Adaptive Denoising of Signals with Shift-Invariant Structure

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Abstract

We study the problem of discrete-time signal denoising, following the line of research initiated by [Nem91] and further developed in [JN09, JN10, HJNO15, OHJN16]. Previous papers considered the following setup: the signal is assumed to admit a convolution-type linear oracle – an unknown linear estimator in the form of the convolution of the observations with an unknown time-invariant filter with small ℓ_2-norm. It was shown that such an oracle can be “mimicked” by an efficiently computable non-linear convolution-type estimator, in which the filter minimizes the Fourier-domain ℓ_∞-norm of the residual, regularized by the Fourier-domain ℓ_1-norm of the filter. Following [OHJN16], here we study an alternative family of estimators, replacing the ℓ_∞-norm of the residual with the ℓ_2-norm. Such estimators are found to have better statistical properties; in particular, we prove sharp oracle inequalities for their ℓ_2-loss. Our guarantees require an extra assumption of approximate shift-invariance: the signal must be κ-close, in ℓ_2-metric, to some shift-invariant linear subspace with bounded dimension s. However, this subspace can be completely unknown, and the remainder terms in the oracle inequalities scale at most polynomially with s and κ. In conclusion, we show that the new assumption implies the previously considered one, providing explicit constructions of the convolution-type linear oracles with ℓ_2-norm bounded in terms of parameters s and κ.

1 Introduction

We study the problem of signal denoising in discrete time. Let C(Z) be the space of all complex functions on Z, and for n ∈ Z_+ := \{0, 1, 2, ...\}, let C_n(Z) be composed of all functions whose domain is included in

\[ D_n := \{-n, ..., n\}. \]

One is interested in estimating a signal x on a finite domain D_{m+n}, m, n ∈ Z_+, from noisy observations

\[ y_\tau = x_\tau + \sigma \zeta_\tau, \quad \tau ∈ D_{m+n}. \]

Here, \( \zeta_\tau \sim \mathcal{CN}(0, 1) \) are i.i.d. standard complex-valued Gaussian random variables, meaning that the mutual distribution of its real and imaginary parts is \( \mathcal{N}(0, I_2) \). Specifically, one can consider the two following tasks (see Fig. 1):

- **full recovery**, where estimation is required on the whole observation domain D_{m+n};
- **partial recovery**, where the signal is only required to be estimated on subdomain D_n ⊆ D_{m+n}, preferably with n ≥ cm for some constant c > 0.\(^1\)

In what follows, we restrict attention to partial recovery, returning to full recovery in Section 3.2. Symbols C, c, C_i, c_i with i = 0, 1, ... are generic constants that can recovered from the proofs.

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Vast literature has motivated the focus on the general class of linear estimators – those linear in observations $y$. Linear estimators have been extensively studied in various forms, being both theoretically attractive and easy to use in practice. In particular, under rather general assumptions about the set $\mathcal{X} \subset \mathbb{C}(\mathbb{Z})$ of possible signals, a linear minimax estimator is nearly minimax on $\mathcal{X}$, with respect to the pointwise loss and the $\ell_2$-loss, see [IK84, DLM90]. Besides, if $\mathcal{X}$ can be specified in a computationally tractable way, a near-minimax linear estimator can be efficiently computed by solving a convex optimization problem, see [JN17] and references therein. The last observation suggests the following approach to the denoising problem: given a computationally tractable set $\mathcal{X}$, one can simply compute a near-minimax linear estimator by “feeding” $\mathcal{X}$ to a convex optimization algorithm. The strength of this approach, however, comes at a price: the set $\mathcal{X}$ must be known to the statistician (although is not required to be of a simple predefined form). Moreover, this approach is not robust to model misspecification: a near-minimax estimator for some $\mathcal{X}$ can have a poor quality for some other $\mathcal{X}$. Although this difficulty can be addressed by the various adaptive model selection procedures [Lep91, LMS97, Tsy08, Joh11, GL+11], see also an overview in [L+15], these procedures usually impose strong structural assumptions on $\mathcal{X}$, assuming it to be known up to a small set of hyper-parameters, for example, the order and the magnitude of derivative of the underlying regression function. One exception is a recent work [L+15], where the authors devise a general adaptation scheme which can handle, for example, inhomogeneous and anisotropic smoothness of the signal. However, their approach results in a non-convex optimization problem, and therefore is not practical.

An alternative, more robust approach to the denoising problem with unknown $\mathcal{X}$ has been developed in [Nem91, JN09, JN10, HJNO15, OHJN16]. Here, instead of restricting the class of signals directly, requiring a specification of $\mathcal{X}$, one restricts the class of possible estimators. Namely, one introduces linear time-invariant, or convolution-type estimators, associated to filters $\phi \in \mathbb{C}_m(\mathbb{Z})$ as follows:

$$\hat{x}_t = [y * \phi]_t := \sum_{\tau \in \mathbb{Z}} \phi_{\tau} y_{t-\tau} = \sum_{\tau \in D_m} \phi_{\tau} y_{t-\tau}.$$  \hspace{1cm} (2)

In fact, convolution-type estimators are ubiquitous in nonparametric estimation, arising as near-optimal linear estimators in the setting of the known $\mathcal{X}$ – for instance, classical kernel estimators belong to this class [HJNO15]. The breakthrough of [Nem91] was the realization that such estimators can still be used as oracles in the context of adaptive estimation, where the structure of the signal, as given by $\mathcal{X}$, is unknown, and a near-optimal linear estimator is unavailable. To present this idea, we first have to introduce the related definition from [HJNO15].

**Definition 1.1 (Simple signals).** Given parameters $m, n \in \mathbb{Z}_+$, $\rho \geq 1$, and $\theta \geq 0$, signal $x \in \mathbb{C}(\mathbb{Z})$ is called $(m, n, \rho, \theta)$-simple if there exists a filter $\phi^o \in \mathbb{C}_m(\mathbb{Z})$ which satisfies

$$\|\phi^o\|_2 \leq \frac{\rho}{\sqrt{2m + 1}}.$$  \hspace{1cm} (3)
The class of \((m,n,\rho,\theta)\)-simple signals for a given quadruple \((m,n,\rho,\theta)\) is denoted \(S_{m,n}(\rho,\theta)\).

Noting that the pointwise mean-squared error of the convolution-type oracle estimator \(\hat{\phi}^o = \phi^o \ast y\) related to \(\phi^o\) can be decomposed as

\[
E|x_{\tau} - [\phi^o \ast y]_{\tau}|^2 = \sigma^2 E|\phi^o \ast \zeta|_{\tau}^2 + |x_{\tau} - [\phi^o \ast y]_{\tau}|^2,
\]

and using that \(\phi^o\) is fixed, we immediately arrive at the following bounds on the pointwise risks:\(^2\)

\[
[E|x_{\tau} - [\phi^o \ast y]_{\tau}|^2]^{1/2} \leq \frac{\sigma\sqrt{1 + \theta^2\rho}}{\sqrt{2m + 1}}, \quad \tau \in D_{m+n};
\]

as a consequence, we also obtain a bound on \(\ell_2\)-risk:\(^3\)

\[
[E\|x - \phi^o \ast y\|_{n,2}^2]^{1/2} \leq \kappa_{m,n} \sigma \sqrt{1 + \theta^2\rho}.
\]

where \(\| \cdot \|_{n,2}\) is the \(\ell_2\)-norm given by

\[
\|x\|_{n,2} = \left(\sum_{\tau \in D_{n}} |x_{\tau}|^2\right)^{1/2}.
\]

In other words, simple signals are those for which there exists a linear estimator, invariant in \((m + n)\)-radius of the origin, that uses observations in \(m\)-neighbourhood of a point, and attains a pointwise risk of order \(m^{-1/2}\). Parameters \(\rho\) and \(\theta\) provide the fine control of the risks; in particular, \(\theta\) specifies the bias-variance balance.

Now, assume that the only prior information about the signal is that it belongs to the class \(S_{m,n}(\rho,\theta)\) for some values of \((m,n,\rho,\theta)\). As we have just seen, this implies the existence of a convolution-type linear estimator \(\hat{\phi}^o = \phi^o \ast y\) with good statistical performance. However, the estimator itself is unavailable, and a legitimate question is whether one can still attain similar statistical performance. A reasonable way to achieve that would be through mimicking the form of the oracle estimator, that is, introducing a non-linear estimator \(\hat{\phi} = \hat{\phi}^o \ast y\), and inferring \(\hat{\phi}\) from data. As such, oracle characterization (3)–(4) can be used as a guide when selecting a data-dependent filter \(\hat{\phi}^o \ast y\). Namely, one could minimize the observable proxy to (4), the error \(y - \varphi \ast y\), over the set of filters satisfying (3).

The outlined approach seems reasonable, but its implementation is not straightforward. First, it is unclear what loss should be used. Second, and perhaps more importantly, it is not guaranteed that (3) provides a “correct” choice of the class of filters from the statistical perspective. Indeed, while the associated \(\ell_2\)-ball certainly contains the oracle filter, this ball can be too large, which would result in a high statistical price. As a simple illustration of these challenges, fix \(\ell_2\)-norm to measure the residual, and consider the following estimator:

\[
\hat{\phi} \in \text{Argmin}_{\phi \in C_{m}(\mathbb{Z})}\left\{\|y - \phi \ast y\|_{n,2} : \|\phi\|_2 \leq \frac{\rho}{\sqrt{2m + 1}}\right\}.
\]

As natural as this estimator appears after the above discussion, we are not aware of any theoretical results for its statistical performance, and this fact seems to be related to the \(\ell_2\)-norm constraint.

\(^2\)Bounds in high probability can also be obtained; we postpone them to Section 2 to keep presentation simple.

\(^3\)While we can bound the risk on \(D_{m+n}\), we avoid it since estimator \([\phi^o \ast y]_{m+n}\) uses observations on \(D_{2m+n}\).
Indeed, suppose that one would like to obtain an oracle inequality for the $\ell_2$-loss of $\hat{\phi}$. The feasibility of $\phi^o$ in the above problem implies that
\begin{equation*}
\|x - \hat{\phi} \ast y\|_{n,2}^2 \leq \|x - \phi^o \ast y\|_{n,2}^2 + 2\sigma \langle \xi, x - \phi^o \ast y \rangle_n + 2\sigma \langle -\xi, x - \hat{\phi} \ast y \rangle_n,
\end{equation*}
and then one is left with controlling the two cross terms corresponding to $\phi^o$ and $\hat{\phi}$, so that they grow at most as fast as $\kappa_{m,n}$ with $m, n$, cf. (6). While the first of these terms can be bounded rather easily, the second one pose a challenge because of the randomness of $\hat{\phi}$. In particular, one runs into difficulty when trying to control the quadratic in $\xi$ term $\sigma^2 \langle \xi, \hat{\phi} \ast \xi \rangle_n$, since with any fixed probability,
\begin{equation*}
\sup_{\|\phi\|_{m,2} \leq \sqrt{\frac{m}{n}} + 1} \langle \xi, \phi \ast \xi \rangle_n = \frac{\rho \|\xi\|_{n,2}^2}{\sqrt{2m + 1}} \geq c \rho \kappa_{m,n} \sqrt{2n + 1}.
\end{equation*}

In spite of the outlined difficulties, adaptive convolution-type estimators with non-trivial performance guarantees can indeed be obtained. The key idea, dating back to [Nem91], is to pass to a new oracle with a sharper characterization than (3), which would allow one to restrict the search space. Namely, it can be easily shown that (m, n, $\rho$, $\theta$)-simplicity of $x$ implies the existence of a new oracle $\phi^o \in C_{2m}(\mathbb{Z})$ with a twice larger support, which satisfies an analogue of (4),
\begin{equation*}
|x_r - [\phi^o \ast x]_r| \leq \frac{2\sqrt{2\sigma \theta \rho^2}}{\sqrt{4m + 1}}, \quad r \in D_n, \tag{7}
\end{equation*}
and the following counterpart of (3):
\begin{equation*}
\|F_{2m}[\phi^o]\|_1 \leq \frac{2\rho^2}{\sqrt{4m + 1}}, \tag{8}
\end{equation*}
where $F_n$ is the unitary Discrete Fourier transform $C_n(\mathbb{Z}) \to C_n(\mathbb{Z})$:
\begin{equation*}
([F_n x])_k = \frac{1}{\sqrt{2n + 1}} \sum_{\tau \in D_n} \exp \left( \frac{2\pi ik \tau}{2n + 1} \right) x_\tau, \quad k \in D_n.
\end{equation*}
While the new bounds are inflated by the additional factor of $\rho$, and the size of the neighborhood in which the error is controlled is somewhat restricted, the bound (8) is essentially stronger than its counterpart (3) which would only allow for $\|F_n[\phi^o]\|_1 \leq \rho$. Based on this observation, the works [Nem91, JN09, HJNO15] introduced a class of adaptive convolution-type estimators with provable guarantees of statistical performance. These uniform-fit estimators correspond to filters which minimize the uniform-norm Fourier-residual $\|F_n[y - y \ast \varphi]\|_{n,\infty}$, constrained or penalized by the $\ell_1$-norm of the Fourier transform of the filter. Such estimators can be efficiently computed since the corresponding filters are given as optimal solutions to well-structured convex optimization problems – namely, second-order cone problems. As for their statistical performance, the price for adaptation for them – that is, the suboptimality factor compared to the initial oracle $\phi^o$ – was proved to be $C \rho^3 / \sqrt{\log(m + n)}$ in the case of pointwise loss; with the lower bound $c \rho^3 / \log m$ when $m \geq c'n$, see [HJNO15, Theorems 2 and 5]. While the choice of $\ell_1$ regularization is quite natural in view of (8), the choice of $\ell_\infty$-norm for the residual is dictated by technical considerations, providing simpler control of the pointwise risk at the expense of a large suboptimality factor. Moreover, the choice of $\ell_\infty$-fit seems to be artificial given that the statistical performance of estimators is quantified with $\ell_2$-loss. As such, one may ask the following question:

\textit{Can one obtain rigorous performance guarantees for $\ell_1$-regularized least-squares estimators, in which one minimizes the $\ell_2$-loss $\|y - y \ast \varphi\|$?}
Contributions. Here we propose a new family of regularized least-squares estimators, obtained by minimizing an \( \ell_2 \)-norm residual, constrained or penalized by the \( \ell_1 \)-norm of the filter in the Fourier domain. Similarly to the uniform-fit estimators, the new estimators can be efficiently computed using convex optimization. We prove sharp oracle inequalities for the \( \ell_2 \)-loss of these estimators which lead to the improved price for adaptation compared with the uniform-fit estimators: \( C(\rho^2 + \rho \sqrt{\log(m+n)}) \) for the case of pointwise loss, and \( C(\rho + \sqrt{\log(m+n)}) \) for the case of \( \ell_2 \)-loss. These oracle inequalities hold under an assumption that the the signal is approximately shift-invariant, meaning that \( x \in C(\mathbb{Z}) \) can be represented as the sum of two components:

- a component in a small-dimensional shift-invariant linear subspace \( S \) of \( C(\mathbb{Z}) \);
- a residual component which is controlled in \( \ell_2 \)-norm.

The remainder terms in the oracle inequalities explicitly depend on the subspace dimension \( s = \dim(S) \) and the magnitude \( \kappa \) of the residual component. Importantly, the shift-invariant subspace, as well as the decomposition itself, is assumed to be unknown. The approximate shift-invariance assumption is non-trivially related to the previously introduced concept of simplicity: in fact, all examples of simple signals introduced so far satisfy small-order ordinary difference equations, see [JN10], and as such, belong to small-dimensional shift-invariant subspaces. We further investigate this relation, showing that approximate shift-invariance implies simplicity with parameters \( \rho, \theta \) moderately depending on \( s, \kappa \). Using these developments, we obtain new results for the the denoising harmonic oscillations – sums of complex sinusoids with arbitrary (and unknown) frequencies. The known approaches to this particular denoising problem [BTR13, TBR13] are essentially based on the ideas from sparse recovery. As such, they require structured sparsity assumptions [DB13]; in the case of signals with line spectra, this boils down to frequency separation assumptions. In contrast, the approach based on convolution-type estimators allows to attain near-optimal statistical rates without imposing any such assumptions.

The preliminary versions of some results presented here have appeared in [OHJN16].

Organization of further sections. In the remainder of Section 1, we introduce the necessary notation. Section 2 contains the main results of this work: there, we present the new estimators and sharp oracle inequalities for their \( \ell_2 \)-loss, and then use these inequalities to derive guarantees for \( \ell_2 \)-loss and pointwise loss. We then extend the results to filtering – estimation with one-sided filters – and prediction of the signal beyond the observation domain. In Section 3, we study the relationship between the classes of approximately shift-invariant and simple signals. Finally, we consider the application of our results to the denoising of harmonic oscillations, comparing our approach agains the state of the art. The technical proofs are deferred to the appendix to simplify the presentation.

1.1 Notation

Asymptotic notation. \( \log(x) \) denotes the natural logarithm. Symbols \( c, C, \) and \( C' \), sometimes with integer subscripts, stand for absolute constants whose exact values can be recovered from the proofs. We use the “big O” notation: for two functions \( f, g \) of the same argument, \( f = \mathcal{O}(g) \) means that there exists a constant \( C \geq 0 \) such that \( f \leq Cg \) for any possible value of the argument; \( f = \Omega(g) \) is the same as \( g = \mathcal{O}(f) \). Besides, \( f = \widetilde{\mathcal{O}}(g) \) means that \( f \leq g \) holds up to a logarithmic factor in the common argument of \( f \) and \( g \), and is equivalent to \( g = \widetilde{\Omega}(f) \).
Matrices, vectors, and signal slices. We follow the “Matlab convention” for matrices: $[A, B]$ and $[A; B]$ denote, correspondingly, the horizontal and vertical concatenations of two matrices of compatible dimensions. Unless explicitly stated otherwise, all vectors are column vectors. Given a signal $x \in \mathcal{C}(\mathbb{Z})$ and $n_1, n_2 \in \mathbb{Z}$ such that $n_1 \leq n_2$, we define the “slicing” map

$$x_{n_1}^{n_2} := [x_{n_1}; \ldots; x_{n_2}]. \quad (9)$$

Filters. Recall that $\mathcal{C}(\mathbb{Z})$ is the linear space of all two-sided complex sequences, and $\mathcal{C}_m(\mathbb{Z})$ is the space of sequences whose support is contained in $D_n := \{-n, \ldots, n\}$. With a slight abuse of the term, we call the smallest $m \in \mathbb{Z}^+$ such that $\varphi \in \mathcal{C}_m(\mathbb{Z})$ the bandwidth of a filter $\varphi$ and denoted $W(\varphi)$. In addition to bilateral filters $\mathcal{C}_m(\mathbb{Z})$, we introduce the following classes of filters:

- one-sided filters $\mathcal{C}_m^+(\mathbb{Z})$ and $\mathcal{C}_m^-(\mathbb{Z})$, whose support is contained in
  $$D_m^+ := \{0, \ldots, m\} \; \text{or} \; D_m^- = \{-m, \ldots, 0\}.$$  

- shifted filters $\mathcal{C}_m^h(\mathbb{Z})$ with shift $h \in \mathbb{Z}$, whose support is contained in
  $$D_m^h := \{h, \ldots, h + m\};$$

note that $h < 0$ corresponds to bilateral filters with lobes of different length, $h = 0$ and $h = -m$ to one-sided filters, and $h > 0$ to predictive filters which allow for extrapolation with “horizon” $h$ beyond the observation domain.

Note that “slicing mapping” (9) allows to identify $\mathcal{C}_m(\mathbb{Z})$, $\mathcal{C}_m^+(\mathbb{Z})$, $\mathcal{C}_m^-(\mathbb{Z})$, $\mathcal{C}_m^h(\mathbb{Z})$ with complex vector spaces $\mathbb{C}^N$ of appropriate dimension.

Convolution. The discrete convolution of $\varphi * \psi \in \mathcal{C}(\mathbb{Z})$ of $\varphi, \psi \in \mathcal{C}(\mathbb{Z})$ is formally defined as

$$[\varphi * \psi]_t := \sum_{\tau \in \mathbb{Z}} \varphi_{\tau} \psi_{t-\tau};$$

Clearly, the convolution is a commutative operation. It is convenient to identify a filter $\varphi$ with its Laurent series $\varphi(z) = \sum_j \varphi_j z^j$. Then, $\varphi * \psi$ corresponds to the product $\varphi(z)\psi(z)$, and therefore

$$W(\varphi * \psi) \leq W(\varphi) + W(\psi).$$

Yet another way to view convolution is by introducing the forward shift operator $\Delta$ on $\mathcal{C}(\mathbb{Z})$,

$$[\Delta x]_t = x_{t-1},$$

and its inverse, the backward shift $\Delta^{-1}$. The convolution can then be expressed as $\varphi * \psi = \varphi(\Delta)\psi$.

Convolution-type estimates. Given $\varphi \in \mathcal{C}(\mathbb{Z})$ with $W(\varphi) < \infty$, and observations $y = (y_t)$, we can associate with $\varphi$ the following estimate of $x \in \mathcal{C}(\mathbb{Z})$:

$$\hat{x} = \varphi * y = \varphi(\Delta)y.$$

In fact, such estimate is simply a kernel estimate over the grid $\mathbb{Z}$ with a finitely supported discrete kernel $\varphi$. Note that the notions we use here have counterparts in signal processing: convolution-type linear estimates with bilateral filters correspond to linear interpolation, those with one-sided filters to linear filtering, and those with $h$-predictive filters – to linear prediction.
Norms in the time and Fourier domains. Given $p \geq 1$ and $n \in \mathbb{Z}_+$, we introduce the semi-norms on $C(\mathbb{Z})$ defined by

$$\|x\|_{n,p} := \left( \sum_{\tau \in D_n} |x_\tau|^p \right)^{1/p},$$

with the natural interpretation for $p = +\infty$. When such notation is unambiguous, we also use $\| \cdot \|_p$ to denote the “usual” $\ell_p$-norm on $C(\mathbb{Z})$, e.g. $\|x\|_p = \|x\|_{n,p}$ whenever $W(x) \leq n$. Besides, we define the unitary Discrete Fourier Transform (DFT) operator $F_n : C_n(\mathbb{Z}) \to C_n(\mathbb{Z})$ by

$$(F_n[x])[k] = \frac{1}{\sqrt{2n+1}} \sum_{\tau \in D_n} \exp\left( -\frac{i 2\pi k \tau}{2n + 1} \right) x_\tau, \quad k \in D_n,$$

and introduce the seminorms associated with the standard $p$-norms of the DFT of the signal:

$$\|x\|_{n,p}^F := \left( \sum_{k \in D_n} |(F_n[x])[k]|^p \right)^{1/p}.$$  \hspace{1cm} (10)

Note that the unitarity of $F_n$ implies that $F_n[\zeta]$ follows the same law as $[\zeta]_{-n}$, and also the Parseval identities: for any $\varphi, \psi \in C(\mathbb{Z})$ and $m \in \mathbb{Z}_+$, denoting $\langle \varphi, \psi \rangle := \sum_{\tau \in \mathbb{Z}} \bar{\varphi}_\tau \psi_\tau$ and $\langle \varphi, \psi \rangle_m := \sum_{\tau \in D_m} \bar{\varphi}_\tau \psi_\tau$, $\varphi_\tau$ being the complex conjugate of $\varphi_\tau$, one has

$$\langle \varphi, \psi \rangle_m = (F_m \varphi, F_m \psi), \quad \|x\|_{n,2} = \|x\|_{n,2}^F.$$  \hspace{1cm} (11)

Unilateral Fourier transform and norms. Extra notation is required for the prediction setting. We introduce the norm $\| \cdot \|_{n,p}^+$, unilateral DFT operator $F_n^+$ acting on $C_n^+(\mathbb{Z})$, the corresponding Fourier-domain norms $\| \cdot \|_{n,p}^+$, and the dot product $\langle \varphi, \psi \rangle_n^+$. All these objects are defined similarly to their two-sided counterparts, except that $D_n$ must be replaced with $D_n^+$, and $\sqrt{2n+1}$ with $\sqrt{n+1}$.

2 Main results

Preliminaries. Recall that we are interested in the task of partial recovery: estimate the signal on $D_n$ from noisy observations (1) on $D_{m+n}$ using a filter $\varphi \in C_m(\mathbb{Z})$. Without the loss of generality, assume for a moment that $m = 2m_0$ for some $m_0 \in \mathbb{Z}$, and that $x$ is $(m_0, n, \rho, \theta)$-simple (see Definition 1.1). As we have already mentioned, this would imply the existence of an oracle filter $\varphi^o \in C_m(\mathbb{Z})$ with a small $\ell_1$-norm and small pointwise risk on $D_n$. In particular,

$$\|\varphi^o\|_{m,1}^F \leq \frac{\varrho}{\sqrt{2m+1}}, \quad \varrho := 2\rho^2,$$  \hspace{1cm} (12)

and

$$|x_\tau - [\varphi^o * x]_\tau| \leq \frac{\sqrt{2} \sigma \varrho}{\sqrt{2m+1}}, \quad \tau \in D_n,$$  \hspace{1cm} (13)

cf. (3) and (4), which results in

$$[E|x_\tau - [\varphi^o * y]_\tau|^2]^{1/2} \leq \frac{\sigma \sqrt{1 + 2q^2 \varrho}}{\sqrt{2m+1}}, \quad \tau \in D_n.$$  \hspace{1cm} (14)
\[ \mathbb{E}[\|x - \varphi^o \ast y\|_{n,2}^2]^{1/2} \leq \sigma \kappa_{m,n} \sqrt{1 + 2\theta^2 \rho}, \]  

(15)

where we define the ratio 
\[ \kappa_{m,n} := \sqrt{\frac{2n + 1}{2m + 1}}. \]

Unfortunately, this “ideal” estimator is unavailable. We now present adaptive estimators that are able to “mimic” the statistical properties of \( \varphi^o \), as given by (14)-(15), whenever it exists.

**Construction of estimators.** Given \( m, n \in \mathbb{Z}_+ \) and \( \rho > 0 \), let \( \hat{\varphi}_{\text{con}} \) be defined as follows:

\[ \hat{\varphi}_{\text{con}} = \text{Argmin}_{\varphi \in \mathbb{C}_m(\mathbb{Z})} \left\{ \|y - \varphi \ast y\|_{n,2}^2 : \|\varphi\|_{m,1} \leq \frac{\rho}{\sqrt{2m + 1}} \right\}, \]  

(Con)

We refer to \( \hat{x} = \hat{\varphi}_{\text{con}} \ast y \) as the constrained (least-squares) estimator. In the sequel, we will prove a sharp oracle inequality for the constrained estimator, which states that the \( \ell_2 \)-loss of this estimator is comparable to the \( \ell_2 \)-loss of any filter \( \varphi \) feasible to (Con), and in particular, to \( \varphi^o \) provided that \( \rho = \bar{\rho} \). However, this requires the knowledge of \( \rho \), or a non-trivial upper bound on it, which is not always achievable in practice. Hence, we also consider the penalized estimator \( \hat{x} = \hat{\varphi}_{\text{pen}} \ast y \), where for some \( \lambda > 0 \), filter \( \hat{\varphi}_{\text{pen}} \) is defined by

\[ \hat{\varphi}_{\text{pen}} \in \text{Argmin}_{\varphi \in \mathbb{C}_m(\mathbb{Z})} \left\{ \|y - \varphi \ast y\|_{n,2}^2 + \sigma^2 \lambda^2 (2m + 1) \|\varphi\|_{m,1}^2 \right\}. \]  

(Pen)

As we see, instead of the knowledge of \( \rho \), some knowledge of noise variance \( \sigma^2 \) is required to properly tune this estimator. In fact, we will show that as long the penalization parameter \( \lambda \) is set as an absolute constant, the \( \ell_2 \)-loss of the penalized estimator enjoys essentially the same bound as the constrained estimator with the “optimal” choice of \( \rho \) – the one balancing the norm and bias of the oracle in the best possible way. Hence, the practical recommendation is to use (Pen) when possible, i.e. whenever \( \sigma^2 \) is known or can be estimated from data.

### 2.1 Oracle inequalities for \( \ell_2 \)-loss

To analyze the adaptive estimators, we need the following assumption:\(^4\)

**Assumption 2.1 (Approximate shift-invariance).** \( x \in \mathbb{C}(\mathbb{Z}) \) admits a decomposition

\[ x = x^S + \varepsilon. \]

Here, \( x^S \in S \), where \( S \) is some shift-invariant linear subspace of \( \mathbb{C}(\mathbb{Z}) \) with \( s := \text{dim}(S) \leq 2n + 1 \), and \( \varepsilon \) is bounded in the \( \ell_2 \)-norm: for some \( \kappa \geq 0 \) one has

\[ \|\Delta^{-\tau} \varepsilon\|_{n,2} \leq \kappa \sigma, \quad \tau \in \mathbb{D}_m. \]  

(16)

In other words, Assumption 2.1 states the existence of a shift-invariant linear subspace \( S \), \( \Delta S \subseteq S \) with controlled dimension, to which the signal is close in \( \ell_2 \)-norm. Importantly, the decomposition of the signal, as well as the subspace \( S \), can be completely unknown to the statistician. Besides, Assumption 2.1 merits some further remarks.

\(^4\)Recall that the lag operators \( \Delta \) and \( \Delta^{-1} \) on \( \mathbb{C}(\mathbb{Z}) \) are defined by \( [\Delta x]_t = x_{t-1} \) and \( [\Delta^{-1} x]_t = x_{t+1} \).
Remark 2.1. Letting the signal to be close, in $\ell_2$-norm, to a shift-invariant subspace, instead of simply belonging to the subspace, is essential. It significantly extends the set of signals to which our theory applies, allowing to address the nonparametric setting. For example, signals which are close to discrete-time polynomials, which satisfy homogeneous linear difference equations, and hence belong to a small shift-invariant subspaces, are Sobolev-smooth functions sampled over the uniform grid [JN10]. Some other examples will be considered in Sections 3–3.2.

Remark 2.2. Assumption 2.1 looks similar to signal simplicity according to Definition 1.1, which also postulates some kind of “invariance” of the signal claiming that there exists a time-invariant filter which reproduces the signal on a certain interval. However, the actual relationship between the two notions is rather intricate, and will be investigated in Section 3.

We now present oracle inequalities which relate the $\ell_2$-loss of adaptive filter $\hat{\varphi}$ to the loss of any feasible solution $\varphi$ to the corresponding optimization problem. These inequalities, interesting for their own sake, will be used later on to obtain performance guarantees for the proposed estimators in $\ell_2$-loss and the pointwise loss. We first state the result for the constrained estimator.

Theorem 2.1. Let $m,n \in \mathbb{Z}_+$, and let $\hat{\varphi}_{\text{con}}$ be an optimal solution to (Con) with some $\bar{\rho} > 1$. Suppose that Assumption 2.1 holds with some $(s, \varkappa)$, and let $\varphi$ be any feasible solution to (Con). Then for any $0 < \alpha \leq 1$, the following holds with probability at least $1 - \alpha$: $\hat{\varphi}_{\text{con}}$ satisfies

$$
\|x - \hat{\varphi}_{\text{con}} \ast y\|_n,2 \leq \|x - \varphi \ast y\|_n,2 + C\sigma(Q_0)^{1/2},
$$

where

$$Q_0 = Q_0(\bar{\rho}, \varkappa, \kappa_{m,n}, \alpha) := \bar{\rho}(\kappa_{m,n}^2 + 1) \log[(m + n) / \alpha] + \bar{\rho}\varkappa \sqrt{\log[1/\alpha]} + s. \tag{18}
$$

The counterpart of Theorem 2.1 for the penalized estimator is as follows.

Theorem 2.2. Let $m,n \in \mathbb{Z}_+$, and let $\hat{\varphi}_{\text{pen}}$ be an optimal solution to (Pen) with some $\lambda > 0$. Suppose that Assumption 2.1 holds with some $(s, \varkappa)$, and let $\varphi$ be any feasible solution to (Pen). Then for any $0 < \alpha \leq 1$, the following holds with probability at least $1 - \alpha$: $\hat{\varphi}_{\text{pen}}$ satisfies

$$
\|x - \hat{\varphi}_{\text{pen}} \ast y\|_n,2 \leq \|x - \varphi \ast y\|_n,2 + \sigma(\lambda \bar{\rho} + C_1 \lambda^{-1} Q_1 + C_2 Q_2^{1/2}),
$$

where

$$Q_1 = Q_1(\varkappa, \kappa_{m,n}, \alpha) = (\kappa_{m,n}^2 + 1) \log[(m + n) / \alpha] + \varkappa \sqrt{\log[1/\alpha]} + 1,
Q_2 = Q_2(\bar{\rho}, s, \varkappa, \alpha) = \bar{\rho} \log[1/\alpha] + \varkappa \sqrt{\log[1/\alpha]} + s.
$$

In particular, when choosing $\lambda = Q_1^{1/2}$, we get

$$
\|x - \hat{\varphi}_{\text{pen}} \ast y\|_n,2 \leq \|x - \varphi \ast y\|_n,2 + C\sigma(Q_1^{1/2} + Q_2^{1/2}).
$$

Discussion of the results. Note that the choice $\lambda = \sqrt{C_1 Q_1 / \bar{\rho}}$ would result in the same remainder term (of the order $\sqrt{\bar{\rho}}$) as for the constrained estimator with the “optimal” constraint parameter $\bar{\rho} = \rho$. Clearly, this choice cannot be implemented since parameter $\bar{\rho}$ is unknown. Nevertheless, Theorem 2.2 provides us with an implementable choice$^5$ of $\lambda$ that still results in an oracle inequality, at the expense of a larger remainder term which scales as $\bar{\rho}$. As a result of this suboptimal choice of $\lambda$, oracle inequality (19) essentially loses its sharpness when applied for a simple signal. Indeed, in this case we can only hope that the oracle loss itself scales as $\rho$, cf. (15). However, in the main application that we have in mind – simple signals – one is interested in the bounds on the overall risk of $\hat{\varphi}$ compared to that of $\varphi^0$, and not just the remainder term. As we are about to demonstrate in the next section, the loss of sharpness in this case is not crucial.
2.2 Corollaries for simple signals

In continuation of the above discussion, assume that in addition to Assumption 2.1, one has $x \in S_{m_0,n}(\rho, \theta)$ with $m = 2m_0$. Then, it is not hard to prove the following $(1 - \alpha)$-confidence bound for the $\ell_2$-loss of the oracle $\varphi \in C_m(Z)$ (see Appendix B):

$$\|x - \varphi^o \ast y\|_{n,2} \leq 4\sigma_0 \kappa_{n,m} \rho^2 (1 + \sqrt{2\theta} + \sqrt{\log(1/\alpha)}).$$

(20)

Together with Theorem 2.1, this bound implies the following result:

**Corollary 2.1.** Assume that $x \in S_{m_0,n}(\rho, \theta)$ with $\rho \geq 1/\sqrt{2}$, and let $m = 2m_0 > 0$. Moreover, suppose that Assumption 2.1 holds with some $(s, \kappa)$, and let $\hat{\varphi}_{\text{con}}$ and $\hat{\varphi}_{\text{pen}}$ be, correspondingly, optimal solutions to (Con) with $\bar{\rho} = 2\rho^2$ and to (Pen) with $\lambda$ chosen as in the premise of Theorem 2.2. Then, for any $0 < \alpha \leq 1$, with probability at least $1 - \alpha$ one has

$$\|x - \hat{\varphi}_{\text{con}} \ast y\|_{n,2} \leq 4\sigma_0 \kappa_{n,m} \rho^2 (1 + \sqrt{2\theta} + \sqrt{\log(1/\alpha)}) + C \sigma \{\rho \sqrt{(\kappa_{n,m}^2 + 1) \log((m + n)/\alpha)} + \kappa \sqrt{\log(1/\alpha)} + \sqrt{s}\},$$

(21)

$$\|x - \hat{\varphi}_{\text{pen}} \ast y\|_{n,2} \leq 4\sigma_0 \kappa_{n,m} \rho^2 (1 + \sqrt{2\theta} + \sqrt{\log(1/\alpha)}) + C \sigma \{\rho \sqrt{(\kappa_{n,m}^2 + 1) \log((m + n)/\alpha)} + (\kappa + 1) \sqrt{\log(1/\alpha)} + \sqrt{s}\}.$$

(22)

As a consequence,

$$[\mathbb{E}\|x - \hat{\varphi}_{\text{con}} \ast y\|_{n,2}^2]^{1/2} \leq C \sigma \{\kappa_{n,m} \rho^2 (1 + \theta) + \rho \sqrt{(\kappa_{n,m}^2 + 1) \log((m + n)/\alpha)} + \kappa \sqrt{\log(1/\alpha)} + \sqrt{s}\},$$

(23)

$$[\mathbb{E}\|x - \hat{\varphi}_{\text{pen}} \ast y\|_{n,2}^2]^{1/2} \leq C \sigma \{\kappa_{n,m} \rho^2 (1 + \theta) + \rho^2 \sqrt{(\kappa_{n,m}^2 + 1) \log((m + n)/\alpha)} + \kappa \sqrt{\log(1/\alpha)} + \sqrt{s}\}.$$

(24)

As we see, the bounds for both estimators coincide up to a logarithmic factor. Moreover, as we show in Section 3, if $x$ belongs to a shift-invariant subspace (that is, $\kappa = 0$), then $x \in S_{m,n}(\rho, \theta)$ with $\theta = 0$ and $\rho = O(\sqrt{s})$, so that the right-hand sides in (23) and (24) are both bounded from above with

$$Cs(\kappa_{n,m} \sqrt{\log((m + n)/\alpha)} + 1).$$

One should realize, however, that Theorems 2.1–2.2 per se are not tied to the particular choice of oracle characterized by (12)–(13), and moreover, do not assume at all that the signal is simple. Hence, the oracle inequalities would remain useful in case where the bound $\rho = O(\sqrt{s})$ is unavailable – e.g. in the situation considered in Section 2.3 when dealing with one-sided filters. On the other hand, if there exists, by chance, an oracle with smaller $\rho$, adaptive estimators are guaranteed to be competitive with it.

**Guarantees for pointwise loss.** Under the premise of Corollary 2.1, i.e. when the signal is simple according to Definition 1.1 and as well satisfies Assumption 2.1, one can also bound the pointwise loss of the adaptive estimators on a subdomain of $D_n$.

**Proposition 2.1.** Suppose that the premise of Corollary 2.1 holds with $n \geq m_0$ (recall that $m = 2m_0 > 0$), and fix $0 < \alpha \leq 1$. Let $\hat{\varphi}_{\text{con}}$ be an optimal solution to (Con) with $\bar{\rho} = 2\rho^2$, and let $\hat{\varphi}_{\text{pen}}$ be an optimal solution to (Pen) with $\lambda$ chosen as in the premise of Theorem 2.2. Then, for any fixed $t \in D_{n-m_0}$, the following holds with probability at least $1 - \alpha$:

---

5In contrast to that of $\bar{\rho}$, the knowledge of $\kappa$ is a reasonable assumption, since in practice one usually fixes $\kappa$ in advance to ensure the desired bias-variance ratio, see also Proposition 3.1.
As a consequence, we remind that additional notation for the prediction setting has been introduced in Section 1.1. We now present an extension of the results obtained in the previous section to the prediction setting (as such, we also cover filtering since the latter corresponds to prediction with \( \hat{m} \)). This filter can be used as well to estimate the signal to the right of \( y \) horizon setting (as such, we also cover filtering since the latter corresponds to prediction with \( \hat{m} \)).

So far, we only considered the setting of interpolation, whereas in the case of uniform-fit estimators, one has to fit a separate filter for any target point, see [HJNO15]. Note that the computation of \( \hat{\varphi}_{\text{pen}} \) once \( \hat{\varphi} \) has been obtained, can be done in time \( \tilde{O}(m+n) \) via Fast Fourier transform. Given a signal that lives on the “global” domain \( D_N \), the above observation allows to dramatically reduce the computational price of the proposed estimators as compared to the uniform-fit ones, by applying them in a “blockwise” manner: dividing the overall domain to the blocks of size \( m+n \) with \( n=cm \), and computing only one adaptive filter for each block. The optimal block size can be chosen adaptively via the Lepski-Goldenschlueger method, see [HJNO15, Theorem 6].

### 2.3 Filtering and prediction

So far, we only considered the setting of interpolation, where one estimates the signal by tuning a two-sided filter \( \hat{\varphi} \in \mathcal{C}_m(\mathbb{Z}) \). Meanwhile, the cases of filtering \( \varphi \in \mathcal{C}^+_m(\mathbb{Z}) \) and prediction \( \varphi \in \mathcal{C}^h_m(\mathbb{Z}) \) can also be of interest, either on their own right, or when full recovery is required, since two-sided filters cannot be used near the borders of the observation domain. We now present an extension of the results obtained in the previous section to the prediction setting (as such, we also cover filtering since the latter corresponds to prediction with \( h=0 \)).

We remind that additional notation for the prediction setting has been introduced in Section 1.1.

The prediction setting can be summarized as follows, see Fig. 2. We assume that one is given a horizon \( h \in \mathbb{Z}^+ \) and observations \( (y_t) \) on the interval \([-m-n-h,\ldots,0] \), and our first objective would be to estimate \( x \) with small \( \ell_2 \)-loss on the interval \([-n,0] \) by taking convolution of \( y \) with a predictive filter \( \hat{\varphi} \in \mathcal{C}^h_m(\mathbb{Z}) \) fitted from these observations (later on, we will also see that this filter can be used as well to estimate the signal to the right of \( t=0 \)). We consider adaptive filters \( \hat{\varphi} \) given as optimal solutions to the following optimization problems:

\[
\hat{\varphi}_{\text{con}} \in \text{Argmin}_{\varphi \in \mathcal{C}^h_m(\mathbb{Z})} \left\{ \left\| \Delta^n[y-\varphi*y] \right\|_{n,2}^2 : \left\| \Delta^{-h}[\varphi] \right\|_{m,1}^F \leq \frac{\bar{c}}{\sqrt{m+1}} \right\}, \tag{Con'}
\]

\[
\hat{\varphi}_{\text{pen}} \in \text{Argmin}_{\varphi \in \mathcal{C}^h_m(\mathbb{Z})} \left\{ \left\| \Delta^n[y-\varphi*y] \right\|_{n,2}^2 + \sigma^2 \lambda^2 (m+1) \left[ \left\| \Delta^{-h}[\varphi] \right\|_{m,1}^F \right]^2 \right\}. \tag{Pen'}
\]

---

\(^6\) More precisely, we only need observations on two intervals \([-m-n-h,-h] \) and \([-n,0] \) with combined length at most \( m+2n \). As a consequence, we do not have to pay an extra logarithmic factor in \( h \) in the risk bounds.
Theorems 2.1–2.2. Then, the following statement holds:

The class of \( S \in \text{shift-invariant subspace} \) Assumption 2.2.

with obvious adjustments, provided that Assumption 2.1 is replaced with the following one:

\[ h < n \]

we illustrate prediction with \( h \) since otherwise the observation domain becomes disconnected.

Signal is not observed in the violet region, yet we are able to estimate it (pointwise) in that region as guaranteed by Proposition 2.2.

A close inspection of the proofs of Theorems 2.1–2.2 shows that those results remain valid, with obvious adjustments, provided that Assumption 2.1 is replaced with the following one:

**Assumption 2.2.** \( x \in C(Z) \) admits a decomposition \( x = x^S + \varepsilon \), where \( x^S \) belongs to a shift-invariant subspace \( S \in C(Z) \) with dimension \( s \leq n + 1 \), and \( \varepsilon \) can be bounded as follows:

\[
\|\Delta^T\varepsilon\|_{m,2} \leq \kappa \sigma, \quad 0 \leq \tau \leq h + m.
\]  

(27)

In order to guarantee the existence of a predictive oracle \( \varphi^o \in C_m^h(Z) \) with small \( \ell_1 \)-norm in the Fourier domain and good statistical properties, we have to accordingly modify Definition 1.1, see also [HJNO15, Section 3.2].

**Definition 2.1 (Predictable signals).** Given parameters \( m, n, h, \rho \geq 1, \) and \( \theta \geq 0 \), signal \( x \in C(Z) \) is called \((m, n, h, \rho, \theta)\)-predictable if there exists a filter \( \varphi^o \in C_m^h(Z) \) that satisfies

\[
\|\varphi^o\|_2 \leq \frac{\rho}{\sqrt{m+1}}, \quad \text{and} \quad |x_\tau - [\varphi^o * x]\tau| \leq \frac{\sigma \theta \rho}{\sqrt{m+1}}, \quad -m-n-h \leq \tau \leq 2h.
\]  

(28)

The class of \((m, n, h, \rho, \theta)\)-predictable signals for a given tuple \((m, n, h, \rho, \theta)\) is denoted \( P_{m,n}^h(\rho, \theta) \).

Now, suppose that \( x \in P_{n,0,n}^h(\rho, \theta) \), and let \( \varrho = 2 \rho^2 \), \( m = 2m_0 \), and \( h = 2h_0 \). Then, one can easily verify (see [HJNO15, Proposition 8]) the existence of a suitable oracle for \((\text{Con}^*)\) and \((\text{Pen}^*)\), namely, a filter \( \varphi^o \in C^h_m(Z) \) with the following properties analogous to (12) and (13):

\[
\|\Delta^h[\varphi^o]\|_{m,1}^+ \leq \frac{\theta}{\sqrt{m+1}}, \quad \text{and} \quad |x_\tau - [\varphi^o * x]\tau| \leq \frac{\sqrt{2} \sigma \theta \varrho}{\sqrt{m+1}}, \quad -n \leq \tau \leq h.
\]

Extensions of Corollary 2.1 and Proposition 2.1 to the prediction setting are now straightforward.

**Proposition 2.2.** Assume that \( x \in P_{n,0,n}^h(\rho, \theta) \), and let \( \varrho = 2 \rho^2 \), \( m = 2m_0 \in \mathbb{Z}^{++} \), \( h = 2h_0 \in \mathbb{Z}^{++} \). Moreover, suppose that Assumption 2.1 holds with some \((s, x)\), and let \( \hat{\varphi}_\text{con} \) and \( \hat{\varphi}_\text{pen} \) be, correspondingly, optimal solutions to \((\text{Con}^*)\) and \((\text{Pen}^*)\) with parameters as in the premises of Theorems 2.1–2.2. Then, the following statement hold:

1. Quantities \( \|\Delta^n[x - \hat{\varphi}_\text{con} * y]\|_{n,2}^+ \) and \( \|\Delta^n[x - \hat{\varphi}_\text{pen} * y]\|_{n,2}^+ \) enjoy the same bounds (21)–(24), up to a multiplicative constant, as their counterparts in the interpolation setting.
2. Whenever $n \geq m_0$, quantities

$$|x_t - [\hat{\varphi}_{\text{con}} * y]_t|, \quad |x_t - [\hat{\varphi}_{\text{pen}} * y]_t|, \quad h_0 - n + m_0 \leq t \leq h_0$$

also enjoy the same bounds (25)–(26) as their counterparts in the interpolation setting.

**Extrapolation.** Note that according to the second claim of Proposition 2.2, if the signal is predictable, adaptive filters $\hat{\varphi}_{\text{con}}$ and $\hat{\varphi}_{\text{pen}}$ allow to extrapolate it: from observations $(y_\tau)$ on the interval

$$-m - n - 2h_0 \leq \tau \leq 0,$

one is able to estimate $x_t$ at point $t = h_0$, by fitting a filter $\hat{\varphi} \in C_{m_0}^2(Z)$ and evaluating $[\hat{\varphi} * y]_{h_0}$.

3 **Shift-invariance and simplicity**

Conditions of the results of Section 2 merit some discussion. The primary question is how signal simplicity is related to Assumption 2.1 about proximity of the signal to a shift-invariant subspace. Below we present results which shed some light on this relationship. As we will see in an instant, signals which are uniformly close to shift-invariant subspaces are simple to interpolate. We also show that an important subset of such signals – those which are close to solutions of homogeneous linear difference equations whose solutions are prohibited from increasing or decreasing exponentially fast – are also easy to filter and predict.

We start with an auxiliary result which allows us to concentrate on the case of exact shift-invariance, i.e. when the signal belongs to a shift-invariant subspace without any approximation.

**Proposition 3.1.** Suppose that $x \in S_{m,n}(\rho, \theta)$, cf. Definition 1.1, with $\rho \geq 1$. Then, $\tilde{x} := x + \varepsilon$ with $\varepsilon$ satisfying

$$\|\Delta^\tau \varepsilon\|_{m,\infty} \leq \frac{\kappa \sigma}{\sqrt{2n + 1}}, \quad -m - n \leq \tau \leq m + n \quad (29)$$

belongs to $S_{m,n}(\rho, \tilde{\theta})$ with

$$\tilde{\theta} = \theta + \frac{2\kappa}{\kappa_{m,n}}.$$

Similarly, assume that $x \in P_{m,n}^h(\rho, \theta)$, cf. Definition 2.1, with $\rho \geq 1$, and suppose that

$$\|\Delta^\tau \varepsilon\|_{m,\infty} \leq \frac{\kappa \sigma}{\sqrt{n + 1}}, \quad -h \leq \tau \leq m + n + 2h. \quad (30)$$

Then, $\tilde{x}$ belongs to $P_{m,n}^h(\rho, \tilde{\theta})$ with

$$\tilde{\theta} = \theta + \frac{2\sqrt{2}\kappa}{\kappa_{m,n}}.$$

Observe that if $x$ is an element of a shift-invariant subspace, then $\tilde{x} = x + \varepsilon$ from the premise of Proposition 3.1 satisfies Assumption 2.1 in case of (29) and Assumption 2.2 in case of (30). On the other hand, quite naturally, requirements (29)–(30) in this case are are stronger than the corresponding conditions (16), (27) in Assumptions 2.1, 2.2: to ensure simplicity. we need proximity to the subspace to be measured in $\ell_{\infty}$-norm, rather than in $\ell_2$-norm.

**Remark 3.1.** As we are about to see, signals $x$ belonging to shift-invariant subspaces $S$ can be equivalently derived as the discretized solutions to homogeneous ordinary differential equations. Namely,

$$x_\tau = f(\tau/N), \quad \tau = 0, 1, ..., N - 1,$$
where \( f : [0, 1] \to \mathbb{R} \) satisfies
\[
p \left( \frac{d}{du} \right) f(u) = 0, \quad u \in [0, 1]
\]
for some polynomial \( p(z) \) with degree \( \dim(S) \). As such, one can show that in case of (29) or (30), \( \tilde{x} \) corresponds to a function \( \tilde{f} \) which satisfies a Hölder-type condition
\[
\sup_{u \in [0,1]} \left| p \left( \frac{d}{du} \right) \tilde{f}(u) \right| \leq L,
\]
whereas (16), (27) correspond to Sobolev-type conditions in which the sup-norm is replaced with the \( L_2 \)-norm, see [Nem00, Lemma 4.3.1].

### 3.1 Exact shift-invariance

As we have already mentioned in Sections 1 and 2.3, shift-invariant subspaces of \( \mathbb{C}(\mathbb{Z}) \) are closely related to homogeneous linear difference equations with constant coefficients. In fact, shift-invariant subspaces with fixed dimension are exactly the solution sets of such equations with fixed order. This is formally stated by the following simple proposition\(^7\) whose proof we provide in Appendix B.

**Proposition 3.2.** Solution set of a difference equation

\[
[p(\Delta)x]_t = \sum_{\tau=0}^{s} p_\tau x_{t-\tau} \equiv 0, \quad t \in \mathbb{Z}, \tag{31}
\]

with a polynomial \( p(z) = 1 + p_1 z + \ldots + p_s z^s \) is a shift-invariant subspace of \( \mathbb{C}(\mathbb{Z}) \) with \( \dim(S) = s \).

Conversely, any shift-invariant subspace \( S \) of \( \mathbb{C}(\mathbb{Z}) \) with \( \dim(S) = s \) is the solution set of a difference equation of the form (31) with \( \deg(p) = s \). Moreover, such polynomial is unique if normalized by \( p(0) = 1 \).

Note that the set of solutions of (31) with a fixed polynomial \( p(z) \) is spanned by exponential polynomials determined by the roots of \( p(z) \). Specifically, let for \( k = 1, \ldots, r \leq s \) the numbers \( z_k \) be the distinct roots of \( p(z) \) with the corresponding multiplicities \( m_k \), and choose \( \omega_k \in \mathbb{C} \) such that \( z_k = e^{i\omega_k} \). Then the solutions to (31) can be expressed as

\[
x_t = \sum_{k=1}^{r} q_k(t)e^{i\omega_k t}, \tag{32}
\]

where \( q_k(\cdot) \) are arbitrary polynomials with \( \deg(q_k) = m_k - 1 \). For instance, discrete-time polynomials of degree \( s - 1 \) satisfy (31) with \( p(z) = (1 - z)^s \). Another important example is that of harmonic oscillations

\[
x_t = \sum_{k=1}^{s} C_k e^{i\omega_k t}, \quad \omega_k \in [0, 2\pi] \tag{33}
\]

which satisfy (31) with \( p(z) = \prod_{k=1}^{s}(1 - e^{i\omega_k} z) \). The set of harmonic oscillations with fixed frequencies \( \omega_1, \ldots, \omega_s \) and varying complex amplitudes \( C_k \) form an \( s \)-dimensional shift-invariant subspace depending on the frequencies.

\(^7\)Analogues of Proposition 3.1 are known in the continuous setting [AK64]; the result for the discrete-time setting follows from those for Abelian groups, see [Lai79], [Szé82], but we are not aware of any elementary proof.
Interpolation for general shift-invariant subspaces. As the next result shows, elements of arbitrary low-dimensional shift-invariant subspaces of \( C(\mathbb{Z}) \) (equivalently, solutions of (31) exponential polynomials \((32)\) are always simple. More precisely, such signals can always be interpolated by a filter \( \phi \in \mathcal{C}_m(\mathbb{Z}) \) with moderate \( \ell_2 \)-norm which exactly reproduces the signal:

**Proposition 3.3.** Let \( \mathcal{S} \) be a shift-invariant subspace of \( C(\mathbb{Z}) \) with dimension \( s \leq m + 1 \), and \( x \in \mathcal{S} \). Then, there exists a filter \( \phi^o \in \mathcal{C}_m(\mathbb{Z}) \), which only depends on \( \mathcal{S} \) and not on \( x \), such that \( x_t = [\phi^o \ast x]_t \) for any \( t \in \mathbb{Z} \), and

\[
\|\phi^o\|_2 \leq \sqrt{\frac{2s}{2m+1}}.
\]  

(34)

As such, for any \( n \in \mathbb{Z}_+ \) one has \( x \in \mathcal{S}_{m,n}(\rho,0) \) with \( \rho = \sqrt{2s} \).

Note that the bound \( \rho = O(\sqrt{s}) \) attained by the oracle filter \( \phi^o \) from Proposition 3.3 is the best one could hope for provided that \( \phi^o \) is only allowed to depend on the subspace but not on the signal itself. Indeed, the better dependency \( \rho(s) \) would contradict the minimax risk bound \( \sigma O(\sqrt{s/n}) \) for a subspace that holds for all possible estimators, not only convolution-type ones, see e.g. [Joh11]. One should note, however, that feasible filter \( \varphi \) in Theorems 2.1–2.2 is allowed to depend on \( x \), and hence \( \varphi \) might have a smaller risk than as implied by Proposition 3.3. Then, we are guaranteed to “mimic” the statistical properties of this filter, in the sense of the results obtained in Section 2, whenever it exists, and the price of adaptation is guaranteed to be upper-bounded in terms of its norm.

**Prediction of generalized harmonic oscillations.** On the other hand, it is clear that Assumption 2.1 is not sufficient to imply predictability of \( x \) in the sense of Definition 2.1, i.e. when one is allowed to use only unilateral filters. One can see, for instance, that already the signals coming from the parametric family

\[
X_\alpha = \{ x \in C(\mathbb{Z}) : x_\tau = \beta \alpha^\tau, \beta \in \mathbb{C} \},
\]

for given \( \alpha : |\alpha| > 1 \), which form a one-dimensional shift-invariant subspace of \( C(\mathbb{Z}) \) defined by \((1 - \alpha \Delta)x \equiv 0\), cannot be estimated consistently at \( t = 0 \) using only observations on the left of \( t \), and thus do not satisfy Definition 2.1. Of course, this difficulty with \( X_\alpha \) is due to the instability of solutions of a difference equation which do not remain bounded when \( \tau \to +\infty \). Meanwhile, “stable” signals – decaying exponents, harmonic oscillations, and their products – are predictable. More generally, suppose again that \( x \) belongs to a shift-invariant subspace, or, equivalently, satisfies a difference equation (31) with characteristic polynomial \( p(z) \). When \( p(z) \) has at least one root \( z_k \) such that \(|z_k| < 1\), the solution set of (31) contains signals unbounded as \( \tau \to \infty \), which cannot be estimated by a “causal” filter \( \phi \in \mathcal{C}_m^h(\mathbb{Z}) \) with \( h \geq 0 \). On the other hand, when \( p(z) \) has a root \( z_k \) with \(|z_k| > 1\), the set of solutions to (31) contains signals unbounded as \( \tau \to - \infty \), which cannot be estimated by any “anti-causal” filter \( \phi \in \mathcal{C}_m^h(\mathbb{Z}) \) with \( h \leq -m \).

In view of the above, it is interesting to consider the case where all roots of \( p(z) \) have unit modulus. In this case, the solutions of (31) are exponential polynomials (32) with

\[
\omega_1, \ldots, \omega_s \in [0, 2\pi];
\]

we call such signals generalized harmonic oscillations as they are sums of polynomially-modulated complex sinusoids. This class of signals has already been studied in [JN13], where the authors showed that a harmonic oscillation with \( s \) frequencies are simple in the sense of filtering, i.e. can reproduced by a small-norm one-sided filter \( \phi^o \in \mathcal{C}_m^+(\mathbb{Z}) \). Namely, such signals belong to any class \( \mathcal{P}_{m,n}(\rho,0) \) with any \( n \in \mathbb{Z}_+ \), whenever \( m \) is large enough, and \( \rho = O(s^{3/2}) \), as formally stated below.
Proposition 3.4 ([JN13, Lemma 6.1]). Suppose that all roots of $p(z) = 1 + p_1 z + \ldots + p_s z^s$ satisfy $|z_k| = 1$. Then there is an absolute constant $c$ such that for any $m \geq c s^2 \log s$, one can construct a filter $\phi^o \in C^*_m(Z)$, which only depends on $p(z)$, such that any solution to (31) satisfies $x_t = [\phi^o * x]_t$ for any $t \in Z$, and

$$\|\phi^o\|_2 \leq C \sqrt{\frac{s^3 \log(s+1)}{m+1}}.$$  \hfill (35)

We show an improvement over this result. Its proof, given in Appendix B, uses some complex-analytical techniques, and can be of independent interest.

Proposition 3.5. Under the premise of Proposition 3.4, one can replace (35) with

$$\|\phi^o\|_2 \leq C \sqrt{\frac{s^2 \log(m s+1)}{m+1}}.$$ \hfill (36)

Moreover, when dealing with “ordinary” harmonic oscillation given by (33), the bound (36) can be further improved under the additional condition that the frequencies $\omega_1, \ldots, \omega_s$ are well-separated, see [DB13, TBR13, CFG14]. Namely, assume that all the roots $z_k = e^{i \omega_k}$ of $p(z)$ are simple, let $|x - y|$ be the wrap-around metric on $[0, 2\pi]$, and consider the minimal frequency separation $\delta_{\min}$ defined as

$$\delta_{\min} := \min_{1 \leq j \neq k \leq s} |\omega_j - \omega_k|.$$ \hfill (37)

The following result shows that whenever $\delta_{\min}$ is large enough, the bound $\rho = O(s)$ can be improved to $\rho = O(\sqrt{s})$.

Proposition 3.6. For some $\nu > 1$, let

$$\delta_{\min} \geq \frac{2 \pi \nu}{m+1}.$$ \hfill (38)

Then there exists a filter $\phi^o \in C^*_m(Z)$ satisfying $x_t = [\phi^o * x]_t = 0$ for any $t \in Z$, and such that

$$\|\phi^o\|_2 \leq \sqrt{\frac{Q s}{m+1}}, \quad \text{where} \quad Q = \frac{\nu + 1}{\nu - 1}.$$  

In particular, whenever $\delta_{\min} \geq 4 \pi / n$, one has

$$\|\phi^o\|_2 \leq \sqrt{\frac{3 s}{m+1}}.$$  

3.2 Full recovery of harmonic oscillations

To illustrate the results obtained in Section 3.1, let us consider the problem of full recovery of (ordinary) harmonic oscillations. Namely, we are asked to estimate on $D_N$ a harmonic oscillation

$$x_t = \sum_{k=1}^s \alpha_k e^{i \omega_k t}, \quad t \in D_N, \quad N \in \mathbb{Z}_+,$$

without the knowledge of frequencies $\omega_1, \ldots, \omega_s$. Since estimation is required on the entire $D_N$, we will measure the statistical performance of an estimator $\hat{x}$ of $x$ via the mean-square error\(^8\):

$$\text{Risk}(\hat{x}, x) := (2N + 1)^{-1/2} \mathbb{E}\|\hat{x} - x\|_{N,2}^{1/2}.$$  

\(^8\)The results of this section can be generalized, in a straightforward manner, for the risk measured by the width of the confidence interval for $\ell_2$-loss. We omit this generalization in order to simplify the presentation.
Note that if the frequencies were known, the ordinary least-squares estimator would satisfy
\[
\sup_{S_{\omega_1, \ldots, \omega_s}} \text{Risk}(\hat{x}, x) \leq C \sigma \sqrt{\frac{s}{2N + 1}},
\]
where \( S_{\omega_1, \ldots, \omega_s} \) is the set of harmonic oscillations corresponding to the fixed tuple of frequencies \( \omega_1, \ldots, \omega_s \). On the other hand, when the frequencies are unknown, one has the lower bound [TBR13, Theorem 2]
\[
\sup_{x \in S^{(s)}} \text{Risk}(\hat{x}, x) \geq c \sigma \sqrt{\frac{s \log(N + 1)}{2N + 1}}, \tag{39}
\]
where \( S^{(s)} \) is the set of all harmonic oscillations with no more than \( s \) frequencies. Moreover, the bound (39) is in fact attained on a subspace with separated frequencies in the sense of (38), indicating that the general case is not harder, from the statistical viewpoint, than that of well-separated frequencies. As such, one could hope to match (39) by some adaptive estimator in \( \omega_1, \ldots, \omega_s \), whether the frequencies are restricted to be well-separated or not. However, to the best of our knowledge, the only estimator known to match (39), called Atomic Soft Thresholding (AST) and studied in [BTR13, TBR13], only does so in the case of well-separated frequencies, see [TBR13, Theorem 1]. As such, the question whether the lower bound (39) can be matched in the general case, is still open.

A crucial step towards bridging this gap has been taken in [HJNO15] where it was suggested to use one-sided version of a uniform-fit estimator jointly with Proposition 3.4, exploiting that (35) holds for (generalized) harmonic oscillations without any frequency separation assumptions. In particular, fitting a “left” uniform-fit estimator \( \hat{\varphi} \in C_N^-(Z) \) on \( D_N^- \), and a “right” estimator \( \hat{\varphi} \in C_N^+(Z) \) on \( D_N^+ \), and using the bound \( C\rho^3 \sqrt{\log N} \) on the price of adaptation for such estimators\(^9\), one obtains for such construction the correct \( s \) and \( N \) rate
\[
\sup_{x \in S^{(s)}} \text{Risk}(\hat{x}, x) \leq C \sigma \sqrt{\frac{s^{3/2} \log(N + 1)}{2N + 1}}.
\]
As we see, the price to pay is a polynomial factor in \( s \), and additional assumption \( N \geq cs^2 \log s \) needed for Proposition 3.4.

Using the results presented in this paper, we immediately improve the dependence on \( s \) in the above bound by replacing one-sided uniform-fit estimators with estimators of the form (Con), used together with Theorem 2.2 and the improved bound (36) instead of (35):
\[
\sup_{x \in S^{(s)}} \text{Risk}(\hat{x}, x) \leq C \sigma \sqrt{\frac{s^4 \log^2(N + 1)}{2N + 1}}. \tag{40}
\]
Note that while this estimator requires the knowledge of \( s \) in advance (AST does not), this requirement can be circumvented by using (Pen\(^*\)) instead of (Con\(^*\)), at the expense of an additional logarithmic factor. Moreover, (40) can be further improved if the signal frequencies are restricted to be well-separated. In this case, using Proposition 3.6, the same estimator satisfies
\[
\sup_{x \in S^{(s)}_{\text{sep}}} \text{Risk}(\hat{x}, x) \leq C \sigma \sqrt{\frac{s^2 + s \log(N + 1)}{2N + 1}} \tag{41}
\]
where the supremum is taken over the set of harmonic oscillations with no more than \( s \) frequencies and pairwise separation at least \( 4\pi/(N + 1) \).

\(^9\)This bound holds for both two-sided and one-sided uniform-fit estimators, see [HJNO15].
Finally, let us describe how the results obtained so far can be combined, resulting in the state-of-the-art estimator of harmonic oscillations which improves over the bound (40) in the general case while still preserving (41). Namely, consider the following procedure:

1. Pick some $M \leq N$, and divide the observation domain $D_N$ into the large central subdomain $D_M$ and the smaller subdomains $D^+ := D_{N-M}^M$ and $D^- := D_{N-M}^-$.  
2. Estimate the signal on $D_M$ with a two-sided filter $\hat{\varphi} \in C_{N-M}^N(Z)$, on $D^+$ with a one-sided filter $\hat{\varphi}^+ \in C_{M+N}^M(Z)$, and on $D^-$ with a one-sided filter $\hat{\varphi}^- \in C_{M+N}^-N(Z)$. 
3. Choose $M$ to minimize the total bound on Risk over $D_N$.

Direct calculations lead to the following choice of $M$:

$$C_1 s \log(N + 1) \leq \frac{M + 1}{N - M + 1} \leq C_2 s \log(N + 1),$$

and the resulting bound is

$$\sup_{x \in S^s} \text{Risk}(\hat{x}, x) \leq C \sigma \sqrt{\frac{s^3 \log(N + 1) + s^2 \log^2(N + 1)}{2N + 1}}.$$  

(42)

These calculations are provided in Appendix B.

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A Proof of oracle inequalities

In this section, we prove our main results – sharp oracle inequalities for regularized least-squares estimators (Theorems 2.1–2.2). First, let us present some additional notation and technical tools to be used in the proofs.

**Additional notation.** $\text{Re}(z)$ and $\text{Im}(z)$ denote, correspondingly, the real and imaginary parts of a complex number $z \in \mathbb{C}$, and $\overline{z} = \text{Re}(z) - i \text{Im}(z)$ denotes the complex conjugate of $z$. We denote $A^T$ the transpose of a complex-valued matrix $A$, and $A^H$ its conjugate transpose. We denote $\overline{A}$ the conjugation of $A$ without transposition. We denote $A^{-1}$ the inverse of $A$ whenever it is guaranteed to exist. We denote $\text{Tr}(A)$ the trace of a matrix $A$, $\text{det}(A)$ its determinant, $\|A\|_F$ the Frobenius norm, and $\|A\|_{\text{op}}$ the operator norm. We denote $\lambda_{\text{max}}(A)$ and $\lambda_{\text{min}}(A)$ the maximal and minimal eigenvalues of a Hermitian matrix $A$. We denote $\text{Diag}(a)$ the diagonal matrix formed from a vector $a \in \mathbb{C}^n$. We denote $I$ the identity matrix, sometimes with a subscript indicating its size. We denote $\langle \cdot, \cdot \rangle$ the Hermitian scalar product: for two complex vectors $a, b$ of the same dimension, $\langle a, b \rangle = a^H b$. We denote $\langle x, y \rangle_n := \langle [x]_n^n, [y]_n^n \rangle_n$ for $x, y \in \mathbb{C}(Z)$.

In what follows, we associate linear maps $\mathbb{C}^n(Z) \to \mathbb{C}^{n'}(Z)$ with matrices in $\mathbb{C}^{(2n+1) \times (2n'+1)}$. 

A.1 Technical tools

**Convolution matrices.** We use various matrix-vector representations of discrete convolution.

- Given \( y \in \mathbb{C}(\mathbb{Z}) \), we associate to it an \((2n + 1) \times (2m + 1)\) matrix

\[
T(y) = \begin{bmatrix}
    y_{-n+m} & \cdots & y_{-n} & \cdots & y_{-n-m} \\
    \vdots & \ddots & \vdots & \ddots & \vdots \\
    y_m & \cdots & y_0 & \cdots & y_{-m} \\
    \vdots & \ddots & \vdots & \ddots & \vdots \\
    y_{n+m} & \cdots & y_n & \cdots & y_{n-m}
\end{bmatrix},
\]

such that \([\varphi \ast y]_n = T(y)[\varphi]_{-m}^m\) for \( \varphi \in \mathbb{C}_m(\mathbb{Z}) \). Its squared Frobenius norm satisfies

\[
\|T(y)\|_F^2 = \sum_{\tau \in \mathbb{D}_m} \|\Delta^\tau y\|_{n,2}^2.
\]

- Given \( \varphi \in \mathbb{C}_m(\mathbb{Z}) \), consider an \((2n + 1) \times (2m + 2n + 1)\) matrix

\[
M(\varphi) = \begin{bmatrix}
    \varphi_m & \cdots & \varphi_{-m} & 0 & \cdots & 0 \\
    0 & \varphi_m & \cdots & \varphi_{-m} & 0 & \cdots & 0 \\
    \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    0 & \cdots & 0 & \varphi_m & \cdots & \varphi_{-m}
\end{bmatrix},
\]

such that for \( y \in \mathbb{C}(\mathbb{Z}) \) one has \([\varphi \ast y]_n = M(\varphi)[y]_{-m-n}^{m+n}\), and

\[
\|M(\varphi)\|_F^2 = (2n + 1)\|\varphi\|_{m,2}^2.
\]

- Given \( \varphi \in \mathbb{C}_m(\mathbb{Z}) \), consider the following circulant matrix of size \(2m + 2n + 1\):

\[
C(\varphi) = \begin{bmatrix}
    \varphi_0 & \cdots & \varphi_{-m} & 0 & \cdots & 0 & \varphi_m & \cdots & \varphi_1 \\
    \varphi_1 & \varphi_0 & \cdots & \varphi_{-m} & 0 & \cdots & 0 & \varphi_m & \cdots & \varphi_2 \\
    \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
    \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
    \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
    \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
    0 & \cdots & 0 & \varphi_m & \cdots & \varphi_0 & \varphi_{-m} & 0 & \cdots & 0 \\
    \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    \varphi_{-1} & \cdots & \varphi_{-m} & 0 & \cdots & 0 & \varphi_m & \cdots & \varphi_0
\end{bmatrix}.
\]

Note that \(C(\varphi)[y]_{-m-n}^{m+n}\) is the circular convolution of \([y]_{-m-n}^{m+n}\) and the zero-padded filter

\[
\tilde{\varphi} := [\varphi]_{-m-n}^{m+n} = [0; \ldots; \varphi_{-m}; \ldots; \varphi_m; 0; \ldots; 0],
\]
that is, convolution of the periodic extensions of \([y]_{m-n}^{m+n}\) and \(\tilde{\varphi}\) evaluated on \(D_{m+n}\). Hence, by the diagonalization property of the DFT operator one has

\[
C(\varphi) = \sqrt{2m + 2n + 1}F_{m+n}^H\text{diag}(F_{m+n}[\tilde{\varphi}])F_{m+n}.
\]  

(48)

Besides, note that

\[
||C(\varphi)||_F^2 = (2m + 2n + 1)||\varphi||_{m,2}^2.
\]

Deviations bounds for quadratic forms. We use simple probabilistic facts listed below.

- Let \(\zeta \sim \mathcal{CN}(0, I_n)\) be a standard complex Gaussian vector, meaning that \(\zeta = \xi_1 + i\xi_2\) where \(\xi_1\) and \(\xi_2\) are two independent draws from \(\mathcal{N}(0, I_n)\). We use a simple bound

\[
P\left\{ ||\zeta||_\infty \leq \sqrt{2\log n + 2u} \right\} \geq 1 - e^{-u}
\]

(49) which can be verified directly using that \(||\xi_1||_2^2 \sim \lambda_2^2\).

- The following deviation bounds for \(||\zeta||_2^2 \sim \lambda_2^2\), are due to [LM00, Lemma 1]:

\[
P\left\{ \frac{||\zeta||_2^2}{2} \leq n + \sqrt{2nu + u} \right\} \geq 1 - e^{-u},
\]

\[
P\left\{ \frac{||\zeta||_2^2}{2} \geq n - \sqrt{2nu} \right\} \geq 1 - e^{-u}.
\]

(50)

By simple algebra we obtain an upper bound for the norm:

\[
P\left\{ ||\zeta||_2 \leq \sqrt{2n + \sqrt{2u}} \right\} \geq 1 - e^{-u}.
\]

(51)

- Further, let \(K\) be an \(n \times n\) Hermitian matrix with the vector of eigenvalues \(\lambda = [\lambda_1; \ldots; \lambda_n]\). Then the real-valued quadratic form \(\zeta^H K \zeta\) has the same distribution as \(\xi^T B \xi\), where \(\xi = [\xi_1; \xi_2] \sim \mathcal{N}(0, I_{2n})\), and \(B\) is a real \(2n \times 2n\) symmetric matrix with the vector of eigenvalues \(\lambda; \lambda\). We have \(\text{Tr}(B) = 2\text{Tr}(K)\), \(||B||_F^2 = 2||K||_F^2\) and \(||B|| = ||K|| \leq ||K||_F\), where \(\| \cdot \| \) and \(\| \cdot \|_F\) denote, correspondingly, the spectral and Frobenius norms of a matrix. Invoking again [LM00, Lemma 1] (a close inspection of the proof shows that the assumption of positive semidefiniteness can be relaxed), we have

\[
P\left\{ \frac{\zeta^H K \zeta}{2} \leq \text{Tr}(K) + (u + \sqrt{2u})||K||_F \right\} \geq 1 - e^{-u}.
\]

(52)

Further, when \(K\) is positive semidefinite, we have \(||K||_F \leq \text{Tr}(K)\), whence

\[
P\left\{ \frac{\zeta^H K \zeta}{2} \leq \text{Tr}(K)(1 + \sqrt{u})^2 \right\} \geq 1 - e^{-u}.
\]

(53)

Reformulation of approximate shift-invariance The following reformulation of Assumption 2.1 will be convenient for our purposes.

\[
\text{There exists an \(s\)-dimensional vector subspace \(S_n\) of } \mathbb{C}^{2n+1} \text{ and an idempotent Hermitian } (2n + 1) \times (2n + 1) \text{ matrix } \Pi_{S_n} \text{ of rank } s - \text{ projector on } S_n - \text{ such that}
\]

\[
\| (I_{2n+1} - \Pi_{S_n}) [\Delta^\tau x]_{-n}^n \|_2 \leq \sigma \varepsilon, \quad \tau \in D_m,
\]

(54)

where \(I_{2n+1}\) is the identity matrix of size \(2n + 1\).
A.2 Proof of Theorem 2.1

Step 1°. Let \( \varphi^o \in C_m(Z) \) be any filter satisfying the constraint in (Con). Then,

\[
\|x - \hat{\varphi} \ast y\|_{n,2}^2 \leq \|(1 - \varphi^o) \ast y\|_{n,2}^2 - \sigma^2 \|\xi\|_{n,2}^2 - 2\sigma \text{Re}(\zeta, x - \hat{\varphi} \ast y)_n
\]

\[
= \|x - \varphi^o \ast y\|_{n,2}^2 - 2\sigma \text{Re}(\zeta, x - \hat{\varphi} \ast y)_n + 2\sigma \text{Re}(\zeta, x - \varphi^o \ast y)_n. \tag{55}
\]

Let us bound \( \delta^{(1)} \). Denote for brevity \( I := I_{2n+1} \), and recall that \( \Pi S_n \) is the projector on \( S_n \) from (54). We have the following decomposition:

\[
\delta^{(1)} = \left( \sigma \text{Re}([\zeta]_{-n}, \Pi S_n [x - \hat{\varphi} \ast y]_{-n}) \right) + \sigma \text{Re}([\zeta]_{-n}, (I - \Pi S_n)[x - \hat{\varphi} \ast x]_{-n})
\]

\[
- \sigma^2 \text{Re}([\zeta]_{-n}, (I - \Pi S_n)[\hat{\varphi} \ast [\zeta]_{-n}]). \tag{56}
\]

One can easily bound \( \delta_1^{(1)} \) under the premise of the theorem:

\[
\left| \delta_1^{(1)} \right| \leq \sigma \|\Pi S_n [\zeta]_{-n}\|_2 \|\Pi S_n [x - \hat{\varphi} \ast y]_{-n}\|_2
\]

\[
\leq \sigma \|\Pi S_n [\zeta]_{-n}\|_2 \|x - \hat{\varphi} \ast y\|_{n,2}. \tag{57}
\]

Note that \( \Pi S_n [\zeta]_{-n} \sim C \mathcal{N}(0, I_2) \), and by (51) we have

\[
P \left\{ \|\Pi S_n [\zeta]_{-n}\|_2 \geq \sqrt{2s + \sqrt{2n}} \right\} \leq e^{-u},
\]

which gives the bound

\[
P \left\{ \left| \delta_1^{(1)} \right| \leq \sigma \|x - \hat{\varphi} \ast y\|_{n,2} \left( \sqrt{2s + \sqrt{2\log[1/\alpha_1]} } \right) \right\} \geq 1 - \alpha_1. \tag{57}
\]

Step 2°. We are to bound the second term of (56). To this end, note first that

\[
\delta_2^{(1)} = \sigma \text{Re}([\zeta]_{-n}, (I - \Pi S_n)[x]_{-n}) - \sigma \text{Re}([\zeta]_{-n}, (I - \Pi S_n)[\hat{\varphi} \ast x]_{-n}).
\]

By (54), \( \|(I - \Pi S_n)[x]_{-n}\|_2 \leq \sigma \kappa \), thus with probability 1 - \( \alpha \),

\[
\left| \langle [\zeta]_{-n}, (I - \Pi S_n)[x]_{-n} \rangle \right| \leq \sigma \kappa \sqrt{2\log[1/\alpha]}. \tag{58}
\]

On the other hand, using the notation defined in (43), we have \( \hat{\varphi} \ast x]_{-m} = T(x)[\hat{\varphi}]_{-m} \), so that

\[
\langle [\zeta]_{-n}, (I - \Pi S_n)[\hat{\varphi} \ast x]_{-n} \rangle = \langle [\zeta]_{-n}, (I - \Pi S_n)T(x)[\hat{\varphi}]_{-m} \rangle.
\]

Note that \( T(x)_\tau = [\Delta^\tau x]_{-n} \) for the columns of \( T(x) \), \( \tau \in D_m \). By (54), we have

\[
(I - \Pi S_n)T(x) = T(\varepsilon),
\]

and by (44),

\[
\|(I - \Pi S_n)T(x)\|_F^2 = \|T(\varepsilon)\|_F^2 = \sum_{\tau \in D_m} \|\Delta^\tau \varepsilon\|_{n,2}^2 \leq (2m + 1)\sigma^2 \kappa^2.
\]
Due to (53) we conclude that
\[ \| T(x)^H(I - \Pi_{S_n})[\zeta]_{-n}^m \|_2^2 \leq 2(2m + 1)\sigma^2 x^2(1 + \sqrt{\log[1/\alpha]})^2 \]
with probability at least 1 - \alpha. Since
\[ |\langle [\zeta]_{-n}^m, (I - \Pi_{S_n})T(x)[\hat{\varphi}]_{-m}^n \rangle| \leq \frac{\overline{\varphi}}{2m + 1} \| T(x)^H(I - \Pi_{S_n})[\zeta]_{-n}^m \|_2, \]
we arrive at the bound with probability 1 - \alpha:
\[ |\langle [\zeta]_{-n}^m, (I - \Pi_{S_n})T(x)[\hat{\varphi}]_{-m}^n \rangle| \leq \sqrt{2}\sigma \sqrt{\varphi}(1 + \sqrt{\log[1/\alpha]}). \]

Along with (58) this results in the bound
\[ P \left\{ |\delta_s^{(1)}| \leq \sqrt{2}\sigma \sqrt{\varphi}(\varphi + 1)(1 + \sqrt{\log[1/\min(\alpha_2, \alpha_3)]}) \right\} \geq 1 - \alpha_2 - \alpha_3. \] (59)

**Step 3**. Let us rewrite \( \delta_s^{(1)} \) as follows:
\[ \delta_s^{(1)} = \sigma^2 \Re\langle [\zeta]_{-n}^m, (I - \Pi_{S_n})M(\hat{\varphi})[\zeta]_{-m}^n \rangle = \sigma^2 \Re\langle [\zeta]_{-n}^m, QM(\hat{\varphi})[\zeta]_{-m}^n \rangle, \]
where \( M(\hat{\varphi}) \in \mathbb{C}^{(2m+1)\times(2m+2n+1)} \) is defined by (45), and \( Q \in \mathbb{C}^{(2m+2n+1)\times(2n+1)} \) is given by
\[ Q = \begin{bmatrix} O_{m,2n+1} \\ I - \Pi_{S_n} \\ O_{m,2n+1} \end{bmatrix}. \]

(Hereafter we denote \( O_{m,n} \) the \( m \times n \) zero matrix.) Now, by the definition of \( \hat{\varphi} \) and since the mapping \( \varphi \mapsto M(\varphi) \) is linear,
\[ \delta_s^{(1)} = \frac{\sigma^2}{2}(\langle [\zeta]_{-m}^n \rangle^H QM(\hat{\varphi}) + M(\hat{\varphi})^H Q^H) [\zeta]_{-m}^n \]
\[ \leq \frac{\sigma^2 \overline{\varphi}}{2\sqrt{2m + 1}} \max_{u \in \mathbb{C}_m(\mathbb{Z}), \|u\|_{m,1}^2 \leq 1} \langle [\zeta]_{-m}^n \rangle^H K_1(\hat{\varphi}) [\zeta]_{-m}^n \]
\[ = \frac{\sigma^2 \overline{\varphi}}{\sqrt{2m + 1}} \max_{j \in \mathbb{Z}_m} \max_{\theta \in [0,2\pi]} \frac{1}{2} \langle [\zeta]_{-m}^n \rangle^H K_1(e^{i\theta}u^j) [\zeta]_{-m}^n, \]
where \( u^j \in \mathbb{C}_m(\mathbb{Z}) \), and \( \|u^j\|_{m,1}^2 = \|e^{jH}e^j\|_{m,1} \) being the discrete Dirac pulse centered at \( j \in \mathbb{Z} \). Indeed, \( \langle [\zeta]_{-m}^n \rangle^H K_1(\hat{\varphi}) [\zeta]_{-m}^n \) is clearly a convex function of the argument \( u \) as a linear function of \( \langle \Re(u) ; \Im(u) \rangle \); as such, it attains its maximum over the set
\[ B_{m,1} = \{ u \in \mathbb{C}_m(\mathbb{Z}) : \|u\|_{m,1}^2 \leq 1 \} \] (60)
at one of the extremal points \( e^{i\theta}u^j \), \( \theta \in [0,2\pi] \), of this set. It can be directly verified that
\[ K_1(e^{i\theta}u) = K_1(u) \cos \theta + K_2(u) \sin \theta, \]
where the Hermitian matrix \( K_2(u) \) is given by
\[ K_2(u) = i \left( QM(u) - M(u)^HQ^H \right). \]
Denoting $q^l_j(\zeta) = \frac{1}{2}([\zeta]^{m+n}_{-m-n})^H K_l(u) [\zeta]^{m+n}_{-m-n}$ for $l = 1, 2$, we have

$$\max_{\theta \in [0,2\pi]} \frac{1}{2}([\zeta]^{m+n}_{-m-n})^H K_1(e^{i\theta} u^j) [\zeta]^{m+n}_{-m-n} = \max_{\theta \in [0,2\pi]} q^l_1(\zeta) \cos \theta + q^l_2(\zeta) \sin \theta = \sqrt{|q^l_1(\zeta)|^2 + |q^l_2(\zeta)|^2} \leq \sqrt{2} \max(|q^l_1(\zeta)|, |q^l_2(\zeta)|).$$

(61)

By simple algebra and using (46), we get for $l = 1, 2$:

$$\text{Tr}[K_l(u^j)]^2 \leq 4 \text{Tr}[M(u^j) M(u^j)^H] = 4(2n + 1)\|u^j\|^2_{m,2} \leq 4(2n + 1).$$

Now let us bound $\text{Tr}[K_l(u)]$, $l = 1, 2$, on the set $B_{m,1}$ cf. (60). One can verify that for the circulant matrix $C(u)$, cf. (47), it holds:

$$QM(u) = RC(u),$$

where $R = QQ^H$ is an $(2m + 2n + 1) \times (2m + 2n + 1)$ projection matrix of rank $s$ defined by

$$R = \begin{bmatrix} O_{m,m} & O_{m,n+1} & O_{m,m} \\ O_{n+1,m} & I - \Pi S_n & O_{n+1,m} \\ O_{m,m} & O_{m,n+1} & O_{m,m} \end{bmatrix}.$$  

Hence, denoting $\| \cdot \|_{\text{op}}$ and $\| \cdot \|_{\text{nuc}}$ the operator and nuclear matrix norms, we can bound $\text{Tr}[K_l(u)]$, $l = 1, 2$, as follows:

$$|\text{Tr}[K_l(u)]| \leq 2|\text{Tr}[RC(u)]| \leq 2\|R\|_{\text{op}} \|C(u)\|_{\text{nuc}} \leq 2\|C(u)\|_{\text{nuc}} = 2\sqrt{2m + 2n + 1}\|\tilde{u}\|^F_{m+n+1},$$

(62)

where in the last transition we used the Fourier diagonalization property (48). Recall that $u \in \mathcal{C}_m(Z)$, hence $F_{m+n}[u]$ is the Discrete Fourier transform of the zero-padded filter $\tilde{u} = [0; \ldots; 0; [u]_{m-n}; 0; \ldots; 0] \in \mathcal{C}^{2m+2n+1}.$

The following lemma, interesting in its own right, controls the inflation of the $\ell_1$-norm of the DFT of a filter after zero padding. The proof, presented later on, relies to the fact that the normalized $\ell_1$-norm of the Dirichlet kernel of order $N$ grows not faster than $\log N$.

**Lemma A.1 (\ell_1-norm of the DFT after zero-padding).** For any $u \in \mathcal{C}_m(Z)$, one has

$$\|u\|^F_{m+n+1} \leq \|u\|^F_{m,1} \sqrt{1 + \kappa_{m,n}^2} \log(m + n + 1) + 3].$$

Combining this lemma with (62) we arrive at

$$|\text{Tr}[K_l(u^j)]| \leq 2\sqrt{2m + 1} (\kappa_{m,n}^2 + 1) \log[2m + 2n + 1] + 3, \quad l = 1, 2.$$

By (52) we conclude that for any fixed pair $(l, j) \in \{1, 2\} \times D_m$, with probability $\geq 1 - \alpha$, 

$$|q^l_j(\zeta)| \leq |\text{Tr}[K_l(u^j)]| + \|K_l(u^j)\|_F (1 + \sqrt{\log[2/\alpha]})^2.$$  

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With $\alpha_0 = 2(2m + 1)\alpha$, by the union bound together with (60) and (61) we get
\[
\mathbb{P} \left\{ \delta_3^{(1)} \leq 2\sqrt{2}\sigma^2 \theta \left[ (\kappa_{m,n}^2 + 1)(\log(2m + 2n + 1) + 3) + \kappa_{m,n}(1 + \sqrt{\log[4(2m + 1)/\alpha_0]^2}) \right] \right\} \\
\geq 1 - \alpha_0.
\] (63)

**Step 4°.** Bounding $\delta^{(2)}$ is relatively easy since $\varphi^o$ does not depend on the noise. We decompose
\[
\delta^{(2)} = \sigma \Re(\zeta, x - \varphi^o \ast x)_n - \sigma^2 \Re(\zeta, \varphi^o \ast \zeta)_n.
\]
Note that $\Re(\zeta, x - \varphi^o \ast x)_n \sim \mathcal{N}(0, \|x - \varphi^o \ast x\|_{n,2}^2)$, therefore, with probability $\geq 1 - \alpha$,
\[
\Re(\zeta, x - \varphi^o \ast x)_n \leq \sqrt{2 \log[1/\alpha]} \|x - \varphi^o \ast x\|_{n,2}.
\] (64)

On the other hand, defining
\[
\theta = \sqrt{2m + 1}||\varphi^o||_{m,1}^F,
\]
we have
\[
\|x - \varphi^o \ast x\|_{n,2} \leq \|x - \varphi^o \ast y\|_{n,2} + \sigma \|\varphi^o \ast \zeta\|_{n,2} \leq \|x - \varphi^o \ast y\|_{n,2} + \sqrt{2}\sigma\sqrt{\kappa_{m,n}}(1 + \sqrt{\log[1/\alpha]})
\] (65)

with probability $1 - \alpha$. Indeed, one has
\[
\|\varphi^o \ast \zeta\|_{n,2}^2 = \|M(\varphi^o)[\zeta]_{m+n}^2,\]
where for $M(\varphi^o)$ by (46) we have
\[
\|M(\varphi^o)\|_F^2 = (2n + 1)||\varphi^o||_{m,2}^2 \leq \kappa_{m,n}^2\theta^2.
\] (66)

Using (53) we conclude that, with probability at least $1 - \alpha$,
\[
\|\varphi^o \ast \zeta\|_{n,2}^2 \leq 2\kappa_{m,n}^2\theta^2(1 + \sqrt{\log[1/\alpha]})^2,
\] (67)

which implies (65). Using (64) and (65), we get that with probability at least $1 - \alpha_4 - \alpha_5$,
\[
\Re(\zeta, x - \varphi^o \ast x)_n \leq \sqrt{2 \log[1/\min(\alpha_4, \alpha_5)]} \left[ \|x - \varphi^o \ast y\|_{n,2} + \sqrt{2}\sigma\sqrt{\kappa_{m,n}}(1 + \sqrt{\log[1/\min(\alpha_4, \alpha_5)]}) \right]
\] (68)

\[
\leq \|x - \varphi^o \ast y\|_{n,2} \sqrt{2 \log[1/\min(\alpha_4, \alpha_5)]} + 2\sigma\sqrt{\kappa_{m,n}}(1 + \sqrt{\log[1/\min(\alpha_4, \alpha_5)]})^2.
\]

Now, the (indefinite) quadratic form
\[
\Re(\zeta, \varphi^o \ast \zeta)_n = \frac{1}{2}([\zeta]_{m+n}^{m+n})^H K_0(\varphi^o)[\zeta]_{m+n}^{m+n},
\]
where
\[
K_0(\varphi^o) = \begin{bmatrix}
O_{m,2m+2n+1} & M(\varphi^o) \\
M(\varphi^o)^T & O_{m,2m+2n+1}
\end{bmatrix},
\]
whence (cf. **Step 3°)
\[
|\text{Tr}[K_0(\varphi^o)]| \leq 2(2n + 1)\|\varphi^o\|
\]

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Let us bound $|\varphi^0|$. Let $e^0$ be the discrete centered Dirac vector in $\mathbb{R}^{2m+1}$, and note that $\|F_m[e^0]\|_\infty = 1/\sqrt{2m + 1}$. Then,
\[
|\varphi^0| = |\langle [\varphi^0]^m_{-m}, e^0 \rangle| \leq \|\varphi^0\|_m \|F_m[e^0]\|_\infty \leq \frac{\theta}{2m + 1},
\]
whence $|\text{Tr}[K_0(\varphi^0)]| \leq 2\kappa_{m,n}^2\theta$. On the other hand, by (66),
\[
\|K_0(\varphi^0)\|_F^2 \leq 4\|M(\varphi^0)\|_F^2 \leq 4\kappa_{m,n}^2\theta^2.
\]
Hence by (52),
\[
P \left\{ -\text{Re}(\zeta, \varphi^0 \ast \zeta) \leq 2\kappa_{m,n}^2 \theta + 2\kappa_{m,n} \theta (1 + \sqrt{2\log(1/\alpha)})^2 \right\} \geq 1 - \alpha_6. \tag{69}
\]

**Step 5.** Let us combine the bounds obtained in the previous steps with initial bound (55). For any $\alpha \in (0, 1)$, putting $\alpha_i = \alpha/4$ for $i = 0, 1, 6$, and $\alpha_j = \alpha/16$, $2 \leq j \leq 5$, by the union bound we get that with probability $\geq 1 - \alpha$,
\[
\|x - \tilde{\varphi} \ast y\|_{n,2}^2 \leq \|x - \varphi^0 \ast y\|_{n,2}^2 + 2\delta^{(2)} - 2\delta^{(1)} \tag{68}
\]
\[
\text{by (68) \quad} \leq \|x - \varphi^0 \ast y\|_{n,2}^2 + 2\sigma \|x - \varphi^0 \ast y\|_{n,2} \sqrt{2\log(16/\alpha)} \tag{67}
\]
\[
\text{by (68)\textendash}(69) \quad + 4\sigma^2 \theta \left[ \kappa_{m,n}^2 + 2\kappa_{m,n} \left( 1 + \sqrt{2\log(16/\alpha)} \right)^2 \right] \tag{69}
\]
\[
\text{by (57)} \quad + 2\sigma \|x - \tilde{\varphi} \ast y\|_{n,2} (\sqrt{2s} + \sqrt{2\log(16/\alpha)}) \tag{57}
\]
\[
\text{by (59)} \quad + 2\sqrt{2}\sigma^2 \theta (\tilde{\varphi} + 1)(1 + \sqrt{\log(16/\alpha)}) \tag{59}
\]
\[
\text{by (63)} \quad + 4\sqrt{2}\sigma^2 \theta \left[ (\kappa_{m,n}^2 + 1)(\log(2m + 2n + 1) + 3) + \kappa_{m,n} \left( 1 + \sqrt{\log(16(m + 1)/\alpha)} \right)^2 \right] \tag{73}
\]
Now, denote $c_\alpha := \sqrt{2\log(16/\alpha)}$ and let
\[
u(\alpha) = 2(\sqrt{2} + c_\alpha), \tag{71}
\]
\[
v_1(\alpha) = 4 \left[ \kappa_{m,n}^2 + 2\kappa_{m,n} \left( 1 + c_\alpha \right)^2 \right], \tag{72}
\]
\[
v_2(\alpha) = 4\sqrt{2} \left[ (\kappa_{m,n}^2 + 1)(\log(2m + 2n + 1) + 3) + \kappa_{m,n} \left( 1 + \sqrt{\log(16(m + 1)/\alpha)} \right)^2 \right]. \tag{73}
\]
In this notation, (70) becomes
\[
\|x - \tilde{\varphi} \ast y\|_{n,2}^2 \leq \|x - \varphi^0 \ast y\|_{n,2}^2 + 2\sigma(\sqrt{2s} + c_\alpha) (\|x - \tilde{\varphi} \ast y\|_{n,2} + \|x - \varphi^0 \ast y\|_{n,2}) \tag{74}
\]
\[
\quad + u(\alpha)\sigma^2 (\bar{\varphi} + 1) \kappa + (v_1(\alpha) + v_2(\alpha))\sigma^2 \bar{\varphi},
\]
which implies, by completing the squares, that
\[
\|x - \tilde{\varphi} \ast y\|_{n,2} \leq \|x - \varphi^0 \ast y\|_{n,2} + 2\sigma(\sqrt{2s} + c_\alpha) + \sigma \sqrt{u(\alpha) (\bar{\varphi} + 1) \kappa + (v_1(\alpha) + v_2(\alpha)) \bar{\varphi}}.
\]
Finally, let us simplify this bound. Note that
\[
\text{while on the other hand,} \tag{75}
\]
\[
u_1(\alpha) + v_2(\alpha) \leq 4\sqrt{2}(\kappa_{m,n}^2 + 1)(\log(2m + 2n + 1) + 4) + 4.5(4\sqrt{2} + 8)\kappa_{m,n} \log(16(2m + 1)/\alpha)
\]
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Hence we arrive at
\[
\|x - \hat{\varphi} \ast y\|_{n,2} \leq \|x - \varphi^o \ast y\|_{n,2} + 2\sigma \left( \sqrt{\hat{\varphi}W_\alpha + \sqrt{(\hat{\varphi} + 1)c_\alpha}x + \sqrt{2s} + c_\alpha} \right),
\]
where we introduced
\[
V_\alpha := 2(1 + 4\kappa_{m,n})^2 \log \left[110(m + n + 1)/\alpha \right].
\]
The bound (17) of the theorem follows from (77) after some straightforward simplifications. □

A.3 Proof of Theorem 2.2

Denote \( \hat{\varphi} = \sqrt{2m + 1}\|\hat{\varphi}\|_{F,1}^2 \), and let \( \varphi = \sqrt{2m + 1}\|\varphi^o\|_{F,1}^2 \) for a feasible solution \( \varphi^o \) to (Pen).

In the rest, we will use the same notation as in the previous proof. Due to feasibility of \( \varphi^o \), we have the following counterpart of (55):
\[
\|x - \hat{\varphi} \ast y\|_{n,2}^2 + \lambda^2 \sigma^2 \varrho^2 \leq \|x - \varphi^o \ast y\|_{n,2}^2 - 2\delta^{(1)} + 2\delta^{(2)} + \lambda^2 \sigma^2 \varrho^2.
\]
Thus, repeating Steps 1°–4° of the previous proof, we obtain a counterpart of (74):
\[
\|x - \hat{\varphi} \ast y\|_{n,2}^2 + \lambda^2 \sigma^2 \varrho^2 \leq \|x - \varphi^o \ast y\|_{n,2}^2 + 2\sigma(\|x - \varphi^o \ast y\|_{n,2} + \|x - \hat{\varphi} \ast y\|_{n,2})(\sqrt{2s} + c_\alpha) + u(\alpha)\sigma^2 \kappa + v_1(\alpha)\sigma^2 \varrho + \lambda^2 \sigma^2 \varrho^2 + [u(\alpha)\kappa + v_2(\alpha)]\sigma^2 \varrho, \tag{79}
\]
with \( u(\alpha) \), \( v_1(\alpha) \), and \( v_2(\alpha) \) given by (71)–(73). We now consider two cases as follows.

**Case (a).** First, assume that
\[
\|x - \hat{\varphi} \ast y\|_{n,2}^2 \leq \|x - \varphi^o \ast y\|_{n,2}^2 + 2\sigma(\|x - \varphi^o \ast y\|_{n,2} + \|x - \hat{\varphi} \ast y\|_{n,2})(\sqrt{2s} + c_\alpha) + u(\alpha)\sigma^2 \kappa + v_1(\alpha)\sigma^2 \varrho + \lambda^2 \sigma^2 \varrho^2. \tag{80}
\]
In this case, clearly,
\[
\|x - \hat{\varphi} \ast y\|_{n,2} \leq \|x - \varphi^o \ast y\|_{n,2} + 2\sigma(\sqrt{2s} + c_\alpha) + \sigma(\sqrt{u(\alpha)\kappa + v_1(\alpha)\varrho} + \lambda \varrho) \tag{81}
\]

**Case (b).** Suppose, on the contrary, that (80) does not hold, we then conclude from (79) that
\[
\hat{\varphi} \leq \lambda^{-2}(u(\alpha)\kappa + v_2(\alpha)),
\]
and
\[
u(\alpha)\hat{\varphi} \kappa + v_2(\alpha)\hat{\varphi} \leq \lambda^{-2}(u(\alpha)\kappa + v_2(\alpha))^2.
\]
When substituting the latter bound into (79), we obtain
\[
\|x - \hat{\varphi} \ast y\|_{n,2} \leq \|x - \varphi^o \ast y\|_{n,2} + 2\sigma(\sqrt{2s} + c_\alpha) + \sigma(\sqrt{u(\alpha)\kappa + v_1(\alpha)\varrho} + \lambda^{-1}(u(\alpha)\kappa + v_2(\alpha)) + \lambda \varrho),
\]
which is also satisfied in **Case (a)** due to (81).

Finally, using (75), (76), and the bound
\[
v_1(\alpha) \leq 4(1 + \kappa_{m,n})^2(1 + c_\alpha)^2
\]
which directly follows from (72), we get that
\[
\|x - \hat{\varphi} \ast y\|_{n,2} \leq \|x - \varphi^o \ast y\|_{n,2} + \sigma(\lambda \varrho + 4\lambda^{-1}(c_\alpha \kappa + V_\alpha)) + 2\sigma \left( \sqrt{\hat{\varphi}W_\alpha + \sqrt{c_\alpha}\kappa} + \sqrt{2s} + c_\alpha \right),
\]
with \( V_\alpha \) is given by (78), and \( W_\alpha = (1 + \kappa_{m,n})^2(1 + c_\alpha)^2 \). The bound (19) of the theorem follows by simplifying the above bound in a straightforward manner. □
Proof of Lemma A.1

Let us prove that the bound of the theorem, 
\[ \|u\|_{m+n,1}^F \leq \sqrt{1 + \kappa_{m,n}^2 \log(m + n + 1) + 3}, \]
holds on the unit ball 
\[ \mathcal{B}_{m,1} = \{ u \in C_m(\mathbb{Z}) : \|u\|_{m,1}^F \leq 1 \}; \]
then, the statement will follow by the homogeneity of the norm \( \| \cdot \|_{m+n,1}^F \). We assume that \( n \geq 1 \) (otherwise the statement of the lemma is trivial).

First of all, function \( \|u\|_{m+n,1}^F \) is convex on \( \mathcal{B}_{m,1} \), so its maximum over this set is attained at one the extreme points \( u^j \in C_m(\mathbb{Z}) \) which are given by 
\[ F_{m}[u^j]_{m} = e^{i\theta}e^j \] where \( e^j \) is the discrete Dirac pulse centered at \( j \in \mathbb{Z} \), and \( \theta \in [0,2\pi] \). Note that 
\[ u^j_\tau = \frac{1}{\sqrt{2m + 1}} \exp \left( i \left( \theta + \frac{2\pi\tau}{2m + 1} \right) \right), \]
hence, for \( \gamma_{m,n} := \sqrt{(2m + 2n + 1)(2m + 1)} \) we obtain 
\[ \|u^j\|_{m+n,1}^F = \frac{1}{\gamma_{m,n}} \sum_{k \in D_{m+n}} \left| \sum_{\tau \in D_m} 2\pi i \tau \left( \frac{j}{2m + 1} - \frac{k}{2m + 2n + 1} \right) \right| \]
\[ = \frac{1}{\gamma_{m,n}} \sum_{k \in D_{m+n}} |\text{DirKer}_m(\omega_{jk})|, \quad \text{where} \quad \omega_{jk} := 2\pi \left( \frac{j}{2m + 1} - \frac{k}{2m + 2n + 1} \right), \]
where \( \text{DirKer}_m(\cdot) \) is the Dirichlet kernel of order \( m \):
\[ \text{DirKer}_m(\omega) := \begin{cases} \sin ((2m + 1)\omega/2) / \sin (\omega/2), & \omega \neq 2\pi l, \\ 2m + 1, & \omega = 2\pi l. \end{cases} \]

Hence,
\[ \gamma_{m,n}\|u^j\|_{m+n,1}^F \leq \max_{\theta \in [0,2\pi]} \left\{ \Sigma_{m,n}(\theta) := \sum_{k \in D_{m+n}} |\text{DirKer}_m(\frac{2\pi k}{2m + 2n + 1} + \theta)| \right\}. \quad (82) \]

For any \( \theta \in [0,2\pi] \), the summation in (82) is over the \( \theta \)-shifted regular \( (2m + 2n + 1) \)-grid on the unit circle. The contribution to the sum \( \Sigma_{m,n}(\theta) \) of the two closest to \( x = 1 \) points of this grid is at most \( 2(2m + 1) \). On the other hand, for the remaining points, we can use the bound 
\[ \text{DirKer}_m(\omega) \leq \frac{1}{|\sin(\omega/2)|} \leq \frac{\pi}{\min(\omega, 2\pi - \omega)}. \]

Finally, note that \( f(\omega) = \frac{\pi}{\omega} \) decreases on \( [\frac{2\pi}{2m + 2n + 1}, \pi] \) (recall that \( n \geq 1 \)). These considerations result in the following estimate:
\[ \Sigma_{m,n}(\theta) \leq 2 \left( 2m + 1 + \sum_{k=1}^{m+n+1} \frac{2m + 2n + 1}{2k} \right). \]

Using the bound \( H_n \leq \log n + 1 \) for harmonic numbers, we arrive at 
\[ \Sigma_{m,n}(\theta) \leq 2(2m + 1) + (2m + 2n + 1) [\log(m + n + 1) + 1] \leq (2m + 2n + 1) [\log(m + n + 1) + 3], \]
and the lemma is proved. \( \square \)
B  Technical proofs

B.1  Proof of relation (20)

From (13) it follows that
\[ \| x - \varphi^o \ast x \|_{n,2} \leq \sqrt{2} \kappa_{m,n} \sigma \theta. \]

On the other hand,
\[ \| \varphi^o \ast \xi \|_{n,2}^2 = \langle \xi, M(\varphi^o) \xi \rangle_n, \]
where \( M(\varphi) \) is defined by (45). Bounding \( \| \varphi^o \|_2 \leq \| \varphi^o \|_{F_{m,1}} \) via (12), and using (46), we obtain
\[ \| M(\varphi^o) \|_F^2 = (2n + 1) \| \varphi^o \|_2^2 \leq \kappa_{m,n}^2 \rho^2. \]

Deviation bound (53) now implies, for any \( 0 < \alpha \leq 1 \), that with probability at least \( 1 - \alpha \),
\[ \| \varphi^o \ast \xi \|_{n,2} \leq \sqrt{2} \kappa_{m,n} \rho (1 + \sqrt{\log[1/\alpha]}), \]
and we arrive at (20). \( \square \)

B.2  Proof of Proposition 2.1

We only give the proof for the contrained estimator \( \hat{\varphi} = \hat{\varphi}_{\text{con}} \); the penalized estimator can be treated analogously. First, for \( t \in \mathbb{Z} \) we decompose
\[ \| [x - \hat{\varphi} \ast y]_t \| = \| [(\varphi^o + (1 - \varphi^o)) \ast (x - \hat{\varphi} \ast y)]_t \|
\leq \| [\varphi^o \ast (x - \hat{\varphi} \ast y)]_t \| + \| [(1 - \varphi) \ast (1 - \varphi^o) \ast x]_t \| + \sigma |[\hat{\varphi} \ast \xi]_t| + \sigma |[\hat{\varphi} \ast \varphi^o \ast \xi]_t| \]
\[ := \delta^{(1)} + \delta^{(2)} + \delta^{(3)} + \delta^{(4)}. \] (84)

For the remainder of the proof, let \( t \in D_{n-m_0} \). Then, we have
\[ \delta^{(1)} \leq \| \varphi^o \|_2 \| \Delta^{-t} [x - \hat{\varphi} \ast y] \|_{m_0,2}
\leq \frac{\rho}{\sqrt{m+1}} \| x - \hat{\varphi} \ast y \|_{n,2}. \]

Using the bound of Corollary 2.1 with \( \tilde{\varrho} = 2 \rho^2 \), we conclude that with probability \( \geq 1 - \alpha/3 \),
\[ \delta^{(1)} \leq \frac{C \sigma \rho}{\sqrt{m+1}} \left[ \kappa_{m,n} \rho^2 (1 + \theta + \sqrt{\log[1/\alpha]}) + \rho \sqrt{\left( \kappa_{m,n}^2 + 1 \right) \log[(m+n)/\alpha] + \sqrt{\log[1/\alpha]} + s} \right]. \]

It remains to make sure that the remaining terms are dominated by \( \delta^{(1)} \). We get
\[ \delta^{(2)} \leq (1 + \| \hat{\varphi} \|_1) \| \Delta^{-t} [(1 - \varphi^o) \ast x] \|_{m_0,\infty}
\leq (1 + 2 \rho^2) \frac{\sigma \theta \rho}{\sqrt{m+1}}
\leq \frac{C \kappa_{m,n} \rho^3 \sigma \theta}{\sqrt{m+1}}, \]
where the last transition is due to \( n \geq m_0 \). Further, by the Parseval's identity,
\[ \delta^{(3)} = \sigma |\langle F_m[\hat{\varphi}], F_m[\Delta^{-t} \xi] \rangle|
\leq \sigma \| \hat{\varphi} \|_{m,1} \| \Delta^{-t} \xi \|_{m,\infty} \]
As a precursory remark, note that if a finite-dimensional subspace $S$ is shift-invariant, i.e. $\Delta S \subseteq S$, then necessary $\Delta$ is a bijection on $S$, and $\Delta S = S$. Indeed, when restricted on $S$, $\Delta$ obviously is a linear transformation with a trivial kernel, and hence a bijection.

1. To prove the direct statement, note that the solution set of (31) with $\deg(p(\cdot)) = s$ is a shift-invariant subspace of $C(\mathbb{Z})$ – let us call it $S'$. Indeed, if $x \in C(\mathbb{Z})$ satisfies (31), so does $\Delta x$, so $S'$ is shift-invariant. To see that $\dim(S') = s$, note that $x \mapsto x^t$ is a bijection $S' \to C^s$: under this map arbitrary $x^t \in C^s$ has a unique preimage. Indeed, as soon as one fixes $x^t$, (31) uniquely defines the next samples $x_{s+1}, x_{s+2}, \ldots$ (note that $p(0) \neq 0$); dividing (31) by $\Delta^s$, one can retrieve the remaining samples of $x$ since $\deg(p(\cdot)) = s$ (we used that $\Delta$ is bijective on $S$).

\[
\leq \frac{2\rho^2}{\sqrt{m+n}} \sigma \sqrt{2 \log \left[3(2m+1)(2n-2m_0+1)/\alpha\right]},
\]

where the last inequality, holding with probability $\geq 1 - \alpha/3$, is due to (49). Finally, observe that with probability $\geq 1 - \alpha/3$ it holds

\[
\|\Delta^{-t}[\phi^o \ast \zeta]\|_{m,2} \leq \|\phi^o \ast \zeta\|_{n,2} \leq \sqrt{2} \kappa_{m,n} \rho \left(1 + \sqrt{\log[3/\alpha]}\right),
\]

cf. (83). Therefore, we have for $\delta^{(4)}$:

\[
delta^{(4)} \leq \sigma \left\|\Delta^{-t}[\phi^o \ast \zeta]\right\|_{m,2} \leq \sigma \left\|\Delta^{-t}[\phi^o \ast \zeta]\right\|_{m,2} \leq \delta^{(4)} \leq \sigma \left\|\phi^o \ast \zeta\right\|_{n,2} \leq \sqrt{2} \kappa_{m,n} \rho \left(1 + \sqrt{\log[3/\alpha]}\right)
\]

with probability $\geq 1 - \alpha/3$. Substituting the bounds for $\delta^{(k)}$, $k = 1, \ldots, 4$, into (84), we arrive at the claim. $\square$

### B.3 Proof of Proposition 3.1

Let $\phi^o \in C_m(\mathbb{Z})$ be an oracle for $x \in S_{m,n}(\theta, \rho)$, and let us use it to estimate $\tilde{x} = x + \varepsilon$. Then,

\[
|\tilde{x}_t - [\phi^o \ast \tilde{x}]_t| \leq |x_t - [\phi^o \ast x]_t| + |\varepsilon_t| + |[\phi^o \ast \varepsilon]_t|, \quad t \in D_{m+n}.
\]

The first term in the right-hand side is at most $\frac{\sigma \theta \rho}{\sqrt{2m+n}}$ since $x \in S_{m,n}(\theta, \rho)$. The second term can be bounded by (29) directly: using that $\rho \geq 1$,

\[
|\varepsilon_t| \leq \frac{\sigma \kappa}{\sqrt{2m+n}} \leq \frac{\sigma \kappa \rho}{\kappa_m \sqrt{2m+n}} \leq \frac{\sigma \kappa \rho}{\sqrt{2m+n}} \leq t \in D_{m+n}.
\]

The last term is controlled by the Cauchy-Schwarz inequality: for any $\tau \in D_{m+n}$,

\[
|\phi^o \ast \varepsilon|_\tau \leq \left\|\phi^o\right\|_2 \left\|\Delta^{-\tau} \varepsilon\right\|_{m,2} \leq \frac{\sigma \kappa \rho}{\kappa_m \sqrt{2m+n}}.
\]

The proof for the case of prediction is obtained in the same manner; the only adjustments are $t$ on which the pointwise error must be controlled and a different scaling factor

\[
\sqrt{\frac{m+n}{n+1}} \leq \sqrt{\frac{4m+2}{2n+1}} = \sqrt{2} \kappa_{m,n}.
\]

### B.4 Proof of Proposition 3.2

As a precursory remark, note that if a finite-dimensional subspace $S$ is shift-invariant, i.e. $\Delta S \subseteq S$, then necessary $\Delta$ is a bijection on $S$, and $\Delta S = S$. Indeed, when restricted on $S$, $\Delta$ obviously is a linear transformation with a trivial kernel, and hence a bijection.
2°. To prove the converse, first note that any polynomial $p(\cdot)$ with $\deg(p(\cdot)) = s$ and such that $p(0) = 1$ is uniquely expressed via its roots $z_1, ..., z_s$ as
\[ p(z) = \prod_{k=1}^{s} (1 - z/z_k). \]

Since $S$ is shift-invariant, we have $\Delta S = S$ as discussed above, i.e. $\Delta$ is a bijective linear operator on $S$. Let us fix some basis $e = [e^1; ..., e^s]$ on $S$ and denote $A$ the $s \times s$ matrix of $\Delta$ in this basis, that is, $\Delta(e^j) = \sum_{i=1}^{s} a_{ij} e^i$. Moreover, by the Jordan theorem basis $e$ can be chosen such that $A$ is upper-triangular. Then, any vector $x \in S$ satisfies $q(\Delta)x \equiv 0$, where
\[ q(\Delta) = \prod_{i=1}^{s} (\Delta - a_{ii}) = \det(\Delta I - A) \]
is the characteristic polynomial of $A$. Note that $\det A = \prod_{i=1}^{s} a_{ii} \neq 0$ since $\Delta$ is a bijection. Hence, choosing $p(\Delta) = q(\Delta)\det A$, we obtain
\[ \prod_{i=1}^{s} (1 - \Delta c_i) x \equiv 0 \]
for some complex coefficients $c_i \neq 0$. This means that $S$ is contained in the solution set $S'$ of (31) with $\deg(p(\cdot)) = s$ and such that $p(0) = 1$. Note that by 1°, $S'$ is also a shift-invariant subspace of dimension $s$, thus $S$ and $S'$ must coincide. Finally, uniqueness of $p(\cdot)$ follows from the fact that $q(\cdot)$ is a characteristic polynomial of $A$. \[ \square \]

B.5 Proof of Proposition 3.3

Let $\Pi_{S_m}$ be the Euclidean projection on $S_m := S \cap C_m^+(\mathbb{Z})$. Since $\dim(S_m) \leq s$, one has
\[ \|\Pi_{S_m}\|_2^2 = \text{Tr}(\Pi_{S_m}) \leq s. \]
As such, there is a $j \in \{0, ..., m\}$ such that the $j$-th row $\pi = [\Pi_{S_m}]_j$ of $\Pi_{S_m}$ satisfies
\[ \|\pi\|_2 \leq \sqrt{\frac{s}{m+1}} \leq \sqrt{\frac{2s}{2m+1}}. \]
On the other hand, since $\Pi_{S_m}$ is the projector on $S_m$, one has $x_j - \langle \pi, x_m \rangle = 0$ for any $x \in S$. Hence, using that $\Delta S = S$, for any $k \in \mathbb{Z}$ we have
\[ x_t - \langle \pi, x_{t-j+m} \rangle = 0, \quad t \in \mathbb{Z}. \]
Finally, let $\phi^{\circ} \in C_m(\mathbb{Z})$ be obtained by augmenting $\pi$ with zeroes in such a way that the $j$-th entry of $\pi$ becomes the central entry of $\phi^{\circ}$. Obviously, $\phi^{\circ} \in C_m(\mathbb{Z})$; on the other hand,
\[ \|\phi^{\circ}\|_2 \leq \sqrt{\frac{2s}{2m+1}} \quad \text{and} \quad x_t - [\phi^{\circ} * x]_t = 0, \quad t \in \mathbb{Z}. \]
\[ \square \]

B.6 Proof of Proposition 3.5

Note that to prove the theorem we have to exhibit a vector $q \in C^{n+1}$ of small $\ell_2$-norm and such that the polynomial $1 - q(z) = 1 - \sum_{i=0}^{n} q_i z^i$ is divisible by $p(z)$, i.e., that there is a polynomial $r(z)$ of degree $n - s$ such that
\[ 1 - q(z) = r(z)p(z). \]
Indeed, this would imply that
\[ x_t - [q * x]_t = [1 - q(\Delta)]x_t = r(\Delta)p(\Delta)x_t = 0 \]
due to \( p(\Delta)x_t = 0 \).

Our objective is to prove inequality
\[ \|q\|_2 \leq C's\sqrt{\frac{\log(n)}{n}}. \]

So, let \( \theta_1, ..., \theta_s \) be complex numbers of modulus 1 – the roots of the polynomial \( p(z) \). Given \( \delta = 1 - \epsilon \in (0, 1) \), let us set \( \bar{\delta} = 2\delta/(1 + \delta) \), so that \( \frac{\bar{\delta}}{\delta} - 1 = 1 - \bar{\delta} > 0. \) (85)

Consider the function
\[ \bar{q}(z) = \prod_{i=1}^{s} \frac{z - \theta_i}{\delta z - \theta_i}. \]

Note that \( \bar{q}(\cdot) \) has no singularities in the circle
\[ B = \{ z : |z| \leq 1/\bar{\delta} \}; \]
besides this, we have \( \bar{q}(0) = 1 \). Let \( |z| = 1/\bar{\delta} \), so that \( z = \bar{\delta}^{-1}w \) with \( |w| = 1 \). We have
\[ \frac{|z - \theta_i|}{|\delta z - \theta_i|} = \frac{1}{\bar{\delta}} \frac{|w - \delta \theta_i|}{|w - \frac{1}{\bar{\delta}} \theta_i|}. \]

We claim that when \( |w| = 1 \), \( |w - \bar{\delta} \theta_i| \leq |w - \frac{1}{\bar{\delta}} \theta_i| \).

Indeed, assuming w.l.o.g. that \( w \) is not proportional to \( \theta_i \), consider triangle \( \Delta \) with the vertices \( A = w, B = \bar{\delta} \theta_i \) and \( C = \frac{1}{\bar{\delta}} \theta_i \). Let also \( D = \theta_i \). By (85), the segment \( AD \) is a median in \( \Delta \), and \( \angle CDA \geq \frac{\pi}{2} \) (since \( D \) is the closest to \( C \) point in the unit circle, and the latter contains \( A \) ), so that \( |w - \bar{\delta} \theta_i| \leq |w - \bar{\delta} \theta_i| \).

As a consequence, we get
\[ z \in B \Rightarrow |\bar{q}(z)| \leq \delta^{-s}, \] (86)
whence also
\[ |z| = 1 \Rightarrow |\bar{q}(z)| \leq \delta^{-s}. \] (87)

Now, the polynomial \( p(z) = \prod_{i=1}^{s} (z - \theta_i) \) on the boundary of \( B \) clearly satisfies
\[ |p(z)| \geq \left[ \frac{1}{\delta} - 1 \right]^{s} = \left[ \frac{1 - \delta}{2\delta} \right]^{s}, \]
which combines with (86) to imply that the modulus of the holomorphic in \( B \) function
\[ r(z) = \left[ \prod_{i=1}^{s} (\delta z - \theta_i) \right]^{-1} \]
is bounded with \( \delta^{-s} \left[ \frac{1 - \delta}{2\delta} \right]^{-s} = \left[ \frac{2}{1 - \delta} \right]^{s} \) on the boundary of \( B \). It follows that the coefficients \( r_j \) of the Taylor series of \( r \) satisfy
\[ |r_j| \leq \left[ \frac{2}{1 - \delta} \right]^{s} \bar{\delta}^j, \quad j = 0, 1, 2, ... \]
When setting
\[ q^\ell(z) = p(z)r^\ell(z), \quad r^\ell(z) = \sum_{j=1}^{\ell} r_j z^j, \tag{88} \]
for \(|z| \leq 1\), utilizing the trivial upper bound \(|p(z)| \leq 2^s\), we get
\[
|q^\ell(z) - \bar{q}(z)| \leq |p(z)||r^\ell(z) - \bar{r}(z)|
\leq 2^s \left[ \frac{2}{1 - \delta} \right]^s \sum_{j=\ell+1}^{\infty} |r_j|
\leq \left[ \frac{4}{1 - \delta} \right]^s \frac{\delta^{\ell+1}}{1 - \delta}.
\tag{89} \]
Note that \(q^\ell(0) = p(0)r^\ell(0) = p(0)\bar{r}(0) = 1\), that \(q^\ell\) is a polynomial of degree \(\ell + s\), and that \(q^\ell\) is divisible by \(p(z)\). Besides this, on the unit circumference we have, by (89),
\[
|q^\ell(z)| \leq |\bar{q}(z)| + \left[ \frac{4}{1 - \delta} \right]^s \frac{\delta^{\ell+1}}{1 - \delta}
\leq \delta^{-s} + \left[ \frac{4}{1 - \delta} \right]^d \frac{\delta^{\ell+1}}{1 - \delta},
\tag{90} \]
where we used (87). Now,
\[
\delta = \frac{2\delta}{1 + \delta} = \frac{2 - 2\epsilon}{2 - \epsilon} = \frac{1 - \epsilon}{1 - \epsilon/2} \leq 1 - \epsilon/2 \leq e^{-\epsilon/2},
\]
and
\[
\frac{1}{1 - \delta} = \frac{1 + \delta}{1 - \delta} = \frac{2 - \epsilon}{\epsilon} \leq \frac{2}{\epsilon}.
\]
We can upper-bound \(R\):
\[
R = \left[ \frac{4}{1 - \delta} \right]^s \frac{\delta^{\ell+1}}{1 - \delta} \leq \frac{2^{2s+1}}{\epsilon^{s+1}} e^{-\epsilon/2}
\]
Now, given positive integer \(\ell\) and positive \(\alpha\) such that
\[
\frac{\alpha}{\ell} \leq \frac{1}{4},
\tag{91} \]
let \(\epsilon = \frac{\alpha}{2\ell s}\). Since \(0 < \epsilon \leq \frac{1}{8}\), we have \(-\log(\delta) = -\log(1 - \epsilon) \leq 2\epsilon = \frac{\alpha}{4s}\), implying that \(\delta \leq e^{-\epsilon/2} = e^{-\frac{\alpha}{4s}}\), and
\[
R \leq \left[ \frac{8\ell s}{\alpha} \right]^{s+1} \exp\left\{ -\frac{\alpha}{4s} \right\}.
\]
Now let us put
\[
\alpha = \alpha(\ell, s) = 4s(s + 2) \log(8\ell s);
\]
observe that this choice of \(\alpha\) satisfies (91), provided that
\[
\ell \geq O(1)s^2 \log(s + 1)
\tag{92} \]
with properly selected absolute constant \(O(1)\). With this selection of \(\alpha\), we have \(\alpha \geq 1\), whence
\[
R \left[ \frac{\alpha}{\ell} \right]^{-1} \leq \exp\left\{ -\frac{\alpha}{4s} \right\} \left[ \frac{8\ell s}{\alpha} \right]^{s+1} \frac{\ell}{\alpha}.
\]

\[ \leq \exp\left\{ -\frac{\alpha}{4s}\right\} [8\ell s]^{s+2} \leq \exp\{-s\log(8\ell s)\} \exp\{(s+2)\log(8\ell s)\} = 1, \]

that is,
\[ R \leq \frac{\alpha}{\ell} \leq \frac{1}{4}. \tag{93} \]

Furthermore,
\[
\begin{align*}
\delta^{-s} &= \exp\{-s \log(1 - \epsilon)\} \leq \exp\{2s\} = \exp\{\frac{s}{2}\} \leq 2, \\
\delta^{-2s} &= \exp\{-2s \log(1 - \epsilon)\} \leq \exp\{4s\} = \exp\{2\alpha\} \leq 1 + \exp\{\frac{1}{2}\} \frac{2\alpha}{\ell} \leq 1 + \frac{4\alpha}{\ell}. \tag{94}
\end{align*}
\]

When invoking (90) and utilizing (94) and (93) we get
\[
\frac{1}{2\pi} \int_{|z|=1} |q^\ell(z)|^2 |dz| \leq \delta^{-2s} + 2\delta^{-s} R + R^2 \leq 1 + 4\frac{\alpha}{\ell} + 4R + \frac{1}{4} R \leq 1 + 10\frac{\alpha}{\ell}.
\]

On the other hand, denoting by \( q_0, q_1, \ldots, q_{\ell+s} \) the coefficients of the polynomial \( q^\ell \) and taking into account that \( \bar{q}_0 = q^\ell(0) = 1 \), we have
\[
1 + \sum_{i=1}^{\ell+s} |q_i|^2 = |q_0|^2 + \ldots + |q_{\ell+s}|^2 = \frac{1}{2\pi} \int_{|z|=1} |q^\ell(z)|^2 |dz| \leq 1 + 10\frac{\alpha}{\ell}. \tag{95}
\]

We are done: when denoting \( n = \ell + s \), and \( q(z) = \sum_{i=1}^{n} q_j z^j \), we have the vector of coefficients \( q = [0; q_1; \ldots; q_n] \in \mathbb{C}^{n+1} \) of \( q(z) \) such that, by (95),
\[
\|q\|_2 \leq \frac{40\alpha(s+2) \log[8s(n-s)]}{n-s},
\]

and such that the polynomial \( q^\ell(z) = 1 + q(z) \) is divisible by \( p(z) \) due to (88). \qed

### B.7 Proof of Proposition 3.6

As in the proof of Proposition 3.3, consider the projector \( \Pi_{S_m} \) onto the subspace \( S_m \) (the restriction of \( S \) to coordinates \( 0, \ldots, m \)), but now let \( \phi^\ell \in \mathbb{C}_{m+1}^m(Z) \) correspond to the last row of \( \Pi_{S_m} \). As in the proof of Proposition 3.3, we see that \( x_t = [\phi^\ell * x]_t \) for any \( t \in Z \), and it remains to bound \( \|\phi^\ell\|_2 \). Note that the premise of the proposition is in fact equivalent to the assumption that \( S_m \) is spanned by the vectors
\[
\left\{ v(\omega) : |v(\omega)|_t = \frac{e^{i\omega_t t}}{\sqrt{m+1}}, \quad t \in D_m^+ \right\}, \quad \omega \in \{\omega_1, \ldots, \omega_s\}.
\]

Hence, the projector \( \Pi_{S_m} \) can be written as
\[
\Pi_{S_m} = V (V^H V)^{-1} V^H,
\]

where \( V \) is an \((m+1) \times s\) Vandermonde matrix with columns \( v(\omega_k) \), \( k = 1, \ldots, s \). Note that since \( s \leq m+1 \), and \( \omega_k, k = 1, \ldots, s \) are distinct, matrix \( V \) has full column rank.

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Now, in order to bound $||\phi^o||_2$ from above, it suffices to separate $\lambda_{\min}(V^H V)$, the minimal
eigenvalue of $V^H V$, from zero. Indeed, suppose that $\lambda_{\min}(V^H V) > 0$, and write
$$\Pi s_m = U U^H,$$
where $U = [U_1 \cdots U_s]$ is the unitary normalization of $V$:
$$U = [U_1 \cdots U_s] = V (V^H V)^{-1/2}, \quad U^H U = I_s.$$
Let $u = [u_1, ..., u_s]$ be the last row of $U$, and $v$ that of $V$. One has $\phi = u U^H = \sum_{k=1}^s u_k [U_k]^H$, and hence, $||\phi^o||_2^2 = ||u||_2^2$. On the other hand, writing $u = v (V^H V)^{-1/2}$, we arrive at
$$||u||_2^2 \leq \frac{||v||_2^2}{\lambda_{\min}(V^H V)} \leq \frac{s}{(m + 1) \lambda_{\min}(V^H V)},$$
the last transition being due to the bound \(1/\sqrt{m+1}\) on the absolute values of the elements of $V$.

Finally, let us exploit the bound on the condition number of a Vandermonde matrix (see \cite{Moi15}):

**Lemma B.1** (Theorem 2.3 in \cite{Moi15}). For $\delta_{\min}$ given by (37), we have
$$\frac{\lambda_{\max}(V^H V)}{\lambda_{\min}(V^H V)} \leq \left( m - \frac{2\pi}{\delta_{\min}} \right)^{-1} \left( m + \frac{2\pi}{\delta_{\min}} \right).$$

We clearly have $||V||_{\text{op}} \geq 1$, and hence $\lambda_{\max}(V^H V) \geq 1$. Together with (38), this results in
$$\frac{1}{\lambda_{\min}(V^H V)} \leq \frac{\nu + 1}{\nu - 1},$$
whence the necessary bound on $||\phi^o||_2$ follows. \(\square\)

### B.8 Proof of the bound (42) for the “composite” estimator from Section 3.2

Recall that
$$c_1 s \log(N+1) \leq \frac{M+1}{N-M+1} \leq c_2 s \log(N+1). \quad (96)$$

W.l.o.g. assume that $\sigma = 1$, $N$ and $M$ are even, $s \geq 3$, and $C_1 \geq 1$; as a result, $M \geq N/2$.

Recall also the partition of the domain $D_N$ into subdomains $D_M$, $D^+$, $D^-$, and let $\hat{\varphi}$, $\hat{\varphi}^+$, $\hat{\varphi}^-$ be the corresponding adaptive filters for these subdomains:

- $\hat{\varphi} \in C_{N-M}(\mathbb{Z})$ is an optimal solution to (Con) with $m = N - M$, $n = M$, and
  $$\bar{\varphi} = 2(\sqrt{2}s)^2 = 4s;$$

- $\hat{\varphi}^+ \in C^+_{M+N}(\mathbb{Z})$ is an optimal solution to (Con\*) applied to the shifted observations $\Delta^N y$ with $h = 0$, $m = N + M$, $n = N - M$, and
  $$\bar{\varphi}^+ = \bar{\varphi}^+ := 2C^2 s^2 \log((2N+1)s+1),$$
  $C$ being the constant in (36);

- $\hat{\varphi}^- \in C^-_{M+N}(\mathbb{Z})$ is an optimal solution to (Con\*) applied to $\Delta^{-N} y$ with $h = -(M + N)$ and the same $m$, $n$, and $\bar{\varphi}$ as in the previous case.

Correspondingly, let $\hat{x}$ be defined pointwise as
$$\hat{x}_t = \begin{cases} 
[\hat{\varphi} * y]_t, & t \in D_M; \\
[\hat{\varphi}^+ * y]_t, & t \in D^+(\mathbb{Z}); \\
[\hat{\varphi}^- * y]_t, & t \in D^-(\mathbb{Z}). 
\end{cases}$$
1°. From Proposition 3.5 along with (12)–(15), we obtain the existence of \( \varphi^+ \in \mathcal{C}^+_M(Z) \) which satisfies \( x_t - [\varphi^+ * x]_t = 0 \) for any \( t \in D^+ \), and

\[
\|\varphi^+\|_{M+N,1}^F \leq \frac{\varphi^+}{\sqrt{M+N+1}} \leq \frac{C_1 s^2 \log(N+1)}{\sqrt{M+N+1}},
\]

implying that

\[
E\left\| [x - \varphi^+ * y]_M \right\|_2^N \leq C_2 \left( \frac{N - M + 1}{N + M + 1} \right) s^4 \log^2(N + 1).
\]

Let \( \hat{\varphi}^+ \) be as defined above, and denote

\[
\kappa_+ := \sqrt{\frac{N - M + 1}{N + M + 1}}.
\]

Then, using that \( \kappa_+ \leq 1 \), Proposition 2.1 implies

\[
E\left\| y_M \right\|_2^N \leq C_3 \log^2(N) \left( \kappa_+^2 s^4 + s^2 \right).
\]

We can repeat this argument almost verbatim for \( \hat{\varphi}^- \), arriving at

\[
E\left\| [x - \hat{\varphi}^- * y]_M \right\|_2^N \leq 2 C_3 \log^2(N) \left( \kappa_+^2 s^4 + s^2 \right).
\]

(97)

2°. Similarly, as follows from Proposition 3.3, there exists a filter \( \varphi \in \mathcal{C}_{N-M}(Z) \) such that \( x_t - [\varphi * x]_t = 0 \) for any \( t \in D_M \), and

\[
\|\varphi\|_{N-M,1}^F \leq \frac{4s}{\sqrt{2N - 2M + 1}}.
\]

Let \( \hat{\varphi} \) be as defined above, and denote

\[
\kappa := \sqrt{\frac{M + 1}{N - M + 1}}.
\]

Proceeding as in 1° but this time using Theorem 2.1, we obtain

\[
E\|x - \hat{x}\|_N^2 \leq C_4 \left( s^2 \kappa^2 + s \left( 1 + \kappa^2 \right) \log N \right) \leq C_5 \kappa^2 \left( s^2 + s \log N \right),
\]

where the last transition is due to \( \kappa^2 \geq 1 \).

3°. It remains to combine (97) and (98). Doing so, and using that \( M + 1 \geq c(N + M + 1) \), we arrive at

\[
E\|x - \hat{x}\|_N^2 \leq C s^2 \log^2(N) + C'' \left( \frac{M + 1}{N - M + 1} \left( s^2 + s \log N \right) + \frac{N - M + 1}{M + 1} N^2 \log^2 N \right).
\]

The choice of \( M \) according to (96) minimizes the right-hand side, and we obtain (42). \( \square \)
C Why estimators (Con) and (Pen) cannot be analyzed as Lasso?

Despite striking similarity with Lasso and Dantzig selector [Tib96, CT07, BRT09], the proposed least-squares estimators are of quite different nature. First of all, minimization in these procedures is aimed to recover a filter but not the signal itself, and this filter is not sparse unless for harmonic oscillations with frequencies on the DFT grid. Second, the equivalent of “regression matrices” involved in these procedures cannot be assumed to satisfy the usual “restricted incoherency” conditions usually imposed to prove statistical properties of “classical” \( \ell_1 \)-recoveries (see [BVDG11, Chapter 6] for a comprehensive overview of these conditions). Moreover, being constructed from the noisy signal itself, these matrices depend on the noise, which poses some extra difficulties in the analysis of the properties of these estimators, in particular, leading to the necessity of Assumption 2.1. Let us briefly illustrate these difficulties.

Let \( m = n \) for simplicity, and, given \( y \in C(\mathbb{Z}) \), let \( T(y) \) be the \((2n+1) \times (2n+1)\) “convolution matrix” as defined by (43) such that for \( \varphi \in C_n(\mathbb{Z}) \) one can write \([\varphi \ast y]_n = T(y)[\varphi]_n\). When denoting \( f = F_n[\varphi] \), the optimization problem in (Con) can be recast as a “standard” \( \ell_1 \)-constrained least-squares problem with respect to \( f \):

\[
\min_{f \in C_n^{2n+1}} \left\{ \|y - A_n f\|_{2,2}^2 : \|f\|_1 \leq \frac{\bar{\sigma}}{\sqrt{2n+1}} \right\},
\]

where \( A_n = T(y)F_n^H \). Observe that \( f^o = F_n[\varphi^o] \) is feasible for (99), so that

\[
\|y - A_n \hat{f}\|_{2,2}^2 \leq \|y - A_n f^o\|_{2,2}^2,
\]

where \( \hat{f} = F_n[\hat{\varphi}] \), and

\[
\|x - A_n \hat{f}\|_{2,2}^2 - \|x - A_n f^o\|_{2,2}^2 \leq 2\sigma \left( \text{Re}(\zeta, x - A_n f^o) - \text{Re}(\zeta, x - A_n \hat{f}) \right)_n \\
\leq 2\sigma |\zeta, A_n(f^o - \hat{f})|_n \\
\leq 2\sigma \|A_n^H[\zeta]_n\|_\infty \|f^o - \hat{f}\|_1 \\
\leq 4\sigma \|A_n^H[\zeta]_n\|_\infty \frac{\bar{\sigma}}{\sqrt{2n+1}}.
\]

In the “classical” situation, where \([\zeta]_n\) is independent of \( A_n \) (see, e.g., [JN00]), one has

\[
\|A_n^H[\zeta]_n\|_\infty \leq c_\alpha \sqrt{\log n} \max_j \|A_n\|_j \leq c_\alpha \sqrt{n \log n} \max_{i,j} |A_{ij}|,
\]

where \( c_\alpha \) is a logarithmic in \( \alpha^{-1} \) factor. This would rapidly lead to the bound (17). In the case we are interested in, where \( A_n \) incorporates observations \([y]_n\), and thus depends on \([\zeta]_n\), curbing the cross term is more involved and explicitly requires Assumption 2.1.

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