Coherent states, $6j$ symbols and properties of the next to leading order asymptotic expansions

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Abstract

We present the first complete derivation of the well-known asymptotic expansion of the $SU(2)$ $6j$ symbol using a coherent state approach, in particular we succeed in computing the determinant of the Hessian matrix. To do so, we smear the coherent states and perform a partial stationary point analysis with respect to the smearing parameters. This allows us to transform the variables from group elements to dihedral angles of a tetrahedron resulting in an effective action, which coincides with the action of first order Regge calculus associated to a tetrahedron. To perform the remaining stationary point analysis, we compute its Hessian matrix and obtain the correct measure factor. Furthermore, we expand the discussion of the asymptotic formula to next to leading order terms, prove some of their properties and derive a recursion relation for the full $6j$ symbol.

1 Introduction

Spin foam models [1–4] are candidate models for quantum gravity invented as a generalization of Feynman diagrams to higher dimensional objects. Their popularity is rooted in the fact that they were well adapted to describe 3D Quantum Gravity theories such as the Ponzano-Regge [5, 6] or the Turaev-Viro model [7]. To examine whether these models are a quantum theory of 4D General Relativity, in particular whether one obtains Gravity in a semi-classical limit is an active area of topical research. One of the strongest positive implications comes from the asymptotic analysis of single simplices in spin foam models: A first attempt to compute the asymptotic expansion of the amplitude associated to a 4-simplex in the Barrett-Crane model [8] can be found in [9, 10]. This was continued for the square of (the Euclidean and Lorentzian) $6j$ and $10j$ symbols in [11], whereas the most recent asymptotic results for modern spin foam models, i.e. the EPRL-model [12] or the FK-model [13], were obtained using a coherent state approach [14–19]:

The basic amplitudes of the spin foam model are their vertex amplitudes ($SU(2)$ $6j$ symbols in the 3D Ponzano Regge model). They are defined in a representation theoretic way and can be constructed from coherent states of the underlying Lie group [20] as a multidimensional integral to which the stationary point approximation is applicable [21]. This method has proven to be very efficient in determining the dominating phase in the asymptotic formula as well as the geometric interpretation of the contributions to the asymptotic expansion in spin foam models [14–19]. In 3D, on the points of stationary phase, $6j$ symbols are geometrically interpreted as tetrahedra, their dominating phase given by the Regge action [22, 23], a discrete version of General Relativity on a triangulation. Similar results were proven by this method for the 4-simplex [9,14–18] in spin foam models. Until today, this is still one of the most promising evidences that spin foam models are viable Quantum Gravity theories.

Despite this success, the coherent state approach fails to produce the full amplitude. It has not yet been possible to compute the so-called measure factor, a proportionality constant (depending on the representation labels) in the asymptotic expansion, which is given by the determinant of the matrix of
second derivatives, i.e. the Hessian matrix, evaluated on the stationary point. This failure even applies to the simplest spin foam model in 3D, the Ponzano-Regge model [6], whose vertex amplitude is the $SU(2)$ $6j$ symbol. To the authors’ best knowledge the Ponzano and Regge formula [5] has not yet been obtained this way; we can only refer to numerical results in [19]. This is particularly troubling for the coherent state approach, since the full asymptotic formula for $SU(2)$ $6j$ symbols introduced in [5] has been proven in many different ways, for example, by geometric quantization [24], Bohr-Sommerfeld approach [25], Euler-MacLaurin approximation [26] or the character integration method [11].

The source of the problem is the size of the Hessian matrix and the lack of immediate geometric formulas for its determinant. For the $6j$ symbol, for example, this matrix is 9 dimensional and its entries are basis dependent. This is a major drawback of the coherent state approach, in particular, since the full expansion is necessary to discuss and examine the properties of spin foams models of Quantum Gravity. To obtain this measure factor and compare it to other approaches [27, 28], the complete asymptotic expansion is indispensable. This is an important open issue for 4D spin foam models.

Our approach to overcome this problem can be seen (as we will show in Appendix H) as a combination of the coherent state approach [14–19] and the propagator kernel method [29]. It inherits nice geometric properties from the coherent state analysis with a similar geometric interpretation of the points of stationary phase. Moreover, the Hessian matrix is always described in terms of geometrical quantities and, most importantly, its determinant can be computed for the $6j$ symbol.

In addition to the computation of the asymptotic formula of the $6j$ symbol [5], our approach allows us to propose a new way to compute higher order corrections to the asymptotic expansion. These corrections have already been discussed in [30,31]: it was conjectured that the asymptotic expansion has an alternating form

$$\{6j\} = A_0 \cos \left( \sum \left( j_i + \frac{1}{2} \right) \theta_i + \frac{\pi}{4} \right) + A_1 \sin \left( \sum \left( j_i + \frac{1}{2} \right) \theta_i + \frac{\pi}{4} \right) + \ldots ,$$

where $A_n$ are consecutive higher order corrections and homogeneous functions in $j + \frac{1}{2}$. Our method allows us to prove this conjecture to any order in the asymptotic expansion.

### 1.1 Coherent states and integration kernels

The coherent state approach is based on the following principle: Invariants (under the action of the group) can be constructed by integration of a tensor product of vectors (living in the tensor product of vector spaces of irreducible representations) over the group, i.e. group averaging. Since the invariant subspace of the tensor product of three representations of $SU(2)$ is one-dimensional, the invariant is uniquely defined up to normalization. However, in order to apply the stationary point analysis the vectors in the construction above cannot be chosen arbitrarily. The choice, from which the method takes its name, is the coherent states class, which consists of eigenvectors of the generators of rotations with highest eigenvalues [20]. Although these states are very effective in obtaining the dominating phase of the amplitude, the associated Hessian matrix turns out to be very complicated. This problem occurs since the action is not purely imaginary, which is also related to the problem of choice of phase for the coherent states which has not yet been fully understood.

Both latter problems disappear if, instead of eigenstates with maximal eigenvalues, we take null eigenvectors for a generator of rotations $L$. Since this vector is trivially invariant with respect to rotations generated by $L$, the phase problem disappears. Similarly the contraction of invariants can be expanded in terms of an action that actually is purely imaginary. There is a trade-off, though: The quantity of stationary points increases and their geometric interpretation becomes more complicated. Moreover, frequently there exist no such eigenvectors for certain representations, (half-integer spins for $SU(2)$) and their tensor product gives thus vanishing invariants.

The solution to these issues comes from the simple observation that null eigenvectors can be obtained by the integration of a coherent state, pointing in direction perpendicular to the axis of $L$, over the rotations generated by $L$. Like that the geometric interpretation usually obtained when using coherent states is restored. Furthermore, if we first perform the partial stationary phase approximation with respect to the additional circle variables, we obtain a purely imaginary action. In the special case of the $6j$ symbol, our construction allows us to write the invariant purely in terms of edge lengths and dihedral angles of a tetrahedron, in particular we perform a variable transformation from group elements to dihedral angles of the tetrahedron. The resulting phase of the integral is given be the first order Regge action [32].
1.2 Relation to discrete Gravity

Regge calculus [22, 23] is a discrete version of General Relativity defined upon a triangulation of the manifold. Influenced by Palatini’s formulation, a first order Regge calculus was derived in [32], in which both edge lengths and dihedral angles are considered as independent variables and their respective equations of motion are first order differential equations. Additional constraints on the angles have to be imposed in order to reobtain their geometric interpretation once the equations of motion for the angles have been solved. Our derivation of the Ponzano-Regge formula shows astonishing similarity to this procedure. Moreover, from our calculation one can deduce a suitable measure for first order (linearized) Quantum Regge calculus, such that the expected Ponzano-Regge factor \( \frac{1}{\sqrt{V}} \) appears, which naturally leads to a triangulation invariant measure [27].

Another version of 4D Regge calculus was explored in [33] with areas of triangles and (a class of) dihedral angles as fundamental and independent variables. Several local constraints guarantee that the geometry of a 4-simplex is uniquely determined. These variables were chosen in the pursuit to better understand the relation between discrete gravity and 4D spin foam models. The latter are based on a similar paradigm as the Ponzano-Regge or the Turaev-Viro models [5, 7] in 3D, yet enhanced by the implementation of the simplicity constraints from the Plebanski formulation of General Relativity [34]. Area-angle variables as a discretization of Plebanski rather than Einstein-Hilbert formulation were conjectured to be more suitable to describe the semi-classical limit of those models.

Although it is known that the asymptotic limit of the amplitude of a 4-simplex for 4D gravity models is proportional to the cosine of the Regge action [15–17], the proportionality factor still remains unknown. We hope that the method presented in this work can help in filling the gap.

1.3 Problem of the next to leading order (NLO) and complete asymptotic expansion

The asymptotic expansion for the \( SU(2) 6j \) symbol, in particular for the next to leading order (NLO), is still a scarcely examined issue, since it is very non-trivial to write the (NLO) contributions in a compact form. Steps forward in this direction can be found in [30, 31, 35], where the latter gives the complete expansion in the isosceles case of the 6j symbol.

The stationary point analysis applied in this work allows for a natural extension in a Feynman diagrammatic approach. From this approach the full expansion can be computed in principle, however in a very lengthy way. We derive a recursion relations of the Ward-Takesaki type, which is surprisingly similar to the one invented in [36,37] however in very different context, that, basically can be used in the asymptotic expansion to derive the NLO in a more concise way. Moreover, we can show explicitly that the consecutive terms in the expansion (1.1) are of the conjectured ‘sin/cos’ form.

1.4 Organization of the paper

This paper is organized as follows: In section 2 we will present our modified coherent states, how to use them to construct invariants and how to contract these invariants to compute spin network amplitudes. The contracted invariants will be used to define an action for the stationary point analysis, which will be examined whether it allows for the same geometric interpretation on its stationary points as other coherent state approaches [14–19]. Its symmetries as well as the group generated by the symmetry transformations will be discussed. Section 3 deals with the partial stationary point analysis with respect to the introduced circle variables. This will allow us to write the amplitude, after a variable transformation, purely in terms of angle variables, which will be identified as exterior dihedral angles of a polyhedron. In section 4 we focus on the example of the 6j symbol. After another variable transformation, we obtain the action of first order Regge calculus and perform the remaining stationary point analysis. Eventually we obtain the asymptotic formula from [5]. In section 5 we prove the conjecture from [30, 31] that the full asymptotic expansion is of alternating form (1.1) and derive the recursion relations for the full 6j symbol. We conclude with a discussion of the results and an outlook in section 6.

We would like to point out that several results of this paper have been obtained by tedious calculations which we did not include in its main part to improve readability. Interested readers are welcome to look them up in the appendices.

\(^1\)The vanishing of the angle Gram matrix on flat spacetime implies the existence of the flat \( n \)-simplex with the given angles.
2 Modified coherent states, spin-network evaluations and symmetries

In this section we are going to present the modified coherent states, how to construct the spin-network evaluation from them and that they allow for the same geometric interpretation in the stationary point analysis as similar coherent state approaches. Furthermore the symmetries of the action will be investigated.

Consider a three-valent spin network, i.e. a graph with three-valent nodes carrying SU(2) intertwiners and edges carrying irreducible representations of SU(2). For each edge of the spin network we introduce a (fiducial) orientation such that each node of the network can be denoted as the ‘source’ s(e) or the ‘target’ t(e) of the edge e. Later in this work we intend to give a geometrical meaning to the spin network, in terms of polyhedra, triangles, etc. so we denote the set of nodes by F and the set of edges by E, which will become the set of triangles / faces and set of edges of the triangulation respectively. This dual identification is not always possible but we restrict our attention to the case of planar (spherical) graphs, where such notions are natural.

2.1 Intertwiners from modified coherent states

Intertwiners are invariant vectors (with respect to the action of the group) in the tensor product of vector spaces associated to irreducible representations of that group. In the case of 3 irreducible representations of SU(2) the space of invariants is one dimensional and, moreover, there is a unique choice for the invariant for a given cyclic order of representations [38,39].

Suppose $\xi \in V_{j_1} \otimes \cdots \otimes V_{j_n}$ is a vector in the tensor product of vector spaces of representations, then

$$\int_{SU(2)} dU U \xi$$  \hspace{1cm} (2.1)

is invariant under the action of SU(2). If $\xi$ is chosen in a clever way, such an invariant is non-trivial. In the following we present a choice which has the advantage that the method of stationary phase can be directly applied.

For every face $f$, which is bounded by three edges, we choose a cyclic order of these edges $(j_{fe_1}, j_{fe_2}, j_{fe_3})$, labelled by the carried representations. These choices influence the orientation of the spin network [38,39] and are used to define and determine the sign of its amplitude, see also appendix A. We introduce the following intertwiners for every face $f$:

$$C_f = \int_{SU(2)} dU_f U_f \frac{1}{2\pi} \prod_j \phi_{ji}^{j_{fe_1}} f_f \left( \{\phi_{fe}\}_{e \in F} \right) \prod_{e \in F} \left( O_{\phi_{fe}} |^{1/2} \right)^{2j_e} ,$$  \hspace{1cm} (2.2)

where $f_f$ is a function of the three angles $\phi_{fe}$, $e \subset f$, $|^{1/2}$ is the basic state of the fundamental representation and $O_{\phi}$ is a rotation matrix on $\mathbb{R}^2$:

$$O_{\phi} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} .$$  \hspace{1cm} (2.3)

As mentioned above, (2.2) is invariant under the action of SU(2).

Before moving on, we would like to outline the key differences between the approach described above and the usual coherent state approach [15–17,19].

- Coherent states of SU(2) are labelled by vectors in $\mathbb{R}^3$. On the stationary point with satisfied reality conditions, one obtains the geometric interpretation that for every face these three vectors form the edge vectors of a triangle. Later on we will prove the same geometric interpretation for the invariant $C_f$.

- Furthermore we smear the coherent state by a rotation, which is the key ingredient of our approach. In addition to the stationary point analysis with respect to the $\{U_f\}$, we will also perform a stationary point analysis for the smearing angles $\{\phi_{fe}\}$. Clearly, this will result in more stationary points contributing to the final amplitude. To suppress their contributions, we introduce modifiers $f_f$ which will be described in the next section.
2.1.1 Prescription of the modifiers

In order to make (2.2) complete, we have to describe the function \( f \).

For every face \( f \) we choose three vectors \( v_e \) \((e \subset f)\) on \( \mathbb{R}^2 \) with norms \( j_e \) such that \( \sum_{e \subset f} v_e = 0 \), i.e. they form a triangle with edge lengths \( j_e \). The vectors are ordered anti-clockwise, their choice is unique up to Euclidean transformations, i.e. rotations and translations.

Let us denote the edges (in cyclic order) by 1, 2, 3. The angles (counted clockwise) between the vectors \( v_k \) and \( v_j \) are denoted by \( 2(\psi_{kj} - \pi) \), where \( 2(\psi_{kj} - \pi) \) is the \( SO(3) \) angle taking values in \((0, \pi)\) for \((k,j) \in \{(2,1), (3,2), (1,3)\}\). Due to the ordering, the \( SU(2) \) angles \( \psi_{21}, \psi_{32} \) and \( \psi_{13} \) are positive and smaller than \( 2\pi \), in fact, one can also check that \( \psi_{kj} \) is in \((\pi, 2\pi)\). This choice contributes an overall sign to the invariant, to be more precise, there are two different choices of cyclic order giving two invariants that may differ by a sign factor. This will be discussed in more detail in appendix A.2.1. In particular we compare them to the intertwiner introduced in [38,39]. The angles \( \psi_{kj} \) satisfy the relation

\[
\psi_{21} + \psi_{32} + \psi_{13} = 4\pi.
\]

We introduce a function \( f(x \mod 2\pi, y \mod 2\pi) \) such that

- it is equal to 1 in the neighbourhood of \( x = \psi_{21}, y = \psi_{32} \),
- it is equal to zero in the neighbourhood of points

\[
(x, y) = \pm(\psi_{21} + \pi, \psi_{32}), \pm(\psi_{21}, \psi_{32} + \pi), \pm(\psi_{21} + \pi, \psi_{32} + \pi), (-\psi_{21}, -\psi_{32}).
\]

Hence, we define

\[
f_f(\phi_{f1}, \phi_{f2}, \phi_{f3}) = f(\phi_{f2} - \phi_{f1}, \phi_{f3} - \phi_{f2}).
\]

2.1.2 The spin network evaluation

Given the definition of invariants in (2.2) it is straightforward to define the evaluation of a given spin network: The intertwiners are contracted with each other according to the combinatorics of the network. The resulting amplitude has to be normalized, i.e. divided by the product of norms of our intertwiners, see section 3.4. It is, however, not sufficient in order to agree with the canonical definition [38, 39]. The remaining sign ambiguity will be resolved in Appendix A.

As in the standard coherent state approach the amplitude (contraction of intertwiners) then reads:

\[
(-1)^s \int \prod_{f \in F} dU_f \prod_{e \subset f} \frac{d\phi_e}{2\pi} \prod_{f} f_f(\{\phi_e\}_{e \subset f}) \prod_{e \in E} \epsilon\left(U_{e(e)}O_{\phi_{e(e)}}|\frac{1}{2}\rangle, U_{e(e)}O_{\phi_{e(e)}}|\frac{1}{2}\rangle\right)^{2j_e},
\]
where $s$ is the sign factor as prescribed in [38, 39] (see Appendix A), and $\epsilon(\cdot, \cdot)$ is an invariant bilinear form defined by

$$
\epsilon(|1/2\rangle, |1/2\rangle) = \epsilon(| - 1/2\rangle, | - 1/2\rangle) = 0, \quad \epsilon(|1/2\rangle, | - 1/2\rangle) = -\epsilon(| - 1/2\rangle, |1/2\rangle) = 1.
$$

(2.8)

The choice of the orientation of edges, faces and the sign factor prescription will be described in appendix A.1. To perform the stationary point analysis we rewrite (part of) the integral kernel as an exponential function and define the 'action' $S$. From (2.7) one can deduce that

$$
S = \sum_e S_e,
$$

(2.9)

where the action $S_e$ (labelled by the edge $e$) is given by:

$$
S_e = 2j_e \ln \epsilon \left( U_{s(e)} O_{\phi_{s(e)}}, |1/2\rangle, U_{t(e)} O_{\phi_{t(e)}}, |1/2\rangle \right).
$$

(2.10)

### 2.2 The action

In order to examine the geometric meaning of the action on its points of stationary phase, let us introduce the following geometric quantities. For each face $f \in F$ we introduce vectors $n_f$ (as traceless Hermitian matrices, which can be naturally identified with vectors in $\mathbb{R}^3$) defined by:

$$
n_f = U_f H U_f^{-1},
$$

(2.11)

where

$$
H = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.
$$

(2.12)

For each pair $\{f, e\}$ with $e \subset f$, we define vectors $B_{fe}$ (also as traceless matrices):

$$
B_{fe} = j_e (2U_f O_{\phi_{fe}} |1/2\rangle \langle 1/2| O_{\phi_{fe}}^{-1} U_f^{-1} - I)
= U_f O_{\phi_{fe}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} O_{\phi_{fe}}^{-1} U_f^{-1}.
$$

(2.13)

Note that the length of $B_{fe}$ is equal to $j_e$.

We can already deduce that $\Re S \leq 0$. The stationary point analysis contains the conditions $\partial S = 0$ and $\Re S = 0$. These are as follows:

- The reality condition is satisfied if and only if

$$
U_{s(e)} O_{\phi_{s(e)}}, |1/2\rangle \perp U_{t(e)} O_{\phi_{t(e)}}, |1/2\rangle,
$$

(2.14)

where $\perp$ means perpendicular in the $SU(2)$ invariant scalar product. This is equivalent to $B_{s(\cdot)e} = -B_{t(\cdot)e}$.

- Using both the reality condition and the definition of $B_{fe}$, we obtain from the variation of $S$ with respect to $U_f$:

$$
X \frac{\partial S_e}{\partial U_f} = \begin{cases} \text{Tr} \ X B_{fe} & e \subset f \\ 0 & e \not\subset f \end{cases},
$$

(2.15)

where $X$ is a generator of the Lie algebra. Hence the action is stationary with respect to $U_f$ if:

$$
\sum_{e \subset f} B_{fe} = 0.
$$

(2.16)

- Similarly we obtain for the variation of $S$ with respect to $\phi_{fe}$ (again using the reality condition):

$$
\frac{\partial S_e}{\partial \phi_{fe}} = \begin{cases} \text{Tr} \ n_f B_{fe} & e = e' \subset f \\ 0 & \text{otherwise} \end{cases}.
$$

(2.17)

So the condition from variation with respect to $\phi$ is

$$
\forall e \subset f \ n_f \perp B_{fe}.
$$

(2.18)

Before we discuss the geometric meaning of the just derived conditions, we first have to examine the symmetries of the action to determine the amount of stationary points and their relations.
2.3 Symmetry transformations of the action

There exist several variable transformations that only change $e^S$ by a sign such that a stationary point is transformed into another stationary point. Some of the transformations below are continuous so the stationary points form submanifolds of orbits under the action of these symmetries. We will explain the geometric interpretation of these orbits in section 2.5, and show that these orbits are isolated for many spin networks, e.g. the $6j$ symbol.

The above mentioned transformations are as follows:

- **$u$-symmetry:**
  \[ \forall u \in SU(2), \quad \forall f \in F, \quad U_f \rightarrow uU_f \] (2.19)
  applied to all $U_f$ simultaneously preserves $e^S$. This is the only symmetry which has to be applied to all group elements simultaneously showing that one of the $SU(2)$ integrations in (2.7) is redundant (gauge).

- **$o_f$-rotation:**
  For a chosen face $f$ and $\phi$,
  \[ U_f \rightarrow U_fO_\phi, \quad \forall e \subset f \phi_{fe} \rightarrow \phi_{fe} - \phi \] (2.20)
  preserves $e^S$, in fact, each $e^{S_e}$ is preserved.

- **$-u_f$-symmetry:**
  For any chosen face $f \in F$,
  \[ U_f \rightarrow (-1)U_f \] (2.21)
  preserves $e^S$ because for every face $\sum_{e \subset f} j_e$ is an integer.

- **$r_f$-reversal transformation:**
  For any chosen face $f$,
  \[ U_f \rightarrow U_f \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \forall e \subset f \phi_{fe} \rightarrow -\phi_{fe}. \] (2.22)

Because
\[ D^{-1}O_\phi D = O_{-\phi}, \quad D^{1/2} = i^{1/2}, \] (2.23)
$e^S$ is multiplied by
\[ i^{\sum_{e \subset f} j_e} = (-1)^{\sum_{e \subset f} j_e}. \] (2.24)

Let us notice that $2j_e \in \mathbb{Z}$ and $\sum_{e \subset f} j_e$ is an integer,
• $-o_{fe}$ transformation:
  For any chosen pair $e \subset f$
  \[
  \phi_{fe} \rightarrow \phi_{fe} + \pi \tag{2.25}
  \]
  This multiplies the integrated term by $(-1)^{2j_e}$.

  Note that the transformations $o_f$, $-u_f$ $-o_{fe}$, $r_f$ are restricted to variables associated to one face. They transform the functions $f_f$ as follows:

  • $o_f$ shifts all angles $\phi_{fe}$ on $f$ by an angle $\phi$:
    \[
    f_f'(\{\phi_{fe}\}) = f_f(\{\phi_{fe} + \phi\}) = f_f(\{\phi_{fe}\}) \tag{2.26}
    \]
    since $f_f$ only depends on differences of angles.

  • $-o_{fe}$
    \[
    f_f'(\{\phi_{fe}\}) = f_f(\{\phi_{fe} + \delta_{ee'}\pi\}) \tag{2.27}
    \]

  • $r_f$
    \[
    f_f'(\{\phi_{fe}\}) = f_f(\{-\phi_{fe}\}) \tag{2.28}
    \]

  To sum up, the functions $f_f$ are preserved by $u_-$, $-u_f$- and $o_f$-transformations, since the first two do not affect the angles $\phi$ and the last one translates all angles by a constant.

  In addition to that, let us also define an additional transformation $c$, which we call parity transformation:
  \[
  \forall f : \quad U_f \rightarrow U_f \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right) \tag{2.29}
  \]
  It transforms the integral into its complex conjugate due to the fact that
  \[
  \bar{U} = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right)^{-1} U \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right) \tag{2.30}
  \]
  and the $f_f$, the matrix $O_{\phi}$ and the vectors $|\pm 1/2\rangle$ are real.

  In the next section we will examine which group is generated by the transformations, i.e. the symmetry group of the action.

2.4 Groups generated by symmetry transformations

The transformations described in 2.3 generate a group $\hat{G}$ with the following relations:

\[
\begin{align*}
  u_{(-1)} &= \prod_f (-u_f), \\
  \forall f, \quad r_f^2 &= (-u_f), \quad (-u_f)^2 = 1, \quad o_f(2\pi) = 1, \\
  \forall e \subset f, \quad (-o_{fe})^2 &= 1, \\
  \forall f, \quad o_f(\pi) \prod_{e \subset f} (-o_{fe}) &= 1. \tag{2.31}
\end{align*}
\]

and all its elements commute besides $u$ (that form $SU(2)$) and

\[
\forall f, \quad r_f o_f(\alpha) r_f^{-1} = o_f(-\alpha). \tag{2.32}
\]

The group generated by all transformations except $u$ is denoted by $G$.

In $\hat{G}$ (resp. $G$), there is a normal subgroup generated by the transformations $u$, $o_f$, $-u_f$ (resp. $o_f$, $-u_f$), which preserves the modifiers $f_f$. We denote these subgroups by $\hat{H}$ (and $H$ respectively); their quotient groups are given by

\[
K = \hat{G}/\hat{H} = G/H. \tag{2.33}
\]

This is an Abelian group generated by

\[
\forall e \subset f, \quad [r_f], [-o_{fe}] \tag{2.34}
\]
with relations
\[ \forall f \prod_{e \subset f} [-\alpha_{fe}] = 1, \ |r_f|^2 = |\alpha_{fe}|^2 = 1, \]  
which show that \( K \) is isomorphic to \( \mathbb{Z}_2^{3|F|} \).

In the next two sections, we will discuss the geometric interpretation of the points of stationary phase.

2.5 Geometric lemma

Our goal in this section is to describe the geometric interpretation of the stationary point orbits introduced in section 2.2. In particular, we will show how these points are related to the standard stationary point interpretation in the coherent state method.

**Lemma 1.** For every set of vectors \( B_{fe} \) of length \( j_e \) satisfying
\[ \forall e B_{s(e)e} = -B_{t(e)e} \, , \]
\[ \forall f \sum_{e \subset f} B_{fe} = 0 \, , \]  
there exist \( \phi_{fe} \) and \( U_f \) being a point of stationary phase with vectors \( B_{fe} \). Moreover, all these points are related via \( G \) transformations.

**Proof.** For every \( f \) we can choose the unit vector \( n_f \) perpendicular to all \( B_{fe} \) (for all \( e \subset f \)). Such a normal is only determined up to a sign. Let us choose \( U_f \) such that
\[ n_f = U_f H U_f^{-1} \, . \]  
Such a choice always exists, but it is not unique. \( U_f \) is only determined up to the transformation
\[ U_f \to U_f D \phi \]  
since \( D \), defined in (2.22), stabilizes \( H \) up to a sign:
\[ DHD^{-1} = -H \, . \]  
This is called the \( D_\infty \) group.

The vectors \( U_f^{-1}B_{fe}U_f \) are orthogonal to \( H \). The operators \( U_f^{-1}B_{fe}U_f \) are thus real and we can choose their eigenvectors with positive eigenvalues as
\[ \begin{pmatrix} \cos \phi_{fe} \\ \sin \phi_{fe} \end{pmatrix} \, . \]  
Hence \( \phi_{fe} \) is fixed (up to \( \pi \)).

It is straightforward to check that this construction gives a stationary point, in fact, each stationary point with vectors \( B_{fe} \) must be constructed in this way. The ambiguities in the choices above are all related by \( o_f \)-, \( -u_f \)-, \( -\alpha_f \)- and \( r_f \)-transformations, i.e. \( G \)-transformations. \( \square \)

For every face \( f \) on the stationary point \( \sum_{e \subset f} B_{fe} = 0 \) and given the definition of \( f_f \) in section 2.1.1, there is a unique choice (up to \( o_f \) transformations) of the stationary point angles \( \phi_{fe} \) such that \( f_f \) is nonzero. In the neighbourhood of those stationary point \( f_f = 1 \), whereas around all remaining ones at least one of the functions \( f_f \) is zero:

**Lemma 2.** For given vectors \( B_{fe} \) satisfying (2.36), there exists only one orbit (orbit of the action of the group \( \tilde{H} \)) of stationary points of the action, such that
\[ \prod f_f(\{\phi_{fe}\}) \neq 0 \, , \]  
and in the neighbourhood of this orbit
\[ \prod f_f(\{\phi_{fe}\}) = 1 \, . \]
Note that the normals to the faces change sign under \( r_f \) transformations:
\[
n_f \rightarrow -n_f \quad .
\] (2.43)

Under the \( c \)-transformation, the \( B_{fe} \) are inverted, but the normals to the faces are not affected, i.e. they behave as pseudovectors.
\[
B_{fe} \rightarrow -B_{fe} \quad ,
\]
\[
n_f \rightarrow n_f \quad .
\] (2.44)

In the next section we will specify the definition of the normals \( n_f \).

## 2.6 Normal vectors to the faces

We will now give a precise geometric definition of \( n_f \) (normal to the face). To simplify notation we will omit the subscript \( \phi \) in \( O_{\phi_{fe}} \). Note that
\[
n_f = U_f O_{fe} H O_{fe}^{-1} U_f^{-1}
\]
for any edge \( e \subset f \), since \( O_{fe} \) and \( H \) commute.

Take two consecutive edges \( e_1, e_2 \subset f \) and their respective edge vectors \( B_{fe_1} \):
\[
B_{fe_1} = U_f O_{fe_1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} O_{fe_1}^{-1} U_f^{-1} \quad ,
\] (2.46)
\[
B_{fe_2} = U_f O_{fe_2} (O_{fe_1}^{-1} O_{fe_2}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (O_{fe_1}^{-1} O_{fe_2})^{-1} O_{fe_1}^{-1} U_f^{-1} \quad .
\] (2.47)

Rotating all three vectors by \( U_f O_{fe_1} \) one obtains (the rotated vectors are denoted by \( B'_{fe_1}, n'_f \)):
\[
n'_f = H, \quad B'_{fe_1} = j_{e_1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_{fe_2} = O_{fe_1}^{-1} O_{fe_2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (O_{fe_1}^{-1} O_{fe_2})^{-1} .
\] (2.48)

For a stationary point with non-vanishing modifier \( f_f \), \( O_{fe_1}^{-1} O_{fe_2} \) describes the rotation by the \( SO(3) \) angle \( 0 < 2\psi_{12} - \pi < \pi \). We thus conclude:
\[
n_f \cdot (B_{fe_1} \times B_{fe_2}) = n'_f \cdot (B'_{fe_1} \times B'_{fe_2}) > 0 \quad ,
\] (2.49)
where we regard \( n_f \) and \( B_{fe} \) as vectors using the natural identification of hermitian matrices with \( \mathbb{R}^3 \) (tracial scalar product). Condition (2.49) fixes the sign of \( n_f \) and also completes the geometric interpretation of the points of stationary phase.

## 2.7 Interpretation of planar (spherical) spin-networks as polyhedra

In the last section we obtained an interpretation of the stationary points in terms of a set of vectors \( B_{fe} \) satisfying closure conditions for every face \( f \)
\[
\sum_{e} B_{fe} = 0 \quad .
\] (2.50)

However, these conditions do not specify a unique reconstruction of the according surface dual to the spin network. In fact, already each triangle allows for two different configurations of \( B_{fe} \) vectors. Therefore, we will here describe a method to reconstruct the surface from \( B_{fe} \) vectors for the spherical case:

Let us draw the graph on the sphere (on the plane) as described in appendix A.1. From the possible ways of drawing it, which in the case of 2-edge irreducible spin networks is in one-to-one correspondence with the orientation of the spin network, we have to choose one. In the case of 2-edge irreducible graphs the polyhedra obtained from different choices only differ by orientation. In addition to nodes and edges, there is also a natural notion of two-cells. The latter are defined as areas bounded by loops of edges. We are mainly interested in the dual picture that in this case is a triangulation of the sphere. Thus there is a unique identification of the vertices in the dual picture. A cyclic ordering of the edges for each \( f \) is inherited from the orientation of the sphere.
In the following, we will construct an immersion (not an embedding) of this triangulation of the sphere into \( \mathbb{R}^3 \), such that every edge \( e \) is given by \( B_{f(e)}(e) \) (with the right orientation).

Let us choose one vertex \( v_0 \). Every other vertex \( v' \) can be connected to \( v_0 \) by a path

\[
v_0, e_0, v_1, e_1, \ldots, v'.
\]

(2.51)

Every edge \( e_i \) in the sequence belongs to two faces. Exactly one of these faces is such that \( v_i, v_{i+1} \) are the consecutive vertices w.r.t the cyclic order of the face. We denote this face by \( f_i \) (see figure 3). We introduce the vector

\[
\tilde{v}' = \sum_i B_{f_i} e_i.
\]

(2.52)

One can prove that this vector does not depend on the chosen path. To see this, let us consider a basic move that consists of replacing \( v_i, e_i, v_{i+1} \) by \( v_i, e, v, e', v_{i+1} \) where all three vertices belong to the same face \( f \). Using the property

\[
\sum_{e \subset f} B_{fe} = 0 \quad (2.53)
\]

and the proper orientation, one can show that the vector \( \tilde{v}' \) is invariant with respect to this move. In fact any two paths can be transformed into one another by a sequence of these basic moves (or their inverses) due to the fact that the graph is spherical. A different choice of \( v_0 \) gives a translated surface. It is straightforward to check that

\[
\tilde{v}_b - \tilde{v}_a = B_{fe},
\]

(2.54)

where \( v_a \) and \( v_b \) are vertices joint by the edge \( e \) and \( f \) is the face such that \((v_a, v_b)\) is the pair of consecutive vertices in the cyclic order of \( f \).

Let us notice that from three vectors \( B_{fe} \) satisfying the closure condition one can form a triangle in two ways (see figure 4), but only that one depicted on the left appears in the reconstruction discussed here. Moreover the direction of the normal to the face coincides with the orientation inherited by the face from the cyclic order of its edges.

For non-planar graphs, in general, we can only reconstruct the universal cover of the surface.

Before we continue with the stationary point analysis for the angles \( \phi_{fe} \) in the next section, let us briefly summarize the results of section 2: We have introduced a class of modified coherent states for irreducible representations of \( SU(2) \), which contain an additional smearing parameter, and presented how to construct invariants from them. From the contraction of these invariants (according to the spin network) an effective action has been derived, whose points of stationary phase allow for the same geometrical interpretation as the standard coherent states [19, 20]. The amount of stationary points is significantly increased by the smearing parameters, yet they are all related by symmetry transformations of the action; a certain set of them can be suppressed by the prescribed modifiers. Eventually, we have depicted a way to reconstruct a triangulation from planar spin networks.
3 Variable transformation and final form of the integral

In this section we focus on the stationary point analysis with respect to the angles \( \phi_{fe} \), which is the key modification in comparison to previously used coherent state approaches, see also section 2.1. This analysis allows us to obtain an effective action for \( S_e \) associated to the edge \( e \) in terms of a phase, which we will identify as the angle between the normals of the faces sharing the edge \( e \). Furthermore we are able to expand the effective action for \( S_e \) in orders of \( \frac{1}{j} \) and initiate the discussion of next-to-leading order contributions.

3.1 Partial integration over \( \phi \) and the new action

Suppose that we have a non-degenerate configuration, i.e.

\[ \forall_e n_{s(e)} \cdot n_{t(e)} \neq \pm 1 \]  

(3.1)

Then the partial stationary point analysis with respect to all \( \phi_{fe} \) can be performed. Its result will be the sum over the contribution from all stationary points with respect to \( \phi_{fe} \) for a given configuration of \( B_{fe} \) vectors, but for fixed \( U_f \) (so also fixed \( n_f \)).

3.1.1 Stationary points for \( S_e \)

In this section we will explain the contribution to the integral from the stationary point of the action \( S_e \) with respect to \( \phi_{s(e)e}, \phi_{t(e)e} \). The \( f_s(e), f_t(e) \) terms can be ignored, since they are equal to 1 around the stationary point.

We can separately consider terms corresponding to each edge

\[ \frac{1}{4\pi^2} \int d\phi_{s(e)e} d\phi_{t(e)e} \epsilon (U_{s(e)e} O_{s(e)e}|1/2), U_{t(e)e} O_{t(e)e}|1/2) \]  

(3.2)

and perform the stationary point analysis that gives the asymptotic result of the integration over \( \phi_{s(e)e}, \phi_{t(e)e} \). The stationary point with respect to \( \phi_{t(e)e} \) and \( \phi_{s(e)e} \) is given by the conditions

\[ U_{s(e)e} O_{s(e)e}|1/2 \perp U_{t(e)e} O_{t(e)e}|1/2 \]  

(3.3)

which is equivalent to

\[ U = O_{s(e)e}^{-1} U_{s(e)e}^{-1} U_{t(e)e} O_{t(e)e} = (-1)^{s} e^{-i\tilde{\theta}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]  

(3.4)

where \( \tilde{\theta} \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) and \( \tilde{s} \in \{0, 1\} \) are uniquely determined by this equation. In section 3.1.2 we will show that \( 2\tilde{\theta} \) can be interpreted as the angle enclosed by the normal vectors \( n_{s(e)} \) and \( n_{t(e)} \) (w.r.t. the axis \( B_{t(e)e} \)). Hence, \( S_e \) on the stationary point is of the following form:

\[ S_e = 2j_e \ln \epsilon(\cdots) = 2j_e \tilde{\theta} + i 2j_e \pi \tilde{s} \]  

(3.5)
As already discussed in section 2.5, each stationary point is characterized by the existence of $B_{t(c)}e = -B_{s(c)}e$ orthogonal to both $n_{s(e)}$ and $n_{t(e)}$ (see also stationary point conditions in section 2.2). There exist two such configurations that differ by a sign of $B_{t(c)}e$.

For every configuration one has 4 stationary points that can be obtained from one another by $-o_{t(c)}$- and $-o_{t(c)}$-transformations. In case $j_c$ is an integer the contributions from the two stationary points are equal, see also section 2.3.

Contributions from $B_{t(e)}$ configurations with opposite signs are related by complex conjugation.

### 3.1.2 Geometric interpretation of the angle $\bar{\theta}$

The missing piece of the description above is the exact value of the angle $\bar{\theta}$. Here we will provide a geometric interpretation of this angle and its relation to the angle between faces. Let us recall:

\begin{align}
B_{s(c)}e &= j_{c}U_{s(c)}O_{s(c)}e \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) O_{s(e)}^{-1} U_{s(e)}^{-1} \\
n_{s(c)} &= U_{s(c)}O_{s(e)}e HO_{s(e)}^{-1} U_{s(e)}^{-1} \\
n_{t(e)} &= U_{t(e)}O_{t(e)}e HO_{t(e)}^{-1} U_{t(e)}^{-1} = e^{i\bar{\theta} O_{t(c)}e} n_{s(c)}e^{-i\bar{\theta} O_{t(c)}e}
\end{align}

The angle $2\bar{\theta}$ is the angle by which one needs to rotate $n_{s(e)}$ around the axis $B_{t(e)}e$ to obtain $n_{t(e)}$. We will denote this $SO(3)$ angle by

\[
\theta = 2\bar{\theta}, \quad \theta \in (-\pi, \pi).
\]

This remaining ambiguity of the sign factor $\bar{s}$ will be resolved in appendix A.

### 3.2 Partial integration over $\phi$

We introduce new variables

\[
\phi_{1} = \phi_{s(e)} - \phi_{s(c)}^{0}, \quad \phi_{2} = \phi_{t(c)} - \phi_{t(e)}^{0},
\]

where $\phi_{s(c)}^{0}$ and $\phi_{t(e)}^{0}$ denote the stationary points. Then using (3.4), we can write the action as:

\[
\frac{1}{4\pi^{2}} \int d\phi_{1} d\phi_{2} (-1)^{2j_{c}\bar{s}} \left( e^{i\bar{\theta} \cos \phi_{1} \cos \phi_{2} + e^{-i\bar{\theta} \sin \phi_{1} \sin \phi_{2}} \right)^{2j_{c}},
\]

where we integrate over $\phi_{i}$. By splitting the terms in the bracket in real and imaginary part, we obtain:

\[
\frac{\cos \bar{\theta} \left( \cos \phi_{1} \cos \phi_{2} + \sin \phi_{1} \sin \phi_{2} \right) + i \sin \bar{\theta} \left( \cos \phi_{1} \cos \phi_{2} - \sin \phi_{1} \sin \phi_{2} \right)}{\cos(\phi_{1} - \phi_{2}) \cos(\phi_{1} + \phi_{2})}.
\]

We define new variables

\[
\alpha := \phi_{1} - \phi_{2}, \quad \beta := \phi_{1} + \phi_{2},
\]

and the Jacobian for this transformation is given by:

\[
\left| \frac{\partial \alpha \partial \beta}{\partial \phi_{1} \partial \phi_{2}} \right| = \left| \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right| = 2.
\]

Hence equation (3.11) becomes:

\[
\frac{1}{8\pi^{2}} \int d\alpha d\beta (-1)^{2j_{c}\bar{s}} \left( \cos \bar{\theta} \cos \alpha + i \sin \bar{\theta} \cos \beta \right)^{2j_{c}} = \frac{1}{8\pi^{2}} \int d\alpha d\beta (-1)^{2j_{c}\bar{s}} \exp \left\{ 2j_{c} \ln \left( \cos \bar{\theta} \cos \alpha + i \sin \bar{\theta} \cos \beta \right) \right\},
\]

where $S_{c} = S_{e} + i 2j_{c}\pi\bar{s}$.
3.2.1 Expansion around stationary points

Given the definitions from the previous section, we compute the expansion of the following expression:

\[ \frac{1}{8\pi^2} \int d\alpha d\beta (-1)^{2j_e} e^{S'_e}. \]  

(3.16)

The stationary point is given by \( \alpha = \beta = 0 \), which corresponds to \( \phi_i = 0 \), i.e. \( \phi_{\nu(e)} = \phi_{\nu(e)}^0 \), \( \phi_{\mu(e)} = \phi_{\mu(e)}^0 \).

In this point the action associated to the edge \( e \) becomes:

\[ S'_e = 2j_e \ln \left( e^{i\bar{\theta}} \right) = i2j_e \tilde{\theta}_e. \]  

(3.17)

In order to compute the first order contribution, one has to consider the matrix of second derivatives (evaluated on the point of stationary phase):

\[ \frac{\partial^2 S'_e}{\partial \alpha^2} = -2j_e \cos \bar{\theta} \cos \alpha, \]  

(3.18)

\[ \frac{\partial^2 S'_e}{\partial \alpha \partial \beta} = 0 = \frac{\partial^2 S'_e}{\partial \beta \partial \alpha}, \]  

(3.19)

\[ \frac{\partial^2 S'_e}{\partial \beta^2} = -2ij_e \frac{\sin \bar{\theta} \cos \beta}{\cos \bar{\theta} \cos \alpha + i \sin \bar{\theta} \cos \beta}. \]  

(3.20)

Around the stationary point the action can be expanded (up to second order in the variables \( \alpha, \beta \)):

\[ S'_e = i2j_e \tilde{\theta} + \frac{1}{2} \left( \begin{array}{cc} -2j_e \cos \bar{\theta} e^{-i\bar{\theta}} & 0 \\ 0 & -2ij_e \sin \bar{\theta} e^{-i\bar{\theta}} \end{array} \right) \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) + \cdots. \]  

(3.21)

In order to correctly perform the stationary phase approximation, it is indispensable to state the right branch of the square root, here for \( \tilde{\theta} \in (-\frac{\pi}{2}, \frac{\pi}{2}) \):

\[ \sqrt{\cos \bar{\theta} e^{-i\bar{\theta}}} = \sqrt{|\cos \bar{\theta}| e^{-i\frac{\pi}{2}} \tilde{\theta}} \bigg| \begin{array}{c} e^{-i\frac{\pi}{2}} \tilde{\theta} \in (-\frac{\pi}{2}, 0) \\ e^{i\frac{\pi}{2}} \tilde{\theta} \in (0, \frac{\pi}{2}) \end{array} \bigg. \]  

(3.22)

Let us notice that

\[ \text{sgn} \sin \theta = \text{sgn} \sin \tilde{\theta} \quad \text{for} \quad \tilde{\theta} \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right). \]  

(3.23)

Hence, the leading order contribution from the stationary point is:

\[ \frac{1}{8\pi^2} \frac{2\pi(1)^{2j_e}}{\sqrt{2j_e}} e^{i2\tilde{\theta}(j_e + \frac{1}{2})} \left( 1 + O(\frac{1}{j_e}) \right). \]  

(3.24)

In the next section we will show an improvement of this result.

3.2.2 The total expansion of the edge integral

Let us introduce a number (see appendix H for a motivation of its origin)

\[ C_j = \frac{1}{4} \Gamma(2j + 1) \left( \frac{\pi}{2} \right)^{j_e}. \]  

(3.25)

We can multiply (3.24) by \( \frac{C_j}{C_j} = 1 \) and use the expansion \( \frac{1}{4} \Gamma(2j + 1) = \sqrt{\pi j_e} \left( 1 + O(\frac{1}{j_e}) \right) \) derived in appendix D.2.1 to write the result as

\[ C_j \left( \frac{(-1)^{j_e}}{4\sqrt{2\pi j_e}} e^{i(\bar{\theta}(j_e + \frac{1}{2}) - \frac{\pi}{2} \text{sgn}(\sin \theta))} \left( 1 + O(\frac{1}{j_e}) \right) \right). \]  

(3.26)
By \( s_c \) we denoted the sign factor

\[
s_c = \begin{cases} 
0 & \text{if } j_c \text{ integer}, \\
0 & \text{if } j_c \text{ half-integer and } \bar{s} = 0 \\
1 & \text{if } j_c \text{ half-integer and } \bar{s} = 1 
\end{cases}
\]  

(3.27)

We will determine the sign \( s_c \) in appendix A.

We introduce new ‘length’ parameters

\[
l_c := j_c + \frac{1}{2},
\]

(3.28)

and using the fact that

\[
\frac{(-1)^s}{4\sqrt{2\pi l_c |\sin \theta|}} = \frac{(-1)^s}{4\sqrt{2\pi j_c |\sin \theta|}} \left( 1 + O(j^{-1}) \right)
\]

we can express (3.26) in terms of \( l_c \).

Before we move on, we would like to present a first glimpse at the next-to-leading order contribution: As it will be shown in section 5.2.4 by application of the stationary point analysis (3.26) and the recursion relation (5.29), the contribution (including next-to-leading order (NLO)) from the integral of \( e^{S_c} \) over \( \phi s(e), \phi t(e) \) is given by

\[
C_{j_c} \frac{(-1)^s}{4\sqrt{2\pi l_c |\sin \theta|}} e^{i(j_c \theta - \frac{1}{4} \text{sgn} \sin \theta - \frac{\pi}{4} \cot \theta)} \left( 1 + O\left( \frac{1}{l_c^2} \right) \right),
\]

(3.29)

where \( \theta \in (-\pi, \pi) \) is the angle by which one has to rotate \( n_{s(e)} \) around \( B_{t(e)} \) to obtain \( n_{t(e)} \).

### 3.3 New form of the action

In the previous sections we have computed the contribution of one point of stationary phase with respect to the angles \( \phi f \). From section 3.2 we can also conclude that having one stationary point all others are obtained by application of transformations from \( \tilde{G} \) that keep \( U_f \) fixed. These are given by compositions of

\[
(-u_f)\alpha_f(\pi), \quad -\alpha_f \forall f.
\]

(3.30)

However, only the orbit generated by the group of \( (-u_f)\alpha_f(\pi) \) from a non-trivial stationary point contributes, since all other stationary points are suppressed by the modifiers \( f \). Therefore it is sufficient to compute the number of these stationary points. The group generated by \( (-u_f)\alpha_f(\pi) \) is equal to \( 2^{[F]} \) and acts freely on the stationary points; the countability of the orbit is thus \( 2^{[F]} \).

Around the stationary orbit, the integral is hence of the form:

\[
(-1)^{s_{[F]}} \int \prod_j dU_f \prod_{e} (-1)^{s_e} C_{j_e} \frac{1}{4\sqrt{2\pi (j_e + \frac{1}{2}) |\sin \theta_e|}} e^{-i \frac{1}{4} \text{sgn} \sin \theta_e} e^{i(j_e + \frac{1}{2}) \theta_e - \frac{\pi}{4(l_e + \frac{1}{2})} \cot \theta_e},
\]

(3.31)

where \( \theta_e \) is the angle between \( n_{s(e)} \) and \( n_{t(e)} \) with the sign determined by left hand rule with respect to \( B_{t(e)} \). In the neighbourhood of the stationary point this definition is meaningful. The value of the product \( \prod_e (-1)^{s_e} \) is discussed in appendix A.2.2. We use new ‘length’ parameters introduced in section 3.2.2

\[
l_e := j_e + \frac{1}{2}
\]

(3.32)

and perform a change of variables

\[
U_f \to n_f
\]

(3.33)

which is worked out in appendix B.1. The correct integral measure is given by:

\[
\mu = \frac{1}{2\pi} \delta(|n|^2 - 1) \, dn_1 \, dn_2 \, dn_3
\]

(3.34)

Thus, we can write the integral (integrating out \( \alpha_f \) and \( -u_f \) gauges) as:

\[
\frac{(-1)^s(-1)^{\sum_{e} s_e e^{-i \frac{1}{4} \sum_{e} \text{sgn}_e}} \prod_{e} C_{j_e}}{2^{[F]}} \int \prod_{f \in F} \delta(|n_f|^2 - 1) \, d^3 n_f \frac{1}{\sqrt{\prod_{e} |\sin \theta_e|}} e^{i \sum_e (l_e \theta_e + s_j^e(\theta_e))}.
\]

(3.35)
On the stationary

\[ S^s_1(\theta_e) = -\frac{1}{8\epsilon_c} \cot \theta_e + \ldots. \]  

(3.36)

The only present symmetry that has to be discussed is a \( u \)-symmetry, which is implemented by \( SO(3) \) rotations:

\[ n_f \rightarrow un_fu^{-1} \]  

(3.37)

If the configurations of the vectors \( B_{fe} \) is rigid then the stationary \( u \)-orbit is isolated.\(^2\)

### 3.3.1 \( c \) transformation as parity transformation

Furthermore, we would like to point out that given one orbit of stationary phase, we can always construct a different one via parity transformation of the \( B_{fe} \) vectors (see also section 2.5 about \( c \) transformations). After integrating out gauges these two points are related by

\[ n_f' = n_f, \]

\[ B_{fe}' = -B_{fe}, \]  

(3.38)

so also the angles are related by \( \theta_e' = -\theta_e \) (\( n_f \) are preserved as pseudovectors). Finally, we see that the asymptotic contribution from the parity related stationary orbits is just the complex conjugate of the original one, such that the complete expansion is real.

In order to provide the correct expression of the action before performing the remaining stationary point analysis, it is necessary to compute the normalization of the intertwiners, the so-called ‘Theta’ graph.

### 3.4 Normalization - ‘Theta’ graph

We need to compute the self-contraction of the invariants \( C_f \) using the (in this case) symmetric bilinear form \( \epsilon \) (as a generalization of the anti-symmetric form \( \epsilon \) of spin \( 1/2 \) to arbitrary representations). Its special properties allow us to relate the \( \epsilon \) product \( (\cdot, \cdot) \) to the scalar product on \( SU(2) \):

\[ (C_f, C_f) = \langle \bar{C}_f, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} C_f \rangle = \langle C_f, C_f \rangle, \]  

(3.39)

since \( C_f \) is real and \( SU(2) \) invariant. The integral of the contraction of the intertwiner with itself is given by:

\[ \int_{SU(2)^2 \times S^2} dU_1dU_2 \prod_i \frac{d\phi_1}{2\pi} \prod_i \frac{d\phi_{i2}}{2\pi} f_1(\{\phi_1\})f_2(\{\phi_{i2}\}) \prod_i \left(1/2\right) |O_{\phi_1}^{-1}U_1^{-1}U_2O_{\phi_{i2}}^{-1}1/2|^{2j_i}. \]  

(3.40)

Its stationary point conditions are:

- \( B_{i1} = -B_{i2} \).
- \( \sum_i B_{i1} = \sum_i B_{i2} = 0 \).

As the ‘Theta’ graph itself is an evaluation of a spin network its effective action have the same transformations on the action as described in 2.3.

The \( u \) symmetry can be ruled out just by dropping the integration over \( U_1 \). Then one is left with the group \( G \) generated by the transformations

\[ r_f, r_f, -u_2, o_{f1}, o_{f2}, -o_{f1,i}, -o_{f2,i} \]  

(3.41)

On the stationary \( H \) orbits, i.e. the normal subgroup of \( G \) generated by \( \{o_f, -u_f\} \), these transformations act as the group \( K = G/H \), which gives \( Z^4_2 \times Z^4_2 \).

This group acts freely on the stationary \( H \) orbits and as before the modifiers suppress all but one of the \( H \) orbits. If we take \( f_1 = f_2 = 1 \) and restrict ourselves to the case where \( \sum j_i \) is even (all \( j_i \) integer) then the action is invariant with respect to all transformations, thus every stationary orbit contribute the same \( 1/3 \) of the overall result.\(^3\)

\(^2\)Rigid means that the only deformations of the configuration of the edges with given lengths are rotations. For an isolated orbit, there exists a neighbourhood of the orbit that does not intersect any other orbit.

\(^3\)In the case when \( \sum j_e \) is not even, or some \( j_e \) are not integer, this choice leads to vanishing invariant.
The computation of the full expansion of the theta graph in the even case also gives an expansion on the stationary orbit in the presence of $f_i$. This is briefly discussed in the next section.

3.4.1 **Theta graph for integer spins and $\sum j$ even**

We will derive the complete expansion for $\sum j$ even. We need to compute

$$\prod_i C_j (C_{000}^{j_1j_2j_3})^2$$

(3.42)

where $C_j$, (see also appendix D.2.1) is the normalization of the $|0\rangle$ vector.

In appendix D.3 we show (following [40]) that the theta graph $(C_{000}^{j_1j_2j_3})^2$ is equal to

$$\frac{1}{2\pi S}(1 + O\left(\frac{1}{l}\right))$$

(3.43)

where $S$ is the area of the triangle with edges $j_i + \frac{1}{2}$.

3.5 **Final formula**

Let us state the final formula normalized by the square roots of the ‘Theta’ diagrams. Those are equal to:

$$(-1)^{s_f} 2^{-7/2} \sqrt{\frac{\Pi_{e \in F} C_{1e}}{\pi s_f}} \left(1 + O\left(\frac{1}{l}\right)\right),$$

(3.44)

where $s_f$ is a sign factor necessary to be consistent with [38, 39] that will be derived in A.2.1.

To summarize the various calculations of this chapter, the contraction of normalized intertwiners has the following asymptotic expansion after the stationary phase approximation for the angles $\phi_{fe}$ has been performed and the asymptotic expansion from (3.44) has been inserted:

$$\frac{(-1)^{s_f + \sum_j s_j + \sum_e s_e} e^{i\frac{\pi}{4} \sum_j \sin \theta_j}}{2^{1/2} |E|^{-1/2} |F|^{-1/2} \prod_{e \in E} l_e^{1/2}} \prod_{f \in F} S_f^{1/2}$$

(3.45)

$$\int_{\prod_{e \in E} \delta'(|n_f|^2 - 1)} d^3n_f \frac{1}{\prod_{e} \sin \theta_e} e^{i \sum_{e} (l_e \theta_e - \frac{\pi}{4} \cot \theta_e)}.$$

As it will be shown in appendix A, $s + \sum_j s_j + \sum_e s_e = 0 \mod 2$ and thus the term

$$(-1)^{s_f + \sum_j s_j + \sum_e s_e}$$

(3.46)

in the integral can be omitted.

This is the contribution up to next-to-leading order. It is straightforward to generalize it to higher order due to the complete expansion of the edge amplitude (section 5.2) and the expansion of ‘Theta’ diagrams (appendix D.3).

In the next section we will focus our attention on the specific example of the $6j$ symbol. After another variable transformation to the set of exterior dihedral angles of the tetrahedron has been performed, we obtain the action of flat first order Regge Calculus, i.e. Regge Calculus in which both edge lengths and dihedral angles are considered as independent variables. The stationary point conditions (with respect to the dihedral angles) will reduce the action to ordinary Regge calculus, such that the geometry is entirely described by the set of edge lengths, where angles on the stationary point agree with the angles given for a tetrahedron built from the lengths. We will perform the stationary point analysis, in particular compute the determinant of the Hessian matrix, and obtain the correct asymptotic expression for the $SU(2)$ $6j$ symbol [5].
4 Analysis of 6j symbol and first order Regge Calculus

In this section, we will perform the remaining integrations via stationary phase approximation starting from (3.45) in the case of the 6j symbol. As we are restricting the discussion to a specific spin network, we introduce the following notations:

This spin network consists of 4 faces \( f \), which we will simply count by \( i \in \{1, \ldots, 4\} \), and 6 edges \( e \), which we will denote by \( ij, i < j \), i.e. the faces sharing it. On the stationary point with respect to \( \{\phi_f\} \), we have two configurations of \( B_{fe} \), which we will label accordingly as \( B_{ij} \) and similarly \( \theta_{ij} \) using the convention that \( \theta_{ij} \) is the angle at the edge \( l_{ij} \).

In [19] it has been shown that the 6j symbol can be interpreted as a tetrahedron on the points of stationary phase (for non-degenerate configurations). In section 2.2 we have shown that our approach gives the same interpretation. Hence, we can assume that for one stationary point, the normals to the faces \( n_i \) of the tetrahedron are outward pointing and the \( B_{ij} \) vectors are oriented such that \( \theta_{ij} \in (0, \pi) \). For the second stationary orbit, described by \( B_{ij} = -B_{ij} \), the angles are negative, hence this contributes the complex conjugate.

In order to perform the remaining stationary point analysis, it is necessary to perform another variable transformations from normals of faces \( n_i \) to angles between these normals \( \theta_{ij} \) followed by integrating out gauge degrees of freedom corresponding to \( u \) transformations:

\[
\begin{align*}
n_i \to \theta_{ij}.
\end{align*}
\]  

This transformation is performed in appendix B.2 in great detail, and we obtain the following relation:

\[
\prod_i d^3 n_i \delta(|n_i|^2 - 1) \to \prod_{ij} d\theta_{ij} \prod_{ij} |\sin \theta_{ij}| \delta(\text{det} \tilde{G}) ,
\]

where \( \tilde{G} \) denotes the angle Gram matrix (for exterior dihedral angles) of a tetrahedron with components \( G_{ij} = \cos(\theta_{ij}) \), with \( \theta_{ii} = 0 \). Using (4.2) and simplifying (3.45) for the case of the 6j symbol, we obtain in the neighbourhood of the stationary point:

\[
\frac{e^{-i\pi/4} \prod_i l_{ij}^{1/2}}{2\pi^3} \sqrt{\prod_{i<j} |\sin \theta_{ij}|} \int \prod_{i<j} d\theta_{ij} |\sin \theta_{ij}| \delta(\text{det} \tilde{G}) \frac{1}{\prod_{i<j} \sqrt{|\sin \theta_{ij}|}} e^{i \sum_{i<j} \left( l_{ij}\theta_{ij}^2 - \frac{8}{\pi} \cot \theta_{ij} \right)} .
\]

Let us consider one of the stationary points for which \( \sin \theta_{ij} > 0 \). The second one contributes the complex conjugate of the first because two points (orbits) are related by \( e \) (parity) transformations:

\[
\frac{i}{4\pi^2} \frac{|l|^{1/2}}{\prod_{i<j} l_{ij}^{1/2}} \int \prod_{i<j} d\rho d\theta_{ij} \prod_{i<j} \sqrt{|\sin \theta_{ij}|} e^{i \sum_{i<j} \left( l_{ij}\theta_{ij}^2 - \frac{8}{\pi} \cot \theta_{ij} \right) - |l|\rho \text{det} \tilde{G}} ,
\]

where \( |l|^2 := \sum_{i<j} l_{ij}^2 \) and \( \rho \) is a Lagrange multiplier.

It is worth to examine the action in (4.4) in more detail: This function of edge lengths \( l_{ij} \) and angles \( \theta_{ij} \) is known as the action for ‘first order’ Regge Calculus [32]. We will comment on this further in section 4.4.

In the next section we will perform a stationary phase approximation for the integrations over the angles \( \theta_{ij} \). We will use the improved action \( \sum_{i<j} l_{ij} \theta_{ij} \), where we regard higher order corrections as the vertices of a Feynman diagram expansion, and the resulting points of stationary phase will correspond to perturbed stationary points obtained previously from the stationary point analysis w.r.t. the \( SU(2) \) group elements \( U_f \) in section 2.

4.1 Stationary point analysis

The stationary point conditions for the action (4.4) are:

- Derivative with respect to \( \theta_{ij} \):

\[
l_{ij} - |l|\rho \frac{\partial \text{det} \tilde{G}}{\partial \theta_{ij}} = 0 .
\]
We will prove that the inverse of the kinetic term is equal to 4.

\[ (4.2.1) \text{Propagator} \]

Using the results of appendix C.2, we see that (4.15) is equal to the identity. This gives

\[ c = \frac{1}{3} V S_i S_j \sin \theta_{ij} . \]

\[ \text{where } c \text{ is a constant (defined in Lemma 7 in appendix C).} \]

On the other hand \( \det \tilde{G} = 0 \) holds, where a (single) null eigenvector of \( \tilde{G} \) is given by the vector of areas of the triangles \( (S_1, \ldots, S_4) \) (of the tetrahedron)\(^5\).

The quadratic order in the expansion around the stationary point, which we also call the kinetic term, is given by:

\[ \frac{\partial \det \tilde{G}}{\partial \theta_{ij}} = -2 \frac{\det' \tilde{G}}{\sum_k S_k^2} S_i S_j \sin \theta_{ij} = -2 \frac{\det' \tilde{G}}{\sum_k S_k^2} l_{ij} , \]

where \( \det' \tilde{G} = \sum_i \tilde{G}_{ii}^* \) and \( \tilde{G}_{ii}^* \) is the \( (i, i) \)th minor of \( \tilde{G} \). \( \det' \tilde{G} \) is computed in appendix C.1.2:

\[ \det' \tilde{G} = \frac{3}{27} (\sum_i S_i^2) \prod S_i^4 . \]

Using (4.8) and (4.9), we solve (4.5) for the Lagrange multiplier \( \rho \):

\[ \rho = -\frac{2^2}{3^2 V^5} \sum l . \]

The quadratic order in the expansion around the stationary point, which we also call the kinetic term, i.e. the Hessian matrix of the action, is given by:

\[ \mathcal{H} := -i|l| \left( \begin{array}{cc} 0 & \frac{\partial \det \tilde{G}}{\partial \theta_{km}} \\ \frac{\partial \det \tilde{G}}{\partial \theta_{km}} & \rho \frac{\partial \det \tilde{G}}{\partial \theta_{km}} \end{array} \right) . \]

To complete the stationary point analysis, we have to compute the determinant of its inverse evaluated on the stationary point.

\[ (4.2) \text{Propagator and Hessian} \]

Let us introduce a function of lengths \( l \):

\[ \lambda = |l| \rho = -\frac{2^2}{3^2 V^5} \sum l . \]

It is of scaling dimension 1 with respect to \( l \).

\[ (4.2.1) \text{Propagator} \]

We will prove that the inverse of the kinetic term is equal to

\[ \mathcal{H}^{-1} = i \left( \begin{array}{cc} \frac{1}{|l|} & \frac{1}{\rho} \\ \frac{1}{\rho} & 1 \end{array} \right) = \frac{1}{|l|} \left( \begin{array}{cc} \frac{c}{|l|} & \frac{1}{\rho} \\ \frac{1}{\rho} & \frac{c}{|l|} \end{array} \right) \left( -i |l| \right) \left( \begin{array}{cc} 0 & \frac{\partial \det \tilde{G}}{\partial \theta_{km}} \\ \frac{\partial \det \tilde{G}}{\partial \theta_{km}} & \rho \frac{\partial \det \tilde{G}}{\partial \theta_{km}} \end{array} \right) . \]

This gives

\[ |l| \left( \begin{array}{cc} \frac{1}{|l|} & \frac{1}{\rho} \\ \frac{1}{\rho} & \frac{c}{|l|} \end{array} \right) \left( \begin{array}{cc} \frac{\partial \det \tilde{G}}{\partial \theta_{km}} & c \frac{\partial \det \tilde{G}}{\partial \theta_{km}} \\ \frac{\partial \det \tilde{G}}{\partial \theta_{km}} & \frac{\partial \det \tilde{G}}{\partial \theta_{km}} \end{array} \right) + \frac{1}{|l|} \left( \begin{array}{cc} \frac{1}{\rho} & \frac{\partial \det \tilde{G}}{\partial \theta_{km}} \\ \frac{\partial \det \tilde{G}}{\partial \theta_{km}} & \rho \frac{\partial \det \tilde{G}}{\partial \theta_{km}} \end{array} \right) \left( \begin{array}{cc} \frac{c}{|l|} & \frac{1}{\rho} \\ \frac{1}{\rho} & \frac{c}{|l|} \end{array} \right) \left( \begin{array}{cc} \frac{\partial \det \tilde{G}}{\partial \theta_{km}} & c \frac{\partial \det \tilde{G}}{\partial \theta_{km}} \\ \frac{\partial \det \tilde{G}}{\partial \theta_{km}} & \frac{\partial \det \tilde{G}}{\partial \theta_{km}} \end{array} \right) \end{array} \right) , \]

using the results of appendix C.2, we see that (4.15) is equal to the identity.

\(^4\)The point of stationary phase w.r.t. the angles \( \theta_{ij} \) is only a small perturbation in comparison to the stationary point w.r.t. group elements.

\(^5\)This illustrates that \( \det \tilde{G} = 0 \) imposes the closure of the flat tetrahedron.
4.2.2 Hessian

Similar to the angle Gram matrix discussed in the previous section, \( \det \frac{\partial \theta_{ij}}{\partial e_{kl}} = 0 \) in the case of a flat tetrahedron. This is due to the fact that given a set of dihedral angles of a flat tetrahedron, the tetrahedron is only defined up to rotations and uniform scaling of its edge lengths. Hence, the null eigenvector of the matrix \( \frac{\partial \theta_{ij}}{\partial e_{kl}} \) is given by the edge vector \( \vec{l} := (l_{12}, \ldots, l_{34})^T \). We rewrite the matrix \( \mathcal{H}^{-1} \) in the basis in which its second row is parallel to \( \vec{l} \) and the next ones are perpendicular to \( \vec{l} \):

\[
\begin{pmatrix}
\frac{1}{\pi^2} & \frac{1}{\pi} & \frac{1}{\pi} & \cdots & \cdots \\
\frac{1}{\pi} & \frac{1}{\pi} & \frac{1}{\pi} & \cdots & \cdots \\
0 & 0 & \frac{\partial \theta_{ij}}{\partial e_{kl}} & \cdots & \cdots \\
\vdots & 0 & \frac{\partial \theta_{ij}}{\partial e_{kl}} & \ddots & \ddots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}
\]

(4.16)

The determinant of \( (-\mathcal{H}^{-1}) \) is thus equal to

\[
\det(-\mathcal{H}^{-1}) = \frac{1}{\pi^2} \left( \frac{1}{|\vec{l}|} \right)^2 \det' \frac{\partial \theta_{ij}}{\partial l_{kl}}.
\]

(4.17)

Since \( \lambda \) is of scaling dimension 1 (with respect to edge lengths), \( l_{ij} \frac{\partial \theta}{\partial e_{kl}} = \lambda \). More details and the tedious calculation of \( \det' \frac{\partial \theta_{ij}}{\partial l_{kl}} \) can be found in appendix C:

\[
\det' \frac{\partial \theta_{ij}}{\partial l_{kl}} = \frac{3}{2} \left[ \frac{|\vec{l}|}{|\vec{V}|} \right]^3 V^3.
\]

(4.18)

Combining all these results, we obtain:

\[
\det(-\mathcal{H}^{-1}) = -i \frac{1}{\pi^2} \left( \frac{2}{3^{3/2}} \sqrt{\prod_i S_i^2} \right)^2 3^{3/2} \frac{|\vec{l}|}{2} \prod_i S_i V^3 = -i \frac{1}{2 \pi^2} \prod_i S_i V^3,
\]

(4.19)

and hence

\[
\sqrt{|\det(-\mathcal{H}^{-1})|} = \frac{1}{\sqrt{2 \pi^2}} \prod_i S_i V^{3/2}.
\]

(4.20)

Since \( \mathcal{H}^{-1} \) is antihermitian, it has only imaginary (and nonzero) eigenvalues. Therefore it is important to count the number of \(+i\mathbb{R}\) and \(-i\mathbb{R}\) eigenvalues in order to pick the right branch of \( \sqrt{|\det(-\mathcal{H}^{-1})|} \). The number of positive and negative imaginary eigenvalues is constant on the connected components of parameter spaces. For oriented tetrahedra (one of the two components) it can be computed in the equilateral case, i.e. all \( l_{ij} \) are equal. This was done in appendix D.4, then \( \mathcal{H}^{-1} \) has 4 \(+i\mathbb{R}\) eigenvalues and 3 \(-i\mathbb{R}\). Finally, we conclude:

\[
\frac{1}{\sqrt{|\det(-\mathcal{H})|}} = e^{-\pi i \frac{3}{2} V^{3/2}} \sqrt{|\det(-\mathcal{H}^{-1})|} = \frac{1}{\sqrt{2 \pi^2}} e^{-\pi i \frac{3}{2}} \prod_i S_i V^{3/2}.
\]

(4.21)

The last step is to combine all the previous results to obtain the final formula for the asymptotics of the 6j symbol.

4.3 Final Result

In this section, we will combine the results of the previous calculations step by step. First we perform the stationary point analysis for (4.4):

\[
i \frac{\pi}{\sqrt{2}} \frac{|\vec{l}|}{\pi} \prod_{i<j} S_i^{1/2} \prod_{i<j} \sqrt{\sin \theta_{ij}} \left( \frac{2\pi}{\sqrt{\det(-\mathcal{H})}} \right)^3 e^{i \left( \sum_{ij} \left( l_{ij} \theta_{ij} - \frac{1}{\pi} \right) \cot \theta_{ij} \right)}.
\]

(4.22)

This is equivalent to the Schl"afli identity in 3D: \( \sum_{ij} l_{ij} \partial \theta_{ij} = 0 \).
where $S_1$ is the NLO contribution. As a next step, we substitute $\sin \theta_{ij} = \frac{3}{2} l_{ij} V_{S_{i}S_{j}}$ (for $\sin \theta_{ij} > 0$) and (4.21) in (4.22):

$$
\frac{2^{\frac{3}{2}} |l| \prod S_{i}^{l/2}}{\prod l_{ij}^{l/2}} \frac{3}{2} \frac{\prod l_{ij}^{l/2} V^{3}}{\prod S_{i}^{2}} e^{\frac{i}{\sqrt{3}} \frac{\pi}{2} |l| V \frac{1}{2}} e^{i \left( \sum_{l} l_{ij} \theta_{ij} + S_{1} \right)} = \frac{1}{2} \sqrt{\frac{12}{\pi}} V e^{\frac{i}{\sqrt{12}} \pi \frac{4}{4} e^{i \left( \sum_{l} l_{ij} \theta_{ij} + S_{1} \right)}},
$$

(4.23)

As previously discussed, the full contribution comes from two stationary points, which are related by parity transformations. Eventually, we obtain:

$$
\frac{1}{\sqrt{12} \pi V} \left( \cos \left( \sum_{l} l_{ij} \theta_{ij} + \frac{\pi}{4} + S_{1} \right) \right) + O \left( |l|^{-2} \right),
$$

(4.24)

as in [5]. In the formula above, we implicitly assumed that $S_1$ is real. This property will be proven in section 5.

### 4.4 First order Regge calculus

A first order formulation of Regge Calculus [32, 43] is a discretization of General Relativity defined on the triangulation of the manifold in which both edge lengths and dihedral angles are considered as independent variables. Its introduction was motivated by Palatini’s formulation of Relativity where equations of motion are first order differential equations. Its action in 3D is given by

$$
S_{R[l]} = \sum_{e} l_{e} \epsilon_{e}, \quad \epsilon_{e} = 2\pi - \sum_{\tau \supset e} \theta_{e}^{(\tau)}
$$

(4.25)

where $l_{e}$ denotes the length of the edge $e$, $\theta_{e}^{(\tau)}$ denotes the dihedral angle at edge $e$ in the tetrahedron $\tau$. By $\epsilon_{e}$ we denote the deficit angle at edge $e$. For every tetrahedron an additional constraint is imposed, namely

$$
\det \tilde{G} = 0
$$

(4.26)

that enter the action via a Lagrange multiplier [32]. $\tilde{G}$ is the angle Gram matrix of the tetrahedron. One can eliminate the $\theta_{e}^{(\tau)}$ variables by partially solving the equations of motion (given by variations with respect to $\theta_{e}^{(\tau)}$), then

$$
\theta_{e}^{(\tau)} = \theta_{e}^{(\tau)} (l)
$$

(4.27)

turns out to be the dihedral angle at the edge $e$ for a discrete geometry determined by the edge lengths \{l_{e}\}.

Our derivation of the $6j$ symbol asymptotics follows the same idea. It also suggests a suitable measure in the path integral quantization for (linearized) first order Regge calculus in order to reobtain the factor $\frac{1}{\sqrt{V}}$ from Ponzano-Regge asymptotics. We also hope that our methods might be applied in the 4D case, where a similar action, motivated by the construction of modern spin foam models, was proposed in [33]. Furthermore, the present results could naturally provide and motivate a triangulation independent measure for first order Regge calculus following the approach in [27]. Examining first order and area-angle (quantum) Regge calculus in 4D might also give new insights into possible measures for 4D spin foam models.
5 Properties of the next to leading order and complete asymptotic expansion

So far, we dealt with the asymptotic expansion of a spherical spin network evaluation in the leading order approximation and managed to work out the example of the $6j$ symbol. However, our method allows us to derive, in principle, the full asymptotic expansion of the evaluation by the higher order stationary point analysis, e.g. we have already mentioned the next-to-leading order (NLO) corrections to the contribution from edges of the spin network (on the stationary points) in section 3.2. Such corrections improve the asymptotic behaviour in particular for small spins. Therefore we will apply our formalism in this section to derive new insights on the NLO corrections (to the $6j$ symbol).

NLO order corrections to the asymptotic formula of the $\text{SU}(2)$ $6j$ symbol have been thoroughly discussed in [30, 31]. In particular, the authors found evidence that the leading contributions in the expansion in $1/l$ are purely real and oscillating as $\cos(S_R + \frac{\pi}{4})$, whereas the next order term (also purely real) behaves like $\sin(S_R + \frac{\pi}{4})$, where $S_R$ denotes the Regge action for the tetrahedron. Furthermore, this behaviour is conjectured to be alternating for consecutive orders.

We will refer to this behaviour introduced in [30, 31] as “Dupuis-Livine” (DL) property and we will show that it holds for the full expansion of the asymptotics of any evaluation of spin networks, satisfying certain generic conditions, for example the $6j$ symbol in the non-degenerate case. Furthermore we will derive a new recursion relation for the $6j$ symbol which can be applied to obtain a simpler form of the next to leading order correction to the Ponzano-Regge formula.

5.1 Properties of the Dupuis-Livine form

In this section we will give a definition to the Dupuis-Livine form and also discuss some of its basic properties.

Consider an asymptotic expansion in the variables $\{j\}$ of the following form

$$\sum_i A_k(\{j\}) e^{i \sum j_i \theta_i}, \quad (5.1)$$

where $A_k$ is a homogeneous function in all variables $j$ of degree $k + \beta$. It can be rewritten in terms of the variables $\{l\}$ (with $l = j + \frac{1}{2}$):

$$\sum_i \tilde{A}_k(\{l\}) e^{i \sum l_i \theta_i}, \quad (5.2)$$

where $\tilde{A}_0 = e^{-\frac{i}{2} \sum \theta_i} A_0$.

We will say that it has the Dupuis-Livine (DL) property, if it can be written as

$$\tilde{A}_0(\{l\}) e^{i \sum l_i \theta_i} \sum_k B_k, \quad (5.3)$$

where $i^k B_k$ is a real and homogeneous function of degree $k$. Note that if we write this expansion in the form

$$\tilde{A}_0(l) e^{i \sum l_i \theta_i + S}, \quad (5.4)$$

then $S$ also has DL form (and starts with degree 1). Furthermore, suppose that two asymptotic series $f_1$ and $f_2$ have the DL property then also

$$f_1 f_2, \quad \frac{1}{f_1}, \quad (5.5)$$

have this property. In particular the last two relations are very useful for our discussion, since they allow us to examine the full expansion of the evaluation of the spin network in steps: first we examine the contributions from the edges, i.e. the partial integrations over the $\phi_{fe}$, then the normalization factors until we eventually discuss the full expansion.

5.2 Partial integration over $\phi$

In this subsection, we will examine whether the contributions from the partial integration over $\phi$ have the DL property. We will prove it by using a recurrence relation similar to Bonnet’s formula for Legendre polynomials. Therefore, it will be necessary to introduce some technical definitions, from which we are able to derive recursion relations.

22
5.2.1 Weak equivalence

Let \( \psi_i = f_i e^{S} \), where \( S = kS_{-1} + \ldots, \Re S_{-1} \leq 0 \) and \( f_i \) grows at most polynomially in \( k \) and admits a power series expansion in \( k \).

**Definition 1.** \( \psi_1 \) is weakly equivalent to \( \psi_2 \) around the point \( x_0 \),

\[
\psi_1 \equiv \psi_2 \quad ,
\]

(5.6)

if the expansion in \( k \) of the integral of both around \( x_0 \) is the same.

If \( \psi = fe^{S} \) then

\[
L^* \psi \equiv 0
\]

(5.7)

where

\[
L^* \psi = L \psi + (\text{div } L) \psi
\]

(5.8)

and \( L \) is a vector field.

5.2.2 Equivalences and recursion relations

Let us introduce

\[
L_\pm = \cos \tilde{\theta} \sin \alpha \frac{\partial}{\partial \alpha} \pm i \sin \tilde{\theta} \sin \beta \frac{\partial}{\partial \beta} \quad ,
\]

(5.9)

\[
A_\pm = \cos \tilde{\theta} \cos \alpha \pm i \sin \tilde{\theta} \cos \beta
\]

(5.10)

that we regard as vector fields and functions of the variables \( \alpha, \beta \). It is straightforward to calculate

\[
\text{div } L_\pm = A_\pm \quad ,
\]

(5.11)

and

\[
L_+ A_+ = L_- A_- = \frac{1}{2} A_+^2 + \frac{1}{2} A_-^2 - \cos 2\tilde{\theta}
\]

\[
L_- A_+ = L_+ A_- = A_+ A_- - 1
\]

(5.12)

Starting from \( L^*_+ A_+^k \equiv 0 \) and using the above identities, we derive the following relation (see appendix D.1 for more details):

\[
- \frac{(k+2)^2}{k+1} (A_+)^{k+2} + 2(k+1) \cos 2\tilde{\theta} A_+^k - (k-1)A_+^{k-2} \equiv 0
\]

(5.13)

Therefore we introduce the following quantity:

\[
\tilde{P}_l = \frac{1}{C_j} A_+^{2j} \quad ,
\]

(5.14)

where \( l = j + \frac{1}{2} \) and \( C_j \) is given by (3.25):

\[
C_j = \frac{1}{4^j} \frac{\Gamma(2j+1)}{\Gamma(j+1)^2}
\]

(5.15)

Furthermore \( C_j \) admits a complete expansion in \( j \), see also appendix D.2.1:

\[
C_j = \frac{1}{\sqrt{\pi j}} \left( 1 + O \left( \frac{1}{j} \right) \right)
\]

(5.16)

Moreover, one can show that

\[
C_{j+1} = \frac{2j+1}{2j+2} C_j \quad \text{and} \quad C_{j-1} = \frac{2j}{2j-1} C_j
\]

(5.17)
Combining (5.14), (5.15) and (5.17) with (5.13) and substituting $k = 2j$ in (5.13) we obtain:

$$0 \equiv 2C_j \left[ -\frac{(2j+2)(j+1)2j+1}{2j+1} \frac{2j+1}{2j+2} \tilde{P}_{j+1} + (2j+1) \cos 2\tilde{\theta} \tilde{P}_j - \frac{2j-1}{2j-1} \tilde{P}_{j-1} \right] =$$

$$= 2C_j \left[ -\left(l + \frac{1}{2}\right) \tilde{P}_{j+1} + 2l \cos 2\tilde{\theta} \tilde{P}_j - \left(l - \frac{1}{2}\right) \tilde{P}_{j-1} \right] .$$

But $C_j$ admits a nonzero asymptotic expansion, thus

$$-\left(l + \frac{1}{2}\right) \tilde{P}_{j+1} + 2l \cos 2\tilde{\theta} \tilde{P}_j - \left(l - \frac{1}{2}\right) \tilde{P}_{j-1} \equiv 0$$

around any stationary point. With the definitions given here, (3.15), i.e. the amplitude associated to one edge, becomes:

$$\frac{1}{8\pi^2} \int d\alpha d\beta A_2 j = \frac{1}{8\pi^2} \int d\alpha d\beta \tilde{P}_l ,$$

which establishes the connection to our previous calculations.

Let us notice that (5.19) is exactly Bonnet’s recursion formula for Legendre polynomials.

### 5.2.3 Total expansion and DL property

Over any stationary point we have shown that the integral of $\tilde{P}_l$ can be expanded as

$$(-1)^s \sum_{k \geq 0} e^{i\theta} A_k(\theta) + O(l^{-\infty}) ,$$

where $\theta = 2\tilde{\theta}$ is now the $SO(3)$ angle and $s$ is a sign factor that comes from the $SU(2)$ angle\(^7\). Moreover we know from the previous section that

$$-\left(l + \frac{1}{2}\right) \tilde{P}_{l+1} + 2l \cos \theta \tilde{P}_l - \left(l - \frac{1}{2}\right) \tilde{P}_{l-1} \equiv 0 .$$

Applying the asymptotic form to the recursion relations, we obtain:

**Lemma 3.** For every $m \geq 0$

$$\sum_{k \leq m} (2 \beta_{m+1-k} + \beta_{m-k}) A_k i^{m+1-k} \sin \left(\theta - \frac{\pi}{2} (m-k)\right) = 0 ,$$

where

$$\beta_k^m = \frac{(-k - \frac{1}{2})_m}{m!} \in \mathbb{R} ,$$

and

$$\left(a\right)_m = a \cdot \left(a - 1\right) \cdots \left(a - m + 1\right) , \quad \left(a\right)_0 = 1 .$$

We will prove the lemma in appendix D.2.

Consider the case where $k = m$ in (5.23). For any $m \geq 0$ one obtains that

$$2\beta_m^m + \beta_0^m = -2 \left(m + \frac{1}{2}\right) + 1 = -2m ,$$

such that (5.23) can be rewritten in the following way:

$$2mA_m i \sin(\theta) = \sum_{k < m} (2 \beta_{m+1-k}^k + \beta_{m-k}^k) A_k i^{m+1-k} \sin \left(\theta - \frac{\pi}{2} (m-k)\right) .$$

Let us introduce

$$B_m := A_m i^{-m} e^{\frac{\pi}{4} \text{sign} \sin \theta} .$$

\(^7\) Values of the integral for $\theta$ and $\theta + \pi$ differ by the factor $(-1)^s$. This restricts $A_k$ to be of the form described above.
From the asymptotics of the integrations over \( \hat{P} \) follows that \( B_0 \in \mathbb{R} \) and (5.27) can be rewritten as

\[
2m A_m i^{-m} e^{\frac{1}{2} \pi \text{sign}(\theta) \sin(\theta)} \sin(\theta) = \sum_{k < m} (2\beta_{m+1-k}^k + \beta_{m-k}^k) A_k i^{-k} e^{i \frac{1}{2} \pi \text{sign}(\theta) \sin(\theta)} \sin \left( \theta - \frac{\pi}{2} (m-k) \right) \]

\[
\iff 2m B_m \sin(\theta) = \sum_{k < m} (2\beta_{m+1-k}^k + \beta_{m-k}^k) B_k \sin \left( \theta - \frac{\pi}{2} (m-k) \right) .
\]

(5.29)

This implies that all \( B_k \in \mathbb{R} \) and it proves that the asymptotic terms (in the connected component \( -e^S \)) are of the form

\[
\hat{A}_k \in i^k \mathbb{R} \quad \text{for} \quad k > 0 .
\]

(5.30)

This proves that the contributions from the integration over \( \phi \) evaluated on the points of stationary phase are of DL form.

5.2.4 The total expansion of the original integral

We know that the total expansion of the original integral around the stationary point is of the form given in (3.26). Using the recurrence relation (5.29) we can compute its next-to-leading order:

\[
C_f \left[ \frac{(-1)^r}{4\sqrt{2\pi} |\sin \theta|} e^{i \left( \theta - \frac{1}{2} \pi \text{sign}(\sin \theta) \right) \cot \theta} \left( 1 + O \left( \frac{1}{r^2} \right) \right) \right] .
\]

(5.31)

As a next step, we will examine whether the normalization factors computed from the self-contraction of intertwiners is of DL form as well.

5.3 Different forms of intertwiners and DL property

To examine whether the normalization factors satisfy the DL property, we will construct different forms of invariants. Since the (three-valent) intertwiner is unique, all new constructions are proportional to the original one.

Let \( U_i \) be distinct group elements from a sufficiently small neighbourhood of the identity. Let

\[
\tilde{C}_{U_i,f} = \int dU \int \prod \frac{d\phi_i}{2\pi} f(\phi_1, \phi_2, \phi_3)
\]

\[
UU_1 O_{\phi_1} |l/2\rangle^{2i_{s1}} \otimes U U_2 O_{\phi_2} |l/2\rangle^{2i_{s2}} \otimes U U_3 O_{\phi_3} |l/2\rangle^{2i_{s3}}
\]

be the new invariant, where \( f \) is such a function that it is constant in the neighbourhood of the angles, which satisfy the stationary phase conditions, i.e. where all

\[
B_i = j_{s_i} U U_1 O_{\phi_i} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (U U_1 O_{\phi_i})^{-1} \]

(5.33)

sum to zero. We will choose \( U_i \) in such a way (described below) that such points are separated. In such a case we can choose \( f \) to be nonzero around only one of them.

Let us now describe \( U_i \). For given three vectors \( B_i \) in the plane perpendicular to \( H = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \) (see also section 2 for more details) such that

\[
\sum B_i = 0 , \]

(5.34)

we choose \( U_i \) in the neighborhood of identity such that \( U_i H U_i^{-1} \perp B_i \). There are many such choices which will be used in the sequel.

Let us take contraction of such a \( C_{U_i,f} \) with the intertwiner \( C_f \), obtained with the help of modifiers.

\[
(C_{U_i,f}, C_f) \]

(5.35)

Due to the definition of \( C_{U_i,f} \), there is only one \( -u \) and \( a_f \) orbit of stationary points on which \( f \) and \( f' \) are nonzero. These are given by the conditions

\[
B_{fe_i} = -B_i , \quad n_f \perp B_{fe_i} , \quad U U_i H U_i^{-1} U^{-1} \perp B_{fe_i} .
\]

(5.36)
Hence, on the stationary point $U$ is of the form

$$U = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}. \quad (5.37)$$

If we choose $U_i$ in such a way that

$$U_i H U_i^{-1} \cdot H \neq \pm 1, \quad (5.38)$$

i.e. the two normal vectors are not (anti)parallel, then also

$$UU_i H U_i^{-1} \cdot H \neq \pm 1. \quad (5.39)$$

This guarantees that this configuration is non-degenerate, such that the partial integration over $\phi$ (see section 5.2) and the fixing of the $\alpha_f$ and $u_f$ symmetry can be performed. Following the same method as presented in section 5.2 we prove that the asymptotic expansion of $(C_{U_i,f}, C_{f})$ has the DL property. Similar considerations apply to $(C_{U_1^{i},f}, C_{U_2^{i},f}) \quad (5.40)$ if $U_1^i H(U_1^i)^{-1} \cdot U_2^i H(U_2^i)^{-1} \neq \pm 1$.

Finally, using the uniqueness of the intertwiner, we obtain

$$(C_f, C_f) = \pm \frac{(C_{f}, C_{U_2^{i},f^2})}{(C_{U_1^{i},f^1}, C_{U_2^{i},f^2})}. \quad (5.41)$$

As a product of functions whose asymptotic expansion is of DL form, it follows directly that (5.41) is of DL form, too.

### 5.4 Leading order expansion and a recursion relation for the $6j$ symbol

In the two previous sections we have shown that both the contributions from partial integrations over $\phi$ and the normalization factors satisfy the DL property. Hence using properties explained in appendix E we have proven the conjecture from [30,31].

In this section we will discuss the next-to-leading order expansion for the $6j$ symbol. Therefore we do a brief recap of the results of section 4.

From the stationary point (with outward pointing normals) we have contributions from the Hessian, i.e. the kinetic term, and higher order terms, which are computed using a Feynman diagrammatic approach:

$$\propto \frac{1}{\sqrt{\det(-H)}} e^{i \sum_{ij} l_{ij} \theta_{ij} + S_1}, \quad (5.42)$$

where $S_1$ are the evaluations of the connected Feynman diagrams of the expansion in $\{\theta, \rho\}$ evaluated on the stationary point of the action $i \sum l_{ij} \theta_{ij}$, using $-H^{-1}$ as the propagator of this theory. We are interested only in $|l|^{-1}$ contributions, the respective Feynman rules are briefly discussed in appendix G.

The expansion up to the next to leading order is of the form (see also section 4.3):

$$\frac{1}{\sqrt{12\pi V}} e^{i (\sum_{ij} l_{ij} \theta_{ij} - \frac{\pi}{4} \cos \theta_{ij}) + S_1} = \frac{1}{2 \sqrt{12\pi V}} e^{i (\sum_{ij} l_{ij} \theta_{ij} + S_1)} \quad (5.43)$$

where $S_1$ is of order $|l|^{-1}$. The full contribution comes from two stationary point that are related via parity transformation, see also section 3.3.1; their contributions are related by complex conjugation. Hence, we obtain up to $|l|^{-1}$:

$$\frac{1}{\sqrt{12\pi V}} \left( \cos \left( \sum_{ij} l_{ij} \theta_{ij} + \frac{\pi}{4} + S_1 \right) + O(|l|^{-2}) \right) \quad (5.44)$$

The next to leading order expansion is briefly described in appendix G. Although, this method is algorithmically more involved than the method proposed in [30,31], the final expression is also more geometric. We will now derive a recursion relation for the full $6j$ symbol using a similar idea as in [36,37] that, we hope, can serve to compute the NLO expansion in more concise way.
5.4.1 Recursion relation for $6j$ symbols

In this section we derive a recursion relation for the whole $6j$ symbol. First, let us introduce a multiplication operator

$$N(l) = \sqrt{\prod_i \Theta_i(l)}$$

(5.45)

where $\Theta_i$ is normalization (of a three-valent intertwiner) computed from the Theta graph. Furthermore we define the operator $T_{ij}^v$ via its action on a function of edge lengths $\{l\}$:

$$T_{ij}^v C(l) = \left(1 + \frac{1}{2v}l_{ij}\right) C(\{l_{km} + v\delta_{(ij)(km)}\}) .$$

(5.46)

We assume that $T_{ii}^v = 1$.

As a next step, recall the definition of $\tilde{P}_l$ (5.14) and its recursion relation (5.19). The latter can be written as follows:

$$\cos \theta \tilde{P}_l \equiv \left(\frac{1}{2} + \frac{1}{4l}\right) \tilde{P}_{l+1} + \left(\frac{1}{2} - \frac{1}{4l}\right) \tilde{P}_{l-1} .$$

(5.47)

and we can write the non-normalized $6j$ amplitude as

$$Z''(l) = \int \prod d\theta_{ij} \prod \sin \theta_{ij} \prod \tilde{P}_{l_{ij}}(\theta_{ij}) \delta(\det \tilde{G}) .$$

(5.48)

In order to derive the recursion relation, we insert an additional $\det \tilde{G}$ into (5.48):

$$\int \prod d\theta_{ij} \det \tilde{G} \prod \sin \theta_{ij} \prod \tilde{P}_{l_{ij}}(\theta_{ij}) \delta(\det \tilde{G}) = 0 ,$$

(5.49)

since $\det \tilde{G}$ is constrained to vanish. Similar to [36,37], $\det \tilde{G}$ can be expanded as a sum over perturbations:

$$\det \tilde{G} = \sum_{\sigma \in S_4} \text{sgn} \sigma \frac{1}{16} \sum_{\vec{v} \in \{-1,1\}^4} e^{iv_{\sigma i}} ,$$

(5.50)

with the convention that $\theta_{ij} = \theta_{ji}$ and $\theta_{ii} = 0$. Using (5.50), equation (5.49) can be rewritten as:

$$\sum_{\sigma \in S_4} \text{sgn} \sigma \frac{1}{16} \sum_{\vec{v} \in \{-1,1\}^4} \prod_i T_{ij}^v, Z''(l) = 0 .$$

(5.51)

On the other hand, we know from previous calculations that

$$\{6j\} \equiv N^{-1} Z''(l) + c.c + O(l^{-\infty}) ,$$

(5.52)

such that we can summarize both (5.51) and (5.52) into the following recursion relation for the $6j$ symbol:

$$\text{det} \left[ \frac{T_{ij}^1 + T_{ij}^{-1}}{2} \right] N\{6j\} \equiv 0 .$$

(5.53)

where $T_{ij}^v$ is defined as in (5.46).

Another useful form is the following

$$\sum_{\sigma \in S_4} \text{sgn} \sigma \frac{1}{16} \sum_{\vec{v} \in \{-1,1\}^4} N(l + v_{\sigma i}) N(l) \left(\prod_i T_{ij}^v\right) \{6j\} \equiv 0 ,$$

(5.54)

since the expansion of $N(l + v_{\sigma i}) N(l)$ is straightforward to compute. We have to point out though that the coefficients in this formula are not rational, yet they allow for nice a asymptotic expansion. Thus they should in principle allow for the computation of the higher order expansions of the $6j$ symbol.

\footnote{The recursion relation has been verified numerically for several $6j$ symbols.}
6 Discussion and outlook

Coherent state approaches are the only available tools so far to successfully compute the asymptotic expansion of spin foam models [14–19], which gives us a first, and yet, very incomplete understanding of the relation of spin foam models to gravity. The strength and beauty of this approach is its clear geometrical interpretation and straightforward computation of the dominating phase of the expansion, which is identified as the Regge action of the examined triangulation. Despite these successes, the approach usually fails in the computation of the determinant of the Hessian matrix, which provides the normalization to the path integral and, more importantly, a measure on the space of geometries.

To overcome this drawback, we have introduced modified coherent states, i.e. states labelled by null eigenvectors with respect to a generator of rotations, smeared perpendicular to the axis of rotation. We have shown that these states allow for the same geometrical interpretation as the usual SU(2) coherent states and presented a method to deal with the (due to the smearing) increased number of stationary points. This allowed us to derive the well-known asymptotic expansion of the SU(2) 6j symbol [5] entirely, by computing its amplitude in the stationary phase approximation, first with respect to the smearing parameters and second, after a variable transformation, with respect to the dihedral angles of the tetrahedron. In the process, we have discovered that the resulting amplitude is proportional to the action of the first order formulation of Regge calculus, a result that supports the conjecture given in [33] that 4D spin foam models can be better described by angle and area variables instead of only edge lengths, the fundamental variables of ordinary Regge calculus. This result could also stimulate new work following the ideas of [27] to obtain an invariant path integral measure (under Pachner moves [44, 45]) for first order Regge calculus and to compare it to spin foam models.

In addition to this result, we also extended the calculation to the next to leading order correction for the 6j symbol. We have been able to prove the conjecture presented in [30,31] that the higher order corrections are alternatingly oscillating with the cosine or the sine of the Regge action, and furthermore we can, in principle, calculate the asymptotic expansion up to arbitrary order. Despite this success, we are not able to present the next-to-leading order in a short and concise way. This is a nuisance of all known derivations of next-to-leading order expansion, see for example [30, 31]. However, we derived a recursion relation for the 6j symbol, very similar in nature to the one in [36,37], that can in principle be used to obtain more concise form of the next to leading order term.

The main goal of this work was not the derivation of known results, but to develop and advertise a new coherent state method, which is capable of challenging the determination of the measure in spin foam models [14–18]. The computation of the full asymptotic expansion (even only up to leading order) would not only increase the understanding of spin foam models, but could also give a measure on the space of geometries, which could be compared to the proposed measure in [27]. Given such a measure, one would be able to examine which geometries dominate the spin foam transition amplitudes in the various models, which could also be used to exclude some of them. Our successful and complete derivation of the asymptotic expansion of the SU(2) 6j symbol is a good start, however the method still has to prove itself by tackling more complicated models. Therefore, two issues have to be overcome:

The first problem is to extend the presented coherent state approach to groups with non-unique intertwiners. Our calculations are heavily based on the fact that the intertwiner of three irreducible representations of SU(2) is unique, which simplified the construction of our model. The only 4D spin foam model with unique intertwiners is the Barrett-Crane model [8], which has already been ruled out as a viable quantum gravity theory. Nevertheless, our calculations presented in this work can be applied and can lead to interesting new insights [46].

The second problem is common to all coherent state approaches to spin foam models so far; all the known calculations are restricted to one simplex of the triangulation. To extract the asymptotic expansion for larger triangulations and to examine possible invariances under (local) changes of the triangulation like Pachner moves is still an open issue. In this work, before computing the asymptotic expansion of the 6j symbol, we have kept the discussion as general as possible. It would be interesting to examine, whether the relation to the first order formulation of Regge calculus can also be found in larger triangulations or whether one obtains modifications, which could be understood as quantum gravity effects.

At the end we would like to point out that the application of our method to the case of the non-compact group SL(2, R) is rather straightforward and we leave the determination of the 3D Lorentzian 6j symbol for future investigations.
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A Spin network evaluation and sign convention

This appendix is devoted to the sign issue. We will show how one can determine the total sign of our formula using the prescription of [38,39].

A.1 Penrose prescription for spherical graph

In this section we will describe a canonical way to evaluate spherical (planar) spin networks. Let us draw it on the 2-sphere such that no edges intersect; if the spin network is 2-line irreducible there are two distinct ways to do so, which differ by orientation. The result of the evaluation does not depend on this choice. For every node of the graph (a face in the dual picture) we have a natural cyclic order inherited from the orientation of the sphere. In the second step we choose any ordering of nodes (faces). This gives a natural orientation of the edges; they start in nodes lower in the order and end in nodes higher in the order. We draw the graph on the plane as shown on figure 5 such that the order of the nodes is preserved and the order of legs in every node is consistent with the cyclic order obtained above.

In the third step we count the number of crossings $s$ of half-integer edges with each other. The spin network is evaluated by contracting invariants, given for every node, by using the $\epsilon$ bilinear form oriented according to the edge orientation inherited from the nodes. The ordering of the legs is as in figure 6:

$$\prod_e \epsilon^{\alpha_e}_{A_{\alpha(e)}A_{\alpha(e)\alpha}} \prod_v \epsilon^{\lambda_{ve_1}A_{ve_2}A_{ve_3}}.$$  \hfill (A.1)

These invariants are described in [38,39] (see also section 3.4). One can show that the result does not depend on the made choices.
Figure 6: Nodes are in the right order and for each intertwiner the legs are in the right cyclic order. The number of crossings for half-integer edges is $s = 2$.

### A.2 Sign factors and spin structure

In this section we will show how to compute the sign factor for spherical graphs. First of all, let us notice that in the case that all $j$ are integers, the sign disappears completely. We will prove now that this is also the case in general. Explicitly we will prove that (see 3.5 for the definitions)

$$s + \sum_f s_f + \sum_s e = 0 \mod 2.$$  \hfill (A.2)

#### A.2.1 Sign factor in the intertwiner

In this section we compute the sign $s_f$. In order to do this, we compare our invariant with the one from [38,39] (given for a fixed order of $j_1, j_2, j_3$). The dual of the latter is given on vectors $\xi_1^{2j_1} \otimes \xi_2^{2j_2} \otimes \xi_3^{2j_3}$ by the formula

$$(-1)^{j_1+j_3-j_2} C \epsilon(\xi_1, \xi_2)^{j_1+j_2-j_3} \epsilon(\xi_2, \xi_3)^{j_2+j_3-j_1} \epsilon(\xi_3, \xi_1)^{j_1+j_3-j_2}$$  \hfill (A.3)

with normalization $C > 0$ [38,39].

The contraction of (A.3) with our invariant is given by:

$$(-1)^{j_1+j_3-j_2} C \int \frac{d\phi_1}{(2\pi)^3} \frac{d\phi_2}{(2\pi)^3} \frac{d\phi_3}{(2\pi)^3} \epsilon(v_{\phi_1}, v_{\phi_2})^{j_1+j_2-j_3} \epsilon(v_{\phi_2}, v_{\phi_3})^{j_2+j_3-j_1} \epsilon(v_{\phi_3}, v_{\phi_1})^{j_1+j_3-j_2},$$  \hfill (A.4)

where

$$v_\phi = \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}, \quad \epsilon(v_\phi, v_{\phi'}) = \sin(\phi' - \phi),$$  \hfill (A.5)

and we skipped the integration over $U$, since (A.3) is invariant. Let us recall our notation:

$$\psi_{ij} = \phi_i - \phi_j.$$  \hfill (A.6)

After a change of variables

$$(\phi_1, \phi_2, \phi_3) \to (\psi_{21}, \psi_{32}, \psi_{13})$$  \hfill (A.7)

and performing one trivial integration over $\phi_1$, (A.4) is equal to

$$(-1)^{j_1+j_3-j_2} C \int \frac{d\psi_{21}}{(2\pi)^2} \frac{d\psi_{32}}{(2\pi)^2} \sin(\psi_{21})^{j_1+j_2-j_3} \sin(\psi_{32})^{j_2+j_3-j_1} \sin(\psi_{13})^{j_1+j_3-j_2},$$  \hfill (A.8)

with the constraint $\psi_{21} + \psi_{32} + \psi_{13} = 0$.

As the expression is real (since $j_i + j_k - j_l$ is an integer), in the asymptotic limit it is dominated by the stationary point (maxima of the integral) of the action

$$(j_1 + j_2 - j_3) \ln |\sin \psi_{21}| + (j_2 + j_3 - j_1) \ln |\sin \psi_{32}| + (j_1 + j_3 - j_2) \ln |\sin \psi_{13}| + \rho(\psi_{21} + \psi_{32} + \psi_{13})$$  \hfill (A.9)

where $\rho$ is a Lagrange multiplier and $\psi_{21}, \psi_{32}, \psi_{13}$ are treated as independent variables. The stationary point condition reads

$$(j_i + j_k - j_l) \cot \psi_{ij} = \rho.$$  \hfill (A.10)
Now we can use the fact that
\[ \cot \psi_{32} \cot \psi_{21} + \cot \psi_{13} \cot \psi_{32} + \cot \psi_{21} \cot \psi_{13} = 1 \]  
(A.11)
to obtain
\[ \rho^2 = \frac{(j_1 + j_2 - j_3)(j_2 + j_3 - j_1)(j_1 + j_3 - j_2)}{j_1 + j_2 + j_3}. \]  
(A.12)
Furthermore, we see that
\[ \cot^2 \psi_{32} = \frac{(j_1 + j_2 - j_3)(j_1 + j_3 - j_2)}{(j_2 + j_3 - j_1)(j_1 + j_2 + j_3)} = \frac{j_1^2 - (j_2 - j_3)^2}{(j_2 + j_3)^2 - j_1^2}. \]  
(A.13)
Hence, we compute that
\[ \cos 2\psi_{32} = \cot^2 \psi_{32} - 1 = \frac{j_1^2 - j_2^2 - j_3^2}{2j_2j_3}, \]  
(A.14)
\[ \sin 2\psi_{32} = \frac{2\cot \psi_{32}}{\cot^2 \psi_{32} + 1} = \pm \frac{A}{j_2j_3}, \]  
(A.15)
where \( A \) is the area of the triangle with edge lengths \( j_1, j_2, j_3 \).
Thus, in general, we have
\[ 2\psi_{21}, 2\psi_{32}, 2\psi_{13} \mod 2\pi \]  
(A.16)
are oriented (i.e., incorporate sign) angles of the triangle on the plane with edges \( (j_1, j_2, j_3) \).
In the presence of a function \( f_f \), only one of those stationary points contributes. Since the Jacobian is real, the only contribution to the sign is given by the value of the integral in the stationary point. We know that \( \psi_{ij} \in (\pi, 2\pi) \) for consecutive pair of edges \( (ij) \) (see 2.1.1), thus \( \sin \psi_{ij} < 0 \) and the total sign is
\[ (-1)^{j_1+j_3-j_2} (-1)^{j_1+j_2-j_3} (-1)^{j_2+j_3-j_1} (-1)^{j_1+j_3-j_2} = (-1)^{j_2} \]  
(A.17)
As already discussed above, this is a relative sign of our invariant with respect to the invariant described in [38,39].

A.2.2 The sign \( \sum s_e \)

In the stationary point we can write (see 3.1.1 and 3.1.2 for the derivation)
\[ U_{s(e)}^{-1} U_{t(e)} = (-1)^s \hat{O}_{s(e)e} e^{-i \hat{\theta}_{s(t)\ell(e)}} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \hat{O}_{s(e)e}^{-1}, \]  
(A.18)
where we assumed that \( \hat{\theta}_{s(t)\ell(e)} \in (-\pi, \pi) \). It is straightforward to check that
\[ U_{t(e)}^{-1} U_{s(e)} = (-1)^{s+1} \hat{O}_{t(e)e} e^{-i \hat{\theta}_{s(t)\ell(e)}} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \hat{O}_{s(e)e}^{-1}, \]  
(A.19)
where \( \hat{\theta}_{t(e)s(e)} = -\hat{\theta}_{s(t)\ell(e)} \in (-\pi, \pi) \). Thus in general we have
\[ U_{f}^{-1} U_{f'} = (-1)^{s+c_e} \hat{O}_{f'e} e^{-i \hat{\theta}_{f'f}} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \hat{O}_{f'e}^{-1}, \]  
(A.20)
where
\[ c_e = \begin{cases} 0 & f = s(e) \text{ and } f' = t(e) \\ 1 & f = t(e) \text{ and } f' = s(e) \end{cases}. \]  
(A.21)
By a cycle we denote an assignment of a number \( \{0, 1\} \) to every edge such that
\[ \forall f \sum_{e \subset f} c_e = 0 \mod 2. \]  
(A.22)
The set of cycles is denoted by $Z_1$. Abusing the notation, we will also say that the cycle is formed by edges with $c_e = 1$. Let us notice that such edges form a disjoint sum of loops that we will denote by $c_i$.

For every cycle $c$ holds
\[
\prod_i \left( \prod_j U_{f_j}^{-1} U_{f_{j+1}} \right) = 1 ,
\]
where \( \{f_j f_{j+1}\} \) are consecutive pair of faces in the cycle $c_i$ (in the correctly chosen order).

Thus, we can write
\[
(-1)^{\tilde{s}(c)} = \prod_{e\in\{ff'\}\subset c} O_{fe} e^{-i\tilde{\theta}_{ff'} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) O_{f'e}^{-1} } ,
\]
where we used the same order of multiplications as before. The equations (A.23) translate into the set of equations satisfied by $\tilde{s}_e$:
\[
\forall c \in Z_1 \sum_{e\in c} \tilde{s}_e(c) = \tilde{s}(c) \mod 2 .
\]

Given a solution for the $\tilde{s}_e$, $U_f$ can be reconstructed up to a $U$ transformation. The solutions $\{\tilde{s}_e\}$ are not completely determined, but the residual symmetry is given by $\text{Ran} \partial$ where
\[
\partial : C_0 \rightarrow C_1 ,
\]
is a boundary operator\(^9\). Those correspond exactly to $-U_f$ transformations.

We are interested in $3.5$
\[
\sum s_e = \sum_{e\in c} \tilde{s}_e(c) ,
\]
where $c$ is the cycle formed by all edges that are half-integer.

### A.2.3 Sign of basic cycles in spherical case

In this section we will compute the sign factor $\tilde{s}(c)$ for cycles consisting of only a single loop. Every other cycle can be uniquely written as a sum (as the $\mathbb{Z}_2$ module) of such disjoint cycles.

Let us take such a cycle. The cycle is described by the sequence of consecutive faces and edges. The value of $(-1)^{\tilde{s}(c)}$ is thus equal to
\[
\prod_{(ff')\in c} O_{fe} e^{-i\tilde{\theta}_{ff'} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) O_{f'e}^{-1} } .
\]

All parameters (i.e $\phi_{fe}$ and $\tilde{\theta}_{ff'}$) can be continuously deformed, i.e. there exists a map
\[
[0,1] \ni t \rightarrow \{\phi_{fe}^t, \tilde{\theta}_{ff'}^t\}
\]
such that
\[
\forall e\in f \phi_{fe}^0 = \phi_{fe} , \quad \forall f \tilde{\theta}_{ff'}^0 = \tilde{\theta}_{ff'} ,
\]
that satisfies the following conditions:

- the image of (A.28) in $SO(3)$ is always the identity,
- at the end all deformed $SU(2)$ angles $\tilde{\theta}_{ff'}^t$ are equal to 0
- for every face $f$ with ordered pair of edges $e, e'$ (neighbours in the cycle), the difference $\phi_{fe}^t - \phi_{fe'}^t \in (\frac{\pi}{2}, \frac{3\pi}{2})$ modulo $2\pi$ during the whole deformation process. In fact, it is larger than $\pi$ if order of edges agrees with the orientation of the face and smaller if it does not.

\(^9\)This is because $C_1 = \ker \partial^* \oplus \text{Ran} \partial$
The final stage of the deformation will be denoted by
\[ \forall e \in f \tilde{\theta}_{ff'} = \tilde{\theta}_{f'f'}, \quad \forall ff', \phi'_{fe} = \phi_{f'e} . \] (A.31)

Up to 2-dimensional homotopies, there are two possible final stages of such deformations. They differ by orientation of the cycle (loop) drawn on the plane. We assume that the faces are ordered in agreement with total orientation.

The cycle before and after the deformation is shown in figure 7. The process is shown on the figure 8 on example of a single-loop cycle around the vertex. In the end we obtain
\[ \prod_{\{ff'\} \in c} (-1)^{c_e} O_{fe} e^{-i \tilde{\theta}_{ff'} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} O_{f'e}^{-1} e = (-1)^{C_e} \prod_{\{ff'\}} O'_{fe} O'^{-1}_{f'e} \prod_e \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \] (A.32)

where \( C_e \) is the number of edges with \( c_e = 1 \) because \( O'_{fe} \) commutes with \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). The images of \( O_{fe} \) (and related \( SO(3) \) angles \( \pi(\phi'_{fe}) \)) satisfy (see figure 9)
\[ \sum_e \pi(\phi'_{fe}) - \pi(\phi'_{f'e}) = - \sum_{\{ee'\} \subset f} \pi(\phi'_{fe}) - \pi(\phi'_{f'e}) = -(n-2)\pi , \] (A.33)

where \( n \) is the number of faces meeting in the cycle \( c \).

Using prescription 2.1.1 for \( \phi_{fe} - \phi_{f'e} \), the fact that \( SU(2) \) is the double cover of \( SO(3) \) and continuity of the deformation we obtain (modulo \( 2\pi \))
\[ \sum_f \phi'_{fe} - \phi'_{f'e} = \sum_f \frac{\pi(\phi'_{fe}) - \pi(\phi'_{f'e})}{2} - \pi = \left( -n + \frac{n}{2} + 1 \right) \pi . \] (A.34)

Thus
\[ \prod_{\{ff'\}} O'_{fe} O'^{-1}_{f'e} \prod_e \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -1 . \] (A.35)

Figure 7: (A) Cycle before deformation. (B) Cycle after deformation.

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To sum up, we obtained for a given cycle $c$

$$\sum e(c) \tilde{s}_e = C_e + 1 \mod 2.$$  \hspace{1cm} (A.36)

Since the cycle is oriented in the same way as the faces, $C_e$ is the number of edges oriented according to the cycle.

### A.2.4 Other method of computation

Let us consider an arbitrary cycle $c$. Let us draw it on the graph $G$ as in figure 6. We will denote by $s(c)$ the number of crossings in the cycle. For any node (face) $f$ we also denote

$$f(c) = \begin{cases} 0 & \text{if the middle leg edge of } f \text{ does not belong to } c \\ 1 & \text{if the middle leg edge of } f \text{ belongs to } c \end{cases}$$ \hspace{1cm} (A.37)

In the following, we will present another method of how to compute $\sum e_v(c) \tilde{s}_e$ for a basic cycle $c$. First we will prove:

**Lemma 4.** For a single loop cycle $c$ in a spherical network, the quantity

$$C_e + \sum f(c) + s(c) \mod 2$$ \hspace{1cm} (A.38)

**Proof.** Any two graphs can be transformed into one another by a sequence of basic moves

- One of the Reidemeister moves [47,48] for the edge (see figure 10 for example). It only changes $s(c)$ by an even number.
- Transposition of two consecutive nodes belonging to the cycle (see figure 11). In the move shown in the figure

$$C'_e = C_e \pm 1, \quad s(c)' = s(c) + 3 \ ,$$ \hspace{1cm} (A.39)

and all $f(c)$ remain unchanged.
Figure 9: (A) example of the cycle with orientations shown, (B) \( \pi(\phi_{fe}) - \pi(\phi'_{fe}) \) and \( \alpha = \pi - (\pi(\phi_{fe}) - \pi(\phi'_{fe})) \)

Figure 10: Part of the graph changed by the move. Example of a Reidemeister move.

- Cyclic permutation of the legs of a node \( f \) (figure 12). In this case

\[
f(c) + s(c)
\] (A.40)

is preserved.

Thus \( C_e + \sum_f f(c) + s(c) \mod 2 \) is invariant. \( \square \)

We see that \( (C_e + 1) + \sum_f f(c) + s(c) \) does not depend on the chosen graph \( G \), hence, we can choose the most convenient one (see figure 13). For this particular choice

\[
C_e = 1, \quad \forall_{f \in e} f(c) = 0, \quad s(c) = 0
\] (A.41)

and thus \( (C_e + 1) + \sum_f f(c) + s(c) = (1 + 1) + 0 = 0 \mod 2 \) and

\[
\sum_{e} c(e) \tilde{s}_e = C_e + 1 = \sum_f f(c) + s(c) \mod 2.
\] (A.42)
Figure 11: Part of the graph changed by the move (edges not belonging to the cycle are not drawn). Two consecutive nodes in the cycle transposed.

Figure 12: Part of the graph changed by the move. Cyclic change of the order of legs.

**A.2.5 Sign of the general cycle in spherical case**

Let us state now a few properties of \( f(c) \) and \( s(c) \) useful in the sequel.

For two cycles \( c \) and \( c' \) we denote a cycle by \( c + c' \) if it satisfies the following property:

\[
\forall e, \ (c + c')(e) = c(e) + c'(e) \mod 2.
\]  
(A.43)

We have for two disjoint cycles \( c \) and \( c' \)

\[
s(c + c') = s(c) + s(c') \mod 2 ,
\]  
(A.44)

\[
\forall f, f(c + c') = f(c) + f(c') \mod 2 .
\]  
(A.45)

We can now write every cycle \( c \) in the spherical case as a sum of disjoint single-loop cycles \( c_\alpha \), such that
\[ c = \sum_{\alpha} c_{\alpha} : \]
\[ \sum_{e} c(e) \tilde{s}_e = \sum_{e} \left( \sum_{\alpha} c_{\alpha}(e) \right) \tilde{s}_e = \sum_{\alpha} \left( \sum_{f} f(c_{\alpha}) + s(c_{\alpha}) \right) = \]
\[ = \sum_{f} f \left( \sum_{\alpha} c_{\alpha} \right) + s \left( \sum_{\alpha} c_{\alpha} \right) = \sum_{f} f(c) + s(c) \mod 2 . \]  

(A.46)

A.2.6 Final sign formula

Let us notice that in the case when \( c \) is the cycle of all half-integer spins we have
\[ s = s(c) , \quad \forall f s_f = f(c) , \quad \sum s_e = \sum_{e} c(e) \tilde{s}_e , \]
\[ (A.47) \]
since if we denote the spin of the middle leg edge of \( f \) by \( j_f \) then \( f(c) = 2j_f \mod 2 \). Finally
\[ s + \sum_{f} s_f + \sum s_e = 2 \left( \sum_{f} f(c) + s(c) \right) = 0 \mod 2 . \]  

(A.48)

B Changes of variables and their Jacobians

Let the Lie group \( G \) act transitively on the manifold \( S \) and let
\[ \chi : G \to \mathbb{R} \]  
\[ (B.1) \]
be a homomorphism. There exists at most one measure \( \mu \) (up to scaling) on \( S \) such that
\[ g^* \mu = \chi(g) \mu . \]  

(B.2)

Let
\[ H \cong S_1 \to S_2 \]  
\[ (B.3) \]
where \( S_1 \) is a principal Lie group bundle with the structure group \( H \) and the base space \( S_2 \). Any (pseudo-)\( k \)-form \( \mu_2 \) on \( S_2 \) can be uniquely represented by a (pseudo-) \( k \)-form \( \mu_1 \) on \( S_1 \) that satisfies
\[ h^* \mu_1 = \mu_1 , \quad \forall h \in H \]
\[ \mu_1 \perp \partial_\xi = 0 \quad \forall \partial_\xi \in \mathfrak{h} , \]  

(B.4)  

(B.5)

where \( \mathfrak{h} \) is the Lie algebra of \( H \) and \( \perp \) is contraction of the (pseudo-) form with the vector on the first site. Any form \( \mu_1 \) determines the form \( \mu_2 \) on \( S_2 \). The integration over \( S_2 \) is the integration over any section of the projection map \( S_1 \to S_2 \).
Such a form satisfying conditions (B.4) and (B.5) can be obtained from the $H$ invariant form $\mu$ on $S_1$ via the formula
\[
\mu_2 = \mu \perp \bigwedge_{\xi \text{ basis } h} \partial_\xi .
\] (B.6)

In case of a compact group $H$ it is related to the measure obtained by integration over the fibers, called $\mu_{f_H}$, as follows
\[
\mu_2 = (\mu_H \perp \bigwedge \partial_\xi)\mu_{f_H} ,
\] (B.7)
where $\mu_H$ is the normalized Haar measure on $H$.

Let $M \subset S$ be a submanifold described locally by a set of independent equations $f_a$. For any measure (form) $\mu$ on $S$ we can define a measure (form) $\mu_{f_a}$ on $M$ by the following integration prescription: Let $g \in C^0(M)$ and $\tilde{g}$ be any continuous extension to $S$ then
\[
\int_M \mu_{f_a} = \int_M \prod \delta(f_a) \tilde{g}\mu .
\] (B.8)

Let $M$ be a section of the bundle $H \subset S_1 \to S_2$ described by equations $f_a$, then we can compare the just described measures on $M$ and $S_2$ since $M \to S_2$ is a diffeomorphism of $M$ onto $S_2$:
\[
\mu_{f_a} = (\det \partial_\xi f_a)^{-1} \mu \perp \bigwedge \partial_\xi .
\] (B.9)

Indeed, we can choose local coordinates such that $S_1 = M \times H$ and the zero section is described by $f_a = 0$. We have
\[
\mu = (\mu \perp \bigwedge \partial_\xi) \wedge \bigwedge d\xi .
\] (B.10)

By extending the function $g$ constantly along fibers from the zero section, we obtain:
\[
\int_M g\mu_{f_a} = \int_{S_1} \prod \delta(f_a) \tilde{g}\mu = \int_M \tilde{g} \left( \int_H \prod \delta(f_a) \bigwedge d\xi \right) (\mu \perp \bigwedge \partial_\xi)
= \int_{S_2} g (\det \partial_\xi f_a)^{-1} (\mu \perp \bigwedge \partial_\xi) .
\] (B.11)

### B.1 Change of variables $u_i \to n_i$ (integrating out gauge)

Let us remind from section 3.3 that
\[
S^2 = SU(2)/S^1
\] (B.12)
given by the right action of $S^1$ on $SU(2)$. The sphere $S^2$ can either be represented by unit vectors $|n_i|^2 = 1$ or traceless $2 \times 2$ matrices $n_i$ with the condition
\[
\frac{1}{2} \text{Tr} n_in_i = 1 .
\] (B.13)

Then the quotient map is given by
\[
n_i(u) = uHu^{-1} ,
\] (B.14)
where $H = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. The group $SU(2)$ acts on $S^2$ (as a left action on the quotient) by
\[
n_i \to u n_i u^{-1} .
\] (B.15)

The Haar measure from $SU(2)$ can be integrated over the fibers giving the invariant measure $\mu$ on the sphere with total volume 1.

Another invariant measure is
\[
\delta(|n|^2 - 1) d n_1 d n_2 d n_3 .
\] (B.16)
Since there is only one invariant measure up to scale, both are related by a scaling transformation:
\[
\mu = c\delta(|n|^2 - 1) d n_1 d n_2 d n_3 .
\] (B.17)
The constant is fixed by requiring:
\[
1 = \int_{S^2} \mu = c \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \int_0^\infty r^2 \delta(r^2 - 1) d r = 2\pi c ,
\] (B.18)
thus
\[
\mu = \frac{1}{2\pi} \delta(|n|^2 - 1) d n_1 d n_2 d n_3 .
\] (B.19)
B.2 Variables $\theta$ in the flat tetrahedron

Let us consider two sets of variables

$$N = (\vec{n}_1, \ldots, \vec{n}_{m+1})$$  \hspace{1cm} (B.20)

where $\vec{n}_i$ are $m$ vectors with exactly one dependency, i.e. every subset of $m$ vectors forms a basis. Let

$$M = N^T N, \ m_{ij} = \vec{n}_i \vec{n}_j, \ i \leq j$$  \hspace{1cm} (B.21)

where $M$ is a symmetric positive $(m+1) \times (m+1)$ matrix, which is degenerate with exactly one null eigenvector, whose entries are all non vanishing.

On $N$ there exists a left action of $O(m)$, $\vec{n}_i \rightarrow O\vec{n}_i$. The matrix $M$ is $O(m)$ invariant, so the parameters of this action can be regarded as supplementary to $M$. The vector fields of this action will be denoted by $L_{ab}$. The map $N \rightarrow M$ is an $O(m)$ principal bundle.

In the following, our goal is to compare the pseudo-form

$$\mu_1 = \left| \bigwedge_{i=1 \ldots m+1, a=1 \ldots m} d n^a_i \bigwedge_{a < b} L_{ab} \right|$$  \hspace{1cm} (B.22)

with the form

$$\mu_2 = \left| \delta(\det M) \bigwedge_{i \leq j} d m_{ij} \right| .$$  \hspace{1cm} (B.23)

Let us notice that both $\mu_1$ and $\mu_2$ are measures on $M$.

There are additional transformations parametrized by $U \in GL(m+1)$

$$\tilde{N} = NU, \ \tilde{M} = U^T MU,$$  \hspace{1cm} (B.24)

which commute with the $O(m)$ action on $N$. The measure $\mu_1$ is $\chi$ covariant with respect to this action, where

$$\chi(U) = |\det^m(U) := |\det(U)|^m \ \forall U \in GL(m+1)$$  \hspace{1cm} (B.25)

Furthermore, we have

- $\mu_2$ is invariant for $U \in O(m+1)$,
- for transformations of the form $U = \text{diag} \lambda_i$ (diagonal matrix) the measure $\mu_2$ transforms as

$$\left| \prod \lambda_i \right| \prod_{i \leq j} \lambda_i \lambda_j \right| \mu_2 ,$$  \hspace{1cm} (B.26)

so it is also $\chi$ covariant as rotations and scaling generate the whole group.

Hence the two measures $\mu_1$ and $\mu_2$ differ by a constant $c$ as $GL(m+1)$ acts transitively on $M$. This constant can be computed for a special value of $N$:

$$n^a_i = \left\{ \begin{array}{ll}
\delta^a_i, & i \leq m \\
0, & i = m+1 
\end{array} \right.$$  \hspace{1cm} (B.27)

In this choice

$$d m_{ij} = \left\{ \begin{array}{ll}
d n^i_j + d n^i_j, & i, j \leq m \\
d n^i_{m+1}, & j = m+1 \\
0, & i = j = m+1 
\end{array} \right.$$  \hspace{1cm} (B.28)

Moreover

$$d n^a_i \perp L_{cd} = (L_{cd} \vec{n}_i)^a = \delta_{ac} \delta_{id} - \delta_{ad} \delta_{ic} .$$  \hspace{1cm} (B.29)

The following equalities hold:

$$2^m \left| \bigwedge_{i=1 \ldots m+1, a=1 \ldots m} d n^a_i \right| = \left| \bigwedge_{1 \leq j \leq i \leq m+1, j \neq m+1} d n^i_j \bigwedge_{d m_{ij}} \bigwedge_{1 \leq i \leq m+1} d n^i_j \right| ,$$  \hspace{1cm} (B.30)
but since \( d m_{ij} \perp L_{ab} = 0 \):

\[
2^m \left| \bigwedge_{j=1\ldots m+1, a=1\ldots m} d n_i^a \perp \bigwedge_{a<b} L_{ab} \right| = \left| \bigwedge_{1\leq j \leq m+1, j \neq m+1} d m_{ij} \right| = \delta(m_{m+1,m+1}) \left| \bigwedge_{(i,j) : i \leq j} d m_{ij} \right| .
\]

Moreover in this case

\[
M_{ij} = \begin{cases} 
\delta_{ij}, & i \leq m \\
0, & i = m + 1 
\end{cases}
\]

and so \( \frac{\partial \det M}{\partial m_{m+1,m+1}} = 1 \). Eventually

\[
2^m \left| \bigwedge_{j=1\ldots m+1, a=1\ldots m} d n_i^a \perp \bigwedge_{a<b} L_{ab} \right| = \delta(\det M) \left| \bigwedge_{i \leq j} d m_{ij} \right| .
\]

### B.2.1 Integration over the \( SO(3) \) fiber

In the case \( m = 3 \) we are interested in integrating over the fiber, however not the whole \( O(3) \) but only over one connected component with respect to \( SO(3) \). This is due to the \( u \) transformation symmetry corresponds to \( SO(3) \) not \( O(3) \) (see also section 2.3).

In this section we continue to compute the correct constant in front of the measure. We will now consider a fibration with the group \( SO(m) \) that can still be described locally by a projection \( N \mapsto M \).

This is, however, enough because we are only interested in local variables.

In general we have [49]

\[
\int_{SO(m)} \left| \bigwedge_{a<b} L_{ab} \right| = \prod_{k=2}^{n} \frac{2\pi^\frac{k}{2}}{\Gamma \left( \frac{k}{2} \right)} ,
\]

so from (B.7) our measure integrated over the fibre is equal to

\[
\int_{SO(m)} \left| \bigwedge_{i=1\ldots m+1, a=1\ldots m} d n_i^a \right| = \frac{1}{2^m} \prod_{k=2}^{m} \frac{2\pi^\frac{k}{2}}{\Gamma \left( \frac{k}{2} \right)} \delta(\det M) \left| \bigwedge_{i \leq j} d m_{ij} \right| .
\]

If we impose the condition \( |\vec{n}_i| = 1 \), we integrate over a set of unit vectors. This implies for \( M \) that we have to skip \( d m_{ii} \) in the measure and we define \( m_{ij} = \cos \theta_{ij} \). In these new angle variables the measure takes the form

\[
\frac{1}{2^m} \prod_{k=2}^{m} \frac{2\pi^\frac{k}{2}}{\Gamma \left( \frac{k}{2} \right)} \delta(\det \tilde{G}) \prod_{i<j} \left| \sin \theta_{ij} \right| \left| d \theta_{ij} \right| ,
\]

where \( \tilde{G} \) is the Gram matrix with the convention

\[
\tilde{G}_{ij} = \cos \theta_{ij}, \quad \theta_{ii} = 0 .
\]

In case \( n = 3 \) we have

\[
\pi^2 \delta(\det \tilde{G}) \prod_{i<j} \left| \sin \theta_{ij} \right| \left| d \theta_{ij} \right| .
\]

### B.3 Variables \( \theta / l \) in the spherical constantly curved tetrahedron

Let us consider the spaces of matrices

\[
\tilde{N} = \{ N \in M_n(\mathbb{R}) : \ \det N > 0 \}
\]

and

\[
\tilde{M} = \{ M \in M_n(\mathbb{R}) : M > 0 \} .
\]

We have a fibration with the group \( SO(n) \) (via left action on \( \tilde{N} \))

\[
\tilde{N} \rightarrow \tilde{M}, \quad M = N^T N .
\]
We can compare forms
\[ \mu_1 = \left| \bigwedge d n_i \right| \right|, \]
\[ \mu_2 = (\det M)^{-1/2} \left| \bigwedge_{i<j} d m_{ij} \right| \] \hspace{1cm} (B.42)

As in section B.2 there is an action of $SL(n)$ by
\[ N \to NU, \quad M \to U^T M U . \] \hspace{1cm} (B.43)

We can check that both measures are $\chi = |\det^n|$ covariant. Since $SL(n)$ acts transitively on matrices with positive determinant, we have
\[ \mu_1 = c \mu_2 . \] \hspace{1cm} (B.44)

Checking for $N = I$ gives $c = 1$.

Let us notice that $n_i \cdot n_i = m_{ii}$. On the surface $m_{ii} = 1$ we can introduce angle variables $\cos \theta_{ij} = m_{ij}$ and obtain
\[ (\det \tilde{G})^{-1/2} \prod \sin \theta_{ij} \wedge d \theta_{ij} = \pm \prod \delta(m_{ii} - 1) \wedge d n_i \] \hspace{1cm} (B.45)

B.4 Determinant $\det \frac{\partial}{\partial l}$ for constantly curved simplices

We denote the length Gram matrix by $G$ and the angle Gram matrix by $\tilde{G}$. The dimension is equal to $n-1$ and we are working in $\mathbb{R}^n$ on the sphere with radius 1.

Our goal is to prove the following formulas for the $(n-1)$-dimensional curved simplex. It was first proposed in [41] and checked using an algebraic manipulator. Now we are presenting the complete derivation.

Lemma 5. The following formulas hold for a spherical $(n-1)$-simplex:
\[ \det \frac{\partial \theta_{ij}}{\partial l_{km}} = (-1)^n \prod \sin l_{ij} \prod \sin \theta_{ij} \left( \frac{\det \tilde{G}}{\det G} \right)^{n+1} , \] \hspace{1cm} (B.46)

and for $n = 4$
\[ \det \frac{\partial \theta_{ij}}{\partial l_{km}} = - \det \frac{\partial \theta_{ij}}{\partial l'_{km}} = - \frac{\det \tilde{G}}{\det G} . \] \hspace{1cm} (B.47)

Where we used standard convention that the angle $\theta_{ij}$ is the angle on the hinge obtained by leaving out indices $i$ and $j$. The length of the opposite edge, i.e. the edge connecting vertices $i$ and $j$, is denoted by $l'_{ij}$ and in 3D, $l_{ij}$ is the length of the edge at which the angle sits.

B.4.1 Outline of the proof

We compute how the measure $\bigwedge d l'_{ij}$ transforms under the the change of variables
\[ \theta_{ij} \to l'_{ij} . \] \hspace{1cm} (B.48)

In fact, introducing variables $m_{ij} = \cos \theta_{ij}$ and $m'_{ij} = \cos l_{ij}$, we have (in the right order)
\[ \prod \sin \theta_{ij} \bigwedge d \theta_{ij} = \prod \delta(m_{ii} - 1) \bigwedge d m_{ij} \] \hspace{1cm} (B.49)
\[ \prod \sin l'_{ij} \bigwedge d l'_{ij} = \prod \delta(m'_{ii} - 1) \bigwedge d m'_{ij} \] \hspace{1cm} (B.50)

Both measures on the left hand side are on
\[ \tilde{M}_1 = \{ M \in \tilde{M} : \forall_i m_{ii} = 1 \} , \] \hspace{1cm} (B.51)

where we introduced the notation $\tilde{M} = GL_+(n)$ for simplicity.
B.4.2 Computation

There is an action of the group of diagonal matrices
\[
D = \{ d \in GL_+(n): d_{ij} = \lambda_i \delta_{ij}, \quad \lambda_i > 0 \}
\]
(B.52)
on \tilde{M} given by
\[
M \rightarrow d^T M d
\]
(B.53)
A basis for the Lie algebra \( \mathfrak{d} \) of the group \( D \) is given by
\[
\partial \xi_i = E_{ii} \quad \text{(matrices with only one nonzero entry being the } i\text{-th element on the diagonal equal to 1)}.
\]
We have a fibration
\[
\tilde{M} \rightarrow \tilde{M}/D
\]
(B.54)
and let \( \tilde{M}_1 \subset \tilde{M} \) be a cross section given by the equations
\[
\forall i \quad m_{ii} = 1.
\]
(B.55)
Let us introduce maps
\[
\psi_1: \tilde{M}_1 \rightarrow \tilde{M}/D, \quad M \mapsto [M],
\]
\[
\psi_2: \tilde{M} \rightarrow \tilde{M}, \quad M \mapsto M^{-1}.
\]
(B.56)
Acting with \( \psi_2 \) on matrix transformed as in (B.53), we have
\[
\psi_2(d^T M d) = (d^{-1})^T \psi_2(M)(d^{-1})
\]
(B.57)
such that there is a map
\[
[\psi_2]: \tilde{M}/D \rightarrow \tilde{M}/D.
\]
(B.58)
Let us notice that the composition \( \psi := \psi_1^{-1}[\psi_2] \psi_1 \) transforms \( \tilde{M}_1 \) into \( \tilde{M}_1 \).
We define measures
\[
\mu_1 = \prod_i \delta (m_{ii} - 1) \bigwedge_{i < j} d m_{ij} = \bigwedge_{i < j} d m_{ij},
\]
\[
\mu = (\det M)^{-\frac{n+1}{2}} \bigwedge_{i < j} d m_{ij},
\]
(B.59)
\[
\mu_{\tilde{M}/D} = \mu \bigwedge_i \partial \xi_i,
\]
where \( \partial \xi_i \) is the basis of the Lie algebra \( \mathfrak{d} \)
\[
\partial \xi_i m_{kl} = (\delta_{ik} + \delta_{il}) m_{kl}.
\]
(B.60)
Let us notice that according to section B.2 \( \mu \) is \( SL(n) \) invariant (where it acts as \( D \)) thus the pullback
\[
\psi_2^* \mu = c \mu
\]
(B.61)

\[
\psi_2^* \mu_{\tilde{M}/D} = (-1)^n \mu_{\tilde{M}/D}
\]
(B.62)
\[
\psi_1^* \mu_{\tilde{M}/D} = [\det \partial \xi_i (m_{jj} - 1)] (\det M)^{-\frac{n+1}{2}} \mu_1 = 2^n (\det M)^{-\frac{n+1}{2}} \mu_1.
\]
(B.63)
Combining all transformations we obtain
\[
\mu_1 = (-1)^n \left( \frac{\det M}{\det \psi(M)} \right)^{\frac{n+1}{2}} \psi^* \mu_1
\]
(B.64)
Finally, we obtain
\[ \wedge d\theta_{ij} = (-1)^n \prod \sin l'_{ij} \left( \frac{\det \tilde{G}}{\det G} \right)^{\frac{n+1}{2}} \wedge dl'_{ij}. \] (B.65)

So eventually
\[ \det \frac{\partial \theta_{ij}}{\partial l'_{km}} = (-1)^n \prod \sin l'_{ij} \left( \frac{\det \tilde{G}}{\det G} \right)^{\frac{n+1}{2}}. \] (B.66)

**B.4.3 Further simplifications for n = 4**

We can simplify the above formula using equalities from [42]:
\[ \det \tilde{G} = \frac{(\det G)^{n-1}}{\prod G^*_{ii}}, \quad (\sin \theta_{ij})^2 = \frac{\det G \det G^*_{ij}}{G^*_{kk} G^*_{jj}}, \] (B.67)

where \( G^*_{ii} \) is the \( ii \)-element of the minor matrix, \( G^*_{ij} \) is the \( G \) matrix without \( i \)th and \( j \)th rows and columns. Eventually we obtain
\[ \det \frac{\partial \theta_{ij}}{\partial l'_{km}} = (-1)^n \frac{\prod \sin l'_{ij} \prod (G^*_{kk})^{\frac{n+1}{2}}}{(\det G)^{\frac{n(n-1)(n+1)}{2}} \prod \sqrt{\det G_{(ij)}}} \frac{1}{\prod (G^*_{kk})^{\frac{n+1}{2}} (\det G)^{\frac{n+1}{2}}}. \] (B.68)

After simplification it is equal to
\[ \det \frac{\partial \theta_{ij}}{\partial l'_{km}} = \left( \prod \frac{\sin l'_{ij}}{\sqrt{\det G_{(ij)}}} \right) \frac{\det \tilde{G}}{\det G}. \] (B.69)

Note that this simplification only holds for \( n = 4 \). Since
\[ \det \frac{\partial l'_{ij}}{\partial l'_{km}} = -1 \] (B.70)
we obtain
\[ \det \frac{\partial \theta_{ij}}{\partial l_{km}} = - \det \frac{\partial \theta_{ij}}{\partial l'_{km}}. \] (B.71)

**C Technical computations of determinants**

In this section we will prove several technical results.

Let us introduce the notation
\[ \det' M = \sum_i M_{ii}^*. \] (C.1)

It is an invariant of the matrix and, moreover, in the case when the matrix is symmetric and has one null eigenvector, it is the determinant of the matrix restricted to the space perpendicular to that null eigenvector.

Let us also remind some general facts
\[ l_{ij} \frac{\partial \theta_{ij}}{\partial l_{kl}} = l_{kl} \frac{\partial \theta_{ij}}{\partial l_{kl}} = 0, \quad \frac{\partial \theta_{ij}}{\partial l_{kl}} = \frac{\partial \theta_{ij}}{\partial l_{kl}}, \] (C.2)
\[ l_{ij} = \lambda \frac{\partial \det \tilde{G}}{\partial \theta_{ij}}, \quad \lambda = \frac{\alpha^2}{3^4 V^4}, \] (C.3)
\[ \det' \tilde{G} = \frac{3^4}{2^4} \left( \sum_i S_i^2 \right) V^4 \prod S_i^2, \] (C.4)

which are proven for completeness in appendix C.1. Our results are (see appendix C.3):
Lemma 6. For the flat tetrahedron holds
\[ \det \frac{\partial \theta_{ij}}{\partial l_{kl}} = \frac{3^3}{2^5} \prod S_l^2 V^3. \] (C.5)

Moreover, in appendix C.2 we prove:

Lemma 7. Let
\[ \lambda = -\frac{2^2 \prod S_l^2}{3^5 V^3}. \] (C.6)

For the flat tetrahedron holds
\[ \frac{\partial \lambda}{\partial l_{ij}} \frac{\partial \det \tilde{G}}{\partial \theta_{kl}} + \frac{\partial \theta_{ij}}{\partial l_{mn}} \lambda \frac{\partial^2 \det \tilde{G}}{\partial \theta_{mn} \partial \theta_{kl}} = \delta_{(ij)(kl)}, \] (C.7)
\[ \exists c \frac{\partial \det \tilde{G}}{\partial \theta_{kl}} c + \lambda \frac{\partial \lambda}{\partial l_{ij}} \frac{\partial \det \tilde{G}}{\partial \theta_{ij}} \frac{\partial \theta_{ij}}{\partial l_{mn}} \frac{\partial \theta_{mn}}{\partial \theta_{kl}} = 0, \] (C.8)
\[ \frac{\partial \lambda}{\partial l_{mn}} \frac{\partial \det \tilde{G}}{\partial \theta_{mn}} = 1. \] (C.9)

C.1 General knowledge

We know that \( \det \tilde{G} = 0 \) for a geometric set of \( \theta \)'s. Moreover, the null eigenvector is given by
\[ (S_1, S_2, S_3, S_4), \] (C.10)
where \( S_i \) denote the areas of the triangles of the tetrahedron. The computation of \( \det' \tilde{G} \) can be found in appendix C.1.2. We have
\[ \frac{\partial \det \tilde{G}}{\partial \theta_{ij}} = -\frac{2^2 \prod S_l^2}{3^5 V^3} l_{ij} \] (C.11)

In addition to that, we also have
\[ 0 = \frac{\partial \det \tilde{G}}{\partial l_{kl}} = \frac{\partial \det \tilde{G}}{\partial \theta_{ij}} \frac{\partial \theta_{ij}}{\partial l_{kl}} = \lambda l_{ij} \frac{\partial \theta_{ij}}{\partial l_{kl}}. \] (C.12)

We know that \( \theta \) has scaling dimension 0, thus
\[ l_{kl} \frac{\partial \theta_{ij}}{\partial l_{kl}} = 0. \] (C.13)

C.1.1 Expressing \( \frac{\partial \theta_{ij}}{\partial l_{kl}} \) in terms of \( l_{ij} \)

Here we recall several well-known facts for flat simplices of arbitrary dimension using the notation of \( l'_{ij} \) from appendix B.4, see also [41, 50] for more details. Let \( M \) be the following matrix
\[ M = \begin{bmatrix} 0 & 1 & \ldots & 1 \\ 1 & l'_{11} & \ldots & l'_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & l'_{n1} & \ldots & l'_{nn} \end{bmatrix}, \] (C.14)
where \( l'_{11} = \ldots = l'_{nn} = 0 \) and \( l'_{ij} = l'_{ji} \). Then we have:
\[ V^2 = \frac{(-1)^{n-1}}{2^{n(n-1)/2}} \det M, \quad S_l^2 = \frac{(-1)^{n-2}}{2^{n-1(n-2)/2}} M_{ii}^*, \] (C.15)
\[ \cos \theta_{ij} = \frac{M_{ij}^*}{\sqrt{M_{ii}^* M_{jj}^*}}. \] (C.16)
In three dimension we also have
\[ \sin^2 \theta_{ij} = \left( \frac{3}{2} \right)^2 \frac{V^2 l^2_{ij}}{S_i^2 S_j^2}, \]  
so in the case \( \theta_{ij} \in (0, \pi) \) we can write:
\[ \frac{\partial \theta_{ij}}{\partial l_{kl}} = -\frac{1}{\sin \theta_{ij}} \frac{\partial \cos \theta_{ij}}{\partial l_{kl}} = \frac{2}{3} l_{ij} \frac{\partial}{\partial l_{kl}} \frac{M^*_{ij}}{\sqrt{M^*_i M^*_j}}, \]
\[ (C.17) \]
This, in principle, allows us to compute \( \frac{\partial \theta_{ij}}{\partial l_{kl}} \) and all other derivatives in terms of lengths.

### C.1.2 Computation of \( \det' \tilde{G} \)

Let us start with the spherical case, i.e. a tetrahedron with constant non vanishing (positive) curvature – a curved tetrahedron on the unit sphere. In this case we define \( \tilde{l}_{ij} := \epsilon l_{ij} \) and \( \theta_{ij} := \theta(\epsilon l_{kl}) \) and take the limit \( \epsilon \to 0 \) in order to reobtain the flat case. The angles \( \theta_{ij} \) have a limit as the angles of flat tetrahedron \( (\theta_{ij}) \) with lengths \( l_{ij} \).

First, let us notice that
\[ \det G = \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & G & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 1 \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 - \frac{1}{2} \epsilon^2 l_{ij}^2 + O(\epsilon^4) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 1 \end{pmatrix}, \]
\[ (C.19) \]
\[ = \frac{1}{8} \epsilon^6 \det \begin{pmatrix} 0 & 1 & \cdots & \cdots \\ 1 & l_{ij}^2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \cdots & 0 \end{pmatrix} + O(\epsilon^8). \]
\[ (C.20) \]
We can compute \( \det' \tilde{G} = \sum_i \tilde{G}_{ii}^* \) using the following identity from [42] \((n = 4, \text{i.e. } D = 3)\):
\[ \tilde{G}_{ii}^* = \frac{(\det G)^{n-2}}{\prod G_{ii}^*}, \]
\[ (C.21) \]
 obtaining
\[ \det' \tilde{G} = \sum_i \tilde{G}_{ii}^* = \left( \sum_i G_{ii}^* \right)^{n-2} \frac{V^4}{\prod S_i^2} + O(\epsilon^2). \]
\[ (C.22) \]
So in the flat case
\[ \det' \tilde{G} = \frac{3^4}{2^2} \left( \sum_i S_i^2 \right) \frac{V^4}{\prod S_i^2}. \]
\[ (C.23) \]

### C.2 Collection of results

Let us prove the following useful formulas:
\[ \frac{\partial \lambda}{\partial l_{ij}} \frac{\partial \det \tilde{G}}{\partial \theta_{kl}} + \sum_{m<n} \frac{\partial \theta_{ij}}{\partial l_{mn}} \frac{\partial^2 \det \tilde{G}}{\partial \theta_{mn} \partial \theta_{kl}} = \frac{\partial}{\partial l_{ij}} \left( \lambda \frac{\partial \det \tilde{G}}{\partial \theta_{kl}} \right) = \delta_{(ij), (kl)}, \]
\[ (C.24) \]
since we know, due to the Schlaffi identity, that \( \frac{\partial \theta_{ij}}{\partial l_{kl}} = \frac{\partial \theta_{kl}}{\partial l_{ij}} \). This also implies that
\[ \sum_{m<n} \frac{\partial \theta_{mn}}{\partial l_{ij}} \frac{\partial^2 \det \tilde{G}}{\partial \theta_{mn} \partial \theta_{kl}} = \delta_{(ij), (kl)} - \frac{\partial \lambda}{\partial l_{ij}} \frac{\partial l_{kl}}{\partial \lambda}. \]
\[ (C.25) \]
We will now prove that there exists such a \( c \) that
\[ \frac{\partial \det \tilde{G}}{\partial \theta_{kl}} c + \lambda \frac{\partial \lambda}{\partial l_{ij}} \frac{\partial^2 \det \tilde{G}}{\partial \theta_{ij} \partial \theta_{kl}} = 0. \]
\[ (C.26) \]
Because the range of the matrix $\frac{\partial \theta_{ij}}{\partial l_{kl}}$ is the whole space perpendicular to the vector $\vec{l} = (l_{ij})$ and the vector $\frac{\partial \det \tilde{G}}{\partial \theta_{kl}}$ is proportional to $\vec{l}$, it is enough to compute

$$
\left( \frac{\partial \det \tilde{G}}{\partial \theta_{kl}} + \lambda \frac{\partial l_{ij}}{\partial l_{mn}} \frac{\partial \theta_{ij}}{\partial \theta_{kl}} \right) \frac{\partial \theta_{kl}}{\partial l_{mn}} = \frac{\partial \lambda}{\partial l_{ij}} \frac{\partial \theta_{ij}}{\partial l_{mn}} = \lambda \frac{\partial \lambda}{\partial l_{ij}} \frac{\partial \theta_{ij}}{\partial l_{mn}} = \lambda \frac{\partial \lambda}{\partial \theta_{kl}} \frac{\partial \theta_{ij}}{\partial l_{mn}} 
$$

$$
(C.27)
$$

On the other hand we know that, since $\lambda$ is of scaling dimension 1, $l_{ij} \frac{\partial \lambda}{\partial l_{ij}} = \lambda$, and thus

$$
\lambda \frac{\partial \lambda}{\partial l_{ij}} \left( \frac{\delta_{(ij)(mn)}}{\lambda} - l_{ij} \frac{\partial \lambda}{\partial l_{mn}} \lambda \right) = \frac{\partial \lambda}{\partial l_{ij}} \frac{\partial \theta_{ij}}{\partial l_{mn}} = 0 .
$$

(C.28)

Let us also remind that:

$$
\frac{\partial \lambda}{\partial l_{mn}} \frac{\partial \det \tilde{G}}{\partial \theta_{mn}} = \frac{\partial \lambda}{\partial l_{mn}} \frac{l_{mn}}{\lambda} = 1 
$$

(C.29)

C.3 Computation of $\text{det}' \frac{\partial \theta_{ij}}{\partial l_{ij}}$

In this section we will prove that

$$
\text{det}' \frac{\partial \theta_{ij}}{\partial l_{ij}} = \frac{3^3 |l|^2}{2^5 \prod S_i^2} V^3 
$$

(C.30)

We will start from the formula valid for a spherical tetrahedron (Lemma 5):

$$
\text{det} \frac{\partial \theta_{ij}}{\partial l_{ij}} = \frac{\text{det} \tilde{G}}{\text{det} G} 
$$

(C.31)

As mentioned above we set $\tilde{l}_{ij} = e l_{ij}$ and $\theta_{ij}' = \theta(e l_{ij})$ and take the limit $\epsilon \to 0$ in the end. In this limit the angles converge to the angles of a flat tetrahedron with lengths $l_{ij}$.

Let us remind that

$$
\text{det} G = \frac{1}{8} \epsilon^6 \text{det} \begin{pmatrix} \epsilon^6 & \epsilon^6 & \cdots & \epsilon^6 \\ \epsilon^6 & \epsilon^6 & \cdots & \epsilon^6 \\ \cdots & \cdots & \cdots & \cdots \\ \epsilon^6 & \epsilon^6 & \cdots & \epsilon^6 \end{pmatrix} + O(\epsilon^8) 
$$

(C.32)

Let us notice that because $G$ (in the spherical case) is a function of $\cos e l_{ij}$, its expansion around $\epsilon = 0$ is an analytic function in $\epsilon^2$ and not only in $\epsilon$. The same holds for the matrix $\tilde{G}$ since it is

$$
\tilde{G}_{ij} = \frac{1}{\sqrt{G_{ii}^*}} G_{ij}^* \frac{1}{\sqrt{G_{jj}^*}} 
$$

(C.33)

where $G_{ij}^*$ is the cofactor matrix of $G$ and $\sqrt{G_{ii}^*}$ is $\epsilon^3$ times an analytic function in $\epsilon^2$.

Hence we know that for the vector $\vec{S} = (S_1, S_2, S_3, S_4)$ (the single null eigenvector in the limit $\epsilon = 0$)

$$
\tilde{G} \vec{S} = O(\epsilon^2) , \ (\vec{S}, \tilde{G} \vec{S}) = O(\epsilon^2) ,
$$

(C.34)

then also $\text{det} \tilde{G} = O(\epsilon^2)$ and

$$
\text{det} \tilde{G} = \text{det}' \tilde{G} \frac{(\vec{S}, \tilde{G} \vec{S})}{|\vec{S}|^2} + O(\epsilon^4) .
$$

(C.35)

Moreover

$$
\frac{\partial}{\partial \epsilon} \frac{(\vec{S}, \tilde{G} \vec{S})}{|\vec{S}|^2} = 2 \frac{(\vec{S}, \tilde{G} \vec{S})}{|\vec{S}|^2} + O(\epsilon^3) .
$$

(C.36)
We have
\[
\frac{\partial}{\partial \epsilon} \frac{(S, \hat{G}S)}{|S|^2} = \sum_{(ij)} l_{ij} \frac{\partial}{\partial \epsilon} \frac{(S, \hat{G}S)}{|S|^2} = \frac{(S, \sum_{(ij)} l_{ij} \partial_{\epsilon} \hat{G}S)}{|S|^2} = -2 \frac{\sum_{(km)(ij)} S_k S_m \sin \theta_{km} l_{ij} \frac{\partial \theta_{km}}{\partial l_{ij}}}{|S|^2} = -3 V \frac{\sum_{km} l_{km} \frac{\partial \theta_{km}}{\partial l_{ij}}}{|S|^2} \cdot \tag{C.37}
\]
Similarly, we know that \( \sum_{ij} \frac{\partial \theta_{km}}{\partial l_{ij}} l_{ij} = O(\epsilon^2) \) and \( \sum_{ijkm} l_{km} \frac{\partial \theta_{km}}{\partial l_{ij}} l_{ij} = O(\epsilon^2) \), so
\[
\det \frac{\partial \theta_{km}}{\partial l_{ij}} = \epsilon^{-6} \det \frac{\partial \theta_{km}}{\partial l_{ij}} = \epsilon^{-6} \det \frac{\theta_{km}}{\partial l_{ij}} \sum_{(ij)(km)} l_{km} \frac{\partial \theta_{km}}{\partial l_{ij}} l_{ij} \cdot \tag{C.38}
\]
Eventually, we have
\[
\epsilon^{-6} \det \frac{\theta_{km}}{\partial l_{ij}} \sum_{ijkm} \frac{\partial \theta_{km}}{\partial l_{ij}} l_{ij} |l|^2 = 12 \epsilon^{-6} \det \theta_{km} \frac{V}{C} |S|^2 \sum_{km} l_{km} \frac{\partial \theta_{km}}{\partial l_{ij}} + O(\epsilon^{-3}) \cdot \tag{C.39}
\]
and so
\[
\det \frac{\partial \theta_{km}}{\partial l_{ij}} = \frac{12 |l|^2}{C} \frac{V}{|S|^2} \det \theta_{km} + O(\epsilon) \cdot \tag{C.40}
\]
Now we can use the identities from appendix C.1.2
\[
\det \theta_{km} = \frac{3^4}{2^2} \left( \sum_i S_i^2 \right)^2 V^4 \prod_i S_i^2 \cdot \det C = 8(3!)^2 V^2 \cdot \tag{C.41}
\]
Finally, in the limit \( \epsilon \to 0 \), \( (\theta_{ij} = \lim \theta_{ij}^\epsilon) \):
\[
\det \frac{\partial \theta_{km}}{\partial l_{ij}} = \frac{3^3}{2^3} \frac{|l|^2}{\prod_i S_i^2} V \cdot \tag{C.42}
\]
\section{D Technical computations}
In this appendix we give some explicit computations needed in the main body of the paper.
\subsection{D.1 Weak equivalences}
In the following we will use the notation introduced in section 5.2.2. We can compute
\[
0 \equiv L_+ A^k_+ = k(L_+ A_+) A^{k-1}_+ + A^{k+1}_+ = \left( \frac{k}{2} + 1 \right) A^{k+1}_+ + \frac{k}{2} A^2 A^{k-1}_+ - k \cos 2 \theta A^{k-1}_+ \cdot \tag{D.1}
\]
such that
\[
A^2 A^{k-1}_+ \equiv - \frac{k + 2}{k} A^{k+1}_+ + 2 \cos 2 \theta A^{k-1}_+ \cdot \tag{D.2}
\]
Similarly, we can derive an identity by acting on \( A^k_+ \) with \( L_-^* \):
\[
0 \equiv L_- A^k_+ = (k + 1) A_- A^k_+ - k A^{k-1}_+ \cdot \tag{D.3}
\]
\[\Rightarrow A_- A^k_+ = \frac{k}{k + 1} A^{k-1}_+ \cdot \tag{D.4}\]
By acting again on (D.4) we obtain:
\[
L_- (A_- A^k_+) = \frac{1}{2} A^{k+2}_+ + \left( k + \frac{3}{2} \right) A^2 A^k_+ - k A_- A^{k-1}_+ - \cos 2 \theta A^k_+ \equiv 0 \cdot \tag{D.5}\]
Hence using (D.2) and (D.4) we have
\[
0 \equiv \frac{1}{2} A^{k+2}_+ + \left( k + \frac{3}{2} \right) \left( -\frac{k + 3}{k + 1} A^{k+2}_+ + 2 \cos 2\theta \ A^k_+ \right) - k \frac{k + 1}{k} A^k_+ - \cos 2\theta \ A^k_+ .
\] (D.6)

\[\text{D.2 Proof of the lemma}\]
In this section we will prove the following lemma:

**Lemma 3.** For every \( m \geq 0 \)
\[
\sum_{k \leq m} (2\beta_{m+1-k}^k + \beta_{m-k}^k) A_k i^{m+1-k} \sin \left( \theta - \frac{\pi}{2} (m-k) \right) = 0 ,
\] (D.7)
where
\[
\beta_m^k = \frac{(-k - \frac{1}{2})^m}{m!} \in \mathbb{R} ,
\] (D.8)
and
\[
(a)_{m} = a \cdot (a-1) \ldots \cdot (a-m+1) , \quad (a)_{0} = 1 .
\] (D.9)

To do so, we need:

**Lemma 8.** The following equality holds:
\[
\frac{1}{(l \pm 1)_{k+\frac{\pi}{2}}} = \sum_{m \geq k} \frac{(-1)^{m-k} \beta_m^k}{im + \frac{\pi}{2}} .
\] (D.10)

**Proof.**
\[
\frac{1}{(l \pm 1)_{k+\frac{\pi}{2}}} = \sum_{n=0}^{\infty} \frac{(-k - \frac{1}{2})^n}{n!} (l \pm 1)_{k+\frac{\pi}{2}} (n+1)^n = \sum_{n=0}^{\infty} \frac{(-k - \frac{1}{2})^n}{n!} \frac{1}{l^{k+n+\frac{\pi}{2}}} (l \pm 1)^n
\]
\[
= \sum_{m \geq k} \frac{(1)^{m-k} \beta_m^k}{(m-k)!} .
\]
\(\Box\)

In the following we will use Lemma 8 to prove Lemma 3:

**Proof of Lemma 3.** In any stationary point we have by Lemma 8
\[
\hat{P}_{l+1} \equiv \sum_{k \geq 0} e^{i(\pm 1)\theta} A_k(\theta) \equiv \sum_{k \geq 0} e^{i\theta} \sum_{m \geq k} \frac{(-1)^{m-k} \beta_m^k}{im + \frac{\pi}{2}} A_k(\theta) e^{\pm i\theta} .
\] (D.11)

We thus have
\[
l(\hat{P}_{l+1} + \hat{P}_{l-1} - 2 \cos \theta \hat{P}_l) \equiv \sum_{k \geq 0} e^{i\theta} A_k \left( \sum_{m \geq k} \beta_m^k \sin \left( \theta - \frac{\pi}{2} (m-k) \right) \cos \left( \theta - \frac{\pi}{2} (m-k) \right) \right) .
\] (D.12)

A simple algebraic manipulation gives
\[
e^{i\theta} + (-1)^{m-k} e^{-i\theta} = \begin{cases} 2 \cos \theta & \text{if } m = k \\ 2^{m-k} \cos \left( \theta - \frac{\pi}{2} (m-k) \right) & \text{if } m > k \end{cases}
\] (D.13)
such that we obtain:
\[
l(\hat{P}_{l+1} + \hat{P}_{l-1} - 2 \cos \theta \hat{P}_l) \equiv \sum_{k \geq 0} e^{i\theta} A_k \sum_{m \geq k} \frac{2\beta_m^k \sin \left( \theta - \frac{\pi}{2} (m-k) \right)}{im + \frac{\pi}{2}} \cos \left( \theta - \frac{\pi}{2} (m-k) \right) .
\] (D.14)
We also have
\[ \frac{1}{2} (\tilde{P}_{t+1} - \tilde{P}_{t-1}) = \sum_{k \geq 0} e^{i\theta} A_k \sum_{m \geq k} \frac{\beta_{m-k}}{l^{m+\frac{1}{2}}} m+1-k \sin \left( \theta - \frac{\pi}{2} (m-k) \right) . \]  \hspace{1cm} (D.15)

By combining (D.14) and (D.15), we obtain for the full recursion relation (5.22):
\[ \sum_{k \geq 0} e^{i\theta} A_k \sum_{m \geq k} \frac{2^{\beta_{m+1-k} + \beta_{m-k}}}{l^{m+\frac{1}{2}}} m+1-k \sin \left( \theta - \frac{\pi}{2} (m-k) \right) \]
\[ = \sum_{m \geq 0} \frac{e^{i\theta}}{l^{m+\frac{1}{2}}} \left( \sum_{k \leq m} (2^{\beta_{m+1-k} + \beta_{m-k}}) l^{m+1-k} A_k \sin \left( \theta - \frac{\pi}{2} (m-k) \right) \right) = O(l^{-\infty}) . \]  \hspace{1cm} (D.16)

Thus every single term must be zero. That ends the proof.

**D.2.1 Expanding** \( C_j \)

In the following we will expand the normalization factor \( C_j = \frac{1}{j!} \) up to \( O(\frac{1}{j}) \). Therefore we use Stirling’s series for the logarithm of the factorial:
\[ \ln n! = n \ln n - n + \frac{1}{2} \ln(2\pi n) + O \left( \frac{1}{n} \right) . \]  \hspace{1cm} (D.17)

Hence
\[ \ln(C_j) = \ln \left( \frac{1}{\sqrt{\pi j}} \right) + O \left( \frac{1}{j} \right) . \]  \hspace{1cm} (D.18)

Therefore we obtain:
\[ C_j = \frac{1}{\sqrt{\pi j}} e^{O(\frac{1}{j})} = \frac{1}{\sqrt{\pi j}} \left( 1 + O \left( \frac{1}{j} \right) \right) . \]  \hspace{1cm} (D.19)

Moreover, since \( \ln n! \) admits a complete expansion (neglecting the first terms) in powers of \( \frac{1}{n} \), also \( C_j \) can be completely expanded in powers of \( \frac{1}{j} \). The same is true for an expansion in \( l \).

**D.3 Theta graph**

In this section we explain the result that the theta graph \((C_{000}^{j_1j_2j_3})^2\) is equal to
\[ \frac{1}{2\pi S} \left( 1 + O \left( \frac{1}{l^2} \right) \right) . \]  \hspace{1cm} (D.20)

From [40] we have
\[ C_{000}^{j_1j_2j_3} = (-1)^g \frac{g!}{(g-j_1)!(g-j_2)!(g-j_3)!} \sqrt{\frac{(2g-2j_1)!(2g-2j_2)!(2g-2j_3)!}{(2g+1)!}} , \]  \hspace{1cm} (D.21)

where \( 2g = j_1 + j_2 + j_3 \). We compute the expansion of \( \ln(C_{000}^{j_1j_2j_3}) \) using the Stirling’s formula:
\[ \ln(n!) = n \ln n - n + \frac{1}{2} \ln n + \frac{1}{2} \ln 2\pi + \frac{1}{12n} + O(n^{-2}) , \]  \hspace{1cm} (D.22)

obtaining
\[ \ln \left( (-1)^g C_{000}^{j_1j_2j_3} \right) = -\frac{1}{4} \ln \left( \frac{2\pi^2}{16} (l_1 + l_2 + l_3)(-l_1 + l_2 + l_3)(l_1 - l_2 + l_3)(l_1 + l_2 - l_3) \right) \]
\[ + O(l^{-2}) . \]  \hspace{1cm} (D.23)

This is exactly
\[ -\frac{1}{4} \ln 4\pi^2 S^2 + O(l^{-2}) , \]  \hspace{1cm} (D.24)

where \( S \) is the area of the triangle with edge lengths \( l_i \). We conclude that the theta graph \((C_{000}^{j_1j_2j_3})^2\) is equal to
\[ \frac{1}{2\pi S} \left( 1 + O \left( \frac{1}{l^2} \right) \right) . \]  \hspace{1cm} (D.25)
D.4 Kinetic term in equilateral case

Let us introduce

$$M_\lambda = \begin{pmatrix}
0 & a & a & a & a & a \\
-\lambda & c - \lambda & c - \lambda & c - \lambda & c - \lambda & c - \lambda \\
-\lambda & c - \lambda & b - \lambda & c - \lambda & c - \lambda & c - \lambda \\
-\lambda & c - \lambda & c - \lambda & -\lambda & b - \lambda & c - \lambda \\
-\lambda & c - \lambda & c - \lambda & c - \lambda & b - \lambda & c - \lambda \\
-\lambda & c - \lambda & c - \lambda & c - \lambda & c - \lambda & -\lambda
\end{pmatrix},$$

(D.26)

where \( a = -\sqrt{\frac{64}{81}} \), \( b = \sqrt{\frac{3}{4}} \) and \( c = \frac{1}{2} \sqrt{3} \). In the equilateral case (all \( l \) equal to 1) the kinetic term is of the form

$$-i M_0.$$

(D.27)

Let us note that

$$\det M_\lambda = \det M_0 \neq 0,$$

(D.28)

and all \( M_\lambda \) are symmetric. Thus all of them have the same number of positive and negative eigenvalues.

Matrix \( M_\lambda \) for \( \lambda = c \) is similar (have the same determinant) by simultaneous permutation of rows and columns to the matrix

$$M' = \begin{pmatrix}
b - c & -c & 0 & 0 & 0 & 0 & a \\
-\lambda & b - c & 0 & 0 & 0 & 0 & a \\
0 & 0 & b - c & -c & 0 & 0 & a \\
0 & 0 & -c & b - c & 0 & 0 & a \\
0 & 0 & 0 & 0 & b - c & -c & a \\
0 & 0 & 0 & 0 & -c & b - c & a \\
a & a & a & a & a & a & 0
\end{pmatrix}.$$

(D.29)

The matrix \( M' \) restricted to its first 6 rows and columns has 3 positive and 3 negative eigenvalues. Applying the min-max principle [51] to \( M' \) and \(-M'\) shows that \( M' \) has at least three positive and three negative eigenvalues. Together with the fact that determinant is positive it shows that there are 4 positive and 3 negative eigenvalues.

Hence, the matrix of kinetic term has 4 \(-i \mathbb{R}_+\) eigenvalues and 3 \(i \mathbb{R}_+\) and the same is true for matrix \((-H)^{-1}\).

E Dupuis-Livine form and stationary points

In this section we will prove the following lemma:

Lemma 9. Suppose that the integral is of the form as

$$\int d\theta \left( \frac{i^n}{l^k} e^S \right),$$

(E.1)

where \( S(\theta) \) has an asymptotic expansion around the isolated stationary point of \( S_{-1}(\theta) \) of the form

$$S = S_{-1} + S_0 + S_1 + \ldots,$$

(E.2)

and \( i^n S_k \in \mathbb{R} \) is a homogeneous function of order \(-k\) in \( l \). Then the contribution to the expansion of the integral from this stationary point has the DL property.

Proof. Let us consider the contribution from the isolated stationary point of \( S_{-1} \). They are of the form

$$\frac{1}{\sqrt{H}} e^\tilde{S},$$

(E.3)

where \( \tilde{S} \) is given by the contraction of all connected Feynman diagrams. They are made up of vertices, given by the derivatives of \( S_{\geq 0} \), connected by the propagator \( H \), which is the inverse to \((-1)\) times the matrix of second derivatives of \( S_{-1} \).

$$H = (-\partial^2 S_{-1})^{-1}, \quad \mathcal{H} = \det (-\partial^2 S_{-1}).$$

(E.4)
Their contribution is computed by contracting the vertices $V_k$ with propagators $H$. Since vertices are obtained from derivatives of $S_m$, $m \geq 0$, the homogeneous degree $\text{deg} V_k$ of this vertex is thus $m$ and the matrix elements of $i^{\text{deg} V_k} V_k$ are real. Similarly $iH$ is a real matrix and is of degree 1.

To conclude, the total contraction is thus of degree

$$\sum_k \text{deg} V_k + n \quad ,$$

(E.5)

where $n$ is the number of propagators in the diagram. Moreover, the complete contraction multiplied by

$$i^{\sum_k \text{deg} V_k + n}$$

(E.6)

is again real as a contraction of real matrices. This proves that expansion is still of DL form.

\section*{F Stationary point analysis}

In the paper we use an advanced version of the stationary point analysis. This appendix is intended to explain the details of this method.

\textbf{Lemma 10.} Let $S(x) = i(S_{-1} + S_0) + \sum_{i>0} S_i$ be an asymptotic expansion of the action such that

- $S_i$ is of homogeneous degree $-i$ in $j$,
- $S_0$ and $S_{-1}$ are real
- $S_{-1} + S_0$ is homogeneous in $l = j + \frac{1}{2}$

and let $x_0$ be an isolated stationary point of $S_{-1}$. Then there is an asymptotic expansion of the contribution to the integral

$$\int dx e^{S} \quad (F.1)$$

from the neighbourhood of $x_0$ given as follows:

We can write the asymptotic expansion of $S$ in homogeneous terms in $l$ as

$$S = i\tilde{S}_{-1} + S'_0 + \sum_{i>0} \tilde{S}_i \quad ,$$

(F.2)

where $\tilde{S}_{-1} = S_{-1} + S_0$. Let $x_1$ be the stationary point of $\tilde{S}_{-1}$ obtained by perturbation of $x_0$ (there is exactly one such stationary point if the matrix of second derivatives of $\tilde{S}_{-1}$ is non-degenerate). The asymptotic expansion of the integral is equal to

$$\frac{1}{\sqrt{\det(-H)}} i^{\sum_{i>1} A_i} \quad (F.3)$$

where $H$ is the matrix of second derivatives of $\tilde{S}_{-1}$ and $A_{-1} = \tilde{S}_{-1}$ evaluated on $x_1$. The terms $A_i$ for $i \geq 0$ are homogeneous functions of order $-i$ in $l$ and can be obtained from the Feynman diagram expansion with the propagator $(-H)^{-1}$ and interaction vertices given by derivatives of $\tilde{S}_i$ for $i \geq 0$.

The same fact applies when the isolated point is replaced by the isolated orbit of the symmetry group of the action.

The second fact concerns with integration over only a part of the variables:

\textbf{Lemma 11.} Let $S(x,y) = iS_{-1}(x,y) + \sum_{i>0} S_i$ has an isolated stationary point $(x_0,y_0)$ with a non-degenerate matrix of second derivatives $H$ with the property

$$H = \begin{pmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{pmatrix}, \quad H_{yy} \text{ invertible} \quad .$$

(F.4)

Then there exists a function $y(x)$ such that (in the neighbourhood of stationary point)

$$\nabla_x S_{-1} + \frac{\partial y}{\partial x} \nabla_y S_{-1} = 0 \quad (F.5)$$

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and the asymptotic expansion of $\int dx \, e^S$ is equal to asymptotic expansion of

$$\int dx \, e^\tilde{S},$$

where $\tilde{S}$ is obtained by asymptotic expansion of the integral $e^\tilde{S} = \int dy \, e^\tilde{S}$.

G Feynman diagrams

In this subsection we are interested in the next to leading order in the expansion of the $6j$ symbol. We will derive expressions for $S_1$ in terms of Feynman diagrams. Vertices in this expansion consist of derivatives of

$$- \sum_i \frac{1}{2} \ln \sin \theta_{ij}, \quad \frac{i}{8l_{ij}} \cot \theta_{ij},$$

and higher than second derivatives of $|l| \rho \det \tilde{G}$ with respect to $\rho$ and $\theta_{ij}$. Each propagator contributes a weight $|l|^{-1}$. We only evaluate closed diagrams, so if the diagram is made up of vertices of valency $n_k$, i.e. the $n_k$-th derivative of a function with weight $|l|^{\alpha_k}$, then the scaling behaviour of the whole diagram is as

$$|l|^{-\sum_k (\alpha_k - \frac{n_k}{2})}.$$

The only vertices that can contribute up to order $|l|^{-1}$ are thus

| Vertex | $-\frac{1}{2} \ln \sin \theta_{ij}$ | $-\frac{1}{2} \cot \theta_{ij}$ | $-\frac{i}{2} \frac{\partial}{\partial \rho} \cot \theta_{ij}$ | $-\frac{1}{4} \ln \cot \theta_{ij}$ | $\frac{i}{|l|} \rho \det \tilde{G}$ | $\frac{i}{|l|} \frac{\partial^3}{\partial \rho^3} \rho \det \tilde{G}$ |
|--------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|---------------------------------|---------------------------------|
| Valency | 0 | 1 | 2 | 3 | 4 |
| Order   | $|l|^0$ | $|l|^{-1/2}$ | $|l|^{-1}$ | $|l|^{-1}$ | $|l|^{-1/2}$ | $|l|^{-1}$ |

Note that the only diagram that is real (up to the order $|l|^{-1}$) is just the first vertex (being of order 0). Furthermore, this is also the only contribution of order $|l|^0$. All other diagrams are purely imaginary and of order $|l|^{-1}$.

H Relation to spin-network kernel formula

We will prove that (for $j \in \mathbb{Z}$)

$$\int_0^\pi \frac{d\phi_1}{\pi} \left( e^{i\theta} \cos \phi_1 \cos \phi_2 + e^{-i\theta} \sin \phi_1 \sin \phi_2 \right)^{2j} = \frac{1}{4^j} \binom{2j}{j} \left( e^{i2\theta} \cos^2 \phi_2 + e^{-i2\theta} \sin^2 \phi_2 \right)^j.$$

(H.1)

It is straightforward to check that

$$e^{i2\theta} \cos^2 \phi_2 + e^{-i2\theta} \sin^2 \phi_2 = \cos 2\theta + i \sin 2\theta \cos 2\phi_2.$$

(H.2)

In this way we obtain the formula from [29]. To prove (H.1), we use the following formulas:

$$2 \int_0^{\pi/2} d\phi \ \sin 2\alpha \phi \cos 2\beta \phi = \frac{\Gamma(\alpha + \frac{1}{2}) \Gamma(\beta + \frac{1}{2})}{\Gamma(\alpha + \beta + 1)},$$

(H.3)

$$\Gamma(n + 1) = n!, \quad \Gamma \left( n + \frac{1}{2} \right) = \frac{(2n)!}{4^n n! \sqrt{\pi}}.$$  

(H.4)

In the case that $k, l \in \mathbb{N}$, these formulas can be simplified to:

$$\int_0^\pi d\phi \ \sin 2k \phi \cos 2l \phi = \frac{(2k)! (2l)! \pi}{4^{k+l} k! l! (k + l)!},$$

(H.5)

$$\int_0^\pi d\phi \ \sin 2k+1 \phi \cos 2l+1 \phi = 0.$$  

(H.6)
Expanding the left hand side of (H.1) and using these formulas we have:

\[ \int_0^\pi \frac{d\phi_1}{\pi} \left( e^{i\theta} \cos \phi_1 \cos \phi_2 + e^{-i\theta} \sin \phi_1 \sin \phi_2 \right)^{2j} = \quad \text{(H.7)} \]

\[ \sum_{n=0}^{2j} \binom{2j}{n} e^{i(2j-2n)\theta} \cos^{2j-n} \phi_1 \cos^n \phi_2 \int_0^\pi \frac{d\phi_1}{\pi} \cos^{2j-n} \phi \sin^n \phi \quad \text{.} \quad \text{(H.8)} \]

This is equal to \((k := 2n)\)

\[ \sum_{k=0}^{j} \binom{2j}{2k} e^{i(j-2k)\theta} \cos^{2j-2k} \phi_1 \sin^{2k} \phi_2 \frac{1}{\pi} \frac{(2k)!(2j-2k)!}{4^k!(j-k)!j!} . \quad \text{(H.9)} \]

The factors in \(j\) and \(k\) can be rewritten in terms of binomial coefficients

\[ \binom{2j}{2k} \frac{(2j-2k)!}{4^k!(j-k)!j!} = \frac{(2j)!}{4^k!(j-k)!j!} = \frac{1}{4^j} \binom{2j}{j} \binom{j}{k} , \quad \text{(H.10)} \]

such that we obtain the final result:

\[ \frac{1}{4^j} \binom{2j}{j} \sum_{k=0}^{j} \binom{j}{k} \left( e^{i\theta} \cos \phi_2 \right)^{j-k} \left( e^{-i\theta} \sin \phi_2 \right)^k = \]

\[ = \frac{1}{4^j} C_j \left( e^{i\theta} \cos \phi_2 + e^{-i\theta} \sin \phi_2 \right)^j . \quad \text{(H.11)} \]

This explains the occurrence of \(C_j\) in our formulas, which is absent in integral kernel approach \([29]\).
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