THE COMPACT SUPPORT PROPERTY FOR
MEASURE-VALUED DIFFUSIONS

ROSS G. PINSKY

Abstract. The purpose of this article is to give a rather thorough understanding of the compact support property for measure-valued diffusion processes corresponding to semi-linear equations of the form

\[ u_t = Lu + \beta u - \alpha u^p \] in \( \mathbb{R}^d \times (0, \infty) \), \( p \in (1, 2] \);

\[ u(x, 0) = f(x) \] in \( \mathbb{R}^d \);

\[ u(x, t) \geq 0 \] in \( \mathbb{R}^d \times [0, \infty) \).

In particular, we shall investigate how the interplay between the underlying motion (the diffusion process corresponding to \( L \)) and the branching affects the compact support property. In [6], the compact support property was shown to be equivalent to a certain analytic criterion concerning uniqueness of the Cauchy problem for the semilinear parabolic equation related to the measured valued diffusion. In a subsequent paper [7], this analytic property was investigated purely from the point of view of partial differential equations. Some of the results obtained in this latter paper yield interesting results concerning the compact support property. In this paper, the results from [7] that are relevant to the compact support property are presented, sometimes with extensions. These results are interwoven with new results and some informal heuristics. Taken together, they yield a fairly comprehensive picture of the compact support property. Inter alia, we show that the concept of a measure-valued diffusion hitting a point can be investigated via the compact support property, and suggest an alternate proof of a result concerning the hitting of points by super-Brownian motion.

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1. Introduction and Statement of Results

The purpose of this article is to give a rather thorough understanding of the compact support property for measure-valued diffusion processes. In particular, we shall investigate how the interplay between the underlying motion and the branching affects the compact support property. In [6], the compact support property was shown to be equivalent to a certain analytic criterion concerning uniqueness of the Cauchy problem for the semilinear parabolic equation related to the measured valued diffusion. In a subsequent paper [7], this analytic property was investigated purely from the point of view of partial differential equations. Some of the results obtained in this latter paper yield interesting results concerning the compact support property. In this paper, the results from [7] that are relevant to the compact support property are presented, sometimes with extensions. These results are interwoven with new results and some informal heuristics. Taken together, they yield a fairly comprehensive picture of the compact support property. Inter alia, we show that the concept of a measure-valued diffusion hitting a point can be investigated via the compact support property and suggest an alternate proof of a result concerning the hitting of points by super-Brownian motion.

We begin by defining the measure-valued diffusions under study. Let

\[ L = \frac{1}{2} \sum_{i,j=1}^{d} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i}, \]

where \( a_{i,j}, b_i \in C^{\alpha}(R^d) \) and \( \{a_{i,j}\} \) is strictly elliptic; that is, \( \sum_{i,j=1}^{d} a_{i,j}(x)\nu_i\nu_j > 0 \), for all \( x \in R^d \) and \( \nu \in R^d - \{0\} \). Let \( Y(t) \) denote the diffusion process corresponding to the generalized martingale problem for \( L \) on \( R^d \) [12], and denote corresponding probabilities by \( P_{\cdot} \). Denote the lifetime of \( Y(t) \) by \( \tau_\infty \). One has \( \tau_\infty = \lim_{n \to \infty} \tau_n \), where \( \tau_n = \inf\{t \geq 0 : |Y(t)| \geq n\} \). Recall that \( Y(t) \) is called non-explosive (or conservative) if \( P_x(\tau_\infty < \infty) = 0 \), for some,
or equivalently all, \( x \in \mathbb{R}^d \); otherwise \( Y(t) \) is called explosive. The process \( Y(t) \) serves as the underlying motion of the measure-valued diffusion.

The branching mechanism is of the form \( \Phi(x, z) = \beta(x)z - \alpha(x)z^p \), where \( p \in (1, 2] \), \( \beta \) is bounded from above, \( \alpha > 0 \), and \( \alpha, \beta \in C^\kappa(\mathbb{R}^d) \), for some \( \kappa \in (0, 1] \). A finite measure-valued diffusion \( X(t) = X(t, \cdot) \) is the Markov process defined uniquely via the following log-Laplace equation:

\[
E(\exp(-<f, X(t)>)|X(0) = \mu) = \exp(-<u_f(\cdot, t), \mu>),
\]

for \( f \in C^+_c(\mathbb{R}^d) \), the space of compactly supported, nonnegative, continuous functions on \( \mathbb{R}^d \), and for finite initial measures \( \mu \), where \( u_f \) is the minimal positive solution to the evolution equation

\[
\begin{align*}
 u_t &= Lu + \beta u - \alpha u^p \quad \text{in } \mathbb{R}^d \times (0, \infty); \\
 u(x, 0) &= f(x) \quad \text{in } \mathbb{R}^d; \\
 u(x, t) &\geq 0 \quad \text{in } \mathbb{R}^d \times [0, \infty).
\end{align*}
\]

The measure for the process started from \( \mu \) will be denoted by \( P_\mu \), and its expectation operator will be denoted by \( E_\mu \).

We recall the compact support property.

**Definition.** Let \( \mu \in M_F(D) \) be compactly supported. The measure-valued process corresponding to \( P_\mu \) possesses the compact support property if

\[
P_\mu( \bigcup_{0 \leq s \leq t} \{ \text{supp } X(s) \text{ is bounded} \}) = 1, \text{ for all } t \geq 0.
\]

**Remark.** The parameter \( \beta \) may be thought of as the mass creation parameter (see the discussion of the particle process approximation to the measure-valued diffusion at the end of this section). **Without further mention, it will always be assumed in this paper that \( \beta \) is bounded from above.** It is possible to extend the construction of the measure-valued process to certain \( \beta \) which are unbounded from above; namely, to those \( \beta \) for which the generalized principal eigenvalue of the operator \( L + \beta \) is finite. However, the resulting
process has paths which are mutually absolutely continuous with respect to another measure-valued diffusion whose mass creation parameter $\beta$ is bounded from above [6]. Thus, the compact support property will hold for the former process if and only if it holds for the latter one.

There are four objects, corresponding to four different underlying probabilistic effects, which can influence the compact support property:

1. $L$, the operator corresponding to the underlying motion;
2. $\beta$, the mass creation parameter of the branching mechanism;
3. $\alpha$, the nonlinear component of the branching mechanism, which can be thought of as the variance parameter if $p = 2$;
4. $p$, the power of the nonlinearity, which is the scaling power and is connected to the fractional moments of the offspring distribution in the particle process approximation to the measure-valued diffusion.

(For (2), (3) and (4) above, see the discussion of the particle process approximation to the measure-valued diffusion at the end of this section.)

We shall see that both $L$ and $\alpha$ play a large role in determining whether or not the compact support property holds; $\beta$ and $p$ play only a minor role.

In [6], the compact support property was shown to be equivalent to a uniqueness property for solutions to (1.2).

**Theorem EP1.** The compact support property holds for one, or equivalently all, nonzero, compactly supported initial measures $\mu$ if and only if there are no nontrivial solutions to (1.2) with initial data $f \equiv 0$.

**Remark 1.** We emphasize that the uniqueness property in Theorem EP1 concerns all classical solutions to (1.2), with no growth restrictions. If one restricts to mild solutions—solutions which solve an integral equation involving the linear semigroup corresponding to the operator $L$—then, for example, uniqueness holds in this class if $\alpha, \beta$ and the coefficients of $L$ are
bounded [11]; yet, these conditions certainly do not guarantee uniqueness in the class of all positive, classical solutions (see Theorem 1 below).

Remark 2. In fact, the proof of Theorem EP1 shows that there exists a maximal solution \( u_{\text{max}} \) to (1.2) with initial data \( f = 0 \), and

\[
P_{\mu} \left( \bigcup_{0 \leq s \leq t} \{ \text{supp } X(s) \text{ is bounded} \} \right) = \exp \left( - \int_{R^d} u_{\text{max}}(x,t) \mu(dx) \right),
\]

for compactly supported \( \mu \). This shows that when the compact support property fails, the onset of the failure is gradual; that is, as a function of \( t \),

\[
P_{\mu} \left( \bigcup_{0 \leq s \leq t} \{ \text{supp } X(s) \text{ is bounded} \} \right)
\]

is continuous and equal to 1 at time \( t = 0 \). This behavior is in contrast to the behavior of the measure-valued process corresponding to the semilinear operator \( u_t = \Delta u - u \log u \), investigated recently in [9]. This process is obtained as a weak limit as \( p \to 1 \) of the processes corresponding to the semilinear operators \( u_t = \Delta u + \frac{1}{p-1}u - \frac{1}{p-1}u^p \).

Unlike the measure-valued diffusions defined above, this process is immortal; that is, \( P_{\mu}(X(t) = 0) = 0 \), for all \( t \geq 0 \). Furthermore, it is shown that

\[
P_{\mu} \left( \bigcup_{0 \leq s \leq t} \{ \text{supp } X(s) \text{ is bounded} \} \right) = 0, \text{ for all } t > 0.
\]

Thus, the onset of the failure of the compact support property is instantaneous. Theorem EP1 is not valid for this process. Indeed, the proof of Theorem EP1 requires the fact that a maximal solution exists for (1.2) with initial condition \( f = 0 \). The existence of such a maximal solution is essentially equivalent to the existence of a universal, a priori upper bound on all solutions to (1.2); that is, the existence of a finite function \( M(x,t) \) on \( R^d \times (0, \infty) \) such that

\[
u(x,t) \leq M(x,t), \text{ for all } (x,t) \in R^d \times (0, \infty), \text{ and all solutions } u \text{ to (1.2)}.
\]

Such a universal a priori upper bound does not exist for the equation \( u_t = \Delta u - u \log u \). In [13], a more or less necessary and sufficient condition on the nonlinear term (independent of the operator \( L \)) is given for the existence of such a bound.

Remark 3. Theorem EP1 suggests a parallel between the compact support property for measure-valued diffusions and the non-explosion property for
diffusion processes. Indeed, the non-explosion property for the diffusion process \( Y(t) \) corresponding to the operator \( L \) is equivalent to the nonexistence of nontrivial, bounded positive solutions to the linear Cauchy problem with 0 initial data:

\[
\begin{align*}
  u_t &= Lu \quad \text{in } \mathbb{R}^d \times (0, \infty); \\
  u(x,0) &= 0 \quad \text{in } \mathbb{R}^d; \\
  u &\geq 0 \quad \text{in } \mathbb{R}^d \times (0, \infty).
\end{align*}
\]

(1.4) (For one direction of this result, note that \( u(x,t) \equiv \mathcal{P}_x(\tau_\infty \leq t) \) serves as a nontrivial solution to (1.4) in the explosive case.) It is natural for bounded, positive solutions to be the relevant class of solutions in the linear case and for positive solutions to be the relevant class of solutions in the semilinear case. Indeed, by Ito’s formula, the probabilities for certain events related to \( Y(t) \) are obtained as bounded, positive solutions to the linear equation, and by the log-Laplace equation, the negative of the logarithm of the probability of certain events related to \( X(t) \) can be obtained as positive solutions to the semilinear equation.

The class of operators \( L \) satisfying the following assumption will play an important role.

**Assumption 1.** For some \( C > 0 \),

1. \( \sum_{i,j=1}^n a_{ij}(x)\nu_i\nu_j \leq C|\nu|^2(1 + |x|^2), \quad x, \nu \in \mathbb{R}^d; \)
2. \( |b(x)| \leq C(1 + |x|), \quad x \in \mathbb{R}^d. \)

The next theorem culls some results from [7] and applies them to the probabilistic setting at hand.

**Theorem EP2.** Let \( p \in (1,2] \) and let the coefficients of \( L \) satisfy Assumption [7].

1. There is no nontrivial solution to (1.4); thus, the diffusion process \( Y(t) \) does not explode.
(2) If
\[ \inf_{x \in \mathbb{R}^d} \alpha(x) > 0, \]
then there is no nontrivial solution to (1.2) with initial data \( f = 0 \);
thus, the compact support property holds for \( X(t) \).

**Proof of part (1).** For the proof that there are no nontrivial solutions to (1.4) if Assumption \( 1 \) holds, see [7, Proposition 5 and Remark 1 following it]. Non-explosiveness of \( Y(t) \) then follows from the parenthetical sentence following (1.4) above.

**Proof of part (2).** For the proof that there are no nontrivial solutions to (1.2), see [7, Theorem 2]. The fact that the compact support property holds then follows from Theorem EP1 above. \( \square \)

The conditions in Assumption 1 are classical conditions which arise frequently in the theory of diffusion processes. Theorem EP2 shows that if the coefficients of \( L \) obey this condition and if the branching coefficient \( \alpha \) is bounded away from zero, then everything is well behaved—neither can the underlying diffusion process explode nor can the measure-valued diffusion fail to possess the compact support property.

The following result shows that the compact support property can fail if \( \inf_{x \in \mathbb{R}^d} \alpha(x) = 0 \). It also demonstrates that the effect of \( \alpha \) on the compact support property cannot be studied in isolation, but in fact depends on the underlying diffusion.

**Theorem 1.** Let \( p \in (1, 2] \) and let
\[ L = A(x) \Delta, \]
where \( C_0^{-1}(1 + |x|)^m \leq A(x) \leq C_0(1 + |x|)^m, \ m \in [0, 2], \]
for some \( C_0 > 0 \).

(1) If
\[ \alpha(x) \geq C_1 \exp(-C_2|x|^{2-m}), \]
for some \( C_1, C_2 > 0 \), then the compact support property holds for \( X(t) \).
(2) If
\[
\alpha(x) \leq C \exp(-|x|^{2-m+\epsilon}) \quad \text{and} \quad \beta(x) \geq -C(1 + |x|)^{2-m+2\delta},
\]
for some $C, \epsilon > 0$ and some $\delta < \epsilon$, then the compact support property does not hold for $X(t)$.

Remark. By Theorem EP1, to prove Theorem 1, it is necessary and sufficient to show that if $\alpha$ is as in part (1) of the theorem, then there is no nontrivial solution to (1.2) with initial data $f = 0$, while if $\alpha$ is as in part (2) of the theorem, then there is such a nontrivial solution. In the case that $L = \frac{1}{2}\Delta$, and for part (2), $\beta \geq 0$, this result was obtained in [7, Theorem 7]. An alternative, more purely probabilistic proof which does not rely on Theorem EP1 can be found in [15] for the case $L = \frac{1}{2}\Delta$ and $\beta = 0$.

A heuristic, qualitative understanding of Theorem 1 is given at the end of this section.

As a complement to Theorem 1 we note the following result [5, 7].

**Theorem EP3.** Let $p \in (1, 2]$.

(1) Let $d \geq 2$ and let
\[
L = A(x)\Delta, \quad \text{where} \quad A(x) \geq C(1 + |x|)^m, \quad \text{for some} \quad C > 0 \quad \text{and} \quad m > 2.
\]
Assume that
\[
\sup_{x \in \mathbb{R}^d} \alpha(x) < \infty \quad \text{and} \quad \beta \geq 0.
\]
Then the compact support property does not hold for $X(t)$.

(2) Let $d = 1$ and let
\[
L = A(x) \frac{d^2}{dx^2}, \quad \text{where} \quad A(x) \geq C(1 + |x|)^m, \quad \text{for some} \quad C > 0 \quad \text{and} \quad m > 1 + p.
\]
Assume that
\[
\sup_{x \in \mathbb{R}^d} \alpha(x) < \infty \quad \text{and} \quad \beta \geq 0.
\]
Then the compact support property does not hold for $X(t)$. 
(3) Let \( d = 1 \) and let
\[
L = A(x) \frac{d^2}{dx^2}, \quad \text{where} \quad A(x) \leq C(1 + |x|)^m, \quad \text{for some} \ C > 0, \quad \text{and} \ m \leq 1 + p.
\]
Assume that
\[
\inf_{x \in \mathbb{R}^d} \alpha(x) > 0 \quad \text{and} \quad \beta \leq 0.
\]
Then the compact support property holds for \( X(t) \).

Remark. It follows from Theorem EP3 that if \( d = 1 \) and \( L = (1 + |x|)^m \frac{d^2}{dx^2} \), with \( m \in (2, 3] \), and say \( \alpha = 1 \) and \( \beta = 0 \), then the compact support property will depend on the particular choice of \( p \in (1, 2] \).

Proof. By Theorem EP1, it is necessary and sufficient to show that under the conditions of parts (1) and (2), there exists a nontrivial solution to (1.2), while under the conditions of part (3) there does not. In the case that \( \alpha = 1 \) (or equivalently, any positive constant) and \( \beta = 0 \), this follows from [7, Theorem 5]. To extend this to \( \alpha \) and \( \beta \) as in the statement of the theorem, one appeals to a comparison result which we state below [7, Proposition 4].

Comparison Result. Assume that
\[
\beta_1 \leq \beta_2
\]
and
\[
0 < \alpha_2 \leq \alpha_1.
\]
If uniqueness holds for (1.2) with initial data \( f = 0 \) when \( \beta = \beta_2 \) and \( \alpha = \alpha_2 \), then uniqueness also holds when \( \beta = \beta_1 \) and \( \alpha = \alpha_1 \). Thus, from Theorem EP1, if the compact support property holds for \( \beta = \beta_2 \) and \( \alpha = \alpha_2 \), then it also holds for \( \beta = \beta_1 \) and \( \alpha = \alpha_1 \).

Theorem [7] and Theorem EP3 demonstrate the effect of the underlying diffusion \( Y(t) \) on the compact support property in the case that \( L \) is a scalar function times the Laplacian. We now consider more generally the effect of
the underlying diffusion process on the compact support property. We begin with the following result which combines [7, Theorem 3] with Theorem EP1.

**Theorem EP4.** Let \( p \in (1, 2] \) and assume that the underlying diffusion process \( Y(t) \) explodes. Assume in addition that

\[
\inf_{x \in \mathbb{R}^d} \frac{\beta(x)}{\alpha(x)} > 0.
\]

Then there exists a nontrivial solution to (1.2) with initial data \( f = 0 \); thus, by Theorem EP1, the compact support property does not hold.

In particular, if \( Y(t) \) is explosive and \( \sup_{x \in \mathbb{R}^d} \alpha < \infty \), then a sufficient condition for the compact support property to fail is that

\[
\inf_{x \in \mathbb{R}^d} \beta(x) > 0.
\]

What about the case that \( \beta = 0 \)? This is the case of critical branching (see the discussion of the particle process approximation to the measure-valued diffusion at the end of this section). We will prove the following result.

**Theorem 2.** Let \( p \in (1, 2] \) and assume that the diffusion process \( Y(t) \) is "strongly" explosive in the sense that

\[
(1.5) \quad \sup_{x \in \mathbb{R}^d} E_x \tau_\infty < \infty.
\]

Assume in addition that \( \sup_{x \in \mathbb{R}^d} \alpha(x) < \infty \) and that

\[
(1.6) \quad \beta > -\frac{1}{\sup_{x \in \mathbb{R}^d} E_x \tau_\infty}.
\]

Then the measure-valued diffusion \( X(t) \) does not possess the compact support property.

The next result shows that the restriction \( \sup_{x \in \mathbb{R}^d} \alpha(x) < \infty \) in Theorem 2 is essential.

**Proposition 1.** Let \( p \in (1, 2] \). Let \( m \in (-\infty, \infty) \),

\[
L = (1 + |x|^m) \Delta \text{ in } \mathbb{R}^d,
\]
\(\beta = 0\) and
\[
\alpha(x) = (1 + |x|)^m - 2.
\]

Then the compact support property holds for the measure-valued diffusion \(X(t)\). However, if \(m > 2\) and \(d \geq 3\), the diffusion process \(Y(t)\) explodes and \(\sup_{x \in \mathbb{R}^d} E_x \tau_\infty < \infty\).

**Proof.** There is no nontrivial, nonnegative solution to the elliptic equation
\[
\Delta W - \frac{1}{(1 + |x|)^2} W = 0 \text{ in } \mathbb{R}^d
\]
(see, for example [7, Theorem 6]). From this it follows that uniqueness holds for (1.2) with initial data \(f = 0\); indeed, if uniqueness did not hold, and \(u(x, t)\) were a nontrivial solution, then it would follow from the maximum principle that \(u(x, t)\) is increasing in \(t\). Then \(W(x) = \lim_{t \to \infty} u(x, t)\) would be a nontrivial solution to the above stationary elliptic equation. (For details, see [7, Theorem 4-(ii)].) Since uniqueness holds for (1.2), it follows from Theorem EP1 that the compact support property holds.

For a proof that \(Y(t)\) explodes if \(m > 2\) and \(d \geq 3\), see, for example, [7, Proposition 5 and Remark 1 following it]). The function \(g(x) = E_x \tau_\infty \leq \infty\) is the minimal positive solution to the equation \(Lg = -1\) in \(\mathbb{R}^d\). It satisfies \(g(x) = \lim_{n \to \infty} g_n(x)\), where \(g_n(x) = E_x \tau_n\) solves \(Lg_n = -1\), for \(|x| < n\), and \(g_n(x) = 0\), for \(|x| = n\). Since everything is radial, a direct calculation can be made, which shows that \(\sup_{x \in \mathbb{R}^d} g(x) < \infty\).

\[\square\]

In the sequel, we will show that condition (1.5) is not necessary in order that the compact support property fail when, say \(\beta = 0\) and \(\alpha = 1\). We will also demonstrate how \(p\) and \(b\) can affect the compact support property. In order to accomplish this, we first need to discuss how the concept of a measure-valued diffusion hitting a point can be formulated and understood in terms of the compact support property. This last point is of independent interest.
Let \( R_t = \text{cl}(\bigcup_{s \in [0,t]} \text{supp}(X(s))) \) and let \( R = \text{cl}(\bigcup_{s \geq 0} \text{supp}(X(s))) \). The random set \( R \) is called the range of \( X = X(\cdot) \). A path of the measure-valued diffusion is said to hit a point \( x_0 \in \mathbb{R}^d \) if \( x_0 \in R \).

If \( X(t) \) becomes extinct with probability one, that is, \( P_\mu(X(t) = 0 \text{ for all large } t) = 1 \), or more generally, if \( X(t) \) becomes locally extinct with probability one, that is, \( P_\mu(X(t, B) = 0 \text{ for all large } t) = 1 \), for each bounded \( B \subset \mathbb{R}^d \), then \( x_0 \in R \) if and only if \( x_0 \in R_t \) for sufficiently large \( t \). Thus, we have:

\[
\text{(1.7)}
\]

If \( X(t) \) suffers local extinction with probability one, then

\[
P_\mu(\text{X hits } x_0) > 0 \text{ if and only if there exists a } t > 0 \text{ such that}
\]

\[
P_\mu(\bigcup_{0 \leq s \leq t} \{\text{supp}(X(s)) \text{ is not compactly embedded in } \mathbb{R}^d - \{x_0\}\}) > 0.
\]

Now although we have assumed in this paper that the underlying state space is \( \mathbb{R}^d \), everything goes through just as well on an arbitrary domain \( D \subset \mathbb{R}^d \). Of course now, the compact support property is defined with respect to the domain \( D \), and the underlying diffusion will explode if it hits \( \partial D \) in finite time. In particular, Theorem EP1 still holds with \( \mathbb{R}^d \) replaced by \( D \).

In light of the above observations, consider a measure-valued diffusion \( X(t) \) corresponding to the log-Laplace equation \( \text{(1.2)} \) on \( \mathbb{R}^d \) with \( d \geq 2 \). The underlying diffusion process \( Y(t) \) on \( \mathbb{R}^d \) corresponds to the operator \( L \) on \( \mathbb{R}^d \). Let \( \hat{Y}(t) \) denote the diffusion process on the domain \( D = \mathbb{R}^d - \{x_0\} \) with absorption at \( x_0 \) and corresponding to the same operator \( L \). (Note that if \( x_0 \) is polar for \( Y(t) \), then \( Y(t) \) and \( \hat{Y}(t) \) coincide when started from \( x \neq x_0 \). In fact, \( x_0 \) is always polar under the assumptions we have placed on the coefficients of \( L \).) Let \( \hat{X}(t) \) denote the measure-valued diffusion corresponding to the log-Laplace equation \( \text{(1.2)} \), but with \( \mathbb{R}^d \) replaced by \( D = \mathbb{R}^d - \{x_0\} \). It follows from \( \text{(1.7)} \) that if \( X(t) \) suffers local extinction with probability one, then the measure-valued diffusion \( X(t) \) hits the point \( x_0 \) with positive probability if and only if \( \hat{X}(t) \) on \( \mathbb{R}^d - \{x_0\} \) does not possess the compact support property. Furthermore, the above discussion shows that
even if \( X(t) \) does not suffer local extinction with probability one, a sufficient condition for \( X(t) \) to hit the point \( x_0 \) with positive probability is that \( \hat{X}(t) \) on \( R^d - \{x_0\} \) does not possess the compact support property.

Consider now the following semilinear equation in the punctured space \( R^d - \{0\} \).

\[
\begin{align*}
    u_t &= \frac{1}{2} \Delta u - u^p \text{ in } (R^d - \{0\}) \times (0, \infty); \\
    u(x, 0) &= 0 \text{ in } R^d - \{0\}; \\
    u &\geq 0 \text{ in } (R^d - \{0\}) \times [0, \infty).
\end{align*}
\] (1.8)

By Theorem EP1, the measure-valued diffusion corresponding to the semilinear equation \( u_t = \frac{1}{2} \Delta u - u^p \) in \( R^d - \{0\} \) will possess the compact support property if and only if (1.8) has a nontrivial solution. The following theorem was recently proved in [14].

**Theorem P.** Let \( p > 1 \) and \( d \geq 2 \).

1. If \( d < \frac{2p}{p-1} \), then there exists a nontrivial solution to (1.8).
2. If \( d \geq \frac{2p}{p-1} \), then there is no nontrivial solution to (1.8).

Using the method employed in [14] to prove Theorem P, we will prove the following extension in the case \( d < \frac{2p}{p-1} \).

**Theorem 3.** Let \( p > 1 \) and \( d \geq 2 \). Let \( X(t) \) denote the measure-valued diffusion on the punctured space \( R^d - \{0\} \) corresponding to the semilinear equation

\[
\begin{align*}
    u_t &= \frac{1}{2} \Delta u + \beta u - u^p \text{ in } (R^d - \{0\}) \times (0, \infty); \\
    u(x, 0) &= 0 \text{ in } R^d - \{0\}; \\
    u &\geq 0 \text{ in } (R^d - \{0\}) \times [0, \infty).
\end{align*}
\] (1.9)

Assume that

\[ d < \frac{2p}{p-1}. \]
Let \( \beta_0 = \frac{d(p - 1) - 2p}{(p - 1)^2} < 0. \)

(1) If \( \beta(x) \geq \frac{\beta_0 + \kappa}{|x|^2} \), for some \( \kappa \in (0, -\beta_0] \),
then there exists a nontrivial solution to (1.9); hence, the compact support property does not hold for \( X(t) \).

(2) If
\[
\limsup_{x \to 0} |x|^2 \beta(x) < \beta_0,
\]
then there is no nontrivial solution to (1.9); hence, the compact support property holds for \( X(t) \).

Remark. The restriction \( \kappa \leq -\beta_0 \) is made to insure that \( \beta \) is bounded from above.

We have the following corollary of Theorem P and Theorem 3.

**Corollary 1.** Let \( X(t) \) denote the measure-valued diffusion on all of \( \mathbb{R}^d \) corresponding to the semilinear equation
\[
u_t = \frac{1}{2} \Delta u + \beta u - u^p \text{ in } \mathbb{R}^d \times (0, \infty).
\](1)
If \( \beta \geq 0 \) and \( d < \frac{2p}{p-1} \), then \( X(t) \) hits any point \( x_0 \) with positive probability;
(2) If \( \beta \leq 0 \) and \( d \geq \frac{2p}{p-1} \), then \( X(t) \) hits any point \( x_0 \) with probability 0;
(3) If \( \beta \leq 0 \), \( d < \frac{2p}{p-1} \) and \( \beta \) has a singularity at the origin such that
\[
\limsup_{x \to 0} |x|^2 \beta(x) < \frac{d(p - 1) - 2p}{(p - 1)^2},
\]
then \( X(t) \) hits 0 with probability 0.

**Proof.** When \( \beta = 0 \), part (1) follows immediately from Theorem P, Theorem EP1 and the discussion preceding Theorem P. For \( \beta \geq 0 \), one appeals to the comparison result appearing above after the proof of Theorem EP3.
When $\beta = 0$, it is well-known that the super-Brownian motion in the statement of the corollary suffers extinction with probability one. By comparison, this also holds when $\beta \leq 0$ [6]. Thus, part (2) follows from Theorem P, Theorem EP1 and the discussion preceding Theorem P, while part (3) follows from Theorem 3, Theorem EP1 and the discussion preceding Theorem P.

Remark 1. When $\beta = 0$, the results in parts (1) and (2) of Corollary 1 state that critical, super-Brownian motion hits a point with positive probability if $d < \frac{2p}{p-1}$, and with zero probability if $d \geq \frac{2p}{p-1}$. This result can be found in [3] for the case $p = 2$. We are unaware of a reference when $p \in (1, 2)$, although the result is certainly “known”; results in a similar vein for $p \in (1, 2)$ can be found in [10, section 5]. See also [8]. The proof of parts (1) and (2) of Corollary 1 (via Theorem P) is quite different from the proof in [3].

Remark 2. Theorem P gives an example where the measure-valued diffusion does not possess the compact support property even though the underlying diffusion process does not explode. Indeed, the underlying diffusion is Brownian motion in $\mathbb{R}^d - \{0\}$. Since singletons are polar for multi-dimensional Brownian motion, this process does not explode. However for $d < \frac{2p}{p-1}$, the compact support property fails for the measure-valued process.

Remark 3. In order to obtain an example of the phenomenon occurring in Remark 2 when the state space is the whole space, and in order to see how the parameter $p$ and the mass creation parameter $\beta$ can affect the compact support property when the state space is the whole space, we convert the set-up in Theorem 3 and Theorem P to the state space $\mathbb{R} = (-\infty, \infty)$ by considering just the radial variable, $r = |x|$, and then making a change of variables, say, $z = \frac{1}{r} - r$. One obtains an operator $L = \frac{1}{2} a(z) \frac{d^2}{dz^2} + b(z) \frac{d}{dz}$.
on $\mathbb{R}$, where

\[ a(z) \sim z^4 \text{ as } z \to \infty, \quad \lim_{z \to -\infty} a(z) = 1; \]

\[ b(z) \sim \frac{3 - d}{2} z^3, \text{ as } z \to \infty, \quad \lim_{z \to -\infty} b(z) = 0. \]

Also, one has $\alpha = 1$. One has $\beta = 0$ in Theorem P and one has

\begin{equation}
\beta(z) \geq \frac{4(\beta_0 + \kappa)}{\left((z^2 + 4) \frac{d}{2} - z\right)^2} \quad \text{in part (1) of Theorem 3,} \tag{1.10a}
\end{equation}

\begin{equation}
\limsup_{z \to \infty} \frac{1}{4} \left((z^2 + 4) \frac{d}{2} - z\right)^2 \beta(z) < \beta_0 \quad \text{in part (2) of Theorem 3,} \tag{1.10b}
\end{equation}

It follows then that $\liminf_{z \to \infty} \frac{\beta(z)}{z^2} \geq \beta_0 + \kappa$ in part (1) of Theorem 3 and $\limsup_{z \to \infty} \frac{\beta(z)}{z^2} < \beta_0$ in part (2) of Theorem 3.

Consider first the phenomenon mentioned in Remark 2 with regard to Theorem P. After the change of variables, the state space is $\mathbb{R}$ and the operator $L$ depends on the parameter $d$. The diffusion corresponding to $L$ is nonexplosive for all $d$, as it inherits this property from the original process before the change of variables. Also $\alpha = 1$ and $\beta = 0$. However, if $d < \frac{2p}{p+1}$, then the compact support property fails.

With the same setup as in the previous paragraph, we also see how the parameter $p$ affects the compact support property—the property will hold if and only if $d \geq \frac{2p}{p+1}$.

Now consider the above change of variables applied to Theorem 3. We see that if $d < \frac{2p}{p+1}$, then $\beta$ affects the compact support property—it holds if $\beta$ satisfies (1.10b) and does not hold if $\beta$ satisfies (1.10a). Note that (1.10) demonstrates that an unbounded change in $\beta$ is necessary here to affect the compact support property.

The discussion at the end of Remark 3 leads us to pose the following question.

**Open Question 1.** Can a bounded change in the mass creation parameter $\beta$ affect the compact support property?
The particle process approximation to the measure-valued diffusion, which we discuss in some detail below, along with the remarks on the probabilistic intuition for the role of branching in Theorem 1, which follows that discussion, might suggest that in the case that $\alpha = 1$ and $\beta = 0$, the compact support property for the measure-valued diffusion is equivalent to the non-explosion property for the underlying diffusion. Remark 2 following Corollary 1 shows that one of these inclusions is not true. Theorem 2 shows that if the explosion requirement is replaced by the strong explosion condition (1.5), the other inclusion becomes true. Whether this inclusion is true in its original form we leave as an open question.

**Open Question 2.** If $\alpha = 1$, $\beta = 0$ and the underlying diffusion process explodes, does the compact support property necessarily fail?

**Remark 1.** This is an interesting question from the point of view of partial differential equations. In light of Theorem EP1, the question is whether nonuniqueness for the linear Cauchy problem $u_t = Lu$ guarantees nonuniqueness for the semilinear Cauchy problem $u_t = Lu - u^p$ with vanishing initial data.

**Remark 2.** Note that if the answer to Open Question 1 is negative, then by Theorem EP4, the answer to Open Question 2 is affirmative.

Theorems 1-3 will be proved successively in the sections that follow.

We now turn to an intuitive probabilistic understanding of the role of the branching in Theorem 1. We recall the particle process approximation to the measure-valued diffusion in the case that $\alpha$ and $\beta$ are bounded. Consider first the case $p = 2$, the case in which the offspring distribution has finite variance. Let $\hat{R}^d = R^d \cup \{\Delta\}$ denote the one-point compactification of $R^d$. One may consider the diffusion process $Y(t)$ to live on $\hat{R}^d$; if $Y(t)$ does not explode, then it never reaches $\Delta$, while if $Y(t)$ does explode, then it enters the state $\Delta$ upon leaving $R^d$, and remains there forever. For each
positive integer \( n \), consider \( N_n \) particles, each of mass \( \frac{1}{n} \), starting at points \( y_i^{(n)}(0) \in \mathbb{R}^d, i = 1, 2, \ldots, N_n \), and performing independent branching diffusion according to the process \( Y(t) \), with branching rate \( cn, c > 0 \), and spatially dependent branching distribution \( \{ p_k^{(n)}(y) \}_{k=0}^{\infty} \), where

\[
\sum_{k=0}^{\infty} k p_k^{(n)}(y) = 1 + \frac{\gamma(y)}{n} + o\left( \frac{1}{n} \right), \text{ as } n \to \infty;
\]

\[
\sum_{k=0}^{\infty} (k - 1)^2 p_k^{(n)}(y) = m(y) + o(1), \text{ as } n \to \infty, \text{ uniformly in } y,
\]

with \( m, \gamma \in C^\alpha(\mathbb{R}^d) \) and \( m(y) > 0 \). Let \( N_n(t) \) denote the number of particles alive at time \( t \) and denote their positions by \( \{ Y_i^{(n)}(t) \}_{i=1}^{N_n(t)} \). Denote by \( \mathcal{M}_F(\mathbb{R}^d) (\mathcal{M}_F(\hat{\mathbb{R}}^d)) \) the space of finite measures on \( \mathbb{R}^d (\hat{\mathbb{R}}^d) \). Define an \( \mathcal{M}_F(\hat{\mathbb{R}}^d) \)-valued process \( X_n(t) \) by \( X_n(t) = \frac{1}{n} \sum_{i=1}^{N_n(t)} \delta_{Y_i^{(n)}(t)}(\cdot) \). Note that \( X_n \) is càdlàg. Denote by \( P^{(n)} \) the probability measure corresponding to \( \{X_n(t), 0 \leq t < \infty \} \) on \( D([0, \infty), \mathcal{M}_F(\hat{\mathbb{R}}^d)) \), the space of càdlàg paths with the Skorohod topology.

Assume that \( m(y) \) and \( \gamma(y) \) are bounded from above. One can show that if \( w - \lim_{n \to \infty} X_n(0) = \mu \in \mathcal{M}_F(\mathbb{R}^d) \), then \( P^{(n)} = w - \lim_{n \to \infty} P^{(n)} \) exists in \( D([0, \infty), \mathcal{M}_F(\hat{\mathbb{R}}^d)) \). Furthermore, the measure \( P^{(n)} \) restricted to \( D([0, \infty), \mathcal{M}_F(\mathbb{R}^d)) \) satisfies (1.1) and (1.2) with \( \beta(y) = c\gamma(y), \alpha(y) = \frac{1}{2} cm(y) \) and \( p = 2 \) (see [1]). Denoting this restriction by \( P^{(n)}|_{\mathbb{R}^d} \), it then follows that \( P_\mu = P^{(n)}|_{\mathbb{R}^d} \). In fact, one can show that \( P_\mu \) is supported on the space of continuous paths, \( C([0, \infty), \mathcal{M}_F(\mathbb{R}^d)) \).

One should think of \( \beta \) and \( \alpha \) as the mass creation and the variance parameters respectively of the branching.

For the case that \( p \in (1, 2) \), one cooks up a sequence of distributions \( \{ p_k^{(n)}(y) \}_{n=1}^{\infty} \) for which the generating functions \( \Phi^{(n)}(s; y) = \sum_{k=0}^{\infty} p_k^{(n)}(y)s^k \) satisfy \( \lim_{n \to \infty} n^p \left( \Phi^{(n)}(1 - \frac{\lambda}{n}; y) - (1 - \frac{\lambda}{n}) \right) = \alpha(y)\lambda^p - \beta(y)\lambda \). These offspring distributions \( \{ p_k^{(n)}(y) \}_{n=1}^{\infty} \) will possess all moments smaller than \( p \). Again, \( \beta \) can be thought of as the mass creation parameter. As above, each particle is given mass \( \frac{1}{n} \), but in the present case, the branching rate is \( n^{p-1} \). The
same construction and conclusion as above holds, although in this case the
paths are not continuous, but only càdlàg \[2\].

With the above set-up, we can now give some intuition concerning The-
orem 1. Consider two particular cases of the above construction. In both
cases we will assume that at time 0 there are \(n\) particles, all positioned at
\(y = 0\); that is, \(N_n = n\) and \(y_i^{(n)} = 0\), for \(i = 1, 2, \ldots n\). Then the initial
measure, both for the approximating process and the limiting one, will be
\(\mu = \delta_0\). We will also assume that the diffusion \(Y(t)\) does not explode.

The first case is the completely trivial case in which there is no branching
at all. This degenerate case corresponds to \(\beta = \alpha = 0\) (and thus does
not actually fit into the above set-up). In this case, \(X_n(t)\) is a random
probability measure with \(n\) atoms of mass \(\frac{1}{n}\) positioned at \(n\) IID points,
distributed according to the distribution of \(Y(t)\). Thus, by the law of large
numbers, \(X(t) = w - \lim_{n \to \infty} X_n(t)\) is the deterministic measure dist(\(Y(t)\)).
Since dist(\(Y(t)\)) is not compactly supported for \(t > 0\), it follows that the
compact support property does not hold for \(X(t)\) in this trivial case.

Now consider the case of critical, binary branching; that is, \(p_0^{(n)} = p_2^{(n)} = \frac{1}{2}\). Letting \(c = 1\), it then follows that \(\beta = 0\) and \(\alpha = \frac{1}{2}\). In such a case,
discrete, deterministic branching may be used in place of the exponentially
distributed branching described above: the branching will occur determin-
istically at integral multiples of \(\frac{1}{n}\) instead of according to an exponential
random variable with parameter \(n\). This deterministic mechanism will be
more convenient in what follows.

Consider the process at, say, time \(t = 1\). At this time, the process has
undergone \(n\) generations of branching. The \(N_n(1)\) particles \(\{Y_i^{(n)}(1)\}_{i=1}^{N_n(1)}\)
may be grouped in the following way: for each \(j = 1, 2, \ldots n\), let \(M_j^{(n)}(1)\)
denote the number of descendants of \(y_j^{(n)}\) and let \(\{Z_{j,i}^{(n)}(1)\}_{i=1}^{M_j^{(n)}(1)}\)
denote their positions. Then \(N_n(1) = \sum_{j=1}^{n} M_j^{(n)}(1)\) and \(\bigcup_{j=1}^{n} \bigcup_{i=1}^{M_j^{(n)}(1)} Z_{j,i}^{(n)}(1) = \)
\{Y_i^{(n)}(1)\}_{i=1}^{N_n^{(1)}} \). Let

$$W_j^{(n)}(1) = \frac{1}{n} \sum_{i=1}^{M_j^{(n)}(t)} \delta_{Z_i^{(n)}(1)}(\cdot), \quad j = 1, \ldots, n.$$ \hspace{1cm} (1.11)

Note that the \{W_j^{(n)}(1)\}_{j=1}^{n} are independent point measures, and that the approximate measure valued diffusion, \(X_n(1)\), is given by

$$X_n(1) = \frac{1}{n} \sum_{j=1}^{n} W_j^{(n)}(1).$$

The random variable \(M_j^{(n)}(1)\) is just the number of individuals alive in the \(n\)-th generation of a critical, binary branching process starting from a single individual at the zeroth generation. Denoting such a branching process by \(\{U_n\}_{n=0}^{\infty}\), standard results \[1\] give

$$P(U_n > 0) \sim \frac{2}{n}, \quad \text{as } n \to \infty;$$ \hspace{1cm} (1.11) \hspace{1cm} \lim_{n \to \infty} P(U_n > z | U_n > 0) = \exp(-2z), \quad \text{for } z \geq 0.$$

In particular, from (1.11) it follows that the distribution of the number of nonzero \(W_j^{(n)}(1)\) is approximately \(Bin(n, \frac{2}{n})\); thus, by the Poisson approximation, as \(n \to \infty\), the contribution to \(X_n(1)\) will come from approximately a \(\text{Poiss}(2)\) number of nonzero measures from among the \{\(W_j^{(n)}(1)\)\}_{j=1}^{n}.

In particular, for example, with probability about \(2 \exp(-2)\) there will be exactly one nonzero random variable from among \{\(W_j^{(n)}(1)\)\}_{j=1}^{n}. Conditioned on this event, denote the unique nonzero random variable by \(W_j^{(n)}(1)\).

It follows easily from (1.11) that

$$\liminf_{n \to \infty} P\left(\frac{M_{jn}(s)}{n} \geq C \frac{M_{jn}(1)}{n} \geq K \right) > 0, \quad \text{for all } C, K > 0 \text{ and } s \in (0, 1].$$ \hspace{1cm} (1.12)

Now (1.12) is perhaps misleading, as it may suggest that if one fixes a time \(s < 1\) and some \(c_1 > 0\), then the probability that at time 1 there are at least \(c_1 n\) particles with a common ancestor at time \(s\) remains bounded away from 0 as \(n \to \infty\). Note that if this were to occur, then if one looked at the contribution to \(X_n(1)\) coming from these particular particles with
a common ancestor at time $s$, and let $n \to \infty$, the law of large numbers would come into play as it did above in the degenerate case and cause $X_n(1)$ to be dominated by $\delta \text{dist}(Y(1-s))$, for some $\delta > 0$. Since the distribution of $\delta \text{dist}(Y(1-s))$ is not compactly supported, regardless of the choice of $L$, one would conclude that the compact support property does not hold when $\beta = 0$ and $\alpha = \frac{1}{2}$, for all $L$. This is of course false. The above fallacy is a well-known paradox in the theory of branching processes—if a critical, binary, branching process has $n$ individuals alive at the $n$-th generation, then the probability that they had a common ancestor at time $\rho_n$ goes to 0 at $n \to \infty$, for all $\rho < 1$.

The above discussion suggests that one way for the compact support property to break down is for the branching mechanism to be spatially dependent and to decay sufficiently fast as $|x| \to \infty$ so that the law of large numbers will come into play. Furthermore, the faster the diffusion is, the more quickly individual particles that begin together become statistically uncorrelated, so one might expect that the stronger the diffusion, the weaker the threshold on the decay rate in order for the compact support property to break down. If the diffusion process $Y(t)$ corresponds to the operator $L = \frac{1}{2}(1 + |x|)^m \Delta$, for $m \in [0, 2]$, then for $m = 0$ one obtains Brownian motion, while for $m \in (0, 2]$, one obtains a time-changed Brownian motion with the diffusion sped up, the speed increasing in $m$. Theorem one shows that for such an underlying diffusion, for any $\epsilon > 0$, the rate $\exp(-|x|^{2-m+\epsilon})$ is sufficiently fast to cause the compact support property to fail, but the rate $\exp(-|x|^{2-m})$ is not fast enough.

2. Proof of Theorem 1

Proof of part (1). Let $u(x, t)$ be any solution of 1.2 with initial data $g = 0$. By Theorem EP1, we need to show that $u \equiv 0$. Define $\hat{U}(x, t)$ through the equality $u(x, t) = \hat{U}(x, t) \exp(\lambda(1 + |x|^2)^{\frac{m}{2+m}}(t+\delta))$, for some $\lambda, \delta > 0$. Then
\begin{align}
(2.1) \quad & \exp(-\lambda(1+|x|^2)^{2-m}(t+\delta))A(x)\Delta u = \\
& A(x) \bigg( \Delta \hat{U} + 2\lambda(2-m)(1+|x|^2)^{-\frac{m}{2}}(t+\delta)x \cdot \nabla \hat{U} \bigg) \\
& + A(x) \bigg( (t+\delta)^2 \lambda^2(2-m)^2(1+|x|^2)^{-m}|x|^2 \\
& + (t+\delta)\lambda d(2-m)(1+|x|^2)^{-\frac{m}{2}} - (t+\delta)\lambda(2-m)m(1+|x|^2)^{-\frac{m}{2}-1}|x|^2 \bigg) \hat{U} \\
\end{align}

Also, using the bound on \(\alpha\) in the statement of the theorem, we have

\begin{align}
(2.2) \quad & \exp(-\lambda(1+|x|^2)^{2-m}(t+\delta)) \big( \beta u - \alpha u^p - u_t \big) = \\
& \left( \beta - \lambda(1+|x|^2)^{2-m}(t+\delta) \right) \hat{U} - \alpha \exp((p-1)\lambda(1+|x|^2)^{2-m}(t+\delta)) \hat{U}^p. \\
\end{align}

Since \(u\) is a solution to (1.2), the sum of the left hand sides of (2.1) and (2.2) is equal to 0. Consider now the sum of the terms on the right hand sides of (2.1) and (2.2) with the variable \(t\) restricted by \(0 \leq t \leq \delta\). By assumption, \(\alpha(x) \geq C_1 \exp(-C_2|x|^2^{m})\). Thus, the coefficient of \(\hat{U}^p\) will be bounded away from 0 if

\begin{equation}
(2.3) \quad \lambda \delta = \frac{C_2}{p-1}.
\end{equation}

In the case that \(m = 2\), the coefficient of \(\hat{U}\) is bounded from above. Otherwise, the two unbounded terms in the coefficient of \(\hat{U}\) are \(A(x)(t+\delta)^2 \lambda^2(2-m)^2(1+|x|^2)^{-m}|x|^2\) and \(-\lambda(1+|x|^2)^{2-m}\). By assumption, there exists a \(C_4 > 0\) such that \(A(x) \leq C_4(1+|x|^2)^{\frac{m}{2}}\). Thus, in order to guarantee that the coefficient of \(\hat{U}\) is bounded from above, it suffices to have \(\lambda = C_4(2\delta)^2 \lambda^2(2-m)^2\). Using (2.3) to substitute for \((\lambda \delta)^2\) on the right hand side above, we have

\begin{equation}
(2.4) \quad \lambda = 4C_4(2-m)^2C_2^2(p-1)^{-2}.
\end{equation}

With \(\lambda\) and \(\delta\) chosen as in (2.3) and (2.4), it then follows that

\begin{align}
(2.5) \quad & A(x) \bigg( \Delta \hat{U} + 2\lambda(2-m)(1+|x|^2)^{-\frac{m}{2}}(t+\delta)x \cdot \nabla \hat{U} \bigg) \\
& + c_1 \hat{U} - c_2 \hat{U}^p \geq 0, \text{ for } (x, t) \in \mathbb{R}^n \times [0, \delta], \text{ for some } c_1, c_2 > 0.
\end{align}
Let \( M_{R,K}(x,t) = (1 + |x|)^{2} (R - |x|)^{-\frac{2}{p-1}} \exp(K(t+1)), \) for \((x,t) \in B_R \times (0,\infty)\). In [7, proof of Theorem 2], it was shown that for any operator \( \mathcal{L} \) satisfying the conditions of Theorem EP2, there exists a \( K > 0 \) such that for all \( R > 0 \)

\[
(2.6) \quad \mathcal{L}M_{R,K}(x,t) \leq 0, \ (x,t) \in B_R \times (0,\infty),
\]

where \( B_R \) denotes the ball of radius \( R \) centered at the origin. In particular the operator \( \mathcal{A} = A(x)\left(\Delta + 2\lambda(2 - m)(1 + |x|^2)^{-\frac{m}{2}}(t + \delta)x \cdot \nabla\right) \) satisfies the conditions of Theorem EP1, except for the fact that it is time inhomogeneous. The time inhomogeneity causes no problem since we are only considering \( t \in [0,\delta] \) and since for fixed \( x \), everything in sight is uniformly bounded for \( t \in [0,\delta] \). Thus the proof of (2.6) shows that

\[
(2.7) \quad \mathcal{A}M_{R,K}(x,t) \leq 0, \ (x,t) \in B_R \times (0,\delta).
\]

Since \( 0 = \hat{U}(x,0) \leq M_{R,K}(x,0) \) and \( \hat{U}(y,t) \leq M_{R,K}(y,t) = \infty \), for \( y \in \partial B_R \), it follows from (2.5), (2.7) and the maximum principle for semilinear equations [7, Proposition 1] that

\[
(2.8) \quad \hat{U}(x,t) \leq M_{R,K}(x,t), \text{ for } (x,t) \in B_R \times [0,\delta].
\]

Letting \( R \to \infty \), we conclude that \( \hat{U}(x,t) = 0 \), for \((x,t) \in \mathbb{R}^d \times (0,\delta) \). Thus, we also have \( u(x,t) = 0 \), for \((x,t) \in \mathbb{R}^d \times [0,\delta] \). Since \( u \) satisfies a time homogeneous equation, we conclude that in fact \( u(x,t) = 0 \), for \((x,t) \in \mathbb{R}^d \times [0,\infty) \). This completes the proof of part (1).

**Proof of part (2).** By Theorem EP1, we need to show that there is a nontrivial solution to (1.2) with initial data \( g = 0 \). Let \( u(x,t) \) be any solution of (1.2) with initial data \( g = 0 \). Define \( \hat{U}(x,t) \) through the equality \( u(x,t) = \hat{U}(x,t) \exp(\lambda(1 + |x|^2)^{-\frac{m}{2}}(t + \delta)), \) where \( \kappa \in (\delta,\epsilon) \), and \( \epsilon \) and \( \delta \) are as in the statement of the theorem. Now (2.1) and (2.2) hold with the following changes: (i) \( m \) is replaced by \( m - \kappa \); (ii) \( t + \delta \) is replaced by \( 1 \); (iii) the term \( \lambda(1 + |x|^2)^{-\frac{m}{2}} \) in (2.2) is deleted. As before, the sum of the
left hand sides of (2.1) and (2.2) is equal to 0. By assumption, \( \alpha(x) \leq C \exp(-|x|^2 - m + \epsilon) \). Thus, the coefficient of \( \hat{U}^p \) in the amended version of (2.2) is bounded from above, since \( \kappa < \epsilon \). The coefficient of \( \hat{U} \) in the amended version of (2.2) is \( \beta \), and by assumption, there exists a \( C_5 > 0 \) such that \( \beta \geq -C_5(1 + |x|^2)^{\frac{2}{2} - m + \kappa} \). The coefficient of \( \hat{U} \) in the amended version of (2.1) is \( A(x) \left( \Delta \hat{U} + 2\lambda(2 - m + \kappa)(1 + |x|^2)^{-\frac{m + \kappa}{2}} x \cdot \nabla \hat{U} \right) \). By assumption, there exists a \( C_3 > 0 \) such that \( A(x) \geq C_3(1 + |x|^2)^\frac{m}{2} \). It is easy to check that by choosing \( \lambda \) sufficiently large, the factor multiplying \( A(x) \) above will be bounded from below by \( \frac{2C_3}{C_5}(1 + |x|^2)^{\frac{2m + 2\kappa}{2} - 1 - \frac{m + \kappa}{2}} \). (For \( |x| \leq \frac{1}{2} \), use the second term in the parentheses, and for \( |x| > \frac{1}{2} \) use the first and third terms.) Thus, the coefficient of \( \hat{U} \) in the amended version of (2.1) is greater or equal to \( \beta \geq C_6 > 0 \) and \( \alpha \leq C_7 \), and that uniqueness for the original equation is equivalent to uniqueness for (2.9). We will show below that the diffusion process corresponding to the operator \( A(x) \left( \Delta + 2\lambda(2 - m + \kappa)(1 + |x|^2)^{-\frac{m + \kappa}{2}} x \cdot \nabla \right) \) explodes. Thus, it follow from Theorem EP4 that uniqueness does not hold for (2.9). Consequently, uniqueness does not hold for the original equation with initial data \( g = 0 \); thus, by Theorem EP1, the compact support property does not hold.

The above analysis shows that \( \hat{U} \) satisfies the equation

\[
\hat{U}_t = A(x) \left( \Delta \hat{U} + 2\lambda(2 - m + \kappa)(1 + |x|^2)^{-\frac{m + \kappa}{2}} x \cdot \nabla \hat{U} \right) + \hat{\beta} \hat{U} - \hat{\alpha} \hat{U}^p;
\]

\( \hat{U}(x, 0) = 0 \),

where \( \hat{\beta} \geq C_6 > 0 \) and \( \hat{\alpha} \leq C_7 \), and that uniqueness for the original equation is equivalent to uniqueness for (2.9). We will show below that the diffusion process corresponding to the operator \( A(x) \left( \Delta + 2\lambda(2 - m + \kappa)(1 + |x|^2)^{-\frac{m + \kappa}{2}} x \cdot \nabla \right) \) explodes. Thus, it follow from Theorem EP4 that uniqueness does not hold for (2.9). Consequently, uniqueness does not hold for the original equation with initial data \( g = 0 \); thus, by Theorem EP1, the compact support property does not hold.

It remains to show that the diffusion corresponding to the operator \( A(x) \left( \Delta + 2\lambda(2 - m + \kappa)(1 + |x|^2)^{-\frac{m + \kappa}{2}} x \cdot \nabla \right) \) explodes. The diffusion in question is a time change of the diffusion corresponding to \( \Delta + 2\lambda(2 - m + \kappa)(1 + |x|^2)^{-\frac{m + \kappa}{2}} x \cdot \nabla \).
It is not hard to show that since $A(x) \geq C_3(1+|x|^2)^{\frac{m}{2}}$, explosion will occur for the diffusion in question if it occurs for the diffusion corresponding to the operator $C_3(1+|x|^2)^{\frac{m}{2}}\left(\Delta + 2\lambda(2-m+\kappa)(1+|x|^2)^{-\frac{m-\kappa}{2}}x\cdot \nabla\right)$.

This latter operator is radially symmetric, and its radial component is of the form $p(r) \frac{d^2}{dr^2} + q(r) \frac{d}{dr}$, with $p(r)$ satisfying $c_1(1+r)^m \leq p(r) \leq c_2(1+r)^m$, for all $r \geq 0$, and $q(r)$ satisfying $c_3(1+r)^{1+\kappa} \leq q(r) \leq c_4(1+r)^{1+\kappa}$, for $r > 1$, where $c_1, c_2, c_3, c_4 > 0$. This diffusion explodes by the Feller criterion [12].

\[ \square \]

3. Proof of Theorem \[2\]

\textit{Proof.} We recall the $h$-transform theory for measure-valued diffusions, developed in [3, Section 2]. Let $X(t)$ be the measure-valued diffusion corresponding to the triplet $(L, \beta, \alpha)$, and let $h(x)$ be a twice differentiable, positive function. The $h$-transform $L^h$ of the operator $L$ is defined by $L^hf = \frac{1}{h}L(fh)$. Equivalently, $L^h = L + a \frac{\nabla h}{h}$. Let $L_0^h = L + a \frac{\nabla h}{h}$ so that

\begin{equation}
L^h = L_0^h + \frac{Lh}{h} \tag{3.1}
\end{equation}

Then the measure-valued diffusion $X^h(t)$ corresponding to the triplet $(L_0^h, \beta + \frac{Lh}{h}, \alpha h^{p-1})$ is almost surely, mutually absolutely continuous with respect to $X(t)$ and the Radon-Nikodym derivative is equal to the deterministic, time-independent function $h$; that is $\frac{dX^h(t)}{dX(t)}(x) = h(x)$. From this, it follows in particular that $X(t)$ possesses the compact support property if and only if $X^h(t)$ does.

We utilize this $h$ transform as follows. Let $X(t)$ correspond to the triplet $(L, \beta, \alpha)$, as in the statement of the theorem. Let $g(x) = E_x \tau_\infty$. By assumption, $g$ is finite. In fact, $g$ is the minimal positive solution to the equation $Lg = -1$ in $R^d$. (To see this, let $g_n$ solve $Lg_n = -1$ in the ball of radius $n$ centered at the origin, with $g_n = 0$ on the boundary of the ball. By Ito’s formula, $g_n(x) = E_x \tau_n$. Then $g(x) = \lim_{n \to \infty} g_n(x)$, and the minimality of $g$
follows from the maximum principal and the vanishing Dirichlet boundary data on the boundary of the ball.) Let $M = \sup_{x \in \mathbb{R}^d} g(x)$. By assumption, $M < \infty$. For $\delta > 0$, define $h(x) = M + \delta - g(x)$. Let $X^h(t)$ be the measure-valued diffusion corresponding to the triplet $(L^h_0, \beta + \frac{Lh}{h}, \alpha h)$. We have $\sup_{x \in \mathbb{R}^d} h(x) = M + \delta$ and $Lh = 1$. Since $\alpha$ is bounded, so is $\alpha h$. Also, by [12], it follows that if $\delta$ is chosen sufficiently small, then
\[
\inf_{x \in \mathbb{R}^d} (\beta(x) + \frac{Lh(x)}{h(x)}) = \inf_{x \in \mathbb{R}^d} (\beta(x) + \frac{1}{M + \delta}) > 0.
\]
Thus, if we show that the diffusion process corresponding to $L^h_0$ explodes, then it follows from Theorem EP4 that $X^h(t)$ does not possess the compact support property. Consequently, by the above $h$-transform theory for measure-valued diffusions, neither does $X(t)$ possess the compact support property. Thus, to complete the proof of the theorem, it suffices to show that the diffusion process corresponding to $L^h_0$ explodes.

Let $p(t, x, y) = P_x(Y(t) \in dy)$ denote the transition kernel for the semigroup corresponding to $L$. Let $Y_0(t)$ denote the diffusion process corresponding to the operator $L^h_0$, and denote the corresponding probabilities and expectations for the process starting from $x$ by $P^0_x$ and $E^0_x$. Let $p_0(t, x, y) = P^0_x(Y_0(t) \in dy)$ denote the transition kernel for the semigroup corresponding to $L^h_0$. Finally, let $p^h(t, x, y)$ denote the transition kernel for the semigroup corresponding to $L^h$. In light of (3.1) and the fact that $\frac{Lh}{h} = \frac{1}{h}$, the Feynman-Kac formula gives
\[
p^h(t, x, y) = E^0_x \exp\left(\int_0^t \frac{1}{h(Y_0(s))} ds\right) 1_{dy}(Y_0(t)).
\]
In particular then,
\[
p^h(t, x, y) \geq p_0(t, x, y).
\]
On the other hand, it follows from $h$-transform theory for diffusions [12, Theorem 4.1.1] that
\[
p^h(t, x, y) = p(t, x, y) \frac{h(y)}{h(x)}.
\]
From (3.3) and (3.4) along with the fact that $h$ is bounded and bounded away from 0, we conclude that

$$p_0(t, x, y) \leq C p(t, x, y), \text{ for some } C > 0.$$ 

Since

$$E_x \tau_\infty = E_x \int_0^\infty 1_{R^d}(Y(s))ds = \int_0^\infty P_x(Y(s) \in R^d) = \int_0^\infty \int_{R^d} p(t, x, y)dy,$$

it follows from (3.5) that

$$\int_0^\infty \int_{R^d} p(t, x, y)dy < \infty.$$ 

Letting $\tau^0_\infty$ denote the explosion time for $Y^0(t)$, we conclude from (3.5)-(3.7) that

$$E_x^{0, \tau^0_\infty} = \int_0^\infty \int_{R^d} p_0(t, x, y)dy < \infty.$$ 

In particular then, the process $Y^0(t)$ explodes.

\[\square\]

4. Proof of Theorem 3

Proof of part (1). We will show that uniqueness does not hold for (1.9). Then by Theorem EP1, the compact support property does not hold. By the comparison result stated after the proof of Theorem EP3, we may assume that $\beta = \frac{\beta_0 + \kappa}{|x|^2}$. Since the problem is now radially symmetric, it suffices to show that there exists a nontrivial solution to the radially symmetric equation

$$u_t = u_{rr} + \frac{d-1}{r} u_r - u^p, \quad r \in (0, \infty), \quad t > 0;$$

$$u(r, 0) = 0, \quad r \in (0, \infty);$$

$$u \geq 0, \quad r \in (0, \infty), \quad t \geq 0.$$ 

The function $W(x) = \frac{\kappa^{p-1}}{r^{p-1}} r^{-\frac{p-2}{p-1}}$ is a positive, stationary solution of the parabolic equation $u_t = u_{rr} + \frac{d-1}{r} u_r + \frac{\beta_0 + \kappa}{r^2} u - u^p$ in $(0, \infty)$. By [13, Theorem 2-ii], the fact that there exists a nontrivial positive, stationary solution
guarantees that uniqueness does not hold for the corresponding parabolic equation with initial data 0; that is, uniqueness does not hold for (4.1). Actually, the result in [13] is for equations with domain $\mathbb{R}^d$, $d \geq 1$, whereas the domain here is $(0, \infty)$. One can check that the proof also holds in a half space, but more simply, one can make the change of variables $z = \frac{1}{x} - x$, which converts the problem to all of $R$.

Proof of part (2). We will show that uniqueness holds for (1.9). Then by Theorem EP1, the compact support property holds. For $\epsilon$ and $R$ satisfying $0 < \epsilon < 1$ and $R > 1$, and for some $l \in (0, 1]$, define

\begin{equation}
\phi_{R, \epsilon}(x) = ((|x| - \epsilon)(R - |x|))^{-\frac{2}{p-1}} (1 + |x|)^{\frac{2}{p-1}} (1 + \frac{\epsilon^l}{|x|^l} R^{\frac{2}{p-1}}).
\end{equation}

Also, for $R$ and $\epsilon$ as above, and some $\gamma > 0$, define

\begin{equation}
\psi_{R, \epsilon}(x, t) = \phi_{R, \epsilon}(x) \exp(\gamma(t + 1)).
\end{equation}

Note that $\psi_{R, \epsilon}(x, 0) > 0$, for $|x| \in (\epsilon, R)$, and $\psi_{R, \epsilon}(x, t) = \infty$, for $|x| = \epsilon$ and $|x| = R$. We will show that for all sufficiently large $R$ and all sufficiently small $\epsilon$, and for $\gamma$ sufficiently large and $l$ sufficiently small, independent of those $R$ and $\epsilon$, one has

\begin{equation}
\frac{1}{2} \Delta \psi_{R, \epsilon} + \beta \psi_{R, \epsilon} - \psi_{R, \epsilon}^p - (\psi_{R, \epsilon})_t \leq 0, \quad \text{for } \epsilon < |x| < R \text{ and } t > 0.
\end{equation}

It then follows from the maximum principle for semi-linear equations [7, Proposition 1] that every solution $u(x, t)$ to (1.9) satisfies

\begin{equation}
u(x, t) \leq \psi_{R, \epsilon}(x, t), \quad \text{for } \epsilon < |x| < R \text{ and } t \in [0, \infty).
\end{equation}

Substituting (4.2) and (4.3) in (4.5), letting $\epsilon \to 0$, and then letting $R \to \infty$, we conclude that $u(x, t) \equiv 0$. Thus, it remains to show (4.4).
From now on we will use radial coordinates, writing $\phi(r)$ for $\phi(x)$ with $|x| = r$ and similarly for $\psi$. We have

\begin{equation}
\exp(-\gamma(t + 1)) \psi_{R,\epsilon} = -\left(\frac{2}{p - 1}\right)n((r - \epsilon)(R - r))^{-\frac{2}{p - 1} - 1}(R + \epsilon - 2r)(1 + r)^{\frac{2}{p - 1} - 1} (1 + \epsilon^l R^\frac{2}{p - 1}) \\
+ \left(\frac{2}{p - 1}\right)n((r - \epsilon)(R - r))^{-\frac{2}{p - 1} - 1}(1 + r)^{\frac{2}{p - 1} - 1} (1 + \epsilon^l R^\frac{2}{p - 1}) \\
- l((r - \epsilon)(R - r))^{-\frac{2}{p - 1} - 1}(1 + r)^{\frac{2}{p - 1} - 1} \epsilon^l R^\frac{2}{p - 1},
\end{equation}

and

\begin{equation}
\exp(-\gamma(t + 1))((r - \epsilon)(R - r))^{-\frac{2}{p - 1} - 1} \left(\frac{1}{2}\right) \psi_{R,\epsilon} = \\
\left(\frac{1}{p - 1}\right)n\left(\frac{2}{p - 1} + 1\right)(R + \epsilon - 2r)^2(1 + r)^{\frac{2}{p - 1} - 1} (1 + \epsilon^l R^\frac{2}{p - 1}) \\
+ \left(\frac{2}{p - 1}\right)n(r - \epsilon)(R - r)(1 + r)^{\frac{2}{p - 1} - 1} (1 + \epsilon^l R^\frac{2}{p - 1}) \\
- \left(\frac{2}{p - 1}\right)n^2(r - \epsilon)(R - r)(R + \epsilon - 2r)(1 + r)^{\frac{2}{p - 1} - 1} (1 + \epsilon^l R^\frac{2}{p - 1}) \\
+ l\left(\frac{2}{p - 1}\right)n(r - \epsilon)(R - r)(R + \epsilon - 2r)(1 + r)^{\frac{2}{p - 1} - 1} \epsilon^l r^{\frac{2}{p - 1}} R^\frac{2}{p - 1} \\
+ \left(\frac{1}{p - 1}\right)n\left(\frac{2}{p - 1} - 1\right)((r - \epsilon)(R - r))^2(1 + r)^{\frac{2}{p - 1} - 2} (1 + \epsilon^l R^\frac{2}{p - 1}) \\
- l\left(\frac{2}{p - 1}\right)n((r - \epsilon)(R - r))^2(1 + r)^{\frac{2}{p - 1} - 1} \epsilon^l r^{\frac{2}{p - 1}} R^\frac{2}{p - 1} \\
+ \frac{l(l + 1)}{2}((r - \epsilon)(R - r))^2(1 + r)^{\frac{2}{p - 1} - 2} \epsilon R^{\frac{2}{p - 1}}. 
\end{equation}
Using (4.3), (4.6) and the fact that \( \frac{2}{p-1} + 2 = \frac{2p}{p-1} \), we have

\[
\exp(-\gamma(t+1))((r - \epsilon)(R - r))^{-\frac{2}{p-1}} \times \\
\left( \frac{d - 1}{2r} (\psi_{R,\epsilon})_r + \beta \psi_{R,\epsilon} - \psi_{R,\epsilon}^p - (\psi_{R,\epsilon})_t \right) = \\
- \left( \frac{2}{p-1} \right) \frac{d - 1}{2r} (r - \epsilon)(R - r)(R + \epsilon - 2r)(1 + r)^{\frac{2}{p-1}} (1 + \frac{\epsilon l}{r l R^\frac{2}{p-1}}) \\
\frac{d - 1}{2r} (r - \epsilon)(R - r)^2 (1 + r)^{\frac{2}{p-1}} (1 + \frac{\epsilon l}{r l R^\frac{2}{p-1}}) \\
- \left( \frac{2}{p-1} \right) \frac{d - 1}{2r} (r - \epsilon)(R - r)^2 (1 + r)^{\frac{2}{p-1}} (1 + \frac{\epsilon l}{r l R^\frac{2}{p-1}}) \\
+ (\beta - \gamma)(r - \epsilon)(R - r)^2 (1 + r)^{\frac{2}{p-1}} (1 + \frac{\epsilon l}{r l R^\frac{2}{p-1}}) \\
- (1 + r)^{\frac{2p}{p-1}} (1 + \frac{\epsilon l}{r l R^\frac{2}{p-1}}) \exp((p - 1)\gamma(t + 1)).
\]

We will show that for all sufficiently large \( R \) and sufficiently small \( \epsilon \), and for sufficiently large \( \gamma \) and sufficiently small \( l \), independent of those \( R \) and \( \epsilon \), the sum of the right hand sides of (4.7) and (4.8) is non-positive. This will then prove (4.4).

We will denote the seven terms on the right hand side of (4.7) by \( J_1 - J_7 \), and the five terms on the right hand side of (4.8) by \( I_1 - I_5 \). Note that the terms that are positive are \( J_1, J_2, J_4, J_5, J_7 \) and \( I_2 \). (Since \( \beta \) is bounded from above, \( I_4 \) is negative for \( \gamma \) sufficiently large.) In what follows, \( M \) will denote a positive number that can be made as large as one desires by choosing \( \gamma \) sufficiently large. Consider first those \( r \) satisfying \( r \geq cR \), where \( c \) is a fixed positive number. For \( r \) in this range, we have \( |I_5| \geq MR^{\frac{2}{p-1}+2}(1+\epsilon l R^\frac{2}{p-1}) \).

It is easy to see that for \( M \) sufficiently large, \( |I_5| \) dominates each of the positive terms, uniformly over large \( R \) and small \( \epsilon \), and thus (since \( M \) can be made arbitrarily large) also the sum of all of the positive terms. Now consider those \( r \) for which \( \delta_0 \leq r \leq C \), for some constants \( 0 < \delta_0 < C \). For \( r \) in this range, we have \( |I_4| \geq MR^2(1+\epsilon l R^\frac{2}{p-1}) \), and it is easy to see that for \( M \) sufficiently large, \( |I_4| \) dominates each of the positive terms, uniformly over large \( R \) and small \( \epsilon \), and thus, also the sum of all of the positive terms.
One can also show that the transition from $r$ of order unity to $r$ of order $R$ causes no problem. Thus, we conclude that for any fixed $\delta_0 > 0$, for all $l \in (0, 1]$ and $\gamma$ sufficiently large, the sum of the right hand sides of (4.7) and (4.8) is negative for all large $R$ and small $\epsilon$.

We now turn to the case that $\epsilon \leq r \leq \delta_0$. (Note that at $r = \epsilon$, all the terms vanish except $J_1$ and $I_5$. Using the fact that $2p^{-1} + 2 = 2p^{-1}$, it is easy to see that for sufficiently large $\gamma$, $|I_5(\epsilon)|$ dominates $J_1(\epsilon)$, uniformly over all large $R$ and small $\epsilon$. However, when $r$ is small, but on an order larger than $\epsilon$, the analysis becomes a lot more involved.) In the sequel, whenever we say that a condition holds for $\gamma$ or $M$ sufficiently large, or for $l$ sufficiently small, we mean that it holds independent of $R$ and $\epsilon$.

Clearly, $J_5 \leq |I_4|$ if $\gamma$ is sufficiently large. We now show that for $\gamma$ sufficiently large, $J_2 \leq |I_4| + |I_5|$, for $\epsilon \leq r \leq \delta_0$. (We are reusing $|I_4|$ here. Later we will reuse $|I_5|$. This is permissible because $\gamma$ can be chosen as large as we like.) To show this inequality, it suffices to show that for $M$ sufficiently large,

\[(4.9) \quad (r - \epsilon)R \leq M(r - \epsilon)^2 R^2 + M(1 + \frac{\epsilon}{\gamma^{p-1} R^{p-1}})^{-1}, \text{ for } r \in [\epsilon, \delta_0].\]

A trivial calculation shows that the left hand side of (4.9) is less than the first term on the right hand side if $r \geq \epsilon + \frac{1}{RM}$. If $r \in [\epsilon, \epsilon + \frac{1}{RM}]$, then the left hand side of (4.9) is less than or equal to $\frac{1}{M}$ while the second term on the right hand side is greater than $M$. We conclude that (4.9) holds with $M \geq 1$.

Since $I_2$ has the factor $(r - \epsilon)^2$, while $I_1$ has the factor $(r - \epsilon)$, and since $\frac{R - \epsilon}{R + \epsilon - 2\epsilon}$ can be made arbitrarily close to one by choosing $R$ sufficiently large, it follows that for any $\eta > 0$, we can guarantee that $I_2 \leq \eta |I_1|$, for $r \in [\epsilon, \delta_0]$, if we choose $\delta_0$ sufficiently small. Note that $J_4 \leq \frac{2l}{\gamma^{p-1}} |I_1|$. Thus, given any $\zeta > 0$, if we choose $\delta_0$ and $l$ sufficiently small, we will have $I_2 + J_4 \leq \zeta |I_1|$. To complete the proof, we will show that

\[(4.10) \quad J_1 + J_7 + (1 - \zeta)I_1 + I_4 + I_5 \leq 0, \text{ for } r \in [\epsilon, \delta_0],\]
for sufficiently large $\gamma$ and sufficiently small $l$ and $\delta_0$, uniformly over large $R$ and small $\epsilon$. By the assumption on $\beta$, there exists an $\eta_0 > 0$ such that if $\delta_0$ is chosen sufficiently small, then

\begin{equation}
I_4 \leq (\beta_0 - \eta_0)(1 - \frac{\epsilon}{r})(R - r)^2(1 + r)^{\frac{2}{p-1}}(1 + \frac{\ell}{r^l}R^{\frac{2}{p-l}}),
\end{equation}

for $r \in [\epsilon, \delta_0]$,

where $\beta_0$ is as in the statement of the theorem. Since $\frac{R + \epsilon - 2r}{R - r}$ can be made arbitrarily close to one by choosing $R$ sufficiently large, we also have

\begin{equation}
I_1 \leq -\frac{(d - 1)}{(p - 1)}(1 - \frac{\epsilon}{r})(R - r)^2(1 + r)^{\frac{2}{p-1}}(1 + \frac{\ell}{r^l}R^{\frac{2}{p-l}}), \text{ for } r \in [\epsilon, \delta_0].
\end{equation}

Since

\begin{equation}
J_7 = \frac{l(l + 1)}{2}((r - \epsilon)(R - r))^2(1 + r)^{\frac{2}{p-1}} \frac{\ell}{r^l} R^{\frac{2}{p-l}},
\end{equation}

and since $\frac{R + \epsilon - 2r}{R + \epsilon - 2r}$ can be made arbitrarily close to one by choosing $R$ sufficiently large, it follows that for any $\tau > 0$, we can choose $l$ sufficiently small so that

\begin{equation}
J_7 \leq \tau J_1, \text{ for } r \in [\epsilon, \delta_0],
\end{equation}

uniformly over large $R$ and small $\epsilon$. Using (4.11)-(4.13) along with the fact that

\begin{equation}
J_1 \leq \frac{p + 1}{(p - 1)^2} (R - r)^2(1 + r)^{\frac{2}{p-1}}(1 + \frac{\ell}{r^l} R^{\frac{2}{p-l}})
\end{equation}

and that $\beta_0 = \frac{d(p - 1) - 2p}{(p - 1)^2}$, it follows that

\begin{equation}
J_1 + J_7 + (1 - \zeta) I_1 + I_4 \leq \left( \frac{\tau(p + 1) + \zeta(d - 1)(p - 1)}{(p - 1)^2} - \eta_0 \left( \frac{1}{2} + \frac{\zeta}{2} \right) + C\frac{\epsilon}{\tau} \right) \times
\end{equation}

\begin{equation}
(R - r)^2(1 + r)^{\frac{2}{p-1}}(1 + \frac{\ell}{r^l} R^{\frac{2}{p-l}}), \text{ for } r \in [\epsilon, \delta_0],
\end{equation}

for some $C > 0$, uniformly over large $R$ and small $\epsilon$. By picking $\tau$ and $\zeta$ sufficiently small, we have $\frac{\tau(p + 1) + \zeta(d - 1)(p - 1)}{(p - 1)^2} - \eta_0 \left( \frac{1}{2} + \frac{\zeta}{2} \right) < 0$. Thus, in order
to show (4.10), it suffices to show that

\[
\frac{\epsilon}{r} R^2 \leq M \left(1 + \frac{\epsilon^l}{r^l R^{p-1}}\right)^{p-1}, \quad r \in [\epsilon, \delta_0],
\]

for sufficiently large \(M\). But since \(l(p - 1) \leq 1\), the right hand side of (4.15) is greater or equal to \(M \frac{\epsilon}{r} R^2\).

\[\square\]

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Department of Mathematics, Technion—Israel Institute of Technology, Haifa, 32000, Israel

E-mail address: pinsky@math.technion.ac.il