LARGE DEVIATIONS FOR BROWNIAN MOTION IN A RANDOM SCENERY.

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Abstract. We prove large deviations principles in large time, for the Brownian occupation time in random scenery \( \int_0^t \xi(B_s) \, ds \). The random field is constant on the elements of a partition of \( \mathbb{R}^d \) into unit cubes. These random constants, say \( \{\xi(j), j \in \mathbb{Z}^d\} \), consist of i.i.d. bounded variables, independent of the Brownian motion \( \{B_s, s \geq 0\} \). This model is a time-continuous version of Kesten and Spitzer’s random walk in random scenery. We prove large deviations principles in “quenched” and “annealed” settings.

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1 Introduction.

We study the large time asymptotics of random additive functionals of Brownian motion \( \int_0^t \xi(B_s) \, ds \), where the random field \( \{\xi(x), x \in \mathbb{R}^d\} \) is independent of the Brownian motion \( \{B_s, s \geq 0\} \). We consider the case where \( \xi \) is a random constant, say \( \xi(i) \), on the \( i \)th cube of a partition of \( \mathbb{R}^d \) into unit cubes. The sequence \( \{\xi(i), i \in \mathbb{Z}^d\} \) consists of i.i.d. bounded random variables with common law \( \nu_i = \nu \), and we assume for convenience that \( |\xi(0)| \leq 1 \) and \( E_{\nu}[\xi(0)] = 0 \).

This is related to one of Kesten-Spitzer’s models of random walk in random scenery: let \( \{X_i, i \in \mathbb{N}\} \) be a sequence of \( \mathbb{Z}^d \)-valued i.i.d. random vectors with mean 0 and finite non-singular covariance matrix \( \Sigma \), and define \( S_n = X_1 + \cdots + X_n \). Let \( \{\xi(i), i \in \mathbb{Z}^d\} \) be i.i.d. random variables independent of the \( \{X_i, i \in \mathbb{N}\} \), with mean 0 and finite variance \( \sigma^2 \). Kesten and Spitzer showed in [13] that in dimension

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1, the following weak convergence in law (over both randomness) holds

\[ \frac{1}{n^{3/4}} \sum_{k=1}^{[nt]} \xi(S_k) \xrightarrow{\text{law}} \Delta_t, \]

where \( \Delta_t \) is a non-Gaussian, self-similar process of order \( 3/4 \) with stationary increments. When \( d = 2 \), Bolthausen [4] established that

\[ \frac{1}{\sqrt{n \log(n)}} \sum_{k=1}^{[nt]} \xi(S_k) \xrightarrow{\text{law}} \frac{\sigma}{\sqrt{\pi \det(\Sigma)^{1/4}}} B_t, \]

where \( E[N(0)] \) is the expected number of visits to the origin of the (transient) random walk \( \{S_n, n \in \mathbb{N}\} \). Our interest was to understand how these super-diffusive scaling would reflect in the large deviation speed rates. The use of Brownian motion, rather than random walks, has technical advantages: on one hand, we have the spectral analysis and the classical estimates for Schrödinger semi-groups at our disposal, and on the other hand, we have a clean scaling property.

Some Large Deviations estimates for \( \frac{1}{t} \int_0^t \xi(B_s)ds \) were obtained in [20] for the annealed case. In particular the speed rate was obtained in dimension 1, but not the rate functional. Besides, in \( d > 1 \), not even the correct speed was discovered.

We now give some heuristics to explain the correct speed rates in estimating, for any real \( y \), the probability of the event \( A \triangleq \{(\xi, B) : \langle L_{t}, \xi \rangle \approx y \} \) where \( L_t = \frac{1}{t} \int_0^t \delta_{B_s} ds \) is the occupation measure of Brownian motion, and \( \langle \cdot, \cdot \rangle \) is the duality bracket between measures and functions. By scale invariance, we have for any \( r > 0 \), \( B_{r^2 s} \xrightarrow{\text{law}} rB_s \). Thus, we have to find the probability of the event

\[ A = \left\{ (\xi, B) : \left\langle L_{\frac{r^2}{t}}, \xi_r \right\rangle \approx y \right\}, \quad (1) \]

where

\[ \xi_r(x) \triangleq \xi(rx) = \sum_{i \in \mathbb{Z}^d} \xi(i) \mathbb{1}_{Q_i(\mathbb{R})}(x), \quad \text{and} \quad Q_i(\mathbb{R}) \triangleq \frac{i}{r} + \left[-\frac{1}{2r} : \frac{1}{2r}\right]^d. \quad (2) \]

Thus, the Brownian scale invariance has allowed us to “coarse-grain” the field. Indeed, we think of \( \{\xi(i), i \in \mathbb{Z}^d\} \) as our microscopic description and introduce the empirical density \( \xi_r \), which represents coarse graining over about \( r^d \) sites.

Now, a Large Deviations Principle (LDP) holds for the field \( \xi_r \) integrated against continuous functions with compact support (see e.g. [4]). In other words, for any \( y > 0 \), \( \varphi \in \mathcal{C}_c(\mathbb{R}^d) \), and \( r \) large

\[ \frac{1}{r^d} \log \otimes_{i \in \mathbb{Z}^d} \nu_i \left[ \left\langle \xi_r, \varphi \right\rangle \approx y \right] \approx -\inf_{w: |w(x)| \leq 1} \left\{ I(u) : \int_{\mathbb{R}^d} u(x)\varphi(x) dx \approx y \right\}. \quad (3) \]
with
\[ I(u) \triangleq \int_{\mathbb{R}^d} H(u(x))dx, \quad \text{and} \quad H(x) \triangleq \sup_y \{ xy - \log E_\nu[\exp(y\xi(0))] \}. \] (4)

On the other hand, the Donsker-Varadhan theory provides a LDP for the occupation measure $L_{t/r^2}$ in the weak topology.

Thus, when we average with respect to both randomness, i.e. in the annealed case, it is natural to look for a LDP by a contraction principle (cf. [6]). Assume for a moment that we are entitled to do so. Then, the correct speed appears as one equals $t/r^2$ with $r^d$, i.e. as one equals the speed rates for each marginal LDP. Thus, this yields the correct speed $t^{d/(d+2)}$. Moreover, the rate function is
\[ J(y) = \inf \{ I(u) + \mathcal{L}(\mu) : \langle \mu, u \rangle = y \}, \]
where $\mathcal{L}$ is the rate function for the Brownian occupation measure. However, when using a contraction principle, we face two problems. First, the map $(u, \mu) \mapsto \int ud\mu$ is not continuous in the product of the weak topologies. The remedy is to regularize the field: if $\{\psi_\delta\}$ is an approximate identity, one first has to replace $\xi_r$ by $\psi_\delta * \xi_r$. Second, the LDP for the Brownian occupation measure is a weak one, i.e. the upper bound is only valid for compact sets. The standard trick, which has first been used by Donsker and Varadhan [8], is to replace the Brownian motion by a process for which we have a “full” LDP, for instance the Brownian motion on the torus $T(A)$ of side $A$. This compactification is possible in our situation, since we show that if we integrate first with respect to the law of $\xi_r$,
\[ \mathbb{P} \left[ \langle L^{A}_{t/r^2}, \psi_\delta \ast \xi_r \rangle \geq y \right] \leq e^{-r^d F_A(L^{A}_{t/r^2})}, \text{ with } F_A(\mu) = \inf \{ I_A(u) : \langle \mu, \psi_\delta \ast u \rangle \geq y \}, \]
where $L^A$ is the occupation measure for the Brownian on $T(A)$, and $I_A$ has the same expression as $I$ in (4) with $T(A)$ instead of $\mathbb{R}^d$. The upper bound follows then from Varadhan’s integral lemma and coincides with the lower bound.

Another standard way to obtain a LDP is to use Gärtner-Ellis method, i.e. to look for the asymptotics of the log-Laplace transform
\[ \frac{t}{r^2} \log \tilde{E}_0 \left[ \exp \left( \frac{\alpha}{r^2} \int_0^t \xi(B_s) \, ds \right) \right] = \frac{t}{r^2} \log \tilde{E}_0 \left[ \exp \left( \alpha \int_0^{t/r^2} \xi_r(B_s) \, ds \right) \right], \]
where $\tilde{E}_0$ denotes the annealed law. By Feynman-Kac formula, this behavior is related to the (annealed) behavior of the principal eigenvalue of the random operator $-\frac{1}{2} \triangle - \alpha \xi_r$. Similar quantities have been thoroughly studied both in the annealed and in the quenched setting, for different kinds of potential $\xi$ and different scaling $r$: for instance Sznitman [21], Merkl & Wüthrich [17] [19] [18] for the case of a Poissonian potential; Gärtner & Molchanov [11] [12], Biskup & König [3] for the i.i.d case; Gärtner & König [4], Gärtner, König & Molchanov [10] for more general potentials. This method leads to a LD upper bound which is necessarily convex,
being defined as a Legendre transform. However, the functional $J$ is not convex in general, and this method is doomed to fail.

What about the case with a fixed field $\xi$, i.e. the quenched case? First, note that if $\sigma(R)$ denotes the Brownian exit time from a cube of radius $R$, then by classical results, there is a constant $C$ such that

$$P_0 \left[ \sigma \left( \frac{R t}{r^2} \right) \leq \frac{t}{r^2} \right] \leq C \exp \left( -\frac{R^2 t}{2 r^2} \right).$$

Hence, we can restrict everything to a box $Q$ of size $Rt/r^2$. Now, to establish estimates holding $\xi$-almost surely, a pattern of the scaled field $\bar{\xi}_r$ (on a macroscopic domain) should persist as we take $t$ to infinity. By a Borel-Cantelli argument, this happens as soon as $r^d = \log(t/r^2)$. Indeed, the cost for $\psi_\delta * \bar{\xi}_r$ to look like a definite profile $u$ on a unit cube, is of order $\exp(-r^d I(u))$. Since the smoothed empirical density $\psi_\delta * \bar{\xi}_r$ is almost independent on the different cubes of a partition of $Q(Rt/r^2)$ into unit cubes, the probability that in one of the element of the partition, $\psi_\delta * \bar{\xi}_r$ is close to $u$ is of order

$$1 - \left( 1 - \exp \left( -r^d I(u) \right) \right)^{(r^2 t)^d}.$$  

This is almost 1, if $r^d = \log(t/r^2)$, whose root we call $r_t$, and $I(u) < d$. Now, forcing the Brownian motion to stay in a unit cube during a time $t/r_t^2$ costs of the order of $\exp(-c t/r_t^2)$. Thus, we have the heuristic speed $t/r_t^2$ with $r_t^d \triangleq \log(t/r_t^2)$. Following this strategy and optimizing over all admissible profiles $u$, we obtain

$$P_0 \left[ \langle L_{t/r_t^2}; \bar{\xi}_r \rangle \approx y \right] \geq \exp \left( -\frac{\bar{\xi}_r(t)}{r_t} \right),$$

where

$$\bar{\xi}_r(t) = \inf_{u,\mu} \{ L(\mu) : \langle \mu, u \rangle = y, I(u) < d \}.$$

The quenched large deviations upper bound is obtained using Gärtner-Ellis method. As already mentioned, we are led to study the almost sure behavior of the principal eigenvalue of the random Schrödinger operator $-\frac{1}{2} \Delta - \alpha \bar{\xi}_r$ with boundary Dirichlet conditions on $Q(Rt/r_t^2)$. In the case of a Poissonian potential, this study has been carried out by Merkl & Wüthrich [19]. We rely here on a localization lemma borrowed from Gärtner and König [9], which has also been crucial in the papers [3], [1], [19]. According to this lemma, the principal eigenvalue is close to the minimum of the principal eigenvalues of the same operators over boxes of fixed size forming a partition of $Q(Rt/r_t^2)$. This leads to a quenched upper bound with a rate functional $J$ which is convex.

The problem is now to identify $\mathcal{J}_1$ with $J$. It is easy to check that $J$ is the greatest convex minorant of $\mathcal{J}_1$. However, the convexity of $\mathcal{J}_1$ could not be established. Hence, we use an approach developed in [1], and we convexify through a sequence of scenarios: the $n$-th one corresponds to partitioning $[0,T]$ into $n$ time intervals, in each of which the Brownian motion goes fast to a region where the field $\psi_\delta * \bar{\xi}_r$ has a fixed deterministic profile, and stays there during this time interval. To each scenario corresponds a lower bound of the type

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{r_t^2}{t} \log P_0 \left[ \frac{r_t^2}{t} \int_0^{t/r_t^2} \psi_\delta * \bar{\xi}_r(B_s) ds \approx y \right] \geq -\mathcal{J}_n(y).$$
The family of functions $J_n$ is decreasing, and satisfies for any $y_1, y_2$ and $\lambda \in ]0, 1[$

$$
\lambda J_n(y_1) + (1 - \lambda) J_n(y_2) \geq J_{2n}(\lambda y_1 + (1 - \lambda) y_2).
$$

Thus, the limit $J(y) \triangleq \lim_{n \to \infty} J_n(y)$ is convex. This enables us to identify $J$ with the upper bound $J$.

Though we restrict ourselves to the i.i.d case, the crucial assumptions are that the rescaled field $\bar{\xi}_r$ is bounded and satisfies a LDP.

The paper is organized as follows. In section 2, we introduce the notations and state the main results. In section 3, we prove the LDP for the annealed case. In section 4, we establish the LDP for the quenched case. Section 5 gathers the proof of some technical lemmas.

2 Notations and results.

The random scenery. Let $\{\xi(j), j \in \mathbb{Z}^d\}$ be a family of i.i.d random variables with values in $\mathbb{R}$. We denote by $\mathbb{P} = \otimes_j \nu_j$ the law of the environment. Expectation with respect to $\mathbb{P}$ is denoted by $E$. We assume that

$$
\mathbb{P}\text{-a.s. } -1 \leq \xi(0) \leq 1 \quad E[\xi(0)] = 0, \quad \text{and } E[\xi(0)^2] \neq 0. \quad (7)
$$

We will denote $m \triangleq \text{essinf}(\xi(0))$, and $M \triangleq \text{esssup}(\xi(0))$.

Let $\Lambda$ be the log-Laplace transform of $\xi(0)$:

$$
\forall \alpha \in \mathbb{R}, \quad \Lambda(\alpha) \triangleq \log E[e^{\alpha \xi(0)}]. \quad (8)
$$

$\Lambda$ is convex, everywhere finite by (7). Moreover, since $E(\xi(0)) = 0$, $\Lambda(\alpha) \geq 0$, and $\Lambda(0) = 0$. Let $H$ be the Legendre transform of $\Lambda$:

$$
\forall y \in \mathbb{R}, \quad H(y) \triangleq \sup_{\alpha \in \mathbb{R}} (\alpha y - \Lambda(\alpha)). \quad (9)
$$

$H$ is convex, takes positive values, is increasing on $\mathbb{R}^+$, decreasing on $\mathbb{R}^-$. $H(0) = -\inf \Lambda = 0$. $H(y) = +\infty$ for $y \notin [m, M]$, and $H(y) < \infty$ for $y \in ]m, M[$.

If $Q(A) \triangleq [-\frac{A}{2}; +\frac{A}{2}]^d$, let $\mathcal{M}(Q(A))$ (resp. $\mathcal{M}_1(Q(A))$, $\mathcal{M}_1^0(Q(A))$) be the set of finite signed measures on $Q(A)$ (resp. the set of probability measures on $Q(A)$, the set of probability measures with compact support included in $Q(A)$), endowed with the topology of weak convergence (i.e the topology defined by duality against continuous and bounded test functions). For all $r > 0$, let $\bar{\xi}_r$ be the function defined by

$$
\bar{\xi}_r(x) \triangleq \xi([rx]) \quad \text{where } [rx] \text{ is the integer part of } x. \quad (10)
$$

$\{\bar{\xi}_r(x), x \in Q(A)\}$ are then random variables with values in

$$
\mathcal{B}_1(A) = \{u \in L_\infty(Q(A)), \|u\|_\infty \leq 1\}.
$$

$\mathcal{B}_1(A)$ will be viewed as the subspace of $\mathcal{M}(Q(A))$ of measures whose density with respect to Lebesgue measure belongs to $\mathcal{B}_1(A)$.

A key result is the following large deviations principle (see for instance [2]).
Lemma 2.1 For all $A > 0$, let us define the rate function $I_A$ on $\mathcal{B}_1(A)$ by

$$ I_A(u) = \int_{Q(A)} H(u(x)) \, dx $$

$I_A$ is convex, lower semi-continuous.

When $r \to \infty$, $\xi_r$ satisfies a LDP on $\mathcal{B}_1(A)$, with good rate function $I_A$ and speed $r^d$; i.e. for all measurable subset $F$ of $\mathcal{B}_1(A)$,

$$ -\inf_{u \in F} I_A(u) \leq \lim_{r \to \infty} \frac{1}{r^d} \log \mathbb{P}(\xi_r \in F) \leq \lim_{r \to \infty} \frac{1}{r^d} \log \mathbb{P}(\xi_r \in F) \leq -\inf_{u \in F} I_A(u). \quad (12) $$

For $A = \infty$, $I_A$ and $\mathcal{B}_1(A)$ will simply be denoted by $I$ and $\mathcal{B}_1$.

The Brownian motion in random scenery. Let $\{B_t, t \in \mathbb{R}^+\}$ be a $d$-dimensional Brownian motion, independent of the random field $\xi$. $E_x$ denotes expectation under the Wiener measure starting from $x$. For $t > 0$, let $L_t \triangleq \frac{1}{t} \int_0^t \delta_{B_s} \, ds$ be the Brownian occupation measure. From Donsker-Varadhan theory, $L_t$ satisfies a weak LDP in $\mathcal{M}_1(\mathbb{R}^d)$, with speed $t$, and rate function

$$ \mathcal{L}(\mu) = \begin{cases} \frac{1}{2} \int \left\| \nabla \left( \frac{\sqrt{d\mu}}{dx} \right) \right\|^2 \, dx, & \text{if } \mu \ll dx, \\ +\infty, & \text{otherwise}. \end{cases} $$

When $\mu$ is a measure, and $u$ is a function, $\langle \mu, u \rangle \triangleq \int u \, d\mu$. Our main interest in this paper is large deviations estimates for the random additive functional $\langle L_t, \xi \rangle$ under the “quenched” measure $P_0$, and the “annealed” one $P_0 \triangleq \mathbb{E}(P_0)$. Before describing our results, we need more notations.

In all the sequel, when $D$ is a domain of $\mathbb{R}^d$, $\mathcal{C}^\infty_c(D)$ is the space of infinitely differentiable functions with compact support in $D$. $H^1_0(D)$ is the Sobolev space obtained by closure of $\mathcal{C}^\infty_c(D)$ under the norm

$$ \|f\|^2 = \int_D f^2(x) \, dx + \int_D \|\nabla f\|^2 \, dx. $$

When $V : D \to \mathbb{R}$ is a bounded measurable function, we will write $\lambda(V, D)$ for the principal eigenvalue of the operator $-1/2\Delta - V$, with Dirichlet boundary condition on $D$.

$$ \lambda(V, D) \triangleq \inf \left\{ \frac{1}{2} \int_D \|\nabla f\|^2 \, dx - \int_D V(x) f^2(x) \, dx : f \in H^1_0(D), \int_D f^2(x) \, dx = 1 \right\}. $$

For $V \equiv 0$, and $D = Q(1)$, $\lambda(V, D)$ will be denoted by $\lambda_1(d)$.

The annealed large deviations principle. For any $y \in \mathbb{R}$, let us define

$$ I(y) \triangleq \inf \left\{ I(u) + \mathcal{L}(\mu) : u \in \mathcal{B}_1, \mu \in M^0(\mathbb{R}^d), \langle \mu, u \rangle = y \right\}. \quad (13) $$
Let \( \tilde{\mathcal{I}} \) be the greatest lower semi-continuous minorant of \( \mathcal{I} \):

\[
\tilde{\mathcal{I}}(y) \triangleq \lim_{\epsilon \to 0} \inf_{|z-y| < \epsilon} \mathcal{I}(z). \tag{14}
\]

**Theorem 2.2** Assume (7). Then, for any measurable subset \( F \) of \( \mathbb{R} \),

\[
\limsup_{t \to \infty} \frac{1}{t^{d+2}} \log \tilde{P}_0 [\langle L_t, \xi \rangle \in F] \leq - \inf_{y \in F} \tilde{\mathcal{I}}(y), \tag{15}
\]

\[
\liminf_{t \to \infty} \frac{1}{t^{d+2}} \log \tilde{P}_0 [\langle L_t, \xi \rangle \in F] \geq - \inf_{y \in \bar{F}} \tilde{\mathcal{I}}(y). \tag{16}
\]

\( \tilde{\mathcal{I}} : \mathbb{R} \to [0, +\infty] \) is lower semi-continuous, increasing on \( \mathbb{R}^+ \), decreasing on \( \mathbb{R}^- \). \( \tilde{\mathcal{I}}(0) = 0 \), \( \tilde{\mathcal{I}}(y) < \infty \) for \( y \in ]m; M[ \), \( \tilde{\mathcal{I}}(y) = \infty \) for \( y \notin [m; M] \). Moreover, for \( d \leq 4 \),

\[
\liminf_{y \to 0} \frac{\tilde{\mathcal{I}}(y)}{|y|^{\frac{4}{2+d}}} > 0. \tag{17}
\]

**Remark.** For \( d \leq 4 \), \( \lim_{y \to 0} \tilde{\mathcal{I}}(y)/|y|^{4/(2+d)} \in ]0, +\infty[ \). We will not prove this fact, since our interest is to show that \( \tilde{\mathcal{I}} \) is not convex in dimension \( d = 3 \) and \( d = 4 \), and (17) is enough for that purpose. Actually, \( \frac{4}{2+d} < 1 \) for \( d = 3, 4 \). Hence, if \( \tilde{\mathcal{I}} \) is convex, \( \tilde{\mathcal{I}}(y) = +\infty \) for any \( y \neq 0 \). This contradicts the fact that \( \tilde{\mathcal{I}} \) is finite on \( ]m, M[ \).

**The quenched large deviations principle.** For any \( \alpha \in \mathbb{R} \), let

\[
l(\alpha) \triangleq \inf_{u \in B_1} \{ \lambda(\alpha u, \mathbb{R}^d) : I(u) \leq d \}, \tag{18}
\]

and for any \( y \in \mathbb{R} \)

\[
J(y) \triangleq \sup_{\alpha \in \mathbb{R}} \{ \alpha y + l(\alpha) \}. \tag{19}
\]

Then,

**Theorem 2.3** Assume (7). Let us define \( r(t) \) by the relation \( t = r^2(t) \exp(r^d(t)) \). Then, \( \mathbb{P} \)-a.s., for any measurable subset \( F \) of \( \mathbb{R} \),

\[
\lim_{t \to \infty} \frac{r^2(t)}{t} \log P_0 [\langle L_t, \xi \rangle \in F] \leq - \inf_{y \in F} J(y), \tag{20}
\]

\[
\lim_{t \to \infty} \frac{r^2(t)}{t} \log P_0 [\langle L_t, \xi \rangle \in F] \geq - \inf_{y \in \bar{F}} J(y). \tag{21}
\]

\( J : \mathbb{R} \to [0, +\infty] \) is convex, lower semi-continuous, increasing on \( \mathbb{R}^+ \), decreasing on \( \mathbb{R}^- \). \( J(0) = 0 \), \( J(y) < \infty \) for any \( y \in ]m, M[ \), \( J(y) = \infty \) for \( y \notin [m; M] \).
3 Annealed Bounds.

In section 3.1, we regularize the field. We prove the annealed LD lower bound in section 3.2, and the corresponding upper bound in section 3.3. In all the sequel, we set for convenience \( \tau = t/r^2 \).

3.1 Smoothing the field.

Let \( \psi \) be a rotationally invariant, nonnegative, smooth function with support in \( Q(1) \) and integral 1. For \( \delta > 0 \), let \( \psi_\delta(x) \triangleq \psi(x/\delta)/\delta^d \). We denote by \( * \) the convolution, that is \( u * v(x) = \int_{\mathbb{R}^d} u(x-y)v(y)dy \).

**Lemma 3.1** For any \( \epsilon > 0 \),

\[
\lim_{\delta \to 0} \lim_{\tau \to \infty} \frac{1}{\tau} \log \sup_{u \in B_1} P_0 (| \langle L_\tau, \psi_\delta * u - u \rangle | > \epsilon) = -\infty, \tag{22}
\]

and

\[
\lim_{\delta \to 0} \lim_{\tau \to \infty} \frac{1}{\tau} \log \tilde{P}_0 (| \langle L_\tau, \psi_\delta * \tilde{\xi}_r - \tilde{\xi}_r \rangle | > \epsilon) = -\infty. \tag{23}
\]

**Proof.** In view of the classical fact

\[
\lim_{R \to \infty} \lim_{\tau \to \infty} \frac{1}{\tau} \log P_0 (\sigma(R\tau) < \tau) = -\infty, \tag{24}
\]

the result (22) follows as soon as we show that

\[
\lim_{\delta \to 0} \lim_{\tau \to \infty} \frac{1}{\tau} \log \sup_{u \in B_1} P_0 (A_{\text{reg}}(\tau, \delta, u)) = -\infty, \tag{25}
\]

where

\[
A_{\text{reg}}(\tau, \delta, u) \triangleq \{ B_r : | \langle L_\tau, \psi_\delta * u - u \rangle | > \epsilon, \sigma(R\tau) \geq \tau \}. \tag{26}
\]

We only estimate the probability of the event \( A(\tau, \delta, u) \triangleq \{ B_r : | \langle L_\tau, \psi_\delta * u - u \rangle | > \epsilon, \sigma(R\tau) \geq \tau \} \), and the remaining part of \( A_{\text{reg}}(\tau, \delta, u) \) can be dealt with similarly.

By Chebychev’s inequality, we have for any \( a > 0 \)

\[
P_0(A(\tau, \delta, u)) \leq E_0 \left[ \exp \left( a \int_0^\tau (\psi_\delta * u - u) (B_s) ds \right) \mathbb{1}_{\sigma(R\tau) > \tau} \right] e^{-\tau a \epsilon}. \tag{27}
\]

Using classical bounds (see e.g. Theorem 3.1.2, p.93 of [21]), there is \( c(d) > 0 \) such that

\[
P_0(A(\tau, \delta, u)) \leq c(d) (1 + (\tau \lambda(a(\psi_\delta * u - u), Q(R\tau)))^{d/2}) e^{-\tau (a \epsilon + \lambda(a(\psi_\delta * u - u), Q(R\tau)))}, \tag{28}
\]

Note that when \( u \in B_1 \),

\[
\lambda(a(\psi_\delta * u - u), Q(R\tau)) \leq 2a + \lambda_1(d)/(R\tau)^2.
\]

Moreover, we prove in section 3 the following result.
Lemma 3.2 For any \( u \in B_1(\mathbb{R}^d) \), \( \mu \in \mathcal{M}_1(\mathbb{R}^d) \), \( \delta > 0 \) and \( \epsilon_1 > 0 \), we have

\[
| \int (\psi_\delta * u - u) \, d\mu | \leq \frac{c_0 \delta^2}{\epsilon_1} \mathcal{L}(\mu) + 2\epsilon_1.
\]  

(28)

Thus, for any \( a > 0 \),

\[
P_0(A(\tau, \delta, u)) \leq \left( 1 + \left( \frac{2a\tau + \lambda_1(d)}{R^2 \tau} \right)^{d/2} \right) e^{-\tau \left( \frac{\epsilon^2}{16c_0 \delta^2} + \frac{\lambda_1(d)}{2R^2 \tau^2} \right)}.
\]  

(29)

We set \( \epsilon_1 = \epsilon / 4 \), and \( a = \frac{\epsilon_1}{2c_0 \delta \tau} \). We have

\[
\sup_{u \in B_1} P_0(A(\tau, \delta, u)) \leq \left( 1 + \left( \frac{\epsilon \tau}{4c_0 \delta^2} + \frac{\lambda_1(d)}{R^2 \tau} \right)^{d/2} \right) \exp \left( -\tau \left( \frac{\epsilon^2}{16c_0 \delta^2} + \frac{\lambda_1(d)}{2R^2 \tau^2} \right) \right).
\]  

(30)

Hence, we also have

\[
\tilde{P}_0 \left[ \langle L_\tau, \psi_\delta * \bar{\xi}_r \rangle - \bar{\xi}_r \geq \epsilon \mid \sigma(R\tau) > \tau \right] \leq \left( 1 + \left( \frac{\epsilon \tau}{4c_0 \delta^2} + \frac{\lambda_1(d)}{R^2 \tau} \right)^{d/2} \right) \exp \left( -\tau \left( \frac{\epsilon^2}{16c_0 \delta^2} + \frac{\lambda_1(d)}{2R^2 \tau^2} \right) \right).
\]  

(31)

Equations (22) and (23) follow.

3.2 The annealed lower bound.

By Lemma 3.1, we can now replace \( \langle L_\tau, \bar{\xi}_r \rangle \) by \( \langle L_\tau, \psi_\delta * \bar{\xi}_r \rangle \). The aim of regularizing is the following.

Lemma 3.3 For any \( A > 0 \), the function \((\mu, u) \in \mathcal{M}_1(Q(A)) \times B_1(A) \mapsto \langle \mu, \psi_\delta * u \rangle\), is continuous in the product of weak topologies.

The proof of this lemma is given in section 5. Since \((L_\tau, \bar{\xi}_r)\) satisfies a LDP in the product of weak topologies, we immediately get the LD lower bound:

Lemma 3.4 Let \( r \) and \( \tau \) be such that \( \tau = r^d \). Then, for any \( y \in \mathbb{R} \) and any \( \epsilon > 0 \),

\[
\lim_{\epsilon \to 0} \liminf_{\tau \to \infty} \frac{1}{\tau} \log \tilde{P}_0 \left[ \mid \langle L_\tau, \bar{\xi}_r \rangle - y \mid < \epsilon \right] \geq -\tilde{I}(y),
\]

where \( \tilde{I}(y) \) is defined in (14).

Proof: From

\[
\tilde{P}_0 \left[ \mid \langle L_\tau, \bar{\xi}_r \rangle - y \mid < \epsilon \right] \geq \tilde{P}_0 \left[ \langle L_\tau, \psi_\delta * \bar{\xi}_r \rangle - y \mid < \frac{\epsilon}{2} \right] - \tilde{P}_0 \left[ \mid \langle L_\tau, \psi_\delta * \bar{\xi}_r - \bar{\xi}_r \rangle \mid > \frac{\epsilon}{2} \right],
\]

(32)
We now prove the annealed upper bound. Note that by assumption

\[ 3.3 \text{ The annealed upper bound.} \]

and Lemma 3.1, it is enough to prove that for any \( y \in \mathbb{R} \),

\[
\lim_{\epsilon \to 0} \liminf_{\delta \to 0} \liminf_{\tau \to \infty} \frac{1}{\tau} \log \tilde{P}_0 \left[ |\langle \xi_\delta, \psi \rangle - y | < \epsilon \right] \geq -\tilde{J}(y). \tag{32}
\]

Let \( A, \epsilon, \delta \) be fixed positive numbers. Let \( \mu_0 \in \mathcal{M}_1(Q(A)) \), and \( u_0 \in \mathcal{B}_1(A) \) be such that \( |\langle \mu_0, \psi_\delta * u_0 \rangle - y | < \frac{\epsilon}{2} \). By Lemma 3.3, one can then find a weak neighborhood \( V_1(\mu_0) \) of \( \mu_0 \), and a weak neighborhood \( V_2(u_0) \) of \( u_0 \), such that \( \forall \mu \in V_1, \forall u \in V_2, |\langle \mu, \psi_\delta * u \rangle - y | < \epsilon \). Hence,

\[
\tilde{P}_0 \left[ |\langle \xi_\delta, \psi \rangle - y | < \epsilon \right] \geq \tilde{P}_0 [L_\tau \in V_1; \bar{\xi}_\tau \in V_2] = P_0 [L_\tau \in V_1] \mathbb{P} [\bar{\xi}_\tau \in V_2], \tag{33}
\]

by independence. Applying now the LDP for \( L_\tau \) and \( \bar{\xi}_\tau \), we get for \( \tau = r^d \),

\[
\liminf_{\tau \to \infty} \frac{1}{\tau} \log \tilde{P}_0 \left[ |\langle \xi_\delta, \psi \rangle - y | < \epsilon \right] \geq -\mathcal{L}(\mu_0) - I_A(u_0). \tag{34}
\]

Taking the supremum over admissible \((\mu_0, u_0)\) leads to

\[
\liminf_{\tau \to \infty} \frac{1}{\tau} \log \tilde{P}_0 \left[ |\langle \xi_\delta, \psi \rangle - y | < \epsilon \right] \geq -\inf \left\{ J_{A,\delta}(z) : |z - y| < \frac{\epsilon}{2} \right\}, \tag{35}
\]

where

\[
J_{A,\delta}(z) \triangleq \inf \left\{ \mathcal{L}(\mu) + I_A(u) : u \in \mathcal{B}_1(A), \mu \in \mathcal{M}_1(Q(A)), \langle \mu, \psi_\delta * u \rangle = z \right\}.
\]

Now, it is easy to see that

\[
\limsup_{A \to \infty} \inf_{|z - y| < \frac{\epsilon}{2}} J_{A,\delta}(z) \leq \inf_{|z - y| < \frac{\epsilon}{2}} J_\delta(z),
\]

where \( J_\delta(z) \triangleq \inf \left\{ \mathcal{L}(\mu) + I(u) : \mu \in \mathcal{M}_1(\mathbb{R}^d), u \in \mathcal{B}_1, \langle \mu, \psi_\delta * u \rangle = z \right\}. \tag{36}
\]

We now take \( \delta \) to 0. In view of Lemma 3.2, it is easy to see that

\[
\limsup_{\delta \to 0} \inf_{|z - y| < \frac{\epsilon}{2}} J_\delta(z) \leq \inf_{|z - y| < \frac{\epsilon}{2}} J(z). \tag{37}
\]

This ends the proof of (32) and of Lemma 3.4. \( \square \)

### 3.3 The annealed upper bound.

We now prove the annealed upper bound. Note that by assumption \( \|\xi_\delta\|_\infty \leq 1 \), so that \( |\langle \xi_\delta, \xi_\delta \rangle| \leq 1 \). Hence, it is enough to prove the weak large deviations upper bound. By Lemma 3.1, the problem is thus reduced to prove that

\[
\lim_{\epsilon \to 0} \limsup_{\delta \to 0} \limsup_{\tau \to \infty} \frac{1}{\tau} \log \tilde{P}_0 \left[ |\langle L_\tau, \psi \rangle - y | < \epsilon \right] \leq -\tilde{J}(y). \tag{38}
\]

In contrast with the lower bound, we cannot obtain (38) by contraction, since \( L_\tau \) does not satisfy a full LDP. We begin by the following lemma.
Lemma 3.5 Let \( r \) and \( \tau \) be such that \( \tau = r^d \). Then for any \( y > 0 \), for any \( \epsilon > 0 \),
\[
\lim_{\delta \to 0} \limsup_{\tau \to \infty} \frac{1}{\tau} \log \mathbb{P}_0 \left[ \langle L_\tau, \psi_\delta * \xi_\tau \rangle \geq y \right] \leq - \inf_{z \geq y - 2\epsilon} I(z).
\]

Proof. Let \( y, \epsilon > 0 \) be fixed. \( P_0 \)-a.s, for any \( a > 0 \),
\[
\mathbb{P} \left[ \langle L_\tau, \psi_\delta * \xi_\tau \rangle \geq y \right] \leq \exp(-a\tau y) \mathbb{E} \left[ \exp(a\tau \langle \psi_\delta * L_\tau, \xi_\tau \rangle) \right]
= \exp(-a\tau y) \mathbb{E} \left[ \exp \left( a\tau \sum_{i \in \mathbb{Z}^d} \xi(i) \int_{Q_i(1/\tau)} \psi_\delta * L_\tau(x) \, dx \right) \right]
= \exp(-a\tau y) \exp \left( \sum_{i \in \mathbb{Z}^d} \Lambda(a\tau \int_{Q_i(1/\tau)} \psi_\delta * L_\tau(x) \, dx) \right).
\]

Since \( \tau = r^d = 1/|Q_i(1/\tau)| \), we have by convexity of \( \Lambda \)
\[
\Lambda(a\tau \int_{Q_i(1/\tau)} \psi_\delta * L_\tau(x) \, dx) \leq \tau \int_{Q_i(1/\tau)} \Lambda(a\psi_\delta * L_\tau(x)) \, dx.
\]

Hence, \( P_0 \)-a.s., \( \forall a > 0 \),
\[
\mathbb{P} \left[ \langle L_\tau, \psi_\delta * \xi_\tau \rangle \geq y \right] \leq \exp(-a\tau y) \exp \left( \tau \int_{\mathbb{R}^d} \Lambda(a\psi_\delta * L_\tau(x)) \, dx \right).
\]

Let \( A \) be a large number which will be sent to infinity later. We cover \( \mathbb{R}^d \) by boxes \( \{Q_i(A), i \in \mathbb{Z}^d\} \) of diameter \( A \). The center of \( Q_i(A) \) is \( iA \). We get
\[
\int_{\mathbb{R}^d} \Lambda(a\psi_\delta * L_\tau(x)) \, dx = \int_{Q_0(A)} \sum_{i \in \mathbb{Z}^d} \Lambda(a\psi_\delta * L_\tau(x + iA)) \, dx.
\]

Now, by Hölder’s inequality, we have for any \( x, y \geq 0 \), \( \Lambda(x) + \Lambda(y) \leq \Lambda(x + y) \). Thus,
\[
\int_{\mathbb{R}^d} \Lambda(a\psi_\delta * L_\tau(x)) \, dx \leq \int_{Q_0(A)} \Lambda(a \sum_{i \in \mathbb{Z}^d} \psi_\delta * L_\tau(x + iA)) \, dx.
\]

\( \psi_\delta \) being rotationally invariant, \( \sum_i \psi_\delta * L_\tau(x + iA) = \frac{1}{\tau} \int_0^\tau \sum_i \psi_\delta(|x + iA - B_s|) \, ds \). For \( \delta < A/2 \), there is at most one non-vanishing term among \( \{\psi_\delta(|x + iA - B_s|), i \in \mathbb{Z}^d\} \), whose index \( i \) is such that \( |x + iA - B_s| = d_A(x_A, B_s^A) \), where \( d_A \) denotes the Riemannian metric on the torus \( \mathcal{T}(A) \) of diameter \( A \), \( x_A \) and \( B_s^A \) being the projection of \( x \) and \( B_s \) on \( \mathcal{T}(A) \). Setting \( \psi_\delta^A : (x_A, y_A) \in \mathcal{T}(A) \times \mathcal{T}(A) \to \psi_\delta(d_A(x_A, y_A)) \), we have thus proved that \( P_0 \)-a.s, for any positive \( a, A \), and \( \delta < A/2 \),
\[
\mathbb{P} \left[ \langle L_\tau, \psi_\delta * \xi_\tau \rangle \geq y \right] \leq \exp(-a\tau y) \exp(\tau \int_{\mathcal{T}(A)} \Lambda(a\psi_\delta^A * L_\tau^A(x)) \, dx).
\]

Taking now the infimum in \( a > 0 \) yields \( P_0 \)-a.s, \( \forall A > 0 \), \( \forall \delta < A/2 \),
\[
\mathbb{P} \left[ \langle L_\tau, \psi_\delta * \xi_\tau \rangle \geq y \right] \leq \exp \left( -\tau F_A^A(y; L_\tau^A) \right),
\]

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where
\[ \forall \mu \in \mathcal{M}_1(\mathcal{T}(A)), \quad F_A^\delta(y; \mu) \triangleq \sup_{a > 0} \left\{ ay - \int_{\mathcal{T}(A)} \Lambda(a \psi_A^* \mu(x)) \, dx \right\}. \]

Note that \( \mu \mapsto F_A^\delta(y; \mu) \) is nonnegative and lower semi-continuous as the supremum of continuous functions. Moreover, \( L_A^\delta \) satisfies a full LDP with the good rate function \( \mathcal{L}_A \). Hence, integrating with respect to \( E_0 \), and applying Varadhan’s integral lemma (see for instance Lemma 4.3.6 in [6]), yields
\[ \limsup_{\tau \to \infty} \frac{1}{\tau} \log \tilde{P}_0 \left[ \langle L_\tau, \psi_\delta * \bar{\xi}_r \rangle \geq y \right] \leq - \inf_{z \geq y} \bar{\mathcal{I}}_{\delta,A}(z), \]

We consider now the limit \( A \) to infinity.

**Lemma 3.7** Let \( \mathcal{I}_\delta \) be defined by (36). For any positive \( y \) and \( \epsilon \),
\[ \inf_{z \geq y - \epsilon} \mathcal{I}_\delta(z) \leq \limsup_{A \to \infty} \inf_{z \geq y} \bar{\mathcal{I}}_{\delta,A}(z). \]

The proof of Lemma 3.7 is given in section 5.

We now take \( \delta \) to 0. It follows from Lemma 3.2 that
\[ \inf_{z \geq y - 2\epsilon} \mathcal{J}(z) \leq \liminf_{\delta \to 0} \inf_{z \geq y - \epsilon} \mathcal{I}_\delta(z). \]

We come now to the properties of the rate functional.

**Lemma 3.8** Let \( \tau \) and \( r \) be such that \( \tau = r^d \). Then, for any \( y \in \mathbb{R} \),
\[ \lim_{\epsilon \to 0} \limsup_{\tau \to \infty} \frac{1}{\tau} \log \tilde{P}_0 \left[ \left| \langle L_\tau, \bar{\xi}_r \rangle - y \right| < \epsilon \right] \leq - \tilde{\mathcal{J}}(y). \]

Moreover, \( \tilde{\mathcal{J}} \) satisfies the properties listed in Theorem 2.2.
Proof. Let us first prove the properties of $\tilde{J}$. Since $\mathcal{L}$ and $I$ take positive values, the same holds for $J$ and $\tilde{J}$. Also, $\tilde{J}$ is lower semi-continuous by definition. Since $H(0) = 0$, taking $u \equiv 0$ in the infimum defining $J$ (see (13)), we get $J(0) \leq I(0) + \inf \{ \mathcal{L}(\mu) : \mu \in M_0^1(\mathbb{R}^d) \} = 0$. For any $y \in [m, M]$, taking $u \triangleq y 1_{Q(1)}$, and $\mu \in M_0^1(Q(1))$ in the infimum defining $J$, leads to $J(y) \leq H(y) + \lambda_1(d) < \infty$. Now if we assume that $J(y)$ is finite, one can find $\mu$ and $u$ such that $\langle \mu, u \rangle = y$, $\mathcal{L}(\mu) < \infty$ and $J(u) < \infty$. Hence $d\mu$-a.e., $u(x) \in [m, M]$. $\mathcal{L}(\mu)$ being finite, $\mu \ll d\mu$, and one therefore gets $y = \langle \mu, u \rangle \in [m, M]$.  

We prove (17) when $d \leq 4$. For any $A > 0$, we perform the change of variables, 

$$u_A(x) = u(Ax), \quad d\mu_A = A^d d\mu (Ax) dx,$$

and obtain 

$$J(y) = \inf_{u, \mu} \inf_{A > 0} \left\{ A^d I(u) + A^{-2} \mathcal{L}(\mu) : u \in B_1, \mu \in M_0^1(\mathbb{R}^d), \langle \mu, u \rangle = y \right\} \leq \left( \frac{2}{d} \right)^{\frac{d}{2}} \left( 1 + \frac{d}{2} \right) \inf_{u, \mu} \left\{ I(u) \frac{2}{d} \mathcal{L}(\mu) \frac{d}{2} : u \in B_1, \mu \in M_0^1(\mathbb{R}^d), \langle \mu, u \rangle = y \right\}.$$ 

Now, there is $C > 0$ such that for any $x \in \mathbb{R}$, $H(x) \geq C x^2$. Hence, for another constant $C$, 

$$J(y) \geq C \inf_{u, \mu} \left\{ \left( \int y^2 u^2(x) dx \right)^{\frac{2}{d}} \mathcal{L}(\mu) \frac{d}{2} : u \in B_1, \mu \in M_0^1(\mathbb{R}^d), \langle \mu, u \rangle = 1 \right\}$$ 

$$= Cy^{\frac{d}{2}} \inf_{u, \mu} \left\{ \| u \|_{\mathcal{L}(\mu)}^{\frac{d}{2}} \mathcal{L}(\mu) \frac{d}{2} : u \in B_1, \mu \in M_0^1(\mathbb{R}^d), \langle \mu, u \rangle = 1 \right\}.$$ 

It remains to prove that the infimum is strictly positive. When $d \leq 4$, and for any $\mu \in M_0^1(\mathbb{R}^d)$ such that $\mathcal{L}(\mu) < \infty$, we have by a Nash type inequality (see for instance lemma 5 in (4)) that $\| du / dx \|_2 \leq C\mathcal{L}(\mu)^{\frac{d}{2}}$. Hence, for any $u$ such that $\langle \mu, u \rangle = 1$, 

$$1 = \langle \mu, u \rangle \leq \| u \| \left\| \frac{du}{dx} \right\|_2 \leq C \| u \|_2 \mathcal{L}(\mu)^{\frac{d}{2}}.$$ 

This yields (17).

Let us now prove the monotonicity of $J$ (and thus of $\tilde{J}$) on $\mathbb{R}^+$. Let $0 < y_1 \leq y_2$. We can assume that $J(y_2) < \infty$. Let then $\eta > 0$, $\mu_2 \in M_0^1(\mathbb{R}^d)$, $u_2 \in B_1$ be such that $\langle \mu_2, u_2 \rangle = y_2$ and $\mathcal{L}(\mu_2) + I(u_2) \leq J(y_2) + \eta$. Let us define $\alpha \triangleq y_1 / y_2 \in [0, 1]$, $\mu_1 \triangleq \mu_2$, $u_1 \triangleq \alpha u_2$. Then $\mu_1 \in M_0^1(\mathbb{R}^d)$, $\langle \mu_1, u_1 \rangle = \alpha y_2 = y_1$. Hence, $J(y_1) \leq \mathcal{L}(\mu_1) + I(u_1) \leq \mathcal{L}(\mu_2) + \alpha I(u_2)$, by convexity of $I$. Therefore, $J(y_1) \leq J(y_2) + \eta$ for any $\eta > 0$. 

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We now turn to the proof of (41) for \( y > 0 \); the negative case can be treated similarly. Choose \( \epsilon > 0 \) such that \( 3\epsilon < y \), then
\[
\lim_{\tau \to \infty} \frac{1}{\tau} \log \tilde{P}_0 \left[ |\langle L_{\tau}, \xi_{\tau} \rangle - y| < \epsilon \right] \leq \lim_{\tau \to \infty} \frac{1}{\tau} \log \tilde{P}_0 \left[ |\langle L_{\tau}, \xi_{\tau} \rangle - y| \geq y - \epsilon \right]
\]
\[
\leq - \inf_{z \geq y - 3\epsilon} J(z) \text{ by Lemma 3.3,}
\]
\[
\leq - \inf_{\|z-y\|<3\epsilon} J(z) \text{, since } J \text{ is increasing on } \mathbb{R}^+.
\]

4 Quenched bounds

4.1 Quenched upper bound.

The task at hand in this section is to prove (20) of Theorem 2.3. Note that by assumption (7), \( P_{\text{a.s.}} \forall t, |\langle L_t, \xi \rangle| \leq 1 \). Therefore it is enough to prove the weak large deviations upper bound (i.e. the upper bound for compact sets). Using regularization of the field (lemma 3.1), Brownian scaling, equation (24) and the Gärtner-Ellis method, the problem is reduced to study the large time asymptotics of
\[
\Gamma_{\tau} \left( \alpha \psi^\beta \ast \xi_{\beta \tau}, Q(R\tau) \right) \triangleq E_0 \left[ \exp \left( \int_{0}^{\tau} \alpha \psi^\beta \ast \xi_{\beta \tau} (B_s) \, ds \right) ; \sigma_{R\tau} > \tau \right],
\]
where as before \( \tau = t/r^2 \). The next lemma gives the asymptotical behavior along some subsequences.

Lemma 4.1 Let \( \beta > 1 \), and let \((\tau_n)\) and \((r_n)\) be defined by \( \tau_n = \exp(r_n^d) = \beta^n \). Then \( \forall \delta > 0, \forall R > 0, \forall \alpha \in \mathbb{R}, \mathbb{P}\text{-a.s.}, \)
\[
\lim_{n \to \infty} \frac{1}{\tau_n} \log \Gamma_{\tau_n} \left( \alpha \psi^\beta \ast \xi_{\beta \tau_n}, Q(R\tau_n) \right) \leq -l(\alpha),
\]
where \( l(\alpha) \) is defined by (18).

Before entering the proof, we note that Lemma 4.1 is enough to prove the weak large deviations upper bound. Indeed, it follows from the continuity of \( l \) (see Lemma 5.1) and the bound \( \| \psi^\beta \ast \xi_r \|_\infty \leq 1 \), that we can make the "\( \mathbb{P}\text{-a.s.} \)" in the preceding lemma independent of \( \alpha \). By standard arguments (see for instance Theorem 4.5.3 of [6]), we have the weak upper bound along subsequences of the form \( t_n^\beta = \log(\beta)^{2/d} n^{2/d} \beta^n \) with \( \beta > 1 \). Thus, for any \( \beta > 1 \), we have \( \mathbb{P}\text{-a.s.}, \) that for any \( K \) compact,
\[
\limsup_{n \to \infty} \frac{(r_n^\beta)^2}{t_n^\beta} \log P_0 \left[ \langle L_{t_n^\beta}, \xi \rangle \in K \right] \leq - \inf_{y \in K} J(y),
\]

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The result follows now from the fact that for \( t \in [t_n^\beta, t_{n+1}^\beta] \),
\[
\left| \langle L_t, \xi \rangle - \langle L_{t_n^\beta}, \xi \rangle \right| \leq 2 \frac{t_{n+1}^\beta - t_n^\beta}{t_n^\beta} \leq 2(\beta - 1) + 2\beta \frac{C(d)}{n}, \tag{44}
\]
and the lower semi-continuity of \( J \).

**Proof of Lemma 4.1.** In all the sequel, \( R, \beta, \delta \) and \( \alpha \) are fixed. For convenience, we will often not mention the dependence in \( n \) of \( \tau \) and \( r \).

**Step 1.** We begin to reduce the problem on boxes of fixed size, using Lemma 4.6 of [3]. First of all, note that as in the proof of Lemma 3.1, \( \mathbb{P} \)-a.s,
\[
\Lambda_r (\alpha \psi_\delta \ast \tilde{\xi}_r, Q(R\tau)) \leq C \left( 1 + (\tau \lambda(\alpha \psi_\delta \ast \tilde{\xi}_r, Q(R\tau)))^{d/2} \right) e^{-\tau \lambda(\alpha \psi_\delta \ast \tilde{\xi}_r, Q(R\tau))}.
\]
Now, we cover \( Q(R\tau) \) with \( N \triangleq (1 + \lceil \frac{R\tau}{A} \rceil)^d \) boxes of diameter \( A \) whose centers we denote by \( x_i, i = 1, \ldots, N \). By Lemma 4.6 of [3], there is constant \( C \) such that for any \( \alpha, R, r, \tau, A \) with \( R\tau \geq A \),
\[
\lambda(\alpha \psi_\delta \ast \tilde{\xi}_r, Q(R\tau)) \geq \min_{1 \leq k \leq N} \lambda(\alpha \psi_\delta \ast \tilde{\xi}_r, Q_k(A)) - \frac{C}{A^2},
\]
where we have used the notation \( Q_k(A) \triangleq x_k + Q(A) \). Since
\[
\lambda(\alpha \psi_\delta \ast \tilde{\xi}_r, Q(R\tau)) \leq \frac{\lambda_1(d)}{(R\tau)^2} + |\alpha|,
\]
we are led to
\[
\lim_{\tau \to \infty} \frac{1}{\log \tau} \log \Lambda_r (\alpha \psi_\delta \ast \tilde{\xi}_r, Q(R\tau)) \leq \frac{C}{A^2} - \lim_{\tau \to \infty} \min_{1 \leq k \leq N} \{ \lambda (\alpha \psi_\delta \ast \tilde{\xi}_r, Q_k(A)) \}. \tag{45}
\]

**Step 2.** We have now to estimate the \( \mathbb{P} \)-a.s. behavior of the minimum in the above expression. The proof of the following lemma is done in section [3].

**Lemma 4.2** Let \( r^d = \log(\tau) \) with \( \tau = \beta^n \). Then, for any positive \( \delta, A, R \), and any reals \( \alpha, x \), we have
\[
\lim_{n \to \infty} \frac{1}{\log(\tau)} \log \mathbb{P} \left[ \min_{1 \leq k \leq N} \{ \lambda (\alpha \psi_\delta \ast \tilde{\xi}_r, Q_k(A)) \} \leq x \right] \leq d - \inf_{u \in B_1(A+1)} \{ I_{A+1}(u) : \lambda(\alpha \psi_\delta \ast u, Q(A+1)) \leq x \}. \tag{46}
\]
Hence, for any \( \epsilon > 0 \) and \( n \) sufficiently large,
\[
\mathbb{P} \left[ \min_{1 \leq k \leq N} \lambda (\alpha \psi_\delta \ast \tilde{\xi}_r, Q_k(A)) \leq x \right] \leq \exp \left( \log(\tau) \left( d - \inf_{u \in B_1(A+1)} \{ I_{A+1}(u) : \lambda(\alpha \psi_\delta \ast u, Q(A+1)) \leq x \} + \epsilon \right) \right).
\]
It follows from Borel-Cantelli lemma that for any fixed $\alpha, \delta, A, R$,
\[
d < \inf_{u \in B_1(A+1)} \{ I_{A+1}(u) : \lambda(\alpha \psi_{\delta} * u, Q(A+1)) \leq x \}
\]
\[
\Rightarrow \lim_{n \to \infty} \min_{1 \leq k \leq N} \{ \lambda(\alpha \psi_{\delta} * \xi_k, Q_k(A)) \} > x, \ P - \text{a.s.} \tag{47}
\]

We now prove that
\[
x < \inf_{u \in B_1(A+1)} \{ \lambda(\alpha \psi_{\delta} * u, Q(A+1)) : I_{A+1}(u) \leq d \} \tag{48}
\]
\[
\Rightarrow \inf_{u \in B_1(A+1)} \{ I_{A+1}(u) : \lambda(\alpha \psi_{\delta} * u, Q(A+1)) \leq x \} > d. \tag{49}
\]
Indeed, let $x$ satisfy (48). For all $u$ such that $\lambda(\alpha \psi_{\delta} * u, Q(A+1)) \leq x$, we have then $I_{A+1}(u) > d$; in other words,
\[
\inf_{u \in B_1(A+1)} \{ I_{A+1}(u) : \lambda(\alpha \psi_{\delta} * u, Q(A+1)) \leq x \} \geq d, \tag{50}
\]
with strict inequality if the infimum is reached, which is actually the case. Indeed, it is proved in Lemma 5.2 that $\{ u \in B_1(A+1) ; \lambda(\alpha \psi_{\delta} * u, Q(A+1)) \leq x \}$ is compact in weak topology. Since $I_{A+1}$ is lower semi-continuous in weak topology, $I_{A+1}$ reaches its minimum value on any compact set.

From (47) and (49), we get for any $\alpha, A, R$, that $P$-a.s,
\[
\lim_{n \to \infty} \min_{1 \leq k \leq N} \{ \lambda(\alpha \psi_{\delta} * \xi_k, Q_k(A)) \} \geq \inf_{u \in B_1(A+1)} \{ \lambda(\alpha \psi_{\delta} * u, Q(A+1)) : I_{A+1}(u) \leq d \}. \tag{51}
\]

**Step 3.** We show that for any $A > 0$,
\[
\inf_{u \in B_1(A)} \{ \lambda(\alpha \psi_{\delta} * u, Q(A)) : I_A(u) \leq d \} \geq \inf_{u \in B_1} \{ \lambda(\alpha \psi_{\delta} * u, \mathbb{R}^d) : I(u) \leq d \}. \tag{52}
\]
Let $u_0 \in B_1(A)$ be such that $I_A(u_0) \leq d$. Let us consider the function $\tilde{u}_0 \in B_1$ defined by $\tilde{u}_0 \triangleq u_0 \mathbb{I}(Q(A))$. Since $H(0) = 0$, we have $I(\tilde{u}_0) = I_A(u_0) \leq d$. Therefore,
\[
\inf_{u \in B_1} \{ \lambda(\alpha \psi_{\delta} * u, \mathbb{R}^d) : I(u) \leq d \} \leq \lambda(\alpha \psi_{\delta} * \tilde{u}_0, \mathbb{R}^d) \leq \lambda(\alpha \psi_{\delta} * u_0, Q(A)).
\]

Let us now prove that $\forall \delta > 0$,
\[
\inf_{u \in B_1} \{ \lambda(\alpha \psi_{\delta} * u, \mathbb{R}^d) : I(u) \leq d \} \geq l(\alpha). \tag{53}
\]
Indeed, by convexity of $H$,
\[
I(\psi_{\delta} * u) = \int H(\psi_{\delta} * u) \leq \int \psi_{\delta} * H(u) = \int H(u) = I(u).
\]
Therefore,
\[
\inf_{u \in B_1} \{ \lambda(\alpha \psi_{\delta} * u, \mathbb{R}^d) : I(u) \leq d \} \geq \inf_{u \in B_1} \{ \lambda(\alpha \psi_{\delta} * u, \mathbb{R}^d) : I(\psi_{\delta} * u) \leq d \} \geq l(\alpha).
\]

**Step 4.** Lemma 4.1 is then proved by putting (43), (51), (52) and (53) together, and by letting $A$ tend to infinity along subsequences in (43).
4.2 Quenched lower bound.

In this section, we prove \([21]\) of Theorem 2.3.

Step 1. Almost sure behavior of the field.

Lemma 4.3 Let \(A > 0\) be fixed, and let \(u \in B_1(A)\) be such that \(I_A(u) < d\). Let \(\beta > 1\) and let us define \(\tau_n, r_n\) by \(\tau_n = e^{d^n} = \beta^n\). Then, for any positive \(\delta\) and \(\epsilon\), we have \(\mathbb{P}\)-a.s., that for \(n\) sufficiently large, there is a box \(Q_k\left(\frac{|Ar_n|}{r_n}\right) \subset Q(\tau_n/\log(\tau_n))\) such that

\[
\|\psi_\delta * \bar{\xi}_r - \psi_\delta * u_k\|_{\infty, Q_k\left(\frac{|Ar_n|}{r_n}\right)} < \epsilon,
\]

where \(u_k\) denotes the translation of \(u\) in the box \(Q_k\left(\frac{|Ar_n|}{r_n}\right)\).

Proof. Let us note \(A_r \triangleq \frac{|Ar|}{r} \approx A\) for large \(r\). Define

\[
K \triangleq \left\{ k \in \mathbb{Z}^d : Q_k(A_r) \subset Q(\tau/\log(\tau)) \right\},
\]

and let \(\hat{K}\) be the subset of \(K\) corresponding to multi-integers with even coordinates. Note that as soon as \(\delta + 1/r < A_r\), the functions \(\{\psi_\delta * \bar{\xi}_r|_{Q_k(A_r)} ; k \in \hat{K}\}\) are independent. Moreover, \(A_r\) being an integer multiple of \(1/r\), they also have the same law. Therefore,

\[
\mathbb{P}\left[ \forall k \in K, \|\psi_\delta * \bar{\xi}_r - \psi_\delta * u_k\|_{\infty, Q_k(A_r)} \geq \epsilon \right] \\
\leq \mathbb{P}\left[ \forall k \in \hat{K}, \|\psi_\delta * \bar{\xi}_r - \psi_\delta * u_k\|_{\infty, Q_k(A_r)} \geq \epsilon \right] \\
\leq \mathbb{P}\left[ \|\psi_\delta * \bar{\xi}_r - \psi_\delta * u\|_{\infty, Q(A)} \geq \epsilon \right]_{|\hat{K}|} \\
\leq \mathbb{P}\left[ \|\psi_\delta * \bar{\xi}_r - \psi_\delta * u\|_{\infty, Q(A)} \geq \epsilon \right]_{|\hat{K}|}.
\]

Now, \(\left\{ v \in B_1(A) ; \|\psi_\delta * v - \psi_\delta * u\|_{\infty, Q(A)} < \epsilon \right\}\) is an open neighborhood of \(u\) in weak topology. Thus, by the LDP of \(\bar{\xi}_r\),

\[
\lim_{r \to \infty} \frac{1}{r^d} \log \mathbb{P}\left[ \|\psi_\delta * \bar{\xi}_r - \psi_\delta * u\|_{\infty, Q(A)} < \epsilon \right] \geq -I_A(u).
\]

Let \(\eta > 0\) be such that \(d - I_A(u) > \eta\). For \(r\) sufficiently large and \(\tau = e^{r^d}\), we have then

\[
\mathbb{P}\left[ \forall k \in K, \|\psi_\delta * \bar{\xi}_r - \psi_\delta * u_k\|_{\infty, Q_k(A_r)} \geq \epsilon \right] \leq \left( 1 - e^{-r^d(I_A(u)+\eta)} \right)_{|\hat{K}|} \\
\approx e^{-\left(\frac{d}{2} d^d - I_A(u) - \eta\right)} \left( \frac{d^d - I_A(u) - \eta}{(\log(r))^d} \right).
\]

Taking \(\tau_n = \beta^n\) for some \(\beta > 1\), the result follows from Borel Cantelli lemma.

\[
\]

Step 2. A first lower bound.
Lemma 4.4 Let us define for \( y \in \mathbb{R} \)

\[
\mathcal{J}_1(y) \triangleq \inf_{\mu \in \mathcal{P}_1^d(\mathbb{R}^d)} \inf_{u \in \mathcal{B}_1(\mathbb{R}^d)} \{ \mathcal{L}(\mu) : \langle \mu, u \rangle = 0, I(u) < \delta \}.
\] (54)

Let \( \beta > 1 \), and as before \( \tau_n = e^{\beta n} \). Then, \( \mathbb{P} \)-a.s., \( \forall y \in \mathbb{R}, \forall \epsilon > 0, \)

\[
\lim_{\delta \to 0, \delta \in Q} \lim_{n \to \infty} \frac{1}{\tau_n} \log P_0 \left[ \left| \left\langle L_{\tau_n}, \psi_\delta \ast \bar{\xi}_{\tau_n} \right\rangle - y \right| \leq \epsilon \right] \geq -\mathcal{J}_1(y).
\]

Moreover, let \( \mathcal{J}_1^{**} \) be the double Legendre transform of \( \mathcal{J}_1 \), then \( \mathcal{J}_1^{**} = J \).

**Proof.** Let \( \beta > 1, \delta > 0, A > 0 \) and fix \( u \) such that \( I_{A} (u) < \delta \). Let \( k \) be the index of the box of size \( A r_n \) associated by Lemma 1.3 to \( \delta \) and \( \epsilon/4 \). The center of this box is denoted by \( x_k = k A r_n \). \( \theta \) will denote the shift on the Brownian trajectories, and \( \sigma(D) \) will denote the exit time of \( D \).

\[
P_0 \left[ \left| \langle L_\tau, \psi_\delta \ast \bar{\xi}_\tau \rangle - y \right| \leq \epsilon \right] \geq P_0 \left[ \left| B_{\frac{\tau}{\log(\tau)}} - x_k \right| \leq 1; \left| \frac{1}{\tau} \int_{0}^{\frac{\tau}{\log(\tau)}} \psi_\delta \ast \bar{\xi}_\tau(B_s) \, ds \right| \leq \epsilon \right] \leq \psi(1) \left( 1 - \frac{1}{\log(\tau)} \right); \left| \frac{1}{\tau} \int_{0}^{\frac{\tau}{\log(\tau)}} \psi_\delta \ast \bar{\xi}_\tau(B_s) \, ds - y \right| \leq \frac{\epsilon}{2} \right], (55)
\]

Applying the Markov property at time \( \tau / \log(\tau) \) yields,

\[
P_0 \left[ \left| \langle L_\tau, \psi_\delta \ast \bar{\xi}_\tau \rangle - y \right| \leq \epsilon \right] \geq P_0 \left[ \left| B_{\frac{\tau}{\log(\tau)}} - x_k \right| \leq 1; \left| \frac{1}{\tau} \int_{0}^{\frac{\tau}{\log(\tau)}} \psi_\delta \ast \bar{\xi}_\tau(B_s) \, ds \right| \leq \frac{\epsilon}{2} \right] \inf_{|x - x_k| \leq 1} P_x \left[ \sigma(Q_k(A_\tau)) \geq \tau \left( 1 - \frac{1}{\log(\tau)} \right); \left| \frac{1}{\tau} \int_{0}^{\frac{\tau}{\log(\tau)}} \psi_\delta \ast \bar{\xi}_\tau(B_s) \, ds - y \right| \leq \frac{\epsilon}{2} \right].
\]

Now, on \( \sigma(Q_k(A_\tau)) \geq \tau \left( 1 - \frac{1}{\log(\tau)} \right), \)

\[
\left| \frac{1}{\tau} \int_{0}^{\tau \left( 1 - \frac{1}{\log(\tau)} \right)} \psi_\delta \ast \bar{\xi}_\tau(B_s) \, ds - \frac{1}{\tau} \int_{0}^{\tau \left( 1 - \frac{1}{\log(\tau)} \right)} \psi_\delta \ast u_k(B_s) \, ds \right| \leq \left( 1 - \frac{1}{\log(\tau)} \right) \frac{\epsilon}{4}.
\]

Thus, for \( \tau \) sufficiently large \( \left( \frac{1}{\log(\tau)} < \frac{\epsilon}{4} \right), \)

\[
\inf_{|x - x_k| \leq 1} P_x \left[ \sigma(Q_k(A_\tau)) \geq \tau \left( 1 - \frac{1}{\log(\tau)} \right); \left| \frac{1}{\tau} \int_{0}^{\tau \left( 1 - \frac{1}{\log(\tau)} \right)} \psi_\delta \ast \bar{\xi}_\tau(B_s) \, ds - y \right| \leq \frac{\epsilon}{2} \right] \geq \inf_{|x - x_k| \leq 1} P_x \left[ \sigma(Q_k(A_\tau)) \geq \tau \left( 1 - \frac{1}{\log(\tau)} \right); \left| \frac{1}{\tau} \int_{0}^{\tau \left( 1 - \frac{1}{\log(\tau)} \right)} \psi_\delta \ast u_k(B_s) \, ds - y \right| \leq \frac{\epsilon}{4} \right] \geq \inf_{|x| \leq 1} P_x \left[ \sigma(Q(A_\tau)) \geq \tau \left( 1 - \frac{1}{\log(\tau)} \right); \left| \langle L_{\tau \left( 1 - \frac{1}{\log(\tau)} \right)}, \psi_\delta \ast u \rangle - y \right| \leq \frac{\epsilon}{8} \right].
\]
By the LDP lower bound for the Brownian occupation measure, we get then
\[
\lim_{\tau \to \infty} \frac{1}{\tau} \log \inf_{|x-x_k| \leq 1} P_x \left[ \sigma(Q_k(A_\tau)) \geq \tau \left( 1 - \frac{1}{\log(\tau)} \right) \left| \frac{1}{\tau} \int_0^{\tau (1 - \frac{1}{\log(\tau)})} \psi \ast \xi_{\tau} \right| d\tau - y \right] \leq \frac{\epsilon}{2}
\]
\[
\geq - \inf_{\mu \in \mathcal{M}_0^0(\mathbb{Q}(A))} \left\{ \mathcal{L}(\mu) : \left| \langle \mu, \psi \ast u \rangle - y \right| < \frac{\epsilon}{8} \right\}.
\]

For the other term in (55), since \(\frac{1}{\tau} \int_0^{\tau / \log(\tau)} \psi \ast \xi_{\tau} \) \(d\tau \leq \frac{1}{\log(\tau)}\), we have for \(\tau\) sufficiently large \((\frac{1}{\log(\tau)} \leq \frac{\epsilon}{2})\),
\[
P_0 \left[ \left| B_{\tau / \log(\tau)} - x_k \right| \leq 1; \frac{1}{\tau} \int_0^{\tau / \log(\tau)} \psi \ast \xi_{\tau} \right) d\tau \leq \frac{\epsilon}{2} \right]
\]
\[
= P_0 \left[ \left| B_{\tau / \log(\tau)} - x_k \right| \leq 1 \right]
\]
\[
= \int_{\|y - x_k\| \leq 1} \exp \left( -\frac{\|y\|^2}{2\tau / \log(\tau)} \right) \left( \frac{2\pi \tau}{\log(\tau)} \right)^d dy
\]
\[
\geq \frac{C(d)}{(2\pi \tau / \log(\tau))^{d/2}} \exp \left( -\frac{(\tau / \log(\tau) + 1)^2}{2\tau / \log(\tau)} \right) .
\]

Putting (53), (54), (57) together, we have that for any \(A > 0, u \in \mathcal{B}_1(A)\) with \(I_A(u) < d, \delta > 0, \epsilon > 0, \mathbb{P}\)-a.s., for any \(y \in \mathbb{R}\),
\[
\lim_{\tau \to \infty} \frac{1}{\tau} \log P_0 \left[ \left| \langle L_{\tau}, \psi \ast \xi_{\tau} \rangle - y \right| \leq \epsilon \right] \geq - \inf_{\mu \in \mathcal{M}_0^0(\mathbb{Q}(A))} \left\{ \mathcal{L}(\mu) : \left| \langle \mu, \psi \ast u \rangle - y \right| < \frac{\epsilon}{8} \right\}.
\]

We would like to take the supremum over \(u\) in the preceding expression. Since the “\(\mathbb{P}\)-a.s.” depends on \(u\), we have to restrict ourselves to a countable subset of \(\mathcal{B}_1(A)\).

**Lemma 4.5** For any \(A > 0\), there exists a countable subset \(\mathcal{D}\) of \(\mathcal{B}_1(A)\), such that for any \(u \in \mathcal{B}_1(A)\), there is a sequence \((u_n)\) in \(\mathcal{D}\) satisfying

1. \(\lim_{n \to \infty} I_A(u_n) \leq I_A(u)\).
2. \(\forall \mu \in \mathcal{M}_0^0(\mathbb{Q}(A)), \forall \delta > 0, \lim_{n \to \infty} \langle \mu, \psi \ast u_n \rangle = \langle \mu, \psi \ast u \rangle\).

The proof of this lemma is given in section 5. Lemma 4.3 implies that \(\forall \mu \in \mathcal{M}_0^0(\mathbb{Q}(A))\), and \(\forall \delta > 0, \forall \epsilon > 0, \forall y \in \mathbb{R}\),
\[
\inf_{u \in \mathcal{D}} \left\{ I_A(u) : \left| \langle \mu, \psi \ast u \rangle - y \right| < \epsilon \right\} = \inf_{u \in \mathcal{B}_1(A)} \left\{ I_A(u) : \left| \langle \mu, \psi \ast u \rangle - y \right| < \epsilon \right\} .
\]

Thus, taking the supremum over \(u \in \mathcal{D}\) in (58), we obtain that
\[
\lim_{\tau \to \infty} \frac{1}{\tau} \log P_0 \left[ \left| \langle L_{\tau}, \psi \ast \xi_{\tau} \rangle - y \right| \leq \epsilon \right]
\]
\[
\geq - \inf_{\mu \in \mathcal{M}_0^0(\mathbb{Q}(A))} \left\{ \mathcal{L}(\mu) : \exists u \in \mathcal{D} \text{such that} \left| \langle \mu, \psi \ast u \rangle - y \right| < \frac{\epsilon}{8} \text{ and } I_A(u) < d \right\}
\]
\[
\geq - \inf_{\mu \in \mathcal{M}_0^0(\mathbb{Q}(A))} \left\{ \mathcal{L}(\mu) : \inf_{u \in \mathcal{B}_1(A)} \left\{ I_A(u) : \left| \langle \mu, \psi \ast u \rangle - y \right| < \frac{\epsilon}{8} \right\} < d \right\}.
\]
We now take $\delta$ to $0$, $\delta \in \mathbb{Q}^+$. By Lemma \[3.32\],

$$\lim_{\delta \to 0, \delta \in \mathbb{Q}^+} \inf_{\mu \in M_1^0(Q(A))} \left\{ \mathcal{L} \left( \mu \right) : \inf_{u \in B_1(A)} \left\{ I_A(u) : |\langle \mu, \psi_\delta * u \rangle - y | < \frac{\epsilon}{8} \right\} < d \right\} \leq \inf_{\mu \in M_1^0(Q(A))} \left\{ \mathcal{L} \left( \mu \right) : \inf_{u \in B_1(A)} \left\{ I_A(u) : |\langle \mu, u \rangle - y | < \frac{\epsilon}{8} \right\} < d \right\}. $$

We have thus proved that $\forall A > 0$, $\forall \epsilon > 0$, $\mathbb{P}$-a.s., $\forall y \in \mathbb{R}$,

$$\lim_{\delta \to 0, \delta \in \mathbb{Q}^+} \lim_{\tau \to \infty} \frac{1}{\tau} \log P_0 \left[ |\langle L_\tau, \psi_\delta * \xi_\tau \rangle - y | \leq \epsilon \right] \geq - \inf_{\mu \in M_1^0(Q(A))} \inf_{u \in B_1(A)} \left\{ \mathcal{L} \left( \mu \right) : I_A(u) < d, |\langle \mu, u \rangle - y | < \frac{\epsilon}{8} \right\} \leq \mathcal{J}_1(y).$$

Taking $A$ to infinity, it is easy to see that

$$\lim_{A \to \infty, A \in \mathbb{Q}, \mu \in M_1^0(Q(A))} \inf_{u \in B_1(A)} \left\{ \mathcal{L} \left( \mu \right) : I_A(u) < d, |\langle \mu, u \rangle - y | < \frac{\epsilon}{8} \right\} = \inf_{\mu \in M_1^0(\mathbb{R}^d)} \inf_{u \in B_1(\mathbb{R}^d)} \left\{ \mathcal{L} \left( \mu \right) : I(u) < d, |\langle \mu, u \rangle - y | < \frac{\epsilon}{8} \right\} \leq \mathcal{J}_1(y).$$

We now prove that $\mathcal{J}_1^*(y) = J(y)$. Since $J = (-l)^*$, it is enough to prove that $\mathcal{J}_1^* = -l$. It follows from the large deviations estimates that $-\mathcal{J}_1 \leq -J$, so that $\mathcal{J}_1^* \leq J^* = (-l)^* = -l$, since $-l$ is convex continuous (cf. Lemma \[3.34\]). Hence, it remains to prove that $-l \leq \mathcal{J}_1^*$. A direct computation yields

$$\mathcal{J}_1^*(\alpha) = - \inf_{u \in B_1} \left\{ \lambda(\alpha u, \mathbb{R}^d) : I(u) < d \right\},$$

which is almost $-l$, except for the strict inequality “$I(u) < d$”, which we treat now.

Let $\alpha \in \mathbb{R}$ and $t$ be such that $t < -l(\alpha)$. It follows from the definition of $l$ that $\exists u \in B_1, I(u) \leq d$, and $\mu \in M_1^0(\mathbb{R}^d)$, such that $\mathcal{L}(\mu) - \alpha \langle \mu, u \rangle < -t$. Let $A > 0$ and let us consider $d\mu_A(x) = A^{d_u} \mu (Ax) dx$, and $u_A(uA)$. Then $\mu_A \in M_1^0(\mathbb{R}^d)$, $u_A \in B_1$, $\langle \mu_A, u_A \rangle = \langle \mu, u \rangle$, $\mathcal{L}(\mu_A) = A^2 \mathcal{L}(\mu)$, and $I(u_A) = A^{-d} I(u) \leq A^{-d} d < d$ for any $A > 1$. Hence, for any $A > 1$,

$$-\mathcal{J}_1^*(\alpha) \leq \mathcal{L}(\mu_A) - \alpha \langle \mu_A, u_A \rangle = A^2 \mathcal{L}(\mu) - \alpha \langle \mu, u \rangle. \quad (59)$$

Therefore, $-\mathcal{J}_1^*(\alpha) \leq \mathcal{L}(\mu) - \alpha \langle \mu, u \rangle < -t$. Since $t$ can be chosen arbitrarily in $]-\infty; -l(\alpha)[$, the result follows.

**Step 3. A sequence of lower bounds.**

For $y \in \mathbb{R}$, and $p \in \mathbb{N}^*$, let us define

$$D_p(y) \triangleq \left\{ \left( \bar{\alpha}, \bar{u}, \bar{\mu} \right) \in [0, 1]^p \times B_1^p \times M_1^0(\mathbb{R}^d)^p : \sum_{j=1}^p \alpha_j = 1, \sum_{j=1}^p \alpha_j \langle \mu_j, u_j \rangle = y, \forall j, I(u_j) < d \right\},$$

$$J_p(\bar{\alpha}, \bar{\mu}) \triangleq \sum_{j=1}^p \alpha_j \mathcal{L}(\mu_j), \quad \text{and} \quad J_p(y) \triangleq \inf_{(\bar{\alpha}, \bar{u}, \bar{\mu}) \in D_p(y)} J_p(\bar{\alpha}, \bar{\mu}).$$


Lemma 4.6 Let $\beta > 1$, and let us define $\tau_n$ and $r_n$ by $\tau_n = e^{-\beta n}$. Then, for every $\epsilon > 0$, $P$-a.s., $\forall p \in \mathbb{N}$, $\forall y \in \mathbb{R}$,

$$
\lim_{\delta \to 0, \delta \in \mathbb{Q}} \lim_{n \to \infty} \frac{1}{\tau_n} \log P_0 \left[ \left| \langle L_{\tau_n}, \psi_{\delta} * \tilde{\xi}_{\tau_n} \rangle - y \right| \leq \epsilon \right] \geq -\mathbb{E}(y).
$$

Proof. The proof follows the same lines as step 2. Let $\beta > 1$, $\epsilon > 0$, $A > 0$, $(\tilde{a}, \tilde{y}) \in [0, 1]^p \times \mathbb{R}^p$, $\sum \alpha_j = 1, \sum \alpha_j y_j = y$, and $u_1, \cdots, u_p$ such that $I_A(u_i) < d$ be fixed. Let $k_i$ be the indices of the boxes of size $A_{r_n}$ associated by Lemma 4.3 to the $u_i, \delta$ and $\epsilon/6$. We divide the time interval $[0, \tau]$ in $p$ time intervals $[\tau_{i-1}, \tau_i]$, where $\tau_i = \sum_{j=1}^i \alpha_j \tau$. In the $i$-th time interval, we force the Brownian motion to go fast (i.e., in time of order $\Delta_i = \alpha_i \tau / \log(\tau)$), from a neighborhood of 0 to a neighborhood of $k_i A_r$, to remain in $Q_{k_i}(A_r)$ during $\alpha_i \tau - 2\Delta_i$, and to return in a neighborhood of 0 in time $\Delta_i$. We have then

$$
P_0 \left[ \left| \langle L_{\tau}; \psi_{\delta} * \tilde{\xi}_r \rangle - y \right| < \epsilon \right] \geq \prod_{i=1}^p U_i V_i W_i,
$$

where

$$
U_i = \inf_{|\xi| \leq 1} \mathbb{E}_i \left[ \left| \frac{1}{\tau} \int_0^{\Delta_i} \psi_{\delta} * \tilde{\xi}_r(B_s) ds \right| < \epsilon \frac{\tau}{3p} \right],
$$

$$
V_i = \inf_{x \in Q(1)} \mathbb{E}_i \left[ \sigma(Q(A_r)) > \alpha_i \tau - 2\Delta_i; \left| \frac{1}{\tau} \int_0^{\alpha_i \tau - 2\Delta_i} \psi_{\delta} * u_i(B_s) ds - \alpha_i y_i \right| < \epsilon \frac{\tau}{6p} \right],
$$

$$
W_i = \inf_{x \in Q_{k_i}(A_r)} \mathbb{E}_i \left[ \left| \frac{1}{\tau} \int_0^{\Delta_i} \psi_{\delta} * \tilde{\xi}_r(B_s) ds \right| < \epsilon \frac{\tau}{3p}; |B_{\Delta_i} - k_i A_r| \leq 1 \right].
$$

Exactly as in step 2, we can prove that

$$
\lim_{\tau \to \infty} \frac{1}{\tau} \log(U_i) \geq 0, \quad \lim_{\tau \to \infty} \frac{1}{\tau} \log(W_i) \geq 0,
$$

$$
\lim_{\tau \to \infty} \frac{1}{\tau} \log(V_i) \geq -\inf_{\mu \in \mathbb{M}_\beta(Q(A_r))} \left\{ \alpha_i \mathbb{L}(\mu_i) : \alpha_i \left| \langle \mu_i, \psi_{\delta} * u_i \rangle - y_i \right| < \epsilon \frac{\tau}{6p} \right\}.
$$

Taking now the supremum over $u_i \in \mathcal{D}$, we get

$$
\lim_{\tau \to \infty} \frac{1}{\tau} \log P_0 \left[ \left| \langle L_{\tau}; \psi_{\delta} * \tilde{\xi}_r \rangle - y \right| < \epsilon \right] \geq -\inf_{\bar{\mu} \in \mathbb{M}_\beta(Q(A_r))} \left\{ J_p(\bar{\alpha}, \bar{\mu}) : \inf_{u_i \in \mathcal{D}} \max I_A(u_i) < d \right\},
$$

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where we have denoted \( \mathcal{D}_i = \left\{ u \in \mathcal{D}; \alpha_i |\langle \mu_i, \psi z u \rangle - y_i | < \frac{\epsilon}{6p} \right\} \). Since
\[
\inf_{u_i \in \mathcal{D}_i} \max_i I_A(u_i) = \max_i \inf_{u_i \in \mathcal{D}_i} I_A(u),
\]
the above infimum becomes by Lemma 4.5
\[
- \inf_{\check{\mu}, \check{\gamma}} \inf_{\check{\mu}, \check{\gamma}} \left\{ J_p(\check{\alpha}, \check{\mu}) : \alpha_i |\langle \mu_i, u_i \rangle - y_i | < \frac{\epsilon}{6p}; I_A(u_i) < d \right\}.
\]
Taking \( \delta \) to 0, we obtain
\[
- \inf_{\check{\mu}, \check{\gamma}} \left\{ J_p(\check{\alpha}, \check{\mu}) : \alpha_i |\langle \mu_i, u_i \rangle - y_i | < \frac{\epsilon}{6p}; I_A(u_i) < d \right\}
\geq - \inf_{\check{\mu}, \check{\gamma}} \inf_{\check{\mu}, \check{\gamma}} \left\{ J_p(\check{\alpha}, \check{\mu}) : \langle \mu_i, u_i \rangle = y_i, I_A(u_i) < d \right\}.
\]
Optimizing in \((\check{\alpha}, \check{y})\), we are led to
\[
- \inf_{\check{\alpha} \in [0, 1]^d} \inf_{\check{\mu}, \check{\gamma}} \left\{ J_p(\check{\alpha}, \check{\mu}) : \sum_{i=1}^p \alpha_i = 1, \sum_{i=1}^p \alpha_i \langle \mu_i, u_i \rangle = y, I_A(u_i) < d \right\}.
\]
The proof of Lemma 4.6 follows after taking \( A \to \infty \), and noting that the infimum over \( \alpha \in [0, 1]^d \) is the same as taking \( \alpha \in [0, 1]^d \).

**Step 4. Conclusion.**
From Lemma 4.6, Lemma 3.1, and (14), it is straightforward to see that if \( r(t) \) is defined as in Theorem 2.3, then we have \( \mathbb{P}\text{-a.s.} \), for any \( y \in \mathbb{R}, \epsilon > 0, p \in \mathbb{N}, \)
\[
\lim_{t \to \infty} \frac{r^2(t)}{t} \log P_0 \left[ |\langle L_t, \xi \rangle - y | < \epsilon \right] \geq -J_p(y).
\]
We now take \( p \) to \( \infty \).

**Lemma 4.7.**
\begin{enumerate}
\item \( \forall p \in \mathbb{N}, \forall y \in \mathbb{R}, J_{p+1}(y) \leq J_p(y). \)
\item \( \forall p \in \mathbb{N}, \forall \alpha \in [0, 1], \forall y_1, y_2 \in \mathbb{R}, \)
\[
J_{2p}(\alpha y_1 + (1 - \alpha) y_2) \leq \alpha J_p(y_1) + (1 - \alpha) J_p(y_2).
\]
\item Let \( \mathcal{J}(y) \triangleq \lim_{p \to \infty} J_p(y) \), and \( \mathcal{J} (y) \triangleq \sup_{s > 0} \inf_{y - |z| \leq \epsilon} \mathcal{J}(z) \) the greater lower semi-continuous minorant of \( \mathcal{J} \). Then \( \mathcal{J} = J. \)
\item \( \mathbb{P}\text{-a.s.}, \forall y \in \mathbb{R}, \)
\[
\lim_{\epsilon \to 0} \lim_{t \to \infty} \frac{r^2(t)}{t} \log P_0 \left[ |\langle L_t, \xi \rangle - y | \leq \epsilon \right] \geq -J(y).
\]
\end{enumerate}
Proof of 1. For any \((\vec{a}, \vec{u}, \vec{\mu}) \in D_p(y)\), and any \(\nu\) with \(L(\nu) < \infty\), we set \(\vec{\beta} \triangleq (\vec{a}, 0)\), \(\vec{w} \triangleq (\vec{u}, 0)\) and \(\vec{\pi} \triangleq (\vec{\mu}, \nu)\). We note that \((\vec{\beta}, \vec{w}, \vec{\pi}) \in D_{p+1}(y)\). Thus,
\[
J_{p+1}(y) \leq J_{p+1}(\vec{\beta}, \vec{\pi}) = J_p(\vec{a}, \vec{\mu}).
\]
Taking the infimum over \(D_p(y)\) yields \(J_{p+1}(y) \leq J_p(y)\).

Proof of 2. In the same way, let \(\alpha \in [0, 1]\) and \(y_1, y_2 \in \mathbb{R}\) be fixed. For any \((\vec{\beta}_1, \vec{u}_1, \vec{\mu}_1) \in D_p(y_1)\), and any \((\vec{\beta}_2, \vec{u}_2, \vec{\mu}_2) \in D_p(y_2)\), we set \(\vec{\lambda} \triangleq (\alpha \vec{\beta}_1, (1-\alpha) \vec{\beta}_2)\), \(\vec{v} \triangleq (\vec{u}_1, \vec{u}_2)\), and \(\vec{\nu} \triangleq (\vec{\mu}_1, \vec{\mu}_2)\). Note that \((\vec{\lambda}, \vec{v}, \vec{\nu}) \in D_{2p}(\alpha y_1 + (1-\alpha) y_2)\). Thus,
\[
J_{2p}(\alpha y_1 + (1-\alpha) y_2) \leq J_{2p}(\vec{\lambda}, \vec{v}) = \alpha J_p(\vec{\beta}_1, \vec{\mu}_1) + (1-\alpha) J_p(\vec{\beta}_2, \vec{\mu}_2).
\]
Taking the infimum over elements of \(D_p(y_1)\) and \(D_p(y_2)\), leads to (61).

Proof of 3 and 4. Taking \(p\) to \(\infty\) in (60) yields that \(\mathbb{P}\)-a.s, for any \(y \in \mathbb{R}\) and \(\epsilon > 0\),
\[
\lim_{t \to \infty} \frac{r^2(t)}{t} \log P_0 [\|L_t, \xi\| \leq \epsilon] \geq -\tilde{J}(y). \tag{62}
\]
From this, it follows easily that \(\mathbb{P}\)-a.s, \(\forall y \in \mathbb{R},\)
\[
\lim_{\epsilon \to 0} \lim_{t \to \infty} \frac{r^2(t)}{t} \log P_0 [\|L_t, \xi\| \leq \epsilon] \geq -\tilde{J}(y) \tag{63}
\]
From the large deviations upper bound, we have then \(\tilde{J} \geq J\). On the other hand, we also have \(\tilde{J} \leq J \leq \tilde{J}_1\), so that \(\tilde{J}^{**} \leq \tilde{J}_1^{**} = J(y)\) by Lemma 4.4. Taking \(p\) to \(\infty\) in (61), we obtain that \(\tilde{J}\) is convex. Thus, \(\tilde{J}\) is convex and lower semi-continuous. Thus, \(\tilde{J} = \tilde{J}^{**}\), and \(\tilde{J} \leq J\). Finally, \(\tilde{J} = J\).

5 Technical Lemmas

Proof of Lemma 3.2. We can assume that \(\mathcal{L}(\mu) < \infty\). Let then \(\varphi = \sqrt{d\mu/dx}\), and as \(u\) is bounded by 1,
\[
|\int (\psi_\delta * u - u) d\mu| \leq \int_{\mathbb{R}^d} |\psi_\delta * \varphi^2(x) - \varphi^2(x)| dx. \tag{64}
\]
Now, for any \(\epsilon_1 > 0\), and any \(x \in \mathbb{R}^d\)
\[
\int \psi_\delta(y)|\varphi^2(x - y) - \varphi^2(x)| dy \leq \frac{1}{2\epsilon_1} \int \psi_\delta(y) (\varphi(x - y) - \varphi(x))^2 dy
+ \epsilon_1 (\psi_\delta * \varphi^2(x) + \varphi^2(x)). \tag{65}
\]
Also,
\[
|\varphi(x - y) - \varphi(x)| = |\int_0^1 \nabla \varphi(x - ty). y dt| \leq ||y|| \left(\int_0^1 ||\nabla \varphi(x - ty)||^2 dt\right)^{1/2}. \tag{66}
\]
Thus,
\[
\int_{\mathbb{R}^d} |\psi_\delta * \varphi^2(x) - \varphi^2(x)| dx \leq \frac{1}{2\epsilon_1} \int_0^1 dt \int_{\mathbb{R}^d} dy \psi_\delta(y) ||y||^2 \int_{\mathbb{R}^d} dx ||\nabla \varphi(x - ty)||^2 + 2\epsilon_1 \\
\leq \int \psi_\delta(y) ||y||^2 dy \int_{\mathbb{R}^d} ||\nabla \varphi(x)||^2 dx + 2\epsilon_1.
\] (67)

There is a constant \( c_0 \) such that \( \int \psi_\delta(y) ||y||^2 dy = c_0\delta^2 \), and the result follows. \( \blacksquare \)

**Proof of Lemma 3.3.**

Let \((u_n)\) be a sequence converging weakly to \( u \in \mathcal{B}_1(A) \), and \((\mu_n)\) a sequence converging weakly to \( \mu \in \mathcal{M}_1(Q(A)) \). We think of \( u_n \) and \( u \) as vanishing outside \( Q(A) \). For any \( \delta > 0 \), \((\psi_\delta * u_n)\) is an equicontinuous, uniformly bounded sequence converging pointwise to \( \psi_\delta * u \). By Ascoli-Arzelà, we have

\[
\lim_{n \to \infty} \| \psi_\delta * u_n - \psi_\delta * u \|_{\infty, \mathcal{Q}(A)} = 0.
\]

The result follows then from the inequality

\[
| \langle \mu_n, \psi_\delta * u_n \rangle - \langle \mu, \psi_\delta * u \rangle | \leq \| \psi_\delta * u_n - \psi_\delta * u \|_{\infty, \mathcal{Q}(A)} + | \langle \mu_n, \psi_\delta * u \rangle - \langle \mu, \psi_\delta * u \rangle |.
\]

\( \blacksquare \)

**Proof of Lemma 3.6.**

Set

\[
\mathcal{J}_1(y) = \sup_{a > 0} \left\{ ay - \int \Lambda(af(x)) dx \right\},
\]

\[
\mathcal{J}_2(y) = \inf_{u \in \mathcal{B}_1(A)} \left\{ I_A(u) : \int_{\mathcal{Q}(A)} f(x)u(x) dx \geq y \right\}.
\]

Note that

\[
\mathcal{J}_2(y) = \inf_{u \in \mathcal{B}_1(A)} \sup_{a > 0} \left\{ I_A(u) + a \left( y - \int_{\mathcal{Q}(A)} f(x)u(x) dx \right) \right\}.
\]

Inverting the infimum and the supremum in the preceding expression, we obtain \( \mathcal{J}_1(y) \). Hence \( \mathcal{J}_2(y) \geq \mathcal{J}_1(y) \), and \( \mathcal{J}_2(y) \) and \( \mathcal{J}_1(y) \) are dual optimization problems.

Since \( \forall a > 0, \Lambda(a) \leq aM \), it follows from the definition of \( \mathcal{J}_1 \) that \( \forall a > 0, y \leq \frac{\mathcal{J}_1(y)}{a} + M \). Hence if \( \mathcal{J}_1(y) < \infty \), then \( y \leq M \). In other words, for \( y > M \), \( +\infty = \mathcal{J}_1(y) \leq \mathcal{J}_2(y) \).

For \( y < M \), note that \( \mathcal{J}_2(y) \leq I_A(u \equiv y) = |\mathcal{Q}(A)|H(y) < \infty \). Moreover, the infimum in \( \mathcal{J}_2 \) is actually a minimum. Actually, \( \mathcal{C}_y = \{ u \in \mathcal{B}_1(A) ; \langle f, u \rangle \geq y \} \) is compact in weak topology. Indeed, let \((u_n)\) be a sequence in \( \mathcal{C}_y \). It follows from Banach-Alaoglu theorem that \( u_n \) converges weakly to \( u \in \mathcal{B}_1(A) \). Hence \( \langle f, u_n \rangle \to \langle f, u \rangle \), and \( u \in \mathcal{C}_y \). \( I_A \) being lower semi-continuous, the infimum of \( I_A \) on \( \mathcal{C}_y \) is a minimum, as soon as \( \mathcal{C}_y \) is not empty, which is actually the case, since

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We denote by \( \tilde{u} \) and we can as well assume that

\[
\text{Theorem 6.7 in [14]}\] is satisfied by \( u \equiv z \), for \( z \in [y; M] \). The identity between \( J_1(y) \) and \( J_2(y) \) follows then from standard results in convex optimization.

We have thus proved that \( J_1 = J_2 \), except on \( y = M \). But, note that \( J_1 \) and \( J_2 \) are obviously increasing on \( \mathbb{R}^+ \). \( J_1 \) is clearly lower semi-continuous, and the same is true for \( J_2 \). Indeed, let \( (y_n) \) a sequence converging to \( y \), and let \( L \) be such that \( \liminf_{n \to \infty} J_2(y_n) < L \). We can then find a (sub)sequence \((u_n) \) in \( B_1(A) \), with \( \langle f, u_n \rangle \geq y_n \), and \( I_A(u_n) < L \) for sufficiently large \( n \). \( B_1(A) \) being weakly compact, there exists \( u \in B_1(A) \) such that \( u_n \) converges weakly to \( u \). Hence \( \langle f, u_n \rangle \to \langle f, u \rangle \), so that \( u \in C_y \). Therefore, \( J_2(y) \leq I_A(u) \leq \liminf_{n \to \infty} I_A(u_n) \) by lower continuity of \( I_A \). Hence \( J_2(y) \leq L \) for any \( L > \liminf_{n \to \infty} J_2(y_n) \).

By lower semi-continuity and monotonicity, we have that

\[
J_1(M) \leq \liminf_{y_n \nearrow M} J_1(y_n) \leq \limsup_{y_n \nearrow M} J_1(y_n) \leq J_1(M),
\]

and the same holds true for \( J_2 \). Hence \( J_1(M) = J_2(M) \).

**Proof of Lemma 3.7.**

We can assume that there is \( L < \infty \) such that

\[
\limsup_{A \to \infty} \inf_{z \geq y} \tilde{J}_{\delta,A}(z) = L.
\]

For sufficiently large \( A \), let \( \mu_A, u_A \in \mathcal{M}_1(\mathcal{F}(A)) \times \mathcal{B}_1(Q(A)) \) be such that

\[
I_A(u_A) + \mathcal{L}_A(\mu_A) < L + \frac{1}{A} \quad \text{and} \quad \int_{Q(A)} u_A \psi^A \, \mu_A \geq y.
\]  

(68)

Note that changing \( u_A \) on \( \partial Q(A) \) does not change anything in the above expression, and we can as well assume that \( u_A \equiv 0 \) on \( \partial Q(A) \). We extend \( u_A \) outside \( Q(A) \) by periodization. Following Lemma 3.5 of [14], it is possible to translate both \( \mu_A \) and \( u_A \) by the same amount –we still call \( \mu_A, u_A \) the translates– in such a way that

\[
\mu_A(\partial Q(A)) \leq \frac{2d}{\sqrt{A}},
\]  

(69)

where \( \partial Q(A) = \bigcup_{i=1}^{d} \left\{ -\frac{A}{2} \leq x_i \leq -\frac{A}{2} - \sqrt{A} \right\} \cup \{ \frac{A}{2} - \sqrt{A} \leq x_i \leq \frac{A}{2} \}. \) And there is a measure \( \tilde{\mu}_A \) with Dirichlet boundary on \( Q_0(A) \) such that

\[
|\mathcal{L}_A(\mu_A) - \mathcal{L}_A(\tilde{\mu}_A)| \leq \frac{2d}{\sqrt{A}}, \quad \text{and} \quad \tilde{\mu}_A |_{Q(A) \setminus \partial_A Q(A)} = \mu_A |_{Q(A) \setminus \partial_A Q(A)}.
\]  

(70)

We denote by \( \tilde{u}_A \) the function vanishing on \( \partial_A Q(A) \), and equal to \( u_A \) on \( Q(A) \setminus \partial A Q(A) \). Note that \( I_A(\tilde{u}_A) \leq I_A(u_A) \) and

\[
|\int \psi^A \ast u_A d\mu_A - \int \psi^A \ast \tilde{u}_A d\tilde{\mu}_A| \leq |\psi^A \ast u_A|_{L^\infty} \mu_A(\partial A Q(A)) \leq \frac{2d}{\sqrt{A}}
\]  

(71)
Thus, \( \int \psi \ast \bar{u} \, d\mu \geq y - \epsilon \) for \( \epsilon > 2d/\sqrt{A} \), and \( \mu, \bar{u} \in M_1^0(\mathbb{R}^d) \times \mathcal{B}_1 \). This completes the proof.

\[ \]  

Proof of Lemma 4.2

\[
\mathbb{P} \left[ \min_{1 \leq k \leq N} \{ \lambda \left( \alpha \psi \ast \xi_r, Q_k(A) \right) \} \leq x \right] \leq \sum_{1 \leq k \leq N} \mathbb{P} \left[ \lambda \left( \alpha \psi \ast \xi_r, Q_k(A) \right) \leq x \right].
\]

Note that if \( A \) is a multiple integer of \( 1/r \), all the random variables appearing in the sum have the same law. Since \( Q_k(A) \subset \left[ \frac{|kA|}{r} + Q\left( \frac{|kA|}{r} + \frac{2}{r} \right) \right] \), we have for any \( r > 2 \),

\[
\mathbb{P} \left[ \min_{1 \leq k \leq N} \{ \lambda \left( \alpha \psi \ast \xi_r, Q_k(A) \right) \} \leq x \right] \\
\leq \left( \frac{B + A}{A} \right)^d \mathbb{P} \left[ \lambda \left( \alpha \psi \ast \xi_r, Q\left( \frac{|kA|}{r} + \frac{2}{r} \right) \right) \leq x \right] \\
\leq \left( \frac{B + A}{A} \right)^d \mathbb{P} \left[ \lambda \left( \alpha \psi \ast \xi_r, Q(A + 1) \right) \leq x \right]
\]

If we impose the relation \( \tau = \exp(d^d) \), the result follows from the LDP for \( \xi_r \) (lemma 2.1), since by Lemma 5.2, \( \{ u \in \mathcal{B}_1(A + 1), \lambda \left( \alpha \psi \ast u, Q(A + 1) \right) \leq x \} \) is closed.

Proof of Lemma 4.3.

For any \( n \in \mathbb{N} \), let us partition \( Q(A) \) into \( N_n \) dyadic cubes of order \( n \), denoted by \( I_j^{(n)} \). Let

\[
\mathcal{D}_n \triangleq \left\{ \sum_{j=1}^{N_n} \frac{k_j}{2^n} \mathbb{1}_{I_j^{(n)}}, k_j \in [-2^n, 2^n] \cap \mathbb{Z} \right\}, \quad \text{and} \quad \mathcal{D} \triangleq \cup_n \mathcal{D}_n.
\]

For any real \( x \), let \( \lfloor x \rfloor \) be the nearest integer of \( x \) in the interval \( [0, x] \), i.e.

\[
\lfloor x \rfloor = \left\{ \begin{array}{ll}
k & \text{if } 0 \leq k \leq x < k + 1, \\
k & \text{if } k - 1 < x \leq k \leq 0.
\end{array} \right.
\]

We associate to any \( u \in \mathcal{B}_1(A) \), the function \( u_n \) of \( \mathcal{D}_n \) defined by

\[
u_n \triangleq \sum_{j=1}^{N_n} \frac{2^n \bar{u}_j^{(n)}}{2^n} \mathbb{1}_{I_j^{(n)}} , \quad \text{where} \quad \bar{u}_j^{(n)} \triangleq \frac{1}{|I_j^{(n)}|} \int_{I_j^{(n)}} u(x) \, dx.
\]

Since \( H \) is convex, increasing on \( \mathbb{R}^+ \), decreasing on \( \mathbb{R}^- \), we get

\[
I_A(u_n) = \sum_{j=1}^{N_n} H \left( \frac{2^n \bar{u}_j^{(n)}}{2^n} \right) |I_j^{(n)}| \leq \sum_{j=1}^{N_n} H(\bar{u}_j^{(n)}) |I_j^{(n)}| \leq I_A(u).
\]

Moreover,

\[
|\langle \psi \ast \mu, u_n \rangle - \langle \psi \ast \mu, u \rangle |
\leq \sum_{j=1}^{N_n} \left( \frac{2^n \bar{u}_j^{(n)}}{2^n} - \bar{u}_j^{(n)} \right) \int_{I_j^{(n)}} \psi \ast \mu(x) \, dx + \left| \sum_{j=1}^{N_n} \int_{I_j^{(n)}} \psi \ast \mu(x)(\bar{u}_j^{(n)} - u(x)) \, dx \right|
\leq \frac{1}{2^n} + \left| \sum_{j=1}^{N_n} \int_{I_j^{(n)}} u(x) \int_{I_j^{(n)}} (\psi \ast \mu(y) - \psi \ast \mu(x)) \, dy \right|
\leq \frac{1}{2^n} + \omega(\psi \ast \mu, \frac{1}{2^n}) \int_{Q(A)} u(x) \, dx,
\]

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where $\omega(\psi_\delta \ast \mu, \frac{1}{n})$ is the modulus of continuity of $\psi_\delta \ast \mu$ on $Q(A)$. ■

**Lemma 5.1** $l$ is concave and continuous, and takes negative values. Moreover, $l(0) = 0$. $J$ is convex and lower semi-continuous, increasing on $\mathbb{R}^+$, decreasing on $\mathbb{R}^-$. $J(0) = 0$, $J = \infty$ outside $[m, M]$, and $J$ is finite on $]m, M[$.

**Proof.** $l$ is concave as the infimum of affine functions. $I(0) = 0 \leq d$. Hence, $l(\alpha) \leq \lambda(0, \mathbb{R}^d) = 0$. Moreover, $l(0) = \lambda(0, \mathbb{R}^d) = 0$. Since $H(y) = +\infty$ for $y \notin [m, M]$, when $u$ is such that $I(u) \leq d$, one also has $m \leq u(x) \leq M$ d-a.s.. It follows easily that

$$-\alpha M \leq l(\alpha) \leq 0 \text{ for } \alpha \geq 0, \quad \text{and} \quad -\alpha m \leq l(\alpha) \leq 0 \text{ for } \alpha \leq 0. \quad (72)$$

The continuity of $l$ is then a consequence of its concavity, and of the fact that it is everywhere finite.

Now, $J$ is lower semi-continuous and convex as supremum of affine functions. $J(y) \geq 0, y + l(0) = 0$. $J(0) = \sup_{\alpha \in \mathbb{R}} l(\alpha) = 0$. One deduces from (72) that

$$J(y) \geq \max \left\{ \sup_{\alpha \leq 0} \{ \alpha(y - m) \}, \sup_{\alpha \geq 0} \{ \alpha(y - M) \} \right\} \geq +\infty \text{ for } y \notin [m, M]$$

Let us prove that $J$ is finite on $]m, M[$. Since $J \leq J_1$, it is enough to prove that $J_1$ is finite on $]m, M[$. But for any $y \in [m, M]$, $H(y) < \infty$, and one can find $\epsilon > 0$ such that $H(y)e^d < d$. Thus, $u \triangleq y \mathbb{I}_{Q(\epsilon)}$ is such that $I(u) < d$, so that $J_1(y) \leq L(\mu)$ for any $\mu \in \mathcal{M}_1(Q(\epsilon))$.

It remains to prove the monotonicity of $J$. For $y \in \mathbb{R}^+$, $\sup_{\alpha \leq 0} \{ \alpha y + l(\alpha) \} \leq 0 \leq J(y)$. Hence $J(y) = \sup_{\alpha \geq 0} \{ \alpha y + l(\alpha) \}$, and $J$ is increasing on $\mathbb{R}^+$.

**Lemma 5.2.**

\forall A > 0, \forall \delta > 0, \forall x \in \mathbb{R}, \forall \alpha \in \mathbb{R}, \{ u \in \mathcal{B}_1(A) ; \lambda(\alpha \psi_\delta u, Q(A)) \leq x \} \text{ is compact in weak topology.}

**Proof.** Since $\mathcal{B}_1(A)$ is weakly compact, it is enough to prove that $u \in \mathcal{B}_1(A) \mapsto \lambda(\alpha \psi_\delta u, Q(A))$ is lower semi-continuous. Let then $(u_n)$ be a sequence in $\mathcal{B}_1(A)$ weakly converging to $u$, and let $L > \liminf_{n \to \infty} \lambda(\alpha \psi_\delta u_n, Q(A))$. By definition of $\lambda(\alpha \psi_\delta u_n, Q(A))$, one can then find a (sub)sequence of probability measures $(\mu_n) \in \mathcal{M}_1(Q(A))$ such that for sufficiently large $n$, $\mathcal{L}(\mu_n) + \alpha \langle \mu_n, \psi_\delta u_n \rangle < L$. For such $n$, $\mathcal{L}(\mu_n) \leq L + |\alpha|$, and there exists $\mu \in \mathcal{M}_1(Q(A))$, and a subsequence $(n_k)$ such that $\mu_{n_k}$ converges weakly to $\mu$. It follows then from Lemma 3.3, and the lower semi-continuity of $\mathcal{L}$ that

$$\lambda(\alpha \psi_\delta u, Q(A)) \leq \mathcal{L}(\mu) + \alpha \langle \mu, \psi_\delta u \rangle \leq \liminf_{k \to \infty} \mathcal{L}(\mu_{n_k}) + \alpha \langle \mu_{n_k}, \psi_\delta u_{n_k} \rangle \leq L.$$ 

The proof is completed as $L$ tends to $\liminf_{n \to \infty} \lambda(\alpha \psi_\delta u_n, Q(A))$. ■
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