Towards the Baum-Connes’ Analytical Assembly Map for the Actions of Discrete Quantum Groups

by
Debashish Goswami
and
A. O. Kuku;
The Abdus Salam International Centre for Theoretical Physics
(Mathematics Section)
Strada Costiera 11, trieste 34014, Italy.
e-mail : goswamid@hotmail.com,
kuku@ictp.trieste.it
Address for correspondence :
Prof. A. O. Kuku, Mathematics Section, The Abdus Salam I.C.T.P.
Strada Costiera 11, Trieste 34014, Italy.
Phone : 0039 - 040 - 2240267, Fax :0039 - 040 - 224163
Running head : Baum-Connes Map for Quantum Groups
Abstract

Given an action of a discrete quantum group (in the sense of Van Daele, Kustermans and Effros-Ruan) $\mathcal{A}$ on a $C^*$-algebra $\mathcal{C}$, satisfying some regularity assumptions resembling the proper $\Gamma$-compact action for a classical discrete group $\Gamma$ on some space, we are able to construct canonical maps $\mu_i^\epsilon$ ($\mu^\epsilon$ respectively) ($i = 0, 1$) from the $\mathcal{A}$-equivariant K-homology groups $KK^\mathcal{A}_i(\mathcal{C}, \mathcal{C}')$ to the K-theory groups $K_i(\hat{\mathcal{A}}_\epsilon)$ ($K_i(\hat{\mathcal{A}})$ respectively), where $\hat{\mathcal{A}}_\epsilon$ and $\hat{\mathcal{A}}$ stand for the quantum analogues of the reduced and full group $C^*$-algebras (c.f. [11], [6]). We follow the steps of the construction of the classical Baum-Connes map (c.f. [1], [2], [13], [14]), although in the context of quantum group the nontrivial modular property of the invariant weights (and the related fact that the square of the antipode is not identity) has to be taken into serious consideration, making it somewhat tricky to guess and prove the correct definitions of relevant Hilbert module structures.

Key words: Baum-Connes Conjecture, Discrete Quantum Group, Equivariant KK-Theory

AMS Subject classification numbers : 19K35, 46L80, 81R50
1 Introduction

The famous conjecture made by Paul Baum and Alain Connes has given birth to one of the most interesting areas of research in both classical and noncommutative geometry, topology, K-Theory etc. Let us very briefly recall the main statement of this conjecture (c.f. [1], [2], and also [13], [14] for a nice and easily accessible account). Given a locally compact group $G$, and a locally compact Hausdorff space $X$ equipped with a $G$-action such that $X$ is proper and $G$-compact (see for example [13] and the references therein for various equivalent formulation of these concepts), there are canonical maps $\mu_r^i : KK^G_i(C_0(X), C^r) \rightarrow K_i(C^*(G))$, and $\mu_i : KK^G_i(C_0(X), C^r) \rightarrow K_i(C^*(G))$, for $i = 0, 1$, where $C_0(X)$ is the commutative $C^*$-algebra of continuous complex-valued functions on $X$ vanishing at infinity, $C^r(G)$ and $C^*(G)$ are respectively the reduced and free groups $C^*$-algebras, and $KK^G$ denotes the Kasparov’s equivariant KK-functor. In particular, $KK^G(C_0(X), C^r)$ is identified with the $G$-equivariant K-homology of $X$, and thus is essentially something geometric or topological, whereas the object $K_i(\cdot)$ on the right hand side involves the reduced or free group algebras, which are analytic in some sense. Now, let $EG$ be the universal space for proper actions of $G$. The definition of proper $G$-actions and explicit constructions in various cases of interest can be found in [13], [14] and the references therein. The equivariant K-homology of $EG$, say $RK^G_i(EG)$, $i = 0, 1$, can be defined as the inductive limit of $KK^G_i(C_0(X), C^r)$, over all possible locally compact, $G$-proper and $G$-compact subsets $X$ of the universal space $EG$. Since the construction of $KK^G_i$ and $K_i$ commute with the procedure of taking an inductive limit, it is possible to define $\mu^r_i, \mu^i$ on the equivariant K-homology $RK^G_i(EG)$, and the conjecture of Baum-Connes states that $\mu^r_i, i = 0, 1$ are isomorphisms of abelian groups. This conjecture admits certain other generalizations, such as the Baum-Connes conjecture with coefficients (which seems to be false from some recent result announced by M. Gromov, see [13] for references), but we do not want to discuss those here. However, we would like to point out that the Baum-Connes conjecture has already been verified for many classical groups, using different methods and ideas from many diverse areas of mathematics, and has given birth to many new and interesting tools and techniques in all these areas. In fact, the truth of this conjecture, if established, will prove many other famous conjectures in topology, geometry and K-theory.

Now, in last two decades, the theory of quantum groups has become another fast-growing branch of mathematics and mathematical physics. Motivated by examples coming from physics, as well as some fundamental math-
ematical problems (e.g. to develop a good theory of duals for noncommutative topological groups), many mathematicians including Drinfeld, Jimbo, Woronowicz and others have formulated and studied the concept of quantum groups, which is a far-reaching generalization of classical topological groups. On the other hand, with the pioneering efforts of Connes (see [4]), followed by himself and many other mathematicians, a powerful generalization of classical differential and Riemannian geometry has emerged under the name of noncommutative geometry, which has had, since its very beginning, very close connections with $K$-theory too. Furthermore, Baaj and Skandalis ([3]) have been able to construct an analogue of equivariant $KK$-theory for the actions of quantum groups, as natural extension of Kasparov’s equivariant $KK$-theory. This motivates one to think of a possibility of generalizing the Baum-Connes construction in the framework of quantum groups. In the present article, we make an attempt towards this generalization. As we have already mentioned : there are two steps in the classical formulation of the Baum-Connes conjecture. First of all, one has to define the maps $\mu_i^r$ for $G$-compact and $G$-proper actions. Then in the second step, one defines the universal space for proper action of the group, and then more importantly, tries to build explicit good models for this universal space to show that it can be approximated in a suitable sense by its subsets having $G$-proper and $G$-compact actions, thereby defining the maps $\mu_i^r$ by inductive limit. What we have been able to achieve in our work here is essentially the first step, for a class of quantum groups called discrete quantum groups (which are indeed generalizations of discrete groups). However, a definition of proper action of quantum groups has been already proposed in [3], and we hope that it may be possible to achieve the second step starting from this definition, thereby actually formulating (and then verifying in some cases, if possible) Baum-Connes conjecture for discrete quantum groups. But we would like to postpone that task for later work.

We would also like to mention one thing. We have restricted ourselves within the framework of discrete quantum groups not only because it is technically easier to do so, but also because, in fact, the classical Baum-Connes conjecture is very interesting and nontrivial for discrete groups, and in some sense most of the difficult cases belong to them. Of course, if our programme seems to go through satisfactory for discrete quantum groups, we would like to take up more general locally compact quantum groups in future. It should be noted that in the quantum case, discreteness does not imply the unimodularity of the haar weight, and thus even for discrete quantum groups, one has to be very careful about the choices of left or right invariant weights as well as the appropriate role of the modular operator, as
we shall see.

Let us conclude this section with some useful notational convention. For a Hilbert space \( \mathcal{H} \), and some pre-C*-algebra \( \mathcal{B} \subseteq \mathcal{B}(\mathcal{H}) \), we shall denote the multiplier algebra of the norm-closure of \( \mathcal{B} \) by \( \mathcal{M}(\mathcal{B}) \). For two Hilbert spaces \( \mathcal{H}_1, \mathcal{H}_2 \) and some bounded operator \( X \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) = \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \), we denote by \( X_{12} \) the operator \( X \otimes 1_{\mathcal{H}_2} \) on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2 \), and denote by \( X_{13} \) the operator \( (1_{\mathcal{H}_1} \otimes \Sigma)(X \otimes 1_{\mathcal{H}_2})(1_{\mathcal{H}_1} \otimes \Sigma) \) on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2 \), where \( \Sigma : \mathcal{H}_2 \otimes \mathcal{H}_2 \to \mathcal{H}_2 \otimes \mathcal{H}_2 \) flips the two copies of \( \mathcal{H}_2 \). For two vectors \( \xi, \eta \in \mathcal{H}_1 \) we define a map \( T_{\xi\eta} : \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_2) \) by setting \( T_{\xi\eta}(A \otimes B) := < \xi, A\eta > \), where \( A \in \mathcal{B}(\mathcal{H}_1), B \in \mathcal{B}(\mathcal{H}_2) \), and extend this definition to the whole of \( \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \) in the obvious way. It is easy to see that \( T_{\xi\eta}(X^*) = (T_{\eta\xi}(X))^* \), and \( T_{\xi\xi}(X) \) is nonnegative operator if \( X \) is. In fact, \( T_{\xi\xi} \) is a completely positive map.

For some Hilbert space \( \mathcal{H} \), we denote by \( \mathcal{B}_0(\mathcal{H}) \) the C*-algebra of compact operators on \( \mathcal{H} \), and by \( \mathcal{L}(E) \) the C*-algebra of adjointable linear maps on a Hilbert \( A \)-module \( E \). Furthermore, for a von Neumann algebra \( \mathcal{B} \subseteq \mathcal{B}(\mathcal{H}) \), and some Hilbert space \( \mathcal{H}' \), we introduce the following notation : for \( \eta \in \mathcal{H}' \), \( X \in \mathcal{B}(\mathcal{H}') \otimes \mathcal{B} \equiv \mathcal{L}(\mathcal{H}' \otimes \mathcal{B}) \), \( X\eta := X(\eta \otimes 1_{\mathcal{B}}) \in \mathcal{H}' \otimes \mathcal{B} \). Note that we have denoted by \( \mathcal{H}' \otimes \mathcal{B} \) the Hilbert von Neumann module obtained from the algebraic \( \mathcal{B} \)-module \( \mathcal{H}' \otimes_{\text{alg}} \mathcal{B} \) by completing this algebraic module in the strong operator topology inherited from \( \mathcal{B}(\mathcal{H}, \mathcal{H}' \otimes \mathcal{H}) \), where we have identified an element of the form \( (\xi \otimes b), \xi \in \mathcal{H}', b \in \mathcal{B} \), with the operator which sends a vector \( v \in \mathcal{H} \) to \( (\xi \otimes bv) \in \mathcal{H}' \otimes \mathcal{H} \). It is easy to see that \( \mathcal{H}' \otimes \mathcal{B} \) is isomorphic as a Hilbert von Neumann module with \( \{ X \in \mathcal{B}(\mathcal{H}, \mathcal{H}' \otimes \mathcal{H}) : Xc = (1 \otimes c)X, \forall c \in \mathcal{B}' \} \), where \( \mathcal{B}' \) denotes the commutant of \( \mathcal{B} \) in \( \mathcal{B}(\mathcal{H}) \). Similarly, for a possibly nonunital C*-algebra \( \mathcal{A} \), we can complete the algebraic \( \mathcal{B} \)-module \( \mathcal{H}' \otimes_{\text{alg}} \mathcal{A} \) in the locally convex topology coming from the strict topology on \( \mathcal{M}(\mathcal{A}) \), so that the completion becomes in a natural way a locally convex Hilbert \( \mathcal{M}(\mathcal{A}) \)-module, to be denoted by \( \mathcal{H}' \otimes \mathcal{M}(\mathcal{A}) \). It is also easy to see that if \( X \in \mathcal{M}(\mathcal{B}_0(\mathcal{H}') \otimes \mathcal{A}), \eta \in \mathcal{H}' \), then we have \( X\eta \equiv X(\eta \otimes 1) \in \mathcal{H}' \otimes \mathcal{M}(\mathcal{A}) \).

If \( \mathcal{B}_1, \mathcal{B}_2 \) are two von Neumann algebras, \( \mathcal{H}' \) is a Hilbert space, and \( \rho : \mathcal{B}_1 \to \mathcal{B}_2 \) is a normal *-homomorphism, then it is easy to show that \( (id \otimes \rho) : \mathcal{H}' \otimes_{\text{alg}} \mathcal{B}_1 \to \mathcal{H}' \otimes_{\text{alg}} \mathcal{B}_2 \) admits a unique extension (to be denoted again by \( (id \otimes \rho) \)) from the Hilbert von Neumann module \( \mathcal{H}' \otimes \mathcal{B}_1 \) to the Hilbert von Neumann module \( \mathcal{H}' \otimes \mathcal{B}_2 \). Furthermore, one has that \( (id \otimes \rho)(X\eta) = (id \otimes \rho)(X)\eta \) for \( X \in \mathcal{B}(\mathcal{H}') \otimes \mathcal{B}, \eta \in \mathcal{H}' \). By very similar arguments one can also prove that if \( \mathcal{A}_1, \mathcal{A}_2 \) are two C*-algebras, and \( \pi : \mathcal{A}_1 \to \mathcal{A}_2 \) is a nondegenerate *-homomorphism (hence extends uniquely as a unital strictly continuous *-homomorphism from \( \mathcal{M}(\mathcal{A}_1) \) to \( \mathcal{M}(\mathcal{A}_2) \)), then \( (id \otimes \pi) : \mathcal{H}' \otimes_{\text{alg}} \mathcal{A}_1 \to \mathcal{H}' \otimes_{\text{alg}} \mathcal{A}_2 \) is a completely positive map.
$\mathcal{H}' \otimes_{\text{alg}} \mathcal{A}_1 \to \mathcal{H}' \otimes_{\text{alg}} \mathcal{A}_2$ admits a unique extension (to be denoted by the same notation) from $\mathcal{H}' \otimes \mathcal{M}(\mathcal{A}_1)$ to $\mathcal{H}' \otimes \mathcal{M}(\mathcal{A}_2)$, which is continuous in the locally convex topologies coming from the respective strict topologies. We also have that $(id \otimes \pi)(X \eta) = (id \otimes \pi)(X) \eta$, for $X \in \mathcal{M}(\mathcal{B}_0(\mathcal{H}' \otimes \mathcal{A})), \eta \in \mathcal{H}'$.

2 Preliminaries on discrete quantum groups

We briefly discuss the theory of discrete quantum groups as developed in [10], [6], [12], [8] and other relevant references to be found there. Let us fix an index set $I$ (possibly uncountable), and let $\mathcal{A}_0 := \bigoplus_{\alpha \in I} \mathcal{A}_\alpha$ be the algebraic direct sum of $\mathcal{A}_\alpha$’s, where for each $\alpha$, $\mathcal{A}_\alpha = M_{n_\alpha}$ is the finite dimensional $C^*$-algebra of $n_\alpha \times n_\alpha$ matrices with complex entries, and $n_\alpha$ is some positive integer. Let us denote by $M \equiv M(\mathcal{A}_0)$ the unital $C^*$-algebra consisting of all collections $(a_\alpha)_{\alpha \in I}$ with $a_\alpha \in \mathcal{A}_\alpha$ for each $\alpha$, and $\sup_\alpha \|a_\alpha\| < \infty$. The algebra operations are taken to be the obvious ones; i.e. $(a_\alpha) + (b_\alpha) := (a_\alpha + b_\alpha)$, $(a_\alpha)(b_\alpha) := (a_\alpha b_\alpha)$ and $(a_\alpha)^* := (a_\alpha^*)$. Similarly, denote by $M(\mathcal{A}_0 \otimes \mathcal{A}_0)$ the $C^*$-algebra consisting of all collections of the form $(a_\alpha \otimes b_\beta)$ where $\alpha, \beta$ varies over $I$, and equip $M(\mathcal{A}_0 \otimes \mathcal{A}_0)$ with the obvious $C^*$-algebra structure. Let us now assume that there is a unital $C^*$-homomorphism $\Delta : M(\mathcal{A}_0) \to M(\mathcal{A}_0 \otimes \mathcal{A}_0)$ which satisfies the following:

(i) For $a, b \in \mathcal{A}_0$, we have

$$T_1(a \otimes b) := \Delta(a)(1 \otimes b) \in \mathcal{A}_0 \otimes_{\text{alg}} \mathcal{A}_0,$$

and

$$T_2(a \otimes b) := (a \otimes 1)\Delta(b) \in \mathcal{A}_0 \otimes_{\text{alg}} \mathcal{A}_0;$$

(ii) $T_1, T_2 : \mathcal{A}_0 \otimes_{\text{alg}} \mathcal{A}_0 \to \mathcal{A}_0 \otimes_{\text{alg}} \mathcal{A}_0$ are bijections;

(iii) $\Delta$ satisfies the coassociativity in the sense that

$$(a \otimes 1 \otimes 1)(\Delta \otimes id)(\Delta(b)(1 \otimes c)) = (id \otimes \Delta)((a \otimes 1)\Delta(b))(1 \otimes 1 \otimes c),$$

for $a, b, c \in \mathcal{A}_0$. As explained in the relevant references mentioned above, $(\Delta \otimes id), (id \otimes \Delta)$ admit extensions as $C^*$-homomorphisms from $M(\mathcal{A}_0 \otimes \mathcal{A}_0)$ to $M(\mathcal{A}_0 \otimes \mathcal{A}_0 \otimes \mathcal{A}_0)$ (we denote these extensions by the same notation) and the condition (iii) translates into $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$.

The above conditions essentially constitute the definition of a discrete quantum group (for details see [10], [8] and [11]). Let us recall from [10] and [8] some of the important properties of our discrete quantum group $\mathcal{A}_0$. It is remarkable that it is possible to deduce from (i) to (iii) the existence of a canonical antipode $S : \mathcal{A}_0 \to \mathcal{A}_0$ satisfying $S(a)^* = a$ and other usual
properties of the antipode of a Hopf algebra. Furthermore, there exists a counit \( \epsilon : \mathcal{A}_0 \to \mathbb{C} \). For details of the constructions of these maps and their properties we refer to [10].

We shall call an arbitrary collection \( (a_\alpha)_{\alpha \in I} \), with \( a_\alpha \in \mathcal{A}_\alpha \forall \alpha \), the “algebraic multiplier” of \( \mathcal{A}_0 \). The set of all algebraic multipliers of \( \mathcal{A}_0 \), denoted by \( \mathcal{M}_{\text{alg}}(\mathcal{A}_0) \), is obviously a \( * \)-algebra, with pointwise multiplication and adjoint, i.e. for \( U = (u_\alpha), V = (v_\alpha) \in \mathcal{M}_{\text{alg}}(\mathcal{A}_0) \), \( UV := (u_\alpha v_\alpha)_\alpha \), and \( U^* := (u^*_\alpha) \). Clearly, any element of \( \mathcal{A}_0 \) can be viewed as an element of \( \mathcal{M}_{\text{alg}}(\mathcal{A}_0) \), by thinking of \( a \in \mathcal{A}_0 \) as \( (a_\alpha) \), where \( a_\alpha \) is the component of \( a \) in \( \mathcal{A}_\alpha \). It is easy to see that \( Ua, aU \in \mathcal{A}_0 \) for \( a \in \mathcal{A}_0, U \in \mathcal{M}_{\text{alg}}(\mathcal{A}_0) \). We can give a similar definition of algebraic multiplier of \( \mathcal{A}_0 \otimes \mathcal{A}_0 \), which will be any collection of the form \( M \equiv (m_{\alpha\beta})_{\alpha,\beta \in I} \), with \( m_{\alpha\beta} \in \mathcal{A}_\alpha \otimes \mathcal{A}_\beta \). In fact, since by [10] there is a bijection of the index set \( I \), say \( \alpha \mapsto \alpha' \), such that \( S(e_\alpha) = e_{\alpha'}, S(e_{\alpha'}) = e_\alpha \), we can define \( S(X) \) for \( X = (x_\alpha)_I \in \mathcal{M}_{\text{alg}}(\mathcal{A}_0) \) by \( S(X) := X' = (x'_{\alpha'})_I \) where \( x'_{\alpha'} = S(x_{\alpha'}) \). Similarly, to define \( \Delta(X) \) for \( X = (x_\alpha) \in \mathcal{M}_{\text{alg}}(\mathcal{A}_0) \), we note that (c.f. [10]) for fixed \( \alpha, \beta \in I \), there is a finite number of \( \gamma \in I \) such that \( \Delta(e_\gamma)(e_\alpha \otimes e_\beta) \) is nonzero. Thus, \( \Delta(X) \) can be defined as the element \( Y \in \mathcal{M}_{\text{alg}}(\mathcal{A}_0 \otimes \mathcal{A}_0) \) such that \( Y = (y_{\alpha\beta}) \), where \( y_{\alpha\beta} = \sum_\gamma \Delta(x_\gamma)\Delta(e_\gamma)(e_\alpha \otimes e_\beta) \). For algebraic multipliers \( A, B \) of \( \mathcal{A}_0 \) and \( L \) of \( \mathcal{A}_0 \otimes \mathcal{A}_0 \), it is clear that \( \Delta(A) = L \) if and only if \( \Delta(Aa) = L\Delta(a) \forall a \in \mathcal{A}_0 \), and \( S(A) = B \) if and only if \( S(Aa) = S(a)B \) for \( a \in \mathcal{A}_0 \).

Let \( \mathcal{K} \) be the smallest Hilbert space containing the algebraic direct sum \( \bigoplus_{\alpha \in I} \mathcal{K}_\alpha \equiv \bigoplus_\alpha \mathcal{C}^{n_\alpha} \), i.e. \( \mathcal{K} = \{ (f_\alpha)_{\alpha \in I} : f_\alpha \in \mathcal{K}_\alpha = \mathcal{C}^{n_\alpha}, \sum_\alpha \| f_\alpha \|^2 < \infty \} \), where the possibly uncountable sum \( \sum_\alpha \) means the limit over the net consisting of all possible sums over finite subsets of \( I \). Let us consider the canonical imbedding of \( \mathcal{A}_0 \) in \( \mathcal{B}(\mathcal{K}) \), with \( \mathcal{A}_\alpha \) acting on \( \mathcal{C}^{n_\alpha} \). Let \( \mathcal{A} \) be the completion of \( \mathcal{A}_0 \) under the norm-topology inherited from \( \mathcal{B}(\mathcal{K}) \). Let us fix some matrix units \( e_{ij}^\alpha, i, j = 1, ..., n_\alpha \) for \( \mathcal{A}_\alpha = M_{n_\alpha} \), w.r.t. some fixed orthonormal basis \( e_i^\alpha, i = 1, ..., n_\alpha \), of \( \mathcal{C}^{n_\alpha} \), and thus \( \mathcal{A} \) is the \( C^* \)-algebra generated by \( e_{ij}^\alpha \). s. It is also clear that any element of \( \mathcal{M}_{\text{alg}}(\mathcal{A}_0) \) can be viewed as a possibly unbounded operator on \( \mathcal{K} \), with the domain containing the algebraic direct sum of \( \mathcal{K}_\alpha \)'s. Similarly, elements of \( \mathcal{M}_{\text{alg}}(\mathcal{A}_0 \otimes \mathcal{A}_0) \) can be thought of as possibly unbounded operators on \( \mathcal{K} \otimes \mathcal{K} \) with suitable domain.

Let us denote by \( \mathcal{A}'_0 \) the set of all linear functionals on \( \mathcal{A}_0 \) having “finite support”, i.e. they vanish on \( \mathcal{A}_\alpha \)'s for all but finite many \( \alpha \in I \). It is clear
that any \( f \in A'_0 \) can be identified as a functional on \( M_{\text{alg}}(A_0) \), by defining \( f((a_\alpha)I) := \sum_{\alpha \in I} f(a_\alpha) \equiv \sum_{I_0} f(a_\alpha) \), where \( I_0 \) is the finite set of \( \alpha \)'s such that for \( \alpha \)'s not belonging to \( I_0 \), \( f|_{A_\alpha} = 0 \). With this identification, \( f(1) \) makes sense for any \( f \in M_{\text{alg}}(A_0) \). Let us denote by \( e_\alpha \) the identity of \( A_\alpha = M_{n_\alpha} \), which is a minimal central projection in \( A_0 \). For any subset \( I_1 \) of \( I \) we denote by \( e_{I_1} \) the direct sum of \( e_\alpha \)'s for \( \alpha \in I_1 \). It is clear that a functional \( f \) on \( A_0 \) is in \( A'_0 \) if and only if there is some finite \( I_1 \) such that \( f(a) = f(e_{I_1}a) \) for all \( a \in A_0 \).

We say that a linear functional \( \phi \) (not necessarily with finite support) on \( A_0 \) is left invariant if we have \( (id \otimes \phi)((b \otimes 1)\Delta(a)) = b\phi(a) \) for all \( a, b \in A_0 \), or equivalently, \( \phi((\omega \otimes id)(\Delta(a))) = \omega(1)\phi(a) \) for all \( a \in A_0 \), \( \omega \in A'_0 \). Similarly, a linear functional \( \psi \) on \( A_0 \) is called right invariant if \( (\psi \otimes id)((1 \otimes b)\Delta(a)) = \psi(a)b \) for all \( a, b \in A_0 \). Let us now recall some of the main results regarding left and right invariant functionals as proved in \([10]\). It is shown in \([10]\) that up to constant multiples, there is a unique left invariant functional, and same thing is true for right invariant functionals, although in general (unless \( S^2 = id \)) left and right invariant functionals are not the same. Moreover, for each \( \alpha \in I \), there is a positive invertible element \( K_\alpha \in A_\alpha \) such that the positive functional \( \phi \) defined by

\[
\phi(x) = Tr_\alpha(K_\alpha^{-1}x)
\]

for \( x \in A_\alpha \), (where \( Tr_\alpha \) is the trace on the algebra \( A_\alpha \) of \( n_\alpha \times n_\alpha \) matrices) and extended on \( A_0 \) by linearity, is left invariant. We get a right invariant positive functional \( \psi \) by replacing \( K_\alpha^{-1} \) by \( c_\alpha K_\alpha \) for some positive constant \( c_\alpha \), i.e.

\[
\psi(x) := c_\alpha Tr_\alpha(K_\alpha x),
\]

for \( x \in A_\alpha \). Furthermore, \( S^2(a) = K_\alpha^{-1}aK_\alpha \) for \( a \in A_\alpha \). If we define a possibly unbounded positive invertible operator \( K \) on \( K \) by setting \( K|_{K_\alpha} = K_\alpha \) for each \( \alpha \), then it is easy to see that \( \psi(a) = c_\alpha \phi(K^2a) = c_\alpha \phi(aK^2) \) for \( a \in A_\alpha \). Now, observe that \( a \mapsto \phi(S(a)) \) is right invariant, hence there is some constant \( c \) such that \( \psi = c\phi \circ S \). From the results of \([3]\) it follows that there is a “modular operator” \( \delta \), which can be thought of as a collection \( (\delta_\alpha)_{\alpha \in I} \) such that \( \delta_\alpha \in A_\alpha \) for each \( \alpha \) (i.e. \( \delta \in M_{\text{alg}}(A_0) \)), and we also have that \( \Delta(\delta) = \delta \otimes \delta \), \( S(\delta) = \delta^{-1} \), \( S(\delta^{-1}) = \delta \), in the sense described earlier; and furthermore, \( \phi(S(a)) = \phi(a\delta_\alpha) \) for all \( a \in A_\alpha \). Thus, for \( a \in A_\alpha \), \( c_\alpha \phi(a\delta_\alpha) = \psi(a) = c\phi(S(a)) = c\phi(a\delta_\alpha) \). Since \( c \) is clearly nonzero, we conclude that \( \delta_\alpha = c^{-1}c_\alpha K_\alpha^2 \) for each \( \alpha \). Let us now argue that \( c \) is positive, which will show the positivity of \( \delta_\alpha \). Since \( \psi \) is by construction a positive
functional, we need to prove that \( \phi \circ S \) is positive too. However, for \( a \in \mathcal{A}_0 \), \( \phi(S(a^*a)) = \phi(S(a)S(a^*)) = \phi(S(a)S^2(S(a)^*)) = \phi(S(a)K^{-1}S(a)^*K) \). From the definition of \( \phi \) in terms of trace on each finite dimensional component, it is clear that \( \phi(S(a)K^{-1}S(a)^*K) = \phi(K^{1/2}S(a)K^{-1}S(a)^*K^{1/2}) \geq 0 \).

So, \( c \) is positive, and hence so is the operator \( \delta_\alpha \) for each \( \alpha \). Let \( \theta_\alpha := \delta_\alpha \) for each \( \alpha \), and let \( \theta \) be the unbounded positive operator on \( \mathcal{K} \) defined by \( \theta|_{\mathcal{K}_\alpha} = \theta_\alpha \).

Let us fix some \( \alpha \) now. From [10], note that there is some index \( \beta \) such that \( S(\mathcal{A}_\alpha) = \mathcal{A}_\beta \), and in particular \( S(e_\alpha) = e_\beta \). Since \( \delta_\alpha \) is a finite dimensional positive invertible matrix, all its eigenvalues are strictly positive. Similar thing is true for \( \delta_\beta, \delta_\alpha^{-1}, \delta_\beta^{-1} \) too. Thus, we can choose a holomorphic function \( g \) defined on an open set of the complex plane containing the union of the spectrum of the matrices \( \delta_\alpha, \delta_\beta, \delta_\alpha^{-1}, \delta_\beta^{-1} \) such that \( g(\delta_\alpha) = \theta_\alpha, g(\delta_\alpha^{-1}) = \theta_\alpha^{-1} \). As the restriction of \( S \) on \( \mathcal{A}_\alpha \), say \( S_\alpha \), is a linear map on a finite dimensional space, it is norm-continuous, and furthermore, \( S(x^n) = S(x)^n \) for any positive integer \( n, x \in \mathcal{A}_\alpha \), from which it is easy to see that \( S(\theta_\alpha) = S(g(\delta_\alpha)) = g(S(\delta_\alpha)) = g(\delta_\beta^{-1}) = \theta_\beta^{-1} = S(e_\alpha)\theta^{-1} \). Similarly, \( S(\theta_\alpha^{-1}) = \theta_\beta \). Since this is true for any \( \alpha \), we conclude that \( S(\theta) = \theta^{-1} \) and \( S(\theta^{-1}) = \theta \). By a very similar argument we can prove that \( \Delta(\theta) = \theta \otimes \theta \).

Furthermore, from our discussion it is also clear that \( S^2(a) = \theta^{-1}a\theta \) for \( a \in \mathcal{A}_0 \). Let us summarize these facts here:

(a) There exists a positive (possibly unbounded) invertible operator \( \theta \) on \( \mathcal{K} \), with its domain containing all \( \mathcal{K}_\alpha \)'s, with \( \theta_\alpha = \theta|_{\mathcal{K}_\alpha} \in \mathcal{A}_\alpha \) satisfying \( \Delta(\theta) = (\theta \otimes \theta), S(\theta) = \theta^{-1} \), and \( S(\theta^{-1}) = \theta \).

(b) \( S^2(a) = \theta^{-1}a\theta \) for all \( a \in \mathcal{A}_0 \).

(c) We can choose a positive faithful left invariant functional (to be referred to as left haar measure later on) \( \phi \) and a positive faithful right invariant functional (to be referred to as right haar measure) \( \psi \) such that \( \psi(a) = \phi(a\theta^2) = \phi(\theta^2a) \) for \( a \in \mathcal{A}_0 \).

(d) \( \phi(S^2(a)) = \phi(a), \psi(S^2(a)) = \psi(a) \) for all \( a \in \mathcal{A}_0 \), where \( \phi, \psi \) as in (c).

Note that we may have to multiply the left and right invariant functionals \( \phi \) and \( \psi \) we constructed earlier by some positive constant in order to make them satisfy the property (c) above.

We say that a unitary element in \( \mathcal{M}(\mathcal{L}(\mathcal{H} \otimes \mathcal{A})) \equiv \mathcal{M}(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}) \) is a unitary representation of the discrete quantum group \( \mathcal{A} \) if \( (id \otimes \Delta)(U) = U_{12}U_{13} \), and \( (id \otimes S)(U) = U^* \). Note that the second equality has to be understood in the sense of the definition of \( S \) on the algebraic multiplier,
Proof :-

Let us now extend the definition of $\phi$ and $\psi$ on a larger set than $\mathcal{A}_0$ as follows. For a nonnegative element $a \in \mathcal{M}(\mathcal{A}) \subseteq \mathcal{B}(\mathcal{K})$, we define $\phi(a)$ as the limit of $\phi_J(a)$, whenever this limit exists as a finite number, and where $J$ is any finite subset of $I$. $\phi_J(.) \triangleq \phi(e_J) = \phi(e_J)$, and the limit is taken over the net of finite subsets of $I$ partially ordered by inclusion. Similarly, we set $\psi(a) = \lim_J \psi(e_Ja)$ whenever the limit exists as a finite number. Since a general element $a \in \mathcal{M}(\mathcal{A})$ can be canonically written as a linear combination of four nonnegative elements, and extend the definition of $\phi$ on $\mathcal{M}(\mathcal{A})$ by linearity. For any nonnegative $X \in \mathcal{M}(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A})$ (where $\mathcal{H}$ is some Hilbert space), we define $(id \otimes \phi)(X)$ as the limit in the weak-operator topology (if it exists as a bounded operator) of the net $(id \otimes \phi_J)(X)$ over finite subsets $J \subseteq I$, and extend this definition for a general $X \in \mathcal{M}(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A})$ in the usual way. Similar definition will be given for $(id \otimes \psi)$.

**Lemma 2.1** If we choose $\mathcal{H} = \mathcal{K}$ in the above, and take any $a \in \mathcal{M}(\mathcal{A})$ such that $\phi(a)$ is finite, then $(id \otimes \phi)(\Delta(a)) = \phi(a)1_{\mathcal{M}(\mathcal{A})}$.

Proof :-

For any nonnegative $X \in \mathcal{M}(\mathcal{A} \otimes \mathcal{A})$, and any positive operator $P \in \mathcal{A}_0 \otimes \mathcal{A}_0$, with $0 \leq P \leq 1$, such that $P$ and $X$ commute, it is easy to see that $(id \otimes \phi_J)(PX) \leq (id \otimes \phi_J)(X)$, for any finite subset $J$ of $I$. By choosing large enough $J$ one can ensure that $P \leq (1 \otimes e_J)$, so that $(id \otimes \phi_J)(PX) = (id \otimes \phi)(PX)$ ($PX$ is in $\mathcal{A}_0 \otimes \mathcal{A}_0$, so $(id \otimes \phi)(PX)$ makes sense). So, for any vector $\xi \in \mathcal{K}$, $\sup_P < \xi, (id \otimes \phi)(PX)\xi > \leq \sup_J < \xi, (id \otimes \phi_J)(X)\xi >$, where the supremum in the left hand side is taken over all positive $P \in \mathcal{A}_0 \otimes \mathcal{A}_0$ with $P \leq 1$, and commutes with $X$. On the other hand, for fixed finite subsets $J, K$, $(e_K \otimes e_J)$ is one such $P$, and thus $\sup_P < \xi, (id \otimes \phi)(PX)\xi > \geq < e_K \xi, (id \otimes \phi_J)(X) e_K \xi >$, and taking limit over $K$, we conclude that $\sup_P < \xi, (id \otimes \phi)(PX)\xi > \geq \sup_J < \xi, (id \otimes \phi_J)(X)\xi >$, which proves that they are equal, and hence $(id \otimes \phi)(X)$ exists as a bounded operator if and only if the weak-operator limit of $(id \otimes \phi)(P_X)$ exists as a bounded operator, over any net $P_\nu$ of nonnegative operators in $\mathcal{A}_0 \otimes \mathcal{A}_0$, commuting with $X$, and such that $P_\nu \uparrow 1$. Using this fact, we see that for nonnegative $a \in \mathcal{M}(\mathcal{A})$, $(id \otimes \phi)(\Delta(a)) = \lim_{J,K} (id \otimes \phi)((e_K \otimes 1)\Delta(e_J)\Delta(a)) = \lim_{J,K} (id \otimes \phi)((e_K \otimes 1)\Delta(\Delta(e_J)))$, where $J, K$ are varied over all finite subsets of $I$, and we have used the fact that $e_K$’s are central projections, and thus $\Delta(e_J)$ commutes.
with $\Delta(a)$. From the above expression, by using the left invariance of $\phi$ on $\mathcal{A}_0$, and then taking limits, the desired result follows.

We remark that an analogous fact is true for $\psi$.

We shall now define a $\ast$-algebra structure on $\mathcal{A}'_0$, and then identify $\mathcal{A}_0$ with suitable elements of $\mathcal{A}'_0$, thereby equipping $\mathcal{A}_0$ with this new $\ast$-algebra structure, and then consider suitable $C\ast$-completions. This will give rise to the analogues of the full and reduced group $C\ast$-algebra in the framework of discrete quantum groups. Following [8] and others, we define $f \ast g$ for $f, g \in A_0$ by $(f \ast g)(a) := (f \otimes g)(\Delta(a))$, $a \in A_0$. Note that since $f, g$ have finite supports, there is some finite subset $J$ of $I$ such that $(f \otimes g)(\Delta(a)) = (f \otimes g)((e_J \otimes e_J)\Delta(a))$, and since $(e_J \otimes e_J)\Delta(a) \in A_0 \otimes_{\text{alg}} A_0$, $f \ast g$ is well defined. We also define an adjoint by $f^\ast(a) := \bar{\theta} - 2S^{-1}(f(a))$, $a \in A_0$. We now define for each $a \in A_0$, an element $\psi_a \in A'_0$ by $\psi_a(b) := \psi(ab)$. It is easy to verify the following by using standard formulae involving $\Delta$ and $S$.

**Proposition 2.2** For $a, b \in A_0$, $\psi_a \ast \psi_b = \psi_{a \ast b}$, where $a \ast b := (id \otimes \psi)((1 \otimes b)((id \otimes S^{-1})(\Delta(a))))) = (\phi \otimes \psi)((a \otimes 1)((S \otimes id)(\Delta(b))))$. Furthermore, $\psi_a^\ast = \psi_{a^\ast}$, where $a^\ast := \theta^{-2}S^{-1}(a^\ast)$.

We denote by $\hat{A}_0$ the set $A_0$ equipped with the $\ast$-algebra structure given by $(a, b) \mapsto a \ast b, a \mapsto a^\ast$ described by the above proposition. There are two different natural ways of making $\hat{A}_0$ into a $C\ast$-algebra, and thus we obtain the so-called reduced $C\ast$-algebra $\hat{A}_r$ and the free or full $C\ast$-algebra $\hat{A}_f$. This is done in a similar way as in the classical case: one can realize elements of $\hat{A}_0$ as bounded linear operators on the Hilbert space $L^2(\phi)$ (the GNS-space associated with the positive linear functional $\phi$, see [11] and [6] for details) and complete $\hat{A}_0$ in the norm inherited from the operator-norm of $B(L^2(\phi))$ to get $\hat{A}_r$. The definition of $\hat{A}$ is slightly more complicated and involves the realization of $\hat{A}_0$ as elements of the Banach $\ast$-algebra $L^1(\phi)$ (see [8] and other relevant references) and then taking the associated universal $C\ast$-completion. However, it is not important for us how the explicit constructions of these two $C\ast$-algebras are done; we refer to [11], [6] for that; all we need is that $\hat{A}_0$ is dense in both of them in the respective norm-topologies. It should also be mentioned that exactly as in the classical case, there is a canonical surjective $C\ast$-homomorphism from $\hat{A}$ to $\hat{A}_r$. 


3 Construction of the analytic assembly map

In this section, we shall show how one can construct an analogue of the Baum-Connes analytic assembly map for the action of the discrete quantum group \( \mathcal{A}_0 \) on some \( \mathcal{C}^\ast \)-algebra, under some additional assumptions on the action, which may be called “properness and \( \mathcal{A} \)-compactness”, since these assumptions are actually weaker than having a proper and \( G \)-compact action in the classical situation of an action by a group \( G \). Our construction is analogous to that described in, for example, [13],[14], for the discrete group. We essentially translate that into our noncommutative framework step by step, and verify that it really goes through. However, in case \( S^2 \) is not identity, it is somewhat tricky to give the correct definition of \( \hat{\mathcal{A}}_0 \)-valued inner product, and prove the required properties, as one has to suitably incorporate the modular operator \( \delta \).

Let \( \mathcal{C} \) be a \( \mathcal{C}^\ast \)-algebra (possibly nonunital). Assume furthermore that there is an action of the quantum group \( \mathcal{A} \) on it, given by \( \Delta : \mathcal{C} \to \mathcal{M}(\mathcal{C} \otimes \mathcal{A}) \), which is coassociative \( \mathcal{C}^\ast \)-homomorphism, and assume also that there is a dense \( \ast \)-subalgebra \( \mathcal{C}_0 \) of \( \mathcal{C} \) such that the following conditions are satisfied:

A1 \( \Delta_{\mathcal{C}}(c)(c' \otimes 1) \in \mathcal{C}_0 \otimes_{\text{alg}} \mathcal{A}_0 \) for all \( c, c' \in \mathcal{C}_0 \);
A2 \( \Delta_{\mathcal{C}}(c)(1 \otimes a) \in \mathcal{C}_0 \otimes_{\text{alg}} \mathcal{A}_0 \) for all \( c \in \mathcal{C}_0, a \in \mathcal{A}_0 \);
A3 There is a positive element \( h \in \mathcal{C}_0 \) such that

\[
(id \otimes \phi)(\Delta_{\mathcal{C}}(h^2)) = 1,
\]

or equivalently \( (id \otimes \phi)(\Delta_{\mathcal{C}}(h^2)(c \otimes 1)) = c, \forall c \in \mathcal{C}_0 \).

Remark 3.1 In the classical situation, when \( \mathcal{A} \) is \( C_0(G) \) for some discrete group, and \( \mathcal{A}_0 = C_c(G) \), \( \mathcal{C} = C_0(X) \) for some locally compact Hausdorff space \( X \) equipped with an \( G \)-action such that \( X \) is \( G \)-compact and \( G \)-proper, one can take \( \mathcal{C}_0 = C_c(X) \), and it is easy to verify that with this choice of \( \mathcal{C}_0 \), the conditions A1, A2, A3 are satisfied. It is rather straightforward to see A1 and A2. The construction of a positive function \( h \) satisfying A3 can be found in [13]. Thus, the above conditions are in some sense noncommutative generalization of \( G \)-proper and \( G \)-compact actions.

Remark 3.2 Let us note that the above assumptions are indeed satisfied in a typical situation, namely for nice “quantum quotient spaces” corresponding to “quantum subgroups” of the discrete quantum group \( \mathcal{A} \). Indeed, from [11],
the existence of an element $h$ as in $A_3$ above follows, if we take $C = A$, and $C_0 = A_0$. The same thing will trivially hold if $C$ is taken to be a direct sum of finitely many copies of $A$, with the natural action of $A$. More importantly, there is a natural generalization of the notion of subgroups and quotient spaces for quantum groups, which is by now more or less well-known and standard in this theory (see, for example, [3] for these concepts in the context of compact quantum groups, and note that for more general quantum groups they can be easily extended). It is not difficult to see, by using the fact that our assumptions $A_1, A_2, A_3$ are valid for $C = A, C_0 = A_0$, that the same thing will be true if we take $C$ to be the quotient by some compact (i.e. finite dimensional in this case) quantum subgroup of $A$. This is of particular interest in view of the fact that one of the models for the universal space for proper actions of a classical discrete second countable group involves some kind of “infinite join” of some set constructed out of disjoint union over all possible quotient spaces by finite subgroups of the group. Thus, if a similar construction can be done in the noncommutative framework starting from the definition of proper actions as proposed in [3], then it seems very likely that using the techniques of the present article an analytic assembly map can be defined on the “quantum universal space” for “quantum proper action”, and hence a precise formulation of the Baum-Connes conjecture for discrete quantum groups will turn into a reality.

Now, our aim is to construct maps $\mu_i: KK^A(C, C') \to KK^i(\hat{C}, \hat{A}) \equiv K_i(\hat{A})$, and $\mu^*_i: KK^A(C, C') \to KK^i(\hat{C}', \hat{A}_r) \equiv K_i(\hat{A}_r)$, for $i = 0, 1$, i.e. even and odd cases. For simplicity let us do it for $i = 1$ only, the other case can be taken care of by obvious modifications. We have chosen the convention of [13] to treat separately odd and even cases, instead of treating both of them on the same footing as in the original work of Kasparov or in [7]. This is merely a matter of notational simplicity. For the definition and properties of equivariant KK groups $KK^A(\ldots)$, we refer to the paper by Baaj and Skandalis ([3]) (with the easy modifications of their definitions to treat odd and even cases separately).

Let $(U, \pi, F)$ be a cycle (following [3]) in $KK^A(C, C')$, i.e.\(\begin{align*}
(\text{i}) & \quad U \in \mathcal{L}(\mathcal{H} \otimes A) \equiv \mathcal{M}(\mathcal{B}_0(\mathcal{H}) \otimes A) \text{ is a unitary representation of } A, \text{ where } \mathcal{H} \text{ is a separable Hilbert space, i.e. } U \text{ is unitary and } (id \otimes \Delta)(U) = U_{12}U_{13}, (id \otimes S)(U) = U^*; \\
(\text{ii}) & \quad \pi: C \to B(\mathcal{H}) \text{ is a nondegenerate } \ast\text{-homomorphism such that } (\pi \otimes id)(\Delta_C(a)) = U(\pi(a) \otimes 1)U^*, \forall a \in C; \\
(\text{iii}) & \quad F \in B(\mathcal{H}) \text{ is self-adjoint, } [F, \pi(c)], \pi(c)(F^2 - 1) \in B_0(\mathcal{H}) \forall c \in C, \text{ and} \end{align*}\)
\[(F \otimes 1) - U(F \otimes 1)U^* \in B_0(H) \otimes A.\]

We say that a cycle \((U, \pi, F)\) is equivariant (or \(F\) is equivariant) if \(U(F \otimes 1)U^* = F \otimes 1\). We say that \(F\) is properly supported if for any \(c \in C_0\), there are finitely many \(c_1, \ldots c_k, b_1, \ldots, b_k \in C_0\) and \(A_1, \ldots A_k \in B(H)\) (all depending on \(c\)) such that \(F\pi(c) = \sum_i \pi(c_i)A_i\pi(b_i)\).

Before we proceed further, let us make the following convention: we canonically embed \(A\) in the set of bounded operators on \(K\), as described before, and for any element \(A \in B(H)\), we shall denote by \(\tilde{A}\) the element \(A \otimes 1_K\) in \(B(H \otimes K)\).

**Theorem 3.3** Given a cycle \((U, \pi, F)\), we can find a homotopy-equivalent cycle \((U, \pi, F')\) such that \((U, \pi, F')\) is equivariant and \(F'\) is properly supported.

**Proof**: 
Since \(\pi\) is nondegenerate, we can choose a net \(e_\nu\) of elements from \(C_0\) such that \(\pi(e_\nu)\) converges to the identity of \(B(H)\) in the strict topology, i.e. in the strong \(*\)-topology. Now, let \(X_\nu := \pi(e_\nu)^*U\pi(h)\tilde{F}\pi(h)U^*\pi(e_\nu) = \pi(e_\nu)^*(\pi \otimes id)(\Delta_C(h))U\tilde{F}U^*(\pi \otimes id)(\Delta_C(h))\pi(e_\nu)\). Since by our assumption \(\pi(\epsilon_\nu)\Delta_C(h) \in \mathcal{C}_0 \otimes_{alg} A_0\), and similar thing is true for \(\Delta_C(h)\epsilon_\nu\), it is easy to see that \(X_\nu\) is of the form \(X_\nu = \sum_j (\pi(c_j) \otimes a_j)(U\tilde{F}U^*)(\pi(c'_j) \otimes a'_j)\), for some finitely many \(c_j, c'_j \in \mathcal{C}_0\) and \(a_j, a'_j \in A_0\). Choosing a suitably large enough finite subset \(I_1\) of \(I\), we can assume that all the \(a_j, a'_j\)'s are in the support of \(e_{I_1}\), and hence it is easy to see that \(X_\nu \in B(H) \otimes_{alg} (e_{I_1}A_0e_{I_1})\), so \((id \otimes \phi)(X_\nu)\) is finite. Similarly, \((id \otimes \phi)(\pi\epsilon_\nu)^*U\pi(h^2)U^*\pi(e_\nu)\) is finite, and by assumption \(A_3\), is equal to \((\pi(e_\nu^*e_\nu) \otimes 1)\). Now, from the operator inequality \(-\|F\|1 \leq F \leq \|F\|1\), we get the operator inequality

\[-\pi(\epsilon_\nu)^*U\pi(h^2)U^*\pi(\epsilon_\nu)\|F\| \leq X_\nu \leq \pi(\epsilon_\nu)^*U\pi(h^2)U^*\pi(\epsilon_\nu)\|F\|;\]

from which it follows after applying \((id \otimes \phi)\) that

\[-\pi(e_\nu^*e_\nu)\|F\| \leq (id \otimes \phi)X_\nu \leq \pi(e_\nu^*e_\nu)\|F\|.\]

Since \(\pi(e_\nu^*e_\nu) \rightarrow 1_{B(H)}\) in the strong operator topology, one can easily prove by the arguments similar to those in [13] that \((id \otimes \phi)(X_\nu)\) converges in the strong operator topology of \(B(H)\), and let us denote this limit by \(F'\). It is also easy to see that in fact \(F' = (id \otimes \phi)(U(\pi(h)F\pi(h) \otimes 1)U^*)\), where we have used the extended definition of \((id \otimes \phi)\) on \(\mathcal{M}(B_0(H) \otimes A)\) as discussed in the previous section.
Fix some \( c \in \mathcal{C}_0 \). Clearly we have \( F'\pi(c) = (id \otimes \phi)(U\pi(h)\tilde{F}\pi(h)U^*\pi(c)) \).
Now, note that \( U\pi(h)\tilde{F}\pi(h)U^*\pi(c) = (\pi \otimes id)(\Delta_C(h))(U\tilde{F}\nu^* \otimes id)((\Delta_C(h)(c \otimes 1)) \).
Since \( \Delta_C(h)(c \otimes 1) \in \mathcal{C}_0 \otimes \mathrm{alg} \mathcal{A}_0 \), we can write it as a finite sum of the form \( \sum_{i,j,\alpha} x_{ij}^\alpha \otimes e_{ij}^\alpha \), where \( x_{ij}^\alpha \in \mathcal{C}_0 \), and \( e_{ij}^\alpha \)'s are the matrix units of \( \mathcal{A}_\alpha \), as described in the previous section, and \( \alpha \) in the above sum varies over some finite set \( T \), say, with \( i, j = 1, \ldots, n_\alpha \). Thus, \( U\pi(h)\tilde{F}\pi(h)U^*\pi(c) = \sum_{\alpha,i,j}(\pi \otimes id)(\Delta_C(h))(1 \otimes e_{ij}^\alpha) (Fx_{ij}^\alpha \otimes 1) \).
Since for each \( \alpha, i, j, \Delta_C(h)(1 \otimes e_{ij}^\alpha) \in \mathcal{C}_0 \otimes \mathrm{alg} \mathcal{A}_0 \), we can write \( \Delta_C(h)(1 \otimes e_{ij}^\alpha) \) as a finite sum of the form \( \sum x_p \otimes a_p \)
with \( x_p \in \mathcal{C}_0, a_p \in \mathcal{A}_0 \), and hence \( U\pi(h)\tilde{F}\pi(h)U^*\pi(c) \) is clearly a finite sum of the form \( \sum_k \pi(c_k)A_k \pi(c'_k) \otimes a_k \), with \( c_k, c'_k \in \mathcal{C}_0, A_k \in \mathcal{B}(\mathcal{H}) \) and \( a_k \in \mathcal{A}_0 \).
From this it follows that \( F' \) is properly supported.

It is easy to show the equivariance of \( F' \). Indeed, \( U(F' \otimes 1)U^* = (id \otimes id \otimes \phi)((id \otimes \Delta)(U\pi(h)\tilde{F}\pi(h)U^*)) \) by using the fact that \((id \otimes \Delta)(U) = U_{12}U_{13} \) and \( \Delta \) is a *-homomorphism. Now, since it is easy to see using what we have proved in the earlier section that \((id \otimes id \otimes \phi)((id \otimes \Delta)(X)) = (id \otimes \phi)(X) \otimes 1 \), for \( X \in \mathcal{M}(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}) \), from which the equivariance of \( F' \) follows.

Finally, we can verify that \( \pi(c)(F - F') \) is compact for \( c \in \mathcal{C}_0 \), hence for all \( c \in \mathcal{C} \), by very similar arguments as in [3], adapted to our framework in a suitable way. We omit this part of the proof, which is anyway straightforward.

Let us make some more notational convention and note some simple but useful facts. Since the von Neumann algebra generated by \( \mathcal{A} \) in \( \mathcal{B}(\mathcal{K}) \) is the direct sum of matrix algebras \( \bigoplus_{\alpha \in I} \mathcal{A}_\alpha = \bigoplus_{\alpha} \mathcal{M}_{n_\alpha}, \) where \( \bigoplus \) has been used to denote the weak (or equivalently strong) operator closure of the algebraic direct sum, it is clear that for any \( X \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{A}'' = \bigoplus_{\alpha} \mathcal{B}(\mathcal{H}) \otimes \mathcal{M}_{n_\alpha}, \) so in particular for \( X \in \mathcal{M}(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}) \), we can write \( X = \sum_{\alpha,i,j} X_{ij}^\alpha \otimes \mathcal{E}_{ij}^\alpha \) as a strongly convergent sum, with \( X_{ij}^\alpha \in \mathcal{B}(\mathcal{H}) \).
For \( \xi \in \mathcal{H} \), it is clear that \( X\xi = \sum_{\alpha,i,j} X_{ij}^\alpha \xi \otimes \mathcal{E}_{ij}^\alpha \in \mathcal{H} \otimes \mathcal{A}'' \), where we recall that \( \mathcal{H} \otimes \mathcal{A}'' \) is the smallest Hilbert von Neumann \( \mathcal{A}'' \)-module generated by the algebraic right \( \mathcal{A} \)-module \( \mathcal{H} \otimes \mathrm{alg} \mathcal{A} \). Recall from the Introduction that \( X\xi \in \mathcal{H} \otimes \mathcal{M}(\mathcal{A}) \), and \((id \otimes \Delta)(X\xi) = (id \otimes \Delta)(X)\xi \forall \xi \in \mathcal{H}, X \in \mathcal{M}(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}) \).

Furthermore, for any finite subset \( J \subseteq I \), it is clear that \( X_J := (1 \otimes e_J)X = \sum_{\alpha \in I, i,j=1,1 \ldots, n_\alpha} X^\alpha_{ij} \).
Now, recall that there is a bijection of the index set \( I \), say \( \alpha \to \alpha' \), such that \( S(e_\alpha) = e_{\alpha'}, S(e_{\alpha'}) = e_\alpha \). In particular, \( S(e_\alpha + e_{\alpha'}) = e_\alpha + e_{\alpha'} \). Thus, given any finite \( J \subseteq I \), we can enlarge \( J \) suitably such that \( S(e_J) = e_J \). Since \( e_J \)'s are central projections, it is easy to see that \((id \otimes S)(U_J) = U_J^* = (U_J)^* \) whenever \( S(e_J) = e_J \). Now, for \( \xi \in \mathcal{H}_0 \), say \( \xi = \pi(c)\eta, c \in \mathcal{C}_0, \eta \in \mathcal{H} \), we have that \( \pi(h)U\xi = ((\pi \otimes id)((h \otimes
1) $\Delta_C(c)U\eta = ((1 \otimes e_J)(\pi \otimes id)((h \otimes 1)\Delta_C(c))U)\eta$, where we have chosen some finite subset $J$ of $I$ by using the fact that $(h \otimes 1)\Delta_C(c) \in C_0 \otimes A_0$, and if necessary by enlarging $J$ suitably, assumed that $S(e_J) = e_J$. Thus, we can write $(\pi(h)U)\xi$ as a finite sum over some set indexed by $p$ (say) of the form $\sum_p \pi(h)U_1^{(p)} \otimes U_2^{(p)}$, where $U_1^{(p)} \in B(H), U_2^{(p)} \in A_0$, and we also have $\sum_p U_1^{(p)*} \otimes U_2^{(p)} = \sum_p U_1^{(p)} \otimes S(U_2^{(p)})$.

Let $\mathcal{H}_0 := \pi(C)\mathcal{H}$. By the fact that $F'$ is properly supported, it is clear that $F'\mathcal{H}_0 \subseteq \mathcal{H}_0$. We now equip $\mathcal{H}_0$ with a right $A_0$-module structure. Define

$$(\xi,a) := (id \otimes \psi_{\theta^{-1}S(a)\theta^{-2}}(U))\xi,$$

for $\xi \in \mathcal{H}_0, a \in A_0$. It is useful to note that for $c \in B(\mathcal{H}) \otimes_{alg} A_0$, $(id \otimes \psi_{\theta^{-1}S(a)\theta^{-2}}(c) = (id \otimes (\psi_{\theta a} \circ S^{-1}))(c)$ by simple calculation using the properties of $\psi$ and $\theta$ described in the previous section. By taking suitable limit, it is easy to extend this for $c \in M(B_0(\mathcal{H}) \otimes A)$, in particular for $U$. So we also have that $\xi.a = (id \otimes \psi_{\theta a} \circ S^{-1})(U)\xi$.

**Proposition 3.4** $(\xi,a).b = \xi.(a * b)$ for $a, b \in A_0, \xi \in \mathcal{H}_0$. That is, $(\xi,a) \mapsto \xi.a$ is indeed a right $A_0$-module action.

**Proof:**

Choosing finite subsets $J, K$ of $I$ such that $\theta^{-1}S(a)\theta^{-2} \in supp(e_K), \theta^{-1}S(b)\theta^{-2} \in supp(e_J)$, we have that

$$(\xi,a).b$$

$$= \sum_{\alpha \in J, i,j=1,...,\alpha} U_1^{(\alpha)}(\xi,a)\psi_{\theta^{-1}S(b)\theta^{-2}}(e_{ij}^{(\alpha)})$$

$$= \sum_{\alpha \in J, i,j=1,...,\alpha, \beta \in K, k,l=1,...,\beta} U_1^{(\alpha)}U_2^{(\beta)}\pi_{\theta^{-1}S(b)\theta^{-2}S(\theta a)\theta^{-2}}(e_{ij}^{(\alpha)}e_{kl}^{(\beta)}S(S(\theta a)\theta^{-2}))$$

$$= (id \otimes \psi_{\theta^{-1}S(b)\theta^{-2}} \otimes \psi_{\theta^{-1}S(a)\theta^{-2}})(U_{12}U_{13})\xi$$

$$= (id \otimes \psi_{\theta^{-1}S(b)\theta^{-2}} \otimes \psi_{\theta^{-1}S(a)\theta^{-2}})((id \otimes (\Delta))(U))\xi$$

$$= (id \otimes (\psi_{\theta^{-1}S(b)\theta^{-2}} \otimes \psi_{\theta^{-1}S(a)\theta^{-2}}))(U)\xi$$

$$= (id \otimes \psi_{\theta^{-1}S(b)\theta^{-2}S(a)\theta^{-2}})(U)\xi.$$

Now, by a straightforward calculation using the properties of $\psi$, $S$ and $\theta$ one can verify that $(\theta^{-1}S(b)\theta^{-2} \ast \theta^{-1}S(a)\theta^{-2}) = \theta^{-1}S(a * b)\theta^{-2}$, which completes the proof.

For $\xi, \eta \in \mathcal{H}_0$, say of the form $\xi = \pi(c_1)\xi', \eta = \pi(c_2)\eta'$, it is clear that $T_{\xi\eta}(U)$ is an element of $A_0$, since $(\pi(c_1)U\pi(c_2) = (\pi \otimes id)((c_1^1 \otimes 1)\Delta_C(c_2))U$, where $c_1^1 \in C_0$ and $c_2^1 \in C_0$. Hence, $T_{\xi\eta}(U)$ is a right $A_0$-module action.
which belongs to \((\pi(C_0) \otimes_{\text{alg}} A_0)\mathcal{M}(B_0(H) \otimes A) \subseteq B(H) \otimes_{\text{alg}} A_0\). We define

\[
\langle \xi, \eta \rangle_{\hat{A}_0} := \theta^{-1}T_{\xi\eta}(U) \in \hat{A}_0
\]

identifying \(A_0\) as the *-algebra \(\hat{A}_0\) described earlier.

We shall show that \(H_0\) with the above right \(\hat{A}_0\)-action and the \(\hat{A}_0\)-valued bilinear form \(< \cdot, \cdot >_{\hat{A}_0}\) is indeed a pre-Hilbert \(\hat{A}_0\)-module. However, instead of proving it directly, we shall prove it by embedding \(H_0\) into the free pre-Hilbert \(\hat{A}_0\)-module \(F_0 := H \otimes_{\text{alg}} \hat{A}_0\) (with the natural \(\hat{A}_0\)-action given by \((\xi \otimes a) :\xi \otimes (a * b), \xi \in H, a, b \in A_0 \equiv \hat{A}_0\)), and showing that the pull back of the natural \(\hat{A}_0\)-valued inner product of \(F_0\) (which is given by \(< \xi \otimes a, \eta \otimes b > := < \xi, \eta > a \ast b, \xi, \eta \in H, a, b \in A_0 \equiv \hat{A}_0\) coincides with 
\(< \cdot, \cdot >_{\hat{A}_0}\).

Define \(\Sigma : H_0 \to F_0\) by

\[
\Sigma(\xi) := ((\pi(h) \otimes \theta^{-1})U)\xi,
\]

for \(\xi \in H_0\). Note that by writing \(\xi = \pi(c)\xi'\) for some \(\xi' \in H, c \in C_0\), we have that \(((\pi(h) \otimes 1)U)\xi = ((\pi \otimes id)((h \otimes 1)\Delta(c))U)\xi',\) and since \((\pi \otimes id)((h \otimes 1)\Delta(c))U \in \pi(C_0) \otimes_{\text{alg}} A_0\), the range of \(\Sigma\) is clearly in \(H \otimes_{\text{alg}} A_0\). We now prove that \(\Sigma\) is in fact a module map and preserves the bilinear form \(< \cdot, \cdot >_{\hat{A}_0}\) on \(H_0\).

**Proposition 3.5** For \(\xi, \eta \in H_0, a \in A_0\), we have that

(i) \(\Sigma(\xi, a) = \Sigma(\xi)a\).

(ii) \(< \Sigma(\xi), \Sigma(\eta) > = < \xi, \eta >_{\hat{A}_0}\).

**Proof**:

(i) Choose suitable finite set indexed by \(p\) such that \((\pi(h)U)\xi = \sum_p \pi(h)U_1^{(p)} \otimes U_2^{(p)}\), where \(U_1^{(p)} \in B(H), U_2^{(p)} \in A_0\), and also \(\sum_p U_1^{(p)} \otimes U_2^{(p)} = \sum_p U_1^{(p)} \otimes S(U_2^{(p)}).\) Using the facts that \(\Delta(\theta) = \theta \otimes \theta, S^{-1}(\theta) = \theta^{-1}\) and that \(\psi(b\theta) = \psi(\theta)b\forall b \in A_0\), and also the easily verifiable relation \(\psi_a \circ S^{-1} = \psi_{\theta^{-1}S(a)\theta^{-1}}\) for \(a \in A_0\), we have that

\[
\Sigma(\xi)a
\]

\[
= \sum_p \pi(h)U_1^{(p)} \otimes (id \otimes \psi_{\theta^{-1}S(a)\theta^{-1}}) (\Delta(\theta^{-1}U_2^{(p)}))
\]

\[
= \sum_p \pi(h)U_1^{(p)} \otimes (id \otimes \psi_{\theta^{-1}S(a)\theta^{-1}}) (\Delta(U_2^{(p)}))
\]

\[
= (\pi(h) \otimes \theta^{-1} \otimes \psi_{\theta^{-1}S(a)\theta^{-2}})(id \otimes \Delta)(U\xi))
\]
(ii) Choosing suitable finite index sets as explained before, such that \((\pi(h) \otimes 1)U \xi = \sum_p U_1^{(p)} \otimes U_2^{(p)}\), with \(\sum_p U_1^{(p)} \otimes S(U_2^{(p)}) = \sum_p U_1^{(p)*} \otimes U_2^{(p)*}\), and similarly for \((\pi(h) \otimes 1)U \eta\) with the index \(p\) replaced by say \(q\), we can write

\[
< \Sigma(\xi), \Sigma(\eta) > = \sum_{p,q} < U_1^{(p)} \xi, \pi(h^2)U_1^{(q)} \eta > (\theta^{-1}U_2^{(p)})^* (\theta^{-1}U_2^{(q)})
\]

\[
= \sum_{p,q} < \xi, U_1^{(p)*} \pi(h^2)U_1^{(q)} \eta > (\theta^{-2}S^{-1}(\theta^{-1})S^{-1}(U_2^{(p)*})) (\theta^{-1}U_2^{(q)})
\]

\[
= \sum_{p,q} < \xi, U_1^{(p)*} \pi(h^2)U_1^{(q)} \eta > (\theta^{-1}S^{-1}(U_2^{(p)*})) (\theta^{-1}U_2^{(q)})
\]

\[
= \sum_{p,q} < \xi, U_1^{(p)*} \pi(h^2)U_1^{(q)} \eta > \theta^{-1}(U_2^{(p)*} \otimes U_2^{(q)})
\]

\[
= \sum_{p,q} < \xi, U_1^{(p)} \pi(h^2)U_1^{(q)} \eta > (\phi \otimes \theta^{-1})(U_2^{(p)} \otimes 1)(S \otimes id)(\Delta(U_2^{(q)}))...(1),
\]

using the fact that \(\sum_p U_1^{(p)*} \otimes U_2^{(p)*} = \sum_p U_1^{(p)} \otimes S(U_2^{(p)})\) and the simple observation that \((\theta^{-1}x)*(\theta^{-1}y) = \theta^{-1}(x*y)\). Now,

\[
\sum_p \pi(h^2)U_1^{(q)} \eta \otimes ((S \otimes id)(\Delta(U_2^{(q)})))
\]

\[
= (\pi(h^2) \otimes S \otimes id)((id \otimes \Delta)(U \eta))
\]

\[
= (\pi(h^2) \otimes S \otimes id)((id \otimes \Delta)(U \eta))
\]

\[
= (\pi(h^2) \otimes id \otimes id)((U^*)_{12}U_{13})\eta....(2).
\]

Thus, from (1) and (2), \(< \Sigma(\xi), \Sigma(\eta) > = (T_{\xi_0} \otimes \phi \otimes \theta^{-1})(U(\pi(h^2) \otimes 1)U^* \otimes 1)U_{13}) = \theta^{-1}T_{\xi_0}(U)\), since \((id \otimes \phi)(U\pi(h^2)U^*) = 1\). This completes the proof.

Note that from the above proposition it follows in particular that \(< \xi, \eta \alpha >_{\tilde{A}_0} = < \Sigma(\xi), \Sigma(\eta \alpha) > = < \Sigma(\xi), \Sigma(\eta) \alpha > = < \Sigma(\xi), \Sigma(\eta) > \alpha = < \xi, \eta \alpha >_{\tilde{A}_0} \alpha\). Similarly, \(< \xi, \eta \xi >_{\tilde{A}_0} = < \eta, \xi >_{\tilde{A}_0}\), and \(< \xi, \xi >\) is a non-negative element in the \(*\)-algebra \(\tilde{A}_0\), since \(< ., >\) on \(\mathcal{F}_0\) is a nonnegative definite form.
Given any $C^*$-algebra which contains $A_0$ as a dense $*$-subalgebra, we can complete $F_0$ w.r.t. the corresponding norm to get a Hilbert $C^*$-module in which $F_0$ sits as a dense submodule. Let us denote by $F$ and $F_r$ the Hilbert $\hat{A}$ and $\hat{A}_r$-modules respectively obtained in the above mentioned procedure, by considering $A_0$ as dense $*$-subalgebra of $\hat{A}$ and $\hat{A}_r$ respectively. The corresponding completions of $H_0$ will be denoted by $E$ and $E_r$ respectively. By construction, $\Sigma$ extends to an isometry from $E$ to $F$ and also from $E_r$ to $F_r$. We denote both these extensions by the same notation $\Sigma$, as long as no confusion arises. Clearly, $E \cong \Sigma E \subseteq F$ as closed submodule, and similar statement will be true for $E_r$ and $F_r$.

Let us now compute the explicit form of $\Sigma^*$. Fix $\xi, \eta \in H_0$ and $a \in A_0$. Using the easy observation that $(\theta^{-1}x)^* = \theta^{-1}x^*$ for $x \in A_0$, we have that

$$< \Sigma(\xi), \eta \otimes a > = \sum_p < \pi(h)U_1^{(p)}\xi, \eta > (\theta^{-1}S^{-1}(U_2^{(p)^*})* a$$

$$= \sum_p < \xi, U_1^{(p)^*}\pi(h)\eta > (\theta^{-1}S^{-1}(U_2^{(p)^*}))* a$$

$$= (\theta^{-1}T_{\xi,\pi(h)\eta}(U))* a,$$

using the fact that $\sum_p U_1^{(p)^*} \otimes S^{-1}(U_2^{(p)^*}) = \sum_p U_1^{(p)} \otimes U_2^{(p)}$. Now, $\theta^{-1}T_{\xi,\pi(h)\eta}(U)* a = < \xi, \pi(h)\eta >_{A_0} * a = < \xi, (\pi(h)\eta)a >_{A_0}$. Thus,

$$\Sigma^*(\eta \otimes a) = (\pi(h)\eta)a = (id \otimes \psi_{\theta-1}s_{(a)\theta-1})(U)\pi(h)\eta.$$

Let us now prove the following important result.

**Theorem 3.6** Let $T \in \mathcal{B}(H)$ be equivariant, i.e. $U(T \otimes 1)U^* = T \otimes 1$, and also assume that it satisfies the following condition which is slightly weaker than being properly supported:

For $c \in C_0$, one can find $c_1, ..., c_m \in C_0, A_1, ..., A_m \in \mathcal{B}(H)$ (for some integer $m$) such that $T\pi(c) = \sum_k \pi(c_k)A_k$.

Then we have the following:

(i) $T(\xi a) = (T\xi)a \forall a \in A_0$, and thus $T$ is a module map on the $\hat{A}_0$-module $H_0$. Furthermore, if $T$ is self-adjoint in the sense of Hilbert space, then $< \xi, T\eta >_{\hat{A}_0} = < T\xi, \eta >_{\hat{A}_0}$ for $\xi, \eta \in H_0$.

(ii) $T$ is continuous in the norms of $E$ as well as $E_r$, thus admits continuous extensions on both $E$ and $E_r$. We shall denote these extensions by $T$ and $T_r$ respectively.

(iii) If $T\pi(h)$ is compact in the Hilbert space sense, i.e. in $B_0(H)$, then $T$ and $T_r$ are compact in the Hilbert module sense.
Proof:

(i) is obvious from the definition of the right $\hat{A}_0$ action, the definition of $<\cdots>_\hat{A}_0$, and the equivariance of $T$. Let us prove (ii) and (iii) only for $T$, as the proof for $T_e$ will be exactly the same. In fact, it is enough to show that $\Sigma T\Sigma^*$ is continuous on $F$, and is compact if $T\pi(h)$ is compact in the Hilbert space sense. Let us introduce the following notation: for $X \in \mathcal{M}(B_0(\mathcal{H}) \otimes \mathcal{A}), a \in \mathcal{A}_0, \eta \in \mathcal{H}$, define $X * b := (id \otimes id \otimes \psi_b \circ S^{-1}) \cdot ((id \otimes \Delta)(X))$, and $X * (\eta \otimes a) := (X * a) \eta$. Note that clearly $X * a \in \mathcal{M}(B_0(\mathcal{H}) \otimes \mathcal{A})$, so $(X * a) \eta$ makes sense. Now, we observe using the equivariance of $T$ and the explicit formula for $\Sigma^*$ derived earlier that for $\eta \in \mathcal{H}, a \in \mathcal{A}_0$,

$$
\Sigma T\Sigma^*(\eta \otimes a) = (\pi(h) \otimes 1)U\beta,
$$

where $\beta \in \mathcal{H}$ is given by $\beta = (id \otimes \psi_{\theta^{-1}S(a)\theta^{-2}})(U(T\pi(h))\eta)$. Now, by using the fact that $(id \otimes \Delta)(U) = U_{12}U_{13}$, it follows by a straightforward computation that

$$(1 \otimes \theta^{-1})U\beta = (id \otimes \theta^{-1} \circ (\psi_{\theta a} \circ S^{-1}))(\pi(h) \otimes 1)U(T\pi(h))\eta.$$ 

But $\psi_{\theta a}(S^{-1}(b)) = \psi(\theta a S^{-1}(b)) = \psi(a S^{-1}(b) \theta) = \psi(a S^{-1}(\theta^{-1} b)) = (\psi_a \circ S^{-1})((\theta^{-1})\theta^{-1}),$ and hence $(id \otimes \theta^{-1} \circ (\psi_{\theta a} \circ S^{-1}))(\pi(h) \otimes 1)U(T\pi(h))\eta) = (id \otimes id \circ (\psi_a \circ S^{-1}))(\pi(h) \otimes 1)U(T\pi(h))\eta).$ From this, it is clear that

$$(\Sigma T\Sigma^*)(\eta \otimes a) = ((\pi(h) \otimes \theta^{-1}U(T\pi(h) \otimes 1)) \ast (\eta \otimes a).$$

Now, note that $T\pi(h) = \sum_{k=1}^m \pi(c_k)A_k$, for some $c_1, \ldots, c_m \in \mathcal{C}_0, A_1, \ldots, A_m \in \mathcal{B}(\mathcal{H})$, and so we have $(\pi(h) \otimes \theta^{-1}U(T\pi(h) \otimes 1) = \sum_{k=1}^m (1 \otimes \theta^{-1})(\pi \otimes id)((h \otimes 1)\Delta \mathcal{C}(c_k))U(A_k \otimes 1).$ But $(h \otimes 1)\Delta \mathcal{C}(c_k)$ is in $\mathcal{C}_0 \otimes_{\text{alg}} \mathcal{A}_0$ for each $k = 1, \ldots, m$, and thus $(\pi(h) \otimes \theta^{-1})U(T\pi(h) \otimes 1) \in \mathcal{B}(\mathcal{H}) \otimes_{\text{alg}} \mathcal{A}_0$ clearly. Choosing some large enough finite subset $J$ of $I$ such that $(\pi(h) \otimes \theta^{-1})U(T\pi(h) \otimes 1) = \sum_{\alpha \in J} B_{ij}^\alpha \otimes \phi_{ij}^\alpha$, (with $B_{ij}^\alpha \in \mathcal{B}(\mathcal{H})$), it is clear that $\Sigma T\Sigma^* = \sum_{\alpha \in J} \sum_{i,j=1}^m B_{ij}^\alpha \otimes \phi_{ij}^\alpha$, where for $x \in \mathcal{A}_0, L_x : \mathcal{A}_0 \to \mathcal{A}_0$ with $L_x(a) = x \ast a$. As $L_x$ is a norm-continuous map on $A$, the above finite sum shows that $\Sigma T\Sigma^*$ indeed admits a continuous extension on the Hilbert $\hat{A}$-module $F$. This proves (ii).

Furthermore, since $K(\mathcal{H} \otimes \hat{A}) \cong \mathcal{B}(\mathcal{H}) \otimes \hat{A}$, where $K(E)$ means the set of compact (in the Hilbert module sense) opearators on the Hilbert module $E$, it is easy to see that $\Sigma T\Sigma^*$ is compact on $F$ if $B_{ij}^\alpha$’s are compact on the Hilbert space $\mathcal{H}$. Now, $B_{ij}^\alpha = (id \otimes \phi_{ij}^\alpha)(((\pi(h) \otimes \theta^{-1})U(T\pi(h) \otimes 1)) = \ldots$
\(\pi(h) (id \otimes \phi_i^\alpha)(1 \otimes \theta^{-1}) U T \pi(h)\), where \(\phi_i^\alpha\) is the functional on \(A_0\) which is 0 on all \(e_{kl}^\beta\) except \(\beta = \alpha, (kl) = (ij)\), with \(\phi_i^\alpha(e_{ij}^\alpha) = 1\). It follows that \(B_{ij}^\alpha\)'s are all compact if \(T \pi(h)\) is so, which completes the proof.

Now, let us come to the construction of the Baum-Connes maps \(\mu_1 : KK_1^\Lambda(C, C^{'}) \to KK_1(C^{'}, \hat{A})\) and \(\mu_1' : KK_1^\Lambda(C, C^{'}) \to KK_1(C^{'}, \hat{A}_r)\). Let us do it only for \(\mu_1\), as the case of \(\mu_1'\) is similar, and in fact \(\mu_1'\) will be the composition of \(\mu_1\) and the canonical map from \(KK_1(C^{'}, \hat{A})\) to \(KK_1(C^{'}, \hat{A}_r)\) induced by the canonical surjective \(C^*\)-homomorphism from \(\hat{A}\) to \(\hat{A}_r\). Note that an element of \(KK_1(C^{'}, \hat{A})\) is given by the suitable homotopy class \([E, L]\) of a pair of the form \((E, L)\), where \(E\) is a Hilbert \(\hat{A}\)-module and \(L \in L(E)\) (the set of adjointable \(\hat{A}\)-linear maps on \(E\)) such that \(L^* = L, L^2 - 1\) is compact in the sense of Hilbert module. For more details, see for example [7].

**Theorem 3.7** Given a cycle \((U, \pi, F) \in KK_1^\Lambda(C, C^{'})\), let \(F' \equiv F'_1\) be the equivariant and properly supported operator as constructed in [3,3] with a given choice of \(h\) as in that theorem. Then the continuous extension of \(F'_h\) on the Hilbert module \(E\) (as described by the Theorem [3,4]), to be denoted by say \(F'_{h'}\), satisfies the conditions that \((F'_h)^* = F'_{h'}\) (as module map), and \((F'_{h'})^2 - I\) is compact on \(E\). Define

\[\mu_1((U, \pi, F)) := [E, F'_h] \in KK_1(C^{'}, \hat{A}) \cong K_1(\hat{A}).\]

In fact, \([E, F'_h]\) is independent (upto operatorial homotopy) of the choice of \(h\).

**Proof :**
Since \(F'_h\) is equivariant and properly supported, it is clear that \(T_h := (F'_h)^2 - 1\) is equivariant and for any \(c \in C_0\), there are finitely many \(c_1, \ldots, c_m \in C_0, A_1, \ldots, A_m \in B(H)\) such that \(T_h \pi(c) = \sum_k \pi(c_k) A_k\). Furthermore, by the Theorem [3,3], we have that \(\pi(c) T_h\), and hence \(T_h \pi(c)\) is compact operator on \(H\) for every \(c \in C\). So, in particular, \(T_h \pi(h)\) is compact. By Theorem [3,6] it follows that the continuous extension of \(T_h\) on \(E\) is compact in the sense of Hilbert modules. Furthermore, the fact that \((F'_h)^* = F'_{h'}\) is clear from (i) of the Theorem [3,6]. So, \([E, F'_h] \in KK_1(C^{'}, \hat{A})\). Furthermore, as we can see from the proof of the Theorem [3,3], \((F'_h - F'_{h'}) \pi(c) \in B_0(H) \forall c \in C_0\), and so for \(h, h'\) satisfying \(A_3\), we have \((F'_h - F'_{h'}) \pi(c) \in B_0(H)\), and hence by Theorem [3,6], \(F'_h - F'_{h'}\) is compact in the Hilbert module sense. Thus, for each \(t \in [0, 1]\), setting \(F(t) := t F'_{h'} + (1 - t) F'_h\), we have that \(F(t)^2 - I\) is compact on \(E\), and this gives a homotopy in \(KK_1(C^{'}, \hat{A})\) between \([E, F'_h]\) and \([E, F'_{h'}]\).
Acknowledgement:

D. Goswami would like to express his gratitude to I.C.T.P. for a visiting research fellowship during January-August 2002, and to A.O. Kuku and the other organisers of the “School and Conference on Algebraic K Theory and Its Applications” at I.C.T.P. (Trieste) in July 2002. He would also like to thank T. Schick for sending some relevant preprint, and A. Valette and I. Chatterjee for giving useful information regarding some manuscript (yet to be published) by A. Valette.

References

[1] P. Baum, A. Connes, Geometric $K$-theory for Lie groups and foliations, *Enseign. Math. (2)* **46** (2000), no. 1-2, 3–42.

[2] P. Baum, A. Connes and N. Higson, Classifying space for proper actions and $K$-theory of group $C^*$-algebras, “$C^*$-algebras: 1943–1993”, (San Antonio, TX, 1993), 240–291, Contemp. Math., **167**, Amer. Math. Soc., Providence, RI, 1994.

[3] S. Baaj and G. Skandalis, $C^*$-algbras de Hopf et thorie de Kasparov `equivariante, *K-Theory* **2** (1989), no. 6, 686-721.

[4] A. Connes, “Noncommutative Geometry”, Academic Press (1994).

[5] D. A. Ellwood, A New Characterisation of Principal Actions, *J. Funct. Anal.* **173** (2000), 49-60.

[6] E. G. Effros and Z-J. Ruan, Discrete quantum groups. I. The Haar measure, *Internat. J. Math.* **5**(1994), no. 5, 681-723.

[7] K. K. Jensen and K. Thomsen, “Elements of $KK$-Theory”, Mathematics : Theory and Applications, Birkhäuser Boston, Inc., Boston, MA (1991).
[8] J. Kustermans, The analytic structure of an algebraic quantum group, preprint, available at funct-an/9707010.

[9] P. Podles, Symmetries of quantum spaces. Subgroups and quotient spaces of quantum SU(2) and SO(3) groups, Comm. Math. Phys. 170 (1995), no. 1, 1–20.

[10] A. Van Daele, Discrete quantum groups, J. Algebra 180 (1996), no. 2, 431-444.

[11] A. Van Daele, Multiplier Hopf *-algebras with positive integrals: A laboratory for locally compact quantum groups, preprint, available at math.OA/0205285.

[12] A. Van Daele, Multiplier Hopf algebras, Trans. Amer. Math. Soc. , 342 (1994), no. 2, 917–932.

[13] A. Valette, “Introduction to the Baum-Connes Conjecture”, ETH (Zurich) Lecture Notes, Bikhäuser, to appear (also available at http://www.math.ethz.ch/~indira/BC.dvi.)

[14] A. Valette, “On the Baum-Connes assembly map for discrete groups, with an appendix by Dan Kucerovsky”, available at http://www.unine.ch/math/preprints/preprints.html.