Persistent energy flow for a stochastic wave equation model in nonequilibrium statistical mechanics

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Dedicated to Elliott Lieb on the occasion of his 80th birthday.

Abstract

We consider a one-dimensional partial differential equation system modeling heat flow around a ring. The system includes a Klein-Gordon wave equation for a field satisfying spatial periodic boundary conditions, as well as Ornstein-Uhlenbeck stochastic differential equations with finite rank dissipation and stochastic driving terms modeling heat baths.

There is an energy flow around the ring. In the case of a linear field with different (fixed) bath temperatures, the energy flow can persist even when the interaction with the baths is turned off. A simple example is given.

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1 Introduction

We consider the following system of coupled partial differential equations:

\[
\begin{align*}
\partial_t \phi(x,t) &= \pi(x,t) \\
\partial_t \pi(x,t) &= (\partial_x^2 - 1)\phi(x,t) - g(\phi(x,t)) - \eta r(t) \cdot \alpha(x) \\
dr_i(t) &= -(r_i(t) - \eta \langle \alpha_i, \pi(t) \rangle) dt + \sqrt{T_i} d\omega_i(t) \quad i = 1, 2.
\end{align*}
\]

In these equations \( \phi \) is a field and \( \pi \) is the corresponding momentum field; both satisfy periodic boundary conditions with \( x \in [0, 2\pi] \). The non-linear term \( g \) is assumed to be bounded Lipschitz. The functions (actually distributions) \( \alpha = (\alpha_1, \alpha_2) \) are fixed, real, and assumed to have Fourier coefficients \( \hat{\alpha}_i(n) \propto n^\theta \) for some \(-1/2 < \theta < 1/4\), made precise below. (The brackets \( \langle \cdot, \cdot \rangle \) stand for the \( L^2[0, 2\pi] \)-inner product or distributional-test function pairing.) The vector \( r = (r_1, r_2) \in \mathbb{R}^2 \) is an artifact of the baths, and \( \eta \) is a coupling constant controlling the strength of the coupling of the
baths to the field. Finally, $\omega = (\omega_1, \omega_2)$ is 2-dimensional standard Brownian motion, and $T_1$ and $T_2$ are the bath temperatures. The system of equations (1) serves as a model of heat conduction around the ring.

We will show, for the linear setting with $g \equiv 0$ and with the system in a stationary non-equilibrium state, i.e. with the temperatures $T_1$ and $T_2$ differing, that there can be a stationary energy flow—a current—around the ring and, moreover, that this current persists even as the bath couplings are turned off, $\eta \to 0$. This phenomenon was noted but not proved in a previous article [18] with Y. Wang. The goals of the present article are to provide a proof of the claim and to give an explicit formula for the current. Parts of the analysis can be carried out assuming a non-linearity $g$ in the wave equation, and so the non-linearity is retained in the analysis to the extent possible, with a view toward understanding its effect on this persistent current.

The model itself has its origins in the heat flow models studied by Eckmann, Pillet, and Rey-Bellet [5, 6], who considered a finite chain of non-linear oscillators coupled at either end to free fields that model heat baths. Assuming a judicious choice of couplings and Gaussian-distributed random initial conditions on the baths, these authors showed that, after integrating out the bath field equations, one is left with Markovian stochastic equations. They established existence of a unique stationary state exhibiting heat flow and entropy production for these equations. See also [2, 3, 4, 14, 15, 16] for further developments of the model. The equations above, (1), are an adaptation of the models to an infinite-dimensional setting where the chain of oscillators has been replaced by a wave equation.

Even in the case where $g$ is cubic, $g(y) = y^3$ and thus unbounded, and the $\alpha$’s are chosen appropriately, the equations of motion (1) are known to have global in time solutions in spaces of low regularity, that is, with $\phi(\cdot, t)$ in $H^\gamma$, for any $\gamma > 1/3$ ($\gamma = 1$ corresponding to the usual energy norm) [17]. For the linear problem $g = 0$, the equations are of Ornstein-Uhlenbeck form. They have a unique invariant Gaussian measure (regardless of whether the temperatures are equal or not), and this measure is supported on fields $(\phi, \pi)$ such that $\phi(x)$ is almost surely Hölder continuous with index $< 1/2$ [18]. For bounded, Lipschitz $g$, the equations of motion with ultra-violet cutoffs have a unique invariant measure for each cutoff. By a soft argument, these measures have a weak* limit as the cutoff is removed, but it is not known whether the limiting measure(s) is invariant with respect to the above equations except in the equilibrium case $T_1 = T_2$, where a Gibbs state for the field is readily seen to be invariant [12, 20, 17, 19].

We turn now to the principal result. Let $\hat{\alpha}_i(n)$ be the $n^{th}$ Fourier coefficient of $\alpha_i$,

$$\hat{\alpha}_i(n) \equiv \int_0^{2\pi} e^{-inx} \alpha_i(x) \, dx. \quad (2)$$

**Assumptions.** We assume that there exist positive constants $c_1$ and $c_2$ and another constant $\theta$, $-1/2 < \theta < 1/4$ such that for all $n \neq 0$,

$$c_1 |n|^\theta \leq |\hat{\alpha}_i(n)| < c_2 |n|^\theta \quad i = 1, 2 \quad (3)$$
and
\[
    c_1(|\hat{\alpha}_1(n)|^2 + |\hat{\alpha}_2(n)|^2) \leq |\hat{\alpha}_1^2(n) + \hat{\alpha}_2^2(n)|^2. \tag{4}
\]

By way of explanation, the equations of motion are such that high frequency Fourier modes of the field couple rather weakly to the noise; however, the dissipation of these modes also is weak. Eq. (3) insures that the modes couple sufficiently strongly to the dissipation to counterbalance the driving terms, thereby maintaining stationarity. (The situation is to be contrasted with stochastic heat equations and equations of hydrodynamics with viscosity, where high frequency modes dissipate very rapidly \[1, 11, 7, 8, 9\].) The second condition, Eq. (4), insures that degeneracy of mode frequencies is broken with the coupling \(\alpha\)'s turned on. As will be seen, the difference of frequencies between modes is the primary factor in controlling the correlations between field modes.

The classical energy flow through \(x \in [0, 2\pi]\) is given by \(\pi(x)\partial_x \phi(x)\). We will consider this energy flow averaged over the ring, i.e., \(\frac{1}{2\pi} \int_0^{2\pi} \pi(x)\partial_x \phi(x) \, dx\). A non-zero expectation of this quantity in a stationary state signals a net circular current. We project the equations of motion Eq. (1) onto the subspace spanned by \(\{e^{inx}\}, |n| \leq M\), and we let \(E_M\) denote expectation with respect to the invariant measure for the system in this subspace. Although it is in fact a long calculation, we state the principal result as a theorem:

**Theorem 1.1** For the linear system \(g \equiv 0\), we have
\[
    \lim_{\eta \to 0} \lim_{M \to \infty} E_M \left[ \frac{1}{2\pi} \int_0^{2\pi} \pi(x)\partial_x \phi(x) \, dx \right] = -\frac{(T_1 - T_2)}{2} \frac{\pi}{\eta} \sum_{n: n > 0} \left( \frac{n}{n^2 + 1} |\hat{\alpha}_1^2(n) + \hat{\alpha}_2^2(n)|^2 + (|\hat{\alpha}_1^2(n)|^2 + |\hat{\alpha}_2^2(n)|^2) \right). \tag{5}
\]

The theorem requires explanation. First, we emphasize that the result concerns only the linear problem \(g = 0\). Second, for the equilibrium state or for a non-equilibrium limiting stationary state, \(\pi(x)\) is in \(H^{\gamma-1}\), \(\phi(x)\) in \(H^\gamma\), with \(\gamma < 1/2\) almost surely, so that the pointwise heat flow \(\pi(x)\partial_x \phi(x)\) makes no sense; however, the product does make sense for the equations with an ultraviolet cutoff \(M\). There is persistent current flow for the finite-dimensional ultraviolet cutoff systems. The principal result here is the uniform convergence of this current, \(M \to \infty\) in \(\eta \to 0\), and indeed that the current has a limit. It is this uniformity that we investigate here in detail, while including the effect of non-linearity where possible. Finally, we remark that when the coupling functions are turned off, the linear equations of motion certainly are not ergodic, and indeed invariant measures other than Gibbs states are possible. Of particular interest is how different non-equilibrium states are singled out in this limit, \(\eta \to 0\).
The existence of such a current was announced in joint work with Y. Wang [18]. What is new here is an account of the analysis behind the computation of the current, as well as a more explicit form for it in terms of the Fourier series coefficients of the coupling functions. We also provide an example below in which the coupling \( \alpha \)'s are \( \delta \)-functions at two points \( x_0, x_1 \in [0, 2\pi) \) – pictorially the situation in which the heat reservoirs are attached to a vibrating ring at just two points. The persistent current in this case shows a surprising discontinuous behavior as a function of the separation \( x_1 - x_0 \) of the two contact points.

The result here, for a linear system, is in the spirit of the 1966 paper by Rieder, Lieb, and Lebowitz [10] on stationary states for harmonic crystals. These authors considered a stationary state for a linear chain of oscillators with thermal reservoirs at either end maintained in general at different temperatures. They found that the expected kinetic energy of the \( j \)th oscillator is essentially constant in \( j \) in the interior of the chain (at the mean temperature), but that it dips as a function of \( j \) near the hotter end, with then a large discontinuous positive spike at the end oscillator. Similarly, the expected kinetic energy grows in \( j \) in a neighborhood of the cooler end with a large discontinuous dip at the end oscillator. They also determined the expected energy flow, which is constant throughout the chain but which exhibits anomalous behavior when the coupling is large. Presumably these effects disappear with non-linearity, as might well happen to the current in our model if the non-linearity effectively impedes energy flow around the ring.

We remind the reader that the existence of a stationary state invariant under the evolution equations for the non-linear \((g \neq 0)\), nonequilibrium system remains an open question. One might attempt to construct an ultraviolet limit of stationary measures for \( g \neq 0, M \rightarrow \infty \), in such a way that the expected current is fixed and finite. This would suggest how \( g \) is to be renormalized (if indeed renormalization is necessary) in order to achieve this limit.

### 1.1 Example

We conclude this introduction with a simple example in which the coupling distributions, the \( \alpha \)'s, are chosen to be \( \delta \)-functions, in particular

\[
\alpha_1(x) = \delta(x), \quad \alpha_2(x) = c\delta(x - x_1)
\]

with \( x_1 \) chosen \( 0 < x_1 \leq \pi \), and \( c \) chosen \( 0 < c < 1 \). In this case \( \hat{\alpha}_1(n) = 1 \) and \( \hat{\alpha}_2(n) = c e^{-inx_1} \), and the growth (with \( \theta = 0 \)) and non-degeneracy conditions on the \( \alpha \)'s, Eqs. (3,4), are seen to be satisfied. The limiting current is then given by

\[
C(x_1) \equiv \frac{T_1 - T_2}{2\pi} \sum_{n : n \geq 1} \frac{nc^2 \sin(2nx_1)}{(n^2 + 1)(1 + c^4 + 2c^2 \cos(2nx_1)) + (1 + c^2)^2},
\]

which is a conditionally convergent series. When the \( \delta \)-functions are localized at diametrically opposite points, i.e., at 0 and at \( x_1 = \pi \), this expected limiting current
$C(x_1)$ is zero, as one expects just for symmetry reasons. It is perhaps strange that $C(x_1)$ also vanishes for $x_1 = \pi/2$. As seen numerically, $C(x_1)$ exhibits discontinuous jumps at both $\pi/2$ and $\pi$ but is otherwise continuous in $x_1$. We have no physical explanation for this behavior.

2 Calculation of the expected current

2.1 Notation and eigenfunctions

Let

$$A(\eta) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \partial_x^2 - 1 & 0 & -\eta\alpha_1 & -\eta\alpha_2 \\ 0 & \eta\langle \alpha_1 & -1 & 0 \\ 0 & \eta\langle \alpha_2 & 0 & -1 \end{pmatrix}. \quad (8)$$

In terms of $A(\eta)$, the linear equations of motion Eq. (1) read

$$d\Phi(t) = A(\eta)\Phi(t) + \sqrt{T}d\omega, \quad (9)$$

with $\Phi(t)$ the four-vector $\Phi(t) = (\phi(x,t), \pi(x,t), r_1(t), r_2(t))^t$.

We will utilize the eigenfunctions and eigenvalues of $A(\eta)$ obtained via perturbation theory, $A(\eta)$ being a rank 2 perturbation of $A(0)$. (See particularly the appendix of [19] for an account of the estimation of these eigenfunctions and their eigenvalues.) Note that $A(\eta)$ is not normal for $\eta \neq 0$. We denote the right (column) eigenvectors of $A(\eta)$ by $e_n(\eta) = (e_n\phi(\eta), e_n\pi(\eta), e_n r_1(\eta))^t$, the last component being an abbreviation for the two $r$ components. The subscript $n$ is a triple, $n = (\pm, n, \sigma)$, $n \geq -1$ being an integer, $\sigma = 1, 2$. The eigenvalue $\lambda_n(\eta)$ corresponding to $e_n(\eta)$ is equal to

$$\lambda_n(\eta) = \lambda_{\pm, n, \sigma}(\eta) = \pm i(n^2 + 1)^{1/2} - \frac{\eta^2 \mu_{n, \sigma}}{2(\pm i(n^2 + 1)^{1/2} + 1) + O(\eta^4 n^{4\sigma-2} \ln n)} \quad (10)$$

for $n$ large (where by abuse of notation, $O(\eta^4) = O(\eta)O(n^\theta)$, for example). The eigenvalue shifts $\mu_{n, \sigma}$, $\sigma = 1, 2$ are the eigenvalues of the $2 \times 2$-matrix

$$M(n) \equiv \begin{pmatrix} |\hat{\alpha}_1^2(n)| & |\hat{\alpha}_2^2(n)| \\ \hat{\alpha}_1^2(n) + \hat{\alpha}_2^2(n) & |\hat{\alpha}_1^2(n)| + |\hat{\alpha}_2^2(n)| \end{pmatrix} \quad (11)$$

and are given by

$$\mu_{n, \sigma} = |\hat{\alpha}_1(n)|^2 + |\hat{\alpha}_2(n)|^2 - (-1)^\sigma |\hat{\alpha}_1(n)^2 + \hat{\alpha}_2(n)^2|, \ \sigma = 1, 2. \quad (12)$$
For $n = -1$, the two corresponding eigenvalues of $A(\eta)$ are simply $\lambda_{-\sigma} = -1 + \mathcal{O}(\eta^2)$; the $\pm$ plays no role. For the unperturbed $\eta = 0$ eigenfunctions, we take, for $n > 0$,

$$e_{\pm, n, 1}(0) = \frac{1}{2\sqrt{\pi}}(\mp i(n^2 + 1)^{-1/2}\cos(nx + \psi_n), \cos(nx + \psi), 0)^t$$

$$e_{\pm, n, 2}(0) = \frac{1}{2\sqrt{\pi}}(\mp i(n^2 + 1)^{-1/2}\sin(nx + \psi_n), \sin(nx + \psi), 0)^t,$$  \hspace{1cm} (13)

where $\psi_n$ is the phase defined by

$$e^{i\psi_n} = \frac{\hat{\alpha}^2_1(n) + \hat{\alpha}^2_2(n)}{|\hat{\alpha}^2_1(n) + \hat{\alpha}^2_2(n)|}.$$  \hspace{1cm} (14)

For $n = 0$, $e_{\pm, 0} = \frac{1}{2\sqrt{\pi}}(\mp i, 1, 0)^t$; these eigenfunctions are not degenerate and there is no $\sigma$ index. For $n = -1$, $e_{-1, 1} = (0, 0, 1, 0)$ and $e_{-1, 2} = (0, 0, 0, 1)$, the $\pm$ index is not in play. The momentum components of these functions, projected into the subspace of $L^2(0, 2\pi)$ spanned by $e^{\pm inx}$, are eigenvectors of $M(n)$ above.

Similarly, we have left eigenvectors (linear functionals) $\{f_n(\eta)\}$ of $A(\eta)$, their unperturbed versions taking the form

$$f_{\pm, n, 1}(0) = \frac{1}{2\sqrt{\pi}}(\pm i(n^2 + 1)^{1/2}\cos(nx + \psi_n), \cos(nx + \psi), 0)^t$$

$$f_{\pm, n, 2}(0) = \frac{1}{2\sqrt{\pi}}(\pm i(n^2 + 1)^{1/2}\sin(nx + \psi_n), \sin(nx + \psi), 0)^t;$$  \hspace{1cm} (15)

with analogous expressions for $f_{\pm, 0}$ and $f_{-1, \sigma}$. Both the $e_n(\eta)'s$ and the $f_n(\eta)'s$ are normalized so that the real inner product $\langle f_n(\eta), e_n(\eta) \rangle = 1$ and so that, for $n \geq 0$, $e_n(\eta)$ has $L^2(0, 2\pi)$-norm equal to 1/2. We emphasize that we are particularly concerned with the large $n$ behavior of these functions. Also, here and throughout, we are regarding these $f_n$'s as linear functionals so that, e.g.,

$$\Phi(f_n(\eta)) \equiv \int_0^{2\pi} (f_n, \phi(\eta, x)\phi(x) + f_n, \pi(\eta, x)\pi(x))dx + f_n, \eta \cdot r.$$  \hspace{1cm} (16)

We have that the momentum components of $e_n(\eta)$ and $e^*_n(0)$ differ by a small amount:

**Lemma 2.1** (Cf. [19]) *For* $n = (\pm, n, \sigma)$

$$e_{n, \pi}(\eta) - e_{n, \pi}(0) = \sum_{k} a_{n, k}(\eta) e_{k, \pi}(0)$$  \hspace{1cm} (17)

where the Fourier coefficients $a_{n, k}(\eta)$ are estimated

$$a_{n, k}(\eta) = \left\{ \begin{array}{ll}
\mathcal{O}(\eta^4 n^{4\theta - 2} \ln^2 n) & k = n \text{ or } k = (\mp, n, \sigma) \\
\mathcal{O}(\eta^2 n^{2\theta - 1} \ln n) & k = (\pm, n, \sigma'), \text{ or } k = (\mp, n, \sigma'), \sigma' \neq \sigma \\
\mathcal{O}(\eta^2 n^{2\theta - 1} \ln n) & k = (\pm, k, \sigma'), \text{ with } k \neq n.
\end{array} \right.$$  \hspace{1cm} (18)
Moreover, we also have that the imaginary parts of these coefficients satisfy

\[ \Im \alpha_{n,k}(\eta) = \begin{cases} \mathcal{O}(\eta^4 n^{10-3} \ln^2 n) & k = n \text{ or } k = (\mp, n, \sigma) \\ \mathcal{O}(\eta^2 n^{2\theta-2} \ln n) & k = (\pm, n, \sigma') \text{ or } k = (\mp, n, \sigma'), \quad \sigma' \neq \sigma \\ \mathcal{O}\left(\frac{n^2 k^\theta n^{\theta-1}}{n^2 - k^2}\right) & k = (\pm, k, \sigma'), \text{ with } k \neq n. \end{cases} \tag{19} \]

These estimates are uniform with respect to an ultraviolet cutoff \( M \).

Remark. In these estimates and throughout this article one encounters sums of the sort \( \sum_{k > 0; k \neq n} \frac{k^\theta}{n^2 - k^2} \). A convenient method for estimating these sums is to break the sum up into three pieces: 1) an “infrared” part with \( k < n/2 \) and where the denominator is simply replaced by \( n^2/2 \), the \( k \)-sum then giving \( \mathcal{O}(n^{\theta+1}/n^2) \); 2) a “near” part where \( n/2 \leq k < 3n/2 \), where the numerator is treated as a constant \( 3n^\theta/2 \) and the denominator is replaced by \( n|n - k| \) with the resulting \( \sum_{k:n/2 \leq k < 3n/2} \frac{1}{|n-k|} = \mathcal{O}(\ln n) \), so that the near part then gives \( \mathcal{O}(n^{\theta-1} \ln n) \); and 3) an “ultraviolet” part with \( k \geq 3n/2 \), where the summand is estimated simply by \( n^{\theta-2} \) with resulting sum \( \mathcal{O}(n^{\theta-2}) \). Thus this example would give, overall, \( \mathcal{O}(n^{\theta-1} \ln n) \), furnishing some explanation for the mysterious logarithms in our estimates.

### 2.2 Averaged expected energy flow

In this section, we give an expression for the expectation with respect to the invariant measure of the energy flow \( \pi(x) \partial_x \phi(x) \) through a point \( x \in [0, 2\pi] \) averaged over the circle. We assume an ultraviolet cut-off \( M \); Fourier modes of the field with frequency \( |n| > M \) are simply set to zero, but all of our estimates are uniform with respect to the cutoffs, so the cutoff \( M \) is not explicitly expressed. Thus we consider

\[
E \left[ \frac{1}{2\pi} \int_0^{2\pi} \pi(x) \partial_x \phi(x) dx \right] = \frac{1}{2\pi} \sum_m \sum_n \langle \epsilon_{m,\pi}(\eta), \partial_x \epsilon_{n,\phi}(\eta) \rangle E[\Phi(f_m(\eta))^* \Phi(f_n(\eta))],
\]

where on the right side we have employed eigenfunction expansions for \( \pi(x) \) and \( \phi(x) \) using the eigenfunctions of \( A(\eta) \). The inner product is simply the \( L^2(0, 2\pi) \)-inner product involving the \( \pi \) and \( \phi \) components of the eigenfunctions. In order to prove the theorem, we will show that the diagonal terms, i.e. terms with \( m = n \) in this double sum, do not contribute in the limit \( \eta \to 0 \), nor do terms \( m, n \) with \( m \neq n \). Only the near-resonant terms, terms such as \( m = (\pm, n, 1) \) and \( n = (\pm, n, 2) \), contribute in the limit, taking the form stated in the theorem.
Diagonal terms

We begin by considering the diagonal terms in the above equation (20), terms with \( n = m \). The estimates Eqs. (18, 19) above and the fact that \( e_n, \phi(\eta) = \frac{1}{\lambda_n(\eta)} e_n, \pi(\eta) \) from its eigenvalue equation imply that

\[
\langle e_n, \pi(\eta), \partial_x e_n, \phi(\eta) \rangle = \frac{2i}{\lambda_n} \langle R e_n, \pi(\eta), \partial_x \Im e_n, \pi(\eta) \rangle = \mathcal{O}(\eta^2 n^{2\theta - 2} \ln n) + \mathcal{O}(\eta^4 n^{4\theta - 2}).
\] (21)

Now in the linear theory, \( E[|\Phi(f_n)|^2] \) is bounded by the supremum of the temperatures, uniformly in \( \eta \) small, see [18, 19], so that \( \langle e_n, \pi(\eta), \partial_x e_n, \phi(\eta) \rangle E[|\Phi(f_n)|^2] \) is certainly summable over \( n \) provided the exponent \( \theta < \frac{1}{4} \) as is assumed; the sum clearly vanishes in the limit \( \eta \to 0 \). (In a non-linear theory, \( g \neq 0 \), we would also need \( E[|\Phi(f_n)|^2] \) to be bounded or have suitably slow growth in \( n \), but this is not known except in equilibrium, \( T_1 = T_2 \).)

Off-diagonal terms

We next consider the non-resonant off-diagonal terms \( m \neq n \) with \( m \neq n \) or the \( \pm \)'s differing in the double sum Eq. (20). The above estimates of the lemma, Eqs. (17, 18), imply that

\[
\langle e_m, \pi(\eta), \partial_x e_n, \phi(\eta) \rangle = \mathcal{O}\left(\eta^2 \frac{m^\theta n^\theta}{m^2 - n^2}\right) + \mathcal{O}\left(\eta^2 \frac{m^{\theta + 1} n^{\theta - 1}}{m^2 - n^2}\right) + \mathcal{O}(\eta^4 m^\theta n^{\theta - 1} \sum_{k: k \neq m, n} \mathcal{O}\left(\frac{k^{2\theta + 1}}{(k^2 - m^2)(k^2 - n^2)}\right)), \ m \neq n.
\] (22)

The last term on the right side of this equation can be estimated using the Schwarz inequality (regarding one of the functions as \( k^\theta/(k^2 - m^2) \)):

\[
\mathcal{O}(\eta^4 m^\theta n^{\theta - 1}) \sum_{k: k \neq m, n} \mathcal{O}\left(\frac{k^{2\theta + 1}}{(k^2 - m^2)(k^2 - n^2)}\right) = \mathcal{O}(\eta^4 m^{2\theta - 1} n^{2\theta - 1}).
\] (23)

Thus

\[
\langle e_m, \pi(\eta), \partial_x e_n, \phi(\eta) \rangle = \mathcal{O}\left(\eta^2 \frac{m^\theta n^\theta}{m^2 - n^2}\right) + \mathcal{O}\left(\eta^2 \frac{m^{\theta + 1} n^{\theta - 1}}{m^2 - n^2}\right) + \mathcal{O}(\eta^4 m^{2\theta - 1} n^{2\theta - 1}).
\] (24)

We also need an estimate on the correlation \( E[\Phi(f_m(\eta))^* \Phi(f_n(\eta))] \). Simply differentiating this expression with respect to time and using stationarity, we obtain the
identity

\[ E[\Phi(f_m(\eta))^{*}\Phi(f_n(\eta))] = \frac{1}{\lambda^*_m(\eta) + \lambda_n(\eta)} \left( E[\Phi(f_m(\eta))^{*}\langle f_n(\eta), g(\phi) \rangle] + E[\langle f_m(\eta), g(\phi) \rangle^{*}\Phi(f_n(\eta))] \right) \]

\[ - \frac{f_m^{*}\cdot T f_n(\eta)}{\lambda^*_m(\eta) + \lambda_n(\eta)}, \] (25)

the last term in the third line being an Ito-term. For this term \( f_m^{*}\cdot T f_n(\eta) \) are two-dimensional vectors, and \( T \) is to be interpreted as a \( 2 \times 2 \)-diagonal matrix, \( T = \text{Diag}(T_1, T_2) \). For the linear theory, the first term on the right side is absent. The component \( f_m^{*}\cdot T f_n(\eta) = \eta \langle f_m, \sigma \rangle/(\lambda_m(\eta) + 1) \) is of order \( \mathcal{O}(\eta^{1/2}) \); a similar estimate holds for \( f_m^{*}\cdot T f_n(\eta) \). We therefore have that

\[ E[\Phi(f_m(\eta))^{*}\Phi(f_n(\eta))] = \mathcal{O}\left( \eta^2 m^{\theta-1} n^{\theta-1} \right). \] (26)

This equation, combined with Eq. (24) above, gives the estimate

\[ \langle e_{m,\pi}(\eta), \partial_x e_{n,\phi}(\eta) \rangle E[\Phi(f_m(\eta))^{*}\Phi(f_n(\eta))] = \mathcal{O}\left( \eta^4 \left( \frac{m^2 n^2}{(m^2 - n^2)(m - n)} \right) \right) \]

\[ + \mathcal{O}\left( \eta^4 \left( \frac{m^2 n^2}{(m^2 - n^2)(m - n)} \right) \right) + \mathcal{O}\left( \eta^4 \left( \frac{m^2 n^2}{(m^2 - n^2)(m - n)} \right) \right) \] (27)

for \( m \neq n \), at least in the situation where \( m \neq n \). This quantity is certainly summable over \( m \) and \( n \), \( m \neq n \) using Young’s inequality, the sum going to zero, \( \eta \to 0 \).

The “antiresonant” terms in the double sum with \( m = n \), but with different choices of the \( \pm \) and no restriction on the \( \sigma \)’s, have matrix elements \( \langle e_{m,\pi}(\eta), \partial_x e_{n,\phi}(\eta) \rangle \) which can be \( \mathcal{O}(1) \) in \( n \). But the covariance factors are \( \mathcal{O}(\eta^4 n^{2\theta-3}) \), from the identity Eq. (25), with the denominator \( (\lambda^*_m(\eta) + \lambda_n(\eta)) \) being \( \mathcal{O}(n) \). So again the products of the matrix element and covariance factors for these terms are summable, the sum going to zero, \( \eta \to 0 \).

Remark: A term of the sort \( E[\Phi(f_m(\eta))^{*}\langle f_n(\eta), g(\phi) \rangle] \) appearing in the identity Eq. (25) is no worse than \( \kappa^2 \mathcal{O}(n^{-1/2}) \), where \( \kappa^2 \) is a uniform bound on variances \( E[|\Phi(f_n(\eta))|^2] \) and \( g \) is Lipschitz, a remark potentially of utility in analyzing the non-linear problem.

These off-diagonal terms in the double sum also include terms with matrix elements and field correlations involving the eigenvectors \( e_{-1,\sigma}(\eta) \) and \( f_{-1,\sigma}(\eta) \), corresponding to eigenvalues near \(-1\). In terms of the \( r \)-components of \( e_{-1,\sigma}(\eta) \) that are \( \mathcal{O}(1) \) (actually nearly normalized), the \( \phi \)-component is given by

\[ e_{-1,\sigma,\phi}(\eta) = -\eta e_{-1,\sigma,r}(\eta) \cdot (\lambda^2_{-1,\sigma}(\eta) - \partial_x^2 + 1)^{-1} \alpha. \] (28)
The coefficient estimates Eq. (18) then imply
\[
\langle e_{n,\pi}(\eta), \partial_x e_{-1,\sigma,\phi}(\eta) \rangle = O(\eta^{\theta-1})
\]
\[
\langle e_{-1,\sigma,\pi}(\eta), \partial_x e_{n,\phi}(\eta) \rangle = O(\eta^{\theta-2})
\]
\[
\langle e_{-1,\sigma,\pi}(\eta), \partial_x e_{-1,\sigma',\phi}(\eta) \rangle = O(\eta^2)
\]  
(29)

and, from the identity Eq. (25),
\[
E[\Phi(f_n(\eta))\Phi(f_{-1,\sigma}(\eta))] = O(\eta^{\theta-1})
\]
\[
E[\Phi(f_{-1,\sigma}(\eta))\Phi(f_{-1,\sigma'}(\eta))] = O(1).
\]  
(30)

The resulting products such as \( \langle e_{n,\pi}(\eta), \partial_x e_{-1,\sigma,\phi}(\eta) \rangle E[\Phi(f_n(\eta))\Phi(f_{-1,\sigma}(\eta))] \) are clearly summable in \( n \) and go to zero, \( \eta \to 0 \), hence do not contribute in the limit to the expected current.

**Near-resonant terms**

Finally we investigate the near-resonant terms in the double sum of the sort
\[
\langle e_{\pm,n,1,\pi}(\eta), \partial_x e_{\pm,n,2,\phi}(\eta) \rangle E[\Phi(f_{\pm,n,1}(\eta))\Phi(f_{\pm,n,2}(\eta))].
\]  
(31)

We have by the Fourier coefficient estimates, Eq. (18), that
\[
\langle e_{\pm,n,1,\pi}(\eta), \partial_x e_{\pm,n,2,\phi}(\eta) \rangle = \frac{n}{\lambda_{\pm,n,2}(\eta)} \left( \langle e_{\pm,n,1,\pi}(0), e_{\pm,n,1,\pi}(0) \rangle + \langle e_{\pm,n,2,\pi}(0), e_{\pm,n,2,\pi}(\eta) - e_{\pm,n,2,\pi}(0) \rangle \right)
\]
\[
+ \langle e_{\pm,n,1,\pi}(\eta) - e_{\pm,n,1,\pi}(0), e_{\pm,n,1,\pi}(0) \rangle
\]
\[
+ \frac{1}{\lambda_{\pm,n,2}(\eta)} \langle (e_{\pm,n,1,\pi}(\eta) - e_{\pm,n,1,\pi}(0)), \partial_x (e_{\pm,n,2,\pi}(\eta) - e_{\pm,n,2,\pi}(0)) \rangle
\]
\[
= \frac{n}{2\lambda_{\pm,n,2}(\eta)} + O(\eta^4 n^{2\theta-2} \ln^2 n).
\]  
(32)

We have used \( \partial_x e_{\pm,n,2,\pi}(0) = ne_{\pm,n,1,\pi}(0) \) and \( \partial_x e_{\pm,n,1,\pi}(0) = -ne_{\pm,n,2,\pi}(0) \). Recall as well that the \( e_n(0) \)'s are normalized so that \( \langle e_{\pm,n,1,\pi}(0), e_{\pm,n,1,\pi}(0) \rangle = 1/2 \), and that \( \lambda_n(\eta) = \lambda_n(0) + O(\eta^2 n^{2\theta-1}) \) with \( \lambda_n(0) = \pm i(n^2 + 1)^{1/2} \). We summarize:
\[
\langle e_{\pm,n,1,\pi}(\eta), \partial_x e_{\pm,n,2,\phi}(\eta) \rangle = \pm \frac{n}{2i(n^2 + 1)^{1/2}} + O(\eta^2 n^{2\theta-2}) + O(\eta^4 n^{4\theta-2} \ln^2 n).
\]  
(33)

It remains to analyze the covariance factor \( E[\Phi(f_{\pm,n,1}(\eta))\Phi(f_{\pm,n,2}(\eta))] \) in Eq. (31). Again, we assume that there is no non-linearity in the identity Eq. (25) so that
\[
E[\Phi(f_{\pm,n,1}(\eta))\Phi(f_{\pm,n,2}(\eta))] = -\frac{f_{\pm,n,1,r}(\eta) \cdot T f_{\pm,n,2,r}(\eta)}{\lambda_{\pm,n,1}(\eta) + \lambda_{\pm,n,2}(\eta)}
\]
\[
= -\frac{\eta^2 \langle f_{\pm,n,1,\pi}(\eta), \alpha \rangle \cdot T \langle f_{\pm,n,2,\pi}(\eta), \alpha \rangle}{(\lambda_{\pm,n,1}(\eta) + \lambda_{\pm,n,2}(\eta))(\lambda_{\pm,n,1}(\eta) + 1)(\lambda_{\pm,n,2}(\eta) + 1)}
\]  
(34)
Moreover, via two-dimensional matrix algebra, for \( \eta \) \( D \) with \( \langle \) \( \rangle \) Here, \( \text{coefficient estimates Eq. (18), which are also satisfied by the moment components Eq. (10):} \)

\[
\begin{align*}
\langle f_{\pm,n,1,\pi}(\eta), \alpha \rangle^* \cdot T \langle f_{\pm,n,2,\pi}(\eta), \alpha \rangle &= \left( \frac{T_1 - T_2}{2} \right) \langle f_{\pm,n,1,\pi}(\eta), \alpha \rangle^* \cdot D \langle f_{\pm,n,2,\pi}(\eta), \alpha \rangle \\
&+ \left( \frac{T_1 + T_2}{2} \right) \langle f_{\pm,n,1,\pi}(\eta), \alpha \rangle^* \cdot \langle f_{\pm,n,2,\pi}(\eta), \alpha \rangle, \quad (35)
\end{align*}
\]

with \( D \) the two-dimensional diagonal matrix \( \text{Diag}(1, -1) \). But again, using the Fourier coefficient estimates Eq. (18), which are also satisfied by the momentum components \( \{f_{n,\pi}(\eta)\} \) of the \( f_n \)’s, we find this equal to

\[
\left( \frac{T_1 - T_2}{2} \right) \langle f_{\pm,n,1,\pi}(0), \alpha \rangle^* \cdot D \langle f_{\pm,n,2,\pi}(0), \alpha \rangle + \mathcal{O}(\eta^2 n^{9\gamma - 1} \ln n). \quad (36)
\]

Here, \( \langle f_{\pm,n,1,\pi}(\eta), \alpha \rangle \) and \( \langle f_{\pm,n,2,\pi}(\eta), \alpha \rangle \) have nearly zero dot product (exactly zero for \( \eta = 0 \)), so that the \( (T_1 + T_2) \) part is represented in the small remainder terms. Moreover, via two-dimensional matrix algebra,

\[

\nu_n \equiv \langle f_{\pm,n,1,\pi}(0), \alpha \rangle^* \cdot D \langle f_{\pm,n,2,\pi}(0), \alpha \rangle = \frac{\Im (\hat{\alpha}_2^2(n)\hat{\alpha}_1^2(n))}{|\hat{\alpha}_1^2(n) + \hat{\alpha}_2^2(n)|}. \quad (37)
\]

(The notation \( \nu_n \) is temporarily introduced to facilitate handling of already complicated expressions.) Thus the numerator factor is given by

\[

\langle f_{\pm,n,1,\pi}(\eta), \alpha \rangle^* \cdot T \langle f_{\pm,n,2,\pi}(\eta), \alpha \rangle = \left( \frac{T_1 - T_2}{2} \right) \nu_n + \mathcal{O}(\eta^2 n^{9\gamma - 1} \ln n). \quad (38)
\]

The denominator of Eq. (34) is expanded with the aid of the eigenvalue expansion Eq. (10):

\[
\eta^2 \left( \lambda_{\pm,n,1}(\eta)^* + \lambda_{\pm,n,2}(\eta) \right) \left( \lambda_{\pm,n,1}(\eta)^* + 1 \right) \left( \lambda_{\pm,n,2}(\eta) + 1 \right) = -2 \left( \mu_{n,1} + \mu_{n,2} \right) \pm i(n^2 + 1)^{1/2}(\mu_{n,1} - \mu_{n,2}) + \mathcal{O}(\eta^2 n^{-2} \ln n). \quad (39)
\]

Multiplying the factors (38) and (39) in (34) and noting that \( \nu_n = \mathcal{O}(n^{9\gamma}) \), we obtain

\[
E[\Phi(f_{\pm,n,1}(\eta))^*\Phi(f_{\pm,n,2}(\eta))] = \left( \frac{T_1 - T_2}{2} \right) \left( \frac{2\nu_n}{2} \right) \left( \left( \mu_{n,1} + \mu_{n,2} \right) \pm i(n^2 + 1)^{1/2}(\mu_{n,1} - \mu_{n,2}) \right) + \mathcal{O}(\eta^2 n^{-2} \ln n). \quad (40)
\]
Equation (33) and the previous Eq.(40) show that
\[
\langle e_{\pm, n, \pi}^{(n)}(\eta), \partial_{x}e_{\pm, n, 2, \phi}^{(n)}(\eta) \rangle E[\Phi(f_{\pm, n, 1}(\eta))^{*}\Phi(f_{\pm, n, 2}(\eta))]
\]
\[
= -\frac{T_{1} - T_{2}}{2} \sum_{n} \frac{\nu_{n}}{(n^{2} + 1)^{1/2}} \left( (n^{2} + 1)^{1/2}(\mu_{n,1} - \mu_{n,2}) \mp i(\mu_{n,1} + \mu_{n,2}) \right)
\]
\[
+ \mathcal{O}(\eta^{2} n^{-2} \ln n) + \mathcal{O}(\eta^{2} n^{2g-3}) + \mathcal{O}(\eta^{4} n^{4g-3} \ln^{2} n).
\] (41)

**Symmetries**

In the limit, \( \eta \to 0 \) the following symmetries obtain:
\[
\langle e_{\pm, n, \pi}^{(n)}(0), \partial_{x}e_{\pm, n, 2, \phi}^{(n)}(0) \rangle E[\Phi(f_{\pm, n, 1}(0))^{*}\Phi(f_{\pm, n, 2}(0))]
\]
\[
= \left( \langle e_{\pm, n, 2, \pi}^{(n)}(0), \partial_{x}e_{\pm, n, 1, \phi}^{(n)}(0) \rangle E[\Phi(f_{\pm, n, 2}(0))^{*}\Phi(f_{\pm, n, 1}(0))] \right)^{*}
\]
\[
= \left( \langle e_{\mp, n, 2, \pi}^{(n)}(0), \partial_{x}e_{\mp, n, 1, \phi}^{(n)}(0) \rangle E[\Phi(f_{\mp, n, 2}(0))^{*}\Phi(f_{\mp, n, 1}(0))] \right).
\] (42)

The first relation involving an integration by parts in the matrix element and using that \( e_{n, \phi}^{(n)}(0) = \frac{1}{\nu_{n}(0)} e_{n, \pi}^{(n)}(0) \). The third line follows from the facts that \( e_{\pm, n, \sigma, \phi}^{(n)}(0) = -e_{\mp, n, \sigma, \phi}^{(n)}(0) \) and an integration by parts so that
\[
\langle e_{\mp, n, 2, \pi}^{(n)}(0), \partial_{x}e_{\mp, n, 1, \phi}^{(n)}(0) \rangle = \langle e_{\pm, n, 1, \pi}^{(n)}(0), \partial_{x}e_{\pm, n, 2, \phi}^{(n)}(0) \rangle,
\] (43)
and
\[
E[\Phi(f_{\mp, n, 2}(0))^{*}\Phi(f_{\mp, n, 1}(0))] = E[\Phi(f_{\pm, n, 1}(0))^{*}\Phi(f_{\pm, n, 2}(0))],
\] (44)
from Eq.(34). This simplifies the residual part of the double sum in Eq.(20); the expected current is, in any case, real.

In summary, the diagonal and off-diagonal terms and the remainder part of the near resonant terms in (41) in the double sum Eq.(20) are uniformly summable, and these sums go to zero, \( \eta \to 0 \). With the symmetry of the surviving terms, we have the final result
\[
\lim_{\eta \to 0} E \left[ \frac{1}{2\pi} \int_{0}^{2\pi} \pi(x) \partial_{x} \phi(x) \, dx \right]
\]
\[
= \frac{1}{2\pi} \lim_{\eta \to 0} \sum_{m} \sum_{n} \langle e_{m, \pi}^{(n)}(\eta), \partial_{x}e_{n, \phi}^{(n)}(\eta) \rangle E[\Phi(f_{m}(\eta))^{*}\Phi(f_{n}(\eta))]
\]
\[
= -\frac{T_{1} - T_{2}}{2} \sum_{n; n > 0} \frac{4n\nu_{n}(\mu_{n,1} - \mu_{n,2})}{(n^{2} + 1)(\mu_{n,1} - \mu_{n,2})^{2} + (\mu_{n,1} + \mu_{n,2})^{2}}
\]
\[
= -\frac{T_{1} - T_{2}}{2} \sum_{n; n > 0} \frac{n\Im(\hat{\alpha}_{1}^{*2}(n)\hat{\alpha}_{2}^{*2}(n))}{(n^{2} + 1)|\hat{\alpha}_{1}^{*2}(n)|^{2} + |\hat{\alpha}_{1}^{*2}(n)| + |\hat{\alpha}_{2}^{*2}(n)|^{2}}
\] (45)

by the definition of the \( \mu_{n, \sigma} \)’s and \( \nu_{n} \). This concludes the proof of the theorem. ■
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