CORE ENTROPY OF POLYNOMIALS WITH A CRITICAL POINT OF MAXIMAL ORDER

DOMINGO GONZÁLEZ* AND GAMALIEL BLÉ

División Académica de Ciencias Básicas
Universidad Juárez Autónoma de Tabasco
Carr. Cunduacán-Jalpa Km 1
Cunduacán Tabasco, México

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ABSTRACT. This paper discusses some properties of the topological entropy of the systems generated by polynomials of degree \( d \) with two critical points. A partial order in the parameter space is defined. The monotonicity of the topological entropy of postcritically finite polynomials of degree \( d \) acting on Hubbard tree is generalized.

1. Introduction. One of the biggest issues in holomorphic dynamics is to understand the global dynamics of the polynomials of degree \( d \). The family of quadratic polynomials has been studied by many researchers, hence there are many known results which are relevant, \([5, 2, 9, 10]\). However, for higher degrees very little is known.

Given a function \( f \) defined on a compact set \( X \) to itself, the topological entropy \( h(f) \) is the measure of the complexity degree of the dynamics of \( f \) and it is a way of measuring the chaos in the system, \([13]\). More precisely, it measures the growing rate of the number of orbits of \( f \). It is known that given a polynomial \( P_d : \mathbb{C} \to \mathbb{C} \) of degree \( d \), \( h(P_d) = \log d \). Even more, the entropy of \( P_d \) restricted to its filled Julia set is the same as the entropy of \( P_d \) restricted to its Julia set, and is equal to \( \log d \), \([8]\).

The entropy for real polynomial functions was studied by Milnor, Thurston, Tresser and Douady among others, they studied the quadratic and cubic family, proved the monotonicity of entropy, \([12, 13, 4]\). On the other hand, Radulescu studied the behavior of the entropy for a real quartic family obtained by composition of two logistic maps, using symbolic dynamics, \([15]\). Thurston generalized the entropy concept defined for invariant intervals in the real case to the entropy study of a polynomial restricted to its Hubbard tree, which keeps the dynamics information of \( P_d \). This is called the core entropy, \([17, 18]\).

The core entropy can be used as a tool to classify the different dynamics of a polynomial family in its parameter space. For the quadratic polynomials it was proved that the core entropy grows through the veins of the Mandelbrot set, \( \mathcal{M} \),

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* Corresponding author.
Recently, Tiozzo proved the continuity of the core entropy varying the external argument, [19]. In this paper we calculate the core entropy of a polynomial family $P_d$ of degree $d$ with one parameter. The family is

$$P_a(z) = z^{d-1}(z + \frac{da}{d-1}). \quad (1.1)$$

The polynomial function $P_a(z)$ has two critical points, 0 that is a fixed point and $-a$ that is a free point. The parameter space of this family was studied by Roesch in [16]. She studies the connectedness locus of this family and gives a description of the closures of the hyperbolic components and the Mandelbrot copies contained in this set.

The goal of this work is to generalize the results obtained for quadratic and cubic families in [8]. We give a description of the dynamics in the family $P_d$, showing the behavior of the entropy in Hubbard’s trees. For this purpose, we define the following partial order in the parameter space. We say that $a \prec b$ if the polynomial $P_a$ is dynamically included in $P_b$, and we obtain the next result:

**Main Theorem 1.** Let $a_1, a_2 \in \mathbb{C}$ such that $P_{a_1}, P_{a_2} \in P_d$ and their critical points are periodic. If $a_2 \prec a_1$ then $h(H(a_2), P_{a_2}) \leq h(H(a_1), P_{a_1})$, where $H(a_1)$ and $H(a_2)$ denote the Hubbard trees of $P_{a_1}$ and $P_{a_2}$, respectively.

For the proof of this result, we use the external rays and the conjugation of the dynamic of polynomials of degree $d$ with multiplication by $d$.

2. **Topological entropy.** In this section we state the definition and some properties of the topological entropy for a function $f$ defined on a compact set $X$ to itself, where $X$ is a topological compact set. These results and properties can be consulted in [1, 4].

If $\alpha$ and $\beta$ are two open covers of $X$, the refinement cover of $\alpha$ and $\beta$ is the union of all sets $A \cap B$, $A \in \alpha$ and $B \in \beta$ and its denoted by $\alpha \vee \beta$.

**Definition 2.1.** The entropy of an open cover $\alpha$ is defined as

$$H(\alpha) = \log(N(\alpha)),$$

where $N(\alpha)$ is the infimum number of open sets in any finite subcover. It can be seen that $H(\alpha) \geq 0$, and that the equality is achieved if and only if $X \in \alpha$.

The limit

$$h(f, \alpha) = \lim_{n \to \infty} \frac{H(\alpha \vee f^{-1} \alpha \vee \ldots \vee f^{-n+1} \alpha)}{n}$$

exists and it is called the topological entropy of $f$ relative to the cover $\alpha$ and satisfies

$$0 \leq h(f, \alpha) \leq H(\alpha).$$

The number

$$h(f) = \sup_{\alpha} h(f, \alpha),$$

is called the topological entropy of $f$, where the supremum is taken over all open covers of $X$. 

2.1. Properties of the entropy.

1. If \( f : X \to X \) is a continuous function on a compact metric space \( X \), then 
\[
h(f^k) = kh(f),
\]
for all integers \( k > 0 \).
2. If \( f : X \to X \) is a homeomorphism, then \( h(f) = h(f^{-1}) \).
3. If \( f : X \to X \) and \( g : Y \to Y \) are topologically conjugate continuous functions, then \( h(f) = h(g) \).
4. If \( X = X_1 \cup X_2 \) with \( X_i \) compact and \( f \)-invariant for \( i = 1, 2 \), then 
\[
h(X, f) = \sup\{h(X_1, f), h(X_2, f)\}.
\]

3. Polynomial family of degree \( d \). Let us consider a polynomial family of degree \( d \geq 3 \) that has two critical points and one of them is a fixed critical point of maximum multiplicity. It is described modulo affine conjugacy as the polynomial family
\[
P_a(z) = z^{d-1}(z + \frac{da}{d-1}),
\]
where the parameter \( a \in \mathbb{C} \) and \( P_a \) has two critical points, 0 is a critical point of maximum multiplicity and \(-a\) is a critical point of simple multiplicity. The topological properties of the closures of the hyperbolic components and Mandelbrot copies in the parameter space for this family are analyzed in [16].

The filled Julia set \( K_a \) of \( P_a \) consists of the points whose orbits do not go to infinity and the Julia set \( J_a \) of \( P_a \) is the boundary of \( K_a \).

For \( d \) fixed, the connectedness locus of the family \( P_d \) is defined as 
\[
\mathcal{C} = \{ a \in \mathbb{C} : J_a \text{ is connected} \}.
\]

The parameter space \( \mathbb{C} \) is divided into two regions, \( \mathcal{C} \) and \( W_\infty \), where \( W_\infty \) is the set of all the parameters for which the free critical point \(-a\) is attracted to \( \infty \). In the Figure 1 it is shown the connectedness locus for \( d = 4 \) and \( d = 5 \).

![Figure 1. Connectedness locus for \( d = 4 \) and \( d = 5 \).](image-url)

In the connectedness locus, there is a special set of parameters, the set of parameters \( a \in \mathcal{C} \) for which \( P_a \) is post-critically finite, that is, the critical orbits are finite. This set is denoted by \( \mathcal{C}^0 \).
Note that $K_a \neq J_a$ because $K_a$ contains the basin of attraction of 0,
$$B_a = \{z \in \mathbb{C} : P_a^n(z) \to 0\}.$$ 
Let us denote by $B_a$ the immediate basin of attraction of zero, this is the component of $B_a$ containing 0. By the maximum modulus principle, $B_a$ is topologically a disk. Moreover, if $-a \notin B_a$, then the polynomial $P_a|B_a$ is conjugated to $z^{d-1}$ in $\mathbb{D}$, otherwise $B_a = B_a$.

By Böttcher’s Theorem, for the super-attracting fixed points $p = 0$, $\infty$ there are neighborhoods $V_a^p$, $W_a^p$ of $p$ such that $P_a(V_a^p) \subset V_a^p$, and conformal isomorphisms $\varphi_a^p : V_a^p \to W_a^p$, such that the following diagrams are commutative,

$$
\begin{array}{ccc}
V_a^\infty & \xrightarrow{P_a} & V_a^\infty \\
\varphi_a^\infty & \downarrow & \varphi_a^\infty \\
W_a^\infty & \xrightarrow{z^d} & W_a^\infty \\
\end{array}
\quad
\begin{array}{ccc}
V_a^0 & \xrightarrow{P_a} & V_a^0 \\
\varphi_a^0 & \downarrow & \varphi_a^0 \\
W_a^0 & \xrightarrow{z^{d-1}} & W_a^0 \\
\end{array}
$$

That is
\[
\varphi_a^0 \circ P_a = (\varphi_a^0)^{d-1} \text{ in } V_a^0 \quad \text{and} \quad \varphi_a^\infty \circ P_a = (\varphi_a^\infty)^d \text{ in } V_a^\infty,
\]
with $\varphi_a^\infty$ tangent to the identity, in a neighborhood of $\infty$ and $\varphi_a^0$ tangent to $z \to \lambda(a)z$ around zero, where $\lambda(a)$ is a $(d-2)$-th root of $\frac{da}{d-1}$, [5]. On the other hand, if $a \in \mathbb{C}^0$ then $J_a$ is locally connected, and then the set $V_a^\infty$ is the set $\mathbb{C} \setminus K_a$, [16].

**Definition 3.1.** Set $\psi = (\varphi_a^\infty)^{-1}$. For $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, we define the external ray of argument $\theta$ by
$$R_a(\theta) = \psi(\{re^{\pi i \theta} : r > 1\}).$$

3.1. The parameter space. Since $P_{\tau a}(\tau z) = \tau(P_a(z))$, where $\tau = e^{\frac{\pi i}{d}}$, the rotation is the only conformal conjugation between two polynomials $P_a$, $P_{a'} \in \mathcal{P}_d$. Moreover, $P_a$ is conjugated to the polynomial $P_{\pi}$ by the complex conjugation $\sigma(z) = \overline{z}$, [16]. In this way, a fundamental domain for the study of the polynomial family $P_a$ is
$$S = \{a \in \mathbb{C} : 0 \leq \arg(a) \leq \frac{1}{2(d-1)}\}.$$

**Definition 3.2.** The main connected component $W_0 \subset \mathcal{C}$ is the set
$$\{a \in \mathbb{C} : -a \notin B_a\}.$$ 

According to Milnor, there are four possible types of bounded hyperbolic components in the polynomials family with two critical points [10]. For the family $\mathcal{P}_d$, we have three component types, the main component $W_0$ is the only adjacent component and there are other components of the capture type or disjoint type. In particular, we can classify the capture components in terms of the iterated to reach the basin of zero.

**Definition 3.3.** A capture component is said to have depth $i$, if it is a connected component of the set $W_i = \{a \in \mathbb{C} : P_a^i(-a) \in B_a \text{ and } P_a^{i-1}(-a) \notin B_a\}$. 

The connectedness locus $\mathcal{C}$ is compact and connected. Moreover, $W_\infty$ is isomorphic to $\mathbb{C} \setminus \mathbb{D}$, [16]. In fact, the isomorphism
$$\phi_\infty : \mathbb{C} \setminus \mathcal{C} \to \mathbb{C} \setminus \mathbb{D}$$
is defined as \( \phi_\infty(a) = \varphi_\infty(P_a(-a)) \), where \( \varphi_\infty \) is the Böttcher coordinate of \( P_a \) at infinity, for \( a \in \mathbb{C} \setminus \mathcal{C} \). From \( \varphi_\infty \) we can define the external rays \( R_a(\theta) \) to \( K_a \) and with \( \phi_\infty \) we can define the external rays \( R_{C}(\alpha) \) to \( C \). In particular, if \( \alpha \in \mathbb{Q} \) the external ray \( R_C(\alpha) \) lands in the parameter \( a \in \mathcal{C} \) or in the root of hyperbolic component containing \( a \). In the dynamic plane of \( P_a \), the external ray \( R_a(\alpha) \) lands in \( P_a(-a) \) or in the root of the Fatou component containing \( P_a(-a) \), [16].

4. Hubbard trees. Let \( a \in \mathcal{C} \), since \( P_a \) has a super-attracting fixed point at 0, the interior \( \text{int}(K_a) \) of the filled Julia set is not empty. Every component \( U \) of \( \text{int}(K_a) \) is a bounded Fatou component, with closure \( \overline{U} \) homeomorphic to the closed disc \( \overline{D} \).

Also, by Sullivan’s Theorem every component \( U \subset \text{int}(K_a) \) is eventually periodic, [2]. Therefore if \( -a \) is pre-periodic then \( U \) is mapped in the fixed Fatou component \( B_a \) and if \( -a \) is periodic, then \( U \) is sent to the component \( B_a \) or in the periodic component \( U_{-a} \) containing \( -a \). The center \( c(U) \) of the component \( U \) is the unique inverse image of 0 or \( -a \) in \( U \). In particular, \( c(B_a) = 0 \).

Given any bounded Fatou component \( U \), there is a homeomorphism \( \phi: \overline{U} \rightarrow \overline{D} \) with \( \phi(c(U)) = 0 \). A radial arc is an arc in \( U \) obtained as \( \phi^{-1}(r e^{i\theta}) : 0 \leq r \leq 1 \). Since \( \phi \) is unique, except for a rotation of \( \overline{D} \), the radial arcs are well defined.

An embedded arc in \( K_a \) is any subset of \( K_a \) which is homeomorphic to the closed interval \([0, 1] \subset \mathbb{R} \). An embedded arc \( I \) is regulated if for every bounded Fatou component \( U \), the intersection \( I \cap \overline{U} \) is either empty, a point, or it consists of radial arcs in \( \overline{U} \).

The following Lemma is shown in [5, 20] for quadratic polynomials but the argument is valid for polynomials of degree \( d \).

**Lemma 4.1.** Given any two points \( x, y \in K_a \), there exists a unique regulated arc \( I \in K \) with endpoints \( x, y \). Thus, if \( \eta \) is any embedded arc in \( K_a \), which connects \( x \) with \( y \), then \( I \cap J_a \subset \eta \cap J_a \).

We denote the regulated arc \( I \) from previous Lemma by \([x, y] \). The open arc \((x, y)\) is defined as \([x, y] \setminus \{x, y\} \). Similarly we can define a semi open arc \([x, y] \). In general, given a finite set of points \( \{x_1, x_2, \ldots, x_n\} \subset K_a \) there exists a unique connected smallest set \([x_1, x_2, \ldots, x_n] \subset K_a \) which consists of regulated arcs containing these points. This set is a topological tree and it is called the regulated tree generated by \( \{x_1, x_2, \ldots, x_n\} \), [5].

Given a polynomial \( P_a \) with \( a \in \mathcal{C}^0 \) we define the Hubbard tree of \( P_a \) as the smallest regulated tree that contains the orbits of the critical points and it is denoted by \( H(a) \). Some important points that characterize a Hubbard tree are the following:

Given a Hubbard tree \( H(a) \), a point \( x \in H(a) \) is called endpoint if \( H(a) \setminus \{x\} \) is connected. The set of endpoints is denoted by \( \partial H(a) \) and the number of endpoints is denoted by \( N(a) \), this is, \( N(a) = \text{Card} \partial H(a) \).

A point \( x \in H(a) \) is called a branching point if \( H(a) \setminus \{x\} \) has more than two components. The set of branching points is denoted by \( \text{Br}(H(a)) \).

A point \( x \in H(a) \) is a vertex of \( H(a) \) if \( x \) belongs to the orbit of a critical point or \( x \in \text{Br}(H(a)) \). We denote by \( V(H(a)) \), the set of vertices, which divides the Hubbard tree \( H(a) \) in a finite number of open topological intervals. The closure \( I_j \) of these intervals are called edges of the Hubbard tree. Two points \( x, y \in V(H(a)) \) are adjacent if \((x, y) \cap V(H(a)) = \emptyset \).

Directly of definition we have the following results, see [14].
Lemma 4.3. Let \( \eta \) be a regulated arc that does not contain critical points, except in its endpoints, then \( P_\alpha|_\eta \) is injective and \( P_\alpha(\eta) \) is a regulated arc.

Lemma 4.4. Let \( a \in \mathbb{C}^0 \) and \( \deg(P_a) = d \). If \( N(a) \) is the number of endpoints of \( H(a) \) and \( 0 \not\in \partial(H(a)) \), then the set \( \{ P_a(-a), \ldots, P_a^{N(a)}(-a) \} \) is exactly the set of end points of the Hubbard tree \( H(a) \).

Proof. Assume that \( a \neq 0 \) and that the free critical point \(-a\) is an endpoint of \( H(a) \) that is not fixed. Since any non extreme point that is mapped to an end point must be a critical point, and the critical point 0 is fixed, then there is no non extreme point that can be mapped to \(-a\). Thus we conclude that the preimage of \(-a\) is an end point and therefore \(-a\) is periodic and its orbit is exactly the set of end points.

Now assume that \( a \neq 0 \) and the free critical point \(-a\) is not an end point of \( H(a) \). Since \( P_a \) is locally injective except in the critical points and the critical point 0 is a fixed point, then the only non extreme point that is mapped to an end point is the critical point \(-a\). Therefore, the end points of \( H(a) \) are generated by the first iterations of the critical point \(-a\).

5. Entropy on the Hubbard tree. By the Remark 4.2, the Hubbard tree \( H(a) \) of a polynomial \( P_a \) with postcritical finite set is invariant. Consequently, we can talk about the entropy of \( P_a \) restricted to \( H(a) \) and we will denote it by \( h(H(a), P_a) \).

5.1. Small copies of the Mandelbrot set. Starting from the main component \( W_0 \), across any direction \( t \in \mathbb{Z} \) periodic under the multiplication by \( d - 1 \), there is a small copy of the Mandelbrot set directly attached to \( W_0 \), [16]. We denote the center of this copy by \( a(t) \), then the small copy of the Mandelbrot set can be represented by \( P_{a(t)} \ast \mathcal{M} \).

Let us assume that the angle \( \alpha \in \mathbb{Q} \) and the critical point \(-a(\alpha)\) has period \( k \), then the Hubbard tree \( H(a(\alpha)) \) consists of \( k \) radial edges coming out from a central vertex located in the zero, connecting a point of the orbit to the free critical point \(-a(\alpha)\). Since the dynamics of polynomial \( P_{a(\alpha)} \) over \( H(a(\alpha)) \) is conjugated to multiplication by \( d - 1 \) in these edges, \( h(H(a(\alpha)), P_{a(\alpha)}) = 0 \).

Any polynomial \( P_b \) with \( a \in P_{a(t)} \ast \mathcal{M} \) has the form \( P_{a(t)} \perp P_c \) where \( P_c \) is a quadratic polynomial with \( c \in \mathcal{M} \) and \( \perp \) denote the tuning, [3]. The filled Julia set \( K_b \) of \( P_b \) can be obtained from the filled Julia set \( K_{a(t)} \) as follows: the closure of each component \( U \) from the interior of \( K_{a(t)} \) is replaced by a copy of \( K_c \), [3].

Lemma 5.1. Let \( P_b = P_a \perp P_c \) be the tuning of \( P_c \) over \( P_a \). Then \( h(H(b), P_b) = \sup \{ h(H(a), P_a), \frac{1}{k} h(H(c), P_c) \} \) where \( k \) is the period of the critical point \(-a\) of \( P_a \).

Proof. The set \( H(b) \) consists of two parts, \( H(b) = A \cup B \) where \( A \) are \( k \) copies of the Hubbard tree \( H(c) \) and \( B \) is the Hubbard tree of \( P_a \) removing the \( k \) regulated arcs containing the points of the periodic orbit. Since \( P_b \) is conjugated to \( P_c \), in each copy of \( H(c) \), then \( kh(A, P_b) = h(H(c), P_c) \) so \( h(A, P_b) = \frac{k}{k} h(H(c), P_c) \). In this way we conclude that \( h(H(b), P_b) = \sup \{ h(B, P_a), \frac{1}{k} h(H(c), P_c) \} \).

To complete the proof, we must show that \( h(B, P_a) = h(H(a), P_a) \). Applying a similar idea to the previous one about the tree, we can write \( H(a), H(a) = A' \cup B \), where \( A' \) is the set of the regulated arcs containing the critical orbit of \( P_a \) and \( B \) is the Hubbard tree of \( P_a \) taking out the \( k \) regulated arcs of the critical orbit. Since the \( k \)-th iteration of \( P_a \) restricted to any of these regulated arcs is conjugated to \( z^d \)
in the interval and this one has entropy zero, then \( h(A', P_a) = 0 \). Hence \( h(B, P_a) = h(H(a), P_a) \). Therefore \( h(H(b), P_b) = \sup\{h(H(a), P_a), \frac{1}{b}h(H(c), P_c)\} \).

\( \square \)

**Remark 5.2.** Since \( h(H(a), P_a) = 0 \) we have that \( h(H(b), P_b) = \frac{1}{b}h(H(c), P_c) \).

5.2. **Capture component.** Any copy of the Mandelbrot set \( P_{a(t)} \ast M \) has in its tips a capture component \( C \) directly attached to \( P_{a(t)} \ast M \). [16].

Let \( b \) be any tip of \( P_{a(t)} \ast M \), then \( P_b = P_{a(t)} \perp P_c \) where \( a(t) \) is the center of a hyperbolic component of a copy of the Mandelbrot set and \( c \) is the tip corresponding in the Mandelbrot set. In a similar way to Lemma 5.1 we have that \( h(H(b), P_b) = \frac{1}{b}h(H(c), P_c) \).

**Proposition 5.3.** Let \( a_0 \) be the center of any capture component \( C \). Let \( b \in \partial C \) such that the free critical point \(-b\) is pre-periodic for \( P_b \). Then \( h(H(b), P_b) = h(H(a_0), P_{a_0}) \).

**Proof.** Given a parameter \( b \in \partial C \) such that the free critical point \(-b\) is strictly pre-periodic for \( P_b \), there exists the smallest integer \( n \) such that \( P_b^n(-b) \) is periodic and \( P_b^n(-a) \in \partial B_b \). We denote by \( k \) the period of \( P_b^n(-b) \), let \( l \leq n \) be the smallest integer such that \( P_b^k(-b) \in \partial B_b \), then
\[
B_b \cap H(b) = \{ P_b^k(-b), P_b^{k+1}(-b), \ldots, P_b^n(-b), P_b^{n+1}(-b), P_b^{n+k}(-b) \}.
\]
That is, the orbit of \( P_b^k(-a) \) consists of \( n + k - l \) points and all of them are in the boundary of \( B_b \). Let \( T \) be the tree generated by the regulated arcs connecting the orbit of \( P_b^k(-b) \), which has \( n + k - l \) edges with a common vertex, the super attracting critical point 0. Let \( b_i \) be the edge that connects 0 and \( P_b^l(-b) \) for \( l \leq i \leq n + k - 1 \). Then \( P_b(b_i) = b_{i+1} \) for \( l \leq i < n + k - 1 \) and \( P_b(b_{n+k-1}) = b_n \).

Therefore \( h(T, P_b) = 0 \). On the other hand, we can obtain the Hubbard tree \( H(b) \) from the Hubbard tree \( H(a_0) \) replacing the critical point 0, by the regulated tree \( T \) described previously. As \( h(T, P_b) = 0 \) then
\[
h(H(b), P_b) = \sup\{h(H(a_0), P_{a_0}), h(T, P_b)\} = h(H(a_0), P_{a_0}).
\]
\( \square \)

**Corollary 5.4.** Let \( C \) be a capture component such that \( C \cap M_0 = \{b\} \) where \( M_0 \) is a copy of the Mandelbrot set, then the Hubbard tree from the center of \( C \) has the same entropy as the Hubbard tree of \( P_b \).

5.3. **External rays in the Hubbard tree.** The entropy on the Hubbard tree for a given polynomial \( P_a \) can be studied in a similar way as Douady in [4]. Using the semi-conjugation given by Böttcher’s theorem we have that the entropy of \( P_a \) in the Hubbard tree is the same that the entropy of the map \( z \mapsto dz \) restricted to a \( T_J \subset \mathbb{T} \). This set can be obtained by the following diagram,

\[
\begin{array}{ccc}
\mathbb{T} & \overset{\rho}{\rightarrow} & \mathbb{T} \\
\gamma_a \downarrow & & \gamma_a \\
J_a & \overset{P_a}{\rightarrow} & J_a,
\end{array}
\]

where \( \gamma_a : \mathbb{T} \rightarrow J_a \) is the surjective and continuous map given by the Carathéodory’s Theorem. Thus, \( h(H(a), P_a) = h(\gamma_a^{-1}(H(a) \cap J_a), \rho) \). To determine the set \( T_J \), we will analyze the external rays properties at \( J_a \). [5, 6].
Since 0 is a super attracting fixed point, we give a description of the external rays that land on the Hubbard tree using [8]. Let \( P_a \) be a polynomial with \( a \in \mathbb{C}^0 \). Given \( x \in H(a) \), we define \( \theta(x) \) as the angles of the external rays that land on \( x \), if \( x \in J_a \), or the angles of the external rays that land in the root of the Fatou component containing \( x \), if \( x \notin J_a \). We denote by \( R_a(\theta(x)) \) the external ray with angle \( \theta \) and landing on \( H(a) \). If there is more than one external ray that lands on \( x \) or in the root of Fatou component that contains \( x \) we write it as \( R_a(\theta(x)^-), R_a(\theta(x)^+), R_a(\theta(x)^0), R_a(\theta(x)^2), \) etc.

For a fixed \( a \in \mathbb{C}^0 \), we denoted by \( H = H(a) \) the Hubbard tree corresponding to \( P_a \). Also, \( H_1 = P_a^{-1}(H) \) and \( H_{n+1} = P_a^{-1}(H_n) \). Note that \( H \subset H_1 \subset H_2 \subset \ldots \).

**Definition 5.5.** Let \( a \in \mathbb{C}^0 \) and \([x, y]\) be a regulated arc such that \([x, y]\cap[-a, 0] = \emptyset\). We say that \( x \prec y \) if \( x \in (0, y) \) and \( \theta(x) \neq \theta(y) \).

Let \( P_a \) be a polynomial of degree \( d \) with \( a \in \mathbb{C}^0 \) and let \([x, y]\) be a regulated arc that does not contain critical points. Let \( \{\theta(x)^-, \theta(x)^+, \theta(y)^-, \theta(y)^+\} \) be the angles of the four external rays, such that \([\theta(x)^-, \theta(y)^-]\) and \([\theta(x)^+, \theta(y)^+]\) are the smallest intervals that contain all the external rays that land on \([x, y]\). The set
\[
\{R_a(\theta(x)^-), R_a(\theta(y)^-), R_a(\theta(y)^+), R_a(\theta(x)^+)\}
\]
is called the external rays associated to \([x, y]\). When only one ray lands in \( y \), we have that \( \theta(y)^- = \theta(y)^+ = \theta(y) \), in this case the external rays associated to \([x, y]\) are denoted by \( \{R_a(\theta(x)^-), R_a(\theta(x)^+), R_a(\theta(y))\} \).

From the definition of the associated external rays, we have the next result.

**Lemma 5.6.** Let \( a \in \mathbb{C}^0 \) and \( x, y \in H(a) \). If \( x \prec y \), then
\[
\{R_a(\theta(x)^-), R_a(\theta(y)^-), R_a(\theta(y)^+), R_a(\theta(x)^+)\}
\]
are in a positive cyclic order. Thus, \( \prec \) is a partial order.

5.4. **Intervals of characteristic angles.** Let \( P_a \) be a post-critically finite polynomial and let \( H \) be the Hubbard tree associated to \( P_a \). Suppose that the free critical point \(-a\) is not an endpoint of \( H \).

We can describe the set \( H_1 \setminus H \) in a unique way as the finite union of semi open regulated arcs. Explicitly
\[
H_1 \setminus H = \bigcup_{k=1}^{n} I_k,
\]
where \( I_k = (x_k, y_k] \), \( x_k \in V(H_1) \) and \( y_k \in \partial H_1 \).

For \( x_k \neq 0 \), let \( \{R_a(\theta(x_k)^-), R_a(\theta(y_k)^-), R_a(\theta(y_k)^+), R_a(\theta(x_k)^+\} \) be the external rays associated to \([x_k, y_k]\), we define the interval \( U_k = (\theta(x_k)^-, \theta(x_k)^+) \).

On the other hand, let \( H \cap \partial B_a = \{z_1, z_2, \ldots, z_m\} \) be the set of points on \( H \) that intersects the boundary of the immediate basin \( B_a \) of zero. The external rays that land on these points \( z_l \) are denoted by \( R_a(z_l)^\pm \) for \( 1 \leq l \leq m \), see the Figure 2. We define the subset of the circle
\[
\nu(a) = \mathbb{T} \setminus \bigcup_{l=1}^{m} (\epsilon_l^-, \epsilon_l^+).
\]

We define the interval of the characteristic angles \( \mathcal{U}(a) \) associated to the Hubbard tree \( H \) as
\[
\mathcal{U}(a) = \nu(a) \bigcup \bigcup_{k=1}^{n} U_k.
\]
Remark 5.7. Any two regulated arcs of $H_1 \setminus H$ have empty intersection or if $I_k \cap I_l \neq \emptyset$ then $U_k = U_l$ or $U_k \cap U_l = \emptyset$.

Example 5.8. In the Figure 2 we show the Julia set and the angles associated to $H(a)$, for a polynomial of degree 4 with parameter $a = 1.100389 + 0.605217i$.

Figure 2. Filled Julia set and intervals of angles of $P_a$.

The set $H \cap \overline{B}_a$ has two points, which are in an orbit of period two with angles $\{\frac{3}{15}, \frac{1}{15}\}$ and $\{\frac{12}{15}, \frac{15}{15}\}$. And $\nu(a) = (\frac{4}{15}, \frac{12}{15}) \cup (\frac{1}{15}, \frac{3}{15})$. Thus, $H_1 \setminus H$ consists of a unique regulated arc which is disjoint of $\overline{B}_a$ and its external arguments are contained in the arc $(\frac{49}{60}, \frac{3}{60})$. Therefore

$$U(a) = \nu(a) \cup U_1 = (\frac{4}{15}, \frac{12}{15}) \cup (\frac{1}{15}, \frac{3}{15}) \cup (\frac{49}{60}, \frac{3}{60}).$$

Lemma 5.9. Given a polynomial $P_a$ with $a \in \mathbb{C}$, we have that an external ray $R_a(\theta)$ lands on $H$ if and only if the orbit of $\theta$ under $\rho(\theta) = d\theta$ does not get inside $U$.

Proof. First we are going to prove the necessary condition. Since $H$ is invariant, $P_a(H) = H$. If $R_a(\theta)$ lands on $H$ then $R_a(\rho(\theta))$ also lands on $H$, this ensures that the orbit of $\theta$ never intersects $U(a)$.

For the sufficient condition, let us assume that $R(\theta)$ lands in $x \in J_a \setminus H$. We define

$$\alpha^+ = \inf\{\alpha \geq \theta : R_a(\alpha) \text{ lands on } H\}$$

and

$$\alpha^- = \sup\{\alpha \leq \theta : R_a(\alpha) \text{ lands on } H\}.$$ 

Since $H$ is closed and $J_a$ is locally connected, the rays with angles $\alpha^+, \alpha^-$ must land on $H$ and $\theta \in (\alpha^+, \alpha^-)$. We are going to prove that there exists an integer $k \geq 0$ such that $\rho^k(\alpha^+, \alpha^-) \subset U(a)$. Since that $H \subset H_1 \subset H_2 \subset \ldots$ is dense on $J_a$, then the image of $(\alpha^+, \alpha^-)$ under iteration by $\rho$ must get inside $\gamma_a^{-1}(H_1)$. It means, there exists $k \in \mathbb{N}$ such that $\rho^k(\alpha^+, \alpha^-) \cap \gamma_a^{-1}(H_1) \neq \emptyset$, where $\gamma_a : \mathbb{T} \to J_a$ is the
Carathéodory map. Since $H_1 = P^{-1}_a(H) = H \cup (H_1 \setminus H)$, we have that $P^k(\alpha^+, \alpha^-)$ intersects to $\gamma^{-1}_a(H_1 \setminus H)$ before getting inside $\gamma^{-1}_a(H)$. By the construction of $U(a)$, $P^k(\alpha^+, \alpha^-) \subset U(a)$. Since $\theta \in (\alpha^+, \alpha^-)$, we conclude that the orbit of $\theta$ get inside in $U(a)$.

6. Monotonicity of the entropy. In this section we are going to analyze and compare the entropy of polynomials restricted to Hubbard trees (core entropy) for polynomials with periodic critical point $-a$. To obtain this, we propose a way to compare Hubbard trees following the ideas in [7].

We start introducing an equivalence relation on Hubbard trees, such that any equivalence class is characterized by the dynamics in the vertex set.

**Definition 6.1.** Let $H$ be the Hubbard tree associated to the polynomial $P_a$, let $\mathcal{P}$ be the partition of $H$ that has five elements $\{c_1\}, \{c_2\}, T_0, T_1, T_2$ (some of them might be empty), such that $P_a|_{T_i}$ is a homeomorphism for all $i$. Thus $c_1, c_2 \in T_0$ and for $i = 1, 2$, $c_i \in T_i$ unless $T_i = \emptyset$. $\mathcal{P}$ is called the partition associated to $H$, where $c_1 = 0$ and $c_2 = -a$.

With this partition we can define the itinerary for each point of $H$, as follows:

**Definition 6.2.** The itinerary of a point $z \in H$ is the infinite sequence $\tau(z) = (\tau_n(z))_{n=1}^{\infty} \in \{0, 1, 2, C_1, C_2\}$ given by

$$
\tau_n(z) = \begin{cases} j, & \text{if } P_a^{n-1}(z) \in T_j, \\ c_i, & \text{if } P_a^{n-1}(z) = c_i. \end{cases}
$$

Using this definition we will give a characterization of the Hubbard trees, which allows us to define an equivalence relation.

Let $\tau(z)$ be the itinerary of a point $x$ of a Hubbard tree. We denote by $\tau^?(z)$ the itinerary that is obtained when we interchange in the sequence the symbols 1 and 2.

**Definition 6.3.** Two Hubbard trees $H(a_1), H(a_2)$, are equivalent if there exists a bijection $\xi : V(a_1) \to V(a_2)$ between the vertex set of $H(a_1)$ and $H(a_2)$, such that $\xi$ conjugates $P_{a_1}|_{V(a_1)}$ with $P_{a_2}|_{V(a_2)}$. Thus, $v, \tilde{v} \in V(a_1)$ are adjacent vertex if and only if $\xi(v), \xi(\tilde{v})$ are also adjacent vertices. Similarly, $\tau(v) = \tau(\xi(v))$ for all $v \in V(a_1)$ or $\tau(v) = \tau^?(\xi(v))$.

**Definition 6.4.** Let $x$ be a periodic point. If there exists an $i \in \{1, 2\}$ and $\tilde{x} \in orb(x) \cap (c_i, f(c_i) \setminus [c_1, c_2])$ such that $\text{orb}(x) \subset G_{\tilde{x}}(c_i)$, then the point $x$ is called the $v_i$-characteristic point of $\text{orb}(x)$. Where $G_{\tilde{x}}(c_i)$ is the connected component of $H \setminus \{x\}$ that contains $c_i$. A point $z$ is characteristic if it is periodic and is a $v_i$-characteristic point, for $i = 1$ or $i = 2$.

**Definition 6.5.** Let $(H, P_a, \mathcal{P}), (\tilde{H}, P_{\tilde{a}}, \tilde{\mathcal{P}})$ be two Hubbard trees with free periodic critical point, let $n$ be the period of $-\tilde{a}$. We will say that $(H, P_a, \mathcal{P})$ contains dynamically the Hubbard tree $(\tilde{H}, P_{\tilde{a}}, \tilde{\mathcal{P}})$ if there exists an embedding $\iota : \tilde{H} \to H$, a characteristic point $z \in H$ and a bijection $\xi : \tilde{V} \to \text{orb}(z) \cup \iota(\tilde{V} \setminus \text{orb}(\tilde{a}))$ such that:

i) $\xi(P_{\tilde{a}}(-\tilde{a})) = z$;

ii) $\xi \circ P_{\tilde{a}}|_{\text{orb}(\tilde{a})} = P_a \circ \xi|_{\text{orb}(\tilde{a})}$;

iii) $\iota(\tilde{H}) = [\text{orb}(z)] \cup \{0\}$, where $[\text{orb}(z)]$ is the tree generated by the orbit of $z$ on $H$. 
iv) For any endpoint $P_a^i(z) \in \mathcal{H}$, $P_a^i(z) \in T_j$ if and only if $P_a^i(-\bar{a}) \in \text{cl}(\tilde{T}_j)$.

v) If $\tilde{v} \neq \tilde{w} \in \tilde{V}$ are adjacent in $\tilde{H}$, then $\tilde{\xi}(\tilde{v})$ and $\tilde{\xi}(\tilde{w})$ are adjacent as vertices of $\{\text{orb}(z)\} \subset H$.

On the other hand, if $\tilde{\xi}(\tilde{v})$, $\tilde{\xi}(\tilde{w})$ are adjacent as vertex of $\{\text{orb}(z)\}$, then $\tilde{v}$, $\tilde{w}$ are adjacent or there exists an unique branching point $\tilde{c}_i \in \text{orb}(-\bar{a}) \cap \tilde{H}$, such that $(\tilde{v}, \tilde{w}) \cap \tilde{V} = \tilde{c}_i$.

Using the previous definition we define a partial order in $\mathbb{C}$ as follows: Let $a_1, a_2 \in \mathbb{C}$, we will say that $a_2 \prec a_1$ if the Hubbard tree $H(a_2)$ is inside dynamically in the Hubbard tree $H(a_1)$, and the Hubbard trees $H(a_1)$ and $H(a_2)$ are not in the same equivalence class.

**Remark 6.6.** If $a_2 \prec a_1$ then $N(a_2) \leq N(a_1)$.

Since the forcing orbit lemma is also valid in a copy of the Mandelbrot set (see [11]), we have the following:

**Lemma 6.7.** Let $M_0 \subset \mathcal{C}$ be a small Mandelbrot set copy intersecting $W_0$, $a_1, a_2 \in M_0$ and $P_{a_1}, P_{a_2}$ polynomials with periodic critical points. If the hyperbolic component containing $a_2$ split $a_1$ of $W_0$ then $a_2 \prec a_1$.

**Lemma 6.8.** Let $a_1, a_2 \in \mathbb{C}$. If $a_2 \prec a_1$, $N(a_1) = N(a_2)$ and $z \in H(a_1)$ is the characteristic point given by Definition 6.5, then

$$H(a_1) \cong H(a_2) \cong [P_{a_1}(z), P_{a_1}^2(z), \ldots, P_{a_1}^N(z)],$$

where $\cong$ means they are topologically homeomorphic.

**Proof.** We denote the number of endpoints by $N(a_1) = N(a_2) = N$ and $P_{a_i}$ by $P$. By the property iv) from Definition 6.5, the external rays that land on $z$ in the tree $H(a_1)$ are the same external rays that land on $P_{a_2}(-a_2)$ in the tree $H_{a_2}$. Thus $z, P(z), P^2(z), \ldots, P^{N-1}(z)$ are exactly the endpoints of $[z, P(z), P^2(z), \ldots, P^{N-1}(z)]$

and therefore

$$H(a_2) \cong [P(z), P(z), \ldots, P^N(z)].$$

Now let us prove by induction that there are not branching points and critical points in the intervals $(P^k(z), P^k(P(-a)))$ for $k = 0, 1, \ldots, N - 1$. Note that, $0, -a \notin (z, P(-a))$ and there are not branching points in $(x,0)$, otherwise $N(a_1) > N(a_2)$. Now, assume that there are not branching points or critical points in $(P^k(z), P^k(P(-a)))$ for all $k \leq k_0 \leq N - 1$. Since $P$ is injective in each regulated arc that does not contain a critical point, it follows that $(P^{k_0+1}(z), P^{k_0+1}(P(-a)))$ does not contain any branching point.

On the other hand, if $c_i \in (P^{k_0+1}(z), P^{k_0+1}(P(-a)))$ given that $N(a_1) = N(a_2)$, there is $k_0 + 1 < l < N$ such that $P^l(z) \in (c_i, P^{k_0+1}(P(-a)))$, this means that $P^{k_0+1}(z)$ is not an endpoint of $[P(z), P^2(z), \ldots, P^N(z)]$ and this is a contradiction.

Since, there are not branching points in $(P^k(z), P^k(P(-a)))$ for $k = 1, 2, \ldots, N$ we have that $H(a_1) \cong [P(z), P^2(z), \ldots, P^N(z)]$.

**Lemma 6.9.** Let $P = P_a$ be as in the previous Lemma, let $x, y \in H \cap J_a$. Assume that $P^i(x) \prec P^i(y)$ for $0 \leq i \leq k$ then the following affirmations are valid:
1) If $P^k(y) \prec y'$ in $H_1$, then $P^k(x) \prec x'$ in $H_1$;
2) If $y' \prec P^k(y)$ in $H_1$, then $x' \prec P^k(x)$ in $H_1$,
where $y'$ is any preimage of $P(y)$ different from $y$ and $x'$ is the preimage of $P(x)$ such that $[x', y']$ does not contain any critical points.
Let us prove the first affirmation by contradiction. Assume that \( P^k(y) \prec y' \) and \( x' \prec P^k(x) \) in \( H_1(a) \), then we have
\[
x' \prec P^k(x) \leq P^k(y) \prec y',
\]
from this, it follows that \( P^k(x), P^k(y) \subset [x, y'] \), then \( P^k \) restricted to \( [x', y'] \) is a homeomorphism onto itself, which is a contradiction with the expansion of \( P|_{J_a} \) since \( x', y' \in J_a \).

The proof of 2) is obtained by the same argument.

**Theorem 6.10.** Let \( a_1, a_2 \in \mathbb{C}^0 \). If \( a_2 \prec a_1 \) and \( N(a_1) = N(a_2) \), then \( U(a_1) \subset U(a_2) \).

**Proof.** Note that \( \nu(a_1) = \nu(a_2) \). Furthermore, \( U(a_1) \) and \( U(a_2) \) are disjoint unions of intervals of angles. Each one of these intervals is associated with a regulated arc of the trees \( H_1(a_1), H_1(a_2) \), respectively. We will prove that for each interval of angles \( U \subset U(a_1) \) limited by the external rays not landing on \( B_a \), there exists an interval of angles \( U' \subset U(a_2) \), such that \( U \subset U' \).

We will denote the number of endpoints as \( N(a_1) = N(a_2) = N \) and \( P_{a_1} \) by \( P \).

Let \( H_1(a_1) \setminus H(a_1) = \bigcup_{k=1}^{N} I_k \) be a finite union of regulated arcs. For any regulated arc \( I_k \) such that \( I_k \cap \partial B_a = \emptyset \), we have two cases:

1) If \( I_k \) has the form \( (P^{k_1}(-a), -P^{k_2}(-a)) \), for \( 1 \leq k_1, k_2 \leq N \), then \( P^{k_1}(-a) \prec -P^{k_2}(-a) \), \( (k_1 \neq k_2) \).

If
\[
R_a(\theta(P^{k_1})), R_a(\theta(-P^{k_2})), R_a(\theta(-P^{k_2})^{+}), R_a(\theta(P^{k_1})^{+})
\]
are the external rays associated to the regulated arc \( (P^{k_1}(-a), -P^{k_2}(-a)) \), then
\[
(\theta(P^{k_1}(-a))^{−}, \theta(P^{k_1}(-a))^{+}) \subset U(a_1)
\]
is an interval of angles of \( U(a_1) \). By Lemma 6.9, \( P^{k_1}(z) \not\prec -P^{k_2}(z) \), to see this, if \( k_1 > k_2 \) we apply the first case and if \( k_1 < k_2 \) we apply the second case. Since the rays landing on \( P^{k_1}(z) \) in the tree \( H_1(a_1) \) are exactly the same ones than the rays landing on \( P_{a_2}^{k_1}(z) \) in the tree \( H_1(a_2) \), we have as before,
\[
(\theta(P^{k_1}(z))^{−}, \theta(P^{k_1}(z))^{+}) \subset U(a_2),
\]
which is an interval of angles of \( U(a_2) \).

By Lemma 6.8, \( H(a_1) \cong [P_{a_1}(z), P_{a_1}^2(z), \ldots, P_{a_1}^N(z)] \) and \( P^{k_1}(z) \prec P^{k_1}(-a) \). Then \( P^{k_1}(-a), -P^{k_2}(z) \in [P_{k_1}(z), -P^{k_1}(-a)] \). Hence,
\[
(\theta(P^{k_1}(z))^{−}, \theta(P^{k_1}(z))^{+}) \subset (\theta(P^{k_1}(-a))^{−}, \theta(P^{k_1}(-a))^{+}).
\]

2) In any other case, \( I_k \) must have the form \( (w, -P^{k_1}(-a)) \) for some \( 0 \leq k_1 \leq N \), where \( w \notin \text{Orb}(-a) \). Since \( H(a_1) \cong [P_{a_1}(z), P_{a_1}^2(z), \ldots, P_{a_1}^N(z)] \), then
\[
(w, -P^{k_1}(-a)) \cap [P_{a_1}(z), P_{a_1}^2(z), \ldots, P_{a_1}^N(z)] = \{w\}.
\]
From this result, we have that the interval of angles \( U_k \) corresponding to \( I_k \) is also an interval of angles of \( U(a_2) \).

From 1) and 2) we conclude the proof. □
Next we show the result that compare the topological entropy in the Hubbard tree, of two comparable polynomials in $C^0$.

**Theorem 6.11.** Let $a_2$ and $a_1$ be two parameters that represents polynomials of degree $d$ in $C^0$ each with periodic critical points, if $a_2 \prec a_1$ and $N(a_2) = N(a_1)$ then $h(H(a_2), P_{a_2}) \leq h(H(a_1), P_{a_1})$.

**Proof.** By Theorem 6.10 we know that $\mathcal{U}(a_1) \subset \mathcal{U}(a_i)$ and then

$$
\mathcal{T} \setminus \bigcup_{k=1}^{\infty} \rho^{-k}(\mathcal{U}(a_2)) \subset \mathcal{T} \setminus \bigcup_{k=1}^{\infty} \rho^{-k}(\mathcal{U}(a_1)).
$$

Therefore, we have the following inequality of the entropy,

$$
h(\mathcal{T} \setminus \bigcup_{k=1}^{\infty} \rho^{-k}(\mathcal{U}(a_2)), \rho) \leq h(\mathcal{T} \setminus \bigcup_{k=1}^{\infty} \rho^{-k}(\mathcal{U}(a_1)), \rho).
$$

Since we have the following conjugation between $P_{a_i}$ and the function $\rho$

$$
\begin{array}{ccc}
\mathcal{T} \setminus \bigcup_{k=1}^{\infty} \rho^{-k}(\mathcal{U}(a_i)) & \xrightarrow{\rho} & \mathcal{T} \setminus \bigcup_{k=1}^{\infty} \rho^{-k}(\mathcal{U}(a_i)) \\
\gamma_{a_i} & H(a_i) & P_{a_i} \xrightarrow{\gamma_{a_i}} H(a_i),
\end{array}
$$

we have that

$$
h(H(a_i), P_{a_i}) = h(\mathcal{T} \setminus \bigcup_{k=1}^{\infty} \rho^{-k}(\mathcal{U}(a_i)), \rho).
$$

Therefore

$$
h(H(a_2), P_{a_2}) \leq h(H(a_1), P_{a_1}).$$

\[\square\]

In the case of the polynomials with parameters $a_2 \prec a_1$ whose Hubbard trees have different number of endpoints, we will calculate the core entropy using the Markov matrix associated to the Hubbard tree.

If $P_a$ is a hyperbolic polynomial, the restriction of $P_a$ to any edge $e_j$ of $H(a)$ is injective. In addition $P_a(e_j)$ is the union of edges $e_1, \ldots, e_l$ of $H(a)$. If $H(a)$ has edges $e_1, \ldots, e_n$, we define the Markov matrix $M(a)$ associated to the $H(a)$ as $M_i = 1$ if $e_j \subset P_a(e_i)$ and 0 otherwise. By definition, the matrix $M$ has dimension $n \times n$ and has entries 0 and 1. By Perron-Frobenius’ Theorem, the spectral radius $\lambda$ of $M$ is greater or equal than 1, [1].

**Proposition 6.12.** Let $M_0 \subset \mathcal{C}$ be a Mandelbrot copy and let $a \in M_0$ be a parameter such that $P_a$ has periodic critical points. If $H(a)$ is its Hubbard tree, then $h(H(a), P_a) = \log \lambda$.

**Proof.** Let $\mathcal{P}$ be the partition of $H(a)$ whose elements are the $l$ edges $e_i$ of $H(a)$. $\mathcal{P}$ satisfies that $e_i \cap e_j$ is either an empty set or has only one point, for all $1 \leq i < j \leq k$. Moreover, $\partial \mathcal{P}$ is the set of all endpoints of $e_i$, these are the vertices of $H(a)$. For any $k \geq 1$ we define the partition $P_a^k \ast S$ of $H(a)$ given by the set $\partial \mathcal{P}^k \ast S = P_a^{-k}(\partial S)$.

If $S$ and $S'$ are two partitions of $H(a)$, we define a new partition $S \lor S'$ of $H(a)$ given by $\partial S \lor \partial S'$. 

Consider the partition
\[ \bigvee^n S = S \lor P_a \lor SV_1 \lor \cdots \lor P_a^{n-1} \lor S. \]

By Lemma 4.4, the free critical point \(-a\) is not a branch point. Moreover, \(-a\) is not a precritical point. Thus, the number of branch points on \(H(a)\) is invariant under \(P_a^{n-1}\). Hence \(\partial \bigvee S\) is given by the union of all \(n\) preimages of points in the orbit of \(-a\) in \(H(a)\), zero and the branch points of \(H(a)\). From the entropy definition, we have
\[ h(H(a), P_a, S) = \lim_{n \to \infty} \frac{1}{n} \log \text{card} \bigvee^n S. \]

**Claim 6.13.** \( h(H(a), P_a, S) = h(H(a), P_a). \)

The proof of this affirmation is obtained using the same arguments as in Lemma 1 in [4].

By the construction of \(P_a^k \lor S\), we have that
\[ \partial S \subset \partial(P_a \lor S) \subset \cdots \subset \partial(P_a^{n-1} \lor S) \text{ and } \bigvee^n S = (P_a^{n-1}) \lor S. \]

If \(S'\) is a finer partition than \(S\), we define the Markov vector of \(v_{S'} = (v_1, \ldots, v_k)\) associated to \(S'\), taking \(v_i\) as the number of elements in \(S'\) that belong to edge \(e_i\) of \(S\). Observe that \(v_2 = (1, 1, \ldots, 1)\) and \(v_{P_a \lor S} = M^T \cdot v_S\) where \(M^T\) is the transpose matrix of the Markov matrix \(M\) of \(H(a)\). Thus, \(v_{\bigvee^n S} = (M^T)^{n-1} c \cdot v_S\). Using the norm \(\|v\| = \sum v_i\), for any vector \(v = (v_1, \ldots, v_n)\), such that \(v_2 > 0\), it is known that there exist constants \(0 < c_1 < c_2\), such that \(c_1 \cdot \lambda^n \leq \|(M^T)^n \cdot v\| \leq c_2 \cdot \lambda^n\).

Since \(\|v_S\| = \text{card} S\), we have \(c_1 \cdot \lambda^n \leq \text{card} \bigvee^n S \leq c_2 \cdot \lambda^n\).

Taking logarithm and dividing by \(n\) we have
\[ \frac{\log c_1 \cdot \lambda^n}{n} \leq \frac{\log \text{card} \bigvee^n S}{n} \leq \frac{\log c_2 \cdot \lambda^n}{n}. \]

If we let \(n\) tends to \(\infty\) then we obtain that \(h(H(a), P_a, S) = \log \lambda\). \(\square\)

In a general setting, let \(M\) be a \(n \times n\) matrix with entries 0 and 1, and spectral radius \(\lambda \geq 1\). Let \(X\) be a compact connected space and \(f : X \to X\) be a continuous map. The set \(A = \{X_1, X_2, \ldots, X_n\}\) of compact subsets of \(X\), such that int \((X_i)\) is not empty and the \(X_i\)'s mutually disjoint, it is called an over-Markov packing with matrix \(M\), if \(X_j \subset f(X_i)\) when \(M_{ij} = 1\).

**Proposition 6.14.** If \(A = \{X_1, X_2, \ldots, X_n\}\) is an over-Markov packing with matrix \(M\), then \(h(X, f) \geq \log \lambda\), where \(\lambda\) is the spectral radius of \(M\).

**Main Theorem 2.** Let \(a_2 < a_1\) be parameters in a Mandelbrot copy \(M_0\) intersecting \(W_0\), such that \(P_{a_1}\) and \(P_{a_2}\) are hyperbolic polynomials and postcritically finite, then \(h(H(a_2), P_{a_2}) \leq h(H(a_1), P_{a_1})\).

**Proof.** If \(N(a_1) = N(a_2)\) then we have the result by Theorem 6.11.

Hence we can assume that \(N(a_2) < N(a_1)\). By Remark 6.6 the number of endpoints is not decreasing. Moreover, without loss of generality, we can assume that does not exist a hyperbolic parameter \(c \in C^0\), such that \(c_2 \prec c \prec c_1\) and \(N(c_2) < N(c) < N(c_1)\). Let \(H(a_1), H(a_2)\) be the Hubbard trees and let \(z\) be the characteristic point given by Definition 6.5.
Let \( H(z) \) be the regulated tree generated by the orbit of \( z \). By definition \( H(z) \) is a subtree of \( H(a_1) \) and is not invariant under \( P_{a_1} \). We take as vertices of \( H(z) \), the union of \( \text{orb}(z) \) and the branch points of \( H(z) \). There is a bijection between the vertices of \( H(z) \) and the vertices of \( H(a_2) \). Moreover, by Definition 6.5, the tree \( H(z) \) is topologically homeomorphic to the Hubbard tree \( H(a_2) \).

Let \( e_1, e_2, \ldots, e_n \) be edges of \( H(a_2) \) and \( M(a_2) \) the Markov matrix associated to \( H(a_2) \). Then \( h(H(a_2), P_{a_2}) = \log \lambda \), where \( \lambda \) is the spectral radius of \( M \). Since \( H(z) \) is homeomorphic to \( H(a_2) \), we can label the edges of \( H(z) \) by \( e_1^z, e_2^z, \ldots, e_n^z \). These edges satisfy the condition that \( P_{a_1}(e_i^z) \supset e_j^z \), whenever \( M_{ij} = 1 \). Hence the set \( E_z = \{ e_1^z, e_2^z, \ldots, e_n^z \} \) is an over-Markov packing with matrix \( M(a_2) \). Then \( h(H(a_1), P_{a_1}) \geq \log \lambda \). Therefore \( h(H(a_1), P_{a_1}) \geq h(H(a_2), P_{a_2}) \).

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*E-mail address:* domingo.gonzalez@ujat.mx

*E-mail address:* gble@ujat.mx