DIRECT LINEARIZATION OF EXTENDED LATTICE BSQ SYSTEMS

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ABSTRACT. The direct linearization structure is presented of a “mild” but significant generalization of the lattice BSQ system. Some of the equations in this system were recently discovered in [J. Hietarinta, J. Phys A: Math. Theor. 44 (2011) 165204] through a search of a class of three-component systems obeying the property of multidimensional consistency. We show that all the novel equations arising in this class follow from one and the same underlying structure. Lax pairs for these systems are derived and explicit expressions for the $N$-soliton solutions are obtained from the given structure.

1. INTRODUCTION

The lattice Boussinesq (BSQ) equation

\[
p - q + u_{n+1,m+1} - u_{n+2,m} - p - q + u_{n,m+2} - u_{n+1,m+1} \\
= (p - q + u_{n+1,m+2} - u_{n+2,m+1})(2p + q + u_{n,m+1} - u_{n+2,m+2}) \\
- (p - q + u_{n,m+1} - u_{n+1,m})(2p + q + u_{n,m} - u_{n+2,m+1}),
\]

(1.1)
appeared in [24] as a first higher-rank case in what was coined the “lattice Gel’fand-Dikii (GD)” hierarchy of equations. These are equations on the space-time lattice, labelled by an integer $N$ associated with the order (or equivalently the number of effective components) of the system. Whilst for $N = 2$ the equation in this class is the lattice potential Korteweg-de Vries (KdV) equation, which is an equation associated with a quadrilateral stencil of vertices, the lattice BSQ appears in this class for $N = 3$ and is defined on the following 9-point stencil:
The notation employed in (1.1) is illustrated in this Figure: $u = u_{n,m}$ denotes the dependent variable of the lattice points labelled by $(n, m) \in \mathbb{Z}^2$; $p, q$ are continuous lattice parameters associated with the grid size in the directions of the lattice given by the independent variables $n$ and $m$ respectively, and for the sake of clarity we prefer to use a notation with elementary lattice shifts denoted by
\[ u = u_{n,m} \mapsto \tilde{u} = u_{n+1,m} , \quad u = u_{n,m} \mapsto \hat{u} = u_{n,m+1} . \]

Thus, as a consequence of this notation, we have also
\[ \tilde{u} = u_{n+1,m+1} , \quad \hat{u} = u_{n+2,m+1} , \quad \tilde{u} = u_{n+1,m+2} , \quad \hat{u} = u_{n+2,m+2} . \]

Together with the lattice BSQ, also lattice versions of the modified BSQ (MBSQ) and Schwarzian BSQ equations have been found, cf. [21, 18, 19, 30] as well as their semi-continuum limits (leading to differential-difference analogues of these equations). The lattice BSQ equation has (re)gained considerable attention in recent years, w.r.t. similarity reductions [30, 28], soliton solutions [30] and its Lagrangian structure [16].

In the recent paper [11] a one-parameter generalization was presented of this system (after a point transformation on the dependent variables), as well as of two other 3-component BSQ systems, comprising the lattice modified BSQ (MBSQ) and Schwarzian BSQ (SBSQ) systems, through systematic search based on the multidimensional consistency property. Subsequently, in [13] soliton solutions for this extended BSQ system were derived. The main structural feature that emerged from the latter study was that the dispersion relation defining the relevant discrete exponential functions involve a cubic of the form
\[ \omega^3 - k^3 - \beta(\omega^2 - k^2) = 0 , \]
where $k$ is like the wave number of the solution, whilst $\omega$ the corresponding angular frequency. Noting that the parameter $\beta$ corresponds to the deformation of the 3-component system, the limit $\beta \to 0$ would lead to the original lattice BSQ system, in which case the dispersion would involve a cube root of unity for the parameter $\omega$ (i.e., $\omega \to \exp(2\pi i/3)k$ in this limit).

In the present study, we generalize the latter relation to an arbitrary cubic, i.e. we consider a dispersion of the form
\[ G(\omega, k) := \omega^3 + \alpha_2 \omega^2 + \alpha_1 \omega - (k^3 + \alpha_2 k^2 + \alpha_1 k) = 0 , \quad (1.3) \]
and show that not only the extended lattice BSQ of [11], but also the generalizations of the 3-component systems of lattice MBSQ and SBSQ, which remained as yet unidentified, arise from one and the same structure. We will derive this structure, in the case of the extended dispersion relation (1.3), following the direct linearization scheme developed for the case that $\alpha_1 = \alpha_2 = 0$ in [24], and subsequently derive from this the main equations as well as the relations between
the various fields arising in the extended lattice BSQ systems of [11]. In particular we derive the following coupled system

\[ v_a \overset{\sim}{w}_b = \frac{Q_{p,q}(u, \tilde{u}, \hat{u}, \hat{\tilde{u}})}{Q_a P_b \hat{u} - P_a Q_b \tilde{u}}, \quad (1.4a) \]

\[ v_a \overset{\sim}{w}_b = P_a u - P_b \tilde{u}, \quad (1.4b) \]

in which the quadrilateral multilinear function \( Q \) is given by

\[ Q_{p,q}(u, \tilde{u}, \hat{u}, \hat{\tilde{u}}) := P_a P_b (u \tilde{u} + \tilde{u} \hat{u}) - Q_a Q_b (u \hat{u} + \hat{u} \tilde{u}) + G(-p, -q)(\hat{\tilde{u}} u + u \hat{\tilde{u}}), \quad (1.4c) \]

and in which

\[ P_a = \sqrt{-G(-p, -a)}, \quad Q_a = \sqrt{-G(-q, -a)}, \quad G(p, q) = p^3 - q^3 + \alpha_2(p^2 - q^2) + \alpha_1(p - q). \]

The system, by elimination of \( v_a \) and \( w_b \) leads to the following 9-point equation for \( u \)

\[ \frac{Q_{p,q}(u, \tilde{u}, \hat{u}, \hat{\tilde{u}})}{Q_a P_b \hat{u} - P_a Q_b \tilde{u}} = \frac{(Q_a u - Q_b \tilde{u})(P_a \hat{u} - P_b \hat{\tilde{u}})(Q_a P_b \hat{u} - P_a Q_b \tilde{u})}{(Q_a u - Q_b \tilde{u})(Q_a \hat{u} - Q_b \hat{\tilde{u}})(Q_a P_b \hat{u} - P_a Q_b \tilde{u})}. \quad (1.5) \]

Eq. (1.5) is the extended form of the BSQ analogue of the \((Q3)_0\) equation, and which reduces to the analogous equation given in [21] by setting \( \alpha_1 = \alpha_2 = 0 \). It is related by a point transformation to an equation given already in [19, 30], and which constitutes the rank \( 3 \) analogue of the NQC equation of [25] (associated with the lattice KdV equations). In [21] the derivation of a BSQ analogue of the full \( Q3 \) equation in the ABS list was presented, but it gives rise to a rather complicated coupled system. Thus, even though those results can in principle be readily extended to the more general dispersion relation \((1.3)\) as well, we will refrain from doing so in the present paper. As a concrete spin-off of the derivations presented here, we derive also the Lax pairs for the extended BSQ system, whilst \( N \)-soliton solutions in the Cauchy matrix form can be readily inferred from the direct linearization structure.

2. Direct linearization and constitutive system

In this section we present the constitutive relations of the direct linearization scheme for a class of lattice systems generalizing the lattice Gel’fand-Dikii (GD) hierarchy. The members of this hierarchy are labelled by an integer \( N \), which in the case of [21] were associated with the corresponding roots of unity \( \omega = \exp(2\pi i/N) \), the case of \( N = 2 \) corresponding to the lattice KdV systems, whilst the case \( N = 3 \) yielding the lattice BSQ systems. In the present paper the root of unity is extended to a dispersion function \( \omega_j(k) \) associated with the roots of arbitrary fixed \( N^{th} \) order polynomials.

2.1. Infinite matrix scheme. The starting point of the construction is the following integral

\[ C = \sum_{j=1}^{N} \int_{\Gamma_j} d\lambda_j(k) \rho_k c_k c_{-\omega_j(k)}^{\sigma - \omega_j(k)}, \quad (2.1) \]
over a yet unspecified set of contours or arcs $\Gamma_j$ in the complex plane of a variable $k$, and in which for $N \geq 2$ a positive integer, $\omega_j(k)(j = 1, 2, \cdots, N - 1)$ and $\omega_N(k) = k$ are the distinct roots of the following algebraic relation

$$G(\omega, k) := g(\omega) - g(k) = 0 \quad \text{where} \quad g(k) = \sum_{j=1}^{N} \alpha_j k^j$$

is a monic (i.e. $\alpha_N = 1$) polynomial with coefficients $\alpha_j$. The factors $\rho_k$, $\sigma_{k'}$ are discrete exponential functions of $k$, $k'$ respectively, given by

$$\rho_k(n, m) = (p + k)^n(q + k)^m \rho_k(0, 0) \quad \sigma_{k'}(n, m) = (p - k')^{-n}(q - k')^{-m} \sigma_{k'}(0, 0), \quad \text{(2.3)}$$

the infinite-component vector $c_k = (k^j)_{j \in \mathbb{Z}}$ denotes the column vector of a basis of monomials in the variable $k$, and $c_{k'}'$ denotes the transpose of $c_{k'}$. The integration measures $d\lambda_j(k)$ remain unspecified, but we assume that basic operations (such as differentiations w.r.t. parameters or applying shifts in the variables $n$ and $m$) commute with the integrations.

In accordance with the notation introduced in section 1, we denote shifts over one unit in the variables $n$ and $m$ respectively by the operations $\sim$ and $\hat{\sim}$, which implies for the factors $\rho_k$, $\sigma_{k'}$

$$\hat{\rho}_k = \rho_k(n + 1, m) = (p + k)\rho_k \quad \hat{\sigma}_{k'} = \sigma_{k'}(n + 1, m) = (q - k')^{-1}\sigma_{k'}.$$ 

These relations imply the following linear relations for the matrix $C$

$$C(p - t\Lambda) = (p + \Lambda)C \quad \hat{C}(q - t\Lambda) = (q + \Lambda)C \quad \text{(2.4)}$$

where we have introduced the matrices $\Lambda$ and $t\Lambda$ which are defined by their actions on the vector $c_k$, and on its transposed vector, as follows:

$$\Lambda c_k = k c_k \quad \Lambda c_{k'}' = k' c_{k'}'. \quad \text{(2.5)}$$

Besides, from the definition of $C$ together with (2.5), one has

$$g(\Lambda) = C g(-t\Lambda) \quad \text{(2.6a)}$$

and as a consequence, using the fact that $g(\omega) - g(k) = \prod_{j=1}^{N} (\omega - \omega_j(k))$, also

$$\prod_{j=1}^{N} (\omega_j(-p) - \Lambda) = C \prod_{j=1}^{N} (\omega_j(p) + t\Lambda), \quad \text{(2.6b)}$$

leading to

$$\prod_{j=1}^{N} (\omega_j(-p) - \Lambda) \hat{C} = C \prod_{j=1}^{N-1} (\omega_j(p) + t\Lambda), \quad \text{(2.6c)}$$

(and a similar relation to (2.6c) with $p$ replaced by $q$ and $\hat{C}$ by $\hat{C}$). The following ingredients determine the structure:

i) A Cauchy kernel $\Omega$ defined by the relation $\Omega \Lambda + t\Lambda \Omega = O$, where $O$ is a rank 1 projection matrix, obeying $O^2 = O$.

ii) A matrix $U$ obeying the relation $U = C - U \Omega C$. 


iii) A vector \( u_k \) defined by \( u_k = \rho_k(c_k - U\Omega c_k) \).

In terms of these objects the following sets of relations involving the shift \( \sim \) can be derived:

\[
\begin{align*}
\bar{U} (p - \hbar \Lambda) & = (p + \Lambda)U - \bar{U} O U, \quad (2.7a) \\
\bar{U} \left[ \prod_{j=1}^{N-1} (\omega_j (-p) + \hbar \Lambda) \right] & = \left[ \prod_{j=1}^{N-1} (\omega_j (-p) - \Lambda) \right] \bar{U} \\
+ U \sum_{j=0}^{N-2} \left[ \prod_{l=1}^{j} (\omega_l (-p) + \hbar \Lambda) \right] O \left[ \prod_{l=j+2}^{N-1} (\omega_l (-p) - \Lambda) \right] \bar{U}, \\
\bar{U} \sum_{j=1}^{N} \alpha_j (-\hbar \Lambda)^j u_k & = \sum_{j=1}^{N} \alpha_j \Lambda^j u_k - U \sum_{j=1}^{N} \alpha_j \sum_{l=0}^{j-1} (-\hbar \Lambda)^l O \Lambda^{j-1-l} U, \quad (2.7c)
\end{align*}
\]

The derivation of (2.7) follows the same procedure as in [24], and they reduce to the relations obtained there in the special case that \( \omega_j(k) = \exp(2\pi ij/N)k \), i.e. the case that all coefficients \( \alpha_j = 0, j = 1, \ldots, N - 1 \) in (2.2). By virtue of the covariance of the dynamics in terms of the variables \( n \) and \( m \), similar relations to (2.7a) and (2.7b) hold with the shift \( \sim \) replaced by \( \sim \) while replacing \( p \) by \( q \). The associated linear problems (Lax pairs) are derived in terms of the object \( u_k \), for which we have the following set of constitutive relation, [24],

\[
\begin{align*}
\tilde{u}_k & = (p + \Lambda)u_k - \bar{U} O u_k, \quad (2.8a) \\
- \left[ \prod_{j=1}^{N} (\omega_j (-p) - k) \right] u_k & = \left[ \prod_{j=1}^{N-1} (\omega_j (-p) - \Lambda) \right] \tilde{u}_k \\
+ U \sum_{j=0}^{N-2} \left[ \prod_{l=1}^{j} (\omega_l (-p) + \hbar \Lambda) \right] O \left[ \prod_{l=j+2}^{N-1} (\omega_l (-p) - \Lambda) \right] \tilde{u}_k, \\
\sum_{j=1}^{N} \alpha_j \Lambda^j u_k & = \sum_{j=1}^{N} \alpha_j \Lambda^j u_k - U \sum_{j=1}^{N} \alpha_j \sum_{l=0}^{j-1} (-\hbar \Lambda)^l O \Lambda^{j-1-l} u_k, \quad (2.8c)
\end{align*}
\]

and similar relations to (2.8a) and (2.8b) with the shift \( \sim \) replaced by \( \sim \) while replacing \( p \) by \( q \).

The relation (2.7b) can be easily derived from the formal relation

\[
U \frac{g(-p) - g(-\hbar \Lambda)}{-p + \hbar \Lambda} = \frac{g(\Lambda) - g(-p)}{p + \Lambda} \bar{U} \\
+ U \left[ \Omega \frac{g(-p) - g(\Lambda)}{-p + \Lambda} - \frac{g(-p) - g(-\hbar \Lambda)}{p - \hbar \Lambda} \right] \Omega \bar{U},
\]

whilst (2.7c) follows from

\[
U g(-\hbar \Lambda) = g(\Lambda)U - U(\Omega g(\Lambda) - g(-\hbar \Lambda)\Omega)U.
\]
Similarly, the linear relation for $u_k$ follows from
\[ G(k, -p)u_k = \frac{g(A) - g(-p)}{p + A}u_k + U \left[ \Omega \frac{g(-p) - g(A)}{p + A} - \frac{g(-p) - g(-iA)}{p - iA} \right] \tilde{u}_k, \]
whereas follows from
\[ g(k)u_k = g(A)u_k - U(\Omega g(A) - g(-iA)\Omega)u_k. \]

For any fixed value of $N$, these abstract equations form an infinite set of recurrence relations defining the dynamics in terms of the independent variables $n, m$ on the objects $U$ and $u_k$ taking values in an abstract vector space $V$, and an adjoint vector $\langle e \rangle$ in its dual $V^*$. To give a concrete realization, we can choose a fixed vector $e$ and use the matrices $A$ and $A^t$ to define a natural gradation in this vector space. In terms these, we realize quantities such as $C$ and $U$ as infinite-dimensional matrices. In fact, setting
\[ u_{i,j} := \langle e \rangle \Lambda^i U^t \Lambda^j e, \quad i, j \in \mathbb{Z}, \]
we obtain infinite, i.e., $\mathbb{Z} \times \mathbb{Z}$ matrices, on which $\Lambda$ and $\Lambda^t$ act as index-raising operators acting from the left and from the right. In this realization the matrix $O = \langle e \rangle$ is the projector defined by these vectors, and through the relations
\[ \langle e \rangle c_k = 1, \quad c_k^t e = 1, \]
this defines the “central” entry in the space of $\mathbb{Z} \times \mathbb{Z}$ matrices. We will define concrete objects in terms of the matrix $U$, where we allude to this realization.

3. Lattice BSQ case

In this section we will now apply the structure exhibited in section 2 to the case $N = 3$, which in [24] is the one leading to the BSQ class of systems. We recall that the case $N = 2$ leads to the KdV class of lattice systems, but in that case it can be shown that the extended dispersion relation does not lead to new results. However, in the BSQ case ($N = 3$) it does. We restrict ourselves in this paper primarily to this case of $N = 3$, but it is clear that the general system derived in section 2 can be studied in fairly similar ways for any (integer) $N > 2$. Most of the relations given in this section reduce to the ones given in [30] setting $\alpha_1 = \alpha_2 = 0$.

3.1. Basic objects and their relations: BSQ case ($N = 3$). Let us define
\[ G(\omega, k) := \omega^3 - k^3 + \alpha_2(\omega^2 - k^2) + \alpha_1(\omega - k). \]
Obviously, for $N = 3$ the algebraic relation (2.2) reduces to
\[ G(\omega, k) = 0, \]
and its roots are denoted by $\omega_1(k)$, $\omega_2(k)$ and $\omega_3(k) = k$. The set of relations \(2.7\) take the form:

\[
U \left( p - t^t\Lambda \right) = (p + \Lambda)U - \tilde{U} O U ,
\]

\[
U \left( \omega_1(-p) + t^t\Lambda \right)(\omega_2(-p) + t^t\Lambda) = (\omega_1(-p) - \Lambda)(\omega_2(-p) - \Lambda)\tilde{U} + U \left[ (p - \alpha_2)O - (O \Lambda - t^t\Lambda O) \right] \tilde{U} ,
\]

\[
U \sum_{j=1}^3 \alpha_j(-t^t\Lambda)^j = \sum_{j=1}^3 \alpha_j \Lambda^j U - U \sum_{j=1}^3 \alpha_j \sum_{l=0}^{j-1} (-t^t\Lambda)^l O \Lambda^{j-1-l} U ,
\]

and similar relations to \(3.3a\) and \(3.3b\) with the shift $\sim$ replaced by $\sim$ while replacing $p$ by $q$, as well as

\[
\bar{u}_k = (p + \Lambda)u_k - \tilde{U} O u_k ,
\]

\[
- \prod_{j=1}^3 (\omega_j(-p) - k) u_k = (\omega_1(-p) - \Lambda)(\omega_2(-p) - \Lambda)\bar{u}_k + U \left[ (p - \alpha_2)O - (O \Lambda - t^t\Lambda O) \right] \bar{u}_k ,
\]

\[
\sum_{j=1}^3 \alpha_j k^j u_k = \sum_{j=1}^3 \alpha_j \Lambda^j u_k - U \sum_{j=1}^3 \alpha_j \sum_{l=0}^{j-1} (-t^t\Lambda)^l O \Lambda^{j-1-l} u_k ,
\]

(and similar relations to \(3.3a\) and \(3.4b\) with the shift $\sim$ replaced by $\sim$ while replacing $p$ by $q$). For the sake of obtaining from the constitutive relations \(3.3\) closed-form equations, we introduce the following objects:

\[
v_a := 1 - t^t e (a + \Lambda)^{-1} U e , \quad w_b := 1 + t^t e U (b - t^t\Lambda)^{-1} e ,
\]

\[
s_a := a - t^t e (a + \Lambda)^{-1} U t^t \Lambda e , \quad t_b := -b + t^t e U (b - t^t\Lambda)^{-1} e ,
\]

\[
r_a := a^2 - t^t e (a + \Lambda)^{-1} U t^2 \Lambda^2 e , \quad z_b := b^2 + t^t e \Lambda^2 U (b - t^t\Lambda)^{-1} e ,
\]

but in particular the following object:

\[
s_{a,b} := t^t e (a + \Lambda)^{-1} U (b - t^t\Lambda)^{-1} e .
\]

Then for these objects and $u_{i,j}$ defined in \(2.9\) we have the following relations

\[
p \tilde{u}_{i,j} - \tilde{u}_{i,j+1} = pu_{i,j} + u_{i+1,j} - \tilde{u}_{i,0}u_{0,j} ,
\]

\[
(p^2 - \alpha_2 p + \alpha_1)u_{i,j} + (p - \alpha_2)u_{i,j+1} + u_{i,j+2} = (p^2 - \alpha_2 p + \alpha_1)\tilde{u}_{i,j} - (p - \alpha_2)\tilde{u}_{i+1,j} + \tilde{u}_{i+2,j} + (p - \alpha_2)u_{i,0}\tilde{u}_{0,j} - u_{i,0}\tilde{u}_{1,j} + u_{i,1}\tilde{u}_{0,j} ,
\]

as well as

\[
1 + (p - a)s_{a,b} - (p - b)\tilde{s}_{a,b} = \tilde{v}_a w_b ,
\]

\[
(p + a + b - \alpha_2) + pb s_{a,b} - pa \tilde{s}_{a,b} = s_a \tilde{w}_b - v_a t_b + (p - \alpha_2)v_a \tilde{w}_b ,
\]

\[
(p + a + b - \alpha_2) + pb s_{a,b} - pa \tilde{s}_{a,b} = s_a \tilde{w}_b - v_a t_b + (p - \alpha_2)v_a \tilde{w}_b ,
\]
and

\[ \tilde{s}_a = (p + u_0)\tilde{v}_a - (p - a)v_a, \]  
\[ t_b = (p - b)\tilde{w}_b - (p - \tilde{u}_0)w_b, \]  
(3.9a)

and

\[ \tilde{r}_a = p\tilde{s}_a - (p - a)s_a + \tilde{v}_au_{0,1}, \]  
\[ z_b = (p - b)t_b - pt_b + \tilde{u}_{1,0}w_b, \]  
(3.10a)

and

\[ r_a = p\tilde{v}_a - (p - \tilde{u}_0 - \alpha_2)s_a - ((p - \alpha_2)(p - \tilde{u}_0) + \tilde{u}_{1,0} + \alpha_1)v_a, \]  
\[ \tilde{z}_b = p_bw_b + (p + u_0 - \alpha_2)t_b - ((p - \alpha_2)(p + u_0) + u_{0,1} + \alpha_1)\tilde{w}_b, \]  
(3.11a)

where we have used a simplified notation \( u_0 = u_{0,0} \) and \( p_a, p_b \) are defined as

\[ p_a = \frac{G(-p,-a)}{a - p} = (p^2 + ap + a^2) - \alpha_2(p + a) + \alpha_1, \quad p_b = \frac{G(-p,-b)}{b - p} = (p^2 + bp + b^2) - \alpha_2(p + b) + \alpha_1. \]  
(3.12)

All relations (3.8)-(3.11), which are in a slightly different form have already appeared in [30], also hold for their hat-\( q \) counterparts obtained by replacing the shift \( \tilde{\cdot} \) by the shift \( \hat{\cdot} \) whilst replacing the parameter \( p \) by \( q \).

From equations (3.7a), (3.9a), (3.11a) and their hat-q counterparts one can get a further set of relations

\[ (p + q - \tilde{u}_0 + \frac{s_a}{v_a} - \alpha_2)(p + q - \tilde{u}_0 - \tilde{u}_0) = p_a\tilde{v}_a - q_a\tilde{v}_a, \]  
(3.13a)

\[ p - q + \tilde{u}_0 - \tilde{u}_0 = (p - a)\tilde{v}_a - (q - a)\tilde{v}_a, \]  
(3.13b)

and similarly, from (3.7a), (3.9b) and (3.11b),

\[ (p + q + u_0 - \frac{t_b}{w_b} - \alpha_2)(p + q + \tilde{u}_0 - \tilde{u}_0) = p_b\tilde{w}_b - q_b\tilde{w}_b, \]  
(3.14a)

\[ p - q + \tilde{u}_0 - \tilde{u}_0 = (p - b)\tilde{w}_b - (q - b)\tilde{w}_b. \]  
(3.14b)

3.2. Closed-form lattice equations. One can immediately get two closed-form lattice equations. One is composed of (3.9a), its hat-q counterpart and (3.13a), i.e.,

\[ \tilde{s}_a = (p + u_0)\tilde{v}_a - (p - a)v_a, \quad \tilde{s}_a = (q + u_0)\tilde{v}_a - (q - a)v_a, \]  
(3.15a)

\[ (p + q - \tilde{u}_0 + \frac{s_a}{v_a} - \alpha_2)(p + q + \tilde{u}_0 - \tilde{u}_0) = p_a\tilde{v}_a - q_a\tilde{v}_a. \]  
(3.15b)

\[ 1 \]We note in passing that the yet unidentified case (A) of 3-component BSQ type systems which was presented in the recent paper [11] can be identified with eqs. (5.3.7a) together with (5.3.14) of [30], where, up to a point transformation. \( x \) can be identified with \( v_a \), \( z \) with \( u \) and \( y \) with \( s_a \). By duality this case of [11] can also be identified with eqs. (5.3.7b) and (5.3.16) of [30] identifying \( x \) with \( w_{\beta} \), \( z \) with \( u \) and \( t_{\beta} \).
Another is composed of (3.9b), its hat-$q$ counterpart and (3.14b), i.e.,
\[
t_b = (p - b)\tilde{w}_b - (p - \tilde{u}_0)w_b, \quad t_b = (q - b)\tilde{w}_b - (q - \tilde{u}_0)w_b, \quad (3.16a)
\]
\[
(p + q + u_0 - \frac{\tilde{t}_b}{\tilde{w}_b} - \alpha_2)(p - q + \tilde{u}_0 - \tilde{u}_0) = p\tilde{w}_b - q\tilde{w}_b. \quad (3.16b)
\]

To get further closed form lattice systems we take $i = j = 0$ in (3.7) and its hat-$q$ counterpart and get
\[
p\tilde{u}_0 - \tilde{u}_{0,1} = pu_0 + u_{1,0} - \tilde{u}_0u_0, \quad q\tilde{u}_0 - \tilde{u}_{0,1} = qu_0 + u_{1,0} - \tilde{u}_0u_0, \quad (3.17a)
\]
\[
(p^2 - \alpha_2p + \alpha_1)u_0 + (p - \alpha_2)u_{0,1} + u_{0,2}
\]
\[
= (p^2 - \alpha_2p + \alpha_1)\tilde{u}_0 - (p - \alpha_2)\tilde{u}_{1,0} + \tilde{u}_{2,0} + (p - \alpha_2)u_0\tilde{u}_0 - u_0\tilde{u}_{1,0} + u_{0,1}\tilde{u}_0, \quad (3.17b)
\]
\[
(q^2 - \alpha_2q + \alpha_1)u_0 + (q - \alpha_2)u_{0,1} + u_{0,2}
\]
\[
= (q^2 - \alpha_2q + \alpha_1)\tilde{u}_0 - (q - \alpha_2)\tilde{u}_{1,0} + \tilde{u}_{2,0} + (q - \alpha_2)u_0\tilde{u}_0 - u_0\tilde{u}_{1,0} + u_{0,1}\tilde{u}_0. \quad (3.17c)
\]

Let us focus on (3.17b) and (3.17c). By subtraction one can delete $u_{0,2}$, and to delete $u_{2,0}$ from the remains one can make use of (3.7a) where we take $i = 1$, $j = 0$. Then, after some algebra we can reach to
\[
\frac{-G(-p, -q)}{p - q + \tilde{u}_0 - \tilde{u}_0} = \frac{G(-p, -q)}{q - p} + (p + q - \alpha_2)(u_0 - \tilde{u}_0) - u_0\tilde{u}_0 + \tilde{u}_{1,0} + u_{0,1}. \quad (3.18)
\]

This equation together with (3.17a) composes of a closed-form system which is viewed as the extended three-component lattice BSQ equation. (We make a precise connection with the corresponding system of [11] in the next section).

Next, by the transformation (if $a \neq b$)
\[
s_{a,b} = S_{a,b} - \frac{1}{b - a} \quad (3.19)
\]
we rewrite the relation (3.8) and its hat-$q$ counterpart in the form
\[
(p - a)S_{a,b} - (p - b)\tilde{S}_{a,b} = \tilde{v}_aw_b, \quad (q - a)S_{a,b} - (q - b)\tilde{S}_{a,b} = \tilde{v}_aw_b, \quad (3.20a)
\]
\[
p_bS_{a,b} - p_a\tilde{S}_{a,b} = s_aw_b - v_aw_b + (p - \alpha_2)v_aw_b, \quad (3.20b)
\]
\[
q_bS_{a,b} - q_a\tilde{S}_{a,b} = s_aw_b - v_aw_b + (q - \alpha_2)v_aw_b. \quad (3.20c)
\]

Then, from (3.20b) deleting $\tilde{t}_b$ by using (3.14a) and $\tilde{s}_a$ by using the hat-$q$ version of (3.9a) and also making use of (3.14b) we get
\[
(q - a)v_aw_b = \frac{p_b\tilde{v}_aw_b - q_b\tilde{v}_aw_b}{(p - b)\tilde{w}_b - (q - b)\tilde{w}_b} \tilde{v}_aw_b + p_a\tilde{S}_{a,b} - p_b\tilde{S}_{a,b}. \quad (3.21)
\]

Using (3.20a) once again we can rewrite (3.21) to the following desired form
\[
v_aw_b = \frac{p_b}{(p - a)\tilde{w}_b - (q - a)\tilde{w}_b} \tilde{v}_aw_b + \frac{G(-a, -b)}{(p - a)(q - a)} \tilde{S}_{a,b}. \quad (3.22)
\]

The system composed of (3.22) and (3.20a) can be viewed as the extended three-component lattice MBSQ/SBSQ system, in the sense that the extended versions of both the 9-point lattice MBSQ of [24] as well as of the 9-point Schwarzian BSQ lattice can be obtained from this system.
by elimination of two of the three variables. An (equivalent) alternative form of the extended three-component lattice MBSQ/SBSQ equation consists of (3.20a) together with the following equation

\[ v_a \tilde{w}_b = w_b \frac{p_a}{p-b} \tilde{w}_b v_a - \frac{q_a}{q-b} \tilde{w}_b \tilde{v}_a + \frac{G(-a,-b)}{(p-b)(q-b)} S_{a,b} . \]  

(3.23)

\[ \tilde{S}_{a,b} \] can be derived similar to (3.22). We note that (3.22) and (3.23) can be transformed to each other in the view of (3.20a). In fact, noting that the relation \( G(-p,-b) - G(-p,-a) = G(-a,-b) \), a substraction of (3.23) from (3.22) gives

\[ \frac{\tilde{S}_{a,b}}{(p-a)(q-a)} - \frac{S_{a,b}}{(p-b)(q-b)} = \frac{\tilde{w}_a \tilde{v}_a w_b}{(p-a)(p-b)} - \frac{\tilde{w}_b \tilde{v}_a w_b}{(q-a)(q-b)} , \]

which, by replacing \( \tilde{v}_a w_b \) and \( \tilde{w}_a w_b \) using (3.20a), further reduces to

\[ \frac{\tilde{S}_{a,b}}{(p-a)(q-a)} = \frac{(p-a) \tilde{w}_b \tilde{S}_{a,b} - (q-a) \tilde{w}_b \tilde{S}_{a,b}}{(p-b) \tilde{w}_b - (q-b) \tilde{w}_b} , \]

and this is an identity as is evident from (3.20a).

Finally, to derive the coupled system (1.4) we rewrite (3.21) in the form

\[ (q-a) v_a \tilde{w}_b = \frac{p_b \tilde{v}_a \tilde{w}_b - q_b \tilde{v}_a \tilde{w}_b}{(p-b) \tilde{v}_a \tilde{w}_b - (q-b) \tilde{v}_a \tilde{w}_b} \tilde{v}_a w_b + p_a \tilde{S}_{a,b} - p_b \tilde{S}_{a,b} \]

\[ = \frac{p_b[(p-a) \tilde{S}_{a,b} - (p-b) \tilde{S}_{a,b}] - q_b[(q-a) \tilde{S}_{a,b} - (q-b) \tilde{S}_{a,b}]}{(p-b)(q-a) \tilde{S}_{a,b} - (q-b)(p-a) \tilde{S}_{a,b}} \]

\[ \times [(q-a) S_{a,b} - (q-b) S_{a,b}] + p_a \tilde{S}_{a,b} - p_b \tilde{S}_{a,b} , \]  

(3.24)

and introduce the variable \( u \) by setting

\[ \tilde{S}_{a,b} = \left( \frac{(p-a) P_b}{(p-b) P_a} \right)^n \left( \frac{(q-a) Q_b}{(q-b) Q_a} \right)^m u . \]  

(3.25)

Furthermore, (1.5) follows from (1.4) by making use of the equality

\[ \langle v_a \tilde{w}_b \rangle = \frac{\tilde{v}_a w_b}{\tilde{v}_a \tilde{w}_b} \frac{\tilde{v}_a w_b}{\tilde{v}_a \tilde{w}_b} = \frac{\tilde{v}_a w_b}{\tilde{v}_a w_b} \frac{\tilde{v}_a w_b}{\tilde{v}_a w_b} . \]  

(3.26)

We have now all the ingredients in place to make the identifications with the systems exhibited in [11].

4. Deformation of the extended lattice BSQ-type equations

In this section, we identify the extended closed-form lattice BSQ equations obtained in the previous section, with the four extended lattice BSQ-type equations (A-2), (B-2), (C-3) and (C-4) of ref. [11] by means of simple point transformations.
Case A-2: 
Starting from (3.15) and introducing point transformation
\[ v_a = \frac{x}{x_a}, \quad u_0 = z - z_0, \quad s_a = \frac{1}{x_a} (y - v_a y_a), \]
(4.1)
where
\[ x_a = (p - a)^{-n}(q - a)^{-m} c_1, \]
(4.2a)
\[ z_0 = (c_3 - p)n + (c_3 - q)m + c_2, \]
(4.2b)
\[ y_a = x_a (z_0 - c_3), \]
(4.2c)
and \( c_1, c_2, c_3 \) are constants, (3.15) yields
\[ \tilde{y} = \tilde{x} - x, \quad \hat{y} = \hat{x} - x, \]
(4.3a)
\[ y = x \tilde{z} - b_0 x + \frac{G(-p, -a) \tilde{x} + G(-q, -a) \hat{x}}{\tilde{z} - \hat{z}}, \]
(4.3b)
where \( b_0 = -\alpha_2 + 3c_3 \). This is the (A-2) equation given in ref. [11] and the parameter \( b_0 \) can be removed by a transformation [11].

Alternatively, employing the point transformation
\[ w_b = \frac{x}{x_b}, \quad u_0 = z - z_0, \quad t_b = \frac{1}{x_b} (y - w_by_b), \]
(4.4)
where
\[ x_b = (-p + b)^n(-q + b)^m c_1, \]
(4.5a)
\[ z_0 = -(c_3 + p)n - (c_3 + q)m + c_2, \]
(4.5b)
\[ y_b = x_b (z_0 - c_3), \]
(4.5c)
and \( c_1, c_2, c_3 \) are constants, we find
\[ y = \tilde{z} x - \tilde{x}, \quad y = \hat{z} x - \hat{x}, \]
(4.6a)
\[ \tilde{y} = \tilde{z} x - b'_0 \tilde{x} + \frac{G(-p, -b) \tilde{x} + G(-q, -b) \hat{x}}{\tilde{z} - \hat{z}}, \]
(4.6b)
where \( b'_0 = \alpha_2 + 3c_3 \). This equation is related to (4.3) by the transformation
\[ p \to -p, \quad q \to -q, \quad n \to -n, \quad m \to -m, \quad \alpha_2 \to -\alpha_2, \quad b \to -a, \]
which corresponds to the reversal symmetry of (A-2), (see [11]).

Case B-2: 
Now we look at the extended three-component lattice BSQ equation (3.17a) and (3.18). Making transformation
\[ x = u_0 - x_0, \]
(4.7a)
\[ y = u_{1,0} - x_0 u_0 + y_0, \]
(4.7b)
\[ z = u_{0,1} - x_0 u_0 + z_0, \]
(4.7c)
with

\begin{align}
x_0 &= np + mq + c_1, \\
y_0 &= \frac{1}{2}(np + mq + c_1)^2 + \frac{1}{2}(np^2 + mq^2 + c_2) + c_3, \\
z_0 &= \frac{1}{2}(np + mq + c_1)^2 - \frac{1}{2}(np^2 + mq^2 + c_2) - c_3,
\end{align}

and constants \(c_j, (j = 1, 2, 3)\), we have

\begin{align}
\tilde{z} &= x\bar{x} - y, \quad \hat{z} = x\bar{x} - y, \\
z &= x\hat{x} - \hat{z} - \alpha_2(\bar{x} - x) - \alpha_1 - \frac{G(-p, q)}{\bar{x} - \bar{x}}.
\end{align}

This is the (B-2) in Ref. [11] and the parameter \(\alpha_1\) can be removed by transformation [11].

**Case C-3:**

Next, we come to the extended three-component lattice MBSQ/SBSQ equation (3.20a) and (3.22). By the transformation

\begin{align}
S_{a,b} &= \left(\frac{p-a}{p-b}\right)^n \left(\frac{q-a}{q-b}\right)^m x, \\
v_a &= (p-a)^n(q-a)^m y, \\
w_b &= (p-b)^n(q-b)^m z,
\end{align}

then we have

\begin{align}
x - \bar{x} &= \tilde{y}z, \quad x - \hat{x} = \tilde{y}z, \\
\tilde{y}z &= z\frac{G(-p,-b)\tilde{y}y + G(-q,-b)\tilde{z}y}{\tilde{z} - \tilde{z}} + G(-a,-b)\bar{x}.
\end{align}

Similarly, by the same transformation from (3.20a) and (3.23) we have

\begin{align}
x - \bar{x} &= \tilde{y}z, \quad x - \hat{x} = \tilde{y}z, \\
\tilde{z}y &= z\frac{G(-p,-a)\tilde{z}y + G(-q,-a)\tilde{y}y}{\tilde{z} - \tilde{z}} + G(-a,-b)x.
\end{align}

This gives (C-3) equation in [11]. (4.11) and (4.12) are related by

\begin{align}
n \to -n, \quad m \to -m, \quad y \to z, \quad z \to -y, \quad a \leftrightarrow b
\end{align}

which gives the reversal symmetry of (C-3) [11]. Besides, as we mentioned before, (4.11b) and (4.12b) are also connected through

\begin{align}
z\tilde{y}y - \tilde{z}y = \tilde{x} - x,
\end{align}

which holds in light of (4.11a).

**Case C-4:**

The identification of Case (C-4) of [11] is somewhat indirect and relies on the important observation that (4.11) and (4.12) share the same solution (4.10). Thus, from these realtions we have
\[ x - \bar{x} = \bar{y}z, \quad x - \bar{x} = \bar{y}z, \quad (4.13a) \]
\[ \bar{y} \bar{z} = z \frac{P_{a,b} \bar{zy} - Q_{a,b} \bar{zy}}{\bar{z} - \bar{z}} + G_{a,b}(\bar{x} + x), \quad (4.13b) \]

where
\[ P_{a,b} = -\frac{1}{2}(G(-p, -b) + G(-p, -a)), \quad (4.14a) \]
\[ Q_{a,b} = -\frac{1}{2}(G(-q, -b) + G(-q, -a)), \quad (4.14b) \]
\[ G_{a,b} = \frac{G(-a, -b)}{2}. \quad (4.14c) \]

Consider the following transformation
\[ x = \frac{x_1 - G_{a,b}}{2G_{a,b}(x_1 + G_{a,b})}, \quad y = \frac{y_1}{x_1 + G_{a,b}}, \quad z = \frac{z_1}{x_1 + G_{a,b}}. \quad (4.15) \]

Imposing this transformation on (4.13), (4.13a) yields
\[ x_1 - \bar{x}_1 = \bar{y}_1 \bar{z}_1, \quad (4.16) \]
which further gives
\[ \bar{x}_1 = \frac{\bar{x}_1 \bar{z}_1 - \bar{z}_1 \bar{x}_1}{\bar{z}_1 - \bar{z}_1}. \quad (4.17) \]

Meanwhile, (4.13b) multiplied by \((G_{a,b} + x_1)(G_{a,b} + \bar{x}_1)\) yields
\[ y_1 \bar{z}_1 = \Delta + x_1 \bar{x}_1 - G_{a,b}^2, \quad (4.18) \]
where
\[ \Delta = \frac{(G_{a,b} + \bar{x}_1)z_1(P_{a,b} \bar{z}_1 \bar{y}_1 - Q_{a,b} \bar{z}_1 \bar{y}_1)}{(\bar{x}_1 \bar{z}_1 - \bar{x}_1 \bar{z}_1) + G_{a,b}(\bar{z}_1 - \bar{z}_1)}. \quad (4.19) \]

Substituting (4.17) in to \(\Delta\) we then have
\[ y_1 \bar{z}_1 = \bar{z}_1 \frac{P_{a,b} \bar{z}_1 \bar{y}_1 - Q_{a,b} \bar{z}_1 \bar{y}_1}{\bar{z}_1 - \bar{z}_1} + x_1 \bar{x}_1 - G_{a,b}^2. \quad (4.20) \]

(4.16) and (4.20) constitute the (C-4) equation of [11].

In conclusion, we have in this section identified all deformed equations of extended lattice BSQ type of [11] with equations in section 3 arising there from the direct linearization scheme of section 2. We note that, whilst the results in [11] were obtained from a systematic search programme, the equations that are found from the classification conditions come somewhat as separate cases through the elimination of choices and the implementation of certain symmetries. However, search does not really reveal the interconnections between the variables of the different cases, and, in contrast, that is what the DL approach provides. Thus, deriving all equations from the single framework all the interconnections between the variables of the various systems are revealed. In the next section we show how explicit soliton solutions are obtained systematically from the DL scheme outlined in section 2 and applied to the choices of variables associated with the extended BSQ systems.
5. Soliton solutions

In the case of soliton solutions, we start from the following explicit expression for the infinite matrix \( C \)

\[
C = \sum_{j=1}^{3} \sum_{j'=1}^{N_j} \Lambda_{j,j'} \rho_{k,j,j'} \ c_{j,j'} \ t c_{-\omega_j(k_{j,j'}) \sigma - \omega_j(k_{j,j'})},
\]

which can be obtained by choosing in (2.1) the integration contours \( \Gamma_j \) in the complex plane of the variable \( k \) to be Jordan curves surrounding simple poles at locations \( k_{j,j'} \) in the complex plane, with residues \( \Lambda_{j,j'} \) of the measures \( d\lambda_j(k) \), assuming the latter to be meromorphic in \( k \).

In this case we can derive the following expression for the infinite matrix \( U \)

\[
U = \sum_{j=1}^{3} \sum_{j'=1}^{N_j} \Lambda_{j,j'} u_{k,j,j'} \ t c_{-\omega_j(k_{j,j'}) \sigma - \omega_j(k_{j,j'})}.
\]

in which the infinite-component vector \( u_k \) obeys the linear equation:

\[
u_k + \sum_{j=1}^{3} \sum_{j'=1}^{N_j} \Lambda_{j,j'} u_{k,j,j'} \rho_k \frac{\sigma - \omega_j(k_{j,j'})}{k - \omega_j(k_{j,j'})} = \rho_k c_k.
\]

Eqs. (5.7a) and (5.8) are derived from the basic definitions in the Appendix.

By setting in eq. (5.3) \( k = k_{j,j'} \), where \( j' = 1, 2, 3, i' = 1, \ldots, N_{j'} \), the equation becomes a linear system for the quantities \( u_{k,j,j'} \), with a finite block-Cauchy matrix of the form:

\[
M_{i,j} = (M_{(i,i') \cdot (j,j')})_{i,j=1,2,3; i'=1,\ldots, N_i; j'=1,\ldots, N_j} \equiv \frac{\rho_k \Lambda_{j,j'} \sigma - \omega_j(k_{j,j'})}{k - \omega_j(k_{j,j'})},
\]

which is a \( 3 \times 3 \) block matrix with rectangular blocks of size \( N_i \times N_j \). We use the capital compound indices \( I = (i, i') \) and \( J = (j, j') \) to simplify the notation.

We can make the soliton solutions now explicit by assuming that the coefficients \( \Lambda_{j,j'} \) are chosen such that the matrix \( 1 + M \) is invertible, in which case we can make the following identifications for the quantities introduced in (2.9) and (3.6) respectively:

\[
u_{i,j} = t s \ K^{ij} (1 + M)^{-1} K^i r, \quad s_{a,b} = t s \ (-b 1 + K')^{-1} (1 + M)^{-1} (a 1 + K)^{-1} r
\]

in which \( K \) and \( K' \) are the block diagonal matrices:

\[
K = \text{diag} (k_{1,1}, \ldots, k_{1,N_1}; k_{2,1}, \ldots, k_{2,N_2}; k_{3,1}, \ldots, k_{3,N_3}),
\]

\[
K' = \text{diag} (-\omega_1(k_{1,1}), \ldots, -\omega_1(k_{1,N_1}); -\omega_2(k_{2,1}), \ldots, -\omega_2(k_{2,N_2}); -\omega_3(k_{3,1}), \ldots, -\omega_3(k_{3,N_3})),
\]

and where \( r \) and \( t s \) are the vectors:

\[
r = \text{diag} (\rho_{k_{1,1}}, \ldots, \rho_{k_{1,N_1}}; \rho_{k_{2,1}}, \ldots, \rho_{k_{2,N_2}}; \rho_{k_{3,1}}, \ldots, \rho_{k_{3,N_3}})^T,
\]

\[
t s = \text{diag} (\sigma - \omega_1(k_{1,1}), \ldots, \sigma - \omega_1(k_{1,N_1}); \sigma - \omega_2(k_{2,1}), \ldots, \sigma - \omega_2(k_{2,N_2}); \sigma - \omega_3(k_{3,1}), \ldots, \sigma - \omega_3(k_{3,N_3})).
\]
The soliton solutions of the other relevant variables, defined in (3.5), are obtained in a similar way, and lead to the expressions

\[ v_a := 1 - ts(1 + M)^{-1}(a1 + K)^{-1}r, \quad w_b := 1 + ts(-b1 + K')^{-1}(1 + M)^{-1}r, \]
\[ s_a := a - tsK'(1 + M)^{-1}(a1 + K)^{-1}r, \quad t_b := -b + ts(-b1 + K')^{-1}(1 + M)^{-1}Kr, \]
\[ r_a := a^2 - tsK'^2(1 + M)^{-1}(a1 + K)^{-1}r, \quad z_b := b^2 + ts(-b1 + K')^{-1}(1 + M)^{-1}K^2r. \]

Note that all these expressions (5.5) and (5.8) can be made quite explicit after choosing the various parameters of the solution. Finally the relevant \( \tau \)-function associated with these soliton solutions is given by

\[ \tau := \det (1 + M) \quad (5.9) \]

and it can be shown that it obeys the following \textit{trilinear} equation:

\[ (3p^2 - 2a2p + a1)\tau^2 + (3q^2 - 2a2q + a1)\tau^2 - (p - q)^2\tau^2 = [(p^2 + pq + q^2) - a2(p + q) + a1] (\tau^2 + \tau^2) . \quad (5.10) \]

This trilinear equation is derived from a combination of (3.13), (3.15) and (3.12) together with the identifications

\[ v_p = \frac{\tau}{\tau}, \quad w_p = \frac{\tau}{\tau}, \quad v_q = \frac{\tau}{\tau}, \quad w_q = \frac{\tau}{\tau}, \quad (q - p)S_{p,q} = \frac{\tau}{\tau} . \quad (5.11) \]

The explicit expressions (5.8) can be subsequently inserted in the formulae of the previous section identifying the various dependent variables of the systems of [11], and thus we get the explicit multi-soliton solutions for the latter. Furthermore, the identifications (5.11) can be used to express these quantities in terms of the \( \tau \)-function. It is to be expected that the trilinear equation could be cast in bilinear form, but at the expense of having to introduce additional \( \tau \)-functions (e.g. by introducing functions obtained by applying a shift on (5.9) in a lattice direction other than the ones appearing in the equation itself), but we will not pursue that line of investigation here.

6. Lax representations

The relations (3.4) are the basis for the derivation of Lax pairs for the lattice equations list in section 3. In fact, by selecting specific components of these vectors, or combinations thereof, we can constitute the basic vector functions in terms of which we obtain the relevant linear problems. This the DL scheme will provide some of the Lax pairs directly. In some cases we find it more convenient to rely on the 3D-consistency of the systems to produce the Lax pairs, along the lines of the papers [20, 8], cf. also [30, 28] for the BSQ case.

Case A-2:

For the (A-2) equation (3.15), we define an eigenvector of the form(cf. [24])

\[ \phi = \left( ((a + A)^{-1}u_k)_0, (u_k)_0, (Au_k)_0 \right)^T. \]
Here for a infinite-component column vector $\mathbf{v}$, $(\mathbf{v})_0$ stands for the 0-th component of $\mathbf{v}$. In that case one can derive the following Lax relation $\phi = L^{A2}\phi$ with:

$$L^{A2} = \begin{pmatrix}
p - a & \hat{v}_a & 0 \\
0 & p - \hat{u}_0 & 1 \\
g(k, -a) \hat{v}_a & p - \hat{u}_0 & 1
\end{pmatrix}$$

in which $* = (p - \hat{u}_0)(p - \alpha_2 + a/v_a) - p_0\hat{v}_a/v_a$, and $M^{A2}$ is obtained from $L^{A2}$ by replacing $p$ by $q$ and $\hat{u}$ by $\hat{v}$. The compatibility condition of the Lax pair, i.e. the matrix equation $L^{A2}M^{A2} = M^{A2}L^{A2}$ leads to the relations $L^{A2}M^{A2} = M^{A2}L^{A2}$. These three equations constitute a system which is equivalent to the extended lattice BSQ equation (3.17a) and (3.18).

The latter relation follow readily from (3.9a).

**Case B-2:**

For the extended lattice BSQ equation (3.17a) and (3.18), then one can define vector

$$\psi = ((u_k)_0, (A_0)_0, (w_1 - \lambda)(w_2 - \lambda)(u_k)_0)^T,$$

then from (3.4), we obtain

$$\tilde{w} = L^{BSQ}\phi,$$

$$\tilde{u} = M^{BSQ}\phi,$$

where

$$L^{BSQ} = \begin{pmatrix}
p - \hat{u}_0 & 1 & 0 \\
-\hat{u}_1, 0 & p + \alpha_2 & 1 \\
* & -u_0, 1 + 2\alpha_2 u_0 & p - 2\alpha_2 + u_0
\end{pmatrix},$$

in which $* = G(k, -p) - (p - \hat{u}_0)(p - \alpha_2)(p + u_0) + u_0, 1 - (p + u_0 - 2\alpha_2)(\hat{u}_1, 0 + \alpha_1)$, and where $M^{BSQ}$ is obtained from $L^{BSQ}$ by replacing $p$ by $q$ and $\hat{u}$ by $\hat{v}$. The compatibility $\tilde{L}^{BSQ}M^{BSQ} = M^{BSQ}\tilde{L}^{BSQ}$ leads to the equations

$$\hat{u}_1, 0 - \hat{u}_1, 0 = (p - q + \hat{u}_0 - \hat{u}_0)\hat{u}_0 - p\hat{u}_0 + q\hat{u}_0$$

and

$$\hat{u}_0, 1 - \hat{u}_0, 1 = (p - q + \hat{u}_0 - \hat{u}_0)u_0 - p\hat{u}_0 + q\hat{u}_0,$$

together with eq. (3.18). These three equations constitute a system which is equivalent to the extended lattice BSQ equation (3.17a) and (3.18).

In the following cases, we give Lax pairs which are derived on the basis of the 3D consistency property for the systems of three coupled equation which we derived in section 4. In the generic case for a 3-component system containing variables $(x, y, z)$, we would need to set $(x = g/f, y = h/f, z = k/f)$ and constitute the vector is $\phi = (g, h, k, f)^T$ leading to a Lax pair in terms of $4 \times 4$ matrices (see, e.g., (6.23) and (6.31) below). However in some cases the $4 \times 4$ situation
simplifies to a Lax pair in terms of $3 \times 3$ matrices for the 3-component system (e.g., in the cases of the A-2 and B-2 systems).

**Case A-2 revisited:**

For the extended three-component lattice equation (4.3), we extend the lattice equation into a third dimension by introducing a new variable $l$ associated with a new shift $\hat{r}$ and a new complex parameter $r$

\[
\begin{align*}
\tilde{y} &= z\tilde{x} - x, \quad \tilde{y} = z\tilde{x} - x, \quad \tilde{y} = z\tilde{x} - x, \\
y &= x\tilde{z} - b_0 x + \frac{-G(-p_a - a)\tilde{x} + G(-q, -a)\tilde{x}}{\tilde{z} - \tilde{x}}, \\
y &= x\tilde{z} - b_0 x + \frac{-G(-p_a - a)\tilde{x} + G(-q, -a)\tilde{x}}{\tilde{z} - \tilde{x}}, \\
y &= x\tilde{z} - b_0 x + \frac{-G(-p_a - a)\tilde{x} + G(-q, -a)\tilde{x}}{\tilde{z} - \tilde{x}}, \\
y &= x\tilde{z} - b_0 x + \frac{-G(-p_a - a)\tilde{x} + G(-q, -a)\tilde{x}}{\tilde{z} - \tilde{x}}.
\end{align*}
\] (6.6a)

From (6.6a), it is easy to derive

\[
\begin{align*}
\tilde{x} &= \frac{\tilde{x} - x}{\tilde{z} - \tilde{x}}, \quad \tilde{z} = \tilde{x} - \tilde{x}, \quad \tilde{x} = \tilde{x} - \tilde{x}, \\
\tilde{y} &= \frac{\tilde{z} - \tilde{z}}{\tilde{z} - \tilde{z}}, \quad \tilde{y} = \tilde{z} - \tilde{z}, \quad \tilde{y} = \tilde{z} - \tilde{z}.
\end{align*}
\] (6.6b)

By the 3D-consistency, setting $\tilde{x} = \frac{\phi_1}{\phi_0}, \quad \tilde{z} = \frac{\phi_2}{\phi_0}, \quad \tilde{y} = \frac{\phi_3}{\phi_0}$, then the Lax pair is

\[
\tilde{\phi}_{11} = L_{11}\phi_{11}, \quad \tilde{\phi}_{11} = M_{11}\phi_{11},
\] (6.8)
in which

\[
\phi_{11} = \left( \begin{array}{c} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{array} \right), \quad L_{11} = \left( \begin{array}{ccc} \tilde{z} & 0 & -1 & 0 \\ -\tilde{x} & -1 & 0 & 0 \\ 0 & G(-p_a - a)\tilde{x} & -\tilde{z} & 0 \\ 0 & 0 & -(\frac{y}{x} + b_0) & 0 \end{array} \right),
\] (6.9)

where the matrix $M_{11}$ is the hat-$q$ version of $L_{11}$. The compatibility gives the relations:

\[
\begin{align*}
\tilde{x} &= \frac{\tilde{x} - x}{\tilde{z} - \tilde{z}}, \quad x = \frac{\tilde{z} - \tilde{z}}{\tilde{z} - \tilde{z}}, \quad \tilde{y} = \frac{\tilde{z} - \tilde{z}}{\tilde{z} - \tilde{z}}, \\
\tilde{y} &= \frac{\tilde{z} - \tilde{z}}{\tilde{z} - \tilde{z}}.
\end{align*}
\] (6.10)

together with eq. (4.3b). The two equations (6.10) are consequences of the 3-component system which constitutes the case (A-2). We note that there is a zero column appearing in $L_{11}$ as well as in $M_{11}$, which indicates $\tilde{y}$ is not necessary to be used to generate a Lax pair. Let us remove the last columns and rows in $L_{11}$ and $M_{11}$, and examine the remains:

\[
\tilde{\phi}_{12} = L_{12}\phi_{12}, \quad \tilde{\phi}_{12} = M_{12}\phi_{12},
\] (6.11)
in which

\[
\phi_{12} = \left( \begin{array}{c} \varphi_0 \\ \varphi_1 \\ \varphi_2 \end{array} \right), \quad L_{12} = \left( \begin{array}{ccc} \tilde{z} & 0 & -1 \\ -\tilde{x} & -1 & 0 \\ 0 & G(-p_a - a)\tilde{x} & -\tilde{z} \end{array} \right),
\] (6.12)
where the matrix $M_{12}$ is the hat-$q$ version of $L_{12}$. The compatibility of (6.11) gives (6.10) and (4.3b) as well.

**Case B-2 revisited:**
Similarly, for the extended three-component lattice BSQ equation (4.9), by considering the third dimension of this lattice equation, then we have

$$
\tilde{z} = x\tilde{x} - y, \quad \tilde{\zeta} = x\tilde{x} - y, \quad \bar{\nu} = x\bar{\nu} - y,
$$

(6.13a)

$$
z = x\tilde{y} - \tilde{y} - \alpha_2(x - \tilde{x}) - \alpha_1 - \frac{G(-p, -q)}{\bar{x} - \tilde{x}},
$$

(6.13b)

$$
z = x\bar{\nu} - \bar{\nu} - \alpha_2(\bar{x} - \tilde{x}) - \alpha_1 - \frac{G(-p, -r)}{\bar{x} - \tilde{x}},
$$

(6.13c)

$$
z = x\tilde{x} - \tilde{y} - \alpha_2(\tilde{x} - \bar{x}) - \alpha_1 - \frac{G(-r, -q)}{\tilde{x} - \bar{x}}.
$$

(6.13d)

From (6.13a), it is easy to derive

$$
\tilde{\zeta} = \frac{\tilde{y} - \bar{\nu}}{\tilde{x} - \bar{x}}, \quad \bar{\nu} = \frac{\tilde{y} - \bar{\nu}}{\tilde{x} - \bar{x}}, \quad \tilde{z} = \frac{\tilde{y} - \bar{\nu}}{\tilde{x} - \bar{x}},
$$

(6.14a)

$$
\tilde{z} = \frac{ \tilde{y} - \tilde{y} \tilde{x} + \tilde{y} \bar{x}}{\tilde{x} - \bar{x}}, \quad \tilde{\zeta} = \frac{ \tilde{y} - \tilde{y} \tilde{x} + \tilde{y} \bar{x}}{\tilde{x} - \bar{x}}, \quad \tilde{\nu} = \frac{ \tilde{y} - \tilde{y} \tilde{x} + \tilde{y} \bar{x}}{\tilde{x} - \bar{x}}.
$$

(6.14b)

By the 3D-consistency, setting $\bar{x} = \frac{\tilde{\phi}_1}{\phi_0}, \tilde{y} = \frac{\tilde{\phi}_2}{\phi_0}, \tilde{z} = \frac{\tilde{\phi}_3}{\phi_0}$, then the Lax pair is

$$
\tilde{\phi}_{21} = L_{21}\phi_{21}, \quad \tilde{\phi}_{21} = M_{21}\phi_{21},
$$

(6.15)

in which

$$
\phi_{21} = \left(\begin{array}{c}
\phi_0 \\
\phi_1 \\
\phi_2 \\
\phi_3
\end{array}\right), \quad L_{21} = \left(\begin{array}{cccc}
\tilde{x} & -1 & 0 & 0 \\
\tilde{y} & 0 & -1 & 0 \\
\ast & z - \alpha_2x + \alpha_1 & \alpha_2 - x & 0 \\
0 & \tilde{y} & -\tilde{x} & 0
\end{array}\right),
$$

(6.16)

with $\ast = -(z - \alpha_2x + \alpha_1)\tilde{x} + \tilde{y}(x - \alpha_2) + G(-p, -r)$, and where the matrix $M_{21}$ is the hat-$q$ version of $L_{21}$. The compatibility yields the relations

$$
\tilde{\zeta} = \frac{\tilde{y} - \bar{\nu}}{\tilde{x} - \bar{x}}, \quad x = \frac{\tilde{z} - \tilde{\zeta}}{\tilde{x} - \bar{x}}
$$

(6.17)

together with the relation (4.9b). Similar to the case (A-2) the Lax matrices $L_{21}$ and $M_{21}$ can degenerate to $3 \times 3$ case by removing their last rows and columns. The results are

$$
\tilde{\phi}_{22} = L_{22}\phi_{22}, \quad \tilde{\phi}_{22} = M_{22}\phi_{22},
$$

(6.18)

in which

$$
\phi_{22} = \left(\begin{array}{c}
\phi_0 \\
\phi_1 \\
\phi_2
\end{array}\right), \quad L_{22} = \left(\begin{array}{ccc}
\tilde{x} & -1 & 0 \\
\tilde{y} & 0 & -1 \\
\ast & z - \alpha_2x + \alpha_1 & \alpha_2 - x
\end{array}\right),
$$

(6.19)

with $\ast$ is defined as in (6.16) and the matrix $M_{22}$ is the hat-$q$ version of $L_{22}$. The compatibility yields the relations (6.17) and (4.9b).
**Case C-3:**
For this case, which includes the extended three-component lattice MBSQ/SBSQ equation \([4.11]\), we are led by considering a third dimension of this lattice equation to the following relations:

\[
x - \tilde{x} = \tilde{y} z , \quad x - \tilde{x} = \bar{y} z , \quad x - \bar{x} = \bar{y} z , \quad (6.20a)
\]
\[
y\tilde{z} = z \frac{-G(\tilde{p} - a)\bar{y} + G(\tilde{q} - a)\tilde{y}}{\bar{z} - \tilde{z}} + G(-a, -b)\tilde{x} , \quad (6.20b)
\]
\[
\bar{y}\tilde{z} = z \frac{-G(\bar{p} - a)\tilde{y} + G(-r, -a)\tilde{y}}{\bar{z} - \tilde{z}} + G(-a, -b)\bar{x} , \quad (6.20c)
\]
\[
y\tilde{z} = z \frac{-G(-r, -a)\bar{y} + G(-q, -a)\bar{y}}{\bar{z} - \tilde{z}} + G(-a, -b)\tilde{x} . \quad (6.20d)
\]

From \((6.20a)\), it is easy to derive

\[
\tilde{x} = \frac{\tilde{x}z - \bar{x}z}{\bar{z} - \tilde{z}} , \quad \bar{x} = \frac{\bar{x}z - \tilde{x}z}{\bar{z} - \tilde{z}} , \quad \tilde{z} = \frac{\tilde{x}z - \bar{x}z}{\bar{z} - \tilde{z}} , \quad (6.21a)
\]
\[
y = \frac{\bar{x} - \tilde{x}}{\bar{z} - \tilde{z}} = -z\tilde{y} - \bar{y} , \quad \bar{y} = \frac{\bar{x} - \tilde{x}}{\bar{z} - \tilde{z}} = -z\tilde{y} - \bar{y} , \quad \tilde{y} = \bar{x} - \tilde{x} = -z\tilde{y} - \bar{y} . \quad (6.21b)
\]

By the 3D-consistency, setting \(\bar{x} = \frac{\psi_1}{\psi_0}, \quad \tilde{y} = \frac{\psi_3}{\psi_0}, \quad \tilde{z} = \frac{\psi_3}{\psi_0}\), then we have the Lax pair

\[
\tilde{\psi}_1 = L_3\psi_1, \quad \tilde{\psi}_1 = M_3\tilde{\psi}_1 , \quad (6.22)
\]

in which \(\psi_1 = (\psi_0, \psi_1, \psi_2, \psi_3)^T\) and

\[
L_3 = \frac{1}{z} \begin{pmatrix}
-\tilde{z} & 0 & 0 & 1 \\
0 & -\bar{z} & 0 & \bar{x} \\
0 & -G(-a, -b)\bar{y} & -G(-r, -a)\tilde{y} & G(-p, -a)\bar{y} + G(-a, -b)\tilde{y} \\
0 & -G(\tilde{p} - a)\bar{y} & -G(\tilde{q} - a)\tilde{y} & G(-p, -a)\bar{y} + G(-a, -b)\tilde{y}
\end{pmatrix} , \quad (6.23)
\]

where the matrix \(M_3\) is the hat-q version of \(L_3\). The compatibility yields now:

\[
\tilde{\psi}_1 = \frac{\tilde{x}z - \bar{x}z}{\bar{z} - \tilde{z}} , \quad \tilde{\psi}_1 = -z\frac{\tilde{y} - \bar{y}}{\bar{z} - \tilde{z}} \quad (6.24)
\]

together with the relation \([4.11b]\). They constitute the case (C-3) of the extended BSQ family.

**Alternate Case C-3:**
In a similar way as in the previous case, for the extended three-component lattice MBSQ/SBSQ equation \([4.12]\), by considering a third dimension where we impose this lattice system, we have

\[
x - \tilde{x} = \tilde{y} z , \quad x - \tilde{x} = \bar{y} z , \quad x - \bar{x} = \bar{y} z , \quad (6.25a)
\]
\[
y\tilde{z} = z \frac{-G(\tilde{p} - a)\bar{y} + G(\tilde{q} - a)\tilde{y}}{\bar{z} - \tilde{z}} + G(-a, -b)x , \quad (6.25b)
\]
\[
\bar{y}\tilde{z} = z \frac{-G(\bar{p} - a)\tilde{y} + G(-r, -a)\tilde{y}}{\bar{z} - \tilde{z}} + G(-a, -b)x , \quad (6.25c)
\]
\[
y\tilde{z} = z \frac{-G(-r, -a)\bar{y} + G(-q, -a)\bar{y}}{\bar{z} - \tilde{z}} + G(-a, -b)x . \quad (6.25d)
\]
Obviously, the identities (6.21) are still hold. By using 3D-consistency, and setting \( \bar{x} = \frac{\mu_1}{\mu_0}, \bar{y} = \frac{\mu_2}{\mu_0}, \bar{z} = \frac{\mu_3}{\mu_0} \), then from (6.21), (6.29c) and (6.29d) we have the Lax pair

\[
\tilde{\psi}_2 = L_4 \psi_2, \quad \tilde{\psi}_2 = M_4 \psi_2, \quad (6.26)
\]

in which \( \psi_2 = (\mu_0, \mu_1, \mu_2, \mu_3)^T \) and

\[
L_4 = \frac{1}{z} \begin{pmatrix}
-\bar{z} & 0 & 0 & 1 \\
0 & -\bar{z} & 0 & \bar{x} \\
z\bar{y} & 0 & -\bar{z} & 0 \\
-G(-a, -b) \bar{x} \bar{y} & 0 & -G(-r, -a) \bar{x} \bar{y} & G(-p, -a) \bar{x} \bar{y} + G(-a, -b) \bar{x} \bar{y}
\end{pmatrix}, \quad (6.27)
\]

where the matrix \( M_4 \) is the hat-\( q \) version of \( L_4 \). The compatibility now produces the relations

\[
\tilde{x} = \frac{\bar{x} \bar{z} - \bar{x} \bar{z}}{\bar{z} - \bar{z}}, \quad x = \frac{\bar{x} \bar{y} - \bar{x} \bar{y}}{\bar{y} - \bar{y}}, \quad \tilde{y} = -\frac{\bar{y} - \bar{y}}{\bar{y} - \bar{y}} \quad (6.28)
\]

together with (4.125). Once again this system is a consequence of the original lattice system.

**Case C-4:**

For this case, which includes the extended three-component lattice MBSQ/SBSQ equation (4.16) and (4.20), we have the following 3D-consistent relations:

\[
x_1 - \bar{x}_1 = \bar{y}_1 z_1, \quad x_1 - \bar{x}_1 = \bar{y}_1 z_1, \quad x_1 - \bar{x}_1 = \bar{y}_1 z_1, \quad (6.29a)
\]

\[
y_1 \bar{z}_1 = z_1 \left( P_{a,b} \bar{z}_1 \bar{y}_1 - Q_{a,b} \bar{z}_1 \bar{y}_1 \right) + x_1 \bar{x}_1 - G_{a,b}^2, \quad (6.29b)
\]

\[
y_1 \bar{z}_1 = z_1 \left( R_{a,b} \bar{z}_1 \bar{y}_1 - Q_{a,b} \bar{z}_1 \bar{y}_1 \right) + x_1 \bar{x}_1 - G_{a,b}^2, \quad (6.29c)
\]

\[
y_1 \bar{z}_1 = z_1 \left( R_{a,b} \bar{z}_1 \bar{y}_1 - Q_{a,b} \bar{z}_1 \bar{y}_1 \right) + x_1 \bar{x}_1 - G_{a,b}^2, \quad (6.29d)
\]

where \( R_{a,b} = -\frac{1}{2} (G(-r, -b) + G(-r, -a)) \). From (6.29a) we have same consistent forms as (6.21) with \( (x, y, z) \) replaced by \( (x_1, y_1, z_1) \). Setting \( \bar{x}_1 = \frac{\mu_1}{\mu_0}, \bar{y}_1 = \frac{\mu_2}{\mu_0}, \bar{z}_1 = \frac{\mu_3}{\mu_0} \), we have the Lax pair

\[
\tilde{\psi}_3 = L_5 \psi_3, \quad \tilde{\psi}_3 = M_5 \psi_3, \quad (6.30)
\]

in which \( \psi_3 = (\nu_0, \nu_1, \nu_2, \nu_3)^T \) and

\[
L_5 = \frac{1}{z_1} \begin{pmatrix}
-\bar{z}_1 & 0 & 0 & 1 \\
0 & -\bar{z}_1 & 0 & \bar{x}_1 \\
z_1 \bar{y}_1 & 0 & -\bar{z}_1 & 0 \\
G_{a,b}^2 \bar{y}_1 - \bar{z}_1 \bar{x}_1 & \bar{y}_1 \bar{x}_1 & -P_{a,b} \bar{y}_1 \bar{z}_1 + \bar{z}_1 \bar{x}_1 - G_{a,b}^2 \bar{y}_1
\end{pmatrix}, \quad (6.31)
\]

where the matrix \( M_5 \) is the hat-\( q \) version of \( L_5 \). Now the compatibility yields

\[
x_1 = \frac{\bar{x}_1 \bar{y}_1 - \bar{x}_1 \bar{y}_1}{\bar{y}_1 - \bar{y}_1}, \quad \tilde{x}_1 = \frac{\bar{x}_1 z_1 - \bar{x}_1 \bar{z}_1}{\bar{z}_1 - \bar{z}_1}, \quad \tilde{y}_1 = -\frac{\bar{y}_1 - \bar{y}_1}{\bar{z}_1 - \bar{z}_1} \quad (6.32)
\]

together with the relation (4.20). They constitute the case (C-4) of the extended BSQ family.
Since the realization that the notion of multidimensional consistency is a key criterion for integrability of partial difference equations, the search for new integrable equations on the 2D lattice using this notion as the defining property has led to many novel results. The ABS classification [2], of single-variable quadrilateral lattice equations has demonstrated that such systems are highly restricted (up to equivalence), but the non-scalar case remains still to be largely explored. The search in [11] of 2- and 3-component systems of a form inspired by earlier results from the BSQ family of lattice equations [24, 19, 28] seems to confirm the assertion that the realm of such discrete equations has only few members, but the examples are rich and important. The parameter-extensions of the BSQ systems established in [11] are noteworthy because the extra parameter freedom allows their soliton solutions to exhibit some new features, cf. [12, 13]. On the other hand, the “search methodology” exploited in [11] does not reveal all there is to know about these equations, and this is supplemented by the structural approach which we took in the present paper. Thus, by extending the DL approach of [24] to include more general types of dispersion relations, we have been able in the present work to identify all the variables and their interconnections of the extended BSQ systems of [11]. In fact, this amounts to establishing Miura type (i.e., non-auto Bäcklund) transformations between these various systems (A-2), (B-3), (C-3) and (C-4), which involve also point transformations. Furthermore, the DL structure can be used effectively to yield explicit multi-soliton type solutions for all these systems by specifying in an appropriate way the contours and integration measures which define some of the main objects (e.g., in (2.1)). For more general measures and contours the DL also can yield wider classes of solutions, such as inverse scattering type solutions or scaling reductions. To explore further, and make explicit, the latter line of work on the basis of the present results, will be the subject of future investigations.

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Appendix

In this Appendix we derive some integral formulae from the general abstract structure displayed in section 2. Starting from (2.1), i.e.,

\[ C = \sum_{j=1}^{N} \int_{\Gamma_j} d\lambda_j(k) \rho_k c_k c^\dagger_{-\omega_j(k)} \sigma_{-\omega_j(k)} k , \]

and the defining relation for the vectors \( u_k \), i.e.,

\[ u_k + \rho_k U \Omega c_k = \rho_k c_k , \]

we can show that

\[ U = \sum_{j=1}^{N} \int_{\Gamma_j} d\lambda_j(k) u_k c^\dagger_{-\omega_j(k)} \sigma_{-\omega_j(k)} k \]
is equivalent to the defining relation
\[ U = C - U \Omega C. \] (7.4)

In fact,
\[ (7.3) - (7.1) = U - C = \sum_{j=1}^{N} \int_{\Gamma_j} d\lambda_j(k) \left( u_k - \rho_k c_k \right) c_{-\omega_j(k)}^{\ell} \sigma_{-\omega_j(k)} \].

Using (7.2) it is then
\[ U - C = - U \Omega \sum_{j=1}^{N} \int_{\Gamma_j} d\lambda_j(k) \rho_k c_k c_{-\omega_j(k)}^{\ell} \sigma_{-\omega_j(k)} \]
\[ = - U \Omega C. \]

On the other hand, starting from (7.1) and using (7.2), one has
\[ C = \sum_{j=1}^{N} \int_{\Gamma_j} d\lambda_j(k) u_k c_{-\omega_j(k)}^{\ell} \sigma_{-\omega_j(k)} + U \Omega \sum_{j=1}^{N} \int_{\Gamma_j} d\lambda_j(k) \rho_k c_k c_{-\omega_j(k)}^{\ell} \sigma_{-\omega_j(k)} \]
\[ = \sum_{j=1}^{N} \int_{\Gamma_j} d\lambda_j(k) u_k c_{-\omega_j(k)}^{\ell} \sigma_{-\omega_j(k)} + U \Omega C, \] (7.5)

which yields (7.3) if (7.4) holds.

Then, quite similar to the derivation in (7.5), if starting from (5.1), i.e.,
\[ C = \sum_{j=1}^{3} N_j \sum_{j'=1}^{3} \Lambda_{j,j'} \rho_{k_{j,j'}} c_{k_{j,j'}} \left( u_{k_{j,j'}} c_{-\omega_j(k_{j,j'})}^{\ell} \sigma_{-\omega_j(k_{j,j'})} \right), \] (7.6)
and using (7.2) and (7.4), we immediately have
\[ U = \sum_{j=1}^{3} N_j \sum_{j'=1}^{3} \Lambda_{j,j'} u_{k_{j,j'}} c_{-\omega_j(k_{j,j'})}^{\ell} \sigma_{-\omega_j(k_{j,j'})}, \] (7.7)
which is (7.7). Finally, for \( u_k \) we have
\[ u_k - \rho_k c_k = - \rho_k U \Omega c_k \]
\[ = - \rho_k \sum_{j=1}^{3} N_j \sum_{j'=1}^{3} \Lambda_{j,j'} u_{k_{j,j'}} c_{-\omega_j(k_{j,j'})}^{\ell} \sigma_{-\omega_j(k_{j,j'})} \frac{k - \omega_j(k_{j,j'})}{k - \omega_j(k_{j,j'})} \Omega c_k \]
\[ = - \rho_k \sum_{j=1}^{3} N_j \sum_{j'=1}^{3} \Lambda_{j,j'} u_{k_{j,j'}} \sigma_{-\omega_j(k_{j,j'})} \frac{k - \omega_j(k_{j,j'})}{k - \omega_j(k_{j,j'})} \Omega c_k \]
which gives (5.3), where we have successfully made use of
\[ \Lambda c_k = k c_k, \quad c_{k'}^{\ell} \Lambda = k' c_{k'}^{\ell}, \quad \Lambda \Omega + \Omega \Lambda = O, \quad O = e \quad e \]
and \( c_k^{\ell} O c_k = 1. \)
References

[1] Ablowitz, M. J., and F. J. Ladik. “A nonlinear difference scheme and inverse scattering.” *Studies in Applied Mathematics* 55 (1976): 213–29; “On the solution of a class of nonlinear partial difference equations.” ibid. 57 (1977): 1–12.

[2] Adler, V. E., A. I. Bobenko, and Yu. B. Suris. “Classification of integrable equations on quad-graphs, the consistency approach.” *Communications in Mathematical Physics* 233 (2003): 513–43.

[3] Adler, V. E., A. I. Bobenko, and Yu. B. Suris. “Classification of integrable discrete equations of octahedron type.” [arXiv:1011.3527](https://arxiv.org/abs/1011.3527) *International Mathematics Research Notices* Vol. 2011, No.083, (68 pp) doi:10.1093/imrn/rnr083.

[4] Atkinson, J., J. Hietarinta, and F. Nijhoff. “Seed and soliton solutions of Adlers lattice equation.” *Journal of Physics A: Mathematical and Theoretical* 40 (2007): F1–F8.

[5] Atkinson, J., and F.W. Nijhoff. “Solutions of Adlers lattice equation associated with 2-cycles of the Bäcklund transformation.” *Journal of Nonlinear Mathematical Physics* 15, Suppl. 3 (2007): 34–42.

[6] Atkinson, J., J. Hietarinta, and F. Nijhoff. “Soliton solutions for Q3.” *Journal of Physics A: Mathematical and Theoretical* 41, no. 14 (2008): 142001.

[7] Atkinson, J., and F. W. Nijhoff. “A constructive approach to the soliton solutions of integrable quadrilateral lattice equations.” *Communications in Mathematical Physics* 299, no. 2 (2010): 283–304.

[8] Bobenko, A. I., and Yu. B. Suris. “Integrable systems on quad-graphs.” *International Mathematics Research Notices* 11 (2002): 573–611.

[9] Bobenko, A. I., and Yu. B. Suris. *Discrete Differential Geometry*. Graduate Studies in Mathematics 98. Providence: American Mathematical Society, 2008.

[10] Date, E., M. Jimbo, and T. Miwa. “Method for generating discrete soliton equations. I-V.” *Journal of the Physical Society of Japan* 51 (1982): 4116–31, ibid. 52 (1983): 388-93, 761–71.

[11] Hietarinta, J. “Boussinesq-like multi-component lattice equations and multi-dimensional consistency.” *Journal of Physics A: Mathematical and Theoretical* 44, no. 16 (2011): 165204 (22 pp).

[12] Hietarinta, J., and D.-J. Zhang, “Multisoliton solutions to the lattice Boussinesq equation.” *Journal of Mathematical Physics* 51 (2010): 033505 (12 pp).

[13] Hietarinta, J., and D.-J. Zhang, “Soliton taxonomy for a modification of the lattice Boussinesq equation.” *SIGMA* 7 2011: no.061 (14 pp).

[14] Hirota, R. “Nonlinear partial difference equations I-III.” *Journal of the Physical Society of Japan* 43 (1977): 1424–33, 2074–89.

[15] Hirota, R. “Discrete analogue of a generalized Toda equation.” *Journal of the Physical Society of Japan* 50 (1981): 3785–91.

[16] Lobb, S., and F. W. Nijhoff. “Lagrangian multiform structure for the lattice Gel’fand-Dikii hierarchy.” *Journal of Physics A: Mathematical and Theoretical* 43 (2010): 072003.

[17] Maruno, K., and K. Kajiwara. “The discrete potential Boussinesq equation and its multisoliton solutions.” *Applicable Analysis* 89, no. 4 (2010): 593–600.

[18] Nijhoff, F. W. “On some “Schwarzian Equations” and their Discrete Analogues.” in: Eds. A.S. Fokas and I.M. Gel’fand, *Algebraic Aspects of Integrable Systems: In memory of Irene Dorfman*, (Birkhäuser Verlag, 1996), pp. 237–60.

[19] Nijhoff, F. W. “Discrete Painlevé Equations and Symmetry Reduction on the Lattice.” in: Eds. A.I. Bobenko and R. Seiler, *Discrete Integrable Geometry and Physics*, (Oxford Univ. Press, 1999), pp 209–34.

[20] Nijhoff, F. W. “Lax pair for the Adler (lattice Krichever-Novikov) system.” *Phys. Lett.* 297A (2002): 49–58.

[21] Nijhoff, F. W. “A higher-rank version of the Q3 equation.” [arXiv:1104.1166](https://arxiv.org/abs/1104.1166) 2011.

[22] Nijhoff, F. W., J. Atkinson, and J. Hietarinta. “Soliton solutions for ABS lattice equations: I Cauchy matrix approach.” *Journal of Physics A: Mathematical and Theoretical* 42, no. 40, (2009): 404005.

[23] Nijhoff, F. W., J. Atkinson. “Elliptic N-soliton solutions of ABS lattice equations.” *International Mathematics Research Notices* Vol. 2010, no. 20, 3837–95.

[24] Nijhoff, F. W., V. G. Papageorgiou, H. W. Capel, and G. R. W. Quispel. “The lattice Gel’fand-Dikii hierarchy.” *Inverse Problems* 8 (1992): 597–621.
[25] Nijhoff, F. W., G. R. W. Quispel, and H. W. Capel. “Direct linearization of nonlinear difference-difference equations.” *Physics Letters* 97A (1983): 125–8.

[26] Nijhoff, F. W., and A. J. Walker. “The discrete and continuous Painlevé VI hierarchy and the Garnier systems.” *Glasgow Mathematical Journal* 43A (2001): 109–23.

[27] Quispel, G. R. W., F. W. Nijhoff, H. W. Capel, and J. van der Linden. “Linear integral equations and nonlinear difference-difference equations.” *Physica* 125A (1984): 344–80.

[28] Tongas A. S., and F. W. Nijhoff. “The Boussinesq integrable system. Compatible lattice and continuum structures.” *Glasgow Mathematical Journal* 47A (2005): 205–19.

[29] Ovsienko, V., R. Schwartz, and S. Tabachnikov. “The Pentagram map: a discrete integrable system.” *Communications in Mathematical Physics* 299, no. 2, (2011): 409–46.

[30] Walker, A. *Similarity Reductions and Integrable Lattice Equations*, PhD Thesis University of Leeds, (2001).

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