Non-linear Realization of $PSU(2, 2|4)$ on the Light-Cone

Sudarshan Ananth,
Institute for Fundamental Theory,
Department of Physics, University of Florida
Gainesville FL 32611, USA

Lars Brink,
Department of Fundamental Physics
Chalmers University of Technology,
S-412 96 Göteborg, Sweden

Sung-Soo Kim and Pierre Ramond
Institute for Fundamental Theory,
Department of Physics, University of Florida
Gainesville FL 32611, USA

Abstract

The symmetries of the $\mathcal{N} = 4$ SuperYang-Mills theory on the light-cone are discussed, solely in terms of its physical degrees of freedom. We derive explicit expressions for the generators of the $PSU(2, 2|4)$ superalgebra, both in the free theory, and to all orders in the gauge coupling of the classical theory. We use these symmetries to construct its Hamiltonian, and show that it can be written as a quadratic form of a fermionic superfield.
1 Introduction

No-go theorems show that massless fields of helicity higher than two cannot interact locally with gravity, and respect Lorentz invariance [1]. However on the light-cone, Bengtsson, Bengtsson and Brink [2] used algebraic consistency to build Lorentz-invariant cubic interactions of massless particles with arbitrary helicities. Although algebraically demanding, this light-cone approach seems natural to construct Lorentz invariant interactions, as infinite towers of massless higher-spin fields [3] have emerged in possible generalizations of $\mathcal{N} = 1$ Supergravity in eleven dimensions.

Supergravity in eleven dimensions [4], when reduced to four space-time dimensions, becomes the maximally supersymmetric $\mathcal{N} = 8$ Supergravity. Its light-cone formulation in superspace is not entirely known, as its four- and higher-point interactions have not yet been written in terms of one light-cone superfield.

A purpose of this paper is to devise algebraic techniques for finding these higher point functions. This is quite complicated, but fortunately, there is a much simpler maximally supersymmetric, but challenging theory, in which one can study these questions: $\mathcal{N} = 4$ SuperYang-Mills in four dimensions. It is very similar in structure to $\mathcal{N} = 8$ supergravity. Both are elegantly described on the light-cone by one superfield containing only physical degrees of freedom, and both oxidize [5, 6] naturally to their higher-dimensional progenitors: $\mathcal{N} = 4$ to its ten-dimensional $\mathcal{N} = 1$ parent, and $\mathcal{N} = 8$ to $\mathcal{N} = 1$ supergravity in eleven dimensions, producing superspace descriptions without auxiliary fields (although this has been shown only to first order in $\kappa$ for supergravity).

The $\mathcal{N} = 4$ Yang-Mills theory, where the four-point function is well-known, is a natural testing ground for developing algebraic tools to generate higher-point interactions. We expect that the techniques we have uncovered will enable us to determine the full $\mathcal{N} = 8$ supergravity Lagrangian.

We first review the light-cone description of the $\mathcal{N} = 4$ theory, and display the classical symmetries of its free action, which act linearly on the superfield. These include not only the superPoincaré transformations, but also the superconformal symmetries, which combine with an internal $SU(4)$ to form the $PSU(2, 2 | 4)$ superalgebra.

The action for $\mathcal{N} = 4$ Yang-Mills is well known, written in terms of a single chiral superfield that encapsulates its physical degrees of freedom: one helicity-one gauge field, four helicity one-half fermions and their conjugates, and six helicity zero scalar fields. The same light-cone formulation was used to prove its ultraviolet finiteness [7].

Space-time symmetries split into two types; kinematical symmetries which are not altered by interactions, and dynamical symmetries which are realized non-linearly on the fields. In a Lorentz invariant field theory, these dynamical symmetries are the light-cone time translation generated by the Hamiltonian, and the boosts. In supersymmetric theories, the supersymmetries also split, with the dynamical supersymmetries acting as the “square-roots” of the Hamiltonian. The same split occurs in the superconformal transformations.
The construction of these non-linear transformations was initiated in reference [2], order by order in the coupling constant. When a gauge theory is expressed solely in terms of its physical degrees of freedom, Poincaré invariance is not manifest, and remains to be checked. Bengtsson et al formed Ansätze for both boosts and Hamiltonian variations, and verified closure of the Super-Poincaré algebra to first order in the coupling constant $g$. They reproduced the well-known cubic interaction of the $\mathcal{N} = 4$ theory, and showed how closure requires several superfields linked by an antisymmetric $f^{abc}$, where $a, b, c$ label the superfields.

We complete their program to order $g^2$. We first show that the algebraic constraints fix the form of the dynamical supersymmetry transformations uniquely, with no order $g^2$ corrections. From the fact that the conformal group is simple, its kinematical symmetries, together with the form of the dynamical supersymmetry transformations, fully determine the entire $PSU(2,2|4)$ with all classical interactions included! We show how these transformations generate the complete classical Hamiltonian, including the four-point interaction. The full antisymmetry of the structure functions and their Jacobi identities is required by the algebra. In this light-cone language, implementation of the space-time symmetries requires gauge symmetries, and naturally reconstructs the required Lie algebra structures.

Finally, we show that the Hamiltonian of $\mathcal{N} = 4$ Yang-Mills can be written as a quadratic form of a fermionic superfield. This fermionic superfield is simply the dynamical supersymmetry variation of the original superfield.

In future publications, we hope to extend these techniques to derive the form of the quartic and higher-point interactions in $\mathcal{N} = 8$ supergravity.

2 Light-Cone Formulation: Review

2.1 Notation

With the space-time metric $(-,+,+,+,\ldots)$, the light-cone coordinates and their derivatives are

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3) ; \quad \partial^\pm = \frac{1}{\sqrt{2}}(-\partial_0 \pm \partial_3) ; \quad (2.1)$$

$$x = \frac{1}{\sqrt{2}}(x_1 + i x_2) ; \quad \bar{\partial} = \frac{1}{\sqrt{2}}(\partial_1 - i \partial_2) ; \quad (2.2)$$

$$\bar{x} = \frac{1}{\sqrt{2}}(x_1 - i x_2) ; \quad \partial = \frac{1}{\sqrt{2}}(\partial_1 + i \partial_2) , \quad (2.3)$$

so that

$$\partial^+ x^- = \partial^- x^+ = -1 ; \quad \bar{\partial} x = \partial \bar{x} = +1 . \quad (2.4)$$
In four dimensions, massless particles with helicity can be described by a complex field, and its complex conjugate of opposite helicity. Particles with no helicity are described by real fields.

The particle content of the $\mathcal{N} = 4$ Yang-Mills theory is best described in ten dimensions, where the $\mathcal{N} = 1$ supermultiplet contains eight vectors and eight spinors of the little group $SO(8)$. Under the decomposition

$$SO(8) \supset SO(2) \times SO(6),$$

these give

$$8_v = 6_0 + 1_1 + 1_{-1}, \quad 8_s = 4_{1/2} + 4_{-1/2},$$

where the subscripts denote the $SO(2)$ helicity, showing that the $\mathcal{N} = 4$ theory contains six scalar fields, a vector field and four spinor fields and their conjugates. To describe them in compact notation, we introduce anticommuting Grassmann variables $\theta^m$ and $\bar{\theta}^m$,

$$\{ \theta^m, \theta^n \} = \{ \bar{\theta}^m, \bar{\theta}^n \} = 0,$$

which transform as the spinor representations of $SO(6) \sim SU(4)$,

$$\theta^m \sim 4_{1/2}; \quad \bar{\theta}^m \sim \bar{4}_{-1/2},$$

where $m, n, \cdots = 1, \ldots, 4$. Their derivatives are written as

$$\bar{\partial}_m = \frac{\partial}{\partial \theta^m}; \quad \partial^m = \frac{\partial}{\partial \bar{\theta}^m},$$

and satisfy

$$\{ \partial^m, \bar{\theta}_n \} = \delta^m_n; \quad \{ \bar{\partial}_m, \theta^n \} = \delta^*_m_n.$$

### 2.2 Superfield Action

All the physical degrees of freedom of the $\mathcal{N} = 4$ theory can be captured in a single complex superfield $\Phi$.

$$\phi(y) = \frac{1}{\partial^+} A(y) + \frac{i}{\sqrt{2}} \theta^m \theta^n \overline{\chi}_{mn}(y) + \frac{1}{12} \theta^m \theta^n \theta^p \theta^q \epsilon_{mpnq} \partial^+ \bar{A}(y)$$

$$+ \frac{i}{\partial^+} \theta^m \bar{\chi}_m(y) + \frac{\sqrt{2}}{6} \theta^m \theta^n \theta^p \epsilon_{mpnq} \chi^q(y).$$

In this notation, the eight original gauge fields $A_i, i = 1, \ldots, 8$ appear as

$$A = \frac{1}{\sqrt{2}} (A_1 + i A_2), \quad \bar{A} = \frac{1}{\sqrt{2}} (A_1 - i A_2),$$

while the six scalar fields are written as antisymmetric $SU(4)$ bi-spinors.
\[ C^m 4 = \frac{1}{\sqrt{2}} (A_{m+3} + i A_{m+6}) , \quad \overline{C}^m 4 = \frac{1}{\sqrt{2}} (A_{m+3} - i A_{m+6}) , \quad (2.13) \]

for \( m \neq 4 \); complex conjugation is akin to duality,

\[ \overline{C}_{mn} = \frac{1}{2} \epsilon_{mnpq} C^{pq} . \quad (2.14) \]

The fermion fields are denoted by \( \chi^m \) and \( \bar{\chi}^m \). All these fields carry adjoint indices of the gauge group (not shown here), and are local in the modified light-cone coordinates

\[ y = (x, \bar{x}, x^+, y^- \equiv x^- - \frac{i}{\sqrt{2}} \theta^m \bar{\theta}_m) . \quad (2.15) \]

In this particular light-cone formulation called \( LC_2 \), all the unphysical degrees of freedom have been integrated out, leaving only the physical ones.

We introduce chiral derivatives,

\[ d^m = -\partial^m + \frac{i}{\sqrt{2}} \theta^m \bar{\theta}^+ ; \quad \bar{d}_n = -\bar{\partial}_n + \frac{i}{\sqrt{2}} \bar{\theta}_n \partial^+ , \quad (2.16) \]

which satisfy the anticommutation relations

\[ \{ d^m , \bar{d}_n \} = -i \sqrt{2} \delta^m_0 \partial^+ . \quad (2.17) \]

One verifies that \( \phi \) and its complex conjugate \( \bar{\phi} \) satisfy the chiral constraints

\[ d^m \phi = 0 ; \quad \bar{d}_m \bar{\phi} = 0 , \quad (2.18) \]

as well as the “inside-out” relations

\[ \bar{d}_m d^m \phi = \frac{1}{2} \epsilon_{mnpq} d^p d^q \bar{\phi} , \quad (2.19) \]

\[ d^m d^m \bar{\phi} = \frac{1}{2} \epsilon_{mnpq} \bar{d}_p \bar{d}_q \phi . \quad (2.20) \]

The Yang-Mills action is then simply

\[ \int d^4 x \int d^4 \theta d^4 \bar{\theta} \mathcal{L} , \quad (2.21) \]

where

\[ \mathcal{L} = -\bar{\phi} \frac{\Box}{\sqrt{2}} \phi + \frac{4g}{3} f^{abc} \left( \frac{1}{\partial^2} \bar{\phi}^a \phi^b \bar{\phi}^c + \frac{1}{\partial^2} \phi^a \bar{\phi}^b \bar{\phi}^c \right) \]
Grassmann integration is normalized so that \( \int d^4 \theta \theta_1 \theta_2 \theta_3 \theta_4 = 1 \), and \( f^{abc} \) are the structure functions of the Lie algebra. It is straightforward to verify that this action reproduces the component action for the \( \mathcal{N} = 4 \) theory, by simply performing the Grassmann integrals. In this form, SuperPoincaré invariance is far from obvious, the price for having eliminated the unphysical degrees of freedom.

\section{Symmetries of \( \mathcal{N} = 4 \) SuperYang-Mills}

The \( \mathcal{N} = 4 \) theory is invariant under transformations generated by the superalgebra \( PSU(2,2|4) \). Its 30 bosonic generators describe the conformal group \( SO(4,2) \) and an internal \( SU(4) \); its 32 fermionic generators consist of four supersymmetries and superconformal symmetries and their conjugates.

Some of the \( PSU(2,2|4) \) generators have been discussed in papers by Beltisky \textit{et al} [9], who have used the above formalism to study the integrability properties of the dilatation operator. In this paper we construct the full superconformal algebra in its light-cone form. Since the theory is ultraviolet finite, this symmetry survives quantization, and is therefore of great interest. The SuperPoincaré subalgebra of \( PSU(2,2|4) \) is well-known, its generators having been constructed in references [2, 5]. In the following section, we review their construction, and build the rest of the superconformal algebra (on the light supercone).

\subsection{SuperPoincaré Algebra}

The light-cone Poincaré generators split into two types, kinematical and dynamical. Dynamical generators are those which contain the “light-cone time derivative”. Three of the momenta are kinematical

\[
p^+ = -i \partial^+ , \quad p = -i \partial , \quad \bar{p} = -i \bar{\partial} ,
\]

while the fourth, the light-cone Hamiltonian, shown here without interactions,

\[
p^- = -i \frac{\partial \bar{\partial}}{\partial^+} ,
\]

is dynamical. The Lorentz generators include the kinematical transverse space rotation

\[
j = x \bar{\partial} - \bar{x} \partial + \frac{1}{2} ( \theta^p \bar{\partial}_p - \bar{\theta}_p \partial^p ) - \lambda ,
\]

where

\[
\lambda = \frac{i}{4 \sqrt{2} \partial^+} (d^p \bar{d}_p - \bar{d}_p d^p ) ,
\]

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\]

where

\[
\lambda = \frac{i}{4 \sqrt{2} \partial^+} (d^p \bar{d}_p - \bar{d}_p d^p ) ,
\]
measures the helicity of the superfield ($\lambda = +1$ for a chiral superfield and $-1$ for an anti-chiral superfield). This form ensures that the chirality constraints are preserved. Under a transverse space rotation, this generator acts as a differential operator on the chiral superfield

$$\delta \phi = i \omega j \phi, \quad \delta \bar{\phi} = -i \omega j \bar{\phi}. \quad (3.5)$$

Acting on $\phi$, the remaining kinematical generators read

$$j^+ = i x \partial^+, \quad \bar{j}^+ = i \bar{x} \partial^+, \quad j^- = i x^- \partial^+ - \frac{i}{2} (\theta^p \bar{\partial}_p + \bar{\theta}_p \partial^p) + i. \quad (3.6)$$

These generators preserve chirality, since

$$[j^+, y^+] = -i y^-, \quad [j^+, d^m] = \frac{i}{2} d^m, \quad [j^+, \bar{d}_m] = \frac{i}{2} \bar{d}_m. \quad (3.7)$$

The dynamical Lorentz generators are the boosts, given by

$$j^- = i x \partial^+ - i x^- \partial + i \left(\theta^p \partial_p - \lambda - 1\right) \partial^+, \quad \bar{j}^- = i \bar{x} \partial^+ - i x^- \bar{\partial} + i \left(\bar{\theta}_p \partial^p + \lambda + 1\right) \bar{\partial}^+. \quad (3.8)$$

They also preserve chirality because

$$[j^-, d^m] = \frac{i}{2} \bar{d}_m \frac{\partial}{\partial^+}, \quad [j^-, \bar{d}_m] = \frac{i}{2} \bar{d}_m \frac{\partial}{\partial^+}, \quad (3.9)$$

and satisfy the Poincaré algebra. In particular

$$[j^-, j^+] = -i j^+-j, \quad [j^-, j^+] = i j^- . \quad (3.10)$$

Half of the supersymmetry generators

$$q^+_m = -\partial_m + \frac{i}{\sqrt{2}} \theta^m \partial^+; \quad \bar{q}^+_n = \bar{\partial}_n - \frac{i}{\sqrt{2}} \bar{\theta}_n \partial^+, \quad (3.11)$$

are kinematical. They satisfy

$$\{ q^+_m, \bar{q}^+_n \} = i \sqrt{2} \delta^m_n \partial^+, \quad (3.12)$$

and anticommute with the chiral derivatives

$$\{ q^+_m, \bar{d}_n \} = \{ d^m, \bar{q}^+_n \} = 0. \quad (3.13)$$
The other half are dynamical, obtained by boosting the kinematical supersymmetries

\[ q^-_m \equiv i [ j^- , q^+_m ] = \frac{\partial}{\partial x^+} q^+_m , \quad \bar{q}^-_m \equiv i [ j^- , \bar{q}^+_m ] = \frac{\partial}{\partial \bar{x}^+} \bar{q}^+_m . \] (3.14)

These are the “square-roots” of the light-cone Hamiltonian, in the sense that

\[ \{ q^-_m , \bar{q}^-_n \} = i \sqrt{2} \delta^m_n \frac{\partial \bar{\theta}}{\partial x^+} . \] (3.15)

All operators have been constructed so as to be Hermitian with respect to the quadratic form

\[ (\phi , \xi ) \equiv 2i \int d^4x d^4\theta d^4\bar{\theta} \bar{\phi} \frac{1}{\partial^+} \xi , \] (3.16)

where \( \phi \) and \( \xi \) are chiral superfields. In the fully interacting theory, the kinematical superPoincaré generators do not change, still acting linearly on the superfields, but the dynamical generators act non-linearly, as we shall discuss in section 4.

### 3.2 Superconformal Algebra

In this section, we complete the construction of the \( PSU(2, 2|4) \) generators. Our procedure is to start from the simplest kinematical generator in the conformal algebra, and generate the rest by commutation.

The easiest starting point is the “plus” component of the conformal transformations,

\[ K^+ = 2i x \bar{x} \partial^+ . \] (3.17)

For ease of algebra, we continue to work at \( x^+ = 0 \), where \( K^+ \) is kinematical. Since we already know \( j^{+ -} \), the commutation relation

\[ [ K^+ , p^- ] = -2i D + 2i j^{+ -} , \] (3.18)

yields the dilatation generator,

\[ D = i \left( x^- \partial^+ - x \bar{\partial} - \bar{x} \partial - \frac{1}{2} \theta \frac{\partial}{\partial \theta} - \frac{1}{2} \bar{\theta} \frac{\partial}{\partial \bar{\theta}} \right) , \] (3.19)

which satisfies

\[ [ x , D ] = i x ; \quad [ d^m , D ] = -\frac{1}{2} d^m . \] (3.20)
By boosting $K^+$, we obtain the kinematical generators $K$ and $\bar{K}$,

$$K = i [ j^-, K^+] = 2i\bar{x} \left( x^\alpha \partial^\alpha - x \tilde{\partial} - \theta \frac{\partial}{\partial \theta} + \lambda \right), \quad (3.21)$$

$$\bar{K} = i [ \bar{j}^-, K^+] = 2i\bar{x} \left( x^\alpha \partial^\alpha - \bar{x} \partial - \bar{\theta} \frac{\partial}{\partial \bar{\theta}} - \bar{\lambda} \right), \quad (3.22)$$

where $\lambda$ is the helicity counter of the previous section. These generators do not change the chirality of the superfields on which they act, since

$$[ K, d^m ] = 0 ; \quad [ \bar{K}, d^m ] = 2i \bar{x} d^m . \quad (3.23)$$

The supersymmetry generators are now augmented by new “conformal supersymmetries”, easily obtained from their normal counterparts by

$$[ K^+, q_m^- ] = -\sqrt{2} ( i \sqrt{2} \bar{x} q_m^+ ) = -\sqrt{2} \bar{s}_m^+ . \quad (3.24)$$

These, together with their complex conjugates,

$$[ K^+, \bar{q}_n^- ] = \sqrt{2} (-i \sqrt{2} x \bar{q}_n^+ ) = \sqrt{2} \bar{s}_n^+ , \quad (3.25)$$

are kinematical. We note that $K^+$ and these supercharges should contain a dynamical term, but it is multiplied by the parameter $x^+$, which we have set to zero without loss of generality.

In a similar fashion, the dynamical conformal supersymmetries are obtained by boosting

$$s_m^- = i [ j^-, s_m^+ ] = i \sqrt{2} \left( x^\alpha \partial^\alpha - x \tilde{\partial} - \theta \frac{\partial}{\partial \theta} + \lambda + 1 \right) \frac{1}{\partial^+} q_m^+ , \quad (3.26)$$

$$\bar{s}_n^- = i [ \bar{j}^-, \bar{s}_n^+ ] = -i \sqrt{2} \left( x^\alpha \partial^\alpha - \bar{x} \partial - \bar{\theta} \frac{\partial}{\partial \bar{\theta}} - \bar{\lambda} + 1 \right) \frac{1}{\partial^+} \bar{q}_n^+ .$$

Like the supersymmetry transformations, the conformal supersymmetry generators $s$ act as square roots of the conformal translations. Using these expressions, we verify closure of the algebra by checking that the anticommutators

$$\{ s_m^+, \bar{s}_n^+ \} = \sqrt{2} \delta_n^m K^+ ; \quad \{ s_m^-, \bar{s}_n^- \} = \sqrt{2} \delta_n^m \bar{K} ; \quad \{ s_m^+, \bar{s}_n^+ \} = \sqrt{2} \delta_n^m K . \quad (3.27)$$

yield exactly the same expressions as previously obtained for $K^+$, $K$ and $\bar{K}$. The superconformal algebra is completed by calculating the dynamical conformal generator $K^-$ from

$$K^- = i [ j^-, K ] , \quad (3.28)$$
with the result

$$K^- = 2i \left( x^+ \partial^+ - \bar{x} \partial - \theta \frac{\partial}{\partial \theta} - \lambda + 1 \right)$$

$$\times \left( x^+ \partial^+ - \bar{x} \partial - \theta \frac{\partial}{\partial \theta} + \lambda + 1 \right) \frac{1}{\partial^+}. \quad (3.29)$$

The consistency of this expression is checked by calculating the anticommutator,

$$\sqrt{2} \delta^m_n K^- = \{ s^m_\pm , s^-_n \} . \quad (3.30)$$

This completes the construction of the superconformal algebra at the free level.

We note for completeness a host of commutation relations in light-cone form

$$\{ q^m_+ , \bar{q}_{+n} \} = - \sqrt{2} \delta^m_n p^+ ; \quad \{ q^-_m , \bar{q}_{-n} \} = - \sqrt{2} \delta^m_n p^- ;$$

$$[ s^m_+ , p] = - \sqrt{2} q^m_+ ; \quad [ s^m_-, \bar{p}] = \sqrt{2} q^m_- . \quad (3.31)$$

Finally, the anticommutators of the conformal and normal supersymmetries yield new operators, $J^m_n$,

$$\{ q^m_+ , s^-_n \} = - i \delta^m_n (D + j^+- i j) + 2 J^m_n , \quad (3.32)$$

which generate the $SU(4)$ Lie algebra

$$[ J^m_n , J^p_q ] = \delta^m_q J^p_n - \delta^p_n J^m_q , \quad (3.33)$$

and commute with the $SO(4,2)$ generators. The bosonic generators we have constructed generate

$$SO(4,2) \times SU(4) \sim SO(4,2) \times SO(6) ; \quad (3.34)$$

together with the fermionic generators, they form the entire $PSU(2,2|4)$ algebra.

4 Non-linear Realizations

The superPoincaré algebra contains three types of dynamical transformations, those generated by the light-cone Hamiltonian $p^-$, by the boosts $j^-$ and $\bar{j}^-$, and by the supersymmetries $q_-$ and $\bar{q}_-$. In the interacting theory, these transformations act non-linearly on the superfields, while preserving the commutation relations of the Super-Poincaré algebra. The form of these transformations determines the fully interacting super-Poincaré invariant action. This decomposition is also true for the conformal transformations contained in the full $PSU(2,2|4)$. 

9
4.1 Old Results

Bengtsson et al [2] devised in 1982 a systematic procedure for finding these non-linear transformations order by order in the coupling constant $g$. They expanded the dynamical transformations of the fields as a power series in $g$,

\[
\begin{align*}
\delta_{p-} \phi &= \delta^0_{p-} \phi + \delta^g_{p-} \phi + \delta^{g^2}_{p-} \phi + \cdots , \\
\delta_{q-} \phi &= \delta^0_{q-} \phi + \delta^g_{q-} \phi + \delta^{g^2}_{q-} \phi + \cdots , \\
\delta_{j-} \phi &= \delta^0_{j-} \phi + \delta^g_{j-} \phi + \delta^{g^2}_{j-} \phi + \cdots ,
\end{align*}
\]  

(4.1)

where the superscript denotes the order of the variation. Since the kinematical transformations remain unaltered with no order $g$ corrections, much information is gained from the commutation relations

\[
\begin{align*}
[\delta_j, \delta_{p-}] \phi &= 0 , \\
[\delta_{j+}, \delta_{p-}] \phi &= i \delta_{p-} \phi ,
\end{align*}
\]  

(4.2)

(which determine the helicity and the number of $\partial^+$ in the variations) as well as from requiring that

\[
[\delta_{j-}, \delta_{p-}] \phi = 0 ,
\]  

(4.3)

holds order by order in $g$. This allowed Bengtsson et al to determine these non-linear transformations to first order in $g$. They also determined the non-linear dynamical supersymmetry transformations by boosting

\[
\begin{align*}
\delta_{q-} \phi &= i [\delta_{q+}, \delta_{j-}] \phi , \\
\delta_{q+} \phi &= i [\delta_{q+}, \delta_{j-}] \phi .
\end{align*}
\]  

(4.4)

Due to algebraic complications, the authors did not proceed beyond the first order in coupling.

In detail, their method relied on formulating ansätze for the order $g$ Hamiltonian variation,

\[
\delta^g_{p-} \phi = - i g \partial^+ \mu \left[ \partial^a \partial^+ \rho \phi \partial^b \partial^+ \sigma \phi \right] ,
\]  

(4.5)

as well as for the boosts

\[
\delta^g_{j-} \phi = - x \delta^g_{p-} \phi + \delta^g_{\text{spin}} \phi ,
\]  

(4.6)

where the latter is the “spin” change.

Requiring closure of the commutators (4.2) to order $g$ yields, keeping in mind that the variations act only on the superfields,

\[
a + b = 1 , \quad \mu + \rho + \sigma = 0 .
\]  

(4.7)

A simple calculation showed that the order $g$ commutation relation (4.3) could not be satisfied unless there were several fields that entered the variation antisymmetrically. This was the first indication of the gauge structure function $f^{abc}$, which implied several fields, labeled with extra indices $a, b, c$. They
assumed that it was the completely antisymmetric 3-form. Once these assumptions are made, the vanishing of the commutator can be achieved for $\mu = -1$.

To summarize, the variation that satisfies Poincaré invariance to order $g$ is then

$$\delta^g_{\mu^-} \phi^a = -ig f^{abc} \frac{1}{\partial^+}(\bar{\partial} \phi^b \partial^+ \phi^c).$$  \hspace{1cm} (4.8)

In a similar fashion, they obtain the non-linear contribution to the boosts by requiring

$$[\delta_{j^-}, \delta_{p^-}]^g \phi = 0,$$  \hspace{1cm} (4.9)

which implies, after a lengthy calculation, that

$$\delta^g_{j^-} \phi^a = -x \delta^g_{\mu^-} \phi^a + ig f^{abc} \left\{ \left( \frac{\partial}{\partial \theta} \frac{\partial}{\partial \bar{\theta}} - 1 \right) \phi^b \partial^+ \phi^c \right\}.$$  \hspace{1cm} (4.10)

Finally, by boosting the kinematical supersymmetries, they obtained the dynamical ones

$$\delta^g_{\bar{q}^-} \phi^a = -g f^{abc} \frac{1}{\partial^+} \left( \frac{\partial}{\partial \theta} \bar{\phi}^b \partial^+ \phi^c \right),$$  \hspace{1cm} (4.11)

$$\delta^g_{q^-} \phi^a = g f^{abc} \frac{(d)^4}{48 \partial^+^3} \left( \frac{\partial}{\partial \bar{\theta}} \bar{\phi}^b \partial^+ \phi^c \right),$$  \hspace{1cm} (4.12)

where $(d)^4 \equiv \epsilon_{mnpq} d^m d^n d^p d^q$. Note that these transformations do not contain transverse derivatives. The authors used the variations to generate the cubic interaction vertex from the kinetic two-point function, but did not extend their method to higher order in $g$. As a result, their procedure fell short of showing that $f^{abc}$ is a three-form which satisfies the Jacobi identity, and of deriving the four-point function using algebraic means. In the next section, we complete their program.

### 4.2 New Results: Symmetries

The method described in the previous section is very generic, and does not make use of supersymmetry. Yet in supersymmetric theories, the Hamiltonian is a derived concept (as if the square-root of time were taken). Furthermore, Bengtsson et al had to make inspired guesses for both the Hamiltonian and boosts separately.

In this paper, we restrict our attention to the $\mathcal{N} = 4$ theory which has a much larger invariance group, namely $PSU(2,2|4)$. Unlike the superPoincaré symmetry, it is a simple Lie superalgebra. This means that it suffices to know one bosonic kinematical conformal transformation and the form of the non-linearly realized supersymmetry to reconstruct the whole algebra for the interacting case, and therefore the fully interacting classical action!
For that reason, we only have to determine the dynamical supersymmetry
to order $g$ and higher. The construction proceeds in several steps. First we
show that considerations of chirality, dimensional analysis, proper helicity and
simple commutators restrict the first order dynamical supersymmetry to be of
the form

$$
\delta^g_{\bar{q}^-} \phi^a = - g f^{abc} \frac{1}{\partial^{+(2\nu+1)}} \left\{ \bar{d} \partial^+ \nu \phi^b \partial^+ (\nu+1) \phi^c \right\}.
$$

(4.13)

Here

$$
f^{abc} = - f^{acb}.
$$

(4.14)

$\bar{d}$ and $\bar{q}^-$ are interchangeable because of the antisymmetry of the structure
function, and $\nu$ is a free parameter, to be fixed by the algebra. So as not to
interrupt the flow of our arguments, the details of these calculations are relegated
to Appendix A.

We take the conjugate of this expression, and use the “inside-out” relation to find

$$
\delta^g_{\bar{q}^+} \phi^a = - g f^{abc} \frac{(d)^4}{48 \partial^{+(2\nu+3)}} \left\{ d \partial^+ \nu \bar{\phi}^b \partial^+ (\nu+1) \bar{\phi}^c \right\}.
$$

(4.15)

The next step is to evaluate the anticommutator

$$
\{ \delta_{\bar{q}^+}^g, \delta_{\bar{q}^-}^g \}^g \phi^a = - \sqrt{2} \delta^g_{\bar{q}^+} \delta^g_{\bar{q}^-} \phi^a,
$$

(4.16)

to first order in $g$. Use of chirality and the relation

$$
\bar{q}^+ = \bar{d} - i \sqrt{2} \theta \partial^+,
$$

(4.17)

lead to

$$
\delta_{\bar{q}^+} \phi^a = - i \frac{\partial \bar{\theta}}{\partial^+} \phi^a - i g f^{abc} \left\{ \frac{1}{\partial^{+(2\nu+1)}} \left( \partial^+ (\nu) \bar{\phi}^b \partial^+ (\nu+1) \phi^c \right) \right. \\
+ \left. \frac{(d)^4}{48 \partial^{+(2\nu+3)}} \left( \partial^+ (\nu) \bar{\phi}^b \partial^+ (\nu+1) \phi^c \right) \right\} + \mathcal{O}(g^2).
$$

(4.18)

This is of course the light-cone Hamiltonian. In Appendix A, we show that the
dynamical supersymmetry variation does not extend beyond order $g$. Thus the
classical Hamiltonian extends only up to order $g^2$. It is too cumbersome here
to derive it from the anticommutator, and we will obtain it in a much simpler
way from the action.

Because $\bar{K}$ is kinematical, void of order $g$ corrections, we can now derive the
form of the non-linear boosts using the conformal group commutator

$$
[\delta_{\bar{K}}, \delta_{\bar{q}^-}] \phi^a = - 2 i \delta_{\bar{q}^-} \phi^a.
$$

(4.19)
A straightforward computation yields

\[ \delta g \phi^a = -\bar{x} \delta_p \phi^a - \frac{i g (d)^4 f^{abc}}{48 \partial^{(2\nu+3)}} \left\{ (\bar{\theta} \partial \partial + \nu - 1) \partial^+ \nu \bar{\phi}^b \bar{\phi}^+ (\nu + 1) \phi^c \right\}, \]

neglecting the order \( g^2 \) contributions. We are now in a position to verify that

\[ [\delta_{\bar{J}^-}, \delta_p^-] \phi^a = 0. \quad (4.20) \]

Evaluating this commutator to order \( g \) yields terms proportional to \( \nu \), such as

\[ [\delta_{\bar{J}^-}, \delta_p^-] \phi^a = \nu g f^{abc} \partial^+ \left\{ \partial^+ (\partial^+ (\nu - 1) \bar{\phi}^b \partial^+ (\nu + 1) \phi^c) - 2 \bar{\phi}^b (\partial^+ \nu \bar{\phi}^b \partial^+ (\nu + 1) \phi^c) \right\} + \cdots, \quad (4.21) \]

which require \( \nu = 0 \). Hence by using the superconformal algebra and chirality we have arrived at the unique realization of the \( PSU(2,2|4) \) algebra.

The procedure to obtain the remaining transformations is straightforward. In particular, the superconformal transformations are obtained through the commutator,

\[ \delta g \phi^a = \frac{1}{\sqrt{2}} \left[ \delta_{\bar{J}^-}, \delta_{\bar{K}^-} \right] \phi^a = \sqrt{2i \bar{x}} \delta_{\bar{J}^-} \phi^a \]

\[ = \sqrt{2i \bar{x}} g f^{abc} \frac{1}{\partial^+} \left( \partial^+ \nu \phi^b \partial^+ \phi^c \right), \quad (4.22) \]

The total antisymmetry of the \( f^{abc} \) and the Jacobi identities are obtained by requiring closure of the algebra. For instance, we calculate the conformal generator \( K^- \) in two independent ways, from the commutator

\[ \delta_{K^-} = i \left[ \delta_{\bar{J}^-}, \delta_K \right], \]

or from the anticommutator

\[ \delta_{K^-} = \frac{1}{4\sqrt{2}} \left\{ \delta_{\bar{J}^-}, \delta_{\bar{K}^-} \right\}. \quad (4.23) \]

The total antisymmetry of the \( f^{abc} \) and the Jacobi identities are obtained by requiring closure of the algebra. For instance, we calculate the conformal generator \( K^- \) in two independent ways, from the commutator

\[ \delta_{K^-} = i \left[ \delta_{\bar{J}^-}, \delta_K \right], \]

or from the anticommutator

\[ \delta_{K^-} = \frac{1}{4\sqrt{2}} \left\{ \delta_{\bar{J}^-}, \delta_{\bar{K}^-} \right\}. \quad (4.24) \]

Matching these two equations yields the Jacobi identity for the structure constants. In this gauge, space-time and internal symmetries are inextricably linked; conformal invariance requires the gauge symmetry of Yang-Mills theories.

Finally we note that the full dynamical supersymmetry operation can be written in the form

\[ \delta_{\bar{q}^-} \phi^a = \frac{1}{\partial^+} \left\{ (\partial^+ \nu - g f^{abc} \phi^c) \delta_{\bar{q}^-} \phi^c \right\}, \quad (4.25) \]
suggesting a covariant derivative structure

\[ \mathcal{D}^{ab} = \bar{\partial} \delta^{ab} - g f^{abc} \partial^c, \]  

although there are no known symmetries beyond the superconformal symmetry.

4.3 The Hamiltonian

In this section, we show how to use these transformations to derive the fully interacting Hamiltonian starting from just its kinetic term. The action is of course well known, so that we will learn nothing new from this procedure except a methodology we expect to apply to other problems.

It is easy to obtain the Hamiltonian from the light-cone action for \( N = 4 \) SuperYang-Mills, Eq. (2.22)

\[ H = \int d^4 x \, d^4 \theta \, \bar{d} \theta \left\{ \frac{\bar{\phi}^a}{\partial^+} \phi^b \bar{\phi}^c \bar{\phi}^d \bar{\phi}^e \left[ \frac{2 \bar{\delta} \phi}{\partial^+} \phi + \frac{g}{3} f^{abc} \left( \frac{1}{\partial^+} \phi^b \bar{\partial} \phi^c + \frac{1}{\partial^+} \phi^a \bar{\partial} \phi^c \right) \right] \right\}, \]

\[ \equiv H^0 + H^g + H^{g^2}. \]

We require that supersymmetry variations leave the Hamiltonian invariant,

\[ \delta \bar{q}^{-} H = 0. \]

Expanding in the coupling constant \( g \) leads to three conditions

\[ \delta^0 \bar{q}^{-} H^0 = 0, \]

\[ \delta^g \bar{q}^{-} H^0 + \delta^0 \bar{q}^{-} H^g = 0, \]

\[ \delta^g \bar{q}^{-} H^g + \delta^0 \bar{q}^{-} H^{g^2} = 0, \]

which offer a systematic procedure for finding \( H^g \) and then \( H^{g^2} \), starting from \( \delta \bar{q}^{-} \) and the free Hamiltonian \( H^0 \).

The lowest order condition trivially vanishes, but we can use the second to infer the form of \( H^g \). We first evaluate

\[ \delta^g \bar{q}^{-} H^0 = \delta^g \bar{q}^{-} \left\{ \int \bar{\phi} \frac{2 \bar{\delta} \phi}{\partial^+} \phi^a \right\}. \]

A series of simple algebraic steps which use integration by parts of both \( d \) and \( \partial^+ \), as well as chirality, lead to

\[ \delta^g \bar{q}^{-} \left\{ \int \bar{\phi} \frac{2 \bar{\delta} \phi}{\partial^+} \phi^a \right\} = 2 g f^{abc} \int \bar{\phi} \, d \phi \cdot \bar{\partial} \phi^a \phi^b \phi^c. \]
Using Eq. (4.32), we obtain

\[ \delta^0_{q} \bar{H}^g = -2g \int f^{abc} \bar{\phi} \, d\bar{\phi} \, \frac{\partial}{\partial \phi^a} \phi^a , \]  

(4.36)

which tells us the general structure of \( H^g \). Rather than directly rewriting this expression as the sum of three variations, one on each superfield, we consider the variation

\[ \delta^0_{q} \left\{ g f^{abc} \int \frac{1}{\partial^+} \phi^a \bar{\phi}^b \bar{\partial} \phi^c \right\} , \]  

(4.37)

which yields three terms, (one of which is trivially zero),

\[ g f^{abc} \int \frac{1}{\partial^+} \phi^a \bar{\phi}^b \bar{\partial} \phi^c + g f^{abc} \int \frac{1}{\partial^+} \phi^a \bar{\phi}^b \bar{\partial} \phi^c + g f^{abc} \int \frac{1}{\partial^+} \phi^a \bar{\phi}^b \bar{\partial} \phi^c . \]  

(4.38)

and express the first term in the above expression in two different ways. One is to act \( \bar{\partial} \) on the first \( \phi^a \) and integrate by parts the \( \bar{\partial} \); the other is to use duality on \( \bar{\phi} \), followed by partial integrations on both \( \bar{\partial} \) and \( \partial^+ \). Comparing the two resulting expressions yields

\[ g f^{abc} \int \frac{1}{\partial^+} \phi^a \bar{\partial} \phi^b \bar{\partial} \phi^c = \frac{1}{2} \left\{ g f^{abc} \int \frac{1}{\partial^+} \phi^a \bar{\partial} \phi^b \bar{\partial} \phi^c \right\} . \]  

(4.39)

Hence the variation

\[ \delta^0_{q} \left\{ g f^{abc} \int \frac{1}{\partial^+} \phi^a \bar{\phi}^b \bar{\partial} \phi^c \right\} = \frac{3}{2} g f^{abc} \int \frac{1}{\partial^+} \phi^a \bar{\phi}^b \bar{\partial} \phi^c , \]  

(4.40)

leading to the already known three-point function. One can also show that the variation of the complex conjugate of the 3-point vertex vanishes

\[ \delta^0_{q} \left\{ g f^{abc} \int \frac{1}{\partial^+} \bar{\phi}^a \phi^b \bar{\partial} \phi^c \right\} = 0 . \]  

(4.41)

Having obtained the three-point function, we can now vary it to generate the four-point function, using Eq. (4.33). The full three-point function has two parts, one that contains the transverse derivative \( \partial \), and its complex conjugate that contains \( \bar{\partial} \). However, we know that \( H^g \) does not contain any transverse derivatives, since it is obtained from the supersymmetries at order \( g \) which have no transverse derivatives. Hence consistency requires that

\[ \delta^g_{q} \left\{ H^g \right\} = 0 , \]  

(4.42)
since $\delta^0_{q^-}$ contains no $\partial$. The proof proceeds in two steps. We first show that

$$\delta^0_{q^-} H^g_{\bar{\partial}} = i \int \frac{1}{\partial^+} \bar{\phi}^a \left[ \delta^g_{-q^+}, \delta^g_{-p^+} \right] \bar{\phi}^a \bigg|_{\bar{\partial}\text{-part}},$$

(4.43)
after many algebraic manipulations detailed in Appendix B.

The algebraic requirement that the supersymmetries commute with the Hamiltonian, implies to order $g^2$ that

$$\left[ \delta^0_{q^-}, \delta^g_{p^-} \right] + \left[ \delta^g_{q^-}, \delta^g_{p^-} \right] = 0.$$  

(4.44)
The Hamiltonian variation contains both $\partial$ and $\bar{\partial}$ parts, while $\delta^0_{q^-}$ involves only $\partial$. Hence this equation breaks up into two separate equations. The first is

$$\left[ \delta^g_{q^-}, \delta^g_{p^-} \right] \bigg|_{\bar{\partial}} = 0,$$

(4.45)
which as we show in the same Appendix, is satisfied as long as the structure functions are totally antisymmetric and obey the Jacobi identity. This satisfies the consistency requirement (4.42). The second relation

$$\left[ \delta^0_{q^-}, \delta^g_{p^-} \right] + \left[ \delta^g_{q^-}, \delta^0_{p^-} \right] \bigg|_{\bar{\partial}} = 0,$$

(4.46)
can now be used to extract the form of $H^g_{p^-}$ from Eq.(4.33). The three-point variation $\delta^g_{q^-} H^g_{\bar{\partial}}$ can be expressed as

$$\delta^g_{q^-} H^g_{\bar{\partial}} = i \int \frac{1}{\partial^+} \bar{\phi}^a \left[ \delta^g_{q^-}, \delta^g_{p^-} \right] \phi^a \bigg|_{\partial\text{-part}}$$

$$= -i \int \frac{1}{\partial^+} \bar{\phi}^a \left[ \delta^0_{q^-}, \delta^g_{p^-} \right] \phi^a$$

$$= -i \int \left\{ \frac{1}{\partial^+} \bar{\phi}^a \delta^0_{q^-} \phi^a - \frac{1}{\partial^+} \bar{\phi}^a \delta^g_{p^-} \bar{\phi}^a \delta^g_{q^+} \phi^a \right\}$$

$$= -i \int \left\{ \frac{1}{\partial^+} \bar{\phi}^a \delta^0_{q^-} \phi^a + \frac{1}{\partial^+} \delta^0_{q^-} \bar{\phi}^a \delta^g_{p^-} \phi^a \right\}$$

$$= -\delta^0_{q^-} \left\{ i \int \frac{1}{\partial^+} \bar{\phi}^a \delta^g_{p^-} \phi^a \right\},$$

(4.47)
where we have used Eq.(4.44). Hence the four-point function,

$$H^g = i \int \frac{1}{\partial^+} \bar{\phi}^a \delta^g_{p^-} \phi^a = -\frac{i}{4\sqrt{2}} \int \frac{1}{\partial^+} \bar{\phi}^a \left\{ \delta^g_{q^-}, \delta^g_{q^-} \right\} \phi^a.$$  

(4.48)
This completes the construction of the classical Hamiltonian.
5 Hamiltonian as a Quadratic Form

Our algebraic formulation of the $\mathcal{N} = 4$ SuperYang-Mills theory enables us to write its Hamiltonian in a particularly suggestive form. We note that the free Hamiltonian

$$H^0 = \int d^4x d^4\theta d^4\bar{\theta} \frac{2}{\partial^2 + 1} \phi^a,$$  
(5.1)

can be rewritten as a quadratic form

$$H^0 = \frac{1}{2\sqrt{2}} (\mathcal{W}_0, \mathcal{W}_0),$$  
(5.2)

using the inner product notation of Eq.(3.16), where

$$\mathcal{W}_0^a = \frac{\partial}{\partial \bar{q} + \phi^a},$$  
(5.3)

is a fermionic superfield, the free dynamical supersymmetry variation of the superfield ($SU(4)$ spinor indices are summed over). The proof is straightforward, and requires integration by parts and the use of the inside-out property of the superfields.

In order to generalize this simple formula to the fully interacting Hamiltonian, we note that the three-point function can be expressed in the suggestive form

$$4g f^{abc} \int \frac{1}{\partial^\dagger} \phi^a \phi^b \phi^c = \frac{i}{\sqrt{2}} g f^{abc} \int \frac{\partial}{\partial^\dagger} \phi^a (\bar{q} + m) \frac{1}{\partial^\dagger} (d^m \phi^b \phi^c).$$  
(5.4)

The detailed proof of this identity is left to Appendix B. Similarly, the two terms that describe the four-point interaction can be rewritten as

$$\int \frac{i}{\sqrt{2}} g f^{abc} f^{ade} \left\{ \frac{1}{\partial^\dagger} (\phi^b \phi^+ \phi^c) \frac{1}{\partial^\dagger} (\bar{\phi}^d \phi^+ \phi^c) + \frac{1}{2} \phi^d \phi^c \phi^d \phi^e \right\} = \int \frac{i}{\sqrt{2}} g f^{abc} f^{ade} \left\{ \frac{1}{\partial^\dagger} (d^m \phi^b \phi^+ \phi^c) \frac{1}{\partial^\dagger} (d^m \phi^d \phi^+ \phi^e) \right\},$$  
(5.5)

also proved in the same Appendix.

With their help, it is easy to see that the fully interacting Hamiltonian can be expressed as a quadratic form

$$H = \frac{1}{2\sqrt{2}} (\mathcal{W}_0^a, \mathcal{W}_0^a).$$  
(5.6)
where now

\[
W^a = \frac{\partial}{\partial^+} \bar{q}^a \phi^a - g f^{abc} \frac{1}{\partial^+} (\bar{d}^b \partial^+ \phi^c),
\]

(5.7)

is the complete (classical) dynamical supersymmetry variation. The power of supersymmetry allows for this simple rewriting of the fully interacting Hamiltonian.

6 Conclusions

We have used purely algebraic techniques to reconstruct the \(\mathcal{N} = 4\) Yang-Mills theory on the light-cone. Knowledge of the dynamical supersymmetry transformations, which are of first order in the coupling, suffice to fix the full symmetry of the theory, and to write the Hamiltonian as a quadratic form. The simple answers we found in this exercise suggest several lines of inquiry which we are presently pursuing.

We conjecture that, in the quantum theory, the structure of the Hamiltonian remains a quadratic form, except that \(W^a\) picks up quantum corrections of order \(\bar{\hbar}\) and higher.

As the full quantum action is generated by the Dirac-Feynman path integral, this simple quadratic form suggests that we seek a change of variables from the superfields \(\phi^a\) to the fermionic superfields \(W^a\). In addition, we see that field configurations for which \(W^a = 0\) have vanishing energy, and their study should prove interesting.

Finally, we plan to apply the same techniques to \(\mathcal{N} = 8\) supergravity, and generate the classical action through the supersymmetry transformations. We intend to use these techniques to derive the hitherto unknown four- and higher-point functions in terms of chiral superfields.

7 Acknowledgements

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APPENDIX A : The Supersymmetry Variation

In this Appendix, we present two results. First, we construct the supersymmetry variation to order $g$ based on various checks. Secondly, we show that it is not possible to build a variation at order $g^2$ with the requisite properties, thus proving that the supersymmetry variations in this theory do not extend beyond order $g$.

Before attempting to establish the form of the dynamical supersymmetry variation, we list the algebraic constraints on its structure. We start with the commutator involving the kinematical $\bar{\jmath}^+$ and $\bar{q}_{-n}$,

$$[\delta_{\bar{\jmath}^+}, \delta_{\bar{q}_{-n}}] = -i \delta_{\bar{q}_{-n}}. \quad (A-1)$$

Kinematical generators do not involve any $g$-dependent contributions, thus implying that

$$[\delta_{\bar{\jmath}^+}, \delta^{(g, g^2)}_{\bar{q}_{-n}}] = 0 . \quad (A-2)$$

We deduce that $\delta^{(g, g^2)}_{\bar{q}_{-n}}$ cannot have a $\bar{\partial}$, since $\bar{\jmath}^+$ contains an $\bar{x}$. We also note that,

$$[\delta_{\jmath^+}, \delta^{(g, g^2)}_{\bar{q}_{-n}}] = 0 , \quad (A-3)$$

which (by the same argument) rules out the possibility of $\bar{\partial}$ in the variation. Hence there are no transverse space derivatives in $\delta^{(g, g^2)}_{\bar{q}_{-n}}$.

Turning to dimensional analysis, and keeping track of factors of $\hbar$, we find that

$$\delta_{\bar{q}_{-}} \phi \propto g^A \hbar^B \phi^C \quad (A-4)$$

subject to the (mass-)dimensional constraint

$$A - 2B - C = -1 . \quad (A-5)$$

Since the left-hand side of Eq. (A-4) carries a lower spinor index, the right-hand side necessarily contains one of the following quantities:

$$\bar{d}_m ; \quad \bar{q}_m ; \quad (d)^3_m = \epsilon_{mnpq} d^n d^p d^q ; \quad \{q (d)^2\}_m = \epsilon_{mnpq} q^n d^p d^q \quad (A-6)$$

$$\{d (q)^2\}_m = \epsilon_{mnpq} d^n q^p q^q ; \quad (q)^3_m = \epsilon_{mnpq} q^n q^p q^q .$$

We ignore these expressions and any factors of $\partial^+$ when using dimensional analysis since they all have a zero mass dimension which is the quantity we have tracked to obtain Eq. (A-5). For the classical theory, we set $B = 0$
and see that the term proportional to $g$ involves two superfields while the term proportional to $g^2$ contains three.

We now set $\hbar = 1$ and list the various (length-)dimensions and helicities of variables that occur in this theory:

| Variable | Helicity ($h$) | Dimension ($D$) |
|----------|----------------|-----------------|
| $\phi$   | $+1$           | $0$             |
| $\bar{\phi}$ | $-1$         | $0$             |
| $x$      | $+1$           | $1$             |
| $\bar{x}$ | $-1$           | $1$             |
| $\partial$ | $+1$         | $-1$            |
| $\bar{\partial}$ | $-1$    | $-1$            |
| $d^m$    | $+1/2$         | $-1/2$          |
| $\bar{d}^m$ | $-1/2$     | $-1/2$          |
| $q^n$    | $+1/2$         | $-1/2$          |
| $\bar{q}^n$ | $-1/2$     | $-1/2$          |

From the lowest order variation,

$$\delta^0_{\bar{q}_{-m}} \phi = \frac{\partial}{\partial x^m} \bar{q}_m \phi,$$  \hfill (A-7)

we infer that the dynamical supersymmetry variations have a length dimension of $-\frac{1}{2}$ and a helicity of $\frac{3}{2}$. Another requirement is that these variations respect the chirality of the superfield they act on. Indeed the lowest order variation, trivially commutes with the chiral derivatives.

**Variation at order $g$**

Having established general guidelines we now study possible candidates for the variation at order $g$. First consider an Ansatz with two chiral superfields (and a derivative)

$$\delta^g_{\bar{q}_{-m}} \phi \propto \bar{d}_m \phi \phi,$$  \hfill (A-8)

and ask that it leave chirality invariant. In other words, we want

$$\{ \delta^g_{\bar{q}_{-m}} , d^n \} = 0.$$

(A-9)

Since $\{ \bar{d}_m , d^n \} = -i \sqrt{2} \partial^+ \delta^m_n$ we obtain

$$\{ \delta^g_{\bar{q}_{-m}} , d^n \} \phi = -i \sqrt{2} g \partial^+ \phi \phi \delta^m_n,$$  \hfill (A-10)

which is non-zero. We are thus forced to introduce an antisymmetric function $f^{abc}$ into Eq. (A-8) and have the $\bar{d}_m$ act only on one $\phi$. We also introduce factors
of $\partial^+$ into the relation to ensure that antisymmetry does not automatically render it zero,

$$
\delta^g_{\bar{q}^{-m}} \phi^a \propto f^{abc} \frac{1}{\partial^{+(2\nu+1)}} \left( \bar{d}_m \partial^{+\nu} \phi^b \partial^{+(\nu+1)} \phi^c \right), \quad (A-11)
$$

with $f^{abc} = - f^{acb}$. Note at this stage that our requirements are insufficient to prove antisymmetry between the $a$ & $b$ indices or between $a$ & $c$. This will be proven by algebraic means described in Appendix B. The factor of $\frac{1}{\partial^{+(2\nu+1)}}$ in the denominator balances the factors of $\partial^+$ introduced into the numerator.

The anticommutator of this modified Ansatz with $d^n$ now reads,

$$
\{ \delta^g_{\bar{q}^{-m}} , d^n \} \phi^a = - i \sqrt{2} g f^{abc} \partial^{+(\nu+1)} \partial^{+(\nu+1)} \phi^c \delta^n_m = 0 , \quad (A-12)
$$

by antisymmetry. This analysis thus leads us to the following form,

$$
\delta^g_{\bar{q}^{-m}} \phi^a \propto g f^{abc} \frac{1}{\partial^{+(2\nu+1)}} \left( \partial^{+\nu} \bar{d}_m \phi^b \partial^{+(\nu+1)} \phi^c \right). \quad (A-13)
$$

Instead of beginning with Eq. (A-8), we could equally well have started with

$$
\delta^g_{\bar{q}^{-m}} \phi \propto \bar{q}_m \phi \phi , \quad (A-14)
$$

which is manifestly chiral, but it ruins the anticommutator,

$$
\{ \delta^g_{\bar{q}^{-m}} , \delta^g_q \} = 0 . \quad (A-15)
$$

Restoring this anticommutator, requires the introduction of the antisymmetric structure function and once again, leads to result (A-13). Fixing the value of $\nu$ (and the constant of proportionality) requires algebraic computation and is presented in section 4.

A second candidate for the variation at order $g$, is one proportional to $g \phi \bar{\phi}$. This Ansatz has helicity 0 and is manifestly non-chiral. It requires the introduction of three chiral derivatives to reach the target of $h = \frac{3}{2}$. However the inside-out relation tells us that,

$$
(d)^3_m \bar{\phi} \sim \bar{d}_m \phi , \quad (A-16)
$$

making this Ansatz proportional to the one in Eq. (A-8).

Finally, we consider the totally antichiral Ansatz, proportional to $g \bar{\phi} \bar{\phi}$. Characterized by $h = -2$, it requires seven chiral derivatives to achieve a helicity of $\frac{3}{2}$. The inside-out relations again render this expression proportional to the very first Ansatz.

Thus at order $g$, the only viable Ansatz for the dynamical supersymmetry variation contains two chiral superfields and reads,

$$
\delta^g_{\bar{q}^{-m}} \phi^a = - g f^{abc} \frac{1}{\partial^{+(2\nu+1)}} \left( \partial^{+\nu} \bar{d}_m \phi^b \partial^{+(\nu+1)} \phi^c \right). \quad (A-17)
$$
Variation at order $g^2$

The requirements on the supersymmetry variation at order $g^2$ are that it involve three superfields, have a length dimension of $-\frac{1}{2}$, a helicity of $\frac{3}{2}$, and contain no transverse derivatives. We offer a “proof by exhaustion” that such an object does not exist.

We start by studying the case where the variation is proportional to three chiral superfields: $g^2 \phi \phi \phi$. This expression has $D = 0$ and $h = 3$. The single lower spinor index is again introduced using,

$$\bar{d}; \bar{q}; (d)^3; (d)^2 q; d (q)^2; (q)^3$$  \hspace{1cm} (A-18)

These spinor expressions carry either helicities of $-\frac{1}{2}$ or $+\frac{3}{2}$ and when introduced into $g^2 \phi \phi \phi$ result in a net helicity of $\frac{5}{2}$ or $\frac{9}{2}$. Achieving a net helicity of $\frac{3}{2}$ thus requires transverse derivatives (in this case $\bar{\partial}$). Since this would violate the commutator,

$$[j^+, \bar{q}_- n] = [x \partial^+, \bar{q}_- n] = 0 ,$$  \hspace{1cm} (A-19)

the Ansatz is ruled out.

Other possible starting points involve mixtures of chiral and anti-chiral superfields. Consider a variation proportional to: $g^2 \phi \bar{\phi} \bar{\phi}$ (this expression is manifestly non-chiral, something we will not worry about yet). The Ansatz has a helicity of $+1$ and $D = 0$. Since we require a lower spinor index, we could introduce any of the following,

$$\bar{d}; \bar{q}; (d)^3; q (d)^2; (q)^2 d; (q)^3 .$$  \hspace{1cm} (A-20)

The spinor expressions have helicities of either $-\frac{1}{2}$ (which must then be accompanied by a $\partial$) or $+\frac{3}{2}$ (which needs a $\bar{\partial}$). Either way, the Ansatz is not permitted. The other mixed Ansatz, proportional to $g^2 \phi \bar{\phi} \phi$ is ruled out by the same reasoning.

Finally, we consider the antichiral Ansatz: $g^2 \bar{\phi} \bar{\phi} \phi$ with $h = -3$ and $D = 0$. Helicity matching using just chiral derivatives turns the expression into one of the previous cases (since it would take nine chiral derivatives).

Thus any supersymmetry variation at order greater than $g$ necessarily contains one or more transverse space derivatives based on helicity and dimensional-requirements. The presence of a transverse derivative always ruins the commutator with either $j^+$ or $\bar{j}^+$ thus proving that supersymmetry variations in this theory, end at order $g$. The complete supersymmetry variation is rather simple:

$$\delta q_- \phi^a = \frac{1}{\partial^+} \left\{ (\partial \delta^{ab} - g f^{abc} \partial^+ \phi^c) \delta q_+ \phi^b \right\} .$$  \hspace{1cm} (A-21)

We note that it is always possible to alter a given Ansatz, using the index-less objects: $\epsilon_{mnpq} d^m d^o d^p d^q$ and $\epsilon_{mnpq} d_m d_n d_p d_q$. However, these simply act on
the superfields, conjugating them (according to the inside-out relations) and thus reproduce one of the cases already covered.

Our techniques are equally applicable to the larger $\mathcal{N} = 8$ Supergravity theory in four dimensions. However, the dynamical supersymmetry variations in that theory pick up an infinite set of corrections (at every order in $\kappa$) making the theory itself more difficult to build. Order by order in $\kappa$, it is still easy to constrain the structure of the variation based on the various considerations discussed here. The supergravity variations do contain transverse derivatives and evade our arguments regarding Equations (A-2) and (A-3). This is due to the presence of two transverse derivatives in the action for that theory allowing structures wherein the contributions from the two derivatives cancel each other. In reference [6] we discuss the possible form of the variations in the $\mathcal{N} = 8$ theory at order $\kappa^2$.

APPENDIX B : Mathematical Details

Explicit proof: $\delta_{\dot{q}^-} (3$-pt function)$ \big|_{\partial$-part} = 0$

The detailed proof for Eq. (4.42) proceeds in two steps. First we show that the supersymmetry variation on the $\partial$-dependent three-point function

$$H^g_{\dot{\varphi}} \equiv - \frac{4}{3} g f^{abc} \int \frac{1}{\partial^+} \phi^a \bar{\phi}^b \partial \bar{\phi}^c,$$

(B-1)

can be expressed as

$$\delta_{\dot{q}^-} H^g_{\dot{\varphi}} = 2i \int \frac{1}{\partial^+} \phi^a \left[ \delta_{\dot{q}^-}, \delta_{\dot{p}^-} \right] \bar{\phi}^a \big|_{\partial$-part} .$$

(B-2)

we then show that

$$\left[ \delta_{\dot{q}^-}, \delta_{\dot{p}^-} \right] \bar{\phi}^a \big|_{\partial$-part} = 0,$$

(B-3)

thus implying that

$$\delta_{\dot{q}^-} H^g_{\dot{\varphi}} = 0 .$$

(B-4)

The starting point is to rewrite the three-point function as the product of a chiral superfield and the order $g$ Hamiltonian variation,

$$H^g_{\dot{\varphi}} = i \frac{4}{3} \int \frac{1}{\partial^+} \phi^a \delta^g_{\dot{p}^-} \bar{\phi}^a \big|_{\partial} .$$

(B-5)
The order $g$ supersymmetry variation on the three-point function is then

$$\delta^g_q H^g_{\partial} = \frac{4}{3} \int \left\{ \frac{1}{\partial^+} \delta^g_{q-} \phi^a \delta^g_{p-} \bar{\phi}^a + \frac{1}{\partial^+} \phi^a \delta^g_{q-} \delta^g_{p-} \bar{\phi}^a \right\} |_{\partial}. \quad (\text{B-6})$$

Work with the second term: expand $\delta^g_{p-} \bar{\phi}$ explicitly and recombine terms, keeping $\delta^g_{q-} \bar{\phi}$ explicit. This leads to

$$i \int \frac{1}{\partial^+} \phi^a \delta^g_{q-} \delta^g_{p-} \bar{\phi}^a \bigg|_{\partial} \text{-part}$$

$$= g f^{abc} \int \frac{1}{\partial^+} \phi^a \left[ \partial \delta^g_{q-} \bar{\phi}^b \partial^+ \bar{\phi}^c + \partial \phi^b \partial^+ \delta^g_{q-} \bar{\phi}^c \right],$$

$$= g f^{abc} \int \left\{ \partial (\frac{1}{\partial^+} \phi^a \bar{\phi}^c) \delta^g_{q-} \bar{\phi}^b + \partial^+ (\frac{1}{\partial^+} \phi^a \partial \bar{\phi}^c) \delta^g_{q-} \bar{\phi}^b \right\}, \quad (\text{B-7})$$

$$= 2 g f^{abc} \int \frac{1}{\partial^+} \phi^a \partial \delta^g_{q-} \bar{\phi}^b,$$

$$= 2 i \int \frac{1}{\partial^+} \delta^g_{q-} \phi^a \delta^g_{p-} \bar{\phi}^a \bigg|_{\partial} \text{-part},$$

where we have made use of the inside-out relations in the last three steps. Thus, the variation of the three-point function can be written as

$$\delta^g_{q-} H^g_{\partial} = 2 i \int \frac{1}{\partial^+} \phi^a \delta^g_{q-} \delta^g_{p-} \bar{\phi}^a \bigg|_{\partial} \text{-part}. \quad (\text{B-8})$$

The similarity of the derivative structure between $\delta^g_{p-} \bar{\phi}^a$ and $\delta^g_{q-} \bar{\phi}^a$:

$$\delta^g_{p-} \bar{\phi}^a = -i g f^{abc} \frac{1}{\partial^+} (\partial \bar{\phi}^b \partial^+ \bar{\phi}^c); \quad \delta^g_{q-} \bar{\phi}^a = -g f^{abc} \frac{1}{\partial^+} (q_+ \bar{\phi}^b \partial^+ \bar{\phi}^c), \quad (\text{B-9})$$

implies that

$$\int \frac{1}{\partial^+} \phi^a \delta^g_{p-} \delta^g_{q-} \bar{\phi}^a \bigg|_{\partial} = 2 \int \frac{1}{\partial^+} \phi^a \delta^g_{q-} \delta^g_{p-} \bar{\phi}^a \bigg|_{\partial},$$

$$= -2 \int \frac{1}{\partial^+} \phi^a \delta^g_{q-} \delta^g_{p-} \bar{\phi}^a \bigg|_{\partial}, \quad (\text{B-10})$$

and leads to

$$\delta^g_{q-} H^g_{\partial} = -2 i \int \frac{1}{\partial^+} \phi^a \delta^g_{p-} \delta^g_{q-} \bar{\phi}^a \bigg|_{\partial} \text{-part}. \quad (\text{B-11})$$

Equating Eqs. (\text{B-8}) and (\text{B-11}) gives

$$\delta^g_{q-} H^g_{\partial} = i \int \frac{1}{\partial^+} \phi^a [\delta^g_{q-}, \delta^g_{p-}] \bar{\phi}^a \bigg|_{\partial} \text{-part}. \quad (\text{B-12})$$

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The next step is to compute

\[ i \left[ \delta_q^g, \delta_p^g \right] \bar{\phi}^a \bigg|_{\partial-\text{part}} = \]

\[= g^2 f^{abc} f^{bde} \frac{1}{\partial^+} \left\{ \partial \bar{\phi}^c (q + \bar{\phi}^d \partial^+ \bar{\phi}^e) \right\} + g^2 f^{abc} f^{cde} \frac{1}{\partial^+} \left\{ \partial \bar{\phi}^b (q + \bar{\phi}^d \partial^+ \bar{\phi}^e) \right\} , \]

\[ - g^2 f^{abc} f^{bde} \frac{1}{\partial^+} \left\{ \frac{q^+}{\partial^+} (\partial \bar{\phi}^d \partial^+ \bar{\phi}^e) \partial^+ \bar{\phi}^c \right\} - g^2 f^{abc} f^{cde} \frac{1}{\partial^+} \left\{ q^+ \bar{\phi}^h (\partial \bar{\phi}^d \partial^+ \bar{\phi}^e) \right\} . \]

Expanding the first and third terms yields

\[ g^2 f^{abc} f^{bde} \frac{1}{\partial^+} \left\{ \partial \bar{\phi}^c q + \bar{\phi}^d \partial^+ \bar{\phi}^e \right\} . \] (B-14)

Switching the \( b \) and \( d \) indices in the last term of Eq. (B-13), allows us to combine it with the second term as

\[ g^2 (f^{abc} f^{cde} - f^{ade} f^{cbe}) \frac{1}{\partial^+} \left\{ \partial \bar{\phi}^c q + \bar{\phi}^d \partial^+ \bar{\phi}^e \right\} . \] (B-15)

The antisymmetry of the structure constants \( f^{abc} \), and the Jacobi identity

\[ f^{abc} f^{cde} + f^{ade} f^{cbe} + f^{ace} f^{cbd} = 0 , \] (B-16)

are necessary to further simplify this equation to

\[ g^2 f^{ace} f^{cbd} \frac{1}{\partial^+} \left\{ \partial \bar{\phi}^c q + \bar{\phi}^d \partial^+ \bar{\phi}^e \right\} . \] (B-17)

Eq. (B-17) is then reduced to two terms

\[ g^2 f^{abc} f^{bde} \frac{1}{\partial^+} \left\{ \partial \bar{\phi}^c q + \bar{\phi}^d \partial^+ \bar{\phi}^e \right\} + g^2 f^{ace} f^{cbd} \frac{1}{\partial^+} \left\{ \partial \bar{\phi}^c q + \bar{\phi}^d \partial^+ \bar{\phi}^e \right\} \]

\[ = g^2 \left( f^{ace} f^{cdb} + f^{ace} f^{cbd} \right) \frac{1}{\partial^+} \left\{ \partial \bar{\phi}^c q + \bar{\phi}^d \partial^+ \bar{\phi}^e \right\} \]

\[ = 0 . \] (B-18)

We have therefore shown that

\[ \delta_q^g H_0^g = 0 . \] (B-19)

**Useful identities**

We present here, a series of useful identities which have been used at various stages of this paper.

**Identity 1**

For any function \( X(\phi) \) of chiral superfields and its conjugate \( \bar{X}(\bar{\phi}) \),
\[
\int \bar{X} \frac{1}{\partial^+} X = \frac{i}{4\sqrt{2}} \int \frac{d^m}{\partial^+} \bar{d}_m \partial^+ X .
\] (B-20)

Proving this identity is rather simple: put \(\frac{\partial^+}{\partial^+}\) on the \(\bar{X}\) and rewrite the \(\partial^+\) in the numerator as \(\frac{i}{4\sqrt{2}}\{\bar{d}_m, d^m\}\). Then integration by parts with respect to \(\bar{d}_m\) yields the identity.

**Identity 2**

\[
f^{abc} \int \frac{1}{\partial^+} \bar{\phi}^a \phi^b X^c = 0 .
\] (B-21)

Use the inside-out relation on \(\phi^b\) followed by integration by parts with respect to \((d+)^a\). Then swap indices \(a\) and \(b\) and use the antisymmetry of \(f^{abc}\) to obtain Eq. (B-21). As a corollary,

\[
f^{abc} \int \frac{1}{\partial^+} \bar{\phi}^a \phi^b X^c = - f^{abc} \int \frac{1}{\partial^+} \bar{\phi}^a \phi^b X^c.
\] (B-22)

**3-pt function identity**

\[
f^{abc} \int \bar{\phi}^a \frac{\partial}{\partial^+ \bar{q}} + m (d^m \bar{\phi}^b \partial^+ \bar{\phi}^c) = \frac{4i\sqrt{2}}{3} f^{abc} \int \frac{1}{\partial^+} \bar{\phi}^a \phi^b \phi^c .
\] (B-23)

This identity is essential to show that the Hamiltonian is a quadratic form. It can be verified by using the explicit forms of \(\bar{q}^+\) and \(d\), partial integrations, and the inside-out relations. The procedure is as follows (the integral is omitted).

- The first step is to replace \(\bar{q}^+\) and \(d\) by \(\bar{\theta}\) and \(\frac{\partial}{\partial^+ m}\), respectively:

\[
f^{abc} \bar{\phi}^a \frac{\partial}{\partial^+ m} \bar{q}_m (d^m \bar{\phi}^b \partial^+ \bar{\phi}^c) = i \sqrt{2} f^{abc} \bar{\phi}^a \frac{\partial}{\partial^+ \bar{\theta}} \left( \bar{\theta} \frac{\partial}{\partial \theta} \bar{\phi}^b \partial^+ \bar{\phi}^c \right) .
\] (B-24)

- Perform the partial integration with respect to \(\partial^+\) and \(\frac{\partial}{\partial^+ m}\):

\[
4i\sqrt{2} f^{abc} \frac{1}{\partial^+} \bar{\phi}^a \bar{\phi}^b \partial^+ \bar{\phi}^c - i \sqrt{2} f^{abc} \frac{1}{\partial^+} \bar{\phi}^a \bar{\phi}^b \partial^+ \bar{\phi}^c - i \sqrt{2} f^{abc} \bar{\phi}^a \frac{\partial}{\partial \theta} \bar{\phi}^b \partial^+ \bar{\phi}^c
\] (B-25)

and call the resulting terms \(I, II,\) and \(III,\) respectively.
• Work with term $I$: integration by parts with respect to $\partial^+$ yields

$$4i\sqrt{2} f^{abc} \frac{1}{\partial^+} \phi^a \partial^b \partial \bar{\phi}^c - 4i\sqrt{2} \frac{1}{\partial^{+2}} \phi^a \partial^b \partial^c \partial \bar{\phi}^b .$$  

The second term vanishes thanks to Eq. (B-21). Term $I$ is then,

$$I = 4i\sqrt{2} f^{abc} \frac{1}{\partial^+} \phi^a \partial^b \partial \bar{\phi}^c .$$  

$\quad$ (B-27)

• Impose the chiral conditions on term $II$:

$$d\phi = 0 \quad \Rightarrow \quad -\frac{\partial}{\partial \theta} \phi = \frac{i}{\sqrt{2}} \theta \partial^+ \phi ,$$  

$$d\bar{\phi} = 0 \quad \Rightarrow \quad -\frac{\partial}{\partial \bar{\theta}} \bar{\phi} = \frac{i}{\sqrt{2}} \bar{\theta} \partial^+ \bar{\phi} ,$$  

These imply that

$$II = -i\sqrt{2} f^{abc} \frac{\theta}{\partial \theta} \partial \bar{\phi}^b \frac{1}{\partial^+} \phi^a \partial \bar{\phi}^c \partial^+ \bar{\phi}^c .$$  

Combining $II$ and $III$ gives us

$$-i\sqrt{2} f^{abc} \left[ \frac{\theta}{\partial \theta} + \frac{\bar{\theta}}{\partial \bar{\theta}} \right] \partial \bar{\phi}^b \frac{1}{\partial^+} \phi^a \partial \bar{\phi}^c \partial^+ \bar{\phi}^c .$$  

$\quad$ (B-31)

• Use the inside-out relations on $\partial^c$ followed by the commutation relation

$$[d_m, \frac{\partial}{\partial \theta} + \frac{\bar{\theta}}{\partial \bar{\theta}}] = \bar{q} + m ,$$  

to obtain

$$II + III = -2i\sqrt{2} f^{abc} \phi^a \frac{\partial}{\partial \theta} \left( \frac{\bar{\theta}}{\partial \bar{\theta}} \partial^+ \bar{\phi}^b \partial^+ \bar{\phi}^c \right) ,$$  

$\quad$ (B-33)

• Equating Eq. (B-24) and the sum of terms $I$, $II$ and $III$ proves the 3-pt function identity.

4-pt function identity

$$\int g^2 f^{abc} f^{ade} \left\{ \frac{1}{\partial^+} (\phi^b \partial^+ \phi^c) \frac{1}{\partial^+} (\bar{\phi}^d \partial^+ \bar{\phi}^e) + \frac{1}{2} \phi^b \phi^c \phi^d \phi^e \right\}$$  

$$= \int \frac{i}{\sqrt{2}} g^2 f^{abc} f^{ade} \frac{1}{\partial^+} (d_m \phi^b \partial^+ \phi^c) \frac{1}{\partial^{+2}} (d_m \bar{\phi}^b \partial^+ \bar{\phi}^c) .$$  

$\quad$ (B-34)
We have not yet succeeded in producing an analytic proof of this identity, but we have a proof in terms of the component fields. We checked explicitly that the four-scalar interaction, after integration over the Grassmann variables,

\[
\int -\frac{1}{6} g^2 f^{abc} f^{ade} \frac{1}{\partial^+} (\tilde{C}^{\mu n}_{\rho} \partial^+ C^{\tau n c}) \frac{1}{\partial^+} (C^{\rho m d} \partial^+ \tilde{C}^{\tau m e}) ,
\]

is fully reproduced by both sides of Eq. (B-34). Recall that \(a, b, c, d, e\) are gauge indices.

For calculational convenience, we specialize to \(SU(2)\). Then this expression splits into three (one for each value of the summed over gauge index \(a\)), each one containing different fields. We set \(C_{mn}^1 \equiv D_{mn}^1\), and \(C_{mn}^2 \equiv E_{mn}\), and track down terms that involve specific bi-spinor indices such as \(\bar{D}_{12} D_{12}\) terms. Then Eq. (B-35) becomes

\[
\int \frac{2}{3} g^2 \frac{1}{\partial^+} (\bar{D}_{12} \partial^+ E^{12}) \frac{1}{\partial^+} (E^{12} \partial^+ \bar{D}_{12}) .
\]

This term exactly matches the \(\bar{D}_{12} D_{12}\) terms which come from the components

\[
\int -\frac{1}{8} g^2 f^{abc} f^{ade} \frac{1}{\partial^+} (\partial^+ \tilde{C}^{\mu n}_{\rho} b C^{\tau n c}) \frac{1}{\partial^+} (\partial^+ \tilde{C}^{\tau m e} d C^{\rho m d})
\]

\[
-\frac{1}{16} g^2 f^{abc} f^{ade} C^{\rho m b} C^{\tau p e} C_{mn}^1 d C^{\rho m d}.
\]

The algebra is lengthy and not particularly revealing, although the result is non-trivial. Once we have shown it holds for a particular component, we can use the kinematical supersymmetry variations to produce the other terms, and thus show the veracity of this claim for all components.

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