UNIMODULAR $L_{\infty}$-ALGEBRAS

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Abstract. We give a new short proof that the wheeled operad of unimodular Lie algebras is Koszul and use this to explicitly construct its minimal resolution. A representation of this resolution in a finite dimensional vector space $V$ we call a unimodular $L_{\infty}$-algebra. Such a structure corresponds to a homological vector field on $V$ together with an invariant measure. We present explicit formulae for homotopy transferred structures, define the deformation complex and give a cohomological obstruction to the extension of an arbitrary structure of finite dimensional $L_{\infty}$-algebra to a structure of unimodular $L_{\infty}$-algebra.

1. Introduction

The theory of operads and props is an effective tool for constructing homotopy theories of many algebraic structures. Given a particular structure (e.g. associative algebras, Lie algebras, bialgebras), the first step in the construction would be to find an operad or prop $P$ whose representations are in one-to-one correspondence with all possible structures of this type. The second step would then be to construct a minimal resolution $P_{\infty}$ of $P$. This is often difficult to carry through unless $P$ is Koszul, in which case $P_{\infty}$ is the cobar construction on the Koszul dual of $P$. The philosophy is that a generic representation of $P_{\infty}$ is the correct up-to-homotopy version of the structure one began with. This program has successfully been applied to many types of algebraic structures, for example, there is an operad $Lie$ whose representations are precisely Lie algebras. Moreover, $Lie$ is Koszul so it is possible to compute its minimal resolution $Lie_{\infty}$ explicitly. When considering representations of $Lie_{\infty}$ one gets $L_{\infty}$-algebras, a notion which by now has a well-established place in mathematics and mathematical physics. The aim of this paper is to construct the homotopy theory of unimodular Lie algebras according to the program just described. The unimodularity condition implies that this can not be done within the category of ordinary operads or props. We show, however, that in the category of wheeled operads, introduced in [Mer07], unimodularity can be appropriately handled.

The geometric interpretation of an $L_{\infty}$-algebra structure on a vector space $V$ is as a homological vector field $Q$ in a formal neighbourhood of $0$ in $V$ which vanishes at $0$. We show that, with this perspective, a structure of unimodular $L_{\infty}$-algebra is equivalent to a pair $(Q, \rho_f)$ where $\rho_f$ is a $Q$-invariant measure. Such pairs have recently been obtained by P. Mnëv in [Mnë06] via Batalin-Vilkovisky quantization of a certain extended BF-theory. Our results imply that the Feynman integrals in [Mnë06] assemble precisely into a representation of a minimal resolution of the wheeled operad of unimodular Lie algebras, and hence indeed have homotopy theoretical meaning.

A few words on notation and conventions. Our ground field is denoted $\mathbb{k}$ and is of characteristic zero. The set $\{1, \ldots, n\}$ is denoted $[n]$ and $\Sigma_n$ is the group of bijections of $[n]$. The trivial representation of $\Sigma_n$ is denoted $\mathbb{1}_n$, while the sign representation is denoted $\text{sgn}_n$. For a permutation $\sigma$, $\text{sgn} \sigma$, and also $(-1)^\sigma$, denote the sign of the permutation. For $[p_1 + \cdots + p_n] = I_1 \sqcup \cdots \sqcup I_n$, with $|I_j| = \{i_{j,1} < \cdots < i_{j,p_j}\}$,

$$\sigma_{I_1,\ldots,I_n} = \begin{pmatrix} 1 & \cdots & p_1 & p_1 + 1 & \cdots & p_1 + p_2 & \cdots & p_1 + \cdots + p_n \\ i_1,1 & \cdots & i_{1,p_1} & i_{2,1} & \cdots & i_{2,p_2} & \cdots & i_{n,p_n} \end{pmatrix}. \quad (1)$$
The abbreviation dg stands for differential graded and all differentials are of degree +1. The term ‘corolla’ refers to a graph with a single vertex.

The paper is organized as follows. In Section 2 we recall the theory of wheeled properads and operads, the bar and cobar constructions and Koszulness. In Section 3 we introduce the wheeled operad of unimodular Lie algebras \( \mathcal{UL} \) and show that it is Koszul. This gives us explicitly its minimal resolution \( \mathcal{UL}_{\infty} \) in terms of generators and differential. In Section 4 we give explicit formulae for the transferred structure of unimodular \( L_\infty \)-algebra. In Section 5 we interpret a structure of unimodular \( L_\infty \)-algebra as a homological vector field together with an invariant measure. In Section 6 we define the deformation complex of a unimodular \( L_\infty \)-algebra and show that we can extend the structure to a unimodular \( L_\infty \)-algebra if and only if the characteristic class vanishes. Finally, for the sake of completeness, we address the topic of characteristic classes of unimodular associative algebras in an Appendix.

2. Recollection of the theory of wheeled operads

2.1. Decorated graphs and wheeled operads.

Wheeled operads are particular cases of wheeled properads. We find it easier to define this more general notion first and then focus on the special case at hand. For reference on properads and their Koszulness, see [Val07]. For their wheeled extensions, see [MMS07].

We denote with \( \mathfrak{G} \) the class of directed, connected, labeled graphs and by \( \mathfrak{G}_e(m, n) \) the subclass of graphs with \( m \) outgoing, and \( n \) ingoing legs. By ‘labeled’ we mean that \( G \in \mathfrak{G}_e(m, n) \) is equipped with bijections \( \text{out}(G) \rightarrow [m] \) and \( \text{in}(G) \rightarrow [n] \), where \( \text{out}(G) \) and \( \text{in}(G) \) denote the set of outgoing respectively ingoing legs. We draw graphs directed from top to bottom.

Example.

For a graph \( G \in \mathfrak{G}_e \), we denote its set of vertices with \( \mathfrak{v}(G) \). Given a vertex \( v \), its set of outgoing legs and edges is denoted \( \text{out}(v) \) and its set of ingoing legs and edges is denoted \( \text{in}(v) \). The biarity of a vertex \( v \) is \( (|\text{out}(v)|, |\text{in}(v)|) \). We have \( \mathfrak{v}(G) = \mathfrak{v}_c(G) \sqcup \mathfrak{v}_{nc}(G) \), where \( \mathfrak{v}_c(G) \) is the set of cyclic vertices, i.e. those which lie on a closed directed path, and \( \mathfrak{v}_{nc}(G) \) is the set of non-cyclic vertices.

The edges of \( G \) is denoted \( \mathfrak{e}(G) \) and, as for vertices, \( \mathfrak{e}(G) = \mathfrak{e}_c(G) \sqcup \mathfrak{e}_{nc}(G) \).

Recall that a dg \( \Sigma \)-bimodule \( \mathcal{E} \) is a collection \( \{\mathcal{E}(m, n)\} \), for \( m, n \geq 0 \), of dg vector spaces with compatible left \( \Sigma_m \)- and right \( \Sigma_n \)-actions.

Given a \( k \)-vertex graph \( G \) and a \( \Sigma \)-bimodule \( \mathcal{E} \), let \( v \in \mathfrak{v}(G) \) with \( |\text{out}(v)| = p \) and \( |\text{in}(v)| = q \). The vector space \( \mathbb{k}[\text{Bij}([p], \text{out}(v))] \) spanned by all bijections \( [p] \rightarrow \text{out}(v) \) is naturally a right \( \Sigma_p \)-module. Likewise \( \mathbb{k}[\text{Bij}([in(v)], [q])] \) is naturally a left \( \Sigma_q \)-module. We define the space of decorations of the vertex \( v \) by \( \mathcal{E} \) as

\[
\mathcal{E}(\text{out}(v), \text{in}(v)) = \mathbb{k}[\text{Bij}([p], \text{out}(v))] \otimes_{\Sigma_p} \mathcal{E}(p, q) \otimes_{\Sigma_q} \mathbb{k}[\text{Bij}([\text{in}(v)], [q])].
\]

The space of decorations of the graph \( G \) by \( \mathcal{E} \) is defined as

\[
G(\mathcal{E}) = \left( \bigotimes_{v \in \mathfrak{v}(G)} \mathcal{E}(\text{out}(v), \text{in}(v)) \right)_{\text{Aut } G},
\]
where
\[ \bigotimes_{v \in V(G)} E(out(v), in(v)) = \left( \bigoplus_{\gamma \in \bij([k], V(G))} \bigotimes_{i=1}^k E(out(\gamma(i)), in(\gamma(i))) \right) \Sigma_k \]
is the unordered tensor product and \( \text{Aut} \, G \) is the group of automorphisms of \( G \) which fix legs. For \( G \in \mathfrak{S}_n(m, n) \), relabeling gives \( G(\mathcal{E}) \) a left \( \Sigma_m \)- and a right \( \Sigma_n \)-action. An element of \( G(\mathcal{E}) \), for a \( k \)-vertex graph \( G \) is essentially a pair \((G, [e_1 \otimes \cdots \otimes e_k])\) of the graph and an equivalence class of a tensor product of elements of \( \mathcal{E} \).

For any connected subgraph \( H \subset G \), we denote with \( G/H \) the graph in which all vertices and edges of \( H \) has been contracted into a single vertex. Given a \( \Sigma \)-bimodule \( \mathcal{E} \), let \( \{\mu_G : G(\mathcal{E}) \to \mathcal{E}\}_{G \in \mathfrak{S}_c} \) be a collection of equivariant linear maps. For a connected \( H \) there is an induced map \( \mu_H : G(\mathcal{E}) \to G/H(\mathcal{E}) \), which equals \( \mu_H \) on the vertices of \( H \) and the identity elsewhere.

**Definition.** A **wheeled properad** is a \( \Sigma \)-bimodule \( \mathcal{P} \) together with a collection of equivariant linear maps \( \{\mu_G : G(\mathcal{P}) \to \mathcal{P}\}_{G \in \mathfrak{S}_c} \) such that, for any connected subgraph \( H \subset G \), \( \mu_G = \mu_{G/H} \circ \mu_H^G \).

A **wheeled operad** is a wheeled properad \( \mathcal{P} \) such that if \( m \geq 2 \), then \( \mathcal{P}(m, n) = 0 \).

From the equations satisfied by the collection \( \{\mu_G\} \) it follows that it is determined by the subcollection of those maps \( \mu_G \) for which \( G \) is either a two vertex graph without directed cycles or a corolla with a loop.

Given a wheeled operad \( \mathcal{P} \) its **operadic part**, \( \mathcal{P}_o \), is defined by
\[ \mathcal{P}_o(n) = \mathcal{P}(1, n). \]
It is clear that the \( \Sigma \)-module \( \{\mathcal{P}_o(n)\} \) is an ordinary operad. The **wheeled part** of \( \mathcal{P} \), \( \mathcal{P}_w \), is defined by
\[ \mathcal{P}_w(n) = \mathcal{P}(0, n) \]
and is, obviously, a right \( \mathfrak{S}_c \)-module.

The inclusion of genus zero graphs into \( \mathfrak{S}_c \) induces a forgetful functor from wheeled properads to ordinary properads. This functor has an adjoint \( \mathcal{P} \mapsto \mathcal{P}^{\circ} \), called **wheelification**. The wheelification of an ordinary operad is a wheeled operad.

Dually, consider a dg \( \Sigma \)-bimodule \( \mathcal{E} \) and a collection of equivariant linear maps \( \{\Delta_G : \mathcal{E} \to G(\mathcal{E})\}_{G \in \mathfrak{S}_c} \). For any connected subgraph \( H \subset G \) there is an induced map \( \Delta_H^G : G/H(\mathcal{E}) \to G(\mathcal{E}) \), which equals \( \Delta_H \) on the vertex of \( G/H \) corresponding to \( H \) and the identity elsewhere.

**Definition.** A **wheeled cooperad** is a \( \Sigma \)-bimodule \( \mathcal{C} \) together with a collection of linear maps \( \{\Delta_G : \mathcal{C} \to G(\mathcal{C})\}_{G \in \mathfrak{S}_c} \) such that, for any connected subgraph \( H \subset G \), \( \Delta_G = \Delta_H^G \circ \Delta_{G/H} \).

A **wheeled cooperad** is a wheeled cooperad \( \mathcal{C} \) such that if \( m \geq 2 \), then \( \mathcal{P}(m, n) = 0 \).

Again, \( \{\Delta_G\} \) is determined by the subcollections of maps \( \Delta_G \) for which \( G \) is either a two vertex graph without directed cycles or a corolla with a loop.

**Example.** Given a dg \( \Sigma \)-bimodule \( \mathcal{E} \), the **free wheeled properad** on \( \mathcal{E} \), \( \mathcal{F}^{\circ}(\mathcal{E}) \), has as underlying dg \( \Sigma \)-bimodule
\[ \mathcal{F}^{\circ}(m, n) = \bigoplus_{G \in \mathfrak{S}_c(m, n)} G(\mathcal{E}), \]
with differential \( \partial_G \) induced by that of \( \mathcal{E} \). An element of \( G(\mathcal{F}^{\circ}(\mathcal{E})) \) is a graph with vertices decorated by graphs decorated by \( \mathcal{E} \). The structure of wheeled properad on \( \mathcal{F}^{\circ}(\mathcal{E}) \) forgets this double decoration. Note that \( \mathfrak{S}_c \) contains the two exceptional graphs \( \downarrow \) and \( \bigcirc \) without vertices.

The **cofree wheeled cooperad** on \( \mathcal{E} \), \( \mathcal{F}^{\bigcirc}(\mathcal{E}) \), has the same underlying dg \( \Sigma \)-bimodule as \( \mathcal{F}^{\circ}(\mathcal{E}) \). The morphism \( \Delta_G \) on a corolla decorated by some graph \( G' \) is the sum of decorations of \( G \) by connected subgraphs of \( G' \) such that when forgetting the double decoration one gets \( G' \).
Note that if \( m \geq 2 \) implies \( \mathcal{E}(m, n) = 0 \), then \( \mathcal{F}_c(\mathcal{E}) \) is a wheeled operad and \( \mathcal{F}_c(\mathcal{E}) \) is a wheeled cooperad.

**Example.** Given a finite dimensional dg vector space \((V, d)\), let \( \mathcal{E}nd_c^\mathcal{V} \) be defined by

\[
\mathcal{E}nd_c^\mathcal{V}(m, n) = \begin{cases} 
\text{Hom}(V^{\otimes n}, \mathbb{k}) & m = 0, \\
\text{Hom}(V^{\otimes n}, V) & m = 1, \\
0 & \text{otherwise}.
\end{cases}
\]

It is a dg \( \Sigma \)-bimodule with the usual grading, differential and \( \Sigma \)-action. For any \( \text{wheeled endomorphism operad} \) \( P \), the \( \text{coderivation} \) \( \partial \) is a morphism of dg wheeled operads \( A \).

**Definition.** This \( \Sigma \)-bimodule with the usual grading, differential and \( \Sigma \)-action. For any \( \text{wheeled endomorphism operad} \) \( P \) of \( \mathbb{P} \), the \( \text{coderivation} \) \( \partial \) is a morphism of dg wheeled operads.

It is a morphism of \( \text{wheeled operad} \) \( P \) of \( \mathbb{P} \). Hence, for any \( \text{wheeled operad} \) \( P \) of \( \mathbb{P} \), the \( \text{wheeled endomorphism operad} \) \( P \) of \( \mathbb{P} \) is either a two vertex graph without directed cycles or a corolla with a loop defines a morphism \( \mu : \mathcal{F}_c(\mathcal{E}) \rightarrow P \).

**Definition.** An \( \text{augmentation} \) of a wheeled operad \( \mathcal{P} \) is a morphism \( \mathcal{P} \rightarrow I^\mathcal{P} \). An \( \text{augmented wheeled operad} \) is a wheeled operad together with an augmentation. Given an augmented wheeled operad \( \mathcal{P} \), the kernel of the augmentation is denoted \( \mathcal{P} \) and is called the \( \text{augmentation ideal} \).

**Definition.** The \( \text{wheeled suspension} \) of a dg \( \Sigma \)-bimodule \( \mathcal{E} \), denoted \( w\mathcal{E} \), is defined as

\[
w\mathcal{E}(m, n) = \mathcal{E}(m, n)[2m - n] \otimes \text{sgn}_n
\]

and the \( \text{wheeled desuspension} \), \( w^{-1}\mathcal{E} \), as

\[
w^{-1}\mathcal{E}(m, n) = \mathcal{E}(m, n)[n - 2m] \otimes \text{sgn}_n.
\]

Given a dg wheeled properad \( \mathcal{P} \), the collection of maps \( \mu_G \) for which \( G \) is either a two vertex graph without directed cycles or a corolla with a loop defines a morphism \( \mu : \mathcal{F}_c(\mathcal{P}) \rightarrow \mathcal{P} \). This \( \mu \) induces a degree one morphism \( \mathcal{F}_c(\mathcal{P}) \rightarrow \mathcal{P} \), which in turn determines a degree one coderivation \( \partial_\mu \) of \( \mathcal{F}_c(\mathcal{P}) \). One can check that the equations satisfied by the \( \mu_G \) are equivalent with \( \partial_\mu \) being a coderivation.
Definition. The wheeled bar construction on a dg wheeled properad $\mathcal{P}$ is defined as
\[(B^\odot(\mathcal{P}), \partial_B) = (\mathcal{F}_c^\odot(w\mathcal{P}), \partial_{\mathcal{P}} + \partial_\mu).\]

Dually, given a dg wheeled coproperad $\mathcal{C}$, the collection $\Delta_G$ for which $G$ is either a two vertex graph without directed cycles or a corolla with a loop defines a morphism $\Delta : \mathcal{C} \to \mathcal{F}_c^\odot(w^{-1}\mathcal{C})$. This induces a degree one morphism $w^{-1}\mathcal{C} \to \mathcal{F}_c^\odot(w^{-1}\mathcal{C})$, which determines a degree one derivation $\partial_\Delta$ of $\mathcal{F}_c^\odot(w^{-1}\mathcal{C})$. One checks that the equations satisfied by the $\Delta_G$ are equivalent with $\partial_\Delta$ being a differential.

Definition. The wheeled cobar construction on a dg wheeled coproperad $\mathcal{C}$ is defined as
\[(\Omega^\otimes(\mathcal{C}), \partial_\Omega) = (\mathcal{F}_c^\otimes(\mathcal{P}), \partial_{\mathcal{P}} + \partial_\Delta).\]

2.3. Quadratic wheeled operads and wheeled Koszulness.

Definition. A wheeled operad $\mathcal{P}$ is quadratic if it has a presentation
\[\mathcal{P} = \mathcal{F}_c^\odot(\mathcal{E})/(R),\]
where $\mathcal{E}$ is non-zero only for $(m, n) = (1, 2)$ and $R$ is a subspace of $\mathcal{F}_c^\odot(\mathcal{E})(1, 3) \oplus \mathcal{F}_c^\otimes(\mathcal{E})(0, 1)$.

Note that any quadratic wheeled operad is naturally augmented by projection onto the part spanned by the exceptional graphs.

Example. Let $\mathcal{E}(1, 2) = \mathbb{I}_2$ and denote its generator by
\[
\begin{array}{c}
\uparrow \\
1 \\
\downarrow \\
2 \\
\end{array}
\]
Then the wheelification of the operad $\mathcal{Com}$ of commutative associative algebras has the presentation
\[\mathcal{Com}^\odot = \mathcal{F}_c^\odot \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}\!
\right).
\]

Definition. The Čech dual of a $\Sigma$-bimodule $\mathcal{E}$, denoted $\check{\mathcal{E}}$, is defined by
\[\check{\mathcal{E}}(m, n) = \text{sgn}_m \otimes \mathcal{E}(m, n)^* \otimes \text{sgn}_n.\]

Definition. The wheeled Koszul dual operad of the quadratic wheeled operad $\mathcal{P} = \mathcal{F}_c^\odot(\mathcal{E})/(R)$, is defined as
\[\mathcal{P}^! = \mathcal{F}_c^\odot(\check{\mathcal{E}})/(R^\perp).\]
Here $R^\perp$ is the kernel of the composition
\[\mathcal{F}_c^\odot(\check{\mathcal{E}})(1, 3) \oplus \mathcal{F}_c^\otimes(\check{\mathcal{E}})(0, 1) \xrightarrow{\sim} (\mathcal{F}_c^\odot(\mathcal{E})(1, 3) \oplus \mathcal{F}_c^\otimes(\mathcal{E})(0, 1))^* \to R^*.\]

The operadic part of the wheeled Koszul dual operad of a wheeled operad $\mathcal{P}$ is the ordinary Koszul dual operad of the operadic part of $\mathcal{P}$, i.e.
\[(\mathcal{P}^!)_o = (\mathcal{P}_o^!)^1.\]

The wheeled part equals the quotient of the wheeled part of the wheeled completion of $\mathcal{P}_o^!$ by the ideal generated by $R_w^\perp$,
\[(\mathcal{P}^!)_w = ((\mathcal{P}_o^!)_w)/(R_w^\perp).\]

Definition. The wheeled Koszul dual cooperad of the quadratic wheeled operad $\mathcal{P}$ is defined as
\[\mathcal{P}^! = H^0(B^\odot(\mathcal{P}), \partial_B).\]
One can show that $\mathcal{P}^!$ is a wheeled subcooperad of $B^\odot(\mathcal{P})$ and that $\mathcal{P}^! = (\mathcal{P}^!)^*$.

**Definition.** A quadratic wheeled operad $\mathcal{P}$ is **wheeled Koszul** if the inclusion $$(\mathcal{P}, 0) \longrightarrow (B^\odot(\mathcal{P}), \partial_B)$$ is a quasi-isomorphism, i.e. induces an isomorphism on cohomology.

The wheeled operad $\mathcal{P}$ is wheeled Koszul if and only if $\mathcal{P}^!$ is wheeled Koszul. Moreover, $\mathcal{P}$ is wheeled Koszul if and only if the natural projection $$(\Omega^\odot(\mathcal{P}^!), \partial_\Omega) \longrightarrow (\mathcal{P}, 0)$$ is a quasi-isomorphism. Hence, if $\mathcal{P}$ is Koszul, then $(\Omega^\odot(\mathcal{P}^!), \partial_\Omega)$ is a free minimal resolution of $\mathcal{P}$. In this case we write $\mathcal{P}_\infty = (\Omega^\odot(\mathcal{P}^!), \partial_\Omega)$.

**Definition.** An ordinary Koszul operad $\mathcal{P}$ is **stably Koszul** if the wheelification of the quasi-isomorphism $(\Omega(\mathcal{P}^!), \partial_\Omega) \longrightarrow (\mathcal{P}, 0)$ is still a quasi-isomorphism.

In [Mer07] it is shown that $\mathcal{L}ie$ is stably Koszul. We shall use this fact in the proof of wheeled Koszulness for the wheeled operad of unimodular Lie algebras, $\mathcal{U}Lie$.

### 3. Unimodular Lie algebras

Any Lie algebra $(\mathfrak{g}, [-, -])$ is endowed with a canonical representation in itself, the *adjoint representation* $\text{ad} : \mathfrak{g} \longrightarrow \text{End}(\mathfrak{g})$, defined by $\text{ad}(x)(y) = [x, y]$. A finite dimensional Lie algebra $\mathfrak{g}$ is **unimodular** if $\text{tr}(\text{ad } x) = 0$ for all $x \in \mathfrak{g}$. Important examples of unimodular Lie algebras include semi-simple Lie algebras. If we choose a basis $\{e_\alpha\}$ for $\mathfrak{g}$, the Lie algebra structure is determined by the structure constants $L^\gamma_{\alpha\beta}$ defined by $[e_\alpha, e_\beta] = L^\gamma_{\alpha\beta} e_\gamma$.

Unimodularity is then expressed as $L^\beta_{\alpha\beta} = 0$ for all $\alpha$.

Unimodular Lie algebras are representations of a quadratic wheeled operad, $\mathcal{U}Lie = \mathcal{F}^\odot(\mathcal{E})/(R)$, where $\mathcal{E}(1, 2) = \text{sgn}_2$ and $R$ is spanned by the Jacobi relation and the trace condition. We denote the generator of $\mathcal{E}(1, 2)$ by

![Diagram of E(1, 2)](image)

so that

$$\mathcal{U}Lie = \left( \begin{array}{c} 1 \\ 2 \\ \mathcal{F}^\odot \left( \begin{array}{c} 1 \\ 2 \\ 1 \\ 3 \\ 1 \\ 2 \\ -2 \\ +2 \\ +3 \\ +1 \end{array} \right) \right).$$

We will show that $\mathcal{U}Lie$ is wheeled Koszul by determining $(\Omega^\odot(\mathcal{U}Lie^!), \partial)$ and showing that its cohomology equals $\mathcal{U}Lie$. Since $\mathcal{U}Lie^! = (\mathcal{U}Lie^!)^*$, our first step will be to determine $\mathcal{U}Lie^!$.

**Lemma 1.** The wheeled Koszul dual operad of $\mathcal{U}Lie$ is $\mathcal{Com}^\odot$.

**Proof.** We have $\mathcal{U}Lie^! = \mathcal{L}ie$ so that $(\mathcal{U}Lie^!)^! = \mathcal{L}ie^! = \mathcal{Com}$. Next, since $R_w$ spans all of $\mathcal{F}^\odot(\mathcal{E})(0, 1)$, $R_w^\perp = 0$ and $(\mathcal{U}Lie^!)_w = \mathcal{Com}_w^\odot$. Hence, $\mathcal{U}Lie^! = \mathcal{Com}^\odot$. □
Since $\mathcal{ULie}_\alpha = \mathcal{Lie}$ and $\mathcal{Lie}$ is Koszul, we have that $\Omega^\omega(\mathcal{ULie}_\alpha) = \mathcal{Lie}_\infty$. Hence, only the wheeled part of $\Omega^\omega(\mathcal{ULie}_\omega)$ will concern us. Since $\text{Com}_w(n) = \mathbb{I}_n$ for $n \geq 1$, $\Omega^\omega(\mathcal{ULie}_\omega)$ has a single skew-symmetric generator in each arity $n \geq 1$. We denote it

$$1_{\text{w}}^n.$$ 

Considering the cooperad structure in $(\text{Com}_\omega)^*$, the differential $\partial$ applied to this generator has terms of the form

$$\begin{array}{c}
\text{i}\text{j} \\
\text{k}\text{l}
\end{array}$$

where $I \sqcup J = [n]$ and $0 \leq |I| \leq n - 2$, and $\begin{array}{c}n \end{array}$.

Hence, to determine $(\Omega^\omega(\mathcal{ULie}_\omega), \partial)$, we need to find signs in front of these terms such that $\partial^2 = 0$.

**Proposition 1.** $(\Omega^\omega(\mathcal{ULie}_\omega), \partial)$ is free on generators

$$\left\{ \begin{array}{c}1 \end{array}^\infty \right\}_{n=2} \text{ and } \left\{ \begin{array}{c}n \end{array}^\infty \right\}_{n=1},$$

skew-symmetric and of degrees $2 - n$ and $-n$ respectively. The differential is defined by

$$\partial \begin{array}{c}1 \end{array}^n = \sum_{I \sqcup J = [n], 1 \leq |I| \leq n-2} (-1)^{\sigma_{I,J} + |I||J|} \begin{array}{c}I \end{array} \begin{array}{c}J \end{array},$$

$$\partial \begin{array}{c}1 \end{array}^n = \sum_{I \sqcup J = [n], 0 \leq |I| \leq n-2} (-1)^{\sigma_{I,J} + |I||J|} \begin{array}{c}I \end{array} \begin{array}{c}J \end{array} + \begin{array}{c}n \end{array}.$$ 

**Proof.** We must show that $\partial^2 = 0$. Since $\Omega^\omega(\mathcal{ULie}_\omega) = \mathcal{Lie}_\infty$, only the generators of the second type concern us. Consider first the sum

$$\sum_{I \sqcup J = [n+1]} (-1)^{\sigma_{I,J'} + |I||J'|} \begin{array}{c}I \end{array} \begin{array}{c}J' \end{array} = \sum_{J' \sqcup I' = [n+1]} (-1)^{\sigma_{I,J'} + |I'||J'|} \begin{array}{c}I' \end{array} \begin{array}{c}J' \end{array} + \sum_{J' \sqcup I' = [n+1]} (-1)^{\sigma_{I',J} + |I'||J|} \begin{array}{c}I' \end{array} \begin{array}{c}J \end{array}.$$ 

If $n + 1 \in I'$ we let $I = I' \setminus \{n+1\}$, $J = J'$ and note that $I \sqcup J = [n]$ is such that $(-1)^{\sigma_{I,J'}} = (-1)^{\sigma_{I',J} + |J|}$. If instead $n + 1 \in J'$ we let $I = I'$, $J = J' \setminus \{n+1\}$ and note that $I \sqcup J = [n]$ is such that $(-1)^{\sigma_{I',J}} = (-1)^{\sigma_{I,J'}}$. Hence, the above sum equals

$$\sum_{I \sqcup J = [n]} (-1)^{\sigma_{I,J} + |I||J| + |J|} \begin{array}{c}I \end{array} \begin{array}{c}J' \end{array}^n + (-1)^{\sigma_{I,J} + |I||J| + |I|} \begin{array}{c}J \end{array} \begin{array}{c}I \end{array}.$$

This gives us that
\[
\partial I = \sum_{P \sqcup Q = I} (-1)^{\sigma_{P, Q} + |P||Q| + 1} P + \sum_{P \sqcup Q = I} (-1)^{\sigma_{P, Q} + |P||Q| + |P|} P
\]
\[
\sum_{P \sqcup Q = I} (-1)^{|P| + 1} - \sum_{P \sqcup Q = J} (-1)^{|P| + 1} \sigma_{P, Q} + |P||Q| + 1.
\]

With the same reasoning, considering the sum
\[
\sum_{I' \sqcup J' = [n+1]} (-1)^{\sigma_{I', J'} + |I'||J'| + 1}
\]
we get that
\[
\partial I = \sum_{I \sqcup J = [n]} (-1)^{\sigma_{I, J} + |I||J| + 1} I + (-1)^{\sigma_{I, J} + |I||J| + 1} J
\]
\[
\sum_{I \sqcup J = [n]} (-1)^{\sigma_{I, J} + |I||J| + 1} P + \sum_{P \sqcup Q = I} (-1)^{\sigma_{P, Q} + |P||Q| + 1} P
\]
\[
+ \sum_{I \sqcup J = [n]} (-1)^{\sigma_{I, J} + |I||J| + |P|} P
\]
\[
+ \sum_{I \sqcup J = [n]} (-1)^{\sigma_{I, J} + |I||J| + |Q|} Q
\]
\[
\sum_{I \sqcup J = [n]} (-1)^{\sigma_{I, J} + |I||J| + |P|} + \sum_{I \sqcup J = [n]} (-1)^{\sigma_{I, J} + |I||J| + |Q|} + \sum_{I \sqcup J = [n]} (-1)^{\sigma_{I, J} + |I||J| + |P|} + \sum_{I \sqcup J = [n]} (-1)^{\sigma_{I, J} + |I||J| + |Q|}
\]

However, considering \( I' = J \) and \( J' = I \), since
\[
(-1)^{\sigma_{I', J'}} = (-1)^{\sigma_{J, I}} = (-1)^{|I||J| + \sigma_{I, J}}
\]
we have
\[
(-1)^{\sigma_{I', J'} + |I'||J'| + |I'||J'| + 1} = (-1)^{\sigma_{I, J} + |I||J| + 1} = (-1)^{\sigma_{I, J} + |I||J| + 1} I + (-1)^{\sigma_{I, J} + |I||J| + 1} J
\]
and the terms cancel pair-wise so that
\[
\partial I = \sum_{I \sqcup J = [n]} (-1)^{\sigma_{I, J} + |I||J| + 1} I + (-1)^{\sigma_{I, J} + |I||J| + 1} J
\]
\[
\sum_{I \sqcup J = [n]} (-1)^{\sigma_{I, J} + |I||J| + 1} P + \sum_{P \sqcup Q = I} (-1)^{\sigma_{P, Q} + |P||Q| + 1} P
\]
\[
+ \sum_{I \sqcup J = [n]} (-1)^{\sigma_{I, J} + |I||J| + |P|} P
\]
\[
+ \sum_{I \sqcup J = [n]} (-1)^{\sigma_{I, J} + |I||J| + |Q|} Q
\]
\[
\sum_{I \sqcup J = [n]} (-1)^{\sigma_{I, J} + |I||J| + |P|} + \sum_{I \sqcup J = [n]} (-1)^{\sigma_{I, J} + |I||J| + |Q|} + \sum_{I \sqcup J = [n]} (-1)^{\sigma_{I, J} + |I||J| + |P|} + \sum_{I \sqcup J = [n]} (-1)^{\sigma_{I, J} + |I||J| + |Q|}
\]

What remains to show is the vanishing of
\[
\sum_{I \sqcup J = [n]} \sum_{P \sqcup Q = I} (-1)^{\sigma_{I, J} + |I||J| + |P||Q| + 1} P + \sum_{I \sqcup J = [n]} \sum_{P \sqcup Q = I} (-1)^{\sigma_{I, J} + |I||J| + |P||Q| + 1} P
\]
\[
+ \sum_{I \sqcup J = [n]} \sum_{P \sqcup Q = I} (-1)^{\sigma_{I, J} + |I||J| + |P||Q| + 1} P
\]
\[
+ \sum_{I \sqcup J = [n]} \sum_{P \sqcup Q = J} (-1)^{\sigma_{I, J} + |I||J| + |P||Q| + 1} P
\]
To see that the first sum in this expression equals zero, let \( I' = P \sqcup J, \quad J' = Q, \quad P' = P \) and \( Q' = J \), so that \( I = P \sqcup J \). We get

\[
(-1)^{\sigma_{I',J'} + \sigma_{P,Q'}} = (-1)^{\sigma_I + \sigma_{P,Q}} = (-1)^{\sigma_I + J + \sigma_{P,Q}} = (-1)^{|I| + J + \sigma_{P,Q}} = (-1)^{|I| + J + \sigma_{P,Q}},
\]

and hence

\[
(-1)^{\sigma_{I',J'} + \sigma_{P,Q'}} + |I'| + |J'| + |P'| + |Q'| + 1 \chi = (-1)^{\sigma_I + \sigma_{P,Q}} + |I| + |J| + |P| + |Q| + 1 \chi,
\]

so that the terms cancel pair-wise. Next, in the fourth sum, consider \( I' = P, \quad J' = Q \sqcup J, \quad P' = Q \) and \( Q' = J \). We have

\[
(-1)^{\sigma_{I',J'} + \sigma_{P,Q'}} = (-1)^{\sigma_{I',J'} + \sigma_{Q,J}} = (-1)^{\sigma_{I,J} + \sigma_{Q,J}} = (-1)^{\sigma_{I,J} + \sigma_{P,Q}},
\]

so that

\[
(-1)^{\sigma_{I',J'} + \sigma_{P,Q'}} + |I'| + |J'| + |P'| + |Q'| + 1 \chi = (-1)^{\sigma_I + \sigma_{P,Q}} + |I| + |J| + |P| + |Q| + 1 \chi
\]

\[
= (-1)^{\sigma_I + \sigma_{P,Q}} + |P| + |Q| + 1 \chi = (-1)^{\sigma_I + \sigma_{P,Q}} + |P| + |Q| + 1 \chi.
\]

Hence, the fourth sum cancels the second. \( \square \)

**Theorem 1.** The wheeled operad \( \mathcal{ULie} \) is wheeled Koszul.

**Proof.** We know that \( H(\Omega^{\leq}(\mathcal{ULie}^1)) = H(\mathcal{Lie}_\infty) = \mathcal{Lie} \). Hence, we need to show that the cohomology of \( C = \Omega^{\leq}(\mathcal{ULie}^1) \) equals \( \mathcal{ULie}_w \).

The graphs of genus one span a subcomplex of \( C \) isomorphic to \( (\mathcal{Lie}_\infty^1) \) which, since \( \mathcal{Lie} \) is stably Koszul, equals \( \mathcal{Lie}_w^1 \). Considering the inclusion of \( \mathcal{Lie}_w^1 \) into \( C \), we see that if we show that the cohomology of \( C_0 = C/(\mathcal{Lie}_w^1) \) equals \( \mathbb{K}^1 \), then we are done, because then the long exact sequence in cohomology reduces to

\[
0 \longrightarrow \mathbb{K}^1 \longrightarrow \mathcal{Lie}_w^1 \longrightarrow H^0(C) \longrightarrow 0
\]

and

\[
H(C) = \mathcal{Lie}_w^1 \cong \mathcal{ULie}_w.
\]

Now, the subcomplex of \( C_0 \) consisting of decorated trees without root leg whose root vertex has only one incoming leg is isomorphic to \( \mathcal{Lie}_w^{\geq 1} \) and the quotient is isomorphic to \( \mathcal{Lie}_w^{\geq 2} \). Considering the long exact sequence in cohomology induced by the inclusion of this subcomplex into \( C_0 \), that is

\[
0 \longrightarrow H^{-2}(C_0) \longrightarrow \mathcal{Lie}_w^{2} \longrightarrow \mathbb{K}^1 \oplus \mathcal{Lie}_w^{1} \longrightarrow H^{-1}(C_0) \longrightarrow 0,
\]

we conclude that \( H(C_0) = \mathbb{K}^1 \). \( \square \)
Corollary 1. The minimal resolution $\mathcal{UL}ie_\infty$ equals $\Omega^G(\mathcal{UL}ie^G)$ as given in Proposition 3.

Remark. The proof that $H(C_0) = k^+$ is essentially the same as that of the claim in the proof of Theorem 4.1.1 in [Mer07]. As a second remark we note that since $\mathcal{UL}ie^G = \text{Com}^G$, the above proof gives a new and shorter proof of the theorem that $\text{Com}^G$ is wheeled Koszul proven in [MMS07].

Definition. A structure of unimodular $L_\infty$-algebra on a finite dimensional dg vector space $V$ is a representation of $\mathcal{UL}ie_\infty$ in $V$. By the corollary above, this is equivalent with a finite dimensional dg vector space $V$ together with two families of morphisms

$$\{\ell_n : \wedge^n V \to V\}_{n=2}^\infty$$

with $|\ell_n| = 2 - n$ and

$$\{q_n : \wedge^n V \to k\}_{n=1}^\infty$$

with $|q_n| = -n$, satisfying the equations

$$\partial\ell_n = \sum_{I \cup J = [n]} (-1)^{\sigma_I + |I||J|} \ell_{|I|+1}(\text{id} \otimes \cdots \otimes \ell_{|J|})\sigma_{I,J}$$

and

$$\partial q_n = \sum_{I \cup J = [n]} (-1)^{\sigma_I + |I||J|} q_{|I|+1}(\text{id} \otimes \cdots \otimes \ell_{|J|})\sigma_{I,J} + \text{tr}_{n+1} \ell_{n+1}.$$

4. Transfer of structure

Assume we have a diagram

$$V \xrightarrow{f} U \xleftarrow{g} V$$

where $V$ is a unimodular $L_\infty$-algebra, $U$ is a finite dimensional dg vector space and $h$ is a cochain homotopy between $fg$ and $\text{id}_V$, i.e. a degree $-1$ map satisfying

$$fg - \text{id}_V = d_V h + hd_V.$$

In this situation we give, explicitly, an induced structure of unimodular $L_\infty$-algebra on $U$.

To handle signs we apply the parity change functor to $\mathcal{UL}ie_\infty$, $\Pi\mathcal{UL}ie_\infty$ is the free wheeled operad on symmetric generators

$$\left\{\left(\begin{array}{c}
1 \\
n
\end{array}\right)\right\}_{n=2}^\infty \text{ and } \left\{\left(\begin{array}{c}
1 \\
n
\end{array}\right)\right\}_{n=1}^\infty,$$

of degrees 1 and 0 respectively. The differential is given on generators by

$$\partial \left(\begin{array}{c}
1 \\
n
\end{array}\right) = - \sum_{I \cup J = [n]} \left(\begin{array}{c}
l \\
|I|\end{array}\right) \left(\begin{array}{c}
1 \\
j
\end{array}\right),$$

(1)

$$\partial \left(\begin{array}{c}
1 \\
n
\end{array}\right) = \sum_{I \cup J = [n]} \left(\begin{array}{c}
l \\
|I|\end{array}\right) \left(\begin{array}{c}
1 \\
j
\end{array}\right) + \left(\begin{array}{c}
1 \\
n
\end{array}\right).$$

(2)

Let $\mathfrak{G}_1$ denote the class of directed trees with root leg, $\mathfrak{G}_0$ the class of directed trees without root leg and $\mathfrak{S}_1^G$ the class of graphs obtained by grafting the root leg of a tree to one of its ingoing legs. Let $\mathfrak{G}_1(n)$ denote the subclass of trees with $n$ ingoing legs and similarly for $\mathfrak{G}_0(n)$ and $\mathfrak{S}_1^G(n)$. For any tree $G \in \mathfrak{G}_1(n)$ we define a map $\theta_G : V[1]^{\otimes n} \to V[1]$ by decorating the root vertex $v$ with $\ell_{|\text{in}(v)|}$ and every other vertex $v$ with $h\ell_{|\text{in}(v)|}$. For any tree $G \in \mathfrak{G}_0(n)$ we define a morphism $\theta_G : V[1]^{\otimes n} \to k$ by decorating the root vertex $v$ with $q_{|\text{in}(v)|}$ and every other vertex $v$ with
For a graph $G \in \mathfrak{G}_1^\infty(n)$, we define $\theta_G : V[1]^\otimes n \to k$ by decorating all vertices $v$ with $h\ell_{\mathrm{in}(v)}$. Next we define a morphism $\Pi\mathcal{U}C_{\infty} \to \mathcal{E}nd_{U[1]}^\infty$ by

$$
\begin{align*}
1^\otimes n \mapsto \sum_{G \in \mathfrak{G}_1(n)} g\theta_G f^\otimes n,
\end{align*}
$$

Theorem 2. The above morphism $\Pi\mathcal{U}C_{\infty} \to \mathcal{E}nd_{U[1]}^\infty$ is a morphism of dg wheeled operads and hence defines a unimodular $L_\infty$-algebra structure on $U$.

Proof. For a graph $G$ and an edge $e \in \mathfrak{e}(G)$, let $\theta_{G,e}^\otimes$ denote the morphism that differs from $\theta_G$ in the respect that the vertex $v$ from which $e$ starts is decorated with $fg\ell_{\mathrm{in}(v)}$ instead of $h\ell_{\mathrm{in}(v)}$. For the equality (1) we need to show that

$$
\sum_{G \in \mathfrak{G}_1(n)} \partial g\theta_G f^\otimes n = \sum_{G \in \mathfrak{G}_1(n)} dg\theta_G f^\otimes n + g\theta_G f^\otimes n d_\otimes
$$

equals

$$
- \sum_{G_1 \in \mathfrak{G}_1(n+1)} \sum_{G_2 \in \mathfrak{G}_1(j)} \sum_{G = i+j} g\theta_{G_1} (f \otimes \ldots \otimes f \otimes g\theta_{G_2} (f \otimes \ldots \otimes f)) = - \sum_{G \in \mathfrak{G}_1(n)} \sum_{e \in \mathfrak{e}(G)} g\theta_{G,e}^\otimes f^\otimes n.
$$

Since $f$ and $g$ commute with differentials, this would follow, by considering one tree at a time, from

$$
\theta_G d_\otimes = -d\theta_G + \sum_{e \in \mathfrak{e}(G)} (-\theta_{G,e}^\otimes + \theta_{G,e}^\otimes) - \sum_{(G',e')} \theta_{G',e'}^\otimes.
$$

This in turn follows by commuting the differential in $\theta_G d_\otimes$ from the right to the left. Considering one vertex of $G$ at a time, we note that for a non-root vertex

$$
h\ell_k d_\otimes = -hd\ell_k - \sum h\ell_{i+1} (\id \otimes \ldots \otimes \ell_j) \sigma = dh\ell_k - fg\ell_k + \ell_k - \sum h\ell_{i+1} (\id \otimes \ldots \otimes \ell_j) \sigma,
$$

while for the root vertex

$$
\ell_k d_\otimes = -d\ell_k - \sum \ell_{i+1} (\id \otimes \ldots \otimes \ell_j) \sigma.
$$

For the equality (2) we need to show that

$$
\sum_{G \in \mathfrak{G}_1 \cup \mathfrak{G}_1^\infty(n)} \partial g\theta_G f^\otimes n = - \sum_{G \in \mathfrak{G}_1 \cup \mathfrak{G}_1^\infty(n)} \theta_G f^\otimes n d_\otimes
$$

equals

$$
\sum_{G_1 \in \mathfrak{G}_1 \cup \mathfrak{G}_1^\infty(n+1)} \sum_{G_2 \in \mathfrak{G}_1(j)} \sum_{G = i+j} \sum_{G \in \mathfrak{G}_1(n+1)} \sum_{e \in \mathfrak{e}(G)} g\theta_{G_1} (f \otimes \ldots \otimes f \otimes g\theta_{G_2} (f \otimes \ldots \otimes f)) + \sum_{G \in \mathfrak{G}_1(n+1)} \tr_{n+1} (g\theta_G (f \otimes \ldots \otimes f))
$$

$$
= \left( \sum_{G \in \mathfrak{G}_1(n)} \sum_{e \in \mathfrak{e}(G)} g\theta_{G,e}^\otimes f^\otimes n \right) \theta_{G,e}^\otimes f^\otimes n
$$

$$
= \sum_{G \in \mathfrak{G}_1 \cup \mathfrak{G}_1^\infty(n)} \theta_{G,e}^\otimes f^\otimes n.
$$
This would follow from
\[ -\theta_G d_\varnothing = \sum_{e \in e(G)} (\theta^0_{G,e} - \theta^d_{G,e}) + \sum_{(G',e') \in (\mathcal{G}_0,\mathcal{G}_1)(n)} \theta^d_{G',e'} \]

which again follows from commuting the differential from right to left. Considering one vertex at a time, we have for a non-root, non-cyclic vertex that
\[ -h \tilde{\ell}_k d_\varnothing = -dh \tilde{\ell}_k + f g \tilde{\ell}_k - \tilde{\ell}_k + \sum h \tilde{\ell}_{i+1} (id \otimes \cdots \otimes \tilde{\ell}_j) \sigma. \]

For the root vertex of a \( G \in \mathcal{G}_0(n) \) we have
\[ -\tilde{q}_k d_\varnothing = \sum \tilde{q}_{i+1} (id \otimes \cdots \otimes \tilde{\ell}_j) \sigma + \text{tr}_{k+1}(\tilde{\ell}_{k+1}). \]

Finally, for a cyclic vertex \( v_k \) of a \( G \in \mathcal{G}_1(n) \),
\[ -h \tilde{\ell}_k ((d \otimes id \otimes \cdots \otimes id) + \cdots + (id \otimes \cdots \otimes d \otimes id)) \]
\[ = h \tilde{\ell}_k (id \otimes \cdots \otimes id \otimes d) + hd \tilde{\ell}_k + \sum h \tilde{\ell}_{i+1} (id \otimes \cdots \otimes \tilde{\ell}_j) \sigma. \]

The first term gives a term \( dh \tilde{\ell}_k \) for the previous vertex \( v_{k'} \) in the cycle and like-wise, the above equality for the next vertex in the cycle gives a term \( dh \tilde{\ell}_k \) to our expression for \( v_k \). Hence, we get
\[ hd \tilde{\ell}_k + \sum h \tilde{\ell}_{i+1} (id \otimes \cdots \otimes \tilde{\ell}_j) \sigma + dh \tilde{\ell}_k \]
\[ = f g \tilde{\ell}_k - \tilde{\ell}_k + \sum h \tilde{\ell}_{i+1} (id \otimes \cdots \otimes \tilde{\ell}_j) \sigma + dh \tilde{\ell}_k. \]

**Example.** As an illustration of the proof given above, we will show the required equality (2) in the case \( n = 2 \) for the induced structure graphically. The following is a legend for our graphical notations:

\[ \begin{align*}
\tilde{q}_k &= \begin{array}{c}
\uparrow
\end{array}^k \\
h &= \begin{array}{c}
\downarrow
\end{array} \\
fg &= \begin{array}{c}
\circlearrowleft
\end{array} \\
\tilde{\ell}_k &= \begin{array}{c}
\downarrow
\end{array}^k \\
d &= \begin{array}{c}
\downarrow
\end{array}
\end{align*} \]

Ignoring the \( f : s \) decorating the leaves, the left hand side of (2) for the induced structure is
\[ \partial \begin{pmatrix}
\uparrow
\end{pmatrix} + \begin{pmatrix}
\uparrow
\end{pmatrix} + \begin{pmatrix}
\uparrow
\end{pmatrix} + \begin{pmatrix}
\uparrow
\end{pmatrix} + \begin{pmatrix}
\uparrow
\end{pmatrix}, \]

while the right hand side is
\[ \begin{pmatrix}
\uparrow
\end{pmatrix} + \begin{pmatrix}
\uparrow
\end{pmatrix} + \begin{pmatrix}
\uparrow
\end{pmatrix} + \begin{pmatrix}
\uparrow
\end{pmatrix} + \begin{pmatrix}
\uparrow
\end{pmatrix} + \begin{pmatrix}
\uparrow
\end{pmatrix} + \begin{pmatrix}
\uparrow
\end{pmatrix}. \]

We show the computation of the two first terms of the left hand side,
\[ \partial \begin{pmatrix}
\uparrow
\end{pmatrix} = \begin{pmatrix}
\uparrow
\end{pmatrix} + \begin{pmatrix}
\uparrow
\end{pmatrix}, \]
Proceeding in the same manner with the remaining terms we obtain
\[ \partial \begin{array}{c} \uparrow \downarrow \\
\end{array} = \begin{array}{c} \uparrow \downarrow \\
\end{array} - \begin{array}{c} \uparrow \downarrow \\
\end{array} + \begin{array}{c} \uparrow \downarrow \\
\end{array} - \begin{array}{c} \uparrow \downarrow \\
\end{array} + \begin{array}{c} \uparrow \downarrow \\
\end{array} - \begin{array}{c} \uparrow \downarrow \\
\end{array}, \]

which shows the wanted equality.

5. Differential geometric interpretation

Given a dg vector space \( g \), it is well-known that structures of \( L_\infty \)-algebra on \( g \) are in one-to-one correspondence to degree 1 square-zero coderivations of the cofree graded cocommutative coalgebra without counit on \( g[1] \). We consider its dual, the completed free graded commutative algebra without unit on \( g[1] \), as the ideal of the basepoint in the ring of functions \( \mathcal{O}_g[1] = \prod_{n \geq 0} \mathcal{O}^n g[1] \) on the formal graded pointed manifold \( (g[1], 0) \). We denote \( \text{Der} \mathcal{O}_g[1] \) by \( T_g[1] \) and refer to its elements as vector fields. A degree 1 coderivation of \( \mathcal{O}^\geq 1 g[1] \) corresponds, by dualizing, to a degree 1 derivation of \( \mathcal{O}_g[1] \) which vanish at 0. Hence, an \( L_\infty \)-structure on \( g \) is given by a vector field \( Q \in T^1 \mathcal{O}_g[1] \) such that \( Q^2 = 0 \) and \( Q|_0 = 0 \). Such vector fields are called homological.

Let now \( \{\ell_n\}, \{q_n\} \) be a unimodular \( L_\infty \)-algebra. The collection \( \{\tilde{q}_n\} \) assemble into a morphism \( \mathcal{O}^\geq 1 g[1] \to \mathbb{R} \). On the dual side, this determines a degree zero element \( f \) in the ideal of 0 in \( \mathcal{O}_g[1] \). We will want to express the equations satisfied by \( \{\ell_n\} \) and \( \{q_n\} \) in terms of \( Q \) and \( f \). This we will do using the divergence of the canonical volume form on \( g[1] \) along \( Q \). The volume forms are elements of the Berezinian of \( T_g[1] \) which we now describe. For reference, see [BL77b, BL77a, Lei80, Man97].

Let \( A \) be a graded ring and \( M \) a free \( A \)-module. If we choose a basis \( \{e_\alpha\} \) of \( M \) such that \( |e_\alpha| \) is even for \( 1 \leq \alpha \leq p \) and odd for \( p + 1 \leq \alpha \leq n \), then the matrix of an even map \( f : M \to M \) is of the form
\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]
where all entries in \( A \) and in \( D \) are even, while all entries in \( B \) and in \( C \) are odd. One can show that such a map is invertible if and only if \( A \) and \( D \) are both invertible, and moreover that, in this case, the Berezinian,
\[
\text{Ber}(f) = \frac{\det(A - BD^{-1}C)}{\det D},
\]
From now on we let \( L \) be any degree zero function and \( \rho = g\Delta_x \) a volume form. We have
\[
\text{div}_{\epsilon f} V = (-1)^{|\epsilon|} |V^\alpha| \partial_\alpha V^\alpha + V^\alpha \partial_\alpha (\epsilon^j g) g^{-1} e^{-f} = (-1)^{|\epsilon|} |V^\alpha| \partial_\alpha V^\alpha + V^\alpha \partial_\alpha f + V^\alpha \partial_\alpha gg^{-1} = \text{div}_\rho V + V(f).
\]
From now on we let \( \rho_f = e^f \Delta_x \), \( \text{div} = \text{div}_{\Delta_x} \) and \( \text{div}_f = \text{div}_{\rho_f} \). It follows from the above that \( L_Q(\rho_f) = \text{div}_f Q \cdot \rho_f = (\text{div}_Q + Q(f)) \rho_f \). Thus, \((Q, f)\) is a unimodular \( L_\infty\)-structure if and only if \( L_Q(\rho_f) = 0 \), i.e. if and only if \( \rho_f \) is \( Q \)-invariant.

6. The deformation complex

The graded vector space \( T_{\mathfrak{g}[1]} \) is a graded Lie algebra with commutator bracket. Concerning \( \text{div}_\rho : T_{\mathfrak{g}[1]} \to \mathcal{O}_{\mathfrak{g}[1]} \) we have the following well-known lemma.
**Theorem 3.** The definition of the deformation complex is motivated by the following theorem.

**Proof.** From Lemma 2 follows that $\text{div}_f[V_1, V_2] = V_1(\text{div}_f V_2) - (-1)^{|V_1||V_2|} V_2(\text{div}_f V_1)$ holds.

Given an $L_\infty$-algebra $(\mathfrak{g}, Q)$, $Q$ is a differential on $\mathcal{O}_{\mathfrak{g}[1]}$ and $[Q, -]$ is a differential on $T_{\mathfrak{g}[1]}$.

**Lemma 3.** If $(\mathfrak{g}, Q, f)$ is a unimodular $L_\infty$-algebra, then

$$\text{div}_f : T_{\mathfrak{g}[1]} \longrightarrow \mathcal{O}_{\mathfrak{g}[1]}$$

is a morphism of dg vector spaces.

**Proof.** By Lemma 2 and the equality $\text{div}_f Q = 0$ we have

$$\text{div}_f ([Q, V]) = Q(\text{div}_f V) - (-1)^{|V|} V(\text{div}_f Q) = Q(\text{div}_f V).$$

□

**Definition.** Given a unimodular $L_\infty$-algebra $(\mathfrak{g}, Q, f)$, its deformation complex is defined as

$$C(\mathfrak{g}, Q, f) = \text{cone}(\text{div}_f)[-1],$$

that is, $C(\mathfrak{g}, Q, f) = T_{\mathfrak{g}[1]} \oplus \mathcal{O}_{\mathfrak{g}[1]}[-1]$ with differential

$$\partial(V + sg) = [Q, V] - sQ(g) + s\text{div}_f V.$$

We can extend the Lie bracket on $T_{\mathfrak{g}[1]}$ to all of $C(\mathfrak{g}, Q, f)$ by letting $[V, sg] = (-1)^{|V|} sV(g)$ and $[sg_1, sg_2] = 0$. With this convention, the differential of the deformation complex satisfies

$$\partial(V + sg) = [Q, V] - sQ(g) + s\text{div}_f V = \text{sdiff}_f V + [Q, V] + [Q, sg] = \text{sdiff}_f V + [Q, V + g],$$

i.e.

$$\partial = \text{sdiff}_f + [Q, -].$$

From Lemma 2 follows that $\text{sdiff}_f$ is a degree 1 derivation of the bracket on $C(\mathfrak{g}, Q, f)$. Since $[Q, -]$ is also a degree 1 derivation, it follows that the deformation complex is a dg Lie algebra. The definition of the deformation complex is motivated by the following theorem.

**Theorem 3.** Let $(\mathfrak{g}, Q, f)$ be a unimodular $L_\infty$-algebra. Pairs $(V, g)$ such that $(\mathfrak{g}, Q + V, f + g)$ is again a unimodular $L_\infty$-algebra are in one-to-one correspondence with Maurer-Cartan elements in $C(\mathfrak{g}, Q, f)$.

**Proof.** A Maurer-Cartan element $V - sg$ of $C(\mathfrak{g}, Q, f)$ satisfies

$$\partial(V - sg) + \frac{1}{2} [V - sg, V - sg] = \text{sdiff}_f V + [Q, V] - [Q, sg] + V^2 - [V, sg]$$

$$= QV + VQ + V^2 + s(\text{div}_f V + Q(g) + V(g)) = 0,$$

i.e.

$$QV + VQ + V^2 = 0,$$

$$\text{div}_f V + (Q + V)(g) = 0.$$

On the other hand, $(Q + V, f + g)$ is a unimodular $L_\infty$-structure if and only if

$$(Q + V)^2 = Q^2 + QV + VQ + V^2 = QV + VQ + V^2 = 0,$$

$$\text{div}_f(V + Q) = \text{div}_f Q + \text{div}_f V + (V + Q)(g) = \text{div}_f V + (V + Q)(g) = 0.$$

□
7. The Characteristic Class

We briefly address the question of when a given $L_\infty$-algebra $(\mathfrak{g}, Q)$ can be extended to a unimodular $L_\infty$-algebra. Lemma 2 implies that 

$$Q(\text{div}Q) = \frac{1}{2}\text{div}([Q, Q]) = 0,$$

so that divQ determines a cohomology class $[\text{div}Q] \in H^1(O_{\mathfrak{g}[1]}, Q)$.

**Definition.** Given an $L_\infty$-algebra $(\mathfrak{g}, Q)$, the cohomology class $[\text{div}Q]$ is called its **characteristic class**.

**Proposition 2.** An $L_\infty$-algebra can be extended to a unimodular $L_\infty$-algebra if and only if its characteristic class vanishes.

**Proof.** The characteristic class vanishes if and only if $\text{div}Q = Q(-f)$ for some $f$. If so, then $\text{div}_f Q = \text{div}Q + Q(f) = 0$ so that $(\mathfrak{g}, Q, f)$ is unimodular $L_\infty$.

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**Appendix A. Unimodular associative algebras and characteristic classes**

**A.1. Unimodular associative algebras.**

An (associative) algebra $(A, \mu)$ is said to be unimodular if, for all $x \in A$,

$$\text{tr} \mu(x, -) = \text{tr} \mu(-, x) = 0.$$ Unimodular algebras are algebras over the wheeled operad

$$\mathcal{U}_{\text{Ass}} = \mathcal{F}^\circ \left( \begin{array}{c} \uparrow \uparrow \uparrow \\ \downarrow \downarrow \downarrow \end{array} \right),$$

and since $\mathcal{U}_{\text{Ass}}^! = \text{Ass}^\circ$ is wheeled Koszul [MMS07], we can construct $\mathcal{U}_{\text{Ass}_\infty}$ explicitly.

**Proposition 3.** The minimal resolution of $\mathcal{U}_{\text{Ass}}$ is a free wheeled operad on generators

$$\left\{ \begin{array}{c} \uparrow \uparrow \uparrow \uparrow \\ \downarrow \downarrow \downarrow \downarrow \end{array} \right\}_{n \geq 2} \quad \text{and} \quad \left\{ \begin{array}{c} \downarrow \downarrow \downarrow \downarrow \\ \uparrow \uparrow \uparrow \uparrow \\ \downarrow \downarrow \downarrow \downarrow \end{array} \right\}_{m+n \geq 1}$$

of degrees $2-n$ and $-m-n$ respectively. The second type of generator satisfies the following cyclic symmetry

$$(-1)^{m+1} \zeta = (-1)^{n+1} \xi,$$

where $\zeta = (1 2 \ldots m)$ and $\xi = (m+1 m+2 \ldots m+n)$. The differential is defined by

$$\partial \begin{array}{c} \uparrow \uparrow \uparrow \uparrow \\ \downarrow \downarrow \downarrow \downarrow \end{array} = \sum_{j=0}^{n-1} \sum_{i=0}^{n-j} (-1)^{i+j+1+n} \begin{array}{c} \uparrow \uparrow \uparrow \uparrow \\ \downarrow \downarrow \downarrow \downarrow \end{array}$$

and

$$\partial \begin{array}{c} \downarrow \downarrow \downarrow \downarrow \\ \uparrow \uparrow \uparrow \uparrow \\ \downarrow \downarrow \downarrow \downarrow \end{array} = \sum_{i=2}^{m-1} \sum_{j=0}^{m-1} (-1)^{i+m+n+1} \begin{array}{c} \uparrow \uparrow \uparrow \uparrow \\ \downarrow \downarrow \downarrow \downarrow \end{array} + \sum_{i=2}^{n-1} \sum_{j=0}^{n-1} (-1)^{i+m+n} \begin{array}{c} \uparrow \uparrow \uparrow \uparrow \\ \downarrow \downarrow \downarrow \downarrow \end{array} + \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \begin{array}{c} \downarrow \downarrow \downarrow \downarrow \\ \uparrow \uparrow \uparrow \uparrow \\ \downarrow \downarrow \downarrow \downarrow \end{array}.$$
A.2. Characteristic classes.

An $A_{\infty}$-algebra, i.e. a representation of $\mathcal{A}ss_{\infty}$ in a vector space $V$, corresponds to a degree one element

$$\Gamma = \{\Gamma_n : V[1]^{\otimes n} \to V[1]\}_{n \geq 1}$$

in the weight graded Lie algebra

$$C(V, V) = \{C^p_n(V, V) = \text{Hom}^p(V[1]^{\otimes n}, V[1])\}_{n \geq 1}$$

equipped with the Gerstenhaber bracket, such that $[\Gamma, \Gamma] = 0$. Consider next the right $C(V, V)$-module $\text{Cyc}(V, \mathbb{k})$, defined as

$$\{\text{Cyc}^p_{m,n}(V, \mathbb{k}) = \text{Hom}^p((V[1]^{\otimes m})_{m} \otimes (V[1]^{\otimes n})_{n}, \mathbb{k})\}_{m,n \geq 1}.$$  

The module structure induces a Lie algebra structure on $g = C(V, V) \oplus \text{Cyc}(V, \mathbb{k})$ and the fact that $[\Gamma, \Gamma] = 0$ implies that $\partial \Gamma = [\Gamma, -]$ is a differential on $g$ such that $\text{Cyc}(V, \mathbb{k})$ is a subcomplex. Moreover, $\Gamma$ determines an element $\Gamma^\odot$ in $\text{Cyc}^1(V, \mathbb{k})$ as follows,

$$\Gamma_{m,n}^\odot = m \sum_{i=1}^{m} \sum_{j=1}^{n} \text{tr}_{m+1}(\Gamma_{m+n+1})(((-1)^{m+1} \zeta)^i((-1)^{n+1} \xi)^j).$$

Here, $\zeta$ and $\xi$ are as above.

**Proposition 4.** For any $A_{\infty}$-algebra $(V, \Gamma)$ we have $\partial \Gamma \Gamma^\odot = 0$.

**Proof.** For $m, n \geq 1$ this is Proposition-Definition 6.7.2 in [MMS07]. For $m + n = 1$ the proof is exactly the same, i.e. expand the expressions $\partial \text{tr}_1(\Gamma_n)$ and $\partial \text{tr}_n(\Gamma_n)$ and skew symmetrize cyclically. □

Hence $\Gamma$ determines a class $[\Gamma^\odot]$ in $H^1(\text{Cyc}(V, \mathbb{k}), \partial \Gamma)$ called the **unimodular cyclic characteristic class**. The following proposition is proven in exactly the same way as Theorem 6.7.3 in [MMS07].

**Proposition 5.** A structure of $A_{\infty}$-algebra on $V$ can be extended to a structure of unimodular $A_{\infty}$-algebra if and only if the unimodular cyclic characteristic class vanishes.

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