^+ = \eta,
\]

\[
G^- = b, \quad \tilde{G}^- = \{Q, b\xi\} = -bZ + \xi T_{RNS},
\]

\[
J^{++} = c\eta, \quad J = bc + \xi \eta, \quad J^{--} = b\xi,
\]

where \( Q = \int j_{BRST} \) is the standard BRST charge of the N=1 superstring and
\( T_{RNS} \) is the sum of the stress tensors for the RNS matter and ghost variables. Since \( Q = \int G^+ \) and \( \eta_0 = \int \tilde{G}^+ \), this explains the relationship of the action in the previous section with the actions constructed in this section using the N=4 superconformal generators.

Finally, a third critical N=2 superconformal field theory is given by a hybrid version of the superstring which describes in \( D = 4 \) superspace the superstring compactified on a six-dimensional manifold [32]. As will be reviewed in the following section, the open superstring field theory action [33] constructed from this N=2 superconformal field theory is manifestly \( D = 4 \) super-Poincaré covariant and includes all sectors of the superstring.

### 4 Open Superstring Field Theory

For any compactification of the open superstring to four dimensions which preserves at least N=1 \( D = 4 \) spacetime supersymmetry, there exists a field
redefinition that maps the RNS worldsheet variables to a set of hybrid variables which include the four-dimensional Green-Schwarz variables \([x^m, \theta^\alpha, \bar{\theta}^\dot{\alpha}]\) for \(m = 0\) to \(3\) and \(\alpha, \dot{\alpha} = 1\) to \(2\). The complete set of hybrid variables is given by \([x^m, \theta^\alpha, \bar{\theta}^\dot{\alpha}, p_\alpha, \bar{p}_{\dot{\alpha}}, \rho]\) plus a twisted \(\mathbb{N}=2\) \(c=9\) superconformal field theory which describes the compactification manifold. \([p_\alpha, \bar{p}_{\dot{\alpha}}]\) are the conjugate momenta to the superspace variables \([\theta^\alpha, \bar{\theta}^\dot{\alpha}]\) satisfying the OPE’s \(p_\alpha(y)\theta^\beta(z) \to \delta^\beta_\alpha(y-z)^{-1}\) and \(\bar{p}_{\dot{\alpha}}(y)\bar{\theta}^\dot{\beta}(z) \to \delta^{\dot{\beta}}_{\dot{\alpha}}(y-z)^{-1}\), and \(\rho\) is a chiral boson with background charge \(-1\) satisfying the OPE \(\rho(y)\rho(z) \to -\log(y-z)\).

The field redefinition from RNS to hybrid variables maps the twisted \(\mathbb{N}=4\) superconformal generators of (7) to the manifestly \(D=4\) super-Poincaré covariant generators

\[
T = -\frac{1}{2} \partial x^m \partial x_m - p_\alpha \partial \theta^\alpha - \bar{p}_{\dot{\alpha}} \partial \bar{\theta}^\dot{\alpha} - \frac{1}{2} \partial \rho \partial \rho - \frac{1}{2} \partial^2 \rho + T_C,
\]

\[
G^+ = d^\alpha d_\alpha e^\rho + G^+_C, \quad G^+ = \left[ \int e^{-\rho + H_C}, G^+ \right], \quad (8)
\]

\[
G^- = \bar{d}^{\dot{\alpha}} \bar{d}_{\dot{\alpha}} e^{-\rho} + G^-_C, \quad G^- = \left[ \int e^{\rho - H_C}, G^- \right],
\]

\[
J^{++} = e^{-\rho + H_C}, \quad J = -\partial \rho + \partial H_C, \quad J^{--} = e^{\rho - H_C},
\]

where

\[
d_\alpha = p_\alpha + \frac{i}{2} \bar{\theta}^\dot{\alpha} \sigma^{m}_{\alpha a} \partial x_m - \frac{1}{4} (\bar{\theta})^2 \partial \theta_\alpha + \frac{1}{8} \theta_\alpha \partial (\bar{\theta})^2,
\]

\[
\bar{d}_{\dot{\alpha}} = \bar{p}_{\dot{\alpha}} + \frac{i}{2} \theta^\alpha \sigma^m_{\dot{\alpha} a} \partial x_m - \frac{1}{4} (\theta)^2 \partial \bar{\theta}_{\dot{\alpha}} + \frac{1}{8} \bar{\theta}_{\dot{\alpha}} \partial (\theta)^2,
\]

are spacetime supersymmetric combinations of the fermionic momenta and \([T_C, G^+_C, G^-_C, J_C = \partial H_C]\) are the twisted \(c=9\) \(\mathbb{N}=2\) superconformal generators representing the compactification.

In the RNS formalism, one needs to choose a picture for each off-shell state in the string field theory action. As discussed in [33], this choice is replaced in the hybrid formalism by restricting the \(\rho\) charge for any off-shell state to be \((-1, 0, +1)\). So the \(\mathbb{U}(1)\)-neutral off-shell string field \(\Phi\) can be written in an \(\mathbb{N}=1\) \(D=4\) super-Poincaré invariant manner as

\[
\Phi = \Phi_{-1} + \Phi_0 + \Phi_1
\]

where \(\Phi_n\) are string fields carrying \(n\) units of \(\rho\) charge and \(-n\) units of Calabi-Yau \(\mathbb{U}(1)\) charge. As will be seen later, the four-dimensional super-Yang-Mills
multiplet is contained in $\Phi_0$ and the Calabi-Yau chiral and anti-chiral moduli are contained in $\Phi_1$ and $\Phi_{-1}$.

Because of the different amounts of $\rho$ charge in $G^+$ and $\tilde{G}^+$, the linearized gauge invariance $\delta \Phi = \int G^+ \Lambda + \int \tilde{G}^+ \tilde{\Lambda}$ generalizes in the hybrid formalism to

$$
\delta \Phi_{-1} = G_{1+}^- \Lambda_{-2} + G_{0+}^- \Lambda_{-1} + \tilde{G}_{2+}^- \Lambda_0 + \tilde{G}_{1+}^- \Lambda_1,
$$

$$
\delta \Phi_0 = G_{1+}^0 \Lambda_{-1} + G_{0+}^0 \Lambda_0 + \tilde{G}_{2+}^0 \Lambda_1 + \tilde{G}_{1+}^0 \Lambda_2,
$$

$$
\delta \Phi_1 = G_{1+}^1 \Lambda_0 + G_{0+}^1 \Lambda_1 + \tilde{G}_{2+}^1 \Lambda_2 + \tilde{G}_{1+}^1 \Lambda_3
$$

where

$$
G_{1+}^1 = \int d^\alpha d^\beta e^\rho, \quad G_{0+}^1 = \int G_{\alpha\beta},
$$

$$
\tilde{G}_{1+}^- = \int [ \int e^{2\rho-HC} G_{\alpha\beta} ], \quad \tilde{G}_{2+}^- = \int e^{-2\rho-H\bar{C}} \bar{d}^\alpha \bar{d}^\beta,
$$

and $\Lambda_n$ are gauge parameters carrying $n$ units of $\rho$ charge and $-1-n$ units of Calabi-Yau charge. Note that the cohomology of $\tilde{G}_{2+}^-$ and $G_{1+}^+$ is trivial since $\tilde{G}_{2+}^- (\frac{1}{4} e^{2\rho-HC} \bar{\theta}^\alpha \bar{\theta}_\alpha) = 1$ and $G_{1+}^+ (\frac{1}{4} e^{-\rho \theta^\alpha \theta_\alpha}) = 1$. So the gauge transformations generated by $\Lambda_3$ and $\Lambda_{-2}$ allow one to gauge-fix

$$
\Phi_1 = (\frac{1}{4} e^{2\rho-HC} \bar{\theta}^\alpha \bar{\theta}_\alpha) \Omega_{-1}, \quad \Phi_{-1} = (\frac{1}{4} e^{-\rho \theta^\alpha \theta_\alpha}) \Omega_0
$$

where $\Omega_{-1} = \tilde{G}_{1+}^- \Phi_1, \quad \Omega_0 = G_{1+}^+ \Phi_{-1}$.

One can easily check that the linearized equations of motion

$$
\tilde{G}_{2+}^+ G_{1+}^+ \Phi_{-1} + \tilde{G}_{2+}^- G_{0+}^+ \Phi_0 + \tilde{G}_{2+}^- G_{1+}^+ \Phi_1 = 0,
$$

$$
(\tilde{G}_{2+}^- G_{1+}^+ + \tilde{G}_{2+}^- G_{2+}^+ ) \Phi_0 + \tilde{G}_{2+}^- G_{0+}^+ \Phi_1 + \tilde{G}_{2+}^- G_{1+}^+ \Phi_1 = 0,
$$

$$
G_{1+}^+ G_{0+}^+ \Phi_{-1} + G_{1+}^+ G_{1+}^+ \Phi_0 + G_{1+}^+ G_{2+}^+ \Phi_1 = 0,
$$

are invariant under the linearized gauge invariances of (9) and therefore generalize the $\int G^+ \int \tilde{G}^+ \Phi = 0$ equation in the hybrid formalism. It was shown in [33] that the cohomology of these equations of motion up to the gauge invariances of (9) correctly reproduces the RNS cohomology.

To find the non-linear version of these equations of motion and gauge invariances, it will be useful to see how (11) and (9) describe the massless
sector of the uncompactified superstring which corresponds to $D = 10$ super-Yang-Mills. When written in $\mathbb{N}=1$ $D = 4$ superspace, $D = 10$ super-Yang-Mills is described by the superfields

$$v(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}), \quad \omega^j(x^\mu, \theta^{\alpha}), \quad \bar{\omega}_j(x^\mu, \bar{\theta}^{\dot{\alpha}})$$

(12)

for $\mu = 0$ to $9$, $(\alpha, \dot{\alpha}) = 1$ to $2$, and $j = 1$ to $3$, where $\omega^j$ and $\bar{\omega}_j$ are chiral and anti-chiral superfields satisfying $\bar{\partial}^{\dot{\alpha}} \omega^j = \partial_\alpha \bar{\omega}_j = 0$. As usual, the $\mathbb{N}=1$ $D = 4$ supersymmetric derivatives are defined by $\partial_\alpha = \frac{\partial}{\partial \theta^\alpha} - i \frac{1}{2} \sigma^m \theta^\alpha \partial_m$ and $\bar{\partial}\dot{\alpha} = \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i \frac{1}{2} \bar{\sigma}_m \bar{\theta}^{\dot{\alpha}} \bar{\partial}_m$ where $m = 0$ to $3$. The $\theta^\alpha \sigma^m \partial_m$ component of $v$ describes the four-dimensional polarizations of the Yang-Mills gauge field while the $\theta$-independent components of $\omega^j$ and $\bar{\omega}_j$ describe the other six polarizations of the gauge field.

The $\mathbb{N}=1$ $D = 4$ superfields of (12) appear in the string field as

$$\Phi_0 = v(x, \theta, \bar{\theta}) + \ldots, \quad \Omega_{-1} = \frac{1}{2} \epsilon_{jkl} e^{-\rho} \psi^j \psi^k \omega^l(x, \theta) + \ldots, \quad \Omega_0 = \psi^j \bar{\omega}_j(x, \bar{\theta}) + \ldots,$$

where $\psi^j$ and $\bar{\psi}_j$ are the worldsheet fermions in the internal directions. Using $G_C^+ = \psi^j \bar{\partial}\bar{x}_j$, $G_C^- = \bar{\psi}_j \partial x^j$ and $J_C = \psi^j \bar{\psi}_j$ where $x^j = x_{3+j} + i x_{6+j}$ and $\bar{x}_j = x_{3+j} - i x_{6+j}$, one can check that the linearized equations of motion of (11) imply that

$$\bar{\partial}^2 \bar{\omega}_j + \partial_j \bar{D}^2 v + \epsilon_{jkl} \bar{\partial}^k \omega^l = 0,$$

(13)

$$(\partial_j \bar{\partial}^j + D^\alpha \bar{D}^2 D_\alpha) v + \partial_j \omega^j + \bar{\partial}^j \bar{\omega}_j = 0,$$

$$D^2 \omega^j + \bar{\partial}^j D^2 v + \epsilon^{jkl} \partial_k \bar{\omega}_l = 0,$$

which are the desired equations for these massless superfields. These equations are invariant under the linearized gauge transformations

$$\delta v = D^2 s + \bar{D}^2 \bar{s}, \quad \delta \omega^j = -\bar{\partial}^j \bar{D}^2 \bar{s}, \quad \delta \bar{\omega}_j = -\partial_j D^2 s,$$

(14)

which come from choosing $\Lambda_{-1} = e^{-\rho} s(x, \theta, \bar{\theta})$ and $\Lambda_2 = e^{2\rho - H C} \bar{s}(x, \theta, \bar{\theta})$ in (11).

The non-linear versions of (13) and (14) can be obtained by covariantizing the four-dimensional superspace derivatives and six-dimensional spacetime derivatives as

$$\nabla_\alpha = e^{-v} D_\alpha e^v, \quad \nabla_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}}, \quad \nabla_j = e^{-v} (\partial_j + \bar{\omega}_j) e^v, \quad \nabla^j = \bar{\partial}^j - \omega^j.$$

(15)
These covariant derivatives satisfy the identities

$$ F_{\alpha\beta} = F_{\dot{\alpha}\dot{\beta}} = F_{\alpha j} = F_{\dot{\alpha} j} = 0 $$

where $F_{AB} = \{\nabla_A, \nabla_B\}$, and transform as $\delta \nabla_A = [\nabla_A, \sigma]$ under the gauge-transformation

$$ \delta e^\nu = \bar{\sigma} e^\nu + e^\nu \sigma, \quad \delta \omega^j = - \partial^j \sigma + [\omega^j, \sigma], \quad \delta \bar{\omega}_j = - \partial_j \bar{\sigma} - [\bar{\omega}_j, \bar{\sigma}] $$

where $\bar{\sigma} = D^2 s$ and $\sigma = \bar{D}^2 \bar{s}$ for arbitrary $s(x, \theta, \bar{\theta})$ and $\bar{s}(x, \theta, \bar{\theta})$.

In terms of these field strengths, the non-linear equations of motion for ten-dimensional super-Yang-Mills are

$$ 2 \{ \nabla^\alpha, W_\alpha \} = F_j^j, \quad 2 \{ \nabla^\alpha, F_j^j \} = \epsilon^{jkl} F_{kl}, \quad 2 \{ \bar{\nabla}^{\dot{\alpha}}, F_{\dot{\alpha} j} \} = \epsilon_{jkl} F^{kl}. $$

where $W_\alpha = [\nabla_\alpha, \{\nabla_\alpha, \nabla_\dot{\alpha}\}] = \bar{D}_\alpha \bar{D}^\dot{\alpha}(e^{-\nu} D_\alpha e^\nu)$ is the four-dimensional chiral field strength. The action which produces these equations of motion is

$$ S = \frac{1}{2} \int d^5 x [ -2 \int d^2 \theta W^\alpha W_\alpha \]$$

$$ + \int d^4 \theta \left( (e^{-\nu} \partial_j e^\nu)(e^{-\nu} \bar{\partial}^j e^\nu) - \int_0^1 dt (e^{-\nu} \partial e^\nu) \{ e^{-\nu} \partial_j e^\nu, e^{-\nu} \bar{\partial}^j e^\nu \} \right)$$

$$ + 2 \int d^4 \theta \left( (\bar{\partial}^j e^\nu) \bar{\omega}_j e^\nu + e^\nu \omega^j (\partial_j e^{-\nu}) + e^{-\nu} \bar{\omega}_j e^\nu \omega^j \right)$$

$$ + \int d^2 \theta \epsilon_{jkl} (\omega^j \omega^k \omega^l + \frac{2}{3} \omega^j \omega^k \omega^l) + \int d^2 \bar{\theta} \epsilon^{jkl} (\bar{\omega}_j \bar{\omega}_k \bar{\omega}_l - \frac{2}{3} \bar{\omega}_j \bar{\omega}_k \bar{\omega}_l) \] \right).$$

Using intuition from the point-particle example, it is now straightforward to guess the non-linear versions of the superstring equations of motion and gauge invariances of (11) and (9). In analogy with (13), one first defines the covariantized operators

$$ G_1^+ = e^{-\Phi_0} G_1^+ e^{\Phi_0}, \quad \bar{G}_2^+ = \bar{G}_{-2}^+, \quad \bar{G}_{-1}^+ = \bar{G}_{-1}^+ - \Omega_{-1}, $$

where $\Omega_0 \equiv G_0^+ \Phi_{-1}$ and $\Omega_{-1} \equiv \bar{G}_{-2}^+ \Phi_1$. Like their point-particle counterparts in (13) and (16), these covariantized operators satisfy the identities

$$ \{ G_1^+, G_1^+ \} = \{ \bar{G}_{-1}^+, \bar{G}_{-2}^+ \} = \{ G_1^+, G_0^+ \} = \{ \bar{G}_{-2}^+, \bar{G}_{-1}^+ \} = 0 $$

$$ (20) $$
and transform as $\delta G_A = [G_A, \Sigma]$ under the gauge transformations

$$\delta e^{\phi_0} = \Sigma e^{\phi_0} + e^{\phi_0}\Sigma, \quad \delta\Omega_{-1} = -\tilde{G}^+_1\Sigma + [\Omega_{-1}, \Sigma], \quad \delta\Omega_0 = -G^+_0\Sigma - [\Omega_0, \Sigma]$$

(21)

where $\tilde{\Sigma} = G^+_1\Lambda_{-1}$ and $\Sigma = \tilde{G}^+_2\Lambda_2$.

A natural string generalization of the point-particle equations of motion in equation (17) is

$$\mathcal{S}_0 = \int dt (e^{\phi_0}(\tilde{G}^+_1\Lambda_0 + \tilde{G}^+_{-1}\Lambda_1) e^{-\phi_0} + e^{\phi_0}(\tilde{G}^+_2\Lambda_0 + \tilde{G}^+_{-2}\Lambda_1) e^{-\phi_0})$$

where $\tilde{\Sigma} = \tilde{G}^+_1\Lambda_{-1}$ and $\Sigma = \tilde{G}^+_2\Lambda_2$.

(22)

These equations can be combined with the identities of (20) to imply that

$$\left( G_1^+ + G_0^+ + \tilde{G}^+_{-1} + \tilde{G}^+_{-2} \right)^2 = 0,$$

(23)

which is the natural generalization of $(Q + A)^2 = 0$ for the Chern-Simons-like action.

In addition to the gauge invariances of equation (21), the equations of motion implied by (23) are also invariant under

$$\delta e^{\phi_0} = e^{\phi_0}(G_0^+\Lambda_0 + \tilde{G}^+_{-1}\Lambda_1),$$

(24)

$$\delta\Omega_{-1} = \tilde{G}^+_2(G_1^+\Lambda_0 + G_0^+\Lambda_1), \quad \delta\Omega_0 = G_1^+(e^{\phi_0}(\tilde{G}^+_{-1}\Lambda_0 + \tilde{G}^+_{-2}\Lambda_1) e^{-\phi_0}).$$

Unlike the gauge transformations of (21), these gauge transformations have no super-Yang-Mills counterpart since there is no massless contribution to $\Lambda_0$ or $\Lambda_1$.

Starting from the point-particle action of (18), one can guess that the open superstring field theory action is

$$S = \frac{1}{2\lambda^2} Tr \langle$$

$$e^{-\phi_0} G_1^+ e^{\phi_0} e^{-\phi_0} \tilde{G}^+_1 e^{\phi_0} - \int_0^1 dt (e^{-\phi_0} \partial_t e^{\phi_0}) \{ e^{-\phi_0} G_1^+ e^{\phi_0}, e^{-\phi_0} \tilde{G}^+_1 e^{\phi_0} \}$$

$$+ (e^{-\phi_0} G_0^+ e^{\phi_0}) (e^{-\phi_0} \tilde{G}^+_1 e^{\phi_0}) - \int_0^1 dt (e^{-\phi_0} \partial_t e^{\phi_0}) \{ e^{-\phi_0} G_0^+ e^{\phi_0}, e^{-\phi_0} \tilde{G}^+_1 e^{\phi_0} \}$$

$$+ 2 \left( (G_{-1}^+ e^{-\phi_0}) \Omega_{-1} e^{\phi_0} + e^{\phi_0} \Omega_{-1} (G_0^+ e^{-\phi_0}) + e^{-\phi_0} \Omega_0 e^{\phi_0} \Omega_{-1} \right)$$

$$-(\Omega_{-1} \tilde{G}^+_1 \Phi_1 - \frac{2}{3} \Omega_{-1} \Omega_{-1} \Phi_1) + (\Omega_0 G_0^+ \Phi_1 - \frac{2}{3} \Omega_0 \Omega_0 \Phi_1) \rangle$$

(25)
using the normalization definition that
\[ \langle (\theta)^2 (\bar{\theta})^2 e^{-\rho \psi^j \psi^k \psi^l} \rangle = \epsilon^{jkl}. \]

To show that the superstring field theory action of (25) is correct, one can check that its linearized equations of motion and gauge invariances reproduce the on-shell conditions of (11) and (9) for physical vertex operators, and that the cubic term in the action produces the three-point tree-level scattering amplitude. Note that both the $W^\alpha W_\alpha$ and WZW actions for $\nu$ in the first two lines of (18) are replaced by WZW actions in the second and third lines of (25). Note also that the chiral and anti-chiral $F$-terms in the last line of (18) are replaced by the two terms in the last line of (25). Using the gauge invariance $\delta \Phi_1 = \tilde{\Gamma}^+ \Lambda_3$ and $\delta \Phi_{-1} = G^+_1 \Lambda_2$, one can choose the gauge of (10) and write these “$D$-terms” as the chiral and anti-chiral “$F$-terms”

【26】

\[ T r \langle - \Omega_{-1} \tilde{G}_{-1}^+ \Omega_{-1} + \frac{2}{3} \Omega_{-1} \Omega_{-1} \Omega_{-1} \rangle_F + T r \langle \Omega_0 G^+_0 \Omega_0 + \frac{2}{3} \Omega_0 \Omega_0 \Omega_0 \rangle_{\tilde{F}} \]

where one defines the normalization of $\langle \rangle_F$ and $\langle \rangle_{\tilde{F}}$ by

\[ \langle (\theta)^2 e^{-3\rho+2HC} \rangle_F = \frac{1}{4}, \quad \langle (\bar{\theta})^2 e^{HC} \rangle_{\tilde{F}} = \frac{1}{4}. \]

Since $\tilde{G}^+$ is the inverse of the $\xi$ zero mode, turning $D$-terms into $F$-terms is like going from the large to the small RNS Hilbert space. The $F$-terms in (26) are expected to satisfy non-renormalization theorems similar to those satisfied by the point-particle $F$-terms in the last line of (18).

One can include both the GSO(+) and GSO(−) sectors in the action of (25) by allowing $(\theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ and $(p_\alpha, \bar{p}^{\dot{\alpha}})$ to be integer moded in the GSO(+) sector and half-integer moded in the GSO(−) sector. As in the NS action of (17) (11), one needs to include extra $2 \times 2$ matrices on the string fields $(\Phi_{-1}, \Phi_0, \Phi_1)$ and on the operators $(G^+_1, G^+_0, \tilde{G}^+_1, \tilde{G}^+_2)$ to account for the “wrong” statistics in the GSO(−) sector (33). Although $N = 1$ spacetime supersymmetry is broken after including the GSO(−) sector since $(\theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ no longer has zero modes, it has been conjectured by Yoneya that the action contains a hidden $N=2$ spacetime supersymmetry which is restored after the tachyon condenses (36) (37).

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References

[1] E. Witten, Nucl. Phys. B268 (1986) 253.
[2] D. Friedan, E. Martinec and S. Shenker, Nucl. Phys B271 (1986) 93.
[3] E. Witten, Nucl. Phys. B276 (1986) 291.
[4] C. Wendt, Nucl. Phys. B314 (1989) 209.
[5] J. Greensite and F.R. Klinkhamer, Nucl. Phys. B281 (1987) 269.
[6] C.R. Preitschopf, C.B. Thorn and S. Yost, Nucl. Phys. B337 (1990) 363.
[7] I. Ya. Aref’eva, P.B. Medvedev and A.P. Zubarev, Nucl. Phys. B341 (1990) 464.
[8] N. Berkovits, M. Hatsuda and W. Siegel, Nucl. Phys. B371 (1992) 434, hep-th/9108021.
[9] I. Ya. Aref’eva, A.S. Koshelev, D.M. Belov and P.B. Medvedev, hep-th/0011117.
[10] A. Sen, JHEP 9912 (1999) 027, hep-th/9911116.
[11] N. Berkovits, A. Sen and B. Zwiebach, Nucl. Phys. B587 (2000) 147, hep-th/0002211.
[12] A. Sen, private communication.
[13] K. Ohmuri, hep-th/0102085.
[14] I. Ya. Aref’eva and P.B. Medvedev, Phys. Lett. B212 (1988) 299.
[15] E. Witten, Commun. Math. Phys. 92 (1984) 455.
[16] N. Berkovits and C.T. Echevarria, Phys. Lett. B478 (2000) 343, hep-th/9912120.
[17] N. Berkovits, JHEP 0004 (2000) 012, hep-th/0001084.
[18] P.J. De Smet and J. Raeymaekers, JHEP 0005 (2000) 051, hep-th/0003220.
[19] A. Iqbal and A. Naqvi, hep-th/0004015.
[20] J. David, JHEP 0010 (2000) 004, hep-th/0007238.
[21] K. Ohmori, hep-th/0104230.
[22] V.A. Kostelecky and S. Samuel, Nucl. Phys. B336 (1990) 263.
[23] A. Sen and B. Zwiebach, JHEP 0003 (2000) 002, hep-th/9912249.
[24] E. Witten, hep-th/9207094.
[25] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Comm. Math. Phys. 165 (1994) 311, hep-th/9309140.
[26] N. Berkovits and C. Vafa, Nucl. Phys. B433 (1995) 123, hep-th/9407190.
[27] H. Ooguri and C. Vafa, Nucl. Phys. B367 (1991) 83.
[28] S.K. Donaldson, Comm. Math. Phys. 93 (1984) 453.
[29] V.P. Nair and J. Schiff, Phys. Lett. B246 (1990) 423.
[30] N. Berkovits and C. Vafa, Mod. Phys. Lett. A9 (1994) 653, hep-th/9310170.
[31] N. Berkovits, Nucl. Phys. B420 (1994) 332, hep-th/9308129.
[32] N. Berkovits, Nucl. Phys. B431 (1994) 258, hep-th/9404162.
[33] N. Berkovits, Nucl. Phys. B450 (1995) 90, hep-th/9503099.
[34] N. Marcus, A. Sagnotti and W. Siegel, Nucl. Phys. B224 (1983) 159.

[35] N. Berkovits and C.T. Echevarria, work in progress.

[36] T. Yoneya, Nucl. Phys. B576 (2000) 219, hep-th/9912253.

[37] T. Yoneya, private communication.