Boundary Sine-Gordon Interactions at the Free Fermion Point

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We study bosonization of the sine-Gordon theory in the presence of boundary interactions at the free fermion point. In this way we obtain the boundary S-matrix as a function of physical parameters in the boundary sine-Gordon Lagrangian. The boundary S-matrix can be matched onto the solution of Ghoshal and Zamolodchikov, thereby relating the formal parameters in the latter solution to the physical parameters in the lagrangian.
1. Introduction

It is now understood that for the integrable quantum field theories in two dimensions, certain boundary interactions which preserve the integrability are possible\[^{[1]}\]. Apart from the theoretical interest, these models have interesting applications for example to 1D impurity problems \[^{[2]}\] and edge excitations in fractional quantum Hall states \[^{[3],[4]}\].

In \[^{[1]}\], boundary scattering matrices were computed for the sine-Gordon model. There, a general solution to the boundary Yang-Baxter equation and the crossing-unitarity conditions was obtained which depends on two formal parameters. An important open problem in this work was the relation of these formal parameters to the physical parameters in the lagrangian. In this paper we obtain this relation at the free fermion point of the sine-Gordon theory, and see that even here this relation is somewhat non-trivial\[^{1}\]. (In applications to the fractional quantum Hall effect, the free fermion point corresponds to filling fraction $\nu = 1/2$.) Our computation involves understanding bosonization in the presence of boundary interactions, which to our knowledge has not been studied before.

2. Bosonization with Boundary Interactions

The sine-Gordon theory with an integrable boundary interaction is defined by the Euclidean action

$$S = \frac{1}{4\pi} \int dx dt \left( \frac{1}{2} \left( \partial_z \Phi \partial_{\bar{z}} \Phi - 4\lambda \cos \hat{\beta} \Phi \right) - \frac{g}{4\pi} \int dt \cos \left( \frac{\hat{\beta}(\Phi - \phi_0)/2}{2} \right) \right), \quad (2.1)$$

where $z = (t + ix)/2$, $\bar{z} = (t - ix)/2$. Here, $\lambda$ and $g$ are dimensionful parameters, $\hat{\beta}$ is the coupling and $\phi_0$ is a constant parameter\[^{2}\]. The boundary is taken to be the time axis $-\infty < t < \infty$ at $x = 0$.

One may view the above action as a boundary perturbation of a conformal field theory. Conformal field theories with boundary conditions that preserve conformal symmetry were studied by Cardy\[^{[6]}\]. Setting $g = \lambda = 0$, and requiring the boundary terms to vanish in the variation of the resulting conformal action yields the “free” boundary condition: $\partial_z \Phi = \partial_{\bar{z}} \Phi$ at $x = 0$. In the conformal limit, $\Phi = \phi(z) + \bar{\phi}(\bar{z})$, and this implies

$$\text{free : } \phi(z) = \bar{\phi}(\bar{z}) - \sigma \quad (x = 0), \quad (2.2)$$

\[^{1}\] After this work was completed we learned that similar results were obtain in \[^{[5]}\].

\[^{2}\] $\hat{\beta}$ is related to the conventional coupling $\beta$ by $\hat{\beta} = \beta / \sqrt{4\pi}$, and $\phi_0$ in this paper differs from the conventions in \[^{[1]}\] by $\phi_0 \to \sqrt{4\pi} \phi_0$. 

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where $\sigma$ is a constant which represents data not specified by the action (2.1).

Consider now the limit $g \to \infty$, where the bulk is still conformal with $\lambda = 0$. In this limit, vanishing of the boundary terms which arise upon varying the action requires $\sin(\tilde{\beta}(\Phi - \phi_0)/2) = 0$. This preserves conformal invariance, and in terms of the chiral components of the scalar field implies the ‘fixed’ boundary condition:

$$\text{fixed : } \phi(z) = -\overline{\phi}(\overline{z}) + \phi_0 + \frac{2\pi n}{\tilde{\beta}} \quad (x = 0), \quad (2.3)$$

where $n$ is an integer.

For arbitrary $g$, the theory interpolates between the above free and fixed boundary conditions. This model was further studied in [7], and when $\lambda = 0$ studied in [8]. Interestingly, when $\tilde{\beta} = \sqrt{2}$, the model is conformally invariant for all $g$ (when $\lambda = 0$)[9].

It is well-known that at $\tilde{\beta} = 1$, the bulk theory is equivalent to free massive charged fermions[10][11]. We now describe how to extend this bosonization to include the boundary interaction. Consider first the situation where the bulk is massless. Let $\psi_{\pm}(z)$, $\overline{\psi}_{\pm}(\overline{z})$ be the left and right chiral components of the Dirac fermion with $U(1)$ charge $\pm 1$. The bosonisation relations read $\psi_{\pm}(z) = \exp(\pm i\phi(z))$, $\overline{\psi}_{\pm}(\overline{z}) = \exp(\mp i\overline{\phi}(\overline{z}))$. With these bosonisation relations, $\lambda$ is identified with $-m$, the soliton mass. The relative minus sign in the exponent for left versus right comes from the fact that the $U(1)$ charge in the massless limit is $-i(\oint dz \frac{1}{2\pi i} \partial_z \phi - \frac{1}{2\pi i} \partial_{\overline{z}} \overline{\phi})$. Thus, in terms of the fermions, the fixed and free boundary conditions read at $x = 0$:

$$\text{free : } \psi_{\pm} = e^{\mp i\phi} \overline{\psi}_{\mp}$$

$$\text{fixed : } \psi_{\pm} = e^{\pm i\phi_0} \overline{\psi}_{\mp} \quad . \quad (2.4)$$

One sees that free boundary conditions break $U(1)$ symmetry whereas fixed boundary conditions preserve it.

We formulate the fermionic action as a perturbation of the free boundary condition. The action which enforces the free boundary condition on the fermions is the following:

$$S_{\text{free}} = -\frac{1}{4\pi} \int dx dt \frac{1}{2} \left( \psi_+ \partial_z \psi_- + \psi_- \partial_z \psi_+ + \overline{\psi}_+ \partial_{\overline{z}} \overline{\psi}_- + \overline{\psi}_- \partial_{\overline{z}} \overline{\psi}_+ \right)$$

$$- \frac{i}{8\pi} \int dt \left( e^{i\sigma} \psi_+ \overline{\psi}_+ + e^{-i\sigma} \psi_- \overline{\psi}_- \right) . \quad (2.5)$$

$^3$ These bosonisation relations depend explicitly on the exact form of the Dirac action. However with these conventions, charge conjugation is implemented simply by taking $\Phi \to -\Phi$.  

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Namely, the boundary terms in the variation of the above action yield (2.4).

One can now express the boundary perturbation in terms of the fermion fields. The chiral components of the free boson have the following standard mode expansions in the conformal limit:

\[-i\phi(z) = \sum_{n \neq 0} \frac{\alpha_n}{n} z^{-n} - \alpha_0 \log z - \tilde{\alpha}_0\]
\[-i\tilde{\phi}(\bar{z}) = \sum_{n \neq 0} \frac{\bar{\alpha}_n}{n} \bar{z}^{-n} - \bar{\alpha}_0 \log \bar{z} - \bar{\tilde{\alpha}}_0\]

with \([\alpha_n, \alpha_m] = [\bar{\alpha}_n, \bar{\alpha}_m] = n\delta_{n,-m}, \ [\alpha_0, \tilde{\alpha}_0] = [\bar{\alpha}_0, \bar{\tilde{\alpha}}_0] = 1\). Since we view the interpolating theory as a perturbation of the free boundary condition, \(\phi, \tilde{\phi}\) must satisfy \(\partial_z \phi(z) = \partial_{\bar{z}} \tilde{\phi}(\bar{z})\) at \(x = 0\), which implies \(\alpha_n = \bar{\alpha}_n\), leaving no constraints on \(\bar{\alpha}_0, \bar{\tilde{\alpha}}_0\). On the boundary one has:

\[\Phi = \phi(z) + \tilde{\phi}(\bar{z}) = 2\phi - i(\tilde{\alpha}_0 - \bar{\alpha}_0)\]
\[= 2\tilde{\phi} + i(\bar{\alpha}_0 - \tilde{\alpha}_0).\]

Therefore at \(x = 0\),

\[\cos\left(\frac{\Phi - \phi_0}{2}\right) = (\psi_+ a_- + \overline{\psi}_- a_+) e^{-i\phi_0/2} + (a_+ \psi_- + a_- \overline{\psi}_+) e^{i\phi_0/2}\]

where \(a_{\pm}\) are zero mode operators:

\[a_{\pm} = \frac{1}{4} \exp(\pm i(\pi/2 + \phi(z) - \overline{\phi}(\bar{z}))/2) = \frac{1}{4} \exp(\pm (i\pi/2 + \alpha_0 - \bar{\alpha}_0)/2).\]

The additional factors of \(\pi/2\) arise in combining the \(\psi\)’s and the \(a\)’s via \(\exp(A) \exp(B) = \exp(1/2[A, B]) \exp(A + B)\). This leads us to consider the complete action

\[S = S_{\text{free}} + S_{\text{mass}} + S_{\text{int}}.\]

where the interpolating term is

\[S_{\text{int.}} = -\frac{g}{4\pi} \int dt \left[ (\psi_+ a_- + \overline{\psi}_- a_+) e^{-i\phi_0/2} + (a_+ \psi_- + a_- \overline{\psi}_+) e^{i\phi_0/2} \right] - \frac{1}{2\pi} \int dt \ a_+ \partial_+ a_-\]

\[\quad (2.11)\]

See for example [12]
and $S_{\text{mass}}$ is the bulk mass term $-\frac{im}{4\pi} \int dxdt (\psi_- \bar{\psi}_+ - \bar{\psi}_- \psi_+)$. The action (2.10) is completely analogous to the Ising case considered in [1]. From the action and the dimension of the fermion fields one sees that $g$ has dimensions of $\sqrt{\text{mass}}$.

Varying the action with respect to the fermion fields and the zero modes $a_{\pm}$, the boundary terms, upon elimination of the zero modes, yield the following interpolating boundary conditions at $x = 0$:

$$
\psi_+ + e^{i(\phi_0 - \sigma)} \psi_- - e^{-i\sigma} \bar{\psi}_- - e^{i\phi_0} \bar{\psi}_+ = 0
$$

$$
i\partial_t (\psi_- - e^{i\sigma} \bar{\psi}_+) - g^2 (\psi_+ e^{i\phi_0} - \psi_-) = 0
$$

$$
i\partial_t (\psi_+ - e^{-i\sigma} \bar{\psi}_-) + g^2 (\psi_+ - e^{i\phi_0} \bar{\psi}_+) = 0.
$$

By taking the $g \to 0$ ($g \to \infty$) limit, we recover the free (fixed) boundary conditions.

3. Derivation of the Boundary S-Matrix

Generally, given a theory with particles $A^\dagger_a(\theta)$, an integrable boundary perturbation produces scattering via the boundary S-matrix, $R_{ab}(\theta)$:

$$
A^\dagger_a(\theta)B = R_{ab}(\theta)A^\dagger_b(-\theta)B,
$$

where $B$ is the boundary operator. In our case, the theory has two particles, $A^\dagger_+(\theta)$ (solitons) and $A^\dagger_-(\theta)$ (anti-solitons), and the above equation reads

$$
A^\dagger_+(\theta)B = P^+(\theta)A^\dagger_+(-\theta)B + Q^+(\theta)A^\dagger_-(-\theta)B;
$$

$$
A^\dagger_-(\theta)B = Q^-(\theta)A^\dagger_+(-\theta)B + P^-(\theta)A^\dagger_-(-\theta)B.
$$

$P^+(\theta)$ and $P^-(\theta)$ represent soliton-soliton and anti-soliton-anti-soliton scattering whereas $Q^+(\theta)$ and $Q^-(\theta)$ represent soliton-anti-soliton scattering. Because the boundary interaction, in general, is not U(1) preserving, the appearance of $Q^+(\theta)$ and $Q^-(\theta)$ is expected. Only when $g \to \infty$ should $Q^+(\theta)$ and $Q^-(\theta)$ vanish.

To derive the pieces of the boundary S-matrix, we interpret the boundary conditions in (2.12) to vanish when acting on $B$:

$$
\left[\psi_+ + e^{i(\phi_0 - \sigma)} \psi_- - e^{-i\sigma} \bar{\psi}_- - e^{i\phi_0} \bar{\psi}_+\right] B = 0 \quad \text{etc.}
$$
We then substitute the following mode expansions for the fermions:

\[ \psi_+ = \sqrt{m} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi i} e^{\theta/2} \left( A_-(\theta) e^{-m(ze^\theta + ze^{-\theta})} - A_+^\dagger(\theta) e^{m(ze^\theta + ze^{-\theta})} \right), \]

\[ \bar{\psi}_+ = -i\sqrt{m} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} e^{-\theta/2} \left( A_-(\theta) e^{-m(ze^\theta + ze^{-\theta})} - A_+^\dagger(\theta) e^{m(ze^\theta + ze^{-\theta})} \right), \]  

where \( \psi_+ = \psi_-^\dagger \) and \( \bar{\psi}_+ = \bar{\psi}_-^\dagger \). The \( A \)'s satisfy the following anti-commutation relations:

\[ \{ A_+(\theta), A_+^\dagger(\theta') \} = \{ A_-(\theta), A_-^\dagger(\theta') \} = 4\pi^2 \delta(\theta - \theta'). \]  

The normalization of the mode expansions is fixed by insisting the two point functions (in the bulk) satisfy

\[ \langle 0 | \bar{\psi}_-(z, \bar{z}) \psi_+ | 0 \rangle = -2imK_0(mr), \]

\[ \langle 0 | \bar{\psi}_-(z, \bar{z}) \psi_+ | 0 \rangle = 2m \sqrt{\frac{\bar{z}}{z}} K_1(mr), \]  

where \( r/2 = \sqrt{z\bar{z}} \) and the \( K \)'s are standard modified Bessel functions. Performing the substitutions, we find

\[ P^+(\theta) = \left( \cosh(\theta) - \frac{\gamma}{2} \cosh(\theta + i\phi_0) \right) / D(\theta), \]

\[ P^-(\theta) = \left( \cosh(\theta) - \frac{\gamma}{2} \cosh(\theta - i\phi_0) \right) / D(\theta), \]

\[ Q^+(\theta) = -\frac{i}{2} e^{-i\sigma} \sinh(2\theta) / D(\theta), \]

\[ Q^-(\theta) = -\frac{i}{2} e^{i\sigma} \sinh(2\theta) / D(\theta), \]  

where \( D(\theta) \) and \( \gamma \) are given by

\[ \gamma = g^2/m, \]

\[ D(\theta) = i\gamma \cosh \left( \frac{\theta + i\phi_0 + i\pi/2}{2} \right) \sinh \left( \frac{\theta - i\phi_0 - i\pi/2}{2} \right) - \cosh^2(\theta). \]  

We see that \( Q^+(\theta) \) and \( Q^-(\theta) \) differ by a phase. To remove the phase, we apply a U(1) gauge transformation to \( A_+^\dagger \) and \( A_- \):

\[ A_+^\dagger \to e^{-i\sigma/2} A_+^\dagger, \quad A_-^\dagger \to e^{i\sigma/2} A_-^\dagger. \]  

This leaves \( P^+(\theta) \) and \( P^-(\theta) \) invariant, but takes \( Q^+(\theta) \to e^{i\sigma} Q^+(\theta) \) and \( Q^-(\theta) \to e^{-i\sigma} Q^-(\theta) \). Then

\[ Q^+(\theta) = Q^-(\theta) = -\frac{i}{2} \sinh(2\theta) / D(\theta). \]
In what follows, we assume this gauge choice has been made.

If \( \phi_0 = 0 \), we expect charge conjugation symmetry to be restored, as indeed it is. We also see that the R-matrix is independent of the sign of g. Changing the sign of g is equivalent to the shift \( \phi_0 \to \phi_0 + 2\pi \). But the subsequent shift \( \Phi \to \Phi + 2\pi \) restores the boundary term and leaves the bulk terms unchanged, leading to the independence from the sign of g.

Taking the appropriate limits, we can find the boundary S-matrix in the free and fixed cases. For the free case we find

\[
\text{free : } P^\pm(\theta) = -\text{sech}(\theta), \quad Q^\pm(\theta) = i \tanh(\theta).
\]

In this limit, charge conjugation symmetry, \( \Phi \to -\Phi \), is respected. As a result, \( P^+(\theta) \) equals \( P^-(\theta) \). For the fixed case we obtain

\[
\text{fixed : } P^\pm(\theta) = -\cosh\left(\frac{\theta \pm i\phi_0 - i\pi/2}{2}\right) / \cosh\left(\frac{\theta \mp i\phi_0 + i\pi/2}{2}\right), \quad Q^\pm(\theta) = 0.
\]

As expected the soliton-antisoliton scattering amplitudes vanish in this limit where the boundary interaction preserves the U(1) charge.

4. Analytic Structure of the Boundary S-Matrix

Like the bulk S-matrix, the poles of the boundary S-matrix, R, provide information on bound states. However in the case of R, the situation is more complicated. Poles in R can either be indicative of bulk bound states interacting with the boundary or of boundary bound states, which are effectively excitations of the ground boundary state. In the former case, poles in R will both be found at \( \theta = i\pi/2 \) and at \( i\pi/2 - \theta_b \), where \( \theta_b \) is the location of the pole in S corresponding to the bulk bound state. Such poles necessarily imply the addition of zero-momentum states to the boundary state \( |B\rangle \). In the latter case, the poles may appear anywhere in the region \( 0 \leq \theta \leq i\pi/2 \). If the pole in this case appears at \( \theta = i\pi/2 \), a zero-momentum state will, as before, be found in \( |B\rangle \). Because the bulk S-matrix at the free-fermion point has no pole structure, we are only faced with the latter situation.
The poles in the range $0 < \theta < i\pi/2$ appearing in the free-fermion boundary S-matrix depend on the parameters $\gamma$ and $\phi_0$. There are four areas in this parameter space:

\begin{enumerate}
  \item $\gamma/2 \cos \phi_0 > 1$, $\cos \phi_0 < 1$: \quad $u = \sin^{-1} \left( \gamma/4 - (\gamma^2/16 - \gamma/2 \cos \phi_0 + 1)^{1/2} \right)$; 
  \item $\gamma/2 \cos \phi_0 > 1$, $\cos \phi_0 = 1$, $\gamma < 1$: \quad $u = \sin^{-1} (\gamma/2 - 1)$; 
  \item $\gamma/2 \cos \phi_0 > 1$, $\cos \phi_0 = 1$, $\gamma \geq 1$: \quad no poles; 
  \item $0 < \gamma/2 \cos \phi_0 \leq 1$: \quad no poles; 
\end{enumerate}

(4.1)

where $u = -i\theta$ is the location of the poles. In the free and fixed cases, this pole structure reduces to

\begin{align*}
\text{fixed:} \quad & u = \pi/2 - |\phi_0|, \quad 0 < |\phi_0| < \pi/2; \\
\text{free:} \quad & u = \pi/2.
\end{align*}

(4.2)

In all cases the poles are simple, and we are not burdened with interpreting more complicated resonance phenomena.

The energy, $E_\alpha$, of these boundary bound states, $B_\alpha$, is given by

$$E_\alpha - E_0 = m \cos(u_\alpha),$$

(4.3)

where $m$ is the soliton mass, $E_0$ is the energy of the boundary ground state, and $u_\alpha$ is the location of the pole in the boundary S-matrix corresponding to the state. We expect the boundary bound state $B_\alpha$ to be stable if $E_\alpha - E_0 < m$; that is, if there is no chance of the boundary emitting a zero-momentum soliton. We thus see all the boundary states to be stable.

When boundary bound states are present, scattering off the boundary (3.1) needs to be described by the more general equation

$$A^\dagger_\alpha(\theta)B_\alpha = R^{b\beta}_{a\alpha}(\theta)A^\dagger_b(-\theta)B_\beta,$$

(4.4)

where $\{B_\alpha\}$ are all the possible boundary states. By energy conservation, we expect $R^{a\alpha}_{b\beta} = 0$ if $E_\alpha \neq E_\beta$. Thus for all but the free case and the fixed case with $\phi_0 = 0$, the boundary states are not coupled. With such independence, we expect $R^{b\alpha}_{a\alpha} = R^{b\beta}_{a\beta}$ as the constraints among the free-fermi fields (2.12) must annihilate $B_\alpha$ for every $\alpha$.

The interpretation of the boundary states in these non-degenerate cases is straightforward. Classically, the asymptotic behaviour of the ground state in the bulk is described
by $\Phi \to (2n+1)\pi$ as $x \to -\infty$ (the bulk classical potential is $V(\Phi) = \frac{|\lambda|}{8\pi} \cos(\Phi)$; recall $\lambda = -m$ and so $\lambda$ is negative). The difference in energy between these states arises from a boundary term. Provided $\gamma$ is large enough, the boundary term forces $\Phi \sim \phi_0$ at $x = 0$. If $\phi_0$ is in the range $0 < \phi_0 < \pi/2$, the boundary bound state can be viewed as the state where $\Phi$'s asymptotic value is $-\pi$ (for the ground state the asymptotic value of $\Phi$ is $\pi$). If $-\pi/2 < \phi_0 < 0$ the situation is reversed: the ground state corresponds to $\Phi \to -\pi$ and the excited boundary state to $\Phi \to \pi$ as $x \to -\infty$. If however $\pi/2 < \phi_0 < \pi$, there is no boundary excited state: $\Phi$ is too near either of the bulk vacua values $\pm \pi$ at the boundary to drift over to the other as $x \to -\infty$.

In the fixed degenerate case (i.e. $\phi_0 = 0$), the interpretation is much the same. $\phi_0 = 0$ is exactly between the bulk ground state field configurations $\Phi = \pm \pi$. Thus $\Phi \to \pm \pi$ as $x \to -\infty$ are energetically equivalent. Because of this degeneracy there is the possibility that $R^{b_\beta}_{a\alpha}$ does not vanish if $\alpha \neq \beta$. To derive these cross-scattering matrix elements, one would presumably solve

$$f(\psi_{\pm}, \overline{\psi}_{\pm})B_\alpha = \sum_{\beta \neq \alpha} R^{b_\beta}_{a\alpha}(\theta) A^1_b(-\theta) B_\beta,$$

where $f$ is a combination of fields appearing in $(2.12)$. Thus the matrix elements diagonal in $\alpha$ and $\beta$ remain unchanged. With the presence of scattering between boundary states, the unitarity condition that must be satisfied is

$$R^{b_\beta}_{a\alpha}(\theta) R^{c_\gamma}_{b_\beta}(-\theta) = \delta_{ac} \delta_{\alpha \gamma}.$$  

(4.6)

Given that we know in our case (as is easily checked) the diagonal matrix elements satisfy unitarity by themselves, it is straightforward to check that unitarity forces $R^{a_\alpha}_{b_\beta} \propto \delta_{\alpha \beta}$. Thus we still have no scattering between boundary states. The construction of the two boundary states $B_\alpha$ in this case will include a zero momentum soliton (if the pole appears in $P_+$) or anti-soliton (if the pole appears in $P_-$).

In the free (degenerate) case the interpretation of the boundary state differs. Considering the massless limit, the twofold degenerate ground states are characterized by asymptotic behaviour $\Phi \to \pm \pi$ as $t \to \infty$ and $\Phi \to \mp \pi$ as $t \to -\infty$. These two regions of different regions of $\Phi$ are separated by a domain wall which can be thought of as a zero momentum particle present in the boundary state. This description is analogous to the description of semi-infinite Ising model at criticality in [1].

Our discussion has not yet exhausted all the poles in the physical strip $0 < \theta < i\pi$. For every pole in the range $0 < \theta < i\pi/2$, there is a corresponding pole in the range $i\pi/2 < \theta < i\pi$. These additional poles correspond to scattering in the cross channel.
5. Matching Ghoshal and Zamolodchikov’s Solution

We can now match our boundary S-matrix to that of Ghoshal and Zamolodchikov [1], thereby relating the formal parameters in the latter to the physical parameters in the action (2.10). At the free-fermion point (corresponding to the limit \( \lambda \to 1 \) in the notation of [1]), the solution of the boundary Yang-Baxter equation yields

\[
\begin{align*}
P^+(\theta) &= \cos(\xi - i\theta)R(\theta); \\
P^-(\theta) &= \cos(\xi + i\theta)R(\theta); \\
Q^+(\theta) &= -i\frac{k_+}{2}\sinh(2\theta)R(\theta); \\
Q^-(\theta) &= -i\frac{k_-}{2}\sinh(2\theta)R(\theta); \\
\end{align*}
\]

(5.1)

where \( k_\pm \) and \( \xi \) are free parameters. Using the aforementioned gauge transformation, we set \( k_+ = k_- = k \). From the results of the last section we thus see that the phase \( \sigma \) removed from \( k_\pm \) in [1] is nothing other than the phase which specifies the conformal free boundary condition.

The function \( R(\theta) \) is determined through the boundary unitarity and boundary cross-unitarity equations. In the free-fermion limit these constraints yield

\[
R(\theta) = \frac{1}{\cos \xi} \sigma(\eta, -i\theta)\sigma(i\vartheta, -i\theta),
\]

(5.2)

where the parameters \( \eta \) and \( \vartheta \) are related to \( k \) and \( \xi \) through the equations

\[
\begin{align*}
\cos(\eta) \cosh(\vartheta) &= -\frac{1}{k} \cos(\xi); \\
\cos^2(\eta) + \cosh^2(\vartheta) &= 1 + \frac{1}{k^2}.
\end{align*}
\]

(5.3)

The function \( \sigma(x, u) \) is found to be

\[
\sigma(x, u) = \cos x \left( 2\cos(\pi/4 + x/2 - u/2) \cos(\pi/4 - x/2 - u/2) \right)^{-1}.
\]

(5.4)

With some algebra one can show that this solution of Ghoshal and Zamolodchikov maps onto our derivation of the boundary scattering matrix with the following identification of the free parameters:

\[
\begin{align*}
k &= \left( 1 - \gamma \cos(\phi_0) + \gamma^2/4 \right)^{-1/2}; \\
\cos \xi &= k \left( 1 - \frac{\gamma}{2} \cos \phi_0 \right); \\
\sin \xi &= -k\frac{\gamma}{2} \sin \phi_0.
\end{align*}
\]

(5.5)

5 The denominator of (5.23) in [1] should be \( \Pi^2(x, \pi/2)\Pi^2(-x, -\pi/2) \), and (5.24) should read \( \sigma(x, u)\sigma(x, -u) = \cos^2 x[\cos(x + \lambda u) \cos(x - \lambda u)]^{-1} \).
In the limit $\gamma \to 0$, we see that $k = 1$ and $\xi = 0$ yielding in Ghoshal and Zamolodchikov’s solution, $P^+(\theta) = P^-(\theta)$ and $Q^+(\theta) = Q^-(\theta)$, in accordance with charge conjugation symmetry. In the limit of fixed boundary condition, $\gamma \to \infty$ we see that $k = 0$ and

$$\xi = \phi_0 + (2n + 1)\pi,$$

(5.6)

forcing their amplitudes $Q_{\pm}$ to vanish in accordance with U(1) charge conservation. We point out that the relationship between $\xi$ and $\phi_0$ is not in exact accordance with that conjectured in [1] (see 5.27). There, it was taken that when $\phi_0$ vanished, $\xi$ was forced to vanish in order to ensure $P^+(\theta) = P^-(\theta)$. But $\xi = \pi$ also sets $P^+(\theta) = P^-(\theta)$.

It should be emphasized that (5.6) depends strongly on the conventions used in writing the bulk Dirac action. Taking $\gamma_\mu \to -\gamma_\mu$ (where $\gamma_\mu$ are the gamma matrices used in the action) leads to the relationship as conjectured in [1], namely

$$\xi = \phi_0 + 2n\pi.$$  

(5.7)

In this case $\lambda = m$ and the poles in $R(\theta)$ shift to reflect the ground states in the bulk are now $\Phi = 2n\pi$.

For both (5.6) and (5.7), taking $\Phi \to -\Phi$ implements charge conjugation. However it is easy enough to construct Dirac actions where this action does not implement charge conjugation exactly, for example where the bosonisation relations are $\psi_{\pm} = e^{\pm i\phi}$, $\overline{\psi}_{\pm} = \pm ie^{\mp i\phi}$. In this case the relation between $\phi_0$ and $\xi$ takes the form $\phi_0 = \xi \pm \pi/2$. Conventions of this sort are used in [1]. However these authors have taken the relation between $\xi$ and $\phi_0$ as given in (5.7). This has led them to find a ‘boundary anomaly’. This anomaly should vanish when the correct relation corresponding to their conventions between $\phi_0$ and $\xi$ is used.

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